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Behaviour at Time $t=0$ of the Solutions of Semi-linear Evolution Equations

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We investigate the regularity at time $t=0$ of the solutions of linear and semi-linear evolutions equations (including the Stokes and Navier–Stokes equations). Necessary and sufficient conditions on the data for an arbitrary order of regularity are given (the classical "compatibility conditions"). In the case of the Stokes and Navier–Stokes equations the compatibility conditions which we find are *global* conditions on the data. The presentation given here seems to improve and generalize the known results even in the simplest case of linear evolution equations.

We are considering, here, semi-linear evolution equations of the type

$$\frac{du}{dt}(t) + Au(t) = g(t, u), \qquad 0 < t < T, \tag{0.1}$$

$$u(0) = u_0 \tag{0.2}$$

and we are interested in determining in what sense the initial condition (0.2) is achieved, i.e., to determine the best space in which

$$u(t) \to u_0 \qquad \text{as} \quad t \to 0. \tag{0.3}$$

We assume that A is a linear unbounded operator in a Hilbert space H with domain $D(A)$, and that g is a smooth function on $[0, T] \times D(A)$, dominated in some sense by A. The precise assumptions are described later on.

The realization of the convergence (0.3) in a space smaller than H is a problem connected with the regularity near $t=0$ of the solutions of (0.1), (0.2) and is not in general expected even in the simple case of the linear heat equation. However, it is known for linear equations ($g(t, u) \equiv g(t)$) that if u_0 and g satisfy some *compatibility conditions*, then u is more regular at time $t=0$ and the convergence (0.3) is achieved in a space better than H, cf. Friedman [5], Ladyzhenskaya, Solonnikov, and Uralceva [8], Rauch and Massey [11], and Smale [12].

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Our task is to determine the compatibility conditions which guarantee the regularity near $t=0$ of the solutions of (0.1), (0.2). Actually we will show that under appropriate assumptions, some *very simple* and natural necessary conditions of regularity are also sufficient. Some examples are given and in particular our results apply to the evolution Navier–Stokes equations for which they seem to be new. In the case of Navier–Stokes equations (or even of the linearized Stokes equations) the compatibility conditions on u_0 are of a *global* type and are totally unusual. Besides the intrinsic interest of this result we had some other motivations in doing this work which will be developed elsewhere (in particular the extension of the results of [3] from the space periodic case to the case of the flow in a domain with boundary, and the numerical approximation of the evolution Navier–Stokes equations). We believe that the method presented here is general and applies to many other nonlinear evolution equations.

1. Preliminaries

1.1 *Notation*

Let H be a real Hilbert space, and A a linear self-adjoint operator from $D(A)$ into H, where $D(A)=$ the domain of A is a subspace of H. We assume that A is an isomorphism from $D(A)$ (equipped with the graph norm) onto H, that A^{-1} is a linear compact operator in H and that

$$(Au, u) > 0, \qquad \forall u \in D(A), \quad u \neq 0. \tag{1.1}$$

One can then define the powers A^{α} of A for any $\alpha \in \mathbb{R}$, with domain $D(A^{\alpha})$; $D(A^{\alpha})$ is a Hilbert space for the norm $|A^{\alpha}u|_H$ and A^{α} is an isomorphism from $D(A^{\alpha})$ onto H. We set

$$V_{\alpha}=D(A^{\alpha/2}), \qquad \forall \alpha \in \mathbb{R}, \tag{1.2}$$

and in particular $V=V_1$, $H=V_0$. For $\alpha>0$ the space $V_{-\alpha}$ can be identified with the dual $(V_{\alpha})'$ of V_{α}. We denote by $|\cdot|_{\alpha}$ the norm in V_{α}; $|\cdot|$ the norm in H.

It is well known that for $T>0$, and f and u_0 given, satisfying

$$f \in L^2(0, T; V'), \qquad u_0 \in H, \tag{1.3}$$

the initial value problem

$$\frac{du}{dt}(t)+Au(t)=f(t), \qquad 0<t<T, \tag{1.4}$$

$$u(0)=u_0 \tag{1.5}$$

possesses a unique solution u which satisfies

$$u \in L^2(0, T; V) \cap C([0, T]; H) \tag{1.6}$$

which implies

$$u(t) \to u_0 \text{ in } H \text{ as } t \to 0. \tag{1.7}$$

If furthermore

$$f \in L^2(0, T; H), \qquad u_0 \in V, \tag{1.8}$$

then u satisfies

$$u \in L^2(0, T; D(A)) \cap C([0, T]; V) \tag{1.9}$$

and

$$u(t) \to u_0 \text{ in } V \text{ as } t \to 0. \tag{1.10}$$

Now if $u_0 \in V_m$, $m \geqslant 3$, it is not true, in general, that $u(t) \in V_m$ for $t > 0$ and that $u(t) \to u_0$ in V_m as $t \to 0$. The necessary and sufficient conditions that u_0 and f must satisfy in order that $u(t) \to u_0$ in a norm stronger than the norm of V are given in [8, 12] for special classes of operators A. We intend here to extend these results to general operators A and then to nonlinear equations of evolution.

