Lectures on Elliptic and Parabolic Equations in Sobolev Spaces

N. V. Krylov

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Preface

These lectures concentrate on some basic facts and ideas of the modern theory of linear elliptic and parabolic partial differential equations (PDEs) in Sobolev spaces. We hope to show that this theory is based on some general and extremely powerful ideas and some *simple* computations. The main objects of study are the Cauchy problem for parabolic equations and the first boundary-value problem for elliptic equations, with some guidelines concerning other boundary-value problems such as the Neumann or oblique derivative problems or problems involving higher order elliptic operators acting on the boundary. The presentation has been chosen in such a way that after having followed the book the reader should acquire a good understanding of a wide variety of results and techniques.

These lecture notes appeared as the result of a two-quarter or a one-semester graduate course I gave at Moscow State University and the School of Mathematics, University of Minnesota, over a number of years and differ significantly from previous drafts. This book also includes some parts of the initial draft and as a whole is most appropriate for a two-quarter or a one-year course. Naturally, one cannot expect that in such a short course all important issues of the theory of elliptic and parabolic equations can be covered. Actually, even the area of second-order *elliptic* partial differential equations is so wide that one cannot imagine a book, let alone a textbook, of reasonable size covering all bases. Restricting further only to the theory of solvability in the *Sobolev* function spaces and *linear* equations still does not make the task realistic. Because of that we will only be concerned with some basic facts and ideas of the modern theory of linear elliptic and parabolic equations in Sobolev spaces. We refer the interested reader to the books [**7**], [**9**], and [**15**], which are classical texts and reference books in elliptic and parabolic PDEs, and the literature therein for additional information on the subject.

I have been educated as a probabilist who in his early stages of research came across the necessity of using some PDE results and realized, with some deep disappointment mixed with astonishment, that at that time there were no simple *introductory* books about the modern theory available to a wide audience. The situation is slightly better now, forty-five years later, but yet by now, as has been pointed out above, the theory became so wide that it is impossible to have just one simple introductory book available to a wide audience. Indeed, one does have several introductory books in different areas of the theory including the book I wrote on Hölder space theory (see [**12**]) and the current textbook on Sobolev space theory.

As with almost any graduate textbook, this one is written for myself and my graduate students who, as I think, should know at least that much of the theory in order to be able to work on problems related to my own interests. That is why the choice of these "basic facts and ideas" and the exposition is by no means exhaustive but rather reflects the author's taste and, in part, his view on what he should have known to be able to work in some areas of mathematics such as the theory of random diffusion processes. I also hope that the contents of the book will be useful to other graduate students and scientists in mathematics, physics, and engineering interested in the theory of partial differential equations.

Comments on the structure of the book

We start with the $\mathcal{L}_2$ theory of elliptic second-order equations in the whole space, first developing it for the Laplacian on the basis of the Fourier transform. Then we go to the $\mathcal{L}_2$ theory for equations with variable coefficients by using partitions of unity, the method of "freezing the coefficients", the method of a priori estimates, and the method of continuity. This is done in Chapter 1.

In Chapter 2 we deal with the $\mathcal{L}_2$ theory of second-order parabolic equations along similar lines. As far as parabolic equations are concerned, in these notes we only concentrate on equations in the whole space and the Cauchy problem.

After that, in Chapter 3, we present some tools from real analysis, helping to pass from $\mathcal{L}_2$ theory to $\mathcal{L}_p$ theory with $p \neq 2$.

In Chapter 4 we derive basic $\mathcal{L}_p$ estimates first for parabolic and then for elliptic equations. The estimates for the elliptic case turn out to follow immediately from the estimates for parabolic equations. On the other hand, for elliptic equations such estimates can be derived directly and we outline how to do this in Section 4.1 and in a few exercises (see Exercises 1.1.5, 1.3.23, and 4.3.2 and the proof of Theorem 4.3.7).

Chapter 5 is devoted to the $\mathcal{L}_p$ theory of elliptic and parabolic equations with continuous coefficients in the whole space. Chapter 6 deals with the same issues but for equations with VMO coefficients, which is quite a new development (only about 16 years old; compare it with the fundamental papers [**1**] of 1959–64).

Chapter 7 is the last one where we systematically consider parabolic equations. There the solvability of parabolic equations with VMO coefficients is proved in Sobolev spaces with mixed norms. Again as in Chapter 2, everything is done only for the equations in the whole space or for the Cauchy problem for equations whose coefficients are only measurable in the time variable. We return to this problem only briefly in Section 13.5 for equations with coefficients independent of time.

Starting from Chapter 8, we only concentrate on elliptic equations in

$$\mathbb{R}^d = \{x = (x^1, ..., x^d) : x^i \in (-\infty, \infty)\}$$

or in domains $\Omega \subset \mathbb{R}^d$. It is worth noticing, however, that almost everything proved for the elliptic equations in Chapter 8 is easily shown to have a natural version valid for parabolic equations in $\mathbb{R} \times \Omega$. Also the Cauchy problem in $(0, \infty) \times \Omega$ can be treated very similarly to what is done in Section 5.2. We do not show how to do that. Here we again run into choosing between what basic facts your graduate students should know and what are other very interesting topics in PDEs. Anyway, the interested reader can find further information about parabolic equations in [**14**], [**15**], and [**18**].

The reader can consult the table of contents as to what issues are investigated in the remaining chapters. We will only give a few more comments.

Chapters 12 and 13 can be studied almost independently of all previous with the exception of Chapter 3. There are many reasons to include their contents in a textbook, although it could be that this is the first time this is done. I wanted my graduate students to be exposed to equations in the function spaces of Bessel potentials since the modern theory of stochastic partial differential equations is using them quite extensively.

Some important topics are scattered throughout the book, the most notable are:

- Equations in divergence form; see Sections 4.4, 8.2, and 13.6.
- Boundary-value problems involving boundary differential operators; see Exercises 1.1.11 and 13.3.15 and Sections 9.3 and 12.3.

- Elliptic equations with measurable coefficients in two dimensions including the Neumann problem; see Exercises 1.4.7, 1.4.8, 1.4.9, 1.6.7, 8.2.6, 8.2.11, 11.5.5, and 11.6.5.

These notes are designed as a textbook and contain about 271 exercises, a few of which (almost all of the 63 exercises marked with an *) are used in the main text. These are the simplest ones. However, many other exercises are quite difficult, despite the fact that their solutions are almost always short. Therefore, the reader should not feel upset if he/she cannot do them even after a good deal of thinking. Perhaps, hints for them should have been provided right after each exercise. We do give the hints to the exercises but only at the end of each chapter just to give the readers an opportunity to test themselves.

Some exercises which are not used in the main text are put in the main text where the reader has enough knowledge to solve them and thus learn more. Some other exercises less directly connected with the text are collected in optional subsections.

The theorems, lemmas, remarks, and such which are part of the main units of the text are numbered serially in a single system that proceeds by section. Exercise 1.7.6 is the sixth numbered unit in the seventh section in the first chapter. In the course of Chapter 1, this exercise is referred to as Exercise 7.6, and in the course of the seventh section of chapter 1 it is referred to as Exercise 6. Similarly and independently of these units formulas are numbered and cross-referenced.

Basic notation

A complete reference list of notation can be found in the index at the end of the book. We always use the summation convention and allow constants denoted by N, usually without indices, to vary from one appearance to another even in the same proof. If we write $N = N(...)$, this means that N depends only on what is inside the parentheses. Usually, in the parentheses we list the objects that are fixed. In this situation one says that the constant N is *under control*. By domains we mean general open sets. On some occasions, we allow ourselves to use different symbols for the same objects, for example,

$$u_{x^i} = \frac{\partial u}{\partial x^i} = D_i u, \quad u_x = \operatorname{grad} u = \nabla u, \quad u_{xx} = (u_{x^i x^j}).$$

Any d-tuple $\alpha = (\alpha_1, ..., \alpha_d)$ of integers $\alpha_k \in \{0, 1, 2...\}$ is called a multi-index. For a multi-index α, $k, j \in \{1, ..., d\}$, and $\xi = (\xi^1, ..., \xi^d) \in \mathbb{R}^d$ we denote

$$D_{kj} u = D_j D_k u = u_{x^k x^j}, \quad |\alpha| = \alpha_1 + ... + \alpha_d,$$

$$D^\alpha = D_1^{\alpha_1} \cdot \ldots \cdot D_d^{\alpha_d}, \quad \xi^\alpha = (\xi^1)^{\alpha_1} \cdot \ldots \cdot (\xi^d)^{\alpha_d}.$$

We also use the notation $Du = u_x$ for the gradient of u, $D^2u = u_{xx}$ for the matrix of second-order derivatives of u, and $D^n u$ for the set of all nth order derivatives of u. These $D^n u(x)$ for each x can be considered as elements of a Euclidean space of appropriate dimension. By $|D^n u(x)|$ we mean any fixed norm of $D^n u(x)$ in this space.

In the case of parabolic equations we work with

$$\mathbb{R}^{d+1} = \{(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^d\}.$$

For functions $u(t, x)$ given on subdomains of $\mathbb{R}^{d+1}$ we use the above notation only for the derivatives in x and denote

$$\partial_t u = \frac{\partial u}{\partial t} = u_t, \quad \partial_t D_k u = u_{tx^k} = u_{x^k t},$$

and so on.

If Ω is a domain in $\mathbb{R}^d$ and $p \in [1, \infty]$, by $\mathcal{L}_p(\Omega)$ we mean the set of all Lebesgue measurable complex-valued functions f for which

$$\|f\|_{\mathcal{L}_p(\Omega)} := \Big(\int_\Omega |f(x)|^p \, dx \Big)^{1/p} < \infty$$

with the standard extension of this formula if $p = \infty$. We also define

$$\mathcal{L}_p = \mathcal{L}_p(\mathbb{R}^d).$$

One knows that $\mathcal{L}_p(\Omega)$ is a Banach space. In the cases when f and g are measurable functions defined in the same domain $D \subset \mathbb{R}^d$, we write "$f = g$ in D" to mean that f equals g *almost everywhere* in D with respect to Lebesgue measure.

By $C_0^\infty(\Omega)$ we mean the set of all infinitely differentiable functions on Ω with compact support (contained) in Ω. By the *support* of a function we mean the closure of the set where the function is different from zero. We call a subset of $\mathbb{R}^d$ *compact* if it is closed and bounded. Of course, saying "compact support" is the same as saying "bounded support" and we keep "compact" just to remind us that we are talking about *closed* sets. We set

$$C_0^\infty = C_0^\infty(\mathbb{R}^d).$$

For $k \in \{0, 1, 2, ...\}$, by $C^k(\Omega)$ we denote the set of all k times continuously differentiable functions on Ω with finite norm

$$\|u\|_{C^k(\Omega)} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{C(\Omega)},$$

where

$$\|u\|_{C(\Omega)} = \sup_{x \in \Omega} |u(x)|.$$

As usual, $C(\Omega) = C^0(\Omega)$ and we drop Ω in $C^k(\Omega)$ and $\mathcal{L}_p(\Omega)$ if $\Omega = \mathbb{R}^d$. The subset of C^k consisting of functions on $\mathbb{R}^d$ with compact support is denoted C_0^k. In particular, C_0 is the set of continuous functions with compact support.

By $C^k(\bar{\Omega})$ we denote the subset of $C^k(\Omega)$ consisting of all functions u such that u and $D^\alpha u$ extend to functions continuous in $\bar{\Omega}$ (the closure of Ω) whenever $|\alpha| \le k$. For these extensions we keep the same notation u and $D^\alpha u$, respectively.

If Ω is an unbounded domain, by $C_0^k(\bar{\Omega})$ we mean the subset of $C^k(\bar{\Omega})$ consisting of functions vanishing for $|x|$ sufficiently large. Mainly, the notation $C_0^k(\bar{\Omega})$ will be used for $\Omega = \mathbb{R}^d$ and $\Omega = \mathbb{R}^d_+$, where

$$\mathbb{R}^d_+ = \{(x^1, x') : x^1 > 0, x' = (x^2, ..., x^d) \in \mathbb{R}^{d-1}\}.$$

Generally, speaking about functions on $\mathbb{R}^d$, we mean Lebesgue measurable functions.

To the instructor

We begin with an $\mathcal{L}_2$ theory of second-order elliptic equations of the type

$$a^{ij}(x)u_{x^ix^j}(x) + b^i(x)u_{x^i}(x) + c(x)u(x) = f(x), \quad x \in \mathbb{R}^d,$$

where the coefficients are assumed to be bounded and the matrix $a = (a^{ij})$ symmetric and uniformly positive definite. If the matrix a is uniformly continuous and c is sufficiently large negative, we show that the equation is solvable for $f \in \mathcal{L}_2$. Much later in Section 11.6 on the basis of L_p theory, the restriction on c is replaced with $c \le -\delta$, where $\delta > 0$ is any constant.

Then the general scheme set out in Chapter 1 is repeated several times in the succeeding text in various settings without going into all the minor details each time.

Chapter 3 contains all the tools from real analysis that we use. Here is a major difference between these notes and one of the previous drafts. At some stage, I was presenting the $\mathcal{L}_p$ theory on the basis of the Calderón-Zygmund and Marzinkiewicz theorems interpolating between $p = 1$ and $p = 2$. The present course is based on using the Fefferman-Stein theorem on sharp functions. One could use this theorem along with Stampacchia's interpolation theorem (and Marzinkiewicz's interpolation theorem) to interpolate between $p = 2$ and $p = \infty$, which is done in some texts and in one of the drafts of these lectures. However, there is a shorter way to achieve the goal. The point is that it is possible to obtain *pointwise* estimates of the sharp function $u^{\#}_{xx}$ of the second-order derivatives u_{xx} of the unknown solution u through the maximal function of the right-hand side of the equation. This fact has actually been very well known for quite a long time and allows one to get the $\mathcal{L}_p$ estimates of u_{xx} for $p > 2$ just by referring to the Fefferman-Stein and Hardy-Littlewood maximal function theorems.

A disadvantage of this approach is that the students are not exposed to such very powerful and beautiful tools as the Calderón-Zygmund, Stampacchia, and Marzinkiewicz theorems. On the other hand, there are two advantages. First, the Fefferman-Stein theorem is much more elementary than the Calderón-Zygmund theorem (see Chapter 3). Second, the approach based on the pointwise estimates allows us to prove existence theorems for equations with VMO coefficients with almost the same effort as in the case of equations with continuous coefficients.

Despite the fact that, as has been mentioned before, the $\mathcal{L}_p$ theory for elliptic equations can be developed independently of the theory of parabolic equations, in the main text we prefer to derive elliptic estimates from parabolic ones for the following reasons. In the first place, we want the reader to have some insight into the theory of parabolic equations. Secondly, to estimate the $\mathcal{L}_2$ oscillation of u_{xx} over the unit ball B_1 centered at the origin for a C_0^∞ function u, through the maximal function of Δu we split $f := \Delta u$ into two parts: $f = h + g$ where $h \in C_0^\infty$ and $h = f$ in the ball of radius 2 centered at the origin. Then we define v and w as solutions of the equations $\Delta v = h$ and $\Delta w = g$. The trouble is that to find appropriate v and w is not so easy. Say, if $d = 1$ and we want the equations $\Delta v = h$ and $\Delta w = g$ to be satisfied in the whole space, quite often v and v will be unbounded. In the parabolic case this difficulty does not appear because we can find v and w as solutions of the Cauchy problem with zero initial condition, when the initial condition is given for t lying outside the domain where we are estimating the $\mathcal{L}_2$ oscillation of the solution.

One more point worth noting is that one can have a short course on elliptic equations, and after going through Chapter 1, go directly to Chapters 8–10 if one is only interested in the case $p = 2$. If $d = 2$, one can also include Chapter 11. Adding to this list Chapter 3 and Section 4.1 would allow the reader to follow all the material in full generality apart from what concerns parabolic equations and equations with VMO coefficients. In this case one also has to follow the proof of Lemma 6.3.8 in order to get control on the $\mathcal{L}_p$ norm of solutions. Finally, doing Exercise 4.3.2 allows one to include the results of Chapter 6 related to the elliptic equations with VMO coefficients.

If one wants to give a course containing both Hölder space and Sobolev space theories, then one can start with part of the present notes, use the possibility of obtaining basic $C^{2+\alpha}$ estimates by doing Exercises 4.3.2, 4.3.3, and 10.1.8, and then continue lecturing on Hölder space theory following one's favorite texts. By the way, this switch to Hölder space theory can be done right after Chapter 1 if only elliptic equations are to be treated. For parabolic equations this switch is possible after going through Chapters 1 and 2 and doing Exercises 4.3.5, 4.3.6, 10.1.9, and 10.1.10.

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Nicolai Krylov, Minneapolis, August 2007

Chapter 1

Second-order elliptic equations in $W_2^2(\mathbb{R}^d)$

One of the most important elliptic operators acting on functions $u(x), x \in \mathbb{R}^d$, is Laplace's operator given by the formula

$$\Delta u = u_{x^1x^1} + \ldots + u_{x^dx^d}.$$

The function Δu is called the Laplacian of u.

Given a domain $D \subset \mathbb{R}^d$ and a function $f(x), x \in \mathbb{R}^d$, one may try to follow what is usually done in the theory of ordinary differential equations in one space dimension and set up the goal of finding explicitly the general solution of the equation

$$\Delta u(x) = f(x), \quad x \in D.$$

It turns out that generally this is impossible.

On few occasions, however, one can find a solution explicitly. For instance, if $d = 2$, $f \equiv -1$, $D = (-\pi/2, \pi/2)^2$ and we are interested in finding explicitly a solution vanishing on the boundary of D, then, as is known from undergraduate school, the solution is given by the formula

$$u(x, y) = \sum_{n,m=1}^{\infty} c_{nm} \cos(2n - 1)x \cos(2m - 1)y,$$

where

$$c_{nm} = 64\pi^{-4}(-1)^{n+m}(2n - 1)^{-1}(2m - 1)^{-1}[(2n - 1)^2 + (2m - 1)^2]^{-1}.$$

Now one may try to answer some simple questions about the solution like: Is it true that $u \geq 0$? Is u continuously differentiable in D and how many derivatives does it possess?

Looking at this formula, it is either impossible or very hard to answer these questions. Later, by using different means, we will see that indeed $u \geq 0$ and u is infinitely differentiable in D.

Thus, there are no explicit solutions in the general case and often, even if one can find an explicit formula, some fundamental properties of the solution are still hard to establish.

These are the main reasons why other approaches to studying partial differential equations were developed.

The goal of this chapter is to show how some simple and natural computations lead to the necessity of introducing Sobolev spaces and investigating their properties and then to an $\mathcal{L}_2$ theory of elliptic equations. The general scheme set out in this chapter will be used a few times in the future.

1. The simplest equation $\lambda u - \Delta u = f$

To explain how and why Sobolev spaces naturally appear, we start with investigating the solvability of the simplest equation

$$\lambda u - \Delta u = f \tag{1}$$

in $\mathbb{R}^d$ with an "arbitrary" right-hand side f and fixed $\lambda > 0$.

Notice a few simple properties of equation (1).

1. Lemma. *Let $u \in C_0^2$ be a solution of (1) in $\mathbb{R}^d$. Then*

$$\lambda^2\|u\|_{\mathcal{L}_2}^2 + 2\lambda \sum_{j=1}^{d} \|u_{x^j}\|_{\mathcal{L}_2}^2 + \sum_{j,k=1}^{d} \|u_{x^j x^k}\|_{\mathcal{L}_2}^2 = \|f\|_{\mathcal{L}_2}^2. \tag{2}$$

Proof. We use some properties of the Fourier transform, which is defined by

$$F(u)(\xi) = \tilde{u}(\xi) = c_d \int_{\mathbb{R}^d} e^{-i\xi\cdot x} u(x)\,dx, \quad c_d = (2\pi)^{-d/2}.$$

By using integration by parts, one easily proves that

$$i\xi^k \tilde{u}(\xi) = c_d \int_{\mathbb{R}^d} e^{-i\xi\cdot x} u_{x^k}(x)\,dx, \quad -\xi^k\xi^l \tilde{u}(\xi) = c_d \int_{\mathbb{R}^d} e^{-i\xi\cdot x} u_{x^k x^l}(x)\,dx.$$

Hence, (1) implies that $\tilde{f} = (\lambda + |\xi|^2)\tilde{u}$ and $|\tilde{f}|^2 = (\lambda + |\xi|^2)^2|\tilde{u}|^2$. The latter means that

$$\lambda^2|\tilde{u}(\xi)|^2 + 2\lambda\sum_{j=1}^{d}|i\xi^j\tilde{u}(\xi)|^2 + \sum_{j,k=1}^{d}|-\xi^j\xi^k\tilde{u}(\xi)|^2 = |\tilde{f}(\xi)|^2.$$

By integrating this over $\mathbb{R}^d$ and using Parseval's identity ($\|\tilde{g}\|_{\mathcal{L}_2} = \|g\|_{\mathcal{L}_2}$), we arrive at (2). The lemma is proved.

2. Exercise. Prove (2) by raising (1) to the second power and integrating by parts. (Be careful: u is only in C_0^2.)

The following exercise will be mentioned in the hint to Exercise 4.8, that concerns the solvability of elliptic equations with measurable coefficients in $\mathbb{R}^2$.

3. Exercise. By using the Fourier transform, prove that, if $d = 2$ and $u \in C_0^2$, then

$$\int_{\mathbb{R}^d} \det u_{xx}\,dx = 0.$$

It might be of interest that this result holds in any dimension (see Exercise 2.1).

4. Exercise*. Let $\ell_1, \dots, \ell_d$ form an orthonormal basis in $\mathbb{R}^d$ and let u be a twice continuously differentiable function. Prove that

$$\Delta u = \sum_{k,i,j} u_{x^ix^j}\ell_k^i\ell_k^j,$$

which is to say that the operator Δ is invariant under orthogonal transformations.

5. Exercise. Let B be the open ball of radius one centered at the origin and let u be a three times continuously differentiable function in $\bar{B}$. Assume that $u = 0$ on the boundary ∂B of B and prove that

$$\int_B (|\Delta u|^2 - \sum_{i,j}|u_{x^ix^j}|^2)\,dx = (d-1)\int_{\partial B}|u_x \cdot n|^2\,dS,$$

where n is the unit outer normal to ∂B ($n(x) = x$) and dS is the element of the surface measure.

In this chapter we use the following theorem only for $p = 2$ and for this case a different proof is outlined in Exercise 12.

6. Theorem. *Let $\lambda > 0$. Then the set $L := (\lambda - \Delta)C_0^\infty$ is everywhere dense in $\mathcal{L}_p$ for any $p \in [1, \infty)$.*

The proof is based on the following.

7. Lemma. *Let $\lambda > 0$ and let u be a bounded from above twice continuously differentiable function on $\mathbb{R}^d$ satisfying*

$$\Delta u - \lambda u \geq 0$$

in $\mathbb{R}^d$. Then $u \leq 0$. In particular, if u is bounded and $\Delta u - \lambda u = 0$, then $\pm u \leq 0$, so that $u \equiv 0$.

Proof. Assume that $u > 0$ at some points. Set $\zeta(x) = \cosh(\varepsilon|x|)$, where $\varepsilon > 0$ is a small constant. Bearing in mind Taylor's series, one easily proves that ζ is infinitely differentiable. Next the function $v = u/\zeta$ satisfies $(\Delta - \lambda)(\zeta v) \geq 0$, that is,

$$\zeta \Delta v + 2\zeta_{x^j} v_{x^j} + cv \geq 0, \tag{3}$$

where

$$c = \Delta\zeta - \lambda\zeta = \zeta\left\{\varepsilon^2 + (d-1)|x|^{-1}\varepsilon \tanh(\varepsilon|x|) - \lambda\right\}.$$

Since $\tanh|x| \leq |x|$, we have $c < 0$ for all small ε (say $\varepsilon^2 < \lambda/d$). Actually, the above computation makes perfect sense only if $x \neq 0$. But the conclusion that $c = \Delta\zeta - \lambda\zeta < 0$ is valid for all x due to the continuity of ζ and its derivatives.

By assumption, $v > 0$ at some points and the positive part of v tends to zero as $|x| \to \infty$. Therefore, v attains its maximum value at a point $x_0 \in \mathbb{R}^d$ and $v(x_0) > 0$. The first derivatives of v vanish at x_0 and the pure second-order derivatives are nonpositive at this point. This definitely contradicts (3) since $v(x_0) > 0$. We have the desired contradiction and the lemma is proved.

8. Exercise. Give an example of unbounded twice continuously differentiable function $u > 0$ such that $\Delta u - u \geq 0$ in $\mathbb{R}^d$.

9. Remark. In the case that $\lambda = 0$ it turns out that if u is a bounded from above twice continuously differentiable function on $\mathbb{R}^d$ satisfying $\Delta u = 0$ in $\mathbb{R}^d$, then u is a constant. The same is true if we replace $\Delta u = 0$ with $\Delta u \geq 0$ and assume that $d = 1$ or $d = 2$. However if $d \geq 3$, there are smooth bounded functions u such that $\Delta u \geq 0$.

Sometimes one needs the first part of the following version of Lemma 7 in the form of *the maximum principle*. Recall that

$$\mathbb{R}^d_+ = \{x = (x^1, x') : x^1 > 0, x' \in \mathbb{R}^{d-1}\}.$$

10. Exercise. Let Ω be a bounded or unbounded domain in $\mathbb{R}^d$ and let u be bounded and continuous in $\bar{\Omega}$ and twice continuously differentiable in Ω. Let $\Delta u - \lambda u \geq 0$ in Ω, where the constant $\lambda > 0$. Prove that under these conditions

$$u \leq \sup_{\partial\Omega} u_+ \quad \text{in} \quad \bar{\Omega} \quad (\sup_{\emptyset} \ldots := 0).$$

Also prove Hopf's lemma: For $\Omega = \mathbb{R}^d_+$ under the above conditions, if $x_0 \in \partial\Omega$ is such that $u(x_0) = \sup_{\partial\Omega} u_+ > 0$ and $u_{x^1}(x_0)$ exists, then $u_{x^1}(x_0) \leq -\sqrt{\lambda}u(x_0) < 0$.

The following exercise can be used in investigating uniqueness of solutions to some boundary-value problems in $\mathbb{R}^d_+$.

11. Exercise. Let $\Omega = \mathbb{R}^d_+$, a constant $\lambda \geq 0$, and let u be a bounded and twice continuously differentiable function given on Ω. Assume that $\Delta u - u \geq 0$ in Ω and

$$u_{x^1} + \sum_{j\geq 2} \beta^j u_{x^j} + \sum_{j,k\geq 2} \alpha^{jk} u_{x^j x^k} - \lambda u \geq 0$$

on $\partial\Omega$, where β^j and α^{jk} are constant and (α^{jk}) is a nonnegative symmetric matrix. Prove that $u \leq 0$ in $\bar{\Omega}$.

Proof of Theorem 6. If the assertion is wrong, then by the Hahn-Banach theorem there is a linear functional on $\mathcal{L}_p$ vanishing on $(\lambda - \Delta)C_0^\infty$. Then by Riesz's representation theorem there is a function $g \in \mathcal{L}_q$ with $q = p/(p-1)$ and $\|g\|_{\mathcal{L}_q} \neq 0$ such that

$$\int_{\mathbb{R}^d} g(x)(\lambda u(x) - \Delta u(x))\, dx = 0$$

for all $u \in C_0^\infty$. Since, for any $y \in \mathbb{R}^d$, the function $u(y - x)$ belongs to C_0^∞, we have

$$\int_{\mathbb{R}^d} g(x)(\lambda u(y-x) - \Delta u(y-x))\, dx = 0 \tag{4}$$

for all $u \in C_0^\infty$ and $y \in \mathbb{R}^d$. Here the left-hand side happens to be

$$(\lambda - \Delta)(g * u)(y),$$

which follows from the well-known rules of differentiating integrals with respect to parameters. These rules also imply that $g * u$ is infinitely differentiable for any locally integrable g, in particular, for $g \in \mathcal{L}_q$. For $g \in \mathcal{L}_q$ by using Hölder's inequality, we see that $g * u$ is bounded. Thus, by Lemma 7 we conclude $g * u = 0$ and

$$\int_{\mathbb{R}^d} gu\, dx = 0 \tag{5}$$

for any $u \in C_0^\infty$. Finally, we use the well-known fact from integration theory that if g is locally integrable and (5) holds for any $u \in C_0^\infty$, then $g = 0$ (a.e.). This is the desired contradiction and the theorem is proved.

Now imagine that we have to solve (1) with an $f \in \mathcal{L}_2$. Then one can proceed as follows. Since $\bar{L} = \mathcal{L}_2$ (Theorem 6), there is a sequence $u^n \in C_0^2$ such that, for

$$f^n := \lambda u^n - \Delta u^n \tag{6}$$

we have $\|f - f^n\|_{\mathcal{L}_2} \to 0$. Furthermore (6) and Lemma 1 imply that

$$\lambda^2 \|u^n - u^m\|_{\mathcal{L}_2}^2 + 2\lambda \sum_{j=1}^{d} \|u^n_{x^j} - u^m_{x^j}\|_{\mathcal{L}_2}^2$$
$$+ \sum_{j,k=1}^{d} \|u^n_{x^j x^k} - u^m_{x^j x^k}\|_{\mathcal{L}_2}^2 = \|f^n - f^m\|_{\mathcal{L}_2}^2.$$

Since the sequence f_n converges in $\mathcal{L}_2$, it is a Cauchy sequence:

$$\|f^n - f^m\|_{\mathcal{L}_2} \to 0$$

as $n, m \to \infty$. It follows that $u^n, u^n_{x^j}, u^n_{x^j x^k}$ are also Cauchy sequences in $\mathcal{L}_2$. The completeness of $\mathcal{L}_2$ implies that there exist the $\mathcal{L}_2$ limits

$$v = \lim_{n\to\infty} u^n, \quad v_j = \lim_{n\to\infty} u^n_{x^j}, \quad v_{jk} = \lim_{n\to\infty} u^n_{x^j x^k}.$$

Now set by definition

$$v_{x^j x^k} = v_{jk}$$

putting aside for a while the justification of that notation. Then, from (6) we conclude that (a.e.)

$$\lambda v - \Delta v = f. \tag{7}$$

This is a natural way to come to the necessity of introducing Sobolev spaces and generalized solutions of (7).

12. Exercise. The above proof of Theorem 6 uses Lemma 7, the proof of which is based on considering maximum points of certain functions. This way does not work for higher-order or nonelliptic operators with constant coefficients. In connection with this take $p = 2$ in Theorem 6. Use the fact that the Fourier transform can be defined preserving Parseval's identity for all $u \in \mathcal{L}_2$ as the limit in $\mathcal{L}_2$ of $\tilde{u}_n$, where $u_n \in C_0^\infty$ and $u_n \to u$ in $\mathcal{L}_2$.

(i) Prove that if $f \in \mathcal{L}_2$ and $h \in \mathcal{L}_1$, then $f * h \in \mathcal{L}_2$ and $c_d F(f * h) = \tilde{f}\tilde{h}$.

(ii) Derive that the left-hand side in (4) belongs to $\mathcal{L}_2$ and its Fourier transform is

$$c_d^{-1}\tilde{g}(\lambda + |\xi|^2)\tilde{u}.$$

Conclude that $\tilde{g} = 0$, $g = 0$.

13. Exercise. Let $m \geq 1$ be an integer and let a^α be some (complex) numbers, not all of which are zero, given for any multi-indices α such that $|\alpha \leq m$. Consider the operator

$$L = \sum_{|\alpha| \leq m} a^\alpha D^\alpha$$

and prove that the set LC_0^∞ is everywhere dense in $\mathcal{L}_p$ for any $p \in [2, \infty)$. You may like to first prove the following fact: If v is an infinitely differentiable function such that v and each of its derivatives of any order are bounded and belong to $\mathcal{L}_2$, then $F(D^\alpha v)(\xi) = i^{|\alpha|}\xi^\alpha \tilde{v}(\xi)$ for any multi-index α.

14. Remark. As is easy to see from its proof, Lemma 1 remains valid for any $\lambda \in \mathbb{R}$. The same is true for Theorem 6 at least if $p \geq 2$ as is seen from Exercise 12 for $p = 2$ and from Exercise 13 for more general p.

2. Integrating the determinants of Hessians (optional)

The following three exercises do not have much to do with the main subject of these lectures. They are related to the remarkable properties of Jacobians and two very powerful tools often used in the theory of *nonlinear* PDEs. The author could not resist the temptation of discussing them once Exercise 1.3 has been proposed.

1. Exercise. Let Ω be a connected bounded domain with smooth boundary and let $F, G : \Omega \to \mathbb{R}^d$ be $C^1(\Omega)$ mappings such that

$$F = G \quad \text{on} \quad \partial\Omega.$$

Observe that for large $t > 0$ the mappings $F_t = F(x) + tx$ and $G_t = G(x) + tx$ are one-to-one, map $\partial\Omega$ into the boundary of $F_t(\Omega)$ and $G_t(\Omega)$, and therefore

$$\operatorname{Vol} F_t(\Omega) = \operatorname{Vol} G_t(\Omega).$$

Express this equality in terms of $\partial F_t/\partial x$ and $\partial G_t/\partial x$, then use that the determinants of these matrices are polynomials in t, and conclude that

$$\int_\Omega \det \frac{\partial F}{\partial x}\, dx = \int_\Omega \det \frac{\partial G}{\partial x}\, dx.$$

For Ω being a large ball, $F = u_x$, and $G = 0$, this exercise explains without computations why the result of Exercise 1.3 is true.

2. Exercise. For the domain from the preceding exercise show that there is no $C^1(\bar{\Omega})$ function $G : \bar{\Omega} \to \partial\Omega$ such that $G(x) = x$ on $\partial\Omega$.

Extend the result to $C(\bar{\Omega})$ functions G if $\Omega = B_1 := \{x \in \mathbb{R}^d : |x| < 1\}$.

3. Exercise (Brouwer's fixed point theorem). Let $f : \bar{B}_1 \to \bar{B}_1$ be a continuous mapping. We suggest the reader prove that f has fixed points in $\bar{B}_1$ (where $f(x) = x$) by using Exercise 2. You may like to start by assuming the contrary and on the basis of considering the line passing through x and $f(x)$, construct a mapping G as in Exercise 2.

Upon mapping each convex closed bounded set onto B_1, extend the result to all continuous mappings of convex closed bounded sets.

4. Exercise (Fan Ky minimax theorem). Let X, Y be closed bounded convex subsets of $\mathbb{R}^d$ and $f(x,y)$ a real-valued function defined on $X \times Y$ such that

(i) $f(x,y)$ is concave with respect to $y \in Y$ for each $x \in X$,

(ii) $f(x,y)$ is convex with respect to $x \in X$ for each $y \in Y$.

Prove that then

$$\min_{x\in X} \max_{y\in Y} f(x,y) = \max_{y\in Y} \min_{x\in X} f(x,y). \tag{1}$$

3. Sobolev spaces $W_p^k(\Omega)$

Recall that if Ω is a domain (open set) in $\mathbb{R}^d$ and $k \in \{0,1,2,...\}$, we denote by $C^k(\bar{\Omega})$ the subset of $C^k(\Omega)$ consisting of functions u such that u and $D^\alpha u$ extend to functions continuous in $\bar{\Omega}$ (the closure of Ω) whenever $|\alpha| \le k$. For these extensions we keep the same notation u and $D^\alpha u$, respectively.

1. Definition. Let $p \in [1,\infty)$, $k \in \{0,1,2,...\}$ and let Ω be a domain in $\mathbb{R}^d$. For functions $u \in C^k(\bar{\Omega})$ define

$$\|u\|_{W_p^k(\Omega)} = \sum_{|\alpha|\le k} \|D^\alpha u\|_{\mathcal{L}_p(\Omega)}.$$

For a function $u \in \mathcal{L}_p(\Omega)$ we write $u \in W_p^k(\Omega)$ if there exists a sequence $u^n \in C^k(\bar{\Omega})$ such that $\|u^n - u\|_{\mathcal{L}_p(\Omega)} \to 0$,

$$\|u^n\|_{W_p^k(\Omega)} < \infty, \quad \|D^\alpha u^n - D^\alpha u^m\|_{\mathcal{L}_p(\Omega)} \to 0 \quad \forall |\alpha| \le k$$

as $n,m \to \infty$. We call any such sequence u^n *a defining sequence for* u. Denote $W_p^k = W_p^k(\mathbb{R}^d)$. The spaces $W_p^k(\Omega)$ are called *Sobolev spaces*.

Obviously the function v constructed before (1.7) belongs to W_2^2.

2. Definition. If $u \in W_p^k(\Omega)$ and u^n is its defining sequence, define *the generalized derivatives* $D^\alpha u$ for multi-indices α with $|\alpha| \le k$ by

$$D^\alpha u = \mathcal{L}_p\text{-}\lim_{n\to\infty} D^\alpha u^n. \tag{1}$$

Of course, we have to show that Definition 2 makes sense.

3. Lemma. *If $u \in W_p^k(\Omega)$ and $|\alpha| \le k$, then $D^\alpha u$ exists and is independent of the choice of defining sequence.*

Proof. Since $D^\alpha u^n$ is a Cauchy sequence in $\mathcal{L}_p(\Omega)$, the limit in (1) exists. To prove that it is unique, we take a test function $\phi \in C_0^\infty(\Omega)$ (that is, ϕ is an infinitely differentiable function with compact support in Ω) and notice that integrating by parts yields

$$\int_\Omega \phi D^\alpha u^n \, dx = (-1)^{|\alpha|} \int_\Omega u^n D^\alpha \phi \, dx.$$

We pass to the limit noticing that by Hölder's inequality, if $g^n \to g$ in $\mathcal{L}_p(\Omega)$ and $h \in \mathcal{L}_q(\Omega)$ with $q = p/(p-1)$, then

$$\Big| \int_\Omega g^n h \, dx - \int_\Omega gh \, dx \Big| \le \|g^n - g\|_{\mathcal{L}_p(\Omega)} \|h\|_{\mathcal{L}_q(\Omega)} \to 0.$$

Then by virtue of (1) for any defining sequence, we have

$$\int_\Omega \phi D^\alpha u \, dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi \, dx. \tag{2}$$

Now if u_1^n and u_2^n are two defining sequences for u and v_1 and v_2 are the corresponding right-hand sides of (1), then by (2)

$$\int_\Omega \phi v_1 \, dx = \int_\Omega \phi v_2 \, dx, \quad \int_\Omega \phi (v_1 - v_2) \, dx = 0$$

for all $\phi \in C_0^\infty(\Omega)$. Since $v_1 - v_2 \in \mathcal{L}_p$, it follows that $v_1 - v_2 = 0$ (a.e.), which is exactly what is asserted. The lemma is proved.

The last part of the above proof shows that the following definition is unambiguous.

4. Definition. Let α be a multi-index and let v and h be locally integrable functions on Ω such that

$$\int_\Omega \phi h \, dx = (-1)^{|\alpha|} \int_\Omega v D^\alpha \phi \, dx \tag{3}$$

for all $\phi \in C_0^\infty(\Omega)$. Then we call the function h the D^α *derivative of v in the sense of distributions*, or the D^α *Sobolev derivative*, or else the D^α *generalized derivative* and write $h = D^\alpha v$.

Observe that Sobolev derivatives are the usual locally integrable functions and not just distributions.

The fact that (2) holds for any $\phi \in C_0^\infty(\Omega)$ means that $D^\alpha u$ is the derivative of u in the sense of distributions and the notions of generalized derivatives introduced in Definitions 2 and 4 agree if $u \in W_p^k(\Omega)$. It is also worth noting that as we have seen, (3) defines $D^\alpha v$ uniquely (up to almost everywhere).

Since for any smooth function u the usual derivative $D^\alpha u$ satisfies (2), we have the following.

5. Corollary. *If u is a k times continuously differentiable function and $u \in W_p^k(\Omega)$, then, for any multi-index α with $|\alpha| \le k$, the usual derivative $D^\alpha u$ coincides with the Sobolev one (a.e.).*

In the following exercise and many times in the future we use local versions of various function spaces. Unless specified otherwise, they are introduced in the same way as, say, $W_{p,loc}^k(\Omega)$, which is the space of functions u on Ω such that $u\zeta \in W_p^k(\Omega)$ for any $\zeta \in C_0^\infty(\Omega)$. In particular, $C_{loc}^\infty(\Omega)$ is the set of functions u defined on Ω and such that $u\zeta \in C_0^\infty(\Omega)$ for any $\zeta \in C_0^\infty(\Omega)$.

6. Exercise. Let $g \in \mathcal{L}_p(\Omega)$, $\phi_n \in C_{loc}^\infty(\Omega)$,

$$\|\phi_n - g\|_{\mathcal{L}_p(\Omega)} \to 0 \quad \text{and} \quad \|D_1\phi_n - D_1\phi_m\|_{\mathcal{L}_p(\Omega)} \to 0$$

as $n, m \to \infty$. Prove that the generalized derivative $D_1 g$ belongs to $\mathcal{L}_p(\Omega)$ and $\|D_1\phi_n - D_1 g\|_{\mathcal{L}_p(\Omega)} \to 0$. (Warning: The function g need not belong to $W_p^1(\Omega)$.)

7. Definition. For $u \in W_p^k(\Omega)$ define

$$\|u\|_{W_p^k(\Omega)} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{\mathcal{L}_p(\Omega)}, \quad [u]_{W_p^k(\Omega)} = \sum_{|\alpha| = k} \|D^\alpha u\|_{\mathcal{L}_p(\Omega)}.$$

Notice that if u^n is a defining sequence for $u \in W_p^k(\Omega)$, then

$$\|u - u^n\|_{W_p^k(\Omega)} \to 0, \quad \|u^n\|_{W_p^k(\Omega)} \to \|u\|_{W_p^k(\Omega)}.$$

8. Exercise*. Let $u \in W_p^k(\Omega)$ and $|\alpha| + |\beta| \le k$. Prove that

$$D^\alpha u \in W_p^{k-|\alpha|}(\Omega), \quad D^\beta(D^\alpha u) = D^{\alpha+\beta} u, \quad \|D^\alpha u\|_{W_p^{k-|\alpha|}(\Omega)} \le \|u\|_{W_p^k(\Omega)}.$$

9. Exercise*. Let $u \in W_p^k(\Omega)$, $a \in C^n(\bar\Omega)$, and $|\alpha| \le k - n$. Prove that $aD^\alpha u \in W_p^n(\Omega)$ and

$$\|aD^\alpha u\|_{W_p^n(\Omega)} \le N(d, n)\|a\|_{C^n(\bar\Omega)}\|u\|_{W_p^k(\Omega)}.$$

10. Exercise*. Let $\psi : \Omega \to \Omega'$ be a one-to-one mapping of a domain $\Omega \subset \mathbb{R}^d$ onto a domain $\Omega' \subset \mathbb{R}^d$. Assume that $k \geq 1$, $\psi \in C^k(\bar{\Omega})$, and $\psi^{-1} \in C^k(\bar{\Omega}')$ and prove that

$$u \in W_p^k(\Omega) \Longleftrightarrow u(\psi^{-1}(\cdot)) \in W_p^k(\Omega').$$

We remind the reader that by $C_0^k(\bar{\Omega})$ we mean the subset of $C^k(\bar{\Omega})$ consisting of all functions u vanishing for $x \in \Omega$ with sufficiently large $|x|$. This notation will be used most often for $\Omega = \mathbb{R}^d$ and $\Omega = \mathbb{R}^d_+$, where

$$\mathbb{R}^d_+ = \{(x^1, x') : x^1 > 0, x' = (x^2, \ldots, x^d) \in \mathbb{R}^{d-1}\}$$

with an obvious modification of this notation if $d = 1$. Assertion (ii) of the following theorem will be improved in Theorem 8.5 (iii).

11. Theorem. *(i) The space $W_p^k(\Omega)$ with norm $\|\cdot\|_{W_p^k(\Omega)}$ is a Banach space. (ii) Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d_+$. Then the set $C_0^k(\bar{\Omega})$ is dense in $W_p^k(\Omega)$.*

Proof. (i) As usual we need only prove that $W_p^k(\Omega)$ is complete. Let u^n be a Cauchy sequence in $W_p^k(\Omega)$ and, for each n, let v^{nj} be defining sequences for u^n. Then by definition

$$\|u^n - v^{nj}\|_{W_p^k(\Omega)} \to 0$$

as $j \to \infty$. Hence, for each n, there exists $j(n)$ such that

$$\|u^n - v^n\|_{W_p^k(\Omega)} \leq 1/n,$$

where $v^n = v^{nj(n)}$.

Observe that

$$\|v^n - v^m\|_{W_p^k(\Omega)} \leq \|v^n - u^n\|_{W_p^k(\Omega)} + \|u^n - u^m\|_{W_p^k(\Omega)}$$

$$+ \|u^m - v^m\|_{W_p^k(\Omega)} \leq 1/n + 1/m + \|u^n - u^m\|_{W_p^k(\Omega)} \to 0 \qquad (4)$$

as $n, m \to \infty$. In particular,

$$\|v^n - v^m\|_{\mathcal{L}_p(\Omega)} \to 0$$

and there is a function $v \in \mathcal{L}_p(\Omega)$ such that $v^n \to v$ in $\mathcal{L}_p(\Omega)$. This and (4) mean by definition that $v \in W_p^k(\Omega)$. Furthermore, $v^n \to v$ in $W_p^k(\Omega)$ by the definition of $D^\alpha v$, $|\alpha| \leq k$, and

$$\lim_{n\to\infty} \|v - u^n\|_{W_p^k(\Omega)} \leq \lim_{n\to\infty} (\|v - v^n\|_{W_p^k(\Omega)} + 1/n) = 0.$$

This proves (i).

Since $C^k(\bar\Omega)\cap W_p^k(\Omega)$ is dense in $W_p^k(\Omega)$ by definition, to prove (ii), it suffices to show that any

$$u\in C^k(\bar\Omega)\cap W_p^k(\Omega)$$

can be approximated in W_p^k by elements of $C_0^k(\bar\Omega)$. Take any $\zeta\in C_0^\infty$ such that $\zeta(x)=1$ for $|x|\le 1$ and let $u^n(x)=u(x)\zeta(x/n)$. Notice that, by the Leibnitz rule for $|\alpha|\le k$,

$$D^\alpha u^n(x)=\zeta(x/n)D^\alpha u(x)+\sum_{\substack{|\beta+|\gamma|=|\alpha|\\|\gamma|\ge 1}}c^{\beta\gamma}(D^\beta u(x))n^{-|\gamma|}(D^\gamma\zeta)(x/n),$$

where $c^{\beta\gamma}$ are certain constants. Hence by the triangle inequality and the dominated convergence theorem, for a constant N independent of n,

$$\|D^\alpha u^n-D^\alpha u\|_{\mathcal{L}_p(\Omega)}\le\|(1-\zeta(\cdot/n))D^\alpha u\|_{\mathcal{L}_p(\Omega)}+Nn^{-1}\|u\|_{W_p^k(\Omega)},$$

$$\|(1-\zeta(\cdot/n))D^\alpha u\|^p_{\mathcal{L}_p(\Omega)}=\int_\Omega|1-\zeta(x/n)|^p|D^\alpha u(x)|^p\,dx\to 0.$$

It follows that $u^n\to u$ in $W_p^k(\Omega)$. The theorem is proved.

12. Exercise. Take $d=1$ and prove that $C_0^\infty((0,1))$ is *not* dense in $W_2^1((0,1))$.

13. Exercise*. Let u^n be a Cauchy sequence in $W_p^k(\Omega)$, $u\in\mathcal{L}_p(\Omega)$ and let $u^n\to u$ in $\mathcal{L}_p(\Omega)$. Prove that $u\in W_p^k(\Omega)$ and $u^n\to u$ in $W_p^k(\Omega)$.

A version of the following corollary of Theorem 11 (i) will be stated in Corollary 8.1.3, which in turn is used in the hint to Exercise 11.1.10.

14. Corollary. *Let $1<p<\infty$, $k\in\{0,1,2,...\}$, and let U be a bounded subset of $W_p^k(\Omega)$. Then for any sequence $u_n\in U$ there exist a subsequence $u_{n'}$ and a function $u\in W_p^k(\Omega)$ such that the sequence $D^\alpha u_{n'}$ converges weakly in $\mathcal{L}_p(\Omega)$ to $D^\alpha u$ whenever $|\alpha|\le k$.*

Proof. Let $\mathcal{L}_p(\Omega,\mathbb{R}^m)$ be the space of $\mathbb{R}^m$-valued functions with natural norm. Take $m=1+d+...+d^k$ and observe that by Theorem 11 (i) the set

$$M:=\{(D^\alpha u,|\alpha|\le k):u\in W_p^k(\Omega)\}$$

is a closed linear subset of $\mathcal{L}_p(\Omega,\mathbb{R}^m)$. Then recall that, if F is a bounded set in $\mathcal{L}_p(\Omega,\mathbb{R}^m)$ and $f_n\in F$, $n=1,2,...$, then there exists a subsequence $f_{n'}$ and $f\in\mathcal{L}_p(\Omega,\mathbb{R}^m)$ such that $f_{n'}\to f$ weakly in $\mathcal{L}_p(\Omega,\mathbb{R}^m)$. We also know that linear subspaces of Banach spaces are closed in the weak topology.

These three facts combined imply that there is a subsequence

$$(D^\alpha u_{n'}, |\alpha| \le k) \in \{(D^\alpha v, |\alpha| \le k) : v \in U\}$$

converging weakly in $\mathcal{L}_p(\Omega, \mathbb{R}^m)$ to a $(D^\alpha u, |\alpha| \le k) \in M$. This is exactly what is asserted.

15. Remark. For $1 < p < \infty$, the set M in the above proof is actually isomorphic to $W_p^k(\Omega)$ and as a linear subspace of a reflexive space it is reflexive itself. It follows that $W_p^k(\Omega)$ is reflexive too. This observation is not used in the future.

Our discussion before (1.7) proves the existence part in the following result in which $\lambda > 0$ is a fixed number.

16. Theorem. *For any $f \in \mathcal{L}_2$ there exists a unique $u \in W_2^2$ such that $\lambda u - \Delta u = f$ (a.e.). In addition, for any $u \in W_2^2$ we have*

$$\lambda^2 \|u\|_{\mathcal{L}_2}^2 + 2\lambda \sum_{j=1}^d \|u_{x^j}\|_{\mathcal{L}_2}^2 + \sum_{j,k=1}^d \|u_{x^j x^k}\|_{\mathcal{L}_2}^2 = \|\lambda u - \Delta u\|_{\mathcal{L}_2}^2, \tag{5}$$

where by continuity one can take $\lambda = 0$ as well.

The uniqueness follows from (5), which in turn follows from Lemma 1.1 and Theorem 11 (ii).

17. Exercise. Let $\Omega = \mathbb{R}_+^d$, $u \in W_p^1(\Omega)$, $j \ge 2$, and $\phi \in C_0^\infty$ (not $C_0^\infty(\Omega)$). Prove that

$$\int_\Omega u D_j \phi \, dx = -\int_\Omega \phi D_j u \, dx.$$

The following exercise is used in the proof of Theorem 10.5.1.

18. Exercise*. Prove that if $u \in W_p^1(\Omega)$ and $p \ge 1$, then $u_\pm, |u| \in W_p^1(\Omega)$ and

$$(u_+)_x = u_x I_{u>0}, \quad (|u|)_x = u_x \operatorname{sign} u$$

(a.e.). Conclude that if $u, v \in W_p^1(\Omega)$, then $\max(u, v) \in W_p^1(\Omega)$.

19. Exercise. Prove that if $u \in W_p^1(\Omega)$, then $u_x I_{u=0} = 0$ (a.e.). Conclude that $u_x I_{u=c} = 0$ (a.e.) for any constant c.

20. Exercise*. Prove that if $u \in W_p^k(\mathbb{R}_+^d)$, then for almost any $x^1 > 0$ we have

$$u(x^1, \cdot) \in W_p^k(\mathbb{R}^{d-1}).$$

21. Exercise*. By using defining sequences, prove that if $u \in W_p^k(\mathbb{R}_+^d)$, α is a multi-index with $\alpha_1 = 0$, and $v(x) := u(|x^1|, x')$, then

$$D^\alpha v(x) = (D^\alpha u)(|x^1|, x')$$

in $\mathbb{R}^d$, where $D^\alpha v$ is the derivative in the sense of Definition 4.

22. Exercise (Hard if $1 < p < 2$). Prove that if $d = 1$, $u \in W_p^2((a, b))$ and $p > 1$, then $|u|^p \in W_1^2((a, b))$.

23. Exercise. (i) Let B be the open unit ball centered at the origin and let u be a twice continuously differentiable function on $\bar{B}$. Assume that $u = 0$ on ∂B. Set $f = \Delta u$ and prove that

$$\|u\|^2_{\mathcal{L}_2(B)} + \sum_i \|u_{x^i}\|^2_{\mathcal{L}_2(B)} \leq 4\|f\|^2_{\mathcal{L}_2(B)}.$$

(ii) Given an integer n, denote by P_n the set of polynomials of x of degree $\leq n$ and let A be the operator $A : P_n \to P_n$ given by the formula

$$Ap = \Delta[(1 - |x|^2)p].$$

Conclude from (i) that A is invertible.

(iii) Use (i), (ii), and Exercise 1.5 to show that for any $f \in \mathcal{L}_2(B)$ there is a function $u \in W_2^2(B)$ such that $\Delta u = f$ in B and

$$\|u\|^2_{\mathcal{L}_2(B)} + \sum_i \|u_{x^i}\|^2_{\mathcal{L}_2(B)} + \sum_{i,j} \|u_{x^i x^j}\|^2_{\mathcal{L}_2(B)} \leq 5\|f\|^2_{\mathcal{L}_2(B)}.$$

4. Second-order elliptic differential operators

In this section we show how to prove the solvability of general second-order elliptic differential equations in $\mathbb{R}^d$ by using the method of continuity and the method of a priori estimates both introduced by S.N. Bernstein in the beginning of the 20th century.

1. Definition. Let $a^{ij}(x), b^i(x), c(x)$ be *real-valued* measurable functions on $\mathbb{R}^d$ defined for $i, j = 1, ..., d$. Assume that $a^{ij} = a^{ji}$. The expression

$$L = a^{ij} D_{ij} + b^i D_i + c$$

is called a *second-order elliptic differential operator* if there is a constant $\kappa > 0$ called *a constant of ellipticity* such that, for all $x, \xi \in \mathbb{R}^d$, we have

$$\kappa^{-1}|\xi|^2 \geq a^{ij}(x)\xi^i\xi^j \geq \kappa|\xi|^2. \tag{1}$$

2. Exercise*. Derive from (1) and the symmetry of a^{ij} that the a^{ij} are bounded.

3. Exercise*. Assuming that a, b, c are bounded prove that $L : u \to Lu$ is a continuous operator from W_p^2 to $\mathcal{L}_p$.

Now we present the method of continuity.

4. Theorem. *Let L be a second-order elliptic differential operator with bounded coefficients. Assume that there are constants $\lambda, N_0 \in (0, \infty)$ such that, for any $u \in C_0^2$ and $t \in [0, 1]$, we have*

$$\|u\|_{W_2^2} \le N_0 \|L_t u\|_{\mathcal{L}_2}, \tag{2}$$

where

$$L_t = (1 - t)(\Delta - \lambda) + tL.$$

Then for any $f \in \mathcal{L}_2$ there exists a unique $u \in W_2^2$ such that $Lu = f$. Furthermore, (2) holds for any $u \in W_2^2$ and $t \in [0, 1]$.

Proof. That (2) holds for any $u \in W_2^2$ and $t \in [0, 1]$ follows directly from Exercise 3 and Theorem 3.11. The uniqueness in W_2^2 of the solution of $Lu = f$ and even $L_t u = f$ for any $t \in [0, 1]$ follows from (2).

To prove the existence, take a point $t \in [0, 1]$ and call it "good" if for any $f \in \mathcal{L}_2$ there exists a unique $u \in W_2^2$ such that $L_t u = f$. Let T be the set of all good points. Notice that $0 \in T$ by Theorem 3.16.

Obviously, we only need to prove that $1 \in T$ and to do that, it suffices to prove that there is an $\varepsilon > 0$ such that, for any $t_0 \in T$,

$$[t_0 - \varepsilon, t_0 + \varepsilon] \cap [0, 1] \subset T. \tag{3}$$

Fix a $t_0 \in T$ and $f \in \mathcal{L}_2$. For $u \in W_2^2$ and $t \in \mathbb{R}$ consider the equation

$$L_{t_0} v = f + (t_0 - t)(Lu - \Delta u + \lambda u). \tag{4}$$

Since $t_0 \in T$, there is a unique $v \in W_2^2$ satisfying equation (4). Introduce an operator Q_t by defining

$$v = Q_t u.$$

Then Q_t is an operator mapping W_2^2 into itself. It turns out that, if $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ and ε is chosen appropriately, then Q_t is a contraction in W_2^2.

Indeed, if $u^1, u^2 \in W_2^2$, then

$$L_{t_0} Q_t u^i = f + (t_0 - t)(Lu^i - \Delta u^i + \lambda u^i),$$

$$L_{t_0}(Q_t u^1 - Q_t u^2) = (t_0 - t)(L - \Delta + \lambda)(u^1 - u^2),$$

which owing to (2) implies that

$$\|Q_t u^1 - Q_t u^2\|_{W_2^2} \le N_0|t_0 - t| \cdot \|(L - \Delta + \lambda)(u^1 - u^2)\|_{\mathcal{L}_2}$$

$$\le N_0|t_0 - t|N_1\|u^1 - u^2\|_{W_2^2},$$

where N_1 is independent of u^i and t. By taking $\varepsilon = (2N_0N_1)^{-1}$, we get that Q_t is a $1/2$ contraction in W_2^2 if $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$. By Banach's fixed point theorem, for each such t, there exists a $u \in W_2^2$ such that $Q_t u = u$, which means

$$L_{t_0}u = f + (t_0 - t)(Lu - \Delta u + \lambda u), \quad L_t u = f.$$

Thus for any $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ and $f \in \mathcal{L}_2$ there is a solution $u \in W_2^2$ of $L_t u = f$. We know that the solution is unique if in addition $t \in [0, 1]$. This finishes proving (3) and the theorem.

5. Remark. One can restate assumption (2) in the following way. We require the estimate

$$\|u\|_{W_2^2} \le N\|f\|_{\mathcal{L}_2}$$

to hold whenever $u \in W_2^2$ is a solution of $L_t u = f$. Since the existence of such u is not known a priori, estimate (2) is usually called *an a priori* estimate.

6. Exercise. One could take $\lambda = 0$ in Theorem 4 if one knew that the equation $\Delta u = f$ is uniquely solvable in W_2^2 for any $f \in \mathcal{L}_2$. In connection with this prove that there is no finite constant N such that for any $u \in C_0^2$ we have $\|u\|_{\mathcal{L}_2} \le N\|\Delta u\|_{\mathcal{L}_2}$.

Theorem 4 reduces proving the solvability of elliptic equations to obtaining a priori estimates. In the future we will see that it is possible to obtain them if the coefficients of L are uniformly *continuous* or, more generally, are of class VMO. There is a remarkable exception when $d \le 2$. The following set of exercises is aimed at proving the a priori estimate of type (2) for $d = 2$ and *measurable* coefficients.

7. Exercise. Let $A = (a^{ij})$ and $U = (u_{ij})$ be 2×2 symmetric matrices. Assume that

$$\mu|\xi|^2 \le a^{ij}\xi^i\xi^j \le \nu|\xi|^2 \tag{5}$$

for all $\xi \in \mathbb{R}^2$, where $\mu > 0$ and $\nu > 0$ are some constants. Prove that

$$\frac{1}{2\mu^2}\Big(\sum_{i,j=1}^{2} a^{ij}u_{ij}\Big)^2 \ge \frac{\mu^2}{2\nu^2}\sum_{i,j=1}^{2} u_{ij}^2 + \det U.$$

8. Exercise. Let $d = 2$, $a^{ij}(x)$ be measurable functions on $\mathbb{R}^2$ satisfying $a^{ij} = a^{ji}$ and condition (5) for all $x, \xi \in \mathbb{R}^2$, where $\mu > 0$ and $\nu > 0$ are some constants. For a $\lambda > 0$ define

$$Lu = L_\lambda u = a^{ij} u_{x^i x^j} - \lambda(a^{11} + a^{22})u.$$

Prove that, for any $u \in C_0^2$,

$$\lambda^2 \|u\|_{\mathcal{L}_2}^2 + 2\lambda \sum_{j=1}^{2} \|u_{x^j}\|_{\mathcal{L}_2}^2 + \sum_{j,k=1}^{2} \|u_{x^j x^k}\|_{\mathcal{L}_2}^2 \leq \frac{\nu^2}{\mu^4} \|Lu\|_{\mathcal{L}_2}^2. \tag{6}$$

9. Exercise (Two-dimensional case). Under the conditions of Exercise 8 prove that for any $f \in \mathcal{L}_2$ there is a unique $u \in W_2^2$ satisfying $Lu = f$.

The reader can find a continuation of this series in Exercise 6.7 where we add lower-order terms, Exercise 8.2.11 for equations in half spaces, Exercise 11.5.5 with $\lambda \operatorname{tr} a$ replaced with λ and equations in $\mathbb{R}^2$ for large λ or in smooth bounded domains with any $\lambda \geq 0$, Exercise 11.6.5 for equations in $\mathbb{R}^2$ with λ small, and Exercise 8.2.6 concerning the Neumann problem.

5. Multiplicative inequalities

Theorem 3.16 settles the issue of solvability of equations $\lambda u - \Delta u = f$ in W_2^2. To get prepared for treating more general elliptic equations by using the ideas from Section 4, we need the following multiplicative inequalities. Everywhere in this section

$$\Omega = \mathbb{R}^d \quad \text{or} \quad \Omega = \mathbb{R}^d_+.$$

1. Theorem. *For any $p \in [1, \infty)$ and $u \in W_p^2(\Omega)$ we have*

$$\|u_x\|_{\mathcal{L}_p(\Omega)} \leq N \|u\|_{\mathcal{L}_p(\Omega)}^{1/2} \|u_{xx}\|_{\mathcal{L}_p(\Omega)}^{1/2}, \tag{1}$$

where N is independent of u.

Proof. By virtue of Theorem 3.11 we need only prove (1) for $u \in C_0^2(\bar{\Omega})$. Let $\zeta \in C_0^\infty(\mathbb{R})$ be a function of one variable with $\zeta(0) = 1$, $\zeta'(0) = 0$. Take $u \in C_0^2(\bar{\Omega})$ and denote by e_1 the first basis vector. Notice that

$$\big(u_{x^1}(x+te_1)\zeta(t)\big)_t = u_{x^1x^1}(x+te_1)\zeta(t) + \big(u(x+te_1)\zeta'(t)\big)_t - u(x+te_1)\zeta''(t),$$

$$u_{x^1}(x) = -\int_0^\infty u_{x^1x^1}(x+te_1)\zeta(t)\,dt + \int_0^\infty u(x+te_1)\zeta''(t)\,dt.$$

Hence by Minkowski's inequality

$$\|u_{x^1}\|_{\mathcal{L}_p(\Omega)} \le \int_0^\infty (|\zeta(t)| + |\zeta''(t)|)\big(\|u_{xx}(\cdot + te_1)\|_{\mathcal{L}_p(\Omega)} + \|u(\cdot + te_1)\|_{\mathcal{L}_p(\Omega)}\big)\, dt,$$

where the right-hand side equals a constant times

$$\|u_{xx}\|_{\mathcal{L}_p(\Omega)} + \|u\|_{\mathcal{L}_p(\Omega)}$$

if $\Omega = \mathbb{R}^d$ and less than this quantity if $\Omega = \mathbb{R}^d_+$. Thus,

$$\|u_{x^1}\|_{\mathcal{L}_p(\Omega)} \le N(\|u_{xx}\|_{\mathcal{L}_p(\Omega)} + \|u\|_{\mathcal{L}_p(\Omega)})$$

and the same estimate obviously also holds for u_{x^i} with any $i = 2, ..., d$.

Now, take a constant $c > 0$ and put $u(cx)$ in place of u in the just proved inequality:

$$\|u_x\|_{\mathcal{L}_p(\Omega)} \le N(\|u_{xx}\|_{\mathcal{L}_p(\Omega)} + \|u\|_{\mathcal{L}_p(\Omega)}).$$

Then we get

$$c\|u_x\|_{\mathcal{L}_p(\Omega)} \le N(c^2\|u_{xx}\|_{\mathcal{L}_p(\Omega)} + \|u\|_{\mathcal{L}_p(\Omega)}).$$

Upon dividing through by c and minimizing with respect to $c > 0$, we arrive at (1). The theorem is proved.

Quite often instead of (1) we need its corollary, which we actually have obtained in the end of the preceding proof.

2. Corollary. *For any $p \in [1, \infty)$ and $\varepsilon > 0$ there exists a constant N such that, if $u \in W_p^2(\Omega)$, then*

$$\|u_x\|_{\mathcal{L}_p(\Omega)} \le \varepsilon\|u_{xx}\|_{\mathcal{L}_p(\Omega)} + N\|u\|_{\mathcal{L}_p(\Omega)}.$$

In addition, one can take $N = N_0\varepsilon^{-1}$, where N_0 is independent of ε.

3. Exercise. Show that if Ω is a bounded domain, then the assertion of Theorem 1 is wrong.

4. Corollary. *Let $b^1, ..., b^d$, and c be bounded measurable functions on $\mathbb{R}^d$. Then there exists $\lambda_0 \ge 1$, depending only on d and bounds of $b^1, ..., b^d$, and c, such that for any $\lambda \ge \lambda_0$ and any $f \in \mathcal{L}_2$ in W_2^2 there is a unique solution of the equation*

$$Lu - \lambda u := \Delta u + b^i u_{x^i} + cu - \lambda u = f.$$

Furthermore, for any $u \in W_2^2$ and $\lambda \ge \lambda_0$

$$\lambda\|u\|_{\mathcal{L}_2} + \lambda^{1/2}\|u_x\|_{\mathcal{L}_2} + \|u_{xx}\|_{\mathcal{L}_2} \le N\|Lu - \lambda u\|, \tag{2}$$

where the constant N depends only on d and the bounds of $b^1, \dots, b^d$, and c.

Indeed, for $t \in [0, 1]$ set

$$L_t u = \Delta u - \lambda u + t(b^i u_{x^i} + cu)$$

$$= (1-t)(\Delta - \lambda)u + t(\Delta u - \lambda u + b^i u_{x^i} + cu).$$

Then by Theorem 4.4 we only need to check that (2) holds with L_t in place of $L - \lambda$ for any $t \in [0, 1]$. However, by Lemma 1.1

$$\lambda \|u\|_{\mathcal{L}_2} + \lambda^{1/2} \|u_x\|_{\mathcal{L}_2} + \|u_{xx}\|_{\mathcal{L}_2} \leq N_1 \|\Delta u - \lambda u\|_{\mathcal{L}_2}$$

$$\leq N_1 \|L_t u\|_{\mathcal{L}_2} + N_2 \|u_x\|_{\mathcal{L}_2} + N_3 \|u\|_{\mathcal{L}_2},$$

where $N_1 = N_1(d)$ and N_2 and N_3 depend only on the bounds of $b^1, \dots, b^d$, and c and d. By Corollary 2

$$N_2 \|u_x\|_{\mathcal{L}_2} \leq (1/2) \|u_{xx}\|_{\mathcal{L}_2} + N_4 \|u\|_{\mathcal{L}_2},$$

so that

$$(\lambda - N_3 - N_4) \|u\|_{\mathcal{L}_2} + \lambda^{1/2} \|u_x\|_{\mathcal{L}_2} + (1/2) \|u_{xx}\|_{\mathcal{L}_2} \leq N_1 \|L_t u - \lambda u\|_{\mathcal{L}_2}.$$

This implies our a priori estimate for $\lambda \geq 2(N_3 + N_4) + 1 =: \lambda_0$ since then $\lambda - N_3 - N_4 \geq (1/2)\lambda$.

5. Exercise. Let $d = 1$ and let L be a second-order elliptic operator with bounded measurable coefficients. Prove that there exists $\lambda_0 \geq 1$ such that for any $\lambda \geq \lambda_0$ and any $f \in \mathcal{L}_2$ in W_2^2 there is a unique solution of the equation $Lu - \lambda u = f$.

The following exercise is generalized in Exercise 13.3.20.

6. Exercise*. Prove by induction on n that for any $n \in \{0, 1, 2, \dots\}$, $k \in \{0, 1, 2, \dots, n\}$, and $u \in W_p^n(\Omega)$

$$[u]_{W_p^k(\Omega)} \leq N \|u\|_{\mathcal{L}_p(\Omega)}^{1-\gamma} [u]_{W_p^n(\Omega)}^{\gamma}, \tag{3}$$

where $\gamma = k/n$ and N is independent of u.

Exercise 6 and Young's inequality

$$a^{1-\gamma} b^{\gamma} \leq (1-\gamma)a + \gamma b, \quad a, b \geq 0, \gamma \in [0, 1],$$

imply the following.

7. Corollary. *For any $p \in [1,\infty)$, $n \in \{0,1,2,...\}$, and $u \in W_p^n(\Omega)$,*

$$\|u\|_{W_p^n(\Omega)} \leq N(\|u\|_{\mathcal{L}_p(\Omega)} + [u]_{W_p^n(\Omega)}), \tag{4}$$

where N is independent of u.

It also follows from (3) and (4) that

$$\|u\|_{W_p^k(\Omega)} \leq N(\|u\|_{\mathcal{L}_p(\Omega)} + [u]_{W_p^k(\Omega)})$$

$$\leq N(\|u\|_{\mathcal{L}_p(\Omega)} + \|u\|_{\mathcal{L}_p(\Omega)}^{1-\gamma}\|u\|_{W_p^n(\Omega)}^{\gamma}),$$

and since obviously

$$\|u\|_{\mathcal{L}_p(\Omega)} \leq \|u\|_{\mathcal{L}_p(\Omega)}^{1-\gamma}\|u\|_{W_p^n(\Omega)}^{\gamma},$$

we arrive at the first inequality in (5) below.

8. Corollary. *For any $p \in [1,\infty)$, $n \in \{0,1,2,...\}$, $k \in \{0,1,2,...,n\}$, $\varepsilon > 0$, and $u \in W_p^n(\Omega)$,*

$$\|u\|_{W_p^k(\Omega)} \leq N\|u\|_{\mathcal{L}_p(\Omega)}^{1-\gamma}\|u\|_{W_p^n(\Omega)}^{\gamma}$$

$$\leq N\varepsilon\|u\|_{W_p^n(\Omega)} + N\varepsilon^{-\gamma/(1-\gamma)}\|u\|_{\mathcal{L}_p(\Omega)}, \tag{5}$$

where $\gamma = k/n$ and N are independent of u and ε.

The second inequality for $\gamma = 1$ and $\varepsilon < 1$ is trivial because the right-hand side is infinite unless $u = 0$. For $\gamma = 1$ and $\varepsilon \geq 1$ it is also trivial and in this case we see that one cannot replace $N\varepsilon$ with another ε. If $\gamma \in [0,1)$, the second inequality follows from Young's inequality and the observation that

$$Na^{1-\gamma}b^{\gamma} = (N^{1/(1-\gamma)}\varepsilon^{-\gamma/(1-\gamma)}a)^{1-\gamma}(\varepsilon b)^{\gamma}, \quad \gamma \in [0,1), \varepsilon > 0.$$

6. Solvability of elliptic equations with continuous coefficients

We deal with an operator L given on functions defined in $\mathbb{R}^d$. For convenience of future references we collect what we need in the following.

1. Assumption. The operator L is a second-order elliptic differential operator with constant of ellipticity $\kappa > 0$ (see Definition 4.1) independent of x. The coefficients of L are *bounded and measurable and the leading coefficients are uniformly continuous* on $\mathbb{R}^d$. More precisely there is a constant K and an increasing function $\omega(\varepsilon)$, $\varepsilon \geq 0$, such that $\omega(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$, and for all $x, y \in \mathbb{R}^d$ and $i, j = 1, ..., d$ we have

$$|a^{ij}(x)|, |b^i(x)|, |c(x)| \leq K, \quad |a^{ij}(x) - a^{ij}(y)| \leq \omega(|x-y|).$$

Throughout the whole section we suppose that Assumption 1 is satisfied.

2. Lemma. *Additionally assume that the a^{ij} are constant. Then there exist constants $\lambda_0 \geq 1$ and N_0, depending only on K, κ, and d, such that, for any $\lambda \geq \lambda_0$ and $u \in C_0^2$, we have*

$$\lambda\|u\|_{\mathcal{L}_2} + \lambda^{1/2}\|u_x\|_{\mathcal{L}_2} + \|u_{xx}\|_{\mathcal{L}_2} \leq N_0\|Lu - \lambda u\|_{\mathcal{L}_2}. \tag{1}$$

Proof. By repeating the argument from Corollary 5.4, we reduce the general situation to the one in which $L = a^{ij}D_{ij}$. Then we observe that by Theorem 5.1

$$\lambda^{1/2}\|u_x\|_{\mathcal{L}_2} \leq N\lambda^{1/2}\|u\|_{\mathcal{L}_2}^{1/2}\|u_{xx}\|_{\mathcal{L}_2}^{1/2} \leq N\lambda\|u\|_{\mathcal{L}_2} + N\|u_{xx}\|_{\mathcal{L}_2},$$

so that we only need to prove that

$$\lambda\|u\|_{\mathcal{L}_2} + \|u_{xx}\|_{\mathcal{L}_2} \leq N_0\|Lu - \lambda u\|_{\mathcal{L}_2}. \tag{2}$$

Now we use a change of coordinates in order to reduce L to Δ and apply Lemma 1.1. We know that there exists a $d \times d$ symmetric matrix σ such that $a = (a^{ij}) = \sigma^2$. Notice that, since the eigenvalues of a belong to $[\kappa, \kappa^{-1}]$, the eigenvalues of σ belong to $[\kappa^{1/2}, \kappa^{-1/2}]$.

Define $v(x) = u(\sigma x)$. Then

$$v_{x^i}(x) = u_{x^k}(\sigma x)\sigma^{ki}, \quad v_{x^i x^j}(x) = u_{x^k x^r}(\sigma x)\sigma^{ki}\sigma^{rj}, \quad \Delta v(x) = (Lu)(\sigma x).$$

Also observe that

$$\|\Delta v - \lambda v\|_{\mathcal{L}_2}^2 = \det\sigma^{-1}\|Lu - \lambda u\|_{\mathcal{L}_2}^2 \leq N\|Lu - \lambda u\|_{\mathcal{L}_2}^2,$$

and, since there are obvious formulas expressing $u_x(x)$ and $u_{xx}(x)$ in terms of v_x and v_{xx}, also

$$\|u\|_{\mathcal{L}_2} \leq N\|v\|_{\mathcal{L}_2}, \quad \|u_x\|_{\mathcal{L}_2} \leq N\|v_x\|_{\mathcal{L}_2}, \quad \|u_{xx}\|_{\mathcal{L}_2} \leq N\|v_{xx}\|_{\mathcal{L}_2}.$$

This and Lemma 1.1 yield that for $\lambda > 0$

$$\lambda^2\|u\|^2_{\mathcal{L}_2} \le N\lambda^2\|v\|^2_{\mathcal{L}_2} \le N\|\Delta v - \lambda v\|^2_{\mathcal{L}_2} \le N\|Lu - \lambda u\|^2_{\mathcal{L}_2},$$

$$\|u_{xx}\|^2_{\mathcal{L}_2} \le N\|v_{xx}\|^2_{\mathcal{L}_2} \le N\|\Delta v - \lambda v\|^2_{\mathcal{L}_2} \le N\|Lu - \lambda u\|^2_{\mathcal{L}_2}.$$

The lemma is proved.

3. Lemma. *There exists an $\varepsilon = \varepsilon(d, \kappa, K, \omega) > 0$ such that if $u \in C_0^2$ has support in a ball of radius ε, then estimate (1) holds for any $\lambda \ge \lambda_0$ with $2N_0$ in place of N_0, where λ_0 and N_0 are taken from Lemma 2.*

Proof. Without loss of generality we assume that the ball in question is centered at the origin, so that $u(x) = 0$ for $|x| \ge \varepsilon$, where a small $\varepsilon > 0$ is to be specified later. Since ε is small, $Lu(x)$ should be close to $L_0u(x)$, where the operator L_0 is defined by "freezing" the leading coefficients at the origin:

$$L_0u(x) = a^{ij}(0)D_{ij}u(x) + b^i(x)D_iu(x) + c(x)u(x).$$

In fact,

$$|L_0u(x) - Lu(x)| \le N_1\omega(\varepsilon)|D^2u(x)|,$$

where N_1 depends only on d. Furthermore, by Lemma 2, for $\lambda \ge \lambda_0$,

$$\lambda\|u\|_{\mathcal{L}_2} + \lambda^{1/2}\|u_x\|_{\mathcal{L}_2} + \|u_{xx}\|_{\mathcal{L}_2} \le N_0\|L_0u - \lambda u\|_{\mathcal{L}_2}$$

$$\le N_0\|Lu - \lambda u\|_{\mathcal{L}_2} + N_2\omega(\varepsilon)\|u_{xx}\|_{\mathcal{L}_2}. \tag{3}$$

where N_2 depends only on K, κ, and d. Upon choosing ε so that $N_2\omega(\varepsilon) \le 1/2$ and collecting like terms in (3), we get our assertion. The lemma is proved.

4. Theorem. *There exist constants $\lambda_0 \ge 1$ and N_0, depending only on K, κ, ω, and d, such that estimate (1) holds true for any $u \in W_2^2$ and $\lambda \ge \lambda_0$. Furthermore, for any $\lambda \ge \lambda_0$ and $f \in \mathcal{L}_2$, there exists a unique $u \in W_2^2$ satisfying $Lu - \lambda u = f$.*

Proof. Having in mind the method of continuity (Theorem 4.4), we first concentrate on a priori estimates for $u \in C_0^2$. As in the proof of Lemma 2 it suffices to prove (2).

Take a $\zeta \in C_0^\infty$ with

$$\|\zeta\|_{\mathcal{L}_2} = 1$$

and with support in the ball $B_\varepsilon = \{x : |x| < \varepsilon\}$, where $\varepsilon > 0$ is taken from Lemma 3. Observe that

$$1 = \int_{\mathbb{R}^d} \zeta^2(x-y)\,dy,$$

$$u^2_{x^i x^j}(x) = \int_{\mathbb{R}^d} u^2_{x^i x^j}(x)\zeta^2(x-y)\,dy. \tag{4}$$

These formulas are usually associated with the term "a partition of unity". In (4)

$$\begin{aligned} u^2_{x^i x^j}(x)\zeta^2(x-y) &= \big[(u(x)\zeta(x-y))_{x^i x^j} - u_{x^i}(x)\zeta_{x^j}(x-y) \\ &\qquad - u_{x^j}(x)\zeta_{x^i}(x-y) - u(x)\zeta_{x^i x^j}(x-y)\big]^2 \\ &\le 2\big[(u(x)\zeta(x-y))_{x^i x^j}\big]^2 + N(|u_x(x)|^2 + |u(x)|^2)\eta(x-y), \end{aligned}$$

with

$$\eta = |\zeta_x|^2 + |\zeta_{xx}|^2 \in \mathcal{L}_1.$$

Hence, by integrating through (4) with respect to x, we find

$$\|u_{xx}\|^2_{\mathcal{L}_2} \le N \int_{\mathbb{R}^d} \|(\zeta(\cdot - y)u)_{xx}\|^2_{\mathcal{L}_2}\,dy + N(\|u_x\|^2_{\mathcal{L}_2} + \|u\|^2_{\mathcal{L}_2}).$$

Define λ_0 and N_0 as in Lemma 3. Then by this lemma

$$\|(\zeta(\cdot - y)u)_{xx}\|^2_{\mathcal{L}_2} \le 4N_0^2\|(L-\lambda)(\zeta(\cdot - y)u)\|^2_{\mathcal{L}_2}.$$

Therefore,

$$\|u_{xx}\|^2_{\mathcal{L}_2} \le N \int_{\mathbb{R}^d} \|(L-\lambda)(\zeta(\cdot - y)u)\|^2_{\mathcal{L}_2}\,dy + N(\|u_x\|^2_{\mathcal{L}_2} + \|u\|^2_{\mathcal{L}_2}).$$

We obtained an estimate of u_{xx} through $(L-\lambda)(\zeta(\cdot - y)u)$. Now we get ζ outside L. Notice that

$$\begin{aligned} (L-\lambda)(u(x)\zeta(x-y)) &= \zeta(x-y)(L-\lambda)u(x) + 2a^{ij}(x)u_{x^i}(x)\zeta_{x^j}(x-y) \\ &\qquad + u(x)(a^{ij}(x)\zeta_{x^i x^j}(x-y) + b^i(x)\zeta_{x^i}(x-y)), \end{aligned}$$

$$\begin{aligned} |(L-\lambda)(u(x)\zeta(x-y))|^2 &\le 2|\zeta(x-y)(L-\lambda)u(x)|^2 \\ &\qquad + N(|u_x(x)|^2 + |u(x)|^2)\eta(x-y). \end{aligned}$$

It follows that

$$\int_{\mathbb{R}^d} \|(L-\lambda)(\zeta(\cdot - y)u)\|_{\mathcal{L}_2}^2 \, dy \le 2\|(L-\lambda)u\|_{\mathcal{L}_2}^2 + N(\|u_x\|_{\mathcal{L}_2}^2 + \|u\|_{\mathcal{L}_2}^2).$$

$$\|u_{xx}\|_{\mathcal{L}_2}^2 \le N\big(\|(L-\lambda)u\|_{\mathcal{L}_2}^2 + \|u_x\|_{\mathcal{L}_2}^2 + \|u\|_{\mathcal{L}_2}^2\big). \tag{5}$$

Similarly,

$$\begin{aligned}\lambda^2\|u\|_{\mathcal{L}_2}^2 &= \lambda^2 \int_{\mathbb{R}^d} \|\zeta(\cdot - y)u\|_{\mathcal{L}_2}^2 \, dy \\ &\le 4N_0^2 \int_{\mathbb{R}^d} \|(L-\lambda)(\zeta(\cdot - y)u)\|_{\mathcal{L}_2}^2 \, dy \\ &\le N\big(\|(L-\lambda)u\|_{\mathcal{L}_2}^2 + \|u_x\|_{\mathcal{L}_2}^2 + \|u\|_{\mathcal{L}_2}^2\big).\end{aligned}$$

By combining this with (5) and Corollary 5.2, we find

$$\lambda\|u\|_{\mathcal{L}_2} + \|u_{xx}\|_{\mathcal{L}_2} \le N_1\|Lu - \lambda u\|_{\mathcal{L}_2} + (1/2)\|u_{xx}\|_{\mathcal{L}_2} + N_2\|u\|_{\mathcal{L}_2},$$

where N_i depend only on K, κ, ω, and d.

To finish proving (2) with $2N_1$ in place of N_0, it only remains to take $\lambda \ge 2N_2$, so that $N_0 \le \lambda/2$ (do not forget that $\lambda \ge \lambda_0$ with λ_0 from Lemma 3).

By inspecting the above argument, one easily sees that λ_0 and N_0 can be taken to be the same for $L_t = (1-t)(\Delta - 1) + tL$ in place of L with any $t \in [0,1]$. Therefore, the method of continuity is applicable and the theorem is proved.

We will see much later in Theorem 11.6.2 that, if $c \le 0$, then one can take any $\lambda_0 > 0$ in Theorem 4 if one allows N_0 to depend on λ. The proof of this fact will be based on an $\mathcal{L}_p$ version of Exercise 6 before which we state the following a priori estimate.

5. Theorem. *There exists a constant N depending only on d, K, κ, and ω such that for any $u \in W_2^2$ and $\lambda \ge 0$ we have*

$$\|u\|_{W_2^2} \le N(\|\lambda u - Lu\|_{\mathcal{L}_2} + \|u\|_{\mathcal{L}_2}).$$

The proof is almost trivial since for $\lambda \ge \lambda_0$ our assertion is contained in Theorem 4, whereas for $\lambda \in [0, \lambda_0]$ we have

$$\|u\|_{W_2^2} \le N\|\lambda_0 u - Lu\|_{\mathcal{L}_2} \le N(\|\lambda u - Lu\|_{\mathcal{L}_2} + (\lambda_0 - \lambda)\|u\|_{\mathcal{L}_2})$$

with $\lambda_0 - \lambda \le \lambda_0$.

6. Exercise*. Assume that, for a particular L (as always satisfying Assumption 1), there exists a constant N such that for any $\lambda \geq 0$ and $u \in W_2^2$ we have

$$\|u\|_{\mathcal{L}_2} \leq N\|\lambda u - Lu\|_{\mathcal{L}_2}$$

with N independent of u and λ. Prove that, for this L, the assertions of Theorem 4 hold true for any $\lambda \geq 0$ rather than $\lambda \geq \lambda_0 \geq 1$.

7. Exercise. Under the conditions of Exercise 4.8 prove that if b^i, $i = 1, 2$, are bounded measurable functions on $\mathbb{R}^2$, then there are constants $\lambda_0 \geq 1$ and N, depending only on μ, ν, and the bounds of b^i, such that for any $\lambda \geq \lambda_0$ and $f \in \mathcal{L}_2$ the equation $L_\lambda u + b^i u_{x^i} = f$ has a unique solution $u \in W_2^2$ and for any $u \in W_2^2$

$$\lambda\|u\|_{\mathcal{L}_2} + \lambda^{1/2}\|u_x\|_{\mathcal{L}_2} + \|u_{xx}\|_{\mathcal{L}_2} \leq N\|L_\lambda u + b^i u_{x^i}\|_{\mathcal{L}_2}.$$

In Exercise 11.6.5 we will see that one can take any $\lambda_0 > 0$ if we allow N to depend on λ.

8. Exercise. Consider the nonlinear equation

$$a^{ij}(x)u_{x^ix^j}(x) + F(u_x(x), u(x), x) - \lambda u(x) = 0, \tag{6}$$

where $F(\alpha, \beta, x)$ is a measurable function of (α, β, x) on $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ such that, for an $f \in \mathcal{L}_2$, we have

$$|F(\alpha, \beta, x)| \leq K(|\alpha| + |\beta| + f(x)),$$

$$|F(\alpha_1, \beta_1, x) - F(\alpha_2, \beta_2, x)| \leq K(|\alpha_1 - \alpha_2| + |\beta_1 - \beta_2|)$$

for all $\alpha, \alpha_i, \beta, \beta_i, x$. Use the method of continuity to prove that there is $\lambda_0 = \lambda_0(d, \kappa, K, \omega)$ such that for any $\lambda \geq \lambda_0$ equation (6) has a unique solution $u \in W_2^2$.

7. Higher regularity of solutions

Here we want to show that if the coefficients of the operator L from Section 6 have bounded derivatives up to order n and the right-hand side f is in W_2^n, then the solutions of $Lu - \lambda u = f$ belong to W_2^{2+n} provided that λ is large enough but yet independently of n. We will again use the method of a priori estimates and the method of continuity.

First of all we need to know that the desired result holds for $L = \Delta$. The simplest way to do this is to use the Fourier transform. However, our goal is to give a presentation which would be applicable to equations in W_p^{2+n} with $p \neq 2$ without much additional effort. Therefore, we prefer an approach based on the Green's function G_λ of $\lambda - \Delta$, which is introduced in the following way.

Let $\lambda > 0$ and as in Section 1 first take $u \in C_0^2$, denote $f = \lambda u - \Delta u$ and write

$$\tilde{u}(\xi) = (\lambda + |\xi|^2)^{-1}\tilde{f}(\xi) = \tilde{f}(\xi)\int_0^\infty e^{-\lambda t}e^{-|\xi|^2 t}\,dt.$$

Then recall that $e^{-|\xi|^2 t}$ is proportional to the Fourier transform of

$$p(t,x) = (4\pi t)^{-d/2}e^{-|x|^2/(4t)} \tag{1}$$

and that the product of the Fourier transforms is proportional to the Fourier transform of the convolution. Then, after observing that the formula

$$\int_{\mathbb{R}^d} p(t,x)\,dx = 1$$

and the computation

$$\int_{\mathbb{R}^d}\int_0^\infty e^{-\lambda t}|p(t,x)|\,dtdx = \int_0^\infty e^{-\lambda t}\int_{\mathbb{R}^d}|p(t,x)|\,dx = \lambda^{-1} < \infty$$

allow us to use Fubini's theorem in finding the Fourier transform of

$$G_\lambda(x) := \int_0^\infty e^{-\lambda t}p(t,x)\,dt, \tag{2}$$

we easily see that

$$F(u) = F(G_\lambda * f). \tag{3}$$

Note that

$$G_\lambda(x) \geq 0, \quad \int_{\mathbb{R}^d} G_\lambda(x)\,dx = \lambda^{-1}. \tag{4}$$

Since also $f \in \mathcal{L}_1$, we have that $G_\lambda * f \in \mathcal{L}_1$ and by uniqueness of the Fourier transform we obtain from (3) that

$$u = G_\lambda * f \tag{5}$$

almost everywhere. However, $f \in C_0$ and the dominated convergence theorem implies that

$$G_\lambda * f(x) = \int_{\mathbb{R}^d} f(x-y)G_\lambda(y)\,dy$$

is a continuous function of x, so that (5) holds on $\mathbb{R}^d$ rather than only almost everywhere.

Also observe that if $f \in C_0^n$ for an integer $n \geq 1$, then the rules of differentiating convolutions show that

$$D^n(G_\lambda * f) = G_\lambda * D^n f. \tag{6}$$

In particular, $G_\lambda * f$ is infinitely differentiable if $f \in C_0^\infty$.

1. Theorem. *(i) The formula*

$$R_\lambda f = G_\lambda * f$$

defines a bounded operator in $\mathcal{L}_p$ for any $p \in [1, \infty]$. Furthermore,

$$\lambda \| R_\lambda f \|_{\mathcal{L}_p} \leq \| f \|_{\mathcal{L}_p}. \tag{7}$$

(ii) For any $f \in \mathcal{L}_2$ the unique solution $u \in W_2^2$ of the equation $\lambda u - \Delta u = f$ is given by $u = R_\lambda f$.

(iii) If $n \geq 1$ and $f \in W_2^n$, then $R_\lambda f \in W_2^{2+n}$ and

$$D^n R_\lambda f = R_\lambda D^n f. \tag{8}$$

Proof. Assertion (i) follows from (4) and Minkowski's inequality, which we have already used a few times.

(ii) Equation (5) says that

$$u = R_\lambda(\lambda u - \Delta u)$$

for $u \in C_0^2$. By the above both sides of this equation are continuous $\mathcal{L}_2$-valued functions on W_2^2, so that the formula holds for any $u \in W_2^2$.

(iii) By (6) equation (8) holds if $f \in C_0^n$. If $f \in W_2^n$ and $f_k \in C_0^n$ is its defining sequence, then for $0 \leq i \leq n$,

$$\begin{aligned} \| D^i R_\lambda f_k - D^i R_\lambda f_m \|_{W_2^2} &= \| R_\lambda (D^i f_k - D^i f_m) \|_{W_2^2} \\ &\leq N \| D^i f_k - D^i f_m \|_{\mathcal{L}_2} \to 0 \end{aligned}$$

as $k, m \to \infty$, where the constant N is independent of k, m. This shows that the $R_\lambda f_k$ form a Cauchy sequence in W_2^{2+n}. By completeness, there is a $v \in W_2^{2+n}$ such that

$$D^i R_\lambda f_k \to D^i v$$

in $\mathcal{L}_2$ for $i \leq 2+n$. In particular, $R_\lambda f_k \to v$ and we discover that $v = R_\lambda f$ due to the continuity of R_λ on $\mathcal{L}_2$. After that it only remains to substitute f_k in place of f in (8) and to pass to the limit. The theorem is proved.

2. Corollary. *Let $n \geq 0$, $\lambda > 0$, and $f \in W_2^n$. Then there exists a unique solution $u \in W_2^{2+n}$ of the equation $\lambda u - \Delta u = f$. Furthermore, for any multi-index α with $|\alpha| \leq n$*

$$\lambda^2 \|D^\alpha u\|_{\mathcal{L}_2}^2 + 2\lambda \sum_{j=1}^{d} \|D^\alpha u_{x^j}\|_{\mathcal{L}_2}^2 + \sum_{j,k=1}^{d} \|D^\alpha u_{x^j x^k}\|_{\mathcal{L}_2}^2 = \|D^\alpha f\|_{\mathcal{L}_2}^2.$$

In particular, there exists a constant N depending only on d and n such that

$$\lambda \|u\|_{W_2^n} + \|u_{xx}\|_{W_2^n} \leq N \|f\|_{W_2^n}.$$

Indeed, uniqueness follows from Theorem 3.16. Furthermore, by Theorem 1 the solution u, found in W_2^2 according to Theorem 3.16, is actually in W_2^{2+n}. After that one can differentiate the equation $\lambda u - \Delta u = f$ and get the estimate again from Theorem 3.16.

3. Exercise. Prove that the norm of λR_λ in any $\mathcal{L}_p$ is 1.

4. Exercise. For $u \in C_0^2$ prove that $|\nabla u| \leq 2(\sup |\Delta u|^{1/2})(\sup |u|^{1/2})$ on $\mathbb{R}^d$ and that the estimate is sharp.

5. Theorem. *Take the operator L from Section 6, an integer $n \geq 1$, and assume that the coefficients a, b, c are in C^n and their norms in C^n are bounded by a constant K_1. Take λ_0 from Theorem 6.4. Then there exists a constant N_0, depending only on $K, K_1, n, \kappa, \omega$, and d, such that, for any $\lambda \geq \lambda_0$ and $u \in W_2^{n+2}$*

$$\lambda \|u\|_{W_2^n} + \|u\|_{W_2^{n+2}} \leq N_0 \|Lu - \lambda u\|_{W_2^n}. \tag{9}$$

Furthermore, for any $\lambda \geq \lambda_0$ and $f \in W_2^n$, there exists a unique $u \in W_2^{n+2}$ satisfying $Lu - \lambda u = f$.

Proof. Recall that if $g \in C^n$ and $u \in W_p^n$, then $gu \in W_2^n$ and

$$\|gu\|_{W_2^n} \leq N \|g\|_{C^n} \|u\|_{W_2^n},$$

where N depends only on d and n (see Exercise 3.9). Also invoke Corollary 2. With these two tools available, one can repeat the proof of Theorem 4.4, using the method of continuity, taking there W_2^n and W_2^{n+2} in place of $\mathcal{L}_2$ and W_2^2, respectively. Then one sees that to prove the theorem, it suffices only to find N_0 such that (9) holds for $u \in C_0^{n+2}$ and $\lambda \geq \lambda_0$.

Take a $u \in C_0^{n+2}$, observe that Lu is n times continuously differentiable, and, for any multi-index α with $|\alpha| \le n$, by the Leibnitz formula write

$$D^\alpha (L - \lambda)u = (L - \lambda)D^\alpha u + \sum_{|\beta| \le |\alpha|+1} c^\beta D^\beta u,$$

where c^β are certain bounded functions. Hence by Theorem 6.4 for $\lambda \ge \lambda_0$

$$\lambda \|D^\alpha u\|_{\mathcal{L}_2} + \|D^\alpha u\|_{W_2^2} \le N\|(L - \lambda)D^\alpha u\|_{\mathcal{L}_2}$$

$$\le N\|D^\alpha (L - \lambda)u\|_{\mathcal{L}_2} + N \sum_{|\beta| \le |\alpha|+1} \|D^\beta u\|_{\mathcal{L}_2}.$$

By summing up over α such that $|\alpha| \le k$, where $k \in \{0, 1, ..., n\}$, we get

$$\lambda \|u\|_{W_2^k} + \|u\|_{W_2^{k+2}} \le N\|(L - \lambda)u\|_{W_2^n} + N\|u\|_{W_2^{k+1}}.$$

The induction on k leads to

$$\lambda \|u\|_{W_2^n} + \|u\|_{W_2^{n+2}} \le N\|(L-\lambda)u\|_{W_2^n} + N\|u\|_{W_2^1} \le N\|(L-\lambda)u\|_{W_2^n} + N\|u\|_{W_2^2}$$

and to obtain (9) it only remains to use Theorem 6.4 again. The theorem is proved.

Here is a result on *global regularity*.

6. Corollary. *Under the assumptions of Theorem* 5 *take a function* $u \in W_2^2$, $\lambda \ge 0$ *and assume that* $Lu - \lambda u \in W_2^n$. *Then* $u \in W_2^{n+2}$ *and*

$$\lambda \|u\|_{W_2^n} + \|u\|_{W_2^{n+2}} \le N(\|Lu - \lambda u\|_{W_2^n} + \|u\|_{\mathcal{L}_2}),$$

where N *depends only on* n, d, K, K_1, κ, *and* ω.

Indeed, denote $\lambda_1 = \lambda_0 + \lambda$ and introduce

$$g := Lu - \lambda_1 u = (Lu - \lambda u) - \lambda_0 u. \tag{10}$$

If, for an $r \in \{0, ..., n - 1\}$, we have $u \in W_2^{r+2}$, then $g \in W_2^{r+1}$ and by Theorem 5 equation (10) has a solution in $W_2^{(r+1)+2}$, which is unique in W_2^2. Since u is a solution of class W_2^2, it follows that $u \in W_2^{r+3}$. An obvious induction on r proves that $u \in W_2^{n+2}$.

Furthermore, for $r = 1, ..., n$ by Theorem 5 we obtain

$$A_r := \lambda \|u\|_{W_2^r} + \|u\|_{W_2^{r+2}} \leq \lambda_1 \|u\|_{W_2^r} + \|u\|_{W_2^{r+2}}$$

$$\leq N\|g\|_{W_2^r} \leq N(\|Lu - \lambda u\|_{W_2^n} + \|u\|_{W_2^r})$$

$$\leq N\|Lu - \lambda u\|_{W_2^n} + NA_{r-1} \leq ... \leq N\|Lu - \lambda u\|_{W_2^n} + NA_0$$

and referring to Theorem 6.5 finishes the argument.

Corollary 6 can be localized providing a *local regularity result.*

7. Theorem. *Under the assumptions of Theorem 5 take two numbers $0 < \rho < R \leq \infty$ and a function $u \in W_2^2(B_R)$. Also take a $\lambda \geq 0$ and assume that $Lu - \lambda u \in W_2^n(B_R)$. Then $u \in W_2^{n+2}(B_\rho)$ and*

$$\lambda \|u\|_{W_2^n(B_\rho)} + \|u\|_{W_2^{n+2}(B_\rho)} \leq N(\|Lu - \lambda u\|_{W_2^n(B_R)} + \|u\|_{W_2^1(B_R)}),$$

where N depends only on $n, d, K, K_1, \kappa, \rho, R$, and ω.

Proof. Take a $\zeta \in C_0^\infty(B_R)$ and notice that

$$L(u\zeta) - \lambda \zeta u = \zeta(Lu - \lambda u) + u(L\zeta - c\zeta) + 2a^{ij} u_{x^i} \zeta_{x^j} =: g.$$

Since $n \geq 1$, we have that $g \in W_2^1$, so that $u\zeta \in W_2^3$ by Corollary 6. This holds for any $\zeta \in C_0^\infty(B_R)$, implying that for $n \geq 2$ we have $g \in W_2^2$. In that case by Corollary 6 we have $u\zeta \in W_2^4$. Proceeding further in this way, we see that $g \in W_2^n$ and $u\zeta \in W_2^{n+2}$ for any $\zeta \in C_0^\infty(B_R)$. In particular, $u \in W_2^{n+2}(B_\rho)$.

This allows us to apply Corollary 6 to the functions $u\zeta_r$, $r = 0, ..., n+1$, where $\zeta_r \in C_0^\infty(B_R)$ are chosen in such a way that $\zeta_0 = 1$ on B_ρ and $\zeta_{r+1} = 1$ on the support of ζ_r. Then for $r = 0, ..., n$ we obtain

$$\lambda \|u\zeta_r\|_{W_2^{n-r}} + \|u\zeta_r\|_{W_2^{n-r+2}} \leq NF + N\|u\zeta_{r+1}\|_{W_2^{n-r+1}}, \tag{11}$$

where

$$F := \|Lu - \lambda u\|_{W_2^n(B_R)}.$$

In particular,

$$\|u\zeta_r\|_{W_2^{n-r+2}} \leq NF + N\|u\zeta_{r+1}\|_{W_2^{n-r+1}}, \quad \|u\zeta_r\|_{W_2^{n-r+2}} \leq NF + N\|u\zeta_{n+1}\|_{W_2^1}.$$

We take here $r = 1$ and use (11) with $r = 0$. Then we get

$$\lambda \|u\zeta_0\|_{W_2^n} + \|u\zeta_0\|_{W_2^{n+2}} \leq NF + N\|u\zeta_1\|_{W_2^{n+1}} \leq NF + N\|u\zeta_{n+1}\|_{W_2^1}.$$

The theorem is proved.

This result will be substantially improved in Theorems 2.4.7 and 9.4.1.

Corollary 6 implies, in particular, that if $\lambda \geq \lambda_0$ and $f \in W_2^n$ for any $n \geq 0$ and if each derivative of a, b, c is bounded, then the unique solution $u \in W_2^2$ of equation $\lambda u - Lu = f$ belongs to W_2^n also for all $n \geq 0$. It turns out that in this case u is infinitely differentiable with each derivative bounded on $\mathbb{R}^d$, or, to be more precise, admits a function that equals u almost everywhere and is infinitely differentiable with each derivative bounded on $\mathbb{R}^d$. In this way we get the classical solvability of the equation $\lambda u - Lu = f$.

This fact is obtained immediately from one of the Sobolev embedding theorems that reads as follows.

8. Theorem. *Let integers $n > k \geq 0$ and assume that $2(n-k) > d$. Then for any $u \in W_2^n$ there exists a unique function $v \in C^k$ such that $v = u$ almost everywhere. Furthermore, there is a constant N independent of u such that*

$$\|v\|_{C^k} \leq N\|u\|_{W_2^n}. \tag{12}$$

Proof. First we claim that it suffices to prove (12) with $v = u \in C_0^n$. Indeed, if (12) is true in this case, then for any $u \in W_2^n$ we take a defining sequence $u_m \in C_0^n$ and observe that by (12) we have

$$\|u_r - u_m\|_{C^k} \leq N\|u_r - u_m\|_{W_2^n} \to 0$$

as $r, m \to \infty$. By the completeness of C^k there exists a $v \in C^k$ such that $u_m \to v$ in C^k. Since $u_n \to u$ in $\mathcal{L}_2$, we have $u = v$ (a.e.). After that it only remains to notice that

$$\|v\|_{C^k} = \lim_{m\to\infty} \|u_m\|_{C^k} \leq N \lim_{m\to\infty} \|u_m\|_{W_2^n} = \|u\|_{W_2^n}.$$

Thus, we assume that $u \in C_0^n$. In that case, for any multi-index α with $|\alpha| \leq n$ we have

$$D^\alpha u(x) = i^{|\alpha|} c_d \int_{\mathbb{R}^d} e^{i\xi\cdot x} \xi^\alpha \tilde{u}(\xi)\, d\xi.$$

Hence

$$\max_{\mathbb{R}^d} |D^\alpha u| \leq c_d \int_{\mathbb{R}^d} |\xi|^{|\alpha|} |\tilde{u}(\xi)|\, d\xi.$$

By using Hölder's inequality and Parseval's identity, we see that the square of the last integral is dominated by the product of

$$\int_{\mathbb{R}^d} |\xi|^{2|\alpha|}(1+\sum_{j=1}^d |\xi^j|^{2n})^{-1}\,d\xi,$$

which is finite if $|\alpha| \le k$, and

$$\int_{\mathbb{R}^d}(1+\sum_{j=1}^d |\xi^j|^{2n})|\tilde{u}(\xi)|^2\,d\xi = \|u\|_{\mathcal{L}_2}^2 + \sum_{j=1}^d \|D_j^n u\|_{\mathcal{L}_2}^2 \le N\|u\|_{W_2^n}^2.$$

The theorem is proved.

Theorem 8 will be generalized to a very large extent in Sections 10.2 and 13.8.

9. Remark. If we have two functions f and g of class $\mathcal{L}_p$ defined in a domain and $f = g$ almost everywhere, then we say that f is a modification of g and vice versa. Speaking about elements of $\mathcal{L}_p$, it is common to say that a function u is, say, continuous if it has a continuous modification. We will always use this stipulation in the future. In this sense Theorem 8 says that under its conditions any $u \in W_2^n$ is continuous.

10. Remark. There is also a "local" version of Theorem 8. Namely, if $2(n-k) > d$, D is a domain in $\mathbb{R}^d$, bounded domain $G \subset \bar{G} \subset D$, and $u \in W_2^n(D)$, then $u \in C^k(G)$ and the norm of u in $C^k(G)$ is less than a constant, independent of u, times the norm of u in $W_2^n(D)$.

This result follows at once from Theorem 8 if one takes a $\zeta \in C_0^\infty(D)$ such that $\zeta = 1$ on G and applies Theorem 8 to $u\zeta$.

11. Exercise*. Let $d = 1$, $\Omega = (0,1)$. For $u \in W_2^1(\Omega)$ prove that

$$|u(x)-u(y)| \le N|x-y|^{1/2}\|u\|_{W_2^1(\Omega)}, \quad |u(x)| \le N\|u\|_{W_2^1(\Omega)}$$

for any $x, y \in \Omega$, where N is independent of u.

8. Sobolev mollifiers

The most important point in Section 7 was that by Theorem 7.1 (iii) the solutions of $\Delta u - \lambda u = f$ are smoother if f is smoother. It turns out that this fact can also be obtained in a different and very general way without using explicit representations of solutions.

In this section we take

$$p \in [1,\infty).$$

By definition a function in W_p^k can be approximated by smooth functions. There is one very powerful unified way to do such approximations by using the Sobolev mollifiers. In particular, this method will allow us to give a criterion for deciding if $u \in W_p^{rk}(\Omega)$ on the basis of knowing its Sobolev derivatives.

We need the following Young's inequalities.

1. Lemma. *(i) Let $g \in \mathcal{L}_p$ and $h \in \mathcal{L}_1$. Then $g * h \in \mathcal{L}_p$ and*

$$\|g * h\|_{\mathcal{L}_p} \leq \|g\|_{\mathcal{L}_p} \|h\|_{\mathcal{L}_1}. \tag{1}$$

(ii) More generally, let $q, r \in [1, \infty)$ and

$$\frac{1}{q} + \frac{1}{r} = \frac{1}{p} + 1. \tag{2}$$

Also let $g \in \mathcal{L}_q$ and $h \in \mathcal{L}_r$. Then

$$\|g * h\|_{\mathcal{L}_p} \leq \|g\|_{\mathcal{L}_r} \|h\|_{\mathcal{L}_q}. \tag{3}$$

Proof. (i) The convolution $g * h(y)$ is

$$\int_{\mathbb{R}^d} g(y - x) h(x)\, dx,$$

which is the "sum" with respect to x of the functions $g(y - x)h(x)$ as functions of y. By Minkowski's inequality (the norm of a sum is less than the sum of norms)

$$\|g * h\|_{\mathcal{L}_p} \leq \|\, |g| * |h|\, \|_{\mathcal{L}_p} \leq \int_{\mathbb{R}^d} |h(x)| \Big(\int_{\mathbb{R}^d} |g(y - x)|^p\, dy \Big)^{1/p} dx,$$

where the last expression is just the right-hand side of (1).

(ii) If $r = 1$, we can use (i). In case that $r > 1$, observe that $1/q > 1/p$ and $p > q$. Similarly, $p \geq r$. Then, by Hölder's inequality

$$|g * h|(x) \leq |g| * |h|(x) = \int_{\mathbb{R}^d} (|g(y - x)|\, |h(y)|^{q/p}) |h(y)|^{1-q/p}\, dy$$

$$\leq \big(|g|^r * |h|^{rq/p} \big)^{1/r}(x) \Big(\int_{\mathbb{R}^d} |h(y)|^{(1-q/p)r/(r-1)}\, dy \Big)^{1-1/r}$$

$$= \big(|g|^r * |h|^{rq/p} \big)^{1/r}(x) \|h\|_{\mathcal{L}_q}^{q(1-1/r)}.$$

Furthermore, by (i)

$$\|(|g|^r * |h|^{rq/p})^{1/r}\|_{\mathcal{L}_p} = \| |g|^r * |h|^{rq/p}\|_{\mathcal{L}_{p/r}}^{1/r}$$

$$\le \| |g|^r\|_{\mathcal{L}_1}^{1/r} \| |h|^{rq/p}\|_{\mathcal{L}_{p/r}}^{1/r} = \|g\|_{\mathcal{L}_r} \|h\|_{\mathcal{L}_q}^{q/p}.$$

Hence,

$$\|g * h\|_{\mathcal{L}_p} \le \|g\|_{\mathcal{L}_r} \|h\|_{\mathcal{L}_q}^{q/p} \|h\|_{\mathcal{L}_q}^{q(1-1/r)}$$

and this is (3) since $q/p + q(1 - 1/r) = 1$. The lemma is proved.

Let $\zeta \in C_0^\infty$. For $\varepsilon > 0$ let $\zeta_\varepsilon(x) = \varepsilon^{-d}\zeta(x/\varepsilon)$ and for any locally integrable u define

$$u^{(\varepsilon)}(x) = u * \zeta_\varepsilon(x) = \varepsilon^{-d} \int_{\mathbb{R}^d} u(y)\zeta((x-y)/\varepsilon)\, dy = \int_{\mathbb{R}^d} u(x - \varepsilon y)\zeta(y)\, dy. \tag{4}$$

2. Lemma. *(i) If u is locally integrable in $\mathbb{R}^d$, then $u^{(\varepsilon)}$ is infinitely differentiable in $\mathbb{R}^d$ and for any multi-index α*

$$D^\alpha u^{(\varepsilon)}(x) = u * D^\alpha \zeta_\varepsilon(x) = \varepsilon^{-d-|\alpha|} \int_{\mathbb{R}^d} u(y)(D^\alpha\zeta)((x-y)/\varepsilon)\, dy. \tag{5}$$

In particular, if $u \in \mathcal{L}_p$, then by Hölder's inequality the $D^\alpha u^{(\varepsilon)}$ are bounded for any $\varepsilon > 0$ and α.

(ii) Let α be a multi-index, u locally integrable in $\mathbb{R}^d$, and let $D^\alpha u$ be the Sobolev D^α derivative of u (see Definition 3.4). Then $D^\alpha u^{(\varepsilon)} = (D^\alpha u)^{(\varepsilon)}(x)$.

(iii) If

$$\int_{\mathbb{R}^d} \zeta\, dx = 1$$

and $u \in \mathcal{L}_p$, then $u^{(\varepsilon)} \in W_p^k$ for any k and $u^{(\varepsilon)} \to u$ in $\mathcal{L}_p$ as $\varepsilon \downarrow 0$. In particular, $C_b^\infty \cap W_p^k$ is dense in W_p^k, where $C_b^\infty = \bigcap_n C^n$.

Proof. Assertion (i) is a standard result in integration theory. Assertion (ii) follows from (5) by the definition of $D^\alpha u$.

To prove (iii), notice that for any bounded continuous function ϕ with compact support and $\varepsilon \in (0, 1)$ the functions $\phi^{(\varepsilon)}$ are bounded by a constant independent of ε. For $\varepsilon \le 1$ they also have support in a ball independent of ε. Indeed, if $\zeta(x) = 0$ and $\phi(x) = 0$ for $|x| \ge R$, then it is not hard to see that, for $|x| \geq R + \varepsilon R$, we have $\phi(y)\zeta((x-y)/\varepsilon) = 0$ for all y, so that $\phi^{(\varepsilon)}(x) = 0$. Furthermore, for any x

$$\phi^{(\varepsilon)}(x) = \int_{\mathbb{R}^d} \phi(x - \varepsilon y)\zeta(y)\,dy \to \phi(x)$$

as $\varepsilon \downarrow 0$ by the dominated convergence theorem. Also if $\phi \in C_0^k$ and $|\alpha| \leq k$, then

$$D^\alpha \phi^{(\varepsilon)}(x) = \int_{\mathbb{R}^d} D^\alpha \phi(x - \varepsilon y)\zeta(y)\,dy \to D^\alpha \phi(x)$$

as $\varepsilon \downarrow 0$ with all functions involved uniformly bounded and vanishing outside the same ball.

By the dominated convergence theorem and the above properties we have $\phi^{(\varepsilon)} \to \phi$ in W_p^k if $\phi \in C_0^k$. Also use Lemma 1, which, along with (5), implies in particular that $u^{(\varepsilon)} \in W_p^k$ for any k. Then by inspecting

$$\overline{\lim_{\varepsilon \downarrow 0}} \|u - u^{(\varepsilon)}\|_{W_p^k} \leq \|u - \phi\|_{W_p^k} + \overline{\lim_{\varepsilon \downarrow 0}} \|\phi - \phi^{(\varepsilon)}\|_{W_p^k} + \overline{\lim_{\varepsilon \downarrow 0}} \|(\phi - u)^{(\varepsilon)}\|_{W_p^k}$$

$$\leq (1 + N(k,d)\|\zeta\|_{\mathcal{L}_1})\|u - \phi\|_{W_p^k}$$

and using the fact that C_0^k is dense in W_p^k, we get (iii). The lemma is proved.

In light of assertion (i) of Lemma 2 the operator $u \to u^{(\varepsilon)}$ is called a *Sobolev mollifier*.

3. Exercise*. Prove that if α is a multi-index $u \in \mathcal{L}_2$ and the Sobolev derivative $D^\alpha u \in \mathcal{L}_2$, then

$$\int_{\mathbb{R}^d} |\xi^\alpha|^2 |\tilde{u}(\xi)|^2\,d\xi \leq N\|D^\alpha u\|_{\mathcal{L}_2}^2,$$

where N is independent of u and $\tilde{u}$ is the Fourier transform of u.

New proof of Corollary 7.2. We know that $u \in W_2^2$ and that we need only show that $u \in W_2^{2+n}$.

Take a $\zeta \in C_0^\infty(\mathbb{R}^d)$ with unit integral. Then by assertions (i) and (ii) of Lemma 2 for any multi-index α with $|\alpha| \leq n$ and $\varepsilon > 0$ we have

$$\lambda u^{\alpha\varepsilon} - \Delta u^{\alpha\varepsilon} = f^{\alpha\varepsilon},$$

where

$$u^{\alpha\varepsilon} = D^\alpha u^{(\varepsilon)}, \quad f^{\alpha\varepsilon} = (D^\alpha f)^{(\varepsilon)}.$$

By assertion (iii) we have $u^{\alpha\varepsilon} \in W_2^2$, so that by Theorem 3.16 for any $\delta > 0$

$$\|D^\alpha[u^{(\varepsilon)} - u^{(\delta)}]\|_{W_2^2} \le N\|(D^\alpha f)^{(\varepsilon)} - (D^\alpha f)^{(\delta)}\|_{\mathcal{L}_2},$$

where N is independent of ε, δ. Since this holds for any α with $|\alpha| \le n$,

$$\|u^{(\varepsilon)} - u^{(\delta)}\|_{W_2^{2+n}} \le N \sum_{|\alpha| \le n} \|(D^\alpha f)^{(\varepsilon)} - (D^\alpha f)^{(\delta)}\|_{\mathcal{L}_2}.$$

The right-hand side here tends to zero as $\varepsilon, \delta \downarrow 0$ by Lemma 2 (iii), so that the sequence of smooth functions $u^{(1/j)}$ is Cauchy in W_2^{2+n}. In addition, $u^{(1/j)} \to u$ in $\mathcal{L}_2$ as $j \to \infty$ again by Lemma 2 (iii). By referring to the definition of W_2^{2+n}, we obtain what we needed.

4. Exercise*. Let u be a continuous function on $\mathbb{R}^d$ and let the generalized derivative u_{x^1} also be a continuous function. By inspecting the proof of Lemma 2 (iii), prove that u_{x^1} is the classical derivative of u with respect to x^1.

Assertion (i) of the following theorem turns out to be true for smooth domains (see Exercise 8.4.5). Assertion (iii) generalizes assertion (ii) of Theorem 3.11. We suggest the reader further generalize assertions (i) and (iii) of Theorem 5 for the simplest case of Lipschitz domains, namely Lipschitz half spaces, in Exercise 8. The proofs of assertions (i) through (iii) are based on mollifying the function u extended as zero outside Ω, but since near the boundary the mollified functions may have "bad" behavior, we shift them "outwards".

5. Theorem. *Assume that $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d_+$. Then*

(i) A function $u \in \mathcal{L}_p(\Omega)$ belongs to $W_p^k(\Omega)$ if and only if its Sobolev derivatives $D^\alpha u$ exist and belong to $\mathcal{L}_p(\Omega)$ for any multi-index α with $|\alpha| \le k$.

(ii) For any $\zeta \in C_0^\infty$, which integrates to one and, in case $\Omega = \mathbb{R}^d_+$, additionally satisfies $\zeta(x) = 0$ if $x^1 \ge 0$, we have in the notation before Lemma 2 that

$$(uI_\Omega)^{(\varepsilon)} \to u$$

in $W_p^k(\Omega)$ as $\varepsilon \downarrow 0$ for any $u \in W_p^k(\Omega)$.

(iii) The set of functions which are infinitely differentiable in $\bar{\Omega}$ and vanish for large $|x|$ is dense in $W_p^k(\Omega)$.

(iv) For ζ as in (ii) there is a constant $N = N(d, k, \zeta)$ such that, for

any $u \in \mathcal{L}_p$ and constant M.

$$
\begin{gathered}
u \in W_p^k, \quad \|u\|_{W_p^k} \le M \Longrightarrow \|u^{(\varepsilon)}\|_{W_p^k} \le NM \quad \forall \varepsilon > 0, \\
p > 1, \quad \|u^{(\varepsilon)}\|_{W_p^k} \le M \quad \forall \varepsilon \in (0,1) \Longrightarrow u \in W_p^k, \quad \|u\|_{W_p^k} \le NM.
\end{gathered} \tag{6}
$$

Proof. (i) The "only if" part follows from Definition 3.2. To prove the "if" part, take $u \in \mathcal{L}_p(\Omega)$ such that $D^\alpha u \in \mathcal{L}_p(\Omega)$ for $|\alpha| \le k$ and define

$$
v = uI_\Omega, \quad v^\alpha = I_\Omega D^\alpha u.
$$

Also take ζ as in assertion (ii). Then $v \in \mathcal{L}_p$ and the $v^{(\varepsilon)}$ are infinitely differentiable, have bounded derivatives and belong to W_p^m for any m by Lemma 2. In case $\Omega = \mathbb{R}^d$, the same lemma shows that $D^\alpha v^{(\varepsilon)} \to D^\alpha u$ in $\mathcal{L}_p$ if $|\alpha| \le k$. Hence $v^{(1/n)}$ is a Cauchy sequence in W_p^k and $v^{(1/n)} \to u$ in $\mathcal{L}_p$, so that by definition $u \in W_p^k$.

