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Some properties on the surfaces of vector fields and its application to the Stokes and Navier–Stokes problems with mixed boundary conditions

Tujin Kim [a,*], Daomin Cao [b]

[a] Institute of Mathematics, Academy of Sciences, Pyongyang, Democratic People's Republic of Korea
[b] Institute of Applied Mathematics, AMSS, Chinese Academy of Sciences, Beijing 100190, PR China

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ABSTRACT

In this paper we are concerned with the stationary and non-stationary Stokes and Navier–Stokes problems with mixed boundary conditions involving velocity, pressure, rotation, stress and normal derivative of velocity together. Relying on the relations among strain, rotation, normal derivative of velocity and shape of boundary surface, we get variational formulations of the Stokes and Navier–Stokes problems with the mixed boundary conditions. Also, we study existence and uniqueness of solutions to the problems.

1. Introduction

As mathematical models of flows of incompressible viscous Newtonian fluids the Stokes equations

$$-\nu\Delta v+\nabla p=f,\qquad \nabla\cdot v=0\quad\text{in }\Omega,\tag{1.1}$$

$$\frac{\partial v}{\partial t}-\nu\Delta v+\nabla p=f,\qquad \nabla\cdot v=0\quad\text{in }\Omega\tag{1.2}$$

and Navier–Stokes equations

$$-\nu\Delta v+(v\cdot\nabla)v+\nabla p=f,\qquad \nabla\cdot v=0\quad\text{in }\Omega,\tag{1.3}$$

$$\frac{\partial v}{\partial t}-\nu\Delta v+(v\cdot\nabla)v+\nabla p=f,\qquad \nabla\cdot v=0\quad\text{in }\Omega\tag{1.4}$$

* Corresponding author. Tel.: +850 23814410.
E-mail addresses: arirangip@star-co.net.kp (T. Kim), dmcao@amt.ac.cn (D. Cao).

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are used. For these systems, different natural and artificial boundary conditions are considered. For example on solid walls, homogeneous Dirichlet condition $v = 0$ is used. On a free surface a Neumann condition $2\nu\varepsilon(v)n - pn = 0$ may be useful (cf. [28] and references therein). Here and in what follows $\varepsilon(v)$ denotes the so-called strain tensor with the components $\varepsilon_{ij}(v) = \frac{1}{2}(\partial_{x_i}v_j + \partial_{x_j}v_i)$ and n is the outward normal unit vector.

The Navier slip-with-friction boundary conditions $v \cdot n = 0$, $[\varepsilon(v)n + \alpha v] \cdot \tau = 0$, where and in what follows τ is vector tangent to the boundary, are also used for simulations of flows in the presence of rough boundaries (cf. [47,48,60] and references therein). Combination of the normal component of the velocity and the tangential component of the friction (slip condition for uncovered fluid surfaces) or the tangential component of the velocity and the normal component of the friction (condition for in/out-stream surfaces) (cf. [61]) are frequently used. At the outlet of a channel the boundary condition $\nu\frac{\partial v}{\partial n} - pn = \sigma$ (cf. [20–22,52,53] and references therein) or $\nu\varepsilon(v)n - pn = 0$ (cf. [54,53] and references therein) also is used. Rotation boundary condition has been fairly extensively studied over the past several years. (cf. [14–19,25,43]). Of course, on flat portions of the boundary the rotation boundary condition $v_n|_\Gamma = 0$, $\operatorname{rot} v \times n|_{\Gamma_3} = \phi/\nu$ and the Navier slip condition $v_n|_\Gamma = 0$, $\nu\varepsilon_{n\tau}(v)|_\Gamma = \phi$ are equivalent. (cf. Section 6 in [15], Remark 1.1 in [16]). Also, on a part of boundary one deals with the total pressure (Bernoulli's pressure) $\frac{1}{2}|v|^2 + p$ (cf. [23,24]) or static pressure p (cf. [4,59]). It is also known that the total stress $\sigma^{\text{tot}}(v, p) \cdot n$ on the boundary is a natural boundary condition, where $\sigma^{\text{tot}}(v, p) = -\left(p + \frac{1}{2}|v|^2\right) + 2\nu\varepsilon(v)$ (see [38,39]).

In practice we deal with a mixture of some kind of boundary conditions. There are vast literatures for the Stokes and Navier–Stokes problem with mixed boundary conditions and several variational formulations are used for them.

First, we briefly outline main points of variational formulations for the Stokes and Navier–Stokes problems with mixed boundary conditions.

In the case of mixed boundary conditions of Dirichlet condition and $\nu\frac{\partial v}{\partial n} - pn = \sigma$, for variational formulation of the problem the bilinear form

$$a(v, u) = (\nabla v, \nabla u)_{\mathbf{L}_2(\Omega)} \quad \text{for } v, u \in \mathbf{H}^1(\Omega), \tag{1.5}$$

which is reduced from $-(\Delta v, u)_{\mathbf{L}_2(\Omega)}$ by integration by parts, is used (cf. [9,21,54,56,67]).

When one deals with the mixed boundary conditions of velocity and the component of strain $\varepsilon(u)n$, for variational formulation of the problem the bilinear form

$$a(v, u) = 2\Sigma_{i,j}(\varepsilon_{ij}(v), \varepsilon_{ij}(u))_{L_2(\Omega)} \quad \text{for } v, u \in \mathbf{H}^1(\Omega) \tag{1.6}$$

is used and Korn's inequality is on the base, because in the process of change from $-(\Delta v, u)_{\mathbf{L}_2(\Omega)}$ to $2\Sigma_{i,j}(\varepsilon_{ij}(v), \varepsilon_{ij}(u))_{L_2(\Omega)}$ components of strain are reflected on the boundary integral (see (2.16)). In this way different problems are studied (cf. [7,8,10,34,36,38–42,55,61,62,66]). In [36] another equivalent variational formulation, where strain, pressure, velocity and rotation are unknown functions, also is given.

On the other hand, when one deals with the mixed boundary conditions of the Dirichlet condition and the condition for pressure or rotation on a part of boundary, the bilinear form

$$a(v, u) = (\operatorname{rot} v, \operatorname{rot} u)_{\mathbf{L}_2(\Omega)} \quad \text{for } v, u \in \mathbf{H}^1(\Omega) \tag{1.7}$$

is used for variational formulation of the problem. Because, when $-(\Delta v, u)_{\mathbf{L}_2(\Omega)}$ is reduced to (1.7), is reflected rotation in the boundary integral on the part where flow is tangent and on the part where flow is orthogonal to the boundary the boundary integral disappear, and so by integration by parts $(\nabla p, u)$ pressure is reflected in variational formulation (cf. (2.13), (3.6)). In this case equivalence between the norms $\|v\|_{\mathbf{H}^1(\Omega)}$ and $\|\operatorname{rot} v\|_{\mathbf{L}_2(\Omega)}$ under some conditions (see Theorem A.1 in [31], Lemma 2 in [49]) is on the base and in this way many problems are studied (cf. [1,6,12,13,23–25,29,31,32,49,58,59]). In Section 1 of [29], the bilinear form (1.5) instead of (1.7) is used since two bilinear forms (1.5) and (1.7) for polygon or polyhedral domain under some boundary conditions are equal (cf. [33]). When one deals with the boundary condition for pressure or rotation on a part of boundary, there are other variational formulations using three unknown functions v, p and ω, where $\omega = \operatorname{rot} v$, (cf. [5,2–4]) for the 2-dimensional case and v, p and vector potential for the 3-dimensional case (cf. [44]).

No one considered mixed boundary problems for the Stokes and Navier–Stokes equations with stress and pressure conditions together.

In the present paper, we are concerned with the systems (1.1)–(1.4) with mixed boundary conditions involving Dirichlet, pressure, rotation, stress and normal derivative of velocity together.

Mainly relying on the bilinear form (1.6), we reflect all these boundary conditions into variational formulations of problems. Using the variational formulations we prove existence and uniqueness of weak solutions to boundary value problems under conditions for boundary surface. For the initial boundary value problems we prove existence of weak solutions to the problems without assumption for form of boundary surfaces.

This paper consists of 5 sections and an Appendix.

In Section 2, relations among strain, rotation, normal derivative of vector field and shape of boundary surface are considered. Theorem 2.1 gives relations among strain, rotation of vector fields given near a surface and the shape operator of surface, i.e. the matrix in the second fundamental form of surface, (the curvature of boundary for 2-D case), when the vector fields are tangent to the surface. The relation (2.1) of Theorem 2.1, a generalization for 3-D case of Lemma 2.1 in [30] for 2-D, is known (cf. [14]). Theorem 2.2 gives a relation among strain, normal derivative, divergence of vector fields and the mean curvature of the surface when the vector fields are normal to the surface. In the case of divergence free vector fields Theorem 2.2 implies Theorem 1 in [11], but our proof is very simple and different from one in [11]. As a corollary we get equalities of three bilinear forms for vector fields v, u when the vector fields are tangent or orthogonal to parts of boundary which are pieces of plane or straight line and on the other parts of boundary are equal to zero. This implies equalities of the bilinear forms (1.5)–(1.7) for divergence free vector fields under such conditions.

In Section 3, first, the variational formulations based on the bilinear form (1.6) for the stationary Stokes and Navier–Stokes problems with a mixture of 7 kinds of boundary conditions are considered. Also, as an example, one based on the bilinear form (1.5) for the Stokes problem with a mixture of 6 kinds of boundary conditions is given.

In Section 4 using the variational formulations based on the bilinear form (1.6), we study existence and uniqueness of solutions to the stationary Stokes and Navier–Stokes problems under convexity assumption for the connected components of boundary given the conditions for rotation, pressure and normal derivative of velocity (Theorems 4.1–4.4). Relying to the variational formulation based on the bilinear form (1.5), under convexity assumption for the connected components of boundary given the conditions for rotation, pressure and concavity assumption of the connected components of boundary for the Navier slip-with-friction condition, we can get similar results (cf. Theorem 4.5, Remark 4.3).

In Section 5, we study first existence of solutions to the time dependent Navier–Stokes problem with 7 kinds of boundary condition including total pressure, total stress and so on. Using the variational formulations based on the bilinear form (1.6), by elliptic regularization and a transformation of unknown functions we connect the problem with elliptic problem. In this way we prove existence of solutions to the time dependent Navier–Stokes problems without convexity assumption for the connected components of boundary given the conditions for rotation, pressure and normal derivative of velocity (Theorem 5.1). For Stokes problem unique existence of solution is proved (Theorem 5.5).

In the Appendix for convenience of readers we consider the bilinear form by the shape operator used in previous sections.

Throughout this paper we will use the following notation.

Let Ω be a connected bounded open subset of R^l, $l = 2, 3$. $\partial\Omega \in C^{0,1}$, $\partial\Omega = \cup_{i=1}^{N} \overline{\Gamma}_i$, $\Gamma_i \cap \Gamma_j = \varnothing$ for $i \neq j$, $\Gamma_i = \bigcup_j \Gamma_{ij}$, where Γ_{ij} are connected open subsets of $\partial\Omega$ and $\Gamma_{ij} \in C^2$. Let $n(x)$ and $\tau(x)$ be, respectively, outward normal and tangent unit vectors at x in $\partial\Omega$. When X is a Banach space, $\mathbf{X} = X^l$. Let $W_\alpha^k(\Omega)$ be Sobolev spaces, $H^1(\Omega) = W_2^1(\Omega)$, and so $\mathbf{H}^1(\Omega) = \{H^1(\Omega)\}^l$.

An inner product and norm in the space $\mathbf{L}_2(\Omega)$ is denoted by $(\cdot\,,\cdot)$ and $\|\cdot\|$; and $\langle\cdot\,,\cdot\rangle$ means the duality pairing between a Sobolev space X and its dual one. Also, $(\cdot\,,\cdot)_{\Gamma_i}$ is an inner product in the $\mathbf{L}_2(\Gamma_i)$ or $L_2(\Gamma_i)$; and $\langle\cdot\,,\cdot\rangle_{\Gamma_i}$ means the duality pairing between $\mathbf{H}^{\frac{1}{2}}(\Gamma_i)$ and $\mathbf{H}^{-\frac{1}{2}}(\Gamma_i)$ or between $H^{\frac{1}{2}}(\Gamma_i)$ and $H^{-\frac{1}{2}}(\Gamma_i)$. The inner product and norms in R^l, respectively, are denoted by $(\cdot\,,\cdot)_{R^l}$ and $|\cdot|$. Sometimes $a \cdot b$ or ab is used for inner product in R^l between a and b.

For convenience, in the case that $l = 2$, $y = (y_1(x_1, x_2),\ y_2(x_1, x_2))$ is identified with $\bar{y} = (y_1,\ y_2,\ 0)$, and so $\operatorname{rot} y = \operatorname{rot} \bar{y}$. Thus, for $y = (y_1,\ y_2)$ and $v = (v_1,\ v_2)$, $\operatorname{rot} y \times v$ is the 2-D vector consisted of the first two components of $\operatorname{rot} \bar{y} \times \bar{v}$.

2. Properties on the boundary of vector fields

Let Γ be a surface (curve for $l = 2$) of C^2 and v be a vector field of C^2 on a domain of R^l near Γ. In this paper the surfaces concerned by us are pieces of boundary of 3-D or 2-D bounded connected domains, and so we can assume the surfaces are oriented.

Theorem 2.1. *Suppose that $v \cdot n|_\Gamma = 0$. Then, on the surface Γ the following holds.*

$$(\varepsilon(v)n, \tau)_{R^l} = \frac{1}{2}(rot\ v \times n, \tau)_{R^l} - (S\tilde{v}, \tilde{\tau})_{R^{l-1}}, \tag{2.1}$$

$$(rot\ v \times n, \tau)_{R^l} = \left(\frac{\partial v}{\partial n}, \tau\right)_{R^l} + (S\tilde{v}, \tilde{\tau})_{R^{l-1}}, \tag{2.2}$$

$$(\varepsilon(v)n, \tau)_{R^l} = \frac{1}{2}\left(\frac{\partial v}{\partial n}, \tau\right)_{R^l} - \frac{1}{2}(S\tilde{v}, \tilde{\tau})_{R^{l-1}}, \tag{2.3}$$

where $\varepsilon(v)$ denotes the matrix with the components $\varepsilon_{ij}(v)$, S is the shape operator of the surface Γ (the matrix (A.1)) for $l = 3$ and the curvature of Γ for $l = 2$, and $\tilde{v}, \tilde{\tau}$ are expressions of the vectors v, τ in a local curvilinear coordinates on Γ.