1.2. *Hypotheses*

Besides the scale of spaces V_α, we introduce a family of Hilbert spaces E_m, $m \in \mathbb{N}$, with

$$E_{m+1} \subset E_m, \ \forall m \in \mathbb{N}, \text{ the injection being continuous, and } E_0 = H \tag{1.11}$$

$$\begin{aligned}&V_m \text{ is a closed subspace of } E_m, \ \forall m \in \mathbb{N}, \text{ the norm induced}\\ &\text{by } E_m \text{ on } V_m \text{ being equivalent to } |\cdot|_m.\end{aligned} \tag{1.12}$$

We are also given a linear operator $\mathscr{A}$ satisfying

$$\mathscr{A} \text{ is continuous from } E_{m+2} \text{ into } E_m, \ \forall m \geqslant 0, \tag{1.13}$$

$$\mathscr{A} \text{ is an isomorphism from } E_{m+2} \cap V \text{ onto } E_m, \ \forall m \geqslant 0, \tag{1.14}$$

$$\mathscr{A} u = Au, \ \forall u \in V_m, \ m \geqslant 2.^1 \tag{1.15}$$

[1] In the applications the E_m's are Sobolev type spaces, $\mathscr{A}$ is the differential operator, and A the abstract operator associated to $\mathscr{A}$; cf. the examples.

We denote by G the inverse A^{-1} of A which is linear continuous from V_α onto $V_{\alpha+2}$, $\forall \alpha \in \mathbb{R}$. Because of (1.11), (1.14), and (1.15),

$$G\mathscr{A}u = u, \qquad \forall u \in E_{m+2} \cap V, \quad m \geqslant 0, \tag{1.16}$$

while, in general $G\mathscr{A}u \neq u$, for an arbitrary u in E_{m+2}, $m \geqslant 0$.

We then have the following scheme

$$\begin{array}{l} E_m \subset E_1 \subset E_0 \\ \cup \quad\ \cup \quad \| \\ V_m \subset V \subset H \subset V_{-1} = V' \subset V_{-m} = V'_m. \end{array} \tag{1.17}$$

EXAMPLE 1. We show how the previous assumptions are satisfied in the easy case of the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = f \text{ in } \Omega \times (0, T), \tag{1.18}$$

where Ω is a bounded open set of $\mathbb{R}^n$ with boundary Γ. New examples are described in Section 2.3.

We assume that Γ is a C^∞ manifold of dimension $n-1$ and we set $H = L^2(\Omega)$, $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$, $Au = -\Delta u$, $\forall u \in D(A)$, and $E_m = H^m(\Omega)$, $\forall m \geqslant 0$, $\mathscr{A}u = -\Delta u$, $\forall u \in H^2(\Omega)$. The operator A is an isomorphism from $D(A)$ onto H because of the regularity theory for elliptic boundary value problems (cf. Agmon, Douglis, and Nirenberg [1]), and assumption (1.14) also follows from [1]. It is clear that V_1 is a closed subspace of E_1, $V_2 = D(A)$ is a closed subspace of E_2: assumption (1.12) for $m > 2$ follows then from this remark, the definition of A and (1.14). All the other assumptions are trivial or well known. The spaces V_m and E_m are different for $m \geqslant 3$. For instance $V_4 = D(A^2) = \{u \in E_4 = H_4(\Omega),\ u = \Delta u = 0$ on $\Gamma\}$.

1.3. *The Successive Derivatives of u*

For any integer m, we denote by W_m the space

$$W_m = \left\{ v \in C([0,T]; E_m), \frac{d^j v}{dt^j} \in C([0,T]; E_{m-2j}), j = 1,\dots, l \right\},$$

where l is the integer part of $m/2$.

The following remark will be useful.

LEMMA 1.1. *We assume that* (1.8) *is satisfied and that, for some* $m \geqslant 2$,

$$f \in W_{m-2}, \tag{1.19}$$

and the solution u of (1.4), (1.5) *satisfies*

$$u \in C([0, T]; E_m). \tag{1.20}$$

Then

$$u \in W_m. \tag{1.21}$$

Proof. Due to (1.15), Eq. (1.4) can be rewritten

$$\frac{du}{dt} + \mathscr{A} u = f. \tag{1.22}$$

Because of (1.19), (1.20), and (1.13),

$$\frac{du}{dt} = f - \mathscr{A} u \in C([0, T]; E_{m-2}).$$

Applying then the operator $d/dt - \mathscr{A}$ to each member of (1.22), we get[2]

$$\frac{d^2u}{dt^2} - \mathscr{A}^2 u = \left(\frac{d}{dt} - \mathscr{A}\right) f.$$

Whence, with (1.19), (1.20), and (1.13)

$$\frac{d^2u}{dt^2} = \mathscr{A}^2 u + \left(\frac{d}{dt} - \mathscr{A}\right) f \in C([0, T]; E_{m-4}).$$

More generally, with $\delta = d/dt$,

$$-(-\delta)^{j+1} + \mathscr{A}^{j+1} = \alpha_j(\delta, \mathscr{A})(\delta + \mathscr{A}) = (\delta + \mathscr{A})\, \alpha_j(\delta, \mathscr{A}), \tag{1.23}$$

where

$$\alpha_j(\delta, \mathscr{A}) = \sum_{i=0}^{j} (-\delta)^i \mathscr{A}^{j-i} \tag{1.24}$$

and therefore, for $j = 0,..., l-1$,

$$(-1)^{j+1} \frac{d^{j+1}u}{dt^{j+1}} = \mathscr{A}^{j+1} u - \alpha_j\left(\frac{d}{dt}, \mathscr{A}\right) f, \tag{1.25}$$

and we infer from (1.19), (1.20), and (1.13) that the right-hand side of (1.25) belongs to $C([0, T]; E_{m-2j-2})$ and the result follows. ∎

[2] The time derivatives are understood in the sense of vector-valued distributions.