In the remaining case that $\Omega = \mathbb{R}_+^d$ the same argument is applicable if we restrict our attention to $x^1 > 0$. Indeed, for those x and $|\alpha| \le k$, due to the fact that $\zeta((x-y)/\varepsilon) \ne 0$ only if $y^1 > x^1$ and, in particular, $\zeta((x-\cdot)/\varepsilon) \in C_0^\infty(\Omega)$, we have

$$
\begin{aligned}
D^\alpha v^{(\varepsilon)}(x) &= \varepsilon^{-d-|\alpha|} \int_{\mathbb{R}^d} v(y)(D^\alpha \zeta)((x-y)/\varepsilon)\,dy \\
&= \varepsilon^{-d-|\alpha|} \int_\Omega v(y)(D^\alpha \zeta)((x-y)/\varepsilon)\,dy \\
&= \varepsilon^{-d} \int_\Omega \zeta((x-y)/\varepsilon) D^\alpha u(y)\,dy = v^\alpha * \zeta_\varepsilon(x).
\end{aligned}
$$

In addition $v^\alpha \in \mathcal{L}_p$, so that $v^\alpha * \zeta_\varepsilon \to v^\alpha$ in $\mathcal{L}_p$ and $v^\alpha * \zeta_{1/n}$ is a Cauchy sequence in $\mathcal{L}_p$ whereas $D^\alpha v^{(1/n)}$ $(= v^\alpha * \zeta_{1/n}$ on $\Omega)$ is a Cauchy sequence in $\mathcal{L}_p(\Omega)$. Finally, $v^{(1/n)} \to v$ in $\mathcal{L}_p$; in particular, $v^{(1/n)} \to v = u$ in $\mathcal{L}_p(\Omega)$. By definition $u \in W_p^k$ and assertion (i) is proved.

Assertion (ii) has actually been proved in the above argument. Assertion (iii) also follows easily from the above. Indeed $(uI_\Omega)^{(\varepsilon)}$ is infinitely differentiable in $\bar{\Omega}$, has bounded derivatives, belongs to W_p^k by Lemma 2, and converges to u in $W_p^k(\Omega)$. In addition, by using cut-off functions as in the proof of Theorem 3.11, we can approximate in $W_p^k(\Omega)$ the smooth functions $(uI_\Omega)^{(\varepsilon)}$ by smooth ones vanishing for large $|x|$.

To prove (iv), notice that the first implication in (6) follows from the formula $D^\alpha(u^{(\varepsilon)}) = (D^\alpha u)^{(\varepsilon)}$ and Lemma 1 estimating norms of convolutions. To prove the second one, observe that, for $|\alpha| \leq k$, the $D^\alpha u^{(1/n)}$ are bounded sequences in $\mathcal{L}_p$. Since $p > 1$, there is a subsequence $n' \to \infty$ and functions $u^\alpha \in \mathcal{L}_p$ such that $D^\alpha u^{(1/n')} \to u^\alpha$ weakly in $\mathcal{L}_p$. In particular, $\|u^\alpha\|_{\mathcal{L}_p} \leq M$ and for any $\phi \in C_0^\infty$

$$(-1)^{|\alpha|} \int_{\mathbb{R}^d} u^{(1/n')} D^\alpha \phi \, dx = \int_{\mathbb{R}^d} \phi D^\alpha u^{(1/n')} \, dx \to \int_{\mathbb{R}^d} \phi u^\alpha \, dx.$$

Since $u^{(1/n')} \to u$ in $\mathcal{L}_p$, the first expression tends to

$$(-1)^{|\alpha|} \int_{\mathbb{R}^d} u D^\alpha \varphi \, dx.$$

Hence by definition $u^\alpha = D^\alpha u$, which along with the above estimate $\|u^\alpha\|_{\mathcal{L}_p} \leq M$ and assertion (i), proves (iv). The theorem is proved.

6. Exercise*. Let $u \in W^k_p$ be even (or odd) with respect to x^1. Prove that the derivatives of order $\leq k$ which do not contain differentiations in x^1 are even (respectively, odd) with respect to x^1.

7. Exercise*. Let $u(x) = 0$ for $|x| \geq 1$ and let u be Lipschitz continuous: $|u(x) - u(y)| \leq \alpha|x - y|$, where α is a constant. Prove that $u \in W^1_p$ for any p and $\|u_x\|_{\mathcal{L}_p} \leq N(d)\alpha$.

8. Exercise. Let $\Omega = \{x : x^1 > f(x')\}$ where f is a Lipschitz continuous function: $|f(x') - f(y')| \leq \alpha|x' - y'|$, where α is a constant. Prove that in this situation assertions (i) and (iii) of Theorem 5 are still valid.

9. Singular-integral representation of u_{xx}

By Theorem 7.1 if $u \in C_0^2$, then for any $\lambda > 0$ we have

$$u(x) = \int_{\mathbb{R}^d} (\lambda u(x - y) - \Delta u(x - y)) G_\lambda(y) \, dy, \tag{1}$$

where G_λ is given by an explicit formula from which it follows that, as $\lambda \downarrow 0$,

$$G_\lambda(y) \uparrow G_0(y) := \frac{1}{(4\pi)^{d/2}} \int_0^\infty t^{-d/2} e^{-|y|^2/4t} \, dt = N(d) \frac{1}{|y|^{d-2}},$$

where the last equality is obtained by a change of variables. As is easy to see, $N(d) < \infty$ if and only if $d \geq 3$. In that case G_0 is locally summable and by the dominated convergence theorem we obtain from (1) that for any $u \in C_0^2$

$$u(x) = N(d) \int_{\mathbb{R}^d} \frac{1}{|y|^{d-2}} f(x-y)\,dy, \tag{2}$$

where $f = -\Delta u$. This is a representation of u by means of the classical *Newtonian potential* of f.

In the sequel we take $d \geq 3$. Again owing to the dominated convergence theorem, one can differentiate (2) and see that if $u \in C_0^3$, then $f \in C_0^1$ and

$$u_{x^1}(x) = N(d) \int_{\mathbb{R}^d} \frac{1}{|y|^{d-2}} f_{x^1}(x-y)\,dy.$$

We represent $\mathbb{R}^d$ as $\{y = (y^1, y') : y^1 \in \mathbb{R}, y' \in \mathbb{R}^{d-1}\}$ and write

$$\int_{\mathbb{R}^d} \frac{1}{|y|^{d-2}} f_{x^1}(x-y)\,dy = \int_{\mathbb{R}^{d-1}} \left(\int_{-\infty}^{\infty} \frac{1}{|y-x|^{d-2}} f_{y^1}(y)\,dy^1 \right) dy'.$$

If $y' \neq x'$, the function $|y-x|^{-(d-2)}$ is an infinitely differentiable function of y^1 and one can transform the inside integral by integrating by parts, thus finding that for almost all y' it equals

$$(d-2) \int_{-\infty}^{\infty} \frac{y^1 - x^1}{|y-x|^d} f(y)\,dy^1.$$

Observe that

$$\int_{\mathbb{R}^d} \left| \frac{y^1 - x^1}{|y-x|^d} f(y) \right| dy \leq \int_{\mathbb{R}^d} \frac{1}{|y-x|^{d-1}} |f(y)|\,dy < \infty$$

since f has compact support and

$$\int_{|y-x|<R} \frac{1}{|y-x|^{d-1}}\,dy < \infty \tag{3}$$

for any R. This allows us to write the repeated integral which we obtain after integrating by parts as a usual integral over $\mathbb{R}^d$ and leads to the formula

$$u_{x^1}(x) = N(d)(d-2) \int_{\mathbb{R}^d} \frac{y^1 - x^1}{|y-x|^d} f(y)\,dy = N_1(d) \int_{\mathbb{R}^d} \frac{y^1}{|y|^d} f(y+x)\,dy.$$

Similarly for any $i = 1, ..., d$

$$u_{x^i}(x) = N_1(d) \int_{\mathbb{R}^d} \frac{y^i}{|y|^d} f(y+x)\,dy.$$

For the same reasons as above one can differentiate this formula with respect to x one more time and find

$$u_{x^ix^j}(0) = N_1(d)\int_{\mathbb{R}^d}\frac{y^i}{|y|^d}f_{y^j}(y)\,dy.$$

However, if we now try to go the same way as above integrating by parts, we will not be able to rewrite this expression as a usual integral over $\mathbb{R}^d$ because this time we will have d in place of $d-1$ in (3) and the integral will diverge. This is the reason why we take a $\zeta\in C_0^\infty$ depending only on $|y|$ and such that $\zeta(0)=1$ and write

$$u_{x^ix^j}(0) = N_1(d)\int_{\mathbb{R}^d}\frac{y^i}{|y|^d}(f(y)-f(0)\zeta(y))_{y^j}\,dy + f(0)N_{ij}, \tag{4}$$

where

$$N_{ij} := N_1(d)\int_{\mathbb{R}^d}\frac{y^i}{|y|^d}\zeta_{y^j}(y)\,dy.$$

Now we integrate by parts with respect to y^j and notice that

$$-\frac{\partial}{\partial y^j}\frac{y^i}{|y|^d} = \frac{y^iy^jd-\delta^{ij}|y|^2}{|y|^{d+2}} =: K_{ij}(y),$$

$$|K_{ij}(y)|\le\frac{N}{|y|^d},\quad |f(y)-f(0)\zeta(y)| = |f(y)-f(0)+f(0)(1-\zeta(y))|\le N|y|.$$

We also use the fact that $f(y)-f(0)\zeta(y)$ has compact support. Then we easily transform (4) into

$$u_{x^ix^j}(0) = N_1(d)\int_{\mathbb{R}^d}K_{ij}(y)(f(y)-f(0)\zeta(y))\,dy + f(0)N_{ij}. \tag{5}$$

with the integral converging in the usual sense. In particular,

$$\int_{\mathbb{R}^d}K_{ij}(y)(f(y)-f(0)\zeta(y))\,dy = \lim_{r\downarrow 0}\int_{|y|\ge r}K_{ij}(y)(f(y)-f(0)\zeta(y))\,dy.$$

It turns out that for any $r>0$

$$\int_{|y|\ge r}K_{ij}(y)\zeta(y)\,dy = 0.$$

By using the fact that ζ is radially symmetric, to prove this, it suffices to show that the integral of K_{ij} over spheres centered at the origin vanishes. If

$i \neq j$, this is obvious because K_{ij} is anti-symmetric in y^i. If $i = j$, then the said integrals are obviously independent of i and their sum is zero because

$$\sum_i (y^i)^2 d - |y|^2 d = 0.$$

Thus,

$$u_{x^i x^j}(0) = N_1(d) \lim_{r \downarrow 0} \int_{|y| \geq r} K_{ij}(y) f(y)\, dy + f(0) N_{ij}.$$

Similarly, for any x, if $u \in C_0^3$,

$$u_{x^i x^j}(x) = N_1(d) \mathcal{K}_{ij} f(x) + f(x) N_{ij}, \tag{6}$$

where

$$\mathcal{K}_{ij} f(x) = \lim_{r \downarrow 0} \int_{|y| \geq r} K_{ij}(y) f(x + y)\, dy$$

is one of the so-called singular-integral operators. That it is singular is reflected, in particular, in the fact that the usual estimate

$$\|k * f\|_{\mathcal{L}_p} \leq \|k\|_{\mathcal{L}_1} \|f\|_{\mathcal{L}_p}$$

does not allow us to estimate the $\mathcal{L}_p$ norm of $\mathcal{K}_{ij} f$ through the $\mathcal{L}_p$ norm of f since the K_{ij} are not integrable near the origin, let alone near infinity.

A quite discouraging fact is that (6) does not seem to allow us to get the estimate even if $p = 2$, when we know that the estimate exists from Theorem 3.16.

There is a theory of singular-integral operators (see, for instance, [**19**]) in which one shows that they are well defined and bounded on $\mathcal{L}_p$ for any $p \in (1, \infty)$ and this along with (6) yields the estimate

$$\|u_{xx}\|_{\mathcal{L}_p} \leq N(d, p) \|\Delta u\|_{\mathcal{L}_p}.$$

We are going to avoid using the theory of singular-integral operators and obtain this estimate in a somewhat different way.

1. Exercise. Show that $N_{ij} = -\delta^{ij} d^{-1}$.

To give the reader some feeling about certain issues arising in the theory of singular integrals, we give the following.

2. Exercise. Take $k \neq j$ and for $\varepsilon \in (0, 1)$ define $K_{kj\varepsilon}(x) = K_{kj}(x)$ if $\varepsilon < |x| < \varepsilon^{-1}$ and $K_{kj\varepsilon}(x) = 0$ otherwise. Take a $u \in C_0^\infty$, set $f = -\Delta u$, and prove that in the $\mathcal{L}_2$ sense $N_1(d) K_{kj\varepsilon} * f \to u_{x^k x^j}$.

3. Exercise*. Take $d \geq 3$, an $f \in C_0^\infty$, define u by (2) and show that u is infinitely differentiable, $\Delta u = -f$ in $\mathbb{R}^d$, and $\|u_{xx}\|_{\mathcal{L}_2} \leq N(d)\|f\|_{\mathcal{L}_2}$.

10. Hints to exercises

1.5. The integrand on the left is the divergence of the vector-field F given by

$$F^i = u_{x^i}\Delta u - u_{x^j}u_{x^j x^i}$$

and on the boundary we have $u_x = (u_x \cdot n)n$,

$$(F, n) = (u_x, n)(\Delta u - (u_{xx}n) \cdot n).$$

To compute $\Delta u - (u_{xx}n)\cdot n$ at a point $x_0 \in \partial B$, take $d-1$ unit mutually orthogonal vectors $\ell_1, ..., \ell_{d-1}$ orthogonal to $n(x_0)$, so that owing to Exercise 1.4

$$\Delta u - (u_{xx}n) \cdot n = \sum_{k=1}^{d-1} u_{x^i x^j}\ell_k^i \ell_k^j$$

at x_0. Then, after differentiating the relation $u_x = (u_x \cdot n)n$ at x_0 along ℓ_k, use the equality $u_x \cdot \ell_k = 0$ and show that at this point

$$u_{x^i x^j}\ell_k^j = \ell_k^i(u_x \cdot n) + n^i(u_{x^j x} \cdot n)\ell_k^j, \quad u_{x^i x^j}\ell_k^i\ell_k^j = u_x \cdot n.$$

By the way, the latter identity is known as the Meusnier theorem.

1.8. Try e^{x^1}.

1.10. To prove the first assertion, show that the points where u/ζ takes its positive maximum value can only belong to $\partial\Omega$, and then let $\varepsilon \downarrow 0$. To prove the second, either consider the function

$$u(x) - u(x_0)\exp(-x^1\sqrt{\lambda})$$

or, what will be used in the hint to Exercise 1.11, observe that Lemma 1.7 holds true even if we allow $|u|$ to grow exponentially at infinity and then consider $ue^{\delta x^1} - u(x_0)$ with appropriate $\delta > 0$.

1.11. Follow the hint to Exercise 1.10 to make λ strictly positive and preserve $u \leq \sup_{\partial\Omega} u_+$. Then work with $u/\cosh(\varepsilon|x'|)$ and use Exercise 1.10.

1.12. (i) Use a simple corollary of Minkowski's inequality: $\|f * h\|_{\mathcal{L}_p} \leq \|f\|_{\mathcal{L}_p}\|h\|_{\mathcal{L}_1}$ (see Lemma 8.1); (ii) observe that for $u_\varepsilon(x) = \varepsilon^{-d}u(x/\varepsilon)$, we have $\tilde{u}_\varepsilon(\xi) \to \int u\,dx$ as $\varepsilon \downarrow 0$.

1.13. To prove the fact mentioned in the statement, consider $F(\phi_n D^\alpha v)$, where $\phi_n = \phi(x/n)$, $\phi \in C_0^\infty$, $\phi(x) = 1$ for $|x| \leq 1$. Then send n to infinity and use the continuity of the Fourier transform in $\mathcal{L}_2$. Then while proving the denseness, use part of the proof of Theorem 1.6 and the facts that $v := g * u$ and all its derivatives of any order are in $\mathcal{L}_2$ and are bounded if $g \in \mathcal{L}_q$, $q = p/(p-1)$ (≤ 2), and $u \in C_0^\infty$. Then you will arrive at $\sigma\tilde{v} = 0$, where

$$\sigma(\xi) = \sum_{|\alpha|\leq m} a^\alpha i^{|\alpha|}\xi^\alpha$$

is the so-called *characteristic polynomial of L*. Finally, observe that $\sigma(\xi) \neq 0$ (a.e.) since at least one of $a^\alpha \neq 0$ (the set where $\sigma = 0$ cannot have density points).

2.2. Show that the vectors $G_{x^1}, ..., G_{x^d}$ are linearly dependent, set $F(x) = x$ and use Exercise 2.1. For the second part use approximations by polynomials and observe that $g(x)/|g(x)| \in \partial B_1$, whenever $g(x) \neq 0$.

2.3. If there are no fixed points, then introduce a mapping G, which sends $x \in \bar{B}_1$ into the point $G(x)$ on ∂B_1 such that x lies between $G(x)$ and $f(x)$ on the straight segment joining those points.

2.4. Reduce the general situation to the one where f is *strictly* concave in y and strictly convex in x. Then consider the function bringing $x \in X$ to the unique point $y(x) \in Y$ maximizing $f(x, y)$ over Y. From the uniqueness of $y(x)$ deduce its continuity in x. Similarly introduce $x(y)$ by minimizing $f(x, y)$ over X. Then from Exercise 2.3 obtain that the mapping $x \to x(y(x))$ has a fixed point $x_0 \in X$. Set $y_0 = y(x_0)$ and prove that $x_0 = x(y_0)$,

$$f(x_0, y_0) = \min_X f(x, y_0) = \max_Y f(x_0, y),$$

$$\max_Y \min_X f(x, y) \geq f(x_0, y_0) \geq \min_X \max_Y f(x, y).$$

This yields a one-sided inequality in (2.1). Show that the opposite inequality is true regardless of whether the requirements (i) and (ii) are imposed or not.

3.6. Observe that there exists a $g_1 \in \mathcal{L}_p$ such that $\|D_1\phi_n - g_1\|_{\mathcal{L}_p} \to 0$ and use the fact that

$$\int_\Omega g D_1\psi \, dx = \lim_{n\to\infty} \int_\Omega \phi_n D_1\psi \, dx = -\lim_{n\to\infty} \int_\Omega (D_1\phi_n)\psi \, dx = \int_\Omega g_1\psi \, dx$$

to conclude that $D_1 g = g_1$.

3.13. There is a $v \in W^k_p(\Omega)$ such that $u^n \to v$ in $W^k_p(\Omega)$. Find $\|v - u\|_{\mathcal{L}_p(\Omega)}$.

3.17. Use the definition of $W^1_p(\Omega)$.

3.18. First prove that if $u \in W^1_p(\Omega)$, then for any $\varepsilon > 0$

$$u_\varepsilon := F_\varepsilon(u) \in W^1_p(\Omega),$$

where $F_\varepsilon(t) = ([t^2+\varepsilon^2]^{1/2} - \varepsilon)I_{t>0}$, and compute $(u_\varepsilon)_x$. After that use Exercise 3.13. You may also like to notice that the functions $F_\varepsilon(t)$ are continuously differentiable on $\mathbb{R}$, $0 \leq F_\varepsilon(t) \uparrow t_+$ as $\varepsilon \downarrow 0$.

3.19. What is $(u_-)_x$?

3.20. If $\|\phi_n\|_{\mathcal{L}_p} \to 0$, then for a subsequence n' we have that $\phi_{n'}(x^1, \cdot) \to 0$ in $\mathcal{L}_p(\mathbb{R}^{d-1})$ for almost all x^1.

3.23. (i) For $d \geq 2$ multiply both parts of $f = \Delta u$ by $(2 - |x|^2)u$ and integrate by parts. (ii) P_n is a linear finite-dimensional space. (iii) Use approximations of f by polynomials.

4.6. Use dilations.

4.7. Notice that $a^{ij}u_{ij} = \operatorname{tr} AU$ and the trace of a matrix is invariant under orthogonal transformations.

4.8. Notice that $Lu = a^{ij}v_{ij}$, where $v_{ij} = u_{x^ix^j} - \lambda\delta^{ij}u$ and use Exercises 1.3 and 4.7.

4.9. Consider $(1-t)(\Delta - 2\lambda) + tL$.

6.6. Use the method of continuity moving λ and Theorems 6.4 and 6.5.

7.3. Prove that the norm is independent of λ and use that $\lambda R_\lambda f = f + R_\lambda \Delta f$ for smooth f.

7.4. Use (7.2) and (7.5) to show that $|\nabla u| \le \lambda^{-1/2} \sup |\lambda u - \Delta u|$. Then minimize with respect to λ. To prove the sharpness, take $d = 1$.

8.3. From Lemma 8.2 and Parseval's identity derive that

$$\int_{\mathbb{R}^d} |\xi^\alpha|^2 |\widetilde{u^{(\varepsilon)}}(\xi)|^2 \, d\xi = N \|D^\alpha u^{(\varepsilon)}\|_{\mathcal{L}_2}^2 \le N \|D^\alpha u\|_{\mathcal{L}_2}^2,$$

where

$$\widetilde{u^{(\varepsilon)}}(\xi) = N \tilde{u}(\xi) \tilde{\zeta}(\varepsilon\xi).$$

You may first like to prove part of the above for $u \in C_0^\infty$.

8.6. Use mollifiers.

8.7. Use mollifiers.

8.8. Take ζ with support in $2\alpha|x'| < -x^1$.

9.1. What is $\sum_i N_{ii}$?

9.2. First show that

$$K_{kj\varepsilon} * f(x) = \int_{\varepsilon^{-1} > |y| > \varepsilon} K_{ij}(y)[f(x+y) - f(x)\zeta(y)] \, dy$$

and conclude that $N_1(d) K_{kj\varepsilon} * f \to u_{x^k x^j}$ on $\mathbb{R}^d$. Then introduce R as a number such that $f(x) = 0$ for $|x| > R$ and show that $|K_{kj\varepsilon} * f(x)|$ is bounded independently of ε if $|x| \le R+1$ by repeating the argument which led to (9.5). To prove the uniform boundedness of $|K_{kj\varepsilon} * f(x)|$ for $|x| \ge R+1$, show that $|K_{kj\varepsilon} * f(x)| \le N|x|^{-d}$.

9.3. First work with $G_\lambda f$.

Second-order parabolic equations in $W_2^{1,k}(\mathbb{R}^{d+1})$

1. The simplest equation $u_t + a^{ij}(t)D_{ij}u - \lambda u = f$

In this section we consider the parabolic equation

$$u_t + Lu - \lambda u = f, \tag{1}$$

in

$$\mathbb{R}^{d+1} = \{(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^d\},$$

where $\lambda > 0$ is a fixed number and for each $t \in \mathbb{R}$

$$L = L(t) = a^{ij}(t)D_{ij} \tag{2}$$

is an elliptic operator in the sense of Definition 1.4.1 with constant of ellipticity κ independent of t. The coefficients a^{ij} are assumed to depend only on t and to be at least measurable. Quite often we use the notation $\partial_t u$ instead of u_t, so that (1) can be written as

$$(\partial_t + L - \lambda)u = f.$$

If Ω is a domain in $\mathbb{R}^{d+1}$ and

$$r = 0, 1, \quad k \in \{0, 1, 2, ...\},$$

we denote by $C^{r,k+2r}(\Omega)$ the set of bounded continuous functions $u(t,x)$, $(t,x) \in \Omega$, such that their derivatives $D^\alpha \partial_t^\rho u$ are bounded and continuous in Ω for

$$|\alpha| + 2\rho \leq k + 2r, \quad \rho \leq r. \tag{3}$$

In other words, if $r = 0$, then we (only) require $D^\alpha u$ to be bounded and continuous for $|\alpha| \leq k$. If $r = 1$, then the definition requires

$$D^\alpha u \quad \forall |\alpha| \leq k+2, \quad \partial_t D^\beta u \quad \forall |\beta| \leq k$$

to be bounded and continuous. One can say that we require the derivatives of order $\leq k+2r$ to be bounded and continuous if we do not take more than one derivative in t and evaluate the order of a derivative by adding up the order of the derivative in x with factor two the order of the derivative in time.

As before $C^{r,k+2r}(\bar\Omega)$ is the subset of $C^{r,k+2r}(\Omega)$ consisting of functions u such that u and, generally, $D^\alpha \partial_t^\rho u$ extend to functions continuous in $\bar\Omega$ whenever (3) is satisfied. If Ω is unbounded, the sets $C_0^{r,k+2r}(\Omega)$ and $C_0^{r,k+2r}(\bar\Omega)$ are the subsets of $C^{r,k+2r}(\Omega)$ and $C^{r,k+2r}(\bar\Omega)$, respectively, consisting of functions vanishing for large $|t| + |x|$.

Warning. Quite often in the literature one denotes by $C_0^{r,k+2r}(\Omega)$ the subset of $C^{r,k+2r}(\Omega)$ consisting of functions with compact support in Ω. Observe that our definition is different. On the other hand, in these lectures, as is usual in the literature, $C_0^\infty(\Omega)$ is the space of infinitely differentiable function in Ω with compact support.

For $T \in [-\infty, \infty)$ introduce

$$\mathbb{R}_T^{d+1} = \{(t,x) \in \mathbb{R}^{d+1} : t > T\}.$$

1. Lemma. *Let $k \geq 0$, $u \in C_0^{1,k+2}(\mathbb{R}_T^{d+1})$ satisfy (1) in $\mathbb{R}_T^{d+1}$. Then for any $n \in \{0, 1, ..., k\}$*

$$\|D^n u_t\|_{\mathcal{L}_2(\mathbb{R}_T^{d+1})} + \|D^n u_{xx}\|_{\mathcal{L}_2(\mathbb{R}_T^{d+1})} + \sqrt{\lambda}\|D^n u_x\|_{\mathcal{L}_2(\mathbb{R}_T^{d+1})}$$

$$+\lambda\|D^n u\|_{\mathcal{L}_2(\mathbb{R}_T^{d+1})} \leq N(d,\kappa,k)\|D^n f\|_{\mathcal{L}_2(\mathbb{R}_T^{d+1})}.$$

Proof. One can apply the operator D^n to both parts of (1) and then one sees that we need only prove the lemma for $k = 0$.

In that case we use the Fourier transform with respect to x. Denoting by $\tilde{v}(t,\xi)$ the Fourier transform of $v(t,x)$ with respect to x, we find

$$\tilde{u}_t(t,\xi) = (\lambda + A(t,\xi))\tilde{u}(t,\xi) + \tilde{f}(t,\xi),$$

where

$$A(t,\xi) = a^{ij}(t)\xi^i\xi^j.$$

Since for each ξ, $\tilde{u}(t,\xi)$ has compact support, we have

$$\tilde{u}(t,\xi) = -\int_t^\infty e^{-\lambda(s-t)-A_{ts}(\xi)}\tilde{f}(s,\xi)\,ds, \tag{4}$$

where

$$A_{ts}(\xi) = \int_t^s A(r,\xi)\,dr.$$

Next we use the inequality $A(t,\xi) \geq \kappa|\xi|^2$ to get

$$|\tilde{u}(t,\xi)| \leq \int_t^\infty e^{-(\lambda+\kappa|\xi|^2)(s-t)}|\tilde{f}(s,\xi)|\,ds = \int_{-\infty}^0 e^{(\lambda+\kappa|\xi|^2)s}|\tilde{f}(t-s,\xi)|\,ds.$$

The last expression is the convolution of

$$|\tilde{f}(t,\xi)| \quad \text{and} \quad I_{t<0}e^{(\lambda+\kappa|\xi|^2)t}.$$

Therefore, after observing that

$$|\tilde{u}(t,\xi)|I_{t\geq T} \leq \int_{-\infty}^0 e^{(\lambda+\kappa|\xi|^2)s}|\tilde{f}(t-s,\xi)|I_{t\geq T}\,ds$$

$$\leq \int_{-\infty}^0 e^{(\lambda+\kappa|\xi|^2)s}|\tilde{f}(t-s,\xi)|I_{t-s\geq T}\,ds$$

and using Minkowski's inequality, we obtain

$$\int_T^\infty |\tilde{u}(t,\xi)|^2\,dt \leq \Big(\int_{-\infty}^0 e^{(\lambda+\kappa|\xi|^2)s}\,ds\Big)^2 \int_T^\infty |\tilde{f}(t,\xi)|^2\,dt,$$

$$\int_T^\infty (\lambda+\kappa|\xi|^2)^2|\tilde{u}(t,\xi)|^2\,dt \leq \int_T^\infty |\tilde{f}(t,\xi)|^2\,dt.$$

Integrating with respect to ξ and using Parseval's identity finally yield

$$\kappa^2\sum_{i,j}\|u_{x^ix^j}\|^2_{\mathcal{L}_2(\mathbb{R}^{d+1}_T)} + 2\kappa\lambda\sum_i\|u_{x^i}\|^2_{\mathcal{L}_2(\mathbb{R}^{d+1}_T)} + \lambda^2\|u\|^2_{\mathcal{L}_2(\mathbb{R}^{d+1}_T)} \leq \|f\|^2_{\mathcal{L}_2(\mathbb{R}^{d+1}_T)}$$

and this proves the desired estimates for u_{xx}, u_x, u. The estimate for u_t is obtained from the formula $u_t = f + \lambda u - a^{ij}D_{ij}u$. The lemma is proved.

2. Exercise. By using parabolic dilations, that is, the changes of coordinates $(t,x) \to (c^2t, cx)$ where $c \neq 0$ is an arbitrary constant, prove that there is no constant N such that for all $u \in C_0^\infty(\mathbb{R}^{d+1})$ we have

$$\|u\|_{\mathcal{L}_2(\mathbb{R}^{d+1})} \leq N\|\Delta u + u_t\|_{\mathcal{L}_2(\mathbb{R}^{d+1})}.$$

3. Remark. By using (4), one can derive an integral representation of u in terms of f as in a similar situation of elliptic equations in Section 1.7. For $s > t$ define

$$A_{ts} = \int_t^s a(r)\,dr, \quad B_{ts} = A_{ts}^{-1}, \quad \sigma_{ts} = A_{ts}^{1/2}.$$

Observe that the matrices A_{ts} are nondegenerate so that B_{ts} is well defined and

$$\kappa^{-1}(s-t)|\xi|^2 \geq A_{ts}^{ij}\xi^i\xi^j \geq \kappa(s-t)|\xi|^2,$$

$$\kappa^{-1}(s-t)^{-1}|\xi|^2 \geq B_{ts}^{ij}\xi^i\xi^j \geq \kappa(s-t)^{-1}|\xi|^2.$$

Introduce

$$p(t,s,x) = I_{s>t}(4\pi)^{-d/2}(\det B_{ts})^{1/2}\exp(-(B_{ts}x,x)/4).$$

Note that

$$p(t,s,x) = I_{s>t}(4\pi)^{-d/2}(\det \sigma_{ts})^{-1}\exp(-|\sigma_{ts}^{-1}x|^2/4),$$

which allows one to use the change of variables $x \to \sigma_{ts}x$, and find that for $s > t$

$$\int_{\mathbb{R}^d} p(t,s,x)\,dx = 1, \quad F(p(t,s,\cdot)) = c_d e^{-A_{ts}(\xi)}.$$

Then as in Section 1.7 from (4) we find that, if $u \in C_0^{1,2}(\mathbb{R}_T^{d+1})$, then

$$u = \mathcal{G}_\lambda(\lambda u - u_t - Lu) \tag{5}$$

in $\mathbb{R}_T^{d+1}$, where

$$\mathcal{G}_\lambda f(t,x) = \int_0^\infty e^{-\lambda r}\int_{\mathbb{R}^d} f(t+r, x-y)p(t,t+r,y)\,dydr. \tag{6}$$

4. Remark. We have derived (5) for $\lambda > 0$. Actually, the formula is true for any $\lambda \in \mathbb{R}$ as long as $u \in C_0^{1,2}(\mathbb{R}_T^{d+1})$. Indeed, take a $\mu \in \mathbb{R}$ and denote $v(t,x) = e^{-\mu t}u(t,x)$. Obviously $v \in C_0^{1,2}(\mathbb{R}_T^{d+1})$ and

$$(\partial_t + L - 1)v(t,x) = e^{-\mu t}(\partial_t + L - 1 - \mu)u(t,x) =: e^{-\mu t}g(t,x).$$

Therefore, by (5) applied with $\lambda = 1$ we get that in $\mathbb{R}_T^{d+1}$

$$e^{-\mu t}u(t,x) = \int_0^\infty e^{-r}\int_{\mathbb{R}^d} e^{-\mu(t+r)}p(t,t+r,y)g(t+r,x-y)\,dydr.$$

Upon cancelling $e^{-\mu t}$ in both parts, we come to (5) where λ is replaced with $\mu + 1$ which is an arbitrary number in $\mathbb{R}$.

5. Remark. For each t formula (6) represents the function $\mathcal{G}_\lambda f(t,x)$ as the "sum" with respect to y, r of

$$e^{-\lambda r}f(t+r, x-y)p(t,t+r,y).$$

It follows by Minkowski's inequality that for any $p \in [1,\infty)$

$$\begin{aligned}\|\mathcal{G}_\lambda|f|(t,\cdot)\|_{\mathcal{L}_p} &\le \int_0^\infty\int_{\mathbb{R}^d}\|e^{-\lambda r}f(t+r,\cdot-y)p(t,t+r,y)\|_{\mathcal{L}_p}\,dydr\\ &= \int_0^\infty e^{-\lambda r}\int_{\mathbb{R}^d}p(t,t+r,y)\|f(t+r,\cdot)\|_{\mathcal{L}_p}\,dydr\\ &= \int_0^\infty e^{-\lambda r}\|f(t+r,\cdot)\|_{\mathcal{L}_p}\,dr,\end{aligned}$$

where the last expression is again a "sum". Therefore, for

$$g(t) = \|\mathcal{G}_\lambda|f|(t,\cdot)\|_{\mathcal{L}_p}, \quad h(t) = \|f(t,\cdot)\|_{\mathcal{L}_p}$$

and any $q \in [1,\infty)$ we have

$$\begin{aligned}&\left(\int_{\mathbb{R}_T}\left(\int_{\mathbb{R}^d}|\mathcal{G}_\lambda f(t,x)|^p\,dx\right)^{q/p}dt\right)^{1/q} \le \|g\|_{\mathcal{L}_q(\mathbb{R}_T)}\\ &\le \int_0^\infty e^{-\lambda r}\|h(\cdot+r)\|_{\mathcal{L}_q(\mathbb{R}_T)}\,dr \le \|h\|_{\mathcal{L}_q(\mathbb{R}_T)}\int_0^\infty e^{-\lambda r}\,dr\\ &= \lambda^{-1}\left(\int_{\mathbb{R}_T}\left(\int_{\mathbb{R}^d}|f(t,x)|^p\,dx\right)^{q/p}dt\right)^{1/q}.\end{aligned}$$

In particular, for $q = p$,

$$\|\mathcal{G}_\lambda f\|_{\mathcal{L}_p(\mathbb{R}^{d+1}_T)} \le \lambda^{-1}\|f\|_{\mathcal{L}_p(\mathbb{R}^{d+1}_T)}.$$

Now follows a counterpart of Theorem 1.1.6.

6. Theorem. *The set $(\partial_t + \Delta - \lambda)C_0^\infty(\mathbb{R}^{d+1})$ is dense in $\mathcal{L}_p(\mathbb{R}^{d+1})$ for any $p \in [1, \infty)$.*

The proof of this theorem, which is set as an exercise, is almost identical to that of Theorem 1.1.6 and is based on the following version of the maximum principle for parabolic equations.

7. Lemma. *Let u be a bounded from above function on $\mathbb{R}^{d+1}$ having continuous derivatives u_t, u_x, and u_{xx}. Assume that $u_t + \Delta u - \lambda u \ge 0$ in $\mathbb{R}^{d+1}$. Then $u \le 0$. In particular, if u is bounded and $u_t + \Delta u - \lambda u = 0$ in $\mathbb{R}^{d+1}$, then $\pm u \le 0$, so that $u \equiv 0$.*

Proof. As in the proof of Lemma 1.1.7 we use an auxiliary function and introduce $\zeta(t, x) = [\cosh(\varepsilon t)]\cosh(\varepsilon|x|)$, where $\varepsilon > 0$ is small. Next we observe that the function $v = u/\zeta$ satisfies $(\partial_t + \Delta - \lambda)(\zeta v) \ge 0$, that is,

$$\zeta(v_t + \Delta v) + 2\zeta_{x^j}v_{x^j} + cv \ge 0, \tag{7}$$

where

$$c = \zeta_t + \Delta\zeta - \lambda\zeta = \zeta\Big\{\varepsilon\tanh(\varepsilon t) + \varepsilon^2 + (d-1)|x|^{-1}\varepsilon\tanh(\varepsilon|x|) - \lambda\Big\}.$$

Since $\tanh|x| \le |x|$ and $\tanh|x| \le 1$, we have $c < 0$ for all small ε. After that assuming that $u > 0$ at some points leads to a contradiction with (7) in the same way as in the proof of Lemma 1.1.7. The lemma is proved.

8. Remark. It turns out that if $\lambda = 0$ and u is a bounded function on $\mathbb{R}^{d+1}$ having continuous derivatives u_t, u_x, and u_{xx} such that $u_t + \Delta u = 0$ in $\mathbb{R}^{d+1}$, then u is constant.

9. Exercise. Give an example showing that for $\lambda = 0$ the first assertion of Lemma 7 is false. Also give an example showing that for $\lambda > 0$ the assertion is wrong if we allow u to be unbounded from above.

10. Exercise. Notice that Lemma 7 contains Lemma 1.1.7 which is seen if one takes u independent of t. In connection with this derive Lemma 1.6.2 from Lemma 1 and multiplicative inequalities.

11. Exercise (One-dimensional case). Let $d = 1$ and let $a(t, x)$ be a positive measurable function bounded from above and away from zero. Prove that for $u \in C_0^{1,2}(\mathbb{R}^2_T)$ and $\lambda \ge 0$ it holds that

$$\lambda\|u\|_{\mathcal{L}_2(\mathbb{R}^2)} + \|u_t\|_{\mathcal{L}_2(\mathbb{R}^2)} + \|u_{xx}\|_{\mathcal{L}_2(\mathbb{R}^2)} \le N\|au_{xx} + u_t - \lambda u\|_{\mathcal{L}_2(\mathbb{R}^2)},$$

where N depends only on the bounds of a and a^{-1}.

2. Sobolev spaces $W_p^{r,k+2r}(\Omega)$

In this section $p \in [1,\infty)$, $k \in \{0,1,2,...\}$, Ω is a domain in $\mathbb{R}^{d+1}$, and unless explicitly stated otherwise

$$r = 0 \quad \text{or} \quad r = 1.$$

1. Definition. For functions $u \in C^{r,k+2r}(\bar{\Omega})$ define

$$\|u\|_{W_p^{r,k+2r}(\Omega)} = \sum_{\substack{|\alpha|+2\rho\le k+2r,\\ \rho\le r}} \|D^\alpha \partial_t^\rho u\|_{\mathcal{L}_p(\Omega)}. \tag{1}$$

For a function $u \in \mathcal{L}_p(\Omega)$ we write $u \in W_p^{r,k+2r}(\Omega)$ if there exists a sequence $u^j \in C^{r,k+2r}(\bar{\Omega})$ such that $\|u^j - u\|_{\mathcal{L}_p(\Omega)} \to 0$,

$$\|u^j\|_{W_p^{r,k+2r}(\Omega)} < \infty, \quad \|D^\alpha\partial_t^\rho u^j - D^\alpha\partial_t^\rho u^m\|_{\mathcal{L}_p(\Omega)} \to 0$$

as $j, m \to \infty$ for all multi-indices α and integers ρ satisfying

$$|\alpha| + 2\rho \le k + 2r, \quad \rho \le r. \tag{2}$$

We call any such sequence u^j *a defining sequence for* u. Define $W_p^{r,k+2r} = W_p^{r,k+2r}(\mathbb{R}^{d+1})$. The spaces $W_p^{r,k+2r}(\Omega)$ are called *parabolic Sobolev spaces*.

As before for $u \in W_p^{r,k+2r}(\Omega)$ the Sobolev derivatives $D^\alpha\partial_t^\rho u$ are introduced, the space provided with the norm (1) becomes a Banach space, and the set $C_0^{r,k+2r}(\bar{\mathbb{R}}_T^{d+1})$ turns out to be dense in $W_p^{r,k+2r}(\mathbb{R}_T^{d+1})$. Many other properties of $W_p^k(\Omega)$ also translate almost literally to $W_p^{r,k+2r}(\Omega)$. We will be using them without detailed proofs.

Informally, $k+2r$ is the number of (Sobolev) derivatives the elements of $W_p^{r,k+2r}(\Omega)$ possess provided that each derivative in t counts as two derivatives in x and we only allow zero or one derivative in t (see condition (2)).

2. Exercise*. Assume (2) and prove that the operator $D^\alpha\partial_t^\rho$ continuously maps $W_p^{r,k+2r}$ into $W_p^{r-\rho,k+2r-|\alpha|-2\rho}$.

3. Exercise*. Prove that $u \in W_p^{r,k+2r}(\mathbb{R}_T^{d+1})$ if and only if its Sobolev derivatives $D^\alpha\partial_t^\rho u$ are in $\mathcal{L}_p(\mathbb{R}_T^{d+1})$ whenever (2) holds.

4. Exercise*. Prove that $u \in W_p^{r,k+2r}(\mathbb{R}_0^{d+1})$ if and only if for the function $v(t,x) := u(|t|,x)$ we have $v \in W_p^{r,k+2r}$.

5. Exercise*. Prove that $u \in W_p^{1,k+2}(\mathbb{R}_T^{d+1})$ if and only if $u_t, u_{xx}, u \in W_p^{0,k}(\mathbb{R}_T^{d+1})$.

The spaces $W_p^{r,k+2r}$ are called anisotropic because a different number of derivatives with respect to different variables is involved. Here is an embedding theorem for some such spaces. This theorem will be used later only for proving Corollary 3.4.

6. Theorem. *Let $r,k \in \{1,2,...\}$, $\rho,\chi \in \{0,1,2,...\}$, $u \in \mathcal{L}_2(\mathbb{R}^{d+1})$. Assume that the Sobolev derivatives $\partial_t^r u, D_i^k u$, $i = 1,...,d$, are in $\mathcal{L}_2(\mathbb{R}^{d+1})$. Also assume that*

$$\frac{2\chi+d}{2k}+\frac{2\rho+1}{2r}<1. \tag{3}$$

Then the usual derivatives $\partial_t^\rho D^\chi u$ exist, are bounded and continuous, and

$$\sup_{\mathbb{R}^{d+1}} |\partial_t^\rho D^\chi u| \le N(\|u\|_{\mathcal{L}_2(\mathbb{R}^{d+1})} + \|\partial_t^r u\|_{\mathcal{L}_2(\mathbb{R}^{d+1})} + \|D^k u\|_{\mathcal{L}_2(\mathbb{R}^{d+1})}) =: J,$$

where N is independent of u.

Proof. Denote by $\tilde{u}(\tau,\xi)$ the Fourier transform of u with respect to (t,x). Then (a.e.)

$$u(t,x) = c_{d+1}\int_{\mathbb{R}^{d+1}} e^{i(\xi\cdot x+\tau t)}\tilde{u}(\tau,\xi)\,d\xi d\tau. \tag{4}$$

It turns out that the right-hand side of (4) is a continuous function of (t,x). Owing to the dominated convergence theorem, to prove this, it suffices to prove that $|\tilde{u}(\tau,\xi)|$ is integrable. Since we need a stronger fact below, we are going to prove that

$$I(\chi,\rho) := \int_{\mathbb{R}^{d+1}} |\xi|^\chi|\tau|^\rho|\tilde{u}(\tau,\xi)|\,d\xi d\tau \le J.$$

By Hölder's inequality

$$I(\chi,\rho) \le I_1^{1/2} I_2^{1/2},$$

where

$$I_1 = \int_{\mathbb{R}^{d+1}} (|\tau|^{2r} + |\xi|^{2k} + 1)|\tilde{u}(\tau,\xi)|^2\,d\xi d\tau$$

is dominated by J by Exercise 1.8.3 and

$$I_2 = \int_{\mathbb{R}^{d+1}} \frac{|\xi|^{2\chi}|\tau|^{2\rho}}{|\tau|^{2r} + |\xi|^{2k} + 1}\, d\xi d\tau = N \int_0^\infty \Big(\int_0^\infty \frac{r^\theta \tau^{2\rho}}{\tau^{2r} + r^{2k} + 1}\, dr \Big)\, d\tau,$$

and $\theta = 2\chi + d - 1$. We split the domain of integration with respect to τ in the last integral into two parts: over $(0, 1)$ and over $(1, \infty)$. The first domain is bounded and on it

$$\frac{r^\theta \tau^{2\rho}}{\tau^{2r} + r^{2k} + 1} \le \frac{r^\theta}{r^{2k} + 1},$$

which has finite integral over $(0, \infty)$ since owing to (3) we have

$$\theta - 2k = 2\chi + d - 2k - 1 < -1.$$

The remaining part is dealt with by observing that

$$\int_1^\infty \Big(\int_0^\infty \frac{r^\theta \tau^{2\rho}}{\tau^{2r} + r^{2k} + 1}\, dr \Big)\, d\tau \le \int_1^\infty \Big(\int_0^\infty \frac{r^\theta \tau^{2\rho}}{\tau^{2r} + r^{2k}}\, dr \Big)\, d\tau,$$

where the last expression after changing variables $r = s\tau^{r/k}$ turns out to be

$$\int_1^\infty \tau^{r/k - 2r + \theta r/k + 2\rho} \Big(\int_0^\infty \frac{s^\theta}{1 + s^{2k}}\, ds \Big)\, d\tau = N \int_1^\infty \tau^{r/k - 2r + \theta r/k + 2\rho}\, d\tau.$$

One easily checks that (3) is equivalent to $r/k - 2r + \theta r/k + 2\rho < -1$. Therefore, the last integral is finite, $I_2 < \infty$ and combining this with the estimate of I_1, we see that indeed $I(\chi, \rho) \le J$.

Now coming back to (4) and applying the above result with $\chi = \rho = 0$, we see that u is continuous and $|u| \le J$. If $\chi \ge 1$, since $I(1, 0) \le J$, by the dominated convergence theorem we can differentiate the right-hand side of (4) with respect to x and see that u_x is continuous and $|u_x| \le J$. By induction we conclude that all derivatives $D^\chi u$ exist, are continuous, and are dominated by J. Then we differentiate the formula expressing $D^\chi u$ through the Fourier transform, which for $|\alpha| \le \chi$ yields

$$\partial_t^\rho D^\alpha u(t, x) = i^{|\alpha| + m} c_{d+1} \int_{\mathbb{R}^{d+1}} e^{i(\xi \cdot x + \tau t)} \xi^\alpha \tau^\rho \tilde{u}(\tau, \xi)\, d\xi d\tau$$

and the result follows. The theorem is proved.

7. Exercise. Let $r, k \in \{1, 2, \ldots\}$, $\rho, \chi \in \{0, 1, 2, \ldots\}$, $u \in \mathcal{L}_2(\mathbb{R}^{d+1})$. Assume that the Sobolev derivatives $\partial_t^r u, D^k u$ are in $\mathcal{L}_2(\mathbb{R}^{d+1})$. Prove that if $\rho/r + \chi/k \le 1$, then the Sobolev derivatives $\partial_t^\rho D^\chi u \in \mathcal{L}_2(\mathbb{R}^{d+1})$.

As in a similar situation in Section 1.3, one derives from Lemma 1.1 and Theorem 1.6 the validity of the following result for $T = -\infty$.

8. Theorem. *Let $\lambda > 0$. Then for any $f \in \mathcal{L}_2(\mathbb{R}_T^{d+1})$ there exists a unique $u \in W_2^{1,2}(\mathbb{R}_T^{d+1})$ such that*

$$u_t + \Delta u - \lambda u = f.$$

Furthermore, for any $u \in W_2^{1,2}(\mathbb{R}_T^{d+1})$ it holds that

$$\|u_t\|_{\mathcal{L}_2(\mathbb{R}_T^{d+1})} + \|u_{xx}\|_{\mathcal{L}_2(\mathbb{R}_T^{d+1})} + \lambda\|u\|_{\mathcal{L}_2(\mathbb{R}_T^{d+1})} \leq N\|u_t + \Delta u - \lambda u\|_{\mathcal{L}_2(\mathbb{R}_T^{d+1})}, \tag{5}$$

where N is independent of λ and u.

Proof. For $T > -\infty$ estimate (5) follows from Lemma 1.1 and the definition of $W_2^{1,2}(\mathbb{R}_T^{d+1})$. This estimate implies uniqueness of solutions in $W_2^{1,2}(\mathbb{R}_T^{d+1})$. To prove existence, it suffices to solve the equation in $W_2^{1,2}$ with $fI_{t>T}$ in place of f and observe that the restrictions to $\{t \geq T\}$ of solutions from $W_2^{1,2}$ are obviously in $W_2^{1,2}(\mathbb{R}_T^{d+1})$. The theorem is proved.

9. Remark. In (5) and on a few similar occasions in the future we drop the term

$$\sqrt{\lambda}\|u_x\|_{\mathcal{L}_2(\mathbb{R}_T^{d+1})}$$

on the left only for brevity. This term can be recovered from (5) by using multiplicative inequalities.

10. Remark. The solvability of equations

$$u_t + Lu - \lambda u = f \tag{6}$$

in $W_2^{1,2}(\mathbb{R}_T^{d+1})$ will be derived below from Theorem 8 and the method of continuity. We need to use this method even if $T = -\infty$ because, so far, we do not know that $(\partial_t + L - \lambda)C_0^\infty(\mathbb{R}^{d+1})$ is dense in $\mathcal{L}_2(\mathbb{R}^{d+1})$.

Actually, we will prove the solvability of (6) in $W_2^{1,k+2}(\mathbb{R}_T^{d+1})$ and for that we need an integral representation of solutions from Remark 1.3. One particular case of it will be of special interest. Take the function p from (1.7.1) and define an operator $\mathcal{G}_\lambda^0$ by

$$\mathcal{G}_\lambda^0 f(t,x) = \int_0^\infty \int_{\mathbb{R}^d} f(t+s, x+y)e^{-\lambda s}p(s,y)\,dyds. \tag{7}$$

This is a particular case of the operator $\mathcal{G}_\lambda$ introduced in Remark 1.3 and corresponding to the operator L from (1.2) with $a^{ij} = \delta^{ij}$.

11. Lemma. *Let $\lambda > 0$ and $p \in [1,\infty)$. We assert the following.*

(i) The operator $\mathcal{G}_\lambda$ is a bounded operator acting from $\mathcal{L}_p(\mathbb{R}_T^{d+1})$ into $\mathcal{L}_p(\mathbb{R}_T^{d+1})$ and its norm is not greater than λ^{-1}.

(ii) For any $u \in W_p^{1,2}(\mathbb{R}_T^{d+1})$ we have

$$u = \mathcal{G}_\lambda(\lambda u - Lu - u_t). \tag{8}$$

(iii) If $k \geq 0$ and $f \in W_2^{0,k}(\mathbb{R}_T^{d+1})$, then $\mathcal{G}_\lambda^0 f \in W_2^{1,k+2}(\mathbb{R}_T^{d+1})$ and

$$\lambda\mathcal{G}_\lambda^0 f - \Delta\mathcal{G}_\lambda^0 f - \partial_t\mathcal{G}_\lambda^0 f = f \quad in \quad \mathbb{R}_T^{d+1} \text{ (a.e.)}.$$

Proof. Assertion (i) is contained in Remark 1.5

By (i) both parts of (8) are continuous $\mathcal{L}_p(\mathbb{R}_T^{d+1})$-valued functions on $W_p^{1,2}(\mathbb{R}_T^{d+1})$. Therefore, (8) follows from the fact that it holds on $C_0^{1,2}(\mathbb{R}_T^{d+1})$ by Remark 1.3 and $C_0^{1,2}(\mathbb{R}_T^{d+1})$ is dense in $W_p^{1,2}(\mathbb{R}_T^{d+1})$. This proves (ii).

(iii) By Theorem 8, in $W_2^{1,2}(\mathbb{R}_T^{d+1})$ there exists a solution of

$$\lambda u - \Delta u - u_t = f$$

and by (ii) this is $\mathcal{G}_\lambda^0 f$. This proves (iii) for $k = 0$ and also proves that

$$\|\mathcal{G}_\lambda^0 f\|_{W_2^{1,2}(\mathbb{R}_T^{d+1})} \leq N\|f\|_{\mathcal{L}_2(\mathbb{R}_T^{d+1})}, \tag{9}$$

where N is independent of f.

The rules of differentiating integrals show that if $f \in C_0^{0,k}(\mathbb{R}_T^{d+1})$, then $\mathcal{G}_\lambda^0 f \in C^{0,k}(\mathbb{R}_T^{d+1})$ and for any multi-index α with $|\alpha| \leq k$

$$D^\alpha\mathcal{G}_\lambda^0 f = \mathcal{G}_\lambda^0 D^\alpha f.$$

This and (9) imply that for $f \in C_0^{0,k}(\mathbb{R}_T^{d+1})$ we have

$$\|\mathcal{G}_\lambda^0 f\|_{W_2^{1,k+2}(\mathbb{R}_T^{d+1})} \leq N\|f\|_{W_2^{0,k}(\mathbb{R}_T^{d+1})}$$

where N is independent of f. By using the completeness of $W_2^{1,k+2}(\mathbb{R}_T^{d+1})$, one easily concludes from the above estimate that indeed $\mathcal{G}_\lambda^0 f \in W_2^{1,k+2}(\mathbb{R}_T^{d+1})$ for any $f \in W_2^{0,k}(\mathbb{R}_T^{d+1})$. The lemma is proved.

12. Theorem. *Let $\lambda > 0$ and let L be the operator from Section 1, $k \geq 0$, $f \in W_2^{0,k}(\mathbb{R}_T^{d+1})$. Then there exists a unique solution $u \in W_2^{1,k+2}(\mathbb{R}_T^{d+1})$ of equation (1.1). Furthermore, for any $u \in W_2^{1,k+2}(\mathbb{R}_T^{d+1})$ and $\lambda \geq 0$ we have*

$$\|u_t\|_{W_2^{0,k}(\mathbb{R}_T^{d+1})} + \|u_{xx}\|_{W_2^{0,k}(\mathbb{R}_T^{d+1})} + \lambda\|u\|_{W_2^{0,k}(\mathbb{R}_T^{d+1})}$$

$$\leq N(d,\kappa,k)\|u_t + Lu - \lambda u\|_{W_2^{0,k}(\mathbb{R}_T^{d+1})}. \tag{10}$$

Proof. Estimate (10) follows immediately for $\lambda > 0$ from Lemma 1.1 and the definition of $W_2^{0,k}(\mathbb{R}_T^{d+1})$. For $\lambda = 0$ we get it by continuity.

If $L = \Delta$, the first assertion is part of Lemma 11. For arbitrary L it only remains to use the method of continuity and to consider the family of operators $\tau L + (1-\tau)\Delta$, $\tau \in [0,1]$. The theorem is proved.

13. Exercise (One-dimensional case). In the setting of Exercise 1.11 fix a $\lambda > 0$ and prove that for any $f \in \mathcal{L}_2(\mathbb{R}_T^2)$ there exists a unique $u \in W_2^{1,2}(\mathbb{R}_T^2)$ satisfying

$$au_{xx} + u_t - \lambda u = f \quad \text{in} \quad \mathbb{R}_T^2.$$

3. Parabolic equations with continuous coefficients

Here we are considering the equation

$$u_t + Lu - \lambda u = f \tag{1}$$

in $\mathbb{R}_T^{d+1}$, where $T \in [-\infty, \infty)$ and

$$Lu(t,x) = a^{ij}(t,x)u_{x^ix^j}(t,x) + b^i(t,x)u_{x^i}(t,x) + c(t,x)u(t,x).$$

We assume that the coefficients a, b, and c are measurable functions of $(t,x) \in \mathbb{R}^{d+1}$ and for some constants $\kappa > 0$ and $K \in (0,\infty)$ for all values of the arguments and $\xi \in \mathbb{R}^d$ it holds that

$$|b| + |c| \leq K, \quad c \leq 0, \quad \kappa^{-1}|\xi|^2 \geq a^{ij}\xi^i\xi^j \geq \kappa|\xi|^2,$$

where $b = (b^1, ..., b^d)$. We also assume that there exists an increasing function $\omega(\varepsilon)$, $\varepsilon \geq 0$, such that $\omega(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ and for all $t \in \mathbb{R}$, $x, y \in \mathbb{R}^d$, and $i,j = 1, ..., d$ we have

$$|a^{ij}(t,x) - a^{ij}(t,y)| \leq \omega(|x-y|).$$

1. Theorem. *There exist constants $\lambda_0 \geq 1$ and N_0 depending only on K, κ, ω, and d such that the estimate*

$$\lambda \|u\|_{\mathcal{L}_2(\mathbb{R}^{d+1}_T)} + \lambda^{1/2}\|u_x\|_{\mathcal{L}_2(\mathbb{R}^{d+1}_T)} + \|u_{xx}\|_{\mathcal{L}_2(\mathbb{R}^{d+1}_T)} + \|u_t\|_{\mathcal{L}_2(\mathbb{R}^{d+1}_T)}$$

$$\leq N_0\|Lu + u_t - \lambda u\|_{\mathcal{L}_2(\mathbb{R}^{d+1}_T)}$$

holds for any $\lambda \geq \lambda_0$ and $u \in W^{1,2}_2(\mathbb{R}^{d+1}_T)$. Furthermore, for any $\lambda \geq \lambda_0$ and $f \in \mathcal{L}_2(\mathbb{R}^{d+1}_T)$ there exists a unique $u \in W^{1,2}_2(\mathbb{R}^{d+1}_T)$ satisfying (1) in $\mathbb{R}^{d+1}_T$.

The proof of this theorem is just a repetition of the proof of Theorem 1.6.4 for elliptic equations. Of course, while "freezing" the coefficients, we only freeze the variable x and not t, so that we start with the formulas

$$u^2_{xx}(t,x) = \int_{\mathbb{R}^d} u^2_{xx}(t,x)\zeta^2(x-y)\,dy, \quad u^2_t(t,x) = \int_{\mathbb{R}^d} u^2_t(t,x)\zeta^2(x-y)\,dy.$$

Next we derive a "global regularity result".

2. Theorem. *Let $k \geq 1$ be an integer and assume that, for each t, the coefficients $a(t,\cdot), b(t,\cdot), c(t,\cdot)$ are in C^k and their norms in C^k are bounded by a constant K_1.*

Then for the constant λ_0 from Theorem 1 and $N = N(k, d, K, K_1, \kappa, \omega)$ the estimate

$$\lambda \|u\|_{W^{0,k}_2(\mathbb{R}^{d+1}_T)} + \lambda^{1/2}\|u_x\|_{W^{0,k}_2(\mathbb{R}^{d+1}_T)} + \|u_{xx}\|_{W^{0,k}_2(\mathbb{R}^{d+1}_T)} + \|u_t\|_{W^{0,k}_2(\mathbb{R}^{d+1}_T)}$$

$$\leq N\|Lu + u_t - \lambda u\|_{W^{0,k}_2(\mathbb{R}^{d+1}_T)} \tag{2}$$

holds for any $\lambda \geq \lambda_0$ and $u \in W^{1,k+2}_2(\mathbb{R}^{d+1}_T)$. Furthermore, for any $\lambda \geq \lambda_0$ and $f \in W^{0,k}_2(\mathbb{R}^{d+1}_T)$ there exists a unique $u \in W^{1,k+2}_2(\mathbb{R}^{d+1}_T)$ satisfying (1) in $\mathbb{R}^{d+1}_T$.

Proof. We can use the method of continuity starting with the operator $\Delta + \partial_t - \lambda$ and using Lemma 2.11. Then we see that we need only prove the a priori estimate (2). As usual we may assume that $u \in C^{1,k+2}_0(\bar{\mathbb{R}}^{d+1}_T)$.

After that it only suffices to repeat the proof of Theorem 1.7.5 bearing on elliptic equations and to obtain (2) after differentiating the relation $Lu + u_t - \lambda u =: f$ with respect to x. The theorem is proved.

Here is a counterpart of Corollary 1.7.6.

3. Corollary. *Under the assumptions of Theorem 2 take a function $u \in W_2^{1,2}(\mathbb{R}_T^{d+1})$, $\lambda \geq 0$, set $f := Lu + u_t - \lambda u$, and assume that $f \in W_2^{0,k}(\mathbb{R}_T^{d+1})$. Then $u \in W_2^{1,k+2}(\mathbb{R}_T^{d+1})$ and*

$$\lambda \|u\|_{W_2^{0,k}(\mathbb{R}_T^{d+1})} + \|u\|_{W_2^{1,k+2}(\mathbb{R}_T^{d+1})} \leq N(\|f\|_{W_2^{0,k}(\mathbb{R}_T^{d+1})} + \|u\|_{\mathcal{L}_2(\mathbb{R}_T^{d+1})}), \tag{3}$$

where N depends only on k, d, K, K_1, κ, and ω

Indeed, denote $\lambda_1 = \lambda_0 + \lambda$ and

$$g := Lu + u_t - \lambda_1 u = f - \lambda_0 u. \tag{4}$$

If, for an $r \in \{0, ..., k-1\}$, we have $u \in W_2^{1,r+2}(\mathbb{R}_T^{d+1})$, then $g \in W_2^{1,r+1}(\mathbb{R}_T^{d+1})$ and by Theorem 2 equation (4) has a solution in $W_2^{1,(r+1)+2}(\mathbb{R}_T^{d+1})$, which is unique in $W_2^{1,2}(\mathbb{R}_T^{d+1})$. Since u is a solution of class $W_2^{1,2}(\mathbb{R}_T^{d+1})$, it follows that $u \in W_2^{1,r+3}(\mathbb{R}_T^{d+1})$. An obvious induction on r proves that $u \in W_2^{1,k+2}(\mathbb{R}_T^{d+1})$.

Next, for $r = 1, ..., k$ by Theorem 2 we obtain

$$\lambda \|u\|_{W_2^{0,r}(\mathbb{R}_T^{d+1})} + \|u\|_{W_2^{1,r+2}(\mathbb{R}_T^{d+1})} \leq \lambda_1 \|u\|_{W_2^{0,r}(\mathbb{R}_T^{d+1})} + \|u\|_{W_2^{1,r+2}(\mathbb{R}_T^{d+1})}$$

$$\leq N\|g\|_{W_2^{0,r}(\mathbb{R}_T^{d+1})} \leq N(\|f\|_{W_2^{0,k}(\mathbb{R}_T^{d+1})} + \|u\|_{W_2^{1,(r-1)+2}(\mathbb{R}_T^{d+1})})$$

and again the induction on r finishes the argument. Of course, at the last induction step, as in Corollary 1.7.6, one uses a parabolic counterpart of Theorem 1.6.5 which the reader can easily state and prove.

In the following corollary, which is not used later, we do not use the parameter λ because we can always call $c - \lambda$ a new c. We discuss the regularity of u in both x and t.

4. Corollary. *Take a function $u \in W_2^{1,2}$, an integer $k \geq 0$, set $f := Lu + u_t$ and assume that*

$$D^\alpha \partial_t^\rho f \in \mathcal{L}_2(\mathbb{R}^{d+1}), \quad \forall \alpha, \rho : |\alpha| + 2\rho \leq 2k. \tag{5}$$

Also assume that

$$D^\alpha \partial_t^\rho (a, b, c) \in C(\mathbb{R}^{d+1}), \quad \forall \alpha, \rho : |\alpha| + 2\rho \leq 2k. \tag{6}$$

Then

$$D^\alpha \partial_t^\rho u \in \mathcal{L}_2(\mathbb{R}^{d+1}), \quad \forall \alpha, \rho : |\alpha| + 2\rho \leq 2(k + 1). \tag{7}$$

In particular, if all derivatives of f are in $\mathcal{L}_2(\mathbb{R}^{d+1})$ and each derivative of a, b, c is bounded and continuous, then u is infinitely differentiable and every derivative of u is in $\mathcal{L}_2(\mathbb{R}^{d+1}) \cap C(\mathbb{R}^{d+1})$.

Proof. The first assertion is trivial if $k = 0$. If we know that it holds for $k = 0, ..., n$ and (5) and (6) are satisfied with $n + 1$ in place of k, then by Corollary 3 the fact that (5) and (6) hold with $\rho = 0$ and $k = n + 1$ implies that (7) with $k = n + 1$ is true if we restrict ρ there to be 0 or 1, that is,

$$D^\alpha u \in \mathcal{L}_2(\mathbb{R}^{d+1}), \quad \forall\alpha : |\alpha| \leq 2(n+1) + 2, \tag{8}$$

$$D^\alpha u_t \in \mathcal{L}_2(\mathbb{R}^{d+1}), \quad \forall\alpha : |\alpha| \leq 2(n+1). \tag{9}$$

Then set $v = u_t$ and observe that $v = f - Lu$, where $f_t \in \mathcal{L}_2(\mathbb{R}^{d+1})$ since (5) holds with $k = n + 1 \geq 1$ and

$$\partial_t Lu \in \mathcal{L}_2(\mathbb{R}^{d+1})$$

because of (9) and (6) with $k = n + 1$. Thus we can differentiate through $v = f - Lu$ with respect to t to conclude that

$$v_t + Lv = g, \tag{10}$$

where

$$g = f_t - a^{ij}_t u_{x^i x^j} - b^i_t u_{x^i} - c_t u.$$

We want to apply our induction hypothesis to (10). Observe that, of course, (6) holds with $k = n$, so that the left-hand side of (10) is suitable for applying the hypothesis. Furthermore, if

$$|\alpha| + 2\rho \leq 2n, \tag{11}$$

then $|\alpha| + 2(\rho + 1) \leq 2(n + 1)$ and by assumption

$$D^\alpha \partial^\rho_t f_t = D^\alpha \partial^{\rho+1}_t f \in \mathcal{L}_2(\mathbb{R}^{d+1}).$$

Again if (11) holds, then for some constants $c^{r\beta\gamma}$

$$D^\alpha \partial^\rho_t (a^{ij}_t u_{x^i x^j}) = \sum_{r \leq \rho, \beta+\gamma=\alpha} c^{r\beta\gamma} (D^\beta \partial^r_t a^{ij}_t) D^\gamma \partial^{\rho-r}_t u_{x^i x^j}.$$

Here the $D^\beta \partial^r_t a^{ij}_t = D^\beta \partial^{r+1}_t a^{ij}$ are bounded since

$$|\beta| + 2(r+1) \leq |\alpha| + 2(\rho + 1) \leq 2(n + 1)$$

and (6) has been supposed to hold for $k = n+1$. Also, for some δ such that $|\delta| = |\gamma| + 2$ we have

$$D^\gamma \partial_t^{\rho-r} u_{x^i x^j} = D^\delta \partial_t^{\rho-r} u,$$

where

$$|\delta| + 2(\rho - r) \le |\gamma| + 2 + 2\rho \le 2n + 2.$$

By the induction hypothesis $D^\gamma \partial_t^{\rho-r} u_{x^i x^j} \in \mathcal{L}_2(\mathbb{R}^{d+1})$. Similarly,

$$D^\alpha \partial_t^\rho (b_t^i u_{x^i}), D^\alpha \partial_t^\rho (c_t u) \in \mathcal{L}_2(\mathbb{R}^{d+1}).$$

It follows that $D^\alpha \partial_t^\rho g$ is in $\mathcal{L}_2(\mathbb{R}^{d+1})$ whenever α and ρ satisfy (11). By the induction hypothesis

$$D^\alpha \partial_t^\rho v \in \mathcal{L}_2(\mathbb{R}^{d+1}), \quad D^\alpha \partial_t^{\rho+1} u \in \mathcal{L}_2(\mathbb{R}^{d+1})$$

as long as $|\alpha| + 2\rho \le 2(n+1)$, that is,

$$|\alpha| + 2(\rho + 1) \le 2(n+2).$$

This proves all inclusions in (7) with $k = n+1$ apart from the ones with $\rho = 0$. However, these we have already obtained in (8). This shows that the first assertion of the corollary follows by induction.

The second assertion follows directly from the first one and Theorem 2.6. The corollary is proved.

4. Local or interior estimates

Here we discuss "local" versions of the regularity results for parabolic equations. The corresponding results for elliptic equations are obtained by taking u and the coefficients of L independent of t. We take an operator L as in Section 3.

We set $B_r(x)$ to be the open ball in $\mathbb{R}^d$ of radius r centered at x,

$$B_r = B_r(0), \quad Q_r(t,x) = (t, t+r^2) \times B_r(x), \quad Q_r = Q_r(0,0).$$

We continue to use Theorem 3.2. If u and the coefficients of L are independent of t (elliptic case), the following corollary is part of Theorem 1.7.7.

1. Corollary. *Let the assumptions of Theorem 3.2 be satisfied and let* $r \in (0, \infty)$ *be a fixed number. Take a function* u *such that* $u \in W_2^{1,2}(Q_s)$ *for all* $s \in (0, r)$ *and assume that*

$$f := Lu + u_t \in W_2^{0,k}(Q_s)$$

for all $s \in (0, r)$. *Then* $u \in W_2^{1,k+2}(Q_s)$ *for any* $s \in (0, r)$.

Indeed, if $k = 0$, the assertion is trivial and we may assume that $k \geq 1$. In that case take an $s < r$ and a $\zeta \in C_0^\infty(\mathbb{R}^{d+1})$ such that $\zeta = 1$ on Q_s and $\zeta(t, x) = 0$ if $t \geq 0$ and $(t, x) \notin Q_{s_1}$, where $s_1 = s + (r - s)/2$. Then

$$(L + \partial_t)(\zeta u) = \zeta f + u(L\zeta + \zeta_t - c\zeta) + 2a^{ij}\zeta_{x^i}u_{x^j} =: g.$$

Since the first derivatives in x of the coefficients are bounded and $u \in W_2^{0,2}(Q_{s_1})$, we have that $g \in W_2^{0,1}(\mathbb{R}_0^{d+1})$. Also $\zeta u \in W_2^{1,2}(\mathbb{R}_0^{d+1})$. It follows by Corollary 3.3 that $\zeta u \in W_2^{1,3}(\mathbb{R}_0^{d+1})$. The arbitrariness of ζ now allows us to conclude that $u \in W_2^{1,3}(Q_s)$ for any $s \in (0, r)$. If $k \geq 2$, then $g \in W_2^{0,2}(\mathbb{R}_0^{d+1})$ and we keep going as above.

2. Remark. For Corollary 1 to be true we do not need any smoothness of the coefficients of L outside Q_r. It suffices to assume that for $t \in (0, r^2)$ we have $a(t, \cdot), b(t, \cdot), c(t, \cdot) \in C^k(B_r)$ and that the norms of $a(t, \cdot), b(t, \cdot), c(t, \cdot)$ in $C^k(B_r)$ are bounded for $t \in (0, r^2)$.

Indeed, observe that the result is local in the sense that it suffices to have it true for any $R < r$ in place of r. Now fix an $R < r$ and find an $\mathbb{R}^d$-valued C_0^∞ function $\pi(x)$ with range belonging to B_r and such that $\pi(x) = x$ if $|x| \leq R$ and redefine a, b, c using the formula $g \to g(t, \pi(x))$. Then the new coefficients will satisfy the assumption in Corollary 1 with R in place of r and hence its assertion also holds with R in place of r. This is enough, as we have pointed out above.

Recall that the way the local versions of function spaces are defined is described before Exercise 1.3.6.

3. Corollary. *Let* $r \in (0, \infty)$ *be a fixed number and assume that* $a, b, c \in C_{loc}^\infty(Q_r)$. *Take a function* u *such that* $u \in W_2^{1,2}(Q_r)$ *and assume that*

$$f := Lu + u_t \in C_{loc}^\infty(Q_r).$$

Then $u \in C_{loc}^\infty(Q_r)$.

Indeed, by Corollary 1, applied to cylinders with closure lying inside Q_r, we have $u \in W_{2,loc}^{1,k+2}(Q_r)$ for any k. In particular, $u_t \in W_{2,loc}^{0,k}(Q_r)$ for any k.

If $\rho \in \{1, 2, \ldots\}$ and we already know that

$$\partial_t^m u \in W_{2,loc}^{0,k}(Q_r)$$

for any k and $m \le \rho$, then we can differentiate $\rho - 1$ times the equation $u_t = f - Lu$ with respect to t and find that

$$\partial_t^\rho u = \partial_t^{\rho-1} f + \sum_{m \le \rho-1} \alpha_m^{ij} \partial_t^m u_{x^i x^j} + \sum_{m \le \rho-1} \alpha_m^i \partial_t^m u_{x^i} + \sum_{m \le \rho-1} \alpha_m \partial_t^m u =: g,$$

and the $\alpha_m^{ij}, \alpha_m^i, \alpha_m$ are certain functions of class $C_{loc}^\infty(Q_r)$. By the induction hypotheses $g_t \in W_{2,loc}^{0,k}(Q_r)$ for any k, that is,

$$\partial_t^{\rho+1} u \in W_{2,loc}^{0,k}(Q_r)$$

for any k. Hence, by induction all derivatives of u with respect to (t, x) are in $\mathcal{L}_{2,loc}(Q_r)$ and for any $\zeta \in C_0^\infty(Q_r)$ all derivatives of ζu are in $\mathcal{L}_2(\mathbb{R}^{d+1})$. By Theorem 1.7.8 all derivatives of ζu are bounded and continuous in $\mathbb{R}^{d+1}$ and this is exactly what we claimed.

The following results present local or interior versions of (3.3) and are used in many places, in particular, in passing from $p = 2$ to arbitrary $p \in (1, \infty)$ and in investigating the interior smoothness of solutions of parabolic or elliptic equations in nonsmooth domains.

4. Lemma. *Let $0 < r < R < \infty$, $\lambda \ge 0$, $u \in W_2^{1,2}(Q_s)$ for all $s \in (0, R)$. Set $f := Lu + u_t - \lambda u$. Then*

$$\lambda \|u\|_{\mathcal{L}_2(Q_r)} + \|u\|_{W_2^{1,2}(Q_r)} \le N\big(\|f\|_{\mathcal{L}_2(Q_R)} + \{1 + (R-r)^{-2}\}\|u\|_{\mathcal{L}_2(Q_R)}\big), \tag{1}$$

where $N = N(d, K, \kappa, \omega)$.

Proof. Assume that we know that (1) is true with s in place of R whenever $s \in (r, R)$. Then letting $s \uparrow R$ would yield (1) as is. For each $s \in (r, R)$ we have $u \in W_2^{1,2}(Q_s)$. Therefore, we may concentrate on proving (1) under the additional assumption that $u \in W_2^{1,2}(Q_R)$. In that case for obvious reasons we may assume that $u \in C^{1,2}(\bar{Q}_R)$. Then, let $\chi(s)$ be an infinitely differentiable function on $\mathbb{R}$ such that $\chi(s) = 1$ for $s \le 0$ and $\chi(s) = 0$ for $s \ge 1$. For $m = 0, 1, 2, \ldots$ introduce $(r_0 = r)$

$$r_m = r + (R - r)\sum_{j=1}^m 2^{-j}, \quad \xi_m(x) = \chi\big(2^{m+1}(R-r)^{-1}(|x| - r_m)\big),$$

$$\eta_m(t) = \chi\big(2^{2m+2}(R-r)^{-2}(t - r_m^2)\big), \quad \zeta_m(t, x) = \xi_m(x)\eta_m(t).$$

As is easy to check, for

$$Q(m) = Q_{r_m} = (0, r_m^2) \times B_{r_m}$$

it holds that

$$\zeta_m = 1 \quad \text{on} \quad Q(m), \quad \zeta_m = 0 \quad \text{on} \quad Q_R \setminus Q(m+1).$$

Also (observe that $N2^{m+1} = N_1 2^m$ with $N_1 = 2N$)

$$|\zeta_{mx}| \leq N2^m(R-r)^{-1} \leq N(1 + 2^{2m}(R-r)^{-2}), \quad |\zeta_{mt}| \leq N2^{2m}(R-r)^{-2},$$

$$|\zeta_{mxx}| \leq N2^{2m}(R-r)^{-2}. \tag{2}$$

Next, the function $\zeta_m u$ is in $W_2^{1,2}(\mathbb{R}_0^{d+1})$ and satisfies

$$L(\zeta_m u) + (\zeta_m u)_t - \lambda\zeta_m u = \zeta_m f + u((L-c)\zeta_m + \zeta_{mt}) + 2a^{ij}\zeta_{mx^i}u_{x^j}.$$

By Corollary 3.3

$$A_m := \lambda\|\zeta_m u\|_{\mathcal{L}_2(Q(m+1))} + \|\zeta_m u\|_{W_2^{1,2}(Q(m+1))}$$

$$\leq N\|f\|_{\mathcal{L}_2(Q_R)} + NB + NC_m, \tag{3}$$

where

$$B = \|u\|_{\mathcal{L}_2(Q_R)}, \quad C_m = \|u((L-c)\zeta_m + \zeta_{mt}) + 2a^{ij}\zeta_{mx^i}u_{x^j}\|_{\mathcal{L}_2(Q_R)}.$$

Observe that by the above-mentioned properties of ζ_m

$$C_m \leq N(1 + 2^{2m}(R-r)^{-2})B + N2^m(R-r)^{-1}\|u_x\|_{\mathcal{L}_2(Q(m+1))},$$

where by interpolation inequalities for any $t \in (0, R^2)$, $\varepsilon > 0$, and $B(m) := B_{r_m}$

$$\|u_x(t,\cdot)\|^2_{\mathcal{L}_2(B(m+1))} \leq \|(\zeta_{m+1}u)_x(t,\cdot)\|^2_{\mathcal{L}_2}$$

$$\leq \varepsilon^2(R-r)^2 2^{-2m}\|\zeta_{m+1}u(t,\cdot)\|^2_{W_2^2(B(m+2))}$$

$$+ N\varepsilon^{-2}2^{2m}(R-r)^{-2}\|u(t,\cdot)\|^2_{\mathcal{L}_2(B_R)},$$

so that

$$\|u_x\|_{\mathcal{L}_2(Q(m+1))} \leq \varepsilon(R-r)2^{-m}A_{m+1} + N\varepsilon^{-1}2^m(R-r)^{-1}B.$$

Hence (3) yields for $\varepsilon \in (0,1)$ (with ε, perhaps, different from the one above)

$$A_m \le N\|f\|_{\mathcal{L}_2(Q_R)} + N(1+\varepsilon^{-1}2^{2m}(R-r)^{-2})B + \varepsilon A_{m+1}.$$

Now we take $\varepsilon = 1/8$ and get

$$\varepsilon^m A_m \le N\varepsilon^m\|f\|_{\mathcal{L}_2(Q_R)} + N\varepsilon^m(1+\varepsilon^{-1}2^{2m}(R-r)^{-2})B + \varepsilon^{m+1}A_{m+1},$$

$$A_0 + \sum_{m=1}^{\infty} \varepsilon^m A_m \le N\|f\|_{\mathcal{L}_2(Q_R)} + N(1+(R-r)^{-2})B + \sum_{m=1}^{\infty} \varepsilon^m A_m. \tag{4}$$

Here the series converges because owing to (2),

$$A_m \le N(1+4^m(R-r)^{-2})\|u\|_{W_2^{1,2}(Q_R)}.$$

Therefore upon cancelling like terms in (4), we see that A_0 is less than the right-hand side of (1). Since its left-hand side is obviously less than A_0, the lemma is proved.

5. Exercise*. Prove (2).

6. Exercise. In the situation of Lemma 4 assume that a is independent of x and $b = 0$ and $c = 0$. Then prove that

$$\lambda\|u\|_{\mathcal{L}_2(Q_r)} + \|u_{xx}\|_{\mathcal{L}_2(Q_r)} + \|u_t\|_{\mathcal{L}_2(Q_r)} \le N(d,\kappa)\big(\|f\|_{\mathcal{L}_2(Q_R)} + (R-r)^{-2}\|u\|_{\mathcal{L}_2(Q_R)}\big).$$

Under additional assumptions one can get a higher interior regularity. If u and the coefficients of L are independent of t (elliptic case), the following result is stronger than Theorem 1.7.7.

7. Theorem. *Let $0 < r < R < \infty$, $\lambda \ge 0$, $u \in W_2^{1,2}(Q_s)$ for all $s \in (0,R)$. Suppose that the assumptions of Theorem 3.2 are satisfied and*

$$f := Lu + u_t - \lambda u \in W_2^{0,k}(Q_s)$$

for all $s \in (0,R)$. Then $u \in W_2^{1,k+2}(Q_r)$ and

$$\lambda\|u\|_{W_2^{0,k}(Q_r)} + \|u\|_{W_2^{1,k+2}(Q_r)} \le N\big(\|f\|_{W_2^{0,k}(Q_R)} + \|u\|_{\mathcal{L}_2(Q_R)}\big), \tag{5}$$

where $N = N(r,R,k,d,K,K_1,\kappa,\omega)$.

Proof. By Corollary 1 we have $u \in W_2^{1,k+2}(Q_s)$ for any $s < R$. In the proof of (5) we use the induction on k. If $k = 0$, then (5) follows from Lemma 4. Assume that the theorem is true for $k = n \geq 0$ and its assumptions are satisfied for $k = n+1$. Then for $v = u_{x^1}$ (which is in $W_2^{1,n+2}(Q_s)$, $0 < s < R$, by the above) we have

$$Lv + v_t - \lambda v = f_{x^1} - a^{ij}_{x^1} u_{x^i x^j} - b^i_{x^1} u_{x^i} - c_{x^1} u =: g$$

and $g \in W_2^{0,n}(Q_s)$ for any $s \in (0, R)$. By the induction hypothesis, for $R_1 = r + (R - r)/4$,

$$\lambda \|v\|_{W_2^{0,n}(Q_r)} + \|v\|_{W_2^{1,n+2}(Q_r)} \leq N\big(\|f_{x^1}\|_{W_2^{0,n}(Q_R)} + \|u_{xx}\|_{W_2^{0,n}(Q_{R_1})}$$

$$+ \|u_x\|_{W_2^{0,n}(Q_{R_1})} + \|u\|_{W_2^{0,n}(Q_{R_1})}\big).$$

By the induction hypothesis,

$$\|u_{xx}\|_{W_2^{0,n}(Q_{R_1})} + \|u_x\|_{W_2^{0,n}(Q_{R_1})} + \|u\|_{W_2^{0,n}(Q_{R_1})}$$

$$\leq N\|u\|_{W_2^{1,n+2}(Q_{R_1})} \leq N\big(\|f\|_{W_2^{0,n}(Q_R)} + \|u\|_{\mathcal{L}_2(Q_R)}\big).$$

Hence

$$\lambda \|v\|_{W_2^{0,n}(Q_r)} + \|v\|_{W_2^{1,n+2}(Q_r)} \leq N\big(\|f\|_{W_2^{0,n+1}(Q_R)} + \|u\|_{\mathcal{L}_2(Q_R)}\big).$$

By repeating the proof for other first-order derivative of u, we conclude

$$\lambda \|u_x\|_{W_2^{0,n}(Q_r)} + \|u_x\|_{W_2^{1,n+2}(Q_r)} \leq N\big(\|f\|_{W_2^{0,n+1}(Q_R)} + \|u\|_{\mathcal{L}_2(Q_R)}\big). \tag{6}$$

Furthermore by Lemma 4

$$\lambda \|u\|_{\mathcal{L}_2(Q_r)} + \|u_t\|_{\mathcal{L}_2(Q_r)} \leq N\big(\|f\|_{W_2^{0,n+1}(Q_R)} + \|u\|_{\mathcal{L}_2(Q_R)}\big). \tag{7}$$

This and (6) yield the needed estimate for the first term on the left in (5) with $k = n + 1$.

Next, it is not hard to see that

$$\|u\|_{W_2^{1,n+3}(Q_r)} \leq N\big(\|u\|_{\mathcal{L}_2(Q_R)} + \|u_t\|_{\mathcal{L}_2(Q_r)} + \|u_x\|_{W_2^{1,n+2}(Q_r)}\big).$$

By combining this with (6) and (7), we come to the conclusion that (5) holds with $k = n + 1$. The theorem is proved.

Before we prove interior estimates for derivatives in the sup norm, we give the reader the following.

8. Exercise*. Assume that $p \in (1,\infty)$, $k \in \{0,1,2,...\}$, $u \in W_p^{1,k+2}(Q_R)$, and prove that the function of t given by

$$\|u(t,\cdot)\|_{W_p^k(B_R)}$$

is in $W_p^1((0,R^2))$ and

$$|\partial_t\|u(t,\cdot)\|_{W_p^k(B_R)}| \leq \|\partial_t u(t,\cdot)\|_{W_p^k(B_R)} \quad \text{(a.e.)}.$$

One of the main applications of the following theorem is when $f \equiv 0$.

9. Theorem. *Let $k, m \in \{0,1,2,...\}$ be such that $2(k-m) > d$ and let the assumptions of Theorem 3.2 be satisfied. Also let $0 < r < R < \infty$, $u \in W_2^{1,2}(Q_s)$ and*

$$f := Lu + u_t \in W_2^{0,k}(Q_s)$$

for any $s < R$. Then the derivatives $D^\alpha u$ with $|\alpha| \leq m$ are continuous in Q_r and

$$\sup_{Q_r,|\alpha|\leq m} |D^\alpha u| \leq N\big(\|f\|_{W_2^{0,k}(Q_R)} + \|u\|_{\mathcal{L}_2(Q_R)}\big), \tag{8}$$

where $N = N(k,K,K_1,d,\omega,r,R,\kappa)$. Furthermore, if $m \geq 2$ and the derivatives $D^\alpha f$ are bounded on Q_r for $|\alpha| \leq m-2$, then

$$\sup_{Q_r,|\alpha|\leq m-2} |D^\alpha u_t| \leq N\big(\sup_{Q_r,|\alpha|\leq m-2} |D^\alpha f| + \|f\|_{W_2^{0,k}(Q_R)} + \|u\|_{\mathcal{L}_2(Q_R)}\big),$$

where $N = N(k,K,K_1,d,\omega,r,R,\kappa)$.

Proof. The second assertion follows immediately from the first one and the fact that

$$D^\alpha u_t = D^\alpha f - D^\alpha Lu.$$

To prove the first assertion, observe that by Theorem 7 we have $u \in W_2^{1,k+2}(Q_s)$ for any $s < R$ and, obviously, if we prove our theorem with R replaced with a smaller number, we will also prove it as stated. Therefore, we assume that $u \in W_2^{1,k+2}(Q_R)$.

In that case, as usual, it suffices to concentrate on $u \in C^{1,k+2}(\bar{Q}_R)$ (cf. the proof of Theorem 1.7.8). By Exercises 8 and 1.7.11, for $r_1 = r+(R-r)/2$, we have

$$\sup_{[0,r^2]} \|u(t,\cdot)\|_{W_2^k(B_{r_1})} \leq N\big(\|u_t\|_{W_2^{0,k}(Q_{r_1})} + \|u\|_{W_2^{0,k}(Q_{r_1})}\big) \leq N\|u\|_{W_2^{1,k+2}(Q_{r_1})},$$

where the last expression is estimated by the right-hand side of (8) by Theorem 7. Using embedding theorems (see Remark 1.7.10) yields (8).

The theorem is proved.

10. Exercise. Let $\gamma \in [0, \infty)$, $u \in C^{\infty}_{loc}(\mathbb{R}^{d+1})$. Assume that u is a *caloric function* in $\mathbb{R}^{d+1}_0$, that is, $\Delta u + u_t = 0$ in $\mathbb{R}^{d+1}_0$, and assume that

$$\lim_{\rho \to \infty} \frac{1}{\rho^{d+2+2\gamma}} \int_{Q_\rho} |u|^2 \, dx dt = 0,$$

which holds, in particular, if $\gamma > 0$ and $|u(t, x)| \leq N(1 + |x| + \sqrt{t})^{\delta}$ with a constant N and $\delta \in [0, \gamma)$. Show that $\partial_t^n D^{\alpha} u = 0$ in $\mathbb{R}^{d+1}_0$ whenever $2n + |\alpha| \geq \gamma$, so that

$$u(t, x) = \sum_{n, \alpha} c^{n, \alpha} t^n x^{\alpha},$$

where $c^{n, \alpha} = 0$ if $2n + |\alpha| \geq \gamma$. In particular, (i) if $\gamma = 0$, then $u = 0$, and (ii) bounded or sublinear caloric functions in $\mathbb{R}^{d+1}_0$ are constant (take $\gamma = 1$).

Conclude that if a function $v = v(x)$ is *harmonic* in $\mathbb{R}^d$, that is, satisfies $\Delta v = 0$ in $\mathbb{R}^d$ and $|v(x)| \leq N(1 + |x|^{\delta})$, then $v(x)$ is a polynomial in x of degree at most δ. In particular, bounded harmonic functions are constant.

5. The Cauchy problem

We take $T \in (0, \infty)$, an operator L as in Section 3, and investigate solvability of the Cauchy problem

$$u_t = Lu + f \quad \text{in} \quad \Omega = \Omega_T := (0, T) \times \mathbb{R}^d \tag{1}$$

with initial condition

$$u(0, x) = g(x) \quad \text{in} \quad \mathbb{R}^d. \tag{2}$$

We will assume that $f \in \mathcal{L}_2(\Omega)$ and look for solutions $u \in W^{1,2}_2(\Omega)$. Such functions can be changed on a set of Lebesgue $\mathbb{R}^{d+1}$ measure zero and therefore, condition (2) does not make sense as is. However, for $u \in W^{1,2}_2(\Omega)$ one can uniquely (up to the almost everywhere sense in $\mathbb{R}^d$) define what $u(0, x)$ is as the trace of u on the hyperplane $\{0\} \times \mathbb{R}^d$ (see Section 11.7) and then one can make sense of (2). Nevertheless, then there is an issue of describing the set of g for each of which there is a $u \in W^{1,2}_2(\Omega)$ satisfying (2). This issue is not hard in the framework of $W^{1,2}_p(\Omega)$ spaces with $p = 2$ and the space of traces turns out to be W^1_2 (see Exercise 5). However, if $p \neq 2$, this space is the Slobodetskii space $W^{2-2/p}_p$ and studying it is beyond the scope of these lectures.

Since we want to construct an $\mathcal{L}_2$ theory in such a way that the $\mathcal{L}_p$ theory in later chapters could use the same arguments as in the case $p = 2$, we will adopt an implicit approach based on the observation that if one can make sense of (2) and the problem (1)–(2) has a solution $u \in W_2^{1,2}(\Omega)$, then there exists at least one $v \in W_2^{1,2}(\Omega)$ such that $v(0,x) = g(x)$, so that $u(0,x) = v(0,x)$ (for instance, $v \equiv u$). Therefore, we can replace (2) with

$$(u - v)(0, x) = 0.$$

Now the issue is to define what the condition $w(0,x) = 0$ means for a $W_2^{1,2}(\Omega)$ function.

We introduce the following definition in which

$$\mathbb{R}_T^{d+1,c} = (-\infty, T) \times \mathbb{R}^d.$$

1. Definition. For $T \in (0,\infty)$ and a function $u \in W_p^{1,2}(\Omega)$ we write

$$u \in \overset{0}{W}{}_p^{1,2}(\Omega)$$

if for the function v defined as $v(t,x) = u(t,x)$ for $t \in (0,T)$, $v(t,x) = 0$ for $t \leq 0$, we have $v \in W_p^{1,2}(\mathbb{R}_T^{d+1,c})$.

Equation (1) contains the derivative u_t with an opposite sign compared to the equation from Theorem 3.1. Therefore, it is convenient to adapt Theorem 3.1 to the new situation. The following result is obtained from Theorem 3.1 just by changing t to $-t$.

2. Theorem. *For the constants λ_0 and N from Theorem 3.1, any $\lambda \geq \lambda_0$ and $u \in W_2^{1,2}(\mathbb{R}_T^{d+1,c})$ we have*

$$\lambda \|u\|_{\mathcal{L}_2(\mathbb{R}_T^{d+1,c})} + \|u\|_{W_2^{1,2}(\mathbb{R}_T^{d+1,c})} \leq N \|Lu - u_t - \lambda u\|_{\mathcal{L}_2(\mathbb{R}_T^{d+1,c})}.$$

Furthermore, for any $\lambda \geq \lambda_0$ and $f \in \mathcal{L}_2(\mathbb{R}_T^{d+1,c})$ there exists a unique $u \in W_2^{1,2}(\mathbb{R}_T^{d+1,c})$ such that $Lu - u_t - \lambda u = f$ in $\mathbb{R}_T^{d+1,c}$.

Here is the existence and uniqueness theorem for the Cauchy problem.

3. Theorem. *For any $f \in \mathcal{L}_2(\Omega)$ and $v \in W_2^{1,2}(\Omega)$ there exists a unique $u \in W_2^{1,2}(\Omega)$ satisfying (1) and such that*

$$u - v \in \overset{0}{W}{}_2^{1,2}(\Omega). \tag{3}$$

Furthermore, there is a constant N, depending only on d, T, K, κ, and ω, such that

$$\|u\|_{W_2^{1,2}(\Omega)} \le N(\|f\|_{\mathcal{L}_2(\Omega)} + \|v\|_{W_2^{1,2}(\Omega)}). \tag{4}$$

Proof. Observe that $u \in W_2^{1,2}(\Omega)$ satisfies (1) and (3) if and only if $w := u - v$ is in $\overset{0}{W}{}_2^{1,2}(\Omega)$ and satisfies

$$w_t = Lw + g, \quad g := Lv - v_t + f.$$

Furthermore, if (4) holds with $w, g, 0$ in place of u, f, v, respectively, then it also holds as is. It follows that we may concentrate on the case where $v = 0$.

In that case take λ_0 from Theorem 3.1, set

$$f'(t,x) = -e^{-\lambda_0 t} f(t,x) I_{[0,T]}(t),$$

observe that

$$f' \in \mathcal{L}_2(\mathbb{R}_T^{d+1,c}),$$

and define $w \in W_p^{1,2}(\mathbb{R}_T^{d+1,c})$ as the unique solution of

$$Lw - w_t - \lambda_0 w = f'. \tag{5}$$

For $t \in (0,T)$ we also set

$$u(t,x) = e^{\lambda_0 t} w(t,x).$$

Then, obviously, $u \in W_p^{1,2}(\Omega)$ and since in light of (5) in Ω we have

$$Lw - e^{-\lambda_0 t}(e^{\lambda_0 t} w)_t = f',$$

we see that u satisfies (1).

Next, note that, using Theorem 2, with $T = 0$ there, leads to the conclusion that $w(t,x) = 0$ for $t \le 0$. Hence u satisfies (3) with $v = 0$.

Estimate (4) follows immediately from the definition of u and Theorem 2 providing an estimate for w. Therefore, it only remains to prove uniqueness.

If we have two functions satisfying (1) and (3), then denoting their difference by $\bar{u}$ and introducing

$$\bar{w}(t,x) = e^{-\lambda_0 t}\bar{u}(t,x) I_{[0,T]}(t),$$

in light of Definition 1, we have

$$\bar{u} I_{[0,T]} \in W_2^{1,2}(\mathbb{R}_T^{d+1,c}), \quad \bar{w} \in W_2^{1,2}(\mathbb{R}_T^{d+1,c}).$$

Furthermore, in Ω

$$L\bar{w} - \bar{w}_t - \lambda_0 \bar{w} = e^{-\lambda_0 t}(L\bar{u} - \bar{u}_t) = 0.$$

That the first expression here vanishes for $t < 0$ is obvious because then $\bar{w} = 0$. By Theorem 2 we have $\bar{w} = 0$, $\bar{u} = 0$, and the theorem is proved.

4. Remark. Denote by u^T the solution in Ω_T constructed in Theorem 3. If f is given in $(0,\infty) \times \mathbb{R}^d$ and $f \in \mathcal{L}_2(\Omega_T)$ for all $T \in (0,\infty)$, then, since $\Omega_T \subset \Omega_S$ for $S \geq T$, by uniqueness $u^T = u^S$ in Ω_T and, therefore, we have a single function $u(t,x)$ defined for $t > 0, x \in \mathbb{R}^d$ satisfying

$$u_t = Lu + f \tag{6}$$

in $(0,\infty) \times \mathbb{R}^d$ and such that $u \in W_2^{1,2}(\Omega_T)$ for all $T \in (0,\infty)$.

In particular, if $v = 0$ and we define $u(t,x)$ for negative t as zero, then by definition the extended function, which we still call u, satisfies

$$u \in W_2^{1,2}((-\infty,T) \times \mathbb{R}^d)$$

for all $T \in (0,\infty)$. Obviously, it also satisfies (6) in $\mathbb{R}^{d+1}$ if we extend f as zero for negative t.

5. Exercise. (i) For $u \in C^{1,2}(\bar{\mathbb{R}}_0^{d+1})$ such that $u(t,x) = 0$ for large $t + |x|$ show that

$$\|u(0,\cdot)\|_{W_2^1} \leq N\|u\|_{W_2^{1,2}(\mathbb{R}_0^{d+1})},$$

where N is independent of u.

(ii) Show that if $g \in C_0^\infty(\mathbb{R}^d)$, then there exists a $u \in W_2^{1,2}(\mathbb{R}_0^{d+1})$ which is infinitely differentiable in $\bar{\mathbb{R}}_0^{d+1}$ and satisfies $u(0,\cdot) = g$ and

$$\|u\|_{W_2^{1,2}(\mathbb{R}_0^{d+1})} \leq N\|g\|_{W_2^1},$$

where N is independent of g.

6. Hints to exercises

1.10. Take a $\zeta \in C_0^\infty(\mathbb{R})$, consider $\zeta(t/R)u(x)$, and then send R to infinity.

1.11. Observe that

$$a^{-1}|au_{xx} + u_t - \lambda u|^2 \geq \varepsilon(|u_{xx}|^2 + |u_t - \lambda u|^2) + 2u_{xx}(u_t - \lambda u).$$

Then integrate through this inequality, and either use Parseval's identity or integration by parts and the formula $2u_{xx}u_t = 2(u_x u_t)_x - (|u_x|^2)_t$.

2.4. While proving the "only if" part, first use Exercise 2.3 to show that $v \in W_p^{1,k+2}$ if $u \in C_0^{1,k+2}(\bar{\mathbb{R}}_0^{d+1})$.

2.5. Use mollifiers and the results of Section 1.5 while treating u_x.

2.7. Use mollifiers.

4.6. Use dilations.

4.8. It suffices to concentrate on $k = 0$, in which case consider

$$\Big(\int_{B_R} |u(t,x)|^p \, dx + \varepsilon \Big)^{1/p}$$

first for smooth u, then for general u, and finally let $\varepsilon \downarrow 0$.

4.10. First observe that it suffices to prove that $D^\alpha u = 0$ for $|\alpha| \geq \gamma$. Then show that for $\rho = 1$ and for any $\rho > 0$

$$\rho^{d+2+2|\alpha|} |D^\alpha u(0,0)|^2 \leq N \int_{Q_\rho} |u|^2 \, dxdt.$$

5.5. (i) Express the Fourier transform of $u(0,\cdot)$ through the Fourier transform of $f := \Delta u + u_t - u$. (ii) Use the Fourier transform to define u as the solution of the Cauchy problem $u_t = \Delta u - u$ in $\mathbb{R}^{d+1}_0$ with initial data $u(0,x) = g(x)$ and prove what is required.

Chapter 3

Some tools from real analysis

In this chapter $(\Omega, \mathcal{F}, \mu)$ is a complete measure space with a σ-finite measure μ, such that

$$\mu(\Omega) = \infty.$$

Let $\mathcal{F}^0$ be the subset of $\mathcal{F}$ consisting of all sets A such that $\mu(A) < \infty$. By $\mathbb{L}$ we denote a fixed dense subset of $\mathcal{L}_1(\Omega) = \mathcal{L}_1(\Omega, \mathcal{F}, \mu)$. For any $A \in \mathcal{F}$ we set

$$|A| = \mu(A).$$

For $A \in \mathcal{F}^0$ and functions f summable on A we use the notation

$$f_A = \fint_A f\, \mu(dx) := \frac{1}{|A|} \int_A f(x)\, \mu(dx) \quad \left(\frac{0}{0} := 0\right)$$

for the average value of f over A. We write $f \in \mathcal{L}_{1,loc}(\Omega)$ if $fI_A \in \mathcal{L}_1(\Omega)$ for any $A \in \mathcal{F}^0$.

1. Exercise*. Prove that $\mathcal{L}_1(\Omega) \cap \mathcal{L}_p(\Omega)$ is dense in $\mathcal{L}_p(\Omega)$ for any $p \in (1, \infty)$.

1. Partitions and stopping times

1. Definition. Let $\mathbb{Z} = \{n : n = 0, \pm 1, \pm 2, ...\}$ and let $(\mathbb{C}_n, n \in \mathbb{Z})$ be a sequence of partitions of Ω each consisting of countably many disjoint sets $C \in \mathbb{C}_n$ and such that $\mathbb{C}_n \subset \mathcal{F}^0$ for each n. For each $x \in \Omega$ and $n \in \mathbb{Z}$ there exists (a unique) $C \in \mathbb{C}_n$ such that $x \in C$. We denote this C by $C_n(x)$.

We call the sequence $(\mathbb{C}_n, n \in \mathbb{Z})$ *a filtration of partitions* if the following conditions are satisfied.

(i) The elements of partitions are "large" for big negative n's and "small" for big positive n's:

$$\inf_{C \in \mathbb{C}_n} |C| \to \infty \quad \text{as} \quad n \to -\infty, \quad \lim_{n \to \infty} f_{C_n(x)} = f(x) \quad \text{(a.e.)} \quad \forall f \in \mathbb{L}.$$

(ii) The partitions are nested: for each n and $C \in \mathbb{C}_n$ there is a (unique) $C' \in \mathbb{C}_{n-1}$ such that $C \subset C'$.

(iii) The following regularity property holds: for any n, C, and C' as in (ii) we have

$$|C'| \leq N_0 |C|,$$

where N_0 is a constant independent of n, C, C'.

Observe that since the elements of partition $\mathbb{C}_n$ become large as $n \to -\infty$, we have $N_0 > 1$. One obvious but useful property of filtrations is that if $n \leq m$ and $C_n \in \mathbb{C}_n$, $C_m \in \mathbb{C}_m$, then $C_n \cap C_m$ is either C_m or empty.

2. Example. The simplest example of a filtration of partitions in the case $\Omega = \mathbb{R}^d$ with Lebesgue measure μ is given by *dyadic cubes*, that is, by

$$\mathbb{C}_n = \{C_n(i_1, ..., i_d), i_1, ..., i_d \in \mathbb{Z}\},$$

where

$$C_n(i_1, ..., i_d) = [i_1 2^{-n}, (i_1 + 1)2^{-n}) \times ... \times [i_d 2^{-n}, (i_d + 1)2^{-n}).$$

In this case, to satisfy requirement (i) in Definition 1, one can take $\mathbb{L}$ as the set of continuous functions with compact support.

3. Example. Another example, frequently used in the theory of parabolic equations, is given by *parabolic dyadic cubes* in

$$\Omega = \mathbb{R}^{d+1} = \{(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^d\},$$

namely,

$$\mathbb{C}_n = \{[i_0 4^{-n}, (i_0 + 1)4^{-n}) \times C_n(i_1, ..., i_d), i_0, i_1, ..., i_d \in \mathbb{Z}\}.$$

Here again μ is Lebesgue measure but in $\mathbb{R}^{d+1}$.

4. Definition. Let $\mathbb{C}_n$, $n \in \mathbb{Z}$, be a filtration of partitions of Ω.

(i) Let $\tau = \tau(x)$ be a function on Ω with values in $\{\infty, 0, \pm 1, \pm 2, ...\}$. We call τ *a stopping time* (relative to the filtration) if, for each $n = 0, \pm 1, \pm 2, ...$ the set

$$\{x : \tau(x) = n\}$$

is either empty or else is the union of some elements of $\mathbb{C}_n$.

(ii) For a function $f \in \mathcal{L}_{1,loc}(\Omega)$ and $n \in \mathbb{Z}$, we denote

$$f_{|n}(x) = \fint_{C_n(x)} f(y)\,\mu(dy).$$

We read $f_{|n}$ as "f given $\mathbb{C}_n$", continuing to borrow the terminology from probability theory. If we are also given a stopping time τ, we let

$$f_{|\tau}(x) = f_{|\tau(x)}(x)$$

for those x for which $\tau(x) < \infty$ and $f_{|\tau}(x) = f(x)$ otherwise.

The simplest example of a stopping time is given by $\tau(x) \equiv 0$.

5. Exercise*. Take an $f \in \mathcal{L}_2(\Omega)$ and $n \in \mathbb{Z}$ and consider the problem of finding the best approximation in $\mathcal{L}_2(\Omega)$ of f by functions that are constant on each element of $\mathbb{C}_n$. Prove that the best approximation is given by $f_{|n}$.

To define a stopping time, we usually choose, for each $n \in \mathbb{Z}$, elements $C_{n1}, C_{n2}, ... \in \mathbb{C}_n$ and define $\tau = n$ on each of them. Of course, we need to have all C_{ni} disjoint for different n, i. If the union of all C_{ni} is not Ω, then one lets $\tau = \infty$ on the complement of the union. The most important property of $f_{|n}$ is that, obviously, it is constant on each $C \in \mathbb{C}_n$, so that, for any $x \in C \in \mathbb{C}_n$, we have

$$\int_C f_{|n}(y)\,\mu(dy) = |C| f_{|n}(x) = \int_C f(y)\,\mu(dy).$$

We are going to use the following simple properties of the objects introduced above.

6. Lemma. *Let $\mathbb{C}_n$, $n \in \mathbb{Z}$, be a filtration of partitions of Ω.*

(i) Let $f \in \mathcal{L}_{1,loc}(\Omega)$, $f \geq 0$, and let τ be a stopping time. Then

$$\int_\Omega f_{|\tau}(x) I_{\tau<\infty}\,\mu(dx) = \int_\Omega f(x) I_{\tau<\infty}\,\mu(dx), \tag{1}$$

$$\int_\Omega f_{|\tau}(x)\,\mu(dx) = \int_\Omega f(x)\,\mu(dx). \tag{2}$$

(ii) Let $g \in \mathcal{L}_1(\Omega)$, $g \geq 0$, and let $\lambda > 0$ be a constant. Then

$$\tau(x) := \inf\{n : g_{|n}(x) > \lambda\} \quad (\inf \emptyset := \infty) \tag{3}$$

is a stopping time. Furthermore, we have

$$0 \leq g_{|\tau}(x) I_{\tau<\infty} \leq N_0 \lambda, \quad |\{x : \tau(x) < \infty\}| \leq \lambda^{-1} \int_\Omega g(x) I_{\tau<\infty}\, \mu(dx). \tag{4}$$

Proof. (i) Equation (2) follows immediately from (1) since $f_{|\tau} = f$ on the set where $\tau = \infty$. Then notice that, for any $n \in \mathbb{Z}$ and set Γ which is the union of some elements $C_i \in \mathbb{C}_n$, obviously

$$\int_\Gamma f_{|n}\, \mu(dx) = \sum_i \int_{C_i} f_{|n}\, \mu(dx) = \sum_i \int_{C_i} f\, \mu(dx) = \int_\Gamma f\, \mu(dx).$$

Hence,

$$\int_\Omega f_{|\tau} I_{\tau<\infty}\, \mu(dx) = \sum_{n=-\infty}^{\infty} \int_{\{\tau=n\}} f_{|n}\, \mu(dx)$$

$$= \sum_{n=-\infty}^{\infty} \int_{\{\tau=n\}} f\, \mu(dx) = \int_\Omega f I_{\tau<\infty}\, \mu(dx).$$

(ii) First notice that $\tau > -\infty$ since $g_{|n} \to 0$ as $n \to -\infty$ due to the assumptions that $g \in \mathcal{L}_1(\Omega)$ and $|C_n| \to \infty$ whenever $C_n \in \mathbb{C}_n$. Next, observe that

$$C_n(x) \subset C_m(x)$$

for all $m \leq n$ since the partitions are nested. It follows that, if $y \in C_n(x)$, then

$$C_m(y) = C_m(x), \quad g_{|m}(y) = g_{|m}(x), \quad \forall m \leq n.$$

By adding that

$$\tau(x) = n \iff g_{|n}(x) > \lambda, \quad g_{|m}(x) \leq \lambda \quad \forall m < n,$$

we conclude that the set $\{\tau = n\}$ contains $C_n(x)$ along with each x. Therefore, $\{\tau = n\}$ is indeed the union of some elements of $\mathbb{C}_n$.

To prove the first relation in (4), it suffices to notice that, if $\tau(x) = n$, then

$$g_{|\tau}(x) = g_{|n}(x) = \frac{1}{|C_n(x)|} \int_{C_n(x)} g(y)\, \mu(dy) \leq N_0 \frac{1}{|C_{n-1}(x)|} \int_{C_n(x)} g(y)\, \mu(dy)$$

$$\leq N_0 \frac{1}{|C_{n-1}(x)|} \int_{C_{n-1}(x)} g(y)\,\mu(dy) = N_0 g_{|n-1}(x) \leq N_0\lambda.$$

The second inequality in (4) follows from Chebyshev's inequality and (i):

$$|\{x : \tau < \infty\}| = |\{x : g_{|\tau} I_{\tau<\infty} > \lambda\}|$$

$$\leq \lambda^{-1} \int_\Omega g_{|\tau} I_{\tau<\infty}\,\mu(dx) = \lambda^{-1} \int_\Omega g I_{\tau<\infty}\,\mu(dx).$$

The lemma is proved.

For g, λ, and τ as in Lemma 6 we have the following *Riesz-Calderón-Zygmund decomposition*: $g = \alpha + \beta$, where

$$\alpha = g - g_{|\tau}, \quad \beta = g_{|\tau} = g_{|\tau} I_{\tau<\infty} + g I_{\tau=\infty},$$

β is "small" in the sense that, as can be seen (see Exercise 2.7 below), $\beta \leq N_0\lambda$ (a.e.), the set where $\alpha \neq 0$ is under control having measure less than $\lambda^{-1}\|g\|_{\mathcal{L}_1(\Omega)}$, and $\alpha_{|\tau} = 0$.

It is worth clarifying the structure of $g_{|\tau}$ in Lemma 6. First we take any m so largely negative that

$$g_{|m} \leq \lambda \quad \text{on} \quad \Omega.$$

Let C_j, $j = 1, 2, \ldots$, be the set of all "cubes" in the family $\mathbb{C}_m$, so that

$$\Omega = \bigcup_j C_j.$$

Then we divide each "cube" C_j into smaller "cubes" $C_{jk} \in \mathbb{C}_{m+1}$, so that $C_j = \bigcup_k C_{jk}$ and we look for those

$$C_{jk} \quad \text{on which} \quad g_{|m+1} > \lambda.$$

We set those aside and set $\tau = m + 1$ and accordingly $g_{|\tau} = g_{|m+1}$ on them. With the *remaining* "cubes" on which $g_{|m+1} \leq \lambda$ we proceed in the same way splitting each of them into smaller "cubes" $C_{jkp} \in \mathbb{C}_{m+2}$ and defining $\tau = m+2$ and $g_{|\tau} = g_{|m+2}$ on those C_{jkp} on which $g_{|m+2} > \lambda$. By continuing in this way, we define τ and $g_{|\tau}$ on a subset of Ω, which may not coincide with Ω (it is just empty if $g \leq \lambda$ everywhere). The remaining set is the set of points x at which $g_{|n}(x) \leq \lambda$ for all n and we let $\tau = \infty$ and $g_{|\tau} = g$ at those points.

7. Exercise. Let $f \in \mathcal{L}_{1,loc}(\Omega)$ and let τ and σ be stopping times such that $\tau \le \sigma$. Prove that $(f_{|\tau})_{|\sigma} = (f_{|\sigma})_{|\tau} = f_{|\tau}$ and by using the fact that $|g_{|\sigma}| \le (|g|)_{|\sigma}$, conclude that
$$\int_\Omega |f_{|\tau} - f_{|\sigma}|\,\mu(dx) \le \int_\Omega |f_{|\tau} - f|_{|\tau}\,\mu(dx).$$
The inequality can be used for doing Exercise 2.16.

2. Maximal and sharp functions

Here we keep using the notation and the definitions from Section 1 and the introduction to the chapter.

2:1. The Fefferman-Stein theorem. Define *the maximal function* of f by
$$\mathcal{M}f(x) = \sup_{n<\infty} |f|_{|n}(x),$$
so that $\mathcal{M}f = \mathcal{M}|f|$.

Notice that Lemma 1.6 implies the following.

1. Corollary (Maximal inequality). *For* $\lambda > 0$ *and nonnegative* $g \in \mathcal{L}_1(\Omega)$, *the* maximal inequality *holds:*
$$|\{x : \mathcal{M}g(x) > \lambda\}| \le \lambda^{-1} \int_\Omega g(x) I_{\mathcal{M}g>\lambda}\,\mu(dx). \tag{1}$$

Indeed, for τ as in (1.3), we have
$$\{x : \mathcal{M}g(x) > \lambda\} = \{x : \tau(x) < \infty\}.$$

2. Remark. Our interest in estimating $|\{\mathcal{M}g > \lambda\}|$ as in Corollary 1 is based on the following formula valid for any $f \ge 0$ in light of Fubini's theorem:
$$\int_\Omega f(x)\,\mu(dx) = \int_\Omega \Big(\int_0^{f(x)} dt\Big)\,\mu(dx) = \int_\Omega \Big(\int_0^\infty I_{f(x)>t}\,dt\Big)\,\mu(dx)$$
$$= \int_0^\infty \Big(\int_\Omega I_{f(x)>t}\,\mu(dx)\Big)\,dt = \int_0^\infty |\{x : f(x) > t\}|\,dt. \tag{2}$$

3. Corollary. *Let* $p \in (1,\infty)$, $g \in \mathcal{L}_1(\Omega)$, $g \ge 0$. *Then*
$$\|\mathcal{M}g\|_{\mathcal{L}_p(\Omega)} \le q\|g\|_{\mathcal{L}_p(\Omega)},$$
where $q = p/(p-1)$.

Indeed, from (2), (1), and Fubini's theorem we conclude that, for any finite constant $\nu > 0$,

$$\|\nu \wedge \mathcal{M}g\|^p_{\mathcal{L}_p(\Omega)} = \int_0^\infty |\{x : \nu \wedge \mathcal{M}g(x) > \lambda^{1/p}\}| \, d\lambda$$

$$= \int_0^{\nu^p} |\{x : \mathcal{M}g(x) > \lambda^{1/p}\}| \, d\lambda \leq \int_\Omega g(x) \Big(\int_0^{\nu^p} \lambda^{-1/p} I_{\mathcal{M}g(x) > \lambda^{1/p}} \, d\lambda \Big) \, \mu(dx)$$

$$= \int_\Omega g(x) \Big(\int_0^{(\nu \wedge \mathcal{M}g(x))^p} \lambda^{-1/p} \, d\lambda \Big) \, \mu(dx) = q \int_\Omega (\nu \wedge \mathcal{M}g)^{p-1} g \, \mu(dx).$$

This and $g \in \mathcal{L}_1(\Omega)$ imply that $\|\nu \wedge \mathcal{M}g\|_{\mathcal{L}_p(\Omega)} < \infty$. Then upon using Hölder's inequality, we get

$$\|\nu \wedge \mathcal{M}g\|^p_{\mathcal{L}_p(\Omega)} \leq q\|g\|_{\mathcal{L}_p(\Omega)} \|\nu \wedge \mathcal{M}g\|^{p-1}_{\mathcal{L}_p(\Omega)}, \quad \|\nu \wedge \mathcal{M}g\|_{\mathcal{L}_p(\Omega)} \leq q\|g\|_{\mathcal{L}_p(\Omega)}$$

and it only remains to let $\nu \to \infty$ and use Fatou's theorem.

Next we extend Corollary 3 to $g \in \mathcal{L}_p(\Omega)$.

4. Theorem. *For any $p \in (1, \infty)$ and $g \in \mathcal{L}_p(\Omega)$*

$$\|\mathcal{M}g\|_{\mathcal{L}_p(\Omega)} \leq q\|g\|_{\mathcal{L}_p(\Omega)}.$$

Proof. Since

$$\mathcal{M}g = \mathcal{M}|g| \quad \text{and} \quad \|g\|_{\mathcal{L}_p(\Omega)} = \| \, |g| \, \|_{\mathcal{L}_p(\Omega)},$$

we may concentrate on $g \geq 0$.

In that case take an increasing sequence of $A_m \in \mathcal{F}^0, m \geq 1$, such that $A_m \uparrow \Omega$ and introduce $g^m(x) = g(x) I_{A_m}$. Then $g^m \in \mathcal{L}_1(\Omega)$ and

$$\|\mathcal{M}g^m\|_{\mathcal{L}_p(\Omega)} \leq q\|g^m\|_{\mathcal{L}_p(\Omega)} \leq q\|g\|_{\mathcal{L}_p(\Omega)}$$

by Corollary 3. It only remains to use Fatou's theorem along with the observation that for any x and n, since $|C_n(x)| < \infty$, we have

$$(g^m)_{|n}(x) \to g_{|n}(x) \quad \text{as} \quad m \to \infty,$$

which implies

$$g_{|n}(x) \le \varliminf_{m\to\infty} \sup_r (g^m)_{|r}(x), \quad \mathcal{M}g \le \varliminf_{m\to\infty} \mathcal{M}g^m.$$

The theorem is proved.

Let $f \in \mathcal{L}_{1,loc}(\Omega)$. Define *the sharp function of* f by

$$f^{\#}(x) = \sup_{n<\infty} \fint_{C_n(x)} |f(y) - f_{|n}(y)|\, \mu(dy). \tag{3}$$

5. Exercise*. Let $f \in \mathcal{L}_{1,loc}(\Omega)$. Prove that for any constant c and $Q \in \mathcal{F}^0$

$$\fint_Q \fint_Q |f(y) - f(z)|\, dydz \le 2 \fint_Q |f(y) - c|\, dy.$$

Also show that

$$f^{\#}(x) \le \sup_{n<\infty} \fint_{C_n(x)} \fint_{C_n(x)} |f(y) - f(z)|\, \mu(dy)\mu(dz) \le 2f^{\#}(x),$$

and using this derive that $(|f|)^{\#}(x) \le 2f^{\#}(x)$.

