Theoret. Comput. Fluid Dynamics (1990) 1: 303–325

Theoretical and Computational Fluid Dynamics
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On Some Control Problems in Fluid Mechanics

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Communicated by M.Y. Hussaini

Abstract. The issue of minimizing turbulence in an evolutionary Navier–Stokes flow is addressed from the point of view of optimal control. We derive theoretical results for various physical situations: distributed control, Bénard-type problems with boundary control, and flow in a channel. For each case that we consider, our results include the formulation of the problem as an optimal control problem and proof of the existence of an optimal control (which is not expected to be unique). Finally, we describe a numerical algorithm based on the gradient method for the corresponding cost function. For readers who are not interested in the mathematical details and the mathematical justifications, a nontechnical description of our results is included in Section 5.

Introduction

It is common knowledge that one of the most mysterious aspects of fluid mechanics is understanding the behavior of turbulent flows. A related problem, although with a more materialistic flavor, is the question of *controlling turbulence* inside a flow. By the latter we mean that we consider a flow in a given physical domain, with known initial configuration, and that we can act upon the fluid through various devices (body forces, boundary values, temperature, etc.); we then address the issue of determining the optimal action upon the system in order to *minimize turbulence* within the flow. This is a fundamental problem from the application point of view, and it also offers some nontrivial mathematical interest.

In this paper we present a systematic approach to the mathematical formulation and resolution of the problem of minimizing turbulence. Naturally, we are led to consider problems that are best formulated in terms of optimal control theory: the set of *controls* will be all the admissible acting functions, the *state of the system*, uniquely determined for each element of the control set, will be the solution of the system of partial differential equations modeling the evolution of the flow in time, and the *cost function* will be some quadratic functional involving the *vorticity* in the fluid, such that its minimum corresponds to the minimum possible vorticity within the bounds of the other relevant constraints. For a more precise account of the methods and results of optimal control theory in the presence of partial differential equations, the reader is referred to Lions (1969). We consider two-dimensional flows, basically because the Navier–Stokes equations are not known to be well-posed in three dimensions. In each case we proceed along the following lines: we formulate a problem of optimal control, the solution of which corresponds to the least-turbulent flow (the one with the smallest vorticity); we then prove the existence of an optimal control, and address the crucial problem of *characterizing* such an optimal control, by deriving the first-order necessary optimality conditions associated with the problem. Once the optimality conditions are available, we write down the

Gradient Algorithm for each minimization problem, and sketch the study of the associated numerical problems.

The most arduous mathematical difficulties stem from the nonlinearity of the state equation, i.e., the equation giving the state of the system as a function of the control; as a consequence, the optimization problems are nonconvex, and this creates some technical difficulties in studying the existence and uniqueness of an optimal control. Furthermore, the optimality conditions must be derived from differentiability arguments that require some rather technical proofs.

Before describing the paper, we want to mention some other interesting problems in the control of turbulence. Within the range of the methods exposed in this paper lies the problem of *maximizing* turbulence, which is of great practical interest for instance in combustion. A much more complicated situation is the problem of three-dimensional flows, for which the state equation may not be well-posed; however, we believe that our methods apply as well, at least heuristically, to the three-dimensional case.

We now give a detailed description of the content of this article. In the first section we solve what can be viewed as a "model problem," that of minimizing turbulence in a Navier–Stokes flow by *distributed controls*; the equations we consider are

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u + \nabla p = f + \nu\Delta u \quad \text{in } \Omega, \\ \nabla\cdot u = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on the boundary } \partial\Omega, \\ u|_{t=0} = u_0 \quad \text{in } \Omega, \end{cases} \tag{1}$$

where $u = (u_1, u_2)$ is the velocity vector, p is the pressure, $\nu > 0$ is the kinematic viscosity, and Ω, the domain occupied by the fluid, is a two-dimensional open set with a smooth boundary; the system of equations (1) is the classical mathematical description of Newton's law of motion and conservation of mass for an incompressible viscous fluid (see, e.g., Landau and Lifschitz, 1966). In this case the *control* is the right-hand side of the momentum equation in (1), i.e., the volume forces f; this amounts to assuming that we act upon the system by changing the body forces applied to it. This is obviously rather unrealistic, but turns out to be a good introduction to more interesting, and more complicated, situations. This model problem contains all the basic features of general control problems that arise more naturally in controlling turbulence, and the exposition of the method and results is easier, as a consequence of its relative simplicity.

The optimal control problem is formulated as follows:

$$\text{(P)} \quad \begin{cases} \text{Find a control } f \text{ minimizing the cost function} \\ \qquad J(f) = \frac{1}{2}\displaystyle\int_0^T\int_\Omega |f(x,t)|^2\,dx\,dt + \frac{1}{2}\int_0^T\int_\Omega |\nabla\times u_f(x,t)|^2\,dx\,dt, \\ \text{where } u_f \text{ is the solution of (1) associated with } f \text{ and } \nabla\times u_f \text{ is the curl of } u_f. \end{cases} \tag{1a}$$

In the expression for $J(f)$, the term $\int_0^T\int_\Omega|\nabla\times u_f|^2\,dx\,dt$ speaks for itself, and is the relevant one. It measures the turbulence in the flow through the L^2-norm in space and time of the vorticity; as for $\int_0^T\int_\Omega|f|^2\,dx\,dt$, it is a concession to the demands of mathematical rigour. This coercivity term is necessary in order to prove the existence of an optimal control, and appears, possibly in a modified form, in all the situations that we consider.

For Problem (P), we obtain the following results:

- —the existence of an optimal control, which may not be unique,
- —the determination of the first-order necessary conditions of optimality, which are obtained in a straightforward manner upon differentiation of the functional $J(f)$; they involve in the language of optimal control, the so-called adjoint state corresponding to the adjoint of a linearized version of system (1) (while u_f itself is the state and (1) is the state equation in the terminology of optimal control);

—as a consequence of the optimality conditions, we obtain an expression for the "gradient" of J that provides explicit directions of descent for J, paving the way for a numerical study of Problem (P).

In Section 2 we focus our attention on a different system. We consider a fluid in a driven square cavity, and introduce another physical quantity, namely, the temperature inside the fluid; the temperature is allowed to change by means of a heat flux through the bottom of the cavity, and the control is now the boundary value of the temperature at the bottom of the cavity. On the vertical sides of the square, the temperature is kept constant (see Figure 1). In this situation the turbulence is created by the driving velocity at the top of the cavity, and the role of the control is to fight this effect by cooling the fluid from below. This is a Bénard-type problem, and the state of the system, determined by the velocity vector, the pressure, and the temperature, is given by the Boussinesq equations (see Foias *et al.*, 1987)

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = -\rho(\tau_1 - \tau)e_2 \quad \text{in } Q_T, \\ \dfrac{\partial \tau}{\partial t} + (u\cdot\nabla)\tau - \kappa\Delta\tau = 0 \quad \text{in } Q_T, \\ \nabla\cdot u = 0 \quad \text{in } Q_T, \qquad u(\cdot,0),\ \tau(\cdot,0) \quad \text{are given}, \\ u = u_0 \quad \text{on } \partial\Omega, \qquad \tau = e \quad \text{on } \partial\Omega. \end{cases} \tag{2}$$

In the equations above, the control is the prescribed temperature e at the bottom of the cavity. A natural formulation for the optimal control problem associated with this situation is:

$$(\mathrm{Q}_1)\quad \begin{cases} \text{Find } e \text{ minimizing the cost function} \\ \qquad K_1(e) = \frac{1}{2}\displaystyle\int_0^T\int_\Omega |\nabla\times u_e|^2\,dx\,dt + \frac{1}{2}\int_0^T\int_{\Gamma_2} |e|^2\,dx\,dt, \\ \text{where } u_e \text{ is the velocity vector inside the fluid, and a solution of (2).} \end{cases} \tag{2a}$$

Unfortunately, serious difficulties occur when we try to solve Problem (Q_1) directly; in particular, the vorticity need not be square-integrable, so that $K_1(e)$ need not be defined for every e. In order to overcome these difficulties, which arise mainly because of the limited smoothness of the boundary of Ω, we study a modified version of (Q_1).

Essentially, the modifications consist in changing slightly the geometry of the physical domain, and adapting the functional K_1 so as to have a well-posed problem. As in Section 1, we prove the existence of an optimal control, and derive the first-order necessary conditions that characterize it; next, we confine our results to the physically important case of controls consisting in a *finite* number of acting functions, the intensity of which can be modulated in time. This is a very realistic assumption that leads to relatively simple conditions of optimality. Finally, we adapt the numerical algorithms exposed in Section 1 to this new situation.

In the third section we deal with yet another problem. We consider a fluid in a square cavity, kept at given temperatures at the top and the bottom, and act upon the system by imposing various values of the velocity at the top of the cavity (see Figure 2). This velocity v is now the control, and must be chosen so as to minimize the turbulent effect of the temperature gradient. The equations of the state of the system are very similar to those in Section 2, except that we now impose periodicity conditions in the longitudinal direction instead of assigning specific boundary values. Needless to say, this is not the most important change.

We then study the problem of optimal control that naturally comes to mind:

$$(\mathrm{R})\quad \begin{cases} \text{Find } v \text{ minimizing the functional} \\ L(v) = \frac{1}{2}\displaystyle\int_0^T\int_{\Gamma_1} v^2(x,t)\,d\Gamma\,dt + \frac{1}{2}\int_0^T\int_\Omega |\nabla\times u_v|^2(x,t)\,dx\,dt, \\ \text{where } u_v \text{ is the velocity vector inside the fluid.} \end{cases} \tag{2b}$$

We derive, at least formally, the first-order necessary conditions for an optimal control, the rigourous proof of which is available through suitable modifications along the lines of Section 2.

The fourth section is devoted to the study of the control of a flow in a channel. The flow is driven by a gradient of pressure and the turbulence is somehow measured by the stresses at the boundary. The flow is periodic in the horizontal direction, and the control is the boundary value of the velocity field at the top and the bottom of the channel. The minimization problem is now to:

$$\left\{\begin{array}{l}\text{Find } \varphi \text{ minimizing the functional} \\ \qquad M(\varphi) = \frac{1}{2}\int_0^T \int_\Gamma |D(u)|^2 \, d\sigma \, dt + \alpha \|\varphi\|_X^2, \\ \text{where } D(u) = \frac{1}{2}(\nabla u + \nabla u^T) \text{ is the tensor of deformation rate and } X \text{ is the space of the} \\ \text{boundary velocity } \varphi. \end{array}\right. \tag{3}$$

After deriving the first-order optimality conditions, we write down the formulation of the Gradient Algorithm for this particular problem, in the manner of the previous sections.

Section 5 contains some comments and summarizes our main results.

1. A Model Distributed Control Problem: Control of the Forcing Term

In this first section we describe and study a problem of distributed control for the minimization of turbulence in a Navier–Stokes fluid.

We consider a system of Navier–Stokes equations in an open set Ω of $\mathbb{R}^2$:

$$\left\{\begin{array}{l}\dfrac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = f + \nu \Delta u \quad \text{in } \Omega, \\ \nabla \cdot u = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on the boundary } \partial\Omega, \\ u|_{t=0} = u_0 \quad \text{in } \Omega, \end{array}\right. \tag{1.0.1}$$

where the forcing term f is the control. Our goal is to determine the existence and characterization of those f that minimize the turbulence in Ω. This particular choice of control is not of the greatest practical interest, for we generally think of acting on the system through boundary controls. Nevertheless, we find it enlightning to expose the method in this rather simple case, before considering more involved situations. Also, it will become clear that other, more interesting, problems can be reduced to this model problem, or a slightly modified version thereof.