Proof. As a generalization of Lemma 2.1 in [30] for 2-D, formula (2.1) is known in the form

$$2\nu\,(\varepsilon(v)n, \tau)_{R^l} = \nu(\operatorname{rot} v \times n, \tau)_{R^l} - 2\nu v \cdot \frac{\partial n}{\partial \tau}$$

(cf. [14], (6.3) in [15]). Since $v \cdot \frac{\partial n}{\partial \tau} = (S\tilde{v}, \tilde{\tau})_{R^{l-1}}$ (cf. (A.6)), we use the expression above.

Let $J(v) = (\frac{\partial v_j}{\partial x_i})$ for vector field $v(x) = (v_1(x), v_2(x), v_3(x))$ denoted in orthogonal coordinates x.

Since

$$(J(v)^T - J(v))n = \text{rot } v \times n, \qquad J(v)^T n = \frac{\partial v}{\partial n},$$

we get

$$(\text{rot } v \times n, \tau) = \left(\frac{\partial v}{\partial n}, \tau\right) - (J(v)n, \tau). \tag{2.4}$$

On the other hand, since $\Gamma \in C^2$, $n(x) \in C^1$. Let us make a vector field on a neighborhood of Γ with thickness small enough, vectors of which equal to $n(x)$ on the normal line at $x \in \Gamma$. Denote the vector field again by $n(x)$. Then, from $v(x) \cdot n(x)|_\Gamma = 0$, we know

$$\tau \cdot \nabla(v \cdot n)|_\Gamma = 0. \tag{2.5}$$

$\nabla(v \cdot n) = J(v)n + J(n)v$, and from (2.5) we get

$$\tau \cdot J(v)n = -\tau \cdot J(n)v \quad \text{on } \Gamma. \tag{2.6}$$

Since

$$\tau \cdot J(n)v = (S\tilde{v}, \tilde{\tau})_{R^{l-1}} \quad \text{on } \Gamma \tag{2.7}$$

(cf. Lemma A.1), from (2.4), (2.6), (2.7) we have (2.2). Formulas (2.1) and (2.2) imply (2.3). □

Theorem 2.2. *On the surface Γ the following holds.*

$$(\varepsilon(v)n, n)_{R^l} = \left(\frac{\partial v}{\partial n}, n\right)_{R^l}. \tag{2.8}$$

If $v \cdot \tau|_\Gamma = 0$, then

$$(\varepsilon(v)n, n)_{R^l} = \left(\frac{\partial v}{\partial n}, n\right)_{R^l} = -(k(x)v, n)_{R^l} - \text{div}_\Gamma\, v_\tau + \text{div}\, v, \tag{2.9}$$

where $k(x) = \text{div}\, n(x)$, v_τ is the tangential component of v and div_Γ is the divergence of a tangential vector field in the tangential coordinate system on Γ.

Proof. Let $n(x)$ be the vector field expanded on a domain near Γ as the proof of Theorem 2.1. Since

$$\varepsilon(v)n = \frac{1}{2}J(v)^T n + \frac{1}{2}J(v)n = \frac{1}{2}\frac{\partial v}{\partial n} + \frac{1}{2}J(v)n,$$
$$(J(v)n, n)_{R^l} = \sum_{j=1}^{l}\sum_{i=1}^{l}\frac{\partial v_j}{\partial x_i}n_j n_i = \left(\frac{\partial v}{\partial n}, n\right)_{R^l},$$

we have (2.8). By Lemma 7 in [50]

$$\left(\frac{\partial v}{\partial n}, n\right)_{R^l} + (k(x)v, n)_{R^l} + \text{div}_\Gamma\, v_\tau = \text{div}\, v \quad \text{on } \Gamma. \tag{2.10}$$

From (2.8) and (2.10) we have (2.9). □

Remark 2.1. $k(x) = \text{div}\, n(x) = Tr(S(x)) = 2 \cdot$ mean curvature.
Since $\partial\Omega \in C^{0,1}$, $\Gamma \in C^2$, elements of S belong to $C(\bar{\Gamma})$ and so is $k(x)$.

Let $\partial\Omega = \cup_{i=1}^3 \bar{\Gamma}_i$ and $\mathbf{H}^1_\Gamma(\Omega) = \{u \in \mathbf{H}^1(\Omega) : u|_{\Gamma_1} = 0, u \cdot \tau|_{\Gamma_2} = 0, u \cdot n|_{\Gamma_3} = 0\}$.

Corollary 2.3. *Assume that Γ_{ij} for $i = 2, 3$ are pieces of plain (straight segments for 2-D). Then,*

$$(\nabla v, \nabla u) = (rot\ v, rot\ u) + (\text{div}\, v, \text{div}\, u) = 2(\varepsilon(v), \varepsilon(u)) - (\text{div}\, v, \text{div}\, u) \quad \forall v, u \in \mathbf{H}^1_\Gamma(\Omega), \tag{2.11}$$

where $(\varepsilon(v), \varepsilon(u)) = \sum_{ij}(\varepsilon_{ij}(u), \varepsilon_{ij}(u))$.

Proof. By density of smooth functions, it is enough to prove for $v, u \in C^2(\bar{\Omega})$.

By integration by parts, we get

$$-(\Delta v, u)_\Omega = (\nabla v, \nabla u) - \left(\frac{\partial v}{\partial n}, u\right)_{\Gamma_2\cup\Gamma_3}. \tag{2.12}$$

On the other hand, using the facts that

$$-\Delta v = \mathrm{rot}\,\mathrm{rot}\, v - \mathrm{grad}(\mathrm{div}\, v),$$
$$(\mathrm{rot}\, v, u) - (v, \mathrm{rot}\, u) = -(v \times n, u)_{\partial\Omega},$$

we get

$$\begin{aligned} -(\Delta v, u)_\Omega &= (\mathrm{rot}\, v, \mathrm{rot}\, u)_\Omega + (\mathrm{div}\, v, \mathrm{div}\, u) - (\mathrm{rot}\, v \times n, u)_{\Gamma_2\cup\Gamma_3} - (\mathrm{div}\, v, u\cdot n)_{\Gamma_2\cup\Gamma_3} \\ &= (\mathrm{rot}\, v, \mathrm{rot}\, u)_\Omega + (\mathrm{div}\, v, \mathrm{div}\, u) - (\mathrm{rot}\, v \times n, u)_{\Gamma_3} - (\mathrm{div}\, v, u\cdot n)_{\Gamma_2}, \end{aligned} \tag{2.13}$$

where $u\cdot\tau\,|_{\Gamma_2} = 0$, $u\cdot n\,|_{\Gamma_3} = 0$ were used. Since $S \equiv 0$ on Γ_3, by (2.2)

$$(\mathrm{rot}\, v \times n, u)_{\Gamma_3} = \left(\frac{\partial v}{\partial n}, u\right)_{\Gamma_3}. \tag{2.14}$$

Since $k(x) = \mathrm{div}\, n(x) \equiv 0$ on Γ_2 and $v\cdot\tau|_{\Gamma_2} = 0$, by multiplying (2.10) with $(u\cdot n)$ we get

$$(\mathrm{div}\, v, u\cdot n)_{\Gamma_2} = \left(\frac{\partial v}{\partial n}, u\right)_{\Gamma_2}. \tag{2.15}$$

From (2.12)–(2.15), the first equality of (2.11) follows.

Also, using

$$\Delta v_i = 2\mathrm{div}\, \varepsilon_i(v) - \mathrm{div}\,\frac{\partial v}{\partial x_i}, \quad \text{where } \varepsilon_i(v) = (\varepsilon_{i1}(v), \ldots, \varepsilon_{il}(v)),$$
$$(\Delta v_i, u_i) = -2(\varepsilon_i(v), \nabla u_i) + 2(\varepsilon_i(v)\cdot n, u_i)_{\partial\Omega} - \left(\mathrm{div}\,\frac{\partial v}{\partial x_i}, u_i\right),$$

by the fact that the tensor $\varepsilon(v)$ is symmetric we have

$$\begin{aligned} -(\Delta v, u)_\Omega &= 2(\varepsilon(v), \varepsilon(u)) + (\nabla(\mathrm{div}\, v), u) - 2(\varepsilon(v)n, u)_{\Gamma_2\cup\Gamma_3} \\ &= 2(\varepsilon(v), \varepsilon(u)) - (\mathrm{div}\, v, \mathrm{div}\, u) - 2(\varepsilon(v)n, u)_{\Gamma_2\cup\Gamma_3} + (\mathrm{div}\, v, u\cdot n)_{\Gamma_2}. \end{aligned} \tag{2.16}$$

By (2.3) and (2.8), we have

$$-2(\varepsilon(v)n, u)_{\Gamma_3} = -\left(\frac{\partial v}{\partial n}, u\right)_{\Gamma_3} \tag{2.17}$$

and

$$-2(\varepsilon(v)n, u)_{\Gamma_2} = -2\left(\frac{\partial v}{\partial n}, u\right)_{\Gamma_2}. \tag{2.18}$$

From (2.15)–(2.18), we have

$$-(\Delta v, u)_\Omega = 2(\varepsilon(v), \varepsilon(u)) - (\mathrm{div}\, v, \mathrm{div}\, u) - \left(\frac{\partial v}{\partial n}, u\right)_{\Gamma_2\cup\Gamma_3}. \tag{2.19}$$

Thus, (2.19) and (2.12) imply the second equality of (2.11). □

Remark 2.2. For polygon or polyhedron the first equality of (2.11) follows from Theorem 4.1 in [33].

3. Variational formulation of the stationary Stokes and Navier–Stokes problems with mixed boundary conditions

We consider the Stokes equations

$$-\nu\Delta v + \nabla p = f, \qquad \nabla\cdot v = 0 \quad \text{in } \Omega \tag{3.1}$$

and the Navier–Stokes equations

$$-\nu\Delta v + (v\cdot\nabla)v + \nabla p = f, \qquad \nabla\cdot v = 0 \quad \text{in } \Omega \tag{3.2}$$

with the following boundary conditions

$$\begin{aligned}
&(1)\ v|_{\Gamma_1} = h_1,\\
&(2)\ v_\tau|_{\Gamma_2} = 0, \qquad -p|_{\Gamma_2} = \phi_2,\\
&(3)\ v_n|_{\Gamma_3} = 0, \qquad \operatorname{rot} v \times n|_{\Gamma_3} = \phi_3/\nu,\\
&(4)\ v_\tau|_{\Gamma_4} = h_4, \qquad (-p + 2\nu\varepsilon_{nn}(v))|_{\Gamma_4} = \phi_4,\\
&(5)\ v_n|_{\Gamma_5} = h_5, \qquad 2(\nu\varepsilon_{n\tau}(v) + \alpha v_\tau)|_{\Gamma_5} = \phi_5, \quad \alpha : \text{a matrix},\\
&(6)\ (-pn + 2\nu\varepsilon_n(v))|_{\Gamma_6} = \phi_6,\\
&(7)\ v_\tau|_{\Gamma_7} = 0, \qquad \left(-pn + \nu\frac{\partial v}{\partial n}\right)\Big|_{\Gamma_7} = \phi_7,
\end{aligned} \tag{3.3}$$

or

$$\begin{aligned}
&(1)\ v|_{\Gamma_1} = h_1,\\
&(2)\ v_\tau|_{\Gamma_2} = 0, \qquad -\left(p + \frac{1}{2}|v|^2\right)\Big|_{\Gamma_2} = \phi_2,\\
&(3)\ v_n|_{\Gamma_3} = 0, \qquad \operatorname{rot} v \times n|_{\Gamma_3} = \phi_3/\nu,\\
&(4)\ v_\tau|_{\Gamma_4} = h_4, \qquad \left(-p - \frac{1}{2}|v|^2 + 2\nu\varepsilon_{nn}(v)\right)\Big|_{\Gamma_4} = \phi_4,\\
&(5)\ v_n|_{\Gamma_5} = h_5, \qquad 2(\nu\varepsilon_{n\tau}(v) + \alpha v_\tau)|_{\Gamma_5} = \phi_5, \quad \alpha : \text{a matrix},\\
&(6)\ \left(-pn - \frac{1}{2}|v|^2 n + 2\nu\varepsilon_n(v)\right)\Big|_{\Gamma_6} = \phi_6,\\
&(7)\ v_\tau|_{\Gamma_7} = 0, \qquad \left(-pn - \frac{1}{2}|v|^2 n + \nu\frac{\partial v}{\partial n}\right)\Big|_{\Gamma_7} = \phi_7,
\end{aligned} \tag{3.4}$$

where $v_n = v \cdot n$, $v_\tau = v - (v \cdot n)n$, $\varepsilon_n(v) = \varepsilon(v)n$, $\varepsilon_{nn}(v) = (\varepsilon(v)n, n)_{R^l}$, $\varepsilon_{n\tau}(v) = \varepsilon(v)n - \varepsilon_{nn}(v)n$ and h_i, ϕ_i, α_{ij} (components of matrix α) are given functions or vectors of functions.

Let $\partial\Omega = \cup_{i=1}^{7} \bar{\Gamma}_i$, $\mathbf{V}(\Omega) = \{u \in \mathbf{H}^1(\Omega) : \operatorname{div} u = 0,\ u|_{\Gamma_1} = 0,\ u_\tau|_{(\Gamma_2\cup\Gamma_4\cup\Gamma_7)} = 0,\ u \cdot n|_{(\Gamma_3\cup\Gamma_5)} = 0\}$ and $\mathbf{V}_{\Gamma 237}(\Omega) = \{u \in \mathbf{H}^1(\Omega) : \operatorname{div} u = 0,\ u_\tau|_{(\Gamma_2\cup\Gamma_7)} = 0,\ u \cdot n|_{\Gamma_3} = 0\}$.

First, let us consider variational formulations based on the bilinear form $(\varepsilon(v), \varepsilon(u))$.