6. Exercise. Prove that

$$|f_{|n+1}(x) - f_{|n}(x)| \le N_0 f^{\#}(x).$$

7. Exercise*. Let $f \in \mathcal{L}_{1,loc}(\Omega)$. Prove the Lebesgue differentiating theorem: $f_{|n} \to f$ μ-almost everywhere. Conclude that $|f| \le \mathcal{M}f$ almost everywhere.

8. Exercise. Let $f \in \mathcal{L}_{1,loc}(\Omega)$. Prove another version of the Lebesgue differentiating theorem: $|f - f(x_0)|_{|n}(x_0) \to 0$ for μ-almost all x_0.

Obviously $f^{\#}(x) \le 2\mathcal{M}f(x)$. It turns out that f and hence $\mathcal{M}f$ are also controlled by $f^{\#}$.

The following result is false for locally integrable functions.

9. Lemma. *For* $\alpha = (2N_0)^{-1}$, *any constant* $c > 0$, *and* $f \in \mathcal{L}_1(\Omega)$, *we have*

$$|\{x : |f(x)| \ge c\}| \le \frac{4}{c} \int_\Omega I_{\mathcal{M}f(x) > \alpha c} f^{\#}(x)\, \mu(dx)$$

and if $f \ge 0$, *then one can replace* $4/c$ *with* $2/c$.

Proof. Exercise 5 shows that it suffices to prove the second assertion of the lemma. Introduce

$$\tau(x) = \inf\{n : f_{|n}(x) > c\alpha\}.$$

Use Lemma 1.6 (ii) to get that $f_{|\tau} \le c/2$ if $\tau < \infty$ and also use the fact that $f_{|n} \to f$ (a.e.) (see Exercise 7). Then we find that (a.e.)

$$\{x : f(x) \ge c\} = \{x : f(x) \ge c, \tau(x) < \infty\}$$

$$= \{x : f(x) \ge c, f_{|\tau}(x) \le c/2\} \subset \{x : |f(x) - f_{|\tau}(x)| \ge c/2\}.$$

By Chebyshev's inequality and Lemma 1.6

$$|\{x : |f(x)| \ge c\}| \le (2/c) \int_{\mathbb{R}^d} |f(x) - f_{|\tau}(x)|\, \mu(dx)$$

$$= (2/c) \int_{\mathbb{R}^d} |f - f_{|\tau}|_{|\tau}(x)\, \mu(dx) = (2/c) \int_{\mathbb{R}^d} |f - f_{|\tau}|_{|\tau}(x) I_{\tau<\infty}\, \mu(dx),$$

where in the equalities we also used the fact that

$$|f - f_{|\tau}|_{|\tau}(x) = |f(x) - f_{\tau}(x)| = 0$$

when $\tau(x) = \infty$.

Finally, if at an x we have $\tau(x) = n$, then (recall that if $\tau(x) = n$ and $y \in Q_n(x)$, then $\tau(y) = n$)

$$|f - f_{|\tau}|_{|\tau}(x) = \fint_{Q_n(x)} |f(y) - f_{|\tau}(y)|\, \mu(dy)$$

$$= \fint_{Q_n(x)} |f(y) - f_{|n}(y)|\, \mu(dy) \le f^{\#}(x).$$

Now it only remains to notice that

$$\{\tau(x) < \infty\} = \{\mathcal{M}f(x) > c\alpha\}.$$

The lemma is proved.

10. Theorem (Fefferman-Stein). *Let $p \in (1, \infty)$. Then for any $f \in \mathcal{L}_p(\Omega)$ we have*

$$\|f\|_{\mathcal{L}_p(\Omega)} \le N \|f^{\#}\|_{\mathcal{L}_p(\Omega)},$$

where $N = (2q)^p N_0^{p-1}$, $q = p/(p-1)$.

Proof. As in the beginning of the proof of Corollary 3 (with $\nu = \infty$) we get from Lemma 9 that if $f \in \mathcal{L}_1(\Omega)$, then

$$\|f\|^p_{\mathcal{L}_p(\Omega)} \leq N \int_\Omega f^\#(\mathcal{M}f)^{p-1}\,\mu(dx).$$

By using Hölder's inequality, we obtain

$$\|f\|^p_{\mathcal{L}_p(\Omega)} \leq N\|f^\#\|_{\mathcal{L}_p(\Omega)}\|\mathcal{M}f\|^{p-1}_{\mathcal{L}_p(\Omega)}.$$

If in addition $f \in \mathcal{L}_p(\Omega)$, then it only remains to use Theorem 4 and check that the resulting constant is right.

If we only have $f \in \mathcal{L}_p(\Omega)$, then it suffices to take $f_n \in \mathcal{L}_1(\Omega) \cap \mathcal{L}_p(\Omega)$ converging to f in $\mathcal{L}_p(\Omega)$ and observe that $f_n^\# \leq (f - f_n)^\# + f^\#$ and

$$\|(f - f_n)^\#\|_{\mathcal{L}_p(\Omega)} \leq 2\|\mathcal{M}(f - f_n)\|_{\mathcal{L}_p(\Omega)} \leq 2q\|f - f_n\|_{\mathcal{L}_p(\Omega)} \to 0.$$

The theorem is proved.

11. Remark. This theorem is quite puzzling from the following point of view. In the definition (3) only the integrals of the *first* power of f are involved and the sup there is certainly not attained or approximated when $n \to \infty$. Then how $f^\#$ can control the integral of the pth power of f may look mysterious since $p > 1$ and generally the integrals of $|f|^p$ over sets of finite measure are *not* controlled by the integrals of $|f|$.

12. Remark. By Hölder's inequality, for any $p \in [1, \infty]$

$$f^\#(x) \leq \sup_{n<\infty} \Big(\fint_{C_n(x)} |f(y) - f_{|n}(y)|^p\,dy\Big)^{1/p} =: f_p^\#(x).$$

Observe that for $1 < p < q < \infty$ and $f \in \mathcal{L}_q(\Omega)$

$$\|f_p^\#\|_{\mathcal{L}_q(\Omega)} \leq N(p,q)\|f^\#\|_{\mathcal{L}_q(\Omega)} \tag{4}$$

since $f_p^\# \leq 2(\mathcal{M}(|f|^p))^{1/p}$ and

$$\|(\mathcal{M}(|f|^p))^{1/p}\|_{\mathcal{L}_q(\Omega)} = \|(\mathcal{M}(|f|^p))\|^{1/p}_{\mathcal{L}_{q/p}(\Omega)} \leq N\|\,|f|^p\|^{1/p}_{\mathcal{L}_{q/p}(\Omega)} = N\|f\|_{\mathcal{L}_q(\Omega)}.$$

It is a remarkable fact that typically (see Exercise 15) estimate (4) also holds for $q = \infty$ (the John-Nirenberg theorem, not used in these lectures).

2:2. Exercises (optional). We are very close to beautiful and powerful results from real analysis. Although they are not used in the lectures, the author could not help himself but had to at least mention them.

13. Definition. If $f \in \mathcal{L}_{1,loc}(\Omega)$, then we write $f \in BMO$ (the dyadic John-Nirenberg class) if and only if $f^{\#}$ is a bounded function. In that case we say that f has bounded mean oscillation and set

$$[f]_{BMO} = \sup_{\Omega} f^{\#}.$$

14. Exercise. On the real line consider the filtration of dyadic intervals and prove that $\log|x| \in BMO$.

15. Exercise. Take $p \in (1, \infty)$ and assume that for a constant $N_1 < \infty$ and all n and $C_n \in \mathbb{C}_n$ we have

$$\sum_{m=-\infty}^{n} \frac{|C_n|^{p-1}}{|C_{mn}|^{p-1}} \leq N_1,$$

where C_{mn} is defined as the element of $\mathbb{C}_m$ containing C_n. Observe that typically (say for dyadic partitions) for $C \in \mathbb{C}_n$, $C' \in \mathbb{C}_{n-1}$, and $C \subset C'$ the ratio $|C|/|C'|$ is less than a constant $\alpha < 1$ independent of n. In such a case, obviously one can take $N_1 = (1 - \alpha^{p-1})^{-1}$.

Prove that (the John-Nirenberg theorem)

$$\sup_{\Omega} f_p^{\#} \leq N[f]_{BMO},$$

where N is independent of f.

16. Exercise. By mimicking the proof of Lemma 9 for any stopping time σ, *non-negative* $g \in \mathcal{L}_1(\Omega)$, and $c > 0$ prove that

$$|\{x : g_{|\sigma}(x) > c\}| \leq \frac{2}{c} \int_{\Omega} I_{\mathcal{M}g(x)>\alpha c} g^{\#}(x)\, \mu(dx)$$

and for any $f \in \mathcal{L}_1(\Omega)$

$$|\{x : \mathcal{M}f(x) > c\}| \leq \frac{4}{c} \int_{\Omega} I_{\mathcal{M}f(x)>\alpha c} f^{\#}(x)\, \mu(dx).$$

17. Exercise. In Exercise 16 assume that $g^{\#} \leq 1$, take $c > 2N_0$ and define

$$\tau = \inf\{n : g_{|n} > c - 2N_0\}.$$

By using that $g_{|\tau} > c - 2N_0$ and also that $g_{|\tau}$ admits an estimate from above (Exercise 6) and proceeding as in Exercise 16, prove that

$$|\{x : \mathcal{M}g(x) > c\}| \leq \frac{1}{N_0} \int_{\Omega} I_{\mathcal{M}g(x)>c-2N_0} g^{\#}(x)\, \mu(dx)$$

$$\leq \frac{1}{N_0} |\{x : \mathcal{M}g(x) > c - 2N_0\}|.$$

Iterate this estimate by choosing $c = kN_0$ and prove the John-Nirenberg inequality

$$|\{x : \mathcal{M}g(x) > c + \lambda\}| \leq Ne^{-\alpha c}|\{x : \mathcal{M}g(x) > \lambda\}|,$$

where $c, \lambda > 0$ and the constants $N < \infty$, $\alpha > 0$ are independent of c, λ, g. Since $g \in \mathcal{L}_1(\Omega)$, we have $|\{x : \mathcal{M}g(x) > \lambda\}| < \infty$ for any λ, say $\lambda = 1$, and then we see that the measure $|\{x : \mathcal{M}g(x) > c + 1\}|$ decreases exponentially fast.

Extend the result to g taking values of any sign. Then you will see that if $f \in BMO \cap \mathcal{L}_1(\Omega)$ and $f^{\#} \leq 1$, then f is almost bounded, in the sense that on the set $\{|f| > 1\}$ the function $\exp(\beta|f|)$ is summable, for any $\beta \in (0, \alpha)$:

$$\int_\Omega I_{|f|>1} e^{\beta|f|}\, \mu(dx) \leq \int_\Omega I_{\mathcal{M}f>1} e^{\beta\mathcal{M}f}\, \mu(dx)$$

$$\leq N(\beta, N_0)|\{x : \mathcal{M}f(x) > 1\}| \leq N(\beta, N_0)\|f\|_{\mathcal{L}_1(\Omega)}.$$

This result is close to be optimal since for $\Omega = \mathbb{R}$ the function $\log(|x| \wedge 1)$ is in BMO relative to the dyadic filtration and Lebesgue measure.

18. Exercise. For $f \in \mathcal{L}_{2,loc}$ introduce the *quadratic variation* of $f_{|n}$, $n = 0, \pm 1, \pm 2...$ by

$$S^n f(x) = \sum_{k=n}^{\infty} (f_{|k+1} - f_{|k})^2(x).$$

Prove that for $m \geq n$

$$S^n_{|n}(f_{|m}) = (|f_{|m} - f_{|n}|^2)_{|n}, \quad S^n_{|n} f = (|f - f_{|n}|^2)_{|n} \leq (f_2^{\#})^2.$$

Polarize the result and show that, if $g \in \mathcal{L}_{2,loc}$, then for

$$S^n(f, g)(x) := \sum_{k=n}^{\infty} (f_{|k+1} - f_{|k})(g_{|k+1} - g_{|k})$$

we have

$$S^n_{|n}(f, g)(x) = (fg - f_{|n} g_{|n})_{|n}(x).$$

19. Exercise (Davis's inequality). We say that a sequence of functions f_n, $n = 0, 1, 2, \ldots$, is *adapted* to the filtration $\mathbb{C}_n$, $n = 0, 1, 2, \ldots$, if for each n, f_n is constant on each $C \in \mathbb{C}_n$. Let $C \in \mathbb{C}_0$, and let f_n, g_n be two nonnegative adapted sequences such that for all nonnegative *bounded* stopping times

$$⨍_C f_\tau\, \mu(dx) \leq ⨍_C g_\tau\, \mu(dx).$$

Also assume that

$$g_{n+1} \leq N_0 g_n, \quad \forall n \geq 0, \quad f_0 \leq g_0.$$

Then prove that

$$⨍_C \sup_{n\geq 0} f_n^{1/2}\, \mu(dx) \leq 3N_0 ⨍_C \sup_{n\geq 0} g_n^{1/2}\, \mu(dx).$$

20. Exercise (Davis's inequality). On the basis of Exercises 19 and 18 conclude that, if $f \in \mathcal{L}_{2,loc}$, then for any $n < \infty$ and $C \in \mathbb{C}_n$

$$\int_C (S^n(f))^{1/2} \mu(dx) \le 3N_0 \int_C \mathcal{M}f \, \mu(dx).$$

21. Exercise (Fefferman's theorem). Take $f, g \in \mathcal{L}_2$ with support in a $C \in \mathbb{C}_0$ and assume that $f_2^\# \le 1$ and either

$$\int_C f \, \mu(dx) = 0 \quad \text{or} \quad \int_C g \, \mu(dx) = 0.$$

Fefferman's theorem says that under this conditions there is a constant N depending only on N_0 such that

$$\int_C fg \, \mu(dx) \le N \int_C \mathcal{M}g \, \mu(dx).$$

In other words, the theorem asserts that BMO is a subspace of the dual to the so-called Hardy space H^1, which is defined as the space of all g with support in C for which $\mathcal{M}g \in \mathcal{L}_1(C)$. A somewhat easier fact that the dual to H^1 is a subspace of BMO was known before C. Fefferman proved his theorem.

As an exercise, we suggest the reader fill in missing details in the following argument. Assume the notation from preceding exercises and set

$$F_k = \sum_{n=k}^{\infty} (f_{|n+1} - f_{|n})^2, \quad G_k = \sum_{n=0}^{k} (g_{|n+1} - g_{|n})^2, \quad G_{-1} = 0.$$

By Exercise 18

$$I := \int_C fg \, \mu(dx) = \int_C \sum_{k=0}^{\infty} (f_{|k+1} - f_{|k})(g_{|k+1} - g_{|k}) \, \mu(dx).$$

By following a method due to C. Hertz, we replace $(f_{|k+1} - f_{|k})$ and $(g_{|k+1} - g_{|k})$ with

$$G_k^{1/4} |f_{|k+1} - f_{|k}| \quad \text{and} \quad G_k^{-1/4} |g_{|k+1} - g_{|k}|,$$

respectively, and use Hölder's inequality to get that I is less than the product of the square roots of

$$J_1 := \sum_{k=0}^{\infty} \int_C G_k^{-1/2} |g_{|k+1} - g_{|k}|^2 \, \mu(dx) = \sum_{k=0}^{\infty} \int_C \frac{G_k - G_{k-1}}{G_k^{1/2}} \, \mu(dx)$$

and

$$J_2 := \int_C \sum_{k=0}^{\infty} G_k^{1/2} |f_{|k+1} - f_{|k}|^2 \, \mu(dx) = \int_C \sum_{k=0}^{\infty} G_k^{1/2} (F_k - F_{k+1}) \, \mu(dx).$$

Since $(G_k - G_{k-1})/G_k^{1/2} \le 2(G_k^{1/2} - G_{k-1}^{1/2})$, we have

$$J_1 \le 2 \int_C G_\infty^{1/2}\, \mu(dx).$$

Next

$$J_2 = \sum_{k=0}^{\infty} \int_C F_k(G_k^{1/2} - G_{k-1}^{1/2})\, \mu(dx) = \int_C (f_{|k+1} - f_{|k})^2 (G_k^{1/2} - G_{k-1}^{1/2})\, \mu(dx)$$

$$+ \sum_{k=0}^{\infty} \int_C F_{k+1}(G_k^{1/2} - G_{k-1}^{1/2})\, \mu(dx),$$

where $(f_{|k+1} - f_{|k})^2 \le N_0^2((f - f_{|k})^2)_{|k} \le N_0^2 f_2^{\#} \le N_0^2$ and

$$\int_C F_{k+1}(G_k^{1/2} - G_{k-1}^{1/2})\, \mu(dx) = \int_C (F_{k+1})_{|k+1}(G_k^{1/2} - G_{k-1}^{1/2})\, \mu(dx)$$

$$= \int_C ((f - f_{|k+1})^2)_{|k+1}(G_k^{1/2} - G_{k-1}^{1/2})\, \mu(dx) \le \int_C (G_k^{1/2} - G_{k-1}^{1/2})\, \mu(dx).$$

Conclude that

$$J_2 \le (1 + N_0^2) \int_C G_\infty^{1/2}\, \mu(dx).$$

A reference to Exercise 20 finishes our comments on Fefferman's theorem.

22. Exercise. Observe that for any series of nonnegative a_k and any integer $m \in \{1, 2, ...\}$

$$\Big(\sum_{k=1}^{\infty} a_k\Big)^m \le m! \sum_{k_1 \le .. \le k_m} a_{k_1} \cdot \ldots \cdot a_{k_m}.$$

Then upon relying on Exercises 18 and 6 and the fact that

$$\big[(f_{|k+1} - f_{|k})^2 (f_{|l+1} - f_{|l})^2\big]_{|k+1} = (f_{|k+1} - f_{|k})^2 \big[(f_{|l+1} - f_{|l})^2\big]_{|k+1}$$

if $l > k$, prove that if $f \in \mathcal{L}_{2,loc}$ and $f_2^{\#} \le 1$, then

$$\big((S^n f)^m\big)_{|n} \le m!(N_0^2 + 1)^m.$$

Conclude that for $\lambda \in (0, (2N_0^2 + 2)^{-1})$ we have

$$\big(e^{\lambda S^n f}\big)_{|n} \le 2.$$

3. Comparing maximal and sharp functions in the Euclidean space

The maximal and sharp functions introduced in Section 2 are related to the underlying filtration of partitions. In particular applications the following classical maximal and sharp functions are used:

$$\mathbb{M}g(t,x) = \sup_{Q\in\mathbb{Q}:(t,x)\in Q} \fint_Q |g(s,y)|\,dyds,$$

$$g^\sharp(t,x) = \sup_{Q\in\mathbb{Q}:(t,x)\in Q} \fint_Q |g(s,y) - g_Q|\,dyds, \tag{1}$$

where $\mathbb{Q}$ is the collection of cylinders

$$Q_r(t,x) = (t, t+r^2) \times B_r(x), \quad (t,x) \in \mathbb{R}^{d+1}, r \in (0,\infty),$$

$$g_Q = \fint_Q g(s,y)\,dyds = \frac{1}{|Q|}\int_Q g(s,y)\,dyds,$$

and $|Q|$ is the Lebesgue measure of $Q \subset \mathbb{R}^{d+1}$. One obtains the corresponding "elliptic" counterparts of these functions by taking g independent of t and one easily sees that

$$\mathbb{M}g(x) = \sup_{B\in\mathbb{B}:x\in B} \fint_B |g(y)|\,dy, \quad g^\sharp(x) = \sup_{B\in\mathbb{B}:x\in B} \fint_B |g(y) - g_B|\,dy,$$

where $\mathbb{B}$ is the collection of balls in $\mathbb{R}^d$,

$$g_B = \fint_B g(y)\,dy = \frac{1}{|B|}\int_B g(y)\,dy,$$

and $|B|$ is the volume of B.

Let $g^\#$ be the dyadic sharp function of g associated with the filtration $(\mathbb{C}_n, n \in \mathbb{Z})$ of parabolic dyadic cubes from Example 1.3. The reader should notice the difference in shapes of the "sharp" symbol in $g^\#$ and $g^\sharp$.

1. Theorem (Fefferman-Stein). *Let $g \in \mathcal{L}_{1,loc}(\mathbb{R}^{d+1})$; then $g^\# \leq Ng^\sharp$, where the constant N is independent of g. In particular, owing to Theorem 2.10, if $p \in (1,\infty)$, $g \in \mathcal{L}_p(\mathbb{R}^{d+1})$, and $f \in \mathcal{L}_p$, then*

$$\|g\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} \leq N\|g^\sharp\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}, \quad \|f\|_{\mathcal{L}_p} \leq N\|f^\sharp\|_{\mathcal{L}_p},$$

where N is independent of f and g.

Proof. For $(t,x)\in\mathbb{R}^{d+1}$, $n\in\mathbb{Z}$, and $C_n(t,x)$ $(\in\mathbb{C}_n)$ we have

$$(t,x)\in C_n(t,x)\subset(t-4^{-n},t+4^{-n})\times B_{2^{-n}2d}(x)=:Q_{(n)}(t,x),$$

$$\fint_{C_n(t,x)}|g(s,y)-g_{C_n(t,x)}|\,dyds$$

$$\leq\fint_{C_n(t,x)}\fint_{C_n(t,x)}|g(s,y)-g(r,z)|\,dydzdsdr$$

$$\leq N_1\fint_{Q_{(n)}(t,x)}\fint_{Q_{(n)}(t,x)}|g(s,y)-g(r,z)|\,dydzdsdr,$$

where

$$N_1=\frac{|Q_{(n)}(t,x)|^2}{|C_n(t,x)|^2}$$

is independent of n,t,x. It follows that

$$\fint_{C_n(t,x)}|g(s,y)-g_{C_n(t,x)}|\,dyds$$

$$\leq 2N_1\fint_{Q_{(n)}(t,x)}|g(s,y)-g_{Q_{(n)}(t,x)}|\,dyds\leq 2N_1g^{\sharp}(t,x),$$

implying the result. The theorem is proved.

The following theorem, one of the Hardy-Littlewood theorems, is used, in particular, in the proof of the Hardy-Littlewood-Sobolev inequality (see Lemma 13.8.5) and, as well as Theorem 1, in constructing the $\mathcal{L}_p$ theory of elliptic and parabolic equations.

2. Theorem. *Let $p\in(1,\infty)$ and $g\in\mathcal{L}_p(\mathbb{R}^{d+1})$ and $f\in\mathcal{L}_p$. Then $\mathbb{M}g\in\mathcal{L}_p(\mathbb{R}^{d+1})$, $\mathbb{M}f\in\mathcal{L}_p$, and*

$$\|\mathbb{M}g\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}\leq N\|g\|_{\mathcal{L}_p(\mathbb{R}^{d+1})},\quad \|\mathbb{M}f\|_{\mathcal{L}_p}\leq N\|f\|_{\mathcal{L}_p},\tag{2}$$

where the constants N are independent of g and f.

Proof. We will only prove the first inequality in (2). The second one is proved similarly.

We may certainly assume that $g \geq 0$ and, as in the case of Theorem 2.4, it suffices to prove that for any $\lambda > 0$ and $g \in \mathcal{L}_1(\mathbb{R}^{d+1})$ we have

$$|A(\lambda)| \leq \frac{3^{d+1}}{\lambda} \int_{\mathbb{R}^{d+1}} I_{A(\lambda)}(t,x) g(t,x)\, dxdt, \tag{3}$$

where

$$A(\lambda) = \{(t,x) : \mathbb{M}g(t,x) > \lambda\},$$

and $|A(\lambda)|$ is the $(d+1)$-dimensional volume of $A(\lambda)$.

We know that for any bounded Borel set B

$$\int_B g(x+y)\, dy$$

is a continuous function of x. It follows easily from this fact that $A(\lambda)$ is an open set. Take a compact set $K \subset A(\lambda)$. Then by the definition of $A(\lambda)$ for any $(t,x) \in K$ there exists a $Q \in \mathbb{Q}$ such that $(t,x) \in Q$ and

$$\int_Q g\, dxdt > \lambda |Q|. \tag{4}$$

By the way, observe that $Q \subset A(\lambda)$.

By the compactness of K, there is a finite collection $Q_1, ..., Q_n \in \mathbb{Q}$ covering K and such that for each $Q = Q_i$ equation (4) holds.

Now we use a Vitali covering argument. If $Q \in \mathbb{Q}$, then define Q^* as the three times dilated Q with the center of dilation being the center of Q. Then denote by $\tilde{Q}_1$ any of Q_i which has the largest volume and set it aside. Next, introduce $\tilde{Q}_2$ as one of the remaining Q_i which has the largest volume between those Q_i that have *no* intersection with $\tilde{Q}_1$. It may happen that no such Q_i exist. Then it is almost obvious that $Q_i \subset \tilde{Q}_1^*$ for any i. If $\tilde{Q}_2$ exists, we proceed further.

If we have already defined $\tilde{Q}_1, ..., \tilde{Q}_k$, then we define $\tilde{Q}_{k+1}$ as one of the cylinders in the family

$$\{Q_1, ..., Q_n\} \setminus \{\tilde{Q}_1, ..., \tilde{Q}_k\} \tag{5}$$

which is disjoint from $\tilde{Q}_1, ..., \tilde{Q}_k$ and has the largest volume between those that are disjoint from $\tilde{Q}_1, ..., \tilde{Q}_k$. In finitely many steps we will come to a k

for which any cylinder in family (5) intersects one of $\tilde{Q}_1, ..., \tilde{Q}_k$ or else the family is empty. In the second case, obviously, for any i

$$Q_i \subset \bigcup_{j=1}^{k} \tilde{Q}_j^*. \tag{6}$$

It turns out that (6) also holds for any i in the first case. Indeed, if, for a fixed i, Q_i has a nonempty intersection with a $\tilde{Q}_j$, then define $r = r(i)$ as the smallest such j and observe that, if $r = 1$, then as has been pointed out above, $Q_i \subset \tilde{Q}_1^*$ and (6) holds. If $r > 1$, then $|Q_i| \leq |\tilde{Q}_r|$ by the choice of $\tilde{Q}_r$ and because Q_i has no intersection with $\tilde{Q}_1, ..., \tilde{Q}_{r-1}$ by the definition of r. Now as above

$$|Q_i| \leq |\tilde{Q}_r|, \tilde{Q}_r \cap Q_i \neq \emptyset \Longrightarrow Q_i \subset \tilde{Q}_r^*,$$

implying (6).

It follows that

$$K \subset \bigcup_{j=1}^{k} \tilde{Q}_j^*.$$

By using (4) for $\tilde{Q}_j$, recalling that they are disjoint, and using that $Q_i \subset A(\lambda)$ for any i, we get

$$|K| \leq \sum_{j=1}^{k} |\tilde{Q}_j^*| = 3^{d+1} \sum_{j=1}^{k} |\tilde{Q}_j|$$

$$\leq 3^{d+1}\lambda^{-1} \sum_{j=1}^{k} \int_{\tilde{Q}_j} g\,dxdt \leq 3^{d+1}\lambda^{-1} \int_{\mathbb{R}^{d+1}} g I_{A(\lambda)}\,dxdt.$$

We thus obtain (3) with K in place of $A(\lambda)$. By taking a sequence of compact sets $K_m \uparrow A(\lambda)$ and passing to the limit, we get (3). The theorem is proved.

Here is a result complementary to Theorem 1.

3. Corollary. *For $p \in (1, \infty)$ there is a constant $N = N(d, p)$ such that, for any $g \in \mathcal{L}_p(\mathbb{R}^{d+1})$ and $f \in \mathcal{L}_p$, we have*

$$\|g^\sharp\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} \leq N\|g^\#\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}, \quad \|f^\sharp\|_{\mathcal{L}_p} \leq N\|f^\#\|_{\mathcal{L}_p}.$$

Indeed, it suffices to observe that $g^\sharp \leq 2\mathbb{M}g$, $f^\sharp \leq 2\mathbb{M}f$ and to use Theorems 2 and 1.

In the following exercises we denote by $\mathcal{M}f$ the maximal function relative to the filtration of dyadic cubes in $\mathbb{R}^d$, so-called *dyadic maximal function of f*.

4. Exercise. For $d = 1$, obviously $\mathcal{M}f(x) \leq \mathbb{M}f(x)$ at any $x \in \mathbb{R}$. Give an example in which $\mathcal{M}f(0) < \infty$ and $\mathbb{M}f(0) = \infty$.

5. Exercise. For $d \geq 1$ prove that, if $\mathcal{M}f \in \mathcal{L}_1 = \mathcal{L}_1(\mathbb{R}^d)$, then $f = 0$.

6. Exercise. For $d \geq 1$ and $f \in \mathcal{L}_{1,loc}$ take an $\alpha > 0$ and let Q_j be the disjoint cubes such that

$$\{x : \mathcal{M}f(x) > \alpha\} = \bigcup_j Q_j.$$

Let B_j be the ball with the same center as Q_j but twice its diameter. Show that there are constants c_1 and c_2 depending only on d such that

$$\{x : \mathbb{M}f(x) > c_1\alpha\} \subset \bigcup_j B_j,$$

$$|\{x : \mathbb{M}f(x) > c_1\alpha\}| \leq c_2 \sum_j |Q_j| = c_2|\{x : \mathcal{M}f(x) > \alpha\}|.$$

By integrating the extreme terms of the last inequality, prove again the second estimate in (2).

4. Hints to exercises

1.5. Observe that for any n, constant c, and $C \in \mathbb{C}_n$

$$\int_C |f - c|^2 \,\mu(dx) = \int_C |f - f_{|n}|^2 \,\mu(dx) + |f_{|n} - c|^2.$$

2.7. It suffices to concentrate on $f \in \mathcal{L}_1(\Omega)$. Then observe that for any $g \in \mathbb{L}$ we have

$$\overline{\lim_{n\to\infty}} |f_{|n} - f| \leq \overline{\lim_{n\to\infty}} |(f - g)_{|n}| + |f - g| \leq \mathcal{M}(f - g) + |f - g|.$$

After that use Corollary 2.1 and the freedom of choice of g.

2.8. The functions $F_n(x, c) = |f - c|_{|n}(x)$ are Lipschitz continuous in c and, for any c by Exercise 2.7, they converge to $|f(x) - c|$ for almost all x. Conclude that they converge to $|f(x) - c|$ for all c at once (μ-a.e.).

2.15. Take an $n < \infty$ and a $C_n \in \mathbb{C}_n$ and apply the Fefferman-Stein theorem to $(f - f_{|n})I_{C_n}$ to obtain

$$\int_{C_n} |f(y) - f_{|n}(y)|^p \,\mu(dy) \leq N \int_{C_n} |f^{\#}(y)|^p \,\mu(dy)$$

$$+N|C_n|\left[\fint_{C_n} |f(y) - f_{|n}(y)|\,\mu(dy)\right]^p \sum_{m=-\infty}^{n-1} \frac{|C_n|^{p-1}}{|C_{mn}|^{p-1}}.$$

2.16. You may need to refer to Exercise 2.5 and notice that $\tau < \sigma$ if $g_{|\sigma} > c$ and on the set $\tau = n$ it holds that $(|g_{|\sigma} - g_{|\tau}|)_{|n} \le (|g - g_\tau|)_{|n}$. For proving that, you may like to refer to Exercise 1.7.

2.18. Use that $f_{|k+1} - f_{|k}$ are mutually orthogonal in $\mathcal{L}_2$. To get the second assertion consider $f + \lambda g$ in place of f.

2.19. Fix a $c > 0$ and set

$$\tau = \inf\{n : g_n > c\}, \quad \sigma = \inf\{n : f_n > c\}.$$

By splitting the set $\{x \in C : \sigma(x) < \infty\}$ into two appropriate pieces, prove that

$$|\{x \in C : \sup_{n\ge 0} f_n > c\}| \le |\{x \in C : \sup_{n\ge 0} g_n > c\}|$$
$$+ |\{x \in C : f_{\tau\wedge\sigma} I_{\sigma<\infty} > c\}|.$$

Then use Chebyshev's inequality and the assumption to obtain

$$|\{x \in C : \sup_{n\ge 0} f_n > c\}| \le |\{x \in C : \sup_{n\ge 0} g_n > c\}|$$
$$+ N_0 c^{-1} \int_C (c \wedge \sup_{n\ge 0} g_n)\,\mu(dx).$$

After that it only remains to substitute c^2 in place of c and integrate with respect to $c \in (0, \infty)$.

There is a subtle point where you use for arbitrary nonnegative stopping times what was assumed for bounded stopping time. You may like to use Fatou's theorem and replace g_n with $\sup_{m\le n} g_m$ from the very beginning.

Chapter 4

Basic $\mathcal{L}_p$ estimates for parabolic and elliptic equations

In this chapter we consider the operator

$$Lu(t,x) = a^{ij}(t)u_{x^i x^j}(t,x) \tag{1}$$

with coefficients depending only on $t \in \mathbb{R}$ and such that $a^{ij} = a^{ji}$ and

$$\kappa^{-1}|\xi|^2 \geq a^{ij}(t)\xi^i\xi^j \geq \kappa|\xi|^2$$

for all $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$, where $\kappa \in (0,1)$ is a fixed constant.

We recall that $B_r(x)$ is the open ball in $\mathbb{R}^d$ of radius r centered at x, $B_r = B_r(0)$, $Q_r(t,x) = (t, t+r^2) \times B_r(x)$, $Q_r = Q_r(0,0)$. Also $\mathbb{B}$ is the collection of open balls in $\mathbb{R}^d$ and $\mathbb{Q}$ is the collection of all $Q_r(t,x)$, $(t,x) \in \mathbb{R}^{d+1}$, $r \in (0,\infty)$. For $B \in \mathbb{B}$ and $Q \in \mathbb{Q}$ by

$$|B|, \quad |Q|$$

we mean the volumes of B and Q, respectively, in the corresponding spaces.

We use the notation

$$u_Q = \fint_Q u(s,y)\,dyds := \frac{1}{|Q|}\int_Q u(s,y)\,dyds$$

for the average value of a function $u(s, y)$ over Q and

$$u_B(t) = \fint_B u(t, y)\, dy := \frac{1}{|B|}\int_B u(t, y)\, dy \tag{2}$$

for the average value of a function $u(t, y)$ over B. Naturally, if u is independent of t, we write u_B instead of $u_B(t)$.

1. Remark. Many times in the future we are going to use the inequalities

$$|u_Q|^p \leq (|u|^p)_Q,$$

$$\fint_Q |u - u_Q|^p\, dxdt \leq 2^{p+1} \fint_Q |u - c|^p\, dxdt,$$

where c is any constant. The first inequality follows from Hölder's inequality and the second one is obtained by adding that

$$u_Q - c = (u - c)_Q, \quad (a + b)^p \leq 2^p(a^p + b^p).$$

Sometimes it is also worth using that

$$\fint_Q |u - u_Q|^2\, dxdt = \fint_Q |u|^2\, dxdt - (u_Q)^2 \leq \fint_Q |u|^2\, dxdt.$$

1. An approach to elliptic equations

Here we describe an idea that one could follow while developing the $\mathcal{L}_p$ theory for elliptic and parabolic equations. Regardless of the fact that we will be doing this somewhat differently, the idea is worth talking about because the reader can see better which difficulties arise and how they will be avoided in the future. Also the reader might not be interested in parabolic equations at this time.

We deal only with the elliptic case assuming that $d \geq 3$. The case $d = 2$ is suggested in Exercise 4. We will be using the tools from real analysis which we learned about in Chapter 3.

1. Lemma. *Let $u \in C_0^\infty$. Then on $\mathbb{R}^d$*

$$(u_{xx})^\sharp \leq N(d)\mathbb{M}^{1/2}(|\Delta u|^2). \tag{1}$$

Proof. Set $f = -\Delta u$. Then by (1.9.6)

$$u_{x^i x^j}(x) = N_1(d)\mathcal{K}_{ij}f(x) + f(x)N_{ij}, \tag{2}$$

where $N_1(d)$, N_{ij} are certain constants, and

$$\mathcal{K}_{ij}f(x) = \lim_{r\downarrow 0}\int_{|y|\geq r} K_{ij}(y)f(x+y)\,dy, \quad K_{ij}(y) = \frac{y^i y^j d - \delta^{ij}|y|^2}{|y|^{d+2}}.$$

Take an $r > 0$ and a $\zeta \in C_0^\infty$ such that $\zeta = 1$ in B_{3r} and $\zeta = 0$ outside B_{4r} and set

$$g = \zeta f, \quad h = (1-\zeta)f,$$

$$v(x) = N(d)\int_{\mathbb{R}^d}\frac{1}{|y|^{d-2}}g(x-y)\,dy, \quad w(x) = N(d)\int_{\mathbb{R}^d}\frac{1}{|y|^{d-2}}h(x-y)\,dy,$$

where $N(d)$ is taken from (1.9.2). Observe that the functions v and w are well defined since f, g, h have compact supports and $d \geq 3$. Also observe that by (1.9.2) we have

$$u = v + w.$$

By Exercise 1.9.3 we know that v, w are infinitely differentiable and

$$\Delta v = -g, \quad \Delta w = -h.$$

Furthermore,

$$\int_{\mathbb{R}^d}|v_{xx}|^2\,dx \leq N\int_{\mathbb{R}^d}|g|^2\,dx \leq N\int_{B_{4r}}|f|^2\,dx.$$

It follows that, for any $x_0 \in B_r$ (see the hint to Exercise 3.1.5 in which set $c = 0$),

$$\fint_{B_r}|v_{xx} - (v_{xx})_{B_r}|\,dx \leq \left(\fint_{B_r}|v_{xx} - (v_{xx})_{B_r}|^2\,dx\right)^{1/2}$$

$$\leq \left(\fint_{B_r}|v_{xx}|^2\,dx\right)^{1/2} \leq N\left(\fint_{B_{4r}}|f|^2\,dx\right)^{1/2} \leq N\mathbb{M}^{1/2}(|f|^2)(x_0). \tag{3}$$

Now we estimate the integral oscillation of w_{xx}. Since $h = 0$ in B_{3r}, by (2) we see that, if $x \in B_r$, then

$$w_{x^ix^j}(x) = \int_{|y|\geq 2r} K_{ij}(y)h(x+y)\,dy,$$

$$w_{x^ix^jx^k}(x) = \int_{|y|\geq 2r} K_{ij}(z)h_{x^k}(x+y)\,dy = -\int_{|y|\geq 2r} K_{ijx^k}(y)h(x+y)\,dy,$$

Furthermore, it is easy to see that

$$|K_{ijx^k}(y)| \leq N|y|^{-d-1}$$

and for $|x| < r, |y| \geq 2r$ we have $2|y| \geq |x+y| \geq r$. Therefore,

$$|w_{x^ix^jx^k}(x)| \leq N\int_{|y|\geq 2r} |x+y|^{-d-1}|h(x+y)|\,dy$$

$$\leq N\int_{|x+y|\geq 2r} |x+y|^{-d-1}|h(x+y)|\,dy = N\int_{|y|\geq r} |y|^{-d-1}|h(y)|\,dy =: NI.$$

We write I in polar coordinates and use the fact that

$$\frac{\partial}{\partial \rho}\int_{B_\rho} |h(y)|\,dy = \int_{\partial B_\rho} |h(y)|\,dS_\rho,$$

where dS_ρ is the surface measure on ∂B_ρ. Then we find that

$$I = N\int_r^\infty \rho^{-d-1}\int_{\partial B_\rho} |h(y)|\,dS_\rho\,d\rho$$

$$= N\int_r^\infty \rho^{-d-2}\int_{B_\rho} |h(y)|\,dy\,d\rho - Nr^{-d-1}\int_{B_r} |h(y)|\,dy$$

$$\leq N\int_r^\infty \rho^{-2} \fint_{B_\rho} |h(y)|\,dy\,d\rho \leq N\mathbb{M}h(x_0)\int_r^\infty \rho^{-2}\,d\rho$$

$$\leq Nr^{-1}\mathbb{M}f(x_0) \leq Nr^{-1}\mathbb{M}^{1/2}(|f|^2)(x_0).$$

Hence,

$$\fint_{B_r} |w_{xx} - (w_{xx})_{B_r}|\,dx \leq Nr\sup_{B_r}|w_{xxx}| \leq N\mathbb{M}^{1/2}(|f|^2)(x_0).$$

Upon combining this with (3), we conclude

$$\fint_{B_r} |u_{xx} - (u_{xx})_{B_r}|\, dx \le N\mathbb{M}^{1/2}(|f|^2)(x_0).$$

This estimate allows shifting the origin. For this reason for any x_0 and any ball B such that $x_0 \in B$ we have

$$\fint_{B} |u_{xx} - (u_{xx})_{B}|\, dx \le N\mathbb{M}^{1/2}(|f|^2)(x_0).$$

By taking the supremum of the left-hand side over all balls B containing x_0, we obtain (1) at point x_0. The lemma is proved.

The following theorem provides the basic $\mathcal{L}_p$ estimate for u_{xx}. We only prove it for $p \ge 2$. One obtains the same result for $p \in (1, 2]$ by duality (see the proof of Theorem 3.8).

2. Theorem. *Let $d \ge 3$, $u \in C_0^\infty$, and $p \in [2, \infty)$. Then*

$$\|u_{xx}\|_{\mathcal{L}_p} \le N(p, d)\|\Delta u\|_{\mathcal{L}_p}.$$

The proof only takes few lines. First observe that for $p = 2$ we know the result from Lemma 1.1.1. If $p > 2$, then by the Fefferman-Stein theorem (see Theorems 3.2.10 and 3.3.1), Lemma 1, and the Hardy-Littlewood theorem

$$\|u_{xx}\|_{\mathcal{L}_p} \le N\|(u_{xx})^\sharp\|_{\mathcal{L}_p} \le N\|\mathbb{M}^{1/2}(|\Delta u|^2)\|_{\mathcal{L}_p}$$

$$= N\|\mathbb{M}(|\Delta u|^2)\|_{\mathcal{L}_{p/2}}^{1/2} \le N\|\,|\Delta u|^2\|_{\mathcal{L}_{p/2}}^{1/2} = N\|\Delta u\|_{\mathcal{L}_p}.$$

3. Exercise. The necessity of considering the sharp function of u_{xx} can be explained by the following. Prove that there is no (finite) constant $N = N(d)$ such that $|u_{xx}(0)| \le N\mathbb{M}^{1/2}(|\Delta u|^2)(0)$ for all $u \in C_0^\infty$ if $d \ge 2$.

4. Exercise. Prove that the result of Theorem 2 holds for $d = 2$ as well. For that, take a function $\zeta \in C_0^\infty(\mathbb{R})$ such that $\zeta(0) = 1$, take a $u \in C_0^\infty(\mathbb{R}^2)$, and set $u_n(x^1, x^2, x^3) = u(x^1, x^2)\zeta(x^3/n)$ with the intention of letting $n \to \infty$.

5. Exercise. By using the John-Nirenberg inequality (see Exercise 3.2.17), prove that if $u \in W_1^1$ and $|\Delta u| \le 1$, then

$$\int_{\mathbb{R}^d} I_{|u_{xx}|>1} e^{\beta |u_{xx}|}\, dx \le N\|u_{xx}\|_{\mathcal{L}_1},$$

where $\beta > 0$ and N $(< \infty)$ are constants depending only on d.

This result is close to being optimal as one sees for $d = 2$ by taking $u = \zeta v$ and

$v(x,y) = xy\log(x^2+y^2)$, where $\zeta \in C_0^\infty$ and $\zeta(x,y) = 1$ for $x^2+y^2 \le 1$. In this case

$$v_x = y\log(x^2+y^2) + \frac{2x^2y}{x^2+y^2},$$

$$v_{xx} = \frac{6xy}{x^2+y^2} - \frac{4x^3y}{(x^2+y^2)^2}.$$

The right-hand side of the last formula is a homogeneous function of degree zero bounded on the unit circle and, hence, bounded. Similarly, u_{yy} is bounded. However,

$$v_{xy} = \log(x^2+y^2) + 2 - \frac{4x^2y^2}{(x^2+y^2)^2}$$

behaves as $\log(x^2+y^2)$ near the origin. Despite the fact that it has a finite $\mathcal{L}_1$ norm near the origin, not only is it unbounded but even

$$\int_{B_1} e^{\beta|v_{xy}|}\,dxdy = \infty$$

if $\beta \ge 1$.

2. Preliminary estimates of L-caloric functions

Recall that we are dealing with the operator L introduced in (0.1). First, we prove a few results somewhat similar to Poincaré's inequality allowing one to estimate the integral oscillation of a function in terms of the integrals of its derivatives.

1. Lemma. *Let $p \in [1,\infty)$, $r \in (0,\infty)$, $u \in C_{loc}^\infty(\mathbb{R}^{d+1})$,*

$$f = (f^1,\dots,f^d), \quad f^i, g \in C_{loc}^\infty(\mathbb{R}^{d+1}).$$

Assume that $u_t + Lu = \operatorname{div} f + g$ in Q_r. Then

$$\int_{Q_r} |u(t,x) - u_{Q_r}|^p\,dxdt \le Nr^p \int_{Q_r} (|u_x|^p + |f|^p + r^p|g|^p)\,dxdt, \qquad (1)$$

where $N = N(d,\kappa,p)$.

Proof. Assume (1) is true for $r = 1$. Substitute $v(t,x) = u(r^2t, rx)$ in (1), written for $r = 1$ and v in place of u, and observe that

$$v_t(t,x) + L^r v(t,x) = r^2(u_t + Lu)(r^2t, rx) = r\operatorname{div}(f(r^2t, r\cdot))(x) + r^2 g(r^2t, rx),$$

where L^r is constructed from $a(r^2t)$. Then the change of variables will lead to (1) as is. Hence, it suffices to prove the lemma only for $r = 1$ and below we assume that $r = 1$.

In that case take a function $\zeta \in C_0^\infty(B_1)$ with unit integral. Then by Poincaré's inequality (which we prove later in Theorem 10.2.5), for any $t \in (0,1)$ and

$$\bar{u}(t) := \int_{B_1} \zeta(y) u(t,y)\, dy,$$

we have

$$\int_{B_1} |u(t,x) - \bar{u}(t)|^p\, dx = \int_{B_1} \Big| \int_{B_1} [u(t,x) - u(t,y)]\zeta(y)\, dy \Big|^p\, dx$$

$$\leq N \int_{B_1}\int_{B_1} |u(t,x) - u(t,y)|^p\, dxdy \leq N \int_{B_1} |u_x(t,x)|^p\, dx. \qquad (2)$$

Then, observe that for any constant c the left-hand side of (1) is less than a constant times (recall that $r = 1$)

$$\int_{Q_1} |u(t,x) - c|^p\, dxdt \leq 2^p \int_{Q_1} |u(t,x) - \bar{u}(t)|^p\, dxdt + 2^p|B_1| \int_0^1 |\bar{u}(t) - c|^p\, dt.$$

By (2) the first term on the right is less than the right-hand side of (1). To estimate the second term, take

$$c = \int_0^1 \bar{u}(t)\, dt.$$

Then by Poincaré's inequality

$$\int_0^1 |\bar{u}(t) - c|^p\, dt \leq N \int_0^1 \Big| \int_{B_1} \zeta u_t\, dx \Big|^p\, dt,$$

where

$$u_t = -(a^{ij} u_{x^i})_{x^j} + \operatorname{div} f + g.$$

Integrating by parts with respect to x and then using Hölder's inequality show that this term is also less than the right-hand side of (1). The lemma is proved.

2. Lemma. *Let $p \in [1,\infty)$. Then there is a constant $N = N(d,p)$ such that for any $r \in (0,\infty)$ and $u \in C_{loc}^\infty(\mathbb{R}^{d+1})$ we have*

$$\int_{Q_r} |u_x(t,x) - (u_x)_{Q_r}|^p\, dxdt \leq N r^p \int_{Q_r} (|u_{xx}|^p + |u_t|^p)\, dxdt, \qquad (3)$$

$$\int_{Q_r} |u(t,x) - u_{Q_r} - x^i(u_{x^i})_{Q_r}|^p \, dxdt$$
$$\leq N r^p \int_{Q_r} (|u_x - (u_x)_{Q_r}|^p + r^p |u_t|^p + r^p |u_{xx}|^p) \, dxdt$$
$$\leq N r^{2p} \int_{Q_r} (|u_{xx}|^p + |u_t|^p) \, dxdt. \tag{4}$$

Proof. To prove (3), it suffices to take $a^{ij} = \delta^{ij}$, introduce $f = Lu + u_t$, note that $L(u_x) + (u_x)_t = f_x$, and apply Lemma 1.

To prove (4), set $v(t,x) = u(t,x) - u_{Q_r} - x^i(u_{x^i})_{Q_r}$, denote by I the left-hand side of (4), and observe that

$$v_{Q_r} = 0, \quad v_x = u_x - (u_x)_{Q_r}, \quad I = \int_{Q_r} |v(t,x) - v_{Q_r}|^p \, dxdt.$$

By taking $g := Lv + v_t$ $(= Lu + u_t)$ and $f \equiv 0$ in Lemma 1, we obtain the first inequality in (4). The second one follows from (3) and the lemma is proved.

Much later (see Section 6.4) we are going to use an improvement of Lemma 2 showing that the power of summability on the right can be taken somewhat smaller than p.

3. Lemma. *Let $q \geq 1$, $\nu \in (1, \infty)$,*

$$\frac{1}{q} < \frac{2}{d+2} + \frac{1}{p}. \tag{5}$$

Then there is a constant $N = N(d, p, q, \nu)$ such that for any $u \in C_0^\infty(\mathbb{R}^{d+1})$ and $\rho \in (0, \infty)$ we have

$$\Big(\fint_{Q_\rho} |u(t,x) - u_{Q_{\nu r}} - x^i(u_{x^i})_{Q_{\nu\rho}}|^p \, dxdt \Big)^{1/p}$$
$$\leq N\rho^2 \Big(\fint_{Q_{\nu\rho}} (|u_{xx}|^q + |u_t|^q) \, dxdt \Big)^{1/q}. \tag{6}$$

Proof. First, observe that an argument based on self-similarity reduces the case of general ρ to the case that $\rho = 1$, the one we confine ourselves to. If $q \geq p$, the result follows from Lemma 2 and Hölder's inequality. Therefore, we assume that $q \leq p$.

Then, take an infinitely differentiable function ζ on $\mathbb{R}^{d+1}$ such that $\zeta = 1$ on Q_1 and $\zeta = 0$ on $\mathbb{R}_0^{d+1} \setminus Q_\nu$, and set

$$f = \Delta u + u_t, \quad v = \zeta(u - u_{Q_\nu} - x^i(u_{x^i})_{Q_\nu}).$$

so that

$$\Delta v + v_t = \zeta f + (u - u_{Q_\nu} - x^i (u_{x^i})_{Q_\nu})(\Delta\zeta + \zeta_t) + 2\zeta_{x^i}(u_{x^i} - (u_{x^i})_{Q_\nu}) =: g.$$

By Remark 2.1.4 we have

$$v(t,x) = -\int_0^\infty \int_{\mathbb{R}^d} g(t+s, x+y)p(s,y)\,dyds,$$

where p is introduced in (1.7.1). Here, if $0 \le t \le \nu^2$, there is no need to integrate with respect to s beyond $[0,\nu^2]$, since $g(r,z) = 0$ for $r \ge \nu^2$. Therefore, upon denoting

$$\bar v(t,x) = |v(t,x)| I_{t\in[0,\nu^2]}, \quad \bar g(s,y) = |g(s,y)| I_{s\in[0,\nu^2]},$$

$$\bar p(s,y) = p(s,y) I_{s\in[0,\nu^2]},$$

we find

$$\bar v(t,x) \le \int_{\mathbb{R}^{d+1}} \bar g(t+s, x+y)\bar p(s,y)\,dyds$$

$$= \int_{\mathbb{R}^{d+1}} \bar g(t-s, x-y)\bar p(-s,y)\,dyds.$$

Now we can apply Lemma 1.8.1 (ii) with

$$r = \frac{pq}{q-p+pq},$$

which satisfies (1.8.2), $r \ge 1$ since $q \le p$, and also

$$rd < d+2, \tag{7}$$

the latter being equivalent to (5). Then we find

$$\|v\|_{\mathcal{L}_p(Q_1)} \le \|\bar v\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} \le \|g\|_{\mathcal{L}_q(Q_\nu)} \|p\|_{\mathcal{L}_r([0,\nu^2]\times\mathbb{R}^d)}. \tag{8}$$

Here by the definition of g and Lemma 2 (just in case, recall that N in (6) is allowed to depend on ν)

$$\|g\|_{\mathcal{L}_q(Q_\nu)} \le N(\|u_{xx}\|_{\mathcal{L}_q(Q_\nu)} + \|u_t\|_{\mathcal{L}_q(Q_\nu)}).$$

Furthermore, changing variables shows that the integral

$$\int_{\mathbb{R}^d} t^{-d/2} e^{-r|x|^2/(4t)}\,dx$$

is finite and independent of $t > 0$. Therefore,

$$\|p\|^r_{\mathcal{L}_r([0,\nu^2]\times\mathbb{R}^d)} = N\int_0^{\nu^2} t^{-rd/2+d/2}\int_{\mathbb{R}^d} t^{-d/2}e^{-r|x|^2/(4t)}\,dxdt$$

$$= N\int_0^{\nu^2} t^{-rd/2+d/2}\,dt < \infty,$$

where the inequality holds since owing to (7) we have $-rd/2 + d/2 > -1$.

Thus, (8) implies that

$$\|v\|_{\mathcal{L}_p(Q_1)} \le N(\|u_{xx}\|_{\mathcal{L}_q(Q_\nu)} + \|u_t\|_{\mathcal{L}_q(Q_\nu)})$$

and it only remains to observe that the left-hand side here coincides with a constant times the left-hand side of (6). The lemma is proved.

4. Exercise. Take $q \ge 1$, $\nu > 1$, and assume that

$$\frac{1}{q} < \frac{1}{d+2} + \frac{1}{p}.$$

Then prove that there is a constant $N = N(d,p,q,\nu)$ such that for any $u \in C_0^\infty(\mathbb{R}^{d+1})$ and $\rho \in (0,\infty)$ we have

$$\Big(\int\!\!\!\!\!\!-_{Q_\rho} |u_x(t,x) - (u_x)_{Q_\rho}|^p\,dxdt\Big)^{1/p} \le N\rho\Big(\int\!\!\!\!\!\!-_{Q_{\nu\rho}} (|u_{xx}|^q + |u_t|^q)\,dxdt\Big)^{1/q}.$$

Now come two results about L-caloric functions. We call a function u L-caloric in a domain Ω if $Lu + u_t = 0$ in Ω. The following theorem may look like a step back from Theorem 2.4.9. The idea behind getting estimates like (9) is that the left-hand side will allow us to estimate the sharp function of u_{xx} as in Section 1 while the right-hand side will generate the maximal function of u_{xx} with as small a coefficient as we wish as will be seen from Theorem 6.

5. Lemma. *Let $m \in \{0,1,2,\dots\}$ and let $u \in C_0^\infty(\mathbb{R}^{d+1})$ be such that $Lu+u_t$ vanishes in Q_2. Then*

$$\max_{Q_1}\big(|D^m u_{xx}|^2 + |D^m u_t|^2\big) \le N\int_{Q_2} |u_{xx}|^2\,dxdt, \tag{9}$$

where $N = N(d,m,\kappa)$.

Proof. By Theorem 2.4.9

$$I := \max_{Q_1} \left(|D^m u_{xx}|^2 + |D^m u_t|^2\right) \leq N\|u\|^2_{\mathcal{L}_2(Q_2)}.$$

Here we can replace u with

$$v := u - u_{Q_2} - x^i (u_{x^i})_{Q_2}$$

without violating the fact that $Lu + u_t$ vanishes in Q_2 or changing the left-hand side. Therefore,

$$I \leq N\|v\|^2_{\mathcal{L}_2(Q_2)},$$

and using Lemma 2 and the observation that $|u_t| \leq N|u_{xx}|$ in Q_2 yields the desired result. The lemma is proved.

In the following theorem having the factor ν^{-2} in (10), which can be made as small as we wish, will be crucial in developing the $\mathcal{L}_p$ theory. Observe that (10) will become false if we drop the term $(u_{xx})_{Q_r}$ on the left. This is seen if we take $L = \Delta$, $u(t,x) = |x|^2 - 2td$, $r = 1$, and ν sufficiently large.

6. Theorem. *Let $\nu \geq 2$ and $r \in (0,\infty)$ be some constants and let $u \in C^\infty_{loc}(\mathbb{R}^{d+1})$ be such that $f := Lu + u_t$ vanishes in $Q_{\nu r}$. Then there is a constant $N = N(d,\kappa)$ such that*

$$\fint_{Q_r} |u_{xx}(t,x) - (u_{xx})_{Q_r}|^2 \, dxdt \leq N\nu^{-2} \fint_{Q_{\nu r}} |u_{xx}|^2 \, dxdt. \tag{10}$$

Proof. Notice that $v(t,x) := u(tr^2, xr)$ satisfies

$$\fint_{Q_r} |u_{xx}(t,x) - (u_{xx})_{Q_r}|^2 \, dxdt = r^{-4} \fint_{Q_1} |v_{xx}(t,x) - (v_{xx})_{Q_1}|^2 \, dxdt,$$

$$\fint_{Q_{\nu r}} |u_{xx}|^2 \, dxdt = r^{-4} \fint_{Q_\nu} |v_{xx}|^2 \, dxdt,$$

and

$$L(tr^2)v(t,x) + v_t(t,x) = r^2 f(tr^2, xr)$$

which vanishes in Q_ν. It follows that if (10) holds for $r = 1$, then it holds for any $r > 0$.

Therefore, in the rest of the proof we assume that $r = 1$ and observe that the left-hand side of (10) with $r = 1$ is obviously less than a constant $N = N(d)$ times

$$\max_{Q_1}(|u_{xxx}|^2 + |u_{txx}|^2).$$

Therefore, we need only prove that

$$\max_{Q_1}(|u_{xxx}|^2 + |u_{txx}|^2) \le N\nu^{-2} \fint_{Q_\nu} |u_{xx}|^2\,dxdt. \tag{11}$$

Observe that the function $w(t,x) = u(t\nu^2/4, x\nu/2)$ satisfies

$$L(t\nu^2/4)w(t,x) + w_t(t,x) = 0$$

in Q_2 and

$$\fint_{Q_\nu} |u_{xx}|^2\,dxdt = 16\nu^{-4} \fint_{Q_2} |w_{xx}|^2\,dxdt,$$

$$\max_{Q_1} |u_{xxx}|^2 = 64\nu^{-6} \max_{Q_{2/\nu}} |w_{xxx}|^2 \le 64\nu^{-6} \max_{Q_1} |w_{xxx}|^2,$$

$$\max_{Q_1} |u_{txx}|^2 \le 156\nu^{-8} \max_{Q_1} |w_{txx}|^2.$$

It follows that if (11) is true with $\nu = 2$, then

$$\max_{Q_1}(|u_{xxx}|^2 + |u_{txx}|^2) \le N\nu^{-6} \max_{Q_1}(|w_{xxx}|^2 + |w_{txx}|^2)$$

$$\le N\nu^{-6} \fint_{Q_2} |w_{xx}|^2\,dxdt = N\nu^{-2} \fint_{Q_\nu} |u_{xx}|^2\,dxdt.$$

Finally, (11) with $\nu = 2$ is indeed true by Lemma 5 and the theorem is proved.

7. Exercise. In Theorem 6 assume that $f = 0$ in Q_1. Drop $(u_{xx})_{Q_r}$ on the left in (10) and prove that the resulting inequality holds for all $\nu \ge 2$ and r such that $\nu r \le 1$ only if $u_{xx}(0,0) = 0$.

3. Solvability of model equations

We remind the reader that we are dealing with operators of form (0.1). The results of this section are based on the following theorem which we state and prove only for parabolic operators. The corresponding result for elliptic operators is obtained by taking u independent of t.

1. Theorem. *There is a constant N, depending only on d and κ, such that for any $u \in W^{1,2}_{2,loc}$, $r \in (0,\infty)$, and $\nu \ge 4$*

$$\begin{aligned}&\fint_{Q_r} |u_{xx}(t,x) - (u_{xx})_{Q_r}|^2\,dxdt\\&\qquad\le N\nu^{d+2}\fint_{Q_{\nu r}} |f|^2\,dxdt + N\nu^{-2}\fint_{Q_{\nu r}} |u_{xx}|^2\,dxdt,\end{aligned}\tag{1}$$

where

$$f := Lu + u_t.$$

Proof. Fix $r \in (0,\infty)$ and $\nu \ge 4$. We may certainly assume that the a^{ij} are infinitely differentiable and have bounded derivatives. Also changing u for large $|t| + |x|$ does not affect (1). Therefore, we may assume that $u \in W_2^{1,2}(\mathbb{R}^{d+1})$ and moreover

$$u \in C_0^\infty(\mathbb{R}^{d+1}).$$

In that case observe that $f \in C_0^\infty(\mathbb{R}^{d+1})$. Also take a $\zeta \in C_0^\infty(\mathbb{R}^{d+1})$ such that $\zeta = 1$ on $Q_{\nu r/2}$ and $\zeta = 0$ outside $(-\nu^2 r^2, \nu^2 r^2) \times B_{\nu r}$ and set

$$g = f\zeta, \quad h = f(1-\zeta).$$

Finally, take $T > 0$ so large that $u(t,x) = 0$ for $t \ge T$ and by using Remark 2.5.4 with reversed time axis, find a function v which belongs to $W_2^{1,2}(\mathbb{R}_S^{d+1})$ for all $S > -\infty$ and such that $v(t,x) = 0$ for $t \ge T$ and

$$Lv + v_t = h \quad \text{in} \quad \mathbb{R}^{d+1}.$$

By Corollary 2.4.3 the function v is infinitely differentiable. Since $h = 0$ in $Q_{\nu r/2}$ and $\nu/2 \ge 2$, by Theorem 2.6 we obtain

$$\begin{aligned}\fint_{Q_r} |v_{xx} - (v_{xx})_{Q_r}|^2\,dxdt &\le N\nu^{-2}\fint_{Q_{\nu r/2}} |v_{xx}|^2\,dxdt\\&\le N\nu^{-2}\fint_{Q_{\nu r}} |v_{xx}|^2\,dxdt.\end{aligned}\tag{2}$$

On the other hand the function $w := u - v \in W_2^{1,2}(\mathbb{R}_0^{d+1})$ satisfies

$$Lw + w_t = g$$

in $\mathbb{R}_0^{d+1}$ and by Theorem 2.2.12

$$\int_{\mathbb{R}_0^{d+1}} |w_{xx}|^2\,dxdt \le N\int_{\mathbb{R}_0^{d+1}} |g|^2\,dxdt \le N\int_{Q_{\nu r}} |f|^2\,dxdt.\tag{3}$$

$$\int_{Q_r} |w_{xx}|^2\,dxdt \leq N \int_{Q_{\nu r}} |f|^2\,dxdt,$$

$$\fint_{Q_r} |w_{xx}|^2\,dxdt \leq N\nu^{d+2} \fint_{Q_{\nu r}} |f|^2\,dxdt.$$

By combining this with (2) and observing that $u = v + w$ and

$$I := \fint_{Q_r} |u_{xx} - (u_{xx})_{Q_r}|^2\,dxdt$$

$$\leq 2 \fint_{Q_r} |w_{xx} - (w_{xx})_{Q_r}|^2\,dxdt + 2 \fint_{Q_r} |v_{xx} - (v_{xx})_{Q_r}|^2\,dxdt$$

$$\leq 2 \fint_{Q_r} |w_{xx}|^2\,dxdt + 2 \fint_{Q_r} |v_{xx} - (v_{xx})_{Q_r}|^2\,dxdt,$$

we get

$$I \leq N\nu^{d+2} \fint_{Q_{\nu r}} |f|^2\,dxdt + N\nu^{-2} \fint_{Q_{\nu r}} |v_{xx}|^2\,dxdt$$

$$\leq N\nu^{d+2} \fint_{Q_{\nu r}} |f|^2\,dxdt + N\nu^{-2} \fint_{Q_{\nu r}} |u_{xx}|^2\,dxdt + N\nu^{-2} \fint_{Q_{\nu r}} |w_{xx}|^2\,dxdt.$$

Here by (3)

$$\fint_{Q_{\nu r}} |w_{xx}|^2\,dxdt \leq N \fint_{Q_{\nu r}} |f|^2\,dxdt$$

and since $\nu \geq 1$, we conclude

$$I \leq N\nu^{d+2} \fint_{Q_{\nu r}} |f|^2\,dxdt + N\nu^{-2} \fint_{Q_{\nu r}} |u_{xx}|^2\,dxdt.$$

The theorem is proved.

2. Exercise. Assume that the a^{ij} are independent of t and notice that equation (1) implies that there is a constant N, depending only on d and κ, such that for any $u \in W^2_{2,loc}$, $r \in (0, \infty)$, and $\nu \geq 4$

$$\fint_{B_r} |u_{xx}(t,x) - (u_{xx})_{B_r}|^2\,dx \leq N\nu^{d+2} \fint_{B_{\nu r}} |Lu|^2\,dx + N\nu^{-2} \fint_{B_{\nu r}} |u_{xx}|^2\,dx.$$

Prove this result even with ν^d in place of ν^{d+2} without using (1) and, for that matter, anything from the above theory of parabolic equations.

3. Exercise. Improve the result of Exercise 2 by showing that under its conditions with the same N it holds that

$$\fint_{B_r} |u_{xx}(t,x) - (u_{xx})_{B_r}|^2\,dx \le N\nu^d \fint_{B_{\nu r}} |Lu - (Lu)_{B_{\nu r}}|^2\,dx$$

$$+ N\nu^{-2} \fint_{B_{\nu r}} |u_{xx} - (u_{xx})_{B_{\nu r}}|^2\,dx.$$

4. Exercise. Take a $\delta > 0$ and show that there are no (finite) $N_1 = N_1(d,\nu)$ and $N_2 = N_2(d)$ such that for all $u \in C_0^\infty$ and $\nu \ge 4$ we have

$$\fint_{B_1} |u_{xx} - (u_{xx})_{B_1}|^2\,dx \le N_1 \fint_{B_\nu} |\Delta u|^2\,dx + N_2\nu^{-2-\delta} \fint_{B_\nu} |u_{xx}|^2\,dx.$$

The following exercises can be used to derive the basic $C^{2+\alpha}$ estimates for elliptic and parabolic equations (see Exercises 10.1.9 and 10.1.10).

5. Exercise. Improve (1) by showing that under the conditions of Theorem 1

$$\fint_{Q_r} |u_{xx}(t,x) - (u_{xx})_{Q_r}|^2\,dxdt$$

$$\le N\nu^{d+2} \fint_{Q_{\nu r}} |f - f_{B_{\nu r}}|^2\,dxdt + N\nu^{-2} \fint_{Q_{\nu r}} |u_{xx}|^2\,dxdt.$$

6. Exercise. Improve (1) by showing that under the conditions of Theorem 1

$$\fint_{Q_r} |u_{xx}(t,x) - (u_{xx})_{Q_r}|^2\,dxdt$$

$$\le N\nu^{d+2} \fint_{Q_{\nu r}} |f - f_{B_{\nu r}}|^2\,dxdt + N\nu^{-2} \fint_{Q_{\nu r}} |u_{xx} - (u_{xx})_{Q_{\nu r}}|^2\,dxdt.$$

Here are the basic a priori $\mathcal{L}_p$ estimates for parabolic and elliptic equations for $p \ge 2$. We extend them for $p \in (1,2)$ in Theorem 8.

7. Theorem. *Let $p \in [2,\infty)$. Then there is a constant $N = N(d,\kappa,p)$ such that*

(i) for any $u \in W_p^{1,2}$ we have

$$\|u_{xx}\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} + \|u_t\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} \le N\|Lu + u_t\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}; \tag{4}$$

(ii) if additionally the coefficients of L are independent of t, then for any $u \in W_p^2$

$$\|u_{xx}\|_{\mathcal{L}_p} \le N\|Lu\|_{\mathcal{L}_p}.$$

Proof. First observe that, if $p = 2$, the result is known from before. Therefore, we assume that $p > 2$ and fix a $\nu \geq 4$ to be specified later. Then, any $Q \in \mathbb{Q}$ is written as $Q = Q_r(t, x)$ and we define a unique Q' by setting $Q' = Q_{\nu r}(t, x)$. Upon observing that by Hölder's inequality

$$\left(\fint_Q |u_{xx} - (u_{xx})_Q| \, dxdt \right)^2 \leq \fint_Q |u_{xx} - (u_{xx})_Q|^2 \, dxdt$$

and applying (1) to the translates of Q_r, we get for any $Q \in \mathbb{Q}$

$$\left(\fint_Q |u_{xx}(t, x) - (u_{xx})_Q| \, dxdt \right)^2$$

$$\leq N\nu^{d+2} \fint_{Q'} |f|^2 \, dxdt + N\nu^{-2} \fint_{Q'} |u_{xx}|^2 \, dxdt,$$

where $f = Lu + u_t$. Notice that for any $(t, x) \in Q$ we have

$$\fint_{Q'} |u_{xx}|^2 \, dxdt \leq \mathbb{M}(|u_{xx}|^2)(t, x), \quad \fint_{Q'} |f|^2 \, dxdt \leq \mathbb{M}(|f|^2)(t, x).$$

It follows that for any $(t, x) \in \mathbb{R}^d$ and $Q \in \mathbb{Q}$ such that $Q \ni (t, x)$ it holds that

$$|u_{xx} - (u_{xx})_Q|_Q \leq N\nu^{(d+2)/2}\mathbb{M}^{1/2}(f^2)(t, x) + N\nu^{-1}\mathbb{M}^{1/2}(|u_{xx}|^2)(t, x),$$

that is,

$$(u_{xx})^{\sharp} \leq N\nu^{(d+2)/2}\mathbb{M}^{1/2}(f^2) + N\nu^{-1}\mathbb{M}^{1/2}(|u_{xx}|^2) \tag{5}$$

on $\mathbb{R}^{d+1}$. By the Hardy-Littlewood theorem (Theorem 3.3.2) and the fact that $p/2 > 1$

$$\|\mathbb{M}^{1/2}(f^2)\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} = \|\mathbb{M}(f^2)\|^{1/2}_{\mathcal{L}_{p/2}(\mathbb{R}^{d+1})}$$

$$\leq N\|f^2\|^{1/2}_{\mathcal{L}_{p/2}(\mathbb{R}^{d+1})} = N\|f\|^{1/2}_{\mathcal{L}_p(\mathbb{R}^{d+1})}.$$

A similar estimate of the second term on the right in (5) and the Fefferman-Stein theorem (see Theorems 3.2.10 and 3.3.1) allow us to conclude that

$$\|u_{xx}\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} \leq N\nu^{(d+2)/2}\|f\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} + N_1\nu^{-1}\|u_{xx}\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}.$$

In the above, all constants N depend only on d, κ, and p and $\nu \geq 4$ is arbitrary. By taking ν such that $N_1\nu^{-1} \leq 1/2$, we finish proving (4) in what concerns u_{xx}. One gets the estimate of u_t from the equation $u_t = f - a^{ij}u_{x^ix^j}$.

Assertion (ii) is proved in the same way on the basis of either Exercise 2 or estimate (1) where we take u independent of t which allow us to replace Q_ρ with B_ρ. The theorem is proved.

Here are the basic existence and uniqueness results for parabolic equations with coefficients depending only on t and for elliptic equations with constant coefficients.

8. Theorem. *Let $p \in (1, \infty)$, $\lambda > 0$, $T \in [-\infty, \infty)$.*

(i) Then for any $f \in \mathcal{L}_p(\mathbb{R}^{d+1}_T)$ there is a unique $u \in W^{1,2}_p(\mathbb{R}^{d+1}_T)$ such that

$$Lu + u_t - \lambda u = f. \tag{6}$$

Furthermore, for an $N = N(d, \kappa, p)$ and any $u \in W^{1,2}_p(\mathbb{R}^{d+1}_T)$ we have

$$\|u_t\|_{\mathcal{L}_p(\mathbb{R}^{d+1}_T)} + \|u_{xx}\|_{\mathcal{L}_p(\mathbb{R}^{d+1}_T)} + \lambda\|u\|_{\mathcal{L}_p(\mathbb{R}^{d+1}_T)} \leq N\|Lu + u_t - \lambda u\|_{\mathcal{L}_p(\mathbb{R}^{d+1}_T)}. \tag{7}$$

(ii) Assume that the coefficients of L are independent of t (that is, they are constant). Then for any $f \in \mathcal{L}_p$ there is a unique $u \in W^2_p$ such that

$$Lu - \lambda u = f.$$

Furthermore, for an $N = N(d, \kappa, p)$ and any $u \in W^2_p$ we have

$$\|u_{xx}\|_{\mathcal{L}_p} + \lambda\|u\|_{\mathcal{L}_p} \leq N\|Lu - \lambda u\|_{\mathcal{L}_p}. \tag{8}$$

Proof. (i) First we consider the case that $T = \infty$. By Theorem 2.1.6 the set $(\partial_t + \Delta - \lambda)C_0^\infty(\mathbb{R}^{d+1})$ is dense in $\mathcal{L}_p(\mathbb{R}^{d+1})$. Therefore, if we knew that the a priori estimate (7) holds, we could first conclude that equation (6) is solvable for $L = \Delta$ as in Theorems 1.3.16 and after that for arbitrary L under consideration by using the method of continuity. Therefore, we need only prove (7).

Case $p \geq 2$. By Theorem 7

$$\|u_{xx}\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} + \|u_t\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} \leq N\|Lu + u_t - \lambda u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} + N\lambda\|u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}$$

and it only remains to notice that by Lemma 2.2.11

$$\lambda\|u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} \leq \|Lu + u_t - \lambda u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}. \tag{9}$$

Case $1 < p \leq 2$. Here we are going to use a duality argument. Observe that again we need only prove (7) and it suffices to do that assuming that $u \in C_0^\infty(\mathbb{R}^{d+1})$. Changing the variable $t \to -t$ shows that the result of case $p \geq 2$ is applicable to the operator $L - \partial_t - \lambda$ in place of $L + \partial_t - \lambda$. Therefore, for $u, v \in C_0^\infty(\mathbb{R}^{d+1})$ by integrating by parts we find

$$\int_{\mathbb{R}^{d+1}} u_{xx}(Lv - v_t - \lambda v)\,dxdt = \int_{\mathbb{R}^{d+1}} v_{xx}(Lu + u_t - \lambda u)\,dxdt$$

$$\leq \|v_{xx}\|_{\mathcal{L}_q(\mathbb{R}^{d+1})}\|f\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} \leq N\|Lv - v_t - \lambda v\|_{\mathcal{L}_q(\mathbb{R}^{d+1})}\|f\|_{\mathcal{L}_p(\mathbb{R}^{d+1})},$$

where

$$f := Lu + u_t - \lambda u \tag{10}$$

and $q = p/(p-1) \geq 2$. Since $(L - \partial_t - \lambda)C_0^\infty(\mathbb{R}^{d+1})$ is dense in $\mathcal{L}_q(\mathbb{R}^{d+1})$, we conclude that for any $g \in \mathcal{L}_q(\mathbb{R}^{d+1})$

$$\int_{\mathbb{R}^{d+1}} u_{xx} g\,dxdt \leq N_1\|g\|_{\mathcal{L}_q(\mathbb{R}^{d+1})}\|f\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}.$$

By a well-known fact from integration theory we infer

$$\|u_{xx}\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} \leq N_1\|f\|_{\mathcal{L}_p(\mathbb{R}^{d+1})},$$

which is part of (7).

Next, for the same reason as above, (9) holds again. One gets the estimate of u_t either similarly to that of u_{xx} or from the equation $u_t = f + \lambda u - Lu$. This proves (7) for $1 < p \leq 2$ and finishes our argument in the case $T = -\infty$.

In the case $T > -\infty$, take $f \in \mathcal{L}_p(\mathbb{R}_T^{d+1})$, introduce $g(t,x) = f(t,x)$ for $t \geq T$ and $g(t,x) = 0$ for $t < T$, and solve the equation

$$Lv + v_t - \lambda v = g$$

in $W_p^{1,2}(\mathbb{R}^{d+1})$. Obviously $v \in W_p^{1,2}(\mathbb{R}_T^{d+1})$ and this proves the existence assertion in (i).

Uniqueness will follow from estimate (7), to prove which we take a $u \in W_p^{1,2}(\mathbb{R}_T^{d+1})$, define f as in (10), and introduce g and v as above. Then according to the result for the case $T = -\infty$, estimate (7) holds with v

in place of u on the left. It only remains to observe that, as follows from Lemma 2.2.11 and the structure of $\mathcal{G}_\lambda$, for almost all (t,x) with $t \geq T$

$$v(t,x) = \mathcal{G}_\lambda g(t,x) = \mathcal{G}_\lambda f(t,x) = u(t,x).$$

Assertion (ii) of the theorem is obtained either as assertion (i) in the case $T = -\infty$ or as in Exercise 9 below. The theorem is proved.

9. Exercise. By continuity, estimate (7) holds with $\lambda = 0$. Derive from this fact alone that (8) for L with constant coefficients holds first for all $\lambda > 0$ and then for $\lambda = 0$ as well.

The following exercise contains a simple but rather unexpected result.

10. Exercise. Let $b(t)$ be a bounded $\mathbb{R}^d$-valued function. Prove that with the *same* constant N as in (7) (in particular, independent of b) for any $u \in W_p^{1,2}(\mathbb{R}_T^{d+1})$ and $\lambda \geq 0$ we have

$$\|u_t\|_{\mathcal{L}_p(\mathbb{R}_T^{d+1})} + \|u_{xx}\|_{\mathcal{L}_p(\mathbb{R}_T^{d+1})} + \lambda\|u\|_{\mathcal{L}_p(\mathbb{R}_T^{d+1})} \leq N\|Lu + b^i u_{x^i} + u_t - \lambda u\|_{\mathcal{L}_p(\mathbb{R}_T^{d+1})}.$$

By using the method of continuity, prove the corresponding existence theorem assuming that $\lambda > 0$.

11. Exercise. Similarly to Exercise 9 derive from Exercise 10 that, if the coefficients of L are independent of t, then with the *same* constant N as in (7) for any $u \in W_p^2$, constant vector $b \in \mathbb{R}^d$, and $\lambda \geq 0$ we have

$$\|u_{xx}\|_{\mathcal{L}_p} + \lambda\|u\|_{\mathcal{L}_p} \leq N\|Lu + b^i u_{x^i} - \lambda u\|_{\mathcal{L}_p}.$$

State and prove the corresponding existence theorem assuming that $\lambda > 0$.

The following theorem says that the solution is independent of the space in which we solve the equation.

12. Theorem. *Let $p, q \in (1,\infty)$, $\lambda > 0$, $T \in [-\infty,\infty)$.*

(i) If $u \in W_p^{1,2}(\mathbb{R}_T^{d+1})$ and

$$Lu + u_t - \lambda u \in \mathcal{L}_q(\mathbb{R}_T^{d+1}),$$

then $u \in W_q^{1,2}(\mathbb{R}_T^{d+1})$.

(ii) Assume that the coefficients of L are independent of t, $u \in W_p^2$, and $Lu - \lambda u \in \mathcal{L}_q$. Then $u \in W_q^2$.

Proof. As is easy to see, owing to uniqueness, assertion (i) means that if

$$f \in \mathcal{L}_p(\mathbb{R}_T^{d+1}) \cap \mathcal{L}_q(\mathbb{R}_T^{d+1})$$

and the functions $u \in W_p^{1,2}(\mathbb{R}_T^{d+1})$ and $v \in W_q^{1,2}(\mathbb{R}_T^{d+1})$ satisfy (6), then $u = v$ in $\mathbb{R}_T^{d+1}$. We see that p and q play symmetric roles and, therefore, may assume that $q \le p$.

Next, take a function u satisfying the assumptions in (i) and set $f = Lu + u_t - \lambda u$. Note that we can confine ourselves to the case that

$$\frac{1}{q} - \frac{1}{p} \le \frac{1}{d+1}. \tag{11}$$

Indeed, one can find a decreasing sequence $q_i \in [q, p]$, $i = 0, ..., m$, where m is finite and depends only on $q^{-1} - p^{-1}$ and d, such that $q_0 = p, q_m = q$, and

$$\frac{1}{q_{i+1}} \le \frac{1}{q_i} + \frac{1}{d+1}.$$

Since $|f|^{q_i} \le |f|^q + |f|^p$, we have that $f \in \mathcal{L}_{q_i}(\mathbb{R}_T^{d+1})$ for all i. Then if (i) is true under assumption (11), by induction we see that $u \in W_{q_i}^{1,2}(\mathbb{R}_T^{d+1})$ for all i and, in particular, for $i = m$.

Assuming (11), take a $\zeta \in C_0^\infty(\mathbb{R}^{d+1})$ such that $\zeta(0, 0) = 1$ and set $\zeta_n(t, x) = \zeta(t/n, x/n)$. Then $u\zeta_n \in W_q^{1,2}(\mathbb{R}_T^{d+1})$ and

$$(L + \partial_t - \lambda)(u\zeta_n) = f\zeta_n + u(L + \partial_t)\zeta_n + 2a^{ij}u_{x^i}\zeta_{nx^j}.$$

By Theorem 8

$$\|u\zeta_n\|^q_{W_q^{1,2}(\mathbb{R}_T^{d+1})} \le N\|f\|^q_{\mathcal{L}_q(\mathbb{R}^{d+1})} + N(I_{1n} + I_{2n}), \tag{12}$$

where the constants N are independent of n and

$$I_{1n} = \|u(L + \partial_t)\zeta_n\|^q_{\mathcal{L}_q(\mathbb{R}_T^{d+1})}, \quad I_{2n} = \|a^{ij}u_{x^i}\zeta_{nx^j}\|^q_{\mathcal{L}_q(\mathbb{R}_T^{d+1})}.$$

Now, owing to Fatou's lemma, to prove that $u \in W_q^{1,2}(\mathbb{R}_T^{d+1})$, it suffices to show that the right-hand side of (12) is bounded by a constant independent of n. If $p = q$, this is obvious since the derivatives of ζ_n are uniformly bounded. If $q < p$, by Hölder's inequality

$$\begin{aligned} I_{2n} &\le Nn^{-q} \int_{\mathbb{R}_T^{d+1}} |u_x(t, x)|^q |\zeta_x(t/n, x/n)|^q \, dxdt \\ &\le Nn^{-q}\|u_x\|^q_{\mathcal{L}_p(\mathbb{R}_T^{d+1})} \Big(\int_{\mathbb{R}_T^{d+1}} |\zeta_x(t/n, x/n)|^{pq/(p-q)} \, dxdt \Big)^{(p-q)/p} \\ &= Nn^{(d+1)(p-q)/p-q}\|u_x\|^q_{\mathcal{L}_p(\mathbb{R}_T^{d+1})} \Big(\int_{\mathbb{R}_T^{d+1}} |\zeta_x(t, x)|^{pq/(p-q)} \, dxdt \Big)^{1-q/p}. \end{aligned}$$

Here $(d+1)(p-q)/p - q \leq 0$ by (11) implying that the sequence I_{2n} is bounded. Similarly, I_{1n} is bounded. Therefore, the right-hand side of (12) is indeed bounded by a constant independent of n, and assertion (i) is proved.

In the same way one proves (ii). Alternatively, one can use Exercise 13. The theorem is proved.

13. Exercise. Derive assertion (ii) of Theorem 12 directly from assertion (i) for $T = 0$ by considering $u(x)e^{-\mu t}$ with an appropriate $\mu > 0$.

4. Divergence form of the right-hand side for the Laplacian

Let $p \in (1,\infty)$, $f_1, ..., f_d, g \in \mathcal{L}_p$, $\lambda > 0$ and consider the equation

$$\Delta u - \lambda u = D_i f_i + g. \tag{1}$$

Generally the right-hand side of (1) is not a function but a distribution and we understand this equation in the sense of distributions. We only consider equations with the operator Δ. Equations with other elliptic operators with constant coefficients can be either treated similarly or reduced to (1) by linear changes of coordinates. The reader will find in Section 13.6 a generalization of the result of this section to divergence form equations with continuous coefficients. Finally, before we go to the main contents of this section, we mention that the results of this section are only used in Section 13.6 and in the proof of Theorem 8.2.8, which in turn is only used in our investigation of the oblique derivative boundary-value problem in Section 9.3.

Recall that a distribution u on $\mathbb{R}^d$ is a linear functional on the space C_0^∞: $u = u(\phi), \phi \in C_0^\infty$, continuous in the sense that if we have $\phi_n, \phi \in C_0^\infty$, $n = 1, 2, ...$, such that $\phi = \phi_n = 0$ outside a ball B and $D^\alpha \phi_n \to D^\alpha \phi$ uniformly on B for any multi-index α, then $u(\phi_n) \to u(\phi)$. By common abuse of notation one writes the value $u(\phi)$ of u at ϕ as

$$\int_{\mathbb{R}^d} \phi(x)u(x)\,dx \quad \text{or} \quad \int_{\mathbb{R}^d} u(x)\phi(x)\,dx.$$

An advantage of this notation is that if u is locally integrable, then the above integrals make sense for any $\phi \in C_0^\infty$ and define a distribution.

If $u(\phi)$ is a distribution and α is a multi-index, then $u(D^\alpha\phi)$ is a distribution as well. This distribution is called $(-1)^{|\alpha|}D^\alpha u$. Thus,

$$\int_{\mathbb{R}^d} \phi(x)D^\alpha u(x)\,dx = (-1)^{|\alpha|}\int_{\mathbb{R}^d} u(x)D^\alpha\phi(x)\,dx.$$

We see that the distributions are infinitely differentiable (in the generalized sense) and since locally integrable functions are distributions, they are also infinitely differentiable. This is the sense in which (1) is understood.

1. Exercise*. Let w be a distribution and $\phi \in C_0^\infty$. Prove that the function

$$\psi(y) := \int_{\mathbb{R}^d} \phi(y+x)w(x)\,dx \tag{2}$$

is infinitely differentiable in the classical sense and for any multi-index α

$$D^\alpha\psi(y) = \int_{\mathbb{R}^d} w(x)D^\alpha\phi(y+x)\,dx = (-1)^{|\alpha|}\int_{\mathbb{R}^d} \phi(y+x)D^\alpha w(x)\,dx.$$

2. Theorem. *Under the above assumptions, equation* (1) *has a solution* $u \in W_p^1$. *This solution satisfies*

$$\|u_x\|_{\mathcal{L}_p} + \lambda^{1/2}\|u\|_{\mathcal{L}_p} \le N(d,p)(\sum_i \|f_i\|_{\mathcal{L}_p} + \lambda^{-1/2}\|g\|_{\mathcal{L}_p}). \tag{3}$$

Furthermore, equation (1) *can only have one solution in* $\mathcal{L}_p$.

Proof. To prove existence, introduce $v_i, v \in W_p^2$ as solutions of

$$\Delta v_i - \lambda v_i = f_i, \quad \Delta v - \lambda v = g.$$

By Theorem 3.8 and interpolation inequalities such solutions exist and

$$\|v_{ixx}\|_{\mathcal{L}_p} + \lambda^{1/2}\|v_{ix}\|_{\mathcal{L}_p} \le N(d,p)\|f_i\|_{\mathcal{L}_p}, \tag{4}$$

$$\|v_x\|_{\mathcal{L}_p} + \lambda^{1/2}\|v\|_{\mathcal{L}_p} \le N(d,p)\lambda^{-1/2}\|g\|_{\mathcal{L}_p}. \tag{5}$$

Obviously, $u := D_i v_i + v$ is in W_p^1 and solves (1). Upon combining (4) and (5), we come to (3).

To prove uniqueness, observe that if we have two solutions in $\mathcal{L}_p$, then their difference $w \in \mathcal{L}_p$ solves (1) with $f_i = g = 0$, that is,

$$\int_{\mathbb{R}^d} (\Delta\phi(x) - \lambda\phi(x))w(x)\,dx = 0$$

for any $\phi \in C_0^\infty$. Substitute here $\phi(y+x)$ in place of $\phi(x)$. Then by Exercise 1, in notation (2) we find that the infinitely differentiable function ψ satisfies $\Delta\psi - \lambda\psi = 0$. Furthermore, Hölder's inequality and the fact that $w \in \mathcal{L}_p$ show that ψ is bounded. By Lemma 1.1.7 we have $\psi = 0$. In particular, $\psi(0) = 0$. The arbitrariness of ϕ now leads to the conclusion that $u = 0$. The theorem is proved.

5. Hints to exercises

1.3. Take $d = 2$ and start with $u(x, y) = xy$.

2.4. Observe that in the notation from the proof of Lemma 2.3

$$v_x(t, x) = -\int_0^\infty \int_{\mathbb{R}^d} g(t + s, x + y)p_y(s, y)\, dyds.$$

Then, use an elementary inequality

$$x^\alpha e^{-\beta x} \le Ne^{-\beta x/2}, \quad \forall x \ge 0,$$

where $\alpha, \beta > 0$ and $N = N(\alpha, \beta)$.

3.2. Follow the proof of Theorem 3.1 but first find a w. For that use Exercise 1.3.23 and a linear transformation to find an ellipsoid $E \supset B_{\nu r}$ and a $w \in W_2^2(E)$ such that $Lw = g$ in E and

$$\|w_{xx}\|_{\mathcal{L}_2(E)} \le N(d, \kappa)\|g\|_{\mathcal{L}_2(E)}.$$

Observe that N is independent of the size of E which is seen from using dilations.

3.3. Consider $u - b^{ij}x^i x^j$ for appropriate constants b^{ij}.

3.4. Start with v such that $\Delta v = 0$ and such that v is a polynomial of the third order.

3.5. Consider $u(t, x) - g(t)$ for an appropriate g.

3.6. Use Exercise 3.5 and consider $u(t, x) - a^{ij}x^i x^j$ for appropriate constants a^{ij}.

3.9. Consider the function $u(x)e^{-\lambda t}$ in $\mathbb{R}_0^{d+1}$.

3.10. Compute $Lv + v_t - \lambda v$ for $v(t, x) = u(t, x + B_t)$, where

$$B(t) = \int_0^t b(s)\, ds.$$

3.11. Use the hint to Exercise 3.9.

Chapter 5

Parabolic and elliptic equations in $W_p^{1,k}$ and W_p^k

Throughout this rather short chapter we fix a $p \in (1, \infty)$. Here we present some results about the solvability of parabolic and elliptic equations in $W_p^{1,2}$ and W_p^2 and about better regularity of solutions in x if the right-hand sides are more regular. Given the basic a priori estimates obtained in Chapter 4, the exposition in the present chapter follows very closely (almost literally) that in Chapters 1 and 2.

1. Better regularity for equations with coefficients independent of x

We take an operator

$$Lu(t, x) = a^{ij}(t) u_{x^i x^j}(t, x)$$

with measurable coefficients depending only on $t \in \mathbb{R}$ and such that $a^{ij} = a^{ji}$ and

$$\kappa^{-1}|\xi|^2 \geq a^{ij}(t)\xi^i\xi^j \geq \kappa|\xi|^2$$

for all $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$, where $\kappa \in (0, 1)$ is a fixed constant.

First we generalize Corollary 1.7.2 and Theorem 2.2.12.

1. Theorem. *(i) Let $\lambda > 0$, $T \in [-\infty, \infty)$, $k \geq 0$, $f \in W_p^{0,k}(\mathbb{R}_T^{d+1})$. Then there exists a unique solution $u \in W_p^{1,k+2}(\mathbb{R}_T^{d+1})$ of the equation*

$$\lambda u - u_t - Lu = f. \tag{1}$$

Furthermore, for any $u \in W_p^{1,k+2}(\mathbb{R}_T^{d+1})$ we have

$$\|u_t\|_{W_p^{0,k}(\mathbb{R}_T^{d+1})} + \|u_{xx}\|_{W_p^{0,k}(\mathbb{R}_T^{d+1})} + \lambda \|u\|_{W_p^{0,k}(\mathbb{R}_T^{d+1})}$$

$$\leq N(d, \kappa, p)\|u_t + Lu - \lambda u\|_{W_p^{0,k}(\mathbb{R}_T^{d+1})}. \tag{2}$$

(ii) Let $\lambda > 0$ and let L be an operator with constant coefficients. Let $k \geq 0$, $f \in W_p^k$. Then there exists a unique solution $u \in W_p^{k+2}$ of the equation

$$\lambda u - Lu = f.$$

Furthermore, for any $u \in W_p^{k+2}$ we have

$$\|u_{xx}\|_{W_p^k} + \lambda \|u\|_{W_p^k} \leq N(d, \kappa, p)\|Lu - \lambda u\|_{W_p^k}. \tag{3}$$

Proof. (i) Basically, we repeat part of the proof of Lemma 2.2.11. If $k = 0$, the result is contained in Theorem 4.3.8 and by Lemma 2.2.11 the $W_p^{1,2}(\mathbb{R}_T^{d+1})$ solution of (1) is given by the formula $u = \mathcal{G}_\lambda f$. It also follows from these results that for any $f \in \mathcal{L}_p(\mathbb{R}_T^{d+1})$

$$\|(\mathcal{G}_\lambda f)_{xx}\|_{\mathcal{L}_p(\mathbb{R}_T^{d+1})} + \|(\mathcal{G}_\lambda f)_t\|_{\mathcal{L}_p(\mathbb{R}_T^{d+1})}$$

$$+\lambda \|(\mathcal{G}_\lambda f)\|_{\mathcal{L}_p(\mathbb{R}_T^{d+1})} \leq N(d, \kappa, p)\|f\|_{\mathcal{L}_p(\mathbb{R}_T^{d+1})}. \tag{4}$$

The rules of differentiating integrals show that if $f \in C_0^{0,k}(\mathbb{R}_T^{d+1})$, then $\mathcal{G}_\lambda f \in C^{0,k}(\mathbb{R}_T^{d+1})$ and for any multi-index α with $|\alpha| \leq k$

$$D^\alpha \mathcal{G}_\lambda f = \mathcal{G}_\lambda D^\alpha f.$$

This and (4) imply that for $f \in C_0^{0,k}(\mathbb{R}_T^{d+1})$ we have

$$\|(D^\alpha \mathcal{G}_\lambda f)_{xx}\|_{\mathcal{L}_p(\mathbb{R}_T^{d+1})} + \|(D^\alpha \mathcal{G}_\lambda f)_t\|_{\mathcal{L}_p(\mathbb{R}_T^{d+1})}$$

$$+\lambda \|(D^\alpha \mathcal{G}_\lambda f)\|_{\mathcal{L}_p(\mathbb{R}_T^{d+1})} \leq N(d, \kappa, p)\|D^\alpha f\|_{\mathcal{L}_p(\mathbb{R}_T^{d+1})}.$$

It follows that

$$\|(\mathcal{G}_\lambda f)_{xx}\|_{W_p^{0,k}(\mathbb{R}_T^{d+1})} + \|(\mathcal{G}_\lambda f)_t\|_{W_p^{0,k}(\mathbb{R}_T^{d+1})}$$

$$+\lambda\|\mathcal{G}_\lambda f\|_{W_p^{0,k}(\mathbb{R}_T^{d+1})} \le N(d,\kappa,p)\|f\|_{W_p^{0,k}(\mathbb{R}_T^{d+1})}. \tag{5}$$

One carries this estimate over to arbitrary $f \in W_p^{0,k}(\mathbb{R}_T^{d+1})$ by using the completeness of $W_p^{1,k+2}(\mathbb{R}_T^{d+1})$ and the denseness of $C_0^{0,k}(\mathbb{R}_T^{d+1})$ in $W_p^{0,k}(\mathbb{R}_T^{d+1})$. Thus, $\mathcal{G}_\lambda f$ is a solution of class $W_p^{1,k+2}(\mathbb{R}_T^{d+1})$. Uniqueness even in the larger class $W_p^{1,2}(\mathbb{R}_T^{d+1})$ is given by Theorem 4.3.8.

Finally, by Lemma 2.2.11 any $u \in W_p^{1,k+2}(\mathbb{R}_T^{d+1})$ is written as $\mathcal{G}_\lambda f$ with

$$f := \lambda u - u_t - Lu \in W_p^{0,k}(\mathbb{R}_T^{d+1}),$$

which along with (5) proves (2).

(ii) Estimate (3) is obtained from (2) by setting $T = \lambda = 0$ *there* and then resurrecting $\lambda > 0$ and substituting $e^{-\lambda t}u(x)$ in place of $u(t,x)$. The existence of $u \in W_p^2$ such that $\lambda u - Lu = f$ with $f \in \mathcal{L}_p$ we know from Theorem 4.3.8. If $f \in W_p^k$, then the mollified functions $u^{(\varepsilon)}$ satisfy $\lambda u^{(\varepsilon)} - Lu^{(\varepsilon)} = f^{(\varepsilon)}$ and $u^{(\varepsilon)} \in W_p^n$ for all n. After that, it only remains to use (3) with $u^{(\varepsilon)}$ in place of u, let $\varepsilon \downarrow 0$ and use Theorem 1.8.5.

The theorem is proved.

2. Equations with continuous coefficients. The Cauchy problem

First we are considering the equation

$$u_t + Lu - \lambda u = f \tag{1}$$

in $\mathbb{R}_T^{d+1}$, where $T \in [-\infty, \infty)$ and

$$Lu(t,x) = a^{ij}(t,x)u_{x^ix^j}(t,x) + b^i(t,x)u_{x^i}(t,x) + c(t,x)u(t,x).$$

We assume that the coefficients a, b, and c are measurable functions of $(t,x) \in \mathbb{R}^{d+1}$, $a^{ij} = a^{ji}$, and for some constants $\kappa > 0$ and $K \in (0,\infty)$ for all values of the arguments and $\xi \in \mathbb{R}^d$ it holds that

$$|b| + |c| \le K, \quad c \le 0, \quad \kappa^{-1}|\xi|^2 \ge a^{ij}\xi^i\xi^j \ge \kappa|\xi|^2,$$

where $b = (b^1, \dots, b^d)$. We also assume that there exists an increasing function $\omega(\varepsilon)$, $\varepsilon \geq 0$, such that $\omega(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ and for all $t \in \mathbb{R}$, $x, y \in \mathbb{R}^d$, and $i, j = 1, \dots, d$ we have

$$|a^{ij}(t,x) - a^{ij}(t,y)| \leq \omega(|x-y|). \tag{2}$$

When the coefficients of L are independent of t, we also consider the equation

$$Lu(x) - \lambda u(x) = f(x) \tag{3}$$

in $\mathbb{R}^d$.

1. Theorem. *There exist constants $\lambda_0 \geq 1$ and N_0 depending only on K, κ, ω, p, and d such that the following hold.*

(i) The estimate

$$\lambda\|u\|_{\mathcal{L}_p(\mathbb{R}^{d+1}_T)} + \lambda^{1/2}\|u_x\|_{\mathcal{L}_p(\mathbb{R}^{d+1}_T)} + \|u_{xx}\|_{\mathcal{L}_p(\mathbb{R}^{d+1}_T)} + \|u_t\|_{\mathcal{L}_p(\mathbb{R}^{d+1}_T)}$$

$$\leq N_0\|Lu + u_t - \lambda u\|_{\mathcal{L}_p(\mathbb{R}^{d+1}_T)}$$

is true for any $\lambda \geq \lambda_0$ and $u \in W^{1,2}_p(\mathbb{R}^{d+1}_T)$. Furthermore, for any $\lambda \geq \lambda_0$ and $f \in \mathcal{L}_p(\mathbb{R}^{d+1}_T)$ there exists a unique $u \in W^{1,2}_p(\mathbb{R}^{d+1}_T)$ satisfying (1) *in $\mathbb{R}^{d+1}_T$.*

(ii) The estimate

$$\lambda\|u\|_{\mathcal{L}_p} + \lambda^{1/2}\|u_x\|_{\mathcal{L}_p} + \|u_{xx}\|_{\mathcal{L}_p} \leq N_0\|Lu - \lambda u\|_{\mathcal{L}_p}$$

is true for any $\lambda \geq \lambda_0$ and $u \in W^2_p$. Furthermore, for any $\lambda \geq \lambda_0$ and $f \in \mathcal{L}_p$ there exists a unique $u \in W^2_p$ satisfying (3).

This theorem is similar to Theorem 2.3.1 and as in the proof of the latter we obtain the result by just repeating the proof of Theorem 1.6.4 for elliptic equations. Of course, in the parabolic case while "freezing" the coefficients, we only freeze the variable x and not t, so that we start with the formulas

$$|u_{xx}(t,x)|^p = \int_{\mathbb{R}^d} |u_{xx}(t,x)|^p \zeta^p(x-y)\,dy,$$

$$|u_t(t,x)|^p = \int_{\mathbb{R}^d} |u_t(t,x)|^p \zeta^p(x-y)\,dy.$$

The proof of the following global regularity result is obtained in the same way as the proofs of Theorems 2.3.2 and 1.7.5.

2. Theorem. *Let $k \geq 1$ be an integer and assume that, for each t, the coefficients $a(t,\cdot), b(t,\cdot), c(t,\cdot)$ are in C^k and their norms in C^k are bounded by a constant K_1. Then the following hold.*

(i) For the constant λ_0 from Theorem 1 and $N = N(k, d, K, K_1, \kappa, \omega, p)$ the estimate

$$\lambda\|u\|_{W_p^{0,k}(\mathbb{R}_T^{d+1})} + \lambda^{1/2}\|u_x\|_{W_p^{0,k}(\mathbb{R}_T^{d+1})} + \|u_{xx}\|_{W_p^{0,k}(\mathbb{R}_T^{d+1})} + \|u_t\|_{W_p^{0,k}(\mathbb{R}_T^{d+1})}$$

$$\leq N\|Lu + u_t - \lambda u\|_{W_p^{0,k}(\mathbb{R}_T^{d+1})} \tag{4}$$

holds for any $\lambda \geq \lambda_0$ and $u \in W_p^{1,k+2}(\mathbb{R}_T^{d+1})$. Furthermore, for any $\lambda \geq \lambda_0$ and $f \in W_p^{0,k}(\mathbb{R}_T^{d+1})$ there exists a unique $u \in W_p^{1,k+2}(\mathbb{R}_T^{d+1})$ satisfying (1) *in $\mathbb{R}_T^{d+1}$.*

(ii) If the coefficients of L are independent of t, for the same λ_0 and N the estimate

$$\lambda\|u\|_{W_p^k} + \lambda^{1/2}\|u_x\|_{W_p^k} + \|u_{xx}\|_{W_p^k} \leq N\|Lu - \lambda u\|_{W_p^k} \tag{5}$$

holds for any $\lambda \geq \lambda_0$ and $u \in W_p^{k+2}$. Furthermore, for any $\lambda \geq \lambda_0$ and $f \in W_p^k$ there exists a unique $u \in W_p^{k+2}$ satisfying (3).

3. Exercise*. Prove (5) by taking $T = 0$ and substituting $e^{-\varepsilon t}u(x)$ into (4) and letting $\varepsilon \downarrow 0$. The emphasis here is on preserving the value of λ_0.

The following corollary is proved in the same way as Corollary 2.3.3. Its "elliptic" part also can be obtained from the "parabolic" part as in Exercise 3.

4. Corollary. *Under the assumptions of Theorem 2 take functions $u \in W_p^{1,2}(\mathbb{R}_T^{d+1})$, $v \in W_p^2$, $\lambda \geq 0$, set*

$$f := Lu + u_t - \lambda u, \quad g := Lv - \lambda v,$$

and assume that $f \in W_p^k(\mathbb{R}_T^{d+1})$. Then $u \in W_p^{1,k+2}(\mathbb{R}_T^{d+1})$ and

$$\lambda\|u\|_{W_p^{0,k}(\mathbb{R}_T^{d+1})} + \|u\|_{W_p^{1,k+2}(\mathbb{R}_T^{d+1})} \leq N(\|f\|_{W_p^{0,k}(\mathbb{R}_T^{d+1})} + \|u\|_{\mathcal{L}_p(\mathbb{R}_T^{d+1})}),$$

where N depends only on k, d, K, K_1, κ, p, and ω.

If in addition the coefficients of L are independent of t and $g \in W_p^k$, then $v \in W_p^{k+2}$ and with the same N

$$\lambda\|v\|_{W_p^k} + \|v\|_{W_p^{k+2}} \leq N(\|g\|_{W_p^k} + \|v\|_{\mathcal{L}_p}).$$

We state the following local regularity result only for parabolic equations. Its elliptic counterpart is obtained by taking u independent of t.

5. Theorem. *Let $0 < r < R < \infty$, $u \in W_p^{1,2}(Q_s)$ for all $s \in (0, R)$, $\lambda \geq 0$. Suppose that the assumptions of Theorem 2 are satisfied and*

$$f := Lu + u_t - \lambda u \in W_p^{0,k}(Q_s)$$

for all $s \in (0, R)$. Then $u \in W_p^{1,k+2}(Q_r)$ and

$$\lambda \|u\|_{W_p^{0,k}(Q_r)} + \|u\|_{W_p^{1,k+2}(Q_r)} \leq N\big(\|f\|_{W_p^{0,k}(Q_R)} + \|u\|_{\mathcal{L}_p(Q_R)}\big),$$

where $N = N(r, R, k, d, K, K_1, \kappa, \omega, p)$.

This theorem is proved in the same way as Theorem 2.4.7. Of course, we first prove it for $k = 0$ by repeating the proof of Lemma 2.4.4 with $W_p^{1,2}$ in place of $W_2^{1,2}$. This is possible because we proved the necessary multiplicative inequalities for arbitrary $p \in [1, \infty)$.

6. Remark. Similarly to Remark 2.4.2, for Theorem 5 to be true, we do not need any smoothness of the coefficients of L outside Q_R. It suffices to assume that for $t \in (0, R^2)$ we have $a(t, \cdot), b(t, \cdot), c(t, \cdot) \in C^k(B_R)$ and that the norms of $a(t, \cdot), b(t, \cdot), c(t, \cdot)$ in $C^k(B_R)$ are bounded for $t \in (0, R^2)$.

Next, Exercise 2.4.8 is also stated for arbitrary p. However, to prove an $\mathcal{L}_p$ counterpart of Theorem 2.4.9, we have to use not only Exercise 2.4.8 but also the following embedding theorem for W_p^k spaces, which we will generalize and prove later as Theorem 10.4.10 (also see Remark 10.4.12).

7. Theorem. *Let $k, m \in \{0, 1, 2, ...\}$ be integers such that $p(k - m) > d$, $R \in (0, \infty)$, $u \in W_p^k(B_R)$. Then $u \in C^m(\bar{B}_R)$ and*

$$\|u\|_{C^m(B_R)} \leq N \|u\|_{W_p^k(B_R)},$$

with N independent of u.

We are now ready to state the following.

8. Theorem. *Let $k, m \in \{0, 1, 2, ...\}$ be integers such that $p(k - m) > d$ and let the assumptions of Theorem 2 be satisfied. Also let $0 < r < R < \infty$, $u \in W_p^{1,2}(Q_s)$ and*

$$f := Lu + u_t \in W_p^{0,k}(Q_s) \quad \textit{for any} \quad s \in (0, R).$$

Then the derivatives $D^\alpha u$ with $|\alpha| \leq m$ are continuous in Q_r and

$$\sup_{Q_r, |\alpha| \leq m} |D^\alpha u| \leq N\big(\|f\|_{W_p^{0,k}(Q_R)} + \|u\|_{\mathcal{L}_p(Q_R)}\big).$$

where $N = N(k, K, K_1, d, \omega, r, R, p, \kappa)$. *Furthermore, if* $m \geq 2$ *and the derivatives* $D^\alpha f$ *are bounded on* Q_r *for* $|\alpha| \leq m-2$, *then*

$$\sup_{Q_r, |\alpha| \leq m-2} |D^\alpha u_t| \leq N\Big(\sup_{Q_r, |\alpha| \leq m-2} |D^\alpha f| + \|f\|_{W_p^{0,k}(Q_R)} + \|u\|_{\mathcal{L}_p(Q_R)}\Big),$$

where $N = N(k, K, K_1, d, \omega, r, R, p, \kappa)$.

The proof of this theorem is almost identical to that of Theorem 2.4.9. The only difference is that instead of the embedding theorems for $p = 2$ we use Theorem 7.

9. Corollary. *Let* $r \in (0, \infty)$ *and assume that* $a, b, c \in C^\infty_{loc}(Q_r)$. *Let* $u \in W_p^{1,2}(Q_r)$ *be such that* $f := Lu + u_t \in C^\infty_{loc}(Q_r)$. *Then* $u \in C^\infty_{loc}(Q_r)$.

Indeed, by Theorem 8 all derivatives in x of u and u_t are continuous in Q_r. This implies that $f - Lu$ has the first derivative in t and this derivative has all derivatives in x continuous in Q_r. Since $u_t = f - Lu$, we see that all derivatives in x of u_{tt} are continuous in Q_r. This provides additional information about the derivatives in t of $f - Lu$ and yields the desired result by induction.

Finally, we state a result for the Cauchy problem. Its proof is obtained by repeating the proof of Theorem 2.5.3 and replacing the power of summability 2 with p. We take a $T \in (0, \infty)$ and set

$$\Omega = \Omega_T = (0, T) \times \mathbb{R}^d.$$

Recall that $\overset{0}{W}{}_p^{1,2}(\Omega)$ was introduced in Definition 2.5.1.

10. Theorem. *For any* $f \in \mathcal{L}_p(\Omega)$ *and* $v \in W_p^{1,2}(\Omega)$ *there exists a unique* $u \in W_p^{1,2}(\Omega)$ *satisfying equation* $u_t = Lu + f$ *in* Ω *and such that*

$$u - v \in \overset{0}{W}{}_p^{1,2}(\Omega).$$

Furthermore, there is a constant $N = N(d, T, K, \kappa, p, \omega)$, *such that*

$$\|u\|_{W_p^{1,2}(\Omega)} \leq N(\|f\|_{\mathcal{L}_p(\Omega)} + \|v\|_{W_p^{1,2}(\Omega)}).$$

The following is just an adaptation of Remark 2.5.4.

11. Remark. Let $f \in \mathcal{L}_p(\Omega_T)$ for all $T \in (0, \infty)$ and $f(t, x) = 0$ for $t \leq 0$. Then on $\mathbb{R}^{d+1}$ there exists a unique function u possessing the following properties:

(i) $u \in W_p^{1,2}((-\infty, T) \times \mathbb{R}^d)$ for all $T \in (0, \infty)$;

(ii) $u_t = Lu + f$ in $\mathbb{R}^{d+1}$;

(iii) $u(t, x) = 0$ for $t < 0$.

Chapter 6

Equations with VMO coefficients

In this chapter we show that solvability theory in Sobolev function spaces can be developed for a rather large class of equations with *discontinuous* coefficients. The general scheme we follow is the same as for continuous coefficients and is based on the pointwise estimates of $u^{\sharp}_{xx}$ through the maximal function of the result of application of an operator to u. It would be a good idea while reading the chapter for the first time to take $q = 2$ and restrict p to the range $p > 2$. In that situation Section 1 can be skipped. The results and methods developed in Section 1 are used only in the case of any $p \in (1, \infty)$ and in the case of parabolic equations in Sobolev spaces with mixed norms in Chapter 7.

The reader who has a deeper interest in various issues of equations with VMO coefficients is referred to [**17**] and the references therein. More recent references can be tracked down by following [**22**].

1. Estimating $\mathcal{L}_q$ oscillations of u_{xx}

We take an operator

$$Lu(t, x) = a^{ij}(t)u_{x^ix^j}(t, x)$$

with measurable coefficients depending only on $t \in \mathbb{R}$ and such that $a^{ij} = a^{ji}$ and

$$\kappa^{-1}|\xi|^2 \geq a^{ij}(t)\xi^i\xi^j \geq \kappa|\xi|^2$$

for all $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$, where $\kappa \in (0, 1)$ is a fixed constant.

First, we state a generalization of Theorem 4.2.6, which was proved for $q = 2$.

1. Theorem. *Let $\nu \geq 2$, $q \in (1, \infty)$, and $r \in (0, \infty)$ be some constants and let $u \in C^{\infty}_{loc}(\mathbb{R}^{d+1})$ be such that $Lu + u_t = 0$ in $Q_{\nu r}$. Then there is a constant $N = N(d, \kappa, q)$ such that*

$$\fint_{Q_r} |u_{xx}(t, x) - (u_{xx})_{Q_r}|^q \, dxdt \leq N\nu^{-q} \fint_{Q_{\nu r}} |u_{xx}|^q \, dxdt.$$

Proof. The parabolic self-similarity reduces proving the theorem to the case where $r = 1$ as in the proof of Theorem 4.2.6. Also for the same reasons as there it is sufficient to prove that

$$\max_{Q_1}(|u_{xxx}|^q + |u_{txx}|^q) \leq N\nu^{-q} \fint_{Q_\nu} |u_{xx}|^q \, dxdt. \tag{1}$$

Again as in the proof of Theorem 4.2.6 we reduce the general situation to the one with $\nu = 2$. Therefore, in the rest of the proof we assume that $r = 1$ and $\nu = 2$.

Next, introduce

$$v = u - u_{Q_2} - x^i (u_{x^i})_{Q_2},$$

observe that $v_t + Lv = u_t + Lu = 0$ in Q_2, and use Theorem 5.2.8, in which we take $r = 1$, $R = 2$, $p = q$, $m = 4$, and k so large that $q(k - 4) > d$. Then we see that

$$\max_{Q_1}(|u_{xxx}|^q + |u_{txx}|^q) = \max_{Q_1}(|v_{xxx}|^q + |v_{txx}|^q) \leq N \int_{Q_2} |v|^q \, dxdt.$$

Upon combining this with Lemma 4.2.2 and using the equation $v_t = -Lv$, we come to (1). The theorem is proved.

The main result of this section is a generalization of Theorem 4.3.1.

2. Theorem. *Let $q \in (1, \infty)$. Then there exists a constant N, depending only on q, d, and κ, such that for any $\nu \geq 4$, $r > 0$, $u \in W^{1,2}_{q,loc}$, we have*

$$(|u_{xx} - (u_{xx})_{Q_r}|^q)_{Q_r} \leq N\nu^{-q}(|u_{xx}|^q)_{Q_{\nu r}} + N\nu^{d+2}(|Lu + u_t|^q)_{Q_{\nu r}}.$$

Proof. The argument below follows very closely the proof of Theorem 4.3.1. This time, however, we will give all details for completeness.

Fix an $r \in (0, \infty)$ and a $\nu \geq 4$. We may certainly assume that $u \in C_0^\infty(\mathbb{R}^{d+1})$ and that the coefficients of L are infinitely differentiable and have bounded derivatives. In that case define

$$f = Lu + u_t.$$

Observe that $f \in C_0^\infty(\mathbb{R}^{d+1})$. Also take a $\zeta \in C_0^\infty(\mathbb{R}^{d+1})$ such that $\zeta = 1$ on $Q_{\nu r/2}$ and $\zeta = 0$ outside $(-\nu^2 r^2, \nu^2 r^2) \times B_{\nu r}$ and set

$$g = f\zeta, \quad h = f(1-\zeta).$$

Finally, take $T > 0$ so large that $u(t,x) = 0$ for $t > T$ and by using Remark 5.2.11 with reversed time axis, define a function v which belongs to all $W_p^{1,2}(\mathbb{R}_S^{d+1})$, $S > -\infty$, $v(t,x) = 0$ for $t > T$ and

$$Lv + v_t = h \quad \text{in} \quad \mathbb{R}^{d+1}.$$

By Corollary 5.2.9 the function v is infinitely differentiable. Since $h = 0$ in $Q_{\nu r/2}$ and $\nu/2 \geq 2$, by Theorem 1

$$(|v_{xx} - (v_{xx})_{Q_r}|^q)_{Q_r} \leq N\nu^{-q} \fint_{Q_{\nu r/2}} |v_{xx}|^q \, dxdt \leq N\nu^{-q} \fint_{Q_{\nu r}} |v_{xx}|^q \, dxdt. \tag{2}$$

On the other hand, the function $w := u - v \in W_q^{1,2}(\mathbb{R}_0^{d+1})$ satisfies

$$Lw + w_t = g$$

and by Theorem 4.3.8

$$\int_{\mathbb{R}_0^{d+1}} |w_{xx}|^q \, dxdt \leq N \int_{\mathbb{R}_0^{d+1}} |g|^q \, dxdt \leq N \int_{Q_{\nu r}} |f|^q \, dxdt. \tag{3}$$

$$(|w_{xx} - (w_{xx})_{Q_r}|^q)_{Q_r} \leq 2^q (|w_{xx}|^q)_{Q_r}$$

$$\leq N r^{-d-2} \int_{Q_r} |w_{xx}|^q \, dxdt \leq N r^{-d-2} \int_{Q_{\nu r}} |f|^q \, dxdt,$$

$$(|w_{xx} - (w_{xx})_{Q_r}|^q)_{Q_r} \leq N\nu^{d+2} \fint_{Q_{\nu r}} |f|^q \, dxdt.$$

By combining this with (2) and observing that $u = v + w$ and

$$I := (|u_{xx} - (u_{xx})_{Q_r}|^q)_{Q_r}$$

$$\leq 2^q (|v_{xx} - (v_{xx})_{Q_r}|^q)_{Q_r} + 2^q (|w_{xx} - (w_{xx})_{Q_r}|^q)_{Q_r},$$

we get

$$I \le N\nu^{-q} \fint_{Q_{\nu r}} |v_{xx}|^q \, dxdt + N\nu^{d+2} \fint_{Q_{\nu r}} |f|^q \, dxdt$$

$$\le N\nu^{-q} \fint_{Q_{\nu r}} |u_{xx}|^q \, dxdt + N\nu^{d+2} \fint_{Q_{\nu r}} |f|^q \, dxdt$$

$$+ N\nu^{-q} \fint_{Q_{\nu r}} |w_{xx}|^q \, dxdt.$$

Here by (3)

$$\nu^{-q} \fint_{Q_{\nu r}} |w_{xx}|^q \, dxdt \le N\nu^{-q} \fint_{Q_{\nu r}} |f|^q \, dxdt$$

and since $\nu \ge 1$, we conclude

$$I \le N\nu^{d+2} \fint_{Q_{\nu r}} |f|^q \, dxdt + N\nu^{-q} \fint_{Q_{\nu r}} |u_{xx}|^q \, dxdt.$$

The theorem is proved.

3. Exercise. Improve the result of Theorem 2 by showing that under its assumptions

$$(|u_{xx} - (u_{xx})_{Q_r}|^q)_{Q_r} \le N\nu^{-q}(|u_{xx} - (u_{xx})_{Q_{\nu r}}|^q)_{Q_{\nu r}} + N\nu^{d+2}(|f - f_{B_{\nu r}}|^q)_{Q_{\nu r}},$$

where $f = Lu + u_t$.

