Nonlinear Problems in Mathematical Physics and Related Topics I

In Honor of Professor
O. A. Ladyzhenskaya

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Edited by M. Sh. Birman, S. Hildebrandt, V. A. Solonnikov, N. N. Uraltseva

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In Honor of Professor O. A. Ladyzhenskaya

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To Olga Aleksandrovna Ladyzhenskaya
with love and admiration

Acknowledgments of the Series Editor

My deep thanks to all who helped to gather, prepare, and publish the first two volumes of the series in two languages in a very short time:

Deputy Director of IDMI, Dean of the Math. Dept. of Novosibirsk State University, Professor **Sergei Goncharov** for his constant moral and administrative support of all my publishing projects, including this series.

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All the authors who responded to me and supported this edition by their excellent new results and who worked hard in order to submit their papers (the most of which were specially written for this series) in an unusually short time (1-3 months) and promptly answered any questions or requests.

Tamara Rozhkovskaya

Series Editor and the Russian Publisher of the series
March, 2002, Novosibirsk and London

Main Topics

The choice of the main topics was motivated by the fields of mathematics in which the world-known results of Professor O. A. Ladyzhenskaya were of the most influence and inspired the development of further investigations

I. Navier-Stokes Equations and Other Mathematical Problems in the Theory of Viscous Fluids

- Compressible and incompressible Navier-Stokes equations
- Mathematical analysis of Navier-Stokes equations in bounded and unbounded domains
- Existence, uniqueness, and regularity of solutions
- Differentiability properties of solutions
- Stability and long time behavior of solutions, asymptotical analysis
- Properties of solutions in domains with nonsmooth boundaries
- Mathematical models of inviscous fluids and non-Newtonian fluids

II. Nonlinear Partial Differential Equations

- Fully nonlinear PDE's and quasilinear PDE's
- Boundary-value problems for nonlinear elliptic equations and systems
- Initial-boundary-value problems for parabolic equations and systems
- Initial-boundary-value problems for hyperbolic equations and systems
- Existence and uniqueness of weak and classical solutions
- Partial regularity and differentiability properties
- Approximation methods for solving problems of mathematical physics
- Properties of function spaces used in the study of PDE's
- Blow-up theorems for quasilinear hyperbolic equations
- Semigroups generated by initial-boundary-value problems
- Calculus of variations and applications to mathematical physics

III. Mathematical Aspects of Computational Analysis

- Approximation methods
- Convergence of numerical methods
- A priori and a posteriori estimates of the accuracy of approximations

The Mathematics Genealogy Project
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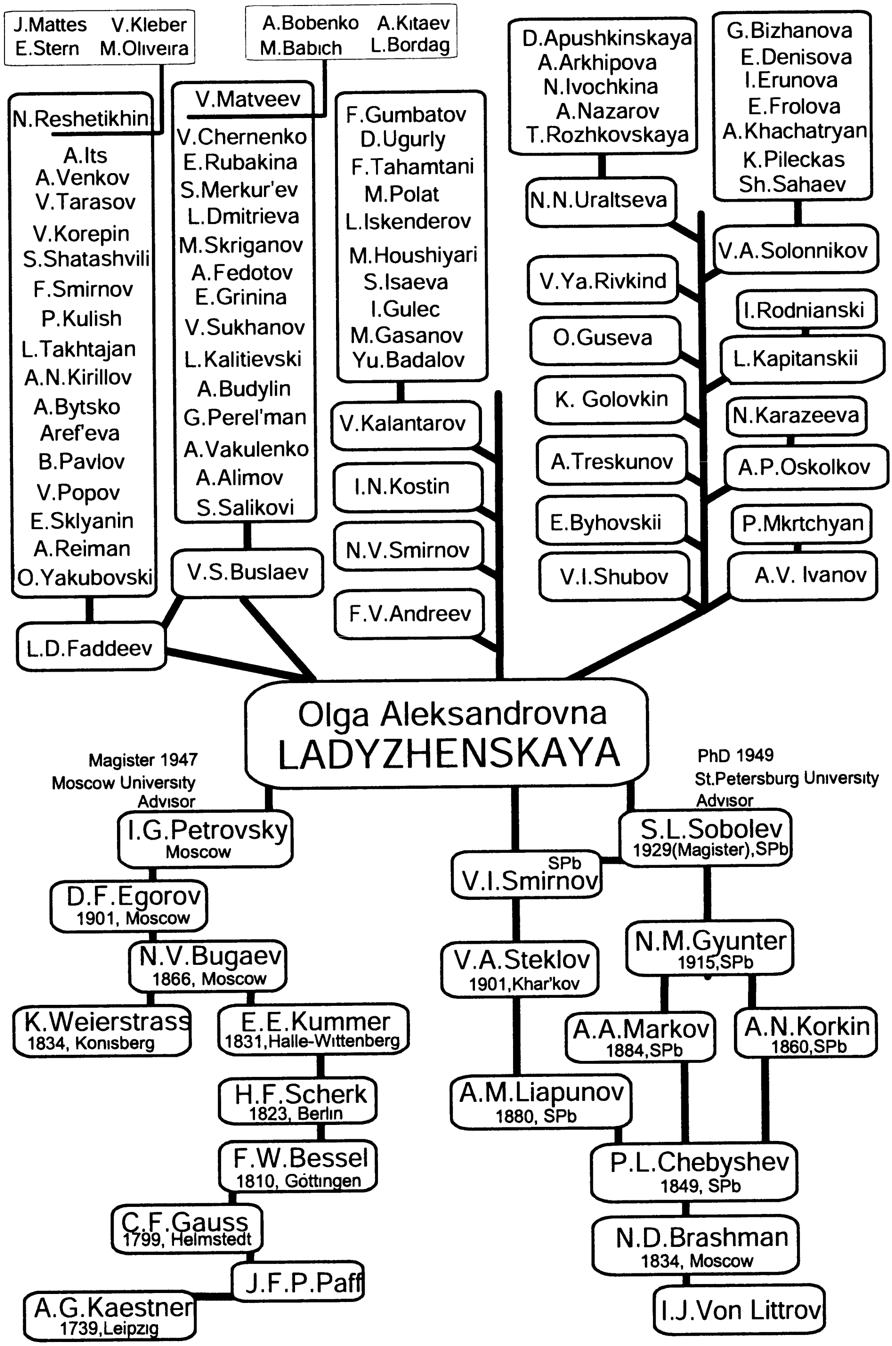
This tree only follows a formal information from this site at the Minnesota State University,where one can find an information about more than 53 000 mathematicians, their advisors and former students

But there would be a large garden if we try to mention all actual learners of Professor O. A. Ladyzhenskaya, all who entered in Mathematics through her seminars and lectures, read her books and papers, absorbed her ideas and continued her results and methods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Contents

Area Formulas for σ-Harmonic Mappings

Giovanni Alessandrini† and Vincenzo Nesi††

To Professor Olga A. Ladyzhenskaya
with our deep admiration

The goal of the present paper is two-fold. First, we review some recent progress concerning generalizations of various classical results, such as sufficient conditions to guarantee univalence of harmonic mappings in dimension two, to certain pairs of elliptic partial differential equations with measurable coefficients. Second, we apply these results to prove new area formulas which are valid for a large class of mappings arising as solutions of these pairs of elliptic partial differential equations. Finally, we briefly discuss some applications to homogenized constants in the context of G-closure problems.

1. Introduction

In this paper, we study mappings U from an open set Ω of the plane R^2 into R^2 whose components u_1 and u_2 are σ-harmonic functions in the sense that they are weak solutions to the divergence form elliptic equation

$$\operatorname{div}(\sigma \nabla u) = 0 \quad \text{in } \Omega, \tag{1.1}$$

where σ is a symmetric, uniformly elliptic matrix with measurable entries. For an overall introduction to the classical subject of the study of the solutions to Eq. (1.1) we refer to **[1]**. Our starting point for this investigation has its origin in applications to homogenization. Let us first introduce some notation

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which is necessary for an appropriate description of such applications and of the results of the present paper. We denote by $\mathcal{M}^s$ the class of real symmetric 2×2-matrices and by $\mathcal{M}^s_K$, $K \geqslant 1$, the subclass of matrices $\sigma = \{\sigma_{ij}\} \in \mathcal{M}^s$ satisfying the uniform ellipticity condition

$$K^{-1}|\xi|^2 \leqslant \sigma_{ij}\xi_i\xi_j \leqslant K|\xi|^2 \quad \text{for every } \xi \in R^2.$$

Let Ω be an open set in R^2. Any given $\sigma \in L^\infty(\Omega, \mathcal{M}^s_K)$ will be referred to as a *conductivity* and a mapping $U \in W^{1,2}_{\text{loc}}(\Omega, R^2)$ is said to be *σ-harmonic* if its components u_1 and u_2 are weak solutions to Eq. (1.1). In places, we shall consider $\Omega = R^2$. Let $Q = (0,1) \times (0,1)$. We often deal with functions that are 1-periodic with respect to each of its variables x_1 and x_2. These functions will be said to be *Q-periodic* or *periodic* for short. This will be indicated by the subscript $\sharp$ in the relevant function spaces. For example,

$$\begin{aligned} L^\infty_\sharp(R^2, \mathcal{M}^s_K) \equiv &\{\sigma \in L^\infty(R^2, \mathcal{M}^s_K) \mid \sigma(x_1+m, x_2+n) = \sigma(x_1, x_2) \\ &\text{for a.e. } (x_1, x_2) \in R^2 \ \forall m, n \in Z\}, \\ W^{1,2}_\sharp(R^2, R^2) \equiv &\{U \in W^{1,2}_{\text{loc}}(R^2, R^2) \mid U(x_1+m, x_2+n) = U(x_1, x_2) \\ &\text{for a.e. } (x_1, x_2) \in R^2 \ \forall m, n \in Z\}. \end{aligned}$$

It is also convenient to define for any 2×2-matrix A

$$W^{1,2}_{\sharp,A}(R^2, R^2) \equiv \{U \in W^{1,2}_{\text{loc}}(R^2, R^2) \mid U - Ax \in W^{1,2}_\sharp(R^2, R^2)\}. \tag{1.2}$$

Our special interest in boundary conditions of periodic type is motivated by homogenization. Let us review some preliminary facts about homogenization theory.

1.1. Connections with homogenization. We recall the definition of effective (or homogenized) conductivity restricting our attention to dimension two. Let $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$ be given, and let Ω be a bounded open and simply connected set with Lipschitz boundary. Let $f \in W^{-1,2}(\Omega, R)$. We set $\sigma_\varepsilon(x) = \sigma(x/\varepsilon)$. Consider the problem

$$-\operatorname{div}(\sigma_\varepsilon(x)\nabla u_\varepsilon(x)) = f \quad \text{in } \Omega, \quad u_\varepsilon \in W^{1,2}_0(\Omega, R).$$

As is known (cf., for example, [**2**]), $u_\varepsilon \rightharpoonup u_0$ in $W^{1,2}(\Omega, R)$, where u_0 is a solution to the following (homogenized) problem:

$$-\operatorname{div}(\sigma_{\text{hom}}\nabla u_0(x)) = f \quad \text{in } \Omega, \quad u_0 \in W^{1,2}_0(\Omega, R).$$

The new (constant) matrix σ_{hom}, called *homogenized conductivity*, belongs to $\mathcal{M}^s_K$ and is determined by the following rule:

$$\forall \xi \in R^2 \quad \langle \sigma_{\text{hom}}\xi, \xi \rangle = \inf \int_Q \langle \sigma(y)\nabla u(y), \nabla u(y) \rangle \, dy, \tag{1.3}$$

where the infimum is taken over u such that $u - \langle \xi, x \rangle \in W^{1,2}_\sharp(R^2, R)$.

This approach was initiated by Spagnolo (cf. [3]–[5]). We are interested in the so-called G-closure problem (cf. Subsec. 1.4 below). Its study really amounts to the study of "periodic" σ-harmonic mappings.

It is convenient to replace the definition (1.3) with the equivalent one:

$$\forall A \in \mathcal{M} \quad \operatorname{tr}(A\sigma_{\text{hom}}A^T) = \inf \int_Q \operatorname{tr}[DU(y)\sigma(y)DU(y)^T]\,dy, \tag{1.4}$$

where the infimum is taken over U such that $U \in W^{1,2}_{\sharp,A}(R^2, R^2)$, $\mathcal{M}$ denotes the set of real 2×2-matrices, and $\mathcal{M}_+$ denotes the subset of such matrices with strictly positive determinant.

We note that the infimum in (1.4) is taken over vector fields rather than functions. We use the notation D (rather than ∇) to denote the gradient of vector-valued mappings. Our convention is that for $F = (f, g)$

$$DF = \begin{pmatrix} f_{x_1} & f_{x_2} \\ g_{x_1} & g_{x_2} \end{pmatrix}.$$

It is obvious that (1.4) implies (1.3). Since the Euler–Lagrange equations corresponding to the variational principle (1.3) are linear, (1.3) implies (1.4). Hence these definitions are equivalent.

Given $A \in \mathcal{M}$, denote by U^A a solution (unique up to an additive constant) to the problem

$$\operatorname{div}[\sigma(y)(DU^A(y))^T] = 0 \quad \text{in } R^2, \quad U^A \in W^{1,2}_{\sharp,A}(R^2, R^2), \tag{1.5}$$

where for any matrix B, div B is the vector whose ith component is the divergence of the vector whose components form the ith column of B.

By (1.5), U^A is a solution to the Euler–Lagrange equations associated to (1.4). Integrating by parts, it is easy to see that

$$\forall A \in \mathcal{M} \quad \sigma_{\text{hom}}A^T = \int_Q \sigma(y)DU^A(y)^T\,dy. \tag{1.6}$$

The auxiliary problem (1.5) is usually called the *cell problem*. Solutions to the problem (1.5) will be called, with a slight abuse of language, *periodic σ-harmonic mappings*. They are unique up to translation by a constant vector.

Now, let us recall some of the main results of [6] which are needed in the analysis of the present paper and which have been the starting point for the applications to homogenization developed in [7]. In Subsec. 1.4 below we outline the main results of the latter work.

In the sequel, $K \geqslant 1$ and $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$ are given.

Theorem 1.1. *Let $A \in \mathcal{M}_+$, and let U^A be a solution to the problem (1.5). Then the following assertions hold:*

(i) *U^A is a homeomorphism from R^2 onto itself,*

(ii) *the following inequality holds* (*cf.* [**6**, Theorem 1]):

$$\det DU^A > 0 \quad a.e. \text{ in } R^2. \tag{1.7}$$

In Secs. 3 and 4, we discuss on further applications to homogenization of our results. The first result is a partial solution to a conjecture made by Milton [**8**]. These are statements about higher integrability of the gradient of σ-harmonic functions. The higher integrability for planar quasiconformal mapping dates back to the work of Bojarski [9] (cf. also [**10, 11**]). It was later extended to σ-harmonic functions in any dimension by Meyers [**12**]. Gehring [**13**] established higher integrability for quasiconformal mappings in any dimension. However, these approaches did not give a precise evaluation of the *best exponent*. Gehring and Milton made conjectures about the best exponent in any dimension for the case of quasiconformal mappings and σ-harmonic functions respectively. Milton made an even more challenging conjecture about integrability of a precise power of $|\nabla u|^{-1}$ for σ-harmonic mappings.

Gehring's conjecture was proved for dimension two by Astala [**14**] in a fundamental advance. Using the latter result in an essential way, Leonetti and Nesi [**15**] confirmed Milton's conjecture in dimension two. However, the result for $|\nabla u|^{-1}$ depends very strongly on boundary conditions and in [**15**], the authors were only able to treat the Dirichlet and Neumann boundary conditions making very heavy use of results by Alessandrini and Magnanini [**16**]. In Theorem 3.1, we extend the result to periodic σ-harmonic mappings which are the most interesting from the point of view of applications to homogenization.

In Proposition 4.1, we represent σ_{hom} in terms of an area formula for suitable associated quasiconformal mappings (cf. Subsec. 1.3 below and Sec. 2 for further details). In fact, this continues the analysis developed in [**15**]. Here, we give the periodic version of that result.

We also prove (cf. Proposition 4.2) that σ-harmonic mappings satisfy the change of variable formula and give some examples in this direction.

1.2. Univalence. We briefly discuss the background to Theorem 1.1. We start by recalling a theorem due to Radó [**17**] (cf. the proof due to Kneser [**18**] and Choquet [**19**]) which states that if U is a harmonic mapping (i.e., σ-harmonic with $\sigma = I$) on a disk B whose boundary data $\varphi = U\big|_{\partial B}$ form a homeomorphism from ∂B onto a closed *convex* curve Γ, then U is univalent. In view of a result by Lewy [**20**], the Jacobian determinant of U does not vanish inside B. We refer to the survey by Duren [**21**] about developments on the study of univalent harmonic mappings in the plane. Let us quote the recent interesting counterexamples by Melas [**22**] and by Laugesen [**23**] to the extension of Radó's theorem to higher dimensions. We also mention generalizations to harmonic mappings between certain Riemannian two-dimensional manifolds (cf. [**24, 25**]) and to mappings whose components are solutions to quasilinear degenerate elliptic p-laplacian-type equations (cf. [**26**]).

Let us note, incidentally, that the quoted paper by Lewy [**20**] gave rise to the related, although different, theory of univalent solutions of systems of the so-called Heinz–Lewy type (cf., for example, Heinz [**27**] and, as a general reference to this topic Schulz [**28**]).

More specifically, regarding the generalization of Radó's theorem to σ-harmonic mappings, a result in this direction was proved by Bauman, Marini, and Nesi [**29**] in the case of a smooth conductivity σ and was already applied to issues of homogenization. In [**6**, Theorem 4], the following version of Radó's theorem for σ-harmonic mappings with nonsmooth σ was obtained.

Theorem 1.2. *Let $\Omega \subset R^2$ be a bounded simply connected open set whose boundary $\partial\Omega$ is a simple closed curve, let $\varphi = (\varphi_1, \varphi_2)$, $\varphi : \partial\Omega \to R^2$, be a homeomorphism from $\partial\Omega$ onto a convex closed curve Γ, and let D be the bounded convex domain bounded by Γ. Let $\sigma \in L^\infty(\Omega, \mathcal{M}_K^s)$, and let $U \in W^{1,2}_{\text{loc}}(\Omega, R^2) \cap C(\overline{\Omega}, R^2)$ be the σ-harmonic mapping whose components are solutions to the Dirichlet problems*

$$\operatorname{div}(\sigma \nabla u_i) = 0 \quad \text{in } \Omega, \quad u_i = \varphi_i \quad \text{on } \partial\Omega, \quad i = 1, 2.$$

Then U is a homeomorphism from $\overline{\Omega}$ onto $\overline{D}$.

In the same vein, in [**6**], we gave a proof to assertion (i) of Theorem 1.1, i.e., we established the univalence of the periodic σ-harmonic mapping U^A solving the problem (1.5). To the best of our knowledge, this is the first available result of univalence in the periodic setting, also in the case of a smooth σ. Our approach is based on the analysis of the structure of the level lines of the σ-harmonic functions obtained by linear combinations of the components of U^A. This analysis relies on arguments and concepts introduced in [**16**], which enable to treat, in a generalized sense, critical points of a σ-harmonic function u and develop the corresponding index calculus also in the case where the conductivity σ is discontinuous and the gradient ∇u is not defined in a pointwise sense.

For the convenience of the reader, these concepts are reviewed in Sec. 2. Moreover, we also state necessary, and some sufficient, conditions for univalence (cf. Theorem 1.3 and Propositions 2.1–2.3). In fact, these are our basic tools in Secs. 3 and 4. The proof of these results can be found in [**6**, Sec. 2 and Theorem 3.1].

Once univalence is proved, it can be seen rather easily, for example by regularization arguments, that U^A satisfies the inequality $\det DU^A \geqslant 0$ almost everywhere.

It seems to be a less trivial task to show that for a locally univalent σ-harmonic mapping U $\det DU$ does not vanish almost everywhere. This is accomplished by the following result.

Theorem 1.3 (cf. [**6**, Theorem 5]). *Let $U \in W^{1,2}_{\text{loc}}(\Omega, R^2)$ be a σ-harmonic mapping with $\sigma \in L^\infty(\Omega, \mathcal{M}_K^s)$ which is locally one-to-one and sense-preserving.*

For every $D \subset\subset \Omega$ we have

$$\log(\det DU) \in \mathrm{BMO}(D). \tag{1.8}$$

As is known from the theory of Muckenhoupt weights (cf., for example, García–Cuerva and Rubio de Francia [**30**]), this result implies that, locally, for some small $\varepsilon > 0$ $(\det DU)^{-\varepsilon}$ is integrable, and assertion (ii) of Theorem 1.1 is valid. The main tools used in the proof of Theorem 1.3 are a reverse Hölder inequality for nonnegative solutions to the adjoint equation for a nondivergence elliptic operator due to Bauman [**31**] (cf. also Fabes and Strook [**32**] and the results by Reimann [**33**] about the transformation rules of the space BMO under quasiconformal mappings). This brings us into another topic which is strictly intertwined to σ-harmonic mappings.

1.3. Connections with quasiregular mappings. We recall that, given $f \in W^{1,2}_{\mathrm{loc}}(\Omega, R^2)$, the *dilatation quotient* for f is defined for almost every $x \in \Omega$ as follows:

$$\mathcal{D}_f(x) = \frac{\max\limits_{|\xi|=1} |\partial_\xi f(x)|}{\min\limits_{|\xi|=1} |\partial_\xi f(x)|}, \tag{1.9}$$

where ∂_ξ denotes the directional derivative in the direction ξ and for a given $K \geqslant 1$, $f \in W^{1,2}_{\mathrm{loc}}(\Omega, R^2)$ is said to be a (sense-preserving) *K-quasiregular* mapping if

$$\mathcal{D}_f(x) \leqslant K, \quad \det Df \geqslant 0 \quad \text{for almost every } x \in \Omega, \tag{1.10}$$

where Df denotes the Jacobian matrix of f. A mapping f is said to be *K-quasiconformal* if, in addition, it is injective. Recall that the following conditions are equivalent to (1.10):

$$\mathrm{tr}(Df Df^T) \leqslant (K + K^{-1}) \det Df \quad \text{a.e. in } \Omega \tag{1.11}$$

or

$$|f_{\bar{z}}| \leqslant \frac{K-1}{K+1} |f_z| \quad \text{a.e. in } \Omega, \tag{1.12}$$

where the standard identification $z = x_1 + ix_2$ is used (we refer to [**34**] as a basic reference for quasiregular mappings in the plane).

Connections between σ-harmonic and quasiregular mappings are various.

First of all, the components u_1 and u_2 of a σ-harmonic mapping U are also the components of quasiregular mappings. In fact, with each σ-harmonic function u (i.e., a solution to Eq. (1.1)) we can associate in a natural fashion, which generalizes the harmonic conjugation, a new function, called *stream function* $\widetilde{u}$, that is a solution to the dual equation

$$\mathrm{div}\left(\frac{\sigma}{\det \sigma} \nabla \widetilde{u}\right) = 0 \quad \text{in } \Omega$$

such that the mapping $f = u + i\tilde{u}$ is K-quasiregular. These facts, which can be traced back to the functional analytic approach for two-dimensional elliptic equations due to Bers and Nirenberg [**10**] (cf. also Bers, John, and Schechter [**11**, Ch. II.6] and Vekua [**35**])) were the starting point for the geometric study of σ-harmonic functions in [**16**] (cf. also the discussion at the beginning of Sec. 2).

Second, we stress the well-known fact that quasiregular mappings are indeed σ-harmonic for some suitable σ. By the Ahlfors–Bers representation [**36**], any quasiregular mapping f can be written as follows:

$$f = F \circ \chi,$$

where F is holomorphic and χ is quasiconformal. Since the components of F are harmonic, the components of f are G-harmonic, where G can be chosen as follows:

$$G(x) = \begin{cases} \det D\chi (D\chi^T D\chi)^{-1} & \text{if } \det D\chi \neq 0, \\ I & \text{if } \det D\chi = 0. \end{cases} \tag{1.13}$$

We note that $G \in L^\infty(\Omega, \mathcal{M}^s_K)$, where K is the supremum of the dilatation of f (which is the same as that of χ), and $\det G = 1$ everywhere.

Another kind of relationship between these two classes of mappings was investigated by the authors [**37**], where we analyzed whether a locally univalent σ-harmonic mapping is locally quasiconformal. We just mention here that it turns out that this is a characteristic property of the conductivity σ rather than of the single mapping at hand (cf. [**37**, Theorem 2.1]); moreover, this property is nongeneric within the class of conductivities with prescribed ellipticity constant K (cf. [**37**, Theorem 4.1]).

1.4. *G*-closure problems. We already mentioned that the so-called G-closure problem is an important topic. This appears in material sciences, optimal design, and in certain issues of the calculus of variation such as quasiconvexification of nonconvex energies.

The simplest nontrivial example is the so-called *two-phase problem*. To describe it, we assume that

$$\sigma(x) = (K\chi_E(x) + K^{-1}(1 - \chi_E(x)))I,$$

where E is a measurable subset of Q. In this case, the G-closure problem can be roughly described as follows. The given data are the conductivities in each phase (KI and $K^{-1}I$) and the *volume fractions* p, $1 - p$ with $p \in [0, 1]$. The unknown is a set E, called the *microgeometry*. As E varies, so does the homogenized matrix σ_{hom}. The goal is to characterize the exact range of σ_{hom} as the measurable set E varies in the family of *all* possible measurable subsets of Q satisfying the constraint $|E| = p$ (more precisely, its closure in the space of symmetric matrices equipped with the natural norm).

The study of the two-phase problem was initiated by Hashin and Shtrikman [**38**] and was completely solved only about twenty years later by Tartar and Murat [**39, 40**] and by Cherkaev and Lurie [**41**]. However, many interesting G-closure problems are still open. For example, the *three-phase problem* in which σ takes three distinct values with prescribed volume fractions attracted considerable attention.

Periodicity is a very convenient setting. However, it is not necessary. On the contrary, the more general setting of arbitrary sequences of matrices satisfying the usual ellipticity condition, inspired very deep progress in this field. The general plan to establish bounds on the effective conductivity is very clearly outlined by Tartar in his fundamental paper [**39**] and also in the less known work [**42**]. The idea is to consider any known differential constraint on the fields coming into play.

We transform this constraint (using the theory of compensated compactness developed by Murat and Tartar [**43**] and [**44**]) into a compactness property for suitable weakly convergent sequences of fields hence establishing necessary conditions that any effective conductivity (or more generally any H-limit in the language of Murat and Tartar [**39**]) must satisfy. This approach has been tremendously successful. In dimension two, and restricting attention to periodic boundary conditions, the essence of the method is to use what (in the slightly different context of multi-well problems) are called the "minor relations". In other words, one takes advantage of the fundamental fact that given any $A \in \mathcal{M}$ and $U \in W^{1,2}_{\sharp,A}(R^2, R^2)$, in addition to the obvious constraint

$$\int_Q DU(x)dx = A,$$

we have

$$\int_Q \det DU(x)dx = \det A. \tag{1.14}$$

The latter is often expressed by saying that $A \to \det A$ is a *null-lagrangian* on the space $W^{1,2}_{\sharp,A}(R^2, R^2)$. The equality (1.14) is a special example of a much more general phenomenon leading to the notion of *quasiconvexity*: a real valued function F on the space of 2×2-matrices is quasiconvex if for any matrix A we have

$$U \in W^{1,2}_{\sharp,A}(R^2, R^2) \Rightarrow \int_Q F(DU)dx \geqslant F(A).$$

By Jensen's inequality, convex functions have this property and if the target space of U has dimension one, the set of quasiconvex functions, reduces itself to the set of convex functions [**45**]. However, if both the domain and the target space have dimension greater than one, there exist quasiconvex functions that are not convex as shown, for example, by (1.14). The compensated compactness developed by Murat and Tartar [**43, 40**] is the natural mathematical tool to find bounds on homogenized coefficients by using the existence of these functions. Due to its elegance, simplicity, and generality, the method has been a tremendous source of stimulus and results in material sciences, optimal design as well as in their connections to certain branches of the calculus of variations.

In its general form, the compensated compactness method works with any quasiconvex function, not just null-lagrangians. However, as observed by Milton [**46**], many different quasiconvex functions may lead to the same bound. Hence the method faces another difficulty, namely that very little is known about the set of quasiconvex functions. In practice, in two dimensional linear conductivity, all the bounds obtained with this approach select only the determinant (or not relevant modifications of it) in the (unknown) class of all quasiconvex functions and use it as efficiently as possible. This is what we will call the *conventional translation method*. Use of different quasiconvex functions (*unconventional translation method*) is in principle possible but, at present, no other efficient candidates are available, at least in dimension two.

In recent years, however, there has been some progress in finding bounds for the G-closure problems. This has been made by using (apparently) different approaches. In the first of such results [**47**], the fact (proved in [**29**]) that suitably chosen σ-harmonic mappings U are sense-preserving ($\det DU \geqslant 0$ a.e.) was used.

In another piece of work [**48**], the use has been made of a recent fundamental advance in the theory of quasiconformal mappings due to Astala [**14**].

In these papers, the strategy is inspired by the compensated compactness approach and it is similar to it. The difference consists in using constraints on the variable $U \in W^{1,2}_{\sharp,A}$ which are valid *only* for those fields U which, in addition, are (in our language) σ-harmonic mappings.

The additional difficulty in this approach is that this constraint must depend only on the a priori information on the pertinent G-closure problem. In [**7**] it is shown that these different contributions can be unified.

Let us recall the main result of [**7**] (cf. Theorem 3.2).

Here $\operatorname{Adj} A$ denote the adjugate of a matrix A.

Given $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$, we set

$$d_m = \operatorname{ess}\inf_{x \in Q} \sqrt{\det \sigma}. \tag{1.15}$$

To state the result, it is convenient to introduce the following definitions.

Let $S \in \mathcal{M}^s$ be positive definite. We set $s = \sqrt{\det S}$. Let $\lambda \geqslant 0$. We define a set of *quasiconformal matrices* and the corresponding function space as follows:

$$m(S,\lambda) \equiv \begin{cases} \{M \in \mathcal{M} : (s^2+\lambda^2)\det M > \lambda \operatorname{tr}(M \operatorname{Adj}(S) M^T)\} & \text{if } \lambda \in [0,s), \\ \{M \in \mathcal{M} : (s^2+\lambda^2)\det M = \lambda \operatorname{tr}(M \operatorname{Adj}(S) M^T)\} & \text{if } \lambda = s, \\ \mathcal{M} & \text{if } \lambda > s. \end{cases} \tag{1.16}$$

$$W(A,\sigma,\lambda) \equiv \{\varphi \in W^{1,2}_{\sharp,A}(R^2,R^2) : \operatorname{Adj} D\varphi(x) \in m(\sigma(x),\lambda) \text{ for a.e. } x \in Q\}. \tag{1.17}$$

We set

$$Q_{d_m} = \{x \in Q : \sqrt{\det \sigma(x)} = d_m\} \tag{1.18}$$

and consider the space

$$\mathcal{B}(A,\sigma,d_m) \equiv \{\varphi \in W^{1,2}_{\sharp,A}(R^2,R^2) : \operatorname{Adj} D\varphi(x) \in m(\sigma(x),d_m) \text{ for a.e. } x \in Q_{d_m}\}. \tag{1.19}$$

It is easy to describe this space by words. It is the subspace of the space $W^{1,2}_{\sharp,A}(R^2,R^2)$ consisting of mappings satisfying the first-order Beltrami equation

$$D\varphi^t D\varphi = G \det D\varphi, \quad G = \left(\frac{\sigma}{\sqrt{\det \sigma}}\right)^{-1}$$

in the set Q_{d_m}.

In the G-closure problems, it is of interest to consider the case where the eigenvalues of σ take a finite number of values. On the other hand, if $Q = Q_{d_m}$, then the G-closure is well known. It is convenient to avoid this special case assuming that $|Q_{d_m}| < 1$. The assumptions of the following theorem are therefore very natural.

Theorem 1.4. *Let $K > 1$ be given. For $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$ we introduce d_m and Q_{d_m} according to* (1.15) *and* (1.18) *respectively. Assume that $|Q_{d_m}| > 0$ and*

$$\operatorname*{ess\,inf}_{Q\setminus Q_{d_m}} \sqrt{\det \sigma} > d_m. \tag{1.20}$$

Then

$$\det \sigma_{\text{hom}} > d_m^2 \tag{1.21}$$

and the homogenized conductivity σ_{hom} satisfies the following variational principle. For every $A \in m(\sigma_{\text{hom}}, d_m)$

$$\frac{\operatorname{tr}(A\sigma_{\text{hom}}A^T) - 2d_m \det A}{\det \sigma_{\text{hom}} - d_m^2} = \inf_{\varphi \in \mathcal{B}(A,\sigma,d_m)} \int_Q \left\{ \chi_{Q_{d_m}}(y) \frac{\operatorname{tr}[D\varphi(y)\sigma(y)D\varphi(y)^T]}{2\det \sigma(y)} \right.$$

$$\left. (1-\chi_{Q_{d_m}}(y)) \frac{\operatorname{tr}[D\varphi(y)\sigma(y)D\varphi(y)^T] - 2d_m \det D\varphi(y)}{\det \sigma(y) - d_m^2} \right\} dy. \tag{1.22}$$

The minimizer of (1.22) *is unique up to a constant vector and is expressed by the formula*

$$\varphi_{U^{B_{d_m}},d_m} = d_m U^{B_{d_m}} + J\widetilde{U}^{B_{d_m}}, \tag{1.23}$$

where $U^{B_{d_m}}$ *is a solution to the problem* (1.5) *when* A *is replaced with*

$$B_{d_m} = \frac{-d_m A + \mathrm{Adj}(A\sigma_{\mathrm{hom}})}{\det \sigma_{\mathrm{hom}} - d_m^2}. \tag{1.24}$$

We remark that the set $\mathcal{B}(A,\sigma,d_m)$ defined by (1.15)–(1.19) is a *closed linear* subspace of $W^{1,2}_{\sharp,A}(R^2,R^2)$.

The minimizer of (1.22) given by formula (1.23) can be thought of as a *constrained minimizer*, to emphasize that it might be different from the minimum of the same functional on the whole space $W^{1,2}_{\sharp,A}(R^2,R^2)$.

As is shown in [**7**, Corollary 3.2], under the assumptions of Theorem 1.4, the minimizer of (1.22) is quasiconformal and we also establish a priori bounds on its dilatation.

For various examples and applications of the above result we refer to [**7**, Sec. 4]. The results of this section should be seen as a continuation of the work of many authors. Let us just mention some very relevant literature.

First, the early work of Keller [**49**], Dykhne [**50**], and Mendelson [**51**] which was already based on the idea of *duality*. In our terminology, these are the first papers in the field of composites, where the idea of *stream function* shows its power. Next, for polycrystal problems the study was initiated in [**52**] and completed later by Francfort and Murat in the interesting paper [**53**] which, unfortunately, has not been fully appreciated.

Several papers, including [**54**]–[**56**], deal with examples using various form of duality. Some other works partly focusing on the case with prescribed volume fraction, used duality in conjunction with more refined arguments (cf. [**57**]–[**59**]). Quite recently, quasiconformal mappings are having an increasing impact on the two-phase polycrystalline problem (cf. [**48**] and [**60**]–[**62**]).

More recently there has been an impulse on problems when more (isotropic) phases are present. This case was considered already by Hashin and Shtrikman and later by Kohn and Milton [**63**] (cf. also [**64**]) and by Zhikov [**65**]. Further progress was made by Lurie and Cherkaev [**66**], Gibiansky and Cherkaev [**67**], Cherkaev [**68, 69**], and Gibiansky and Sigmund [**70**]. The nowadays tighter available bounds in dimension two were established in [**47**] and the best known microgeometries stem, at least for the three-phase problem, from a combination of the work in [**70**] and [**68**] (cf. also [**69**]). For the tightest bounds in dimension two for an arbitrary number of possible anisotropic phases we refer to [**7**, Theorem 4.1].

Finally, we conclude this section recalling the connection between our work and the literature concerning certain problems of optimal design and the more general issue of search for quasiconvex functions.

Following the pioneering work of Kohn and Strang [**71**] (cf. also [**72**]) several authors used the knowledge of the G-closure in certain specific cases to compute the "quasiconvexification" of certain functions. A typical example is a function $f = \min(f_1, f_2)$, where f_1 and f_2 are quadratic functions.

The quasiconvexification can be computed on the basis of results from the G-closure of the two-phase problem. This was studied in [**71**] in the limiting case and in a greater generality by Allaire and Francfort [**73**] and Allaire and Lods [**74**]. Their analysis, in our language, shows that the conventional translation method gives all the necessary information. However, it turns out that, in this particular case, the conventional translation method corresponds to the "polyconvexification" of f and one proves by other means that this coincides with the "rank-one convexification" and hence with the "quasiconvexification."

The study of the quasiconvexification of a function that is a minimum of several (say, three) quadratic functions, reduces again itself to the study of a three-phase G-closure problem. The conventional translation method again delivers the polyconvexification of f. Therefore, since our bounds improve upon it, we are effectively giving a bound on f which is strictly tighter than the bound obtained using the polyconvexification of f. Further investigation on this issue is the subject of ongoing work (cf. [**7**] for details).

2. Preliminaries

The main goal of this section is to recall some notions we need to prove the main results of this paper. We mainly refer the reader to [**16, 6**].

If $\Omega \subset R^2$ is simply connected and $u \in W^{1,2}_{\text{loc}}(\Omega, R)$ is a weak solution to the equation

$$\operatorname{div}(\sigma \nabla u) = 0 \quad \text{in } \Omega, \tag{2.1}$$

then there exists, and is unique up to an additive constant, a function $\widetilde{u} \in W^{1,2}_{\text{loc}}(\Omega, R)$ (called the *stream function associated to* u) such that

$$\nabla \widetilde{u} = J\sigma \nabla u \quad \text{for a.e. } x \in \Omega, \tag{2.2}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The stream function associated to u satisfies in the weak sense the equation

$$\operatorname{div}\left(\frac{\sigma}{\det \sigma} \nabla \widetilde{u}\right) = 0 \quad \text{in } \Omega. \tag{2.3}$$

Setting

$$f = u + i\widetilde{u}, \quad z = x_1 + ix_2, \tag{2.4}$$

we have

$$f_{\bar{z}} = \mu f_z + \nu \overline{f_z} \quad \text{a.e. in } \Omega, \tag{2.5}$$

where the coefficients $\nu, \mu \in L^\infty(\Omega)$ depend (explicitly) only on σ and satisfy the inequalities

$$|\nu| + |\mu| \leqslant \frac{K-1}{K+1} < 1 \quad \text{a.e. in } \Omega.$$

More details can be found in **[16]**. In particular, for any domain $D \subset\subset \Omega$ we have

$$f \in W^{1,2}(D, C), \quad |f_{\bar{z}}| \leqslant \frac{K-1}{K+1}|f_z| \quad \text{a.e. in } D. \tag{2.6}$$

As is known, any f satisfying (2.6) is a K-quasiregular mapping. It is also well known that f is represented in Ω as

$$f = F \circ \chi, \tag{2.7}$$

where χ is K-quasiconformal and F is holomorphic (cf., for example, **[10, 34]**). Consequently, setting $h = \operatorname{Re} F$ and $\widetilde{h} = \operatorname{Im} F$, we obtain the representations

$$u = h \circ \chi, \quad \widetilde{u} = \widetilde{h} \circ \chi. \tag{2.8}$$

Note that the notion of a stream function is invariant under quasiconformal changes of coordinates, i.e., if u is a solution to Eq. (2.1) and $\chi : \Omega \to R^2$ is a quasiconformal mapping, then $v = u \circ \chi^{-1}$ is a solution to the equation

$$\operatorname{div}(\tau \nabla v) = 0 \quad \text{in } G = \chi(\Omega), \tag{2.9}$$

where

$$\tau = \frac{D\chi \sigma D\chi^T}{\det D\chi} \circ \chi^{-1}. \tag{2.10}$$

We note that the stream function $\widetilde{v}$ associated to v by formula (2.9) is given by the formula $\widetilde{v} = \widetilde{u} \circ \chi^{-1}$.

Returning to (2.8), we say that $z_0 \in \Omega$ is a *geometric critical point* if $\nabla h(\chi(z_0)) = 0$.

We observe that geometric critical points are isolated; furthermore, in a small neighborhood of any geometric critical point z_0, the level set $\{u = u(z_0)\}$ is composed by $I + 1$ simple arcs whose pairwise intersection consists of $\{z_0\}$ only. Here, $I = I(z_0, u)$ is the positive integer given by the multiplicity of the zero of $\partial_z h$ at the point $\chi(z_0)$. Such a number is called the *geometric index* of u at z_0. Let $D \subset\subset \Omega$ be an open set such that ∂D contains no geometric critical points. We denote by $I(D, u)$ the sum of the geometric indices of the geometric critical points z_k of u within D. If u is smooth and ∂D is piecewise regular, then the index $I(D,u)$ can be computed by the contour integral

$$I(D,u) = -\frac{1}{2\pi}\int_{\partial D} \mathrm{d}(\arg \nabla u). \tag{2.11}$$

If u_1 and u_2 are solutions to Eq. (2.1), then we fix their stream functions $\widetilde{u}_1$ and $\widetilde{u}_2$ by prescribing $\widetilde{u}_1(0) = \widetilde{u}_2(0) = 0$ and set $\widetilde{U} = (\widetilde{u}_1, \widetilde{u}_2)$, where, without loss of generality, we assume that $0 \in \Omega$. For any fixed $\xi \in R^2$ such that $|\xi| = 1$ we set

$$u = \langle \xi, U\rangle = \xi_1 u_1 + \xi_2 u_2, \quad \widetilde{u} = \langle \xi, \widetilde{U}\rangle = \xi_1\widetilde{u}_1 + \xi_2\widetilde{u}_2, \quad f = u + i\widetilde{u}. \tag{2.12}$$

Clearly, u, $\widetilde{u}$, and f depend on ξ. However, we will not keep track of this dependence in the notation. We stress that for any choice of ξ the function u defined in (2.12) is also a solution to Eq. (2.1) in Ω and $\widetilde{u}$ is its associated stream function. Hence f is K-quasiregular on the same set.

Proposition 2.1 (cf. [6, Proposition 1]). *Let Ω be a connected open set in R^2, let $\sigma \in L^\infty(\Omega, \mathcal{M}_K^s)$, and let $U \in W^{1,2}_{\mathrm{loc}}(\Omega, R^2)$ be σ-harmonic. If for every ξ such that $|\xi| = 1$ the function f is univalent, then U is univalent.*

Proposition 2.2 (cf. [6, Proposition 2]). *Let the assumptions of Theorem 1.1 be satisfied, let $U = U^I$, and let u be defined by (2.12). For every ξ such that $|\xi| = 1$ the function u has no geometric critical points in R^2.*

Proposition 2.3 (cf. [6, Proposition 3]). *Let the assumptions of Theorem 2.1 be satisfied, let $U = U^I$, and let f be defined by (2.12). Then for every ξ such that $|\xi| = 1$ the function f is univalent.*

3. Threshold Exponents

We give a complete (and affirmative) solution to two conjectures due to Milton [8]. These conjectures are discussed in detail in the paper [15] by Leonetti and Nesi. Roughly speaking, they can be stated as follows. Let $\sigma \in L^\infty(\Omega, \mathcal{M}_K^s)$. Then, under reasonable boundary conditions, any σ-harmonic *function* satisfies the condition that $|\nabla u|$ and $|\nabla u|^{-1}$ belong to some *precise* family of L^p-spaces. The original conjectures were posed within a rather general framework and were formulated in any dimension. We focus here on dimension two. In [15], the authors were able to treat the Dirichlet or Neumann boundary conditions relying on previous work by Alessandrini and Magnanini [16]. However, a more satisfactory statement should include periodic boundary conditions. Now we fill that gap. We emphasize that the essential advance which makes possible to answer Milton's questions is due to Astala [14].

For a given vector $\xi \in R^2$, $|\xi| = 1$, we consider a solution u to the problem

$$\operatorname{div}(\sigma(x)\nabla u) = 0, \quad x \in Q, \quad u - \langle \xi, x\rangle \in W^{1,2}_{\sharp}(R^2, R). \tag{3.1}$$

According to the notation introduced in (2.12), we have $u = \langle \xi, U^I\rangle$, where U^I is the σ-harmonic mapping solving (1.5) for $A = I$. Let

$$p_K = \frac{2K}{K-1}, \quad q_K = \frac{2}{K-1}.$$

In fact, these are the *threshold exponents* introduced by Milton. We show that

$$\forall p < p_K \quad |\nabla u| \in L^p(Q), \quad \forall q < q_K \quad |\nabla u|^{-1} \in L^q(Q).$$

In fact, more is true and the precise statement requires the notion of the weak-L^p spaces of Marcinkiewicz (cf., for example, [**75**]). We recall that a measurable function f belongs to $L^p_{\text{weak}}(Q)$, $1 < p < \infty$, if and only if there exists a constant $c > 0$ such that for every measurable set $E \subset Q$ the following inequality holds:

$$\int\limits_E |f| \leqslant c|E|^{1-1/p}.$$

Theorem 3.1. *For every* $\xi \in R^2$, $|\xi| = 1$, *we have*

$$|\nabla u| \in L^{p_K}_{\text{weak}}(Q), \tag{3.2}$$

$$|\nabla u|^{-1} \in L^{q_K}_{\text{weak}}(Q). \tag{3.3}$$

PROOF. Formula (3.2) follows from Theorem 1 in [**15**] (based on Astala's work [**14**]), which provides an optimal form of the local higher integrability property of first derivatives, as obtained by Bojarski [**76, 9**] for quasiregular mappings and by Meyers [**12**] for solutions to elliptic equations in divergence form in any spatial dimension. Formula (3.3) follows from [**15**, Theorems 2 and 3] and the observation that the set of geometric critical points is empty in this case because of Proposition 2.2.

4. Area Formulas

4.1. First area formula: geometric interpretation for functions of σ_{hom}. This is a continuation of the analysis developed in [**15**]. Here we give a periodic version of that result.

Proposition 4.1. *Let* $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$. *Set* $\xi \in R^2$, $|\xi| = 1$. *Let* $U = U^I$ *be a solution to the problem* (1.5) *with* $A = I$, *and let* $\widetilde{u}$ *and* f *be defined according to* (2.12). *For every* $\xi \in R^2$, $|\xi| = 1$, *we have*

$$\langle \sigma_{\text{hom}}\xi, \xi\rangle = |f(Q)|, \tag{4.1}$$

where the expression on the left-hand side represents the **area** *of* $f(Q)$.

REMARK 4.1. This identity is similar to that obtained in the final part of [**15**] except that it now applies directly to σ_{hom}.

PROOF OF PROPOSITION 4.1. By Proposition 2.3, f is univalent and hence quasiconformal. Therefore,

$$|f(Q)| = \int_Q \det Df(x)dx = \int_Q \langle J\nabla u, \nabla \widetilde{u}\rangle = \quad \text{[by (2.2)]}$$
$$= \int_Q \langle J\nabla u, J\sigma\nabla u\rangle = \int_Q \langle \nabla u, \sigma\nabla u\rangle = \quad \text{[by (1.4)]} \quad = \langle \xi, \sigma_{\text{hom}}\xi\rangle.$$

Consequently, (4.1) follows. □

4.2. Second area formula: geometric interpretation for $\det \sigma_{\text{hom}}$ in the two-phase problem. We explore the properties of σ-harmonic mapping U in some special case. We begin with a preliminary result of independent interest.

Proposition 4.2. *Let $\sigma \in L^\infty(\Omega, \mathcal{M}^s_K)$, and let U be a σ-harmonic sense-preserving univalent mapping. Then the change of variable formula holds: for any measurable set $E \subset \Omega$ ($E \subset Q$) and function $\varphi \in L^1(U(\Omega), R)$ ($\varphi \in L^1(U(Q), R)$)*

$$\int_E \varphi(U(x))|\det DU(x)|dx = \int_{U(E)} \varphi(y)dy. \tag{4.2}$$

In particular, the area formula holds.

PROOF. This assertion is a consequence of the Radó–Reicheldorfer theorem (cf., for example, the book by Giaquinta, Modica, and Souček [**77**, Theorem 2, p. 223]). Indeed, our mapping U belongs to $W^{1,p}_{\text{loc}}$ for some $p > 2$, which follows by Meyers' theorem [**12**]. Therefore, it satisfies the Lusin property (cf. [**77**, Theorem 3, p. 223]) and, consequently, the (generalized) change of variable formula applies (cf. [**77**, p. 219]). It remains to check that the Banach indicatrix function is one almost everywhere in our case. This can be proved as follows. First we note that U is differentiable almost everywhere and, consequently, approximately differentiable almost everywhere (cf. [**77**, Theorem 5, p. 200]). Hence the generalized Banach indicatrix function can be interpreted in the classical sense. Therefore, it is identically one by the injectivity of the mapping U. □

Let us give a very elementary but amusing application. For any integer $n \geqslant 2$ we set

$$f_n : B(0,1) \to C, \quad f_n(z) = z + \overline{z}^n/n.$$

It is easy to check that f_n are univalent harmonic mappings. Hence

$$|f_n(B(0,1))| = \int_B |\det Df_n| = \int_B (|\partial_z f|^2 - |\partial_{\bar{z}} f|^2) = \int_B (1 - |z|^{n-1}) = \frac{n-1}{n}\pi.$$

Now we use Proposition 4.2 in a special but interesting case. Assume that $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$ and, in addition, $\det\sigma$ takes only two distinct values, say, d_1^2 and d_2^2. For example, if σ is isotropic (i.e., proportional to the identity) at any point, then we come back to the two-phase problem described in Sec. 1. If σ is not isotropic, the following simplest example is when each eigenvalue of σ takes only two values, usually called principal conductivities of the basic crystals. This is often referred to as the *two-polycrystal problem* because of its physical interpretation. The corresponding G-closure problem has been solved only in the context of unconstrained volume fraction (cf. [**53**]). Substantial progress was made more recently in the case where the volume fraction is prescribed. However, a complete understanding is yet not available. Let us give a geometric interpretation of this problem.

Our starting points are formulas (1.5) and (1.6). We have

$$\det A \det \sigma_{\text{hom}} = d_1^2 \det A + (d_2^2 - d_1^2) \int_{Q_2} \det DU^A(x)dx,$$

where Q_2 denotes the set where $\det\sigma = d_2^2$.

Therefore, $\det\sigma_{\text{hom}} = d_1^2$ if $d_1 = d_2$ and $\det A > 0$ if $d_1 \neq d_2$; moreover,

$$\frac{\det\sigma_{\text{hom}} - d_1^2}{d_2^2 - d_1^2} = \frac{\int_{Q_2} \det DU^A(x)dx}{\det A} = \frac{|U^A(Q_2)|}{\det A} = \frac{|U^A(Q_2)|}{|U^A(Q)|}. \tag{4.3}$$

The last two equalities directly follow from Proposition 4.2. Therefore, the minimization (maximization) problem for $\det\sigma_{\text{hom}}$ is *equivalent* to the problem of finding the "best" σ-harmonic mapping relative to the following criterion: minimize (maximize) with respect to the microgeometry of the (relative) area of $U^A(Q_2)$.

Unfortunately, the area distortion properties of generic σ-harmonic mappings are not yet clear, this is also suggested by examples developed in [**37**]. For this reason, at present, the only bound which can be deduced directly by (4.3) is

$$\min(d_1^2, d_2^2) \leqslant \det\sigma_{\text{hom}} \leqslant \max(d_1^2, d_2^2).$$

A different and more involved application of such a type can be found in [**61**, Sec. 7].

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On a Variational Problem Connected with Phase Transitions of Means in Controllable Dynamical Systems

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Dedicated to Olga Aleksandrovna with whom we discussed these questions in Dilizhan, at Pokrov-Na-Nerli, and during a voyage along canals of Sweden from Stockholm to Göteborg

We study the optimization of the integral of a given smooth function along the distribution determined by a density bounded by given functions from above and from below. Phase transitions are nonsmooth dependence of optimal means, as well as optimal strategies, on parameters. In this variational problem and, in particular, in the case of functions of even number of variables, they necessarily appear by topological reasons, which leads to logarithmic singularities. Two-dimensional variational problems in hydrodynamics and in magnetohydrodynamics are also considered. In these problems, as in the case studied in this paper, singularities are caused by topological reasons which I discussed with great pleasure with O. A. Ladyzhenskaya in Dilizhan in 1973.

The variational problem under consideration consists in optimization (for definiteness, maximization) of the mean value of a smooth function defined on a compact manifold by an appropriate choice of a density of mass distribution that is bounded from above and from below (by two smooth positive functions bounding the unknown density). If, in addition, the initial data of the problem smoothly depend on parameters, then the optimal distribution, as well as the

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optimal mean value, depends on these parameters. But this new dependence is, in general, nonsmooth because the change of parameters can cause the change of the character of the optimal strategy, which, in turn, leads to special singularities of functions expressing the dependence of the optimal distribution and the mean value on exterior parameters (on some "phase transition hypersurface" in the space of parameters).

The necessity to study these phase transitions naturally appears while looking for the control optimizing the long time temporal mean in the theory of controllable dynamical systems, which, in some simplest cases, is reduced to the above variational problem. But, in these problems, the functions bounding the unknown density of distribution have some standard singularities, which leads to additional (comparing with the cases considered here) phase transitions (studied in [**1**]).

The problem about optimal mass distribution is, in a sense, a limiting-simplified form of A. D. Sakharov's minimization problem for the magnetic energy of a magnetic field of a star under the condition that the magnetic field is frozen in an incompressible fluid. At the end of the present paper, we compare problems about topologically inevitable singularities of various origin appearing while solving various variational problems, in particular, those that come from the Euler equation in hydrodynamics, as well as in magnetohydrodynamics.

1. Description of Optimal Distribution in Mean

Below we study the maximum of the mean value of a given goal function $f : M \to \mathbb{R}$ that is assumed to be smooth on an N-dimensional compact Riemannian manifold M.

The mean value is maximized by an appropriate choice of an optimal mass distribution defined on M by an N-form $\rho(x)d^N x$. The unknown density ρ of this form (relative to the Riemannian element of volume $d^N x$) is assumed to be bounded from above and from below by two given positive smooth functions r and R on M so that, at each point x of M, the density ρ satisfies the inequalities

$$0 < r(x) \leqslant \rho(x) \leqslant R(x) < \infty, \quad \text{where } r(x) < R(x).$$

These conditions on the unknown density ρ force us to use for the localization of mass not only the points of M at which the goal function f attains the maximum.

Choosing ρ in an appropriate way, it is required to maximize the mean value of the goal function f along the distribution with density ρ, i.e., the number

$$\widehat{f}[\rho] = \Big(\int\limits_M f(x)\rho(x)d^N x\Big)\Big/\Big(\int\limits_M \rho(x)d^N x\Big).$$

First we fix the total mass (i.e., the integral standing in the denominator). Then we must maximize the integral of f in the numerator by choosing a density ρ of the fixed total mass distribution. The density maximizing the mean value of the function is very special:

Theorem 1. *For a fixed total mass the optimal density ρ coincides, at almost every point, with either its upper bound or its lower bound. Namely, there exists a constant c such that*

$$\rho(x) = r(x) \quad \text{if } f(x) < c, \tag{1}$$

$$\rho(x) = R(x) \quad \text{if } f(x) > c. \tag{2}$$

REMARK. For each value of the constant c formulas (1) and (2) determine some density ρ_c (irrespective to any optimality and total mass).

The total mass of this distribution monotonically decreases if the constant c increases. The mass changes within the following limits: from the integral of the upper bound $\int\limits_M R(x)d^N x$ which is obtained from formula (2) as the total mass for any $c \leqslant \min f$ to the integral of the lower bound which is obtained from formula (1) as the total mass for any $c \geqslant \max f$.

Indeed, the increase of the constant c leads to the decrease of the density ρ_c (from the value $R(x)$ to the value $r(x)$) at the points where the value of the function f is located between the old constant and the new increased constant, which decreases the total mass distribution density ρ_c (by the quantity that is equal to the integral of the positive function $R - r$ over the set, where the value of the goal function f is located between two constants, the old constant and the increased one).

From the monotonicity (and the obvious continuity) of the total mass of distribution with density ρ_c regarded as a function of c, which has been proved, it follows that for any value I of the total mass (located between the value of the integral of $rd^N x$ and that of $Rd^N x$ over M) there exists a unique value of the constant c (between the maximum and minimum of the goal function) such that the total mass of the distribution of density ρ_c is equal to I.

Thereby we have found a monotonically decreasing function

$$I(c) = \int\limits_M \rho_c(x) d^N x$$

and its inverse $c(I)$ which is also a monotone function.

PROOF OF THEOREM 1. We compare an arbitrary distribution ρ with total mass I and the special distribution ρ_c constructed from the value of the constant $c(I)$ corresponding the fixed total mass. (In the case of the maximal or minimal total mass, for the value of the constant c we take respectively the minimum or maximum of the goal function).

Let us prove that the integral of the function f along the special distribution of density ρ_c is greater than that along to any other distribution of density ρ with the same total mass.

Indeed, we consider the difference of these integrals,

$$g = \int\limits_M f(x)(\rho_c(x) - \rho(x))d^N x,$$

and prove that it is positive.

By the assumptions on the density ρ and the choice ((1), (2)) of the density of the special distribution ρ_c, the difference of densities $\rho_c(x) - \rho(x)$ is positive at points where $f(x) > c$ and is negative at points where $f(x) < c$ (under the assumption that it does not vanish for a given c). The integral of this difference of densities vanishes because the total masses of the distributions with densities ρ_c and ρ are the same. Therefore, computing the integral g of the function f, we can increase (or decrease) this function by any constant; the value of an integral of the form g will not change.

Replacing the function f in the formula for g with the equivalent function $f - c$, for g we get the integral of the positive function $(f - c)(\rho_c - \rho)$ (over the Riemannian volume on M). Hence $g > 0$, which proves Theorem 1. □

The case where the integral g vanishes can be easily treated: at the points where $f(x) \neq c$, from $g = 0$ it follows that $\rho_c(x) = \rho(x)$. Hence all optimal (conditionally, for a fixed total mass) distributions have the density ρ_c determined by the conditions (1) and (2).

Now we study the unconstrained optimization problem, maximizing the ratio of integrals $\widehat{f}[\rho_c] = J(c)/I(c)$, where

$$J(c) = \int\limits_M f(x)\rho_c(x)d^N x, \quad I(c) = \int\limits_M \rho_c(x)d^N(x), \tag{3}$$

by an appropriate choice of the constant c. Formula (3) determines a curve (parametrized by the parameter c) on the plane with coordinates (I, J), and we look for a critical point P on this curve at which tan of the angle between the radius-vector and the I-axis is maximal.

Lemma 1. *The tangent to the curve defined in the plane $\{(I, J)\}$ by formula (3) exists at each point c, and* tan *of the angle between this tangent and the I-axis is equal to c.*

PROOF. For the constant c we consider a small (negative below) increment Δc and study increments ΔI and ΔJ of the integrals (3). For definiteness, we assume that the value of c increases. Then the domain where $f(x) > c$ increases, and, in the corresponding increment of domain, the function ρ_c increases from the previous value $r(x)$ to a larger value $R(x)$ and remains the same in other places. Therefore, the increment of the total mass integral I is determined by the formula

$$\Delta I = \int (R(x) - r(x)) d^N(x)$$

(the integral is taken over the small, together with $|\Delta c|$, domain where $(c+\Delta c) < f(x) < c$).

In the same way, the increment of the integral of the goal function J is determined by the formula

$$\Delta J = \int f(x)(R(x) - r(x)) d^N(x)$$

(the integral is taken over the same small, together with $|\Delta c|$, domain).

Thus, the ratio of increments $(\Delta J)/(\Delta I)$ is equal to the mean value of the function f over the domain where this function is concluded between $c - \Delta c$ and c. Therefore, there exists tan of the angle between the curve (3) and the I-axis:

$$\lim_{\Delta c \to 0} (\Delta J)/(\Delta I) = c,$$

which was asserted by the lemma (this proof did not use the smoothness of the functions r and R). □

REMARK. The smoothness of the integrals $I(c)$ and $J(c)$ is not asserted by the lemma and was not used; we will study their singularities in Sec. 2 below.

Corollary 1. *The curve given by formula* (3) *on the plane is convex "upward" (relative to the region where* $J > 0$*).*

PROOF. We already proved that $I(c)$ is monotonically decreasing. Therefore, the curve (3) is uniquely projected onto the I-axis along the J-axis and can be regarded as the graph of a single-valued function $J = H(I)$. Unlike the functions I and J of variable c, this function H has the known derivative $dH/dI = c$ (by Lemma 1). Moreover, this derivative is continuous and monotonically decreasing while the abscissa I of the point of the curve (3) increases, which proves the convexity of this curve. □

From Lemma 1 it also follows that for genuine (unconditional) optimal distribution ρ_c, if it exists, the mean value $\widehat{f}[\rho_c]$ coincides with the value of the constant c because, at an interior point of the curve (3) at which the angle between the radius-vector and the abscissa-axis is maximal, the tangent to the curve should be directed along this radius-vector. The endpoints of this curve cannot be optimal. For example, the left endpoint corresponds to the maximal value c of the goal function and the slope of the tangent (at the point corresponding to the minimal total mass $I(c)$). But the slope of the radius-vector at this point is less than c since the goal function averaged while calculating takes the maximal value not everywhere.

In the same way, the right endpoint of the curve (3) corresponds to the minimal value c of the goal function and the slope of the tangent (at the point corresponding to the maximal total mass $I(c)$). But the slope of this radius-vector at this point is greater than c because the goal function averaged while calculating takes the minimal value not everywhere.

The above arguments are not appropriate in a single case where the goal function is a constant and its mean value is independent of a choice of the density ρ, so all the above arguments are not valid.

Below the goal function, as well as above, will be assumed to be nonconstant.

Corollary 2. *An unconditionally optimal density ρ exists and is unique.*

It is required to show (by Theorem 1) that the function $\widehat{f}[\rho_c]$ of the variable c has one and only one maximum point with respect to c (located between the maximum and minimum of the goal function f).

In terms of the curve (3) which is convex in view of Corollary 1 we can say that critical points of the function $\widehat{f}$ of the variable c are those points P on this curve at which the tangent to the curve is directed along the radius-vector (i.e., where the mean value of f along the distribution $\rho_c d^N x$ is c).

Lemma 2. *Every critical point of the function $\widehat{f}$ of the variable c is a local maximum point.*

PROOF. To the left of the point P at which the curve touches its radius-vector the value of the total mass I is less (consequently, the value c of tan of the angle between the tangent to the curve and the I-axis is greater) than that at the point P. To the right of the point P, conversely, the slope of the tangent to the curve is less than that at the critical point. Consequently, the distance between a point of the curve (3) and the tangent to the curve at the critical point P measured along vertical (as the difference of J-coordinates) has negative derivative with respect to I for values of I less than that at the point P and has positive derivative for greater values of the abscissa I.

This proves that the curve (3) is located (in a neighborhood of the critical point P) below the radius-vector joining 0 with P, i.e., $\widehat{f}$ has a local maximum at the point c corresponding to P. □

The assertion of Corollary 2 immediately follows from Lemma 2 since between two local maxima there is a local minimum (at least, nonstrict). In fact, Lemma 2 asserts more than it was asserted by Corollary 2: not only other maximum points, but also a critical point is unique.

The *existence* of a critical point also follows from the above inequalities because the slope of the curve (3) is *greater* than the slope of the radius-vector at the left endpoint and is less than the slope of the radius-vector at the right endpoint. Hence, on the curve (3) there is a point P at which the tangent is directed along the radius-vector; and tan c of this slope of this tangent is the required critical point of the function $\widehat{f}$.

2. Study of Singularities of the Optimal Mean Value for a Fixed Total Mass

Denote by $D(x)$ the positive difference $R(x) - r(x)$ of the functions bounded the choice of an optimal density ρ. Then the first integral in formula (3) takes the form

$$I(c) = \int_M r(x)\,dx + \int_{f(x)>c} D(x)\,dx \tag{4}$$

(to treat the second integral J, one should replace r with rf and D with Df).

The first term of the sum (4) is independent of c. Therefore, the question on singularities of the dependence of I on c is reduced to the study of the dependence of the second term of the sum (4) on c. A typical example is the case $D \equiv 1$, i.e., the study of volumes of sets of larger or smaller values of the smooth function f.

If c is not a critical value of the goal function f, then the smoothness of the dependence of the integrals (3) on the value of c is obvious (if the functions r and R that bound the density are smooth).

Theorem 2. *For a smooth general state function f of N real variables the volume $V(c)$ of the domain where $f(x) < c$ has the following singularities in a neighborhood of each critical value c of the function f:*

— if N is odd, then the singularity is a root one of degree $N/2$ (the difference between $V(c+\varepsilon)$ and some smooth function of ε is asymptotically equivalent to the power function $L_\pm|\varepsilon|^{N/2}$ for small $|\varepsilon|$, where the constants $L_\pm$ corresponding to $\varepsilon > 0$ and $\varepsilon < 0$ are different);

— if N is even, then the volume $V(c+\varepsilon)$ differs from some smooth function for $\varepsilon = 0$ by an additive term $L_\pm\varepsilon^{N/2}\ln(1/|\varepsilon|)$ with logarithmic asymptotics for small $|\varepsilon|$ (again, the coefficients L_+ and L_- are different depending on the sign of ε).

Although this theorem (and its variant for integrals generalizing volumes of functions that are not necessarily equal to 1 everywhere, as in the case of our functions D and fD) should be a classical result of the standard distribution theory, it is usually not explained to students. That is why I give necessary (elementary) estimates below.

From all this theory we immediately obtain the analysis of the nonsmoothness degree for the curve (3) the tangent of which was used in Sec. 1. For example, for an odd N on the curve there are weak root singularities (in a neighborhood of the points P corresponding to the critical values of the goal function):

$$J(I) = J_0(I) + E_\pm |I - I_0|^{N/2+1}.$$

Here, the function J_0 is smooth and the coefficient of the singular term E depends on the sign of the difference $I - I_0$; the point P has the coordinates $(I_0, J_0(I_0))$.

If the initial data (f, r, R) depend, in addition, on exterior parameters, then the above results allow us to study the phase transitions appearing here, i.e., the nonsmoothness degree with respect to these exterior parameters (of the choice of an optimal strategy ρ, as well as of the resulting optimal mean value $\widehat{f}[\rho]$).

Such considerations were done in [**1**] for a slightly other problem (in which the functions r and R restricting a choice of the density ρ have themselves standard singularities).

Here, I restrict myself to the proof of the above-formulated result on the nonsmoothness of the functions I and J with respect to parameter-argument c.

PROOF OF THEOREM 2. It suffices to consider the standard Morse function $f = x^2 - y^2$ with critical value 0 for $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $N = m + n$.

We consider the contribution to the integral of a neighborhood of the critical point 0 that is given by the localization inequalities $|x| < C$, $|y| < c$:

$$\mathcal{F}(\varepsilon) = \iint\limits_{|x|<C,\ |y|<c,\ f(x,y)<\varepsilon} dx\, dy$$

(hereinafter, we do not use the letter c for the critical value).

Case $\varepsilon > 0$. We set $\mathcal{F}(\varepsilon) = \mathcal{F}(0) + \Omega(\varepsilon)$. Hence $\Omega(\varepsilon)$ is the volume of the domain located between the cone $f = 0$ and the hyperboloid $|x|^2 \leqslant |y|^2 + \varepsilon$ (under the above localization conditions on $|x|$ and $|y|$).

We denote by V_k the volume of the k-dimensional ball of radius 1 in the Euclidean space. Then, stratifying the integration domain in annular layers $y = \text{const}$, for $|y| = r$ and $|x| = R$ on the hyperboloid we find

$$\Omega(\varepsilon) = \int\limits_{r=0}^{c} V_m(R^m - r^m) n V_n r^{n-1} dr$$

(since the sphere of radius r in the Euclidean space has the surface area $nV_n r^{n-1}$ and the annular domain $r < |x| < R$ in $\mathbb{R}^m$ has the volume $V_m(R^m - r^m)$.

By the Newton binomial formula, we find

$$R^m = (r^2 + \varepsilon)^{m/2} = \sum_{s=0}^{\infty} r^{m-2s} C^s_{m/2} \varepsilon^s.$$

The term with $s = 0$ contributes r^m. Therefore, for $\varepsilon > 0$ the volume of the layer located between the cone and the hyperboloid is expressed as follows:

$$\Omega(\varepsilon) = \frac{nV_m V_n}{2} \sum_{s=1}^{\infty} \int_{r=0}^{c} C^s_{m/2} \varepsilon^s r^{m+n-2-2s} du,$$

where $u = r^2$. Denoting the coefficient $nV_m V_n$ by K, we reduce this expression to the form of a power series with respect to ε:

$$\Omega(\varepsilon) = K \sum_{s=1}^{\infty} C^s_{m/2} \varepsilon^s \frac{c^{N-2s}}{N-2s}$$

if N is odd.

If N is even, then the resonance term with $2s = N$ in this sum should be omitted while integrating and be replaced with the logarithmic (regularized at zero) integral

$$\int_{r=r_0}^{c} \frac{du}{u} = 2 \ln(c/r_0)$$

(and adding to the volume Ω, the contribution of the domain $|y| < r_0$ eliminated from the integral while regularization; this contribution does not exceed the product $V_m \varepsilon^{m/2} V_n r_0^n$).

We choose the value of the regularization parameter r_0 later, but just now we consider the case of negative ε.

Case $\varepsilon < 0$. In this case, we consider the domain between the same cone and a hyperboloid of signature that differs from that in the case $\varepsilon > 0$; this domain is given by the inequalities $\delta < f(x, y) < 0$, where $\delta = -\varepsilon > 0$.

Under the same localization assumptions, we denote the volume of this domain by $A(\delta)$, Then for the value of $\mathcal{F}$ for negative argument ε we obtain the expression $\mathcal{F}(\varepsilon) = \mathcal{F}(0) - A(\delta)$.

To integrate over the above-indicated domain, we divide this domain into the conical part where $|x| < |y| = r < \sqrt{\delta}$ and the part which is stratified into the following annula: $R < |x| < r$, $|y| > \sqrt{\delta}$ (now, $R^2 = r^2 + \varepsilon = r^2 - \delta < r^2$ on the hyperboloid).

Then we express the integral as the sum of two terms (corresponding to these two parts):

$$A(\delta) = \int_{r=0}^{\sqrt{\delta}} V_m r^m n V_n r^{n-1} dr + \int_{r=\sqrt{\delta}}^{c} V_m (r^m - R^m) n V_n r^{n-1} dr.$$

Again, using the notation $nV_mV_n = K$ and the binomial expression for $R^m = (r^2 - \delta)^{m/2}$, we write the final expression for this sum $A(\delta)$ as follows:

$$\frac{A(\delta)}{K} = \frac{\delta^{N/2}}{N} - \sum_{s=1}^{\infty} C^s_{m/2}(-\delta)^s \frac{C^{N-2s}}{N-2s} + \sum_{s=1}^{\infty} C^s_{m/2}(-\delta)^s \frac{\delta^{N/2-s}}{N-2s}$$

(with a logarithmic correction for even N).

The intermediate term of this three-term formula gives in the sum $\mathcal{F}(\varepsilon) = \mathcal{F}(0) - A(\delta)$ a contribution which coincides (up to the notation $\varepsilon = -\delta$) with the formula for $\Omega(\varepsilon)$ obtained for $\varepsilon > 0$ earlier.

In addition to this contribution, one more term is obtained from the first and third terms of the sum for $A(\delta)/K$; it is equal to $-K\delta^{N/2}L$, where the coefficient L is expressed as follows:

$$L = 1/N + \sum_{s=1}^{\infty} C^s_{m/2}(-1)^s/(N-2s).$$

If N is odd, then the "difference" between the described left and right asymptotics ($\varepsilon < 0$ and $\varepsilon > 0$) is of order $|\varepsilon|^{N/2}$, as was asserted by the theorem.

If N is even, then for $\varepsilon > 0$ the terms written above are completed by the terms obtained in the regularization:

$$KC^{N/2}_{m/2}\varepsilon^{N/2}(2\ln c/r_0) + \frac{1}{n}\varepsilon^{m/2}r_0^n$$

and for $\varepsilon < 0$ we have other terms of similar origin:

$$KC^{N/2}_{m/2}\varepsilon^{N/2}(2\ln c/r_0) + (-\varepsilon)^{N/2}K\widetilde{L}.$$

If for the regularization parameter we take $r_0 = \sqrt{|\varepsilon|}$, then we obtain a term that is nonsmooth with respect to ε and has an asymptotic of order $\varepsilon^{N/2}\ln\varepsilon$ for $\varepsilon > 0$ and an asymptotic of order $\varepsilon^{N/2}\ln|\varepsilon|$ for $\varepsilon < 0$, which proves Theorem 2 in this case. $\square$

3. Sources of Singularities in Variational Problems

The proved asymptotics show what discontinuities of the derivatives of the dependence of the integrals on parameter c are caused by a critical point of the goal function f (the larger is dimension, the weaker are these discontinuities; roughly speaking, the number of discontinuous derivatives at a critical point of the goal function is equal to one-half the number of its arguments, but it is necessary to take into account that the logarithms appearing if the number of arguments is even).

These singularities of the volume and of the functions I and J of c lead to the same singularities of the dependence of the optimal mean value on additional parameters. This fact is described in **[1]** in detail for problems in the theory of optimization of the mean values in controlled dynamical systems (where similar singularities come from another source, namely, from the fact that the functions r and R themselves may have singularities, whereas they are smooth in our case).

However, the complete analysis of *all* typical singularities in such problems of the theory of the mean values optimization in controlled dynamical systems is yet incomplete even in the case where the number of variables and parameters is small, and, especially, if singularities of different origin begin to compete, as was described in the case of optimal control of loading cement mills in [**2**] and, after that, for the general problem about phase transitions of the mean values in [**1**].

Another interesting source of phase transitions in variational problems are provided by hydrodynamics and magnetohydrodynamis, even in the two-dimensional case. The simplest problem of such a kind (discussed, for example, in [**3**] and in [**4**]) is the minimization problem for the Dirichlet integral

$$\iint (\nabla u)^2 d^2x$$

over the class of all functions u in a given domain that are obtained from an initial smooth function u_0 by area-preserving diffeomorphisms.

The "Euler equation" of this variational problem formally coincides with the Euler equation for a steady flow of a two-dimensional incompressible fluid. But, from the physical point of view, our problem is closer to A. D. Sakharov's problem in magnetohydrodynamics about a steady-state magnetic field of minimal energy in a conducting star; this magnetic field is frozen in an incompressible fluid and is transferred by volume-preserving diffeomorphisms that are realized by fluid flows, whereas, in hydrodynamics, not the velocity field but its vortex field is frozen in (i.e., in the two-dimensional case, not the stream function u but its Laplacian r). The magnetic field of the star is minimized by the hydrodynamic evolution, i.e., by the fluid flow caused by the Lorentz force.

In both cases, the Euler equations of a steady-state field means that the "extremal" function u (independently of its physical meaning which is different in the hydrodynamic and magnetohydrodynamic cases) must commute, i.e., must be functionally dependent with its Laplacian (i.e., their Poisson bracket must vanish identically: $\{u, \Delta u\} \equiv 0$ in a two-dimensional "flow domain" or "star"),

However, among the functions obtained from an initial function u_0 by incompressible diffeomorphisms, a solution to this "commutativity equation" does not necessarily exist. Therefore, solutions of the above variational problem (for most initial conditions u_0) have singularities (so that the minimal steady-state "magnetic fields" u of the corresponding "stars" are nonsmooth).

The study of "hydrodynamic" and "magentohydrodynamic" singularities of fields of minimal energy that arise in such a way is more difficult than the study of the distributions minimizing the mean value of a function which, however, can be regarded as a limiting-simplified version of this magnetohydrodynamic problem (the three-dimensional version of this problem is of special interest from the topological point of view and was already discussed in [**3**] from the point of view of the theory of invariants of knots which even was improved due to this discussion).

If the "star" is a disk and the initial smooth function u_0 vanishes on the circle bounding this disk and has a single critical point inside the disk, say, a nondegenerate maximum, then the optimally transformed function is the "symmetrization" of the initial function. It depends only on the distance from the center of the disk, and the area of all domains where this function has less values is the same as that for the smooth initial function. This symmetrized function is smooth (in the case of such a simple topology of an initial smooth mountain with a single top).

If the initial smooth function, like the mountain Elbrus, has two tops separated by a saddle, then the optimally transformed function seems to have an entire "ditch" of singular points (singularities of type $|x|$) separating both mountain tops and both endpoints of this "ditch" are singularity points of special character (which were not still studied in a rigorous mathematical way).

Hopefully, the "topological" study of typical singularities in nonstandard variational problems, like the simplest example of the optimization problem for distribution considered in the present paper, allows one to be successful in the study of more difficult problems concerning the appearance of singularities of either vector fields, or forms, or tensors which should be optimized; many questions still remain open in this field. But it is extraordinary important to *analyze* phenomena observed in a neighborhood of a singularity point rather than to investigate the question whether optimal objects belong to some function spaces, which, actually, hides the essence of the matter.

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A Priori Estimates for Starshaped Compact Hypersurfaces with Prescribed mth Curvature Function in Space Forms

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Dedicated to Professor Olga A. Ladyzhenskaya

We obtain a priori bounds for solutions of the nonlinear second-order elliptic equation of the geometric problem consisting in finding a compact starshaped hypersurface in a space form whose mth elementary symmetric function of principal curvatures is a given function.

1. Introduction

Let $\mathcal{R}^{n+1}(K)$, $n \geqslant 2$, be a space form of sectional curvature $K = -1, 0$, or $+1$, and let m be an integer, $1 \leqslant m \leqslant n$. In this paper, we establish a priori bounds for solutions of the following geometric problem: under what conditions a given function $\psi : \mathcal{R}^{n+1}(K) \to (0, \infty)$ is the mth mean curvature H_m of a hypersurface M embedded in $\mathcal{R}^{n+1}(K)$ as a graph over a sphere?

Let us formulate the problem more precisely. First we describe the space $\mathcal{R}^{n+1}(K)$ in a form convenient for our purposes. In the Euclidean space R^{n+1}, we fix the origin O and the unit sphere S^n centered at O. Denote by u a point

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on S^n and by (u,ρ) the spherical coordinates in R^{n+1}. The standard metric on S^n induced from R^{n+1} is denoted by e. Let $a = \text{const}$, $0 < a \leqslant \infty$, $I = [0,a)$, and $f(\rho)$ a positive C^∞-function on I such that $f(0) = 0$. In R^{n+1}, we introduce the metric

$$h = d\rho^2 + f(\rho)e. \tag{1.1}$$

For $a = \infty$ and $f(\rho) = \rho^2$ the space (R^{n+1}, h) is the Euclidean space $\equiv R^{n+1}$. For $a = \infty$ and $f(\rho) = \sinh^2 \rho$ the space $(R^{n+1}, h) = \mathcal{R}^{n+1}(-1)$ is the hyperbolic space H^{n+1} with sectional curvature -1 and for $a = \pi/2$, $f(\rho) = \sin^2 \rho$, $(R^{n+1}, h) = \mathcal{R}^{n+1}(1)$ it is the elliptic space S^{n+1}_+ with sectional curvature $+1$. By the mth *mean curvature*, H_m, we understand the normalized elementary symmetric function of order m of principal curvatures $\lambda_1, \dots, \lambda_n$ of M, i.e.,

$$H_m = \frac{1}{\binom{n}{m}} \sum_{i_1 < \quad < i_n} \lambda_{i_1} \cdots \lambda_{i_m}.$$

The above problem can be formulated as follows. Let $\psi(u,\rho)$, $u \in S^n$, $\rho \in I$, be a given positive function. Under what conditions on ψ there exists a smooth hypersurface M given as $(u, z(u))$, $u \in S^n$, $z > 0$, such that

$$H_m(u) = \psi(u, z(u)) \quad \text{on } M? \tag{1.2}$$

In the Euclidean space $R^{n+1}(= \mathcal{R}^{n+1}(0))$, such conditions were found by Bakelman and Kantor [**1**, **2**] and Treibergs and Wei [**3**] for $m = 1$ (the mean curvature case), by Oliker [**4**] for $m = n$ (the Gauss curvature case), and by Caffarelli, Nirenberg, and Spruck [**5**] for $1 < m < n$. Other forms of such conditions for the Gauss curvature case in R^{n+1} were investigated by Delanoë [**6**], Li [**7**], and others. In [**8**], the Gauss curvature case was studied for hypersurfaces in $\mathcal{R}^{n+1}(-1)$ and $\mathcal{R}^{n+1}(1)$. Recently, special curvature functions for convex hypersurfaces in Riemannian manifolds have been considered by Gerhardt [**9**] (cf. references there).

In all investigations of (1.2) in the Euclidean space, a priori C^0-, C^1-, and C^2-estimates for solutions of (1.2) play a central role in the proof of existence. However, except for the C^0-estimates, obtaining these a priori estimates for hypersurfaces in the hyperbolic space $\mathcal{R}^{n+1}(-1)$ and in the elliptic space $\mathcal{R}^{n+1}(1)$ is not straight forward and requires additional efforts. The approach of this paper allows us to obtain a priori C^1-bounds in $\mathcal{R}^{n+1}(K)$ for any $1 \leqslant m \leqslant n$ and $K = -1, 0, 1$. In the case $K = 1$, we also obtain a priori C^2-estimates. The same proof of the C^2-estimate is valid in the case $K = 0$ treated earlier in [**4**] for $m = n$ and in [**5**] for $1 \leqslant m \leqslant n$.

The estimates established in this paper will be used to prove the corresponding existence results. These results will be presented in a forthcoming paper.

2. Preliminaries

2.1. Local formulas. Unless explicitly stated otherwise, all latin indices are in the range $1, \ldots, n$, the sums are taken over this range, and summation over repeated subscripts and superscripts is assumed. Since most of our considerations apply to space forms $\mathcal{R}^{n+1}(K)$, where K can be $-1, 0$ or 1, we will discuss the general case and indicate K explicitly only if necessary.

We consider hypersurfaces in $\mathcal{R}^{n+1}(K)$ that are graphs over S^n. Thus, for a smooth positive function $z(u)$, $u \in S^n$, we denote by $r(u) = (u, z(u))$ the graph M of z.

Throughout the paper, we use covariant differentiation on the sphere S^n and on the hypersurface M. We introduce the notation. We begin with S^n. Let $u^1, \ldots, u^n$ be smooth local coordinates in a coordinate neighborhood $U \subset S^n$, and let $\partial_i = \partial/\partial u^i$, $i = 1, 2, \ldots, n$, be the corresponding local frame of tangent vectors on U such that $e(\partial_i, \partial_j) = e_{ij}$. For a smooth function v on U the first covariant derivative is defined as follows: $v_i \equiv \nabla'_i v = \partial v/\partial u^i$. Put $\nabla' v = e^{ij} v_j \partial_i$, where $e^{ij} = (e_{ij})^{-1}$. For the covariant derivative of $\nabla' v$ we have

$$\nabla'_{\partial_s} \nabla' v = v_{sj} e^{ji} \partial_i + v_j \nabla'_{\partial_s}(e^{ij}\partial_i) = \nabla'_{sj} v e^{jk} \partial_k, \quad v_{sj} = \frac{\partial^2 v}{\partial u^s \partial u^j}$$

or, equivalently,

$$\nabla'_{sj} v = v_{sj} - \Gamma'^i_{sj} v_i,$$

where Γ'^i_{sj} are the Christoffel symbols of the second kind of the metric e. This differentiation is extended to vector-valued functions by differentiating each of the components.

Similarly, if T is a smooth symmetric $(0,2)$-tensor on U with components T_{ij} relative to the dual coframe, then the components of the first covariant derivatives of T on S^n are given by the formula

$$\nabla'_l T_{ij} = \frac{\partial T_{ij}}{\partial u^l} - T_{kj}\Gamma'^k_{lj} - T_{ki}\Gamma'^k_{il}.$$

If M is a hypersurface in $\mathcal{R}^{n+1}(K)$ and g is a metric on M, then the covariant differentiation on M is defined as above but with respect to connection of the metric g. In this case, for a smooth function v on M we denote by $\nabla_i v$ and $\nabla_{ij} v$ the first and second covariant derivatives respectively. The similar notation is used for vector-valued functions and smooth symmetric tensors on$M$.

We introduce the metric and the second fundamental form of M in the case where M is the graph of a smooth positive function z on S^n, i.e., $M = (u, z(u))$, $u \in S^n$. In the spherical coordinates (u, ρ) in $\mathcal{R}^{n+1}(K)$, we set $R = \partial/\partial\rho$. The frame $\partial_1, \ldots, \partial_n, R$ is a local frame along M, and a basis of tangent vectors on M is given by $r_i = \partial_i + z_i R$, $i = 1, \ldots, n$. The metric

$g = g_{ij} du^i du^j$ on M induced from $\mathcal{R}^{n+1}(K)$ has coefficients

$$g_{ij} = f e_{ij} + z_i z_j, \quad \det(g_{ij}) = f^{n-1}(f + |\nabla' z|^2) \det(e_{ij}). \tag{2.1}$$

Obviously, M is an embedded hypersurface. The inverse matrix $(g_{ij})^{-1}$ is given by the formula

$$g^{ij} = \frac{1}{f}\left[e^{ij} - \frac{z^i z^j}{f + |\nabla' z|^2}\right], \quad z^i = e^{ij} z_j. \tag{2.2}$$

The unit normal vector field on M is given by

$$N = \frac{\nabla' z - fR}{\sqrt{f^2 + f|\nabla' z|^2}}. \tag{2.3}$$

The second fundamental form b of M is the normal component of the covariant derivative in $\mathcal{R}^{n+1}(K)$ with respect to connection defined by the metric (1.1). In local coordinates, the coefficients of b are given by the formula (cf. [**8**])

$$b_{ij} = \frac{f}{\sqrt{f^2 + f|\nabla' z|^2}}\left[-\nabla'_{ij} z + \frac{\partial \ln f}{\partial \rho} z_i z_j + \frac{1}{2}\frac{\partial f}{\partial \rho} e_{ij}\right]. \tag{2.4}$$

Due to our choice of the normal, the second fundamental form of a sphere $z = \text{const} > 0$ is positive definite since $\partial f/\partial \rho > 0$ for $\mathcal{R}^{n+1}(K)$.

The principal curvatures of M are the eigenvalues of the second fundamental form relative to the metric g and are the real roots, $\lambda_1, \ldots, \lambda_n$, of the equation

$$\det(b_{ij} - \lambda g_{ij}) = 0$$

or

$$\det(a^i_j - \lambda \delta^i_j) = 0,$$

where

$$a^i_j = g^{ik} b_{kj}. \tag{2.5}$$

The elementary symmetric function of order m, $1 \leqslant m \leqslant n$, of $\lambda = (\lambda_1, \ldots, \lambda_n)$ is as follows:

$$S_m(\lambda) = \sum_{i_1 < \;\; < i_n} \lambda_{i_1} \cdots \lambda_{i_m},$$

i.e., $S_m(\lambda) = F(a^i_j)$, where F is the sum of the principal minors of (a^i_j) of order m. By the above arguments, we have

$$F(a^i_j) \equiv F(u, z, \nabla'_1, \ldots, \nabla'_n z, \nabla'_{11} z, \ldots, \nabla'_{nn} z).$$

Equation (1.2) takes the form

$$F(a^i_j) = \overline{\psi}(u, z(u)). \tag{2.6}$$

Hereinafter, we set $\overline{\psi} \equiv \binom{n}{m}\psi$ for the sake of convenience.

Let Γ be the connected component of $\{\lambda \in R^n \mid S_m(\lambda) > 0\}$ containing the positive cone $\{\lambda \in R^n \mid \lambda_1, \ldots, \lambda_n > 0\}$.

Definition 2.1. A positive function $z \in C^2(S^n)$ is said to be *admissible* for the operator F if for the corresponding hypersurface $M = (u, z(u))$, $u \in S^n$, at every point of M with the normal as in (2.3), the principal curvatures $(\lambda_1, \ldots, \lambda_n)$ belong to Γ.

As is known (cf. [**10**]),

$$S_{m\lambda_i} \equiv \frac{\partial S_m}{\partial \lambda_i} > 0, \quad S_{m\lambda_i\lambda_j} \equiv \frac{\partial^2 S_m}{\partial \lambda_i \partial \lambda_j} > 0 \tag{2.7}$$

for all $\lambda \in \Gamma$, $i \neq j$ (for the first inequality in (2.7) we refer to [**11**]). It is also known that the function $(S_m(\lambda))^{1/m}$ is concave on Γ (cf. [**10**]).

The function

$$q(\rho) \equiv \frac{1}{2f(\rho)} \frac{df(\rho)}{d\rho} \tag{2.8}$$

will play an important role in our construction. We note that, in the case of a sphere of radius c,

$$F(a^i_j) = \binom{n}{m} q^m(\rho)\Big|_{\rho=c}. \tag{2.9}$$

In R^{n+1}, we have $q(\rho) = \rho^{-1}$.

We indicate two basic properties of the function $q(\rho)$. First, it is strictly positive on the interval I (where f is defined). Second, since

$$\frac{\partial q}{\partial \rho} = -\frac{1}{f}, \tag{2.10}$$

it is strictly decreasing on I. From the definition of f for each of the spaces $\mathcal{R}^{n+1}(K)$, it follows that

$$f = \frac{1}{q^2 + K}. \tag{2.11}$$

3. C^0-Estimate

Lemma 3.1. *Let $1 \leqslant m \leqslant n$, and let $\psi(X)$ be a positive continuous function on $\mathcal{R}^{n+1}(K) \setminus \{0\}$. Suppose that there exist two numbers R_1 and R_2, $0 < R_1 < R_2 < a$, such that*

$$\psi(u, \rho) > q^m(\rho), \quad u \in S^n, \ \rho < R_1, \tag{3.1}$$

$$\psi(u, \rho) < q^m(\rho), \quad u \in S^n, \ \rho > R_2. \tag{3.2}$$

Let $z \in C^2(S^n)$ be a solution of Eq. (2.6). Then

$$R_1 \leqslant z(u) \leqslant R_2, \quad u \in S^n. \tag{3.3}$$

In applications, it suffices to use a slightly different form of this estimate.

Lemma 3.2. *Let $1 \leqslant m \leqslant n$, and let $\psi(X)$ be a positive continuous function in the annulus $\overline{\Omega}$: $u \in S^n$, $\rho \in [R_1, R_2]$, $0 < R_1 < R_2 < a$. Suppose that ψ satisfies the conditions*

$$\psi(u, R_1) \geqslant q^m(R_1), \quad u \in S^n, \tag{3.4}$$

$$\psi(u, R_2) \leqslant q^m(R_2), \quad u \in S^n. \tag{3.5}$$

Let $z \in C^2(S^n)$ be a solution of Eq. (2.6), and let $R_1 \leqslant z(u) \leqslant R_2$, $u \in S^n$. Then either $z \equiv R_1$, or $z \equiv R_2$, or

$$R_1 < z(u) < R_2, \quad u \in S^n. \tag{3.6}$$

PROOF OF LEMMA 3.1. Assume that there exists a point $\overline{u} \in S^n$ such that $\max_{S^n} z(u) = z(\overline{u}) > R_2$. At $\overline{u}$, we have $\operatorname{grad} z = 0$ and $\operatorname{Hess}(z) \leqslant 0$. Then, at $\overline{u}$, we have

$$g^{ij} = \frac{1}{f}e^{ij}, \quad b_{ij} = -\operatorname{Hess}(z) + fqe_{ij} \geqslant fqe_{ij},$$

and $a^i_j \geqslant q\delta^i_j$. Consequently,

$$F(a^i_j) = \overline{\psi}(\overline{u}, R_2) \geqslant \tbinom{n}{m} q^m(\rho)_{|\rho=R_2}$$

which contradicts the inequality (3.2). Similarly, $R_1 \geqslant z(u)$. □

The assertion of Lemma 3.2 follows from the strong maximum principle (cf. [**12**, Theorem 1]).

4. C^1-Estimate

Theorem 4.1. *Let $1 \leqslant m \leqslant n$, and let $\psi(X)$ be a positive C^1-function in the annulus $\overline{\Omega}$: $u \in S^n$, $\rho \in [R_1, R_2]$, $0 < R_1 < R_2 < a$. Let $z \in C^3(S^n)$ be an admissible solution of Eq. (2.6) satisfying the inequalities*

$$R_1 \leqslant z(u) \leqslant R_2, \quad u \in S^n. \tag{4.1}$$

In addition, suppose that for all $u \in S^n$ and $\rho \in [R_1, R_2]$ the function ψ satisfies one of the following conditions:

(a) *if the sectional curvature K is equal to 0 or to 1, then*

$$\frac{\partial}{\partial \rho}[\psi(u, \rho) q^{-m}(\rho)] \leqslant 0, \tag{4.2}$$

(b) *if the sectional curvature K is equal to -1, then*

$$\frac{\partial}{\partial \rho}[\psi(u, \rho) f^{m/2}(\rho)] \leqslant 0. \tag{4.3}$$

Then

$$|\operatorname{grad} z| \leqslant C, \tag{4.4}$$

where the constant C depends only on m, n, R_1, R_2, ψ, and $\operatorname{grad}\psi$.

PROOF. Substitute $v(u) = q(\rho)|_{\rho=z(u)}$. Using (2.10), we get

$$v_i = -\frac{z_i}{f}, \quad \nabla'_{ij} v = \frac{1}{f}\left[-\nabla'_{ij} z + \frac{f_\rho}{f} z_i z_j\right],$$

where f_ρ denotes the derivative of f with respect to ρ. Then

$$\nabla'_{ij} v + v e_{ij} = \frac{1}{f}\left[-\nabla'_{ij} z + \frac{f_\rho}{f} z_i z_j + \frac{1}{2} f_\rho e_{ij}\right].$$

Using (2.2), (2.4), and (2.5), we find

$$\begin{aligned} g^{ij} &= \frac{1}{f}\left[e^{ij} - \frac{f v^i v^j}{1 + f|\nabla' v|^2}\right], \quad v^i = e^{ij} v_j, \\ b_{ij} &= \frac{f}{\sqrt{1 + f|\nabla' v|^2}} (\nabla'_{ij} v + v e_{ij}), \\ a^i_j &= \left[\frac{(1 + f|\nabla' v|^2) e^{is} - f v^i v^s}{(1 + f|\nabla' v|^2)^{3/2}}\right] (\nabla'_{sj} v + v e_{sj}). \end{aligned} \tag{4.5}$$

We put

$$p^2 = v^2 + |\nabla' v|^2 + K, \quad P^{is} = p^2 e^{is} - v^i v^s, \quad W_{sj} = \nabla'_{sj} v + v e_{sj}.$$

By (2.11) and (4.1), we have $v \geqslant c > 1$ for $K = -1$ and $v \geqslant c' > 0$ for $K = 0, 1$, where the constants c and c' depend only on R_1 and R_2. Using (2.11) with $q = v$, we write (4.5) in the form

$$a^i_j = \frac{\sqrt{v^2 + K}}{p^3} P^{is} W_{sj}. \tag{4.6}$$

The C^0-bounds of v imply $p \geqslant c = \text{const} >$ on S^n, where c depends only on R_1 and R_2. To estimate $|\nabla' v|$, we estimate the maximum of the function $\varphi = p^2 \eta(v)$, where η is a positive function to be specified later. This will provide us with an estimate for $|\nabla' v|$ and, consequently, for $|\nabla' z|$.

Let $\overline{u} \in S^n$ be a point at which $\max_{S^n} \varphi(u)$ is attained, i.e., $\max_{S^n} \varphi(u) = \varphi(\overline{u})$. Assume that $\overline{u}$ is the origin of a local coordinate system on S^n chosen so that, at $\overline{u}$, the corresponding local frame of tangent vectors to S^n is orthonormal. Then, at $\overline{u}$, the covariant derivatives coincide with the usual derivatives. At $\overline{u}$, we have

$$\varphi_i = 2p\nabla'_i p \eta + p^2 \eta' v_i = 0, \quad \varphi_{ii} (\equiv \nabla'_{ii} \varphi) \leqslant 0, \quad i = 1, 2, \ldots, n, \tag{4.7}$$

where $\eta' \equiv \dfrac{d\eta}{dv}$. The first condition in (4.7) implies

$$p \nabla'_i p = v^s (\nabla'_{si} v + v e_{si}) = v^s W_{si} = -\frac{p^2}{2} \frac{\eta'}{\eta} v_i. \tag{4.8}$$

From (4.6) if follows that, at $\overline{u}$, we have

$$a_i^j = \frac{\sqrt{v^2+K}}{p}\left(e^{ik}W_{kj} + \frac{\eta'}{2\eta}v^i v_j\right). \tag{4.9}$$

The second condition in (4.7), together with (4.8), gives

$$2(\nabla_i' v^s W_{si} + v^s \nabla_i' W_{si}) + \left[\frac{\eta''}{\eta} - 2\left(\frac{\eta'}{\eta}\right)^2\right] p^2 v_i^2 + p^2 \frac{\eta'}{\eta}\nabla_{ii}' v \leqslant 0, \tag{4.10}$$

where $\eta'' = \dfrac{d^2\eta}{dv^2}$.

By the Ricci identity, we have $\nabla_i' W_{si} = \nabla_s' W_{ii}$ on S^n. By this fact, from (4.10) we get

$$2(\nabla_i' v^s W_{si} + v^s \nabla_s' W_{ii}) + \left[\frac{\eta''}{\eta} - 2\left(\frac{\eta'}{\eta}\right)^2\right] p^2 v_i^2 + p^2 \frac{\eta'}{\eta}\nabla_{ii}' v \leqslant 0. \tag{4.11}$$

Differentiating covariantly Eq. (2.6) on S^n, we find

$$F_i^j \nabla_s' a_j^i = \overline{\psi}_s + \overline{\psi}_v v_s, \tag{4.12}$$

where the subscript v at $\overline{\psi}_v$ denotes differentiation with respect to v. Then, we multiply (4.12) by v^s and sum over s. We divide the calculation into several steps.

Using (4.6) and (4.8), at $\overline{u}$, we find

$$v^s \nabla_s' \frac{\sqrt{v^2+K}}{p^3} = \frac{\sqrt{v^2+K}}{p^3}\left[\frac{v}{v^2+K} + \frac{3}{2}\frac{\eta'}{\eta}\right]|\nabla' v|^2$$

and

$$v^s F_i^j \nabla_s' \frac{\sqrt{v^2+K}}{p^3} P^{ik} W_{kj} = \left[\frac{v}{v^2+K} + \frac{3}{2}\frac{\eta'}{\eta}\right] m\overline{\psi}|\nabla' v|^2. \tag{4.13}$$

Using (4.8) again, we get

$$\begin{aligned} v^s \nabla_s' P^{ik} W_{kj} &= 2pp_s v^s e^{ik} W_{kj} - v^s \nabla_s' v^i v^k W_{kj} - v^i v^s \nabla_s' v^k W_{kj} \\ &= -\frac{p^3}{\sqrt{v^2+K}}\frac{\eta'}{\eta}|\nabla' v|^2 a_j^i + \frac{p^2}{2}\left(\frac{\eta'}{\eta}\right)^2 |\nabla' v|^2 v^i v_j - p^2\frac{\eta'}{\eta}\left(\frac{p^2}{2}\frac{\eta'}{\eta} + v\right) v^i v_j. \end{aligned}$$

Since $p^2 = v^2 + |\nabla' v|^2 + K$, we find

$$\begin{aligned} v^s \frac{\sqrt{v^2+K}}{p^3} F_i^j \nabla_s' P^{ik} W_{kj} &= -\frac{\eta'}{\eta}|\nabla' v|^2 m\overline{\psi} \\ &\quad - \frac{\sqrt{v^2+K}}{p}\left[\frac{v^2+K}{2}\left(\frac{\eta'}{\eta}\right)^2 + \frac{\eta'}{\eta}v\right]|Dv|^2, \end{aligned} \tag{4.14}$$

where $|Dv|^2 \equiv F_i^j v^i v_j$.

Multiplying (4.12) by v^s, summing over s, and taking into account (4.13) and (4.14), we get

$$\frac{\sqrt{v^2+K}}{p^3}v^s F^j_i P^{ik}\nabla'_s W_{kj} = \overline{\psi}_s v^s + \left[\overline{\psi}_v - \frac{mv\overline{\psi}}{v^2+K}\right]|\nabla' v|^2$$

$$-\frac{\eta'}{2\eta}m\overline{\psi}|\nabla' v|^2 + \frac{\sqrt{v^2+K}}{p}\left[\frac{v^2+K}{2}\left(\frac{\eta'}{\eta}\right)^2 + \frac{\eta'}{\eta}v\right]|Dv|^2. \quad (4.15)$$

We transform (4.15) as follows. Assuming that $|\nabla' v| \neq 0$ (otherwise the required estimate is obvious), we can rotate the local frame and choose u^1 in such a way that $\nabla' v = v^1\partial_1$ and $v^1 > 0$. At $\overline{u}$, we have $e_{ij} = \delta_{ij}$. From (4.8) it follows that $\nabla'_{1i}v = v_{1i} = W_{1i} = 0$, $i > 1$, at $\overline{u}$. By rotating the frame $\partial_2, \dots, \partial_n$ at $\overline{u}$, we can reduce the matrix (v_{ij}) to the diagonal form at $\overline{u}$. By (4.9), we have

$$a^1_1 = \frac{\sqrt{v^2+K}}{p}\left(v_{11} + v\delta_{11} + \frac{\eta'}{2\eta}v_1^2\right),$$

$$a^i_i = \frac{\sqrt{v^2+K}}{p}(v_{ii} + v\delta_{ii}), \quad i > 1,$$

$$a^i_j = 0, \quad i \neq j.$$

Consequently, the matrix (F^j_i) is diagonal at $\overline{u}$.

The matrix (P^{ik}) is also diagonal at $\overline{u}$, where $P^{11} = p^2 - (v_1)^2 = v^2 + K$ and $P^{ii} = p^2$ for $i > 1$. Thus, (4.15) takes the form

$$\frac{\sqrt{v^2+K}}{p^3}v_1 F^i_j P^{ji}\nabla'_1 W_{ii} = \overline{\psi}_1 v_1 + \left[\overline{\psi}_v - \frac{mv\overline{\psi}}{v^2+K}\right]v_1^2$$

$$-\frac{\eta'}{2\eta}m\overline{\psi}v_1^2 + \frac{\sqrt{v^2+K}}{p}\left[\frac{v^2+K}{2}\left(\frac{\eta'}{\eta}\right)^2 + \frac{\eta'}{\eta}v\right]F^1_1 v_1^2. \quad (4.16)$$

By the choice of the coordinates, the inequality (4.11) takes the form

$$2\Big(\sum_s v_{is}W_{si} + v_1\nabla'_1 W_{ii}\Big) + \left[\frac{\eta''}{\eta} - 2\left(\frac{\eta'}{\eta}\right)^2\right]p^2 v_i^2 + p^2\frac{\eta'}{\eta}v_{ii} \leqslant 0. \quad (4.17)$$

We have $F^i_i > 0$ by the first inequality in (2.7) and $P^{ii} > 0$ by the C^0-bounds (cf. the beginning of this section). Multiplying the inequalities in (4.17) by $F^i_j P^{ji}$ for each $i = 1, 2, \dots, n$ and summing over i, we find

$$\begin{aligned}
&v_1 F^i_j P^{ji} \nabla'_1 W_{ii} \\
&\leqslant -F^i_j P^{ji} \sum_s v_{is} W_{si} - \left[\frac{\eta''}{2\eta} - \left(\frac{\eta'}{\eta}\right)^2\right] p^2 F^1_1 P^{11} v_1^2 - \frac{p^2}{2}\frac{\eta'}{\eta} F^i_j P^{ji} v_{ii} \\
&= -F^i_j P^{ji} W^2_{ii} + \frac{vp^3}{\sqrt{v^2+K}} F^i_j a^j_i - \left[\frac{\eta''}{2\eta} - \left(\frac{\eta'}{\eta}\right)^2\right] p^2 F^1_1 P^{11} v_1^2 \\
&\quad - \frac{p^5}{2}\frac{\eta'}{\eta}\frac{F^i_j a^j_i}{\sqrt{v^2+K}} + \frac{p^2 v}{2}\frac{\eta'}{\eta}\sum_i F^i_j P^{ji} \\
&\leqslant \frac{vm\overline{\psi}p^3}{\sqrt{v^2+K}} - \frac{p^5}{2}\frac{\eta'}{\eta}\frac{m\overline{\psi}}{\sqrt{v^2+K}} - \left[\frac{\eta''}{2\eta} - \left(\frac{\eta'}{\eta}\right)^2\right] p^2 F^1_1 P^{11} v_1^2 + \frac{p^2 v}{2}\frac{\eta'}{\eta}\sum_i F^i_j P^{ji}.
\end{aligned} \tag{4.18}$$

Combining this inequality with (4.16), we get

$$\begin{aligned}
&vm\overline{\psi} - \frac{p^2}{2}\frac{\eta'}{\eta} m\overline{\psi} - \frac{\sqrt{v^2+K}}{p}\left[\frac{\eta''}{2\eta} - \left(\frac{\eta'}{\eta}\right)^2\right] F^1_1 P^{11} v_1^2 + \frac{\sqrt{v^2+K}}{p}\frac{v}{2}\frac{\eta'}{\eta}\sum_i F^i_j P^{ji} \\
&\geqslant \overline{\psi}_1 v_1 + \left[\overline{\psi}_v - \frac{mv\overline{\psi}}{v^2+K}\right] v_1^2 - \frac{\eta'}{2\eta} m\overline{\psi} v_1^2 + \frac{\sqrt{v^2+K}}{p}\left[\frac{v^2+K}{2}\left(\frac{\eta'}{\eta}\right)^2 + \frac{\eta'}{\eta} v\right] F^1_1 v_1^2.
\end{aligned} \tag{4.19}$$

It can be shown that each of the conditions (4.2) and (4.3) implies

$$\overline{\psi}_v - \frac{mv\overline{\psi}}{v^2+K} \geqslant 0. \tag{4.20}$$

The inequality (4.20) will be proved at the end of this section.

Using (4.20), we strengthen the inequality (4.19) by deleting the term with $\overline{\psi}_v - \dfrac{mv\overline{\psi}}{v^2+K}$. In addition, we simplify it by using the equality $P^{11} = v^2 + K$ and collecting the terms. Then (4.19) takes the form

$$\overline{\psi}_1 v_1 \leqslant vm\overline{\psi} - \frac{(v^2+K)m\overline{\psi}\eta'}{2\eta} + \frac{pv\sqrt{v^2+K}}{2}\frac{\eta'}{\eta}\sum_i F^i_i + J\sqrt{v^2+K}\,v_1 F^1_1, \tag{4.21}$$

where

$$J \equiv \frac{\eta'}{\eta}\left[\frac{v(v^2+K)}{2pv_1} - \frac{pv}{2v_1} - \frac{v_1 v}{p}\right] - \frac{v_1(v^2+K)}{2p}\left[\frac{\eta''}{\eta} - \left(\frac{\eta'}{\eta}\right)^2\right].$$

We claim that the function η can be chosen so that $J \leqslant 0$. Without loss of generality, we can assume that $|\nabla' v| \geqslant \max_{S^n}\sqrt{v^2+K}$ at $\overline{u}$. Otherwise, the required estimate is trivial (it suffices to take $\eta \equiv 1$). Under this assumption, we have

$$\frac{pv}{2|\nabla' v|} + \frac{|\nabla' v| v}{p} \leqslant \left(1 + \frac{\sqrt{2}}{2}\right) \max_{S^n} v \ (\equiv A),$$
$$\frac{|\nabla' v|(v^2+K)}{2p} \geqslant \frac{\min_{S^n}(v^2+K)}{2\sqrt{2}} \ (\equiv B).$$

By the C^0-estimates, we have $A > 0$ and $B > 0$. Let η be such that

$$\eta(v) = \exp\left\{Q\frac{B}{A}\exp\left\{-\frac{Av}{B}\right\}\right\},$$

where Q is a positive constant to be specified later. At $\overline{u}$, we have

$$\frac{\eta'}{\eta} = -Q\exp\left\{-\frac{Av}{B}\right\}, \quad \frac{\eta''}{\eta} - \left(\frac{\eta'}{\eta}\right)^2 = Q\frac{A}{B}\exp\left\{-\frac{Av}{B}\right\}$$
$$J = Q\exp\left\{-\frac{Av}{B}\right\}\left\{\left[-\frac{v(v^2+K)}{2pv_1} + \frac{pv}{2v_1} + \frac{v_1 v}{p}\right] - \frac{A}{B}\frac{v_1(v^2+K)}{2p}\right\} < 0.$$

Consequently, the last term on the right-hand side of (4.21) can be deleted.

Consider the remaining terms in (4.21). We have

$$\frac{pv\sqrt{v^2+K}}{2}\frac{\eta'}{\eta}\sum_i F_i^i = -Q\exp\left\{-\frac{Av}{B}\right\}\frac{pv\sqrt{v^2+K}}{2}\sum_i F_i^i.$$

Since $\overline{\psi} > 0$, the sum $\sum_i F_i^i$ admits a positive lower bound depending only on $\overline{\psi}$ (cf. [**13**]). Therefore,

$$\exp\left\{-\frac{Av}{B}\right\}\frac{v\sqrt{v^2+K}}{2}\sum_i F_i^i \geqslant c > 0,$$

where c is a constant depending only on $\overline{\psi}$, R_1, R_2, m, and n. Thus, we can choose Q such that

$$Q\exp\left\{-\frac{Av}{B}\right\}\frac{v\sqrt{v^2+K}}{2}\sum_i F_i^i - \max_{\overline{\Omega}}|\operatorname{grad}\overline{\psi}| \geqslant c_1 > 0,$$

where Q depends only on $\overline{\psi}$, R_1, R_2, m, n, and $|\operatorname{grad}\psi|$. Then the inequality (4.21) takes the form

$$c_1 p \leqslant vm\overline{\psi} + Q\frac{(v^2+K)m\overline{\psi}}{2}\exp\left\{-\frac{Av}{B}\right\},$$

which implies a bound on p at $\overline{u}$. Then

$$\max_{S^n}\varphi \leqslant c_2 < \infty,$$

where c_2 depends only on $\overline{\psi}$, R_1, R_2, m, n, and $|\operatorname{grad}\psi|$. This implies the required estimate (4.4).

To complete the proof, it remains to establish (4.20). Consider the case $K = 1$. We transform the condition (4.2) in Theorem 4.1 as follows. Using (2.10) and (2.11), we get

$$\frac{\partial}{\partial \rho}[\psi q^{-m}] = q^{-m}[\psi_\rho + mq^{-1}(q^2 + K)\psi] \leqslant 0, \tag{4.22}$$

where $\psi_\rho = \dfrac{\partial \psi}{\partial \rho}$. Using the relation $v = q(\rho)$, (2.10), and (2.11), we find

$$\frac{\partial \rho}{\partial v} = -f = -\frac{1}{q^2 + K}, \quad \psi_v = -\frac{\psi_\rho}{v^2 + K}.$$

From (4.22) it follows that

$$-\psi_v + mv^{-1}\psi \leqslant 0. \tag{4.23}$$

Since $K > 0$, (4.23) implies that

$$\overline{\psi}_v - \frac{mv\overline{\psi}}{v^2 + K} > \overline{\psi}_v - \frac{m\overline{\psi}}{v} \geqslant 0, \tag{4.24}$$

and (4.20) is proved.

Let $K = 0$. Arguing as in the case $K = 1$, we conclude that the left-hand side of (4.2) takes the form $\psi_v - mv^{-1}\psi$, which, together with (4.2), implies (4.20).

Finally, consider the case $K = -1$. By (4.3) and the definition of q, we have $-\psi_v(v^2 + K) + mv\psi \leqslant 0$. Theorem 4.1 is proved. □

REMARK 1. In the case $K > 0$, the proof of the estimate for gradient can be completed by setting $\eta \equiv 1$ in (4.19). Therefore, from (4.19) it follows that

$$vm\overline{\psi} \geqslant \overline{\psi}_1 v_1 + \left[\overline{\psi}_v - \frac{v}{v^2 + K} m\overline{\psi}\right] v_1^2. \tag{4.25}$$

Together with (4.24), this estimate leads to the required estimate (4.4).

2. If $m = n$ in Theorem 4.1, then the estimate (4.4) holds without the conditions (4.2) and (4.3) (cf. [**8**]). In this case, it can be shown that $\nabla' v = 0$ at the point where $\max_{S^n} p^2$ is attained. This obviously implies an estimate for $|\nabla' v|$ by the $\max_{S^n}(v^2 + K)$.

5. C^2-Estimate

Let $z \in C^4(S^n)$ be an admissible solution of Eq. (2.6), and let M be a hypersurface in $\mathcal{R}^{n+1}(1)$ given as the graph of z over S^n. In this section, we estimate the maximal principal curvature of M. This estimate, together with the a priori C^0- and C^1-estimates in Secs. 3 and 4, implies an a priori estimate for the C^2-norm of solutions to Eq. (2.6).

Many our arguments remain valid in the case $\mathcal{R}^{n+1}(K)$ for $K = -1, 0, 1$. Therefore, we state and prove some preliminary results for an arbitrary space form. Unfortunately, the proof of Theorem 5.2 is valid only if the sectional curvature K is equal to either 1 or 0.

5.1. More local formulas. It is convenient to use a common framework to model $\mathcal{R}^{n+1}(K)$ in which the hyperbolic space $\mathcal{R}^{n+1}(-1)$ is regarded as the upper sheet of the two-sheeted hyperboloid in the $(n+2)$-dimensional Minkowski space with Lorentz metric and the elliptic space $\mathcal{R}^{n+1}(1)$ is considered as the upper half sphere of S^{n+1} in the Euclidean space R^{n+2}. We combine all three cases (including the Euclidean space) by introducing the space

$$L^{n+2} = \{p = (p_0, p_1, \ldots, p_{n+1}) | \; p_0, p_1, \ldots, p_{n+1} \in R\}$$

with the metric

$$\langle \cdot, \cdot \rangle = K dp_0^2 + dp_1^2 + \ldots dp_{n+1}^2.$$

Then $\mathcal{R}^{n+1}(K)$ is identified with the appropriate hypersurface $\{p \in L^{n+2};$ $\langle p, p \rangle = K, \}$, where $p_0 \geqslant 1$ for $K = -1$ and $p_0 > 0$ for $K = 1$. If $K = 0$, then $R^{(n+1)}(0)$ is the hyperplane $p_0 = 0$.

Let S^n be a unit sphere centered at the origin and lying in the hyperplane $p_0 = 0$ in L^{n+2}. Let $e_0 = (1, 0, \ldots, 0)$. For $K = \pm 1$ we represent the hypersurface M in $\mathcal{R}^{n+1}(K)$ defined by function $z(u)$, $u \in S^n$, as follows:

$$X(u) = c(z(u))e_0 + s(z(u))u, \tag{5.1}$$

where u is treated as a point on S^n and as a unit vector; moreover, $s(\rho) = \sqrt{f(\rho)}$ and $c = \dfrac{ds}{d\rho}$. For $K = 0$ we set $X(u) = z(u)u$.

As in Sec. 2.1, we introduce the local coordinates $u^1, \ldots, u^n$ on M and the corresponding local frame of tangent vectors $X_i = \partial_i X$, $i = 1, \ldots, n$. The unit inward normal N to M (as a submanifold of L^{n+2}) is given by the formula

$$N = \frac{1}{\sqrt{f(z) + |\nabla' z|^2}} (K f(z) e_0 + \nabla' z - c(z) s(z) u). \tag{5.2}$$

We write fundamental equations for hypersurfaces in a space form:

$$\nabla_i N = -b_{is} g^{sk} X_k, \tag{5.3}$$

$$\nabla_{ij} N = -\sum_{s,k} \nabla_j b_{is} g^{sk} X_k - \sum_{s,k} b_{is} g^{sk} b_{kj} N + K b_{ij} X, \tag{5.4}$$

$$\nabla_i b_{jk} = \nabla_k b_{ji}, \tag{5.5}$$

$$\nabla_{ij} X = b_{ij} N - K g_{ij} X, \tag{5.6}$$

$$R_{ijkl} = b_{ik} b_{jl} - b_{il} b_{jk} + K(g_{ik} g_{jl} - g_{il} g_{jk}), \tag{5.7}$$

$$\nabla_l \nabla_k b_{ij} - \nabla_k \nabla_l b_{ij} = \sum_l b_{il} R_{ljkl} + \sum_l b_{jl} R_{likl}, \tag{5.8}$$

where ∇_i and ∇_{ij} denote the covariant differentiation in the metric g on M with respect to some local coordinates on M.

5.2. Estimate for the maximal normal curvature of M. Let $k_1 \geqslant \cdots \geqslant k_n$ be the principal curvatures of M. Since the function $\overline{\psi}$ in (2.6) is positive, we have $\sum_i k_i > 0$ on M and, consequently, $k_1 > 0$.

Lemma 5.1. *Let $1 \leqslant m \leqslant n$, and let $\psi(X)$ be a positive C^2-function in the annulus $\overline{\Omega}$: $u \in S^n$, $\rho \in [R_1, R_2]$, $0 < R_1 < R_2 < a$. Let $z \in C^4(S^n)$ be an admissible solution of Eq. (2.6) in $\mathcal{R}^{n+1}(1)$. Let $\overline{u}$ be a point on M, and let the coordinates $u^1, \dots, u^n$ with origin at $\overline{u}$ in some neighborhood U of $\overline{u}$ be such that the frame $X_1, \dots, X_n$ is orthonormal at $\overline{u}$ in the metric g and the second fundamental form b_{ij} is diagonal at $\overline{u}$. Then, at $\overline{u}$, we have*

$$\overline{\psi}_{II} - \left(1 - \frac{1}{m}\right)\frac{\overline{\psi}_I^2}{\overline{\psi}} \leqslant \sum_i F_i^i \nabla_{ii} b_{11} + b_{11} \sum_i F_i^i b_{ii}^2 - b_{11}^2 m \overline{\psi}$$
$$+ K\left(m\overline{\psi} - b_{11} \sum_i F_i^i\right), \tag{5.9}$$

where

$$\overline{\psi}_I \equiv \nabla_1 \overline{\psi} + \overline{\psi}_z \nabla_1 z, \quad \overline{\psi}_z = \frac{\partial \overline{\psi}}{\partial z},$$

$$\overline{\psi}_{II} \equiv \nabla_{11} \overline{\psi} + 2\nabla_1 \overline{\psi}_z \nabla_1 z + \overline{\psi}_{zz} (\nabla_1 z)^2 + \overline{\psi}_z \nabla_{11} z, \quad \overline{\psi}_{zz} = \frac{\partial^2 \overline{\psi}}{\partial z^2}.$$

PROOF. We begin by computing the first covariant derivative (in the metric g) of Eq. (2.6) with respect to u_1:

$$\sum_{i,j} F_i^j \nabla_1 a_i^j = \overline{\psi}_I. \tag{5.10}$$

The second covariant derivative is given by the formula

$$\sum_{\substack{i,j,k,s \\ i \neq s, j \neq k}} \frac{\partial F_i^j}{\partial a_s^k} \nabla_1 a_s^k \nabla_1 a_j^i + \sum_{i,j} F_i^j \nabla_{11} a_j^i = \overline{\psi}_{II}. \tag{5.11}$$

Note that the metric g is constant with respect to the operator ∇. Therefore, $\nabla_1 a_i^j = \nabla_1 b_{ij}$ at $\overline{u}$. Taking into account that both F_j^i and b_{ij} are diagonal at

$\overline{u}$, from (5.10) we find

$$\sum_i F_i^i \nabla_1 b_{ii} = \overline{\psi}_I \quad \text{at } \overline{u}. \tag{5.12}$$

Similarly, at $\overline{u}$, we have

$$\sum_{\substack{i,j,k,s \\ i\neq s,\ j\neq k}} \frac{\partial F_i^j}{\partial a_s^k} \nabla_1 a_s^k \nabla_1 a_i^j = \sum_{i\neq j} \frac{\partial^2 F}{\partial b_{jj} \partial b_{ii}} [\nabla_1 b_{jj} \nabla_1 b_{ii} - (\nabla_1 b_{ij})^2].$$

Since $F^{1/m}(a_i^j) = S_m^{1/m}(k_1, \ldots, k_n)$, we can use the second inequality in (2.7) to delete the term with the negative sign on the right-hand side. Then

$$\sum_{\substack{i,j,k,s \\ i\neq s,\ j\neq k}} \frac{\partial F_i^j}{\partial a_s^k} \nabla_1 a_s^k \nabla_1 a_j^i \leqslant \sum_{i\neq j} \frac{\partial^2 F}{\partial b_{jj} \partial b_{ii}} \nabla_1 b_{jj} \nabla_1 b_{ii}. \tag{5.13}$$

By the concavity of $F^{1/m}(a_i^j)$, we get

$$\sum_{i\neq j} \frac{\partial^2 F}{\partial b_{jj} \partial b_{ii}} \nabla_1 b_{jj} \nabla_1 b_{ii} \leqslant \left(1 - \frac{1}{m}\right) \frac{1}{F} \left(\sum_i F_i^i \nabla_1 b_{ii}\right)^2 = \left(1 - \frac{1}{m}\right) \frac{\overline{\psi}_I^2}{\overline{\psi}}.$$

The equality on the right-hand side follows from (5.12). This inequality, (5.13), and (5.11) yield

$$\sum_i F_i^i \nabla_{11} b_{ii} \geqslant \overline{\psi}_{II} - \big(1 - \frac{1}{m}\big) \frac{\overline{\psi}_I^2}{\overline{\psi}}. \tag{5.14}$$

Using (5.5) and (5.8), we transform the left-hand side of this inequality as follows:

$$\nabla_{11} b_{ii} = \nabla_{1i} b_{1i} = \nabla_{i1} b_{1i} + \sum_k b_{1k} R_{kii1} + \sum_k b_{ik} R_{k1i1}.$$

By (5.5), we have $\nabla_{i1} b_{1i} = \nabla_{ii} b_{11}$. Using Eq. (5.7), we have at $\overline{u}$

$$\begin{aligned}
\nabla_{11} b_{ii} &= \nabla_{ii} b_{11} + \sum_k b_{1k}(b_{ki} b_{i1} - b_{k1} b_{ii} + K(\delta_{ki}\delta_{i1} - \delta_{k1}\delta_{ii})) \\
&\quad + \sum_k b_{ik}(b_{ki} b_{11} - b_{k1} b_{1i} + K(\delta_{ki}\delta_{11} - \delta_{k1}\delta_{1i})) \\
&= \nabla_{ii} b_{11} + b_{11} b_{ii}^2 - b_{11}^2 b_{ii} + K(b_{11}\delta_{1i}\delta_{1i} - b_{11}\delta_{ii} + b_{ii} - b_{i1}\delta_{1i}).
\end{aligned}$$

Since $\sum_i F_i^i b_{ii} = m\overline{\psi}$ at $\overline{u}$, we get

$$\sum_i F_i^i \nabla_{11} b_{ii} = \sum_i F_i^i \nabla_{ii} b_{11} + b_{11} \sum_i F_i^i b_{ii}^2 - b_{11}^2 m\overline{\psi} + K\Big(m\overline{\psi} - b_{11} \sum_i F_i^i\Big). \tag{5.15}$$

This expression and (5.14) lead to (5.9). □

Theorem 5.2. *Let $1 \leqslant m \leqslant n$, and let $\psi(X)$ be a positive C^2-function in the annulus $\overline{\Omega}$: $u \in S^n$, $\rho \in [R_1, R_2]$, $0 < R_1 < R_2 < a$. Let $z \in C^4(S^n)$ be an admissible solution of Eq.* (2.6) *in $\mathcal{R}^{n+1}(1)$ satisfying the inequalities*

$$R_1 \leqslant z(u) \leqslant R_2, \quad u \in S^n, \tag{5.16}$$

$$|\nabla' z| \leqslant C = \text{const} \quad \text{on } S^n. \tag{5.17}$$

Then

$$\|z\|_{C^2(S^n)} \leqslant C_1, \tag{5.18}$$

where the constant C_1 depends only on m, n, R_1, R_2, C, and $\|\psi\|_{C^2(\overline{\Omega})}$.

PROOF. We estimate the maximal principal curvature of M. This estimate, together with the C^0- and C^1-estimates, implies an estimate for $\|z\|_{C^2(S^n)}$. We use the notation introduced in the proof of Lemma 5.1. We set $\tau(u) = \langle N(u), e_0\rangle$ and $\eta(u) = \langle X(u), e_0\rangle$. By (5.16) and (5.17), the function τ on M is uniformly bounded away from 0 and ∞. Let

$$\omega(u) = \log \frac{b_{11}(u)}{\tau(u)}. \tag{5.19}$$

As above, because of the estimates (5.16) and (5.17), in order to estimate the maximal curvature k_1 on M, it suffices to estimate $\max_M \omega$.

The function ω is similar to the function g in [**5**, Sec. 4], but here we work in elliptic space. In addition, we do not use the special local coordinates, as in [**5**], and this simplifies the computations.

Let $\overline{u} \in M$ be a point where the function ω attains the maximum, and let the coordinates $u^1, \dots, u^n$ be the same as in Lemma 5.1. Then $F_i^j = \dfrac{\partial F}{\partial b_{ii}}$ for $i = j$; otherwise, $F_i^j = 0$. We note that, at $\overline{u}$, the covariant derivatives coincide with the usual derivatives.

At $\overline{u}$, we have $\nabla_i \omega = 0$, which implies

$$\frac{\nabla_i b_{11}}{b_{11}} = \frac{\nabla_i \tau}{\tau}, \quad i = 1, 2, \dots, n, \tag{5.20}$$

$$\nabla_{ii}\omega = \frac{\nabla_{ii} b_{11}}{b_{11}} - \left(\frac{\nabla_i b_{11}}{b_{11}}\right)^2 - \frac{\nabla_{ii}\tau}{\tau} + \left(\frac{\nabla_i \tau}{\tau}\right)^2 \leqslant 0, \quad i = 1, 2, \dots, n. \tag{5.21}$$

Squaring (5.20) and substituting in (5.21), we get

$$\frac{\nabla_{ii} b_{11}}{b_{11}} \leqslant \frac{\nabla_{ii}\tau}{\tau}. \tag{5.22}$$

Using the definitions of τ and η and Eqs. (5.3)–(5.6) (with $K=1$), we find

$$\nabla_i\tau = -b_{ii}\nabla_i\eta, \quad \nabla_{ii}\tau = -\sum_s \nabla_s\eta\nabla_s b_{ii} - \tau b_{ii}^2 + \eta b_{ii}, \quad \nabla_{ii}\eta = \tau b_{ii} - \eta\delta_{ii}.$$

Substituting these expressions into (5.22), we get

$$\frac{\nabla_{ii}b_{11}}{b_{11}} \leqslant -\frac{1}{\tau}\sum_s \nabla_s\eta\nabla_s b_{ii} - b_{ii}^2 + \frac{\eta b_{ii}}{\tau}.$$

At $\overline{u}$, F_j^i is diagonal and $F_i^i > 0$ by (2.7). Multiplying the last inequality by F_i^i, summing over i, and taking into account that $\sum_i F_i^i b_{ii} = m\overline{\psi}$, we find

$$\frac{1}{b_{11}}\sum_i F_i^i\nabla_{ii}b_{11} \leqslant \frac{1}{\tau}\sum_s \nabla_s\eta\nabla_s\overline{\psi} + \frac{m\overline{\psi}\eta}{\tau} - \sum_i F_i^i b_{ii}^2. \tag{5.23}$$

Consider the inequality (5.9) in Lemma 5.1. We use (5.23) to estimate the first term on the right-hand side of (5.9). Also, we strengthen the inequality (5.9) by deleting the term $-b_{11}\sum_i F_i^i$. Then (5.9) takes the form

$$\overline{\psi}_{II} - \left(1-\frac{1}{m}\right)\frac{\overline{\psi}_I^2}{\overline{\psi}} \leqslant \frac{b_{11}}{\tau}\left(\sum_s \nabla_s\eta\nabla_s\overline{\psi} + m\overline{\psi}\eta\right) + m\overline{\psi} - b_{11}^2 m\overline{\psi}. \tag{5.24}$$

Next, we observe that $\max_M |\overline{\psi}_I|$ is bounded by a constant depending only on $\overline{\psi}$, its first derivatives, and C^1-norm of z. Similarly, $\max_M |\overline{\psi}_{II}|$ is bounded by a constant depending on the same quantities, $\|\overline{\psi}\|_{C^2(\overline{\Omega})}$, and $\max_M |\nabla'_{11}z|$. On the other hand, from (2.4) it follows that $|\nabla'_{11}z| < c_2|b_{11}| + c_3$, where the constants c_2 and c_3 depend only on R_1, R_2, and C in (5.17). By (5.24), b_{11} is bounded by a constant depending only on m, n, R_1, R_2, ψ, $\operatorname{grad}\psi$, and $|\operatorname{grad} z|$. By (5.19), $\max_M b_{11}$ is bounded by a constant depending on the same quantities. Then (2.4) implies the required estimate (5.18). □

REMARK. Essentially, the same arguments lead to the estimate (5.18) in the case $K=0$. This case was studied in [4] for $m=n$ and in [5] for $1 \leqslant m \leqslant n$. The necessary modifications consist in replacing $\tau(u)$ by $\tau(u) = \langle N(u), X(u)\rangle$, setting $\eta(u) \equiv 0$, and using Lemma 5.1 with $K=0$. The calculations using Eqs. (5.4), (5.6), and (5.7), should also be adjusted respectively.

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Elliptic Variational Problems with Nonstandard Growth

Michael Bildhauer and Martin Fuchs

Dedicated to O. A. Ladyzhenskaya on her birthday

For a bounded Lipschitz domain a minimization problem is considered over functions of the Orlicz–Sobolev space generated by an N-function A (with Δ_2-property) that have prescribed trace u_0. Regularity results are established. In the vector case $N > 1$, partial $C^{1,\alpha}$-regularity is proved without any additional structural conditions. The results are easily extended to the case of locally minimizing mappings. In the scalar case, the results obtained cover the case of (double) obstacles. Under an additional assumption, the regularity results can be improved (cf. Theorem 3 below which admits the anisotropic two-dimensional vector case).

1. Introduction

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, and a variational integrand $f : \mathbb{R}^{nN} \to \mathbb{R}$ of class $C^2(\mathbb{R}^{nN})$ we consider the autonomous minimization problem

$$J[w] := \int_\Omega f(\nabla w)\, dx \to \min \qquad (\mathcal{P})$$

over mappings $w : \Omega \to \mathbb{R}^N$, $N \geqslant 1$, with given Dirichlet boundary data u_0. Test functions are assumed to belong to a suitable energy class depending on f. In this paper, we always assume that the variational integrand is strictly convex. Thus, we do not touch the quasiconvex case (cf., for example, **[1]**–**[6]**).

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The purpose of this paper is to establish the regularity of minimizers of problem $(\mathcal{P})$ under quite general growth and structural conditions on f. We begin with a brief historical overview.

A.1. Power growth. Having the standard example $f_p(Z) = (1+|Z|^2)^{p/2}$, $1 < p$, in mind, we assume that the upper growth order coincides with the lower growth order, i.e., for some number $p > 1$ and constants c_1, c_2, C, λ, $\Lambda > 0$ and for all $Z, Y \in \mathbb{R}^{nN}$ the integrand f satisfies the following condition (note that the second line of (1) implies the first one):

$$\begin{aligned} &c_1|Z|^p - c_2 \leqslant f(Z) \leqslant C(1+|Z|^p),\\ &\lambda(1+|Z|^2)^{(p-2)/2}|Y|^2 \leqslant D^2f(Z)(Y,Y) \leqslant \Lambda(1+|Z|^2)^{(p-2)/2}|Y|^2. \end{aligned} \tag{1}$$

Owing to the pioneering work of De Giorgi, Moser, and Nash and the results of Ladyzhenskaya and Ural′tseva, the local $C^{1,\alpha}$-regularity of minimizers of problem $(\mathcal{P})$ is well known in the scalar case, and many other authors could be mentioned (cf. [**7**]–[**9**] and [**10**] for a complete overview and a detailed list of references).

For $N > 1$ the two-dimensional case $n = 2$ substantially differs from the case of higher dimension. By a classical result of Morrey, full regularity holds if $n = 2$ (we refer the reader to the first monograph [**11**] on multiple integrals in the calculus of variations, where a detailed list of references can be found).