By Theorems 2.1 and 2.2 for $v \in \mathbf{H}^2(\Omega) \cap \mathbf{V}_{\Gamma 237}(\Omega)$ and $u \in \mathbf{V}(\Omega)$

$$\begin{aligned}
-(\Delta v, u) &= 2(\varepsilon(v), \varepsilon(u)) - 2(\varepsilon(v)n, u)_{\cup_{i=2}^{7}\Gamma_i}\\
&= 2(\varepsilon(v), \varepsilon(u)) + 2(k(x)v, u)_{\Gamma_2} - (\operatorname{rot} v \times n, u)_{\Gamma_3} + 2(S\tilde{v}, \tilde{u})_{\Gamma_3} - 2(\varepsilon_n(v), u)_{\cup_{i=4}^{7}\Gamma_i}\\
&= 2(\varepsilon(v), \varepsilon(u)) + 2(k(x)v, u)_{\Gamma_2} - (\operatorname{rot} v \times u, u)_{\Gamma_3} + 2(S\tilde{v}, \tilde{u})_{\Gamma_3}\\
&\quad - 2(\varepsilon_{nn}(v), u \cdot n)_{\Gamma_4} - 2(\varepsilon_{n\tau}(v), u)_{\Gamma_5} - 2(\varepsilon_n(v), u)_{\Gamma_6} - \left(\frac{\partial v}{\partial n}, u\right)_{\Gamma_7} + (k(x)v, u)_{\Gamma_7}.
\end{aligned} \tag{3.5}$$

Also, for $p \in H^1(\Omega)$ and $u \in \mathbf{V}(\Omega)$ we have

$$(\nabla p, u) = (p, u \cdot n)_{\cup_{i=2}^{7}\Gamma_i} = (p, u \cdot n)_{\Gamma_2} + (p, u \cdot n)_{\Gamma_4} + (pn, u)_{\Gamma_6\cup\Gamma_7}, \tag{3.6}$$

where $u \cdot n|_{\Gamma_3\cup\Gamma_5} = 0$ was used.

Assume that the following holds.

Assumption 3.1. There exists a function $U \in \mathbf{H}^1(\Omega)$ such that

$$\operatorname{div} U = 0, \qquad U|_{\Gamma_1} = h_1, \qquad U_\tau|_{(\Gamma_2\cup\Gamma_7)} = 0, \qquad U \cdot n|_{\Gamma_3} = 0, \qquad U_\tau|_{\Gamma_4} = h_4, \qquad U \cdot n|_{\Gamma_5} = h_5.$$

Also, $f \in \mathbf{V}(\Omega)^*$, $\phi_i \in H^{-\frac{1}{2}}(\Gamma_i)$, $i = 2, 4$, $\phi_i \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_i)$, $i = 3, 5, 6, 7$, $\alpha_{ij} \in L_\infty(\Gamma_5)$, and $\Gamma_1 \neq \varnothing$.

Then, using (3.5), (3.6) we get a variational formulation for problem (3.1), (3.3):

Formulation 3.1. *Find v such that*

$$\begin{aligned}
&v - U \in \mathbf{V}(\Omega),\\
&2\nu(\varepsilon(v), \varepsilon(u)) + 2\nu(k(x)v, u)_{\Gamma_2} + 2\nu(S\tilde{v}, \tilde{u})_{\Gamma_3} + 2(\alpha(x)v, u)_{\Gamma_5} + \nu(k(x)v, u)_{\Gamma_7}\\
&\quad = \langle f, u\rangle + \sum_{i=2,4}\langle\phi_i, u \cdot n\rangle_{\Gamma_i} + \sum_{i=3,5,6,7}\langle\phi_i, u\rangle_{\Gamma_i} \quad \forall u \in \mathbf{V}(\Omega).
\end{aligned} \tag{3.7}$$

Also, we get a variational formulation for problem (3.2), (3.3):

Formulation 3.2. *Find v such that*

$$
\begin{aligned}
& v-U \in \mathbf{V}(\Omega), \\
& 2\nu(\varepsilon(v), \varepsilon(u)) + \langle (v\cdot\nabla)v, u\rangle + 2\nu(k(x)v, u)_{\Gamma_2} + 2\nu(S\tilde{v}, \tilde{u})_{\Gamma_3} + 2(\alpha(x)v, u)_{\Gamma_5} + \nu(k(x)v, u)_{\Gamma_7} \\
& \quad = \langle f, u\rangle + \sum_{i=2,4}\langle \phi_i, u\cdot n\rangle_{\Gamma_i} + \sum_{i=3,5,6,7}\langle \phi_i, u\rangle_{\Gamma_i} \quad \forall u \in \mathbf{V}(\Omega).
\end{aligned}
\tag{3.8}
$$

On the other hand, taking $(v\cdot\nabla)v = \text{rot}\, v\times v + \frac{1}{2}\text{grad}|v|^2$ into account, we get a variational formulation for problem (3.2), (3.4):

Formulation 3.3. *Find v such that*

$$
\begin{aligned}
& v-U \in \mathbf{V}(\Omega), \\
& 2\nu(\varepsilon(v), \varepsilon(u)) + \langle \text{rot}\ v\times v, u\rangle + 2\nu(k(x)v, u)_{\Gamma_2} + 2\nu(S\tilde{v}, \tilde{u})_{\Gamma_3} + 2(\alpha(x)v, u)_{\Gamma_5} + \nu(k(x)v, u)_{\Gamma_7} \\
& \quad = \langle f, u\rangle + \sum_{i=2,4}\langle \phi_i, u\cdot n\rangle_{\Gamma_i} + \sum_{i=3,5,6,7}\langle \phi_i, u\rangle_{\Gamma_i} \quad \forall u \in \mathbf{V}(\Omega).
\end{aligned}
\tag{3.9}
$$

Next, let us consider variational formulations using the bilinear form $(\nabla v, \nabla u)$. We consider the cases that $\Gamma_6=\varnothing$ and $h_i=0,\ i=4,5$, in (3.3) and (3.4). Let $\mathbf{V}_{\Gamma 1-5}(\Omega)=\{u\in \mathbf{H}^1(\Omega) : \text{div}\, u=0,\ u|_{\Gamma_1}=0,\ u_\tau|_{(\Gamma_2\cup\Gamma_4)}=0,\ u\cdot n|_{(\Gamma_3\cup\Gamma_5)}=0\}$ and $\mathbf{V}_{\Gamma 2-5}(\Omega)=\{u\in \mathbf{H}^1(\Omega) : \text{div}\, u=0,\ u_\tau|_{(\Gamma_2\cup\Gamma_4)}=0,\ u\cdot n|_{(\Gamma_3\cup\Gamma_5)}=0\}$. By theorems (2.1) and (2.2) for $v\in \mathbf{H}^2(\Omega)\cap \mathbf{V}_{\Gamma 2-5}(\Omega)$ and $u\in \mathbf{V}_{\Gamma 1-5}(\Omega)$

$$
\begin{aligned}
-(\Delta v, u) &= (\nabla v, \nabla u) - \left(\frac{\partial v}{\partial n}, u\right)_{\partial\Omega} \\
&= (\nabla v, \nabla u) + (k(x)v, u)_{\Gamma_2} - (\text{rot}\ v\times n, u)_{\Gamma_3} + (S\tilde{v}, \tilde{u})_{\Gamma_3} - (\varepsilon_n(v), u)_{\Gamma_4} \\
&\quad - (\varepsilon_n(v), u)_{\Gamma_5} + (S\tilde{v}, \tilde{u})_{\Gamma_5} - \left(\frac{\partial v}{\partial n}, u\right)_{\Gamma_7} \\
&= (\nabla v, \nabla u) + (k(x)v, u)_{\Gamma_2} - (\text{rot}\ v\times n, u)_{\Gamma_3} + (S\tilde{v}, \tilde{u})_{\Gamma_3} - (\varepsilon_{nn}(v), u\cdot n)_{\Gamma_4} \\
&\quad - 2(\varepsilon_{n\tau}(v), u)_{\Gamma_5} - (S\tilde{v}, \tilde{u})|_{\Gamma_5} - \left(\frac{\partial v}{\partial n}, u\right)_{\Gamma_7}.
\end{aligned}
\tag{3.10}
$$

Using (3.10), (3.6) we get a variational formulation for problem (3.1), (3.3) without the condition $v_\tau|_{\Gamma_7}=0$ in (7) of (3.3):

Formulation 3.4. *Find $v\in \mathbf{V}_{\Gamma 2-5}(\Omega)$ such that*

$$
\begin{aligned}
& v|_{\Gamma_1} = h_1, \\
& \nu(\nabla v, \nabla u) + \nu(k(x)v, u)_{\Gamma_2} + \nu(S\tilde{v}, \tilde{u})_{\Gamma_3} + 2(\alpha(x)v, u)_{\Gamma_5} - \nu(S\tilde{v}, \tilde{u})_{\Gamma_5} \\
& \quad = \langle f, u\rangle + \sum_{i=2,4}\langle \phi_i, u\cdot n\rangle_{\Gamma_i} + \sum_{i=3,5,7}\langle \phi_i, u\rangle_{\Gamma_i} \quad \forall u\in \mathbf{V}_{\Gamma 1-5}(\Omega).
\end{aligned}
\tag{3.11}
$$

Similarly, we can obtain variational formulations using the bilinear form $(\nabla v, \nabla u)$ for problems (3.2), (3.3) and problem (3.2), (3.4).

For existence of solutions to the problems of (3.7)–(3.9), coercivity of the quadratic form corresponding to the bilinear form $\nu(\varepsilon(v), \varepsilon(u)) + 2(k(x)v, u)_{\Gamma_2} + 2(S\tilde{v}, \tilde{u})|_{\Gamma_3} + (k(x)v, u)_{\Gamma_7}$ is important (see the next section), and so is coercivity of the quadratic form corresponding to $\nu(\nabla v, \nabla u) + \nu(k(x)v, u)_{\Gamma_2} + \nu(S\tilde{v}, \tilde{u})|_{\Gamma_3} - \nu(S\tilde{v}, \tilde{u})|_{\Gamma_5}$ for (3.11). Thus, in the point of view in coercivity two bilinear forms (1.5) and (1.6) look similar, but relying on the bilinear form (1.5), in variational formulations we cannot include the boundary conditions (6) in (3.3), (3.4). Because now we only know the relation between strain and normal derivative of vector fields on the boundary when the vector fields are tangent or orthogonal to the boundary.

Remark 3.1. Since $S\equiv 0$ on the flat surface Γ, on flat portions of the boundary the rotation boundary condition $v_n|_\Gamma=0$, $\text{rot}\ v\times n|_{\Gamma_3}=\phi/\nu$ and the Navier slip condition $v_n|_\Gamma=0,\ \nu\varepsilon_{n\tau}(v)|_\Gamma=\phi$ are equivalent. In [26] the condition $v_n|_\Gamma=0,\ \text{rot}\ v\times n|_{\Gamma_3}=0$ is called Navier's type slip-without-friction boundary conditions.

Remark 3.2. Boundary condition $\nu\frac{\partial v}{\partial n}-pn=0$ often called "do nothing" or "free outflow" boundary condition, results from variational principle and does not have a real physical meaning but is rather used in truncating large physical domains to smaller computational domains by assuming parallel flow (cf. [20,22]). In the outlet the boundary condition $\nu\varepsilon(v)n-pn=0$ or $\nu\frac{\partial v}{\partial n}-pn=0$ is used. But, in our knowledge it seems not to be known whether two conditions are equivalent or not. By

Theorem 2.2, for divergence free flows orthogonal to the boundary we know that two conditions $\nu\varepsilon(v)n - pn$ and $\nu\frac{\partial v}{\partial n} - pn$ are equivalent in variational formulations above.

In our knowledge, literatures concerned with condition (7) in (3.4) are not known yet. This is concerned with the fact that the corresponding condition (7) in (3.3) as artificial boundary condition is used in the outlet when the flow is almost parallel and so similar to the Stokes flow, but the term $\frac{1}{2}|v|^2$ is derived from initial term $(v\cdot\nabla)v$, we think. To restrict backward flow in the outlet, which guarantees existence of global weak solutions to the Navier–Stokes problem, in some special cases for an outlet boundary condition $\nu\frac{\partial v}{\partial n} - pn + \frac{1}{2}(v\cdot n)^- v = \phi$ is used, where $a^- = (|a| - a)/2$ (cf. [37] and reference therein). When flow is into the domain this boundary condition coincides with (7) in (3.4).

4. Existence and uniqueness of solutions to the stationary problems

Theorem 4.1. *Assume that the surfaces $\Gamma_{2j}, \Gamma_{3j}, \Gamma_{7j}$ are convex and α is positive. Then, under Assumption 3.1 there exists a unique solution to Formulation 3.1 of the stationary Stokes problem with mixed boundary condition (3.3) for any f and ϕ_i, $i = 2$–7.*

Proof. Having in mind Assumption 3.1 and putting $v = w + U$, from Formulation 3.1 we get an equivalent new problem.

Find w such that

$$\begin{aligned} & w \in \mathbf{V}(\Omega), \\ & 2\nu(\varepsilon(w), \varepsilon(u)) + 2\nu(k(x)w, u)_{\Gamma_2} + 2\nu(S\tilde{w}, \tilde{u})_{\Gamma_3} + 2(\alpha(x)w, u)_{\Gamma_5} + \nu(k(x)w, u)_{\Gamma_7} \\ & \quad = -2\nu(\varepsilon(U), \varepsilon(u)) - 2\nu(k(x)U, u)_{\Gamma_2} - 2\nu(S\tilde{U}, \tilde{u})_{\Gamma_3} - 2(\alpha(x)U, u)_{\Gamma_5} - \nu(k(x)U, u)_{\Gamma_7} \\ & \qquad + \langle f, u\rangle + \sum_{i=2,4} \langle \phi_i, u\cdot n\rangle_{\Gamma_i} + \sum_{i=3,5,6,7} \langle \phi_i, u\rangle_{\Gamma_i} \quad \forall u \in \mathbf{V}(\Omega). \end{aligned} \tag{4.1}$$

Define a linear operator $A \in \mathcal{L}(\mathbf{V} \to \mathbf{V}^*)$, where $\mathbf{V}^*$ is the dual space of $\mathbf{V}$, by

$$\langle Aw, u\rangle = 2\nu(\varepsilon(w), \varepsilon(u)) + 2\nu(k(x)w, u)_{\Gamma_2} + 2\nu(S\tilde{w}, \tilde{u})_{\Gamma_3} + 2(\alpha(x)w, u)_{\Gamma_5} + \nu(k(x)w, u)_{\Gamma_7} \quad \forall w, u \in \mathbf{V}(\Omega). \tag{4.2}$$

Then, by Remark 2.2 and Assumption 3.1 we have

$$|\langle Aw, u\rangle| \le K\|w\|_{\mathbf{V}} \cdot \|u\|_{\mathbf{V}} \quad \forall w, u \in \mathbf{V}(\Omega). \tag{4.3}$$

And by Korn's inequality

$$2\nu(\varepsilon(w), \varepsilon(w)) \ge \delta\|w\|^2_{\mathbf{V}} \quad \exists \delta > 0$$

(cf. [63]), Assumption 3.1 and Lemma A.3, we have

$$\langle Aw, w\rangle \ge \beta_1\|w\|^2_{\mathbf{V}}, \quad \exists \beta_1 > 0. \tag{4.4}$$

Define an element $F \in \mathbf{V}^*$ by

$$\begin{aligned} \langle F, u\rangle = & -2\nu(\varepsilon(U), \varepsilon(u)) - 2\nu(k(x)U, u)_{\Gamma_2} - 2\nu(S\tilde{U}, \tilde{u})_{\Gamma_3} - 2(\alpha(x)U, u)_{\Gamma_5} - \nu(k(x)U, u)_{\Gamma_7} \\ & + \langle f, u\rangle + \sum_{i=2,4} \langle \phi_i, u\cdot n\rangle_{\Gamma_i} + \sum_{i=3,5,6,7} \langle \phi_i, u\rangle_{\Gamma_i} \quad \forall u \in \mathbf{V}(\Omega). \end{aligned} \tag{4.5}$$

Thus, in view of (4.3)–(4.5) by the Lax–Milgram theorem we come to our assertion. □

Remark 4.1. Since $n(x) \in \mathbf{C}^1(\bar{\Gamma}_i)$, $u\cdot n \in H^{\frac{1}{2}}(\Gamma_i)$ (cf. Theorem 1.4.4.2 in [46]). Also, $H^{\frac{1}{2}}(\Gamma_i) = H_0^{\frac{1}{2}}(\Gamma_i)$ (cf. Theorem 11.1, Chapter 1 in [57]), and so $\langle \phi_i, u\rangle_{\Gamma_i}$ and $\langle \phi_i, u\cdot n\rangle_{\Gamma_i}$ make sense.

