J. math. fluid mech. 12 (2010) 435–472
1422-6928/10/030435-38

DOI 10.1007/s00021-009-0297-2

Journal of Mathematical Fluid Mechanics

Incompressible Fluids in Thin Domains with Navier Friction Boundary Conditions (I)

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Communicated by V. A. Solonnikov

Abstract. This study is motivated by problems arising in oceanic dynamics. Our focus is the Navier–Stokes equations in a three-dimensional domain Ω_ε, whose thickness is of order $O(\varepsilon)$ as $\varepsilon \to 0$, having non-trivial topography. The velocity field is subject to the Navier friction boundary conditions on the bottom and top boundaries of Ω_ε, and to the periodicity condition on its sides. Assume that the friction coefficients are of order $O(\varepsilon^{3/4})$ as $\varepsilon \to 0$. It is shown that if the initial data, respectively, the body force, belongs to a large set of $H^1(\Omega_\varepsilon)$, respectively, $L^2(\Omega_\varepsilon)$, then the strong solution of the Navier–Stokes equations exists for all time. Our proofs rely on the study of the dependence of the Stokes operator on ε, and the non-linear estimate in which the contributions of the boundary integrals are non-trivial.

Mathematics Subject Classification (2000). 35Q30, 76D03, 76D05.

Keywords. Navier–Stokes equations, Navier friction boundary condition, thin domain, strong solution.

1. Introduction and main result

Our motivation is the studies of fluid flows in meteorology and oceanography. The Navier–Stokes equations are used essentially to describe the dynamics of viscous incompressible fluids in these studies. However, the basic question on the global well-posedness of those equations is still an open challenging problem. It is not known that starting from a large smooth initial data (with respect to some norm), the solution exists for all time and remains regular. Nonetheless, in many cases we do not necessarily consider arbitrary types of domains and arbitrarily large initial data. For examples, the fluids in oceans, great lakes or atmosphere are contained in a three-dimensional (3D) domain with very small thickness compared to its length and width. Therefore it is appropriate to study fluids in such thin domains. There is a vast literature on this subject, the listed references [23, 24, 25, 31, 32, 20, 21, 14, 15, 16, 17], by no mean are complete. The advantage of this approach is that we have the affirmative answers to the question of global well-posedness.

Regarding the conditions imposed on the viscous fluids on the boundary, there is a common no-slip condition, that is the fluids stick to the wall of the boundary. However, in [22] (see also [26]), Navier proposed a condition with which the fluids can slip along the wall, but have some constraints on the stress. Those are called the Navier boundary conditions. Moreover, there are other proposed boundary conditions (see e.g. [19]) which specify such constraints in various situations. Here we focus on one type of those conditions – the Navier friction boundary conditions – in which the friction between the fluid and the wall of the domain is taken into account. These conditions can be considered as the continuum between the no-slip boundary condition (when the friction is infinity) and the slip-condition for inviscid fluids (when the viscosity and friction are vanishing). The Navier friction conditions are also studied for different kinds of fluids such as compressible fluids [12], and non-Newtonian fluids [3, 2]. It is worth mentioning that another direction in studying the oceanic flows and climate models is to modify the Navier–Stokes equations using appropriate physical assumptions. One of those models is the primitive equations (see, e.g., the survey article [33]). Interestingly, the question of global existence of regular solutions to primitive equations has been recently answered [5]. However, the mathematical justifications for those models from the point of view of the Navier–Stokes equations are not strongly established yet.

Our goal has three-folds. The first is to develop the theory of the Navier–Stokes equations in thin domains which can be used in important practical problems. The second is to understand more about the Navier–Stokes equations in various contexts. In particular, we want to see how different conditions on different types of boundaries play on both the viscous term and the non-linear terms in the Navier–Stokes equations. The third is to give justifications for the other models using the pure analysis of the Navier–Stokes equations (see also [32, 1]). Though saying that, we only focus on the first two issues in this paper. We now pass on the mathematical exposition of the problems.

In this article, we consider three dimensional thin domains of the form

$$\hat{\Omega}_\varepsilon = \big\{(x_1, x_2, x_3) : (x_1, x_2) \in \mathbb{R}^2, h_0(x_1, x_2) < x_3 < h_1(x_1, x_2)\big\}, \tag{1.1}$$

where $\varepsilon \in (0, 1]$, $h_0 = h_0^\varepsilon = \varepsilon g_0$, $h_1 = h_1^\varepsilon = \varepsilon g_1$, with g_0 and g_1 being given C^3 scalar functions in $\mathbb{R}^2$ satisfying the following periodicity condition

$$g_i(x' + L\mathbf{e}_j) = g_i(x'), \quad x' = (x_1, x_2) \in \mathbb{R}^2, \ \ i = 0, 1, \ j = 1, 2,$$

where $L > 0$ and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis of $\mathbb{R}^3$.

The initial value problem for the Navier–Stokes equations in $\hat{\Omega}_\varepsilon$ is

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u = -\nabla p + f, \\ \qquad\qquad \text{div } u = 0, \\ \qquad\qquad u(x, 0) = u_0(x), \end{cases} \tag{1.2}$$

where $\nu > 0$ is the kinematic viscosity, $u = u(t, x)$ is the unknown velocity field, $p(t, x)$ is the unknown pressure, $f(t, x)$ is the body force and $u_0(x)$ is the known initial velocity field. The solution $u(t,x)$ is required to satisfy the periodicity condition

$$u(t, x + L\mathbf{e}_j) = u(t,x) \quad \text{for all} \quad t \geq 0,\ x \in \hat{\Omega}_\varepsilon, \quad j = 1,2. \tag{1.3}$$

A vector field u defined on the closure $\overline{\hat{\Omega}_\varepsilon}$ is said to satisfy the Navier friction boundary condition on a portion S of the boundary of $\hat{\Omega}_\varepsilon$ if

$$u \cdot N = 0, \tag{1.4}$$

$$\nu[(Du)N]_{\tan} + \gamma u = 0, \tag{1.5}$$

on S, where N denotes the outward normal vector on the boundary; $\gamma \geq 0$ is the friction coefficient; $[\ \cdot\ ]_{\tan}$ denotes the tangential part of a vector; Du is the symmetric part of the matrix of partial derivatives ∇u, that is

$$Du = \frac{\nabla u + (\nabla u)^*}{2}, \text{ where } (\nabla u)_{ij} = \partial_j u_i, \quad i,j = 1,2,3,$$

and $(\nabla u)^*$ is the transpose matrix of ∇u.

When S is flat, say, a part of a horizontal plane $x_3 = \text{const}$, the conditions (1.4) and (1.5) become the mixed Dirichlet and Robin boundary conditions:

$$u_3 = 0, \quad \nu\partial_3 u_1 + \gamma u_1 = 0, \quad \nu\partial_3 u_2 + \gamma u_2 = 0, \tag{1.6}$$

see the article [13].

When viscosity and friction are ignored ($\nu = 0$ and $\gamma = 0$), the Navier friction conditions deduce to the usual slip-condition (1.4) for viscous fluids. When $\nu > 0$ and $\gamma = 0$, the conditions (1.4) and (1.5) are referred to as the Navier boundary conditions, in the studies [16, 11]. Naively, (1.5) can also be rewritten as $u = \gamma^{-1}\nu[(Du)N]_{\tan}$ which deduces to the Dirichlet condition, $u = 0$, as $\gamma \to \infty$.

By rescaling the x and t variables we assume $\nu = 1$ and $L = 1$ throughout.

Denoting $g = g_1 - g_0$, we assume that

$$\min\{g(x') \ :\ x' \in \mathbb{R}^2\} = c_g > 0. \tag{1.7}$$

The boundary of $\hat{\Omega}_\varepsilon$ is $\hat{\Gamma} = \hat{\Gamma}_0 \cup \hat{\Gamma}_1$, where $\hat{\Gamma}_0$ and $\hat{\Gamma}_1$ are the bottom and the top of $\hat{\Omega}_\varepsilon$:

$$\hat{\Gamma}_i = \{(x_1,x_2,x_3) : (x_1,x_2) \in \mathbb{R}^2, x_3 = h_i(x_1,x_2)\}, \quad i = 0,1.$$

For our study, the solution $u = u(t,x)$ satisfies (1.4) and (1.5) on entire boundary $\hat{\Gamma}$ with possibly different friction coefficients $\gamma_i \geq 0$ on each $\hat{\Gamma}_i$, that is

$$u \cdot N = 0 \text{ on } \hat{\Gamma}, \quad [(Du)N]_{\tan} + \gamma_i u = 0 \text{ on } \hat{\Gamma}_i,\ i = 0,1. \tag{1.8}$$

When ε varies, so does the domain $\hat{\Omega}_\varepsilon$, and the friction coefficients appearing in (1.8) are

$$\gamma_0 = \gamma_{0,\varepsilon} \quad \text{and} \quad \gamma_1 = \gamma_{1,\varepsilon}.$$

In the case $\gamma_{0,\varepsilon}$ and $\gamma_{1,\varepsilon}$ are of order $O(\varepsilon)$ as $\varepsilon \to 0$, the method in [16, 11] can be applied to obtain similar results there. However, to cover a wider range of applications, we consider the following situation:

There is a non-negative number δ such that

$$\gamma_{0,\varepsilon} = O(\varepsilon^\delta) \quad \textit{and} \quad \gamma_{1,\varepsilon} = O(\varepsilon^\delta), \ \textit{as} \ \ \varepsilon \to 0. \tag{1.9}$$

We assume that there is a positive number c_0 such that

$$\gamma_{0,\varepsilon} \le c_0\varepsilon^\delta \quad \text{and} \quad \gamma_{1,\varepsilon} \le c_0\varepsilon^\delta, \quad \text{for all} \quad \varepsilon \in (0,1]. \tag{1.10}$$

For our convenience, we denote by Ω_ε the following domain

$$\Omega_\varepsilon = \big\{(x_1,x_2,x_3) : (x_1,x_2) \in Q_2, h_0(x_1,x_2) < x_3 < h_1(x_1,x_2)\big\}, \tag{1.11}$$

where $Q_2 = (0,1)^2$. The bottom and top boundaries of Ω_ε are denoted by Γ_0 and Γ_1 respectively, and let $\Gamma = \Gamma_0 \cup \Gamma_1$.

Let $L^2_{\rm per}(\hat\Omega_\varepsilon)$, resp. $H^k_{\rm per}(\hat\Omega_\varepsilon), k \ge 1$, be the closure with respect to the norm $\|\cdot\|_{L^2(\Omega_\varepsilon)}$, resp. $\|\cdot\|_{H^k(\Omega_\varepsilon)}$, of the set of all functions $\varphi \in C^\infty(\overline{\hat\Omega_\varepsilon})$ satisfying

$$\varphi(x + \mathbf{e}_j) = \varphi(x) \text{ for all } x \in \hat\Omega_\varepsilon, \ j = 1,2.$$

We use here the same approach as in [28, 16, 11] to study the Navier–Stokes equations with Navier boundary friction conditions. This approach requires the Korn inequality for the functional formulation of the problem (see Section 5 below). Therefore, one considers the following Helmholtz–Leray decomposition

$$L^2_{\rm per}(\hat\Omega_\varepsilon, \mathbb{R}^3) = \tilde H \oplus \tilde H^\perp = H \oplus H_0 \oplus \tilde H^\perp, \tag{1.12}$$

where

$$\tilde H = \big\{u \in L^2_{\rm per}(\hat\Omega_\varepsilon, \mathbb{R}^3) \colon u \text{ satisfies } \nabla\cdot u = 0 \text{ in } \hat\Omega_\varepsilon \text{ and } u\cdot N = 0 \text{ on } \hat\Gamma\big\}, \tag{1.13}$$

$$\tilde H^\perp = \big\{\nabla\phi : \phi \in H^1_{\rm per}(\hat\Omega_\varepsilon)\big\}, \tag{1.14}$$

$$H_0 = \big\{u \in H^1_{\rm per}(\hat\Omega_\varepsilon)^3 : Du = 0 \text{ in } \hat\Omega_\varepsilon \text{ and } u\cdot N = 0 \text{ on } \hat\Gamma\big\}. \tag{1.15}$$

It is shown in [11] that for our particular domains, one has

$$H_0 = \big\{a = (a_1,a_2,0) \in \mathbb{R}^3 : a_1\partial_1 g_i + a_2\partial_2 g_i = 0 \text{ in } \mathbb{R}^2, \text{ for } i = 0,1\big\}. \tag{1.16}$$

Moreover, the uniform Korn inequality on thin domains (see Lemma 2.7 below) is needed in our study. For that purpose, we define the following space Z_0 which contains H_0:

$$Z_0 = \big\{a = (a_1,a_2,0) \in \mathbb{R}^3 : a_1\partial_1 g + a_2\partial_2 g = 0 \text{ in } \mathbb{R}^2\big\}. \tag{1.17}$$

We assume throughout that

$$\big\{\nabla_2 g(x), \ x \in \mathbb{R}^2\big\} \ \ \textit{spans} \ \ \mathbb{R}^2. \tag{1.18}$$

One can see that the domains that satisfy condition (1.18) are *"generic"*. Further discussions on this condition are given in Remark 7.5.

It follows from (1.18) that

$$H_0 = Z_0 = \{0\} \quad \text{and} \quad H = \tilde H. \tag{1.19}$$

Let P denote the orthogonal projection from $L^2_{\mathrm{per}}(\hat{\Omega}_\varepsilon, \mathbb{R}^3)$ onto H. Let

$$V = H^1_{\mathrm{per}}(\hat{\Omega}_\varepsilon, \mathbb{R}^3) \cap H. \tag{1.20}$$

The Stokes operator A is defined as

$$Au = P(-\Delta u), \quad u \in D_A, \tag{1.21}$$

where D_A is the domain of A and is defined by

$$D_A = \big\{u \in H^2_{\mathrm{per}}(\hat{\Omega}_\varepsilon, \mathbb{R}^3) \cap V,\ u \text{ satisfies (1.8)}\big\}. \tag{1.22}$$

In the following we use $\|\cdot\|_{L^2}$, $\|\cdot\|_{H^1}$, …, to denote the L^2, H^1, … norms over Ω_ε. Also, $\langle\cdot,\cdot\rangle$ always denotes the inner product in $L^2(\Omega_\varepsilon)$. The notation $\|\cdot\|_{L^\infty L^2}$ means the norm in $L^\infty\big((0,\infty), L^2(\Omega_\varepsilon)\big)$, the notation ∇_2 denotes the two-dimensional (2D) gradient (∂_1, ∂_2), while ∇^2 denotes the tensor of all second order partial derivatives.

We denote by $M_0 = M_{0,\varepsilon}$ the average operator in the vertical direction (see (3.1)) and $\overline{M}u = (M_0u_1, M_0u_2, 0)$ for $u = (u_1, u_2, u_3)$.

We now state the main result of this paper.

Theorem 1.1 (Main Theorem). *Suppose*

$$3/4 \le \delta \le 1. \tag{1.23}$$

There are positive numbers ε_* and κ such that if $\varepsilon < \varepsilon_*$, $u_0 \in V$ and $f \in L^\infty\big((0,\infty), L^2_{\mathrm{per}}(\hat{\Omega}_\varepsilon)^3\big)$ satisfy that all of the quantities

$$\begin{aligned} U_0 &\stackrel{\mathrm{def}}{=} \|\overline{M}u_0\|^2_{L^2}, & U_1 &\stackrel{\mathrm{def}}{=} \varepsilon\|u_0\|^2_{H^1},\\ F_0 &\stackrel{\mathrm{def}}{=} \|\overline{M}Pf\|^2_{L^\infty L^2},\ \textit{and} & F_1 &\stackrel{\mathrm{def}}{=} \varepsilon\|Pf\|^2_{L^\infty L^2} \end{aligned} \tag{1.24}$$

are less than κ, then the strong solution $u(t) = u(t,\cdot)$ of the Navier–Stokes equations (1.2) *satisfying the Navier boundary conditions* (1.8) *with initial data $u(0,\cdot) = u_0$, exists for all $t \ge 0$. Moreover,*

$$\|u(t)\|^2_{L^2} \le c_1^*(\Lambda_1 e^{-2\alpha_0 t} + \Lambda_2), \tag{1.25}$$

and

$$\|u(t)\|^2_{H^1} \le c_2^*\varepsilon^{-1}(\Lambda_3 e^{-\alpha_0 t} + \Lambda_4), \tag{1.26}$$

for all $t \ge 0$, where

$$\Lambda_1 = U_0 + \varepsilon U_1, \quad \Lambda_2 = F_0 + \varepsilon F_1, \quad \Lambda_3 = U_0 + U_1, \quad \Lambda_4 = F_0 + F_1, \tag{1.27}$$

and the positive constants α_0, c_1^* and c_2^* are independent of ε, u_0, f.

We first note that $\overline{M}u_0$ is a 2D vector field, hence its norm $\|\overline{M}u_0\|_{L^2(\Omega_\varepsilon)}$ is of order $O(\varepsilon)$. Moreover, if u_0 is smooth, as usually assumed in practice, then $\|u_0\|_{H^1(\Omega_\varepsilon)} = O(\varepsilon)$ as well. Therefore, the conditions on U_0 and U_1, and similarly on F_0 and F_1, in Theorem 1.1 are not too strict. In fact, the norms $\|u_0\|_{H^1}$ and $\|Pf\|_{L^\infty L^2}$ are allowed to be large of order $O(\varepsilon^{-1/2})$ as $\varepsilon \to 0$.