However, according to an example of De Giorgi [**12**] (cf. also [**13, 14**] and a recent example in [**15**]), we cannot hope to establish the full result if $n \geqslant 3$ and $N > 1$. In this case, we can expect only partial regularity (i.e., a solution belongs to the class $C^{1,\alpha}$ on some open set $\Omega_0 \subset \Omega$ of full Lebesgue measure). A theorem of this type was proved for any dimension and in a quite general statement by Anzellotti and Giaquinta in [**16**], where the whole scale of integrands up to the limiting linear growth was covered (with some suitable notion of relaxation). In addition, the assumptions on the second-order derivatives (namely, $D^2f(Z) > 0$ for any matrix Z) in [**16**] were much weaker than those stated above.

To keep the historical line, we mention the earlier contributions to the partial regularity theory [**17**]–[**19**] (cf. also [**20, 21**], and [**22**] for a detailed overview).

Under some additional structural conditions (for example, $f(Z) = g(|Z|^2)$) in the vector case, it is possible to improve the partial regularity result and obtain the full $C^{1,\alpha}_{loc}$-regularity of solutions. Such results are mainly connected with the name of Uhlenbeck (cf. [**23**], where the full strength of (1) is not needed, which means that the case of degenerate ellipticity can be treated).

A.2. Anisotropic power growth. Anisotropic variational problems were extensively studied by Marcellini [**24**]–[**29**]. Such problems appear as a result of a natural generalization of (1). To give a typical anysotropic example, we set

$n = 2$, $2 \leqslant p \leqslant q$ and replace f_p with the function

$$f_{p,q}(Z) = (1 + |Z|^2)^{p/2} + (1 + |Z_2|^2)^{q/2}, \quad Z = (Z_1, Z_2) \in \mathbb{R}^{2N}.$$

Then the upper growth order differs from the lower growth order for f. The structural condition (1) is naturally generalized as follows (as above, the growth condition on the second-order derivatives imply the corresponding growth order of f):

$$\begin{aligned} &c_1|Z|^p - c_2 \leqslant f(Z) \leqslant C(1 + |Z|^q), \\ &\lambda(1 + |Z|^2)^{(p-2)/2}|Y|^2 \leqslant D^2 f(Z)(Y, Y) \leqslant \Lambda(1 + |Z|^2)^{(q-2)/2}|Y|^2 \end{aligned} \tag{2}$$

for all Z, $Y \in \mathbb{R}^{nN}$, where c_1, c_2, C, λ, and Λ are positive constants and $1 < p \leqslant q$.

If the difference between p and q is too large, then singularities can occur even in the scalar case (we recall only one well-known example in [**30**]). However, as was shown by Marcellini, under suitable assumptions on p and q, the solution is regular. We note that [**27**] covers the case $N > 1$ under some additional structural condition.

In the general vector case, only a few results are known (we refer to Acerbi and Fusco [**31**] and Passarelli Di Napoli and Siepe [**32**], where partial regularity theorems are obtained under quite restrictive assumptions on p and q excluding any subquadratic growth.)

Under an additional boundedness condition, the above results were improved by Esposito, Leonetti, and Mingione [**33**] and Choe [**34**]. In [**33**], higher integrability (up to a certain limit exponent) was established ($N \geqslant 1$, $2 \leqslant p$) under a quite weak relation between p and q. A result for energy densities $f(Z) = g(|Z|^2)$ can be found in [**34**].

B.1. Growth conditions involving N-functions. As it can be seen from the monograph of Seregin and the second author [**35**], many problems in mathematical physics are not within the framework of power growth models. The theory of Prandtl–Eyring fluids and the theory of plastic materials with logarithmic hardening serve as typical examples. The variational integrands under consideration are of nearly linear growth. For example, we study the logarithmic integrand $f(Z) = |Z| \ln(1 + |Z|)$ which satisfies neither (1) nor (2).

The main results for integrands of logarithmic structure were obtained by Frehse and Seregin [**36**] (full regularity if $n = 2$), Seregin and the second author [**37**] (partial regularity if $n \leqslant 4$), Esposito and Mingione [**38**] (partial regularity for any dimension), and Mingione and Siepe [**39**] (full regularity for any dimension).

B.2. Generalization of the logarithmic integrand. It is natural to generalize the problem assuming that integrands are bounded from above and from below by the same quantity $A(|Z|)$, where $A : [0, \infty) \to [0, \infty)$ is an arbitrary

N-function possessing Δ_2-property (cf. [**40**] for precise definitions). Although this assumption does not imply natural bounds (in terms of A) for the second-order derivatives, it is reasonable, taking into account (1) and (2), to consider the following model. For a given N-function A as above and positive constants c, C, λ, and Λ we assume that f satisfies the condition

$$\begin{aligned} &cA(|Z|) \leqslant f(Z) \leqslant CA(|Z|), \\ &\lambda(1+|Z|^2)^{-\mu/2}|Y|^2 \leqslant D^2 f(Z)(Y,Y) \leqslant \Lambda(1+|Z|^2)^{(q-2)/2}|Y|^2 \end{aligned} \tag{3}$$

for all Z, $Y \in \mathbb{R}^{nN}$ and some real numbers $1 \leqslant \mu$, $1 < q$, chosen in accordance with the logarithmic integrand satisfying (3) with $\mu = 1$ and $q = 1 + \varepsilon$ for any $\varepsilon > 0$. We note that the analogy between this condition and the conditions (1) and (2) is formal. By the requirement $\mu \geqslant 1$, the μ-ellipticity condition (i.e., the first inequality in the second line of (3)) does not provide us with any information about the lower growth order for f in terms of a power function with exponent $p > 1$.

Variational problems with the structural condition (3) and additional balance conditions were first investigated by Osmolovskii and the second author (cf. [**41**], where partial regularity in the vector case was established for $\mu < 4/n$).

If $N = 1$ or if $N > 1$ and $f(Z) = g(|Z|^2)$, full regularity was established by Mingione and the second author [**42**] for $\mu < 1 + 2/n$.

2. Notation and Statements of Results

In this section, we give precise formulations. For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, we consider the minimization problem

$$J[w] = \int_\Omega f(\nabla w)\, dx \to \min \quad u_0 + \overset{\circ}{W}{}^1_A(\Omega;\mathbb{R}^N), \tag{$\mathcal{P}$}$$

where $\overset{\circ}{W}{}_A^1(\Omega;\mathbb{R}^N)$ is the subclass of the Orlicz–Sobolev space $W^1_A(\Omega;\mathbb{R}^N)$ generated by an N-function A (with Δ_2-property) consisting of all mappings $\Omega \to \mathbb{R}^N$ with zero trace and u_0 is a given function of class $W^1_A(\Omega;\mathbb{R}^N)$ with finite energy, i.e., $J[u_0] < \infty$. The energy density f is a function of class $C^2(\mathbb{R}^{nN})$ and satisfies the inequalities

$$c_1 A(|Z|) - c_2 \leqslant f(Z), \tag{4}$$

$$\lambda(1+|Z|^2)^{-\mu/2}|Y|^2 \leqslant D^2 f(Z)(Y,Y) \leqslant \Lambda(1+|Z|^2)^{(q-2)/2}|Y|^2 \tag{5}$$

for all Y, $Z \in \mathbb{R}^{nN}$, where c_1, c_2, λ, and Λ are positive constants and $\mu \in \mathbb{R}$, $q > 1$ are fixed real numbers. We choose $s \geqslant 1$ such that

$$A(t) \geqslant \text{const}\, t^s \quad \text{for all } t \gg 1. \tag{6}$$

We give some examples to show how to choose the parameters μ, s, q.

(i) $f(\nabla u) = |\nabla u| \ln(1 + |\nabla u|)$: $\mu = 1$, $s = 1$, $q = 1 + \varepsilon$.

(ii) $f(\nabla u) = (1 + |\nabla u|^2)^{p/2} + (1 + |\partial_1 u|^2)^{t/2}$ for $1 < p < t$: $\mu = 2 - p$, $s = p$, $q = t$.

(iii) $f(\nabla u) = h(\nabla u) + \Phi_\mu(\nabla u)$, where h is of growth order q and satisfies the inequalities $0 \leqslant D^2 h(Z)(Y, Y) \leqslant \text{const}(1 + |Z|^2)^{(q-2)/2}|Y|^2$,

$$\Phi_\mu(Z) = \int\limits_0^{|Z|} \int\limits_0^{s} (1 + t^2)^{-\mu/2}\, dt\, ds.$$

Since $D^2 h(Z) \geqslant 0$ (i.e., $D^2 h$ may degenerate), the ellipticity estimate (i.e., the left-hand side of (5)) follows from the corresponding inequality for Φ_μ (cf., for example, [**43**]).

Theorem 1 ((s, μ, q)-growth condition). *Let u be the unique solution of problem $(\mathcal{P})$. Suppose that* (4)–(6) *are satisfied and $q < 2 - \mu + \frac{2}{n}s$.*

(i) (scalar case) *If $N = 1$, then $u \in C^{1,\alpha}(\Omega)$ for any $0 < \alpha < 1$.*

(ii) *If $N > 1$ and $f(Z) = g(|Z|^2)$, then assertion* (i) *holds.*

(iii) *In the general vector case $N > 1$, partial $C^{1,\alpha}$-regularity holds.*

Remark 1. (i) The above results can be easily generalized to the case of locally minimizing mappings. Moreover, in the scalar case, these results are also valid in the presence of (double) obstacles.

(ii) The detailed discussion and comparison with known results given in [**43**]–[**46**] show that Theorem 1 provides us with a unified and extended approach to the regularity theory of convex variational problems having nonstandard growth.

We can improve Theorem 1 under the additional condition

$$u_0 \in L^\infty(\Omega; \mathbb{R}^N). \tag{7}$$

Theorem 2. *If u is a solution of problem $(\mathcal{P})$, the conditions* (4), (5), *and* (7) *are satisfied, and $q < 4 - \mu$, then assertions* (i) *and* (ii) *of Theorem* 1 *are valid. If $N > 1$ and $q < 4 - \mu$, then for any $\varkappa \in (q, 4 - \mu)$ and $\Omega' \Subset \Omega$ there is a positive number c such that*

$$\int\limits_{\Omega'} |\nabla u|^\varkappa\, dx \leqslant c < \infty \tag{8}$$

provided that the following structural condition is satisfied:

$$f(Z_1, \ldots, Z_n) = g(|Z_1|, \ldots, |Z_n|), \quad Z \in \mathbb{R}^{nN},$$

g *is increasing in each argument.*

Replacing $q < 4-\mu$ *with* $q < \min\{(2-\mu)n/(n-2), 4-\mu\}$, *for* $n \geqslant 3$ *we obtain assertion* (iii) *of Theorem* 1.

REMARK 2. (i) Again, in the scalar case, we can include obstacles.

(ii) Theorem 2 substantially generalizes the results of [**45**].

(iii) The condition $q < 4-\mu$ agrees with the bound $\mu < 3$ for linear growth problems (cf. [**46**]–[**48**]). The similar constraint $q < 2+p$ was first considered in [**33**], where higher integrability (up to a certain limit exponent) in the anisotropic superquadratic (p,q)-case was established under some additional boundedness condition.

(iv) Let us compare Theorem 1 and Theorem 2 in the anisotropic (p,q)-case, i.e., we suppose that f satisfies (2) with given exponents $1 < p < q < \infty$. Then we have to choose $\mu = 2-p$, and Theorem 1 implies full regularity in the scalar case if

$$q < p(n+2)/n. \tag{a}$$

Thus, the range of admissible exponents becomes smaller if we increase n. This effect can be compensated by assuming (7). Then for $N=1$, according to Theorem 2, full regularity is a consequence of the dimensionless condition

$$q < p+2. \tag{b}$$

We note that for $p \leqslant n$ we have $p\frac{n+2}{n} \leqslant p+2$. Thus, in the case $p \leqslant n$, condition (a) implies condition (b), i.e., Theorem 2 gives better results than Theorem 1. Conversely, for $p > n$ condition (b) implies condition (a). Therefore, Theorem 1 is preferable in this case (despite of the fact that for $p > n$ the condition (7) is always satisfied by Sobolev's embedding theorem).

The same is true in the case $N > 1$ for $f(Z) = g(|Z|^2)$.

In the general vector situation, condition (b) should be replaced with

$$q < \min\{p+2, pn/(n-2)\}. \tag{b'}$$

But the interpretation of the bounds is the same: for small values of p condition (b′) is less restrictive than condition (a) but, in the presence of the weaker bound (b′), we have to require (7) with $f(Z) = g(|Z_1|, \dots, |Z_n|)$ in order to obtain partial regularity.

(v) We do not claim that our bounds imposed on the exponents are the best possible ones. But if we look at μ-elliptic problems with linear growth, where (5) holds with $q=1$ and $\mu > 1$ (cf. [**46, 48**]), then a counterexample to regularity (even in the scalar case) was constructed under the assumption $q = 1 > 4-\mu$, i.e., $\mu > 3$.

EXAMPLE. Consider an example demonstrating the improvements obtained in view of Theorem 2. Let $Z = (Z_1, Z_2) \in \mathbb{R}^{kN} \times \mathbb{R}^{(n-k)N}$, $1 \leqslant k < n$. Assume that exponents $1 < p < \tau < 2$ are fixed and f has the form

$$f(Z) = (1+|Z_1|^2)^{p/2} + (1+|Z_2|^2)^{\tau/2}.$$

In this subquadratic case, it is clear that the estimate

$$\lambda(1+|Z|^2)^{(p-2)/2}|Y|^2 \leqslant D^2 f(Z)(Y,Y) \leqslant \Lambda|Y|^2$$

is the best possible one. Hence no regularity results follow from Theorem 1 if p is close to 1 (even if $(\tau - p)$ becomes very small). Therefore, using the trivial inequality $2 < p+2$ ($q := 2$ and $\mu := 2-p$), Theorem 2 really provides some completely new results.

The following theorem covers the anisotropic vector case in dimension two.

Theorem 3. *Let $n = 2$, and let $1 < s < q < \infty$ be such that* (4)–(6) *hold with $\mu = 2 - s$. If $q < 2s$, then the solution u of problem $(\mathcal{P})$ is smooth on Ω.*

EXAMPLE. In particular, we get regularity in the case

$$f(\nabla u) = |\nabla u|^2 + (1+|\partial_1 u|^2)^{q/2}, \quad q \in (2,4).$$

REMARK 3. The assumption $q < 2s$ of Theorem 3 formally coincides with the "(s,μ,q)-condition" of Theorem 1.

REMARK 4. In Theorems 1–3, we concentrated on the case of integrands of superlinear growth so that the existence of minimizers in appropriate Orlicz–Sobolev spaces is easily established. However, it is possible to consider μ-elliptic integrands of linear growth (cf. [**49**] and [**47**]) or even anisotropic problems of mixed linear/superlinear growth (cf. [**50**]). Of course, one has to take into account suitable generalized minimizers from the space of functions with bounded variation (cf. [**51**] for three formally different approaches leading to the same set of generalized minimizers). A short overview of the regularity results in the case of linear growth can be found in [**48**].

3. Some Remarks on the Proofs of Theorems 1–3

A complete proof of Theorem 1 is contained in [**43, 44**]. The proof of Theorem 3 is based on a lemma due to Frehse and Seregin (cf. [**36**]) and is published in [**52**]. The first assertion of Theorem 2 is contained in [**53**] (cf. also [**46**]).

In this paper, we focus on the general vector case and prove the higher integrabilty assertion (8) of Theorem 2. This is done by refining some ideas of Choe [**34**] combined with an appropriate Caccioppoli-type inequality. We emphasize that we do not require the condition $f(Z) = g(|Z|^2)$, as was done in [**34**], where also the constraint $q < 1+p$ was assumed.

REMARK 5. If (8) is already established, then the partial regularity result immediately follows from ideas expressed in [**44**]. In fact, the blow-up arguments of [**44**] remain valid provided that a Caccioppoli-type inequality and the higher local integrability of the gradient are already verified (we refer to [**45**] for more details).

We like to emphasize again that the condition $q < 4-\mu$ with (7) is sufficient for the local higher integrability of the gradient. To establish partial regularity, we need the additional bound

$$q < (2-\mu)\frac{n}{n-2} \quad \text{if } n \geqslant 3, \tag{$*$}$$

which enters during the blow-up procedure by using an appropriate Caccioppoli-type inequality. Since, the Caccioppoli-type inequality depends on the derivatives of the solution, we cannot expect to improve $(*)$ using (7).

Hereinafter, we assume that the assumptions of Theorem 2 are satisfied. Consider a ball $B_R(x_0) \Subset \Omega$ and an ε-mollification $(u)^\varepsilon$ of u, where $\varepsilon > 0$ is sufficiently small. For any $\delta \in (0,1)$ we set

$$f_\delta(Z) := f(Z) + \delta(1+|Z|^2)^{t/2}, \quad Z \in \mathbb{R}^{nN},$$

with some exponent $t > \max\{2, q\}$, and denote by $v_\varepsilon = v_{\varepsilon,\delta}$ the unique solution of the Dirichlet problem

$$J_\delta[w] := \int\limits_{B_R(x_0)} f_\delta(\nabla w)\, dx \to \min, \quad w \in (u)^\varepsilon_{|B_R(x_0)} + \overset{\circ}{W}{}^1_t(B_R(x_0);\mathbb{R}^N). \quad (\mathcal{P}_\delta)$$

For sufficiently small $\delta = \delta(\varepsilon)$ (cf. [**45**]) the main properties of the regularizing sequence $\{v_\varepsilon\}$ are listed in the following assertion.

Lemma 1. *In the above notation, the following assertions hold.*

(i) $v_\varepsilon \in W^1_{\infty,loc} \cap W^2_{2,loc}(B_R(x_0);\mathbb{R}^N)$.

(ii) $\|v_\varepsilon\|_{W^1_A(B_R(x_0),\mathbb{R}^N)} \leqslant c < \infty$, *where the constant* c *is independent of* ε.

(iii) v_ε *weakly converges to* u *in* $W^1_1(B_R(x_0);\mathbb{R}^N)$ *and a.e. as* $\varepsilon \to 0$.

(iii) $\sup_{B_R(x_0)} |v_\varepsilon| \leqslant \sup_{B_{R+\varepsilon}(x_0)} |u| < \infty$.

(iv) $\delta(\varepsilon) \int\limits_{B_R(x_0)} (1+|\nabla v_\varepsilon|^2)^{t/2}\, dx \to 0$ *as* $\varepsilon \to 0$.

(v) $\int\limits_{B_R(x_0)} f(\nabla v_\varepsilon)\, dx \to \int\limits_{B_R(x_0)} f(\nabla u)\, dx$ *as* $\varepsilon \to 0$.

(vi) $\int\limits_{B_R(x_0)} f_{\delta(\varepsilon)}(\nabla v_\varepsilon)\, dx \to \int\limits_{B_R(x_0)} f(\nabla u)\, dx \quad$ *as* $\varepsilon \to 0$.

Proof. The arguments are standard and can be found, for instance, in [**43**]–[**45**] or [**46**]. We give only two comments. First, since the regularization is done with respect to the exponent $t > \max\{2, q\}$, the discussion of asymptotically regular integrands (cf. [**54**] or a generalization in [**55**, Theorem 5.1]) immediately yields (i). Second, the structural condition $f = g(|Z_1|, \dots |Z_n|)$

with g as above provides the convex hull property (cf. [45] for a detailed proof). Hence (iv) follows from the boundedness of u_0. Note that if (iv) is established, we no longer make use of the L^∞-bounds for u_0 and the structural condition on f. $\square$

By Lemma 1, it is obvious that (8) follows from the following assertion.

Lemma 2. *Under the above assumptions, for any number $\varkappa \in (q, 4-\mu)$ and ball $B_r(x_0)$, $r < R$, there is a constant c depending only on the data, $\sup_{B_R(x_0)} |(u)^\varepsilon|$, r, and $\varkappa$ such that*

$$\int\limits_{B_r(x_0)} |\nabla v_\varepsilon|^\varkappa \, dx \leqslant c < \infty.$$

PROOF. We use the notation $f_\varepsilon = f_{\delta(\varepsilon)}$ for short and adopt the summation convention over the repeated Greek indices $\gamma = 1, \dots, n$ and the repeated Latin indices $i = 1, \dots, N$.

By definition, v_ε is a solution of the Euler equation

$$\int\limits_{B_R(x_0)} \nabla f_\varepsilon(\nabla v_\varepsilon) : \nabla \varphi \, dx = 0 \quad \text{for all } \varphi \in C_0^\infty(B_R(x_0); \mathbb{R}^N).$$

By assertion (i) of Lemma 1, we may differentiate this equation (taking $\partial_\gamma \varphi$ as a test function and integrating by parts). We find

$$\int\limits_{B_R(x_0)} D^2 f_\varepsilon(\nabla v_\varepsilon)(\partial_\gamma \nabla v_\varepsilon, \nabla \varphi) \, dx = 0 \quad \text{for all } \varphi \in C_0^\infty(B_R(x_0); \mathbb{R}^N). \tag{9}$$

For a smooth cut-off function η, the function $\eta^2 \partial_\gamma v_\varepsilon$ is admissible in (9) and we obtain a Caccioppoli-type inequality.

Lemma 3. *There is a real number $c > 0$, independent of ε, such that for any $\eta \in C_0^\infty(B_R(x_0))$, $0 \leqslant \eta \leqslant 1$,*

$$\int\limits_{B_R(x_0)} D^2 f_\varepsilon(\nabla v_\varepsilon)(\partial_\gamma \nabla v_\varepsilon, \partial_\gamma \nabla v_\varepsilon)\eta^2 \, dx \leqslant c \int\limits_{B_R(x_0)} |D^2 f_\varepsilon(\nabla v_\varepsilon)| |\nabla v_\varepsilon|^2 |\nabla \eta|^2 \, dx.$$

Proceeding with the proof of Lemma 2, we fix $\varkappa$ as given there. Then it is possible to define

$$q + \mu - 4 < \alpha := \varkappa + \mu - 4 < 0, \tag{10}$$

where

$$0 < \sigma := 2 + \alpha - \frac{\mu}{2} < 2 + \frac{\alpha - \mu}{2} =: \sigma' \tag{11}$$

since α is negative. Without loss of generality, choosing $\varkappa$ close to $4 - \mu$, we can assume that $|\alpha|$ is sufficiently small so that σ is positive. Alternatively, if

σ is negative, then the second integral on the right-hand side of the inequality (17) below is trivially bounded.

By (11), we can choose a sufficiently large number $k \in \mathbb{N}$ such that $2k\dfrac{\sigma}{\sigma'} < 2k-2$. For a given function $\eta \in C_0^\infty(B_R(x_0))$ such that $0 \leqslant \eta \leqslant 1$, and $\eta \equiv 1$ on $B_r(x_0)$, $|\nabla\eta| \leqslant c/(R-r)$ we introduce the function $\Gamma_\varepsilon = 1 + |\nabla v_\varepsilon|^2$ and recall assertion (i) of Lemma 1. Thus, v_ε is smooth enough to perform the following integration by parts

$$\int\limits_{B_R(x_0)} |\nabla v_\varepsilon|^2 \Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}} \eta^{2k}\,dx = -\int\limits_{B_R(x_0)} v_\varepsilon{}^i \cdot \nabla\Big[\nabla v_\varepsilon{}^i \Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}} \eta^{2k}\Big]\,dx$$

$$\leqslant c \int\limits_{B_R(x_0)} |\nabla^2 v_\varepsilon| \Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}} \eta^{2k}\,dx + c\int\limits_{B_R(x_0)} \Gamma_\varepsilon^{\frac{3+\alpha-\mu}{2}} \eta^{2k-1}|\nabla\eta|\,dx.$$

Here, we used the uniform boundedness of v_ε. If a positive constant M is fixed, then the left-hand side of the last relation is estimated as follows:

$$\int\limits_{B_R(x_0)} |\nabla v_\varepsilon|^2 \Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}} \eta^{2k}\,dx \geqslant c \int\limits_{B_R(x_0)\cap[|\nabla v_\varepsilon|\geqslant M]} \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}} \eta^{2k}\,dx$$

$$\geqslant c \int\limits_{B_R(x_0)} \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}} \eta^{2k}\,dx - c(M).$$

Therefore, the starting inequality takes the form

$$\int\limits_{B_R(x_0)} \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}} \eta^{2k}\,dx \leqslant c\Bigg\{1 + \int\limits_{B_R(x_0)} |\nabla^2 v_\varepsilon| \Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}} \eta^{2k}\,dx$$

$$+ \int\limits_{B_R(x_0)} \Gamma_\varepsilon^{\frac{3+\alpha-\mu}{2}} \eta^{2k-1}|\nabla\eta|\,dx\Bigg\} =: c\{1 + I + II\}. \tag{12}$$

At this point, we emphasize that the choice (10) of α gives

$$2 + (\alpha-\mu)/2 = \varkappa/2 > q/2. \tag{13}$$

For a sufficiently small $\gamma > 0$ Young's inequality yields the following estimate:

$$II \leqslant \gamma \int\limits_{B_R(x_0)} \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}} \eta^{2k}\,dx + \gamma^{-1} \int\limits_{B_R(x_0)} \Gamma_\varepsilon^{-2-\frac{\alpha-\mu}{2}} \Gamma_\varepsilon^{3+\alpha-\mu} \eta^{2k-2} |\nabla\eta|^2\,dx$$

$$\leqslant \gamma \int\limits_{B_R(x_0)} \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}} \eta^{2k}\,dx + \frac{c\gamma^{-1}}{(R-r)^2} \int\limits_{B_R(x_0)} \Gamma_\varepsilon^{1+\frac{\alpha-\mu}{2}} \eta^{2k-2}\,dx. \tag{14}$$

We note that the first integral on the right-hand side of (14) can be absorbed on the left-hand side of (12), whereas the second integral remains uniformly bounded. In fact, if $\mu \geqslant 1$, then $(2-\mu)/2 < 1/2$ and the claim is trivial (recall that α is negative). In the case $\mu < 1$, we can assume that $\nabla f(0) = 0$ (replace f with $\widetilde{f}(Z) = f(Z) - \nabla f(0) : Z$) and $f(0) = 0$. By (5), we have

$$\nabla f(Z) : Z = \int\limits_0^1 D^2 f(Z)(\theta Z)(Z, Z)\, d\theta \geqslant a|Z|^{2-\mu} - b$$

for some real numbers $a > 0$ and b. The equality on the left-hand side, in particular, shows that $\nabla f(Z) : Z \geqslant 0$, and we find

$$f(Z) \geqslant \int\limits_{1/2}^1 \nabla f(\theta Z) : \theta Z\, d\theta \geqslant c_1|Z|^{2-\mu} - c_2 \tag{15}$$

for some other constants $c_1 > 0$ and c_2. Thus, the growth order of f is at least $2-\mu$ if $\mu < 1$, and we obtain the required assertion in view of Lemma 1 (vi).

It remains to obtain an appropriate estimate for I. We note that

$$I \leqslant \gamma \int\limits_{B_R(x_0)} \Gamma_\varepsilon^{-\frac{\mu}{2}} |\nabla^2 v_\varepsilon|^2 \eta^{2k+2}\, dx + \gamma^{-1} \int\limits_{B_R(x_0)} \Gamma_\varepsilon^{2+\alpha-\frac{\mu}{2}} \eta^{2k-2}\, dx =: \gamma I_1 + \gamma^{-1} I_2. \tag{16}$$

Again, we assume that $\gamma > 0$ is sufficiently small and use Young's inequality. By Lemma 3 and assertion (iv) of Lemma 1, we find

$$\begin{aligned}
I_1 &\leqslant \int\limits_{B_R(x_0)} D^2 f_\varepsilon(\nabla v_\varepsilon)(\partial_\gamma \nabla v_\varepsilon, \partial_\gamma \nabla v_\varepsilon)\big(\eta^{k+1}\big)^2\, dx \\
&\leqslant c \int\limits_{B_R(x_0)} \big|D^2 f_\varepsilon(\nabla v_\varepsilon)\big| |\nabla v_\varepsilon|^2 \eta^{2k} |\nabla \eta|^2\, dx \\
&\leqslant \frac{c}{(R-r)^2} \Bigg\{ \int\limits_{B_R(x_0)} \Gamma_\varepsilon^{\frac{q}{2}} \eta^{2k}\, dx + \delta(\varepsilon) \int\limits_{B_R(x_0)} \Gamma_\varepsilon^{\frac{t}{2}} \eta^{2k}\, dx \Bigg\} \\
&\leqslant \frac{c}{(R-r)^2} \Bigg\{ 1 + \int\limits_{B_R(x_0)} \Gamma_\varepsilon^{\frac{q}{2}} \eta^{2k}\, dx \Bigg\}.
\end{aligned}$$

From (16) (recall (13)) it follows that

$$I \leqslant \frac{c\gamma}{(R-r)^2} \Bigg\{ 1 + \int\limits_{B_R(x_0)} \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}} \eta^{2k}\, dx \Bigg\} + \gamma^{-1} \int\limits_{B_R(x_0)} \Gamma_\varepsilon^{2+\alpha-\frac{\mu}{2}} \eta^{2k-2}\, dx. \tag{17}$$

Choosing $\gamma = \widehat{\gamma}(R-r)^2$ with sufficiently small $\widehat{\gamma} > 0$, we find that the first integral on the right-hand side of (17) can be absorbed on the left-hand side of

(12). Therefore, it remains to estimate the second integral. Here, the fact that α is negative and, as a consequence, (11) and the choice of k come into play. For sufficiently small $\widetilde{\gamma} > 0$, using Young's inequality, we finally find

$$\widehat{\gamma}^{-1}(R-r)^{-2} \int_{B_R(x_0)} \Gamma_\varepsilon^{2+\alpha-\frac{\mu}{2}} \eta^{2k-2}\, dx$$
$$\leqslant c\widehat{\gamma}^{-1}(R-r)^{-2}\left\{\widetilde{\gamma} \int_{B_R(x_0)} \Gamma_\varepsilon^{2+\frac{\alpha-\mu}{2}} \eta^{2k}\, dx + \widetilde{\gamma}^{-\frac{\sigma}{\sigma'-\sigma}} |B_R(x_0)|\right\}.$$

We absorb terms by letting $\widetilde{\gamma} = \gamma'\widehat{\gamma}(R-r)^2$, $1 \gg \gamma' > 0$. □

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Existence and Regularity of Solutions of $d\omega = f$ with Dirichlet Boundary Conditions

Bernard Dacorogna

This article is dedicated to Olga A. Ladyzhenskaya for her birthday and in admiration for her mathematical achievements

Given a bounded open set $\Omega \subset \mathbb{R}^n$ and a $(k+1)$-form f satisfying some compatibility conditions, we solve the problem (in Hölder spaces)

$$d\omega = f \text{ in } \Omega\,, \quad \omega = 0 \text{ on } \partial\Omega\,.$$

We consider, in particular, the divergence and the curl operators.

1. Introduction

The goal of the present article is to study the problem

$$d\omega = f \text{ in } \Omega\,, \quad \omega = 0 \text{ on } \partial\Omega\,.$$

or its dual version

$$\delta\omega = g \text{ in } \Omega\,, \quad \omega = 0 \text{ on } \partial\Omega\,,$$

where $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, is a bounded smooth convex set, ω is a k-form ($1 \leqslant k \leqslant n-1$) and d (respectively, δ) denotes the exterior derivative (respectively, the codifferential). We look for solutions in the Hölder class $C^{r,\alpha}$, completely analogous results holding in Sobolev spaces $W^{r,p}$.

In fact, we have in mind two important cases. The first one is as follows:

$$\operatorname{div} \omega = f \text{ in } \Omega\,, \quad \omega = 0 \text{ on } \partial\Omega\,, \tag{1}$$

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where div is the usual divergence operator. Assume that $f \in C^{r,\alpha}(\overline{\Omega})$ and the following compatibility condition holds:

$$\int_\Omega f(x)dx = 0.$$

Then we find $\omega \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ satisfying the above problem.

The second case is $n = 3$ and $f \in C^{r,\alpha}(\overline{\Omega}; \mathbb{R}^3)$ such that

$$\operatorname{div} f = 0 \text{ in } \Omega\,, \quad \langle f; \nu \rangle = 0 \text{ on } \partial\Omega\,,$$

where $\langle \cdot; \cdot \rangle$ denotes the scalar product and ν is the outward unit normal. We prove that there is $\omega \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^3)$ such that

$$\operatorname{curl} \omega = f \text{ in } \Omega\,, \quad \omega = 0 \text{ on } \partial\Omega\,. \tag{2}$$

The general problem under consideration is well known in algebraic topology due to the classical work of De Rham (cf., for example, [**1**]). However, usually, either only manifolds without boundary are considered or the forms have compact support. Moreover, the question of regularity of the solution is not an issue that is discussed.

Because of the relevancy to applications, the particular case of the divergence (including the question of regularity) has received special attention by many analysts. We quote only a few of them that we have been able to trace: Bogovski [**2**], Borchers–Sohr [**3**], Dacorogna–Moser [**4**] (cf. also Dacorogna [**5**]), Dautray–Lions [**6**], Galdi [**7**], Girault–Raviart [**8**], Kapitanskii–Pileckas [**9**], Ladyzhenskaya [**10**], Ladyzhenskaya–Solonnikov [**11**], Necas [**12**], Tartar [**13**], and Von Wahl [**14, 15**].

The case of the curl in dimension 3, which is also useful for applications, was considered, in particular, by Borchers–Sohr [**3**], Dautray–Lions [**6**], Griesinger [**16**], and Von Wahl [**14, 15**].

We present here a different proof that is in the spirit of Dacorogna–Moser [**4**] and that applies to the general case of k-forms. Of course, the ingredients are also very similar to those of, for example, Ladyzhenskaya [**10**] or Von Wahl [**14, 15**]. They differ essentially in the way we fix the boundary data. The proof is self-contained up to the important result on elliptic systems (cf. Theorem 8) which finds its origins in Duff–Spencer [**17**] and Morrey [**18, 19**]. As quoted here, the result is due to Kress [**20**].

We finally comment on possible generalizations of the obtained results.

(1) A completely similar analysis can be carried over to inhomogeneous boundary data.

(2) At the end of Sec. 7, we will explain how one can deal with nonconvex sets. It should be immediately noted that, with no change, we could have assumed that the set $\Omega \subset \mathbb{R}^n$ is starshaped or, more generally, contractible. Moreover, we should observe that, in the particular case of the divergence, no other condition than connectedness is assumed.

(3) The smoothness of the boundary $\partial\Omega$ can also be relaxed but this requires finer regularity results.

(4) As mentioned earlier, analogous results can be obtained by this method for Sobolev spaces instead of Hölder ones.

The article is organized as follows. For the sake of exposition, we first discuss the problems (1) and (2) although both results are particular cases of the general ones contained in Sec. 7.

2. Preliminary Lemma

We start with an elementary lemma. The proof of this lemma can be found in Dacorogna–Moser [4]. This lemma and its consequences established in Sec. 6 will be used to fix the boundary data.

Lemma 1. *Let $r \geqslant 1$ be an integer, let $0 < \alpha < 1$, and let $\Omega \subset \mathbb{R}^n$ be a bounded open set with orientable $C^{r+2,\alpha}$-boundary consisting of finitely many connected components (ν denotes the outward unit normal). Let $c \in C^{r,\alpha}(\overline{\Omega})$. Then there exists $b \in C^{r+1,\alpha}(\overline{\Omega})$ such that*

$$\operatorname{grad} b = c\nu \quad \text{on } \partial\Omega\,.$$

PROOF. If one is not interested in the sharp regularity result, a solution of the problem is given by

$$b(x) = -c(x)\zeta(\operatorname{dist}(x, \partial\Omega)),$$

where $\operatorname{dist}(x, \partial\Omega)$ stands for the distance from x to the boundary and ζ is a smooth function such that $\zeta(0) = 0$, $\zeta'(0) = 1$, and $\zeta \equiv 0$ outside a small neighborhood of 0.

To construct a smoother solution, we proceed as follows. First find a $C^{r+1,\alpha}(\overline{\Omega})$-solution of the problem (cf. [21] or [22])

$$\Delta d = \frac{1}{\operatorname{meas}\Omega} \int\limits_{\partial\Omega} c\, d\sigma \quad \text{in } \Omega\,, \qquad \frac{\partial d}{\partial \nu} = c \quad \text{on } \partial\Omega\,.$$

Let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi(0) = 1$, $\chi'(0) = 0$, and $\chi \equiv 0$ outside a small neighborhood of 0. We define

$$b(x) = d(x) - \chi(\operatorname{dist}(x, \partial\Omega))d(\psi(x)),$$

where $\psi(x) = x - \operatorname{dist}(x, \partial\Omega)\operatorname{grad}(\operatorname{dist}(x, \partial\Omega))$.

It remains to check that b has the claimed property. Indeed, if $x \in \partial\Omega$ (note that $\psi(x) = x$ on $\partial\Omega$), then

$$\begin{aligned}\operatorname{grad} b(x) &= \operatorname{grad} d(x) - \operatorname{grad} d(\psi(x)) D\psi(x)\\ &= \operatorname{grad} d(x) - \operatorname{grad} d(x)[I - \operatorname{grad}(\operatorname{dist}(x,\partial\Omega)) \otimes \operatorname{grad}(\operatorname{dist}(x,\partial\Omega))]\\ &= \operatorname{grad} d(x)[\nu \otimes \nu] = \frac{\partial d}{\partial \nu}\,\nu = c\nu.\end{aligned}$$

□

3. The Case of the Divergence in $\mathbb{R}^n$

Theorem 2. *Let $r \geqslant 0$ be an integer, let $0 < \alpha < 1$, and let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set with orientable $C^{r+3,\alpha}$-boundary consisting of finitely many connected components (ν denotes the outward unit normal). The following conditions are equivalent:*

(i) *$f \in C^{r,\alpha}(\overline{\Omega})$ satisfies the equation*

$$\int_\Omega f(x)dx = 0,$$

(ii) *there exists $\omega \in C^{r+1,\alpha}\left(\overline{\Omega};\mathbb{R}^n\right)$ such that*

$$\operatorname{div}\omega = f \ \text{ in } \Omega\,, \quad \omega = 0 \ \text{ on } \partial\Omega\,,$$

where $\operatorname{div}\omega = \sum_{i=1}^n \dfrac{\partial \omega_i}{\partial x_i}$.

Remark 3. If the set Ω is disconnected, then the result holds if the compatibility condition is understood on each connected component.

Proof of Theorem 2. (ii) $\Rightarrow$ (i) This implication is just the divergence theorem.

(i) $\Rightarrow$ (ii) We split the proof into two steps.

Step 1. We first find $a \in C^{r+2,\alpha}$ (cf. [**21**] or [**22**]) such that

$$\Delta a = f \text{ in } \Omega\,, \quad \frac{\partial a}{\partial \nu} = 0 \text{ on } \partial\Omega\,.$$

Step 2. We write

$$\omega = \operatorname{curl}^* v + \operatorname{grad} a,$$

where $v = (v_{ij})_{1 \leqslant i < j \leqslant n} \in \mathbb{R}^{n(n-1)/2}$,

$$\operatorname{curl}^* v = ((\operatorname{curl}^* v)_1, \ldots, (\operatorname{curl}^* v)_n), \quad (\operatorname{curl}^* v)_i = \sum_{j=1}^{i-1} \frac{\partial v_{ji}}{\partial x_j} - \sum_{j=i+1}^{n} \frac{\partial v_{ij}}{\partial x_j}.$$

Since $\operatorname{div}\operatorname{curl}^* v = 0$, it remains to find $v \in C^{r+2,\alpha}$ such that

$$\operatorname{curl}^* v = -\operatorname{grad} a \quad \text{on } \partial\Omega .$$

An easy computation shows that a solution of this problem is given by

$$\operatorname{grad} v_{ij} = \left(\frac{\partial a}{\partial x_i}\nu_j - \frac{\partial a}{\partial x_j}\nu_i\right)\nu \quad \text{on } \partial\Omega$$

whose solvability is ensured by Lemma 1. This achieves the proof of the theorem.

In order to clarify the link with the more abstract framework of differential forms, we rewrite the proof in this terminology. We consider ω as a 1-form. Therefore, the problem we want to solve is as follows:

$$\delta\omega = f \text{ in } \Omega , \quad \omega = 0 \text{ on } \partial\Omega .$$

We write $\omega = da + \delta v$, where a is a 0-form and v is a 2-form. This leads to the equalities

$$f = \delta\omega = \delta da = \Delta a$$

since $\delta\delta v = 0$, $\Delta a = \delta da + d\delta a$, $\delta a = 0$, and a is a 0-form. (The fact that $\Delta a = \delta da$ makes easier the case of 1-forms ω in comparison with k-forms $k \geqslant 2$.)

We also note that

$$\delta_\nu(da) = \langle \operatorname{grad} a; \nu\rangle = \frac{\partial a}{\partial \nu},$$

which leads to our choice at Step 1.

Now, in order to have $\omega = 0$ on the boundary, it remains to solve (cf. Step 2) the equation

$$\delta v = -da \quad \text{on } \partial\Omega .$$

The idea is to find a solution, via Lemma 1, of the equation

$$\operatorname{grad} v_{ij} = -[d_\nu da]_{ij}\nu = \left(\frac{\partial a}{\partial x_i}\nu_j - \frac{\partial a}{\partial x_j}\nu_i\right)\nu \quad \text{on } \partial\Omega$$

and then to check (as in Lemmas 9 and 10) that such a solution v satisfies the equation $\delta v = -da$ on $\partial\Omega$. □

4. The Case of the Curl in $\mathbb{R}^3$

Theorem 4. *Let $r \geqslant 1$ be an integer, let $0 < \alpha < 1$, and let $\Omega \subset \mathbb{R}^3$ be a bounded convex set with $C^{r+3,\alpha}$-boundary. Let ν denote the outward unit normal. The following conditions are equivalent:*

(i) $f \in C^{r,\alpha}(\overline{\Omega}; \mathbb{R}^3)$ *satisfies the equations*

$$\operatorname{div} f = 0 \text{ in } \Omega , \quad \langle f; \nu\rangle = 0 \text{ on } \partial\Omega ,$$

(ii) *there exists* $\omega \in C^{r+1,\alpha}(\overline{\Omega};\mathbb{R}^3)$ *such that*

$$\operatorname{curl}\omega = f \text{ in } \Omega\,, \quad \omega = 0 \text{ on } \partial\Omega\,,$$

where

$$\operatorname{curl}\omega = \left(\frac{\partial\omega_3}{\partial x_2} - \frac{\partial\omega_2}{\partial x_3}, \frac{\partial\omega_1}{\partial x_3} - \frac{\partial\omega_3}{\partial x_1}, \frac{\partial\omega_2}{\partial x_1} - \frac{\partial\omega_1}{\partial x_2}\right)$$

for $\omega = (\omega_1, \omega_2, \omega_3)$.

PROOF. (ii) $\Rightarrow$ (i) The fact that $\operatorname{div} f = 0$ is obvious. We show that $\langle f;\nu\rangle = 0$ on $\partial\Omega$. For this purpose, we suppose that $\psi \in C^2(\overline{\Omega})$ is an arbitrary function. The integration by parts formula and the facts that $\omega = 0$ on $\partial\Omega$ and $\operatorname{div} f = 0$ lead to the equalities

$$\int_\Omega \langle \operatorname{grad}\psi; f\rangle\, dx = \int_\Omega \langle \operatorname{grad}\psi; \operatorname{curl}\omega\rangle\, dx = 0,$$

$$\int_\Omega \langle \operatorname{grad}\psi; f\rangle\, dx = \int_{\partial\Omega} \psi\,\langle f;\nu\rangle\, d\sigma.$$

In view of these two equalities and the arbitrariness of ψ, we conclude that $\langle f;\nu\rangle = 0$ on $\partial\Omega$.

(i) $\Rightarrow$ (ii) We divide the proof into two steps.

STEP 1. We first find $u \in C^{r+1,\alpha}$, using Theorem 8, that solves the following system:

$$\begin{aligned} \operatorname{curl} u &= f \quad \text{in } \Omega\,, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega\,, \\ u \times \nu &= 0 \quad \text{on } \partial\Omega\,, \end{aligned}$$

where $u \times \nu$ denotes the vector product. Using the notation of the next sections, we are solving, in fact, the problem

$$\begin{aligned} du &= \widetilde{f} \text{ in } \Omega\,, \\ \delta u &= 0 \quad \text{in } \Omega\,, \\ d_\nu u &= 0 \quad \text{on } \partial\Omega\,, \end{aligned}$$

where u and f are regarded as a 1-form and a 2-form respectively and $\widetilde{f} = (f_{23}, -f_{13}, f_{12})$ for $f = (f_{12}, f_{13}, f_{23})$. The compatibility condition for solving this problem is exactly the following:

$$d\widetilde{f} = \operatorname{div} f = 0 \quad \text{in } \Omega\,,$$

$$d_\nu \widetilde{f} = 0 \text{ if and only if } \langle f;\nu\rangle = 0 \quad \text{on } \partial\Omega\,.$$

STEP 2. We set $\omega = u + \operatorname{grad} v$, where $v \in C^{r+2,\alpha}$ solves the equation

$$\operatorname{grad} v = -u \quad \text{on } \partial\Omega\,.$$

Indeed, this is possible by Lemma 1 and by the fact that $u \times \nu = 0$. □

5. Notation and General Results on Differential Forms

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with sufficiently smooth orientable boundary. Denote by ν the outward unit normal. Let $0 \leqslant k \leqslant n$. Consider a k-form $\omega : \Omega \to \mathbb{R}^{\binom{n}{k}}$ (we often identify, by abuse of notations, the form and the vector of $\mathbb{R}^{\binom{n}{k}}$ whose components are those of the form)

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \ldots i_k} \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

(if $k = 0$ then ω is just a function).

For such a form we introduce the following notions.

1) The *exterior derivative*, denoted by $d\omega$, is a $(k+1)$-form defined by

$$d\omega = \sum_{i_1 < \cdots < i_{k+1}} \left(\sum_{\gamma=1}^{k+1} (-1)^{\gamma-1} \frac{\partial \omega_{i_1 \ldots i_{\gamma-1} i_{\gamma+1} \ldots i_{k+1}}}{\partial x_{i_\gamma}} \right) dx_{i_1} \wedge \cdots \wedge dx_{i_{k+1}}$$

(for $k = n$ we set $d\omega = 0$).

2) The *codifferential*, denoted by $\delta\omega$, is a $(k-1)$-form defined by

$$\delta\omega = \sum_{i_1 < \cdots < i_{k-1}} \left(\sum_{j=1}^{n} \varepsilon^j_{i_1 \ldots i_{k-1}} \frac{\partial \omega_{(i_1 \ldots i_{k-1} j)}}{\partial x_j} \right) dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}}$$

(if $k = 0$, then $\delta\omega = 0$ as usual), $(i_1 \ldots i_{k-1} j)$ denotes the kth index rearranged increasingly and

$$\varepsilon^j_{i_1 \ldots i_{k-1}} = \begin{cases} 0 & \text{if } j \in \{i_1, \ldots, i_{k-1}\}, \\ (-1)^{\gamma-1} & \text{if } i_{\gamma-1} < j < i_\gamma \end{cases}$$

(if $k = 1$, then $\varepsilon^j = 1$).

3) The *tangential part*, denoted by $d_\nu\omega$, on the boundary $\partial\Omega$ is a $(k+1)$-form defined by

$$d_\nu\omega = \sum_{i_1 < \cdots < i_{k+1}} \left(\sum_{\gamma=1}^{k+1} (-1)^{\gamma-1} \nu_{i_\gamma} \omega_{i_1 \ldots i_{\gamma-1} i_{\gamma+1} \ldots i_{k+1}} \right) dx_{i_1} \wedge \cdots \wedge dx_{i_{k+1}}$$

(if $k = n$, then $d_\nu\omega = 0$).

4) The *normal part*, denoted by $\delta_\nu\omega$, on $\partial\Omega$ is a $(k-1)$-form defined by

$$\delta_\nu\omega = \sum_{i_1 < \cdots < i_{k-1}} \left(\sum_{j=1}^{n} \varepsilon^j_{i_1 \ldots i_{k-1}} \nu_j \, \omega_{(i_1 \ldots i_{k-1} j)} \right) dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}}$$

(if $k = 0$, then $\delta_\nu\omega = 0$).

Remark 5. (1) One can define the operator δ, equivalently, by duality as follows:

$$\delta\omega = (-1)^{k(n-k)} * d(*\omega).$$

(2) Our definition of the operator δ may differ from the one of some textbooks by a minus sign. Our choice is motivated (cf. (3) below) by the fact that we define for $\omega : \Omega \to \mathbb{R}$ the Laplacian by the formula

$$\Delta\omega = \sum_{i=1}^{n} \frac{\partial^2 \omega}{\partial x_i^2}.$$

Those textbooks that have an opposite sign in the definition of δ define therefore the Laplacian with the opposite sign.

(3) Our definition of the tangential and normal parts of a k-form ω is not the usual one. We have adopted here the definition of Kress [**20**]. For example, Duff–Spencer [**17**] and Morrey [**18, 19**] write the tangential part $t\omega$ and the normal part $n\omega$ both as forms of the same degree as ω so that $\omega = t\omega + n\omega$. However, our definitions and theirs carry similar informations as the following example shows. Indeed, if $\Omega = \{x \in \mathbb{R}^n : x_n > 0\}$ and ω is a k-form, then

$$d_\nu\omega = 0 \Leftrightarrow t\omega = 0 \Leftrightarrow \omega_{i_1 \ldots i_k}(x_1, \ldots, x_{n-1}, 0) = 0, \quad 1 \leqslant i_1 < \cdots < i_k < n,$$
$$\delta_\nu\omega = 0 \Leftrightarrow n\omega = 0 \Leftrightarrow \omega_{i_1 \ldots i_k}(x_1, \ldots, x_{n-1}, 0) = 0, \quad i_k = n,$$

at $x_n = 0$. The terms "tangential" and "normal", which will induce those ("Dirichlet" and "Neumann") used in Theorem 8 and Sec. 7, are not always appropriate. They are only adequate for 1-forms (and totally inadequate for $(n-1)$-forms) since then $d_\nu\omega$ can be identified with $\omega \times \nu$ (i.e., the vector product), while $\delta_\nu\omega$ is the scalar product $\langle \omega ; \nu \rangle$.

Hence the following assertion can be established.

Proposition 6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with orientable C^1-boundary, ν the outward unit normal to $\partial\Omega$, $1 \leqslant k \leqslant n-1$, and $\omega \in C(\overline{\Omega}; \mathbb{R}^{\binom{n}{k}})$ a k-form.*

(i) *If, in addition, ω is of class C^2, then*

$$dd\omega = 0, \quad \delta\delta\omega = 0, \quad \Delta\omega = d\delta\omega + \delta d\omega\,. \tag{3}$$

(ii) *The following identity is valid on $\partial\Omega$:*

$$d_\nu\delta_\nu\omega + \delta_\nu d_\nu\omega = \omega\,. \tag{4}$$

(iii) *If, in addition, ω is of class C^1, then the following version of the divergence theorem holds*:

$$\int_\Omega d\omega\, dx = \int_{\partial\Omega} d_\nu\omega\, d\sigma, \qquad \int_\Omega \delta\omega\, dx = \int_{\partial\Omega} \delta_\nu\omega\, d\sigma.$$

(iv) *The integration by parts formula holds, namely,*

$$\int_{\Omega} \langle \psi; d\varphi \rangle \, dx + \int_{\Omega} \langle \delta\psi; \varphi \rangle \, dx = \int_{\partial\Omega} \langle \psi; d_\nu \varphi \rangle \, d\sigma = \int_{\partial\Omega} \langle \delta_\nu \psi; \varphi \rangle \, d\sigma,$$

where $\psi \in C^1(\overline{\Omega}; \mathbb{R}^{\binom{n}{k}})$, $\varphi \in C^1(\overline{\Omega}; \mathbb{R}^{\binom{n}{k-1}})$, *and the scalar product of two* k*-forms* α *and* β *is defined by*

$$\langle \alpha; \beta \rangle = \sum_{i_1 < \dots < i_k} \alpha_{i_1 \dots i_k} \beta_{i_1 \dots i_k}.$$

We give some examples corresponding to two particular cases considered at the beginning of the present article.

Example 7. (1) Consider the following case of a 1-form ω:

$$\omega = \omega_1 dx_1 + \cdots + \omega_n dx_n.$$

Then

$$d\omega = \sum_{1 \leqslant i < j \leqslant n} \left(\frac{\partial \omega_j}{\partial x_i} - \frac{\partial \omega_i}{\partial x_j} \right) dx_i \wedge dx_j$$

which for $n = 3$ leads to $\operatorname{curl} \omega$ componentwise (up to the sign and the order of the components). Similarly,

$$\delta\omega = \operatorname{div} \omega = \sum_{i=1}^{n} \frac{\partial \omega_i}{\partial x_i}$$

$$d_\nu \omega = \sum_{1 \leqslant i < j \leqslant n} (\omega_j \nu_i - \omega_i \nu_j) \, dx_i \wedge dx_j$$

$$\delta_\nu \omega = \langle \nu; \omega \rangle = \sum_{i=1}^{n} \nu_i \omega_i.$$

In particular, for $n = 2$ we have

$$d\omega = \left(\frac{\partial \omega_2}{\partial x_1} - \frac{\partial \omega_1}{\partial x_2} \right) dx_1 \wedge dx_2,$$

$$\delta\omega = \operatorname{div} \omega = \frac{\partial \omega_1}{\partial x_1} + \frac{\partial \omega_2}{\partial x_2}$$

(the combination of the operators d and δ leads to the anti-Cauchy–Riemann operator).

(2) Consider the following case of a 2-form ω in $\mathbb{R}^3$:

$$\omega = \omega_{12} \, dx_1 \wedge dx_2 + \omega_{13} \, dx_1 \wedge dx_3 + \omega_{23} \, dx_2 \wedge dx_3.$$

We have

$$d\omega = \left(\frac{\partial\omega_{12}}{\partial x_3} - \frac{\partial\omega_{13}}{\partial x_2} + \frac{\partial\omega_{23}}{\partial x_1}\right) dx_1 \wedge dx_2 \wedge dx_3,$$

$$\delta\omega = \left(-\frac{\partial\omega_{12}}{\partial x_2} - \frac{\partial\omega_{13}}{\partial x_3}\right) dx_1 + \left(\frac{\partial\omega_{12}}{\partial x_1} - \frac{\partial\omega_{23}}{\partial x_3}\right) dx_2 + \left(\frac{\partial\omega_{13}}{\partial x_1} + \frac{\partial\omega_{23}}{\partial x_2}\right) dx_3,$$

$$d_\nu\omega = (\nu_3\omega_{12} - \nu_2\omega_{13} + \nu_1\omega_{23})\, dx_1 \wedge dx_2 \wedge dx_3,$$

$$\delta_\nu\omega = (-\nu_2\omega_{12} - \nu_3\omega_{13})\, dx_1 + (\nu_1\omega_{12} - \nu_3\omega_{23})\, dx_2 + (\nu_1\omega_{13} + \nu_2\omega_{23})\, dx_3.$$

More generally, if ω is a 2-form over $\mathbb{R}^n$, then

$$\delta\omega = \sum_{i=1}^{n}\left(\sum_{j=1}^{i-1}\frac{\partial\omega_{ji}}{\partial x_j} - \sum_{j=i+1}^{n}\frac{\partial\omega_{ij}}{\partial x_j}\right) dx_i\,.$$

The following result, for the existence part, is due to Kress [**20**] (cf. also Morrey [**19**, Secs. 7.7 and 7.8]). The regularity follows from standard arguments (cf. Morrey [**19**]).

Theorem 8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex set with $C^{r+1,\alpha}$-boundary, where $r \geqslant 1$ is an integer and $0 < \alpha < 1$, let ν denote the outward unit normal, and let $1 \leqslant k \leqslant n-1$. Let $f \in C^{r,\alpha}(\overline{\Omega}; \mathbb{R}^{\binom{n}{k+1}})$ and $g \in C^{r,\alpha}(\overline{\Omega}; \mathbb{R}^{\binom{n}{k-1}})$ be such that*

$$df = 0, \quad \delta g = 0 \quad \text{in } \Omega\,. \tag{5}$$

DIRICHLET PROBLEM. *If, in addition to* (5), *either*

- $1 \leqslant k \leqslant n-2$ *and* $d_\nu f = 0$ *on* $\partial\Omega$

or

- $k = n-1$ *and* $\displaystyle\int_\Omega f = 0,$

then there exists $u \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^{\binom{n}{k}})$ such that

$$\begin{aligned} du &= f \quad \text{in } \Omega\,,\\ \delta u &= g \quad \text{in } \Omega\,,\\ d_\nu u &= 0 \quad \text{on } \partial\Omega\,. \end{aligned}$$

NEUMANN PROBLEM. *If, in addition to* (5), *either*

- $2 \leqslant k \leqslant n-1$ *and* $\delta_\nu g = 0$ *on* $\partial\Omega$

or

- $k = 1$ *and* $\displaystyle\int_\Omega g = 0,$

then there exists $v \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^{\binom{n}{k}})$ *such that*

$$\begin{aligned} dv &= f \quad \textit{in } \Omega\,, \\ \delta v &= g \quad \textit{in } \Omega\,, \\ \delta_\nu v &= 0 \quad \textit{on } \partial\Omega\,. \end{aligned}$$

6. Generalization of Preliminary Lemma

We have two generalizations of Lemma 1.

Lemma 9. *Let $r \geqslant 1$, $1 \leqslant k \leqslant n-1$ be integers, and let $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with orientable $C^{r+2,\alpha}$-boundary consisting of finitely many connected components (ν denotes the outward unit normal), and let $c \in C^{r,\alpha}(\overline{\Omega}; \mathbb{R}^{\binom{n}{k}})$ be such that*

$$d_\nu c = 0 \quad \textit{on } \partial\Omega\,.$$

Then there exists $b \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^{\binom{n}{k-1}})$ such that

$$db = c \quad \textit{on } \partial\Omega\,.$$

PROOF. First we solve by Lemma 1 (note that for $k = 1$ Lemma 1 and the present lemma are the same) the problem

$$\operatorname{grad} b_{i_1 \ldots i_{k-1}} = [\delta_\nu c]_{i_1 \ldots i_{k-1}}\, \nu.$$

We claim that

$$db = c \quad \text{on } \partial\Omega\,.$$

We first note that the definition of b implies that $db = d_\nu \delta_\nu c$. Combining the above fact with the hypothesis $d_\nu c = 0$ and the identity (4), we find $d_\nu \delta_\nu c + \delta_\nu d_\nu c = d_\nu \delta_\nu c = c$, which is the claimed result. □

The second generalization is the dual of the preceding one and is proved by duality (replacing d by δ and conversely).

Lemma 10. *Let $r \geqslant 1$ and $1 \leqslant k \leqslant n-1$ be integers, and let $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with orientable $C^{r+2,\alpha}$-boundary consisting of finitely many connected components (ν denotes the outward unit normal), and let $c \in C^{r,\alpha}(\overline{\Omega}; \mathbb{R}^{\binom{n}{k}})$ be such that*

$$\delta_\nu c = 0 \quad \textit{on } \partial\Omega\,.$$

Then there exists $b \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^{\binom{n}{k+1}})$ such that

$$\delta b = c \quad \textit{on } \partial\Omega\,.$$

7. The Main Result

Theorem 11. *Let $r \geqslant 1$ and $1 \leqslant k \leqslant n-1$ be integers, and let $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded convex set with $C^{r+3,\alpha}$-boundary, let ν denote the outward unit normal, and let f be a $(k+1)$-form. The following two assertions are equivalent:*

(i) *either*

- $1 \leqslant k \leqslant n-2$ *and* $f \in C^{r,\alpha}(\overline{\Omega}; \mathbb{R}^{\binom{n}{k+1}})$ *is such that*

$$df = 0 \text{ in } \Omega\,, \quad d_\nu f = 0 \text{ on } \partial\Omega$$

or

- $k = n-1$ *and* $f \in C^{r,\alpha}(\overline{\Omega})$ *satisfies the equation*

$$\int_\Omega f(x)dx = 0,$$

(ii) *there exists* $\omega \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^{\binom{n}{k}})$ *such that*

$$d\omega = f \text{ in } \Omega\,, \quad \omega = 0 \text{ on } \partial\Omega\,. \tag{6}$$

Remark 12. The above theorem is trivially valid for $k = 0$, i.e.,

$$\operatorname{grad}\omega = f \text{ in } \Omega \quad \omega = 0 \text{ on } \partial\Omega\,.$$

However, we note that in the sufficiency part of the proof we cannot invoke anymore Theorem 8. A straightforward integration leads immediately to the result. We also note that, in this case, the solution is unique and the regularity holds in spaces C^r as well.

Proof of Theorem 11. (i) $\Rightarrow$ (ii) We divide this proof into two steps.

Step 1. We start by applying Theorem 8 to get $u \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^{\binom{n}{k}})$ solving the system

$$\begin{aligned} du &= f \quad \text{in } \Omega\,,\\ \delta u &= 0 \quad \text{in } \Omega\,,\\ d_\nu u &= 0 \quad \text{on } \partial\Omega\,. \end{aligned}$$

We note that this is possible in view of the compatibility conditions on f.

Step 2. We use Lemma 9 to find $v \in C^{r+2,\alpha}(\overline{\Omega}; \mathbb{R}^{\binom{n}{k}})$ such that

$$dv = -u \quad \text{on } \partial\Omega\,.$$

This is possible since $d_\nu u = 0$ on $\partial\Omega$. Finally, we write $\omega = u + dv$ to obtain the result.

(ii) $\Rightarrow$ (i) We start by discussing the case $k = n-1$. Combining the divergence theorem and (6), we get

$$\int_\Omega f = \int_\Omega d\omega = \int_{\partial\Omega} d_\nu \omega = 0.$$

Consider the case $1 \leqslant k \leqslant n-2$. The first condition, $df = 0$, is obvious. It remains to prove that $d_\nu f = 0$ on $\partial\Omega$. For this purpose, we assume that ψ is any smooth $(k+2)$-form. Using the integration by parts formula, (6), and the fact that $\delta\delta\psi = 0$, we get

$$\int_\Omega \langle \delta\psi; f\rangle\, dx = \int_\Omega \langle \delta\psi; d\omega\rangle\, dx + \int_\Omega \langle \delta\delta\psi; \omega\rangle\, dx = \int_{\partial\Omega} \langle \delta_\nu \delta\psi; \omega\rangle\, d\sigma = 0. \tag{7}$$

We again invoke the integration by parts formula and the fact that $df = 0$ to get

$$\int_\Omega \langle \delta\psi; f\rangle\, dx = \int_\Omega \langle \delta\psi; f\rangle\, dx + \int_\Omega \langle \psi; df\rangle\, dx = \int_{\partial\Omega} \langle \psi; d_\nu f\rangle\, d\sigma. \tag{8}$$

Using (7), (8) and the arbitrariness of ψ, we obtain $d_\nu f = 0$ on $\partial\Omega$, which is the claimed result. $\square$

The dual version of Theorem 11 is the following.

Theorem 13. *Let $r \geqslant 1$ and $1 \leqslant k \leqslant n-1$ be integers, and let $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded convex set with $C^{r+3,\alpha}$-boundary, let ν denote the outward unit normal, and let f be a $(k-1)$-form. The following two conditions are equivalent:*

(i) *either*

- $2 \leqslant k \leqslant n-1$ *and* $f \in C^{r,\alpha}(\overline{\Omega}; \mathbb{R}^{\binom{n}{k-1}})$ *is such that*

$$\delta f = 0 \text{ in } \Omega\,, \quad \delta_\nu f = 0 \text{ on } \partial\Omega$$

or

- $k = 1$ *and* $f \in C^{r,\alpha}(\overline{\Omega})$ *is such that*

$$\int_\Omega f(x)dx = 0,$$

(ii) *there exists* $\omega \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^{\binom{n}{k}})$ *such that*

$$\delta\omega = f \text{ in } \Omega\,, \quad \omega = 0 \text{ on } \partial\Omega\,. \tag{9}$$

Remark 14. The case $k = n$ requires a different treatment (cf. Remark 12).

We conclude the article with some comments on the results when Ω is not necessarily convex. We assume that $\Omega \subset \mathbb{R}^n$ is a bounded connected set with orientable smooth boundary (then ν denotes the outward unit normal) consisting of finitely many connected components.

Definition 15. Let $0 \leqslant k \leqslant n$, and let ψ be a k-form over Ω. The set of *k-harmonic fields with the Dirichlet* (*Neumann*) *boundary condition* is defined as the vector space

$$D_k(\Omega) = \left\{ \begin{array}{l} \psi \in C^0(\overline{\Omega}; \mathbb{R}^{\binom{n}{k}}) \cap C^1(\Omega; \mathbb{R}^{\binom{n}{k}}) : \\ d\psi = 0, \ \delta\psi = 0 \text{ in } \Omega \text{ and } d_\nu \psi = 0 \text{ on } \partial\Omega, \end{array} \right\}$$

respectively,

$$N_k(\Omega) = \left\{ \begin{array}{l} \psi \in C^0(\overline{\Omega}; \mathbb{R}^{\binom{n}{k}}) \cap C^1(\Omega; \mathbb{R}^{\binom{n}{k}}) : \\ d\psi = 0, \ \delta\psi = 0 \text{ in } \Omega \text{ and } \delta_\nu \psi = 0 \text{ on } \partial\Omega. \end{array} \right\}$$

Remark 16. We note that for Ω as above, we always have

$$N_n(\Omega) \simeq D_0(\Omega) \simeq \{0\}, \quad D_n(\Omega) \simeq N_0(\Omega) \simeq \mathbb{R}.$$

Furthermore, if $1 \leqslant k \leqslant n-1$ and the set Ω is convex or, more generally, is contractible, then

$$N_k(\Omega) = D_k(\Omega) \simeq \{0\} \subset \mathbb{R}^{\binom{n}{k}},$$

while for general sets we have

$$\dim D_k(\Omega) = B_{n-k}, \quad \dim N_k(\Omega) = B_k,$$

where B_k are the Betti numbers of Ω (cf. Duff–Spencer [**17**] and Kress [**20**]).

Theorem 11 (respectively, Theorem 13) remains valid for such general sets if we add the following necessary condition:

$$\int_\Omega \langle f; \psi \rangle dx = 0 \quad \forall \psi \in D_{k+1}(\Omega),$$

respectively,

$$\int_\Omega \langle f; \psi \rangle dx = 0 \quad \forall \psi \in N_{k-1}(\Omega).$$

Finally, we note that for $k = n-1$ in Theorem 11 or for $k = 1$ in Theorem 13 we therefore have no new condition. This explains why in Theorem 2 we do not assume that Ω is convex.

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A Singular Perturbation Property of Capillary Surfaces

Robert Finn†

Dedicated to O. A. Ladyzhenskaya, whose achievements have so deeply influenced the mathematics of her time

A discontinuous reversal of limiting behavior for capillary surfaces is considered, that was established for a particular configuration in [**1**]. A class of configurations is postulated, for which it is conjectured that an analogous behavior will be observed. We establish the conjecture in a particular case. We show also that the result of the conjecture, if correct, could not be significantly improved, in the sense that it can be made to fail under an arbitrary small change in the configuration at points distinct from the limiting set.

The material outlined below was completed jointly with Darren Lee. Our starting point is a discontinuous dependence property of solutions of the capillary equation at low gravity, which was exhibited as an example in [**1**]. The equation is for the height $u(x,y)$ of an equilibrium surface interface in a capillary tube of general section Ω, dipped into an infinite bath that is in horizontal rest position $u = 0$ at infinity:

$$\operatorname{div} Tu = Bu, \quad Tu = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \tag{1}$$

over Ω, with

$$\nu \cdot Tu = \cos\gamma \tag{2}$$

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on $\Sigma = \partial\Omega$. Here, ν is the unit exterior normal and γ is the (prescribed) *contact angle* with which the surface S, the graph of $u(x,y)$, meets the vertical cylinder Z over Σ. The *Bond number* $B = \rho g a^2/\sigma$, where ρ is density change across S, g the gravitational acceleration, σ the surface tension, and a a representative length. We may assume (following an eventual normalization) that $0 \leqslant \gamma < \pi/2$. The equations are to be interpreted nondimensionally, in the sense that u, x, and y are the ratios of the actual (physical) coordinates to a.

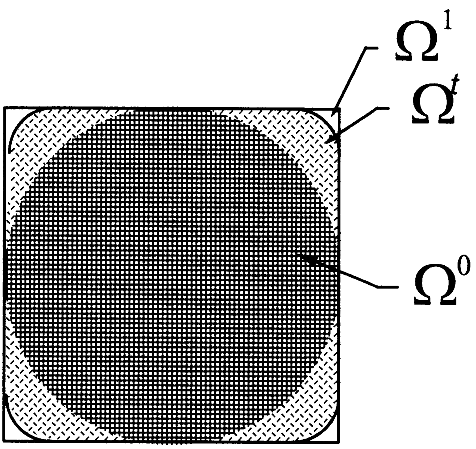


Fig. 1. Approximation to a disk by smoothed squares

In [**1**], the family of configurations indicated in Fig. 1 is considered, in which Ω^t is the domain obtained by smoothing the corners of a square of side length 2 by circular arcs of radius $(1-t)$, $0 \leqslant t \leqslant 1$, Thus, Ω^0 is the inscribed disk and Ω^1 is the square. As is proved in [**1**], *if $B > 0$ and $\pi/4 \leqslant \gamma < \pi/2$, then for each of the unique solutions $u^t(x,y;B)$ of* (1), (2) *in Ω^t, there exist positive constants $C_1(t)$, $C_0(t)$ such that $u^1(x,y;B) - u^t(x,y;B) > C_1(t)/B - C_0(t)$ but $u^1(x,y;B) < u^0(x,y;B)$ for all $B > 0$.*

Thus, although the domains Ω^t tend in $C^{1+\alpha}$-norm to Ω^0 as $t \to 0$, the limiting behavior of $u^1(x,y;B) - u^t(x,y;B)$ as $B \to 0$ reverses discontinuously (with an infinite jump) at $t = 0$. Remarkably, the reversal occurs precisely at the smoothest of all the domains considered.

We wish to characterize domains $\{\Omega\}$ for which analogous behavior occurs. To this effect, we observe that key features in the proof of the assertion just made are the following:

(a) the ratio $|\Sigma|/|\Omega|$ is the same for the square and for the circle,

(b) a smaller ratio is obtained for any t in the interval $0 < t < 1$.

Corresponding geometrical properties appear when the square is replaced with a domain Ω constructed by replacing a countable number of disjoint arcs of a unit circle, each of which subtends an angle less than π at the center, with pairs of segments tangent to the arcs at their endpoints, as in Fig. 2. The domains Ω^t are then constructed by smoothing as above.

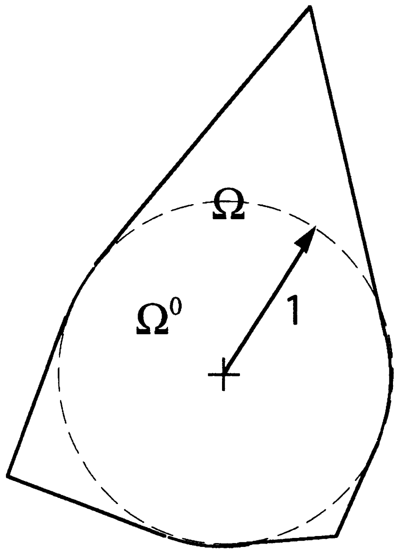


Fig. 2. Construction of domains for conjecture

The equality $|\Sigma^1|/|\Omega^1| = |\Sigma^0|/|\Omega^0|$ is easy to prove; the asserted behavior of $|\Sigma^t|/|\Omega^t|$ follows from [**1**, Theorem 5].

Given a Lipschitz domain Ω, we say that $u(x, y; B)$ is a *variational solution* of (1), (2) in Ω if u is integrable and of class C^2 interior to Ω and

$$\int_\Omega (\nabla\eta \cdot Tu + Bu\eta)\, dx - \oint_\Sigma \eta \cos\gamma = 0 \tag{3}$$

for every test function $\eta \in H^{1,1}(\Omega) \cap L^\infty(\Omega)$. We define analogously variational solutions $U(x, y)$ of the "zero gravity" equation

$$\operatorname{div} TU = 2H \equiv \text{const} \tag{4}$$

subject to (2) on Σ.

Note that the boundary behavior of u or of U does not enter explicitly into the definition; nevertheless, every strict solution is a variational solution, and variational solutions of (1), (2) are uniquely determined, while those of (4), (2) are determined up to an additive constant. As is proved in [**1**], *if a variational solution* $U(x,y)$ *of* (4), (2) *exists in* Ω, $-\infty < m < U(x,y) < M$, *and* $u(x,y;B)$ *is a variational solution of* (1), (2) *in* Ω, *then*

$$\left| u - \frac{|\Sigma|}{B|\Omega|}\cos\gamma \right| < M - m \tag{5}$$

throughout Ω (cf. [**2**] for an earlier version of this result, formulated for strict solutions).

From another direction, a theorem due originally to Siegel [**3**] (a slightly stronger version appears in [**1**]) shows that if $u^0(x,y;B)$ is a variational solution of (1), (2) over the disk Ω^0 on which the construction of Fig. 2 is based and $u(x,y;B)$ is a variational solution of (1), (2) over Ω, then $u^0(x,y;B) > u(x,y;B)$ throughout Ω^0.

For a domain Ω as in Fig. 2, and also for domains Ω^t constructed by smoothing the corners as indicated above, it is known [**4**]–[**7**] that for any γ in $[0,\pi/2]$ there is a unique variational solution of (1), (2) in Ω, which assumes the data (2) strictly on Σ, except at the vertex points of intersecting segments. The solution is bounded at any such vertex if and only if the half angle α at the vertex is such that $\alpha+\gamma \geqslant \pi/2$. For such a domain Ω, let $2\alpha_0$ the smallest of the opening angles at the vertices defined by the pairs of segments. Then a bounded variational solution of (4) in Ω, with (2) satisfied strictly except at the vertices, can be obtained explicitly for any γ in the range $(\pi/2) - \alpha_0 \leqslant \gamma < \pi/2$, as a lower hemisphere centered over the center of the disk and of radius $1/\cos\gamma$. If $\alpha_0+\gamma < \pi/2$, there is no solution of (4), (2), either strict or variational, over Ω.

With these considerations in mind, we make the following

Conjecture. *For a domain* Ω *as in Fig. 2, the discontinuous reversal of limiting behavior of solutions in smoothed domains, as indicated above in the example of the square, will occur whenever* $\alpha_0 + \gamma \geqslant \pi/2$.

An examination of the remarks above shows that the proof of the conjecture reduces to that of existence of bounded variational solutions of (4), (2) in each Ω', in the sense introduced above. In the present paper, we describe a proof for this existence in the particular case of a domain with a single vertex, as indicated in Fig. 3. We base the proof on the following general result, established in [**8**] and [**9**, Chaps. 6 and 7], for existence of a variational solution in a piecewise smooth domain Ω, and on the bound for such solutions given in [**10**].

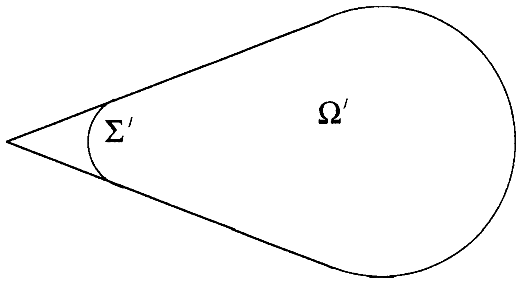


Fig. 3. Single vertex domain Ω; smoothing arc Σ'; smoothed domain Ω'

Theorem. *A variational solution $u(x)$ of (4), (2) exists in Ω if and only if, for every subdomain $\Omega^* \subset \Omega$ such that $\Omega^* \neq \varnothing, \Omega$ and such that Ω^* is bounded on $\Sigma = \partial\Omega$ by subarcs $\Sigma^* \subset \Sigma$ and within Ω by subarcs Γ^* of semicircles of radius $|\Omega|/(|\Sigma|\cos\gamma)$ with the properties*

(i) *the curvature vector of each Γ^* is directed exterior to Ω^*,*

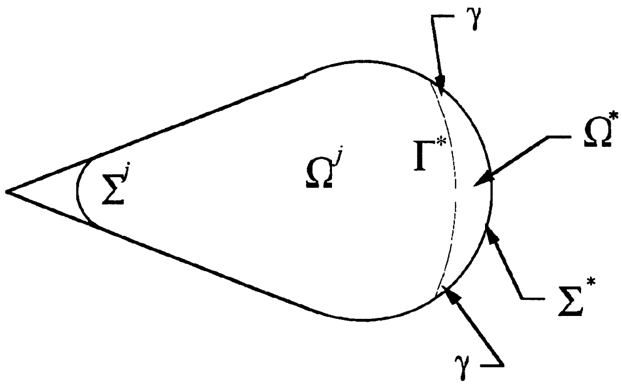
(ii) *each Γ^* meets Σ, either in the angle γ measured within Ω^* or else at re-entrant corner points of Σ,*

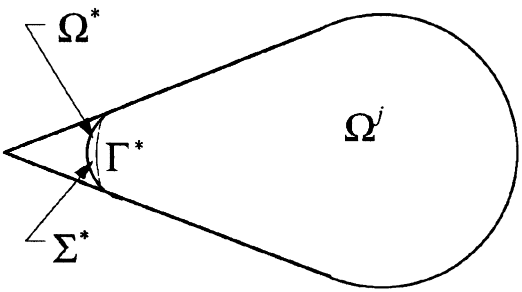
the following relation holds:

$$\Phi(\Omega^*;\gamma) \equiv |\Gamma^*| - |\Sigma^*|\cos\gamma + \tfrac{|\Sigma|}{|\Omega|}|\Omega^*|\cos\gamma > 0. \tag{6}$$

Every such solution is smooth interior to Ω and uniquely determined up to an additive constant.

The arcs Γ^* are referred to as *extremal arcs*; they arise from a "subsidiary variational problem." Using this result, we see that the problem is reduced to finding all extremal arcs Γ^* with the requisite geometrical properties, and evaluating the corresponding Φ. It is not difficult to show that such arcs can occur in only three ways, as indicated in Figs. 4a,b,c. Formal estimates of the values of Φ that occur for the subtended domains yield $\Phi > 0$ in all cases, thus ensuring the existence of the solution, and from this follows, as indicated above, the singular perturbation in the height differences for the solutions. The details of the calculations will be included in a complete exposition, to appear elsewhere.





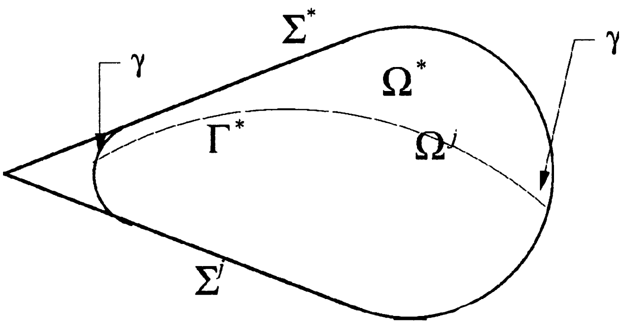


Fig. 4a,b,c. Extremal arc Γ^*. Cases 1, 2, 3

For the general case of domains as indicated in Fig. 2, the problem remains open. But we note that the example of the square was settled affirmatively in [**1**]. That was possible because the particular geometry facilitates the proof of existence of variational solutions of (4), (2) in a smoothed square; in fact that proof had already been given in [**11**], in the context of another investigation. Presumably other particular configurations can also be settled by analogous procedures. The following observations, however, seem worth noting:

Theorem A. *Let Ω be a convex domain that does not have the form indicated in Fig. 2, and for which there exists, for a value γ with $0 \leqslant \gamma < \pi/2$, a bounded variational solution of* (4), (2). *We suppose further that there exists a variational solution $u(x, y; B)$ of* (1), (2) *in Ω for all sufficiently small $B > 0$. Let Ω^0 be a disk in Ω of maximal radius r_0, and let $u^0(x, y; B)$ denote the variational solution of* (1), (2) *over Ω^0 Then there exists a constant $C^0 < 0$ such that, uniformly for all $(x, y) \in \Omega$ and $(\xi, \eta) \in \Omega^0$, there holds $u(x, y; B) - u^0(\xi, \eta; B) = C^0 B^{-1} + O(1)$ as $B \to 0$.*

Theorem B. *With Ω, Ω^0, C^0 as in Theorem* A, *let $\Omega^j \subset \Omega$ be a sequence of domains, such that $\lim_{j\to\infty}(|\Sigma^j|/|\Omega^j|) = 2/r_0$, $\Sigma^j = \partial\Omega^j$. We suppose that all Ω^j admit variational solutions $u^j(x, y; B)$ of* (1), (2) *for all sufficiently small $B > 0$, and we suppose also that all Ω^j admit variational solutions of* (4), (2), *which are bounded above and below, independent of j. Then there exist constants $C^j \to C^0$ such that, for all $(x, y) \in \Omega$ and $(\xi, \eta) \in \Omega^j$ there holds $u(x, y; B) - u^j(\xi, \eta; B) = C^j B^{-1} + O(1)$, as $B \to 0$.*

As an example we refer to the family $\{\Omega'\}$ obtained by smoothing of the square, as described at the beginning of this article, and choose for Ω (in the theorems) the particular domain $\Omega^{1/2}$. We then choose the Ω^j of Theorem B to be those domains of the family $\{\Omega'\}$, for which $t = 1/j$, $j > 2$. These choices satisfy the hypotheses of the theorems; the results of the theorems then show that the kind of discontinuous reversal of limiting behavior described in [**1**] does not occur. Thus, the result asserted in Conjecture above would be in that sense best possible.

Actually, somewhat more precise information is available: under the conditions of the two theorems, one finds $u(x, y) = \lambda B^{-1} + O(11)$ in Ω, with $0 < \lambda < 2/r_0$. There holds $u^0(x, y; B) = (2/r_0)B^{-1} + U^0(x, y) + O(B)$, where

$$U^0(x, y) = \frac{2(1 - \sin^3\gamma)}{3\cos^3\gamma} r_0 - \sqrt{r_0^2 \sec^2\gamma - (x^2 + y^2)}, \tag{7}$$

and $u^j(x, y; B) = \lambda^j B^{-1} + O(1)$, where $\lambda^j = ((2/r_0) - \varepsilon)B^{-1}$ and $\lim_{j\to\infty} \varepsilon^j = 0$. In the particular example just discussed, one has also $\varepsilon^j > 0$.

The stated theorems follow from (5) above, from Theorem 5 in [**1**], according to which $|\Sigma|/|\Omega| < |\Sigma^0|/|\Omega^0| = 2/r_0$, and from the hypothesis

$$\lim_{j\to\infty}(|\Sigma^j|/|\Omega^j|) = 2/r_0.$$

In obtaining the more precise information just indicated, we used also a result of [**2**].

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On Solutions with Fast Decay of Nonstationary Navier–Stokes System in the Half-Space

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Dedicated to Professor O. A. Ladyzhenskaya
on the occasion of her birthday

The Navier–Stokes initial-value problem in the half-space is studied. Employing the asymptotic expansion of solutions, as well as the idea of [**1**], we specify a class of solutions which decay in time more rapidly than observed in general. The class is described in terms of moments and correlations of velocity fields. The existence of such solutions is proved. The same initial-value problem with the Neumann boundary condition as in [**2**] is considered. A class of solutions with fast decay is specified, also in terms of conditions on moments and correlations which, however, are complementary to those on solutions to the standard Navier–Stokes system.

1. Introduction and the Main Results

We study the asymptotic behavior as $t \to \infty$ of solutions u to the Navier–Stokes initial-value problem in the half-space

$$\begin{aligned}
&\partial_t u + \nabla \cdot (u \otimes u) = \Delta u - \nabla p \quad (x \in D^n,\ t > 0),\\
&\nabla \cdot u = 0 \quad (x \in D^n,\ t \geqslant 0), \qquad\qquad\qquad \text{(NSD)}\\
&u|_{\partial D^n} = 0, \quad u|_{t=0} = a,
\end{aligned}$$

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where $n \geqslant 2$ is a fixed integer, $D^n = \mathbb{R}^n_+ = \{x = (x', x_n) = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$ is the upper half-space of $\mathbb{R}^n$, $u = (u_j)_{j=1}^n$ is an unknown velocity, $a = (a_j)_{j=1}^n$ is a given initial velocity, p is an unknown pressure, and the following notation was used:

$$\nabla = (\partial_1, \ldots, \partial_n), \quad \partial_j = \partial/\partial x_j, \quad \partial_t = \partial/\partial t,$$

$$u \otimes v = (u_j v_k)_{j,k=1}^n, \quad \nabla \cdot (u \otimes v) = \sum_{j=1}^n \partial_j (u_j v), \quad \nabla \cdot u = \sum_{j=1}^n \partial_j u_j.$$

We denote by $\boldsymbol{L}^q = (L^q(D^n))^n$, $1 \leqslant q \leqslant \infty$, the Lebesgue spaces of $\mathbb{R}^n$-valued functions. Using the Helmholtz decomposition (cf. [**3**])

$$\boldsymbol{L}^q = \boldsymbol{L}^q_\sigma \oplus \boldsymbol{L}^q_\pi, \quad 1 < q < \infty, \tag{1.1}$$

with

$$\begin{aligned} \boldsymbol{L}^q_\sigma &= \{u \in \boldsymbol{L}^q : \nabla \cdot u = 0, u_n|_{\partial D^n} = 0\}, \\ \boldsymbol{L}^q_\pi &= \{\nabla p \in \boldsymbol{L}^q : p \in L^q_{\text{loc}}(\overline{D}^n)\} \end{aligned} \tag{1.2}$$

and the associated bounded projection $P : \boldsymbol{L}^q \to \boldsymbol{L}^q_\sigma$, we write the problem (NSD) in the form

$$\partial_t u + Au = -P\nabla \cdot (u \otimes u) \quad (t > 0), \quad u(0) = a, \tag{1.3}$$

where $A = -P\Delta$ is the Stokes operator. As was shown in [**3**], the operator $-A$ generates a bounded analytic C_0-semigroup $\{e^{-tA}\}_{t \geqslant 0}$ on $\boldsymbol{L}^q_\sigma$ so that for $a \in \boldsymbol{L}^q_\sigma$ the function $v = e^{-tA}a$ gives a unique L^q-solution to the Stokes system $\partial_t v + Av = 0$ $(t > 0)$, $v(0) = a$. Thus, (1.3) is transformed into the integral equation

$$u(t) = e^{-tA}a - \int_0^t e^{-(t-s)A} P\nabla \cdot (u \otimes u)(s)\,ds. \tag{IED}$$

As is known, there are two notions of a solution to (NSD) or (IED), namely, a weak solution and a strong (regular) solution. A weak solution u exists globally in time for any given $a \in \boldsymbol{L}^2_\sigma$, satisfies (NSD) in the sense of distributions, and satisfies the *energy inequality*

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(s)\|_2^2\,ds \leqslant \|a\|_2^2 \quad \text{for all } t \geqslant 0 \tag{E}$$

(cf. [**4**] for details). Hereinafter, $\|\cdot\|_q$ denotes the norm in L^q. A strong solution exists locally in time, in general, for any $a \in \boldsymbol{L}^n_\sigma$ and globally in time if $\|a\|_n$ is sufficiently small. The solution is continuous in $t \geqslant 0$, takes the values in $\boldsymbol{L}^n_\sigma$, and satisfies the integral equation (IED). In fact, it is of class C^∞ in $x \in D^n$ and $t > 0$. Strong solutions on D^n are discussed in [**5**].

In this paper, we additionally suppose that the initial data a in the problem (NSD) satisfy the condition

$$\int_{D^n} (1 + y_n)|a(y)|dy < \infty. \tag{1.4}$$

To state our result for (NSD), we introduce the notation. For an initial velocity $a = (a', a_n)$, let $v = e^{-tA}a = (v', v_n)$ be the corresponding Stokes flow. The formula of [6] for the Stokes flow is written as follows:

$$v_n(t) = Ue^{-tB}[a_n - S \cdot a'], \quad v'(t) = e^{-tB}[a' + Sa_n] - Sv_n. \tag{1.5}$$

Hereinafter, $B = -\Delta$ is the Laplacian on D^n with the zero boundary condition, $S = (S_1, \dots, S_{n-1})$ are the Riesz transforms on $\mathbb{R}^{n-1}$ (cf. [**7, 8**]) which will be regarded as bounded linear operators on $L^q(D^n)$, $1 < q < \infty$, and U is a bounded linear operator on $L^q(D^n)$, $1 < q < \infty$, defined by the formula

$$\widetilde{Uf}(\xi', x_n) = |\xi'| \int_0^{x_n} e^{-(x_n - y)|\xi'|} \widetilde{f}(\xi', y)dy, \tag{1.6}$$

where

$$\widetilde{f}(\xi', x_n) = \int e^{-ix'\cdot\xi'} f(x', x_n)dx'$$

is the (tangential) Fourier transform on $\mathbb{R}^{n-1}$. The semigroup $\{e^{-tB}\}_{t\geqslant 0}$ is given by

$$e^{-tB}f = e^{t\Delta}f^*|_{D^n} \equiv E_t * f^*|_{D^n}, \quad E_t(x) = (4\pi t)^{-n/2}e^{-|x|^2/4t}, \tag{1.7}$$

where

$$\begin{aligned} f^*(x', x_n) &= \begin{cases} f(x', x_n), & x_n > 0, \\ -f(x', -x_n), & x_n < 0, \end{cases} \\ f_*(x', x_n) &= \begin{cases} f(x', x_n), & x_n > 0, \\ f(x', -x_n), & x_n < 0 \end{cases} \end{aligned} \tag{1.8}$$

for a function f on D^n. We use the same notation E_t to denote the heat kernel defined on different (Euclidean) spaces. Thus, for $x = (x', x_n) \in \mathbb{R}^n$ we write $E_t(x) = E_t(x_1) \cdots E_t(x_n) = E_t(x')E_t(x_n)$, etc. The kernel function $F_t = (F_t^1, \dots, F_t^{n-1})$ of the operator $e^{t\Delta}S$ on $\mathbb{R}^n$ is given by the formula

$$F_t(x) = \pi^{-1/2}E_t(x_n) \int_0^\infty s^{-1/2}\nabla' E_{s+t}(x')ds, \quad \nabla' = (\partial_1, \dots, \partial_{n-1}). \tag{1.9}$$

Using the above notation, we state the first main result.

Theorem 1.1. *Let a satisfy* (1.4). *Suppose that either* $a \in \boldsymbol{L}^2_\sigma$ *and* $n = 3, 4$ *or* $a \in \boldsymbol{L}^n_\sigma \cap \boldsymbol{L}^q$ *for all* $n < q < \infty$ *with sufficiently small* $\|a\|_n$. *Then there is a weak or strong solution* $u = (u', u_n)$ *to the problem* (NSD) *with* $u(0) = a$ *such that*

$$\|u(t)\|_q \leqslant c_q(1+t)^{-\frac{n}{2}(1+1/n-1/q)}. \tag{1.10}$$

Here, $1 < q \leqslant 2$ *for a weak solution and* $1 < q \leqslant \infty$ *for a strong solution. Moreover,*

(i) *the following convergence holds*:

$$t^{\frac{n}{2}(1+1/n-1/q)}\left\|u_n(t) - 2U\partial_n F_t \cdot \Big(\int_{D^n} y_n a'(y)dy + \int_0^\infty\!\!\int_{D^n} (u_n u')dyds\Big)\right\|_q \to 0, \tag{1.11}$$

$$t^{\frac{n}{2}(1+1/n-1/q)}\left\|u'(t) + 2[\partial_n E_t + SU\partial_n F_t]\Big(\int_{D^n} y_n a'(y)dy + \int_0^\infty\!\!\int_{D^n} (u_n u')dyds\Big)\right\|_q \to 0; \tag{1.12}$$

(ii) *the following inequality holds*

$$t^{\frac{n}{2}(1+1/n-1/q)}\|u(t)\|_q \geqslant c_q \quad \textit{for large } t > 0 \tag{1.13}$$

with some $c_q > 0$ *if and only if*

$$\int_{D^n} y_n a'(y)dy + \int_0^\infty\!\!\int_{D^n} (u_n u')dyds \neq 0. \tag{1.14}$$

Theorem 1.1 improves our previous result given in [**9**] and will be proved in Sec. 4. We note that if a belongs to some $\boldsymbol{L}^q_\sigma$ and satisfies (1.4), then

$$\int_{D^n} y_n a_n(y)dy = 0, \tag{1.15}$$

which was not known when [**9**] was published. Formula (1.15) will be used in the proof of the subsequent results and will be proved in Sec. 4 (cf. Lemma 4.1) on the basis of ideas of [**1**].

It seems possible to extend Theorem 1.1 to the case of $n/(n+1) \leqslant q \leqslant 1$ by introducing appropriate L^q-like spaces with exponents $q \leqslant 1$. This is done in [**10**] for the case of Navier–Stokes flows in $\mathbb{R}^n$ with the help of the theory of homogeneous Besov spaces. However, as was noticed in [**11**], the use of Besov spaces seems not so effective in dealing with flows in the half-space. Thus, we would need to employ another kind of spaces. This problem will be discussed in the forthcoming paper [**12**]. We know nothing as to whether there exist examples of flows u for which (1.14) holds. To find such examples, we need systematic study on weighted L^q-estimates for solutions to the problem (NSD).

Estimates of this kind are studied in detail in [**13**] for flows in an exterior domain. The case of the half-space will be examined in the forthcoming paper [**12**].

We give a class of solutions u for which (1.14) (and so (1.13)) breaks down. To do so, we prepare some terminology. The notion below is inspired by [**1**] where solutions with fast decay to the Cauchy problem were treated.

Definition 1.2. A vector field a on D^n satisfies the *tangential parity condition* if

(a) for $j \leqslant n-1$ and $k \leqslant n-1$ with $j \neq k$, $a_j(x', x_n)$ is odd in x_j and is even in x_k,

(b) $a_n(x', x_n)$ is even in each component of x'.

In the above terminology, our second main results are stated as follows.

Theorem 1.3. *Let a satisfy* (1.4), *and let u be the corresponding weak or strong solution given in Theorem* 1.1. *If a satisfies the tangential parity condition, so does u. In this case,*

$$\lim_{t\to\infty} t^{\frac{n}{2}(1+1/n-1/q)}\|u(t)\|_q = 0 \tag{1.16}$$

for all possible values of q.

We note that the tangential parity condition for a and u implies

$$\int_{D^n} y_n a'(y)dy = \int_{D^n} (u_n u')(y,s)dy = 0, \tag{1.17}$$

so (1.14) breaks down and (1.16) is obtained from (1.11) and (1.12). Therefore, in order to prove Theorem 1.3, it suffices to establish the existence of solutions satisfying the tangential parity condition which admit the expansions (1.11) and (1.12).

Definition 1.2 is inspired by [**1**], in which is given a sufficient condition for a Navier–Stokes flow u in $\mathbb{R}^n$ to satisfy

$$\int y_j a_k(y)dy = 0, \quad \int (u_j u_k)(y,s)dy = \lambda(s)\delta_{jk}, \quad j,k = 1,\ldots,n. \tag{1.17$'$}$$

(Hereinafter, integration is performed over $\mathbb{R}^n$ unless otherwise specified.) In view of a result of [**14**], the condition (1.17′) is sufficient for a flow u on $\mathbb{R}^n$ to satisfy (1.16). It should be noticed that our necessary and sufficient condition (1.14) involves only the products of u_n and u', while the corresponding condition given in [**14**] for flows in $\mathbb{R}^n$ involves the products of *all components* of u. For this reason, we need in this paper only the *tangential* parity condition to find flows with fast decay in the half-space. Indeed, as shown in [**1**], for flows in $\mathbb{R}^n$ one needs both a parity condition and a kind of symmetry condition in *all* directions to specify a class of solutions satisfying (1.17′) (cf. Definition 1.6 below).

If u satisfies the tangential parity condition, we can reasonably expect that

$$\|u(t)\|_q \leqslant c_q(1+t)^{-\frac{n}{2}(1+2/n-1/q)} \tag{1.18}$$

for suitable values of q, under additional assumptions on a. The existence of solutions satisfying (1.18) was proved in **[1, 10, 20]** for the Cauchy problem, in which case one can even replace $2/n$ by $3/n$ under an additional assumption on a. However, in our case, the question on the validity of (1.18) remains open since there are no results regarding the second-order asymptotic expansion of solutions on D^n even when they satisfy the tangential parity condition. We discuss (1.18) in the forthcoming paper, employing the method given in **[13]**.

In this paper, we supplement the lower bound result (ii) of Theorem 1.1. Let a Stokes flow $u_0(t) = e^{-tA}a$ satisfy the inequality $\|u_0(t)\|_q \leqslant c_q(1+t)^{-\frac{n}{2}(1-1/q)}$. Then the results of **[3, 5]** ensure the existence of a weak or strong solution u to the problem (NSD) such that

$$\|u(t)\|_q \leqslant c_q(1+t)^{-\frac{n}{2}(1-1/q)} \tag{1.19}$$

for all possible values of q. In this situation, we prove the following assertion.

Theorem 1.4. *Let u be a weak or strong solution to the problem* (NSD). *For large $t>0$ we have*

$$t^{\frac{n}{2}(1-1/q)}\|u(t)\|_q \geqslant c_q \quad \textit{if and only if} \quad t^{\frac{n}{2}(1-1/q)}\|u_0(t)\|_q \geqslant c'_q$$

for all possible values of q.

A similar result holds for flows in $\mathbb{R}^n$, as was shown in **[14]**. In **[14]**, there are some examples of initial data a on $\mathbb{R}^n$ for which the corresponding $u_0(t)$ satisfy the inequality $\|u_0(t)\|_2 \geqslant ct^{-n/4}$. However, for flows in D^n we know nothing about the existence of an initial velocity a satisfying the estimate $c \leqslant t^{n/4}\|u_0(t)\|_2 \leqslant c'$.

Consider the Navier–Stokes system with the Neumann boundary condition

$$\begin{aligned}
&\partial_t u + \nabla\cdot(u\otimes u) = \Delta u - \nabla p \quad (x\in D^n,\ t>0),\\
&\nabla\cdot u = 0 \quad (x\in D^n,\ t\geqslant 0),\\
&\partial_n u'|_{\partial D^n} = 0, \quad u_n|_{\partial D^n} = 0, \quad u|_{t=0} = a.
\end{aligned} \tag{NSN}$$

This problem was studied in **[2]** in bounded domains, and we know that the mathematical structure of (NSN) is simpler than that of (NSD). Indeed, let $\{e^{-tA'}\}_{t\geqslant 0}$ be the semigroup generated by $-A' = \Delta$ with the Neumann boundary condition as described in (NSN). In the notation (1.8), we have

$$\begin{aligned}
(e^{-tA'}a)' &= e^{t\Delta}a'_*|_{D^n} = E_t * a'_*|_{D^n},\\
(e^{-tA'}a)_n &= e^{t\Delta}a^*_n|_{D^n} = E_t * a^*_n|_{D^n}.
\end{aligned} \tag{1.20}$$

Then (NSN) is written as follows:

$$\partial_t u + A'u = -P\nabla\cdot(u\otimes u) \ (t>0),$$

and the associated integral equation takes the form

$$u(t) = e^{-tA'}a - \int_0^t e^{-(t-s)A'} P\nabla \cdot (u \otimes u)(s)ds. \tag{IEN}$$

We will see in Sec. 3 that if u solves (NSN), then $v = (u'_*, u^*_n)$ solves the Cauchy problem for the Navier–Stokes system. In this way, the existence problem of weak or strong solutions to (NSN) or (IEN) is reduced to that of the Cauchy problem, and we can freely invoke known results on weak and strong solutions to the Cauchy problem. Thus, we can apply our results of [**15, 14**] to prove the following assertion.

Theorem 1.5. *Let*

$$F_{\ell,jk}(x,t) = F^1_{\ell,jk}(x,t) + F^2_{\ell,jk}(x,t) \equiv \partial_\ell E_t(x)\delta_{jk} + \int_0^\infty \partial_j \partial_k \partial_\ell E_{s+t}(x)ds \tag{1.21}$$

for $x \in D^n$, $t > 0$, *and* $j, k, \ell = 1, \dots, n$. *Suppose that*

$$c_0 = \sup(1+|y|)^{n+1}|a(y)| < \infty, \quad c_1 = \int_{D^n} |y||a(y)|dy < \infty, \tag{1.4'}$$

and $c_0 + c_1$ *is sufficiently small.*

(i) *There exists a strong solution u to the problem* (NSN) *satisfying*

$$\lim_{t\to\infty} t^{\frac{n}{2}(1+1/n-1/q)} \Big\| u_j(t) + 2(\nabla' E_t) \cdot \int_{D^n} y' a_j(y)dy$$
$$+ 2\sum_{k,\ell=1}^{n-1} F_{\ell,jk}(\cdot,t) \int_0^\infty\!\!\int_{D^n} (u_k u_\ell)dyds + 2F^2_{n,jn}(\cdot,t)\int_0^\infty\!\!\int_{D^n}(u_n u_n)dyds\Big\|_q = 0 \tag{1.22}$$

for $j = 1, \dots, n-1$ *and*

$$\lim_{t\to\infty} t^{\frac{n}{2}(1+1/n-1/q)} \Big\| u_n(t) + 2\sum_{k,\ell=1}^{n-1} F^2_{\ell,nk}(\cdot,t) \int_0^\infty\!\!\int_{D^n} (u_k u_\ell)dyds$$
$$+ 2F_{n,nn}(\cdot,t)\int_0^\infty\!\!\int_{D^n}(u_n u_n)dyds\Big\|_q = 0, \tag{1.23}$$

where $1 \leqslant q \leqslant \infty$. *For* $1 \leqslant q \leqslant 2$ *instead of* $1 \leqslant q \leqslant \infty$, *the same result holds for weak solutions corresponding to an arbitrary* $a \in \boldsymbol{L}^2_\sigma$ *such that*

$$\int_{D^n} (1+|y|)|a(y)|dy < \infty.$$

(ii) *Let u be a weak or strong solution to the problem* (NSN) *treated in* (i). *Then*

$$\lim_{t\to\infty} t^{\frac{n}{2}(1+1/n-1/q)}\|u(t)\|_q = 0 \tag{1.24}$$

if and only if there is $\lambda \geqslant 0$ such that

$$\left(\int_{D^n} (y' \otimes a')dy, \int_0^\infty\!\!\int_{D^n} (u' \otimes u')dyds, \int_0^\infty\!\!\int_{D^n} (u_n u_n)dyds \right) = (0, \lambda I', \lambda), \tag{1.25}$$

where I' is the identity matrix of order $n-1$.

(iii) *Let u be a weak or strong solution to the problem* (NSN) *treated in* (i). *Then*

$$t^{\frac{n}{2}(1+1/n-1/q)}\|u(t)\|_q \geqslant c_q \quad \textit{for large } t > 0 \tag{1.26}$$

with a constant $c_q > 0$ if and only if for all $\lambda \geqslant 0$

$$\left(\int_{D^n} (y' \otimes a')dy, \int_0^\infty\!\!\int_{D^n} (u' \otimes u')dyds, \int_0^\infty\!\!\int_{D^n} (u_n u_n)dyds \right) \neq (0, \lambda I', \lambda). \tag{1.27}$$

We note that $q = 1$ is admitted in Theorem 1.5. Moreover, the condition (1.25) is in striking contrast to (1.14). To specify a class of solutions satisfying (1.24), we introduce the following definition.

Definition 1.6. Let u be a vector field on D^n, and let $v = (u'_*, v_n^*)$. We say that u satisfies

(i) the *parity condition* if each v_j is odd in x_j and is even in each of the other variables,

(ii) the *cyclic symmetry condition* if

$$v_1(x_1, \ldots, x_n) = v_2(x_n, x_1, \ldots, x_{n-1}) = \cdots = v_n(x_2, x_3, \ldots, x_n, x_1).$$

Definition 1.6 for v is due to Brandolese [**1**]. If our solution u to the problem (NSN) satisfies the conditions in Definition 1.6, then

$$\left(\int_{D^n} (y' \otimes a')dy, \int_{D^n} (u' \otimes u')dy, \int_{D^n} (u_n u_n)dy \right) = (0, \lambda(t)I', \lambda(t))$$

for some $\lambda(t)$ such that $0 \leqslant \lambda(t) \leqslant c(1+t)^{-(n+2)/2}$. Therefore, u has the decay property (1.24). We further prove the following assertion.

Theorem 1.7. *Suppose that a satisfies the parity condition and the cyclic symmetry condition Then the corresponding weak or strong solution u to the problem* (NSN) *also satisfies the same conditions for each $t \geqslant 0$. Furthermore, if a satisfies the additional condition*

$$\sup(1+|y|)^{n+3}|a(y)| < \infty, \quad \int_{D^n} |y|^3 |a(y)|dy < \infty, \tag{1.28}$$

then the corresponding strong solution satisfies the estimate

$$\|u(t)\|_q \leqslant c_q(1+t)^{-\frac{n}{2}(1+3/n-1/q)}, \quad 1 \leqslant q \leqslant \infty. \tag{1.29}$$

The estimate (1.29) holds also for a weak solution with $1 \leqslant q \leqslant 2$ if

$$\int_{D^n} (1+|y|)^3 |a(y)| dy < \infty, \quad \int_{D^n} (1+|y|)^2 |a(y)|^2 dy < \infty.$$

The proof is technical and is omitted here. Theorems 1.5 and 1.7 will be proved in Sec. 3.

Finally, we prove Theorem 1.1 in Sec. 4. We deduce (1.11)–(1.12) and (1.13)–(1.14) after preparing necessary lemmas. Expansions (1.11) and (1.12), as well as their proofs, improve those given in [**9**]. The proof of (1.10) is given in [**9**] and is omitted in this paper.

2. Proof of Theorems 1.3 and 1.4

We begin by establishing the following assertion.

Lemma 2.1. *Suppose that a satisfies the tangential parity condition. If both $u(t)$ and $v(t)$ satisfy the tangential parity condition for all $t \geqslant 0$, then the function*

$$w(t) = e^{-tA}a - \int_0^t e^{-(t-s)A} P\nabla \cdot (u \otimes v)(s) ds$$

also satisfies the tangential parity condition for all $t \geqslant 0$.

Proof. First we show that if a satisfies the tangential parity condition, so does the function $w^0(t) = e^{-tA}a$. We note that the operators U and e^{-tB} in (1.5) preserve the parity in tangential directions. Now $a_n - S \cdot a'$ is even in each component of x'. Hence $w_n^0 = Ue^{-tB}[a_n - S \cdot a']$ is even in each component of x'. This implies that $S_j w_n^0$ is odd in x_j and is even in x_k for $k \neq j$, $k \leqslant n-1$. On the other hand, $a_j + S_j a_n$ is odd in x_j and is even in x_k for $k \neq j$, $k \leqslant n-1$. Therefore, if $j \leqslant n-1$, then $w_j^0 + S_j w_n^0 = e^{-tB}[a_j + S_j a_n]$ is odd in x_j and is even in x_k for $k \neq j$, $k \leqslant n-1$. Thus, $w^0(t)$ satisfies the tangential parity condition for all $t \geqslant 0$.

We show that if $u(t)$ and $v(t)$ satisfy the tangential parity condition for all $t \geqslant 0$ and $\nabla \cdot u = \nabla \cdot v = 0$, $u_n|_{\partial D^n} = v_n|_{\partial D^n} = 0$, then the function

$$w^1(t) = -\int_0^t e^{-(t-s)A} P\nabla \cdot (u \otimes v)(s) ds \tag{2.1}$$

satisfies the tangential parity condition for all $t \geqslant 0$. Let N be Green's function of the homogeneous Neumann problem for $-\Delta$ on D^n. We easily see that

$$Nf = Q_n f_*|_{D^n}, \tag{2.2}$$

where Q_n means the convolution by the fundamental solution of the operator $-\Delta$ on $\mathbb{R}^n$. By definition,

$$P\nabla \cdot (u \otimes v) = \nabla \cdot (u \otimes v) + \nabla N[\partial_j \partial_k (u_j v_k)]. \tag{2.3}$$

(Hereinafter, we sometimes employ the summation convention.) We easily see that $\nabla \cdot (u \otimes v)$ satisfies the tangential parity condition. Furthermore, $\partial_j \partial_k (u_j v_k)$ is even in each component of x'. By (2.2), $N[\partial_j \partial_k (u_j v_k)]$ is also even in each component of x'. Hence for $\ell \leqslant n-1$ the derivative $\partial_\ell N[\partial_j \partial_k (u_j v_k)]$ is odd in x_ℓ and is even in x_s for $s \neq \ell$, $s \leqslant n-1$, and $\partial_n N[\partial_j \partial_k (u_j v_k)]$ is even in each component of x'. By (2.3), $P\nabla \cdot (u \otimes v)$ satisfies the tangential parity condition. Hence the function $w^1(t)$ defined by (2.1) satisfies the tangential parity condition. □

Proof of Theorem 1.3. Let $\boldsymbol{L}^q_\tau$, $1 < q < \infty$, be the space of L^q-valued vector fields on D^n satisfying the tangential parity condition. We first establish the *Helmholtz decomposition of vector fields satisfying the tangential parity condition*, i.e.,

$$\boldsymbol{L}^q_\tau = (\boldsymbol{L}^q_\sigma \cap \boldsymbol{L}^q_\tau) \oplus (\boldsymbol{L}^q_\pi \cap \boldsymbol{L}^q_\tau), \quad 1 < q < \infty, \tag{2.4}$$

with $\boldsymbol{L}^q_\sigma$ and $\boldsymbol{L}^q_\pi$ given in (1.1) and (1.2). To do so, we note that the space $\boldsymbol{L}^q_\tau \cap \boldsymbol{C}^\infty_c(D^n)$ is dense in $\boldsymbol{L}^q_\tau$. Indeed, let $m \in \mathbb{N}$, $\chi_m(t) = \chi_{[-m,m]}(t)$ $(t \in \mathbb{R})$, $\eta_m(s) = \chi_{[m^{-1},m]}(s)$ $(s > 0)$, where χ_E is the indicator function of a set $E \subset \mathbb{R}$. If $v \in \boldsymbol{L}^q_\tau$, then the function $v_m(x) = \chi_m(x_1) \cdots \chi_m(x_{n-1}) \eta_m(x_n) v(x', x_n)$ belongs to $\boldsymbol{L}^q_\tau$, has compact support in D^n, and satisfies $\|v_m - v\|_q \to 0$ as $m \to \infty$. Applying the standard radial mollification to each v_m, we get a sequence $\{\overline{v}_m\}$ of vector fields in $\boldsymbol{L}^q_\tau \cap \boldsymbol{C}^\infty_c(D^n)$ such that $\|\overline{v}_m - v\|_q \to 0$ as $m \to \infty$.

Fix $v \in \boldsymbol{L}^q_\tau \cap \boldsymbol{C}^\infty_c(D^n)$ and consider the homogeneous Neumann problem

$$-\Delta p = \nabla \cdot v \quad \text{on } D^n, \quad \partial_n p|_{\partial D^n} = 0.$$

Since $\nabla \cdot v$ is even in each component of x', so is the function

$$p = N[(\nabla \cdot v)] = Q_n[(\nabla \cdot v)_*]|_{D^n}.$$

Therefore, each of the functions $\partial_j p = \partial_j N[(\nabla \cdot v)]$, $j = 1, \ldots, n-1$, is odd in x_j is and even in x_k if $k \neq j$, $k \leqslant n-1$; and $\partial_n p$ is even in each component of x'. Furthermore, a direct calculation using (2.2) gives

$$N[(\nabla \cdot v)] = \sum_{j=1}^{n-1} \partial_j Q_n((v_j)_*)|_{D^n} + \partial_n Q_n((v_n)^*)|_{D^n}.$$

Using the Calderón–Zygmund theory on singular integrals (cf. [**7, 8**]), we obtain the estimate

$$\|\nabla N[(\nabla \cdot v)]\|_q \leqslant c_q \|v\|_q, \quad 1 < q < \infty.$$

Hence the operator $P_\tau v = v + \nabla N[(\nabla \cdot v)]$ defined on $\boldsymbol{L}^q_\tau \cap \boldsymbol{C}^\infty_c(D^n)$ extends uniquely to a bounded projection from $\boldsymbol{L}^q_\tau$ to itself. Actually, P_τ is the restriction to $\boldsymbol{L}^q_\tau$ of the projection $P : \boldsymbol{L}^q \to \boldsymbol{L}^q_\sigma$ associated with the decomposition (1.1). Thus, the decomposition (2.4) is proved.

By Ukai's formula (1.5), the Stokes semigroup $\{e^{-tA}\}_{t \geqslant 0}$ defines a bounded analytic C_0-semigroup on $\boldsymbol{L}^q_\sigma \cap \boldsymbol{L}^q_\tau$ (cf. [**16**] for the definition of the bounded analytic C_0-semigroup). Moreover, Lemma 2.1 and the estimates in [**3, 5**] show that the standard iteration argument for (IED) works within the Banach scale $\boldsymbol{L}^r_\sigma \cap \boldsymbol{L}^r_\tau$, $1 < r < \infty$, and gives us a strong solution to (IED) satisfying the tangential parity condition whenever the initial value a is small in $\boldsymbol{L}^n_\sigma \cap \boldsymbol{L}^n_\tau$. As was shown in [**9**], (1.4) implies that our strong solution satisfies (1.10) for $1 < q \leqslant \infty$. By Theorem 1.1, we conclude that this solution satisfies (1.16).

To get weak solutions for $n = 3, 4$, we have to solve, for each $m \in \mathbb{N}$, the integral equation

$$u_m(t) = e^{-tA} a_m - \int_0^t e^{-(t-s)A} P \nabla \cdot (\overline{u}_m \otimes u_m)(s) ds \tag{2.5}$$

in $C([0,T] : \boldsymbol{L}^2_\tau \cap D(A^{1/2}))$, where $a_m = (I + m^{-1}A)^{-1} a$ and $\overline{u}_m = (I + m^{-1}A)^{-[n/2]-1} u_m$ (cf. [**17**] for details). Thus, we need to show that $a_m \in \boldsymbol{L}^2_\tau$ if $a \in \boldsymbol{L}^2_\tau$. It suffices to verify that $(I + m^{-1}A)^{-1}$ is bounded from $\boldsymbol{L}^2_\tau$ to itself. But this is obvious from the well-known formula (cf. [**16**])

$$(b', b_n) \equiv (1 + m^{-1}A)^{-1} a = m \int_0^\infty e^{-mt} e^{-tA} a dt.$$

Our weak solution u is obtained as the limit of (a subsequence of) $\{u_m\}$ as $m \to \infty$, and so u satisfies the tangential parity condition.

By (1.4), our weak solution u satisfies (1.10) for $1 < q \leqslant 2$. By Theorem 1.1, the solution u satisfies (1.16). $\square$

Remark. A velocity field a satisfying the tangential parity condition and the conditions

$$\nabla \cdot a = 0, \quad a_n|_{\partial D^n} = 0 \tag{2.6}$$

is constructed as follows. Choose $b = (b_1, \ldots, b_n)$ such that $b \in \boldsymbol{L}^r(\mathbb{R}^n)$ for some $1 < r < \infty$ and each b_j is odd in x_j and is even in x_k, $k \neq j$. We define $v = (v_1, \ldots, v_n)$ by the formula

$$\widehat{v}_j(\xi) \equiv \int e^{-ix\,\xi} c_j(x) dx = \Big(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2}\Big) \widehat{b}_k(\xi), \quad j = 1, \ldots, n,$$

in terms of the Fourier transform. Then $v \in \boldsymbol{L}^r(\mathbb{R}^n)$ by the Calderón-Zygmund theory (cf. [**7**, **8**]), $\nabla \cdot v = 0$ on $\mathbb{R}^n$, and each $\widehat{v}_j$ is odd in ξ_j and is even in each of the other variables. Hence $a = v|_{D^n}$ satisfies the tangential parity condition. To verify (2.6), we take $\psi \in C_c^1(\overline{D}^n)$ and consider $\varphi = \psi_*$ which is in $W^{1,r'}(\mathbb{R}^n)$, $1/r' = 1 - 1/r$. The parity condition for each v_j implies that $v_j(x)(\partial_j\varphi)(x)$ is even in x_n for each $j = 1, \ldots, n$. Since $\nabla \cdot v = 0$ in $\mathbb{R}^n$, we get

$$0 = \int v(x) \cdot (\nabla\varphi)(x)dx = 2 \int_{D^n} a(y) \cdot (\nabla\psi)(y)dy.$$

This proves (2.6).

The above construction is typical in the following sense. Let $a \in \boldsymbol{L}_\sigma^r$ for some $1 < r < \infty$ satisfy the tangential parity condition. Define $b = (a'_*, a_n^*)$. Each b_j is odd in x_j and is even in x_k whenever $k \neq j$. Since a satisfies (2.6), for $\psi \in C_c^1(\mathbb{R}^n)$ we have

$$\int b \cdot \nabla\psi dx = \int_{D^n} a(x', x_n) \cdot \nabla\psi(x', x_n)dx + \int_{D^n} a(x', x_n) \cdot \nabla\varphi(x', x_n)dx = 0,$$

with $\varphi(x', x_n) = \psi(x', -x_n) \in C_c^1(\mathbb{R}^n)$. Hence $\nabla \cdot b = 0$ in $\mathbb{R}^n$.

Proof of Theorem 1.4. Let u satisfy (1.19), and let $u_0(t) = e^{-tA}a$. Our argument of Sec. 4 for proving (1.11) and (1.12) can be modified to deduce

$$\left\| u(t) - u_0(t) + 2 \begin{pmatrix} \partial_n E_t + SU\partial_n F_t \cdot \\ -U\partial_n F_t \cdot \end{pmatrix} \int_0^{t/2}\int_{D^n} (u_n u')(y, s)dyds \right\|_q$$
$$\leqslant c_q t^{-\frac{n}{2}(1+1/n-1/q)} \log(1+t)$$

and

$$\left\| \begin{pmatrix} \partial_n E_t + SU\partial_n F_t \cdot \\ -U\partial_n F_t \cdot \end{pmatrix} \int_0^{t/2}\int_{D^n} (u_n u')(y, s)dyds \right\|_q \leqslant c_q t^{-\frac{n}{2}(1+1/n-1/q)} \log(1+t).$$

(The function $\log(1+t)$ appears only if $n = 2$.) Therefore,

$$\|u(t) - u_0(t)\|_q \leqslant c_q t^{-\frac{n}{2}(1+1/n-1/q)} \log(1+t). \tag{2.7}$$

Suppose that $t^{\frac{n}{2}(1-1/q)}\|u_0(t)\|_q \geqslant c_q > 0$ for large $t > 0$. A direct calculation using (2.7) gives

$$t^{\frac{n}{2}(1-1/q)}\|u(t)\|_q \geqslant t^{\frac{n}{2}(1-1/q)}\|u_0(t)\|_q - t^{\frac{n}{2}(1-1/q)}\|u(t) - u_0(t)\|_q \geqslant c'_q > 0$$

for large $t > 0$. Conversely, if $t^{\frac{n}{2}(1-1/q)}\|u(t)\|_q \geqslant c_q > 0$ for large $t > 0$, then (2.7) gives

$$t^{\frac{n}{2}(1-1/q)}\|u_0(t)\|_q \geqslant t^{\frac{n}{2}(1-1/q)}\|u(t)\|_q - t^{\frac{n}{2}(1-1/q)}\|u_0(t) - u(t)\|_q \geqslant c'_q > 0$$

for large $t > 0$. □

3. Proof of Theorems 1.5 and 1.7

In this section, we consider (NSN). A solution to the problem (NSN) on $[0, \infty)$ is a function u such that

$$-\int_0^T \psi'\langle u, \varphi\rangle_{D^n} dt + \int_0^T \psi\langle \nabla u, \nabla\varphi\rangle_{D^n} dt - \int_0^T \psi\langle u \otimes u, \nabla\varphi\rangle_{D^n} dt = \psi(0)\langle a, \varphi\rangle_{D^n} \tag{3.1}$$

for all $T > 0$, $\psi \in C^1([0,T] : \mathbb{R})$ such that $\psi(T) = 0$, and $\varphi \in C_c^1(\overline{D}^n)$ such that $\nabla \cdot \varphi = 0$ in D^n and $\varphi_n|_{\partial D^n} = 0$ (cf. **[18, 4, 19]** for the definition of a solution to (NSD) and (NSN)). Hereinafter, $\langle \cdot, \cdot \rangle_{D^n}$ denotes the duality pairing between distributions and functions on D^n, while $\langle \cdot, \cdot \rangle$ is the duality pairing between distributions and functions on $\mathbb{R}^n$.

Define $v = (u'_*, u_n^*)$ for a solution $u = (u', u_n)$ to (NSN). If $\varphi \in C^1_{0,\sigma}(\mathbb{R}^n)$, then $\langle v, \varphi\rangle = 2\langle u, \varphi\rangle_{D^n}$, $\langle \nabla v, \nabla\varphi\rangle = 2\langle \nabla u, \nabla\varphi\rangle_{D^n}$, and $\langle v \otimes v, \nabla\varphi\rangle = 2\langle u \otimes u, \nabla\varphi\rangle_{D^n}$, where $\varphi = (\varphi', \varphi_n)$ is a vector field on D^n defined by the formulas

$$\varphi'(x', x_n) = \frac{1}{2}[\varphi'(x', x_n) + \varphi'(x', -x_n)],$$
$$\varphi_n(x', x_n) = \frac{1}{2}[\varphi_n(x', x_n) - \varphi_n(x', -x_n)].$$

We note that $\nabla \cdot \varphi = 0$ in D^n, $\varphi_n|_{\partial D^n} = 0$, and (3.1) implies

$$-\int_0^T \psi'\langle v, \varphi\rangle dt + \int_0^T \psi\langle \nabla v, \nabla\varphi\rangle dt - \int_0^T \psi\langle v \otimes v, \nabla\varphi\rangle dt = \psi(0)\langle b, \varphi\rangle \tag{3.2}$$

for all $T > 0$ and $\psi \in C^1([0,T] : \mathbb{R})$ such that $\psi(T) = 0$. Since $\nabla \cdot v = 0$ in $\mathbb{R}^n$, the function v solves the Cauchy problem for the Navier–Stokes system.

Conversely, suppose that v solves the Cauchy problem and has the form $v = (u'_*, u_n^*)$ in terms of some vector field u on D^n. It is easy to see that $\nabla \cdot u = 0$ in D^n and $u_n|_{\partial D^n} = 0$, i.e.,

$$\int_{D^n} u \cdot \nabla\psi dx = 0 \quad \text{for all } \psi \in C_c^1(\overline{D}^n).$$

For $\varphi \in C_c^1(\overline{D}^n)$ such that $\nabla \cdot \varphi = 0$ in D^n and $\varphi_n|_{\partial D^n} = 0$ we set $\varphi = (\varphi'_*, \varphi_n^*)$. A simple calculation shows $\varphi \in C_c^{0,1}(\mathbb{R}^n)$, $\nabla \cdot \varphi = 0$ in $\mathbb{R}^n$, and

$$\langle v, \varphi\rangle = 2\langle u, \varphi\rangle_{D^n}, \quad \langle \nabla v, \nabla\varphi\rangle = 2\langle \nabla u, \nabla\varphi\rangle_{D^n}, \quad \langle v \otimes v, \nabla\varphi\rangle = 2\langle u \otimes u, \nabla\varphi\rangle_{D^n}.$$

By (3.2), u satisfies (3.1), i.e., u solves the problem (NSN).

Now, let $b = (b', b_n)$ be a vector field on $\mathbb{R}^n$ such that $\nabla \cdot b = 0$, b' is even in x_n, and b_n is odd in x_n. If $v = (v', v_n)$ is a solution to the Cauchy problem with $v(0) = b$ obtained by the standard iteration scheme for constructing weak or strong solutions, then v' is even in x_n and v_n is odd in x_n for each fixed $t > 0$. This fact is verified owing to the relation

$$\widehat{v}_j(\xi,t) = e^{-t|\xi|^2}\widehat{b}_j(\xi) - i\int_0^t \xi_\ell e^{-(t-s)|\xi|^2}\Big(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\Big)\widehat{(v_k v_\ell)}(\xi,s)ds,\ j = 1,\dots,n,$$

which is the Fourier transform of the integral equation associated with the Cauchy problem for the Navier–Stokes system.

PROOF OF THEOREM 1.5. Given a satisfying (1.4′), let $b = (a'_*, a^*_n)$. Then $\nabla\cdot b = 0$,

$$c'_0 = \sup(1+|y|)^{n+1}|b(y)| = c_0 < \infty, \quad c'_1 = \int |y||b(y)|dy = 2c_1 < \infty,$$

and we may assume that $c'_0 + c'_1$ is sufficiently small. The results of [**15**, **20**] imply the existence of a strong solution v to the Cauchy problem such that for $j = 1,\dots,n$ and $1 \leqslant q \leqslant \infty$ we have

$$\begin{aligned}\lim_{t\to\infty} t^{\frac{n}{2}(1+1/n-1/q)}\Big\| v_j(t) &+ (\partial_k E_t)(\cdot)\int y_k b_j(y)dy \\ &+ F_{\ell,jk}(\cdot,t)\int_0^\infty\!\!\int (v_k v_\ell)dyds\Big\|_q = 0,\end{aligned} \tag{3.3}$$

where $F_{\ell,jk}(x,t) = F^1_{\ell,jk}(x,t) + F^2_{\ell,jk}(x,t)$ is given in (1.21). As was proved in [**15**], (3.3) is also valid for weak solutions with $1 \leqslant q \leqslant \infty$ replaced by $1 \leqslant q \leqslant 2$ provided that $b \in \boldsymbol{L}^2(\mathbb{R}^n)$, $\nabla\cdot b = 0$, and $\int(1+|y|)|b(y)|dy < \infty$. Since b' and v' are even in x_n and b_n and v_n are odd in x_n, assertion (i) follows from (3.3). Indeed, we have

$$\int (y'\otimes b')dy = 2\int_{D^n}(y'\otimes a')dy, \quad \int y' b_n(y)dy = \int y_n b'(y)dy = \int (v_n v')dy = 0,$$

$$\int (v'\otimes v')dy = 2\int_{D^n}(u'\otimes u')dy, \quad \int y_n b_n(y)dy = 2\int_{D^n} y_n a_n(y)dy = 0.$$

The last result is valid in view of (1.15). Thus, (3.3) can be written in the form (1.22)–(1.23). This proves (i).

To prove (ii), we set

$$a_{jk} = 2\int_{D^n} y_j a_k(y)dy, \quad c_{jk} = 2\int_0^\infty\!\!\int_{D^n}(u_j u_k)dyds$$

and note that

$$\left\|\sum_{\ell=1}^{n-1}(\partial_\ell E_t)(\cdot)a_{\ell j} + \sum_{k,\ell=1}^{n-1} F_{\ell,jk}(\cdot,t)c_{k\ell} + F^2_{n,jn}(\cdot,t)c_{nn}\right\|_q = c_q t^{-\frac{n}{2}(1+1/n-1/q)},$$

$$\left\|\sum_{k,\ell=1}^{n-1} F^2_{\ell,nk}(\cdot,t)c_{k\ell} + F_{n,nn}(\cdot,t)c_{nn}\right\|_q = c'_q t^{-\frac{n}{2}(1+1/n-1/q)} \tag{3.4}$$

since the functions on the left-hand sides are written as $t^{-(n+1)/2}K(xt^{-1/2})$, where K is bounded integrable and uniformly continuous on $\mathbb{R}^n$. Hence it suffices to find conditions on the coefficients a_{jk} and c_{jk} regarded as functions of x under which

$$\sum_{\ell=1}^{n-1}(\partial_\ell E_t)(x)a_{\ell j} + \sum_{k,\ell=1}^{n-1} F_{\ell,jk}(x,t)c_{k\ell} + F^2_{n,jn}(x,t)c_{nn} = 0, \quad j = 1,\ldots,n-1,$$

$$\sum_{k,\ell=1}^{n-1} F^2_{\ell,nk}(x,t)c_{k\ell} + F_{n,nn}(x,t)c_{nn} = 0. \tag{3.5}$$

Taking the Fourier transform of (3.5), we find

$$|\xi|^2\sum_{\ell=1}^{n-1}\xi_\ell(a_{\ell j}+c_{j\ell}) - \xi_j\sum_{k,\ell=1}^{n-1}\xi_k\xi_\ell c_{k\ell} - \xi_j\xi_n^2 c_{nn} = 0, \quad j = 1,\ldots,n-1,$$

$$\sum_{k,\ell=1}^{n-1}\xi_k\xi_\ell c_{k\ell} - |\xi'|^2 c_{nn} = 0, \tag{3.5'}$$

for all $\xi \in \mathbb{R}^n$. The second equation of (3.5′) implies that $c_{jj} = c_{nn} = \lambda$ ($j = 1,\ldots,n-1$) and $c_{k\ell} = \lambda\delta_{k\ell}$ ($k,\ell = 1,\ldots,n-1$) for some $\lambda \geqslant 0$. Thus, the first equation of (3.5′) gives

$$\sum_{\ell=1}^{n-1}\xi_\ell a_{\ell j} = 0, \quad j = 1,\ldots,n-1.$$

This shows that $a_{\ell j} = 0$, and the proof of (ii) is complete.

Finally, we prove (iii). Let $w = (w', w_n)$ be the vector field with components given in (3.5). Suppose that $\|u(t)\|_q \geqslant c_q t^{-\frac{n}{2}(1+1/n-1/q)}$ for large $t > 0$. By (1.22) and (1.23), we see that if (1.26) holds, then

$$\begin{aligned}\|w(t)\|_q &\geqslant \|u(t)\|_q - \|w(t)+u(t)\|_q \geqslant c_q t^{-\frac{n}{2}(1+1/n-1/q)} - o(t^{-\frac{n}{2}(1+1/n-1/q)})\\ &\geqslant c'_q t^{-\frac{n}{2}(1+1/n-1/q)}\end{aligned}$$

for large $t > 0$. By (3.4), we have $w(t) \neq 0$. Hence (1.27) holds. Conversely, if (1.27) holds, the proof of (ii) shows that $w(t) \neq 0$. Thus, (3.4) implies that

$\|w(t)\|_q = c_q t^{-\frac{n}{2}(1+1/n-1/q)}$ for some $c_q > 0$ and, by (1.22) and (1.23), we get

$$\begin{aligned}\|u(t)\|_q &\geqslant \|w(t)\|_q - \|u(t)+w(t)\|_q = c_q t^{-\frac{n}{2}(1+1/n-1/q)} - o(t^{-\frac{n}{2}(1+1/n-1/q)}) \\ &\geqslant c_q' t^{-\frac{n}{2}(1+1/n-1/q)}\end{aligned}$$

for large $t > 0$. Hence u satisfies (1.26), and the proof of (iii) is complete. □

PROOF OF THEOREM 1.7. Let a, u, and v satisfy the parity condition and the cyclic symmetry condition, and let $b = (a_*', a_n^*)$, $\overline{u} = (u_*', u_n^*)$, and $\overline{v} = (v_*', v_n^*)$. It suffices to show that the function w defined by the formula

$$\widehat{w}_j(\xi,t) = e^{-t|\xi|^2}\widehat{b}_j(\xi) - i\int_0^t \xi_\ell e^{-(t-s)|\xi|^2}\Big(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\Big)\widehat{(\overline{u}_k\overline{v}_\ell)}(\xi,s)ds,\ j=1,\dots,n,$$

satisfies the parity condition and the cyclic symmetry condition. But this fact is verified by a straightforward calculation. The details of the proof are left to the reader (cf. [**1**]).