When the surfaces Γ_{2j}, Γ_{3j} and Γ_{7j} are not convex, if the norms of the mean curvature $k(x)$, the matrix α and the shape operator S are small enough, then by Korn's inequality the estimate (4.4) is valid, and so is the assertion.

Theorem 4.2. *Assume Assumption 3.1 and that the surfaces Γ_{2j}, Γ_{3j}, Γ_{7j} are convex and $\|U\|_{\mathbf{L}_3(\Omega)}$ is small enough. Then, there exists a solution to Formulation 3.3 of the stationary Navier–Stokes problem with mixed boundary condition (3.4) for any f and ϕ_i, $i = 2 \sim 7$. If $\|U\|^2_{\mathbf{H}^1}$, $\|f\|_{\mathbf{V}^*}$, $\|\phi_i\|_{H^{-\frac{1}{2}}(\Gamma_i)}$, $i = 2, 4$, $\|\phi_i\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_i)}$, $i = 3, 5, 6, 7$, are small enough, then the solution is unique.*

Proof. In the same way as the proof of Theorem 4.1 we get a new problem equivalent to (3.9).

Find w such that

$$\begin{aligned} & w \in \mathbf{V}(\Omega), \\ & 2\nu(\varepsilon(w), \varepsilon(u)) + \langle \operatorname{rot} w \times w, u\rangle + \langle \operatorname{rot} U \times w, u\rangle + \langle \operatorname{rot} w \times U, u\rangle \\ & \qquad + 2\nu(k(x)w, u)_{\Gamma_2} + 2\nu(S\tilde{w}, \tilde{u})_{\Gamma_3} + 2(\alpha(x)w, u)_{\Gamma_5} + \nu(k(x)w, u)_{\Gamma_7} \\ & \quad = -2\nu(\varepsilon(U), \varepsilon(u)) - \langle \operatorname{rot} U \times U, u\rangle - 2\nu(k(x)U, u)_{\Gamma_2} - 2\nu(S\tilde{U}, \tilde{u})_{\Gamma_3} - 2(\alpha(x)U, u)_{\Gamma_5} \\ & \qquad - \nu(k(x)U, u)_{\Gamma_7} + \langle f, u\rangle + \sum_{i=2,4} \langle \phi_i, u\cdot n\rangle_{\Gamma_i} + \sum_{i=3,5,6,7} \langle \phi_i, u\rangle_{\Gamma_i} \quad \forall u \in \mathbf{V}(\Omega). \end{aligned} \tag{4.6}$$

Define a nonlinear operator $A \in (\mathbf{V} \to \mathbf{V}^*)$ by

$$\begin{aligned}\langle Aw, u\rangle &= 2\nu(\varepsilon(w), \varepsilon(u)) + \langle \mathrm{rot}\, w \times w, u\rangle + \langle \mathrm{rot}\, U \times w, u\rangle + \langle \mathrm{rot}\, w \times U, u\rangle \\ &\quad + 2\nu(k(x)w, u)_{\Gamma_2} + 2\nu(S\tilde{w}, \tilde{u})_{\Gamma_3} + 2(\alpha(x)w, u)_{\Gamma_5} + \nu(k(x)w, u)_{\Gamma_7} \quad \forall w, u \in \mathbf{V}(\Omega).\end{aligned} \tag{4.7}$$

On the other hand, for any $w \in \mathbf{V}(\Omega)$ we have

$$\begin{aligned}&\langle \mathrm{rot}\, w \times w, w\rangle = 0, \qquad \langle \mathrm{rot}\, U \times w, w\rangle = 0, \\ &|\langle \mathrm{rot}\, w \times U, w\rangle| \le \int_\Omega |(\mathrm{rot}\, w \times U) \cdot w|\, dx \le \gamma \|w\|^2_{\mathbf{V}} \cdot \|U\|_{\mathbf{L}_3}.\end{aligned} \tag{4.8}$$

Also, by Korn's inequality

$$2\nu(\varepsilon(w), \varepsilon(w)) \ge \delta\|w\|^2_{\mathbf{V}}. \tag{4.9}$$

Therefore, if $\delta - \gamma\|U\|_{\mathbf{L}_3(\Omega)} = \beta_2 > 0$, then by Assumption 3.1, Lemma A.3, (4.8), (4.9) we have

$$\langle Aw, w\rangle \ge \beta_2\|w\|^2_{\mathbf{V}}, \quad \beta_2 > 0. \tag{4.10}$$

Define an element $F \in \mathbf{V}^*$ by

$$\begin{aligned}\langle F, u\rangle &= -2\nu(\varepsilon(U), \varepsilon(u)) - \langle \mathrm{rot}\, U \times U, u\rangle - 2\nu(k(x)U, u)_{\Gamma_2} - 2\nu(S\tilde{U}, \tilde{u})_{\Gamma_3} - 2(\alpha(x)U, u)_{\Gamma_5} \\ &\quad - \nu(k(x)U, u)_{\Gamma_7} + \langle f, u\rangle + \sum_{i=2,4}\langle \phi_i, u\cdot n\rangle_{\Gamma_i} + \sum_{i=3,5,6,7}\langle \phi_i, u\rangle_{\Gamma_i} \quad \forall u \in \mathbf{V}(\Omega).\end{aligned} \tag{4.11}$$

Now, let us prove that

$$\text{if } w^k \rightharpoonup w \text{ weakly in } \mathbf{V} \text{ as } k \to \infty, \text{ then } \left\langle Aw^k, u\right\rangle \to \langle Aw, u\rangle \ \forall u \in \mathbf{V}. \tag{4.12}$$

First let us prove that

$$\langle \mathrm{rot}\, w^k \times w^k, u\rangle \to \langle \mathrm{rot}\, w \times w, u\rangle \quad \forall u \in \mathbf{V} \text{ as } k \to \infty. \tag{4.13}$$

To this end, let us consider

$$\langle \mathrm{rot}\, w^k \times w^k, u\rangle - \langle \mathrm{rot}\, w \times w, u\rangle = \langle \mathrm{rot}\, w^k \times (w^k - w), u\rangle + \langle \mathrm{rot}\,(w^k - w) \times w, u\rangle. \tag{4.14}$$

Let us estimate the first term on the right-hand side in (4.14).

$$\begin{aligned}\left|\langle \mathrm{rot}\, w^k \times (w^k - w), u\rangle\right| &\le \gamma_1 \int_\Omega |\mathrm{rot}\, w^k| \cdot |(w^k - w)| \cdot |u|\, dx \\ &\le \gamma_2\|\mathrm{rot}\, w^k\|_{\mathbf{L}_2} \cdot \|(w^k - w)\|_{\mathbf{L}_3} \cdot \|u\|_{\mathbf{L}_6}.\end{aligned} \tag{4.15}$$

Since $w^k \to w$ strongly in $\mathbf{L}_3(\Omega)$ and $\mathrm{rot}\, w^k$ is bounded in $\mathbf{L}_2(\Omega)$, by virtue of (4.15) the first term on the right-hand side of (4.14) converges to zero as $k \to \infty$.

By the embedding of the space $\mathbf{V}$ into $\mathbf{L}_6(\Omega)$ we have $w_i u_j \in L_2(\Omega),\ i, j = 1 \sim l$, for any $w, v \in \mathbf{V}$. Also, since $w^k \rightharpoonup w$ weakly in $\mathbf{V}$ as $k \to \infty$, $\mathrm{rot}\, w^k \rightharpoonup \mathrm{rot}\, w$ weakly in $\mathbf{L}_2(\Omega)$. Then, the second term on the right-hand side of (4.14) converges to zero as $k \to \infty$. Thus, we get (4.13).

All terms except $\langle \mathrm{rot}\, w \times w, u\rangle$ in $\left\langle Aw^k, u\right\rangle$ are linear with respect to w, and so it is easy to check their convergence as $k \to \infty$.

Therefore, we prove (4.12).

By (4.10) and (4.12), there exists a solution to (4.6) (cf. Theorem 1.2, Chapter IV in [45]), and therefore (3.9) has a solution.

Let us prove uniqueness. We can get the following estimates.

$$\begin{aligned}|\langle \mathrm{rot}\, w_1 \times u, v\rangle - \langle \mathrm{rot}\, w_2 \times u, v\rangle| &\le \gamma_3\|\mathrm{rot}\, w_1 - \mathrm{rot}\, w_2\|_{\mathbf{L}_2} \cdot \|u\|_{\mathbf{L}_3} \cdot \|v\|_{\mathbf{L}_6} \\ &\le \gamma_4\|w_1 - w_2\|_{\mathbf{V}} \cdot \|u\|_{\mathbf{V}} \cdot \|v\|_{\mathbf{V}} \quad \forall u, v, w_1, w_2 \in \mathbf{V},\end{aligned}$$

$$\|F\|_{\mathbf{V}^*} \le M_1\Big(\|U\|^2_{\mathbf{H}^1} + \|f\|_{\mathbf{V}^*} + \sum_{i=2,4}\|\phi_i\|_{H^{-\frac12}(\Gamma_i)} + \sum_{i=3,5,6,7}\|\phi_i\|_{\mathbf{H}^{-\frac12}(\Gamma_i)}\Big),$$

where M_1 depends on mean curvature, shape operator, ν and α.

Thus, if $\frac{\|F\|_{\mathbf{V}^*}}{\beta_2^2}\gamma_4 < 1$, where β_2 is one in (4.10), then by virtue of Theorem 1.3, Chapter IV in [45] the solution is unique. □

Theorem 4.3. *Assume Assumption 3.1 is valid. Let $\Gamma_2 = \Gamma_4 = \Gamma_6 = \Gamma_7 = \varnothing$, the surfaces Γ_{3j} are convex and $\|U\|_{\mathbf{L}_3(\Omega)}$ is small enough. Then, there exists a solution to Formulation 3.2 of the stationary Navier–Stokes problem with mixed boundary condition (3.3) for any f and $\phi_i,\ i = 3, 5$. If $\|U\|_{\mathbf{H}^1}, \|f\|_{\mathbf{V}^*}, \|\phi_i\|_{\mathbf{H}^{-\frac12}(\Gamma_i)},\ i = 3, 5$, are small enough, the solution is unique.*

Proof. In the same way as the proof of Theorem 4.1 we get a new problem equivalent to (3.8).

Find w such that

$$\begin{aligned}
&w \in \mathbf{V}(\Omega),\\
&2\nu(\varepsilon(w),\varepsilon(u)) + \langle (w\cdot\nabla)w, u\rangle + \langle (U\cdot\nabla)w,u\rangle + \langle (w\cdot\nabla)U,u\rangle + 2\nu(S\tilde{w},\tilde{u})_{\Gamma_3} + 2(\alpha(x)w,u)_{\Gamma_5}\\
&\quad = -2\nu(\varepsilon(U),\varepsilon(u)) - \langle (U\cdot\nabla)U,u\rangle - 2\nu(S\tilde{U},\tilde{u})_{\Gamma_3} - 2(\alpha(x)U,u)_{\Gamma_5} + \langle f,u\rangle + \sum_{i=3,5}\langle \phi_i, u\rangle_{\Gamma_i} \quad \forall u\in \mathbf{V}(\Omega). \qquad (4.16)
\end{aligned}$$

Define a nonlinear operator $A \in (\mathbf{V}\to\mathbf{V}^*)$ by

$$\begin{aligned}
\langle Aw,u\rangle &= 2\nu(\varepsilon(w),\varepsilon(u)) + \langle (w\cdot\nabla)w,u\rangle + \langle (U\cdot\nabla)w,u\rangle + \langle (w\cdot\nabla)U,u\rangle\\
&\quad + 2\nu(S\tilde{w},\tilde{u})_{\Gamma_3} + 2(\alpha(x)w,u)_{\Gamma_5} \quad \forall w,u\in\mathbf{V}(\Omega). \qquad (4.17)
\end{aligned}$$

Using the fact that $w_n = 0$, $v_n = 0$ on any parts of the boundary, we can prove that

$$\langle (w\cdot\nabla)u, v\rangle = -\langle (w\cdot\nabla)v,u\rangle \quad \forall w,v\in\mathbf{V}, u\in\mathbf{H}^1(\Omega),$$

and so we have

$$\langle (w\cdot\nabla)w,w\rangle = 0, \qquad \langle (w\cdot\nabla)U,w\rangle = -\langle (w\cdot\nabla)w,U\rangle.$$

Then,

$$|\langle (U\cdot\nabla)w,w\rangle + \langle (w\cdot\nabla)U,w\rangle| \le \gamma_1\|w\|^2_{\mathbf{V}}\cdot\|U\|_{\mathbf{L}_3}. \qquad (4.18)$$

If $\delta - \gamma_1\|U\|_{\mathbf{L}_3} = \beta_3 > 0$, then by Korn's inequality (4.9), Assumption 3.1, Lemma A.3, (4.17), and (4.18) we have

$$\langle Aw,w\rangle \ge \beta_3\|w\|^2_{\mathbf{V}}, \quad \beta_3 > 0. \qquad (4.19)$$