1.1. The Mathematical Setting of the Problem

We consider an open bounded set Ω of $\mathbb{R}^2$ with a C^2 boundary $\partial\Omega$. Following Temam (1984), we set $\mathcal{X} = \{u \in (C_0^\infty(\Omega))^2, \nabla \cdot u = 0 \text{ in } \Omega\}$, and denote by H (resp. V) the closure of $\mathcal{X}$ in $(L^2(\Omega))^2$ (resp. $(H_0^1(\Omega))^2$). The norm and inner product on H (resp. V) are denoted by $|u|$ and (u, v) (resp. $\|u\|$ and $((u, v))$). The Navier–Stokes equations in Ω can be written under the following form (see Temam, 1984):

$$\left\{\begin{array}{ll}\dfrac{du}{dt} + \nu A u + B(u) = f \text{ in} & Q_T = \Omega \times (0, T), \\ u(\cdot, t) \in V, & \forall t \in (0, T), \\ u(\cdot, 0) = u_0 & \text{in } \Omega. \end{array}\right. \tag{1.1.1}$$

T is a fixed (but arbitrary) strictly positive real number, the forcing term f is in $L^2(0, T; H)$, and the initial datum u_0 is in H; as usual, the kinematic viscosity ν is > 0. The operators A, B are defined as follows: $Au = -P\Delta u$, where Δ is the vector Laplacian, and P is the orthogonal projector from $L^2(\Omega)^2$ onto H, and B is the nonlinear operator, from V into its dual V', such that $\langle B(u), v\rangle_{V' \times V} = \int_\Omega (u \cdot \nabla)u \cdot v \, dx$ for every v in V.

This variational formulation, excluding the pressure, of the Navier–Stokes equations is by now classical. For further use, we recall some classical results related to (1.1.1).

Proposition 1.1.1.

(i) *Let f, u_0 be given in $L^2(0, T; V')$ and H; there exists a unique weak solution u of* (1.1.1), *and u belongs to $C([0, T]; H) \cap L^2(0, T; V)$.*
(ii) *If we assume furthermore that u_0 is in V and that f is in $L^2(0, T; H)$, then u is in $C([0, T]; V) \cap L^2(0, T; (H^2(\Omega))^2)$, and du/dt is in $L^2(0, T; H)$.*

Concerning the nonlinear operator B, we have:

Lemma 1.1.2. *Let $\mathscr{B}(a, b, c)$ be the trilinear form on $V \times V \times V$ defined by*

$$\mathscr{B}(a, b, c) = \int_\Omega \sum_{i,j=1}^{2} a_i \frac{\partial b_j}{\partial x_i} c_j \, dx,$$

and let $B(a) = \mathscr{B}(a, a)$ in V'.

(i) *$\mathscr{B}$ has the following properties:*

$$\begin{aligned}
&(\textit{orthogonality}) && \mathscr{B}(a, b, b) = 0,\\
&\forall (a, b, c) \in V^3, && |\mathscr{B}(a, b, c)| \leq \|a\| \, \|b\| \, \|c\|,\\
&\forall (a, b, c) \in V^3, && |\mathscr{B}(a, b, c)| \leq \sqrt{2}|a|^{1/2} \|a\|^{1/2} \|b\| \, |c|^{1/2} \|c\|^{1/2},\\
&\forall (a, b, c) \in V \times (V \cap H^2(\Omega))^2) \times H, && \mathscr{B}(a, b, c)| \leq C|a|^{1/2} \|a\|^{1/2} \|b\|^{1/2} |Ab|^{1/2} |c|.
\end{aligned}$$

(ii) *$a \to B(a)$ is differentiable from V into V', and we have*

$$\forall b \in V, \qquad B'(a)b = \mathscr{B}(a, b) + \mathscr{B}(b, a).$$

(iii) *Let $B'(a)^*$ denote the adjoint of $B'(a)$ for the duality between V and V', i.e., $\langle B'(a) \cdot b, c \rangle = \langle b, B'(a)^* c \rangle$; we have*

$$\langle B'(a)^* b, c \rangle_{V' X V} = \int_\Omega \sum_{i,j=1}^{2} c_j \left(\frac{\partial a_i}{\partial x_j} b_i - a_i \frac{\partial b_j}{\partial x_i} \right) dx.$$

The proof of assertions (ii) and (iii) is postponed till the end of Section 1; as for (i), the interested reader is referred to Chapter III, Section 3, of Temam (1984). Let us now focus our attention on the control problem we are interested in. As mentioned before, the control is the right-hand side of (1.1.1); in other words, we are acting on the system by prescribing different possible forcing terms, and the space of controls is, naturally, $L^2(0, T; H)$. The expression "to minimize the turbulence in the system" can be given various mathematical interpretation; the criterion we choose is the *L^2-norm of the curl of u*. The smaller this quantity is, the less agitated the fluid in Ω will be. For purely mathematical reasons, i.e., to ensure the well-posedness of the optimization problem, we are led to formulate the control problem in the following terms:

(P) $\left\{\begin{array}{l} \text{Find } f \text{ in } L^2(0, T; H),\ u_f \text{ is the solution of (1.1.1) associated with } f, \text{ minimiz-}\\ \text{ing the cost function} \\ \displaystyle J(f) = \tfrac{1}{2} \int_0^T \int_\Omega |f(x, t)|^2 \, dx \, dt + \tfrac{1}{2} \int_0^T \int_\Omega |\nabla \times u_f(x, t)|^2 \, dx \, dt, \qquad (1.1.2) \\ \text{where } \nabla \times u \text{ denotes the curl of the vector field } u. \end{array}\right.$

Problem (1.1.2) is a nonconvex optimization problem for the mapping $f \mapsto u_f$ is nonlinear; we study the existence of its solution(s) by classical means, using the built-in coercivity of the functional J. Moreover, we state and prove the system of the necessary first-order optimality condition for (P), using the smoothness property of $f \mapsto u_f$. However, we do not have any uniqueness result, mainly because of the nonconvexity of J. For the moment, we state and prove the existence of an optimal pair $(\bar{f}, \bar{u}_f)$:

Theorem 1.1.3. *Let u_0 be given in H; there exists at least an element $\bar{f}$ in $L^2(0, T; H)$, and $\bar{u}$ in $C([0, T], H) \cap L^2(0, T; V)$ such that the functional $J(f)$ attains its minimum at $\bar{f}$, and $\bar{u} = u_{\bar{f}}$ is the solution of* (1.1.1) *with the right-hand side $\bar{f}$.*

Proof. Let f_n be a minimizing sequence for Problem (P), and set $u_n = u_{f_n}$. The sequence (f_n) is bounded in $L^2(0, T; H)$ and, therefore, (u_n) is bounded in $C([0, T]; H) \cap L^2(0, T; V)$. We can then find a pair $(\bar{f}, \bar{u})$ and a subsequence, still denoted by (f_n, u_n), such that

$$\begin{cases} f_n \to \bar{f} & \text{in } L^2(0, T; H) \text{ weakly,} \\ u_n \to \bar{u} & \text{in } L^2(0, T; V) \text{ weakly and in } L^\infty(0, T; H) \text{ *-weakly.} \end{cases} \tag{1.1.3}$$

In particular, $\nabla \times u_n \to \nabla \times \bar{u}$ in $L^2(0, T; H)$ weakly; by lower semicontinuity, we have

$$\iint_{Q_T} |\bar{f}(x, t)|^2 \, dx \, dt \le \varliminf_{n\to\infty} \iint_{Q_T} |f_n(x, t)|^2 \, dx \, dt$$

and

$$\iint_{Q_T} |\nabla \times \bar{u}(x, t)|^2 \, dx \, dt \le \varliminf_{n\to\infty} \iint_{Q_T} |\nabla \times u_n|^2(x, t) \, dx \, dt,$$

which implies

$$\tfrac{1}{2} \iint_{Q_T} |\bar{f}(x, t)|^2 \, dx \, dt + \tfrac{1}{2} \iint_{Q_T} |\nabla \times \bar{u}(x, t)|^2 \, dx \, dt \le \varliminf_{n\to\infty} J(f_n). \tag{1.1.4}$$

One thing remains to be verified, namely that $\bar{u}$ is actually the solution of (1.1.1) with right-hand side $\bar{f}$. This result hinges upon an *a priori* estimate, proven by Temam (1984, see Chapter III, Sections 2–3) for u in a fractional order (in time) Sobolev space. This essential estimate yields a compactness property, namely (in our case) that $u_n \to \bar{u}$ strongly in $L^2(0, T; H)$; this makes passing to the limit in the nonlinear term possible (see Temam, 1984, Chapter III, Sections 3–2) and therefore concludes the proof of Theorem 1.1.3. □

1.2. First-Order Necessary Conditions for the Optimal Pair $(\bar{f}, \bar{u})$

We now proceed to derive the first-order optimality conditions associated with Problem (P). This is done in a straightforward manner by studying the Gâteaux derivative of the functional $J(f)$. As a matter of fact, for every f in $L^2(0, T; H)$, we have,

$$\forall \lambda \in R, \qquad J(\bar{f} + \lambda f) \ge J(\bar{f}),$$

due to the very definition of $\bar{f}$. In particular, we have,

$$\forall \lambda > 0, \qquad \frac{J(\bar{f} + \lambda f) - J(\bar{f})}{\lambda} \ge 0$$

and,

$$\forall \lambda < 0, \qquad \frac{J(\bar{f} + \lambda f) - J(\bar{f})}{\lambda} \le 0,$$

which implies that the derivative at $\lambda = 0$ of the function $\lambda \mapsto J(\bar{f} + \lambda f)$ (which is precisely the Gâteaux derivative of J in the direction of f at $\bar{f}$) vanishes for every f in $L^2(0, T; H)$.

Before stating our main result, we give some auxiliary lemmas:

Lemma 1.2.1. *Let u_0 be in V; the mapping $f \mapsto u_f$, from $L^2(0, T; H)$ into $L^2(0, T; V)$, has a Gâteaux derivative $((\mathscr{D}u_f/\mathscr{D}f)\cdot h)$ in every direction h in $L^2(0, T; H)$. Furthermore, $(\mathscr{D}u_f/\mathscr{D}f)\cdot h = w(h)$ is the solution of the linearized problem*

$$\begin{cases} \dfrac{dw}{dt} + \nu A w + B'(u_f)\cdot w = h \quad \text{in } Q_T. \\ w \in V, \\ w(0) = 0 \quad \text{in } \Omega; \end{cases} \tag{1.2.1}$$

finally, w is in $L^\infty(0, T; V) \cap L^2(0, T; (H^2(\Omega))^2)$.

Lemma 1.2.2. *Let h_1 be given in $L^2(0, T; H)$, and let $w(h_1)$ be defined as above; for every h_2 in $L^2(0, T; H)$, we have*

$$\iint_{Q_T} (h_2 \cdot w(h_1))(x, t)\, dx\, dt = \iint_{Q_T} (\tilde{w}(h_2) \cdot h_1)(x, t)\, dx\, dt,$$

where $\tilde{w}(h_2)$ is the solution of the adjoint linearized problem

$$\begin{cases} -\dfrac{d\tilde{w}}{dt} + \nu A\tilde{w} + B'(u_f)^* \cdot \tilde{w} = h_2 \quad \text{in } Q_T, \\ \tilde{w} \in V, \\ \tilde{w}(T) = 0. \end{cases} \tag{1.2.2}$$

The proof of Lemma 1.2.1 is postponed till the end of Section 1.