To deduce (1.29) under the assumption (1.28), it suffices to show that if $b = (a_*', a_n^*)$ satisfies

$$\sup(1+|x|)^{n+3}|b(x)| < \infty, \qquad \int |x|^3|b(x)|dx < \infty,$$

then the corresponding strong solution v to the Cauchy problem satisfies the inequality

$$\|v(t)\|_q \leqslant c_q(1+t)^{-\frac{n}{2}(1+3/n-1/q)}, \quad 1 \leqslant q \leqslant \infty. \tag{3.6}$$

By [**20**], we have

$$|v(x,t)| \leqslant c(1+|x|)^{\alpha-n-3}(1+t)^{-\alpha/2} \quad \text{for all } 0 \leqslant \alpha \leqslant n+3. \tag{3.7}$$

This implies

$$\|v(t)\|_{n/(n+1),w} \leqslant c(1+t)^{-1}, \quad \|v(t)\|_\infty \leqslant c(1+t)^{-(n+3)/2},$$

where $\|\cdot\|_{p,w}$ is the weak L^p-quasinorm. Thus, we get (3.6) by interpolation. To deduce (3.7), we consider the equation

$$v_j = E_t * b_j - \int_0^t F_{\ell,jk}(t-s) * (v_k v_\ell)(s)ds.$$

The parity condition, together with the cyclic symmetry condition, implies

$$\int x^\gamma b(x) = 0 \quad (|\gamma| \leqslant 2), \qquad \int (v_k v_\ell)(x,t)dx = \lambda(t)\delta_{k\ell},$$
$$\int x_j(v_k v_\ell)(x,t)dx = 0 \quad (j=1,\dots,n).$$

Furthermore,

$$F_{\ell,j\ell} = \partial_j E_t + \int_0^\infty \partial_j \Delta E_{t+s} ds = \partial_j E_t + \int_0^\infty \partial_s \partial_j E_{t+s} ds = \partial_j E_t - \partial_j E_t = 0.$$

Applying Taylor's theorem, we find

$$v_j = \int E_t'(x,y) b_j(y) dy - \int_0^t \int F_{\ell,jk}'(x,y,t-s)(v_k v_\ell)(y,s) dy ds,$$

where

$$E_t'(x,y) = \frac{1}{2} \int_0^1 (1-\theta)^2 \left(\frac{d}{d\theta}\right)^3 E_t(x-y\theta) d\theta,$$

$$F_{\ell,jk}'(x,y,t) = \int_0^1 (1-\theta) \left(\frac{d}{d\theta}\right)^2 F_{\ell,jk}(x-y\theta,t) d\theta.$$

The further proof of (3.7) is standard (cf. [**20**] for details). □

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1, especially, the expansions (1.11) and (1.12). These results improve [**9**, Theorem 3.5], but our arguments in this section are much simpler and more accessible than those given in [**9**]. We begin by proving (1.15).

Lemma 4.1. *Let $a \in L^1$ satisfy $\nabla \cdot a = 0$, $a_n|_{\partial D^n} = 0$, and* (1.4). *Then*

$$\int_{D^n} y_n a_n(y) dy = 0.$$

Proof. The vector field $b = (a_*', a_n^*)$ satisfies $\nabla \cdot b = 0$ in $\mathbb{R}^n$ and $\int (1 + |x_n|)|b(x)| dx < \infty$. Hence the Fourier transform $\widehat{b}(\xi)$ and its derivative $\partial_n \widehat{b}(\xi)$ with respect to ξ_n are bounded and continuous on $\mathbb{R}^n$. Since $\widehat{b}_n$ is odd in ξ_n, it follows that $\widehat{b}_n(\xi', 0) = 0$. Therefore,

$$\widehat{b}_n(\xi', \xi_n) = \widehat{b}_n(\xi', 0) + \xi_n \int_0^1 (\partial_n \widehat{b}_n)(\xi', s\xi_n) ds = \xi_n \int_0^1 (\partial_n \widehat{b}_n)(\xi', s\xi_n) ds.$$

On the other hand, the condition $\nabla \cdot b = 0$ implies $\xi \cdot \widehat{b} = \xi' \cdot \widehat{b}' + \xi_n \widehat{b}_n = 0$. Differentiating the last equality with respect to ξ_n, we find $\xi' \cdot (\partial_n \widehat{b}') + \widehat{b}_n +$

$\xi_n(\partial_n \widehat{b}_n) = 0$. Inserting this expression in the above formula, we get

$$\xi' \cdot (\partial_n \widehat{b}')(\xi', \xi_n) + \xi_n(\partial_n \widehat{b}_n)(\xi', \xi_n) = -\xi_n \int_0^1 (\partial_n \widehat{b}_n)(\xi', s\xi_n)ds.$$

Setting $\xi' = 0$ and dividing the obtained identity by $\xi_n \neq 0$, we find

$$(\partial_n \widehat{b}_n)(0, \xi_n) = -\int_0^1 (\partial_n \widehat{b}_n)(0, s\xi_n)ds \quad \text{for all } \xi_n \neq 0.$$

Letting $\xi_n \to 0$, we find $(\partial_n \widehat{b}_n)(0) = -(\partial_n \widehat{b}_n)(0) = 0$. Since $x_n b_n(x) = x_n a_n^*(x)$ is even in x_n, we have

$$0 = (\partial_n \widehat{b}_n)(0) = -i \int x_n b_n(x)dx = -2i \int_{D^n} y_n a_n(y)dy. \qquad \square$$

Lemma 4.2. *Let $x \in D^n$, $y \in \mathbb{R}^n$, $t > 0$, and let*

$$K(x, y, t) = t^{-(n+1)/2} K^0(xt^{-1/2}, yt^{-1/2}),$$

where $K^0(\xi, \eta)$ is smooth on $\overline{D}^n \times \mathbb{R}^n$ and satisfies the inequality

$$\|\nabla^m_{\xi,\eta} K^0(\cdot, \eta)\|_q \leqslant c_{m,q}$$

for all $m \geqslant 0$ and $\eta \in \mathbb{R}^n$ and some $1 < q \leqslant \infty$. Then

$$\|\partial_t^\ell \nabla^m_{x,y} K(\cdot, y, t)\|_q \leqslant c_{m,\ell,q} t^{-\frac{2\ell+m+1}{2} - \frac{n}{2}(1-\frac{1}{q})}, \quad \ell, m = 0, 1, 2, \ldots, \tag{4.1}$$

uniformly in $y \in \mathbb{R}^n$. Moreover, if

$$I_*(x,t) = \int_0^{t/2}\!\!\int K(x, y, t-s)(u \otimes u)_*(y, s)dyds,$$

$$I^*(x,t) = \int_0^{t/2}\!\!\int K(x, y, t-s)(u \otimes u)^*(y, s)dyds,$$

where u is a weak or strong solution given in Theorem 1.1, then

$$\lim_{t\to\infty} t^{\frac{n}{2}(1+1/n-1/q)} \left\| I_*(t) - 2K(\cdot, 0, t) \int_0^\infty\!\!\int_{D^n} (u \otimes u)(y, s)dyds \right\|_q = 0, \tag{4.2}$$

$$\lim_{t\to\infty} t^{\frac{n}{2}(1+1/n-1/q)} \|I^*(t)\|_q = 0. \tag{4.3}$$

PROOF. We derive only (4.2) because the proof of (4.3) is similar and (4.1) is directly verified. We write

$$
\begin{aligned}
I_*(x,t) &= 2K(x,0,t)\int_0^{t/2}\int_{D^n}(u\otimes u)(y,s)dyds \\
&\quad + \int_0^{t/2}\int [K(x,y,t-s)-K(x,0,t)](u\otimes u)_*(y,s)dyds \\
&= 2K(x,0,t)\int_0^{\infty}\int_{D^n}(u\otimes u)(y,s)dyds - K(x,0,t)\int_{t/2}^{\infty}\int (u\otimes u)_*(y,s)dyds \\
&\quad + \int_0^{t/2}\int [K(x,y,t-s)-K(x,0,t-s)](u\otimes u)_*(y,s)dyds \\
&\quad + \int_0^{t/2}\int [K(x,0,t-s)-K(x,0,t)](u\otimes u)_*(y,s)dyds \\
&\equiv 2K(x,0,t)\int_0^{\infty}\int_{D^n}(u\otimes u)(y,s)dyds + I_1 + I_2 + I_3.
\end{aligned}
$$

Since $||u(s)||_2^2 \leqslant c(1+s)^{-1-n/2}$, it is easy to see that

$$dm \lim_{t\to\infty} t^{\frac{n}{2}(1+1/n-1/q)}||I_1||_q = 0.$$

On the other hand, since $t-s \geqslant t/2$ if $0 \leqslant s \leqslant t/2$, Minkowski's inequality for the integral gives

$$
\begin{aligned}
||I_2||_q &\leqslant ct^{-\frac{n}{2}(1+1/n-1/q)}\int_0^{t/2}\int ||K^0(\cdot, y(t-s)^{-1/2}) - K^0(\cdot,0)||_q |u_*(y,s)|^2 dyds \\
&\equiv ct^{-\frac{n}{2}(1+1/n-1/q)}\int_0^{t/2}\int \varphi_t(y,s)dyds \equiv ct^{-\frac{n}{2}(1+1/n-1/q)}\int_0^{t/2}\psi_t(s)ds.
\end{aligned}
$$

By the assumption on K^0, we have

$$0 \leqslant \varphi_t(y,s) \leqslant c|u_*(y,s)|^2, \quad \lim_{t\to\infty}\varphi_t(y,s) = 0. \tag{4.4}$$

The former of (4.4) is obvious. To show the latter we apply

$$K^0(x, y(t-s)^{-1/2}) - K^0(x,0) = y(t-s)^{-1/2} \cdot \int_0^1 (\nabla_\eta K^0)(x, y(t-s)^{-1/2}\theta)d\theta.$$

By the properties of K^0, we have

$$\begin{aligned}\varphi_t(y,s) &\leqslant |y|(t-s)^{-1/2} \sup_\eta ||\nabla_\eta K^0(\cdot,\eta)||_q |u_*(y,s)|^2 \\ &\leqslant c|y|(t-s)^{-1/2}|u_*(y,s)|^2 \to 0\end{aligned}$$

as $t \to \infty$. This proves the latter assertion of (4.4). From (4.4) we get

$$0 \leqslant \psi_t(s) \leqslant c||u(s)||_2^2 \leqslant c(1+s)^{-1-n/2}, \quad \lim_{t\to\infty} \psi_t(s) = 0. \tag{4.5}$$

Thus, the bounded convergence theorem implies

$$\lim_{t\to\infty} \int_0^T \psi_t(s)ds = 0 \quad \text{for any fixed } T > 0. \tag{4.6}$$

Given $\varepsilon > 0$, we choose $T > 0$ such that

$$\int_T^\infty (1+s)^{-1-n/2}ds < \varepsilon.$$

If $t > 2T$, then from (4.5) it follows that

$$\begin{aligned}\int_0^{t/2} \psi_t(s)ds &= \left(\int_0^T + \int_T^{t/2}\right)\psi_t(s)ds \leqslant \int_0^T \psi_t(s)ds + c\int_T^\infty ||u(s)||_2^2 ds \\ &\leqslant \int_0^T \psi_t(s)ds + c\int_T^\infty (1+s)^{-1-n/2}ds \leqslant \int_0^T \psi_t(s)ds + c\varepsilon.\end{aligned}$$

By (4.6), we find

$$\limsup_{t\to\infty} \int_0^{t/2} \psi_t(s)ds \leqslant c\varepsilon,$$

which proves

$$\lim_{t\to\infty} t^{\frac{n}{2}(1+1/n-1/q)}||I_2||_q = 0.$$

Finally, from

$$K(x,0,t-s) - K(x,0,t) = -s\int_0^1 (\partial_t K)(x,0,t-s\theta)d\theta$$

and (4.1) it follows that

$$\|I_3\|_q \leqslant c_q t^{-1-\frac{n}{2}(1+1/n-1/q)} \int\limits_0^{t/2} s(1+s)^{-1-n/2} ds.$$

Therefore,

$$t^{\frac{n}{2}(1+1/n-1/q)}\|I_3\|_q \leqslant c_q t^{-1} \int\limits_0^t (1+s)^{-1} ds \to 0 \quad \text{as } t \to \infty.$$

Hence we get (4.2), and the proof of Lemma 4.2 is complete. □

We next deduce the desired expansion for $v(t) = e^{-tA}a$.

Lemma 4.3. *Let a satisfy (1.4), and let $v(t) = (v'(t), v_n(t)) = e^{-tA}a$. Then*

$$\|\nabla^k v(t)\|_q \leqslant c_q t^{-\frac{n}{2}(1+\frac{k+1}{n}-\frac{1}{q})} \int\limits_{D^n} y_n |a(y)| dy, \quad k = 0, 1, \quad 1 < q \leqslant \infty. \tag{4.7}$$

Moreover,

$$\lim_{t\to\infty} t^{\frac{n}{2}(1+1/n-1/q)} \left\| v_n(t) - 2U(\partial_n F_t)(\cdot) \cdot \int\limits_{D^n} y_n a'(y) dy \right\|_q = 0, \tag{4.8}$$

$$\lim_{t\to\infty} t^{\frac{n}{2}(1+1/n-1/q)} \left\| v'(t) + 2[(\partial_n E_t)(\cdot) + SU(\partial_n F_t)(\cdot)\cdot] \int\limits_{D^n} y_n a'(y) dy \right\|_q = 0 \tag{4.9}$$

for $1 < q < \infty$. Here, U is given in (1.6) and $F_t = (F_t^k)_{k=1}^{n-1}$ in (1.9).

Proof. By (1.5) and (1.7), we have

$$v_n(t) = Ue^{t\Delta}[a_n^* - (S \cdot a')^*] = Ue^{t\Delta}a_n^* - Ue^{t\Delta}S \cdot (a')^*.$$

We note that

$$\int\limits_{-\infty}^{\infty} a_n^*(y', y_n) dy_n = 0.$$

Hence

$$\begin{aligned} e^{t\Delta}a_n^* &= \int E_t(x'-y')[E_t(x_n - y_n) - E_t(x_n)]a_n^*(y)dy \\ &= -\iint\limits_0^1 E_t(x'-y')(\partial_n E_t)(x_n - y_n\theta) y_n a_n^*(y) dy d\theta. \end{aligned}$$

Applying Minkowski's inequality for the integral, we find

$$\|Ue^{-tB}a_n\|_q \leqslant ct^{-\frac{n}{2}(1+1/n-1/q)} \int_{D^n} y_n|a_n(y)|dy.$$

We next invoke

$$\int y_n a_n^*(y)dy = 2\int_{D^n} y_n a_n(y)dy = 0,$$

which is due to (1.15), and write the above equality in the form

$$\begin{aligned} e^{t\Delta}a_n^* &= -\int_0^1\!\!\int [E_t(x'-y')-E_t(x')](\partial_n E_t)(x_n-y_n\theta)y_n a_n^*(y)dy \\ &\quad -\int_0^1\!\!\int E_t(x')[(\partial_n E_t)(x_n-y_n\theta)-(\partial_n E_t)(x_n)]y_n a_n^*(y)dy. \end{aligned}$$

Thus,

$$\begin{aligned} t^{\frac{n}{2}(1+1/n-1/q)}\|Ue^{-tB}a_n\|_q &\leqslant c\int \|(E_1)(\cdot-y't^{-1/2})-(E_1)(\cdot)\|_q|y_n||a_n^*(y)|dy \\ &+c\int_0^1\!\!\int \|(\partial_n E_1)(\cdot-y_nt^{-1/2}\theta)-(\partial_n E_1)(\cdot)\|_q|y_n||a_n^*(y)|dy \to 0 \end{aligned}$$

as $t\to\infty$. Similarly, by (1.9), we can write $F_t(x)=E_t(x_n)F_t'(x')$, where

$$F_t'(x')=\pi^{-1/2}\int_0^\infty s^{-1/2}\nabla' E_{t+s}(x')ds,$$

to obtain

$$-e^{t\Delta}S\cdot(a')^*=\int_0^1\!\!\int F_t'(x'-y')\cdot(\partial_n E_t)(x_n-y_n\theta)y_n(a')^*(y)dyd\theta.$$

This implies

$$\|-Ue^{-tB}S\cdot a'\|_q \leqslant ct^{-\frac{n}{2}(1+1/n-1/q)}\int_{D^n} y_n|a'(y)|dy.$$

We write the above expression for $-e^{t\Delta}S\cdot(a')^*$ in the form

$$-e^{t\Delta}S\cdot(a')^*=2(\partial_n F_t)(x)\int_{D^n} y_n a'(y)dy$$

$$+\iint_0^1 [F_t'(x'-y')(\partial_n E_t)(x_n - y_n\theta) - (\partial_n F)(x)] \cdot y_n (a')^*(y)dyd\theta$$

and get

$$-e^{t\Delta} S \cdot (a')^* - 2(\partial_n F_t)(x) \int_{D^n} y_n a'(y)dy$$

$$= \iint_0^1 [F_t'(x'-y') - F_t'(x')] \cdot (\partial_n E_t)(x_n - y_n\theta) y_n (a')^*(y)dyd\theta$$

$$+ \iint_0^1 F_t'(x') \cdot [(\partial_n E_t)(x_n - y_n\theta) - (\partial_n E_t)(x_n)] y_n (a')^*(y)dyd\theta.$$

Therefore,

$$t^{\frac{n}{2}(1+1/n-1/q)} \left\| -Ue^{-tB} S \cdot a' - 2U(\partial_n F_t)(\cdot) \int_{D^n} y_n a'(y)dy \right\|_q$$

$$\leqslant c \int \|(F_1')(\cdot - y' t^{-1/2}) - (F_t')(\cdot)\|_q |y_n| |(a')^*(y)| dy$$

$$+ c \iint_0^1 \|(\partial_n E_1)(\cdot - y_n t^{-1/2}\theta) - (\partial_n E_1)(\cdot)\|_q |y_n| |(a')^*(y)| dy \to 0$$

as $t \to \infty$. Combining the above results, we obtain (4.7) with $k = 0$ for $v_n(t)$ and (4.8). Formula (4.7) with $k = 0$ for $v'(t)$ and formula (4.9) are proved in a similar way. To deduce (4.7) with $k = 1$, we use

$$\|\nabla e^{-tA} a\|_q \leqslant c_r t^{-\frac{n}{2}(\frac{1}{r}+\frac{1}{n}-\frac{1}{q})} \|a\|_r, \quad 1 < r < q \leqslant \infty \tag{4.10}$$

to get

$$\|\nabla v(t)\|_q \leqslant ct^{-\frac{n}{2}(\frac{1}{r}+\frac{1}{n}-\frac{1}{q})} \|v(t/2)\|_r \leqslant ct^{-\frac{n}{2}(1+2/n-1/q)} \int_{D^n} y_n |a(y)| dy. \qquad \square$$

The result below is well known. However, we give a proof for the reader's convenience.

Lemma 4.4. *Let Q_n be the fundamental solution of the operator $-\Delta$. Then*

$$\mathcal{F}[\mathrm{sgn}(\cdot)(\partial_n^2 Q_n)](\xi) = -\frac{i|\xi'|\xi_n}{|\xi|^2},$$

$$\mathcal{F}[\mathrm{sgn}(\cdot)(\partial_j \partial_k Q_n)](\xi) = \frac{i\xi_j \xi_k \xi_n}{|\xi'||\xi|^2}, \quad j,k = 1,\dots,n-1, \quad (i = \sqrt{-1}). \tag{4.11}$$

$$\mathcal{F}[\mathrm{sgn}(\cdot)(\partial_j \partial_n Q_n)](\xi) = -\frac{i\xi_j |\xi'|}{|\xi|^2}, \quad j = 1,\dots,n-1,$$

Here, $\mathcal{F}$ is the Fourier transform and $\operatorname{sgn}(\cdot)$ *means multiplication by* $\operatorname{sgn}(x_n)$.

PROOF. We use

$$(\partial_n^2 Q_n)(x) = \int_0^\infty (\partial_n^2 E_t)(x)dt = \int_0^\infty E_t(x')(\partial_n^2 E_t)(x_n)dt,$$

which converges absolutely for $n \geqslant 2$. A direct calculation yields

$$\begin{aligned}\mathcal{F}[\operatorname{sgn}(\cdot)(\partial_n^2 Q_n)](\xi) &= \int_0^\infty e^{-t|\xi'|^2}\Big(\int_{-\infty}^\infty e^{-ix_n\xi_n}\operatorname{sgn}(x_n)(\partial_n^2 E_t)(x_n)dx_n\Big)dt\\ &= -2i\int_0^\infty \sin(x_n\xi_n)\Big(\int_0^\infty e^{-t|\xi'|^2}(\partial_n^2 E_t)(x_n)dt\Big)dx_n\\ &= -2i\int_0^\infty \sin(x_n\xi_n)\partial_n^2\Big(\int_0^\infty e^{-t|\xi'|^2}E_t(x_n)dt\Big)dx_n.\end{aligned}$$

Inserting

$$E_t(x_n) = (2\pi)^{-1}\int_{-\infty}^\infty e^{ix_n\tau}e^{-t\tau^2}d\tau$$

in the above expression and using

$$e^{-\gamma} = \frac{1}{\pi}\int_{-\infty}^\infty \frac{e^{is\gamma}}{1+s^2}ds, \quad \gamma > 0, \tag{4.12}$$

we obtain the first equality of (4.11) because

$$\begin{aligned}\mathcal{F}[\operatorname{sgn}(\cdot)(\partial_n^2 Q_n)](\xi) &= \frac{-i}{\pi}\int_0^\infty \sin(x_n\xi_n)\partial_n^2\Big(\int_{-\infty}^\infty \frac{e^{ix_n\tau}}{|\xi'|^2+\tau^2}d\tau\Big)dx_n\\ &= -i|\xi'|\int_0^\infty \sin(x_n\xi_n)e^{-x_n|\xi'|}dx_n = -\frac{i|\xi'|\xi_n}{|\xi|^2}.\end{aligned}$$

To show the second equality, we use

$$(\partial_j\partial_k Q_n)(x) = \int_0^\infty (\partial_j\partial_k E_t)(x')E_t(x_n)dt$$

which also converges absolutely for $n \geqslant 2$. By (4.12), we get

$$\begin{aligned}
\mathcal{F}[\operatorname{sgn}(\cdot)(\partial_j \partial_k Q_n)](\xi) &= -\xi_j \xi_k \int_0^\infty e^{-t|\xi'|^2} \Big(\int_{-\infty}^{\infty} e^{-ix_n\xi_n} \operatorname{sgn}(x_n) E_t(x_n) dx_n \Big) dt \\
&= 2i\xi_j \xi_k \int_0^\infty \sin(x_n \xi_n) \Big(\int_0^\infty e^{-t|\xi'|^2} E_t(x_n) dt \Big) dx_n \\
&= \frac{i\xi_j \xi_k}{|\xi'|} \int_0^\infty \sin(x_n\xi_n) e^{-x_n|\xi'|} dx_n = \frac{i\xi_j\xi_k\xi_n}{|\xi'||\xi|^2}.
\end{aligned}$$

The third equality is deduced in a similar way. □

Proof of Theorem 1.1 (i). We prove (1.11) and (1.12) under the assumption (1.10). Suppose that a strong solution u satisfies (1.10). We write

$$-\int_0^t e^{-(t-s)A} P\nabla \cdot (u \otimes u)(s) ds = -\Big(\int_0^{t/2} + \int_{t/2}^t \Big) \equiv w(t) + \overline{w}(t).$$

Since $A^{-1/2}P\nabla$ regarded as an operator from $\boldsymbol{L}^q$ to $\boldsymbol{L}^q_\sigma$ is bounded, a direct calculation yields (cf. [**3**])

$$\begin{aligned}
\|\overline{w}(t)\|_q &\leqslant \int_{t/2}^t \|A^{1/2}e^{-(t-s)A}\| \|A^{-1/2}P\nabla \cdot (u\otimes u)(s)\|_q ds \\
&\leqslant c_q \int_{t/2}^t (t-s)^{-1/2} \|u(s)\|_{2q}^2 ds.
\end{aligned}$$

Since $\|u(s)\|_{2q}^2 \leqslant c_q s^{-n(1+1/n-1/(2q))}$ by (1.10), it follows that

$$\begin{aligned}
\|\overline{w}(t)\|_q &\leqslant c_q \int_{t/2}^t (t-s)^{-1/2} (1+s)^{-n(1+1/n-1/(2q))} ds \\
&\leqslant c_q (1+t)^{-n(1+1/n-1/(2q))+1/2}.
\end{aligned}$$

Hence

$$\lim_{t\to\infty} t^{\frac{n}{2}(1+1/n-1/q)} \|\overline{w}(t)\|_q = 0. \tag{4.13}$$

The above proof of (4.13) is not valid for a weak solution because we know nothing about the boundedness of $\|u(t)\|_{2q}$, $q > 1$, for weak solutions. However, for $1 < q < n' = n/(n-1)$, from

$$\langle \overline{w}(t), \varphi\rangle_{D^n} = \int_{t/2}^{t} \langle u \otimes u, \nabla e^{-(t-s)A}\varphi\rangle_{D^n} ds, \quad \varphi \in C_{0,\sigma}^{\infty}(D^n)$$

and (4.10) with $q = \infty$ we find

$$|\langle \overline{w}(t), \varphi\rangle_{D^n}| \leqslant c_q \int_{t/2}^{t} (t-s)^{-\frac{n}{2}(1+1/n-1/q)} \|u(s)\|_2^2 ds \|\varphi\|_{q'}.$$

We note that $n(1+1/n-1/q)/2 < 1$ if $1 < q < n'$. Hence the right-hand side of the above relation is finite. Since $C_{0,\sigma}^{\infty}(D^n)$ is dense in L_σ^q and $(L_\sigma^{q'})^* = L_\sigma^q$, it follows that

$$\|\overline{w}(t)\|_q \leqslant c_q \int_{t/2}^{t} (t-s)^{-\frac{n}{2}(1+1/n-1/q)} (1+s)^{-1-n/2} ds \leqslant c_q (1+t)^{-n/2-\frac{n}{2}(1+1/n-1/q)}$$

and (4.13) holds for $1 < q < n'$. So we need only prove (4.13) for weak solutions in case $q = 2$. This case is proved in [**9**], and we omit details.

By Lemma 4.3 and (4.13), it suffices to deduce

$$\lim_{t\to\infty} t^{\frac{n}{2}(1+1/n-1/q)} \left\| w_n(t) - 2U\partial_n F_t \cdot \int_0^\infty\int_{D^n} (u_n u')(y,s) dy ds \right\|_q = 0, \qquad (1.11')$$

$$\lim_{t\to\infty} t^{\frac{n}{2}(1+1/n-1/q)} \left\| w'(t) + 2[\partial_n E_t + SU\partial_n F_t \cdot] \int_0^\infty\int_{D^n} (u_n u')(y,s) dy ds \right\|_q = 0. \qquad (1.12')$$

We first prove (1.11′). Using (1.5) and Green's function N of the homogeneous Neumann problem for $-\Delta$ (cf. (2.2)), we find

$$\begin{aligned} w_n(t) = &- U \int_0^{t/2} e^{-(t-s)B}[\nabla \cdot (uu_n) - S \cdot \nabla \cdot (uu')](s) ds \\ &- U \int_0^{t/2} e^{-(t-s)B}(\partial_n N[\partial_j \partial_k (u_j u_k)] - S \cdot \nabla' N[\partial_j \partial_k (u_j u_k)])(s) ds. \end{aligned}$$

By (1.7) and (2.2), we see that

$$\begin{aligned} e^{-tB}[\nabla \cdot (uu_n) - S \cdot \nabla \cdot (uu')] = &\nabla' \cdot e^{t\Delta}(u'u_n)^* + \partial_n e^{t\Delta}(u_n u_n)_* \\ &- \nabla' \cdot e^{t\Delta} S \cdot (u'u')^* - \partial_n e^{t\Delta} S \cdot (u_n u')_*, \end{aligned}$$

and

$$e^{-tB}[\partial_n N(\partial_j\partial_k(u_ju_k)) - S\cdot\nabla' N(\partial_j\partial_k(u_ju_k))]$$

$$= \sum_{j,k=1}^{n-1} \partial_n e^{t\Delta}(\partial_j\partial_k Q_n)(u_ju_k)_* + 2\sum_{j=1}^{n-1}\partial_n e^{t\Delta}(\partial_j\partial_n Q_n)(u_ju_n)^*$$

$$+ \partial_n e^{t\Delta}(\partial_n^2 Q_n)(u_nu_n)_* - \sum_{j,k=1}^{n-1}\nabla'\cdot e^{t\Delta}S[\operatorname{sgn}(\cdot)(\partial_j\partial_k Q_n)(u_ju_k)_*]$$

$$- 2\sum_{j=1}^{n-1}\nabla'\cdot e^{t\Delta}S[\operatorname{sgn}(\cdot)(\partial_j\partial_n Q_n)(u_ju_n)^*] - \nabla'\cdot e^{t\Delta}S[\operatorname{sgn}(\cdot)(\partial_n^2 Q_n)(u_nu_n)_*],$$

where $\operatorname{sgn}(\cdot)$ means multiplication by $\operatorname{sgn}(x_n)$. Due to Lemma 4.4 and the relations

$$|\xi|^{-2} = \int_0^\infty e^{-s|\xi|^2}ds, \quad |\xi'|^{-1} = \pi^{-1/2}\int_0^\infty s^{-1/2}e^{-s|\xi'|^2}ds,$$

we see that the kernel functions of the operators involved above are of the form as treated in Lemma 4.2. Indeed, for example, the kernel function K of $\nabla'\cdot e^{t\Delta}S[\operatorname{sgn}(\cdot)(\partial_j\partial_k Q_n)]$ has the Fourier transform

$$\frac{i\xi_j i\xi_k i\xi_n}{|\xi|^2}e^{-t|\xi|^2} = i\xi_j i\xi_k i\xi_n \int_0^\infty e^{-(t+s)|\xi|^2}ds$$

and so $K(x,y,t) = t^{-(n+1)/2}K^0((x-y)t^{-1/2})$, where

$$K^0(x-y) = \int_0^\infty \partial_j\partial_k\partial_n E_{s+1}(x-y)ds.$$

The other terms are treated similarly. By Lemma 4.2, the terms involving the odd extensions, $(u_ju_k)^*$, etc. may be discarded. Therefore, to deduce (1.11′), it suffices to show that

$$(\partial_n e^{t\Delta} + \partial_n e^{t\Delta}(\partial_n^2 Q_n) - \nabla'\cdot e^{t\Delta}S[\operatorname{sgn}(\cdot)(\partial_n^2 Q_n)])c_{nn}$$
$$+ \sum_{j,k=1}^{n-1}(\partial_n e^{t\Delta}(\partial_j\partial_k Q_n) - \nabla'\cdot e^{t\Delta}S[\operatorname{sgn}(\cdot)(\partial_j\partial_k Q_n)])c_{jk} = 0, \quad (4.14)$$

where

$$c_{jk} = 2\int_0^\infty\int_{D^n}(u_ju_k)dyds, \quad j,k = 1,\dots,n, \quad (4.15)$$

and each operator stands for its kernel function of the form $K(x,0,t)$ treated in Lemma 4.2. The kernel functions of these operators are naturally defined on $\mathbb{R}^n$

as odd functions of x_n. By Lemma 4.4, the Fourier transform of the left-hand side of (4.14) is equal to the expression

$$e^{-t|\xi|^2}\left(i\xi_n/|\xi|^2 - i\xi_n/|\xi|^2\right)\left(c_{nn}|\xi'|^2 - \sum_{j,k=1}^{n-1} c_{jk}\xi_j\xi_k\right) = 0.$$

The proof of (1.11′) is complete.

We prove (1.12′). By (1.5), we have

$$\begin{aligned} w'(t) + Sw_n(t) = &- \int_0^{t/2} e^{-(t-s)B}[\nabla\cdot(uu') + S\nabla\cdot(uu_n)](s)ds \\ &- \int_0^{t/2} e^{-(t-s)B}[\nabla' N(\partial_j\partial_k(u_ju_k)) + S\partial_n N(\partial_j\partial_k(u_ju_k))](s)ds. \end{aligned}$$

The functions Sw_n are expanded by (1.11′), and the term involving $SU(\partial_n F_t)$ in (1.12′) is derived from the expansion of Sw_n. It remains to expand the right-hand side above, applying Lemma 4.2. By (1.7), we have

$$\begin{aligned} &e^{-tB}[\nabla\cdot(uu') + S\nabla\cdot(uu_n)] = \nabla' e^{t\Delta}(u'u')^* + \partial_n e^{t\Delta}(u_nu')_* \\ &\quad + S\nabla' e^{t\Delta}(u'u_n)^* + S\partial_n e^{t\Delta}(u_nu_n)_*, \\ &e^{-tB}[\nabla' N(\partial_j\partial_k(u_ju_k)) + S\partial_n N(\partial_j\partial_k(u_ju_k))] \\ &= \sum_{j,k=1}^{n-1} \nabla' e^{t\Delta}\operatorname{sgn}(\cdot)(\partial_j\partial_k Q_n)(u_ju_k)_* + 2\sum_{j=1}^{n-1} \nabla' e^{t\Delta}[\operatorname{sgn}(\cdot)(\partial_j\partial_n Q_n)(u_ju_k)^*] \\ &\quad + \nabla' e^{t\Delta}[\operatorname{sgn}(\cdot)(\partial_n^2 Q_n)(u_nu_n)_*] + \sum_{j,k=1}^{n-1} S\partial_n e^{t\Delta}(\partial_j\partial_k Q_n)(u_ju_k)_* \\ &\quad + 2\sum_{j=1}^{n-1} S\partial_n e^{t\Delta}(\partial_j\partial_n Q_n)(u_ju_n)^* + S\partial_n e^{t\Delta}(\partial_n^2 Q_n)(u_nu_n)_*. \end{aligned}$$

Define c_{jk} by (4.15). By Lemma 4.2, it suffices to show that

$$\begin{aligned} &(S\partial_n e^{t\Delta} + \nabla' e^{t\Delta}[\operatorname{sgn}(\cdot)(\partial_n^2 Q_n)] + S\partial_n e^{t\Delta}(\partial_n^2 Q_n))c_{nn} \\ &+ \sum_{j,k=1}^{n-1} (\nabla' e^{t\Delta}[\operatorname{sgn}(\cdot)(\partial_j\partial_k Q_n)] + S\partial_n e^{t\Delta}(\partial_j\partial_k Q_n))c_{jk} = 0. \end{aligned} \tag{4.16}$$

But (4.16) is deduced from Lemma 4.4. Indeed, the Fourier transform equals

$$(i\xi'/|\xi'|)e^{-t|\xi|^2}(i\xi_n/|\xi|^2 - i\xi_n/|\xi|^2)\left(c_{nn}|\xi'|^2 - \sum_{j,k=1}^{n-1} c_{jk}\xi_j\xi_k\right) = 0.$$

The proof of (1.12′) is complete. □

PROOF OF THEOREM 1.1 (ii). We write

$$W' = (W_1, \ldots, W_{n-1}) = \int_{D^n} y_n a'(y)dy + \int_0^\infty\int_{D^n} (u_n u')(y,s)dyds$$

and observe that

$$\|U\partial_n F_t \cdot W'\|_2^2 = c_0^2|W'|^2 t^{-(n+2)/2} \tag{4.17}$$

for some $c_0 > 0$. This follows from the fact that $\{U\partial_n F_t^1, \ldots, U\partial_n F_t^{n-1}\}$ are mutually orthogonal in $L^2(D^n)$ and have the same L^2-norm. Using this fact and (1.11) with $q = 2$, we easily see that $W' \neq 0$ if and only if $t^{(n+2)/4}\|u_n(t)\|_2 \geqslant c > 0$ for large $t > 0$. Suppose that

$$t^{(n+2)/4}\|u(t)\|_2 \geqslant c > 0 \text{ for large } t > 0, \quad \liminf_{t\to\infty} t^{(n+2)/4}\|u_n(t)\|_2 = 0. \tag{4.18}$$

From

$$c_0|W'| = t^{(n+2)/4}\|U\partial_n F_t \cdot W'\|_2 \leqslant t^{(n+2)/4}(\|u_n(t)\|_2 + \|u_n(t) - U\partial_n F_t \cdot W'\|_2)$$

we get

$$\begin{aligned} c_0|W'| &\leqslant \liminf_{t\to\infty} t^{(n+2)/4}(\|u_n(t)\|_2 + \|u_n(t) - U\partial_n F_t \cdot W'\|_2) \\ &= \liminf_{t\to\infty} t^{(n+2)/4}\|u_n(t)\|_2 + \lim_{t\to\infty} t^{(n+2)/4}\|u_n(t) - U\partial_n F_t \cdot W'\|_2 = 0. \end{aligned}$$

Hence $W' = 0$. This, together with (1.12), implies that $t^{(n+2)/4}\|u(t)\|_2 \to 0$ as $t \to \infty$, which contradicts the assumption (4.18). This proves assertion (ii) for $q = 2$. To treat the case $q \neq 2$, we note that (4.17) implies that $W' = 0$ if and only if $U\partial_n F_t(x) \cdot W' \equiv 0$ as a function of x. Using this, we can proceed as in the proof of Theorem 1.4 to see that assertion (ii) is valid for all possible values of q. □

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Strong Solutions to the Problem of Motion of a Rigid Body in a Navier–Stokes Liquid Under the Action of Prescribed Forces and Torques

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Dedicated to Professor O. A. Ladyzhenskaya on her jubilee, in sincere appreciation of her seminal contribution to the mathematical theory of the Navier–Stokes equations

This paper is devoted to the motion of a rigid body in an infinite Navier–Stokes liquid under the action of external forces and torques. For sufficiently regular data, we prove the existence of a local strong solution to the corresponding initial-boundary-value problem for the system body–liquid.

1. Introduction

The motion of rigid bodies in a liquid is one of the oldest and most classical problems in fluid mechanics. In fact, the first, significant contribution to the field can be dated back to the work of Stokes [**1**], Kirchhoff [**2**], and Thomson (Lord Kelvin) and Tait [**3**], around the second half of the nineteenth century.

Due to the complexity of the problem, a systematic and rigorous mathematical study was initiated only much later, in the wake of the fundamental work

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of Oseen [**4**], Leray [**5, 6**] and Ladyzhenskaya [**7, 8**]. This investigation was further deepened and, under certain aspects, completed as a result of the efforts of several mathematicians including Fujita [**9**], Finn [**10**], Babenko [**11**], and Heywood [**12**] (cf. also [**13, 14**]).

It should be emphasized that all the above work deals with the case where the motion of the body through the liquid is *prescribed*; specifically, it is a *given* translational motion. However, during the past couple of decades, in several branches of engineering research it has been increasingly recognized the importance of studying the motion of particles in a viscous liquid when the motion of the particles is *not* prescribed, thus becoming a further unknown. These researches include manufacturing of short-fiber composites [**15**], separation of macromolecules by electrophoresis, [**16**], flow-induced microstructures [**17**], and blood flow problems [**18**]. In all these issues, the presence of the particles *affects the flow of the liquid*, and this, in turn, *affects the motion of the particles*, so that the problem of determining the flow characteristics is highly coupled. It is just this latter feature that makes any fundamental mathematical problem related to liquid-particle interaction a particularly interesting one.

Over the last few years, mathematicians have become interested in this fascinating and challenging subject, and have started a systematic study, mostly investigating the well-posedness of relevant boundary-value problems [**19**]–[**25**], and initial-boundary-value problems [**26**]–[**32**] (cf. also [**33**]).[1)] However, many fundamental questions that have been completely assessed in the classical case of a body moving by prescribed translational motion, remain still open in this more complicated case. A typical example is given by the motion of a rigid body in an unbounded liquid under the action of given forces and torques. This problem is particularly relevant in the study of sedimentation of rigid particles in a Navier–Stokes liquid (like water), where the external forces reduce to the weight (cf. [**33**]). In such a case, only (global) weak solution à la Leray–Hopf are known [**22, 34**], and the question of existence of more regular (strong) solution is open, even for small time interval.[2)] The difficulty related to this problem is two-fold. On the one hand, the equations of the liquid and those of the body are coupled through "nonlocal" terms, representing the force and torque exerted by the liquid on the body. On the other hand, the fact that the body can move by *arbitrary* rigid motion, that is, non-necessarily by translational motion as in the classical case, produces in the equation of the linear momentum of the liquid a term with an *unbounded* (in space) coefficient involving the angular velocity of the body (cf. $(1.1)_1$). Actually, this latter circumstance by itself

[1)] Of course, there is a fairly rich engineering literature dedicated to the theoretical analysis of particle-liquid interaction. Results, however, are not rigorous and they are all based on formal expansions of the relevant fields.

[2)] Notice that the case of one or more particles moving in a *bounded* container filled with fluid, presents other technical difficulties, and even the existence of *global weak* solutions is still an open question [**26**]–[**32**].

makes the problem already challenging. In fact, even if the motion of the body is a *prescribed, constant* rotation, the existence of local (in time) strong solutions is by no means a trivial matter, and it has been established only very recently by Hishida [**35**], by means of a suitable generalization of the semigroup approach of Fujita and Kato [**36**].

In this paper we study the initial-boundary-value problem for the motion of a rigid body in an unbounded Navier–Stokes liquid, under the action of given external forces and torques. Specifically, we show that such a problem has one strong solution, at least in some time interval $[0, T'[$, where the positive number T' depends only on the initial data and on the physical properties of the body and liquid, such as mass, viscosity, etc. The uniqueness of these solutions deserves separate attention, and will be treated elsewhere.

In order to describe our results, we shall now give the mathematical formulation of the problem.

Assume a rigid body $\mathcal{B}$ is moving in an infinitely extended Navier–Stokes liquid $\mathcal{L}$ under the action of external forces and torques. We suppose that only conservative forces act on $\mathcal{L}$ and that the external forces and torques F and M, respectively, acting on $\mathcal{B}$ are known in an inertial frame of reference. In order to make the region occupied by the fluid time-independent, it is convenient to write the equations of motion of the system $\{\mathcal{B}, \mathcal{L}\}$ in a frame attached to $\mathcal{B}$, with origin in the center of mass of $\mathcal{B}$, and coinciding with an inertial frame at time $t = 0$. In this case, the relevant equations are given by (cf. [**33**])

$$
\left\{
\begin{aligned}
&\left.\begin{aligned}
&\frac{\partial u}{\partial t} = \mathrm{div} T(u,p) - u \cdot \mathrm{grad}\, u + V \cdot \mathrm{grad}\, u - \omega \times u,\\
&\mathrm{div} u = 0
\end{aligned}\right\} \text{ in } \mathcal{D} \times (0,\infty),\\
&u = V \quad \text{at } \Sigma \times (0,\infty),\\
&\lim_{|x|\to\infty} u(x,t) = 0, \quad t \in (0,\infty),\\
&m\frac{d\xi}{dt} + m\omega \times \xi + \int_\Sigma T(u,p)\cdot n = \mathrm{F},\\
&I \cdot \frac{d\omega}{dt} + \omega \times (I \cdot \omega) + \int_\Sigma x \times T(u,p) \cdot n = \mathrm{M},\\
&\xi(0) = \xi_0, \quad \omega(0) = \omega_0,\\
&u(x,0) = u_0(x), \quad x \in \mathcal{D},
\end{aligned}
\right.
\tag{1.1}
$$

with

$$
\mathrm{F}(t) = \frac{1}{\varrho_L} Q^\top(t) F(t), \quad \mathrm{M}(t) = \frac{1}{\varrho_L} Q^\top(t) M(t). \tag{1.2}
$$

In system (1.1), $\mathcal{D}$ is the region occupied by the liquid, exterior to $\mathcal{B}$, Σ is the boundary of $\mathcal{D}$ and $\mathcal{B}$. The velocity and pressure fields of $\mathcal{L}$ are $u = u(x,t)$ and

$\widetilde{p} = \varrho_L p(x,t)$, respectively, with ϱ_L constant density of $\mathcal{L}$. The velocity field associated with the rigid motion of $\mathcal{B}$ is $V(x,t) = \xi(t) + \omega(t) \times x$, where ξ is the velocity of the center of mass of the body and ω its angular velocity. T is the stress tensor of $\mathcal{L}$ whose components are given by

$$T_{ij}(u,p) = \nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - p\delta_{ij},$$

with ν kinematic viscosity coefficient. Moreover, $\varrho_L\, m$ is the mass of the rigid body and $\varrho_L I$ the inertia tensor relative to the center of mass of $\mathcal{B}$. The tensor $Q^\top$ is related to ω in the following way:

$$\begin{cases} \dfrac{dQ^\top}{dt} = \Omega(\omega)Q^\top, \\ Q^\top(0) = \mathrm{I}, \end{cases} \qquad \Omega(\omega) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \tag{1.3}$$

It is well known that Q is proper orthogonal, i.e.,

$$Q^\top(t)Q(t) = Q(t)Q^\top(t) = \mathrm{I}, \det Q(t) = 1 \quad \forall t \in [0,T]. \tag{1.4}$$

Conversely, every matrix Q satisfying (1.3), satisfies also (1.4). From (1.4) we immediately find $|Q^\top u| = |Qu| = |u|$ for all $u \in \mathbb{R}^3$. Moreover,

$$|Q_\omega^\top(t) - Q_\chi^\top(t)| \leqslant \int_0^t |(\omega - \chi)(\tau)| d\tau.$$

If $M = 0$ and $F = m_e g$, with m_e "effective mass" of $\mathcal{B}$ and g acceleration of gravity, Eqs. (1.1)–(1.2) describe the sedimentation (free fall) of $\mathcal{B}$ in $\mathcal{L}$ [**33**]. The steady-state counterpart of this problem, obtained by imposing that u, p, ξ, ω, and Q are time-independent, has been investigated by several authors [**19**]–[**33**]. In particular, Weinberger [**19**]–[**21**] proved the existence result provided that the effective mass of the body is small with respect to the viscosity of the fluid. Existence without restriction on the size of the data was first proved by Serre [**22**] (cf. also [**33**]). The uniqueness issue was investigated by Galdi and Vaidya [**25**] who showed, among other things, that the problem may have multiple solutions, even for vanishingly small data.

In this paper, we are interested in the existence of time-dependent solutions. Specifically, for given F, M, u_0, ξ_0, and ω_0 in suitable function classes, we prove the existence of a strong solution (u,p,ξ,ω,Q) to (1.1)–(1.3) in a time-interval $[0,T'[$ for suitable T'. By "*strong*" solution we mean that u, p, ξ, and ω satisfy the conditions

$$\operatorname*{ess\,sup}_{t\in[0,T']} \int_{\mathcal{D}} \left(|u(x,t)|^2 + |\operatorname{grad} u(x,t)|^2\right) dx + \int_0^{T'} \int_{\mathcal{D}} |D^2 u(x,t)|^2 \, dx\, dt < \infty,$$

$$\int_0^{T'} \int_{\mathcal{D}\cap\{|x|<R\}} \left(\left| \frac{\partial}{\partial t} u(x,t) \right|^2 + |\operatorname{grad} p(x,t)|^2 \right) dx\, dt < \infty \tag{1.5}$$

for all sufficiently large R,

$$\int_0^{T'} \left(|\xi(t)|^2 + |\omega(t)|^2 + \left| \frac{d}{dt}\xi(t) \right|^2 + \left| \frac{d}{dt}\omega(t) \right|^2 \right) dt < \infty,$$

and u, ξ, and ω assume continuously the initial data in the appropriate L^2-norm. From (1.3) and (1.5) it follows, in particular, that Q is Lipschitz continuous in $[0, T']$. By classical methods, one can further prove that u and p are, in fact, of class C^∞ in the space variable [**37, 38**].

As was mentioned, very few results are known for the initial-value problem (1.1)–(1.2) or for related simpler problems. The existence of global weak solutions à la Leray–Hopf is proved in [**22, 34**]. If the motion of $\mathcal{B}$ is prescribed as a constant rotation (which amounts to consider only $(1.1)_{1,2,3,4,8}$ with $V = \omega \times x$, $\omega = \text{const}$), the existence of local in time solutions in a class similar to that defined by the first condition in (1.5), was proved by Hishida [**35**] by a semi-group approach. [3)]

The method we use to show our result is a suitable modification of the classical Faedo–Galerkin method for the standard initial-value problem associated with the Navier–Stokes equation, and it is based on the "invading domains" technique [**8, 12**]. The fundamental estimate in order for this method to produce strong solutions, is obtained by formally multiplying $(1.1)_1$ by $\operatorname{div} T(u,p)$ and by integrating by parts over $\mathcal{D}$. Roughly speaking, this generalizes the classical procedure of obtaining estimates for the Navier–Stokes equation, based on multiplying the equation of linear momentum by Au, where A is the Stokes operator (cf., for example, [**12**]). In order for our method to work, we need a special base of the underlying Hilbert space. While in the classical approach this base is constituted by the eigenfunction of A, in the case at hand, the base is constituted by the eigenfunctions of a suitable self-adjoint, positive operator that involves (as expected) *both* $\mathcal{B}$ and $\mathcal{L}$.

Our method furnishes the existence of strong solution only for a finite time-interval. The outstanding problem that remains to be investigated is that of

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3) For the existence of global weak solutions with ξ and ω prescribed we refer to [**39**].

global in time existence, for small initial data. Unlike the classical Navier–Stokes problem, here the question seems to be complicated by the fact that, as we mentioned earlier, the set of *steady* solutions to (1.1)–(1.2) to which, presumably, the unsteady solution is going to converge for large t, need not be formed by a single element, *even for infinitely large viscosity.* This problem will be the object of a future research.

The paper is organized as follows. After recalling in Sec. 2 some preliminary results, in Sec. 3 we prove a key result relating to the spectrum of a suitable linear operator that is the natural generalization of the Stokes operator to the current problem. Finally, in Sec. 4, we state and prove our main result.

2. Notation and Preliminary Results

In addition to the notation introduced in Introduction, we adopt the following one. As customary, we denote by $\mathbb{N}$ the set of nonnegative integers, and by $\mathbb{R}^3$ the three-dimensional Euclidean space. For $\mathcal{A} \subset \mathbb{R}^3$, we denote by $\delta(\mathcal{A})$ the diameter of $\mathcal{A}$. Given $R > 0$, B_R is the ball of radius R centered at the origin, i.e., $B_R = \{x \in \mathbb{R}^3 : \ |x| < R\}$. For $R > \delta(\mathcal{A})$ we set $\mathcal{A}_R = \mathcal{A} \cap B_R$ and $\mathcal{A}^R = \mathcal{A}\backslash\overline{\mathcal{A}_R}$.

We use standard notation for function spaces (cf. [**40**]). For instance, $L^q(\mathcal{A})$, $W^{m,q}(\mathcal{A})$, $W_0^{m,q}(\mathcal{A})$, etc., denote the usual Lebesgue and Sobolev spaces on the domain $\mathcal{A}$, with norms $\|\cdot\|_{q,\mathcal{A}}$ and $\|\cdot\|_{m,q,\mathcal{A}}$ respectively. Whenever confusion will not arise, we shall omit the subscript $\mathcal{A}$. The trace space on $\partial\mathcal{A}$ for functions from $W^{m,q}(\mathcal{A})$ is denoted by $W^{m-1/q,q}(\partial\mathcal{A})$ and its norm by $\|\cdot\|_{m-1/q,q,\partial\mathcal{A}}$. Classical properties and results related to these spaces can be found, for example, in [**40, 41**]. Likewise, by $L^q(0,T;X)$ and $C([0,T];X)$, X a Banach space, we denote the space of all measurable functions from $[0,T]$ to X, such that

$$\int_0^T \|u(t)\|_X^p \, dt < \infty,$$

and the space of continuous functions from $[0,T]$ to X respectively.

For $u \in W^{1,2}(\mathcal{A})$ we denote by $D(u)$ the symmetric part of $\operatorname{grad} u$, i.e.,

$$D(u) = \frac{1}{2}(\operatorname{grad} u + (\operatorname{grad} u)^{\mathsf{T}}).$$

By C we denote a generic constant whose possible dependence on parameters $\xi_1, \dots, \xi_m$ will be specified whenever it is needed. In such a case, we write $C = C(\xi_1, \dots, \xi_m)$, etc. Sometimes, we shall use the symbol C to denote a constant whose numerical value or dependence on parameters is not essential to our aims. In such a case, C may have several different values in a single computation. For example, we may have, in the same line, $2c \leqslant c$.

We now collect a number of preliminary results. We begin with some well-known inequalities (cf., for example, [**41**]).

Lemma 2.1. *Let $u \in W_0^{1,2}(B_R)$. Then there is a positive constants C independent of u and R such that*

$$\|u\|_{2,B_R} \leqslant CR\|\operatorname{grad} u\|_{2,B_R} \qquad \text{(Poincaré's inequality)}$$

$$\|u\|_{6,B_R} \leqslant C\,\|\operatorname{grad} u\|_{2,B_R} \qquad \text{(Sobolev's inequality)}$$

Moreover, the following assertion holds (cf., for example, [**42**]).

Lemma 2.2. *Let $u \in W_0^{1,2}(B_R)$ with $\operatorname{div} u = 0$ in B_R. Then*

$$\|\operatorname{grad} u\|_{2,B_R} = 2\|D(u)\|_{2,B_R} \qquad \text{(Korn's identity)}$$

A proof of the following assertion can be found in [**41**, p. 43].

Lemma 2.3. *Let $\mathcal{A} \subset \mathbb{R}^3$ be a locally lipschitzian bounded domain, and let $u \in W^{1,2}(\mathcal{A})$. Then the trace of u on $\partial\mathcal{A}$, $u\big|_{\partial\mathcal{A}}$, belongs to $L^2(\partial\mathcal{A})$, and the following inequality holds*:

$$\|u|_{\partial\mathcal{A}}\|^2_{2,\partial\mathcal{A}} \leqslant C(\|u\|^2_{2,\mathcal{A}} + \|u\|_{2,\mathcal{A}}\|\operatorname{grad} u\|_{2,\mathcal{A}}),$$

where C is a positive constant depending only on $\partial\mathcal{A}$.

We complete this section with some results on higher order trace of u on the boundary of $\mathcal{D}_R$. To this end, we denote by (r, φ, θ) a system of spherical coordinates and by $\{e_r, e_\varphi, e_\theta\}$ the corresponding orthonormal base. The components of a vector v in such a base will be denoted by $(v_r, v_\varphi, v_\theta)$. If u is a vector in $W^{1,2}(B_R)$, the components of $\operatorname{grad} u$ in the given base are given by the matrix

$$\begin{bmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{\partial u_\varphi}{\partial r} & \dfrac{\partial u_\theta}{\partial r} \\[2ex] \dfrac{1}{r}\dfrac{\partial u_r}{\partial \varphi} - \dfrac{u_\varphi}{r} & \dfrac{1}{r}\dfrac{\partial u_\varphi}{\partial \varphi} + \dfrac{u_r}{r} & \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \varphi} \\[2ex] \dfrac{1}{r\sin\varphi}\dfrac{\partial u_r}{\partial \varphi} - \dfrac{u_\theta}{r} & \dfrac{1}{r\sin\varphi}\dfrac{\partial u_\varphi}{\partial \theta} - \cot\varphi\,\dfrac{u_\theta}{r} & \dfrac{1}{r\sin\varphi}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{u_r}{r} + \cot\varphi\,\dfrac{u_\varphi}{r} \end{bmatrix}$$

Therefore, if $u \in W_0^{1,2}(B_R) \cap W^{2,2}(B_R)$, the trace $\operatorname{grad} u\big|_{\partial B_R}$ is given by the matrix

$$\begin{bmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{\partial u_\varphi}{\partial r} & \dfrac{\partial u_\theta}{\partial r} \\[2ex] 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

evaluated at $r = R$. If, moreover,

$$\operatorname{div} u \equiv \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\varphi}\frac{\partial}{\partial\varphi}(u_\varphi \sin\varphi) + \frac{1}{r\sin\varphi}\frac{\partial u_\theta}{\partial\theta} = 0 \quad \text{in } B_R$$

we obtain $\dfrac{\partial u_r}{\partial r} = 0$ at $r = R$ and $\operatorname{grad} u\big|_{\partial B_R}$ reduces to

$$\operatorname{grad} u\big|_{\partial B_R} \equiv \begin{bmatrix} 0 & \dfrac{\partial u_\varphi}{\partial r} & \dfrac{\partial u_\theta}{\partial r} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.1}$$

evaluated at $r = R$.

With these considerations in mind, we can prove the first part of the following lemma.

Lemma 2.4. *Let $u \in W^{2,2}(\mathcal{D}_R)$ be such that $u = \xi + \omega \times x$ at Σ for some $\xi, \omega \in \mathbb{R}^3$ and $u = 0$ at ∂B_R. Then the following properties hold.*

(a) $(\omega \times x \cdot \operatorname{grad} u - \omega \times u)\big|_{\partial B_R} = 0$,

(b) $(\omega \times x \cdot \operatorname{grad} u - \omega \times u) \cdot n\big|_\Sigma = \xi \times \omega \cdot n\big|_\Sigma$.

PROOF. To show (a), we have only to prove that

$$\omega \times x \cdot \operatorname{grad} u\big|_{\partial B_R} = 0. \tag{2.2}$$

At the boundary of B_R, we have $x = Re_r$. Therefore, using (2.1), we find

$$\omega \times x \cdot \operatorname{grad} u\big|_{\partial B_R} = R\begin{bmatrix} 0 & \omega_\theta & -\omega_\varphi \end{bmatrix} \begin{bmatrix} 0 & \dfrac{\partial u_\varphi}{\partial r} & \dfrac{\partial u_\theta}{\partial r} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and (2.2) is proved. To show property (b), let $w = u - \xi - \omega \times x$. Clearly,

$$w\big|_\Sigma = 0, \quad \operatorname{div} w = 0 \quad \text{in } \mathcal{D}_R. \tag{2.3}$$

Setting $\widetilde{\Omega}(w) = \operatorname{grad} w - D(w)$, we have

$$\operatorname{grad} u \cdot n = D(w) \cdot n + \widetilde{\Omega}(w) \cdot n + \operatorname{grad}(\omega \times x) \cdot n. \tag{2.4}$$

However, since w satisfies (2.3), it is well known that (cf., for example, [**43**])

$$D(w) \cdot n = \frac{1}{2} \operatorname{curl} w \times n \quad \text{at } \Sigma.$$

Moreover, $\widetilde{\Omega}(w) \cdot n = -\frac{1}{2} \operatorname{curl} w \times n$ and from (2.4) we deduce

$$\omega \times x \cdot \operatorname{grad} u \cdot n = \omega \times x \cdot \operatorname{grad}(\omega \times x) \cdot n = \omega \times (\omega \times x) \cdot n \quad \text{at } \Sigma,$$

from which, taking into account that $u\big|_\Sigma = \xi + \omega \times x$, property (b) follows. $\square$

3. Function Spaces and Key Results

In order to prove our main result, we introduce some suitable function spaces on domains $\mathcal{D}$ and $\mathcal{D}_R$, and prove some related fundamental results.

Let $\mathcal{R}$ be the set of velocity fields in a generic rigid motion, namely,

$$\mathcal{R} = \{\overline{u} \in C^\infty(\mathbb{R}^3) : \overline{u}(x) = \overline{u}_1 + \overline{u}_2 \times x, \quad \overline{u}_1, \overline{u}_2 \in \mathbb{R}^3\}.$$

The vectors $\overline{u}_1$ and $\overline{u}_2$ are the *characteristic vectors* of the rigid motion $\overline{u}$.

We define the space $\mathcal{H}(\mathcal{D})$ as follows:

$$\mathcal{H}(\mathcal{D}) = \{u \in L^2(\mathcal{D}) : \operatorname{div} u = 0 \text{ in } \mathcal{D}, (u - \overline{u}) \cdot n\big|_\Sigma = 0 \text{ for some } \overline{u} \in \mathcal{R}\}.$$

The space $\mathcal{H}(\mathcal{D})$ becomes a Hilbert spaces by introducing the following scalar product [4)]

$$(u, v) \longmapsto m\overline{u}_1 \cdot \overline{v}_1 + \overline{u}_2 \cdot I \cdot \overline{v}_2 + \int_{\mathcal{D}} u \cdot v,$$

where $\overline{u}_i$ and $\overline{v}_i$, $i = 1, 2$, are the characteristic vectors of the rigid motions $\overline{u}$ and $\overline{v}$ associated to u and v respectively.

The space $\mathcal{V}(\mathcal{D})$ is a subspace of $\mathcal{H}(\mathcal{D})$ defined as follows:

$$\mathcal{V}(\mathcal{D}) = \{u \in W^{1,2}(\mathcal{D}) : \operatorname{div} u = 0 \text{ in } \mathcal{D}, u\big|_\Sigma = \overline{u} \text{ for some } \overline{u} \in \mathcal{R}\}.$$

The space $\mathcal{V}(\mathcal{D})$ becomes a Hilbert space endowed with the scalar product

$$(u, v) \longmapsto m\overline{u}_1 \cdot \overline{v}_1 + \overline{u}_2 \cdot I \cdot \overline{v}_2 + \int_{\mathcal{D}} u \cdot v + 2\nu \int_{\mathcal{D}} D(u) : D(v).$$

We next define some function spaces on $\mathcal{D}_R$. We introduce the space

$$\mathcal{H}(\mathcal{D}_R) = \{u \in L^2(\mathcal{D}_R) : \operatorname{div} u = 0 \text{ in } \mathcal{D}_R, u \cdot n\big|_{\partial B_R} = 0,$$
$$(u - \overline{u}) \cdot n\big|_\Sigma = 0 \text{ for some } \overline{u} \in \mathcal{R}\}$$

endowed with the inner product

$$(u, v) \longmapsto m\overline{u}_1 \cdot \overline{v}_1 + \overline{u}_2 \cdot I \cdot \overline{v}_2 + \int_{\mathcal{D}_R} u \cdot v$$

and the space

$$\mathcal{V}(\mathcal{D}_R) = \{u \in W^{1,2}(\mathcal{D}_R) : \operatorname{div} u = 0 \text{ in } \mathcal{D}_R, u\big|_{\partial B_R} = 0, u\big|_\Sigma = \overline{u} \text{ for some } \overline{u} \in \mathcal{R}\}$$

endowed with the inner product

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4) As a rule, we shall omit the infinitesimal volume and surface elements in the integrals.

$$(u,v) \longmapsto 2\nu \int_{\mathcal{D}_R} D(u) : D(v). \tag{3.1}$$

The following properties hold.

Lemma 3.1. *Let $u \in \mathcal{V}(\mathcal{D}_R)$. Then*

$$\begin{aligned}
&\|\operatorname{grad} u\|_{2,\mathcal{D}_R} \leqslant \sqrt{2}\|D(u)\|_{2,\mathcal{D}_R},\\
&\|u\|_{2,\mathcal{D}_R} \leqslant CR\|D(u)\|_{2,\mathcal{D}_R},\\
&\|u\|_{6,\mathcal{D}_R} \leqslant C\|D(u)\|_{2,\mathcal{D}_R},\\
&|\overline{u}_1| + |\overline{u}_2| \leqslant C\|D(u)\|_{2,\mathcal{D}_R},
\end{aligned}$$

where C is a positive constant depending on $\mathcal{B}$ at most. Moreover, $\mathcal{V}(\mathcal{D}_R)$ is compactly embedded in $\mathcal{H}(\mathcal{D}_R)$.

Proof. Since $u \in W^{1,2}(\mathcal{D}_R)$ and $u_{|\Sigma} = \overline{u} \in \mathcal{R}$, we extend u by $\overline{u}$ in $\mathcal{B}$, and obtain an extended function, $\widetilde{u}$, say, that belongs to $W_0^{1,2}(B_R)$ and with $\operatorname{div}\widetilde{u} = 0$ in B_R. By Korn's identity (cf. Lemma 2.2), $\widetilde{u}$ satisfies

$$\|\operatorname{grad}\widetilde{u}\|_{2,B_R}^2 = 2\|D(\widetilde{u})\|_{2,B_R}^2.$$

But $D(\widetilde{u}) = 0$ in $\mathcal{B}$ and $\|\operatorname{grad}\widetilde{u}\|_{2,B_R}^2 = 2|\mathcal{B}||\overline{u}_2|^2 + \|\operatorname{grad} u\|_{2,\mathcal{D}_R}^2$. Therefore,

$$2|\mathcal{B}||\overline{u}_2|^2 + \|\operatorname{grad} u\|_{2,\mathcal{D}_R}^2 = 2\|D(u)\|_{2,\mathcal{D}_R}^2.$$

Using the Sobolev inequality for a ball, as in Lemma 2.1, we find

$$\begin{aligned}
|\mathcal{B}|^{\frac{1}{6}}|\overline{u}_1| &\leqslant \|\overline{u}\|_{6,\mathcal{B}} + \|\overline{u}_2 \times x\|_{6,\mathcal{B}} \leqslant \|\widetilde{u}\|_{6,B_R} + \delta(\mathcal{B})|\mathcal{B}|^{\frac{1}{6}}|\overline{u}_2|\\
&\leqslant C\|\operatorname{grad}\widetilde{u}\|_{2,B_R} + \delta(\mathcal{B})|\mathcal{B}|^{\frac{1}{6}}|\overline{u}_2| \leqslant C(\mathcal{B})\|D(u)\|_{2,\mathcal{D}_R}.
\end{aligned}$$

Finally,

$$\|u\|_{2,\mathcal{D}_R} \leqslant \|\widetilde{u}\|_{2,B_R} \leqslant CR\|\operatorname{grad}\widetilde{u}\|_{2,B_R} \leqslant CR\|D(u)\|_{2,\mathcal{D}_R}.$$

The compact embedding of $\mathcal{V}(\mathcal{D}_R)$ in $\mathcal{H}(\mathcal{D}_R)$ is a consequence of the compact embedding of $W^{1,2}(\mathcal{D}_R)$ in $L^2(D_R)$. $\square$

Lemma 3.2. *Given $f \in \mathcal{H}(\mathcal{D}_R)$, the problem*

$$\begin{cases} \left.\begin{aligned} &-\operatorname{div}T(u,p) = f \\ &\operatorname{div}u = 0 \end{aligned}\right\} \quad in\ \mathcal{D}_R, \\ u = 0 \quad at\ |x| = R, \\ m\overline{f}_1 = -\displaystyle\int_{\Sigma} T(u,p)\cdot n, \\ I\cdot\overline{f}_2 = -\displaystyle\int_{\Sigma} x \times T(u,p)\cdot n \end{cases} \tag{3.2}$$

has a unique solution $(u,p) \in \mathcal{V}(\mathcal{D}_R) \cap W^{2,2}(\mathcal{D}_R) \times W^{1,2}(\mathcal{D}_R)$ with

$$\int_{\mathcal{D}_{R_0}} p = 0 \quad for\ fixed\ R_0 < R.$$

Moreover, the following estimates hold:

$$\begin{aligned} &\nu\|D(u)\|_{2,\mathcal{D}_R} \leqslant C(\mathcal{B})R(\|f\|_2^2 + m|\overline{f}_1|^2 + \overline{f}_2\cdot I\cdot\overline{f}_2)^{1/2}, \\ &\|\overline{u}\|_{3/2,2,\Sigma} \leqslant C(\mathcal{B})\|D(u)\|_{2,\mathcal{D}_R}, \\ &\|D^2u\|_{2,\mathcal{D}_R} + \|\operatorname{grad}p\|_{2,\mathcal{D}_R} \leqslant C(\mathcal{B})(\|f\|_{2,\mathcal{D}_R} + \|D(u)\|_{2,\mathcal{D}_R}), \\ &\|\operatorname{grad}u\|_3 \leqslant C(\mathcal{B})(\|f\|_{2,\mathcal{D}_R}^{1/2}\|D(u)\|_{2,\mathcal{D}_R}^{1/2} + \|D(u)\|_{2,\mathcal{D}_R}), \\ &\|p\|_{2,\mathcal{D}_{R_0}} \leqslant C(\mathcal{B})(\|f\|_{2,\mathcal{D}_R} + \|D(u)\|_{2,\mathcal{D}_R}), \end{aligned} \tag{3.3}$$

where $\overline{u} \in \mathcal{R}$ is the trace of u at Σ.

PROOF. To show the existence part, we follow the ideas of Ladyzhenskaya [**8**]. For each $f \in \mathcal{H}_R(\mathcal{D})$ we consider the problem of finding $u \in \mathcal{V}(\mathcal{D}_R)$ such that

$$2\nu\int_{\mathcal{D}_R} D(u) : D(\varphi) = \int_{\mathcal{D}_R} f\cdot\varphi + m\overline{f}_1\cdot\overline{\varphi}_1 + \overline{f}_2\cdot I\cdot\overline{\varphi}_2 \quad \forall\varphi \in \mathcal{V}(\mathcal{D}_R). \tag{3.4}$$

Using Lemma 3.1, it is easy to show that the right-hand side of (3.4) defines a bounded linear functional in $\mathcal{V}(\mathcal{D}_R)$, and therefore, by Riesz theorem, there exists a function $u \in \mathcal{V}(\mathcal{D}_R)$ satisfying (3.4). In fact, one can show that such a function u is the only solution in $\mathcal{V}(\mathcal{D}_R)$ corresponding to the given f. It is then simple routine to prove that u is a solution to (3.2) with the specified regularity. First of all, it is clear that $u \in W^{1,2}(\mathcal{D}_R)$ and that it satisfies $(3.2)_{2,3}$ and $u = 0$ on ∂B_R. Next, in (3.4), we choose $\varphi \in C_0^\infty(\mathcal{D}_R)$ with $\operatorname{div}\varphi = 0$. We get

$$2\nu\int_{\mathcal{D}_R} D(u) : D(\varphi) = \int_{\mathcal{D}_R} f\cdot\varphi$$

for all $\varphi \in C_0^\infty(\mathcal{D}_R)$ with $\operatorname{div}\varphi = 0$. Therefore (cf. [**8**]), $u \in W^{2,2}(\mathcal{D}_R)$ and there exists $p \in W^{1,2}(\mathcal{D}_R)$ that is unique if, for example, $\int\limits_{\mathcal{D}_{R_0}} p = 0$, such that

$$-\operatorname{div} T(u,p) = f \quad \text{in } \mathcal{D}_R. \tag{3.5}$$

We now choose $\varphi \in \mathcal{V}(\mathcal{D}_R)$ such that $\overline{\varphi} = e_i$. Taking the scalar product in (3.5) by φ and integrating by parts, we find

$$-e_i \cdot \int\limits_{\Sigma} T(u,p)\cdot n + 2\nu \int\limits_{\mathcal{D}_R} D(u) : D(\varphi) = \int\limits_{\mathcal{D}_R} f \cdot \varphi$$

and from (3.4) with $\overline{\varphi}_2 = 0$ we deduce $(3.2)_5$. Analogously, taking $\varphi \in \mathcal{V}(\mathcal{D}_R)$ such that $\overline{\varphi} = e_i \times x$, we obtain $(3.2)_6$.

We show the validity of the estimates (3.3). Replacing u for φ in (3.4) and taking into account Lemmas 2.1 and 3.1, we find

$$\begin{aligned}2\nu\|D(u)\|_2^2 &\leqslant (\|f\|_{2,\mathcal{D}_R}^2 + m|\overline{f}_1|^2 + \overline{f}_2\cdot I\cdot\overline{f}_2)^{1/2}(\|u\|_{2,\mathcal{D}_R}^2 + m|\overline{u}_1|^2 + \overline{u}_2\cdot I\cdot\overline{u}_2)^{1/2}\\ &\leqslant C(\mathcal{B})R(\|f\|_{2,\mathcal{D}_R}^2 + m|\overline{f}_1|^2 + \overline{f}_2\cdot I\cdot\overline{f}_2)^{1/2}\|D(u)\|_{2,\mathcal{D}_R}.\end{aligned}$$

Thus,

$$\nu\|D(u)\|_{2,\mathcal{D}_R} \leqslant C(\mathcal{B})R(\|f\|_{2,\mathcal{D}_R}^2 + m|\overline{f}_1|^2 + \overline{f}_2\cdot I\cdot\overline{f}_2)^{1/2},$$

and $(3.3)_1$ is proved. The estimate $(3.3)_2$ follows from Lemma 3.1. The estimate for the second-order derivatives of u and the gradient of p are obtained from classical regularity results for the Stokes problem (cf. [**8, 41**]) and from the first inequality in Lemma 3.1. □

With the aid of the previous lemma, we are now in a position to prove the main result of this section.

Theorem 3.1. *The problem*

$$\begin{cases}\left.\begin{array}{l}-\operatorname{div} T(a,p) = \lambda a\\ \operatorname{div} a = 0\end{array}\right\} \quad \text{in } \mathcal{D}_R,\\ a = 0 \ \text{at } |x| = R,\\ \lambda m\overline{a}_1 = \int\limits_{\Sigma} T(a,p)\cdot n,\\ \lambda I\cdot\overline{a}_2 = \int\limits_{\Sigma} x \times T(a,p)\cdot n,\end{cases} \tag{3.6}$$

$\lambda \in \mathbb{R}$, $a \in \mathcal{V}(\mathcal{D}_R)$, admits a denumerable number of positive eigenvalues $\{\lambda_{Ri}\}$ clustering at infinity. Moreover, the corresponding eigenfunctions $\{a_{Ri}\}$ are in $W^{2,2}(\mathcal{D}_R)$, and the associated pressure fields $\{p_{Ri}\}$ are in $W^{1,2}(\mathcal{D}_R)$. Finally, the vector fields $\{a_{Ri}, i \in \mathbb{N}\}$ form an orthonormal base in $\mathcal{H}(\mathcal{D}_R)$.

PROOF. We follow the same reasoning to prove the existence of eigenfunctions for the classical Stokes problem (cf., for example, [**8, 44**]). The operator $\Lambda : f \to u$ is linear and continuous from $\mathcal{H}(\mathcal{D}_R)$ into $\mathcal{V}(\mathcal{D}_R)$. Since $\mathcal{V}(\mathcal{D}_R)$ is compactly embedded in $\mathcal{H}(\mathcal{D}_R)$, Λ is a compact operator in $\mathcal{H}(\mathcal{D}_R)$ and it is easily seen that this operator is also self-adjoint and positive definite. Hence $\mathcal{H}(\mathcal{D}_R)$ admits a base of eigenfunctions a_i of Λ with corresponding eigenvalues ν_i verifying $\nu_i > 0$ for all $i \in \mathbb{N}$ and $\nu_i \to 0$ as $i \to \infty$. Therefore, we have

$$2\nu \int\limits_{\mathcal{D}} D(a_i) : D(\varphi) = \lambda_i \left[\int\limits_{\mathcal{D}_R} a_i \cdot \varphi + m\overline{a_i}_1 \cdot \overline{\varphi}_1 + \overline{a_i}_2 \cdot I \cdot \overline{\varphi}_2 \right],$$

for all $\varphi \in \mathcal{V}(\mathcal{D}_R)$, where $\lambda_i = 1/\nu_i$. Proceeding as in the proof of the previous lemma, we prove that a_i satisfies (3.6). □

We end this section with a property of denseness of the spaces $\mathcal{H}(\mathcal{D}_R)$.

Lemma 3.3. *Let $\mathcal{S} = \{R_k, k \in \mathbb{N}\}$ be any increasing and unbounded sequence of positive numbers with $R_0 > \delta(\mathcal{B})$. Then $\bigcup\limits_{R_k \in \mathcal{S}} \mathcal{H}(\mathcal{D}_{R_k})$ is dense in $\mathcal{H}(\mathcal{D})$ and $\bigcup\limits_{R_k \in \mathcal{S}} \{a_{R_k i}, i \in \mathbb{N}\}$[5] is a base in $\mathcal{H}(\mathcal{D})$.*

PROOF. Given $u \in \mathcal{H}(\mathcal{D})$, there exists a sequence $(\varphi_k)_{k \in \mathbb{N}}$ of divergence free functions converging to u in $\mathcal{H}(\mathcal{D})$ such that $\operatorname{supp} \varphi_n \subset \overline{\mathcal{D}_{R_k}}$, $\varphi_k = 0$, in a neighborhood of ∂B_{R_k}, and $\varphi_k = \overline{\varphi_k}$, $\overline{\varphi_k} \in \mathcal{R}$, in a neighborhood of Σ. Since $\varphi_k = 0$, in a neighborhood of ∂B_{R_k}, it can be extended by zero to $\mathcal{D}^{R_k}$. Therefore, there exits $R_{k'} \geqslant R_k$ such that $\varphi_k \in \mathcal{H}(\mathcal{D}_{R_k})$. The fact that $\bigcup\limits_{R_k \in \mathcal{S}} \{a_{Ri}, i \in \mathbb{N}\}$ is a base in $\mathcal{H}(\mathcal{D})$ follows from the previous lemma. □

4. Existence of Strong Solutions

The objective of this section is to prove the following main result.

Theorem 4.1. *Let $\mathcal{B}$ be a rigid body with boundary Σ of class C^2. Let $F, M \in L^2(0,T)$ and $u_0 \in \mathcal{V}(\mathcal{D})$. Then there exist $0 < T^* \leqslant T$ and $u = u(x,t)$, $p = p(x,t)$, $\xi = \xi(t)$, $\omega = \omega(t)$, and $Q = Q(t)$ satisfying* (1.1)–(1.3) *a.e. in $\mathcal{D} \times]0, T^*[$ such that*

$$\begin{aligned}
&u, \operatorname{grad} u \in L^\infty(0,T'; L^2(\mathcal{D})), \quad \operatorname{grad} u, D^2 u \in L^2(0,T'; L^2(\mathcal{D})),\\
&\xi, \omega \in W^{1,2}(0,T'), \quad Q \in W^{2,2}(0,T'),\\
&\partial u/\partial t, \operatorname{grad} p \in L^2(0,T'; L^2(\mathcal{D}_R)) \quad \forall R > \delta(\mathcal{B})
\end{aligned} \tag{4.1}$$

[5] We extend the functions in $\mathcal{H}(\mathcal{D}_R)$ by zero in $\mathcal{D}^R$.

for all $T' < T^*$. Moreover,

$$\begin{aligned} &\xi, \omega, Q \in C([0,T']), \quad \xi(0) = \xi_0, \quad \omega(0) = \omega_0, \quad Q(0) = \mathrm{I}, \\ &u \in C([0,T']; L^2(\mathcal{D}_R)) \quad \forall R > \delta(\mathcal{B}) \quad \textit{with } u(\cdot,0) = u_0(\cdot), \end{aligned} \tag{4.2}$$

where ξ_0 and ω_0 are the characteristic vectors of the rigid motion associated to u_0.

The validity of the above result is proved via the Faedo–Galerkin method and requires several intermediate steps.

Step 1: Construction of approximating solutions. Let $\mathcal{S} = \{R_m, m \in \mathbb{N}\}$ be an increasing and unbounded sequence of positive numbers with $R_0 > \delta(\mathcal{B})$, and let $\{\mathcal{D}_{R_m}, m \in \mathbb{N}\}$ be the corresponding sequence of bounded domains invading $\mathcal{D}$. We choose a sequence of initial velocity functions $\{u_{0R_m}, m \in \mathbb{N}\} \subset \mathcal{V}(\mathcal{D})$ such that $u_{0R_m}(x) = 0$ for $|x| \geqslant R_m$, $\|u_{0R_m}\|_{\mathcal{V}(\mathcal{D})} \leqslant \|u_0\|_{\mathcal{V}(\mathcal{D})}$ for all $m \in \mathbb{N}$ and $u_{0R_m} \to u_0$ in $\mathcal{V}(\mathcal{D})$ as $m \to \infty$.

For each $R \in \mathcal{S}$ the sequence of approximating solutions $(u_{Rk}, \xi_{Rk}, \omega_{Rk})$ is defined by

$$u_{Rk}(x,t) = \sum_{i=1}^{k} c_{Rki}(t) a_{Ri}(x), \; \xi_{Rk}(t) = \sum_{i=1}^{k} \overline{a_{Ri}}_1 c_{Rki}(t), \; \omega_{Rk}(t) = \sum_{i=1}^{k} \overline{a_{Ri}}_2 c_{Rki}(t)$$

with $\{a_{Ri}\}_{i\in\mathbb{N}}$ the base introduced in Theorem 3.1, and the coefficients c_{Rkj} are required to be solutions of the following system of ordinary differential equations ($\partial_t \equiv \partial/\partial t$):

$$\begin{aligned} &\int_{\mathcal{D}_R} \partial_t u_{Rk} \cdot a_{Rj} + m\frac{d\xi_{Rk}}{dt} \cdot \overline{a_{Rj}}_1 + \frac{d\omega_{Rk}}{dt} \cdot I \cdot \overline{a_{Rj}}_2 + 2\int_{\mathcal{D}_R} D(a_{Rj}) : D(u_{Rk}) \\ &= -\int_{\mathcal{D}_R} [u_{Rk} \cdot \operatorname{grad} u_{Rk} \cdot a_{Rj} + \omega_{Rk} \times u_{Rk} \cdot a_{Rj} - V_{Rk} \cdot \operatorname{grad} u_{Rk} \cdot a_{Rj}] \\ &\quad + [m\overline{a_{Rj}}_1 \cdot \xi_{Rk} \times \omega_{Rk} + \varphi_{2j} \cdot (I \cdot \omega_{Rk}) \times \omega_k] + F \cdot Q_{Rk} \cdot \overline{a_{Rj}}_1 \\ &\quad + M \cdot Q_{Rk} \cdot \overline{a_{Rj}}_2, \quad j = 1, \dots, k, \\ &\frac{dQ_{Rk}^{\top}}{dt} = \Omega(\omega_{Rk}) Q_{Rk}^{\top} \end{aligned} \tag{4.3}$$

with

$$c_{Rkj}(0) = c_{0Rkj} := \int_{\mathcal{D}_R} a_{Rj} \cdot u_{0R} + m\overline{a_{Rj}}_1 \cdot \xi_{R0} + \overline{a_{Rj}}_2 \cdot I \cdot \omega_{R0}, \quad j = 1, \dots, k,$$

$$Q_{Rk}^{\top}(0) = \mathrm{I}$$

Equations (4.3) can be equivalently written as follows:

$$\begin{aligned}
&\sum_{i=1}^{k} A_{ij}\frac{dc_{Rki}}{dt} = \sum_{i=1}^{k} B_{ij}c_{Rki} + \sum_{i,l=1}^{k} C_{ijl}c_{Rki}c_{Rkl} + F\cdot Q_{Rk}\cdot \overline{a_{Rj}}_1\\
&\quad + M\cdot Q_{Rk}\cdot \overline{a_{Rj}}_2, \quad j=1,\dots,k, \qquad (4.4)\\
&\frac{dQ_{Rk}^{\top}}{dt} = \sum_{q=1}^{k} c_{Rkq}(t)\Omega(\overline{a_{Rq}}_2)Q_{Rk}^{\top},
\end{aligned}$$

where

$$\begin{aligned}
A_{ij} &= \int_{\mathcal{D}_R} a_i\cdot a_j + m\overline{a_{Ri}}_1\cdot\overline{a_{Rj}}_1 + \overline{a_{Ri}}_2\cdot I\cdot\overline{a_{Rj}}_2 = \delta_{ij},\\
B_{ij} &= -2\int_{\mathcal{D}_R} D(a_i):D(a_j),\\
C_{ijl} &= \int_{\mathcal{D}_R} [a_i\cdot \operatorname{grad} a_j\cdot a_k + a_j\cdot\overline{a_{Ri}}_2\times a_l + m\overline{a_{Rj}}_1\cdot\overline{a_{Ri}}_2\times\overline{a_{Rl}}_1\\
&\quad + \overline{a_{Rj}}_2\cdot(I\cdot\overline{a_{Rl}}_2)\times\overline{a_{Ri}}_2.
\end{aligned}$$

If $F, M\in L^2(0,T)$, system (4.4) has a unique solution $c_{Rk}\in W^{1,2}(0,T_{Rk})$ with $T_{Rk}\leqslant T$.

For each $R\in\mathcal{S}$ and each k we put

$$u_{0Rk} := u_{Rk}(.,0), \quad \xi_{0Rk} := \xi_{Rk}(0), \quad \omega_{0Rk} := \omega_{Rk}(0).$$

Since u_{0Rk} is the projection of u_{0R} in $\operatorname{span}\{a_{R1},\dots,a_{Rk}\}$, we have

$$\begin{aligned}
&\|u_{0Rk}\|_{2,\mathcal{D}_R}^2 + m|\xi_{0Rk}|^2 + \omega_{0Rk}\cdot I\cdot\omega_{0Rk}\\
&\qquad\leqslant \|u_0\|_{2,\mathcal{D}_R}^2 + m|\xi_0|^2 + \omega_0\cdot I\cdot\omega_0 \leqslant \|u_0\|_{\mathcal{V}(\mathcal{D})}^2. \qquad (4.5)
\end{aligned}$$

Moreover, since

$$2\nu\int_{\mathcal{D}} D(a_i):D(a_j) = \lambda_{Ri}\left(m\overline{a_i}_1\cdot\overline{a_j}_1 + \overline{a_i}_2\cdot I\cdot\overline{a_j}_2 + \int_{\mathcal{D}_R} a_i\cdot a_j\right) = \lambda_{Ri}\delta_{ij},$$

we have

$$\begin{aligned}
D(u_{0Rk}) &= \sum_{j=1}^{k} c_{Rkj}(0)D(a_j) = 2\nu\sum_{j=1}^{k}\left(\int_{\mathcal{D}_R}\frac{1}{\lambda_{Rj}}D(u_{0R}):D(a_j)\right)D(a_j)\\
&= \sum_{j=1}^{k}\left(\int_{\mathcal{D}_R}\frac{D(u_{0R}):D(a_j)}{\|D(a_j)\|_{2,\mathcal{D}_R}^2}\right)D(a_j)
\end{aligned}$$

and therefore

$$\|D(u_{0Rk})\|_{2,\mathcal{D}_R} \leqslant \|D(u_{0R})\|_{2,\mathcal{D}_R} \leqslant \|u_0\|_{\mathcal{V}(\mathcal{D})}. \tag{4.6}$$

Step 2: Uniform estimates for the approximating solutions. We prove the fundamental estimates of the approximating solutions in a series of lemmas.

Lemma 4.1. *There exist functions $y_i = y_i(t; \mathcal{B}, \mathcal{L}, F, M, u_0, \xi_0, \omega_0)$, $i = 1, 2$, that are continuous in $t \in [0, T[$, such that*

$$\|u_{Rk}(t)\|^2_{2,\mathcal{D}_R} + m\|\xi_{Rk}(t)|^2 + \omega_{Rk}(t) \cdot I \cdot \omega_{Rk}(t) \leqslant y_1, \tag{4.7}$$

$$\int_0^t \|D(u_{Rk})(\tau)\|_{L^2(\mathcal{D}_R)} d\tau \leqslant y_2. \tag{4.8}$$

PROOF. Multiplying (4.3) by c_{Rkj} and summing over j, we see that u_{Rk}, ξ_{Rk} and ω_{Rk} satisfy the following equation: [6]

$$\frac{1}{2}\frac{d}{dt}(\|u\|_2^2 + m|\xi|^2 + \omega \cdot I \cdot \omega) + 2\nu\|D(u)\|^2$$
$$= \int_{\mathcal{D}_R} (V - u) \cdot \operatorname{grad} u \cdot u + \mathrm{F} \cdot \xi + \mathrm{M} \cdot \omega. \tag{4.9}$$

Integrating by parts and recalling that $\operatorname{div} u = 0$ and that $u = V$ at Σ and $u = 0$ at ∂B_R, we find

$$\int_{\mathcal{D}_R} (V - u) \cdot \operatorname{grad} u \cdot u = 0. \tag{4.10}$$

Moreover, by Lemma 3.1 and by Cauchy inequality, we find

$$\begin{aligned} |\mathrm{F} \cdot \xi| &\leqslant |F|\,|\xi| \leqslant \frac{\nu}{2}\|D(u)\|_2^2 + C(\mathcal{B}, \mathcal{L})|F|^2, \\ |\mathrm{M} \cdot \omega| &\leqslant |M||\omega| \leqslant \frac{\nu}{2}\|D(u)\|_2^2 + C(\mathcal{B}, \mathcal{L})|M|^2. \end{aligned} \tag{4.11}$$

Thus, employing (4.10) and (4.11) into (4.9), we conclude

$$\frac{d}{dt}(\|u\|_2^2 + m|\xi|^2 + \omega \cdot I \cdot \omega) + 2\nu\|D(u)\|_2^2 \leqslant C(\mathcal{B}, \mathcal{L})(|F|^2 + |M|^2).$$

The lemma then follows by integrating the last differential inequality. □

The estimates given in the previous lemma imply that $c_{Rk} \in W^{1,2}(0, T)$. In the next lemma, we deduce an estimate for $\operatorname{div} T(u_{Rk}, p_{Rk})$, where $p_{Rk}(x, t) = \sum_{i=1}^{k} c_{Rki}(t) p_{Ri}(x)$. As a consequence, in view of Lemma 3.2, we will obtain

[6] In order do alleviate the notation, throughout the rest of this section, we shall omit the subscripts.

a uniform bound on the second-order spatial derivatives of the approximating solutions.

Lemma 4.2. *There exist functions* $y_i = y_i(t; \mathcal{B}, \mathcal{L}, F, M, u_0, \xi_0, \omega_0)$, $i = 3, 4$, *that are continuous in* $t \in [0, T^*[$ *with* $0 < T^* \leqslant T$ *such that*

$$\|D(u_{Rk})(t)\|_{2,\mathcal{D}_R} \leqslant y_3, \quad \int_0^t \|D^2 u_{Rk}(t)\|_{L^2(\mathcal{D}_R)} d\tau \leqslant y_4 \tag{4.12}$$

PROOF. Multiplying (4.3) by $\lambda_{Rkj} c_{Rkj}$ and summing over j, we deduce

$$\begin{aligned}
\nu \frac{d}{dt} &\|D(u)\|^2_{2,\mathcal{D}_R} + m|T_1|^2 + T_2 \cdot I \cdot T_2 + \|\operatorname{div} T(u,p)\|^2_{2,\mathcal{D}_R} \\
&= m\omega \times \xi \cdot T_1 + F \cdot T_1 + \omega \times (I \cdot \omega) \cdot T_2 + M \cdot T_2 \\
&\quad + \int_{\mathcal{D}_R} u \cdot \operatorname{grad} u \cdot \operatorname{div} T(u,p) - \int_{\mathcal{D}_R} \xi \cdot \operatorname{grad} u \cdot \operatorname{div} T(u,p) \\
&\quad - \int_{\mathcal{D}_R} (\omega \times x \cdot \operatorname{grad} u - \omega \times u) \cdot \operatorname{div} T(u,p),
\end{aligned} \tag{4.13}$$

where

$$T_1 := \frac{1}{m} \int_\Sigma T(u,p) \cdot n, \quad T_2 := I^{-1} \cdot \int_\Sigma x \times T(u,p) \cdot n.$$

We begin to consider the last integral in (4.13). Integrating by parts and using Lemma 2.4 (a), we find

$$\begin{aligned}
\mathcal{I} &\equiv \int_{\mathcal{D}_R} (\omega \times x \cdot \operatorname{grad} u - \omega \times u) \cdot \operatorname{div} T(u,p) \\
&= \int_\Sigma (\omega \times x \cdot \operatorname{grad} u - \omega \times u) \cdot T(u,p) \cdot n \\
&\quad - 2 \int_{\mathcal{D}_R} \operatorname{grad}(\omega \times x \cdot \operatorname{grad} u - \omega \times u) : D(u) = \mathcal{I}_1 - \mathcal{I}_2
\end{aligned}$$

Setting $\Phi = \omega \times x \cdot \operatorname{grad} u - \omega \times u$ and using again integration by parts and Lemma 2.4 (a), it follows that (with $\partial_i = \partial/\partial x_i$ and ε_{ijk} alternating symbol)

$$\begin{aligned}
\mathcal{I}_2 &= \int_{\mathcal{D}_R} \partial_i \Phi_j (\partial_i u_j + \partial_j u_i) = \int_{\mathcal{D}_R} \partial_i \Phi_j \partial_i u_j + \int_\Sigma \Phi_j \partial_j u_i n_i \\
&= \int_{\mathcal{D}_R} (\varepsilon_{lri} \omega_r \partial_l u_j \partial_i u_j + \varepsilon_{lrs} \omega_R x_s (\partial_l \partial_i u_j) \partial_i u_j)
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\mathcal{D}_R} \operatorname{grad}(\omega \times u) : \operatorname{grad} u + \int_{\Sigma} \Phi \cdot \operatorname{grad} u \cdot n \\
&= \int_{\mathcal{D}_R} \omega \times \operatorname{grad} u_\imath \cdot \operatorname{grad} u_\imath - \int_{\mathcal{D}_R} \operatorname{grad}(\omega \times u) : \operatorname{grad} u \\
&\quad + \int_{\Sigma} \left(\frac{1}{2} |\operatorname{grad} u|^2 \omega \times x \cdot n + \Phi \cdot \operatorname{grad} u \cdot n \right).
\end{aligned}
$$

Moreover, by Lemma 2.4 (b), we get

$$
\mathcal{I}_1 = - \int_{\Sigma} p\xi \times \omega \cdot n + \int_{\Sigma} (n \cdot \operatorname{grad} u \cdot \Phi + \Phi \cdot \operatorname{grad} u \cdot n),
$$

and we conclude

$$
\begin{aligned}
\mathcal{I} =& \int_{\Sigma} \left[n \cdot \operatorname{grad} u \cdot (\omega \times x \cdot \operatorname{grad} u - \omega \times u) - \frac{1}{2} |\operatorname{grad} u|^2 \omega \times x \cdot n - p\xi \times \omega \cdot n \right] \\
&- \int_{\mathcal{D}_R} \omega \times \operatorname{grad} u_\imath \cdot \operatorname{grad} u_\imath + \int_{\mathcal{D}_R} \operatorname{grad}(\omega \times u) : \operatorname{grad} u
\end{aligned}
$$

Hence from this relation and (4.13) we obtain

$$
\begin{aligned}
&m|T_1|^2 + T_2 \cdot I \cdot T_2 + \|\operatorname{div} T(u,p)\|^2_{2,\mathcal{D}_R} + \nu \frac{d}{dt} \|D(u)\|^2_{2,\mathcal{D}_R} \\
&= m\omega \times \xi \cdot T_1 + F \cdot T_1 + \omega \times (I \cdot \omega) \cdot T_2 + M \cdot T_2 \\
&\quad + \left(\int_{\mathcal{D}_R} u \cdot \operatorname{grad} u \cdot \operatorname{div} T(u,p) - \int_{\mathcal{D}_R} \xi \cdot \operatorname{grad} u \cdot \operatorname{div} T(u,p) \right) \\
&\quad - \left(\int_{\Sigma} n \cdot \operatorname{grad} u \cdot (\omega \times x \cdot \operatorname{grad} u) \right. \\
&\quad + \left. \int_{\mathcal{D}_R} \left(\operatorname{grad}(\omega \times u) : \operatorname{grad} u - \omega \times \operatorname{grad} u_i \cdot \operatorname{grad} u_\imath \right) \right) \\
&\quad + \left(\frac{1}{2} \int_{\Sigma} |\operatorname{grad} u|^2 \omega \times x \cdot n + \int_{\Sigma} n \cdot \operatorname{grad} u \cdot (\omega \times u) + \int_{\Sigma} p\xi \times \omega \cdot n \right).
\end{aligned}
$$

Using the estimates obtained in Lemma 3.1, we deduce

$$
\begin{aligned}
\Big|\int\limits_{\mathcal{D}_R} u\cdot \operatorname{grad} u \cdot \operatorname{div} T(u,p)\Big| &\leqslant \|u\|_{6,\mathcal{D}_R}\|\operatorname{grad} u\|_{3,\mathcal{D}_R}\|\operatorname{div} T(u,p)\|_{2,\mathcal{D}_R}\\
&\leqslant \varepsilon\|\operatorname{div} T(u,p)\|^2_{2,\mathcal{D}_R} + C_1(\varepsilon,\mathcal{B},\mathcal{L})\|D(u)\|^4_{2,\mathcal{D}_R}\\
&\quad + C_2(\varepsilon,\mathcal{B},\mathcal{L})\|D(u)\|^6_{2,\mathcal{D}_R},
\end{aligned}
$$

$$
\begin{aligned}
&\Big|\int\limits_{\mathcal{D}_R} \xi\cdot \operatorname{grad} u \cdot \operatorname{div} T(u,p)\Big|\\
&\qquad\leqslant \varepsilon\|\operatorname{div} T(u,p)\|^2_{2,\mathcal{D}_R} + C_3(\varepsilon,\mathcal{B},\mathcal{L})\|D(u)\|^4_{2,\mathcal{D}_R},
\end{aligned}
$$

$$
\begin{aligned}
&\Big|\int\limits_{\mathcal{D}_R} (\operatorname{grad}(\omega\times u) : \operatorname{grad} u - \omega\times \operatorname{grad} u_\imath \cdot \operatorname{grad} u_\imath)\Big|\\
&\qquad\leqslant C_4(\mathcal{B},\mathcal{L})\|D(u)\|^3_{2,\mathcal{D}_R},
\end{aligned}
$$

$$
\begin{aligned}
&|m\mathrm{F}\cdot T_1| + |\mathrm{M}\cdot I\cdot T_2|\\
&\qquad\leqslant \delta_1 m|T_1|^2 + \delta_2 T_2\cdot I\cdot T_2 + C(\delta_1,\delta_2,\mathcal{B})(|F|^2+|M|^2),
\end{aligned}
$$

$$
\begin{aligned}
&|m\omega\times\xi\cdot T_1| + |\omega\times(I\cdot\omega)\cdot T_2|\\
&\qquad\leqslant \delta_1 m|T_1|^2 + \delta_2 T_2\cdot I\cdot T_2 + C(\delta_1,\delta_2,\mathcal{B},\mathcal{L})\|D(u)\|^4_{2,\mathcal{D}_R}.
\end{aligned}
$$

In order to estimate the surface integrals we use the estimate of Lemma 2.3 in a fixed bounded domain $\mathcal{D}_{R_0}$ adjacent to Σ, and the estimate for the second-order derivatives of u proved in Lemma 3.2. We find

$$
\begin{aligned}
&\Big|\int\limits_{\Sigma} |\operatorname{grad} u|^2\omega\times x\cdot n\Big|\\
&\qquad\leqslant C(\mathcal{B},\mathcal{L})\|D(u)\|_{2,\mathcal{D}_R}(\|D(u)\|^2_{2,\mathcal{D}_R} + \|D(u)\|_{2,\mathcal{D}_R}\|D^2u\|_{2,\mathcal{D}_R})\\
&\qquad\leqslant \varepsilon\|\operatorname{div} T(u,p)\|^2_{2,\mathcal{D}_R} + C(\varepsilon,\mathcal{B},\mathcal{L})\|D(u)\|^3_{2,\mathcal{D}_R} + C(\varepsilon,\mathcal{B},\mathcal{L})\|D(u)\|^4_{2,\mathcal{D}_R},
\end{aligned}
$$

$$
\begin{aligned}
&\Big|\int\limits_{\Sigma} n\cdot \operatorname{grad} u\cdot(\omega\times x\cdot \operatorname{grad} u)\Big|\\
&\qquad\leqslant \varepsilon\|\operatorname{div} T(u,p)\|^2_{2,\mathcal{D}_R} + C(\varepsilon,\mathcal{B},\mathcal{L})\|D(u)\|^3_{2,\mathcal{D}_R} + C(\varepsilon,\mathcal{B},\mathcal{L})\|D(u)\|^4_{2,\mathcal{D}_R},
\end{aligned}
$$

$$\left| \int_\Sigma n \cdot \operatorname{grad} u \cdot (\omega \times u) \right| \leqslant C(\mathcal{L}, \mathcal{B}) \| \operatorname{grad} u\|_{\Sigma,2} \|\omega \times (\xi + \omega \times x)\|_{\Sigma,2}$$
$$\leqslant (\|D(u)\|^2_{2,\mathcal{D}_R} + \|D(u)\|_{2,\mathcal{D}_R} \|D^2 u\|_{2,\mathcal{D}_R})^{1/2} \|D(u)\|^3_{2,\mathcal{D}_R},$$
$$\leqslant \varepsilon \| \operatorname{div} T(u,p)\|^2_{2,\mathcal{D}_R} + C(\mathcal{L}, \mathcal{B}) \|D(u)\|^2_{2,\mathcal{D}_R} + C(\mathcal{L}, \mathcal{B}) \|D(u)\|^6_{2,\mathcal{D}_R},$$

$$\left| \int_\Sigma p\xi \times \omega \cdot n \right| \leqslant C(\mathcal{L}, \mathcal{B}) \|p\|_{1/2,2,\Sigma} |\xi| \, |\omega|$$
$$\leqslant C(\mathcal{L}, \mathcal{B}) \|D(u)\|^2_{2,\mathcal{D}_R} (\| \operatorname{div} T(u,p)\|_{2,\mathcal{D}_R} + \|D(u)\|_{2,\mathcal{D}_R})$$
$$\leqslant \varepsilon \| \operatorname{div} T(u,p)\|^2_{2,\mathcal{D}_R} + C(\mathcal{L}, \mathcal{B}) \|D(u)\|^4_{2,\mathcal{D}_R} + C(\mathcal{L}, \mathcal{B}) \|D(u)\|^6_{2,\mathcal{D}_R}.$$

Therefore, choosing $\varepsilon = 1/12$, $\delta_1 = \delta_2 = 1/4$, we get

$$\nu \frac{d}{dt} \|D(u)\|^2_{2,\mathcal{D}_R} + m|T_1|^2 + T_2 \cdot I_{\mathcal{B}} \cdot T_2 + \| \operatorname{div} T(u,p)\|^2_{2,\mathcal{D}_R}$$
$$\leqslant C(\mathcal{B}, \mathcal{L}) (\|D(u)\|^2_{2,\mathcal{D}_R} + \|D(u)\|^3_{2,\mathcal{D}_R} + \|D(u)\|^4_{2,\mathcal{D}_R} \|D(u)\|^6_{2,\mathcal{D}_R}$$
$$+ (|F|^2 + |M|^2))$$

Recalling (4.6), there exists an interval $[0, T_2[$ and functions y_3, z_3 that are continuous in $t \in [0, T_2[$ and that depend also on the data, such that

$$\|D(u)(t)\|^2_{2,\mathcal{D}_R} \leqslant y_3, \quad \int_0^t \| \operatorname{div} T(u,p)(\tau)\|^2_{2,\mathcal{D}_R} d\tau \leqslant z_3. \tag{4.14}$$

Since, by Lemma 3.2, we have

$$\|D^2 u\|_{2,\mathcal{D}_R} \leqslant C(\| \operatorname{div} T(u,p)\|_{2,\mathcal{D}_R} + \|D(u)\|_{2,\mathcal{D}_R},)$$

with C independent of k and R, from (4.14) we infer that there exists a continuous function y_4 of $t \in [0, T_2[$ such that

$$\int_0^t \|D^2 u(\tau)\|^2_{2,\mathcal{D}_R} d\tau \leqslant y_4(t). \qquad \square$$

In the next lemma, we provide estimates for the time-derivative of u.

Lemma 4.3. *There exists a function $y_5 = y_5(t; \mathcal{B}, \mathcal{L}, F, M, u_0, \xi_0, \omega_0, R)$ that is continuous for $t \in [0, T^\sharp[$, with $0 < T^\sharp \leqslant T$, such that*

$$\int_0^t \left(\|\partial_\tau u_{Rk}(\tau)\|^2_{L^2(\mathcal{D}_R)} + \left| \frac{d\xi_{Rk}}{d\tau} \right|^2 + \left| \frac{d\omega_{Rk}}{d\tau} \right|^2 \right) \leqslant y_5(t). \tag{4.15}$$

PROOF. Multiplying (4.3) by $\dfrac{dc_{Rkj}}{dt}$ and summing over j, we get

$$\nu\frac{d}{dt}\|D(u)\|^2_{2,\mathcal{D}_R}+m\left|\frac{d\xi}{dt}\right|^2+\frac{d\omega}{dt}\cdot I\cdot\frac{d\omega}{dt}+\|\partial_t u\|^2_{2,\mathcal{D}_R}$$
$$=m\omega\times\xi\cdot\frac{d\xi}{dt}+F\cdot\frac{d\xi}{dt}+\omega\times(I\cdot\omega)\cdot\frac{d\omega}{dt}+M\cdot\frac{d\omega}{dt}+\int\limits_{\mathcal{D}_R}u\cdot\operatorname{grad}u\cdot\partial_t u$$
$$-\int\limits_{\mathcal{D}_R}\xi\cdot\operatorname{grad}u\cdot\partial_t u+\int\limits_{\mathcal{D}_R}\omega\times u\cdot\partial_t u-\int\limits_{\mathcal{D}_R}\omega\times x\cdot\operatorname{grad}u\cdot\partial_t u.$$

All terms, except the last two, are estimated as in the previous lemma. From Lemma 3.1 we deduce

$$\Big|\int\limits_{\mathcal{D}_R}\omega\times u\cdot\partial_t u\Big|+\Big|\int\limits_{\mathcal{D}_R}\omega\times x\cdot\operatorname{grad}u\cdot\partial_t u\Big|\leqslant C(\mathcal{B})R\|D(u)\|_2^2\|\partial_t u\|_2.$$

Estimate (4.15) is then obtained in the same way as those proved in previous lemmas. □

Step 3: Passage to the limit $k\to\infty$, $R_m\to\infty$. Let $T'<T^*$. Passing to the limit $k\to\infty$ from estimates (4.7)–(4.15) we deduce that for any fixed $R\in\mathcal{S}$, there is a vector field $u_R\in L^\infty(0,T';W^{1,2}(\mathcal{D}_R))$ with $\operatorname{grad}u_R, D^2u_R\in L^2(0,T';L^2(\mathcal{D}_R))$, $\partial_t u_R\in L^2(0,T';L^2(\mathcal{D}_R))$, and two vector-valued functions $\xi_R,\omega_R\in W^{1,2}(0,T')$ such that, for all $\psi\in C_0^\infty([0,T^*])$ and $\varphi\in\mathcal{H}(\mathcal{D}_R)$

$$\int\limits_0^{T'}\Big[\int\limits_{\mathcal{D}_R}\partial_t u_R\cdot\psi\varphi+m\frac{d\xi_R}{dt}\cdot\psi\overline{\varphi}_1+\frac{d\omega_R}{dt}\cdot I\cdot\psi\overline{\varphi}_2\Big]+\int\limits_0^{T'}\Big[2\int\limits_{\Sigma}n\cdot D(u_R)\cdot\overline{\varphi}-\int\limits_{\mathcal{D}_R}\Delta u_R\cdot\varphi\Big]$$
$$=-\int\limits_0^{T'}\int\limits_{\mathcal{D}_R}[u_R\cdot\operatorname{grad}u_R\cdot\psi\varphi+\omega_R\times u_R\cdot\psi\varphi-V_R\cdot\operatorname{grad}u_R\cdot\psi\varphi]$$
$$+\int\limits_0^{T'}[m\overline{\varphi}_1\cdot\xi_R\times\omega_R+\overline{\varphi}_2\cdot(I\cdot\omega_R)\times\omega_R]+F\cdot Q_R\cdot\overline{\varphi}_1+M\cdot Q_R\cdot\overline{\varphi}_2],$$
$$\frac{dQ_R^\top}{dt}=\Omega(\omega_R)Q_R^\top.$$

The sequence $(u_{R_m},\xi_{R_m},\omega_{R_m})_{m\in\mathbb{N}}$, obtained by applying the previous passage to the limit in each domain $\mathcal{D}_{R_m}$, still satisfies the uniform estimates for the approximating solutions. Therefore, there exists $u\in L^\infty(0,T';W^{1,2}(\mathcal{D}))$ with $\operatorname{grad}u, D^2u\in L^2(0,T';L^2(\mathcal{D}))$, $\partial_t u\in L^2(0,T';L^2_{\mathrm{loc}}(\overline{\mathcal{D}}))$, $\xi,\omega\in W^{1,2}(0,T')$

such that for each $\varphi \in \mathcal{H}(\mathcal{D})$ with compact support in $\overline{\mathcal{D}}$ we have

$$\int_0^{T'}\left[\int_{\mathcal{D}}\partial_t u\cdot\psi\varphi + m\frac{d\xi}{dt}\cdot\psi\overline{\varphi}_1 + \frac{d\omega}{dt}\cdot I\cdot\psi\overline{\varphi}_2\right] + \int_0^{T'}\left[2\int_{\Sigma} n\cdot D(u)\cdot\overline{\varphi} - \int_{\mathcal{D}}\Delta u\cdot\varphi\right]$$

$$= -\int_0^{T'}\int_{\mathcal{D}}[u\cdot\operatorname{grad} u\cdot\psi\varphi + \omega\times u\cdot\psi\varphi - V\cdot\operatorname{grad} u\cdot\psi\varphi]$$

$$+\int_0^{T'}[m\overline{\varphi}_1\cdot\xi\times\omega + \overline{\varphi}_2\cdot(I\cdot\omega)\times\omega] + F\cdot Q\cdot\overline{\varphi}_1 + M\cdot Q\cdot\overline{\varphi}_2],$$

$$\frac{dQ^{\top}}{dt} = \Omega(\omega)Q^{\top}. \tag{4.16}$$

From (4.16) we obtain, for each R,

$$\int_0^{T'}\int_{\mathcal{D}_R}\partial_t u\cdot\psi\varphi - \int_{\mathcal{D}_R}\Delta u\cdot\varphi = -\int_0^{T'}\int_{\mathcal{D}_R}[u\cdot\operatorname{grad} u\cdot\psi\varphi + \omega\times u\cdot\psi\varphi - V\cdot\operatorname{grad} u\cdot\psi\varphi] \tag{4.17}$$

for all $\varphi \in C_0^\infty(\mathcal{D}_R)$, and from classical results for the Navier–Stokes equations (cf., for example, [**8**]) there exists p with $\operatorname{grad} p \in L^2(0,T';L^2(\mathcal{D}_R))$ such that $(1.1)_{1-5}$ holds *a.e.* in $\mathcal{D}\times(0,T')$. In order to obtain $(1.1)_6$, we multiply $(1.1)_1$ by $\psi\varphi$, φ equal to e_i in a neighborhood of Σ and equal to zero far from Σ, integrate by parts the term with $\operatorname{div} T(u,p)$, and compare the result with (4.16). Equation $(1.1)_7$ is obtained in a similar way. The continuity of $u(\cdot,t),\xi(t)$ and $\omega(t)$ for $t\in[0,T']$ in the corresponding norm (cf. (4.2)) follows from Lemma 4.3 and from the fact that $(u_{R_mk},\xi_{R_mk},\omega_{R_mk})$ converges to (u,ξ,ω) uniformly in $[0,T']$ in the appropriate topology. Notice that we can only guarantee that $u\in C([0,T'];L^2_{\text{loc}}(\overline{\mathcal{D}}))$.

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The Partially Free Boundary Problem for Parametric Double Integrals

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Dedicated to Professor O. A. Ladyzhenskaya in admiration

We prove the existence of conformally paramaterized minimizers for parametric two-dimensional variational problems subject to partially free boundary conditions. We establish regularity of class $H^{2,2}_{\text{loc}} \cap C^{1,\alpha}$, $0 < \alpha < 1$, up to the free boundary under the assumption that there exists a perfect dominance function in the sense of Morrey.

1. Introduction and the Main Results

In $\mathbb{R}^n$, $n \geqslant 3$, we consider a boundary configuration $\langle \Gamma, \mathcal{S} \rangle$ consisting of a closed Jordan arc Γ and a support surface $\mathcal{S}$. First, we merely assume that $\mathcal{S}$ is a closed arcwise connected set such that any two points on $\mathcal{S}$ can be connected by some rectifiable Jordan arc contained in $\mathcal{S}$ and that Γ is a rectifiable Jordan arc whose endpoints P_1 and P_2 lie on $\mathcal{S}$. We also suppose that the open subarc $\overset{\circ}{\Gamma} := \Gamma - \{P_1, P_2\}$ does not meet $\mathcal{S}$.

Denote by B the parameter domain

$$B := \{w = (u,v) \in \mathbb{R}^2 : u^2 + v^2 < 1, v > 0\}$$

whose boundary ∂B consists of an open interval I on the u-axis and a closed semicircle C with the endpoints $w_1 := (-1, 0)$, $w_2 := (1, 0)$ such that

$$I := \{(u,0) \in \mathbb{R}^2 : |u| < 1\}, \quad C := \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 = 1, v \geqslant 0\}.$$

Let $\mathcal{C}(\Gamma, \mathcal{S})$ be the class of surfaces $X \in H^{1,2}(B, \mathbb{R}^n)$ whose traces $X|_C$ and $X|_I$ satisfy the following conditions:

(i) $X|_C$ *is a continuous weakly monotonic mapping from* C *onto* Γ *such that* $X(w_1) = P_1$ *and* $X(w_2) = P_2$,

(ii) $X(u, 0) \in \mathcal{S}$ *for* $\mathcal{L}^1$*-almost all* $u \in I$.

We note that $\mathcal{C}(\Gamma, \mathcal{S})$ is nonempty.

In this paper, we consider the variational problem

$$\mathcal{F}(X) \to \min \quad \text{in } \mathcal{C}(\Gamma, \mathcal{S}), \tag{$\mathcal{P}$}$$

where $\mathcal{F}$ is a general parametric double integral of the form

$$\mathcal{F}(X) := \int_B F(X, X_u \wedge X_v)\, du\, dv.$$

The Lagrangian $F(x, z)$ is defined for $(x, z) \in \mathbb{R}^n \times \mathbb{B}^n$, where $\mathbb{B}^n$ is the space of bivectors $\zeta = \xi \wedge \eta$ with $\xi, \eta \in \mathbb{R}^n$, i.e., $\zeta = (\zeta^{jk})_{j<k}$, where $\zeta^{jk} := \xi^j \eta^k - \xi^k \eta^j$ for $\xi = (\xi^1, \xi^2, \dots, \xi^n)$ and $\eta = (\eta^1, \eta^2, \dots, \eta^n)$. We can identify $\mathbb{B}^n$ with $\mathbb{R}^N$, $N := n(n-1)/2$. The Lagrange identity $|\xi \wedge \eta|^2 = |\xi|^2 |\eta|^2 - (\xi \cdot \eta)^2$ implies

$$|\xi \wedge \eta| \leqslant \frac{1}{2}(|\xi|^2 + |\eta|^2),$$

and we have equality if and only if $|\xi|^2 = |\eta|^2$ and $\xi \cdot \eta = 0$.

Definition 1.1. A function $F : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}$ is said to be a *parametric Lagrangian* if it is of class $C^0(\mathbb{R}^n \times \mathbb{R}^N)$ and satisfies the condition

$$F(x, tz) = tF(x, z) \text{ for all } t > 0,\ (x, z) \in \mathbb{R}^n \times \mathbb{R}^N. \tag{H}$$

The *associated Lagrangian* $f : \mathbb{R}^n \times \mathbb{R}^{2n} \to \mathbb{R}$ of F is defined as follows:

$$f(x, p) := F(x, p_1 \wedge p_2) \text{ for } p = (p_1, p_2) \in \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}. \tag{1.1}$$

Definition 1.2. A parametric Lagrangian F is said to be *definite* if there are numbers m_1 and m_2 with $0 < m_1 \leqslant m_2$ such that

$$m_1|z| \leqslant F(x, z) \leqslant m_2|z| \text{ for all } (x, z) \in \mathbb{R}^n \times \mathbb{R}^N \tag{D}$$

and is said to be *semi-elliptic* if

$$F(x, z) \quad \text{is convex with respect to } z. \tag{C}$$

Our first result establishes the existence of solutions to the *partially free boundary problem* $(\mathcal{P})$ for parametric double integrals. It will be proved in Sec. 2 using an approximation argument developed in [**1**].

Theorem 1.3. *If* $F \in C^0(\mathbb{R}^n \times \mathbb{R}^N)$ *possesses properties* (H), (D), *and* (C), *then there exists a minimizer* X *of* $\mathcal{F}$ *in* $\mathcal{C}(\Gamma, \mathcal{S})$ *which is conformally parametrized, i.e.,* $X \in \mathcal{C}(\Gamma, \mathcal{S})$ *has the property*

$$\mathcal{F}(X) = \inf_{\mathcal{C}(\Gamma, \mathcal{S})} \mathcal{F}$$

and

$$|X_u|^2 = |X_v|^2, \quad X_u \cdot X_v = 0 \ \textit{a.e. on } B. \tag{1.2}$$

Free boundary problems for parametric double integrals were treated by Lipkin [**2**] who proved the existence of minimizers, modifying a method of Morrey [**3**] and using ideas of Courant [**4**]. However, he did not prove the existence of conformally parametrized minimizers.

In Sec. 3, we prove the Hölder continuity up to the boundary of the parameter domain B if the configuration $\langle \Gamma, \mathcal{S} \rangle$ satisfies a chord-arc condition (cf., for example, [**5**, Sec. 7.5]).

Definition 1.4. A set S in $\mathbb{R}^n$ satisfies a *chord-arc condition* if there are numbers $M \geqslant 1$ and $\delta > 0$ such that any two points $P, Q \in S$ with $|P - Q| \leqslant \delta$ can be connected in S by a rectifiable arc Γ^* whose length $L(\Gamma^*)$ satisfies the inequality $L(\Gamma^*) \leqslant M|P - Q|$. More precisely, in this case, we speak of an (M, δ)*-chord-arc condition.*

Theorem 1.5. *Suppose that* $F \in C^0(\mathbb{R}^n \times \mathbb{R}^N)$ *satisfies* (H) *and* (D). *Let* $X \in \mathcal{C}(\Gamma, \mathcal{S})$ *be a conformally parametrized minimizer of* $\mathcal{F}$ *in* $\mathcal{C}(\Gamma, \mathcal{S})$. *If* $\mathcal{S}$ *satisfies an* (M, δ)*-chord-arc condition, then for any* $w_0 \in B \cup I$ *and* $r > 0$ *we have*

$$\int\limits_{B \cap B_r(w_0)} |\nabla X|^2 \, du \, dv \leqslant \left(\frac{2r}{R}\right)^{2\sigma} \int\limits_{B} |\nabla X|^2 \, du \, dv, \tag{1.3}$$

where $0 < R \leqslant \operatorname{dist}(w_0, C)$ *and* $\sigma \in (0, 1/2]$ *is defined by*

$$\sigma := \min\{m_1 m_2^{-1}(1 + M^2)^{-1}, [2\pi \mathcal{F}(X)]^{-1} m_1 \delta^2\}. \tag{1.4}$$

It follows that $X \in C^{0,\sigma}(\overline{\Omega}, \mathbb{R}^n)$ *for every domain* $\Omega \subset\subset B \cup I$. *In particular, for* $\Omega_{\rho_0} := B \cap B_{\rho_0}(0)$ *we obtain the following estimate for the Hölder seminorm* $[X]_{\sigma, \overline{\Omega}_{\rho_0}}$ *of* X *on* $\overline{\Omega}_{\rho_0}$ *if* $0 < \rho_0 < 1$:

$$[X]_{\sigma, \overline{\Omega}_{\rho_0}} \leqslant c(\sigma)(1 - \rho_0)^{-\sigma} \sqrt{\int\limits_B |\nabla X|^2 \, du \, dv}\,, \tag{1.5}$$

where $c(\sigma) > 0$ *depends only on* σ.

We note that this result ensures the Hölder continuity of any conformally parametrized minimizer X of $\mathcal{F}$ in B and up to the "free part" I of the boundary ∂B, but does not even provide the continuity in the corners $w_1 = (-1, 0)$ and $w_2 = (1, 0)$. This, however, can be obtained by requiring a chord-arc condition on $\langle \Gamma, \mathcal{S} \rangle$ (and not on $\mathcal{S}$ alone).

Theorem 1.6. *Suppose that* $F \in C^0(\mathbb{R}^n \times \mathbb{R}^N)$ *satisfies* (H) *and* (D). *Let* $X \in \mathcal{C}(\Gamma, \mathcal{S})$ *be a conformally parametrized minimizer of* $\mathcal{F}$ *in* $\mathcal{C}(\Gamma, \mathcal{S})$. *If* $\mathcal{S} \cup \Gamma$

satisfies an (M,δ)-chord-arc condition, then for $\sigma \in (0,1/2]$ defined by (1.4) and arbitrary $w_0 \in \mathbb{R}^2$ we have

$$\int_{B\cap B_r(w_0)} |\nabla X|^2 \, du\, dv \leqslant c r^{2\sigma} \int_B |\nabla X|^2 \, du\, dv \quad \text{for all } r > 0. \tag{1.6}$$

This implies $X \in C^{0,\sigma}(\overline{B},\mathbb{R}^n)$ and

$$[X]_{\sigma,\overline{B}} \leqslant c(\sigma)\sqrt{\int_B |\nabla X|^2 \, du\, dv}\,. \tag{1.7}$$

Remark 1.7. Since $X(C) = \Gamma$, the inequality (1.7) implies an L^∞-bound for X of the following kind:

$$\sup_{\overline{B}} |X| \leqslant c(\sigma, \Gamma, \mathcal{D}(X)), \tag{1.8}$$

where

$$\mathcal{D}(X) := \mathcal{D}_B(X) = \frac{1}{2}\int_B |\nabla X|^2 \, du\, dv$$

denotes the Dirichlet integral of X. Let Y be a minimal surface minimizing $\mathcal{D}$ in $\mathcal{C}(\Gamma, \mathcal{S})$, i.e.,

$$\mathcal{D}(Y) = \inf_{\mathcal{C}(\Gamma,\mathcal{S})} \mathcal{D}.$$

Then, by (D), for any conformally parametrized minimizer of $\mathcal{F}$ in $\mathcal{C}(\Gamma,\mathcal{S})$ we have

$$m_1\mathcal{D}(X) = m_1\mathcal{A}(X) \leqslant \mathcal{F}(X) \leqslant \mathcal{F}(Y) \leqslant m_2\mathcal{A}(Y) \leqslant m_2\mathcal{D}(Y),$$

where $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ are the areas of X and Y respectively (cf. the definition before Corollary 1.9). Then

$$\mathcal{D}(X) \leqslant (m_2/m_1) \inf_{\mathcal{C}(\Gamma,\mathcal{S})} \mathcal{D}.$$

Thus, we obtain the a priori bound

$$\sup_{\overline{B}} |X| \leqslant c(\sigma, \Gamma, \mathcal{S}). \tag{1.9}$$

Following Morrey [**6**, pp. 390–394], we introduce the useful notion of a *dominance function*. For the sake of convenience, we repeat the terminology introduced in [**7**].

Definition 1.8. Let $F(x,z)$ be a parametric Lagrangian with the associated Lagrangian $f(x,p) = F(x, p_1 \wedge p_2)$.

(i) A function $G : \mathbb{R}^n \times \mathbb{R}^{2n} \to \mathbb{R}$ is called a *dominance function for F* if it is continuous and satisfies the following two conditions:

$$f(x,p) \leqslant G(x,p) \text{ for any } (x,p) \in \mathbb{R}^n \times \mathbb{R}^{2n}, \tag{D1}$$

$$f(x,p) = G(x,p) \quad \text{if and only if } p \in \Pi_0, \tag{D2}$$

where Π_0 denotes the algebraic surface in $\mathbb{R}^{2n}$ defined by the formula

$$\Pi_0 := \{p = (p_1, p_2) \in \mathbb{R}^{2n} : |p_1|^2 = |p_2|^2,\ p_1 \cdot p_2 = 0\}.$$

(ii) A dominance function G of the parametric Lagrangian F is said to be *quadratic* if

$$G(x, tp) = t^2 G(x,p) \quad \text{for all } t > 0,\ (x,p) \in \mathbb{R}^n \times \mathbb{R}^{2n}, \tag{D3}$$

and *positive definite* if there are two constants μ_1 and μ_2 such that $0 < \mu_1 \leqslant \mu_2$ and

$$\mu_1 |p|^2 \leqslant G(x,p) \leqslant \mu_2 |p|^2 \quad \text{for any } (x,p) \in \mathbb{R}^n \times \mathbb{R}^{2n}. \tag{D4}$$

(iii) A function $G \in C^0(\mathbb{R}^n \times \mathbb{R}^{2n}) \cap C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} - \{0\}))$ is called a *perfect dominance function for the parametric Lagrangian F* if it satisfies (D1)–(D4), the ellipticity condition

$$\begin{gathered} \pi \cdot G_{pp}(x,p)\pi \geqslant \lambda(R_0)|\pi|^2 \text{ for } |x| \leqslant R_0 \text{ and } p, \pi \in \mathbb{R}^{2n}, p \neq 0, R_0 > 0, \\ \text{where } \lambda(R_0) > 0 \text{ depends only on } R_0. \end{gathered} \tag{E}$$

Condition (E) means that

$$G_{p^j_\alpha p^k_\beta}(x,p)\pi^j_\alpha \pi^k_\beta \geqslant \lambda(R_0)\pi^j_\alpha \pi^j_\alpha.$$

(Greek indices run from 1 to 2, Latin indices from 1 to n; repeated indices are to be summed from 1 to 2 or n respectively.)

The associated Lagrangian $f(x,p)$ of a parametric Lagrangian $F(x,z)$ fails to be a dominance function for F since (D2) is not satisfied, as one can see for the area integrand $A(z) := |z|$ with the associated Lagrangian

$$a(p) := A(p_1 \wedge p_2) = \sqrt{|p_1|^2|p_2|^2 - (p_1 \cdot p_2)^2}\,.$$

The Lagrangian $A(z)$ has the perfect dominance function

$$D(p) := \frac{1}{2}|p|^2 = \frac{1}{2}(|p_1|^2 + |p_2|^2), \quad p = (p_1, p_2).$$

We note that the associated function $f(x,p)$ of a parametric Lagrangian $F \not\equiv 0$ can only be differentiable on the set $\mathbb{R}^{2n} - \Pi$, where

$$\Pi := \{p = (p_1, p_2) \in \mathbb{R}^{2n} : p_1 \wedge p_2 = 0\}.$$

A perfect dominance function $G(x,p)$ has at most $p = 0$ as a singular point. But this point will truly be singular, except for some special cases such as $D(p)$ (cf. [7, Proposition 1.7] (Grüter's result)).

For a dominance function G of F we introduce the functional

$$\mathcal{G}(X) := \int_B G(X, \nabla X)\, du\, dv.$$

Then $\mathcal{F}(X) \leqslant \mathcal{G}(X)$ for $X \in H^{1,2}(B, \mathbb{R}^n)$, and $\mathcal{F}(X) = \mathcal{G}(X)$ if and only if (1.2) holds. Correspondingly, we set

$$\mathcal{A}(X) := \int_B A(X_u \wedge X_v)\, du\, dv,$$

and we have $\mathcal{A}(X) \leqslant \mathcal{D}(X)$ with equality if and only if (1.2) is fulfilled.

The next result is the key in proving higher regularity of conformally parametrized minimizers (cf. Theorem 1.12 below).

Corollary 1.9. *Suppose that F is continuous and satisfies* (H), (D), *and* (C). *Let G be an arbitrary dominance function of F. Then for the integrals $\mathcal{F}$ and $\mathcal{G}$ corresponding to F and G the following conditions are satisfied.*

(i) $\inf\limits_{\mathcal{C}(\Gamma,\mathcal{S})} \mathcal{F} = \inf\limits_{\mathcal{C}(\Gamma,\mathcal{S})} \mathcal{G}$.

(ii) *Any minimizer of $\mathcal{G}$ in $\mathcal{C}(\Gamma, \mathcal{S})$ is a conformally parametrized minimizer of $\mathcal{F}$ in $\mathcal{C}(\Gamma, \mathcal{S})$.*

(iii) *Conversely, any conformally parametrized minimizer of $\mathcal{F}$ in $\mathcal{C}(\Gamma, \mathcal{S})$ is a minimizer of $\mathcal{G}$ in $\mathcal{C}(\Gamma, \mathcal{S})$.*

PROOF. (i) By Theorem 1.3, there exists a conformally parametrized minimizer X of $\mathcal{F}$ in $\mathcal{C}(\Gamma, \mathcal{S})$. Then

$$\inf_{\mathcal{C}(\Gamma,\mathcal{S})} \mathcal{G} \leqslant \mathcal{G}(X) = \mathcal{F}(X) = \inf_{\mathcal{C}(\Gamma,\mathcal{S})} \mathcal{F} \leqslant \inf_{\mathcal{C}(\Gamma,\mathcal{S})} \mathcal{G},$$

and the first assertion is proved. In the same way, we obtain (iii).

(ii) If X minimizes $\mathcal{G}$ in $\mathcal{C}(\Gamma, \mathcal{S})$, we get

$$\inf_{\mathcal{C}(\Gamma,\mathcal{S})} \mathcal{F} \leqslant \mathcal{F}(X) \leqslant \mathcal{G}(X) = \inf_{\mathcal{C}(\Gamma,\mathcal{S})} \mathcal{G} \underset{(i)}{=} \inf_{\mathcal{C}(\Gamma,\mathcal{S})} \mathcal{F}.$$

Hence $\mathcal{F}(X) = \inf\limits_{\mathcal{C}(\Gamma,\mathcal{S})} \mathcal{F}$ and $\mathcal{F}(X) = \mathcal{G}(X)$, which implies (1.2). □

In particular, we have $\inf\limits_{\mathcal{C}(\Gamma,\mathcal{S})} \mathcal{A} = \inf\limits_{\mathcal{C}(\Gamma,\mathcal{S})} \mathcal{D}$. We can prove higher regularity if the parametric Lagrangian F possesses a perfect dominance function G. In general, we do not know for which F such a function G exists, but we have at least one modest result in store.

Theorem 1.10. *Let $F^* \in C^2(\mathbb{R}^n \times (\mathbb{R}^N - \{0\}))$ satisfy* (H), (D) *and the ellipticity condition*

$$|z|\, \zeta \cdot F^*_{zz}(x,z)\zeta \geqslant \lambda^* |P_z^\perp \zeta|^2 \text{ for } x \in \mathbb{R}^n, z, \zeta \in \mathbb{R}^N, z \neq 0,$$

for some $\lambda^* > 0$ *and* $P_z^\perp \zeta := \zeta - |z|^{-2}(z \cdot \zeta)^2$ *Then for any* $k > k_0 := \max\{2(m_2 - \lambda^*), -m_1/2\}$ *the parametric Lagrangian* F_k *defined by*

$$F_k(x, z) := kA(z) + F^*(x, z)$$

possesses a perfect dominance function.

The proof is based on Morrey's construction in [**6**], we refer the reader to [**8**] and also to [**7**, Proof of Theorem 1.10].

The arguments used in [**7**] to show higher regularity in the interior were based on the *weak Euler equations,* a variant of which we will also employ in the present case choosing admissible test functions.

Proposition 1.11. *Let* $G \in C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} - \{0\}))$ *be an arbitrary function satisfying* (D3). *Then for any surface* $X \in H^{1,2}(B, \mathbb{R}^n) \cap L^\infty_{\text{loc}}(B, \mathbb{R}^n)$ *and any* $\varphi \in H^{1,2}(B, \mathbb{R}^n) \cap L^\infty_{\text{loc}}(B, \mathbb{R}^n)$ *with* $\operatorname{supp} \varphi \subset\subset B$ *we have*

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}[\mathcal{G}(X + \varepsilon\varphi) - \mathcal{G}(X)] = \delta\mathcal{G}(X, \varphi),$$

where

$$\delta\mathcal{G}(X, \varphi) := \int_B [G_p(X, \nabla X) \cdot \nabla\varphi + G_x(X, \nabla X) \cdot \varphi]\, du\, dv. \tag{1.10}$$

If, in addition, X *is a minimizer of* $\mathcal{G}$ *in* $\mathcal{C}(\Gamma, \mathcal{S})$, *then*

$$\delta\mathcal{G}(X, \varphi) = 0 \text{ for any } \varphi \in \overset{\circ}{H}{}^{1,2}(B, \mathbb{R}^n) \cap L^\infty_{\text{loc}}(B, \mathbb{R}^n). \tag{1.11}$$

According to Morrey (cf. [**6, 8**] for details), every parametric Lagrangian $F(x, z)$ of class $C^2(\mathbb{R}^n \times (\mathbb{R}^N - \{0\}))$ has a dominance function $G_0 \in C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} - \{0\}))$ satisfying (D1)–(D4). If F satisfies (H), (D), and (C), then every conformally parametrized minimizer X of $\mathcal{F}$ in $\mathcal{C}(\Gamma, \mathcal{S})$ is a minimizer of $\mathcal{G}_0$ in $\mathcal{C}(\Gamma, \mathcal{S})$ with $X \in L^\infty_{\text{loc}}(B, \mathbb{R}^n)$. Thus, $\delta\mathcal{G}_0(X, \varphi) = 0$ for all $\varphi \in \overset{\circ}{H}{}^{1,2}(B, \mathbb{R}^n) \cap L^\infty_{\text{loc}}(B, \mathbb{R}^n)$. On the other hand, by [**7**, Lemma 3.5], for any dominance function $G \in C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} - \Pi))$ and F with (D1)–(D3) we have $G_x(x, p) = f_x(x, p)$ and $G_p(x, p) = f_p(x, p)$ for $(x, p) \in \mathbb{R}^n \times \Pi_0$, where $f(x, p) = F(x, p_1 \wedge p_2)$ is the associated Lagrangian of F. In particular, these derivatives exist *and, consequently, we obtain (1.11) for arbitrary dominance functions* $G \in C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} - \Pi))$ *with (D1)–(D3), in particular, for* f *itself if* X *is a conformally parametrized minimizer of* $\mathcal{F}$.