The method of deriving the following corollary will be used while treating parabolic equations in spaces with mixed norms.

4. Corollary. *Let $q \in (1, \infty)$. Then there exists $N = N(d, q, \kappa)$ such that for any $u \in W^{1,2}_q$, $r > 0$, and $\nu \ge 4$ we have*

$$\fint_{(0,r^2)} \fint_{(0,r^2)} \big| \|u_{xx}(t,\cdot)\|_{\mathcal{L}_q} - \|u_{xx}(s,\cdot)\|_{\mathcal{L}_q} \big|^q \, dtds$$

$$\le N\nu^{-q} \fint_{(0,\nu^2 r^2)} \|u_{xx}(t,\cdot)\|^q_{\mathcal{L}_q} \, dt + N\nu^{d+2} \fint_{(0,\nu^2 r^2)} \|(Lu + u_t)(t,\cdot)\|^q_{\mathcal{L}_q} \, dt. \tag{4}$$

Indeed, by the triangle inequality

$$\big|\|u_{xx}(t,\cdot)\|_{\mathcal{L}_q} - \|u_{xx}(s,\cdot)\|_{\mathcal{L}_q}\big|^q \leq \|u_{xx}(t,\cdot) - u_{xx}(s,\cdot)\|^q_{\mathcal{L}_q},$$

so that the left-hand side of (4) is less than

$$I := \fint_{(0,r^2)} \fint_{(0,r^2)} \int_{\mathbb{R}^d} |u_{xx}(t,x) - u_{xx}(s,x)|^q \, dxdtds$$

$$= \fint_{(0,r^2)} \fint_{(0,r^2)} \int_{\mathbb{R}^d} |u_{xx}(t,x+y) - u_{xx}(s,x+y)|^q \, dxdtds,$$

where y is any point in $\mathbb{R}^d$. By taking the average of the extreme terms over $y \in B_r$ we see that

$$I = \fint_{(0,r^2)} \fint_{(0,r^2)} \int_{\mathbb{R}^d} \Big(\fint_{B_r(x)} |u_{xx}(t,z) - u_{xx}(s,z)|^q \, dz \Big) \, dxdtds. \tag{5}$$

Next, since

$$|u_{xx}(t,z) - u_{xx}(s,z)|^q \leq 2^{q-1}|u_{xx}(t,z) - (u_{xx})_{Q_r(0,x)}|^q$$

$$+ 2^{q-1}|u_{xx}(s,z) - (u_{xx})_{Q_r(0,x)}|^q,$$

we have that

$$I \leq 2^q \int_{\mathbb{R}^d} (|u_{xx} - (u_{xx})_{Q_r(0,x)}|^q)_{Q_r(0,x)} \, dx.$$

By Theorem 2 applied to shifted cylinders the last expression is dominated by a constant times

$$\nu^{-q} \int_{\mathbb{R}^d} (|u_{xx}|^q)_{Q_{\nu r}(0,x)} \, dx + \nu^{d+2} \int_{\mathbb{R}^d} (|Lu + u_t|^q)_{Q_{\nu r}(0,x)} \, dx,$$

which similarly to (5) is shown to be equal to

$$\nu^{-q} \fint_{(0,\nu^2 r^2)} \|u_{xx}(t,\cdot)\|^q_{\mathcal{L}_q} \, dt + \nu^{d+2} \fint_{(0,\nu^2 r^2)} \|(Lu + u_t)(t,\cdot)\|^q_{\mathcal{L}_q} \, dt$$

and this yields (4).

2. Estimating sharp functions of u_{xx}

Here we are dealing with an operator

$$Lu(t,x) = a^{ij}(t,x)u_{x^ix^j}(t,x),$$

whose coefficients are assumed to be measurable functions on $\mathbb{R}^{d+1}$ satisfying $a^{ij} = a^{ji}$ and

$$\kappa^{-1}|\xi|^2 \geq a^{ij}(t,x)\xi^i\xi^j \geq \kappa|\xi|^2$$

for all t, x, ξ, where $\kappa > 0$ is a fixed constant.

Recall that $B_r(x)$ is the open ball in $\mathbb{R}^d$ of radius r centered at x, $B_r = B_r(0)$, $Q_r(t,x) = (t, t+r^2) \times B_r(x)$, $Q_r = Q_r(0,0)$, and $\mathbb{Q}$ is the collection of all $Q_r(t,x)$, $(t,x) \in \mathbb{R}^{d+1}$, $r \in (0,\infty)$. Denote

$$\operatorname{osc}_x(a, Q_r(t,x)) = r^{-2}|B_r|^{-2}\int_t^{t+r^2}\int_{y,z\in B_r(x)} |a(s,y) - a(s,z)|\,dydzds,$$

$$a_R^{\sharp(x)} = \sup_{(t,x)\in\mathbb{R}^{d+1}} \sup_{r\leq R} \operatorname{osc}_x(a, Q_r(t,x)).$$

This definition is either naturally modified if a is independent of t as in the elliptic operators or it is kept as is. In that case we sometimes use the notation

$$a_R^{\sharp} = a_R^{\sharp(x)}.$$

1. Remark. We say more about issues related to this notation in the next section. Here we only notice (cf. Exercise 3.2.5) that for any function $\bar{a}(t)$ with values in the set of $d \times d$ matrices

$$\operatorname{osc}_x(a, Q_r(t,x)) \leq 2r^{-2}|B_r|^{-1}\int_t^{t+r^2}\int_{B_r(x)} |a(s,y) - \bar{a}(s)|\,dyds,$$

and in notation (4.0.2)

$$r^{-2}|B_r|^{-1}\int_t^{t+r^2}\int_{y\in B_r(x)} |a(s,y) - a_{B_r(x)}(s)|\,dyds \leq \operatorname{osc}_x(a, Q_r(t,x)). \quad (1)$$

Here is a version of Theorem 1.2 for operators with variable coefficients. Somewhat later, while considering parabolic equations in spaces with mixed norms, we will see that one can take $\alpha = 1$ and $\beta = 2$ in (2) (see Lemma 7.1.1).

2. Lemma. *Let $q, \alpha \in (1, \infty)$, $(t_0, x_0) \in \mathbb{R}^{d+1}$, and $R \in (0, \infty)$. Take a $u \in W^{1,2}_q$ vanishing outside $Q_R(t_0, x_0)$. Then there exists a constant $N = N(d, \kappa, q, \alpha)$ such that, for any $\nu \geq 4$ and $r \in (0, \infty)$, we have*

$$(|u_{xx} - (u_{xx})_{Q_r}|^q)_{Q_r} \leq N\nu^{-q}\mathcal{B}_{\nu r} + N\nu^{d+2}(\mathcal{A}_{\nu r} + \hat{a}^{1/\beta}\mathcal{C}_{\nu r}), \tag{2}$$

where

$$\mathcal{A}_\rho = (|f|^q)_{Q_\rho}, \quad \mathcal{B}_\rho = (|u_{xx}|^q)_{Q_\rho}, \quad \mathcal{C}^\alpha_\rho = \left(|u_{xx}|^{\alpha q}\right)_{Q_\rho},$$

$$\hat{a} = a^{\sharp(x)}_{R \wedge (\nu r)}, \quad f = Lu + u_t, \quad \beta = \alpha/(\alpha - 1).$$

Furthermore, if the coefficients of L are independent of t, then for any $u \in W^2_q$ vanishing outside $B_R(x_0)$, $\nu \geq 4$, and $r \in (0, \infty)$ we have

$$(|u_{xx} - (u_{xx})_{B_r}|^q)_{B_r} \leq N\nu^{-q}\mathcal{B}_{\nu r} + N\nu^{d+2}(\mathcal{A}_{\nu r} + \hat{a}^{1/\beta}\mathcal{C}_{\nu r}),$$

where

$$\mathcal{A}_\rho = (|f|^q)_{B_\rho}, \quad \mathcal{B}_\rho = (|u_{xx}|^q)_{B_\rho}, \quad \mathcal{C}^\alpha_\rho = \left(|u_{xx}|^{\alpha q}\right)_{B_\rho},$$

$$\hat{a} = a^{\sharp}_{R \wedge (\nu r)}, \quad \beta = \alpha/(\alpha - 1), \quad f = Lu.$$

Proof. First, fix a $\nu \geq 4$ and an $r \in (0, \infty)$ and introduce

$$\bar{a}^{ij}(t) = a^{ij}_{B_R(x_0)}(t) \quad \text{if} \quad \nu r \geq R, \quad \bar{a}^{ij}(t) = a^{ij}_{B_{\nu r}}(t) \quad \text{if} \quad \nu r < R.$$

Also let $\bar{L}u = \bar{a}^{ij}u_{x^ix^j}$. Then by Theorem 1.2

$$(|u_{xx} - (u_{xx})_{Q_r}|^q)_{Q_r} \leq N\nu^{-q}\mathcal{B}_{\nu r} + N\nu^{d+2}\left(|\bar{L}u + u_t|^q\right)_{Q_{\nu r}}. \tag{3}$$

Here

$$\int_{Q_{\nu r}} |\bar{L}u + u_t|^q \, dxdt \leq 2^q(I + J),$$

where

$$I = \int_{Q_{\nu r}} |Lu + u_t|^q \, dxdt = N(\nu r)^{d+2}\mathcal{A}_{\nu r},$$

$$J = \int_{Q_{\nu r}} |(L - \bar{L})u|^q \, dxdt = \int_{Q_{\nu r} \cap Q_R(t_0, x_0)} \ldots \leq NJ_1^{1/\alpha}J_2^{1/\beta},$$

$$J_1 = \int_{Q_{\nu r}} |u_{xx}|^{q\alpha} \, dxdt = N(\nu r)^{d+2}\mathcal{C}^\alpha_{\nu r},$$

$$J_2 = \int_{Q_{\nu r}\cap Q_R(t_0,x_0)} |a(t,x) - \bar{a}(t)|^{q\beta}\, dxdt.$$

If $\nu r \geq R$, then we estimate J_2 by the integral over $Q_R(t_0,x_0)$, which owing to (1) is less than

$$NR^{d+2} \fint_{Q_R(t_0,x_0)} |a(t,x) - \bar{a}(t)|\, dxdt \leq N(\nu r)^{d+2} a_R^{\sharp(x)} = N(\nu r)^{d+2}\hat{a}.$$

In the case $\nu r < R$, we estimate J_2 by

$$N(\nu r)^{d+2} \fint_{Q_{\nu r}} |a(t,x) - \bar{a}(t)|\, dxdt \leq N(\nu r)^{d+2} a_{\nu r}^{\sharp(x)} = N(\nu r)^{d+2}\hat{a}.$$

It follows that

$$J \leq N(\nu r)^{d+2}\hat{a}^{1/\beta}\mathcal{C}_{\nu r}$$

and

$$\int_{Q_{\nu r}} |\bar{L}u + u_t|^q\, dxdt \leq N(\nu r)^{d+2}\mathcal{A}_{\nu r} + N(\nu r)^{d+2}\hat{a}^{1/\beta}\mathcal{C}_{\nu r}.$$

By coming back to (3), we get (2).

The second assertion of the lemma is proved in the same way starting from (3) where we drop u_t and replace Q_ρ with B_ρ which is possible since the functions involved are independent of t. The lemma is proved.

3. Remark. A peculiar feature of Lemma 2 is that $\mathcal{C}_{\nu r}$ is not assumed to be finite. The same observation should be kept in mind in the case of Theorem 4.

Here is the main result of this section.

4. Theorem. *Let $q,\alpha \in (1,\infty)$ and $R \in (0,\infty)$. Then there exists a constant $N = N(d,\kappa,q,\alpha)$ such that for any $\nu > 0$ and $u \in W^{1,2}_q$ vanishing outside Q_R we have*

$$(u_{xx})^{\sharp} \leq N\nu^{-1}\mathcal{B}^{1/q} + N\nu^{(d+2)/q}\big(\mathcal{A}^{1/q} + \hat{a}^{1/(\beta q)}\mathcal{C}^{1/q}\big) \tag{4}$$

on $\mathbb{R}^{d+1}$, where $\beta = \alpha/(\alpha-1)$,

$$\mathcal{A} := \mathbb{M}(|f|^q), \quad \mathcal{B} := \mathbb{M}(|u_{xx}|^q), \quad \mathcal{C}^{\alpha} := \mathbb{M}(|u_{xx}|^{\alpha q}),$$

$$f = Lu + u_t, \quad \hat{a} = a_R^{\sharp(x)}.$$

Furthermore, if the coefficients of L are independent of t, then (4) holds on $\mathbb{R}^d$ for any $u \in W^2_q$ vanishing outside B_R.

Proof. Take a $\nu \geq 4$, $(t, x) \in \mathbb{R}^d$, and $Q \in \mathbb{Q}$ such that $Q \ni (t, x)$. Then $Q = Q_r(t_0, x_0)$ for certain r, t_0, x_0. Denote $Q' = Q_{\nu r}(t_0, x_0)$. Then shifting (t_0, x_0) to the origin, we conclude from Lemma 2 that

$$(|u_{xx} - (u_{xx})_Q|^q)_Q \leq N\nu^{-q}(|u_{xx}|^q)_{Q'} + N\nu^{d+2}((|f|^q)_{Q'} + \hat{a}^{1/\beta}(|u_{xx}|^{\alpha q})_{Q'}^{1/\alpha}). \tag{5}$$

Since $(t, x) \in Q' \in \mathbb{Q}$, we have by definition

$$(|u_{xx}|^q)_{Q'} \leq \mathbb{M}(|u_{xx}|^q)(t, x).$$

Similar estimates are true for other terms on the right in (5). It follows that for any $(t, x) \in \mathbb{R}^d$ and $Q \in \mathbb{Q}$ such that $Q \ni (t, x)$ we have

$$(|u_{xx} - (u_{xx})_Q|^q)_Q \leq N\nu^{-q}\mathcal{B}(t, x) + N\nu^{d+2}(\mathcal{A}(t, x) + \hat{a}^{1/\beta}\mathcal{C}(t, x)).$$

Since by Hölder's inequality

$$|u_{xx} - (u_{xx})_Q|_Q \leq [(|u_{xx} - (u_{xx})_Q|^q)_Q]^{1/q},$$

we also have

$$|u_{xx} - (u_{xx})_Q|_Q \leq N\nu^{-1}\mathcal{B}^{1/q}(t, x)$$
$$+ N\nu^{(d+2)/q}\big(\mathcal{A}^{1/q}(t, x) + \hat{a}^{1/(\beta q)}\mathcal{C}^{1/q}(t, x)\big). \tag{6}$$

So far $\nu \geq 4$ and Q (such that $(t, x) \in Q$) were fixed. Now we allow them to vary and observe that (6) is also true for $\nu \in (0, 4)$ since $\mathcal{B}$ is present on the right. After that, upon taking supremums with respect to $Q \in \mathbb{Q}$ such that $Q \ni (t, x)$, we come to (4).

The last assertion of the theorem is proved quite similarly on the basis of the second assertion of Lemma 2. The theorem is proved.

5. Corollary. *By minimizing the right-hand side of* (4) *with respect to* $\nu > 0$, *we see that on* $\mathbb{R}^{d+1}$

$$(u_{xx})^\sharp \leq N\big[\hat{a}^{1/(\beta q)}\mathcal{C}^{1/q} + \mathcal{A}^{1/q}\big]^\mu \mathcal{B}^{(1-\mu)/q}$$
$$\leq N\hat{a}^{\mu/(\beta q)}\mathcal{C}^{\mu/q}\mathcal{B}^{(1-\mu)/q} + N\mathcal{A}^{\mu/q}\mathcal{B}^{(1-\mu)/q},$$

where $\mu = q/(q + d + 2)$. *By noting that* $\mathcal{B} \leq \mathcal{C}$ *and replacing* $\mathcal{B}$ *with* $\mathcal{C}$ *in the first term on the right, we come to*

$$(u_{xx})^\sharp \leq N\mathbb{M}^{\mu/q}(|f|^q)\mathbb{M}^{(1-\mu)/q}(|u_{xx}|^q) + N\hat{a}^{\mu/(\beta q)}\mathbb{M}^{1/(\alpha q)}(|u_{xx}|^{\alpha q}).$$

3. A priori estimates for parabolic and elliptic equations with VMO coefficients

As in Section 5.2, we are considering the equation

$$u_t + Lu - \lambda u = f$$

in $\mathbb{R}^{d+1}_T$, where $T \in [-\infty, \infty)$ and

$$Lu(t, x) = a^{ij}(t, x)u_{x^i x^j}(t, x) + b^i(t, x)u_{x^i}(t, x) + c(t, x)u(t, x).$$

We assume that the coefficients a, b, and c are measurable functions of $(t, x) \in \mathbb{R}^{d+1}$, $a^{ij} = a^{ji}$, and for some constants $\kappa > 0$ and $K \in (0, \infty)$ for all values of the arguments and $\xi \in \mathbb{R}^d$ it holds that

$$|b| + |c| \leq K, \quad c \leq 0, \quad \kappa^{-1}|\xi|^2 \geq a^{ij}\xi^i\xi^j \geq \kappa|\xi|^2,$$

where $b = (b^1, ..., b^d)$. However we replace assumption (5.2.2) with a much weaker one: We assume that there exists an increasing function $\omega(\varepsilon)$, $\varepsilon > 0$, such that $\omega(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ and, for all $\varepsilon > 0$,

$$a_\varepsilon^{\sharp(x)} \leq \omega(\varepsilon).$$

Obviously, this assumption is satisfied if a depends only on t. Also in the case of the coefficients that are uniformly continuous in x, that is, when (5.2.2) is satisfied, we have

$$a_\varepsilon^{\sharp(x)} \leq \omega(2\varepsilon).$$

1. Remark. It may be worth stressing that, actually, we do not need to have $\omega(0+) = 0$. From our arguments it will become clear that we only need $\omega(0+)$ to be sufficiently small and how small it should be is determined by other parameters involved in each particular argument.

When the coefficients of L are independent of t, we also consider the equation

$$Lu(x) - \lambda u(x) = f(x)$$

in $\mathbb{R}^d$.

2. Remark. The $\mathcal{L}_{1,loc}$ functions $f = f(x)$ for which

$$\lim_{R \to 0} f_R^\sharp = 0$$

compose the so-called Sarason class VMO (functions with vanishing mean oscillation). The set of $\mathcal{L}_{1,loc}$ functions $f = f(x)$ such that $f_R^\sharp$ is bounded for

$R \in (0, \infty)$ is the so-called classical John-Nirenberg space BMO (functions with bounded mean oscillation).

3. Exercise. Let h be a Lipschitz continuous function on $\mathbb{R}$, $f \in$ BMO, and $g \in$ VMO. Show that $h(f) \in$ BMO and $h(g) \in$ VMO.

4. Exercise. Obviously if $f = f(x)$ is bounded, then $f \in$ BMO. Show that for $d = 1$ the function $f = \log|x|$ on $\mathbb{R}$, which is not bounded, is in BMO.

5. Remark. VMO functions are not necessarily continuous. It turns out (see, for instance, Exercise 10.2.9) that for $d = 2$ the function $f(x) = \ln^{1/3}(|x| \wedge 1)$ is in VMO. This function is not bounded, but $g(x) = 2+\sin f(x)$ is bounded from above and away from zero, is in VMO (as a Lipschitz continuous function of a VMO function), and is discontinuous at the origin.

Below in this section the above assumptions are supposed to be satisfied.

6. Lemma. *Let $p \in (1, \infty)$. Then there is a constant N depending only on p, d, K, κ, and the function ω, such that for any $u \in W^{1,2}_p$ we have*

$$\|u_{xx}\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} + \|u_t\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}$$

$$\leq N(\|Lu + u_t\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} + \|u_x\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} + \|u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}). \tag{1}$$

Furthermore, if the coefficients of L are independent of t, then for any $u \in W^2_p$

$$\|u_{xx}\|_{\mathcal{L}_p} \leq N(\|Lu\|_{\mathcal{L}_p} + \|u_x\|_{\mathcal{L}_p} + \|u\|_{\mathcal{L}_p}).$$

Proof. On account of the presence of $\|u_x\|_{\mathcal{L}_p}$ and $\|u\|_{\mathcal{L}_p}$ on the right, one may certainly assume that $b \equiv 0$ and $c \equiv 0$. Since $u_t = (Lu + u_t) - Lu$, we only need to estimate u_{xx}.

Next, while proving (1), without losing generality we may assume that $u \in C_0^\infty(\mathbb{R}^{d+1})$. First, fix a number $R_0 \in (0, \infty)$ and assume that u vanishes outside Q_{R_0}. Then set

$$f = Lu + u_t,$$

and use Theorem 2.4 with $\alpha = \alpha(p) \in (1, \infty)$ and $q = q(p) \in (1, \infty)$ such that

$$\alpha q = \frac{p+1}{2} \quad (< p).$$

We also use the Fefferman-Stein theorem on sharp functions (see Theorems 3.2.10 and 3.3.1). Then we obtain

$$\|u_{xx}\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} \leq N\|(u_{xx})^\#\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}$$

$$\leq N\nu^{(d+2)/q}\| [\mathbb{M}(|f|^q)]^{1/q}\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}$$

$$+ N\nu^{(d+2)/q}\omega^{1/(\beta q)}(R_0)\| [\mathbb{M}(|u_{xx}|^{\alpha q})]^{1/(\alpha q)}\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}$$

$$+ N\nu^{-1}\| [\mathbb{M}(|u_{xx}|^q)]^{1/q}\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}.$$

Since $p/(\alpha q) > 1$, by the Hardy-Littlewood maximal function theorem

$$\|\mathbb{M}(|u_{xx}|^{\alpha q})\|_{\mathcal{L}_{p/(\alpha q)}(\mathbb{R}^{d+1})} \leq N\| \, |u_{xx}|^{\alpha q}\|_{\mathcal{L}_{p/(\alpha q)}(\mathbb{R}^{d+1})} = N\|u_{xx}\|^{\alpha q}_{\mathcal{L}_p(\mathbb{R}^{d+1})}.$$

Similar estimates are valid for other terms above, so that after choosing an appropriate ν, we obtain

$$\|u_{xx}\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} \leq N_1\|f\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}$$

$$+ (1/4)\|u_{xx}\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} + N_2\omega^{1/(\beta q)}(R_0)\|u_{xx}\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}.$$

It follows that

$$\|u_{xx}\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} \leq 2N_1\|f\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}$$

if we fix R_0 so that

$$N_2(d, p, \kappa)\omega^{1/(\beta q)}(R_0) \leq 1/4.$$

After (1) is obtained for functions with support in Q_{R_0}, one proves it in the general case by a standard procedure using partitions of unity. Of course, we take a nonnegative $\zeta \in C_0^\infty(\mathbb{R}^{d+1})$ with support in Q_{R_0} and unit norm in $\mathcal{L}_p(\mathbb{R}^{d+1})$, start with the formula

$$|u_{xx}(t, x)|^p = \int_{\mathbb{R}^{d+1}} |u_{xx}(t, x)|^p(t, x)\zeta^p(t - s, x - y)\, dyds,$$

and use the fact that the functions $\zeta(\cdot - (s, y))u$ are supported in translates of Q_{R_0}.

Similarly by using Theorem 2.4 again, one proves the second assertion of the lemma, which is thus proved.

Before we generalize Lemma 6, we give the reader the following.

7. Exercise*. Take a $\zeta \in C_0^\infty(\mathbb{R})$ such that $\zeta \not\equiv 0$ and prove that

$$\int_{\mathbb{R}} |\zeta(y)\cos(\mu y)|^p\, dy$$

is bounded away from zero for $\mu \in \mathbb{R}$.

8. Lemma. *Let $p \in (1, \infty)$. Then there are constants $\lambda_0 \geq 1$ and N, depending only on p, K, κ, d, and the function ω, such that for any $\lambda \geq \lambda_0$ and $u \in W_p^{1,2}(\mathbb{R}^{d+1})$ we have*

$$\lambda \|u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} + \|u\|_{W_p^{1,2}} \leq N \|Lu + u_t - \lambda u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}. \tag{2}$$

Proof. By Lemma 6

$$\|u\|_{W_p^{1,2}} \leq N_1 (\|Lu + u_t\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} + \|u_x\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} + \|u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}).$$

Furthermore, by interpolation inequalities

$$N_1 \|u_x\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} \leq (1/2) \|u\|_{W_p^{1,2}} + N \|u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})},$$

implying that

$$\|u\|_{W_p^{1,2}} \leq N_1 (\|Lu + u_t\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} + \|u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}). \tag{3}$$

Also obviously,

$$\|Lu + u_t\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} \leq \|Lu + u_t - \lambda u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} + \lambda \|u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}.$$

It follows that

$$\|u\|_{W_p^{1,2}} \leq N (\|Lu + u_t - \lambda u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} + (\lambda + 1) \|u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}).$$

If $\lambda \geq \lambda_0 \geq 1$, then $\lambda + 1 \leq 2\lambda$ and it is seen that to prove the lemma, it suffices to prove that

$$\lambda \|u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} \leq N \|Lu + u_t - \lambda u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}. \tag{4}$$

We use a method suggested by S. Agmon to derive (4) from Lemma 6. Consider the space

$$\mathbb{R}^{d+2} = \{(t, z) = (t, x, y) : t, y \in \mathbb{R}, x \in \mathbb{R}^d\}$$

and the function

$$\tilde{u}(t, z) = u(t, x) \zeta(y) \cos(\mu y),$$

where $\mu = \sqrt{\lambda}$ and ζ is a $C_0^\infty(\mathbb{R})$ function, $\zeta \not\equiv 0$. Also introduce the operator

$$\tilde{L}v(t, z) = L(t, x) v(t, z) + v_{yy}(t, z).$$

Finally, set

$$\tilde{B}_r(z_0) = \{|z - z_0| < r\}, \quad \tilde{Q}_r(t_0, z_0) = (t_0, t_0 + r^2) \times \tilde{B}_r(z_0).$$

For any $r \in (0, \infty)$, $(t_0, z_0) \in \mathbb{R}^{d+2}$, and appropriate $\bar{a}(t)$ (see Remark 2.1) we have

$$\int_{\tilde{Q}_r(t_0, z_0)} |a(t, x) - \bar{a}(t)|\, dz dt$$

$$\leq \int_{(t_0, t_0+r^2)} \int_{\substack{|x-x_0|<r \\ |y-y_0|<r}} |a(t, x) - \bar{a}(t)|\, dz dt$$

$$= 2r \int_{Q_r(t_0, x_0)} |a(t, x) - \bar{a}(t)|\, dx dt \leq N r^{d+3} \omega(r).$$

It follows that the quantity $a_R^{\sharp(x)}$ constructed from a considered as a function of (t, z) is less than a constant $N_1 = N_1(d)$ times the original one and hence less than $N_1 \omega(R)$. Therefore, we can apply the above theory to the operator $\tilde{L}$ and in light of (3) applied to $\tilde{u}$ and $\tilde{L}$ we get

$$\|\tilde{u}_{zz}\|_{\mathcal{L}_p(\mathbb{R}^{d+2})} \leq N(\|\tilde{L}\tilde{u} + \tilde{u}_t\|_{\mathcal{L}_p(\mathbb{R}^{d+2})} + \|\tilde{u}\|_{\mathcal{L}_p(\mathbb{R}^{d+2})}). \tag{5}$$

Now, use Exercise 7 and write

$$\|u\|^p_{\mathcal{L}_p(\mathbb{R}^{d+1})} = \mu^{-2p}\Big(\int_{\mathbb{R}} |\zeta(y)\cos(\mu y)|^p\, dy\Big)^{-1} \int_{\mathbb{R}^{d+2}} \big|\tilde{u}_{yy}(t, z)$$

$$- u(t, x)[\zeta''(y)\cos(\mu y) - 2\mu\zeta'(y)\sin(\mu y)]\big|^p\, dz dt$$

$$\leq N\mu^{-2p}\big(\|\tilde{u}_{zz}\|^p_{\mathcal{L}_p(\mathbb{R}^{d+2})} + (\mu^p + 1)\|u\|^p_{\mathcal{L}_p(\mathbb{R}^{d+1})}\big).$$

This and (5) yield

$$\mu^2\|u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} \leq N\|\tilde{L}\tilde{u} + \tilde{u}_t\|_{\mathcal{L}_p(\mathbb{R}^{d+2})} + N(\mu + 1)\|u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}.$$

Since

$$\tilde{L}\tilde{u} + \tilde{u}_t = \zeta\cos(\mu y)[Lu + u_t - \lambda u] + u[\zeta''\cos(\mu y) - 2\mu\zeta'\sin(\mu y)],$$

we have

$$\|\tilde{L}\tilde{u} + \tilde{u}_t\|_{\mathcal{L}_p(\mathbb{R}^{d+2})} \leq N\|Lu + u_t - \lambda u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} + N(\mu + 1)\|u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})},$$

so that

$$\lambda\|u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} \leq N_2\|Lu + u_t - \lambda u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} + N_3(\sqrt{\lambda} + 1)\|u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}.$$

For $\lambda \geq \lambda_0 = 16N_3^2 + 4N_3$ we have

$$N_3\sqrt{\lambda} \leq (1/4)\lambda, \quad N_3 \leq (1/4)\lambda, \quad N_3(\sqrt{\lambda}+1) \leq (1/2)\lambda$$

and we arrive at (4) with $N = 2N_2$. The lemma is proved.

4. Solvability of parabolic and elliptic equations with VMO coefficients. The Cauchy problem

In this section all the assumptions in the beginning of Section 3 are supposed to be satisfied. We take a $T \in [-\infty, \infty)$ and $p \in (1, \infty)$.

1. Theorem. *(i) There are constants $\lambda_0 \geq 1$ and N, depending only on p, K, κ, d, and the function ω, such that for any $\lambda \geq \lambda_0$ and $u \in W_p^{1,2}(\mathbb{R}_T^{d+1})$ we have*

$$\lambda\|u\|_{\mathcal{L}_p(\mathbb{R}_T^{d+1})} + \|u\|_{W_p^{1,2}(\mathbb{R}_T^{d+1})} \leq N\|Lu + u_t - \lambda u\|_{\mathcal{L}_p(\mathbb{R}_T^{d+1})}. \tag{1}$$

Furthermore, for any $\lambda \geq \lambda_0$ and $f \in \mathcal{L}_p(\mathbb{R}_T^{d+1})$ there exists a unique $u \in W_p^{1,2}(\mathbb{R}_T^{d+1})$ such that

$$Lu + u_t - \lambda u = f \quad in \quad \mathbb{R}_T^{d+1}.$$

(ii) If the coefficients of L are independent of t, then with the same λ_0 and N, for any $\lambda \geq \lambda_0$ and $u \in W_p^2$, we have

$$\lambda\|u\|_{\mathcal{L}_p} + \|u\|_{W_p^2} \leq N\|(L-\lambda)u\|_{\mathcal{L}_p}. \tag{2}$$

Furthermore, for any $\lambda \geq \lambda_0$ and $f \in \mathcal{L}_p$ there exists a unique $u \in W_p^2$ such that $(L-\lambda)u = f$.

Proof. (i) If $T = -\infty$, both assertions in (i) follow from Lemma 3.8 and the method of continuity. To treat the case that $T > -\infty$, we need to extract a particular piece of information from the method of continuity outlined in the proof of Theorem 1.4.4.

For $\tau \in [0, 1]$ we introduce the operators $L_\tau = (1-\tau)\Delta + \tau L$ and for a $\lambda \geq \lambda_0$, where λ_0 is taken from Lemma 3.8, and $f \in \mathcal{L}_p(\mathbb{R}^{d+1})$ consider the equation

$$L_\tau u + u_t - \lambda u = f. \tag{3}$$

By the above for any $\tau \in [0, 1]$ and $f \in \mathcal{L}_p(\mathbb{R}^{d+1})$ there exists a unique solution of (3) in $W_p^{1,2}$.

We fix a $\tau_0 \in [0,1]$ and denote by $\mathcal{R}$ the operator sending $f \in \mathcal{L}_p(\mathbb{R}^{d+1})$ into the solution $u \in W_p^{1,2}$. The a priori estimate (3.2) shows that $\mathcal{R}$ is a bounded operator.

Next, we rewrite (3) as

$$L_{\tau_0} u + u_t - \lambda u = f + (\tau_0 - \tau)(L - \Delta)u,$$

or equivalently as

$$u = \mathcal{R}f + (\tau_0 - \tau)\mathcal{R}(L - \Delta)u. \tag{4}$$

We know that $\mathcal{R}(L - \Delta)$ is a bounded operator in $W_p^{1,2}$ with norm bounded independently of τ_0. It follows that for an $\varepsilon > 0$, independent of τ_0, the operator $(\tau_0 - \tau)\mathcal{R}(L - \Delta)$ is a contraction in $W_p^{1,2}$ whenever $|\tau_0 - \tau| \leq \varepsilon$. Therefore, for any τ satisfying this condition, the sequence of u^n defined by $u^0 = 0$,

$$u^{n+1} = \mathcal{R}f + (\tau_0 - \tau)\mathcal{R}(L - \Delta)u^n, \quad n \geq 0,$$

converges in $W_p^{1,2}$ to the solution of (4) or equivalently of (3).

Now, for $T > -\infty$, let us call τ_0 "good" if $\mathcal{R}g = 0$ in $\mathbb{R}_T^{d+1}$ for any $g \in \mathcal{L}_p(\mathbb{R}^{d+1})$ which is zero in $\mathbb{R}_T^{d+1}$. Then, for any τ satisfying the condition $|\tau_0 - \tau| \leq \varepsilon$ and f vanishing in $\mathbb{R}_T^{d+1}$, by induction on n we get that u^n vanishes in $\mathbb{R}_T^{d+1}$, and therefore the solution of (3) vanishes in $\mathbb{R}_T^{d+1}$. Thus, if τ_0 is "good", then all neighboring points are also "good". Since 0 is a "good" point (which follows from Theorem 4.3.8), step by step we obtain that all points in $[0,1]$ are "good".

After that we are ready to prove (1) if $T > -\infty$. We take a $u \in W_p^{1,2}(\mathbb{R}_T^{d+1})$ and by using Exercise 2.2.4, find a $v \in W_p^{1,2}$ such that $u = v$ in $\mathbb{R}_T^{d+1}$. We also set

$$g = (Lu + u_t - \lambda u)I_{t>T}$$

and define w as the unique solution in $W_p^{1,2}$ of the equation

$$Lw + w_t - \lambda w = g.$$

Since $v, w \in W_p^{1,2}$ and

$$L(v - w) + (v - w)_t - \lambda(v - w) = 0$$

in $\mathbb{R}_T^{d+1}$, by the above argument about "good" points we have $u = v = w$ in $\mathbb{R}_T^{d+1}$. Hence

$$\lambda \|u\|_{\mathcal{L}_p(\mathbb{R}_T^{d+1})} + \|u\|_{W_p^{1,2}(\mathbb{R}_T^{d+1})} = \lambda \|w\|_{\mathcal{L}_p(\mathbb{R}_T^{d+1})} + \|w\|_{W_p^{1,2}(\mathbb{R}_T^{d+1})}$$

$$\leq N\|g\|_{\mathcal{L}_p(\mathbb{R}^{d+1})} = N\|Lu + u_t - \lambda u\|_{\mathcal{L}_p(\mathbb{R}^{d+1}_T)},$$

which proves (1) and uniqueness of solutions. We have proved the existence even in $\mathbb{R}^{d+1}$.

(ii) We get (2) either by repeating the proof of Lemma 3.8 or deriving it from (1) in the same way as in Exercise 5.2.3. The second assertion in (ii) is again obtained by the method of continuity. The theorem is proved.

For later use we record a version of Lemma 2.4.4 about local estimates.

2. Theorem. *Let $0 < r < R < \infty$, $\lambda \geq 0$, $u \in W^{1,2}_p(Q_s)$ for all $s \in (0, R)$. Set $f := Lu + u_t - \lambda u$. Then*

$$\lambda\|u\|_{\mathcal{L}_p(Q_r)} + \|u\|_{W^{1,2}_p(Q_r)} \leq N\big(\|f\|_{\mathcal{L}_p(Q_R)} + \{1 + (R-r)^{-2}\}\|u\|_{\mathcal{L}_p(Q_R)}\big),$$

where N depends only on d, K, p, κ, and the function ω.

The proof of this theorem is obtained by repeating the proof of Lemma 2.4.4 using Theorem 1 instead of Corollary 2.3.3.

The following corollary is used in the proof of Lemma 7.1.1 while treating the parabolic equations in spaces with mixed norms.

3. Corollary. *Let $r \in (0, 1]$, $\nu \in (1, \infty)$, $u \in W^{1,2}_{p,loc}$ and assume that in $Q_{\nu r}$ we have $Lu + u_t = 0$, $b = 0$, and $c = 0$. Then*

$$\Big(\fint_{Q_r} |u_{xx}|^p\, dxdt\Big)^{1/p} \leq N \fint_{Q_{\nu r}} |u_{xx}|\, dxdt, \tag{5}$$

where N depends only on d, K, p, κ, ν, and the function ω.

Proof. It turns out that it suffices to prove that if

$$q \in [1, p], \quad \frac{1}{q} < \frac{2}{d+2} + \frac{1}{p}, \quad \mu \in (1, \infty),$$

then

$$\Big(\fint_{Q_r} |u_{xx}|^p\, dxdt\Big)^{1/p} \leq N\Big(\fint_{Q_{\mu r}} |u_{xx}|^q\, dxdt\Big)^{1/q}, \tag{6}$$

where $N = N(d, K, p, q, \omega, \kappa, \mu)$. Indeed, one can find a decreasing sequence $q_i \in [1, p]$, $i = 0, 1, ..., m$, where m depends only on p and d, such that $q_0 = p$, $q_m = 1$, and

$$\frac{1}{q_{i+1}} < \frac{2}{d+2} + \frac{1}{q_i}.$$

Then if (6) is true under the additional assumptions, then the $\mathcal{L}_{q_i}$ average norm of u_{xx} is estimated by the $\mathcal{L}_{q_{i-1}}$ average norm of u_{xx} in an expanded domain of averaging. We can then iterate (6) going along the sequence q_i and we can choose $\mu = \mu(p)$ so close to 1 that, during these finitely many steps, the expanding domains would always be in $Q_{\nu r}$ and (5) would follow.

Therefore, we concentrate on proving (6). Since (6) only involves the values of u in $Q_{\nu r}$, we may assume that $u \in W_p^{1,2}$. In that case introduce $\nu' = \sqrt{\mu}$,

$$v = u - u_{Q_{\mu r}} - x^i(u_{x^i})_{Q_{\mu r}}.$$

Then in $Q_{\nu r}$ we have $Lv + v_t = 0$ so that by Theorem 2 (with $\nu' r$ in place of R) and since $r \leq 1$, we have

$$\int_{Q_r} |u_{xx}|^p \, dxdt = \int_{Q_r} |v_{xx}|^p \, dxdt \leq N r^{-2p} \int_{Q_{\nu' r}} |u - u_{Q_{\mu r}} - x^i(u_{x^i})_{Q_{\mu r}}|^p \, dxdt. \tag{7}$$

By Lemma 4.2.3

$$\Big(\,⨍_{Q_{\nu' r}} |u(t,x) - u_{Q_{\mu r}} - x^i(u_{x^i})_{Q_{\mu r}}|^p \, dxdt\Big)^{1/p}$$

$$\leq N r^2 \Big(\,⨍_{Q_{\mu r}} (|u_{xx}|^q + |u_t|^q) \, dxdt\Big)^{1/q}. \tag{8}$$

Actually, (8) is proved in Lemma 4.2.3 only for $u \in C_0^\infty(\mathbb{R}^{d+1})$. However, one can use a standard approximation (unrelated to the operator L) of $u \in W_p^{1,2}$ in the $W_p^{1,2}$ norm by $C_0^\infty(\mathbb{R}^{d+1})$ functions and observe that, since $q \leq p$, by Hölder's inequality the right-hand sides of (8) for approximations will converge.

After that it only remains to combine (7) and (8) and use the fact that $u_t = -a^{ij} u_{x^i x^j}$ in $Q_{\nu r}$. The corollary is proved.

Now we take $T \in (0,\infty)$ and investigate the solvability of the Cauchy problem

$$u_t = Lu + f \quad \text{in} \quad \Omega = \Omega_T := (0,T) \times \mathbb{R}^d \tag{9}$$

with initial condition

$$u(0,x) = g(x) \quad \text{in} \quad \mathbb{R}^d.$$

We adopt the same approach as in Sections 2.5 and 5.2. The following is just a replica of Theorem 5.2.10 and Remark 5.2.11. The reader understands that no additional comments are needed. We again use the spaces $\overset{0}{W}{}^{1,2}_p(\Omega)$ introduced in Definition 2.5.1.

4. Theorem. *For any $f \in \mathcal{L}_p(\Omega)$ and $v \in W^{1,2}_p(\Omega)$ there exists a unique $u \in W^{1,2}_p(\Omega)$ satisfying (9) in Ω and such that*

$$u - v \in \overset{0}{W}{}^{1,2}_p(\Omega).$$

Furthermore, there is a constant N, depending only on d, T, K, κ, p, and ω, such that

$$\|u\|_{W^{1,2}_p(\Omega)} \leq N(\|f\|_{\mathcal{L}_p(\Omega)} + \|v\|_{W^{1,2}_p(\Omega)}).$$

5. Remark. Let $f \in \mathcal{L}_p(\Omega_T)$ for all $T \in (0, \infty)$ and $f(t, x) = 0$ for $t \leq 0$. Then on $\mathbb{R}^{d+1}$ there exists a unique function u possessing the following properties:

(i) $u \in W^{1,2}_p((-\infty, T) \times \mathbb{R}^d)$ for all $T \in (0, \infty)$;

(ii) $u_t = Lu + f$ in $\mathbb{R}^{d+1}$;

(iii) $u(t, x) = 0$ for $t < 0$.

6. Exercise*. Prove that if $p, q \in (1, \infty)$, $u \in \overset{0}{W}{}^{1,2}_p(\Omega)$ and $Lu + u_t \in \mathcal{L}_q(\Omega)$, then $u \in \overset{0}{W}{}^{1,2}_q(\Omega)$.

5. Hints to exercises

1.3. Use the hint to Exercise 4.3.6.

3.3. See Exercise 3.2.5.

3.4. Show that, for each $a, b \in \mathbb{R}$, $\operatorname{osc}_x(f, (a, b))$ equals $\operatorname{osc}_x(f, (c, d))$ for some $c < d$ such that $d - c = 1$.

4.6. Follow the proof of Theorem 4.3.12.

Chapter 7

Parabolic equations with VMO coefficients in spaces with mixed norms

In this short chapter we are considering the equation

$$u_t + Lu - \lambda u = f$$

in $\mathbb{R}^{d+1}_T$, where $T \in [-\infty, \infty)$ and

$$Lu(t,x) = a^{ij}(t,x)u_{x^ix^j}(t,x) + b^i(t,x)u_{x^i}(t,x) + c(t,x)u(t,x).$$

We assume that the coefficients a, b, and c are measurable functions of $(t,x) \in \mathbb{R}^{d+1}$, $a^{ij} = a^{ji}$, and for some constants $\kappa > 0$ and $K \in (0, \infty)$ for all values of the arguments and $\xi \in \mathbb{R}^d$ it holds that

$$|b| + |c| \leq K, \quad c \leq 0, \quad \kappa^{-1}|\xi|^2 \geq a^{ij}\xi^i\xi^j \geq \kappa|\xi|^2,$$

where $b = (b^1, ..., b^d)$. We also assume that there exists an increasing function $\omega(\varepsilon)$, $\varepsilon > 0$, such that $\omega(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ and, for all $\varepsilon > 0$,

$$a_\varepsilon^{\#(x)} \leq \omega(\varepsilon).$$

Our interest is in estimating mixed norms of solutions such as

$$\left(\int_0^T \left(\int_{\mathbb{R}^d} |u_{xx}|^q\,dx\right)^{p/q} dt\right)^{1/p}.$$

An advantage of having such estimates is that one can extract much more information on $u(t,\cdot)$ for each particular t if p is large. However, for obvious reasons we do not discuss this fact here, only referring the reader to [**21**]. We still give one particular example of using mixed norms in Exercise 10.1.14. We only consider $p \geq q$ and we do not know what happens if $p < q$ in the general case of equations with coefficients measurable in time and VMO in x.

More information on equations in spaces with mixed norms can be found by reading the recent papers [**2**], [**11**], [**10**], as well as references therein and other papers by their authors.

1. Estimating sharp functions of $\|u_{xx}(t,\cdot)\|_{\mathcal{L}_q}$

We start with a version of Lemma 6.2.2. In this section we fix a number

$$q \in (1,\infty).$$

1. Lemma. *Let $b = 0$, $c = 0$. Then there exists a constant $N = N(d,\kappa,q,\omega)$ such that for any $u \in W^{1,2}_q$, $\nu \geq 16$, and $r \in (0,1/\nu]$ we have*

$$(|u_{xx} - (u_{xx})_{Q_r}|^q)_{Q_r} \leq N\nu^{d+2}\mathcal{A}_{\nu r} + N(\nu^{-q} + \nu^{d+2}\hat{a}^{1/2})\mathcal{B}_{\nu r}, \tag{1}$$

where

$$\mathcal{A}_\rho = (|f|^q)_{Q_\rho}, \quad \mathcal{B}_\rho = (|u_{xx}|^q)_{Q_\rho}, \quad \hat{a} = a^{\#(x)}_{\nu r}, \quad f = Lu + u_t.$$

Proof. By having in mind the usual approximations, we may assume that

$$u \in C_0^\infty(\mathbb{R}^{d+1}).$$

According to Remark 6.4.5 (where we change the direction of the time axis), on $\mathbb{R}^{d+1}$ there is a function v such that it belongs to $W^{1,2}_q(\mathbb{R}^{d+1}_T)$ for all $T \in \mathbb{R}$, satisfies

$$Lv + v_t = fI_{Q_\nu},$$

in $\mathbb{R}^{d+1}$, and is such that $v(t,x) = 0$ for $t > 4$. Furthermore, by Exercise 6.4.6 we have $v \in W^{1,2}_p(\mathbb{R}^{d+1}_T)$ for all $T \in \mathbb{R}$ and $p \in (1,\infty)$.

After that we set

$$w = u - v$$

and note for the future that $w \in W^{1,2}_p(\mathbb{R}^{d+1}_T)$ for all $T \in \mathbb{R}$ and $p \in (1,\infty)$.

By Theorem 6.4.4 applied to $\Omega = (0,4) \times \mathbb{R}^d$ we have

$$\int_\Omega |v_{xx}|^q \, dxdt \le N \int_{Q_{\nu r}} |f|^q \, dxdt$$

implying that

$$(|v_{xx}|^q)_{Q_{\nu r}} \le N\mathcal{A}_{\nu r}, \quad (|v_{xx}|^q)_{Q_r} \le N\nu^{d+2}\mathcal{A}_{\nu r}. \tag{2}$$

Next, observe that

$$w \in W^{1,2}_{2q,loc} \subset W^{1,2}_{q,loc}$$

and $Lw + w_t = 0$ in $Q_{\nu r/4}$ and $\nu/4 \ge 4$. Therefore, by Lemma 6.2.2 with $\alpha = 2$

$$(|w_{xx} - (w_{xx})_{Q_r}|^q)_{Q_r} \le N\nu^{-q}(|w_{xx}|^q)_{Q_{\nu r/4}} + N\nu^{d+2}\hat{a}^{1/2}[(|w_{xx}|^{2q})_{Q_{\nu r/4}}]^{1/2}. \tag{3}$$

Actually in Lemma 6.2.2 we required w to have compact support, but this can be easily arranged by multiplying w by a cut-off function which equals 1 on $Q_{\nu r/4}$.

Owing to (2) and the definition of w,

$$(|w_{xx}|^q)_{Q_{\nu r/4}} \le N(|w_{xx}|^q)_{Q_{\nu r}}$$

$$\le N(|w_{xx} + v_{xx}|^q)_{Q_{\nu r}} + N(|v_{xx}|^q)_{Q_{\nu r}} \le N\mathcal{B}_{\nu r} + N\mathcal{A}_{\nu r}. \tag{4}$$

Now we apply Corollary 6.4.3 with $2q$, $\nu r/4$, and 4 in place of p, r, and ν, respectively, noting the fact that the conditions: $Lw + w_t = 0$ in $Q_{\nu r}$ and $\nu r/4 \le 1$, allows us to do that. We also use Hölder's inequality. Then we see that

$$[(|w_{xx}|^{2q})_{Q_{\nu r/4}}]^{1/2} \le N(|w_{xx}|^q)_{Q_{\nu r}}.$$

We estimate the last term using (4) and then infer from (3) that

$$(|w_{xx} - (w_{xx})_{Q_r}|^q)_{Q_r} \le N(\nu^{-q} + \nu^{d+2}\hat{a}^{1/2})(\mathcal{B}_{\nu r} + \mathcal{A}_{\nu r}).$$

To finish proving (1), it only remains to combine this with (2) and observe that

$$(|u_{xx} - (u_{xx})_{Q_r}|^q)_{Q_r} \le N(|v_{xx} - (v_{xx})_{Q_r}|^q)_{Q_r} + N(|w_{xx} - (w_{xx})_{Q_r}|^q)_{Q_r},$$

$$(|v_{xx} - (v_{xx})_{Q_r}|^q)_{Q_r} \le N(|v_{xx}|^q)_{Q_r}.$$

The lemma is proved.

Next, by repeating the argument in Corollary 6.1.4, we get the following.

2. Corollary. *There exists a constant N depending only on d, q, κ, and ω such that for any $u \in W_q^{1,2}$, $r > 0$, and $\nu \geq 16$ satisfying*

$$\nu r \leq 1,$$

we have

$$\fint_{(0,r^2)} \fint_{(0,r^2)} | \|u_{xx}(t,\cdot)\|_{\mathcal{L}_q} - \|u_{xx}(s,\cdot)\|_{\mathcal{L}_q}|^q \, dtds$$

$$\leq N(\nu^{-q} + \nu^{d+2}\hat{a}^{1/2}) \fint_{(0,\nu^2 r^2)} \|u_{xx}(t,\cdot)\|^q_{\mathcal{L}_q} \, dt$$

$$+ N\nu^{d+2} \fint_{(0,\nu^2 r^2)} \|(Lu + u_t)(t,\cdot)\|^q_{\mathcal{L}_q} \, dt.$$

To move further, fix a $u \in W_q^{1,2}$ and set

$$\phi(t) = \|u_{xx}(t,\cdot)\|_{\mathcal{L}_q}, \quad f = Lu + u_t, \quad \psi(t) = \|f(t,\cdot)\|_{\mathcal{L}_q}.$$

For any locally integrable function $\tau(s)$ on $\mathbb{R}$ denote by

$$\mathbb{M}_t \tau(s) \quad \text{and} \quad \tau^{\sharp(t)}(s)$$

the classical maximal and sharp functions of τ, respectively.

3. Lemma. *Let $r_0 \in (0,\infty)$, $b = 0$, $c = 0$. Assume that $u(t,x) = 0$ for $t \notin (0, r_0^2)$. Then for any $\nu \geq 16$ and $R \in (0,1]$, we have*

$$\phi^{\sharp(t)} \leq N\nu^{(d+2)/q} \mathbb{M}_t^{1/q}(\psi^q)$$

$$+ N\left((\nu r_0/R)^{2-2/q} + \nu^{-1} + \nu^{(d+2)/q}\omega^{1/(2q)}(R)\right) \mathbb{M}_t^{1/q}(\phi^q), \tag{5}$$

where $N = N(\omega, d, \kappa, q)$.

Proof. Obviously, Corollary 2 in terms of the functions ϕ and ψ yields

$$\fint_{(0,r^2)} \fint_{(0,r^2)} |\phi(t) - \phi(s)|^q \, dtds$$

$$\leq N\nu^{d+2} \fint_{(0,\nu^2 r^2)} \psi^q(t)\, dt + N(\nu^{-q} + \nu^{d+2}\omega^{1/2}(R)) \fint_{(0,\nu^2 r^2)} \phi^q(t)\, dt$$

if $r \leq R/\nu$ (when $a^{\#(x)}_{\nu r} \leq a^{\#(x)}_{R} \leq \omega(R)$ and $\nu r \leq 1$). This corollary allows shifting the origin. Therefore, for any $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$ and $\beta - \alpha = r^2 \leq R^2/\nu^2$ we have

$$\fint_{(\alpha,\beta)} \fint_{(\alpha,\beta)} |\phi(t) - \phi(s)|^q \, dtds \leq N\nu^{d+2} \fint_{(\alpha,\alpha+\nu^2(\beta-\alpha))} \psi^q(t)\, dt$$

$$+N(\nu^{-q} + \nu^{d+2}\omega^{1/2}(R)) \fint_{(\alpha,\alpha+\nu^2(\beta-\alpha))} \phi^q(t)\, dt.$$

Take a point $t_0 \in \mathbb{R}$ and α and β as above and such that $t_0 \in (\alpha, \beta)$. Then $t_0 \in (\alpha, \alpha + \nu^2(\beta - \alpha))$ and by definition

$$\fint_{(\alpha,\alpha+\nu^2(\beta-\alpha))} \psi^q(t)\, dt \leq \mathbb{M}_t(\psi^q)(t_0), \quad \fint_{(\alpha,\alpha+\nu^2(\beta-\alpha))} \phi^q(t)\, dt \leq \mathbb{M}_t(\phi^q)(t_0).$$

By applying Hölder's inequality, we conclude that

$$\fint_{(\alpha,\beta)} \fint_{(\alpha,\beta)} |\phi(t) - \phi(s)| \, dtds \tag{6}$$

is dominated by the value at t_0 of the right-hand side of (5), whenever $t_0 \in (\alpha, \beta)$ and $\beta - \alpha \leq R^2/\nu^2$. However, if $\beta - \alpha > R^2/\nu^2$, then (6) is dominated by

$$2 \fint_{(\alpha,\beta)} I_{(0,r_0^2)}\phi \, dt \leq 2\Big(\fint_{(\alpha,\beta)} I_{(0,r_0^2)} \, dt\Big)^{1-1/q} \Big(\fint_{(\alpha,\beta)} \phi^q \, dt\Big)^{1/q}$$

$$\leq 2(r_0^2/(\beta-\alpha))^{1-1/q}\mathbb{M}_t^{1/q}(\phi^q)(t_0) \leq 2(\nu r_0/R)^{2-2/q}\mathbb{M}_t^{1/q}(\phi^q)(t_0).$$

In this case (6) is again less than the value at t_0 of the right-hand side of (5). By taking the supremum of (6) over all $\alpha < \beta$ such that $t_0 \in (\alpha, \beta)$, we obtain (5) at t_0. Since t_0 is arbitrary, the lemma is proved.

2. Existence and uniqueness results

Under the assumptions of the chapter we take two numbers $q, p \in (1, \infty)$ such that $p \geq q$.

For $T \in [-\infty, \infty)$ and $S \in (-\infty, \infty]$ such that $S > T$ we denote

$$\mathbb{R}^{d+1}_{T,S} = (T, S) \times \mathbb{R}^d$$

and denote by $\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_{T,S})$ the space of functions $u(t,x)$ on $\mathbb{R}^{d+1}_{T,S}$ such that

$$\|u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_{T,S})} := \left(\int_T^S \left(\int_{\mathbb{R}^d} |u(t,x)|^q\,dx\right)^{p/q} dt\right)^{1/p} < \infty.$$

If $S=\infty$, we drop S in the notation $\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_{T,S})$. If $T=-\infty$, we drop $\mathbb{R}^{d+1}_T$ in the notation $\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_T)$. Observe that

$$\|u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q} = \left(\int_{\mathbb{R}} \|u(t,\cdot)\|^p_{\mathcal{L}_q}\,dt\right)^{1/p}.$$

1. Lemma. *There exists a constant N depending only on p,q,d,κ,K, and the function ω, such that for any $u\in C_0^\infty(\mathbb{R}^{d+1})$,*

$$\|u_{xx}\|_{\mathcal{L}_p\text{-}\mathcal{L}_q} + \|u_t\|_{\mathcal{L}_p\text{-}\mathcal{L}_q} \le N(\|Lu+u_t\|_{\mathcal{L}_p\text{-}\mathcal{L}_q} + \|u_x\|_{\mathcal{L}_p\text{-}\mathcal{L}_q} + \|u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q}). \quad (1)$$

Proof. Observe that we included $\|u_x\|_{\mathcal{L}_p\text{-}\mathcal{L}_q}$ and $\|u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q}$ on the right. Therefore, while proving (1), we may certainly assume that $b\equiv 0$ and $c\equiv 0$. Since $u_t=(Lu+u_t)-Lu$, we only need to estimate u_{xx}. If $p=q$ so that $\mathcal{L}_p\text{-}\mathcal{L}_q=\mathcal{L}_p(\mathbb{R}^{d+1})$, the result is known from Lemma 6.3.6.

In the case $p>q$ we fix a number $r_0>0$ and first assume that

$$u(t,x)=0 \quad \text{for} \quad t\notin(0,r_0^2). \quad (2)$$

Then set $f=Lu+u_t$ and also use other objects introduced before Lemma 1.3. We raise both parts of (1.5) to the power p, integrate over $\mathbb{R}$, and observe that since $p/q>1$, by the Hardy-Littlewood theorem we have

$$\int_{\mathbb{R}} \mathbb{M}_t^{p/q}(\psi^q)(t)\,dt \le N\int_{\mathbb{R}} \psi^p(t)\,dt = N\|f\|^p_{\mathcal{L}_p\text{-}\mathcal{L}_q},$$

$$\int_{\mathbb{R}} \mathbb{M}_t^{p/q}(\phi^q)(t)\,dt \le N\|u_{xx}\|^p_{\mathcal{L}_p\text{-}\mathcal{L}_q}.$$

We also use the Fefferman-Stein theorem (see Theorems 3.2.10 and 3.3.1) and conclude that

$$\|u_{xx}\|_{\mathcal{L}_p\text{-}\mathcal{L}_q} \le N_1\nu^{(d+2)/q}\|f\|_{\mathcal{L}_p\text{-}\mathcal{L}_q}$$
$$+ N_2\big((\nu r_0/R)^{2-2/q} + \nu^{-1} + \nu^{(d+2)/q}\omega^{1/(2q)}(R)\big)\|u_{xx}\|_{\mathcal{L}_p\text{-}\mathcal{L}_q}, \quad (3)$$

whenever $\nu \geq 16$ and $R \leq 1$, where the constants N_i are determined by p, q, d, κ, and the *function* ω. We choose a large $\nu = \nu(N_2, d)$ and a small $R = R(N_2, d, q, \omega)$ so that

$$N_2\big(\nu^{-1} + \nu^{(d+2)/q}\omega^{1/(2q)}(R)\big) \leq 1/4.$$

After ν and R have been fixed, we chose a small $r_0 = r_0(N_2, d, q, \omega)$ so that

$$N_2(\nu r_0/R)^{2-2/q} \leq 1/4.$$

Then (3) implies that

$$\|u_{xx}\|_{\mathcal{L}_p\text{-}\mathcal{L}_q} \leq 2N_1\nu^{(d+2)/q}\|Lu + u_t\|_{\mathcal{L}_p\text{-}\mathcal{L}_q} \tag{4}$$

for any $u \in C_0^\infty(\mathbb{R}^{d+1})$ satisfying (2). We thus have obtained (1) even without the terms $\|u_x\|_{\mathcal{L}_p\text{-}\mathcal{L}_q}$ and $\|u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q}$ on the right of (1).

Now take a nonnegative $\zeta \in C_0^\infty(\mathbb{R})$ such that $\zeta(t) = 0$ if $t \notin (0, r_0^2)$ and

$$\int_{\mathbb{R}} \zeta^p(t)\, dt = 1.$$

take a $u \in C_0^\infty(\mathbb{R}^{d+1})$ and observe that (4) is also true if we shift the t-axis. In particular, (4) is applicable to $u(t,x)\zeta(t - t_0)$. Then we get

$$\int_{\mathbb{R}} \zeta^p(t - t_0)\|u_{xx}(t,\cdot)\|_{\mathcal{L}_q}^p\, dt \leq N\int_{\mathbb{R}} \zeta^p(t - t_0)\|(Lu + u_t)(t,\cdot)\|_{\mathcal{L}_q}^p\, dt$$

$$+ N\int_{\mathbb{R}} |\zeta'(t - t_0)|^p\|u(t,\cdot)\|_{\mathcal{L}_q}^p\, dt.$$

Upon integrating through with respect to t_0 we come to (1). The lemma is proved.

2. Exercise*. Prove that if $u \in C_0^\infty(\mathbb{R}^{d+1})$, then

$$\|u_x\|_{\mathcal{L}_p\text{-}\mathcal{L}_q} \leq N(d)\|u_{xx}\|_{\mathcal{L}_p\text{-}\mathcal{L}_q}^{1/2}\|u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q}^{1/2}.$$

Here is a counterpart of Lemma 6.3.8.

3. Lemma. *There exist constants $\lambda_0 \geq 1$ and N depending only on p, q, d, κ, K, and the function ω, such that if $\lambda \geq \lambda_0$, then for any $u \in C_0^\infty(\mathbb{R}^{d+1})$,*

$$\lambda\|u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q} + \sqrt{\lambda}\|u_x\|_{\mathcal{L}_p\text{-}\mathcal{L}_q} + \|u_{xx}\|_{\mathcal{L}_p\text{-}\mathcal{L}_q} + \|u_t\|_{\mathcal{L}_p\text{-}\mathcal{L}_q}$$

$$\leq N\|Lu + u_t - \lambda u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q}. \tag{5}$$

Proof. We use the same method as in the proof of Lemma 6.3.8, first noticing that the term with u_x on the left in (5) can be eliminated on the account of Exercise 2. As in that proof we convince ourselves that it suffices to establish the estimate

$$\lambda\|u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q} \le N\|Lu + u_t - \lambda u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q}. \tag{6}$$

We introduce the space

$$\mathbb{R}^{d+2} = \{(t,z) = (t,x,y) : t,y \in \mathbb{R}, x \in \mathbb{R}^d\}$$

and the function

$$\tilde{u}(t,z) = u(t,x)\zeta(y)\cos(\mu y),$$

where $\mu = \sqrt{\lambda}$ and ζ is a $C_0^\infty(\mathbb{R})$ function, $\zeta \not\equiv 0$. Also introduce the operator

$$\tilde{L}u(t,z) = L(t,x)u(t,z) + u_{yy}(t,z).$$

For the same reasons as in the proof of Lemma 6.3.8 we can apply Lemma 1 to $\tilde{u}$ and $\tilde{L}$. In addition, in light of Exercise 2 we may not include the norms of the first derivatives of $\tilde{u}$ on the right of the corresponding counterpart of (1). Thus,

$$\|\tilde{u}_{zz}\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+2})} \le N\|\tilde{L}\tilde{u} + \tilde{u}_t\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+2})} + N\|\tilde{u}\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+2})}. \tag{7}$$

By Exercise 6.3.7

$$\alpha = \alpha(\mu) := \int_{\mathbb{R}} |\zeta(y)\cos(\mu y)|^q \, dy$$

is bounded away from zero for $\mu \in \mathbb{R}$. Therefore,

$$\|u\|^p_{\mathcal{L}_p\text{-}\mathcal{L}_q} = \alpha^{-p/q}\mu^{-2p}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}^{d+1}} |\tilde{u}_{yy}(t,z)$$

$$- u(t,x)[\zeta''(y)\cos(\mu y) - 2\mu\zeta'(y)\sin(\mu y)]|^q \, dz\Big)^{p/q} dt$$

$$\le N\mu^{-2p}\big(\|\tilde{u}_{zz}\|^p_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+2})} + (\mu^p + 1)\|u\|^p_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+2})}\big).$$

By combining this with (7), we get

$$\mu^2\|u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q} \le N\|\tilde{L}\tilde{u} + \tilde{u}_t\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+2})} + N(\mu+1)\|u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q}.$$

Here almost obviously

$$\|\tilde{L}\tilde{u} + \tilde{u}_t\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+2})} \le N\|Lu + u_t - \lambda u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q} + N(\mu + 1)\|u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q},$$

so that

$$\lambda\|u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q} \le N\|Lu + u_t - \lambda u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q} + N(\sqrt{\lambda} + 1)\|u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q}.$$

After that, we finish proving (6) as in the proof of Lemma 6.3.8. The lemma is proved.

4. Exercise. Prove that if $L = \Delta$, then one can take $\lambda_0 = 0$ in Lemma 3.

Now the spaces $W^{1,2}_{p,q}(\mathbb{R}^{d+1}_{T,S})$ come naturally. Take a $T \in [-\infty, \infty)$ and $S \in (-\infty, \infty]$ and for functions $u \in C^{1,2}(\bar{\mathbb{R}}^{d+1}_{T,S})$ define

$$\|u\|_{W^{1,2}_{p,q}(\mathbb{R}^{d+1}_{T,S})} = \|u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_{T,S})} + \|u_x\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_{T,S})}$$

$$+ \|u_{xx}\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_{T,S})} + \|u_t\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_{T,S})}.$$

For a function $u \in \mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_{T,S})$ write $u \in W^{1,2}_{p,q}(\mathbb{R}^{d+1}_{T,S})$ if there is a sequence $u^n \in C^{1,2}(\bar{\mathbb{R}}^{d+1}_{T,S})$ such that

$$\|u - u^n\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_{T,S})} \to 0, \quad \|u^n\|_{W^{1,2}_{p,q}(\mathbb{R}^{d+1}_{T,S})} < \infty, \quad \|u^n - u^m\|_{W^{1,2}_{p,q}(\mathbb{R}^{d+1}_{T,S})} \to 0$$

as $n, m \to \infty$. As on a few occasions before, for $u \in W^{1,2}_{p,q}(\mathbb{R}^{d+1}_{T,S})$ the generalized derivatives u_x, u_{xx}, u_t are well defined and belong to $\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_{T,S})$. Then the $W^{1,2}_{p,q}(\mathbb{R}^{d+1}_{T,S})$ norm is introduced naturally and it makes $W^{1,2}_{p,q}(\mathbb{R}^{d+1}_{T,S})$ a Banach space. We write

$$W^{1,2}_{p,q}(\mathbb{R}^{d+1}_{T}) = W^{1,2}_{p,q}(\mathbb{R}^{d+1}_{T,\infty}), \quad W^{1,2}_{p,q} = W^{1,2}_{p,q}(\mathbb{R}^{d+1}).$$

5. Exercise*. Prove that C^{∞}_0 is dense in $W^{1,2}_{p,q}$ and in $\mathcal{L}_p\text{-}\mathcal{L}_q$.

6. Exercise*. For $\lambda > 0$, the operator $\mathcal{G}^0_\lambda$ (see (2.2.7)), and $f \in C^{\infty}_0(\mathbb{R}^{d+1})$ set $u = \mathcal{G}^0_\lambda f$ and prove that

(i) u is infinitely differentiable,

(ii) u satisfies $\Delta u + u_t - \lambda u = -f$, and

(iii) $u \in W^{1,2}_{p,q}$ for any p, q.

7. Exercise*. Prove that $u \in W^{1,2}_{p,q}(\mathbb{R}^{d+1}_0)$ if and only if for the function v defined on $\mathbb{R}^{d+1}$ by $v(t, x) = u(|t|, x)$ we have $v \in W^{1,2}_{p,q}$.

8. Theorem. *(i) With the constants λ_0 and N from Lemma 3, the inequality*

$$\lambda\|u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_T)} + \sqrt{\lambda}\|u_x\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_T)} + \|u_{xx}\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_T)} + \|u_t\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_T)}$$

$$\leq N\|Lu + u_t - \lambda u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_T)} \tag{8}$$

holds for any $u \in W^{1,2}_{p,q}(\mathbb{R}^{d+1}_T)$.

(ii) For any $\lambda \geq \lambda_0$ and $f \in \mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_T)$ there is a unique $u \in W^{1,2}_{p,q}(\mathbb{R}^{d+1}_T)$ such that

$$Lu + u_t - \lambda u = f \quad in \quad \mathbb{R}^{d+1}_T.$$

Proof. First let $T = -\infty$. Then the first assertion is true owing to Exercise 5 and Lemma 3. By Exercise 6 for $\lambda > 0$ the set $(\Delta + \partial_t - \lambda)W^{1,2}_{p,q}$ contains $C_0^\infty(\mathbb{R}^{d+1})$ and hence is dense in $\mathcal{L}_p\text{-}\mathcal{L}_q$. Together with the a priori estimate (8) this proves assertion (ii) for $L = \Delta$. Then in the general case assertion (ii) follows by the method of continuity.

One observation will be used below. For $f \in C_0^\infty(\mathbb{R}^{d+1})$ the function $\mathcal{G}^0_\lambda f$ is the unique solution in $W^{1,2}_{p,q}$ of the equation

$$\Delta u + u_t - \lambda u = -f. \tag{9}$$

If $f_n \to f$ in $\mathcal{L}_p\text{-}\mathcal{L}_q$, the corresponding solutions converge in $\mathcal{L}_p\text{-}\mathcal{L}_q$ owing to the a priori estimate. Also $\mathcal{G}^0_\lambda$ is a continuous operator in $\mathcal{L}_p\text{-}\mathcal{L}_q$ (see Remark 2.1.5). It follows that for any $f \in \mathcal{L}_p\text{-}\mathcal{L}_q$ the unique solution in $W^{1,2}_{p,q}$ of (9) is given by the formula

$$u = \mathcal{G}^0_\lambda f. \tag{10}$$

What is needed below is that if for an $S \in \mathbb{R}$ we have $f(t,x) = 0$ for $t \geq S$, then it follows from (10) that $u(t,x) = 0$ for $t \geq S$, where u is the solution of the equation $Lu + u_t - \lambda u = f$ in $\mathbb{R}^{d+1}$.

Now we are ready to treat the case where $T > -\infty$. One can shift the origin of the t-axis and, therefore, we confine ourselves to the case $T = 0$. Then we follow the proof of Theorem 6.4.1 and exactly as there by using the method of continuity and the above observation obtain that if $f \in \mathcal{L}_p\text{-}\mathcal{L}_q$ and $f(t,x) = 0$ for $t \geq 0$, then $u(t,x) = 0$ for $t \geq 0$.

To prove (i), we take a $u \in W^{1,2}_{p,q}(\mathbb{R}^{d+1}_0)$, define $v(t,x) = u(|t|,x)$ for $(t,x) \in \mathbb{R}^{d+1}$, and observe that $v \in W^{1,2}_{p,q}$ by Exercise 7. Then we set

$$g = (Lu + u_t - \lambda u)I_{t>0}$$

and introduce w as the unique solution in $W^{1,2}_{p,q}$ of the equation

$$Lw + w_t - \lambda w = g.$$

Observe that $v, w \in W^{1,2}_{p,q}$ and

$$L(v - w) + (v - w)_t - \lambda(v - w) = 0$$

in $\mathbb{R}^{d+1}_0$. By the above, $v = w$ in $\mathbb{R}^{d+1}_0$ and by definition $v = u$ in $\mathbb{R}^{d+1}_0$. Hence, by the result for $T = -\infty$,

$$\lambda\|u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_0)} + \sqrt{\lambda}\|u_x\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_0)} + \|u_{xx}\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_0)} + \|u_t\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_0)}$$

$$= \lambda\|w\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_0)} + \sqrt{\lambda}\|w_x\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_0)} + \|w_{xx}\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_0)} + \|w_t\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_0)}$$

$$\le N\|g\|_{\mathcal{L}_p\text{-}\mathcal{L}_q} = N\|Lu + u_t - \lambda u\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\mathbb{R}^{d+1}_0)}.$$

This proves (i) and uniqueness in (ii). We have established the existence of solutions even in $\mathbb{R}^{d+1}$. The theorem is proved.

Finally, we state a counterpart of Theorem 6.4.4 about the Cauchy problem. The reader will obtain its proof after making obvious changes in the proof of Theorem 2.5.3.

For $T \in (0, \infty)$ set

$$\Omega = (0, T) \times \mathbb{R}^{d+1}$$

and write $u \in \overset{0}{W}{}^{1,2}_{p,q}(\Omega)$ if for the function v defined on $\mathbb{R}^{d+1}_{-\infty,T}$ by $v(t, x) = u(t, x)$ for $t \in (0, T)$ and $v(t, x) = 0$ for $t \le 0$ we have $v \in W^{1,2}_{p,q}(\mathbb{R}^{d+1}_{-\infty,T})$.

9. Theorem. *For any $f \in \mathcal{L}_p\text{-}\mathcal{L}_q(\Omega)$ and $v \in W^{1,2}_{p,q}(\Omega)$ there exists a unique $u \in W^{1,2}_{p,q}(\Omega)$ satisfying*

$$u_t = Lu + f$$

in Ω and such that $u - v \in \overset{0}{W}{}^{1,2}_{p,q}(\Omega)$. Furthermore, there is a constant N depending only on $T, p, q, d, \kappa, K, \omega$, such that

$$\|u\|_{W^{1,2}_{p,q}(\Omega)} \le N(\|f\|_{\mathcal{L}_p\text{-}\mathcal{L}_q(\Omega)} + \|v\|_{W^{1,2}_{p,q}(\Omega)}).$$

10. Exercise. Assume that the coefficients of L are independent of x and by using duality, prove that all the results of this section are also true for $1 < p \le q$.

3. Hints to exercises

2.4. Any λ_0 can be reduced to zero by using dilations.

2.6. (ii) See Lemma 2.2.11. (iii) See Remark 2.1.5.

Chapter 8

Second-order elliptic equations in $W_p^2(\Omega)$

Starting from this chapter (apart from Exercises 10.1.10, 10.1.12, and 10.1.14 and Section 13.5) we only concentrate on elliptic equations in domains $\Omega \subset \mathbb{R}^d$. It is worth noticing, however, that almost everything valid for the elliptic equations is easily shown to have a natural version valid for parabolic equations in $\mathbb{R} \times \Omega$. Also the Cauchy problem in $(0, \infty) \times \Omega$ can be treated very similarly to what has been done in Section 5.2. The interested reader will find further information about parabolic equations in [**14**] and [**18**].

In this chapter

$$p \in (1, \infty)$$

is a fixed number, unless explicitly specified otherwise. We are considering a second-order elliptic operator L given by

$$Lu(x) = a^{ij}(x)u_{x^i x^j}(x) + b^i(x)u_{x^i}(x) + c(x)u(x).$$

We suppose that Assumption 1.6.1 is satisfied, that is, the coefficients a, b, and c are measurable functions on $\mathbb{R}^d$, $a^{ij} = a^{ji}$, and for some constants $\kappa > 0$ and $K \in (0, \infty)$ for all values of the arguments and $\xi \in \mathbb{R}^d$ it holds that

$$|b| + |c| \le K, \quad c \le 0, \quad \kappa^{-1}|\xi|^2 \ge a^{ij}\xi^i\xi^j \ge \kappa|\xi|^2,$$

where $b = (b^1, ..., b^d)$. We also assume that there is a function $\omega(\varepsilon)$, $\varepsilon \ge 0$, such that $\omega(0+) = 0$, and for all $i, j = 1, ..., d$, $x, y \in \mathbb{R}^d$, we have

$$|a^{ij}(x) - a^{ij}(y)| \le \omega(|x - y|).$$

We are not going to use the results in Chapters 6 and 7. In this connection, it is important to note for the reader who did not follow Chapters 2–7 that one can take $p = 2$ in what follows and only rely on the results from Chapter 1.

As a matter of fact, the key result that is needed is the following.

1. Theorem. *There exist constants $\lambda_0 \geq 1$ and N, depending only on p, K, κ, ω, and d, such that the estimate*

$$\lambda\|u\|_{\mathcal{L}_p} + \lambda^{1/2}\|u_x\|_{\mathcal{L}_p} + \|u_{xx}\|_{\mathcal{L}_p} \leq N\|Lu - \lambda u\|_{\mathcal{L}_p}$$

holds true for any $u \in W_p^2$ and $\lambda \geq \lambda_0$. Furthermore, for any $\lambda \geq \lambda_0$ and $f \in \mathcal{L}_p$, there exists a unique $u \in W_p^2$ satisfying $Lu - \lambda u = f$.

This theorem is Theorem 1.6.4 if $p = 2$ and Theorem 5.2.1 (ii) if $p \in (1, \infty)$. In this chapter we apply Theorem 1 to investigate *the first boundary value problem*:

$$Lu - \lambda u = f \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \partial\Omega,$$

where Ω is a smooth domain in $\mathbb{R}^d$, f is a given function in Ω, g is a given function on the *boundary* $\partial\Omega$ of Ω, and λ is a large constant.

1. Spaces of functions vanishing on the boundary

To explain what we mean by boundary data, we introduce the following Sobolev spaces of functions "vanishing" on the boundary. Let $k \in \{1, 2, ...\}$.

1. Definition. For $p \in [1, \infty)$ by $\overset{0}{W}{}_p^1(\Omega)$ we mean the subset of $W_p^1(\Omega)$ consisting of all functions for each of which there is a defining sequence vanishing on $\partial\Omega$. For $k \in \{2, 3, ...\}$, let

$$\overset{0}{W}{}_p^k(\Omega) = \overset{0}{W}{}_p^1(\Omega) \cap W_p^k(\Omega).$$

The norm in $\overset{0}{W}{}_p^k(\Omega)$, $k \in \{1, 2, ...\}$, is taken to be the same as in $W_p^k(\Omega)$.

2. Theorem. *The $\overset{0}{W}{}_p^k(\Omega)$ are Banach spaces, which are closed subspaces of $W_p^k(\Omega)$.*

Proof. First take $k = 1$ and let $u_n \in \overset{0}{W}{}_p^1(\Omega)$ be a Cauchy sequence in $W_p^1(\Omega)$ and let v_{nm} be defining sequences for u_n in $W_p^1(\Omega)$, vanishing on $\partial\Omega$. Then there exists a sequence $m(n)$ such that

$$\|u_n - v_{nm(n)}\|_{W_p^1(\Omega)} \leq 1/n.$$

It follows that $v_{nm(n)}$ is a Cauchy sequence in $W^1_p(\Omega)$ and since this space is complete, there is a $u \in W^1_p(\Omega)$ such that $v_{nm(n)} \to u$ in $W^1_p(\Omega)$. Also $v_{nm(n)}$ vanish on $\partial\Omega$, so that $u \in \overset{0}{W}{}^1_p(\Omega)$. Finally, obviously $u_n \to u$ in $W^1_p(\Omega)$, thus proving the completeness of $\overset{0}{W}{}^1_p(\Omega)$ or, equivalently, the closedness of $\overset{0}{W}{}^1_p(\Omega)$ in $W^1_p(\Omega)$.

In the general case if $u_n \in \overset{0}{W}{}^k_p(\Omega)$ is a Cauchy sequence in $\overset{0}{W}{}^k_p(\Omega)$, then it is a Cauchy sequence in $W^k_p(\Omega)$ therefore converging in $W^k_p(\Omega)$ to a $v \in W^k_p(\Omega)$ and in $W^1_p(\Omega)$ to a $u \in \overset{0}{W}{}^1_p(\Omega)$. Convergence in $W^k_p(\Omega)$ implies convergence in $\mathcal{L}_p(\Omega)$, so that

$$u = v \in \overset{0}{W}{}^k_p(\Omega).$$

The theorem is proved.

In the same way that Corollary 1.3.14 is proved, we get the following fact to be used in the hint to Exercise 11.1.10.

3. Corollary. *Let $1 < p < \infty$ and $k \in \{0, 1, 2, ...\}$ and let U be a bounded subset of $\overset{0}{W}{}^k_p(\Omega)$. Then for any sequence $u_n \in U$ there exist a subsequence $u_{n'}$ and a function $u \in \overset{0}{W}{}^k_p(\Omega)$ such that the $D^\alpha u_{n'}$ converge weakly in $\mathcal{L}_p(\Omega)$ to $D^\alpha u$ whenever $|\alpha| \le k$.*

4. Exercise*. Prove that if $u \in W^2_p$ and $u(x) = u(|x^1|, x')$, then $u_{x^1} \in \overset{0}{W}{}^1_p(\mathbb{R}^d_+)$.

5. Exercise. Prove that $u \in \overset{0}{W}{}^1_p(\mathbb{R}^d_+)$ if and only if for the function v defined by $v(x) = u(x)$ for $x \in \mathbb{R}^d_+$ and $v(x) = 0$ otherwise, we have $v \in W^1_p$. In either case

$$\|u\|_{W^1_p(\mathbb{R}^d_+)} = \|v\|_{W^1_p}.$$

6. Exercise*. By using the hint to Exercise 5, prove that $C_0^\infty(\mathbb{R}^d_+)$ is dense in $\overset{0}{W}{}^1_p(\mathbb{R}^d_+)$. Also show that $C_0^\infty(\mathbb{R}^d_+)$ is *not* dense in $\overset{0}{W}{}^2_p(\mathbb{R}^d_+)$.

7. Exercise. Give an example of a domain Ω and a function u such that for v defined according to $v(x) = u(x)$ for $x \in \Omega$ and $v = 0$ outside Ω we have $v \in W^1_p$, but $u \notin \overset{0}{W}{}^1_p(\Omega)$.

The following exercise shows that if $u \in \overset{0}{W}{}^1_p(\mathbb{R}^d_+)$, then u approaches 0 on average near $x^1 = 0$.

8. Exercise. Prove that if $p \in [1, \infty)$, $u \in \overset{0}{W}{}_p^1(\mathbb{R}_+^d)$ and $\varepsilon > 0$, then

$$\frac{1}{\varepsilon}\int_{[0,\varepsilon]\times\mathbb{R}^{d-1}} |u|^p\,dx \le p^{-1}\varepsilon^{p-1}\int_{[0,\varepsilon]\times\mathbb{R}^{d-1}} |u_{x^1}|^p\,dx.$$

9. Exercise (Hardy's inequality). Prove that if $p \in (1, \infty)$ and $u \in \overset{0}{W}{}_p^1(\mathbb{R}_+^d)$, then for $v(x) = (x^1)^{-1}u(x)$ and $q = p/(p-1)$

$$\|v\|_{\mathcal{L}_p(\mathbb{R}_+^d)} \le q\|u_{x^1}\|_{\mathcal{L}_p(\mathbb{R}_+^d)}.$$

2. Equations in half spaces

In this section we take

$$\Omega = \mathbb{R}_+^d = \{(x^1, x') : x^1 > 0, x' = (x^2, ..., x^d) \in \mathbb{R}^{d-1}\}$$

with an obvious modification of this notation if $d = 1$. We start with a natural characterization of spaces $\overset{0}{W}{}_p^k(\Omega)$ for $k = 1, 2$.

1. Lemma. *Let $k = 1, 2$. A function $u \in \overset{0}{W}{}_p^k(\Omega)$ if and only if, for*

$$\bar{u}(x) := u(|x^1|, x')\operatorname{sign} x^1,$$

we have $\bar{u} \in W_p^k$. Also

$$\|u\|_{W_p^k(\Omega)} \le \|\bar{u}\|_{W_p^k} \le 2\|u\|_{W_p^k(\Omega)} \tag{1}$$

whenever $u \in \overset{0}{W}{}_p^k(\Omega)$ or $\bar{u} \in W_p^k$.

Proof. To prove the "if" part, first let $k = 1$, assume $\bar{u} \in W_p^1$, and take a defining sequence v_n for $\bar{u}$. Then $-v_n(-x^1, x')$ is a defining sequence for $-\bar{u}(-x^1, x') = \bar{u}(x)$ and

$$\bar{v}_n(x) = (v_n(x) - v_n(-x^1, x'))/2$$

is a defining sequence for $\bar{u}(x)$ as well. Obviously, $\bar{v}_n = 0$ for $x^1 = 0$,

$$\|\bar{v}_n - u\|_{\mathcal{L}_p(\Omega)} = \|\bar{v}_n - \bar{u}\|_{\mathcal{L}_p(\Omega)} \le \|\bar{v}_n - \bar{u}\|_{\mathcal{L}_p} \to 0,$$

$$\|D_j\bar{v}_n - D_j\bar{v}_m\|_{\mathcal{L}_p(\Omega)} \le \|D_jv_n - D_j\bar{v}_m\|_{\mathcal{L}_p} \to 0$$

as $n, m \to \infty$ and $j = 1, ..., d$. Hence $u \in \overset{0}{W}{}_p^1(\Omega)$ and

$$\|u\|_{W_p^1(\Omega)} = \lim_{n\to\infty}\|\bar{v}_n\|_{W_p^1(\Omega)} \le \lim_{n\to\infty}\|\bar{v}_n\|_{W_p^1} = \|\bar{u}\|_{W_p^1}.$$

The same argument obviously works for $k = 2$.

To prove the "only if" part, again first take $k = 1$. Let $u \in \overset{0}{W}{}^1_p(\Omega)$ and $w_n \in C^1(\bar{\Omega})$ be its defining sequence such that $w_n = 0$ for $x^1 = 0$. Denote $\bar{w}_n(x) = w_n(|x^1|, x') \operatorname{sign} x^1$. As is easy to see, $\bar{w}_n \in C^1$ and

$$D_j \bar{w}_n(x) = (\operatorname{sign} x^1)^{\delta_{1j}} (D_j w_n)(|x^1|, x') \operatorname{sign} x^1. \tag{2}$$

Furthermore, formula (2) and the choice of w_n show that $D_j \bar{w}_n$ is a Cauchy sequence in $\mathcal{L}_p$, and obviously $\bar{w}_n \to \bar{u}$ in $\mathcal{L}_p$. Hence, $\bar{u} \in W^1_p$. Upon passing to the limit in (2), we get

$$D_j \bar{u}(x) = (\operatorname{sign} x^1)^{\delta_{1j}} (D_j u)(|x^1|, x') \operatorname{sign} x^1, \tag{3}$$

which immediately implies the second inequality in (1) and finishes the proof of the "only if" part for $k = 1$.

If $u \in \overset{0}{W}{}^2_p(\Omega)$, then we already know that $\bar{u} \in W^1_p$ and we only need to show that the second-order generalized derivatives of $\bar{u}$ are in $\mathcal{L}_p$. We claim that

$$D_{ij} \bar{u}(x) = (\operatorname{sign} x^1)^{\delta_{1i} + \delta_{1j}} (D_{ij} u)(|x^1|, x') \operatorname{sign} x^1. \tag{4}$$

To prove the claim, notice that if $w \in C^2_0(\bar{\Omega})$, then integrating by parts shows that, for each $\phi \in C^\infty_0$,

$$\int_{\mathbb{R}^d} \phi(x)(D_{11} w)(|x^1|, x') \operatorname{sign} x^1 \, dx$$

$$= \int_{x^1 > 0} \phi D_{11} w \, dx - \int_{x^1 > 0} \phi(-x^1, x') D_{11} w(x) \, dx$$

$$= - \int_{x^1 > 0} D_1 \phi D_1 w \, dx - \int_{x^1 > 0} (D_1 \phi)(-x^1, x') D_1 w(x) \, dx$$

$$= - \int_{\mathbb{R}^d} D_1 \phi(x)(D_1 w)(|x^1|, x') \, dx.$$

By taking instead of w a sequence $w_n \in C^2_0(\bar{\Omega})$ defining u as an element of $W^2_p(\Omega)$, passing to the limit, and using (3), we find

$$\int_{\mathbb{R}^d} \phi(x)(D_{11} u)(|x^1|, x') \operatorname{sign} x^1 \, dx$$

$$= - \int_{\mathbb{R}^d} D_1 \phi(x)(D_1 u)(|x^1|, x') \, dx = \int_{\mathbb{R}^d} \bar{u} D_{11} \phi \, dx.$$

This proves (4) for $i = j = 1$. For other derivatives the argument is similar and since the right-hand side of (4) is in $\mathcal{L}_p$, the lemma is proved.

Recall Theorem 1.8.5 (ii) which shows that the Sobolev mollifiers of $\bar{u}$ converge to $\bar{u}$ in W^k_p and Lemma 1.8.2 which says that they are infinitely differentiable. If one takes a symmetric mollifying kernel ζ, then $\bar{u}^{(\varepsilon)}$ will be odd with respect to x^1 and thus vanishing on $\partial\Omega$. This and Lemma 1 lead to the following.

2. Corollary. *For $k = 1, 2$ and any $u \in \overset{0}{W}{}^k_p(\Omega)$ there exists a sequence $u_n \in C^n(\bar{\Omega}) \cap W^n_p(\Omega)$ such that $u_n = 0$ on $\partial\Omega$ and $u_n \to u$ in $W^k_p(\Omega)$.*

3. Exercise. Give an example showing that Lemma 1 is wrong if $k = 3$. On the other hand, as the reader will see, doing Exercise 10.2.7, the assertion of Corollary 2 holds for all k.

Regarding the ways in which continuations across $\partial\Omega$ are related to the original functions, we give the reader the following.

4. Exercise*. Prove that $u \in W^1_p(\Omega)$ if and only if for the function $v(x) = u(|x^1|, x')$ we have $v \in W^1_p$. In either case

$$v_{x^1}(x) = u_{x^1}(|x^1|, x')\,\mathrm{sign}\, x^1, \quad v_{x^i}(x) = u_{x^i}(|x^1|, x'), \quad i \geq 2, \tag{5}$$

$$\|u\|_{W^1_p(\Omega)} \leq \|v\|_{W^1_p} \leq 2\|u\|_{W^1_p(\Omega)}.$$

5. Exercise*. By using Exercise 1.3.21, (5), and Lemma 1, prove that if $u \in W^2_p(\mathbb{R}^d_+)$ and $u_{x^1} \in \overset{0}{W}{}^1_p(\mathbb{R}^d_+)$, then $v(x) := u(|x^1|, x')$ is in W^2_p.

6. Exercise. Let $d = 2$ and assume that the conditions of Exercise 1.4.8 are satisfied. As in this exercise define L and prove that for $f \in \mathcal{L}_2(\mathbb{R}^2_+)$ there exists a unique $u \in W^2_2(\mathbb{R}^2_+)$ such that $Lu = f$ in $\mathbb{R}^2_+$ and $u_{x^1} \in \overset{0}{W}{}^1_2(\mathbb{R}^2_+)$. Also prove that estimate (1.4.6) holds with $\mathcal{L}_2(\mathbb{R}^2_+)$ in place of $\mathcal{L}_2$.

7. Theorem. *Let $\lambda > 0$, $L = a^{ij}D_{ij}$ with a^{ij} being constant. Then, for any $f \in \mathcal{L}_p(\Omega)$, there is a unique $u \in \overset{0}{W}{}^2_p(\Omega)$ satisfying $Lu - \lambda u = f$ in Ω. For this u we have*

$$\lambda\|u\|_{\mathcal{L}_p(\Omega)} + \lambda^{1/2}\|u_x\|_{\mathcal{L}_p(\Omega)} + \|u_{xx}\|_{\mathcal{L}_p(\Omega)} \leq N_0\|Lu - \lambda u\|_{\mathcal{L}_p(\Omega)}, \tag{6}$$

where N_0 depends only on d, p, and the constant of ellipticity κ of L.

Proof. Of course the first thing to be done is a linear change of coordinates as in the proof of Lemma 1.6.2 so as to transform the operator $L - \lambda$ into $\Delta - \lambda$. It may happen that under this transformation the half space $\Omega = \mathbb{R}^d_+$ will take a different position. But then an appropriate orthogonal change of variables brings the image of $\mathbb{R}^d_+$ back to its original position and it turns out that the operator Δ is *invariant* under orthogonal transformations (see Exercise 1.1.4). Also it is absolutely trivial to understand how linear nondegenerate transformations preserving $\Omega = \mathbb{R}^d_+$ affect the functions in $W^2_p(\Omega)$. All this convinces us that we may concentrate on $L = \Delta$.

In that case extend $f(x)$ for $x^1 < 0$ by the formula $f(x) = -f(-x^1, x')$ and let $u \in W^2_p$ be the unique solution of

$$\Delta u - \lambda u = f,$$

which exists by Theorem 0.1. As is easy to see, the function $-u(-x^1, x')$ satisfies the same equation, so that by uniqueness $u(x) = -u(-x^1, x')$ or

$$u(x) = u(|x^1|, x') \operatorname{sign} x^1.$$

The fact that $u \in W^2_p$ and Lemma 1 imply that $u \in \overset{\circ}{W}{}^2_p(\Omega)$ and (6) holds. The uniqueness of solution in $\overset{\circ}{W}{}^2_p(\Omega)$ also follows directly from Theorem 0.1 and Lemma 1. The theorem is proved.

The following result is used only in Section 9.3 bearing on the oblique derivative boundary problem.

8. Theorem. *Suppose that the assumptions of Theorem 7 are satisfied. Take $f_1, ..., f_d, g \in \mathcal{L}_p(\Omega)$. Then in $\overset{\circ}{W}{}^1_p(\Omega)$ there exists a unique solution of the equation*

$$Lu - \lambda u = D_i f_i + g \tag{7}$$

in the sense of distributions on Ω (cf. Section 4.4). Furthermore, for this solution

$$\|u_x\|_{\mathcal{L}_p(\Omega)} + \lambda^{1/2}\|u\|_{\mathcal{L}_p(\Omega)} \le N_0\Big(\sum_i \|f_i\|_{\mathcal{L}_p(\Omega)} + \lambda^{-1/2}\|g\|_{\mathcal{L}_p(\Omega)}\Big). \tag{8}$$

where N_0 depends only on d, p, and κ.

Proof. As in the proof of Theorem 7 we may concentrate on $L = \Delta$. In that case extend (or redefine) f_i, g for negative x^1 by using the formulas

$$f_1(x) = f_1(|x^1|, x'), \quad f_i(x) = f_i(|x^1|, x') \operatorname{sign} x^1, \quad i = 2, ..., d,$$

$$g(x) = g(|x^1|, x') \operatorname{sign} x^1$$

and consider (7) in $\mathbb{R}^d$. By Theorem 4.4.2 there exists a unique solution $u \in W^1_p \subset W^1_p(\Omega)$ and this solution satisfies (8). One easily checks that

$-u(x^1, x')$ also satisfies the same equation. By Lemma 1 we have $u \in \overset{0}{W}{}^1_p(\Omega)$, and it only remains to prove uniqueness.

Observe that if we have two solutions in $\overset{0}{W}{}^1_p(\Omega)$, then their difference, denoted by w, satisfies (7) with $f_i = g = 0$ and $w \in \overset{0}{W}{}^1_p(\Omega)$. Extend w for negative x^1 as an odd function and keep the notation w for the extended function. Then by Lemma 1 we have $w \in W^1_p$ and as in the proof of Theorem 4.4.2, to prove that $w = 0$, it suffices to prove that $\Delta w - \lambda w = 0$ in the sense of distributions on $\mathbb{R}^d$, that is,

$$I(\phi) := \int_{\mathbb{R}^d} (\lambda\phi - \Delta\phi) w \, dx = 0 \quad \forall \phi \in C_0^\infty.$$

The fact that w is odd with respect to x^1 implies that $I(\tilde{\phi}) = -I(\phi)$, where $\tilde{\phi}(x) = \varphi(-x^1, x')$. Hence

$$2I(\phi) = I(\hat{\phi}),$$

where $\hat{\phi}(x) = \phi(x) - \phi(-x^1, x')$.

Since $w \in W^1_p$,

$$I(\hat{\phi}) = \int_{\mathbb{R}^d} (\lambda\hat{\phi} w + \hat{\phi}_{x^i} w_{x^i}) \, dx$$

and since $\hat{\phi}$ is odd with respect to x^1,

$$2I(\phi) = I(\hat{\phi}) = 2J(\hat{\phi})$$

where

$$J(\psi) = \int_{\mathbb{R}^d_+} (\lambda\psi w + \psi_{x^i} w_{x^i}) \, dx.$$

Now we observe that $\hat{\phi}(0, x') = 0$, so that $\hat{\phi} \in \overset{0}{W}{}^1_q(\Omega)$, where $q = p/(p-1)$. By Exercise 1.6 there is a sequence of $\phi_n \in C_0^\infty(\mathbb{R}^d_+)$ such that $\phi_n \to \hat{\phi}$ in $W^1_q(\mathbb{R}^d_+)$. By definition

$$J(\phi_n) = \int_{\mathbb{R}^d_+} (\lambda\phi_n w + \phi_{nx^i} w_{x^i}) \, dx = \int_{\mathbb{R}^d_+} (\lambda\phi_n - \Delta\phi_{nx^i}) w \, dx = 0.$$

Furthermore, by Hölder's inequality $J(\phi_n) \to J(\hat{\phi})$. Thus,

$$2I(\phi) = I(\hat{\phi}) = 2J(\hat{\phi}) = 0$$

and the theorem is proved.

The following exercise shows that if a function given on $\partial\Omega$ has at least one extension into Ω as a $W^1_p(\Omega)$ function, then as an extension one can take a solution of $\Delta u - \lambda u = 0$.

9. Exercise. Given $g \in W^1_p(\Omega)$ and $\lambda > 0$, prove that there exists a unique $u \in W^1_p(\Omega)$ such that $\Delta u - \lambda u = 0$ and $u - g \in \overset{0}{W}{}^1_p(\Omega)$. Also show that for this solution

$$\|u_x\|_{\mathcal{L}_p(\Omega)} + \sqrt{\lambda}\|u\|_{\mathcal{L}_p(\Omega)} \le N(d,p)(\|g_x\|_{\mathcal{L}_p(\Omega)} + \sqrt{\lambda}\|g\|_{\mathcal{L}_p(\Omega)}).$$

Now follows a counterpart of Theorem 1.6.4.

10. Theorem. *There exist constants $\lambda_0 \ge 1$ and $N_0 < \infty$, depending only on K, κ, p, ω, and d, such that, for any $\lambda \ge \lambda_0$ and $u \in \overset{0}{W}{}^2_p(\Omega)$ estimate* (6) *holds. Furthermore, for any $\lambda \ge \lambda_0$ and $f \in \mathcal{L}_p(\Omega)$, there exists a unique $u \in \overset{0}{W}{}^2_p(\Omega)$ satisfying $Lu - \lambda u = f$ in Ω.*

The proof of this theorem can be obtained by repeating our basic arguments from Section 1.6. Therefore, we only point out the most important steps. The method of continuity and Theorem 7 reduce our task to proving a priori estimate (6). For constant a^{ij} and $b^i = c = 0$ the result is stated as Theorem 7. After that, we treat the case of constant a^{ij} and general b, c on the basis of multiplicative inequalities from Section 1.5. Our next step is to derive (6) for functions with small support. At this moment we use the uniform continuity of a which allows us to use the method of freezing the coefficients. Then the localization method from the proof of Theorem 1.6.4 based on the same formula (1.6.4) with p in place of 2, i.e.,

$$|u_{x^ix^j}(x)|^p = \int_{\mathbb{R}^d} |u_{x^ix^j}(x)\zeta(x-y)|^p\,dy$$

(with the integral over $\mathbb{R}^d$!), yields (6). This finishes our comments on the proof of Theorem 10.

11. Exercise. If $p = d = 2$, no continuity assumptions on the coefficients are needed. Prove that in Exercise 1.6.7 one can replace W^2_2 and $\mathcal{L}_2$ with $\overset{0}{W}{}^2_2(\mathbb{R}^2_+)$ and $\mathcal{L}_2(\mathbb{R}^2_+)$, respectively.

3. Domains of class C^k. Equations near the boundary

In what follows in this chapter we deal only with smooth domains Ω. Recall that by $B_r(z)$ we denote the open ball of radius r centered at z.

1. Definition. Let $k \in \{1, 2, ...\}$ and Ω be a *bounded* domain in $\mathbb{R}^d$. We write $\Omega \in C^k$ (or $\partial\Omega \in C^k$) and say that the domain Ω is of class C^k if there are numbers $K_0, \rho_0 > 0$ such that for any point $z \in \partial\Omega$ there exists a one-to-one mapping ψ of $B_{\rho_0}(z)$ onto a domain $D^z \subset \mathbb{R}^d$ such that

(i) $D^z_+ := \psi(B_{\rho_0}(z) \cap \Omega) \subset \mathbb{R}^d_+$ and $\psi(z) = 0$,

(ii) $\psi(B_{\rho_0}(z) \cap \partial\Omega) = D^z \cap \{y \in \mathbb{R}^d : y^1 = 0\}$,

(iii) $\psi \in C^k(\bar{B}_{\rho_0}(z))$, $\psi^{-1} \in C^k(\bar{D}^z)$ and

$$|\psi|_{C^k(B_{\rho_0}(z))} + |\psi^{-1}|_{C^k(D^z)} \le K_0.$$

We say that the diffeomorphism ψ *straightens or flattens the boundary* near z.

2. Exercise. Show that in $\mathbb{R}^3 = \{(x, y, z)\}$ the domain $x^2 + y^2 + z^2 < 1$ is of class C^k for any k.

Below we fix a domain $\Omega \in C^k$ with $k \ge 2$. We also fix a function $\zeta \in C_0^\infty$ such that $\zeta(x) = 0$ for $|x| \ge \rho_0/2$ and $0 \le \zeta \le 1$ everywhere. Of course ρ_0 comes from Definition 1. Finally, we fix a $z \in \partial\Omega$ and take the objects associated with z from Definition 1.

For functions v and $\hat{v}$ defined in $B_{\rho_0}(z) \cap \Omega$ and D^z_+, respectively, we write

$$v(x) = \hat{v}(y) \tag{1}$$

if this equality holds for

$$y = \psi(x). \tag{2}$$

This equality will also be used to introduce a function $\hat{v}$ on D^z_+ if we are given a function v on $B_{\rho_0}(z) \cap \Omega$, and vice versa. Generally, if in a formula in this section we have both x and y, we will alway assume that they are related by (2).

Set

$$\zeta^z(x) = \zeta(x - z), \quad \hat{\zeta}^z(y) = \zeta^z(x).$$

Let $v \in C^k(B_{\rho_0}(z) \cap \Omega)$. In the domain D^z_+ define the function $\hat{v}(y)$ by (1). Obviously, in $B_{\rho_0}(z) \cap \Omega$

$$v_{x^i}(x) = \hat{v}_{y^r}(y)\psi^r_{x^i}(x), \quad v_{x^i x^j}(x) = \hat{v}_{y^r}(y)\psi^r_{x^i x^j}(x) + \hat{v}_{y^r y^l}(y)\psi^r_{x^i}(x)\psi^l_{x^j}(x).$$

3. Exercise*. By passing to $\mathcal{L}_p$ limits in the above formulas, prove that they are also true if $v \in W^2_p(B_{\rho_0}(z) \cap \Omega)$.

This exercise leads to the following result.

4. Lemma. *Let $z \in \partial\Omega$ and let $v, \hat{v}$ be functions on $B_{\rho_0}(z) \cap \Omega$ and D^z_+, respectively, related by* (2). *Then*

$$\zeta^z v \in \overset{0}{W}{}^k_p(B_{\rho_0}(z) \cap \Omega) \iff \hat{\zeta}^z \hat{v} \in \overset{0}{W}{}^k_p(D^z_+).$$

In addition, for $r = 0, 1, ..., k$

$$\|\zeta^z v\|_{W^r_p(B_{\rho_0}(z)\cap\Omega)} \le N\|\hat{\zeta}^z \hat{v}\|_{W^r_p(D^z_+)} \le N\|\zeta^z v\|_{W^r_p(B_{\rho_0}(z)\cap\Omega)},$$

where $N = N(d, p, k, K_0, \zeta)$. Furthermore, the assertion is also true if we replace $\overset{0}{W}{}^k_p(...)$ with $W^k_p(...)$.

5. Exercise*. Prove Lemma 4.

A rough idea of how to solve the equation $Lu = f$ in Ω is to solve it locally in a neighborhood of each boundary point by straightening the relevant piece of the boundary. In connection with this we need to understand how the mapping ψ affects the equation.

Let $v \in C^2(B_{\rho_0}(z) \cap \Omega)$. Assume that in $B_{\rho_0}(z) \cap \Omega$

$$Lv(x) := a^{ij}(x)v_{x^ix^j}(x) + b^i(x)v_{x^i}(x) + c(x)v(x) = f(x). \tag{3}$$

From the above computations we see that (recall that $y = \psi(x), \hat{v}(y) = v(x)$ and notice $\tilde{L}$ and not $\hat{L}$)

$$\tilde{L}^z\hat{v}(y) := \tilde{a}^{rl}(y)\hat{v}_{y^ry^l}(y) + \tilde{b}^r(y)\hat{v}_{y^r}(y) + \tilde{c}(y)\hat{v}(y) = \hat{f}(y), \tag{4}$$

where

$$\tilde{a}^{rl}(y) = a^{ij}(x)\psi^r_{x^i}(x)\psi^l_{x^j}(x), \quad \tilde{b}^r(y) = a^{ij}(x)\psi^r_{x^ix^j}(x) + b^i(x)\psi^r_{x^i}(x),$$

$$\tilde{c}(y) = c(x), \quad \hat{f}(y) = f(x).$$

6. Lemma. *(i) For any ψ and D^z from Definition* 1 *and $y_1, y_2 \in D^z$ we have*

$$|\tilde{a}^{rl}(y)| + |\tilde{b}^r(y)| + |\tilde{c}(y)| \le N, \quad |\tilde{a}^{rl}(y_1) - \tilde{a}^{rl}(y_2)| \le \tilde{\omega}(|y_1 - y_2|),$$

where $N = N(d, K_0, K)$ and the function $\tilde{\omega}(t)$ is expressed only in terms of ω, d, K_0, and K and tends to zero as $t \downarrow 0$.

(ii) A function $v \in W^k_p(B_{\rho_0}(z) \cap \Omega)$ satisfies (3) *in $B_{\rho_0}(z) \cap \Omega$ if and only if we have $\hat{v} \in W^k_p(D^z_+)$ and $\hat{v}$ satisfies* (4) *in D^z_+.*

(iii) The operator $\tilde{L}$ is elliptic in D^z, that is, with a $\tilde{\kappa} = \tilde{\kappa}(d, K_0, \kappa) > 0$

$$\tilde{a}^{ij}(y)\theta^i\theta^j \ge \tilde{\kappa}|\theta|^2, \quad \forall\theta \in \mathbb{R}^d, y \in D^z.$$

Proof. The first assertion is obvious. The second one has been checked out in one direction for smooth functions. For $W^k_p(B_{\rho_0}(z)\cap\Omega)$ functions one gets the result upon considering smooth functions converging to v in $W^k_p(B_{\rho_0}(z)\cap\Omega)$.

To prove the remaining part of (ii), one could try to change variables back in (4). However, it suffices to recall (1), *define* $g = Lv$, notice that by what has been already proved, $\tilde{L}^z\hat{v} = \hat{g}$, where $\hat{g}(y) = g(x)$, and finally recall that $\tilde{L}^z\hat{v} = \hat{f}$, so that $\hat{f} = \hat{g}$ and $g = f$.

To prove the last assertion of the lemma, we notice that

$$\tilde{a}^{rl}(y)\theta^r\theta^l = a^{ij}(x)(\theta\cdot\psi)_{x^i}(x)(\theta\cdot\psi)_{x^j}(x) \geq \kappa|(\theta\cdot\psi)_x(x)|^2,$$

and for $\phi := \psi^{-1}$ we have

$$\psi^r_{x^i}(x)\phi^i_{y^j}(y) = \delta^r_j, \quad (\theta\cdot\psi)_{x^i}(x)\phi^i_{y^j}(y) = \theta^j,$$

so that $|\theta| \leq NK_0|(\theta\cdot\psi)_x|$. The lemma is proved.

Next, we fix a function $\eta \in C^\infty_0$, such that $\eta(x) = 1$ for $|x| \leq \rho_0/2$ and $\eta(x) = 0$ for $|x| \geq 3\rho_0/4$ and $0 \leq \eta \leq 1$ everywhere, and we define

$$\eta^z(x) = \eta(x - z), \quad \hat{\eta}^z(y) = \eta^z(x),$$

$$\hat{L}^z(y) = \hat{\eta}^z(y)\tilde{L}^z(y) + (1 - \hat{\eta}^z(y))\Delta.$$

Of course, formally speaking, $\tilde{L}^z(y)$ and $\hat{\eta}^z(y)\tilde{L}^z(y)$ are not defined if $y \notin \bar{D}^z$. However $\hat{\eta}^z(y) = 0$ for those y and we define the product to be zero. The operator $\hat{L}^z$ is elliptic together with $\tilde{L}$ and since the support of ζ belongs to the set where $\eta = 1$, we have

$$\hat{L}^z(\hat{\zeta}^z\cdot) = \tilde{L}^z(\hat{\zeta}^z\cdot). \tag{5}$$

One of the advantages of considering $\hat{L}^z$ is that this operator is defined for all $y \in \mathbb{R}^d$. This and Theorem 2.10 allow us to find a $\lambda_0 \geq 1$ depending only on $K_0, K, \kappa, \rho_0, p, d$, and ω such that for $\lambda \geq \lambda_0$ (and any $z \in \partial\Omega$) the operator

$$\lambda - \hat{L}^z : \overset{0}{W}{}^2_p(\mathbb{R}^d_+) \to \mathcal{L}_p(\mathbb{R}^d_+)$$

is invertible. We denote $\hat{\mathcal{R}}^z_\lambda$ its inverse and define

$$\Psi : w = w(y) \to \Psi w(x) = w(\psi(x)),$$

$$\Psi^{-1} : v = v(x) \to \Psi^{-1}v(y) = v(\psi^{-1}(y)),$$

$$R_\lambda^z : f = f(x) \to R_\lambda^z f(x) = \Psi \hat{\mathcal{R}}_\lambda^z \Psi^{-1}[\eta^z f](x),$$

where, according to the facts that η^z vanishes outside of $B_{\rho_0}(z)$ and the values of f outside of Ω are irrelevant, by $\Psi^{-1}[\eta^z f](y)$ we certainly mean the function

$$\eta^z(\psi^{-1}(y))f(\psi^{-1}(y)) \quad \text{for} \quad y \in D_+^z$$

and zero outside D_+^z. In terms of Ψ and Ψ^{-1} we have

$$\hat{v} = \Psi^{-1}v, \quad v = \Psi\hat{v}$$

and formulas (4) and (5) imply that

$$\hat{L}^z\Psi^{-1}(\zeta^z v) = \tilde{L}^z\Psi^{-1}(\zeta^z v) = \Psi^{-1}L(\zeta^z v),$$

$$\Psi\hat{L}^z\Psi^{-1}(\zeta^z v) = L(\zeta^z v), \quad \hat{L}^z(\hat{\zeta}^z\hat{v}) = \Psi^{-1}L\Psi(\hat{\zeta}^z\hat{v}). \tag{6}$$

7. Theorem. *(i) If $\zeta^z v \in \overset{0}{W}{}_p^2(\Omega)$, then in $B_{\rho_0}(z) \cap \Omega$ for any $\lambda \geq \lambda_0$ we have*

$$\zeta^z v = R_\lambda^z(\lambda - L)(\zeta^z v).$$

(ii) There is a constant N depending only on $K_0, K, \kappa, \rho_0, p, d, \omega$, and $\|\zeta\|_{C^2}$ such that for $\lambda \geq \lambda_0$ and $f \in \mathcal{L}_p(\Omega)$ we have

$$\zeta^z R_\lambda^z f \in \overset{0}{W}{}_p^2(B_{\rho_0}(z) \cap \Omega),$$

$$\lambda\|\zeta^z R_\lambda^z f\|_{\mathcal{L}_p(\Omega)} + \lambda^{1/2}\|\zeta^z R_\lambda^z f\|_{W_p^1(\Omega)} + \|\zeta^z R_\lambda^z f\|_{W_p^2(\Omega)} \leq N\|f\|_{\mathcal{L}_p(B_{\rho_0}(z)\cap\Omega)}. \tag{7}$$

Proof. (i) Define $w = \zeta^z v$, $f = \lambda w - Lw$. Then by Lemmas 4 and 6 we have $\hat{w} \in \overset{0}{W}{}_p^2(D_+^z)$ and by (6)

$$\hat{f} = \lambda\hat{w} - \hat{L}^z\hat{w}$$

in D_+^z. Actually this equality holds in $\mathbb{R}_+^d$ if we continue $\hat{f}, \hat{w}$ as zero outside D_+^z. Also $\eta^z f = f$, so that

$$\hat{f} = \Psi^{-1}f = \Psi^{-1}[\eta^z f].$$

For $y \in D_+^z$ and $x = \psi^{-1}(y) \in B_{\rho_0}(z) \cap \Omega$ this yields

$$\hat{w}(y) = \hat{\mathcal{R}}_\lambda^z \hat{f}(y) = \hat{\mathcal{R}}_\lambda^z \Psi^{-1}[\eta^z f](y),$$

$$w(x) = \Psi \hat{w}(x) = \Psi \hat{\mathcal{R}}_\lambda^z \Psi^{-1}[\eta^z f](x) = R_\lambda^z f(x).$$

(ii) The inclusion

$$\zeta^z R_\lambda^z f \in \overset{0}{W}{}_p^2(B_{\rho_0}(z) \cap \Omega)$$

is known from Lemma 4, which also implies that the left-hand side of (7) is less than

$$I := N(\lambda \|\hat{\mathcal{R}}_\lambda^z \Psi^{-1}[\eta^z f]\|_{\mathcal{L}_p(\mathbb{R}_+^d)} + \lambda^{1/2} \|\hat{\mathcal{R}}_\lambda^z \Psi^{-1}[\eta^z f]\|_{W_p^1(\mathbb{R}_+^d)}$$

$$+ \|\hat{\mathcal{R}}_\lambda^z \Psi^{-1}[\eta^z f]\|_{W_p^2(\mathbb{R}_+^d)})$$

$$\leq N(\lambda \|\hat{\mathcal{R}}_\lambda^z \Psi^{-1}[\eta^z f]\|_{\mathcal{L}_p(\mathbb{R}_+^d)}$$

$$+ \lambda^{1/2} \|D\hat{\mathcal{R}}_\lambda^z \Psi^{-1}[\eta^z f]\|_{\mathcal{L}_p(\mathbb{R}_+^d)} + \|D^2\hat{\mathcal{R}}_\lambda^z \Psi^{-1}[\eta^z f]\|_{\mathcal{L}_p(\mathbb{R}_+^d)}),$$

where the inequality is true since $\lambda \geq \lambda_0 \geq 1$. In light of Theorem 2.10 and Lemma 4 we have

$$I \leq N \|\Psi^{-1}[\eta^z f]\|_{\mathcal{L}_p(\mathbb{R}_+^d)} \leq N \|\eta^z f\|_{\mathcal{L}_p(B_{\rho_0}(z) \cap \Omega)} \leq N \|f\|_{\mathcal{L}_p(B_{\rho_0}(z) \cap \Omega)}.$$

The theorem is proved.

8. Remark. Assertion (i) of Theorem 7 says that if we want to solve the equation $\lambda u - Lu = f$ in $\overset{0}{W}{}_p^2(\Omega)$ and we know in advance that the solution vanishes outside $B_{\rho_0/2}(z) \cap \Omega$, then the solution in $B_{\rho_0/2}(z) \cap \Omega$ is given by $R_\lambda^z f$.

9. Exercise*. If $f \in \mathcal{L}_p(B_{\rho_0}(z) \cap \Omega)$, then as we have seen

$$\zeta^z R_\lambda^z f \in \overset{0}{W}{}_p^2(B_{\rho_0}(z) \cap \Omega).$$

Prove that $(\lambda - L)R_\lambda^z f = f$ in $B_{\rho_0/2}(z) \cap \Omega$.

4. Partitions of unity and the regularizer

Take a domain $\Omega \in C^2$. In the previous section we saw that some results concerning elliptic operators near $\partial\Omega$ can be obtained by straightening the boundary. We need a procedure allowing us to get certain results for entire Ω from similar results obtained for half spaces and the whole space.

Take a function $\xi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \le \xi \le 1$,

$$\xi(x) = 0 \quad \text{for} \quad |x| \ge \rho_0/2, \quad \xi(x) = 1 \quad \text{for} \quad |x| \le \rho_0/4.$$

Next, take points $z_1, z_2, \ldots \in \partial\Omega$ so that

$$|z_i - z_j| \ge \rho_0/8$$

for $i \neq j$ and the whole of $\partial\Omega$ is covered by $B_{\rho_0/8}(z_i)$. The number n of the points z_i needed to carry out this construction is finite. Observe that n can be estimated through ρ_0, d and $\operatorname{diam}\Omega$.

Define

$$\xi^i(x) = \xi(x - z_i).$$

To complete the system of these functions, one can find a function $\xi^0 \in C_0^\infty(\mathbb{R}^d)$ such that $0 \le \xi^0 \le 1$,

$$\xi^0(x) = 0 \quad \text{for} \quad x \in \Omega \quad \text{with} \quad \operatorname{dist}(x, \partial\Omega) \le \rho_0/16,$$

$$\xi^0(x) = 1 \quad \text{if} \quad x \in \Omega \quad \text{and} \quad \operatorname{dist}(x, \partial\Omega) \ge \rho_0/8.$$

One can manage to do this, for instance, by mollifying the indicator of

$$\Omega \setminus \{x : \operatorname{dist}(x, \partial\Omega) \le 3\rho_0/32\}.$$

Notice that

$$\sum_{i \ge 1} (\xi^i(x))^2 \ge 1 \quad \text{if} \quad x \in \bar{\Omega} \quad \text{and} \quad \operatorname{dist}(x, \partial\Omega) \le \rho_0/8.$$

Therefore, the function

$$\bar{\xi} = \sum_{i \le n} (\xi^i)^2$$

is greater than 1 in $\bar{\Omega}$. Also, $\bar{\xi}$ and its every derivative of any order are bounded in $\bar{\Omega}$ by a number depending only on d, n, ρ_0, and the order of the derivative. Finally, define

$$\zeta^i = \xi^i \bar{\xi}^{-1/2}, \quad i \ge 0.$$

and notice that, in Ω, all ζ^i are infinitely differentiable and

$$\sum_{i\le n}(\zeta^i)^2 = 1.$$

We have constructed a so-called partition of unity in Ω.

In Section 3 we constructed the operators R_λ^z for $\lambda \ge \lambda_0 \ge 1$ and $z \in \partial\Omega$. Let

$$R_\lambda^{(i)} = R_\lambda^{z_i}, \quad i = 1, 2, \dots, n.$$

We increase λ_0 if needed so as to be able to apply Theorem 0.1 for $\lambda \ge \lambda_0$ and we let $R_\lambda^{(0)}$ be the inverse operator to $\lambda - L : W_p^2 \to \mathcal{L}_p$ which according to Theorem 0.1 satisfies

$$\lambda\|R_\lambda^{(0)}f\|_{\mathcal{L}_p} + \lambda^{1/2}\|R_\lambda^{(0)}f\|_{W_p^1} + \|R_\lambda^{(0)}f\|_{W_p^2} \le N\|f\|_{\mathcal{L}_p}. \tag{1}$$

A naive idea concerning how to solve $\lambda u - Lu = f$ in Ω is to define

$$u = \begin{cases} R_\lambda^{(0)}f & \text{in} \quad \Omega \setminus \bigcup_i B_{\rho_0/2}(z_i), \\ R_\lambda^{(i)}f & \text{in} \quad B_{\rho_0/2}(z_i) \setminus \bigcup_{j\le i-1} B_{\rho_0/2}(z_j). \end{cases}$$

However, this function u is just discontinuous, although by Exercise 3.9 it satisfies the equation in Ω apart from the boundaries of the pieces where it is defined by different formulas.

