Computers and Mathematics with Applications 71 (2016) 2389–2406

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Optimal control problems for the Navier–Stokes system coupled with the k-ω turbulence model

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ARTICLE INFO

Article history:
Available online 11 November 2015

Keywords:
Optimal control
Turbulence modeling
k-ω model

ABSTRACT

Optimal control of fluid-dynamics systems has gained attention in the last several years from the scientific community because of its potential use in design of new engineering devices and optimization of existing ones. Many research works have extensively studied the optimal control for the system of Navier–Stokes but the problem of turbulence in these works is usually not taken into account because of the many difficulties arising from the numerical implementation and solution of the optimality system. In this work turbulence is considered by coupling the k-ω two-equation turbulence model with the averaged Navier–Stokes system. The complete optimality system is derived and the existence of a weak solution proven. Some numerical examples are reported.

1. Introduction

In the last several years optimal control theory has gained attention in the research field of fluid dynamics. Many theoretical and numerical works have been performed to study existence and smoothness of solutions and feasibility of numerical algorithms to solve optimal control problems. In literature there exists a wide range of studies on optimization techniques, like continuous, discrete, global and local optimization or constrained optimization for partial differential equations, linear feedback control and others, for example see [1–3] and for a review on distributed and boundary optimal control and on numerical approaches to solve these problems see also [4] and references therein. Also the study of the optimal control for the Navier–Stokes system has been extensive. In this framework boundary, distributed and shape optimal controls have been studied and applied to this system for the computation of optimal velocity profiles, forces acting on the fluid and shape of the domain, see Refs. [5–8]. However, in almost all these studies the effect of turbulence on the flow field has been neglected.

In this work we consider the Reynolds averaged Navier–Stokes system for the computation of the average velocity and total pressure field $(\mathbf{u}, p)$ coupled with k-ω turbulence model [9]. In particular we consider the following set of equations

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nabla \cdot [(\nu + \nu_t)\mathbf{S}(\mathbf{u})] = \mathbf{f}, \tag{2}$$

$$(\mathbf{u} \cdot \nabla)k - \nabla [(\nu + \sigma_k \nu_t) \cdot \nabla k] = S_k - \beta^* k\,\omega, \tag{3}$$

$$(\mathbf{u} \cdot \nabla)\omega - \nabla [(\nu + \sigma_\omega \nu_t) \cdot \nabla \omega] = \alpha S_\omega - \beta \omega^2, \tag{4}$$

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http://dx.doi.org/10.1016/j.camwa.2015.10.003
0898-1221/

where $\mathbf{f}$ is the force acting on the flow, p the total fluid pressure, ν the kinematic viscosity of the fluid and $\mathbf{S}$ the deformation tensor

$$\mathbf{S}(\mathbf{u}) := \nabla\mathbf{u} + \nabla\mathbf{u}^T. \tag{5}$$

The model coefficients and functions are [9]

$$\sigma_k = 0.6 \qquad \sigma_\omega = 0.5$$
$$\beta^* = 0.09 \qquad \alpha = \frac{13}{25}$$
$$\beta = 0.0708.$$

In order to complete the system (1)–(4) the production term of the turbulence kinetic energy and its dissipation are usually modeled as

$$S_k = \nu_t \mathbf{S}(\mathbf{u}) : \nabla\mathbf{u} = \frac{1}{2}\nu_t \mathbf{S}^2(\mathbf{u}) \tag{6}$$

$$S_\omega = \frac{\omega}{k}\nu_t \mathbf{S}(\mathbf{u}) : \nabla\mathbf{u} = \frac{1}{2}\mathbf{S}^2(\mathbf{u}), \tag{7}$$

with $\mathbf{S}^2(\mathbf{u}) = \mathbf{S}(\mathbf{u}) : \mathbf{S}(\mathbf{u})$ since $\nabla\mathbf{u}$ is symmetric and $\nu_t = k/\omega$. The key quantity ν_t is the turbulent or eddy viscosity which has to be defined in the turbulence model.

Standard regular Navier–Stokes solutions in (1)–(4) have the derivatives of the velocity field square integrable but not necessarily bounded. The k and ω equations have the typical pattern of the diffusion–reaction equations and therefore their solutions can be constrained inside a precise interval limited by the roots of the equation defined only by the right-hand-side nonlinear terms in (6)–(7). For example in an infinite medium (4), with no advection and diffusion term, becomes

$$\frac{\alpha}{2}\mathbf{S}^2(\mathbf{u}) - \beta\,\omega^2 = 0, \tag{8}$$

which has solution $\omega = \pm\sqrt{\alpha \mathbf{S}^2(\mathbf{u})/(2\beta)}$. Only the positive root should be considered but if $\nabla\mathbf{u}$ is not bounded, then $\mathbf{S}(\mathbf{u})$ and ω are unbounded. In order to keep Navier–Stokes solutions in standard functional classes and have turbulent fields bounded in well defined intervals we must regularize the modeling of the turbulence sources. Therefore we assume

$$S_k = \min\left[\frac{1}{2}\nu_t \mathbf{S}^2(\mathbf{u}),\, \beta^* k_{max,v}\omega\right] \tag{9}$$

$$S_\omega = \min\left[\frac{1}{2}\mathbf{S}^2(\mathbf{u}),\, \frac{\omega^2_{max,v}\beta}{\alpha}\right], \tag{10}$$

where $k_{max,v}$ and $\omega_{max,v}$ are positive constants. This notation points out that $k_{max,v}$ and $\omega_{max,v}$ will be proved to be limits for k and ω fields while the label v suggests that these are volumetric bounds. The source model in (9)–(10) assures that, in the case of unbounded gradient velocity, the dissipation terms can cope with the turbulence sources and keep k and ω limited.

In k-ω model $\nu_t = k/\omega$, so if ω vanishes ν_t becomes singular. The existence of regular solutions of (3)–(4) when ν_t is an unbounded function is difficult to prove [10]. For this reason in the rest of the paper we assume

$$\nu_t = \min\left[\frac{k}{\omega},\, \nu_{max}\right]. \tag{11}$$

The constants $k_{max,v}$, $\omega_{max,v}$ and ν_{max} can be chosen as large as needed in order to assure the regularity of the problem together with the accuracy of the physical solution. By doing so the solution of Navier–Stokes equations remains unchanged while only the turbulence source terms are modeled to avoid singularities.

We use a near-wall approach for the solution of the turbulence problem, so the RANS equations are integrated throughout the viscous layer where near-wall boundary conditions are imposed. By using Taylor expansion for the turbulence variables, with respect to the distance from the wall δ, we obtain for the tangential component v of the velocity and for the turbulence variables [11]

$$v = \frac{\sigma_w}{\mu}\delta \qquad k = a_1\delta^2 \qquad \omega = \frac{2\,\nu}{\beta^*\delta^2}, \tag{12}$$

with a_1 constant and σ_w the stress on the boundary.

The first step to obtain the optimality system is to choose an objective for our optimal control problem. In this work we study two problems, a velocity matching profile and a turbulence enhancement or reduction problem,

$$\mathcal{J}(\mathbf{v}, k, f) = a\,\frac{1}{2}\int_\Omega (\mathbf{u} - \mathbf{u}_d)^2\; d\mathbf{x} + b\,\frac{1}{2}\int_\Omega (k - k_d)^2\; d\mathbf{x} + \frac{\lambda}{2}\int_\Omega \mathbf{f}^2\, d\mathbf{x}, \tag{13}$$

where b is a non negative constant and a and λ positive. If $b = 0$, the objective functional can be used in a velocity matching profile problem, while $a \approx 0$ is to be used for a turbulence reduction or enhancement problem. The choice of the regularization parameter λ is a key point for the numerical solution of the problem because high values of λ can result in a poor control, while low ones usually lead to convergence issues due to the enlargement of the control space $\mathbf{f}$ to the space of distributions. The two first integrals are usually referred to as *cost functionals* because they measure the difference between our objective and what we have actually achieved.

The paper is organized as follows: in the next section the mathematical model is studied and the existence of a solution to the RANS-k-ω system is proven by taking some hypotheses on the fields. In the following section the control problem is analyzed and the existence of an optimal control solution is proven. Then the Lagrange multiplier method is presented for the case of interest and the final optimality system is derived. In the final section numerical results obtained from the numerical solution of the optimality system are reported for some test cases.

2. Mathematical model

2.1. Notations

Before introducing the mathematical model, let us briefly recall some notations about functional spaces used in this paper. We denote by $H^s(\mathcal{O}),\ s \in \Re$, the standard Sobolev space of order s with respect to the set $\mathcal{O}$, which is either the flow domain $\Omega \subset \Re^n$ ($n = 2, 3$), or its boundary Γ, or part of its boundary. Whenever m is a non-negative integer, the inner product over $H^m(\mathcal{O})$ is denoted by $(f, g)_m$ and (f, g) denotes the inner product over $H^0(\mathcal{O}) = L^2(\mathcal{O})$. Hence, we associate with $H^m(\mathcal{O})$ its natural norm $\|f\|_{m,\mathcal{O}} = \sqrt{(f, f)_m}$. Whenever possible, we will neglect the domain label in the norm. For details on these spaces, one can consult [12,13].

For vector-valued functions and spaces, we use boldface notation. For example, $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^n$ denotes the space of $\Re^n$-valued functions such that each component belongs to $H^s(\Omega)$. Of special interest is the space

$$\mathbf{H}^1(\Omega) = \left\{ v_i \in L^2(\Omega) \;\middle|\; \frac{\partial v_i}{\partial x_j} \in L^2(\Omega) \quad \text{for } i, j = 1, 2, 3 \right\}$$

equipped with the norm $\|\mathbf{v}\|_1 = (\sum_{i,j}(\|v_i\|_1^2 + \|\partial v_i/\partial x_j\|_1^2))^{1/2}$. We define the space

$$\mathbf{V}(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) \mid \nabla \cdot \mathbf{u} = 0\}.$$

For $\Gamma_s \subset \Gamma$ with nonzero measure, we also consider the subspace

$$\mathbf{H}^1_{\Gamma_s}(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) \mid \mathbf{v} = \mathbf{0} \quad \text{on } \Gamma_s\}.$$

Also, we write $\mathbf{H}^1_0(\Omega) = \mathbf{H}^1_\Gamma(\Omega)$. Let $(\mathbf{H}^1_{\Gamma_s})^*$ denote the dual space of $\mathbf{H}^1_{\Gamma_s}$. Note that $(\mathbf{H}^1_{\Gamma_s})^*$ is a subspace of $\mathbf{H}^{-1}(\Omega)$, where the latter is the dual space of $\mathbf{H}^1_0(\Omega)$. The duality pairing between $\mathbf{H}^{-1}(\Omega)$ and $\mathbf{H}^1_0(\Omega)$ is denoted by $\langle \cdot, \cdot \rangle$.

Let $\mathbf{g}$ be an element of $\mathbf{H}^{1/2}(\Gamma)$. It is well known that $\mathbf{H}^{1/2}(\Gamma)$ is a Hilbert space with norm

$$\|\mathbf{g}\|_{1/2,\Gamma} = \inf_{\mathbf{v}\in\mathbf{H}^1(\Omega);\ \gamma_\Gamma \mathbf{v}=\mathbf{g}} \|\mathbf{v}\|_1,$$

where γ_Γ denotes the trace mapping $\gamma_\Gamma : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}(\Gamma)$. We let $(\mathbf{H}^{1/2}(\Gamma))^*$ denote the dual space of $\mathbf{H}^{1/2}(\Gamma)$ and $\langle \cdot, \cdot \rangle_\Gamma$ denote the duality pairing between $(\mathbf{H}^{1/2}(\Gamma))^*$ and $\mathbf{H}^{1/2}(\Gamma)$. From the definition of the dual norm, we have

$$\|\mathbf{s}\|_{-1/2,\Gamma} = \sup_{\mathbf{g}\in\mathbf{H}^{1/2}(\Gamma);\ \mathbf{g}\neq\mathbf{0}} \frac{\langle \mathbf{s}, \mathbf{g}\rangle_\Gamma}{\|\mathbf{g}\|_{1/2}} = \sup_{\mathbf{v}\in\mathbf{H}^1(\Omega);\ \mathbf{v}\neq\mathbf{0}} \frac{\langle \mathbf{s}, \gamma_\Gamma \mathbf{v}\rangle_\Gamma}{\|\mathbf{v}\|_1}.$$

Since the pressure is only determined up to an additive constant by the Navier–Stokes system with velocity boundary conditions, we define the space of square integrable functions having zero mean over Ω as

$$L^2_0(\Omega) = \left\{ p \in L^2(\Omega) \middle| \int_\Omega p\, d\mathbf{x} = 0 \right\}.$$

In order to define a weak form of the Navier–Stokes-k-ω equations, we introduce the continuous bilinear forms

$$a(\nu; \mathbf{u}, \mathbf{v}) = \frac{1}{2}\int_\Omega \nu\, \mathbf{S}(\mathbf{u}) : \mathbf{S}(\mathbf{v})\, d\mathbf{x} \quad \forall\, \mathbf{u} \in \mathbf{H}^1(\Omega),\ \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega) \tag{14}$$

and

$$b(\mathbf{v}, q) = -\int_\Omega q\, \nabla \cdot \mathbf{v}\, d\mathbf{x} \quad \forall\, q \in L^2_0(\Omega),\ \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega) \tag{15}$$

and the trilinear form

$$c(\mathbf{w}; \mathbf{u}, \mathbf{v}) = \frac{1}{2}\left[\int_\Omega [(\mathbf{w}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v}\, d\mathbf{x} - \int_\Omega [(\mathbf{w}\cdot\nabla)\mathbf{v}]\cdot\mathbf{u}\, d\mathbf{x}\right] \quad \forall\, \mathbf{w} \in \mathbf{V}(\Omega), \mathbf{u} \in \mathbf{H}^1(\Omega), \mathbf{v} \in \mathbf{H}^1_0(\Omega). \tag{16}$$

The above definitions of the continuous bilinear forms are also valid for scalars with the appropriate monodimensional operators, for example we define the form (14)

$$a(\nu; u, v) = \int_{\Omega} \nu \, \nabla u \cdot \nabla v \, d\mathbf{x} \quad \forall \, u \in H^1(\Omega), \; \forall \, v \in H^1(\Omega). \tag{17}$$

Obviously, given any $\nu \in L^{\infty}(\Omega)$, $a(\cdot, \cdot)$ is a continuous bilinear form on $\mathbf{H}^1(\Omega) \times \mathbf{H}^1_0(\Omega)$ and $b(\cdot, \cdot)$ is a continuous bilinear form on $\mathbf{H}^1(\Omega) \times L^2_0(\Omega)$; also $c(\cdot; \cdot, \cdot)$ is a continuous trilinear form on $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1_0(\Omega)$. For details concerning the function spaces we have introduced, one may consult [12,14] while for details about the bilinear and trilinear forms and their properties see [14,15].