The result we obtain is the same as that in [16, 11] though our boundary conditions are more involved. Because of the presence of friction coefficients with large sizes compared to the commonly assumed order $O(\varepsilon)$, the contribution of the boundary terms are non-trivial. For instance, $\|u\|_{H^2} \le C_\varepsilon \|Au\|_{L^2}$, where C_ε is not bounded as $\varepsilon \to 0$. However, our estimate in Proposition 5.5 below shows that under our assumptions, those contributions are manageable. Moreover, the non-linear estimate cannot be obtained by the same way as in [11]. Our estimate in Section 6 combines the approaches in [11] and [32]. Furthermore, by using a 2D-like estimate for the products of 2D and 3D functions (see [4]), we obtain the border case $\|u_0\|_{H^1(\Omega_\varepsilon)} = O(\varepsilon^q)$ with $q = -1/2$ which was missed in some other previous works (e.g. [31, 32]). On the other hand, our result assumes the smallness of U_0 and F_0 defined in (1.24), which is not required in other similar works on spherical domains with the free boundary condition [32] and on periodic domains [17]. This is due to the more involved boundary conditions on non-trivial boundaries which yield only the weak Poincaré inequality and trace estimates as well as cause the lack of orthogonality for the tri-linear terms.

This article is organized as follows. In Section 2, we recall without proofs various auxiliary inequalities which will be used throughout. In Section 3, we recall the definitions of averaging operators and their basic properties. The interpretation of the boundary conditions and their effects on different norm estimates are presented in Section 4. In Section 5, we establish some linear estimates related to the Stokes operator. In particular, we obtain the explicit estimate of $\|u\|_{H^2}$ in terms of ε, $\|Au\|_{L^2}$, and $\|u\|_{H^1}$. Furthermore, the identity in Lemma 5.2 which shows the contribution of the boundary terms in that estimate will also be used later in the estimate of the tri-linear term. We prove the main non-linear estimate in Section 6. This estimate depends heavily on the friction coefficients γ_0 and γ_1. In Section 7, we prove the Main Theorem 1.1 on the global existence of strong solutions of the Navier–Stokes equations. The Appendices contain the proofs of technical results used in other sections.

2. Auxiliary inequalities

We recall some auxiliary inequalities for thin domains (for the proofs, see e.g. [11]). First are Poincaré inequalities and trace theorems.

Lemma 2.1 (Poincaré–Trace I). *For $\phi \in H^1(\Omega_\varepsilon)$, one has*

$$\|\phi\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon^{1/2}\|\phi\|_{L^2(\Gamma_i)} + C\varepsilon\|\partial_3\phi\|_{L^2(\Omega_\varepsilon)}, \quad i = 0, 1, \tag{2.1}$$

$$\|\phi\|_{L^2(\Gamma)} \le C\varepsilon^{-1/2}\|\phi\|_{L^2(\Omega_\varepsilon)} + C\varepsilon^{1/2}\|\partial_3\phi\|_{L^2(\Omega_\varepsilon)}, \tag{2.2}$$

where $C > 0$ is independent of ε.

Lemma 2.2 (Poincaré–Trace II). *Let $\phi \in H^1(\Omega_\varepsilon)$ satisfy*

$$\int_{h_0(x_1,x_2)}^{h_1(x_1,x_2)} \phi(x_1,x_2,y_3)dy_3 = 0, \quad (x_1,x_2) \in (0,1)^2. \tag{2.3}$$

Then

$$\|\phi\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \|\partial_3 \phi\|_{L^2(\Omega_\varepsilon)}, \tag{2.4}$$

$$\|\phi\|_{L^2(\Gamma)} \leq C\varepsilon^{1/2} \|\partial_3 \phi\|_{L^2(\Omega_\varepsilon)}, \tag{2.5}$$

where $C > 0$ is independent of ε.

The following Poincaré-like inequality is proved in [11].

Lemma 2.3. *If u is in $H^1_{\mathrm{per}}(\hat{\Omega}_\varepsilon)^3$ and satisfies* (1.4) *on $\hat{\Gamma}$, then*

$$\|u_3\|_{L^2} \leq C\varepsilon \|u\|_{H^1}, \tag{2.6}$$

where $C > 0$ does not depend on ε.

The following are the Ladyzhenskaya and Agmon inequalities for our thin domains (for the proofs, see e.g. [31]).

Lemma 2.4 (Ladyzhenskaya inequalities). (i) *Let $\phi \in H^1(\Omega_\varepsilon)$ be independent of the third variable, then*

$$\|\phi\|_{L^4(\Gamma)} \leq C\varepsilon^{-1/2} \|\phi\|_{L^2}^{1/2} \|\phi\|_{H^1}^{1/2}, \tag{2.7}$$

$$\|\phi\|_{L^4} \leq C\varepsilon^{-1/4} \|\phi\|_{L^2}^{1/2} \|\phi\|_{H^1}^{1/2}. \tag{2.8}$$

(ii) *Suppose $\phi \in H^1(\Omega_\varepsilon)$, then*

$$\|\phi\|_{L^6} \leq C\varepsilon^{-1/6} \|\phi\|_{H^1}^{2/3} \Big\{ \varepsilon^{-1/2} \|\phi\|_{L^2} + \varepsilon^{1/2} \|\partial_3 \phi\|_{L^2} \Big\}^{1/3}. \tag{2.9}$$

Assume additionally that $\int_{h_0(x')}^{h_1(x')} \phi(x',y_3)dy_3 = 0$ for all $x' \in (0,1)^2$, then one has

$$\|\phi\|_{L^6} \leq C\|\phi\|_{H^1}, \tag{2.10}$$

and the interpolating inequality

$$\|\phi\|_{L^r} \leq C\varepsilon^{3/r-1/2} \|\phi\|_{H^1} \text{ for } r \in [2,6]. \tag{2.11}$$

Lemma 2.5 (Agmon inequalities). *Suppose $\phi \in H^2(\Omega_\varepsilon)$, then*

$$\|\phi\|_{L^\infty} \leq C\varepsilon^{-1/2} \|\phi\|_{L^2}^{1/4} \|\phi\|_{H^2}^{1/2} \Big\{ \|\phi\|_{L^2} + \varepsilon \|\partial_3 \phi\|_{L^2} + \varepsilon^2 \|\partial_3 \partial_3 \phi\|_{L^2} \Big\}^{1/4}. \tag{2.12}$$

If, in addition, $\int_{h_0(x')}^{h_1(x')} \phi(x',y_3)dy_3 = 0$ for all $x' \in (0,1)^2$, then

$$\|\phi\|_{L^\infty} \leq C\|\partial_3 \phi\|_{L^2}^{1/4} \|\phi\|_{H^2}^{1/2} (\|\partial_3 \phi\|_{L^2} + \varepsilon \|\partial_3 \partial_3 \phi\|_{L^2})^{1/4}. \tag{2.13}$$

In our study of the Stokes operator and the functional formulation of the Navier–Stokes equations below, we need the Korn inequality (see e.g. [28, 11]).

Lemma 2.6 (Korn inequality). *For each $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that one has*

$$\|Du\|_{L^2} \le \|u\|_{H^1} \le C_\varepsilon \|Du\|_{L^2}, \tag{2.14}$$

for any $u \in H^1_{\rm per}(\hat\Omega_\varepsilon)^3 \cap H_0^\perp$ that satisfies (1.4) on $\hat\Gamma$.

Note that the constant C_ε in (2.14) may depend on ε and may blow up when $\varepsilon \to 0$. However, with further restrictions on u, those constants can be bounded uniformly when ε is sufficiently small as in the following uniform Korn inequality (for the proof, see [11]).

Lemma 2.7 (Uniform Korn inequality). *There is $\varepsilon_0 \in (0, 1]$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and $u \in H^1_{\rm per}(\hat\Omega_\varepsilon)^3 \cap Z_0^\perp$ satisfying (1.4) on $\hat\Gamma$, one has*

$$\|Du\|_{L^2} \le \|u\|_{H^1} \le c_1 \|Du\|_{L^2}, \tag{2.15}$$

where c_1 is a positive constant independent of ε.

One can find examples of thin domains which do not satisfy (1.18) and the uniform Korn inequality (2.15) fails for $u \in H^1_{\rm per}(\hat\Omega_\varepsilon)^3 \cap (H_0^\perp \setminus Z_0^\perp)$.

However, under the assumption (1.18), the spaces H_0 and Z_0 are trivial, therefore the uniform Korn inequality (2.15) holds for all $u \in H^1_{\rm per}(\hat\Omega_\varepsilon)^3$ satisfying (1.4) on the boundary $\hat\Gamma$.

3. Averaging operator

As in [16, 11], we will use the following averaging operators M_0 and M in our analysis to take into account the thinness of the domain. Their properties are presented below. The proofs of those properties can be found in [11].

First, we define an averaging operator M_0 on $L^2_{\rm per}(\hat\Omega_\varepsilon)$ by

$$M_0\phi(x_1, x_2) = \frac{1}{\varepsilon g(x_1, x_2)} \int_{h_0(x_1,x_2)}^{h_1(x_1,x_2)} \phi(x_1, x_2, y_3) dy_3, \quad (x_1, x_2) \in \mathbb{R}^2, \tag{3.1}$$

for $\phi \in L^2_{\rm per}(\hat\Omega_\varepsilon)$. Then $M_0\phi$ is independent of x_3 and one can verify that M_0 is an orthogonal projection on $L^2_{\rm per}(\hat\Omega_\varepsilon)$. Hence

$$\|M_0\phi\|^2_{L^2} + \|\phi - M_0\phi\|^2_{L^2} = \|\phi\|^2_{L^2}. \tag{3.2}$$

For our convenience, we denote

$$\begin{aligned}\psi = (\psi_1, \psi_2) &= \nabla_2 h_0 + \frac{x_3 - h_0}{g}\nabla_2 g = \nabla_2 h_1 - \frac{h_1 - x_3}{g}\nabla_2 g \\ &= \frac{1}{\varepsilon g}\big\{(x_3 - h_0)\nabla_2 h_1 + (h_1 - x_3)\nabla_2 h_0\big\}.\end{aligned} \tag{3.3}$$

Note that

$$\psi\big|_{\hat\Gamma_i} = \nabla_2 h_i, \quad i = 0, 1, \quad \partial_3\psi = (1/g)\nabla_2 g, \tag{3.4}$$

$$|\psi| \le C\varepsilon, \quad |\partial_3\psi| \le C, \quad |\nabla_2\psi| \le C\varepsilon, \quad |\nabla^2\psi| \le C. \tag{3.5}$$

Lemma 3.1. *Let $\phi \in H^1_{\mathrm{per}}(\hat\Omega_\varepsilon)$ and $j = 1, 2$, then*

$$\partial_j M_0\phi = M_0(\partial_j\phi) + M_0(\psi_j\partial_3\phi). \tag{3.6}$$

Consequently, for $m = 1, 2$, and $\phi \in H^m_{\mathrm{per}}(\hat\Omega_\varepsilon)$,

$$\|M_0\phi\|_{H^m(\Omega_\varepsilon)} \le C(m)\|\phi\|_{H^m(\Omega_\varepsilon)}, \tag{3.7}$$

where positive number $C(m)$ is independent of ε.

Let $u \in \tilde H$ and $\bar v = (M_0u_1, M_0u_2)$. Then

$$\nabla_2 \cdot \bar v = -\bar v \cdot \frac{1}{g}\nabla_2 g. \tag{3.8}$$

Note that $M_0u = (M_0u_1, M_0u_2, M_0u_3)$ does not necessarily satisfy the slip condition (1.4) on $\hat\Gamma$ even when $u = (u_1, u_2, u_3)$ does. For that reason, as in [11] we define the following operator M which is more suitable for our study.

For $u = (u_1, u_2, u_3) \in L^2_{\mathrm{per}}(\hat\Omega_\varepsilon)^3$, let $\bar v = (M_0u_1, M_0u_2)$ and

$$Mu = (\bar v, \bar v\cdot\psi) = (M_0u_1, M_0u_2, \psi_1 M_0u_1 + \psi_2 M_0u_2). \tag{3.9}$$

Thanks to (3.9), (3.8), (3.4) and Lemma 3.1, one has

Lemma 3.2. *If u is in $\tilde H$, so is Mu.*

If $u \in H^1_{\mathrm{per}}(\hat\Omega_\varepsilon)^3$ then $v = Mu = (\bar v, v_3)$ satisfies

$$|v_3| \le C\varepsilon|\bar v|, \quad |\partial_3 v_3| \le C|\bar v|, \quad |\nabla_2 v_3| \le C\varepsilon\big(|\bar v| + |\nabla_2\bar v|\big) \quad \text{in } \hat\Omega_\varepsilon; \tag{3.10}$$

consequently,

$$|v| \le C|\bar v|, \quad |\nabla v| \le C\big(|\bar v| + |\nabla_2\bar v|\big) \quad \text{in } \hat\Omega_\varepsilon. \tag{3.11}$$

Combining with Lemma 3.1, we have

Lemma 3.3. *Let $m = 0, 1, 2$, and $u \in H^m_{\mathrm{per}}(\hat\Omega_\varepsilon)$, then*

$$\|Mu\|_{H^m(\Omega_\varepsilon)} \le C(m)\|u\|_{H^m(\Omega_\varepsilon)}. \tag{3.12}$$

Let $u \in H^1_{\rm per}(\hat\Omega_\varepsilon)^3$ and $v = Mu = (\bar v, v_3)$. Though v depends on x_3, one still has the following Ladyzhenskaya-type inequalities.

$$\|\bar v\|_{L^4} \le C\varepsilon^{-1/4}\|\bar v\|_{L^2}^{1/2}\|\bar v\|_{H_1}^{1/2} \le C\varepsilon^{-1/4}\|u\|_{L^2}^{1/2}\|u\|_{H_1}^{1/2}, \tag{3.13}$$

$$\|v_3\|_{L^4} \le C\varepsilon\|\bar v\|_{L^4} \le C\varepsilon^{3/4}\|u\|_{L^2}^{1/2}\|u\|_{H_1}^{1/2}. \tag{3.14}$$

For $u \in H^2_{\rm per}(\hat\Omega_\varepsilon)^3$,

$$\|\nabla_2 \bar v\|_{L^4} \le C\varepsilon^{-1/4}\|u\|_{H^1}^{1/2}\|u\|_{H_2}^{1/2}, \tag{3.15}$$

$$\|\nabla_2 v_3\|_{L^4} \le C\varepsilon\|\nabla_2 \bar v\|_{L^4} \le C\varepsilon^{3/4}\|u\|_{H^1}^{1/2}\|u\|_{H_2}^{1/2}, \tag{3.16}$$

$$\|\partial_3 v_3\|_{L^4} \le C\|\bar v\|_{L^4} \le C\varepsilon^{-1/4}\|u\|_{L^2}^{1/2}\|u\|_{H_1}^{1/2}. \tag{3.17}$$

Summing up, we obtain

Lemma 3.4. *One has*

$$\|Mu\|_{L^4} \le C\varepsilon^{-1/4}\|u\|_{L^2}^{1/2}\|u\|_{H^1}^{1/2}, \quad \text{for } u \in H^1_{\rm per}(\hat\Omega_\varepsilon)^3, \tag{3.18}$$

$$\|\nabla(Mu)\|_{L^4} \le C\varepsilon^{-1/4}\|u\|_{H^1}^{1/2}\|u\|_{H^2}^{1/2}, \quad \text{for } u \in H^2_{\rm per}(\hat\Omega_\varepsilon)^3. \tag{3.19}$$

Next, we need the estimates for $w = u - Mu$. We already established the Poincaré inequality for w in [11]:

Lemma 3.5. *Let u be in $H^1_{\rm per}(\hat\Omega_\varepsilon)^3$ and satisfy boundary condition* (1.4)*. Let $w = u - Mu$, then*

$$\|w\|_{L^2} \le C\varepsilon\|\nabla w\|_{L^2}. \tag{3.20}$$

4. Navier friction boundary conditions

The following lemma shows the properties of the vector fields satisfying the Navier friction boundary conditions (see [3, 9, 11]). Note that, in this paper, we denote by $\partial/\partial\tau$ the Gateaux derivative with respect to vector $\tau \in \mathbb{R}^3$ not necessarily having unit length.