Therefore, we slightly modify this idea following F. Browder, who suggested a universal method which works in many situations and in function spaces in domains or on smooth manifolds.

1. Lemma. *Let $\lambda \ge \lambda_0$, $u \in \overset{0}{W}{}_p^2(\Omega)$ and $\lambda u - Lu = f$. Then in Ω*

$$u = \sum_{i\le n} \zeta^i R_\lambda^{(i)}(\zeta^i f - L^i u), \tag{2}$$

where

$$L^k u := u(a^{ij}\zeta^k_{x^i x^j} + b^i\zeta^k_{x^i}) + 2a^{ij}\zeta^k_{x^i}u_{x^j} \quad (= L(\zeta^k u) - \zeta^k Lu).$$

Proof. It should be said that in (2) we have the terms $R_\lambda^{(i)}(\dots)$ which, formally, are *not* defined everywhere in Ω. But we multiply $R_\lambda^{(i)}(\dots)$ by the function ζ^i which vanishes outside the set where $R_\lambda^{(i)}(\dots)$ is defined, and we define their product to be zero there.

Next, clearly $\zeta^0 u \in W_p^2$ and by the definition of $R_\lambda^{(0)}$ we have

$$\zeta^0 u = R_\lambda^{(0)}(\lambda(\zeta^0 u) - L(\zeta^0 u)).$$

Hence by Theorem 3.7 (i)

$$u = \sum_{i \leq n} \zeta^i(\zeta^i u) = \sum_{i \leq n} \zeta^i R_\lambda^{(i)}(\lambda(\zeta^i u) - L(\zeta^i u)) = \sum_{i \leq n} \zeta^i R_\lambda^{(i)}(\zeta^i f - L^i u).$$

The lemma is proved.

2. Definition. By a *regularizer* of the operator $\lambda - L$ we mean the operator

$$\boldsymbol{R}_\lambda f = \sum_{i \leq n} \zeta^i R_\lambda^{(i)}(\zeta^i f).$$

From (2) we see that the regularizer almost gives us the inverse to $\lambda - L$. The only operator which interferes is

$$u \to \sum_{i \leq n} \zeta^i R_\lambda^{(i)} L^i u,$$

but as we will see, this operator is "weaker" than the regularizer.

Partitions of unity allow one to carry over many results valid for half spaces to smooth domains. Here we show this by proving an interpolation inequality similar to the one from Corollary 1.5.8.

3. Theorem. *Let $k \in \{1, 2, \ldots\}$, $r \in \{0, 1, 2, \ldots, k\}$, $\Omega \in C^k$. Then for any $\varepsilon > 0$ and $u \in W_p^k(\Omega)$ we have*

$$\|u\|_{W_p^r(\Omega)} \leq N\|u\|_{\mathcal{L}_p(\Omega)}^{1-\gamma}\|u\|_{W_p^k(\Omega)}^{\gamma}$$

$$\leq N(\varepsilon\|u\|_{W_p^k(\Omega)} + \varepsilon^{-\gamma/(1-\gamma)}\|u\|_{\mathcal{L}_p(\Omega)}), \tag{3}$$

where $\gamma = r/k$ and N depends only on $d, p, \rho_0, K_0, k,$ and $\operatorname{diam}\Omega$.

Proof. By using Lemma 3.4 and Corollary 1.5.8 and denoting $\theta^i = (\zeta^i)^2$ so that $\sum_i \theta^i = 1$ on Ω, we get that for $i \geq 1$

$$\|\theta^i u\|_{W_p^r(\Omega)} \leq N\|\hat{\theta^i}\hat{u}\|_{W_p^r(\mathbb{R}_+^d)}$$

$$\leq N\varepsilon\|\hat{\theta^i}\hat{u}\|_{W_p^k(\mathbb{R}_+^d)} + N\varepsilon^{-\gamma/(1-\gamma)}\|\hat{\theta^i}\hat{u}\|_{\mathcal{L}_p(\mathbb{R}_+^d)}$$

$$\leq N\varepsilon\|\theta^i u\|_{W_p^k(\Omega)} + N\varepsilon^{-\gamma/(1-\gamma)}\|\theta^i u\|_{\mathcal{L}_p(\Omega)}$$

$$\le N\varepsilon\|u\|_{W^k_p(\Omega)} + N\varepsilon^{-\gamma/(1-\gamma)}\|u\|_{\mathcal{L}_p(\Omega)}.$$

Similar inequalities hold for $i = 0$. Summing them up and noticing that

$$\|u\|_{W^r_p(\Omega)} \le \sum_{i\le n} \|\theta^i u\|_{W^r_p(\Omega)}$$

lead to the inequality between the extreme terms in (3). Upon minimizing with respect to $\varepsilon > 0$, we get the first inequality in (3) if $\gamma \in [0,1)$. The case that $\gamma = 1$ is treated as in Corollary 1.5.8. The theorem is proved.

4. Exercise*. By using partitions of unity, extend Corollary 2.2: For any $\Omega \in C^2$, $k = 1, 2$, and $u \in \overset{0}{W}{}^k_p(\Omega)$, there exists a sequence $u_n \in C^2(\bar{\Omega})$ vanishing on $\partial\Omega$ and such that $u_n \to u$ in $W^k_p(\Omega)$. This exercise is further generalized in Exercise 10.2.7.

5. Exercise. By using partitions of unity, extend Theorem 1.8.5 (i) for smooth domains: For $k \in \{1, 2, ...\}$ and $\Omega \in C^k$ we have $u \in W^k_p(\Omega)$ if and only if the generalized derivatives $D^\alpha u \in \mathcal{L}_p(\Omega)$ whenever $|\alpha| \le k$.

6. Exercise. For $r = d = 1, k = 2$, and $\Omega = (-1, 1)$ show that the inequality between the extreme terms in (3) fails to hold for certain u and $\varepsilon > 0$ if we take $[u]_{W^2_p(\Omega)}$ in place of $\|u\|_{W^2_p(\Omega)}$. Compare this with Corollary 1.5.2.

5. Solvability of equations in domains for large λ

We deal with $\Omega \in C^2$, $n, z_1, ..., z_n$, λ_0, and $\boldsymbol{R}_\lambda$ introduced in Section 4 for $\lambda > \lambda_0$. Instead of solving the equation

$$\lambda u - Lu = f \tag{1}$$

in $\overset{0}{W}{}^2_p(\Omega)$, we want to solve equation (4.2). First of all we need to know that any solution of the latter equation is indeed a solution of (1).

1. Lemma. *If $\lambda \ge \lambda_0$, $f \in \mathcal{L}_p(\Omega)$, and $u \in \overset{0}{W}{}^1_p(\Omega)$ is a solution of (4.2), then $u \in \overset{0}{W}{}^2_p(\Omega)$. Furthermore, there exists a constant $\lambda_1 \ge 1$, depending only on d, p, ω, K, κ, K_0, ρ_0, and* diam Ω, *such that if, in addition, $\lambda \ge \lambda_1$, then u satisfies (1) in Ω.*

Proof. Since the L^i are first-order operators, we have $\zeta^i f - L^i u \in \mathcal{L}_p(\Omega)$ for $i > 0$ and $\zeta^0 f - L^0 u \in \mathcal{L}_p$. Hence

$$\zeta^i R^{(i)}_\lambda(\zeta^i f - L^i u) \in \overset{0}{W}{}^2_p(\Omega)$$

by Exercise 3.9 and

$$R_\lambda^{(0)}(\zeta^0 f - L^0 u) \in W_p^2, \quad \zeta^0 R_\lambda^{(0)}(\zeta^0 f - L^0 u) \in \overset{0}{W}{}_p^2(\Omega)$$

by the definition of $R_\lambda^{(0)}$. It follows that $u \in \overset{0}{W}{}_p^2(\Omega)$.

Next, denote $g = \lambda u - Lu$. Then by Lemma 4.1 equality (4.2) holds with g in place of f, and to finish the proof, we need only show that if λ is large, then $g = f$. This is the same as showing that if λ is large and $h \in \mathcal{L}_p(\Omega)$ and $\boldsymbol{R}_\lambda h = 0$ in Ω, then $h = 0$ in Ω.

By Exercise 3.9 we have

$$(\lambda - L)R_\lambda^{(i)}(\zeta^i h) = \zeta^i h$$

on the set in Ω where $\zeta^i \neq 0$. Therefore, from $\boldsymbol{R}_\lambda h = 0$ we find

$$0 = (\lambda - L)\boldsymbol{R}_\lambda h = \sum_{i=0}^n \zeta^i(\lambda - L)R_\lambda^{(i)}(\zeta^i h) - \sum_{i=0}^n L^i R_\lambda^{(i)}(\zeta^i h)$$

$$= \sum_{i=0}^n (\zeta^i)^2 h - \sum_{i=0}^n L^i R_\lambda^{(i)}(\zeta^i h) = h - T_\lambda h,$$

where

$$T_\lambda h := \sum_{i=0}^n L^i R_\lambda^{(i)}(\zeta^i h).$$

To finish the proof, it suffices to show that for λ large the operator T_λ is a contraction in $\mathcal{L}_p(\Omega)$.

The expression $L^i v$ can be written as the sum of products of bounded functions times terms either like τv or first derivatives of τv, where the τ are smooth functions vanishing on the same set as ζ^i. It follows from (4.1) and Theorem 3.7 that

$$\|T_\lambda h\|_{\mathcal{L}_p(\Omega)} \leq N\lambda^{-1/2} \sum_{i=1}^n \|\zeta^i h\|_{\mathcal{L}_p(B_{\rho_0}(z_i)\cap\Omega)} + N\lambda^{-1/2}\|\zeta^0 h\|_{\mathcal{L}_p}$$

$$\leq N\lambda^{-1/2}\|h\|_{\mathcal{L}_p(\Omega)}.$$

Since the last constant N does not depend on λ, the lemma is proved.

2. Lemma. *There exists a constant $\lambda_1 \geq 1$, depending only on d, p, ω, K, κ, K_0, ρ_0, and* $\operatorname{diam}\Omega$, *such that for any $\lambda \geq \lambda_1$ and $f \in \mathcal{L}_p(\Omega)$ there exists a unique solution $u \in \overset{0}{W}{}^1_p(\Omega)$ of equation (4.2). Furthermore, this solution satisfies*

$$\lambda \|u\|_{\mathcal{L}_p(\Omega)} + \lambda^{1/2}\|u\|_{W^1_p(\Omega)} \leq N\|f\|_{\mathcal{L}_p(\Omega)},$$

where N depends only on d, p, ω, K, κ, K_0, ρ_0, and $\operatorname{diam}\Omega$.

Proof. We will use the contraction principle in $\overset{0}{W}{}^1_p(\Omega)$. Accordingly, to prove this lemma, it suffices to show that for large λ, $f \in \mathcal{L}_p(\Omega)$, $v \in \overset{0}{W}{}^1_p(\Omega)$, and

$$u = \sum_{i \leq n} \zeta^i R^{(i)}_\lambda(\zeta^i f - L^i v)$$

we have $u \in \overset{0}{W}{}^1_p(\Omega)$ and

$$\lambda \|u\|_{\mathcal{L}_p(\Omega)} + \lambda^{1/2}\|u\|_{W^1_p(\Omega)} \leq N(\|f\|_{\mathcal{L}_p(\Omega)} + \|v\|_{W^1_p(\Omega)}). \tag{2}$$

We know from the above that $u \in \overset{0}{W}{}^1_p(\Omega)$. Next, notice that owing to Theorem 3.7 and (4.1),

$$\lambda\|u\|_{\mathcal{L}_p(\Omega)} \leq N \sum_{i \leq n} \lambda \|\zeta^i R^{(i)}_\lambda(\zeta^i f - L^i v)\|_{\mathcal{L}_p(\Omega)}$$

$$\leq N \sum_{i \leq n} \|\zeta^i f - L^i v\|_{\mathcal{L}_p(\Omega)} \leq N(\|f\|_{\mathcal{L}_p(\Omega)} + \|v\|_{W^1_p(\Omega)}).$$

In the same way we get

$$\lambda^{1/2}\|u\|_{W^1_p(\Omega)} \leq N(\|f\|_{\mathcal{L}_p(\Omega)} + \|v\|_{W^1_p(\Omega)}).$$

These two estimates yield (2) and the lemma is proved.

By combining Lemmas 1 and 2 and considering $u - g$ in place of u, we get the following result about the solvability of the Dirichlet problem for elliptic equations in smooth domains. Here we understand the Dirichlet condition: $u = g$ on $\partial\Omega$ as

$$u - g \in \overset{0}{W}{}^1_p(\Omega). \tag{3}$$

This approach is borrowed from [1]. Later in Section 11.7 we will see that the relation $u = g$ on $\partial\Omega$ makes perfect sense if both u and g are in $W^1_p(\Omega)$

and in Section 13.7 we will see that sometimes one can tell what functions on the boundary can be continued inside the domain as $W_p^1(\Omega)$ functions.

3. Theorem. *Take λ_1 to be the largest of the constants called λ_1 in Lemmas 1 and 2. Then for any $\lambda \geq \lambda_1$, $f \in \mathcal{L}_p(\Omega)$ and $g \in W_p^2(\Omega)$ there exists a unique function $u \in W_p^2(\Omega)$ satisfying the equation*

$$\lambda u - Lu = f$$

in Ω and such that (3) *holds.*

4. Remark. Under the conditions of Theorem 3, actually,

$$u - g \in \overset{0}{W}{}_p^2(\Omega).$$

This is a straightforward consequence of the definition of $\overset{0}{W}{}_p^k(\Omega)$ and the fact that $u, g \in W_p^2(\Omega)$.

5. Remark. The above proof of solvability of elliptic equations in domains uses partitions of unity which makes it possible to carry out the same proof in the case of elliptic equations on manifolds with or without boundary.

We will later see in Theorem 11.3.2 that Theorem 3 holds for any λ satisfying $c - \lambda \leq 0$. To do this, we will use the maximum principle and the a priori estimate from the following theorem.

6. Theorem. *There exists a constant N, depending only on d, p, ω, K, κ, K_0, ρ_0, and* $\operatorname{diam}\Omega$, *such that for any $\lambda \geq 0$ and $u \in \overset{0}{W}{}_p^2(\Omega)$ we have*

$$\|u\|_{W_p^2(\Omega)} \leq N(\|\lambda u - Lu\|_{\mathcal{L}_p(\Omega)} + \|u\|_{\mathcal{L}_p(\Omega)}). \tag{4}$$

In addition if $\lambda \geq \lambda_1$, where λ_1 is taken from Theorem 3, *then*

$$\lambda\|u\|_{\mathcal{L}_p(\Omega)} \leq N\|\lambda u - Lu\|_{\mathcal{L}_p(\Omega)}. \tag{5}$$

Proof. The second assertion follows immediately from Lemmas 1 and 2. To prove the first one, notice that, for $f := \lambda_1 u - Lu$, from Theorem 3.7 (ii) and (4.1), we get that

$$\|u\|_{W_p^2(\Omega)} \leq \sum_{i \leq n} \|\zeta^i R_{\lambda_1}^{(i)}(\zeta^i f - L^i u)\|_{W_p^2(\Omega)}$$

$$\leq N \sum_{i \leq n} \|\zeta^i f - L^i u\|_{\mathcal{L}_p(\Omega)} \leq N(\|f\|_{\mathcal{L}_p(\Omega)} + \|u\|_{W_p^1(\Omega)}).$$

By Lemma 2 the last expression is less than

$$N\|f\|_{\mathcal{L}_p(\Omega)} = N\|\lambda_1 u - Lu\|_{\mathcal{L}_p(\Omega)}$$

$$\leq N(\|\lambda u - Lu\|_{\mathcal{L}_p(\Omega)} + |\lambda_1 - \lambda|\,\|u\|_{\mathcal{L}_p(\Omega)}). \tag{6}$$

This yields (4) if $0 \leq \lambda \leq \lambda_1$. However if $\lambda > \lambda_1$, then $|\lambda_1 - \lambda| \leq \lambda$ and the last term in (6) can be dropped due to (5). The theorem is proved.

The a priori estimate (4) with $\lambda = 0$ is one of the most important results of the theory. Notice that to derive it, we actually do not need to know too much about model equations. Anyway, the detailed estimates involving λ are not needed, although they are extensively used for proving existence theorems. In connection with this we give the reader the following exercise.

7. Exercise. The arguments in this chapter are valid as long as Theorem 0.1 is true for a $p \in (1, \infty)$ and any $\lambda \geq \lambda_0$. Instead assume that the estimate

$$\|u_{xx}\|_{\mathcal{L}_p} \leq N\|\Delta u\|_{\mathcal{L}_p}$$

holds for any $u \in C_0^2$ with N independent of u. Prove that estimate (4) with $\lambda = 0$ holds for (i) $\Omega = \mathbb{R}^d$, (ii) $\Omega = \mathbb{R}^d_+$, (iii) $\Omega \in C^2$.

8. Exercise*. Assume that $\Omega \in C^2$ and, for a particular L (satisfying Assumption 1.6.1), there exists a constant N such that for any $\lambda \geq 0$ and $u \in \overset{0}{W}{}^2_p(\Omega)$ we have

$$\|u\|_{\mathcal{L}_p(\Omega)} \leq N\|\lambda u - Lu\|_{\mathcal{L}_p(\Omega)}$$

with N independent of u and λ. Prove that the assertion of Theorem 3 holds true for any $\lambda \geq 0$.

In the following exercise the reader will show that the solutions of equations $Lu = f$ depend continuously on L and f.

9. Exercise*. Let $\Omega \in C^2$ and L_n, $n = 1, 2, \ldots$, be a sequence of second-order elliptic operators satisfying Assumption 1.6.1 with the same κ, K, and with the same function ω. Let a_n, b_n, c_n be the coefficients of L_n. Assume that

$$a_n, b_n, c_n \to a, b, c$$

as $n \to \infty$ almost everywhere in Ω. Take λ_1 from Theorem 3 and take a $\lambda \geq \lambda_1$. Finally let $f_n, f \in \mathcal{L}_p(\Omega)$, $f_n \to f$ in $\mathcal{L}_p(\Omega)$ and let $u_n, u \in \overset{0}{W}{}^2_p(\Omega)$ satisfy

$$\lambda u_n - L_n u_n = f_n, \quad \lambda u - Lu = f.$$

Prove that $u_n \to u$ in $W^2_p(\Omega)$.

6. Hints to exercises

1.4. The condition just means that u is symmetric relative to x^1. Use mollifiers with symmetric kernels and observe that mollified symmetric about the x^1-axis functions are also symmetric.

1.5. To prove the "if" part, take $\zeta(x) = 0$ if $x^1 \le 0$ in the proof of Theorem 1.8.5. To prove the "only if" part, you may like to use the generalization of Corollary 1.3.5 for continuous and piecewise continuously differentiable functions.

1.8. Dilations show that it suffices to concentrate on $\varepsilon = 1$. Then observe that for smooth u vanishing on $\partial\mathbb{R}^d_+$ and $q = p/(p-1)$

$$u(x^1, x') = \int_0^{x^1} u_{x^1}(y^1, x')\, dy^1, \quad |u(x^1, x')| \le (x^1)^{1/q} \left(\int_0^1 |u_{x^1}(y^1, x')|^p \, dy^1 \right)^{1/p}.$$

1.9. Use Minkowski's inequality after observing that as in the hint to Exercise 1.8 for smooth u

$$v(x) = \int_0^1 u_{x^1}(\theta x^1, x')\, d\theta.$$

Then use Fatou's lemma.

2.4. Part of (2.5) for $i \ge 2$ is known from Exercise 1.3.21. To prove the remaining part, first assume that u is smooth and use Definition 1.3.4 and integration by parts.

2.5. By Lemma 2.1 with $k = 1$ and (2.5) we have $v_{x^1x^1} \in \mathcal{L}_p$. The remaining second derivatives of v are in $\mathcal{L}_p$ by Exercise 1.3.21.

2.6. Use Exercises 2.5 and 1.4. Observe that the method you use does not work if we only know the solvability for equations with *continuous* coefficients in the whole space.

2.9. The function $w = u - g$ satisfies $\Delta w - \lambda w = \lambda g - D_i g_{x^i}$.

5.8. Use the method of continuity with respect to λ and Theorem 5.6.

5.9. Use the formula $(\lambda - L_n)(u - u_n) = (L - L_n)u + f - f_n$.

Chapter 9

Second-order elliptic equations in $W_p^k(\Omega)$

The purpose of this chapter is to investigate to what extent better regularity of data results in better regularity of solutions. As in Chapter 8 we fix a $p \in (1, \infty)$ and a second-order elliptic differential operator

$$L = a^{ij} D_{ij} + b^i D_i + c$$

(see Definition 1.4.1) with constant of ellipticity $\kappa > 0$ (independent of x) and satisfying Assumption 1.6.1 with a constant K and a modulus of continuity $\omega(\varepsilon)$. As in Chapter 8 everything is based on Theorem 8.0.1 and therefore the reader interested only in the case $p = 2$ can study this chapter right after going through Chapter 8.

1. Finite differences. Better regularity of solutions in $\mathbb{R}^d_+$ for model equations

In half spaces the result of Theorem 1.7.5 is also true but we need a different technique for proving it.

The following result shows that the generalized derivatives can be defined as derivatives in the $\mathcal{L}_p$ sense. For $j = 1, 2, \dots, d$ and $h \neq 0$ define

$$\delta_{jh} f(x) = (f(x + e_j h) - f(x))/h.$$

where e_j is the jth basis vector in $\mathbb{R}^d$. Recall that

$$[u]_{W_p^k(\Omega)} = \sum_{|\alpha|=k} \|D^\alpha u\|_{\mathcal{L}_p(\Omega)}.$$

1. Theorem. *Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d_+$ and $k \geq 1$.*

(i) If $f \in W_p^k(\Omega)$, then for any $h > 0$ and $j = 1, 2, \dots, d$

$$[\delta_{jh} f]_{W_p^{k-1}(\Omega)} \leq [f]_{W_p^k(\Omega)} \tag{1}$$

and

$$\delta_{jh} f \to D_j f \quad \text{in} \quad W_p^{k-1}(\Omega) \quad \text{as} \quad h \downarrow 0.$$

(ii) If f is locally summable in Ω and for a $j \in \{1, 2, \dots, d\}$ we have

$$\sup_{h \in (0,1)} \|\delta_{jh} f\|_{W_p^{k-1}(\Omega)} \leq N_0, \tag{2}$$

then $D_j f \in W_p^{k-1}(\Omega)$ and

$$\|D_j f\|_{W_p^{k-1}(\Omega)} \leq N_0. \tag{3}$$

In particular, if $f \in \mathcal{L}_p(\Omega)$ and (2) holds for any $j \in \{1, 2, \dots, d\}$, then by Theorem 1.8.5 we have $f \in W_p^k(\Omega)$.

Proof. (i) For any $\phi \in C_0^k(\bar{\Omega})$ and $h > 0$,

$$\phi(x + e_j h) - \phi(x) = h \int_0^1 \phi_{x^j}(x + t e_j h)\, dt. \tag{4}$$

Hence by Minkowski's inequality (the norm of a "sum" is less than the sum of norms) and the fact that for any f

$$\|f(\cdot + e_j h)\|^p_{\mathcal{L}_p(\Omega)} = \int_\Omega |f(x + e_j h)|^p\, dx \leq \int_\Omega |f(x)|^p\, dx,$$

we obtain

$$[\delta_{jh}\phi]_{W_p^{k-1}(\Omega)} \leq \int_0^1 [\phi_{x^j}(\cdot + t e_j h)]_{W_p^{k-1}(\Omega)}\, dt$$

$$\leq [\phi_{x^j}]_{W_p^{k-1}(\Omega)} \leq [\phi]_{W_p^k(\Omega)}.$$

Once the inequality between the extreme terms is established for a dense subset of $W_p^k(\Omega)$, inequality (1) is proved.

On account of (1) and the dominated convergence theorem, we find that, for any $\phi \in C_0^k(\bar{\Omega})$,

$$\overline{\lim_{h\downarrow 0}} \|f_{x^j} - \delta_{jh} f\|_{W_p^{k-1}(\Omega)} \leq 2\|f - \phi\|_{W_p^k(\Omega)}$$

$$+ \overline{\lim_{h\downarrow 0}} \|\phi_{x^j} - \delta_{jh}\phi\|_{W_p^{k-1}(\Omega)} = 2\|f - \phi\|_{W_p^k(\Omega)}.$$

The last term can be made arbitrarily small and this finishes proving assertion (i).

(ii) Since $\delta_{jh} D^\alpha f$ is bounded in $\mathcal{L}_p(\Omega)$ for $|\alpha| \leq k-1$, there is a sequence $h(m) \downarrow 0$ as $m \to \infty$ such that $\delta_{jh(m)} D^\alpha f$ converges weakly in $\mathcal{L}_p(\Omega)$ to a function $g^{j\alpha}$. Upon observing that for any $\phi \in C_0^\infty(\Omega)$ and all large m we have $\phi(\cdot - h(m)e_j) \in C_0^\infty(\Omega)$ and

$$\int_\Omega \phi D^\alpha \delta_{jh(m)} f\, dx = (-1)^{|\alpha|} \int_\Omega (D^\alpha \phi)\delta_{jh(m)} f\, dx$$

$$= (-1)^{|\alpha|} \int_{\mathbb{R}^d} (D^\alpha \phi)\delta_{jh(m)} f\, dx$$

$$= (-1)^{|\alpha|+1} \int_{\mathbb{R}^d} f \delta_{j,-h(m)} D^\alpha \phi\, dx$$

$$= (-1)^{|\alpha|+1} \int_\Omega f \delta_{j,-h(m)} D^\alpha \phi\, dx.$$

we get that

$$\int_\Omega \phi g^{j\alpha}\, dx = (-1)^{|\alpha|+1} \int_\Omega f D_j D^\alpha \phi\, dx.$$

Hence $g^{j\alpha} = D^\alpha D_j f \in \mathcal{L}_p(\Omega)$ and by Theorem 1.8.5 we have $D_j f \in W_p^{k-1}(\Omega)$. To get (3), it only remains to use the fact that norms of weak limits are less than the liminfs of norms. The theorem is proved.

2. Corollary. *If $\Omega = \mathbb{R}^d_+$, $k \in \{2, 3, ...\}$, and $u \in \overset{0}{W}{}_p^k(\Omega)$, then $D_j u \in \overset{0}{W}{}_p^{k-1}(\Omega)$ for $j = 2, ..., d$.*

Indeed, obviously $\delta_{jh} u \in \overset{0}{W}{}_p^{k-1}(\Omega)$ and $\delta_{jh} u \to D_j u$ in $W_p^{k-1}(\Omega)$ by Theorem 1 (i). It only remains to recall that $\overset{0}{W}{}_p^{k-1}(\Omega)$ is a Banach space.

3. Corollary. *In notation $u^{(\varepsilon)}$ introduced in* (1.8.4) *we have*

$$\|u^{(\varepsilon)} - u\|_{\mathcal{L}_p} \le N\varepsilon\|u_x\|_{\mathcal{L}_p},$$

where $N = N(\zeta)$, provided that ζ integrates to one.

Indeed,

$$|u^{(\varepsilon)}(x) - u(x)| \le \int_{\mathbf{R}^d} |u(x - \varepsilon y) - u(x)|\,|\zeta(y)|\,dy,$$

so that the assertion follows from Minkowski's inequality and Theorem 1 (i) stated as

$$\|u(\cdot - z) - u\|_{\mathcal{L}_p} \le |z|\,\|u_x\|_{\mathcal{L}_p}.$$

4. Remark. By passing to the limit, one proves that (4) holds true for almost any $x \in \Omega$ if $\phi \in W^1_p(\Omega)$. Of course, if in addition both parts turn out to be continuous, then they coincide everywhere. It follows that if $u \in W^1_p(\Omega)$ is continuous and its generalized derivatives are continuous, then they are just the usual derivatives.

5. Exercise. One can improve Corollary 3. Show that if ζ is *even* and integrates to one and $u \in W^2_p$, then

$$\|u^{(\varepsilon)} - u\|_{\mathcal{L}_p} \le N\varepsilon^2\|u_{xx}\|_{\mathcal{L}_p}.$$

6. Exercise. The result of Corollary 3 is rather rough. Prove that if ζ is even and integrates to one, then

$$\|u^{(\varepsilon)} - u\|_{\mathcal{L}_p} = o(\varepsilon)$$

as $\varepsilon \downarrow 0$ whenever $u \in W^1_p$.

7. Exercise. Prove that, if $u \in \mathcal{L}_2$ and

$$\int_0^1 \|u^{(\varepsilon)} - u\|^2_{\mathcal{L}_2}\varepsilon^{-3}\,d\varepsilon \le M^2,$$

then $u \in W^1_2$ and

$$\|u_x\|_{\mathcal{L}_2} \le N(M + \|u\|_{\mathcal{L}_2}),$$

where N is independent of M and u.

8. Exercise. If we write $o(\varepsilon)$ from Exercise 6 as $\varepsilon\gamma(\varepsilon)$, then by definition $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$. Exercise 7 shows that $\varepsilon^{-1}\gamma^2(\varepsilon)$ is integrable near zero. One may think that this implies that

$$\gamma(\varepsilon) \le N\varepsilon^\delta$$

for a $\delta > 0$. Fix a $\delta > 0$ and give an example of $u \in W^1_2$ and even $\zeta \in C_0^\infty$ with $\int \zeta\, dx = 1$ showing that

$$\varepsilon^{-\delta}\gamma(\varepsilon) \to \infty \quad \text{as} \quad \varepsilon \to 0.$$

The following theorem shows that better regularity of f implies better regularity of solutions of the model equation $\lambda u - \Delta u = f$ in $\mathbb{R}^d_+$. Before that, we give the reader an exercise showing that the term with $\lambda^{k/2}$ in (5) cannot be eliminated in contrast with the case $\Omega = \mathbb{R}^d$.

9. Exercise. For $d = 1$ show that there is no finite constant N such that for all solutions $u \in \overset{0}{W}{}^3_p(\mathbb{R}_+)$ of the equation $u'' - u = f$ we have

$$[u]_{W^3_p(\mathbb{R}_+)} \le N[f]_{W^1_p(\mathbb{R}_+)}.$$

Compare this with Theorem 10.

10. Theorem. *Let $\Omega = \mathbb{R}^d_+$, $\lambda > 0$, $k \in \{0, 1, 2, ...\}$, and $f \in W^k_p(\Omega)$. Then there is a unique $u \in \overset{0}{W}{}^{k+2}_p(\Omega)$ satisfying $\lambda u - \Delta u = f$ in Ω. For this u*

$$\begin{aligned} &\lambda\|u\|_{\mathcal{L}_p(\Omega)} \le N\|f\|_{\mathcal{L}_p(\Omega)}, \\ &[u]_{W^{k+2}_p(\Omega)} \le N([f]_{W^k_p(\Omega)} + \lambda^{k/2}\|f\|_{\mathcal{L}_p(\Omega)}), \end{aligned} \tag{5}$$

where N is independent of λ and f.

Proof. Bearing in mind dilations, without loss of generality we assume $\lambda = 1$. In that case the right-hand side of the second inequality in (5) is equivalent to $\|f\|_{W^k_p(\Omega)}$ (see Corollary 1.5.7).

Now we use induction on k. For $k = 0$ we have the result from Theorem 8.2.7. Assume that the theorem holds for a given k and the assumptions are satisfied with $k + 1$ in place of k. We need only prove that the solution

$$u \in \overset{0}{W}{}^1_p(\Omega) \cap W^{k+2}_p(\Omega)$$

is actually in $W^{k+3}_p(\Omega)$ and the second estimate in (5) holds with $k + 1$ in place of k and $N\|f\|_{W^{k+1}_p(\Omega)}$ on the right.

For $h > 0$ and $j = 2, ..., d$, we have $\delta_{jh}u \in \overset{0}{W}{}^{k+2}_p(\Omega)$ and

$$(1 - \Delta)\delta_{jh}u = \delta_{jh}f.$$

Hence by the assumption

$$\sum_{j=2}^{d} \sum_{|\alpha|=k+2} \|D^\alpha \delta_{jh} u\|_{\mathcal{L}_p(\Omega)} \le N \|\delta_{jh} f\|_{W_p^k(\Omega)}.$$

It follows by Theorem 1 that $D_j D^\alpha u \in \mathcal{L}_p(\Omega)$ and

$$\sum_{j=2}^{d} \sum_{|\alpha|=k+2} \|D_j D^\alpha u\|_{\mathcal{L}_p(\Omega)} \le N \|f\|_{W_p^{k+1}(\Omega)}. \tag{6}$$

Now since we already know that $u \in W_p^{k+2}(\Omega)$, owing to Theorem 1.8.5 it only remains to prove that one can include the term with $j = 1$ into the left-hand side of (6). Actually, the only derivative of the $(k + 3)$rd order missing in (6) is $D_1^{k+3} u$. It turns out that it can be estimated just from the equation $u - \Delta u = f$ itself. Indeed notice that owing to (6) we have

$$D_j^2 u \in W_p^{k+1}(\Omega), \quad j = 2, ..., d.$$

Hence the equation implies

$$D_1^2 u = u - f - \sum_{j=2}^{d} D_j^2 u \in W_p^{k+1}(\Omega),$$

$$D_1^{k+3} u = D_1^{k+1}(u - f) - \sum_{j=2}^{d} D_1^{k+1} D_j^2 u$$

with everything on the right belonging to $\mathcal{L}_p(\Omega)$ and having the $\mathcal{L}_p(\Omega)$ norms controlled by the right-hand side of (6). The theorem is proved.

11. Exercise. Let $\Omega = \mathbb{R}_+^d$. Assume that we are given an operator

$$L = a^{ij} D_{ij} + b^i D_i + c$$

with $a, b, c \in C^k$ and $a^{11} \ge 1$. Assume that for whatever reason (there is no restriction on ellipticity of L) we know that

$$\|u\|_{W_p^2(\Omega)} \le N(\|Lu\|_{\mathcal{L}_p(\Omega)} + \|u\|_{\mathcal{L}_p(\Omega)})$$

for any $u \in \overset{\circ}{W}{}_p^2(\Omega)$ with N independent of u. Prove that

$$u \in \overset{\circ}{W}{}_p^2(\Omega), \quad Lu \in W_p^k(\Omega) \Longrightarrow u \in \overset{\circ}{W}{}_p^{k+2}(\Omega).$$

2. Equations in domains

The following theorem is similar to Theorem 1.7.5.

1. Theorem. *Let $\Omega = \mathbb{R}^d_+$ and $k \in \{1, 2, ...\}$. Assume that the coefficients a, b, c are in $C^k(\Omega)$ and their norms in $C^k(\Omega)$ are bounded by a constant K_1. Take λ_0 from Theorem 8.2.10. Then there exists a constant N, depending only on $K, K_1, k, \kappa, p, \omega$, and d, such that, for any $\lambda \geq \lambda_0$ and $u \in \overset{0}{W}{}^{k+2}_p(\Omega)$,*

$$\lambda \|u\|_{\mathcal{L}_p(\Omega)} \leq N \|Lu - \lambda u\|_{\mathcal{L}_p(\Omega)},$$
$$\|u\|_{W^{k+2}_p(\Omega)} \leq N(\|Lu - \lambda u\|_{W^k_p(\Omega)} + \lambda^{k/2} \|Lu - \lambda u\|_{\mathcal{L}_p(\Omega)}). \tag{1}$$

Furthermore, for any $\lambda \geq \lambda_0$ and $f \in W^k_p(\Omega)$, there exists a unique $u \in \overset{0}{W}{}^{k+2}_p(\Omega)$ satisfying $Lu - \lambda u = f$ in Ω.

Proof. As in the proof of Theorem 1.7.5, due to Theorem 1.10 and the method of continuity, it suffices to prove estimates (1). The first one of them follows from Theorem 8.2.10 which also contains the whole result for $k = 0$, the value excluded in the statement of the present theorem just because if $k = 0$, the statement is weaker than Theorem 8.2.10. Anyway, by having in mind induction on k, we may assume that we are given a $k \geq 1$ such that the theorem is true with $k - 1$ in place of k, which is indeed the case at least if $k = 1$.

Take a $u \in \overset{0}{W}{}^{k+2}_p(\Omega)$, a $j \in \{2, ..., d\}$ and use Corollary 1.2 to obtain that $v := D_j u \in \overset{0}{W}{}^{k+1}_p(\Omega)$.

Next, define $f = Lu - \lambda u$ and notice that by the Leibnitz formula

$$D_j f = (L - \lambda) v + \sum_{|\beta| \leq 2} c^\beta D^\beta u,$$

where the c^β are certain $C^{k-1}(\Omega)$ functions with their $C^{k-1}(\Omega)$ norms bounded by a constant depending only on d, k, K_1.

By the induction hypothesis

$$\|D_j u\|_{W^{k+1}_p(\Omega)} = \|v\|_{W^{k+1}_p(\Omega)} \leq N(\|D_j f\|_{W^{k-1}_p(\Omega)} + \|u\|_{W^{k+1}_p(\Omega)})$$
$$+ N\lambda^{(k-1)/2}(\|D_j f\|_{\mathcal{L}_p(\Omega)} + \|u\|_{W^2_p(\Omega)}).$$

Here by the result for $k = 0$ we have

$$\|u\|_{W^2_p(\Omega)} \leq N\|f\|_{\mathcal{L}_p(\Omega)}, \quad \lambda^{(k-1)/2}\|u\|_{W^2_p(\Omega)} \leq N\lambda^{k/2}\|f\|_{\mathcal{L}_p(\Omega)}$$

and by interpolation inequalities

$$\lambda^{(k-1)/2}\|D_j f\|_{\mathcal{L}_p(\Omega)} \le N(\|f\|_{W^k_p(\Omega)} + \lambda^{k/2}\|f\|_{\mathcal{L}_p(\Omega)}).$$

Therefore for $j \ge 2$ and $i = 1, ..., d$

$$\|D_{ij}u\|_{W^k_p(\Omega)} \le N(\|f\|_{W^k_p(\Omega)} + \lambda^{k/2}\|f\|_{\mathcal{L}_p(\Omega)} + \|u\|_{W^{k+1}_p(\Omega)}). \tag{2}$$

Obviously (2) also holds for $i \ge 1$ and $j = 1, ..., d$.

Furthermore, from $Lu - \lambda u = f$ we have

$$D_1^2 u = (a^{11})^{-1}(f + \lambda u - \sum_{ij\ge 2} a^{ij}D_{ij}u - b^i D_i u - cu),$$

where the $W^k_p(\Omega)$ norms of $D_{ij}u$ for at least one of i and j being ≥ 2 are contained in the left-hand side of (2). Therefore,

$$\|D_1^2 u\|_{W^k_p(\Omega)} \le N(\|f\|_{W^k_p(\Omega)} + \lambda\|u\|_{W^k_p(\Omega)} + \lambda^{k/2}\|f\|_{\mathcal{L}_p(\Omega)} + \|u\|_{W^{k+1}_p(\Omega)}).$$

This shows that (2) holds for all i, j if we add $N\lambda\|u\|_{W^k_p(\Omega)}$ on the right. Hence

$$\|u\|_{W^{k+2}_p(\Omega)} \le N(\|u\|_{\mathcal{L}_p(\Omega)} + [u]_{W^{k+2}_p(\Omega)})$$

$$\le N_1(\|f\|_{W^k_p(\Omega)} + \lambda\|u\|_{W^k_p(\Omega)} + \lambda^{k/2}\|f\|_{\mathcal{L}_p(\Omega)} + \|u\|_{W^{k+1}_p(\Omega)}).$$

It only remains to use the induction hypothesis:

$$\|u\|_{W^{k+1}_p(\Omega)} \le N(\|f\|_{W^{k-1}_p(\Omega)} + \lambda^{(k-1)/2}\|f\|_{\mathcal{L}_p(\Omega)}),$$

interpolation inequalities showing that

$$N_1\lambda\|u\|_{W^k_p(\Omega)} \le (1/2)\|u\|_{W^{k+2}_p(\Omega)} + N\lambda^{1+k/2}\|u\|_{\mathcal{L}_p(\Omega)},$$

and the first estimate in (1). The theorem is proved.

The following corollary is a counterpart of Corollary 1.7.6 and is proved in the same way with the only difference that one considers $u - g$ in place of u and uses Theorem 8.5.6 in place of Theorem 1.6.5.

2. Corollary. *Under the conditions of Theorem* 1 *assume that*

$$u - g \in \overset{0}{W}{}^2_p(\Omega), \quad g \in W^{k+2}_p(\Omega), \quad Lu \in W^k_p(\Omega). \tag{3}$$

Then $u - g \in \overset{0}{W}{}^{k+2}_p(\Omega)$ *and*

$$\|u\|_{W^{k+2}_p(\Omega)} \le N(\|Lu\|_{W^k_p(\Omega)} + \|u\|_{\mathcal{L}_p(\Omega)} + \|g\|_{W^{k+2}_p(\Omega)}), \tag{4}$$

where N_0 *depends only on* K, K_1, k, κ, p, ω, *and* d.

The *global regularity result* in Corollary 2 can be extended to arbitrary smooth domains.

3. Theorem. *Let* $k \in \{1, 2, ...\}$, $\Omega \in C^{k+2}$, K_0 *and* ρ_0 *be the constants from Definition* 8.3.1. *Assume that the coefficients* a, b, c *are in* $C^k(\Omega)$ *and their norms in* $C^k(\Omega)$ *are bounded by a constant* K_1. *Then there exists a constant* N, *depending only on* $d, p, K_0, \rho_0, \operatorname{diam}\Omega, K_1, \kappa, k$, *and the function* ω, *such that, if conditions* (3) *are satisfied, then* $u \in W^{k+2}_p(\Omega)$ *and* (4) *holds.*

Proof. Obviously we need only consider the case $g = 0$. We use induction. Assume that for an $r \in \{0, ..., k-1\}$ we know that $u \in W^{r+2}_p(\Omega)$ and (4) holds with r in place of k. In light of Theorem 8.5.6 we can take $r = 0$. Now we claim that u is "smoother" inside Ω, that is, for any $\zeta \in C^\infty_0(\Omega)$

$$\|u\zeta\|_{W^{r+3}_p} \le N(\|Lu\|_{W^k_p(\Omega)} + \|u\|_{W^{r+2}_p(\Omega)}). \tag{5}$$

To prove the claim, it suffices to observe that

$$L(u\zeta) = \zeta Lu + u(L\zeta - c\zeta) + 2a^{ij}\zeta_{x^i}u_{x^j} \in W^{r+1}_p$$

and to apply an $\mathcal{L}_p$ analogue of Corollary 1.7.6 in $\mathbb{R}^d$.

To deal with the part of Ω close to the boundary, we flatten it locally. Take a $z \in \partial\Omega$, the corresponding domain D^z, the mapping $\psi : B_{\rho_0}(z) \to D^z$ from Definition 8.3.1, and a function $\xi \in C^\infty_0(B_{\rho_0}(z))$. Then

$$f := L(u\xi) = \xi Lu + 2a^{ij}\xi_{x^i}u_{x^j} + u(L - c)\xi \in W^{r+1}_p(\Omega)$$

by the induction hypothesis. Also define $\hat{u}(y) = u(x)$ for $y = \psi(x)$, take the operator $\tilde{L}$ from (8.3.4) and recall that $\tilde{L}(\hat{u}\hat{\xi}) = \hat{f}$ in D^z_+. The expression $\tilde{L}(\hat{u}\hat{\xi})$ continued as zero outside of D^z_+ belongs to $W^{r+1}_p(\mathbb{R}^d_+)$ since $\hat{\xi} = 0$ near the boundary of D^z. By Corollary 2 we can estimate the $W^{r+3}_p(\mathbb{R}^d_+)$ norm of $\hat{u}\hat{\xi}$ and then by Lemma 8.3.4 we get the estimate

$$\|u\xi\|_{W^{r+3}_p(\Omega)} \le N(\|Lu\|_{W^k_p(\Omega)} + \|u\|_{W^{r+2}_p(\Omega)}). \tag{6}$$

Bearing in mind partitions of unity, we see that (5) and (6) imply

$$\|u\|_{W^{r+3}_p(\Omega)} \le N(\|Lu\|_{W^k_p(\Omega)} + \|u\|_{W^{r+2}_p(\Omega)})$$
$$\le N(\|Lu\|_{W^k_p(\Omega)} + \|u\|_{\mathcal{L}_p(\Omega)}),$$

where at the last step we have used the induction hypothesis. This justifies the possibility of induction and proves the theorem.

The following corollary is just a simple combination of Theorems 8.5.3 and 3.

4. Corollary. *Let the assumptions of Theorem* 3 *be satisfied. Take* λ_1 *from Theorem* 8.5.3. *Then for any* $\lambda \ge \lambda_1$,

$$f \in W^k_p(\Omega), \quad g \in W^{k+2}_p(\Omega)$$

there exists a unique function $u \in W^{k+2}_p(\Omega)$ *satisfying the equation* $\lambda u - Lu = f$ *in* Ω *and such that* $u - g \in \overset{0}{W}{}^1_p(\Omega)$.

3. The oblique derivative problem in $\mathbb{R}^d_+$

In this section

$$d \ge 2, \quad \Omega = \mathbb{R}^d_+, \quad L = a^{ij}D_{ij}$$

and the a^{ij} are constant. Let $\ell \in \mathbb{R}^d$ be a constant vector with $\ell^1 = 1$. Write

$$u_{(\ell)} = u_x \cdot \ell.$$

Our goal is to investigate the solvability of the equation

$$Lu - \lambda u = f$$

with oblique derivative boundary condition

$$u_{(\ell)} = g \quad \text{on} \quad \partial\Omega, \tag{1}$$

where $\lambda > 0$ is a fixed number and $f \in \mathcal{L}_p(\Omega), g \in W^1_p(\Omega)$. In order not to deal at this moment with a somewhat subtle issue of defining what the values are on $\partial\Omega$ of functions in Sobolev classes, we interpret (1) as

$$u_{(\ell)} - g \in \overset{0}{W}{}^1_p(\Omega).$$

Observe that if $\ell^j = 0$, $j = 2, \dots, d$, and $g = 0$, then (1) becomes the Neumann boundary-value condition $u_{x^1}(0, x') = 0$.

We only consider equations with constant coefficients and constant ℓ. Once the theory is developed in this case, one can follow the already familiar pattern and carry the theory over to variable coefficients also adding the lower order term and allowing ℓ to vary. We suggest the reader do a few steps in this direction in Exercises 7 and 6.

The main message of the following lemma is that it is possible to extend functions from the boundary inside Ω in a certain way.

1. Lemma. *Let $g \in W^1_p(\Omega)$ and $\lambda > 0$. Then there exist a constant $N = N(d, p, \ell)$ and a $v \in W^2_p(\Omega)$ such that $v_{(\ell)} - g \in \overset{0}{W}{}^1_p(\Omega)$ and*

$$\|v_{xx}\|_{\mathcal{L}_p(\Omega)} + \sqrt{\lambda}\|v_x\|_{\mathcal{L}_p(\Omega)} + \lambda\|v\|_{\mathcal{L}_p(\Omega)} \le N(\|g_x\|_{\mathcal{L}_p(\Omega)} + \sqrt{\lambda}\|g\|_{\mathcal{L}_p(\Omega)}). \quad (2)$$

Proof. Linear transformations allow us to assume that ℓ is the first basis vector, so that $v_{(\ell)} = v_{x^1}$. Next, introduce $w \in \overset{0}{W}{}^3_p(\Omega)$ as the unique solution of

$$\Delta w - \lambda w = g. \quad (3)$$

By Theorems 8.2.7 and 1.10

$$\sqrt{\lambda}\|w_x\|_{\mathcal{L}_p(\Omega)} + \|w_{xx}\|_{\mathcal{L}_p(\Omega)} \le N\|g\|_{\mathcal{L}_p(\Omega)},$$

$$\|w_{xxx}\|_{\mathcal{L}_p(\Omega)} \le N(\|g_x\|_{\mathcal{L}_p(\Omega)} + \sqrt{\lambda}\|g\|_{\mathcal{L}_p(\Omega)}).$$

It follows that $v := w_{x^1}$ is in $W^2_p(\Omega)$ and satisfies (2). Finally, equation (3) tells us that

$$v_{x^1} - g = \lambda w - w_{x^2x^2} - \dots - w_{x^dx^d},$$

where all terms on the right are in $\overset{0}{W}{}^1_p(\Omega)$ by the definition of w and Corollary 1.2. This proves the lemma.

The following result treats the Neumann problem for the Laplacian.

2. Theorem. *Let $g \in W^1_p(\Omega)$, $f \in \mathcal{L}_p(\Omega)$, $\lambda > 0$. Then there exists a unique $u \in W^2_p(\Omega)$ such that*

$$\Delta u - \lambda u = f \quad in \quad \Omega, \quad u_{x^1} - g \in \overset{0}{W}{}^1_p(\Omega).$$

Furthermore, there exists a constant $N = N(d, p)$ such that

$$\|u_{xx}\|_{\mathcal{L}_p(\Omega)} + \lambda\|u\|_{\mathcal{L}_p(\Omega)} \le N(\|f\|_{\mathcal{L}_p(\Omega)} + \|g_x\|_{\mathcal{L}_p(\Omega)} + \sqrt{\lambda}\|g\|_{\mathcal{L}_p(\Omega)}). \quad (4)$$

This solution is in $W^{k+2}_p(\Omega)$ if $f \in W^k_p(\Omega)$ and $g \in W^{k+1}_p(\Omega)$.

Proof. *Case* $g = 0$. Extend f for negative x^1 so that it becomes an even function with respect to x^1 and by using Theorem 4.3.8 find $u \in W^1_p$ such that

$$\Delta u - \lambda u = f.$$

This equation is invariant under the transformation $x^1 \to -x^1$ and by uniqueness the solution u is also invariant. By Exercise 8.1.4 we have $u_{x^1} \in \overset{0}{W}{}^1_p(\Omega)$. Estimate (4) follows from Theorem 4.3.8.

To prove uniqueness, assume that $f = 0$ and extend u in an even way across $\partial\Omega$. Observe that by Exercise 8.2.5 the extended function, which we again call u, is in W^2_p and, obviously, $\Delta u - \lambda u = 0$ in $\mathbb{R}^d$. By Theorem 4.3.8 we obtain $u = 0$, which finishes the argument in the case under consideration.

General case. By using Lemma 1 we find a $v \in W^2_p(\Omega)$ such that

$$v_{x^1} - g \in \overset{0}{W}{}^1_p(\Omega)$$

and (2) holds. After that, by using the above result, we find a $w \in W^2_p(\Omega)$, such that $w_{x^1} \in \overset{0}{W}{}^1_p(\Omega)$ and

$$\Delta w - \lambda w = \lambda v - \Delta v + f.$$

Then by setting $u = w + v$ and combining the estimate from Theorem 4.3.8 with (2), we prove the existence part of the theorem and (4). Uniqueness is proved above. The theorem is proved.

3. Remark. As we know from Exercise 8.2.6 the above result and its proof are also valid for $d = p = 2$ if we replace $\Delta - \lambda$ with $a^{ij}(x)D_{ij} - \lambda(a^{11} + a^{22})$, where $a(x) = (a^{ij}(x))$ is any measurable bounded uniformly nondegenerate positive matrix-valued function.

Now we give an a priori estimate.

4. Lemma. *Let* $\lambda > 0$, $u \in W^2_p(\Omega)$, *and let*

$$f := Lu - \lambda u, \quad g \in W^1_p(\Omega), \quad u_{(\ell)} - g \in \overset{0}{W}{}^1_p(\Omega).$$

Then there exists a constant $N = N(d, p, \kappa, \ell)$ *such that*

$$\|u_{xx}\|_{\mathcal{L}_p(\Omega)} + \lambda\|u\|_{\mathcal{L}_p(\Omega)} \leq N(\|f\|_{\mathcal{L}_p(\Omega)} + \|g_x\|_{\mathcal{L}_p(\Omega)} + \sqrt{\lambda}\|g\|_{\mathcal{L}_p(\Omega)}). \quad (5)$$

Proof. Linear transformations allow us to assume that $L = \Delta$ (cf. the proof of Theorem 8.2.7). Under these transformations ℓ will change to a new l. However, the fact that ℓ is not parallel to $\partial\Omega$ will also hold for l, which means that $l^1 \neq 0$. By multiplying l by an appropriate constant (and by multiplying g accordingly), we can have $l^1 = 1$. Thus, we assume that

$$L = \Delta, \quad \ell^1 = 1.$$

Next, notice that

$$u_{x^2x^2} + ... + u_{x^dx^d}$$

is invariant under rotations in x' space. An appropriate rotation will make ℓ' parallel to the x^2-axis. Therefore, (if $d \geq 3$) we may assume that

$$\ell^3 = ... = \ell^d = 0.$$

Furthermore, we may assume that $g = 0$. Otherwise take a function v from Lemma 1 and introduce $w = u - v$. Then $w_{(\ell)} \in \overset{0}{W}{}^1_p(\Omega)$ and

$$\Delta w - \lambda w = f + \lambda v - \Delta v$$

in Ω. Therefore, if the result is true when $g = 0$, then

$$\|w_{xx}\|_{\mathcal{L}_p(\Omega)} + \lambda\|w\|_{\mathcal{L}_p(\Omega)} \leq N\|f + \lambda v - \Delta v\|_{\mathcal{L}_p(\Omega)}.$$

Upon combining this with (2), we come to (5).

In the case $g = 0$ introduce $v = u_{(\ell)}$. Then v is in $\overset{0}{W}{}^1_p(\Omega)$ and satisfies the equation

$$\Delta v - \lambda v = f_{(\ell)}$$

understood in the sense of distributions (because $\Delta u - \lambda u = f$ in the sense of distributions). By Theorem 8.2.8

$$\|v_x\|_{\mathcal{L}_p(\Omega)} + \lambda^{1/2}\|v\|_{\mathcal{L}_p(\Omega)} \leq N\|f\|_{\mathcal{L}_p(\Omega)}. \tag{6}$$

Since $\ell^1 = 1$ and $\ell^i = 0$ for $i \geq 3$, we also have

$$v_{x^1} = u_{x^1x^1} + \ell^2 u_{x^1x^2}, \quad v_{x^2} = u_{x^1x^2} + \ell^2 u_{x^2x^2}.$$

Thus,

$$u_{x^1x^1} = v_{x^1} - \ell^2 u_{x^1x^2} = v_{x^1} - \ell^2 v_{x^2} + (\ell^2)^2 u_{x^2x^2} \tag{7}$$

and u satisfies

$$(1 + (\ell^2)^2)u_{x^2x^2} + u_{x^3x^3} + ... + u_{x^dx^d} - \lambda u = \tilde{f}, \tag{8}$$

where

$$\tilde{f} = f - v_{x^1} + \ell^2 v_{x^2} \in \mathcal{L}_p(\Omega).$$

In (8) we have that

$$u(x^1, \cdot) \in W^2_p(\mathbb{R}^{d-1})$$

for almost any $x^1 > 0$ by Exercise 1.3.20 and also $\tilde{f}(x^1, \cdot) \in \mathcal{L}_p(\mathbb{R}^{d-1})$ for almost any $x^1 > 0$. Therefore, for almost any $x^1 > 0$ we can apply Theorem 4.3.8 and find that for almost any $x^1 > 0$

$$\sum_{i,j \geq 2} \int_{\mathbb{R}^{d-1}} |u_{x^i x^j}(x^1, x')|^p \, dx' + \lambda^p \int_{\mathbb{R}^{d-1}} |u(x^1, x')|^p \, dx'$$

$$\leq N \int_{\mathbb{R}^{d-1}} |\tilde{f}(x^1, x')|^p \, dx'.$$

By integrating through this estimate with respect to x^1 and using the definition of $\tilde{f}$ and (6), we find

$$\sum_{i,j \geq 2} \|u_{x^i x^j}\|_{\mathcal{L}_p(\Omega)} + \lambda \|u\|_{\mathcal{L}_p(\Omega)} \leq N \|f\|_{\mathcal{L}_p(\Omega)}. \tag{9}$$

We get estimates of $u_{x^1 x^j}$ for $j = 1$ from (7) and for $j \geq 2$ from (9) and the equation

$$v_{x^j} = u_{x^1 x^j} + \ell^2 u_{x^2 x^j}.$$

Then we obtain (5). The lemma is proved.

Here is the basic result for the oblique derivative boundary-value problem.

5. Theorem. *Let $g \in W^1_p(\Omega)$, $f \in \mathcal{L}_p(\Omega)$, $\lambda > 0$. Then there exists a unique $u \in W^2_p(\Omega)$ such that*

$$Lu - \lambda u = f \quad in \quad \Omega, \quad u_{(\ell)} - g \in \overset{0}{W}{}^1_p(\Omega).$$

Furthermore, there exists a constant $N = N(d, p, \kappa, \ell)$ such that

$$\|u_{xx}\|_{\mathcal{L}_p(\Omega)} + \lambda \|u\|_{\mathcal{L}_p(\Omega)} \leq N(\|f\|_{\mathcal{L}_p(\Omega)} + \|g_x\|_{\mathcal{L}_p(\Omega)} + \sqrt{\lambda} \|g\|_{\mathcal{L}_p(\Omega)}). \tag{10}$$

Proof. Estimate (10) and uniqueness follow from Lemma 4. While proving existence, as above, we may assume that $L = \Delta$, in which case we are going to use Theorem 2 and the method of continuity in the process of moving the first basis vector e_1 to ℓ.

Since we can solve the equation $\Delta v - \lambda v = f$ in $\overset{0}{W}{}^2_p(\Omega)$ and modify g appropriately, without loss of generality we assume that $f = 0$. Next, for $t \in [0, 1]$ introduce,

$$\ell_t = t\ell + (1-t)e_1.$$

Call a $t \in [0, 1]$ "good" if for any $g \in W^1_p(\Omega)$ the problem

$$\Delta u - \lambda u = 0, \quad u_{(\ell_t)} - g \in \overset{0}{W}{}^1_p(\Omega) \tag{11}$$

has a solution u in $W^2_p(\Omega)$. Let I denote the set of all "good" points. This set contains 0 by Theorem 2.

Fix an $s \in I$ and observe that the solution of (11) with $t = s$ is unique by Lemma 4. Therefore, we have a linear operator $T : g \to u$, which maps $W^1_p(\Omega)$ to $W^2_p(\Omega)$. By Lemma 4 the norm of this operator is bounded by a constant independent of s. Therefore, for fixed $g \in W^1_p(\Omega)$ and $t \in [0, 1]$ the operator

$$S_t u := Tg + (s-t)T\big[u_{(\ell)} - u_{x^1}\big]$$

maps $W^2_p(\Omega)$ into itself and for $u, v \in W^2_p(\Omega)$ it holds that

$$\|S_t u - S_t v\|_{W^2_p(\Omega)} \le |s-t| N_0 \|u - v\|_{W^2_p(\Omega)},$$

where N_0 is independent of t, s. Take a $t \in [0, 1]$ so that

$$N_0 |s-t| \le 1/2. \tag{12}$$

Then S_t is a contraction in $W^2_p(\Omega)$ and there exists a $u \in W^2_p(\Omega)$ such that $S_t u = u$, that is,

$$u = T\big[g + (s-t)(u_{(\ell)} - u_{x^1})\big]$$

or else by the definition of T

$$\Delta u - \lambda u = 0, \quad u_{(\ell_s)} - g - (s-t)(u_{(\ell)} - u_{x^1}) \in \overset{0}{W}{}^1_p(\Omega).$$

However,

$$u_{(\ell_s)} - (s-t)(u_{(\ell)} - u_{x^1}) = u_{(\ell_t)},$$

which implies that u solves (11). Thus, if s is "good", so are all points $t \in [0,1]$ satisfying (12). This definitely means that all points of $[0,1]$ are "good". In particular, 1 is a good point and since $\ell_1 = \ell$, we have proved the existence of solutions. The theorem is proved.

6. Exercise. Let $\ell(x)$ be a bounded vector field on $\mathbb{R}^d$ with bounded fist-order derivatives and such that $\ell^1 \equiv 1$ and let L be an elliptic operator satisfying Assumption 1.6.1.

(i) Show that there exist $R, \lambda_0 \in (0,\infty)$ such that the statement of Lemma 4 is still true provided that we restrict λ to $\lambda \geq \lambda_0$, assume that u has support in a ball of radius R, and allow N to depend on ω.

(ii) Show that there exists $\lambda_0 \in (0,\infty)$ such that the statement of Theorem 5 is true provided that we restrict λ to $\lambda \geq \lambda_0$ and allow N to also depend on ω and K.

7. Exercise. In the situation of Exercise 6 take a constant μ and replace the condition $u_{(\ell)} - g \in \overset{0}{W}{}^1_p(\Omega)$ in Theorem 5 with

$$u_{(\ell)} + \mu u - g \in \overset{0}{W}{}^1_p(\Omega).$$

Prove that the statement of Theorem 5 so modified is still true with the additional change that N also depends on μ and $\lambda \geq \lambda_0(d, \kappa, p, \mu, \omega, K, \ell)$.

4. Local regularity of solutions

Theorem 2.3 about better smoothness of solutions in smooth domains admits a local version involving the boundary in the spirit of the better *local regularity result* stated in Theorem 5.2.5. In the following theorem we also give a local version of (2.4). Notice that, if R is large, Theorem 1 contains Theorem 2.3 and, if $B_{3R} \subset \Omega$, then it also contains Theorem 1.7.7 and the elliptic version of Theorem 5.2.5. A version of Theorem 1 for Lu summable to a degree $q > p$ is given later as Theorem 11.2.3. As in the whole chapter we consider an operator L satisfying the conditions from the introduction to the chapter.

1. Theorem. *Let $k \in \{0,1,2,...\}$, $\Omega \in C^{k+2}$, $R > 0$, $z \in \bar{\Omega}$. Denote*

$$\Omega_r = \Omega \cap B_r(z).$$

Assume that the coefficients a, b, c are in $C^k(\Omega_{4R})$ and their norms in $C^k(\Omega_{4R})$ are bounded by a constant K_1. Then

$$\zeta u \in \overset{0}{W}{}^2_p(\Omega_{3R}) \quad \forall \zeta \in C^\infty_0(B_{3R}(z)), \quad Lu \in W^k_p(\Omega_{3R})$$

$$\Longrightarrow \zeta u \in \overset{0}{W}{}^{k+2}_p(\Omega_{3R}) \quad \forall \zeta \in C^\infty_0(B_{3R}(z)). \quad (1)$$

Furthermore, there exists a constant N, depending only on R, K, K_1, K_0, ρ_0, k, κ, p, ω, and d, such that, if the condition of the implication (1) *holds, then*

$$\|u\|_{W_p^{k+2}(\Omega_R)} \le N(\|Lu\|_{W_p^k(\Omega_{2R})} + \|u\|_{\mathcal{L}_p(\Omega_{2R})}). \tag{2}$$

Proof. Notice that $\zeta u \in \overset{0}{W}{}_p^2(\Omega)$ for any $\zeta \in C_0^\infty(B_{2R}(z))$. This allows us to get our first assertion in the same way as in the proof of Theorem 2.3. Of course, to do this formally, we need the coefficients of L to be smooth in the whole of Ω. But we can always change and extend them appropriately outside Ω_{2R} where $\zeta u = 0$ anyway (see Remark 2.4.2). Due to this fact, in the rest of the proof we assume that a, b, c are in $C^k(\Omega)$ and their norms in $C^k(\Omega)$ are bounded by K_1.

While proving (2), notice that we allow N in (2) to depend on R and we can use dilations. Then we see that without loss of generality we can assume $R = 1$. Obviously we may also assume that $z = 0$.

The following is basically a repetition of the proof of Lemma 2.4.4, which was proved for $p = 2$ and $\Omega_R = B_R$. Of course, we are going to use estimate (2.4) available by Theorem 2.3 for $u\zeta$, where we choose ζ in a special way. Let

$$R_m = \sum_{j=0}^{m} 2^{-j}, \quad D_m = \Omega_{R_m}, \quad m = 0, 1, 2, \dots.$$

We need some functions $\zeta_m \in C_0^\infty(\mathbb{R}^d)$ such that $\zeta_m(x) = 1$ in B_{R_m}, $\zeta_m(x) = 0$ outside $B_{R_{m+1}}$ and

$$|\zeta_m|_{C^{k+2}} \le N 2^{m(k+2)} = N\rho^{-m},$$

where

$$\rho = 2^{-k-2} < 1$$

and $N = N(d, k)$. To construct them, take an infinitely differentiable function $h(t)$, $t \in (-\infty, \infty)$, such that $h(t) = 1$ for $t \le 0$, $h(t) = 0$ for $t \ge 1$ and $0 \le h \le 1$. After this define

$$\zeta_m(x) = h(2^{m+1}(|x| - R_m)).$$

Now we put $u\zeta_m$ in (2.4) to get

$$\begin{aligned} \|u\|_{W_p^{k+2}(D_m)} &\le \|u\zeta_m\|_{W_p^{k+2}(\Omega)} \le N(\|L(u\zeta_m)\|_{W_p^k(\Omega)} + \|u\zeta_m\|_{\mathcal{L}_p(\Omega)}) \\ &\le N\|Lu\|_{W_p^k(\Omega_2)} + N_1\rho^{-m}\|u\zeta_{m+1}\|_{W_p^{k+1}(\Omega)} + N\|u\|_{\mathcal{L}_p(\Omega_2)}. \end{aligned} \tag{3}$$

By interpolation inequalities (see Theorem 8.4.3)

$$\|u\zeta_{m+1}\|_{W^{k+1}_p(\Omega)} \le \varepsilon^{1/\gamma}\|u\zeta_{m+1}\|_{W^{k+2}_p(\Omega)} + N\varepsilon^{-1/(1-\gamma)}\|u\|_{\mathcal{L}_p(\Omega_2)}$$

$$\le \varepsilon^{1/\gamma}N_2\rho^{-m}\|u\|_{W^{k+2}_p(D_{m+2})} + N\varepsilon^{-1/(1-\gamma)}\|u\|_{\mathcal{L}_p(\Omega_2)},$$

where $\gamma = (k+1)/(k+2)$, $\varepsilon > 0$ and N_2 and N are independent of m, ε, u. Substitute this into (3), denote

$$\xi = N_1N_2\rho^{-2m}\varepsilon^{1/\gamma},$$

and notice that ξ is an arbitrary positive number along with ε, whereas

$$1+\varepsilon^{-1/(1-\gamma)}\rho^{-m} = 1 + N\xi^{-\gamma/(1-\gamma)}\rho^{-m(\gamma+1)/(1-\gamma)} \le N\xi^{-\gamma/(1-\gamma)}\rho^{-2m/(1-\gamma)}$$

if $\xi \in (0,1)$. Then, for $\xi \in (0,1)$, we obtain

$$\|u\|_{W^{k+2}_p(D_m)} \le N\|Lu\|_{W^k_p(\Omega_2)} + \xi\|u\|_{W^{k+2}_p(D_{m+2})}$$

$$+ N\xi^{-\gamma/(1-\gamma)}\rho^{-2m/(1-\gamma)}\|u\|_{\mathcal{L}_p(\Omega_2)},$$

which in terms of

$$A_m := \|u\|_{W^{k+2}_p(D_m)}, \quad B := \|Lu\|_{W^k_p(\Omega_2)}, \quad C := \|u\|_{\mathcal{L}_p(\Omega_2)}$$

reads as

$$A_m \le NB + \xi A_{m+2} + N\xi^{-\gamma/(1-\gamma)}\rho^{-2m/(1-\gamma)}C.$$

Now let $2\xi = \rho^{4/(1-\gamma)}$. Then

$$\xi^{-\gamma/(1-\gamma)}\rho^{-2m/(1-\gamma)} = N\rho^{-2m/(1-\gamma)} = N(2\xi)^{-m/2}$$

and

$$A_m \le NB + \xi A_{m+2} + N(2\xi)^{-m/2}C,$$

$$\xi^{m/2}A_m \le N\xi^{m/2}B + \xi^{(m+2)/2}A_{m+2} + N2^{-m/2}C,$$

$$\sum_{i=0}^{\infty}\xi^iA_{2i} \le NB\sum_{i=0}^{\infty}\xi^i + \sum_{i=1}^{\infty}\xi^iA_{2i} + NC\sum_{i=0}^{\infty}2^{-i}.$$

Upon collecting like terms in the last inequality, which is possible due to $\xi < 1$ and

$$A_i \le \|u\|_{W^{k+2}_p(\Omega_2)} < \infty,$$

we get $A_0 \leq N(B+C)$, which is (2). The theorem is proved.

2. Remark. It is seen from the above proof that we do not need the whole of $\partial\Omega$ to be of class C^{k+2}. We only needed the portion of it lying in $B_{3R}(z)$ to be smooth, which can be expressed as $\partial\Omega \cap B_{3R}(z) = \partial\Omega_1 \cap B_{3R}(z)$ for an $\Omega_1 \in C^{k+2}$.

3. Remark. Obviously one can replace $2R$ in (2) with χR for any $\chi > 1$ and then N also depends on γ.

4. Remark. We prove later in Remark 10.4.12 that, if $k - d/p > 0$ and $\Omega \in C^k$, then $W_p^k(\Omega) \subset C^{k-d/p-\varepsilon}(\bar{\Omega})$ for any $\varepsilon \in (0, k-d/p]$. It follows that, if the portion of $\partial\Omega$ lying in $B_{3R}(z)$ is infinitely differentiable and if a, b, c, and Lu are infinitely differentiable in $\Omega \cap B_{3R}(z)$ and have bounded derivatives, then u is infinitely differentiable in $\Omega \cap B_R(z)$ and has bounded derivatives.

5. Hints to exercises

1.5. Consider $u(x - \varepsilon y) - 2u(x) + u(x + \varepsilon y)$.

1.6. You may need to use the fact that for any $f \in \mathcal{L}_p$ we have $f(y + \cdot) \to f$ in $\mathcal{L}_p$ as $y \to 0$.

1.7. Observe that

$$\int_0^\infty \varepsilon^{-3} |\tilde{\zeta}(\varepsilon\xi) - 1|^2 \, d\varepsilon$$

behaves like $|\xi|^2$ as $|\xi| \to \infty$.

1.8. For $d = 1$ try a function u with $\tilde{u}(\xi) \sim |\xi|^{-3/2+\delta}$ as $|\xi| \to \infty$.

1.9. Take $f(x) = e^{-\mu x}$ and let $\mu \downarrow 0$.

1.11. Use first-order finite differences in tangential directions to prove by induction that

$$\|u\|_{W_p^{k+2}(\Omega)} \leq N(\|Lu\|_{W_p^k(\Omega)} + \|u\|_{W_p^{k+1}(\Omega)}).$$

3.7. Find an infinitely differentiable bounded function $h(x^1)$ having bounded derivatives and bounded away from zero such that $h'(0) = -\mu h(0)$. Then for $v = u/h$ we have $\ell^j v_{x^j} = g/h$ on $\partial\Omega$ and $Lu - \lambda u = h(\bar{L}v - \lambda v)$, where $\bar{L}\phi := h^{-1}L(h\phi)$ is an elliptic operator satisfying our hypotheses with a slightly modified K.

Chapter 10

Sobolev embedding theorems for $W_p^k(\Omega)$

We understood above how better smoothness of data brings about better smoothness of solutions in terms of Sobolev spaces. Sometimes one is interested in measuring smoothness in terms of the usual rather than generalized derivatives. This happens, for instance, if we want to prove the maximum principle and to extend the existence and uniqueness results as in Theorem 1.6.4 to all λ such that $c - \lambda < 0$.

The results of this chapter will be proved again in a quite different way for the whole space in Section 13.8 where we deal with the spaces H_p^γ which coincide with W_p^γ if $\gamma = 0, 1, 2, \ldots$. However, the most commonly used spaces in the theory of partial differential equations are Sobolev spaces and the author found it hard to resist the temptation to present beautiful ideas leading to embedding theorems for those spaces. Apart from a few exercises at the end of Section 1 the embedding theorems we will discuss are most appropriate for elliptic equations.

S.L. Sobolev himself proved the first embedding theorem in the following way. Take a $u \in C_0^1$ and an infinitely differentiable function $\zeta(t)$, $t \in \mathbb{R}$, with support in $(-1, 1)$ and such that $\zeta(0) = 1$. Fix a unit $y \in \mathbb{R}^d$ and for the function $u(ry)\zeta(r)$ of one variable r write

$$u(0) = -\int_0^\infty \frac{d}{dr}(u(ry)\zeta(r))\,dr = -\int_0^\infty [\zeta(r)y^i u_{x^i}(ry) + u(ry)\zeta'(r)]\,dr.$$

Integrate this equality with respect to y over the unit sphere and use the polar coordinates in $\mathbb{R}^d$. Then

$$u(0) = \int_0^\infty \int_{\partial B_1} [\zeta(r) y^i u_{x^i}(ry) + u(ry)\zeta'(r)]\, dS\, dr$$

$$= N \int_{\mathbb{R}^d} \frac{1}{|x|^{d-1}} [\zeta(|x|)\frac{x^i}{|x|} u_{x^i}(x) + u(x)\zeta'(|x|)]\, dx,$$

$$|u(0)| \leq N \int_{B_1} \frac{1}{|x|^{d-1}} (|u_x(x)| + |u(x)|)\, dx. \tag{1}$$

Next use Hölder's inequality. Take a $p \in (d, \infty)$. Then

$$q := p/(p-1) \in (1, d/(d-1)), \quad (d-1)q < d$$

and

$$\int_{B_1} \frac{1}{|x|^{(d-1)q}}\, dx = N \int_0^1 \frac{r^{d-1}}{r^{(d-1)q}} < \infty.$$

It follows that

$$|u(0)| \leq N(d,p)\|u\|_{W_p^1}.$$

Since one could take any point in $\mathbb{R}^d$ as the origin,

$$\sup_{\mathbb{R}^d} |u| \leq N(d,p)\|u\|_{W_p^1}.$$

After proving this estimate for $u \in C_0^1$, we extend it to any $u \in W_p^1$ and obtain one of the Sobolev embedding theorems

$$p > d \Longrightarrow W_p^1 \subset C.$$

Morrey's Theorem 2.1 improves this result to $W_p^1 \subset C^{1-d/p}$. We say more about the meaning of these inclusions in Remark 1.4 and Exercise 1.5.

1. Exercise*. By shifting the origin in (1) and using Young's inequality prove that if $p, q \in [1, \infty)$, $q \geq p$, and

$$1 - d/p > -d/q,$$

then for any $u \in W_p^1(\mathbb{R}^d)$ we have

$$\|u\|_{\mathcal{L}_q(\mathbb{R}^d)} \leq N\|u\|_{W_p^1(\mathbb{R}^d)}$$

with N independent of u, so that $W_p^1(\mathbb{R}^d) \subset \mathcal{L}_q(\mathbb{R}^d)$. By considering $v(x) = u(|x^1|, x')$, prove that the same result holds if $\mathbb{R}^d$ is replaced with $\mathbb{R}_+^d$.

In Lemma 4.1 we will see that the case $1 - d/p = -d/q$ need not be excluded.

1. Embedding for Campanato and Slobodetskii spaces

1:1. Embeddings in C^α. Let Ω be a domain in $\mathbb{R}^d$. Recall that by $C(\Omega)$ we denote the set of all bounded continuous functions on Ω. This is a Banach space provided with the norm

$$\|u\|_{C(\Omega)} = \sup_{x\in\Omega} |u(x)|.$$

For $\alpha = 0$ we denote $C^\alpha(\bar\Omega) = C(\bar\Omega)$ and for $\alpha \in (0,1)$ we mean by $C^\alpha(\bar\Omega)$ the set of $u \in C(\bar\Omega)$ having finite norm

$$\|u\|_{C^\alpha(\Omega)} = \|u\|_{C(\Omega)} + [u]_{C^\alpha(\Omega)},$$

where ($\alpha \in (0,1)$)

$$[u]_{C^\alpha(\Omega)} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

If $\gamma = k+\alpha$ with $k \in \{0,1,...\}$ and $\alpha \in [0,1)$, by $C^\gamma(\bar\Omega)$ we mean the Hölder space of all k times continuously differentiable functions u on Ω such that $D^\beta u \in C(\bar\Omega)$ if $|\beta| \le k$ and $D^\beta u \in C^\alpha(\bar\Omega)$ if $|\beta| = k$. The space $C^\gamma(\bar\Omega)$ is a (nonseparable if γ is not an integer or Ω is a general unbounded domain) Banach space provided with norm

$$\|u\|_{C^\gamma(\Omega)} = \sum_{|\beta|\le k} \|D^\beta u\|_{C(\Omega)} + \sum_{|\beta|=k} [D^\beta u]_{C^\alpha(\Omega)}$$

if $\alpha \in (0,1)$ and the usual $C^\gamma(\Omega)$ norm if $\gamma \in \{0,1,...\}$.

In the case $\Omega = \mathbb{R}^d$ as usual we drop $\bar\Omega$ in $C^\gamma(\bar\Omega)$.

1. Exercise. Let $\gamma > 0$, $g_n \in C^\gamma$, $n = 1,2,...$, with $\|g_n\|_{C^\gamma} \le M$, where M is a constant. Prove that there is a subsequence $g_{n'}$ and $g \in C^\gamma$ such that $\|g\|_{C^\gamma} \le M$ and $D^\beta g_{n'} \to D^\beta g$ uniformly on each ball as $n' \to \infty$ whenever $|\beta| < \gamma$.

For an open convex set Ω denote by $\rho(\Omega)$ its *interior diameter* which is the largest diameter of open balls contained in Ω. Recall that $B_r(x)$ stands for the open ball of radius r centered at x. Assumption (iii) in the following lemma can be dropped as follows from Exercise 5.

2. Lemma. *(i) Let Ω be a bounded* convex *domain in $\mathbb{R}^d$, let $\kappa < \infty$ and $\alpha \in (0,1]$ be some constants and assume that*

$$\operatorname{diam}\Omega \le \kappa\rho(\Omega). \tag{1}$$

(ii) Let $u \in \mathcal{L}_{1,loc}(\Omega)$ and assume that there exists a constant $M < \infty$ such that

$$\int_{B_\rho(x)} \int_{B_\rho(y)} |u(r) - u(s)|\, drds \le M|x-y|^{\alpha+2d} \tag{2}$$

whenever

$$B_\rho(x), B_\rho(y) \subset \Omega \quad \text{and} \quad 4\rho \le |x-y|.$$

(iii) Finally, let $u \in C^1(\bar\Omega)$.

Then for any $x, y \in \bar\Omega$ we have

$$|u(x) - u(y)| \le NM|x-y|^\alpha, \tag{3}$$

where N depends only on d, α, and κ. In particular, if (i) and (iii) hold, then

$$|u(x) - u(y)|^p \le N^p|x-y|^{p\alpha} \int_\Omega \int_\Omega \frac{|u(r)-u(s)|^p}{|r-s|^{2d+p\alpha}}\, drds, \tag{4}$$

for $p \in [1, \infty)$, where N depends only on d, α, and κ.

Proof. To begin with, we claim that the first assertion implies (4). Indeed, for $r \in B_\rho(x)$ and $s \in B_\rho(y)$, we have $2\rho + |x-y| \ge |r-s|$, so that the left-hand side of (2) is less than

$$(2\rho + |x-y|)^{\alpha+2d/p} \int_{B_\rho(x)} \int_{B_\rho(y)} \frac{|u(r)-u(s)|}{|r-s|^{\alpha+2d/p}}\, drds.$$

Now our claim follows easily if we recall that $\rho \le |x-y|$ and use Hölder's inequality showing that the last integral is less than

$$\left(\int_\Omega \int_\Omega \frac{|u(r)-u(s)|^p}{|r-s|^{2d+p\alpha}}\, drds \right)^{1/p} \omega_d^{2-2/p} \rho^{2d(1-1/p)},$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$.

To prove (3), define

$$K = \sup_{x,y\in\Omega, x\ne y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$

and observe that $K < \infty$ since $0 < \alpha \le 1$, u is Lipschitz continuous, and Ω is bounded.

First, we want to explain the idea of what follows in the case $\Omega = \mathbb{R}^d$, which is, actually, excluded in the formulation of the lemma and will be covered in Remark 3. Assume that $K < \infty$ and (2) is satisfied whenever $B_\rho(x), B_\rho(y) \subset \mathbb{R}^d$ and $4\rho \le |x-y|$. Then take a $\rho > 0$ and for $|x-r|, |s-y| \le \rho$ write

$$|u(x) - u(y)| \le |u(x) - u(r)| + |u(r) - u(s)| + |u(s) - u(y)|$$

$$\le K(|x-r|^\alpha + |s-y|^\alpha) + |u(r) - u(s)|$$

$$\le 2K\rho^\alpha + |u(r) - u(s)|. \tag{5}$$

Assuming that $\rho = \delta|x-y|$, where the constant $\delta \in (0, 1/4]$, we integrate through this estimate over $r \in B_\rho(x)$ and $s \in B_\rho(y)$ and due to (2) find

$$\rho^{2d}|u(x) - u(y)| \le 2K\rho^{\alpha+2d} + NM|x-y|^{\alpha+2d}, \tag{6}$$

$$|u(x) - u(y)| \le (2K\delta^\alpha + NM\delta^{-2d})|x-y|^\alpha,$$

where N depends only on d, which by the definition of K means that

$$K \le 2K\delta^\alpha + N_1 M\delta^{-2d}. \tag{7}$$

Finally, choose any δ so that not only $\delta \le 1/4$ but also $2\delta^\alpha \le 1/2$ which is possible due to $\alpha > 0$. Then

$$K \le 2N_1 M\delta^{-2d}$$

and this is the desired result.

To implement this idea in the case of our domain Ω, we will step inside Ω if the points x, y are too close to the boundary. We start by noticing that, as is easy to see, (2) and (3) are dilation invariant in the sense that, of course, dilations change the constant M in (2) but they change M in (3) in the same way. Furthermore, dilations do not affect the ratio

$$\frac{\rho(\Omega)}{\operatorname{diam}\Omega}.$$

Therefore, without loss of generality we assume that $\rho(\Omega) = 1$ and the open unit ball B centered at the origin lies in Ω.

Next, notice that for any $x \in \Omega$, the smallest open convex set containing B and x is an ice cream cone-like domain $I(x)$ belonging to Ω. The openings of these "cones" depend only on $|x|$ and are obviously bounded from below since $|x| \le \kappa$ on Ω. It follows easily that there exists a $\theta_0 \in (0, \infty)$, depending only on κ, such that for any $x \in \Omega$ and $\rho \in (0, 1]$ there is an x_ρ on the ray tx, $t \in [0, 1]$, such that

$$|x_\rho - x| \le \theta_0\rho \quad \text{and} \quad B_\rho(x_\rho) \subset \Omega.$$

To find such an x_ρ, it suffices to take B_ρ and to move it along the ray tx, $t \geq 0$, until it either covers x (if $x \in B_1$) or touches the boundary of $I(x)$. In the former case $|x_\rho - x| \leq \rho$ and in the latter case $|x_\rho - x| \leq \rho/\sin|\phi|$, where ϕ is the angle between x and any tangent line to B_1 passing through x.

Now as before, we take $x, y \in \Omega$ and set $\rho = \delta|x - y|$, where $\delta > 0$ is a constant to be specified later. Right away we suppose that

$$\delta \leq \frac{1}{4 + 2\theta_0} \quad \text{and} \quad \delta \leq \frac{1}{\kappa} \tag{8}$$

and notice that $\rho \leq \delta\kappa \leq 1$ and for $r \in B_\rho(x_\rho)$ we have

$$|x - r| \leq \rho + |x_\rho - x| \leq (1 + \theta_0)\rho.$$

Also for y_ρ introduced similarly to x_ρ we have

$$|x_\rho - y_\rho| \geq |x - y| - 2\theta_0\rho = (\delta^{-1} - 2\theta_0)\rho \geq 4\rho$$

due to the choice of δ.

After these preparations we get (5) for $r \in B_\rho(x_\rho)$ and $s \in B_\rho(y_\rho)$ with $2(1 + \theta_0)^\alpha K$ in place of $2K$. We integrate it through over $r \in B_\rho(x_\rho)$ and $s \in B_\rho(y_\rho)$ and owing to (2), come to (6) with $2(1 + \theta_0)^\alpha K$ in place of $2K$ and $|x_\rho - y_\rho|$ in place of $|x - y|$. However, there is no need to do the latter replacement since

$$|x_\rho - y_\rho| \leq |x - y| + 2\theta_0\rho = (2\theta_0\delta + 1)|x - y| \leq 3|x - y|.$$

This leads to (7) with $2(1+\theta_0)^\alpha K$ in place of $2K$ and, after choosing δ such that not only (8) holds but also $2(1 + \theta_0)^\alpha\delta^\alpha \leq 1/2$, allows us to finish the proof as above. The lemma is proved.

3. Remark. Lemma 2 holds true for unbounded convex Ω as well if one replaces condition (1) with

$$\overline{\lim_{R\to\infty}} \frac{\operatorname{diam}\Omega_R}{\rho(\Omega_R)} \leq \kappa, \tag{9}$$

where $\Omega_R = \Omega \cap B_R$. This result is obtained by applying Lemma 2 to Ω_R for all large R and recalling that N in (3) depends only on d, α, and κ. In particular, Lemma 2 is valid for any open convex cone, half spaces, and the whole of $\mathbb{R}^d$.

4. Remark. Notice that the integral in (4) is finite if $u \in C^\beta(\bar{\Omega})$, $\beta \in (0, 1]$ and $\beta > \alpha + d/p$. If p is large, one can take α close to β, thus losing only a little bit while replacing the C^β norm with the integral norm related to (4).

It turns out that assumption (iii) in Lemma 2 can be dropped. Then, however, some explanations of its assertions are necessary. Notice that $\mathcal{L}_1$ functions are uniquely defined only up to almost everywhere and the sense of the pointwise estimates (3) and (4) should be explained. If we have two functions f and g given on Ω and $f = g$ almost everywhere in Ω, we say that g is *a modification* of f (and vice versa).

5. Exercise. Prove that the assertions of Lemma 2 hold if only assumptions (i) and (ii) are satisfied and we take an appropriate modification of u in (3) and (4).

6. Remark. If $0 \leq s < 1$, the set of $\mathcal{L}_p(\Omega)$ functions for which

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}}\, dxdy < \infty$$

form the *Slobodetskii space* $W_p^s(\Omega)$. These spaces naturally arise, for instance, as spaces of traces of functions in $W_p^k(\mathbb{R}_+^{d+1})$, $k = 1, 2, ...,$ on $\partial\mathbb{R}_+^{d+1}$.

It is known (see [20]) that for $0 < s < 1$ we have the embeddings

$$H_p^s \subset W_p^s(\mathbb{R}^d)$$

if $p \geq 2$ and

$$W_p^s(\mathbb{R}^d) \subset H_p^s$$

if $1 < p \leq 2$ with proper inclusions if $p \neq 2$, where H_p^s spaces are introduced in Section 13.3.

If we take $s = \alpha + d/p$ and $s < 1$, then Lemma 2 and Exercise 5 say that

$$0 < s - d/p < 1 \Longrightarrow W_p^s(\Omega) \subset C^{s-d/p}(\Omega).$$

Actually this embedding holds whenever $s - d/p > 0$ and s and $s - d/p$ are not integers if the definition of $W_p^s(\Omega)$ is properly extended for $s \geq 1$.

1:2. Exercises (optional). In what follows below in this subsection $|\Gamma|$ is the volume of $\Gamma \subset \mathbb{R}^d$.

7. Exercise. We see that (2) implies (3). The converse is obviously true as well (with a different constant in (2)). Therefore, (2) can be used to define the seminorms

$$[u]_{C^\alpha(\Omega)}.$$

This new definition is very close to the one introduced by Campanato. For $\alpha \in (0, 1]$ the *Campanato space* $\mathcal{L}^\alpha(\Omega)$ is defined as the set of all $u \in \mathcal{L}_1(\Omega)$ such that, for a constant $C < \infty$ and any ball B,

$$\int_{B\cap\Omega} |u(x) - u_{B\cap\Omega}|\, dx \leq C r_B^{d+\alpha},$$

where r_B is the radius of B and

$$u_{B\cap\Omega} := \frac{1}{|B\cap\Omega|}\int_{B\cap\Omega} u(x)\,dx.$$

The smallest such constant C is denoted by $\mathrm{Camp}_\alpha(u)$. Under assumptions (i) and (iii) of Lemma 2 and assuming that $\alpha < 1$, prove that (with the standard definition of $[u]_{C^\alpha(\Omega)}$)

$$[u]_{C^\alpha(\Omega)} \leq N\mathrm{Camp}_\alpha(u,\Omega) \quad \text{and} \quad \mathrm{Camp}_\alpha(u,\Omega) \leq N[u]_{C^\alpha(\Omega)}$$

with N independent of u. Prove that the same is true for $\alpha = 1$ if one replaces $[u]_{C^\alpha(\Omega)}$ with the Lipschitz constant of u.

8. Exercise. By using Exercises 4.3.3 and 7, prove that if $\alpha \in (0,1)$ and $u \in C^{2+\alpha}$, then

$$[u_{xx}]_{C^\alpha} \leq N(d,\alpha)[\Delta u]_{C^\alpha}.$$

9. Exercise. Recall that for $(t,x) \in \mathbb{R}\times\mathbb{R}^d = \mathbb{R}^{d+1}$ and $\rho > 0$ we denote

$$Q_\rho(t,x) = (t, t+\rho^2)\times B_\rho(x).$$

Assume that we have a function $u = u(t,x)$ of class $\mathcal{L}_{1,loc}(\mathbb{R}^{d+1})$ and an $\alpha \in (0,1]$ such that for all $(t,x),(s,y) \in \mathbb{R}^{d+1}$ and $\rho > 0$ we have

$$\int_{Q_\rho(t,x)}\int_{Q_\rho(s,y)} |u(r_1,p_1) - u(r_2,p_2)|\,dr_1dp_1dr_2dp_2 \leq M\langle(t-s,x-y)\rangle^{\alpha+2d+4}$$

whenever $4\rho \leq \langle(t-s,x-y)\rangle$, where M is a constant and

$$\langle(t-s,x-y)\rangle := |t-s|^{1/2} + |x-y|.$$

By mimicking the proof of Lemma 2, show that for all $(t,x),(s,y) \in \mathbb{R}^{d+1}$ it holds that

$$|u(t,x) - u(s,y)| \leq NM\langle(t-s,x-y)\rangle^\alpha,$$

where N is independent of $u,(t,x),(s,y)$.

10. Exercise. Take an operator L as in the beginning of Chapter 4 (with coefficients independent of x) and by using Exercises 4.3.6 and 9, show that if $\alpha \in (0,1)$ and $u \in C_0^\infty(\mathbb{R}^{d+1})$, then for all $(t,x),(s,y) \in \mathbb{R}^{d+1}$

$$|u_{xx}(t,x) - u_{xx}(s,y)| \leq N\sup_{t\in\mathbb{R}}[f(t,\cdot)]_{C^\alpha}\langle(t-s,x-y)\rangle^\alpha,$$

where $N = N(d,\alpha,\kappa)$ and $f = u_t + Lu$. This is a basic Hölder space estimate for parabolic equations. A somewhat unusual feature of it is that the u_{xx} admit a Hölder space estimate even in t without a and f being assumed to satisfy a Hölder condition in t. Of course, if a and f are $C^{\alpha/2}$ Hölder in t uniformly with respect to x, then from the equation $u_t = f - Lu$ we see that u_t is also $C^{\alpha/2}$ Hölder in t and C^α Hölder in x, which gives a standard basic Hölder space estimate for parabolic equations.

11. Exercise. For $\alpha \in (0,1)$ and $u \in C_0^\infty(\mathbb{R}^{d+1})$ prove that

$$|u_{xx}(t,x) - u_{xx}(s,x)| \leq N \sup_{t\in\mathbb{R}}([u_t(t,\cdot)]_{C^\alpha} + [\Delta u(t,\cdot)]_{C^\alpha})|t-s|^{\alpha/2}$$

for all $s,t \in \mathbb{R}$ and $x \in \mathbb{R}^d$, where $N = N(d,\alpha)$.

The following two exercises are similar to Exercises 4.3.10 and 4.3.11. Doing Exercise 13 without using Exercise 12 can be quite challenging.

12. Exercise. Let $b(t)$ be a bounded $\mathbb{R}^d$-valued function. Prove that, if $u \in C_0^\infty(\mathbb{R}^{d+1})$ and $\alpha \in (0,1)$, then with the *same* constant N as in Exercise 10 we have

$$|u_{xx}(t,x) - u_{xx}(t,y)| \leq N \sup_{s\in\mathbb{R}}[f(s,\cdot)]_{C^\alpha}|x-y|^\alpha$$

for all $t \in \mathbb{R}$, $x,y \in \mathbb{R}^d$, where $f = u_t + Lu + b^i u_{x^i}$. By observing that the values of $u(t,\cdot)$ are independent of the values of $f(s,\cdot)$ for $s \leq t$, conclude that

$$|u_{xx}(t,x) - u_{xx}(t,y)| \leq N \sup_{s>t}[f(s,\cdot)]_{C^\alpha}|x-y|^\alpha.$$

By plugging in $u(t,\cdot)e^{-\lambda t}$ in place of $u(t,\cdot)$, prove that the estimate also holds for all $\lambda \geq 0$ with $f = u_t + Lu + b^i u_{x^i} - \lambda u$.

13. Exercise. Derive from Exercise 12 that, if the coefficients of L are independent of t, then with the *same* constant N as in Exercise 12 for any $u \in C_0^\infty$, constant vector $b \in \mathbb{R}^d$, constant $\lambda \geq 0$, and $\alpha \in (0,1)$ we have

$$[u_{xx}]_{C^\alpha} \leq N(d,\alpha)[Lu + b^i u_{x^i} - \lambda u]_{C^\alpha}.$$

14. Exercise. Let $p,q \in (1,\infty]$ (∞ is included) be such that $\alpha := 1 - d/q - 2/p \in (0,1]$. Recall that the spaces $\mathcal{L}_p\text{-}\mathcal{L}_q$ and $W^{1,2}_{p,q}$ are introduced in Section 7.2. By using Exercise 9 and Lemma 4.2.2, show that if $u \in W^{1,2}_{p,q}$, then

$$|u_x(t,x) - u_x(s,y)| \leq N(d,\alpha)(\|u_{xx}\|_{\mathcal{L}_p\text{-}\mathcal{L}_q} + \|u_t\|_{\mathcal{L}_p\text{-}\mathcal{L}_q})\langle(t-s,x-y)\rangle^\alpha.$$

In particular, prove that if u, u_x, u_{xx}, u_t are bounded, then

$$|u_x(t,x) - u_x(s,y)| \leq N(d)(\sup|u_{xx}| + \sup|u_t|)\langle(t-s,x-y)\rangle.$$

2. Embedding $W_p^1(\Omega) \subset C^{1-d/p}(\Omega)$. Morrey's theorem

Here we prove the following celebrated theorem of Ch.B. Morrey. Recall that by $|\Gamma|$ we denote the volume of $\Gamma \subset \mathbb{R}^d$.

1. Theorem (Morrey). *Let Ω be a bounded convex domain in $\mathbb{R}^d$ satisfying condition* (1.1) *with a constant $\kappa < \infty$. Let $M < \infty$, $\alpha \in (0,1]$ be some constants. Finally, let $u, u_x \in \mathcal{L}_1(\Omega)$ and assume that for any ball $B \subset \Omega$ we have*

$$\fint_B |u_x|\,dx \le M r_B^{\alpha-1}, \tag{1}$$

where r_B is the radius of B. Then for any $x, y \in \Omega$ we have

$$|u(x) - u(y)| \le NM|x-y|^\alpha, \tag{2}$$

$$|u(x)| \le NM + \sup_{y\in\Omega} \fint_{\Omega\cap B_1(y)} |u(z)|\,dz, \tag{3}$$

where N depends only on d, α, and κ. In particular, if $p > d$ and $u, u_x \in \mathcal{L}_p(\Omega)$, then (2) *and* (3) *hold with*

$$\alpha = 1 - d/p, \quad M = \|u_x\|_{\mathcal{L}_p(\Omega)}$$

and N depending only on d, p, and κ.

Proof. Of course, as is explained before Exercise 1.5, we are going to prove that there exists a modification of u satisfying (2) and (3). Notice at once that the last assertion of the theorem follows immediately from the first one since by Hölder's inequality

$$\int_B |u_x|\,dx \le \|u_x\|_{\mathcal{L}_p}|B|^{1-1/p}.$$

Also observe that (3) follows from (2) after integrating

$$|u(x)| \le |u(x)-u(y)| + |u(y)|$$

with respect to y.

While proving (2), we start with the case $u \in C^1(\bar\Omega)$ and use Lemma 1.2. Take $x, y \in \Omega$ and $\rho > 0$ such that $B_\rho(x), B_\rho(y) \subset \Omega$. Then for $r \in B_\rho(x)$ and $s \in B_\rho(y)$ the straight segment with ends r, s is in Ω. Also

$$|r-s| \le 2\rho + |x-y|.$$

Hence

$$|u(r)-u(s)| = |(r^i - s^i)\int_0^1 D_i u(tr+(1-t)s)\,dt|$$

$$\le (2\rho + |x-y|)\int_0^1 |Du(tr+(1-t)s)|\,dt. \tag{4}$$

Furthermore, for each $t \in (0,1)$ and $s \in B_\rho(y)$ we have

$$\{tr + (1-t)s : r \in B_\rho(x)\} = B_{t\rho}(tx + (1-t)s) \subset \Omega.$$

By changing variables according to $\xi = tr + (1-t)s$, from (1) we obtain

$$\int_{B_\rho(x)} |Du(tr + (1-t)s)|\, dr = t^{-d} \int_{B_{t\rho}(tx+(1-t)s)} |Du(\xi)|\, d\xi$$

$$\leq t^{-d} NM(t\rho)^{d-1+\alpha} = NMt^{\alpha-1}\rho^{d-1+\alpha},$$

where N depends only on d.

Therefore, for $\rho \leq |x-y|$ the left-hand side of (1.2) is less than

$$3|x-y| \int_0^1 \int_{B_\rho(x)} \int_{B_\rho(y)} |Du(tr + (1-t)s)|\, drdsdt$$

$$\leq NM|x-y|\rho^{d-1+\alpha} \int_0^1 \int_{B_\rho(y)} t^{\alpha-1}\, dsdt = NM|x-y|\rho^{2d-1+\alpha},$$

which is less than $NM|x-y|^{2d+\alpha}$ since $\rho \leq |x-y|$. Referring to Lemma 1.2 yields the desired result for $u \in C^1(\bar{\Omega})$.

In the general case we use the Sobolev mollifiers. Take a nonnegative $\zeta \in C_0^\infty$ with unit integral and satisfying $\zeta(x) = 0$ for $|x| \geq 1$ and for $\varepsilon > 0$ define $\zeta_\varepsilon(x) = \varepsilon^{-d}\zeta(x/\varepsilon)$. Also let

$$u^{(\varepsilon)} = (uI_\Omega) * \zeta_\varepsilon.$$

Take any convex open $\Omega' \subset \Omega$ with $\chi := \operatorname{dist}(\partial\Omega', \partial\Omega) > 0$ and with the constant κ corresponding to Ω' less than the doubled original one. It is not hard to prove that for $\varepsilon < \chi$ we have $u_x^{(\varepsilon)} = (u_x I_\Omega) * \zeta_\varepsilon$ in Ω'. Furthermore, if a ball $B \subset \Omega'$ and $\varepsilon < \chi$,

$$\int_B |u_x^{(\varepsilon)}|\, dx \leq \int_B \Big(\int_{|y|\leq 1} |(u_x I_\Omega)(x - \varepsilon y)|\zeta(y)\, dy \Big)\, dx$$

$$= \int_B \Big(\int_{|y|\leq 1} |u_x(x - \varepsilon y)|\zeta(y)\, dy \Big)\, dx$$

$$= \int_{|y|\leq 1} \Big(\int_{B-\varepsilon y} |u_x(x)|\, dx \Big)\zeta(y)\, dy \leq M|B_1|r_B^{d-1+\alpha},$$

where $B - \varepsilon y$ is the ball B shifted by εy. Therefore, estimate (2) holds for $u^{(\varepsilon)}$ in place of u if $x, y \in \Omega'$ and $\varepsilon < \chi$. Moreover,

$$\|u^{(\varepsilon)}\|_{\mathcal{L}_1(\Omega')} \leq \|u\|_{\mathcal{L}_1(\Omega)}.$$

These estimates and Arzelà's theorem imply that the $u^{(\varepsilon)}$ form a precompact set in $C(\Omega')$. Therefore, there exists a sequence $u^{(\varepsilon(n))}$ with $\varepsilon(n) \to 0$ as $n \to \infty$ which converges uniformly on Ω' to a function $\check{u}$. Obviously, estimate (2) holds for $\check{u}$ in place of u if $x, y \in \Omega'$. On the other hand

$$u^{(\varepsilon)} \to u \quad \text{in} \quad \mathcal{L}_1(\Omega')$$

as $\varepsilon \to 0$. It follows that $\check{u} = u$ almost everywhere in Ω'. This and the continuity of $\check{u}$ shows that if we construct another function $\check{u}$ starting off from a different domain Ω', the results will coincide on the intersection of the domains Ω'. Expanding Ω' to cover all of Ω, we get a continuous function $\check{u}$ defined on Ω, satisfying (2) and equal to u almost everywhere on Ω. This is exactly what we need and the theorem is proved.

2. Remark. To memorize the statement of Morrey's theorem and also to see that its result is optimal in what concerns the Holder exponent of u one can take $u = |x|^\alpha$. Then for balls B centered at the origin, one easily finds that (1) becomes an equality with certain constant M. This shows how to find the power of r_B. At the same time $|x|^\alpha$ admits no better estimate than (2), so that the result is sharp.

3. Remark. Due to Remark 1.3, estimate (2) in Theorem 1 holds true for unbounded convex Ω as well, if one replaces condition (1.1) with (1.9). Notice that (1.9) is satisfied for any open convex cone, half spaces, and the whole of $\mathbb{R}^d$. The last assertion of Theorem 1 for $\Omega = \mathbb{R}^d$ will be proved differently in Theorem 13.8.1.

4. Remark. It is not hard to see that Theorem 1 is also true (with a different constant N_0) for bounded domains Ω such that

(i) a *uniform interior cone condition* holds, that is, there exists a fixed cone

$$Q = \{x \in \mathbb{R}^d_+ : |x'| \leq \theta x^1 \leq \theta^2\}, \quad \theta > 0,$$

and each $x \in \Omega$ is the vertex of a cone $Q(x) \subset \Omega$ congruent to Q;

(ii) there is a constant N such that for each $x, y \in \Omega$ there is a smooth curve ℓ_{xy} of length $\leq N|x - y|$ inside Ω with distance of the curve to $\partial\Omega$ not less than $N^{-1}|x - y|$ and with one end lying in $Q(x)$ at a distance less than $N|x - y|$ from x and the other being in $Q(y)$ at a distance less than $N|x - y|$ from y.

In particular, these conditions are satisfied for so-called bounded Lipschitz domains.

Formula (4) also allows us to prove a particular case of Poincaré's inequality, more general versions of which are given in Exercises 5.2 and 5.3.

5. Theorem. *Let $p \in [1, \infty)$ and let Ω be a convex bounded domain and $u \in W^1_p(\Omega)$. Then*

$$\int_\Omega \int_\Omega |u(x) - u(y)|^p \, dxdy \le 2^{d+1} d^p(\Omega)|\Omega| \int_\Omega |u_x|^p \, dx, \tag{5}$$

where $d(\Omega)$ is the diameter of Ω.

Proof. By substituting x, y in place of r, s in (4) and using Hölder's inequality and the fact that $|x - y| \le d(\Omega)$ we see that the left-hand side of (5) is less than

$$d^p(\Omega) \int_0^1 I(t) \, dt,$$

where

$$I(t) = \int_\Omega \int_\Omega |Du(tx + (1-t)y)|^p \, dxdy.$$

Obviously, $I(t) = I(1-t)$, so that

$$\int_0^1 I(t) \, dt = 2 \int_{1/2}^1 I(t) \, dt.$$

To estimate $I(t)$ for $t \ge 1/2$, we integrate with respect to x for fixed y. As in the proof of Theorem 1 we observe that $w := tx$ runs through the set

$$t\Omega + (1-t)y = \{tz + (1-t)y : z \in \Omega\} \subset \Omega$$

and therefore

$$I(t) = t^{-d} \int_\Omega \Big(\int_{t\Omega + (1-t)y} |u_x(w)|^p \, dw \Big) \, dy$$

$$\le 2^d \int_\Omega \Big(\int_\Omega |u_x(x)|^p \, dx \Big) \, dy = 2^d |\Omega| \int_\Omega |u_x(x)|^p \, dx.$$

Now (5) follows immediately and the theorem is proved.

6. Remark. A more traditional albeit almost equivalent way to write (5) is the following

$$\int_\Omega |u - u_\Omega|^p \, dx \le N(d) d^p(\Omega) \int_\Omega |u_x|^p \, dx, \quad \text{where} \quad u_\Omega = \frac{1}{|\Omega|} \int_\Omega u \, dx. \tag{6}$$

This inequality follows from (5) since by Hölder's inequality

$$|u(x) - u_\Omega| = |\frac{1}{|\Omega|}\int_\Omega (u(x) - u(y))\,dy| \le \Big(\frac{1}{|\Omega|}\int_\Omega |u(x) - u(y)|^p\,dy\Big)^{1/p},$$

so that

$$\int_\Omega |u - u_\Omega|^p\,dx \le \frac{1}{|\Omega|}\int_\Omega\int_\Omega |u(x) - u(y)|^p\,dxdy.$$

On the other hand,

$$|u(x) - u(y)|^p = |(u(x) - u_\Omega) + (u_\Omega - u(y))|^p$$

$$\le 2^p(|u(x) - u_\Omega|^p + |u_\Omega - u(y)|^p),$$

so that in turn (6) implies (5) with a constant factor $N = N(p)$ on the right.

7. Exercise. Generalize Exercise 8.4.4. Let $k, r \in \{1, 2, ...\}$, $r \le k$, and let Ω be a domain in $\mathbb{R}^d$ of class C^k. Prove that for any $u \in \overset{0}{W}{}^r_p(\Omega)$ there exists a sequence $u_m \in C^k(\bar\Omega)$ of functions vanishing on $\partial\Omega$ and satisfying $u_m \to u$ in $W^r_p(\Omega)$.

8. Exercise. Prove that if $u \in W^1_d$, then $u \in \mathsf{VMO}\,(\mathbb{R}^d)$ and

$$[u]_{\mathsf{BMO}\,(\mathbb{R}^d)} \le N\|u_x\|_{\mathcal{L}_d}, \tag{7}$$

where N is independent of u and

$$[u]_{\mathsf{BMO}\,(\mathbb{R}^d)} := \sup_{B\in\mathbb{B}} \frac{1}{|B|}\int_B |u - u_B|\,dx \quad (= \sup_{\mathbb{R}^d} u^\sharp)$$

where $\mathbb{B}$ is the collection of balls in $\mathbb{R}^d$. The reader will find a generalization of this result for functions, having, say, a fractional number of derivatives, in Exercise 13.8.3.

9. Exercise. For $d = 2$ and the function $f(x) = \ln^{1/3}(|x| \wedge 1)$ prove that $f \in W^1_d$ and f is in VMO.

10. Exercise. Prove that if $\alpha \in (0, 1)$ and for any $u \in W^1_p$

$$[u]_{C^\alpha} \le N\|u_x\|_{\mathcal{L}_p}$$

with N independent of u, then $\alpha = 1 - d/p$.

3. The Gagliardo-Nirenberg theorem

We need the following generalization of Hölder's inequality.

1. Exercise*. Prove that, if Ω is a domain in $\mathbb{R}^d$ and $u_1, \dots, u_m$ are measurable functions on Ω and $p_1, \dots, p_m \in [1, \infty]$ satisfy

$$\frac{1}{p_1} + \dots + \frac{1}{p_m} = 1,$$

then

$$\|u_1 \cdot \dots \cdot u_m\|_{\mathcal{L}_1(\Omega)} \le \|u_1\|_{\mathcal{L}_{p_1}(\Omega)} \cdot \dots \cdot \|u_m\|_{\mathcal{L}_{p_m}(\Omega)}.$$

2. Theorem (Gagliardo-Nirenberg). *Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d_+$. Let $u \in W^1_1(\Omega)$. Then $u \in \mathcal{L}_{d/(d-1)}(\Omega)$ and*

$$\|u\|_{\mathcal{L}_{d/(d-1)}(\Omega)} \le \prod_{j=1}^{d} \|D_j u\|^{1/d}_{\mathcal{L}_1(\Omega)}. \tag{1}$$

Proof. It suffices to prove (1) for $C^1_0(\bar{\Omega})$ functions. First we consider the case $\Omega = \mathbb{R}^d$. Use induction on d. If $d = 1$, we have

$$u(t) = -\int_t^\infty u_x(s)\,ds, \quad |u(t)| \le \int_\Omega |u_x(s)|\,ds$$

and this implies (1). This also implies that for appropriate functions of many variables

$$|u(t,y)| \le \int_{(t,\infty)} |u_s(s,y)|\,ds \le \int_{\mathbb{R}} |u_s(s,y)|\,ds =: \phi(y). \tag{2}$$

Next if (1) holds for a $d \ge 1$, then by Hölder's inequality, for $\Omega = \mathbb{R}^{d+1}$,

$$\int_\Omega |u(x)|^{(d+1)/d}\,dx = \int_{\mathbb{R}} \Big(\int_{\mathbb{R}^d} |u(t,y)|^{1/d} |u(t,y)|\,dy \Big)\,dt$$

$$\le \int_{\mathbb{R}} \Big(\int_{\mathbb{R}^d} \phi^{1/d}(y) |u(t,y)|\,dy \Big)\,dt$$

$$\le \int_{\mathbb{R}} \Big[\Big(\int_{\mathbb{R}^d} \phi(y)\,dy \Big)^{1/d} \|u(t,\cdot)\|_{\mathcal{L}_{d/(d-1)}(\mathbb{R}^d)} \Big]\,dt$$

$$\le \Big(\int_{\mathbb{R}^d} \phi(y)\,dy \Big)^{1/d} \int_{\mathbb{R}} \prod_{j=2}^{d+1} \|D_j u(t,\cdot)\|^{1/d}_{\mathcal{L}_1(\mathbb{R}^d)}\,dt.$$

By using Exercise 1 with $m = d$ and $p_j = d$ to estimate the second factor and noting that

$$\int_{\mathbb{R}^d} \phi(y)\,dy = \int_{\mathbb{R}^{d+1}} |u_{x^1}|\,dx,$$

we come to (1) with $d+1$ in place of d. Hence (1) holds for all d.

In the case $\Omega = \mathbb{R}^d_+$ we do not even need induction. We can just use the above result and, while repeating the above computations, observe that in definition (2) of ϕ one can replace $\mathbb{R}$ with $\mathbb{R}_+$. The theorem is proved.

3. Exercise. Iterate (2) and prove that if $u \in W_1^d(\Omega)$, where $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d_+$, then $u \in C$ and with $\alpha = (1, ..., 1)$

$$\|u\|_C \leq \|D^\alpha u\|_{\mathcal{L}_1(\Omega)}.$$

4. General embedding theorems

First we generalize the Gagliardo-Nirenberg theorem. The following theorem is extended to H_p^γ spaces in Theorem 13.8.7. In Lemma 1 below we exclude the value $q = \infty$. This case is partially covered by Exercises 2.8 and 3.3 which show that

$$W_d^1 \subset \mathsf{VMO}\,(\mathbb{R}^d), \quad W_1^d \subset C.$$

1. Lemma. *Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d_+$ and take a $k \in \{1, 2, ...\}$. Let $p \in [1, \infty)$, $m \in \{0, ..., k\}$ and $q \in (0, \infty)$ satisfy*

$$k - \frac{d}{p} = m - \frac{d}{q}. \tag{1}$$

Then $q \geq p$ and for any $u \in W_p^k(\Omega)$ we have

$$[u]_{W_q^m(\Omega)} \leq N[u]_{W_p^k(\Omega)}, \tag{2}$$

with N independent of u. In particular, if

$$1 - \frac{d}{p} = -\frac{d}{q},$$

that is, (1) is satisfied with $k = 1$ and $m = 0$, then

$$\|u\|_{\mathcal{L}_q(\Omega)} \leq N\|u_x\|_{\mathcal{L}_p(\Omega)}. \tag{3}$$

Proof. The case $m = k$ is trivial since then $q = p$. Therefore we assume $m < k$, so that

$$\frac{d}{p} - \frac{d}{q} = k - m > 0 \quad \text{and} \quad q > p.$$

As usual, it suffices to prove (2) for $u \in C_0^k(\bar{\Omega})$. In that case we start with proving (3).

Observe that

$$1 - \frac{d}{p} = -\frac{d}{q} < 0$$

due to $q < \infty$, so that $p < d$, $d \geq 2$, and $q = pd/(d-p)$. Set

$$\gamma = q(d-1)/d = p(d-1)/(d-p)$$

and notice that $1 \leq \gamma < \infty$ (since $p \geq 1$). Introduce

$$v = u \quad \text{if} \quad \gamma = 1, \quad v = |u|^\gamma \quad \text{if} \quad \gamma > 1.$$

Since $u \in C_0^1(\bar{\Omega})$, we have $v \in C_0^1(\bar{\Omega})$ and $v \in W_1^1(\Omega)$. Next, use Theorem 3.2 and Hölder's inequality to conclude

$$\|u\|^\gamma_{\mathcal{L}_q(\Omega)} = \|v\|_{\mathcal{L}_{d/(d-1)}(\Omega)} \leq N\|v_x\|_{\mathcal{L}_1(\Omega)} = N\| \, |u|^{\gamma-1}u_x\|_{\mathcal{L}_1(\Omega)}$$

$$\leq N\| \, |u|^{\gamma-1}\|_{\mathcal{L}_{p/(p-1)}(\Omega)}\|u_x\|_{\mathcal{L}_p(\Omega)} = N\|u\|^{\gamma-1}_{\mathcal{L}_q(\Omega)}\|u_x\|_{\mathcal{L}_p(\Omega)}.$$

Upon cancelling $\|u\|^{\gamma-1}_{\mathcal{L}_q(\Omega)}$ ($< \infty$) in the inequality between the extreme terms, we get (3).

Now if $m = k - 1$, we obtain (1) by noticing that

$$1 - \frac{d}{p} = -\frac{d}{q},$$

so that by (3)

$$[u]_{W_q^m(\Omega)} = \sum_{|\alpha|=m} \|D^\alpha u\|_{\mathcal{L}_q(\Omega)} \leq N \sum_{|\alpha|=m} \|D^\alpha u_x\|_{\mathcal{L}_p(\Omega)}$$

$$\leq N \sum_{|\alpha|=k} \|D^\alpha u\|_{\mathcal{L}_p(\Omega)} \leq N[u]_{W_p^k(\Omega)}.$$

In the general case that $k \geq m + 2$ we fill the gap between m and k moving one step at a time by defining $q_m, \ldots, q_k \in (0, \infty)$ from

$$m - \frac{d}{q} = k - \frac{d}{p} = i - \frac{d}{q_i}, \quad i = m, \ldots, k,$$

and notice that, since $m \le i$, one can indeed find an appropriate $q_i \in (0,\infty)$, which by the beginning of the proof satisfies $q_i \ge p$ since $i \le k$. Obviously, $q_m = q$, $q_k = p$,

$$i + 1 - \frac{d}{q_{i+1}} = i - \frac{d}{q_i},$$

and by the above for $u \in C^k_0(\bar\Omega)$

$$[u]_{W^m_q(\Omega)} = [u]_{W^m_{q_m}(\Omega)} \le N[u]_{W^{m+1}_{q_{m+1}}(\Omega)} \le \dots \le N[u]_{W^{m+i}_{q_{m+i}}(\Omega)}$$

$$\le N[u]_{W^{m+i+1}_{q_{m+i+1}}(\Omega)} \le \dots \le N[u]_{W^k_{q_k}(\Omega)} = N[u]_{W^k_p(\Omega)}.$$

The lemma is proved.

2. Exercise. Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d_+$ and assume that (2) holds for any $u \in C^\infty_0(\Omega)$ and *some* k, m, p, q with a constant independent of u. Then prove that $k \ge m$ and (1) holds.

3. Exercise. Prove that (2) is false no matter which $k > m$ and p, q we take if $\Omega \in C^k$.

4. Theorem. *Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d_+$ or else $\Omega \in C^k$, where $k \in \{1, 2, ...\}$. Let $p \in [1, \infty)$, $m \in \{0, ..., k\}$ and $q \in [p, \infty)$ satisfy*

$$k - \frac{d}{p} \ge m - \frac{d}{q}. \tag{4}$$

Then $W^k_p(\Omega) \subset W^m_q(\Omega)$ in the sense that for any $u \in W^k_p(\Omega)$ we have

$$\|u\|_{W^m_q(\Omega)} \le N\|u\|_{W^k_p(\Omega)}, \tag{5}$$

with N independent of u.

Proof. As in the proof of the interpolation inequalities in Theorem 8.4.3, partitions of unity reduce the case of $\Omega \in C^k$ to $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d_+$.

In that case first assume that (1) holds and observe that, for $i = 0, ..., m$, we have

$$i - \frac{d}{q} = i + (k - m) - \frac{d}{p},$$

so that by Lemma 1

$$\|D^i u\|_{\mathcal{L}_q(\Omega)} \le N\|D^{i+k-m}u\|_{\mathcal{L}_p(\Omega)} \le N\|u\|_{W^k_p(\Omega)}.$$

By summing up these relations over $i = 0, ..., m$, we get (5).

In the remaining case

$$k - \frac{d}{p} > m - \frac{d}{q},$$

and we can find an integer $\hat{m} \geq m$ such that

$$k - \frac{d}{p} > \hat{m} - \frac{d}{q} \geq k - 1 - \frac{d}{p}. \tag{6}$$

Since $q \geq p$, the left inequality in (6) implies that $\hat{m} < k$, $\hat{m} \leq k - 1$, and we can find a $\hat{p} \in (0, \infty)$ such that

$$\hat{m} - \frac{d}{q} = k - 1 - \frac{d}{\hat{p}}.$$

Actually, the right inequality in (6) implies that $\hat{p} \geq p$.