2.2. The associated boundary value problem

We consider the formulation of the direct problem for the Navier–Stokes system (1)–(2) and turbulence equations (3)–(4). A weak formulation of the Navier–Stokes-k-ω system is given as follows

given ν_{max}, $k_{max,v}$ and $\omega_{max,v}$ positive real constants and $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, $\mathbf{g}_u \in \mathbf{H}^1(\Omega)$, $g_k \in H^1(\Omega)$, $g_\omega \in H^1(\Omega)$, find $(\mathbf{u}, p, k, \omega) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times H^1(\Omega) \times H^1(\Omega)$ satisfying

$$\begin{cases} a(\nu + \nu_t; \mathbf{u}, \mathbf{v}) + c(\mathbf{u}; \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \langle \mathbf{f}, \mathbf{v} \rangle & \forall \, \mathbf{v} \in \mathbf{H}^1_0(\Omega) \\ b(\mathbf{u}, q) = 0 & \forall \, q \in L^2_0(\Omega) \\ \langle \mathbf{u}, \mathbf{s} \rangle_\Gamma = \langle \mathbf{g}_u, \mathbf{s} \rangle_\Gamma & \forall \, \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma), \end{cases} \tag{18}$$

$$\begin{cases} c(\mathbf{u}; k, \psi) + a(\nu + \nu_t \sigma_k; k, \psi) = \langle S_k, \psi \rangle - (\beta^* \, k\omega, \psi) & \forall \psi \in H^1_0(\Omega) \\ \langle k, s_k \rangle_\Gamma = \langle g_k, s_k \rangle_\Gamma & \forall \, s_k \in H^{-1/2}(\Gamma) \\ c(\mathbf{u}; \omega, \phi) + a(\nu + \nu_t \sigma_\omega; \omega, \phi) = \langle \alpha \, S_\omega, \phi \rangle - (\beta \, \omega^2, \phi) & \forall \phi \in H^1_0(\Omega) \\ \langle \omega, s_w \rangle_\Gamma = \langle g_w, s_w \rangle_\Gamma & \forall s_w \in \mathbf{H}^{-1/2}(\Gamma), \end{cases} \tag{19}$$

where

$$\nu_t(k, \omega) = \min\left\{ \frac{k}{\omega}, \nu_{max} \right\} \tag{20}$$

$$S_k(\mathbf{u}, k, \omega) = \min\left\{ \nu_t \, \frac{\mathbf{S}^2(\mathbf{u})}{2}, \beta^* \omega \, k_{max,v} \right\} \tag{21}$$

$$S_\omega(\mathbf{u}, k, \omega) = \min\left\{ \frac{\mathbf{S}^2(\mathbf{u})}{2}, \frac{\beta \omega^2_{max,v}}{\alpha} \right\}. \tag{22}$$

Existence and uniqueness results for solutions of the system (18) are contained in the following theorem; see, e.g., [6,14].

Theorem 1. *Let Ω be an open, bounded set with Lipschitz-continuous boundary Γ. Let ν_t be a non-negative function in $L^{\infty}(\Omega)$, $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ and $\mathbf{g}_u \in \mathbf{H}^1(\Omega)$. Then,*

(i) *there exists at least one solution $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$ of (18); and*
(ii) *if*

$$\nu > \nu_0(\Omega, \mathbf{f}, \mathbf{g}_u) \tag{23}$$

for some positive ν_0 whose value is determined by the given data, then the set of solutions of (18) consists of a single element.

Note that solutions of (18) exist for any value of the Reynolds number. However, (ii) implies that uniqueness can be guaranteed only for "large enough" values of ν or for "small enough" data $\mathbf{f}$ and $\mathbf{g}_u$.

For the k-ω turbulence system we have a similar result for the existence of solutions to the nonlinear set of equations (19).

Theorem 2. *Let Ω be an open, bounded set with Lipschitz-continuous boundary Γ. Let $\mathbf{u}$ be in $\mathbf{V}(\Omega)$, g_k and g_ω in $H^1(\Omega) \cap L^{\infty}(\Omega)$ and ν_t, S_k, S_ω as in (20)–(22). Then,*

(i) *there exists at least one solution $(k, \omega) \in H^1(\Omega) \times H^1(\Omega)$ of (19);*
(ii) *let $\omega_{max,v}$ and $k_{max,v}$ be positive real constants, and*

$$k_{sup} = \sup\left\{ \sup_\Gamma\{g_k\}, k_{max,v} \right\} \tag{24}$$

$$\omega_{inf} = \inf\left\{ \inf_\Gamma\{g_\omega\}, \inf_\Omega\{\sqrt{\alpha S_\omega/\beta}\} \right\} \qquad \omega_{sup} = \sup\left\{ \sup_\Gamma\{g_\omega\}, \omega_{max,v} \right\}, \tag{25}$$

then

$$0 \le k \le k_{sup}, \tag{26}$$
$$0 \le \omega_{inf} \le \omega \le \omega_{sup}. \tag{27}$$

Proof. Due to the theorem assumptions the proof of (i) follows from standard techniques, see [10,16,17] for details. Therefore let us suppose that there is a solution $(k, \omega) \in H^1(\Omega) \times H^1(\Omega)$ and prove that $(k, \omega) \in L^\infty(\Omega) \times L^\infty(\Omega)$ with the bounds in (ii).

The proof follows the basic framework for the maximum principle and the material in [10,17]. Let $\phi \in H_0^1(\Omega)$ that can be decomposed as

$$\phi = \phi^+ - \phi^- \qquad \phi^+ = \sup(\phi, 0) \qquad \phi^- = \sup(-\phi, 0). \tag{28}$$

A well known result in [18] states that both ϕ^+ and ϕ^- are in $H_0^1(\Omega)$ and are orthogonal, which means

$$(\phi^+, \phi^-) = (\nabla\phi^+, \nabla\phi^-) = 0. \tag{29}$$

Furthermore if $\phi^- = 0$, then $\phi \ge 0$ or if $\phi^+ = 0$, then $\phi \le 0$.

In order to prove the boundedness of k and ω we define $\widetilde{k}$ and $\widetilde{\omega}$ as

$$\tilde{k} = \begin{cases} k_{inf} & \text{if } k \le k_{inf} \\ k & \text{if } k_{inf} \le k \le k_{sup} \\ k_{sup} & \text{if } k \ge k_{sup}, \end{cases} \qquad \widetilde{\omega} = \begin{cases} \omega_{inf} & \text{if } \omega \le \omega_{inf} \\ \omega & \text{if } \omega_{inf} \le \omega \le \omega_{sup} \\ \omega_{sup} & \text{if } \omega \ge \omega_{inf}, \end{cases} \tag{30}$$

with $k_{inf} = 0$. The $\widetilde{k}$ and $\widetilde{\omega}$ are the same functions as k and ω inside the proposed limits.

Now we introduce a similar k-ω problem where the nonlinear terms are regularized by $\widetilde{k}$ and $\widetilde{\omega}$ and prove that $\widetilde{k} = k$ and $\widetilde{\omega} = \omega$ are indeed solutions of the problem. We consider the following regularized problem

$$\begin{cases} c(\mathbf{u}; k, \psi) + a(\nu + \widetilde{\nu}_t\sigma_k; k, \psi) = \langle \widetilde{S}_k, \psi\rangle - (\beta^* \widetilde{k}\widetilde{\omega}, \psi) & \forall \psi \in H_0^1(\Omega) \\ \langle k, s_k\rangle_\Gamma = \langle g_k, s_k\rangle_\Gamma & \forall s_k \in H^{-1/2}(\Gamma) \\ c(\mathbf{u}; \omega, \phi) + a(\nu + \widetilde{\nu}_t\sigma_\omega; \omega, \phi) = \langle \alpha\, \widetilde{S}_\omega, \phi\rangle - (\beta\, \widetilde{\omega}^2, \phi) & \forall \phi \in H_0^1(\Omega) \\ \langle \omega, s_w\rangle_\Gamma = \langle g_w, s_w\rangle_\Gamma & \forall s_w \in \mathbf{H}^{-1/2}(\Gamma), \end{cases} \tag{31}$$

where $\widetilde{\nu}_t = \nu_t(\widetilde{k}, \widetilde{\omega})$, $\widetilde{S}_k = S_k(\mathbf{u}, \widetilde{k}, \widetilde{\omega})$ and $\widetilde{S}_\omega = S_\omega(\mathbf{u}, \widetilde{k}, \widetilde{\omega})$. The problem (31) is the same as problem (19) inside the proposed interval.

We have that $\omega - \omega_{inf} = (\omega - \omega_{inf})^+ - (\omega - \omega_{inf})^-$ where $()^-$ and $()^+$ are the negative and positive operator functions respectively. Since $(\omega - \omega_{inf})^- \in H_0^1(\Omega)$ we can consider $-(\omega - \omega_{inf})^-$ as test function and obtain

$$c(\mathbf{u}; \omega, -(\omega - \omega_{inf})^-) = c(\mathbf{u}; -(\omega - \omega_{inf})^-, -(\omega - \omega_{inf})^-) = 0 \tag{32}$$
$$a(\nu + \widetilde{\nu}_t\sigma_k; \omega, -(\omega - \omega_{inf})^-) = a(\nu + \widetilde{\nu}_t\sigma_k; -(\omega - \omega_{inf})^-, -(\omega - \omega_{inf})^-). \tag{33}$$

From (22), since $\omega_{inf}^2 \le \inf_\Omega\{\alpha\widetilde{S}_\omega/\beta\}$ and $S_\omega(\mathbf{u}, \widetilde{k}, \widetilde{\omega}) \ge \omega_{inf}^2\beta/\alpha$ for all $\widetilde{k}$ and $\widetilde{\omega}$, we have

$$(\alpha\widetilde{S}_\omega, -(\omega - \omega_{inf})^-) - (\beta\, \widetilde{\omega}^2, -(\omega - \omega_{inf})^-) = (\alpha\widetilde{S}_\omega - \beta\, \omega_{inf}^2, -(\omega - \omega_{inf})^-) \le 0, \tag{34}$$

which implies

$$a(\nu + \widetilde{\nu}_t\sigma_k; (\omega - \omega_{inf})^-, (\omega - \omega_{inf})^-) \le 0, \tag{35}$$

and $(\omega - \omega_{inf})^- = 0$. Therefore we have $0 \le \omega_{inf} \le \omega$.

In a similar way if one uses $(\omega - \omega_{sup})^+ \in H_0^1(\Omega)$ as test function, we obtain

$$c(\mathbf{u}; \omega, (\omega - \omega_{sup})^+) = c(\mathbf{u}; (\omega - \omega_{sup})^+, (\omega - \omega_{sup})^+) = 0 \tag{36}$$
$$a(\nu + \widetilde{\nu}_t\sigma_k; \omega, (\omega - \omega_{sup})^+) = a(\nu + \widetilde{\nu}_t\sigma_k; (\omega - \omega_{sup})^+, (\omega - \omega_{sup})^+). \tag{37}$$

From (22), since $\omega_{sup}^2 \ge \sup_\Omega\{\alpha\widetilde{S}_\omega/\beta\}$ then $S_\omega(\mathbf{u}, \widetilde{k}, \widetilde{\omega}) \le \omega_{sup}^2\beta/\alpha$ for all $\widetilde{k}$ and $\widetilde{\omega}$, we have

$$(\alpha\widetilde{S}_\omega, (\omega - \omega_{sup})^+) - (\beta\, \widetilde{\omega}^2, (\omega - \omega_{sup})^+) = (\alpha\widetilde{S}_\omega - \beta\, \omega_{sup}^2, (\omega - \omega_{sup})^+) \le 0, \tag{38}$$

which implies

$$a(\nu + \widetilde{\nu}_t\sigma_k; (\omega - \omega_{sup})^+, (\omega - \omega_{sup})^+) \le 0, \tag{39}$$

and $(\omega - \omega_{sup})^+ = 0$ or $\omega \le \omega_{sup}$. This implies $\omega_{inf} \le \omega \le \omega_{sup}$ and $\omega = \widetilde{\omega} \in [\omega_{inf}, \omega_{sup}]$.

In a similar way we can prove that the solution k should be equal to $\widetilde{k}$. In fact we can use $-(k-k_{inf})^- \in H_0^1(\Omega)$ as test function. We obtain

$$c(\mathbf{u}; k, -(k-k_{inf})^-) = c(\mathbf{u}; -(k-k_{inf})^-, -(k-k_{inf})^-) = 0 \tag{40}$$

$$a(\nu+\widetilde{\nu}_t\sigma_k; k, -(k-k_{inf})^-) = a(\nu+\widetilde{\nu}_t\sigma_k; -(k-k_{inf})^-, -(k-k_{inf})^-). \tag{41}$$

Since $k_{inf}=0$ and $\widetilde{S}_k \geq 0$ then

$$(\widetilde{S}_k, -(k-k_{inf})^-) - (\beta^*\widetilde{k}\widetilde{\omega}, -(k-k_{inf})^-) = (\widetilde{S}_k, -k^-) \leq 0, \tag{42}$$

which implies

$$a(\nu+\widetilde{\nu}_t\sigma_k; k^-, (k)^-) \leq 0, \tag{43}$$

and $(k)^- = 0$ or $k \geq 0$.