Proof of Lemma 1.2.2. We proceed as follows (dropping momentarily the argument h_1, h_2 in w and $\tilde{w}$):

$$\begin{aligned}
&\iint_{Q_T} h_2 w\, dx\, dt \\
&\quad = \iint_{Q_T} \left(-\frac{d\tilde{w}}{dt} + \nu A\tilde{w} + B'(u_f)^*\tilde{w}\right) \cdot w\, dx\, dt \\
&\quad = \int_\Omega \left(\int_0^T \left(-\frac{d\tilde{w}}{dt} \cdot w\right) dt\right) dx + \int_0^T \left(\int_\Omega \nu A\tilde{w} \cdot w\, dx\right) dt + \int_0^T \left(\int_\Omega B'(u_f)^*\tilde{w} \cdot w\, dx\right) dt \\
&\quad = \text{(using the definition of } B'(u_f)^* \text{ and the fact that } A \text{ is self-adjoint)} \\
&\quad = \int_\Omega \left([-\tilde{w}w]_0^T + \int_0^T \tilde{w}\frac{dw}{dt}\, dt\right) dx + \nu \int_0^T \left(\int_\Omega \tilde{w} \cdot Aw\, dx\right) dt + \int_0^T \left(\int_\Omega \tilde{w} \cdot B'(u_f)w\, dx\right) dt \\
&\quad = \iint_{Q_T} \tilde{w}\left(\frac{dw}{dt} + \nu Aw + B'(u_f)w\right) dx\, dt \\
&\quad = \iint_{Q_T} \tilde{w} \cdot h_1\, dx\, dt,
\end{aligned}$$

and the integrations by parts used above can be easily justified rigorously through a density argument, using the regularity properties of u_f, w, $\tilde{w}$. □

Remark 1.2.3. In a less abstract way, we can express w, $\tilde{w}$ as the respective solutions of the following systems:

$$\begin{cases} \dfrac{\partial w}{\partial t} - \nu\Delta w + (u_f \cdot \nabla)w + (w \cdot \nabla)u_f + \nabla q = h_1 \quad \text{in } Q_T, \\ \nabla \cdot w = 0 \quad \text{in } Q_T, \qquad w = 0 \quad \text{on } \partial\Omega, \\ w(x, 0) = 0 \quad \text{in } \Omega, \end{cases} \tag{1.2.1'}$$

and

$$\begin{cases} -\dfrac{\partial \tilde{w}}{\partial t} - \nu\Delta\tilde{w} + (\nabla u_f)^T w - (u_f \cdot \nabla)\tilde{w} + \nabla\tilde{q} = h_2 \quad \text{in } Q_T, \\ \nabla \cdot \tilde{w} = 0 \quad \text{in } Q_T, \qquad \tilde{w} = 0 \quad \text{on } \partial\Omega, \\ \tilde{w}(x, T) = 0 \quad \text{in } \Omega. \end{cases} \tag{1.2.2'}$$

These equations are just the translation of (1.2.1) and (1.2.2) in terms of the concrete operators; q, $\tilde{q}$ can be interpreted respectively as the linearized and adjoint pressures.

We now state our main result:

Theorem 1.2.4.

(i) *Let* $(\bar{f}, \bar{u})$ *be an optimal pair for Problem* (P); *the following equality holds*

$$\bar{f} + \tilde{w}(\nabla \times (\nabla \times \bar{u})) = 0,$$

where $\tilde{w}(\nabla \times (\nabla \times \bar{u}))$ *is the adjoint state that is the solution of the linearized adjoint problem*

$$\begin{cases} -\dfrac{\partial \tilde{w}}{\partial t} - \nu\Delta\tilde{w} + (\nabla\bar{u})^T\cdot\tilde{w} - (\bar{u}\cdot\nabla)\tilde{w} + \nabla\tilde{q} = \nabla \times (\nabla \times \bar{u}) \quad \textit{in } Q_T, \\ \nabla\cdot\tilde{w} = 0, \qquad \tilde{w} = 0 \quad \textit{on } \partial\Omega, \\ \tilde{w}(x, T) = 0 \quad \textit{in } \Omega. \end{cases} \tag{1.2.3}$$

(ii) $\bar{f}$ *is in* $L^\infty(0, T; V) \cap L^2(0, T; (H^2(\Omega))^2)$.

Remark 1.2.5. If $u_0 = 0$ or, more generally, if $\nabla \times u_0 = 0$, then we find the obvious solution $\bar{f} = 0$; as a matter of fact, in such a case, $\nabla \times u$ is always zero, and therefore, so are $\tilde{w}$ and $\bar{f}$.

Proof of Theorem 1.2.4. Let $(\bar{f}, \bar{u})$ be an optimal pair; denoting, as before, the functional we want to minimize by $J(f)$, we write that the Gâteaux derivative $J'(\bar{f})\cdot h$ of J in the direction of h is zero, for every h in $L^2(0, T; H)$. As a result of Lemma 1.2.1, the chain rule provides the following expression:

$$\begin{aligned} J'(f)\cdot h &= \int\!\!\int_{Q_T} \left[f\cdot h + (\nabla \times u_f)\cdot\left(\nabla \times \frac{Du_f}{Df}\cdot h\right)\right] dx\, dt \\ &= \text{(with the notations above)} \\ &= \int\!\!\int_{Q_T} [f\cdot h + (\nabla \times u_f)\cdot(\nabla \times w(h))]\, dx\, dt, \end{aligned}$$

so that an integration by parts yields

$$\begin{aligned} J'(f)\cdot h &= \int\!\!\int_{Q_T} f\cdot h\, dx\, dt + \int\!\!\int_{Q_T} \nabla \times (\nabla \times u_f)\cdot w(h)\, dx\, dt + \int_0^T \int_{\partial\Omega} (\nabla \times u_f)\cdot(w(h) \times \vec{n})\, d\Gamma\, dt \\ &= \text{(for } w \text{ is zero on } \partial\Omega) \\ &= \int\!\!\int_{Q_T} [f\cdot h + \nabla \times (\nabla \times u_f)\cdot w(h)]\, dx\, dt. \end{aligned} \tag{1.2.4}$$

By mimicking the proof of Lemma 1.2.2, we obtain

$$J'(f)\cdot h = \int\!\!\int_{Q} [f + \tilde{w}(\nabla \times \nabla \times u_f)]\cdot h\, dx\, dt, \tag{1.2.5}$$

where $\tilde{w}(\nabla \times \nabla \times u_f)$ is the solution of (1.2.3).

We now interpret the fact that $\bar{f}$ realizes the minimum of J. If J attains its minimum at $\bar{f}$, then we necessarily have $J'(\bar{f})\cdot h = 0$ for every h in $L^2(0, T; H)$, that is to say

$$\bar{f} + \tilde{w}(\nabla \times (\nabla \times \bar{u})) = 0 \quad \text{in } Q_T. \tag{1.2.6}$$

Part (i) of Theorem 1.2.4 is therefore proven, and part (ii) is an easy consequence of the regularity properties of $\tilde{w}$ (which are similar to those of $w(h)$); $\tilde{w}$ is in $L^\infty(0, T; V) \cap L^2(0, T; (H^2(\Omega))^2)$. (This result can be easily infered from the proof of Lemma 1.2.1). □

We now prove the technical results used in this section:

Proof of Lemma 1.1.2. Assertion (i) is proven by Temam (1984, Chapter III.3); as for (ii), we proceed as follows: we need to prove

$$\operatorname*{Sup}_{\substack{c\in V\\ c\neq 0}} \frac{|(B(b) - B(a) - B'(a)\cdot(b - a), c)|}{\|b - a\|\,\|c\|} \to 0 \qquad \text{as} \quad \|b - a\| \to 0,$$

with $B'(a)\cdot(b-a)=\mathscr{B}(b-a,a)+\mathscr{B}(a,b-a)$. For a, b, c in V, we have

$$B(b)-B(a)-B'(a)\cdot(b-a)=\mathscr{B}(b,b)-\mathscr{B}(a,a)-\mathscr{B}(a,b-a)-\mathscr{B}(b-a,a)$$
$$=\mathscr{B}(b-a,b-a),$$

which yields, using the continuity property of $\mathscr{B}$ (Lemma 1.1.2(i)),

$$\sup_{\substack{c\in V\\ c\neq 0}}\frac{|(B(b)-B(a)-B'(a)\cdot(b-a),c)|}{\|c\|\,\|b-a\|}\leq C\|b-a\| \tag{1.2.7}$$

and (ii) is proven. For (iii), we write

$$\langle B'(a)\cdot b,c\rangle_{V'\times V}=\int_\Omega\sum_{i,j}\left(a_i\frac{\partial b_j}{\partial x_i}c_j+b_i\frac{\partial a_j}{\partial x_i}c_j\right)dx$$
$$=\text{(by integration by parts)}$$
$$=\int_\Omega\sum_{i,j}b_i\left(\frac{\partial a_j}{\partial x_i}c_j-a_j\frac{\partial c_j}{\partial x_j}\right)dx,$$

because $\sum_i(\partial a_i/\partial x_i)=0$, and a, b, c vanish on $\partial\Omega$. This proves assertion (iii). □

Remark 1.2.6. Lemma 1.1.2 still holds true in the three-dimensional case.

Proof of Lemma 1.2.1. We now fix u_0 in V, and let f, h be given in $L^2(0,T;H)$. Using the same notations as previously, we need to prove the following result:

$$\lim_{s\to 0}\left(\frac{|u_{f+sh}-u_f-sw(h)|_{L^2(0,T;V)}}{|s|}\right)=0. \tag{1.2.8}$$

Let us set $R=u_{f+sh}-u_f-sw(h)$: R is the solution of the evolution equation

$$\begin{cases}\dfrac{dR}{dt}+\nu AR+B(u_{f+sh})-B(u_f)-B'(u_f)\cdot sw=0\quad\text{in } Q_T,\\ R\in V,\\ R(0)=0.\end{cases} \tag{1.2.9a}$$

Using Lemma 1.1.2, if we define the function $k(x,t)$ as follows,

$$B(u_{f+sh})-B(u_f)-B'(u_f)\cdot(u_{f+sh}-u_f)=k(x,t),$$

we have

$$\|k(\cdot,t)\|_{V'}\leq C\|n_{f+sh}-u_f\|_V^2, \tag{1.2.9b}$$

and R is the solution of

$$\begin{cases}\dfrac{dR}{dt}+\nu AR+B'(u_f)\cdot R=k(x,t)\quad\text{in } Q_T,\\ R\in V,\qquad R(0)=0.\end{cases} \tag{1.2.10}$$

To estimate $\int_0^T\|R(t)\|^2\,dt$, we rely on some standard techniques, based on the inequality (Lemma 1.1.2(i))

$$\|(B'(u_f)\cdot R,R)\|\leq C\|u\|\,|R|^{1/2}\|R\|^{3/2},$$

to obtain the inequality

$$\int_0^T\|R(t)\|^2\,dt\leq C\int_0^T(|k(\cdot,t)|_{V'}^2)\,dt,$$

where the constant C depends on Ω, ν, u_s, f, h, and can be chosen independently of s for small values of s. As a result of (1.2.9), we obtain

$$\int_0^T\|R(t)\|^2\,dt\leq C\int_0^T\|u_{f+sh}-u_f\|^4\,dt$$

and our claim will follow from the estimate

$$\operatorname*{Sup}_{0\le t\le T} \|u_{f+sh} - u_f\| \le C|s|. \tag{1.2.11}$$

In order to prove (1.2.11), we set $\hat{u} = u_{f+sh} - u_f$; due to the fact that u_0 is in V, we know that both u_{f+sh} and u_f are in $L^\infty(0, T; V) \cap L^2(0, T; (H^2(\Omega))^2)$. Moreover, we can bound their norms in the latter space by a constant independent of s, for s small enough.