Remark 1.1. Under the assumptions of Lemma 1.1 one can compute all the derivatives

$$\frac{d^j u}{dt^j}(t), \qquad j = 1,\dots, l, \quad t \in [0, T],$$

in term of $u(t)$ and the data f. This can be obtained by differentiating $l-1$ times Eq. (1.4) to obtain

$$\begin{aligned} &\frac{d^2u}{dt^2} + \mathscr{A}\frac{du}{dt} = \frac{df}{dt}, \\ &\frac{d^3u}{dt^3} + \mathscr{A}\frac{d^2u}{dt^2} = \frac{d^2f}{dt^2}, \qquad \text{etc.,} \end{aligned} \tag{1.26}$$

and then eliminating $du/dt,\dots, d^{j-1}u/dt^{j-1}$. More directly, with (1.25) we get

$$(-1)^{j+1}\frac{d^{j+1}u}{dt}(t) = \mathscr{A}^{j+1}u(t) - [\varphi_j(f)](t), \qquad j = 0,\dots, l-1, \quad t \in [0, T], \tag{1.27}$$

where we set

$$\varphi_j(f) = \alpha_j\left(\frac{d}{dt}, \mathscr{A}\right)f = \sum_{i=0}^{j}\left(-\frac{d}{dt}\right)^i \mathscr{A}^{j-i}f. \tag{1.28}$$

In particular at $t = 0$

$$\frac{d^{j+1}u}{dt^{j+1}}(0) = (-1)^{j+1}\mathscr{A}^{j+1}u_0 + (-1)^j\psi_j(f) \tag{1.29}$$

where

$$\psi_j(f) = [\varphi_j(f)](0). \tag{1.30}$$

2. Linear Problems

2.1. *Result in the Linear Case*

Theorem 2.1. *The assumptions are those of Sections* 1.1 *and* 1.2 *and we assume that for some* $m \geqslant 2$

$$u_0 \in E_m \cap D(A), \qquad f \in W_{m-2}, \tag{2.1}$$

$$\begin{aligned} \frac{d^l f}{dt^l} &\in L^2(0, T; H) \qquad \text{if} \quad m = 2l+1 \\ &\in L^2(0, T; V') \qquad \text{if} \quad m = 2l. \end{aligned} \tag{2.2}$$

Then a necessary and sufficient condition for the solution of (1.4), (1.5) *to belong to* W_m *is that*

$$\frac{d^j u}{dt^j}(0) \in V \quad \text{for} \quad j = 1,..., l-1,$$

$$\frac{d^l u}{dt^l}(0) \in V \quad \text{if} \quad m = 2l+1, \ \in H \text{ if } m = 2l,$$

the expression of the derivatives $d^j u/dt^j(0)$ *in terms of the data being given by formulas* (1.29) *and* (1.30), *i.e.*,

$$\frac{d^j u}{dt^j}(0) = (-\mathscr{A})^j u_0 + \sum_{i=0}^{j-1} \left(\frac{d}{dt}\right)^i (-\mathscr{A})^{j-i-1} f \Big|_{t=0} \tag{2.4}$$

for $j = 1,..., l$. ∎

Two proofs of the theorem are given in Sections 2.2 and 2.3, the first one simpler, the second one giving more information.

2.2. *Proof of Theorem* 2.1

It is *obvious* that conditions (2.3) are necessary, assuming that f and u_0 satisfy (2.1). Indeed, if $u \in C([0,T]; V)$ and $du/dt \in C([0,T]; E_p)$, $p \geqslant 1$, then $du/dt \in C([0,T]; E_1)$ and, since V is a closed subspace of E_1, $du/dt(t) \in V$ for each t and in particular for $t = 0$. The argument is the same for all the other derivatives of u, except for $d^l u/dt^l$ when $m = 2l$, in which case we replace E_p by E_0 and v by H.

In order to prove that the conditions are sufficient, we differentiate Eq. (1.4) j times, and we consider the following initial value problem for $u^{(j)} = d^j u/dt^j$:

$$\frac{du^{(j)}}{dt} + \mathscr{A} u^{(j)} = f^{(j)}, \tag{2.5}$$

$$u^{(j)}(0) \text{ given by (2.4).} \tag{2.6}$$

For $j = 1,..., l$ (if $m = 2l+1$), or $j = 1,...,l-1$, (if $m = 2l$), we apply (1.9) and get

$$u^{(j)} \in L^2(0, T; D(A)) \cap C([0,T]; V). \tag{2.7}$$

If $m = 2l$, we infer from (2.4) and (2.7) that $u^{(l)}(0) \in H$ and, applying (1.6) we get

$$u^{(l)} \in L^2(0, T; V) \cap C([0,T]; H). \tag{2.8}$$

Then by (2.1), relations (2.5) and hypothesis (1.14) we find for $m = 2l + 1$

$$u^{(l)} \in C([0, T]; V), \qquad u^{(l-1)} \in C([0, T]; E_3), \dots, u \in C([0, T]; E_{2l+1})$$

and for $m = 2l$

$$u^{(l)} \in C([0, T]; H), \qquad u^{(l-1)} \in C([0, T]; E_2), \dots, u \in C([0, T]; E_{2l})$$

Whence the result.