Finally if we use $(k-k_{sup})^+ \in H_0^1(\Omega)$ as test function, we have

$$c(\mathbf{u}; k, (k-k_{sup})^+) = c(\mathbf{u}; (k-k_{sup})^+, (k-k_{sup})^+) = 0 \tag{44}$$

$$a(\nu+\widetilde{\nu}_t\sigma_k; k, (k-k_{sup})^+) = a(\nu+\widetilde{\nu}_t\sigma_k; (k-k_{sup})^+, (k-k_{sup})^+). \tag{45}$$

From (24), since $\beta^* k_{sup}\,\widetilde{\omega} \geq \sup_\Omega\{\widetilde{S}_k\}$ then $\widetilde{S}_k \leq \beta^* k_{sup}\,\widetilde{\omega}$ we have

$$(\widetilde{S}_k, (k-k_{sup})^+) - (\beta\,\widetilde{k}\widetilde{\omega}, (k-k_{sup})^+) = (\widetilde{S}_k - \beta\,k_{sup}\widetilde{\omega}, (k-k_{sup})^+) \leq 0, \tag{46}$$

which implies

$$a(\nu+\widetilde{\nu}_t\sigma_k; (k-k_{sup})^+, (k-k_{sup})^+) \leq 0, \tag{47}$$

and $(k-k_{sup})^+ = 0$ or $k \leq k_{sup}$. This implies $0 \leq k \leq k_{sup}$ and $k = \widetilde{k} \in [0, k_{sup}]$. □

We remark that $\omega_{inf} = \inf\left\{\inf_\Gamma\{g_\omega\}, \inf_\Omega\{\sqrt{\alpha S_\omega/\beta}\}\right\}$ is zero if there is a region where $S^2(\mathbf{u}) = \mathbf{0}$. In this case, which is very usual, we have $k \in [0, k_{sup}]$ and $\omega \in [0, \omega_{sup}]$. With these bounds the ratio $\nu_t = k/\omega$ is non-negative but may be unbounded. The total kinematic viscosity $\nu + \nu_t$ is strictly positive. In order to have $\nu_t \in L^\infty(\Omega)$ the turbulence viscosity must be bounded by ν_{max} when ω vanishes.

By using the previous theorems we can prove an important result of the associated boundary value problem.

Theorem 3. *There exists a solution* $(\mathbf{u}, p, k, \omega)$ *of the associated boundary value problem in* (18)–(22).

Proof. The proof is obtained with standard techniques that can be found in [10]. We briefly describe the most important steps. In order to simplify the notation we assume $\mathbf{g}_u = \mathbf{0}$ and $g_k = g_\omega = 0$. Let $(\mathbf{u}_1, k_1, \omega_1) \in \mathbf{H}_0^1 \times H_0^1 \times H_0^1$ be given. Let $(\mathbf{u}_\eta, p_\eta)$ and (k_η, ω_η) be the state of the following Navier–Stokes-k-ω split problem

$$\begin{array}{ll} a(\nu+\nu_{t1}; \mathbf{u}_\eta, \mathbf{v}) + c(\mathbf{u}_1; \mathbf{u}_\eta, \mathbf{v}) + b(\mathbf{v}, p_\eta) = \langle \mathbf{f}, \mathbf{v}\rangle & \forall\, \mathbf{v} \in \mathbf{H}_0^1(\Omega) \\ b(\mathbf{u}_\eta, q) = 0 & \forall\, q \in L_0^2(\Omega) \\ c(\mathbf{u}_1; k_\eta, \psi) + a(\nu+\nu_{t1}\sigma_k; k_\eta, \psi) = \langle S_{k1}, \psi\rangle - (\beta^*\,k_\eta\,\omega_\eta, \psi) & \forall \psi \in H_0^1(\Omega) \\ c(\mathbf{u}_1; \omega_\eta, \phi) + a(\nu+\nu_{t1}\sigma_\omega; \omega_\eta, \phi) = \langle \alpha\,S_{\omega1}, \phi\rangle - (\beta\,\omega_\eta\,\omega_\eta, \phi) & \forall \phi \in H_0^1(\Omega) \end{array} \tag{48}$$

where $\nu_{t1} = \nu_t(k_1, \omega_1)$, $S_{k1} = S_k(\mathbf{u}_1, k_1, \omega_1)$ and $S_{\omega1} = S_\omega(\mathbf{u}_1, k_1, \omega_1)$. By using standard techniques and Theorem 2 we can prove the existence of a solution of the split system (48). Since $\|\nu_{t1}\|_\infty \leq \nu_{max}$ then $\|\mathbf{u}_\eta\|_1$ and $\|p_\eta\|_0$ are bounded uniformly by the constants C_u and C_p, respectively, for any $\mathbf{u}_1$ and ν_{t1}. By using Theorem 2 also $\|k_\eta\|_1$ and $\|\omega_\eta\|_1$ are uniformly bounded by the constants C_k and C_ω as functions of the given values $k_{max,\upsilon}$ and $\omega_{max,\upsilon}$.

Consider now the following mapping

$$\mathcal{T} : \mathbf{D} = \mathbf{H}_0^1 \times H_0^1 \times H_0^1 \to \mathbf{A} = \mathbf{H}_0^1 \times H_0^1 \times H_0^1 \tag{49}$$

$$\begin{cases} \mathbf{u}_\eta = \mathbf{u}_\eta(\mathbf{u}_1, k_1, \omega_1) \\ k_\eta = k_\eta(\mathbf{u}_1, k_1, \omega_1) \\ \omega_\eta = \omega_\eta(\mathbf{u}_1, k_1, \omega_1). \end{cases}$$

We endow the product space $\mathbf{H}_0^1 \times H_0^1 \times H_0^1$ with the norm $\|(\mathbf{u}_1, k_1, \omega_1)\| = \|\mathbf{u}_1\| + \|k_1\| + \|\omega_1\|$. By using standard techniques it is possible to show that (49) is a continuous mapping with respect to this norm. For similar proofs see [10]. Let R denote the constant $R = C_u + C_k + C_\omega$ and let B_R be the ball of radius R. Since for all $(\mathbf{u}_1, k_1, \omega_1) \in \mathbf{D}$ we have $\|(\mathbf{u}_\eta, k_\eta, \omega_\eta)\| = \|\mathbf{u}_\eta\| + \|k_\eta\| + \|\omega_\eta\| < C_u + C_k + C_\omega = R$ then

$$\mathcal{T}(\mathbf{B}_R) \subset \mathbf{B}_R. \tag{50}$$

Now we can use the Schauder fixed point theorem in order to prove the existence of the fixed point $(\mathbf{u}_1, k_1, \omega_1) = (\mathbf{u}_\eta, k_\eta, \omega_\eta)$ in the mapping $\mathcal{T}$. We recall briefly the theorem. Let $\mathbf{D}$ be a separated topological vector space, $\mathbf{B}_R \subset \mathbf{D}$ a convex subset, and $\mathcal{T}(\mathbf{B}_R) \to \mathbf{B}_R$ a continuous function on $\mathbf{B}_R$, equipped with the topology inherited from $\mathbf{D}$. Also let $\mathcal{T}(\mathbf{B}_R)$ be a compact subset of $\mathbf{B}_R$. Then $\mathcal{T}$ has a fixed point, namely, there exists $\mathbf{x} \in \mathbf{B}_R$ such that $\mathcal{T}(\mathbf{x}) = \mathbf{x}$. The theorem follows from the compactness of $\mathbf{B}_R$, which can be proved with standard techniques, see for example [10]. □

3. Control problem and existence of solutions

We now formulate the model of the optimal control problem. Given the extended boundary functions $g_k \in H^1(\Omega) \cap L^\infty(\Omega)$ and $g_\omega \in H^1(\Omega) \cap L^\infty(\Omega)$ and the positive constants $k_{max,\upsilon}, \omega_{max,\upsilon}, \nu_{max}$ we can define k_{sup}, ω_{sup} and ω_{inf}. The set of all admissible functions k and ω is determined by

$$\mathcal{Q}_{ad} = \left\{ (k, \omega) \in H^1(\Omega) \times H^1(\Omega) \mid 0 \le \omega_{inf} \le \omega \le \omega_{sup} \text{ and } 0 \le k \le k_{sup} \right\}, \tag{51}$$

and set of all admissible functions ν_t by

$$\mathcal{H}_{ad} = \left\{ \nu_t \in L^2(\Omega) \mid \text{ such that } \nu_t \in [0, \nu_{max}] \right\}. \tag{52}$$

The optimal control problem can then be stated in the following way

Given $g_k, g_\omega \in H^1(\Omega) \cap L^\infty(\Omega) \subset \mathcal{Q}_{ad}$ and $\mathbf{g}_u \in \mathbf{H}^1(\Omega)$, find the control $\widehat{\mathbf{f}} \in L^2(\Omega)$ and $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{k}, \widehat{\omega}, \widehat{\nu}_t, \widehat{S}_k, \widehat{S}_\omega)$ such that

$$\mathcal{J}(\widehat{\mathbf{u}}, \widehat{k}, \widehat{\mathbf{f}}) \le \mathcal{J}(\mathbf{u}, k, \mathbf{f}) \tag{53}$$

for all $(\mathbf{u}, p, k, \omega, \nu_t, S_k, S_\omega) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathcal{Q}_{ad} \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ satisfying the constraints (18)–(22) and the objective functional (13).

The admissible set of states and controls is given by

$$\begin{aligned}\mathcal{A}_{ad} = \{&(\mathbf{u}, p, k, \omega, \nu_t, S_k, S_\omega, \mathbf{f}) \in \mathbf{V}(\Omega) \times L^2_0(\Omega) \times \mathcal{Q}_{ad} \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times \mathbf{H}^{-1}(\Omega)\\ &\text{such that } \mathcal{J}(\mathbf{u}, k, \mathbf{f}) < \infty \text{ and } (\mathbf{u}, k, \omega, \mathbf{f}) \text{ satisfies (18)–(22) and (13)}\}.\end{aligned}$$

We now turn to the question of the existence of optimal solutions for the problem in (53).

Theorem 4. *There exists at least one optimal solution $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{k}, \widehat{\omega}, \widehat{\nu}_t, \widehat{S}_k, \widehat{S}_\omega, \widehat{\mathbf{f}}) \in \mathcal{A}_{ad}$ of the optimal control problem (53).*

Proof. The proof follows from standard techniques (see, e.g., [5] or [19]) and here we sketch the main idea. Let $\mathbf{f} = \mathbf{0}$ then we can solve the flow system $(\mathbf{u}, p, k, \omega, \nu_t, S_k, S_\omega, \mathbf{0})$. Since the set of admissible solutions $\mathcal{A}_{ad}$ is not empty and the set of the values assumed by the functional is bounded from below, there exists a minimizing sequence $(\mathbf{u}_m, p_m, k_m, \omega_m, \nu_{tm}, S_{km}, S_{\omega m}, \mathbf{f}_m)$ in $\mathbf{V}(\Omega) \times L^2_0(\Omega) \times \mathcal{Q}_{ad} \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times \mathbf{H}^{-1}(\Omega)$. The sequences $\nu_{tm}, S_{km}, S_{\omega m}$ are uniformly bounded in $L^2(\Omega)$ by construction. If the turbulence source terms $S_{km}, S_{\omega m}$ are bounded, then also sequences k_m, ω_m are uniformly bounded in $\mathbf{H}^1(\Omega)$. Furthermore the functional value for the solution $\mathbf{f} = \mathbf{0}$ is a uniform bound for $\mathbf{u}_m$ and $\mathbf{f}_m$. Using a standard argument, we can extract subsequences $(\mathbf{u}_n, p_n, k_n, \omega_n, \nu_{tn}, S_{kn}, S_{\omega n}, \mathbf{f}_n)$ that converge weakly to $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{k}, \widehat{\omega}, \widehat{\nu}_t, \widehat{S}_k, \widehat{S}_\omega, \widehat{\mathbf{f}})$ [5,10,19]. By standard argument we can pass to the limit inside the linear and the nonlinear terms to prove that this satisfies the constraints. For details on Navier–Stokes one can see [5,10] and for details on turbulence equations [10]. □

In order to compute the optimal solution, we introduce the Lagrange multiplier method and define the optimality system.

4. The Lagrange multiplier method

4.1. Preliminaries

In this section, we show that the Lagrange multiplier technique is well posed and can be used to obtain the first-order necessary condition. Further, the Lagrangian map can be shown to be strictly differentiable for all values of the external force and this allows us to apply the Lagrange multiplier method to a wider range of problems and completes the theoretical treatment of the problem for arbitrary values of the viscosity. Also, this method gives a different and better theoretical insight into the control process, allowing us to write the inequality constraints in a different form.

First, we introduce auxiliary variables that allow us to transform the inequality constraints into equalities and then invoke well-known techniques for equality constrained minimization problems; see, e.g., [20] or [21]. We begin by replacing

$$\nu_t = \min\left\{\frac{k}{\omega}, \nu_{max}\right\} \qquad S_k = \min\left\{\frac{\nu_t}{2}\mathbf{S}^2(\mathbf{u}), \beta^* k_{max,\upsilon}\omega\right\} \qquad S_\omega = \min\left\{\frac{1}{2}\mathbf{S}^2(\mathbf{u}), \frac{\omega^2_{max,\upsilon}\beta}{\alpha}\right\}, \tag{54}$$

by

$$(k-\nu_t\omega)(\nu_{max}-\nu_t)=0 \qquad r_\nu^2-(k-\nu_t\omega)-\omega\,(\nu_{max}-\nu_t)=0 \tag{55}$$

$$\left(\frac{\nu_t}{2}\mathbf{S}^2(\mathbf{u})-S_k\right)(\beta^* k_{max,\nu}\omega-S_k)=0 \qquad r_k^2-\left(\frac{\nu_t}{2}\mathbf{S}^2(\mathbf{u})-S_k\right)-(\beta^* k_{max,\nu}\omega-S_k)=0 \tag{56}$$

$$\left(\frac{1}{2}\mathbf{S}^2(\mathbf{u})-S_\omega\right)\left(\frac{\omega^2_{max,\nu}\beta}{\alpha}-S_\omega\right)=0 \qquad r_\omega^2-\left(\frac{1}{2}\mathbf{S}^2(\mathbf{u})-S_\omega\right)-\left(\frac{\omega^2_{max,\nu}\beta}{\alpha}-S_\omega\right)=0, \tag{57}$$

for some $r_\nu, r_k, r_\omega \in L^2(\Omega)$. Let us consider (55). If $r_\nu^2>0$, then $\nu_t=k/\omega<\nu_{max}$ or $\nu_t=\nu_{max}<k/\omega$ which implies (54), and vice versa if $k/\omega<\nu_{max}$ we have $\nu_t=k/\omega$ and $r_\nu^2=(\nu_{max}-\nu_t)>0$ or $k/\omega>\nu_{max}$ we have $\nu_t=\nu_{max}$ and $r_\nu^2=(k-\nu_{max}\omega)>0$. If $r_\nu^2=0$, then $\nu_t=\nu_{max}=k/\omega$. The same remark is true for the inequalities (56)–(57).