By the first part of the proof

$$\|u\|_{W^m_q(\Omega)} \leq \|u\|_{W^{\hat{m}}_q(\Omega)} \leq N\|u\|_{W^{k-1}_{\hat{p}}(\Omega)}. \tag{7}$$

Furthermore, by (6)

$$k - 1 - \frac{d}{\hat{p}} = \hat{m} - \frac{d}{q} < k - \frac{d}{p}, \quad -\frac{d}{\hat{p}} < 1 - \frac{d}{p},$$

which (recall that $\hat{p} \geq p$) by Exercise 0.1 implies that for $i = 0, ..., k - 1$

$$\|D^i u\|_{\mathcal{L}_{\hat{p}}(\Omega)} \leq N\|D^i u\|_{W^1_p(\Omega)} \leq N\|u\|_{W^k_p(\Omega)}.$$

By summing up these inequalities with respect to $i = 0, ..., k - 1$ and taking into account (7), we came to (5) and the theorem is proved.

5. Corollary. *Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d_+$ or else $\Omega \in C^k$, where $k \in \{1, 2, ...\}$. Let $p \in [1, \infty)$ and assume that $pk \geq d$. Then for any $u \in W^k_p(\Omega)$ we have $u \in \mathcal{L}_q(\Omega)$ for any $q \in [p, \infty)$. Furthermore,*

$$\|u\|_{\mathcal{L}_q(\Omega)} \leq N\|u\|_{W^k_p(\Omega)},$$

where N is independent of u.

Indeed, $k - d/p \geq 0 > -d/q$ for any $q \geq p$.

For $pk \geq d$ complementary information to Corollary 5 is contained in the following.

6. Exercise. Under the conditions of Corollary 5 prove that if $u \in W^k_p(\Omega)$, then $u \in W^1_d(\Omega)$ and, if $\Omega = \mathbb{R}^d$, then $u \in \mathrm{VMO}(\mathbb{R}^d)$.

7. Remark. If $\Omega \in C^k$, then one can replace the assumption $q \geq p$ in Theorem 4 with a weaker one, i.e., $q \geq 1$.

Indeed, if $k - d/p < m$, then q satisfying (4) is less than $q > 0$ satisfying (1) and we can use $W_q^m(\Omega) \supset W_r^m(\Omega)$ for $r \geq q$. In the remaining case, $(k-m)p \geq d$ and

$$D^\alpha u \in W_p^{k-m}(\Omega) \quad \text{if} \quad |\alpha| \leq m \quad \text{and} \quad u \in W_p^k(\Omega),$$

which by Corollary 5 implies that $D^\alpha u \in \mathcal{L}_q(\Omega)$ if $|\alpha| \leq m$ for all large and consequently all $q \geq 1$. The reader will easily complete the argument and obtain (5).

The following exercise shows that one cannot relax the conditions imposed on m, k, q, p in Theorem 4. In particular, the assertion of Remark 7 is false if $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d_+$.

8. Exercise. Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d_+$ and assume that for *some* p, q, m, k inequality (5) holds for any $u \in W_p^k(\Omega)$ with N independent of u. Then prove that $m \leq k$, $q \geq p$, and (4) holds.

It turns out that if the inequality in (4) is strict, then $u(x + \cdot)$ is a Hölder continuous $W_p^m(\Omega)$-valued function in the following situation.

9. Exercise. Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d_+$ and let $1 \leq p \leq q < \infty$, $p < d$, and

$$1 - \frac{d}{p} \geq -\frac{d}{q}, \quad \alpha := 1 - \frac{d}{p} + \frac{d}{q}.$$

Then prove that there exists a constant $N = N(d, q, p)$ such that for any $u \in W_p^1(\Omega)$ and $x \in \mathbb{R}^d_+$ we have

$$\|u(x + \cdot) - u\|_{\mathcal{L}_q(\Omega)} \leq N|x|^\alpha \|u_x\|_{\mathcal{L}_p(\Omega)}.$$

According to Exercise 8 the embedding of W_p^k into all $\mathcal{L}_q$ is impossible if $pk < d$ and by Remark 7 this embedding holds if $pk \geq d$. For $pk > d$ this result admits the following important generalization, which for $k = 1$ is just Morrey's theorem.

10. Theorem. *Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d_+$ or else $\Omega \in C^k$, where $k \in \{1, 2, ...\}$. Assume that*

$$p > 1, \quad k - \frac{d}{p} > 0, \quad \textit{and} \quad k - \frac{d}{p} \quad \textit{is not an integer.}$$

Then

$$W_p^k(\Omega) \subset C^{k-d/p}(\Omega)$$

in the sense that any $u \in W_p^k(\Omega)$ has a continuous modification denoted again by u and such that $u \in C^{k-d/p}(\bar{\Omega})$ and

$$\|u\|_{C^{k-d/p}(\Omega)} \leq N\|u\|_{W_p^k(\Omega)} \tag{8}$$

with N independent of u.

Proof. As before we concentrate on the cases in which $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}_+^d$. It also suffices to consider $u \in C^k(\Omega) \cap W_p^k(\Omega)$. If $m \in \{0, 1, ..., k-1\}$ and $\alpha \in (0, 1)$, then by Morrey's theorem the $C^\alpha(\Omega)$ norm of $D^m u$ is dominated by the $\mathcal{L}_q(\Omega)$ norm of $D^{m+1}u$ for q defined by

$$1 - \frac{d}{q} = \alpha.$$

It follows that

$$\|u\|_{C^{m+\alpha}(\Omega)} \leq N\|u\|_{W_q^{m+1}(\Omega)}. \tag{9}$$

By Theorem 4 the right-hand side of (9) is dominated by the right-hand side of (8) if for an integer $k \geq m+1$ and $p \in [1, \infty)$ we have

$$m + 1 - \frac{d}{q} = k - \frac{d}{p}. \tag{10}$$

Here $m + 1 - d/q = m + \alpha$, so that

$$m + \alpha = k - \frac{d}{p}, \tag{11}$$

and this suggests to us the way to organize the rest of the proof.

First we find an integer m and $\alpha \in (0, 1)$ satisfying (11). This is possible since $k - d/p$ is positive and not an integer. Then we find a $q \in (0, \infty)$ satisfying (10). This is possible since $m + 1 > k - d/p$. By comparing (10) and (11), we see that $\alpha = 1 - d/q$.

Next, we notice that

$$m < k - \frac{d}{p} < k \quad \text{and} \quad m + 1 \leq k,$$

so that Theorem 4 is applicable with $m+1$ in place of m. Upon combining Theorem 4 and (9), we come to (8). The theorem is proved.

11. Remark. Formally the above proof goes the same way even if $p = 1$. The only trouble is that in this case the assumption that $k - d/p$ is not an integer fails to hold. We discuss what happens if $k - d/p$ is an integer in Exercise 12.10.6.

12. Remark. Sometimes the assumption that $k - d/p$ is not an integer turns unpleasant. Therefore it is worth mentioning that if $p \geq 1$ and $\Omega \in C^k$, then

$$k - \frac{d}{p} > 0 \implies W^k_p(\Omega) \subset C^{k-d/p-\varepsilon}(\Omega) \quad \forall \varepsilon \in (0, k - d/p].$$

Indeed if $p > 1$, one can always find an $r \in (1, p)$ such that $k - d/r > k - d/p - \varepsilon \geq 0$ and $k - d/r$ is not an integer. Then

$$W^k_p(\Omega) \subset W^k_r(\Omega) \subset C^{k-d/r}(\Omega) \subset C^{k-d/p-\varepsilon}(\Omega).$$

If $p = 1$, then $W^k_p(\Omega) \subset W^{k-d}_q(\Omega)$ for any $q \geq p = 1$ by Theorem 4 and

$$W^{k-d}_q(\Omega) \subset C^{k-d-d/q}(\Omega)$$

by Theorem 10 if q is large enough.

The same conclusion also holds if $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d_+$. Indeed, in those cases it suffices to concentrate on the intersections of unit balls with Ω, cut off the functions outside of these intersections and notice that

$$\frac{|f(x) - f(y)|}{|x - y|^\alpha} \leq 2 \sup |f|$$

if $|x - y| \geq 1$.

13. Remark. Theorem 10 obviously implies that, if $\Omega \in C^1$, $p > d$, and $u \in \overset{0}{W}{}^1_p(\Omega)$, then the continuous modification of u vanishes on $\partial\Omega$.

It follows from Remark 11.7.9 about the traces of continuous functions in $\overset{0}{W}{}^1_p(\Omega)$ that the above conditions can be relaxed to

$$\Omega \in C^1 \quad \text{and} \quad u \in \overset{0}{W}{}^1_p(\Omega) \cap C(\bar{\Omega}).$$

14. Remark. If $\Omega \in C^k$, then obviously the constants N in Theorems 4 and 10 depend only on $k, m, K_0, \rho_0, d, \operatorname{diam} \Omega$, and p.

15. Remark. The Gagliardo-Nirenberg theorem is true for bounded Lipschitz graph domains due to the fact that $u(x^1 + \psi(x'), x')$ is in $W_1^1(\mathbb{R}^d_+)$ if $u \in W_1^1(\{x^1 > \psi\})$ and ψ satisfies the Lipschitz condition. By Remark 2.4 Morrey's theorem also holds for such domains. Since as above we may fill the gap in the number of derivatives with one at a time, Exercise 1.8.8 characterizing the space $W_p^k(\Omega)$ as the space of functions in $\mathcal{L}_p(\Omega)$ having the generalized derivatives in $\mathcal{L}_p(\Omega)$ shows that Theorems 4 and 10 are actually true for bounded Lipschitz graph domains. We will not use this fact.

5. Compactness of embeddings. Kondrashov's theorem

Condition (4.4) guarantees the embedding of $W_p^k(\Omega)$ into $W_q^m(\Omega)$. There is an important case when the inequality in (4.4) is *strict*. Then it turns out that, if Ω is bounded, then $W_p^k(\Omega)$ is *compactly embedded* into $W_q^m(\Omega)$. The following theorem, called *Kondrashov's theorem*, is most frequently used for $p = q$, $k = 1$, and $m = 0$. It also plays an important role in many other situations (see Exercises 2, 3, and 11.1.10), in particular, in applications to nonlinear equations of which we give an instance in Exercise 11.3.5. In the future the results of this section are only used in unstarred exercises.

1. Theorem. *Let $p \in [1,\infty)$, $k \in \{1,2,...\}$, $m \in \{0,1,...,k-1\}$, $\Omega \in C^k$, and let U be a bounded subset of $W_p^k(\Omega)$. Then U is precompact in $W_q^m(\Omega)$ for any $q \in [1,\infty)$ satisfying the strict inequality in* (4.4)*:*

$$k - \frac{d}{p} > m - \frac{d}{q},$$

so that q is just any number in $[1,\infty)$ if $p(k-m) \geq d$.

Proof. From the definition of the norm in $W_q^m(\Omega)$ it follows that it suffices to prove that the sets

$$U_\alpha := \{D^\alpha u : u \in U\}$$

are precompact in $\mathcal{L}_q(\Omega)$ whenever $|\alpha| \leq m$. In addition, the U_α are bounded subsets of $W_p^{k-m}(\Omega)$. This reduces the case of general m to $m = 0$, the only one we concentrate on in the remaining part of the proof. In particular we have

$$k - \frac{d}{p} > -\frac{d}{q}.$$

Now we reduce general k to $k = 1$. It turns out that there is an $s \in [1,\infty)$ satisfying

$$k - \frac{d}{p} \geq 1 - \frac{d}{s} > -\frac{d}{q}.$$

Indeed, if $k - d/p \geq 1$, then one can take any $s \in [d, \infty)$. However, if $k - d/p < 1$, then we have

$$1 - d \leq k - d \leq k - \frac{d}{p} < 1$$

and one can find $s \in [1, \infty)$ such that $k - d/p = 1 - d/s$ in which case

$$1 - \frac{d}{s} = k - \frac{d}{p} > -\frac{d}{q}.$$

Furthermore, by Remark 4.7 sets bounded in $W_p^k(\Omega)$ are also bounded in $W_s^1(\Omega)$ if

$$k - \frac{d}{p} \geq 1 - \frac{d}{s} \quad \text{and} \quad s \in [1, \infty).$$

It follows that without loss of generality we may assume that U is a bounded set in $W_p^1(\Omega)$ and

$$1 - \frac{d}{p} > -\frac{d}{q}.$$

After these reductions we recall again that $U \subset \mathcal{L}_q(\Omega)$ by Remark 4.7. Next, we need to show that for any $\varepsilon > 0$ there exist finitely many functions

$$f_1, \ldots, f_n \in \mathcal{L}_q(\Omega)$$

which constitute an ε-net in $\mathcal{L}_q(\Omega)$ for U. This is equivalent to showing that for any $\varepsilon > 0$ there exists a precompact set

$$U_\varepsilon \subset \mathcal{L}_q(\Omega) \quad \text{with} \quad \operatorname{dist}_q(U, U_\varepsilon) \leq \varepsilon,$$

that is, such that for any $u \in U$ there is a $v \in U_\varepsilon$ with the property

$$\|u - v\|_{\mathcal{L}_q(\Omega)} \leq \varepsilon.$$

Now we consider the case where $q = p$, which is the case most frequently used in applications. First we assume that all elements of U vanish near $\partial\Omega$, take a nonnegative $\zeta \in C_0^\infty$ with unit integral, and use the notation $u^{(\varepsilon)}$ introduced in (1.8.4). Observe that by Corollary 9.1.3 we have

$$\|u^{(\varepsilon)} - u\|_{\mathcal{L}_p} \leq N\varepsilon,$$

where $N = N(U)$. On the other hand, for any $\varepsilon > 0$ any derivative of $u^{(\varepsilon)}$ is bounded uniformly with respect to $u \in U$ by virtue of (1.8.5) and Hölder's inequality. Hence by Arzelà's theorem

$$U_\varepsilon := \{u^{(\varepsilon)} : u \in U\}$$

is a precompact set in $C(\Omega)$ and therefore in $\mathcal{L}_p(\Omega)$. We saw above that $\text{dist}_p(U, U_\varepsilon) \le N\varepsilon$ and this shows that U is precompact in $\mathcal{L}_p(\Omega)$ if all elements of U vanish near $\partial\Omega$.

In the general case again let $q = p$ and take $\varepsilon > 0$ and a function $\zeta \in C_0^\infty(\Omega)$ such that

$$\|1 - \zeta\|_{\mathcal{L}_r(\Omega)} \le \varepsilon,$$

where $r = ps/(s-p)$ and s is any number such that

$$1 - \frac{d}{p} > -\frac{d}{s} \quad \text{and} \quad s > p.$$

Then by Hölder's inequality and Theorem 4.4, for any $u \in U$,

$$\|u - u\zeta\|_{\mathcal{L}_p(\Omega)} \le \|u\|_{\mathcal{L}_s(\Omega)} \|1 - \zeta\|_{\mathcal{L}_r(\Omega)} \le N\varepsilon \|u\|_{W_p^1(\Omega)} \le N_1\varepsilon$$

with N_1 independent of $u \in U$. Since the set

$$\{u\zeta : u \in U\}$$

is precompact in $\mathcal{L}_p(\Omega)$ by the above, it follows that U is precompact in $\mathcal{L}_p(\Omega)$ as well. This finishes proving the theorem in the case where $q = p$.

For bounded domains, owing to Hölder's inequality, sets precompact in $\mathcal{L}_p(\Omega)$ are precompact in any $\mathcal{L}_q(\Omega)$ with $q \in [1, p]$. Therefore, in the rest of the proof we assume that

$$q > p.$$

Furthermore, by the above, U is precompact in $\mathcal{L}_p$ and as we know, an ε-net in $\mathcal{L}_p$ for U can always be composed of finitely many elements $u_i, i = 1, ..., n(\varepsilon)$, of U. In addition, if there is a constant M such that $|u| \le M$ for all $u \in U$, then the inequality

$$|v - u_i|^q \le |v - u_i|^p (\sup|v| + \sup|u_i|)^{q-p}$$

shows that $u_i, i = 1, ..., n(\varepsilon)$, is a $(2^{q-p}\varepsilon^p M^{q-p})^{1/q}$-net for U in $\mathcal{L}_q$. It follows that U is precompact in $\mathcal{L}_q$ if all its elements are bounded by the same constant.

Now define

$$\chi_m(t) = (-m) \vee t \wedge m$$

and notice that $|\chi_m| \le m$ and by Exercise 1.3.18 we have

$$\|\chi_m(u)\|_{W_p^1(\Omega)} \le \|u\|_{W_p^1(\Omega)},$$

so that

$$U^m := \{\chi_m(u) : u \in U\}$$

is precompact in $\mathcal{L}_p(\Omega)$ and in $\mathcal{L}_q(\Omega)$ by the above. It only remains to observe that

$$\operatorname{dist}_q(U, U^m) \le N m^{1-r/q},$$

where $r > q$ is any number satisfying $d/p - 1 < d/r$ since

$$\|u - \chi_m(u)\|^q_{\mathcal{L}_q(\Omega)} \le \int_\Omega I_{|u|\ge m}|u|^q\,dx \le m^{q-r}\int_\Omega |u|^r\,dx \le N m^{q-r}\|u\|^r_{W_p^1(\Omega)}.$$

This proves the theorem.

2. Exercise (Poincaré's inequality). Let

$$p, q \in [1, \infty), \quad \frac{d}{q} > \frac{d}{p} - 1, \quad \Omega \in C^1.$$

Arguing by contradiction, prove that there exists a constant N such that for any $u \in W_p^1(\Omega)$ we have

$$\int_\Omega |u - u_\Omega|^q\,dx \le N\|u_x\|^q_{\mathcal{L}_p(\Omega)} \quad \text{where} \quad u_\Omega = \frac{1}{|\Omega|}\int_\Omega u\,dx. \tag{1}$$

3. Exercise (Sobolev-Poincaré inequality). Let

$$p \in [1, d), \quad \frac{d}{q} = \frac{d}{p} - 1, \quad \Omega \in C^1.$$

Prove that there exists a constant N such that for any $u \in W_p^1(\Omega)$ estimate (1) holds again.

4. Exercise. Give an example showing that the assertion of Theorem 1 is false if $\Omega = \mathbb{R}^d$.

6. An application of Riesz's theory of compact operators

If L is an operator satisfying Assumption 1.6.1, $p \in (1, \infty)$, and $\Omega \in C^2$, then by Theorems 8.5.3 and 8.5.6 there exists a $\lambda_1 \geq 1$ such that for $\lambda = \lambda_1$ the mapping

$$\lambda - L : \overset{0}{W}{}^2_p(\Omega) \to \mathcal{L}_p(\Omega)$$

is a one-to-one onto mapping with bounded inverse. Denote the inverse by R and consider R as an operator in $\mathcal{L}_p(\Omega)$. Below in this section it is convenient to allow our functions to be complex valued.

Observe that, owing to Theorems 5.1 and 8.5.6, the mapping

$$R : \mathcal{L}_p(\Omega) \to \overset{0}{W}{}^2_p(\Omega) \subset \mathcal{L}_p(\Omega)$$

is compact as a mapping in $\mathcal{L}_p(\Omega)$. By Riesz's theory of compact operators (see [**4**]), there could only exist countably many distinct complex ξ_n, $n = 0, 1, 2, \ldots$, for which the operator

$$\xi_n - R$$

does not have a bounded inverse as an operator from $\mathcal{L}_p(\Omega)$ *onto* $\mathcal{L}_p(\Omega)$. Furthermore, since

$$R^{-1} = \lambda - L$$

is an unbounded operator in $\mathcal{L}_p(\Omega)$, there are indeed infinitely many ξ_n's and

$$\xi_n \to 0 \quad \text{as} \quad n \to \infty.$$

By the way, one of the ξ_n's equals zero since the inverse to R is $\lambda - L$ which is not bounded as an operator in $\mathcal{L}_p(\Omega)$. We renumber ξ_n if needed to have $\xi_0 = 0$.

In addition for each $\xi_n \neq 0$ there exists a $u_n \in \mathcal{L}_p(\Omega)$, $u_n \neq 0$ (maybe not unique even up to a constant factor), such that

$$Ru_n = \xi_n u_n.$$

Thus, for $\xi \notin \{\xi_0, \xi_1, \ldots\}$, the equation

$$\xi u - Ru = v \tag{1}$$

is uniquely solvable in $\mathcal{L}_p(\Omega)$ for any $v \in \mathcal{L}_p(\Omega)$ and

$$\|u\|_{\mathcal{L}_p(\Omega)} \leq N \|v\|_{\mathcal{L}_p(\Omega)}. \tag{2}$$

where N is independent of v. If we take $v = \xi Rf$ with $f \in \mathcal{L}_p(\Omega)$ and define $z = \lambda - \xi^{-1}$ (notice $\xi \neq 0$), then

$$u = Rf + (\lambda - z)Ru, \quad u \in \overset{0}{W}{}_p^2(\Omega), \quad \lambda u - Lu = f + (\lambda - z)u,$$

$$zu - Lu = f. \tag{3}$$

This yields the solvability of (3) in $\overset{0}{W}{}_p^2(\Omega)$. Furthermore, the boundedness of $R : \mathcal{L}_p(\Omega) \to W_p^2(\Omega)$ and (2) imply that

$$\|u\|_{W_p^2(\Omega)} = \|R[f + (\lambda - z)u]\|_{W_p^2(\Omega)}$$

$$\leq N\|f\|_{\mathcal{L}_p(\Omega)} + N\|u\|_{\mathcal{L}_p(\Omega)} \leq N\|f\|_{\mathcal{L}_p(\Omega)},$$

where the constants N are independent of f.

In addition, one can go back from (3) to (1) and then one sees that the solution of (3) in $\overset{0}{W}{}_p^2(\Omega)$ is unique.

Hence, upon defining

$$z_n = \lambda - \xi_n^{-1}, \quad n \geq 1, \quad \sigma = \sigma(L, \Omega) = \{z_n : n \geq 1\},$$

we conclude that $|z_n| \to \infty$ and for every $z \notin \sigma$ the mapping

$$z - L : \overset{0}{W}{}_p^2(\Omega) \to \mathcal{L}_p(\Omega)$$

is a one-to-one onto mapping with bounded inverse. The discrete set σ is called *the spectrum* of L.

Next, for $n \geq 1$, the relations

$$R\mathcal{L}_p(\Omega) = \overset{0}{W}{}_p^2(\Omega) \quad \text{and} \quad u_n = \xi_n^{-1} Ru_n$$

imply that

$$u_n \in \overset{0}{W}{}_p^2(\Omega), \quad (\lambda - L)u_n = \xi_n^{-1} u_n \quad \text{and} \quad z_n u_n = Lu_n.$$

Thus, for $z_n \in \sigma$ there are nonzero solutions $u_n \in \overset{0}{W}{}_p^2(\Omega)$ of $z_n u_n = Lu_n$. These are called *the eigenfunctions* of L corresponding to *the eigenvalues* z_n of L.

1. Exercise. Formally the spectrum $\sigma = \sigma(L, \Omega)$ introduced above may also depend on p. Let us write $\sigma = \sigma(p)$. Prove that, if $u \in \overset{0}{W}{}^2_p(\Omega)$, $u \neq 0$, z is a complex number and $zu = Lu$, then

(i) $z \in \sigma(p)$;

(ii) $u \in \overset{0}{W}{}^2_q(\Omega)$ for any $q \in (1, \infty)$;

(iii) conclude that $z \in \sigma(q)$ and $\sigma(p)$ is independent of p.

7. Hints to exercises

1.5. One can replace r and s in (1.2) with $r - \varepsilon z$ and $s - \varepsilon z$, where $\varepsilon > 0$ and $|z| \leq 1$, provided that $B_\rho(x - \varepsilon z), B_\rho(y - \varepsilon z) \subset \Omega$. By multiplying the resulting inequality by $\zeta(z)$ and integrating with respect to z, prove that $u^{(\varepsilon)}$ satisfy (1.2) in a smaller domain and hence (1.3) and (1.4) hold for $u^{(\varepsilon)}$ in a smaller domain. Then use Arzelà's theorem.

1.7. Observe that $|u(r) - u(s)| \leq |u(r) - c| + |u(s) - c|$ for any constant c and that the left-hand side of (1.2) increases if we take domains larger than $B_\rho(x)$ and $B_\rho(y)$, say containing both. Also take into account Exercise 1.5.

2.7. The cases (i) $k = 1$, (ii) $k = 2$, $r = 1$, (iii) $k, r \geq 2$ are different. In the third one first consider half spaces on the basis of the equation $\Delta u - u = f$. You also might like using Exercise 8.1.8 to show that smooth functions from $\overset{0}{W}{}^r_p(\Omega)$ vanish on the boundary.

2.8. Estimate (2.7) and the denseness of C_0^∞ in W_d^1 imply that for any $\varepsilon > 0$ and $u \in W_d^1$ there is a $v \in C_0^\infty$ such that $[u - v]_{\mathrm{BMO}(\mathbb{R}^d)} \leq \varepsilon$. Then certainly $v \in \mathrm{VMO}$ and

$$\varlimsup_{\rho \downarrow 0} \sup_x \fint_{B_\rho(x)} |u(y) - v(y) - (u - v)_{B_\rho(x)}| \, dy \leq \varepsilon.$$

2.9. Use dilations.

4.3. The set Ω is bounded.

4.8. Use dilations.

4.9. Check that the assertion holds with $q = q_0$ and $q = q_1$, where $q_0 = p$ and q_1 is such that $1 - d/p = -d/q_1$. You may also like noting that it suffices to consider $|x| = 1$.

5.2. One may assume that $u_\Omega = 0$ and $q \geq p$. After that assume that the estimate is false and find u_n such that $\|u_{nx}\|_{\mathcal{L}_p(\Omega)} \to 0$ and $\|u_n\|_{\mathcal{L}_q(\Omega)} = 1$.

5.3. Use Theorem 4.4 for $u - u_\Omega$ and then Exercise 5.2 with $q = p$.

6.1. (i) Is $z - L$ invertible? (ii) Write $\lambda u - Lu = (\lambda - z)u$ and notice that by embedding theorems $(\lambda - z)u \in \mathcal{L}_r(\Omega)$ with $r > p$. Do not forget to use the uniqueness in $\overset{0}{W}{}^2_p(\Omega) \supset \overset{0}{W}{}^2_r(\Omega)$.

Chapter 11

Second-order elliptic equations $Lu - \lambda u = f$ with λ small

Throughout this chapter we deal with operators of the type

$$L = a^{ij}D_{ij} + b^iD_i + c$$

satisfying Assumption 1.6.1 unless explicitly stated otherwise. Recall that one of the requirements of Assumption 1.6.1 is that a^{ij} be uniformly continuous on $\mathbb{R}^d$. We take a

$$p \in (1, \infty).$$

The goal here is to show that elliptic equations $Lu - \lambda u = f$ admit solutions in $\overset{\circ}{W}{}^2_p(\Omega)$ as long as

(i) $L1 \le 0$ and $\lambda = 0$ if Ω is bounded or

(ii) $L1 \le 0$ and the constant $\lambda > 0$ if $\Omega = \mathbb{R}^d$.

The method we follow is based on the maximum principle and embedding theorems. An alternative way in the case of bounded Ω based on the compactness of embeddings of W^k_p into $\mathcal{L}_q$ is outlined in Exercises 1.7 and 1.9.

In the main text the second-order coefficients are assumed to be uniformly continuous if $d \ge 2$. Two-dimensional equations with measurable coefficients are treated in Exercises 5.5 and 6.5.

1. Maximum principle for smooth functions

1:1. Maximum principle. At first sight the maximum principle looks quite modest. However, it turns out that this is one of the most powerful tools of the theory of second-order elliptic equations, linear or nonlinear. In these lectures we will only see its application to proving a priori estimates for equations with $L1 \le 0$.

Observe that the continuity of a^{ij} is not used in this section. On the other hand we assume that

$$L1 = c \le 0.$$

Recall that for a number a

$$a_+ = \frac{1}{2}(|a| + a), \quad a_- = \frac{1}{2}(|a| - a).$$

In the following lemma we use the simple fact that given two symmetric $d \times d$ nonnegative matrices A and B, we have $\operatorname{tr} AB \ge 0$. One can prove this fact by making one of the matrices diagonal.

1. Lemma. *Let Ω be a bounded domain in $\mathbb{R}^d$ and let $u \in C^2_{loc}(\Omega) \cap C(\bar\Omega)$. Assume that $L1 < 0$ and $Lu \ge 0$ in Ω. Then in Ω we have*

$$u \le \max_{\partial\Omega} u_+. \tag{1}$$

Proof. Assume the contrary. Then at some point $x_0 \in \Omega$ we have $u(x_0) = \max_\Omega u(x) > 0$. At this point the matrix u_{xx} is symmetric and nonpositive and $u_x = 0$. Therefore,

$$Lu(x_0) = \operatorname{tr} a(x_0)u_{xx}(x_0) + b(x_0) \cdot u_x(x_0) + c(x_0)u(x_0) \le c(x_0)u(x_0) < 0,$$

which contradicts $Lu \ge 0$, and the lemma is proved.

To generalize Lemma 1 to the case $c \le 0$, we need the so-called *global barriers*.

2. Lemma. *Let $c \le 0$. Then for any $R > 0$ there exists a constant $\mu > 0$, depending only on κ, and $\sup_x |b|$, such that the infinitely differentiable function*

$$v_0(x) = \cosh(\mu R) - \cosh(\mu|x|)$$

satisfies $Lv_0 \le -1$ and $v_0 > 0$ in B_R and $v_0 = 0$ on ∂B_R.

Proof. That v_0 is infinitely differentiable follows from Taylor's formula. Denote $v = \cosh(\mu|x|)$ and $\xi = x|x|^{-1}$ and notice that

$$|\xi| = 1, \quad \operatorname{tr} a \geq a^{ij}\xi^i\xi^j, \quad |\sinh t| \leq \cosh t, \quad v \in C^2(\bar{B}_R).$$

Then for $x \neq 0$ we obtain

$$(L - c)v = \mu^2 \cosh(\mu|x|)a^{ij}\xi^i\xi^j + \mu|x|^{-1}\sinh(\mu|x|)(\operatorname{tr} a - a^{ij}\xi^i\xi^j)$$

$$+ \mu \sinh(\mu|x|)b^i\xi^i \geq \cosh(\mu|x|)(\mu^2\kappa - \mu N) \geq 1$$

under an appropriate choice of $\mu = \mu(\kappa, \sup_x |b|) > 0$. The above inequalities also hold for $x = 0$ since v and its derivatives are continuous. Due to the fact that $c \leq 0$, we have

$$Lv_0 \leq (L - c)v_0 = -(L - c)v \leq -1$$

in B_R, and the lemma is proved.

3. Theorem (Maximum principle). *Let Ω be a bounded domain, $c \leq 0$. If $u \in C^2_{loc}(\Omega) \cap C(\bar{\Omega})$ and $Lu \geq 0$ in Ω, then* (1) *holds in Ω.*

Proof. Without loss of generality we may assume that $0 \in \Omega$. Also notice that, since $c \leq 0$, for any *nonnegative* constant γ we have $L(u - \gamma) = Lu - c\gamma \geq 0$. We can take $\gamma = \max_{\partial\Omega} u$, and then we see that in the remaining part of the proof we may assume that $u \leq 0$ on $\partial\Omega$.

Now set $R = 2\operatorname{diam}\Omega$, and take the function v_0 from Lemma 2. Also define the operator

$$L'w = L(v_0 w).$$

Then L' is an elliptic operator and $L'1 \leq -1$. Also the function $\bar{u} := u/v_0$ is continuous in $\bar{\Omega}$ and

$$L'\bar{u} = Lu \geq 0$$

in Ω and $\bar{u} \leq 0$ on $\partial\Omega$. By Lemma 1 we have $\bar{u} \leq 0$ and $u \leq 0$ in Ω. The theorem is proved.

The following corollary explains why increasing λ does not affect N in estimates of the kind

$$\|u\|_{\mathcal{L}_p} \leq N\|\lambda u - Lu\|_{\mathcal{L}_p}.$$

It will be used in our investigation of equations in $\mathbb{R}^2$ with λ in place of $\lambda(a^{11} + a^{22})$.

4. Corollary. *Let Ω be a bounded domain, $c \le 0$, and let α be a nonnegative function on Ω. Let $u, v \in C^2_{loc}(\Omega) \cap C(\bar\Omega)$, $u = v = 0$ on $\partial\Omega$. Introduce,*

$$f = \alpha u - Lu, \quad h = -Lv$$

and assume that

$$f_+ \le h$$

in Ω. Then $u_+ \le v$ in $\bar\Omega$. In particular, if $g \ge 0$ (typically, $g = 0$) and

$$|f| + g = h$$

in Ω, then $|u| \le v$ in $\bar\Omega$, so that, for any constant N_1,

$$\|v\|_{\mathcal{L}_p(\Omega)} \le N_1 \|h\|_{\mathcal{L}_p(\Omega)} \Longrightarrow \|u\|_{\mathcal{L}_p(\Omega)} \le N_1 \|f\|_{\mathcal{L}_p(\Omega)} + N_1 \|g\|_{\mathcal{L}_p(\Omega)}.$$

Indeed, Theorem 3 implies that $v \ge 0$,

$$\alpha v - Lv \ge \alpha u - Lu, \quad (\alpha - L)(v - u) \ge 0, \quad v \ge u, \quad v \ge u_+.$$

1:2. Exercises (optional).

5. Exercise. Let Ω be a bounded domain, $c \le -\delta$, where $\delta \ge 0$ is a constant, and let $\lambda > -\delta$. Prove that, if $u \in C^2_{loc}(\Omega) \cap C(\bar\Omega)$, then in Ω

$$u \le \frac{1}{\lambda + \delta} \sup_{\Omega}(\lambda u - Lu)_+ + \max_{\partial\Omega} u_+, \quad |u| \le \frac{1}{\lambda + \delta} \sup_{\Omega} |\lambda u - Lu| + \max_{\partial\Omega} |u|.$$

In Theorem 8.1 we extend the maximum principle to W^2_p functions and operators with continuous leading coefficients. It turns out that if $p \ge d$, then the maximum principle also holds without the assumption of continuity. However, if $p < d$ and L has a discontinuity at just one point, the maximum principle can fail for W^2_p functions. In connection with this we give the reader the following exercise.

6. Exercise. Let $d \ge 2$, let $\gamma > d - 2$ be a parameter, and let

$$a^{ij}(x) = \delta^{ij} + \gamma |x|^{-2} x^i x^j$$

for $x \ne 0$ and $a^{ij}(0) = \delta^{ij}$. Let $\Omega = B_1$ and

$$u(x) = 1 - |x|^{\beta},$$

where $\beta = (2 + \gamma - d)(1 + \gamma)^{-1} > 0$. By identifying $D_{ij} u$, prove that $u \in \overset{\circ}{W}{}^2_p(\Omega)$ if

$$1 \le p < \frac{d + \gamma d}{d + \gamma} \tag{2}$$

and $a^{ij} D_{ij} u = 0$ in Ω. Notice that $u(0) = 1$, which is larger than the value of u on $\partial\Omega$. Also notice that the right-hand side of (2) tends to d as $\gamma \to \infty$.

7. Exercise. (i) By using Exercise 10.6.1 prove that if $a^{ij}, b^i, c \in C^1_{loc}(\Omega)$, $c \le 0$, and $\Omega \in C^2$, then for any $f \in \mathcal{L}_p(\Omega)$ there exists a unique $u \in \overset{0}{W}{}^2_p(\Omega)$ solving $Lu = f$. Furthermore,

$$\|u\|_{W^2_p(\Omega)} \le N \|f\|_{\mathcal{L}_p(\Omega)} \tag{3}$$

with N independent of f.

(ii) Assuming that the maximum principle holds for $\overset{0}{W}{}^2_p(\Omega)$ functions if $\Omega \in C^2$ and L satisfies only Assumption 1.6.1 and $c \le 0$, prove then that, for such Ω and L, the assertion in part (i) holds without the assumption that $a^{ij}, b^i, c \in C^1_{loc}(\Omega)$. This yields the solvability of $Lu = f$ at the expense of losing all information about what the constants N depend on in our estimates.

Here is an extension of Exercise 5 for functions in $\overset{0}{W}{}^2_p(\Omega)$. Estimates of $\mathcal{L}_p$ norms of u_+ for $u \in \overset{0}{W}{}^2_p(\Omega)$ through $\mathcal{L}_p$ norms of $(Lu)_-$ for $p \in (1, \infty)$ rather than $p > d$ are given in Theorem 8.1.

8. Exercise. Let $\Omega \in C^2$, L satisfy Assumption 1.6.1, $c \le -\delta$, where $\delta > 0$ is a constant. Prove that, if $u \in \overset{0}{W}{}^2_p(\Omega)$ and $p > d$ (so that by embedding theorems $u \in C^1(\bar\Omega)$ and $u = 0$ on $\partial\Omega$) and in Ω we have

$$-Lu = f \in C^1(\bar\Omega),$$

then

$$u \le \frac{1}{\delta}\sup_\Omega f_+, \quad |u| \le \frac{1}{\delta}\sup_\Omega |f|.$$

Now the reader may try to prove the solvability of equations $\lambda u - Lu = f$ with small λ by using the results of Section 10.5. Again as in the case of Exercise 7 the suggested approach is somewhat short of providing estimates with constants under "good control".

9. Exercise. (i) Let $\Omega \in C^2$, L satisfy Assumption 1.6.1, $c \le -\delta$, where $\delta > 0$ is a constant. Prove that for any $f \in \mathcal{L}_p(\Omega)$ there exists a unique $u \in \overset{0}{W}{}^2_p(\Omega)$ satisfying $Lu = f$ in Ω. Also prove that (3) holds with N independent of f.

(ii) Prove that the result of (i) holds if $\delta = 0$.

In the following exercises we suggest the reader prove a priori estimates for the $\mathcal{L}_p$ norm of solutions in small balls. The proofs are quite instructive, being based on an ingenious combination of Kondrashov's theorem, the maximum principle, and dilations. Unlike what is suggested in the hint to Exercise 9, doing this exercises is possible without using Riesz's theory of compact operators.

10. Exercise. Let L satisfy Assumption 1.6.1. Prove by contradiction that there exist constants $r_0 > 0$ and N depending only on K, κ, ω, d, and p such that, if Ω is a ball of radius $r \le r_0$, then for any $u \in \overset{0}{W}{}^2_p(\Omega)$ and $\lambda \ge 0$ we have

$$\|u\|_{\mathcal{L}_p(\Omega)} \le N r^2 \|\lambda u - Lu\|_{\mathcal{L}_p(\Omega)}. \tag{4}$$

11. Exercise. Derive from Exercise 10 that, if Ω is a small ball, then for any $f \in \mathcal{L}_p(\Omega)$ there is a unique $u \in \overset{\circ}{W}{}^2_p(\Omega)$ satisfying $Lu = f$ in Ω.

12. Exercise. Prove that the result of Exercise 10 remains true if we replace (4) with the assertion that

$$\lambda\|u\|_{\mathcal{L}_p(\Omega)} \leq N\|\lambda u - Lu\|_{\mathcal{L}_p(\Omega)}$$

with the same kind of constant N.

2. Resolvent operator for λ large

Let $\Omega \in C^2$ be a domain in $\mathbb{R}^d$ and recall that $p \in (1, \infty)$. As we know from Theorem 8.5.3 there is a constant $\lambda_1 \geq 1$, depending only on K, κ, ω, $\operatorname{diam}\Omega$, K_0, ρ_0, d, and p, such that, for any $\lambda \geq \lambda_1$ and $f \in \mathcal{L}_p(\Omega)$, the equation

$$\lambda u - Lu = f \tag{1}$$

in the space $\overset{\circ}{W}{}^2_p(\Omega)$ has a unique solution. We denote this solution by $R_\lambda f$ and call the operator

$$R_\lambda : \mathcal{L}_p(\Omega) \to \overset{\circ}{W}{}^2_p(\Omega) \tag{2}$$

the resolvent operator of L. By Theorem 8.5.6 the mapping (2) is a bounded operator for $\lambda \geq \lambda_1$.

By what is said in Section 10.6, the operator R_λ is actually well defined for all complex λ apart from a sequence of numbers z_n such that $|z_n| \to \infty$. However, we are not going to use this information.

The following properties of R_λ for λ large will be instrumental in proving the solvability of (1) with λ small.

1. Theorem. *Let the coefficients of L be infinitely differentiable, $L1 \leq 0$, and $\lambda \geq \lambda_1$. Then*

(i) for any bounded f and any $\alpha \in (0, 1)$ we have $R_\lambda f \in C^{1+\alpha}(\Omega)$, $R_\lambda f = 0$ on $\partial\Omega$, and in Ω

$$|R_\lambda f(x)| \leq R_\lambda|f|(x) \leq \lambda^{-1}\sup_{x\in\Omega}|f(x)|; \tag{3}$$

(ii) there exists an integer m_0, depending only on p and d, and a constant N, depending only on κ, K, ω, d, p, K_0, ρ_0, and $\operatorname{diam}\Omega$, *such that for any $f \in \mathcal{L}_p(\Omega)$ we have*

$$\sup_{x\in\Omega}|R^{m_0}_{\lambda_1}f(x)| \leq N\|f\|_{\mathcal{L}_p(\Omega)}. \tag{4}$$

2. Remark. There is a simple case when one can take $m_0 = 1$ and $\lambda_2 = \lambda_1$ in (4). This happens if $p > d/2$ since by embedding theorems

$$\sup_{x\in\Omega} |R_{\lambda_1} f(x)| \le N\|R_{\lambda_1} f\|_{W^2_p(\Omega)} \le N\|(\lambda_1 - L)R_{\lambda_1} f\|_{\mathcal{L}_p(\Omega)} = N\|f\|_{\mathcal{L}_p(\Omega)}.$$

To prove this theorem, we need the maximum principle and the following version of Theorem 9.4.1 about better *local regularity* of solutions, meaning by better regularity higher power of summability of u_{xx}. In this section we are going to use Theorem 3 only for large R when $B_R \supset \Omega$ so that $\Omega_R = \Omega_{2R} = \Omega$.

3. Theorem. *Let* $\infty > q \ge p$ (> 1), $R > 0$, $z \in \bar{\Omega}$. *Denote*

$$\Omega_r = \Omega \cap B_r(z).$$

Then

$$\zeta u \in \overset{0}{W}{}^2_p(\Omega_{2R}) \quad \forall \zeta \in C_0^\infty(B_{2R}(z)), \quad Lu \in \mathcal{L}_q(\Omega_{2R})$$

$$\Longrightarrow \zeta u \in \overset{0}{W}{}^2_q(\Omega_{2R}) \quad \forall \zeta \in C_0^\infty(B_{2R}(z)). \quad (5)$$

Furthermore, there exists a constant N, *depending only on* R, K, K_0, ρ_0, κ, ω, p, q, *and* d, *such that, if the condition of the implication* (5) *holds, then*

$$\|u\|_{W^2_q(\Omega_R)} \le N(\|Lu\|_{\mathcal{L}_q(\Omega_{2R})} + \|u\|_{\mathcal{L}_p(\Omega_{2R})}). \quad (6)$$

Proof. For $q = p$ the first assertion is trivial and the second one is contained in Theorem 9.4.1. Let $q > p$ and define

$$\gamma = d/(d-1) \quad \text{if} \quad d \ge 2, \quad \gamma = 2 \quad \text{if} \quad d = 1,$$

$$p(n) = \gamma^n p, \quad n = 0, 1, \dots, m-1, \quad p(m) = q, \quad (7)$$

where $m - 1$ is the last n such that $p(n) < q$. Take λ so large (see Theorem 8.5.3) that $\lambda - L$ is invertible as an operator acting from $\overset{0}{W}{}^2_{p(n)}(\Omega)$ onto $\mathcal{L}_{p(n)}(\Omega)$ for $n = 0, \dots, m$. Also take a u such that the condition of the implication (5) holds, take a $\zeta \in C_0^\infty(B_{2R}(z))$, notice that $\zeta u \in \overset{0}{W}{}^2_p(\Omega)$ and denote

$$f = Lu, \quad g = (L - \lambda)(\zeta u) = \zeta f + 2a^{ij} u_{x^i} \zeta_{x^j} + u(L - c - \lambda)\zeta.$$

If $p \geq d$, then by embedding theorems (see Corollary 10.4.5)

$$\eta u \in W^1_r(\Omega) \tag{8}$$

for any $r \in [1, \infty)$ and $\eta \in C_0^\infty(B_{2R}(z))$. In particular, for $n = 1$

$$\eta u \in W^1_{p(n)}(\Omega). \tag{9}$$

If $p < d$, then by embedding theorems (8) holds with

$$r = pd/(d-p) \geq \gamma p = p(1).$$

In any case (9) holds for any $\eta \in C_0^\infty(B_{2R}(z))$ if $n = 1$.

It follows that $g \in \mathcal{L}_{\gamma p}(\Omega)$. By the choice of λ the equation

$$(L - \lambda)w = g$$

has a solution in $\overset{0}{W}{}^2_{\gamma p}(\Omega) \subset \overset{0}{W}{}^2_p(\Omega)$ which in addition is unique in $\overset{0}{W}{}^2_p(\Omega)$. Hence for $n = 1$

$$w = \zeta u \in \overset{0}{W}{}^2_{p(n)}(\Omega) \quad \forall \zeta \in C_0^\infty(B_{2R}). \tag{10}$$

If $p(1) < q$, then by repeating this argument with $p(1)$ in place of p, we get (10) for $n = 2$. In this way we get this inclusion for all n and this proves (5).

To prove (6), we accompany the above argument with estimates. By the choice of λ, for $n \geq 1$ and any $\zeta, \eta \in C_0^\infty(B_{2R}(z))$ such that $\eta = 1$ on the support of ζ, we have

$$\|\zeta u\|_{W^2_{p(n)}(\Omega)} \leq N\|\zeta f + 2a^{ij}u_{x^i}\zeta_{x^j} + u(L - c - \lambda)\zeta\|_{\mathcal{L}_{p(n)}(\Omega)}$$

$$\leq N(\|f\|_{\mathcal{L}_q(\Omega_{2R})} + \|\eta u\|_{W^1_{p(n)}(\Omega)}) \leq N(\|f\|_{\mathcal{L}_q(\Omega_{2R})} + \|\eta u\|_{W^2_{p(n-1)}(\Omega)}).$$

By iterating the inequality between the extreme terms, we obviously get that for any $\zeta \in C_0^\infty(B_{3R/2}(z))$ there is an $\eta \in C_0^\infty(B_{7R/4}(z))$ such that

$$\|\zeta u\|_{W^2_q(\Omega)} \leq N(\|f\|_{\mathcal{L}_q(\Omega_{2R})} + \|\eta u\|_{W^2_p(\Omega)}).$$

Finally, recall that by Theorem 9.4.1

$$\|\eta u\|_{W^2_p(\Omega)} \leq N\|u\|_{W^2_p(\Omega_{7R/4})} \leq N(\|f\|_{\mathcal{L}_p(\Omega_{7R/2})} + \|u\|_{\mathcal{L}_p(\Omega_{7R/2})}).$$

This yields (6) with $7R/2$ in place of $2R$. However, obviously Theorem 9.4.1 is also true with any number > 1 in place of 2. Then on the right in the above inequality one can take $2R$ ($> 7R/4$) in place of $7R/2$ and get (6) in its original form. The theorem is proved.

Proof of Theorem 1. (i) Denote $u = R_\lambda f$. Then $u \in \overset{0}{W}{}^2_p(\Omega)$ and (1) holds. Since $f \in \mathcal{L}_q(\Omega)$ for any $q > 1$, Theorem 3, applied to $L - \lambda$ in place of L, implies that $u \in \overset{0}{W}{}^2_q(\Omega)$ for any $q > 1$, which by embedding theorems implies that $u \in C^{1+\alpha}(\bar{\Omega})$ for any $\alpha \in (0, 1)$ and further by Remark 10.4.13 that $u = 0$ on $\partial\Omega$.

The proof of (3) is based on the following claim:

$$f \in C_0^\infty \Longrightarrow u := R_\lambda f, v := R_\lambda |f| \in C^2_{loc}(\Omega). \tag{11}$$

If (11) holds and $f \in C_0^\infty$, then we can finish the proof in the following way. For $M = \lambda^{-1} \sup |f|$ we have

$$(\lambda - L)M \geq \lambda M \geq |f| = (\lambda - L)v \geq (\lambda - L)u.$$

Hence by the maximum principle for smooth functions, we get

$$M \geq R_\lambda |f| \geq R_\lambda f.$$

Replacing f with $-f$ leads to (3) for smooth f.

Assuming (11), for arbitrary bounded f we define $f^{(\varepsilon)}$ as the Sobolev mollifiers of $f I_\Omega$. Then

$$\sup |f^{(\varepsilon)}| \leq \sup |f|$$

and since $f^{(\varepsilon)} \to f$ as $\varepsilon \to 0$ in $\mathcal{L}_q(\Omega)$ for any $q > 1$, we have

$$R_\lambda f^{(\varepsilon)} \to R_\lambda f, \quad R_\lambda |f^{(\varepsilon)}| \to R_\lambda |f|$$

in $\mathcal{L}_p(\Omega)$ and by Theorem 3 in $W^2_q(\Omega)$ for any $q > 1$, in particular, uniformly on Ω. This allows us to pass to the limit in (3) with $f^{(\varepsilon)}$ in place of f and it shows that it only remains to prove claim (11).

For $f \in C_0^\infty$, the function $|f|$ is Lipschitz continuous and by Exercise 1.8.7 we have $f \in W^1_q(\Omega)$ for any $q \geq 1$. Therefore, by taking into account embedding theorems, we see that now our task is only to prove that, for any large q, we have $u \in W^3_{q,loc}(\Omega)$ whenever $f \in W^1_q(\Omega)$. Since we already know that $u \in W^2_q(\Omega)$, the desired result follows directly from Theorem 9.4.1 or Theorem 5.2.5.

(ii) If $p > d/2$, we are done owing to Remark 2. If $p \le d/2$, we introduce $\gamma = d/(d-1)$, $p(n) = \gamma^n p$ and set

$$u_0 = f, \quad u_n = R^n_{\lambda_1} f, \quad n \ge 1.$$

Observe that, for $n \ge 0$, we have

$$\lambda_1 u_{n+1} - L u_{n+1} = u_n,$$

so that $u_{n+1} \in \overset{0}{W}{}^2_p(\Omega)$ and

$$\|u_{n+1}\|_{W^2_p(\Omega)} \le N\|u_n\|_{\mathcal{L}_p(\Omega)} \le N\|u_n\|_{\mathcal{L}_{p(n)}(\Omega)}.$$

By Theorem 3

$$\|u_{n+1}\|_{W^2_{p(n)}(\Omega)} \le N\|u_n\|_{\mathcal{L}_{p(n)}(\Omega)} + N\|u_{n+1}\|_{\mathcal{L}_p(\Omega)}.$$

Hence

$$\|u_{n+1}\|_{W^2_{p(n)}(\Omega)} \le N\|u_n\|_{\mathcal{L}_{p(n)}(\Omega)}, \tag{12}$$

and by embedding theorems

$$\|u_{n+1}\|_{\mathcal{L}_{p(n+1)}(\Omega)} \le N\|u_n\|_{\mathcal{L}_{p(n)}(\Omega)}.$$

Iterating this yields that for $n \ge 0$

$$\|u_n\|_{\mathcal{L}_{p(n)}(\Omega)} \le N\|u_0\|_{\mathcal{L}_{p(0)}(\Omega)} = N\|f\|_{\mathcal{L}_p(\Omega)},$$

where the constants N depend on the data as in the statement of the theorem and they also depend on n. Now we fix an $n = n(p, d)$ so that $p(n) > d/2$ and from (12) and embedding theorems conclude that

$$\sup_{x\in\Omega} |u_{n+1}(x)| \le N\|u_{n+1}\|_{W^2_{p(n)}(\Omega)} \le N\|f\|_{\mathcal{L}_p(\Omega)},$$

which shows that (4) holds with $m_0 = n + 1$. The theorem is proved.

4. Exercise. It turns out that under the assumptions of Theorem 1 one can take any $\lambda \ge \lambda_1$ in place of λ_1 in (4). To show this, prove that, if $\lambda \ge \mu \ge \lambda_1$ and $f \ge 0$, then $R_\lambda f \le R_\mu f$.

3. Solvability of equations in smooth domains

First we prove an a priori estimate for solution of $\lambda u - Lu = f$ with $\lambda > 0$.

1. Lemma. *Let $\Omega \in C^2$, $\nu > 0$, and $L1 \le 0$. Then there exists a constant N depending only on $\nu, \kappa, K, \omega, d, p, K_0, \rho_0$, and* $\operatorname{diam}\Omega$, *such that for any $\lambda \ge \nu$ and $u \in \overset{0}{W}{}^2_p(\Omega)$*

$$\|u\|_{W^2_p(\Omega)} \le N\|(\lambda - L)u\|_{\mathcal{L}_p(\Omega)}. \tag{1}$$

Proof. If $\lambda \ge \lambda_1$ with λ_1 taken from Theorem 8.5.6, the result is known from that theorem. Therefore we will only concentrate on

$$\nu \le \lambda < \lambda_1.$$

Without loss of generality (see Exercise 8.4.4) we may assume that $u \in C^2(\bar{\Omega})$ and $u = 0$ on $\partial\Omega$. Furthermore, we may assume that the coefficients of L are infinitely differentiable. In this case define

$$f = \lambda u - Lu$$

and observe that in the notation of Section 2 we have

$$\lambda_1 u - Lu = (\lambda_1 - \lambda)u + f, \quad u = (\lambda_1 - \lambda)R_{\lambda_1}u + R_{\lambda_1}f,$$

and by induction on n

$$u = [(\lambda_1 - \lambda)R_{\lambda_1}]^n u + \sum_{i=0}^{n-1}[(\lambda_1 - \lambda)R_{\lambda_1}]^i R_{\lambda_1}f,$$

where n is any integer ≥ 1. We thus have the beginning of the von Neumann series.

Introduce the constants N_2 and M_n (which are under control) so that

$$\|R_{\lambda_1}g\|_{\mathcal{L}_p(\Omega)} \le N_2\|g\|_{\mathcal{L}_p(\Omega)} \quad \forall g \in \mathcal{L}_p(\Omega), \quad M_n = \sum_{i=0}^{n-1}(\lambda_1 - \nu)^i N_2^{i+1}.$$

Finally let $|\Omega|$ be the volume of Ω, take m_0 from Theorem 2.1 and denote by N_1 the constant N in (2.4). Then by Theorem 2.1 for $n > m_0$ we obtain

$$\|u\|_{\mathcal{L}_p(\Omega)} \le |\Omega|^{1/p}(\lambda_1 - \nu)^n \sup_{x\in\Omega}|R^{n-m_0}_{\lambda_1}R^{m_0}_{\lambda_1}u(x)| + M_n\|f\|_{\mathcal{L}_p(\Omega)}$$

$$\le |\Omega|^{1/p}\lambda_1^{m_0}(1 - \nu/\lambda_1)^n \sup_{x\in\Omega}|R^{m_0}_{\lambda_1}u(x)| + M_n\|f\|_{\mathcal{L}_p(\Omega)}$$

$$\leq N_3(1-\nu/\lambda_1)^n\|u\|_{\mathcal{L}_p(\Omega)} + M_n\|f\|_{\mathcal{L}_p(\Omega)}.$$

where $N_3 = |\Omega|^{1/p}\lambda_1^{m_0}N_1$. Now fix $n > n_0$ such that

$$N_3(1-\nu/\lambda_1)^n \leq 1/2.$$

Then we find

$$\|u\|_{\mathcal{L}_p(\Omega)} \leq N\|f\|_{\mathcal{L}_p(\Omega)}$$

and to get (1), it only remains to refer to Theorem 8.5.6. The lemma is proved.

The following result concludes our investigation of solvability of elliptic equations in bounded smooth domains and W_p^2 spaces. The corresponding result for W_p^{k+2} spaces follows from Corollary 9.2.2. The reader who did Exercise 1.9 is already familiar with the first statement of Theorem 2.

2. Theorem. *Let $\Omega \in C^2$, $L1 \leq 0$. Then for any $\lambda \geq 0$, $f \in \mathcal{L}_p(\Omega)$, and $g \in W_p^2(\Omega)$ there exists a unique $u \in W_p^2(\Omega)$ such that*

$$\lambda u - Lu = f \quad in \quad \Omega, \quad u - g \in \overset{0}{W}{}_p^2(\Omega).$$

Furthermore, there exists a constant N, depending only on d, p, K, κ, ω, $\operatorname{diam}\Omega$, and K_0, ρ_0 associated with Ω (see Definition 8.3.1), such that for any $v \in \overset{0}{W}{}_p^2(\Omega)$ and $\lambda \geq 0$ we have

$$\|v\|_{W_p^2(\Omega)} \leq N\|\lambda v - Lv\|_{\mathcal{L}_p(\Omega)}.$$

Proof. As always the case of general g is reduced to the case $g = 0$ by replacing the unknown function u with $u - g$. In that case the method of continuity and Theorem 8.5.6 show that we only have to prove that there exists a constant N like the one in the statement of the theorem such that for any $v \in \overset{0}{W}{}_p^2(\Omega)$ and $\lambda \geq 0$ we have

$$\|v\|_{W_p^2(\Omega)} \leq N\|\lambda v - Lv\|_{\mathcal{L}_p(\Omega)}. \tag{2}$$

To do this step, we use Lemma 1 and the global barrier v_0 from Lemma 1.2. Without loss of generality we may assume that $\Omega \subset B_{2R}$, where $R = \operatorname{diam}\Omega$ and take v_0 from Lemma 1.2 corresponding to $4R$ in place of R. Then we introduce a new operator L' by the formula

$$L'u = v_0^{-1}L(v_0u).$$

Observe that in B_{2R} the coefficients of L' satisfy Assumption 1.6.1 with new K, κ, and ω depending only on the original ones, d, and R. We also notice that in $\Omega \subset B_{2R}$ due to the construction of v_0 we have $L'1 \leq -\delta$ for a constant $\delta > 0$ depending only on κ, K, R, and d.

By Lemma 1 applied to the operator $L'' = L' + \delta$, we find

$$\|v\|_{W^2_p(\Omega)} \leq N\|vv_0^{-1}\|_{W^2_p(\Omega)} \leq N\|(\lambda+\delta)vv_0^{-1} - L''(vv_0^{-1})\|_{\mathcal{L}_p(\Omega)}$$

$$= N\|(\lambda v - v_0 L'(vv_0^{-1}))v_0^{-1}\|_{\mathcal{L}_p(\Omega)} = N\|(\lambda v - Lv)v_0^{-1}\|_{\mathcal{L}_p(\Omega)},$$

where the last expression is less than the right-hand side of (2). The theorem is proved.

We can now prove the maximum principle for W^2_p functions vanishing on the boundary. More general functions are treated in Theorem 8.1.

3. Theorem (Maximum principle). *Let* $\Omega \in C^2$, $L1 \leq 0$. *Then there exists a constant* N *depending only on* $\operatorname{diam}\Omega, d, K, \kappa, \omega$, *and* p *such that for any* $u \in \overset{0}{W}{}^2_p(\Omega)$ *and* $\lambda \geq 0$ *we have*

$$\|u_+\|_{\mathcal{L}_p(\Omega)} \leq N\|(\lambda u - Lu)_-\|_{\mathcal{L}_p(\Omega)}. \tag{3}$$

In particular, if $u \in \overset{0}{W}{}^2_p(\Omega)$ *and* $Lu \geq 0$ *in* Ω, *then* $u \leq 0$ *in* Ω.

Proof. Obviously we need only prove (3). Recall Exercise 8.4.4 which says that the set of functions in $C^2(\bar{\Omega})$ vanishing on $\partial\Omega$ is dense in $\overset{0}{W}{}^2_p(\Omega)$. It shows that we may concentrate on such u. Furthermore without loss of generality one can assume that the coefficients of L are infinitely differentiable. Finally, we may assume that $\Omega \subset B_R$ with $R = 2\operatorname{diam}\Omega$.

After these simplifications fix a $\lambda \geq 0$, denote

$$f = (\lambda u - Lu)_+,$$

observe that $f \in C(\bar{\Omega})$, and take a sequence of smooth functions $f_n \geq 0$ defined in B_R such that

$$f \leq f_n \leq f + 1/n$$

in Ω and $f_n \to 0$ in $\mathcal{L}_p(B_R \setminus \Omega)$. Observe that considering B_R instead of Ω has the purpose of ensuring that N in (3) is independent of the structure of $\partial\Omega$. Let $v_n \in \overset{0}{W}{}^2_p(B_R)$ be solutions of

$$\lambda v_n - Lv_n = f_n.$$

Notice that the existence of v_n is guaranteed by Theorem 2, which also implies that $v_n \in \overset{\circ}{W}{}^2_q(B_R)$ for any $q > 1$. For $q > d$, embedding theorems show that $v_n \in C(\bar{B}_R)$ and $v_n = 0$ on ∂B_R (see Remark 10.4.13). Furthermore, by interior regularity results $v_n \in W^k_{p,loc}(B_R)$ for any k. Hence, v_n is infinitely differentiable in B_R.

Next, $(\lambda - L)v_n \geq 0$ in B_R and

$$(\lambda - L)(v_n - u) \geq f_n - f \geq 0$$

in Ω, so that by the maximum principle for smooth functions we have $v_n \geq 0$ in B_R and $v_n \geq u$, $v_n \geq u_+$ in Ω. Therefore,

$$\|u_+\|_{\mathcal{L}_p(\Omega)} \leq \|v_n\|_{\mathcal{L}_p(B_R)} \leq N\|(\lambda - L)v_n\|_{\mathcal{L}_p(B_R)} = N\|f_n\|_{\mathcal{L}_p(B_R)}$$

and (3) follows when $n \to \infty$. The theorem is proved.

4. Remark. It turns out that if $p \geq d$, then the constant N in (3) can be chosen to be independent of ω, which is the modulus of continuity of a^{ij} (A.D. Alexandrov's theorem).

5. Exercise. Let $F(t)$ be a continuous function on $\mathbb{R}$ satisfying $|F(t)| \leq |t|^\alpha$, where $\alpha > 1$ is a constant satisfying $2p > (1 - \alpha^{-1})d$, so that

$$W^2_p(\Omega) \subset \mathcal{L}_{\alpha p}(\Omega).$$

Fix an $f \in \mathcal{L}_p(\Omega)$ and consider the equation $\Delta u + F(u) = f$ in $\Omega = B_1$ with zero boundary condition.

(i) Show that, for any $M \in [0, \infty)$, the set

$$\Gamma_M = \{u \in \overset{\circ}{W}{}^2_p(\Omega) : \|u\|_{W^2_p(\Omega)} \leq M\}$$

is a closed compact convex subset of $\mathcal{L}_{\alpha p}(\Omega)$.

(ii) Define an operator $T : v \to Tv$ for $v \in \mathcal{L}_{\alpha p}(\Omega)$ by requiring

$$Tv \in \overset{\circ}{W}{}^2_p(\Omega), \quad \Delta Tv + F(v) = f.$$

Prove that, for any $M \geq 0$, the operator T is a continuous operator from Γ_M to $\mathcal{L}_{\alpha p}(\Omega)$.

(iii) Take M and $\|f\|_{\mathcal{L}_p(\Omega)}$ small enough and prove that $T : \Gamma_M \to \Gamma_M$. On the basis of the Schauder-Tikhonov theorem (see [4]) conclude that in this case the equation

$$\Delta u + F(u) = f \tag{4}$$

has a solution in $\overset{\circ}{W}{}^2_p(\Omega)$.

(iv) Prove that *any* solution in $\overset{\circ}{W}{}^2_p(\Omega)$ of (4) is infinitely differentiable if f and F are infinitely differentiable and $\|f\|_{\mathcal{L}_p(\Omega)}$ is not necessarily small.

4. The way we proceed further

Here we want to explain some ideas about how the equation $Lu = f$ in $\mathbb{R}^d$ will be investigated in the following sections. We assume that $L1 \leq -1$ and all the coefficients of L are smooth. If u is smooth and bounded and $|Lu| \leq \alpha$, where α is a constant, then

$$L(u - \alpha) = Lu - \alpha L1 \geq 0,$$

which by Exercise 5.2 implies that $u \leq \alpha$. The same holds for $-u$. Hence for any x

$$|u(x)| \leq \sup_{\mathbb{R}^d} |Lu|. \tag{1}$$

Let $F := \{Lu : u \in C_0^\infty\}$. Then F is a subset of the space X of all continuous functions tending to zero at infinity. Next, fix an x and consider the functional $R(x)$ defined on F by the formula

$$R(x)f = u(x) \quad \text{if} \quad f = -Lu \quad \text{for a} \quad u \in C_0^\infty.$$

Owing to (1), this is a well-defined linear functional on F with norm less than 1. By the Hahn-Banach theorem it admits an extension to all of X and by Riesz's representation theorem there is a measure G_x (perhaps changing sign) with the total variation not bigger than 1 and such that

$$R(x)f = \int_{\mathbb{R}^d} f(y)\, G_x(dy), \quad u(x) = -\int_{\mathbb{R}^d} Lu(y)\, G_x(dy) \tag{2}$$

for all $f \in F$ and $u \in C_0^\infty$. We will see later that F is actually dense in X. For now we assume this fact and proceed further with our explanation. This fact, the second relation in (2), and Exercise 5.2, showing that $u \geq 0$ if $Lu \leq 0$, seem to imply that G_x is a positive measure. This is the so-called Green's measure of L. The denseness should also imply that, in a certain sense, $R(x)f$ satisfies

$$LRf = -f \quad \text{for any} \quad f \in X.$$

Next, owing to Theorem 8.0.1, the corresponding $\mathcal{L}_p$ counterpart of the a priori estimate from Theorem 1.6.5 and the existence result from Exercise 1.6.6 are available. Therefore, to prove the solvability of $Lu = -f$ in W_p^2, it suffices to prove the estimate $\|u\|_{\mathcal{L}_p} \leq N\|Lu\|_{\mathcal{L}_p}$ or

$$\|Rf\|_{\mathcal{L}_p} \leq N\|f\|_{\mathcal{L}_p}. \tag{3}$$

To prove (3), we represent f as the sum, actually integral, of $\zeta^z f(x)$ with respect to z, where $\zeta^z(x) = \zeta(z - x)$, $\zeta \in C_0^\infty$ and $\operatorname{supp}\zeta \subset B_1$. An advantage of this representation is that, for each z, $R(\zeta^z f)$ satisfying the equation

$$Lv = -\zeta^z f$$

decreases as $|z - x| \to \infty$ exponentially fast due to Lemma 5.3. Therefore, to estimate the norm in $\mathcal{L}_p$ of $v = R(\zeta^z f)$, it suffices to estimate its norm over the support of $\zeta^z f$. Since v is small on the boundary of the ball $B_r(z)$ with r large, a simple argument (see the proof of Lemma 6.1) shows that it suffices to estimate the $\mathcal{L}_p(B_r(z))$ norm of the solution of $Lw = -\zeta^z f$ in $B_r(z)$ with *zero*, rather than small, boundary data. This estimate follows directly from Theorem 3.2.

After we estimate $R(\zeta^z f)$ for each z, we notice that due to the linearity of R, Rf is the "sum" of $R(\zeta^z f)$ with respect to z, and up to a small detail we use the fact that the norm of a sum is less than the sum of norms.

One of the main difficulties we encounter in implementing this idea is that we do not know that

$$LR(\zeta^z f) = -\zeta^z f$$

even if $f = -Lu \in F$. However, we do know that equations $Lv - \lambda v = f$ are uniquely solvable if λ is large. This allows us to prove (3) first for $L - \lambda$ in place of L with λ large and then to use the fact that the resulting constant N that we get is independent of λ, as will be seen from the proof of Theorem 6.2.

5. Decay at infinity of solutions of $Lu = f$ in $\mathbb{R}^d$

In the following lemmas we denote any constant satisfying

$$|a^{ij}|, |b^i| \leq K_1$$

on $\mathbb{R}^d$ by K_1. We introduce this constant in order to emphasize that some quantities are independent of lower bounds of c.

1. Lemma. *If $\Omega = B_r$, $u \in C^2_{loc}(\Omega) \cap C(\bar{\Omega})$, $L1 \leq -1$ and $Lu \geq 0$ in Ω, then*

$$u(x) \leq \frac{\cosh(\varepsilon|x|)}{\cosh(\varepsilon r)} \max_{\partial\Omega} u_+.$$

where $\varepsilon > 0$ depends only on K_1 and d.

Proof. Define $v = \cosh(\varepsilon|x|)$. As we have pointed out already in the proof of Lemma 1.1.7, v is an infinitely differentiable function. Define $\xi = x|x|^{-1}$ and notice that $|\xi| = 1$, $\sinh t \leq t\cosh t$ and $\sinh t \leq \cosh t$ if $t \geq 0$. Then for $x \neq 0$ we obtain

$$(L - c)v = \varepsilon^2 \cosh(\varepsilon|x|)a^{ij}\xi^i\xi^j + \varepsilon|x|^{-1}\sinh(\varepsilon|x|)(\operatorname{tr} a - a^{ij}\xi^i\xi^j)$$

$$+ \varepsilon \sinh(\varepsilon|x|)b^i\xi^i$$

$$\leq N(d, K_1)(\varepsilon^2 + \varepsilon)\cosh(\varepsilon|x|).$$

It follows that $(L - c)v \leq v/2$ if ε is chosen appropriately and $x \neq 0$. Actually, this inequality is also true for $x = 0$ since v is a C^∞ function. We conclude $Lv \leq (c + 1/2)v \leq -v/2 < 0$ in Ω.

Next, for the operator $L'w = L(wv)$ which is an elliptic operator, we have $L'1 = Lv < 0$. Furthermore, $L'(u/v) = Lu \geq 0$. Hence, by Lemma 1.1

$$u/v \leq \max_{\partial\Omega}(u_+/v) = \frac{1}{\cosh(\varepsilon r)}\max_{\partial\Omega} u_+,$$

which is exactly what is asserted. The lemma is proved.

2. Exercise*. Let u be a *bounded* from above C^2_{loc} function in $\mathbb{R}^d$ satisfying $Lu \geq 0$ and $L1 \leq -\delta$, where δ is a constant, $\delta > 0$. Then prove that $u \leq 0$. Also give an example showing that this assertion is false if u is allowed to be unbounded.

3. Lemma. *Let $\Omega = \mathbb{R}^d \setminus B_r$, and let u be a continuous function bounded from above in $\bar{\Omega}$. Assume that u is twice continuously differentiable in Ω, $L1 \leq -1$, and $Lu \geq 0$ in Ω. Then in Ω*

$$u(x) \leq e^{\delta(r-|x|)}\max_{\partial\Omega} u_+,$$

where $\delta > 0$ depends only on K_1 and d.

Proof. Define $v_1 = e^{-\delta|x|}$ and $\xi = x/|x|$. By straightforward computations relying on the inequality $a^{ij}\xi^i\xi^j \leq \operatorname{tr} a$, we get

$$v_1^{-1}(L - c)v_1 = \delta|x|^{-1}(a^{ij}\xi^i\xi^j - \operatorname{tr} a) + \delta^2 a^{ij}\xi^i\xi^j - \delta b^i\xi^i$$

$$\leq \delta^2 a^{ij}\xi^i\xi^j - \delta b^i\xi^i \leq 1/2$$

if δ is chosen appropriately. Hence $Lv_1 < 0$. Next take ε from Lemma 1, take a $\gamma > 0$ and define $v_2 = \cosh(\varepsilon|x|)$ and $v = v_1 + \gamma v_2$.

Since $Lv < 0$, we can apply the argument in the proof of Lemma 1 for the ring $r \le |x| \le R$, where R is a fixed number to be sent to infinity later. Then we obtain that for $r \le |x| \le R$

$$u/v \le \max\big(\max_{|x|=r}(u_+/v), \max_{|x|=R}(u_+/v)\big) \le \max_{|x|=r}(u_+/v) + \max_{|x|=R}(u_+/v).$$

Here the last term tends to zero as $R \to \infty$ because u_+ is bounded and $v(x) \ge \gamma\cosh(\varepsilon|x|)$. Therefore, for any $\gamma > 0$ and $|x| \ge r$,

$$u(x) \le (e^{-\delta|x|} + \gamma\cosh(\varepsilon|x|))(e^{-\delta r} + \gamma\cosh(\varepsilon r))^{-1}\max_{|x|=r} u_+.$$

Finally, upon letting $\gamma \downarrow 0$, we get the result. The lemma is proved.

In the same way that Corollary 1.4 follows from Theorem 1.3, Exercise 2 implies the following.

4. Corollary. *Let $\delta > 0$ be a constant, $c \le -\delta$, and let α be a nonnegative function on $\mathbb{R}^d$. Let $u, v \in C^2_{loc}(\mathbb{R}^d)$, let u, v be bounded and let*

$$(\alpha u - Lu)_+ \le -Lv$$

in $\mathbb{R}^d$. Then $u_+ \le v$. In particular, if $g \ge 0$ and

$$|\alpha u - Lu| + g = -Lv$$

in $\mathbb{R}^d$, then $|u| \le v$, so that, for any constant N_1,

$$\|v\|_{\mathcal{L}_p} \le N_1\|Lv\|_{\mathcal{L}_p} \implies \|u\|_{\mathcal{L}_p} \le N_1\|\alpha u - Lu\|_{\mathcal{L}_p} + N_1\|g\|_{\mathcal{L}_p}.$$

Corollary 4 allows one to prove the solvability of elliptic equations with measurable coefficients in Sobolev spaces for $p = d = 2$. Interestingly enough the results of the following exercise fail to hold if $p \ne 2$. However, they still hold provided that $|p - 2|$ is less than a small constant depending on the ellipticity constant.

5. Exercise. Let $d = 2$ and let $a = (a^{ij})$ be a (measurable) 2×2 symmetric matrix-valued function defined on $\mathbb{R}^2$ and satisfying

$$\kappa^{-1}|\xi|^2 \ge a^{ij}(x)\xi^i\xi^j \ge \kappa|\xi|^2$$

for all $x, \xi \in \mathbb{R}^2$. Let b^1, b^2, and c be (measurable) functions on $\mathbb{R}^2$ satisfying $|b^i| + |c| \le K$, $c \le 0$. Prove the following.

(i) There exist constants $\lambda_1 \ge 1$ and N, depending only on κ and K, such that for any $\lambda \ge \lambda_1$ and $u \in W^2_2(\mathbb{R}^2)$ we have

$$\lambda\|u\|_{\mathcal{L}_2} \le N\|\lambda u - Lu\|_{\mathcal{L}_2}.$$

As in Theorem 1.6.5 derive from this that

$$\|u\|_{W_2^2} \leq N\|\lambda u - Lu\|_{\mathcal{L}_2}$$

and by the method of continuity prove that for any $f \in \mathcal{L}_2$ and $\lambda \geq \lambda_1$ there is a unique $u \in W_2^2$ such that $\lambda u - Lu = f$.

(ii) If $\Omega \in C^2$ is a domain in $\mathbb{R}^2$, then there exist constants $\lambda_1 \geq 0$ and N, depending only on κ, K, K_0, ρ_0, and $\operatorname{diam}\Omega$, such that for any $\lambda \geq \lambda_1$ and $u \in \overset{0}{W}{}_2^2(\Omega)$ we have

$$\|u\|_{W_2^2(\Omega)} \leq N\|\lambda u - Lu\|_{\mathcal{L}_2(\Omega)}.$$

Furthermore, for any $f \in \mathcal{L}_2(\Omega)$ and $\lambda \geq \lambda_1$ there is a unique $u \in \overset{0}{W}{}_2^2(\Omega)$ such that $\lambda u - Lu = f$.

(iii) The result of (ii) holds with $\lambda_1 = 0$.

6. Equations in $\mathbb{R}^d$ with λ small

We start by deriving from Theorem 3.2 an a priori estimate for solutions of $Lu = f$ in $\mathbb{R}^d$ with f having support in B_1.

1. Lemma. *Take a constant $\lambda \geq 0$ and bounded infinitely differentiable functions u and f. Assume that $L1 \leq -1$ and f vanishes outside B_1. Also assume that the coefficients of L are infinitely differentiable. Finally, assume that*

$$\lambda u - Lu = f.$$

Then there exists a constant $\gamma > 0$, depending only on the bounds of $|a^{ij}|$ and $|b^i|$ and d, and there exists a constant N depending only on K, κ, ω, d, and p such that

$$\|u/v\|_{\mathcal{L}_p} \leq N\|f\|_{\mathcal{L}_p},$$

where $v(x) = e^{-\gamma|x|}$.

Proof. Fix a $q > d$ and using Theorem 3.2, define $h \in W_q^2(B_4)$ as the unique solution of

$$\lambda h - Lh = 0 \quad \text{in} \quad B_4 \quad \text{such that} \quad w := h - u \in \overset{0}{W}{}_q^2(B_4).$$

By regularity results h is infinitely differentiable in $\bar{B}_4$ and $h = u$ on ∂B_4 (see Remark 10.4.13). Hence w is infinitely differentiable in $\bar{B}_4$, vanishes on ∂B_4, and satisfies

$$\lambda w - Lw = -f.$$

Notice that $\lambda u - Lu = 0$ outside B_1 so that with δ defined in Lemma 5.3 we have

$$|u(x)| \leq e^{-\delta(|x|-2)} \max_{|x|=2} |u| \quad \text{for} \quad |x| \geq 2, \tag{1}$$

and by the maximum principle

$$|h| \leq \max_{|x|=4} |u|$$

in B_4. Now we claim that to prove the lemma, it suffices to prove that

$$|w(x)| \leq N\|\lambda w - Lw\|_{\mathcal{L}_p} \quad \text{for} \quad |x| = 2. \tag{2}$$

Indeed, if (2) holds, then

$$\max_{|x|=2} |u| \leq \max_{|x|=2} |h| + \max_{|x|=2} |w| \leq \max_{|x|=4} |u| + N\|f\|_{\mathcal{L}_p}$$

$$\leq e^{-2\delta} \max_{|x|=2} |u| + N\|f\|_{\mathcal{L}_p},$$

implying that

$$\max_{|x|=2} |u| \leq e^{-2\delta} \max_{|x|=2} |u| + N\|f\|_{\mathcal{L}_p}, \quad \max_{|x|=2} |u| \leq N\|f\|_{\mathcal{L}_p}.$$

Coming back to (1) and taking $\gamma = \delta/2$, we get that

$$\|u/v\|_{\mathcal{L}_p(B_2^c)} \leq N\|f\|_{\mathcal{L}_p}.$$

The remaining part of the norm is also bounded by $N\|f\|_{\mathcal{L}_p}$ since $|u| \leq |h| + |w|$, by Theorem 3.2 we have

$$\|w\|_{\mathcal{L}_p(B_4)} \leq N\|f\|_{\mathcal{L}_p}.$$

and

$$\max_{B_4} |h| \leq \max_{|x|=4} |u| \leq e^{-2\delta} \max_{|x|=2} |u| \leq N\|f\|_{\mathcal{L}_p}.$$

Thus indeed we need only prove (2).

First we reduce (2) to the case $\lambda = 0$ by using the maximum principle. Recall that $q > d$ is fixed in the beginning of the proof and use Theorem 3.2 to conclude that there exists a unique $\psi \in \overset{0}{W}{}^2_q(B_4)$ satisfying

$$-L\psi = |f|.$$

Since $-L(\psi \pm w) \geq 0$, by Theorem 3.3 we have $\psi \geq \pm w$ and $\psi \geq |w|$. It follows that to prove (2), it suffices to show that

$$|\psi(x)| \leq N\|L\psi\|_{\mathcal{L}_p} \quad \text{for} \quad |x| = 2.$$

Take a point x_0 with $|x_0| = 2$ and observe that by embedding theorems we have

$$|\psi(x_0)| \leq N\|\psi\|_{W_q^2(B_{1/2}(x_0))}.$$

Next, if $q \leq p$, we replace q with p here and, as well as in the case $q > p$, use the local regularity result from Theorem 2.3. Then we find

$$\|\psi\|_{W_q^2(B_{1/2}(x_0))} \leq N\|L\psi\|_{\mathcal{L}_q(B_1(x_0))} + N\|\psi\|_{\mathcal{L}_p(B_1(x_0))}.$$

Here the first term on the right is zero since $L\psi = f$ vanishes outside of B_1 and the second term is less than $N\|f\|_{\mathcal{L}_p}$ by Theorem 3.2. The lemma is proved.

Now comes the main result of our investigation of elliptic second-order equations in the whole space. It is stated only for W_p^2 spaces. The corresponding result for W_p^{k+2} spaces follows from Theorem 9.2.3.

2. Theorem. *Let $L1 \leq -\delta$ where $\delta > 0$ is a constant. Then for any $f \in \mathcal{L}_p$ there exists a unique $u \in W_p^2$ such that $Lu = f$. Moreover, there exists a constant N depending only on $\delta, d, K, \kappa, \omega$, and p such that for any $u \in W_p^2$ and $\lambda \geq 0$ we have*

$$\|u\|_{W_p^2} \leq N\|\lambda u - Lu\|_{\mathcal{L}_p}. \tag{3}$$

Proof. Needless to say, we fix d, K, κ, ω, and p and only consider the operators L satisfying Assumption 1.6.1 (with the above parameters) and such that $L1 \leq -\delta$. By dividing all the coefficients of L by δ, we reduce the general situation to the one with $\delta = 1$. Therefore, we assume that $\delta = 1$.

Next, owing to Theorem 8.0.1, the following $\mathcal{L}_p$ counterpart of the a priori estimate from Theorem 1.6.5 holds true:

$$\|u\|_{W_p^2} \leq N_0(\|\lambda u - Lu\|_{\mathcal{L}_p} + \|u\|_{\mathcal{L}_p}),$$

where u is any W_p^2 function, λ any nonnegative number, and N_0 depends only on d, p, K, κ, and ω. After that, naturally, an $\mathcal{L}_p$ counterpart of the existence result from Exercise 1.6.6 is available. That is to say, to prove the theorem, it suffices to prove that the following claim $A(\mu)$ holds with $\mu = 0$.

$A(\mu)$: *There exists a constant N (depending only on d, K, κ, ω, and p, that are fixed above) such that, for any $\lambda \geq \mu$, $u \in W_p^2$, and any operator*

L satisfying Assumption 1.6.1 *(with the above parameters) and such that* $L1 \le -1$, *we have*

$$\|u\|_{\mathcal{L}_p} \le N\|\lambda u - Lu\|_{\mathcal{L}_p}. \tag{4}$$

By virtue of Theorem 8.0.1 we know that $A(\mu)$ holds for $\mu \ge \lambda_0$. We are going to gradually relax this inequality to $\mu \ge 0$. Notice that, for the same reasons as above, for any μ and not only $\mu = 0$, claim $A(\mu)$ implies that for any $\lambda \ge \mu$ and $f \in \mathcal{L}_p$, there exists a unique $u \in W^2_p$ satisfying $\lambda u - Lu = f$. Finally, before starting to prove that claim $A(0)$ holds true, we observe that, obviously, it suffices to concentrate on $u \in C_0^\infty$ and the operators with infinitely differentiable coefficients whose derivatives are bounded.

Now, assume that $A(\mu)$ holds for a $\mu \ge 0$, take an operator L with infinitely differentiable coefficients whose derivatives are bounded, take a $\lambda \ge \mu$ and a $u \in C_0^\infty$. Introduce $f = \lambda u - Lu$. Our first goal is to estimate N in (4).

Let ζ be a C_0^∞ function with unit integral and support in B_1. Define $\zeta^z(x) = \zeta(x-z)$ and let, for any $z \in \mathbb{R}^d$, $w^{(z)} \in W^2_p$ be the unique solution of

$$\lambda w^{(z)} - Lw^{(z)} = \zeta^z f. \tag{5}$$

Such functions $w^{(z)}$ exist owing to $A(\mu)$. Since the coefficients of L and f are infinitely differentiable with bounded derivatives and f has compact support, the regularity results imply that, for any $z \in \mathbb{R}^d$, the function $w^{(z)}(x)$ and each of its derivatives in x are bounded as a function of x. These bounds are in terms of W^k_p norms of $\zeta^z f$ and therefore are uniform with respect to z. Furthermore, for any $k = 1, 2, ...$, there are constants N such that

$$\|w^{(y)} - w^{(z)}\|_{W^{k+2}_p} \le N\|(\zeta^y - \zeta^z)f\|_{W^k_p} \le N|y - z|$$

for all y and z. It follows by embedding theorems that $w^{(z)}$ and all its derivatives in x are Lipschitz continuous functions of z. Also $\zeta^z f(x) = 0$ and hence $w^{(z)}(x) = 0$ for all x if $|z|$ is large enough because f has compact support. Therefore, the definition

$$w = \int_{\mathbb{R}^d} w^{(z)}\,dz$$

makes sense and defines w as an infinitely differentiable and bounded function of x. Integrating through in (5), we find that

$$\lambda w - Lw = f,$$

which by Exercise 5.2 yields $w = u$.

Hence, by Hölder's inequality, for v taken from Lemma 1 and $v^z(x) = v(x-z)$,

$$|u(x)|^p \le \int_{\mathbb{R}^d} |w^{(z)}(x)/v^z(x)|^p \, dz \, \|v\|^p_{\mathcal{L}_q} = N_1^p \int_{\mathbb{R}^d} |w^{(z)}(x)/v^z(x)|^p \, dz, \quad (6)$$

where $q = p/(p-1)$ and N_1 depends only on K, p, and d. In addition, by Lemma 1 we have

$$\int_{\mathbb{R}^d} |w^{(z)}(x)/v^z(x)|^p \, dx \le N_2^p \int_{\mathbb{R}^d} |\zeta^z f|^p \, dx,$$

where N_2 is the constant called N in Lemma 1 (independent of λ and μ). This and (6) yield

$$\|u\|_{\mathcal{L}_p} \le N_3 \|\lambda u - Lu\|_{\mathcal{L}_p}, \quad N_3 = N_1 N_2 \|\zeta\|_{\mathcal{L}_p}.$$

We conclude that,

if $A(\mu)$ holds, then one can take $N = N_3$ in (4).

The most important message of this conclusion is that N_3 is independent of μ (as long as $\mu \ge 0$, so that $\lambda \ge \mu$ satisfies $\lambda \ge 0$, which is required in Lemma 1).

Now it turns out that, if $A(\mu)$ holds, then $A(\mu - \gamma)$ holds as well if $\gamma = (2N_3)^{-1}$. Indeed, if $A(\mu)$ holds and $\lambda \ge \mu - \gamma$, then for any $u \in W^2_p$

$$\|u\|_{\mathcal{L}_p} \le N_3 \|(\lambda + \gamma)u - Lu\|_{\mathcal{L}_p} \le N_3 \|\lambda u - Lu\|_{\mathcal{L}_p} + (1/2)\|u\|_{\mathcal{L}_p},$$

so that $A(\mu - \gamma)$ and $A((\mu - \gamma)_+)$ hold with $2N_3$ in place of N in (4). In addition, by the above result, once $A((\mu-\gamma)_+)$ holds, it holds with $N = N_3$. Finally, upon starting with $\mu = \lambda_0$ and going from μ to $(\mu - \gamma)_+$, we reach $\mu = 0$ in finitely many steps, which shows that $A(0)$ holds and the theorem is proved.

3. Corollary. *Let $L1 \le -1$, $\lambda \ge 0$, and $u \in W^2_p$. Then*

$$\|u_\pm\|_{\mathcal{L}_p} \le N \|(\lambda u - Lu)_\pm\|_{\mathcal{L}_p} \quad (7)$$

with the same N as in (4).

Indeed, it suffices to prove this for $u \in C_0^\infty$ and L with smooth coefficients. Also $a_- = (-a)_+$, which shows that we may concentrate on the sign $+$. In that case notice that

$$f := (\lambda u - Lu)_+$$

is a continuous function with compact support. Take a $\zeta \in C_0^\infty$ such that $\zeta \geq 1$ on the support of f and for any $n \geq 1$ find $f_n \in C_0^\infty$ such that $f \leq f_n \leq f + \zeta/n$. After this define w_n as the unique W_p^2 solution of

$$(\lambda - L)w_n = f_n.$$

By regularity results w_n is bounded and infinitely differentiable. By the maximum principle for smooth functions $w_n \geq 0$. Also $(\lambda - L)(w_n - u) \geq 0$. Hence $w_n - u \geq 0$ by Exercise 5.2. We conclude $u_+ \leq w_n$ and (7) with $+$ follows from (4) by applying it to w_n in place of u and letting $n \to \infty$.

4. Corollary (Maximum principle). *Let $L1 \leq -\delta$, where $\delta > 0$ is a constant, $u \in W_p^2$ and $Lu \geq 0$. Then $u \leq 0$.*

Indeed, assuming that $\delta = 1$ does not restrict generality, and in this case for the $+$ sign and $\lambda = 0$ the norm on the right of (7) vanishes, so that $u_+ = 0$.

5. Exercise. By using Exercise 5.5 and repeating the above arguments, prove that all results of this section are true in the two dimensional case if $a = (a^{ij})$ is a (measurable) 2×2 symmetric matrix-valued function defined on $\mathbb{R}^2$ and satisfying

$$\kappa^{-1}|\xi|^2 \geq a^{ij}(x)\xi^i\xi^j \geq \kappa|\xi|^2$$

for all $x, \xi \in \mathbb{R}^2$, and b^1, b^2, and c are (measurable) functions on $\mathbb{R}^2$ satisfying $|b^i| + |c| \leq K$, $c \leq -\kappa$.

The following exercises are intended to justify some arguments in Section 4. It is always assumed that

$$L1 \leq -\delta,$$

where $\delta > 0$ is a constant.

6. Exercise. Let $2p > d$, $q = p/(p-1)$. Notice that any $u \in W_p^2$ is a continuous function (more precisely has a continuous modification) so that speaking about $u(x)$ for any particular x makes perfect sense. Prove the following.

(i) For any $x \in \mathbb{R}^d$ and $u \in W_p^2$ we have $|u(x)| \leq N\|Lu\|_{\mathcal{L}_p}$ with N independent of x, u.

(ii) For any $x \in \mathbb{R}^d$ there exists a $g(x, \cdot) \in \mathcal{L}_q$ such that

$$u(x) = -\int_{\mathbb{R}^d} g(x, y)Lu(y)\,dy \quad \forall u \in W_p^2. \tag{8}$$

(iii) For any $x \in \mathbb{R}^d$, the conditions $g(x, \cdot) \in \mathcal{L}_q$ and (8) define $g(x, \cdot)$ uniquely.

This function $g(x, y)$ is called *the Green's function* of L.

7. Exercise. Prove that for any $x \in \mathbb{R}^d$ we have $g(x, y) \geq 0$ (a.e.). By the way, for operators with coefficients which are only continuous, there are no "usual" pointwise estimates of g, neither from above nor from below through the Green's function of the Laplacian.

8. Exercise. Define

$$Rf(x) = \int_{\mathbb{R}^d} g(x, y) f(y)\, dy.$$

Prove that for any $r \in (1, \infty)$ and $f \in \mathcal{L}_r$, we have $R|f| \in W_r^2$ and $LRf = -f$.

9. Exercise (Hard). Define

$$g_\rho(x, y) = \frac{1}{|B_\rho(y)|} \int_{B_\rho(y)} g(x, z)\, dz$$

where $|B_\rho(y)|$ is the volume of $B_\rho(y)$. Prove that, for any $r > 0$, $p \in (1, \infty)$, and almost any $y \in \mathbb{R}^d$, the functions $g_\rho(x, y)$ converge as $\rho \downarrow 0$ at least along a subsequence uniformly in $B_r^c(y)$ and in $W_p^2(B_r^c(y))$ to a function $\bar{g}(x, y)$ satisfying $L(x)\bar{g}(x, y) = 0$ (a.e.). Notice that by the Lebesgue differentiation theorem, for each x we have $\bar{g}(x, y) = g(x, y)$ for almost all y. Therefore, one can take $\bar{g}$ in place of g from the very beginning and then one has $L(x)g(x, y) = 0$ for almost every x and y.

7. Traces of $W_p^k(\Omega)$ functions on $\partial\Omega$

The issue of defining traces of W_p^k functions on hypersurfaces arises in many situations. An immediate use of traces is in stating and proving the maximum principle for W_p^2 functions not necessarily vanishing on the boundary. For such functions we need to explain the meaning of the inequality $u|_{\partial\Omega} \leq 0$.

Also above almost exclusively we have only dealt with the Dirichlet boundary condition which is interpreted in Theorem 8.5.3 as $u - g \in \overset{0}{W}{}_p^1(\Omega)$. In Section 9.3 a similar meaning was given to the oblique derivative boundary condition. What follows below will allow us to give these conditions the most natural sense.

1. Exercise*. Let $p \in [1, \infty)$ and let $u(t)$ be a *continuous* $\mathcal{L}_p(\mathbb{R}^{d-1})$-valued function defined for $t \in [0, 1]$. For each $t \in [0, 1]$ denote by $u^{(\varepsilon)}(t)$ a mollification of $u(t)$ as an element of $\mathcal{L}_p(\mathbb{R}^{d-1})$ by using a $C_0^\infty(\mathbb{R}^{d-1})$ function with unit integral. Prove the following

(i) The $\mathcal{L}_p(\mathbb{R}^{d-1})$-valued functions $u^{(\varepsilon)}(t)$ are equicontinuous as functions on $[0, 1]$ and, as $\varepsilon \to 0$, converge in $\mathcal{L}_p(\mathbb{R}^{d-1})$ uniformly with respect to $t \in [0, 1]$.

(ii) On $[0, 1] \times \mathbb{R}^{d-1}$ there is a Borel measurable function v such that $v(t, \cdot) = u(t)$ (equality in the sense of $\mathcal{L}_p(\mathbb{R}^{d-1})$) for any $t \in [0, 1]$.

For any domain Ω, any set $\Gamma \subset \bar{\Omega}$ and $g \in C(\bar{\Omega})$ we denote the restriction or trace of g on Γ by $\mathcal{T}_\Gamma g$. Remember that as everywhere in the chapter

$$p \in (1, \infty)$$

unless explicitly stated otherwise. In this section we assume that $d \geq 2$.

2. Theorem. *Let $\Omega = \mathbb{R}^d_+$, $q = p/(p-1)$, and $e := (1, 0, \ldots, 0)$. Then for any $t \geq 0$ there exists a unique operator $\mathcal{T}_{te+\partial\Omega}$, which is a bounded operator from $W^1_p(\Omega)$ to $\mathcal{L}_p(\partial\Omega)$ and which coincides with $\mathcal{T}_{te+\partial\Omega}$ as defined before the theorem on the set*

$$C(\bar{\Omega}) \cap W^1_p(\Omega).$$

For any $x^1 \geq 0$ one can choose such a realization of $\mathcal{T}_{x^1e+\partial\Omega}u(x')$ that the function $\mathcal{T}_{x^1e+\partial\Omega}u(x')$ as a function of x on $\bar{\Omega}$ becomes Borel measurable and becomes a modification of u. Furthermore, for any $u \in W^1_p(\Omega)$

$$\|\mathcal{T}_{se+\partial\Omega}u - \mathcal{T}_{te+\partial\Omega}u\|_{\mathcal{L}_p(\partial\Omega)} \leq |s-t|^{1/q}\|u_{x^1}\|_{\mathcal{L}_p(\Omega)} \leq |s-t|^{1/q}\|u\|_{W^1_p(\Omega)}, \tag{1}$$

$$\|\mathcal{T}_{te+\partial\Omega}u\|_{\mathcal{L}_p(\partial\Omega)} \leq 2\|u\|_{W^1_p(\Omega)}. \tag{2}$$

Proof. First we prove (1) and (2) for $u \in C(\bar{\Omega}) \cap W^1_p(\Omega)$ for which $\mathcal{T}_{te+\partial\Omega}u$ is set to be $u(t, \cdot)$ by definition.

To do this, we show that for any $t > s \geq 0$ and $\phi \in C^\infty_0(\mathbb{R}^{d-1})$

$$\int_{\mathbb{R}^{d-1}} \phi(x')[u(t, x') - u(s, x')]\, dx' = \int_s^t \int_{\mathbb{R}^{d-1}} \phi(x') D_1 u(r, x')\, dx' dr. \tag{3}$$

Notice that, due to the continuity of integrals and the continuity and boundedness of u, both parts of (3) are continuous functions of s, t. Therefore we may concentrate on $t > s > 0$. In that case we consider the Sobolev mollifiers of u. These are infinitely differentiable functions in Ω converging to u in the W^1_p norm on $[s, \infty) \times \mathbb{R}^{d-1}$, or even on Ω if the kernel is chosen properly (see Theorem 1.8.5). They also converge to u uniformly on each closed compact subset of Ω again owing to the continuity of u. Finally, for smooth u, (3) is just the Newton-Leibnitz formula, so it holds for the mollified u. The above argument allows us to pass to the limit from the mollified to the original function u, which proves (3).

Now we take the sups of both parts in (3) with respect to ϕ satisfying

$\|\phi\|_{\mathcal{L}_q(\partial\Omega)} = 1$ and use Hölder's inequality to show that the right-hand side of (3) is less than

$$\int_s^t \|D_1 u(r,\cdot)\|_{\mathcal{L}_p(\partial\Omega)}\,dr \le (t-s)^{1/q}\Big(\int_s^t \|D_1 u(r,\cdot)\|^p_{\mathcal{L}_p(\partial\Omega)}\,dr\Big)^{1/p}$$

$$\le (t-s)^{1/q}\|u_{x^1}\|_{\mathcal{L}_p(\Omega)}.$$

Then we get that (1) holds indeed. Furthermore,

$$\|u(t,\cdot)\|_{\mathcal{L}_p(\partial\Omega)} \le \|u(s,\cdot)-u(t,\cdot)\|_{\mathcal{L}_p(\partial\Omega)} + \|u(s,\cdot)\|_{\mathcal{L}_p(\partial\Omega)}.$$

By integrating this with respect to $s \in (t, t+1)$ and using Hölder's inequality, we also get (2).

The rest of the proof is almost trivial. Indeed, the set $C(\bar{\Omega}) \cap W_p^1(\Omega)$ is dense in $W_p^1(\Omega)$ and if

$$u_n \in C(\bar{\Omega}) \cap W_p^1(\Omega), \quad u_n \to u \in W_p^1(\Omega) \quad \text{in} \quad W_p^1(\Omega), \tag{4}$$

then u_n is a Cauchy sequence in $W_p^1(\Omega)$ and by (2) the sequence $u_n(t,\cdot)$ is a Cauchy sequence in $\mathcal{L}_p(\partial\Omega)$ for each $t \ge 0$. Then it converges in $\mathcal{L}_p(\partial\Omega)$ to a limit, say $w(t,\cdot)$. This limit will be the same (as an element of $\mathcal{L}_p(\partial\Omega)$) if we take a different sequence v_n satisfying (4) since one can apply the above argument to the sequence $u_1, v_1, u_2, v_2, \ldots$. Therefore, the notation

$$\mathcal{T}_{te+\partial\Omega}u = w(t,\cdot)$$

makes perfect sense. For $\mathcal{T}_{te+\partial\Omega}u$ so defined, one obtains both (1) and (2) by passing to the limit in the corresponding inequalities valid for u_n.

By (1) and Exercise 1 (ii) the function $\mathcal{T}_{te+\partial\Omega}u(x')$ admits a version which is jointly measurable in (t,x') and by $\mathcal{T}_{te+\partial\Omega}u(x')$ we always mean one of these versions. Finally, that $\mathcal{T}_{x^1e+\partial\Omega}u$ is a modification of u follows from the fact that by Fubini's and Fatou's theorems we have

$$\int_\Omega |w-u|^p\,dx = \int_0^\infty \|w(t,\cdot)-u(t,\cdot)\|^p_{\mathcal{L}_p(\partial\Omega)}\,dt$$

$$\le \varliminf_{n\to\infty}\int_0^\infty \|u_n(t,\cdot)-u(t,\cdot)\|^p_{\mathcal{L}_p(\partial\Omega)}\,dt = \lim_{n\to\infty}\|u_n-u\|^p_{\mathcal{L}_p(\Omega)} = 0.$$

The theorem is proved.

3. Remark. Theorem 2 admits an obvious version with $\mathbb{R}^d$ and $\mathbb{R}^{d-1}$ in place of $\mathbb{R}^d_+$ and $\partial\Omega$, respectively. The proofs of this version is obtained by repeating the above proof.

Theorem 2 shows that the traces of $W_p^1(\mathbb{R}_+^d)$ functions form a subspace in $L_p(\mathbb{R}^{d-1})$. If $p=2$, it is possible to describe this set as $H_2^{1/2}(\mathbb{R}^{d-1})$ (cf. Exercise 12.3.1), where the spaces H_p^γ are introduced in Chapter 13. For general $p \in (1,\infty)$ the set of traces is known to be the Slobodetskii space $W_p^{1-1/p}(\mathbb{R}^{d-1})$. Later in Theorem 13.7.2 we will give a sufficient condition for a function on $\mathbb{R}^{d-1}$ to be a trace of a $W_p^1(\mathbb{R}_+^d)$ function.

Theorem 2 says, in particular, that restricting derivatives of order $\le k-1$ of functions in $W_p^k(\mathbb{R}_+^d)$ on the set $x^1 = 0$ is a continuous operator from $W_p^k(\mathbb{R}_+^d)$ to $\mathcal{L}_p(\mathbb{R}^{d-1})$.

4. Definition. Let $\Omega = \mathbb{R}_+^d$, $k \in \{1,2,...\}$, $u \in W_p^k(\Omega)$, and let α be a multi-index with $|\alpha| \le k-1$. Then $D^\alpha u \in W_p^1(\Omega)$ and

$$\mathcal{T}_{te+\partial\Omega}(D^\alpha u)$$

is well defined. For $t \ge 0$, by $D^\alpha u(t,x')$ we always mean

$$\mathcal{T}_{te+\partial\Omega}(D^\alpha u)(x')$$

and by $D^\alpha u|_{\partial\Omega}$ we mean $\mathcal{T}_{\partial\Omega}(D^\alpha u)$.

5. Corollary. *If $u \in W_p^k(\mathbb{R}_+^d)$, $|\alpha| \le k$, $\alpha = e_1 + \beta$, $\beta = (0,\beta_2,...,\beta_d)$, then for any $t > s \ge 0$ and any function $\phi \in C^\infty([s,t]\times\mathbb{R}^{d-1})$, whose support is a bounded subset of $[s,t]\times\mathbb{R}^{d-1}$, we have*

$$\int_{[s,t]\times\mathbb{R}^{d-1}} uD^\alpha\phi\,dx = (-1)^{|\beta|}\int_{\mathbb{R}^{d-1}} \phi(t,x')D^\beta u(t,x')\,dx'$$

$$-(-1)^{|\beta|}\int_{\mathbb{R}^{d-1}} \phi(s,x')D^\beta u(s,x')\,dx' + (-1)^{|\alpha|}\int_{[s,t]\times\mathbb{R}^{d-1}} \phi D^\alpha u\,dx.$$

Indeed, for defining functions we get the equality just integrating by parts and for arbitrary $u \in W_p^k(\mathbb{R}_+^d)$ the assertion follows from the continuity of

$$\mathcal{T}_{re+\mathbb{R}^{d-1}} : W_p^1(\mathbb{R}_+^d) \to \mathcal{L}_p(\mathbb{R}^{d-1})$$

for each $r \ge 0$ and Hölder's inequality.

The extension of the following result for smooth domains (Theorem 7 (ii)) is used in proving the maximum principle for W_p^2 functions.

6. Theorem. *Let* $\Omega = \mathbb{R}^d_+$, $k \in \{1, 2, ...\}$, $u \in W^k_p(\Omega)$.

(i) If g is a uniformly continuous function given on $\partial\Omega$ and if $u|_{\partial\Omega} \leq g$ (a.e.) on $\partial\Omega$, then there is a sequence $u_n \in C^n(\bar{\Omega}) \cap W^k_p(\Omega)$ such that

$$u_n \leq g + 1/n$$

on $\partial\Omega$ and $u_n \to u$ in $W^k_p(\Omega)$.

(ii) If $u \geq 0$ (a.e.), then $u|_{\partial\Omega} \geq 0$ (a.e. with respect to the surface measure).

Proof. As in Theorem 1.8.5 take a nonnegative $\zeta \in C^\infty_0$ with integral equal to one and such that $\zeta(x) = 0$ for $x^1 \geq 0$ and define

$$v = uI_\Omega.$$

However, this time it is convenient to additionally assume that $\zeta(x) = 0$ for $x^1 \leq -1$ and to take

$$v_\varepsilon(x) := \varepsilon^{-\gamma-(d-1)} \int_{\mathbb{R}^d} v(y)\zeta((x^1 - y^1)/\varepsilon^\gamma, (x' - y')/\varepsilon)\, dy$$

$$= \int_{\mathbb{R}^d} v(x^1 - \varepsilon^\gamma y^1, x' - \varepsilon y')\zeta(y)\, dy, \quad \gamma := (p + d - 1)/(p - 1).$$

In literally the same way as in the proofs of Lemma 1.8.2 and Theorem 1.8.5 one proves that $v_\varepsilon \in C^n(\bar{\Omega}) \cap W^k_p(\Omega)$ for any n and $v_\varepsilon \to u$ in $W^k_p(\Omega)$ as $\varepsilon \downarrow 0$. This immediately implies assertion (ii). It follows also that to prove assertion (i), it only remains to show that, under the assumptions in (i), for any $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$

$$v_\varepsilon \leq g + \delta \quad \text{on} \quad \partial\Omega. \tag{5}$$

For convenience denote $\eta(x) = \zeta(-x)$. Then (recall that $\eta(y^1, y') = 0$ if $y^1 \geq 1$)

$$v_\varepsilon(0, x') = \int_\Omega v(\varepsilon^\gamma y^1, x' + \varepsilon y')\eta(y)\, dy$$

$$= \int_0^1 \Big(\int_{\mathbb{R}^{d-1}} u(\varepsilon^\gamma y^1, x' + \varepsilon y')\eta(y)\, dy'\Big)\, dy^1 \leq I_1 + I_2,$$

where

$$I_1 = \int_0^1 \Big(\int_{\mathbb{R}^{d-1}} g(x' + \varepsilon y')\eta(y)\, dy'\Big)\, dy^1,$$

$$I_2 = \int_0^1 \Big(\int_{\mathbb{R}^{d-1}} |u(\varepsilon^\gamma y^1, x' + \varepsilon y') - u(0, x' + \varepsilon y')|\eta(y)\, dy' \Big)\, dy^1.$$

By (1) and Hölder's inequality

$$I_2 \le N \int_0^1 \|u(\varepsilon^\gamma y^1, x' + \varepsilon \cdot) - u(0, x' + \varepsilon \cdot)\|_{\mathcal{L}_p(\partial\Omega)}\, dy^1$$

$$= N\varepsilon^{(1-d)/p} \int_0^1 \|u(\varepsilon^\gamma y^1, \cdot) - u(0, \cdot)\|_{\mathcal{L}_p(\partial\Omega)}\, dy^1$$

$$\le N\varepsilon^{(1-d)/p+\gamma/q} \int_0^1 (y^1)^{1/q}\, dy^1 = N\varepsilon^{(1-d)/p+\gamma/q} \to 0$$

as $\varepsilon \downarrow 0$.

Finally, using the fact that η integrates to one, we conclude

$$I_1 \le \int_0^1 \Big(\int_{\mathbb{R}^{d-1}} g(x' + \varepsilon y')\eta(y)\, dy' \Big)\, dy^1$$

$$\le g(x') + \sup\{|g(x') - g(z')| : |x' - z'| \le \varepsilon R\}, \tag{6}$$

where the constant R is such that $\eta(x) = 0$ for $|x| \ge R$. The last term in (6) tends to zero as $\varepsilon \downarrow 0$ by the assumption that g is uniformly continuous. This shows how to choose ε to satisfy (5). The theorem is proved.

Theorems 2 and 6 admit modifications for smooth domains. If $\Omega \in C^1$, then the *surface measure* $\sigma_{\partial\Omega}$ is well defined. By $\mathcal{L}_p(\partial\Omega)$ we denote the $\mathcal{L}_p$ space of functions on $\partial\Omega$ relative to $\sigma_{\partial\Omega}$.

7. Theorem. *(i) If $\Omega \in C^1$, then the operator $\mathcal{T}_{\partial\Omega}$ restricted to the set $C(\bar\Omega) \cap W_p^1(\Omega)$ uniquely extends from this set to a bounded operator from $W_p^1(\Omega)$ to $\mathcal{L}_p(\partial\Omega)$.*

(ii) If $k \in \{1, 2, ...\}$, $\Omega \in C^k$, g is a continuous function on $\partial\Omega$, $u \in W_p^k(\Omega)$, and

$$\mathcal{T}_{\partial\Omega} u \le g$$

($\sigma_{\partial\Omega}$-a.e.), then there exists a sequence $u_n \in C^k(\bar\Omega)$ such that $u_n \le g$ on $\partial\Omega$ and $u_n \to u$ in $W_p^k(\Omega)$.

(iii) If $\Omega \in C^1$, $u \in W_p^1(\Omega)$, and $u \ge 0$ (a.e.) in Ω, then $\mathcal{T}_{\partial\Omega} u \ge 0$ ($\sigma_{\partial\Omega}$-a.e.).

Proof. (i) As in the end of the proof of Theorem 2 it suffices to show that, if (4) holds, then u_n is a Cauchy sequence in $\mathcal{L}_p(\partial\Omega)$. Furthermore since $\partial\Omega$ is a bounded set, it suffices to prove that u_n is a Cauchy sequence in $\mathcal{L}_p(\partial\Omega \cap B_{\rho_0/2}(z))$ for any $z \in \partial\Omega$, where ρ_0 is taken from Definition 8.3.1. Next by taking $\zeta \in C_0^\infty(B_{\rho_0}(z))$ which is 1 in $B_{\rho_0/2}(z)$, we reduce our task to proving that $u_n\zeta$ is a Cauchy sequence in $\mathcal{L}_p(\partial\Omega \cap B_{\rho_0}(z))$. We flatten the boundary near z by using the corresponding transformations Ψ and Ψ^{-1} from Section 8.3 and then we get that the sequence of functions $\Psi^{-1}(u_n\zeta)$ (more precisely equal to $\Psi^{-1}(u_n\zeta)$ in D_+^z and extended as zero to the remaining part of $\mathbb{R}_+^d$) is a Cauchy sequence in $W_p^1(\mathbb{R}_+^d)$ (see Lemma 8.3.4). By Theorem 2 the restriction of $\Psi^{-1}(u_n\zeta)$ on $\partial\mathbb{R}_+^d$ is a Cauchy sequence in $\mathcal{L}_p(\partial\mathbb{R}_+^d)$. By using the change of variable formula, we get that $u_n\zeta = \Psi\Psi^{-1}(u_n\zeta)$ is a Cauchy sequence in $\mathcal{L}_p(\partial\Omega \cap B_{\rho_0}(z))$. This proves (i).

(ii) Since the argument is straightforward, we confine ourselves to a few comments only. Once again we can use flattening of the boundary and also partitions of unity and Theorem 6. Although that theorem provides us with approximating sequences as smooth in the usual sense as we like, the transformation Ψ^{-1} only preserves functions of class C^k. On the other hand, Theorem 6 yields functions which are less than $g + 1/n$ on the boundary, but since Ω is bounded, we can always add or subtract constants from our functions, preserving their membership in $W_p^k(\Omega)$.

The same observations prove (iii) on the basis of Theorem 6. The theorem is proved.

8. Definition. If $\Omega \in C^1$, $k \in \{1, 2, ...\}$, α is a multi-index with $|\alpha| \leq k-1$, and $u \in W_p^k(\Omega)$, then $D^\alpha u \in W_p^1(\Omega)$ and $\mathcal{T}_{\partial\Omega}(D^\alpha u)$ is well defined by Theorem 7. By $D^\alpha u|_{\partial\Omega}$ or by $D^\alpha u$ on $\partial\Omega$ we mean $\mathcal{T}_{\partial\Omega}(D^\alpha u)$.

9. Remark. By definition $\overset{0}{W}{}_p^1(\Omega)$ functions can be approximated in $W_p^1(\Omega)$ by $C^1(\bar\Omega)$ functions vanishing on $\partial\Omega$ and thus having zero trace on $\partial\Omega$. Therefore, by Theorems 2 and 7, if $\Omega = \mathbb{R}_+^d$ or $\Omega \in C^1$ and $u \in \overset{0}{W}{}_p^1(\Omega)$, then $u|_{\partial\Omega} = 0$ as an $\mathcal{L}_p(\partial\Omega)$ function. In particular, if additionally $u \in C(\bar\Omega)$, then by definition $u|_{\partial\Omega}$ is just the restriction of u on $\partial\Omega$ and therefore $u \equiv 0$ on $\partial\Omega$.

10. Exercise. Prove that, if $\Omega = \mathbb{R}_+^d$ and $u \in \overset{0}{W}{}_p^2(\Omega)$, then $D_j u = 0$ on $\partial\Omega$ for $j = 2, 3, ..., d$.

The following exercise contains a characterization of $\overset{0}{W}{}_p^1$ spaces in terms of traces.

11. Exercise. Let $\Omega = \mathbb{R}_+^d$ or $\Omega \in C^1$ and $u \in W_p^1(\Omega)$. Then prove that $u \in \overset{0}{W}{}_p^1(\Omega)$ if and only if $u|_{\partial\Omega} = 0$.

12. Exercise. Let $u \in W_p^2(\mathbb{R}_+^d)$. Prove that $D_1 u(0, \cdot)$, understood in the sense of Definition 4, is zero if and only if there is a sequence u_n defining u in $W_p^2(\mathbb{R}_+^d)$ such that $D_1 u_n(0, x') = 0$.

13. Exercise. Complement the result of Exercise 8.1.4. Let $u \in W^2_p(\mathbb{R}^d_+)$ define $v(x) = u(|x^1|, x')$ and prove that $v \in W^2_p$ if and only if $D_1 u(0, \cdot)$, understood in the sense of Definition 4, is zero.

8. The maximum principle in W^2_p. Green's functions

In this section we assume that $L1 \leq 0$.

1. Theorem (Maximum principle). *Let $\Omega \in C^2$. Then there exists a constant N depending only on* $\operatorname{diam}\Omega, d, K, \kappa, \omega$, *and p such that for any $u \in W^2_p(\Omega)$ satisfying $u|_{\partial\Omega} \leq 0$ and $\lambda \geq 0$ we have*

$$\|u_+\|_{\mathcal{L}_p(\Omega)} \leq N\|(\lambda u - Lu)_+\|_{\mathcal{L}_p(\Omega)}. \tag{1}$$

In particular, if $u \in W^2_p(\Omega)$, $u|_{\partial\Omega} \leq 0$, and $Lu \geq 0$ in Ω, then $u \leq 0$ in Ω (a.e.).

Proof. Obviously, it suffices to prove (1). Furthermore, owing to Theorem 7.7, we may assume that $u \in C^2(\bar{\Omega})$. After that it only remains to repeat the proof of Theorem 3.3. The theorem is proved.

2. Corollary. *Let either*

(i) $\Omega_j \in C^2$, $j = 1, 2$, $\Omega_1 \subset \Omega_2$, or

(ii) $\Omega_1 \in C^2$, $\Omega_2 = \mathbb{R}^d$, and $L1 \leq -\delta$ for a constant $\delta > 0$.

Let $f \in \mathcal{L}_p(\Omega_2)$, $f \geq 0$. Denote by u_j the unique solution in $\overset{0}{W}{}^2_p(\Omega_j)$ of $Lu_j = -f$. Then $0 \leq u_1 \leq u_2$ in Ω_1.

Indeed, by Theorem 1 in the case $\Omega_2 \in C^2$ and by Corollary 6.4 in case $\Omega_2 = \mathbb{R}^d$ applied to $-u_j$, we get that $u_j \geq 0$. By Theorem 7.7 this implies $u_2|_{\partial\Omega_1} \geq 0 = u_1|_{\partial\Omega_1}$ and again by Theorem 1 applied to $u_1 - u_2$ in Ω_1 we conclude $0 \leq u_1 \leq u_2$ in Ω_1.

3. Exercise. Assume that $L1 \leq -\delta$ with a constant $\delta > 0$. Then prove that N in (1) is independent of Ω.

4. Exercise. Let $\Omega \in C^2$, $M \geq 0$ be a constant, $u \in W^2_p(\Omega)$ satisfy $u|_{\partial\Omega} \leq M$ and $Lu \geq 0$ in Ω. Prove that $u \leq M$ in Ω.

5. Exercise. Let $\Omega \in C^2$ and $d \geq 2$. Prove that for any $x \in \Omega$ there exists a unique (up to a.e. in Ω) function $g_\Omega(x, y) \geq 0$ satisfying $g_\Omega(x, \cdot) \in \mathcal{L}_q(\Omega)$ for any $q \in [1, d/(d-2))$ and such that, for any $p \in (d/2, \infty)$,

$$u(x) = -\int_\Omega g_\Omega(x, y) Lu(y)\, dy \quad \forall u \in \overset{0}{W}{}^2_p(\Omega).$$

The function $g_\Omega(x, y)$ is called *the Green's function* of L in Ω.

6. Exercise. Prove that if $\Omega_1, \Omega_2 \in C^2$ and $\Omega_1 \subset \Omega_2$, then for any $x \in \Omega_1$ we have $g_{\Omega_1}(x,y) \le g_{\Omega_2}(x,y)$ for almost all $y \in \Omega_1$.

7. Exercise. Let Ω be a bounded domain. Take an expanding sequence of $\Omega_n \in C^2$ such that $\bar{\Omega}_n \subset \Omega_{n+1}$ and $\bigcup_n \Omega_n = \Omega$ and define

$$g_\Omega = \lim_{n\to\infty} g_{\Omega_n}.$$

Prove that the limit is independent of the sequence Ω_n and, for any $p \in (1,\infty)$ and $f \in \mathcal{L}_p(\Omega)$, the function

$$u(x) = \int_\Omega g(x,y)f(y)\,dy$$

is well defined for almost any $x \in \Omega$, belongs to $W^2_{p,loc}(\Omega) \cap \mathcal{L}_p(\Omega)$ and satisfies $Lu = -f$ in Ω.

8. Exercise. Let $L = \Delta$ and $\Omega \in C^2$. Prove that $g_\Omega(x,y) = g_\Omega(y,x)$ for almost all $x, y \in \Omega$.