2.3. *Another Proof*

We give an alternate proof of the sufficiency, i.e., that (2.1)–(2.3) imply that the solution u of (1.4), (1.5) belongs to W_m. Actually, due to Lemma 1.1 we only have to prove that

$$u \in C([0, T]; E_m). \tag{2.9}$$

Another simplification: we can assume that $u_0 = 0$. If we set $\hat{u} = u - u_0$, $\hat{f} = f - \mathscr{A} u_0$, then $\hat{u}_0 = 0$ and $\hat{f}$ satisfy (2.1)–(2.3) and $u \in C([0, T]; E_m)$ if and only if $\hat{u} \in C([0, T]; E_m)$.

The expression w

We set

$$w = \sum_{i=0}^{l-1} \left(-G \frac{d}{dt}\right)^i Gf. \tag{2.10}$$

Since $f \in W_{m-2}$, it is clear with (1.12) and (1.14) that

$$w \in C([0, T]; E_m \cap V). \tag{2.11}$$

Then we infer from assumption (2.3) the following remark:

LEMMA 2.1.

$$w(0) \in V_m. \tag{2.12}$$

Proof. We are going to show that

$$w(0) = G^l \psi_{l-1}(f) \tag{2.13}$$

(cf. the notations in Remark 1.1). Since, by assumption, $\psi_{l-1}(f) \in V$ (if $m = 2l + 1$) or H (if $m = 2l$), and G coincides with A^{-1} on V or H, we find that $w(0) \in V_{2l+1}$ or V_{2l} and (2.12) follows.

In order to prove (2.13), we note that we have the recurrence formula

$$\psi_j(f) = \left(-\frac{d}{dt}\right)^j f(0) + \mathscr{A}\psi_{j-1}(f), \qquad \forall j \geqslant 0 \tag{2.14}$$

(assuming $\psi_{-1}(f) = 0$). Whence

$$G^l\psi_{l-1}(f) = \left(-\frac{d}{dt}G\right)^{l-1} Gf\bigg|_{t=0} + G^l\mathscr{A}\psi_{l-2}(f). \tag{2.15}$$

It follows from (1.14) (cf. (1.28) and (1.30)) that

$$\begin{gathered}\varphi_j(f) \in C([0,T]; E_{m-2j-2}),\ \psi_j(f) \in E_{m-2j-2},\\ j = 0,\dots, l-2,\ l = \left[\frac{m}{2}\right],\end{gathered} \tag{2.16}$$

and because of assumptions (2.1) and (2.3)

$$\psi_{l-2}(f) \in V \cap E_p, \qquad p = 2 \text{ if } m = 2l,\ p = 3 \text{ if } m = 2l+1.$$

But $G\mathscr{A}$ is the identity on $V \cap E_2$ or $V \cap E_3$ and therefore

$$G^l\mathscr{A}\psi_{l-2}(f) = G^{l-1}\psi_{l-2}(f).$$

By reiterating the argument until $j = 0$, we arrive at (2.13). ∎

The function $v = u - w$

The difference $v = u - w$ is solution of an evolution equation which is easy to determine:

$$\begin{aligned}\left(\frac{d}{dt} + A\right)v &= f - \left(\frac{d}{dt} + A\right)w\\ &= f + \sum_{i=0}^{l-1}\left(-G\frac{d}{dt}\right)^{i+1} f - \sum_{i=0}^{l-1} A\left(-G\frac{d}{dt}\right)^i Gf.\end{aligned}$$

Since, by (2.1), $f(t) \in H$, $\forall t \in [0,T]$ and AG is the identity on H, we get

$$\begin{aligned}\frac{dv}{dt} + Av &= f + \sum_{j=1}^{l}\left(-G\frac{d}{dt}\right)^j f - \sum_{j=0}^{l-1}\left(-G\frac{d}{dt}\right)^j f\\ &= \left(-G\frac{d}{dt}\right)^l f\end{aligned}$$

and v is solution to the initial value problem

$$\frac{dv}{dt}+Av=(-G)^l\frac{d^lf}{dt^l}. \tag{2.17}$$

$$v(0)=-w(0). \tag{2.18}$$

By Lemma 2.1, $-w(0)\in V_m$ and, by assumption (2.2), the right-hand side of (2.17) belongs to $L^2(0,T;V_{m-1})$. If we apply the classical theorem on linear evolution equations as in (1.3)–(1.6), with V, H and V' replaced by V_{m+1}, V_m and V_{m-1},[3] we get that

The problem (2.17), (2.18) possesses a unique solution $v\in L^2(0,T;V_{m+1})\cap C([0,T];V_m)$. (2.19)

Since, by (1.13), V_m is a closed subspace of E_m, $v\in C([0,T];E_m)$ and then using (2.11) we conclude that u satisfies (2.9).

The proof of Theorem 2.1 is complete.

2.3. *Examples*

We study an equation associated to a linear operator of the fourth order and the linear Stokes equations. We assume that:

Ω is an open bounded set of $\mathbb{R}^n$, whose boundary Γ is a manifold of class C^{m+2} and of dimension $n-1$. (2.20)

EXAMPLE 2.1.

$$\begin{aligned}&\frac{\partial u}{\partial t}+\Delta^2u=f\text{ in }\Omega\times(0,T),\\&u=\Delta u=0\text{ on }\Gamma\times(0,T),\\&u(0)=u_0.\end{aligned} \tag{2.21}$$

We set $H=L^2(\Omega)$, $V=H^2(\Omega)\cap H_0^1(\Omega)$, $D(A)=\{u\in H^4(\Omega),\ u=\Delta u=0$ on $\Gamma\}$, $Au=\Delta^2u$, $E_m=H^{2m}(\Omega)$, $\mathscr{A}u=\Delta^2u$.