Now we compact all the constraint equations and the functional in two mappings in order to study their differential properties. It is convenient to define the following functional spaces

$$\widehat{\mathbf{B}}_1=\mathbf{H}^1_0(\Omega)\times L^2_0(\Omega)\times \mathbf{L}^2(\Omega)\times \mathcal{Q}_{ad}\times L^2(\Omega)\times L^2(\Omega)^2\times (L^2(\Omega))^3, \tag{58}$$

$$\widehat{\mathbf{B}}_2=\mathbf{H}^{-1}(\Omega)\times L^2_0(\Omega)\times \mathbf{H}^{1/2}(\Gamma)\times H^{-1}(\Omega)\times H^{1/2}(\Gamma)\times H^{-1}(\Omega)\times H^{1/2}(\Gamma)\times L^1(\Omega)^6, \tag{59}$$

$$\widehat{\mathbf{B}}_3=\mathbf{H}^{-1}(\Omega)\times L^2_0(\Omega)\times \mathbf{H}^{1/2}(\Gamma)\times H^{-1}(\Omega)\times H^{1/2}(\Gamma)\times H^{-1}(\Omega)\times H^{1/2}(\Gamma)\times L^1(\Omega)^6, \tag{60}$$

and equip $\widehat{\mathbf{B}}_1, \widehat{\mathbf{B}}_2, \widehat{\mathbf{B}}_3$ with the usual graph norms for the product spaces involved. The functional space $\widehat{\mathbf{B}}_1$ is set for the state variables $\widehat{\mathbf{b}}=(\widehat{\mathbf{u}},\widehat{p},\widehat{\mathbf{f}},\widehat{k},\widehat{\omega},\widehat{\nu_t},\widehat{S_k},\widehat{S_\omega},\widehat{r}_\nu,\widehat{r}_k,\widehat{r}_\omega)$. With this notation the constraints of the problem can be used to form the nonlinear mapping M from $\widehat{\mathbf{B}}_1$ to $\widehat{\mathbf{B}}_3$ defined by

$$M(\widehat{\mathbf{b}})=\widehat{\mathbf{b}}^*$$

if and only if

$$\begin{cases} a(\nu+\nu_t;\widehat{\mathbf{u}},\widehat{\mathbf{v}})+c(\widehat{\mathbf{u}};\widehat{\mathbf{u}},\widehat{\mathbf{v}})+b(\widehat{\mathbf{v}},\widehat{p})-\int_\Omega \widehat{\mathbf{f}}\cdot\widehat{\mathbf{v}}\,d\mathbf{x}=\int_\Omega \mathbf{l}_1\cdot\widehat{\mathbf{v}}\,d\mathbf{x} & \forall\widehat{\mathbf{v}}\in\mathbf{H}^1_0(\Omega)\\ b(\widehat{\mathbf{u}},\widehat{z})=\int_{\widehat{\Omega}} l_2\,\widehat{z}\,d\mathbf{x} & \forall\widehat{z}\in L^2_0(\Omega)\\ \int_\Gamma(\widehat{\mathbf{u}}-\mathbf{g}_u)\cdot\widehat{\mathbf{s}}_u\,ds=\int_\Gamma \mathbf{l}_3\cdot\widehat{\mathbf{s}}_u\,ds & \forall\widehat{\mathbf{s}}_u\in\mathbf{H}^{-1/2}(\Gamma)\\ a(\nu+\nu_t\sigma_k;\widehat{k},\widehat{\phi})+c(\widehat{u};\widehat{k},\widehat{\phi})-\langle S_k,\widehat{\phi}\rangle+\langle\beta^*k\omega,\widehat{\phi}\rangle=\int_\Omega l_4\widehat{\phi}\,d\mathbf{x} & \forall\widehat{\phi}\in H^1_0(\Omega)\\ \int_\Gamma(\widehat{k}-g_k)\widehat{s}_k\,ds=\int_\Gamma l_5\,\widehat{s}_k\,ds & \forall\widehat{s}_k\in\mathbf{H}^{-1/2}(\Gamma)\\ a(\nu+\nu_t\sigma_\omega;\widehat{\omega},\widehat{\psi})+c(\widehat{u};\widehat{\omega},\widehat{\psi})-\alpha\,\langle S_\omega,\widehat{\psi}\rangle+\langle\beta\,\omega^2,\widehat{\psi}\rangle=\int_\Omega l_6\widehat{\psi}\,d\mathbf{x} & \forall\widehat{\psi}\in H^1_0(\Omega)\\ \int_\Gamma(\widehat{\omega}-g_\omega)\widehat{s}_w\,ds=\int_\Gamma l_7\,\widehat{s}_w\,ds & \forall\widehat{s}_w\in\mathbf{H}^{-1/2}(\Gamma)\end{cases} \tag{61}$$

$$\begin{cases} (\widehat{k}-\widehat{\nu_t}\widehat{\omega})(\nu_{max}-\widehat{\nu_t})=l_{\nu 0} & \forall\mathbf{x}\in\Omega\\ \widehat{r}_\nu^2-(\widehat{k}-\widehat{\nu_t}\widehat{\omega})-\widehat{\omega}(\nu_{max}-\widehat{\nu_t})=l_{\nu 1} & \forall\mathbf{x}\in\Omega\\ \left(\frac{\widehat{\nu_t}}{2}\mathbf{S}^2(\widehat{\mathbf{u}})-\widehat{S}_k\right)(\beta^*k_{max,\nu}\widehat{\omega}-\widehat{S}_k)=l_{k0} & \forall\mathbf{x}\in\Omega\\ \widehat{r}_k^2-\left(\frac{\widehat{\nu_t}}{2}\mathbf{S}^2(\widehat{\mathbf{u}})-\widehat{S}_k\right)-(\beta^*k_{max,\nu}\widehat{\omega}-\widehat{S}_k)=l_{k1} & \forall\mathbf{x}\in\Omega\\ \left(\frac{1}{2}\mathbf{S}^2(\widehat{\mathbf{u}})-\widehat{S}_\omega\right)\left(\frac{\omega^2_{max,\nu}\beta}{\alpha}-\widehat{S}_\omega\right)=l_{\omega 0} & \forall\mathbf{x}\in\Omega\\ \widehat{r}_\omega^2-\left(\frac{1}{2}\mathbf{S}^2(\widehat{\mathbf{u}})-\widehat{S}_\omega\right)-\left(\frac{\omega^2_{max,\nu}\beta}{\alpha}-\widehat{S}_\omega\right)=l_{\omega 1} & \forall\mathbf{x}\in\Omega,\end{cases}$$

with $\widehat{\mathbf{b}}^*=(\mathbf{l}_1,l_2,\mathbf{l}_3,l_4,l_5,l_6,l_7,\mathbf{l}_\nu,\mathbf{l}_k,\mathbf{l}_\omega)\in\widehat{\mathbf{B}}_3$. The set of constraint equations in the optimal control problem can be expressed as

$$M(\widehat{\mathbf{b}})=\mathbf{0}.$$

Given $\widehat{\mathbf{b}} \in \mathcal{A}_{ad}$, we define another nonlinear mapping $Q : \widehat{\mathbf{B}}_1 \to \Re \times \widehat{\mathbf{B}}_3$ by $Q(\widehat{\mathbf{b}}) = \widehat{\mathbf{b}}^*$ if and only if

$$\begin{pmatrix} \mathcal{J}(\widehat{\mathbf{u}}, \widehat{k}, \widehat{\mathbf{f}}) - \mathcal{J}(\mathbf{u}_1, k_1, \mathbf{f}_1) \\ M(\widehat{\mathbf{b}}) \end{pmatrix} = \begin{pmatrix} a_1 \\ \widehat{\mathbf{b}}^* \end{pmatrix}. \tag{62}$$

4.2. Differentiability

These mappings are strictly differentiable, as it is shown in the following lemma. We recall the notion of strict differentiability (see [21]). Let X and Y denote Banach spaces, then the mapping $\varphi : X \to Y$ is strictly differentiable at $x \in X$ if there exists a bounded, linear mapping D from X to Y such that for any $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $\|x - x_1\|_X < \delta$ and $\|x - x_2\|_X < \delta$ for $x_1, x_2 \in X$, then

$$\|\varphi(x_1) - \varphi(x_2) - D(x_1 - x_2)\|_Y \le \epsilon \|x_1 - x_2\|_X.$$

The strict derivative D at the point $x \in X$, if it exists, will often be denoted by $D = \varphi'(x)$. The value of this mapping on an element $\widetilde{x} \in X$ will often be denoted by $\varphi'(x) \cdot \widetilde{x}$. In the next theorem we can identify $X = \widehat{\mathbf{B}}_1$ and $Y = \widehat{\mathbf{B}}_2$.

Lemma 1. *Let the nonlinear mappings $M : \widehat{\mathbf{B}}_1 \to \widehat{\mathbf{B}}_2$ and $Q : \widehat{\mathbf{B}}_1 \to \Re \times \widehat{\mathbf{B}}_3$ be defined by (61) and (62), respectively. Then, these mappings are strictly differentiable at the point $\widehat{\mathbf{b}} = (\widehat{\mathbf{u}}, \widehat{p}, \widehat{\mathbf{f}}, \widehat{k}, \widehat{\omega}, \widehat{\nu_t}, \widehat{S}_k, \widehat{S}_\omega, \widehat{r}_\nu, \widehat{r}_k, \widehat{r}_\omega) \in \widehat{\mathbf{B}}_1$ and its strict derivative is given by the bounded linear operator $M'(\widehat{\mathbf{b}}) : \widehat{\mathbf{B}}_1 \to \widehat{\mathbf{B}}_2$, where*

$$M'(\widehat{\mathbf{b}}) \cdot \widetilde{\mathbf{b}} = \overline{\mathbf{b}}$$

for all $\widetilde{\mathbf{b}} = (\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\mathbf{f}}, \widetilde{k}, \widetilde{\omega}, \widetilde{\nu_t}, \widetilde{S}_k, \widetilde{S}_\omega, \widetilde{r}_\nu, \widetilde{r}_k, \widetilde{r}_\omega) \in \widehat{\mathbf{B}}_1$ and $\overline{\mathbf{b}} = (\bar{\mathbf{l}}_1, \bar{l}_2, \bar{\mathbf{l}}_3, \bar{l}_4, \bar{l}_5, \bar{l}_6, \bar{l}_7, \bar{\mathbf{l}}_\nu, \bar{\mathbf{l}}_k, \bar{\mathbf{l}}_\omega) \in \widehat{\mathbf{B}}_2$ if and only if

$$\begin{cases} a(\widetilde{\nu_t}; \widehat{\mathbf{u}}, \widehat{\mathbf{v}}) + a(\nu + \nu_t; \widetilde{\mathbf{u}}, \widehat{\mathbf{v}}) + c(\widetilde{\mathbf{u}}; \widehat{\mathbf{u}}, \widehat{\mathbf{v}}) + c(\widehat{\mathbf{u}}; \widetilde{\mathbf{u}}, \widehat{\mathbf{v}}) + b(\widehat{\mathbf{v}}, \widetilde{p}) - \langle \widetilde{\mathbf{f}}, \widehat{\mathbf{v}} \rangle = \int_{\widehat{\Omega}} \bar{\mathbf{l}}_1 \cdot \widehat{\mathbf{v}}\, d\mathbf{x} & \forall \widehat{\mathbf{v}} \in \mathbf{H}_0^1(\widehat{\Omega}) \\ b(\widetilde{u}, \widehat{z}) = \int_{\widehat{\Omega}} \bar{l}_2 \widehat{z}\, d\mathbf{x} & \forall \widehat{z} \in L_0^2(\widehat{\Omega}) \\ \int_\Gamma (\widetilde{u} - \mathbf{g}_u) \cdot \widehat{s}_u\, ds = \int_\Gamma \bar{\mathbf{l}}_3 \cdot \widehat{s}_u\, ds & \forall \widehat{s}_u \in \mathbf{H}^{-1/2}(\Gamma) \\ a(\widetilde{\nu_t}\sigma_k; \widehat{k}, \widehat{\phi}) + a(\nu + \nu_t\sigma_k; \widetilde{k}, \widehat{\phi}) + c(\widetilde{\mathbf{u}}; \widehat{k}, \widehat{\phi}) + c(\widehat{\mathbf{u}}; \widetilde{k}, \widehat{\phi}) - \langle \widetilde{S}_k, \widehat{\phi} \rangle & \\ \quad + \langle \beta^* \widetilde{k}\widehat{\omega}, \widehat{\phi} \rangle + \langle \beta^* \widehat{k}\widetilde{\omega}, \widehat{\phi} \rangle = \int_\Omega \bar{l}_4 \widehat{\phi}\, d\mathbf{x} & \forall \widehat{\phi} \in H_0^1(\Omega) \\ \int_\Gamma \widetilde{k}\widehat{s}_k\, ds = \int_\Gamma \bar{l}_5 \widehat{s}_k\, ds & \forall \widehat{s}_k \in \mathbf{H}^{-1/2}(\Gamma) \\ a(\widetilde{\nu_t}\sigma_\omega; \widehat{\omega}, \widehat{\psi}) + a(\nu + \nu_t\sigma_\omega; \widetilde{\omega}, \widehat{\psi}) + c(\widetilde{\mathbf{u}}; \widehat{\omega}, \widehat{\psi}) + c(\widehat{\mathbf{u}}; \widetilde{\omega}, \widehat{\psi}) - \alpha\, \langle \widetilde{S}_\omega, \widehat{\psi} \rangle & \\ \quad + \langle \beta\, 2\widehat{\omega}\widetilde{\omega}, \widehat{\psi} \rangle = \int_\Omega \bar{l}_6 \widehat{\psi}\, d\mathbf{x} & \forall \widehat{\psi} \in H_0^1(\Omega) \\ \int_\Gamma \widetilde{\omega}\widehat{s}_w\, ds = \int_\Gamma \bar{l}_7 \widehat{s}_w\, ds & \forall \widehat{s}_w \in \mathbf{H}^{-1/2}(\Gamma) \end{cases} \tag{63}$$

and for the turbulent sources by

$$\begin{cases} (\widetilde{k} - \widetilde{\nu_t}\widehat{\omega} - \widehat{\nu_t}\widetilde{\omega})(\nu_{max} - \widehat{\nu_t}) - \widetilde{\nu_t}(\widehat{k} - \widehat{\nu_t}\widehat{\omega}) = \bar{l}_{\nu 0} & \forall \mathbf{x} \in \Omega \\ 2\,\widetilde{r}_\nu\widehat{r}_\nu - (\widetilde{k} - \widetilde{\nu_t}\widehat{\omega} - \widehat{\nu_t}\widetilde{\omega}) + \widehat{\omega}\widetilde{\nu_t} + \widehat{\nu_t}\widetilde{\omega} = \bar{l}_{\nu 1} & \forall \mathbf{x} \in \Omega \\ \left(\frac{\widetilde{\nu_t}}{2}\mathbf{S}^2(\widehat{\mathbf{u}}) + \widehat{\nu_t}\mathbf{S}(\widetilde{\mathbf{u}}) : \mathbf{S}(\widehat{\mathbf{u}}) - \widetilde{S}_k\right)(\beta^* k_{max,v}\widehat{\omega} - \widehat{S}_k) + \left(\frac{\widehat{\nu_t}}{2}\mathbf{S}^2(\widehat{\mathbf{u}}) - \widehat{S}_k\right)(\beta^* k_{max,v}\widetilde{\omega} - \widetilde{S}_k) = \bar{l}_{k0} & \forall \mathbf{x} \in \Omega \\ 2\widetilde{r}_k\widehat{r}_k - \left(\frac{\widetilde{\nu_t}}{2}\mathbf{S}^2(\widehat{\mathbf{u}}) + \widehat{\nu_t}\mathbf{S}(\widetilde{\mathbf{u}}) : \mathbf{S}(\widehat{\mathbf{u}}) - \widetilde{S}_k\right) - (\beta^* k_{max,v}\widetilde{\omega} - \widetilde{S}_k) = \bar{l}_{k1} & \forall \mathbf{x} \in \Omega \\ (\mathbf{S}(\widetilde{\mathbf{u}}) : \mathbf{S}(\widehat{\mathbf{u}}) - \widetilde{S}_\omega)\left(\frac{\omega_{max,v}^2 \beta}{\alpha} - \widehat{S}_\omega\right) - \widetilde{S}_\omega\left(\frac{1}{2}\mathbf{S}^2(\widehat{\mathbf{u}}) - \widehat{S}_\omega\right) = \bar{l}_{\omega 0} & \forall \mathbf{x} \in \Omega \\ 2\widetilde{r}_\omega\widehat{r}_\omega - \mathbf{S}(\widetilde{\mathbf{u}}) : \mathbf{S}(\widehat{\mathbf{u}}) + 2\widetilde{S}_\omega = \bar{l}_{\omega 1} & \forall \mathbf{x} \in \Omega. \end{cases} \tag{64}$$