Lemma 4.1. *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^3$ such that $\Gamma_* = \partial\Omega \cap \mathcal{O} \ne \emptyset$. Let u belong to $C^1(\overline{\Omega} \cap \mathcal{O}, \mathbb{R}^3)$ and satisfy* (1.4) *on Γ_*. Suppose τ is a tangential vector field on Γ_*. Then*

$$\frac{\partial u}{\partial \tau}\cdot N + \frac{\partial N}{\partial \tau}\cdot u = 0 \quad on \quad \Gamma_*\,. \tag{4.1}$$

If, in addition, u satisfies (1.5) *on Γ_*, and $\check N \in C^1(\overline{\Omega} \cap \mathcal{O}, \mathbb{R}^3)$ such that $\check N\big|_{\Gamma_*} = \pm N$, then one has*

$$\frac{\partial u}{\partial N}\cdot\tau = u\cdot\left\{\frac{\partial N}{\partial \tau} - 2\gamma\tau\right\} \quad on \quad \Gamma_*, \tag{4.2}$$

and

$$N \times (\nabla \times u) = 2N \times \left\{ \check{N} \times \left((\nabla \check{N})^* u \right) - \gamma N \times u \right\} \quad on \quad \Gamma_*. \tag{4.3}$$

Proof. Taking the derivative $\partial/\partial\tau$ of (1.4) we obtain (4.1).

On Γ_*, the relation $[(Du)N] \cdot \tau = -\gamma u \cdot \tau$ yields

$$(\nabla u)N \cdot \tau + (\nabla u)\tau \cdot N = -2\gamma u \cdot \tau,$$

hence

$$\frac{\partial u}{\partial N} \cdot \tau + \frac{\partial u}{\partial \tau} \cdot N = -2\gamma u \cdot \tau. \tag{4.4}$$

Therefore (4.2) follows from (4.1) and (4.4).

Identity (4.23) is similar to Proposition 2 of [3]. We follow the proof presented there. Let $\omega = \nabla \times u$. Suppose $\check{N}\big|_{\Gamma_*} = \sigma N$, where $\sigma = \pm 1$. From the identity $\check{N} \times \nabla(u \cdot \check{N}) = 0$ on Γ_*, we have

$$\begin{aligned} 0 &= \check{N} \times [(\nabla u)^* \check{N}] + \check{N} \times [(\nabla \check{N})^* u] \\ &= \check{N} \times [(Du)\check{N} - (Ku)\check{N}] + \check{N} \times [(\nabla \check{N})^* u], \end{aligned}$$

where $Ku = \frac{1}{2}\{\nabla u - (\nabla u)^*\}$. Since $(Du)\check{N} = \lambda N + [(Du)\check{N}]_{\tan} = \lambda N - \sigma\gamma u$, for some $\lambda \in \mathbb{R}$, and $(Ku)\check{N} = \frac{1}{2}\omega \times \check{N}$, we thus have

$$0 = -\gamma\sigma\check{N} \times u - \frac{1}{2}\check{N} \times (\omega \times \check{N}) + \check{N} \times [(\nabla \check{N})^* u].$$

Therefore

$$N \times (\omega \times N) = \check{N} \times (\omega \times \check{N}) = 2\check{N} \times [(\nabla \check{N})^* u] - 2\gamma N \times u. \tag{4.5}$$

Applying $(N\times)$ to (4.5) and using the identity

$$a \times (a \times (a \times b))) = -|a|^2 (a \times b),$$

one obtains (4.3). $\square$

Our study below requires specific extensions of N to a neighborhood of each $\hat{\Gamma}_i$ or to the whole domain $\hat{\Omega}_\varepsilon$. Therefore we let

$$\hat{N}^i = (-1)^i (\partial_1 h_i, \partial_2 h_i, -1), \qquad i = 0, 1, \tag{4.6}$$

$$N^i = \hat{N}^i / |\hat{N}^i| = (-1)^i (\partial_1 h_i, \partial_2 h_i, -1) / \sqrt{1 + |\nabla_2 h_i|^2}, \qquad i = 0, 1. \tag{4.7}$$

Then N^i, for $i = 0, 1$, is an extension of the outward normal vector field on $\hat{\Gamma}_i$ to the whole space.

For specific tangential vector fields on $\hat{\Gamma}$ and their extensions, we specifically let

$$\tau_j^i = (\mathbf{e}_j + \partial_j h_i \mathbf{e}_3) \Big/ \sqrt{1 + |\partial_j h_i|^2}, \quad i = 0, 1, \quad j = 1, 2. \tag{4.8}$$

On each $\hat{\Gamma}_i$, $i = 0, 1$, the two vectors τ_1^i and τ_2^i form a basis of the tangent space. From (4.6) and (4.7), we have the following estimates in $\hat{\Omega}_\varepsilon$

$$|\nabla N^i|, \, |\nabla \tau_j^i|, \, |\mathbf{e}_3 - N^1|, \, |\mathbf{e}_3 + N^0|, \, |\mathbf{e}_j - \tau_j^i| \le C\varepsilon. \tag{4.9}$$

Lemma 4.2. *Let u belong to $H^2_{\mathrm{per}}(\Omega_\varepsilon)^3$ satisfying* (1.8). *We have*

$$\|\partial_3 u_j\|_{L^2} \le C\varepsilon\|u\|_{H^2} + C\varepsilon^\delta\|u\|_{L^2}, \quad j = 1, 2, \tag{4.10}$$

where $C > 0$ is independent of ε.

Proof. We prove the inequality for the case $j = 1$. Using (2.1), we have

$$\|\partial_3 u_1\|_{L^2} \le C\sqrt{\varepsilon}\|\partial_3 u_1\|_{L^2(\Gamma_0)} + C\varepsilon\|\partial_3\partial_3 u_1\|_{L^2}. \tag{4.11}$$

We write on Γ_0:

$$\begin{aligned}\partial_3 u_1 &= (\nabla u_1)\cdot(\mathbf{e}_3 + N^0) - (\nabla u_1)\cdot N^0\\ &= (\nabla u_1)\cdot(\mathbf{e}_3 + N^0) - \big((\nabla u)^*\mathbf{e}_1\big)\cdot N^0\\ &= (\nabla u_1)\cdot(\mathbf{e}_3 + N^0) - \big((\nabla u)^*(\mathbf{e}_1 - \tau_1^0)\big)\cdot N^0 - \big((\nabla u)^*\tau_1^0\big)\cdot N^0\\ &= (\nabla u_1)\cdot(\mathbf{e}_3 + N^0) - \big((\nabla u)^*(\mathbf{e}_1 - \tau_1^0)\big)\cdot N^0 - \big((\nabla u)N^0\big)\cdot\tau_1^0.\end{aligned}$$

In Lemma 4.1, let $\Gamma_* = \Gamma_0$ and $\tau = \tau_1^0$. Note that $N = N^0$ in (4.2), and we have $\big((\nabla u)N^0\big)\cdot\tau_1^0 = \big((\nabla N^0)\tau_1^0\big)\cdot u - 2\gamma_0\tau_1^0\cdot u$ on Γ_0. Then thanks to (4.9), we obtain $|\partial_3 u_1| \le C\varepsilon|\nabla u| + C(\varepsilon + \gamma_0)|u|$ on Γ_0. Together with (4.11) and (2.2),

$$\begin{aligned}\|\partial_3 u_1\|_{L^2} &\le C\sqrt{\varepsilon}\left\{\varepsilon\left(\frac{1}{\sqrt{\varepsilon}}\|u\|_{H^1} + \sqrt{\varepsilon}\|u\|_{H^2}\right) + (\varepsilon + \gamma_0)\left(\frac{1}{\sqrt{\varepsilon}}\|u\|_{L^2} + \sqrt{\varepsilon}\|u\|_{H^1}\right)\right\}\\ &\quad + C\varepsilon\|\nabla^2 u\|_{L^2} \le C\varepsilon\|u\|_{H^2} + C(\varepsilon + \gamma_0)\|u\|_{L^2}.\end{aligned}$$

Using (1.9) we obtain (4.10) for $j = 1$. The case $j = 2$ is treated similarly. □

Connecting with the averaging operator M, we have the following Poincaré-like inequality.

Lemma 4.3. *Let $u \in D_A$ and $w = u - Mu$, then*

$$\|\nabla w\|_{L^2} \le C\varepsilon\|u\|_{H^2} + C\varepsilon^\delta\|u\|_{L^2}. \tag{4.12}$$

Proof. Let $i, j \in \{1, 2\}$. We estimate $\partial_i w_j$. Using Lemma 3.1,

$$\partial_i w_j = \partial_i u_j - M(\partial_i u_j) - M(\psi_i\partial_3 u_j).$$

By (2.4), (3.5) and (3.2),

$$\|\partial_i w_j\|_{L^2} \le C\varepsilon\|\partial_3\partial_i u_j\|_{L^2} + C\varepsilon\|\partial_3 u_j\|_{L^2} \le C\varepsilon\|u\|_{H^2}.$$

Since $\partial_3 w_3 = -\partial_1 w_1 - \partial_2 w_2$, we have $\|\partial_3 w_3\|_{L^2} \le C\varepsilon\|u\|_{H^2}$ as well.
For $\partial_i w_3$, $i = 1, 2$, we have

$$\|\partial_i w_3\| \le C\sqrt{\varepsilon}\|\partial_i w_3\|_{L^2(\Gamma_1)} + C\varepsilon\|\partial_3\partial_i w_3\|_{L^2}. \tag{4.13}$$

One has on Γ_1:

$$\begin{aligned}
\partial_i w_3 &= (\nabla w_3)\cdot \mathbf{e}_i = (\nabla w_3)\cdot(\mathbf{e}_i - \tau_i^1) + (\nabla w_3)\cdot \tau_i^1\\
&= (\nabla w_3)\cdot(\mathbf{e}_i - \tau_i^1) + (\nabla w)^*\mathbf{e}_3\cdot \tau_i^1\\
&= (\nabla w_3)\cdot(\mathbf{e}_i - \tau_i^1) + (\nabla w)^*(\mathbf{e}_3 - N^1)\cdot \tau_i^1 + (\nabla w)^* N^1\cdot \tau_i^1\\
&= (\nabla w_3)\cdot(\mathbf{e}_i - \tau_i^1) + (\nabla w)^*(\mathbf{e}_3 - N^1)\cdot \tau_i^1 + N^1\cdot(\nabla w)\tau_i^1.
\end{aligned}$$

The last term equals $\{-w\cdot(\nabla N^1)\tau_i^1\}$, thanks to (4.1). Using (4.9) one obtains

$$|\partial_i w_3| \le C\varepsilon(|\nabla w| + |w|) \quad \text{on} \quad \Gamma_1. \tag{4.14}$$

It then follows from (2.2) that

$$\begin{aligned}
\sqrt{\varepsilon}\|\partial_i w_3\|_{L^2(\Gamma_1)} &\le C\varepsilon^{3/2}\big(\|\nabla w\|_{L^2(\Gamma_1)} + \|w\|_{L^2(\Gamma_1)}\big)\\
&\le C\varepsilon\|w\|_{H^1} + C\varepsilon^2\big(\|\nabla w\|_{L^2} + \|\nabla^2 w\|_{L^2}\big)\\
&\le C\varepsilon\|u\|_{H^2}.
\end{aligned}$$

Combining this estimate with (4.13) yields $\|\partial_i w_3\|_{L^2} \le C\varepsilon\|u\|_{H^2}$.

Finally, by Lemma 4.2, $\|\partial_3 w_j\|_{L^2} = \|\partial_3 u_j\|_{L^2} \le C\big(\varepsilon\|u\|_{H^2} + \varepsilon^\delta\|u\|_{L^2}\big)$. Summing up the above estimates for $\|\partial_j w_i\|_{L^2}$, for $i, j = 1, 2, 3$, we obtain (4.12). The proof is complete. □

Lemma 4.4 (Sobolev inequalities). *Let $u \in V$ and $w = u - Mu$. One has*

$$\|w\|_{L^6} \le C\|w\|_{H^1}. \tag{4.15}$$

If $u \in D_A$, then

$$\|\nabla w\|_{L^6} \le C\|u\|_{H^2} + C\varepsilon^{-1+\delta}\|u\|_{L^2}. \tag{4.16}$$

Proof. Using (2.9) with $\phi = w = u - Mu$ combined with (3.20), we obtain (4.15).

Assume $u \in D_A$, then combining (2.9) with $\phi = \partial_i w_j$ and (4.12) yields

$$\begin{aligned}
\|\nabla w\|_{L^6} &\le C\varepsilon^{-1/6}\|\nabla w\|_{H^1}^{2/3}\Big\{\varepsilon^{-1/2}\|\nabla w\|_{L^2} + \varepsilon^{1/2}\|w\|_{H^2}\Big\}^{1/3}\\
&\le C\varepsilon^{-1/6}\|w\|_{H^2}^{2/3}\Big\{\varepsilon^{1/2}\|u\|_{H^2} + \varepsilon^{-1/2+\delta}\|u\|_{L^2}\Big\}^{1/3}.
\end{aligned}$$

Therefore

$$\|\nabla w\|_{L^6} \le C\|u\|_{H^2} + C\varepsilon^{-1/3+\delta/3}\|u\|_{L^2}^{1/3}\|u\|_{H^2}^{2/3}. \tag{4.17}$$

Applying Hölder's inequality yields (4.16). □

Lemma 4.5 (Agmon inequality). *Let $u \in D_A$ and $w = u - Mu$, we have*

$$\|w\|_{L^\infty} \le C\varepsilon^{1/2}\|u\|_{H^2} + C\varepsilon^{\delta/2}\|u\|_{L^2}^{1/2}\|u\|_{H^2}^{1/2}. \tag{4.18}$$

Proof. Using (2.12), (3.20) and (4.12), we obtain

$$\|w\|_{L^\infty} \le C\varepsilon^{-1/2}\|w\|_{L^2}^{1/4}\|w\|_{H^2}^{1/2}\Big\{\|w\|_{L^2} + \varepsilon\|\partial_3 w\|_{L^2} + \varepsilon^2\|\partial_3\partial_3 w\|_{L^2}\Big\}^{1/4}$$

$$\leq C\varepsilon^{-1/2}\Big\{\varepsilon^{1+\delta}\|u\|_{L^2}+\varepsilon^2\|u\|_{H^2}\Big\}^{1/4}\|u\|_{H^2}^{1/2}$$
$$\times\Big\{\varepsilon^{1+\delta}\|u\|_{L^2}+\varepsilon^2\|u\|_{H^2}+\varepsilon\big(\varepsilon^{\delta}\|u\|_{L^2}+\varepsilon\|u\|_{H^2}\big)+\varepsilon^2\|\partial_3\partial_3 w\|_{L^2}\Big\}^{1/4}$$
$$\leq C\varepsilon^{-1/2}\Big\{\varepsilon^{1+\delta}\|u\|_{L^2}+\varepsilon^2\|u\|_{H^2}\Big\}^{1/2}\|u\|_{H^2}^{1/2}$$
$$\leq C\|u\|_{H^2}^{1/2}\Big\{\varepsilon^{\delta}\|u\|_{L^2}+\varepsilon\|u\|_{H^2}\}^{1/2}.$$

Hence we obtain (4.18). □

The following is an useful extension of the upward normal vectors on the boundary. It will be used in the estimates obtained in Sections 5 and 6 below. Let

$$\tilde N^i=(-\partial_1 h_i,-\partial_2 h_i,1)/\sqrt{1+|\nabla_2 h_i|^2},\quad i=0,1. \tag{4.19}$$

$$\tilde N=\frac{x_3-h_0}{\varepsilon g}\tilde N^1+\frac{h_1-x_3}{\varepsilon g}\tilde N^0=\tilde N^0+\frac{x_3-h_0}{\varepsilon g}(\tilde N^1-\tilde N^0). \tag{4.20}$$

Then $\tilde N^0\big|_{\hat\Gamma_0}=\tilde N\big|_{\hat\Gamma_0}=-N$ and $\tilde N\big|_{\hat\Gamma_1}=\tilde N^1\big|_{\hat\Gamma_1}=N$. We also have

$$|\tilde N_j|,\ |\partial_j\tilde N|\leq C\varepsilon,\quad \text{for } j=1,2,\quad |\tilde N_3|,\ |\partial_3\tilde N|,\ |\nabla^2\tilde N|\leq C, \tag{4.21}$$

$$|\tilde N^0-\tilde N^1|\leq C\varepsilon. \tag{4.22}$$

Using the identity (4.3) with $\check N=-N^0$ on $\Gamma_*=\Gamma_0$ and $\check N=N^1$ on $\Gamma_*=\Gamma_1$, one derives

$$N\times(\nabla\times u)=N\times G(u)\quad\text{on}\quad\hat\Gamma, \tag{4.23}$$

where vector field $G(u)$ is defined on $\hat\Omega_\varepsilon$ by

$$G(u)=G^{(1)}(u)-G^{(2)}(u), \tag{4.24}$$

where

$$G^{(1)}(u)=2\tilde N\times[(\nabla\tilde N)^*u]\quad\text{and}\quad G^{(2)}(u)=2\tilde{\tilde N}\times u, \tag{4.25}$$

with

$$\tilde{\tilde N}=\frac{h_1-x_3}{\varepsilon g}\gamma_0\tilde N^0+\frac{x_3-h_0}{\varepsilon g}\gamma_1\tilde N^1. \tag{4.26}$$

Note that

$$G^{(1)}(u)=2\tilde N\times\big[(\nabla\tilde N)^*u\big]=\sum_{m=1}^{3}u_m G_m^{(1)}, \tag{4.27}$$

where

$$G_m^{(1)}=(\tilde N\times\nabla)\tilde N_m=\sum_{i,j,k=1}^{3}\mathbf{e}_i\varepsilon_{ijk}\tilde N_j\partial_k\tilde N_m. \tag{4.28}$$