Finally, we state our main result on higher regularity of solutions X to the minimum problem $(\mathcal{P})$, to be proved in Sec. 4.

Theorem 1.12. *Suppose that* F *has properties* (H), (D), (C), *and possesses a perfect dominance function. Assume that* $\mathcal{S}$ *is an* m*-dimensional submanifold in* $\mathbb{R}^n$, $1 \leqslant m \leqslant n - 1$, *that is of class* C^3 *and satisfies a chord-arc condition. Then any conformally parametrized minimizer of* $\mathcal{F}$ *in* $\mathcal{C}(\Gamma, \mathcal{S})$ *belongs to the class* $H^{2,2}_{\text{loc}}(B \cup I, \mathbb{R}^n) \cap C^{1,\alpha}(B \cup I, \mathbb{R}^n)$ *for some* $\alpha \in (0, 1)$.

We note that already $\mathcal{S} \in C^1$ implies a chord-arc condition for $\mathcal{S}$ if $\mathcal{S}$ is compact, but a noncompact $\mathcal{S}$ might not satisfy a chord-arc condition even if it is of class C^∞. If $\mathcal{S}$ is complete as a Riemannian submanifold of $\mathbb{R}^n$, but not compact, we might regard a chord-arc condition as a regularity assumption on $\mathcal{S}$ at infinity.

Concerning the regularity of minimizers at the "Plateau boundary" C corresponding to Γ we refer to [**9**].

2. Existence of Minimizers

In this section, we prove the existence of conformally parametrized minimizers for the problem $(\mathcal{P})$ as stated in Theorem 1.3.

We choose some point $P_3 \in \overset{\circ}{\Gamma}$ and set $w_3 := (0,1)$. Then we define $\mathcal{C}(\Gamma, \mathcal{S}, P_3)$ as the subclass of surfaces $X \in \mathcal{C}(\Gamma, \mathcal{S})$ satisfying $X(w_3) = P_3$. We note that the surfaces $X \in \mathcal{C}(\Gamma, \mathcal{S}, P_3)$ respect the 3-point-condition

$$X(w_\nu) = P_\nu, \quad \nu = 1,2,3. \tag{2.1}$$

Let us recall the following variant of the Courant–Lebesgue lemma (cf., for example, [**5**, Vol. I, pp. 259]) which is essential for proving the existence of "partially free" minimizers.

Proposition 2.1. *From any sequence $\{X_j\}$ of surfaces $X_j \in \mathcal{C}(\Gamma, \mathcal{S}, P_3)$ with uniformly bounded Dirichlet integrals $\mathcal{D}_B(X_j)$ we can select a subsequence $\{Y_j\}$ such that*

(i) *$\{Y_j\}$ converges weakly in $H^{1,2}(B, \mathbb{R}^n)$ to some $X \in H^{1,2}(B, \mathbb{R}^n)$,*
(ii) *the traces $Y_j|_C$ converge uniformly on C to $X|_C$,*
(iii) *the weak limit X is of class $\mathcal{C}(\Gamma, \mathcal{S}, P_3)$.*

A consequence of this proposition is the following assertion.

Corollary 2.2. *The class $\mathcal{C}(\Gamma, \mathcal{S}, P_3)$ is closed with respect to the weak convergence of sequences in $H^{1,2}(B, \mathbb{R}^n)$.*

Proof of Theorem 1.3. We can essentially argue as in [**1**] or in [**7**, Sec. 2] with the Courant–Lebesgue lemma replaced with Proposition 2.1 above. For the convenience of the reader we present the line of reasoning.

(i) For $\varepsilon > 0$ we consider a functional $\mathcal{F}^\varepsilon : H^{1,2}(B, \mathbb{R}^n) \to \mathbb{R}$ such that $\mathcal{F}^\varepsilon(X) := \mathcal{F}(X) + \varepsilon \mathcal{D}(X)$. Introducing the nonparametric integrand $f^\varepsilon(x,p) := f(x,p) + \varepsilon |p|^2/2$, we have $\mathcal{F}^\varepsilon(X) = \int_B f^\varepsilon(X, \nabla X)\, du\, dv$. By (C), the Lagrangian $f^\varepsilon(x,p)$ is polyconvex and, consequently, quasiconvex in p and satisfies the inequalities

$$\frac{1}{2}\varepsilon |p|^2 \leqslant f^\varepsilon(x,p) \leqslant \frac{1}{2}(m_2 + \varepsilon)|p|^2.$$

Thus, $\mathcal{F}^\varepsilon$ is sequentially weakly lower semicontinuous on $H^{1,2}(B,\mathbb{R}^n)$ (cf. [**10**]) and satisfies the inequalities

$$\varepsilon\mathcal{D}(X) \leqslant \mathcal{F}^\varepsilon(X) \leqslant (m_2+\varepsilon)\mathcal{D}(X) \text{ for any } X \in H^{1,2}(B,\mathbb{R}^n).$$

Let $\{X_j\}$ be a sequence of surfaces $X_j \in \mathcal{C}(\Gamma,\mathcal{S})$ such that

$$\lim_{j\to\infty} \mathcal{F}^\varepsilon(X_j) = \inf_{\mathcal{C}(\Gamma,\mathcal{S})} \mathcal{F}^\varepsilon =: d(\varepsilon).$$

Since $\mathcal{F}^\varepsilon$ is conformally invariant, we can assume that $X_j \in \mathcal{C}(\Gamma,\mathcal{S},P_3)$ for all $j \in \mathbb{N}$. In addition, the sequence of numbers $\mathcal{D}(X_j)$ is bounded by some constant $c(\varepsilon)$ depending on ε. By Proposition 2.1, we can assume that $X_j \rightharpoonup X^\varepsilon$ in $H^{1,2}(B,\mathbb{R}^n)$ for some surface $X^\varepsilon \in \mathcal{C}(\Gamma,\mathcal{S},P_3)$. Therefore,

$$d(\varepsilon) \leqslant \mathcal{F}^\varepsilon(X^\varepsilon) \leqslant \liminf_{j\to\infty} \mathcal{F}^\varepsilon(X_j) = d(\varepsilon).$$

Hence $\mathcal{F}^\varepsilon(X^\varepsilon) = d(\varepsilon)$. Thus, X^ε minimizes $\mathcal{F}^\varepsilon$ in $\mathcal{C}(\Gamma,\mathcal{S})$ and we find that $\partial\mathcal{F}(X^\varepsilon,\eta) = 0$ for every $\eta \in C^1(\overline{B},\mathbb{R}^2)$, where $\partial\mathcal{F}^\varepsilon(X^\varepsilon,\eta)$ is the inner variation of $\mathcal{F}^\varepsilon$ at X^ε in direction of η. Since $\mathcal{F}$ is parameter invariant, $\partial\mathcal{F}(X^\varepsilon,\eta) = 0$. Hence

$$\partial\mathcal{D}(X^\varepsilon,\eta) = 0 \text{ for any } \eta \in C^1(\overline{B},\mathbb{R}^2).$$

This implies the conformality relations

$$|X_u^\varepsilon|^2 = |X_v^\varepsilon|^2, \quad X_u^\varepsilon \cdot X_v^\varepsilon = 0 \text{ a.e. on } B \tag{2.2}$$

(cf., for example, [**4**] or [**5**]), and we have $\mathcal{A}(X^\varepsilon) = \mathcal{D}(X^\varepsilon)$. Assumption (D) implies $(m_1+\varepsilon)\mathcal{D}(X^\varepsilon) \leqslant \mathcal{F}^\varepsilon(X^\varepsilon)$, and for any $Z \in \mathcal{C}(\Gamma,\mathcal{S})$ we have

$$\mathcal{F}^\varepsilon(X^\varepsilon) = d(\varepsilon) \leqslant \mathcal{F}^\varepsilon(Z) \leqslant m_2\mathcal{A}(Z) + \varepsilon\mathcal{D}(Z) \leqslant (m_2+\varepsilon)\mathcal{D}(Z).$$

Since $(m_2+\varepsilon)/(m_1+\varepsilon) \leqslant m_2/m_1$ for $\varepsilon \geqslant 0$, we see that $\mathcal{D}(X^\varepsilon) \leqslant (m_2/m_1)\mathcal{D}(Z)$ for any $Z \in \mathcal{C}(\Gamma,\mathcal{S})$. If we choose Z as a minimal surface $Y \in \mathcal{C}(\Gamma,\mathcal{S})$ with $\mathcal{D}(Y) = \inf_{\mathcal{C}(\Gamma,\mathcal{S})} \mathcal{D}$, then

$$||X^\varepsilon||_{H^{1,2}(B,\mathbb{R}^n)} \leqslant c(m_1,m_2,\Gamma,\mathcal{S}) \tag{2.3}$$

for some number c independent of $\varepsilon > 0$ by virtue of a suitable Poincaré inequality.

(ii) By (2.3) and Proposition 2.1, there is a sequence of numbers $\varepsilon_j > 0$ such that $\varepsilon_j \to 0$ and a surface $X \in \mathcal{C}(\Gamma,\mathcal{S},P_3)$ such that $X^{\varepsilon_j} \rightharpoonup X$ in $H^{1,2}(B,\mathbb{R}^n)$ as $j \to \infty$. Setting $d(0) := \inf_{\mathcal{C}(\Gamma,\mathcal{S})} \mathcal{F}$, we find

$$d(0) \leqslant \mathcal{F}(X) \leqslant \liminf_{j\to\infty} \mathcal{F}(X^{\varepsilon_j})$$

since $\mathcal{F}$ is sequentially weakly lower semicontinuous on $H^{1,2}(B,\mathbb{R}^n)$ (cf. [**10**], and note that $f(x,p)$ is polyconvex with respect to p and satisfies $0 \leqslant f(x,p) \leqslant m_2|p|^2/2$).

Clearly, $d : [0, \infty) \to \mathbb{R}_+$ is nondecreasing. Therefore, $\lim_{\varepsilon \to +0} d(\varepsilon)$ exists and satisfies

$$d(0) \leqslant \lim_{\varepsilon \to +0} d(\varepsilon) = \lim_{\varepsilon \to +0} \mathcal{F}^\varepsilon(X^\varepsilon)$$

because of (2.3),

On the other hand, $\mathcal{F}^\varepsilon(X^\varepsilon) \leqslant \mathcal{F}^\varepsilon(Z)$ for any $Z \in \mathcal{C}(\Gamma, \mathcal{S})$. Hence

$$\lim_{\varepsilon \to +0} \mathcal{F}^\varepsilon(X^\varepsilon) \leqslant \mathcal{F}(Z)$$

and, consequently, $\lim_{\varepsilon \to +0} \mathcal{F}^\varepsilon(X^\varepsilon) = d(0)$. We conclude that

$$d(0) = \inf_{\mathcal{C}(\Gamma, \mathcal{S})} \mathcal{F} = \mathcal{F}(X) = \lim_{\varepsilon \to +0} \mathcal{F}^\varepsilon(X^\varepsilon).$$

Hence X is a minimizer of $\mathcal{F}$ in $\mathcal{C}(\Gamma, \mathcal{S})$.

(iii) It remains to show that X is conformally parametrized. The minimum properties of X^{ε_j} and X imply

$$\begin{aligned} \mathcal{F}(X^{\varepsilon_j}) + \varepsilon_j \mathcal{D}(X^{\varepsilon_j}) = \mathcal{F}^{\varepsilon_j}(X^{\varepsilon_j}) &\leqslant \mathcal{F}^{\varepsilon_j}(X) = \mathcal{F}(X) + \varepsilon_j \mathcal{D}(X) \\ &\leqslant \mathcal{F}(X^{\varepsilon_j}) + \varepsilon_j \mathcal{D}(X). \end{aligned}$$

Hence $\mathcal{D}(X^{\varepsilon_j}) \leqslant \mathcal{D}(X)$ and, consequently, $\limsup_{j \to \infty} \mathcal{D}(X^{\varepsilon_j}) \leqslant \mathcal{D}(X)$. On the other hand, $X^{\varepsilon_j} \rightharpoonup X$ in $H^{1,2}(B, \mathbb{R}^n)$ implies $\mathcal{D}(X) \leqslant \liminf_{j \to \infty} \mathcal{D}(X^{\varepsilon_j})$. Therefore, $\mathcal{D}(X^{\varepsilon_j}) \to \mathcal{D}(X)$ and, finally, $\|X - X^{\varepsilon_j}\|_{H^{1,2}(B,\mathbb{R}^n)} \to 0$. Taking into account (2.2), we arrive at $|X_u|^2 = |X_v|^2$ and $X_u \cdot X_v = 0$ a.e. on B. $\square$

3. Hölder Continuity

First, we state a result which is proved in the same way as Theorem 1.5 in [**7**]. Therefore, we can omit its proof.

Theorem 3.1. *If* $F \in C^0(\mathbb{R}^n \times \mathbb{R}^N)$ *satisfies* (H), (D), *and* (C), *then every conformally parametrized minimizer* X *of* $\mathcal{F}$ *in* $\mathcal{C}(\Gamma, \mathcal{S})$ *satisfies the Morrey condition*

$$\int_{B_r(w_0)} |\nabla X|^2 \, du \, dv \leqslant \left(\frac{r}{R}\right)^{2\gamma} \int_{B_R(w_0)} |\nabla X|^2 \, du \, dv, \quad \gamma := \frac{m_1}{m_2} \tag{3.1}$$

for any $w_0 = (u_0, v_0) \in B$ *and* $0 < r \leqslant R \leqslant \operatorname{dist}(w_0, \partial B)$. *Therefore,* X *is of class* $C^{0,\gamma}(B, \mathbb{R}^n)$.

Proof of Theorem 1.5. For any $w_0 = (u_0, v_0) \in B \cup I$ we define

$$\Phi(r, w_0) := \int_{S_r(w_0)} |\nabla X|^2 \, du \, dv,$$

where $S_r(w_0) := B \cap B_r(w_0)$ and $d_0 := \operatorname{dist}(w_0, C)$.

(i) If $\rho \in (0, d_0)$ and $v_0 \geqslant \rho$, then $B_r(w_0) \subset B$ for any $r \in [0, \rho]$, and Theorem 3.1 with $R := \rho$ yields

$$\Phi(r, w_0) \leqslant (r/\rho)^{2\gamma}\Phi(R, w_0) \text{ if } 0 < r \leqslant \rho. \tag{3.2}$$

(ii) Let $w_0 \in I$. For $0 < r \leqslant \rho \leqslant d_0$ we set $B_r := B_r(w_0)$, $S_r := S_r(w_0)$, and $\Phi(r) := \Phi(r, w_0)$. Introducing the polar coordinates ρ, θ around $w_0 = (u_0, v_0) \cong u_0 + iv_0$ by $w \cong w_0 + \rho e^{i\theta}$ and writing $X(\rho, \theta) = X(w_0 + \rho e^{i\theta})$, we have $|X_\rho|^2 = \rho^{-2}|X_\theta|^2$, $X_\rho \cdot X_\theta = 0$, and

$$\Phi(r) = 2\int_0^r \rho^{-1} \int_0^\pi |X_\theta(\rho, \theta)|^2 \, d\theta d\rho.$$

For almost all $r \in (0, d_0)$ we have

$$\Phi'(r) = 2r^{-1} \int_0^\pi |X_\theta(r, \theta)|^2 \, d\theta,$$

and the limits $Q_1(r) := \lim_{\theta \to \pi - 0} X(r, \theta)$ and $Q_2(r) := \lim_{\theta \to +0} X(r, \theta)$ exist.

Suppose first that

$$\int_0^\pi |X_\theta(r, \theta)|^2 \, d\theta \leqslant \delta^2/\pi.$$

Then

$$|Q_1(r) - Q_2(r)| \leqslant \int_0^\pi |X_\theta(r, \theta)| \, d\theta \leqslant \sqrt{\pi}\Big[\int_0^\pi |X_\theta(r, \theta)|^2 \, d\theta\Big]^{1/2} \leqslant \delta.$$

By an (M, δ)-chord-arc condition on $\mathcal{S}$, there is a curve Γ^* in $\mathcal{S}$ of length $L(\Gamma^*) = l^*$ connecting $Q_1(r)$ and $Q_2(r)$ and satisfying $l^* \leqslant M|Q_1(r) - Q_2(r)|$. We can assume that Γ^* is given by a Lipschitz parametrization $\xi : [0, l^*] \to \mathbb{R}^n$ such that $\Gamma^* = \xi([0, 1])$ and $|\xi'(s)| = 1$ a.e. on $[0, l^*]$. For the representation $\zeta : [\pi, 2\pi] \to \mathbb{R}^n$ of Γ^* defined by $\zeta(\theta) := \xi((\theta - \pi)l^*/\pi)$ we find $|\zeta_\theta(\theta)| = l^*/\pi$ a.e. and $l^* = \int_\pi^{2\pi} |\zeta_\theta| \, d\theta$; moreover,

$$\int_\pi^{2\pi} |\zeta_\theta|^2 \, d\theta = l^{*2}/\pi.$$

Therefore,

$$\int_\pi^{2\pi} |\zeta_\theta|^2 \, d\theta \leqslant M^2 \int_0^\pi |X_\theta(r, \theta)|^2 \, d\theta.$$

We can assume that $\zeta(2\pi) = Q_2(r)$ and $\zeta(\pi) = Q_1(r)$, and we form the harmonic map $H : B_r \to \mathbb{R}^n$ with the absolutely continuous boundary values $\eta(\theta) = H(w_0 + re^{i\theta})$ defined by $\eta(\theta) := X(r,\theta)$ on $[0,\pi]$ and $\eta(\theta) := \zeta(\theta)$ on $[\pi, 2\pi]$. It follows that

$$\int_0^{2\pi} |\eta_\theta|^2 \, d\theta \leqslant (1 + M^2) \int_0^{\pi} |X_\theta(r,\theta)|^2 \, d\theta.$$

On the other hand, we have the well-known inequality

$$\int_{B_r} |\nabla H|^2 \, du \, dv \leqslant \int_0^{2\pi} |\eta_\theta|^2 \, d\theta.$$

So, we arrive at

$$\int_{B_r} |\nabla H|^2 \, du \, dv \leqslant \frac{1}{2}(1 + M^2) r \Phi'(r).$$

We consider a mapping $Y : B \cup B_r \to \mathbb{R}^n$ such that $Y(w) := X(w)$ for $w \in B - B_r$ and $Y(w) := H(w)$ for $w \in B_r$. Let τ be a homeomorphism from $\overline{B}$ onto $\overline{B \cup B_r}$ that maps B conformally onto $B \cup B_r$ keeping w_1, w_2, and w_3 fixed. Then the reparametrization $Z := Y \circ \tau$ lies in $\mathcal{C}(\Gamma, \mathcal{S})$. Hence the minimum property of X implies $\mathcal{F}(X) \leqslant \mathcal{F}(Z)$.

For $\Omega \subset \mathbb{R}^2$ and $V \in H^{1,2}(\Omega, \mathbb{R}^n)$ we set $\mathcal{F}_\Omega(V) := \int_\Omega f(V, \nabla V) \, du \, dv$. It follows that $\mathcal{F}(Z) = \mathcal{F}_{B \cup B_r}(Y)$ because $\mathcal{F}$ is parameter invariant. Hence $\mathcal{F}_{S_r}(X) \leqslant \mathcal{F}_{B_r}(H)$. Since X is conformally parametrized, we have $m_1 \mathcal{D}_{S_r}(X) = m_1 \mathcal{A}_{S_r}(X) \leqslant \mathcal{F}_{S_r}(X)$ and, trivially, $\mathcal{F}_{B_r}(H) \leqslant m_2 \mathcal{A}_{B_r}(H) \leqslant m_2 \mathcal{D}_{B_r}(H)$. Hence

$$\Phi(r) = 2\mathcal{D}_{S_r}(X) \leqslant (m_2/m_1) 2\mathcal{D}_{B_r}(H).$$

Thus, we arrive at

$$\Phi(r) \leqslant 2^{-1} m_1^{-1} m_2 (1 + M^2) r \Phi'(r) \tag{3.3}$$

provided that

$$\int_0^{\pi} |X_\theta(r,\theta)|^2 \, d\theta \leqslant \delta^2/\pi.$$

If, however,

$$\int_0^{\pi} |X_\theta(r,\theta)|^2 \, d\theta > \delta^2/\pi,$$

we immediately have

$$\Phi(r) \leqslant 2\mathcal{D}(X) \leqslant 2m_1^{-1}\mathcal{F}(X) < \delta^{-2}m_1^{-1}[2\pi\mathcal{F}(X)] \int_0^\pi |X_\theta(r,\theta)|^2\,d\theta.$$

Hence

$$\Phi(r) < \delta^{-2}m_1^{-1}\pi\mathcal{F}(X)r\Phi'(r). \tag{3.4}$$

We set $\sigma := \min\{m_1 m_2^{-1}(1+M^2)^{-1}, [2\pi\mathcal{F}(X)]^{-1}m_1\delta^2\}$, and note that σ satisfies $0 < \sigma \leqslant \min\{\gamma, 1/2\}$. From (3.3) and (3.4) we infer that

$$2\sigma\Phi(r) \leqslant r\Phi'(r) \text{ for a.e. } r \in (0,\rho). \tag{3.5}$$

Integrating, we arrive at $\Phi(r) \leqslant (r/\rho)^{2\sigma}\Phi(\rho)$ for all $r \in (0,\rho]$, which means

$$\Phi(r, w_0) \leqslant (r/\rho)^{2\sigma}\Phi(\rho, w_0) \text{ for all } r \in (0,\rho], \tag{3.6}$$

where $0 < \rho \leqslant d_0 = \operatorname{dist}(w_0, C)$.

(iii) We fix some $R \in (0,1)$ and choose an arbitrary point $w_0 = (u_0, v_0) \in B \cup I$ with $R \leqslant \operatorname{dist}(w_0, C)$.

Case 1: $v_0 \geqslant R/2$. Choosing $\rho := R/2$ in (i), from (3.2) and the fact that $0 < \sigma \leqslant \gamma$ we infer that

$$\Phi(r, w_0) \leqslant (2r/R)^{2\sigma} \int_B |\nabla X|^2\,du\,dv \text{ for } 0 < r \leqslant R/2. \tag{3.7}$$

Case 2: $0 \leqslant v_0 \leqslant R/2$ and $v_0 \leqslant r \leqslant R/2$. Then for $w_0^* := (u_0, 0) \in I$ we have $B_r(w_0) \subset B_{2r}(w_0^*)$ and, consequently, $\Phi(r, w_0) \leqslant \Phi(2r, w_0^*)$. By (ii) for $\rho := R$, in particular (3.6), it follows that $\Phi(2r, w_0^*) \leqslant (2r/R)^{2\sigma}\Phi(R, w_0^*)$ and, consequently,

$$\Phi(r, w_0) \leqslant (2r/R)^{2\sigma} \int_B |\nabla X|^2\,du\,dv. \tag{3.8}$$

In particular, we find

$$\Phi(v_0, w_0) \leqslant (2v_0/R)^{2\sigma} \int_B |\nabla X|^2\,du\,dv. \tag{3.9}$$

Case 3: $0 \leqslant v_0 \leqslant R/2$ and $0 < r \leqslant v_0$.

By the choice $\rho := v_0$ in the inequality (3.2) of (i), we obtain $\Phi(r, w_0) \leqslant (r/v_0)^{2\gamma}\Phi(v_0, w_0)$, and, by the inequality (3.9), we conclude that

$$\Phi(r, w_0) \leqslant (r/v_0)^{2\gamma}(2v_0/R)^{2\sigma} \int_B |\nabla X|^2\,du\,dv.$$

Therefore, by $0 \leqslant \sigma \leqslant \gamma$, we have

$$\Phi(r, w_0) \leqslant (2r/R)^{2\sigma} \int_B |\nabla X|^2 \, du \, dv. \tag{3.10}$$

Thus, with (3.7), (3.8), and (3.10), we have verified that in all possible cases

$$\Phi(r, w_0) \leqslant (2r/R)^{2\sigma} \int_B |\nabla X|^2 \, du \, dv \text{ for all } 0 < r \leqslant R/2,$$

for $w_0 \in B \cup I$ and $R \in (0, \operatorname{dist}(w_0, C)]$. If $r > R/2$, then this inequality holds by obvious reasons. Thus, the inequality (1.3) is proved. The final statements follow from Morrey's Dirichlet growth theorem. □

Theorem 1.6 can be obtained by a straightforward modification of the above arguments, and we leave details to the reader.

4. Higher Regularity at the Free Boundary

PROOF OF THEOREM 1.12. *Step* 1. *Interior regularity.* Let X be a conformally parametrized minimizer of $\mathcal{F}$ in $\mathcal{C}(\Gamma, \mathcal{S})$. Assume that G is a perfect dominance function of F. By Proposition 1.11, we have the weak Euler equation

$$\int_B [G_p(X, \nabla X) \cdot \nabla\varphi + G_x(X, \nabla X) \cdot \varphi] \, du \, dv = 0 \tag{4.1}$$

which holds for all test functions $\varphi \in \mathring{H}^{1,2}(B, \mathbb{R}^n) \cap L^\infty_{\mathrm{loc}}(B, \mathbb{R}^n)$.

By Theorem 1.5, we have $X \in C^{0,\sigma}(B \cup I, \mathbb{R}^n)$ for some $\sigma \in (0,1)$, and

$$\int_{B \cap B_r(w_0)} |\nabla X|^2 \, du \, dv \leqslant \left(\frac{2}{R}\right)^{2\sigma} 2\mathcal{D}(X) r^{2\sigma} \text{ for all } r > 0, \tag{4.2}$$

where $w_0 \in B \cup I$ with $0 < R \leqslant d_0 := \operatorname{dist}(w_0, C)$.

Suppose that $\zeta_0 \in I$, $0 < r_0 < \operatorname{dist}(\zeta_0, C)/2$, and $\Omega = \Omega(\zeta_0, 2r_0) := B \cap B_{2r_0}(\zeta_0)$. By straightforward arguments, from (4.2) for $\Omega_\rho(w_0) := \Omega \cap B_\rho(w_0) = B \cap B_{2r_0}(\zeta_0) \cap B_\rho(w_0)$ we find that

$$\int_{\Omega_\rho(w_0)} |\nabla X|^2 \, dudv \leqslant M_0 \rho^{2\sigma} \text{ for } w_0 \in \mathbb{R}^2 \text{ and } \rho > 0, \tag{4.3}$$

where $M_0 = M(d_0, \mathcal{D}(X))$ is independent of r_0 and ρ. By the reasoning of [**7**, Sec. 4], we find

$$X \in H^{2,2}_{\mathrm{loc}}(B, \mathbb{R}^n) \cap C^{1,\nu}(B, \mathbb{R}^n) \quad \text{for some } \nu \in (0,1). \tag{4.4}$$

Step 2. Linearizing the free boundary condition. Pick any $\zeta_0 \in I$ and set $x_0 := X(\zeta_0)$. Then there is a C^3-diffeomorphism g from $\mathbb{R}^n$ onto itself and an open neighborhood U of x_0 such that

$$g(x_0) = 0, \quad g(U) = K := \{y \in \mathbb{R}^n : |y| < 1\},$$
$$g(\mathcal{S}) = \{y \in K : y^j = 0 \text{ for } m+1 \leqslant j \leqslant n\}.$$

For sufficiently small $r_0 > 0$ and $w \in \overline{\Omega} = \overline{\Omega(\zeta_0, 2r_0)}$ we have $X(w) \in U$. Therefore, $Y := g \circ X \in H^{1,2}(B, \mathbb{R}^n) \cap C^{0,\sigma}(B \cup I, \mathbb{R}^n)$ satisfies $Y(w) \in K$ for $w \in \overline{\Omega}$. Since $X(I) \subset \mathcal{S}$, it follows that

$$Y^j(w) = 0 \text{ for } w \in \overline{\Omega} \cap I, \quad m+1 \leqslant j \leqslant n, \tag{4.5}$$

and the chain rule implies $Y \in H^{2,2}_{\text{loc}}(B, \mathbb{R}^n) \cap C^{1,\nu}(B, \mathbb{R}^n)$. Let $h := g^{-1}$ be the inverse of g, and let $H := h'$ be the Jacobian matrix of h. Then we can write $X = h \circ Y$ with $D_u X = H(Y) D_u Y$ and $D_v X = H(Y) D_v Y$. We introduce the new Lagrangian $\widetilde{G} \in C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} - \{0\}))$ by the formula

$$\widetilde{G}(y, q) := \widetilde{G}(y, q_1, q_2) := G(h(y), H(y) q_1, H(y) q_2) \tag{4.6}$$

for $q = (q_1, q_2) \in \mathbb{R}^{2n}$, and set

$$\widetilde{\mathcal{G}}(Z) := \int_B \widetilde{G}(Z, \nabla Z)\, du\, dv \text{ for any } Z \in H^{1,2}(B, \mathbb{R}^n). \tag{4.7}$$

Then $\mathcal{G}(h \circ Z) = \widetilde{\mathcal{G}}(Z)$ and, in particular, $\mathcal{G}(X) = \widetilde{\mathcal{G}}(Y)$. We set $\widetilde{\Gamma} := g(\Gamma)$ and $\widetilde{\mathcal{S}} := g(\mathcal{S})$, and define $\mathcal{C}(\widetilde{\Gamma}, \widetilde{\mathcal{S}})$ accordingly. Suppose that $\varphi \in H^{1,2}(B, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ satisfies the conditions

$$\begin{gathered} \varphi^{m+1}(w) = 0, \ldots, \varphi^n(w) = 0 \text{ for } w \in \overline{\Omega} \cap I, \\ \varphi(w) = 0 \text{ for } w \in B - \Omega. \end{gathered} \tag{4.8}$$

Then for $|\varepsilon| < \varepsilon_0 \ll 1$, we have $Y^\varepsilon := Y + \varepsilon\varphi \in \mathcal{C}(\widetilde{\Gamma}, \widetilde{\mathcal{S}})$ and, consequently, $X^\varepsilon := h(Y^\varepsilon) \in \mathcal{C}(\Gamma, \mathcal{S})$. Hence

$$0 \leqslant \mathcal{G}(X^\varepsilon) - \mathcal{G}(X) = \widetilde{\mathcal{G}}(Y^\varepsilon) - \widetilde{\mathcal{G}}(Y) \text{ for } |\varepsilon| < \varepsilon_0 \ll 1.$$

We note that $\widetilde{G}(y, q)$ is positively homogeneous of degree two with respect to q, and Y satisfies the generalized conformality relations

$$Y_u \cdot \Xi(Y) Y_u = Y_v \cdot \Xi(Y) Y_v, \quad Y_u \cdot \Xi(Y) Y_v = 0 \text{ a.e. on } B, \tag{4.9}$$

where $\Xi : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is a positive definite symmetric matrix-valued function of class C^2 defined by the formula $\Xi := H^T \cdot H$. By Proposition 1.11 which is still applicable, since $\widetilde{G}$ satisfies (D3), we get

$$\int_B [\widetilde{G}_q(Y, \nabla Y) \cdot \nabla \varphi + \widetilde{G}_y(Y, \nabla Y) \cdot \varphi] \, du \, dv = 0 \tag{4.10}$$

for all test functions φ with the properties (4.8). We note that

$$|\nabla X|^2 = Y_u \cdot \Xi(Y) Y_u + Y_v \cdot \Xi(Y) Y_v, \tag{4.11}$$

and for suitable numbers λ_1 and λ_2 such that $0 < \lambda_1 \leqslant \lambda_2$ we have

$$\lambda_1 |\xi|^2 \leqslant \xi \cdot \Xi(y) \xi \leqslant \lambda_2 |\xi|^2 \tag{4.12}$$

for $\xi \in \mathbb{R}^n$ and $y \in K$. From (4.3) it follows that

$$\int_{\Omega_\rho(w_0)} |\nabla Y|^2 \, du \, dv \leqslant M_0' \, \rho^{2\sigma} \text{ for } w_0 \in \mathbb{R}^2 \text{ and } \rho > 0 \tag{4.13}$$

and $M_0' := \lambda_1^{-1} M_0$. Finally, the ellipticity condition (E) for G is transformed to an analogous ellipticity condition for $\widetilde{G}$:

$$\pi \cdot \widetilde{G}_{qq}(y, q) \pi \geqslant \widetilde{\lambda}(R_0) |\pi|^2 \text{ for } |y| \leqslant R_0, \ \pi, q \in \mathbb{R}^{2n}, \ q \neq 0, \tag{4.14}$$

where $\widetilde{\lambda}(R_0) > 0$ depends only on $R_0 > 0$.

Summarizing, we see that Y has analogous properties as X, but the corresponding *variational equation* (4.10) is true for a larger class of test functions φ. This we shall now exploit.

Step 3. *$H^{2,2}$-estimates of Y on $\Omega = \Omega(\zeta_0, 2r_0)$ for $0 < r_0 \ll 1$.* Let us introduce the set Σ_0 of singular points of X:

$$\Sigma_0 := \{w_0 \in B : X_u(w) \wedge X_v(w) = 0\}$$

It is relatively closed in B since $X \in C^1(B, \mathbb{R}^n)$. Thus, $B - \Sigma_0$ is an open set of $\mathbb{R}^2$. From $\Pi \cap \Pi_0 = \{0\}$ it follows that

$$\Sigma_0 = \{w \in B : |\nabla X(w)| = 0\}.$$

Therefore,

$$\Sigma_0 = \{w \in B : |\nabla Y(w)| = 0\}.$$

From (4.10) and $Y \in H^{2,2}_{\text{loc}}(B, \mathbb{R}^n)$ we find

$$-D_\beta [\widetilde{G}_{q^j_\beta}(Y, \nabla Y)] + \widetilde{G}_y(Y, \nabla Y) = 0 \ \text{ a.e. on } B - \Sigma_0,$$

where $u^1 := u$, $u^2 := v$, $D_1 := D_u$, and $D_2 := D_v$. This is symbolically written as follows:

$$-\nabla [\widetilde{G}_q(Y, \nabla Y)] + \widetilde{G}_y(Y, \nabla Y) = 0 \ \text{ a.e. on } B - \Sigma_0.$$

On $B - \Sigma_0$, we can apply the chain rule:

$$\nabla [\widetilde{G}_q(Y, \nabla Y)] = \widetilde{G}_{qq}(Y, \nabla Y) \nabla^2 Y + \widetilde{G}_{qy}(Y, \nabla Y) \nabla Y.$$

This implies

$$\widetilde{G}_{qq}(Y, \nabla Y)\nabla^2 Y = -\widetilde{G}_{qy}(Y, \nabla Y)\nabla Y + \widetilde{G}_y(Y, \nabla Y) \text{ a.e. on } B - \Sigma_0.$$

Since $\widetilde{G}_y(y, q)$ and $\widetilde{G}_{qy}(y, q)$ are positively homogeneous in q of degree two and one respectively, we have

$$|\widetilde{G}_y(y, q)| \leqslant c_0(R_0)|q|^2, |\widetilde{G}_{qy}(y, q)| \leqslant c_1(R_0)|q| \text{ if } |y| \leqslant R_0$$

for some constants $c_0(R_0)$ and $c_1(R_0)$ depending on R_0. Thus, we arrive at the inequality

$$|\widetilde{G}_{qq}(Y, \nabla Y)\nabla^2 Y| \leqslant \text{const}\, |\nabla Y|^2 \text{ a.e. on } B - \Sigma_0.$$

Furthermore, $\widetilde{G}_{qq}(Y, \nabla Y)\nabla^2 Y = \widetilde{G}_{q_k q_s}(Y, \nabla Y) D_k D_s Y$, and we have

$$|\widetilde{G}_{qq}(y, q)| \leqslant c_2(R_0) \text{ if } |y| \leqslant R_0,\ q \neq 0,$$

for some constant $c_2(R_0)$ depending on R_0 since $\widetilde{G}_{qq}(y, q)$ is positively homogeneous in q of degree zero. Thus,

$$\widetilde{G}_{q_2 q_2}(Y, \nabla Y) D_v D_v Y = \widetilde{G}_{qq}(Y, \nabla Y)\nabla^2 Y + R(Y, \nabla Y, \nabla^2 Y),$$

where the remainder term

$$R(Y, \nabla Y, \nabla^2 Y) = -\sum_{(k,s)\neq(2,2)} \widetilde{G}_{q_k q_s}(Y, \nabla Y) D_k D_s Y$$

can be estimated as follows:

$$|R(Y, \nabla Y, \nabla^2 Y)| \leqslant \text{const}\, |\nabla D_u Y| \text{ a.e. on } B - \Sigma_0.$$

From (4.14) it follows that $\widetilde{G}_{q_2 q_2}(Y(w), \nabla Y(w))$ is invertible for all $w \in B - \Sigma_0$, and its inverse is uniformly bounded in norm on $B - \Sigma_0$. Hence we arrive at the estimate

$$|D_v D_v Y|^2 \leqslant \text{const}(|\nabla D_u Y|^2 + |\nabla Y|^4) \text{ a.e. on } B - \Sigma_0. \tag{4.15}$$

On Σ_0, this estimate is trivially satisfied since $\nabla Y = 0$ on Σ_0 and $Y \in H^{2,2}_{\text{loc}}(B, \mathbb{R}^n)$ imply $\nabla^2 Y = 0$ a.e. on Σ_0. Consequently, the inequality (4.15) holds a.e. on B. Now we proceed in a similar way as in [**11**, pp. 64–68] using some refinements developed for the proof of interior regularity (we refer the reader to [**7**, Sec. 4]).

Introduce the notation. For $0 < |k| \ll 1$ and $z(u, v)$ let $\Delta_k z(u, v) := k^{-1}[z(u + k, v) - z(u, v)]$ be the difference quotient in tangential direction (symbolically, we write $D_u = \lim_{k \to 0} \Delta_k$). Let η be some cut-off function on $\Omega(\zeta_0, 2r)$ for $0 < r \leqslant r_0$, i.e., $\eta \in C^\infty(\overline{B})$, $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ on $B_r(\zeta_0)$, $\eta \equiv 0$ on $\overline{B} - B_{2r}(\zeta_0)$, and $|\nabla \eta| \leqslant 2/r$ on the semi-annulus

$$T_{2r} := \{w \in B : r < |w - \zeta_0| < 2r\} = \Omega(\zeta_0, 2r) - \Omega(\zeta_0, r),$$

and $\nabla \eta \equiv 0$ on $\overline{B} - T_{2r}$.

Then

$$\varphi := -\Delta_{-k}(\eta^2 \Delta_k Y), \quad 0 < |k| \ll 1, \tag{4.16}$$

is an admissible test function for the variational equation (4.10) since $Y \in H^{1,2}(B, \mathbb{R}^n) \cap C^0(B \cup I, \mathbb{R}^n)$ satisfies the boundary conditions (4.5) which are not violated by taking tangential difference quotients as in (4.16). Hence $\varphi^{m+1}(w) = 0, \ldots, \varphi^n(w) = 0$ for $w \in \Omega \cup I$. Manipulations of the same kind as in [**7**, specifically (4.17)–(4.22)] lead first to the relation

$$\int_B \eta^2 |\nabla \Delta_k Y|^2 \, du\, dv \leqslant \text{const}\Big[\int_B \eta^2 |\Delta_k Y|^2 (|\nabla Y|^2 + |\nabla Y_k|^2)\, du\, dv + r^{-2}\mathcal{D}(Y)\Big],$$

where $Y_k(u,v) := Y(u+k, v)$, and then to

$$\int_B \eta^2 |\nabla \Delta_k Y|^2 \, du\, dv \leqslant c(r)\mathcal{D}(Y) \text{ for } |k| \ll 1 \text{ and } 0 < r \ll 1, \tag{4.17}$$

where the number $c(r)$ is independent of k. (The main difficulty to be overcome is that $\widetilde{G}_{qq}(y,q)$ is not defined for $q = 0$, and this is achieved by an appropriate perturbation argument carried out in [**7**].) As $k \to 0$, we arrive at the inequality

$$\int_B \eta^2 |\nabla D_u Y|^2 \, du\, dv \leqslant c(r)\mathcal{D}(Y) \text{ for } 0 < r \ll 1. \tag{4.18}$$

As in [**7**, (4.20)], we see that

$$\int_B \eta^2 |\Delta_k Y|^2 |\nabla Y|^2 \, du\, dv \leqslant c(r)\Big[\int_B \eta^2 |\nabla \Delta_k Y|^2 \, du\, dv + \mathcal{D}(Y)\Big],$$

which in conjunction with (4.17) implies

$$\int_B \eta^2 |\Delta_k Y|^2 |\nabla Y|^2 \, du\, dv \leqslant c(r)\mathcal{D}(Y) \text{ for } |k| \ll 1 \text{ and } 0 < r \ll 1,$$

and as $k \to 0$ we find

$$\int_B \eta^2 |D_u Y|^2 |\nabla Y|^2 \, du\, dv \leqslant c(r)\mathcal{D}(Y) \text{ for } 0 < r \ll 1. \tag{4.19}$$

In particular,

$$\int_B \eta^2 |D_u Y|^4 \, du\, dv \leqslant c(r)\mathcal{D}(Y) \text{ for } 0 < r \ll 1.$$

By (4.9) and (4.12), we have

$$|D_v Y|^2 \leqslant (\lambda_2/\lambda_1)|D_u Y|^2.$$

Hence

$$\int_B \eta^2 |\nabla Y|^4 \, du \, dv \leqslant c(r)\mathcal{D}(Y) \text{ for } 0 < r \ll 1. \tag{4.20}$$

Combining (4.15), (4.18), and (4.20) we find

$$\int_B \eta^2 |\nabla^2 Y|^2 \, du \, dv \leqslant c(r)\mathcal{D}(Y) \text{ for } 0 < r \ll 1. \tag{4.21}$$

Thus,

$$\int_{\Omega(\zeta_0,r)} |\nabla^2 Y|^2 \, du \, dv + \int_{\Omega(\zeta_0,r)} |\nabla Y|^4 \, du \, dv \leqslant c(r)\mathcal{D}(Y) \tag{4.22}$$

for $0 < r \ll 1$ since $\eta \equiv 1$ on $\Omega(\zeta_0, r)$. From $X = h(Y)$ we find

$$\nabla X = h'(Y)\nabla Y \text{ and } \nabla^2 X = h'(Y)\nabla^2 Y + h''(Y)\nabla Y \nabla Y,$$

and, by (4.22),

$$\int_{\Omega(\zeta_0,r)} |\nabla^2 X|^2 \, du \, dv \leqslant c(r)\mathcal{D}(X) \text{ for } 0 < r \ll 1.$$

A covering argument leads to the following inequality:

$$\int_{B \cap B_r(0)} |\nabla^2 X|^2 \, du \, dv \leqslant \text{const} \text{ for } 0 < r < 1,$$

if we also use that $X \in H^{2,2}_{\text{loc}}(B, \mathbb{R}^n)$.

Step 4. *Hölder continuity of* ∇X *on* $B \cup I$. Now, instead of (4.16), we take the test function

$$\varphi := -\eta^2 \Delta_{-k} \Delta_k Y, \quad 0 < |k| \ll 1,$$

which is also admissible in (4.10). Using the same manipulations that lead to (4.25) in [7] and taking into account (4.15) above, we find

$$\begin{aligned} &\int_{\Omega(\zeta_0,r)} |\nabla^2 Y|^2 \, du \, dv \\ &\qquad \leqslant \text{const} \Big[\int_{\Omega(\zeta_0,2r)} |\nabla Y|^4 \, du \, dv + r^{-2} \int_{T_{2r}} |D_u Y - C|^2 \, du \, dv \Big], \end{aligned} \tag{4.23}$$

where C can be an arbitrary vector in $\mathbb{R}^n$. We choose C as the mean value

$$\fint_{T_{2r}} D_u Y \, du \, dv$$

of $D_u Y$ over T_{2r}. By the Poincaré inequality, there is a constant K_p such that

$$\int_{T_{2r}} |D_u Y - C|^2 \, du \, dv \leqslant K_p \, r^2 \int_{T_{2r}} |\nabla D_u Y|^2 \, du \, dv.$$

Therefore, from (4.23) it follows that

$$\int_{\Omega(\zeta_0,r)} |\nabla^2 Y|^2 \, du \, dv \leqslant \text{const} \Big[\int_{T_{2r}} |\nabla^2 Y|^2 \, du \, dv + \int_{\Omega(\zeta_0,2r)} |\nabla Y|^4 \, du \, dv \Big].$$

Recall that $T_{2r} = \Omega(\zeta_0, 2r) - \Omega(\zeta_0, r)$. Hole filling and Sobolev's inequality lead to the inequality

$$\int_{\Omega(\zeta_0,r)} |\nabla^2 Y|^2 \, du \, dv \leqslant \theta_0 \Big[\int_{\Omega(\zeta_0,2r)} |\nabla^2 Y|^2 \, du \, dv + k(\delta) r^{2-2\delta} \Big]$$

for $0 < r \leqslant r_0 \ll 1$, $\delta \in (0, 1/2)$, and some constant $k(\delta)$. A standard iteration procedure yields

$$\int_{\Omega(\zeta_0,r)} |\nabla^2 Y|^2 \, du \, dv \leqslant \text{const}(r/r_1)^{2\alpha} \text{ for } 0 < r \leqslant r_1 \tag{4.24}$$

where $\zeta_0 \in I$, $\alpha := -(\log \theta)/(2 \log 2)$, $\theta := \max\{\theta_0, 2^{-2+2\tau}\} \in (0,1)$, $\tau \in (2\delta, 1)$, $r_1 := \min\{r_0, r^*\}$, $r^* := [\theta^{-1} k(\delta)^{-1} (2^\tau - 1)]^{1/(\tau - 2\delta)}$. The interior estimates of [7] give

$$\int_{B_r(\zeta_0)} |\nabla^2 Y|^2 \, du \, dv \leqslant \text{const}(r/r_2)^{2\alpha} \text{ for } 0 < r \leqslant r_2 \text{ and } \zeta_0 \in B, \tag{4.25}$$

where $r_2 := \min\{r_0', r^*\}$ and $r_0' := 4^{-1} \operatorname{dist}(w_0, \partial B)$. Combining the estimates (4.24) and (4.25) and using similar reasoning as in the proof of Theorem 1.5 in Sec. 3, we conclude that for $\zeta_0 \in I$ and $w_0 \in \Omega(\zeta_0, \rho_0)$ we have

$$\int_{B_r(w_0) \cap \Omega(\zeta_0,\rho_0)} |\nabla^2 Y|^2 \, du \, dv \leqslant c(\rho_0) r^{2\alpha} \text{ for all } r > 0 \tag{4.26}$$

where $0 < \rho_0 \ll 1$ (in particular, $\rho_0 < 2^{-1}(1 - |\zeta_0|)$) and $c(\rho_0) \to \infty$ as $\rho_0 \to +0$. By Morrey's Dirichlet growth theorem, from (4.26) we conclude that ∇Y is of class $C^{0,\alpha}$ on $\Omega(\zeta_0, \rho_0)$ for $0 < \rho_0 \ll 1$. By $\nabla X = h'(Y) \nabla Y$ and Theorem 1.5, the same is true for ∇X if w.l.o.g. we assume that $0 < \alpha \leqslant \sigma$. A covering argument finally implies that $X \in C^{1,\alpha}(B \cup I, \mathbb{R}^n)$, as we have claimed. $\square$

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On Evolution Laws Forcing Convex Surfaces to Shrink to a Point

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Dedicated to Professor O. A. Ladyzhenskaya

A general approach to evolution laws forcing convex surfaces to shrink to a point in a finite time is presented. The nontriviality of such a generalization is illustrated by examples.

1. Introduction

We consider an evolution $\{\Gamma_t,\ t > 0\}$ of a closed surface Γ_0 under the law

$$E[\Gamma_t] := E(s(v), \mathbf{k})[\Gamma_t] = f, \quad t > 0, \tag{1.1}$$

where $s(v)$ is equal v or $-v$, $v(M)$ is the normal component of the velocity at a point $M \in \Gamma_t$, $\mathbf{k}(M) = (k_1, \dots, k_n)(M_t)$ is the vector of principal curvatures of Γ_t, $E = E(s, S)$ and $f = f(t)$ are given functions, $(s, S) \in D \subset R^1 \times \mathrm{Sym}(n)$, $n \geqslant 2$, and $\mathrm{Sym}(n)$ is the space of symmetric matrices of order n. Throughout the paper, $\mathbf{k}$ and s are diagonal matrices.

Since (1.1) contains only the normal component of the velocity, the required evolution exists if $v \neq 0$ and E is a monotone function, Without loss of generality, we assume that $v > 0$ and the curvatures of spheres are positive.

In this paper, we describe laws guaranteeing that Γ_t, $t > 0$, either expand indefinitely as $t \to \infty$ or shrink to a point in a finite time. Such a general

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approach was presented in the recent paper [**1**]. The corresponding results in [**1**] will be formulated below.

Many important examples of geometric contractions, starting with flow by mean curvature [**2**], are well known (cf. [**3, 4**]). It turns out that the study of contractions also admits a general approach. The main goal of this paper is to describe such an approach.

While developing the theory of fully nonlinear second-order differential equations, it becomes clear that one of the necessary requirements in this theory is the global monotonicity of E in some domain $D \subset R^1 \times \mathrm{Sym}(n)$ such that $(s,S) \in D$ implies $(s+\sigma\,\mathrm{sgn}(s), S+\xi\times\xi) \in D$ for $\sigma \geqslant 0$, $\xi \in R^n$. In fact, we require the strict monotonicity of E, i.e.,

$$E(s+\sigma, S+\xi\times\xi) > E(s,S), \quad \sigma \geqslant 0,\ \xi \in R^n,\ \sigma + |\xi| > 0, \tag{1.2}$$

if $D \subset R^+ \times \mathrm{Sym}(n)$ and

$$E(s, S+\xi\times\xi) > E(s-\sigma, S) \tag{1.3}$$

if $D \subset R^- \times \mathrm{Sym}(n)$.

We denote by R^+ the set of all positive real numbers and by R^- the set of all negative real numbers. Hereinafter, D is the domain of global monotonicity of E. We always assume that D is convex.

The condition $v > 0$ implies that either $s = v$ or $s = -v$ and the inequalities (1.2) and (1.3) express the opposite types of monotonicity of E with respect to v depending on D, whereas E is always assumed to be positive monotone with respect to S. In this paper, we distinguish the cases (1.2) and (1.3). We say that the variables v and S are *E-cooperating* in the case (1.2) and are *E-competing* in the case (1.3). Using this definiiton, we can explicitly indicate the kind of evolution described by Eq. (1.1).

Proposition 1.1. *If the variables v and $\mathbf{k}$ are E-cooperating, then Eq.* (1.1) *describes expansion. If v and $\mathbf{k}$ are E-competing, then only contractions can happen.*

Indeed, under the assumptions of Proposition 1.1, we always deal with well-posed parabolic problems, which is easily seen by introducing a local parametrization.

By the monotonicity conditions (1.2) and (1.3), we can introduce notions of E-admissible surfaces and E-admissible evolutions that are similar to the correpsonding notions for funcitons in the theory of fully nonlinear elliptic and parabolic equations (cf. [**3**]–[**5**]).

Definition 1.2. A surface Γ is said to be *E-admissible* if it is a closed C^3-surface and there exists a positive function $v \in C^1, v = v(M)$ such that $(s(v),\mathbf{k})(M) \in D$, $M \in \Gamma$. An evolution $\{\Gamma_t, t \in [t_0;t_1)\}$ is said to be *E-admissible* if the surfaces Γ_t are E-admissible for all $t \in [t_0;t_1)$. A solution to Eq. (1.1) is said to be *admissible* if it is an E-admissible evolution.

If an E-admissible evolution (solution) consists of strictly convex surfaces, then we call it a convex-monotone evolution (solution).

For convex-monotone evolutions the law (1.1) can be written in the form

$$G(v,\mathbf{r})[\Gamma_t]=g, \tag{1.4}$$

where $\mathbf{r}[\Gamma_t]=(r_1,\dots,r_n)[\Gamma_t]$ is the vector of radii of the principal curvatures. The notion of E-admissibility can be expressed in terms of $(v,\mathbf{r})$. Therefore, we can introduce the notion of G_r-admissibility. The law (1.4) allows us to associate contractions of convex surfaces with the case of cooperating variables. This observation is the basis of our further consideration. In the case of competing variables, Eq. (1.4) can be written as follows:

$$E(-v,\mathbf{k})=-G(v,(\mathbf{k})^{-1})=-g.$$

We consider only orthogonal invariant couples (D,E). Namely, if $(s,S)\in D$, then $(s,BSB^T)\in D$ and $E(s,S)=E(s,BSB^T)$ for $B\in O(n)$. For diagonal matrices s this means that they are invariant under permutations of diagonal elements.

Since E is strictly monotone with respect to v, we can write Eq. (1.1) in the form

$$v=\widehat{F}(\mathbf{k}). \tag{1.5}$$

The uniformization (1.5) of the evolution law (1.1) is usually said to be a *flow by* $\widehat{F}$. A solution to Eq. (1.5) is also referred to as a *flow*. Using this terminology, we can say that the papers [**2**]–[**4**] and [**8, 9**] deal with geometric flows. The general concept of evolution laws was developed in [**6, 7, 10**] within the framework of the theory of fully nonlinear second-order parabolic equations. We note that for the right-hand side of (1.5) the function $\widehat{F}=F(\mathbf{k})$ was considered in [**2**]–[**4**] and $\widehat{F}=1/F(\mathbf{k})$ in [**8, 9**]. In both cases, F is a positive monotone homogeneous function of $\mathbf{k}$ in some convex domain in R^n. This difference in the statement agrees with Proposition 1.1 and, on the other hand, separates contractions from expansions.

The general form of evolution laws (1.1) provides us with new sources of information (a domain D, the boundary of D, and a function $f(t)$) and new examples.

One of important problems in the theory of fully nonlinear equations is to control the admissibility of solutions. Some general cases where admissibility was controlled by the right-hand side of equations were first considered in [**5**] for fully nonlinear second-order elliptic Hessian equations. The case of parabolic equations was treated in [**6, 7, 10**] in a similar way. To apply the ideas from there to Eq. (1.1), we proceed in the same way as it was done for expansions in [**1**] and associate with a pair $\{D, G\}$ two values (which may be infinite)

$$\underline{g} = \sup_{\partial D} \lim_{(s,S)\to(s^0,S^0)} G(s,S), \quad \overline{g} = \lim_{a\to\infty} G(a, aI),$$

where $(s, S) \in D$, $(s^0, S^0 \in \partial D$ and D is the domain of global monotonicity of G.

It is of interest to study the case

$$\underline{g} < \overline{g} \tag{1.6}$$

(cf. examples in Sec. 2).

To formulate the main results, we introduce the notation

$$E^0 = \frac{\partial E}{\partial s}, \quad E^{ij} = \frac{\partial E}{\partial s_{ij}}, \quad S = (s_{ij}).$$

We note that E^0 and (E^{ij}) are always positive independently of the type of monotonicity of E with respect to v. The notation G^0 and (G^{ij}) has a similar meaning. We denote by $\mathrm{Sym}^+(n)$ the set of positive definite matrices. For the sake of simplicity, we assume that the functions and surfaces under consideration are as smooth as necessary.

Theorem 1.3. *Let $D \subset R^+ \times \mathrm{Sym}^+(n)$, and let Γ_0 be a G_r-admissible surface. Suppose that for any constant $\beta \in (0;1])$ there exists s_β such that $(s_\beta, \beta I) \in D$,*

$$\overline{g} = \lim_{s\to\infty} G(s, \beta I), \tag{1.7}$$

and admissible solutions to Eq. (1.4) *satisfy the inequality*

$$G^0 v \leqslant \mu_1 G^{ii} r_i + \mu_2 \tag{1.8}$$

for some constants μ_1 and μ_2. We also assume that

$$\underline{g} < g(t) < \overline{g}, \quad 0 \leqslant g_t, \quad t \in [0; \overline{T}], \tag{1.9}$$

where $\overline{T}$ is the time of shrinking a sphere that encloses Γ_0 to a point under the law (1.4).

Let G be concave in D. Then there exists a number $T \leqslant \overline{T}$, a point $M \in R^{n+1}$, and a unique G_r-admissible evolution $\{\Gamma_t, t \in [0; T)\}$ satisfying Eq. (1.4) *such that $\Gamma_t \to M$ as $t \to T$.*

The corresponding assertion for expansions from [**1**] can be formulated as follows.

Theorem 1.4. *Suppose that $D \subset R^+ \times \mathrm{Sym}(n)$, $(s, \mathbf{0}) \notin D$, for any s, G is concave in D, and Γ_0 is a G-admissible starshaped surface. Let*

$$\overline{g} = \lim_{a\to\infty} G(s, aS), \quad (s, S) \in D, \tag{1.10}$$

$$G^0 s \geqslant G^{ij} s_{ij}, \quad (s, S) \in D. \tag{1.11}$$

Suppose that g is a constant such that

$$\underline{g} < g < \overline{g}. \tag{1.12}$$

Then there exists a unique G-admissible evolution $\{\Gamma_t, t \in [0;\infty)\}$ satisfying the equation $G(v, \mathbf{k}) = g$. Moreover, the rescaled flow $\{\Gamma_t/R_t\}$ converges to a sphere as $t \to \infty$. Here, $R_t = r(t)$, $G(r', 1/rI) = g$, and $R_0 = 1$.

Theorem 1.4 was first proved for Eq. (1.5) with a homogeneous concave function $F = 1/\widehat{F}$, which corresponds to equality in (1.11) (cf. [**8, 9**]). In the general case considered here, the uniqueness and existence were established in [**1**, Theorem 1.1]. The asymptotic behavior was studied in [**1**] under the assumption that the inequality (1.11) is strict (cf. [**1**, Theorem 1.2]). Recently, Th. Nehring (private communication) proved the asymptotic convergence to a sphere under the condition (1.11)

There are evolutions such that the E-admissibility of a solution is controlled by the strict convexity of the initial surface but not (1.9) and (1.12), i.e., the entire boundary of D is not of interest in this sense and the condition (1.6) becomes unnecessary. In the case of expansion, this fact was observed in [**9**]. As was shown there, if a homogeneous positive monotone function F is concave with respect to $\mathbf{k}$ and is convex with respect to $\mathbf{r}$, then the evolution (1.5) with $\widehat{F} = 1/F(\mathbf{k})$ preserves the strict convexity of the initial surface, whereas the convexity property is not necessary for solutions to be admissible. Here, we present sufficient conditions for contractions to preserve convexity. To this end, we introduce the quantities

$$\underline{c} = \sup_{s>0} \lim_{\alpha \to 0} G(s, \alpha I), \quad \overline{c} = \inf_{s>0} \lim_{a \to \infty} G(s, aI)$$

instead of $\overline{g}$ and $\underline{g}$,

We also need an analog of the inequality (1.6) for $\underline{c}$ and $\overline{c}$.

Theorem 1.5. *Suppose that $D \subset R^+ \times \mathrm{Sym}^+(n)$, G is concave in D and Γ_0 is a G_r-admissible surface. Suppose also that for any constant $\beta \in (0;1])$ there exists s_β such that $(s_\beta, \beta I) \in D$,*

$$\overline{c} \leqslant \lim_{s \to \infty} G(s, \beta I).$$

For some μ_1 and μ_2 the inequality (1.8),

$$E^0 v \leqslant E^{ii} k_i, \tag{1.13}$$

holds on admissible solutions to Eq. (1.4), and

$$\underline{c} < g(t) < \overline{c}, \quad g_t > 0. \tag{1.9'}$$

Suppose that the function v formally defined by (1.5) is concave with respect to $\mathbf{k}$. Then there exists a number $T \leqslant \overline{T}$, a point $M \in R^{n+1}$, and a unique G_r-admissible evolution $\{\Gamma_t, t \in [0;T)\}$ satisfying Eq. (1.4) such that $\Gamma_t \to M$ as $t \to T$.

We note that the boundary of the domain of global monotonicity does not appear in the assumptions of Theorem 1.5. This means that we can consider only a part of the maximal domain of E-admissibility if $\underline{c}$ and $\overline{c}$ are known and are associated with some $D \subset R^+ \times \mathrm{Sym}^+(n)$. The corresponding examples are given in Secs. 2 and 5.

As is known, an evolution of an E-admissible immersed surface under the law (1.1) exists if the initial surface is E-admissible at least for a small time t_1 and the further existence depends on the possibility to establish a priori estimates. To establish the convergence of the flow to a point, we use the following idea from [**4**].

Proposition 1.6. *Let $v[\Gamma_t]$, $k_i[\Gamma_t]$, $i = 1, \ldots, n$, be bounded uniformly in t from below and from above for any finite t by positive constants until the surfaces Γ_t enclose a ball B_ρ with some $\rho > 0$ and the evolution $\{\Gamma_t\}$ subject to the law* (1.1) *remains E-admissible. If the variables v and $\mathbf{k}$ are E-competing, then there exists a number $T \in (0; \overline{T})$, $\overline{T} < \infty$, and a point M such that there is an admissible solution to Eq.* (1.1) *for $t < T$ and $\Gamma_t \to M$ as $t \to T$.*

Under the assumptions of Theorems 1.3 and 1.5, the corresponding estimates are proved in Secs. 4 and 5. Using some parametrizations, we reduce the geometric evolution (1.1) to fully nonlinear second-order parabolic equations (cf. Sec. 3). Section 2 contains examples demonstrating the power of our general approach. Sections 4 and 5 also contain examples, but they illustrate a specific character of the material presented in this paper.

The interest of the author in this topic was initiated by the papers [**3, 4**]. The first version of Theorem 1.4 was proved in cooperation with Prof. F. Tomi and Dr. Th. Nehring in 1999. The participation of the author in the program "Nonlinear Partial Differential Equations," Isaac Newton Institute, Cambridge, 2001, assisted to the further study of this subject.

2. Examples

Our examples are based on well-known properties of elementary symmetric functions and quotients. We introduce the notation

$$H_{m,l}(S) = \frac{\mathrm{tr}_m S}{\mathrm{tr}_l S}, \quad F_{m,l}(S) = H_{m,l}^{1/(m-l)}(S), \quad 0 \leqslant l, m \leqslant n,$$

where $\mathrm{tr}_i S$ is the sum of all principal ith minors of the matrix S. If $l = 0$, we omit the second subscript. For example, we write H_m instead of $H_{m,0}$. Without loss of generality, we assume that $l < m$.

We note that $F_{m,l}$ is a one-homogeneous positive monotone concave function in the cone $C_m = \{S \in \mathrm{Sym}(n) : H_i(S) > 0, i = 1, \ldots, m\}$. We will use the following simple observation in [**1**].

Proposition 2.1. *Let $\{G_i\}$ be positive monotone concave functions in $D \subset R^+ \times \mathrm{Sym}(n)$, let φ_i, be strictly increasing concave functions defined on the ranges of G_i respectively, and let λ_i be positive constants. Then the function $G := \sum_i \lambda_i \varphi_i \circ G_i$ is positive monotone concave in D.*

Using Proposition 2.1, we can consider much more examples within the framework of Theorem 1.3. For example, we can consider the following function from [**1**]:

$$G(s, S) = \log s + \beta \log F_n(S) - \frac{1}{sF_1^\gamma(S)}, \quad \beta, \gamma > 0, \tag{2.1}$$

where $D = D^+ := R^+ \times \mathrm{Sym}^+(n)$.

We begin with the following example:

$$v[\Gamma_t] = F^q_{m,l}(\mathbf{k})[\Gamma_t]. \tag{2.2}$$

The maximal set of admissible surfaces is given by

$$\mathbf{k}(M) \in C_m, \quad M \in \Gamma, \tag{2.3}$$

The surfaces (2.3) are usually referred to as *m-convex*. Analogously to convex-monotone evolutions (cf. Definition 1.2), we can introduce the notion of *m*-convex-monotone evolutions.

Proposition 2.2. *Suppose that the parameter q in* (2.2) *satisfies one of the following relations*:

$$q \in (0; \infty), \quad l = 0, \tag{i}$$

$$q = 1, \quad 0 < l < m. \tag{ii}$$

Then the flow (2.2) *forces an arbitrary strictly convex closed surface Γ_0 to shrink to a point in a finite time and the corresponding solution to* (2.2) *is a unique convex-monotone evolution.*

If $-1 \leqslant q < 0$, then the flow (2.2) *infinitely expands an arbitrary starshaped closed m-convex surface Γ_0, the corresponding solution to* (2.2) *is a unique m-convex evolution, the surfaces Γ_t, $t < \infty$, are starshaped and asymptotically tend to a sphere.*

PROOF. To prove (i), we write Eq. (2.2) on D^+ in the equivalent form

$$G(v, \mathbf{r}) = -\frac{1}{vF^q_{n,n-m}(\mathbf{r})}[\Gamma_t] = -1.$$

It is easy to see that the assumptions of Theorem 1.3 are satisfied by $D = D^+$, $\underline{g} = -\infty$, $\overline{g} = 0$, $\mu_1 = 1/q$, $\mu_2 = 0$ in view of Proposition 2.1 and the properties of $F_{n,n-m}$.

To prove (ii), with Eq. (2.2) we associate the relations

$$G(v, \mathbf{r}) = \frac{-1}{v} F_{n-m,n-l}(\mathbf{r}) = -1, \quad v = F_{m,l}(\mathbf{k})$$

and verify the assumptions of Theorem 1.5.

Indeed, $v = v(\mathbf{k})$ and $G = G(s, \mathbf{s}$ are concave in C_m and $R^+ \times C_{n-l}$ respectively. In particular, G is concave in D^+. Since $q = 1$, the relation (1.13) becomes equality. Hence the assumptions of Theorem 1.5 are satisfied for $\underline{c} = -\infty$, $\overline{c} = 0$, $\mu_1 = 1$, and $\mu_2 = 0$.

To treat expansions, we write Eq. (2.2) in the form

$$G(v, \mathbf{k}) := -\frac{1}{vF^{|q|}_{m,l}(\mathbf{k})} = -1$$

and verify that all the assumptions of Theorem 1.4 are satisfied. This example is taken from [**1**]. □

Equation (2.2) is important in the case $1 \leqslant l < m \leqslant n-1$. Indeed, from the general point of view, we should take $D = R^- \times C_m$, $m < n$, i.e., m-convex-monotone evolutions are admissible. However, we cannot prove contraction to a point in this class. It is reasonable to think that this is impossible (cf. [**11**]). Even if we write these equations in terms of $(v, \mathbf{r})$, the situation does not become better because for the domain of maximal global monotonicity we should take $D \in R^+ \times C_{n-l}$, whereas only the part D^+ corresponds to the set of convex-monotone evolutions. Therefore, unlike Theorem 1.3, we cannot control the convex–monotonicity of admissible evolutions by the right-hand side of Eq. (1.4). Nevertheless, the additional condition that v is concave with respect to $\mathbf{k}$ admits control. In accordance with these arguments, we introduce the notion of "flows preserving convexity."

As far as the author knows, there are no analogs of Proposition 2.2 for $l \geqslant 1, m < n$ in the case of contractions. As for expansions, this phenomenon was discovered in [**9**].

The flow (2.2) can be easily reduced to the form (1.1) with a function E that is 1-homogeneous with respect to v and $\mathbf{k}$. At the first glance, it seems that any equation from Theorem 1.5 can be reduced in such a way. But it is not the case for the evolution laws from Theorem 1.3. For example, the equation

$$v = H_m(\mathbf{k} + \gamma I), \quad \gamma > 0, \tag{2.4}$$

cannot be reduced to a 1-homogeneous form in $\mathbf{k}$. However, by Theorem 1.3, the flow (2.4) determines convex-monotone contractions to a point. To show this, we consider Eq. (2.4) as a special case of the following evolution law:

$$\sum_0^n \frac{a_i}{v^{q_i}} H_i(\mathbf{k}) = c, \quad a_i \geqslant 0, \quad \sum_1^n a_i > 0, \quad q_i > 0. \tag{2.5}$$

Proposition 2.3. *Suppose that $c = c(t)$ and $c_t \leqslant 0$. Then the law (2.5) forces an arbitrary strictly convex closed surface Γ_0 to shrink to a point in a finite time T. The corresponding evolution $\{\Gamma_t, t \in [0; T)\}$ is a unique convex-monotone solution to Eq. (2.5).*

PROOF. We apply Theorem 1.3 to the following equation, which is equivalent to (2.5) on convex-monotone evolutions:

$$G(v,\mathbf{r}) =: -\sum_{0}^{n} \frac{a_i}{v^{q_i}} H_{n-i,n}(\mathbf{r}) = -c. \tag{2.6}$$

For this equation it is natural to choose $D = D^+$, $\underline{g} = -\infty$, $\overline{g} = 0$,

$$\mu_1 = \max_i q_i = \overline{q}, \quad \mu_2 = \frac{\overline{q} a_0}{v_0^{q_0}}, \quad v_0 = \min_{\Gamma_0} v.$$

Since any strictly convex surface is G_r-admissible by definition, μ_2 is well defined. By Proposition 2.1 and the concavity of $F_{n,n-i}$, the function G is concave in D^+. Hence Eq. (2.5) satisfies the assumptions of Theorem 1.3, which proves Proposition 2.3. □

Let us compare the flow (2.4) with

$$v^m = H_m(\mathbf{r} + \gamma I), \quad 0 < m \leqslant n,$$

According to our concept, we deal with expansions because v and $\mathbf{r}$ are competing variables in (2.6) and, consequently, we can use Theorem 1.4.

Proposition 2.4. *For any strictly convex closed initial surface Γ_0 Eq. (2.6) uniquely determines an infinite expansion $\{\Gamma_t, t \in (0;\infty)\}$. The surfaces Γ_t asymptotically converge to a sphere.*

In order to apply Theorem 1.4, we replace (2.6) with the equivalent equation

$$G(v,\mathbf{k}) =: \frac{1}{v^m} \sum_{0}^{m} C_m^k \gamma^{m-k} H_{n-k,n}(\mathbf{k}) = -1$$

and verify the assumptions of Theorem 1.4.

Proposition 2.4 still holds if the power m on the left-hand side is replaced with $q > m$. The case $q < m$ is not covered by Theorem 1.4. In this case, we can only say that such extensions exist, generally speaking, only in a finite time.

To conclude the section, we turn to the example (2.1).

Proposition 2.5. *Let Γ_0 be a strictly convex closed surface, and let G be given by* (2.1).

(a) *Let $\beta, \gamma \in (0;1]$. Then the equation $G(v,\mathbf{k}) = g$, $g = \mathrm{const}$, determines a unique convex-monotone infinite expansion in an infinite time of Γ_0 that asymptotically converges to a sphere.*

(b) *For arbitrary β and γ the equation $G(v,\mathbf{r}) = g$ determines a unique convex-monotone contraction of Γ_0 to a point in a finite time.*

The assertions of Proposition 2.5 follow from Theorems 1.4 and 1.3.

3. Reduction of Geometric Evolution Problems to Problems for Fully Nonlinear Parabolic Equations

Various parametrizations of hypersurfaces lead to various methods of reducing geometric evolution equations to second-order parabolic equations. In the theory of geometric flows (cf. [**2**]–[**4**] and [**8, 9, 11**]), the parametrization by spherical coordinates serves as a basic parametrization, where an evolution $\{\Gamma_t\}$ is regarded as a set Γ_t ordered with respect to t. In this case, the center of coordinate sphere is strictly enclosed by the surfaces Γ_t during the existence of evolution. Thus, if we have contractions to a point, this point is the center. We will not mention this point while obtaining a priori estimates for the geometric characteristics of contractions $\{\Gamma_t\}$ or mean that this point is the center of the ball in Proposition 1.6, if necessary.

Our construction is based on the classic maximum principle for parabolic equations, where only certain points and their neighborhoods are of interest. With such a point we associate one of two local parametrizations depending on the case of cooperating or competing variables v, $\mathbf{s}$.

We first describe a parametrization for competing variables. Let $(M_0) \in \Gamma_{t_0}, t_0 > 0$ be the origin of the Euclidian coordinates, $\{Y = (y, y^{n+1}), y = (y^1, \dots, y^n)\}$, $\nu(M,t)$ the interior normal to Γ_t at a point M, and $\nu(M_0, t_0) = (0, \dots, 1)$. In some neighborhood of (M_0, t_0), we can introduce the following parametrizations of the surfaces Γ_t:

$$\Gamma_t = \{y(t), y^{n+1} = u(y,t)\}, \quad |u_y(0,t_0)| = 0, \quad v = (Y_t, \nu), \tag{3.1}$$

$$\nu = -\frac{(u_1, \dots, u_n, -1)}{\sqrt{1+u_y^2}}. \tag{3.2}$$

Hereinafter, the subscript indicates differentiation, i.e.,

$$u_i = \frac{\partial u}{\partial y^i}, \quad |u_y|^2 = \sum_1^n u_i^2, \quad u_t = \frac{\partial u}{\partial t}.$$

We denote by $u_{yy} = (u_{ij})$ the Hesse matrix of u.

In the case of competing variables $(v, \mathbf{k})$, by a *standard* parametrization we mean a paramterization in (3.1) such that the vector y is independent of t in some neighborhood of the origin and $u_{yy}(0, t_0)$ is a diagonal matrix. We consider a symmetric matrix $\tau[\Gamma_t]$ such that

$$\tau = \sqrt{(g^{ij})}, \quad (g^{ij}) = \frac{1}{\sqrt{1+u_y^2}}\left(\delta^i_j - \frac{u_i u_j}{1+u_y^2}\right).$$

Let $u_{(yy)} := \tau u_{yy} \tau$. In the standard parametrization

$$v = \frac{u_t}{\sqrt{1+u_y^2}} := u_{(t)},$$

the principal curvatures of Γ_t are the eigenvalues of $u_{(yy)}$. By the orthogonal invariance of E, Eq. (1.1) is reduced to the second-order parabolic partial differential equation

$$E(s(v), \mathbf{k}) = E(-u_{(t)}, u_{(yy)}) = f \tag{3.3}$$

in some neighborhood of (M_0, t_0).

To treat the case of cooperating variables $(v, \mathbf{r})$, we need some other standard parametrization. We emphasize that, in this statement, all the surfaces under consideration are assumed to be strictly convex. With (M_0, t_0) we associate the parametrizations (3.1). Let $X(t)$ be the position vector of Γ_t with origin strictly enclosed by theses surfaces. Let $P(t)$ be the Legendre transform associated with $X(t) = X_0 + Y(t)$ as follows:

$$p = u_y, \quad h = (x_0 + y, p) - (x_0^{n+1} + u).$$

Since Γ_t are strictly convex, we can parametrize them as follows:

$$\Gamma_t = \{p, p_{n+1} = h(p,t)\}. \tag{3.4}$$

The expression (3.2) should be replaced with the following:

$$\nu[\Gamma_t] = -\frac{(p_1, \dots, p_n, -1)}{\sqrt{1+p^2}}.$$

In this case, a parametrization in (3.4) is said to be *standard* if p is independent of t in some neighborhood of (M_0, t_0) and $u_{yy}(0, t_0)$ is diagonal. By the properties of Legendre transform, the curvature radii of Γ_t are the eigenvalues of the matrix $h_{(pp)} := \eta h_{pp} \eta$, where $\eta = \eta(p)$ is the inverse to τ, i.e.,

$$\eta^2 = (g_{ij})(p) = \sqrt{1+p^2}(\delta_j^i + p_i p_j).$$

In this case, we have

$$v = (Y_t, \nu) = (X, \nu)_t = \frac{-h_t}{\sqrt{1+p^2}} := h_{(t)}$$

and the parabolic partial differential equation

$$G(-h_{(t)}, h_{(pp)}) = g, \tag{3.5}$$

is locally equivalent to Eq. (1.4).

To construct a priori estimates, we use the following linearizations of the fully nonlinear operators in (3.3) and (3.5):

$$L[w] := -\widehat{E}^0 w_{(t)} + \widehat{E}^{ij} w_{(ij)}, \tag{3.6}$$

where either $\widehat{E} = E$ or $\widehat{E} = G$ depending on the type of parametrization. The meaning of subscripts in the parenthesis is also determined by the type of parametrization.

We use the classical maximum principle in the following form.

Proposition 3.1. *Suppose that* $w', w'' \in C^{2,1}(Q_T)$, $Q_T = \Omega \times (0;T]$, $\Omega \subset R^n$, $w'' > 0$. *Let* $w = w'/w''$ *attain the maximum at some point* $(M_0, t_0) \in Q_T$. *Then at the point* $(M_0, t_0) \in Q_T$, *the following inequality holds*:

$$L[w'] - wL[w''] \leqslant 0. \tag{3.7}$$

In the sequel, we consider equations of the form (1.1) or (1.4) rather than (1.5). However, the condition that v regarded as a function of the principal curvatures is concave (cf. Theorem 1.5) means that (1.5) is included into the consideration. We give a consequence of this condition in the coordinate form for the general equations. This fact will be used in the proof of Theorem 1.5.

Proposition 3.2. *Suppose that* v, *implicitly given by the equality* $E(-v, S) = c$, *is concave with respect to* S *on the set of monotonicity of* E. *For any sufficiently smooth* E*-admissible solution to Eq.* (3.3) *the following inequality holds*:

$$\frac{\partial^2 E}{\partial v^2} v_1^2 + 2\frac{\partial^2 E}{\partial v \partial u_{(ij)}} v_1 u_{(ij)1} + \frac{\partial^2 E}{\partial u_{(ij)} \partial u_{(kl)}} u_{(ij)1} u_{(kl)1} \leqslant 0. \tag{3.8}$$

4. Estimates for Cooperating Variables v and r. Proof of Theorem 1.3

In this section, we deal with the geometric evolution equation (1.4).

Lemma 4.1. *Let* $\{\Gamma_t, t \in [0;t_1]\}$, *be an admissible solution to Eq.* (1.4). *Suppose that*

$$g_t \geqslant 0, \quad t \in (0;t_1]. \tag{4.1}$$

Then

$$v[\Gamma_t] \geqslant \min_{\Gamma_0} v[\Gamma_0] > 0. \tag{4.2}$$

If, in addition, G *is concave in* D, *then*

$$r_i[\Gamma_t] \leqslant \max_{i,\Gamma_0} r_i[\Gamma_0], \quad i = 1, \ldots, n. \tag{4.3}$$

PROOF. We fix a point (M,t) and write Eq. (1.4) in the form (3.5) in a neighborhood of the point (M,t). Since the parametrization is standard (in particular, $p = 0$ at M), we have

$$v_t = -h_{tt}, \quad v_{(ii)} = -h_{iit} - v \tag{4.4}$$

at $(0,t)$. Differentiating Eq. (3.5) with respect to t in a neighborhood of $(0,t)$, we find

$$L[v] = -g_t - v\sum_1^n G^{ii} < 0$$

at the point (M,t). By the minimum principle for parabolic equations, from the last relation we obtain the estimate (4.2).

To obtain (4.3), we suppose that there are $t' \in (0;t_1]$ and $M' \in \Gamma_{t'}$ such that $r_1(M',t') \geqslant r_i[\Gamma_t]$, $t \in (0;t_1]$, $i = 1,\dots,n$. In a neighborhood of the chosen point, we introduce the standard parametrization of the form (3.4) and note that our assumption is equivalent to the fact that $h_{(11)}$ attains the maximal value at $(0;t'$. In addition to (4.4) we also have the following

$$h_{(ii)t} = h_{tii}, \quad h_{(ii)(jj)} - h_{(jj)(ii)} = h_{ii} - h_{jj}, \quad i,j = 1,\dots,n. \tag{4.5}$$

Taking into account (4.4), (4.5) and twice differentiating Eq. (3.5) with respect to p_1, we obtain the following inequality for admissible solutions to Eq. (1.4) at the point $(0;t')$:

$$L[h_{(11)}] \geqslant G^0 v + G^{ii}(h_{11} - h_{ii}) > 0.$$

However, it contradicts the maximum principle for parabolic equations, which proves (4.3). $\square$

The uniqueness of an admissible solution to Eq. (1.4) follows, for example, from the inclusion principle for G_r-admissible evolutions.

Theorem 4.2. *Let $\{\Gamma_t, \widetilde{\Gamma}_t, t \in (0;T)\}$ be G-admissible evolutions of the initial surfaces Γ_0 and $\widetilde{\Gamma}_0$ respectively, and let*

$$G[\Gamma_t] = g, \quad G[\widetilde{\Gamma}_t] \leqslant \widetilde{g}, \quad t \in [0;T). \tag{4.6}$$

Suppose that $\widetilde{\Gamma}_0$ encloses Γ_0, g satisfies and (4.1), and the following inequality holds:

$$g(t) \geqslant \widetilde{g}(t), \quad t \in (0;T). \tag{4.7}$$

Then $\widetilde{\Gamma}_t$ encloses Γ_t for all $t \in (0;T)$.

Proof. By the definition of G_r-admissibility, we have $v_0 = \min v[\Gamma_0] > 0$ and, consequently,

$$d^\varepsilon(t) := \underline{\mathrm{dist}}\{\Gamma_{t+\varepsilon}; \widetilde{\Gamma}_t\} > 0$$

for any small $\varepsilon > 0$. Assume that there exists $\underline{t}$ such that

$$0 < \underline{t} = \min_t\{t < T : d^\varepsilon(t) = 0\}.$$

Then there are t_0, t_1, $0 < t_0$, $t_1 < \underline{t}$ such that the function $d^\varepsilon(t)$ attains the minimum at t_0 on the interval $(0;t_1]$ and its minimal value is positive.

The evolutions $\{\Gamma_{t+\varepsilon}, \widetilde{\Gamma}_t, t \in (0;t_1]\}$ enclose the ball $B_\rho, 0 < \rho << 1$. Let $X^\varepsilon(t)$ and $\widetilde{X}(t)$ be the position vectors of the surfaces $\Gamma_{t+\varepsilon}$ and $\widetilde{\Gamma}_t$ respectively with the origin located at the center of this ball. Then

$$d^\varepsilon(t) = \min_{\nu=1}(X^\varepsilon, \nu - \widetilde{X}, \nu)(t),$$

where X^ε and $\widetilde{X}$ correspond to the position vectors of the points with the interior normal ν. Let M_0^ε and $\widetilde{M}_0$ realize $d^\varepsilon(t_0)$. With M_0^ε we associate the parametrization (3.4). By the choice of $\widetilde{M}_0$, the same parametrization is considered for $\widetilde{\Gamma}_t$ in some neighborhood of $(\widetilde{M}_0, t_0)$. Therefore, the function

$$w = \frac{(\widetilde{h} - h^\varepsilon)(p,t)}{\sqrt{1+p^2}}$$

is well defined in some neighborhood of $(0;t_0)$ and attains the positive minimum at the point $(0,t_0)$. Hence $w_t(0,t_0) \leqslant 0$ and

$$0 \leqslant w_{pp}(0,t_0) = \widetilde{h}_{pp} - h^\varepsilon_{pp} - d^\varepsilon(t_0)I.$$

Replacing the relations (4.6) with their analogs of the form (3.5) and taking into account the strict monotonicity of G, we obtain the following inequalities at $(0,t_0)$:

$$G(-\widetilde{h}_{(t)}, \widetilde{h}_{pp}) \geqslant G(-h^\varepsilon_t, h^\varepsilon_{pp} + d^\varepsilon(t_0)I) > g(t_0+\varepsilon) \geqslant g(t_0),$$

which contradicts the assumptions (4.6) and (4.7). Hence $\underline{t}$ does not exist. Since ε is arbitrary, the theorem is proved. □

To estimate the velocity from above, we adapt the arguments of [**9**] to Eq. (1.4).

Lemma 4.3. *Let* $\{\Gamma_t, t \in [0;t_1]\}$ *be an admissible solution to Eq.* (1.4), *and let* Γ_{t_1} *encloses a ball* B_ρ. *Suppose that a function* G *is concave in* D *and satisfies the conditions* (1.7) *and* (1.8). *Then*

$$v[\Gamma_t] \leqslant C(\rho, \Gamma_0), \quad t \in [0;t_1]. \tag{4.8}$$

PROOF. Without loss of generality, we can assume that μ_1 in (1.8) is sufficiently large and $\delta > 0$ is sufficiently small so that

$$\max_{(0,t_1)} g + \frac{\mu_2 + \delta}{1 + \mu_1} < \overline{g}.$$

Let v_ρ be a solution to the equation

$$G\left(v, \frac{\rho}{2(\mu+1)}\right) = c, \quad c = \max_{(0,t_1)} g + \frac{\mu_2+\delta}{1+\mu_1}. \tag{4.9}$$

Since $c < \overline{g}$, the condition (1.7) is satisfied and v_ρ is uniquely determined by Eq. (4.9). Introduce the notation

$$\overline{v} = \max\left\{ v_\rho; 2\frac{|g_t|R_0}{\delta} \right\},$$

where R_0 is the radius of a sphere enclosing Γ_0. Consider the function

$$w = -\frac{v}{(X,\nu)+\rho/2}, \tag{4.10}$$

where $X(t)$ is the position vector of Γ_t with origin at the center of a given ball B_ρ. By Theorem 4.2, Γ_t encloses B_ρ for all $t \in [0;t_1]$ and $-(X,\nu) \geqslant \rho$, $w > 0$. Assume that w attains the maximal value at a point $(M',t') \in \{\Gamma_t\}$. With (M',t') we associate the standard parametrization (3.4) and Eq. (3.5). Then the function (4.10) takes the form

$$w = \frac{-h_t}{h - \frac{\rho}{2}\sqrt{1+p^2}}$$

in some neighborhood of $(0,t')$; moreover, at the point $(0;t')$, we have

$$0 \geqslant L[w] = -g_t - w(G^0 v + G^{ii}h_{ii}) + \frac{\rho}{2}\sum_1^n G^{ii}. \tag{4.11}$$

Let $v(0,t') > \overline{v}$. Since G is concave with respect to S, we have

$$\begin{aligned} \frac{\mu_2+\delta}{2(\mu_1+1)} &< G(v, \frac{\rho}{2(\mu_1+1)}I) - G(v, h_{(pp)}) \\ &\leqslant -G^{ii}h_{ii} + \frac{\rho}{2(\mu_1+1)}\sum_i G^{ii}. \end{aligned} \tag{4.12}$$

It easy to see that (1.8) and (4.11) are incompatible with (4.12), which implies $v \leqslant \overline{v}$ at the maximum point of w. This leads to (4.8).

Proof of Theorem 1.3. We possess all necessary estimates in order to apply Proposition 1.6, except for an estimate from above for curvatures or, which is the same, an estimate from below for radii of curvature. To derive the estimate, we use the left-hand side of the inequality (1.9). Assume that the assumptions of Lemma 4.3 are satisfied. Let

$$\begin{aligned} &\underline{r} := r_1(M',t') \leqslant r_i(M,t), \ (M,t) \in \{\Gamma_t\}, \\ &\overline{r}_0 := \max_{i,\Gamma_0} r_i[\Gamma_0], \ i = 1,\dots,n. \end{aligned}$$

By (1.9), (4.3), and (4.8), we have

$$\underline{g} < g = G(v,\mathbf{r}) \leqslant G(C(\rho,R_0),\overline{r},\dots,\overline{r},\underline{r})$$

at the point (M',t'), which leads to the desired estimate from below for $\underline{r}$. The theorem is proved. □

To demonstrate the possibilities of Theorem 1.3, we give an example of evolution equations with which some nontrivial domain D is associated, i.e., $D \neq R^+ \times \mathrm{Sym}^+(n)$. Namely,

$$E(-v, \mathbf{k}) := -v^q F_{n,l}(vI - \mathbf{k}) = -g^{q+1}, \quad 0 \leqslant l < n, \quad q \geqslant 0. \tag{4.13}$$

Since we wish to study Eq. (4.13) within the framework of Theorem 1.3, we write (4.13) in the form

$$G(v, \mathbf{r}) := (v^q F_{n,l}(vI - \mathbf{r}^{-1}))^{1/(q+1)} = g.$$

It is obvious that for the domain of global monotonicity of G we should take

$$D = \{(s, S) \in R^+ \times \mathrm{Sym}^+(n) : sI - S^{-1} \in \mathrm{Sym}^+(n)\}.$$

The set of G_r-admissible surfaces consists of all strictly convex closed surfaces.

The function G is concave in D, $\underline{g} = 0$, $\overline{g} = \infty$, and the inequalities (1.7) and (1.8) hold for $\mu_1 = q$ and $\mu_2 = q \max_t g$. Therefore, from Theorem 1.3 we obtain the following assertion.

Corollary. *Let $g > 0$, $g_t \geqslant 0$. Then an arbitrary closed convex is forced to shrink to a point in a finite time under the law* (4.13). *The corresponding admissible solution to* (4.13) *is unique.*

The uniformization of fully nonlinear equations leads, generally speaking, to the loss of information. The evolution law (4.13) can serve as a confirmation of this. It cannot be considered within the framework of the theory of geometric flows, where a single source of information is a strictly fixed function $\widehat{F}$ (cf. (1.5)).

5. Estimates for Competing Variables v and k. Proof of Theorem 1.5

In this section, we deal with Eq. (1.1) for contractions, i.e.,

$$E(s(v), S)[\Gamma_t] = E(-v, \mathbf{k})[\Gamma_t] = f \tag{5.1}$$

in the domain $D \subset R^- \times \mathrm{Sym}(n)$ of global monotonicity of E. For the evolution law (1.1) we consider the local parametrizations (3.1) and (3.2). With a point $(M_0, t_0) \in \{\Gamma_t, t > 0\}$ we associate the standard parametrization (3.1). Instead of (4.4) and (4.5), we take the main relations

$$v_t = u_{tt}, \quad v_{(ii)} = u_{tii} - vu_{ii}^2, \tag{5.2}$$

$$u_{(ii)(jj)} - u_{(jj)(ii)} = u_{ii}u_{jj}(u_{ii} - u_{jj}) \tag{5.3}$$

at the point $(0, t_0)$. The identities (5.2) and (5.3) are infinitesimal versions of the well-known relations in differential geometry. They were proved (in the terms accepted here) in [**12**].

Replacing Eq. (5.1) with the locally equivalent expression (3.3), differentiating the latter in a neighborhood of $(0;t_0)$, and taking into account (5.2), we find

$$L[v] = -E^0 v_t + E^{ii} v_{(ii)} = f_t - vE^{ii}u_{ii}^2 \tag{5.4}$$

at $(0;t_0)$. Let the assumptions of Proposition 3.2 be satisfied. If we differentiate twice with respect to y^1 and take into account (3.8) and (5.3), then we obtain the inequality

$$L[u_{(11)}] \geqslant u_{11}(u_{11}(-E^0 v + E^{ii}u_{ii}) - E^{ii}u_{ii}^2) \tag{5.5}$$

at $(0;t_0)$.

We start to derive a priori estimates with the following simple assertion.

Lemma 5.1. *Let* $\{\Gamma_t, t \in [0;T]\}$ *be an admissible solution to Eq.* (5.1), *and let* $f_t \leqslant 0$. *Then*

$$v[\Gamma_t] \geqslant \min_{\Gamma_0} v[\Gamma_0]. \tag{5.6}$$

Lemma 5.1 follows from the parabolic minimum principle and (5.4).

The estimate (5.6) is formally identical to (4.3). However, Lemma 5.1 covers a larger set of evolution equations because the convexity of E-admissible surfaces is not required. For example, the assumptions of Lemma 5.1 are satisfied by m-convex-monotone solutions to Eq. (2.2) if $m \neq n$.

The author does not know if the inclusion principle remains valid in this general case.

To estimate the curvatures, we need the inequalities (3.8) and (1.13).

Lemma 5.2. *Let the assumptions of Lemma* 5.1 *be satisfied. Suppose that* v *determined by the version* (1.5) *of Eq.* (5.1) *is concave with respect to* $\mathbf{k}$. *Let the inequality* (1.13) *hold. Then an arbitrary admissible solution to Eq.* (5.1) *satisfies the estimate*

$$\frac{k_i}{v}[\Gamma_t] \leqslant \max_{i,\Gamma_0} \frac{k_i}{v}, \quad i = 1, \ldots, n. \tag{5.7}$$

PROOF. Let

$$\max_{i,\{\Gamma_t\}} \frac{\exp(-\varepsilon t)k_i}{v}[\Gamma_t] = \frac{\exp(-\varepsilon t')k_1}{v}(M', t'), \tag{5.8}$$

and let $t' > 0$. With (M', t') we associate the standard parametrization (3.1), (3.2). Then the function

$$w^\varepsilon = \frac{\exp(-\varepsilon t)u_{(11)}}{u_t}(y,t)$$

attains the maximal value at the point $(0,t')$. We use Proposition 3.1 (the maximum principle) with $w = w^\varepsilon$, $w' = \exp(-\varepsilon t)u_{(11)}$, $w'' = u_{(t)}$. Taking into account (5.4) and (5.5), we obtain the inequality

$$0 \geqslant -\frac{1}{v} f_t + \varepsilon E^0 + u_{11}(E^{ii} u_{ii} - E^0 u_t). \tag{5.9}$$

But, the inequality (5.9) contradicts the assumption $f_t \leqslant 0$ and (1.13). Hence $t' = 0$. Since $\varepsilon > 0$ is arbitrary, we obtain the estimate (5.7). □

Let us discuss the assumptions of Theorem 1.5. There are two functions E and G. Respectively, to prove this theorem, we use the results of this section (namely, Lemma 5.2), as well as those of Sec. 4. For the sake of definiteness, we take E in the form

$$E(-v, \mathbf{k}) := -v + F(\mathbf{k}).$$

We could take G as follows:

$$G(v, \mathbf{r}) := v + \widehat{F}(\mathbf{r}^{-1}).$$

However, we do not fix G but only suppose that G satisfies the assumptions of Lemma 4.2 and Eqs. (1.4) and (1.1) are equivalent on the set of E-admissible convex-monotone evolutions.

PROOF OF THEOREM 1.5. We verify the assumptions of Proposition 1.6. The inequality (5.6) provide us with the uniform estimate from below for the velocity with respect to t, and (5.8) guarantees the majorant for the curvatures which depends on the velocity. By the concavity of G in $D \subset G^+$, we can use some results of Sec. 4 (namely, the inclusion principle and the estimates (4.3) and (4.8)). By (1.9′), the surfaces Γ_t, $t > 0$, inherit the G_r-admissibility of the initial surface Γ_0, i.e., the solution to Eq. (1.4) is admissible as long as exists. Thus, the assumptions of Proposition 1.6 are satisfied, which proves Theorem 1.5. □

Corollary. *Let the assumptions of Theorem* 1.5 *be satisfied, and let* $\{\Gamma_t, t > 0\}$ *be an admissible solution to Eq.* (5.1). *If there is* $t = t_1$ *such that* Γ_{t_1} *is strictly convex, then this solution shrinks to a point in a finite time.*

Theorems 1.3 and 1.5 contain an implicit assumption that $v[\Gamma_0] > 0$. For some evolution equations this can means that the set of initial surfaces decreases. We explain this by the following example:

$$E(-v, \mathbf{k}) = -v + F_{m,l}(\mathbf{k}) = f, \quad 0 \leqslant l < m \leqslant n. \tag{5.10}$$

With Eq. (5.10) we associate the equivalent equation

$$G(v, \mathbf{r}) = v - F_{n-m,n-l}(\mathbf{r}) = g \tag{5.11}$$

on the set of convex-monotone evolutions. Here, $g = -f$. The function E is positive monotone in $R^- \times C_m$, and the function v that is formally determined by (5.10) is concave in C_m. On the other hand, the function G is positive monotone concave in D^+. Moreover, the inequalities (1.8) and (1.13) hold. To use Theorem 1.5, we set $\underline{c} = -\infty$ and $\overline{c} = 0$, which leads to the condition $g < 0$.

By Definition 1.2, any strictly convex surface is G_r-admissible. However, not every strictly convex surface is suitable. Indeed, the solution v_0 to the equation obtained from (5.11) at $t = 0$ should be positive. In Theorem 1.5, this fact is sufficient. In the above example, the initial surfaces should satisfy the inequality

$$F_{m\,l}(\mathbf{k})[\Gamma_0] > f(0) \tag{5.12}$$

and be strictly convex.

Thus, Theorem 1.5 implies the following assertion.

Corollary. *Let $f > 0$, let $f_t < 0$, and let Γ_0 be a strictly convex closed surface satisfying* (5.12). *Then there exists T and a unique convex-monotone solution $\{\Gamma_t, t \in [0;T)\}$ such that Γ_t shrink to a point as $t \to T$.*

The above arguments exclude from the consideration Eq. (5.10) with $f = 0$. However, in this case, the assumptions of Proposition 2.2 are satisfied. We note that the choice of G is different in this case, and any strictly convex closed surface can be taken for the initial surface.

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Existence of a Generalized Green Function for Integro-Differential Operators of Fractional Order

Moritz Kassmann and Mark Steinhauer

Dedicated to O. A. Ladyzhenskaya on her birthday

It is a great pleasure and honor for both authors of this paper to contribute to this volume. Throughout their education in analysis, especially in the field of partial differential equations, the authors were in close contact with the ideas and works of O. A. Ladyzhenskaya, in particular through the well-known monographs [1]–[3] written by O. A. Ladyzhenskaya and her former students. The first author spent the academic year 1993/1994 as a graduate student in St.-Petersburg and thereby became acquainted with the famous school of analysis headed by O. A. Ladyzhenskaya.

The existence of a generalized Green function for integro-differential operators of order $\alpha \in (1, 2)$ is proved and a pointwise estimate from above is established.

1. Introduction

As is known, Green functions play a fundamental role in the study of differential operators. The notion of a classical Green function was extended within the framework of weak solutions among others by Stampacchia [4], Widman and Grüter [5, 6], Solonnikov [7, 8], and Dolzmann and Müller [9] in order to investigate the local behavior of weak solutions in the case where differential operators have only weakly regular coefficients. This concept of what we call

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a *generalized Green function* was successfully used in many areas of the theory of partial differential equations.

As linear differential operators of second order have a counterpart in probability theory (namely, Markovian diffusion processes), so do Green functions. The stochastic interpretation of Green functions is given by the transition probability of the underlying process (cf., for example, [**10, 11**]). Since there are many more Markovian processes than only diffusion ones, it is natural to study transition probabilities and Green functions of Markovian nondiffusion-type processes. This question has become very important in recent years since many of these processes (for example, Lévy processes) are getting more and more relevant for modeling. A characteristic but easy to understand example is given by a symmetric stable jump process of fractional order $\alpha \in (1,2)$. The corresponding integro-differential operator is nothing but the $\alpha/2$th power of the Laplacian.

The goal of this paper is to establish the existence of a generalized Green function G for the Dirichlet problem corresponding to a stable-type symmetric process and to derive sharp estimates.

Introduce the notation: $\Omega \subset \mathbb{R}^n$ is a bounded domain; $J(dx,dy)$ is a positive Radon measure on $(\Omega\times\Omega)\setminus \text{diag}$ satisfying the condition

$$\int\limits_{(K\times K)\setminus\,\text{diag}} |x-y|^2 J(dx,dy) < \infty, \quad J(K,\Omega - O) < \infty$$

for all compact sets K and open sets O such that $K \subset O \subset \Omega$; $\widetilde{k}(dx)$ is a positive Radon measure on Ω.

We consider the bilinear form

$$a(u,\varphi) = \int\limits_{\Omega}\int\limits_{\Omega} (u(x)-u(y))(\varphi(x)-\varphi(y))J(dx,dy) + \int\limits_{\Omega} u(x)\varphi(x)\widetilde{k}(dx),$$

where $J(dx,dy)$ is the *jumping measure* of the underlying stochastic process and $\widetilde{k}(dx)$ is the so-called *killing measure* of the domain Ω (cf. [**12**]).

Definition 1.1. *A function $G(\cdot,\cdot):\Omega\times\Omega\to\mathbb{R}\cup\{\infty\}$ is called a generalized Green function of $a(\cdot,\cdot)$ if for all $y\in\Omega$*

$$a(G(\cdot,y),\varphi) = \varphi(y) \quad \forall\varphi\in C_0^\infty(\Omega). \tag{1}$$

Suppose that the measures $J(dx,dy)$ and $\widetilde{k}(dx)$ satisfy the following condition. There exist positive constants λ and Λ and a measurable function k with $k(x,y)=k(y,x)$, $\lambda \leqslant k(x,y) \leqslant \Lambda$ for all $x,y\in\Omega$ such that

$$J(dx,dy) = |x-y|^{-n-\alpha}k(x,y)\,dx\,dy,$$
$$\widetilde{k}(dx) = \Big(2\int\limits_{\mathbb{R}^n\setminus\Omega} |x-y|^{-n-\alpha}k(x,y)\,dy\Big)\,dx.$$

The main result of this paper can be formulated as follows.

Theorem 1.1. *Under the above assumptions, there exists a nonnegative generalized Green function* $G(\cdot,\cdot):\Omega\times\Omega\to\mathbb{R}\cup\{\infty\}$ *that satisfies Eq.* (1) *for given* $y\in\Omega$; *moreover,*

$$G(\cdot,y)\in H^{\alpha/2,2}(\Omega\backslash B_r(y))\cap H_0^{\alpha/2,1}(\Omega)\quad \forall r>0, \tag{2}$$

$$G(\cdot,y)\in L_{\text{weak}}^{n/(n-\alpha)}(\Omega),\quad \|G(\cdot,y)\|_{L_{\text{weak}}^{n/(n-\alpha)}}\leqslant C, \tag{3}$$

$$G(\cdot,y)\in H_0^{\alpha/2,s}(\Omega)\quad \forall s\in[1,n/(n-\alpha/2)), \tag{4}$$

$$G(x,y)\leqslant C\,|x-y|^{\alpha-n}\ \textit{a.e. in}\ (\Omega\times\Omega)\backslash\operatorname{diag}\,. \tag{5}$$

REMARK. Assertion (4) is presumed to be not optimal. It is natural to expect that $G(\cdot,y)\in H_{\text{weak}}^{\alpha/2,n/(n-\alpha/2)}(\Omega)$. Pointwise estimates for G from below can be derived with the help of Harnack's inequality from [**13**]. This will be a topic of further research.

SURVEY OF THE PROBLEM. Classical Green functions for local diffusion operators like the Laplace operator date back to Gauss [**14**] and Green [**15**]. In recent years sharp estimates for classical Green functions were intensively studied for nonlocal operators appearing as generators of stable-type processes, for example, fractional powers of the Laplace operator. A good overview of the potential theory for these operators can be found in papers by Kulczycki [**16, 17**], Chen and Song [**18**], and Bogdan [**19**]. As is known [**18**], the behavior of the classical Green function $G(x,y)$ of a stable-type process in $\mathbb{R}^n$ with index α is as follows: $G(x,y)\sim C|x-y|^{\alpha-n}$. Therefore, the estimates (3) and (5) for the generalized Green function $G(\cdot,y)$ are really optimal in this context.

The study of generalized Green functions requires local a priori estimates for weak solutions. Thus, the results of De Giorgi [**20**], Nash [**21**], and Moser [**22**], which were put within a general framework by Ladyzhenskaya and Uraltseva [**2**], have to be extended to integro-differential operators of fractional order. The possibility of such an extension was recently shown within the classical framework via stochastic methods by Bass and Levin [**23, 24**] who proved a Harnack inequality for smooth functions that are harmonic with respect to stable-type processes. This approach, however, is not suitable in the case of weak solutions.

Global bounds for a weak $H^{\alpha/2}$-solution u of the equation $a(u,\varphi)=(f,\varphi)$ were obtained by Fukushima [**25**]. Local estimates for parabolic nonlocal Dirichlet forms in the whole space $\mathbb{R}^n$ were proved by Komatsu [**26, 27**] who used the ideas of Nash. Tomisaki [**28**] extended Stampacchia's approach to local estimates. Because of the appearance of global terms, this approach does not result in the Caccioppoli and Harnack inequalities that would imply the existence of Green functions as explained in this paper. Both estimates were originally established by the first author [**13**] by analytical methods with the help of Moser's iteration technique. The Caccioppoli estimate obtained in [**13**] will be used to derive pointwise estimates from above for the Green function.

We note that the operators corresponding to jump-diffusion processes, i.e., bilinear forms $\int \nabla u \nabla \varphi \, dx + a(u, \varphi)$, were also studied. The approach is based on the use of the local part stemming from the Laplace operator (the diffusion part) in order to majorize the nonlocal part (cf. Gimbert and P. L. Lions, [**29**], Bensoussan and J. L. Lions [**30**], Garroni and Menaldi [**31**] or the first author [**32**]).

Outline of the paper. We complete the introduction by the notation and some known results that will be used later. Section 2 contains preliminary material for the proof of the estimate (5) and recalls the Caccioppoli inequality established in [**13**]. In Sec. 3, the function G_ρ is defined as a solution to a problem approximating (1) as $\rho \to 0$. Uniform bounds for G_ρ with respect to ρ are obtained, and Theorem 1.1 is proved there.

Notation. We denote by $B_\rho(x_0)$ the open ball $\{x \in \mathbb{R}^n : |x - x_0| < \rho\}$ with center $x_0 \in \mathbb{R}^n$ and radius ρ. If no confusion arises, we write B_ρ instead of $B_\rho(x_0)$. Let $\operatorname{meas}(A) = |A|$ denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$. As usual, the expression $\fint\limits_M v(y)\, dy$ denotes the mean value $\frac{1}{|M|}\Big(\int\limits_M v(y)\, dy\Big)$ over a measurable set $M \subset \mathbb{R}^n$ and $L^p(\Omega)$ is the Lebesgue space of measurable functions f such that $|f|^p$ is integrable. For $p \in [1, \infty)$ we denote by $L^p_{\text{weak}}(\Omega)$ the Banach space of measurable functions $v : \Omega \to \mathbb{R}$ such that the expression

$$[v]_{L^p_{\text{weak}}} := \sup_{t>0} t|\{x \in \Omega : |v(x)| > t\}|^{1/p}$$

is finite. Note that $[v]_{L^p_{\text{weak}}}$ is not a norm on $L^p_{\text{weak}}(\Omega)$ since the triangle inequality is not satisfied. For example, for $\Omega = (0,1)$, $p = 1$, $u(x) = x$, $v(x) = 1 - x$ we have $[u]_{L^1_{\text{weak}}} = 1/4$ and $[v]_{L^1_{\text{weak}}} = 1/4$ but $[u+v]_{L^1_{\text{weak}}} = 1$. In view of this example, we have

$$[u+v]_{L^p_{\text{weak}}(\Omega)} \leqslant 2\{[u]_{L^p_{\text{weak}}(\Omega)} + [v]_{L^p_{\text{weak}}(\Omega)}\}.$$

In other words, $[v]_{L^p_{\text{weak}}}$ is a quasinorm on $L^p_{\text{weak}}(\Omega)$ (cf. [**33**], where, in particular, the above example is given). For an equivalent norm on $L^p_{\text{weak}}(\Omega)$ we refer the reader to [**4**] and [**34**]. However, we will not use this norm here. But the following facts are important for our purposes:

$$L^p(\Omega) \subsetneqq L^p_{\text{weak}}(\Omega), \quad [f]_{L^p_{\text{weak}}(\Omega)} \leqslant \|f\|_{L^p(\Omega)},$$
$$L^p_{\text{weak}}(\Omega) \subset L^{p-\varepsilon}(\Omega), \quad \|f\|_{L^{p-\varepsilon}(\Omega)} \leqslant \Big(\frac{p}{\varepsilon}\Big)^{1/(p-\varepsilon)} |\Omega|^{\varepsilon/[p(p-\varepsilon)]} [f]_{L^p_{\text{weak}}(\Omega)} \qquad (6)$$

for $0 < \varepsilon \leqslant p-1$, which are proved, for example, in [**33**] and [**35**]. A short review of weak Lebesgue spaces and their connection with Green matrices and elliptic systems can be found in [**9**].

We also use Sobolev spaces of fractional order (or Slobodeckij spaces)

$$W^{\beta,p}(\Omega) := \{u \in L^p(\Omega) : \|u\|_{W^{\beta,p}(\Omega)} < \infty\},$$
$$\|u\|^p_{W^{\beta,p}(\Omega)} := \|u\|^p_{L^p(\Omega)} + \int\limits_\Omega\int\limits_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+\beta p}}\,dxdy$$

for $0<\beta<1$, $1 \leqslant p < \infty$. These spaces are Banach spaces and Hilbert spaces for $p=2$. We write $H^\beta(\Omega) = H^{\beta,2}(\Omega) = W^{\beta,2}(\Omega)$. As usual, $H^{\beta,2}_0(\Omega)$ and $W^{\beta,p}_0(\Omega)$ denote the closure of $C^\infty_0(\Omega)$ with respect to the H^β- and $W^{\beta,p}$-norm. We refer to [**36**]–[**38**] for details. In particular, $W^{\beta,p}(\Omega) = F^\beta_{p,2}(\Omega)$ if $0<\beta<1$, $1<p<\infty$, and $\partial\Omega$ is sufficiently regular.

2. Local Bounds of Weak Solutions

In this section, we extend the classical result on local bounds for local bilinear forms to the case of nonlocal bilinear forms. Actually, this work was done in [**13**], where Caccioppoli's inequality and even Harnack's inequality were proved for purely nonlocal integro-differential operators. We review some of the results in [**13**].

Theorem 2.1 (local boundedness, [**13**]). *Let $u \in H^{\alpha/2}_0(\Omega)$ be a solution of the equation $a(u,\varphi)=0$ for all $\varphi \in H^{\alpha/2}_0(\Omega)$. Then there exists a constant $C(\lambda,\Lambda,\alpha,n)$ such that for real $p>1$ and $B_\rho(x) \subset\subset \Omega$ we have*

$$\sup_{B_{\rho/2}(x)} |u(y)| \leqslant C\Big(\fint_{B_\rho(x)} |u(y)|^p\,dy\Big)^{1/p}.$$

It is remarkable that Theorem 2.1 can be proved by Moser's iteration technique [**22**] in the same way as in the case of local differential operators of second order. The behavior of a nonlocal bilinear form $a(\cdot,\cdot)$ is of the same diffusion type as in the case of the Laplace operator. This assertion is a quintessence of [**13**]. The iteration procedure is based on iterating the Caccioppoli inequality for powers of u. The iteration has to be carried out for local and global terms simultaneously. Since the proof of Theorem 2.1 is given in [**13**], we restrict ourselves to obtaining the Caccioppoli inequality.

Theorem 2.2 (Caccioppoli's inequality, [**13**]). *Let $u \in H^{\alpha/2}_0(\Omega)$ be a solution of the equation $a(u,\varphi)=0$ for all $\varphi \in H^{\alpha/2}_0(\Omega)$. Suppose that $x_0 \in \Omega$ and $\rho, R > 0$ are given and satisfy the condition $B_\rho(x_0) \subset B_R(x_0), B_{4R}(x_0) \subset\subset \Omega$.*

Then the following estimate holds:

$$\int_{B_\rho}\int_{B_R}(u(x)-u(y))^2k(x,y)\,dy\,dx+\int_{B_\rho}|u(x)|^2\Big(\int_{\mathbb{R}^n\setminus B_R}k(x,y)\,dy\Big)\,dx \leqslant C(R-\rho)^{-\alpha}\int_{B_{2R-\rho}}|u(y)|^2\,dy. \tag{7}$$

REMARK. For $s=\rho$ and $S=2R-\rho$ the inequality (7) implies the estimate

$$\int_{B_s}\int_{B_s}(u(x)-u(y))^2k(x,y)\,dy\,dx\leqslant C(S-s)^{-\alpha}\int_{B_S}|u(y)|^2\,dy. \tag{8}$$

This means that the nonlocal bilinear form $a(\cdot,\cdot)$ leads to a local Caccioppoli inequality. The only difference from the case of local differential operators of second order is that the norm on the left-hand side is taken in spaces of fractional order. The constant C depends on λ, Λ, n, α, and $|\Omega|$.

PROOF OF THEOREM 2.2. We consider a localization function τ such that $\tau:\Omega\to\mathbb{R}$ is a smooth function possessing the following properties:

$$\tau(x)=1\quad\forall x\in B_\rho(x_0),\qquad \tau(x)=0\quad\forall x\in\Omega\setminus B_R(x_0),$$
$$|\tau(x)-\tau(y)|\leqslant C(R-\rho)^{-1}|x-y|,\qquad \tau(x)\leqslant\tau(y)\quad\forall x:|x-x_0|\geqslant|y-y_0|.$$

By symmetry, we have

$$\begin{aligned}a(u,\tau^2\varphi)&=\int_{B_\rho}\int_{B_\rho}(u(x)-u(y))(\varphi(x)-\varphi(y))k(x,y)\,dy\,dx\\&\quad+2\int_{B_\rho}\int_{B_R\setminus B_\rho}(u(x)-u(y))(\varphi(x)-\tau^2(y)\varphi(y))k(x,y)\,dy\,dx\\&\quad+\int_{B_R\setminus B_\rho}\int_{B_R\setminus B_\rho}(u(x)-u(y))(\tau^2(x)\varphi(x)-\tau^2(y)\varphi(y))k(x,y)\,dy\,dx\\&\quad+2\int_{B_R}\int_{\Omega\setminus B_R}(u(x)-u(y))\tau^2(x)\varphi(x)k(x,y)\,dy\,dx\\&\quad+2\int_{B_R}\tau^2(x)|u(x)|^2\Big(\int_{\mathbb{R}^n\setminus\Omega}k(x,y)\,dy\Big)\,dx.\end{aligned} \tag{9}$$

Choosing $\varphi=u$, taking into account the equality

$$\tau^2(x)\varphi(x)-\tau^2(y)\varphi(y)=\tau^2(x)(\varphi(x)-\varphi(y))+(\tau(x)+\tau(y))(\tau(x)-\tau(y))\varphi(y),$$

using the notation $U(x,y)=(u(x)-u(y))^2k(x,y)$, applying Young's inequality in the form

$$(u(x)-u(y))(\tau(x)+\tau(y))(\tau(x)-\tau(y))u(y) \leqslant \frac{1}{2}U(x,y)(\tau(x)+\tau(y))^2+\frac{1}{2}(\tau(x)-\tau(y))^2|u(y)|^2,$$

and referring to the assumption $|\tau(x)-\tau(y)|^2 \leqslant C\cdot(R-\rho)^{-2}|x-y|^2$, from (9) we finally obtain the estimate

$$\begin{aligned} a(u,\tau^2u) \geqslant & \int_{B_\rho}\int_{B_R} U(x,y)\,dy\,dx + \frac{1}{8}\int_{B_R\setminus B_\rho}\int_{B_R\setminus B_\rho} \tau^2(x)U(x,y)\,dy\,dx \\ & + \int_{B_R} \tau^2(x)|u(x)|^2\Big(\int_{\mathbb{R}^n\setminus B_R} k(x,y)\,dy\Big)\,dx \\ & - C\int_{B_R\setminus B_\rho} |u(y)|^2\Big\{\Big(\int_{B_\rho}(1-\tau(y))^2k(x,y)\,dx\Big) \\ & + \Big(\int_{B_R\setminus B_\rho}(\tau(x)-\tau(y))^2k(x,y)\,dx\Big)\Big\}\,dy - \int_{B_R}\int_{\Omega\setminus B_R}\tau^2(x)|u(y)|^2k(x,y)\,dy\,dx. \end{aligned}$$

For $y\in B_R\setminus B_\rho$ we have

$$\int_{B_\rho}(1-\tau(y))^2k(x,y)\,dx \leqslant C(R-\rho)^{-\alpha}(|y|-\rho)^\alpha\int_{B_\rho}k(x,y)\,dx \leqslant C(R-\rho)^{-\alpha}$$

$$\int_{B_R\setminus B_\rho}(\tau(x)-\tau(y))^2k(x,y)\,dx \leqslant C(R-\rho)^{-2}\int_{B_R\setminus B_\rho}|x-y|^2k(x,y)\,dx \leqslant C(R-\rho)^{-\alpha}.$$

For the last inequality we refer to [**13**]. Finally, we have

$$\begin{aligned} & \int_{B_\rho}\int_{B_R} U(x,y)\,dy\,dx + \frac{1}{8}\int_{B_R\setminus B_\rho}\int_{B_R\setminus B_\rho}\tau^2(x)U(x,y)\,dy\,dx \\ & \quad + \int_{B_R}\tau^2(x)|u(x)|^2\Big(\int_{\mathbb{R}^n\setminus B_R}k(x,y)\,dy\Big)\,dx \\ & \leqslant C(R-\rho)^{-\alpha}\int_{B_R\setminus B_\rho}|u(y)|^2\,dy + \int_{B_R}\int_{\Omega\setminus B_R}|u(y)|^2\tau^2(x)k(x,y)\,dy\,dx. \qquad (10) \end{aligned}$$

So far, the analogy with local diffusion is quite obvious. The main difference is that the following global term appears:

$$(\mathrm{T}) \quad := \int_{B_R} \int_{\Omega\setminus B_R} |u(y)|^2 \tau^2(x) k(x,y)\, dy\, dx. \tag{11}$$

In order to cope with this term, it is necessary to take into account the structure of the bilinear form. As we will see below, using a globalization function $\widetilde{\tau}$, i.e., a function with support outside the ball B_R, one can estimate (T) by a local term.

We note that such an estimate would be possible when dealing with local diffusion operators. The main difference is that the use of localization functions as test functions in local bilinear forms does not lead to any nonlocal terms.

Assume that $x \in \Omega$ and $\widetilde{\rho}, \widetilde{R} > 0$ satisfy the condition $B_{\widetilde{\rho}}(x_0) =: B_{\widetilde{\rho}} \subset B_{\widetilde{R}}$, $B_{4\widetilde{R}} \subset\subset \Omega$. Consider a globalization function $\widetilde{\tau}$ such that $\widetilde{\tau} : \Omega \to \mathbb{R}$ is a smooth function possessing the properties

$$\begin{aligned}
&\widetilde{\tau}(x) = 1 \quad \forall x \in \Omega\setminus B_{\widetilde{R}};\, \widetilde{\tau}(x) = 0 \quad \forall x \in B_{\widetilde{\rho}},\\
&|\widetilde{\tau}(x) - \widetilde{\tau}(y)| \leqslant C(\widetilde{R} - \widetilde{\rho})^{-1}|x-y|,\\
&\widetilde{\tau}(x) \leqslant \widetilde{\tau}(y) \quad \forall \widetilde{x} : |x - x_0| \leqslant |y - x_0|.
\end{aligned}$$

By symmetry, in the same way as in the case (9) and (10), we find

$$\begin{aligned}
&\int_{\Omega\setminus B_{\widetilde{R}}} \int_{\Omega\setminus B_{\widetilde{\rho}}} (u(x) - u(y))^2 k(x,y)\, dy\, dx\\
&\quad + \frac{1}{8} \int_{B_{\widetilde{R}}\setminus B_{\widetilde{\rho}}} \int_{B_{\widetilde{R}}\setminus B_{\widetilde{\rho}}} \widetilde{\tau}^2(x)(u(x) - u(y)) k(x,y)\, dy\, dx\\
&\quad + \int_{\Omega\setminus B_{\widetilde{\rho}}} \widetilde{\tau}^2(x) |u(x)|^2 \Big(\int_{B_{\widetilde{\rho}}} k(x,y)\, dy \Big)\, dx\\
&\leqslant C(\widetilde{R} - \widetilde{\rho})^{-\alpha} \int_{B_{\widetilde{R}}\setminus B_{\widetilde{\rho}}} |u(y)|^2\, dy + \int_{B_{\widetilde{\rho}}} |u(y)|^2 \Big(\int_{\Omega\setminus B_{\widetilde{\rho}}} \widetilde{\tau}^2(x) k(x,y)\, dy \Big)\, dx.
\end{aligned}$$

As a trivial consequence, we have

$$\begin{aligned}
&\int_{\Omega\setminus B_{\widetilde{\rho}}} \widetilde{\tau}^2(y) |u(y)|^2 \Big(\int_{B_{\widetilde{\rho}}} k(x,y)\, dx \Big)\, dy\\
&\leqslant C(\widetilde{R} - \widetilde{\rho})^{-\alpha} \int_{B_{\widetilde{R}}\setminus B_{\widetilde{\rho}}} |u(y)|^2\, dy + C(\widetilde{R} - \widetilde{\rho})^{-\alpha} \int_{B_{\widetilde{\rho}}} |u(y)|^2\, dy.
\end{aligned} \tag{12}$$

The remaining question is whether the term on the left-hand side of (12) dominates the term (T) in (11). Let us try to prove the estimate (11) using

(12) and setting $\widetilde{\rho} = R$:

$$
\begin{aligned}
(\mathrm{T}) &= \int\limits_{B_R}\int\limits_{\Omega\setminus B_R} |u(y)|^2\tau^2(x)k(x,y)\,dy\,dx \\
&= \int\limits_{\Omega\setminus B_{\widetilde{R}}} |u(y)|^2\Big(\int\limits_{B_R}\tau^2(x)k(x,y)\,dx\Big)\,dy + \int\limits_{B_{\widetilde{R}}\setminus B_R} |u(y)|^2\Big(\int\limits_{B_R}\tau^2(x)k(x,y)\,dx\Big)\,dy \\
&\leqslant \int\limits_{\Omega\setminus B_{\widetilde{\rho}}} \widetilde{\tau}^2(y)|u(y)|^2\Big(\int\limits_{B_R} k(x,y)\,dx\Big)\,dy + \int\limits_{B_{\widetilde{R}}\setminus B_R} |u(y)|^2\Big(\int\limits_{B_R\setminus B_\rho} \tau^2(x)k(x,y)\,dx\Big)\,dy \\
&\quad + \int\limits_{B_{\widetilde{R}}\setminus B_R} |u(y)|^2\Big(\int\limits_{B_\rho} k(x,y)\,dx\Big)\,dy \\
&\overset{(12)}{\leqslant} C(\widetilde{R}-R)^{-\alpha}\int\limits_{B_{\widetilde{R}}\setminus B_R} |u(y)|^2\,dy + C(\widetilde{R}-R)^{-\alpha}\int\limits_{B_R}|u(y)|^2\,dy \\
&\quad + C(R-\rho)^{-2}\int\limits_{B_{\widetilde{R}}\setminus B_R} |u(y)|^2\Big(\int\limits_{B_R\setminus B_\rho} |x-y|^{2-\alpha-n}\,dx\Big)\,dy \\
&\quad + C\int\limits_{B_{\widetilde{R}}\setminus B_R} |u(y)|^2(|y|-\rho)^{-\alpha}\,dy \\
&\leqslant C(\widetilde{R}-R)^{-\alpha}\int\limits_{B_{\widetilde{R}}\setminus B_R} |u(y)|^2\,dy + C(\widetilde{R}-R)^{-\alpha}\int\limits_{B_R}|u(y)|^2\,dy \\
&\quad + C(R-\rho)^{-\alpha}\int\limits_{B_{\widetilde{R}}\setminus B_R} |u(y)|^2\,dy\,,
\end{aligned}
$$

We have successfully estimated from above the global term (T) by a local term. In order to use this estimate in (10), we set $\widetilde{R} = 2R-\rho$. Then

$$
\begin{aligned}
&\int\limits_{B_\rho}\int\limits_{B_R} U(x,y)\,dy\,dx + \int\limits_{B_R}\tau^2(x)|u(x)|^2\Big(\int\limits_{\mathbb{R}^n\setminus B_R} k(x,y)\,dy\Big)\,dx \\
&\leqslant C(R-\rho)^{-\alpha}\int\limits_{B_R}|u(y)|^2\,dy + C(R-\rho)^{-\alpha}\int\limits_{B_{2R-\rho}\setminus B_R} |u(y)|^2\,dy.
\end{aligned}
$$

The estimate (7) is proved. $\square$

3. The Regularized Green Function and Uniform Bounds

Theorem 1.1 is proved as follows. At the first step, for $\rho > 0$ we prove the existence of a regularized Green function $G_\rho(\cdot, \cdot)$ satisfying the following equality for $x_0 \in \Omega$:

$$a(G_\rho(\cdot, x_0), \varphi) = \frac{1}{B_\rho(x_0)} \int\limits_{B_\rho(x_0)} \varphi(x)\, dx \qquad \forall \varphi \in C_0^\infty(\Omega).$$

Next, we derive L^q-bounds for G_ρ that are uniform with respect to ρ. Uniform bounds for G_ρ in $H_0^s(\Omega)$ for some $s > 0$ have to be proved. By compactness, we can assert that there exists a subsequence ρ_k such that G_{ρ_k} converges to G. By Proposition 3.3, the pointwise estimate (5) follows from Theorem 2.1.

Propositions 3.1 and 3.3 follow closely the presentation of [**39, 4, 5, 6**]. It is a remarkable fact that the ideas developed for local bilinear forms of type $\int\limits_\Omega \nabla u \nabla \varphi\, dx$ can be extended to the nonlocal forms treated here.

Let $x_0 \in \Omega$ be fixed. The form $a(\cdot, \cdot)$ is a continuous positive definite bilinear form on $H_0^{\alpha/2}(\Omega) \times H_0^{\alpha/2}(\Omega)$. By the Lax–Milgram theorem, there exists a unique function $G_\rho \in H_0^{\alpha/2}(\Omega)$ such that for all $\varphi \in H_0^{\alpha/2}(\Omega)$ we have the representation

$$a(G_\rho, \varphi) = \fint\limits_{B_\rho(x_0)} \varphi(x)\, dx. \tag{13}$$

It is obvious that G_ρ is nonnegative almost everywhere.

Proposition 3.1. *There exists a constant C independent of ρ such that*

$$\|G_\rho\|_{L^{n/(n-\alpha)}_{\text{weak}}(\Omega)} \leqslant C, \tag{14}$$

$$\int\limits_\Omega \int\limits_\Omega \frac{|G_\rho(x) - G_\rho(y)|^2}{|x-y|^{n+\alpha}}\, dy\, dx \leqslant C\rho^{\alpha - n}. \tag{15}$$

PROOF. Let $\Omega_t := \{x \in \Omega\, ;\, G_\rho(x) > t\}$. The function $\varphi(x) := \max\{0, 1/t - 1/G_\rho(x)\}$, $t \in \mathbb{R}_+$, is an admissible test function in (13). Since the expression $(G_\rho(x) - G_\rho(y))(\varphi(x) - \varphi(y))$ is positive for $(x, y) \in (\Omega \backslash \Omega_t) \times \Omega_t$ and $(x, y) \in \Omega_t \times (\Omega \backslash \Omega_t)$, in view of this choice, we have

$$\int\limits_{\Omega_t} \int\limits_{\Omega_t} (G_\rho(x) - G_\rho(y))((G_\rho(y))^{-1} - (G_\rho(x))^{-1}) k(x, y)\, dy\, dx \leqslant \frac{1}{t}.$$

Taking into account the inequality $(a-b)(b^{-1} - a^{-1}) \geqslant (\log a - \log b)^2$ (cf. [**13**] for this and other tricks related to the logarithm and nonlocal bilinear forms) and the Sobolev–Poincaré inequality, we find

$$\int\limits_{\Omega_t}\int\limits_{\Omega_t} \frac{(\log G_\rho(x) - \log G_\rho(y))^2}{|x-y|^{n+\alpha}}\, dy\, dx \leqslant \frac{1}{\lambda t},$$

$$\Big(\int\limits_{\Omega_t} |\log(G_\rho(x)/t)|^{2n/(n-\alpha)}\, dx\Big)^{(n-\alpha)/n} \leqslant \frac{C}{t},$$

$$\operatorname{meas}(\Omega_{2t})^{(n-\alpha)/n} \leqslant (\log 2)^{-2}\frac{2C}{2t} \quad \Rightarrow \quad \operatorname{meas}(\Omega_{2t}) \leqslant Ct^{-n/(n-\alpha)}.$$

Thus, the estimate (14) is proved.

To prove (15), we set $\varphi(x) = G_\rho(x)$ in (13). Then

$$\begin{aligned}\lambda\int\limits_\Omega\int\limits_\Omega \frac{|G_\rho(x) - G_\rho(y)|^2}{|x-y|^{n-\alpha}}\, dy\, dx &\leqslant \fint\limits_{B_\rho} G_\rho(x)\, dx \\ &\leqslant C\rho^{-n}\Big(\int\limits_{B_\rho} |G_\rho(x)|^{2n/(n-\alpha)}\, dx\Big)^{(n-\alpha)/2n}\rho^{[n(n+\alpha)]/2n} \\ &\leqslant C\rho^{(-\alpha+n)/2}\Big(\int\limits_\Omega\int\limits_\Omega \frac{|G_\rho(x) - G_\rho(y)|^2}{|x-y|^{n-\alpha}}\, dy\, dx\Big)^{1/2},\end{aligned}$$

which implies (15). □

Proposition 3.2. *There exists a constant C independent of ρ such that for $p \in [1, n/(n-\alpha/2))$ the following estimate holds:*

$$\int\limits_\Omega\int\limits_\Omega \frac{|G_\rho(x) - G_\rho(y)|^p}{|x-y|^{n+\frac{\alpha}{2}p}}\, dy\, dx \leqslant C. \tag{16}$$

In order to prove Proposition 3.2, we need two technical lemmas. Lemma 3.2 below allows us to use a test function that finally will result in control of the fractional derivative of G_ρ as expressed by (16).

Lemma 3.1. *For $s \in (0,1]$ and $z > 0$ the following estimate holds:*

$$(1 + z^s)^{1/s} \leqslant 2^{(1-s)/s}(1+z). \tag{17}$$

Proof. Let $f(z) := (1+z^s)^{1/s}(1+z)^{-1}$. Then $f(0) = 1$ and $f(1) = 2^{(1-s)/s} > 1$. Computing $f'(z)$, we see that $f'(z) > 0$ for $z < 1$, $f'(z) < 0$ for $z > 1$, and $f'(1) = 0$. □

Lemma 3.2. *For $s \in (0,1]$ and $a, b > 0$ the following estimate holds:*

$$\begin{aligned}&(a-b)(a(1+a^s)^{-1/s} - b(1+b^s)^{-1/s}) \\ &\qquad \geqslant \frac{8}{2^{1/s}}(1-s)^{-2}((1+a)^{(1-s)/2} - (1+b)^{(1-s)/2})^2.\end{aligned} \tag{18}$$

PROOF. First we define two auxiliary functions $f(a) = a(1+a^s)^{-1/s}$ and $g(a) = (1+a)^{(1-s)/2}$. Note that $f'(a) = (1+a^s)^{(-1-s)/s}$ and $g'(a) = \frac{1-s}{2}(1+a)^{(-1-s)/2}$. Using Lemma 3.1, we can prove (18) as follows:

$$\begin{aligned}
f(a) - f(b) &= \int_0^1 f'(b + t(a-b))(a-b)\,dt \\
&= (a-b)\int_0^1 (1 + (b + t(a-b))^s)^{(-1-s)/s}\,dt \\
&\overset{(17)}{\geqslant} (a-b)2^{(s-1)/s}\int_0^1 (1 + b + t(a-b))^{-1-s}\,dt \\
&= (a-b)^{-1}2^{(s-1)/s}\int_0^1 [(1 + b + t(a-b))^{(-1-s)/2}(a-b)]^2\,dt \\
&\geqslant (a-b)^{-1}2^{(s-1)/s}\Big(\int_0^1 (1 + b + t(a-b))^{(-1-s)/2}(a-b)\,dt\Big)^2 \\
&= (a-b)^{-1}2^{(s-1)/s}\Big(\frac{2}{1-s}\Big)^2\Big(\int_0^1 g'(b + t(a-b))(a-b)\,dt\Big)^2 \\
&= (a-b)^{-1}2^{(s-1)/s}\Big(\frac{2}{1-s}\Big)^2(g(a) - g(b))^2 \\
&= (a-b)^{-1}2^{(3s-1)/s}(1-s)^{-2}((1+a)^{(1-s)/2} - (1-b)^{(1-s)/2})^2.
\end{aligned}$$

The lemma is proved. □

PROOF OF PROPOSITION 3.2. For a test function in (13) we take $\varphi(x) = G_\rho(x)(1 + G_\rho(x)^s)^{-1/s}$, where $s \in (0, \alpha/n)$. The bound α/n becomes essential only at the end of the proof when we use an embedding theorem. For our purposes it is important to choose s as small as possible. Note that $\varphi(x) \in H_0^{\alpha/2}(\Omega)$ and $0 \leqslant \varphi(x) < 1$. Such test functions were often used (for instance, by Frehse in several situations) in order to obtain L^q-bounds for Green functions and their derivatives in the case of local differential operators. In our case, Lemma 3.2 replaces the rule of partial integration which is not available here. Thus, using φ in (13) and taking into account (18), we find

$$\int_\Omega\int_\Omega \frac{((1 + G_\rho(x))^{(1-s)/2} - (1 + G_\rho(y))^{(1-s)/2})^2}{|x-y|^{n+\alpha}}\,dy\,dx \leqslant C(s), \tag{19}$$

where $C(s)$ is independent of ρ. The estimate (19) means that the $H^{\alpha/2}$-norm of $(1+G_\rho)^{(1-s)/2}$ is uniformly bounded. Hence the following question arises: In what function space the function G_ρ is uniformly bounded. This question is answered by the following theorem:

Theorem 3.1 (cf., for example, [**40**, Chap. 5.4.3]). *Assume that $\mu > 1$ and $\mu((n-\alpha)/2) < n$. We set $t := \dfrac{n}{\alpha/2+\mu((n-\alpha)/2)}$. Then for all $f \in W^{\alpha/2,2}(\Omega)$ the following estimate holds:*

$$\| \, |f|^\mu \|_{W^{\alpha/2,t}(\Omega)} \leqslant C\|f\|_{W^{\alpha/2,2}(\Omega)}.$$

In order to apply this theorem to $f=(1+G_\rho)^{(1-s)/2}$ with $\mu = 2/(1-s)$, we have to check that $\dfrac{2}{1-s}(\dfrac{n-\alpha}{2}) < n$ if and only if $s < \alpha/n$. It is true because of the choice of s. Using the theorem, we get

$$\|G_\rho\|_{W^{\alpha/2,t}(\Omega)} \leqslant C(s)$$

with

$$t = \frac{n}{\alpha/2+\frac{2}{1-s}((n-\alpha)/2)} = \frac{n(1-s)}{n-\frac{\alpha}{2}(1+s)} < \frac{n}{n-\alpha/2}.$$

We note that for any $p \in (1, n/(n-\alpha/2))$ there exists $s \in (0, \alpha/n)$, namely, $s = \dfrac{n(1-p)+\frac{\alpha}{2}p}{n-\frac{\alpha}{2}p}$, such that $t=p$. This proves (16). □

Using the uniform bounds for G_ρ in Propositions 3.1 and 3.2, we can establish the existence of G. The pointwise estimate (5) should be proved separately. This is the goal of the following assertion.