The function $\hat{u}$ is the solution of

$$\begin{cases} \dfrac{d\hat{u}}{dt} + \nu A\hat{u} + \mathscr{B}(u_f, \hat{u}) + \mathscr{B}(\hat{u}, u_f) + \mathscr{B}(\hat{u}) = sh \quad \text{in } Q_T, \\ \hat{u} \in V, \qquad \hat{u}(0) = 0. \end{cases} \tag{1.2.12}$$

To estimate the norm of $\hat{u}$ in $L^\infty(0, T; V)$, we take two successive steps. Firstly, we take, in (1.2.12), the scalar product with $\hat{u}$ and obtain, using the orthogonality property of $\mathscr{B}$ (Lemma 1.1.2(i)),

$$\frac{1}{2}\frac{d}{dt}|\hat{u}|^2 + \nu\|\hat{u}\|^2 \le |(sh, \hat{u})| + |\mathscr{B}(\hat{u}, u_f, \hat{u})|.$$

The right-hand side of the inequality above is estimated exactly as for (1.2.10), using the upper bound for $|\mathscr{B}(\hat{u}, u_f, \hat{u})|$

$$|\mathscr{B}(\hat{u}, u_f, \hat{u})| \le C\|u_f\|\,|\hat{u}|^{1/2}\|\hat{u}\|^{3/2}.$$

Using Gronwall's Lemma technique, we infer that $\hat{u}$ is in $L^2(0, T; H) \cap L^2(0, T; V)$ and, furthermore, that the norm of $\hat{u}$ in the latter space is bounded by $C|s|$, where C can be chosen so as to depend only on Ω, ν, $\|u_0\|$, $|f|$, $|h|$, T. We now proceed to derive the estimate of $\hat{u}$ in $L^\infty(0, T; V)$; in order to do so, we take the scalar product with $A\hat{u}$ in (1.2.9), and obtain

$$\frac{1}{2}\frac{d}{dt}\|\hat{u}\|^2 + \nu|A\hat{u}|^2 \le \left|\int_\Omega (sh \cdot A\hat{u})\,dx\right| + |\mathscr{B}(u_f, \hat{u}, A\hat{u})| + |\mathscr{B}(\hat{u}, u_f, A\hat{u})| + |\mathscr{B}(\hat{u}, \hat{u}, A\hat{u})|. \tag{1.2.13}$$

The end of the proof once again uses Lemma 1.1.2(i):

$$|\mathscr{B}(a, b, c)| \le C|a|^{1/2}\|a\|^{1/2}\|b\|^{1/2}|Ab|^{1/2}|c| \qquad \text{for } (a, b, c) \text{ in } \quad V \times (H^2(\Omega))^2 \times H.$$

Using this inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\hat{u}\|^2 + \nu|A\hat{u}|^2 \le |s||h||A\hat{u}| + C\|u_f\|\,\|\hat{u}\|^{1/2}|A\hat{u}|^{3/2} + C\|\hat{u}\|\,\|u_f\|^{1/2}|Au_f|^{1/2}|A\hat{u}| + C|\hat{u}|^{1/2}\|\hat{u}\|\,|A\hat{u}|^{3/2}. \tag{1.2.14}$$

We now use Young's inequality, valid for α, $\beta > 0$, $1 < p < +\infty$, $\varepsilon > 0$,

$$\alpha\beta \le \frac{\varepsilon}{p}\alpha^p + \frac{1}{p'\varepsilon^{p'/p}}\beta^{p'} \qquad \left(p' = \frac{p}{p-1}\right),$$

and apply it to all four terms in the right-hand side of (1.2.14), each time creating the term $((\nu/5)|A\hat{u}|^2)$. We therefore obtain a new differential inequality for $\|\hat{u}\|$, with a possibly very large constant,

$$\frac{1}{2}\frac{d}{dt}\|\hat{u}\|^2 + \frac{\nu}{5}|A\hat{u}|^2 \le C(s^2|h|^2 + \|u_f\|^4\|\hat{u}\|^2 + |\hat{u}|\,\|\hat{u}\|\,\|u_f\|\,|Au_f| + |\hat{u}|^2\|\hat{u}\|^4).$$

If we set $y(t) = \|\hat{u}\|^2$, we obtain

$$y' \le Cs^2|h|^2 + gy,$$

where $g(t) = C(\|u_f\|^4 + \|u_f\|\,|Au_f| + |\hat{u}|^2\|\hat{u}\|^2)$.

From the previous result, we know that $\int_0^T g(t)\,dt < +\infty$, and the classical Gronwall's Lemma yields the desired conclusion: for all t in $(0, T)$, $y(t) \le s^2C(t)$, where $C(t)$ is bounded. □

Remark 1.2.7. Along the lines of the previous proof, we can easily see that the mapping $f \mapsto u_f$ is Fréchet differentiable, i.e., in the classical sense, from $L^2(0, T; H)$ into $L^2(0, T; V)$.

1.3. Numerical Algorithms

We now depart somewhat from our previous theoretical study to lay the ground for the computational study of the optimal control problem (1.1.2). We present two classical numerical algorithms for optimization problems, describing them in the actual situation we consider. Due to the nonconvexity of the functional we minimize, the convergence of such algorithms is conditional, depending on a good "initial guess."

We recall the notations, for f in $L^2(Q_T)$, we set

$$J(f) = \tfrac{1}{2} \iint_{Q_T} (|f|^2 + |\nabla \times u_f|^2)\, dx\, dt,$$

where u_f is the velocity field associated with f, and we denote by $\tilde{w}_f$ the adjoint state solution of (1.2.3).

The classical Simple Gradient Algorithm proceeds as follows:

$$\text{(G)} \quad \begin{cases} \text{Given } f_0 \text{ in } L^2(Q_T),\ \rho > 0, \text{ define } f_n \text{ recursively by} \\ \qquad f_{n+1} = f_n - \rho(f_n + \tilde{w}_{f_n}). \end{cases}$$

For a given f_n, u_{f_n} and therefore $\tilde{w}_{f_n}$ are uniquely determined, so that the sequence f_n is well defined. The convergence of the algorithm (G) to a minimum of J (see Appendix 1 for a more careful analysis) stems from the two following observations:

First, if f_n is convergent, then its limit $\bar{f}$ must be a critical point of J, for we then have $\bar{f} = \bar{f} - \rho(\bar{f} + \tilde{w}_{\bar{f}})$, therefore $\bar{f} + \tilde{w}_{\bar{f}} = 0$, precisely the critical-point condition of Theorem 1.2.4. Secondly, if the second derivative of $f \mapsto u_f$ is sufficiently well behaved, the sequence $J(f_n)$ is decreasing for small values of ρ, and the limit $\bar{f}$ of f_n must be a local minimum of J; as a matter of fact, we can write, with the help of Taylor's formula and the formula for $J'(f)\cdot h$ (1.2.5),

$$\begin{aligned} J(f_{n+1}) &= J(f_n - \rho_n(f_n + \tilde{w}_{f_n})) \\ &= J(f_n) - \rho \iint_{Q_T} |f_n + \tilde{w}_{f_n}|^2\, dx\, dt \qquad (+\text{second-order terms in } \rho), \end{aligned}$$

so that, if the second-order terms can be controlled, the sequence $J(f_n)$ is decreasing, and f_n must then converge to a local minimum of J. As we have already pointed out, the nonconvexity of J makes it very hard to ascertain whether a critical point is a *global* minimum.

In the same manner, a more refined algorithm is the Conjugate Gradient Algorithm:

$$\text{(CG)} \quad \begin{cases} \text{Given } f_0,\ \rho > 0, \text{ define } f_n \text{ recursively as follows:} \\ \qquad f_{n+1} = f_n - \rho k_n, \\ \text{where } k_n \text{ is given by} \\ \quad k_0 = f_0 + \tilde{w}_{f_0}, \\ \quad k_n = f_n + \tilde{w}_{f_n} + k_{n-1} \cdot \dfrac{\iint_{Q_T} (f_n - f_{n-1} + \tilde{w}_{f_n} - \tilde{w}_{f_{n-1}}) \cdot (f_n + \tilde{w}_{f_n})\, dx\, dt}{\iint_{Q_T} |f_{n-1} + \tilde{w}_{f_{n-1}}|^2\, dx\, dt}. \end{cases}$$

Once again, if f_j is known for $j \le n$, then u_{f_j}, $\tilde{w}_{f_j}$, and therefore k_{n+1} are determined, and the sequence f_n is well defined. As in the previous case, we easily notice that the limit, if it exists, of f_n must be a critical point of J. Furthermore, the sequence $J(f_n)$ is decreasing, for small values of ρ, when the second derivative of $f \mapsto u_f$ is controlled. This result is explained in detail in Appendix 1. As a concluding remark, we want to make the reader aware of the great complexity of both algorithms. Essentially, be it for (G) or (CG), we have to solve at each step, one system of Navier–Stokes and one parabolic (linear) system to obtain $\tilde{w}_{f_n}$.

2. A Boundary Temperature Control Problem for a Fluid in a Driven Cavity

In this section we focus our attention on a different and more physical problem. We consider a fluid in a driven cavity (see Figure 1), and allow the temperature inside the fluid to vary, by changing its value at the bottom of the cavity.

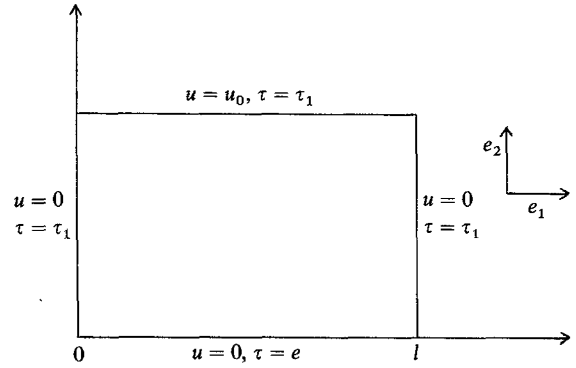

Figure 1

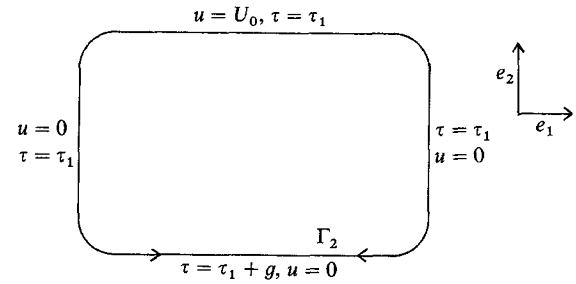

Figure 2

This is a Bénard problem, and the system is governed by the Boussinesq equations; the control is now the temperature e at the bottom $(0, l) \times \{0\}$ of Ω. As opposed to the situation described in Section 1, some nontrivial regularity problems arise. For instance (see, e.g., Serre, 1983), the boundary values of u prevent it from being in $(H^1(\Omega))^2$, if u_0 is constant. A similar problem arises for τ, and we have to reformulate in a suitable way the control problem we want to solve.

2.1. Mathematical Setting of the Problem

We first assume, by "rounding off" the angles in Ω, that Ω is a C^2 open set; furthermore, we assume that the boundary values of u are those of a divergence-free function U_0 in $(H^2(\Omega))^2$, satisfying as well as possible the boundary conditions imposed on u (see Figure 2). As for the temperature, we write $\tau = \tau_1 + g$, where g is a function in $H^1(0, T; H^2(\Omega))$, the trace of which is zero except on the bottom of Ω. The control is then the boundary value of g.