Hence, as in [16, 11], we have

$$|G^{(1)}(u)|\leq C\varepsilon|u|,\quad |\nabla G^{(1)}(u)|\leq C\varepsilon|\nabla u|+C|u|. \tag{4.29}$$

Also, $\tilde{\tilde{N}}\big|_{\Gamma_i} = \gamma_i N$ for $i = 0, 1$ and

$$|\tilde{\tilde{N}}| \le C(\gamma_0 + \gamma_1) \le C\varepsilon^\delta, \quad |\nabla_2 \tilde{\tilde{N}}| \le C\varepsilon^\delta, \quad |\partial_3 \tilde{\tilde{N}}| \le C\varepsilon^{\delta-1}. \tag{4.30}$$

Therefore

$$|\nabla \tilde{\tilde{N}}| \le C\varepsilon^{\delta-1}. \tag{4.31}$$

Consequently, one obtains

$$|G^{(2)}(u)| \le 2|\tilde{\tilde{N}}|\,|u| \le C\varepsilon^\delta |u|, \tag{4.32}$$

$$|\nabla G^{(2)}(u)| \le C(|\nabla u|\,|\tilde{\tilde{N}}| + |u|\,|\nabla \tilde{\tilde{N}}|) \le C\varepsilon^\delta |\nabla u| + C\varepsilon^{\delta-1}|u|. \tag{4.33}$$

Combining (4.24), (4.29), (4.32) and (4.33) yields

Lemma 4.6. *Let $u \in H^1_{\mathrm{per}}(\hat{\Omega}_\varepsilon)^3$ and $G(u)$ be defined by (4.24). We have the following estimates in $\hat{\Omega}_\varepsilon$:*

$$|G(u)| \le C\varepsilon^\delta |u|, \tag{4.34}$$

$$|\nabla G(u)| \le C\varepsilon^\delta |\nabla u| + C\varepsilon^{\delta-1}|u|. \tag{4.35}$$

5. The Stokes operator

For $u \in H^2_{\mathrm{per}}(\Omega_\varepsilon)$ and $v \in H^1_{\mathrm{per}}(\Omega_\varepsilon)$, we have the following Green's formula (see [28, 16]):

$$\begin{aligned} \int_{\Omega_\varepsilon} \Delta u \cdot v dx = \int_{\Omega_\varepsilon} &-2(Du : Dv) + (\nabla \cdot u)(\nabla \cdot v) dx \\ &+ \int_{\partial\Omega_\varepsilon} \big\{2((Du)N)\cdot v - (\nabla \cdot u)(v \cdot N)\big\} d\sigma, \end{aligned} \tag{5.1}$$

where $Du : Dv$ denotes the usual scalar product of the two matrices.

If, in addition, u and v are divergence-free and u satisfies the friction boundary condition (1.8) then

$$-\int_{\Omega_\varepsilon} \Delta u \cdot v dx = 2\int_{\Omega_\varepsilon} (Du : Dv) dx + 2\gamma_0 \int_{\Gamma_0} u \cdot v d\sigma + 2\gamma_1 \int_{\Gamma_1} u \cdot v d\sigma. \tag{5.2}$$

By the Korn inequality (2.14) and the trace estimate (2.2), the bi-linear form

$$E(u, v) = 2\int_{\Omega_\varepsilon} Du : Dv dx + 2\gamma_0 \int_{\Gamma_0} u \cdot v d\sigma + 2\gamma_1 \int_{\Gamma_1} u \cdot v d\sigma.$$

is bounded and coercive on V. Hence there is a bounded operator $A : V \to V'$ such that

$$\langle Au, v \rangle_{V',V} = E(u, v), \quad u, v \in V.$$

We consider A as an unbounded operator on H. Then the domain of A is a subspace D_A such that $Au \in H$ for all $u \in D_A$. By the regularity theory for the

Stokes problem (see e.g. [18, 6, 8]), we obtain that the above definitions of A and D_A are the same as those given by (1.21) and (1.22), respectively. We also have V given in (1.20) is the domain of the fractional operator $A^{\frac{1}{2}}$.

For the nonlinear terms of the Navier–Stokes equations one defines a bi-linear form $B(\cdot,\cdot)$ from $V\times V$ to the dual space V', such that

$$\langle B(u,v),w\rangle_{V',V}=\langle (u\cdot\nabla)v,w\rangle,\quad \text{for all } w\in V. \tag{5.3}$$

The functional formulation of the Navier–Stokes equations is

$$\frac{d}{dt}\langle u,v\rangle+E(u,v)+\langle (u\cdot\nabla)u,v\rangle=\langle f,v\rangle,\quad \text{for all } v\in V, \tag{5.4}$$

or equivalently,

$$\frac{du}{dt}+Au+B(u,u)=Pf, \tag{5.5}$$

in the space V', (for more details, see e.g. [29]).

For $u\in D_A, v\in V$, we have

$$\langle Au,v\rangle=2\langle Du,Dv\rangle+2\gamma_0\langle u,v\rangle_{\Gamma_0}+2\gamma_1\langle u,v\rangle_{\Gamma_1}. \tag{5.6}$$

Also, for $u\in V$,

$$\|A^{\frac{1}{2}}u\|_{L^2}^2=2\langle Du,Du\rangle+2\gamma_0\langle u,u\rangle_{\Gamma_0}+2\gamma_1\langle u,u\rangle_{\Gamma_1}. \tag{5.7}$$

Lemma 5.1. *There are positive numbers c_2 and c_3 such that for all $\varepsilon\in(0,\varepsilon_0]$, one has the following:*

(i) *If $u\in V$ then*

$$\|u\|_{L^2}\le\|u\|_{H^1}\le c_2\|A^{\frac{1}{2}}u\|_{L^2}, \tag{5.8}$$

$$\|A^{\frac{1}{2}}u\|_{L^2}\le c_3\big(\|\nabla u\|_{L^2}+\varepsilon^{(\delta-1)/2}\|u\|_{L^2}\big). \tag{5.9}$$

(ii) *If $u\in D_A$ then*

$$\|A^{\frac{1}{2}}u\|_{L^2}\le c_2\|Au\|_{L^2}. \tag{5.10}$$

Proof. By the uniform Korn inequality (2.15), one has for $u\in V$ that

$$\|u\|_{L^2}\le\|u\|_{H^1}\le c_1\|Du\|_{L^2}\le c_2\|A^{\frac{1}{2}}u\|_{L^2}, \tag{5.11}$$

where $c_2=c_1/\sqrt{2}$. It follows from (5.7), the trace estimate (2.2) and (1.9) that

$$\|A^{\frac{1}{2}}u\|_{L^2}\le C\|\nabla u\|_{L^2}+C\frac{\sqrt{\gamma_0}+\sqrt{\gamma_1}}{\sqrt{\varepsilon}}\|u\|_{L^2}\le C\|\nabla u\|_{L^2}+C\varepsilon^{(\delta-1)/2}\|u\|_{L^2},$$

thus (5.9) follows.

Now, for $u\in D_A$ and $u\neq 0$, one has

$$\|A^{\frac{1}{2}}u\|_{L^2}^2=\langle Au,u\rangle\le c_2\|Au\|_{L^2}\|A^{\frac{1}{2}}u\|_{L^2},$$

thanks to (5.7),(5.6) and (5.8), hence one obtains (5.10). $\square$

The next lemma will be used both in the linear estimates in Corollary 5.3 and Proposition 5.5, as well as in the nonlinear estimate in the next section. It shows that the Navier friction boundary condition introduces extra terms which need extra treatments while one studies the Stokes operator and the nonlinear term of the Navier–Stokes equations.

Lemma 5.2. *Let $u \in D_A$ and $\Phi \in H^1_{\mathrm{per}}(\hat{\Omega}_\varepsilon)^3$. One has*

$$\int_{\Omega_\varepsilon} (\nabla \times (\nabla \times u)) \cdot \Phi dx = \int_{\Omega_\varepsilon} (\nabla \times \Phi) \cdot (\nabla \times u - G(u)) dx + \int_{\Omega_\varepsilon} \Phi \cdot (\nabla \times G(u)) dx, \quad (5.12)$$

where $G(u)$ is given by (4.24).

Proof. Let $\omega = \nabla \times u$, one has

$$\int_{\Omega_\varepsilon} (\nabla \times \omega) \cdot \Phi dx = \int_{\Omega_\varepsilon} \omega \cdot (\nabla \times \Phi) dx + \int_\Gamma (N \times \omega) \cdot \Phi d\sigma.$$

By (4.23),

$$\begin{aligned}\int_\Gamma (N \times \omega) \cdot \Phi d\sigma &= \int_\Gamma (N \times G(u)) \cdot \Phi d\sigma = -\int_\Gamma (\Phi \times G(u)) \cdot N d\sigma \\ &= -\int_{\Omega_\varepsilon} \nabla \cdot (\Phi \times G(u)) dx = -\int_{\Omega_\varepsilon} (\nabla \times \Phi) \cdot G(u) - \Phi \cdot (\nabla \times G(u)) dx.\end{aligned}$$

Therefore (5.12) follows. □

Corollary 5.3. *Let $u \in D_A$, then*

$$\|Au + \Delta u\|_{L^2} \leq C\big(\varepsilon^\delta \|\nabla u\|_{L^2} + \varepsilon^{\delta-1} \|u\|_{L^2}\big). \quad (5.13)$$

Proof. Let $\omega = \nabla \times u$ and $\Phi = Au + \Delta u$. One has $\Phi = \nabla q$, thanks to (1.12) and (1.19). Since $Au \in H$ and $\Phi \in H^\perp$ are orthogonal in $L^2(\Omega_\varepsilon)$, we have

$$\int_{\Omega_\varepsilon} |\Phi|^2 dx = \int_{\Omega_\varepsilon} (Au + \Delta u) \cdot \Phi dx = \int_{\Omega_\varepsilon} \Delta u \cdot \Phi dx = -\int_{\Omega_\varepsilon} \nabla \times \omega \cdot \Phi dx.$$

By virtue of Lemma 5.2 noting that $\nabla \times \Phi = 0$, and (4.35), one obtains

$$\begin{aligned}\int_{\Omega_\varepsilon} |\Phi|^2 dx &= \left| \int_{\Omega_\varepsilon} \Phi \cdot \nabla \times G(u) dx \right| \\ &\leq C \int_{\Omega_\varepsilon} |\Phi| \big(\varepsilon^\delta |\nabla u| + \varepsilon^{\delta-1} |u|\big) dx \\ &\leq C \|\Phi\|_{L^2} \big\{\varepsilon^\delta \|\nabla u\|_{L^2} + \varepsilon^{\delta-1} \|u\|_{L^2}\big\},\end{aligned}$$

hence (5.13) follows. □

Lemma 5.4. *There is $\varepsilon_1 \in (0,1]$ such that if $\varepsilon < \varepsilon_1$ and $u \in H^2_{\rm per}(\hat{\Omega}_\varepsilon)^3$ satisfies the Navier boundary condition* (1.8)*, then*

$$\|\nabla^2 u\|_{L^2} \le C\|\Delta u\|_{L^2} + C\|u\|_{H^1}. \tag{5.14}$$

Moreover, ε_1 is chosen to satisfy $\varepsilon_1 \le \varepsilon_0$, where ε_0 is the positive number occurring in Lemma 2.7.

The proof of Lemma 5.4 is technical and is given in Appendix A. We finally obtain the relation between $\|Au\|_{L^2}$ and $\|u\|_{H^2}$.

Proposition 5.5. *For $\varepsilon < \varepsilon_1$ and $u \in D_A$, we have*

$$C'\|Au\|_{L^2} \le \|u\|_{H^2} \le C\big\{\|Au\|_{L^2} + \varepsilon^{\delta-1}\|u\|_{L^2}\big\}, \tag{5.15}$$

where positive numbers C and C' are independent on ε.

Proof. On one hand, $\|Au\|_{L^2} = \|P(-\Delta u)\|_{L^2} \le \|\Delta u\|_{L^2} \le C\|u\|_{H^2}$. On the other hand, it follows from Lemma 5.4 and Corollary 5.3 that

$$\begin{aligned}
\|u\|_{H^2} &\le C\|u\|_{H^1} + C\|\nabla^2 u\|_{L^2} \le C\|u\|_{H^1} + C\big(\|\Delta u\|_{L^2} + C\|u\|_{H^1}\big)\\
&\le C\|u\|_{H^1} + C\big(\|Au\|_{L^2} + \|Au + \Delta u\|_{L^2}\big)\\
&\le C\|u\|_{H^1} + C\|Au\|_{L^2} + C\big(\varepsilon^{\delta}\|\nabla u\|_{L^2} + \varepsilon^{\delta-1}\|u\|_{L^2}\big)\\
&\le C\|u\|_{H^1} + C\|Au\|_{L^2} + C\varepsilon^{\delta-1}\|u\|_{L^2}.
\end{aligned}$$

Using (5.8) and (5.10) one obtains

$$\begin{aligned}
\|u\|_{H^2} &\le \|A^{\frac{1}{2}}u\|_{L^2} + C\|Au\|_{L^2} + C\varepsilon^{\delta-1}\|u\|_{L^2}\\
&\le C\|Au\|_{L^2} + C\varepsilon^{\delta-1}\|u\|_{L^2}.
\end{aligned}$$

The proof is complete. □

6. Estimate of the tri-linear term

The main result of this section is the following estimate for the tri-linear term $\langle u\cdot\nabla u, Au\rangle$ which is essential to the theory of global strong solutions of the Navier–Stokes equations.

Proposition 6.1. *Given $\alpha > 0$, there is $C_\alpha > 0$ such that for any $\varepsilon \in (0,1]$ and $u \in D_A$, we have*

$$\begin{aligned}
|\langle u\cdot\nabla u, Au\rangle| \le{}& \alpha\|u\|^2_{H^2} + C\varepsilon^{1/2}\|u\|_{H^1}\|u\|^2_{H^2} + C_\alpha\varepsilon^{2\delta}\|u\|^2_{L^2}\|u\|^4_{H^1}\\
&+ C_\alpha\varepsilon^{-2+4\delta/3}\|u\|^2_{L^2}\|u\|^{4/3}_{H^1} + C_\alpha\varepsilon^{-1}\|u\|^2_{L^2}\|u\|^2_{H^1},
\end{aligned} \tag{6.1}$$

where $C > 0$ is independent of ε and α.

The proof of Proposition 6.1 combines the approaches in [32, 16, 11] with the relation in Lemma 5.2 and the estimate in Lemma 6.3 below. We first recall an inequality established in [4] for a general product domain.

Lemma 6.2 ([4]). *Let $\Omega = U\times(-h,0)$ where h is a positive number and $U\subset\mathbb{R}^2$ is open and bounded. Let u, v, w be smooth functions on $\bar\Omega$ and v be independent of x_3. Then*

$$\|vwu\|_{L^1(\Omega)} \le C\|v\|_{L^4(U)}\|w\|^{1/2}_{L^2(\Omega)}\|w\|^{1/2}_{H^1(\Omega)}\|u\|_{L^2(\Omega)}. \tag{6.2}$$

The following is a version of Lemma 6.2 for our thin domain Ω_ε whose bottom and top are not necessarily flat. The proof is given in Appendix B.