Proposition 3.3. *There exists a constant C independent of ρ such that for $x \in \Omega$, $|x-x_0| \geqslant 2\rho$, the following estimate holds:*

$$G_\rho(x) \leqslant C|x-x_0|^{\alpha-n}. \tag{20}$$

PROOF. Let $R := |x-x_0| \geqslant 2\rho$. Consider the case $B_{R/2}(x) \subset \Omega$. Since G_ρ is a solution of the equation $a(u,\varphi)=0$ in $\Omega\backslash B_\rho$ (take test functions φ with support in $\Omega\backslash B_{3\rho/2}$), we can use Theorem 2.1 to get

$$(G_\rho(x))^p \leqslant (\sup_{B_{R/8}(x)} G_\rho(y))^p \leqslant C^p \fint_{B_{R/4}(x)} G_\rho(y)^p\,dy,$$

where we assume that p is less than $n/(n-\alpha)$, so that the L^p-norm can be estimated by the $L^{n/(n-\alpha)}_{\text{weak}}$-norm as follows:

$$\fint_{B_{R/4}(x)} G_\rho(y)^p\,dy \leqslant C(n,p)R^{p(\alpha-n)}[G_\rho]^p_{L^p_{\text{weak}}} \leqslant C(n,p)R^{p(\alpha-n)},$$

which implies $G_\rho(x) \leqslant C(n,p,\lambda,\Lambda)|x-x_0|^{\alpha-n}$. Hence Proposition 3.3 is proved in this case.

If $B_{R/2}(x) \not\subset \Omega$, we consider a domain $\widetilde{\Omega} \supset \Omega$ so large that $B_{R/2}(x) \subset \widetilde{\Omega}$. Since the bilinear form $a(\cdot,\cdot)$ can be extended to $\widetilde{\Omega} \times \widetilde{\Omega}$, we get a function $\widetilde{G_\rho}$. Restricting $\widetilde{G}_\rho$ to Ω, we see that $a(G_\rho - \widetilde{G}_\rho, \varphi) = 0$ in Ω. But $0 = G_\rho \leqslant \widetilde{G}_\rho$ on $\partial\Omega$. Therefore, $(G_\rho - \widetilde{G}_\rho)^+$ is an admissible test function. Hence $(G_\rho - \widetilde{G}_\rho)^+ = 0$ in Ω if and only if $G_\rho \leqslant \widetilde{G}_\rho$ in Ω. Now, the assertion of Proposition 3.3 is true for $\widetilde{G}_\rho$ with the same constants as above (by the previous case). Therefore, it is proved for G_ρ. □

Proof of Theorem 1.1. Having proved the estimates (14) and (16) which are uniform with respect to ρ, it is not difficult to prove Theorem 1.1. For any $s \in [1, n/(n-\alpha/2))$ the function G_ρ is uniformly bounded in $H_0^{\alpha/2,s}(\Omega)$. Therefore, there is a subsequence ρ_k tending to zero and a function G such that

$$G_{\rho_k} \rightharpoonup G \text{ in } H_0^{\alpha/2,s}(\Omega) \quad \forall s \in [1, n/(n-\alpha/2)). \tag{21}$$

The equality (1) follows from the fact that $a(\cdot,\varphi)$ is a continuous linear functional on $H_0^{\alpha/2,s}(\Omega)$ for any $\varphi \in C_0^\infty(\Omega)$. Thus, (1) follows from (21) and Lebesgue's differentiation theorem.

The inclusion $G \in L_{\text{weak}}^{n/(n-\alpha)}(\Omega)$ can be proved as follows. Setting $q := n/(n-\alpha)$, $0 < \varepsilon < q-1$, $\Omega_t := \{x \in \Omega; G(x) > t\}$ and denoting by $|\Omega_t|$ the Lebesgue measure of Ω_t, from the lower semi-continuity of L^p-norms and (6) we find

$$\begin{aligned} \|G\|_{L^{q-\varepsilon}(\Omega_t)} &\leqslant \liminf_{k\to\infty} \|G_{\rho_k}\|_{L^{q-\varepsilon}(\Omega_t)} \\ &\leqslant \liminf_{k\to\infty} (q/\varepsilon)^{1/(q-\varepsilon)} |\Omega_t|^{\varepsilon/(q(q-\varepsilon))} [G_{\rho_k}]_{L^q_{\text{weak}}(\Omega_t)} \\ &\leqslant C(n,\lambda)(q/\varepsilon)^{1/(q-\varepsilon)} |\Omega_t|^{\varepsilon/(q(q-\varepsilon))}. \end{aligned}$$

Since $[f]_{L^p_{\text{weak}}} \leqslant \|f\|_{L^p}$ for all $f \in L^p$, we have

$$t|\Omega_t|^{\frac{1}{q-\varepsilon}} \leqslant C(n,\lambda)(q/\varepsilon)^{1/(q-\varepsilon)} |\Omega_t|^{\varepsilon/(q(q-\varepsilon))} \Rightarrow t|\Omega_t|^{1/q} \leqslant C(n,\lambda)(q/\varepsilon)^{1/(q-\varepsilon)}.$$

The pointwise estimate (5) follows from (20) because a subsequence of G_ρ converges pointwise to G. □

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$\mathbf{L}_q$-Estimates of the First-Order Derivatives of Solutions to the Nonstationary Stokes Problem

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Dedicated to Professor O. A. Ladyzhenskaya
on the occasion of her birthday

For a solution to the nonstationary Stokes problem in the half-space $\mathbb{R}^3_+$ with the external force $f = \nabla \cdot F$, $F \in L_q(\mathbb{R}^3_+ \times (0,T))$, we establish the L_q-estimates for the first-order derivatives of the vector field of velocities and prove that the pressure does not belong to the space $L_q(\mathbb{R}^3_+ \times (0,T))$.

1. Introduction

It is well known that a solution to the initial-boundary-value problem

$$\begin{aligned} &\vec{v}_t - \Delta\vec{v} + \nabla p = \vec{f}, \quad \nabla \cdot \vec{v} = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \\ &\vec{v} = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \qquad \vec{v} = 0 \quad \text{on } \Omega \times \{0\} \end{aligned} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with C^2-boundary $\partial\Omega$ satisfies the estimate

$$\|\vec{v}_t\|_{L_q(\Omega\times\mathbb{R}^+)} + \|\vec{v}\|_{L_q(\mathbb{R}^+,W^2_q(\Omega)} + \|\nabla p\|_{L_q(\Omega\times\mathbb{R}^+)} \leqslant c\|\vec{f}\|_{L_q(\Omega\times\mathbb{R}^+)}, \quad 1 < q < \infty.$$

This result was first proved in [**1**] for an arbitrary finite time interval $(0,T)$.

In the present paper, we establish the estimate

$$\|\nabla\vec{v}\|_{L_q(\Omega\times(0,T))} \leqslant c\|F\|_{L_q(\Omega\times(0,T))} \tag{1.2}$$

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combined with estimates for the pressure assuming that $\vec{f} = \nabla \cdot F$, i.e.,

$$f_m = \sum_{k=1}^{n} \partial_{x_k} F_{km},$$

and $\Omega = \mathbb{R}^n_+ = \{x : x_n > 0\}$. Knowledge about the regularity of the pressure of weak solutions is crucial, for example, for local estimates of weak solutions (where the multiplication by cutoff functions leads to loss of the divergence freedom) for analytical questions (like partial regularity), as well as practical questions (like a posteriori error estimates). Unfortunately, the regularity issue is rather complicated and different from what could be intuitively expected. In particular, we will see that for $\varphi \in C_0^\infty(\Omega)$

$$t \to \int_\Omega \varphi(x) p(x,t)\, dx \in W_q^{-1/(2q)}(\mathbb{R}) \tag{1.3}$$

and not in L_q in general. The estimate (1.3) follows from Theorem 1.1. It is the best possible one in view of Theorem 1.3. For the sake of simplicity, we restrict ourselves to the case $n = 3$.

We also consider a more general problem with nonzero divergence: $\nabla \cdot \vec{v} = g(x,t)$. Let $\Sigma_T = \Omega \times (-\infty, T] = \mathbb{R}^3_+ \times (-\infty, T]$ for $T > 0$. We extend all functions on $\mathbb{R}^3_+ \times [0,T]$ by zero to $t < 0$. We prove all estimates for finite time intervals $(0,T)$ with constants independent of T.

We formulate our main results.

Theorem 1.1. *If $\vec{f} = \nabla \cdot F$, $F \in L_q(\Sigma_T)$, $q > 1$, then there is a unique solution to the problem* (1.1) *that satisfies the inequality* (1.2). *Moreover, we can decompose the velocity and pressure*

$$p(x,t) = p_1(x,t) + \frac{\partial P}{\partial t}, \tag{1.4}$$

where $P(x,t)$ is a harmonic function with respect to x for every t, and

$$\vec{v} = \vec{u} - \nabla P \tag{1.5}$$

so that

$$\begin{aligned} \|\nabla \vec{v}\|_{L_q(\Sigma_T)} + \|\vec{u}\|_{W_q^{1/2}((-\infty,T),L_q(\mathbb{R}^3_+))} + \|\partial_t \vec{u}\|_{L_q((0,T),\dot{W}_q^{-1}(\mathbb{R}^3_+))} + \|p_1\|_{L_q(\Sigma_T)} \\ + \|P|_{x_3=0}\|_{\dot{W}_q^{2-1/q,1-1/(2q)}(\mathbb{R}^2\times(-\infty,T))} \leqslant c\|F\|_{L_q(\Sigma_T)}. \end{aligned} \tag{1.6}$$

Remark 1.2. Formula (1.3) is an easy consequence of Theorem 1.1. The function spaces are defined in Sec. 2. The estimate of $\|\nabla \vec{v}\|_{L_q}$ follows from [**2**]. It is similar to well-known estimates for the heat equation. We note that the regularity with respect to t in (1.4)–(1.6) is considerably weaker than that for the heat equation:

$$\vec{v} \in \dot{W}_q^1(\mathbb{R}; \dot{W}_q^{-1}(\mathbb{R}^3_+))$$

if $F \in L_q$. On the other hand, it does not follow from the fact that $\vec{v} \in \dot{W}_q^{1,1/2}(\Sigma_T)$ obtained in [**2**].

Theorem 1.3. *Let $\Omega = \mathbb{R}^3_+$. The estimate in Theorem 1.1 is sharp in the following sense.*

(a) *Let $\eta \in W_q^{1-1/(2q)}(-\infty, T)$ have support in $[0,T]$. There exist $F \in L_q(\Omega \times (0,T))$, $\vec{\varphi} \in (C_0^\infty(\mathbb{R}^3_+))^3$, and $h \in W_q^1(0,T)$ such that*

$$\eta(t) = h(t) + \int\limits_{\mathbb{R}^3_+} \vec{u}(x,t) \cdot \vec{\varphi}(x)\, dx. \tag{1.7}$$

(b) *Let $\eta \in W_q^{-1/(2q)}(-\infty, T)$ be supported in $[0,T]$. There exist $F \in L_q(\Omega \times (0,T))$ and $\varphi \in C_0^\infty(\Omega)$ such that*

$$\eta(t) = \int\limits_{\mathbb{R}^3_+} p(x,t)\varphi(x)\, dx. \tag{1.8}$$

Remark 1.4. Formula (1.7) is a simple consequence of Theorem 1.1 and formula (1.8).

Theorem 1.5. *The solution to the problem*

$$\begin{aligned} &\vec{v}_t - \Delta\vec{v} + \nabla p = \nabla F, \quad \nabla\cdot\vec{v} = g \quad \text{in } \mathbb{R}^3_+ \times (0,T), \\ &\vec{v}\big|_{t=0} = 0, \quad \vec{v}\big|_{x_3=0} = 0, \end{aligned} \tag{1.9}$$

where $g, F \in L_q(\mathbb{R}^3_+ \times (0,T))$, satisfies the inequality

$$\|\nabla\vec{v}\|_{L_q(\Sigma_T)} \leqslant c[\|F\|_{L_q(\Sigma_T)} + \|g\|_{L_q(\Sigma_T)}] \tag{1.10}$$

The pressure can be represented in the form

$$p = p_1 + \partial_t P,$$

where both $p_1 \in L_q(\mathbb{R}^3_+ \times (0,T))$ and $P \in L_q((0,T); W_q^2(\mathbb{R}^3_+))$ can be estimated in terms of the right-hand side of (1.10).

Remark 1.6. The estimates are the same as in $\mathbb{R}^3$.

Remark 1.7. The solution to the Cauchy problem in the entire space

$$\vec{v}_t - \Delta\vec{v} + \nabla p = \nabla\cdot F, \quad \nabla\cdot\vec{v} = 0, \quad \vec{v}\big|_{t=0} = 0$$

satisfies the estimate

$$\|\nabla\vec{v}\|_{L_q(\mathbb{R}^3\times(0,T))} + \|p\|_{L_q(\mathbb{R}^3\times(0,T))} \leqslant c\|F\|_{L_q(\mathbb{R}^3\times(0,T))}$$

which can easily be obtained from the explicit formula for $(\vec{v}, p)$:

$$v_k = \sum_{m=1}^{3} \int_0^t \int_{\mathbb{R}^3} T_{km}(x - y, t - \tau) f_m(y, \tau)\, dy\, d\tau,$$

$$p = \int_{\mathbb{R}^3} \nabla E(x - y) \cdot \vec{f}(y, t)\, dy,$$

where $\vec{f} = \nabla \cdot F$ and

$$T_{km}(x, t) = \delta_{km}\Gamma(x, t) - \frac{\partial^2}{\partial x_k \partial x_m} \int_{\mathbb{R}^3} E(x - y)\Gamma(y, t)\, dy,$$

$E(x) = -\dfrac{1}{4\pi|x|}$ is the fundamental solution of the Laplace equation and $\Gamma(x, t) = (4\pi t)^{-n/2} \exp\left\{ - \dfrac{|x|^2}{4t} \right\}$ is the fundamental solution of the heat equation.

It is reasonable to expect that Theorems 1.1, 1.3, and 1.5 remain valid for rather general domains. The amount of regularity needed for the boundary is not clear at all. One can expect that the estimates hold for C^1-boundaries. We plan to return to this question later.

The authors would like to thank Prof. Nochetto for his interest to this work. which was a motivation to explain the (unexpected) regularity properties of the pressure in the simplest possible context.

The L_q-estimates for the gradient of $\vec{v}$ were obtained in [2] by methods of the abstract semigroup theory, including bounded imaginary powers of the Stokes operator. This result demonstrated the power of the abstract semigroup theory. However, it seems difficult to control the pressure by semigroup methods. The goal of this paper is to derive stronger estimates by decomposing the corresponding operators into simpler operators. Using this procedure, we will have sharp control of all the terms occurring in the equations.

2. Notation. Function Spaces and Norms

The standard Lebesgue spaces and Sobolev spaces are denoted by L_q and W_q^k, $1 \leqslant q \leqslant \infty$, $k \in \mathbb{N}$. The homogeneous part of the norm is denoted by $\dot{W}_p^k$. It contains only the highest-order terms. Let $0 < s < 1$ and $1 < q < \infty$. The Sobolev space $W_q^s(\mathbb{R}^n)$ is equipped with the norm

$$\|u\|^q_{W_q^s(\mathbb{R}^n)} = \|u\|^q_{L_q(\mathbb{R}^n)} + \|u\|^q_{\dot{W}_q^s(\mathbb{R}^n)},$$

where

$$\|u\|^q_{\dot W^s_q} = \int\limits_{\mathbb{R}^n} \|u(\cdot + h) - u(\cdot)\|^q_{L_q} |h|^{-n-sq} dh.$$

Similarly,

$$\|f\|^q_{\dot W^{1+s}_q} = \sum_i \|\partial_i f\|^q_{\dot W^s_q}.$$

In half-spaces and bounded domains with Lipschitz boundaries, we first fix a standard Whitney-type extension to $\mathbb{R}^n$ and introduce the norm as the norm of the extension. We equip the space $\dot W^{l,l/2}_{q,0}(\mathbb{R}^n \times (0,T))$, $0 < l < 2$, with the norm

$$\left(\int\limits_0^T \|u\|^q_{W^l_q(\mathbb{R}^n)} ds + \int\limits_{\mathbb{R}^n} \int\limits_0^T \int\limits_{-\infty}^T \frac{|u(x,t) - u(x,s)|^q}{|t-s|^{1+ql/2}} dt\, ds\, dx\right)^{1/q}.$$

The same norm is used for vector-valued functions.

We set $W^{-s}_q = (W^s_{\frac{q}{q-1},0})^*$, where the space on the right-hand side is the closure of the set of smooth functions in the $W^s_{\frac{q}{q-1}}$-norm. Similarly, we define the space $W^{-s,-s/2}_q$ by duality. By definition, we have

$$W^{s,s/2}_q = L^q([0,T], W^s_q) \cap W^{s/2}_q([0,T], L_q).$$

For the dual space we have

$$L^q([0,T], W^{-s}_q) \oplus L^q(\Omega; W^{-s/2}_q) \subset W^{-s,-s/2}_q.$$

It is easy to see (using localization in the Fourier variables and the Markinciewicz multiplier theorem) that the two sides coincide up to equivalent norms. We use homogeneous spaces with positive and negative indices of regularity without discussion of their properties (cf. [**3**] for a detailed discussion of homogeneous function spaces).

Lemma 2.1. *Let $u \in L_q(\mathbb{R}^n_+)$ be a harmonic function. Then u has trace; moreover,*

$$\|u\|_{L_q(\mathbb{R}^n_+)} \sim \|u|_{x_n=0}\|_{\dot W^{-1/q}_q(\mathbb{R}^{n-1})}.$$

PROOF. Let u be the harmonic extension of u_0 defined by the Poisson kernel. We assume that u_0 decreases sufficiently fast so that the extension is well defined. Then

$$\|u_{x_n}\|_{L_q(\mathbb{R}^n_+)} \sim \|u_0\|_{\dot W^{1-1/q}_q(\mathbb{R}^{n-1})} \sim \|\partial_{x_n} u|_{x_n=0}\|_{\dot W^{-1/q}_q(\mathbb{R}^{n-1})}$$

since

$$|\xi|\widehat{u}_0(\xi) = \partial_{x_n}\widehat{u}|_{x_n=0},$$

where $\widehat{}$ denotes the Fourier transform in $\mathbb{R}^{n-1}$. On the other hand, using this equality, for a given function $f \in \dot{W}_q^{-1/q}(\mathbb{R}^{n-1})$ we define u_0 by the Fourier transform $\widehat{u}_0 = |\xi|^{-1}\widehat{f}$. Then $\partial_{x_n} u\big|_{x_n=0} = f$ for the harmonic extension u of u_0. □

Lemma 2.2. *There exists a map*

$$E : W_q^{2-1/q,1-1/(2q)}(\mathbb{R}^{n-1} \times \mathbb{R}) \to W_q^{2,1}(\mathbb{R}_+^n \times \mathbb{R})$$

such that $Eg\big|_{x_n=0} = g$; *moreover,*

$$||(\partial_t - \Delta)Eg||_{L_q(\mathbb{R}_+^n \times \mathbb{R})} \sim ||g||_{\dot{W}_q^{2-1/q,1-1/(2q)}(\mathbb{R}^{n-1} \times \mathbb{R})}$$

for all $g \in W_q^{2-1/q,1-1/(2q)}(\mathbb{R}^{n-1} \times \mathbb{R})$.

PROOF. This is a consequence of trace theorems and inverse trace theorems. □

3. Proof of Theorem 1.1

The space of test functions is dense in L_q. By a standard approximation procedure, we can assume that F is supported away from $x_3 = 0$. The first assertion below concerns the decomposition of $\vec{f} = \nabla \cdot F$ into the sum of a gradient vector field and a divergence free vector field:

$$\vec{f} = \nabla\Phi + \vec{f}', \tag{3.1}$$

where

$$\Phi(x) = -\int_{\mathbb{R}_+^3} \nabla_y N(x,y) \cdot \vec{f}(y)\,dy = -\int_{\mathbb{R}_+^3} \sum_{i,j=1}^{3} \partial_i N \partial_j F_{ji} dy \tag{3.2}$$

and $N(x,y) = E(x-y) + E(x-y^*)$, $y^* = (y_1, y_2, -y_3)$, is the Green function for the Neumann problem for the Laplace equation in $\mathbb{R}_+^3$. It is easy to see that (at least, for sufficiently regular $\vec{f}$)

$$\nabla \cdot \vec{f}' = 0, \quad f_3'\big|_{x_3=0} = 0. \tag{3.3}$$

Proposition 3.1. *If* $\vec{f} = \nabla F = \left(\sum_{k=1}^{3} \frac{\partial F_{km}}{\partial x_k}\right)_{m=1,2,3}$, $F_{3m}\big|_{x_3=0} = 0$, *then*

$$\vec{f}' = \nabla F',$$

where

$$F'_{km} = F_{km} - \delta_{km}F_{33} + (1-\delta_{3k})\frac{\partial}{\partial x_m}\Big(\sum_{q=1}^{3}\int\limits_{\mathbb{R}^3_+}\frac{\partial N(x,y)}{\partial y_q}F_{kq}(y)\,dy$$

$$+\int\limits_{\mathbb{R}^3_+}\Big(\frac{\partial N}{\partial y_3}F_{3k} - \frac{\partial N}{\partial y_k}F_{33}\Big)\,dy\Big). \tag{3.4}$$

In particular, $F'_{3m} = F_{3m} - \delta_{3m}F_{33}$ *and* $F'_{3m}\big|_{x_3=0} = 0$; *moreover,*

$$\|F'\|_{L_q(\mathbb{R}^3_+)} \leqslant c\|F\|_{L_q(\mathbb{R}^3_+)}. \tag{3.5}$$

PROOF. The equality $\vec{f}' = \nabla F'$ is equivalent to the equality

$$\partial_{x_m}\Phi = \partial_{x_m}F_{33} - \sum_{k=1}^{2}\frac{\partial^2}{\partial x_k\partial x_m}\Big(\sum_{q=1}^{3}\int\limits_{\mathbb{R}^3_+}\frac{\partial N(x,y)}{\partial y_q}F_{kq}(y)\,dy$$

$$+\int\limits_{\mathbb{R}^3_+}\Big(\frac{\partial N}{\partial y_3}F_{3k} - \frac{\partial N}{\partial y_k}F_{33}\Big)\,dy\Big)$$

which follows from the equality

$$\Phi = -\sum_{j=1}^{2}\sum_{k=1}^{3}\frac{\partial}{\partial x_j}\int\limits_{\mathbb{R}^3_+}\frac{\partial N}{\partial y_k}F_{jk}\,dy - \sum_{k=1}^{2}\frac{\partial}{\partial x_k}\int\limits_{\mathbb{R}^3_+}\Big(\frac{\partial N}{\partial y_3}F_{3k} - \frac{\partial N}{\partial y_k}F_{33}\Big)dy + F_{33} \tag{3.6}$$

which, in turn, is equivalent to (3.2). Hence $F'_{3m}\big|_{x_3=0} = 0$. The estimate (3.5) follows from (3.4) and the Calderon–Zygmund theorem. □

Let us pass to the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. We represent $\vec{f} = \nabla F$ in the form (3.1) and incorporate Φ into the pressure. The function Φ given by (3.6) satisfies the estimate

$$\|\Phi\|_{L_q(\Sigma_T)} \leqslant c\|F\|_{L_q(\Sigma_T)}. \tag{3.7}$$

Without loss of generality, we can assume that $\nabla\cdot\vec{f} = 0$ and $F_{3k}\big|_{x_3=0} = 0$ for $k = 1,2,3$. Then the pressure $p(x,t)$ can be regarded as a solution to the Neumann problem

$$\begin{aligned}&\Delta p = 0 \quad \text{in } x_3 > 0,\\ &\frac{\partial p}{\partial x_3} = \Delta v_3 = -\sum_{j=1}^{2}\frac{\partial^2 v_j}{\partial x_3\partial x_j} \quad \text{if } x_3 = 0\end{aligned} \tag{3.8}$$

which is represented by the formula

$$p = -2 \int\limits_{\mathbb{R}^2} E(x' - y', x_3) \sum_{j=1}^{2} \frac{\partial^2 v_j}{\partial y_3 \partial y_j}\Big|_{y_3=0} dy'$$

and $\vec{v}$ can be regarded as a solution to the parabolic initial-boundary-value problem

$$\begin{aligned} &\vec{v}_t - \Delta \vec{v} = \vec{f} - \nabla p \quad \text{in } x_3 > 0, \\ &\vec{v}\big|_{t=0} = 0, \quad \vec{v}\big|_{x_3=0} = 0, \end{aligned} \tag{3.9}$$

which is represented by the formula

$$\vec{v} = \int\limits_0^t \int\limits_{\mathbb{R}^3_+} G(x, y, t - \tau)(\vec{f} - \nabla p) dy\, d\tau, \tag{3.10}$$

where $G(x, y, t) = \Gamma(x - y, t) - \Gamma(x - y^*, t)$. Therefore,

$$\begin{aligned} \sum_{j=1}^{2} \frac{\partial^2 v_j}{\partial x_j \partial x_3}\Big|_{x_3=0} &= -\Delta' \int\limits_0^t \int\limits_{\mathbb{R}^3_+} \mathcal{J}(x' - y', y_3, t - \tau) p(y, \tau) dy d\tau \\ &\quad + \nabla' \cdot \int\limits_0^t \int\limits_{\mathbb{R}^3_+} \mathcal{J}(x' - y', y_3, t - \tau) \vec{f}'(y, \tau)\, dy\, d\tau, \end{aligned}$$

where

$$\begin{aligned} \nabla' &= \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right), \quad \vec{f}' = (f_1, f_2), \\ \mathcal{J}(x', y, t) &= \frac{\partial G(x, y, t)}{\partial x_3}\Big|_{x_3=0} = -2 \frac{\partial \Gamma(x' - y', y_3, t)}{\partial y_3}. \end{aligned}$$

Thus, the boundary condition (3.8) yields the following equation for $\pi = \frac{\partial p}{\partial x_3}\Big|_{x_3=0}$:

$$\begin{aligned} &\pi(x', t) + 2\Delta' \int\limits_0^t \int\limits_{\mathbb{R}^3} \mathcal{J}(x' - y', y_3, t - \tau)\, dy\, d\tau \int\limits_{\mathbb{R}^2} E(y' - z', y_3) \pi(z', \tau)\, dz \\ &\quad = -\nabla' \cdot \int\limits_0^T \int\limits_{\mathbb{R}^3_+} \mathcal{J}(x' - y', y_3, t - \tau) \vec{f}(y, \tau)\, dy\, d\tau =: k(x', t). \end{aligned} \tag{3.11}$$

We write it in the form

$$(I + V)\pi = k.$$

The operator $I+V$ can easily be inverted. Indeed, using the Fourier–Laplace transform

$$\widetilde{u}(s,\xi) = FLu = \int\limits_0^\infty e^{st}dt \int\limits_{\mathbb{R}^2} e^{ix'\xi} u(x',t)dx'$$

with respect to x' and t and taking the relations

$$FL\mathcal{J}(\xi, y_3, s) = e^{-ry_3}, \quad 2FE(\xi, y_3) = \frac{e^{-|\xi|y_3}}{|\xi|},$$

where $r=\sqrt{s+|\xi|^2}$ and F denotes the Fourier transform with respect to x', we reduce (3.11) to the following relation:

$$\widetilde{\pi} - |\xi|^2 \int\limits_0^\infty e^{-(r+|\xi|)y_3} \frac{dy_3}{|\xi|} \widetilde{\pi} = \frac{r}{r+|\xi|}\widetilde{\pi} = \widetilde{k}. \tag{3.12}$$

Hence

$$\widetilde{\pi} = \left(1 + \frac{|\xi|}{r}\right)\widetilde{k} = \left(1 + \frac{|\xi|^2}{|\xi| r}\right)\widetilde{k},$$

i.e.,

$$\begin{aligned}\pi(x',t) &= k(x',t) - 4\Delta' \int\limits_0^t \int\limits_{\mathbb{R}^2} \Gamma(x'-y',0,t-\tau)dyd\tau \int\limits_{\mathbb{R}^2} E(y'-z',0)k(z',\tau)dz' \\ &= k - 4\nabla'\Gamma * \nabla' E *_1 k = (1+V)^{-1}k,\end{aligned}$$

where $*_1$ denotes the convolution with respect to (x,t) and $*$ denotes the convolution with respect to $x' \in \mathbb{R}^2$. This equation can be obtained from the explicit formula for the solution to the problem (1.1) (cf. [**4**, formula (3.11)]).

The convolution operators

$$\nabla'\Gamma * f = \nabla' \int\limits_0^t \int\limits_{\mathbb{R}^2} \Gamma(x'-y',0,t-\tau)f(y',\tau)\,dy'\,d\tau,$$

$$\nabla' E *_1 f = \nabla' \int\limits_{\mathbb{R}^2} E(x'-y',0)f(y')\,dy'$$

are singular "parabolic" and "elliptic" operators bounded in L_q. Estimates in Besov spaces follow immediately. In particular, we have

$$\|(I+V)^{-1}f\|_{\dot{W}_q^{1-\frac{1}{q},\frac{1}{2}-\frac{1}{2q}}(\mathbb{R}^2\times(0,T))} \leqslant c\|f\|_{W_q^{1-\frac{1}{q},\frac{1}{2}-\frac{1}{2q}}(\mathbb{R}^2\times(0,T))}. \tag{3.13}$$

For the function k defined in (3.11) we have

$$k(x',t) = 2\sum_{\alpha,\beta=1}^{2} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \int_0^t \int_{\mathbb{R}^3_+} \frac{\partial \Gamma(x'-y', y_3, t-\tau)}{\partial y_3} F_{\alpha\beta} dy\, d\tau$$

$$2\sum_{\beta=1}^{2} \frac{\partial}{\partial x_\beta} \int_0^t \int_{\mathbb{R}^3_+} \frac{\partial \Gamma(x'-y', y_3, t-\tau)}{\partial y_3} \frac{\partial F_{3\beta}(y,\tau)}{\partial y_3} dy\, d\tau$$

The last integral is equal to the expression

$$F_{3\beta} - \sum_{\alpha=1}^{2} \int_0^t \int_{\mathbb{R}^3_+} \frac{\partial \Gamma(x'-y', y_3, t-\tau)}{\partial y_\alpha} \frac{\partial F_{3\beta}}{\partial y_\alpha}\, dy\, d\tau$$

$$- \frac{\partial}{\partial t} \int_0^t \int_{\mathbb{R}^3_+} \Gamma(x'-y', y_3, t-\tau) F_{3\beta}(y,\tau) dy\, d\tau.$$

Since $F_{3\beta}$ vanishes at the boundary, we find

$$\begin{aligned}
k(x',t) &= \sum_{\beta=1}^{2} \frac{\partial^2 k_{0\beta}}{\partial t \partial x_\beta} + \sum_{\alpha,\beta=1}^{2} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} k_{\alpha\beta}(x',t), \\
k_{0\beta}(x',t) &= -2 \int_0^t \int_{\mathbb{R}^3_+} \Gamma(x'-y', y_3, t-\tau) F_{3\beta}(y,\tau) dy\, d\tau, \\
k_{\alpha\beta}(x',t) &= 2 \int_0^t \int_{\mathbb{R}^3_+} \frac{\partial \Gamma(x'-y', y_3, t-\tau)}{\partial x_\alpha} F_{3\beta}(y,\tau) dy\, d\tau \\
&\quad + 2 \int_0^t \int_{\mathbb{R}^3_+} \frac{\partial \Gamma(x'-y', y_3, t-\tau)}{\partial x_3} F_{\alpha\beta}(y,\tau)\, dy\, d\tau.
\end{aligned} \tag{3.14}$$

Since the function

$$\mu(x,t) = \int_0^t \int_{\mathbb{R}^3_+} \Gamma(x-y, t-\tau) F(y,\tau) dy\, d\tau$$

satisfies the inequality

$$\int_0^T \left(\|\mu_t\|^q_{L_q(\mathbb{R}^3_+)} + \sum_{i,j=1}^{3} \|\frac{\partial^2 \mu}{\partial x_i \partial x_j}\|^q_{L_q(\mathbb{R}^3_+)} \right) dt \leqslant c \int_0^T \|F\|^q_{L_q(\mathbb{R}^3_+)}\, d\tau,$$

we have

$$\|k_{0\beta}\|_{W_q^{2-1/q,1-1/(2q)}(\mathbb{R}^2\times(0,t))} + \sum_{\alpha,\beta=1}^{2} \|k_{\alpha\beta}\|_{W_q^{1-1/q,1/2-1/(2q)}(\mathbb{R}^2\times(0,t))}$$
$$\leqslant c\Big(\int\limits_0^T \|F\|^q_{L_q(\mathbb{R}^3_+)}\,dt\Big)^{1/q}$$

in view of trace theorems.

We set

$$q_1 = (1+V)^{-1}\sum_{\alpha,\beta=1}^{2}\frac{\partial^2 k_{\alpha\beta}}{\partial x_\alpha \partial x_\beta} = \sum_{\alpha\beta=1}^{2}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}(1+V)^{-1}k_{\alpha\beta},$$

$$p_1 = 2\int\limits_{\mathbb{R}^2} E(x'-y', x_3)q_1(y',t)\,dy'$$
$$= 2\sum_{\alpha,\beta=1}^{2}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\int\limits_{\mathbb{R}^2} E(x'-y',x_3)(1+V)^{-1}k_{\alpha\beta}dy',$$

$$P = \sum_{\beta=1}^{2} 2\partial_{x_\beta}\int\limits_{\mathbb{R}^2} E(x'-y',x_3)k_{0\beta}(y',t)dy'.$$

By the well-known estimates for the simple layer potential, we have

$$\int\limits_0^T \|p_1\|^q_{L_q(\mathbb{R}^3_+)}\,dt \leqslant c\sum_{\alpha,\beta=1}^{2}\int\limits_0^T \|(1+V)^{-1}k_{\alpha\beta}\|^q_{\dot{W}_q^{1-1/q}(\mathbb{R}^2)}\,dt$$
$$\leqslant c\sum_{\alpha,\beta=1}^{2}\|k_{\alpha\beta}\|^q_{W_q^{1-1/q,1/2-1/(2q)}(\mathbb{R}^2\times(0,T))} \leqslant c\|F\|^q_{L_q(\mathbb{R}^3\times(0,T))}, \tag{3.15}$$

$$\|P\|_{\dot{W}_q^{2-1/q,1-1/(2q)}(\partial\Sigma_T)} \leqslant c\sum_{\beta=1}^{2}\|k_{0\beta}\|_{\dot{W}_q^{2-1/q,1-1/(2q)}(\mathbb{R}^2\times(-\infty,T))}$$
$$\leqslant c\|F\|_{L_q(\mathbb{R}^3\times(0,T))}, \tag{3.16}$$

which proves the statement of Theorem 1.1 concerning the pressure.

We set $\vec{w} = \nabla P$ and write (1.1) in the form

$$\vec{u}_t - \Delta\vec{u} = \vec{f} - \nabla p_1, \quad x_3 > 0,$$
$$\vec{u}\big|_{t=0} = 0, \quad \vec{u}\big|_{x_3=0} = -\vec{w}\big|_{x_3=0},$$

where $\vec{u} = \vec{v} + \vec{w}$. We write $\vec{u} = \vec{\bar{u}} + \vec{\tilde{u}}$, where

$$\vec{\bar{u}}_t - \Delta\vec{\bar{u}} = \vec{f} - \nabla p_1, \quad x_3 > 0, \qquad \vec{\bar{u}}\big|_{t=0} = 0, \quad \vec{\bar{u}}\big|_{x_3=0} = 0$$

and

$$\vec{\tilde{u}}_t - \Delta\vec{\tilde{u}} = 0, \quad x_3 > 0, \qquad \vec{\tilde{u}}\big|_{t=0} = 0, \quad (\vec{\tilde{u}} - \vec{w})\big|_{x_3=0} = 0.$$

Then

$$\|\vec{\tilde{u}}\|_{L_q((0,T)\times \dot{W}^1_q(\mathbb{R}^3_+))} + \|\partial_t\vec{\tilde{u}}\|_{L_q((0,T),W^{-1}_q(\mathbb{R}^3_+))} \leqslant c\left(\|F\|_{L_q(\mathbb{R}^3_+\times(0,T))} + \|p_1\|_{L_q(\mathbb{R}^3_+\times(0,T))}\right).$$

The full estimate for $\vec{\tilde{u}}$ follows by interpolation.

Let U be a solution to the problem

$$U_t - \Delta U = 0, \quad x_3 > 0,$$
$$U\big|_{t=0} = 0, \quad U\big|_{x_3=0} = -P\big|_{x_3=0}.$$

Then

$$\|U\|_{W^{2,1}_q(\mathbb{R}^3_+\times(0,T))} \leqslant c\|P\big|_{x_3=0}\|_{\dot{W}^{2-1/q,1-1/(2q)}_q(\mathbb{R}^2\times(-\infty,T))}$$

and, by interpolation, we have

$$\|U\|_{\dot{W}^{1/2}_q((-\infty,T),\dot{W}^1_q(\mathbb{R}^3_+))} \leqslant c\|P\big|_{x_3=0}\|_{\dot{W}^{2-1/q,1-1/(2q)}_q(\mathbb{R}^2\times(-\infty,T))}.$$

For $i=1,2$ we have $\tilde{u}_i = \partial_i U$ and, since P is harmonic,

$$\partial_{x_3}\widehat{P}|_{x_3=0} = |\xi|\widehat{P}\big|_{x_3=0} =: \mathcal{F}(|D|P),$$

where $|D|$ is the corresponding operator. Thus, $\tilde{u}_3 = |D|U$ with the obvious interpretation and we obtain the desired estimate

$$\begin{aligned}
&\|\vec{u}\|_{L_q((0,T),\dot{W}^1_q(\mathbb{R}^3_+))} + \|\vec{u}\|_{\dot{W}^{1/2}_q((-\infty,T),L_q(\mathbb{R}^3_+))} + \left\|\frac{\partial}{\partial t}\vec{u}\right\|_{L_q((0,T),W^{-1}_q(\mathbb{R}^3_+))}\\
&\leqslant c\|P\big|_{x_3=0}\|_{W^{2-1/q,1-1/(2q)}_q(\mathbb{R}^2\times(-\infty,T))} + c\|p_1\|_{L_q(\mathbb{R}^3_+\times(0,T))}\\
&\leqslant c\|F\|_{L_q(\mathbb{R}^3_+\times(0,T))};
\end{aligned} \tag{3.17}$$

here the estimates (3.15), (3.16) and Proposition 6.1 were used.

From the inequalities (3.15), (3.16), and (3.17) it follows that

$$\|\nabla\vec{v}\|_{L_q(\mathbb{R}^3_+\times(0,T))} \leqslant c\|F\|_{L_q(\mathbb{R}^3_+\times(0,T))}.$$

Since $\vec{w}$ is a harmonic vector field, we have

$$\|\nabla\vec{w}\|^q_{L_q(\mathbb{R}^3_+\times(0,T))} \leqslant c\int\limits_0^T \|\vec{w}\|^q_{\dot{W}^{1-1/q}_q(\mathbb{R}^2)}dt \leqslant c\|F\|^q_{L_q(\mathbb{R}^3_+\times(0,T))}.$$

The last inequality yields (1.2). □

4. Proof of Theorem 1.3

PROOF OF THEOREM 1.3. We begin with assertion (b).

Let $\eta \in W_q^{-1/(2q)}(-\infty, T)$ be supported in $(0,T)$. We claim that the map

$$(L_q)^{3\times 3} \ni F \to p\big|_{\mathbb{R}^2\times(0,T)} \in W_q^{-1/q,-1/(2q)}(\mathbb{R}^2 \times (0,T)) \tag{4.1}$$

is surjective. Assuming that this claim is true, we can write $p = p_1 + p_2$, where

$$p_1\big|_{x_3=0} \in L_q((0,T); W_q^{-1/q}(\mathbb{R}^2)), \qquad p_2\big|_{x_3=0} \in W_q^{-1/2q}((0,T); L_q(\mathbb{R}^2)).$$

On the other hand,

$$(p_1 + p_2)_{x_3=0} = p\big|_{x_3=0} \in \dot{W}_q^{-1/q,-1/(2q)}(\mathbb{R}^2 \times (-\infty, T)).$$

We fix a smooth harmonic function φ in L_q and choose $\psi \in C_0^\infty(\mathbb{R}^3_+)$ such that

$$\int\limits_{\mathbb{R}^3_+} \psi\varphi\, dx = 1.$$

If $p\big|_{x_3=0} = \eta(t)\varphi(x)\big|_{x_3=0}$, then

$$\eta(t) = \int\limits_{\mathbb{R}^3_+} \psi(x)p(t,x)\, dx,$$

which implies assertion (b).

To prove assertion (a), we consider a function $\eta \in W_q^{1-2/q}(-\infty, T)$ supported in $[0,T]$. The derivative $\overline{\eta} = \eta'$ belongs to $W_q^{-2/q}(-\infty, T)$ and has the same support as η. We construct F as above with $\overline{\eta}$ instead of η. We choose a vector field Φ with components in C_0^∞ such that

$$\int_{\mathbb{R}^3_+} \nabla \cdot \Phi(x) p\, dx = \overline{\eta}(t).$$

Then

$$\frac{d}{dt}\int_{\mathbb{R}^3_+} u(x,t)\Phi(x)dx = \overline{\eta}(t) - \int_{\mathbb{R}^3_+} \nabla u \cdot \nabla\Phi\, dx - \int_{\mathbb{R}^3_+} \sum_{i,j=1}^{3} F_{ij}\partial_i u^j\, dx,$$

where the second and third terms on the right-hand side belong to $L_q(-\infty, T)$. This completes the proof of assertion (a).

It remains to verify the surjectivity of (4.1). In the notation in (3.11), we must show that the mapping

$$(L_q)^{3\times 3} \ni F \to k \in \dot{W}_q^{-1-1/q,-1/2-1/(2q)}(\mathbb{R}^2 \times (-\infty, T)) \tag{4.2}$$

is surjective. Let $k \in \dot{W}_q^{-1-1/q,-1/2-1/(2q)}(\mathbb{R}^2 \times (-\infty, T))$ have support in $0 \leqslant t \leqslant T$. For a given $k_{0\beta} \in W_q^{2-1/q,1-1/(2q)}(\mathbb{R}^2 \times (-\infty, T))$ we find $F_{3\beta} \in L_q(\mathbb{R}^3_+ \times$

$(0,T))$ such that $k_{0\beta}$ and $F_{3\beta}$ are connected by formula (3.14). It is easy to find $F_{\alpha\beta}$ such that $k_{\alpha\beta}=0$. Similarly, for a given $k_{\alpha\beta}\in W_q^{1-1/q,1/2-1/(2q)}(\mathbb{R}^2\times(-\infty,T))$ we find $F_{\alpha\beta}\in L_q(\mathbb{R}^3_+\times(0,T))$ such that they are connected by formula (3.14); moreover, $F_{3i}=F_{i3}=0$ for $1\leqslant i\leqslant 3$.

Thus, it suffices to show that each $k\in \dot W_q^{-1-1/q,-1/2-1/(2q)}(\mathbb{R}^2\times(-\infty,T))$ can be written in the form

$$k=\partial_t k_0+\Delta k_1,$$

where $k_0,k_1\in W_q^{1-1/q,1/2-1/(2q)}(\mathbb{R}^2\times(-\infty,T))$. Let $\rho\in C_0^\infty(\mathbb{R})$ be a nonnegative function that is identically equal to 1 on $[-1,1]$. We set

$$\chi(\xi,\tau)=\rho(|\xi|^2/\tau).$$

We define k_0 by the Fourier multiplier $(i\tau)^{-1}\rho(|\xi|^2/\tau)$ and k_1 by the Fourier multiplier $-|\xi|^{-2}(1-\rho(|\xi|^2/\tau))$ applied to k. By the Markinciewicz multiplier theorem, we arrive at the required assertion. □

5. Proof of Theorem 1.5

PROOF OF THEOREM 1.5. By Theorem 1.1, we can assume that $F=0$. We define Φ by the formulas

$$\Delta\Phi=g,\quad \partial_{x_3}\Phi=0\quad\text{at } x_3=0,$$

$$\vec u=\vec v-\nabla\Phi,\quad p=p_1+\partial_t\Phi-\Delta\Phi.$$

It suffices to establish the inequality

$$||\nabla\vec u||_{L_q}\leqslant c||g||_{L_q}. \tag{5.1}$$

We have

$$\begin{aligned}&\partial_t\vec u-\Delta\vec u+\nabla p_1=0,\quad \nabla\cdot\vec u=0,\\ &u_3\big|_{x_3}=0\quad u_1|_{x_3}=-\partial_1\Phi\big|_{x_3=0},\quad u_2\big|_{x_3}=-\partial_2\Phi_{x_3=0}.\end{aligned}\tag{5.2}$$

As above, we have

$$\frac{\partial p_1}{\partial x_3}\bigg|_{x_3=0}=\Delta u_3\big|_{x_3=0}=-\sum_{j=1}^2\frac{\partial^2 u_j}{\partial x_3\partial x_j}\bigg|_{x_3=0},$$

$$\begin{aligned}u_j(x,t)=&\int_0^t\int_{\mathbb{R}^3_+}G(x,y,t-\tau)\partial_{x_j}p_1\,dy\,d\tau\\ &+2\int_0^t\int_{\mathbb{R}^2}\frac{\partial\Gamma(x'-y',x_3,t-\tau)}{\partial x_3}\partial_{x_j}\Phi\,dy'\,d\tau,\quad j=1,2,\end{aligned}$$

$$(1+V)\frac{\partial p_1}{\partial x_3}\Big|_{x_3=0} = -(\partial_t - \Delta')\sum_{j=1}^{2}\frac{\partial}{\partial x_j}k_j,$$

where

$$k_j = 2\int_0^t\int_{\mathbb{R}^2}\partial_{x_j}\Gamma(x'-y',0,t-s)\Phi(y',0,s)dy'\,ds.$$

Let Q be a harmonic function such that

$$(1+V)\partial_{x_3}Q\big|_{x_3=0} = \sum_{j=1}^{2}\frac{\partial}{\partial x_j}k_j.$$

Then

$$\begin{aligned}||Q||_{L_q((0,T),\dot{W}_q^2(\mathbb{R}^3_+))} &\leqslant ||D^{-1}\partial_{x_3}Q\big|_{x_3=0}||_{L_q((0,T),\dot{W}_q^{2-1/q}(\mathbb{R}^2))}\\ &\leqslant c||k_j||_{L_q((0,T),\dot{W}_q^{2-1/q}(\mathbb{R}^2))} \leqslant c||\Phi||_{L_q((0,T),\dot{W}_q^{2-1/q}(\mathbb{R}^2))}.\end{aligned}$$

We can write $p_1 = p_2 + \partial_t Q$, where $p_2 = -\Delta' Q \in L_q$. The required assertion follows from Lemma 6.2 below. □

6. Appendix

We prove auxiliary assertions. Consider the parabolic initial-boundary-value problem

$$\begin{aligned}&u_t - \Delta u = \nabla\cdot F \quad \text{for } x_3>0,\ t>0,\\ &u\big|_{t=0}=0, \quad u\big|_{x_3=0}=\varphi,\end{aligned} \tag{6.1}$$

where $F \in (L_q(\mathbb{R}^3\times(0,T)))^3$ and $\varphi \in \dot{W}_q^{1-1/q,1/2-1/(2q)}(\mathbb{R}^2\times(-\infty,T))$ has support in $t>0$.

Proposition 6.1. *There exists a unique solution to the problem* (6.1) *such that*

$$\begin{aligned}||\nabla u||_{L_q(\mathbb{R}^3_+\times(0,T))} \leqslant c\big(&||F||_{L_q(\mathbb{R}^3_+\times(0,T))}\\ &+||\varphi||_{\dot{W}_q^{1-1/q,1/2-1/(2q)}(\mathbb{R}^2\times(-\infty,T))}\big).\end{aligned} \tag{6.2}$$

PROOF. The estimate (6.2) follows from the following explicit formula for $u(x,t)$:

$$u(x,t) = \int_0^t\int_{\mathbb{R}^3_+}G(x,y,t-\tau)\nabla F\,dy\,d\tau - 2\int_0^t\int_{\mathbb{R}^2}\frac{\partial\Gamma(x'-y',x_3,t-\tau)}{\partial x_3}\vec{\varphi}\,dy'\,d\tau$$

$$= -\int_0^t \nabla G(x,y,t-\tau)F(y,\tau)\,dy\,d\tau - 2\int_0^t\int_{\mathbb{R}^2} \frac{\partial\Gamma(x'-y',x_3,t-\tau)}{\partial x_3}\varphi\,dy'\,d\tau, \tag{6.3}$$

where $G(x,y,t) = \Gamma(x-y,t) - \Gamma(x-y^*,t)$, $y^* = (y_1,y_2,-y_3)$, is the Green function for the problem (6.1) in the half-space, $\varphi_0(x',t) = \varphi(x',t)$ for $t > 0$, and $\varphi_0 = 0$ for $t < 0$. The heat potentials in (6.3) were estimated in [**5, 6**], and (6.2) follows immediately. □

Lemma 6.2. *Let*

$$F \in (L_q(\mathbb{R}^3_+ \times \mathbb{R}_+))^3, \quad Q \in L_q(\mathbb{R}_+; \dot W^1_q(\mathbb{R}^3_+)), \quad g \in L_q(\mathbb{R}^3_+, \dot W_q^{1-1/q}(\mathbb{R}^2)),$$
$$u_t - \Delta u = \nabla F + \partial_t Q, \quad u = g \ \textit{at}\ \{x_3 = 0\}, \quad u = 0 \ \textit{at}\ t = 0.$$

Then

$$\|\nabla u\|_{L_q(\mathbb{R}^3_+\times\mathbb{R}+)} \leqslant c(\,\|F\|_{L_q(\mathbb{R}^3_+\times\mathbb{R}+)} + \|Q\|_{L_q(\dot W^1_q)} + \|g\|_{L_q(\mathbb{R}+\times \dot W_q^{1-1/q}(\mathbb{R}^2))}).$$

PROOF. There exists a harmonic function Φ such that $\Phi = g$ at $x_3 = 0$. Setting $v = u - \varphi$, we reduce the problem to the case $g = 0$. We reflect all the variables to the entire space $\mathbb{R}^n \times \mathbb{R}$. Then we should establish estimates in the entire space. The Fourier multipliers $\tau/(i\tau + |\xi|^2)$ and $\xi_i\xi_j/(i\tau + |\xi|^2)$ define bounded operators on L_q. This completes the proof. □

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Two Sufficient Conditions for the Regularity of Lateral Boundary for the Heat Equation

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To O. A. Ladyzhenskaya
great mathematician and wonderful person

The one-dimensional heat equation in the domain $x > x(t)$, $t \geqslant 0$, is considered. We prove the following fact: if the lateral boundary is "Hölder" regular for the heat equation $u_t = \nu^2 u_{xx}$ for at least one $\nu > 0$, then it is regular for the equation with any $\nu > 0$. The proof is based on another condition of regularity somewhat close to the exterior cone condition for Laplace's equation.

1. Introduction and the Main Results

Let $T \in (0, \infty)$, and let $x(t)$, $t \geqslant 0$, be a real-valued function given on $[0, T]$. We define

$$Q = Q(x(\cdot)) = \{(t, x) : t \in (0, T), x > x(t)\}$$

and assume that Q is an open set, which always holds if $x(t)$ is a continuous function and which, actually, means that $x(t)$ is upper semicontinuous. Denote by $\partial' Q$ the parabolic boundary of Q i.e., $\partial Q \setminus \{(0, x) : x > x(0)\}$. We take a constant $\nu \in (0, \infty)$ and consider the following boundary-value problem:

$$u_t(t, x) + \tfrac{1}{2}\nu^2 u_{xx}(t, x) = 0, \quad (t, x) \in Q, \tag{1.1}$$

$$u = g \quad \text{on} \quad \partial' Q. \tag{1.2}$$

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We assume that g is a bounded continuous function. By a solution u of the problem (1.1), (1.2) we mean a probabilistic or Perron's solution.

One of the main goals of this paper is to prove the following result: *If a point $(t_0, x(t_0))$, $t_0 \in [0,T)$, of the lateral boundary of Q is "Hölder" regular (1.1), (1.2) for at least one $\nu > 0$, then it is regular for (1.1), (1.2) with any $\nu > 0$.*

Theorem 1.1. *Let $\nu, \nu_0 \in (0,\infty)$, $t_0 \in [0,T)$, and let v be a probabilistic or Perron's solution of the problem (1.1), (1.2) with ν_0 instead of ν and g_0 instead of g, where g_0 is a nonnegative bounded continuous function such that $g_0(T,x) > 0$ for some $x > x(T)$. Suppose that for $\lambda \in (0,1]$ and all $x \in (0,1]$ we have*

$$v(t_0, x + x(t_0)) \leqslant x^{\lambda}. \tag{1.3}$$

Let u be a probabilistic or Perron's solution of the problem (1.1), (1.2) with a bounded continuous function g. Then

$$\lim_{\substack{(t,x)\to(t_0 x(t_0))\\(t,x)\in Q}} u(t,x) = g(t_0, x(t_0)). \tag{1.4}$$

Parabolic equations in noncylindrical domains have been considered since long ago and many important results are known for them. We refer the interested reader to [**1**] and a very extensive bibliography there. Most of the literature treats the case where the boundary is $(1/2+)$ Hölder. In this case, it is possible to obtain Hölder estimates up to the boundary if g is Hölder continuous. However, in author's investigation of stochastic partial differential equations, a situation appeared in which $x(t)$ is a trajectory of a Brownian motion that typically is only $(1/2-)$ Hölder continuous. Therefore, quite interesting was to understand even when a point $(t_0, x(t_0))$ is *regular* in the usual sense, i.e., when (1.4) is satisfied.

There is a Wiener type criterion (cf. [**2**]) for the regularity of boundary points for parabolic equations. However, it is not clear how to apply it if $x(t)$ is a Brownian trajectory. The best tractable conditions for the caloric regularity of a point on the lateral boundary are expressed in terms of the Khinchin law of iterated logarithm or the more general Kolmogorov–Petrovskii criterion. In particular, they imply that for $\varepsilon \in (-1,1)$ the point $(0,0)$ on the curve $x(t) = -(1+\varepsilon)\nu\sqrt{2t\ln|\ln t|}$ is regular if $\varepsilon < 0$ and is irregular if $\varepsilon > 0$. This shows that if, instead of (1.3), we just assume that (1.4) holds with v instead of u, then the conclusion of Theorem 1.1 becomes false, in general, for $\nu < \nu_0$.

The proof of Theorem 1.1 is based on two ingredients. The first one is quite standard and is similar to the exterior cone condition for elliptic equations. Actually, it is closer to the sufficient condition from [**3**] for the regularity of boundary points for two-dimensional elliptic equations with measurable coefficients. The condition in [**3**] consists of the requirement that in each neighborhood of the boundary point there exist two concentric discs centered at the point with the distance between their boundaries proportional to the radius of the inside disc and such that their boundaries are connected by part of the complement of the domain. Such results are easier to prove by using the probabilistic approach rather than the general result in [**2**]. We use the following fact: *The solution is continuous at a point* $(t_0, x(t_0))$ *on the lateral boundary if there is a parabola* $t \geqslant t_0 + a^2(x - x(t_0))^2$ *with the pole at this point such that the boundary has common points with the (interior of) parabola in any small neighborhood of the pole.*

Theorem 1.2. *Suppose that for* $t_0 \in [0, T)$ *and* $c_0 < \infty$

$$\overline{\lim_{h \downarrow 0}} \frac{x(t_0 + h) - x(t_0)}{\sqrt{h}} \geqslant -c_0. \tag{1.5}$$

Let g *be a bounded continuous function. Then the probabilistic or Perron's solution* u *of* (1.1), (1.2) *satisfies* (1.1) *and* (1.4).

We note that this theorem has little to do with the law of iterated logarithm, which says that $(t_0, x(t_0))$ is regular if

$$\underline{\lim_{h \downarrow 0}} \frac{x(t_0 + h) - x(t_0)}{\sqrt{2t \ln |\ln t|}} > -\nu$$

and some monotonicity assumptions on $x(t)$ are satisfied. Theorem 1.2 is proved in Sec. 2.

The second ingredient in the proof of Theorem 1.1 is provided by the following law of square root.

Theorem 1.3. *Under the assumptions of Theorem* 1.1, *there is a constant* $c_0 \in (0, \infty)$ *depending only on* ν_0 *and* λ (*cf. Remark* 4.1 *below*) *such that the inequality* (1.5) *holds.*

Theorem 1.1 immediately follows from Theorems 1.2 and 1.3.

We prove Theorem 1.3 in Sec. 4. We note that if the left-hand side of (1.5) were too big negative, then there would exist a parabola $t \geqslant t_0 + a^2(x - x(t_0))^2$ with large a such that its sufficiently small piece near the pole is inside of Q. Then the barrier from Sec. 3 would imply that the opposite inequality holds in (1.3) for small $x > 0$ with a $\lambda > 0$, which tends to zero as $a \to \infty$.

2. Proof of Theorem 1.2

As was mentioned, we prefer to use a probabilistic argument to prove Theorem 1.2. To understand this argument, the reader only needs to know that a Brownian motion is a strong Markov process.

Let C be the space of continuous real-valued functions on $[0, \infty)$, and let $\nu > 0$. For $t, x \in \mathbb{R}$, $s \geqslant 0$, and $y(\cdot) \in C$ we define

$$\begin{aligned}
&\sigma_s(t, x, y(\cdot)) = \inf\{r \geqslant s : (t + r, x + \nu y(r)) \notin Q\},\\
&\tau_s(t, x, y(\cdot)) = \inf\{r > s : (t + r, x + \nu y(r)) \notin Q\},\\
&\tau(t, x, y(\cdot)) = \tau_0(t, x, y(\cdot)) = \inf\{r > 0 : (t + r, x + \nu y(r)) \notin Q\}.
\end{aligned}$$

From the fact that $(t + s, x + \nu y(r))$ is a continuous function of $(t, x, y(\cdot))$ it is quite easy to get that $\sigma(t, x, y(\cdot))$ is a lower semicontinuous function and hence a Borel function of $(t, x, y(\cdot))$. It is obvious that

$$\tau_s(t, x, y(\cdot)) = \lim_{r \downarrow s} \sigma_r(t, x, y(\cdot)), \quad s \geqslant 0.$$

Therefore, $\tau_s(t, x, y(\cdot))$ is also a Borel function of $(t, x, y(\cdot))$.

Let B_t be a standard one-dimensional Wiener process. For $t, x \in \mathbb{R}$ define

$$\tau(t, x) = \inf\{r > 0 : (t + r, x + \nu B_r) \notin Q\} = \tau(t, x, B)$$

as the first exit time of the process $(t + r, x + \nu B_r)$, $r > 0$, from Q. It is obvious that $\tau(t, x) \leqslant T - t$. It is important to emphasize that the infimum is taken over $r > 0$ rather than $r \geqslant 0$. By the above discussion, $\tau(t, x)$ is a random variable and

$$u(t, x) := Eg(t + \tau(t, x), x + \nu B_{\tau(t,x)}) \tag{2.1}$$

is a Borel function of (t, x). The function u is known as the probabilistic solution of the problem (1.1), (1.2).

Since B_t is a strong Markov process, for any box $A := (a, b) \times (c, d) \subset Q$ and $(t, x) \in A$ we have

$$u(t, x) = Eu(t + \gamma(t, x), x + \nu B_{\gamma(t,x)}),$$

where $\gamma(t, x) = \inf\{s > 0 : (t + s, x + \nu B_s) \notin A\}$. Therefore, from [4] it follows that u is infinitely differentiable in A and satisfies Eq. (1.1) there. Since $A \subset Q$ is arbitrary, u satisfies (1.1) in Q.

Now, the only issue is that of the boundary values. Of course, as $t \uparrow T$ we have $T - t \geqslant \tau(t, x) \to 0$ and $u(t, x) \to g(t, x)$ for $x \geqslant x(T)$ in view of the continuity of g. However, proving (1.4) requires more work.

For any $h \in (0, T - t_0)$ we have

$$P(x(t_0) + \nu B_h \leqslant x(t_0 + h)) \leqslant P(\tau(t_0, x(t_0)) \leqslant h).$$

By (1.5), we can choose $h \in (0, T - t)$ as small as we like to satisfy $x(t_0 + h) \geqslant x(t_0) - 2c_0\sqrt{h}$. Then

$$\begin{aligned}
P(\nu B_h \leqslant -2c_0\sqrt{h}) &= P(x(t_0) + \nu B_h \leqslant x(t_0) - 2c_0\sqrt{h})\\
&\leqslant P(x(t_0) + \nu B_h \leqslant x(t_0 + h)) \leqslant P(\tau(t_0, x(t_0)) \leqslant h).
\end{aligned}$$

Here, the first expression is equal to $P(\nu B_1 \leqslant -2c_0)$. Therefore, it is independent of h and is strictly positive. Hence

$$P(\tau(t_0, x(t_0)) = 0) = \lim_{h \downarrow 0} P(\tau(t_0, x(t_0)) \leqslant h) > 0,$$

which implies that $P(\tau(t_0, x(t_0)) = 0) = 1$ by Blumenthal's zero-one law.

We prove that not only $\tau(t_0, x(t_0)) = 0$ (a.s.) but also $\tau(t, x) \to 0$ in probability as $(t, x) \to (t_0, x(t_0))$. Since g is continuous and we have (2.1), this is more than enough to prove (1.4).

We repeat a standard argument from the theory of Markov processes (cf., for example, [5]) taking into account that, although the process $(t+s, x+\nu B_s)$, $s > 0$, is not strong Feller, its resolvent is. Define $v(t, x) = E\tau(t, x)$ and note that, by the Markov property, for $s \geqslant 0$ we have

$$Ev(t + s, x + \nu B_s) = E\tau_s(t, x) - s, \tag{2.2}$$

where $\tau_s(t, x) = \tau_s(t, x, B)$. By the argument at the beginning of this section, $v(t, x)$ is a Borel function of (t, x). Therefore, taking a number $\lambda > 0$, from (2.2) we find

$$E\tau_s(t, x) = s + \frac{1}{\sqrt{2\pi s}} \int\limits_{\mathbb{R}} v(t + s, x + \nu y) e^{-y^2/(2s)}\, dy,$$

$$\begin{aligned} v_\lambda(t, x) &:= \lambda \int\limits_0^\infty e^{-\lambda s} E\tau_s(t, x)\, ds \\ &= \lambda^{-1} + \lambda \int\limits_0^\infty \int\limits_{\mathbb{R}} v(t + s, x + \nu y) e^{-\lambda s} \frac{1}{\sqrt{2\pi s}} e^{-y^2/(2s)}\, dyds. \end{aligned} \tag{2.3}$$

We note that $v(t, x) = 0$ for $t \notin [0, T]$, v is bounded and, as for any L_1-function, $v(t + \cdot, x + \cdot)$ is continuous with respect to (t, x) in the $L_1(A)$-sense for any compact set $A \subset \mathbb{R}^2$. Therefore, (2.3) implies that $v_\lambda(t, x)$ is a continuous function of (t, x). Furthermore, it is obvious that $\tau_s(t, x) \downarrow \tau(t, x)$ as $s \downarrow 0$. Hence the formula

$$v_\lambda(t, x) = \int\limits_0^\infty e^{-s} E\tau_{s/\lambda}(t, x)\, ds$$

shows that $v_\lambda(t, x) \downarrow v(t, x)$ as $\lambda \to \infty$, which, along with the continuity of v_λ, implies that v is upper semicontinuous and, in particular,

$$v(t_0, x(t_0)) \geqslant \varlimsup_{(t,x) \to (t_0, x(t_0))} v(t, x).$$

Here, the left-hand side is zero by the above argument and, consequently, $\tau(t, x) \to 0$ in probability as $(t, x) \to (t_0, x(t_0))$. Theorem 1.2 is proved.

3. Barrier Function

The main technical tool of this paper is a barrier function which will be constructed in this section. It should be pointed out that, in our particular case, such barrier functions can be constructed in a simpler way using the fact that the boundaries are pieces of parabolas (cf., for example, [**6**]). However, we prefer to give a different construction. The advantage of this construction is that it is of a general nature and carries over to different curvilinear boundaries and different problems. For instance, functions of the same kind were used by Weinberger [**7**, Sec. 2.4] to find conditions on functions $t(x)$ that are needed for the boundary $t = t(x)$ of the domain $D = \{(t,x) : t < t(x)\}$ to be regular for the equation $u_t + u_{xx} = 0$ in D. Here, regularity can be understood as the continuity up to the boundary of certain derivatives of solutions. For the same purpose and the same equation in $\{(t,x) : t > t(x)\}$, functions of the same kind were also used in [**8**, Sec. 2] and, for that matter, fully nonlinear degenerate elliptic equations were studied.

The constructions of this section are quite elementary although slightly technical and may be well known. However, the author could not find in the literature the results needed and this is the reason to give them with complete proof here. Let $p(t,x) = I_{t>0}\dfrac{1}{\sqrt{4\pi t}}\, e^{-x^2/(4t)}$ be the fundamental solution of the heat equation $u_t = u_{xx}$. For $c \geqslant 0$ and $\alpha \in (0,1]$ we define

$$v(t,x) = \sqrt{4\pi}\int_0^1 s^{(\alpha-1)/2} p(s-t, x + c\sqrt{s})\, ds \tag{3.1}$$

as a simple-layer caloric potential of certain mass distributed over the parabola $\Gamma := \{(t,x) : t \in [0,1], x = -c\sqrt{t}\}$.

Lemma 3.1. (i) *v is a continuous function defined in $\mathbb{R}^2_+ = \{(t,x) : t \geqslant 0, x \in \mathbb{R}\}$.*

(ii) *v is infinitely differentiable and satisfies the equation $v_t + v_{xx} = 0$ in $\mathbb{R}^2_+ \setminus \Gamma$.*

(iii) *The restrictions of v to the regions $\{(t,x) : 0 < t < 1, x \geqslant -c\sqrt{t}\}$ and $\{(t,x) : 0 < t < 1, x \leqslant -c\sqrt{t}\}$ are infinitely differentiable.*

(iv) *If $\alpha \in (0,1)$, then there is a finite constant $q = q(c,\alpha) > 0$ such that $v(0,x) - v(0,0) \sim -qx^\alpha$ as $x \downarrow 0$.*

Proof. (i) We denote by $g(t,x,s)$ the integrand in (3.1). If $(t,x) \to (t_0,x_0)$, then $g(t,x,s) \to g(t_0,x_0,s)$ for almost any $s \in (0,1)$ (actually, for any $s \neq t_0$). Furthermore, in both cases $s > |s-t|$ and $s \leqslant |s-t|$, we have

$$|g(t,x,s)| \leqslant s^{(\alpha-1)/2}|s-t|^{-1/2} \leqslant s^{\alpha/2-1} + |s-t|^{\alpha/2-1},$$

which shows that, for $p > 1$, $\int_0^1 |g(t,x,s)|^p \, ds$ is a bounded function of (t,x). Therefore, in addition to continuity, g is uniformly integrable and assertion (i) follows.

(ii) The fact that v is infinitely differentiable in $\mathbb{R}^2_+ \setminus \Gamma$ with respect to x follows from a similar argument and the fact that the derivatives of $g(t,x,s)$ with respect to x are bounded in s as long as (t,x) stays at a positive distance from Γ.

By a straightforward computation, one can check that $v_t + v_{xx} = 0$ in $\mathbb{R}^2_+ \setminus \Gamma$. From this equation it follows that v is infinitely differentiable in $\mathbb{R}^2_+ \setminus \Gamma$ with respect to (t,x). This proves (ii).

(iii) Since Γ is smooth apart from its extreme points and its tangent lines are not orthogonal to the t-axis, the regularity results for parabolic equations imply that v is infinitely differentiable in the regions in question if and only if its trace on Γ is infinitely differentiable apart from the extreme points of Γ. The regularity results we have alluded to, actually, follow right away if one observes that the function $v(t, x + c\sqrt{t})$, $t, x > 0$, satisfies the heat equation with a drift term which is bounded (and smooth) whenever t is bounded away from 0. We see that to prove (iii), it suffices to show that the function

$$\varphi(x) := v(c^{-2}x^2, x) \tag{3.2}$$

is infinitely differentiable in $(-c, 0)$. However, after changing variables $s = c^{-2}x^2r^{-2}$, $r > 0$, and noticing that (remember $x \leqslant 0$)

$$\rho(s,x) := \frac{(x + c\sqrt{s})^2}{4(s - c^{-2}x^2)} = \frac{c^2}{4}\frac{(1 - r^{-1})^2}{r^{-2} - 1} = \frac{c^2}{4}\frac{1 - r}{1 + r},$$

we find

$$\begin{aligned}
\varphi(x) &= \int_{c^{-2}x^2}^{1} \frac{1}{\sqrt{s - c^{-2}x^2}} e^{-\rho(s,x)} s^{(\alpha-1)/2} \, ds \\
&= 2|x|^\alpha c^{-\alpha} \int_{|x|/c}^{1} \frac{1}{\sqrt{1 - r^2}} e^{-c^2(1-r)/(4+4r)} r^{-1-\alpha} \, dr.
\end{aligned}$$

This implies that φ is infinitely differentiable in $(-c, 0)$ and proves (iii).

(iv) To prove (iv), we make the change of variables $s \to x^2 s$. Then

$$v(0,x) - v(0,0) = x^\alpha \int_0^{x^{-2}} [e^{-(1+c\sqrt{s})^2/(4s)} - e^{-c^2/4}] s^{\alpha/2-1} \, ds$$

and we get the result with

$$q = e^{-c^2/4} \int_0^\infty [1 - e^{-1/(4s) - c/\sqrt{4s}}] s^{\alpha/2-1}\, ds < \infty.$$

The lemma is proved. □

For special c the trace of v on Γ is infinitely differentiable up to its extreme point $(0,0)$. Before proving this fact, we prove the following assertion.

Lemma 3.2. (i) *For any $\alpha \in (0,1)$ there exists a unique solution $c = c(\alpha) \in (0,1)$ of the equation*

$$\varphi(c,\alpha) := \int_0^1 \varkappa(c,r) r^{-\alpha}\, dr = \frac{1}{\alpha}, \tag{3.3}$$

where

$$\varkappa(c,r) := \left[\frac{1}{\sqrt{1-r^2}} e^{c^2 r/(2+2r)} - 1\right] r^{-1}.$$

(ii) *The function $c(\alpha)$ is continuous, strictly decreasing, $c(\alpha) \to 0$ as $\alpha \uparrow 1$, and $c(\alpha) \to \infty$ as $\alpha \downarrow 0$. Furthermore, $c(\alpha)$ has the inverse $\alpha(c)$ defined for $c > 0$, $\alpha(c)$ is strictly decreasing, $0 < \alpha(c) < 1$ for $c \in (0,\infty)$, $\alpha(c) \downarrow 0$ as $c \to \infty$, $\alpha(0+) = 1$.*

(iii) *$1 - \alpha(c) \sim c/\sqrt{\pi}$ as $c \downarrow 0$ and $\alpha(c) \sim e^{-c^2/4} c/\sqrt{4\pi}$ as $c \to \infty$, where $a \sim b$ means $a/b \to 1$.*

Proof. (i) For fixed α the function $\varphi(c,\alpha)$ is a strictly increasing function of c tending to infinity as $c \to \infty$. Hence $\varphi(c,\alpha) > \alpha^{-1}$ for large c. It turns out that $\varphi(c,\alpha) < \alpha^{-1}$ for small $c > 0$. To prove this fact, we note that

$$\varphi(0+,\alpha) = \int_0^1 \left[\frac{1}{\sqrt{1-r^2}} - 1\right] r^{-1-\alpha}\, dr =: \varphi(\alpha).$$

It is obvious that $\varphi(\alpha)$ is an increasing function of α and α^{-1} is a decreasing one. In addition, near 0 we have $\varphi(\alpha) < \alpha^{-1}$ since, as $\alpha \downarrow 0$, the left-hand side is bounded and the right-hand side goes to infinity. Finally, making the change of variables $r = 1/\cosh s$, $s \geqslant 0$, we find the expression

$$\varphi(1) = \int_0^1 \left[\frac{1}{\sqrt{1-r^2}} - 1\right] r^{-2}\, dr = \int_0^\infty e^{-s}\, ds = 1, \tag{3.4}$$

which coincides with the value of α^{-1} at $\alpha = 1$. It follows that $\varphi(0+,\alpha) = \varphi(\alpha) < \alpha^{-1}$ for $\alpha \in (0,1)$, and the above argument proves that for each $\alpha \in (0,1)$ there is a unique constant $c > 0$ such that $\varphi(c,\alpha) = \alpha^{-1}$.

(ii) Note that φ is differentiable in (c,α) and

$$\varphi_c(c,\alpha) = c\int\limits_0^1 \frac{1}{(1+r)\sqrt{1-r^2}} e^{c^2r/(2+2r)} r^{-\alpha}\, dr > 0,$$

$$\varphi_\alpha(c,\alpha) = \int\limits_0^1 \varkappa(c,r) r^{-\alpha} \ln(1/r)\, dr > 0.$$

Furthermore, we know how to find the derivatives of implicit functions. Since $\varphi(c(\alpha),\alpha) = \alpha^{-1}$ and $(\alpha^{-1})' < 0$, the above analysis shows that $c(\alpha)$ is smooth and $c'(\alpha) < 0$. Hence $c(\alpha)$ is a continuous strictly decreasing function of α.

The fact that $c(\alpha) \to \infty$ as $\alpha \downarrow 0$ is obvious because of (3.3). To show that $c(\alpha) \to 0$ as $\alpha \uparrow 1$, it suffices to go back to (3.4). The corresponding properties of $\alpha(c)$ are now obvious.

(iii) For any $\delta > 1$ we have $1+x \leqslant e^x \leqslant 1+\delta x$ if $x \geqslant 0$ and x is sufficiently small. Taking into account that $0 \leqslant c^2r/(2+2r) \leqslant c^2/2$ and using (3.3), for sufficiently small $c > 0$ we get

$$\frac{c^2}{2}\,\Psi(\alpha) + \varphi(\alpha) \leqslant \frac{1}{\alpha} \leqslant \frac{c^2}{2}\,\delta\Psi(\alpha) + \varphi(\alpha),$$

where $\alpha = \alpha(c)$ and

$$\Psi(\alpha) = \int\limits_0^1 \frac{1}{(1+r)\sqrt{1-r^2}} r^{-\alpha}\, dr.$$

Recalling that $\varphi(1) = 1$ by (3.4), we find

$$\frac{c^2}{2}\,\Psi(\alpha) + \varphi'(\overline{\alpha})(\alpha-1) \leqslant \frac{1}{\alpha} - 1 \leqslant \frac{c^2}{2}\,\delta\Psi(\alpha) + \varphi'(\overline{\alpha})(\alpha-1), \tag{3.5}$$

where $\alpha < \overline{\alpha} < 1$. Next, $\varphi'(\overline{\alpha}) \to \varphi'(1)$ as $c \downarrow 0$ and computations similar to those which led us to (3.4) show that

$$\varphi'(1) = \int\limits_0^1 \left[\frac{1}{\sqrt{1-r^2}} - 1\right] r^{-2} \ln(1/r)\, dr = \int\limits_0^\infty e^{-s} \ln\cosh s\, ds$$

$$= \int\limits_0^1 \ln(x + 1/x)\, dx - \ln 2 = \pi/2 - 1.$$

Hence (3.5) implies

$$\lim_{c\downarrow 0} \frac{c^2\Psi(\alpha)}{2(1-\alpha)} = \pi/2. \tag{3.6}$$

Finally, $\Psi(\alpha) = \Sigma(\alpha) + 1/(1-\alpha)$, where

$$\Sigma(\alpha) = \int_0^1 \left[\frac{1}{(1+r)\sqrt{1-r^2}} - 1\right] r^{-\alpha}\, dr \to \Sigma(1)$$

and $\Sigma(1)$ is a finite constant. Therefore, we can write (3.6) as follows:

$$\lim_{c\downarrow 0} \frac{c^2}{(1-\alpha)^2}[(1-\alpha)\Sigma(\alpha) + 1] = \pi.$$

The first part of assertion (iii) is proved.

An estimate showing that $\alpha(c)$ decreases extremely fast as $c \to \infty$ can be obtained in the following way. The rough estimates $(1-r^2)^{-1/2} \geqslant 1$, $r^{-1-\alpha} \geqslant 1$, $r/(1+r) \geqslant r/2$, and Eq. (3.3) imply that for all $c \geqslant 0$ (and always $\alpha = \alpha(c)$) we have

$$\frac{1}{\alpha} \geqslant \int_0^1 (e^{c^2 r/4} - 1)\, dr = 4\,\frac{e^{c^2/4} - 1}{c^2} - 1.$$

To improve this estimate, we write

$$\varphi(c,\alpha) = \int_0^1 \frac{1}{\sqrt{1-r^2}}\big[e^{c^2 r/(2+2r)} - 1\big] r^{-1-\alpha}\, dr + \varphi(\alpha)$$

and observe that $\varphi(\alpha)$ is bounded for small α or, equivalently, large c. Using $r/(1+r) \leqslant \varepsilon/(1+\varepsilon) < 1/2$ for $0 \leqslant r \leqslant \varepsilon < 1$ and the inequality $e^x - 1 \leqslant xe^x$, $x \geqslant 0$, we see that for any $\varepsilon \in (0,1)$

$$\begin{aligned}\varphi(c,\alpha) &= \int_\varepsilon^1 \frac{1}{\sqrt{1-r^2}}\big[e^{c^2 r/(2+2r)} - 1\big] r^{-1-\alpha}\, dr + o(e^{c^2/4}/c^2)\\ &= \int_\varepsilon^1 \frac{1}{\sqrt{1-r^2}} e^{c^2 r/(2+2r)} r^{-1-\alpha}\, dr + o(e^{c^2/4}/c^2)\end{aligned}$$

as $c \to \infty$. By the mean value theorem, we conclude that

$$e^{-c^2/4}\varphi(c,\alpha) = \gamma(\varepsilon,c) \int_\varepsilon^1 \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1-r}} e^{c^2 r/(2+2r) - c^2/4}\, dr + o(c^{-2}),$$

where $\gamma(\varepsilon, c) \to 1$ as $\varepsilon \uparrow 1$ uniformly in c. Making the change of variables $(1-r)/(1+r) = s^2$ and using again the mean value theorem, we conclude that

$$e^{-c^2/4}\alpha^{-1} = e^{-c^2/4}\varphi(c,\alpha) = \gamma_1(\varepsilon, c)2 \int\limits_0^{\sqrt{(1-\varepsilon)/(1+\varepsilon)}} e^{-s^2/(4\sigma^2)}\, ds + o(c^{-2}),$$

where $\sigma^2 = 1/c^2$ and γ_1 has the same property as γ. This implies the second part of (iii) almost immediately. □

Lemma 3.3. *If $\alpha \in (0,1)$ and $c = c(\alpha)$ are the same as in Lemma 3.2, then the function φ introduced in (3.2) and considered only for $x \in (-c, 0]$, is infinitely differentiable on $(-c, 0]$ and is concave in a neighborhood of 0.*

PROOF. For $y \in (0,1)$ we have

$$\int\limits_y^1 \frac{1}{\sqrt{1-r^2}} e^{-c^2(1-r)/(4+4r)} r^{-1-\alpha}\, dr = e^{-c^2/4} \int\limits_y^1 r^{-1-\alpha}\, dr$$

$$+ \int\limits_y^1 \left[\frac{1}{\sqrt{1-r^2}} e^{-c^2(1-r)/(4+4r)} - e^{-c^2/4}\right] r^{-1-\alpha}\, dr = e^{-c^2/4}\alpha^{-1}[y^{-\alpha} - 1]$$

$$+ e^{-c^2/4} \int\limits_y^1 \varkappa(c,r) r^{-\alpha}\, dr = -e^{-c^2/4} \int\limits_0^y \varkappa(c,r) r^{-\alpha}\, dr + e^{-c^2/4}\alpha^{-1} y^{-\alpha}$$

It follows that $\varphi(x) = 2e^{-c^2/4}[\alpha^{-1} - \psi(|x|/c)]$, where for $y \in [0,1)$ we have

$$\psi(y) := y^{\alpha} \int\limits_0^y \varkappa(c,r) r^{-\alpha}\, dr = y \int\limits_0^1 \varkappa(c, yr) r^{-\alpha}\, dr \tag{3.7}$$

and the second equality is obtained by the change of the variable $r \to yr$. We observe that

$$\varkappa(c,r) = \frac{1}{\sqrt{1-r^2}}\left[(e^{c^2r/(2+2r)} - 1)r^{-1} - (\sqrt{1-r^2} - 1)r^{-1}\right],$$

which implies that $\varkappa(c,r)$ is infinitely differentiable for $r \in [0,1)$. By the second equality in (3.7) and the relation between ψ and φ, the function φ is infinitely differentiable for $x \in (-c, 0]$. By (3.7), we have

$$\psi''(0) = 2 \int\limits_0^1 \varkappa_r(c,0) r^{1-\alpha}\, dr = 2\alpha^{-1}\varkappa_r(c,0).$$

Some elementary but tedious computations based on Taylor's formulas for $(1-r^2)^{-1/2}$ with two terms and e^y with four terms show that $\varkappa_r(c,0) = (c^2-2)^2/8$. Therefore, for $c \neq \sqrt{2}$ the function ψ is convex and φ is concave near 0. If $c = \sqrt{2}$, then $\varkappa_{rr}(c,0) = 4/3$, $\varkappa_r(c,r) \sim 4r/3$ near 0 and $\psi'' \geqslant 0$ near 0. $\square$

We formulate the main result of this section.

Lemma 3.4. *Let $c \in (0,\infty)$, $c \neq \sqrt{2}$, $t_0, x_0 \in \mathbb{R}$. For $a > 0$ denote $G_a(t_0,x_0) = G_{c,a}(t_0,x_0) = \{(t,x) : t_0 + a > t > t_0, -c\sqrt{t-t_0} < x - x_0 < 2\sqrt{a}\}$. Let $u(t,x)$ be a bounded continuous function defined in $\widetilde{G}_a(t_0,x_0) := \overline{G}_a(t_0,x_0) \setminus (t_0,x_0)$ and satisfying the inequality*

$$u_{xx} + u_t \leqslant 0 \tag{3.8}$$

in $G_a(t_0,x_0)$ in the classical sense. Suppose that $u \geqslant 0$ and $u \not\equiv 0$ in $G_a(t_0,x_0)$. Then there exists a constant $\delta > 0$ such that

$$u(t_0,x) \geqslant \delta(x-x_0)^{\alpha(c)}$$

for $0 < x - x_0 \leqslant \sqrt{a}$, where $\alpha(c)$ is introduced in Lemma 3.2.

Proof. Notice that the function

$$u(at + t_0, x\sqrt{a} + x_0)$$

satisfies the assumption of the lemma with $t_0 = x_0 = 0$ and $a = 1$. Furthermore, the assertion of the lemma is also easily rewritten in terms of this new function. Therefore, without losing generality we assume that $t_0 = x_0 = 0$ and $a = 1$, so that u satisfies (3.8) in $G := G_1(0,0)$.

By the Harnack inequality, $u(t,1) > 0$ in a closed right neighborhood of 0. Then simple barriers show that for any $s_0 \in (0,1)$ sufficiently close to 0 there exists an $\varepsilon > 0$ such that $u(s_0,x) \geqslant \varepsilon(x + c\sqrt{s_0})$ for all $x \in [-c\sqrt{s_0}, 1]$ and $u(t,1) \geqslant \varepsilon$ for $t \in [0,s_0]$. We fix appropriate $s_0 \in (0,1/4]$ and $\varepsilon > 0$ and take $\alpha = \alpha(c) \in (0,1)$. Let v and φ be the same as in (3.1) and (3.2) respectively.

We continue $\varphi(x)$ for $x > 0$ in such a way that the continued function (use the same notation φ) becomes smooth on $(-c,\infty)$ and vanishes for $x \geqslant 1$. We introduce the barrier function $\psi(t,x) = x^2 - c^2 t$. Note that $\psi = 0$ on the part of the lateral boundary of G where $x = -c\sqrt{t}$ and $\psi_{xx} + \psi_t = 2 - c^2 \neq 0$ everywhere. Therefore, there exists a finite constant $K \geqslant 0$ such that for $\overline{v}(t,x) := \varphi(x) - v(t,x) + K(2-c^2)^{-1}\psi(t,x)$ we have

$$\overline{v}_{xx} + \overline{v}_t = \varphi_{xx}(x) + K \geqslant 0 \tag{3.9}$$

in $G \cap \{x > -c/2\}$. Recalling that $s_0 \leqslant 1/4$, we see that (3.9) holds in the domain $D := \{(t,x) : 0 < t < s_0, -c\sqrt{t} < x < 1\} \subset G$. We note that $\overline{v}(s_0,x)$ is a smooth function vanishing at $x = -c\sqrt{s_0}$. Therefore, there is a constant $\gamma > 0$ such that

$$\gamma\overline{v}(s_0,x) \leqslant \varepsilon(x + c\sqrt{s_0}) \leqslant u(s_0,x).$$

Reducing $\gamma > 0$, if necessary, we can achieve the inequality $\gamma\overline{v}(t,1) \leqslant u(t,1)$ for all $t \in [0, s_0]$. Then the inequality $\gamma\overline{v} \leqslant u$ holds on the parabolic boundary of D and $\gamma\overline{v} \leqslant u$ everywhere in $\overline{D} \setminus (0,0)$ because of (3.8), (3.9), and the maximum principle. In particular, for $0 < x \leqslant 1$, we have

$$\begin{aligned} u(0,x) &\geqslant \gamma(\varphi(x) - v(0,x) + K(2-4c)^{-1}\psi(0,x)) \\ &= \gamma(\varphi(x) - \varphi(0)) + \gamma(v(0,0) - v(0,x)) + \gamma K(2-4c)^{-1}x^2. \end{aligned} \tag{3.10}$$

Taking into account that φ is smooth near 0 and that $v(0,0) - v(0,x) \sim qx^\alpha$, $q > 0$, $\alpha \in (0,1)$, we see that the right-hand side of (3.10) is equivalent to the expression $\gamma q x^\alpha$ for $x \downarrow 0$. By the Harnack inequality, $u(0,x) > 0$ for $x \in (0,2)$. □

REMARK 3.5. One can get estimates for $u(t_0,x)$ from above as well. Let $c \in (0,\infty)$, $a > 0$, $t_0, x_0 \in \mathbb{R}$. Let $u(t,x)$ be a bounded continuous function given in $\widetilde{G}_a(t_0,x_0)$ and satisfying $u_{xx} + u_t \geqslant 0$ in $G_a(t_0,x_0)$ in the classical sense. Suppose that $u \leqslant 0$ for $x - x_0 = -c\sqrt{t-t_0}$ if $t_0 + a \geqslant t \geqslant t_0$. Using the maximum principle and the fact that $\varphi'' \leqslant 0$ near the origin, it is easy to see that there exists a constant $K > 0$ such that $u(at+t_0, x\sqrt{a}+x_0) \leqslant K(\varphi(x) - v(t,x))$ in the intersection of $\overline{G}$ with a neighborhood of $(0,0)$. Hence there exists a constant N such that $u(t_0,x) \leqslant N(x-x_0)^{\alpha(c)}$ for $0 < x - x_0 \leqslant \sqrt{a}$, where $\alpha(c)$ is introduced in Lemma 3.2.

4. Proof of Theorem 1.3

We note that the function $u(t,x) = v(t, \nu_0 x/\sqrt{2})$ satisfies the equation $u_t + u_{xx} = 0$ in $R := \{(t,x) : t \in (0,T), x > y(t)\}$, $y(t) := \nu_0^{-1}x(t)\sqrt{2}$. We take any $c \in (0,\infty)$ such that

$$\alpha(c) < \lambda \tag{4.1}$$

We note that the point $(t_0, y(t_0))$ is on the parabolic boundary of R and claim that, no matter how small $a > 0$ is,

$$G_{c,a}(t_0, y(t_0)) \not\subset R.$$

Indeed, in the opposite case, there exists $a > 0$ such that for any $c' \in (0,c)$ the function u is continuous in $\widetilde{G}_{c',a}(t_0,y_0)$ and satisfies the equation $u_t + u_{xx} = 0$ in $G_{c',a}(t_0,y(t_0))$. By Lemma 3.4, for $c' \neq \sqrt{2}$, we have

$$0 < \lim_{x\downarrow 0} \frac{u(t_0, x + y(t_0))}{x^{\alpha(c')}} = \lim_{x\downarrow 0} \frac{v(t_0, \nu_0 x\sqrt{2} + x(t_0))}{x^{\alpha(c')}} =: I.$$

However, if $\alpha(c') < \lambda$, then $I = 0$ by the condition (1.3). We get a contradiction since $\alpha(c) < \lambda$ and, by the continuity of $\alpha(c)$, one can indeed choose $c' \in (0,c)$, $c' \neq \sqrt{2}$, so that $\alpha(c') < \lambda$.

Our claim just proved is equivalent to saying that, for any $a \in (0, T - t_0)$, there exists $h \in (0, a)$ such that

$$y(t_0) - c\sqrt{h} \leqslant y(t_0 + h), \quad x(t_0) - c\nu_0\sqrt{h/2} \leqslant x(t_0 + h).$$

Hence

$$\overline{\lim_{h\downarrow 0}} \frac{x(t_0 + h) - x(t_0)}{\sqrt{h}} \geqslant -c\nu_0/\sqrt{2}. \tag{4.2}$$

Theorem 1.3 is proved.

REMARK 4.1. The inequality (4.2) holds whenever $c > 0$ and the condition (4.1) is satisfied. The latter can be written as $c > c(\lambda)$, where $c(1) = c(1-) = 0$ and, for other $\lambda \in (0, 1)$, $c(\lambda)$ is the function introduced in Lemma 3.2.

It follows that (1.5) holds with $c_0 = \nu_0 c(\lambda)/\sqrt{2}$. By the way, this value cannot be improved in general. Indeed, take $\nu_0 = \sqrt{2}$, $t_0 = x_0 = 0$, $c > 0$, and $x(t) = -c\sqrt{t}$. By Remark 3.5, the estimate (1.3) holds with $\lambda = \alpha(c)$ if $g = 0$ on the lateral boundary. Furthermore, in this case, $c(\lambda) = c$ and (1.5) becomes equality for $c_0 = \nu_0 c(\lambda)/\sqrt{2}$.

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Bound State Asymptotics for Elliptic Operators with Strongly Degenerated Symbols

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We dedicate this paper to O. A. Ladyzhenskaya on the occasion of her birthday with our admiration, warmest wishes, and gratitude

We study the rate of accumulation of eigenvalues at the edge of the essential spectrum of Schrödinger-type operators $|P(i\nabla)|^\gamma - V(x)$, where γ is a positive number, on $L^2(\mathbb{R}^d)$ in the case where the kinetic energy strongly degenerates at some nontrivial minimal Fermi surface $P(\xi) = 0$.

1. Introduction

Let $P : \mathbb{R}^d \to \mathbb{R}$ be a smooth function taking both negative and positive values. Assume that P has no critical points in a neighborhood of the non-trivial set $M_0 = \{\xi \in \mathbb{R}^d : P(\xi) = 0\}$ and $|P(\xi)| > c|\xi|^s$ for some positive number s and sufficiently large $|\xi|$. Let V be a non-negative potential. We study the negative spectrum of operators of type

$$|P(i\nabla)|^\gamma - V(x), \tag{1.1}$$

where γ is some positive number. The symbol of the differential part of this operator takes the minimal value not at a single point but on a submanifold M_0. This leads to the high instability of the lower edge of the spectrum of $|P(i\nabla)|^\gamma$. A perturbation by some negative potential usually leads to infinitely many negative eigenvalues. We calculate the rate of accumulation of these

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eigenvalues at the point zero. Our main result is to reduce this problem to the spectral analysis of the integral operator

$$(Iu)(\eta) = \int_{x\in\mathbb{R}^d} \int_{\xi\in M_0} V(x)e^{ix(\eta-\xi)}u(\xi)\, d\xi\, dx \tag{1.2}$$

on $L_2(M_0)$ with respect to the measure induced on M_0 by $|\nabla P(\xi)|^{-1}d\xi$.

Our approach is straightforward. We study the integral operator with the Birman–Schwinger kernel

$$\frac{V(x)}{(|P(\xi)|^\gamma+\tau)^{1/2}(|P(\eta)|^\gamma+\tau)^{1/2}} \quad \tau>0, \tag{1.3}$$

as $\tau\to 0$. Unlike the case of virtual bound states for one- and two-dimensional Schrödinger operators, where the Birman–Schwinger kernel diverges by a term of rank one [**1**, **2**], the kernel function (1.3) contains a diverging part of infinite rank. This part can be described by means of the integral operator (1.2). The evaluation of the corresponding spectral asymptotics of the family (1.3) is not quite trivial and requires certain classes of families of compact operators introduced by Safronov [**3**].

Operators of type (1.2) already appeared in [**4**], where the asymptotic behavior of the scattering phases for the operator

$$P(i\nabla) - V(x) \tag{1.4}$$

near a fixed energy has been studied. The similarity of formula (2.15) in [**4**] and our results (4.14)–(4.15) indicates a certain connection between the accumulation of the negative spectrum of (1.1) and the scattering data of (1.4).

In Sec. 2, we give necessary facts from the theory of compact operators. In particular, we focus on certain operator-valued functions with compact values [?]. In Sec. 3, we formulate an abstract operator-theoretical version of our result. In the final section, we apply this result to operators of type (1.1).

The abstract version of our result opens a new line of applications. We mention only one of them. Assume that A is a non-negative operator in some infinite-dimensional Hilbert space $\mathcal{H}$ and the lower bound b of it is an isolated eigenvalue of infinite multiplicity. Consider the operator

$$B(V) = -\frac{d^2}{dx^2}\otimes A - V$$

in $\mathcal{G} = L^2(\mathbb{R})\otimes\mathcal{H}$ with some appropriate symmetric perturbation V. Although the point b itself is not an eigenvalue of $B(0)$ on $\mathcal{G}$, the lower edge b of the spectrum of $B(0)$ is highly unstable and $B(V)$ might have infinitely many negative eigenvalues.

For example, if A is the Schrödinger operator with constant magnetic field in the dimension $d=2$, then, adding free motion into the third direction, it is possible to obtain the Schrödinger operator, denoted by $B(V)$, with constant magnetic field and some electric potential V for $d = 3$. The asymptotics of accumulation of the negative eigenvalues in this case was calculated by Sobolev [**5**, **6**].

Here, we restrict ourself to the abstract result and operators of type (1.1). Further applications will be published elsewhere.

2. Preliminaries

2.1. Let H_1 and H_2 be separable Hilbert spaces. For a linear closed mapping $T : H_1 \mapsto H_2$ we denote by $D(T)$, T^*, $\rho(T)$, and $\sigma(T)$ the domain, the adjoint, the resolvent set, and the spectrum of T respectively. If $T = T^*$, then $E_T(\Omega)$ denotes the spectral projection of T with respect to the Borel set Ω.

Further, denote by $B(H_1, H_2)$ the Banach algebra of all bounded linear operators from H_1 to H_2 and by $S_\infty(H_1, H_2)$ the ideal of compact operators in $B(H_1, H_2)$. If $T \in S_\infty(H_1, H_2)$, then $\{s_n(T)\}$ denotes the non-increasing sequence of the singular values of T, appearing according to their multiplicities. We set

$$n(s, T) = \operatorname{card}\{s_n(T) : s_n(T) > s\}, \quad s > 0.$$

Then

$$\Sigma_p(H_1, H_2) = \{T \in S_\infty(H_1, H_2) : |T|_p = \sup_{s>0} s^p n(s, T) < \infty\}$$

defines a two-sided quasi-normed sub-ideal in $S_\infty(H_1, H_2)$ for any given positive p. If the choice of the underlying spaces is not important for our purposes, we do not indicate them in the notation.

If $T = T^* \in S_\infty$, then $-\lambda_n^-(T)$ and $\lambda_n^+(T)$ are the non-increasing sequences of the absolute values of the negative and positive eigenvalues of T, which occur according to their multiplicities. We set

$$n_\pm(s, T) = \operatorname{card}\{\lambda_n^\pm(T) : \lambda_n^\pm(T) > s\}, \quad s > 0.$$

It is obvious that $n(s, T) = n_+(s, T) + n_-(s, T)$, $s > 0$, and $T = T^* \in S_\infty$. For $T = T^* \in \Sigma_p$, $p > 0$, the functionals

$$\Delta_p^\pm(T) := \limsup_{s \to +0} s^p n_\pm(s, T), \quad \delta_p^\pm(T) := \liminf_{s \to +0} s^p n_\pm(s, T)$$

are well defined and finite. If $T \in \Sigma_p$, $S \in \Sigma_p$, and $Q \in B$, then $TS \in \Sigma_{2p}$, $ST \in \Sigma_{2p}$, and $TQ \in \Sigma_p$, $QT \in \Sigma_p$ with $\max\{|TQ|_p, |QT|_p\} \leqslant \|Q\|_B^p |T|_p$.

2.2. We indicate some generalizations of the operator ideals Σ_p and Σ_p^0. In particular, we consider not individual operators but rather an operator function

$$\mathcal{T} : \mathbb{R}_+ \to S_\infty. \tag{2.1}$$

Definition 2.1. We say that the operator family $\mathcal{T}$ *belongs to the class* $\mathcal{S}^p$, $0 < p < \infty$, if the functional

$$\sup_{s>0} s^p n(1, \alpha\mathcal{T}(s)) \tag{2.2}$$

is finite for all $\alpha > 0$.

The classes $\mathcal{S}^p$ are linear sets. For $\mathcal{T} \in \mathcal{S}^p$ one can introduce the functionals

$$\Delta_p(\mathcal{T}) := \limsup_{s\to 0} s^p n(1, \mathcal{T}(s)), \quad \delta_p(\mathcal{T}) := \liminf_{s\to 0} s^p n(1, \mathcal{T}(s)). \tag{2.3}$$

If $\mathcal{T}(s) = (\mathcal{T}(s))^*$ for all $s > 0$, then it is possible to define the quantities

$$\Delta_p^\pm(\mathcal{T}) := \limsup_{s\to 0} s^p n_\pm(1, \mathcal{T}(s)), \quad \delta_p^\pm(\mathcal{T}) := \liminf_{s\to 0} s^p n_\pm(1, \mathcal{T}(s)). \tag{2.4}$$

The linear subset of operators $\mathcal{T} \in \mathcal{S}^p$ satisfying the equality

$$\Delta_p(\alpha\mathcal{T}) = 0 \quad \text{for all } \alpha > 0$$

is denoted by $\mathcal{S}_0^p$. The classes $\mathcal{S}^q$ and $\mathcal{S}_0^q$ are invariant with respect to the conjugation $\mathcal{T}^*(\tau) := (\mathcal{T}(\tau))^*$.

2.3. Let us describe the basic properties of the classes $\mathcal{S}^p$ and $\mathcal{S}_0^p$ (cf. [**3**] for details).

Lemma 2.2. *Suppose that* $\mathcal{T} = \mathcal{T}^* \in \mathcal{S}^p$ *and* $\mathcal{T}_0 = \mathcal{T}_0^* \in \mathcal{S}_0^p$, $p > 0$. *Then*

$$\lim_{t\to 1} \Delta_p^\pm(t\mathcal{T}) = \Delta_p^\pm(\mathcal{T}) \quad \Rightarrow \quad \Delta_p^\pm(\mathcal{T} + \mathcal{T}_0) = \Delta_p^\pm(\mathcal{T}),$$
$$\lim_{t\to 1} \delta_p^\pm(t\mathcal{T}) = \delta_p^\pm(\mathcal{T}) \quad \Rightarrow \quad \delta_p^\pm(\mathcal{T} + \mathcal{T}_0) = \delta_p^\pm(\mathcal{T}).$$

The following two statements provide sufficient conditions for the inclusion $\mathcal{T} \in \mathcal{S}_0^p$.

Lemma 2.3. *Suppose that* $\mathcal{T}$ *is an operator family of type* (2.1) *and* $\mathcal{T}(s)$ *converges in* S_∞ *as* $s \to 0$. *Then* $\mathcal{T} \in \mathcal{S}_0^p$ *for any* $p > 0$.

Lemma 2.4. *Let* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *be operator families of type* (2.1). *Suppose that* $\mathcal{T}_1 \in \mathcal{S}^p$ *and* $\mathcal{T}_2(s)$ *converges in* S_∞ *as* $s \to 0$. *Let* $\mathcal{T}(s) = \mathcal{T}_1(s)\mathcal{T}_2(s)$ *for all* $s > 0$. *Then*

$$\Delta_p(\varepsilon\mathcal{T}_1) \to 0 \ \textit{as}\ \varepsilon \to 0 \quad \Rightarrow \quad \mathcal{T} \in \mathcal{S}_0^p.$$

PROOF. From Ky-Fan's inequality it follows that

$$\Delta_p(\alpha\mathcal{T}) \leqslant \Delta_p(\varepsilon\mathcal{T}_1) + \Delta_p(\varepsilon^{-1}\alpha\mathcal{T}_2)$$

for all $\varepsilon > 0$. Since $\Delta_p(\varepsilon^{-1}\alpha\mathcal{T}_2) = 0$ for arbitrary $\alpha > 0$, we have

$$\Delta_p(\alpha\mathcal{T}) \leqslant \Delta_p(\varepsilon\mathcal{T}_1) \to 0 \quad \text{as } \varepsilon \to 0.$$

Hence $\mathcal{T} \in \mathcal{S}_0^p$. $\square$

2.4. We note that the results of the previous section extend to logarithmic asymptotics. We say that $\mathcal{T} \in \mathcal{S}^{\ln}$ if

$$\sup_{s>0}(1+\ln(1+s^{-1}))^{-1}n(1,\alpha\mathcal{T}(s))$$

is finite for all $\alpha > 0$. The corresponding asymptotic limits $\Delta_{\ln}$, $\delta_{\ln}$, $\Delta_{\ln}^{\pm}$, and $\delta_{\ln}^{\pm}$ are defined by replacing the factor s^q in (2.3), (2.4) with $(1+|\ln s|)^{-1}$. Moreover, $\mathcal{T} \in \mathcal{S}_0^{\ln}$ if $\mathcal{T} \in \mathcal{S}^{\ln}$ and $\Delta_{\ln}(\alpha\mathcal{T}) = 0$ for all $\alpha > 0$. Then assertions similar to Lemmas 2.2–2.4 are valid for this logarithmic class.

3. Abstract Result

3.1. Let A be a self-adjoint operator on the separable Hilbert space H, and let $\min\sigma(A) = 0$. We also impose some conditions on the spectral decomposition of the initial operator A near the spectral point 0.

We fix a sufficiently small $\delta > 0$ and put $\Delta = [0,\delta)$. Consider a strictly monotone increasing continuous function φ from $[0,\nu)$ onto Δ such that

$$\varphi(\lambda) = \lambda^{\gamma}(1+o(1)) \quad \text{as } \lambda \to +0, \quad \gamma \geqslant 1.$$

Put $\Lambda = (-\nu,\nu)$. Let G be an infinite-dimensional Hilbert space. We set

$$G_\Lambda := \oplus\int_\Lambda G d\lambda, \quad \Theta := \oplus\int_\Lambda \varphi(|\lambda|)d\lambda.$$

Equivalently, these objects can be understood as follows:

$$G_\Lambda = L_2(\Lambda)\otimes G, \quad \Theta = [\varphi(|\cdot|)]\otimes \mathbb{I},$$

where $[\varphi(|\cdot|)]$ is the operator of multiplication by the function $\varphi(|\cdot|)$ in $L_2(\Lambda)$ and $\mathbb{I}$ is the identity operator on G.

Assume that the following condition is satisfied.

Condition 3.1. The operator $E_A(\Delta)A$ on $E_A(\Delta)H$ is unitarily equivalent to the operator Θ on G_Λ.

We denote by $F : E_A(\Delta)H \to G_\Lambda$ an operator that establishes the unitary equivalence between $E_A(\Delta)A$ and Θ. For a fixed $f \in E_A(\Delta)H$ the notion $J_\lambda f := (Ff)(\lambda)$ is well defined for almost all $\lambda \in \Delta$.

3.2. We describe admissible perturbations of the operator A. Let W be an operator on H such that $E_A(\Delta)H \subset D(W)$ and $WE_A(\Delta)$ is bounded.

Definition 3.2. We say that W is *$\varkappa$-smooth with respect to A at zero*, $\varkappa \in (0,1]$, if there exists a bounded operator $X(W)$ from H to G such that

$$\operatorname*{ess\,sup}_{\lambda\in\Lambda}|\lambda|^{-\varkappa}\|X(W)f - J_\lambda(WE_A(\Delta))^*f\|_G \leqslant C\|f\|_H \tag{3.1}$$

for all $f \in H$.

In this case, the operator $X(W) : H \to G$ is given by the limit

$$X(W)f := \text{ess} \lim_{\lambda \to 0} J_\lambda (W E_A(\Delta))^* f. \tag{3.2}$$

3.3. Let $W_k : H \to H$ be $\varkappa$-smooth with respect to A at zero, $\varkappa \in (0,1]$, $k = 1, \dots, n$. Assume that $D(W_k) \supset D((A + \mathbb{I})^{1/2})$. Let

$$W_k (A + \mathbb{I})^{-1/2} \in S_\infty \ , \quad k = 1, \dots, n. \tag{3.3}$$

Moreover, let $v_{j,k}$ be a set of bounded operators in H satisfying $v_{j,k} = v_{k,j}^*$ for all $k, j = 1, \dots, n$. Consider the quadratic form

$$\|(A + \mathbb{I})^{1/2} f\|^2 - \sum_{j,k=1}^{n} (v_{j,k} W_j f, W_k f), \quad f \in D(A^{1/2}).$$

This form is semi-bounded from below, is closed on $D(A^{1/2})$, and defines a self-adjoint operator $B + \mathbb{I}$ in H. The difference of the resolvents of B and A is compact. Hence the negative spectrum of B is discrete and bounded from below, the respective eigenvalues are of finite multiplicity and can accumulate to zero only. Put

$$N(\tau, B) = \operatorname{rank} E_B((-\infty, -\tau)), \quad \tau > 0.$$

We study the asymptotic behavior of $N(\tau, B)$ as $\tau \to 0$. To this end, we introduce the following additional condition on the strength of the perturbation.

Condition 3.3. For some $p \in (0, \infty)$

$$K_k = X(W_k) \in \Sigma_{2p}(H, G) \quad \text{for all } k = 1, \dots, n. \tag{3.4}$$

Put

$$K = \sum_{j,k=1}^{n} K_j v_{j,k} K_k^*. \tag{3.5}$$

Then Condition 3.3 implies $K \in \Sigma_p(G)$.

3.4. We formulate our main abstract result.

Theorem 3.4. *Suppose that Condition 3.1 is satisfied for some* $1 \leqslant \gamma < 3$, *the operators* W_k, $k = 1, \dots, n$, *satisfy* (3.3), (3.4) *and are* $\varkappa$*-smooth at zero for some* $\varkappa \in ((\gamma - 1)/2, 1]$. *Then the following asymptotic formulas hold:*

$$\begin{aligned} \limsup_{\tau \to +0} \psi^p(\tau) N(\tau, B) &= \Delta_p^{(+)}(K), \\ \liminf_{\tau \to +0} \psi^p(\tau) N(\tau, B) &= \delta_p^{(+)}(K), \end{aligned} \tag{3.6}$$

where

$$\psi(\tau) = \frac{\gamma \sin(\pi\gamma^{-1})}{2\pi}\tau^{1-1/\gamma}, \quad \gamma > 1,$$
$$\psi(\tau) = 2^{-1}\left|\ln|\tau|\right|^{-1}, \quad \gamma = 1.$$

The remaining part of this section is devoted to the proof of Theorem 3.4.

3.5. Consider the case $\gamma > 1$. By the arguments of Sec. 2.4, the proof for $\gamma = 1$ is similar. Put

$$\mathcal{Z}_k(\tau) = W_k(A+\tau)^{-1/2}, \quad \mathcal{Y}(\tau) = \sum_{j,k=1}^{n} \mathcal{Z}_j^*(\tau) v_{j,k} \mathcal{Z}_k(\tau). \tag{3.7}$$

By (3.3), the operator $\mathcal{Y}$ is compact and self-adjoint in H. By the Birman–Schwinger principle [**1**], we have

$$N(\tau, B) = n_+(1, \mathcal{Y}(\tau)), \quad \tau > 0. \tag{3.8}$$

If we show that

$$\mathcal{Y} \in \mathcal{S}^q, \quad q = (1-\gamma^{-1})p, \tag{3.9}$$

$$\begin{aligned} \Delta_q^+(\mathcal{Y}) &= c^p(\gamma)\Delta_p^+(K), \\ \delta_q^+(\mathcal{Y}) &= c^p(\gamma)\delta_p^+(K), \\ c(\gamma) &:= \frac{2\pi}{\gamma\sin(\pi\gamma^{-1})}, \end{aligned} \tag{3.10}$$

then we complete the proof of Theorem 3.4.

3.6. Let $[\eta_\tau]$ be the operator of multiplication by the function $\eta_\tau(\lambda) := (\varphi(|\lambda|)+\tau)^{-1/2}$, $\tau > 0$, on $L_2(\Lambda)$. Let $\mathcal{K}_k(\tau) : H \to G_\Lambda = L_2(\Lambda) \otimes G$ be the operator families

$$\mathcal{K}_k(\tau) = [\eta_\tau] \otimes X(W_k), \quad k = 1, \ldots, n, \quad \tau > 0. \tag{3.11}$$

We set

$$\mathcal{K}(\tau) = \sum_{j,k=1}^{n} \mathcal{K}_j(\tau) v_{j,k} \mathcal{K}_k^*(\tau), \quad \tau > 0. \tag{3.12}$$

Finally, let $\mathcal{L}$ denote the operator family

$$\mathcal{L}(\tau) = F^*\mathcal{K}(\tau)F \oplus \mathbb{O} \quad \text{on } H = E_A(\Delta)H \oplus E_A(\mathbb{R}\backslash\Delta)H,$$

where $F : E_A(\Delta)H \to G_\lambda$ is the unitary operator introduced in Sec. 3.1. By Lemma 2.2, Eqs. (3.9) and (3.10), and thereby Theorem 3.4, immediately follow from the following two assertions.

Lemma 3.5. *Under the assumptions of Theorem* 3.4, $\mathcal{L} \in \mathcal{S}^q$ *and*

$$\Delta_q^+(\alpha\mathcal{L}) = c^p(\gamma)\alpha^p\Delta_p^+(K), \quad \delta_q^+(\alpha\mathcal{L}) = c^p(\gamma)\alpha^p\delta_p^+(K) \tag{3.13}$$

for $q = (1-\gamma^{-1})p$, $\gamma > 1$, *and* $\alpha > 0$.

Lemma 3.6. *Under the assumptions of Theorem* 3.4,

$$\mathcal{L} - \mathcal{Y} \in \mathcal{S}_0^q \quad \textit{with} \quad q = (1-\gamma^{-1})p, \quad \gamma > 1. \tag{3.14}$$

3.7. To prove Lemma 3.5, we note that (3.11) and (3.12) imply

$$\begin{aligned} s_l(\mathcal{K}_k(\tau)) &= \|\eta_\tau\|_{L_2(\Lambda)} s_l(K_k), \quad l \in \mathbb{N}, \\ \lambda_l^\pm(\mathcal{K}(\tau)) &= \|\eta_\tau\|_{L_2(\Lambda)}^2 \lambda_l^\pm(K), \quad l \in \mathbb{N}. \end{aligned}$$

Hence

$$\begin{aligned} n(s, \mathcal{K}_k(\tau)) &= n(s, \|\eta_\tau\|_{L_2(\Lambda)} K_k), \\ n_\pm(s, \mathcal{K}(\tau)) &= n_\pm(s, \|\eta_\tau\|_{L_2(\Lambda)} K), \end{aligned}$$

for arbitrary $s > 0$. This can be written as follows:

$$\begin{aligned} \tau^q n(1, \alpha\mathcal{K}_k(\tau)) &= \tau^q \|\eta_\tau\|_{L_2(\Lambda)}^{2p} \{\omega_\tau^{2p} n(\omega_\tau, \alpha K_k)\}, \\ \tau^q n(1, \alpha\mathcal{K}(\tau)) &= \tau^q \|\eta_\tau\|_{L_2(\Lambda)}^{2p} \{\varpi_\tau^p n(\varpi_\tau, \alpha K)\}, \end{aligned} \tag{3.15}$$

where $\omega_\tau = \|\eta_\tau\|_{L_2(\Lambda)}^{-1}$ and $\varpi_\tau = \|\eta_\tau\|_{L_2(\Lambda)}^{-2}$ Further, $\omega_\tau \to 0$ and $\varpi_\tau \to 0$ as $\tau \to 0$,

$$\sup_{\tau>0} \tau^q \|\eta_\tau\|_{L_2(\Lambda)}^{2p} = C < \infty, \quad \lim_{\tau\to 0} \tau^q \|\eta_\tau\|_{L_2(\Lambda)}^{2p} = c^p(\gamma).$$

The relations (3.15) imply $\mathcal{K}_k \in \mathcal{S}^q$, $\mathcal{K} \in \mathcal{S}^q$, and

$$\Delta_q(\alpha\mathcal{K}_k) = c^p(\gamma)\alpha^{2p}\Delta_{2p}(K_k), \quad \delta_q(\alpha\mathcal{K}_k) = c^p(\gamma)\alpha^{2p}\delta_{2p}(K_k), \tag{3.16}$$

$$\Delta_q^+(\alpha\mathcal{K}) = c^p(\gamma)\alpha^p \; \Delta_p^+(K) \;, \quad \delta_q^+(\alpha\mathcal{K}) = c^p(\gamma)\alpha^p \; \delta_p^+(K) \tag{3.17}$$

for all $\alpha > 0$. For the nontrivial parts of $\mathcal{K}(\tau)$ and $\mathcal{L}(\tau)$ that are unitarily equivalent, we find $\mathcal{L} \in \mathcal{S}^q$ and (3.13), which completes the proof of Lemma 3.5.

3.8. To prove Lemma 3.6, we use the decomposition

$$\mathcal{Z}_k(\tau) = \check{\mathcal{Z}}_k(\tau) + \hat{\mathcal{Z}}_k(\tau), \tag{3.18}$$

where $\check{\mathcal{Z}}_k(\tau) = \mathcal{Z}_k(\tau)E_A(\Delta)$. Then the family $\mathcal{Y}$ can be represented as follows:

$$\mathcal{Y}(\tau) = \check{\mathcal{Y}}(\tau) + \widetilde{\mathcal{Y}}(\tau) + \widetilde{\mathcal{Y}}^*(\tau) + \widehat{\mathcal{Y}}(\tau), \tag{3.19}$$

where

$$\check{\mathcal{Y}}(\tau) = \sum_{j,k=1}^{n} \check{\mathcal{Z}}_j^*(\tau) v_{j,k} \check{\mathcal{Z}}_k(\tau), \tag{3.20}$$

$$\widehat{\mathcal{Y}}(\tau) = \sum_{j,k=1}^{n} \widehat{\mathcal{Z}}_j^*(\tau) v_{j,k} \widehat{\mathcal{Z}}_k(\tau), \tag{3.21}$$

$$\widetilde{\mathcal{Y}}(\tau) = \sum_{j,k=1}^{n} \check{\mathcal{Z}}_j^*(\tau) v_{j,k} \widehat{\mathcal{Z}}_k(\tau). \tag{3.22}$$

It is convenient to divide the verification of the inclusion (3.14) into the following consecutive steps:

$$\mathcal{L} - \check{\mathcal{Y}} \in \mathcal{S}_0^q, \tag{3.23}$$

$$\mathcal{Y} - \check{\mathcal{Y}} \in \mathcal{S}_0^q, \tag{3.24}$$

where $q = (1 - \gamma^{-1})p$.

3.9. Let us prove the inclusion (3.23). From (3.20) it follows that

$$\begin{aligned}\check{\mathcal{Y}}(\tau) &= \sum_{j,k=1}^{n} F^*(\mathcal{Q}_j(\tau) + \mathcal{K}_j(\tau)) v_{j,k} (\mathcal{Q}_k^*(\tau) + \mathcal{K}_k^*(\tau))F \\ &= \mathcal{L}(\tau) + \mathcal{U}(\tau) + \mathcal{V}(\tau) + \mathcal{V}^*(\tau),\end{aligned} \tag{3.25}$$

where $\mathcal{Q}_k(\tau) := F\check{\mathcal{Z}}_k^*(\tau) - \mathcal{K}_k(\tau)$ and

$$\mathcal{U}(\tau) := \sum_{j,k=1}^{n} F^* \mathcal{Q}_j(\tau) v_{j,k} \mathcal{Q}_k^*(\tau) F, \quad \mathcal{V}(\tau) := \sum_{j,k=1}^{n} F^* \mathcal{Q}_j(\tau) v_{j,k} \mathcal{K}_k^*(\tau) F.$$

By Condition 3.1 and the relations (3.7) and (3.18), we have

$$\check{\mathcal{Z}}_k^*(\tau) = F^*[\eta]_\tau F(W_k E_A(\Delta))^*.$$

In view of (3.11), the operators $\mathcal{Q}_k(\tau) \in S_\infty(H, G_\Lambda)$ act as follows:

$$\mathcal{Q}_k(\tau) f = [\eta_\tau] F(W_k E_A(\Delta))^* f - [\eta_\tau] \otimes X(W_k) f = [\eta_\tau] R_k f,$$

where the operator $R_k : H \to G_\Lambda$, $R_k = F(W_k E_A(\Delta))^* - \mathbb{I} \otimes X(W_k)$ is independent of τ. From (3.1) it follows that

$$\|J_\lambda R_k f\|_G \leqslant c_0 |\lambda|^{\varkappa} \|f\|_H, \quad \lambda \in \Lambda.$$

Therefore, for $\tau_1 > \tau_2 > 0$ we have

$$\|(\mathcal{Q}_k(\tau_1) - \mathcal{Q}_k(\tau_2))f\|^2_{G_\Lambda} = \int_\Lambda \|(\eta_{\tau_1}(\lambda) - \eta_{\tau_2}(\lambda))J_\lambda R_k f\|^2_G d\lambda$$

$$\leqslant c_1\|f\|^2_H \int_\Lambda |\lambda|^{2\varkappa}|\eta_{\tau_1} - \eta_{\tau_2}|^2 d\lambda \leqslant c_2\|f\|^2_H \tau_1^\theta \int_0^\infty \lambda^{2\varkappa-\gamma}(\lambda^\gamma+1)^{-2}d\lambda,$$

where $\theta = (2\varkappa - \gamma + 1)\gamma^{-1} > 0$. We conclude that the operator families $\mathcal{Q}_k(\tau)$ converge in S_∞ as $\tau \to 0$.

3.10. We begin with the following remark. By Lemma 2.3, for the operator family $\mathcal{Q}_k$ defined in Sec. 3.9 we have $\mathcal{Q}_k \in \mathcal{S}_0^q$. By the relations (3.16), Lemma 2.2, and the identity $\check{\mathcal{Z}}_k(\tau) = (\mathcal{Q}_k(\tau) + \mathcal{K}_k(\tau))^* F$, $\tau > 0$, we have

$$\check{\mathcal{Z}}_k \in \mathcal{S}^q, \quad \Delta_q(\alpha\check{\mathcal{Z}}_k) = c^p(\gamma)\alpha^{2p}\Delta_{2p}(K_k). \tag{3.26}$$

To prove (3.24), it suffices to show that the terms (3.21) and (3.22) belong to $\mathcal{S}_0^q$. It is easy to see that the families

$$\hat{\mathcal{Z}}_k(\tau) = W_k E_A(\mathbb{R}\setminus\Delta)(A+\tau)^{-1/2}, \quad \Delta = [0,\delta), \quad \tau > 0,$$

have limits in S_∞ as $\tau \to 0$, $k = 1, \dots, n$. Hence $\hat{\mathcal{Y}}(\tau)$ also converges in S_∞. Thus, $\hat{\mathcal{Y}} \in \mathcal{S}_0^q$ by Lemma 2.3. At the same time, $\hat{\mathcal{Z}}_j^*(\tau)v_{j,k}$ converges in S_∞, and (3.26), together with Lemma 2.4, implies $\widetilde{\mathcal{Y}} \in \mathcal{S}_0^q$. This completes the proof of Lemma 3.6. Hence Theorem 3.4 is proved.

4. Applications

4.1. Consider an operator $A = \varphi^*[a]\varphi$ in the Hilbert space $H = L_2(\mathbb{R}^d)$, where φ is the Fourier transform and $[a]$ is multiplication by a real-valued function $a(\xi)$, $\xi \in \mathbb{R}^d$.

Condition 4.1. The function $a(\xi)$ is continuous,

$$\min_{\xi\in\mathbb{R}^d} a(\xi) = 0,$$

and there exist $r, c, C > 0$ such that

$$a(\xi) \geqslant c|\xi|^r \quad \text{for all } |\xi| > C. \tag{4.1}$$

From (4.1) it follows that $D(A) \subseteq H^r(\mathbb{R}^d)$. The spectrum of A coincides with the image of $a(\xi)$. Thus, $\sigma(A) = [0, +\infty)$. For sufficiently small $\delta > 0$ we consider the pre-image Ξ of the interval $\Delta = [0, \delta)$ under the mapping a. We impose the following condition on the structure of the minimum of the symbol a.

Condition 4.2. There exists a function $P \in C^\infty(M, \mathbb{R})$ and a number $\gamma > 1/2$ such that

$$a(\xi) = |P(\xi)|^\gamma, \quad \nabla P(\xi) \neq 0 \quad \text{for all } \xi \in \Xi.$$

4.2. In accordance with Sec. 3.1, we reduce the decomposition of the operator A to a suitable direct integral. Put $\nu = \delta^{1/\gamma}$, $\Lambda = (-\nu, \nu)$ and define the (compact) surface $M_\lambda = \{\xi \in \mathbb{R} : P(\xi) = \lambda\}$, $\lambda \in \Lambda$. The Lebesgue measure $d\xi$ on $\mathbb{R}^d$ induces the measure dM_λ on M_λ. We set $d\mu_\lambda := |\nabla P|^{-1} dM_\lambda$ and $G(\lambda) := L_2(M_\lambda, d\mu_\lambda)$ for $\lambda \in \Lambda$. The measures $d\xi$ and $d\mu_\lambda \otimes d\lambda$ coincide on Ξ. By Condition 4.2, we have $af|_{M_\lambda} = |\lambda|^\gamma f|_{M_\lambda}$ for all $\lambda \in \Lambda$ and all functions f on $\mathbb{R}^d$. Hence

$$[a] = \oplus \int\limits_\Lambda |\lambda|^\gamma d\lambda \quad \text{on } L_2(\Xi) = \oplus \int\limits_\Lambda G(\lambda) d\lambda. \tag{4.2}$$

The direct integral on the right-hand side of (4.2) can be identified with the space $G_\Lambda := L_2(\Lambda) \otimes G$, $G = G(0)$. To justify this assertion, we note that the manifolds M_λ are diffeomorphic. Indeed, substituting $\lambda \in \Lambda$ into the equation $P(\xi) = \lambda$, we see that each point of M_λ moves along the trajectory determined by the normal vectors to M_λ. The translation along these trajectories defines a family of unitary operators $U(\lambda) : G(\lambda) \to G(0)$, $\lambda \in \Lambda$. (To make these operators to be unitary, one should use the corresponding Jacobian.) Finally, let $\Upsilon : L_2(\Xi) \to G_\Lambda$ be a unitary operator that acts as follows:

$$(\Upsilon f)(\lambda) = U(\lambda)(f|_{M_\lambda}), \quad \lambda \in \Lambda, \quad f \in L_2(\Xi).$$

Put $F = \Upsilon \varphi$. This isometry between $E_A(\Delta)H$ and G_Λ reduces $E_A(\Delta)A$ to the operator $\Theta = [|\lambda|^\gamma] \otimes \mathbb{I}$, i.e.,

$$J_\lambda A f = |\lambda|^\gamma J_\lambda f, \quad \lambda \in \Lambda, \quad f \in E_A(\Delta)H, \tag{4.3}$$

where $J_\lambda f$ is the value of Ff at the point $\lambda \in \Lambda$.

4.3. For $\varepsilon > 1$ we denote by $\langle x \rangle_\varepsilon$ the operator of multiplication by $(1+|x|)^{-\varepsilon/2}$. The image of the operator $\varphi \langle \langle x \rangle_\varepsilon$ on $L_2(\mathbb{R}^d)$ coincides with the Sobolev class $H^{\varepsilon/2}(\mathbb{R}^d)$, and the image of $\varphi E_A(\Delta)\langle x \rangle_\varepsilon$ is $[\chi_\Xi] H^{\varepsilon/2}(\mathbb{R}^d)$, where $[\chi_\Xi]$ denotes the operator of multiplication by the characteristic function χ_Ξ of the open set Ξ. For $M_\lambda \in \Xi$, by the trace embedding theorem, the mappings $J_\lambda E_A(\Delta)\langle x \rangle_\varepsilon$, $\lambda \in \Lambda$, can be extended to a Hölder continuous family of compact operators from $L_2(\mathbb{R}^d)$ to G:

$$||(J_{\lambda_1} E_A(\Delta)\langle x \rangle_\varepsilon - J_{\lambda_2} E_A(\Delta)\langle x \rangle_\varepsilon) f||_G \leqslant C|\lambda_1 - \lambda_2|^\varkappa ||f||_{L_2(\mathbb{R}^d)} \tag{4.4}$$

for all $\lambda_1, \lambda_2 \in \Lambda$ and $\varkappa = \min\{(\varepsilon - 1)/2, 1\}$.

For the multi-index $\beta \in \mathbb{N}^d$ we introduce the operator

$$W_\beta = \langle x\rangle_\varepsilon D^\beta, \quad D^\beta = (-i)^{|\beta|}\frac{\partial^{|\beta|}}{\partial x_1^{\beta_1}\cdots\partial x_d^{\beta_d}}.$$

Note that for a fixed $\beta \in \mathbb{N}^d$ the operators $[\xi^\beta]_\lambda$ of multiplication by ξ^β on $L_2(M_\lambda, d\mu_\lambda)$ are uniformly bounded for all $\lambda \in \Lambda$. Moreover, the family of operators $Q(\lambda) = U(\lambda)[\xi^\beta]_\lambda U^*(\lambda)$ is smooth in $\lambda \in \Lambda$. For $J_\lambda(W_\beta E_A(\Delta))^* = Q(\lambda)J_\lambda E_A(\Delta)\langle x\rangle_\varepsilon$, $\lambda \in \Lambda$, the bound (4.4) implies the estimate

$$\|(J_{\lambda_1}(W_\beta E_A(\Delta))^* - J_{\lambda_2}(W_\beta E_A(\Delta))^*)f\|_G \leqslant C_1|\lambda_1-\lambda_2|^\varkappa\|f\|_{L_2(\mathbb{R}^d)}. \quad (4.5)$$

Hence the mapping

$$X(W_\beta) : f \mapsto \lim_{\lambda\to 0} J_\lambda(W_\beta E_A(\Delta))^*$$

regarded as a mapping from $L_2(\mathbb{R}^d)$ to G is well defined and is bounded; moreover,

$$\|X(W_\beta)f - J_\lambda(W_\beta E_A(\Delta))^* f\|_G \leqslant C_1|\lambda|^\varkappa\|f\|_{L_2(\mathbb{R}^d)}, \quad \lambda \in \Lambda, \quad (4.6)$$

where $\varkappa = \min\{(\varepsilon-1)/2, 1\}$. Hence the operator W_β is $\varkappa$-smooth with respect to A.

4.4. We perturb the operator A by differential expressions of the form

$$V := \sum_{\alpha,\beta} D^\alpha\langle x\rangle_\varepsilon v_{\alpha,\beta}\langle x\rangle_\varepsilon D^\beta, \quad |\alpha|+|\beta|<r, \quad (4.7)$$

where r is given in (4.1) and $\varepsilon > 1$. Put $\theta = x/|x| \in \mathbb{S}^{d-1}$. Assume that

$$v_{\alpha,\beta}(x) = \overline{v_{\beta,\alpha}(x)} \in L_\infty(\mathbb{R}^d), \quad (4.8)$$

$$v_{\alpha,\beta}(x) = w_{\alpha,\beta}(\theta)(1+o(1)) \quad \text{as } |x|\to\infty, \quad w_{\alpha,\beta} \in C^\infty(\mathbb{S}^{d-1}). \quad (4.9)$$

Following (3.5), we set $K_\alpha = X(W_\alpha)$, where $W_\alpha = \langle x\rangle_\varepsilon D^\alpha$ for $\alpha \in \mathbb{N}^d$ and

$$K = \sum_{\alpha,\beta\in\mathbb{N}^d\, |\alpha|+|\beta|<r} K_\alpha v_{\alpha,\beta}K_\beta^*.$$

The spectral properties of this operator were discussed in [4]. To describe this result, we introduce

$$b(x,\xi) = \sum_{\alpha,\beta} w_{\alpha,\beta}(\theta)\xi^\alpha\xi^\beta, \quad \xi \in \mathbb{R}^d. \quad (4.10)$$

Put $M = M_0$. Let $\nu(\zeta)$ denote the normal unit vector to M at the point $\zeta \in M$, and let T_ζ be the hyperplane of all $\xi \in \mathbb{R}^d$ that are orthogonal to $\nu(\zeta)$. For $\zeta \in M$ and $\xi \in T_\zeta \cap \mathbb{S}^{d-1}$ we set

$$\Omega(\zeta,\xi) = (2\pi)^{-1}|\nabla P(\zeta)|^{-1}\int_0^\pi b(\nu(\zeta)\cos t + \xi\sin t, \zeta)\sin^{\varepsilon-2} t\,dt$$

and $(\Omega(\zeta,\xi))_+ = \max\{0, \Omega(\zeta,\xi)\}$. According to [**4**, Lemma 1], the following assertions holds.