All the assumptions concerning the spaces and operators are clearly satisfied, the main one, (1.14), following from [1].

For $m=2$, if u_0 and f satisfy (2.1) and (2.2), then (2.3) is automatically satisfied and Theorem 2.1 gives that $u\in C([0,T];D(A))$ which is well

[3] If we identify V_m with its dual, then V_{m-1} can be identified with the dual of V_{m+1}, with an appropriate pairing.

known. For $m=4$, assuming (2.1) and (2.2), the necessary and sufficient condition for u to be in $C([0,T];H^8(\Omega))$ is that

$$-\Delta^2 u_0 + f(0) = 0 \text{ on } \Gamma. \tag{2.22}$$

The left-hand side of (2.22) must belong to $H^4(\Omega)\cap H_0^1(\Omega)$ and, in particular, need not belong to $D(A)$.

EXAMPLE 2.2. *Stokes problem.* The unknown functions are $u=(u_1,\ldots,u_n)$ are p solutions to

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \nu\Delta u + \operatorname{grad} p = f \text{ in } \Omega\times(0,T),\\
&\operatorname{div} u = 0 \text{ in } \Omega\times(0,T),\\
&u = 0 \text{ on } \Gamma\times(0,T),\\
&u(x,0) = u_0(x), \qquad x\in\Omega.
\end{aligned} \tag{2.23}$$

We set (cf. [14]])

$$H = \{u\in L^2(\Omega)^n,\ \operatorname{div} u = 0,\ u\cdot\mathbf{N} = 0 \text{ on } \Gamma\},$$

$\mathbf{N}$ the unit outward normal on Γ,

$$V = \{u\in H_0^1(\Omega)^n,\ \operatorname{div} u = 0\},\ D(A) = H^2(\Omega)^n\cap V.$$

We denote by P *the orthogonal projector in* $L^2(\Omega)^n$ *onto* H, and for $u\in D(A)$, we set

$$Au = -\nu P\Delta u. \tag{2.24}$$

It follows from the Cattabriga–Solonnikov–Vorovich–Yudovich theorem [2, 12, 14], that A is an isomorphism from $D(A)$ onto H. We set also $E_m = H^m(\Omega)^n\cap H$ and again for $u\in E_m$, $m\geqslant 2$, $\mathscr{A}u = -\nu P\Delta u$. The details of the proofs of the assumptions of Sections 1.1 and 1.2 can be found in [14]. In particular (1.13) follows (cf. [14, p. 13]) from the fact that

$$P \text{ is linear continuous from } H^m(\Omega)^n \text{ into } H^m(\Omega)^n\cap H,\quad \forall m\geqslant 0. \tag{2.25}$$

We recall [14] that the orthogonal $H^\perp$ of H in $L^2(\Omega)^n$ is

$$H^\perp = \{u = \operatorname{grad} q,\ q\in H^1(\Omega)\}, \tag{2.26}$$

and then by applying the operator P to both members of the first relation in

(2.23), the pressure p disappears and we are left with the functional form of (2.23):

$$\frac{du}{dt}+Au=f,\qquad u(0)=u_0 \tag{2.27}$$

(we assume as usual that $Pf = f$).

Theorem 2.1 applies. For $m=2$, $l=1$, we recover a clasical result: if $u_0\in D(A)$, $f\in L^2(0,T;H)$, $df/dt\in L^2(0,T;V')$, then (2.27) possesses a unique solution u which belongs in particular to $C([0,T];D(A))$.

For $m=3$ or 4 we get the following:

THEOREM 2.2. *We assume that* $u_0\in H^3(\Omega)^n\cap V$, $f\in C([0,T];H^1(\Omega)^n\cap H)$, $df/dt\in L^2(0,T;H)$; *then the solution* u *of* (2.23) (*or* (2.27)) *belongs to* $C([0,T];H^3(\Omega)^n\cap V)$ *if and only if*

$$\nu P\Delta u_0+f(0)=0 \text{ on } \Gamma. \tag{2.28}$$

If $u_0\in H^4(\Omega)^n\cap V$, $f\in C([0,T];H^2(\Omega)^n\cap H)$, $df/dt\in C([0,T];H)$, $d^2f/dt^2\in L^2(0,T;V')$, *then* (2.28) *is necessary and sufficient for* u *to belong to* $C([0,T];H^4(\Omega)^n\cap V)$.

Proof. In both cases, applying Theorem 2.1, we just have to write that $du(0)/dt\in V$. Using (2.17), we find

$$\frac{du}{dt}(0)=\nu P\Delta u_0+f(0) \tag{2.29}$$

which already belongs to $H^1(\Omega)^n\cap H$ (if $m=3$) or $H^2(\Omega)^n\cap H$ (if $m=4$). Then expression (2.29) belongs to V if and only if its trace on Γ vanishes. ▮

Remark 2.3. Let q be the solution to the Neumann problem

$$\Delta q=0 \text{ in } \Omega,\qquad \frac{\partial q}{\partial \mathbf{N}}=\Delta u_0\cdot\mathbf{N} \text{ on } \Gamma. \tag{2.30}$$

Then (cf. [14]), $Au_0=\Delta u_0-\nabla q$ and (2.28) amounts to

$$\nu\Delta u_0-\nu\nabla q+f(0)=0 \text{ on } \Gamma. \tag{2.31}$$

This is equivalent to a condition given by Heywood [6].