9. Hints to exercises

1.5. Denote by M the right-hand side of the first inequality and find $(\lambda - L)(u - M)$.

1.7. You need only prove that 0 does not belong to the spectrum of L. If it does, then for any large q there exists a nonzero $u \in \overset{0}{W}{}^2_q(\Omega)$ satisfying $Lu = 0$. This is impossible in case (ii) by the assumption. In (i) by using interior regularity results derive that

$$u \in \overset{0}{W}{}^3_{q,loc}(\Omega) \cap \overset{0}{W}{}^2_q(\Omega),$$

which by embedding theorems implies that $u \in C^2_{loc}(\Omega) \cap C(\bar{\Omega})$ and $u = 0$ on $\partial\Omega$. Conclude that $u = 0$.

1.8. Take λ_1 from Theorem 8.5.3 and take operators L_n with smooth coefficients, solve the equations $\lambda_1 u_n - L_n u_n = f + \lambda_1 u$ in $\overset{0}{W}{}^2_p(\Omega)$, prove that $u_n \in C^2_{loc}(\Omega)$, apply Exercise 1.5 to u_n and use Exercise 8.5.9.

1.9. (i) As in the hint to Exercise 1.7 it suffices to show that, if $u \in \overset{0}{W}{}^2_{2d}(\Omega)$ and $Lu = 0$, then $u = 0$. Then Exercise 1.8 helps. (ii) Use the operator L' from the proof of Theorem 1.3.

1.10. Assume the contrary. Then there exist sequences r_n, λ_n, L_n, u_n such that

$$0 < r_n \le 1/n, \quad \lambda_n \ge 0,$$

the operators L_n satisfy the same conditions as L with the same K, κ, and ω, $u_n \in \overset{0}{W}{}^2_p(B_{r_n})$ and

$$\|u_n\|_{\mathcal{L}_p(B_{r_n})} > n r_n^2 \|\lambda_n u_n - L_n u_n\|_{\mathcal{L}_p(B_{r_n})}.$$

Denote $v_n(x) = u_n(x r_n)$ and derive that

$$\|v_n\|_{\mathcal{L}_p(B_1)} > n\|\mu_n v_n - M_n v_n\|_{\mathcal{L}_p(B_1)},$$

where $\mu_n = r_n^2\lambda_n$ and the M_n are elliptic operators whose coefficients tend to some constants along a subsequence. Conclude that the μ_n are bounded, and assuming that

$$\|v_n\|_{\mathcal{L}_p(B_1)} = 1, \tag{1}$$

prove that along a subsequence the v_n tend weakly in $W_p^2(B_1)$ to a function $v \in \overset{0}{W}{}_p^2(B_1)$ (see Corollary 8.1.3) satisfying

$$\|v\|_{\mathcal{L}_p(B_1)} = 1, \quad \bar{a}^{ij}D_{ij}v - \mu v = 0,$$

where $\bar{a}^{ij}$ is a constant positive matrix and $\mu > 0$. Finally, prove that v is infinitely differentiable in $\bar{\Omega}$ and $v = 0$ in contradiction to (1).

1.12. Proceed as in the hint to Exercise 1.10.

3.5. (i) Apply Kondrashov's theorem and Corollary 8.1.3. (ii) You may like to use an argument similar to the one in the very end of the proof of Theorem 10.5.1. (iii) Observe that

$$\|Tv\|_{W_p^2(\Omega)} \le N_0(\|f\|_{\mathcal{L}_p(\Omega)} + \|v\|^{\alpha}_{W_p^2(\Omega)});$$

then take $\varepsilon > 0$ so small that $M > N_0(\varepsilon + M^{\alpha})$. (iv) It suffices to prove that u is bounded. If $p > d/2$, the boundedness follows from embedding theorems. If $p < d/2$, iterate:

$$u \in \mathcal{L}_{\alpha p}(\Omega) \Longrightarrow u \in \overset{0}{W}{}_p^2(\Omega) \Longrightarrow u \in \mathcal{L}_{\alpha p^*}(\Omega),$$

where $p^* = pd/(\alpha d - 2\alpha p) > p$.

5.2. Divide all the coefficients of L by δ.

5.5. (i) From Exercise 1.6.7 we know that the estimate holds with $u\mu \operatorname{tr} a - Lu$ in place of $\lambda u - Lu$ for large μ. Corollary 5.4 allows us to replace $\mu \operatorname{tr} a$ with λ if $\lambda \ge \mu \operatorname{tr} a$, the only trouble being that one has to find v satisfying its assumptions with, say, $g = 0$. To this end notice that from the very beginning one may assume that a, b, c are infinitely differentiable and $u \in C_0^\infty$. In that case $|\lambda u - Lu|$ is Lipschitz continuous and by the $\mathcal{L}_p$ theory there is a $v \in W_p^3$, $p > 1$, satisfying $v\mu \operatorname{tr} a - Lv = |\lambda u - Lu|$. By embedding theorems $v \in C^2(\mathbb{R}^2)$.

Observe that one can also avoid using the $\mathcal{L}_p$ theory by finding for μ large a $v \in W_2^2$ which is infinitely differentiable in $\mathbb{R}^2$ and satisfies

$$\varepsilon\zeta + |\lambda u - Lu| \ge v\mu \operatorname{tr} a - Lv \ge |\lambda u - Lu|$$

in $\mathbb{R}^2$ for as small ε as we wish and a $\zeta \in C_0^\infty$. Notice that by embedding theorems for W_2^n ($p = 2$) we have $v \in C(\mathbb{R}^2)$. (ii) The result of (i) opens up the possibility of repeating almost literally our treatment of equations in smooth domains. Notice that for the operators of type $L - \mu \operatorname{tr} a$ this is impossible since flattening the boundary breaks the special structure of them. (iii) Remark 2.2 shows that the proof of Lemma 3.1 is valid for $p = 2$.

7.1. (ii) By multiplying $u(t)$ by indicators of balls in $\mathbb{R}^{d-1}$, reduce the case of arbitrary p to $p = 1$. Then find ε_k such that

$$\sup_{t\in[0,1]} \|u(t) - u^{(\varepsilon_k)}(t)\|_{\mathcal{L}_1(\mathbb{R}^{d-1})} \leq k^{-2},$$

observe that, for any $t \in [0, 1]$,

$$\sum_k |u^{(\varepsilon_{k+1})}(t) - u^{(\varepsilon_k)}(t)| < \infty$$

(a.e.), the limit of $u^{(\varepsilon_k)}(t)$ as $k \to \infty$ exists almost everywhere in $\mathbb{R}^{d-1}$, and denote by v this limit.

7.10. Recall Corollary 8.2.2.

7.11. The "only if" part is just Remark 7.9. To prove the "if" part, use Lemma 8.2.1, guess what the derivatives of $\bar{u}$ should be and prove your guess by using Exercise 1.3.17 and Corollary 7.5.

7.12. The "if" part follows from Exercise 7.11 and the fact that $D_1 u_n \in \overset{0}{W}{}^1_p(\mathbb{R}^d_+)$. To prove the "only if" part, introduce $v(x) = u(|x^1|, x')$, guess what the derivatives of v should be and prove your guess by using Exercise 1.3.17 and Corollary 7.5. You will conclude that $v \in W^2_p$. Then use mollifiers.

7.13. The "only if" part follows from Exercise 8.1.4. To prove the "if" part, use Exercise 7.12 combined with the fact that the $u_n(|x^1|, x')$ are twice continuously differentiable in $\mathbb{R}^d$.

8.5. If $p > d/2$, by embedding theorems the mapping $\mathcal{L}_p(\Omega) \ni f \to u(x)$, where u is the unique solution in $\overset{0}{W}{}^2_p(\Omega)$ of $Lu = f$, is a continuous linear functional on $\mathcal{L}_p(\Omega)$. Then use the Riesz representation theorem to determine $g(x, \cdot)$ and after that use the maximum principle to show that $g \geq 0$.

8.7. It suffices to concentrate on $f \geq 0$. In that case the functions

$$u_n = \int_{\Omega_n} g_{\Omega_n}(x, y) f(y)\, dy$$

are nonnegative and increase with n and are majorated by the solution $v \in \overset{0}{W}{}^2_p(B)$ of the equation $Lv = -f$ in a ball $B \supset \Omega$. After this use interior regularity results.

8.8. Use the Green's formula

$$\int_\Omega u \Delta v\, dx = \int_\Omega v \Delta u\, dx$$

valid for all $u, v \in C^2(\bar{\Omega})$ vanishing on $\partial\Omega$.

Chapter 12

Fourier transform and elliptic operators

We have already used the Fourier transform in Chapter 1. Here we will use it more systematically. Therefore, we recall some basics facts.

For a function $g(x)$ given on $\mathbb{R}^d$ its Fourier transform $F(g) = \tilde{g}$ is defined by the formula

$$F(g)(\xi) = \tilde{g}(\xi) = c_d \int_{\mathbb{R}^d} e^{-ix\,\xi} g(x)\,dx, \quad c_d = \frac{1}{(2\pi)^{d/2}}.$$

By $C_0^\infty = C_0^\infty(\mathbb{R}^d)$ we denote the space of all infinitely differentiable complex-valued functions on $\mathbb{R}^d$ with compact support. Spaces $\mathcal{L}_p$ are now spaces of complex-valued functions; in particular, $\mathcal{L}_2$ is the Hilbert space of square summable functions with the inner product

$$(f, g) = \int_{\mathbb{R}^d} f\bar{g}\,dx. \tag{1}$$

Generally we use notation (1) each time when $|fg| \in \mathcal{L}_1$.

We need the following facts.

- If $g \in \mathcal{L}_1$, then $\tilde{g}$ is well defined, continuous, and bounded:

$$|\tilde{g}(\xi)| \le c_d \int_{\mathbb{R}^d} |g(x)|\,dx.$$

Furthermore, if $g \in \mathcal{L}_1$ and $\tilde{g} = 0$, then $g = 0$ (a.e.).

- If $g \in C_0^\infty$, then, for any multi-index α,

$$D^\alpha \tilde{g}(\xi) = (-i)^{|\alpha|} F(\cdot^\alpha g)(\xi), \quad i^{|\alpha|}\xi^\alpha \tilde{g}(\xi) = F(D^\alpha g)(\xi),$$

which implies, in particular, that $\tilde{g} \to 0$ faster than $|\xi|^{-n}$ for any $n > 0$ as $|\xi| \to \infty$.

- $\tilde{g}$ can be defined for any function $g \in \mathcal{L}_2$ in such a way that $\tilde{g} \in \mathcal{L}_2$ and for any $f, g \in \mathcal{L}_2$ Parseval's identity holds:

$$(f, g) = \int_{\mathbb{R}^d} f\bar{g}\, dx = \int_{\mathbb{R}^d} \tilde{f}\bar{\tilde{g}}\, d\xi = (\tilde{f}, \tilde{g}).$$

- Parseval's identity also holds if $f \in \mathcal{L}_1$, g is bounded, $g \in \mathcal{L}_2$, and $\tilde{g} \in \mathcal{L}_1$, which is proved by using the above for $n \wedge f \vee (-n)$ and using the dominated convergence theorem as $n \to \infty$.
- For any $g \in \mathcal{L}_2$ we have

$$g(x) = c_d \int_{\mathbb{R}^d} e^{ix\cdot\xi} \tilde{g}(\xi)\, d\xi =: F^{-1}(\tilde{g})$$

almost everywhere, where the right-hand side is understood in the $\mathcal{L}_2$ sense.

If $g \in \mathcal{L}_1$, then $\tilde{g}$ is not necessarily integrable. For instance, if $d = 1$ and $g = e^{-|x|}\operatorname{sign} x$, then $\tilde{g}(\xi) = c\xi/(1 + |\xi|^2)$, where c is a constant.

Also as a reminder recall that, for $g \in \mathcal{L}_2$, its Fourier transform $\tilde{g}$ is defined as the $\mathcal{L}_2$ limit of the Fourier transforms $\tilde{g}_n$ for any sequence $g_n \to g$ in $\mathcal{L}_2$.

1. The space $\mathcal{S}$

1. Definition. By $\mathcal{S}$ we denote the set of all infinitely differentiable complex-valued functions ψ on $\mathbb{R}^d$ such that, for any multi-indices α and β, $x^\alpha D^\beta \psi(x)$ is bounded on $\mathbb{R}^d$. For a sequence $\psi_n \in \mathcal{S}$ and $\psi \in \mathcal{S}$ we write

$$\psi_n \xrightarrow{\mathcal{S}} \psi \quad \text{if} \quad \sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta(\psi_n(x) - \psi(x))| \to 0$$

for any multi-indices α and β.

2. Exercise*. Prove that if $\psi \in \mathcal{S}$, then $x^\alpha D^\beta \psi, D^\beta(x^\alpha \psi) \in \mathcal{S}$. Furthermore, if $\psi_n \xrightarrow{\mathcal{S}} \psi$, then $\psi_n \to \psi$ in $\mathcal{L}_p$ for any $p \in [1, \infty]$ and

$$x^\alpha D^\beta \psi_n \xrightarrow{\mathcal{S}} x^\alpha D^\beta \psi, \quad D^\beta(x^\alpha \psi_n) \xrightarrow{\mathcal{S}} D^\beta(x^\alpha \psi).$$

3. Lemma. *(i) We have $F(\mathcal{S}) = \mathcal{S}$ and, if $\psi_n \xrightarrow{\mathcal{S}} \psi$, then $\tilde{\psi}_n \xrightarrow{\mathcal{S}} \tilde{\psi}$.*

(ii) We have $F^{-1}(\mathcal{S}) = \mathcal{S}$ and, if $\psi_n \xrightarrow{\mathcal{S}} \psi$, then $F^{-1}(\psi_n) \xrightarrow{\mathcal{S}} F^{-1}(\psi)$.

Proof. To prove that $F(\mathcal{S}) = \mathcal{S}$ and $F^{-1}(\mathcal{S}) = \mathcal{S}$, it suffices to prove that $F(\mathcal{S}) \subset \mathcal{S}$ and $F^{-1}(\mathcal{S}) \subset \mathcal{S}$. Furthermore, since F and F^{-1} are quite similar, we may and will only concentrate on F. Notice that, for any $\psi \in \mathcal{S}$,

$$\sup_{\xi \in \mathbb{R}^d} |\tilde{\psi}(\xi)| \leq c_d \int_{\mathbb{R}^d} |\psi(x)|\, dx$$

$$\leq c_d \sup_{x \in \mathbb{R}^d} |(1+|x|^2)^d \psi(x)| \int_{\mathbb{R}^d} \frac{dx}{(1+|x|^2)^d} = N \sup_{x \in \mathbb{R}^d} |(1+|x|^2)^d \psi(x)|.$$

Hence from the formula

$$\xi^\alpha D^\beta \tilde{\psi} = (-i)^{|\alpha|+|\beta|} F(D^\alpha(\cdot^\beta \psi))$$

we get

$$\sup_{\xi \in \mathbb{R}^d} |\xi^\alpha D^\beta \tilde{\psi}| \leq N \sup_{x \in \mathbb{R}^d} |(1+|x|^2)^d D^\alpha (x^\beta \psi)|,$$

which along with Exercise 2 implies (i). The lemma is proved.

4. Exercise*. (i) Prove that for any $\psi \in \mathcal{S}$ there is a sequence $\psi_n \in C_0^\infty$ such that $\psi_n \xrightarrow{\mathcal{S}} \psi$. (ii) Let $\psi(x) = e^{-|x|^2/2}$ and let $\zeta \in C_0^\infty$ be such that $\zeta(x) = 1$ for $|x| \leq 1$. Define $\psi_n(x) = \psi(x)\zeta(x/n)$ and prove that $\tilde{\psi}_n(\xi) \to N e^{-|\xi|^2/2}$ uniformly in $\xi \in \mathbb{R}^d$, where N is a constant.

2. The notion of elliptic differential operator

1. Definition. Let $m \geq 1$ be an integer and let a^α be some (complex) numbers given for any multi-indices α such that $|\alpha| \leq m$. The operator

$$L = \sum_{|\alpha| \leq m} a^\alpha D^\alpha$$

is called an mth *order (differential) operator with constant coefficients*. It is called (mth order) *strongly elliptic* if both

$$\sum_{|\alpha|=m} a^\alpha \xi^\alpha \neq 0 \quad \text{for} \quad \xi \in \mathbb{R}^d \setminus \{0\}, \qquad \sum_{|\alpha| \leq m} a^\alpha i^{|\alpha|} \xi^\alpha \neq 0 \quad \text{for} \quad \xi \in \mathbb{R}^d.$$

The polynomial

$$\sigma(\xi) = \sigma_L(\xi) = \sum_{|\alpha| \leq m} a^\alpha i^{|\alpha|} \xi^\alpha$$

is called *the characteristic polynomial of* L. The operator

$$\sum_{|\alpha|=m} a^{\alpha} D^{\alpha}$$

is called *the principal part of* L. The operator

$$L^{*} := \sum_{|\alpha|\leq m} \bar{a}^{\alpha}(-1)^{|\alpha|} D^{\alpha}$$

is called the *operator formally adjoint* for L.

2. Remark. Obviously, $\sigma(\xi) = e^{-ix\cdot\xi} L e^{ix\cdot\xi}$, $\bar{\sigma}(\xi) = e^{-ix\cdot\xi} L^{*} e^{ix\cdot\xi}$.

The following is a simple consequence of properties of the Fourier transform.

3. Lemma. *If* L *is an mth order operator with constant coefficients and* $\psi \in \mathcal{S}$, *then* $L\psi \in \mathcal{S}$ *and* $\widetilde{Lg}(\xi) = \sigma(\xi)\tilde{g}(\xi)$.

4. Example. The characteristic polynomial of Laplace's operator

$$\Delta = \frac{\partial^2}{(\partial x^1)^2} + \dots + \frac{\partial^2}{(\partial x^d)^2}$$

is $-|\xi|^2$, which is zero for $\xi = 0$. Therefore Δ is *not* a strongly elliptic operator although it is elliptic in the sense of Definition 1.4.1.

In Definition 10 we introduce homogeneous elliptic operators which embrace Laplace's operator. It is worth noting that strongly elliptic operators in the sense of our definition are also elliptic in the broader sense of the book by L. Bers, F. John, and M. Schechter [**3**]. In this book one can also find the definitions of strongly elliptic (different from above) and properly elliptic operators. Some properties of homogeneous elliptic operators are collected in Exercises 9.4 and 9.5.

The convenience of the current definition can be seen if one tries to solve the equation

$$Lu = f$$

in the whole space. Indeed, formally we have $\sigma(\xi)\tilde{u} = \tilde{f}$, and since $\sigma \neq 0$, $\tilde{u} = \sigma^{-1}\tilde{f}$, which allows us to find u by using the inverse Fourier transform.

5. Example. The operator $1 - \Delta$ is strongly elliptic, for its characteristic polynomial equals $1 + |\xi|^2$.

6. Example. For two operators with constant coefficients L_1 and L_2 we have $\sigma_{L_1L_2} = \sigma_{L_1}\sigma_{L_2}$ (see Remark 2). Therefore if the operators L_1 and L_2 are strongly elliptic, so is their product L_1L_2. In particular, the operator $(1-\Delta)^k$ is strongly elliptic for any integer $k \geq 1$.

7. Example. The operator $\Delta + b^k D_k - 1$ (recall the summation convention) for any constant $b \in \mathbb{R}^d$ and the operators $d/dx - 1$ and $d^3/(dx)^3 + 1$ for $d = 1$ are strongly elliptic.

8. Lemma. *If L is an mth order strongly elliptic operator, then there exists a constant $\kappa > 0$, called the constant of ellipticity, such that*

$$|\sum_{|\alpha|\leq m} a^\alpha i^{|\alpha|}\xi^\alpha| \geq \kappa(1+|\xi|^2)^{m/2} \quad \forall \xi \in \mathbb{R}^d. \tag{1}$$

Proof. Observe that the function

$$f(t,\xi) = |\sum_{|\alpha|\leq m} a^\alpha i^{|\alpha|} t^{m-|\alpha|}\xi^\alpha|$$

is positive-homogeneous and continuous, and $f > 0$ on the unit sphere $t^2 + |\xi|^2 = 1$ in $\mathbb{R}^{d+1}$. Therefore, $f \geq \kappa$ on the sphere for a constant $\kappa > 0$. This implies

$$f(t,\xi) \geq \kappa(t^2 + |\xi|^2)^{m/2}$$

everywhere, so that

$$f(1,\xi) \geq \kappa(1+|\xi|^2)^{m/2}$$

and the lemma is proved.

9. Exercise*. Prove that (1) implies the strong ellipticity of L.

10. Definition. We call an operator

$$L = \sum_{|\alpha|=m} a^\alpha D^\alpha$$

homogeneous elliptic (differential) of the mth order if

$$\sum_{|\alpha|=m} a^\alpha \xi^\alpha \neq 0 \qquad \forall \xi \in \mathbb{R}^d \setminus \{0\}.$$

11. Exercise. Prove that if L is an mth order homogeneous elliptic operator and $m = 1$, then $d \le 2$. Also prove that if $m = 1$ and $d = 2$, then there is a linear change of coordinates such that the operator L takes the form $c\bar{\partial}$ where c is a constant and $\bar{\partial}$ is *the Cauchy-Riemann operator*

$$\bar{\partial} := \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

12. Exercise. The operator $\bar{\partial}$ is a first-order *homogeneous* elliptic operator. Prove that if L is a first-order *strongly elliptic* operator, then $d = 1$.

13. Exercise. Prove that if the coefficients a^α of an mth order strongly elliptic differential operator are real and $d \ge 2$, then m is even.

3. Comments on the oblique derivative and other boundary-value problems. Instances of pseudo-differential operators

Let $d \ge 2$ and $\Omega = \mathbb{R}^d_+$. Let $\ell \in \mathbb{R}^d$ be a constant vector with $\ell^1 = 1$. In Section 9.3 we have already studied some properties of the oblique derivative boundary-value problem:

$$\Delta u - \lambda u = f, \quad u \in W^2_p(\Omega), \quad u_{(\ell)} - h \in \overset{0}{W}{}^1_p(\Omega).$$

In this section we want to apply a different method based on the Fourier transform in tangential directions x' and in order to better understand the nature of arising difficulties, we will not pay too much attention to the lack of rigorousness of our arguments (see however Exercise 13.3.14).

Introduce

$$Mv = \sum_{j\ge2} \ell^j v_{x^j}.$$

Let u be a "nice" function on Ω and for a fixed $\lambda > 0$ define

$$f = \lambda u - \Delta u \quad \text{in} \quad \Omega, \quad h = (u_{x^1} + Mu)|_{\partial\Omega} \quad \text{on} \quad \partial\Omega. \tag{1}$$

First of all either extend f to $\mathbb{R}^d$ in any reasonable way and find u_0 from the equation

$$\lambda u_0 - \Delta u_0 = f \quad \text{in} \quad \mathbb{R}^d, \tag{2}$$

or just solve this equation in Ω with zero boundary condition on $\partial\Omega$. Then $v = u - u_0$ satisfies

$$\lambda v - \Delta v = 0 \quad \text{in} \quad \Omega, \quad v|_{\partial\Omega} = g \quad \text{on} \quad \partial\Omega \tag{3}$$

with a function $g = g(x')$ subject to (notice $Mv = Mg$ on $\partial\Omega$)

$$(v_{x^1} + Mg)|_{\partial\Omega} = h - h_0, \quad h_0 := (u_{0x^1} + Mu_0)|_{\partial\Omega} \quad \text{on} \quad \partial\Omega. \tag{4}$$

It turns out that one can eliminate v from (4). To show this, define F_{d-1} to be the Fourier transform with respect to x'. For functions $w(x)$, $x \in \mathbb{R}^d_+$, and $\xi' \in \mathbb{R}^{d-1}$ write

$$\tilde{w}(x^1, \xi') = F_{d-1}w(x^1, \cdot)(\xi').$$

Then (3) implies that for any $x^1 \geq 0$ and $\xi' \in \mathbb{R}^{d-1}$

$$D_1^2\tilde{v}(x^1, \xi') - (|\xi'|^2 + \lambda)\tilde{v}(x^1, \xi') = 0.$$

For each $\xi' \in \mathbb{R}^{d-1}$ this is an ordinary linear equation with constant coefficients and we do not expect $|\tilde{v}(x^1, \xi')|$ to go to infinity as $x^1 \to \infty$. Hence we find that

$$\tilde{v}(x^1, \xi') = \tilde{g}(\xi')\exp(-(|\xi'|^2 + \lambda)^{1/2}x^1).$$

It follows that

$$\tilde{v}_{x^1}(0, \xi') = -(|\xi'|^2 + \lambda)^{1/2}\tilde{g}(\xi')$$

and the condition in (4) becomes

$$q\tilde{g} = \tilde{h}_0 - \tilde{h}, \quad q = q(\xi') := (|\xi'|^2 + \lambda)^{1/2} - i\ell' \cdot \xi'. \tag{5}$$

Now let us consider (1) as a system of conditions with f and h given, say $h = 0$, and with u unknown. Then the natural way to find u and investigate its properties is the following. First find u_0 from (2); then define g according to

$$\tilde{g} = q^{-1}(F_{d-1}(u_{0x^1} + Mu_0) - \tilde{h}).$$

By the way, dividing by q brings no trouble since, as is easy to see, $|q| \geq \lambda^{1/2}$. After that we solve (3) and, finally, set $u = v + u_0$.

It turns out that this way can be rather easily justified in W_2^2 spaces; however even then we need new spaces of functions $H_2^{3/2}$ and $H_2^{1/2}$ on $\partial\Omega$ (see Exercise 1). If $p \neq 2$, then yet new boundary spaces appear, the so-called Slobodetskii spaces $W_p^{2-1/p}$ and $W_p^{1-1/p}$. But even if we tried to work only in Sobolev spaces, proving that g defined by (5) is in $W_p^2(\partial\Omega)$ given that $h_0 - h \in W_p^1(\partial\Omega)$ is beyond the scope of the theory of differential operators because equation (5) considered as an equation about g and not $\tilde{g}$ is not a differential equation. Here we are dealing with the operator

$$g \to F_{d-1}^{-1}(qF_{d-1}g)$$

which is an example of a *pseudo-differential* operator. The solvability theory for such operators in Sobolev spaces is presented in Chapter 13.

The above treatment of problem (1) can be repeated almost word for word if we take

$$Mu = \sum_{j\geq 2} \ell^j u_{x^j} + \sum_{j,k\geq 2} a^{jk} u_{x^i x^j}.$$

In that case the only difference is that q in (5) becomes

$$(|\xi'|^2 + \lambda)^{1/2} - i\ell' \cdot \xi' + \sum_{j,k\geq 2} a^{jk}\xi^j\xi^k \tag{6}$$

and if we want to solve equation (5), we need the matrix (a^{jk}) to be nonnegative, so that new $|q|$ will be bounded away from zero. We return to such boundary-value problems in Exercise 13.3.15.

1. Exercise. Prove that if $u \in W_2^2(\Omega)$, then

$$\int_{\partial\Omega} (1 + |\xi|^2)^3 |\tilde{g}(\xi')|^2 \, d\xi' < \infty, \tag{7}$$

where $g(x') = u(0, x')$. Also prove that if (7) holds, then there is a $u \in W_2^2(\Omega)$ such that $g(x') = u(0, x')$. The set of functions g for which (7) holds is denoted by $H_2^{3/2}(\mathbb{R}^{d-1})$.

4. Pseudo-differential operators

Lemmas 2.3 and 2.8 call for the following generalization of differential operators.

1. Definition. Let $\mu \in \mathbb{R}$ and let $\sigma(\xi)$ be an infinitely differentiable complex-valued function on $\mathbb{R}^d$ such that, for each multi-index α, the estimate

$$|D^\alpha \sigma(\xi)| \leq N^\alpha (1 + |\xi|^2)^{(\mu - |\alpha|)/2} \tag{1}$$

holds with a constant N^α independent of $\xi \in \mathbb{R}^d$. Then we call σ *a symbol of order* μ. If in addition, for all $\xi \in \mathbb{R}^d$,

$$|\sigma(\xi)| \geq \kappa(1 + |\xi|^2)^{\mu/2}, \tag{2}$$

where $\kappa > 0$ is a constant, then the symbol is called *elliptic* and κ is called *a constant of ellipticity of* σ. Sometimes to emphasize that N^α and κ are related to σ, we write $N^\alpha = N^\alpha(\sigma, \mu)$, $\kappa = \kappa(\sigma)$. The symbol $\bar{\sigma}(\xi)$ is called *the symbol adjoint* to σ.

2. Remark. We write $N^{\alpha}(\sigma, \mu)$ and not just $N^{\alpha}(\sigma)$ because, actually, any symbol of order ν is also a symbol of order μ for any $\mu \geq \nu$. However, the constants N^{α} in (1) may change if we change μ.

Obviously, any derivative of a symbol is also a symbol with the order of the result equal to the order of the original symbol reduced by the order of the derivative. The simplest example of symbols is given by polynomials of ξ. One more series of examples is given by the functions

$$(1 + |\xi|^2)^{\gamma/2}, \tag{3}$$

where $\gamma \in \mathbb{R}$, which are of order γ. One may try to prove this by estimating the derivatives of $(1 + |\xi|^2)^{\gamma}$, that is, by definition. An easier way is to consider the positive-homogeneous function

$$f(t, \xi) := (t^2 + |\xi|^2)^{\gamma}$$

and to use the following almost obvious lemma.

3. Lemma. *Let $g(\eta)$ be a positive-homogeneous function of order γ given on $\mathbb{R}^n$: $g(\lambda\eta) = \lambda^{\gamma} g(\eta)$, $\lambda > 0$. If g is bounded on the unit sphere, then*

$$|g(\eta)| \leq N|\eta|^{\gamma}$$

for $\eta \neq 0$ with a constant N independent of η. If it is continuously differentiable at any point $\eta \neq 0$, then its first-order partial derivatives are positive-homogeneous functions of order $\gamma - 1$ bounded on the unit sphere, so that

$$|D_j g(\eta)| \leq N|\eta|^{\gamma-1}$$

for $\eta \neq 0$ with a constant N independent of η.

We have already met some symbols involving $(1 + |\xi|^2)^{1/2}$ in our treatment of boundary-value problems; see (3.5) and (3.6).

Lemma 3 allows one to obtain very many other examples of symbols. Symbols also arise as the results of some other operations.

4. Lemma. *(i) Let σ be an elliptic symbol of order μ. Then σ^{-1} is an elliptic symbol of order $-\mu$.*

(ii) For $k = 1, 2$, let σ_k be symbols of order μ_k. Then $\sigma_1\sigma_2$ is a symbol of order $\mu_1 + \mu_2$.

Proof. (i) For $\alpha = 0$, (2) implies that

$$|\sigma^{-1}| \leq N(1 + |\xi|^2)^{-\mu/2}.$$

By differentiating $\sigma\sigma^{-1} = 1$, we get

$$0 = \sum_{|\beta|+|\gamma|=|\alpha|} c^{\beta\gamma}(D^\beta\sigma)D^\gamma\sigma^{-1}, \quad D^\alpha\sigma^{-1} = -\sigma^{-1} \sum_{\substack{|\beta|+|\gamma|=|\alpha| \\ |\gamma|<|\alpha|}} c^{\beta\gamma}(D^\beta\sigma)D^\gamma\sigma^{-1},$$

where the $c^{\beta\gamma}$ are some constants. Hence, if

$$|D^\gamma\sigma^{-1}| \leq N(1 + |\xi|^2)^{-(\mu+|\gamma|)/2}$$

whenever $|\gamma| < |\alpha|$, then

$$|D^\alpha\sigma^{-1}| \leq N(1 + |\xi|^2)^{-\mu/2} \sum_{\substack{|\beta|+|\gamma|=|\alpha| \\ |\gamma|<|\alpha|}} (1 + |\xi|^2)^{(\mu-|\beta|)/2}(1 + |\xi|^2)^{-(\mu+|\gamma|)/2}.$$

Since the right-hand side equals $N(1 + |\xi|^2)^{-(\mu+|\alpha|)/2}$, it only remains to use induction on $|\alpha|$.

Assertion (ii) follows directly from the definition and Leibnitz's formula. The lemma is proved.

5. Exercise*. For integers $n \geq 0$ introduce

$$N_n(\sigma, \mu) = \max_{|\alpha|\leq n} N^\alpha(\sigma, \mu)$$

and under the conditions in Lemma 4 (i) by analyzing the above proof, show that $N_n(\sigma^{-1}, \mu) \leq M$, where the constant M depend only on d, n, $N_n(\sigma, \mu)$, and $\kappa(\sigma)$. Also show that under the conditions in Lemma 4 (ii) we have

$$N_n(\sigma_1\sigma_2, \mu_1 + \mu_2) \leq NN_n(\sigma_1, \mu_1)N_n(\sigma_2, \mu_2),$$

where the constant N depends only on n and d.

6. Exercise*. Prove that if $\sigma = \sigma(x)$ is a symbol and $\psi \in \mathcal{S}$, then $\sigma\psi \in \mathcal{S}$ and furthermore the operator $\psi \to \sigma\psi$ is continuous in the sense of convergence in $\mathcal{S}$.

Exercise 6 and Lemma 1.3 show that the following definition makes sense.

7. Definition. Let σ be a symbol of order μ. The operator

$$L : \mathcal{S} \to \mathcal{S}$$

defined by the formula

$$Lg = F^{-1}(\sigma\tilde{g}) = F^{-1}(\sigma F(g))$$

is called *the pseudo-differential operator of order μ with symbol σ*. In connection with this we also write $\sigma = \sigma_L$. If σ is elliptic, L is called strongly elliptic. The constant of ellipticity of σ is also called the *constant of ellipticity of L*. Naturally, we use the notation

$$N^\alpha(L,\mu) = N^\alpha(\sigma_L,\mu), \quad \kappa(L) = \kappa(\sigma_L), \quad N_n(L,\mu) = N_n(\sigma,\mu).$$

If L is strongly elliptic, by L^{-1} we denote the pseudo-differential operator with symbol $1/\sigma$. In the general case the operator L^*, defined as a pseudo-differential operator with symbol $\bar{\sigma}$, is said to be *formally adjoint* to L.

8. Exercise. Pseudo-differential operators are translation invariant in the following sense. Prove that for any constant $y \in \mathbb{R}^d$, $g \in \mathcal{S}$, and pseudo-differential operator L we have $L(g(\cdot - y)) = (Lg)(\cdot - y)$.

The following exercise is used in the proof of Theorem 13.9.1.

9. Exercise*. For fixed $y \in \mathbb{R}^d$ introduce the operator $T_y : g \to T_y g$ by $T_y g(x) = g(y - x)$. Let L be a pseudo-differential operator, and let $g \in \mathcal{S}$. Use the fact that $F(\bar{h})(\xi) = \overline{F(h)}(-\xi)$ and $F(T_y g)(\xi) = e^{-i\xi\cdot y}\tilde{g}(-\xi)$ and prove that

$$L^*\overline{T_y g} = \overline{T_y(Lg)}.$$

10. Example. If $\gamma \in \mathbb{R}$, then, as has been pointed out above, formula (3) defines an elliptic symbol of order γ. The operator which corresponds to it is denoted by

$$(1 - \Delta)^{\gamma/2}.$$

It plays a very important role in the theory.

Observe that if γ is an even integer, say $\gamma = 2k$, $k = 1, 2, \dots$, then (unless the notation is misleading) $(1 - \Delta)^{\gamma/2}$ should naturally be $(1 - \Delta)^k$ and the latter can also be defined as the product of k factors each of which is $1 - \Delta$. The natural question arises as to whether these two definitions agree.

The answer to this question is positive because the Fourier transforms of the results of the action of both operators on $g \in \mathcal{S}$ is the same, i.e., $(1 + |\xi|^2)^k\tilde{g}$.

11. Remark. If $\sigma(\xi)$ is a polynomial in ξ of order m, i.e.,

$$\sigma(\xi) = \sum_{|\alpha|\leq m} a^\alpha i^{|\alpha|}\xi^\alpha,$$

where the a^α are some constants, then its corresponding operator L is a differential operator and the characteristic polynomial of L, introduced in Definition 2.1, coincided with σ according to Lemma 2.3. In particular, differential operators with constant coefficients are pseudo-differential. In that case also both Definitions 2.1 and 7 agree in what concerns L^*.

12. Exercise*. Let $\phi, \psi \in \mathcal{S}$. Prove that $\phi * \psi \in \mathcal{S}$ and for any pseudo-differential operator L we have $L(\phi * \psi) = (L\phi) * \psi = \phi * L\psi \in \mathcal{S}$.

13. Exercise*. Let $\phi \in \mathcal{S}$ and let L be a pseudo-differential operator with symbol σ. Prove that

$$\int_{\mathbb{R}^d} L\phi\, dx = \sigma(0) \int_{\mathbb{R}^d} \phi\, dx.$$

14. Remark. By Exercise 6 and Lemma 1.3 the operator L is continuous on $\mathcal{S}$ as a superposition of three continuous operators F, σ, F^{-1}. Furthermore, if $\psi_n \to \psi$ in $\mathcal{S}$, then $\psi_n \to \psi$ in $\mathcal{L}_p$ for any $p \in [1, \infty]$. It follows that if $\psi_n \to \psi$ in $\mathcal{S}$, then $L\psi_n \to L\psi$ in $\mathcal{L}_p$ for any $p \in [1, \infty]$.

15. Remark. If $\psi \in \mathcal{S}$, then ψ is a symbol of any order. The pseudo-differential operator L with symbol ψ is proportional to the convolution with $F^{-1}(\psi)$, i.e.,

$$Lg = c_d g * F^{-1}(\psi),$$

since $F(Lg) = \psi\tilde{g}$.

Conversely, the operator $L_\psi : g \to g * \psi$ is a pseudo-differential operator with symbol $c_d^{-1}\tilde{\psi}$. Observe that the formally adjoint L_ψ^* has symbol $c_d^{-1}\bar{\tilde{\psi}}$ and is given by the convolution with the function $\bar{\psi}(-x)$.

The following remark is used in the proof of Theorem 13.7.2.

16. Remark. It turns out that for $\gamma > 0$, $\delta \geq 0$, $\nu \in \mathbb{R}$, and $\varepsilon > 0$, the function

$$\sigma_\nu(\xi) := \varepsilon^\delta (1 + |\xi|^2)^{\nu/2} \exp(-\varepsilon(1 + |\xi|^2)^{\gamma/2})$$

is a symbol of order $\nu - \delta\gamma$, with $N_n(\sigma, \nu - \delta\gamma)$ majorated by a constant depending only on γ, δ, ν, d, and n and, in particular, independent of ε.

To prove this, observe that by induction on $n = 0, 1, 2, \dots$ one easily shows that for $\xi \neq 0$ any multi-index with $|\alpha| = n$ we have

$$D^\alpha \exp(-|\xi|^\gamma) = \exp(-|\xi|^\gamma) \sum_{j=0}^{n} \eta_j^\alpha(\xi),$$

where the η_j^α are some positive-homogeneous functions of order $j\gamma - n$ infinitely differentiable on the unit sphere. In particular ($x^m e^{-x}$ is bounded for $x \geq 0$ if $m \geq 0$), for any $\delta \geq 0$

$$|D^\alpha \exp(-|\xi|^\gamma)| \leq N \exp(-|\xi|^\gamma) \sum_{j=0}^{n} |\xi|^{j\gamma - n} \leq N|\xi|^{-n-\delta\gamma},$$

where N depends only on n, δ, d, and γ. For $\nu \in \mathbb{R}$ by the Leibnitz formula and Lemma 3 we get that

$$\big|D^\alpha\big(|\xi|^\nu \exp(-|\xi|^\gamma)\big)\big| = \Big| \sum_{|\tau|+|\theta|=n} c_\alpha^{\tau\theta} (D^\tau |\xi|^\nu) D^\theta \exp(-|\xi|^\gamma)\Big| \leq N|\xi|^{\nu-n-\delta\gamma},$$

where N depends only on ν, γ, d, δ, and n, and the $c_\alpha^{\tau\theta}$ are some constants.

By applying this result to $\mathbb{R}^{d+1} = \{(t, \xi) : t \in \mathbb{R}, \xi \in \mathbb{R}^d\}$, we find that

$$\big|D_\xi^\alpha\big[(t^2 + |\xi|^2)^{\nu/2} \exp(-(t^2 + |\xi|^2)^{\gamma/2})\big]\big| \leq N(t^2 + |\xi|^2)^{(\nu-n-\delta\gamma)/2},$$

whence by changing the variables and replacing t and ξ with $\varepsilon^{1/\gamma}$ and $\varepsilon^{1/\gamma}\xi$, respectively, we obtain that for any $\varepsilon > 0$

$$\varepsilon^{\nu/\gamma}\big|D^\alpha\big[(1 + |\xi|^2)^{\nu/2} \exp(-\varepsilon(1 + |\xi|^2)^{\gamma/2})\big]\big|$$

$$= \big|D^\alpha\big[(\varepsilon^{2/\gamma} + |\varepsilon^{1/\gamma}\xi|^2)^{\nu/2} \exp(-(\varepsilon^{2/\gamma} + |\varepsilon^{1/\gamma}\xi|^2)^{\gamma/2})\big]\big|$$

$$\leq N\varepsilon^{n/\gamma}(\varepsilon^{2/\gamma} + |\varepsilon^{1/\gamma}\xi|^2)^{(\nu-n-\delta\gamma)/2} = N\varepsilon^{\nu/\gamma-\delta}(1 + |\xi|^2)^{(\nu-n-\delta\gamma)/2},$$

$$\big|D^\alpha\big[\varepsilon^\delta(1 + |\xi|^2)^{\nu/2} \exp(-\varepsilon(1 + |\xi|^2)^{\gamma/2})\big]\big| \leq N(1 + |\xi|^2)^{(\nu-n-\delta\gamma)/2},$$

where N depends only on n, d, δ, ν, and γ. This is exactly what has been asserted.

The following lemma bears on the unique solvability of the equation $Lu = f$ in $\mathcal{S}$.

17. Lemma. *If L is a strongly elliptic pseudo-differential operator, then $L\mathcal{S} = \mathcal{S}$ and the mapping $L : \mathcal{S} \to \mathcal{S}$ is one-to-one and L^{-1} is its inverse.*

Proof. Take $\psi \in \mathcal{S}$ and define $u = L^{-1}\psi$. Then $\tilde{u} = \tilde{\psi}/\sigma$ and $\sigma\tilde{u} = \tilde{\psi}$, so that by definition $Lu = \psi$. The uniqueness of such $u \in \mathcal{S}$ follows from the fact that, if $u \in \mathcal{S}$ or even $u \in \mathcal{L}_1$ and $\tilde{u} = 0$, then $u = 0$. The lemma is proved.

18. Theorem. *For $k = 1, 2$ let L_k be pseudo-differential operators of order μ_k with symbol σ_k. Then $L_1 L_2$ is a pseudo-differential operator of order $\mu_1 + \mu_2$ with symbol $\sigma_1\sigma_2$. In particular, $L_1 L_2 = L_2 L_1$.*

Indeed, it suffices to recall Lemma 4 and to use the formula

$$L_1 L_2 g = F^{-1}(\sigma_1 F(L_2 g)) = F^{-1}(\sigma_1\sigma_2\tilde{g}).$$

The following lemma explains why L^* is called the formal adjoint for L.

19. Lemma. *If $g, \phi \in \mathcal{S}$ and L is a pseudo-differential operator, then*

$$(Lg, \phi) = (g, L^*\phi).$$

This is proved by using the Fourier transform and definitions:

$$(Lg, \phi) = \int_{\mathbb{R}^d} (Lg)\bar{\phi}\, dx = \int_{\mathbb{R}^d} \sigma\tilde{g}\bar{\tilde{\phi}}\, d\xi = \int_{\mathbb{R}^d} \tilde{g}\overline{\bar{\sigma}\tilde{\phi}}\, d\xi = (g, L^*\phi).$$

Our main goal is to study strongly elliptic *differential* operators with variable coefficients. Their natural generalization would be pseudo-differential operators with symbol depending on x. There exists a powerful and beautiful theory of such operators. However, in this course we only need pseudo-differential operators with "constant coefficients" and only as a tool to achieve our modest goals.

5. Green's functions

In this section L is a *strongly elliptic* pseudo-differential operator of order $\mu \in \mathbb{R}$ with symbol σ.

By Lemma 4.17 for any $\phi \in \mathcal{S}$ the solution of class $\mathcal{S}$ of the equation $Lu = \phi$ is given by $u = L^{-1}f$, which means that $\tilde{u} = \sigma^{-1}\tilde{\phi}$. Since the Fourier transform of a convolution is a constant times the product of the Fourier transforms of the factors, it is natural to hope that the function u has the form

$$u(x) = \int_{\mathbb{R}^d} \phi(y)G(x-y)\, dy = \int_{\mathbb{R}^d} \phi(x-y)G(y)\, dy, \tag{1}$$

where

$$G(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{1}{\sigma(\xi)} e^{iy\cdot\xi}\, d\xi. \tag{2}$$

1. Definition. A generalized function (in other words, a distribution) $G(x)$ is called a *Green's function* of the equation $Lu = \phi$ in the whole space if for any $\phi \in C_0^\infty$ we have $G * \phi \in \mathcal{S}$ and

$$L(G * \phi) = \phi, \tag{3}$$

that is, if the unique solution in $\mathcal{S}$ of the equation $Lu = \phi$ is given by $u = G * \phi$ or equivalently by (1).

If f is a distribution and $\phi \in C_0^\infty$, then we denote the value of f on ϕ by

$$\langle f, \phi \rangle$$

and we are going to use the following without discussion.

- Definition of a distribution f as a (complex-valued) linear functional on the space of complex-valued functions of class C_0^∞ continuous in the sense that if $\phi, \phi_n \in C_0^\infty$ have supports in the same ball and $\phi_n \to \phi$ uniformly on $\mathbb{R}^d$ together with every derivative of any order, then $\langle f, \phi_n \rangle \to \langle f, \phi \rangle$. We write $\langle f, \phi \rangle$ formally as

$$\langle f, \phi \rangle = \int_{\mathbb{R}^d} f(x)\bar{\phi}(x)\, dx.$$

- The distribution called the delta-function at the origin is defined as

$$\langle \delta_0, \phi \rangle = \bar{\phi}(0).$$

- The definition of the convolution of a distribution f with $\phi \in C_0^\infty$:

$$f * \phi(x) = \langle f, \bar{\phi}(x - \cdot) \rangle.$$

- Any locally integrable function is a distribution (by the dominated convergence theorem).
- Every distribution is infinitely differentiable in the generalized sense:

$$(-1)^{|\alpha|} \langle f, D^\alpha \phi \rangle$$

is a distribution, which by definition is called $D^\alpha f$.

- (See Exercise 4.4.1.) If $\phi \in C_0^\infty(\mathbb{R}^d)$ and f is a distribution, then the function $\langle f, \phi(x - \cdot)\rangle$ is infinitely differentiable and for any multi-index α

$$D^\alpha \langle f, \phi(x - \cdot)\rangle = \langle D^\alpha f, \phi(x - \cdot)\rangle = \langle f, (D^\alpha \phi)(x - \cdot)\rangle.$$

- If f_n is a sequence of distributions, we say that it converges to a distribution f if $\langle f_n, \phi\rangle \to \langle f, \phi\rangle$ for any test function $\phi \in C_0^\infty$. In this case it holds that $D^\alpha f_n \to D^\alpha f$ for any multi-index α.
- The notion of distribution over a domain $D \subset \mathbb{R}^d$ as a linear functional on $C_0^\infty(D)$ continuous in the sense that if $\phi, \phi_n \in C_0^\infty(D)$ have supports in the same compact set $K \subset D$ and $\phi_n \to \phi$ uniformly on D together with every derivative of any order, then $\langle f, \phi_n\rangle \to \langle f, \phi\rangle$.
- The notion of restriction of distributions over $\mathbb{R}^d$ to a domain $D \subset \mathbb{R}^d$, when we only consider $\langle f, \phi\rangle$ for $\phi \in C_0^\infty(D)$ extending such ϕ as zero outside D.

2. Exercise*. Let $D \subset \mathbb{R}^d$ be a domain, let f be a continuous locally bounded function on D, and let the distribution f_{x^1} coincide in the sense of distributions on D with a continuous function g. Prove that f is continuously differentiable in D with respect to x^1 and the usual derivative f_{x^1} equals g in D.

3. Remark. If L is a *differential* operator, then LG is a distribution and (3) means that $(LG) * \varphi = \varphi$, that is, for any x

$$\int_{\mathbb{R}^d} (LG)(y)\bar{\varphi}(x - y)\, dy = \varphi(x),$$

which for $x = 0$ means that

$$\int_{\mathbb{R}^d} (LG)(y)\bar{\varphi}(y)\, dy = \varphi(0), \tag{4}$$

that is, $LG = \delta_0$. Somewhat later (see Section 13.1) we will see that one can define MG for any pseudo-differential operator M and it turns out (Theorem 13.1.7) that $LG = \delta_0$ even if L is a pseudo-differential operator. Generally, (3) means that

$$\sigma F(G * \varphi) = \tilde{\varphi}. \tag{5}$$

By the way, if we knew that $G \in \mathcal{L}_1$, then (5) would imply that $\tilde{G} = c_d \sigma^{-1}$.

4. Exercise*. Prove that if a Green's function exists, then it is unique (as a distribution).

5. Exercise. The function $\sigma(\xi) \equiv 1$ is a strongly elliptic symbol of order 0. Find its Green's function.

6. Exercise. The function $\sigma(\xi) = (1 + |\xi|^2)^{-1}$ is a strongly elliptic symbol of order -2. According to Definition 4.7 the corresponding pseudo-differential operator is $(1 - \Delta)^{-1}$. By using (5), prove that its Green's function is given by

$$\langle G, \varphi \rangle = (1 - \Delta)\bar{\varphi}(0).$$

Exercises 5 and 6 provide examples showing that generally the following fact is wrong if $\mu < 0$.

7. Lemma. *Let $\mu > 0$ and let G be a Green's function of (a strongly elliptic pseudo-differential operator) L. Take a $\phi \in C_0^\infty$ and set $\phi_n(x) = \phi(nx)$. Then $\phi_n G \to 0$ as $n \to \infty$ in the sense of distributions.*

Proof. We have to show that $\langle G, \phi_n f \rangle \to 0$ as $n \to \infty$ for any $f \in C_0^\infty$. Set $\psi_n(x) = \bar{\phi}(-nx)\bar{f}(-x)$ and $u_n = G * \psi_n$. Then by the definition of convolution

$$\langle G, \phi_n f \rangle = u_n(0).$$

By definition $u_n \in \mathcal{S}$ and by (5) its Fourier transform is

$$\tilde{u}_n = \sigma^{-1}\tilde{\psi}_n.$$

By Lemma 1.3 we have $\tilde{u}_n \in \mathcal{S}$ and $u_n = F^{-1}(\sigma^{-1}\tilde{\psi}_n)$. By writing what it means at $x = 0$, we find

$$\langle G, \phi_n f \rangle = u_n(0) = c_d \int_{\mathbb{R}^d} \sigma^{-1}(\xi)\tilde{\psi}_n(\xi)\, d\xi.$$

Next, by changing variables $-nx \to x$, we find that

$$\tilde{\psi}_n(\xi) = c_d \int_{\mathbb{R}^d} e^{-ix\cdot\xi}\bar{\phi}(-nx)\bar{f}(-x)\, dx = n^{-d} g_n(\xi/n),$$

where

$$g_n(\xi) = c_d \int_{\mathbb{R}d} e^{ix\cdot\xi}\bar{\phi}(x)\bar{f}(x/n)\, dx.$$

Therefore,

$$\langle G, \phi_n f \rangle = c_d \int_{\mathbb{R}^d} \sigma^{-1}(n\xi) g_n(\xi)\, d\xi.$$

Finally, it is easy to check that $f(\cdot/n)\phi \to f(0)\phi$ in $\mathcal{S}$, hence

$$g_n \to \bar{f}(0)\bar{\tilde{\phi}}$$

in $\mathcal{S}$ and, in particular, in $\mathcal{L}_1$. Since σ^{-1} is bounded and $\sigma^{-1}(n\xi) \to 0$ at each point $\xi \neq 0$, by invoking the dominated convergence theorem, we conclude

$$\varlimsup_{n\to\infty} |\langle G, \phi_n f\rangle| \le N \lim_{n\to\infty} \int_{\mathbb{R}^d} |g_n(\xi) - \bar{f}(0)\bar{\tilde{\phi}}(\xi)|\, d\xi$$

$$+ c_d |f(0)| \lim_{n\to\infty} \int_{\mathbb{R}^d} |\sigma^{-1}(n\xi)|\, |\tilde{\phi}(\xi)|\, d\xi = 0.$$

The lemma is proved.

8. Example. Let $c > 0$ be a constant. By Lemma 4.17 for any $f \in C_0^\infty$ there is a unique $u \in \mathcal{S}$ such that $(c - \Delta)u = f$. By Theorem 1.7.1 (ii), u has the form $G_c * f$. Hence the Green's function of $c - \Delta$ is given by

$$G(x) = \int_0^\infty \frac{1}{(4\pi t)^{d/2}} e^{-\frac{1}{4t}|x|^2 - ct}\, dt. \tag{6}$$

We know that for odd d the function $G(x)$ from (6) admits a closed form; see Exercises 7.9 and 7.10.

9. Exercise. Let $d \ge 3$ and for the function G given by (6) prove that

$$\lim_{x\to 0} |x|^{d-2} G(x) = \int_0^\infty \frac{1}{(4\pi t)^{d/2}} e^{-\frac{1}{4t}}\, dt = 4^{-1}\pi^{-d/2}\Gamma(d/2 - 1).$$

10. Exercise. Let $d = 2$ and for the function G given by (6) prove that

$$\lim_{x\to 0} (\ln|x|)^{-1} G(x) = -2^{-1}\pi^{-1}.$$

11. Exercise (Structure of a general distribution). Let f be a distribution and let $\phi \in C_0^\infty$.

(i) Arguing by contradiction, prove that there exist an $m = 1, 2, \ldots$ and a constant N such that for any $\psi \in C_0^\infty$ we have

$$|\langle f, \phi\psi\rangle| \le N \max_{k\le m} \sup_{x\in\mathbb{R}^d} |D^k \psi(x)|.$$

(ii) By using (i), Theorem 1.7.8, and Theorem 1.7.5, which implies that

$$\|\psi\|_{W_2^{2n}} \le N\|(\Delta - 1)^n \psi\|_{\mathcal{L}_2},$$

show that there is an $n = 1, 2, \ldots$ and $g \in \mathcal{L}_2$ such that $f\phi = (1-\Delta)^n g$. Slightly informally speaking, this means that the restriction of a distribution on any bounded domain is written as $(1-\Delta)^n g$ for a $g \in \mathcal{L}_2$.

6. Existence of Green's functions

We continue to study a strongly elliptic pseudo-differential operator L of order $\mu \in \mathbb{R}$ with symbol σ. It is natural to look for G in the form (5.2). However, if $\sigma^{-1} \notin \mathcal{L}_1 + \mathcal{L}_2$, the meaning of the integral in (5.2) should be explained. First we prove that Green's functions always exist. The proof is based on the fact that for $u, f \in \mathcal{S}$ the equations $Lu = f$ and

$$(1-\Delta)^k Lu = (1-\Delta)^k f$$

are equivalent, where k is an integer.

1. Lemma. *(i) Assume that $\mu > d$ and define*

$$G(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{1}{\sigma(\xi)} e^{ix\cdot\xi}\, d\xi.$$

Then G is a bounded function, hence a distribution. Furthermore, G is the Green's function for L and for any integer $r = 0, 1, 2, \ldots$

$$|x|^{2r} G(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} (-\Delta)^r \Big(\frac{1}{\sigma(\xi)}\Big)\, d\xi. \tag{1}$$

In particular, $|x|^{2r}G(x)$ is bounded and $G \in \mathcal{L}_1$.

(ii) For general μ take any $k = 0, 1, 2, \ldots$ such that $2k + \mu > d$ and define G' as the Green's function of $(1-\Delta)^k L$, that is,

$$G'(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{1}{\sigma(\xi)(1+|\xi|^2)^k} e^{ix\cdot\xi}\, d\xi.$$

Set

$$G(x) = (1-\Delta)^k G'(x).$$

Then the distribution G is the Green's function for L.

Proof. (i) That G is a bounded function follows from the fact that $\sigma^{-1} \in \mathcal{L}_1$. Since σ^{-1} is a symbol of order $-\mu < -d$, all its derivatives decrease at infinity faster than $|\xi|^{-d-\varepsilon}$, where $\varepsilon > 0$. This allows us to integrate by parts on the right of (1) bringing $(-\Delta)^r$ to $e^{ix\cdot\xi}$. Upon observing that

$$(-\Delta_\xi)^r e^{ix\cdot\xi} = |x|^{2r} e^{ix\cdot\xi},$$

where the subscript ξ means that Δ is applied with respect to the variable ξ, we conclude that the right-hand side of (1) equals $|x|^{2r}G(x)$ indeed.

Next, we prove that G is the Green's function of L. Take an $f \in C_0^\infty$ and find the unique

$$u \in \mathcal{S}$$

such that $Lu = f$. Then $\tilde{u} = \sigma^{-1}\tilde{f}$. Here we can use the above-mentioned property of the Fourier transform of a convolution and find

$$u(x) = \int_{\mathbb{R}^d} G(y)f(x-y)\,dy.$$

Thus $u = G * f \in \mathcal{S}$ and $L(G * f) = f$, meaning that G is the Green's function of L.

(ii) Since $G' * f \in \mathcal{S}$ and $(1-\Delta)^k f \in C_0^\infty$ for any $f \in C_0^\infty$, we have

$$G * f = ((1-\Delta)^k G' * f) = G' * ((1-\Delta)^k f) \in \mathcal{S}.$$

Finally,

$$G * f = G' * ((1-\Delta)^k f) = (1-\Delta)^k [G' * f]$$

and by the above

$$L(G * f) = L(1-\Delta)^k [G' * f] = f.$$

The lemma is proved.

Formula (3) below explains the sense in which (5.2) is true.

2. Lemma. *For any multi-indices α and β, in the sense of distributions*

$$x^\beta D^\alpha G(x) = \frac{i^{|\alpha|+|\beta|}}{(2\pi)^d} \lim_{R\to\infty} \int_{|\xi|\le R} e^{ix\cdot\xi} D^\beta\left[\xi^\alpha \sigma^{-1}(\xi)\right] d\xi. \tag{2}$$

In particular (as $\alpha = \beta = 0$),

$$G(x) = \frac{1}{(2\pi)^d} \lim_{R\to\infty} \int_{|\xi|\le R} \frac{1}{\sigma(\xi)} e^{ix\cdot\xi}\, d\xi. \tag{3}$$

Proof. According to (5.5) and the definition of convolution, for any $\varphi \in C_0^\infty$ we have

$$\langle G, \bar{\varphi}(0-\cdot)\rangle = G * \varphi(0) = F^{-1}(\sigma^{-1}\tilde{\varphi})(0) = c_d \int_{\mathbb{R}^d} \sigma^{-1}(\xi)\tilde{\varphi}(\xi)\,d\xi.$$

Changing notation, we rewrite the result as

$$\langle G, \varphi \rangle = c_d \int_{\mathbb{R}^d} \sigma^{-1}(\xi)\bar{\tilde{\varphi}}(\xi)\, d\xi.$$

Hence for $u(x) = x^\beta$ and $v(\xi) = \xi^\alpha$

$$\langle uD^\alpha G, \varphi \rangle = (-1)^{|\alpha|}\langle G, D^\alpha(u\varphi)\rangle$$

$$= i^{|\alpha|-|\beta|}c_d \int_{\mathbb{R}^d} \sigma^{-1}vD^\beta\bar{\tilde{\varphi}}\, d\xi = i^{|\alpha|+|\beta|}c_d \int_{\mathbb{R}^d} \bar{\tilde{\varphi}}D^\beta[\sigma^{-1}v]\, d\xi,$$

where the last relation is obtained by integrating by parts (recall that $\tilde{\varphi} \in \mathcal{S}$). Now we use the dominated convergence theorem and Parseval's identity and for

$$G_R^{\alpha\beta}(x) := c_d \int_{|\xi|\le R} e^{ix\cdot\xi}D^\beta[v\sigma^{-1}]\, d\xi = F^{-1}(I_{B_R}D^\beta[v\sigma^{-1}])(x)$$

we find that

$$(-i)^{|\alpha|+|\beta|}c_d^{-1}\langle uD^\alpha G, \varphi\rangle = \lim_{R\to\infty}\int_{\mathbb{R}^d}\bar{\tilde{\varphi}}I_{B_R}D^\beta[\sigma^{-1}v]\, d\xi$$

$$= \lim_{R\to\infty}\int_{\mathbb{R}^d}\tilde{G}_R^{\alpha\beta}(\xi)\bar{\tilde{\varphi}}(\xi)\, d\xi = \lim_{R\to\infty}\int_{\mathbb{R}^d}G_R^{\alpha\beta}(x)\bar{\varphi}(x)\, dx.$$

In short

$$\langle uD^\alpha G, \varphi\rangle = i^{|\alpha|+|\beta|}\lim_{R\to\infty} c_d \int_{\mathbb{R}^d} G_R^{\alpha\beta}(x)\bar{\varphi}(x)\, dx$$

for any $\varphi \in C_0^\infty$, which by definitions means that (2) holds in the sense of distributions. The lemma is proved.

3. Corollary. *If $\sigma(\xi)$ depends only on $|\xi|$, then by virtue of* (3) *we have $G(x) = G(Tx)$ for any orthogonal transformation T, so that $G(x)$ depends only on $|x|$.*

4. Exercise. By using (2), prove that if $2|\beta| > d$ and $\mu = 0$, then $x^\beta G(x) \in \mathcal{L}_2$.

5. Exercise. Prove that in the sense of distributions

$$\delta_0(x) = \frac{1}{(2\pi)^d}\lim_{R\to\infty}\int_{|\xi|\le R} e^{ix\cdot\xi}\, d\xi.$$

7. Estimating G and its derivatives

In this section in the framework of Section 6 we investigate smoothness properties of the Green's functions.

7:1. Differentiability of G and estimates of its derivatives. Our first goal is to show that outside the origin G coincides with a smooth function.

1. Theorem. *In $\mathbb{R}^d \setminus \{0\}$ the distribution $G(x)$ coincides with an infinitely differentiable function denoted in a common abuse of notation again by $G(x)$. For any multi-index α and $n > 0$ we have*

$$|D^\alpha G(x)| = o(|x|^{-n})$$

as $|x| \to \infty$. Furthermore, for any multi-indices α and β such that

$$\mu + |\beta| > d + |\alpha| \tag{1}$$

in the sense of distributions we have

$$x^\beta D^\alpha G(x) = \frac{i^{|\alpha|+|\beta|}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} D^\beta \Big[\frac{\xi^\alpha}{\sigma(\xi)}\Big]\, d\xi, \tag{2}$$

where the right-hand side is a bounded and continuous function.

Proof. First we prove that both parts of (2) coincide as distributions and the right-hand side of (2) is a bounded and continuous function. By Lemma 6.2 in the sense of distributions

$$\frac{i^{|\alpha|+|\beta|}}{(2\pi)^d} \int_{|\xi|<R} e^{ix\cdot\xi} D^\beta \Big[\frac{\xi^\alpha}{\sigma(\xi)}\Big]\, d\xi \to x^\beta D^\alpha G(x) \tag{3}$$

as $R \to \infty$. Next, observe that $\xi^\alpha \sigma^{-1}$ is a symbol of order $|\alpha| - \mu$, and hence

$$|D^\beta[\xi^\alpha \sigma^{-1}]| \le N(1 + |\xi|)^{|\alpha|-\mu-|\beta|}.$$

It follows that in (3)

$$|D^\beta[\xi^\alpha \sigma^{-1}]| \le N(1 + |\xi|)^{|\alpha|-\mu-|\beta|},$$

and the last expression here is integrable over $\mathbb{R}^d$ owing to (1). Therefore, the left-hand side of (3) *uniformly* on $\mathbb{R}^d$ converges to the right-hand side of (2), which is, in particular, bounded and continuous. The uniform convergence implies the convergence in the sense of distributions and this yields (2). We have proved the third assertion of the theorem.

Next, for any integer $r = 0, 1, 2, \ldots$ the function $|x|^{2r}$ is a polynomial in x. It follows that if $\mu + 2r > d + |\alpha|$, then

$$|x|^{2r} D^{\alpha} G(x) = \frac{i^{|\alpha|}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} (-\Delta)^r \Big[\frac{\xi^{\alpha}}{\sigma(\xi)} \Big] \, d\xi, \tag{4}$$

where the right-hand side is bounded and continuous. Since we can take r as large as we want, outside the origin the distributions $D^{\alpha}G$ coincide with continuous functions for any α. By using Exercise 5.2, we conclude that the distribution G coincides outside the origin with an infinitely differentiable function. That $|D^{\alpha}G(x)| = o(|x|^{-2r})$ as $|x| \to \infty$ for any $r > 0$ follows from (4) as well. The theorem is proved.

2. Corollary. *If L is a differential operator, then $LG(x) = 0$ for $|x| \neq 0$ in the usual rather than generalized sense.*

Indeed, by taking arbitrary $\varphi \in C_0^{\infty}(\mathbb{R}^d \setminus \{0\})$ and substituting it into (5.4), we easily conclude that $LG = 0$ outside the origin.

We generalize this corollary in Theorem 13.1.7.

3. Exercise*. By using Exercise 4.5 show that for $\mu + |\beta| > d + |\alpha|$ the right-hand side of (2) is dominated by a constant depending only on d, $|\alpha|$, $|\beta|$, μ, $N_{|\beta|}(\sigma, \mu)$, and $\kappa(\sigma)$.

The following result on sharp estimates of $G(x)$ and its derivatives near zero plays an important role in many situations. It shows that if $d > \mu$, then the behavior of G and its derivatives near zero is the same as for the function $1/|x|^{d-\mu}$. It also shows that in the general case if $|\alpha| = \mu$ (so that μ is an integer), then the kernel $K(x) := D^{\alpha}G(x)$ satisfies

$$|K(x)| \leq \frac{N}{|x|^d}, \quad |D_j K(x)| \leq \frac{N}{|x|^{d+1}}$$

for any $j = 1, \ldots, d$. Such estimates are crucial in the Sobolev space theory of elliptic equations as well as in Hölder space theory. Notice that for $\mu = 2$ and $\alpha = 0$ the assertions of Theorem 4 below are quite reasonable as Exercises 5.9 and 5.10 show. It is worth mentioning that the stated dependence of N in (5) on the data is not optimal and better results in this direction can be found in [**19**].

4. Theorem. *Let α be a multi-index such that $d + |\alpha| - \mu > 0$. Then, for $x \neq 0$,*

$$|D^{\alpha}G(x)| \leq N \frac{1}{|x|^{d+|\alpha|-\mu}}. \tag{5}$$

The same estimate is true for $d + |\alpha| - \mu = 0$ *for* $|x| \leq 1/2$ *if we replace the right-hand side with* $N \log(1/|x|)$. *Finally, if* $d + |\alpha| - \mu < 0$, *then* $|D^\alpha G(x)|$ *is bounded by a constant* N. *In all cases the constants* N *depend only on* d, $|\alpha|$, μ, r, $N_r(L, \mu)$, *and* $\kappa(L)$, *where* r *is the least nonnegative integer such that*

$$\mu + r > d + |\alpha|. \tag{6}$$

Proof. Notice that in light of Theorem 1 and Exercise 3 we need to show (5) only for small $|x|$, say $|x| \leq 1/2$. Take a multi-index β such that $|\beta| = r$, where r is the least nonnegative integer satisfying (6), and use Theorem 1. Also take a function $\zeta \in C_0^\infty(\mathbb{R}^d)$ such that $\zeta(x) = 1$ for $|x| \leq 1$ and $\zeta(x) = 0$ for $|x| \geq 2$, and on the basis of the identity

$$1 = \zeta(|x|\xi) + [1 - \zeta(|x|\xi)]$$

represent the integral in (2) as the sum of two integrals, with the first integral containing $\zeta(|x|\xi)$ and the second one containing $1 - \zeta(|x|\xi)$. Finally, in the first integral integrate by parts, bringing all derivatives (with respect to ξ) from $\xi^\alpha \sigma^{-1}$ to

$$\zeta(|x|\xi) \exp(ix \cdot \xi),$$

and notice that

$$\left|D_\xi^\beta [\zeta(|x|\xi) \exp(ix \cdot \xi)]\right| \leq N|x|^{|\beta|} I_{|x|\cdot|\xi| \leq 2} = N|x|^r I_{|x|\cdot|\xi| \leq 2}.$$

For $d + |\alpha| - \mu > 0$ this yields

$$|x^\beta D^\alpha G(x)| \leq \frac{N|x|^r}{(2\pi)^d} \int_{|\xi| \leq 2/|x|} \frac{|\xi^\alpha|}{|\sigma(\xi)|} \, d\xi + N \int_{|\xi| > 1/|x|} \frac{1}{|\xi|^{\mu - |\alpha| + r}} \, d\xi$$

$$\leq N|x|^r \int_0^{2/|x|} \frac{1}{\rho^{\mu - |\alpha|}} \rho^{d-1} \, d\rho + N \int_{1/|x|}^\infty \frac{1}{\rho^{\mu - |\alpha| + r}} \rho^{d-1} \, d\rho = \frac{N|x|^r}{|x|^{d + |\alpha| - \mu}},$$

where the constants N depend on the data as stated in the theorem, which is shown as in Exercise 3. In short,

$$|x^\beta| \, |D^\alpha G(x)| \leq \frac{N|x|^r}{|x|^{d + |\alpha| - \mu}}.$$

We sum up the left-hand sides of this estimate with respect to β such that $|\beta| = r$. We also observe that

$$|x|^r = \left(\sum_{j=1}^d |x^j|^2\right)^{r/2} \leq d^{r/2} \sum_{j=1}^d |x^j|^r \leq d^{r/2} \sum_{|\beta| = r} |x^\beta|.$$

Then we obtain

$$|x|^r|D^\alpha G(x)| \le \frac{N|x|^r}{|x|^{d+|\alpha|-\mu}},$$

which yields (5) for $d+|\alpha| > \mu$.

If $d+|\alpha| = \mu$, we get the result by noticing that

$$D^\alpha G(x) = -\int_1^{1/|x|} \frac{\partial}{\partial\rho} D^\alpha G(\rho x)\,d\rho + D^\alpha G(\frac{x}{|x|}).$$

Finally if $d+|\alpha|-\mu < 0$, it suffices to refer to Theorem 1 with $\beta = 0$. The theorem is proved.

5. Corollary. *If $\mu > 0$, then the Green's function $G \in \mathcal{L}_1$.*

To prove the corollary, it is tempting to take $\alpha = 0$ in (5) and use the fact that $G(x)$ decreases at infinity faster than any power of $|x|^{-1}$. However, we are talking not about the usual function with which the Green's function G coincides outside the origin but about the Green's function itself, which is a distribution. Say, δ_0 coincides with 0 outside the origin but yet $\delta_0 \notin \mathcal{L}_1$.

So, let us define $\hat{G}(x)$ as $G(x)$ for $x \ne 0$ and let $\hat{G}(0)$ be any fixed constant. By the above argument $\hat{G} \in \mathcal{L}_1$ and we have to prove that $G = \hat{G}$ in the sense of distributions. For that, take any $\phi \in C_0^\infty$ such that $\phi(x) = 1$ near the origin. Also take an $f \in C_0^\infty$. Then, for $\phi_n(x) = \phi(nx)$, by Lemma 5.7 we have

$$\langle G, f\rangle = \lim_{n\to\infty} \langle G, (1-\phi_n)f\rangle.$$

Since $\hat{G} \in \mathcal{L}_1$ and the $1-\phi_n$ are uniformly bounded and tend to 1 apart from the origin, by the dominated convergence theorem

$$(\hat{G}, f) = \lim_{n\to\infty} (\hat{G}, (1-\phi_n)f).$$

Finally, $\langle G, (1-\phi_n)f\rangle = (\hat{G}, (1-\phi_n)f)$ for any n by Theorem 1. The corollary is proved.

The following corollary will be used for proving that $LG = \delta_0$ once we understand what LG is.

6. Corollary. *If $\mu > 0$ and $\phi \in \mathcal{S}$, then $L^*\phi \in \mathcal{S}$ and*

$$\int_{\mathbb{R}^d} G(x)\overline{L^*\phi}(x)\,dx = (G, L^*\phi) = \bar{\phi}(0).$$

Indeed, by Parseval's identity (see the list of properties of the Fourier transform in the introduction of the chapter)

$$(G, L^*\phi) = \int_{\mathbb{R}^d} \tilde{G}(\xi)\overline{F(L^*\phi)(\xi)}\, d\xi.$$

Here $\tilde{G}$ is found from Remark 5.3 to be $c_d\sigma^{-1}$ and by definition $F(L^*\phi) = \bar{\sigma}\tilde{\phi}$, so that

$$(G, L^*\phi) = c_d \int_{\mathbb{R}^d} \bar{\tilde{\phi}}(\xi)\, d\xi = c_d \overline{\int_{\mathbb{R}^d} \tilde{\phi}(\xi)\, d\xi} = \overline{F^{-1}F\phi}(0) = \bar{\phi}(0).$$

7. Remark. The above properties of G allow us to define $G * \varphi$ for a class of functions much wider than C_0^∞. In Chapter 13 we will see that one can extend $G * \varphi$ treated as $L^{-1}\varphi$ to a rather large class of distributions.

7:2. Exercises (optional). The reader will acquire some useful information and develop technical skills while doing the following exercises, which otherwise are not used in the main body of the lectures.

8. Exercise. Prove that if $\mu > 0$, then

$$\int_{\mathbb{R}^d} G(x)\, dx = \sigma^{-1}(0).$$

9. Exercise. Prove that if $d = 1$ and $L = c - \partial^2/(\partial x)^2$, where $c > 0$ is a constant, then $G(x) = (2\sqrt{c})^{-1}e^{-|x|\sqrt{c}}$.

By comparing (5.6) with the result of Exercise 9, we obtain that for $r > 0$ and $c > 0$

$$\int_0^\infty \frac{1}{(4\pi t)^{1/2}} e^{-\frac{1}{4t}r^2 - ct}\, dt = \frac{1}{2\sqrt{c}} e^{-r\sqrt{c}}. \tag{7}$$

In particular, for $c = 1$ and $r = 2$ we come to the relation

$$\int_0^\infty \frac{1}{t^{1/2}} e^{-\frac{1}{t} - t}\, dt = \sqrt{\pi} e^{-2},$$

which we propose the reader try to prove differently.

10. Exercise. By differentiating (7) with respect to r and using (5.6), prove that for all *odd* d and $c > 0$ the Green's function $G(x)$ of $c - \Delta$ satisfies $G(x) = g(|x|)$, where for $r > 0$ the function g is given by

$$g(r) = N(d, c)\left(\frac{1}{r}\frac{d}{dr}\right)^{(d-1)/2} e^{-r\sqrt{c}}.$$

In particular, show that for $d = 3$ and $x \neq 0$

$$G(x) = \frac{1}{4\pi|x|} e^{-\sqrt{c}|x|}.$$

11. Exercise. Assume that the symbol $\sigma(\xi)$ has order $\mu > 1$ and depends only on $|\xi|$: $\sigma(\xi) = \tau(|\xi|)$. Then as we know, $G(x)$ depends only on $|x|$: $G(x) = g(|x|)$. By taking $x = (r, 0, ..., 0)$, $\alpha = 0$, and $\beta = (n, 0, ..., 0)$ in (2) with sufficiently large n, prove that for $r > 0$

$$g(r) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} e^{irt} \frac{1}{\tau(\sqrt{|\eta|^2 + t^2})} \, dt \right) d\eta.$$

In particular, show that

$$\int_{\mathbb{R}^{d-1}} \left| \int_{\mathbb{R}} e^{irt} \frac{1}{\tau(\sqrt{|\eta|^2 + t^2})} \, dt \right| d\eta < \infty.$$

12. Exercise. For $d = 1$ the strongly elliptic operator $L = D + 1$ has a Green's function, which satisfies the equation $G' + G = 0$ outside the origin. By using the fact that G goes to zero as $|x| \to \infty$, show that $G(x) = 0$ if $x < 0$ and $G(x) = ce^{-x}$ if $x > 0$, where c is a constant. Finally find c.

13. Exercise. Let r be defined as in Theorem 4. Assume that r is odd: $r = 2q - 1$, $q \geq 1$, and prove that

$$|x|^{2q} D^\alpha G(x) = \frac{i^{|\alpha|} i x^k}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} (-\Delta)^{q-1} [\xi^\alpha \sigma^{-1}(\xi)]_{\xi^k} \, d\xi,$$

where the factors of x^k are bounded functions.

8. Boundedness of the zeroth-order pseudo-differential operators in $\mathcal{L}_p$

The main result of this section will be used in the H_p^γ theory of elliptic equations. Before stating it, we prove the following.

1. Lemma. *For any $f \in \mathcal{L}_1$ and $\rho, \nu > 0$ we have*

$$\rho^\nu \int_{|y| \geq \rho} |y|^{-d-\nu} |f(y)| \, dy \leq N \mathbb{M} f(x), \tag{1}$$

where $\mathbb{M}f$ is the classical maximal function of f introduced in Section 3.3, *N depends only on d and ν, and x is any point such that $|x| < \rho$.*

Proof. For $|y| \geq \rho$ we have

$$|y|^{-d-\nu} = (d + \nu) \int_{|y|}^\infty \frac{1}{t^{d+\nu+1}} \, dt = (d + \nu) \int_\rho^\infty I_{|y| \leq t} \frac{1}{t^{d+\nu+1}} \, dt.$$

Therefore, by Fubini's theorem the integral on the left-hand side of (1) equals

$$(d + \nu)\rho^\nu \int_\rho^\infty \frac{1}{t^{d+\nu+1}} \Big(\int_{\rho \leq |y| \leq t} |f(y)| \, dy \Big) \, dt$$

and it only remains to observe that for $t \geq \rho > |x|$

$$\int_{\rho \leq |y| \leq t} |f(y)|\,dy \leq \int_{|y| \leq t} |f(y)|\,dy \leq N t^d \mathbf{M} f(x),$$

where N is the volume of the unit ball, and

$$\int_\rho^\infty \frac{1}{t^{\nu+1}}\,dt = \nu^{-1}\rho^{-\nu}.$$

The lemma is proved.

2. Theorem. *Let L be a zeroth-order pseudo-differential operator. Then for any $f \in \mathcal{S}$ and $p \in [2, \infty)$ we have*

$$\|Lf\|_{\mathcal{L}_p} \leq N \|f\|_{\mathcal{L}_p}, \tag{2}$$

where the constant N depends only on d, p, and $N_{d+2}(L, 0)$.

Proof. First, recall that C_0^∞ is dense in $\mathcal{S}$ (Exercise 1.4), the operator L is continuous in $\mathcal{S}$ (Remark 4.14) and the convergence in $\mathcal{S}$ implies the convergence in $\mathcal{L}_p$ (Exercise 1.2). Then we see that while proving the theorem, we may concentrate on $f \in C_0^\infty$.

If $p = 2$, then by definition and Parseval's identity

$$\|Lf\|_{\mathcal{L}_2} = \|\sigma_L \tilde{f}\|_{\mathcal{L}_2} \leq \sup |\sigma_L| \cdot \|\tilde{f}\|_{\mathcal{L}_2} = \sup |\sigma_L| \cdot \|f\|_{\mathcal{L}_2}.$$

Passing to the case that $p > 2$, we observe that without loss of generality we may assume that $|\sigma_L| \geq 1$. Indeed, almost obviously, it suffices to prove (2) for $L + c$ in place of L for a nonnegative constant

$$c \leq \sup |\sigma_L| + 1,$$

and if we take $c = \sup |\sigma_L| + 1$, then the symbol $\sigma_L + c$ of $L + c$ satisfies $|\sigma_L + c| \geq 1$.

Under this assumption L^{-1} is well defined as a zeroth-order pseudo-differential operator, and it possesses a Green's function G. Hence if $f \in C_0^\infty$ and $Lf = \phi$, then $f = L^{-1}\phi$ and $\phi = G * f$. In short

$$Lf = G * f.$$

Now the idea is to obtain a pointwise estimate of the classical sharp function of $G * f$ through the classical maximal function of f. Fix an $f \in \mathcal{S}$, denote

$$u = Lf = G * f,$$

take a $\rho > 0$, take a $\zeta \in C_0^\infty$ such that $\zeta(x) = 1$ for $|x| \leq 2$ and $\zeta(x) = 0$ for $|x| \geq 3$, and set $\zeta_\rho(x) = \zeta(x/\rho)$. Also set

$$g_\rho = f\zeta_\rho, \quad h_\rho = (1 - \zeta_\rho)f, \quad v_\rho = G * g_\rho, \quad w_\rho = G * h_\rho,$$

so that $u = v_\rho + w_\rho$.

By the above case of $p = 2$ and Hölder's inequality

$$\fint_{B_\rho} |v_\rho - v_{\rho(B_\rho)}|\, dx \leq \Big(\fint_{B_\rho} |v_\rho - v_{\rho(B_\rho)}|^2\, dx\Big)^{1/2}$$

$$\leq \Big(\fint_{B_\rho} |v_\rho|^2\, dx\Big)^{1/2} \leq N\rho^{-d/2}\|v_\rho\|_{\mathcal{L}_2}$$

$$\leq N\rho^{-d/2}\|g_\rho\|_{\mathcal{L}_2} \leq N\Big(\fint_{B_{3\rho}} |f|^2\, dx\Big)^{1/2} \leq N\mathbb{M}^{1/2}(|f|^2)(x_0), \tag{3}$$

where x_0 is any point in B_ρ (or for that matter in $B_{3\rho}$). Next, for $|x| \leq \rho$ the support of $h_\rho(x - \cdot)$ lies outside B_ρ and by Theorem 7.1

$$w_\rho(x) = \int_{\mathbb{R}^d} G(y)h_\rho(x - y)\, dy = \int_{\mathbb{R}^d} G(x + y)h_\rho(-y)\, dy,$$

$$w_\rho(0) = \int_{\mathbb{R}^d} G(y)h_\rho(-y)\, dy,$$

$$w_\rho(x) - w_\rho(0) = \int_{\mathbb{R}^d} [G(x + y) - G(y)]h_\rho(-y)\, dy. \tag{4}$$

In the last integral we can restrict the domain of integration to $|y| \geq 2\rho$ since $1 - \zeta_\rho(-y) = 0$ otherwise. In that domain the balls $B_\rho(y)$ are at distance $|y| - \rho \geq (1/2)|y|$ from the origin and, for $|x| \leq \rho$, the straight segment $tx + y$, $t \in [0, 1]$, stays at distance at least $(1/2)|y|$ from the origin. By the mean value theorem and Theorem 7.4 (with $\mu = 0$, $|\alpha| = 1$, and $r = d + 2$) for an appropriate $\theta \in [0, 1]$ we have

$$|G(x + y) - G(y)| \leq |x| \cdot |G_y(\theta x + y)| \leq N\rho|y|^{-d-1}.$$

Hence,

$$|w_\rho(x) - w_\rho(0)| \leq N\rho \int_{|y|\geq 2\rho} |y|^{-d-1}|f(y)|\, dy$$

with the constant N depending on the data as stated in the theorem.

By using Lemma 1, we conclude $|w_\rho(x) - w_\rho(0)| \leq N\mathbb{M}f(x_0)$ if $|x_0| \leq \rho$ and

$$\int\!\!\!\!-_{B_\rho} |w_\rho(x) - w_{\rho(B_\rho)}|\,dx \leq 2 \int\!\!\!\!-_{B_\rho} |w_\rho(x) - w_\rho(0)|\,dx$$

$$\leq N\mathbb{M}f(x_0) \leq N\mathbb{M}^{1/2}(|f|^2)(x_0),$$

where the last relation follows from Hölder's inequality.

Upon combining this with (3) and recalling that $u = v_\rho + w_\rho$, we see that

$$\int\!\!\!\!-_{B_\rho} |u(x) - u_{(B_\rho)}|\,dx \leq N\mathbb{M}^{1/2}(|f|^2)(x_0). \tag{5}$$

Obviously, the origin here can be replaced with any point, so that, actually,

$$\int\!\!\!\!-_{B} |u(x) - u_{(B)}|\,dx \leq N\mathbb{M}^{1/2}(|f|^2)(x_0)$$

for any ball B as long as $x_0 \in B$. It follows by definition that

$$u^\sharp \leq N\mathbb{M}^{1/2}(|f|^2)$$

on $\mathbb{R}^d$. Now for $p > 2$ we use Theorems 3.3.1 and 3.3.2 and conclude

$$\|u\|_{\mathcal{L}_p} \leq N\|u^\sharp\|_{\mathcal{L}_p} \leq N\|\mathbb{M}^{1/2}(|f|^2)\|_{\mathcal{L}_p} = N\|\mathbb{M}(|f|^2)\|^{1/2}_{\mathcal{L}_{p/2}}$$

$$\leq N\|f^2\|^{1/2}_{\mathcal{L}_{p/2}} = N\|f\|_{\mathcal{L}_p}.$$

The theorem is proved.

Next, by using duality, we extend Theorem 2 to the full range $p \in (1,\infty)$. In the following theorem r can be taken to be the least integer $> d/2$ as we suggest the reader prove doing Exercises 13.3.17 and 13.3.18 (also see [**19**]).

3. Theorem. *Let L be a zeroth-order pseudo-differential operator. Then for any $f \in \mathcal{S}$ and $p \in (1,\infty)$ we have*

$$\|Lf\|_{\mathcal{L}_p} \leq N\|f\|_{\mathcal{L}_p}, \tag{6}$$

where the constant N depends only on d, p, and $N_{d+2}(L,0)$.

Proof. In light of Theorem 2 we may concentrate of $p \in (1,2)$ when $q := p/(p-1) \in (2,\infty)$.

By Lemma 4.19 and Theorem 2 for any $f, \phi \in \mathcal{S}$ we have

$$|(Lf,\phi)| = |(f, L^*\phi)| \le \|f\|_{\mathcal{L}_p}\|L^*\phi\|_{\mathcal{L}_q} \le N\|f\|_{\mathcal{L}_p}\|\phi\|_{\mathcal{L}_q}.$$

Since, as is well known,

$$\|Lf\|_{\mathcal{L}_p} = \sup\{|(Lf,\phi)| : \phi \in \mathcal{S}, \|\phi\|_{\mathcal{L}_q} = 1\},$$

we come to (6) and the theorem is proved.

The following exercise will be generalized to a very large extent in Section 13.3.

4. Exercise. Let L be a strongly elliptic pseudo-differential operator of order m, where m is a nonnegative *even* integer. Prove that for $p \in (1,\infty)$ and $f \in \mathcal{S}$, we have

$$\|Lf\|_{W_p^m} \le N\|f\|_{\mathcal{L}_p}, \quad \|f\|_{W_p^m} \le N\|Lf\|_{\mathcal{L}_p},$$

where N is independent of f.

9. Operators related to the Laplacian

9:1. The operators $(1-\Delta)^{\gamma/2}$, Cauchy's operator, the Riesz and Hilbert transforms, the Cauchy-Riemann operator. In this section we collect some important properties of the operators $(1-\Delta)^{\gamma/2}$, $\gamma \in \mathbb{R}$, which are used in the future. We also discuss some properties of other operators such as Δ, more general homogeneous elliptic differential operators, Cauchy's operator, the Hilbert transform, Riesz transforms, and the Cauchy-Riemann operator $\bar\partial = \partial/\partial\bar z$. Recall how the operators $(1-\Delta)^{\gamma/2}$ have been introduced in Example 4.10.

1. Definition. By $(1-\Delta)^{\gamma/2}$ we mean the pseudo-differential operator of order γ with symbol $(1+|\xi|^2)^{\gamma/2}$.

The operators $(1-\Delta)^{\gamma/2}$ play a very important role in the theory since by Theorem 8.3 the operator $L(1-\Delta)^{-\gamma/2}$ is, basically, a bounded operator in $\mathcal{L}_p$ if L is a pseudo-differential operator of order γ and if it is strongly elliptic, then the same is true for $L^{-1}(1-\Delta)^{\gamma/2}$.

There is a particular case when $\gamma = 1$.

2. Definition. The pseudo-differential operator $\mathcal{K}$ with symbol $-(1+|\xi|^2)^{1/2}$ is called *Cauchy's operator*.

Actually, a more standard definition of Cauchy's operator is that it is an operator with symbol $-|\xi|$ (see Exercise 7). But in our setting we do not consider $|\xi|$ as a symbol because it is not infinitely differentiable at the origin. Still the reader will be able to get a quite substantial bit of information about this operator by doing Exercise 7.

Obviously, $\mathcal{K}$ is a strongly elliptic first-order operator. Here is a reflection of this fact generalized in Theorem 13.3.10 which, by the way, is proved without using Theorem 3.

3. Theorem. *Let $p \in (1,\infty)$, and let $f, g \in \mathcal{S}$. Then $\mathcal{K}^{-1}f, \mathcal{K}^{-1}Df, \mathcal{K}g \in \mathcal{S}$ and*

$$\|\mathcal{K}^{-1}f\|_{W_p^1} \leq N\|f\|_{\mathcal{L}_p}, \quad \|g\|_{W_p^1} \leq N\|\mathcal{K}g\|_{\mathcal{L}_p}, \tag{1}$$

$$\|\mathcal{K}g\|_{\mathcal{L}_p} \leq N\|g\|_{W_p^1}, \tag{2}$$

where N is independent of f and g.

Proof. The second estimate in (1) is obtained from the first one by setting $f = \mathcal{K}g$. To prove the first one, it suffices to observe that $\mathcal{K}^{-1}$ and $D\mathcal{K}^{-1}$ are pseudo-differential operators of order zero, so that by Theorem 8.3

$$\|\mathcal{K}^{-1}f\|_{W_p^1} \leq N(\|\mathcal{K}^{-1}f\|_{\mathcal{L}_p} + \|D\mathcal{K}^{-1}f\|_{\mathcal{L}_p}) \leq N\|f\|_{\mathcal{L}_p}.$$

To prove (2), observe that (by Theorem 4.18)

$$\mathcal{K}g = (1-\Delta)\mathcal{K}^{-1}g = u - \sum_{j=1}^{d} D_j v_j,$$

where

$$u := \mathcal{K}^{-1}g, \quad v_j := D_j\mathcal{K}^{-1}g = \mathcal{K}^{-1}D_j g,$$

so that by (1)

$$\|\mathcal{K}g\|_{\mathcal{L}_p} \leq \|u\|_{\mathcal{L}_p} + \sum_{j=1}^{d}\|D_j v_j\|_{\mathcal{L}_p} \leq N\big(\|g\|_{\mathcal{L}_p} + \sum_{j=1}^{d}\|D_j g\|_{\mathcal{L}_p}\big).$$

The theorem is proved.

The following two exercises are applicable to the Cauchy-Riemann operator $\bar{\partial}$ and the operators $d^n/(dx)^n$, $n = 1, 2, \ldots$, if $d = 1$.

4. Exercise (Homogeneous elliptic operators). Let $m \geq 1$ be an integer and let L be a homogeneous elliptic differential operator (see Definition 2.10). Prove that for any $p \in (1,\infty)$ and $u \in W_p^1$ we have

$$[u]_{W_p^m} \leq N\|Lu\|_{\mathcal{L}_p}, \tag{3}$$

where N is independent of u.

You may like to start by using Exercise 8.4 to derive that for any $u \in \mathcal{S}$

$$\|u\|_{W_p^{2m}} \leq N\|(LL^* + 1)u\|_{\mathcal{L}_p} \leq N\|LL^*u\|_{\mathcal{L}_p} + N\|u\|_{\mathcal{L}_p},$$

where N is independent of u. Then write

$$\|L^*Lu\|_{W_p^m} \leq N\|L(L^*L)u\|_{\mathcal{L}_p} + N\|Lu\|_{\mathcal{L}_p}. \tag{4}$$

Additional hints are given at the end of the chapter.

5. Exercise (Homogeneous elliptic operators). For the operator from Exercise 4 prove that LC_0^∞ is dense in $\mathcal{L}_p$ for any $p \in (1, \infty)$. Give an example showing that the latter is false in general if $p = 1$.

6. Exercise (Riesz transforms). For $\phi \in \mathcal{S}$, $j = 1, ..., d$, introduce the Riesz transforms by

$$\mathcal{R}_j\phi = F^{-1}(\zeta^j F\phi),$$

where $\zeta^j(\xi) = \xi^j|\xi|^{-1}$ $(0 \cdot 0^{-1} := 0)$. If $d = 1$, then there is only one such operator and it is called the Hilbert transform, denoted by $\mathcal{H}$.

Also take an $\eta \in C^\infty(\mathbb{R}^d)$ such that $\eta = 0$ near the origin and $\eta(\xi) = 1$ for $|\xi| \geq 1$. For $\varepsilon > 0$ introduce $\mathcal{R}_j^\varepsilon$ by the same formula as above with $\zeta^j(\xi)\eta(\xi/\varepsilon)$. Prove that $\mathcal{R}_j^\varepsilon\phi \to \mathcal{R}_j\phi$ in $\mathcal{L}_2$ as $\varepsilon \downarrow 0$ for any $\phi \in \mathcal{S}$. By using Theorem 8.3 and dilations, prove that for any $\varepsilon > 0$, $p \in (1, \infty)$, and $\phi \in \mathcal{S}$,

$$\|\mathcal{R}_j^\varepsilon\phi\|_{\mathcal{L}_p} \leq N\|\phi\|_{\mathcal{L}_p},$$

where N is independent of ϕ *and* ε. Conclude that

$$\|\mathcal{R}_j\phi\|_{\mathcal{L}_p} \leq N\|\phi\|_{\mathcal{L}_p}, \tag{5}$$

where N is independent of ϕ. Estimate (5) allows one to uniquely extend $\mathcal{R}_j$ from a dense subset $\mathcal{S}$ of $\mathcal{L}_p$ to the whole of $\mathcal{L}_p$. Show that for this extension

$$\sum_{j=1}^d \mathcal{R}_j^2 = 1, \quad \|\phi\|_{\mathcal{L}_p} \leq N\sum_{j=1}^d \|\mathcal{R}_j\phi\|_{\mathcal{L}_p} \tag{6}$$

with N independent of $\phi \in \mathcal{L}_p$.

7. Exercise (Homogeneous Cauchy operator). For $\phi \in \mathcal{S}$ set

$$\mathcal{K}_0\phi = F^{-1}(\zeta F\phi),$$

where $\zeta(\xi) = -|\xi|$. Prove that $\mathcal{K}_0\phi \in \mathcal{L}_2$ and moreover, for $p \in (1, \infty)$,

$$\|\mathcal{K}_0\phi\|_{\mathcal{L}_p} \leq N\|D\phi\|_{\mathcal{L}_p}, \quad \|D\phi\|_{\mathcal{L}_p} \leq N\|\mathcal{K}_0\phi\|_{\mathcal{L}_p},$$

where N is independent of ϕ.

8. Exercise (Laplacian). (i) By observing that $\mathcal{K}_0^2 = -\Delta$, prove again the basic L_p estimate:

$$\|D^2\phi\|_{\mathcal{L}_p} \leq N\|\Delta\phi\|_{\mathcal{L}_p}$$

for all $\phi \in \mathcal{S}$ and $p \in (1,\infty)$ with N independent of ϕ.

(ii) Without using Exercise 5, prove that for any $n \in \{1,2,...\}$ the set $\Delta^n\mathcal{S}$ is dense in $\mathcal{L}_p$ ($p \in (1,\infty)$). Using this, show first that, for $k,n \in \{1,2,...\}$, the set $(1-\Delta)^k\Delta^n\mathcal{S}$ is dense in $\mathcal{L}_p$ and then that the set $\Delta^n\mathcal{S}$ is dense in W_p^k.

(iii) As we know, $\|D^2\phi\|_{\mathcal{L}_p} \leq N\|\Delta\phi\|_{\mathcal{L}_p}$ for any $\phi \in W_p^2$ and (ii) shows that the set ΔW_p^2 is dense in $\mathcal{L}_p$. However, this does not lead to the invertibility of the mapping

$$\Delta : W_p^2 \to \mathcal{L}_p.$$

In connection with this let $d \geq 3$, choose a number α such that

$$d < \alpha p < d + 2p, \quad 2 < \alpha < d,$$

set $f(x) = (1+|x|^2)^{-\alpha/2}$, and prove that $f \in \mathcal{L}_p$ but there is no $u \in W_p^2$ satisfying the equation $\Delta u = f$.

9. Remark. By Theorem 4.18 we have that for any $\gamma,\nu \in \mathbb{R}$

$$(1-\Delta)^{\gamma+\nu} = (1-\Delta)^{\gamma}(1-\Delta)^{\nu}.$$

10. Remark. The operator $(1-\Delta)^{\gamma/2}$ is defined on $\mathcal{S}$ and is formally self-adjoint by Lemma 4.19, that is, if $g,\phi \in \mathcal{S}$, then

$$((1-\Delta)^{\gamma/2}g,\phi) = (g,(1-\Delta)^{\gamma/2}\phi).$$

In particular, for $(1-\Delta)^{-\gamma/2}g$ in place of g,

$$(g,\phi) = ((1-\Delta)^{-\gamma/2}g,(1-\Delta)^{\gamma/2}\phi). \tag{7}$$

11. Remark. From Corollary 7.5 and Theorem 7.1 we know that for $\phi \in C_0^\infty$ and $\gamma > 0$ the function $(1-\Delta)^{-\gamma/2}\phi$ is represented as the convolution of ϕ and an infinitely differentiable outside zero integrable function G. It turns out that G can be found almost explicitly. Indeed, take $\phi \in \mathcal{S}$, then

$$F((1-\Delta)^{-\gamma/2}\phi)(\xi) = (1+|\xi|^2)^{-\gamma/2}\tilde{\phi}(\xi) = c\tilde{\phi}(\xi)\int_0^\infty t^{\gamma/2}e^{-t}e^{-|\xi|^2t}\,\frac{dt}{t},$$

where c is an appropriate constant. Then as in Section 1.7 we recall that $e^{-|\xi|^2t}$ is proportional to the Fourier transform of

$$p(t,x) = (4\pi t)^{-d/2}e^{-|x|^2/(4t)}$$

and that the product of the Fourier transforms is proportional to the Fourier transform of the convolution. The computation

$$\int_{\mathbb{R}^d}\int_0^\infty t^{\gamma/2}e^{-t}|p(t,x)|\,\frac{dt}{t}\,dx = \int_0^\infty t^{\gamma/2}e^{-t}\int_{\mathbb{R}^d}|p(t,x)|\,dx\,\frac{dt}{t}$$

$$= \int_{\mathbb{R}^d}|p(1,x)|\,dx\int_0^\infty t^{\gamma/2}e^{-t}\,\frac{dt}{t} < \infty$$

allows us to use Fubini's theorem in finding the Fourier transform of the function

$$g(x) := \int_0^\infty t^{\gamma/2}e^{-t}p(t,x)\,\frac{dt}{t},$$

which is positive and integrable:

$$\int_{\mathbb{R}^d} g(x)\,dx = \int_0^\infty t^{\gamma/2}e^{-t}\,\frac{dt}{t} < \infty.$$

One sees that $F((1-\Delta)^{-\gamma/2}\phi)(\xi) = F(G*\phi)$ for any $\phi \in \mathcal{S}$ and not only for $\phi \in C_0^\infty$, where

$$G(x) = c\int_0^\infty t^{\gamma/2}e^{-t}p(t,x)\,\frac{dt}{t} \tag{8}$$

and c is an appropriate constant. In this way we arrive at formula (9) below, which explains why the Green's function G is also called *the kernel of* $(1-\Delta)^{-\gamma/2}$.

12. Theorem. *If $\gamma > 0$, then for any $\phi \in \mathcal{S}$ we have*

$$(1-\Delta)^{-\gamma/2}\phi(x) = \int_{\mathbb{R}^d} G(x-y)\phi(y)\,dy \tag{9}$$

and G is a function depending only on $|x|$. Furthermore, $G \ge 0$ and $\|G\|_{\mathcal{L}_1} = 1$.

To prove the second assertion, one can either use properties of the gamma function and, by following the way we get (8), find the constant c in (8), or just notice that integrating (9) and using Exercise 4.13, we obtain

$$\int_{\mathbb{R}^d}\phi\,dx = \int_{\mathbb{R}^d}(1-\Delta)^{-\gamma/2}\phi\,dx = \int_{\mathbb{R}^d} G\,dx\int_{\mathbb{R}^d}\phi\,dx,$$

which implies that indeed $c > 0$, $G \ge 0$, and $\|G\|_{\mathcal{L}_1} = 1$.

From Theorem 12 and the well-known inequality (see Lemma 1.8.1)

$$\|f * g\|_{\mathcal{L}_p} \le \|f\|_{\mathcal{L}_1} \|g\|_{\mathcal{L}_p},$$

we get the following for $\nu = 0$.

13. Corollary. *If $\gamma, \nu \in \mathbb{R}$ and $\gamma > \nu$, then for any $\phi \in \mathcal{S}$ and $p \in [1, \infty]$ we have*

$$\|(1-\Delta)^{-\gamma}\phi\|_{\mathcal{L}_p} \le \|(1-\Delta)^{-\nu}\phi\|_{\mathcal{L}_p}.$$

The assertion in the case of general ν follows from the particular case since $(1-\Delta)^{-\gamma} = (1-\Delta)^{\nu-\gamma}(1-\Delta)^{-\nu}$ and $\nu - \gamma < 0$.

9:2. Exercises (optional).

14. Exercise. The Riesz transforms $\mathcal{R}_j$ and the Hilbert transform $\mathcal{H}$ admit singular-integral representations similar to the ones obtained in Section 1.9 for $u_{x^i x^j}$. As a matter of fact, in the notation of Section 1.9 we have $u_{x^j x^k} = -\mathcal{R}_j\mathcal{R}_k f$. In this connection prove that for $\phi \in C_0^\infty$ (a.e.)

$$\mathcal{R}_j\phi(x) = \omega_d \lim_{\varepsilon \downarrow 0} \int_{|y| \ge \varepsilon} \frac{y^j \phi(x-y)}{|y|^{d+1}}\,dy, \quad \mathcal{H}\phi(x) = \omega_1 \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{\phi(x-y)}{y}\,dy,$$

where the limits exist at each x and the ω_d are appropriate constants. You may like to show first that

$$\lim_{\varepsilon \downarrow 0} \int_{|y| \ge \varepsilon} \frac{y^j \phi(x-y)}{|y|^{d+1}}\,dy = \int_{\mathbb{R}^d} \frac{y^j(\phi(x-y) - \phi(x))}{|y|^{d+1}}\,dy =: \psi_j(x)$$

and then proceeding as in Remark 11 that

$$\psi_j(x) = N \int_0^\infty \frac{1}{s^{(d+3)/2}} \int_{\mathbb{R}^d} (\phi(x-y) - \phi(x)) y^j e^{-|y|^2/(4s)}\,dyds.$$

After that, a little extra effort shows that

$$N \int_0^R \frac{1}{s^{(d+3)/2}} \int_{\mathbb{R}^d} \phi(x-y) y^j e^{-|y|^2/(4s)}\,dyds \xrightarrow{\mathcal{L}_2} \psi_j(x) \tag{10}$$

as $R \to \infty$, where N is an appropriate constant. As in Remark 11 this leads to the fact that in $\mathcal{L}_2$

$$N\tilde{\phi}(\xi)\xi^j \int_0^R \frac{1}{s^{1/2}} e^{-s|\xi|^2}\,ds \to \tilde{\psi}_j(\xi),$$

which immediately yields the desired result: $\omega_d\tilde{\psi}_j = F(\mathcal{R}_j\phi)$.

15. Exercise. By using (8) and Exercise 7.10, prove that if $\gamma \geq 1$ and $d + 2 - \gamma$ is an odd integer ≥ 1, then for the Green's function $G(x)$ of $(1 - \Delta)^{\gamma/2}$ we have $G(x) = g(|x|)$, where for $r > 0$ the function g is defined by

$$g(r) = N(d, \gamma)\left(\frac{1}{r}\frac{d}{dr}\right)^{(d+1-\gamma)/2} e^{-r}.$$

In particular, if $d = 2$, then for $x \neq 0$ the Green's function of Cauchy's operator equals $N|x|^{-1}e^{-|x|}$, where N is a constant.

16. Exercise. Prove that, for $\gamma > 0$, there is *no* constant $N < \infty$ such that

$$\|(1 - \Delta)^{\gamma/2}\phi\|_{\mathcal{L}_p} \leq N\|\phi\|_{\mathcal{L}_p}$$

for all $\phi \in C_0^\infty$, so that the operator $(1 - \Delta)^{\gamma/2}$ cannot be extended from C_0^∞ to a bounded operator in $\mathcal{L}_p$.

17. Exercise. Assume that we know that there is a constant N such that

$$\|\phi_{xx}\|_{\mathcal{L}_p} \leq N\|\Delta\phi\|_{\mathcal{L}_p}$$

for any $\phi \in \mathcal{S}$. Of course, we know this from previous results. We suggest the reader, using Corollary 13 alone, prove that with the same N we have

$$\|\phi_{xx}\|_{\mathcal{L}_p} \leq 2N\|\lambda\phi - \Delta\phi\|_{\mathcal{L}_p}$$

for any $\phi \in \mathcal{S}$ and $\lambda \geq 0$.

18. Exercise. Find c in (8) from knowing that $\|G\|_{\mathcal{L}_1} = 1$ and

$$\int_{-\infty}^{\infty} \exp(-|x|^2/4)\, dx = \sqrt{4\pi}.$$

By Theorem 7.4 we know that G is bounded if $\gamma > d$ and it is infinitely differentiable for $x \neq 0$ and any $\gamma > 0$. We also know that if $d > \gamma > 0$, then

$$|G(x)| \leq N|x|^{\gamma - d}$$

for $x \neq 0$, and if $\gamma = d$, then the same estimate holds for $|x| \leq 1/2$ if we replace the right-hand side with $N\log(1/|x|)$. In all cases the constant N is independent of x.

The behavior at infinity is different as Theorem 7.1 shows. The reader will make a more precise investigation of this behavior doing the following exercise.

19. Exercise. Prove that, for G from (8), any $\varepsilon \in (0, 1)$, multi-index α, and $|x| \geq 1$, we have

$$|D^\alpha G(x)| \leq Ne^{-(1-\varepsilon)|x|},$$

where N is independent of x.

We have seen above that pseudo-differential operators of strictly negative order are represented as convolution operators. It turns out that, if the order is strictly positive, they can also be integral operators with the exception of just differential operators.

20. Exercise. Prove that if $\gamma \in (0,2)$ and $\phi \in \mathcal{S}$, then

$$(1-\Delta)^{\gamma/2}\phi(x) = c\int_0^\infty t^{-\gamma/2}\big[e^{-t}T_t\phi(x) - \phi(x)\big]\,\frac{dt}{t},$$

where

$$T_t\phi(x) := \int_{\mathbf{R}^d} p(t, x-y)\phi(y)\,dy$$

and c is an appropriate constant.

21. Exercise. Observe that

$$\int_{\mathbf{R}^d} p(t,x-y)\phi(y)\,dy - \phi(x) = \int_{\mathbf{R}^d} p(1,y)(\phi(x+\sqrt{t}\,y) - \phi(x))\,dy$$

$$= \int_{\mathbf{R}^d} p(1,y)\sqrt{t}\,y^i\int_0^1 \phi_{x^i}(x + r\sqrt{t}\,y)\,drdy$$

and prove that

$$\int_{\mathbf{R}^d} |T_t\phi(x) - \phi(x)|\,dx \le N\sqrt{t}\int_{\mathbf{R}^d}|\phi_x|\,dx,$$

where N is independent of ϕ and t. By using this and Exercise 20, prove that, for $\phi \in \mathcal{S}$ and $\gamma \in (0,1)$,

$$\|(1-\Delta)^{\gamma/2}\phi\|_{\mathcal{L}_1} \le N(\|\phi\|_{\mathcal{L}_1} + \|\phi_x\|_{\mathcal{L}_1}),$$

where N is independent of ϕ.

22. Exercise. Prove that for any $\gamma, \varepsilon \in (0,1)$, there is a constant N depending only on d, γ, and ε such that, for any $\phi, \psi \in \mathcal{S}$, we have

$$\Big|\int_{\mathbf{R}^d}\phi\psi\,dx\Big| \le N\|\psi\|_{W_1^1}\sup_{x\in\mathbf{R}^d}\int_{\mathbf{R}^d}|\phi(y)|e^{-(1-\varepsilon)|x-y|}|x-y|^{\gamma-d}\,dy.$$

10. An embedding lemma

In this section $\gamma > 0$ and G is the kernel of $(1-\Delta)^{-\gamma/2}$. We prove a few facts which will later in Section 13.8 allow us to generalize Morrey's theorem for functions with fractional derivatives.

1. Lemma. *If* $0 < \gamma < d+1$, *then*

$$|G(x) - G(y)| \le NR(|x|,|y|), \tag{1}$$

where N *is a constant and*

$$R(s,t) := \begin{cases} |s^{\gamma-d} - t^{\gamma-d}| & \text{if } \gamma \ne d, \\ |\log(s/t)| & \text{if } \gamma = d. \end{cases}$$

Proof. The function G is radially symmetric. Define $g(s)$ so that $G(x) = g(|x|)$. Denote $s = |x|$ and $t = |y|$ and without loss of generality assume $t \geq s$. By Theorem 7.4 for $r = |x| > 0$

$$|g'(r)| = |x|^{-1}|x^i G_{x^i}(x)| \leq N r^{\gamma-d-1}.$$

Therefore,

$$|G(x) - G(y)| = |g(s) - g(t)| = \Big| \int_s^t g'(r)\, dr \Big| \leq N \int_s^t r^{\gamma-d-1}\, dr,$$

which is (1). The lemma is proved.

Here is the main result of this section.

2. Lemma. *Let $p \in (1, \infty]$ and let*

$$0 < \delta := \gamma - d/p < 1.$$

Then there is a constant N such that, for any $\phi \in \mathcal{S}$ and $x, y \in \mathbb{R}^d$,

$$|\phi(x)| \leq N\|(1-\Delta)^{\gamma/2}\phi\|_{\mathcal{L}_p},$$

$$|\phi(x) - \phi(y)| \leq N|x-y|^{\delta}\|(1-\Delta)^{\gamma/2}\phi\|_{\mathcal{L}_p}. \tag{2}$$

Proof. Denote $\psi = (1-\Delta)^{\gamma/2}\phi$. Then $\psi \in \mathcal{S}$, by Lemma 4.17, we have $\phi = (1-\Delta)^{-\gamma/2}\psi$, and Theorem 9.12 says that $\phi = G * \psi$. Hence by Hölder's inequality

$$|\phi(x)| \leq \|G\|_{\mathcal{L}_q}\|\psi\|_{\mathcal{L}_p},$$

where $q = p/(p-1)$ if $p < \infty$ and $q = 1$ if $p = \infty$. If $q = 1$, then $\|G\|_{\mathcal{L}_q}=1$. If $q > 1$, then the function G^q is integrable at infinity due to Theorem 7.1. Near the origin, by Theorem 7.4 we have

$$|G(x)| \leq N + N\frac{1}{|x|^{d-\gamma}}\log(1/|x|),$$

and G^q is integrable near the origin since the inequality $q(\gamma - d) > -d$ is equivalent to $\gamma - d/p > 0$.

To prove the second estimate in (2), we notice that by shifting the origin, we may assume that $y = 0$. Now use Lemma 1 and Hölder's inequality to get

$$\int_{\mathbb{R}^d} |G(z-x) - G(z)|^q\, dz \leq NJ(x) := N\int_{\mathbb{R}^d} R^q(|z-x|, |z|)\, dz,$$

$$|\phi(x) - \phi(0)| \leq \int_{\mathbb{R}^d} |G(z-x) - G(z)|\,|\psi(z)|\,dz \leq NJ^{1/q}(x)\|\psi\|_{\mathcal{L}_p}.$$

It is easily seen that $J(x)$ is rotation invariant with respect to x and, therefore, it depends only on $|x|$. Changing variables $z \to z|x|$ yields

$$J(x) = |x|^{q(\gamma-d)+d} J(e),$$

where e is a unit vector. One checks easily that $q(\gamma - d) + d = q\delta$ and then sees that it only remains to prove that $J(e) < \infty$.

The function $R^q(|z-e|, |z|)$ is integrable over $\{z : |z| \leq 10\}$ for the same reason as above since the probable singularities are not too high. At infinity $R^q(|z - e|, |z|)$ is integrable due to the fact that the increments of power functions decrease one power faster than the original functions. Namely, if

$$1 \leq t - 1 \leq s \leq t \leq s + 1,$$

then, for a constant N_0,

$$R(s,t) = N_0 \int_s^t r^{\gamma-d-1}\,dr \leq N_0 s^{\gamma-d-1}$$

$$= N_0 (s/t)^{\gamma-d-1} t^{\gamma-d-1} \leq N_0 2^{1+d-\gamma} t^{\gamma-d-1}$$

since

$$s/t \geq (t-1)/t = 1 - 1/t \geq 1/2.$$

It follows that, if $|t-s| \leq 1$ and $t, s \geq 2$, then $R(s,t) \leq N t^{\gamma-d-1}$. Therefore,

$$\int_{|z|\geq 3} R^q(|z-e|, |z|)\,dz \leq N \int_{|z|\geq 3} |z|^{q(\gamma-d-1)}\,dz$$

$$= N \int_3^\infty t^{q(\gamma-d-1)+d-1}\,dt < \infty,$$

where the last conclusion is due to the fact that the inequality

$$q(\gamma - d - 1) + d < 0$$

holds since it is equivalent to $\gamma - d/p < 1$. The lemma is proved.

3. Exercise. (i) Under the conditions of Lemma 2 prove that for any $\phi \in \mathcal{S}$, $\lambda > 0$, and $x, y \in \mathbb{R}^d$,

$$|\phi(x) - \phi(y)| \leq N|x - y|^{\gamma - d/p}\|(\lambda - \Delta)^{\gamma/2}\phi\|_{\mathcal{L}_p},$$

where N is the same constant as in (2).

(ii) By taking $p = 2$, derive that, for $d/2 < \gamma < 1 + d/2$ and $\delta = \gamma - d/2$,

$$|\phi(x) - \phi(y)| \leq N|x - y|^{\delta}\Big(\int_{\mathbb{R}^d} |\xi|^{2\gamma}|\tilde{\phi}(\xi)|^2\,d\xi\Big)^{1/2}. \tag{3}$$

(iii) Prove that there is *no* (finite) constant N such that (3) holds for any $\phi \in \mathcal{S}$ and $x, y \in \mathbb{R}^d$ if we take $\delta \neq \gamma - d/2$ or if $\delta = \gamma - d/2$ but $\gamma > 1 + d/2$.

4. Exercise. If $\gamma = 1$ and $p = 2$, then the norm on the right in (2) is equivalent to $\|\phi\|_{\mathcal{L}_2} + \|\phi_x\|_{\mathcal{L}_2}$. Using this, prove that, if $\gamma = d = 1$ and $p = 2$, then there is *no* constant N such that (2) holds for any $\phi \in \mathcal{S}$ and $x, y \in \mathbb{R}^d$ if we take $\delta > \gamma - d/p = 1/2$.

5. Exercise. Lemma 2 turns out to be false if $\gamma - d/p = 0$. Prove that if $p \in (1, \infty)$ and $\gamma = d/p$, then for any $\phi \in \mathcal{S}$ and $\rho > 0$

$$\fint_{B_\rho}\fint_{B_\rho} |\phi(x) - \phi(y)|\,dxdy \leq N\|F\|_{\mathcal{L}_{p/(p-1)}}\|(1 - \Delta)^{\gamma/2}\phi\|_{\mathcal{L}_p}$$

$$\leq N\|(1 - \Delta)^{\gamma/2}\phi\|_{\mathcal{L}_p},$$

where the constants N are finite and independent of ϕ and ρ and

$$F(z) := \int_{B_1 \times B_1} \big|\,|x - z|^{\gamma - d} - |y - z|^{\gamma - d}\big|\,dxdy \leq N(1 + |z|)^{\gamma - d - 1}.$$

6. Exercise. Lemma 2 turns out to be false if $\gamma - d/p = 1$. Prove that, for any $p \in (1, \infty]$, there is a constant N such that for any $\phi \in \mathcal{S}$ and $x, y \in \mathbb{R}^d$

$$|\phi(x + y) - 2\phi(x) + \phi(x - y)| \leq |y|N\|(1 - \Delta)^{\gamma/2}\phi\|_{\mathcal{L}_p},$$

where $\gamma = 1 + d/p$. By the way, the space of bounded continuous functions ϕ such that

$$|\phi(x + y) - 2\phi(x) + \phi(x - y)| \leq N|y|$$

holds for a constant N and for all x, y is called *the Zygmund space* $\mathcal{C}^1$. This space is wider than C^1. For instance, in one space dimension the Weierstrass function

$$\sum_{n \geq 1} 2^{-n}\sin(2^n x)$$

belongs to $\mathcal{C}^1$.

11. Hints to exercises

2.13. Take two linearly independent vectors ξ, η and consider $p(\xi + t\eta)$ as a polynomial in real t.

3.1. Denote $h = u - \Delta u$ and define $v \in \overset{0}{W}{}_2^2(\Omega)$ by the equation $h = v - \Delta v$. Then set $w = u - v$ and observe that $\tilde{w}(x^1, \xi') = \tilde{g}(\xi')\exp(-x^1(1 + |\xi'|^2)^{1/2})$.

4.13. What is $\tilde{\psi}(0)$?

5.2. For real-valued $\zeta \in C_0^\infty$ with unit integral, $\zeta_n(x) := n^d\zeta(nx)$, and $f_n(x) := \langle f, \zeta_n(x - \cdot)\rangle$, we have that the f_n are well defined, infinitely differentiable and

$$f_{nx^1}(x) = \langle g, \zeta_n(x - \cdot)\rangle$$

in a domain slightly smaller than D. Then use the continuity of g to conclude that the right-hand sides converge uniformly on compact subsets of D to g and, by using a standard theorem from calculus, conclude that the uniform limit of f_{nx^1} is f_{x^1}.