Proposition 4.3. *Let operators* K_α, $\alpha \in \mathbb{N}^d$, *and* K *be defined as above, and let* $\varepsilon > 1$. *We set* $p = (d-1)(\varepsilon-1)^{-1}$. *Then*

$$K_\alpha \in \Sigma_{2p}, \quad K \in \Sigma_p, \quad \Delta_p^+(K) = g(\varepsilon, d), \tag{4.11}$$

where

$$g(\varepsilon, d) := (d-1)^{-1}(2\pi)^{-\varepsilon p} \int\limits_M dM(\zeta) \int\limits_{T_\zeta \cap \mathbb{S}^{d-1}} (\Omega(\zeta,\xi))_+^p \, d\xi. \tag{4.12}$$

4.5. By (4.7), (4.8), and (4.9), the quadratic form

$$\|A^{1/2}f\|^2 - \sum_{|\alpha|+|\beta|<r} (v_{\alpha,\beta}D^\beta f, D^\alpha f) \tag{4.13}$$

is semi-bounded from below and is closed on $D(A^{1/2})$. Let B be the self-adjoint operator associated with the quadratic form (4.13).

The difference between the resolvents of the operators A and B is compact. Hence the negative spectrum of B is discrete, bounded from below, and can accumulate only to the point zero. Let $\{\lambda_n\}$ be the non-decreasing sequence of the negative eigenvalues of B, the multiplicities of the entrees equal to the multiplicities of the respective eigenvalues. Then the following assertion is a direct consequence of (4.3), (4.6), (4.11), (4.12), and Theorem 3.4.

Theorem 4.4. *Suppose that Conditions* 4.1, 4.2, (4.8), *and* (4.9) *hold and the operator* B *is defined by the quadratic form* (4.13). *Let* $\varepsilon > 1$ *and* $1 \leqslant \gamma < \max\{\varepsilon, 3\}$. *We set* $p = (d-1)(\varepsilon-1)^{-1}$. *Then the following asymptotic formulas hold*:

$$\lim_{n\to\infty} n|\lambda_n|^{(1-\gamma^{-1})p} = c^p(\gamma) g(\varepsilon, d), \quad \gamma > 1, \tag{4.14}$$

$$\lim_{n\to\infty} n|\ln|\lambda_n||^{-p} = 2^p g(\varepsilon, d), \quad \gamma = 1, \tag{4.15}$$

where $g(\varepsilon, d)$ *is given in* (4.12) *and*

$$c(\gamma) := \frac{2\pi}{\gamma \sin \pi\gamma^{-1}}, \quad \gamma > 1.$$

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Nonlocal Problems for Quasilinear Parabolic Equations

Gary M. Lieberman

To Academician Olga Aleksandrovna Ladyzhenskaya on the occasion of her birthday

We study a class of quasilinear parabolic equations with nonlocal initial conditions. The initial conditions are a generalization of periodicity with respect to time and include conditions studied by other authors, which can be used to study inverse problems and problems arising in reactor theory.

Introduction

In 1979, Kerefov [**1**] studied some simple (one-dimensional) parabolic boundary value problems with the initial condition replaced by the nonlocal condition

$$u(x,0) = \beta(x)u(x,T)$$

for a positive constant T and a continuous function β with value in the interval $(0,1]$. In 1982, Vabishchevich [**2**] considered the corresponding higher dimensional problem for self-adjoint parabolic equations, assuming that $0 \leqslant \beta \leqslant 1$, as a means for regularizing an inverse heat conduction problem. Two years later, Chabrowski [**3**] then considered boundary value problems for linear second-order parabolic equations with the initial condition replaced by the condition

$$u(x,0) = \sum_{j=1}^{k} \beta^j(x)u(x,t_j) + g(x) \tag{0.1}$$

as a generalization of the special case considered by Kerefov and Vabishchevich. Here, k is a positive integer and each t_j is in $(0,T)$,

$$\sum_{j=1}^{k} |\beta^j| \leqslant 1, \tag{0.2}$$

and the possibility $k=\infty$ is allowed as long as $\inf t_j > 0$. Such problems have applications in reactor theory [**4**] and in inverse problems [**5**].

In 1994, Lin [**6**] studied some semilinear parabolic equations with (0.1) satisfied with (0.2) replaced by the existence of a positive constant μ (satisfying some additional conditions) such that

$$\sum_{j=1}^{k} \|\beta^j\|_\infty \exp(-\mu t_j) < 1; \tag{0.3}$$

no hypothesis on $\inf t_j$ is needed when $k=\infty$. Lin claimed that this condition is weaker than Chabrowski's (which is certainly true if $k=1$ or if all β^j have their maximum at the same point), but it is useful to point out here that this claim is incorrect. For example, let β^1 and β^2 be two nonnegative functions with disjoint supports, each having a maximum of 1. Then (0.2) is satisfied but (0.3) fails for $k=2$ if t_1 and t_2 are sufficiently small.

A more general nonlocal initial condition was introduced by Chabrowski [**7**]. To describe this condition, we assume that ω is a bounded domain in $\mathbb{R}^n$ and we fix a positive constant T. If we write $\Omega = \omega \times (0,T)$, then Chabrowski's condition can be written as

$$u(\cdot,0) = Bu + g \quad \text{on } \omega \tag{0.4}$$

for some linear mapping $B\colon C(\overline{\Omega}) \to C(\overline{\omega})$ satisfying a suitable analog of (0.2), which we describe in detail below. (Chabrowski made some other assumptions on B which we also describe below.) Note that this condition includes (0.1) by defining

$$Bu(x) = \sum \beta^j(x)u(x,t_j).$$

In particular, the classical initial condition $u(\cdot,0) = g$ on ω is included if we take B to be the zero operator.

Our goal here is to prove existence and regularity of solutions to a large class of parabolic equations with Dirichlet boundary conditions on $S\Omega = \partial\omega \times (0,T)$

and the nonlocal initial condition (0.4). Thus (following the notation in [**8**]), we assume that functions a^{ij}, a, and φ are given, we define the operator P by

$$Pu = -u_t + a^{ij}(X, u, Du)D_{ij}u + a(X, u, Du), \tag{0.5}$$

and we study the problem

$$Pu = 0 \text{ in } \Omega, \quad u = \varphi \text{ on } S\Omega, \quad u(\cdot, 0) = Bu + g \text{ in } \omega. \tag{0.6}$$

The introduction of a general linear mapping B will allow us to state our hypotheses simply but the obvious applications are to the special case (0.1). Even for this condition, we improve the results of Kerefov, Vabishchevich, Chabrowski, and Lin in several ways: We consider quasilinear differential equations rather than linear ones (or semilinear in the case of [**6**]) and we make fairly weak hypotheses on the coefficients in the equations; we also weaken the hypothesis $\sum |\beta^j| \leqslant 1$ when considering (0.1); and we show that our solutions are globally smooth (all the previously mentioned works only considered globally continuous solutions). Of course, in order for the solutions to be smooth, we must assume slightly more regularity for the boundary and initial data so our existence results will not be complete generalizations of the results mentioned before but this shortcoming is more than compensated by our other, weaker hypotheses on the data. Our proof is based on the classical approach to the usual initial-boundary value problem described, for example in [**9**] and [**8**], i.e., we prove certain *a priori* estimates for solutions of a family of related problems. Because these references provide a thorough analysis of the combination of differential equation and boundary condition, we focus on the new ingredient, i.e., the nonlocal initial condition, referring to other sources for details with respect to the differential equation and boundary condition.

We begin in Sec. 1 with a brief discussion of some hypotheses on the mapping B and how these hypotheses relate to ones on the functions β^j in (0.1). The basic element of our proof is a comparison principle, which we prove in Sec. 2 along with a number of L^∞-estimates for nondivergence structure equations. Gradient estimates appear in Sec. 3, and the final Hölder gradient estimates, along with a suitable existence theory, appear in Sec. 4. We provide some examples in Sec. 5.

1. Hypotheses for the Mapping B

In order for our solutions to be sufficiently smooth, we shall assume that the mapping B maps smooth functions to smooth functions. Specifically, we assume that B maps $C(\overline{\Omega})$ into $C(\overline{\omega})$ (as already remarked) and that there is a constant $\alpha \in (0,1)$ such that B maps $H_{1+\theta}(\Omega)$ into $H_{1+\theta}(\overline{\omega})$ for any $\theta \in (0, \alpha)$. (We refer the reader to [**8**] for a definition of $H_{1+\theta}(\Omega)$, which is denoted by $H^{1+\theta,(1+\theta)/2}$ in [**9**]. The space $H_{1+\theta}(\omega)$ is the Hölder space $C^{1,\theta}(\overline{\omega})$.) For (0.1), this hypothesis is satisfied if each β^j is in $H_{1+\alpha}(\omega)$ and (in case $k = \infty$) $\sum |\beta^j|_{1+\alpha} < \infty$.

Next, we assume that any extension of the boundary function φ satisfies the prescribed initial condition. In other words,

$$\varphi(\cdot,0) = B\varphi + g \quad \text{on } \partial\omega, \tag{1.1}$$

$$Bw = 0 \quad \text{on } \partial\omega \tag{1.2}$$

for any $w \in C(\overline{\Omega})$ which vanishes on $S\Omega$. The first condition for (0.1) is just

$$\varphi(\cdot,0) = \sum \beta^j \varphi(\cdot,t_j) + g \quad \text{on } S\Omega,$$

while the second condition is always satisfied in this case.

Condition (1.2) is, except for the choice of functions w, the same as condition (ii) of [**7**, Theorem 1], which states that (1.2) holds for all w of the form

$$w(x,t) = \int_\omega G(x,t;z,0)\varphi(z)\,dz$$

or

$$w(x,t) = \int_{\omega\times(0,t)} G(x,t;z,\tau)f(z,\tau)\,dz\,d\tau$$

with $\varphi \in L^2(\omega)$ and f continuous on $\overline{\Omega}$, where G is Green's function corresponding to some second-order linear parabolic operator L, defined by $Lu = -u_t + a^{ij}D_{ij}u + b^iD_iu + cu$, with Hölder coefficients a^{ij}, b^i, and c such that $c \leqslant 0$. Clearly our condition implies Chabrowski's. On the other hand, we can write any continuous function on $\overline{\Omega}$ which vanishes on $S\Omega$ as the uniform limit of a sequence of smooth functions (w_k) with $w_k = 0$ on $S\Omega$. If we set

$$v_k = \int_\omega G(x,t;z,0)w_k(z,0)\,dz,$$

$$u_k = \int_{\omega\times(0,t)} G(x,t;z,\delta)Lw_k(z,\delta)\,dz\,d\delta,$$

we see that $w_k = v_k + u_k$. So, under Chabrowski's hypotheses, (1.2) holds with w_k in place of w. Sending $k \to \infty$, we see that Chabrowski's condition implies ours.

For our comparison principle, it will be convenient to introduce the positive and negative parts of the mapping B. To this end, we write u^+ and u^- for the positive and negative parts of u, i.e., $u^\pm = \max\{\pm u, 0\}$. We then define

$$B^+u(x) = \sup\{Bv(x) : 0 \leqslant v \leqslant u^+\} - \sup\{Bv(x) : 0 \leqslant v \leqslant u^-\},$$
$$B^-u(x) = Bu(x) - B^+u(x).$$

Then $B^\pm$ are linear mappings with $B^+w \geqslant 0 \geqslant B^-w$ if $w \geqslant 0$. We also define $|B|$ by $|B|u = B^+u - B^-u$.

Let us now consider the hypotheses from [7, Lemma 2]. We first remark that, at least in Sec. 1 of [7], the mapping B is not explicitly assumed to be linear; however, Chabrowski uses the additivity condition $B(u+w) = Bu+Bw$ several times without comment, and this condition (along with continuity of B) easily implies that B is linear. Next, using the notation 1 to denote the constant function with value 1, we can write Chabrowski's hypothesis (C_1) as $B1 \leqslant 1$. Then (C_2) states that if $x \in \overline{\omega}$ is a point such that $B1(x) < 1$ and if $u \in C(\overline{\Omega})$ is nonnegative, then $Bu(x) \geqslant 0$, and (C_3) states that if $x \in \overline{\omega}$ is a point such that $B1(x) = 1$ and if $u \in C(\overline{\Omega})$, then there is a point $Y \in \Omega$ such that $Bu(x) \leqslant u(Y)$. If $u \in C(\overline{\Omega})$ is nonnegative and $x \in \overline{\omega}$ is a point such that $B1(x) = 1$, then we note that $v = 1 - u/(\sup u)$ satisfies $v \leqslant 1$, then $Bv(x) \leqslant \sup_\Omega v \leqslant 1 = B1(x)$, so $B(1 - u/\sup u)(x) \leqslant B1(x)$, which implies that $B(-u/\sup u)(x) \leqslant 0$ and hence $Bu(x) \geqslant 0$. It follows that $Bu \geqslant 0$ on $\overline{\omega}$, so $B^+ = B$ and $B^- = 0$. In particular, $B = |B|$ so $|B|1 \leqslant 1$ and, if $0 \leqslant v < 1$ in Ω, then $|B|v < 1$. These conditions will reappear in Corollary 2.2 below.

We point out that the Riesz representation theorem implies that B can be represented by a Borel measure. In other words, as in [7], for each $x \in \omega$, there is a signed measure β^x such that $Bu(x) = \int_\Omega u\, d\beta^x$. For our purposes, this representation will not be particularly useful but the interested reader may recast our results in terms of this measure. Furthermore, Chabrowski assumed this signed measure to have support in a compact subset of $\overline{\omega} \times (0, T]$. We shall not need this assumption. In fact, we can consider operators of the type

$$Bu(x) = \int_\omega u(y, 0) b(x, y)\, dy \tag{1.3}$$

if b is a sufficiently smooth function satisfying some additional hypotheses which we describe later. (Note in particular that the implication $|B|v < 1$ if $0 \leqslant v < 1$ in Ω holds if $\int |b(x, y)|\, dy < 1$ for all $x \in \overline{\omega}$. In addition, conditions (1.1) and (1.2) hold if $\varphi(x, 0) = g(x)$ and $b(x, y) = 0$ for $x \in \partial\omega$ and $y \in \omega$.)

2. Comparison and Maximum Principles

Our first step is a general comparison principle for linear problems. (Recall that $B\Omega = \omega \times \{0\}$). We say that a function w satisfies MP on Ω if either w is nonpositive or w has a positive maximum on $B\Omega$. For example, if the nondivergence operator L is defined by

$$Lw = -w_t + a^{ij} D_{ij} w + b^i D_i w + cw$$

with $c \leqslant 0$, and if $Lw \geqslant 0$ in Ω and $w \leqslant 0$ on $S\Omega$, then w satisfies MP. In addition, if the divergence operator L is defined by

$$Lw = -w_t + D_i(a^{ij} D_j w + b^i D^i w) + c^i D_i w + dw$$

with $D_i b^i + d \leqslant 0$ in the weak sense and if $Lw \geqslant 0$ in Ω and $w \leqslant 0$ on $S\Omega$, then w satisfies MP.

Lemma 2.1. *Let u satisfy* (0.4). *Suppose that there are functions w_1 and w_2 such that $w_1 - u$ and $u - w_2$ satisfy MP. Suppose also that*

$$g \geqslant -B^- w_2 - B^+ w_1 + w_1(\cdot, 0) \tag{2.1a}$$

$$g \leqslant -B^- w_1 - B^+ w_2 + w_2(\cdot, 0) \tag{2.1b}$$

for all $x \in \omega$. Suppose finally that, for each $x \in \omega$,

$$|B|1(x) < 1. \tag{2.2}$$

Then $w_1 \leqslant u \leqslant w_2$ in Ω.

Proof. Set

$$M_1 = \inf_\Omega (u - w_1), \qquad M_2 = \sup_\Omega (u - w_2).$$

We wish to show that $M_1 \geqslant 0 \geqslant M_2$, and we argue by contradiction.

If $M_1 < 0$, then, by MP, there is a point $y \in \omega$ such that $(u - w_1)(y, 0) = M_1$. Because

$$\begin{aligned} M_1 &= Bu(y) - w_1(y, 0) + g(y) \\ &= B^-(u - w_2)(y) + B^+(u - w_1)(y) + B^- w_2(y) + B^+ w_1(y) - w_1(y, 0) + g(y), \end{aligned}$$

it follows from (2.1a) that

$$M_1 \geqslant a_1 M_1 - b_1 M_2 \tag{2.3}$$

for nonnegative constants a_1 and b_1 satisfying $a_1 + b_1 < 1$, specifically, $a_1 = B^+1(y)$ and $b_1 = -B^-1(y)$. Similarly, if $M_2 > 0$, then there are nonnegative constants a_2 and b_2 satisfying $a_2 + b_2 < 1$ such that

$$M_2 \leqslant a_2 M_2 - b_2 M_1. \tag{2.4}$$

We now consider three cases. If $M_1 < 0$ and $M_2 \leqslant 0$, then (2.3) implies that $M_1 \geqslant a_1 M_1$, which cannot happen. Similarly, if $M_1 \geqslant 0$ and $M_2 > 0$, then (2.4) yields the contradiction. If $M_2 > 0 > M_1$, then we use (2.3) and (2.4) to see that $(1 - a_1)M_1 \geqslant -b_1 M_2$ and $(1 - a_2)M_2 \leqslant -b_2 M_1$, so

$$M_1 \geqslant \frac{-b_1}{1 - a_1} M_2 \geqslant \frac{-b_1}{1 - a_1} \frac{-b_2}{1 - a_2} M_1 = a^* M_1$$

with $a^* \in [0, 1)$, which cannot happen. □

Conditions (2.1) and (2.2) for (0.1) are just

$$g(x) \geqslant \sum_{j=1}^{k} \beta_-^j(x) w_2(x,t_j) - \sum_{j=1}^{k} \beta_+^j(x) w_1(x,t_j) + w_1(x,0),$$

$$g(x) \leqslant \sum_{j=1}^{k} \beta_-^j(x) w_1(x,t_j) - \sum_{j=1}^{k} \beta_+^j(x) w_2(x,t_j) + w_2(x,0),$$

$$\sum_{j=1}^{k} |\beta^j(x)| < 1$$

for all $x \in \omega$.

We can relax (2.2) via the strong maximum principle. We say that a function w satisfies SMP if w is constant or if w does not attain a positive maximum value on $\overline{\Omega} \setminus B\Omega$. The conditions listed above for MP also imply that w satisfies SMP.

Corollary 2.2. *Let u satisfy* (0.4). *Suppose that there are functions w_1 and w_2 such that $w_1 - u$ and $u - w_2$ satisfy SMP. Suppose also that conditions* (2.1a,b) *hold and that*

$$0 \leqslant v < 1 \quad \text{in } \Omega \quad \text{implies} \quad |B|v < 1 \quad \text{on } B\Omega. \tag{2.5}$$

Then $w_1 \leqslant u \leqslant w_2$ in Ω.

PROOF. We follow the proof of Lemma 2.1. In this case, (2.3) holds with $a_1 = B^+v(y)$ and $b_1 = -B^-v(y)$ for

$$v = \max\left\{\frac{(w_1-u)^+}{M_1}, \quad \frac{(u-w_2)^+}{M_2}\right\},$$

so a_1 and b_1 are nonnegative and $a_1 + b_1 < 1$. Similarly (2.4) holds with nonnegative a_2 and b_2 satisfying $a_2 + b_2 < 1$. From the analysis of Lemma 2.1, we see that $M_1 \geqslant 0 \geqslant M_2$ as required. □

Note that (2.5) holds for (0.1) if $\sum_{j=1}^{k} |\beta^j(x)| \leqslant 1$ for all x.

The hypothesis (2.2) can also be relaxed if we invoke the parabolic analog of the generalized maximum principle (Theorem 10 of Chap. 2 from [**10**]) in an appropriate manner. We illustrate this observation with an *a priori* estimate which improves that in [**6**]. In this regard, for any $\mu \in \mathbb{R}$, we define the function e_μ by $e_\mu(X) = e^{\mu t}$ and we define the operator B_μ by $B_\mu u = B(e_\mu u)$.

Corollary 2.3. *Suppose that L is of divergence form with bounded, time-independent coefficients. Suppose also that the minimum eigenvalue of the matrix $[a^{ij}]$ is bounded away from zero. Let μ_0 be the first eigenvalue of the* *operator* L_0 *on* $W_0^{1,2}(\omega)$, *defined by*

$$L_0 u = D_i(a^{ij} D_j u + b^i u) + c^i D_i u + du, \tag{2.6}$$

let $\mu_0 > 0$, *and let* $\mu \in (0, \mu_0)$. *Suppose that* u *satisfies* $Lu = 0$ *in* Ω, $u = 0$ *on* $S\Omega$, *and* (0.4). *Suppose finally that, for each* $x \in \omega$ *and any function* v *which is time-independent, we have*

$$|B_{-\mu}| v(x) < v(x). \tag{2.7}$$

Then there is a constant C *determined only by* μ *and the coefficients of* L *such that* $|u| \leqslant C \sup |g|$ *in* Ω.

PROOF. First note from the implication (a) $\Rightarrow$ (b) of [**11**, Theorem 1] that the solution of $L_0 v + \mu v = 0$ in ω, $v = 1$ on $\partial\omega$ is nonnegative, and then [**12**, Corollary 8.1] implies that v is positive in $\overline{\omega}$.

We now set $\overline{L} = L + \mu$, $\overline{u} = \exp(\mu t) u$. Then $\overline{L}\overline{u} = 0$ and $\overline{L} v \leqslant 0$ in Ω, and $\overline{u}(\cdot, 0) = B_{-\mu}\overline{u} + g$ in $\overline{\omega}$. Because v is positive, there are constants $G_2 \leqslant 0 \leqslant G_1$ such that $G_2 v \leqslant g \leqslant G_1 v$ in ω.

Next, we set

$$\overline{c}^i = c^i + b^i + \frac{1}{v} a^{ij} D_j v + \frac{1}{v} a^{ji} D_j v$$

and we define the operator L^* by

$$L^* w = -w_t + D_i(a^{ij} D_j w) + \overline{c}^i D_i w$$

We now show that $\overline{u} - w_1$ and $w_2 - \overline{u}$ satisfy a suitable variant of MP for $w_i = G_i v$. To this end, we set $\overline{w}_1 = (\overline{u} - w_1)/v$ and $m_1 = \inf_{B\Omega} \overline{w}_1$. Since u and v are in L^∞, we can use $w = (\overline{w}_1 - m_1)_+$ as test function in the weak form of the inequality $L^* \overline{w}_1 \geqslant 0$ to see that

$$\int_{\omega \times \{\tau\}} w^2\, dx + \int_{\omega \times (0,\tau)} |Dw|^2\, dX \leqslant C \int_{\omega \times (0,\tau)} w(1 + |Dv|)|Dw|\, dX$$

and then Cauchy's inequality gives

$$\int_{\omega \times \{\tau\}} w^2\, dx \leqslant C \int_{\omega \times (0,\tau)} w^2\, dX + C \sup_{\omega \times (0,\tau)} w^2.$$

Next, we note that $\overline{L}([\overline{u} - (m_1 + G_1) v]_+) \geqslant 0$, so

$$\begin{aligned} \sup_{\omega \times (0,\tau)} w^2 &\leqslant C \sup_{\omega \times (0,\tau)} ([\overline{u} - (m_1 + G_1) v]_+)^2 \\ &\leqslant C \int_{\omega \times (0,\tau)} ([\overline{u} - (m_1 + G_1) v]_+)^2\, dX \leqslant C \int_{\omega \times (0,\tau)} w^2\, dX, \end{aligned}$$

and therefore

$$\int_{\omega\times\{\tau\}} w^2\,dx \leqslant C \int_{\omega\times(0,\tau)} w^2\,dX.$$

Gronwall's inequality then implies that $w = 0$ so $\overline{w}_1 \geqslant m_1$. A completely similar argument shows that $\overline{w}_2 \leqslant m_2 = \sup_{B\Omega} \overline{w}_2$.

Finally, we abbreviate $M_1 = \inf(\overline{u} - w_1)$ and $M_2 = \sup(w_2 - u)$. As in the proof of Lemma 2.1, let us suppose that $M_1 < 0$. Then there is a point $y \in \omega$ such that $\overline{u}(y,0) - G_1 v(y) = M_1 v(y)$. Since $B^+_{-\mu} v \geqslant 0$ and $B^-_{-\mu} v \leqslant 0$, we have

$$\begin{aligned} M_1 v(y) &\geqslant -B^-_{-\mu}(u - G_2 v)(y) - B^+_{-\mu}(u - G_1 v)(y) \\ &\geqslant -B^-_{-\mu}(M_2 v)(y) - B^+_{-\mu}(M_1 v)(y) \end{aligned}$$

and hence there are nonnegative constants a_1, a_2, b_1, and b_2 satisfying $a_1+b_1 < 1$ and $a_2+b_2 < 1$ such that $M_1 \geqslant a_1 M_1 - b_1 M_2$ if $M_1 < 0$ and $M_2 \leqslant a_2 M_2 - b_2 M_2$ if $M_2 > 0$. It follows that $M_1 \geqslant 0$ and $M_2 \leqslant 0$, so $G_2 v \leqslant u \leqslant G_1 v$, which easily gives the desired estimate. □

Note that the inequality (2.7) only needs to hold for the particular function v described in the proof. On the other hand, for (0.1), this inequality is clear if $\sum_{j=1}^{k} \beta^j(x)\exp(-\mu t_j) < 1$ for all $x \in \omega$.

A corresponding statement is true for equations in nondivergence form as we see by invoking [**13**, Theorem 2] in place of [**11**, Theorem 1] and [**12**, Corollary 8.1].

We also point out a useful comparison theorem for quasilinear equations.

Corollary 2.4. *Suppose that a^{ij} is independent of z and a is nonincreasing with respect to z. Let u, w_1, and w_2 be functions in $C^{2,1}(\Omega) \cap C^0(\overline{\Omega})$ such that $Pw_1 \geqslant Pu \geqslant Pw_2$ in Ω, $w_1 \leqslant u \leqslant w_2$ on $S\Omega$. Suppose also that u satisfies condition (0.4), that conditions (2.1) and (2.5) are satisfied for all $x \in \omega$. If $Pw_1 > Pu > Pw_2$ or if a^{ij} and b^i are locally Lipschitz with respect to p, then $w_1 \leqslant u \leqslant w_2$ in Ω.*

PROOF. The argument in [**14**, Theorem 10.1] (cf. also [**8**, Theorem 9.1]) shows that $w_1 - u$ and $u - w_2$ satisfy SMP and then we use Corollary 2.2. □

We also note some *a priori* estimates for quasilinear equations. The first one is quite simple.

Lemma 2.5. *Let P be defined by (0.5). Suppose that there are constants A_0 and M_0 such that*

$$a(X,z,p)\,\mathrm{sgn}\,z \leqslant A_0 \tag{2.8}$$

for $|z| \geqslant M_0$. *Suppose that* $Pu = 0$ *in* Ω *and* u *satisfies* (0.4). *Suppose also that there are constants* $\mu > 0$ *and* $\rho \in (0,1)$ *such that* $|B_\mu|1 \leqslant \rho$. *Then*

$$\sup_\Omega |u| \leqslant \max\left\{\frac{A_0}{\mu}, \frac{\sup_\omega |g|}{1-\rho}, \sup_{S\Omega} |u|\right\} e^{\mu T} + M_0. \tag{2.9}$$

PROOF. Let A be a constant such that

$$A > \max\left\{\frac{A_0}{\mu}, \frac{\sup_\omega |g|}{1-\rho}, \sup_{S\Omega} |u|\right\}.$$

Define $v_i = (-1)^i Ae^{\mu t}$. If we define $\overline{P}$ by

$$\overline{P}w = -w_t + a^{ij}(X, u, Dw)D_{ij}w + a(X, u, Dw),$$

it is easy to see that $\overline{P}v_1 > \overline{P}u$ where $u \leqslant -M_0$ and $\overline{P}v_2 < \overline{P}u$ where $u \geqslant M_0$. Setting $w_i = (-1)^i \max\{Ae^{\mu t}, M_0\}$, we infer that $w_1 - u$ and $u - w_2$ satisfy MP so Lemma 2.1 yields $w_1 \leqslant u \leqslant w_2$, which implies (2.9). □

We consider some estimates in terms of the trace $\mathcal{T}$ of the matrix $[a^{ij}]$.

Lemma 2.6. *Define* P *by* (0.5). *Suppose that* u *satisfies* $Pu = 0$ *in* Ω *and* (0.4) *in* ω. *Suppose also that there are positive constants* $\rho < 1$, M_0, *and* L *such that*

$$|B|1 \leqslant \rho, \tag{2.10}$$

$|u| \leqslant M_0$ *on* $S\Omega$, *and*

$$a(X, z, p) \operatorname{sgn} z \leqslant \frac{|p|\mathcal{T}(X, z, p)}{R} \tag{2.11}$$

whenever $|z| \geqslant M_0$ *and* $|p| \geqslant L$, *where* $R = \operatorname{diam}\omega$. *Then*

$$\sup_\Omega |u| \leqslant \max\left\{M_0, \frac{|g|_0}{1-\rho}\right\} + 2RL. \tag{2.12}$$

PROOF. The argument in [**15**, Theorem 3] shows that there is a function $v\colon \omega \to [0, 2LR]$ such that $w_1 - u$ and $u - w_2$ satisfy MP for $w_i = (-1)^i(v + M)$ provided the constant M is no less than M_0. If also $M \geqslant |g|_0/(1-\rho)$, then (2.1) holds. If we now take $M = \max\{M_0, |g|_0/(1-\rho)\}$, then Lemma 2.1 gives (2.12). □

Note that if $g \equiv 0$ and if we replace (2.10) by (2.5), then this proof yields

$$\sup_\Omega |u| \leqslant M_0 + 2RL.$$

A simple modification of this lemma gives an estimate if $a(X, z, p) \operatorname{sgn} z$ grows with z provided the growth is appropriately controlled.

Lemma 2.7. *Define P by (0.5). Suppose that u satisfies $Pu = 0$ in Ω and (0.4) in ω. Suppose also that there are positive constants $\rho < 1$, $K > 2R$, M_0, and L_0 such that (2.10) holds, $|u| \leqslant M_0$ on $S\Omega$, and*

$$\sup_{M_0 \leqslant |z| \leqslant K|p|} a(X,z,p)\operatorname{sgn} z \leqslant \inf_{|z| \geqslant M_0} \frac{|p|\mathcal{T}(X,z,p)}{R} \tag{2.13}$$

whenever $|p| \geqslant L_0$. Then there is a constant $C(K,R)$ such that

$$\sup_{\Omega} |u| \leqslant C \max\left\{M_0, \frac{|g|_0}{1-\rho}\right\} + 2RL_0. \tag{2.14}$$

Proof. We take $L = \max\{L_0, |u|_0/K\}$ to infer from Lemma 2.6 that

$$\sup_{\Omega} |u| \leqslant \max\left\{M_0, \frac{|g|_0}{1-\rho}\right\} + 2RL.$$

If $L = L_0$, then (2.14) is clear. Otherwise, we have

$$\sup_{\Omega} |u| \leqslant \max\left\{M_0, \frac{|g|_0}{1-\rho}\right\} + \frac{2R}{K} \sup_{\Omega} |u|,$$

which gives (2.14) with $C = K/(K-2R)$. □

This time, if $g \equiv 0$ and (2.5) replaces (2.10), we infer that

$$\sup_{\Omega} |u| \leqslant CM_0 + 2RL.$$

From the preceding lemma, we can infer an estimate even if $|B|1(x) \geqslant 1$ for some x but under slightly stronger conditions on the coefficients of P.

Lemma 2.8. *Define P by (0.5). Suppose that u satisfies $Pu = 0$ in Ω and (0.4) in $\overline{\omega}$. Suppose also that there are positive constants $\rho < 1$ and μ such that*

$$|B_{-\mu}|1 \leqslant \rho. \tag{2.15}$$

Suppose further that there are positive constants $\varepsilon < 1$, M_0, K, and L_0 such that $|u| \leqslant M_0$ on $S\Omega$, $K > 2R$, and

$$a(X, e^{-\mu t}z, e^{-\mu t}p)\operatorname{sgn} z \leqslant \varepsilon e^{-\mu t} \frac{|p|\mathcal{T}(X, ze^{-\mu t}, pe^{-\mu t})}{R}, \tag{2.16a}$$

$$\mathcal{T}(X, ze^{-\mu t}, pe^{-\mu t}) \geqslant \frac{\mu K R}{1-\varepsilon} \tag{2.16b}$$

whenever $|z| \geqslant M_0$ and $|p| \geqslant L_0$. Then there is a constant $C(K,R)$ such that

$$\sup_{\Omega} |u| \leqslant C \max\{e^{\mu T} M_0, \frac{|g|_0}{1-\rho}\} + 2RL_0. \tag{2.17}$$

PROOF. Define $\overline{P}$ by $\overline{P}w = -w_t + \overline{a}^{ij}(X, w, Dw)D_{ij}w + \overline{a}(X, w, Dw)$ with

$$\overline{a}^{ij}(X,z,p) = a^{ij}(X, e^{-\mu t}z, e^{-\mu t}p), \quad \overline{a}(X,z,p) = e^{\mu t}a(X, e^{-\mu t}z, e^{-\mu t}p) + \mu z.$$

Then $\overline{u} = ue^{\mu t}$ satisfies the equation $\overline{P}\overline{u} = 0$ in Ω and the condition $\overline{u}(\cdot, 0) = B_{-\mu}\overline{u} + g(x)$ in ω, so the hypotheses of Lemma 2.7 hold for $\overline{u}$ (with $e^{\mu T}M_0$ in place of M_0 and $\overline{P}$ in place of P). The conclusion of that lemma along with the easy estimate $|u| \leqslant |\overline{u}|$ in Ω gives (2.17). □

Note that (2.16) follows from the more readily verified conditions

$$a(X,z,p)\operatorname{sgn} z \leqslant \varepsilon\frac{|p|\mathcal{T}(X,z,p)}{R}, \quad \mathcal{T}(X,z,p) \geqslant \frac{\mu K R}{1-\varepsilon}$$

for $|z| \geqslant e^{\mu T}M_0$ and $|p| \geqslant e^{\mu T}L_0$.

3. Gradient Estimates

The considerations of the previous section also apply to the usual methods for proving gradient estimates for solutions. We shall give the details for some simple estimates and refer the reader to the appropriate results in [**8**] for the other estimates.

First, we prove a boundary gradient estimate for equations which are close to being uniformly parabolic; the class of equations we consider is essentially the same as the class of regularly parabolic equations studied by Edmunds and Peletier [**16**]. To this end, we write $d(x)$ for the distance from $x \in \omega$ to $\partial\omega$. We also define the seminorm

$$[\psi]_1' = \sup\{|\psi(x,t) - \psi(y,t)| : x \in \omega,\ y \in \partial\omega,\ 0 < t < T\}.$$

Lemma 3.1. *Let α and ζ be constants with $0 < \alpha \leqslant \zeta \leqslant 1$. Let $u \in C^{2,1}(\Omega) \cap C(\overline{\Omega})$ be a solution of $Pu = 0$ in Ω, (0.4) in ω, and $u = \varphi$ on $S\Omega$ with $\varphi \in H_{1+\alpha}$ satisfying the compatibility condition (1.1) and $\partial\omega \in H_{1+\zeta}$. Suppose that there are positive constants μ and p_0 such that*

$$1 + |p|^{2-\zeta}\Lambda(X, u(X), p) + |a(X, u(X), p)| \leqslant \mu\mathcal{E}(X, u(X), p) \tag{3.1}$$

for all $X \in \Omega$ and all $p \in \mathbb{R}^n$ with $|p| \geqslant p_0$. Suppose finally that there are nonnegative constants G, R, Φ, $\kappa \geqslant 1$ and $\rho < 1$ such that

$$|B|\psi(x) \leqslant \frac{\rho}{\kappa}\sup\{\psi(z,t) : d(z) \leqslant \kappa d(x), 0 < t < T\}, \tag{3.2a}$$

$$[B\varphi]_1' \leqslant \Phi, \tag{3.2b}$$

$$[g]_1' \leqslant G \tag{3.2c}$$

for all nonnegative functions $\psi \in C(\overline{\Omega})$ *with* $\psi = 0$ *on* $S\Omega$. *Then there is a constant* C, *determined only by* μ, ρ, R, p_0, G, H, Φ, ζ, α, $\sup|u|$, *and* $|\varphi|_{1+\alpha}$ *such that*

$$\sup_{X\in\Omega} \frac{|u(X)-u(Y)|}{|X-Y|} \leqslant C \tag{3.3}$$

for all $Y = (y, t_0)$ *with* $0 \leqslant t_0 \leqslant T$.

PROOF. As in the proof of [**8**, Lemma 10.4], there is a regularized distance $\overline{d}$ to $\partial\omega$. In particular, we can determine $\overline{d}$ so that $|D^2\overline{d}| \leqslant Cd^{\zeta-2}$, and $(1+\rho)/(2\rho) \geqslant \overline{d}/d \geqslant 1$. We also note that there is a function $\overline{\varphi}$ which is continuous in $\overline{\Omega} \times [0,1]$ and C^∞ in $\overline{\Omega} \times ((0,1) \cup (-1,0))$ such that $\overline{\varphi}(Z,0) = \varphi(Z)$ for all $Z \in S\Omega$. In addition, if we write the argument of $\overline{\varphi}$ as (X, x^0) and use subscripts to denote differentiation with respect to x^i, $i = 0, \ldots, n$, we see that $|\overline{\varphi}_i(X, x^0)| \leqslant C$ for $i = 0, \ldots, n$. We shall write Φ_0 for the supremum of $\overline{\varphi}_0$. Moreover,

$$|\overline{\varphi}_{ij}(X, x^0)| + |\overline{\varphi}_t(X, x^0)| \leqslant C(x^0)^{\alpha-2}.$$

Now, for an increasing, concave C^2 function f and a positive constant F to be chosen, let

$$w_1 = \overline{\varphi}\left(X, -\frac{f(\overline{d})}{F}\right) - f(\overline{d}), \quad w_2 = \overline{\varphi}\left(X, \frac{f(\overline{d})}{F}\right) + f(\overline{d}).$$

As was shown in [**8**, Sec. 10.1], for all sufficiently large F, there is a constant $R_F \leqslant R$ so that $w_1 - u$ and $u - w_2$ satisfy MP in $\{\overline{d} < R_F\}$ and $f(R_F) \geqslant 2\sup|u-\varphi|$. Since f is increasing, it follows that $w_1 - u$ and $u - w_2$ satisfy MP in Ω if $F \geqslant 2\Phi_0$. Hence we only need to verify conditions (2.1a,b). To this end, we note that

$$-B^- w_2(x) - B^+ w_1(x) = -B\varphi(x) - B^-(w_2 - \varphi)(x) - B^+(w_1 - \varphi)(x).$$

Then

$$w_2 - \varphi = \overline{\varphi}\left(X, \frac{f(\overline{d})}{F}\right) + f(\overline{d}) - \varphi \leqslant \left(1 + \frac{\Phi_0}{F}\right) f(\overline{d}),$$

$$w_1 - \varphi = \overline{\varphi}\left(X, -\frac{f(\overline{d})}{F}\right) - f(\overline{d}) - \varphi \geqslant -\left(1 + \frac{\Phi_0}{F}\right) f(\overline{d}).$$

It follows that

$$-B^-(w_2 - \varphi)(x) - B^+(w_1 - \varphi)(x) \leqslant \left(1 + \frac{\Phi_0}{F}\right) B[f(\overline{d})](x).$$

In addition, $f'' \leqslant 0$ and $f' \geqslant 0$, so $d(z) < \kappa d(x)$ implies that

$$f(\overline{d}(z)) \leqslant \frac{1+\rho}{2\rho}\, \kappa f(\overline{d}(x))$$

and hence

$$-B^- w_2(x) - B^+ w_1(x) \leqslant -B\varphi(x) + \left(1 + \frac{\Phi_0}{F}\right) \frac{1+\rho}{2} f(\bar{d}(x)).$$

Moreover,

$$\begin{aligned} &-B\varphi(x) + w_1(x,0) \\ &\quad = B\varphi(y) - B\varphi(x)c + \overline{\varphi}\left((x,0), \frac{f(\bar{d}(x))}{F}\right) - \varphi(y,0) + g(y) - f(\bar{d}(x)) \\ &\quad \leqslant C\bar{d}(x) + g(x) + \left(-1 + \frac{\Phi_0}{F}\right) f(\bar{d}(x)). \end{aligned}$$

Therefore, we have

$$\begin{aligned} &-B^- w_2(x) - B^+ w_1(x) + w_1(x,0) \\ &\qquad \leqslant g(x) + C\bar{d}(x) + f(\bar{d}(x)) \left[-1 + \frac{\Phi_0}{F} + \left(1 + \frac{\Phi_0}{F}\right) \frac{1+\rho}{2}\right]. \end{aligned}$$

By taking F sufficiently large (determined only by Φ_0 and ρ), we obtain

$$-1 + \frac{\Phi_0}{F} + \left(1 + \frac{\Phi_0}{F}\right) \frac{1+\rho}{2} \leqslant \frac{\rho - 1}{3}.$$

With F now fixed, the form of f given in [**8**] shows that we can arrange

$$C\bar{d} + \frac{\rho - 1}{3} f(\bar{d}) \leqslant 0$$

and thus we infer (2.1a). The proof of (2.1b) is entirely similar. Lemma 2.1 shows that $w_1 \leqslant u \leqslant w_2$, and we are done. □

In the special case of condition (0.1), hypothesis (3.2a) reduces to

$$\sum_{j=1}^{k} |\beta^j| \leqslant \rho, \quad \sum_{j=1}^{k} |\beta^j(x) - \beta^j(y)| \leqslant \rho_1 |x - y| \tag{3.4}$$

for all $x \in \omega$ and $y \in \partial\omega$ with constants $\rho \in [0,1)$ and $\rho_1 \geqslant 0$.

It follows from [**8**, Theorem 10.4] that condition (3.1) provides a local boundary gradient estimate and hence we can modify conditions (3.2) slightly so that the small multiplier for the supremum in (3.2a) only multiplies the supremum for t close to zero. Such a condition is certainly true when condition (0.1) is in force and (3.4) holds with arbitrary nonnegative constants ρ and ρ_1.

Corollary 3.2. *Suppose that all the hypotheses of Lemma 3.1 hold except that* (3.2a) *is only true for functions which vanish for t greater than some constant $T^* \in (0,T)$. Suppose also that there is a $\rho_1 \geqslant 0$ such that*

$$[B\psi]_1' \leqslant \rho_1 [\psi]_1' \tag{3.5}$$

for all $\psi \in C(\overline{\Omega})$. *Then there is a constant* C *determined by* μ, ρ, R, p_0, G, H, Φ, ζ, α, $\sup|u|$, $|\varphi|_{1+\alpha}$, ρ_1, *and* T^* *such that* (3.3) *holds.*

PROOF. We note from [**8**, Theorem 10.4] that there is a constant C such that (3.3) holds for $t_0 \geqslant T^*/2$. Next, we let k be an increasing, Lipschitz function on $[0,T]$ which is one on $[0,T^*/2]$ and zero on $[T^*,T]$, and we define the operator B^* by $B^*w = B[kw]$. It follows that u satisfies the condition

$$u(x,0) = B^*u(x) + g^*(x),$$

with $g^*(x) = B[(1-k)u](x) + g(x)$ satisfying condition (3.2c) with a known constant. Moreover, we know that φ satisfies the condition $\varphi(\cdot,0) = B^*\varphi + g^*$ on $\partial\omega$ by virtue of condition (1.2). The argument in Lemma 3.1 then gives the desired result. □

We can replace condition (3.1) by any of the hypotheses from [**8**, Chap. 10]. We shall quote the appropriate results in Sec. 5.

For global gradient estimates, we use the method of Serrin [**17**], as described in [**8**, Sec. 11.1]. Suppose that a^{ij} can be decomposed as

$$a^{ij}(X,z,p) = a_*^{ij}(X,z,p) + \frac{1}{2}[p_i f_j(X,z,p) + p_j f_i(X,z,p)]$$

for functions a_*^{ij} and f_i which are differentiable with respect to x, z, p and that the matrix $[a_*^{ij}]$ is positive-definite with minimum eigenvalue λ_*. Then define the operators δ and $\overline{\delta}$ by

$$\delta h(X,z,p) = h_z(X,z,p) + \frac{p}{|p|^2}\cdot h_x(X,z,p),$$
$$\overline{\delta} h(X,z,p) = p \cdot h_p(X,z,p),$$

set $\mathcal{E}(X,z,p) = a^{ij}(X,z,p)p_ip_j$, and suppose that the numbers

$$A_\infty = \limsup_{|p|\to\infty} \sup_{X\in\Omega, z\in[m,M]} \frac{1}{\mathcal{E}}\left(\frac{|p|^2}{2\lambda_*}\sum_{i,j=1}^{n}(\overline{\delta}a_*^{ij})^2 + (\overline{\delta}-1)\mathcal{E}\right),$$

$$B_\infty = \limsup_{|p|\to\infty} \sup_{X\in\Omega, z\in[m,M]} \frac{1}{\mathcal{E}}(\delta\mathcal{E} + (\overline{\delta}-1)a),$$

$$C_\infty = \limsup_{|p|\to\infty} \sup_{X\in\Omega, z\in[m,M]} \frac{1}{\mathcal{E}}\left(\frac{|p|^2}{2\lambda_*}\sum_{i,j=1}^{n}(\delta a_*^{ij})^2 + \delta a\right)$$

(where $m = \inf u$ and $M = \sup u$) are finite. If

$$\min\{A_\infty, C_\infty, B_\infty + 2|A_\infty C_\infty|^2\} \leqslant 0, \tag{3.6}$$

then there is a smooth function $\psi : [m,M] \to \mathbb{R}$ such that $\overline{v} = \psi'(u)^2|Du|^2$ satisfies a relation of the form

$$-\overline{v}_t + a^{ij}D_{ij}\overline{v} + b^iD_i\overline{v} + c\overline{v} \geqslant 0$$

with b^i locally bounded and $c \leqslant 0$, on the set $|Du| \geqslant L$ for some constant L determined by the limit behavior for A_∞, B_∞, and C_∞. Moreover, ψ' is bounded away from zero on $[m, M]$. The function ψ is completely determined by the numbers A_∞, B_∞, C_∞, and $\operatorname{osc} u$. A particularly useful quantity is

$$K_0 = \sup \psi' / \inf \psi'.$$

We then have the following basic gradient estimate, using DBw as an abbreviation for the gradient of Bw.

Lemma 3.3. *Let $u \in C^{2,1}(\Omega)$ with $Du \in C(\overline{\Omega})$, and let $Pu = 0$ in Ω. Suppose that P is parabolic at u and that the quantities A_∞, B_∞, and C_∞ defined above are finite . Suppose also that condition* (3.6) *holds and that there are nonnegative constants A_1, ρ_1, and g_1 such that*

$$|DBw| \leqslant A_1 \sup_\Omega |w| + \rho_1 \sup_\Omega |Dw| \tag{3.7}$$

for all $w \in C(\overline{\Omega})$ with $Dw \in L^\infty(\Omega)$ and

$$|Dg| \leqslant g_1. \tag{3.8}$$

If $K_0\rho_1 < 1$, then there is a positive constant c_1, determined only by A_∞, B_∞, C_∞, $\operatorname{osc} u$, A_1, ρ_1, g_1, and $\sup_{\mathcal{P}\Omega} |Du|$, and the limit behavior described above, such that $|Du| \leqslant c_1 v$ in Ω.

PROOF. If $\overline{v}$ takes on its maximum M_1 in Ω or on $S\Omega$, then the result is clear from the maximum principle for $\overline{v}$.

If $\overline{v}$ takes on its maximum at some point $(x, 0) \in B\Omega$, then

$$\overline{v}(x,0) = \left(\psi'(u(x,0))|DBu(x) + Dg(x)|\right)^2 \leqslant \left(K_0\rho_1 \sup[\psi'|Du|] + C\right)^2$$

and hence

$$M_1 \leqslant (K_0\rho_1(M_1)^{1/2} + C)^2,$$

which provides an upper bound for $\overline{v}$ because $K_0\rho_1 < 1$. □

It is useful to observe two situations in which the smallness condition on ρ_1 can be relaxed. First, we note that c in the differential equation for $\overline{v}$ satisfies $\limsup c/\mathcal{E} < 0$.

Corollary 3.4. *Suppose that u and P are as in Lemma 3.3 and suppose there are nonnegative constants A_1, ρ_1, g_1, and μ such that* (3.8) *holds and*

$$|DB_{-\mu} w| \leqslant A_1 \sup |w| + \rho_1 \sup |Dw| \tag{3.9}$$

for all $w \in c(\overline{\Omega})$ with $Dw \in L^\infty(\Omega)$. If $2K_0\rho_1 < 1$ and if there are constants L_0 and $\varepsilon_0 > 0$ such that $\mathcal{E} \geqslant 2\mu/\varepsilon_0$ and $c \leqslant -\varepsilon_0\mathcal{E}$ for $|p| \geqslant L_0$, then there is a constant c_2, determined by the same quantities as for c_1, and also by μ and L_0, such that $|Du| \leqslant c_2$ in Ω.

PROOF. The function $w = e^{2\mu t}\overline{v}$ satisfies the differential inequality

$$-w_t + a^{ij}D_{ij}w + b^i D_i w + (c+2\mu)w \geqslant 0,$$

and the coefficient $c+2\mu$ is nonpositive for $|Du| \geqslant L_0$. In addition, if we define u_μ by $u_\mu(x,t) = e^{\mu t}u(x,t)$, we have

$$w(x,0) = (\psi'(u(x,0))|DB_{-\mu}u_\mu(x) + Dg(x)|)^2 \leqslant (K_0\rho_1 \sup[\psi'|Du_\mu|] + C)^2 .$$

From this inequality and the maximum principle for w, we infer the desired estimate as before. □

If a local gradient bound is known for the equation $Pu = 0$, then the argument of Corollary 3.2 yields a global gradient bound provided condition (3.7) is replaced by

$$|DBw| \leqslant \rho_1 \sup_{\omega\times(0,T^*)} |Dw| + A_1[\sup_{\Omega}|w| + \sup_{\omega\times[T^*,T]} |Dw|]$$

for some $T^* \in [0,T]$. For general operators satisfying the hypotheses of Lemma 3.3, the usual additional hypothesis which implies a local gradient bound is that $\mathcal{E}$ grows faster than some positive power of $|p|$ (cf., for example, [**8**, Theorem 11.13(a)]), so Corollary 3.4 is the stronger result in this case. On the other hand, for some operators, a local gradient bound can be obtained under weaker hypotheses than those included in this corollary. We mention, in particular, the prescribed mean curvature operator

$$Pu = -u_t + \left(\delta^{ij} - \frac{D_iuD_ju}{1+|Du|^2}\right) D_{ij}u + H(x,u)$$

with $H_z \leqslant 0$. Ecker and Huisken [**18**, Theorem 2.3] proved a local gradient bound for this operator, and it is easy to check that $\mathcal{E}$ remains bounded as $|p| \to \infty$.

We close this section with another situation, complementary to that in Corollary 3.4, which implies a global gradient bound.

Lemma 3.5. *Suppose that there are nonnegative constants A_1, μ, $\rho < 1$, and L such that*

$$\frac{|p|^2}{\lambda_*}\sum_{i,j=1}^{n}(\delta a_*^{ij})^2 + \delta a \leqslant \mu \tag{3.10}$$

for $|p| \geqslant L$ and

$$|DB_\mu w| \leqslant \rho \sup|Dw| + A_1 \sup|w| \tag{3.11}$$

for all $w \in C(\overline{\Omega})$ with $Dw \in L^\infty(\Omega)$. Then

$$|Du| \leqslant C(\mu T, L, \sup_{S\Omega}|Du|, A_1, \rho). \tag{3.12}$$

PROOF. We note that $\overline{v} = \exp(-\mu t)|Du|^2$ satisfies the differential inequality

$$-\overline{v}_t + a^{ij} D_{ij}\overline{v} + b^i D_i\overline{v} + c\overline{v} \geqslant 0$$

with the coefficient $c \leqslant 0$ wherever $|Du| \geqslant L$. We now follow the proof of Lemma 3.3 to infer (3.12). □

4. Hölder Gradient Estimates and Existence Results

The final step in our program of *a priori* estimates is an estimate on the Hölder norm of the gradient. Since this estimate is known for solutions of quasilinear parabolic equations in a local form, we state it with minimal proof.

Lemma 4.1. *Let $\partial\omega \in H_{1+\alpha}$ for some $\alpha \in (0,1)$. Let $u \in C^{2,1}(\Omega)\cap C^0(\overline{\Omega})$ be a solution of $Pu = 0$ in Ω, (0.4) for $x \in \omega$, and $u = \varphi$ on $S\Omega$ with $|u|+|Du| \leqslant K$ for some nonnegative constant K. Suppose that a^{ij} is Lipschitz with respect to x, z, p and that a^{ij} and a are continuous with respect to all their arguments. Specifically, suppose that there are constants $\Lambda_K > 0$ and μ_K such that*

$$a^{ij}(X,z,p)\xi_i\xi_j \geqslant (1/\Lambda_K)|\xi|^2, \tag{4.1a}$$

$$|a^{ij}(X,z,p)| \leqslant \Lambda_K, \tag{4.1b}$$

$$K[|a^{ij}_x(X,z,p)| + |a^{ij}_z(X,z,p)||p|] + |a(x,z,p)| \leqslant \mu_K \tag{4.1c}$$

for $X \in \Omega$ and $|z|+|p| \leqslant K$. Suppose also that there are nonnegative constants G_α and Φ_α such that

$$[g]_{1+\alpha} \leqslant G_\alpha, \quad [\varphi]_{1+\alpha} \leqslant \Phi_\alpha. \tag{4.2}$$

Then there is a constant $\theta \in (0,\alpha]$ (determined only by n, Λ_K, $\sup|a^{ij}_p|K$, and $\mu_K R/K$) such that if, for every $\varepsilon \in (0,1)$, there are constants $E(\varepsilon)$ and $A_1(\varepsilon)$ such that

$$[DBw]_\theta \leqslant \varepsilon[Dw]_{\theta,\omega\times(0,E(\varepsilon))} + \rho[Dw]_{\theta,B\Omega} + A_1(\varepsilon)\left([Dw]_{\theta,\omega\times(E,T)} + |w|_0\right) \tag{4.3}$$

for any $w \in H_{1+\alpha}$, then there is a constant C, determined only by n, α, Λ_K, $\mu_K R/K$, Ω, A_1, ρ and E such that

$$[Du]_\theta \leqslant C(K + \mu_K + \Phi_\alpha + G_\alpha). \tag{4.4}$$

PROOF. We first observe that

$$[Du]_{\theta,B\Omega} \leqslant \varepsilon[Du]_{\theta,\omega\times(0,E(\varepsilon))} + \rho[Du]_{\theta,B\Omega} + A_1(\varepsilon)\left([Du]_{\theta,\omega\times(E,T)} + |u|_0\right) + G_\alpha,$$

so

$$[Du]_{\theta,B\Omega} \leqslant \frac{\varepsilon}{1-\rho}[Du]_{\theta,\omega\times(0,E(\varepsilon))} + \frac{A_1(\varepsilon)}{1-\rho}\left([Du]_{\theta,\omega\times(E,T)} + |u|_0\right) + \frac{G_\alpha}{1-\rho}.$$

Next, the Hölder gradient estimates in Theorem12.3 and Lemmata 12.6 and 12.8 of [**8**] imply that there are constants $\theta \in (0, \alpha]$ (determined only by n, α, and Λ_K) and C_0 (determined also by Ω) such that

$$[Du]_\theta \leqslant C_0 \left(K + \mu_K + \Phi_\alpha + [Du]_{\theta,B\Omega}\right).$$

If we choose $\varepsilon = (1-\rho)/(2+2C_0)$, then we also have that

$$[Du]_{\theta,\omega\times(E(\varepsilon),T)} \leqslant C_1(K + \mu_K + \Phi_\alpha)$$

with C_1 determined also by E. By our estimate on $[Du]_{\theta,B\Omega}$, we have

$$[Du]_\theta \leqslant C_2(K + \mu_K + \Phi_\alpha + G_\alpha) + \frac{1}{2}[Du]_\theta,$$

which implies the asserted estimate. □

Note that the mapping B defined by (1.3) satisfies (4.3) with $\varepsilon = 0$ if $b(\cdot, y) \in H_{1+\alpha}$ with $[b(\cdot, y)]_\alpha$ bounded independent of y.

For our existence program, we first need an existence theorem for linear problems in the correct function space. Some existence theorems appear in [**3, 7, 6**], which are easily extended to our hypotheses but they only provide globally continuous solutions. Here, we give a simple existence result which provides smooth solutions for linear problems directly.

Theorem 4.2. *Let $\partial\omega \in H_{1+\alpha}$ for some $\alpha \in (0,1)$, and let a^{ij}, b^i, c, and f be in H_α with $c \leqslant 0$ and $a^{ij}\xi_i\xi_j \geqslant \lambda|\xi|^2$ for some positive constant λ. Suppose that conditions (4.2) and (1.1) hold. Suppose further that, for any $\varepsilon \in (0,1)$, there are positive constants $E(\varepsilon)$ and $A_1(\varepsilon)$ such that (4.3) holds with $\theta = \alpha$. if there is a constant $\rho \in (0,1)$ such that (2.10) holds, then there is a solution of*

$$-u_t + a^{ij}D_{ij}u + b^iD_iu + cu = f \text{ in } \Omega, \quad u = \varphi \text{ on } S\Omega, \tag{4.5a}$$

$$u(\cdot, 0) = Bu + g \quad \text{on } \omega \tag{4.5b}$$

in $C^{2,1}(\Omega) \cap C^0(\overline{\Omega})$. Moreover, $u \in H_{1+\alpha}(\Omega)$ and there is a constant C determined only by n, α, Ω, λ, $|a^{ij}|_\alpha$, $|b^i|_\alpha$, $|c|_\alpha$, ρ and the functions E and A_1 such that

$$|u|_{1+\alpha} \leqslant C(\Phi_\alpha + G_\alpha + |f|_\alpha). \tag{4.6}$$

Proof. We start by arguing as in [**6**, Theorem 2.1]. Define the operator J by writing $u = Jv$ for the solution of the initial-boundary value problem (4.5a) with $u(\cdot, 0) = Bv + g$ in ω. From the maximum principle (cf. also Lemma 2.1), we see that J is a contraction on $C^0(\overline{\Omega})$, so it has a unique fixed point, which is the unique solution of (4.5). More exactly, let us take $u_0 = 0$ and define $u_m = J^m u_0$. Then we define

$$U_m = |u_m - u_{m-1}|_0$$

for m a positive integer and $U_0 = U_1/\rho$ to conclude that $U_m \leqslant \rho^m U_0$. Next, we define

$$H_m = [D(u_m - u_{m-1})(\cdot,0)]_\alpha, \quad H_m^* = [D(u_m - u_{m-1})(\cdot,0)]_\alpha$$

for m a positive integer and $H_0 = H_1/\rho$. It follows from the Schauder estimate [**8**, Proposition 4.25] that $H_m^* \leqslant KH_m$ for some K determined only by n, α, Ω, λ, $|a^{ij}|_\alpha$, $|b^i|_\alpha$, $|c|_\alpha$. Moreover,

$$[D(u_{m-1} - u_{m-2})]_{\alpha,\omega\times(E(\varepsilon),T)} \leqslant CA_1(\varepsilon)|u_{m-1} - u_{m-2}|_0,$$

and hence $H_m \leqslant (\rho + K\varepsilon)H_{m-1} + CA_1(\varepsilon)[U_{m-1} + U_m]$. We now choose ε so that $r = \rho + K\varepsilon < 1$. It follows from our previous estimate on U_m that $H_m \leqslant rH_{m-1} + CU_0\rho^m$ and hence, by induction, $H_m \leqslant H_0 r^m + CU_0 mr^m$, and $H_m^* \leqslant KH_0 r^m + CU_0 mr^m$. Because $\sum mr^m$ converges, we conclude that (u_m) is a Cauchy sequence in $H_{1+\alpha}$ so it has a limit in $H_{1+\alpha}$, which is the desired solution. □

Note that the constant C in (4.6) actually depends on a^{ij}, b^i and c through much weaker norms, but we shall not use that fact here.

With this existence theorem in hand, we obtain the following conditional existence theorem for quasilinear problems.

Theorem 4.3. *Suppose that a^{ij} and a are Hölder continuous with respect to X, z, p with exponent $\alpha \in (0,1)$ and that $\partial\Omega \in H_{1+\alpha}$. Let $\varphi \in H_{1+\alpha}(\Omega)$, let $g \in H_{1+\alpha}(\omega)$, and let $B\colon C(\overline{\Omega}) \to C(\overline{\omega})$ be linear. Suppose also that, for every $\varepsilon \in (0,1)$ and $\theta \in (0,\alpha]$ there are constants $A_1(\varepsilon,\theta)$ and $E(\varepsilon,\theta)$ such that* (4.3) *holds and that condition* (1.1) *holds. For $\tau \in (0,1)$, define Q_τ by*

$$Q_\tau w = -w_t + a^{ij}(X,w,Dw)D_{ij}w + \tau a(X,w,Dw). \tag{4.7}$$

If there are constants $\rho \in (0,1)$ and $\mu \geqslant 0$ such that (2.15) *holds and if there is a constant C such that any solution of*

$$Q_\tau u = 0 \quad in\ \Omega, \tag{4.8a}$$

$$u = \tau\varphi \quad on\ S\Omega, \tag{4.8b}$$

$$u(\cdot,0) = \tau Bu + \tau g \quad in\ \omega, \tag{4.8c}$$

with $\tau \in [0,1]$, satisfies the estimate $|u|_0 + |Du|_0 \leqslant C$, then there is a solution to $Pu + \mu(v-u) = 0$ in Ω, $u = \varphi$ on $S\Omega$, and (0.4) *in ω.*

PROOF. The proof is essentially the same as that of [**19**, Lemma 5.1] except that $u = T(v,\tau)$ is the solution of

$$-u_t + a^{ij}(X,v,Dv)D_{ij}u + \tau a(X,v,Dv) = 0 \quad \text{in } \Omega,$$

along with (4.8b,c). The $H_{1+\theta}$-estimate in that lemma follows from Lemma 4.1. The theorem is proved. □

5. Examples

We now apply our method to a large range of equations. The choice of examples is essentially the standard one (cf. [**8**, Sec. 12.5], [**19**, Sec. 5], [**20**, Sec. 5], etc.). Throughout this section, we assume that Ω, φ, μ, α, B, and g satisfy the hypotheses of Theorem 4.3. We also assume that conditions (3.2) hold (or, more generally, that the modified version of these conditions in Corollary 3.2 is satisfied) and that a^{ij} and a are smooth enough that the Hölder gradient estimate of the previous section applies.

5.1. Mean curvature equations. Suppose that

$$a^{ij}(X,z,p) = (1+|p|^2)^{\sigma/2}\left[\delta^{ij} - \frac{p_i p_j}{1+|p|^2}\right],$$
$$a(X,z,p) = (1+|p|^2)^{\sigma/2} H(X,z)$$

for some real constant σ and H a continuous function. We distinguish three cases.

If $\sigma < 0$, we suppose that (2.10) is valid for some $\rho \in (0,1)$ and that there are nonnegative constants H_0 and M_0 such that $\operatorname{sgn} z H(X,z) \leqslant H_0$ if $|z| \geqslant M_0$. These hypotheses imply an L^∞-bound by Lemma 2.5. Next, we suppose that the mean curvature H' of $\partial\omega$ satisfies the inequality

$$(n-1)H'(x) \geqslant H(X,\varphi(X)) \tag{5.1}$$

for all $X=(x,t) \in S\Omega$ and that φ is independent of t. As in example (ii) of [**8**, Sec. 12.6], these conditions imply a boundary gradient estimate by modifying [**8**, Theorem 10.9] along the lines described after our Corollary 3.2. Finally, a global gradient estimate follows from Lemma 3.5. In fact, we can weaken our conditions somewhat if $-1 \leqslant \sigma < 0$ to allow φ to be uniformly Lipschitz with respect to t. For $\sigma = -1$ the boundary gradient estimate follows if we replace (5.1) by

$$(n-1)H'(x) \geqslant H(X,\varphi(X)) + \Phi,$$

where Φ denotes the Lipschitz constant of φ with respect to t. For $\sigma > -1$, we replace (5.1) by

$$(n-1)H'(x) > H(X,\varphi(X)).$$

The details are in [**19**, Corollary 2.7] or [**8**, Corollary 10.11].

If $\sigma = 0$, we suppose that (2.15) holds for some $\rho \in (0,1)$ and some $\mu \geqslant 0$. We also suppose that there are constants $\varepsilon \in (0,1]$ and $M_0 \geqslant 0$ such that $2\mu R^2 < (1-\varepsilon)(n-1)$ and

$$\operatorname{sgn} z H(X,z) \leqslant \varepsilon\frac{n-1}{R} \tag{5.2}$$

for all $|z| \geqslant M_0$. Then Lemma 2.8 provides the L^∞-bound, and a boundary gradient estimate follows from (5.1) by [**8**, Theorem 10.9]. A global gradient estimate holds if $H_z \leqslant 0$ by the local gradient estimate of Ecker and Huisken [**18**, Theorem 2.3] and the remarks following Corollary 3.4.

For $\sigma > 0$, we suppose that (2.15) holds for some $\rho \in (0,1)$ and some $\mu \geqslant 0$. We also assume that there are constants $\varepsilon \in (0,1)$ and M_0 such that (5.2) holds for $|z| \geqslant M_0$. Since $|p|\Lambda \to \infty$ as $|p| \to \infty$, Lemma 2.8 again provides an L^∞-bound, and [**8**, Theorem 10.9] gives the boundary gradient estimate. Finally, a global gradient estimate holds (cf. [**8**, p. 257]) if $H_z < 0$.

5.2. Uniformly parabolic equations in nondivergence form. We say that P is uniformly parabolic if there is a constant μ_0 such that $\Lambda \leqslant \mu_0\lambda$. We suppose also that there are constants $\theta_1 > 0$ and σ such that

$$\lambda(X,z,p) \geqslant \theta_1(1+|p|)^\sigma.$$

There are several ranges of σ to consider.

If $\sigma < 0$, we assume that there are positive constants $\rho < 1$ and μ such that $|B_\mu|1 \leqslant \rho$ and we suppose condition (2.8) holds. Then an L^∞-bound follows from Lemma 2.5. For $\sigma = 0$, an L^∞-bound follows from Lemma 2.8 if there are constants $\varepsilon \in (0,1)$, $\rho \in (0,1)$, and $\mu \geqslant 0$ such that (2.15) holds and

$$a(X,z,p)\operatorname{sgn} z \leqslant \varepsilon\theta_1\frac{n|p|}{R}, \quad \frac{2\mu R^2}{(1-\rho)(1-\varepsilon)} < n\theta_1$$

for all sufficiently large z and p. For $\sigma > 0$, we have an L^∞-bound if (2.15) holds and

$$a(X,z,p)\operatorname{sgn} z \leqslant \varepsilon\theta_1(1+|p|)^\sigma\frac{n|p|}{R}, \quad \varepsilon < 1.$$

If $\sigma \geqslant -2$, then the assumption $|a| = O(\mathcal{E})$ as $|p| \to \infty$ yields a boundary gradient estimate by Lemma 3.1. If $\sigma < -2$, we also assume that φ is time-independent, in which case a boundary gradient estimate follows by modifying [**8**, Corollary 10.5] along the lines of Corollary 3.2.

For the global gradient estimate, we assume that

$$\begin{gathered}\Lambda = O(|p|^\sigma),\\ |a| + |p|^2\left|\bar\delta a^{ij}\right| + \left|\bar\delta a\right| = O(|p|^{\sigma+2}),\\ \left|\delta a^{ij}\right| = o(|p|^\sigma), \quad \delta a \leqslant o(|p|^{\sigma+2}).\end{gathered}$$

If a modulus of continuity estimate is known, we can replace $o(|p|^{\sigma+2})$ by $O(|p|^{\sigma+2})$ in this last inequality; such an estimate is known if $\Lambda|p|^2 + |a| = O(|p|^{\sigma+2})$ provided $\sigma = 0$ ([**21**, Theorem 3.2]) or $0 < \sigma < 1$ ([**22**, Theorem X.11]). Because a local gradient bound is known under these conditions (cf. [**8**, Sec. 11.7]), a global gradient bound is true if condition (2.15) is satisfied. In this case, there are no restrictions on ρ and μ.

5.3. False mean curvature equations. Finally, we assume that

$$a^{ij}(X,z,p) = (1+|p|^2)^{\sigma/2}(\delta^{ij} + p_i p_j),$$
$$a(X,z,p) = (1+|p|^2)^{(\sigma+1)/2} H(X,z)$$

for $\sigma \in \mathbb{R}$ and H satisfying $\operatorname{sgn} zH(X,z) \leqslant A_0$ if $|z| \geqslant M_0$ for some nonnegative constants A_0 and M_0. Note that the ratio of maximum to minimum eigenvalues for $[a^{ij}]$ is $1+|p|^2$, just as for the mean curvature equations. The distinction is that the eigenspace for the maximum eigenvalue is parallel to p in this example while it was orthogonal to p in the mean curvature case.

If $\sigma < -2$, we suppose that condition (2.10) holds and then Lemma 2.5 gives an L^∞-estimate. If we also assume that φ is time-independent when $\sigma < -4$, then Lemma 3.1 and [**8**, Corollary 10.5] provide a boundary gradient estimate because $|a| = O(\mathcal{E})$. A global gradient estimate then follows from Lemma 3.5.

If $\sigma = -2$, we suppose that condition (2.15) holds with $1 > 2\mu R^2$. Then Lemma 2.8 gives an L^∞-bound (since (2.16) holds with ε arbitrarily small) while Lemma 3.1 gives a boundary gradient estimate and Lemma 3.5 gives a global gradient bound.

If $\sigma > -2$, we suppose that condition (2.15) holds with $\rho \in (0,1)$ and $\mu \geqslant 0$ arbitrary. The L^∞-bound follows from 2.8, and the boundary gradient estimate follows from Corollary 3.2. A global gradient bound follows from Corollary 3.4.

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Boundary Feedback Stabilization of a Vibrating String with an Interior Point Mass

Walter Littman and Stephen W. Taylor

Dedicated to Olga Ladyzhenskaya on the occasion of her birthday

We study the boundary stabilization of a vibrating string with an interior point mass, zero Dirichlet condition at the left end and velocity feedback at the right end. Assuming finite energy initially, we show that the energy to the right of the point mass decays like C/t while that of the point mass decays like $C/\sqrt{t}$. The energy to the left of the point mass approaches zero but at no specific rate.

1. Introduction

For a finite vibrating string with zero Dirichlet condition at the left end point, velocity feedback at the right end point, and no point mass, it is known that the energy decays exponentially. On the other hand, if there is a point mass at the right end, then the energy decays to zero but at no specific rate [**1**]. Yet, if the initial data have *some additional* smoothness than finite energy, then the energy decays at a rational rate [**1**].

If the point mass is in the interior, Hansen and Zuazua [**2**] have shown that the energy decays exponentially if the velocity feedback is applied at both ends. They also show that with Dirichlet conditions on the left, velocity feedback at the right end, and some additional smoothness of the initial data, it follows from [**1**] that the energy decays rationally.

We improve on the result of Hansen and Zuazua by showing (Theorem 1) that the energy to the right of the mass decays like C/t, that of the mass decays like $C/\sqrt{t}$, and that to the left of the mass decays to zero, but at no specific rate. According to Theorem 3, if the initial data to the left of the mass have one additional degree of smoothness, then the total energy decays like C/t.

The system under investigation consists of two strings of length l_1 and l_2 respectively. In their rest states, the strings occupy the intervals $\Omega_1 = (-l_1, 0)$ and $\Omega_2 = (0, l_2)$ of the x-axis respectively. At the origin, each string is tied to a particle of mass M whose displacement away from the x-axis at time t is given by $z(t)$. The transverse displacements of the strings are given by u and v. In this model, the densities ρ_1, ρ_2 and the tensions σ_1, σ_2 are assumed constant.

The equations satisfied by the system are listed below (more details are given in [**2**]):

$$\begin{array}{ll}
\rho_1 u_{tt} = \sigma_1 u_{xx}, & x \in \Omega_1,\ t > 0, \\
\rho_2 v_{tt} = \sigma_2 v_{xx}, & x \in \Omega_2,\ t > 0, \\
M z_{tt} + \sigma_1 u_x(0,t) - \sigma_2 v_x(0,t) = 0, & t > 0, \\
u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), & x \in \Omega_1, \\
v(x,0) = v^0(x),\ v_t(x,0) = v^1(x), & x \in \Omega_2, \\
z(t) = u(0,t) = v(0,t), & t > 0.
\end{array} \tag{1}$$

We assume that the velocity feedback occurs at the end $x = l_2$, while at the other end $x = -l_1$ we simply have a Dirichlet boundary condition. The boundary conditions are thus

$$\begin{array}{ll}
u(-l_1, t) = 0, & t > 0, \\
\sigma_2 v_x(l_2, t) + \gamma v_t(l_2, t) = 0, & t > 0,
\end{array} \tag{2}$$

where γ is positive.

2. A Representation of the Solution

We can simplify the exposition by scaling the space variable x separately for $x < 0$ and for $x > 0$ so that the wave speed of each wave equation becomes unity. This is achieved by considering a new variable

$$\widetilde{x} = \begin{cases} x(\rho_1/\sigma_1)^{1/2}, & x < 0, \\ x(\rho_2/\sigma_2)^{1/2}, & x \geqslant 0. \end{cases}$$

Thus, we may consider without loss of generality the following system (tildes have been removed)

$$\begin{array}{ll}
u_{tt} = u_{xx}, & x \in \Omega_1, t > 0, \\
v_{tt} = v_{xx}, & x \in \Omega_2, t > 0, \\
M z_{tt} + \mu_1 u_x(0,t) - \mu_2 v_x(0,t) = 0, & t > 0, \\
u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), & x \in \Omega_1, \\
v(x,0) = v^0(x),\ v_t(x,0) = v^1(x), & x \in \Omega_1, \\
z(t) = u(0,t) = v(0,t), & t > 0,
\end{array} \tag{3}$$

where the modified tensions are given by $\mu_1 = (\rho_1\sigma_1)^{1/2}$, $\mu_2 = (\rho_2\sigma_2)^{1/2}$. The boundary conditions now become

$$\begin{aligned} &u(-l_1,t) = 0, && t > 0, \\ &\mu_2 v_x(l_2,t) + \gamma v_t(l_2,t) = 0, && t > 0, \end{aligned} \tag{4}$$

and the total mechanical energy simplifies to

$$\begin{aligned} \mathcal{E}(t) = \frac{M}{2}|z_t(t)|^2 + \frac{\mu_1}{2}\int_{-l_1}^{0} |u_t(x,t)|^2 + |u_x(x,t)|^2\,dx \\ + \frac{\mu_2}{2}\int_{0}^{l_2} |v_t(x,t)|^2 + |v_x(x,t)|^2\,dx. \end{aligned} \tag{5}$$

We define the finite energy space

$$\mathcal{H} = \left\{ \begin{pmatrix} U^0 \in H^1(\Omega_1), \\ V^0 \in H^1(\Omega_2), \\ Z^0 \in \mathbb{R}, \\ U^1 \in L^2(\Omega_1), \\ V^1 \in L^2(\Omega_2), \\ Z^1 \in \mathbb{R} \end{pmatrix} : \begin{array}{l} U^0(-l_1) = 0, \\ U^0(0) = V^0(0) = Z^0 \end{array} \right\}$$

and equip $\mathcal{H}$ with the norm

$$\begin{aligned} \|(U^0,V^0,Z^0,U^1,V^1,Z^1)\| = \Bigg(\mu_1 \int_{-l_1}^{0} |U^1(x)|^2 + |U^0_x(x)|^2\,dx \\ + M|Z^1|^2 + \mu_2 \int_{0}^{l_2} |V^1(x)|^2 + |V^0_x(x)|^2\,dx\Bigg)^{1/2}. \end{aligned}$$

It is easy to see that $\mathcal{H}$ is a Hilbert space, and we define on $\mathcal{H}$ the operator $\mathcal{A}$, with domain $D(\mathcal{A}) = \{(U^0,V^0,Z^0,U^1,V^1,Z^1) \in \mathcal{H} \cap (H^2(\Omega_1) \times H^2(\Omega_2) \times \mathbb{R} \times H^1(\Omega_1) \times H^1(\Omega_2) \times \mathbb{R}) : U^1(-l_1) = 0,\ U^1(0) = V^1(0) = Z^1,\ \mu_2 V^0_x(l_2) + \gamma V^1(l_2) = 0\}$ given by

$$\mathcal{A}(U^0,V^0,Z^0,U^1,V^1,Z^1) = (U^1,V^1,Z^1,U^0_{xx},V^0_{xx},(\mu_2 V^0_x(0) - \mu_1 U_x(0))/M).$$

As is mentioned in **[2]**, it is easy to check that $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup of contractions $T(t)$ on $\mathcal{H}$ (the Lumer Phillips Theorem, which is stated in **[3]**, can be used to deduce this). The *finite energy solutions* of (3), (4) are then given by

$$(u(\cdot,t),v(\cdot,t),z(t),u_t(\cdot,t),v_t(\cdot,t),z_t(t)) = T(t)(u^0,v^0,z^0,u^1,v^1,z^1).$$

A convenient way to analyze the decay of solutions of (3), (4) is to use the fact that the solution of each of the wave equations is a sum of two waves, one moving to the left and the other to the right:

$$\begin{aligned} u(x,t) &= F(x+t+l_1) - G(t-x), \qquad x \in \Omega_1,\, t \geqslant 0, \\ v(x,t) &= H(x+t) - E(t-x+l_2), \qquad x \in \Omega_2,\, t \geqslant 0. \end{aligned} \tag{6}$$

The regularity of the functions F, G, H, and E is easily checked because we have

$$\begin{aligned} F'(x+t+l_1) &= \frac{1}{2}(u_x(x,t) + u_t(x,t)), \\ G'(x+t+l_1) &= \frac{1}{2}(u_x(x,t) - u_t(x,t)), \end{aligned}$$

etc. and thus for finite energy solutions, the functions F, G, H, and E are locally in H^1. Moreover, because of the strong continuity of the semigroup, it follows that $z(t) = F(t+l_1) - G(t) = H(t) - E(t+l_2)$ is C^1. If the initial data happen to be in the domain of $\mathcal{A}$, then it is easily seen that all of these functions have one more derivative, i.e., F, G, H, and E are locally in H^2 and $z(t) = F(t+l_1) - G(t) = H(t) - E(t+l_2)$ is C^2.

In terms of these functions, the energy (5) may be written as follows:

$$\begin{aligned} \mathcal{E}(t) =& \mu_1 \int_{-l_1}^{0} |F'(t-x)|^2 + |G'(t-x)|^2 \, dx \\ &+ \mu_2 \int_{0}^{l_2} |E'(t+x)|^2 + |H'(t+x)|^2 \, dx \\ &+ \frac{1}{2} M \left| \frac{d}{dt}(F(t+l_1) - G(t)) \right|^2 . \end{aligned} \tag{7}$$

The initial conditions of (3) imply that the functions F, G, H, and E satisfy, modulo some irrelevant arbitrary constants,

$$F(s) = \frac{1}{2}u^0(s-l_1) + \frac{1}{2}\int_{0}^{s-l_1} u^1(\sigma)\,d\sigma, \quad 0 < s < l_1,$$

$$G(s) = -\frac{1}{2}u^0(-s) + \frac{1}{2}\int_{0}^{-s} u^1(\sigma)\,d\sigma, \quad 0 < s < l_1,$$

$$H(s) = \frac{1}{2}v^0(s) + \frac{1}{2}\int_0^s v^1(\sigma)\,d\sigma, \quad 0 < s < l_2, \tag{8}$$

$$E(s) = -\frac{1}{2}v^0(l_2 - s) + \frac{1}{2}\int_0^{l_2 - s} v^1(\sigma)\,d\sigma, \quad 0 < s < l_2.$$

The values of these functions for other positive values of their arguments are found by solving a system of differential–delay equations, which are obtained from the conditions at $x = 0$ of (3) and the boundary conditions (4). Specifically, for $s > 0$,

$$\begin{aligned}
&G(s + l_1) = F(s),\\
&F(s + l_1) + E(s + l_2) = G(s) + H(s),\\
&H'(s + l_2) = qE'(s),\\
&M(F(s + l_1) - G(s))_{ss} = \mu_2(H'(s) + E'(s + l_2))\\
&\qquad\qquad\qquad\qquad\quad - \mu_1(F'(s + l_1) + G'(s)),
\end{aligned} \tag{9}$$

where $q = (\gamma - \mu_2)/(\gamma + \mu_2)$. There are two points that should be mentioned before we proceed. The first is that $|q| < 1$ which will be very important in the sequel. The second is that Eqs. (9) require more smoothness than finite energy solutions possess. However, we use (9) only to construct explicit formulas for the Laplace transforms (cf. (10) below) of F', G', H', and E'. But because (9) are valid for initial data that are in the domain of $\mathcal{A}$, a standard density argument may be used to show that the formulas for our Laplace transforms are valid for all finite energy solutions. Since the argument is straightforward, we shall omit any further details of it.

We define

$$\begin{aligned}
f(\lambda) &= \int_0^\infty e^{-\lambda t}F'(t)\,dt, & f^1(\lambda) &= \int_0^{l_1} e^{-\lambda t}F'(t)\,dt,\\
g(\lambda) &= \int_0^\infty e^{-\lambda t}G'(t)\,dt, & g^1(\lambda) &= \int_0^{l_1} e^{-\lambda t}G'(t)\,dt,\\
h(\lambda) &= \int_0^\infty e^{-\lambda t}H'(t)\,dt, & h^1(\lambda) &= \int_0^{l_2} e^{-\lambda t}H'(t)\,dt,\\
e(\lambda) &= \int_0^\infty e^{-\lambda t}E'(t)\,dt, & e^1(\lambda) &= \int_0^{l_2} e^{-\lambda t}E'(t)\,dt.
\end{aligned} \tag{10}$$

Note that the Laplace transforms in Eqs. (10) all exist for $\operatorname{Re}\lambda > 0$. This is because for finite energy solutions, F', G', H', and E' have L^2-norms on any bounded subinterval of $(0,\infty)$ that are bounded by constants that depend on the length, but not the location, of the subinterval. The inversion of these transforms will be important to us. One could use an inverse Laplace transform approach, but we prefer to extend the functions F', G', H', and E' to be zero on $(-\infty,0)$ and interpret these transforms as Fourier transforms. Thus, if $\lambda=\sigma+i\xi$ and $\sigma>0$, then $f(\lambda)$ is just the Fourier transform of $e^{-\sigma t}F'(t)$.

Formally, Eqs. (9) imply that

$$\begin{aligned}
0&=g(\lambda)-g^1(\lambda)-f(\lambda)e^{-l_1\lambda},\\
0&=e^{l_1\lambda}(f(\lambda)-f^1(\lambda))-g(\lambda)-h(\lambda)+e^{l_2\lambda}(e(\lambda)-e^1(\lambda)),\\
0&=h(\lambda)-h^1(\lambda)-e^{-l_2\lambda}qe(\lambda),\\
0&=M(\lambda e^{l_1\lambda}(f(\lambda)-f^1(\lambda))-F'(l_1)+G'(0)-\lambda g(\lambda))\\
&\quad-\mu_2(h(\lambda)+e^{l_2\lambda}(e(\lambda)-e^1(\lambda))\\
&\quad+\mu_1(e^{l_1\lambda}(f(\lambda)-f^1(\lambda))+g(\lambda)).
\end{aligned}\tag{11}$$

The solutions of these equations are easily found. First we define

$$Q(\lambda)=\frac{1+qe^{-2l_2\lambda}}{1-qe^{-2l_2\lambda}},\tag{12}$$

$$\begin{aligned}
S(\lambda)&=(1-qe^{-2l_2\lambda})^{-1}([M\lambda+\mu_1+Q(\lambda)\mu_2]\\
&\qquad-e^{-2l_1\lambda}[M\lambda-\mu_1+Q(\lambda)\mu_2])^{-1},
\end{aligned}\tag{13}$$

and now we may write the solutions of the system (11) as follows:

$$\begin{aligned}
f(\lambda)&=\{f^1(\lambda)[M\lambda+\mu_1+\mu_2-qe^{-2l_2\lambda}(M\lambda+\mu_1-\mu_2)]\\
&\quad+g^1(\lambda)e^{-l_1\lambda}[M\lambda-\mu_1+\mu_2-qe^{-2l_2\lambda}(M\lambda-\mu_1-\mu_2)]\\
&\quad+2\mu_2h^1(\lambda)e^{-l_1\lambda}+2q\mu_2e^1(\lambda)e^{-(l_1+l_2)\lambda}\\
&\quad+M(F'(l_1)-G'(0))e^{-l_1\lambda}(1-qe^{-2l_2\lambda})\}S(\lambda),
\end{aligned}\tag{14}$$

$$\begin{aligned}
g(\lambda)&=\{g^1(\lambda)[M\lambda+\mu_1+\mu_2-qe^{-2l_2\lambda}(M\lambda+\mu_1-\mu_2)]\\
&\quad+f^1(\lambda)e^{-l_1\lambda}[M\lambda+\mu_1+\mu_2-qe^{-2l_2\lambda}(M\lambda+\mu_1-\mu_2)]\\
&\quad+2\mu_2h^1(\lambda)e^{-2l_1\lambda}+2q\mu_2e^1(\lambda)e^{-2l_1\lambda}\\
&\quad+M(F'(l_1)-G'(0))e^{-2l_1\lambda}(1-qe^{-2l_2\lambda})\}S(\lambda),
\end{aligned}\tag{15}$$

$$\begin{aligned}
h(\lambda)&=\{h^1(\lambda)[M\lambda+\mu_1+\mu_2-e^{-2l_1\lambda}(M\lambda-\mu_1+\mu_2)]\\
&\quad+e^1(\lambda)qe^{-l_2\lambda}[M\lambda+\mu_1+\mu_2-e^{-2l_1\lambda}(M\lambda-\mu_1+\mu_2)]\\
&\quad+2\mu_1qf^1(\lambda)e^{-(l_1+2l_2)\lambda}+2q\mu_1g^1(\lambda)e^{-2l_2\lambda}\\
&\quad-qM(F'(l_1)-G'(0))e^{-2l_2\lambda}(1-e^{-2l_1\lambda})\}S(\lambda),
\end{aligned}\tag{16}$$

$$\begin{aligned} e(\lambda) = \{ & e^1(\lambda)[M\lambda + \mu_1 + \mu_2 - e^{-2l_1\lambda}(M\lambda - \mu_1 + \mu_2)] \\ & + h^1(\lambda)e^{-l_2\lambda}[M\lambda + \mu_1 - \mu_2 - e^{-2l_1\lambda}(M\lambda - \mu_1 - \mu_2)] \\ & + 2\mu_1 q f^1(\lambda)e^{-(l_1+l_2)\lambda} + 2\mu_1 g^1(\lambda)e^{-l_2\lambda} \\ & - M(F'(l_1) - G'(0))e^{-l_2\lambda}(1 - e^{-2l_1\lambda})\}S(\lambda). \end{aligned} \tag{17}$$

Lemma 1. *$Q(\lambda)$ and $S(\lambda)$ satisfy the following inequalities.*

1. $\dfrac{1 - |q|e^{-2l_2 \operatorname{Re}\lambda}}{1 + |q|e^{-2l_2 \operatorname{Re}\lambda}} \leqslant \operatorname{Re} Q(\lambda) \leqslant \dfrac{1 + |q|e^{-2l_2 \operatorname{Re}\lambda}}{1 - |q|e^{-2l_2 \operatorname{Re}\lambda}}$.

2. *There exists a constant C such that if* $\operatorname{Re}\lambda > 0$, *then*

$$|S(\lambda)| < \frac{C}{1 + |\lambda|}(1 - e^{-2l_1 \operatorname{Re}\lambda})^{-1}$$

PROOF. A simple calculation shows that

$$\operatorname{Re} Q(\sigma + i\xi) = \frac{1 - q^2 e^{-4l_2\sigma}}{1 - 2qe^{-2l_2\sigma}\cos\xi + q^2 e^{-4l_2\sigma}}.$$

Hence

$$\frac{1 - |q|e^{-2l_2\sigma}}{1 + |q|e^{-2l_2\sigma}} \leqslant \operatorname{Re} Q(\sigma + i\xi) \leqslant \frac{1 + |q|e^{-2l_2\sigma}}{1 - |q|e^{-2l_2\sigma}},$$

which proves the first inequality. But $|q| < 1$, so $\operatorname{Re} Q(\sigma + i\xi) > 0$ in the set $\sigma > \dfrac{\log|q|}{2l_2}$, which contains the right half-plane. Hence

$$|\operatorname{Re}(M\lambda + \mu_1 + Q(\lambda)\mu_2)| > |\operatorname{Re}(M\lambda - \mu_1 + Q(\lambda)\mu_2)|$$

if $\operatorname{Re}\lambda \geqslant 0$. It is clear that

$$\operatorname{Im}(M\lambda + \mu_1 + Q(\lambda)\mu_2) = \operatorname{Im}(M\lambda - \mu_1 + Q(\lambda)\mu_2),$$

so

$$|M\lambda + \mu_1 + Q(\lambda)\mu_2| > |M\lambda - \mu_1 + Q(\lambda)\mu_2| \tag{18}$$

if $\operatorname{Re}\lambda \geqslant 0$, and thus

$$|S(\lambda)| \leqslant \left|(1 - qe^{-2l_2\lambda})^{-1}(M\lambda + \mu_1 + Q(\lambda)\mu_2)^{-1}\right|(1 - e^{-2l_1 \operatorname{Re}\lambda})^{-1}.$$

The second inequality in the statement of the lemma follows easily from this.

3. Energy Decay Estimates

In this section, we analyze the decay of energy of the string–mass system. We find that the energy of the string to the right of the particle (i.e., the part of the string corresponding to the interval $(0, l_2)$ of the x-axis) decays uniformly when the initial data have finite energy. A similar decay rate holds for the remainder of the energy, but this requires an extra derivative in the initial data for the part of the string to the left of the particle.

In order to prove our results, it is useful to consider for $m = 1, 2, \ldots$, the following approximations $S_m(\lambda)$ to $S(\lambda)$:

$$S_m(\lambda) = (1 - qe^{-2l_2\lambda})^{-1} \sum_{k=0}^{m} \frac{e^{-2kl_1\lambda}[M\lambda - \mu_1 + Q(\lambda)\mu_2]^k}{[M\lambda + \mu_1 + Q(\lambda)\mu_2]^{k+1}}. \tag{19}$$

Lemma 2. $\lim_{m\to\infty} S_m(\lambda) = S(\lambda)$, *with uniform convergence on sets of the form* $\{\lambda : \operatorname{Re}\lambda \geqslant \sigma\}$, *where* $\sigma > 0$.

PROOF. The stated convergence properties of the geometric series are an immediate consequence of the inequality (18).

We will be able to simplify our proofs of energy decay by considering only solutions of the string–mass systems for which the initial velocity of the point mass is zero. We will achieve this simplification by subtracting an eigenfunction solution from the solution of (3) and (4). The fact that this can be done is due to the fact that the eigenvector components corresponding to the velocity and displacement of the point mass cannot vanish. We prove this in the following lemma.

Lemma 3. *If* $(U^0, V^0, Z^0, U^1, V^1, Z^1)$ *is an eigenvector of* $\mathcal{A}$, *then* Z^0 *and* Z^1 *are non-zero real numbers.*

PROOF. Hansen and Zuazua prove in [2] that the eigenvalues of $\mathcal{A}$ lie in the left half-plane. If λ is such an eigenvalue, then $Z^1 = \lambda Z^0$ and $U^0(0) = Z^0$. Thus, if one of Z^0 or Z^1 vanished then they would both vanish and U^0 would satisfy

$$U^0_{xx} = \lambda^2 U^0, \quad U^0(-l_1) = U^0(0) = 0.$$

This has only trivial solutions because λ is in the left half-plane. Hence $U^0 = 0$. Thus, V^0 must satisfy the conditions

$$V^0_{xx} = \lambda^2 V^0, \quad V^0_x(0) = V^0(0) = 0,$$

which implies that V^0 is also zero. It is now easy to see that all of the components of the eigenvector are zero, which is impossible.

We now consider the energy of different parts of the solution. Specifically, we write $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$, where

$$\begin{aligned}\mathcal{E}_1(t) &= \frac{\mu_1}{2} \int_{-l_1}^{0} |u_t(x,t)|^2 + |u_x(x,t)|^2\, dx \\ &= \mu_1 \int_{-l_1}^{0} |F'(t-x)|^2 + |G'(t-x)|^2\, dx,\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_2(t) &= \frac{M}{2}|z_t(t)|^2 = \frac{1}{2}M|\frac{d}{dt}(F(t+l_1)-G(t))|^2, \\
\mathcal{E}_3(t) &= \frac{\mu_2}{2}\int_0^{l_2} |v_t(x,t)|^2 + |v_x(x,t)|^2\,dx \\
&= \mu_2 \int_0^{l_2} |E'(t+x)|^2 + |H'(t+x)|^2\,dx.
\end{aligned} \tag{20}$$

Thus, $\mathcal{E}_1$ is the energy of the part of the string to the left of the point mass, $\mathcal{E}_2$ is the energy of the point mass, and $\mathcal{E}_3$ is the energy of the part of the string to the right of the mass. We know that the total energies of finite energy solutions of the system do not decay uniformly with respect to the energy norm, but the following theorem shows that $\mathcal{E}_2(t)$ and $\mathcal{E}_3(t)$ do decay uniformly as $t \to \infty$. This is obviously because the dissipative boundary condition directly affects the string to the right of the mass and this is apparently enough to deprive the mass of energy as well.

We note that by lack of uniform decay we mean the following: Let $b(t)$ ($t > 0$) be any positive continuous function which approaches zero as t approaches infinity. Then for some appropriately chosen initial data of unit energy, the energy of the system will exceed $b(t)$ at an infinite sequence of times that approach infinity (cf. [**1**]).

Theorem 1. *There exists a constant C such that all finite energy solutions of the string–mass system satisfy, for $t > 0$:*

$$\mathcal{E}_3(t) \leqslant C\mathcal{E}(0)/t, \tag{21}$$

$$\mathcal{E}_2(t) \leqslant C\mathcal{E}(0)/\sqrt{t}. \tag{22}$$

PROOF OF THEOREM 1. We break the proof into three Parts. In Parts 1 and 2, we establish (21) and in Part 3 we establish (22). We may assume throughout the proof that the initial velocity of the point mass vanishes because, by Lemma 3 this may be achieved by subtracting a solution of the system that is constructed from an eigenfunction. Hence we assume without loss of generality that the term $F'(l_1) - G'(0)$ that appears in Eqs. (14), (15), (16), and (17) is zero.

PART 1. We start by analyzing $H'(t)$ which, for $\sigma > 0$, is $e^{\sigma t}$ times the inverse Fourier transform of $h(\sigma + i\xi)$. Specifically,

$$e^{-\sigma t}H'(t) = \lim_{R\to\infty} \frac{1}{2\pi}\int_{-R}^{R} h(\sigma + i\xi)e^{i\xi t}\,d\xi, \tag{23}$$

which converges in $L^2(\mathbb{R})$ because $\xi \to h(\sigma + i\xi)$ is in $L^2(\mathbb{R})$. We rewrite $H'(t)$ as a sum of "good" and "bad" parts, $H'(t) = H_g^1(t) + H_b^1(t)$, where

$$e^{-\sigma t} H_g'(t) = \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} h_g(\sigma + i\xi) e^{i\xi t}\, d\xi,$$

$$e^{-\sigma t} H_b'(t) = \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} h_b(\sigma + i\xi) e^{i\xi t}\, d\xi,$$

and

$$h_g(\lambda) = \frac{h^1(\lambda) + qe^{-l_2\lambda} e^1(\lambda)}{1 - qe^{-2l_2\lambda}}, \tag{24}$$

$$\begin{aligned} h_b(\lambda) &= S(\lambda) w(\lambda), \\ w(\lambda) &= 2q\mu_1 e^{-2l_2\lambda}(e^{-l_1\lambda} f^1(\lambda) + g^1(\lambda)) \\ &\quad - 2q\mu_2(1 - e^{-2l_1\lambda})(1 - qe^{-2l_2\lambda})^{-1}(h^1(\lambda) + qe^{-l_2\lambda} e^1(\lambda)). \end{aligned} \tag{25}$$

Before continuing with the proof we verify that indeed $h = h_g + h_b$. We may assume that $F'(l_1) - G'(0) = 0$, and thus

$$\begin{aligned} h(\lambda) = \{&[M\lambda + \mu_1 + \mu_2 - e^{-2l_1\lambda}(M\lambda - \mu_1 + \mu_2)](h^1(\lambda) + e^1(\lambda) qe^{-l_2\lambda}) \\ &+ 2\mu_1 qe^{-2l_2\lambda}(f^1(\lambda) e^{-l_1\lambda} + g^1(\lambda))\} S(\lambda). \end{aligned}$$

But

$$\begin{aligned} &[M\lambda + \mu_1 + \mu_2 - e^{-2l_1\lambda}(M\lambda - \mu_1 + \mu_2)] S(\lambda) \\ &= \frac{[M\lambda + \mu_1 + \mu_2 - e^{-2l_1\lambda}(M\lambda - \mu_1 + \mu_2)](1 - qe^{-2l_2\lambda})^{-1}}{M\lambda + \mu_1 + Q(\lambda)\mu_2 - e^{-2l_1\lambda}(M\lambda - \mu_1 + Q(\lambda)\mu_2)} \\ &= \left[1 + \frac{\mu_2 - Q(\lambda)\mu_2 - e^{-2l_1\lambda}(1 - Q(\lambda))\mu_2}{M\lambda + \mu_1 + Q(\lambda)\mu_2 - e^{-2l_1\lambda}(M\lambda - \mu_1 + Q(\lambda)\mu_2)}\right](1 - qe^{-2l_2\lambda})^{-1} \\ &= (1 - qe^{-2l_2\lambda})^{-1} + S(\lambda)(1 - Q(\lambda))\mu_2(1 - e^{-2l_1\lambda}) \\ &= (1 - qe^{-2l_2\lambda})^{-1} - 2q\mu_2(1 - e^{-2l_1\lambda})(1 - qe^{-2l_2\lambda})^{-1} S(\lambda). \end{aligned}$$

Hence

$$\begin{aligned} h(\lambda) &= (h^1(\lambda) + e^1(\lambda) qe^{-l_2\lambda})(1 - qe^{-2l_2\lambda})^{-1} \\ &\quad - 2q\mu_2(1 - e^{-2l_1\lambda})(1 - qe^{-2l_2\lambda})^{-1} S(\lambda)(h^1(\lambda) + e^1(\lambda) qe^{-l_2\lambda}) \\ &\quad + 2\mu_1 qe^{-2l_2\lambda}(f^1(\lambda) e^{-l_1\lambda} + g^1(\lambda)) S(\lambda) \\ &= h_g(\lambda) + h_b(\lambda). \end{aligned}$$

We analyze the "good" functions in this part of the proof, and we leave the "bad" functions to Part 2 of the proof. The reason for writing the functions in this way is that the decay properties of H_g' are easily seen and the convergence properties of the integral defining H_b' are better than those of the corresponding integral for H'.

An explicit formula for $H_g'(t)$ exists. In order to write this, we first define:

$$H_c'(t) = \begin{cases} H'(t), & t \in [0, l_2], \\ 0, & t \in (-\infty, 0) \cup (l_2, \infty), \end{cases}$$

$$E_c'(t) = \begin{cases} E'(t), & t \in [0, l_2], \\ 0, & t \in (-\infty, 0) \cup (l_2, \infty), \end{cases}$$

$$F_c'(t) = \begin{cases} F'(t), & t \in [0, l_1], \\ 0, & t \in (-\infty, 0) \cup (l_1, \infty), \end{cases}$$

$$G_c'(t) = \begin{cases} G'(t), & t \in [0, l_1], \\ 0, & t \in (-\infty, 0) \cup (l_1, \infty), \end{cases}$$

Noting that by (10), h^1 and e^1 are the Laplace transforms of H_c' and E_c' respectively, we observe that

$$H_g'(t) = \sum_{k=0}^{\infty} q^k [H_c'(t - 2kl_2) + qE_c'(t - (2k+1)l_2)].$$

Owing to the fact that H_c' and E_c' have support in $[0, l_2]$,

$$\left(\int_{2ml_2}^{2(m+1)l_2} |H_g'(t)|^2 \, dt \right)^{1/2} \leqslant |q|^m (\|H_c'\|_{L^2(0,l_2)} + \|E_c'\|_{L^2(0,l_2)}). \tag{26}$$

We note that, because $|q| < 1$, the term in the energy corresponding to H_g' decays exponentially with time. We return to this estimate at the end of Part 2 of the proof. This completes Part 1 of the proof.

Before starting Part 2 of the proof of Theorem 1, we state and prove an inequality that we will need in Part 2.

Lemma 4. *If ν, θ, μ, and k are real and $k > 0$, $|\nu| < \mu$, then*

$$\left| \frac{(\nu + i\theta)^k}{(\mu + i\theta)^{k+1}} \right|^2 \leqslant \frac{1}{(k+1)(\mu^2 - \nu^2)}$$

PROOF. Differentiating the expression with respect to θ shows that a maximum occurs at $\theta = 0$ if $k\mu^2 - (k+1)\nu^2 \leqslant 0$ and at $\theta^2 = k\mu^2 - (k+1)\nu^2$ if $k\mu^2 - (k+1)\nu^2 > 0$. In both cases, one easily sees that the inequality is satisfied. This completes the proof of the lemma.

PROOF OF THEOREM 1. PART 2. By (10) and Plancherel's identity, if $\sigma > 0$ then the L^2-norms of $f^1(\sigma + i\xi)$, $g^1(\sigma + i\xi)$, $h^1(\sigma + i\xi)$, and $e^1(\sigma + i\xi)$, as functions of ξ, are no greater than the L^2-norms of $F_c'(t)$, $G_c'(t)$, $H_c'(t)$, and $E_c'(t)$ respectively. This, and the estimate of $S(\lambda)$ in Lemma 1, show that the integral defining $H_b'(t)$ converges absolutely. Thus, we may write

$$H_b'(t) = \frac{e^{\sigma t}}{2\pi} \int_{-\infty}^{\infty} h_b(\sigma + i\xi)e^{i\xi t}\, d\xi = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} h_b(\lambda)e^{\lambda t}\, d\lambda.$$

We would like to deform the contour defining H_b' so that it ends up as a line parallel to the imaginary axis in the left half-plane. But unfortunately this is impossible because, unlike h_g, h_b has a sequence of poles converging to the imaginary axis. These poles in fact correspond to the sequence of eigenvalues converging to the imaginary axis which has been investigated in [**2**]. However, there is a way to get around this problem. We define $h_{b,m}(\lambda) = S_m(\lambda)w(\lambda)$ (cf. (25)). Next, we note that

$$S_m(\lambda) - S(\lambda) = e^{-2(m+1)\lambda l_1} \left[\frac{M\lambda - \mu_1 + Q(\lambda)\mu_2}{M\lambda + \mu_1 + Q(\lambda)\mu_2}\right]^{m+1} S(\lambda).$$

This shows that if $t < 2(m+1)l_1$ then

$$\int_{\sigma - i\infty}^{\sigma + i\infty} (h_{b,m}(\lambda) - h_b(\lambda))e^{\lambda t}\, d\lambda \to 0$$

as $\sigma \to \infty$. Since the expression is independent of σ, it must be zero for $t < 2(m+1)l_1$. Thus, for t in this range, we may use $h_{b,m}(\lambda)$ instead of $h(\lambda)$ in the formula for $H_b'(t)$:

$$H_b'(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} h_{b,m}(\lambda)e^{\lambda t}\, d\lambda. \tag{27}$$

We now consider γ satisfying

$$0 < \gamma < \delta_1 = \min(\mu_1/(2M), -\log(|q|)/(4l_2))$$

but we shall further restrict γ below. We shift the contour in (27) until it becomes the line $\operatorname{Re}\lambda = -\gamma$. Thus we may write

$$H_b'(t) = \sum_{k=0}^{m} \zeta_k(t)e^{-\gamma(t - 2kl_1)},$$

where

$$\zeta_k(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} w(-\gamma + i\xi)e^{(t - 2l_1)i\xi} \frac{[M(-\gamma + i\xi) - \mu_1 + Q(-\gamma + i\xi)\mu_2]^k}{[M(-\gamma + i\xi) + \mu_1 + Q(-\gamma + i\xi)\mu_2]^{k+1}}\, d\xi.$$

By Lemma 1, since $\gamma < -\log(|q|)/(4l_2)$, we must have

$$\operatorname{Re} Q(-\gamma + i\xi) > \rho = \frac{1 - |q|^{1/2}}{1 + |q|^{1/2}} \quad \text{for all } \xi \in \mathbb{R}.$$

Applying Lemma 4, we see that if $0 < \gamma < \delta_2 = \min(\delta_1, \rho\mu_2/(2M))$, then

$$\frac{|M(-\gamma+i\xi) - \mu_1 + Q(-\gamma+i\xi)\mu_2|^k}{|M(-\gamma+i\xi) + \mu_1 + Q(-\gamma+i\xi)\mu_2|^{k+1}}$$
$$\leqslant [4\mu_1(k+1)(\operatorname{Re} Q(-\gamma+i\xi)\mu_2 - M\gamma)]^{-1/2} < [2\rho\mu_1\mu_2(k+1)]^{-1/2}.$$

Hence, by Plancherel's identity, there is a constant B such that

$$\|\zeta_k\|_{L^2(\mathbb{R})} \leqslant C(\mathcal{E}(0))^{1/2}(k+1)^{-1/2}.$$

Thus,

$$\left(\int_{2ml_1}^{2(m+1)l_1} |H_b'(t)|^2\right)^{1/2} \leqslant B(\mathcal{E}(0))^{1/2} \sum_{k=0}^{m} \frac{e^{-2\gamma(m-k)l_1}}{\sqrt{k+1}}.$$

In order to estimate the sum on the right-hand side of this inequality, we use the fact that $e^{-\gamma l_1(m-k)}/\sqrt{k+1}$, as a function of k, is concave up. Thus, if its value for $k = 0$ is no greater than its value for $k = m$ then its value for all $0 \leqslant k \leqslant m$ is no greater than its value for $k = m$. That is, if m is large enough to make

$$e^{-\gamma l_1 m} \leqslant \frac{1}{\sqrt{m+1}},$$

then $e^{-\gamma l_1(m-k)}/\sqrt{k+1} \leqslant 1/\sqrt{m+1}$ and thus

$$\frac{1}{\sqrt{k+1}} \leqslant \frac{e^{\gamma l_1(m-k)}}{\sqrt{m+1}}.$$

Consequently,

$$\left(\int_{2ml_1}^{2(m+1)l_1} |H_b'(t)|^2\right)^{1/2} \leqslant B(\mathcal{E}(0))^{1/2} \sum_{k=0}^{m} \frac{e^{-\gamma(m-k)l_1}}{\sqrt{m+1}}$$
$$\leqslant B(\mathcal{E}(0))^{1/2} \frac{1}{(1 - e^{-\gamma l_1})\sqrt{m+1}}. \tag{28}$$

Thus, by (26) and (28),

$$\int_0^{l_2} |H'(t+x)|^2\,dx \leqslant \frac{\text{const}}{t}\mathcal{E}(0). \tag{29}$$

The analysis for E' is similar. This completes Part 2 of the proof and establishes (21).

PROOF OF THEOREM 1. PART 3. Here we establish (22). It is easy to see that the Laplace transform of $z'(t) = F'(t+l_1) - G'(t)$ is given by

$$j(\lambda) = e^{\lambda l_1}(f(\lambda) - f_1(\lambda)) - g(\lambda).$$

Equations (14) and (15) may be used to show that (keeping in mind that we may assume that $F'(l_1) - G'(0) = 0$)

$$\begin{aligned} j(\lambda) = \{&2\mu_1(qe^{-2\lambda l_2} - 1)(f_1(\lambda)e^{-\lambda l_1} + g_1(\lambda)) \\ &+ (1 - e^{-2\lambda l_1})(2\mu_2(h_1(\lambda) + qe^{-\lambda l_2}e_1(\lambda)))\}S(\lambda). \end{aligned} \tag{30}$$

The form of this expression allows a treatment that is identical to that for $h_b(\lambda)$ in Part 2. Thus, $z'(t)$ satisfies an estimate analogous to (29), i.e.,

$$\int_0^{l_2} |z'(t+x)|^2\, dx \leqslant \frac{\text{const}}{t}\mathcal{E}(0). \tag{31}$$

But

$$\begin{aligned} l_2 z'(t)^2 &= \int_0^{l_2} \frac{d}{dx}[(x-l_2)(z'(t+x)^2]\, dx \\ &= \int_0^{l_2} z'(t+x)^2\, dx + 2\int_0^{l_2} (x-l_2)z'(t+x)z''(t+x)\, dx \\ &\leqslant \int_0^{l_2} z'(t+x)^2\, dx + 2l_2\Big(\int_0^{l_2} z'(t+x)^2\, dx\Big)^{1/2}\Big(\int_0^{l_2} z''(t+x)^2\, dx\Big)^{1/2}. \end{aligned} \tag{32}$$

Also, by the last equation of (9),

$$Mz''(s) = \mu_2(H'(s) + E'(s+l_2)) - \mu_1(F'(s+l_1) + G'(s)),$$

which yields the estimate

$$\int_0^{l_2} z''(t+x)^2\, dx \leqslant \text{const}\,\mathcal{E}(0). \tag{33}$$

Hence we see from (31), (32), and (33) that

$$z'(t)^2 \leqslant \frac{\text{const}}{\sqrt{t}}\mathcal{E}(0), \tag{34}$$

which completes the proof of (22).

We now consider what happens if the initial data have an extra order of smoothness between the endpoint $x = -l_1$ and the point mass at $x = 0$. In this situation, Hansen and Zuazua **[2]** prove that the corresponding open-loop system is exactly controllable in a finite time interval. We show here that the total energy in our closed loop system decays like a constant divided by time. We make this precise by defining a subspace $\mathcal{Y}$ which contains $(U^0, V^0, Z^0, U^1, V^1, Z^1) \in \mathcal{H}$ such that

$$U^0 \in H^2(\Omega_1),\ U^1 \in H^1(\Omega_1),\ U^1(-l_1) = 0,\ U^1(0) = Z^1$$

and we equip $\mathcal{Y}$ with a norm given by

$$\|(U^0, V^0, Z^0, U^1, V^1, Z^1)\|_{\mathcal{Y}} = \|(U^0, V^0, Z^0, U^1, V^1, Z^1)\| + \left(\int_{-l_1}^{0} |U^0_{xx}(x)|^2 + |U^1_x(x)|^2 \right)^{1/2}.$$

It is easy to check that $(\mathcal{Y}, \|\,\|_{\mathcal{Y}})$ is a Banach space. The space $\mathcal{Y}$ is also an invariant subspace of the semigroup $T(t)$. In fact, more is true:

Theorem 2. *$T(t)\mathcal{Y} \subset \mathcal{Y}$ and the restriction of $T(t)$ to $\mathcal{Y}$ is a strongly continuous semigroup on $\mathcal{Y}$.*

Remark. Because of this, $\mathcal{Y}$ is said to be an *$\mathcal{A}$-admissible subspace of $\mathcal{H}$*, where $\mathcal{A}$ is the infinitesimal generator of $T(t)$ (cf. Pazy **[3]** for a discussion of this concept).

PROOF. Hansen and Zuazua's proof of Proposition 2.5 in **[2]** is also applicable here and shows that $T(t)\mathcal{Y} \subset \mathcal{Y}$ and that $T(t)$ is strongly continuous on $\mathcal{Y}$. The semigroup property is obvious because it holds on $\mathcal{H}$.

If $y \in \mathcal{Y}$, then we put $\mathcal{J}(t) = \|T(t)y\|^2_{\mathcal{Y}}$. Obviously, $\mathcal{J}(t)$ depends on y but the notation does not indicate this dependence.

Theorem 3. *Suppose that $y \in \mathcal{Y}$ and $\mathcal{J}(t) = \|T(t)y\|^2_{\mathcal{Y}}$. Let $\mathcal{E}$ be the energy associated with the initial data y. Then there exists a constant C independent of y such that*

$$\mathcal{E}(t) \leqslant C\mathcal{J}(0)/t. \tag{35}$$

PROOF. By Theorem 1, it is sufficient to show that

$$\mathcal{E}_1(t) \leqslant C\mathcal{J}(0)/t, \tag{36}$$

$$\mathcal{E}_2(t) \leqslant C\mathcal{J}(0)/t. \tag{37}$$

The method of proof is similar to that of Theorem 1. We start with

$$e^{-\sigma t} F'(t) = \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} f(\sigma + i\xi) e^{i\xi t}\, d\xi, \tag{38}$$

which, like (23), converges in $L^2(\mathbb{R})$ if $\sigma > 0$ (recall that $F'(t)$ is defined to be zero for $t < 0$). But

$$\frac{e^{\sigma t}}{2\pi} \int_{-R}^{R} f(\sigma + i\xi) e^{i\xi t}\, d\xi = \frac{1}{2\pi i} \int_{\Gamma(\sigma,R)} f(\lambda) e^{\lambda t}\, d\lambda,$$

where $\Gamma(\sigma, R)$ is any contour in the right half-plane that starts at $\sigma - iR$ and ends at $\sigma + iR$. The region of analyticity of f is the same as that of S. We see then that f is analytic in a neighborhood of the origin because $S(0) = 2\mu_1(1-q)^{-1}$ and S is meromorphic. Because of this, we may allow $\Gamma(\sigma, R)$ to pass to the left of the origin and for our purposes it will be sufficient to choose $\varepsilon > 0$ sufficiently small and set $\Gamma(\sigma, R)$ to be the union of

- the portion of the circle that is contained in the left half-plane and is centered at the origin with radius ε,
- the line segments $(\sigma - iR, \sigma - i\varepsilon)$, $(\sigma - i\varepsilon, -i\varepsilon)$, $(i\varepsilon, \sigma + i\varepsilon)$, $(\sigma + i\varepsilon, \sigma + iR)$.

Hence we may now write

$$F'(t) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\Gamma(\sigma,R)} f(\lambda) e^{\lambda t}\, d\lambda,$$

and the limit converges in $L^2_{loc}(\mathbb{R})$.

Now we make use of the extra smoothness of the initial data in the interval Ω_1 and integrate the formulas for $f^1(\lambda)$ and $g^1(\lambda)$ by parts to obtain

$$\begin{aligned} f^1(\lambda) &= (f^2(\lambda) + F'(0) - F'(l_1)e^{-l_1\lambda})/\lambda, \\ g^1(\lambda) &= (g^2(\lambda) + G'(0) - G'(l_1)e^{-l_1\lambda})/\lambda, \end{aligned} \tag{39}$$

where

$$f^2(\lambda) = \int_0^{l_1} e^{-\lambda t} F''(t)\, dt, \quad g^2(\lambda) = \int_0^{l_1} e^{-\lambda t} G''(t)\, dt.$$

As in the proof of Theorem 1, we may assume without loss of generality that $F'(l_1) = G'(0)$. Further, the Dirichlet boundary condition at $x = -l_1$ implies that $G'(l_1) = F'(0)$. These equations and (39) may be used to rearrange (14), yielding

$$f(\lambda) = \frac{F'(0)}{\lambda} + f_b(\lambda), \tag{40}$$

where

$$f_b(\lambda) = \frac{S(\lambda)}{\lambda}\{f^2(\lambda)[M\lambda + \mu_1 + \mu_2 - qe^{-2l_2\lambda}(M\lambda + \mu_1 - \mu_2)]$$

$$+ g^2(\lambda)e^{-l_1\lambda}[M\lambda - \mu_1 + \mu_2 - qe^{-2l_2\lambda}(M\lambda - \mu_1 - \mu_2)]$$
$$+ 2\mu_2\lambda e^{-l_1\lambda}(h^1(\lambda) + qe^1(\lambda)e^{-l_2\lambda}) + 2\mu_1 qe^{-(l_1+2l_2)\lambda}G'(0)\}. \tag{41}$$

Using the fact that

$$\lim_{R\to\infty} \int\limits_{\Gamma(\sigma,R)} \frac{e^{\lambda t}}{\lambda}\, d\lambda = 0,$$

we see that

$$F'(t) = \lim_{R\to\infty} \frac{1}{2\pi i} \int\limits_{\Gamma(\sigma,R)} f_b(\lambda)e^{\lambda t}\, d\lambda = \frac{1}{2\pi i} \int\limits_{\Gamma(\sigma,\infty)} f_b(\lambda)e^{\lambda t}\, d\lambda,$$

and the last integral converges absolutely. The remainder of the analysis of $F'(t)$ is almost identical to that of $H_b'(t)$ in Theorem 1, so we omit the details and conclude that

$$\int\limits_{-l_1}^{0} |F'(t-x)|^2\, dx \leqslant \frac{\text{const}}{t}\mathcal{J}(0). \tag{42}$$

The analysis for G' is similar. This establishes (36).

Because of (36), it is easy to see that the estimate (33) may be improved to give

$$\int\limits_{0}^{l_2} z''(t+x)^2\, dx \leqslant \text{const}\, \frac{\mathcal{J}(0)}{t}, \tag{43}$$

which in turn gives an improvement of the estimate (34):

$$z'(t)^2 \leqslant \text{const}\, \frac{\mathcal{J}(0)}{t}. \tag{44}$$

This establishes (37) and completes the proof of the theorem.

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On Direct Lyapunov Method in Continuum Theories

Mariarosaria Padula

Dedicated to the brilliant mathematician Olga Ladyzheskaya with sincere and deep esteem

Let S_b be a basic motion. We consider two aspects of the direct Lyapunov method of stability theory. The first one is related to the control of perturbations of S_b in terms of the data (stability in mean), and the second one is related to an asymptotic decay to zero for perturbation. First, for a Lyapunov functional we take the difference between the total energy of a given flow and that of the basic flow. An algorithm for computing the norm of perturbation (in a certain space) is demonstrated by three examples. We also propose the useful technique based on the general variational formulation. The algorithm consists in the choice of a test function. Precisely, we note that different test functions can be used for the same formulation and provide us with different informations. We show how to choose the test function in three examples.

Introduction

The basic equations of a motion of a fluid are written as follows:

$$\begin{aligned} &\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \\ &\rho \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla \cdot \mathbf{T} + \rho \mathbf{f} = 0, \\ &\rho \varepsilon_t = \chi \nabla \cdot \mathbf{q} + \mathbf{T} : \mathbf{D}, \end{aligned} \tag{0.1}$$

where ρ is the density, $\mathbf{v}$ is the velocity, ε is the internal energy, $\mathbf{T}$ is the stress tensor, $\mathbf{D}$ is the velocity deformation tensor, $\mathbf{q}$ is the heat flux, $\mathbf{f}$ is the external

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force, and χ is the heat diffusivity. To Eqs. (0.1) we add suitable boundary and initial conditions. To obtain smooth solutions, we also assume that the initial data satisfy some compatibility conditions at the boundary; otherwise, rather strong singularities can be generated by the abrupt change in boundary conditions.

We denote by S_b a basic stationary flow and by S a nonsteady flow obtained by perturbing S_b. The difference $s = S - S_b$ is called a perturbation of S_b.

To study nonlinear stability problems, we use the direct Lyapunov method which provides a control, at every time moment, of a given norm of the perturbation s. This method does not require any knowledge of a solution except for the existence and regularity of a solution S at all time instants. The choice of norms and how to understand the notion of regularity depend on the problem under consideration. This method consists in constructing a suitable Lyapunov functional $\mathcal{F}(s)$ which is equivalent to the norm of s and satisfies the inequality

$$\frac{d\mathcal{F}(s(t))}{dt} \leqslant 0.$$

If this inequality is strict, then the asymptotic stability holds.

Constructing the best Lyapunov functional is still an open question. Lucky cases are given by the study of stability of the steady state of an incompressible viscous fluid in rigid domains, where $\mathcal{F}(s)$ is the kinetic energy of the flow perturbation. This method is known as the *energy method*. However, such a procedure fails in many general cases, for example, in the case of compressible heat-conducting fluids or fluids with free surface. In this paper, we propose a naive method of constructing a Lyapunov functional in the case where the basic flow is the rest. For the Lyapunov functional we take a linear combination of energies involved into a particular problem. We prove that the rest state S_b is stable provided that the energy attains a strict minimum at S_0. The stability is asymptotical for dissipative systems.

The two goals are independently reached by different tools. In Sec. 1, we show that the hypothesis on the total energy to attain a strict minimum at S_b is sufficient for the stability of a fluid in some cases. In Sec. 2, we propose a variational formulation that provides us with a simple proof of the asymptotic (exponential) decay for dissipative fluids.

We explain such a procedure only for the rest state. However, this construction can be applied to a steady flow [**1, 2**].

In Sec. 1, we study the stability in mean in the three cases: isothermal viscous gases, heat-conducting viscous gases, and a fluid in a circular vertical tube bounded from below by a rigid disk.

In Sec. 2, we study exponential stability in the following cases: isothermal viscous gases, heat-conducting viscous gases, and horizontal layer of fluid bounded from below by a rigid plane, with periodicity conditions on the horizontal variables. The case of a rotating drop of a fluid with surface tension can be found in [**2**].

In Sec. 1, our investigation was inspired by the paper of Arnold [**3**], and the stability we obtain is not asymptotic. In Sec. 2, we follow the method introduced in [**4**].

More general extensions of steady (not rest) motions were studied in [**1, 5**].

1. Stability in Mean

Throughout the section, we consider viscous fluids for the sake of generality. To treat the inviscid case, it suffices to assume that the viscosity coefficients are equal to zero.

1.1. Isothermal viscous fluid. In a bounded domain Ω, the motion is determined by the velocity $\mathbf{v}$ and density ρ which form a solution of the problem

$$\begin{aligned} &\rho(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}) = \nabla\cdot\mathbf{T} + \rho\nabla U,\\ &\rho_t + \nabla\cdot\rho\mathbf{v} = 0, \quad x\in\Omega, \quad t>0,\\ &\mathbf{v} = 0 \quad \text{on } \partial\Omega,\\ &\mathbf{v}|_{t=0} = \mathbf{v}_0(x), \quad x\in\Omega,\\ &\rho(x,0) = \rho_0(x), \end{aligned} \tag{1.1}$$

where $\mathbf{T}(\mathbf{v},\rho) = (-k\rho + \lambda\nabla\cdot\mathbf{v})\mathbf{I} + 2\mu\mathbf{D}$, $2D_{ij} = \left(\dfrac{\partial v_i}{\partial x_j} + \dfrac{\partial v_j}{\partial x_i}\right)_{i,j=1,2,3}$ is the stress tensor, $\mu > 0$ and $\lambda + 2\mu/3$ are the shear and bulk viscosity coefficients satisfying the inequality $3\lambda + 2\mu \geqslant 0$, and $k = R_*\theta$ is proportional to the constant temperature θ. In order for the rest state to be a basic one, we consider an external force with potential U.

The problem (1.1) admits a solution corresponding to the rest state

$$\mathbf{v}_b = 0, \quad \ln\rho_b = -\frac{U}{k}.$$

We assume that the total energy

$$\mathcal{E}(t) = \int\limits_\Omega \left\{\frac{1}{2}v^2 + \rho\ln\rho - \rho U\right\}dx$$

has a strict minimum at $\mathcal{S}_b$. In this case, the thermodynamic potential is given by the Helmholtz free energy $\varphi = k\ln\rho$, and ρ_b is a strict minimum of φ. Let us prove that the basic solution is stable. We choose the initial data such that

$$\int\limits_\Omega \rho_0 dx = M.$$

We recall that the solution of the problem (1.1) is subject to the conservation law of mass and the kinetic energy balance

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \rho dx = 0, \\ &\frac{d}{dt}\left\{\int_\Omega \rho\frac{v^2}{2}dx + k\int_\Omega \rho\ln\rho dx\right\} + \mathcal{D}(t) = \int_\Omega \rho\mathbf{v}\cdot\nabla U dx, \end{aligned} \tag{1.2}$$

where

$$\mathcal{D}(t) = \mu\int_\Omega |\nabla v|^2 dx + (\lambda+\mu)\int_\Omega |\nabla\cdot v|^2 dx.$$

Multiplying the equation $0 = -k\nabla\rho_b + \rho_b\nabla U$ by $\rho\mathbf{v}$, we deduce the following relation expressing the energy balance:

$$\frac{d}{dt}k\int_\Omega \rho\ln\rho_b dx = \int_\Omega \rho\mathbf{v}\cdot\nabla U dx. \tag{1.3}$$

Subtracting (1.3) from (1.2) and taking into account $(1.1)_2$, we get

$$\frac{d}{dt}\left\{\int_\Omega \rho\frac{v^2}{2}dx + k\int_\Omega [\rho(\ln\rho - \ln\rho_b) - (\rho-\rho_b)]dx\right\} + \mathcal{D}(t) = 0. \tag{1.4}$$

We write the integrand of the second integral on the left-hand side of (1.4) as the Taylor polynomial of second degree in a neighborhood of the point ρ_b:

$$k\int_\Omega \{\rho(\ln\rho - \ln\rho_b) - (\rho-\rho_b)\}dx = k\int_\Omega \frac{1}{2\tilde\rho}(\rho-\rho_b)^2 dx. \tag{1.5}$$

This formula yields the natural norm determining the distance from the perturbation of ρ_b, i.e., the L^2-norm.

If $\mathcal{D}$ vanishes, then we can deduce only stability in mean. Otherwise, (1.5) is not sufficient for obtaining the asymptotic decay. This situation will be considered in Sec. 2.

1.2. Heat-conducting viscous fluid. *In a bounded domain Ω, the motion is determined by the velocity $\mathbf{v}$, temperature θ, and density ρ which form a solution of the problem*

$$\begin{aligned} &\rho(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}) = \nabla\cdot\mathbf{T} + \rho\nabla U, \\ &c_v\rho(\theta_t + (\mathbf{v}\cdot\nabla)\theta) = \chi\Delta\theta - R_*\theta\rho\nabla\cdot\mathbf{v} + \frac{1}{2}\mathbf{T}:\mathbf{D}, \\ &\rho_t + \nabla\cdot\rho\mathbf{v} = 0, \quad x\in\Omega, \quad t>0, \\ &\mathbf{v} = 0, \quad \mathbf{n}\cdot\nabla\theta = 0 \quad on\ \partial\Omega, \\ &\mathbf{v}|_{t=0} = \mathbf{v}_0(x), \quad \theta_{t=0} = \theta_0(x), \quad \rho(x,0) = \rho_0(x), \end{aligned} \tag{1.6}$$

where $\mathbf{T}(\mathbf{v},\theta,\rho) = (-k\theta\rho + \lambda\nabla\cdot\mathbf{v})\mathbf{I} + 2\mu\mathbf{D}$ *is the stress tensor,* $\mu > 0$ *and* $\lambda + 2\mu/3$ *are the sshear and bulk viscosity coefficients satisfying the inequality* $3\lambda + 2\mu \geqslant 0$*, and* k *is the gas constant.*

The problem (1.6) admits a solution corresponding to the rest state

$$\mathbf{v}_b = 0, \quad \Delta\theta_b = 0, \quad \ln\rho_b = -\frac{U}{k}.$$

We consider only the case $\theta_b = \text{const}$ (for a more general case cf. [6] and [7]).

Let the initial data satisfy the condition

$$\int_\Omega \rho_0 dx = M.$$

Suppose that the energy

$$\mathcal{E}(t) = \int_\Omega \left\{\frac{1}{2}v^2 + cv(\theta - \theta\ln\theta) + k\theta\rho\ln\rho - \rho U\right\}dx,$$

attains a strict minimum at $\mathcal{S}_b$.

The quantity $\varphi = \varepsilon - \theta_b\eta = c_v\theta + k\theta_b\ln\rho - c_v\theta_b\ln\theta$ looks like the Helmholtz free energy and attains the minimum at ρ_b, θ_b. Let us prove that the basic solution is stable.

We recall that the solution of the problem (1.6) is subject to the conservation law of mass and the balance of the sum of kinetic and internal energy $\varepsilon = c_v\theta$:

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \rho dx = 0, \\ &\frac{d}{dt}\left\{\int_\Omega \rho\frac{v^2}{2}dx + c_v\int_\Omega \rho\theta dx\right\} = \int_\Omega \rho\mathbf{v}\cdot\nabla U dx. \end{aligned} \tag{1.7}$$

If θ_b is a constant, then we have the equality

$$\frac{d}{dt}\int_\Omega k\theta_b\rho\ln\rho_b dx = \int_\Omega \theta_b\rho\mathbf{v}\cdot\nabla U dx. \tag{1.8}$$

Unfortunately, it is not sufficient for our purposes, and we need a new balance law. To this end, we use the so called *general equation of heat transfer*

$$\rho\theta\left(\frac{\partial\eta}{\partial t} + \mathbf{v}\cdot\nabla\eta\right) = \mathbf{T}\cdot\nabla\mathbf{v} + \nabla\cdot(\chi\nabla\theta). \tag{1.9}$$

If there is no viscosity or thermal conduction, the right-hand side vanishes and we obtain the *equation of conservation of entropy* for an ideal fluid.

In particular,

$$\frac{d}{dt}\int_\Omega \rho\eta dx = \int_\Omega \frac{1}{\theta}\mathbf{T}\cdot\nabla\mathbf{v}dx + \int_\Omega \frac{\chi(\nabla\theta)^2}{\theta^2}dx + \int_\Omega \nabla\cdot\frac{\chi\nabla\theta}{\theta}dx. \tag{1.10}$$

Multiplying (1.10) by θ_b and subtracting it from $(1.7)_2$, we find

$$\frac{d}{dt}\left\{\int\limits_{\Omega}\rho\frac{v^2}{2}dx+\int\limits_{\Omega}[\rho(\varepsilon-\theta_b\eta)-k\rho\ln\rho_b-k(\rho-\rho_b)]dx\right\}\leqslant 0. \tag{1.11}$$

Let us consider the function

$$\rho\Phi=\rho[c_v(\theta-\theta_b)+k\theta_b(\ln\rho-\ln\rho_b)-c_v\theta_b(\ln\theta-\ln\theta_b)]-k\theta_b(\rho-\rho_b).$$

It is easy to check that the time-derivative of the integral of $\rho\Phi$ over Ω coincides with the time-derivative of the integral of $\rho\varphi$ over Ω. We compute the Taylor polynomial of second degree in a neighborhood of the point ρ_b, θ_b for the function $\rho\Phi$:

$$[\rho(\varepsilon-\theta_b\eta)-k\rho\ln\rho_b-k(\rho-\rho_b)]=\frac{1}{2}\left(\frac{1}{\bar{\rho}}(\rho-\rho_b)^2+\frac{1}{\bar{\theta}}(\theta-\theta_b)^2\right). \tag{1.12}$$

As above, the L^2-norm is the natural norm for computing the distance from perturbation of ρ_b, θ_b.

1.3. A compressible fluid in a capillary tube. Let Ω_t be a part of a vertical circular tube bounded from below by a rigid disk Σ and from above by a free upper surface Γ_t. Denote by S the lateral surface. The curve obtained as the intersection of Γ_t and S is denoted by $\mathcal{C}$.