Lemma 6.3. *Suppose $v, w\in H^1_{\mathrm{per}}(\hat\Omega_\varepsilon)$, v is independent of x_3. Then*

$$\|vw\|_{L^2} \le C\varepsilon^{-1/2}\|v\|^{1/2}_{L^2}\|v\|^{1/2}_{H^1}\|w\|^{1/2}_{L^2}\big(\|w\|^{1/2}_{L^2}+\|\nabla_2 w\|^{1/2}_{L^2}+\varepsilon^{1/2}\|\partial_3 w\|^{1/2}_{L^2}\big). \tag{6.3}$$

Subsequently,

$$\|vw\|_{L^2} \le C\varepsilon^{-1/2}\|v\|^{1/2}_{L^2}\|v\|^{1/2}_{H^1}\|w\|^{1/2}_{L^2}\|w\|^{1/2}_{H^1}. \tag{6.4}$$

Proof of Proposition 6.1. Let $\omega=\nabla\times u$, $v=Mu$ and $w=u-v$. We have

$$\begin{aligned}
&-\int_{\Omega_\varepsilon}(u\cdot\nabla)u\cdot Au\,dx = \int_{\Omega_\varepsilon}\Big\{u\times(\nabla\times u)-\frac12\nabla|u|^2\Big\}\cdot Au\,dx\\
&=\int_{\Omega_\varepsilon}(w\times\omega)\cdot Au\,dx+\int_{\Omega_\varepsilon}(v\times\omega)\cdot Au\,dx\\
&=\int_{\Omega_\varepsilon}(w\times\omega)\cdot Au\,dx+\int_{\Omega_\varepsilon}(v\times\omega)\cdot(Au+\Delta u)dx-\int_{\Omega_\varepsilon}(v\times\omega)\cdot\Delta u\,dx\\
&=J_1+J_2+J_3.
\end{aligned}$$

Estimate of J_1. By Agmon's inequality (4.18) and Hölder's inequality

$$\begin{aligned}
|J_1| &\le \|w\|_{L^\infty}\|\omega\|_{L^2}\|Au\|_{L^2}\\
&\le C\|u\|_{H^1}\|u\|_{H^2}\big(\varepsilon^{1/2}\|u\|_{H^2}+\varepsilon^{\delta/2}\|u\|^{1/2}_{L^2}\|u\|^{1/2}_{H^2}\big)\\
&\le C\varepsilon^{1/2}\|u\|_{H^1}\|u\|^2_{H^2}+\alpha\|u\|^2_{H^2}+C_\alpha\varepsilon^{2\delta}\|u\|^2_{L^2}\|u\|^4_{H^1}.
\end{aligned}$$

Estimate of J_2. Let $\Phi=v\times\omega$. Applying (3.11) and (6.4), we have

$$\begin{aligned}
\|\Phi\|_{L^2} &\le C\varepsilon^{-1/2}\|v\|^{1/2}_{L^2}\|v\|^{1/2}_{H^1}\|\omega\|^{1/2}_{L^2}\|\omega\|^{1/2}_{H^1}\\
&\le C\varepsilon^{-1/2}\|u\|^{1/2}_{L^2}\|u\|_{H^1}\|u\|^{1/2}_{H^2}.
\end{aligned} \tag{6.5}$$

Using (6.5) and Corollary 5.3:

$$\begin{aligned}
|J_2| &\le \|\Phi\|_{L^2}\|Au+\Delta u\|_{L^2}\\
&\le C\varepsilon^{-1/2}\|u\|^{1/2}_{L^2}\|u\|_{H^1}\|u\|^{1/2}_{H^2}\big\{\varepsilon^\delta\|\nabla u\|_{L^2}+\varepsilon^{\delta-1}\|u\|_{L^2}\big\}.
\end{aligned}$$

Note that

$$\|u\|_{H^1}^2 \le C\|A^{\frac{1}{2}}u\|_{L^2}^2 = C\langle Au, u\rangle \le C\|u\|_{L^2}\|Au\|_{L^2} \le C\|u\|_{L^2}\|u\|_{H^2}. \tag{6.6}$$

Using (6.6) and Hölder's inequality gives

$$\begin{aligned}
\varepsilon^{-1/2+\delta}\|u\|_{L^2}^{1/2}\|u\|_{H^1}^2\|u\|_{H^2}^{1/2} &\le C\varepsilon^{-1/2+\delta}\|u\|_{L^2}\|u\|_{H^1}\|u\|_{H^2}\\
&\le \alpha\|u\|_{H^2}^2 + C_\alpha\varepsilon^{-1+2\delta}\|u\|_{L^2}^2\|u\|_{H^1}^2,\\
\varepsilon^{-3/2+\delta}\|u\|_{L^2}^{3/2}\|u\|_{H^1}\|u\|_{H^2}^{1/2} &\le \alpha\|u\|_{H^2}^2 + C_\alpha\varepsilon^{-2+4\delta/3}\|u\|_{L^2}^2\|u\|_{H^1}^{4/3}.
\end{aligned}$$

Hence

$$|J_2| \le \alpha\|u\|_{H^2}^2 + C_\alpha\varepsilon^{-1+2\delta}\|u\|_{L^2}^2\|u\|_{H^1}^2 + C_\alpha\varepsilon^{-2+4\delta/3}\|u\|_{L^2}^2\|u\|_{H^1}^{4/3}. \tag{6.7}$$

Estimate of J_3. By Lemma 5.2, one has

$$\begin{aligned}
J_3 &= \int_{\Omega_\varepsilon}\Phi\cdot\nabla\times\omega dx\\
&= \int_{\Omega_\varepsilon}\nabla\times\Phi\cdot\omega dx - \int_{\Omega_\varepsilon}\nabla\times\Phi\cdot G(u)dx + \int_{\Omega_\varepsilon}\Phi\cdot\nabla\times G(u)dx\\
&= J_{3,1} + J_{3,2} + J_{3,3}.
\end{aligned}$$

We estimate $J_{3,3}$ first. By Lemma 4.6,

$$\begin{aligned}
|J_{3,3}| &\le C\|\Phi\|_{L^2}\big(\varepsilon^\delta\|u\|_{H^1} + \varepsilon^{\delta-1}\|u\|_{L^2}\big)\\
&\le C\varepsilon^{-1/2}\|u\|_{L^2}^{1/2}\|u\|_{H^1}\|u\|_{H^2}^{1/2}\big(\varepsilon^\delta\|u\|_{H^1} + \varepsilon^{\delta-1}\|u\|_{L^2}\big).
\end{aligned}$$

Similar to the last estimate of J_2, one derives

$$|J_{3,3}| \le \alpha\|u\|_{H^2}^2 + C_\alpha\varepsilon^{-1+2\delta}\|u\|_{L^2}^2\|u\|_{H^1}^2 + C_\alpha\varepsilon^{-2+4\delta/3}\|u\|_{L^2}^2\|u\|_{H^1}^{4/3}. \tag{6.8}$$

We estimate $J_{3,2}$ next. Note that

$$\begin{aligned}
\nabla\times\Phi &= (\omega\cdot\nabla)v - (v\cdot\nabla)\omega + \omega(\nabla\cdot v) - v(\nabla\cdot\omega)\\
&= (\omega\cdot\nabla)v - (v\cdot\nabla)\omega.
\end{aligned}$$

Hence

$$|\nabla\times\Phi| \le C(|v|\ |\nabla^2 u| + |\nabla v|\ |\nabla u|). \tag{6.9}$$

It follows that

$$|J_{3,2}| \le C\varepsilon^\delta\int_{\Omega_\varepsilon}|\nabla\times\Phi||u|dx \le C\varepsilon^\delta\int_{\Omega_\varepsilon}\big(|\nabla v||\nabla u| + |v||\nabla^2 u|\big)|u|dx.$$

For the last integral, we apply (3.11), (6.4) and (6.6) to obtain

$$\begin{aligned}
\varepsilon^\delta\int_{\Omega_\varepsilon}|\nabla v||u||\nabla u|dx &\le C\varepsilon^\delta\int_{\Omega_\varepsilon}|\nabla_2\bar v||u||\nabla u|dx \le C\varepsilon^\delta\|\ |\nabla_2\bar v|\ |u|\ \|_{L^2}\|\nabla u\|_{L^2}\\
&\le C\varepsilon^{-1/2+\delta}\|v\|_{H^1}^{1/2}\|v\|_{H^2}^{1/2}\|u\|_{L^2}^{1/2}\|u\|_{H^1}^{1/2}\|u\|_{H^1}
\end{aligned}$$

$$\begin{aligned}&\le C\varepsilon^{-1/2+\delta}\|u\|_{L^2}\|u\|_{H^1}\|u\|_{H^2}\\&\le \alpha\|u\|_{H^2}^2 + C_\alpha\varepsilon^{-1+2\delta}\|u\|_{L^2}^2\|u\|_{H^1}^2,\end{aligned}$$

and

$$\begin{aligned}\varepsilon^\delta\int_{\Omega_\varepsilon}|v||u||\nabla^2 u|dx \le C\varepsilon^\delta\int_{\Omega_\varepsilon}|\bar v||u||\nabla^2 u|dx &\le C\varepsilon^\delta\|\ |\bar v|\ |u|\ \|_{L^2}\|\nabla^2 u\|_{L^2}\\&\le C\varepsilon^{-1/2+\delta}\|u\|_{L^2}\|u\|_{H^1}\|u\|_{H^2}\\&\le \alpha\|u\|_{H^2}^2 + C_\alpha\varepsilon^{-1+2\delta}\|u\|_{L^2}^2\|u\|_{H^1}^2.\end{aligned}$$

Therefore

$$|J_{3,2}| \le \alpha\|u\|_{H^2}^2 + C_\alpha\varepsilon^{-1+2\delta}\|u\|_{L^2}^2\|u\|_{H^1}^2. \tag{6.10}$$

We now estimate $J_{3,1}$. Letting $v = \bar v + v_3\mathbf{e}_3$, we have

$$\begin{aligned}J_{3,1} &= \int_{\Omega_\varepsilon}(\omega\cdot\nabla v)\cdot\omega - (v\cdot\nabla)\omega\cdot\omega dx = \int_{\Omega_\varepsilon}(\omega\cdot\nabla v)\cdot\omega dx\\&= \int_{\Omega_\varepsilon}\Big((\nabla\times v)\cdot\nabla v\Big)\cdot\omega dx + \int_{\Omega_\varepsilon}\Big((\nabla\times w)\cdot\nabla v\Big)\cdot\omega dx\\&= J_{3,1}^{(1)} + J_{3,1}^{(2)}.\end{aligned}$$

For $J_{3,1}^{(1)}$, we have

$$\begin{aligned}J_{3,1}^{(1)} &= \int_{\Omega_\varepsilon}\Big((\nabla\times\bar v)\cdot\nabla\bar v\Big)\cdot\omega dx + \int_{\Omega_\varepsilon}(\nabla\times\bar v)\cdot\nabla(v_3\mathbf{e}_3)\cdot\omega dx\\&\quad + \int_{\Omega_\varepsilon}\Big(\nabla\times(v_3\mathbf{e}_3)\cdot\nabla v\Big)\cdot\omega dx\\&= K_1 + K_2 + K_3.\end{aligned}$$

Since $\nabla\times\bar v$ is collinear to $\mathbf{e}_3$ and $\partial_3\bar v = 0$, we have $K_1 = 0$.
The second term K_2 is estimated by using (3.10) and Lemma 6.3

$$\begin{aligned}|K_2| &\le \int_{\Omega_\varepsilon}|\partial_2 v_1 - \partial_1 v_2||\partial_3 v_3||\omega_3|dx\\&\le C\int_{\Omega_\varepsilon}|\partial_2 v_1 - \partial_1 v_2||\omega_3||\bar v|dx\\&\le C\big(\varepsilon^{-1/2}\|v\|_{H^1}^{1/2}\|v\|_{H^2}^{1/2}\|u\|_{H^1}^{1/2}\|u\|_{H^2}^{1/2}\big)\|v\|_{L^2}\\&\le \alpha\|u\|_{H^2}^2 + C_\alpha\varepsilon^{-1}\|u\|_{L^2}^2\|u\|_{H^1}^2.\end{aligned}$$

For the third term K_3, one notes from (3.10) that

$$|\nabla\times(v_3\mathbf{e}_3)| \le C\varepsilon\big(|\bar v| + |\nabla_2\bar v|\big), \tag{6.11}$$

hence combining with (3.11), Ladyzhenskaya's inequalities (3.13) and (3.15), we obtain

$$|K_3| \le C\varepsilon \| \,|\bar v| + |\nabla_2 \bar v|\, \|_{L^4}^2 \|u\|_{H^1} \le C\varepsilon\big(\varepsilon^{-1/2}\|u\|_{H^1}\|u\|_{H^2}\big)\|u\|_{H^1}$$
$$= C\varepsilon^{1/2}\|u\|_{H^1}^2\|u\|_{H^2} \le C\varepsilon^{1/2}\|u\|_{H^1}\|u\|_{H^2}^2.$$

Thus

$$|J_{3,1}^{(1)}| \le |K_1| + |K_2| + |K_3|$$
$$\le \alpha\|u\|_{H^2}^2 + C\varepsilon^{1/2}\|u\|_{H^1}\|u\|_{H^2}^2 + C_\alpha\varepsilon^{-1}\|u\|_{L^2}^2\|u\|_{H^1}^2. \tag{6.12}$$

For $J_{3,1}^{(2)}$, we have

$$\begin{aligned}
J_{3,1}^{(2)} &= \int_{\Omega_\varepsilon} (\nabla \times w \cdot \nabla \bar v)\cdot \omega dx + \int_{\Omega_\varepsilon} \Big(\nabla \times w \cdot \nabla (v_3\mathbf{e}_3)\Big)\cdot \omega dx\\
&= \int_{\Omega_\varepsilon} (\nabla \times w \cdot \nabla \bar v)\cdot \nabla\times(\bar v + v_3\mathbf{e}_3 + w) dx + \int_{\Omega_\varepsilon} \Big(\nabla \times w \cdot \nabla (v_3\mathbf{e}_3)\Big)\cdot \omega dx\\
&= \int_{\Omega_\varepsilon} (\nabla \times w \cdot \nabla \bar v)\cdot \nabla\times(v_3\mathbf{e}_3) dx + \int_{\Omega_\varepsilon} (\nabla \times w \cdot \nabla \bar v)\cdot \nabla\times w dx\\
&\quad + \int_{\Omega_\varepsilon} \Big(\nabla \times w \cdot \nabla (v_3\mathbf{e}_3)\Big)\cdot \omega dx\\
&= K_1' + K_2' + K_3'.
\end{aligned}$$

Using (3.11) and (6.11) to estimate K_1', we have

$$|K_1'| \le C\varepsilon \int_{\Omega_\varepsilon} |\nabla \bar v|(|\nabla \bar v| + |\bar v|)|\nabla w| dx \le C\varepsilon \| \,|\nabla \bar v|(|\nabla \bar v| + |\bar v|)\, \|_{L^2}\|\nabla w\|_{L^2}$$
$$\le C\varepsilon^{1/2}\|u\|_{H^1}\|u\|_{H^2}\|\nabla w\|_{L^2} \le C\varepsilon^{1/2}\|u\|_{H^1}\|u\|_{H^2}^2,$$

Applying (6.4) and (4.12) yields

$$\begin{aligned}
|K_2'| &\le C\varepsilon^{-1/2}\|v\|_{H^1}^{1/2}\|v\|_{H^2}^{1/2}\|\nabla w\|_{L^2}^{1/2}\|w\|_{H^2}^{1/2}(\varepsilon\|u\|_{H^2} + \varepsilon^\delta\|u\|_{L^2})\\
&\le C\varepsilon^{1/2}\|u\|_{H^1}\|u\|_{H^2}^2 + \varepsilon^{-1/2+\delta}\|u\|_{L^2}\|u\|_{H^1}\|u\|_{H^2}\\
&\le C\varepsilon^{1/2}\|u\|_{H^1}\|u\|_{H^2}^2 + \alpha\|u\|_{H^2}^2 + C_\alpha\varepsilon^{-1+2\delta}\|u\|_{L^2}^2\|u\|_{H^1}^2.
\end{aligned}$$

By (3.10), Lemma 6.3 and (6.6), we obtain

$$\begin{aligned}
|K_3'| &\le C\int_{\Omega_\varepsilon} |\nabla w|(\varepsilon|\nabla_2 \bar v| + |\bar v|)|\nabla u| dx\\
&\le C\varepsilon^{1/2}\|\nabla w\|_{L^2}^{1/2}\|w\|_{H^2}^{1/2}\|v\|_{H^1}^{1/2}\|v\|_{H^2}^{1/2}\|u\|_{H^1}\\
&\quad + C\varepsilon^{-1/2}\|\nabla w\|_{L^2}^{1/2}\|w\|_{H^2}^{1/2}\|v\|_{L^2}^{1/2}\|v\|_{H^1}^{1/2}\|u\|_{H^1}\\
&\le C\varepsilon^{1/2}\|u\|_{H^1}\|u\|_{H^2}^2 + \alpha\|u\|_{H^2}^2 + C_\alpha\varepsilon^{-1}\|u\|_{L^2}^2\|u\|_{H^1}^2.
\end{aligned}$$

Thus

$$|J_{3,1}^{(2)}| \le |K_1'| + |K_2'| + |K_3'| \le C\varepsilon^{1/2}\|u\|_{H^1}\|u\|_{H^2}^2 + \alpha\|u\|_{H^2}^2 + C_\alpha\varepsilon^{-1}\|u\|_{L^2}^2\|u\|_{H^1}^2. \tag{6.13}$$

Summing up the estimates for J_1, J_2, $J_{3,1}^{(1)}$, $J_{3,1}^{(2)}$, $J_{3,2}$ and $J_{3,3}$, we obtain

$$\begin{aligned}|\langle u\cdot\nabla u, Au\rangle| &\le \alpha\|u\|_{H^2}^2 + C\varepsilon^{1/2}\|u\|_{H^1}\|u\|_{H^2}^2 + C_\alpha\varepsilon^{2\delta}\|u\|_{L^2}^2\|u\|_{H^1}^4\\ &\quad + C_\alpha\varepsilon^{-2+4\delta/3}\|u\|_{L^2}^2\|u\|_{H^1}^{4/3} + C_\alpha\varepsilon^{-1}\|u\|_{L^2}^2\|u\|_{H^1}^2.\end{aligned}$$

The proof is complete. □

Consequently, we have the estimate of the tri-linear term in terms of $\|Au\|_{L^2}$ and $\|A^{\frac12}u\|_{L^2}$.