2.4. *Comments*

(i) Provided the spaces V_m are properly defined, the assumption that A is self-adjoint is not necessary for Theorem 2.1 (using the proof in Section 2.2).

(ii) Under some appropriate assumptions, Theorem 2.1 and the proof in Section 2.2 extend to the case where the operator A depends on t (with $D(A(t)) \equiv D(A(0))$ independent of t). The result is essentially the same, except that expression (2.4) of the derivatives $d^ju(0)/dt^j$ is not valid anymore.

(iii) Under the assumptions of Theorem 2.1, but with (2.3) not satisfied, the proof in Section 2.3 allows us to split the difference $v = u - w$ into the sum of two terms v_1, v_2, respectively solutions of

$$\begin{aligned} &\frac{dv_1}{dt} + Av_1 = (-G)^l \frac{d^lf}{dt^l}, \\ &v_1(0) = 0. \end{aligned} \tag{2.32}$$

$$\begin{aligned} &\frac{dv_2}{dt} + Av_2 = 0, \\ &v_2(0) = -w(0). \end{aligned} \tag{2.33}$$

It is still true that $w \in C([0, T]; E_m)$, $v_1 \in C([0, T]; V_m)$ but v_2 is not in $C([0, T]; V_m)$ since $w(0) \notin V_m$. The expansion of v_2 in the basis of eigenfunctions of A gives us, in this case, precise information on the singularity of u at $t = 0$.

(iv) The proof of Theorem 2.1 given in Section 2.3 generalizes that of Smale [12]. Example 2.2 and points (i) and (ii) above answer questions in [12].

3. Nonlinear Problems

3.1. *Hypotheses*

In this section we consider nonlinear evolution equations of the type

$$\frac{du}{dt}(t) + Au(t) = g(t, u(t)), \qquad 0 < t < T \tag{3.1}$$

$$u(0) = u_0, \tag{3.2}$$

all the assumptions of Sections 1.1 and 1.2 on A and H being maintained.

Although more general equations can be handled, we will assume for simplicity that

$$g(t, u) = f(t) - B(u), \tag{3.3}$$

where $B(u) = B(u, u)$ and $B(u, v)$ is a bilinear continuous operator from $V \times V$ into V' and furthermore[4]

$$(B(u, v), v) \geqslant 0, \qquad u, v \in V, \tag{3.4}$$

$$B(E_{m+1} \times E_{m+1}) \subset E_m \qquad \text{for} \quad m \geqslant 1, \tag{3.5}$$

$(B(u, v), v)$ can be extended as a bilinear continuous form on $E_{s_1} \times E_{s_2} \times E_{s_3}$, with $s_1 + s_2 + s_3 \geqslant \frac{3}{2}$ if $s_i \neq \frac{3}{2}$, $\forall i$, and $s_1 + s_2 + s_3 > \frac{3}{2}$, otherwise. (3.6)

It is well known that under these assumptions, for f and u_0 given

$$f \in L^2(0, T; H),\ u_0 \in V, \tag{3.7}$$

there exist T_*, $0 < T_* \leqslant T$ and a unique function u

$$u \in L^2(0, T_*; D(A)) \cap C([0, T_*]; V) \tag{3.8}$$

which satisfies (3.1) and (3.2). By changing the notation, we can assume that $T_* = T$.

3.2. *The Result in the Nonlinear Case*

We have the following analog of Theorem 2.1:

THEOREM 3.1. *We assume that the hypotheses of Sections* 1.1 *and* 1.2 *and* (3.3)–(3.7) *are satisfied and that for some* $m \geqslant 2$

$$u_0 \in E_m \cap D(A), \qquad f \in W_{m-2} \tag{3.9}$$

$$\left.\begin{aligned} \frac{d^l f}{dt^l} &\in L^2(0, T; H) \qquad \text{if} \quad m = 2l + 1 \\ &\in L^2(0, T; V') \qquad \text{if} \quad m = 2l. \end{aligned}\right. \tag{3.10}$$

[4] These assumptions are satisfied by the nonlinear Navier–Stokes operator and related operators, cf. Section 3.3.

Then a necessary and sufficient condition for the solution u of (3.1) *and* (3.2) *to belong to* W_m *is that*[5]

$$\begin{aligned} &\frac{d^j u}{dt^j}(0) \in V \qquad \text{for} \quad j = 1,\dots, l-1, \\ &\frac{d^l u}{dt^l}(0) \in V \qquad \text{if} \quad m = 2l+1, \ \in H \text{ if } m = 2l. \end{aligned} \tag{3.11}$$

Proof. (i) We can prove as in Theorem 2.1 that conditions (3.11) are necessary, provided we check that the derivatives

$$u^{(j)}(0) \in E_{m-2j}, j = 1,\dots, l, \tag{3.12}$$

when (3.9) is satisfied

By successive differentiations of (3.1) we find

$$\frac{du^{(j)}}{dt} + \mathscr{A} u^{(j)} + \beta^{(j)} = f^{(j)} \tag{3.13}$$

where we set $\varphi^{(j)} = d^j\varphi/dt^j$, $\beta(t) = B(u(t))$ so that

$$\beta^{(j)} = \sum_{i=0}^{j} \binom{i}{j} B(u^{(j-i)}, u^{(i)}). \tag{3.14}$$

Now by (1.13), (3.5) and (3.9), $\beta(0) = B(u(0)) \in E_{m-1}$ and

$$u'(0) = -\mathscr{A} u(0) - B(u(0)) + f(0) \in E_{m-2}.$$

Similarly, if $m \geqslant 4$,

$$u''(0) = -\mathscr{A} u'(0) - B(u'(0), u(0)) - B(u(0), u'(0)) + f'(0) \in E_{m-4}.$$