Remark 2.1.1. (i) The time regularity of g is designed to ensure the well-posedness of the Bénard problem.

(ii) At the upper corners, the value of u cannot be specified; similarly, the conditions on g mean that its boundary value has compact support in Γ_2, or equivalently, $g \in H^1(0, T; H_0^{3/2}(\Gamma_2))$.

We now write the Boussinesq equations in Ω:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = -\rho(\tau_1 - \tau)e_2 \quad \text{in } Q_T, \\[2mm] \dfrac{\partial \tau}{\partial t} + (u\cdot\nabla)\tau - \kappa\Delta\tau = 0 \quad \text{in } Q_T, \\[2mm] \nabla\cdot u = 0 \quad \text{in } Q_T, \qquad u(\cdot, 0), \tau(\cdot, 0) \quad \text{are given}, \\[2mm] u = U_0 \quad \text{on } \partial\Omega, \qquad \tau = \tau_1 + g \quad \text{on } \partial\Omega. \end{cases} \tag{2.1.1}$$

ν, ρ, κ are > 0, and T is arbitrary but fixed.

Before writing down the control problem to be considered, we reformulate (2.1.1), introducing $\hat{u} = u - U_0$, $\hat{\tau} = \tau - (\tau_1 + g)$. We obtain

$$\begin{cases} \dfrac{\partial \hat{u}}{\partial t} - \nu\Delta\hat{u} + (\hat{u}\cdot\nabla)\hat{u} + (U_0\cdot\nabla)\hat{u} + (\hat{u}\cdot\nabla)U_0 - \rho\hat{\tau}e_2 + \nabla p = \varphi, \\[2mm] \dfrac{\partial \hat{\tau}}{\partial t} + (\hat{u}\cdot\nabla)\hat{\tau} + (U_0\cdot\nabla)\hat{\tau} - \kappa\Delta\hat{\tau} + (\hat{u}\cdot\nabla)(\tau_1 + g) = \psi, \\[2mm] \nabla\cdot\hat{u} = 0, \qquad \hat{u} = 0 \quad \text{on } \partial\Omega, \qquad \hat{\tau} = 0 \quad \text{on } \partial\Omega, \\[2mm] \hat{u}(0), \hat{\tau}(0) \quad \text{given}, \end{cases} \tag{2.1.2}$$

where

$$\begin{cases} \varphi = \rho g e_2 - (U_0 \cdot \nabla) U_0 + \nu \Delta U_0, \\ \psi = -\dfrac{\partial}{\partial t}(\tau_1 + g) + \kappa \Delta(\tau_1 + g) - (U_0 \cdot \nabla)(\tau_1 + g), \end{cases}$$

and the control problem is

$$\text{(Q)} \quad \begin{cases} \text{Find } g \text{ in } H^1(0, T; H_0^{3/2}(\Gamma_2)) \text{ minimizing the functional} \\ \qquad K(g) = \frac{1}{2} \displaystyle\int_0^T \int_\Omega |\nabla \times u_g|^2 \, dx \, dt + \frac{1}{2} \|g\|^2_{H^1(0,T; H_0^{3/2}(\Gamma_2))}, \\ \text{where } u_g = \hat{u} + U_0, \ \hat{u} \text{ is the solution of (2.1.2) associated with } g. \end{cases} \tag{2.1.3}$$

We first state an existence result for the Boussinesq equations (2.1.1):

Proposition 2.1.2. *There exists a unique solution* (u, τ) *of the system* (2.1.1); *moreover, if we assume that* $u(0)$ *(resp.* $\tau(0)$*) is given in* $(H^1(\Omega))^2$ *(resp.* $H^1(\Omega)$*), then* u *(resp.* τ*) belongs to* $L^\infty(0, T; (H^1(\Omega))^2 \cap L^2(0, T; (H^2(\Omega))^2)$ *(resp.* $L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$*).*

Sketch of the proof. We prove the analogous result for $\hat{u}$, $\hat{\tau}$, associated with (2.1.2). The latter system can be rewritten (see Foias *et al.*, 1987) as a differential equation in the space $V \times H_0^1(\Omega)$ (V is the same space as in Section 1). Still denoting by P the orthogonal projector from $(L^2(\Omega))^2$ onto H, we write (2.1.1) as

$$\frac{dY}{dt} + AY + B(Y) + RY = (p_\varphi, \psi) \tag{2.1.4}$$

where

$$AY = (-\nu P \Delta \hat{u}; -\kappa \Delta \hat{\tau}), \qquad B(Y) = ((\hat{u} \cdot \nabla) \hat{u}; 0),$$

$$R(Y) = ((U_0 \cdot \nabla) \hat{u} + (\hat{u} \cdot \nabla) U_0 - \rho \hat{\tau} e_2; (\hat{u} \cdot \nabla) \hat{\tau} + (U_0 \cdot \nabla) \hat{\tau} + (\hat{u} \cdot \nabla)(\tau_1 + g)).$$

In particular, the assumptions upon Ω, g, U_0 ensure that $(P\varphi \cdot \psi)$ is in $L^2(0, T; H \times L^2(\Omega))$, and the proof now proceeds classically, using *a priori* estimates and compactness (see, e.g., Foias *et al.*, 1987; Temam, 1984). □

For the problem of control, we state without proof an existence and regularity result:

Proposition 2.1.3.

(i) *There exists at least an optimal control* $\bar{g}$ *minimizing* $K(g)$.
(ii) *The functional* K *is Gâteaux differentiable, and its derivative vanishes at* $\bar{g}$.

Proposition 2.1.3 is proven in the same way as Theorem 1.1.3 and Lemma 1.2.1, and provides us with first-order necessary conditions for an optimal control g. It is however obvious that the expression of K' is not very simple, because of the term $\|g\|^2_{H^1(0,T; H_0^{3/2}(\Gamma_2))}$. In the next subsection, we specialize a little more the type of control that we have in mind, in order to make the first-order optimality conditions more convenient.

For the moment, we give the expression of the derivative (in the Fréchet sense) of the function $g \mapsto u_g$:

Lemma 2.1.4. *Let* g *be in* $H^1(0, T; H_0^{3/2}(\Gamma_2))$; *the function* $g \mapsto (u_g, \tau_g)$ *is differentiable as a function with values in* $L^2(0, T; (H^1(\Omega))^2 \times H^1(\Omega))$. *Furthermore, its derivative* $D(u_g, \tau_g)/Dg$ *associates, with every* h, *the solution* $(w(h), \sigma(h))$ *of*

$$\begin{cases} \dfrac{\partial w}{\partial t} - \nu\Delta w + (u_g\cdot\nabla)w + (w\cdot\nabla)u_g - \rho\sigma e_2 + \nabla q = 0 \quad \text{in } Q_T, \\ \dfrac{\partial \sigma}{\partial t} + (u_g\cdot\nabla)\sigma + (w\cdot\nabla)\tau_g - \kappa\Delta\sigma = 0 \quad \text{in } Q_T, \\ w \in V, \qquad \sigma \in H^1(\Omega), \qquad \sigma = h \quad \text{on } \partial\Omega, \\ w(\cdot, 0) = 0, \qquad \sigma(\cdot, 0) = 0, \end{cases} \tag{2.1.5}$$

where $(u_g, \tau_g) = (\hat{u}_g, \hat{\tau}_g) + (U_0, \tau_1 + g)$, *and* $(\hat{u}_g, \hat{\tau}_g)$ *is the solution of* (2.1.2) *associated with the control* g.

Proof. The differentiability of $g \mapsto (u_g, \tau_g)$ is proven through the use of $(\hat{u}_g, \hat{\tau}_g)$, by the same method as that of Section 1; once this is done, the expression of (2.1.5) is obtained in a formal manner directly from the original set of equations for (u, τ) (i.e., (2.1.1)). □

2.2. A Particular Choice for the Control Set; System of Optimality Conditions

We now restrict the set of controls to those functions of the form $g(x, t) = \sum_{i=1}^{N} g_i(t)\varphi_i(x)$, where the g_i's are in $H^1(0, T)$, and the φ_i's are mutually orthogonal in $H_0^{3/2}(\Gamma_2)$; furthermore, we assume that the φ_i's are fixed, so that the controls are now really the g_i's. This is, from an application-oriented point of view, a very sensible assumption. There exists, on the lower boundary Γ_2 of Ω, a finite number of "acting functions," and the intensity of each of these functions is modulated in time. The control problem (Q) is now equivalent to

$$(\mathrm{Q}_1) \quad \begin{cases} \text{Find } (g_i)_{1\le i\le N} \text{ in } (H^1(0, T))^N \text{ minimizing the cost function} \\ K_1((g_i)) = \frac{1}{2}\displaystyle\int_0^T \sum_{i=1}^{n} \beta_i^2(g_i^2 + g_i'^2)(t)\,dt + \frac{1}{2}\int_0^T\int_\Omega |\nabla \times u_g|^2\,dx\,dt, \\ \text{where } u_g \text{ is as before, and } 0 < \beta_i = \|\varphi_i\|_{H_0^{3/2}(\Gamma_2)}. \end{cases} \tag{2.2.1}$$

This new problem has the enormous advantage over (Q) that it provides us with readable first-order optimality conditions, as opposed to the general condition $K'(\bar{g}) = 0$, the interpretation of which is rather delicate.

Our results are summed up in the following theorem:

Theorem 2.2.1. *Let* $(\bar{g}_i)_{1\le i\le N}$ *be an optimal control for problem* (2.2.1), *and let* $(\bar{u}, \bar{\tau})$ *be the corresponding optimal state; then the following equalities hold true for* $1 \le i \le N$:

$$\begin{cases} -\bar{g}_i''(t) + \bar{g}_i(t) = \dfrac{\kappa}{\beta_i^2}\displaystyle\int_{\Gamma_2} \frac{\partial\tilde{\tau}}{\partial\nu}\cdot\varphi_i\,d\Gamma \quad \text{in } (0, T), \\ \bar{g}_i'(0) = \bar{g}_i'(T) = 0, \end{cases} \tag{2.2.2}$$

where $(\tilde{w}, \tilde{\tau})$ *is the solution of the adjoint linearized problem*

$$\begin{cases} -\dfrac{\partial\tilde{w}}{\partial t} - \nu\Delta\tilde{w} - (\bar{u}\cdot\nabla)\tilde{w} + (\nabla\bar{u})^t\tilde{w} + \tilde{\tau}\nabla\bar{\tau} + \nabla\tilde{q} = \nabla \times (\nabla \times \bar{u}) \quad \text{in } Q_T, \\ -\dfrac{\partial\tilde{\tau}}{\partial\tau} - \kappa\Delta\tilde{\tau} - \bar{u}\cdot\nabla\tilde{\tau} - \rho(\tilde{w}\cdot e_2) = 0 \quad \text{in } Q_T, \\ \tilde{w}(T) = 0, \qquad \tilde{\tau}(T) = 0, \\ \tilde{w} \in V, \qquad \tilde{\tau} \in H_0^1(\Omega). \end{cases} \tag{2.2.3}$$