Define $F\in\mathbf{V}^*$ by

$$\begin{aligned}
\langle F,u\rangle &= -2\nu(\varepsilon(U),\varepsilon(u)) - \langle (U\cdot\nabla)U,u\rangle - 2\nu(S\tilde{U},\tilde{u})_{\Gamma_3} - 2(\alpha(x)U,u)_{\Gamma_5}\\
&\quad + \langle f,u\rangle + \sum_{i=3,5}\langle\phi_i,u\rangle_{\Gamma_i} \quad \forall u\in\mathbf{V}(\Omega). \qquad (4.20)
\end{aligned}$$

Now, by virtue of (4.19), (4.20) in the same way as the proof of Theorem 4.2 we have existence of solution. Uniqueness is also proved as proof of Theorem 4.2. □

Theorem 4.4. *Assume Assumption 3.1 and that the surfaces Γ_{2j}, Γ_{3j}, Γ_{7j} are convex and $\|U\|_{\mathbf{H}^1(\Omega)}$ is small enough. Then, when f and ϕ_i, $i = 2$–7, are small enough, there exists a unique solution to Formulation 3.3 of the stationary Navier–Stokes problem with mixed boundary condition (3.3) in a neighborhood of U in $\mathbf{H}^1(\Omega)$.*

Proof. In the same way as the proof of Theorem 4.1 we get a new problem equivalent to (3.8).
Find w such that

$$\begin{aligned}
&w\in\mathbf{V}(\Omega),\\
&2\nu(\varepsilon(w),\varepsilon(u)) + \langle (w\cdot\nabla)w,u\rangle + \langle (U\cdot\nabla)w,u\rangle + \langle (w\cdot\nabla)U,u\rangle\\
&\quad + 2\nu(k(x)w,u)_{\Gamma_2} + 2\nu(S\tilde{w},\tilde{u})_{\Gamma_3} + 2(\alpha(x)w,u)_{\Gamma_5} + \nu(k(x)w,u)_{\Gamma_7}\\
&\quad = -2\nu(\varepsilon(U),\varepsilon(u)) - \langle (U\cdot\nabla)U,u\rangle - 2\nu(k(x)U,u)_{\Gamma_2} - 2\nu(S\tilde{U},\tilde{u})_{\Gamma_3}\\
&\quad\quad - 2(\alpha(x)U,u)_{\Gamma_5} - \nu(k(x)U,u)_{\Gamma_7} + \langle f,u\rangle + \sum_{i=2,4}\langle\phi_i, u\cdot n\rangle_{\Gamma_i} + \sum_{i=3,5,6,7}\langle\phi_i,u\rangle_{\Gamma_i} \quad \forall u\in\mathbf{V}(\Omega). \qquad (4.21)
\end{aligned}$$

Define operators $A, B\in(\mathbf{V}\to\mathbf{V}^*)$ and an element $F\in\mathbf{V}^*$, respectively, by

$$\begin{aligned}
\langle Aw,u\rangle &= 2\nu(\varepsilon(w),\varepsilon(u)) + \langle (U\cdot\nabla)w,u\rangle + \langle (w\cdot\nabla)U,u\rangle + 2\nu(k(x)w,u)_{\Gamma_2}\\
&\quad + 2\nu(S\tilde{w},\tilde{u})_{\Gamma_3} + 2(\alpha(x)w,u)_{\Gamma_5} + \nu(k(x)w,u)_{\Gamma_7} \quad \forall w,u\in\mathbf{V}(\Omega),\\
\langle Bw,u\rangle &= \langle (w\cdot\nabla)w,u\rangle \quad \forall w,u\in\mathbf{V}(\Omega),\\
\langle F,u\rangle &= -2\nu(\varepsilon(U),\varepsilon(u)) - \langle (U\cdot\nabla)U,u\rangle - 2\nu(k(x)U,u)_{\Gamma_2} - 2\nu(S\tilde{U},\tilde{u})_{\Gamma_3}\\
&\quad - 2(\alpha(x)U,u)_{\Gamma_5} - \nu(k(x)U,u)_{\Gamma_7} + \langle f,u\rangle + \sum_{i=2,4}\langle\phi_i,u\cdot n\rangle_{\Gamma_i} + \sum_{i=3,5,6,7}\langle\phi_i,u\rangle_{\Gamma_i} \quad \forall u\in\mathbf{V}(\Omega).
\end{aligned}$$

Then, we can rewrite (4.21) as follows.

$$\langle Aw,u\rangle = \langle F,u\rangle - \langle Bw,u\rangle \quad \forall u\in\mathbf{V}(\Omega). \qquad (4.22)$$

We can know that

$$|\langle Aw, u\rangle| \le m\|w\|_{\mathbf{V}} \cdot \|u\|_{\mathbf{V}}, \qquad \langle Aw, w\rangle \ge \beta_4 \|w\|_{\mathbf{V}}^2, \quad \exists \beta_4 > 0,\ \forall w, u \in \mathbf{V}(\Omega). \tag{4.23}$$

Also

$$|\langle Bw_1 - Bw_2, u\rangle| \le \delta M\|w_1 - w_2\|_{\mathbf{V}} \cdot \|u\|_{\mathbf{V}} \quad \forall w_i \in \mathscr{O}_M(0_{\mathbf{V}}) \equiv \{v \in \mathbf{V} : \|v\|_{\mathbf{V}} \le M\},\ \forall u \in \mathbf{V}(\Omega), \tag{4.24}$$

$$\|F\|_{\mathbf{V}^*} \le M_1\Big(\|U\|_{\mathbf{H}^1}^2 + \|f\|_{\mathbf{V}^*} + \sum_{i=2,4} \|\phi_i\|_{H^{-\frac{1}{2}}(\Gamma_i)} + \sum_{i=3,5,6,7} \|\phi_i\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_i)}\Big), \tag{4.25}$$

where M_1 depends on mean curvature, shape operator, ν and α.

By the Lax–Milgram theorem and (4.23), for any fixed $z \in \mathscr{O}_M(0_{\mathbf{V}})$ there exists a unique solution to the problem

$$Aw = F - Bz, \tag{4.26}$$

and by (4.23), (4.24) the solution w to (4.26) is estimated as follows.

$$\|w\|_{\mathbf{V}} \le \frac{1}{\beta_4}(\|F\|_{\mathbf{V}^*} + \|Bz\|_{\mathbf{V}^*}) \le \frac{1}{\beta_4}(\|F\|_{\mathbf{V}^*} + \delta M^2).$$

Thus, if $\|F\|_{\mathbf{V}^*}$ and M are small enough, then the map $(z \to w)$ is strict contract in $\mathscr{O}_M(0_{\mathbf{V}})$, and so there exists a unique solution to (4.22) in $\mathscr{O}_M(0_{\mathbf{V}})$, which shows our assertion. □

Remark 4.2. For special cases of (3.3) such that $\Gamma_i = \emptyset,\ i = 2, 3, 7$ and $\alpha = 0$, existence of local weak solutions (i.e. under small data) to the Navier–Stokes systems was proved before (cf. Theorem 2.2 in [61], Theorem 3.2 in [64]). When $h_1 = 0$, for very special cases smallness of ϕ_i is not necessary (see Theorem 3.7 in [20]). In the point of view in mathematics, why global weak solution is not getting is due to that one cannot get $\langle (w \cdot \nabla)w, w\rangle = 0$ when $v \cdot n \ne 0$ on parts of boundary. For physical point of view, which is concerned with backward flow on the boundary, refer to [52,37].

In the same way as Theorem 4.1 we get

Theorem 4.5. *Assume that $\Gamma_6 = \varnothing$, the surfaces Γ_{2j}, Γ_{3j} are convex and Γ_{5j} are concave. Then, under Assumption 3.1 without the condition $U_\tau\,|_{\Gamma_7} = 0$ there exists a unique solution to Formulation 3.4 of the stationary Stokes problem with mixed boundary condition (3.3) for any f and $\phi_i,\ i = 2$–$5, 7$.*

Remark 4.3. Assuming that the surfaces Γ_{2j}, Γ_{3j} are convex and Γ_{5j} are concave, we can obtain existence and uniqueness of solution to problems formulated similarly to Formulation 3.4 for problems (3.2), (3.3) and problem (3.2), (3.4).

5. Existence and uniqueness of solutions to the non-stationary problems

Let $Q = \Omega \times (0, T)$, $\Sigma_i = \Gamma_i \times (0, T)$.

We consider the Navier–Stokes problem

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla)v + \nabla p = f, \quad \nabla \cdot v = 0 \quad \text{in } Q \\ v(0) = v_0 \end{cases} \tag{5.1}$$

with boundary conditions (3.4). In this case $h_i,\ \phi_i$ are functions of x, t on Σ_i.

Let

$$\mathbf{\Lambda}(Q) = \{u \in \mathbf{C}^2(\bar{Q}) : \operatorname{div} u = 0, u\,|_{\Sigma_1} = 0, u_\tau\,|_{(\Sigma_2 \cup \Sigma_4 \cup \Sigma_7)} = 0, u \cdot n\,|_{(\Sigma_3 \cup \Sigma_5)} = 0\},$$
$$\mathbf{V}(Q) = L_2(0, T; \mathbf{V}(\Omega)),$$
$$\mathbf{W}(Q) = \{\text{the completion of } \mathbf{\Lambda}(Q) \text{ in the space } \mathbf{H}^1(Q)\}.$$

Assume the following holds.

Assumption 5.1. There exists a function $U \in \mathbf{H}^1(Q) \cap \mathbf{L}_\infty(Q)$ such that

$$\operatorname{div} U = 0, \qquad U\,|_{\Sigma_1} = h_1, \qquad U_\tau\,|_{(\Sigma_2 \cup \Sigma_7)} = 0, \qquad U \cdot n\,|_{\Sigma_3} = 0, \qquad U_\tau\,|_{\Sigma_4} = h_4, \qquad U \cdot n\,|_{\Sigma_5} = h_5.$$

Also, $f \in L_2(0, T; \mathbf{V}(\Omega)^*),\ \phi_i \in L_2(0, T; H^{-\frac{1}{2}}(\Gamma_i)),\ i = 2, 4,\ \phi_i \in L_2(0, T; \mathbf{H}^{-\frac{1}{2}}(\Gamma_i)),\ i = 3, 5, 6, 7,\ \alpha_{ij} \in L_\infty(0, T; L_\infty(\Gamma_5))$, $v_0 - U(x, 0) \in H(\Omega)$, where $H(\Omega)$ is closure of $\mathbf{V}(\Omega)$ in $\mathbf{L}_2(\Omega)$, and $\Gamma_1 \ne \emptyset$.

Taking $(v \cdot \nabla)v = \operatorname{rot} v \times v + \frac{1}{2}\operatorname{grad}|v|^2$ into account, we get a variational formulation for problem (5.1), (3.4):

Formulation 5.1. *Find v such that*

$$\begin{aligned}
& v-U\in \mathbf{V}(Q),\\
& -\int_Q\left(v,\frac{\partial u}{\partial t}\right)dt+\int_0^T 2\nu(\varepsilon(v),\varepsilon(u))\,dt+\int_0^T\langle rot\ v\times v,u\rangle\,dt\\
&\quad +\int_0^T 2\nu(k(x)v,u)_{\Gamma_2}\,dt+\int_0^T 2\nu(S\tilde v,\tilde u)_{\Gamma_3}\,dt+\int_0^T 2(\alpha(t,x)v,u)_{\Gamma_5}\,dt+\int_0^T \nu(k(x)v,u)_{\Gamma_7}\,dt\\
&\ =(v_0,u(0))+\int_0^T\langle f,u\rangle\,dt+\int_0^T\sum_{i=2,4}\langle\phi_i,u\cdot n\rangle_{\Gamma_i}\,dt\\
&\quad +\int_0^T\sum_{i=3,5,6,7}\langle\phi_i,u\rangle_{\Gamma_i}\,dt\quad \forall u\in\mathbf{\Lambda}(Q)\ with\ u(x,T)=0.
\end{aligned}\tag{5.2}$$

Theorem 5.1. *Under Assumption 5.1 there exists a solution to Formulation 5.1 of the non-stationary Navier–Stokes problem with mixed boundary condition (3.4) such that*

$$\operatorname*{ess\,sup}_{t\in(0,T)}\|v\|_{\mathbf{L}_2(\Omega)}\le c$$

for any U, f and $\phi_i,\ i=2$–7.

For the proof of Theorem 5.1, we transform problem (5.2) to equivalent one. Taking into account Assumption 5.1 and putting $v=z+U$, from Formulation 5.1 we get an equivalent new problem:

Find w such that

$$\begin{aligned}
& z\in\mathbf{V}(Q),\\
& -\int_Q\left(z,\frac{\partial u}{\partial t}\right)dt+\int_0^T 2\nu(\varepsilon(z),\varepsilon(u))\,dt+\int_0^T\langle \mathrm{rot}\,z\times z,u\rangle\,dt\\
&\quad +\int_0^T(\mathrm{rot}\,z\times U+\mathrm{rot}\,U\times z,u)\ dt+\int_0^T 2\nu(k(x)z,u)_{\Gamma_2}\,dt\\
&\quad +\int_0^T 2\nu(S\tilde z,\tilde u)_{\Gamma_3}\,dt+\int_0^T 2(\alpha(t,x)z,u)_{\Gamma_5}\,dt+\int_0^T\nu(k(x)z,u)_{\Gamma_7}\,dt\\
&\ =\int_0^T\left(U,\frac{\partial u}{\partial t}\right)dt-\int_0^T 2\nu(\varepsilon(U),\varepsilon(u))\,dt-\int_0^T(\mathrm{rot}\,U\times U,u)_{\Omega(t)}\,dt\\
&\quad -\int_0^T 2\nu(k(x)U,u)_{\Gamma_2}\,dt-\int_0^T 2\nu(S\tilde U,\tilde u)_{\Gamma_3}\,dt-\int_0^T 2(\alpha(x)U,u)_{\Gamma_5}\,dt-\int_0^T\nu(k(x)U,u)_{\Gamma_7}\\
&\quad +\int_0^T\langle f,u\rangle\,dt+\int_0^T\sum_{i=2,4}\langle\phi_i,u\cdot n\rangle_{\Gamma_i}\,dt+\int_0^T\sum_{i=3,5,6,7}\langle\phi_i,u\rangle_{\Gamma_i}\,dt+(v_0,u(0))\\
&\quad \forall u\in\mathbf{\Lambda}(Q)\ \text{with}\ u(x,T)=0.
\end{aligned}\tag{5.3}$$

In (5.3) let us make again changes of the unknown functions by $w=e^{k_1t}z$ where k_1 is a constant to be determined in Lemma 5.2 later. Then we have

$$\begin{aligned}
& -\int_Q z\frac{\partial u}{\partial t}\,dxdt=-\int_Q e^{-k_1t}w\frac{\partial u}{\partial t}\,dxdt=-\int_Q w\frac{\partial\hat u}{\partial t}\,dxdt-k_1\int_Q w\hat u\,dxdt,\\
& \int_Q U\frac{\partial u}{\partial t}\,dxdt=\int_Q Ue^{k_1t}e^{-k_1t}\frac{\partial u}{\partial t}\,dxdt=\int_Q Ue^{k_1t}\left(\frac{\partial\hat u}{\partial t}+k_1\hat u\right)dxdt,
\end{aligned}$$

where $\hat u=e^{-k_1t}u$.