Moreover, the strict derivative of Q at a point $\widehat{\mathbf{b}} \in \widehat{\mathbf{B}}_1$ is given by the bounded linear operator $Q'(\widehat{\mathbf{b}}) : \widehat{\mathbf{B}}_1 \to \Re \times \widehat{\mathbf{B}}_2$, where

$$Q'(\widehat{\mathbf{b}}) \cdot \widetilde{\mathbf{b}} = (\bar{a}, \bar{\mathbf{l}}_1, \bar{l}_2, \bar{\mathbf{l}}_3, \bar{l}_4, \bar{l}_5, \bar{l}_6, \bar{l}_7, \bar{\mathbf{l}}_\nu, \bar{\mathbf{l}}_k, \bar{\mathbf{l}}_\omega) \tag{65}$$

for all $\widetilde{\mathbf{b}} \in \widehat{\mathbf{B}}_1$ *and* $(\bar{a}, \bar{\mathbf{l}}_1, \bar{l}_2, \bar{\mathbf{l}}_3, \bar{l}_4, \bar{l}_5, \bar{l}_6, \bar{l}_7, \bar{\mathbf{l}}_\nu, \bar{\mathbf{l}}_k, \bar{\mathbf{l}}_\omega) \in \Re \times \widehat{\mathbf{B}}_2$ *if and only if*

$$\begin{pmatrix} \mathcal{J}'(\widehat{\mathbf{u}}, \widehat{\mathbf{f}}, \widehat{k}, \widehat{\omega}) \cdot (\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\mathbf{f}}, \widetilde{k}, \widetilde{\omega}, \widetilde{\nu_t}, \widetilde{S_k}, \widetilde{S_\omega}, \widetilde{r}_\nu, \widetilde{r}_k, \widetilde{r}_\omega) \\ M'(\widehat{\mathbf{b}}) \cdot (\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\mathbf{f}}, \widetilde{k}, \widetilde{\omega}, \widetilde{\nu_t}, \widetilde{S_k}, \widetilde{S_\omega}, \widetilde{r}_\nu, \widetilde{r}_k, \widetilde{r}_\omega) \end{pmatrix} \begin{pmatrix} \bar{a} \\ (\bar{\mathbf{l}}_1, \bar{l}_2, \bar{\mathbf{l}}_3, \bar{l}_4, \bar{l}_5, \bar{l}_6, \bar{l}_7, \bar{\mathbf{l}}_\nu, \bar{\mathbf{l}}_k, \bar{\mathbf{l}}_\omega) \end{pmatrix},$$

where

$$\mathcal{J}'(\widehat{\mathbf{b}}) \cdot \widetilde{\mathbf{b}} = a \int_\Omega (\widehat{\mathbf{u}} - \mathbf{u}_d) \cdot \widetilde{\mathbf{u}}\, d\mathbf{x} + b \int_\Omega \left(\widehat{k} - k_d\right) \widetilde{k}\, d\mathbf{x} + \lambda \int_{\mathbf{x}} \widehat{\mathbf{f}} \cdot \widetilde{\mathbf{f}}\, d\mathbf{x}.$$

Proof. The linearity of the operator M' is obvious and its boundedness follows from the continuity of the forms $a(\cdot; \cdot, \cdot)$, $b(\cdot, \cdot)$, and $c(\cdot; \cdot, \cdot)$. Likewise, the linearity and boundedness of the operator Q' are obvious. The fact that M' is the strict derivative of the mapping M also follows from the continuity of the trilinear form $c(\cdot; \cdot, \cdot)$ and bilinear form $a(\cdot; \cdot, \cdot)$. Indeed, given $\widehat{\mathbf{b}} = (\widehat{\mathbf{u}}, \widehat{p}, \widehat{\mathbf{f}}, \widehat{k}, \widehat{\omega}, \widehat{\nu_t}, \widehat{S_k}, \widehat{S_\omega}, \widehat{r_\nu}, \widehat{r_k}, \widehat{r_\omega}) \in \widehat{\mathbf{B}}_1$ we have that for any $\epsilon > 0$ and $\widehat{\mathbf{b}}_1, \widehat{\mathbf{b}}_2$ in $\widehat{\mathbf{B}}_1$, such that $\|\widehat{\mathbf{b}} - \widehat{\mathbf{b}}_1\|_{\widehat{\mathbf{B}}_1} < \delta$ and $\|\widehat{\mathbf{b}} - \widehat{\mathbf{b}}_2\|_{\widehat{\mathbf{B}}_1} < \delta$, with appropriate $\delta = \delta(\epsilon)$ we obtain

$$\|M(\widehat{\mathbf{b}}_1) - M(\widehat{\mathbf{b}}_2) - M'(\widehat{\mathbf{b}}) \cdot (\widehat{\mathbf{b}}_1 - \widehat{\mathbf{b}}_2)\|_{\widehat{\mathbf{B}}_2} \le \epsilon \|\widehat{\mathbf{b}}_1 - \widehat{\mathbf{b}}_1\|_{\widehat{\mathbf{B}}_1}.$$

The procedure is standard and the interested reader can see [5,6,22,23] for similar proofs. Thus, the mapping M is strictly differentiable on all of $\widehat{\mathbf{B}}_1$ and its strict derivative is given by M'.

Using the strict differentiability of the mapping M it is then easy to show that the mapping Q is also strictly differentiable and that its strict derivative is given by Q'. □

In order to prove the closure of the range of M' we need a result that claims the existence of the solution with a convection–diffusion equation of the following type

$$-\nabla \cdot (A\nabla T) + (\mathbf{u} \cdot \nabla)T + bT = f \quad \text{in } \Omega \tag{66}$$

$$T = T_1 \quad \text{on } \Gamma. \tag{67}$$

This operator does not satisfy the coercivity property due to the presence of the convective term and therefore the usual Lax–Milgram setting cannot be applied. Nevertheless, it is possible to claim the existence of the state solution for the non-coercive elliptic case if the velocity field $\mathbf{u}$ is in $L^2(\Omega)$ [24,25]. This existence result is obtained not in the Lax–Milgram setting but by using a Leray–Schauder Topological Degree argument.

Theorem 5. *Let $N_* = N$ when $N \ge 3$, $N_* \in]2, \infty[$ when $N = 2$. Consider (66) with $b \in L^{N_*/2}(\Omega)$, $b \ge 0$ a.e. on Ω, $\mathbf{u} \in \mathbf{L}^{N_*}(\Omega)$, and $f \in (H^{-1}(\Omega))$. If A is a function which satisfies these two properties:*

1. *$\exists\, \alpha_A > 0$ such that $A(x)\xi \cdot \xi \ge \alpha_A|\xi|^2$ for a.e. $x \in \Omega$ and for all $\xi \in \Re$;*
2. *$\exists\, \Lambda_A > 0$ such that $|A(x)| \le \Lambda_A$ for a.e. $x \in \Omega$;*

Then, there exists a unique solution $T \in H^1(\Omega)$ of (66).

Proof. The proof of this result is based on a Leray–Schauder Topological Degree argument and can be found in [25]. □

We note that the Navier–Stokes system in (1)–(2) with Dirichlet boundary conditions has at least one solution $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega)$. The Sobolev compact embedding theorem implies $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^q(\Omega)$ which holds for $1 \le q < \infty$ when $N = 2$ and for $1 \le q \le 6$ when $N = 3$. The velocity solution $\mathbf{u} \in \mathbf{H}^1(\Omega)$ verifies the hypothesis in Theorem 5 both with $N = 2$ and with $N = 3$.

We remark also that, if one needs to use the Lax–Milgram setting, existence can be proven by assuming some condition on the velocity field $\mathbf{u}$. For instance, in the case of fully Dirichlet boundary conditions one can have coercivity with

$$-\frac{1}{2}\nabla \cdot \mathbf{u} + b \ge 0. \tag{68}$$

This condition of additional regularity on the velocity field is not needed in the Leray–Lions setting [24].

Next, we prove some further properties of the derivatives of the mappings M and Q. It is worthwhile to note that r_ν, r_k and r_ω are zero when the differential equations in k and ω satisfy both limits at the same time. This is not a problem for the optimization if this happens over points or boundary regions with zero measure but it may be a problem if this is verified over domain with positive measure. For this reason let us introduce the following subsets

$$\Omega_\nu = \left\{ \mathbf{x} \in \Omega \text{ such that } \nu_t = \nu_{max} = k/\omega \right\} \tag{69}$$

$$\Omega_{S_k} = \left\{ \mathbf{x} \in \Omega \text{ such that } S_k = \nu_t \mathbf{S}^2(\mathbf{u})/2 = \beta^* k_{max,\upsilon}\, \omega \right\} \tag{70}$$

$$\Omega_{S_\omega} = \left\{ \mathbf{x} \in \Omega \text{ such that } S_\omega = \mathbf{S}^2(\mathbf{u})/2 = \beta \omega^2_{max,\upsilon}/\alpha \right\}. \tag{71}$$

We use these sets to assure the validity of the Lagrangian multiplier technique around the region where the minimum point should be searched.

Lemma 2. *Let $\widehat{\mathbf{b}} \in \widehat{\mathbf{B}}_1$ denote a solution of the optimal control problem. Then, if the region $\Omega_\nu \cup \Omega_{S_k} \cup \Omega_{S_\omega}$ has zero measure, we have*

(i) *the operator $M'(\widehat{\mathbf{b}})$ has closed range in $\widehat{\mathbf{B}}_2$;*
(ii) *the operator $Q'(\widehat{\mathbf{b}})$ has closed range in $\Re \times \widehat{\mathbf{B}}_2$;*
(iii) *the operator $Q'(\widehat{\mathbf{b}})$ is not onto $\Re \times \widehat{\mathbf{B}}_2$.*

Proof. In order to show (i) we split the system (63)–(64) into three parts: the Navier–Stokes, the k-ω model and turbulence source constraint derivative operator system. Let us consider the Navier–Stokes derivative operator in (63) with $\nu_t \in L^\infty(\Omega)$ and $\nu + \nu_t > 0$. The question of the closeness of the range of the Navier–Stokes operator defined in $M' : \widehat{\mathbf{B}}_1 \to \widehat{\mathbf{B}}_2$ reduces to the like question for the inhomogeneous Stokes operator $\widetilde{\mathbf{S}} : \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega) \times \mathbf{H}^1(\Omega) \cap \mathbf{L}^2_0(\Omega) \to \mathbf{H}^{-1}(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{1/2}(\Gamma(\alpha))$ defined as $\widetilde{\mathbf{S}} \cdot (\widetilde{\mathbf{w}}, \widetilde{\mathbf{p}}) = (\widetilde{\mathbf{l}}_1, \widetilde{l}_2, \widetilde{\mathbf{l}}_3)$ if and only if

$$\begin{cases} \nu a(\widetilde{\mathbf{w}}, \mathbf{v}) + b(\mathbf{v}, \widetilde{p}) - (\mathbf{v}, \widetilde{\mathbf{f}}) = \langle \widetilde{\mathbf{l}}_1, \mathbf{v} \rangle & \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega) \\ b(\widetilde{\mathbf{w}}, z) = (\widetilde{l}_2, z) & \forall z \in L^2(\Omega) \\ \displaystyle\int_\Gamma (\widetilde{\mathbf{w}} - \mathbf{g}) \cdot \mathbf{s}\, ds = \int_\Gamma \widetilde{\mathbf{l}}_3 \cdot \mathbf{s}\, ds & \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma), \end{cases} \tag{72}$$

where $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$. The fact that the operator $\widetilde{\mathbf{S}}$ has closed range in $\mathbf{H}^{-1}(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{1/2}(\Gamma)$ follows easily from well-known results for the Stokes equations; see, e.g., [14]. We can then conclude that the operator $\widetilde{\mathbf{S}}$ has closed range in $\widehat{\mathbf{B}}_2$, and, since the operator $M'(\mathbf{b})$ is a compact perturbation of the operator $\widetilde{\mathbf{S}}$, we have, from the Fredholm theory, that $M'(\mathbf{b})$ itself has closed range in $\widehat{\mathbf{B}}_2$.