Proof. We compute the derivative $K_1'((\bar{g}_i)\cdot h)$ for h in $(H^1(0, T))^N$:

$$K_1'((\bar{g}_i))\cdot h = \int_0^T \sum_{i=1}^{N} \beta_i^2(h_i\bar{g}_i + h_i'\bar{g}_i')\,dt + \int_0^T\int_\Omega (\nabla \times \bar{u})\cdot(\nabla \times w)\,dx\,dt$$

$$\left((w, \sigma) \text{ defined by (2.1.5), with } \sigma = \sum_{i=1}^{N} h_i(t)\varphi_i(x) \text{ on } \Gamma_2\right)$$

$$= \text{(using integration by parts, and } w = 0 \text{ on } \partial\Omega)$$

$$= \int_0^T \left(\sum_{i=1}^{n} \beta_i^2(h_i\bar{g}_i + h_i'\bar{g}_i') + \int_\Omega \nabla \times (\nabla \times \bar{u})\cdot w\, dx\right) dt$$

$$= \int_0^T \left(\sum_{i=1}^{n} \beta_i^2(h_i\bar{g}_i + h_i'\bar{g}_i') + \int_\Omega \left(w\left(-\frac{\partial \tilde{w}}{\partial t} - \nu\Delta\tilde{w} - (\bar{u}\cdot\nabla)\tilde{w} + (\nabla\bar{u})^t\tilde{w} + \tilde{\tau}\nabla\bar{\tau} + \nabla\tilde{q}\right)\right.\right.$$
$$\left.\left. + \sigma\left(-\frac{\partial\tilde{\tau}}{\partial t} - \kappa\Delta\tilde{\tau} - \bar{u}\cdot\nabla\tilde{\tau} - \rho\tilde{w}\cdot e_2\right)\right) dx\right) dt$$

$$= \text{(by integration by parts)}$$

$$= \int_0^T \left(\sum_{i=1}^{N} \beta_i^2(h_i\bar{g}_i + h_i'\bar{g}_i') - \kappa\int_{\Gamma_2} \frac{\partial\tilde{\tau}}{\partial\nu}\left(\sum_{i=1}^{N} h_i(t)\varphi_i(x)\, d\Gamma\right)\right) dt.$$

If we equate $K_1'((\bar{g}_i))\cdot h$ to zero for every h in $(H^1(0, T))^N$, we then easily obtain the N uncoupled first-order necessary conditions

$$\begin{cases} -\bar{g}_i'' + \bar{g}_i = \dfrac{\kappa}{\beta_i^2}\displaystyle\int_{\Gamma_2} \frac{\partial\tilde{\tau}}{\partial\nu}\varphi_i\, d\Gamma & \text{in } (0, T), \\ \bar{g}_i'(0) = \bar{g}_i'(T) = 0, \end{cases} \tag{2.2.4}$$

which proves Theorem 2.2.1. □

Remark 2.2.2. If we assume the same regularity for u, τ as before, then the first-order conditions for a control $\sum_{i=1}^N g_i(t)\varphi_i(x)$ with the g_i's now in $L^2(0, T)$, would have the following expression:

$$\bar{g}_i(t) = \frac{\kappa}{\beta_i^2}\int_{\Gamma_2} \frac{\partial\tilde{\tau}}{\partial\nu}\tilde{g}_i\, d\Gamma. \tag{2.2.5}$$

Remark 2.2.3. The optimal control $(\bar{g}_i)$ actually belongs to $(H^2(0, T))^N$, as is easily seen from (2.2.4).

2.3. Numerical Algorithms

We now briefly sketch the modifications of the algorithms presented in Section 1 that are relevant to the numerical study of (2.1.3), with the type of controls considered in Section 2.2. Referring to Appendix 1, we need only interpret what is the gradient of K_1. In the simpler case of Remark 2.2.2, the gradient of K_1 is just $\nabla_{L^2}K_1(g) = (\beta_i^2 g_i - \kappa\int_{\Gamma_2}(\partial\tilde{\tau}/\partial\nu)\varphi_i\, d\Gamma)_{1\le i\le N}$; as a matter of fact, we have

$$K_1'(g)\cdot h = \int_0^T \sum_{i=0}^{N}\left[\beta_i^2 g_i(t)h_i(t)\, dt - \kappa h_i(t)\left(\int_{\Gamma_2}\frac{\partial\tilde{\tau}}{\partial\nu}\varphi_i\, d\Gamma\right)\right] dt$$
$$= \langle h, \nabla_{L^2}K_1\rangle_{(L^2(0,T))^N\times(L^2(0,T))^N}.$$

In the case that g is in $(H^1(0, T))^N$, we have

$$\nabla_{H^1}K_1 = (\beta_i^2 g_i + \mu_i(t))_{1\le i\le N},$$

where $\mu_i(t)$ is the solution of

$$\begin{cases} \mu_i - \mu_i'' = -\kappa\displaystyle\int_{\Gamma_2}\frac{\partial\tilde{\tau}}{\partial\nu}\varphi_i\, d\Gamma & \text{in } (0, T), \\ \mu_i'(0) = \mu_i'(T) = 0. \end{cases} \tag{2.3.1}$$

The fact that $K_1'(g)\cdot h = \langle h, \nabla_{H^1}K\rangle_{(H^1(0,T))^N\times(H^1(0,T))^N}$ is verified by straightforward integrations by parts. The algorithms of Section 1.3 are thus transformed accordingly.

The procedure for the Gradient Algorithm is:

Step 1: Choose $(g_1^n, \ldots, g_N^n)$ in $(H^1(0, T))^N$.
Step 2: Solve (2.1.1) with $g^n = \sum_{i=1}^N g_i^n(t)\varphi_i(x)$ and determine u^n, p^n, τ^n.
Step 3: Solve (2.2.3) and determine $\tilde{w}^n$, $\tilde{\tau}^n$, $\tilde{q}^n$.
Step 4: Solve (2.3.1) and determine $\nabla_{H^1} K_1(g^n)$.
Step 5: Write $(g_1^{n+1}, \ldots, g_N^{n+1}) = (g_1^n, \ldots, g_N^n) - \lambda \nabla_{H^1} K_1(g^n)$, $\lambda > 0$.
Step 6: Iterate steps 1–5.

We can also write the Conjugate Gradient Algorithm. Note that, in the algorithm above, λ (the parameter of descent) is constant. This can be refined, as it is done in the Appendix 1 for instance.

3. Another Problem of Boundary Control in a Cavity

In this section we sketch the study of a problem akin to the previous one, although somewhat different in spirit. This time we consider a fluid in a cavity Ω, maintained at given temperatures τ_0, τ, respectively at the bottom and the top of Ω, and the control is chosen to be the boundary value of the velocity u at the top of Ω (see Figure 3). Moreover, we impose a periodicity condition to u and τ in the x direction.

Under the same type of restrictions as in Section 2, our results can be justified rather easily; here, we prefer to derive heuristically the system of optimality conditions, exemplifying how the method works in different situations.

To begin, we describe the optimal control problem we are interested in. For each v in a suitable space, we define u_v as the solution of the Bénard problem (compare with Section 2):

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = -\rho(\tau_1 - \tau_0) e_2 \quad \text{in } Q_T, \\[2mm] \dfrac{\partial \tau}{\partial t} + (u \cdot \nabla)\tau - \kappa \Delta \tau = 0 \quad \text{in } Q_T, \\[2mm] \nabla \cdot u = 0 \quad \text{in } Q_T, \qquad u(\cdot, 0),\ \tau(\cdot, 0) \quad \text{are given}, \\[2mm] u = 0 \quad \text{on } \Gamma_2, \qquad u = v \quad \text{on } \Gamma_1, \qquad u|_{\Gamma_3} = u|_{\Gamma_4}, \\[2mm] \tau = \tau_0 \quad \text{on } \Gamma_2, \qquad \tau = \tau_1 \quad \text{on } \Gamma_1, \qquad \tau|_{\Gamma_3} = \tau|_{\Gamma_4}. \end{cases} \tag{3.1}$$

We then study the minimization problem:

$$\text{(R)} \quad \begin{cases} \text{Find } v \text{ minimizing the functional} \\[2mm] L(v) = \frac{1}{2} \displaystyle\int_0^T \int_{\Gamma_1} v^2(x, t)\, d\Gamma\, dt + \frac{1}{2} \int_0^T \int_\Omega |\nabla \times u_v|^2(x, t)\, dx\, dt, \\[2mm] \text{where } (u_v, \tau_v) \text{ is the solution of (3.1) associated with } v. \end{cases} \tag{3.2}$$

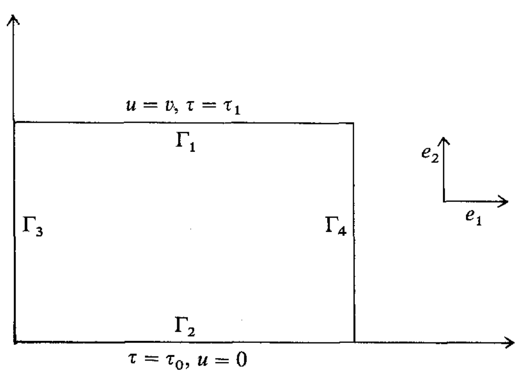

Figure 3

Remark 3.1. To obtain rigorous proofs of the formal argument that we present, we may want to modify L as in Section 2, to ensure that u has the suitable smoothness properties; however, the main lines of the method apply without change.

To obtain the optimality (critical-point) condition, we study the Gâteaux derivative of L in a direction h. First, we need to find the expression of the derivative of the mapping $v \mapsto (u_v, \tau_v)$; as before, it is given by $(w(h), \sigma(h))$, the solution of

$$\begin{cases} \dfrac{\partial w}{\partial t} - \nu\Delta w + (u_v \cdot \nabla)w + (w\cdot\nabla)u_v + \nabla q = 0 \quad \text{in } Q_T, \\ \dfrac{\partial \sigma}{\partial t} + (w\cdot\nabla)\tau_v + (u_v\cdot\nabla)\sigma - \kappa\Delta\sigma = 0 \quad \text{in } Q_T, \\ \nabla\cdot w = 0, \qquad w(\cdot, 0) = 0, \qquad \sigma(\cdot, 0) = 0, \\ w = 0 \quad \text{on } \Gamma_2, \qquad w = h \quad \text{on } \Gamma_1, \qquad w|_{\Gamma_3} = w|_{\Gamma_4}, \\ \sigma = 0 \quad \text{on } \Gamma_0 \text{ and } \Gamma_1, \qquad \sigma|_{\Gamma_3} = \sigma|_{\Gamma_4}. \end{cases} \tag{3.3}$$

As for the adjoint state $(\tilde{w}, \tilde{\sigma})$, it is the solution of

$$\begin{cases} -\dfrac{\partial \tilde{w}}{\partial t} - \nu\Delta\tilde{w} - (u_v\cdot\nabla)\tilde{w} + (\nabla u_v)^t\tilde{w} + \tilde{\sigma}\nabla\tau_v + \nabla\tilde{q} = \nabla\times(\nabla\times u_v) \quad \text{in } Q_T, \\ -\dfrac{\partial\tilde{\sigma}}{\partial t} - (u_v\cdot\nabla)\tilde{\sigma} - \kappa\Delta\tilde{\sigma} = 0 \quad \text{in } Q_T, \\ \nabla\cdot\tilde{w} = 0 \quad \text{in } Q_T, \qquad \tilde{w}(\cdot, T) = 0, \qquad \tilde{\sigma}(\cdot, T) = 0, \\ \tilde{w}|_{\Gamma_3} = \tilde{w}|_{\Gamma_4}, \qquad \tilde{\sigma}|_{\Gamma_3} = \tilde{\sigma}|_{\Gamma_4}, \qquad \tilde{w} = 0 \quad \text{on } \Gamma_1 \text{ and } \Gamma_2, \qquad \tilde{\sigma} = 0 \quad \text{on } \Gamma_1 \text{ and } \Gamma_2. \end{cases} \tag{3.4}$$