5.9. After changing variables, find

$$G(x) = |x|^{2-d} \int_0^\infty \frac{1}{(4\pi t)^{d/2}} e^{-\frac{1}{4t} - ct|x|^2}\, dt.$$

5.10. By using l'Hôpital's rule, prove that the limit in question is 2π times the limit in Exercise 5.9 with $d = 4$.

5.11. In (i) argue by contradiction and find $\psi_m \in C_0^\infty$, $m = 1, 2, \ldots$, such that

$$1 = |\langle f, \phi\psi_m\rangle| \geq m \max_{k \leq m} \sup_{x \in \mathbb{R}^d} |D^k \psi_m(x)|.$$

Then $\phi\psi_m \to 0$ uniformly along with every derivative. In (ii) we have

$$|\langle f, \phi(\Delta - 1)^{-n}\psi\rangle| \leq N\|\psi\|_{\mathcal{L}_2},$$

which shows that $\langle f, \phi(\Delta - 1)^{-n}\psi\rangle = (g, \psi)$ for a $g \in \mathcal{L}_2$ and

$$\langle f, \phi\psi\rangle = \langle g, (\Delta - 1)^n \psi\rangle$$

for any $\psi \in C_0^\infty$.

6.5. See Exercise 5.5 and Lemma 6.2.

7.8. Write what (5.5) means at $\xi = 0$.

7.9. Use Corollary 7.2 and Exercise 7.8.

7.11. Integrate by parts.

7.13. Integrate by parts in (7.4) with q in place of r.

8.4. Use Theorems 8.3 and 5.1.1.

9.4. Continue (9.4) as

$$\|(M + 1)u\|_{W_p^m} \leq N\|L(M + 1)u\|_{\mathcal{L}_p} + N\|u\|_{W_p^m},$$

$$\|v\|_{W_p^m} \leq N\|Lv\|_{\mathcal{L}_p} + N\|(M + 1)^{-1}v\|_{W_p^m},$$

where $M = L^*L$, $u, v \in \mathcal{S}$. Since, $(M+1)^{-1}$ and, for any multi-index α with $|\alpha| \leq m$, $D^\alpha (M+1)^{-1}$ are zeroth-order pseudo-differential operators, the last norm is controlled by $\|v\|_{\mathcal{L}_p}$, which almost yields (9.3).

9.5. For the first part repeat part of the proof of Theorem 1.1.6 and use (9.3). For the second part take $d = 1$ and try $Lu = u'$.

9.6. Use Fatou's lemma. The second relation in (9.6) follows from the first one and (9.5).

9.7. To prove the first required estimate, set $\mathcal{K}_\varepsilon \phi = F^{-1}(\zeta_\varepsilon F\phi)$, where $\zeta_\varepsilon(\xi) = -(\varepsilon^2 + |\xi|^2)^{1/2}$, use dilations and Theorem 9.3 and let $\varepsilon \to 0$. Then extend $\mathcal{K}_0$ to W^1_p and to prove the second required estimate, observe that $D_j = -i\mathcal{R}_j \mathcal{K}_0$.

9.8. (ii) Assuming the contrary, find functions $u^{(\varepsilon)}$ which are bounded for each $\varepsilon > 0$ and satisfy $\Delta u^{(\varepsilon)} = 0$. Then use Exercise 2.4.10. (iii) By the previous theory a possible solution should be infinitely differentiable since f is. Then prove (1.9.2) by using the assumption, implying that $u \in \mathcal{L}_p$, and considering $u\zeta_R$ for appropriate ζ_R, and in the term containing $u_x \cdot \zeta_{Rx}$ integrate by parts. Finally, show that $u \notin \mathcal{L}_p$.

9.14. The extra effort consists in observing that the left-hand side in (9.10) is dominated by $N|x|^{-d} \max|\phi|$ for large $|x|$, with N independent of R.

9.16. Assume the contrary and derive then that one could take $n\gamma/2$ with any $n = 1, 2, \ldots$ in place of $\gamma/2$. After that, take n so large that $n\gamma/2 \geq 1$ and from Corollary 9.13 derive that

$$\|(1-\Delta)\phi\|_{\mathcal{L}_p} \leq N\|\phi\|_{\mathcal{L}_p}.$$

Now use dilations.

9.17. First take $\lambda = 1$; then use dilations.

9.19. Notice $(1-\varepsilon)(t + |x|^2/(4t)) \geq (1-\varepsilon)|x|$. Also you may like to prove that

$$D^\alpha p(t,x) = t^{-|\alpha|/2} q(xt^{-1/2}) p(t,x),$$

where $q = q(y)$ is a polynomial of the variable $y \in \mathbb{R}^d$.

9.20. You may like to use the fact that

$$e^{-t} T_t \phi(x) = \phi(x) + \int_0^t e^{-s} T_s (\Delta - 1)\phi(x)\, ds.$$

9.22. Use (9.7) and Exercises 9.21 and 9.19.

10.6. Take $x = 0$ and use the inequality

$$J(z,y) := |G(z+y) - 2G(z) + G(z-y)| \leq R(|z+y|, |z|) + R(|z-y|, |z|)$$

if $|z| \leq 10|y|$ and

$$J(z,y) \leq N|y|^2 \sup\{|G_{xx}(x)| : |x| \geq 9|z|/10\}$$

for $|z| \geq 10|y|$.

Chapter 13

Elliptic operators and the spaces H_p^γ

The goal of this chapter is to understand how to solve pseudo-differential strongly elliptic equation $Lu = f$ not only for $f \in \mathcal{S}$ as in Lemma 12.4.17 but for f in larger classes such as $\mathcal{L}_p$. To do this, first of all we have to define L on a set larger than $\mathcal{S}$. Here the framework of distributions seems appropriate since distributions are infinitely differentiable in the generalized sense.

Observe that, if L is a strongly elliptic pseudo-differential operator, from $Lu = f$ it should follow that $u = L^{-1}f$ where L^{-1} is also a strongly elliptic operator. Therefore, it is natural to look for solutions in the set of distributions each of which is representable as Lh, where L is a pseudo-differential operator and $h \in \mathcal{L}_p$. Of course, as usual in the theory of distributions, we define Lh by using duality.

The reader will see that after introducing the right definitions, the general theory is constructed almost effortlessly on the basis of the previous chapter.

1. The space $\mathcal{H}$

The first assertion in the following lemma is a simple consequence of the definition of generalized derivatives.

1. Lemma. *If g is a distribution and L is a differential operator with constant coefficients, then, for any $\phi \in C_0^\infty$, we have $\langle Lg, \phi\rangle = \langle g, L^*\phi\rangle$. The same equality holds if $g, \phi \in \mathcal{S}$ and L is a pseudo-differential operator.*

The second assertion is proved in Lemma 12.4.19.

Lemma 1 will serve as a model for defining pseudo-differential operators on $h \in \mathcal{L}_p$. Namely by Lh we will mean the distribution defined by

$$(h, L^*\phi).$$

But first we need to show that this is indeed a distribution. After that we want to define what $M(Lh)$ is if M is a pseudo-differential operator. Of course, the trouble is that it is not immediately clear that the definition

$$M(Lh) := (ML)h$$

is consistent in the sense that, if $L_1h_1 = L_2h_2$, then $(ML_1)h_1 = (ML_2)h_2$. We address both issues in the following theorem. Set

$$\mathcal{L}_{[1,\infty]} = \mathcal{L}_1 + \mathcal{L}_\infty := \{f + h : f \in \mathcal{L}_1, h \in \mathcal{L}_\infty\}, \quad \mathcal{L}_{(1,\infty)} = \bigcup_{p\in(1,\infty)} \mathcal{L}_p.$$

Observe that the decomposition $g = gI_{|g|\leq 1} + gI_{|g|>1}$ shows that

$$\mathcal{L}_{(1,\infty)} \subset \mathcal{L}_{[1,\infty]}.$$

2. Theorem. *(i) Let L be a pseudo-differential operator and let $h \in \mathcal{L}_{[1,\infty]}$. Then the mapping*

$$\phi \to \int_{\mathbf{R}^d} h(x)L\phi(x)\,dx \tag{1}$$

is well defined and is continuous on $\mathcal{S}$. In particular, restricted to C_0^∞, it is a distribution.

(ii) If L is strongly elliptic, then for distinct $h \in \mathcal{L}_{[1,\infty]}$ the distributions (1) *are distinct.*

(iii) If $h_1, h_2 \in \mathcal{L}_{[1,\infty]}$ and L_1, L_2, M are pseudo-differential operators and

$$\int_{\mathbf{R}^d} h_1(x)L_1\phi(x)\,dx = \int_{\mathbf{R}^d} h_2(x)L_2\phi(x)\,dx \tag{2}$$

for all $\phi \in C_0^\infty$, then

$$\int_{\mathbf{R}^d} h_1(x)L_1M\phi(x)\,dx = \int_{\mathbf{R}^d} h_2(x)L_2M\phi(x)\,dx \tag{3}$$

for all $\phi \in \mathcal{S}$.

Proof. (i) Since $L\phi \in \mathcal{S} \subset \mathcal{L}_q$ for any $\phi \in \mathcal{S}$ and $q \in [1,\infty]$, and by Hölder's inequality,

$$\int_{\mathbf{R}^d} |h(x)|\,|L\phi(x)|\,dx \leq \|h\|_{\mathcal{L}_p}\|L\phi\|_{\mathcal{L}_q}, \tag{4}$$

the right-hand side of (1) is well defined. Estimate (4) and Remark 12.4.14 also prove that the mapping (1) is continuous on $\mathcal{S}$. This proves (i).

(iii) By (i) both parts of (2) are continuous on $\mathcal{S}$ and by Exercise 12.1.4, for any $\psi \in \mathcal{S}$, there is a sequence $\psi_n \in C_0^\infty$ such that $\psi_n \xrightarrow{\mathcal{S}} \psi$. It follows that (2) holds for any $\phi \in \mathcal{S}$. Since $M\phi$ is just another element of $\mathcal{S}$, we have (3) indeed.

(ii) If $h_1, h_2 \in \mathcal{L}_{[1,\infty]}$ and

$$\int_{\mathbf{R}^d} h_1(x)L\phi(x)\,dx = \int_{\mathbf{R}^d} h_2(x)L\phi(x)\,dx$$

for all $\phi \in C_0^\infty$, then by (iii) with $M = L^{-1}$ we have

$$\int_{\mathbf{R}^d} h_1(x)\psi(x)\,dx = \int_{\mathbf{R}^d} h_2(x)\psi(x)\,dx$$

for all $\psi \in \mathcal{S}$. Hence $h_1 = h_2$ in the sense of distributions and $h_1 = h_2$ (a.e.). The theorem is proved.

Theorem 2 shows that the following definition makes sense and, in particular, that the distribution Lg introduced by formula (6) below is uniquely determined by L and g regardless of which M and h satisfy (5).

3. Definition. For a distribution $g(\phi) = \langle g, \phi\rangle$ we write $g \in \mathcal{H}$ if there exist $p \in (1,\infty)$, a pseudo-differential operator M, and $h \in \mathcal{L}_p$ such that, for any $\phi \in C_0^\infty$,

$$\langle g, \phi\rangle = \int_{\mathbf{R}^d} h(x)\overline{M^*\phi}(x)\,dx. \tag{5}$$

If a distribution $g \in \mathcal{H}$ is given by (5) and L is a pseudo-differential operator, then we define the distribution $Lg \in \mathcal{H}$ by the formula

$$\langle Lg, \phi\rangle = \int_{\mathbf{R}^d} h(x)\overline{M^*L^*\phi}(x)\,dx, \quad \phi \in C_0^\infty. \tag{6}$$

4. Remark. By taking $M = 1$ and $h = g$ in (5), we see that $\mathcal{L}_p \subset \mathcal{H}$ for any $p \in (1,\infty)$. In particular, $\mathcal{S} \subset \mathcal{H}$. It is also worth noting that $\mathcal{H}$ is *not* a vector space.

5. Remark. We have already introduced L on $\mathcal{S} \subset \mathcal{H}$ in Definition 12.4.7. Therefore, to justify the notation Lg, we have to prove that, if $g \in \mathcal{S}$, then the old definition of Lg agrees with the new one. But if $g \in \mathcal{S}$, we can take $h = g$ and $M = 1$ in (6) and then the right-hand side of (6) becomes $(g, L^*\phi)$ which is equal to the old (Lg, ϕ) by Lemma 1.

6. Exercise*. Each $g \in \mathcal{H}$ is a distribution and as such has derivatives in the generalized sense. On the other hand, differentiating is a pseudo-differential operation. Prove that we do not get different distributions if we use these two ways. In other words, prove that if $g \in \mathcal{H}$, $L = D_1$, and Lg is defined according to Definition 3, then for any $\phi \in C_0^\infty$

$$\langle g, D_1\phi\rangle = \langle Lg, \phi\rangle.$$

Here are a few examples of generalized functions of class $\mathcal{H}$.

7. Theorem. *Let L be a strongly elliptic pseudo-differential operator and let G be its Green's function. Then $G \in \mathcal{H}$ and $LG = \delta_0$. In particular, $\delta_0 \in \mathcal{H}$.*

Proof. If the order μ of L is large enough, then its Green's function G is bounded and belongs to $\mathcal{L}_1$. In particular, $G \in \mathcal{L}_2 \subset \mathcal{H}$ and Corollary 12.7.6 and Definition 3 with $g = G = h$ and $M = 1$ show that

$$LG = \delta_0.$$

One obtains the same result for any μ by observing (see Lemma 12.6.1) that $G = (1 - \Delta)^k G'$, where G' is the Green's function of $(1 - \Delta)^k L$ and k is sufficiently large, so that

$$LG = (L(1 - \Delta)^k)G' = \delta_0.$$

The theorem is proved.

8. Exercise. By Exercise 12.4.8 pseudo-differential operators are translation invariant on $\mathcal{S}$. Prove that they are also translation invariant on $\mathcal{H}$.

9. Exercise (Cf. Theorem 12.9.12). Take a $g \in \mathcal{H}$ and a number $\gamma > 0$ and assume that $(1 - \Delta)^{\gamma/2} g = h \in \mathcal{L}_p$. Let G be the Green's function of $(1 - \Delta)^{\gamma/2}$, which is in $\mathcal{L}_1$, so that $G * h$ is well defined. Prove that $g = G * h$, in the sense of distributions.

2. Some properties of the space $\mathcal{H}$

Assertion (iii) of the following theorem extends Lemma 12.4.17 to $\mathcal{H}$. The theorem itself is a straightforward consequence of the definitions.

1. Theorem. *(i) If L is a pseudo-differential operator, then $L\mathcal{H} \subset \mathcal{H}$.*

(ii) If the symbols of two pseudo-differential operators coincide, then the operators coincide on $\mathcal{H}$. For instance, if L_1, L_2 are pseudo-differential operators, then $L_1L_2 = L_2L_1$ on $\mathcal{H}$. In particular, for any multi-index α and pseudo-differential operator L, we have $D^\alpha L = LD^\alpha$.

(iii) If L is strongly elliptic, then $L^{-1}L = LL^{-1}$ on $\mathcal{H}$, $L\mathcal{H} = \mathcal{H}$, the mapping $L : \mathcal{H} \to \mathcal{H}$ is one-to-one and L^{-1} is its inverse. In particular, for any $f \in \mathcal{H}$, there is a unique $u \in \mathcal{H}$ such that $Lu = f$.

2. Remark. If $g \in \mathcal{L}_{(1,\infty)}$ ($\subset \mathcal{H}$ by Remark 1.4) and $\phi \in C_0^\infty$, then by definition

$$\langle g, \phi\rangle = \int_{\mathbb{R}^d} g\bar{\phi}\, dx,$$

which along with (1.6), where we take $M = 1$, yields

$$\langle Lg, \phi\rangle = \int_{\mathbb{R}^d} g\overline{L^*\phi}\, dx \tag{1}$$

for any $\phi \in C_0^\infty$. In particular, if the symbol of L is real valued, then L is formally self-adjoint.

3. Remark. If $g \in \mathcal{H}$, L is a strongly elliptic pseudo-differential operator and $Lg \in \mathcal{L}_{(1,\infty)}$, then

$$\langle g, \phi\rangle = \int_{\mathbb{R}^d} (Lg(x))\overline{(L^{-1})^*\phi}(x)\, dx$$

for any $\phi \in C_0^\infty$.

Indeed, let $h = Lg$ and use Theorem 1 and Remark 2 to get

$$\langle g, \phi\rangle = \langle L^{-1}Lg, \phi\rangle = \langle L^{-1}h, \phi\rangle = \int_{\mathbb{R}^d} h(x)\overline{(L^{-1})^*\phi}(x)\, dx.$$

4. Remark. Upon taking h and M in place of g and L in Remark 2 and comparing the result with (1.5), we see that the right-hand side of (1.5) equals (Mh, ϕ). Hence $g = Mh$ and one could equivalently define $\mathcal{H}$ as the collection of Mh where M runs through the set of all pseudo-differential operators, h through $\mathcal{L}_{(1,\infty)}$, and Mh is determined according to (1). Actually, there is a shorter description given in Theorem 5.

Recall that, if for a distribution f there is a function $g \in \mathcal{L}_p$ such that for all $\phi \in C_0^\infty$ we have

$$\langle f, \phi \rangle = \int_{\mathbb{R}^d} g(x)\bar{\phi}(x)\,dx,$$

then we write $f \in \mathcal{L}_p$ and identify f with g. In the following theorem the most important assertions are (i) and (ii).

5. Theorem. *(i) If M is a pseudo-differential operator of order γ, then the pseudo-differential operator $(1-\Delta)^{-\gamma/2}M$ is a bounded operator in $\mathcal{L}_p$ for any $p \in (1,\infty)$ with the norm controlled by a constant depending only on γ, d, p, and $N_{d+2}(M,\gamma)$.*

(ii) If M is a strongly elliptic pseudo-differential operator of order γ, then the pseudo-differential operator $(1-\Delta)^{\gamma/2}M^{-1}$ is a bounded operator in $\mathcal{L}_p$ for any $p \in (1,\infty)$.

(iii) We have

$$\mathcal{H} = \bigcup_{\gamma\in\mathbb{R}} \bigcup_{p\in(1,\infty)} (1-\Delta)^{\gamma/2}\mathcal{L}_p = \bigcup_{n=m}^{\infty} \bigcup_{p\in(1,\infty)} (1-\Delta)^{n}\mathcal{L}_p, \tag{2}$$

where m is any nonnegative integer.

Proof. (i) According to (1) for any $\phi \in C_0^\infty$ and $g \in \mathcal{L}_p$ we have

$$\langle (1-\Delta)^{-\gamma/2}Mg, \phi\rangle = \int_{\mathbb{R}^d} g\overline{(1-\Delta)^{-\gamma/2}M^*\phi}\,dx.$$

Here by Theorem 12.8.3

$$\|(1-\Delta)^{-\gamma/2}M^*\phi\|_{\mathcal{L}_q} \le N_0\|\phi\|_{\mathcal{L}_q},$$

where N_0 depends only on d, p, and $N_{d+2}((1-\Delta)^{-\gamma/2}M^*, 0)$. The latter constant is estimated in terms of γ, d, and $N_{d+2}(M,\gamma)$ according to Exercise 12.4.5.

By Hölder's inequality

$$|\langle (1-\Delta)^{-\gamma/2}Mg, \phi\rangle| \le N_0\|g\|_{\mathcal{L}_p}\|\phi\|_{\mathcal{L}_q},$$

where $q = p/(p-1)$. We see that for a fixed $g \in \mathcal{L}_p$ the function

$$\langle (1-\Delta)^{-\gamma/2}Mg, \phi\rangle$$

is a linear bounded functional defined on a linear dense subset of $\mathcal{L}_q$. By Riesz's representation theorem there exists an $f \in \mathcal{L}_p$ such that $\|f\|_{\mathcal{L}_p} \leq N_0\|g\|_{\mathcal{L}_p}$ and

$$\langle (1-\Delta)^{-\gamma/2}Mg, \phi\rangle = \int_{\mathbb{R}^d} f(x)\bar{\phi}(x)\,dx.$$

Hence (by what is said before the theorem) $(1-\Delta)^{-\gamma/2}Mg = f \in \mathcal{L}_p$ and

$$\|(1-\Delta)^{-\gamma/2}Mg\|_{\mathcal{L}_p} \leq N_0\|g\|_{\mathcal{L}_p}$$

for any $g \in \mathcal{L}_p$. This proves (i).

(ii) Under the conditions in (i) the operator $(1-\Delta)^{\gamma/2}M^{-1}$ is a zeroth order pseudo-differential operator and one can repeat the above argument almost literally.

(iii) By Theorem 1 the unions in (2) belong to $\mathcal{H}$ and the second union is a subset of the first one. On the other hand, by Remark 4 if $g \in \mathcal{H}$, then there exist an $h \in \mathcal{L}_{(1,\infty)}$ and a pseudo-differential operator M such that

$$g = Mh.$$

If γ is the order of M and n is a nonnegative integer satisfying $n \geq m$ and $n \geq \gamma$, then $2n$ is also an order of M and

$$g = (1-\Delta)^n f, \quad \text{where} \quad f = (1-\Delta)^{-n}Mh \in \mathcal{L}_{(1,\infty)}$$

by (i). The theorem is proved.

6. Exercise. If g is a distribution on $\mathbb{R}^d$ and T is a nondegenerate $d \times d$ matrix, then the distribution $h(x) = g(Tx)$ is defined according to

$$\langle h, \phi\rangle = |\det T|^{-1}\int_{\mathbb{R}^d} g(x)\bar{\phi}(T^{-1}x)\,dx.$$

Check that the right-hand side is indeed a distribution and if $g \in \mathcal{H}$, then $h \in \mathcal{H}$.

3. The spaces H_p^γ

3:1. Definition, solvability of elliptic equations in H_p^γ, the equality $H_p^\gamma = W_p^\gamma$. Theorem 2.1 asserts the solvability of strongly elliptic equations with constant coefficients. In order to go to variable coefficients, we need some quantitative estimates on solutions. This makes it natural to introduce appropriate Banach spaces.

Take

$$p \in (1, \infty)$$

and recall that $\mathcal{L}_p \subset \mathcal{H}$ and $(1-\Delta)^{\gamma/2} : \mathcal{H} \to \mathcal{H}$ for any $\gamma \in \mathbb{R}$. This allows us to define the following subsets of $\mathcal{H}$.

1. Definition. For $p \in (1, \infty)$ and $\gamma \in \mathbb{R}$ denote $H_p^\gamma = (1-\Delta)^{-\gamma/2}\mathcal{L}_p$ and for a distribution $g \in H_p^\gamma$ define

$$\|g\|_{H_p^\gamma} = \|(1-\Delta)^{\gamma/2} g\|_{\mathcal{L}_p}. \tag{1}$$

Spaces H_p^γ are called *spaces of Bessel potentials.*

If $\gamma = 2$, we know from Theorem 4.3.8 that $H_p^2 = W_p^2$. Thus, the elements of H_p^2 have two Sobolev derivatives. We will see from Theorem 12 that for any $n \in \{1, 2, ...\}$ the elements of H_p^n have n Sobolev derivatives. Analogously for any γ, we think of H_p^γ as the space of functions having γ derivatives summable to the pth power, no matter that "the number" γ does not have to be an integer and can also be *negative*.

2. Remark. If $\|g\|_{H_p^\gamma} = 0$, then $(1-\Delta)^{\gamma/2} g = 0$ and since $g \in \mathcal{H}$, we have $g = 0$. Therefore, (1) defines a norm on H_p^γ.

3. Remark. According to Remark 2.3 if $g \in H_p^\gamma$ and if we denote $h = (1-\Delta)^{\gamma/2} g \in \mathcal{L}_p$, then

$$\langle g, \phi \rangle = \int_{\mathbb{R}^d} h(x)(1-\Delta)^{-\gamma/2}\bar{\phi}(x)\, dx, \tag{2}$$

for any $\phi \in C_0^\infty$. Conversely, by Definition 1.3, if (2) holds for any $\phi \in C_0^\infty$ with $h \in \mathcal{L}_p$, then $g \in \mathcal{H}$,

$$(1-\Delta)^{\gamma/2} g = h, \quad g = (1-\Delta)^{-\gamma/2} h$$

and since $h \in \mathcal{L}_p$, we have $g \in H_p^\gamma$.

Thus, for a distribution g we have $g \in H_p^\gamma$ if and only if there exists $h \in \mathcal{L}_p$ such that (2) holds for any $\phi \in C_0^\infty$. In that case

$$(1-\Delta)^{\gamma/2} g = h, \quad (1-\Delta)^{-\gamma/2} h = g, \quad \|g\|_{H_p^\gamma} = \|h\|_{\mathcal{L}_p}.$$

4. Remark. If in (2) we have $g = g_n$ and $h = h_n$ and $h_n \to 0$ in $\mathcal{L}_p$, then by Hölder's inequality, applied to the right-hand side of (2), $\langle g_n, \phi \rangle \to 0$. In other words, if $g_n \in H_p^\gamma$ and $\|g_n\|_{H_p^\gamma} \to 0$, then $g_n \to 0$ in the sense of distributions. Thus, convergence in H_p^γ implies convergence in the sense of distributions.

5. Remark. We know that $\mathcal{S} \subset \mathcal{L}_p$ and $(1-\Delta)^{-\gamma/2}\mathcal{S} = \mathcal{S}$. Therefore, for any γ

$$\mathcal{S} = (1-\Delta)^{-\gamma/2}\mathcal{S} \subset (1-\Delta)^{-\gamma/2}\mathcal{L}_p = H_p^\gamma.$$

6. Remark. If $g_n \xrightarrow{\mathcal{S}} g$, then by the continuity of $(1-\Delta)^{\gamma/2}$ on $\mathcal{S}$ also

$$(1-\Delta)^{\gamma/2} g_n \xrightarrow{\mathcal{S}} (1-\Delta)^{\gamma/2} g, \quad (1-\Delta)^{\gamma/2} g_n \to (1-\Delta)^{\gamma/2} g$$

in $\mathcal{L}_p$. It follows that

$$g_n \to g \quad \text{in} \quad H_p^\gamma \quad \text{if} \quad g_n \xrightarrow{\mathcal{S}} g.$$

7. Theorem. *(i) H_p^γ is a Banach space.*

(ii) $\mathcal{S} \subset H_p^\gamma$ and C_0^∞ is dense in H_p^γ.

(iii) For any $\gamma, \nu \in \mathbb{R}$ the operator $(1-\Delta)^{\gamma/2}$ defined on $\mathcal{H}$ is an isometric operator from H_p^ν onto $H_p^{\nu-\gamma}$.

Proof. (i) As is easy to see, we only need to prove that H_p^γ is complete. If g_n is a Cauchy sequence in H_p^γ, then $(1-\Delta)^{\gamma/2} g_n$ is a Cauchy sequence in $\mathcal{L}_p$ and since $\mathcal{L}_p$ is complete, there is an $h \in \mathcal{L}_p$ such that

$$h_n := (1-\Delta)^{\gamma/2} g_n \to h$$

in $\mathcal{L}_p$. By Theorem 2.1, there exists a $g \in \mathcal{H}$ such that $(1-\Delta)^{\gamma/2} g = h$ and $g = (1-\Delta)^{-\gamma/2} h$, which by definition yields that $g \in H_p^\gamma$. Finally, by definition

$$\|g_n - g\|_{H_p^\gamma} = \|h_n - h\|_{\mathcal{L}_p} \to 0.$$

(ii) The fact that $\mathcal{S} \subset H_p^\gamma$ has already been noted in Remark 5. Furthermore, by definition, the mapping

$$(1-\Delta)^{-\gamma/2} : \mathcal{L}_p \to H_p^\gamma$$

is an isometry between $\mathcal{L}_p$ and H_p^γ. Therefore, since $\mathcal{S}$ is dense in $\mathcal{L}_p$, the set $(1-\Delta)^{-\gamma/2}\mathcal{S} = \mathcal{S}$ is dense in H_p^γ. Finally, to show how to approximate any $\psi \in \mathcal{S}$ in H_p^γ norm by $\psi_n \in C_0^\infty$, take any $\psi_n \in C_0^\infty$ such that $\psi_n \xrightarrow{\mathcal{S}} \psi$ (see Exercise 12.1.4). Then

$$(1-\Delta)^{\gamma/2}\psi_n \xrightarrow{\mathcal{S}} (1-\Delta)^{\gamma/2}\psi$$

and

$$(1-\Delta)^{\gamma/2}\psi_n \to (1-\Delta)^{\gamma/2}\psi$$

in $\mathcal{L}_p$, which is equivalent to $\psi_n \to \psi$ in H_p^γ.

(iii) The mapping

$$(1-\Delta)^{\nu/2} : H_p^\nu \to \mathcal{L}_p$$

is an isometry between H_p^ν and $\mathcal{L}_p$ and

$$(1-\Delta)^{(\gamma-\nu)/2} : \mathcal{L}_p \to H_p^{\gamma-\nu}$$

is an isometry between $\mathcal{L}_p$ and $H_p^{\gamma-\nu}$. Hence and due to the formula

$$(1-\Delta)^{(\gamma-\nu)/2}(1-\Delta)^{\nu/2} = (1-\Delta)^{\gamma/2}$$

(Theorem 2.1 (ii)), the mapping

$$(1-\Delta)^{-\gamma/2} : H_p^\nu \to H_p^{\nu-\gamma}$$

is an isometry between H_p^ν and $H_p^{\nu-\gamma}$. The theorem is proved.

From assertions (iii) in Theorems 7 and 2.1 we get the following.

8. Corollary. *For any $\gamma \in \mathbb{R}$ and $f \in H_p^\gamma$ there exists a unique $u \in \mathcal{H}$ solving the equation $(1-\Delta)u = f$. For this solution we have $u \in H_p^{\gamma+2}$ and $\|u\|_{H_p^{\gamma+2}} = \|f\|_{H_p^\gamma}$. Furthermore, if $q \in (1,\infty)$, $\nu \in \mathbb{R}$, $v \in H_q^\nu$, and $(1-\Delta)v = f$, then $v = u \in H_p^{\gamma+2}$.*

The simplicity of obtaining the last assertion of this corollary came from our (almost axiomatic) investigation of equations in $\mathcal{H}$. We had proved a similar result before in Theorem 4.3.12 in a much more technical way.

9. Corollary. *If $\gamma \le \nu$ and $g \in H_p^\nu$, then $g \in H_p^\gamma$ and $\|g\|_{H_p^\gamma} \le \|g\|_{H_p^\nu}$. In particular, $H_p^\gamma \subset \mathcal{L}_p$ for $\gamma > 0$.*

Indeed, there is a sequence of $\phi_n \in C_0^\infty$ such that $\|g-\phi_n\|_{H_p^\nu} \to 0$. Then ϕ_n is a Cauchy sequence in H_p^ν. By Corollary 12.9.13, ϕ_n is a Cauchy sequence in H_p^γ. Hence it converges in H_p^γ to a distribution, which by Remark 4 has to be g. Therefore, $g \in H_p^\gamma$ indeed. Finally, again by Corollary 12.9.13

$$\|g\|_{H_p^\gamma} = \lim_{n\to\infty} \|\phi_n\|_{H_p^\gamma} \le \lim_{n\to\infty} \|\phi_n\|_{H_p^\nu} = \|g\|_{H_p^\nu}.$$

Here is a fundamental result of the theory, which is an extension of assertion (iii) of Theorem 7.

10. Theorem. *Let M be a pseudo-differential operator of order $\mu \in \mathbb{R}$ and let $\nu \in \mathbb{R}$. Then the operator M defined on $\mathcal{H}$ maps H_p^ν into $H_p^{\nu-\mu}$ and*

$$\|Mu\|_{H_p^{\nu-\mu}} \le N\|u\|_{H_p^\nu} \tag{3}$$

for any $u \in H_p^\nu$, where N depends only on μ, d, p, and $N_{d+2}(M,\mu)$. Furthermore, if M is strongly elliptic, then the operator M maps H_p^ν onto $H_p^{\nu-\mu}$ is a one-to-one mapping and for any $u \in H_p^\nu$ we have

$$\|u\|_{H_p^\nu} \le N\|Mu\|_{H_p^{\nu-\mu}}, \tag{4}$$

where N depends only on μ, d, p, $N_{d+2}(M,\mu)$, and $\kappa(M)$. Finally, if $q \in (1,\infty)$, $\gamma \in \mathbb{R}$, $u \in H_p^\gamma$, $v \in H_q^\nu$, and $Mu = Mv$, then $u = v$.

Proof. By definition and Theorem 2.5 we have

$$\|Mu\|_{H_p^{\nu-\mu}} = \|(1-\Delta)^{(\nu-\mu)/2}Mu\|_{L_p} = \|[(1-\Delta)^{-\mu/2}M](1-\Delta)^{\nu/2}u\|_{L_p}$$

$$\le N\|(1-\Delta)^{\nu/2}u\|_{L_p} = N\|u\|_{H_p^\nu},$$

where N depends only on μ, d, p, and $N_{d+2}(M,\mu)$. In particular, M defined on $\mathcal{H}$ maps H_p^ν into $H_p^{\nu-\mu}$.

If M is strongly elliptic, by applying the above result to the pseudo-differential operator M^{-1} of order $-\mu$, we obtain that it maps $H_p^{\nu-\mu}$ into H_p^ν and we have (4), where the stated dependence of N follows from Exercise 12.4.5.

The last assertion of the theorem follows from the unique solvability of the equation $Mu = f$ in $\mathcal{H}$. The theorem is proved.

11. Corollary. *Let $\gamma \in \mathbb{R}$ and $n \in \{1, 2, ...\}$. Then $u \in H_p^{\gamma+n}$ if and only if $D^\alpha u \in H_p^\gamma$ for any multi-index α with $|\alpha| \le n$. Furthermore, for any $u \in H_p^{\gamma+n}$*

$$\|u\|_{H_p^{\gamma+n}} \le N\sum_{|\alpha|\le n}\|D^\alpha u\|_{H_p^\gamma}, \quad \sum_{|\alpha|\le n}\|D^\alpha u\|_{H_p^\gamma} \le N\|u\|_{H_p^{\gamma+n}}, \tag{5}$$

where N is a constant independent of u.

To prove this, notice that the second estimate in (5) and the "only if" part of the first assertion follow directly from Theorem 10 and the fact that D^α is a pseudo-differential operator of order n if $|\alpha| \le n$. In the proof of the remaining assertions one can use an obvious induction on n and, therefore, we may assume that $n = 1$.

Next, for any $u \in \mathcal{H}$ we have

$$u = (1 - \Delta)v = v - D_i D_i v \tag{6}$$

where $v = (1 - \Delta)^{-1} u$. If $u, D_i u \in H_p^\gamma$, $i = 1, \ldots, d$, then $v \in H_p^{\gamma+1}$ and

$$D_i D_i v = [D_i(1 - \Delta)^{-1}] D_i u \in H_p^{\gamma+1}$$

by Theorem 10 since $(1 - \Delta)^{-1}$ and $D_i(1 - \Delta)^{-1}$ are pseudo-differential operators of order -1. This and (6) show that $u \in H_p^{\gamma+1}$ and prove the "if" part in the first assertion. By using estimate (3), we also get the first estimate in (5).

The following important and highly nontrivial result of the theory is just a specification of Corollary 11 for $\gamma = 0$ and is also based on Theorem 1.8.5 describing W_p^n in terms of generalized derivatives. A deceiving simplicity of the way we obtain it comes from the fact that we did our homework well. Recall that in the whole section $p \in (1, \infty)$.

12. Theorem. *For $n = 1, 2, \ldots$ we have $u \in H_p^n$ if and only if $u \in W_p^n$. Furthermore,*

$$\|u\|_{H_p^n} \leq N \|u\|_{W_p^n}, \quad \|u\|_{W_p^n} \leq N \|u\|_{H_p^n},$$

where N is a constant independent of u.

13. Exercise*. Prove that for any $g \in \mathcal{H}_p^{-1}$ there exist $f_0, \ldots, f_d \in \mathcal{L}_p$ such that

$$g = f_0 + \sum_j D_j f_j \tag{7}$$

and

$$\sum_{j=0}^{d} \|f_j\|_{\mathcal{L}_p} \leq N \|g\|_{H_p^{-1}},$$

where the constant N is independent of g. Also prove that if (7) holds with $f_0, \ldots, f_d \in \mathcal{L}_p$, then $g \in \mathcal{H}_p^{-1}$ and

$$\|g\|_{H_p^{-1}} \leq N \sum_{j=0}^{d} \|f_j\|_{\mathcal{L}_p},$$

where the constant N is independent of the f_j's.

3:2. Exercises (optional). In this subsection as well as in the whole section $p \in (1, \infty)$.

14. Exercise. We propose the reader justify some arguments of Section 12.3 for $W_p^2(\mathbb{R}_+^d)$ functions, where $d \geq 2$ and as usual

$$\mathbb{R}_+^d = \{x = (x^1, x') : x^1 > 0, x' = (x^2, ..., x^d) \in \mathbb{R}^{d-1}\}.$$

Take a $u \in W_p^2(\mathbb{R}_+^d)$, set $f = \Delta u - u$ and define $v \in \overset{\circ}{W}{}_p^2(\mathbb{R}_+^d)$ as the unique solution of

$$\Delta v - v = f$$

in $\mathbb{R}_+^d$. Recall that, as we know, $u(0, \cdot) \in W_p^1(\mathbb{R}^{d-1})$ and

$$u_{x^1}(0, \cdot), v_{x^1}(0, \cdot) \in \mathcal{L}_p(\mathbb{R}^{d-1}),$$

denote by $\mathcal{K}_{d-1}$ the Cauchy operator in $\mathbb{R}^{d-1}$, that is, the pseudo-differential operator with symbol $-(1 + |\xi'|^2)^{1/2}$, and prove that on $\mathbb{R}^{d-1}$

$$u_{x^1}(0, \cdot) = \mathcal{K}_{d-1} u(0, \cdot) + v_{x^1}(0, \cdot). \tag{8}$$

15. Exercise. We suggest the reader investigate the solvability of the problem

$$\Delta u - u = f \quad \text{in} \quad \mathbb{R}_+^d, \quad u_{x^1} + \sum_{j=2}^{d} \ell^j u_{x^j} + \sum_{j,k=2}^{d} a^{jk} u_{x^j x^k} = g \quad \text{on} \quad \partial\mathbb{R}_+^d. \tag{9}$$

Let $d \geq 2$, let $a = (a^{jk})_{j,k=2}^d$ be a constant symmetric strictly positive matrix, and let $\ell \in \mathbb{R}^d$ be a constant vector with $\ell^1 = 1$. Consider the pseudo-differential operator M on $\mathbb{R}^{d-1} = \{x' = (x^2, ..., x^d)\}$ with symbol

$$(1 + |\xi'|^2)^{1/2} - i\ell' \cdot \xi' + \sum_{j,k \geq 2} a^{jk} \xi^j \xi^k.$$

In other words

$$Mf = -\mathcal{K}_{d-1} f - \sum_{j=2}^{d} \ell^j f_{x^j} - \sum_{j,k=2}^{d} a^{jk} f_{x^j x^k}.$$

(i) Prove that for any $\gamma \in \mathbb{R}$ and $g \in H_p^\gamma(\mathbb{R}^{d-1})$ there exists a unique $v \in H_p^{\gamma+2}(\mathbb{R}^{d-1})$ such that $Mv = -g$. Show that for any $v \in H_p^{\gamma+2}(\mathbb{R}^{d-1})$ we have

$$\|v\|_{H_p^{\gamma+2}(\mathbb{R}^{d-1})} \leq N \|Mv\|_{H_p^\gamma(\mathbb{R}^{d-1})},$$

where N is independent of v.

(ii) Take an $f \in \mathcal{L}_p(\mathbb{R}_+^d)$, $g \in \mathcal{L}_p(\mathbb{R}^{d-1})$ and define $w \in \overset{\circ}{W}{}_p^2(\mathbb{R}_+^d)$ as the unique solution of

$$\Delta w - w = f$$

in $\mathbb{R}_+^d$. Set $g_1(x') = w_{x^1}(0, x')$ and $v = M^{-1}(g_1 - g)$, observe that $v \in W_p^2(\mathbb{R}^{d-1})$, and prove that there exists a unique $z \in W_p^2(\mathbb{R}_+^d)$ such that

$$\Delta z - z = 0$$

in $\mathbb{R}_+^d$ and $z(0, x') = v(x')$. By using Exercise 14, prove that $u := z + w \in W_p^2(\mathbb{R}_+^d)$ and u satisfies (9). Once more using Exercise 14, prove that there is only one $u \in W_p^2(\mathbb{R}_+^d)$ satisfying (9).

16. Exercise. Sometimes it is hard to recognize whether a function u is in H_p^γ, for a $\gamma < 0$. Prove that if u has support in B_ρ, where $\rho \in (0, \infty)$, and

$$|u(x)| \leq N_0 |x|^{-\nu}, \quad \nu < d, \quad 0 < (\nu + \gamma)p < d, \quad \gamma < 0,$$

then $u \in H_p^\gamma$ and $\|u\|_{H_p^\gamma}$ is less than a constant depending only on d, p, ρ, ν, γ, N_0. Observe that generally such a $u \notin \mathcal{L}_p$, because one need not have $\nu p < d$, and one cannot use the trivial embedding $\mathcal{L}_p \subset H_p^\gamma$.

By using Corollary 11 generalize the result and prove that if $n \in \{0, 1, ...\}$, $\gamma \in \mathbb{R}$, $\gamma \leq n$, u has support in B_ρ,

$$|D^\alpha u(x)| \leq N_0 |x|^{-\nu}, \quad \forall |\alpha| \leq n, \quad \nu < d,$$

and

$$\text{either} \quad \gamma < n \quad \text{and} \quad 0 < (\nu + \gamma - n)p < d, \quad \text{or} \quad \gamma = n \quad \text{and} \quad \nu p < d,$$

then $u \in H_p^\gamma$ and $\|u\|_{H_p^\gamma}$ is estimated by a constant depending only on d, p, ρ, ν, γ, n, N_0.

The following two exercises are aimed at proving that in Theorem 12.8.3 the constant $N_{d+2}(L, 0)$ can be replaced with $N_r(L, 0)$, where r is the least integer $> d/2$.

17. Exercise. Let G be the Green's function of a zeroth-order strongly elliptic operator L with symbol σ and let r be the least integer $> d/2$. Let $k \in \{0, 1, ...\}$ satisfy $k < d$. By using Exercise 16, prove that there is a constant N depending only on d, k, $N_r(L, 0)$, and $\kappa(L)$ such that for any $\rho > 0$

$$\rho^{d-k} \int_{|y| \geq \rho} |y|^k G^2(y)\, dy = I_\rho := \int_{|y| \geq 1} |y|^k G_\rho^2(y)\, dy \leq N,$$

where G_ρ is the Green's function of the operator with symbol $\sigma(\xi/\rho)$.

You may start by introducing $\chi = 2$ if d is odd, $\chi = 3$ if d is even and observing that $k \leq d - 1 \leq 2r - \chi$ and

$$I_\rho \leq N \int_{\mathbb{R}^d} \frac{|y|^{2r}}{(1 + |y|^2)^{\chi/2}} G_\rho^2(y)\, dy \leq N \sum_{|\beta| = r} \int_{\mathbb{R}^d} \frac{1}{(1 + |y|^2)^{\chi/2}} |\tilde{g}_{\beta,\rho}(y)|^2\, dy,$$

where $g_{\beta,\rho}(\xi) = D^\beta(\sigma^{-1}(\xi/\rho))$ and $g_{\beta,\rho} \in \mathcal{L}_2$, so that $\tilde{g}_{\beta,\rho}$ is well defined. After that, prove that for any g first in $\mathcal{S}$ and then in $\mathcal{L}_2$ we have

$$\int_{\mathbf{R}^d} \frac{1}{(1+|y|^2)^{\chi/2}} |\tilde{g}(y)|^2\,dy = \|g\|^2_{H_2^{-\chi/2}} \le N\|g\zeta\|^2_{H_2^{-\chi/2}} + N\|(1-\zeta)g\|^2_{\mathcal{L}_2},$$

where $\zeta \in C_0^\infty$ is such that $\zeta = 1$ in B_1 and $\zeta = 0$ outside B_2.

18. Exercise. Complete the missing details in the following argument. After all reductions made in the proof of Theorem 12.8.2, having in mind (12.8.4) and (12.8.3), prove that (12.8.5) holds with a constant N depending only on $N_0(L,0)$, d, and the supremum over $\rho > 0$ and $x \in B_\rho$ of

$$\rho^{-1} \int_{|y| \ge 2\rho} |y|^{d+1} |G(x+y) - G(y)|^2\,dy = I_\rho(x/\rho)$$

$$:= \int_{|y|\ge 2} |y|^{d+1} |G_\rho(x/\rho + y) - G_\rho(y)|^2\,dy,$$

where G_ρ is the function G constructed from the operator L_ρ with symbol $\sigma(\xi) = \sigma(\xi/\rho)$.

Introduce r as the least integer $> d/2$ and set $\chi = 0$ if d is odd and $\chi = 1$ if d is even and prove that

$$I_\rho(x) \le N \int_{|y|\ge 2} \frac{|y|^{2r}}{(1+|y|^2)^{\chi/2}} |G_\rho(x+y) - G_\rho(y)|^2\,dy$$

$$\le N \sum_{|\alpha|=r} \int_{|y|\ge 2} \frac{1}{(1+|y|^2)^{\chi/2}} |y^\alpha G_\rho(x+y) - y^\alpha G_\rho(y)|^2\,dy.$$

Here we are interested in $|x| \le 1$ and

$$y^\alpha = (x+y)^\alpha + \sum_{\substack{\beta+\gamma=\alpha,\\ |\beta|\ge 1}} c_\alpha^{\beta\gamma} x^\beta y^\gamma,$$

where the $c_\alpha^{\beta\gamma}$ are certain constants. In addition, $|\gamma| \le r-1$, $2|\gamma| - \chi < d$, and

$$\int_{|y|\ge 2} \frac{|y|^{2|\gamma|}}{(1+|y|^2)^{\chi/2}} G_\rho^2(x+y)\,dy \le \int_{|y|\ge 1} |y|^{2|\gamma|-\chi} G_\rho^2(y)\,dy$$

is under control by Exercise 17. Next for $|\alpha| = r$

$$\int_{\mathbf{R}^d} \frac{1}{(1+|y|^2)^{\chi/2}} |(y+x)^\alpha G_\rho(x+y) - y^\alpha G_\rho(y)|^2\,dy$$

$$= N \int_{\mathbf{R}^d} \frac{1}{(1+|y|^2)^{\chi/2}} |\tilde{g}_{\alpha,x}(y)|^2\,dy = N\|g_{\alpha,x}\|^2_{H_2^{-\chi/2}},$$

where $g^{\alpha,x}(\xi) = (e^{i\xi\cdot x} - 1)D^\alpha(\sigma(\xi/\rho))$. Conclude the argument as in Exercise 17 thereby proving that the constant $N_{d+2}(L,0)$ in Theorems 12.8.2 and 12.8.3 can be replaced with $N_r(L,0)$.

19. Exercise (Finite differences). Use the notation $\delta_{jh}f$ introduced before Theorem 9.1.1, take $\gamma \in \mathbb{R}$, and prove that

(i) if $f \in H_p^\gamma$, then

$$\|\delta_{jh}f\|_{H_p^{\gamma-1}} \leq N\|f\|_{H_p^\gamma},$$

where N is independent of h, f, j and $\delta_{jh}f \to D_j f$ in $H_p^{\gamma-1}$ as $h \to 0$;

(ii) if $f \in H_p^{\gamma-1}$ and for $j = 1, ..., d$ we have

$$\sup_{h\in(0,1)} \|\delta_{jh}f\|_{H_p^{\gamma-1}} \leq N_0 < \infty,$$

then $f \in H_p^\gamma$ and

$$\|f\|_{H_p^\gamma} \leq N\|f\|_{H_p^{\gamma-1}} + NN_0,$$

where N is independent of f and N_0.

Here is a generalization of the multiplicative inequalities from Exercise 1.5.6.

20. Exercise (Multiplicative inequalities). Take $0 < \nu < \gamma < \infty$, set $\theta = \nu/\gamma$, and by using Theorem 10 and proving the fact that for $\varepsilon > 0$ and

$$\sigma_\varepsilon(\xi) := \frac{(1 + |\xi|^2)^{\nu/2}}{\varepsilon^{-1/(1-\theta)} + \varepsilon^{1/\theta}(1 + |\xi|^2)^{\gamma/2}}$$

we have $N_{d+2}(\sigma_\varepsilon, 0) \leq N$, where N is independent of ε, show that

$$\|f\|_{H_p^\nu} \leq N\|f\|_{\mathcal{L}_p}^{1-\theta}\|f\|_{H_p^\gamma}^{\theta}$$

for any $f \in H_p^\gamma$ with N independent of f.

21. Exercise. Let $d = 3$ and $u \in H_2^2$. Prove Agmon's inequality: $u \in C$ and

$$\|u\|_C \leq N\|u_x\|_{\mathcal{L}_2}^{1/2}\|u_{xx}\|_{\mathcal{L}_2}^{1/2},$$

where N is independent of u.

22. Exercise. It is not true that $\mathcal{L}_q \subset \mathcal{L}_p$ for $q \in (1, p)$ because $\mathbb{R}^d$ is unbounded. However, let $u \in H_p^\gamma$, assume that u has compact support and that $q \in (1, p)$ and then prove that $u \in H_q^\gamma$ at least if $\gamma \geq 0$.

23. Exercise. Prove that the delta function $\delta_0 \in H_p^{-\gamma}$ if $\gamma p > (p-1)d$. Also by using Theorem 1.7, show that if G is the Green's function of a strongly elliptic pseudo-differential operator of order ν, then $G \in H_p^{\nu-\gamma}$, whenever $\gamma p > (p-1)d$.

24. Exercise. There are distributions concentrated not at one point but on surfaces. Let $d \geq 2$, $x' = (x^1, ..., x^{d-1})$ and $g(x')$ be a function of class $H_2^\gamma(\mathbb{R}^{d-1})$ with $\gamma < 0$. Consider the distribution $\check{g}(x) = g(x')\delta(x^d)$, whose action on $\phi \in C_0^\infty(\mathbb{R}^d)$ is defined by

$$(\check{g}, \phi) = \int_{\mathbb{R}^{d-1}} g(x')\bar{\phi}(0, x')\, dx'$$

and prove that $\check{g} \in H_2^{\gamma-1/2}(\mathbb{R}^d)$ and

$$\|\bar{g}\|_{H_2^{\gamma-1/2}(\mathbb{R}^d)} = N\|g\|_{H_2^\gamma(\mathbb{R}^{d-1})},$$

where N is independent of g.

25. Exercise. By using the Hahn-Banach theorem and Riesz's representation theorem, prove that a distribution g belongs to H_p^γ if and only if there is a (finite) constant N such that

$$\langle g, \phi\rangle \leq N\|\phi\|_{H_q^{-\gamma}}$$

for any $\phi \in C_0^\infty$, where $q = p/(p-1)$. In that case $\|g\|_{H_p^\gamma} \leq N$.

26. Exercise. While doing Exercise 25, the reader has shown that if $g \in H_p^\gamma$ and $\phi \in C_0^\infty$, then

$$\langle g, \phi\rangle \leq \|g\|_{H_p^\gamma}\|\phi\|_{H_q^{-\gamma}}.$$

Conclude that $\langle g, \phi\rangle$ originally defined on C_0^∞ extends by continuity to $H_q^{-\gamma}$, so that each $g \in H_p^\gamma$ can be identified with a continuous linear functional on $H_q^{-\gamma}$ and in this sense the dual to $H_q^{-\gamma}$ contains H_p^γ.

On the other hand, Exercise 25 allows one to conclude that one can identify the dual to $H_q^{-\gamma}$ with a subset of H_p^γ.

Hence, the dual to $H_q^{-\gamma}$ can be identified with H_p^γ. In particular, the spaces H_p^γ are reflexive.

27. Exercise. In Exercise 26 an operator mapping $H_q^{-\gamma}$ onto the dual to H_p^γ is mentioned. In connection with this, show that H_2^γ is a Hilbert space with the scalar product

$$((1-\Delta)^{\gamma/2}u, (1-\Delta)^{\gamma/2}v).$$

Therefore, one can use the well-known standard way of identifying Hilbert spaces with their duals, in this way identifying H_2^γ with its dual. Show that this way and the above described way to identify $H_2^{-\gamma}$ with the dual of H_2^γ produce absolutely different results unless $\gamma = 0$.

28. Exercise. Show that, if $g \in H_p^\gamma$, then

$$\|g\|_{H_p^\gamma} = \sup_{\phi\in C_0^\infty, \phi\neq 0} \frac{\langle g, \phi\rangle}{\|\phi\|_{H_q^{-\gamma}}},$$

where $q = p/(p-1)$.

29. Exercise. Prove that H_p^γ is not a closed subset of $\mathcal{L}_p$ for $\gamma > 0$.

4. Higher-order elliptic differential equations with continuous coefficients in H^γ_p

In this section we prove only a very basic result about existence and uniqueness of solutions for elliptic *differential* equations in the whole space. The reader can find much more information and also information on higher-order parabolic equations and systems of equations in domains in [**1**] and [**18**].

In this section

$$p \in (1,\infty), \quad m \in \{1,2,...\}.$$

Let $a^\alpha(x)$ be some (complex-valued) measurable functions given for $|\alpha| \leq m$, $x \in \mathbb{R}^d$. Assume that for a constant $\kappa > 0$ and all $\xi, x \in \mathbb{R}^d$ we have

$$\kappa^{-1}(1+|\xi|^m) \geq |\sum_{|\alpha|\leq m} a^\alpha(x) i^{|\alpha|}\xi^\alpha| \geq \kappa(1+|\xi|^m),$$

so that the operator

$$L = L(x) = \sum_{|\alpha|\leq m} a^\alpha(x) D^\alpha$$

is *uniformly* strongly elliptic. For real λ we also define

$$L_\lambda = \sum_{|\alpha|\leq m} a^\alpha(x)\lambda^{m-|\alpha|}D^\alpha.$$

Observe that if $L = a^{ij}D_{ij} - 1$, then

$$L_\lambda = a^{ij}D_{ij} - \lambda^2.$$

Since

$$\kappa^{-1}(|\lambda|^m + |\xi|^m) \geq |\sum_{|\alpha|\leq m} a^\alpha(x)\lambda^{m-|\alpha|} i^{|\alpha|}\xi^\alpha| \geq \kappa(|\lambda|^m + |\xi|^m),$$

the assertions of the following lemma about existence and uniqueness of solutions follow directly from Theorem 3.10. The last assertion also follows from Theorem 3.10 if $\lambda = 1$ and then is extended to all $\lambda \neq 0$ by using dilations.

1. Lemma. *Let a^α be independent of x and take a $\lambda \in \mathbb{R}\setminus\{0\}$. Then for any $f \in \mathcal{L}_p$ there exists a unique $u \in W^m_p$ such that $L_\lambda u = f$. Furthermore, if $u \in W^m_p$ and $Lu \in \mathcal{L}_q$ with a $q \in (1,\infty)$, then $u \in W^m_q$. Finally, there exists a constant $N = N(d,m,p,\kappa)$ (independent of λ) such that for any $u \in W^m_p$*

$$\sum_{n=0}^{m} |\lambda|^{m-n} \|D^n u\|_{\mathcal{L}_p} \le N \|L_\lambda u\|_{\mathcal{L}_p}. \tag{1}$$

Below in this subsection we impose the following.

2. Assumption. There exists an increasing function $\omega(\varepsilon)$, $\varepsilon \ge 0$, such that $\omega(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$ and for all multi-indices α with $|\alpha| \le m$ and all $x, y \in \mathbb{R}^d$ we have

$$|a^\alpha(x) - a^\alpha(y)| \le \omega(|x-y|).$$

Here is a counterpart of Lemma 1.6.3.

3. Lemma. *There exists an $\varepsilon = (\varepsilon, d, m, p, \omega) > 0$ such that if $u \in W_p^m$ has support in a ball of radius ε, then* (1) *holds with $2N$ in place of N.*

Proof. We may assume that the ball in question is centered at the origin. Set

$$L_\lambda^0 = \sum_{|\alpha| \le m} \lambda^{m-|\alpha|} a^\alpha(0) D^\alpha.$$

Then by Lemma 1 we have

$$\sum_{n=0}^{m} |\lambda|^{m-n} \|D^n u\|_{\mathcal{L}_p} \le N \|L_\lambda^0 u\|_{\mathcal{L}_p} \le N \|L_\lambda u\|_{\mathcal{L}_p} + N_1 \omega(\varepsilon) \sum_{n=0}^{m} |\lambda|^{m-n} \|D^n u\|_{\mathcal{L}_p}$$

and it suffices to choose ε so that $N_1 \omega(\varepsilon) \le 1/2$. The lemma is proved.

Now comes an a priori estimate for equations with variable coefficients and general $u \in W_p^m$. We take an operator

$$B = \sum_{|\beta| \le m-1} b^\beta(x) D^\beta$$

whose coefficients $b^\beta(x)$ are measurable and satisfy

$$|b^\beta(x)| \le K$$

for all β, where K is a fixed constant.

4. Lemma. *There exist constants $\lambda_0 > 0$ and N depending only on d, p, m, κ, K, and the function ω such that for any λ satisfying $|\lambda| \ge \lambda_0$ and $u \in W_p^m$ we have*

$$\sum_{|\alpha| \le m} |\lambda|^{m-n} \|D^n u\|_{\mathcal{L}_p} \le N \|L_\lambda u + Bu\|_{\mathcal{L}_p}. \tag{2}$$

Proof. First if we know that the result is true for $B \equiv 0$, then

$$\sum_{|\alpha|\le m} |\lambda|^{m-n}\|D^n u\|_{\mathcal{L}_p} \le N_0\|L_\lambda u\|_{\mathcal{L}_p} \le N\|L_\lambda u + Bu\|_{\mathcal{L}_p} + N_1\sum_{n=0}^{m-1}\|D^n u\|_{\mathcal{L}_p}.$$

Here on the left the powers of λ in front of $\|D^n u\|_{\mathcal{L}_p}$ for $n \le m-1$ are larger than 1. It follows that, if we make sure that $\lambda_0 \ge 2N_1 + 1$, then (2) for $|\lambda| \ge \lambda_0$ would follow with $2N_0$ in place of N.

Thus, in the rest of the proof we assume that $B \equiv 0$. We use the same method as in the proof of Theorem 1.6.4 and take a nonnegative $\zeta \in C_0^\infty$ such that

$$\|\zeta\|_{\mathcal{L}_p} = 1$$

and the support of ζ is in a ball of radius ε with ε taken from Lemma 3. Then we set $\zeta^y(x) = \zeta(x-y)$, take a multi-index α such that $|\alpha| = n \le m$ and observe that for $n \ge 1$ and $|\lambda| \ge 1$

$$|\lambda|^{m-n}|D^\alpha(\zeta^y u) - \zeta^y D^\alpha u|$$

$$\le N|\lambda|^{-1}\eta^y|\lambda|^{m-(n-1)}\sum_{j\le n-1}|D^j u| \le N|\lambda|^{-1}\eta^y v. \tag{3}$$

where

$$\eta^y(x) = \eta(x-y), \quad \eta = \sum_{n=0}^{m}|D^n\zeta|, \quad v = \sum_{k=0}^{m}|\lambda|^{m-k}|D^k u|.$$

Obviously, the inequality between the extreme terms in (3) is also true if $|\alpha| = n = 0$.

It follows, in particular, that for $n = 0, 1, ..., m$

$$|L_\lambda(\zeta^y u)| \le \zeta^y|L_\lambda u| + N|\lambda|^{-1}\eta^y v, \quad \zeta^y|D^n u| \le |D^n(\zeta^y u)| + N|\lambda|^{-1}\eta^y v.$$

We also notice that for each $f \in \mathcal{L}_p$

$$\int_{\mathbb{R}^d}|f(x)|^p\,dx = \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|\zeta^y(x)f(x)|^p\,dx\Big)\,dy.$$

By combining this with Lemma 3, we conclude that for $n \le m$

$$(|\lambda|^{m-n}\|D^n u\|_{\mathcal{L}_p})^p = |\lambda|^{p(m-n)}\int_{\mathbb{R}^d}|D^n u(x)|^p\,dx$$

$$= |\lambda|^{p(m-n)} \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} |\zeta^y D^n u|^p \, dx \Big) \, dy$$

$$\leq N|\lambda|^{p(m-n)} \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} |D^n(\zeta^y u)|^p \, dx \Big) \, dy + N|\lambda|^{-p} \int_{\mathbb{R}^d} \|\eta^y v\|^p_{\mathcal{L}_p} \, dy$$

$$\leq N \int_{\mathbb{R}^d} \|L_\lambda(\zeta^y u)\|^p_{\mathcal{L}_p} \, dy + N|\lambda|^{-p}\|v\|^p_{\mathcal{L}_p}$$

$$\leq N \int_{\mathbb{R}^d} \|\zeta^y L_\lambda u\|^p_{\mathcal{L}_p} \, dy + N|\lambda|^{-p}\|v\|^p_{\mathcal{L}_p}$$

$$= N\|L_\lambda u\|^p_{\mathcal{L}_p} + N|\lambda|^{-p}\|v\|^p_{\mathcal{L}_p}.$$

In short

$$|\lambda|^{m-n}\|D^n u\|_{\mathcal{L}_p} \leq N\|L_\lambda u\|_{\mathcal{L}_p} + N|\lambda|^{-1}\|v\|_{\mathcal{L}_p}.$$

By summing up these estimates on $n = 0, 1, \dots, m$, we get

$$A := \sum_{n=0}^{m} |\lambda|^{m-n}\|D^n u\|_{\mathcal{L}_p} \leq N\|L_\lambda u\|_{\mathcal{L}_p} + N_1|\lambda|^{-1}A$$

and we obtain (2) with $B \equiv 0$ by taking $|\lambda| \geq 2N_1$. The theorem is proved.

We can now prove a result about solvability of equations with variable coefficients. Recall that Assumption 2 is supposed to hold.

5. Theorem. *Take λ_0 from Lemma 4. Then for any λ satisfying $|\lambda| \geq \lambda_0$ and $f \in \mathcal{L}_p$ there exists a unique $u \in W^m_p$ satisfying*

$$L_\lambda u + Bu = f.$$

Proof. We take and fix a λ such that $|\lambda| \geq \lambda_0$ and notice that the assertion about uniqueness follows from Lemma 4.

In the proof of the existence we will again apply the method of continuity. However, it may happen that our operator L is given as $a(x)(\Delta - 1)$, where $a(x)$ is a complex-valued function satisfying $|a(x)| = 1$ and such that $a(x) = 1$ for some x and $a(x) = -1$ for a different x. Then the "naive" way of joining L with $\Delta - 1$ as $tL + (1-t)(\Delta - 1)$ does not produce a family of uniformly strongly elliptic operators. Therefore, we use a different family of operators.

For $t \in [0, \infty]$ (including ∞) define

$$\zeta(t, x) = \frac{tx}{t + |x|}, \quad a^\alpha(t, x) = a^\alpha(\zeta(t, x)), \quad L^t_\lambda = \sum_{|\alpha| \leq m} a^\alpha(t, x)\lambda^{m-|\alpha|}D^\alpha$$

(of course, $\zeta(\infty, x) := x$) and let T be the set of all points $t \in [0, \infty]$ for which the statement of the theorem is true with L_λ^t in place of L_λ. Observe that $a^\alpha(0, x) = a^\alpha(0)$. Therefore, the set T is not empty by Lemma 1.

Next, we claim that

$$|\zeta(t, x) - \zeta(t, y)| \le |x - y| \tag{4}$$

for any t, x, y. Obviously, we need only show that $|\mathrm{grad}_x \zeta(t, x)| \le 1$ for $x \ne 0$. For $x \ne 0$ and unit l we have

$$|l \cdot \mathrm{grad}_x \zeta(t, x)| = \left|\frac{tl}{t + |x|} - \frac{tx}{(t + |x|)^2} \frac{l \cdot x}{|x|}\right|$$

$$\le \frac{t}{t + |x|} + \frac{t|x|}{(t + |x|)^2} \le \frac{t}{t + |x|} + \frac{|x|}{t + |x|} = 1.$$

This proves (4).

Inequality (4) shows that for any t

$$|a^\alpha(t, x) - a^\alpha(t, y)| \le \omega(x - y),$$

so that by Lemma 4 for any λ satisfying $|\lambda| \ge \lambda_0$ and $u \in W_p^m$ we have

$$\sum_{|\alpha| \le m} |\lambda|^{m-n} \|D^n u\|_{\mathcal{L}_p} \le N \|L_\lambda^t u + Bu\|_{\mathcal{L}_p},$$

with N independent of u and t. This immediately implies that the set T is closed.

Now to prove that $T = [0, \infty]$, it suffices to show that $T \cap [0, \infty)$ is open in the relative topology of $[0, \infty)$. Take any $t_0 \in T \cap [0, \infty)$ and define the linear operator $\mathcal{R} : \mathcal{L}_p \to W_p^m$ by introducing $\mathcal{R}f$ as the solution of the equation $L_\lambda^{t_0} u = f$. By assumption $\mathcal{R}$ is well defined and by Lemma 4 it is bounded. To prove that for $t \in [0, \infty)$ sufficiently close to t_0 the equation $L^t u = f$ is solvable, we write this equation as

$$L_\lambda^{t_0} u = f + (L_\lambda^{t_0} - L_\lambda^t)u, \quad u = \mathcal{R}f + \mathcal{R}(L_\lambda^{t_0} - L_\lambda^t)u,$$

and we show that the operator $\mathcal{R}(L_\lambda^{t_0} - L_\lambda^t)$ is a contraction in W_p^m if $t \in [0, \infty)$ is close to t_0.

By the above

$$\|\mathcal{R}(L_\lambda^{t_0} - L_\lambda^t)u\|_{W_p^m} \le N \|(L_\lambda^{t_0} - L_\lambda^t)u\|_{L_p}.$$

where N is independent of $u \in W_p^m$ and t. Next,

$$\|(L_\lambda^{t_0} - L_\lambda^t)u\|_{L_p} \le N\|u\|_{W_p^m} \max_{|\alpha|\le m} \sup_{\mathbb{R}^d} |a^\alpha(t_0,x) - a^\alpha(t,x)|.$$

Here by using the obvious estimate $|\zeta_t(t,x)| \le 1$, we get that

$$|a^\alpha(t_0,x) - a^\alpha(t,x)| \le \omega(|\zeta(t_0,x) - \zeta(t,x)|) \le \omega(|t_0 - t|),$$

so that the above estimates show that $\mathcal{R}(L_\lambda^{t_0} - L_\lambda^t)$ is a contraction indeed if t is close to t_0. The theorem is proved.

6. Remark. There is a theory of solvability of higher-order elliptic and parabolic equations and systems of equations in domains (see [1], [**18**]). This theory has a few features in common with what we have seen in these lectures. However, at times things may look quite unusual. In the general setting of the Dirichlet boundary-value problem even for second-order homogeneous elliptic equations with *complex* coefficients, new difficulties arise which do not exist in the real-valued case. For instance, for $d = 2$ for the Bitsadze equation

$$\frac{\partial^2}{(\partial \bar z)^2} u = 0$$

with the homogeneous elliptic operator

$$\frac{\partial^2}{(\partial \bar z)^2} = \frac{1}{4}\Big(\frac{\partial^2}{(\partial x)^2} + 2i\frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{(\partial y)^2}\Big)$$

any function of the form

$$u(x,y) = f(z)(1 - |z|^2) = f(z)(1 - z\bar z)$$

with analytic f is a solution which equals zero on the boundary of the unit disk (recall that $(\partial/\partial \bar z)f(z) = 0$ for any analytic function). In this situation we lose uniqueness.

5. Second-order parabolic equations. Semigroups (optional)

This section consists of the following two exercises aimed at investigating second-order parabolic equations with coefficients independent of time by using the theory of analytic semigroup. They also show that the spectrum of a second-order elliptic operator with real coefficients, basically, lies near the negative part of the real axis. The reader can obtain further information on applications of the semigroup approach to parabolic equation, for instance, from [**2**], [**5**], and [**16**].

1. Exercise. Let

$$L = a^{jk}(x)D_{jk} + b^j(x)D_j + c(x)$$

be a second-order elliptic operator with *real-valued* bounded a^{jk}, b^j, c and with uniformly continuous a^{jk} satisfying the condition

$$a^{jk}(x)\xi^j\xi^k \geq \kappa|\xi|^2$$

for all $x, \xi \in \mathbb{R}^d$, where $\kappa > 0$ is a constant. For a fixed $\theta \in [0, \pi)$ and $\nu_0 \geq 0$ introduce a subset of the complex plane by

$$\Gamma_{\theta,\nu_0} = \{\nu : |\nu| \geq \nu_0, |\arg \nu| \leq \theta\}.$$

Prove that there is a constant $\nu_0 > 0$ such that for any $\nu \in \Gamma_{\theta,\nu_0}$ and $f \in \mathcal{L}_p$ there exists a unique solution $u \in W_p^2$ of the equation

$$Lu - \nu u = f.$$

Furthermore, for this solution

$$\|u\|_{W_p^2} + |\nu|\,\|u\|_{\mathcal{L}_p} \leq N\|f\|_{\mathcal{L}_p},$$

where N is independent of ν and f.

Doing the following exercise requires a deeper knowledge of functional analysis than the other parts of the lectures.

2. Exercise. Under the conditions of Exercise 1 for $\nu \in \Gamma_{\theta,\nu_0}$ denote by $\mathcal{R}_\nu$ the operator which sends $f \in \mathcal{L}_p$ into $u \in W_p^2$ such that $\nu u - Lu = f$. Show that, as operators on $\mathcal{L}_p$, the operators $\mathcal{R}_\nu$ satisfy

$$\mathcal{R}_\nu - \mathcal{R}_\mu = (\mu - \nu)\mathcal{R}_\mu\mathcal{R}_\nu, \quad \forall \mu, \nu \in \Gamma_{\theta,\nu_0}$$

(Hilbert's identity). Derive from this that $\mathcal{R}_\nu$ is an analytic function on Γ_{θ,ν_0} with values in the space of operators $\mathcal{L}_p \to \mathcal{L}_p$. By taking $\theta \in (\pi/2, \pi)$, prove that for complex $t \neq 0$ with $|\arg t| < \theta - \pi/2$ and $f \in \mathcal{L}_p$ the integral in the $\mathcal{L}_p$ sense in the counterclockwise direction

$$T_t f = \frac{1}{2\pi i}\int_{\partial\Gamma_{\theta,\nu_0}} e^{tz}\mathcal{R}_z f\, dz$$

is well defined and defines $T_t f$ as an analytic function of t with values in $\mathcal{L}_p$. Furthermore, $T_t f \in W_p^2$. Show that (the strong derivative in t in $\mathcal{L}_p$) $dT_t f/dt = LT_t f$ so that $u(t,x) := T_t f(x)$ for real t satisfies the parabolic equation

$$\partial u/\partial t = Lu, \quad t > 0,$$

for any $f \in \mathcal{L}_p$. Finally, show that $\|T_t f - f\|_{\mathcal{L}_p} \to 0$ as $t \downarrow 0$ for any $f \in \mathcal{L}_p$.

6. Second-order divergence type elliptic equations with continuous coefficients

In this section we fix $p \in (1, \infty)$ and in $\mathbb{R}^d$ consider the equation

$$Lu - \lambda u = D_i f^i + g \quad (= \sum_{i=1}^d D_i f^i + g), \tag{1}$$

where

$$Lu = Lu(x) = D_i(a^{ij}(x)D_j u(x) + a^i(x)u(x)) + b^i(x)D_i u(x) + c(x)u(x).$$

We assume that all coefficients and f^i and g are *real valued* and, for some constant $K, \kappa > 0$ and all i, j, on $\mathbb{R}^d$

$$|a^{ij}|, |a^i|, |b^i|, |c| \leq K, \quad a^{rk}\xi^r\xi^k \geq \kappa|\xi|^2, \quad \forall \xi \in \mathbb{R}^d.$$

We also assume that there exists a function $\omega(\varepsilon)$, $\varepsilon > 0$ such that $\omega(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$ and for all i, j and $x, y \in \mathbb{R}^d$ such that $|x - y| \leq \varepsilon$ we have

$$|a^{ij}(x) - a^{ij}(y)| \leq \omega(\varepsilon).$$

We take $f^i, g \in \mathcal{L}_p$ and look for solution of (1) in the class $W_p^1 = H_p^1$. First of all observe that, in this situation, by Corollary 3.11 the right-hand side of (1) belongs to H_p^{-1}. Also $Du, u \in \mathcal{L}_p$ and again by Corollary 3.11 the left-hand side of (1) is also in H_p^{-1}. Thus, (1) makes perfect sense.

1. Lemma. *Assume that the a^{ij} are constant, $a^i = b^i = c \equiv 0$, and $\lambda > 0$. Then for any $f^1, ..., f^d, g \in \mathcal{L}_p$ there exists a unique $u \in W_p^1$ satisfying* (1). *Furthermore, for this solution*

$$\lambda^{1/2}\|u\|_{\mathcal{L}_p} + \|Du\|_{\mathcal{L}_p} \leq N(\lambda^{-1/2}\|g\|_{\mathcal{L}_p} + \sum_{j=1}^d \|f^i\|_{\mathcal{L}_p}),$$

where N depends only on d, p, κ, and K.

This lemma follows immediately from Theorem 4.4.2 after an appropriate linear change of coordinate transforming the operator

$$(1/2)(a^{ij} + a^{ji})D_i D_j \quad (= a^{ij} D_{ij})$$

into Δ.

2. Lemma. *Assume that the a^{ij} are constant. Then there exists a constant $\lambda_0 > 0$, depending only on d, p, κ, and K, such that for any $\lambda \geq \lambda_0$ and $f^1, ..., f^d, g \in \mathcal{L}_p$ there exists a unique $u \in W_p^1$ satisfying* (1). *Furthermore, for this solution*

$$\lambda^{1/2}\|u\|_{\mathcal{L}_p} + \|Du\|_{\mathcal{L}_p} \leq N\big(\lambda^{-1/2}\|g\|_{\mathcal{L}_p} + \sum_{j=1}^{d} \|f^i\|_{\mathcal{L}_p}\big), \tag{2}$$

where N depends only on d, p, κ, and K.

Proof. We start with proving the a priori estimate (2) assuming that a solution $u \in W_p^1$ already exists. Write

$$a^{ij} D_{ij} u - \lambda u = (g - b^j D_j u - cu) + D_j(f^j - a^j u).$$

By Lemma 1, for $\lambda > 0$,

$$\lambda^{1/2}\|u\|_{\mathcal{L}_p} + \|Du\|_{\mathcal{L}_p} \leq N_1\big(\lambda^{-1/2}(\|g\|_{\mathcal{L}_p} + \|Du\|_{\mathcal{L}_p}) + \sum_j \|f^j\|_{\mathcal{L}_p} + \|u\|_{\mathcal{L}_p}\big).$$

By confining $\lambda_0 > 0$ to satisfy $N_1\lambda_0^{-1/2} \leq 1/2$, we come to (2) with $N = 2N_1$ if $\lambda \geq \lambda_0$.

Now we prove the first assertion of the lemma. First fix a $\lambda \geq \lambda_0$ and take a $u \in W_p^1$. Then $(L - \lambda)u \in H_p^{-1}$ and by Exercise 3.13 and since $W_p^1 = H_p^1$, we see that (2) implies that for any $u \in H_p^1$

$$\|u\|_{H_p^1} \leq N\|(L - \lambda)u\|_{H_p^{-1}}$$

if $\lambda \geq \lambda_0$, where N depends only on d, p, κ, λ_0, and K. In particular, if we introduce the family of operators

$$L^t = tL + (1 - t)\Delta,$$

then for $t \in [0, 1]$, $\lambda \geq \lambda_0$, and $u \in H_p^1$

$$\|u\|_{H_p^1} \leq N\|(L^t - \lambda)u\|_{H_p^{-1}}$$

with N independent of t and u. Furthermore, for $t, s \in [0, 1]$ and $u \in H_p^1$

$$\|(L^t - L^s)u\|_{H_p^{-1}} = |t - s|\, \|(L - \Delta)u\|_{H_p^{-1}} \leq N|t - s|\, \|u\|_{H_p^1}.$$

After that, it only remains to apply the method of continuity using the fact that for $t = 0$ the operator $L^0 - \lambda : H_p^1 \to H_p^{-1}$ is invertible by Lemma 1. The lemma is proved.

The following main result of this section says that the assertions of Lemma 2 are true in the general case and not only for constant coefficients a^{ij}.

3. Theorem. *There exists a constant $\lambda_0 > 0$, depending only on d, p, κ, ω, and K, such that for any $\lambda \geq \lambda_0$ and $f^1, ..., f^d, g \in \mathcal{L}_p$ there exists a unique $u \in W_p^1$ satisfying (1). Furthermore, for this solution*

$$\lambda^{1/2}\|u\|_{\mathcal{L}_p} + \|Du\|_{\mathcal{L}_p} \leq N(\lambda^{-1/2}\|g\|_{\mathcal{L}_p} + \sum_{j=1}^{d}\|f^i\|_{\mathcal{L}_p}), \tag{3}$$

where N depends only on d, p, κ, and K.

Proof. As in the proof of Lemma 2 it suffices to prove (3) assuming that the solution $u \in W_p^1$ already exists. We do it first assuming that the support of u belongs to a ball of radius $\varepsilon > 0$. Without loss of generality we may assume that the ball in question is centered at the origin. Set

$$L_0 v = D_i(a_0^{ij} D_j v + a^i v) + b^i D_i v + cv,$$

where $a_0^{ij} = a^{ij}(0)$, and observe that

$$(L_0 - \lambda)u = g + D_i(f^i + (a_0^{ij} - a^{ij})D_j u).$$

By Lemma 2

$$\lambda^{1/2}\|u\|_{\mathcal{L}_p} + \|Du\|_{\mathcal{L}_p} \leq N_1(\|g\|_{\mathcal{L}_p} + \sum_i \|f^i\|_{\mathcal{L}_p} + \omega(\varepsilon)\|Du\|_{\mathcal{L}_p}).$$

It follows that if we choose $\varepsilon > 0$ such that $N_1\omega(\varepsilon) \leq 1/2$, then we indeed have (3) with $N = 2N_1$ whenever the solution $u \in W_p^1$ has support in a ball of radius $\varepsilon > 0$.

In the general case as usual we apply partitions of unity. Take a nonnegative $\zeta \in C_0^\infty$ with support in the ball of radius ε centered at the origin and having the $\mathcal{L}_p$ norm equal to 1. Then define

$$\zeta^y(x) = \zeta(x - y), \quad u^y = \zeta^y u$$

and observe that, for each $y \in \mathbb{R}^d$,

$$(L - \lambda)u^y = \hat{g}^y + D_i \hat{f}^{iy},$$

where

$$\hat{g}^y = g\zeta^y - f^i D_i\zeta^y + a^{ij}D_i\zeta^y D_j u + u(a^i + b^i)D_i\zeta^y,$$

$$\hat{f}^{iy} = f^i\zeta^y + ua^{ij}D_j\zeta^y.$$

Since the support of u^y is in a ball of radius ε, by the above we have

$$\lambda^{p/2}\|u^y\|_{\mathcal{L}_p}^p + \|Du^y\|_{\mathcal{L}_p}^p \leq N(\lambda^{-p/2}\|\hat{g}^y\|_{\mathcal{L}_p}^p + \sum_i \|\hat{f}^{iy}\|_{\mathcal{L}_p}^p). \tag{4}$$

Here

$$|\hat{g}^y| \leq N(g + \sum_i |f^i| + |Du| + |u|)\eta^y, \quad |\hat{f}^{iy}| \leq N(|f^i| + |u|)\eta^y,$$

where $\eta^y(x) = \eta(x-y)$ and $\eta = \zeta + |D\zeta|$. Also

$$|\zeta^y Du| \leq |Du^y| + |u|\eta^y,$$

so that by (4) for $\lambda \geq \lambda_0 \geq 1$

$$\lambda^{p/2}\|u^y\|_{\mathcal{L}_p}^p + \|\eta^y Du\|_{\mathcal{L}_p}^p$$

$$\leq N(\lambda^{-p/2}\|g\eta^y\|_{\mathcal{L}_p}^p + \sum_i \|f^i\eta^y\|_{\mathcal{L}_p}^p + \lambda^{-p/2}\|\eta^y Du\|_{\mathcal{L}_p}^p + \|\eta^y u\|_{\mathcal{L}_p}^p).$$

Integrating through this relation with respect to $y \in \mathbb{R}^d$ yields

$$\lambda^{p/2}\|u\|_{\mathcal{L}_p}^p + \|Du\|_{\mathcal{L}_p}^p$$

$$\leq N_2(\lambda^{-p/2}\|g\|_{\mathcal{L}_p}^p + \sum_i \|f^i\|_{\mathcal{L}_p}^p + \lambda^{-p/2}\|Du\|_{\mathcal{L}_p}^p + \|u\|_{\mathcal{L}_p}^p),$$

which for λ_0 satisfying $\lambda_0^{p/2} \geq 2N_2$ and $\lambda \geq \lambda_0$ leads to

$$\lambda^{p/2}\|u\|_{\mathcal{L}_p}^p + \|Du\|_{\mathcal{L}_p}^p \leq 2N_2(\lambda^{-p/2}\|g\|_{\mathcal{L}_p}^p + \sum_i \|f^i\|_{\mathcal{L}_p}^p).$$

This estimate is equivalent to (3) and the theorem is proved.

7. Nonzero Dirichlet condition and traces

While studying second-order elliptic equations in $\mathbb{R}^d_+$ or in bounded domains, one might become interested in the nonzero Dirichlet condition. In Section 8.5 and in other places we interpreted the condition that $u = g$ on $\partial\Omega$ as $u - g \in \overset{0}{W}{}^2_p(\Omega)$ assuming that the boundary data g is continued from the boundary into Ω as a $W^2_p(\Omega)$ function. However, sometimes it is desirable to have sufficient conditions in terms of the behavior of g only on $\partial\Omega$, which would guarantee that such a continuation exists.

Here we consider $p \in (1,\infty)$,

$$\Omega = \mathbb{R}^d_+$$

and give some of these sufficient conditions using Remark 12.4.16. The idea of continuing a function g given on $\partial\Omega$ inside Ω is to solve the problem

$$u_{x^1} = \mathcal{K}_{d-1}u \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \partial\Omega,$$

where $\mathcal{K}_{d-1}$ is Cauchy's operator in $d-1$ dimensional space of x'.

1. Exercise. Show that if $u \in W^1_2(\Omega)$ and $u_{x^1} = \mathcal{K}_{d-1}u$ in Ω, then u is infinitely differentiable in Ω and

$$\Delta u - u = 0 \quad \text{in} \quad \Omega.$$

2. Theorem. *Let $\gamma \in \{1,2,...\}$, $\varepsilon > 0$, and let $g \in H^{\gamma+\varepsilon-1/p}_p(\partial\Omega)$. Then there exists a $u \in W^\gamma_p(\Omega)$ such that $T_{\partial\Omega}u = g$ and*

$$\|u\|_{W^\gamma_p(\Omega)} \leq N\|g\|_{H^{\gamma+\varepsilon-1/p}_p(\partial\Omega)},$$

where N is independent of g.

Proof. Since $\mathcal{S}(\partial\Omega)$ is dense in any $H^\nu_p(\partial\Omega)$, it suffices to construct a mapping Ψ bringing $\mathcal{S}(\partial\Omega)$ into the set of infinitely differentiable functions on $\bar{\Omega}$ having all derivatives bounded in $\bar{\Omega}$ and such that

$$\Psi g(0,x') = g(x'), \quad g \in \mathcal{S}(\partial\Omega), x' \in \partial\Omega,$$

and for any $\gamma = 1,2,...$ and $g \in \mathcal{S}(\partial\Omega)$

$$\|\Psi g\|_{W^\gamma_p(\Omega)} \leq N\|g\|_{H^{\gamma+\varepsilon-1/p}_p(\partial\Omega)},$$

with N independent of g. We construct such an operator by defining

$$\sigma_{(x^1)}(\xi') = \exp(-x^1(1+|\xi'|^2)^{1/2})$$

and viewing here $x^1 \geq 0$ as a parameter and $\sigma_{(x^1)}(\xi')$ as a symbol in $\partial\Omega$. Then we take a $\zeta \in C_0^\infty(\mathbb{R})$ such that $\zeta(t) = 1$ for $|t| \leq 1$ and $\zeta(t) = 0$ for $t \geq 2$ and set

$$\Psi g(x) = \zeta(x^1)\Phi g(x), \quad \Phi g(x) = L_{(x^1)}g(x') = F_{d-1}^{-1}(\sigma_{(x^1)}F_{d-1}g)(x'),$$

where $\mathcal{F}_{d-1}$ and $\mathcal{F}_{d-1}^{-1}$ are the Fourier and the inverse Fourier transforms with respect to the x' variable. In other words,

$$\Phi g(x) = c_{d-1}\int_{\partial\Omega} e^{ix'\cdot\xi' - x^1(1+|\xi'|^2)^{1/2}}\tilde{g}(\xi')\,d\xi',$$

where $\tilde{g}$ is the Fourier transform of g as a function of $d-1$ variables.

The fact that $\tilde{g} \in \mathcal{S}(\partial\Omega)$ allows us to differentiate with respect to x inside the integral and shows that all derivatives of Φg are continuous and bounded in $\bar{\Omega}$. Furthermore, if $\alpha = (\alpha^1, \alpha')$, then

$$D^\alpha\Phi g(x) = c_{d-1}(x^1)^{-\delta}\int_{\partial\Omega} e^{ix'\cdot\xi'}(-1)^{\alpha^1}i^{|\alpha'|}\sigma^\alpha_{(x^1)}(\xi')\tilde{g}(\xi')\,d\xi',$$

where $\delta \geq 0$ is a constant and

$$\sigma^\alpha_{x^1}(\xi') := (\xi')^{\alpha'}(x^1)^\delta(1+|\xi'|^2)^{\alpha^1/2}e^{-x^1(1+|\xi'|^2)^{1/2}},$$

which is a product of $(\xi')^{\alpha'}$ and a function, that is, a symbol of order $\alpha^1 - \delta$ according to Remark 12.4.16. This along with Exercise 12.4.5 shows that, for any $\delta \geq 0$ and $x^1 > 0$, $\sigma^\alpha_{(x^1)}$ is a symbol of order $|\alpha| - \delta$ with

$$N_n(\sigma^\alpha_{(x^1)}, |\alpha| - \delta)$$

dominated by a constant depending only on $|\alpha|$, δ, n, and d. It follows that, for $|\alpha| \leq \gamma$,

$$\|D^\alpha\Phi g\|^p_{\mathcal{L}_p([0,2]\times\mathbb{R}^{d-1})} = \int_0^2 (x^1)^{-p\delta}\|L_{\sigma^\alpha_{(x^1)}}g\|^p_{\mathcal{L}_p(\mathbb{R}^{d-1})}\,dx^1$$

$$\leq N\int_0^2 (x^1)^{-p\delta}\,dx^1\|g\|^p_{H_p^{\gamma-\delta}(\mathbb{R}^{d-1})}.$$

We take $\delta = 1/p - \varepsilon$, assuming without loss of generality that $\varepsilon < 1/p$. In that case the integral with respect to x^1 converges, so that

$$\|\Phi g\|^p_{W_p^\gamma([0,2]\times\mathbb{R}^{d-1})} \leq N\|g\|^p_{H_p^{\gamma+\varepsilon-1/p}(\mathbb{R}^{d-1})},$$

where N is independent of g. It only remains to notice that, obviously,

$$\|\Psi g\|_{W_p^\gamma(\Omega)} \leq N\|\Phi g\|_{W_p^\gamma([0,2]\times\mathbb{R}^{d-1})}.$$

The theorem is proved.

3. Remark. A trivial case of Theorem 2 occurs when $\varepsilon = 1/p$, in which case it says that functions of class $W_p^n(\partial\Omega)$, $n = 1, 2, ...$, can be continued in Ω as $W_p^n(\Omega)$ functions.

A similar result can be established for the Neumann problem.

4. Exercise. Prove that if $g \in H_p^{1+\varepsilon-1/p}(\partial\Omega)$, then there is a function $u \in W_p^2(\mathbb{R}_+^d)$ such that $\mathcal{T}_{\partial\Omega}u_{x^1} = g$. In addition,

$$\|u\|_{W_p^2(\Omega)} \leq N\|g\|_{H^{1+\varepsilon-1/p}(\partial\Omega)},$$

where N is independent of g. You might like to define u starting with the expression

$$\int_{\mathbb{R}^{d-1}} e^{ix'\cdot\xi'-x^1(1+|\xi'|^2)^{1/2}}(1+|\xi'|^2)^{-1/2}\tilde{g}(\xi')\,d\xi'$$

and again use Remark 12.4.16.

5. Remark. Theorem 2 is true with $\varepsilon = 0$ if $p \geq 2$, but it is not true if $\varepsilon = 0$ and $p \in (1, 2)$. If $p = 2$ and $\varepsilon = 0$, the result is sharp, which is shown by using the Fourier transform (cf. Exercises 2.5.5 and 12.3.1). As we have pointed out before, the space of traces of W_p^n functions composes Slobodetskii spaces. We refer the reader to [20] and [6] for these and further results.

6. Exercise. In this exercise we suggest the reader improve Theorem 2 for $\varepsilon = 0$ only when $p = 2$ but γ may be fractional. Let $\gamma \in [1/2, 3/2)$, and let h be a function of $x' = (x^1, ..., x^{d-1})$ of class $H_2^{\gamma-1/2}(\mathbb{R}^{d-1})$. Introduce the (generalized) function $g = g(x')$ by

$$g = 2\mathcal{K}_{d-1}h.$$

Let $\breve{g}(x) = g(x')\delta(x^d)$ be the distribution on $\mathbb{R}^d$ acting on $\phi \in C_0^\infty$ as

$$\langle\breve{g}, \phi\rangle := \int_{\mathbb{R}^{d-1}} g(x')\bar{\phi}(0, x')\,dx'.$$

Prove that $\breve{g} \in H_2^{\gamma-2}$ and define v as a unique solution in H_2^γ of

$$\Delta v - v = \breve{g}.$$

Show that v is infinitely differentiable away from the hyperplane $x^1 = 0$ and

$$\lim_{x^1\to 0}\int_{\mathbb{R}^{d-1}} |h(x') - v(x^1, x')|^2\,dx' = 0.$$

It is worth noting that the last formula can be obtained from the formula for the jump of the normal derivative of a double layer potential.

8. Sobolev embedding theorems for H_p^γ spaces

We introduced the spaces H_p^γ of (generalized) functions given on the whole of $\mathbb{R}^d$ only. For half spaces and generally for smooth domains there is a way to introduce H_p^γ extending functions from $\mathbb{R}_+^d$ to $\mathbb{R}^d$ by using Hestenes's formulas (see Section 7.12 of [**9**]). In this way one obtains a generalization of the results following below.

Any distribution, in particular, any $g \in \mathcal{H}$, is infinitely differentiable in the generalized sense. Sometimes one is interested in the usual rather than generalized derivatives. In this section we show that under appropriate conditions the distribution of class H_p^γ have continuous modifications and in this sense H_p^γ is embedded into the space of continuous functions. By the way, recall that we say that a function is *a modification* of a distribution if the function and the distribution coincide as distributions.

Another kind of embedding of H_p^γ, this time into H_q^ν, is also discussed in this section.

In this section

$$p \in (1, \infty).$$

A version of the following embedding theorem for functions in domains is proved in Theorem 10.4.10.

1. Theorem. *If $p\gamma > d$ and $\gamma - d/p$ is not an integer, then $H_p^\gamma \subset C^{\gamma - d/p}$. More precisely, for each $g \in H_p^\gamma$ there exists $\check{g} \in C^{\gamma-d/p}$ such that $g = \check{g}$ in the sense of distributions and*

$$\|\check{g}\|_{C^{\gamma-d/p}} \le N \|g\|_{H_p^\gamma},$$

where the constant N is independent of g.

Proof. It suffices to prove that, for any $g \in C_0^\infty$, we have

$$\|g\|_{C^{\gamma-d/p}} \le N \|g\|_{H_p^\gamma}, \tag{1}$$

where the constant N is independent of g. Indeed, by Theorem 3.7, for each $g \in H_p^\gamma$ there exists a sequence $g_n \in C_0^\infty$ such that

$$\|g - g_n\|_{H_p^\gamma} \to 0$$

as $n \to \infty$. In particular g_n is a Cauchy sequence in H_p^γ. Owing to (1), it is also a Cauchy sequence in $C^{\gamma-d/p}$. Since the latter space is complete, there is $\breve{g} \in C^{\gamma-d/p}$ such that

$$\|\breve{g} - g_n\|_{C^{\gamma-d/p}} \to 0.$$

Furthermore, on account of (1)

$$\|\breve{g}\|_{C^{\gamma-d/p}} = \lim_{n\to\infty} \|g_n\|_{C^{\gamma-d/p}} \le N \lim_{n\to\infty} \|g_n\|_{H_p^\gamma} = N\|g\|_{H_p^\gamma}.$$

Finally, $g_n \to g$ in the sense of distributions by Remark 3.4 and $g_n \to \breve{g}$ uniformly on $\mathbb{R}^d$ and, in particular, in the sense of distributions. Hence, $g = \breve{g}$ in the sense of distributions.

If $0 < \gamma - d/p < 1$, we know (1) from Lemma 12.10.2. In the general case write $\gamma - d/p = k + \alpha$ with $k = 0, 1, 2, \dots$ and $\alpha \in (0,1)$. Then, for $|\beta| \le k$ by the above result and Corollary 3.11

$$\|D^\beta g\|_{C^\alpha} \le N\|D^\beta g\|_{H_p^{\alpha+d/p}} \le N\|g\|_{H_p^{|\beta|+\alpha+d/p}}.$$

It only remains to notice that, by Corollary 3.9, the last norm is less than $\|g\|_{H_p^\gamma}$ since $|\beta| + \alpha + d/p \le \gamma$. The theorem is proved.

2. Exercise (Morrey's embedding theorem). Prove that if $p > d$, then

$$[u]_{C^\alpha} \le N\|u_x\|_{\mathcal{L}_p},$$

where $\alpha = 1 - d/p$ and N is independent of u. Also prove that the estimate is false for any other α. For domains we know the same result from Theorem 10.2.1.

3. Exercise. Generalize the result of Exercise 10.2.8 by proving that if $\gamma = d/p$ and $g \in H_p^\gamma$, then $g \in \mathrm{VMO}$ and

$$\sup_{\mathbb{R}^d} g^\sharp \le N\|g\|_{H_p^\gamma},$$

where N is independent of g.

4. Exercise. By using Exercises 12.9.4 and 3.19, show that for $d = 2$, if $\gamma \in \mathbb{R}$ and $g \in H_p^\gamma$ and $\bar\partial g \in H_p^\gamma$, then $g \in H_p^{\gamma+1}$ and

$$\|Dg\|_{H_p^\gamma} \le N\|\bar\partial g\|_{H_p^\gamma}$$

with N independent of g and γ. Conclude that if $g \in \mathcal{H}$, a domain $D \subset \mathbb{R}^2$, and $\bar\partial g \in C^\infty(D)$, then $g \in C^\infty(D)$.

To investigate embeddings of H_p^γ into H_q^ν, we need the following Hardy-Littlewood-Sobolev inequality.

5. Lemma. *Denote $g(x) = 1/|x|^{d-\nu}$ and assume*

$$\nu > 0, \quad p, q \in (1, \infty), \quad \frac{1}{q} = \frac{1}{p} - \frac{\nu}{d}.$$

Then for any $f \in \mathcal{L}_p$ we have $f * g \in \mathcal{L}_q$ and

$$\|f * g\|_{\mathcal{L}_q} \leq N \|f\|_{\mathcal{L}_p}, \tag{2}$$

where N is independent of f.

Proof. A natural way to prove (2) is to use Young's inequality (see Lemma 1.8.1):

$$\|f * g\|_{\mathcal{L}_q} \leq \|g\|_{\mathcal{L}_r} \|f\|_{\mathcal{L}_p} \quad \text{if} \quad \frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1. \tag{3}$$

However, $r = d/(d-\nu)$ and $g^r(x) = |x|^{-d}$, so that $\|g\|_{\mathcal{L}_r} = \infty$. Therefore, we use a different idea of L. Hedberg, which shows that for radial g the estimate (3) is not that good.

It suffices to concentrate on $f \geq 0$ and

$$0 < \|f\|_{\mathcal{L}_p} < \infty.$$

In that case, for an $R > 0$ to be specified later, write

$$f * g(x) = \int_{|y| \leq R} f(x-y) g(y)\, dy + \int_{|y| > R} f(x-y) g(y)\, dy =: I_1(x) + I_2(x).$$

Next we need the maximal function $\mathbb{M}f$ introduced in (3.3.1). Notice that, obviously, $\nu < d$ and, for any $y \in \mathbb{R}^d$,

$$g(y) = (d-\nu)^{-1} \int_{|y|}^{\infty} r^{\nu-d-1}\, dr = (d-\nu)^{-1} \int_0^{\infty} I_{|y| \leq r} r^{\nu-d-1}\, dr.$$

It follows that

$$I_1(x) = N \int_0^{\infty} r^{\nu-d-1} \Big(\int_{|y| \leq r \wedge R} f(x-y)\, dy \Big)\, dr.$$

Next, for $S = r \wedge R$

$$\int_{|y| \leq r \wedge R} f(x-y)\, dy = N \frac{S^d}{|B_S|} \int_{|y| \leq S} f(x-y)\, dy \leq N (r \wedge R)^d \mathbb{M}f(x),$$

$$\int_0^{\infty} r^{\nu-d-1} (r \wedge R)^d\, dr = R^{\nu} \int_0^{\infty} r^{\nu-d-1} (r \wedge 1)^d\, dr,$$

where the last integral is finite due to $0 < \nu < d$. It follows that

$$I_1(x) \leq N\mathbb{M}f(x) \int_0^\infty r^{\nu-d-1}(r \wedge R)^d \, dr = N\mathbb{M}f(x)R^\nu.$$

Furthermore, for $p' := p/(p-1)$, we have $(\nu - d)p' + d = -p'd/q$ and

$$\int_{|y|>R} g^{p'}(y) \, dy = N \int_R^\infty r^{-p'd/q-1} \, dr = NR^{-p'd/q}.$$

Hence, by Hölder's inequality

$$I_2(x) \leq \Big(\int_{|y|>R} g^{p'}(y) \, dy \Big)^{1/p'} \|f\|_{\mathcal{L}_p} = NR^{-d/q}\|f\|_{\mathcal{L}_p}.$$

Thus,

$$f * g(x) \leq N(\mathbb{M}f(x)R^\nu + R^{-d/q}\|f\|_{\mathcal{L}_p}).$$

Now we choose R so that the terms in the parentheses become equal, which happens for

$$R^{-\nu-d/q} = (\mathbb{M}f(x))\|f\|_{\mathcal{L}_p}^{-1},$$

where $\nu + d/q = d/p \neq 0$. Then we obtain

$$f * g(x) \leq N(\mathbb{M}f(x))^{p/q}\|f\|_{\mathcal{L}_p}^{1-p/q}, \quad \|f * g\|_{\mathcal{L}_q} \leq N\|\mathbb{M}f\|_{\mathcal{L}_p}^{p/q}\|f\|_{\mathcal{L}_p}^{1-p/q},$$

and to get (2), it only remains to note that $0 < p/q = 1 - p\nu/d < 1$ and to use Theorem 3.3.2. The lemma is proved.

6. Remark. The above estimate of I_2 also obviously holds if $p = 1$. However as we know $\|\mathbb{M}f\|_{\mathcal{L}_1} = \infty$ unless $f = 0$.

7. Theorem. *Let $\gamma, \mu \in \mathbb{R}$, $\mu \leq \gamma$, $p, q \in (1, \infty)$, and*

$$\gamma - \frac{d}{p} = \mu - \frac{d}{q}$$

so that automatically $q \geq p$. Then $H_p^\gamma \subset H_q^\mu$ and, for any $u \in H_p^\gamma$,

$$\|u\|_{H_q^\mu} \leq N\|u\|_{H_p^\gamma}, \tag{4}$$

where N is independent of u.

Proof. Without loss of generality we assume that $\mu < \gamma$. Furthermore, since

$$\|u\|_{H_q^\mu} = \|(1-\Delta)^{\gamma/2}u\|_{H_q^{\mu-\gamma}}, \quad \|u\|_{H_p^\gamma} = \|(1-\Delta)^{\gamma/2}u\|_{\mathcal{L}_p},$$

we may and will assume that $\gamma = 0$. Finally, since $\mathcal{S}$ is dense in $\mathcal{L}_p$ and H_p^μ is complete, it follows easily that to prove the theorem, it suffices only to prove (4) for $u \in \mathcal{S}$.

Now let $\nu := -\mu$. Then $\nu > 0$ and $\nu = d/p - d/q < d$. By Theorem 12.9.12, for $u \in \mathcal{S}$,

$$\|u\|_{H_q^\mu} = \|u * G\|_{\mathcal{L}_q} \leq \||u| * G\|_{\mathcal{L}_q},$$

where G is defined by (12.9.8) with ν in place of γ. To finish proving (4), it only remains to use Lemma 5 and Theorem 12.7.4, the latter showing that $|G(x)| \leq N/|x|^{d-\nu}$. The theorem is proved.

9. Sobolev mollifiers

Here we study some properties of convolutions on $\mathcal{H}$ and its subsets H_p^γ, $\gamma \in \mathbb{R}$, $p \in (1, \infty)$.

Let $\zeta \in C_0^\infty$. By Remark 12.4.15, the operator

$$L_\zeta : \phi \to \phi * \zeta$$

is a pseudo-differential operator on $\mathcal{S}$. By Definition 1.3 it extends to $\mathcal{H}$. On the other hand, as for any distribution, for $f \in \mathcal{H}$, the function $f * \zeta$ makes perfect sense. In the following theorem we show, in particular, that the extension of L_ζ is the same as the convolution. Recall that for $n = 0, 1, 2, \ldots$, we let C^n be the set of all n times continuously differentiable functions on $\mathbb{R}^d$ having all derivatives up to the nth order bounded on $\mathbb{R}^d$.

1. Theorem. *Let $\zeta \in C_0^\infty$ and $f \in \mathcal{H}$. Then*

(i) in the sense of distributions

$$L_\zeta f = f * \zeta;$$

(ii) we have $f * \zeta \in C^n \cap \mathcal{H}$ for any n;

(iii) for any pseudo-differential operator L, we have

$$L(f * \zeta) = (Lf) * \zeta;$$

(iv) if $f \in \mathcal{L}_{(1,\infty)}$, then $f * \zeta \in \mathcal{L}_{(1,\infty)} \subset \mathcal{H}$ and for any pseudo-differential operator L

$$L(f * \zeta) = f * L\zeta$$

in the sense of distributions, where $f * L\zeta$ is the usual convolution of an $\mathcal{L}_p$ function with an S function.

Proof. (i) First assume that $f \in \mathcal{L}_{(1,\infty)}$. Then since, for $\phi \in C_0^\infty$,

$$\langle f, \phi \rangle = \int_{\mathbb{R}^d} f(x)\bar{\phi}(x)\,dx,$$

by definition we have

$$\langle L_\zeta f, \phi \rangle = \int_{\mathbb{R}^d} f(x)\overline{L_\zeta^* \phi}(x)\,dx.$$

Here (see Remark 12.4.15)

$$L_\zeta^* \phi(x) = \int_{\mathbb{R}^d} \bar{\zeta}(y - x)\phi(y)\,dy,$$

so that

$$\langle L_\zeta f, \phi \rangle = \int_{\mathbb{R}^d} f(x)\Big(\int_{\mathbb{R}^d} \zeta(y - x)\bar{\phi}(y)\,dy\Big)\,dx. \tag{1}$$

Next observe that the function

$$\int_{\mathbb{R}^d} |\zeta(y - x)\phi(y)|\,dy$$

is bounded and has compact support and $|f|$ is locally integrable. It follows that we can use Fubini's theorem to transform the right-hand side of (1). We see that

$$\langle L_\zeta f, \phi \rangle = \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} f(x)\zeta(y - x)\,dx\Big)\bar{\phi}(y)\,dy = (f * \zeta, \phi),$$

which yields the result if $f \in \mathcal{L}_{(1,\infty)}$.

In the general case, by Theorem 2.5 there exist an integer $n \ge 0$ and $h \in \mathcal{L}_{(1,\infty)}$, such that $f = (1 - \Delta)^n h$. Hence by the above

$$L_\zeta f = L_\zeta(1 - \Delta)^n h = (1 - \Delta)^n L_\zeta h = (1 - \Delta)^n (h * \zeta) = h * ((1 - \Delta)^n \zeta),$$

where by definition

$$h * ((1 - \Delta)^n \zeta)(x) = \int_{\mathbb{R}^d} h(y)[(1 - \Delta)^n \zeta](x - y)\,dy$$

$$= \int_{\mathbb{R}^d} h(y)(1 - \Delta_y)^n [\zeta(x - y)]\,dy = \langle (1 - \Delta)^n h, \bar{\zeta}(x - \cdot)\rangle = f * \zeta(x).$$

This proves (i). This also easily implies (iii) since

$$L(f * \zeta) = LL_\zeta f = L_\zeta(Lf) = (Lf) * \zeta.$$

In (ii), the fact that $f * \zeta \in \mathcal{H}$ follows from (i) and the fact that pseudo-differential operators preserve $\mathcal{H}$. Furthermore,

$$D^\alpha(f * \zeta) = f * (D^\alpha \zeta)$$

for any multi-index α. In addition, if $f \in \mathcal{L}_{(1,\infty)}$, then by Hölder's inequality $f * D^\alpha\zeta$ is bounded. In this case all derivatives of $f * \zeta$ are bounded. In the general case the boundedness of the derivatives of $f * \zeta$ follows from the fact that for appropriate n and $h \in \mathcal{L}_{(1,\infty)}$ we have

$$f * \zeta = ((1-\Delta)^n h) * \zeta = (1-\Delta)^n(h * \zeta)$$

with $h * \zeta$ having all derivatives bounded, by the particular case.

To prove assertion (iv), it suffices to observe that by definition, assertion (iii), and Exercise 12.4.9 (in the notation of that exercise)

$$L(f * \zeta)(x) = (Lf) * \zeta(x) = \langle Lf, \overline{T_x\zeta}\rangle = \int_{\mathbb{R}^d} f(y)\overline{L^*\overline{T_x\zeta}}(y)\,dy$$

$$= \int_{\mathbb{R}^d} f(y)(L\zeta)(x-y)\,dy = f * L\zeta(x).$$

The theorem is proved.

Theorem 3.7 implies that one can approximate any element $g \in H_p^\gamma$ by infinitely differentiable functions. There is one important unified way to do such approximations by mollifying g. Take a real-valued function $\zeta \in C_0^\infty$ such that

$$\int_{\mathbb{R}^d} \zeta\,dx = 1.$$

For $\varepsilon > 0$ define $\zeta_\varepsilon(x) = \varepsilon^{-d}\zeta(x/\varepsilon)$. Any $g \in \mathcal{H}$ is a distribution. Therefore, the following definition makes sense:

$$g^{(\varepsilon)}(x) = g * \zeta_\varepsilon(x) = \langle g, \zeta_\varepsilon(x - \cdot)\rangle, \quad x \in \mathbb{R}^d.$$

If g is a locally summable function, this definition coincides with (1.8.4). The operators $g \to g^{(\varepsilon)}$ defined on $\mathcal{H}$ are the so-called *Sobolev mollifiers*.

For Sobolev spaces the following is proved in Theorem 1.8.5.

2. Theorem. *If $g \in H_p^\gamma$, then $g^{(\varepsilon)} \in H_p^\gamma$, $g^{(\varepsilon)} \in C^n$ for any n, and*

$$\|g^{(\varepsilon)}\|_{H_p^\gamma} \le \|g\|_{H_p^\gamma} \int_{\mathbf{R}^d} |\zeta|\, dx. \tag{2}$$

Furthermore, $g^{(\varepsilon)} \to g$ in H_p^γ as $\varepsilon \downarrow 0$.

Proof. We know from Theorem 1 that $g^{(\varepsilon)} \in \mathcal{H} \cap C^n$ and

$$(1-\Delta)^{\gamma/2} g^{(\varepsilon)} = h^{(\varepsilon)},$$

where $h = (1-\Delta)^{\gamma/2} g \in \mathcal{L}_p$. Since

$$\|h^{(\varepsilon)}\|_{\mathcal{L}_p} = \|h * \zeta_\varepsilon\|_{\mathcal{L}_p} \le \|h\|_{\mathcal{L}_p} \|\zeta_\varepsilon\|_{\mathcal{L}_1} \le \|h\|_{\mathcal{L}_p} \|\zeta\|_{\mathcal{L}_1},$$

we get that $h^{(\varepsilon)} \in \mathcal{L}_p$, $g^{(\varepsilon)} \in H_p^\gamma$, and (2) holds.

To prove the last assertion of the theorem, we notice that, for any bounded continuous function ϕ with compact support and $\varepsilon \in (0,1)$, the functions $\phi^{(\varepsilon)}$ are bounded by a constant independent of ε, have support in a ball independent of ε, and for any x satisfy

$$\phi^{(\varepsilon)}(x) = \int_{\mathbf{R}^d} \phi(x - \varepsilon y)\zeta(y)\, dy \to \phi(x)$$

as $\varepsilon \downarrow 0$ by the dominated convergence theorem. By the same theorem and above properties we have $\phi^{(\varepsilon)} \to \phi$ in $\mathcal{L}_p$. Then by inspecting

$$\varlimsup_{\varepsilon \downarrow 0} \|h - h^{(\varepsilon)}\|_{\mathcal{L}_p} \le \|h - \phi\|_{\mathcal{L}_p} + \varlimsup_{\varepsilon \downarrow 0} \|\phi - \phi^{(\varepsilon)}\|_{\mathcal{L}_p} + \varlimsup_{\varepsilon \downarrow 0} \|(\phi - h)^{(\varepsilon)}\|_{\mathcal{L}_p}$$

$$\le (1 + \|\zeta\|_{\mathcal{L}_1}) \|h - \phi\|_{\mathcal{L}_p}$$

and using the fact that the set of bounded continuous functions with compact support is dense in $\mathcal{L}_p$, we conclude that $h^{(\varepsilon)} \to h$ in $\mathcal{L}_p$. This is to say that

$$(1-\Delta)^{\gamma/2} g^{(\varepsilon)} \to (1-\Delta)^{\gamma/2} g$$

in $\mathcal{L}_p$ and $g^{(\varepsilon)} \to g$ in H_p^γ. The theorem is proved.

10. Hints to exercises

1.9. Observe that for $\phi \in C_0^\infty$

$$\langle g, \phi\rangle = \langle (1-\Delta)^{-\gamma/2}h, \phi\rangle = (h, (1-\Delta)^{-\gamma/2}\phi)$$

and $(1-\Delta)^{-\gamma/2}\phi = G * \phi$.

3.13. Use (3.6).

3.14. Both parts of (3.8) can be viewed as the results of applying certain continuous operators to u and for $u \in C_0^2(\mathbb{R}_+^d)$ conclude the argument by using the Fourier transform with respect to the x' variable as in Section 12.3.

3.15. Observe that, even though we only know that $u(0,\cdot) \in W_p^1(\mathbb{R}^{d-1})$, we also know that $W_p^1(\mathbb{R}^{d-1}) = H_p^1(\mathbb{R}^{d-1})$ and

$$\sum_{j,k=2}^{d} a^{jk}u_{x^jx^k}(0,\cdot) \in H_p^{-1}(\mathbb{R}^{d-1}).$$

While proving the assertions about z, Remark 7.3 might help.

3.16. We only comment on the first part of the exercise. We have $u \in H_p^\gamma$ if and only if $G * u \in \mathcal{L}_p$, where G is the Green's function of $(1-\Delta)^{-\gamma/2}$. Observe that for $|x| \geq 2\rho$,

$$|G * u(x)| \leq N(d,\gamma,n,\rho)|x|^{-m}\|u\|_{\mathcal{L}_1}$$

for any m. For $|x| \leq 2\rho$ the function $G*u(x)$ is dominated by a constant $N_0N(d,\gamma)$ times

$$\int_{\mathbb{R}^d} |x-y|^{-(d+\gamma)}|y|^{-\nu}\,dy = c|x|^{-\gamma-\nu},$$

where c is a finite constant, and $|x|^{-\gamma-\nu}I_{|x|\leq 2\rho} \in \mathcal{L}_p$.

3.17. If $\chi = 2$, so that $2r-2 < d$, use the inequality

$$\|D^\alpha(g\zeta)\|_{H_2^{-1}} \leq N\|g\zeta\|_{\mathcal{L}_2}, \quad |\alpha| = 1,$$

and if $\chi = 3$, use the first part of Exercise 3.16 and the estimate

$$\|D^\alpha(g\zeta)\|_{H_2^{-3/2}} \leq N\|g\zeta\|_{H_2^{-1/2}}.$$

3.19. Use Theorem 9.1.1 with $k = 1$ and $(1-\Delta)^{(\gamma-1)/2}f$ in place of f.

3.20. To estimate $N_{d+2}(\sigma_\varepsilon, 0)$, first prove that for the derivative of any order n we have

$$\Big|\Big(\frac{s^\nu}{1+s^\gamma}\Big)^{(n)}\Big| \leq N_n s^{-n},$$

where N_n is independent of $s \geq 0$. By taking $s = \varepsilon^\delta t$, $\delta = \nu^{-1}(1-\theta)^{-1}$, conclude that

$$\Big|\Big(\frac{t^\nu}{\varepsilon^{-1/(1-\theta)} + \varepsilon^{1/\theta}t^\gamma}\Big)^{(n)}\Big| \leq N_n t^{-n}.$$

Then use the formula

$$D^\alpha[h(\tau(\xi))] = \sum_{|\beta|+|\gamma|=|\alpha|} c^{\beta,\gamma} h^{(|\beta|)}(\tau(\xi))\eta_\gamma(\xi)$$

where $h = h(t)$ is a function of one variable t, τ is a symbol of order 1, the $c^{\beta,\gamma}$ are some constants unrelated to h or τ, and the η_γ are symbols of order $1 - |\gamma|$ unrelated to h.

3.21. First use the Fourier transform to prove the inequality with $\|u_x\|_{\mathcal{L}_2} + \|u_{xx}\|_{\mathcal{L}_2}$ on the right and then use dilations. To do the first step, consider two sets of integration in the inverse Fourier transform: $|\xi| \le 1$ and its complement. Also observe that $|\xi|^{-1}$ and $|\xi|^{-2}$ are square integrable at the origin and at infinity, respectively.

3.22. If $u = (1-\Delta)^{-\gamma/2}h$, then $h \in \mathcal{L}_p$, $h = (1-\Delta)^m(G * u)$, where m is large and G is the kernel of $(1-\Delta)^{\gamma/2-m}$. Derive from this that $h(x)$ decreases exponentially fast as $|x| \to \infty$.

3.23. We have a candidate for h in the formula

$$\langle \delta_0, \phi \rangle = \int_{\mathbf{R}^d} h(x)(1-\Delta)^{\gamma/2}\bar{\phi}\, dx.$$

In the case of general G use Theorem 1.7.

3.24. First take $g \in C_0^\infty(\mathbf{R}^{d-1})$ and prove and use the formula

$$\langle \check{g}, (1-\Delta)^{\gamma/2-1/4}\phi \rangle = \int_{\mathbf{R}^{d-1}} g(x')((1-\Delta)^{\gamma/2-1/4}\bar{\phi})(0,x')\, dx'$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}^d} \tilde{g}(\xi')(1+|\xi|^2)^{\gamma/2-1/4}\bar{\tilde{\phi}}(\xi)\, d\xi$$

to find the $H_2^{\gamma-1/2}$ norm of $\check{g}$. To prove the second equality above, write down what $\tilde{f}(0,\xi')$ is.

3.25. On the set $\Psi = (1-\Delta)^{-\gamma/2}C_0^\infty$ define the function $h((1-\Delta)^{-\gamma/2}\phi) = \langle g, \phi \rangle$ and observe that Ψ is dense in $\mathcal{L}_p$ and $h(\psi) \le N\|\psi\|_{\mathcal{L}_q}$ if $\psi \in \Psi$.

3.27. If $\gamma > 0$, the standard way, by using Riesz's representation theorem, produces a function belonging to at least $\mathcal{L}_2$. The one described in Exercise 3.26, generally, yields a distribution.

3.28. Recall that if $p \in (1,\infty)$, $h \in \mathcal{L}_p$, and A is any dense subset of $\mathcal{L}_q$ with $q = p/(p-1)$, then

$$\|h\|_{\mathcal{L}_p} = \sup_{\phi \in A, \phi \ne 0} \frac{(h,\phi)}{\|\phi\|_{\mathcal{L}_q}}.$$

3.29. If it is, then $H_p^\gamma = \mathcal{L}_p$ since $S \subset H_p^\gamma$. Then $(1-\Delta)^{-n\gamma/2}\mathcal{L}_p = \mathcal{L}_p$ for all $n = 1, 2, \ldots$ and hence for all $n = -1, -2, \ldots$. Now use Exercise 3.23.

5.1. First assume that $b^j = c \equiv 0$. Then prove that for $|\arg \nu| \le \theta$ and $|\nu| = 1$ and real t and $\xi \in \mathbb{R}^d$ such that $t^2 + |\xi|^2 = 1$ we have

$$\inf_x |a^{jk}\xi^j\xi^k + \nu t^2| > 0$$

and conclude that for all $x, \xi \in \mathbb{R}^d$ and $\nu \in \Gamma_{\theta,1}$

$$|a^{jk}\xi^j\xi^k + \nu| \ge \kappa_1(1 + |\xi|^2),$$

with $\kappa_1 > 0$ independent of ξ, x, ν. After that use Theorem 4.5 and Lemma 4.4.

5.2. Use the fact that, by definition, $z\mathcal{R}_z = L\mathcal{R}_z + 1$. Also use Jordan's lemma to show that for $|t| \le 1$

$$T_t f = \frac{1}{2\pi i}\int_{\partial\Gamma_{\theta,\nu_0}} e^z t^{-1}\mathcal{R}_{z/t} f\, dz.$$

This allows one to get an estimate $\|T_t f\|_{\mathcal{L}_p} \le N\|f\|_{\mathcal{L}_p}$, $|t| \le 1$, $|\arg t| \le \theta - \pi/2$, and, while proving that $\|T_t f - f\|_{\mathcal{L}_p} \to 0$ as $t \downarrow 0$, concentrate on $f \in W_p^2$ when $\mathcal{R}_z f = z^{-1}\mathcal{R}_z L f + z^{-1} f$. A complete argument can be found in [**5**].

7.1. Express u in Ω in terms of its values on $\partial\Omega$ by using the Fourier transform.

7.6. The hint to Exercise 3.24 helps find $\tilde{v}$.

8.2. Use dilations.

8.3. Use Exercise 12.10.5 and the hint to Exercise 10.2.8.

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