Let $\Sigma\in R^2$ be a disk, $C=|\Sigma|h$ volume, $\mu>0$ and λ shear and bulk viscosity coefficients such that $3\lambda+2\mu\geqslant 0$, $\alpha>0$ surface tension, g gravity acceleration, and p_0 external pressure. We denote by x_* the variables in Σ and by ∇_* the derivatives with respect to these variables. It is required to find a domain $\Omega_t=\{(x_*,x_3,t):x_*\in\Sigma,0<x_3<\zeta(x_*,t)\}$ with given Σ and a solution of the system

$$\begin{aligned}
&\rho_t+\nabla\cdot(\rho\mathbf{u})=0,\\
&\rho\mathbf{v}_t+\rho\mathbf{u}\cdot\nabla\mathbf{u}-\nabla\cdot\mathbf{T}=-\rho g\nabla x_3,\\
&\mathbf{u}\cdot\mathbf{n}(x_*,\zeta)|_{\Gamma_t}=\zeta_t n_3,\quad \mathbf{n}\cdot\mathbf{T}(x_*,\zeta)|_{\Gamma_t}=(\alpha\mathcal{H}-p_0)\mathbf{n}(x_*,\zeta)|_{\Gamma_t},\\
&\nu\cdot\nabla_*\zeta|_{\partial\Sigma}=\beta\sqrt{1+|\nabla_*\zeta|^2}|_{\partial\Sigma},\\
&\mathbf{u}(x_*,0,t)|=0,\quad \int\limits_{\Omega_t}\rho dx=M,
\end{aligned} \tag{1.13}$$

where $\mathbf{T}$ is the stress tensor, $\mathbf{T}=-k\rho+2\mu\mathbf{D}+\lambda\nabla\cdot\mathbf{u}\mathbf{I}$ with $(\mathbf{u},\rho,\zeta)$ defined in Ω_t. In the Cartesian coordinate, the double mean curvature of the surface is expressed as follows:

$$\mathcal{H}=\nabla_*\left(\frac{\nabla_*\zeta}{\sqrt{g}}\right),$$

where $g = 1 + |\nabla_* \zeta|^2$ *and* $\mathbf{n} \equiv (-\nabla_* \zeta, 1)\dfrac{1}{\sqrt{g}}$. *Here,* $\mathbf{n}$ *is the unit outward normal to* Γ_t *and* ν *is the unit outward normal to* $\mathcal{C}$. *Furthermore, from the geometrical point of view,* β *can be regarded as the cosine of the contact angle* γ *between* Γ_t *and the cylindrical lateral surface* S.

We note that the condition $(1.13)_6$ is not commonly accepted. Indeed, in the classical statement (cf. [**8**]), the density on the free surface is usually given. In our case, on one side, for any constitutive law the pressure on the rest state with positive density is given. On the other hand, as we know, no uniqueness results are proved.

We begin by constructing an exact solution corresponding to the rest state in a domain Ω, where $\zeta = \zeta(x_*)$ is unknown. Substituting $\mathbf{u} = 0$ in the momentum equation $(1.13)_2$, we conclude that the density has the form

$$\rho_b = \rho_* \exp\left(-\frac{g x_3}{k}\right),$$

where ρ_* is the integration constant. The condition that the total mass is given in the domain $\Sigma \times (0, \zeta(x_*))$ is expressed by the formula

$$\rho_* = \frac{Mg}{k} \int\limits_{\Sigma} \left(1 - \exp\left(-\frac{g\zeta}{k}\right)\right) dx, \tag{1.14}$$

where ζ is unknown. On the free surface, the condition $(1.13)_4$ expressing the continuity of pressure should be satisfied. The unknown is $\zeta_b = \zeta_b(x_*)$. For compressible capillary problems there is not so much literature. We quote a very recent paper by Finn [**9**] where this problem is correctly formulated. Following [**9**] we consider the following elliptic boundary-value problem:

$$\begin{aligned} \mathcal{H} &= \frac{\rho_e g}{\alpha}\zeta + \frac{d \ln \rho_b}{d\zeta}\frac{1}{\sqrt{g}} + \Lambda, \\ \nu \cdot \frac{\nabla_* \zeta}{\sqrt{g}} &= \beta. \end{aligned} \tag{1.15}$$

In this problem, the density on Γ_b is given, whereas Λ is the Lagrange parameter to be determined by the mass constraint.

The possibility of such a formulation for a motion different from the rest state is still an open question. Therefore, we can consider only the case where the density distribution is constant on the boundary.

If the density is a constant on Γ_t then, according to the capillarity theory, the pressure p_b is given by the formula $p_b(x) = k\rho_b$ because the system (1.13) admits a solution corresponding to the rest state with density distribution given by (1.14). The boundary Γ_b of the basic configuration Ω_b, is computed by looking for solutions of the boundary value problem

$$\begin{aligned} &\mathcal{H}(\zeta) - p_0 = 0, \quad \zeta \in \Sigma, \\ &\nu \cdot \nabla_* \zeta = \beta\sqrt{1 + |\nabla_* \zeta|^2}, \quad \zeta \in \mathcal{C}, \end{aligned} \tag{1.16}$$

such that $|\Omega_b| = 4\pi/3$. For large β there exists an infinite set of stationary solutions! The energy is given by the formula

$$\mathcal{E}(t) = \int\limits_{\Omega_t} \left\{\frac{1}{2}v^2 + k\rho \ln \rho + \rho g x_3\right\} dx + \alpha|\Gamma_t| - \beta\alpha \int\limits_{\partial\Sigma} \int\limits_0^{\zeta} \rho_b(x_3) dx_3 ds,$$

where ds is the arc element of circumference on $\partial\Sigma$.

Using the results of [**10**], we can deduce the following energy identity:

$$\frac{d\mathcal{E}}{dt} + D = 0 \tag{1.17}$$

if the integrals have meaning. Again, we find the L^2-norm of the perturbation σ of the density ρ_b and we need to find a norm of the perturbation η of ζ.

Naturally, the above arguments are purely formal if we have no existence theorem. Twenty years ago, Pukhnachov and Solonnikov [**12**] proved that, in the case of a dynamic contact angle, there are no solutions with finite Dirichlet integral if the slip conditions are satisfied. We could formally perform the computation below for $\beta = 0$ which corresponds to the assumption $\gamma = \pi/2$. In this case, the trivial solution $\zeta_b = \text{const}$ is possible. In the sequel, we assume that $\mathcal{E}(t)$ exists and has a strict minimum at $\mathcal{S}_b$. The Dirichlet integral is obviously convergent in the case of periodic boundary conditions (cf. [**11**]). In other cases, the situation is more delicate and the question remains open.

For $\zeta_b = \zeta_b(x_*)$ (1.17) is also satisfied by $\mathcal{F} = \mathcal{E} - \alpha|\Gamma_b|$. It is easy to check that $\mathcal{F}$ is a suitable Lyapunov functional. Indeed, the Taylor expansion gives

$$\begin{aligned} |\Gamma_t| - |\Gamma_b| &= \int\limits_{\Sigma} [\sqrt{1 + |\nabla_*(\eta + \zeta_b)|^2} - \sqrt{1 + |\nabla_* \zeta_b|^2}] dx_* \\ &= \int\limits_{\Sigma} \left[\frac{\nabla_* \zeta_b}{\sqrt{1 + |\nabla_* \zeta_b|^2}} \nabla_* \eta + \frac{1}{\sqrt{1 + |\nabla_* \overline{\zeta}|^2}^{\,3}} \frac{|\nabla_* \eta|^2}{2}\right] dx_*, \end{aligned} \tag{1.18}$$

where $\overline{\zeta}$ is some value of ζ between ζ_b and $\zeta_b + \eta$. Under the assumption $\zeta_b = \text{const}$, we obtain a control for the $W_2^1(\Sigma)$-norm of η if a regular solution exists for all time moments.

In the general case, setting

$$\int\limits_{\Sigma} F(\zeta) dx_* = \alpha|\Gamma_t|,$$

we modify the energy by subtracting the term $\int_\Sigma F(\zeta_b)dx_*$. Furthermore, we assume that

$$\int_\Sigma [F(\zeta) - F(\zeta_b)]dx_* > 0.$$

For $F(\zeta) - F(\zeta_b) > 0$ this means that ζ_b is a stationary point of $F(\zeta)$. By the Taylor expansion of the second degree, we have

$$F(\zeta) - F(\zeta_b) = F'(\zeta_b)\eta + \frac{1}{2}F''(\bar{\zeta})\eta^2.$$

In the functional sense, the property of minimum is much more difficult. As was proved in [**2**], the difference of surface energy terms bounds from above the $W_2^1(\Sigma)$-norm of η under the above assumption that the energy functional attains its minimum. Therefore, in a more general case, it remains to prove that $F(\zeta) - F(\zeta_b) > 0$ bounds from above a suitable norm of η. We propose this as a possible way for studying nonlinear stability.

2. Asymptotic Decay

In this section, we prove the exponential decay to zero of the norms of all perturbations (the L^2-norm in Ω and the W_2^1-norm in Σ) in the case of a viscous fluid. We apply a variational approach. For a general fluid we state the following variational problem:

$$\begin{aligned}
&[(\rho\ln\rho,\psi) + (\rho\mathbf{v},\varphi) + (\rho\varepsilon,\pi)]|_\tau^t - \int_\tau^t [(\rho\ln\rho,\psi_t) + (\rho\mathbf{v},\varphi_t) + (\rho\varepsilon,\pi_t)]ds \\
&\quad - \int_\tau^t [(\rho\ln\rho\mathbf{v},\nabla\psi) + (\rho\mathbf{v}\cdot\nabla\varphi,\mathbf{v}) + (\rho\varepsilon\mathbf{v},\nabla\pi)]ds \\
&\quad = \int_\tau^t \Big[\int_{\Gamma_t} [\mathcal{H} - p_e\Big]\varphi\cdot\mathbf{n}d\sigma]ds + \int_\tau^t \Big[\int_{\Gamma_t} \pi\mathcal{Q}\cdot\mathbf{n}d\sigma\Big]ds \\
&\quad - \int_\tau^t [(\rho\nabla\cdot\mathbf{v},\psi) + (\mathbf{T},\nabla\varphi) + (\rho\mathbf{f},\varphi) + \chi(\mathbf{q},\cdot\nabla\pi) + (\mathbf{T}:\mathbf{D},\pi)]ds, \qquad (2.1)
\end{aligned}$$

where $(\cdot,\cdot)$ denotes the inner product in L^2 and the test functions ψ, φ, π are taken in suitable spaces. In fact, (2.1) contains the boundary conditions on Γ_t

$$\mathbf{v}\cdot\mathbf{n} = \zeta_t n_3, \quad \mathbf{t}\cdot\mathbf{Tn} = 0, \quad \mathbf{n}\cdot\mathbf{Tn} = \mathcal{H} - p_e, \quad \chi\mathbf{q}\cdot\mathbf{n} = \mathcal{Q}\cdot\mathbf{n}. \qquad (2.2)$$

It is easy to derive the energy (kinetic and total) equations.

2.1. Isothermal viscous fluid. In this case, the equality (2.1) holds with $\pi = 0$ and $\Gamma_t = \varnothing$. Therefore,

$$(\rho\mathbf{v}, \varphi)\Big|_\tau^t - \int_\tau^t (\rho\mathbf{v}, \varphi_t) ds - \int_\tau^t [(\rho\mathbf{v} \cdot \nabla\varphi, \mathbf{v})] ds$$
$$= -\int_\tau^t [\mu(\mathbf{D}\mathbf{v}, \nabla\varphi) + (\lambda + \mu)(\nabla \cdot \mathbf{v}, \nabla \cdot \varphi) - k(\rho, \nabla \cdot \varphi) + (\rho\mathbf{f}, \varphi)] ds; \quad (2.3)$$

moreover, we have (1.4) for $\varphi = \mathbf{v}$, which provides us with a dissipative (spatially, temporally) term for $\mathbf{v}$. To find a temporally dissipative term for the density perturbation σ, we use (2.3), where $\psi = 0$ and φ is a solution of the boundary value problem treated in the following assertion.

Lemma 2.1. *Given the fields* $(\mathbf{v}, \sigma)$ *such that*

$$\int_\Omega \sigma dx = 0, \quad \sigma_t = -\nabla(\rho\mathbf{v}) \in L^2(0, \infty; L^2(\Omega)),$$

there exists a vector field $\varphi \in L^\infty(0, \infty; \overset{\circ}{W}{}_2^1(\Omega))$ *such that* $\varphi_t \in L^\infty(0, \infty; L^2(\Omega))$ *and* φ *is a solution of the problem*

$$\begin{aligned} &\nabla \cdot (\rho_b\varphi) = \sigma \quad \text{in } \Omega, \\ &\varphi(x_*, 0, t) = 0, \\ &\varphi \cdot \mathbf{n} = 0, \quad x \in \partial\Omega; \end{aligned} \quad (2.4)$$

moreover, there exists a constant c_1 *depending on* ρ_b, ρ, *and* Ω *and a constant* c_2 *depending on* Ω *such that the following estimates hold*:

$$||\varphi_t||_{L^2(\Omega)} \leqslant c_1||\nabla\mathbf{v}||_{W_2^1(\Omega)}, \quad ||\nabla\varphi||_{L^2(\Omega)} \leqslant c_2||\sigma||_{L^2(\Omega)}. \quad (2.5)$$

For $\mathbf{f} = \nabla U = -k\nabla \ln \rho_b$ and $\rho = \rho_b + \sigma$ we have

$$- k(\rho, \nabla \cdot \varphi) + (\rho\mathbf{f}, \varphi) = -k(\rho_b, \nabla \cdot \varphi) - k(\rho_b \nabla \ln \rho_b, \varphi)$$
$$- k(\sigma, \nabla \cdot \varphi) - k(\sigma\nabla \ln \rho_b, \varphi) = -k\left(\frac{\sigma}{\rho_b}, \nabla \cdot \rho_b\varphi\right) = -k\int_\Omega \frac{\sigma^2}{\rho_b} dx. \quad (2.6)$$

Hence

$$k\int_\tau^t \int_\Omega \frac{\sigma^2}{\rho_b} dx ds - (\rho\mathbf{v}, \varphi)\Big|_\tau^t = \int_\tau^t [-(\rho\mathbf{v}, \varphi_t) - (\rho\mathbf{v} \cdot \nabla\varphi, \mathbf{v})$$
$$+ \mu(\mathbf{D}\mathbf{v}, \nabla\varphi) + (\lambda + \mu)(\nabla \cdot \mathbf{v}, \nabla \cdot \varphi)] ds. \quad (2.7)$$

We add (2.7) multiplied by a constant to (1.4):

$$\left\{\int_\Omega \rho\frac{v^2}{2}dx + k\int_\Omega \{\rho(\ln\rho - \ln\rho_b) - (\rho-\rho_b)\}dx - a(\rho\mathbf{v},\varphi)\right\}\Big|_\tau^t$$
$$+\int_\tau^t \mathcal{D}(s)ds + ak\int_\tau^t\int_\Omega \frac{\sigma^2}{\rho_b}dxds = a\int_\tau^t\{-(\rho\mathbf{v},\varphi_t) - (\rho\mathbf{v}\cdot\nabla\varphi,\mathbf{v})$$
$$+\mu(\mathbf{D}\mathbf{v},\nabla\varphi) + (\lambda+\mu)(\nabla\cdot\mathbf{v},\nabla\cdot\varphi)\}ds. \tag{2.8}$$

The functional

$$H = \int_\Omega \rho\frac{v^2}{2}dx + k\int_\Omega \{\rho(\ln\rho - \ln\rho_b) - (\rho-\rho_b)\}dx - a(\rho\mathbf{v},\varphi)$$

is positive definite for small a. For a sufficiently small a and regular bounded flows the right-hand side of (2.8) can be absorbed into the expression

$$K = \int_\tau^t \mathcal{D}(s)ds + ak\int_\tau^t\int_\Omega \frac{\sigma^2}{\rho_b}ds$$

which represents the (temporally) dissipative term. Since H and K are equivalent, Eq. (2.8) gives the desired exponential decay for regular bounded perturbations (cf. [**4**]) in view of Gronwall's lemma.

2.2. Heat-conducting viscous fluid. In this case, the identity (2.1) holds with $\Gamma_t = \varnothing$ and is reduced to the following identity:

$$[(\rho\ln\rho,\psi) + (\rho\mathbf{v},\varphi) + (\rho\varepsilon,\pi)]|_\tau^t - \int_\tau^t[(\rho\ln\rho,\psi_t) + (\rho\mathbf{v},\varphi_t) + (\rho\varepsilon,\pi_t)]ds$$
$$-\int_\tau^t[(\rho\ln\rho\mathbf{v},\nabla\psi) + (\rho\mathbf{v}\cdot\nabla\varphi,\mathbf{v}) + (\rho\varepsilon\mathbf{v},\nabla\pi)]ds$$
$$= -\int_\tau^t[(\rho\nabla\cdot\mathbf{v},\psi) + (\mathbf{T},\nabla\varphi) + (\rho\mathbf{f},\varphi) + \chi(\mathbf{q},\cdot\nabla\pi) + (\mathbf{T}:\mathbf{D},\pi)]ds \tag{2.9}$$

We set $\tilde\theta = \theta + \theta_b$. Choosing $\varphi = \mathbf{v}$ and $\psi = \pi = 0$, we deduce the kinetic energy identity, whereas, choosing $\varphi = 0$, $\psi = 0$ and $\pi = \theta/\tilde\theta$ we deduce (1.10). To include a dissipative term for σ, we proceed in a similar way as in Subsec. 2.1. Then we obtain a decay estimate (cf. [**6**]).

2.3. Incompressible heavy fluid on horizontal layer. Much more different is the case with a free surface because there are two variables ρ and ζ that satisfy the transport (hyperbolic) equation. However, the approach is the same and only some modifications of the choice of a test function φ are required. In this paper, we deal with a horizontal layer of heavy incompressible fluid assuming the periodicity conditions on the horizontal variables. The compressible case was extensively treated in [**5, 11**], the heat-conducting case was investigated in [**13**], and the problem with dynamical contact angle is in preparation.

The variational formulation can be written as follows:

$$(\rho\mathbf{v},\varphi)\Big|_\tau^t - \int_\tau^t [(\rho\mathbf{v},\varphi_t) + (\rho\mathbf{v}\cdot\nabla\varphi,\mathbf{v})]ds = -\int_\tau^t [\nu(\mathbf{D}\mathbf{v},\nabla\varphi) - (p,\nabla\cdot\varphi)$$

$$+ (\rho\nabla U,\varphi)]ds + \int_\tau^t \Big[\int_{\Gamma_s}[\alpha\mathcal{H} - p_e]\varphi\cdot\mathbf{n}d\sigma\Big]ds. \tag{2.10}$$

We set $\zeta = h+\eta$, where η is a perturbation of the basic height h. We denote by $W_\natural^{1,\infty}(\Sigma)$ and $W_\natural^{1,2}(\Omega)$ the subspaces of the usual Sobolev spaces of functions satisfying the periodicity condition with respect to x_1 and x_2. Taking $\varphi = v$, we obtain Eq. (1.17) with $\rho = 1$. Now, for a test function we take the solution of the problem treated in the following lemma.

Lemma 2.2. *Given a field $\eta \in L^\infty(0,T;W_\natural^{1,\infty}(\Sigma))$ and a domain Ω_t, there exists a vector field $\varphi \in L^\infty(0,\infty;W_\natural^{1,2}(\Omega_t))$ such that $\varphi_t \in L^\infty(0,\infty;L_\natural^2(\Omega_t))$ and φ is a solution of the problem*

$$\begin{aligned} &\nabla\cdot\varphi = 0, \quad x\in\Omega_t, \\ &\varphi(x_*,0,t) = 0, \quad \varphi\cdot\mathbf{n}|_{\Gamma_t} = n_3\eta(x_*,t) = -\frac{\eta(x_*,t)}{\sqrt{1+|\nabla_*\eta|^2}}; \end{aligned} \tag{2.11}$$

moreover, there exist two constants c_1 and c_2 depending on Ω_t such that the following estimates hold:

$$\|\varphi_t\|_{L^2(\Omega_t)} \leqslant c_1\|\eta_t\|_{L^2(\Sigma)}, \qquad \|\nabla\varphi\|_{L^2(\Omega_t)} \leqslant c_2\|\eta\|_{W_2^1(\Sigma)}. \tag{2.12}$$

Taking for φ the solution to this problem, we obtain the dissipative term

$$-\int_{\Omega_t}\mathbf{v}\cdot\varphi dx\Big|_\tau^t - \int_\tau^t\Big[\int_\Sigma(\alpha\mathcal{H} - p_0)\eta dx_*\Big]ds + \int_\tau^t\Big[\int_\Sigma g\zeta\eta dx_*\Big]ds$$

$$= \int_\tau^t\Big[\int_{\Omega_s}\mathbf{v}\cdot\frac{d\varphi}{ds}dx\Big]ds - \int_\tau^t\Big[\int_{\Omega_s}(2\mu\mathbf{D}\cdot\nabla\varphi - p\nabla\cdot\varphi)dx\Big]ds. \tag{2.13}$$

We have

$$-\int\limits_{\Sigma}(\alpha\mathcal{H}-p_0)\eta dx_* + \int\limits_{\Sigma} g\zeta\eta dx_* = \alpha\int\limits_{\Sigma}\frac{|\nabla_*\eta|^2}{\sqrt{1+|\nabla_*\eta|^2}}dx_* + \int\limits_{\Sigma} g\eta^2 dx_* = D_\eta,$$

Adding (1.17) and (2.13) multiplied by a positive constant γ, we find

$$\Big(\frac{1}{2}\int\limits_{\Omega_s}|\mathbf{v}|^2dx+\alpha\frac{1}{2}\int\limits_{\Sigma}(\sqrt{1+|\nabla_*\eta|^2}-1)dx_*+\frac{1}{2}\int\limits_{\Sigma}g\eta^2dx_*\Big)\Big|_\tau^t-\gamma\int\limits_{\Omega_s}\mathbf{v}\cdot\varphi dx\Big|_\tau^t$$

$$2\mu\int\limits_\tau^t\|\mathbf{D}(s)\|^2_{L^2(\Omega_s)}ds+\gamma\alpha\int\limits_\tau^t D_\eta(s)ds=\gamma\int\limits_\tau^t\int\limits_{\Omega_s}\mathbf{v}\cdot\frac{d\varphi}{ds}dxds-2\mu\gamma\int\limits_\tau^t\int\limits_{\Omega_s}\mathbf{D}\cdot\nabla\varphi dxds. \tag{2.14}$$

Now, we introduce the generalized energy

$$\mathcal{E}(t)=\frac{1}{2}\int\limits_{\Omega_t}|\mathbf{v}|^2dx+\alpha\frac{1}{2}\int\limits_{\Sigma}(\sqrt{1+|\nabla_*\eta|^2}-1)dx_*+\frac{1}{2}\int\limits_{\Sigma}g\eta^2dx_*-\gamma\int\limits_{\Omega_t}\mathbf{v}\cdot\varphi dx,$$

which becomes equivalent to the L^2-norm for $\mathbf{v}$ and the $W^1_2(\Sigma)$-norm for η, if γ is sufficiently small. From (2.13) it follows that

$$\mathcal{E}(s)\Big|_\tau^t+2\mu\int\limits_\tau^t\|\mathbf{D}(s)\|^2_{L^2(\Omega_s)}ds+\gamma\alpha\int\limits_\tau^t\mathcal{D}_\eta(s)ds\leqslant\gamma\int\limits_\tau^t(\|\mathbf{v}\|_{L^2(\Omega_s)}\|\varphi_s\|_{L^2(\Omega_s)}$$

$$+\|\mathbf{v}\|_{L^3(\Omega_s)}\|\nabla\varphi\|_{L^2(\Omega_s)}\|\nabla\mathbf{v}\|_{L^2(\Omega_s)}+2\gamma\mu\|\nabla\mathbf{v}\|_{L^2(\Omega_s)}\|\nabla\varphi\|_{L^2(\Omega_s)})ds. \tag{2.15}$$

Therefore, using embedding theorems, Lemma 2.2, and the smallness of γ, we conclude that

$$\mathcal{E}(s)|_\tau^t+a\int\limits_\tau^t(\|\mathbf{D}(s)\|^2_{L^2(\Omega_s)}+\mathcal{D}_\eta(s))ds\leqslant 0, \tag{2.16}$$

for some $a>0$. Since $\mathcal{E}\leqslant c(\|\mathbf{D}\|^2_{L^2(\Omega_s)}+\mathcal{D}_\eta)$, the inequality (2.16) guarantees the exponential decay for $\mathcal{E}$.

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The Fourier Coefficients of Stokes' Waves

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Dedicated to Professor O. A. Ladyzhenskaya

It is common to formulate the Stokes wave problem as Nekrasov's nonlinear integral equation to be satisfied by a periodic function θ which gives the angle between the tangent to the wave and the horizontal. The function θ is odd for symmetric waves. In that case, numerical calculations using spectral methods reveal the coefficients in the sine series of θ to form a sequence of positive terms that converges monotonically to zero. In this paper, we prove that the Fourier sine coefficients of θ form a log-convex sequence that converges monotonically to zero. In harmonic analysis there are many very beautiful theorems about the behavior of functions whose Fourier sine series form a convex monotone sequence tending to zero.

1. Introduction

A known approach to the Stokes wave problem involves its formulation as Nekrasov's integral equation (cf. (2.10) below) to be satisfied by a function ϑ on $(0, \pi)$. If a solution ϑ is extended as an odd 2π-periodic function on $\mathbb{R}$, it gives the angle between the free surface of the wave and the horizontal. As is known, for $\nu > 0$ the extension is real-analytic on $\mathbb{R}$ and the wave profile is smooth. But if $\nu = 0$, then $\lim_{x \searrow 0} \vartheta(x) = \pi/6$ and ϑ represents Stokes' wave of greatest height which has a corner with a contained angle of 120° at its highest point [**1**].

If the Fourier sine coefficients ϑ_k of a solution ϑ are calculated numerically, then they form a monotone decreasing sequence of positive numbers tending

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to zero. In this paper, we show that even more is true: $\{\vartheta_k\}$ is a log-convex sequence converging to 0.

For a singular solution ϑ to Eq. (2.10) with $\nu = 0$ we denote by ϑ' its classical derivative almost everywhere (not the weak derivative since the periodic extension of ϑ has a jump at 0). We explain how the Fourier cosine coefficients of ϑ' form a concave negative sequence converging to zero. A complete proof of the second assertion lies beyond the scope of the present paper and depends on observations about the Stokes-highest-wave problem from a forthcoming article [**2**].

2. Stokes' Waves and Nekrasov's Equation

In hydrodynamics, a Stokes wave is a steady irrotational, two-dimensional, infinitely deep, incompressible flow with a free surface that is periodic in the horizontal direction when gravity acts vertically downwards. Such a flow is described by a conservative velocity field, which is stationary relative to a coordinate frame moving with the wave speed. All parameters except for one can be normalized, so that, in dimensionless coordinates, the wavelength is 2π, the velocity of steady propagation is unity, and the only remaining unspecified physical parameter, denoted by g, is the Froude–number–squared. We are interested only in Stokes' waves which are symmetrical about a vertical line through a crest and for which there is one crest and one trough per wavelength (consequently, each streamline has one crest and one trough per wavelength).

Let an ideal incompressible fluid occupy a region $D = \{(X,Y) : Y < \eta(X)\}$ in the plane of the complex variable $Z = X + iY$. The Stokes-wave problem consists of determining a free surface $S := \{(X, \eta(X)) : X \in \mathbb{R}\}$ and a complex velocity potential $w(Z) = x(Z) + iy(Z)$ that is analytic in D and satisfies the boundary conditions

$$\frac{1}{2}\left|\frac{dw}{dZ}\right|^2 + gY = 0, \quad y = 0 \quad \text{if } Y = \eta(X), \tag{2.1a}$$

$$\eta(X) = \eta(X + 2\pi), \quad \frac{dw}{dZ}(Z) = \frac{dw}{dZ}(Z + 2\pi), \tag{2.1b}$$

$$\begin{aligned} &\eta(-X) = \eta(X),\\ &x(-X + iY) = -x(X + iY),\\ &y(-X + iY) = y(X + iY), \end{aligned} \tag{2.1c}$$

$$\left|\frac{dw}{dZ}\right| \neq 0 \text{ on } D, \quad \frac{dw}{dZ} \to 1 \text{ as } Y \to -\infty, \tag{2.1d}$$

$$0 < \arg \frac{dw}{dZ} < \frac{\pi}{2} \text{ on } D, \quad \eta'(X) < 0 \text{ for } X \in (0, \pi). \tag{2.1e}$$

Here, $g > 0$ is a parameter to be determined as a part of the solution. Later we will consider *singular solutions* to the problem (2.1a)–(2.1e) satisfying the additional condition

$$\eta(0) = 0. \tag{2.2}$$

Singular solutions correspond to the so-called *Stokes wave of greatest height* [**1**] in which the symmetric wave has a stagnation point at the (unique) highest point per wavelength which lies on a line of symmetry [**3**]–[**7**].

We introduce the notation

$$S^- = S \cap \{0 < X < \pi\}, \quad D^- = D \cap \{Z : 0 < X < \pi\} \text{ in the } Z\text{-plane},$$
$$\Omega^- = \{w: \ 0 < x < \pi, \ -\infty < y < 0\} \text{ in the } w\text{-plane}.$$

A solution to the problem (2.1) gives rise to a conformal mapping $Z \mapsto w(Z)$ that takes D^- onto Ω^- and a complex velocity field

$$V(w) \equiv \frac{dw}{dZ}(Z(w)) \equiv \exp(p + i\vartheta) \tag{2.3}$$

that is holomorphic on Ω^-, where p and ϑ are harmonic functions on Ω^-. Hence the free-boundary problem (2.1) is equivalent to the following boundary-value problem for V:

$$V \text{ is holomorphic}, \quad 0 < \arg V < \pi/2 \text{ on } \Omega^-, \tag{2.4a}$$
$$\frac{1}{2}|V(x - i0)|^2 + gY(x - i0) = 0, \quad 0 < x < \pi/2, \tag{2.4b}$$
$$\vartheta(0, y) = \vartheta(\pi, y) = 0, \quad -\infty < y < 0, \tag{2.4c}$$
$$p(x, y) \to 0, \quad \vartheta(x, y) \to 0 \quad \text{as} \quad y \to -\infty, \tag{2.4d}$$

where $Z(w) = X + iY$ satisfies the equation

$$\frac{dZ(w)}{dw} = \frac{1}{V} \quad \text{in } \Omega^-. \tag{2.4e}$$

If (2.2) holds, then

$$V(0) = Z(0) = 0. \tag{2.5}$$

Equation (2.4e) gives

$$\frac{\partial Y}{\partial x} \equiv -e^{-p} \sin \vartheta,$$

which, along with (2.4b), yields

$$\left(\frac{\partial}{\partial x} \left(e^{2p}\right) - 2ge^{-p} \sin \vartheta \right)\Big|_{y=0} = 0.$$

Hence

$$e^{3p(x)} = 3g\Big(\nu + \int\limits_0^x \sin\vartheta(s)\,ds\Big), \tag{2.6}$$

where $\nu \geqslant 0$ is a constant referred to as *Nekrasov's parameter*. Therefore,

$$\frac{\partial p}{\partial x}(x) = \frac{1}{3}\frac{\sin\vartheta(x)}{\nu + \int\limits_0^x \sin\vartheta(s)ds} \tag{2.7}$$

for $y = 0$. (If the singularity condition (2.5) holds, then Nekrasov's parameter is equal to zero and

$$e^{3p(x)} = 3g\int\limits_0^x \sin\vartheta(s)\,ds \text{ and } \frac{\partial p}{\partial x}(x) = \frac{\sin\vartheta(x)}{3\int\limits_0^x \sin\vartheta(s)ds} \tag{2.8}$$

for $y = 0$.) Since $p_x = \vartheta_y$, we get the following boundary-value problem for a bounded harmonic function ϑ in the half-strip Ω^-:

$$\frac{\partial\vartheta}{\partial y} = \frac{1}{3}\frac{\sin\vartheta(x)}{\nu + \int\limits_0^x \sin\vartheta(s)ds},$$

$$0 < \vartheta(x) < \pi/2, \quad 0 < x < \pi \ y = 0, \tag{2.9a}$$

$$\vartheta = 0, \quad x \in \{0, \pi\} \ -\infty < y < 0. \tag{2.9b}$$

Solving these equations with respect to ϑ, we can find p, g as a solution to the boundary-value problem

$$p = \frac{1}{3}\log\Big\{3g\Big(\nu + \int\limits_0^x \sin\vartheta(s)ds\Big)\Big\}, \quad 0 < x < \pi, \ y = 0, \tag{2.9c}$$

$$\frac{\partial p}{\partial x} = 0, \quad x \in \{0, \pi\}, \ -\infty < y < 0, \tag{2.9d}$$

$$p \text{ is bounded from above and } p \to 0 \text{ as } y \to -\infty. \tag{2.9e}$$

It is obvious that if $p + i\vartheta$ is smooth on the set $\operatorname{cl}\Omega^- \setminus \{0\}$ and satisfies (2.9), then $V = \exp(p + i\vartheta)$ is a solution to the problem (2.4). We also note that the function $\vartheta(x) = \vartheta(x, 0)$ satisfies *Nekrasov's integral equation* [**8**]–[**10**]

$$\vartheta(x) = \frac{1}{3}\int\limits_0^\pi K(x,t)\frac{\sin\vartheta(t)}{\nu + \int\limits_0^t \sin\vartheta(\tau)d\tau}dt, \quad 0 < \vartheta(x) < \pi/2, \ x \in (0, \pi), \tag{2.10}$$

where

$$K(x,t) = \frac{1}{\pi}\log\left|\frac{\tan x + \tan t}{\tan x - \tan t}\right|.$$

It is easy to see that any solution ϑ of Eq. (2.10) gives rise to a solution of the problem (2.9), and *vice versa*. Various local and global aspects of the existence theory for Nekrasov's equation [8] can be found in [10]–[16].

3. Analytic Continuation of the Velocity Field

Suppose that $p + i\vartheta$, ν is a solution to the problem (2.9). We consider the problem of determining an analytic function $\omega = \omega_1 + i\omega_2$ such that

$$\frac{d\omega}{dw} = \frac{g}{i}\left(\frac{V}{\omega} - \frac{1}{V}\right) \quad \text{on} \quad \Omega^-, \tag{3.1a}$$

$$\omega(x - i0) = -2gY(x - i0) > 0, \quad 0 < x < \pi. \tag{3.1b}$$

As above, $V = \exp(p + i\vartheta)$ and, because of the connection with (2.4), this problem is equivalent to the following problem in the Z-plane:

$$\frac{d\omega}{dZ} = \frac{g}{i}\left(\frac{V^2(w(Z))}{\omega} - 1\right) \quad \text{on} \quad D^-, \tag{3.2a}$$

$$\omega(X + i\eta(X)) = -2g\eta(X), \quad 0 < X < \pi. \tag{3.2b}$$

The first theorem coming back to [17] and [6] is a shortened version of a result in [2].

Theorem 1. *The problem* (3.1) *has a unique solution on* Ω^-,

$$|\omega| > 0, \quad -\pi/2 < \arg\omega < 0 \quad on\ \Omega^-. \tag{3.3}$$

Moreover, ω *has a continuous extension to* $\operatorname{cl}\Omega^-$ *that is real for* $x = \pi$, *and* $\arg\omega(x + iy)$ *converges pointwise* (*and boundedly*) *to a function of* iy, $y \in (-\infty, 0)$, *as* $x \searrow 0$. *We write*

$$\Phi(y) = -\arg\omega(iy), \quad y < 0.$$

PROOF. We set

$$\omega(x - i0) \equiv -2gY(x - i0), \quad 0 < x < \pi. \tag{3.4}$$

From this equation and Eqs. (2.4e) and (2.4b) it follows that

$$\frac{\partial\omega}{\partial x} = 2ge^{-p}\sin\vartheta, \quad \omega = e^{2p}, \quad 0 < x < \pi,\ y = 0. \tag{3.5}$$

Hence

$$\frac{\partial\omega}{\partial x} = \frac{g}{i}\left(\frac{e^{p+i\vartheta}}{e^{2p}} - e^{-p-i\vartheta}\right),$$

which gives

$$\frac{\partial\omega}{\partial x} = \frac{g}{i}\left(\frac{V}{\omega} - \frac{1}{V}\right), \quad 0 < x < \pi,\ y = 0.$$

Therefore, the function ω in (3.4) satisfies (3.1a) on the real line. By Cauchy's theorem, the problem (3.1) has a unique solution defined in a neighborhood of the interval $(0,\pi)$ of the real line. Since $\omega_{2,y}=\omega_{1,x}>0$ on the open interval $(0,\pi)$, every point $x_0\in(0,\pi)$ has a neighborhood where ω satisfies (3.3).

We show that ω can be analytically continued to Ω^- by working in the Z-plane. We have seen that ω^* defined by $\omega^*(Z)=\omega(w(Z))$ is analytic in a neighborhood of every point of S^- and satisfies (3.3) there. The right-hand side of Eq.(3.2a) is analytic in $\omega\in\mathbb{C}\setminus\{0\}$ and holomorphic in $Z\in D^-$; moreover, the numerator and denominator of this function do not vanish on $\mathbb{C}\times D^-$ simultaneously. By the general theory of analytic differential equations, every solution of Eq. (3.2a) can have only irregular points corresponding to the singular value $\omega^*=0$.

For every $X_0+iY_0\in S^-$, $X_0\in(0,\pi)$, $Y_0=\eta(X_0)$, the ray $\operatorname{Re}Z=X_0$, $\operatorname{Im}Z<Y_0$ belongs to D^- and the union of all such rays covers D^-. It suffices to show that ω^* can be analytically continued along each of these rays and that the continuation satisfies Eq. (3.3).

Along such a ray, (3.2) leads to the Cauchy problem for the ordinary differential equation

$$\frac{d\omega^*}{dY}=g\left(\frac{V^2}{\omega^*}-1\right),\quad Y<Y_0, \tag{3.6}$$

and $\omega^*(X_0+iY_0)=-2gY_0$.

Let (Y_1,Y_0) be the maximal interval of existence of an analytic solution of (3.6) with

$$|\omega^*(X_0+iY)|>0,\quad -\pi/2<\arg\omega^*(X_0+iY)<0. \tag{3.7}$$

Multiplying both sides of Eq. (3.6) by ω^*, we find

$$\frac{d(\omega^*)^2}{dY}=2g\left(V^2-\omega^*\right) \tag{3.8}$$

for $Y_1<Y<Y_0$, which yields

$$\left|\frac{d}{dY}\left(\omega^*(X,Y)^2\right)\right|\leqslant 2g\left(|V(X,Y)|^2+1+|\omega^*(X,Y)^2|\right). \tag{3.9}$$

Hence $\omega^*(X,Y)$ remains bounded as $Y\searrow Y_1$ if $Y_1\neq-\infty$. We set $\omega^*=|\omega^*|\exp i\theta^*$. If $Y_1\neq-\infty$, then

$$\left|\omega^*(X_0+iY)\right|^2\sin(2\theta^*(X_0+iY))\to 0 \text{ as } Y\searrow Y_1 \tag{3.10}$$

and for $X=X_0$ and $Y_1<Y<Y_0$ we have

$$\frac{d}{dY}\left(|\omega^*|^2\sin(2\theta^*)\right)=2g\left(|V|^2\sin(2\vartheta)-|\omega^*|\sin((\theta^*)\right). \tag{3.11}$$

Since $|V|>0$ and $0<\vartheta<\pi/2$ in D^-, from Eq. (3.7) it follows that $|\omega^*|^2\sin(2\theta^*)$ is a negative increasing function, which contradicts (3.10). Hence $Y_1=-\infty$, and the existence of a solution to (3.6) on Ω^- is proved.

Since $\omega^*(X+iY)$ lies in the fourth quadrant for all $X+iY \in \Omega^-$, from (3.9) if follows that it converges uniformly on compact Y-intervals as $X \searrow 0$ to a continuous complex-valued function $\omega^*(0+iY)$ satisfying (3.8) for almost all $Y < Y_0 = \eta(0)$. It is immediately follows that the zeros of $\omega^*(0+iY)$, $Y < \eta(0)$, are isolated and, consequently, $\arg\omega^*(X+iY)$ converges pointwise almost everywhere and boundedly to $\arg\omega^*(0+iY)$ as $X \searrow 0$.

Let $(\pi, Y_\pi) = (\pi, \eta(\pi)) \in S$. The above arguments can be repeated with $X_0 = \pi$ to obtain the global existence for $Y < Y_\pi < 0$ of a solution of (3.8). Since $V(\pi+iY)$ is real for $\pi+iY \in D^-$ and $\omega^*(\pi+iY_\pi)$ is positive and real, this solution is obviously positive and real for $Y \in (Y_1, Y_\pi)$ for some $Y_1 < Y_\pi$ if $X = \pi$. By (3.6), we have $(\omega^*)'(\pi+iY_\pi) = 0$. Differentiating (3.6) with respect to Y $(' = d/dY)$, we find

$$\omega'' + \frac{gV^2\omega'}{\omega^2} = \frac{2gVV'}{\omega}, \ X = \pi. \tag{3.12}$$

However, $\big(d/dY\big)V(\pi+iY) > 0$ for all $Y \leqslant Y_\pi$. Therefore, $(\omega^*)''(Y_\pi) > 0$. Hence we can assume that $\omega^*(\pi, Y)$ is decreasing on (Y_1, Y_π). However, by (3.12), the only critical points of $\omega^*(\pi+iY)$ are minima. Hence $\omega^*(\pi+iY)$ is a real-valued decreasing function on $(-\infty, Y_\pi)$. It is easy to see that this solution is a continuous extension to $\mathrm{cl}(D^-) \cap \{X = \pi\}$ of ω^* on D^-.

Analogous results for ω are obvious. □

REMARK. Since $X = 0$ implies $\big(d/dY\big)V(iY) < 0$ for $Y < \eta(0)$, the above arguments fail if we want to prove that ω^* is real for $X = 0$. Indeed, for $X = 0$ from the above arguments it follows that $\omega^*(iY)$ is concave and increasing on any interval (Y_1, Y_0) on which it is real. Hence $\omega^*(iY)$ has a zero $Y^* \in (-\infty, Y_0)$. It can be shown that $\omega^*(iy)$ is complex on $(-\infty, Y^*)$. If Nekrasov's parameter is equal to zero, which is equivalent to the equality $(X_0, Y_0) = (0,0)$, then $Y^* = Y_0 = 0$ and $\omega^*(0+iY)$ is complex on $(-\infty, 0)$.

We are ready to prove the existence of a harmonic continuation of the function $\vartheta : \Omega^- \to \mathbb{C}$.

Theorem 2. *Let (p, ϑ) satisfy (2.9). Then ϑ has a harmonic continuation to the strip Ω that satisfies the boundary conditions*

$$\vartheta(0,y) = \Theta(y), \quad 0 < y < \infty, \qquad \vartheta(0,y) = 0, \quad -\infty < y < 0, \tag{3.13}$$

$$\vartheta(\pi, y) = 0, \quad -\infty < y < \infty. \tag{3.14}$$

Here, Θ is defined by $\Theta(y) = \Phi(-y)$, $y > 0$, where $\Phi(y)$ is the same as in Theorem 1. In particular, $0 \leqslant \Theta(y) \leqslant \pi/2$,

$$\vartheta(x) = \frac{1}{2\pi}\int_0^\infty \frac{\Theta(y)\sin x}{\cosh y - \cos x}dy. \tag{3.15}$$

Hence

$$0 < \vartheta(x) < \pi/4, \quad x \in (0, \pi), \tag{3.16}$$

PROOF. Recall that $\Omega = \{w = x + iy : 0 < x < \pi\}$ and that $\Phi : (-\infty, 0] \to [0, \pi/2]$ is defined by

$$\Phi(y) := -\arg\ \omega(iy), \quad -\infty < y < 0. \tag{3.17}$$

We note that

$$2\log|V| = 2p = \log\omega, \quad 0 < x < \pi, \quad y = 0.$$

Therefore, the holomorphic function

$$\vartheta - ip + \frac{1}{2} i\log\omega : \Omega^- \mapsto \mathbb{C}$$

is continuous on $\mathrm{cl}\,\Omega^- \setminus \{0\}$ and takes only real values on the real axis. Hence the real part $\vartheta - \dfrac{1}{2}$ arg ω has an even harmonic continuation to the strip Ω. On the other hand, the holomorphic function $\log\omega : \Omega^- \mapsto \mathbb{C}$ takes only real values on the real axis and its imaginary part has an odd harmonic continuation to Ω. We conclude that the function $\vartheta^* : \Omega \mapsto \mathbb{R}$ defined by

$$\vartheta^*(x, y) = \vartheta(x, y), \quad (x, y) \in \Omega^-,$$
$$\vartheta^*(x, y) = \vartheta(x, -y) - \arg\omega(x, -y), \quad (x, y) \in \Omega \setminus \Omega^-$$

is harmonic in the whole strip Ω. It remains to note that ϑ^* vanishes if $x = \pi$ and if $x = 0$ for $y < 0$.

The relation (3.15) follows from the Green function representation of a bounded harmonic function satisfying the Dirichlet conditions on the strip Ω. Since

$$\frac{1}{2\pi} \int_0^\infty \frac{\sin x}{\cosh y - \cos x} dy = \frac{1}{2},$$

(3.16) follows from (3.15) because $0 \leqslant \Theta \leqslant \pi/2$. This completes the proof of the theorem. □

Note that the expression

$$\coth((x + iy)/2) = \frac{\sin x}{\cosh y - \cos x} - i\frac{\sinh y}{\cosh y - \cos x}$$

defines an analytic function. We recall the following formulas based on (3.15):

$$\vartheta(x, 0) = \frac{1}{2\pi} \int_0^\infty \frac{\sin x}{\cosh t - \cos x} \Theta(t)\, dt, \tag{3.18}$$

$$\vartheta_x(x,0) = \frac{1}{2\pi}\int_0^\infty \Big(\frac{\sin x}{\cosh t - \cos x}\Big)_x \Theta(t)\,dt$$

$$= \frac{1}{2\pi}\int_0^\infty \Big(\frac{-\sinh t}{\cosh t - \cos x}\Big)_t \Theta(t)\,dt$$

$$= \frac{1}{2\pi}\int_0^\infty \frac{\sinh t}{\cosh t - \cos x}\Theta'(t)\,dt, \tag{3.19}$$

$$\vartheta_{xx}(x,0) = \frac{-1}{2\pi}\int_0^\infty \frac{\sinh t \sin x}{(\cosh t - \cos x)^2}\Theta'(t)\,dt. \tag{3.20}$$

4. The Main Result on Fourier Coefficients

Let ϑ denote a solution of Nekrasov's equation (2.10) for some $\nu \geqslant 0$, and let ϑ_k be the kth Fourier sine coefficient of ϑ:

$$\vartheta_k = \frac{2}{\pi}\int_0^\pi \vartheta(x)\sin kx\,dx, \quad k \in \mathbb{N}.$$

Theorem 3. *The sequence $\{\vartheta_k\}$ is a monotonically decreasing sequence of positive numbers; moreover, if $K = N/p + M/q$, $1/p + 1/q = 1$, $M, N, K \in \mathbb{N}$, then*

$$\vartheta_K \leqslant \vartheta_N^{1/p}\vartheta_M^{1/q} \leqslant \frac{\vartheta_N}{p} + \frac{\vartheta_M}{q}. \tag{4.1}$$

In other words, the sequence $\{\vartheta_k\}$ is log-*convex and, consequently, is convex and decreasing to* 0.

PROOF. Let Ω denote the strip $(x,y) \in (0,\pi) \times \mathbb{R}$, and let $\Theta(y)$ denote the restriction to the axis $x = 0$ of the harmonic extension to Ω of the harmonic function ϑ on Ω^- corresponding to the extreme wave. Then $\Theta(y) = 0$ for $y < 0$ and $0 < \Theta(y) < \frac{1}{2}\pi$ for $y > 0$. Consider the boundary-value problem

$$\Delta u = 0 \text{ in } \Omega, \tag{4.2}$$

$$u(0,y) = \Theta(y), \quad u(\pi,y) = 0. \tag{4.3}$$

The solution can be explicitly written using the Green's function

$$G(x,t) = \frac{1}{2\pi}\frac{\sin x}{\cosh t - \cos x}, \quad x \in (0,\pi),\ t \in \mathbb{R}.$$

We have

$$u(x,y)=\frac{1}{2\pi}\int_0^\infty \frac{\sin x}{\cosh(y-t)-\cos x}\Theta(t)\,dt. \tag{4.4}$$

Since u and ϑ coincide on $y=0$ and

$$0\leqslant \frac{\sin x}{\cosh t-\cos x}\leqslant \frac{\sin x}{1-\cos x}=\coth(x/2),$$

on the basis of Fubini's theorem we find

$$\int_0^\pi \vartheta(x,0)\sin kx\,dx=\frac{1}{2\pi}\int_0^\pi\int_0^\infty \frac{\sin x\sin kx}{\cosh t-\cos x}\Theta(t)\,dt\,dx$$

$$=\frac{1}{2\pi}\int_0^\infty\int_0^\pi \frac{2e^t\sin x\sin kx}{e^{2t}+1-2e^y\cos x}dx\,\Theta(t)dt=\frac{1}{2}\int_0^\infty e^{-kt}\Theta(t)\,dt. \tag{4.5}$$

Then (4.1) follows from Hölder's inequality and Young's inequality. □

REMARK. Formula (4.5) can be derived from Green's formula

$$\int_{\partial\Omega+} u\frac{\partial v}{\partial\eta}\,dS=\int_{\partial\Omega+} v\frac{\partial u}{\partial\eta}\,dS$$

with $v(x,y)=e^{-ky}\sin kx$.

Suppose that ϑ is a nonzero solution of (2.10) with $\nu=0$. Then (cf. [**4**]–[**6**]) $\lim_{t\searrow 0}\vartheta(t)=\pi/6$, the classical derivative ϑ' of ϑ on $(0,\pi)$ is integrable, and the following relation holds:

$$\frac{2}{\pi}\int_0^\pi \vartheta'\cos kx\,dx=-\frac{2}{\pi}\vartheta(0+)+k\vartheta_k=-\frac{1}{3}+k\vartheta_k. \tag{4.6}$$

By the Riemann–Lebesgue lemma, we have

$$k\vartheta_k\to 1/3 \text{ as } k\to\infty. \tag{4.7}$$

The sequence of the second differences of $\{k\vartheta_k\}$ is

$$\{\Delta_k^2\}=\big\{k(\vartheta_k-2\vartheta_{k+1}+\vartheta_{k+2})-2(\vartheta_{k+1}-\vartheta_{k+2})\big\}$$

and, by formula (4.5), can be written as follows:

$$\frac{1}{\pi}\int_0^\infty \big\{ke^{-kt}\big(1-e^{-t}\big)^2-2e^{-kt}(e^{-t}-e^{-2t})\big\}\Theta(t)\,dt$$

$$=\frac{1}{\pi}\int_0^\infty -((e^{-kt})(1-e^{-t})^2)'\Theta(t)\,dt=\frac{1}{\pi}\int_0^\infty (e^{-kt})(1-e^{-t})^2\Theta'(t)\,dt.$$

If we could show that the sequence of cosine coefficients of ϑ' is concave, then the negativity of ϑ' and the monotonicity of ϑ on $[0,\pi]$ would follow from [**18**, I, Ch. V, (1.5)]. Clearly, $\Theta' < 0$ on $(0,\infty)$ implies the concavity of $\{k\vartheta_k\}$. More generally, the following assertion holds.

Lemma 4. *Suppose that there exists $d > 0$ such that Θ is decreasing on $[0,d]$ and $\Theta(t) \leqslant \Theta(d)$ on $[d,\infty)$. Then*

$$\Delta_k^2 \leqslant 0 \quad \text{for all } k \text{ such that } t_k \in [0,d]$$

and, in particular, for all sufficiently large k. If $d \geqslant \log 3 \approx 1.098$, then $\{k\vartheta_k\}$ is concave.

PROOF. We note the expression

$$ke^{-kt}(1-e^{-t})^2 - 2e^{-kt}(e^{-t}-e^{-2t}) = -\big((e^{-kt})(1-e^{-t})^2\big)'$$

regarded as a function of t has the zero mean value on $(0,\infty)$, has one zero t_k at which

$$e^{-t_k} = \frac{k}{k+2} \in [1/3, 1],$$

is negative on $[0,t_k)$, and is positive on (t_k,∞). Suppose that $t_k \in [0,d]$. Then

$$\begin{aligned}
\Delta_k^2 &= \frac{1}{\pi}\int_0^\infty \big\{ke^{-kt}(1-e^{-t})^2 - 2e^{-kt}(e^{-t}-e^{-2t})\big\}\Theta(t)\,dt \\
&= \frac{1}{\pi}\Big(\int_0^{t_k} + \int_{t_k}^\infty\Big)\big\{ke^{-kt}(1-e^{-t})^2 - 2e^{-kt}(e^{-t}-e^{-2t})\big\}\Theta(t)\,dt \\
&\leqslant \frac{\Theta(t_k)}{\pi}\int_0^{t_k}\big\{ke^{-kt}(1-e^{-t})^2 - 2e^{-kt}(e^{-t}-e^{-2t})\big\}\,dt \\
&+ \frac{\Theta(t_k)}{\pi}\int_{t_k}^\infty\big\{ke^{-kt}(1-e^{-t})^2 - 2e^{-kt}(e^{-t}-e^{-2t})\big\}\,dt = 0.
\end{aligned}$$

If $d \geqslant \log 3$, then $t_k \in [0,d]$ for all k and the required assertion follows. □

Theorem 5 (cf. [**2**]). *There exists a solution ϑ of Nekrasov's equation with $\nu = 0$ such that Θ is monotone on $(0,\infty)$.*

The proof of this theorem is long and far beyond the scope of the present article.

Corollary 6. *The function ϑ in Theorem 5 is monotone and convex on $(0,\pi)$, the sine coefficients of ϑ are* log*-convex, and the cosine coefficients of ϑ' are concave.*

PROOF. The assertion immediately follows from (3.15), (3.18), (3.19), and (3.20). □

5. Remarks

So far, the result of [2] does not say that all solutions of (2.10) with $\nu = 0$ gives rise to a monotone function Θ. Nevertheless, the following general consequences of (3.15) are of interest for $\nu > 0$.

Lemma 7. *Suppose that ϑ is a solution of* (2.10) *with $\nu \geqslant 0$. Then*
(a) $\vartheta_x(0,x) < 0$, $x \in [\pi/2, \pi]$,
(b) $x_1 \in (0, \pi/2)$ *and* $x \in (\pi - x_1, \pi]$ *imply* $\vartheta(x_1) > \theta(x)$,
(c) $-2\cos^2 \frac{1}{2} x \vartheta(x) < \sin x \vartheta_x(x,0) - \cos x \vartheta(x,0) < 0$ *on* $(0,\pi]$ *and, if* $\Theta(y) \to \pi/3$ *as* $y \to 0$, *then* $\sin x \vartheta_x(x,0) - \cos x \vartheta(x,0)$ *is a continuous negative function equal to* $-\pi/6$ *at* $x = 0$.

PROOF. Assertion (a) holds since

$$\vartheta_x(x,0) = \frac{1}{2\pi}\int_0^\infty \Big(\frac{\sin x}{\cosh t - \cos x}\Big)_x \Theta(t)\,dt = \frac{1}{2\pi}\int_0^\infty \frac{\cosh t \cos x - 1}{(\cosh t - \cos x)^2}\Theta(t)dt.$$

This expression is negative if $x \in [\pi/2, /\pi]$ because $\Theta > 0$. Since $\theta > 0$, assertion (b) follows from the relation

$$\begin{aligned}\vartheta(x_1, 0) &= \frac{1}{2\pi}\int_0^\infty \frac{\sin x_1}{\cosh t - \cos x_1}\Theta(t)\,dt \geqslant \frac{1}{2\pi}\int_0^\infty \frac{\sin x}{\cosh t - \cos x}\Theta(t)\,dt \\ &= \vartheta(x,0)\end{aligned}$$

under the above choice of x and x_1.

(c) We note that

$$\begin{aligned}\sin x\Big(\frac{\sin x}{\cosh t - \cos x}\Big)_x - \cos x\Big(\frac{\sin x}{\cosh t - \cos x}\Big) &= \frac{-\sin^3 x}{(\cosh t - \cos x)^2} \\ &> \frac{-2\sin x \cos^2 \frac{1}{2}x}{\cosh t - \cos x}\end{aligned}$$

for $x \in (0, \pi]$. This leads to the first part of (c). To prove the second part, we note that

$$\frac{1}{\pi}\int_0^\infty \frac{\sin^3 x}{(\cosh t - \cos x)^2}dt = \frac{1}{\pi}\big((\pi - x)\cos x + \sin x\big).$$

Hence the integrand on the left-hand side behaves itself like that of an approximate identity as $x \to 0$. In particular, if $\Theta(t) \to \pi/3$ as $t \to 0$, then

$\sin x\ \vartheta_x(x,0) - \cos x\ \vartheta(x,0)$ is a continuous negative function equal to $-\pi/6$ at $t = 0$. □

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A Geometric Regularity Estimate via Fully Nonlinear Elliptic Equations

Reiner Schätzle

Dedicated to Olga A. Ladyzhenskaya on her birthday

We prove that integral n-varifolds μ in codimension 1 with $H_\mu \in L^p_{\text{loc}}(\mu)$, $p > n$, $p \geqslant 2$, have quadratic tilt-excess decay $\text{tiltex}_\mu(x, \varrho, T_x\mu) = O_x(\varrho^2)$ for μ-almost all x. This regularity estimate is used to establish a general convergence procedure for hypersurfaces Σ_j with interior E_j whose mean curvatures are given by the trace of ambient Sobolev functions $\vec{\mathbf{H}}_{\Sigma_j} = u_j \nu_{E_j}$ on Σ_j, where ν_{E_j} denotes the inner normal of Σ_j.

1. Quadratic Tilt-Excess Decay

Allard proved as a special case of Theorem 8.16 in [**1**] that integral n-varifolds μ with weak mean curvature $H_\mu \in L^p_{\text{loc}}(\mu)$, $p > n$, $p \geqslant 2$, are regular at points with unit density $\theta^n(\mu, x_0) = 1$. In his proof, he showed that the tilt-excess defined by

$$\text{tiltex}_\mu(x, \varrho, T) := \varrho^{-n} \int\limits_{B_\varrho(x)} \|T_\xi\mu - T\|^2 d\mu(\xi), \tag{1.1}$$

for notions in geometric measure theory (cf. [**2**] or [**3**]), decays for suitable planes $T = T_\varrho$ with a power in the radius ϱ near points of unit-density. This implies that the tangent plane is Hölder continuous. Hence the varifold is regular in a neighborhood. The assumption of unit-density ensures that the varifold consists of only one layer and excludes interaction between different layers. Actually, varifolds need not be the union of regular graphs in case of higher density even for $H_\mu \in L^\infty(\mu)$ as an example of Brakke in [**4**, § 6.1] shows. Instead if the weak second fundamental form is assumed to be $A_\mu \in L^p_{\mathrm{loc}}(\mu)$, $p > n$, then μ is the graph of a Q-valued function (cf. [**5**]).

On the other hand, Brakke improved the tilt-excess decay estimate in the higher density case by using a blow-up technique to the nearly optimal estimate that

$$\operatorname{tiltex}_\mu(x, \varrho, T_x\mu) = o_x(\varrho^{2-\varepsilon}) \tag{1.2}$$

for any $\varepsilon > 0$ and for μ-almost all x if $H_\mu \in L^2_{\mathrm{loc}}(\mu)$ (cf. [**4**, Theorems 5.5 and 5.6]). Even for smooth submanifolds, only quadratic decay of the tilt-excess is in general true, as the second fundamental form gives a lower bound.

The goal of this section is to derive the optimal quadratic decay in the case of codimension 1.

Theorem 1.1 (quadratic tilt-excess decay, [**6**, Theorem 5.1]). *Let μ be an integral n-varifold in $\Omega \subseteq \mathbb{R}^{n+1}$ with $H_\mu \in L^p_{\mathrm{loc}}(\mu)$, $p > n$, $p \geqslant 2$. Then for μ-almost all $x \in \operatorname{spt}\mu$ the tilt-excess decays quadratically, i.e.,*

$$\operatorname{tiltex}_\mu(x, \varrho, T_x\mu) = O_x(\varrho^2). \tag{1.3}$$

Proof. We assume $\Omega = U \times \mathbb{R}$ and write $x = (y, t) \in \mathbb{R}^n \times \mathbb{R}$. The upper and lower height functions $\varphi_\pm : U \to [-\infty, \infty]$ of μ are defined by

$$\varphi_+(y) := \sup\{t \mid (y,t) \in \operatorname{spt}\mu\}, \quad \varphi_-(y) := \inf\{t \mid (y,t) \in \operatorname{spt}\mu\} \tag{1.4}$$

for $y \in U$, where we set $\varphi_+(y) = -\infty$ and $\varphi_-(y) = +\infty$ if $\operatorname{spt}\mu \cap (\{y\} \times \mathbb{R}) = \varnothing$.

First we consider the case $H_\mu = 0$, i.e., μ is stationary, and claim that φ_+ is a C^2-viscosity subsolution of the minimal surface equation

$$-\nabla\left(\frac{\nabla\varphi_+}{\sqrt{1 + |\nabla\varphi_+|^2}}\right) \leqslant 0. \tag{1.5}$$

By the definition of viscosity solutions (cf. [**7**]–[**9**]), this means that any C^2-function ψ such that $\varphi_+ - \psi$ has an interior maximum, say at $y \in U$, satisfies the inequality

$$-\nabla\left(\frac{\nabla\psi}{\sqrt{1 + |\nabla\psi|^2}}\right)(y) \leqslant 0.$$

Indeed, otherwise,

$$-\nabla\left(\frac{\nabla\psi}{\sqrt{1 + |\nabla\psi|^2}}\right) \geqslant \tau > 0$$

in a neighborhood of y and $\operatorname{graph}(\psi + (\varphi_+ - \psi)(y))$ touches $\operatorname{spt}\mu$ from above at $(y, \varphi_+(y))$, which is impossible according to the maximum principle of Solomon and White [**10**], and (1.5) is proved.

Now, we use a theorem on fully nonlinear elliptic equations due to Caffarelli [**11**] and Trudinger [**12**] (cf. also [**8**, Lemma 7.8] and [**9**]) which states that subsolutions of uniformly elliptic equations with right-hand side in L^n are touched from above by paraboloids or likewise have second order superdifferentials almost everywhere. We apply this theorem to a sup-convolution of order 1 of φ_+, i.e.,

$$\varphi_+^\varepsilon(y) := \sup_z \Big(\varphi(z) - \frac{1}{\varepsilon}|y-z| \Big).$$

By a standard procedure for sup-convolutions, φ_+^ε is a subsolution of (1.5) as well (cf. [**8**, § 5.1]).As φ_+^ε is clearly Lipschitz, this equation is now uniformly elliptic for φ_+^ε, and we can apply Caffarelli's and Trudinger's theorem. At points $(y, \varphi_+(y))$, where μ has a nonvertical tangent plane, a cone lies on top of spt μ more precisely

$$spt\ \mu \subseteq (y, \varphi_+(y)) + \{(y,t) \mid t \leq C|y| \}$$

for some $C < \infty$. Hence $\varphi_+(y) = \varphi_+^\varepsilon(y)$ for small ε, and superdifferentials of φ_+^ε at y are superdifferentials of φ_+ as well. This yields

$$\sup_{B_\varrho(y)\cap[\varphi_\pm \in \mathbb{R}]} (\varphi_+ - l_y) \leqslant C_y \varrho^2 \quad \text{almost everywhere on } [\varphi_+ \in \mathbb{R}]$$

for some affine function l_y depending on y. By symmetry for φ_-, we conclude

$$||\varphi_\pm - l_y||_{L^\infty(B_\varrho(y)\cap[\varphi_\pm \in \mathbb{R}])} \leqslant C_y \varrho^2 \quad \text{almost everywhere on } [\varphi_+ = \varphi_-].$$

Using a Cacciopoli-type inequality, as $H_\mu \in L^2_{\text{loc}}(\mu)$ (cf. [**4**, Theorem 5.5] or [**3**, Lemma 22.2]), combined with a covering argument, we arrive at (1.3).

If we have only $H_\mu \in L^p_{\text{loc}}(\mu)$, $p > n$, a refinement of the Brakke blow-up technique (cf. [**6**, Lemma 3.1]) yields

$$-F(\nabla\varphi_+, D^2\varphi_+) \leqslant u$$

for some fully nonlinear elliptic operator F and some $u \in L^n_{\text{loc}}(U)$. The conclusion follows in the same way as above, by using results on quasilinear and fully nonlinear elliptic equations in [**11**] and [**13**]–[**16**]. □

2. C^2-Approximation

Theorem 1.1 can be considered as a regularity estimate for varifolds. In contrast to the slightly weaker estimate (1.2), the optimal estimate (1.3) suffices now to obtain what we call a C^2-approximation described as follows.

First, proceeding along standard technics (cf. [**9**, Propositions 3.4 and 3.5] and [**17**, Theorem 4.20]) and using Aleksandrov's theorem on twice differentiability of convex functions, we see that the height functions have second order approximate differentials almost everywhere on $[\varphi_\pm \in \mathbb{R}]$. We even get that the height functions $\varphi_\pm$ are twice differentiable when restricted to a suitable subset of the coincidence set $[\varphi_+ = \varphi_-]$ of full measure. By Whitney's extension theorem (cf. [**18**]), there exists $\psi \in C^2$ satisfying

$$D^\alpha \varphi_\pm = D^\alpha \psi \quad \text{for } |\alpha| \leqslant 2 \tag{2.1}$$

on a subset $Q \subseteq [\varphi_+ = \varphi_-]$ arbitrarily close to full measure. We assume that $\theta^n(\mu) = \theta_0$ on a set of large measure, which is true locally near points x, where $\theta^n(\mu)$ is approximately continuous. Then we call the weighted smooth graph

$$\mu_\psi := \theta_0 \mathcal{H}^n \lfloor \operatorname{graph} \psi$$

a C^2-*approximation* of μ. This is justified as follows.

More precisely, we consider $x = (y, \varphi_\pm(y)) = (y, \psi(y))$ with $y \in Q$ of full density in Q. From (2.1) we see that the tangent planes

$$T_x \mu = T_x \mu_\psi$$

coincide. This is also true for Lipschitz-approximations as tangent planes are terms involving only first order derivatives. Next we prove that also the mean curvatures coincide for the approximation μ_ψ for x as above almost everywhere. As the mean curvature involves second order derivatives, this is why we call μ_ψ a C^2-approximation of μ.

We choose $\chi \in C_0^\infty(B_1^{n+1}(0))$ rotationally symmetric with $0 \leqslant \chi \leqslant 1$ and $\chi \equiv 1$ on $B_{1/2}^{n+1}(0)$ and put $\chi_\varrho(\xi) := \chi(\varrho^{-1}(\xi - x))$. Assuming that x is a Lebesgue point of $\vec{\mathbf{H}}_\mu$, we calculate the mean curvature for $\widetilde{\mu} = \mu, \mu_\psi$ via

$$\begin{aligned}
\lim_{\varrho \downarrow 0} (\omega_n \varrho^n)^{-1} \delta\widetilde{\mu}(\chi_\varrho) &= -\lim_{\varrho \downarrow 0} (\omega_n \varrho^n)^{-1} \int\limits_{B_\varrho^{n+1}(x)} \chi_\varrho \vec{\mathbf{H}}_{\widetilde{\mu}} \, \mathrm{d}\widetilde{\mu} \\
&= -\omega_n^{-1} \theta_0 \vec{\mathbf{H}}_{\widetilde{\mu}}(x) \int\limits_{T_x \widetilde{\mu} \cap B_1^{n+1}(0)} \chi \, \mathrm{d}\mathcal{L}^n.
\end{aligned}$$

We recall

$$\delta\widetilde{\mu}(\chi_\varrho) = \int\limits_{B_\varrho^{n+1}(x)} D\chi_\varrho(\xi) T_\xi \widetilde{\mu} \, \mathrm{d}\widetilde{\mu}(\xi).$$

Multiplying by the normal $\nu(x)$ at $T_x\widetilde{\mu}$, we denote the difference by

$$I_\varrho := \varrho^{-n} (\delta\mu(\chi_\varrho) - \delta\mu_\psi(\chi_\varrho)) \nu(x).$$

Abbreviating

$$R_{\varrho, \widetilde{\mu}} := \varrho^{-n} \int\limits_{B_\varrho^{n+1}(x) - \Sigma_0} D\chi_\varrho(\xi)(T_\xi \widetilde{\mu} - T_x \widetilde{\mu}) \nu(x) \, \mathrm{d}\widetilde{\mu}(\xi),$$

where

$$\Sigma_0 := \{(z, \varphi_\pm(z)) \mid z \in Q \subseteq [\varphi_+ = \varphi_- = \psi], \theta^n(\mu, (z, \varphi_\pm(z))) = \theta_0\},$$

and using $T_x\widetilde{\mu}\nu(x) = 0$, we find $I_\varrho = R_{\varrho,\mu} - R_{\varrho,\mu_\psi}$.

We estimate

$$
\begin{aligned}
|R_{\varrho,\widetilde{\mu}}| &\leqslant C\varrho^{-n-1} \int\limits_{B_\varrho^{n+1}(x)-\Sigma_0} \|T_\xi\widetilde{\mu} - T_x\widetilde{\mu}\| \,\mathrm{d}\widetilde{\mu}(\xi) \\
&\leqslant C\varrho^{-1}(\varrho^{-n}\widetilde{\mu}(B_\varrho^{n+1}(x) - \Sigma_0))^{1/2}\Big(\varrho^{-n}\int\limits_{B_\varrho^{n+1}(x)} \|T_\xi\widetilde{\mu} - T_x\widetilde{\mu}\|^2 \,\mathrm{d}\widetilde{\mu}(\xi)\Big)^{1/2} \\
&\leqslant C\varrho^{-1}\omega(\varrho)^{1/2}\,\mathrm{tiltex}_{\widetilde{\mu}}(x,\varrho,T_x\widetilde{\mu})^{1/2}
\end{aligned}
$$

with $\omega(\varrho) \to 0$ for $\varrho \to 0$, since $y \in Q$ has full density in Q and $\theta^n(\mu)$ is approximately continuous at x. For $\widetilde{\mu} = \mu$, we have quadratic decay of the tilt-excess at x almost everywhere by Theorem 1.1, whereas such a decay is immediate for $\widetilde{\mu} = \mu_\psi$ since $D^2\psi \in C^0$. Under this assumption, we have $|R_{\varrho,\widetilde{\mu}}| \leqslant C\omega(\varrho)^{1/2}$ which yields

$$H_\mu(x)\nu(x) = H_{\mu_\psi}(x)\nu(x)$$

and, by [**4**, Theorem 5.8],

$$H_\mu(x) = H_{\mu_\psi}(x)$$

apart from a set of measure zero. This justifies the term C^2-approximation.

Recalling (2.1), we calculate

$$
\begin{aligned}
&\nabla\Big(\frac{\nabla\varphi_+}{\sqrt{1+|\nabla\varphi_+|^2}}\Big)(y) = \nabla\Big(\frac{\nabla\psi}{\sqrt{1+|\nabla\psi|^2}}\Big)(y) \\
&= \vec{\mathbf{H}}_{\mu_\psi}(y,\psi(y))\frac{(-\nabla\psi(y),1)}{\sqrt{1+|\nabla\psi(y)|^2}} = \vec{\mathbf{H}}_\mu(y,\varphi_+(y))\frac{(-\nabla\varphi_+(y),1)}{\sqrt{1+|\nabla\varphi_+(y)|^2}}.
\end{aligned}
$$

Combining with Aleksandrov's maximum principle, we conclude that φ_+ is a viscosity subsolution of the above equation which can be considered as a weak maximum principle.

We summarize:

Theorem 2.1 (weak maximum principle, [**6**, Theorem 6.1]). *Suppose that μ is an integral n-varifold in $\Omega \subseteq \mathbb{R}^{n+1}$ with $H_\mu \in L^p_{\mathrm{loc}}(\mu)$, $p > n$, $p \geqslant 2$, $\Omega := U \times \mathbb{R}$, $U \subseteq \mathbb{R}^n$ is open, $\operatorname{spt}\mu \subseteq U \times]-1,1[$, and $\varphi_+ : U \to [-\infty,\infty[$ is the upper height function of μ.*

Then φ_+ is twice approximately differentiable $\mathcal{L}^n$-almost everywhere on $[\varphi_\pm \in \mathbb{R}]$ and the approximate differentials satisfy the equation

$$\vec{\mathbf{H}}_\mu(y,\varphi_+(y)) = \nabla\Big(\frac{\nabla\varphi_+}{\sqrt{1+|\nabla\varphi_+|^2}}\Big)(y)\frac{(-\nabla\varphi_+(y),1)}{\sqrt{1+|\nabla\varphi_+(y)|^2}} \tag{2.2}$$

for $\mathcal{L}^n$-almost all $y \in [\varphi_+ \in \mathbb{R}]$. Moreover φ_+ is a $W^{2,p}$-viscosity subsolution of

$$-\nabla\left(\frac{\nabla\varphi_+}{\sqrt{1+|\nabla\varphi_+|^2}}\right) \leqslant \vec{\mathbf{H}}_\mu(.,\varphi_+)\frac{(\nabla\varphi_+,-1)}{\sqrt{1+|\nabla\varphi_+|^2}} \quad \text{in } U, \tag{2.3}$$

where the right-hand side is extended arbitrarily on $U - [\varphi_+ \in \mathbb{R}]$ to a function still in $L^p_{\mathrm{loc}}(U)$. □

3. Application to Free Boundaries

In this section, we want to apply our abstract results in geometric measure theory of the previous sections to a free boundary problem in a physical application.

In models for melting and solidification of materials such as the Stefan problem, the Gibbs Thomson law

$$H_\Gamma = u$$

states that on the free boundary Γ where the liquid and solid phase meet the melting temperature u equals the mean curvature of the free boundary and is hence determined by the geometry of the free boundary.

For proving existence of solutions for problems which include the Gibbs Thomson law, a procedure for passing to the limit in the Gibbs Thomson law is required. This convergence procedure is justified by a theorem of Reshetnyak **[19]** if the area of the free boundary is preserved when passing to the limit, such as this is the case in the Stefan problem with Gibbs Thomson law with absolute minimizations (cf. **[20]**).

Without assuming the preservation of the area of the free boundary, this was successfully done by Ilmanen for the Allen–Cahn equation in **[21]**, by Hutchinson, Padilla and Tonegawa for the phase field equations in **[22]**–**[24]**, by Soner for the Stefan problem with kinetic undercooling in **[25]**, and by Chen for the Cahn–Hilliard equation in **[26]**.

Still open are the convergence procedures for the Stefan problem with Gibbs Thomson law with local minimizations, for mean curvature flow in BV-context and the Mullins–Sekerka problem (cf. **[27, 28]**).

Here we use the general results of the previous sections to establish a general convergence procedure. We consider hypersurfaces $\Sigma_j = \partial E_j \subseteq \mathbb{R}^{n+1}$ with interior E_j and ambient Sobolev functions u_j corresponding to the temperature. We write the Gibbs Thomson law in the form

$$\vec{\mathbf{H}}_{\Sigma_j} = u_j \nu_{E_j} \quad \text{on } \Sigma_j, \tag{3.1}$$

where ν_{E_j} denotes the inner normal of ∂E_j, and assume the bounds

$$\|u_j\|_{W^{1,q}}, \mathcal{H}^n(\Sigma_j) \leqslant \Lambda,$$

together with the convergences

$$\begin{aligned} u_j \to u &\quad \text{weakly in } W^{1,q}_{\text{loc}}, \\ \chi_{E_j} \to \chi_E &\quad \text{strongly in } L^1_{\text{loc}}, \\ \mathcal{H}^n \lfloor \Sigma_j \to \mu &\quad \text{weakly as varifolds.} \end{aligned}$$

Putting

$$1 - \frac{n+1}{q} =: -\frac{n}{p},$$

by the Sobolev trace mapping on the boundary, we find $H_{\Sigma_j} \in L^p_{\text{loc}}(\mathcal{H}^n \lfloor \Sigma_j)$. We assume that $\frac{1}{2}(n+1) < q < (n+1)$ and $q \geqslant \frac{4}{3}$ if $n = 1$ in order to get

$$n < p < \infty, \quad p \geqslant 2.$$

Hence our general results of the previous sections apply. Then the Gibbs Thomson law is satisfied in the limit.

Theorem 3.1 (cf. [**29**, Theorem 1.1]). *Under the above assumptions, μ is an integral n-varifold in $\mathbb{R}^{n+1}$ with locally bounded first variation and*

$$H_\mu \in L^p_{\text{loc}}(\mu), \tag{3.2}$$

E is a set of finite perimeter with reduced boundary $\partial^* E \subseteq \operatorname{spt} \mu$. The mean curvature vector satisfies

$$\vec{\mathbf{H}}_\mu = u\nu_E \quad \mu\text{-almost everywhere}, \tag{3.3}$$

where $\nu_E = \dfrac{\nabla \chi_E}{|\nabla \chi_E|}$ denotes the generalized normal of $\partial^* E$ which is put equal to 0 outside $\partial^* E$.

Proof. First combining a monotonicity formula (cf. [**29**, Lemma 2.1]) with an estimate of the constant of the Sobolev trace mapping by Meyers and Ziemer (cf. [**30**, Theorem 4.7] 30 and [**31**, Theorem 5.12.4]), 31 we conclude that H_{Σ_j} is bounded in $L^p_{\text{loc}}(\mathcal{H}^n \lfloor \Sigma_j)$. This implies (3.2) and, by Allard's integral compactness theorem, that μ is an integral varifold.

Next, we consider the height functions $\varphi_{j,\pm}$ of Σ_j and $\varphi_\pm$ of μ. As $\Sigma_j \to \operatorname{spt} \mu$ in Hausdorff–distance, the height functions converge

$$\varphi_+ = \lim_{j\to\infty}{}^* \varphi_{+,j} := \sup\{\limsup_{k\to\infty} \varphi_{+,j_k}(y_k) \mid j_k \to \infty, y_k \to y\}$$

and analogously

$$\varphi_- = \lim_{j\to\infty *} \varphi_{-,j}.$$

Our considerations are local near a point $x_0 \in \operatorname{spt} \mu$, where the tangent plane $T_{x_0}\mu$ exists and is not vertical and with $\theta_0 := \theta^n(\mu, x_0) \in \mathbb{N}$. We assume that the set E lies above x_0. Then E (respectively, E^c) lies below x_0 depending on whether θ_0 is even or odd (cf. [**29**, Lemma 3.2]).

The weak maximum principle Theorem 2.1 (cf. (2.3)) applied to $\varphi_{\pm,j}$ combined with (3.1) and then passing to the limit with u_j and the above convergences of the height functions (cf. [**7**, Remark 6.3]) yields that φ_+ is a $W^{2,p}$-viscosity subsolution of the inequality

$$-\nabla\left(\frac{\nabla\varphi_+}{\sqrt{1+|\nabla\varphi_+|^2}}\right) \leqslant -u(\cdot,\varphi_+) \tag{3.4}$$

and φ_- is a $W^{2,p}$-viscosity supersolution of the inequality

$$-\nabla\left(\frac{\nabla\varphi_-}{\sqrt{1+|\nabla\varphi_-|^2}}\right) \geqslant \begin{cases} -u(\cdot,\varphi_-), & \theta_0 \text{ is odd,} \\ u(\cdot,\varphi_-), & \theta_0 \text{ is even.} \end{cases} \tag{3.5}$$

Proceeding again along the standard technique (cf. [**9**, Propositions 3.4 and 3.5] and [**17**, Theorem 4.20] we see that these inequalities are also satisfied pointwise almost everywhere by the approximate differentials of $\varphi_\pm$.

On the other hand, by Theorem 2.1 (cf. (2.2)), these approximate differentials satisfy

$$\vec{\mathbf{H}}_\mu(y,\varphi_\pm(y)) = \nabla\left(\frac{\nabla\varphi_\pm}{\sqrt{1+|\nabla\varphi_\pm|^2}}\right)(y)\frac{(-\nabla\varphi_\pm(y),1)}{\sqrt{1+|\nabla\varphi_\pm(y)|^2}} \tag{3.6}$$

almost everywhere.

When θ_0 is odd, (3.4)–(3.6) imply

$$\vec{\mathbf{H}}_\mu(y,\varphi_\pm(y)) = u(y,\varphi_\pm(y))\frac{(-\nabla\varphi_\pm(y),1)}{\sqrt{1+|\nabla\varphi_\pm(y)|^2}}$$

almost everywhere on $[\varphi_+ = \varphi_-]$, which yields (3.3) after a covering argument, as

$$\nu_E(y,\varphi_\pm(y)) = \frac{(-\nabla\varphi_\pm(y),1)}{\sqrt{1+|\nabla\varphi_\pm(y)|^2}},$$

since E lies above x_0 and E^c lies below x_0 if θ_0 is odd.

When θ_0 is even, we calculate the first variation with (3.1) as

$$\delta(\mathcal{H}^n\lfloor\Sigma_j)(\eta) = \int \operatorname{div}(u_j\eta)\chi_{E_j} \quad \text{for } \eta \in C_0^1$$

and passing to the limit

$$\delta\mu(\eta) = \int \operatorname{div}(u\eta)\chi_E \quad \text{for } \eta \in C_0^1.$$

Therefore, $\vec{\mathbf{H}}_\mu = 0$ outside $\partial^* E$. Now $(y,\varphi_\pm(y)) \notin \partial^* E$ for $y \in [\varphi_+ = \varphi_-]$, as the same phase meets at x_0 since θ_0 is even. This yields $\nu_E(y,\varphi_\pm(y)) = 0$ by our convention and again (3.3). □

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On Eigenvalue Estimates for the Weighted Laplacian on Metric Graphs

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Dedicated to Olga Aleksandrovna Ladyzhenskaya
to whom I am indebted for my formation as a mathematician

It is shown that the eigenvalues of the equation $-\lambda\Delta u = Vu$ on a graph $\mathbf{G}$ of finite total length $|\mathbf{G}|$, where $V \in \mathsf{L}^1(\mathbf{G})$ is nonnegative, under appropriate boundary conditions satisfy the inequality $n^2\lambda_n \leqslant |\mathbf{G}| \int\limits_{\mathbf{G}} V dx$ independently of geometry of a given graph. Applications and generalizations of this result are also discussed.

1. Introduction

A *differential operator on a metric graph* $\mathbf{G}$ is a family of differential expressions on the edges of this graph supplemented with suitable matching conditions at some vertices and with some boundary conditions at the remaining vertices. In particular, for the Laplacian Δ the differential expression is $\Delta u = u''$ and the matching conditions at vertices are the Kirchhoff conditions coming from the theory of electric networks. The main goal of this paper is to study the behavior of eigenvalues for the problem

$$\begin{gathered} -\lambda\Delta u = Vu, \\ u(x_0) = 0, \quad u'(v) = 0, \quad v \in \partial\mathbf{G} \setminus \{x_0\}, \end{gathered} \tag{1.1}$$

on graphs of finite total length $|\mathbf{G}|$. In the problem (1.1), V is a given real-valued measurable weight function on $\mathbf{G}$ and $x_0 \in \mathbf{G}$ is a fixed vertex. Note that the set $\partial\mathbf{G} \setminus \{x_0\}$ can be empty.

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One can consider a similar problem with zero boundary conditions at several vertices, $u(x_1) = \ldots = u(x_r) = 0$. However, this gives nothing new: by the variational principle, any estimate for the problem (1.1) implies the same estimate for the r-point problem. The accurate setting of the problem uses quadratic forms (cf. Sec. 3 for details). The way to insert the spectral parameter in (1.1) is motivated by technical reasons.

We formulate our main result on the problem (1.1). Let $\pm\lambda_n^{\pm}$ denote the positive and negative eigenvalues of this problem.

Theorem 1.1. *Let* $\mathbf{G}$ *be a connected graph of finite total length,* $x_0 \in \mathbf{G}$ *an arbitrary vertex of this graph, and* $V = \overline{V} \in L^1(\mathbf{G})$. *Then the eigenvalues of the problem* (1.1) *satisfy the inequality*

$$\lambda_n^{\pm} \leqslant \frac{|\mathbf{G}| \int_{\mathbf{G}} V_{\pm} dx}{n^2} \quad \forall n \in \mathbb{N}, \tag{1.2}$$

where $2V_{\pm} = |V| \pm V$, *which implies the Weyl-type asymptotics*

$$n\sqrt{\lambda_n^{\pm}} \to \pi^{-1} \int_{\mathbf{G}} \sqrt{V_{\pm}(x)}dx, \quad n \to \infty. \tag{1.3}$$

The simplest example of a metric graph is a single segment $[0, L] \subset \mathbb{R}$. The basic results for this case were obtained by Birman and the author [**1**] as far back as in 1971, as a particular case of the general result for many dimensions (cf. also [**2**]). Namely, as was shown, the eigenvalues of the equation $-\lambda u'' = Vu$ on the interval $(0, L)$, under the Dirichlet boundary conditions at its endpoints, satisfy the asymptotics (1.3) and the inequality

$$n^2\lambda_n^{\pm} \leqslant CL \int_0^L V_{\pm} dx$$

with some absolute constant C. The problem of the best possible constant in the inequality was not discussed in [**1, 2**], although the sharp estimate with $C = 1/4$ is easily derived from Theorem 2.2 of [**3**]. For the zero boundary condition only at one point, as in (1.1), the constant $C = 1$ in (1.2) is unimprovable even in the case of a segment.

The most important feature of the estimate (1.2) for $\mathbf{G} = [0, L]$ is the fact that it is uniform with respect to $V \in \mathsf{L}^1(0, L)$. This is useful for various applications. Theorem 1.1 shows that this estimate extends to arbitrary graphs of finite total length, with the same constant as in the case of a single interval. The very possibility of such an extension might look problematic since the eigenvalues depend on the combinatorial structure of the graph which can be quite diverse.

One can give a qualitative explanation of this effect. If $V \geqslant 0$, the eigenvalues of the problem (1.1) can be expressed in terms of approximative characteristics of the unite ball of the Sobolev space $\mathsf{H}^1(\mathbf{G}, x_0)$ as a compact set in the weight space $\mathsf{L}^2(\mathbf{G}, V)$. Indication of the point x_0 in the first notation reflects the boundary condition $u(x_0) = 0$. The points of a graph $\mathbf{G}$ lie "closer to each other" than the points of the segment $[0, |\mathbf{G}|]$. Therefore, one should expect that the dispersion of the values of a given function $u \in \mathsf{H}^1(\mathbf{G}, x_0)$ is smaller than that of a function $v \in \mathsf{H}^1(0, |\mathbf{G}|)$ having the same L^2-norm of the derivative. As a consequence, it is easier to approximate the class $\mathsf{H}^1(\mathbf{G}, x_0)$ than $\mathsf{H}^1(0, |\mathbf{G}|)$. Eventually, this leads to the estimate (1.2).

This, rather naive argument gives no clue to the proof. Our arguments are based on Theorem 2.1 which has combinatorial nature and can be regarded as a far extended generalization of Theorem 4.1 in [**3**].

We compare our results with the recent results obtained by Evans, Harris, and Lang [**4**] who studied the behavior of approximation numbers of the Hardy-type integral operators in the spaces L^p on trees. If $p = 2$, the approximation numbers coincide with the singular numbers, and this is a link between our corresponding results. The most important results of [**4**] are two-sided estimates and an asymptotic formula for the approximation numbers. Their inequality is similar to (1.2) but without the sharp constant. Moreover, the asymptotics (1.3) could be derived from the results of [**4**]. The authors of [**4**] do not discuss possible applications. Their techniques (cf. [**4**, Sec. 3]) is based on a certain combinatorial construction which is substantially different from ours.

We describe the structure of the paper. In Sec. 2, we give the necessary information about graphs and trees and prove Theorem 2.1 which is our main technical tool. In Sec. 3, we define the Sobolev space $\mathsf{H}^1(\mathbf{G}, x_0)$ and give the variational formulation of the problem (1.1). We also state two auxiliary assertions (cf. Theorems 3.1 and 3.2) on approximation on graphs. They are used in the proof of Theorem 1.1. The proofs of all three theorems are given in Sec. 4. In Sec. 5, we discuss the obtained result and give its application to the estimates of singular numbers for integral operators on graphs. In Sec. 6, we present an analog of Theorem 1.1 in the case of higher-order equations.

The Hilbert space structure is unnecessary in applications of Theorem 2.1. It can also be applied to piecewise-polynomial approximation of Sobolev spaces $\mathsf{W}^{l,p}$ on graphs and trees. This leads to estimates of approximation numbers of embedding of $\mathsf{W}^{l,p}$ in the weight spaces $\mathsf{L}^p(\mathbf{G}, V)$ on graphs of finite total length. The results can also be applied to graphs and trees of infinite total length, in the spirit of [**4, 5**]. This material will be presented elsewhere.

2. Graphs and Trees. Partitions of a Tree

Let $\mathbf{G}$ be a graph with the set of vertices $\mathcal{V} = \mathcal{V}(\mathbf{G})$ and the set of edges $\mathcal{E} = \mathcal{E}(\mathbf{G})$. Each edge e of a metric graph is regarded as a non-degenerate line segment of finite length $|e|$. The quantity

$$|\mathbf{G}| = \sum_{e \in \mathcal{E}(\mathbf{G})} |e|$$

is called the *total length* of the graph $\mathbf{G}$. In this paper, we always assume $|\mathbf{G}| < \infty$. Even though, the number of edges of $\mathbf{G}$ can be infinite. The distance $\rho(x, y)$ between any two points $x, y \in \mathbf{G}$, and thus the metric topology on $\mathbf{G}$, is introduced in a natural way. The natural measure dx on $\mathbf{G}$ is induced by the Lebesgue measure on the edges. For a measurable set $E \subset \mathbf{G}$ the measure of E is denoted by $|E|$. This notation is consistent with the above notation $|e|$ and $|\mathbf{G}|$.

We always consider connected graphs, unless otherwise stipulated. We do not exclude graphs with multiple joins. In order to avoid unnecessary complications, we exclude graphs with loops. Recall that a loop is an edge whose endpoints coincide.

For vertices v and w the notation $w \sim v$ means that there exists an edge $e \in \mathcal{E}$ whose endpoints are v and w. The *connectedness* of a graph means that for any two different vertices $v, w \in \mathcal{V}$ there exists a finite sequence $\{v_k\}_{0 \leqslant k \leqslant m}$ of vertices such that $v_0 = v$, $v_m = w$, and $v_k \sim v_{k-1}$ for each $k = 1, \dots, m$. The *combinatorial distance* $\rho_{\text{comb}}(v, w)$ is defined as the minimal possible m in this construction. We define $\rho_{\text{comb}}(v, v) = 0$ for any $v \in \mathcal{V}$. The *degree* $d(v)$ of a vertex v is the total number of edges incident to v. The vertices v with $d(v) = 1$ form the boundary $\partial\mathbf{G}$ of the graph $\mathbf{G}$. We suppose that $d(v) < \infty$ for each $v \in \mathcal{V}$. It is often convenient to treat an arbitrary point $x \in \mathbf{G}$ as a vertex. We set $d(x) = 2$ for any $x \notin \mathcal{V}(\mathbf{G})$ and write $v \sim x$ if $v \in \mathcal{V}(\mathbf{G})$ is one of the endpoints of the edge containing x. This remark concerns also the point x_0 appearing in (1.1) which actually can be an arbitrary point in $\mathbf{G}$.

Given a subgraph $G \subset \mathbf{G}$, we denote by $d_G(v)$ the degree of a vertex v with respect to G.

A metric graph $\mathbf{G}$ is compact if and only if $\#\mathcal{E}(\mathbf{G}) < \infty$. Any metric graph can be represented as the union of an expanding family of compact subgraphs,

$$\mathbf{G} = \bigcup_{m=1}^{\infty} \mathbf{G}^{(m)}, \quad \mathbf{G}^{(1)} \subset \mathbf{G}^{(2)} \subset \dots; \quad \#\mathcal{E}(\mathbf{G}^{(m)}) < \infty \quad \forall m \in \mathbb{N}. \tag{2.1}$$

Indeed, choose an arbitrary vertex $x_0 \in \mathbf{G}$ and define the subgraph $\mathbf{G}^{(m)}$ as follows. A vertex $v \in \mathcal{V}(\mathbf{G})$ belongs to $\mathcal{V}(\mathbf{G}^{(m)})$ if and only if $\rho_{\text{comb}}(x_0, v) \leqslant m$, and an edge $e \in \mathcal{E}(\mathbf{G})$ belongs to $\mathcal{E}(\mathbf{G}^{(m)})$ if and only if both endpoints of it lie in $\mathcal{V}(\mathbf{G}^{(m)})$. By construction, the graph $\mathbf{G}^{(m)}$ is connected and compact. It is evident that $\{\mathbf{G}^{(m)}\}$ is an expanding family covering the whole $\mathbf{G}$.

Let $\mathbf{G}$ be a compact graph, and let $G, G_1, \dots, G_n$ be (connected) subgraphs of $\mathbf{G}$. We say that the subgraphs $G_1, \dots, G_n$ form a *partition* or a *splitting* of G if $G_1 \cup \dots \cup G_n = G$ and $|G_i \cap G_j| = 0$ for any $i, j \in \{1, \dots, n\}$, $i \neq j$. If G

and G_1 are subgraphs of $\mathbf{G}$ and $G_1 \subset G$ respectively, then it is possible to find connected subgraphs $G_2, \ldots, G_n$ that, together with G_1, form a partition of G (the property of *complementability*).

A pair $\{G, x\}$, where $G \subset \mathbf{G}$ is a subgraph and $x \in G$ is a fixed point, is called a *punctured subgraph.* If $\{G_j, x_j\}$, $j = 1, \ldots, n$, are punctured subgraphs such that $G = G_1 \cup \ldots \cup G_n$ is a partition of a subgraph G, then we say that G is split into the union of punctured subgraphs.

Our auxiliary result (cf. Theorem 2.1) which serves as a basis for the proof of Theorem 1.1, concerns trees rather than arbitrary graphs. Recall that a *tree* is a connected graph without cycles, loops, and multiple joins. So, let $\mathbf{G} = \mathbf{T}$ be a tree. For any two points $x, y \in \mathbf{T}$ there exists a unique simple path in $\mathbf{T}$ connecting x with y. We denote it by $\langle x, y\rangle$.

What was said above about partitions of graphs, applies to trees. Note that if $T = T_1 \cup T_2$ is a partition, then the intersection $T_1 \cap T_2$ consists of exactly one point.

Let Φ be a non-negative function defined on the set of all subtrees T of a given tree $\mathbf{T}$. We call Φ *continuous* if $\Phi(T) \to \Phi(T_0)$ as $|T \triangle T_0| \to 0$. Recall that $T \triangle T_0 = (T \setminus T_0) \cup (T_0 \setminus T)$ is the symmetric difference of the sets T and T_0.

We call a continuous function Φ *superadditive* and write $\Phi \in \mathsf{S}(\mathbf{T})$ if for any subtree $T \subset \mathbf{T}$ and partition $T = T_1 \cup \ldots \cup T_n$ we have

$$\Phi(T_1) + \ldots + \Phi(T_n) \leqslant \Phi(T). \tag{2.2}$$

By complementability, (2.2) implies that any superadditive function is monotone:

$$T_1 \subset T \Longrightarrow \Phi(T_1) \leqslant \Phi(T).$$

Any finite Borel measure μ on $\mathbf{T}$ without atoms (i.e., points of positive measure) generates a continuous superadditive function $\Phi(T) = \mu(T)$. A more general example follows from Hölder's inequality:

$$\Phi_1, \Phi_2 \in \mathsf{S}(\mathbf{T}),\ \alpha_1, \alpha_2 > 0,\ \alpha_1 + \alpha_2 = 1 \Longrightarrow \Phi_1^{\alpha_1}\Phi_2^{\alpha_2} \in \mathsf{S}(\mathbf{T}). \tag{2.3}$$

Let $\{T, x\}$ be a punctured subtree of $\mathbf{T}$. The tree T splits in a unique way into the union of subtrees $\Theta_j \subset T$, $j = 1, \ldots, d_T(x)$, rooted at x and such that $d_{\Theta_j}(x) = 1$ for each j. We call it the *canonical partition* of the punctured subtree $\{T, x\}$. Given a function $\Phi \in \mathsf{S}(\mathbf{T})$, we define the function $\widetilde{\Phi}(T, x)$ of punctured subtrees,

$$\widetilde{\Phi}(T, x) = \max_{1 \leqslant j \leqslant d_T(x)} \Phi(\Theta_j). \tag{2.4}$$

It is obvious that $\widetilde{\Phi}(T, x) \leqslant \Phi(T)$.

Let $T = \mathbf{T}$. Each subtree Θ_j in the canonical partition of $\{\mathbf{T}, x\}$ is completely determined by the initial edge $\langle x, v\rangle$, $v \sim x$. We denote this subtree by $\Theta_{\langle x, v\rangle}$. For $T = \mathbf{T}$ the definition (2.4) takes the form

$$\widetilde{\Phi}(\mathbf{T},x)=\max_{v\sim x}\Phi(\Theta_{\langle x,v\rangle}).$$

We derive estimates for eigenvalues from the following assertion on superadditive functions of subtrees.

Theorem 2.1. *Let* $\mathbf{T}$ *be a compact metric tree, and let* $\Phi\in S(\mathbf{T})$. *Then for any* $n\in\mathbb{N}$ *the tree* $\mathbf{T}$ *can be split into the union of punctured subtrees* $\{T_j,x_j\}$, $j=1,\dots,k$, *in such a way that* $k\leqslant n$ *and*

$$\max_{j=1,\dots,k}\widetilde{\Phi}(T_j,x_j)\leqslant(n+1)^{-1}\Phi(\mathbf{T}). \tag{2.5}$$

We begin with the following assertion.

Lemma 2.2. *Let* $\mathbf{T}$ *be a compact metric tree, and let* $\Phi\in S(\mathbf{T})$. *Then for any* ε, $0<\varepsilon<\Phi(\mathbf{T})$, *there exists a partition* $\mathbf{T}=T\cup T'$ *such that*

$$\Phi(T')\leqslant\Phi(\mathbf{T})-\varepsilon \tag{2.6}$$

and for the only point $x\in T\cap T'$ *the following inequality holds:*

$$\widetilde{\Phi}(T,x)\leqslant\varepsilon. \tag{2.7}$$

PROOF. Without loss of generality, we can assume that $\Phi(\mathbf{T})=1$. For any vertex $v_0\in\partial\mathbf{T}$ we have $\widetilde{\Phi}(\mathbf{T},v_0)=\Phi(\mathbf{T})=1$. There is a unique vertex $v_1\sim v_0$. Now, we choose the vertices $v_2\sim v_1,\dots,v_{k+1}\sim v_k,\dots$ as follows. If v_k is already chosen, we define v_{k+1} as a vertex that differs from v_{k-1} and

$$\Phi(\Theta_{\langle v_k,v_{k+1}\rangle})=\max_{w\sim v_k,w\neq v_{k-1}}\Phi(\Theta_{\langle v_k,w\rangle})=\widetilde{\Phi}(\Theta_{\langle v_k,v_{k+1}\rangle},v_k). \tag{2.8}$$

If there are several vertices $w\sim v_k$ at which the maximum in the middle term of (2.8) is attained, then any of them can be chosen as v_{k+1}. The described procedure is always finite and terminates when we arrive at a vertex $v_m\in\partial\mathbf{T}$. On the path $\mathcal{P}=\langle v_0,v_m\rangle$, we introduce the natural ordering, i.e., $y\succeq x$ means that $x\in\langle v_0,y\rangle$. We write $y\succ x$ if $y\succeq x$ and $y\neq x$.

Let $x\in\mathcal{P}$ be not a vertex of $\mathbf{T}$. Then $v_{k-1}\prec x\prec v_k$ for some $k=1,\dots,m$. Introduce the notation $T_x^+=\Theta_{\langle x,v_k\rangle}$ and $T_x^-=\Theta_{\langle x,v_{k-1}\rangle}$. We also define subtrees $T_x^\pm$ for $x=v_0,\dots,v_m$ as follows:

$$T_{v_k}^-=T_{\langle v_k,v_{k-1}\rangle},\quad k=1,\dots,m,$$

$$T_{v_k}^+=\bigcap_{v_{k-1}\prec x\prec v_k}T_x^+=\bigcup_{v\sim v_k,v\neq v_{k-1}}T_{\langle v_k,v\rangle},\quad k=0,\dots,m-1.$$

Finally, $T_{v_0}^-=\{v_0\}$, $T_{v_m}^+=\{v_m\}$ are degenerate subtrees. For any $x\in\mathcal{P}$, $\mathbf{T}=T_x^+\cup T_x^-$ is a partition of the punctured tree $\{\mathbf{T},x\}$, and $T_x^+\cap T_x^-=\{x\}$.

The function $F(x)=\Phi(T_x^+)$ is well defined on $\mathcal{P}$ and is continuous everywhere except possibly for the vertices $v_0,\dots,v_m$. By construction, we have

$$\widetilde{\Phi}(T_x^+,x)=F(x),\quad x\neq v_0,\dots,v_m,$$

$$\lim_{x \prec v_k, x \to v_k} F(x) = F(v_k),$$

$$\lim_{x \succ v_k, x \to v_k} F(x) = \Phi(T_{\langle v_k, v_{k+1} \rangle}) = \widetilde{\Phi}(T^+_{v_k}, v_k) \leqslant F(v_k).$$

The second equality in the last line follows from (2.8). It is clear that $F(x)$ is non-increasing along the path $\mathcal{P}$. Moreover, $0 = F(v_m) < \varepsilon < F(v_0) = 1$. Therefore, there exists a point $x \in \mathcal{P}$ such that $\widetilde{\Phi}(T_x, x) \leqslant \varepsilon \leqslant F(x)$. We set $T = T^+_x$ and $T' = T^-_x$. The inequality (2.7) is satisfied and (2.6) holds by superadditivity: $\Phi(T') \leqslant 1 - \Phi(T) = 1 - F(x) \leqslant 1 - \varepsilon$. □

Proof of Theorem 2.1. Let $n = 1$. Then we apply Lemma 2.2 with $\varepsilon = \Phi(\mathbf{T})/2$. Let $\mathbf{T} = T \cup T'$ be the corresponding partition. Then $\widetilde{\Phi}(T', x) \leqslant \Phi(T') \leqslant \Phi(\mathbf{T})/2$. Consider the canonical partition of the punctured tree $\{\mathbf{T}, x\}$. Each subtree of this partition is contained in either T or T'. Therefore,

$$\widetilde{\Phi}(\mathbf{T}, x) \leqslant \max\bigl(\widetilde{\Phi}(T, x), \widetilde{\Phi}(T', x)\bigr) \leqslant \Phi(\mathbf{T})/2.$$

Thus, (2.5) with $k = n = 1$ is satisfied for $T_1 = \mathbf{T}$ and $x_1 = x$.

We proceed by induction. Suppose that the required assertion is already proved for $n = n_0 - 1$. Let $\mathbf{T} = T \cup T'$ be the partition constructed according to Lemma 2.2 for $\varepsilon = (n_0 + 1)^{-1}\Phi(\mathbf{T})$. Then

$$\Phi(T') \leqslant n_0(n_0 + 1)^{-1}\Phi(\mathbf{T}).$$

By the induction hypothesis, there exists a splitting of T' into the union of the family of punctured subtrees $\{T_j, x_j\}$, $j = 1, \dots, k$, such that $k \leqslant n_0 - 1$ and for each j we have

$$\widetilde{\Phi}(T_j, x_j) \leqslant n_0^{-1}\Phi(T') \leqslant (n_0 + 1)^{-1}\Phi(\mathbf{T}).$$

Adding to this family the punctured subtree $\{T_{k+1}, x_{k+1}\} = \{T, x\}$, we obtain the desired partition of $\mathbf{T}$ for $n = n_0$. □

3. Variational Setting of the Problem. Reduction to the Case of Trees

3.1. Sobolev spaces on a graph. We denote by $\|\cdot\|_p$, $1 \leqslant p \leqslant \infty$, the norm in the space $\mathsf{L}^p(\mathbf{G})$ and by $\mathsf{L}_+(\mathbf{G})$ the cone of all non-negative elements in $\mathsf{L}^1(\mathbf{G})$.

We say that a function u on $\mathbf{G}$ *belongs to the Sobolev space* $L^{1,2}(\mathbf{G})$ if u is continuous on $\mathbf{G}$, the restriction of u to each edge e belongs to $\mathsf{H}^1(e)$, and $u' \in \mathsf{L}^2(\mathbf{G})$. The functional $\|u'\|_2$ defines a semi-norm on $\mathsf{L}^{1,2}(\mathbf{G})$ that vanishes on the one-dimensional subspace of constant functions.

Let $\mathbf{G}$ be a graph of finite total length, and let ξ, x be two arbitrary points of $\mathbf{G}$. Choose a simple path $\mathcal{L}$ in $\mathbf{G}$ connecting ξ with x. Let the length of this path be t_0. Parametrizing $\mathcal{L}$ by the path length, we can regard the restriction $u \restriction \mathcal{L}$ as a function on the line segment $[0, t_0]$. From the equality

$$u(x) - u(\xi) = \int_0^{t_0} u'(t)dt$$

it follows that

$$|u(x) - u(\xi)|^2 \leqslant t_0 \int_0^{t_0} |u'(t)|^2 dt \leqslant |\mathbf{G}| \int_{\mathbf{G}} |u'(x)|^2 dx. \tag{3.1}$$

This shows that any function $u \in \mathsf{L}^{1,2}(\mathbf{G})$ lies in the Hölder class of order $1/2$.

A *step function* v on $\mathbf{G}$ is a function w that takes only a finite number of different values, each on a connected subset of $\mathbf{G}$. We denote by $\mathrm{Step}(\mathbf{G})$ the linear space (non-closed linear subspace of $\mathsf{L}^\infty(\mathbf{G})$) of all step functions on $\mathbf{G}$.

The following assertion on the approximation of functions $u \in \mathsf{L}^{1,2}(\mathbf{G})$ by step-functions will be used in the proof of Theorem 1.1. Let punctured subgraphs $\{G_j, x_j\}$, $j = 1, \dots, k$, form a partition of the graph $\mathbf{G}$. With this partition we associate the linear operator

$$P : u \mapsto v = \sum_{j=1}^{k} u(x_j)\chi_j, \tag{3.2}$$

where χ_j is the characteristic function of the set G_j. It is clear that the operator P acts from $\mathsf{L}^{1,2}(\mathbf{G})$ into $\mathrm{Step}(\mathbf{G})$ and its rank is less than or equal to k.

Theorem 3.1. *Let $\mathbf{G}$ be a compact graph, and let $V \in L_+(\mathbf{G})$. Then for any $n \in \mathbb{N}$ there exists a partition of $\mathbf{G}$ into punctured subgraphs $\{G_j, x_j\}$, $j = 1, \dots, k$, such that $k \leqslant n$ and for the corresponding operator P given by* (3.2) *we have*

$$\int_{\mathbf{G}} |u - Pu|^2 V dx \leqslant \frac{|\mathbf{G}| \int_{\mathbf{G}} V dx}{(n+1)^2} \|u'\|_2^2 \quad \forall u \in L^{1,2}(\mathbf{G}). \tag{3.3}$$

The assumption that the graph $\mathbf{G}$ is compact is important in the proof. To exhaust the general case, we need one more statement.

A graph $\mathbf{G}$ of finite total length is not necessarily a compact metric space. Let $\overline{\mathbf{G}}$ be its compactification. Any function $u \in \mathsf{L}^{1,2}(\mathbf{G})$ is uniformly continuous on $\mathbf{G}$ and, consequently, admits a unique continuous extension to $\overline{\mathbf{G}}$. We keep the same symbol u for the extended function.

Theorem 3.2. *Let $\mathbf{G}$ be a graph of finite total length, and let $V \in L_+(\mathbf{G})$. Then for any $n \in \mathbb{N}$ there exist points $\overline{x}_1, \dots, \overline{x}_k \in \overline{\mathbf{G}}$ such that $k \leqslant n$ and the* *following inequality holds*:

$$\int_{\mathbf{G}} |u^2| V dx \leqslant \frac{|\mathbf{G}| \int_{\mathbf{G}} V dx}{(n+1)^2} \|u'\|_2^2 \tag{3.4}$$

for any function $u \in L^{1,2}(\mathbf{G})$ *satisfying the conditions* $u(\overline{x}_1) = \ldots = u(\overline{x}_k) = 0$.

Proofs of Theorems 3.1 and 3.2 are given in the next section, before the proof of Theorem 1.1.

3.2. Space $\mathsf{H}^1(\mathbf{G}, x_0)$ and operators $\mathbf{B}_V$. Let a point $x_0 \in \mathbf{G}$ be given. It is convenient to assume that x_0 is a vertex. Consider the Hilbert space

$$\mathsf{H}^1(\mathbf{G}, x_0) = \{u \in \mathsf{L}^{1,2}(\mathbf{G}) : u(x_0) = 0\}$$

equipped with the scalar product

$$(u, v)_{\mathsf{H}^1(\mathbf{G}, x_0)} = (u', v')_{\mathsf{L}_2(\mathbf{G})}.$$

The inequality (3.1) (with $\xi = x_0$) shows that this scalar product is non-degenerate.

Let V be a function from the space $\mathsf{L}^1(\mathbf{G})$. Consider the quadratic form

$$\mathbf{b}_V[u] = \int_{\mathbf{G}} |u|^2 V dx. \tag{3.5}$$

From the inequality it follows that (3.1) (again, with $\xi = x_0$) that $\mathbf{b}_V$ is bounded in the space $\mathsf{H}^1(\mathbf{G}, x_0)$, i.e.,

$$|\mathbf{b}_V[u]| \leqslant |\mathbf{G}| \|u'\|_2^2 \int_{\mathbf{G}} |V(x)| dx \quad \forall u \in \mathsf{H}^1(\mathbf{G}, x_0). \tag{3.6}$$

Therefore, the quadratic form $\mathbf{b}_V[u]$ generates a bounded linear operator, say $\mathbf{B}_V$, in the space $\mathsf{H}^1(\mathbf{G}, x_0)$. It is easy to see that the operator $\mathbf{B}_V$ is compact. Actually, compactness will automatically follow from the estimates we obtain in the next section. This operator is self-adjoint provided that the function V is real-valued and it is non-negative provided that $V \geqslant 0$ a.e. As usual, it is natural to identify the spectrum of the problem (1.1) with the spectrum of the operator $\mathbf{B}_V$. Still, we recall the corresponding argument.

The Laplacian $-\Delta$ on $\mathbf{G}$, with the boundary conditions as in (1.1), is defined as the self-adjoint operator in $\mathsf{L}^2(\mathbf{G})$ associated with the quadratic form $\int_{\mathbf{G}} |u'|^2 dx$ on the domain $\mathsf{H}^1(\mathbf{G}, x_0)$. Given an element $f \in \mathsf{L}^2(\mathbf{G})$, the equality $-\Delta u = f$ under these boundary conditions means that u is a unique function in $\mathsf{H}^1(\mathbf{G}, x_0)$ such that

$$\int_{\mathbf{G}} u' \overline{\varphi'} dx = \int_{\mathbf{G}} f \overline{\varphi} dx \quad \forall \varphi \in \mathsf{H}^1(\mathbf{G}, x_0). \tag{3.7}$$

The Euler–Lagrange equation reduces to $-u'' = f$ on each edge. The continuity of u on the whole $\mathbf{G}$ and the boundary condition $u(x_0) = 0$ follow from the inclusion $u \in \mathsf{H}^1(\mathbf{G}, x_0)$. At each vertex $v \neq x_0$, the solution u meets the natural condition in the sense of Calculus of variations. Namely, let $e_1, \dots, e_{d(v)}$ be the edges that are adjacent to a given vertex v and oriented in such a direction that v is their initial point. Then the condition at v, referred to as *Kirchhoff's condition*, is as follows:

$$(u \restriction e_1)'(v) + \dots + (u \restriction e_{d(v)})'(v) = 0.$$

If, in particular, $d(v) = 1$, this condition takes the form $u'(v) = 0$ which is exactly the boundary condition in (1.1).

The requirement $f \in \mathsf{L}^2(\mathbf{G})$ is unnecessary for the existence of a solution $u \in \mathsf{H}^1(\mathbf{G}, x_0)$ of Eq. (3.7). A solution exists if and only if f is such that the expression on the right-hand side generates a continuous anti-linear functional in the space $\mathsf{H}^1(\mathbf{G}, x_0)$. One has to take into account that the solution of (3.7) for $f \notin \mathsf{L}^2(\mathbf{G})$ does not belong to the domain of the Laplacian considered as an operator in $\mathsf{L}^2(\mathbf{G})$. It is convenient to interpret (3.7) as a weak form of the equation $-\Delta u = f$. In particular, this is the case if $f \in \mathsf{L}^1(\mathbf{G})$ because $\mathsf{H}^1(\mathbf{G}, x_0) \subset \mathsf{C}(\overline{\mathbf{G}})$.

According to the above interpretation, Eq. (1.1) means that

$$\lambda \int_{\mathbf{G}} u'\overline{\varphi'}dx = \int_{\mathbf{G}} Vu\overline{\varphi}dx \quad \forall \varphi \in \mathsf{H}^1(\mathbf{G}, x_0). \tag{3.8}$$

For $V \in \mathsf{L}^1(\mathbf{G})$ we have $Vu \in \mathsf{L}^1(\mathbf{G})$, so that the general scheme applies.

On the other hand, the equation $\mathbf{B}_V u = \lambda u$ means that

$$\mathbf{b}_V[u, \varphi] := \int_{\mathbf{G}} Vu\overline{\varphi}dx = \lambda \int_{\mathbf{G}} u'\overline{\varphi'}dx \quad \forall \varphi \in \mathsf{H}^1(\mathbf{G}, x_0).$$

Comparing this with (3.8), we see that the eigenpairs for both equations are the same.

Below, for the eigenvalues of (1.1) we use the notation $\pm\lambda_n^{\pm}(\mathbf{B}_V)$. If $V \geqslant 0$, we write λ_n instead of λ_n^+.

3.3. Reduction to the case of trees. In order to have the possibility to use Theorem 2.1, it is necessary to reduce the problem to the case of trees. The procedure of reduction (cutting cycles) is rather standard. Nevertheless, we describe it in detail.

Lemma 3.3. *Let* $\mathbf{G}$ *be a compact metric graph. Then there is a compact metric tree* $\mathbf{T}$ *and a continuous mapping* $\tau : \mathbf{T} \to \mathbf{G}$ *such that the operator* $\tau^* : u(x) \mapsto u(\tau(x))$ *defines an isometry from the space* $L^{1,2}(\mathbf{G})$ *onto a subspace of finite codimension in* $L^{1,2}(\mathbf{T})$.

PROOF. Let $\mathbf{G}$ be a compact connected graph that is not a tree. If $e_0 = \langle v, w\rangle \in \mathcal{E}(\mathbf{G})$ is any edge that is a part of a cycle in $\mathbf{G}$, then the subgraph $\mathbf{G}_{e_0} = \mathbf{G}\setminus\mathrm{Int}(e_0)$, with $\mathcal{V}(\mathbf{G}_{e_0}) = \mathcal{V}(\mathbf{G})$ and $\mathcal{E}(\mathbf{G}_{e_0}) = \mathcal{E}(\mathbf{G})\setminus\{e_0\}$, is connected.

Choose a point $x \in \mathrm{Int}\, e_0$ and replace it by a pair $\{x_1, x_2\}$ of new vertices. This gives rise to a new graph $\mathbf{G}_1$ whose rigorous definition is as follows. The set of vertices and the set of edges of this graph are defined by the formula

$$\mathcal{V}(\mathbf{G}_1) = \mathcal{V}(\mathbf{G}) \cup \{x_1, x_2\}, \quad \mathcal{E}(\mathbf{G}_1) = \mathcal{E}(\mathbf{G}_{e_0}) \cup \{\langle v, x_1\rangle,\ \langle w, x_2\rangle\}.$$

For any edge $e \in \mathcal{E}(\mathbf{G}_{e_0})$ the endpoints and length of e in $\mathbf{G}_1$ are defined to be the same as in $\mathbf{G}_{e_0}$. We set

$$|\langle v, x_1\rangle|_{\mathbf{G}_1} = |\langle v, x\rangle|_{\mathbf{G}}, \quad |\langle w, x_2\rangle|_{\mathbf{G}_1} = |\langle w, x\rangle|_{\mathbf{G}}.$$

It is oblivous that $|\mathbf{G}_1| = |\mathbf{G}|$.

Now, we define a mapping $\tau_1 : \mathbf{G}_1 \to \mathbf{G}$. Namely, τ_1 is identical on $\mathbf{G}_{e_0}$ and isometrically sends the edge $\langle v, x_1\rangle$ onto $\langle v, x\rangle$ and the edge $\langle w, x_2\rangle$ onto $\langle w, x\rangle$. It is clear that the mapping τ_1 is continuous and the corresponding mapping $\tau_1^* : u(x) \mapsto u(\tau_1(x))$ is an isometry from the space $\mathsf{L}^{1,2}(\mathbf{G})$ into $\mathsf{L}^{1,2}(\mathbf{G}_1)$. Its image consists of all functions $u \in \mathsf{L}^{1,2}(\mathbf{G}_1)$ such that $u(x_1) = u(x_2)$. Therefore, the image is a subspace of codimension 1.

The number of simple cycles in $\mathbf{G}_1$ is smaller than that for the initial graph $\mathbf{G}_0 := \mathbf{G}$. Therefore, repeating this construction, we obtain a finite sequence of graphs $\mathbf{G}_0, \dots, \mathbf{G}_m$ such that the graph $\mathbf{G}_m$ has no cycles and loops (i.e., it is a tree) and a sequence of corresponding mappings $\tau_j : \mathbf{G}_j \to \mathbf{G}_{j-1}$, $j = 1, \dots, m$. The tree $\mathbf{T} = \mathbf{G}_m$ and the mapping $\tau = \tau_1 \circ \tau_2 \circ \dots \circ \tau_m$ satisfy all the requirements of the lemma. $\square$

It is useful to note that

$$\int\limits_{\mathbf{G}} |u|^2 V\,dx = \int\limits_{\mathbf{T}} |\tau^* u|^2 \tau^* V\,dx$$

for any $V \in \mathsf{L}^1(\mathbf{G})$ and $u \in \mathsf{C}(\mathbf{G})$.

4. Proof of the Main Results

First of all, we prove Theorem 3.1 and then derive Theorem 3.2 from it. Theorem 1.1 follows from the latter almost immediately.

PROOF OF THEOREM 3.1. 1. Let $\mathbf{G} = \mathbf{T}$ be a compact tree, $\{T, \xi\}$ its punctured subtree, and $T = \Theta_1 \cup \ldots \cup \Theta_{d_T(\xi)}$ the canonical partition of $\{T, \xi\}$. Applying the inequality (3.1) to each subtree Θ_j, for any $u \in \mathsf{L}^{1,2}(\mathbf{T})$ we get

$$\int\limits_{\Theta_j} |u(x) - u(\xi)|^2 V\,dx \leqslant |\Theta_j| \int\limits_{\Theta_j} V\,dx \int\limits_{\Theta_j} |u'|^2\,dx, \quad j = 1, \dots, d_T(\xi).$$

Hence

$$\int_T |u(x) - u(\xi)|^2 V dx \leqslant \int_T |u'|^2 dx \max_{j=1,\dots,d_T(\xi)} \Big(|\Theta_j| \int_{\Theta_j} V dx\Big). \tag{4.1}$$

Consider a function of subtrees $T \subset \mathbf{T}$,

$$\Phi(T) = \Phi(T; V) = |T|^{1/2} \big(\int_T V dx\big)^{1/2}. \tag{4.2}$$

By (2.3), we have $\Phi \in \mathsf{S}(\mathbf{T})$. The inequality (4.1) can be written as follows:

$$\int_T |u(x) - u(\xi)|^2 V dx \leqslant \widetilde{\Phi}^2(T, \xi) \int_T |u'|^2 dx. \tag{4.3}$$

Here, $\widetilde{\Phi}$ is the function of punctured subtrees associated with the function (4.2) (cf. (2.4)).

Suppose that the tree $\mathbf{T}$ is split into the union of punctured subtrees $\{T_1, x_1\}, \dots, \{T_k, x_k\}$. Let P be the corresponding operator (3.2). From (4.3) it follows that

$$\int_{\mathbf{T}} |u - Pu|^2 V dx = \sum_{j=1}^k \int_{T_j} |u(x) - u(x_j)|^2 V dx \leqslant \sum_{j=1}^k \widetilde{\Phi}^2(T_j, x_j) \int_{T_j} |u'|^2 dx$$
$$\leqslant \|u'\|_2^2 \left(\max_j \widetilde{\Phi}(T_j, x_j)\right)^2 \quad \forall u \in \mathsf{L}^{1,2}(\mathbf{T}).$$

Now we apply Theorem 2.1 to find that for any $n \in \mathbb{N}$ there exists a partition of $\mathbf{T}$ into k punctured subtrees such that $k \leqslant n$ and

$$\int_{\mathbf{T}} |u - Pu|^2 V dx \leqslant (n+1)^{-2} |\mathbf{T}| \int_{\mathbf{T}} V dx \, \|u'\|_2^2 \quad \forall u \in \mathsf{L}^{1,2}(\mathbf{T}).$$

For the case of trees the proof is complete.

2. Let $\mathbf{G}$ be an arbitrary compact graph. Let $\mathbf{T}$ be a compact tree, and let $\tau : \mathbf{T} \to \mathbf{G}$ be a mapping constructed according to Lemma 3.3. Without loss of generality, we assume that the pre-image $\tau^{-1}(x_0)$ consists of a single point.

Let $\{T_j, x'_j\}$, $j = 1, \dots, k$, be a partition of $\mathbf{T}$ such that (3.3) (with $\mathbf{T}$ instead of $\mathbf{G}$) is satisfied. Take $G_j = \tau(T_j)$ and $x_j = \tau(x'_j)$. Since the mapping τ is continuous, each set G_j is closed and connected. Consequently, it is a subgraph of $\mathbf{G}$. The punctured subgraphs $\{G_j, x_j\}$ form a partition of $\mathbf{G}$. An elementary computation shows that (3.3) holds for the graph $\mathbf{G}$ if for P we take the operator (3.2) corresponding to this partition. $\square$

Proof of Theorem 3.2. By the standard limiting argument, the inequality (3.1) extends from the graph $\mathbf{G}$ to its compactification $\overline{\mathbf{G}}$.

Let $\{\mathbf{G}^{(m)}\}_{1\leqslant m<\infty}$ be the family of compact subgraphs of $\mathbf{G}$ such that (2.1) is fulfilled. Fix a number $n \in \mathbb{N}$. For any m let

$$P_m : u \mapsto \sum_{j=1}^{k_m} u(x_j^m)\chi_j^m$$

be the operators (3.2) for the subgraphs $\mathbf{G}^{(m)}$ and the weight functions $V \restriction \mathbf{G}^{(m)}$. These operators depend also on n, but we do not reflect this dependence in the notation.

According to the inequality (3.3), for any m and function $u \in \mathsf{L}^{1,2}(\mathbf{G})$ normalized by the condition $\int\limits_{\mathbf{G}} |u'|^2 dx = 1$ we have

$$\int\limits_{\mathbf{G}^{(m)}} \Big| u - \sum_{j=1}^{k_m} u(x_j^m)\chi_j^m \Big|^2 V dx \tag{4.4}$$
$$\leqslant (n+1)^{-2}|\mathbf{G}^{(m)}| \int\limits_{\mathbf{G}^{(m)}} V dx \int\limits_{\mathbf{G}^{(m)}} |u'|^2 dx \leqslant (n+1)^{-2}|\mathbf{G}| \int\limits_{\mathbf{G}} V dx.$$

For different values of m the numbers k_m may be different. Denote by $k = k(V)$ the minimal number such that $\#\{m : k_m = k\} = \infty$. Thinning out the sequence $\{\mathbf{G}^{(m)}\}$, we can assume that $k_m = k$ for all m. Passing to a subsequence, we find points $\overline{x}_1, \dots \overline{x}_k \in \overline{\mathbf{G}}$ (not necessarily different) such that $x_j^m \to \overline{x}_j$ as $m \to \infty$ for all $j = 1, \dots, k$.

We show that the desired assertion holds at the obtained points. Indeed, if $u \in \mathsf{L}^{1,2}(\mathbf{G})$, $\int\limits_{\mathbf{G}} |u'|^2 dx = 1$, and $u(\overline{x}_1) = \dots = u(\overline{x}_k) = 0$, then for each m we have

$$\Big(\int\limits_{\mathbf{G}^{(m)}} |u|^2 V dx \Big)^{1/2} \leqslant \Big(\int\limits_{\mathbf{G}^{(m)}} |u - P_m u|^2 V dx \Big)^{1/2} + \Big(\int\limits_{\mathbf{G}^{(m)}} \Big|\sum_{j=1}^{k} u(x_j^m)\chi_j^m \Big|^2 V dx \Big)^{1/2} \tag{4.5}$$

If m is large enough, then $|u(x_j^m)| = |u(x_j^m) - u(\overline{x}_j)| < \varepsilon$ for all $j = 1, \dots, k$, where $\varepsilon > 0$ is arbitrarily small. Then

$$\int\limits_{\mathbf{G}^{(m)}} \Big|\sum_{j=1}^{k} u(x_j^m)\chi_j^m \Big|^2 V dx \leqslant \varepsilon^2 \int\limits_{\mathbf{G}^{(m)}} V dx \leqslant \varepsilon^2 \int\limits_{\mathbf{G}} V dx. \tag{4.6}$$

Letting $m \to \infty$ in the inequality (4.5) and taking (4.4) and (4.6) into account, we arrive at the desired inequality (3.4). □

PROOF OF THEOREM 1.1. Let $V \in \mathsf{L}_+(\mathbf{G})$. Fix a number $n \in \mathbb{N}$ and find points $\overline{x}_1, \dots, \overline{x}_k$ according to Theorem 3.2. The subspace

$$\{u \in \mathsf{H}^1(\mathbf{G}, x_0) : u(\overline{x}_1) = \dots = u(\overline{x}_k) = 0\} \subset \mathsf{H}^1(\mathbf{G}, x_0)$$

is of codimension $k \leqslant n$, and for $n > 1$ the inequality

$$\lambda_n(\mathbf{B}_V) \leqslant |\mathbf{G}|n^{-2} \int_{\mathbf{G}} V dx, \quad V \geqslant 0 \tag{4.7}$$

follows from (3.4) by the variational principle. The same inequality for $n = 1$ holds by the estimate (3.6). This completes the proof of (1.2) for $V \geqslant 0$. The assertion for sign-indefinite V follows from (4.7) by the variational principle due to the inequalities $\pm \mathbf{b}_V[u] \leqslant \mathbf{b}_{V_\pm}[u]$.

The asymptotics (1.3) is an almost immediate consequence of the estimate (1.2). Indeed, suppose that the weight function V has compact support, i.e., it vanishes outside a compact subgraph $G \subset \mathbf{G}$. Then we insert additional conditions $u(v) = 0$ at all the vertices $v \in \mathbf{G}$, $v \neq x_0$. Since the number of these conditions is finite, they do not affect the spectral asymptotics. For the new problem the result follows from the well-known asymptotic formula for a single interval. In the general case, we fix $\varepsilon > 0$ and find a compactly supported function V_ε such that $||V - V_\varepsilon||_1 < \varepsilon$. For the operator $\mathbf{B}_{V_\varepsilon}$ the asymptotics (1.3) holds, and for the operator $\mathbf{B}_V - \mathbf{B}_{V_\varepsilon} = \mathbf{B}_{V-V_\varepsilon}$ we have the estimate

$$\lambda_n(|\mathbf{B}_V - \mathbf{B}_{V_\varepsilon}|) \leqslant |\mathbf{G}|\varepsilon n^{-2} \quad \forall n \in \mathbb{N}.$$

Now, the asymptotics (1.3) for the operator $\mathbf{B}_V$ holds by the lemma on the continuity of the asymptotic coefficients (cf. [**2**, Lemma 1.18]). □

5. Complementary Remarks. Applications to Integral Operators on Graphs

5.1. On sharpness of the estimate (1.2)**.** Consider the simplest case, where $\mathbf{G}$ is a single segment $[a, b] \subset \mathbb{R}$, and $V \geqslant 0$. The analog of (1.2) for the eigenvalue problem $-\Lambda u'' = Vu$, $u(a) = u(b) = 0$ is the inequality

$$4n^2\Lambda_n \leqslant (b-a) \int_a^b V dx. \tag{5.1}$$

The sharpness of this inequality was discussed in [**5**, Sec. 3.2]. As was shown, for any $\varepsilon > 0$ and a fixed $n_0 \in \mathbb{N}$ it is possible to find a function $V = V_{\varepsilon,n_0}$ such that the corresponding eigenvalue Λ_{n_0} satisfies the inequality

$$4n_0^2\Lambda_{n_0} \geqslant (1-\varepsilon)(b-a) \int_a^b V dx.$$

So, the constant in (5.1) is sharp.

We now turn to the eigenvalue problem $-\lambda u'' = Vu$, $u'(0) = u(L) = 0$ which is just our problem (1.1) for the graph $\mathbf{G} = [0, L]$ with $x_0 = L$.

Each eigenvalue λ_n of this problem coincides with the eigenvalue Λ_{2n-1} for the equation $-\lambda u'' = V(|x|)u$ on the interval $(-L, L)$ with the zero boundary conditions at both endpoints. Using the above assertion, we find a function $V \geqslant 0$ such that

$$(2n_0 - 1)^2 \lambda_{n_0} = (2n_0 - 1)^2 \Lambda_{2n_0 - 1} \geqslant (1 - \varepsilon) L \int_0^L V dx.$$

Taking $n_0 = 1$, we see that the constant factor "1" in (1.2) is the best possible one. However, for each particular n the factor 1 is probably not sharp.

Note that the inequality (3.1) implies the estimate

$$||\mathbf{B}_V|| \leqslant \operatorname{diam}(\mathbf{G}) \int_{\mathbf{G}} |V| dx, \quad \operatorname{diam}(\mathbf{G}) := \sup\{\rho(x, y) : x, y \in \mathbf{G}\}.$$

It follows that the inequality (1.2) can be replaced by

$$\lambda_n^{\pm} \leqslant \min\left(\frac{|\mathbf{G}|}{n^2}, \operatorname{diam}(\mathbf{G})\right) \int_{\mathbf{G}} V_{\pm} dx.$$

This can be useful while dealing with graphs of small diameter but large total length.

5.2. Singular numbers of the operator $a(x)(-\Delta)^{-1/2}$ in $\mathsf{L}^2(\mathbf{G})$. Recall that the singular numbers (s-numbers) of a compact operator B acting between two Hilbert spaces are defined as non-zero eigenvalues of any of two self-adjoint compact operators $|B| := (B^* B)^{1/2}$ and $|B^*| := (BB^*)^{1/2}$.

According to the Hilbert space theory, for each $u \in \mathsf{H}^1(\mathbf{G}, x_0)$ we have $||u'||_2 = ||(-\Delta)^{1/2} u||_2$. The operator $-\Delta$ was described in Subsec. 3.2. For any $V \in \mathsf{L}_+(\mathbf{G})$ and a nonzero element $u \in \mathsf{H}^1(\mathbf{G}, x_0)$ we have

$$\frac{\int_{\mathbf{G}} |u|^2 V dx}{\int_{\mathbf{G}} |u'|^2 dx} = \left(\frac{||V^{1/2}(-\Delta)^{-1/2} w||_2}{||w||_2}\right)^2, \quad w = (-\Delta)^{1/2} u. \tag{5.2}$$

Taking into account the variational description of the eigenvalues and the s-numbers, from (5.2) we conclude that

$$\lambda_n(\mathbf{B}_V) = s_n^2(V^{1/2}(-\Delta)^{-1/2}) \quad \forall n \in \mathbb{N}.$$

Let $a(x)$ be an arbitrary function from $\mathsf{L}^2(\mathbf{G})$, and let $V(x) = |a(x)|^2$. Then $a(x) = \psi(x) V^{1/2}(x)$, where $|\psi(x)| = 1$ a.e. Multiplication by ψ is a unitary operator. Therefore, the operators $V^{1/2}(x)(-\Delta)^{-1/2}$ and $a(x)(-\Delta)^{-1/2}$ are simultaneously compact and have the same s-numbers. This leads to a useful consequence of the estimate (1.2):

$$s_n\big(a(x)(-\Delta)^{-1/2}\big) \leqslant \frac{|\mathbf{G}|^{1/2}\|a\|_2}{n} \quad \forall n \in \mathbb{N}, \quad a \in \mathsf{L}^2(\mathbf{G}). \tag{5.3}$$

5.3. The case $V \equiv 1$. In this case, $\mathbf{B}_V = (-\Delta)^{-1}$. Thus, $\lambda_n(\mathbf{B}_V) = \lambda_n^{-1}(-\Delta)$ and the relations (1.2) and (1.3) take the form

$$|\mathbf{G}|^2\lambda_n(-\Delta) \geqslant n^2 \quad \forall n \in \mathbb{N}, \tag{5.4}$$

$$\frac{\sqrt{\lambda_n(-\Delta)}}{n} \to \frac{\pi}{|\mathbf{G}|}. \tag{5.5}$$

Even in this simplest case, the estimate (5.4) is informative since it is uniform with respect to all graphs of a given length, independently of their combinatorial structure. By means of standard perturbation arguments, the asymptotics (5.5) extends to the operators $-\Delta + q$ of Sturm–Liouville type with a real-valued bounded potential q. This considerably improves Theorem 5.4 in [**6**]. For such operators on the so-called regular trees another proof of the asymptotics (5.5) was recently given in [**7**].