Substituting these in (5.3), we know that the problem to find a solution to (5.2) is equivalent to the following problem.

Find $w\in\mathbf{V}(Q)$ such that

$$\begin{aligned}
& -\int_0^T\left(w,\frac{\partial\hat u}{\partial t}\right)dt+2\nu\int_0^T(\varepsilon(w),\varepsilon(\hat u))\,dt+\int_0^T e^{-k_1t}(\mathrm{rot}\,w\times w,\hat u)\,dt\\
&\quad +\int_0^T\left(\mathrm{rot}\,w\times U+\mathrm{rot}\,U\times w,\hat u\right)\,dt+\int_0^T 2\nu(k(x)w,u)_{\Gamma_2}\,dt
\end{aligned}$$

$$+\int_0^T 2\nu(S\tilde{w},\tilde{\hat{u}})_{\Gamma_3}\,dt+\int_0^T 2(\alpha(t,x)w,\hat{u})_{\Gamma_5}\,dt+\int_0^T \nu(k(x)w,\hat{u})_{\Gamma_7}\,dt-k_1\int_0^T(w,\hat{u})\,dt$$
$$=\int_Q \bar{U}\left(\frac{\partial\hat{u}}{\partial t}+k_1\hat{u}\right)dxdt-2\nu\int_0^T(\varepsilon(\bar{U}),\varepsilon(\hat{u}))\,dt-\int_0^T(\operatorname{rot}\bar{U}\times U,\hat{u})\,dt$$
$$-\int_0^T 2\nu(k(x)\bar{U},\hat{u})_{\Gamma_2}\,dt-\int_0^T 2\nu(S\tilde{\bar{U}},\tilde{\hat{u}})_{\Gamma_3}\,dt-\int_0^T 2(\alpha(t,x)\bar{U},\hat{u})_{\Gamma_5}\,dt$$
$$-\int_0^T \nu(k(x)U,\hat{u})_{\Gamma_7}+\int_0^T\langle\bar{f},\hat{u}\rangle\,dt+\int_0^T\sum_{i=2,4}\langle\bar{\phi}_i,\hat{u}\cdot n\rangle_{\Gamma_i}\,dt$$
$$+\int_0^T\sum_{i=3,5,6,7}\langle\bar{\phi}_i,\hat{u}\rangle_{\Gamma_i}\,dt+(v_0,u(x,0))\quad\forall\hat{u}\in\mathbf{\Lambda}(Q),\ \text{with }\hat{u}(x,T)=0,\tag{5.4}$$

where $\bar{\gamma}=e^{k_1t}\gamma$ for any γ.

Therefore, for the proof of Theorem 5.1 it is enough to prove the existence of a solution w to Problem (5.4) and

$$\operatorname*{ess\,sup}_{t\in(0,T)}\|w\|\le c.\tag{5.5}$$

To this end, first we will consider the following auxiliary problem by elliptic regularization.

Let m be positive integers and we find functions $w^m\in\mathbf{W}(Q)$ satisfying the following

$$\int_Q\left(\frac{1}{m}\frac{\partial w^m}{\partial t}-w^m\right)\cdot\frac{\partial u}{\partial t}\,dxdt+2\nu\int_0^T(\varepsilon(w^m),\varepsilon(u))\,dt$$
$$+\int_0^T e^{-k_1t}(\operatorname{rot}w^m\times w^m,u)\,dt+\int_0^T\big(\operatorname{rot}w^m\times U+\operatorname{rot}U\times w^m,u\big)\,dt$$
$$+\int_0^T 2\nu(k(x)w^m,u)_{\Gamma_2}\,dt+\int_0^T 2\nu(S\tilde{w^m},\tilde{u})_{\Gamma_3}\,dt+\int_0^T 2(\alpha(t,x)w^m,u)_{\Gamma_5}\,dt$$
$$+\int_0^T\nu(k(x)w^m,u)_{\Gamma_7}\,dt-k_1\int_0^T(w^m,u)\,dt+(w^m(T),u(T))$$
$$=\int_Q\bar{U}\left(\frac{\partial u}{\partial t}+k_1u\right)dxdt-2\nu\int_0^T(\varepsilon(\bar{U}),\varepsilon(u))\,dt-\int_0^T(\operatorname{rot}\bar{U}\times U,u)_{\Omega(t)}\,dt$$
$$-\int_0^T 2\nu(k(x)\bar{U},u)_{\Gamma_2}\,dt-\int_0^T 2\nu(S\tilde{\bar{U}},\tilde{u})_{\Gamma_3}\,dt-\int_0^T 2(\alpha(t,x)\bar{U},u)_{\Gamma_5}\,dt$$
$$-\int_0^T\nu(k(x)\bar{U},u)_{\Gamma_7}\,dt+\int_0^T\langle\bar{f},u\rangle\,dt+\int_0^T\sum_{i=2,4}\langle\bar{\phi}_i,u\cdot n\rangle_{\Gamma_i}\,dt$$
$$+\int_0^T\sum_{i=3,5,6,7}\langle\bar{\phi}_i,u\rangle_{\Gamma_i}\,dt+(v_0,u(x,0))\quad\forall u\in\mathbf{W}(Q).\tag{5.6}$$

For (5.6) we have the following result on the existence and uniqueness of the solution.

Lemma 5.2. *Under Assumption 5.1, for some k_1 independent of m there exists a unique solution to problem (5.6).*

Proof. Define an operator A_m from $\mathbf{W}(Q)$ into its dual space by

$$\langle A_mz,u\rangle=\int_0^T\left(\frac{1}{m}\frac{\partial z}{\partial t},\frac{\partial u}{\partial t}\right)dt-\int_0^T\left(z,\frac{\partial u}{\partial t}\right)dt+2\nu\int_0^T(\varepsilon(z),\varepsilon(u))\,dt+\int_0^T e^{-k_1t}(\operatorname{rot}z\times z,u)\,dt$$
$$+\int_0^T(\operatorname{rot}z\times U+\operatorname{rot}U\times z,u)\,dt+\int_0^T 2\nu(k(x)z,u)_{\Gamma_2}\,dt+\int_0^T 2\nu(S\tilde{z},\tilde{u})_{\Gamma_3}\,dt$$
$$+\int_0^T 2(\alpha(t,x)z,u)_{\Gamma_5}\,dt+\int_0^T\nu(k(x)z,u)_{\Gamma_7}\,dt-k_1\int_0^T(z,u)\,dt+(z(T),u(T))$$
$$\forall z,u\in\mathbf{W}(Q).\tag{5.7}$$

And also define an element $F \in \mathbf{W}(Q)^*$ by

$$\langle F, u\rangle = \int_Q \bar{U}\left(\frac{\partial u}{\partial t} + k_1 u\right) dxdt - 2\nu \int_0^T (\varepsilon(\bar{U}), \varepsilon(u))\, dt - \int_0^T (\operatorname{rot} \bar{U} \times U, u)_{\Omega(t)}\, dt$$
$$- \int_0^T 2\nu(k(x)\bar{U}, u)_{\Gamma_2}\, dt - \int_0^T 2\nu(S\tilde{\bar{U}}, \tilde{u})_{\Gamma_3}\, dt - \int_0^T 2(\alpha(t,x)\bar{U}, u)_{\Gamma_5}\, dt$$
$$- \int_0^T \nu(k(x)\bar{U}, u)_{\Gamma_7}\, dt + \int_0^T \langle \bar{f}, u\rangle\, dt + \int_0^T \sum_{i=2,4} \langle \bar{\phi}_i, u\cdot n\rangle_{\Gamma_i}\, dt$$
$$+ \int_0^T \sum_{i=3,5,6,7} \langle \bar{\phi}_i, u\rangle_{\Gamma_i}\, dt + (v_0, u(x,0)) \quad \forall u \in \mathbf{W}(Q). \tag{5.8}$$

Now, let us consider the existence of a solution to the following problem

$$A_m w^m = F, \tag{5.9}$$

which is equivalent to the existence of a solution to the auxiliary problem (5.6).

For all $z \in \mathbf{W}(Q)$ we have

$$\langle A_m z, z\rangle = \int_0^T \left(\frac{1}{m}\frac{\partial z}{\partial t}, \frac{\partial z}{\partial t}\right) dt - \int_0^T \left(z, \frac{\partial z}{\partial t}\right) dt + 2\nu \int_0^T (\varepsilon(z), \varepsilon(z))\, dt$$
$$+ \int_0^T (\operatorname{rot} z \times U, z)\ dt + \int_0^T 2\nu(k(x)z, z)_{\Gamma_2}\, dt + \int_0^T 2\nu(S\tilde{z}, \tilde{z})_{\Gamma_3}\, dt$$
$$+ \int_0^T 2(\alpha(t,x)z, z)_{\Gamma_5}\, dt + \int_0^T \nu(k(x)z, z)_{\Gamma_7}\, dt - k_1 \int_0^T \|z(t)\|^2\, dt + \|z(T)\|^2, \tag{5.10}$$

where $(\operatorname{rot} z \times z, z) = 0$ and $(\operatorname{rot} U \times z, z) = 0$ were used.

Integrating by parts, we get

$$- \int_Q z\frac{\partial z}{\partial t}\, dxdt = \frac{1}{2}\left(\|z(0)\|^2 - \|z(T)\|^2\right) \quad \text{for } z \in \mathbf{W}(Q). \tag{5.11}$$

By (5.10), (5.11), for all $z \in \mathbf{W}(Q)$ we have

$$\langle A_m z, z\rangle = \int_0^T \frac{1}{m}\left\|\frac{\partial z}{\partial t}\right\|^2 dt + 2\nu \int_0^T \Sigma_{ij}\|\varepsilon_{ij}(z)\|^2\, dt + \int_0^T (\operatorname{rot} z \times U, z)\, dt$$
$$+ \int_0^T 2\nu(k(x)z, z)_{\Gamma_2}\, dt + \int_0^T 2\nu(S\tilde{z}, \tilde{z})_{\Gamma_3}\, dt + \int_0^T 2(\alpha(t,x)z, z)_{\Gamma_5}\, dt$$
$$+ \int_0^T \nu(k(x)z, z)_{\Gamma_7}\, dt - k_1 \int_0^T \|z(t)\|^2\, dt + \frac{1}{2}\left(\|z(0)\|^2 + \|z(T)\|^2\right). \tag{5.12}$$

By Korn's inequality

$$2\int_0^T \Sigma_{ij}\|\varepsilon_{ij}(z)\|^2\, dt \ge c_1 \int_0^T \|z\|^2_{\mathbf{H}^1(\Omega)}\, dt, \quad c_1 > 0. \tag{5.13}$$

By Young's inequality and Assumption 5.1

$$\left|\int_0^T (\operatorname{rot} z \times U, z)\ dt\right| \le \frac{\nu c_1}{2}\int_0^T \|z\|^2_{\mathbf{H}^1(\Omega)}\, dt + c_2 \int_0^T \|z\|^2\, dt. \tag{5.14}$$

On the other hand, by virtue or Remark 2.1 and Assumption 5.1 there exists a constant M such that

$$\|S(x)\|_\infty,\ \|k(x)\|_\infty,\ \|\alpha\|_{L_\infty(0,T;L_\infty(\Gamma_5))} \le M.$$

Therefore, there exists c_3 such that

$$\left|\int_0^T 2(\alpha(t,x)z, z)_{\Gamma_5}\, dt + \int_0^T 2\nu(k(x)z, z)_{\Gamma_2}\, dt + \int_0^T 2\nu(S\tilde{z}, \tilde{z})_{\Gamma_3}\, dt + \int_0^T \nu(k(x)z, z)_{\Gamma_7}\, dt\right|$$
$$\le \int_0^T \left(\frac{\nu c_1}{4}\|z\|^2_{\mathbf{H}^1(\Omega)} + c_3\|z\|^2\right) dt \quad \forall z \in \mathbf{W}(Q) \tag{5.15}$$

(cf. (1), p. 258 in [35] or Theorem 1.6.6 in [27]).

Taking $-k_1$ large enough in (5.12) independently of m, from (5.12)–(5.15) we have,

$$\langle A_m z, z\rangle \geq c_4 \left(\|z\|^2_{\mathbf{W}(Q)}\right), \quad \exists c_4 > 0, \ \forall z \in \mathbf{W}(Q), \tag{5.16}$$

where c_4 depends on m.