Corollary 6.4. *Given $\alpha > 0$, there is $C_\alpha > 0$ such that for any $\varepsilon < \varepsilon_1$ and $u \in D_A$, we have*

$$\begin{aligned}|\langle u\cdot\nabla u, Au\rangle| \le{}& \big\{\alpha + C\varepsilon^{1/2}\|A^{\frac12}u\|_{L^2}\big\}\|Au\|_{L^2}^2 + C_\alpha\varepsilon^{2\delta-2}\|u\|_{L^2}^2\\ &+ C_\alpha\varepsilon^{2\delta-3/2}\|u\|_{L^2}\|A^{\frac12}u\|_{L^2}^2 + C_\alpha\varepsilon^{2\delta}\|u\|_{L^2}^2\|A^{\frac12}u\|_{L^2}^4\\ &+ C_\alpha\varepsilon^{-2+4\delta/3}\|u\|_{L^2}^2\|A^{\frac12}u\|_{L^2}^{4/3} + C_\alpha\varepsilon^{-1}\|u\|_{L^2}^2\|A^{\frac12}u\|_{L^2}^2. \qquad (6.14)\end{aligned}$$

Proof. Thanks to Lemma 5.5, the term $\big\{\alpha+C\varepsilon^{1/2}\|u\|_{H^1}\big\}\|u\|_{H^2}^2$ in (6.1) is bounded by

$$\begin{aligned}&\big\{\alpha + C\varepsilon^{1/2}\|u\|_{H^1}\big\}\|u\|_{H^2}^2\\ &\le C\big\{\alpha + C\varepsilon^{1/2}\|u\|_{H^1}\big\}\big(\|Au\|_{L^2}^2 + \varepsilon^{2\delta-2}\|u\|_{L^2}^2\big)\\ &\le C\{\alpha + C\varepsilon^{1/2}\|u\|_{H^1}\}\|Au\|_{L^2}^2 + C_\alpha\varepsilon^{2\delta-2}\|u\|_{L^2}^2 + C_\alpha\varepsilon^{2\delta-3/2}\|u\|_{H^1}\|u\|_{L^2}^2.\end{aligned}$$

Using (5.8) and then redenoting α and C, one obtains (6.14). □

7. Global solutions

First, we state the usual local existence theorem for strong solutions of the Navier–Stokes equations.

Theorem 7.1. *Suppose $\varepsilon \in (0,1]$ and $u_0 \in V$. Then there exist $T > 0$ and the unique strong solution $u(t)$ of the Navier–Stokes equations* (1.2) *satisfying the boundary conditions* (1.8) *for $t \in (0,T)$ such that $u(0) = u_0$, and*

$$u \in C([0,T), V) \cap L^2(0,T; D_A).$$

Furthermore, if the maximal time interval of the above existence is $[0, T_{\max})$ and $T_{\max}$ is finite, then

$$\lim_{t\to T_{\max}^-}\|u(t)\|_{H^1} = \infty. \qquad (7.1)$$

We recall the Uniform Gronwall Inequality, see [7, 27, 30].

Lemma 7.2 (Uniform Gronwall Inequality). *Let y, g, and h be non-negative functions in $L^1(0,T;\mathbb{R})$, where $0 < T \le \infty$. Assume that y is absolutely continuous on $(0,T)$ and that*

$$\frac{d}{dt}y(t) \le g(t)\,y(t) + h(t), \quad \textit{almost everywhere on } (0,T).$$

Then $y \in L^\infty_{\mathrm{loc}}((0,T),\mathbb{R})$ and one has

$$y(t) \le \left(\frac{1}{t-\tau}\int_\tau^t y(s)\,ds + \int_\tau^t h(s)\,ds\right)\exp\left(\int_\tau^t g(s)\,ds\right), \quad \textit{for } 0 \le \tau < t < T.$$

We will use the following more specific form of the estimate in Corollary 6.4.

Lemma 7.3. *Suppose $\delta \in [3/4,1]$, then there exists $\varepsilon_* \in (0,1]$ such that for any $\varepsilon < \varepsilon_*$ and $u \in D_A$, one has*

$$\begin{aligned}|\langle u\cdot\nabla u, Au\rangle| \le \left\{\frac{1}{4} + d_1\varepsilon^{1/2}\|A^{\frac12}u\|_{L^2}\right\}\|Au\|_{L^2}^2 + d_2\big\{\|u\|_{L^2}^2\|A^{\frac12}u\|_{L^2}^2\big\}\|A^{\frac12}u\|_{L^2}^2 \\ + d_3\big\{1 + \|u\|_{L^2}^2\big\}\varepsilon^{-1}\|A^{\frac12}u\|_{L^2}^2\,,\end{aligned} \tag{7.2}$$

where positive constants d_1, d_2 and d_3 are independent of ε.

Proof. Let $\varepsilon_* = \varepsilon_1$ which is introduced in Lemma 5.4. We first claim that

$$\begin{aligned}\big|\langle u\cdot\nabla u, Au\rangle\big| \le \left\{\frac{1}{4} + d_1'\varepsilon^{1/2}\|A^{\frac12}u\|_{L^2}\right\}\|Au\|_{L^2}^2 + d_2'\big\{\varepsilon^{2\delta}\|u\|_{L^2}^2\|A^{\frac12}u\|_{L^2}^2\big\}\|A^{\frac12}u\|_{L^2}^2 \\ + d_3'\big\{\varepsilon^{2\delta-1} + \varepsilon^{-1+4\delta/3}\|u\|_{L^2}^{4/3} \\ + \|u\|_{L^2}^2 + \varepsilon^{2\delta-1/2}\|u\|_{L^2}\big\}\varepsilon^{-1}\|A^{\frac12}u\|_{L^2}^2,\end{aligned} \tag{7.3}$$

where positive constants d_1', d_2', d_3' are independent of ε.

Indeed, set $\alpha = 1/4$ in (6.14), then the constant C_α there is now specified. Denote the right-hand side of (7.3) by $I + II + III$. Obviously, I and II come from the corresponding terms on the right-hand side of (6.14). The term III is formed by grouping the remaining terms in (6.14) using

$$\varepsilon^{2\delta-2}\|u\|_{L^2}^2 \le C\varepsilon^{2\delta-1}\big(\varepsilon^{-1}\|A^{\frac12}u\|_{L^2}^2\big),$$
$$\varepsilon^{-2+4\delta/3}\|u\|_{L^2}^2\|A^{\frac12}u\|_{L^2}^{4/3} \le C\varepsilon^{-1+4\delta/3}\|u\|_{L^2}^{4/3}\big(\varepsilon^{-1}\|A^{\frac12}u\|_{L^2}^2\big).$$

Therefore, (7.3) is derived. Next, one uses the conditions $\delta \ge 3/4$ and Höder's inequality to obtain

$$II \le C\Big\{\|u\|_{L^2}^2\|A^{\frac12}u\|_{L^2}^2\Big\}\|A^{\frac12}u\|_{L^2}^2,$$

$$
\begin{aligned}
III &\le C\Big\{\varepsilon^{1/2} + \|u\|_{L^2}^{4/3} + \|u\|_{L^2}^2 + \varepsilon\|u\|_{L^2}\Big\}\varepsilon^{-1}\|A^{\frac12}u\|_{L^2}^2 \\
&\le C\Big\{1 + \|u\|_{L^2}^{4/3} + \|u\|_{L^2}^2 + \|u\|_{L^2}\Big\}\varepsilon^{-1}\|A^{\frac12}u\|_{L^2}^2 \\
&\le C\Big\{1 + \|u\|_{L^2}^2\Big\}\varepsilon^{-1}\|A^{\frac12}u\|_{L^2}^2.
\end{aligned}
$$

Thus (7.2) follows. □

Let $u = (u_1, u_2, u_3) \in L^2_{\mathrm{per}}(\hat\Omega_\varepsilon)^3$, we define

$$\overline{M}u = (M_0u_1, M_0u_2, 0). \tag{7.4}$$

One can verify that $\overline{M}$ is an orthogonal projection on $L^2_{\mathrm{per}}(\hat\Omega_\varepsilon)^3$, hence

$$\|u\|_{L^2}^2 = \|\overline{M}u\|_{L^2}^2 + \|(I-\overline{M})u\|_{L^2}^2. \tag{7.5}$$

For $u \in V$, it follows from (2.4), (2.6) and (5.8) that

$$\|(I-\overline{M})u\|_{L^2} \le c_4\varepsilon\|u\|_{H^1} \le c_5\varepsilon\|A^{\frac12}u\|_{L^2}, \tag{7.6}$$

where c_4 and c_5 are positive constants.

Theorem 7.4. *Suppose the conditions* (1.10) *and* (1.23) *hold. There are positive numbers ε_* and κ such that if $\varepsilon < \varepsilon_*$ and $u_0 \in V$, $f \in L^\infty\big((0,\infty), L^2_{\mathrm{per}}(\hat\Omega_\varepsilon)^3\big)$ satisfy that all of the quantities*

$$
\begin{aligned}
U_0 &\overset{\text{def}}{=\!=} \|\overline{M}u_0\|_{L^2}^2, & U_1 &\overset{\text{def}}{=\!=} \varepsilon\|A^{\frac12}u_0\|_{L^2}^2, \\
F_0 &\overset{\text{def}}{=\!=} \|\overline{M}Pf\|_{L^\infty L^2}^2, & F_1 &\overset{\text{def}}{=\!=} \varepsilon\|Pf\|_{L^\infty L^2}^2
\end{aligned}
\tag{7.7}
$$

are less then κ, then the strong solution $u(t)$ of the Navier–Stokes equations (1.2) *satisfying* (1.8) *with initial data u_0 exists for all $t \ge 0$. Moreover,*

$$\|u(t)\|_{L^2}^2 \le c_1^*(\Lambda_1 e^{-t/c_2^2} + \Lambda_2), \tag{7.8}$$

and

$$\|A^{\frac12}u(t)\|_{L^2}^2 \le c_2^*\varepsilon^{-1}(\Lambda_3 e^{-t/c_1^2} + \Lambda_4), \tag{7.9}$$

for all $t \ge 0$, where

$$\Lambda_1 = U_0 + \varepsilon U_1, \quad \Lambda_2 = F_0 + \varepsilon F_1, \quad \Lambda_3 = U_0 + U_1, \quad \Lambda_4 = F_0 + F_1, \tag{7.10}$$

the numbers c_1 are c_2 are defined in Lemmas 2.7 and 5.1 respectively, and the positive constants c_1^* and c_2^* are independent of ε, u_0, f.

Proof. Let $\varepsilon < \varepsilon_* \le 1$ where ε_* is given in Lemma 7.3. Take $\kappa = \min\Big\{1, \frac{d_4}{2}, \frac{d_4}{8C_{11}}\Big\}$, where

$$\sqrt{d_4} = \frac{1}{4d_1}, \tag{7.11}$$

the number d_1 is introduced in Lemma 7.3, and C_{11} is defined in (7.25) below.

We estimate $\|u(t)\|_{L^2}$ first. We have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2+\|A^{1/2}u\|_{L^2}^2 &\le |\langle u,f\rangle| \le |\langle \overline{M}u,\overline{M}Pf\rangle|+|\langle (I-\overline{M})u,(I-\overline{M}Pf\rangle|\\ &\le \|u\|_{L^2}\|\overline{M}Pf\|_{L^2}+c_5\varepsilon\|A^{1/2}u\|_{L^2}\|Pf\|_{L^2}\\ &\le \frac{1}{2}\|A^{1/2}u\|_{L^2}^2+c_2^2\|\overline{M}Pf\|_{L^2}^2+c_5^2\varepsilon^2\|Pf\|_{L^2}^2\\ &\le \frac{1}{2}\|A^{1/2}u\|_{L^2}^2+c_0^2F_0+c_5^2\varepsilon F_1.\end{aligned}$$

Hence

$$\frac{d}{dt}\|u\|_{L^2}^2+\|A^{1/2}u\|_{L^2}^2 \le 2c_0^2F_0+2c_5^2\varepsilon F_1. \tag{7.12}$$

Using (5.8),

$$\frac{d}{dt}\|u\|_{L^2}^2+\frac{1}{c_2^2}\|u\|_{L^2}^2 \le 2c_0^2F_0+2c_5^2\varepsilon F_1. \tag{7.13}$$

By the Gronwall inequality, one obtains

$$\|u(t)\|_{L^2}^2 \le \|u_0\|_{L^2}^2 e^{-t/c_2^2}+2c_2^2(c_2^2F_0+c_5^2\varepsilon F_1). \tag{7.14}$$

Note from (7.5) and (7.6) that

$$\|u_0\|_{L^2}^2 \le k_1 \stackrel{\text{def}}{=\!=} \|\overline{M}u_0\|_{L^2}^2+c_4^2\varepsilon^2\|u_0\|_{H^1}^2 = U_0+c_4^2\varepsilon U_1 \le C_1\Lambda_1,$$

where $C_1=\max\{1,c_4^2\}$. Also

$$2c_2^2(c_2^2F_0+c_5^2\varepsilon F_1)\le C_2\Lambda_2,$$

where $C_2=2c_2^2\max\{c_2^2,c_5^2\}$, thus

$$\|u(t)\|_{L^2}^2 \le c_1^*(\Lambda_1 e^{-t/c_2^2}+\Lambda_2),\quad c_1^*=\max\{C_1,C_2\}, \tag{7.15}$$

and one obtains (7.8). Moreover,

$$\|u(t)\|_{L^2}^2 \le k_2 \stackrel{\text{def}}{=\!=} c_1^*(U_0+\varepsilon U_1+F_0+\varepsilon F_1)\le 4c_1^*\kappa = l_2\kappa, \tag{7.16}$$

where $l_2=4c_1^*$.

For $t\ge 0$, integrating (7.12) from t to $t+1$ yields

$$\begin{aligned}\int_t^{t+1}\|A^{1/2}u(\tau)\|_{L^2}^2d\tau &\le \|u(t)\|_{L^2}^2+\int_t^{t+1}(2c_2^2F_0+2\varepsilon c_5^2F_1)d\tau\\ &\le c_1^*(\Lambda_1 e^{-t/c_2^2}+\Lambda_2)+\frac{C_2\Lambda_2}{c_2^2}.\end{aligned}$$

Let $C_3=c_1^*+(C_2/c_2^2)$, we have

$$\int_t^{t+1}\|A^{1/2}u(\tau)\|_{L^2}^2d\tau \le C_3(\Lambda_1 e^{-t/c_2^2}+\Lambda_2),\quad t\ge 0. \tag{7.17}$$

Consequently,

$$\int_t^{t+1} \|A^{1/2}u(\tau)\|_{L^2}^2 d\tau \le l_3\kappa, \quad t \ge 0, \tag{7.18}$$

where $l_3 = 4C_3$.

Note that the initial data satisfies $\varepsilon\|A^{1/2}u_0\|_{L^2}^2 = U_1 \le \kappa \le \frac{d_4}{2}$. We will show that

$$\varepsilon\|A^{1/2}u(t)\|_{L^2}^2 \le d_4, \quad \text{for all} \quad t \in [0, T_{\max}). \tag{7.19}$$

Assume (7.19) is false. Then there is $T \in (0, T_{\max})$ such that

$$\varepsilon\|A^{1/2}u(t)\|_{L^2}^2 < d_4, \quad \text{for all } t < T, \tag{7.20}$$

and

$$\varepsilon\|A^{1/2}u(T)\|_{L^2}^2 = d_4. \tag{7.21}$$

From the equation on $(0, T_{\max})$ and Lemma 7.3, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|A^{1/2}u\|_{L^2}^2 + \|Au\|_{L^2}^2 \le |\langle (u\cdot\nabla)u, Au\rangle| + |\langle Au, Pf\rangle| \\
&\le \Big\{\frac{1}{4} + d_1\varepsilon^{1/2}\|A^{1/2}u\|_{L^2}\Big\}\|Au\|_{L^2}^2 \\
&\quad + d_2\Big\{\|u\|_{L^2}^2\|A^{1/2}u\|_{L^2}^2\Big\}\|A^{1/2}u\|_{L^2}^2 + d_3\Big\{1 + \|u\|_{L^2}^2\Big\}\varepsilon^{-1}\|A^{1/2}u\|_{L^2}^2 \\
&\quad + \frac{1}{4}\|Au\|_{L^2}^2 + \|Pf\|_{L^2}^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{d}{dt}\|A^{1/2}u\|_{L^2}^2 + \Big\{1 - 2d_1\varepsilon^{1/2}\|A^{1/2}u\|_{L^2}\Big\}\|Au\|_{L^2}^2 \\
&\le 2d_2\Big\{\|u\|_{L^2}^2\|A^{1/2}u\|_{L^2}^2\Big\}\|A^{1/2}u\|_{L^2}^2 + 2d_3\Big\{1 + \|u\|_{L^2}^2\Big\}\varepsilon^{-1}\|A^{1/2}u\|_{L^2}^2 + 2\|Pf\|_{L^2}^2.
\end{aligned}$$

If $t < T$, then using (7.20) and (7.11) gives

$$2d_1\varepsilon^{1/2}\|A^{1/2}u(t)\|_{L^2} \le 2d_1\sqrt{d_4} = 1/2.$$

Combining this with (5.10) and (7.16) yields

$$\begin{aligned}
\frac{d}{dt}\|A^{1/2}u\|_{L^2}^2 + \frac{1}{2c_2^2}\|A^{1/2}u\|_{L^2}^2 &\le 2d_2\Big\{\|u\|_{L^2}^2\|A^{1/2}u\|_{L^2}^2\Big\}\|A^{1/2}u\|_{L^2}^2 \\
&\quad + C_4\varepsilon^{-1}\|A^{1/2}u\|_{L^2}^2 + 2\varepsilon^{-1}F_1,
\end{aligned}$$

where $C_4 = 2d_3(1 + l_2)$.