Now we consider the k-ω system in M'. Since $\widehat{\mathbf{b}}$ is an optimal solution the system reduces to

$$\begin{cases} a(\nu + \nu_t\sigma_k; \widetilde{k}, \widehat{\phi}) + c(\widehat{\mathbf{u}}; \widetilde{k}, \widehat{\phi}) + \langle \beta^* \widehat{\omega}\widetilde{k}, \widehat{\phi} \rangle = \displaystyle\int_\Omega \bar{l}^*_4 \widehat{\phi}\, d\mathbf{x} & \forall \widehat{\phi} \in H^1_0(\Omega) \\ \displaystyle\int_\Omega \bar{l}^*_4 \widehat{\phi}\, d\mathbf{x} = \int_\Omega \bar{l}_4 \widehat{\phi}\, d\mathbf{x} - a(\widetilde{\nu}_t\sigma_k; \widehat{k}, \widehat{\phi}) - c(\widetilde{\mathbf{u}}; \widehat{k}, \widehat{\phi}) - \langle \beta^* \widetilde{\omega}\widehat{k}, \widehat{\phi} \rangle + \langle \widetilde{S}_k, \widehat{\phi} \rangle & \\ \displaystyle\int_\Gamma \widetilde{k}\widehat{s}_k\, ds = \int_\Gamma \bar{l}_5 \widehat{s}_k\, ds & \forall \widehat{s}_k \in \mathbf{H}^{-1/2}(\Gamma) \\ a(\nu + \nu_t\sigma_\omega; \widetilde{\omega}, \widehat{\psi}) + c(\widehat{\mathbf{u}}; \widetilde{\omega}, \widehat{\psi}) + \langle \beta\, 2\widehat{\omega}\widetilde{\omega}, \widehat{\psi} \rangle = \displaystyle\int_\Omega \bar{l}^*_6 \widehat{\psi}\, d\mathbf{x} & \forall \widehat{\psi} \in H^1_0(\Omega) \\ \displaystyle\int_\Omega \bar{l}^*_6 \widehat{\psi}\, d\mathbf{x} = \int_\Omega \bar{l}_6 \widehat{\psi}\, d\mathbf{x} - a(\widetilde{\nu}_t\sigma_\omega; \widehat{\omega}, \widehat{\psi}) - c(\widetilde{\mathbf{u}}; \widehat{\omega}, \widehat{\psi}) + \langle \widetilde{S}_\omega, \widehat{\psi} \rangle & \\ \displaystyle\int_\Gamma \widetilde{\omega}\widehat{s}_w\, ds = \int_\Gamma \bar{l}_7 \widehat{s}_w\, ds & \forall \widehat{s}_w \in \mathbf{H}^{-1/2}(\Gamma). \end{cases} \tag{73}$$

It is possible to show that $\widetilde{\omega}$-equation in (73) has a solution for all $\bar{l}^*_6$ and also that $\widetilde{k}$-equation can be solved for all $\bar{l}^*_4$. In fact since $\nu + \nu_t$ is a positive function in $L^\infty(\Omega)$ and thanks to the Sobolev compact embeddings $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^q(\Omega)$ which holds for $1 \le q < \infty$ when $N = 2$ and for $1 \le q \le 6$ when $N = 3$, we have that $\mathbf{u} \in \mathbf{H}^1(\Omega)$ verifies the hypothesis in Theorem 5 both with $N = 2$ and with $N = 3$.

Finally we focus on the system (64) under the assumption that $\widehat{\mathbf{b}}$ is an optimal solution. From this we have that $\widehat{\mathbf{S}}^2(\widehat{\mathbf{u}})$ is bounded and $\widehat{\nu}_t \in L^\infty(\Omega)$. If we assume that the region $\Omega_\nu \cup \Omega_k \cup \Omega_\omega \cap \Omega$ has a measure zero, then $\widehat{r}_\nu, \widehat{r}_k, \widehat{r}_\omega$ cannot be zero a.e. on the domain Ω. Therefore the equations can be solved a.e. in Ω for all $\mathbf{l}_\nu = (l_{\nu 0}, l_{\nu 1}) \in L^2(\Omega) \times L^2(\Omega)$, $\mathbf{l}_k = (l_{k0}, l_{k1}) \in L^2(\Omega) \times L^2(\Omega)$ and $\mathbf{l}_\omega = (l_{\omega 0}, l_{\omega 1}) \in L^2(\Omega) \times L^2(\Omega)$ as a function of $\widetilde{\nu}_t, \widetilde{r}_{\nu 1}, \widetilde{k}, \widetilde{r}_{k1}$ and $\widetilde{\omega}$ and $\widetilde{r}_{\omega 1}$, respectively.

Starting from (i), the proof of (ii) and (iii) can be found easily by using the standard techniques in [5,26,27]. □

The first-order necessary condition follows easily from the fact that the operator $Q'(\widehat{\mathbf{b}})$ is not onto $\Re \times \widehat{\mathbf{B}}_2$; see, e.g., [7,27].

Theorem 6. *Let $\widehat{\mathbf{b}} \in \widehat{\mathbf{B}}_1$ be a solution of the optimal control problem, then there exists a nonzero Lagrange multiplier $\widehat{\mathbf{b}}_a = (\lambda_1, \widehat{\mathbf{u}}_a, \widehat{p}_a, \widehat{\mathbf{f}}_a, \widehat{k}_a, \widehat{\omega}_a, \widehat{\nu}_{ta}, \widehat{S}_{ka}, \widehat{S}_{\omega a}, \widehat{r}_{\nu a}, \widehat{r}_{ka}, \widehat{r}_{\omega a}) \in \Re \times \widehat{\mathbf{B}}^*_2$ satisfying the Euler equations*

$$\lambda_1 \mathcal{J}'(\widehat{\mathbf{u}}, \widehat{k}, \widehat{\mathbf{f}}) \cdot \widetilde{\mathbf{b}} + \langle \widehat{\mathbf{b}}_a, M'(\widehat{\mathbf{b}}) \cdot \widetilde{\mathbf{b}} \rangle = 0 \quad \forall \widetilde{\mathbf{b}} \in \widehat{\mathbf{B}}_1, \tag{74}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\widehat{\mathbf{B}}_2$ and $\widehat{\mathbf{B}}^*_2$.

4.3. The optimality system

Next, we examine the first-order necessary condition (74) to derive an optimality system from which optimal states and controls may be determined.

Theorem 7. *Let* $\mathbf{b} \in \widehat{\mathbf{B}}_1$ *denote a solution of the optimal control problem. Then, if the region* $\Omega_\nu \cup \Omega_{S_k} \cup \Omega_{S_\omega}$ *has zero measure,* $(\widehat{\mathbf{u}}_a, \widehat{p}_a, \widehat{\mathbf{f}}_a,)$ *are solutions of*

$$\begin{aligned} & b(\widehat{\mathbf{u}}_a, \widetilde{p}) = 0 \\ & a(\nu+\nu_t; \widetilde{\mathbf{u}}, \widehat{\mathbf{u}}_a) + c(\widetilde{\mathbf{u}}; \widehat{\mathbf{u}}, \widehat{\mathbf{u}}_a) + c(\widehat{\mathbf{u}}; \widetilde{\mathbf{u}}, \widehat{\mathbf{u}}_a) + b(\widetilde{\mathbf{u}}, \widehat{p}_a) = -a\int_\Omega \widetilde{\mathbf{u}}\,(\widehat{\mathbf{u}}-\mathbf{u}_d) - c(\widetilde{\mathbf{u}}; \widehat{k}, \widehat{k}_a) - c(\widetilde{\mathbf{u}}; \widehat{\omega}, \widehat{\omega}_a) \\ & \qquad + a(\widehat{r}_{ka}\widehat{\nu}_t + \widehat{r}_{\omega a}; \widehat{\mathbf{u}}, \widetilde{\mathbf{u}}) - a(\widehat{S}_{ka}\widehat{\nu}_t\,(\beta^* k_{max,\nu}\,\widehat{\omega} - \widehat{S}_k) + \widehat{S}_{\omega a}\left(\frac{\beta\omega^2_{max,\nu}}{\alpha} - \widehat{S}_\omega\right); \widehat{\mathbf{u}}, \widetilde{\mathbf{u}}) \\ & \lambda\langle \widehat{\mathbf{f}}_a, \widetilde{\mathbf{f}}\rangle = \langle \widehat{\mathbf{u}}_a, \widetilde{\mathbf{f}}\rangle, \end{aligned} \tag{75}$$

for all $(\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\mathbf{f}})$ *in* $\mathbf{H}^1_0(\Omega) \times L^2_0(\Omega) \times \mathbf{L}^2(\Omega)$, $\widehat{k}_a, \widehat{\omega}_a$ *are solutions of*

$$\begin{aligned} & a(\nu+\nu_t\sigma_k; \widetilde{k}, \widehat{k}_a) + c(\widehat{\mathbf{u}}; \widetilde{k}, \widehat{k}_a) + \langle \beta^*\,\widehat{k}_a\,\widehat{\omega}, \widetilde{k}\rangle = -b\int_\Omega \left(\widehat{k}-k_d\right)\widetilde{k}\,d\mathbf{x} - \langle \widehat{\nu}_a - \widehat{r}_{\nu a}, \widetilde{k}\rangle \\ & a(\nu+\nu_t\sigma_\omega; \widetilde{\omega}, \widehat{\omega}_a) + c(\widehat{\mathbf{u}}; \widetilde{\omega}, \widehat{\omega}_a) + \langle 2\beta\,\widehat{\omega}\,\widetilde{\omega}, \widehat{\omega}_a\rangle \\ & \quad = \langle -\widehat{r}_{\nu a}\widehat{\nu}_t + \widehat{\nu}_a\widehat{\nu}_t(\nu_{max}-\widehat{\nu}_t), \widetilde{\omega}\rangle - \langle \beta^*\,\widehat{k}\,\widehat{k}_a, \widetilde{\omega}\rangle - \langle \widehat{S}_{ka}\left(\frac{\widehat{\nu}_t}{2}\mathbf{S}^2(\widehat{\mathbf{u}}) - \widehat{S}_k\right) - \widehat{r}_{ka}, \beta^* k_{max,\nu}\widetilde{\omega}\rangle, \end{aligned} \tag{76}$$

for all $(\widetilde{k}, \widetilde{\omega})$ *in* $\mathbf{H}^1_0(\Omega) \times \mathbf{H}^1_0(\Omega)$, $\widehat{\nu}_a, \widehat{S}_{ka}, \widehat{S}_{\omega a}$ *are solutions of the following algebraic equations*

$$\begin{aligned} & \widehat{\nu}_a\Big[\widehat{\omega}(\nu_{max}-\widehat{\nu}_t) + (\widehat{k}-\widehat{\nu}_t\widehat{\omega})\Big] = \Big[\nabla\widehat{\mathbf{u}} : \nabla\widehat{\mathbf{u}}_a + \sigma_k\nabla\widehat{k}\cdot\nabla\widehat{k}_a + \sigma_\omega\nabla\widehat{\omega}\cdot\nabla\widehat{\omega}_a\Big] \\ & \qquad\qquad + 2\widehat{r}_{\nu a}\,\widehat{\omega} + \widehat{S}_{ka}\frac{\mathbf{S}^2(\widehat{\mathbf{u}})}{2}(\beta^* k_{max,\nu}\widehat{\omega} - \widehat{S}_k) - \widehat{r}_{ka}\frac{\mathbf{S}^2(\widehat{\mathbf{u}})}{2}, \\ & \widehat{S}_{ka}\Big[(\beta^* k_{max,\nu}\,\widehat{\omega} - \widehat{S}_k) + \left(\frac{\widehat{\nu}_t}{2}\mathbf{S}^2(\widehat{\mathbf{u}}) - \widehat{S}_k\right)\Big] = -\widehat{k}_a + \widehat{r}_{ka}, \\ & \widehat{S}_{\omega a}\Big[\left(\frac{\beta\omega^2_{max,\nu}}{\alpha} - \widehat{S}_\omega\right) + \left(\frac{1}{2}\mathbf{S}^2(\widehat{\mathbf{u}}) - \widehat{S}_\omega\right)\Big] = -\alpha\widehat{\omega}_a + 2\widehat{r}_{\omega a}, \end{aligned} \tag{77}$$

and $\widehat{r}_{\nu a}, \widehat{r}_{ka}, \widehat{r}_{\omega a}$ *satisfy*

$$\widehat{r}_{\nu a}\widehat{r}_\nu = 0 \qquad \widehat{r}_{ka}\widehat{r}_k = 0 \qquad \widehat{r}_{\omega a}\widehat{r}_\omega = 0. \tag{78}$$