Now, let us prove that

$$\text{if } z_k \rightharpoonup z \text{ weakly in } \mathbf{W}(Q) \text{ as } k \to \infty, \ \text{ then } \langle A_m z_k, u\rangle \to \langle A_m z, u\rangle \ \forall u \in \mathbf{W}(Q). \tag{5.17}$$

First, similar to Lemma 3.2 in [58] let us prove that

$$\int_0^T e^{-k_1 t} \left(\operatorname{rot} z_k \times z_k, u\right) \, dt \to \int_0^T e^{-k_1 t} \left(\operatorname{rot} z \times z, u\right) \, dt \quad \forall u \in \mathbf{W}(Q) \text{ as } k \to \infty. \tag{5.18}$$

To this end, let us estimate the following.

$$\begin{aligned} &\int_0^T e^{-k_1 t} \left(\operatorname{rot} z_k \times z_k, u\right) \, dt - \int_0^T e^{-k_1 t} \left(\operatorname{rot} z \times z, u\right) \, dt \\ &\quad = \int_0^T e^{-k_1 t} \left(\operatorname{rot} z_k \times (z_k - z), u\right) \, dt + \int_0^T e^{-k_1 t} \left(\operatorname{rot} (z_k - z) \times z, u\right) \, dt. \end{aligned} \tag{5.19}$$

By the embedding of $H^1(Q)$ in $L_4(Q)$ we have $e^{-k_1 t} zu \in \mathbf{L}_2(Q)$ and as $z_k \rightharpoonup z$ weakly in $\mathbf{W}(Q)$, rot $(z_k - z) \rightharpoonup 0$ weakly in $\mathbf{L}_2(Q)$. Thus, the second integral on the right-hand side of (5.19) converges to zero when $k \to \infty$. Let us consider the first integral on the right-hand side of (5.19). For any $\varepsilon \geq 0$ we can choose $u_\varepsilon \in \mathbf{\Lambda}(Q)$ such that $\|u - u_\varepsilon\|_{\mathbf{W}(Q)} \leq \varepsilon$. Then,

$$\begin{aligned} &\int_0^T e^{-k_1 t} \left(\operatorname{rot} z_k \times (z_k - z), u\right) \, dt \\ &\quad = \int_0^T e^{-k_1 t} \left(\operatorname{rot} z_k \times (z_k - z), u_\varepsilon\right) \, dt + \int_0^T e^{-k_1 t} \left(\operatorname{rot} z_k \times (z_k - z), u - u_\varepsilon\right) \, dt. \end{aligned} \tag{5.20}$$

Since $z_k \to z$ strongly in $\mathbf{L}_2(Q)$ as $k \to \infty$,

$$\left|\int_0^T e^{-k_1 t} \left(\operatorname{rot} z_k \times (z_k - z), u_\varepsilon\right) \, dt\right| \leq C \|\nabla z_k\|_{\mathbf{L}_2(Q)} \|z_k - z\|_{\mathbf{L}_2(Q)} \|u_\varepsilon\|_{\mathbf{L}_\infty(Q)} \to 0 \quad \text{as } k \to \infty. \tag{5.21}$$

Also, since $\{z_k - z\}$ is bounded in $\mathbf{W}(Q)$, we have

$$\begin{aligned} \left|\int_0^T e^{-k_1 t} \left(\operatorname{rot} z_k \times (z_k - z), u - u_\varepsilon\right) \, dt\right| \ &\leq \ C \|\nabla z_k\|_{\mathbf{L}_2(Q)} \|z_k - z\|_{\mathbf{L}_4(Q)} \|u - u_\varepsilon\|_{\mathbf{L}_4(Q)} \\ &\leq C \ \|z_k\|_{\mathbf{W}(Q)} \|z_k - z\|_{\mathbf{W}(Q)} \|u - u_\varepsilon\|_{\mathbf{W}(Q)} \leq C\varepsilon. \end{aligned} \tag{5.22}$$

From (5.20)–(5.22), we know that the first integral on the right-hand side of (5.19) goes to zero when $k \to \infty$, and so we get (5.18).

It is easy to check that other terms in $\langle A_m z_k, u\rangle$ converge when $k \to \infty$. This fact together with (5.17) implies (5.16).

By (5.15) and (5.16), there exists a solution to (5.9) (cf. Theorem 1.2, Chapter IV in [45]), and therefore the assert was proved. □

Lemma 5.3. *If $w^m \in \mathbf{W}(Q)$ are solutions to problem (5.6) with the k_1 in Lemma 5.2, then*

$$\int_Q \frac{1}{m} \frac{\partial w^m}{\partial t} \frac{\partial u}{\partial t} \, dxdt \to 0 \quad \text{as } m \to \infty \ \forall u \in \mathbf{W}(Q). \tag{5.23}$$

Proof. By (5.9), (5.12)–(5.15), we have

$$\int_Q \frac{1}{m} \left|\frac{\partial w^m}{\partial t}\right|^2 dxdt + \frac{\nu c_1}{4} \int_0^T \|w^m\|^2_{\mathbf{H}^1(\Omega)} \, dt + \frac{1}{2} \left(\|w^m(x, 0)\|^2 + \|w^m(x, T)\|^2\right) \leq \left\langle F, w^m\right\rangle. \tag{5.24}$$

On the other hand,

$$\int_Q \bar{U} \frac{\partial w^m}{\partial t} \, dxdt = (\bar{U}, w^m(x, T)) - (\bar{U}, w^m(x, 0)) - \int_Q \frac{\partial \bar{U}}{\partial t} w^m \, dxdt \quad \forall w^m \in \mathbf{W}(Q). \tag{5.25}$$

Taking (5.25) into account in (5.8) and applying Young's inequality to the right-hand side of (5.24), we have

$$\int_Q \frac{1}{m}\left|\frac{\partial w^m}{\partial t}\right|^2 dxdt + \frac{\nu c_1}{8}\int_0^T \|w^m\|^2_{\mathbf{H}^1(\Omega)}\,dt + \frac{1}{8}\left(\|w^m(x,0)\|^2 + \|w^m(x,T)\|^2\right) \le c, \tag{5.26}$$

where c is independent of m and depends on $\bar{U}$, $\bar{\phi}_i$, v_0, $\bar{f}$, S, k and k_1.

Using $\int_Q \frac{1}{m}\left|\frac{\partial w^m}{\partial t}\right|^2 dxdt \le c$, we can get (5.23). Indeed, by Hölder's inequality

$$\left|\int_Q \frac{1}{m}\frac{\partial w^m}{\partial t}\frac{\partial u}{\partial t}\,dxdt\right| \le \frac{1}{\sqrt{m}}\left(\int_Q \left|\frac{1}{\sqrt{m}}\frac{\partial w^m}{\partial t}\right|^2 dxdt \cdot \int_Q \left|\frac{\partial u}{\partial t}\right|^2 dxdt\right)^{\frac{1}{2}}$$

which shows (5.23). □

Lemma 5.4. *If $w^m \in \mathbf{W}(Q)$ are solutions to problem (5.6) guaranteed by Lemma 5.2, then $\{w^m\}$ are precompact in $\mathbf{L}_2(Q)$.*

This Lemma is proved as Lemma 4.1 in [58].

Proof of Theorem 5.1. Let $\{w^m(Q)\}$ be the sequence of solutions to (5.6) guaranteed by Lemma 5.2. By (5.26) $\{w^m(Q)\}$ is bounded in $V(Q)$. Then by Lemma 5.4, we can choose its subsequence $\{w^k(Q)\}$ such that $w^k(Q) \to w \in V(Q)$ strongly in $\mathbf{L}_2(Q)$.

First, let us prove that

$$\int_0^T e^{-k_1 t}\left(\operatorname{rot} w^k \times w^k, u\right) dt \to \int_0^T e^{-k_1 t}\left(\operatorname{rot} w \times w, u\right) dt \quad \forall u \in \mathbf{\Lambda}(Q) \text{ as } k \to \infty. \tag{5.27}$$

We can write

$$\begin{aligned}&\int_0^T e^{-k_1 t}\left(\operatorname{rot} w^k \times w^k, u\right) dt - \int_0^T e^{-k_1 t}\left(\operatorname{rot} w \times w, u\right) dt\\ &\quad = \int_0^T e^{-k_1 t}\left(\operatorname{rot} w^k \times (w^k - w), u\right) dt + \int_0^T e^{-k_1 t}\left(\operatorname{rot}(w^k - w) \times w, u\right) dt.\end{aligned} \tag{5.28}$$

Since $\{\operatorname{rot} w^k\}$ is bounded in $\mathbf{L}_2(Q)$, $w^k \to w$ in $\mathbf{L}_2(Q)$ and $u \in L_\infty$, the first integral on the right hand side of (5.28) converges to zero as $k \to \infty$. Meanwhile, since $e^{-k_1 t} wu \in \mathbf{L}_2(Q)$ and $w_k \rightharpoonup w$ weakly in $\mathbf{V}(Q)$, the second integral on the right-hand side of (5.28) converges to zero either. Thus, (5.27) follows.

It is easy to verify the convergence of other terms in (5.6) when $k \to \infty$. Thus, passing to limit as $k \to \infty$ in (5.6) with k instead of m, by Lemma 5.3, (5.1) we have (5.4). The estimate (5.5) can be obtained in the same way as the proof of (4.2) in [58]. □

Let us consider the Stokes problem

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu\Delta v + \nabla p = f, \quad \nabla\cdot v = 0 \quad \text{in } Q\\ v(0) = v_0 \end{cases} \tag{5.29}$$

with boundary condition (3.3).

Assume that the following holds.

Assumption 5.2. There exists a function $U \in \mathbf{H}^1(Q)$ such that

$$\operatorname{div} U = 0, \qquad U|_{\Sigma_1} = h_1, \qquad U_\tau|_{(\Sigma_2\cup\Sigma_7)} = 0, \qquad U\cdot n|_{\Sigma_3} = 0, \qquad U_\tau|_{\Sigma_4} = h_4, \qquad U\cdot n|_{\Sigma_5} = h_5.$$

Also, $f \in L_2(0,T;\mathbf{V}(\Omega)^*)$, $\phi_i \in L_2(0,T;H^{-\frac{1}{2}}(\Gamma_i))$, $i = 2, 4$, $\phi_i \in L_2(0,T;\mathbf{H}^{-\frac{1}{2}}(\Gamma_i))$, $i = 3, 5, 6, 7$, $\alpha_{ij} \in L_\infty(0,T;L_\infty(\Gamma_5))$, $v_0 - U(x,0) \in H(\Omega)$, where $H(\Omega)$ is closure of $\mathbf{V}(\Omega)$ in $\mathbf{L}_2(\Omega)$, and $\Gamma_1 \ne \emptyset$.

Then, using (3.5), (3.6) we get a variational formulation for problem (5.29), (3.3):

Formulation 5.2. *Find v such that*

$$\begin{aligned}&v - U \in \mathbf{V}(Q),\\ &-\int_0^T \left(v, \frac{\partial u}{\partial t}\right) dt + 2\nu\int_0^T (\varepsilon(v), \varepsilon(u))\,dt + 2\nu\int_0^T (k(x)v, u)_{\Gamma_2}\,dt\end{aligned}$$

$$+2\nu\int_0^T (S\tilde{v},\tilde{u})_{\Gamma_3}\,dt+2\int_0^T(\alpha(t,x)v,u)_{\Gamma_5}\,dt+\int_0^T\nu(k(x)v,u)_{\Gamma_7}$$
$$=(v_0,u(0))+\int_0^T\langle f,u\rangle\,dt+\int_0^T\sum_{i=2,4}\langle\phi_i,u\cdot n\rangle_{\Gamma_i}\,dt+\int_0^T\sum_{i=3,5,6,7}\langle\phi_i,u\rangle_{\Gamma_i}\,dt$$
$$\forall u\in\mathbf{\Lambda}(Q)\ \textit{with}\ u(x,T)=0. \tag{5.30}$$

Theorem 5.5. *Under Assumption 5.2 there exists a unique solution to Formulation 5.2 of the non-stationary Stokes problem with mixed boundary condition (3.3) and the solution belongs to $C([0,T];\mathbf{L}_2(\Omega))$.*

Proof. Put $v=z+U$, $w=e^{k_1t}z$, where k_1 is taken as Lemma 5.2 under consideration of the fact that there is not the nonlinear term. Then, we get the new problem equivalent to Formulation 5.2.

Find $w\in L_2(0,T;\mathbf{V})$ such that

$$-\int_0^T\left(w,\frac{\partial\hat{u}}{\partial t}\right)dt+2\nu\int_0^T(\varepsilon(w),\varepsilon(\hat{u}))\,dt+\int_0^T2\nu(k(x)w,\hat{u})_{\Gamma_2}\,dt+\int_0^T2\nu(S\tilde{w},\tilde{\hat{u}})_{\Gamma_3}\,dt$$
$$+\int_0^T2(\alpha(t,x)w,\hat{u})_{\Gamma_5}\,dt+\int_0^T\nu(k(x)w,\hat{u})_{\Gamma_7}\,dt-k_1\int_0^T(w,\hat{u})\,dt$$
$$=\int_Q k_1\bar{U}\hat{u}\,dxdt-\int_Q\frac{\partial\bar{U}}{\partial t}\hat{u}\,dxdt-2\nu\int_0^T(\varepsilon(\bar{U}),\varepsilon(\hat{u}))\,dt-\int_0^T2\nu(k(x)\bar{U},\hat{u})_{\Gamma_2}\,dt$$
$$-\int_0^T2\nu(S\tilde{\bar{U}},\tilde{\hat{u}})_{\Gamma_3}\,dt-\int_0^T2(\alpha(t,x)\bar{U},\hat{u})_{\Gamma_5}\,dt-\int_0^T\nu(k(x)\bar{U},u)_{\Gamma_7}+\int_0^T\langle\bar{f},\hat{u}\rangle\,dt$$
$$+\int_0^T\sum_{i=2,4}\langle\bar{\phi}_i,\hat{u}\cdot n\rangle_{\Gamma_i}\,dt+\int_0^T\sum_{i=3,5,6,7}\langle\bar{\phi}_i,\hat{u}\rangle_{\Gamma_i}\,dt+\big(v_0-U(x,0),u(x,0)\big)$$
$$\forall\hat{u}\in\mathbf{\Lambda}(Q),\ \hat{u}(x,T)=0, \tag{5.31}$$

where $\bar{\gamma}=e^{k_1t}\gamma$ for γ.

Existence of a solution to (5.31) is proved as Theorem 5.1 without applying the condition $U\in\mathbf{L}_\infty(Q)$ and Lemma 5.4, which are needed for the nonlinear term.

To finish proof, by Assumption 5.2 it is enough to prove that the solutions to (5.31) belong to $C([0,T];H)$ and are unique.