Consider the case $t < \min\{1, T\}$. We use the estimates (7.20) and (7.16) to obtain

$$\begin{aligned}
\frac{d}{dt}\|A^{1/2}u\|_{L^2}^2 + \frac{1}{2c_2^2}\|A^{1/2}u\|_{L^2}^2 &\le 2d_2(k_2d_4)\varepsilon^{-1}\|A^{1/2}u\|_{L^2}^2 + C_4\varepsilon^{-1}\|A^{1/2}u\|_{L^2}^2 + 2\varepsilon^{-1}F_1 \\
&\le C_5\varepsilon^{-1}\|A^{1/2}u\|_{L^2}^2 + 2\varepsilon^{-1}F_1,
\end{aligned}$$

where $C_5 = C_4 + 2d_2 l_2 d_4$.

Noting that $2c_2^2 = c_1^2$ and using Gronwall's inequality, one has

$$\begin{aligned}\|A^{1/2}u(t)\|_{L^2}^2 &\le \|A^{1/2}u_0\|_{L^2}^2 e^{-t/c_1^2} + \varepsilon^{-1}\int_0^t e^{-(t-\tau)/c_1^2}\Big(C_5\|A^{1/2}u(\tau)\|_{L^2}^2 + 2F_1\Big)d\tau\\ &\le \varepsilon^{-1}\Big\{U_1 e^{-t/c_1^2} + C_5C_3(\Lambda_1 e^{-t/c_2^2} + \Lambda_2) + 2c_1^2F_1\Big\},\end{aligned}$$

thus

$$\|A^{1/2}u(t)\|_{L^2}^2 \le \varepsilon^{-1}C_6(\Lambda_3 e^{-t/c_1^2} + \Lambda_4), \quad \text{for } t < \min\{1, T\}, \tag{7.22}$$

where $C_6 = 1 + C_5C_3 + 2c_1^2$.

If $T > 1$, we now consider $t \in [1, T)$. Keeping in mind that (7.20) is still valid, we have

$$\begin{aligned}\frac{d}{dt}\|A^{1/2}u\|_{L^2}^2 &\le 2d_2\Big\{\|u\|_{L^2}^2\|A^{1/2}u\|_{L^2}^2\Big\}\|A^{1/2}u\|_{L^2}^2 + C_4\varepsilon^{-1}\|A^{1/2}u\|_{L^2}^2 + 2\|Pf\|_{L^2}^2\\ &\le g\|A^{1/2}u\|_{L^2}^2 + h,\end{aligned}$$

where

$$\begin{aligned} g &= 2d_2\|u\|_{L^2}^2\|A^{1/2}u\|_{L^2}^2,\\ h &= C_4\varepsilon^{-1}\|A^{1/2}u\|_{L^2}^2 + 2\|Pf\|_{L^2}^2.\end{aligned}$$

Note from (7.17) that

$$\int_{t-1}^t \|A^{1/2}u(\tau)\|_{L^2}^2 d\tau \le C_7(\Lambda_1 e^{-t/c_2^2} + \Lambda_2),$$

where $C_7 = C_6 e^{1/c_2^2}$, and note from (7.16) and (7.18) that

$$\int_{t-1}^t g(\tau)d\tau \le C_8 \overset{\text{def}}{=\!=} 2d_2 l_2 l_3.$$

Also,

$$\int_{t-1}^t h(\tau)d\tau \le \varepsilon^{-1}\{C_4C_7(\Lambda_1 e^{-t/c_2^2} + \Lambda_2) + 2F_1\},$$

thus

$$\int_{t-1}^t h(\tau)d\tau \le \varepsilon^{-1}C_9(\Lambda_1 e^{-t/c_2^2} + \Lambda_4), \quad C_9 \overset{\text{def}}{=\!=} 3 + C_4C_7.$$

Then applying the uniform Gronwall's inequality (Lemma 7.2) yields

$$\|A^{1/2}u(t)\|_{L^2}^2 \le \Big\{C_7(\Lambda_1 e^{-t/c_2^2} + \Lambda_2) + \varepsilon^{-1}C_9(\Lambda_1 e^{-t/c_2^2} + \Lambda_4)\Big\}e^{C_8},$$

hence letting $C_{10} = (C_7 + C_9)e^{C_8}$, one has

$$\|A^{1/2}u(t)\|_{L^2}^2 \le \varepsilon^{-1}C_{10}(\Lambda_1 e^{-t/c_2^2} + \Lambda_4), \quad \text{for } t \in [1, T). \tag{7.23}$$

It follows from (7.22) and (7.23) that

$$\|A^{1/2}u(t)\|_{L^2}^2 \leq C_{11}\varepsilon^{-1}(\Lambda_3 e^{-t/c_1^2} + \Lambda_4), \quad \text{for all} \quad t \in [0, T), \tag{7.24}$$

where

$$C_{11} = \max\{C_6, C_{10}\}. \tag{7.25}$$

Hence

$$\|A^{1/2}u(t)\|_{L^2}^2 \leq 4C_{11}\kappa\varepsilon^{-1}, \quad \text{for all} \quad t \in [0, T), \tag{7.26}$$

and consequently,

$$\varepsilon\|A^{1/2}u(T)\|_{L^2}^2 \leq 4C_{11}\kappa < d_4,$$

which contradicts (7.21). Thus (7.19) must hold true.

As a consequence of (7.26) and (5.8), the norm $\|u(t)\|_{H^1}$ is bounded on $[0, T_{\max})$, which implies $T_{\max} = \infty$, by virtue of (ii) in Theorem 7.1.

Since (7.24) now holds with arbitrary $T > 0$, one obtains (7.9). The proof is complete. □

Proof of Theorem 1.1 (Main Theorem). Note from (5.9) that

$$\|A^{\frac{1}{2}}u_0\|_{L^2}^2 \leq 2c_3^2(\|u_0\|_{H^1}^2 + \varepsilon^{\delta-1}\|u_0\|_{L^2}^2), \text{ hence } \varepsilon\|A^{\frac{1}{2}}u_0\|_{L^2}^2 \leq \tilde{U}_1 \stackrel{\text{def}}{=\!=} 2c_3^2(U_1 + \varepsilon^\delta U_0).$$

Also, one has

$$U_0 + \varepsilon\tilde{U}_1 \leq C(U_0 + \varepsilon U_1) \quad \text{and} \quad U_0 + \tilde{U}_1 \leq C(U_0 + U_1).$$

Then Theorem 1.1 follows from Theorem 7.4 in which U_1 defined in (7.7) is replaced by the above $\tilde{U}_1$. The estimates (1.25) and (1.26) hold with $2\alpha_0 = 1/c_2^2$. The numbers c_1^* and c_2^* in those estimates are adjusted from the constants occurring in (7.8) and (7.9), respectively. □

Remark 7.5. This work focuses on the Navier friction boundary conditions with general friction coefficients, and on the thin domains with non-flat boundaries. It covers the cases when the friction coefficients may assume different values, zero or nonzero, on different portions of the boundary. Therefore our method is aimed to study the problem in this complicated situation, particularly when condition (1.18) for "generic" domains is satisfied. The reason for imposing the conditions (1.18) and (1.19) is to guarantee that the uniform Korn inequality (2.15) can be applied to $u(t)$ for all $t > 0$. In the case of no friction, i.e. $\gamma_0 = \gamma_1 = 0$, the conditions (1.18) and (1.19) are relaxed and one only requires $H_0 = Z_0$, see the treatment in [11]. In particular, the relation $H_0 = Z_0$ holds true when one of the boundaries is flat, for example, when $g_0 = 0$, see [16]. In the case when the friction coefficients satisfy, in addition to the upper bound condition (1.10), some lower bound condition, say, $\gamma_{0,\varepsilon} \geq c_0'\varepsilon^\delta$ and $\gamma_{1,\varepsilon} \geq c_0'\varepsilon^\delta$, where c_0' is a positive constant, then a "generalized" uniform Korn inequality can be established without the condition (1.18). This will be studied in our future work [10] and reported elsewhere, (see also [13] for the case $\gamma_0 = \gamma_1 = \varepsilon$ and the boundaries are flat).

Appendix A. Proof of Lemma 5.4

Proof. First, integration by parts yields

$$\int_{\Omega_\varepsilon} |\nabla^2 u|^2 dx = \int_{\Omega_\varepsilon} |\Delta u|^2 dx + \int_\Gamma \left(\frac{1}{2} \frac{\partial |\nabla u|^2}{\partial N} - \frac{\partial u}{\partial N} \cdot \Delta u \right) d\sigma. \tag{A.1}$$

Denote by I_0 the last integral on the boundary. For each $i = 0, 1$, let

$$N = N^i, \quad \tilde{\tau}_1 = \tau_1^i, \quad \tilde{\tau}_2 = \tilde{N}^i \times \tilde{\tau}_1,$$

see (4.7) and (4.8). Then $\{ \tilde{\tau}_1, \tilde{\tau}_2, N \}$ is an orthonormal frame on the boundary $\hat{\Gamma}$ satisfying

$$|\nabla \tilde{\tau}_1|, |\nabla \tilde{\tau}_2|, |\nabla N|, |\nabla^2 \tilde{\tau}_1|, |\nabla^2 \tilde{\tau}_2|, |\nabla^2 N| \le C\varepsilon, \quad \text{on} \quad \mathbb{R}^3. \tag{A.2}$$

We have

$$\begin{aligned} \frac{1}{2} \frac{\partial |\nabla u|^2}{\partial N} &= \frac{1}{2} \frac{\partial}{\partial N} \left| \frac{\partial u}{\partial \tilde{\tau}_1} \right|^2 + \frac{1}{2} \frac{\partial}{\partial N} \left| \frac{\partial u}{\partial \tilde{\tau}_2} \right|^2 + \frac{1}{2} \frac{\partial}{\partial N} \left| \frac{\partial u}{\partial N} \right|^2 \\ &= \frac{\partial^2 u}{\partial N \partial \tilde{\tau}_1} \cdot \frac{\partial u}{\partial \tilde{\tau}_1} + \frac{\partial^2 u}{\partial N \partial \tilde{\tau}_2} \cdot \frac{\partial u}{\partial \tilde{\tau}_2} + \frac{\partial^2 u}{\partial N \partial N} \cdot \frac{\partial u}{\partial N} \\ &= J_1 + J_2 + J_3. \end{aligned}$$

We consider J_3 first. Suppose $\{Y_1(x), Y_2(x), Y_3(x)\}$ be orthonormal with $Y_i = (Y_{i,1}, Y_{i,2}, Y_{i,3})$ for $i = 1, 2, 3$. Let ϕ be a smooth scalar function. One has

$$\frac{\partial \phi}{\partial Y_j} = (\nabla \phi) \cdot Y_j = \sum_{l=1}^{3} \partial_l \phi Y_{j,l},$$

$$\begin{aligned} \sum_{j=1}^{3} \frac{\partial^2 \phi}{\partial Y_j \partial Y_j} &= \sum_{j,k,l=1}^{3} \partial_k (\partial_l \phi Y_{j,l}) Y_{j,k} = \sum_{j,k,l=1}^{3} \partial_k \partial_l \phi Y_{j,l} Y_{j,k} + \partial_l \phi \partial_k Y_{j,l} Y_{j,k} \\ &= \sum_{j,k,l=1}^{3} \partial_k \partial_l \phi Y_{j,l} Y_{j,k} + \sum_{j,k,l=1}^{3} \partial_l \phi \partial_k (Y_{j,l} Y_{j,k}) - \sum_{j,k,l=1}^{3} \partial_l \phi Y_{j,l} \partial_k Y_{j,k}. \end{aligned}$$

Since $\sum_{j=1}^3 Y_{j,l} Y_{j,k} = \delta_{lk}$ we obtain

$$\sum_{j=1}^{3} \frac{\partial^2 \phi}{\partial Y_j \partial Y_j} = \Delta \phi + 0 - \sum_j \frac{\partial \phi}{\partial Y_j} (\nabla \cdot Y_j).$$

Let $Y_1 = \tilde{\tau}_1$, $Y_2 = \tilde{\tau}_2$, $Y_3 = N$, and $\phi = u_i$, for $i = 1, 2, 3$, we derive

$$\begin{aligned} \frac{\partial^2 u}{\partial N \partial N} = \Delta u &- \left(\frac{\partial^2 u}{\partial \tilde{\tau}_1 \partial \tilde{\tau}_1} + \frac{\partial^2 u}{\partial \tilde{\tau}_2 \partial \tilde{\tau}_2} \right) \\ &- \left\{ \frac{\partial u}{\partial N} (\nabla \cdot N) + \frac{\partial u}{\partial \tilde{\tau}_1} (\nabla \cdot \tilde{\tau}_1) + \frac{\partial u}{\partial \tilde{\tau}_2} (\nabla \cdot \tilde{\tau}_2) \right\}. \end{aligned} \tag{A.3}$$

It follows that

$$J_3 = \Delta u \cdot \frac{\partial u}{\partial N} - \left(\frac{\partial^2 u}{\partial \tilde{\tau}_1 \partial \tilde{\tau}_1} + \frac{\partial^2 u}{\partial \tilde{\tau}_2 \partial \tilde{\tau}_2} \right) \cdot \frac{\partial u}{\partial N} + J_3', \tag{A.4}$$

where

$$J_3' = -\left\{ \frac{\partial u}{\partial N} (\nabla \cdot N) + \frac{\partial u}{\partial \tilde{\tau}_1} (\nabla \cdot \tilde{\tau}_1) + \frac{\partial u}{\partial \tilde{\tau}_2} (\nabla \cdot \tilde{\tau}_2) \right\} \cdot \frac{\partial u}{\partial N}.$$

Thanks to (A.2), J_3' satisfies

$$|J_3'| \leq C\varepsilon |\nabla u|^2. \tag{A.5}$$

We focus on J_1 and J_2 now. Let $\gamma = \gamma_i$ on $\hat{\Gamma}_i$, for $i = 0, 1$. Suppose τ and τ' are unit tangential vectors to the boundary. Elementary calculations give

$$\frac{\partial^2 u}{\partial N \partial \tau} = \nabla u \left(\frac{\partial \tau}{\partial N} \right) + \big\{ (\nabla^2 u) \cdot N \big\} \tau,$$

where the matrix

$$(\nabla^2 u) \cdot N = \left(\sum_{k=1}^{3} \frac{\partial^2 u_i}{\partial x_k \partial x_j} N_k \right)_{i,j=1,2,3}.$$

Note that $\{(\nabla^2 u) \cdot N\}\tau = \{(\nabla^2 u) \cdot \tau\}N$, hence

$$\frac{\partial^2 u}{\partial N \partial \tau} - \frac{\partial^2 u}{\partial \tau \partial N} = \nabla u \left(\frac{\partial \tau}{\partial N} - \frac{\partial N}{\partial \tau} \right). \tag{A.6}$$

We obtain for $j = 1, 2$,

$$J_j = \frac{\partial^2 u}{\partial \tilde{\tau}_j \partial N} \cdot \frac{\partial u}{\partial \tilde{\tau}_j} + J_j', \tag{A.7}$$

where

$$J_j' = \left\{ \nabla u \left(\frac{\partial \tilde{\tau}_j}{\partial N} - \frac{\partial N}{\partial \tilde{\tau}_j} \right) \right\} \cdot \frac{\partial u}{\partial \tilde{\tau}_j}$$

satisfies

$$|J_j'| \leq C\varepsilon |\nabla u|^2. \tag{A.8}$$

Combining the above identities, we derive

$$\begin{aligned} I_0 &= \int_\Gamma J_1 + J_2 + J_3 - \frac{\partial u}{\partial N} \cdot \Delta u \, d\sigma \\ &= \int_\Gamma \frac{\partial^2 u}{\partial \tilde{\tau}_1 \partial N} \cdot \frac{\partial u}{\partial \tilde{\tau}_1} d\sigma + \int_\Gamma \frac{\partial^2 u}{\partial \tilde{\tau}_2 \partial N} \cdot \frac{\partial u}{\partial \tilde{\tau}_2} d\sigma - \int_\Gamma \left(\frac{\partial^2 u}{\partial \tilde{\tau}_1 \partial \tilde{\tau}_1} + \frac{\partial^2 u}{\partial \tilde{\tau}_2 \partial \tilde{\tau}_2} \right) \cdot \frac{\partial u}{\partial N} d\sigma + J \\ &= I_1 + I_2 + I_3 + J, \end{aligned}$$

where $J = \int_\Gamma J_1' + J_2' + J_3' d\sigma$ satisfies

$$|J| \leq C\varepsilon \int_\Gamma |\nabla u|^2 d\sigma \leq C \|\nabla u\|_{L^2}^2 + C\varepsilon^2 \|\nabla^2 u\|_{L^2}^2, \tag{A.9}$$

thanks to (A.5), (A.8) and (2.2).