5.4. Estimates of singular numbers for integral operators on graphs. The above results admit immediate applications to estimates of singular numbers of the integral operators on graphs. We restrict ourselves to the simplest case where the operator acts in $\mathsf{L}^2(\mathbf{G})$ according to the formula

$$\big(\mathbf{K}u\big)(x) = \int_{\mathbf{G}} K(x,y)u(y)dy. \tag{5.6}$$

We suppose that the kernel $K(x,y)$ belongs to the space $\mathsf{L}^2_y(\mathbf{G}, \mathsf{W}^{1,2}_x(\mathbf{G}))$. This means that K is measurable on $\mathbf{G}\times\mathbf{G}$, for almost all $y \in \mathbf{G}$ the function $K(\cdot, y)$ belongs to the space $\mathsf{L}^{1,2}(\mathbf{G})$, and

$$\mathcal{M}(K,\mathbf{G}) := \int_{\mathbf{G}}\int_{\mathbf{G}} \big(|K(x,y)|^2 + |\mathbf{G}|^2|K'_x(x,y)|^2\big)dxdy < \infty.$$

The expression for $\mathcal{M}(K,\mathbf{G})$ is homogeneous with respect to the similitudes of the graph $\mathbf{G}$. This explains why the factor $|\mathbf{G}|^2$ is included.

Theorem 5.1. *Let* $\mathbf{G}$ *be a graph of finite total length. Suppose that* $K \in L^2_y(\mathbf{G}, W^{1,2}_x(\mathbf{G}))$. *Then the s-numbers of the operator* $\mathbf{K}$ *given by* (5.6) *satisfy the inequality*

$$\sum_{n=1}^{\infty} n^2 s_n^2(\mathbf{K}) \leqslant 32|\mathbf{G}|^2\mathcal{M}(K,\mathbf{G}). \tag{5.7}$$

If, in addition, there exists a point $x_0 \in \mathbf{G}$ such that $K(x_0, y) = 0$ for almost all $y \in \mathbf{G}$, then the inequality (5.7) *can be refined as follows:*

$$\sum_{n=1}^{\infty} n^2 s_n^2(\mathbf{K}) \leqslant 8|\mathbf{G}|^2 \int\limits_{\mathbf{G}} \int\limits_{\mathbf{G}} |K_x'(x,y)|^2 dx dy. \tag{5.8}$$

PROOF. We follow the approach of [**8**, Sec. II.10.4].

We start with the proof of the second assertion of Theorem. By assumption, the function $K(\cdot, y)$ belongs to $H^1(\mathbf{G}, x_0)$ for almost all $y \in \mathbf{G}$. This allows one to write

$$K(\cdot, y) = (-\Delta_x)^{-1/2} L(\cdot, y),$$

where $L(\cdot, y) = (-\Delta_x)^{1/2} K(\cdot, y)$ and $-\Delta_x$ is the operator $-\Delta$ (with the boundary conditions as in (1.1); cf. Subsec. 3.2) acting in the variable x. This representation of the kernel yields factorization of the corresponding operator, $\mathbf{K} = (-\Delta_x)^{-1/2}\mathbf{L}$.

From the estimate (5.3) for $a \equiv 1$ we derive $s_n\big((-\Delta_x)^{-1/2}\big) \leqslant |\mathbf{G}| n^{-1}$. The operator $\mathbf{L}$ is of the Hilbert–Schmidt class; moreover,

$$\sum_{n=1}^{\infty} s_n^2(\mathbf{L}) = \int\limits_{\mathbf{G}} \int\limits_{\mathbf{G}} |L(x,y)|^2 dx dy = \int\limits_{\mathbf{G}} \int\limits_{\mathbf{G}} |K_x'(x,y)|^2 dx dy. \tag{5.9}$$

By the Ky Fan inequality (cf. [**8**, Corollary II. 2.2]), we have

$$s_{2n}(\mathbf{K}) \leqslant s_{2n-1}(\mathbf{K}) \leqslant s_n\big((-\Delta)^{-1/2}\big) s_n(\mathbf{L}) \leqslant |\mathbf{G}| n^{-1} s_n(\mathbf{L}).$$

Together with (5.9), this implies (5.8).

Let us turn to the proof of (5.7). The function $K(\cdot, y)$ is continuous on $\mathbf{G}$ for almost all $y \in \mathbf{G}$. Choose a point $x_0 \in \mathbf{G}$ and split the kernel K into the sum $K = K^0 + \widetilde{K}$, where $K^0(x, y) = K(x_0, y)$. The operator $\mathbf{K}^0$ has rank one, and its only non-zero singular number is given by

$$s_1^2(\mathbf{K}^0) = |\mathbf{G}| \int\limits_{\mathbf{G}} |K(x_0, y)|^2 dy \leqslant 2\mathcal{M}(K, \mathbf{G}). \tag{5.10}$$

Let us explain the last inequality. Any function $w \in \mathsf{L}^{1,2}(\mathbf{G})$ takes the mean value $\widehat{w} = |\mathbf{G}|^{-1} \int\limits_{\mathbf{G}} w dx$ at some point $\xi \in \overline{\mathbf{G}}$. Using the inequality (3.1) (which extends from $\mathbf{G}$ to any $\xi \in \overline{\mathbf{G}}$ by continuity), we obtain

$$|w(x_0) - \widehat{w}|^2 \leqslant |\mathbf{G}| \int\limits_{\mathbf{G}} |w'|^2 dx.$$

Furthermore,

$$|\mathbf{G}| |\widehat{w}|^2 \leqslant \int\limits_{\mathbf{G}} |w|^2 dx.$$

Therefore,

$$|\mathbf{G}||w(x_0)|^2 \leqslant 2\int\limits_{\mathbf{G}}(|w|^2+|\mathbf{G}|^2|w'|^2)dx,$$

which implies (5.10).

Since the operator $\mathbf{K}^0$ has rank 1, we have $s_{n+1}(\mathbf{K}) \leqslant s_n(\widetilde{\mathbf{K}})$ for all $n \in \mathbb{N}$. For $\widetilde{\mathbf{K}}$ the inequality (5.8) holds since $\widetilde{K}(x_0, y) = 0$ a.e. Therefore,

$$\sum_{n=2}^{\infty} n^2 s_n^2(\mathbf{K}) \leqslant \sum_{n=1}^{\infty}(n+1)^2 s_n^2(\widetilde{\mathbf{K}}) \leqslant 32|\mathbf{G}|^2 \int\limits_{\mathbf{G}}\int\limits_{\mathbf{G}} |K_x'(x,y)|^2 dxdy. \tag{5.11}$$

Further, (5.8) implies $s_1(\widetilde{\mathbf{K}}) \leqslant \sqrt{8}\mathcal{M}(K,\mathbf{G})$. The inequality (5.7) follows from this fact, (5.11), and (5.10) in view of the triangle inequality $s_1(\mathbf{K}) \leqslant s_1(\widetilde{\mathbf{K}}) + s_1(\mathbf{K}^0) \leqslant 3\sqrt{2}\mathcal{M}(K,\mathbf{G})$. □

Corollary 5.2. *Let the assumptions of* Theorem 5.1 *be satisfied. Then the singular numbers $s_n(\mathbf{K})$ of the operator* (5.6) *satisfy the estimate*

$$s_n(\mathbf{K}) \leqslant \frac{4\sqrt{6}}{n^{3/2}}\mathcal{M}^{1/2}(K,\mathbf{G}) \quad \forall n \in \mathbb{N}, \tag{5.12}$$

and $s_n(\mathbf{K}) = o(n^{-3/2})$.

Proof. Introduce the notation $s_n = s_n(\mathbf{K})$ and $C^2 = 32\mathcal{M}(K,\mathbf{G})$. The estimate (5.12) follows from the inequality

$$\frac{n^3}{3}s_n^2 \leqslant s_n^2\sum_{k=1}^{n}k^2 \leqslant \sum_{k=1}^{n}k^2 s_k^2 \leqslant C^2 \quad \forall n \in \mathbb{N}.$$

The relation $s_n(\mathbf{K}) = o(n^{-3/2})$ follows from the inequality

$$cn^3 s_n^2 \leqslant \sum_{k=[n/2]}^{n} k^2 s_k^2, \quad n > 1,\ c > 0,$$

where the right-hand side tends to zero as $n \to \infty$ by the convergence of the series in (5.7). □

We emphasize that the estimates (5.7), (5.8), and (5.12) are uniform with respect to all graphs of a given length. Specific values of the constants in these estimates are not so important.

In the same way, it is possible to study similar operators (with dy in (5.6) replaced by $d\mu(y)$) acting between the spaces $\mathsf{L}^2(\mathbf{G},\mu)$ and $\mathsf{L}^2(\mathbf{G},\nu)$, where μ and ν are finite Borelian measures. Such operators appear in various applications (cf. [9]). Note that, in the case of trees, an analog of Theorem 5.1 for l times differentiable kernels can be derived from Theorem 6.1 of the next section.

6. Spaces $\mathbf{H}^l(\mathbf{T}, x_0)$ and Operators $\mathbf{B}_{l,V}$

We discuss the higher-order analogs of the space $\mathsf{L}^{1,2}$. Here, a serious obstacle arises since the continuity of derivatives $u', \dots, u^{(l-1)}$ at the vertices should be included in the definition. However, the derivatives of odd order change their sign depending on the orientation on edges. Hence for $l > 1$ the space $\mathsf{L}^{l,2}(\mathbf{G})$ can be well defined only for oriented graphs and for different choice of orientation such spaces are substantially different. For this reason, we define the spaces $\mathsf{L}^{l,2}$ only on trees since for them a natural orientation does exist.

Thus, let $\mathbf{G} = \mathbf{T}$ be a tree of finite total length, and let $x_0 \in \mathbf{T}$ be a fixed vertex (the root). The natural partial ordering on the rooted tree $\{\mathbf{T}, x_0\}$ is introduced as follows:

$$x \preceq y \Longleftrightarrow x \in \langle x_0, y\rangle.$$

Recall that $\langle x_0, y\rangle$ is a unique simple path in $\mathbf{T}$ connecting x_0 with y. We always parametrize the edges of $\mathbf{T}$ in the direction compatible with this partial ordering.

Now we are in a position to define the space $\mathsf{H}^l(\mathbf{T}, x_0)$ for arbitrary $l \in \mathbb{N}$. A function u on $\mathbf{T}$ belongs to $\mathsf{H}^l(\mathbf{T}, x_0)$ if u is continuous on $\mathbf{T}$, the restriction of u to each edge e lies in $\mathsf{H}^l(e)$, the functions $u', \dots u^{(l-1)}$ extend from $\mathbf{T} \setminus \mathcal{V}(\mathbf{T})$ to the whole $\mathbf{T}$ as continuous functions, $u(x_0) = \dots = u^{(l-1)}(x_0) = 0$, and $u^{(l)} \in \mathsf{L}_2(\mathbf{T})$. We consider $\mathsf{H}^l(\mathbf{T}, x_0)$ as the Hilbert space equipped with the scalar product $(u, v)_{\mathsf{H}^l(\mathbf{T},x_0)} = (u^{(l)}, v^{(l)})_{\mathsf{L}^2(\mathbf{T})}$ and the corresponding norm.

Let $\xi, x \in \mathbf{T}$, $\xi \preceq x$. Consider the Taylor polynomial

$$P_{l-1}(t; u, \xi) = \sum_{k=0}^{l-1} \frac{u^{(k)}(\xi) t^k}{k!}.$$

By our agreement about orientation, we have

$$u(x) - P_{l-1}(\rho(\xi, x); u, \xi) = \frac{1}{(l-1)!} \int\limits_{\langle \xi, x\rangle} u^{(l)}(y) \rho^{l-1}(y, x)\, dy.$$

By Cauchy's inequality,

$$((l-1)!)^2 |u(x) - P_{l-1}(\rho(\xi, x); u, \xi)|^2 \leqslant \frac{\rho^{2l-1}(\xi, x)}{2l-1} \int\limits_{\langle \xi, x\rangle} |u^{(l)}(y)|^2 dy. \tag{6.1}$$

Given a function $V \in \mathsf{L}^1(\mathbf{T})$, let $\mathbf{b}_V$ be the corresponding quadratic form (cf. (3.5)). From the inequality (6.1) for $\xi = x_0$ it follows that

$$|\mathbf{b}_V[u]| \leqslant C'(l) |\mathbf{T}|^{2l-1} \|u^{(l)}\|_2^2 \int\limits_{\mathbf{T}} |V(x)| dx \quad \forall u \in \mathsf{H}^l(\mathbf{T}, x_0),$$

$$C'(l) = ((l-1)!)^{-2} (2l-1)^{-1}.$$

Therefore, in $\mathsf{H}^l(\mathbf{T}, x_0)$, the quadratic form $\mathbf{b}_V[u]$ generates a bounded linear operator denoted by $\mathbf{B}_{l,V}$. The pair $\{\lambda, u\}$ corresponds to the problem

$$\lambda(-\Delta)^l u = Vu, \quad u(x_0) = u'(x_0) = \ldots = u^{(l-1)}(x_0) = 0, \qquad (6.2)$$
$$u^{(l)}(v) = \ldots = u^{(2l-1)}(v) = 0 \text{ if } v \in \partial\mathbf{T} \setminus \{x_0\}.$$

For rooted trees, Theorem 1.1 is a particular case of the following assertion.

Theorem 6.1. *Let* $\mathbf{T}$ *be a rooted tree of finite total length, let* x_0 *be a root of* $\mathbf{T}$, *and let* $V = \overline{V} \in L^1(\mathbf{T})$. *Then the eigenvalues of the problem* (6.2) *satisfy the inequality*

$$\lambda_n^\pm \leqslant C(l)\frac{|\mathbf{T}|^{2l-1}\int_{\mathbf{T}} V_\pm dx}{n^{2l}}, \quad C(l) = \frac{l^{2l}}{((l-1)!)^2(2l-1)} \quad \forall n \in \mathbb{N}. \qquad (6.3)$$

Along with the estimate (6.3), *the Weyl-type asymptotics holds,*

$$n\left(\lambda_n^\pm\right)^{\frac{1}{2l}} \to \pi^{-1}\int_{\mathbf{T}}\left(V_\pm(x)\right)^{\frac{1}{2l}}dx, \quad n \to \infty.$$

Outline of the proof. We discuss only the estimate (6.3) in the case of compact trees and $V \in \mathsf{L}_+(\mathbf{T})$ since the rest needs no serious changes compared with the proof of Theorems 3.2 and 1.1 We also suppose that $d(x_0) = 1$. Otherwise, the operator $\mathbf{B}_{l,V}$ splits into the orthogonal sum of similar operators for each of $d(x_0)$ subtrees constituting the canonical partition of the punctured tree $\{\mathbf{T}, x_0\}$ (cf Sec. 2). The estimate (6.3) for $\mathbf{B}_{l,V}$ easily follows from the same estimate for the corresponding parts.

Consider the following function of subtrees $T \subset \mathbf{T}$:

$$\Phi(T) = |T|^{1-1/(2l)}\Bigl(\int_T V dx\Bigr)^{1/(2l)} \qquad (6.4)$$

Let $\widetilde{\Phi}(T, x)$ be the function of punctured subtrees associated with Φ (cf. (2.4)). Suppose that the punctured subtrees $\{T_j, x_j\}, j = 1, \ldots, k$ form a partition of $\mathbf{T}$ and χ_j is the characteristic function of T_j. From (6.1) it is easy to obtain the following inequality for $u \in \mathsf{H}^l(\mathbf{T}, x_0)$:

$$\int_{\mathbf{T}}\Bigl|u - \sum_{j=1}^k P_{l-1}(\rho(x_j, x); u, x_j)\chi_j\Bigr|^2 V dx \leqslant C'(l)\|u^{(l)}\|_2^2 \,\bigl(\max_{j=1,\ldots,k}(\widetilde{\Phi}(T_j, x_j)\bigr)^{2l},$$

provided that all the subtrees $\{T_j, x_j\}$ are oriented coherently to the orientation on $\mathbf{T}$, i.e., $x_j \preceq x$ for any $x \in T_j$.

Theorem 2.1 applies to the function Φ but this does not lead automatically to the inequality (6.3). Indeed, we have to check that all the subtrees $\{T_j, x_j\}$ are properly oriented. For this purpose, we return to Lemma 2.2 whose consequence is Theorem 2.1. The proof of the lemma started with choosing a vertex $v_0 \in \partial\mathbf{T}$. Then a path $\mathcal{P}$ and subtrees T_x^+ for each $x \in \mathcal{P}$ were constructed. The assumption $d(x_0) = 1$ means that $x_0 \in \partial\mathbf{T}$. If we take $v_0 = x_0$, then all the subtrees T_x^+ are oriented coherently to the orientation on $\mathbf{T}$, and the scheme goes through.

Applying the inequality (2.5) to the function Φ introduced by (6.4) and using the variational principle, we find that

$$\lambda_{Nl+1}(\mathbf{B}_{l,V}) \leqslant C'(l)\frac{|\mathbf{T}|^{2l-1}\int_{\mathbf{T}} V\,dx}{N^{2l}} \qquad \forall N \in \mathbb{N}, \quad V \in L_+(\mathbf{T}).$$

The estimate (6.3) for all $n \in \mathbb{N}$ with $C(l) = C'(l)l^{2l}$ follows from here by the monotonicity of eigenvalues in n.

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Potential Theory for the Nonstationary Stokes Problem in Nonconvex Domains

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Dedicated to O. A. Ladyzhenskaya on the occasion of her birthday

Based on the theory of nonstationary hydrodynamic potentials, we construct a solution to the nonstationary Stokes problem in a bounded domain or in an exterior domain with C^2-boundary. For the kernel of the main hydrodynamic potential we take the matrix of "Poisson's kernels" for the exterior of a ball.

1. Introduction

In this paper, we construct a solution to the nonstationary Stokes problem

$$\begin{aligned} &\vec{v}_t - \nu\Delta\vec{v} + \Delta p = 0, \quad \operatorname{div}\vec{v} = 0, \qquad x \in \Omega,\ t \in (0,T), \\ &\vec{v}\big|_{t=0} = 0, \quad \vec{v}\big|_{x\in S} = \vec{a}(x,t) \end{aligned} \tag{1.1}$$

in a bounded domain or an exterior domain $\Omega \subset \mathbb{R}^n$ with C^2-boundary S on the basis of the theory of nonstationary hydrodynamic potentials. This theory was created by Leray [**1**] in the two-dimensional case and by Golovkin [**2**] in the three-dimensional case. But, since for the basic singular solution they took the matrix of "Poisson's kernels" of the problem (1.2) in the half-space (the so-called the "second singular solution"), it was necessary to require the convexity of the domain Ω. In this paper, this restriction is not required because for the

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kernel of the basic hydrodynamic potential we take the matrix of "Poisson's kernels" of the exterior of a ball.

Thus, we consider the problem

$$\begin{aligned} &\vec{v}_t - \nu\Delta\vec{v} + \nabla p = 0, \quad \operatorname{div}\vec{v} = 0, \qquad x \in B,\ t \in (0,T),\\ &\vec{v}\big|_{t=0} = 0, \quad \vec{v}\big|_{x\in S_a} = \vec{a}(x,t), \end{aligned} \tag{1.2}$$

where $B = B_a = \{|x| < a\}$ or $B = B^a = \{|x| > a\}$ and $S_a = \partial B = \{|x| = a\}$ and $\vec{a}(x,t)$ satisfies the condition

$$\vec{a}(x,t)\cdot\vec{n}(x) = 0, \tag{1.3}$$

where $\vec{n}(x) = a^{-1}\vec{x}$ is the outward normal to S_a. For $B = B^a \subset \mathbb{R}^3$ the solution to the problem (1.2) was found by Oseen [**3**]. We only write this solution as a surface potential for the sake of convenience. We also obtain estimates for the kernels of this potential, but the proof of these estimate is omitted because of the limit of place. These estimates are similar to estimates for the "second fundamental solution" (cf. [**4**]) (The potential theory is presented in Sec. 4).

The author dedicates this work to O. A. Ladyzhenskaya whose student he was and with whom he was working for many years.

2. Preliminaries

We give several known relations which will be used while constructing a solution to the problem (1.2).

We consider the case of an exterior domain B^a. Let $G(x,y)$ be Green's function of the Dirichlet problem

$$\Delta u = f, \quad x \in B^a, \qquad u\big|_{S_a} = \varphi, \tag{2.1}$$

and let $N(x,y)$ be Green's function of the Neumann problem

$$\Delta v = f, \quad x \in B^a, \qquad \frac{\partial u}{\partial n}\Big|_{S_a} = \psi \tag{2.2}$$

($\vec{n} = \vec{x}a^{-1}$). Using these functions, it is possible to express the solutions to these problems as the sums of potentials:

$$u(x) = \int\limits_{B^a} G(x,y)f(y)\,dy - \int\limits_{S_a} \frac{\partial G(x,y)}{\partial n_y}\varphi(y)\,dS_y, \tag{2.3}$$

$$v(x) = \int\limits_{B^a} N(x,y)f(y)\,dy + \int\limits_{S_a} N(x,y)\psi(y)\,dS_y. \tag{2.4}$$

For $G(x,y)$ there is a well-known explicit formula

$$G(x,y) = E(x-y) - \left(\frac{a}{|y|}\right)^{n-2} E(x-y^*)$$
$$= E(x-y) - \left(\frac{a}{|x|}\right)^{n-2} E(x^*-y) \quad \text{if } n>2, \tag{2.5}$$
$$G(x,y) = E(x-y) - E(x-y^*) - \frac{1}{2n}\ln\frac{|y|}{a}$$
$$= E(x-y) - E(x^*-y) - \frac{1}{2\pi}\ln\frac{|x|}{a} \quad \text{if } n=2, \tag{2.6}$$

where y^* is the point symmetric to the point y with respect to the sphere S_a, i.e., $y^* = \dfrac{y}{|y|^2}a^2$, and $E(z)$ is the fundamental solution to the Laplace equation:

$$E(z) = \begin{cases} \dfrac{1}{2\pi}\ln|z|, & n=2, \\ -\dfrac{1}{|S_1|(n-2)|z|^{n-2}}, & n>2; \end{cases}$$

$|S_1| = 2\pi^{m/2}/\Gamma(n/2)$ is the surface area of the unit sphere $S_1 = \{|x|=1\}$ in $\mathbb{R}^n$. The function

$$P(x,y) = -\frac{\partial G(x,y)}{\partial n_y} = \frac{|x|^2-a^2}{|S_1|a|x-y|^n} \tag{2.7}$$

is called *Poisson's kernel.* Using this function, we can express the solution to the problem (2.1) with $f=0$ as a surface potential.

We are interested in the problem (2.2) only for $f=0$. Its solution is the surface potential

$$\int\limits_{S_a} N(x,y)\psi(y)\,dS_y$$

whose kernel $N(x,y)$ can be determined by the formula

$$N(x,y) = 2E(x-y) - \frac{1}{a\pi}\int\limits_{S_a} E(x-\xi)\,dS_\xi, \quad n=2, \tag{2.8}$$

$$N(x,y) = 2E(x-y) - (n-2)\int\limits_{|x|}^{\infty} \frac{E(x_r-y)}{r}dr, \quad n>2, \tag{2.9}$$

where $x_r = r\dfrac{x}{|x|}$. In particular, for $n=3$

$$N(x,y) = -\frac{1}{2\pi|x-y|} + \frac{1}{4\pi a}\ln\frac{|x-y|+|x|+a}{|x-y|+|x|-a}.$$

Indeed, it is easy to verify that for $n > 2$

$$N^*(x,y) = -(n-2)\int\limits_{|x|}^{\infty} \frac{E(x_t - y)}{t}\,dt$$

is a harmonic function. Setting $|x| = r$ and passing to the spherical coordinates, we find

$$\begin{aligned}
\Delta_x N^*(x,y) &= \frac{1}{r^{n-1}}\frac{\partial}{\partial r} r^{n-1}\frac{\partial N^*}{\partial r} + \frac{1}{r^2}\Delta_{S_1} N^* \\
&= (n-2)\frac{1}{r^{n-1}}\frac{\partial}{\partial r} r^{n-2} E(x-y) + \frac{n-2}{r^2}\int\limits_r^{\infty} \frac{1}{t^{n-2}}\frac{\partial}{\partial t} t^{n-1}\frac{\partial E(x_t - y)}{\partial t}\,dt \\
&= \frac{n-2}{r^{n-1}}\frac{\partial}{\partial r} r^{n-2} E(x-y) - \frac{n-2}{r}\frac{\partial E(x-y)}{\partial r} - \frac{(n-2)^2}{r^2} E(x-y) = 0,
\end{aligned}$$

where Δ_{S_1} is the spherical part of the Laplace operator. Moreover,

$$\begin{aligned}
\frac{\partial N}{\partial r} &= \frac{2}{|S_1|}\frac{x}{|x|}\cdot\frac{x-y}{|x-y|^n} - \frac{1}{|S_1||x||x-y|^{n-2}} \\
&= \frac{1}{|S_1||x|}\left(\frac{|x|^2 - 2x\cdot y + a^2 + |x|^2 - a^2}{|x-y|^n} - \frac{1}{|x-y|^{n-2}}\right) \\
&= \frac{a}{|x|} P(x,y)
\end{aligned} \tag{2.10}$$

and, since $N(x,y)$ is a harmonic function of x, we have

$$\int\limits_{S_a} P(x,\xi) N(\xi, y)\, dS_\xi = N(x,y). \tag{2.11}$$

For $n = 2$ the second term in (2.8) does not play any essential role because the function $\psi(x)$ in the problem (2.2) with $f = 0$ should satisfy the condition

$$\int\limits_{S_a} \psi\, dS = 0. \tag{2.12}$$

This term was included in $N(x,y)$ in order to preserve formula (2.11). However, instead of (2.10), we have

$$\begin{aligned}
\frac{\partial N(x,y)}{\partial r} &= \frac{a}{|x|}P(x,y) + \frac{1}{2\pi|x|} - \frac{1}{2a\pi}\int\limits_{S_a}\left(P(x,\xi)\frac{a}{|x|} + \frac{1}{2\pi|x|}\right) dS_\xi \\
&= \frac{a}{|x|}P(x,y) - \frac{1}{2\pi|x|}.
\end{aligned} \tag{2.13}$$

In the case of a bounded domain $B_a = \{|x| < a\}$, formulas (2.3) and (2.4) take the form

$$u(x) = \int\limits_{B_a} G(x,y) f(y)\, dy + \int\limits_{S_a} \frac{\partial G(x,y)}{\partial n_y} \varphi(y)\, dS_y,$$

$$v(x) = \int\limits_{B_a} N(x,y) f(y)\, dy - \int\limits_{S_a} N(x,y)\psi(y) dS_y.$$

The function $G(x,y)$ is determined by the same formulas (2.5) and (2.6), Poisson's kernel for the Dirichlet problem is expressed as follows:

$$P(x,y) = \frac{\partial G(x,y)}{\partial n_y} = \frac{a^2 - |x|^2}{|S_1| a |x-y|^n}, \quad x \in B_a, \ y \in S_a.$$

The function $N(x,y)$, $x \in B_a$, $y \in S_a$, can be determined by the relation

$$N(x,y) = \begin{cases} 2E(x-y), & n = 2, \\ 2E(x-y) + (n-2) \displaystyle\int\limits_0^{|x|} \frac{E(x_r - y) - E(y)}{r} dr, & n > 2. \end{cases} \tag{2.14}$$

The integral in (2.14) is a harmonic function of x; moreover,

$$\frac{\partial N}{\partial r} = \begin{cases} -\dfrac{a}{|x|} P(x,y) + \dfrac{1}{2\pi |x|}, & n = 2, \\ -\dfrac{a}{|x|} P(x,y) + \dfrac{1}{|S_1| a^{n-2} |x|}, & n > 2. \end{cases} \tag{2.15}$$

It is clear that for any function $\psi(y)$ satisfying (2.12) the potential

$$-\int\limits_{S_a} N(x,y)\psi(y)\, dS_y$$

is a solution to the problem (2.2) with $f = 0$. The function (2.14) satisfies the relation (2.11) for $y \in S_a$, $x \in B_a$.

Finally, we denote by $\mathbb{P}(x,y,t)$ Poisson's kernel of the parabolic problem

$$\begin{aligned} &u_t - \nu \Delta u = 0, \quad x \in B, \ t > 0, \\ &u\big|_{t=0} = 0, \quad u\big|_{x \in S_a} = \varphi(x,t) \end{aligned} \tag{2.16}$$

in the domain $B = B_a$ or $B = B^a$. A solution u to this problem is the heat potential

$$u(x,t) = \int\limits_0^t \int\limits_{S_a} \mathbb{P}(x,y,t-\tau)\varphi(y,\tau)\, dS_y\, d\tau, \quad x \in B.$$

As is known, $u(x,t)$ can be represented as the potential of a double layer

$$u(x,t)=\int_a^t\int_{S_a}\frac{\partial\Gamma(x-y,t-\tau)}{\partial n_y}\mu(y,\tau)\,dS_y d\tau,$$

where

$$\Gamma(x,t)=\frac{1}{(4\pi\nu t)^{n/2}}\exp\Big\{-\frac{|x|^2}{4\nu t}\Big\}$$

is the fundamental solution to the heat equations and μ and φ are connected by the integral equation on S_a

$$\mu(x,t)\pm 2\nu\int_0^t\int_{S_a}\frac{\partial\Gamma(x-y,t-\tau)}{\partial n_y}\mu(y,\tau)\,dS_y d\tau=\pm 2\nu\varphi(x,t),$$

where the sign "+" corresponds to the exterior problem and the sign "−" corresponds to the interior problem. Consequently,

$$\mathbb{P}(x,y,t)=\pm 2\nu\frac{\partial\Gamma(x-y,t)}{\partial n_y}\pm 2\nu\int_0^t\int_{S_a}\frac{\partial\Gamma(x-\xi,t-\tau)}{\partial n_\xi}R(\xi,y,\tau)\,dS_\xi d\tau,$$

where R is the resolvent of the integral equation for μ. Thus, $\pm 2\nu\dfrac{\partial\Gamma}{\partial n_y}$ is the basic singular part of $\mathbb{P}$.

We give the further useful relations. The expression $\operatorname{div}\vec{f}=\nabla\cdot\vec{f}$ can be written in the form

$$\begin{aligned}\nabla\cdot\vec{f}(x)&=\Big(\vec{n}(x)\frac{\partial}{\partial r}+\nabla'\Big)(\vec{f}'+\vec{n}(x)f_r)=\frac{\partial f_r}{\partial r}+f_r\operatorname{div}\vec{n}(x)+\nabla'\cdot\vec{f}'\\&=\frac{\partial f_r}{\partial r}+(n-1)\frac{f_r}{r}+\nabla'\cdot\vec{f}',\end{aligned}$$

where $r=|x|$, $\vec{n}(x)=\dfrac{\vec{x}}{|x|}=\vec{e}_r(x)$, $f_r=\vec{f}\cdot\vec{e}_r(x)$ is the radial component of $\vec{f}$, $\vec{f}'=\vec{f}-\vec{n}(x)f_r$ is the tangent component of $\vec{f}$ on the sphere $S_{|x|}$, ∇' is the gradient on $S_{|x|}$:

$$\nabla'=(\partial_{x_1},\dots,\partial_{x_n}),\quad \partial_{x_j}=\frac{\partial}{\partial x_j}-n_{ij}\frac{\partial}{\partial r}.$$

By ∇_{S_1} we understand the gradient on the unit sphere. Hence $\nabla'=\dfrac{1}{r}\nabla_{S_1}$.

The expression $\nabla'\cdot\vec{f}'$ is the divergence $\operatorname{div}_{S_{|x|}}\vec{f}'$ of the field $\vec{f}'$ on $S_{|x|}$ defined for an arbitrary manifold S_a in differential geometry. The following formulas hold:

$$\begin{gathered}\nabla' \cdot \vec{f} = (n-1)\frac{f_r}{r} + \nabla' \cdot \vec{f}', \\ \int\limits_{S_r} \varphi(y)\nabla' \cdot \vec{f}' dS_y = -\int\limits_{S_r} \nabla'\varphi \cdot \vec{f} dS_y = -\int\limits_{S_r} \nabla\varphi \cdot \vec{f}' dS_y,\end{gathered} \tag{2.17}$$

$$\int\limits_{S_r} \varphi(y)\nabla' \cdot \vec{f} dS_y = (n-1)\int\limits_{S_r} \varphi f_r r^{-1} dS_y - \int\limits_{S_r} \nabla'\varphi \cdot \vec{f}' dS_y. \tag{2.18}$$

If the functions $f(x,y)$ and $g(x,y)$ defined on S_r depend only on $\cos\gamma_{xy} = \dfrac{x \cdot y}{|x||y|}$ (γ_{xy} is the angle between the vectors $\vec{x}$ and $\vec{y}$), then

$$\int\limits_{S_r} f(\cos\gamma_{xy})g(\cos\gamma_{zy})\, dS_y = \int\limits_{S_r} g(\cos\gamma_{xy})f(\cos\gamma_{zy})\, dS_y. \tag{2.19}$$

The last formula can be regarded as an analog of the theorem about the commutativity of convolution in $\mathbb{R}^n$. In the two-dimensional case, it reduces to the obvious relation

$$\int\limits_0^{2\pi} f(\cos(\varphi - \varphi_x))g(\cos(\varphi_z - \varphi))\, d\varphi = \int\limits_0^{2\pi} g(\cos(\varphi' - \varphi_x))f(\cos(\varphi_z - \varphi'))d\varphi',$$

where $x = r(\cos\varphi_x, \sin\varphi_x)$, $y = r(\cos\varphi, \sin\varphi)$, $z = r(\cos\varphi_z, \sin\varphi_z)$, which is verified with the help of the simple change of variables $\varphi - \varphi_x = \varphi_z - \varphi'$. In the n-dimensional case, formula (2.19) can be proved in the same way if for the plane (y_1, y_2) we take the plane passing through the origin and the point x, z and introduce the polar angle φ. Then

$$\begin{aligned} y &= (\rho(y)\cos\varphi, \rho\sin\varphi, y''), \\ x &= (r\cos\varphi_x, r\sin\varphi_x, 0), \\ z &= (r\cos\varphi_z, r\sin\varphi_z, 0), \end{aligned}$$

where $\rho(y)$ is the length of the projection of y to the plane y_1, y_2 and y'' is the projection of y to the orthogonal $n-2$-dimensional subspace. It is clear that

$$\cos\gamma_{xy} = \frac{\rho}{r}\cos(\varphi - \varphi_x), \quad \cos\gamma_{zy} = \frac{\rho}{r}\cos(\varphi_z - \varphi),$$

and (2.19) is proved with the help of the simple change of integration variable as in the case $n = 2$.

The relation (2.19) will be applied to the kernels $P(x,y)$, $N(x,y)$, and $\mathbb{P}(x,y,t)$ depending on $|x|$, $|y|$, and $\cos\gamma_{xy}$. Finally, we note that

$$\nabla_x \cos\gamma_{xy} = \nabla' \cos\gamma_{xy} = \frac{1}{|x|}(\vec{n}(y) - \vec{n}(x)\cos\gamma_{xy}). \tag{2.20}$$

3. Solution to the Problem (1.2)

We show that the solution to the interior problem (1.2), as well as that to the exterior problem (1.2), has the following structure:

$$\vec{v}(x,t) = \vec{w}(x,t) + \nabla V(x,t), \quad p(x,t) = -\Big(\frac{\partial}{\partial t} - \nu\Delta\Big)V(x,t). \tag{3.1}$$

Here, $\vec{w}$ is a solution to the parabolic problem

$$\begin{aligned} &\vec{w}_t - \nu\Delta\vec{w} = 0, \quad x \in B,\ t > 0,\\ &\vec{w}\big|_{t=0} = 0,\\ &\vec{w}\big|_{x\in S_a} = \vec{a}(x,t) + \vec{n}(x)Q(x,t), \end{aligned} \tag{3.2}$$

whereas Q and V are solutions to the elliptic problems

$$\Delta Q = 0, \quad x \in B, \qquad \frac{\partial Q}{\partial n}\Big|_{S_a} = \nabla' \cdot \vec{a}, \tag{3.3}$$

$$\Delta V = -\operatorname{div}\vec{w}, \quad x \in B, \quad V\Big|_{S_a} = 0. \tag{3.4}$$

In the case of the exterior domain $B = B^a$, it is required that Q and V vanish as $|x| \to \infty$.

Formula (3.1) is equivalent to formula (3) in Sec. 10 of the book by Oseen [**3**] (under the condition (1.3)), although this formula differs from it by form. The function $Q(x,t)$ coincides with the function $P(x,t)$ in [**3**]. It is clear that for any smooth $\vec{a}(x,t)$ the vector field $\vec{v} = \vec{w} + \nabla V$ is continuous in $\Omega \times (0,T) \equiv Q_T$ up to the boundary S_a; moreover, if $\vec{a}(x,0) = 0$, then the field $\vec{v}$ is continuous in the closed cylinder $\overline{\Omega} \times [0,T]$. Furthermore, $\vec{v}(x,t)$ is twice continuously differentiable with respect to x and continuously differentiable with respect to t in Q_T, satisfies, together with the function $p = -\dfrac{\partial V}{\partial t} - \nu\operatorname{div}\vec{w}$, the Stokes equations and the initial and boundary conditions

$$\vec{v}\big|_{t=0} = 0, \quad \vec{v}'\big|_{x\in S_a} \equiv \vec{v} - \vec{n}(\vec{v}\cdot\vec{n})\big|_{x\in S_a} = \vec{a}.$$

The verification of the equality

$$\vec{v}\cdot\vec{n}\big|_{x\in S_a} = \vec{w}\cdot\vec{n} + \frac{\partial V}{\partial n}\Big|_{S_a s} = 0$$

requires more long computations which will be performed for the exterior problem.

Using the above kernels $\mathbb{P}$, G, and N, we can write the functions $\vec{w}$, V, and Q in the form

$$\vec{w}(x,t) = \int_0^t \int_{S_a} \mathbb{P}(x,\xi,t-\tau)(\vec{a}(\xi,\tau) + \vec{n}(\xi)Q(\xi,\tau))\, dS_\xi d\tau, \tag{3.5}$$

$$V(x,t) = -\int_{B_a} G(x,y) \operatorname{div} \vec{w}(y,t)\, dy, \tag{3.6}$$

$$Q(x,t) = \int_{S_a} N(x,\xi)\nabla' \cdot \vec{a}(\xi,t)\, dS_\xi. \tag{3.7}$$

Differentiating (3.6) and integrating by parts, we find

$$\frac{\partial V}{\partial n} + w_r\Big|_{x\in S_a} = -\fint_{S_a} \frac{\partial P(y,x)}{\partial r} w_r dy + \int_{S_a} \frac{P(y,x)}{r} \nabla_{S_1} \cdot \vec{w}' dy, \tag{3.8}$$

where $r = |y|$ and

$$\fint_{B^a} \frac{\partial P(y,x)}{\partial r} w_r dy = \lim_{\varepsilon\to 0} \int_{B^{a+\varepsilon}} \frac{\partial P(y,x)}{\partial r} w_r dy. \tag{3.9}$$

In spite of the strong singularity of the function $\dfrac{\partial P(y,x)}{\partial r}$ at the point x, the limit in (3.9) exists because, in view of (2.10),

$$\frac{\partial P(y,x)}{\partial r} = \frac{1}{a}\frac{\partial}{\partial r} r \frac{\partial N}{\partial r} = -\frac{1}{a}\left((n-2)\frac{\partial N}{\partial r} + \frac{1}{r}\Delta_{S_1} N\right)$$

and, consequently,

$$\fint_{B_a} \frac{\partial P}{\partial r} w_r dy = -\frac{n-2}{a} \int_{B_a} \frac{\partial N(y,x)}{\partial r} w_r dy - \frac{1}{a}\Delta_{S_1} \int_{B_a} N \frac{w_r}{r} dy.$$

We show that the sum of two integrals on the right-hand side of (3.8) vanishes. By (2.16) and (2.20), we have

$$\begin{aligned}
\nabla_{S_1} \cdot \vec{w}' &= \nabla_{S_1} \cdot \vec{w} - (n-1)\vec{n}(y) \cdot \vec{w}(y,t) \\
&= -\vec{n}(y) \int_0^t \int_{S_a} \left(\frac{\partial \mathbb{P}}{\partial \cos\gamma_{y\xi}} \cos\gamma_{y\xi} + (n-1)\mathbb{P} \right) \vec{a}(\xi,\tau)\, dS_\xi d\tau \\
&\quad + \int_0^t \int_{S_a} \left(\frac{\partial \mathbb{P}}{\partial \cos\gamma_{y\xi}} \sin^2\gamma_{y\xi} - (n-1)\mathbb{P}\cos\gamma_{y\xi} \right) Q(\xi,\tau)\, dS_\xi d\tau.
\end{aligned}$$

Hence

$$-\int\limits_{B^a} \frac{\partial P(y,x)}{\partial r} w_r dy + \int\limits_{B^a} \frac{P(y,r)}{r} \nabla_{S_1} \cdot \vec{w}' dy = \Sigma_1 + \Sigma_2, \tag{3.10}$$

where Σ_1 is the sum of integrals containing Q and Σ_2 is the sum of integrals that do not contain Q, i.e.,

$$\Sigma_1 = -\int\limits_{B^a} \frac{\partial P(y,x)}{\partial r} dy \int\limits_0^t \int\limits_{S_a} \mathbb{P} \cos\gamma_{y\xi} Q(\xi,\tau)\, dS_\xi d\tau$$
$$+ \int\limits_{B^a} \frac{P(y,x)}{r} dy \int\limits_0^t \int\limits_{S_a} \Big(\sin^2\gamma_{y\xi} \frac{\partial \mathbb{P}}{\partial \cos\gamma_{y\xi}} - (n-1)\mathbb{P}\cos\gamma_{y\xi}\Big) Q\, dS_\xi d\tau,$$

$$\Sigma_2 = -\int\limits_{B^a} \frac{\partial P(y,x)}{\partial r} \vec{n}(y)\, dy \int\limits_0^t \int\limits_{S_a} \mathbb{P}(y,\xi,t-\tau)\vec{a}(\xi,\tau)\, dS_\xi d\tau$$
$$- \int\limits_{B^a} \frac{P(y,x)}{r} \vec{n}(y) dy \int\limits_0^t \int\limits_{S_a} \Big(\frac{\partial \mathbb{P}}{\partial \cos\gamma_{y\xi}} \cos\gamma_{y\xi} + (n-1)\mathbb{P}\Big) \vec{a}\, dS_\xi d\tau. \tag{3.11}$$

We transform the sum Σ_1 using the following relation which is obtained from (2.19):

$$\int\limits_{B^a} K(|y|,\cos\gamma_{xy})\, dy \int\limits_0^t \int\limits_{S_a} \mathbb{P}(|y|,\cos\gamma_{y\xi},t-\tau) Q(\xi,\tau)\, dS_\xi d\tau$$
$$= \lim_{\varepsilon\to 0} \int\limits_{a+\varepsilon}^{\infty} r^{n-1} dr \int\limits_{S_1} K(r,\cos\gamma_{xy})\, dS_y \int\limits_0^t \int\limits_{S_a} \mathbb{P} Q dS_\xi d\tau$$
$$= a^{1-n} \lim_{\varepsilon\to 0} \int\limits_{a+\varepsilon}^{\infty} r^{n-1} dr \int\limits_0^t d\tau \int\limits_{S_a} \mathbb{P}(r,\cos\gamma_{xy},t-\tau)\, dS_y$$
$$\times \int\limits_{S_a} K(r,\cos\gamma_{y\xi})\, dS_\xi \int\limits_{S_a} N(\xi,\eta) \nabla' \cdot \vec{a}(\eta,\tau)\, dS_\eta, \tag{3.12}$$

where $K = \dfrac{P}{r}$ or $K = \dfrac{\partial P}{\partial r}$. Using (2.11) and again (2.19), we find for $n > 2$

$$\Sigma_1 = -\int\limits_{B^a} \frac{\partial N(y,x)}{\partial r} dy \int\limits_0^t \int\limits_{S_a} \mathbb{P} \cos\gamma_{y\xi} \nabla' \cdot \vec{a}\, dS_\xi d\tau$$
$$+ \int\limits_{B^a} \frac{N(y,x)}{r} dy \int\limits_0^t \int\limits_{S_a} \left(\sin^2\gamma_{y\xi} \frac{\partial \mathbb{P}}{\partial \cos\gamma_{y\xi}} - (n-1)\mathbb{P} \cos\gamma_{y\xi} \right) \nabla_{S_1} \cdot \vec{a}\, dS_\xi d\tau. \tag{3.13}$$

Now, we integrate by parts and take into account (2.20) and the following equality:

$$\frac{\partial}{\partial \cos\gamma} \sin^2\gamma \frac{\partial \mathbb{P}}{\partial \cos\gamma} = \frac{1}{\sin\gamma} \frac{\partial}{\partial\gamma} \sin\gamma \frac{\partial \mathbb{P}}{\partial \gamma} = \Delta_{S_1}\mathbb{P} + (n-3)\cos\gamma \frac{\partial \mathbb{P}}{\partial \cos\gamma},$$

where Δ_{S_1} can be a differential operator with respect to the variable ξ, as well as with respect to y. This leads to the relation

$$\Sigma_1 = \frac{1}{a} \int\limits_{N-a} \frac{\partial N)y,x)}{\partial r} \vec{n}(y)\, dy \int\limits_0^t \int\limits_{S_a} \frac{\partial \mathbb{P} \cos\gamma_{y\xi}}{\partial \cos\gamma_{y\xi}} \vec{a}(\xi,\tau)\, dS_\xi d\tau$$
$$- \frac{1}{a} \int\limits_{B^a} \frac{N(y,x)}{r} \vec{n}(y)\, dy \int\limits_0^t \int\limits_{S_a} \left(\Delta_{S_1}\mathbb{P} - 2\cos\gamma \frac{\partial \mathbb{P}}{\partial \cos\gamma} - (n-1)\mathbb{P} \right) \vec{a}\, dS_\xi d\tau.$$

Finally, we again integrate by parts the last term and use the fact that

$$\frac{1}{r}\Delta_{S_1}(N(y,x)\vec{n}(y)) = \frac{\vec{n}(y)}{r}\Delta_{S_1}N(y,x) - (n-1)\frac{\vec{n}(y)}{r}N(y,x) + \frac{2}{r}\nabla_{S_1}N.$$

We find

$$\Sigma_1 = \frac{1}{a} \int\limits_{B^a} \frac{\partial N(y,x)}{\partial r} \vec{n}(y)\, dy \int\limits_0^t \int\limits_{S_a} \frac{\partial \mathbb{P} \cos\gamma_{y\xi}}{\partial \cos\gamma_{y\xi}} \vec{a}(\xi,\tau)\, dS_\xi d\tau$$
$$- \frac{1}{a} \int\limits_{S_a} \frac{\vec{n}(y)}{r} \Delta_{S1} N(y,x)\, dy \int\limits_0^t \int\limits_{S_a} \mathbb{P} \vec{a}\, dS_\xi d\tau \tag{3.14}$$

since, in view of (2.18), we have

$$-\frac{2}{a}\int\limits_{B^a}\frac{\partial N(y,x)}{\partial r}\vec{n}(y)\,dy\cdot\int\limits_0^t\int\limits_{S_a}\frac{\partial\mathbb{P}\cos\gamma_{y\xi}}{\partial\cos\gamma_{y\xi}}\vec{a}(\xi,\tau)\,dS_\xi d\tau$$

$$+\frac{2}{a}\int\limits_{B^a}\frac{1}{r}N(y,x)\vec{n}(y)\,dy\int\limits_0^t\int\limits_{S_a}\frac{\partial\mathbb{P}}{\partial\cos\gamma_{y\xi}}\cos\gamma_{y\xi}\vec{a}\,dS_\xi d\tau$$

$$+\frac{2}{a}(n-1)\int\limits_{B^a}\frac{\vec{n}(y)}{r}N(y,x)\,dy\int\limits_0^t\int\limits_{S_a}\mathbb{P}\vec{a}\,dS_\xi d\tau=0. \tag{3.15}$$

Now, we add Σ_1 and Σ_2 and take into account that

$$\frac{P(y,x)}{r}=\frac{1}{a}\frac{\partial N(y,x)}{\partial r},\quad \frac{\partial P(y,x)}{\partial r}=\frac{1}{a}\left(r\frac{\partial^2 N}{\partial r^2}+\frac{\partial N}{\partial r}\right) \tag{3.16}$$

and that $N(y,x)$ is a harmonic function of y. We find $\Sigma_1+\Sigma_2=0$, which is required.

In the case $n=2$, we also obtain formula (3.14) but, instead of (3.16), we have

$$\frac{P}{r}-\frac{1}{2\pi ar}=\frac{1}{a}\frac{\partial N}{\partial r},\quad \frac{\partial P}{\partial r}=\frac{1}{a}\left(r\frac{\partial^2 N}{\partial r^2}+\frac{\partial N}{\partial r}\right). \tag{3.17}$$

Because of (2.20), the last term in (3.11) is equal to the expression

$$\int\limits_{B^a}\frac{P(y,x)}{r.}\nabla'\cdot\vec{\omega}'dy,$$

where

$$\vec{\omega}=\int\limits_0^t\int\limits_{S_a}\mathbb{P}(y,\xi,t-\tau)\vec{a}(\xi,\tau)\,dS_\xi d\tau,$$

and, since

$$\int\limits_{S_r}\nabla'\cdot\vec{\omega}'dS_y=0,$$

we can add $-\dfrac{1}{2\pi ar}$ and $\dfrac{P}{r}$ and, in view of (3.17), obtain the equality

$$\Sigma_2 = -\frac{1}{a}\int\!\!\!\!\!\!\diagup_{B^a} \left(r\frac{\partial^2 N}{\partial r^2} + \frac{\partial N}{\partial r}\right)\vec{n}(y)\, dy \int_0^t \int_{S_a} \mathbb{P}\vec{a}\, dS_\xi d\tau$$

$$-\frac{1}{a}\int_{B^a} \frac{\partial N}{\partial r}\vec{n}(y)\, dy \int_0^t \int_{S_a} \left(\frac{\partial \mathbb{P}}{\partial \cos\gamma_{y\xi}}\cos\gamma_{y\xi} + (n-1)\mathbb{P}\right)\vec{a}\, dS_\xi d\tau,$$

which leads to the same result: $\Sigma_1 + \Sigma_2 = 0$.

In the case $B = B_a$, formulas (3.5) and (3.6) remain in force, but, instead of (3.7) and (3.8), we have

$$Q(x,t) = -\int_{S_a} N(x,\xi)\nabla' \cdot \vec{a}(\xi,t)\, dS_\xi,$$

$$\frac{\partial V}{\partial n} + w_r\big|_{x\in S_a} = \int\!\!\!\!\!\!\diagup_{S_a} \frac{\partial P(y,x)}{\partial r} w_r dy - \int_{S_a} \frac{P(y,x)}{r}\nabla' \cdot \vec{w}' dy,$$

where

$$\int\!\!\!\!\!\!\diagup_{B_a} \frac{\partial P(y,x)}{\partial r} w_r dy = \lim_{\varepsilon\to 0} \int_{B_{a-\varepsilon}} \frac{\partial P(y,x)}{\partial r} w_r dy.$$

Therefore, the sum

$$\Sigma_1 + \Sigma_2 = \int\!\!\!\!\!\!\diagup_{B_a} \frac{\partial P(y,x)}{\partial r} w_r dy - \int_{B_a} \frac{P(y,x)}{r}\nabla' \cdot \vec{w}' dy$$

differs from (3.10) by sign. The further arguments are the same as in the case $B = B^a$, $n = 2$ and they lead to the same conclusion: $\Sigma_1 + \Sigma_2 = 0$.

Thus, we have proved that the solution to the exterior problem (1.2), as well as the solution to the interior problem (1.2), can be written in the form (3.1), i.e.,

$$\vec{v}(x,t) = \int_0^t \int_{S_a} \mathcal{H}(x,\xi,t-\tau)\vec{a}(\xi,\tau)\, dS_\xi d\tau, \tag{3.18}$$

where $\mathcal{H}$ is the matrix with entries

$$H_{ik}(x,\xi,t) = P_{ik}(x,\xi,t) - \int_B \frac{\partial G(x,y)}{\partial x_i}\sum_{m=1}^n \frac{\partial P_{mk}(y,\xi,t)}{\partial y_m} dy = P_{ik} + R_{ik}, \tag{3.19}$$

$$
\begin{aligned}
P_{ik}(x,\xi,t) &= \mathbb{P}(x,\xi,t)\delta_{ik} \pm \partial_{\xi_k} \int\limits_{S_a} \mathbb{P}(x,\eta,t) n_i(\eta) N(\eta,\xi)\, dS_\eta \\
&= \mathbb{P}(x,\xi,t)\delta_{ik} \pm \int\limits_{S_a} \mathbb{P}(x,\eta,t) n_i(\eta) \partial_{\xi_k} N(\eta,\xi)\, dS_\eta \equiv P\delta_{ik} \pm P'_{ik},
\end{aligned} \tag{3.20}
$$

$\partial_{\xi_j} = \dfrac{\partial}{\partial \xi_j} - n_j(\xi)\dfrac{\partial}{\partial n_\xi}$, the sign "+" in (3.20) corresponds to the case $B = B^a$ and the sign "−" corresponds to the case $B = B_a$. The kernel $\partial_{\xi_j} N$ is singular, and the last integral in (3.20) is understood as a singular integral.

Formula (3.18) can be written in the form

$$
\vec{v}(x,t) = \int\limits_0^t \int\limits_{S_a} \mathcal{G}(x,\xi,t-\tau)\vec{a}(\xi,\tau)\, dS_\xi d\tau, \tag{3.21}
$$

where

$$
\mathcal{G} = \mathcal{H}(I - \vec{n}(\xi) \otimes \vec{n}(\xi)). \tag{3.22}
$$

For any smooth $\vec{b}(x,t)$, that not necessarily satisfies the condition (1.3), the vector field

$$
\vec{v}(x,t) = \int\limits_0^t \int\limits_{S_a} \mathcal{G}(x,\xi,t-\tau)\vec{b}(\xi,\tau)\, dS_\xi d\tau \tag{3.23}
$$

is a solution to the problem (1.2) with

$$
\vec{a}(\xi,t) = \vec{b}(\xi,t) - \vec{n}(\xi)(\vec{n}(\xi)\cdot\vec{b}(\xi,t)) = \vec{b}'(\xi,t).
$$

Taking $\vec{b}(\xi,t) = \vec{e}_k \delta_\varepsilon(\xi-\eta)\delta_\varepsilon(\tau)$, $k = 1,2,3$, where $\delta_\varepsilon(\xi-\eta)$ and $\delta_\varepsilon(\tau)$ are smooth functions with support in $S_a \cap (|\xi-\eta|) \leqslant \varepsilon)$ and $(\varepsilon, 2\varepsilon)$ tending to $\delta(\xi-\eta)$ and $\delta(\tau)$ as $\varepsilon \to 0$, from (3.23) we obtain, passing to the limit as $x \to z \in S_a$, $\varepsilon \to 0$, the equality

$$
\mathcal{G}(z,\eta,t) = 0 \quad \forall z,\eta \in S_a,\ t > 0. \tag{3.24}
$$

We turn to the formulas for pressure. Integrating by parts, we find

$$
\begin{aligned}
p(x,t) &= -\Big(\frac{\partial}{\partial t} - \nu\Delta\Big) V = \nu \int\limits_B G(x,y)\Delta \operatorname{div} \vec{w}(y,t)\, dy \\
&- \nu \operatorname{div} \vec{w}(x,t) = -\nu \int\limits_{S_a} P(x,y) \operatorname{div} \vec{w}(y,t)\, dS_y, \quad x \in B.
\end{aligned} \tag{3.25}
$$

We note that the solution to the problem (1.1) in the case $\vec{a} \cdot \vec{n} \neq 0$ can be presented in the form

$$\vec{v}(x,t) = \nabla\varphi(x,t) + \vec{u}(x,t), \quad p(x,t) = -\varphi_t(x,t) + q(x,t),$$

where φ is the solution to the Neumann problem

$$\Delta\varphi = 0, \quad \frac{\partial\varphi}{\partial n}\Big|_{S_a} = \vec{a} \cdot \vec{n}$$

and $(\vec{u}, q)$ is the solution to the problem (1.2) with $\vec{a} - \nabla\varphi = \vec{a}_1$ instead of $\vec{a}$. It is clear that $\vec{a}_1 \cdot \vec{n} = 0$.

It is possible to show that the solution to the problem (1.1) in the half-space obtained in [4] also has the structure (3.1).

We indicate some useful properties of the entries G_{ik} of the matrix $\mathcal{G}$.

Proposition 3.1. *For any $t \in (0,T)$, $T < \infty$, the following inequalities hold*:

$$|G_{ik}(x,\xi,t)| \leqslant \frac{c(T)(|x|-a)^\lambda}{t^{(1+\lambda)/2}(|x-\xi|^2+t)^{n/2}}, \tag{3.26}$$

$$|\nabla'_x G_{ik}(x,\xi,t)| \leqslant \frac{c(T)(|x|-a)^\lambda}{t^{(1+\lambda)/2}(|x-\xi|^2+t)^{(n+1)/2}}, \tag{3.27}$$

$$\left|\frac{\partial}{\partial r} G_{ik}(x,\xi,t)\right| \leqslant \frac{c(T)}{\sqrt{t}((|x|-a)^2+t)^{1/2}(|x-\xi|^2+t)^{n/2}}, \tag{3.28}$$

where $x \in B$, $\xi \in S_a$, λ is an arbitrary number in the interval $[0,1]$, $\nabla' = (\partial_{x_1}, \dots, \partial_{x_n})$, $\partial_{x_k} = \dfrac{\partial}{\partial x_k} - n_k(x)\dfrac{\partial}{\partial r}$.

The proof of the estimates (3.26)–(3.28) requires long computations, and we omit it. Owing to these estimates, it is possible to make sense of the potential (3.21) with a continuous function $\vec{a}(\xi,t)$.

Proposition 3.2. *For any continuous $\vec{a}(x,t)$ satisfying the condition* (1.3) *the vector field* (3.21) *is continuous (moreover, if $\vec{a}(x,0) = 0$, then $\vec{v}(x,t)$ is continuous in $\overline{\Omega} \times [0,T]$) and satisfies the boundary condition $\vec{v}(x,t) = \vec{a}(x,t)$.*

PROOF. The convergence of the integral (3.21) follows from the estimate (3.26) which also implies the inequality

$$\begin{aligned}|\vec{v}(x,t)| &\leqslant c \sup_{S_a \times (0,t)} |\vec{a}(\xi,\tau)| \int_{S_a}\int_0^t \frac{(|x|-a)^\lambda dS_\xi d\tau}{(t-\tau)^{(1+\lambda)/2}(|x-\xi|^2+t-\tau)^{n/2}} \\ &\leqslant c \sup_{S_a \times (0,t)} |\vec{a}(\xi,\tau)|, \quad \lambda \in (0,1).\end{aligned}$$

It is clear that $\vec{v}(x,t)$ is continuous in $\Omega \times (0,T)$ and $\vec{v}(x,0) = 0$. It remains to verify the boundary condition. For definiteness, we consider the exterior problem. We have

$$\vec{v}(x,t) - \vec{a}(x_0,t) = \int_0^t d\tau \int_{S_a} \mathcal{G}(x,\xi,t-\tau)(\vec{a}(y,\tau) - \vec{a}(\overline{x},\tau))\, dS_y$$

$$+ \int_0^t d\tau \int_{S_a} \mathcal{G}(x,\xi,t-\tau)\, dS_y \vec{a}(\overline{x},t) - \vec{a}(\overline{x},t) + (\vec{a}(\overline{x},t) - \vec{a}(x_0,t)), \qquad (3.29)$$

where $x \in B^a$, $x_0 \in S_a$, $\overline{x} = \dfrac{x}{|x|}a$. We take an arbitrary $\varepsilon > 0$. If x is sufficiently close to x_0, then $\overline{x}$ is also sufficiently close to x_0, and for $|x - x_0| \leqslant \delta_1(\varepsilon)$ we have $|\vec{a}(\overline{x},t) - \vec{a}(x_0,t)| \leqslant \varepsilon$.

By the above arguments, for $|x - x_0| \leqslant \delta_2(\varepsilon)$ we have

$$\left|\int_0^t \int_{S_a} \mathcal{G}(x,\xi,t-\tau)\, dS_y d\tau \vec{a}(\overline{x},t) - (\vec{a}(\overline{x},t) - \vec{n}(x)(\vec{n}(x)\cdot\vec{a}(\overline{x},t))\right|$$

$$= \left|\int_0^t \int_{S_a} \mathcal{G}(x,\xi,t-\tau)\, dS_y d\tau \vec{a}(\overline{x},t) - \vec{a}(\overline{x},t)\right| \leqslant \varepsilon.$$

Let us show that the first term on the right-hand side of (3.29) becomes small for $|x-x_0| \leqslant \delta_3(\varepsilon)$. Let $\sigma_\rho(\overline{x})$ be the part of S_a cut out by the sphere $|\xi - \overline{x}| = \rho$, and let $q_\rho(\overline{x},t) = \sigma_\rho(\overline{x}) \times (t-\rho^2, t)$. Taking a sufficiently small ρ such that, in addition, $\rho^2 < t$ if $\vec{a}(x,0) \neq 0$ we can reach the situation where

$$\left|\int_{t-\rho^2}^t d\tau \int_{\sigma_\rho(\overline{x})} \mathcal{G}(x,\xi,t-\tau)(\vec{a}(y,\tau) - \vec{a}(\overline{x},t))\, dS_y d\tau\right| \leqslant c \sup_{q_\rho(\overline{x})} |\vec{a}(y,\tau) - \vec{a}(\overline{x},t)| \leqslant \varepsilon$$

in view of the estimate (3.26). This estimate also implies

$$\left|\iint_{\Sigma_t \setminus q_\rho(x)} \mathcal{G}(x,\xi,t-\tau)(\vec{a}(y,\tau) - \vec{a}(\overline{x},t))\, dS_y d\tau\right|$$

$$\leqslant c \sup_{\Sigma_t} |\vec{a}(y,\tau)| \rho^{-n} (|x| - a)^\lambda \int_0^t \frac{d\tau}{(t-\tau)^{(1+\lambda)/2}} \leqslant \varepsilon$$

if $|x - x_0| \leqslant \delta_3(\varepsilon)$. This shows that the field $\vec{v}(x,t)$ is continuous up to S_a and $\vec{v}(x_0,t) = \vec{a}(x_0,t)$ for $x_0 \in S_a$. The proposition is proved. □

Proposition 3.3. *The matrix $\mathcal{G}(x,y,t)$ (cf. (3.22)) satisfies the relation*

$$\mathcal{A}^{-1}\mathcal{G}(\mathcal{A}x, \mathcal{A}\xi, t)\mathcal{A} = \mathcal{G}(x,\xi,t), \qquad (3.30)$$

where $\mathcal{A}$ is an arbitrary orthogonal matrix.

PROOF. Let $\vec{a}(\xi, t)$ be an arbitrary smooth vector field satisfying (1.3) and the condition $\vec{a}(\xi, 0) = 0$. The functions $(\vec{v}, p)$ defined by formulas (3.21) and (3.25) form a solution to the problem (1.2), the functions $\vec{w}(x,t) = \mathcal{A}^{-1}\vec{v}(\mathcal{A}x, t)$, $q(x,t) = p(\mathcal{A}x, t)$ form a solution to the same problem with $\mathcal{A}^{-1}\vec{a}(\mathcal{A}x, t)$ instead of $\vec{a}(x,t)$. It is clear that $\mathcal{A}^{-1}\vec{a}(\mathcal{A}x, t) \cdot \vec{n}(x) = 0$. Therefore,

$$\mathcal{A}^{-1}\vec{v}(\mathcal{A}x, t) = \int_0^t \int_{S_a} \mathcal{G}(x, \xi, t-\tau)\mathcal{A}^{-1}\vec{a}(\mathcal{A}\xi, \tau)\, dS_\xi d\tau.$$

Hence

$$\vec{v}(z,t) = \int_{-}^{t} \int_{S, a} \mathcal{A}\mathcal{G}(\mathcal{A}^{-1}z, \mathcal{A}^{-1}\eta, t-\tau)\mathcal{A}^{-1}\vec{a}(\eta, \tau)\, dS_\eta d\tau,$$

consequently,

$$\mathcal{A}\mathcal{G}(\mathcal{A}^{-1}z, \mathcal{A}^{-1}\eta, t)\mathcal{A}^{-1} = \mathcal{G}(z, \eta, t)$$

which is equivalent to (3.30).

As for pressure, the integral in (3.25) can have meaning if $\vec{a}$ satisfies the Hölder condition with respect to x and t (moreover, with respect to t with exponent $\beta > 1/2$). In this case, $(\vec{v}, p)$ form the classical solution to the problem (1.2); otherwise, the Stokes equations hold in the weak sense. □

4. Hydrodynamic Potential and Solution to the Stokes Problem for Nonconvex Domains

We consider the Stokes problem (1.1) in a bounded domain or in an exterior domain Ω with C^2-boundary S (for definiteness, we assume that Ω is bounded). Since $S \subset C^2$, at each point ξ we can touch S outside Ω by the sphere $S_a(\xi)$ of a fixed radius $a > 0$. It is clear that its center is located at the point $s(\xi) = \xi - an(\xi)$, where $n(\xi)$ is the inward normal to S coinciding with the outward normal $n^0(\xi)$ to $S_a(\xi)$.

By the *hydrodynamic potential* we mean the integral

$$\vec{u}(x,t) = \int_0^t \int_S \mathcal{G}(x - s(\xi), \xi - s(\xi), t-\tau)\vec{\varphi}(\xi, \tau)\, dS_\xi d\tau, \tag{4.1}$$

where $\mathcal{G}(x, y, t)$ is the matrix (3.22) determining the solution to the exterior problem (1.2) and $\vec{\varphi}(\xi, \tau)$ is a vector field satisfying the condition (1.3) $\vec{\varphi} \cdot \vec{n} = 0$.

The inequalities (3.26)–(3.28) allow us to estimate the entries of the matrix

$$\mathcal{G}(x - s(\xi), \xi - s(\xi), t) = \mathcal{K}(x, \xi, t). \tag{4.2}$$

Proposition 4.1. *The following inequalities hold:*

$$|\mathcal{K}(x,\xi,t)| \leqslant \frac{c}{\sqrt{t}(|x-\xi|^2+t)^{n/2}}, \tag{4.3}$$

$$|\nabla\mathcal{K}(x,\xi,t)| \leqslant \frac{c}{\sqrt{t}(|x-\xi|^2+t)^{n/2}(\delta^2(x)+t)^{1/2}}, \tag{4.4}$$

where $\delta(x) = \operatorname{dist}(x,S)$, $x \in \Omega$, $\xi \in S$. *Moreover,*

$$|\mathcal{K}(x,\xi,t)| \leqslant c\Big(\frac{\delta^\lambda(x)}{t^{(1+\lambda)/2}(|x-\xi|^2+t)^{n/2}} + \frac{1}{t^{(1+\lambda)/2}(|x-\xi|^2+t)^{(n-2\lambda)/2}}\Big)$$
$$\forall\lambda \in [0,1]. \tag{4.5}$$

If $x, z \in S$ *and* $|x-z| \leqslant \frac{1}{2}|x-\xi|$, *then*

$$|\mathcal{K}(x,\xi,t) - \mathcal{K}(z,\xi,t)| \leqslant \frac{c|x-z|^\lambda}{t^{(1+\lambda)/2}(|x-\xi|^2+t)^{(n-\lambda)/2}} \quad \forall\lambda \in [0,1]. \tag{4.6}$$

Proof. The estimates (4.3) and (4.4) directly follow from (3.26)–(3.28). It suffices to prove the inequality (4.5) for x sufficiently close to ξ: $|x-\xi| \leqslant a/2$; otherwise, it follows from (4.4). Let $C_\omega(\xi)$ be a cone with vertex $s(\xi)$, whose symmetry axis is parallel to $n(\xi)$ and the aperture angle is equal to ω. This cone cuts out the set $\sigma^0_\omega(\xi)$ on the sphere $S_a(\xi) = \{|\eta - s(\xi)| = a\}$ and the set $\sigma_\omega(\xi)$ on the surface S; moreover, for sufficiently small fixed ω (independently of ξ) the set $\sigma_\omega(\xi)$ is given by the equation

$$y = s(\xi) + n^0(\eta)\mathcal{F}(\eta), \quad \eta \in \sigma^0_\omega, \tag{4.7}$$

where $\mathcal{F}(\eta)$ is a positive function of class $C^2(\sigma^0_\omega(\xi))$, $\mathcal{F}(\eta) \geqslant a$, $\mathcal{F}(\xi) = a$, $\nabla'\mathcal{F}\big|_{\eta=\xi} = 0$. Hence $|\nabla'\mathcal{F}(\eta)| \leqslant c|\eta-\xi|$. Let $x \in \Omega\cap C_\omega(\xi)$, $|x-\xi| \leqslant a/2$, $\eta = s(\xi) + a\dfrac{x-s(\xi)}{|x-s(\xi)|}$, and let y be a point of the surface S determined by Eq. (4.7) (in other words: $y \in S$ and $\eta \in S_a(\xi)$ lie on the line passing through x and $s(\xi)$). It is clear that $|x| - a = |x-y| + |y-\eta|$. Using standard estimates, in the spirit of the potential theory it is easy to verify that $|x-y| \leqslant c\delta(x)$. Moreover, $|y-\eta| = \mathcal{F}(\eta) - \mathcal{F}(\xi) \leqslant c|\xi-\eta|^2 \leqslant c|x-\xi|^2$. Hence $(|x|-a)^\lambda \leqslant c(\delta^\lambda(x) + |x-\xi|^{2\lambda})$. This fact and (3.26) imply (4.5).

Finally, it suffices to verify (4.6) for x and z close to ξ $(x, z \in \sigma_\omega(\xi))$.

Let $\widetilde{x}$ be a point on the line passing through $s(\xi)$ and z such that $|\widetilde{x} - s(\xi)| = |x - s(\xi)|$, and let $x = \mathcal{F}(\eta)$, $z = \mathcal{F}(\eta')$, $\eta, \eta' \in \sigma^0_\omega(\xi)$. We have

$$|\mathcal{K}(x,\xi,t) - \mathcal{K}(\widetilde{x},\xi,t)| \leqslant \big(|\mathcal{K}(x,\xi,t)| + |\mathcal{K}(\widetilde{x},\xi,t)|\big)^{1-\lambda}\Big|\int\limits_{l(x,\widetilde{x})} \nabla'\mathcal{K}(u,\xi,t)\,dl_u\Big|^\lambda$$
$$\leqslant \frac{c|x-\widetilde{x}|^\lambda|\mathcal{F}(\eta)-\mathcal{F}(\xi)|^\lambda}{t^{1/2}(|x-\xi|^2+t)^{n/2+\lambda/2}} \leqslant \frac{c|x-\widetilde{x}|^\lambda}{t^{(1+\lambda)/2}(|x-\xi|^2+t)^{n/2-\lambda/2}},$$

where $l(x, \widetilde{x})$ is the curve on the sphere $|u - s(\xi)| = |x - s(\xi)|$ joining the points x and $\widetilde{x}$. Moreover,

$$|\mathcal{K}(\widetilde{x}, \xi, t) - \mathcal{K}(z, \xi, t))| \leqslant \frac{c|\mathcal{F}(\eta) - \mathcal{F}(\eta')|^{\lambda}}{t^{(1+\lambda)/2}(|x - \xi|^2 + t)^{n/2}}$$
$$\leqslant \frac{c|x - z|^{\lambda}}{t^{(1+\lambda)/2}(|x - \xi|^2 + t)^{(n-\lambda)/2}}.$$

The last two inequalities imply (4.6). □

Corollary. *The kernel $\mathcal{K}$ and the potential (4.1) satisfy the inequalities*

$$\int_0^t \int_{S_a} |\mathcal{K}(x, \xi, t - \tau)|\, dS_\xi d\tau \leqslant c, \tag{4.8}$$

$$|\vec{u}(x, t)| \leqslant c \sup_{\Sigma_t} |\vec{\varphi}(\xi, \tau)|, \quad \Sigma_t = S \times (0, t). \tag{4.9}$$

Proposition 4.2. *For any continuous function $\vec{\varphi}(\xi, t)$ satisfying the condition $\vec{\varphi}(\xi, 0) = 0$ the potential* (4.1) *is continuous in Ω up to the boundary S and satisfies the relation*

$$\lim_{x \to x_0 \in S} \vec{u}(x, t) = \vec{\varphi}'(x_0, t) + \vec{u}(x_0, t), \tag{4.10}$$

where $\vec{\varphi}'(x_0, t) = \vec{\varphi} - \vec{n}(x_0)(\vec{n}(x_0) \cdot \vec{\varphi}(x_0, t))$ and $\vec{u}(x_0, t)$ is the direct value of the potential (4.1) *on S.*

PROOF. We represent the potential $\vec{u}(x, t)$ in the form

$$\vec{u}(x, t) = \vec{u}'(x, t) + \vec{u}''(x, t) \equiv \int_0^t \int_S \mathcal{K}(x, \xi, t - \tau)(\vec{\varphi}(\xi, \tau) - \vec{\varphi}(x_0, t))\, dS_\xi d\tau$$
$$+ \int_0^t \int_S \mathcal{K}(x, \xi, t - \tau)\, dS_\xi d\tau \vec{\varphi}(x_0, t)$$

and show that the first term tends to

$$\int_0^t \int_S \mathcal{K}(x_0, \xi, t - \tau)(\vec{\varphi}(\xi, \tau) - \vec{\varphi}(x_0, t))\, dS_\xi d\tau \equiv \vec{u}'(x_0, t),$$

whereas the second term tends to

$$\int_0^t \int_S \mathcal{K}(x_0, \xi, t - \tau)\, dS_\xi d\tau \vec{\varphi}(x_0, t) + \vec{\varphi}(x_0, t) = \vec{u}''(x_0, t) + \vec{\varphi}(x_0, t)$$

as $x \to x_0$. Thereby (4.10) will be proved. The first assertion is proved in the same way as in the case of the heat potential of a double layer. We use the identity

$$\vec{u}'(x,t) - \vec{u}'(x_0,t)$$
$$= \int_{t-\rho^2}^{t} d\tau \int_{\sigma_\rho(x_0)} (\mathcal{K}(x,\xi,t-\tau) - \mathcal{K}(x_0,\xi,t-\tau))(\vec{\varphi}(\xi,\tau) - \vec{\varphi}(x_0,t))\, dS_\xi$$
$$+ \iint_{\Sigma_t \setminus q_\rho} (\mathcal{K}(x,\xi,t-\tau) - \mathcal{K}(x_0,\xi,t-\tau))(\vec{\varphi}(\xi,\tau) - \vec{\varphi}(x_0,\tau))\, dS_\xi d\tau,$$

where $\sigma_\rho(x_0)$ is the part of the surface S cut out by the sphere $|y - x_0| \leqslant \rho$, $q_\rho = \sigma_\rho(x_0) \times (t-\rho^2, t)$, $\Sigma_t = S \times (0,t)$. By (4.8), the first integral does not exceed the quantity

$$c \sup_{q_\rho} |\vec{\varphi}(\xi,\tau) - \vec{\varphi}(x_0,t)| \leqslant \varepsilon$$

if $\rho = \rho(\varepsilon)$ is small enough, and the second integral is estimated in terms of

$$c \sup_{\Sigma_t} |\vec{\varphi}(\xi,\tau)| \iint_{\Sigma_t \setminus q_\rho} |\mathcal{K}(x,\xi,t-\tau) - \mathcal{K}(x_0,\xi,t-\tau)|\, dS_\xi d\tau.$$

Using (4.4), it is easy to verify that this quantity becomes less than any $\varepsilon > 0$ provided that the point x is sufficiently close to x_0. We now prove that $\vec{u}''(x,t) \to \vec{u}''(x_0,t) + \vec{\varphi}(x_0,t)$ as $x \to x_0$. This fact follows from the relation

$$\lim_{x \to x_0} k_{ij}(x,t) = k_{ij}(x_0,t) + \mu_{ij}(x_0,t),$$

where

$$k_{ij}(x,t) = \int_0^t \int_S K_{ij}(x,\xi,t-\tau)\, dS_\xi d\tau = \int_0^t \int_S G_{ij}(x - s(\xi), \xi - s(\xi), t-\tau)\, dS_\xi d\tau,$$
$$\mu_{ij}(x_0,t) = \delta_{ij} - n_i(x_0) n_j(x_0).$$

We introduce the functions

$$\varkappa_{ij}(x,y,t) = \int_0^t \int_{S_a(y)} G_{ij}(x - s(y), \eta - s(y), t-\tau)\, dS_\eta d\tau,$$

where $y \in S$ and $S_a(y)$ is the sphere $|\eta - s(y)| = a$, and use the relation

$$k_{ij}(x,t) - k_{ij}(x_0,t) - \mu_{ij}(x_0) = (k_{ij}(x,t) - \varkappa_{ij}(x,\overline{x},t) - k_{ij}(x_0,t))$$
$$+ (\varkappa_{ij}(x,\overline{x},z) - \mu_{ij}(\overline{x})) + (\mu_{ij}(\overline{x}) - \mu_{ij}(x_0)), \tag{4.11}$$

where $\overline{x}$ is the nearest for x point of S. By Proposition 3.2 and the continuity of $n_j(\xi)$, the last two terms converge to zero as $x \to x_0$ (because, in this case, $\overline{x} \to x_0$).

To show that the first term also converges to zero, we consider the difference

$$k_{ij}(x,t) - \varkappa_{ij}(x,\overline{x},t) = \int_0^t \int_S G_{ij}(x - s(\xi), an(\xi), t-\tau)\, dS_\xi d\tau$$

$$- \int_0^t \int_{S_a(\overline{x})} G_{ij}(x - \eta + an^0(\eta), an^0(\eta), t-\tau)), dS_\eta d\tau$$

$$= \Big(\int_0^t \int_{\sigma_\alpha(\overline{x})} G_{ij}(x - \xi + an(\xi), an(\xi), t-\tau))\, dS_\xi d\tau$$

$$- \int_0^t \int_{\sigma_\alpha^0(\overline{x})} G_{ij}(x - \eta + an^0(\eta), an^0(\eta), t-\tau))\, dS_\eta d\tau \Big)$$

$$+ \Big(\int_0^t \int_{S \setminus \sigma_\alpha(\overline{x})} G_{ij}(x - s(\xi), an(\xi), t-\tau)\, dS_\xi d\tau$$

$$- \int_0^t \int_{S_a(x) \setminus \sigma_\alpha^0(\overline{x})} G_{ij}(x - s(\overline{x}), an^0(\eta), t-\tau)\, dS_\eta d\tau \Big)$$

$$= X'_{ij}(x,t) + X''_{ij}(x,t),$$

where $\sigma_\alpha^0(x)$, $\sigma_\alpha(x)$ are the sets cut out by the cone $C_\alpha(\overline{x})$ of small angle $\alpha < \omega$ in $S_\alpha(\overline{x})$ and S respectively and $n^0(\eta)$ is the outward normal to $S_a(\overline{x})$. As was shown above, $\sigma_\alpha(\overline{x}) = \{\xi = s(\overline{x}) + n^0(\eta)\mathcal{F}(\eta) \equiv F(\eta), \eta \in \sigma_\alpha^0(\overline{x})\}$, where $F \in C^2(\sigma_\omega^0(\overline{x}))$, $F(\overline{x}) = \overline{x}$. We introduce the orthogonal matrix $\mathcal{A}(\eta)$, $\eta \in \sigma_\omega^0(\overline{x})$, such that $\mathcal{A}(\eta)n^0(\eta) = n(F(\eta))$, $\mathcal{A}(\overline{x}) = I$ and

$$|\mathcal{A}(\eta) - \mathcal{A}(\overline{x})| \leqslant c|\eta - \overline{x}|. \tag{4.12}$$

The matrix $\mathcal{A}(\eta)$ can be defined by the equality

$$\mathcal{A}(\eta) = \mathcal{B}(\eta)\mathcal{C}(\eta),$$

where $\mathcal{B}(\eta) = (b_{jk})$, $\mathcal{C}(\eta) = (c_{jk})$, $j,k = 1,\dots,n$, are also orthogonal matrices; moreover, $c_{nj}(\eta) = n_j^{(0)}(\eta)$, $b_{jn}(\eta) = n_j(F(\eta))$, $j = 1,\dots,n$ ($n_j^{(0)}$ are the components of $n^0(\eta)$). The remaining rows of the matrix $\mathcal{B}$ and columns of the matrix $\mathcal{C}$ are formed by vectors that are orthogonal to $n^0(\eta)$ and $n(F(\eta))$. In coordinates with the axis x_n directed along $n^0(\overline{x})$, those vectors have components

$$c_{mj} = \delta_{jm}\frac{n_n^{(0)}}{\sqrt{n_m^{(0)2}+n_n^{(0)2}}}(1-\delta_{jn}) - \delta_{jn}\frac{n_m^{(0)}}{\sqrt{n_m^{(0)2}+n_n^{(0)2}}}, \quad m=1,\dots,n-1,$$

$$b_{jm} = \delta_{jm}\frac{n_n(F(\eta))}{\sqrt{n_m^2+n_n^2}}(1-\delta_{jn}) - \delta_{jn}\frac{n_m(F(\eta))}{\sqrt{n_m^2+n_n^2}}, \quad m=1,\dots,n-1.$$

It is clear that $\mathcal{B}(\overline{x}) = \mathcal{C}^{-1}(\overline{x})$, $\mathcal{A}(\overline{x}) = I$.

The matrix $\mathcal{X}' = (X'_{ij})$ can be written in the form

$$\begin{aligned}\mathcal{X}' = \int_0^t \int_{\sigma_\alpha^{(0)}(\overline{x})} & [\mathcal{G}(\mathcal{A}(\mathcal{A}^{-1}(x-F(\eta)) + an^0(\eta)), a\mathcal{A}n^{(0)}, t-\tau)J \\ & - \mathcal{G}(x-\eta+an^0(\eta), an^0(\eta), t-\tau)]\, dS_\eta d\tau \\ = \int_0^t \int_{\sigma_\alpha^{(0)}(\overline{x})} & [\mathcal{A}\mathcal{G}(\mathcal{A}^{-1}x - \mathcal{A}^{-1}F(\eta) + an^0(\eta), an^0(\eta), t-\tau)\mathcal{A}^{-1}J \\ & - \mathcal{G}(x-\eta+an^0(\eta), t-\tau)]\, dS_\eta d\tau,\end{aligned} \tag{4.13}$$

where $J = a^{-2}\mathcal{F}(\eta)\sqrt{\mathcal{F}^2(\eta)+|\nabla\mathcal{F}(\eta)|^2}$ is the function satisfying the inequality $|J-1| \leqslant c(|\mathcal{F}(\eta)-\mathcal{F}(\overline{x})| + |\nabla\mathcal{F}|) \leqslant c|\eta-\overline{x}|$. Moreover,

$$(\mathcal{A}^{-1}(x-F(\eta)) - (x-\eta)| \leqslant |(\mathcal{A}^{-1}-I)(x-\xi)| + |\xi-\eta| \leqslant c|x-\xi|^2,$$

where $\xi = F(\eta)$, and, consequently, the expression in the square brackets in (4.13) does not exceed the quantity

$$c\frac{|\overline{x}-\eta|^\lambda}{(t-\tau)^{(1+\lambda)/2}(|x-\eta|^2+t-\tau)^{(n-\lambda)/2}} \leqslant \frac{c}{(t-\tau)^{(1+\lambda)/2}(|x-\eta|^2+t-\tau)^{(n-2\lambda)/2}}.$$

Hence

$$|X'_{ij}(x,t)| \leqslant c\int_0^t \int_{\sigma_\alpha^{(0)}\overline{x})} \frac{dS_\eta d\tau}{(t-\tau)^{(1+\lambda)/2}(|x-\eta|^2+t-\tau)^{(n-2\lambda)/2}} \leqslant c\alpha^\lambda,$$

$$|X'_{ij}(x,t) - X'_{ij}(x_0,t)| \leqslant c\alpha^\lambda.$$

We fix α such that $c\alpha^\lambda$ is less than any given ε. After that, using the estimates (4.3) and (4.4), we can show that $|X''_{ij}(x,t) - X''_{ij}(x_0,t)| \leqslant \varepsilon$ if x and $\overline{x}$ are sufficiently close to each other (this holds if $|x-x_0| \leqslant \delta(\varepsilon)$). Thus,

$$\begin{aligned}&|k_{ij}(x,t) - \varkappa_{ij}(x,\overline{x},t) - k_{ij}(x_0,t) + \varkappa_{ij}(x_0,x_0,t)| \\ &\quad = |k_{ij}(x,t) - \varkappa_{ij}(x,\overline{x},t) - k_{ij}(x_0,t)| \leqslant 2\varepsilon\end{aligned}$$

for $|x - x_0| \leqslant \delta(\varepsilon)$, which is required. The proposition is proved. $\square$

It is clear that the potential (4.1) satisfies the Stokes equations (in the classical or generalized sense) with some function $q(x,t)$ which will not be considered here. Following [**1, 2**], we look for the solution to the problem (1.1) in the form

$$\vec{v}(x,t) = \mathcal{K}[\vec{\varphi}] + \nabla V[\psi], \tag{4.14}$$

$$p(x,t) = q(x,t) - \frac{\partial V}{\partial t}, \tag{4.15}$$

where $\mathcal{K}[\vec{\varphi}]$ is the potential (4.1) and $V[\psi] = -2 \int\limits_S E(x-y)\psi(y,t)\, dS_y$ is the harmonic potential of a single layer. The boundary conditions lead to the system of integral equations

$$\begin{aligned} &\vec{\varphi} + \mathcal{K}'[\vec{\varphi}] + \nabla' V[\psi] = \vec{a}' \equiv \vec{a} - \vec{n}(\vec{a}\cdot\vec{n}), \\ &\psi + \frac{\partial V}{\partial n} + \vec{n}\cdot\mathcal{K}[\vec{\varphi}] = \vec{a}\cdot\vec{n}, \end{aligned} \tag{4.16}$$

where $\mathcal{K}'$, $\vec{n}\cdot\mathcal{K}$, V, $\dfrac{\partial V}{\partial n}$ are the direct values of potentials $\mathcal{K}' = \mathcal{K} - (\vec{n}\cdot\mathcal{K})\vec{n}$, $\vec{n}\cdot\mathcal{K}$, V, $\dfrac{\partial V}{\partial n}$ on S respectively.

The system (4.16) is rather similar to the system obtained in [**1, 2**] for convex domains and allows us to extend the results of [**5, 6**]. to the case of an arbitrary bounded domain with C^2-boundary. In particular, the following assertions hold.

1. If $\vec{a}$ is continuous and $\vec{a}\cdot\vec{n}$ satisfies the Hölder condition with respect to x with exponent $\alpha \in (0, 1/2)$, then the system (4.16) has a continuous solution and

$$\begin{aligned} &\sup_{\Sigma_t} |\vec{\varphi}(\xi,\tau)| + \sup_{\tau<t} |\psi(\cdot,\tau)|_{C^\alpha(S)} \leqslant c(\sup_{\Sigma_t} |\vec{a}'(\xi,\tau)| + \sup_{\tau<t} |\vec{a}\cdot\vec{n}|_{C^\alpha(S)}), \\ &\sup_{\Omega\times(0,T)} |\vec{v}(x,t)| \leqslant c(\sup_{\Sigma_t} |\vec{a}'(\xi,\tau)| + \sup_{\tau<T} |\vec{a}\cdot\vec{n}|_{C^\alpha(S)}), \end{aligned} \tag{4.17}$$

where $\Sigma_t = S \times (0,t)$, $|\cdot|_{C^\alpha(S)}$ is the Hölder norm on S.

2. If $\vec{a} \in L_p(\Sigma_T)$, then $\vec{\varphi}, \psi \in L_p(\Sigma_T)$. If $\vec{a}\cdot\vec{n} = 0$, $\vec{a} \in W^{r,r/2}_{p,0}(\Sigma_T)$, $r \in (0,1)$, then $\vec{\varphi}, \psi \in W^{r,r/2}_{p,0}(\Sigma_T)$ and the following inequalities hold:

$$||\vec{\varphi}||_{W^{r,r/2}_{p,0}(\Sigma_T)} + ||\psi||_{W^{r,r/2}_{p,0}(\Sigma_T)} \leqslant c||\vec{a}||_{W^{r,r/2}_{p,0}(\Sigma_T)}.$$

By $W^{r,r/2}_{p,0}(\Sigma_T)$ we mean the anisotropic Sobolev–Slobodetskii space of functions given on Σ_T and admitting the zero extension to the domain $t < 0$ with the same class. The norm in this space is the $W^{r,r/2}_p(S \times (-\infty, T))$-norm of the extended function.

3. If $\vec{a} \in W_{p,0}^{1-1/p,1/2-1/2p}(\Sigma_T)$, $\vec{a}\cdot\vec{n} = 0$, then

$$\|\vec{v}\|_{W_{p,0}^{1,1/2}(\Omega\times(0,T))} \leqslant c\|\vec{a}\|_{W_{p,0}^{1-1/p,1/2-1/2p}(\Sigma_T)}.$$

These results are proved in the same way as in [**5, 6**]. We note that, using (4.17), we can estimate the maximum of the modulus of the solution $\vec{v}(x,t)$ to the problem

$$\vec{v}_t - \nu\Delta\vec{v} + \nabla p = 0, \quad \operatorname{div}\vec{v} = 0,$$
$$\vec{v}\big|_{t=0} = \vec{v}_0(x), \quad \vec{v}\big|_{x\in S} = \vec{a}(x,t),$$

where $\vec{a}\cdot\vec{n} = 0$, $\vec{v}_0(x)\big|_{x\in S} = \vec{a}(x,0)$. We have (cf. [**5**])

$$\sup_{\Omega\times(0,T)} |\vec{v}(x,t)| \leqslant c\big(\sup_{\Omega}|\vec{v}_0(x)| + \sup_{\Sigma_T}|\vec{a}(x,t)|\big).$$

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Stability of Axially Symmetric Solutions to the Navier–Stokes Equations in Cylindrical Domains

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I dedicate this paper to Professor O. A. Ladyzhenskaya whose methods, ideas, and techniques I use in my papers

Using the result of O. A. Ladyzhenskaya [1], we establish the existence of a global solution, close to axially symmetric solutions, for the Navier–Stokes equations in a cylinder with boundary slip conditions. The solution belongs to a weight Sobolev space and possesses the property that the angular component of velocity, as well as the angular derivatives of cylindrical components of velocity and pressure, is sufficiently small. The uniqueness theorem is also established.

1. Introduction

We consider a motion of a viscous incompressible fluid in a bounded cylindrical domain $\Omega \subset \mathbb{R}^3$ under boundary slip conditions

$$
\begin{aligned}
&v_{,t} + v \cdot \nabla v - \operatorname{div} \mathbb{T}(v,p) = f, \quad \text{in } \Omega^T = \Omega \times (0,T),\\
&\operatorname{div} v = 0 \quad \text{in } \Omega^T = \Omega \times (0,T),\\
&v \cdot \overline{n} = 0,\\
&\overline{n} \cdot \mathbb{T}(v,p) \cdot \overline{\tau}_\alpha = 0, \quad \alpha = 1,2, \quad \text{on } S^T = S \times (0,T),\\
&v\big|_{t=0} = v_0 \quad \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where dot denotes the inner product in $\mathbb{R}^3$, $v = v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t)) \in \mathbb{R}^3$ is the velocity, $p = p(x,t) \in \mathbb{R}$ is the pressure, $f = f(x,t) =$

$(f_1(x,t), f_2(x,t), f_3(x,t)) \in \mathbb{R}^3$ is the external force field, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ denotes the Cartesian coordinates, $\overline{n}$ is the unit outward normal to $S = \partial\Omega$, $\overline{\tau}_1$ and $\overline{\tau}_2$ are unit tangent vectors to S, $\mathbb{T}(v,p)$ is the stress tensor,

$$\mathbb{T}(v,p) = \{\nu(v_{i,x_j} + v_{j,x_i}) - p\delta_{ij}\}_{i,j=1,2,3} \equiv \mathbb{D}(v) - pI, \tag{1.2}$$

where ν is the constant positive viscosity coefficient, $\mathbb{D}(v)$ is the dilatation tensor, and I is the unit matrix.

Our goal is to establish the existence of a global solution to the problem (1.1) that is close to axially symmetric solutions. In this paper, we outline an approach to this problem and sketch the proof (cf. [**2**] for details). To define axially symmetric solutions, we introduce the cylindrical coordinates r, φ, z by the relations $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, $x_3 = z$. Then $\overline{e}_r = (\cos\varphi, \sin\varphi, 0)$, $\overline{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)$, and $\overline{e}_z = (0,0,1)$ are the unit vectors along the curvilinear coordinates r, φ, and z respectively. For any vector w we write $w_r = w \cdot \overline{e}_r$, $w_\varphi = w \cdot \overline{e}_\varphi$, and $w_z = w \cdot \overline{e}_z$.

We denote by Ω a cylinder $\{(r,z) \mid 0 < r < R, -a < z < a\}$, where R and a are given numbers. The axis of symmetry of Ω coincides with the x_3-axis, and the boundary S can be written as $S = S_1 \cup S_2$, where S_1 is parallel to the x_3-axis and is located at the distance $r = R$ from the x_3-axis and between the planes $x_3 = -a$ and $x_3 = a$, whereas S_2 consists of two parts that are perpendicular to the x_3-axis and intersect it at the points $x_3 = -a$ and $x_3 = a$ respectively. The intersection of S_1 and S_2, denoted by L is two circles.

Definition 1.1. A solution to the problem (1.1) is said to be *axially symmetric* if $v(0)_\varphi = v_\varphi = 0$, $f_\varphi = 0$, $v_{r,\varphi} = 0$, $v_{z,\varphi} = 0$, $p_{,\varphi} = 0$, $f_{r,\varphi} = 0$, $f_{z,\varphi} = 0$, $v(0)_{r,\varphi} = 0$, $v(0)_{z,\varphi} = 0$.

As in [**1**], we study the existence of a global solution to the Navier–Stokes equations under the assumption that the initial velocity and external force are large. In accordance with [**1**], it is reasonable to expect that there exist solutions that are close to axially symmetric ones. Respectively, the domains under consideration should be axially symmetric. As in [**1**], we restrict ourselves to cylindrical domains in this paper. More general domains will be treated in other publication. The study of axially symmetric solutions in a cylindrical domain requires appropriate boundary conditions (cf. [**1**]).

In the case of axially symmetric solutions, Eqs. $(1.1)_{1,2,5}$ are reduced to the following system of problems:

$$\begin{aligned}
&\chi_{,t} - \nu\left(\chi_{,rr} + \chi_{,zz} + \frac{1}{r}\chi_{,r} - \frac{1}{r^2}\chi\right) + v_r\chi_{,r} + v_z\chi_{,z} - \frac{v_r}{r}\chi = F_\varphi, \\
&\chi\big|_{t=0} = \chi(0);
\end{aligned} \tag{1.3}$$

$$v_{r,z} - v_{z,r} = \chi, \quad v_{r,r} + v_{z,z} + \frac{v_r}{r} = 0. \tag{1.4}$$

The existence of a global solution to this system can be proved by the method of successive approximations if the boundary conditions are chosen in an appropriate way. In [1], the boundary conditions were as follows:

$$v \cdot \overline{n}\big|_S = 0, \tag{1.5}$$

$$\chi\big|_S = 0. \tag{1.6}$$

We note that (1.5) is exactly $(1.1)_3$ and (1.6) follows from $(1.1)_{3,4}$ in the case of cylindrical domains (cf. Lemma 3.2 below).

By the arguments of [1], we see that vorticity plays a crucial role in the proof of global existence. It is natural to expect that the same situation takes place if we deal with solutions close to axially symmetric ones.

Instead of the problem (1.1), we should consider the following system of problems.

For a given v the vorticity $\alpha = \operatorname{rot} v$ is a solution to the problem

$$\begin{aligned}
&\alpha_{,t} + v \cdot \nabla\alpha - \alpha \cdot \nabla v - \nu\Delta\alpha = F \equiv \operatorname{rot} f \quad \text{in } \Omega^T, \\
&\alpha_{r,r} = -\frac{1}{R^2}v_{z,\varphi} - \frac{1}{R}v_{\varphi,z}, \quad \alpha_z = \frac{2}{R}v_\varphi \quad \text{on } S_1^T, \\
&\alpha_r = 0, \quad \alpha_{z,z} = 0 \quad \text{on } S_2^T, \qquad \alpha_\varphi \equiv \chi = 0 \quad \text{on } S^T, \\
&\alpha\big|_{t=0} = \alpha(0) \quad \text{in } \Omega,
\end{aligned} \tag{1.7}$$

where the boundary conditions $(1.7)_{2,3,4}$ are derived from $(1.1)_{3,4}$ in Lemma 3.2 below.

For a given α the velocity v is a solution to the elliptic problem

$$\operatorname{rot} v = \alpha, \quad \operatorname{div} v = 0 \quad \text{in } \Omega, \quad v \cdot \overline{n} = 0 \quad \text{on } S. \tag{1.8}$$

From (1.7) and (1.8) it follows that the pressure p is a solution to the problem

$$\begin{aligned}
&\Delta p = -\nabla v \cdot \nabla v + \operatorname{div} f \quad \text{in } \Omega, \\
&\frac{\partial p}{\partial n} = f \cdot \overline{n} - \frac{1}{R}v_\varphi^2 + \frac{2\nu}{R^2}v_{\varphi,\varphi} \quad \text{on } S_1, \\
&\frac{\partial p}{\partial n} = f \cdot \overline{n} \quad \text{on } S_2.
\end{aligned} \tag{1.9}$$

However we fail to prove the existence of solutions to the problems (1.7) and (1.8) by the method of successive approximations directly. Indeed, for a given v_m we can find α_{m+1} from (1.7) and then, substituting α_{m+1} for α in (1.8), we obtain v_{m+1}. However, we cannot establish the convergence of the constructed sequence in view of the boundary conditions $(1.7)_2$.

To overcome this obstacle, we introduce three new quantities

$$h = v_{r,\varphi}\overline{e}_r + v_{\varphi,\varphi}\overline{e}_\varphi + v_{z,\varphi}\overline{e}_z, \quad q = p_{,\varphi}, \quad w = v_\varphi \tag{1.10}$$

expressing the distance between the solutions under consideration and the corresponding axially symmetric solution. These quantities are solutions to the following problems (cf. [**2**, Part 3]).

For a given v we have the following problem for h:

$$\begin{aligned} & h_{,t} - \operatorname{div}\mathbb{D}(h) + \nabla q = -v\cdot\nabla h - h\cdot\nabla v + g, \\ & \operatorname{div} h = 0, \qquad h\cdot\overline{n} = 0, \\ & \overline{n}\cdot\mathbb{D}(h)\cdot\overline{\tau}_\alpha = 0, \quad \alpha = 1,2, \\ & h\big|_{t=0} = h(0), \end{aligned} \tag{1.11}$$

where $g = f_{r,\varphi}\overline{e}_r + f_{\varphi,\varphi}\overline{e}_\varphi + f_{z,\varphi}\overline{e}_z$.

For given q, h, and v we have the following problem for w:

$$\begin{aligned} & w_{,t} + v\cdot\nabla w + \frac{v_r}{r}w - \nu\Delta w + \frac{\nu w}{r^2} = \frac{1}{r}q + \frac{2\nu}{r^2}h_r + f_\varphi \quad \text{in } \Omega^T, \\ & w_{,r} = \frac{1}{R}w \quad \text{on } S_1^T, \qquad w_{,z} = 0 \quad \text{on } S_2^T, \\ & w\big|_{t=0} = w(0) \equiv v_\varphi(0) \quad \text{in } \Omega. \end{aligned} \tag{1.12}$$

To proceed by the method of successive approximations, we replace $(1.7)_2$ with the following condition:

$$\alpha_{r,r} = -\frac{1}{R^2}h_z - \frac{1}{R}w_{,z}, \quad \alpha_z = \frac{2}{R}w \quad S_1^T. \tag{1.13}$$

Therefore, the problem (1.7) takes the form

$$\begin{aligned} & \alpha_{,t} + v\cdot\nabla\alpha - \alpha\cdot\nabla v - \nu\Delta\alpha = F \quad \text{in } \Omega^T, \\ & \alpha_{r,r} = -\frac{1}{R^2}h_z - \frac{1}{R}w_{,z}, \quad \alpha_z = \frac{2}{R}w \quad \text{on } S_1^T, \\ & \alpha_r = 0, \quad \alpha_{z,z} = 0 \quad \text{on } S_2^T, \\ & \alpha_\varphi = 0 \quad \text{on } S^T, \\ & \alpha\big|_{t=0} = \alpha(0) \quad \text{in } \Omega, \end{aligned} \tag{1.14}$$

where v, h, w are given functions.

To prove the existence of a global solution to the problem (1.1), we first find a local solution with large time of existence and, after that, extend it step-by-step up to infinity. We note that Eq. $(1.14)_1$ is strongly nonlinear.

Since we are looking for a solution close to axially symmetric solutions, we should assume that v_r, v_z, χ, f_r, f_z, and F_φ are large, whereas h, q, w, and f_φ are small. Expressing the cylindrical coordinates of vorticity in terms of the cylindrical coordinates of velocity, we find

$$\alpha_r = \frac{1}{r}(v_{z,\varphi} - rv_{\varphi,z}), \ \alpha_\varphi = v_{r,z} - v_{z,r} \equiv \chi, \ \alpha_z = \frac{1}{r}[(rv_\varphi)_{,r} - v_{r,\varphi}], \tag{1.15}$$

which implies that α_r and α_z are also small. Similarly, we can show that F_r and F_z are small.

Using the large and small quantities, we can show that the r- and z-component of Eq. $(1.14)_1$ are linear with respect to the large quantities; moreover, the φ-component of Eq. $(1.14)_1$ is equal to Eq. $(1.3)_1$ modulo small quantities. Therefore, we can represent (1.14) as three problems.

1. THE PROBLEM FOR α_r:

$$\begin{aligned}
&\alpha_{r,t} + v\cdot\nabla\alpha_r - \alpha_r v_{r,r} - \frac{\alpha_\varphi}{r}h_r - \alpha_z v_{r,z} + \frac{2\nu}{r^2}(h_{r,z} - h_{z,r}) \\
&\quad + \frac{\nu\alpha_r}{r^2} - \nu\Delta\alpha_r = F_r \quad \text{in } \Omega^T, \\
&\alpha_{r,r} = -\frac{1}{R^2}h_z - \frac{1}{R}w_{,z} \quad \text{on } S_1^T, \qquad \alpha_r = 0 \quad \text{on } S_2^T, \\
&\alpha_r\big|_{t=0} = \alpha_r(0) \quad \text{in } \Omega.
\end{aligned} \tag{1.16}$$

2. THE PROBLEM FOR α_φ:

$$\begin{aligned}
&\alpha_{\varphi,t} + v\cdot\nabla\alpha_\varphi + \frac{v_\varphi}{r}\alpha_r - \alpha\cdot\nabla v_\varphi - \frac{\alpha_\varphi}{r}v_r - \frac{2\nu}{r^2}\left(\frac{1}{r}h_{z,\varphi} - h_{\varphi,z}\right) \\
&\quad + \frac{\nu\alpha_\varphi}{r^2} - \nu\Delta\alpha_\varphi = F_\varphi \quad \text{in } \Omega^T, \\
&\alpha_\varphi = 0 \quad \text{on } S^T, \\
&\alpha_\varphi\big|_{t=0} = \alpha_\varphi(0) \quad \text{in } \Omega.
\end{aligned} \tag{1.17}$$

3. THE PROBLEM FOR α_z:

$$\begin{aligned}
&\alpha_{z,t} + v\cdot\nabla\alpha_z - (\alpha_r v_{z,r} + \alpha_z v_{z,z}) - \frac{\alpha_\varphi}{r}h_z - \nu\Delta\alpha_z = F_z \quad \text{in } \Omega^T, \\
&\alpha_z = \frac{2}{R}w \quad \text{on } S_1^T, \qquad \alpha_{z,z} = 0 \quad \text{on } S_2^T, \\
&\alpha_z\big|_{t=0} = \alpha_z(0) \quad \text{in } \Omega.
\end{aligned} \tag{1.18}$$

2. Notation

For the sake of simplicity, we introduce the notation ($\|u\|_{0,Q} = |u|_{2,Q}$)

$$\begin{aligned}
&|u|_{p,Q} = \|u\|_{L_p(Q)}, \quad Q \in \{\Omega, S, \Omega^T, S^T\},\ p\in[1,\infty], \\
&\|u\|_{s,Q} = \|u\|_{H^s(Q)}, \quad Q\in\{\Omega, S\},\ s\in\mathbb{R}_+, \\
&\|u\|_{s,Q^T} = \|u\|_{W_2^{s,s/2}(Q^T)}, \quad Q\in\{\Omega, S\},\ s\in\mathbb{R}_+,
\end{aligned}$$

We introduce weight spaces with the weight equal to a power of the distance to the axis of symmetry

$$\|u\|_{L_{p,\mu}(Q)} = \Big(\int_Q |u|^p r^{p\mu} dQ\Big)^{1/p}, \ p\in[1,\infty],\ \mu\in\mathbb{R}, \quad Q\in\{\Omega, S, \Omega^T, S^T\},$$

$$|u|_{p,\mu,Q} = \|u\|_{L_{p,\mu}(Q)},$$

$$\|u\|_{H^s_\mu(Q)} = \Big(\sum_{|\alpha|\leqslant s}\int_Q |D^\alpha_x u|^2 r^{2(\mu-s+|\alpha|)}dQ\Big)^{1/2} \equiv \|u\|_{s,\mu,Q}, \quad Q\in\{\Omega,S\},$$

$$\|u\|_{H^{s,s/2}_\mu(Q^T)} = \Big(\sum_{|\alpha|+2a\leqslant s}\int_{Q^T} |D^\alpha_x \partial^a_t u|^2 r^{2(\mu-s+2a+|\alpha|)}dQdt\Big)^{1/2}$$
$$\equiv \|u\|_{s,\mu,Q^T}, \quad Q\in\{\Omega,S\},$$

$$|||u|||_{l,r,Q} = \|u\|_{W^l_r(Q)}, \quad |||u|||_{l,r,Q^T} = \|u\|_{W^{l,l/2}_r(Q^T)}, \quad Q\in\{\Omega,S\},$$

$$\|u\|_{W^s_{p,\mu}(Q)} = \Big(\sum_{|\alpha|\leqslant s}\int_Q |D^\alpha_x u|^p r^{p\mu}dQ\Big)^{1/p} \equiv |||u|||_{s,p,\mu,Q}, \quad Q\in\{\Omega,S\}.$$

$$\|u\|_{W^{s,s/2}_{p,\mu}(Q^T)} = \Big(\sum_{|\alpha|+2a\leqslant s}\int_{Q^T} |D^\alpha_x \partial^a_t u|^p r^{p\mu}dQdt\Big)^{1/p}$$
$$\equiv |||u|||_{s,p,\mu,Q^T}, \quad Q\in\{\Omega,S\}.$$

Finally, we set $\|u\|_{A_\mu(\Omega^T)} = \|u\|_{W^{2,1}_{2,\mu}(\Omega^T)} + \|u_x\|_{W^{2,1}_{2,\mu}(\Omega^T)}$. We define the space $V_{p,\mu}(Q)$ with the norm

$$\|u\|_{V_{p,\mu}(Q)} = \Big(\sum_{|\alpha|\leqslant l}\int_Q |D^\alpha_x u|^p r^{p(\mu-l+|\alpha|)}dQ\Big)^{1/p},$$

where $\mu\in\mathbb{R}$, $p\in[1,\infty)$, $l\in\mathbb{N}$, $Q\in\{\Omega,S\}$.

From [**3**, Lemma 1.5] we obtain the following assertion.

Lemma 2.1. *If $u\in V^l_{p,\beta}(\Omega)$, $\Omega\subset\mathbb{R}^n$, and $s-l+n/p-n/q\leqslant 0$, then*

$$\|u\|_{V^s_{q,\beta+s-l+n/p-n/q}(\Omega)} \leqslant c\|u\|_{V^s_{p,\beta}(\Omega)}. \tag{2.1}$$

From [**4**] we get

Lemma 2.2. *Suppose that $u\in W^{\bar l}_{p,\alpha}(Q)$, $Q\subset\mathbb{R}^{n+1}$, $\bar l=(l_0,l_1,\dots,l_n)$, $1<p<q<\infty$, $\alpha,\beta\in\mathbb{R}_+$, $0<l_i\in\mathbb{Z}$, $0\leqslant\nu_i\in\mathbb{Z}$, $i=0,1,\dots,n$, $l_1=l_2=l_*$,*

$$\varkappa = 1-\Big(\frac1p-\frac1q\Big)\sum_{i=0}^n\frac{1}{l_i}-\sum_{i=0}^n\frac{\nu_i}{l_i}-\frac{1}{l_*}(\alpha-\beta)\geqslant 0, \quad \alpha>\beta,$$

where Q satisfies the $R(\bar l,h_0)$-horn condition. Then $D^{\bar\nu}u\in L_{q,\beta}(Q)$ and the following estimate holds:

$$\|D^{\bar\nu}u\|_{L_{q,\beta}(Q)} \leqslant c_1\delta^\varkappa\|u\|_{L^{\bar l}_{p,\alpha}(Q)} + c_2\delta^{\varkappa-1}\|u\|_{L_{p,\alpha}(Q)}, \tag{2.2}$$

where the constants c_1 *and* c_2 *are independent of* $u, \delta \in (0, h_0)$, $h_0 = h_0(Q)$,

$$\|u\|_{L^l_{p,\alpha}(Q)} = \Big(\sum_{i=0}^{n} \int\limits_Q |\partial^i_{l_i} u|^p r^{p\alpha} dQ\Big)^{1/p}.$$

3. Auxiliary Results

Since we consider the problem (1.1) in cylindrical coordinates, we need

Lemma 3.1 (cf. [**2**, Lemma 2.1, Part 4]). *If* $(1.1)_{3,4}$ *holds on* S, *then*

$$\begin{aligned} &v_r = 0, \quad v_{z,r} = 0, \quad v_{\varphi,r} = \frac{1}{r} v_\varphi \quad \text{on } S_1, \\ &v_{r,z} = 0, \quad v_{\varphi,z} = 0, \quad v_z = 0 \quad \text{on } S_2. \end{aligned} \tag{3.1}$$

Lemma 3.2 (cf. [**2**, Lemma 2.1, Part 4]). *If* $(1.1)_{3,4}$ *holds, then*

$$\begin{aligned} &\alpha_{r,r} = -\frac{1}{R^2} v_{z,\varphi} - \frac{1}{R} v_{\varphi,z}, \quad \alpha_z = \frac{2}{R} v_\varphi \quad \text{on } S_1, \\ &\alpha_r = 0, \quad \alpha_{z,z} = 0 \quad \text{on } S_2, \qquad \alpha_\varphi \equiv \chi = 0 \quad \text{on } S. \end{aligned} \tag{3.2}$$

Since some estimates are based on the energy method, we need

Lemma 3.3. *Suppose that* $\eta = \overline{e}_0 \times \overline{x}$, $\overline{e}_0 = (0,0,1)$, $\overline{x} = (x_1, x_2, x_3)$, Ω *has the axis of symmetry* $\overline{e}_0$. *Let* $v \cdot \overline{n}\big|_S = 0$, $v\big|_{t=0} = v(0)$,

$$\Big|\int\limits_\Omega v(0) \cdot \eta dx\Big| < \infty, \qquad \Big|\int\limits_{\Omega^t} f \cdot \eta dx\, dt'\Big| < \infty.$$

Then a solution v *to the problem* (1.1) *satisfies the identity*

$$\int\limits_\Omega v \cdot \eta dx = \int\limits_\Omega v(0) \cdot \eta dx + \int\limits_{\Omega^t} f \cdot \eta dx\, dt'. \tag{3.3}$$

To formulate the Korn inequality, we introduce

$$E_\Omega(v) = \int\limits_\Omega (v_{i,x_j} + v_{j,x_i})^2 dx, \tag{3.4}$$

where the summation convention over the repeated indices is assumed.

Lemma 3.4. *Let* v *be a solution to the problem* (1.1), *and let*

$$\Big|\int\limits_\Omega r v_\varphi dx\Big| < \infty, \quad E_\Omega(v) < \infty.$$

Then there exists a constant c *independent of* v *such that*

$$\|v\|^2_{1,\Omega} \leqslant c\Big(E_\Omega(v) + \Big|\int\limits_\Omega r v_\varphi dx\Big|^2\Big). \tag{3.5}$$

As we underlined in Sec. 1, only the problem (1.17) is actually nonlinear with respect to the large quantities. However, the nonlinear part can be treated in the same way as in [**1**]. To apply the method of Ladyzhenskaya, we write (1.17) in the form

$$\begin{aligned}
&\chi_{,t} + v_r\chi_{,r} + \frac{v_\varphi}{r}\chi_{,\varphi} + v_z\chi_{,z} + (v_{r,r} + v_{z,z})\chi \\
&= \nu\left[\left(r\left(\frac{\chi}{r}\right)_{,r}\right)_{,r} + \frac{1}{r^2}\chi_{,\varphi\varphi} + \chi_{,zz} + 2\left(\frac{\chi}{r}\right)_{,r}\right] \\
&+ \frac{2\nu}{r^2}\left(-h_{\varphi,z} + \frac{1}{r}h_{z,\varphi}\right) - \frac{1}{r}\left(w_{,z}h_r - w_{,r}h_z + \frac{w}{r}h_z\right) \\
&+ \frac{2}{r}wv_{\varphi,z} + F_\varphi \quad \text{in } \Omega^T, \\
&\chi = 0 \quad \text{on } S^T, \quad \chi\big|_{t=0} = \chi(0) \quad \text{in } \Omega.
\end{aligned} \tag{3.6}$$

Lemma 3.5. *Let* $h \in H^{2,1}_{-1}(\Omega^t)$, $v \in A_{1-\mu}(\Omega^t)$, $w \in L_\infty(0,t;H^1_0\Omega))$, $F_\varphi \in L_{2,-1}(\Omega^t)$, $\chi(0) \in L_{2,-1}(\Omega)$, $\mu \in (1/2, 1/\sqrt{2})$. *If* χ *is a solution to the problem* (3.6) *vanishing on the axis of symmetry, then*

$$\begin{aligned}
&\left|\frac{\chi}{r}\right|_{2,\Omega} + \nu\int_0^t\left\|\frac{\chi}{r}\right\|^2_{1,\Omega}dt' \leqslant c\sup_t\|h_\varphi\|_{1,-1,\Omega}\int_0^t\left\|\frac{\chi}{r}\right\|^2_{1,\Omega}dt' \\
&+ c(1+\sup_t\|w\|^2_{1,0,\Omega})\|h\|^2_{2,-1,\Omega^t} + c\|v\|^2_{A_{1-\mu}(\Omega^t)}\sup_t\|w\|^2_{1,0,\Omega} \\
&+ c|F_\varphi|^2_{2,-1,\Omega^t} + \left|\frac{\chi(0)}{r}\right|^2_{2,\Omega}.
\end{aligned} \tag{3.7}$$

PROOF. Multiplying (3.6_1) by χ/r^2, integrating over $\Omega_\varepsilon = \{x \in \mathbb{R}^3 \mid 0 < \varepsilon < r < R, \varphi \in [0,2\pi], z \in (-a,a)\}$, applying the Hölder and Young inequalities, using embedding theorems (cf. Lemmas 2.1 and 2.2), and passing to the limit as $\varepsilon \to 0$, we obtain (3.7). □

Lemma 3.6. *Suppose that* $v \in A_{1-\mu}(\Omega^T)$, $\mu \in (1/2, 1/\sqrt{2})$, $g \in L_2(\Omega^T)$, $h(0) \in H^1_{-1}(\Omega)$. *If* h *is a solution to the problem* (1.11), *then* (*cf.* [**2**, Part 4])

$$\begin{aligned}
&\|h\|_{2,-1,\Omega^T} + \left(\int_0^T\|q(t')\|^2_{1,-1,\Omega}dt'\right)^{1/2} \\
&\leqslant \varphi(\|v\|_{A_{1-\mu}(\Omega^T)})[|g|_{2,\Omega^T} + |h(0)|_{2,\Omega}] + c\|h(0)\|_{1,-1,\Omega},
\end{aligned} \tag{3.8}$$

where φ *is an increasing positive function.*

Lemma 3.7. *Suppose that* $v \in A_{1-\mu}(\Omega^T)$, $\mu \in (1/2, 1/\sqrt{2})$, $\alpha_\varphi/r \in L_\infty(0,T;L_2(\Omega)) \cap L_2(0,T;H^1(\Omega))$, $h \in H^{2,1}_{-1}(\Omega^T)$, $w \in L_2(0,T;H^{3/2}(S_1))$,

$F' \in L_{2,1-\mu}(\Omega^T)$, $\alpha'(0) \in H^1_{1-\mu}(\Omega)$. *If* α_φ *is a solution to the problem* (1.16) *and* α_z *is a solution to the problem* (1.18), *then* (*cf.* [**2**, Part 5])

$$|||\alpha'|||_{2,2,1-\mu,\Omega^t} \leqslant \varphi(||v||_{A_{1-\mu}(\Omega^t)})\Big[\Big(1+\sup_t\Big|\frac{\alpha_\varphi}{r}\Big|_{2,\Omega}+\Big|\frac{\alpha_\varphi}{r}\Big|_{10/3,\Omega^t}\Big)||h||_{2,1-\mu,\Omega^t}$$

$$+\Big(\int_0^t ||w||^2_{3/2,S_1}dt'\Big)^{1/2}+|F'|_{2,1-\mu,\Omega^t}+||\alpha'(0)||_{1,1-\mu,\Omega}\Big],\quad t\leqslant T, \tag{3.9}$$

where φ *is an increasing positive function and* $\alpha' = (\alpha_\varphi, \alpha_z)$.

Lemma 3.8. *Let* $v \in A_{1-\mu}(\Omega^T)$, $\mu \in (1/2, 1/\sqrt{2})$, $v(0) \in W^2_{2,1-\mu}(\Omega)$, $h \in H^{2,1}_{-1}(\Omega^T)$, $\alpha_\varphi/r \in L_\infty(0,T;L_2(\Omega)) \cap L_2(0,T;H^1(\Omega))$, $w \in H^{2,1}_{1-\mu}(\Omega^T) \cap L_\infty(0,T;H^1_0(\Omega))$, $F_\varphi \in L_{2,1-\mu}(\Omega^T)$, $\alpha_\varphi(0) \in H^1_{1-\mu}(\Omega)$. *If* α_φ *is a solution to the problem* (1.17), *then* (*cf.* [**2**, Part 5])

$$||\alpha_\varphi||_{2,1-\mu,\Omega^t} \leqslant (\varepsilon||v||_{A_{1-\mu}(\Omega^t)}+c(\varepsilon)\sup_t|v|_{2,\Omega}+c||h||_{2,1-\mu,\Omega^t}$$

$$+c|||v(0)|||_{2,2,1-\mu,\Omega}+c)\Big(\sup_t\Big|\frac{\alpha_\varphi}{r}\Big|_{2,\Omega}+\Big(\int_0^t\Big\|\frac{\alpha_\varphi}{r}\Big\|^2_{1,\Omega}dt'\Big)^{1/2}\Big)$$

$$+c(||w||_{2,1-\mu,\Omega^t}||h||_{2,-1,\Omega^t}+\sup_t||w||_{1,0,\Omega}||v||_{A_{1-\mu}(\Omega^t)})$$

$$+c(|F_\varphi|_{2,1-\mu,\Omega^t}+||\alpha_\varphi(0)||_{1,1-\mu,\Omega}+||h||_{2,-1,\Omega^t}),\quad t\leqslant T, \tag{3.10}$$

where $\varepsilon \in (0,1)$ *and* φ *is an increasing positive function.*

Lemma 3.9. *Suppose that* $v \in A_{1-\mu}(\Omega^T)$, $\mu \in (1/2, 1/\sqrt{2})$, $g \in L_2(\Omega^T)$, $f_\varphi \in L_{2,-\mu}(\Omega^T)$, $h(0) \in H^1_{-1}(\Omega)$, $w(0) \in H^1_{1-\mu}(\Omega)$. *If* w *is a solution to the problem* (1.12), *then*

$$||w||_{2,1-\mu,\Omega^t} \leqslant \varphi(||v||_{A_{1-\mu}(\Omega^t)})e^{ct}[|g|_{2,\Omega^t}+||h(0)||_{1,-1,\Omega}$$

$$+|f_\varphi|_{2,-\mu,\Omega^t})+c||w(0)||_{1,1-\mu,\Omega},\quad t\leqslant T, \tag{3.11}$$

where φ *is an increasing positive function.*

Finally, we consider the problem (1.8).

Lemma 3.10. *Suppose that* $\alpha \in W^{2,1}_{2,1-\mu}(\Omega^T)$, $\mu \in (1/2, 1/\sqrt{2})$. *Then there exists a solution* v *to the problem* (1.8) *such that* $v \in A_{1-\mu}(\Omega^T)$ *and the following estimate holds*:

$$||v||_{A_{1-\mu}(\Omega^t)} \leqslant c||\alpha||_{W^{2,1}_{2,1-\mu}(\Omega^t)},\quad t\leqslant T. \tag{3.12}$$

4. Existence

To establish the local existence of solutions to the problem (1.1), we use the following method of successive approximations.

Let v_m be given. Then h_m and q_m are found from the problem

$$\begin{aligned}&h_{m,t}-\operatorname{div}\mathbb{D}(h_m)+\nabla q_m+v_m\cdot\nabla h_m+h_m\cdot\nabla v_m=g\quad\text{in }\Omega^T,\\&\operatorname{div}h_m=0\quad\text{in }\Omega^T,\\&h_m\cdot\overline{n}=0\quad\text{on }S^T,\\&\overline{n}\cdot\mathbb{D}(h_m)\cdot\overline{\tau}_\alpha=0,\quad\alpha=1,2,\quad\text{on }S^T,\\&h_m\big|_{t=0}=h(0)\quad\text{in }\Omega.\end{aligned}\tag{4.1}$$

Let v_m, h_m, q_m be given. Then w_m is found from the problem

$$\begin{aligned}&w_{m,t}+v_m\cdot\nabla w_m+\frac{v_{mr}}{r}w_m-\nu\Delta w_m+\nu\frac{w_m}{r^2}=\frac{q_m}{r}+\frac{2\nu}{r^2}h_{mr}+f_\varphi\ \text{in }\Omega^T,\\&w_{m,r}=\frac{1}{R}w_m\quad\text{on }S_1^T,\qquad w_{m,z}=0\quad\text{on }S_2^T,\\&w_m\big|_{t=0}=w(0)\quad\text{in }\Omega.\end{aligned}\tag{4.2}$$

Let v_m, q_m, h_m, w_m be given. We find α_{rm+1}, $\alpha_{\varphi m+1}$, and α_{zm+1} from the following problems:

$$\begin{aligned}&\alpha_{rm+1,t}+v_m\cdot\nabla\alpha_{rm+1}-\alpha_{rm+1}v_{mr,r}-\frac{\alpha_{\varphi m+1}}{r}h_{mr}-\alpha_{zm+1}v_{mr,z}\\&+\frac{2\nu}{r^2}(h_{mr,z}-h_{mz,r})+\frac{\nu\alpha_{rm+1}}{r^2}-\nu\Delta\alpha_{rm+1}=F_r\quad\text{in }\Omega^T,\\&\alpha_{rm+1,r}=-\frac{1}{R^2}h_{mz}-\frac{1}{R}w_{m,z}\quad\text{on }S_1^T,\qquad\alpha_{rm+1}=0\quad\text{on }S_2^T,\\&\alpha_{rm+1}\big|_{t=0}=\alpha_r(0)\quad\text{in }\Omega;\end{aligned}\tag{4.3}$$

$$\begin{aligned}&\alpha_{\varphi m+1,t}+v_m\cdot\nabla\alpha_{\varphi m+1}+\frac{v_{m\varphi}}{r}\alpha_{rm+1}-\alpha_{m+1}\cdot\nabla v_{m\varphi}-\frac{\alpha_{\varphi m+1}}{r}v_{mr}\\&-\frac{2\nu}{r^2}\left(\frac{1}{r}h_{mz,\varphi}-h_{m\varphi,z}\right)+\frac{\nu\alpha_{\varphi m+1}}{r^2}-\nu\Delta\alpha_{\varphi m+1}=F_\varphi\quad\text{in }\Omega^T,\\&\alpha_{\varphi m+1}=0\quad\text{on }S^T,\\&\alpha_{\varphi m+1}\big|_{t=0}=\alpha_\varphi(0)\equiv\chi(0)\quad\text{in }\Omega;\end{aligned}\tag{4.4}$$

$$\begin{aligned}&\alpha_{zm+1,t}+v_m\cdot\nabla\alpha_{zm+1}-(\alpha_{rm+1}v_{mz,r}+\alpha_{zm+1}v_{mz,z})\\&-\frac{\alpha_{\varphi m+1}}{r}h_{mz}-\nu\Delta\alpha_{zm+1}=F_z\quad\text{in }\Omega^T,\\&\alpha_{zm+1}=\frac{2}{R}w_m\quad\text{on }S_1^T,\qquad\alpha_{zm+1,z}=0\quad\text{on }S_2^T,\\&\alpha_{zm+1}\big|_{t=0}=\alpha_z(0)\quad\text{in }\Omega.\end{aligned}\tag{4.5}$$

At the next step, v_{m+1} is determined from the elliptic problem

$$\operatorname{rot}v_{m+1}=\alpha_{m+1},\quad\operatorname{div}v_{m+1}=0\quad\text{in }\Omega,\quad v_{m+1}\cdot\overline{n}=0\quad\text{on }S^T.\tag{4.6}$$

We start the above procedure of successive approximations with $v_0=0$.

Lemma 4.1. *Suppose that*

$$\begin{aligned} &X_1 = |g|_{2,\Omega^t} + |f_\varphi|_{2,-\mu,\Omega^t} + \|h(0)\|_{1,-1,\Omega} + \|w(0)\|_{1,0,\Omega},\\ &X_2 = X_1 + |F'|_{2,1-\mu,\Omega^t} + |\alpha'(0)|_{2,1-\mu,\Omega},\\ &Y_1 = |\chi(0)|_{2,-1,\Omega} + |F_\varphi|_{2,-1,\Omega^t}, \quad Y_2 = \|\chi(0)\|_{1,1-\mu,\Omega} + |F_\varphi|_{2,1-\mu,\Omega^t}, \end{aligned} \tag{4.7}$$

where $F' = (F_r, F_z)$, $\alpha' = (\alpha_r, \alpha_z)$, $\mu \in (1/2, 1/\sqrt{2})$. *Suppose that there exists a positive constant* A *such that* (4.13) *and* (4.15) *hold. Then for* $t \leqslant T$, *where*

$$T = \frac{1}{c} \ln \frac{A}{c\varphi(A)X_1} \tag{4.8}$$

and φ *is a positive increasing function, and for sufficiently small* X_1 *and* X_2 *there exists a unique solution* v *to the problem* (1.1) *such that* $v \in A_{1-\mu}(\Omega^T)$ *and the following estimate holds*:

$$\|v\|_{A_{1-\mu}(\Omega^t)} \leqslant A. \tag{4.9}$$

PROOF. Let $K_m(t) = \|v_m\|_{A_{1-\mu}(\Omega^t)}$. From Lemmas 3.5–3.9 we have

$$\begin{aligned} \|\alpha_{m+1}\|_{2,2,1-\mu,\Omega^t} \leqslant{}& \varepsilon K_m(t) + c(\varepsilon)\varphi(K_m(t))[Y_1X_1 + e^{ct}X_1 + X_2]\\ &\times [Y_1 + \varphi(K_m(t))e^{ct}X_1 + 1] + c(\varepsilon)Y_1^2 + c(Y_1 + Y_2). \end{aligned} \tag{4.10}$$

Using Lemma 3.10, we find

$$\begin{aligned} K_{m+1}(t) \leqslant{}& \varepsilon K_m(t) + c(\varepsilon)\varphi(K_m(t))[Y_1X_1 + e^{ct}X_1 + X_2]\\ &\times [Y_1 + \varphi(K_m(t))e^{ct}X_1 + 1] + c(\varepsilon)Y_1^2 + c(Y_1 + Y_2). \end{aligned} \tag{4.11}$$

Let

$$K_m(t) \leqslant A, \quad t \leqslant T. \tag{4.12}$$

Take $\varepsilon = 1/3$ and assume that

$$\begin{aligned} &c(1/3)\varphi(A)[Y_1X_1 + e^{ct}X_1 + X_2](Y_1 + \varphi(A)e^{ct}X_1 + 1) \leqslant \frac{1}{3}A,\\ &c(1/3)Y_1^2 + (Y_1 + Y_2) \leqslant \frac{1}{3}A. \end{aligned} \tag{4.13}$$

Then

$$K_{m+1}(t) \leqslant A, \quad t \leqslant T. \tag{4.14}$$

To show that the inequality (4.12) holds for all $m \in \mathbb{N}$, we check that $K_0(t) = 0$ and $K_1(t)$ is determined from (4.6) for $m = 0$, where α_1 is a solution to the problems (4.3)–(4.5) for $m = 0$. Assuming that A is so large that

$$K_1(t) \leqslant A, \quad t \leqslant T, \tag{4.15}$$

we get (4.12) for all $m \in \mathbb{N}$. The convergence and uniqueness is proved in a standard way. □

To establish the existence of a global solution to the problem (1.1), it remains to extend the local solution step-by-step. For this purpose, we assume that the following decay estimates hold:

$$\begin{aligned}\gamma_1(t) &\equiv |g(t)|_{2,-1,\Omega} + |g_t(t)|_{2,\Omega} \leqslant \gamma_1(0)e^{-\delta_1 t},\\ \gamma_2(t) &\equiv |f_\varphi(t)|_{2,-\mu,\Omega} + |f_\varphi(t)|_{2,1,\Omega} \leqslant \gamma_2(0)e^{-\delta_2 t},\\ \gamma_3(t) &\equiv |F'(t)|_{2,1-\mu,\Omega} \leqslant \gamma_3(0)e^{-\delta_3 t},\end{aligned} \tag{4.16}$$

where δ_i, $i = 1,2,3$, are positive constants. Let

$$\begin{aligned}X_3(t) &= \|h(t)\|_{1,-1,\Omega} + \|w(t)\|_{1,0,\Omega} + |\alpha'(t)|_{2,1-\mu,\Omega},\\ X_4(t) &= \|\chi(t)\|_{1,1-\mu,\Omega} + |\chi(t)|_{2,-1,\Omega}.\end{aligned} \tag{4.17}$$

As is proved in [**2**, Part 7], for sufficiently small X_1 and X_2 there exist constants X_* and X_{**}, where X_* is sufficiently small, such that

$$X_3(t) \leqslant X_*, \quad X_4(t) \leqslant X_{**} \quad \text{for all } t \in \mathbb{R}_+. \tag{4.18}$$

By (4.16) and (4.18), the local solution can be extended infinitely. Thus, the following assertion is valid.

Theorem 4.2. *Let X_1 and X_2 be sufficiently small. Suppose that the decay estimate* (4.16) *holds with sufficiently large δ_i and sufficiently small γ_i, $i = 1,2,3$. Then there exists a global solution v to the problem* (1.1) *such that $v \in A_{1-\mu}(\Omega^t)$, $\mu \in (1/2, 1/\sqrt{2})$, $t \in [kT, (k+1)T]$, $k \in T$, where T is the time of local existence. The solution v possesses the following property: $X_3(t)$ remains small for all t and, consequently, v remains close to an axially symmetric solution.*

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