Proof. The first-order necessary condition (74) is equivalent to

$$\begin{aligned} & \lambda_1\Big(a\int_\Omega \widetilde{\mathbf{u}}\,(\widehat{\mathbf{u}}-\mathbf{u}_d)\;d\mathbf{x} + b\int_\Omega \widetilde{k}\left(\widehat{k}-k_d\right)\,d\mathbf{x} + \lambda\int_{\mathbf{x}}\widetilde{\mathbf{f}}\cdot\widehat{\mathbf{f}}\,d\mathbf{x}\Big) \\ & \quad + a(\widetilde{\nu}_t; \widehat{\mathbf{u}}, \widehat{\mathbf{u}}_a) + a(\nu+\nu_t; \widetilde{\mathbf{u}}, \widehat{\mathbf{u}}_a) + c(\widetilde{\mathbf{u}}; \widehat{\mathbf{u}}, \widehat{\mathbf{u}}_a) + c(\widehat{\mathbf{u}}; \widetilde{\mathbf{u}}, \widehat{\mathbf{u}}_a) + b(\widehat{\mathbf{u}}_a, \widetilde{p}) - \langle \widetilde{\mathbf{f}}, \widehat{\mathbf{u}}_a\rangle \\ & \quad + b(\widetilde{\mathbf{u}}, \widehat{p}_a) + \int_\Gamma (\widetilde{u}-\mathbf{g}_u)\cdot\widehat{u}_a\,ds + a(\widetilde{\nu}_t\sigma_k; \widehat{k}, \widehat{k}_a) + a(\nu+\nu_t\sigma_k; \widetilde{k}, \widehat{k}_a) + c(\widetilde{\mathbf{u}}; \widehat{k}, \widehat{k}_a) \\ & \quad + c(\widehat{\mathbf{u}}; \widetilde{k}, \widehat{k}_a) - \langle \widetilde{S}_k, \widehat{k}_a\rangle + \langle \beta^*\,\widetilde{k}\widehat{\omega}, \widehat{k}_a\rangle + \langle \beta^*\,\widehat{k}\widetilde{\omega}, \widehat{k}_a\rangle + \int_\Gamma \widetilde{k}\,\widehat{k}_a\,ds + a(\widetilde{\nu}_t\sigma_\omega; \widehat{\omega}, \widehat{\omega}_a) \\ & \quad + a(\nu+\nu_t\sigma_\omega; \widetilde{\omega}, \widehat{\omega}_a) + c(\widetilde{\mathbf{u}}; \widehat{\omega}, \widehat{\omega}_a) + c(\widehat{\mathbf{u}}; \widetilde{\omega}, \widehat{\omega}_a) - \alpha\,\langle \widetilde{S}_\omega, \widehat{\omega}_a\rangle + \langle \beta\,2\widehat{\omega}\widetilde{\omega}, \widehat{\omega}_a\rangle + \int_\Gamma \widetilde{\omega}\,\widehat{\omega}_a\,ds \\ & \quad + \langle \widehat{\nu}_a, (\widetilde{k}-\widetilde{\nu}_t\widehat{\omega}-\widehat{\nu}_t\widetilde{\omega})(\nu_{max}-\widehat{\nu}_t) - \widetilde{\nu}_t(\widehat{k}-\widehat{\nu}_t\widehat{\omega})\rangle + \langle \widehat{r}_{\nu a}, 2\widetilde{r}_\nu\widehat{r}_\nu - (\widetilde{k}-\widetilde{\nu}_t\widehat{\omega}-\widehat{\nu}_t\widetilde{\omega}) + \widetilde{\nu}_t\widehat{\omega} + \widetilde{\omega}\widehat{\nu}_t\rangle \\ & \quad + \langle \widehat{S}_{ka}, \left(\frac{\widetilde{\nu}_t}{2}\mathbf{S}^2(\widehat{\mathbf{u}}) + \widehat{\nu}_t\mathbf{S}(\widetilde{\mathbf{u}}) : \mathbf{S}(\widehat{\mathbf{u}}) - \widetilde{S}_k\right)(\beta^* k_{max,\nu}\widehat{\omega} - \widehat{S}_k) + \left(\frac{\widehat{\nu}_t}{2}\mathbf{S}^2(\widehat{\mathbf{u}}) - \widehat{S}_k\right)(\beta^* k_{max,\nu}\widetilde{\omega} - \widetilde{S}_k)\rangle \\ & \quad + \langle \widehat{r}_{ka}, 2\widetilde{r}_k\widehat{r}_k - \left(\frac{\widetilde{\nu}_t}{2}\mathbf{S}^2(\widehat{\mathbf{u}}) + \widehat{\nu}_t\mathbf{S}(\widetilde{\mathbf{u}}) : \mathbf{S}(\widehat{\mathbf{u}}) - \widetilde{S}_k\right) - (\beta^* k_{max,\nu}\widetilde{\omega} - \widetilde{S}_k)\rangle + \langle \widehat{r}_{\omega a}, 2\widetilde{r}_\omega\widehat{r}_\omega - \mathbf{S}(\widetilde{\mathbf{u}}) : \mathbf{S}(\widehat{\mathbf{u}})\rangle \\ & \quad + \langle \widehat{r}_{\omega a}, 2\widetilde{S}_\omega\rangle + \langle \widehat{S}_{\omega a}, (\mathbf{S}(\widetilde{\mathbf{u}}) : \mathbf{S}(\widehat{\mathbf{u}}) - \widetilde{S}_\omega)(\frac{\omega^2_{max,\nu}\beta}{\alpha} - \widehat{S}_\omega) - \widetilde{S}_\omega\left(\frac{1}{2}\mathbf{S}^2(\widehat{\mathbf{u}}) - \widehat{S}_\omega\right)\rangle = 0, \end{aligned}$$

for all $\widehat{\mathbf{b}} = (\widehat{\mathbf{u}}, \widehat{p}, \widehat{\mathbf{f}}, \widehat{k}, \widehat{\omega}, \widehat{\nu}_t, \widehat{S}_k, \widehat{S}_\omega, \widehat{r}_\nu, \widehat{r}_k, \widehat{r}_\omega) \in \widehat{\mathbf{B}}_1$. In order to satisfy the integral on the boundary we set homogeneous Dirichlet boundary conditions for the adjoint variables $(\widehat{\mathbf{u}}_a, \widehat{p}_a, \widehat{k}_a, \widehat{\omega}_a)$. Furthermore we are free to choose $\lambda_1 = 1$. By extracting the terms involved in the same variation we obtain (75)–(78). □

From (78) we note that if $\widehat{r}_k \neq 0$, then $\widehat{r}_{ka} = 0$. This is true also for $\widehat{r}_{\omega a}$ and $\widehat{r}_{\nu a}$. Therefore the final adjoint system reduces to

$$
\begin{aligned}
& b(\widehat{\mathbf{u}}_a, \widetilde{p}) = 0, \\
& a(\nu + \nu_t; \widetilde{\mathbf{u}}, \widehat{\mathbf{u}}_a) + c(\widetilde{\mathbf{u}}; \widehat{\mathbf{u}}, \widehat{\mathbf{u}}_a) + c(\widehat{\mathbf{u}}; \widetilde{\mathbf{u}}, \widehat{\mathbf{u}}_a) + b(\widetilde{\mathbf{u}}, \widehat{p}_a) = -a \int_\Omega \widetilde{\mathbf{u}}\,(\widehat{\mathbf{u}} - \mathbf{u}_d)\, d\mathbf{x} \\
& \quad - c(\widetilde{\mathbf{u}}; \widehat{k}, \widehat{k}_a) - c(\widetilde{\mathbf{u}}; \widehat{\omega}, \widehat{\omega}_a) - a(\widehat{S}_{ka}\widehat{\nu}_t\,(\beta^* k_{max,\nu}\,\widehat{\omega} - \widehat{S}_k) + \widehat{S}_{\omega a}\left(\frac{\beta\omega^2_{max,\nu}}{\alpha} - \widehat{S}_\omega\right); \widehat{\mathbf{u}}, \widetilde{\mathbf{u}}), \\
& a(\nu + \nu_t\sigma_k; \widehat{k}_a, \widetilde{k}) + c(\widehat{\mathbf{u}}; \widetilde{k}, \widehat{k}_a) + \langle \beta^*\,\widehat{k}_a\,\widehat{\omega}, \widetilde{k}\rangle = -\langle \widehat{\nu}_a, \widetilde{k}\rangle - b\int_\Omega \left(\widehat{k} - k_d\right)\widetilde{k}\, d\mathbf{x} \\
& a(\nu + \nu_t\sigma_\omega; \widetilde{\omega}, \widehat{\omega}_a) + c(\widehat{\mathbf{u}}; \widetilde{\omega}, \widehat{\omega}_a) + \langle 2\beta\,\widehat{\omega}\widehat{\omega}_a, \widetilde{\omega}\rangle \\
& \quad = \langle \widehat{\nu}_a\,\widehat{\nu}_t(\nu_{max} - \widehat{\nu}_t), \widetilde{\omega}\rangle - \langle \beta^*\,\widehat{k}\,\widehat{k}_a, \widetilde{\omega}\rangle - \langle \widehat{S}_{ka}\left(\frac{\widehat{\nu}_t}{2}\mathbf{S}^2(\widehat{\mathbf{u}}) - \widehat{S}_k\right)\beta^* k_{max,\nu}, \widetilde{\omega}\rangle,
\end{aligned}
\tag{79}
$$

with control

$$
\widehat{\mathbf{f}}_a = \frac{\widehat{\mathbf{u}}_a}{\lambda},
$$

adjoint turbulent viscosity

$$
\widehat{\nu}_a = \frac{\left[\nabla\widehat{\mathbf{u}} : \nabla\widehat{\mathbf{u}}_a + \sigma_k\nabla\widehat{k}\cdot\nabla\widehat{k}_a + \sigma_\omega\nabla\widehat{\omega}\cdot\nabla\widehat{\omega}_a\right] - \widehat{S}_{ka}\frac{\mathbf{S}^2(\widehat{\mathbf{u}})}{2}(\beta^* k_{max,\nu}\widehat{\omega} - \widehat{S}_k)}{r_\nu^2}, \tag{80}
$$

and adjoint turbulence sources

$$
\widehat{S}_{ka} = -\frac{\widehat{k}_a}{r_k^2}, \qquad \widehat{S}_{\omega a} = -\frac{\alpha\widehat{\omega}_a}{r_\omega^2}. \tag{81}
$$

Furthermore in the case in which no bounds are reached and

$$
\widehat{\nu}_t = \frac{k}{\omega}, \qquad \widehat{S}_k = \frac{\widehat{\nu}_t}{2}\mathbf{S}^2(\widehat{\mathbf{u}}), \qquad \widehat{S}_\omega = \frac{1}{2}\mathbf{S}^2(\widehat{\mathbf{u}}), \tag{82}
$$

the adjoint system (80)–(81) simplifies.

The numerical solution of this system of variational equations and inequalities is a rather important question and thus we propose a simple projected gradient algorithm (see [6]). In practice, one cannot solve the system simultaneously and at each iteration the method requires the sequential solution of the turbulence system (1)–(4) and the adjoint system in (79)–(81). We solve the optimality system by using the following steepest descent algorithm:

1. Setup. Set an initial variable state $(\mathbf{v}^0, p^0, k^0, \omega^0)$ satisfying (18) and (19). Compute the functional $\mathcal{J}^0$ in (13). Set $r^0 = 1$.
2. Compute $(\nu_a^i, S_{ka}^i, S_{\omega a}^i)$ from (80)–(81) and solve (79) for the adjoint state $(\mathbf{v}_a^i, p_a^i, k_a^i, \omega_a^i, \nu_a^i)$ at the iteration i.
   (a) Compute the control $\mathbf{f}^i = \mathbf{f}^{i-1} + r^{i,j}\,\mathbf{v}_a^i/\lambda$ at the iteration j.
      i. Compute (20)–(22) for (ν_t, S_k, S_ω) and solve (18)–(19) for the state $(\mathbf{v}^{i,j,n}, p^{i,j,n}, k^{i,j,n}, \omega^{i,j,n})$ at the iteration n.
      ii. Return to step (i) until convergence of the nonlinear RANS-k-ω system is obtained.
   (b) Compute the new functional $\mathcal{J}^{i,j+1}$ in (13).
      - if $\|\mathcal{J}^{i,j+1} - \mathcal{J}^{i,j}\| < toll$ convergence of the optimal control problem reached.
      - else if $\mathcal{J}^{i,j+1} > \mathcal{J}^{i,j}$ set $r^{i,j+1} = 2/3\, r^{i,j}$, return to step (2a) and perform another j iteration.
      - else if $\mathcal{J}^{i,j+1} < \mathcal{J}^{i,j}$ set $r^{i,j+1} = 3/2\, r^{i,j}$, return to step (2) and perform another i iteration.

In the next section we apply this algorithm to some simple numerical cases.

5. Numerical results

In this section we report the results obtained by solving the optimality system introduced in the previous section over a simple geometry with different values of the parameters a, b and λ of the objective functional. We solve the optimality system with a finite element code that implements a multi-grid solver which refines a coarse mesh of the domain many times and allows to solve on very fine grids. We study a two dimensional plane channel with a developing flow for both a velocity matching and a turbulence enhancement problem. Different values of the parameter λ are chosen in the range of

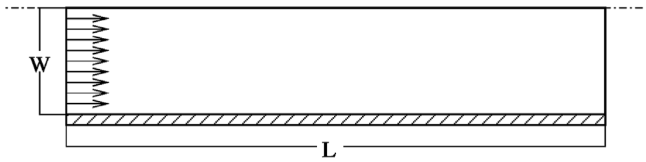

Fig. 1. Plane channel geometry.

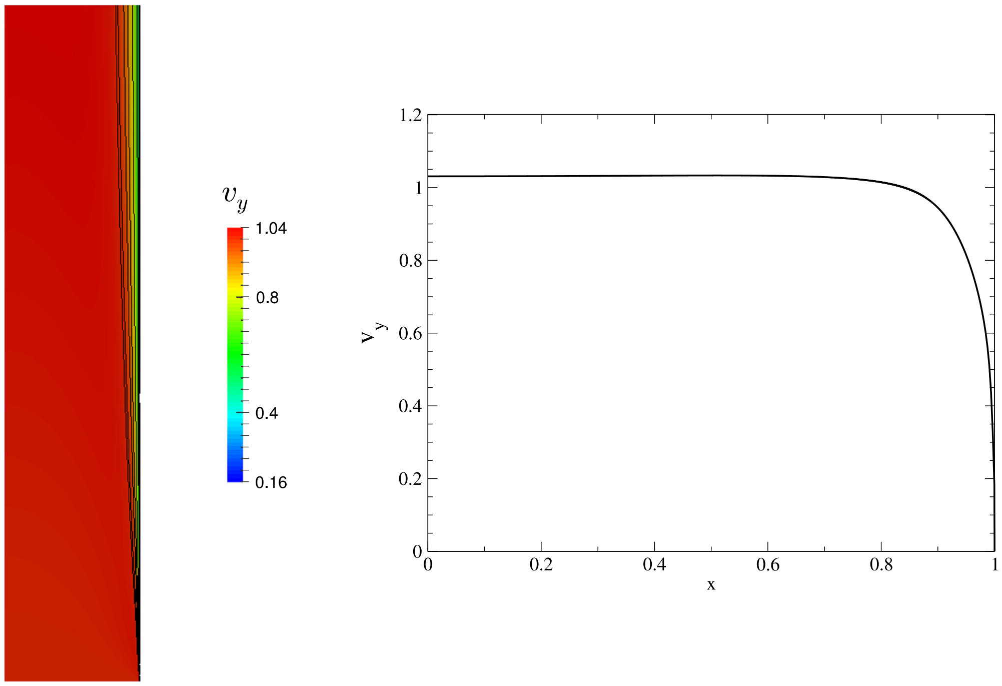

Fig. 2. Axial velocity v_y in the plane channel geometry without control, $Re = 10{,}000$. On the left ten iso-velocity contours on the whole domain, on the right velocity profile on a line at $y = 4$.

10^{-1}–10^{-3}. The plane channel geometry is reported in Fig. 1. The fluid flows from the left to the right. The flow direction, along the horizontal axis, is denoted by y and the transverse axis is denoted by x. On left side, the inlet of the channel, we impose a uniform profile $\mathbf{u} = (u, v) = (0, 1)$. On the outlet we impose outflow boundary conditions while homogeneous Neumann boundary conditions on the symmetry axis for v_y, v_{ay}, k_t, ω_t, k_{at} and ω_{at} are set together with vanishing values for v_x and v_{ax}. On the wall we use the near-wall approach as defined in the introductory section.