If we now write the expression for $L'(v)\cdot h$, we get

$$\begin{aligned} L'(v)\cdot h &= \int_0^T\int_{\Gamma_1} vh\,d\Gamma\,dt + \int_0^T\int_\Omega \nabla\times(\nabla\times u_v)\cdot w(h)\,dx\,dt \\ &= \text{(introducing the adjoint state)} \\ &= \int_0^T\int_{\Gamma_1} vh\,d\Gamma\,dt + \iint_{Q_T}\left(w\cdot\left(-\frac{\partial\tilde{w}}{\partial t} - \nu\Delta\tilde{w} - (u_v\cdot\nabla)\tilde{w} + (\nabla u_v)^T\cdot\tilde{w} + \tilde{\sigma}\nabla\tau_v + \nabla\tilde{q}\right)\right. \\ &\qquad\left. + \sigma\left(-\frac{\partial\tilde{\sigma}}{\partial t} - (u_v\cdot\nabla)\tilde{\sigma} - \kappa\Delta\tilde{\sigma}\right)\right)dx\,dt \\ &= \text{(by integration by parts)} \\ &= \int_0^T\int_{\Gamma_1} vh\,d\Gamma\,dt - \nu\int_0^T\int_{\Gamma_1} h\frac{\partial\tilde{w}}{\partial\nu}\,d\Gamma\,dt, \end{aligned}$$

so that the optimality condition is simply

$$v - \nu\frac{\partial\tilde{w}}{\partial\nu} = 0 \quad \text{in } \Gamma_1\times(0, T). \tag{3.5}$$

As for the numerical algorithm, we propose the following:

Step 1: Choose v^n in $L^2(0, T; \Gamma_1)$.
Step 2: Solve (3.1) with boundary value v^n, and determine u^n, p^n, τ^n.
Step 3: Solve (3.4) and determine $(\tilde{w}^n, \tilde{q}^n, \tilde{\sigma}^n)$.
Step 4: Write $v^{n+1} = v^n - \lambda\nabla L(v^n)$, where $\nabla L(v^n) = v^n - \nu(\partial\tilde{w}^n/\partial\nu)$ and $\lambda > 0$.
Step 5: Iterate steps 1–4.

This algorithm is simpler than that of Section 2, because of the choice of the L^2-norm of v in the cost function $L(v)$; however, we do not know how to prove its convergence in a rigorous way.

4. Boundary Control for a Flow Driven by a Pressure Gradient

Consider the two-dimensional flow in the region shown in Figure 4 and the state equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = 0 \quad \text{in } Q_T, \\ \nabla\cdot u = 0 \quad \text{in } Q_T, \\ u(x, y; 0) = u_0(x, y) \quad \text{in } \Omega, \qquad u(0, y; t) = u(L, y; t), \\ p(0, y; t) = p(L, y; t) + C(t), \\ u(x, 0; t) = \varphi e_1, \qquad u(x, h; t) = \varphi e_1, \end{cases} \tag{4.1}$$

where $C(t)$ is given, the control φ is in $X = H^1(0, T; H^{3/2}_{\text{per}}(\Gamma_1))$ (the subscript "per" stands for "periodic"), and Γ_1 is the horizontal boundary $(0, L) \times \{0, h\}$.

We consider the following problem of optimal control:

$$(\mathrm{S}) \quad \begin{cases} \text{Find } \varphi \text{ minimizing the functional} \\ M(\varphi) = \frac{1}{2}\displaystyle\int_0^T \int_{\Gamma_1} \left[\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)^2 + 2\left(\frac{\partial u_2}{\partial y}\right)^2\right] d\sigma\, dt + \tfrac{1}{2}\|\varphi\|_X^2, \\ \text{where } u = (u_1, u_2) \text{ is the solution of (4.1) associated with } \varphi. \end{cases} \tag{4.2}$$

In the expression of the functional M, the now classical supplementary term $\frac{1}{2}\|\varphi\|_X^2$ is designed for coercivity; as for the function space X, it is chosen so that the functional M is well defined on solutions of (4.1) (see, e.g., Section 2 for similar problems). The relevant term is obviously

$$\int_0^T \int_{\Gamma_1} \left[\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)^2 + 2\left(\frac{\partial u_2}{\partial y}\right)^2\right] d\sigma\, dt,$$

which is none other than the norm of the deformation rate tensor $\frac{1}{2}(\nabla u + \nabla u^T)$ at the boundary, in the space $L^2(0, T; L^2(\Gamma_1))$; in other words, turbulence in the flow is now measured by the value of the stress at the boundary of the channel. We state without proof the following proposition:

Proposition 4.1. *There exists an optimal pair $(\bar{u}, \bar{\varphi})$ for (S), where $\bar{\varphi}$ is an optimal control, and $\bar{u}$ is the corresponding distribution of velocities.*

We now derive the first-order optimality conditions for (S). Following the same lines as in the previous sections, we obtain the next theorem:

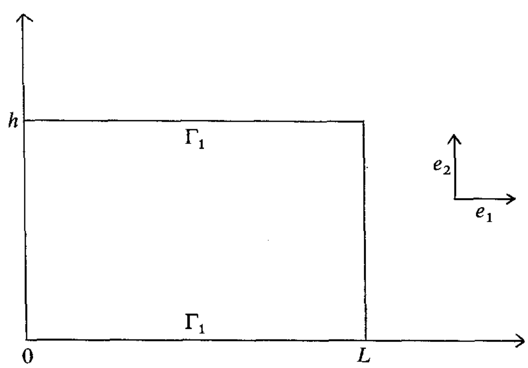

Figure 4

Theorem 4.2. *Let $(\bar{u}, \bar{\varphi})$ be an optimal pair for* (S); *then, for every ψ in X, the following holds*:

$$M'(\bar{\varphi})\cdot\psi \equiv \int_0^T \int_{\Gamma_1} \left[\frac{1}{2}\left(\frac{\partial \bar{u}_1}{\partial y} + \frac{\partial \bar{u}_2}{\partial x}\right)\left(\frac{\partial w_1}{\partial y} + \frac{\partial w_2}{\partial x}\right) + 2\frac{\partial \bar{u}_2}{\partial y}\frac{\partial w_2}{\partial y}\right] dx\, dy\, dt + \langle \bar{\varphi}, \psi\rangle_X = 0,$$

where $w = (w_1, w_2)$ is the solution of the linearized state equation

$$\begin{cases} \dfrac{\partial w}{\partial t} - \nu\Delta w + (\bar{u}\cdot\nabla)w + (w\cdot\nabla)\bar{u} + \nabla q = 0 \quad \textit{in } Q_T, \\ \nabla\cdot w = 0 \quad \textit{in } Q_T, \\ w(x, y; 0) = 0 \quad \textit{in } \Omega, \qquad w(0, y; t) = w(L, y; t), \\ q(0, y; t) = q(L, y; t), \\ w = \psi e_1 \qquad \textit{on} \quad \Gamma_1 \times (0, T). \end{cases} \tag{4.3}$$

Proof. See, e.g., Section 2. □

In order to obtain a tractable expression for the gradient of M, we now confine the admissible controls to functions of the form $\varphi = \sum_{i=1}^{N} \varphi_i(t)h_i(x)$, where the h_i's are "acting functions" belonging to the function space $H^{3/2}_{\text{per}}(\Gamma_1)$. This particular choice corresponds, as already mentioned in Section 2, to the realistic situation of a finite number of acting functions, the intensities of which are modulated in time. The requirement that the h_i's belong to $H^{3/2}_{\text{per}}(\Gamma_1)$ is, beside periodicity, a very mild regularity property.

The expression for $M'(\varphi)\cdot\psi$ now becomes

$$M'(\varphi)\cdot\psi = \int_0^T \int_{\Gamma_1} \left[\frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)\left(\frac{\partial w_1}{\partial y} + \frac{\partial w_2}{\partial x}\right) + \frac{\partial u_2}{\partial y}\frac{\partial w_2}{\partial y}\right] d\sigma\, dt + \sum_{i=1}^{N} \beta_i^2 \int_0^T (\varphi_i(t)\psi_i(t) + \varphi_i'(t)\psi_i'(t))\, dt,$$

where $\beta_i = \|h_i\|_{H^{3/2}_{\text{per}}(\Gamma_1)}$ (the h_i's are chosen mutually orthogonal, as in Section 2).

In order to determine the gradient of M, we introduce the adjoint state $\tilde{w}$, the solution of the linearized adjoint state equation

$$\begin{cases} -\dfrac{\partial \tilde{w}}{\partial t} - \Delta\tilde{w} + (\nabla u^T)\tilde{w} - (u\cdot\nabla)\tilde{w} + \nabla\tilde{q} = 0 \quad \text{in } Q_T, \\ \nabla\cdot\tilde{w} = 0 \quad \text{in } Q_T, \\ \tilde{w}(x, y; T) = 0 \quad \text{in } \Omega, \qquad \tilde{w}(0, y; t) = \tilde{w}(L, y; t), \\ \tilde{q}(0, y; t) = \tilde{q}(L, y; t), \\ \tilde{w} = \begin{pmatrix} \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial y} + \dfrac{\partial u_2}{\partial x}\right) \\ \dfrac{\partial u_2}{\partial y} \end{pmatrix} \qquad \text{on} \quad \Gamma_1 \times (0, T). \end{cases} \tag{4.4}$$

We now transform the expression for $M'(\varphi)\cdot\psi$, using integration by parts as usual, and obtain

$$M'(\varphi)\cdot\psi = 2\int_0^T \int_{\Gamma_1} D(\tilde{w})\cdot\vec{n}\psi\vec{x}\, d\sigma\, dt + \sum_{i=1}^{N} \beta_i^2 \int_0^T (\varphi_i\psi_i + \varphi_i'\psi_i')\, dt,$$

where $D(\tilde{w}) = \frac{1}{2}(\nabla w + \nabla w^T)$. We recall that we have used Green's formula in the form

$$\int_\Omega [\nabla\cdot(D(\tilde{w}))]w = \int_\Omega \tilde{w}[\nabla\cdot(D(w))] + \int_{\Gamma_1} D(\tilde{w})\cdot\vec{n}\cdot w\, d\sigma - \int_{\Gamma_1} D(w)\cdot\vec{n}\cdot\tilde{w}\, d\sigma$$

together with the identity $\nabla \cdot (D(w)) = \frac{1}{2}\Delta w$, for divergence-free vector fields. In particular the following theorem holds:

Theorem 4.3. *Let $(\bar{u}, \bar{\varphi})$ be an optimal pair for* (S), *and let $\tilde{w}$ be the solution of* (4.4) *associated with it; the following conditions of optimality hold*:

$$\begin{cases} 2\displaystyle\int_{\Gamma_1} D(\tilde{w})\cdot\vec{n}\cdot h_i e_i \, d\sigma + \beta_i^2(\bar{\varphi}_i - \bar{\varphi}_i'') = 0 \quad \text{on } (0, T), \\ \bar{\varphi}_i'(0) = \bar{\varphi}_i'(T) = 0, \end{cases} \tag{4.5}$$

for $i = 1, \dots, N$.