To estimate I_1, I_2 and I_3 we need the integration by parts on Γ. By virtue of Lemma A.1 below, we have $I_3 = I_1 + I_2 + I_3'$ with

$$|I_3'| \le C\varepsilon \int_\Gamma \left|\frac{\partial u}{\partial \tilde\tau_1}\cdot\frac{\partial u}{\partial N}\right| + \left|\frac{\partial u}{\partial \tilde\tau_2}\cdot\frac{\partial u}{\partial N}\right| d\sigma \le C\varepsilon\int_\Gamma |\nabla u|^2 d\sigma, \tag{A.10}$$

where the last estimate is due to (A.2).

We now estimate I_1 and I_2. Taking the directional derivative $\partial/\partial\tau'$ of the identity

$$\frac{\partial u}{\partial N}\cdot\tau = \frac{\partial N}{\partial \tau}\cdot u - 2\gamma u\cdot\tau \text{ on } \hat\Gamma, \tag{A.11}$$

see (4.2), gives

$$\frac{\partial^2 u}{\partial\tau'\partial N}\cdot\tau + \frac{\partial u}{\partial N}\cdot\frac{\partial N}{\partial \tau'} = \frac{\partial u}{\partial \tau'}\cdot\frac{\partial N}{\partial\tau} + u\cdot\frac{\partial^2 N}{\partial\tau'\partial\tau} - 2\gamma\frac{\partial u}{\partial\tau'}\cdot\tau - 2\gamma u\cdot\frac{\partial\tau}{\partial\tau'}. \tag{A.12}$$

For $\tau, \tau' \in \{\tilde\tau_1, \tilde\tau_2\}$, we have

$$\left|\frac{\partial^2 u}{\partial\tau'\partial N}\cdot\tau + \gamma\frac{\partial u}{\partial\tau'}\cdot\tau\right| \le C\varepsilon|\nabla u| + C\varepsilon(1+\gamma)|u|. \tag{A.13}$$

Writing $\partial u/\partial\tilde\tau_1$ in the basis $\{\tilde\tau_1, \tilde\tau_2, N\}$ gives

$$\begin{aligned}\frac{\partial^2 u}{\partial\tilde\tau_1\partial N}\cdot\frac{\partial u}{\partial\tilde\tau_1} &= \left(\frac{\partial^2 u}{\partial\tilde\tau_1\partial N}\cdot\tilde\tau_1\right)\left(\frac{\partial u}{\partial\tilde\tau_1}\cdot\tilde\tau_1\right) + \left(\frac{\partial^2 u}{\partial\tilde\tau_1\partial N}\cdot\tilde\tau_2\right)\left(\frac{\partial u}{\partial\tilde\tau_1}\cdot\tilde\tau_2\right)\\ &\quad + \left(\frac{\partial^2 u}{\partial\tilde\tau_1\partial N}\cdot N\right)\left(\frac{\partial u}{\partial\tilde\tau_1}\cdot N\right).\end{aligned}$$

Then using (A.12), one derives

$$\begin{aligned}\frac{\partial^2 u}{\partial\tilde\tau_1\partial N}\cdot\frac{\partial u}{\partial\tilde\tau_1} &= J_4 - 2\gamma\left|\frac{\partial u}{\partial\tilde\tau_1}\cdot\tilde\tau_1\right|^2 - 2\gamma\left|\frac{\partial u}{\partial\tilde\tau_1}\cdot\tilde\tau_2\right|^2 - \frac{\partial^2 u}{\partial\tilde\tau_1\partial N}\cdot\left\{N\left(u\cdot\frac{\partial N}{\partial\tilde\tau_1}\right)\right\}\\ &\le J_4 - \frac{\partial^2 u}{\partial\tilde\tau_1\partial N}\cdot\left\{N\left(u\cdot\frac{\partial N}{\partial\tilde\tau_1}\right)\right\},\end{aligned}$$

where J_4 is bounded, thanks to (A.13), by $|J_4| \le C\varepsilon|\nabla u|^2 + C\varepsilon(1+\gamma)|\nabla u||u|$. We use the integration by parts (Lemma A.1) again to remove the second derivatives of u in the boundary integral to obtain

$$-\int_\Gamma \frac{\partial^2 u}{\partial\tilde\tau_1\partial N}\cdot\left\{N\left(u\cdot\frac{\partial N}{\partial\tilde\tau_1}\right)\right\}d\sigma = \int_\Gamma \frac{\partial u}{\partial N}\cdot\frac{\partial}{\partial\tilde\tau_1}\left\{N\left(u\cdot\frac{\partial N}{\partial\tilde\tau_1}\right)\right\}d\sigma + J_4',$$

where

$$|J_4'| \le C\varepsilon\int_\Gamma\left|\frac{\partial u}{\partial N}\cdot\left\{N\left(u\cdot\frac{\partial N}{\partial\tilde\tau_1}\right)\right\}\right|d\sigma \le C\varepsilon\int_\Gamma|\nabla u||u|d\sigma$$

Also,

$$\int_\Gamma\left|\frac{\partial u}{\partial N}\cdot\frac{\partial}{\partial\tilde\tau_1}\left\{N\left(u\cdot\frac{\partial N}{\partial\tilde\tau_1}\right)\right\}\right|d\sigma \le C\varepsilon\int_\Gamma|\nabla u|(|\nabla u|+|u|)d\sigma.$$

Therefore,

$$I_1 = \int_\Gamma \frac{\partial^2 u}{\partial \tilde\tau_1 \partial N} \cdot \frac{\partial u}{\partial \tilde\tau_1} d\sigma \le C\varepsilon \int_\Gamma |\nabla u|^2 + (1+\gamma)|\nabla u||u| d\sigma.$$

Similarly,

$$|I_2| \le C\varepsilon \int_\Gamma |\nabla u|^2 + (1+\gamma)|\nabla u||u| d\sigma.$$

Summing up and combine with the trace estimate (2.2) we have

$$I_0 \le C\varepsilon \int_\Gamma |\nabla u|^2 + |u|^2 d\sigma \le C\|u\|_{H^1}^2 + C\varepsilon^2 \|\nabla^2 u\|_{L^2}^2. \tag{A.14}$$

It follows that

$$\|\nabla^2 u\|_{L^2}^2 \le \|\Delta u\|_{L^2}^2 + I_0 \le \|\Delta u\|_{L^2}^2 + C\|u\|_{H^1}^2 + C\varepsilon^2 \|\nabla^2 u\|_{L^2}^2. \tag{A.15}$$

For $\varepsilon \in (0, \varepsilon_0]$ sufficiently small we obtain $\|\nabla^2 u\|_{L^2}^2 \le C\|\Delta u\|_{L^2}^2 + C\|u\|_{H^1}^2$, and the proof is complete. □

What remains to be proved is the following integration by parts on the boundary. We recall that $Q_2 = (0,1)^2$.

Lemma A.1. *For two smooth periodic vector fields u, v on $\hat\Gamma$ and a tangential vector field $a(x)$ to Γ, we have*

$$\int_\Gamma \frac{\partial u(x)}{\partial a(x)} \cdot v(x) d\sigma = -\int_\Gamma \frac{\partial v(x)}{\partial a(x)} \cdot u(x) d\sigma - \sum_{i=0,1} \int_{Q_2} u(x) \cdot v(x) \left\{ \frac{d}{dx_1}(a_1 H_i) + \frac{d}{dx_2}(a_2 H_i) \right\} dx', \tag{A.16}$$

where $H_i = \sqrt{1 + |\nabla_2 h_i|^2}$, for $i = 0, 1$. Consequently, for $j = 1, 2$, we have

$$\left| \int_\Gamma \frac{\partial u(x)}{\partial \tilde\tau_j(x)} \cdot v(x) d\sigma + \int_\Gamma \frac{\partial v(x)}{\partial \tilde\tau_j(x)} \cdot u(x) d\sigma \right| \le C\varepsilon \int_\Gamma |u \cdot v| d\sigma. \tag{A.17}$$

Proof. We first have

$$\int_\Gamma \frac{\partial u(x)}{\partial a(x)} \cdot v(x) d\sigma = -\int_\Gamma \frac{\partial v(x)}{\partial a(x)} \cdot u(x) d\sigma + \int_\Gamma \frac{\partial}{\partial a(x)} (u(x) \cdot v(x)) d\sigma.$$

Denote $F = u(x) \cdot v(x)$. One has

$$\int_\Gamma \frac{\partial F(x)}{\partial a(x)} d\sigma = \sum_{i=0,1} \int_{Q_2} (\partial_1 F a_1 + \partial_2 F a_2 + \partial_3 F a_3)(x', h_i(x')) H_i(x') dx'$$

$$= \sum_{i=0,1} \int_{Q_2} \left\{ \left(\frac{d}{dx_1} F(x', h_i(x')) - \partial_3 F \partial_1 h_i \right) a_1 \right.$$
$$\left. + \left(\frac{d}{dx_2} F(x', h_i(x')) - \partial_3 F \partial_2 h_i \right) a_2 + \partial_3 F a_3 \right\} H_i(x') dx'.$$

Integration by parts gives

$$\int_\Gamma \frac{\partial F(x)}{\partial a(x)} d\sigma = - \sum_{i=0,1} \int_{Q_2} F \left\{ \frac{d}{dx_1}(a_1 H_i) + \frac{d}{dx_2}(a_2 H_i) \right\} dx'$$
$$+ \sum_{i=0,1} \int_{Q_2} \partial_3 F (a_3 - a_1 \partial_1 h_i - a_2 \partial_2 h_i) H_i dx'.$$

Since $a \cdot N = 0$ yields $a_3 - a_1 \partial_1 h_i - a_2 \partial_2 h_i = 0$, we have

$$\int_\Gamma \frac{\partial F(x)}{\partial a(x)} d\sigma = - \sum_{i=0,1} \int_{Q_2} F \left\{ \frac{d}{dx_1}(a_1 H_i) + \frac{d}{dx_2}(a_2 H_i) \right\} dx', \tag{A.18}$$

thus (A.16) follows.

For $a = \tilde{\tau}_j$, $j = 1, 2$ one has from (A.2) that $\left| \frac{d}{dx_1}(a_1 H_i) \right|, \left| \frac{d}{dx_2}(a_2 H_i) \right| \leq C\varepsilon$, hence (A.18) yields

$$\left| \int_\Gamma \frac{\partial F(x)}{\partial \tilde{\tau}_j(x)} d\sigma \right| \leq C\varepsilon \int_\Gamma |F| d\sigma, \tag{A.19}$$

and we obtain (A.17). □

Appendix B. Proof of Lemma 6.3

Proof. Let $u \in L^2(\Omega_\varepsilon)$. We have

$$\begin{aligned}
\int_{\Omega_\varepsilon} |v||w||u| dx &= \int_{Q_2} \int_{h_0}^{h_1} |v||w||u| dx_3 dx' \\
&\leq \int_{Q_2} |v| \left(\int_{h_0}^{h_1} |w|^2 dx_3 \right)^{1/2} \left(\int_{h_0}^{h_1} |u|^2 dx_3 \right)^{1/2} dx' \\
&\leq \left(\int_{Q_2} \int_{h_0}^{h_1} |u|^2 dx_3 dx' \right)^{1/2} \left\{ \int_{Q_2} |v|^2 \left(\int_{h_0}^{h_1} |w|^2 dx_3 \right) dx' \right\}^{1/2} \\
&\leq \|u\|_{L^2(\Omega_\varepsilon)} \left(\int_{Q_2} |v|^4 dx' \right)^{1/4} \left\{ \int_{Q_2} \left(\int_{h_0}^{h_1} |w|^2 dx_3 \right)^2 dx' \right\}^{1/4} \\
&= \|u\|_{L^2(\Omega_\varepsilon)} \|v\|_{L^4(Q_2)} \left\{ \int_{Q_2} \eta^2 dx' \right\}^{1/4},
\end{aligned}$$

where

$$\eta = \int_{h_0}^{h_1} w^2 dx_3. \tag{B.1}$$

We recall Ladyzhenskaya–Gagliardo–Nirenberg inequality in Q_2 (see the proof below),

$$\|\eta\|_{L^2(Q_2)} \le \|\eta\|_{L^1(Q_2)} + \|\nabla_2 \eta\|_{L^1(Q_2)}. \tag{B.2}$$

Firstly,

$$\|\eta\|_{L^1(Q_2)} = \int_{Q_2} \left| \int_{h_0}^{h_1} w^2 dx_3 \right| dx' \le \|w\|^2_{L^2(\Omega_\varepsilon)}. \tag{B.3}$$

Secondly,

$$\partial_i \eta = \int_{h_0}^{h_1} 2w\partial_i w dx_3 + \partial_i h_1 w^2(h_1) - \partial_i h_0 w^2(h_0) = \int_{h_0}^{h_1} 2w\partial_i w + \partial_3(\psi_i w^2) dx_3.$$

Hence

$$\begin{aligned}\|\nabla\eta\|_{L^1(Q_2)} &\le C \int_{Q_2}\int_{h_0}^{h_1} |w||\nabla_2 w| + |w|^2 + \varepsilon|w||\partial_3 w| dx_3 dx' \\ &\le C\|w\|^2_{L^2(\Omega_\varepsilon)} + C\|w\|_{L^2(\Omega_\varepsilon)}\|\nabla_2 w\|_{L^2(\Omega_\varepsilon)} + C\varepsilon\|w\|_{L^2(\Omega_\varepsilon)}\|\partial_3 w\|_{L^2(\Omega_\varepsilon)}.\end{aligned}$$

Therefore

$$\|\eta\|_{L^2(Q_2)} \le C\|w\|_{L^2(\Omega_\varepsilon)}\big\{\|w\|_{L^2(\Omega_\varepsilon)} + \|\nabla_2 w\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|\partial_3 w\|_{L^2(\Omega_\varepsilon)}\big\}, \tag{B.4}$$

and we obtain

$$\begin{aligned}\|vwu\|_{L^1(\Omega_\varepsilon)} \le C\|v\|_{L^4(Q_2)}\|u\|_{L^2(\Omega_\varepsilon)}\|w\|^{1/2}_{L^2(\Omega_\varepsilon)} \\ \times \left\{\|w\|_{L^2(\Omega_\varepsilon)} + \|\nabla_2 w\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|\partial_3 w\|_{L^2(\Omega_\varepsilon)}\right\}^{1/2}.\end{aligned}$$

Applying classical Ladyzhenskaya inequality $\|v\|_{L^4(Q_2)} \le C\|v\|^{1/2}_{L^2(Q_2)}\|v\|^{1/2}_{H^1(Q_2)}$, we obtain (6.3). The proof is complete. □

Proof of (B.2). We have

$$\begin{aligned}\int_{Q_2} \eta^2 dx' &= \int_0^1\int_0^1 \eta(x_1,x_2)\eta(x_1,x_2)dx_1dx_2 \\ &\le \int_0^1 \max_{x_1}|\eta(x_1,x_2)| \int_0^1 |\eta(x_1,x_2)|dx_1dx_2 \\ &\le \max_{x_2}\int_0^1 |\eta(x_1,x_2)|dx_1 \int_0^1 \max_{x_1}|\eta(x_1,x_2)|dx_2 \\ &\le \int_0^1 \max_{x_2}|\eta(x_1,x_2)|dx_1 \int_0^1 \max_{x_1}|\eta(x_1,x_2)|dx_2.\end{aligned} \tag{B.5}$$

Integrating the identity

$$\eta(x_1, x_2) = \eta(x_1, y_2) + \int_{y_2}^{x_2} \partial_1\eta(x_1, z_2)dz_2,$$

with respect to y_2 from 0 to 1, we obtain

$$\eta(x_1, x_2) = \int_0^1 \eta(x_1, y_2)dy_2 + \int_0^1 \int_{y_2}^{x_2} \partial_1\eta(x_1, z_2)dz_2dy_2,$$

$$|\eta(x_1, x_2)| \le \int_0^1 |\eta(x_1, y_2)|dy_2 + \int_0^1 |\partial_1\eta(x_1, z_2)|dz_2.$$

Thus

$$\max_{x_2} |\eta(x_1, x_2)| \le \int_0^1 |\eta(x_1, y_2)|dy_2 + \int_0^1 |\partial_1\eta(x_1, z_2)|dz_2. \tag{B.6}$$

Now, integrating with respect to x_1 yields

$$\int_0^1 \max_{x_2} |\eta(x_1, x_2)|dx_1 \le \int_0^1 \int_0^1 |\eta(x_1, y_2)|dy_2dx_1 + \int_0^1 \int_0^1 |\partial_1\eta(x_1, z_2)|dz_2dx_1.$$

Thus

$$\int_0^1 \max_{x_2} |\eta(x_1, x_2)|dx_1 \le \|\eta\|_{L^1(Q_2)} + \|\partial_2\eta\|_{L^1(Q_2)}. \tag{B.7}$$

Similarly,

$$\int_0^1 \max_{x_1} |\eta(x_1, x_2)|dx_2 \le \|\eta\|_{L^1(Q_2)} + \|\partial_2\eta\|_{L^1(Q_2)}. \tag{B.8}$$

Therefore (B.2) follows from (B.5), (B.7) and (B.8). □

Acknowledgment. The author is thankful to Edriss Titi for bringing his attention to Lemma 6.3, and Marta Lewicka and George R. Sell for helpful discussions.

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(accepted: November 19, 2008; published Online First: October 23, 2009)