The most interesting physical variable for this problem is the kinematic viscosity ν. We can choose the Reynolds number $Re = \bar{\mathbf{v}}L/\nu$, based on the half-width of the channel, simply by changing its value. For this test case we impose $Re = 10{,}000$. With this value of Reynolds number the flow is not fully developed at the outlet of the channel and the Poiseuille profile is not recovered. In Fig. 2 on the left, the axial velocity v_y is reported on the whole domain in ten iso-velocity contours, as obtained without any control acting for the test case of developing flow with $Re = 10{,}000$. On the right of the same figure the profile of v_y on a line at $y = 4$ can be seen and the parabolic profile is far from being obtained.

For the velocity matching problem we choose $\mathbf{u}_d = (0, 1)$ and three values of λ, 10^{-1}, 10^{-2} and 10^{-3}. When λ is small, the control can act strongly to achieve a low objective functional, as we show in the following. In Fig. 3 on the left we report the velocity v_y profiles on a line at $y = 4$ obtained with decreasing λ. As λ decreases, the velocity approaches a uniform value of 1. The region where the objective is poorly achieved is near the wall because the velocity is set to zero on the wall. In this region the adjoint variable is higher because of the high source term $(\mathbf{u} - \mathbf{u}_d)$. In Fig. 3 on the right the profile of the adjoint velocity v_{ay} is reported for the three values of λ showing the strong peak near the wall in the boundary layer region and the flat profile in the center-channel region with a negative value that tends to decrease the fluid velocity which in this region is higher than 1. One can see Fig. 2 on the right as the reference result. We can see very different values of v_{ay} obtained with decreasing λ. To understand this behavior we must keep in mind that the control force acting on the velocity is scaled by the value of λ, so a smaller adjoint velocity is needed to obtain the same effect on the fluid velocity. In Fig. 4 ten equally subdivided contours of the adjoint variables $\mathbf{u}_a$ and k_a are reported for $\lambda = 10^{-3}$ to show the effect of convection on the adjoint velocity and the result in term of adjoint turbulence kinetic energy. The k_a seems to be affected mostly by the gradients double product of the velocity and adjoint velocity as a source term through the adjoint viscosity ν_a,

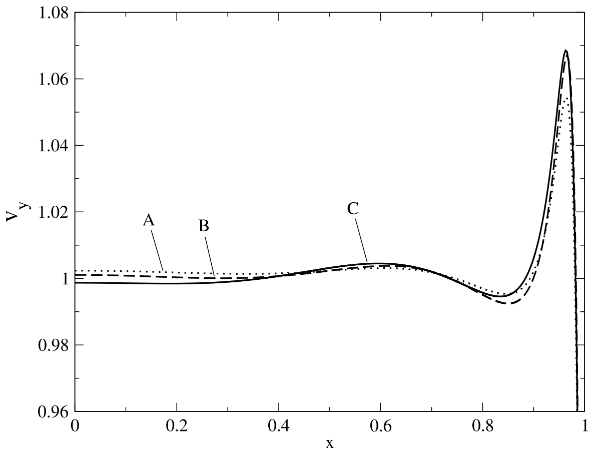

Fig. 3. On the left axial velocity v_y in the plane channel geometry with active control, profile on a line at $y = 4$. On the right adjoint axial velocity profile v_{ay} along the same line. Result (A) is obtained with $\lambda = 10^{-1}$, (B) with $\lambda = 10^{-2}$ and (C) with $\lambda = 10^{-3}$.

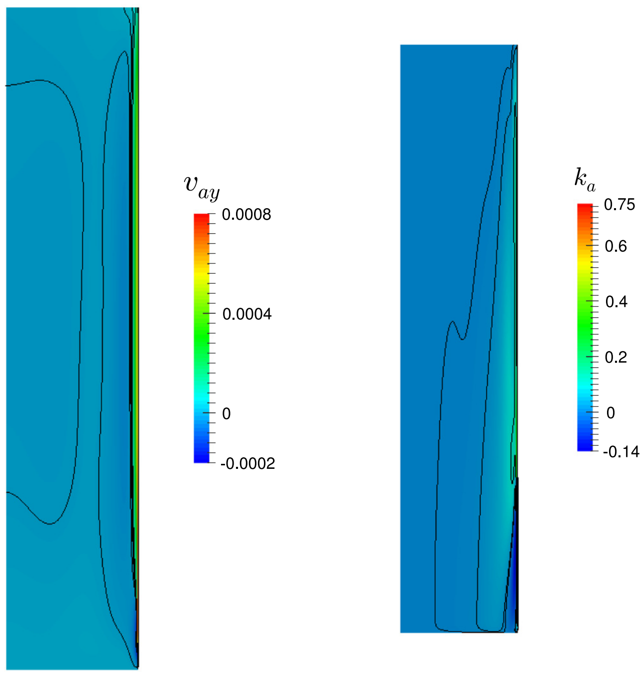

Fig. 4. On the left adjoint axial velocity v_{ay}, on the right adjoint turbulence kinetic energy k_a represented with colors and five line contours for both variables, plane channel with $\lambda = 10^{-3}$.

Table 1
Objective functionals computed with no control and different λ values in the velocity matching profile problem.

λ	∞	10^{-1}	10^{-2}	10^{-3}
$\mathcal{J}(\mathbf{v}, \mathbf{v_a})$	0.015995	0.003803	0.003353	0.003318

although the physical meaning of this variable in the context of velocity matching profile is difficult to be understood. Finally in Table 1 the objective functional is reported as computed in the velocity matching profile problem for the different λ values. As one can see, by decreasing λ a more effective control is attained and the objective functional becomes smaller. Moreover by decreasing λ from 10^{-1} to 10^{-2} a great improvement is obtained while this is less evident when λ is changed from 10^{-2} to 10^{-3}.

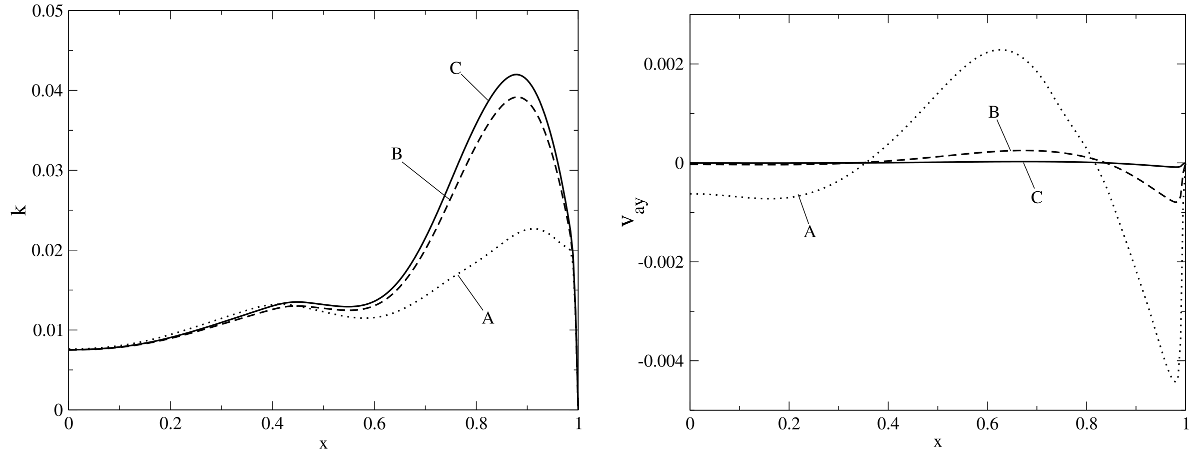

Fig. 5. On the left turbulence energy k in the plane channel geometry with active control, profile on a line at $y = 4$. On the right adjoint axial velocity profile v_{ay} on the same line. Result (A) is obtained with $\lambda = 10^{-1}$, (B) with $\lambda = 10^{-2}$ and (C) with $\lambda = 10^{-3}$.

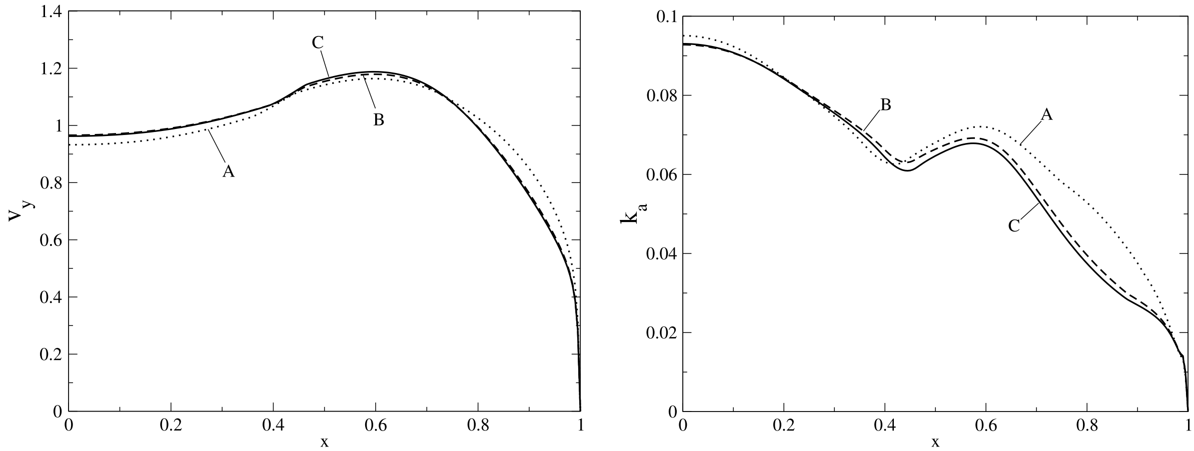

Fig. 6. On the left axial velocity v_y in the plane channel geometry with active control, profile on a line at $y = 4$. On the right adjoint turbulence energy k_a on the same line. Result (A) is obtained with $\lambda = 10^{-1}$, (B) with $\lambda = 10^{-2}$ and (C) with $\lambda = 10^{-3}$.

In order to test the optimal control solver with $a = 10^{-6}$, $b = 1$ in (13) we consider a turbulence enhancement problem and we set $k_d = 0.1$ because the turbulence energy in the test case without control is always smaller than this value. The same three values of λ (10^{-1}, 10^{-2}, 10^{-3}) are employed. In Fig. 5 on the left the turbulence energy k is reported for three values of λ on a line at $y = 4$. On the right of the same figure the axial adjoint velocity is reported on the same line for different values of λ. The control increases the value of the turbulence energy by applying a negative force near the wall and in the center of the channel slowing down the fluid, while it accelerates the motion in the intermediate region with a positive force. The adjoint velocity with decreasing λ behaves in a similar way as pointed out in Fig. 3. The resulting axial velocity profile is reported in Fig. 6 on the left. As expected, the control slows down the fluid near the wall and in the center of the channel while accelerates it in the intermediate region. In this way the gradient of the velocity becomes higher and the turbulence kinetic energy increases. In Fig. 6 we can see the profile of the adjoint variable k_a which, in this test case, is the key variable.

On the right of Fig. 6 the profile of k_a is reported for three values of λ. The physical meaning of k_a in this setting is clear because its main source term is the difference $k - k_d$, so where k_a is higher it means that the objective is far from being attained. By looking at this profile it can be seen that in the center of the channel the turbulence energy is low. When λ decreases, the overall profile improves, except from a small region near the center of the channel where only small changes can be seen. In this test case convective effects are important since the feedback between velocity and turbulence energy is weak. This can be seen in Fig. 7 where we report some plots on the whole domain to understand better the results. In Fig. 7 on the left the turbulence energy k is reported on the whole domain with ten equally subdivided iso-lines for the test case with $\lambda = 10^{-3}$. The highest turbulence kinetic energy is obtained near the outlet of the channel and near the wall where the boundary layer is enlarged by the control action. On the right of the same figure the adjoint turbulence energy k_a is reported on the whole domain as well. On the inlet of the channel near the center the control is ineffective and here k_a

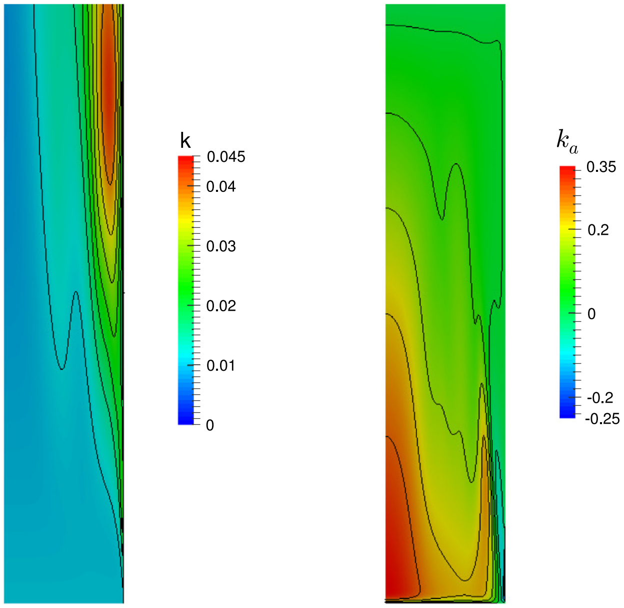

Fig. 7. Turbulence energy k (on the left) and adjoint turbulence kinetic energy k_a (on the right) represented with colors and ten line contours for both variables with $\lambda = 10^{-3}$.

Table 2
Objective functional computed with different λ values in the turbulence enhancement problem.

λ	10^{-1}	10^{-2}	10^{-3}
$\mathcal{J}(k, \mathbf{v_a})$	0.02009	0.01894	0.01870

has a strong peak, while the region where it is smaller is near the outlet and near the wall, where the turbulence energy is higher. By looking at these results it can be easily understood that advection is important for the chosen Reynolds number. Finally we report in Table 2 the objective functional as computed for this test case of turbulence enhancement and different λ values. We can note the decreasing of the functional with the decreasing of λ and the improvement obtained between $\lambda = 10^{-1} \rightarrow 10^{-2}$ with respect to $\lambda = 10^{-2} \rightarrow 10^{-3}$.

6. Conclusion

In this work we have studied a distributed optimal control problem for the Reynolds Averaged Navier–Stokes system coupled with a two-equation turbulence model in a k-ω formulation. The mathematical model has been studied and the existence of solutions of the RANS-k-ω system has been proven. Then the control problem has been considered and the existence of an optimal solution proven. Finally the optimality system has been derived with the Lagrange multiplier method. Numerical results obtained with the implementation of the optimal control solver in a finite element code have been presented and discussed.

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