Proof. Similar to that of Theorem 2.2.1. □

For the sake of completeness, we end this section by a description of the (simple) Gradient Algorithm specialized to Problem (S). From the equality

$$M'(\varphi)\psi = \sum_{i=1}^{N} \beta_i^2 \int_0^T \left(\beta_i^2\varphi_i + \int_{\Gamma_1} 2D(\tilde{w})\cdot\vec{n}\cdot h_i e_1 \, d\sigma\right)\psi_i + \beta_i^2(\varphi_i'\psi_i') \, dt,$$

it follows that the gradient of M at φ, say $\nabla M(\varphi)$ is the N-tuple $(\varphi_1 + \rho_1, \dots, \varphi_N + \rho_N)$, where ρ_i is solution of

$$\begin{cases} \rho_i - \rho_i'' = 2\displaystyle\int_{\Gamma_1} D(\tilde{w})\cdot\vec{n}\cdot h_i e_1 \, d\sigma, \\ \rho_i'(0) = \rho_i'(T) = 0. \end{cases} \tag{4.6}$$

From this expression, we can write the numerical algorithm for the minimization of (M):

Step 1: Choose $(\varphi_1^n, \dots, \varphi_N^n)$ in $H^1(0, T))^N$.
Step 2: Solve (4.1) with boundary data $\{\sum_{i=1}^N \varphi_i^n(t)n_i(x)\}e_1$ and determine u^n, p^n.
Step 3: Solve (4.4) and determine $\tilde{w}^n$, $\tilde{q}^n$.
Step 4: Solve (4.6) for $i = 1, \dots, N$ and determine $\nabla M(\varphi^n)$ as above.
Step 5: Write $(\varphi_1^{n+1}, \dots, \varphi_N^{n+1}) = (\varphi_1^n, \dots, \varphi_N^n) - \lambda\nabla M(\varphi^n)$, $\lambda > 0$.
Step 6: Iterate steps 1–5.

We can write the Congugate Gradient Algorithm similarly; here, we have chosen a constant step λ for the descent. The reader is referred to Appendix 1 for an improvement of this algorithm.

5. Conclusions

In this article we have studied four model problems of control in fluids where the flow is respectively driven by volume forces, a gradient of temperature, and a gradient of pressure. In the first three problems, we try to minimize the turbulence measured by the L^2-norm of the curl vector; in the last example (flow in a channel), the turbulence is measured by the stress at the boundary.

The first example corresponds to the flow in a bounded domain. The governing equations (the state equation in the language of control theory) are the Navier–Stokes equations (1.0.1); the flow is controled by the volume forces f, and the cost function to be minimized is $J(f)$ as given in (1.1.2). Note that the quantity

$$\frac{1}{2}\int_0^T \int_\Omega |f(x, t)|^2 \, dx \, dt$$

has been added to the integral of the curl

$$\frac{1}{2}\int_0^T \int_\Omega |\nabla \times u_f(x, t)|^2 \, dx \, dt$$

for purely technical (mathematical) reasons. The existence of the optimal control is asserted by Theorem 1.1.3.

The necessary conditions of optimality are conditions satisfied by a pair $(\bar{f}, \bar{u})$, for $\bar{f}$ to be the optimal control and $\bar{u}$ to be the corresponding distribution of velocities. The necessary conditions of optimality are given in (1.2.1) where $\tilde{w}(\bar{u})$ is expressed in terms of $\bar{u}$ by solving (1.2.3) (the so-called linearized adjoint state equation).

Finally, Section 1.3 contains numerical procedures to reach an optimal control by the gradient or the conjugate gradient method (see (G) and (CG) in Section 1.3). Here $\tilde{w}_{f_n} = \tilde{w}(u_{f_n})$ is obtained by solving (1.2.3) with $\bar{u}$ replaced by u_{f_n}, i.e., the distribution of velocities corresponding to f_n. Hence, for either algorithms, at each step of the procedure, we have to determine u_{f_n} and $\tilde{w}_{f_n}$, i.e., solve the Navier–Stokes equations (1.0.1) (with $f = f_n$) and the related equation (1.2.3) (with $\bar{u} = u_{f_n}$).

The second example (Section 2) is a Bénard problem in a square cavity whose corner has been smoothed for technical reasons. The governing equations are the thermohydraulic equations (2.1.1), and the control is the distribution of temperatures g at the boundary. The cost function is given in (2.1.3). It is the L^2 integral of the curl vector augmented for technical simplifications by an appropriate quantity involving g. In this case we just prove the existence of an optimal control (Proposition 2.1.2). Then we consider a particular form of this problem (same governing equations, cost function given in (2.2.1). Instead of a continuous distribution of heating at the boundary, we just have a finite number, N, of heat sources and their intensity is precisely the control. The necessary conditions of optimality are given by (2.2.2), in which $\tilde{w}$, $\tilde{\tau}$ are given in terms of $u = u_f$ by (2.2.3) (adjoint state equations) which are similar to the thermohydraulic equations (2.1.1). Moreover, we write down the Gradient Algorithm for the numerical study of the optimal control problem where $\tilde{w}$ is used to determine the expression of the gradient of the cost function (2.2.1), and the main steps of the algorithm are described.

The third example (Section 3) is a variation of the former situation, with the same geometry and governing equations. We now control turbulence in the fluid by the velocity at the upper boundary. The motion is caused by a heating of the fluid from below, and we fight this effect by adjusting the boundary value of the velocity. Details are similar to those in Section 2, and comparable results, including the numerical algorithm, are found. Finally, a fourth example (Section 4) is a flow in a channel, driven by a pressure gradient in the horizontal direction. The corresponding equations are (4.1). The cost function (4.2) measures turbulence by the L^2-norm of the boundary value of the stress tensor or, rather, the deformation rate tensor (once again, a supplementary term is added for mathematical reasons). For the full problem of optimal control, we prove the existence of a minimizer; next, we specialize the set of admissible controls to a time-dependent linear combination of N acting functions. In this case, we derive the necessary conditions of optimality (4.5), and describe the numerical algorithm for the determination of the optimal control and state.

Acknowledgments

This article is an extended version of the lecture of the second author (R.T.) at the workshop of control in fluids organized by ICASE, at Hampton, Virginia, in March 1988. It was written while the first author was at the Department of Mathematics, The Pennsylvania State University, University Park. The authors wish to thank John Kim who suggested to one of us the problem studied in Section 4.

Appendix 1. Convergence of the Numerical Algorithms

For the sake of completeness, we give the standard proof of the convergence of the numerical algorithms described in Section 1.

Lemma A.1. *Let J be a real-valued function on a Hilbert space X with norm $|\cdot|$ (resp. scalar product $\langle\ \rangle$). We make the following assumptions*:

(i) *J is of class C^2 and has a local minimum at a point $\bar{x}$,*
(ii) *there exists a ball B of X, centered at $\bar{x}$, and two reals m, M, such that the following inequalities hold*:

$$\forall u \in B, \quad \forall (x, y) \in X^2, \qquad m|x||y| \le J''(u)(x, y) \le M|x||y|,$$

where $J''(u)(x, y)$ is the bilinear form associated with the second derivative of J.

Then the Gradient and Conjugate Gradient Algorithms with initial value x_0 in B converge to $\bar{x}$.

Proof. We study the case corresponding to the Gradient Algorithm; the Conjugate Gradient is dealt with in a similar fashion. Therefore, we study the convergence of the sequence x_n defined recursively by

$$\begin{cases} x_0 \in B, & x_0 \neq \bar{x}, \\ x_{n+1} = x_n - \rho_n g_n, \end{cases} \tag{A.1}$$

where $g_n = J'(x_n)$ (the "gradient" of J at x_n) and ρ_n is such that

$$J(x_n - \rho_n g_n) = \inf_{\substack{\rho \geq 0 \\ x_n - \rho g_n \in B}} J(x_n - \rho g_n). \tag{A.2}$$

This is a refined version of the algorithm with fixed step in Section 1. The continuity of J ensures that ρ_n is well defined; furthermore, it is easily seen that $\rho_n \neq 0$ if $x_n \neq \bar{x}$, and that we have:

$$\langle J'(x_n - \rho_n g_n), g_n \rangle = 0, \tag{A.3}$$

because the function $\rho \mapsto J(x_n - \rho g_n)$ has a critical point at $\rho = \rho_n$. This can be written as

$$\langle g_n, g_{n+1} \rangle = 0. \tag{A.4}$$

We first proceed to prove a *lower* bound on ρ_n. As a result of (ii), and Taylor's formula, we obtain

$$\begin{aligned} \langle (J'(x_n) - J'(x_{n+1})), x_n - x_{n+1} \rangle &= \langle g_n - g_{n+1}, \rho_n g_n \rangle \\ &\leq \text{(using (ii))} \\ &\leq M |x_n - x_{n+1}|^2 \\ &\leq M \rho_n^2 |g_n|^2, \end{aligned}$$

so that, using (A.4), we have

$$\rho_n |g_n|^2 \leq M \rho_n^2 |g_n|^2$$

or

$$\rho_n \geq \frac{1}{M}. \tag{A.5}$$

We can also derive an *upper* bound on ρ_n as follows:

$$J(x_{n+1}) = J(x_n) + \langle g_n, x_{n+1} - x_n \rangle + \tfrac{1}{2} J''(x_n + \theta(x_{n+1} - x_n))(x_{n+1} - x_n, x_{n+1} - x_n)$$

for some θ in (0, 1). By construction, the sequence $J(x_n)$ is decreasing, so we can write (using (ii))

$$J(x_n) - \rho_n |g_n|^2 + \frac{m}{2} \rho_n^2 |g_n|^2 \leq J(x_n),$$

which yields

$$\rho_n \leq \frac{2}{m}. \tag{A.6}$$

Let us now prove the convergence of x_n to $\bar{x}$. We first remark that (ii) ensures that $\bar{x}$ is the only critical point of J in B. Next, we observe that J is bounded from below in B, from (ii) and Taylor's formula; as a consequence, the sequence $J(x_n)$ is monotonically decreasing and bounded from below, and thus it converges to a real number, say l. From the definition of x_n, the bounds on ρ_n, and Taylor's formula, we derive

$$\begin{aligned} J(x_n) = J(x_{n+1}) &+ J'(x_{n+1})(x_n - x_{n+1}) \\ &+ \tfrac{1}{2} J''(x_{n+1} + \theta'(x_n - x_{n+1}))(x_n - x_{n+1}, x_n - x_{n+1}) \qquad \text{for some } \theta' \text{ in } (0, 1); \end{aligned}$$

using (A.4) and (ii), we obtain

$$J(x_n) - J(x_{n+1}) \geq 0 + \frac{m \rho_n^2}{2} |g_n|^2. \tag{A.7}$$

Once again, the choice of ρ_n proves itself convenient. As $n \to +\infty$, $(J(x_n) - J(x_{n+1})) \to 0$ and, therefore, so does $\rho_n^2 |g_n|^2$, as a result of the lower bound on ρ_n. This shows that g_n converges to 0 in X. In order to conclude the proof, we write Taylor's formula between x_{n+p} and x_n, and obtain

$$J(x_{n+p}) = J(x_n) + \langle g_n, x_{n+p} - x_n \rangle + \tfrac{1}{2} J''(x_n + \theta''(x_{n+p} - x_n))(x_{n+p} - x_n, x_{n+p} - x_n)$$

with θ'' in (0, 1).

As $n \to +\infty$, $(x_{n+p} - x_n)$ stays bounded, whereas $J(x_{n+p}) - J(x_n)$ and $|g_n|$ converge to zero; from (ii), we infer that $|x_{n+p} - x_n| \to 0$, so that x_n is a Cauchy sequence in x. Therefore, x_n converge to some limit x_∞ in x, and an easy argument of continuity and closedness shows that x_∞ is precisely $\bar{x}$, the local minimum of J. □

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