When $w\in L_2(0,T;\mathbf{V})$ is a solution to (5.31), estimating

$$I\equiv\int_0^T\Big[-2\nu(\varepsilon(w),\varepsilon(\hat{u}))-2\nu(k(x)w,u)_{\Gamma_2}-2\nu(S\tilde{w},\tilde{\hat{u}})_{\Gamma_3}$$
$$-2(\alpha(t,x)w,\hat{u})_{\Gamma_5}-\nu(k(x)w,\hat{u})_{\Gamma_7}-k_1(w,\hat{u})+k_1(\bar{U},\hat{u})-\left(\frac{\partial\bar{U}}{\partial t},\hat{u}\right)$$
$$-2\nu(\varepsilon(\bar{U}),\varepsilon(\hat{u}))-2\nu(k(x)\bar{U},\hat{u})_{\Gamma_2}-2\nu(S\tilde{\bar{U}},\tilde{\hat{u}})_{\Gamma_3}-2(\alpha(t,x)\bar{U},\hat{u})_{\Gamma_5}$$
$$-\nu(k(x)U,u)_{\Gamma_7}+\langle\bar{f},\hat{u}\rangle+\sum_{i=2,4}\langle\bar{\phi}_i,\hat{u}\cdot n\rangle_{\Gamma_i}+\sum_{i=3,5,6,7}\langle\bar{\phi}_i,\hat{u}\rangle_{\Gamma_i}\Big]dt\quad\forall\hat{u}\in L_2(0,T;\mathbf{V})$$

we get

$$|I|\le K\|\hat{u}\|_{L_2(0,T;\mathbf{V})}\quad\forall\hat{u}\in L_2(0,T;\mathbf{V}).$$

This means that I is a continuous linear functional on $L_2(0,T;\mathbf{V})$. Thus, there exists a $\overline{F}\in L_2(0,T;\mathbf{V}^*)$ such that

$$I=\int_0^T\langle\overline{F},\hat{u}\rangle\,dt. \tag{5.32}$$

Taking any $\phi\in\mathscr{D}(0,T)$ and $u\in\mathbf{V}\cap C^2(\overline{\Omega})$ and putting $\hat{u}=\phi\cdot u$, by definition of $w'\in\mathscr{D}^*(0,T;V^*)$ (cf. Definition 1.10, Chapter 4, [41]), derivative of w, (5.31), (5.32), Theorem 1.8 in Chapter 4, [41] we have

$$\langle w'(\phi),u\rangle=-\Big\langle\int_0^T w\phi'\,dt,u\Big\rangle=-\int_0^T\langle w,\phi'u\rangle\,dt=\int_0^T\langle\overline{F},\phi u\rangle\,dt=\Big\langle\int_0^T\overline{F}\phi\,dt,u\Big\rangle,$$

which means $w'=\overline{F}\in L_2(0,T;\mathbf{V}^*)$.

Therefore, $w \in C([0,T];H)$ and

$$\int_0^T \langle w, w'\rangle\, dt = \frac{1}{2}\big[\|w(T)\|^2 - \|w(0)\|^2\big]. \tag{5.33}$$

Let w_1, w_2 be solutions to (5.31) corresponding to the same given data and $w = w_1 - w_2$. Then, by (5.31) we have

$$\int_0^T \langle w', \hat{u}\rangle\, dt + 2\nu \int_0^T (\varepsilon(w), \varepsilon(\hat{u}))\, dt + \int_0^T 2\nu (k(x)w, u)_{\Gamma_2}\, dt + \int_0^T 2\nu (S\tilde{w}, \tilde{\hat{u}})_{\Gamma_3}$$
$$+ \int_0^T 2(\alpha(t,x)w, \hat{u})_{\Gamma_5}\, dt + \int_0^T \nu (k(x)w, \hat{u})_{\Gamma_7}\, dt - k_1 \int_0^T (w, \hat{u})\, dt = 0 \quad \text{for } \hat{u} = \phi \cdot u. \tag{5.34}$$

Since the set $\{\hat{u} = \phi \cdot u : \phi \in \mathscr{D}(0,T),\, u \in \mathbf{V} \cap C^2(\overline{\Omega})\}$ is dense in $L_2(0,T;\mathbf{V})$, (5.34) is valid for $\hat{u} = w$. Thus, from (5.33), (5.34) it follows that

$$\|w(T)\|^2 + \beta \int_0^T \|w\|_V^2\, dt \le 0, \quad \beta > 0$$

(cf. (5.13), (5.15)), which shows that $w \equiv 0$, that is, a solution to (5.31) is unique. □

Remark 5.1. We will study unique existence of local solutions to (5.1) with the boundary condition (3.3) in other paper. Some special cases of the problem were studied (cf. [21,22,54,53]).

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Appendix

In this section for convenience of readers we give proof properties of the shape operator of surface.

Lemma A.1. *Let Γ be a surface (curve for $l = 2$) of C^2 and v be a vector field of C^2 on a domain of R^l near Γ. Suppose that $v \cdot n|_\Gamma = 0$. Then, $\tau \cdot J(n(\bar{x}))v = \frac{\partial n}{\partial \tau} \cdot v = (S\tilde{v}, \tilde{\tau})_{R^{l-1}}$ on the surface Γ, where*

$$S = \begin{pmatrix} L & K \\ M & N \end{pmatrix},$$
$$L = \left(e_1, \frac{\partial n}{\partial e_1}\right)_{R^l}, \quad K = \left(e_2, \frac{\partial n}{\partial e_1}\right)_{R^l}, \quad M = \left(e_1, \frac{\partial n}{\partial e_2}\right)_{R^l} \quad \text{and} \quad N = \left(e_2, \frac{\partial n}{\partial e_2}\right)_{R^l}. \tag{A.1}$$

When $l = 3$, the bilinear form $(S\tilde{v}, \tilde{u})_{R^{l-1}}$ for vector u, v tangent to the surface is independent from curvilinear coordinate system which is orthogonal at all points each other.

Proof. In a neighborhood of a point $\bar{x} \in \Gamma$ let us introduce local curvilinear coordinates $(y_1(x_1,x_2,x_3), y_2(x_1,x_2,x_3), y_3(x_1,x_2,x_3))$ such that the lines (y_1), (y_2) and (y_3) are orthogonal at all points each other, the linear (y_3) is the outward normal n to Γ and the surface $y_3 = 0$ coincides with Γ, where $x = (x_1,x_2,x_3)$ is the original orthogonal coordinates. Denote the unit vector of $\nabla_x y_k$ by e_k and let $\tilde{\tau}_{e_k} = (\tau, e_k)_{R^l}$, $\tilde{v}_{e_k} = (v, e_k)_{R^l}$ for $k = 1, 2$.

$$\tau \cdot J(n(\bar{x}))v = \frac{\partial n}{\partial \tau} \cdot v = \sum_i \sum_j \tau_i \frac{\partial n_j}{\partial x_i} v_j = \sum_i \sum_j \tau_i \left(\sum_k \frac{\partial n_j}{\partial y_k}\frac{\partial y_k}{\partial x_i}\right) v_j$$
$$= \sum_k \sum_j \left(\frac{\partial n_j}{\partial y_k} v_j\right) \sum_i \tau_i \frac{\partial y_k}{\partial x_i} = \sum_{k=1}^3 \left(\frac{\partial n}{\partial y_k}, v\right)_{R^l} \cdot (\tau, \nabla y_k)_{R^l}. \tag{A.2}$$

From $\nabla_x y_3(x_1,x_2,x_3) = n$, we get $\frac{\partial n}{\partial y_3} = 0$ on Γ. Thus,

$$\tau \cdot J(n(\bar{x}))v = \sum_{k=1}^2 \left(\frac{\partial n}{\partial y_k}, v\right)_{R^l} \cdot (\tau, \nabla y_k)_{R^l}. \tag{A.3}$$

We get

$$(\tau, \nabla y_k) = |\nabla y_k| \tilde{\tau}_{e_k} \quad \text{for } k = 1, 2. \tag{A.4}$$

Also, since

$$\frac{\partial y_k}{\partial e_k} = (\nabla y_k, e_k) = |\nabla y_k| \quad \text{for } k = 1, 2,$$

we get

$$\frac{\partial n}{\partial y_k} = \frac{\partial n}{\partial e_k} \frac{\partial e_k}{\partial y_k} = \frac{\partial n}{\partial e_k} \frac{1}{|\nabla y_k|} \quad \text{for } k = 1, 2. \tag{A.5}$$

Therefore, from (A.3)–(A.5) we have

$$\begin{aligned} \tau \cdot J(n(\bar{x}))v &= \frac{\partial n}{\partial \tau} \cdot v = \sum_{k=1}^{2} \left(\frac{\partial n}{\partial y_k}, v \right)_{R^l} \cdot (\tau, \nabla y_k)_{R^l} = \sum_{k=1}^{2} \left(v, \frac{\partial n}{\partial e_k} \right)_{R^l} \tilde{\tau}_{e_k} \\ &= \sum_{k=1}^{2} \sum_{r=1}^{2} \left(e_r, \frac{\partial n}{\partial e_k} \right)_{R^l} \tilde{v}_{e_r} \tilde{\tau}_{e_k} = (S\tilde{v}, \tilde{\tau})_{R^2}. \end{aligned} \tag{A.6}$$

For $l = 2$ in the same way we can have the result with $S = (e_\tau, \frac{\partial n}{\partial e_\tau})$, i.e. the curvature of boundary, where e_τ is unit vector tangent to curve Γ.

Let $l = 3$. In a neighborhood of a point $\bar{x} \in \Gamma$ let us introduce new local curvilinear coordinates $(\tilde{y}_1, \tilde{y}_2, y_3)$ as above. Denote the unit vector of $\nabla_x \tilde{y}_k,\ k = 1, 2$ by $\tilde{e}_k$. Let $\tilde{v}_{\tilde{e}_k} = (v, \tilde{e}_k)$ and $\tilde{u}_{\tilde{e}_k} = (u, \tilde{e}_k)$.

Let $T_{\bar{x}}(\Gamma)$ be a tangent space to the manifold Γ at $\bar{x}$. Since $\{e_1, e_2\}$ and $\{\tilde{e}_1, \tilde{e}_2\}$ are orthogonal bases in $T_{\bar{x}}(\Gamma)$, there exists a matrix C such that

$$\begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{pmatrix} = C \begin{pmatrix} e_1 \\ e_2, \end{pmatrix} \quad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}. \tag{A.7}$$

Then,

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = C^{-1} \begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{pmatrix}, \qquad \tilde{v}_e = C^T \tilde{v}_{\tilde{e}} \quad \text{and} \quad \tilde{u}_e = C^T \tilde{u}_{\tilde{e}}. \tag{A.8}$$

Putting $C^{-1} = (c'_{ij})$, by (A.7), (A.8) we have

$$\left(e_i, \frac{\partial n}{\partial e_j} \right)_{R^3} = \left(\sum_l c'_{il} \tilde{e}_l, \sum_k \frac{\partial n}{\partial \tilde{e}_k} \frac{\partial \tilde{e}_k}{\partial e_j} \right)_{R^3} = \sum_l \sum_k c'_{il} \left(\tilde{e}_l, \frac{\partial n}{\partial \tilde{e}_k} \right)_{R^3} c_{kj}. \tag{A.9}$$

Let

$$\tilde{S} = \begin{pmatrix} \tilde{L} & \tilde{K} \\ \tilde{M} & \tilde{N} \end{pmatrix},$$

$$\tilde{L} = \left(\tilde{e}_1, \frac{\partial n}{\partial \tilde{e}_1} \right)_{R^l}, \qquad K = \left(\tilde{e}_2, \frac{\partial n}{\partial \tilde{e}_1} \right)_{R^l}, \qquad M = \left(\tilde{e}_1, \frac{\partial n}{\partial \tilde{e}_2} \right)_{R^l} \quad \text{and} \quad N = \left(\tilde{e}_2, \frac{\partial n}{\partial \tilde{e}_2} \right)_{R^l}.$$

Then, by (A.1) and (A.9) we have

$$S = C^{-1} \tilde{S} C. \tag{A.10}$$

The matrix C is unitary, and so $C^T = C^{-1}$. Therefore, by (A.8) and (A.10)

$$(S\tilde{v}_e, \tilde{u}_e)_{R^2} = \tilde{u}_e^T S \tilde{v}_e = \tilde{u}_{\tilde{e}}^T C C^{-1} \tilde{S} C C^T \tilde{v}_{\tilde{e}} = \tilde{u}_{\tilde{e}}^T \tilde{S} \tilde{v}_{\tilde{e}} = (\tilde{S}\tilde{v}_{\tilde{e}}, \tilde{u}_{\tilde{e}})_{R^2}. \quad \square$$

Remark A.1. When the parameters of curves $(y_1), (y_2)$ are natural one (lengths of curves), coefficients of the second fundamental form of surface Γ are the same with $-L, -K, -M, -N$ and $K = M$ (cf. Lemma 7.2.3 in [65]).

Definition A.2. If a piece of boundary on a neighborhood of $x \in \partial\Omega$ is on the opposite (same) side of the outward normal vector with respect to the tangent plane (line for $l = 2$) at x or coincides with the tangent plane, then the piece of boundary called convex (concave) at x. If at all $x \in \Gamma \subset \partial\Omega$ the boundary convex (concave), then Γ called convex (concave).

Lemma A.3. *If Γ_{ij} are convex (concave), then quadratic forms $(S\tilde{v}, \tilde{v})|_{\Gamma_i}$ and $(k(x)v, v)_{\Gamma_i}$ are positive (negative).*

Proof. Let us consider the case of convexity, since the case of concavity is proved in the same way.

First, let us prove that $(S\tilde{v}, \tilde{v})|_{\Gamma_3}$ is positive.

Let $l = 3$. As the proof of Lemma A.1, in a neighborhood of a point $\bar{x} \in \Gamma$ let us introduce local curvilinear coordinates (ξ, ζ, η) such that the lines (ξ), (ζ), (η) are orthogonal at all points each other, the linear (η) is the outward normal n to Γ and the surface $\eta = 0$ coincides with Γ. We take natural parameters (lengths) of curves as parameters. As the parameters (ξ, ζ) of Γ change to $(\xi + \Delta\xi, \zeta + \Delta\zeta)$, the surface moves away from the plane tangent at $\bar{x} \in \Gamma$ by the deviation $-\frac{1}{2}\left[L(\Delta\xi)^2 + 2M\Delta\xi\Delta\zeta + N(\Delta\zeta)^2 + o((\Delta\xi)^2 + (\Delta\zeta)^2)\right] \cdot n$, where n is the normal unit vector at $\bar{x}$ (cf. Section 7.1 in [65]). Here the fact that $K = M$ was used. If the surface is convex, then the quadratic form $-\left[L(\Delta\xi)^2 + 2M\Delta\xi\Delta\zeta + N(\Delta\zeta)^2\right]$ is negative, and so the matrix S is positive.

By the argument as above the case that $l = 2$ is proved.

Next, let us prove that $(k(x)v, v)_{\Gamma_2}$ is positive. Divergence of $n(x)$ is expressed by $k(x) = (e_u, \frac{\partial n}{\partial s_u}) + (e_v, \frac{\partial n}{\partial s_v}) = L + N$, where the fact that u, v are natural parameters was used (cf. proof of Lemma 5.1 in [51]). Then, by the fact that S is positive, we get $k(x) \geq 0\ \forall x \in \Gamma$, which shows the assert is valid. □

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