The proof continues by induction on j and, using (1.13) and (3.5), shows that

$$\begin{aligned} u^{(j)}(0) = -\mathscr{A} u^{(j-1)}(0) - \sum_{i=0}^{j-1} \binom{i}{j-1} B(u^{(j-1-i)}(0), u^{(i)}(0)) \\ + f^{(j-1)}(0) \in E_{m-2j}, \end{aligned} \tag{3.15}$$

for $j = 1,\dots, l$.

(ii) We now prove the sufficiency of conditions (3.11), as in Section 2.2. For $j = 1$, the equation

$$\frac{du^{(1)}}{dt} + \mathscr{A} u^{(1)} + B(u, u^{(1)}) + B(u^{(1)}, u) = f' \tag{3.16}$$

[5] The expression of the derivatives $d^j u(0)/dt^j$ in term of the data u_0, f, are given by the recurrent formula (3.15).

($u^{(1)} = u'$) together with $u^{(1)}(0)$ given by (3.15) and belonging to V allow us to show that

$$u^{(1)} \in L^2(0, T; D(A)) \cap C([0, T]; V). \tag{3.17}$$

We continue by induction; once we establish that

$$u^{(j)} \in L^2(0, T; D(A)) \cap C([0, T]; V), \qquad j = 0,..., l-1, \tag{3.18}$$

we consider Eqs. (3.13) and (3.14) with $j = l$ and $u^{(l)}(0)$ given by (3.15) and belonging to V (if $m = 2l + 1$) or to H (if $m = 2l$). This allows us to show that

$$\begin{aligned} u^{(l)} &\in L^2(0, T; D(A)) \cap C([0, T]; V) && \text{if} \quad m = 2l + 1 \\ &\in L^2(0, T; V) \cap C([0, T]; H) && \text{if} \quad m = 2l. \end{aligned} \tag{3.19}$$

The next step consists in proving that

$$u^{(j)} \in C([0, T]; D(A)) \qquad \text{for} \quad j = 0,..., l-1. \tag{3.20}$$

For that purpose we write (3.13) and (3.14) in the form

$$\mathscr{A} u^{(j)} = -u^{(j+1)} + \sum_{i=0}^{j} \binom{i}{j} B(u^{(j-i)}, u^{(i)}) + f^{(j)}. \tag{3.21}$$

Because of (3.6), B is a bilinear continuous form on $V \times V$ with value in $V_{-1/2}$. Therefore, by (3.18), for $i = 0,..., j$, and $j = 0,..., l-1$,

$$B(u^{(j-i)}, u^{(i)}) \in C([0, T]; V_{-1/2}). \tag{3.22}$$

Since A is an isomorphism from $V_{3/2}$ onto $V_{-1/2}$, (3.21) implies then that $u^{(j)} \in C([0, T]; V_{3/2})$. Using (3.6) again we get, for the same values of i and j as in (3.22),

$$B(u^{(j-i)}, u^{(i)}) \in C([0, T]; H),$$

and (3.20) follows.

Finally, we show that $u \in W_m$, i.e.,

$$u^{(j)} \in C([0, T]; E_{m-2j}), \qquad j = 1,..., l. \tag{3.23}$$

For $j = l$, this is included in (3.19). For $j = l - 1$, using (3.5) and (3.20) we find that

$$\sum_{i=0}^{j} \binom{i}{j} B(u^{(j-i)}, u^{(i)}) \in C([0, T]; E_1),$$

and then (1.14), (3.21), and (3.19) show that $u^{(l-1)} \in C([0, T]; E_{m-2l+2})$. The proof continuous then by induction for $j = l-2, \ldots, 0$.

Theorem 3.1 is proved. ∎

3.3. *Applications to Navier–Stokes Equation*

Instead of (2.3), we consider now the full Navier–Stokes equations

$$\begin{aligned} &\frac{\partial u}{\partial t} - \nu \Delta u + \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i} + \operatorname{grad} p = f \text{ in } \Omega \times (0, T), \\ &\operatorname{div} u = 0 \text{ in } \Omega \times (0, T), \\ &u = 0 \text{ on } \Gamma \times (0, T), \\ &u(x, 0) = u_0(x). \end{aligned} \tag{3.24}$$

The functional setting is the same as in Section 2.3 and we write

$$B(u, v) = P \sum_{i=1}^{n} \left(u_i \frac{\partial v}{\partial x_i} \right), \qquad B(u) = B(u, u) \tag{3.25}$$

so that (3.24) becomes

$$\begin{aligned} &\frac{du}{dt} + Au + B(u) = f, \\ &u(0) = u_0. \end{aligned} \tag{3.26}$$

Properties (3.4)–(3.6) are well known, cf. [14]. Instead of (3.4) we have

$$(B(u, v), v) = 0, \qquad \forall u, v \in V.$$

Result (3.6) is proved in [4], and (3.5) follows from the fact that $H^m(\Omega)$ is a multiplicative algebra if $m \geqslant 2$ and the dimension of space n is $\leqslant 3$:

$$\begin{aligned} &\text{If } u,\ v \in H^m(\Omega),\ m \geqslant 2,\ n \geqslant 3, \text{ then} \\ &uv \in H^m(\Omega) \text{ and } |uv|_m \leqslant C_m(\Omega)\, |u|_m\, |v|_m. \end{aligned} \tag{3.27}$$

Therefore if u, $v \in H^{m+1}(\Omega)^n$, $m \geqslant 2$, then u_i and $\partial v_j/\partial x \in H^m(\Omega)$, $u_i(\partial v_j/\partial x_i) \in H^m(\Omega)$ and since P maps $H^m(\Omega)^n \cap H$ into itself, $B(u, v)$ is in $H^m(\Omega)^n \cap H$. Properties (3.7) and (3.8) is well known, too; cf. [14, p. 316].

Theorem 3.1 gives *for instance* the following ($m = 3$ or 4):

THEOREM 3.2. *We assume that* $u_0 \in H^3(\Omega)^n \cap V$, $f \in C([0, T]; H^1(\Omega)^n \cap H)$, $df/dt \in L^2(0, T; H)$, $n = 2$ *or* 3. *Then the solution* u *of* (3.24) *(or* (3.26)*) defined on some interval* $[0, T_*]$, $0 < T_* \leqslant T$ *belongs to* $C([0, T_*]; H^3(\Omega)^n \cap V)$ *if and only if*

$$\nu P \Delta u_0 + P\left(\sum_{i=1}^{n} u_i \frac{\partial u_0}{\partial x_i}\right) + f(0) = 0 \text{ on } \Gamma. \tag{3.26}$$

If $u_0 \in H^4(\Omega)^n \cap V$, $f \in C([0, T]; H^2(\Omega)^n \cap H)$, $df/dt \in C([0, T]; H)$, $d^2f/dt^2 \in L^2(0, T; V')$, *then* (3.28) *is necessary and sufficient for* u *to belong to* $C([0, T_*]; H^4(\Omega)^n \cap V)$.

Theorem 3.2 follows immediately from Theorem 3.1.

Remark 3.1. If $u_0 \in C^\infty(\bar{\Omega})^n \cap V$, if $f \in C^\infty(\bar{\Omega} \times [0, T])^n \cap H$ and conditions (3.11) are satisfied for all m, then $u \in C^\infty(\bar{\Omega} \times [0, T])^n$ (T sufficiently small if $n = 3$).[6]

Remark 3.2. If $u_0 \in H$, $f \in C^\infty(\bar{\Omega} \times [0, T])^n \cap H$, the assumptions of Theorem 3.1 are not satisfied but we can prove, using Theorem 3.1, that $u \in C^\infty(\bar{\Omega} \times (0, T])^n$ (T sufficiently small if $n = 3$). We proceed as follows:

— We have a weak solution u of (3.24) or (3.26) in $L^\infty(0, T; H) \cap L^2(0, T; V)$ (cf. [14]). We choose t_0 arbitrarily small such that $u(t_0) \in V$.

— By (3.7) and (3.8) we find that there exists T_*, $t_0 < T_* \leqslant T$ such that the restriction of u to $[t_0, T_*]$ satisfy (3.1). This restriction is still denoted by u.

— We choose $t_1 > t_0$, $t_1 < T_*$, t_1 arbitrarily close to t_0 such that $u(t_1) \in D(A)$. Then (3.26) shows as for (3.12) that $u'(t_1) \in H$. We conclude as in Theorem 3.1 that $u' \in L^2(t_1, T_*; V) \cap C([t_1, T_*]; H)$.

— We choose $t_2 > t_1$, $t_2 < T_*$, $t_2 - t_1$ arbitrarily small such that $u'(t_2) \in V$ and conclude that $u' \in L^2(t_2, T_*; D(A)) \cap C([t_2, T_*]; V)$. We then choose $t_2 < t_3 < T_*$, $t_3 - t_2$ arbitrarily small such that $u'(t_3) \in D(A)$, etc.

Finally we get that $u \in C([t_l, T_*]; E_m)$, for $m = 2l$, t_l arbitrarily close from 0, m arbitrarily large. The regularity follows. The C^∞ regularity in $\Omega \times (0, T)$ was proved by Ladyzhenskaya [8].

Remark 3.3. Theorem 3.1 can be easily extended to more general semilinear equations. We did not present a more general abstract theorem to avoid purely technical difficulties.

[6] Remarks 3.7 and 3.8 of [14, pp. 303, 307] which overlook the compatibility conditions are not correct. To make the result correct one should replace $\bar{Q}$ by Q on p. 303.

Let us consider for instance a nonlinear perturbation of Example 2.1 in Section 2.3.

$$\begin{aligned} &\frac{\partial u}{dt} + \Delta^2 u + u^{2p+1} = f \text{ in } \Omega \times (0, T), \\ &u = \Delta u = 0 \text{ on } \Gamma \times (0, T), \qquad (3.29) \\ &u(0) = u_0, \end{aligned}$$

with $2p + 2 < 2n/(n-2)$ so that $L^{2p+2}(\Omega) \subset H^2(\Omega)$. Then a necessary and sufficient condition for u to be in $C([0, T]; H^8(\Omega))$ is that

$$-\Delta^2 u_0 - u_0^{2p+1} + f(0) = 0 \text{ on } \Gamma$$

(assuming (3.9) and (3.10) with $m = 4$). ∎

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