Lectures on Elliptic Boundary Value Problems

Shmuel Agmon

AMS CHELSEA PUBLISHING
American Mathematical Society • Providence, Rhode Island

Lectures on Elliptic Boundary Value Problems

Lectures on

ELLIPTIC BOUNDARY VALUE PROBLEMS

by

SHMUEL AGMON
Professor of Mathematics
The Hebrew University of Jerusalem

Prepared for publication by B. Frank Jones, Jr.
with the assistance of George W. Batten, Jr.

D. VAN NOSTRAND COMPANY, INC.
PRINCETON, NEW JERSEY
TORONTO NEW YORK LONDON

D. VAN NOSTRAND COMPANY, INC.
120 Alexander St., Princeton, New Jersey (*Principal office*)
24 West 40 Street, New York 18, New York

D. VAN NOSTRAND COMPANY, LTD.
358, Kensington High Street, London, W.14, England

D. VAN NOSTRAND COMPANY (Canada), LTD.
25 Hollinger Road, Toronto 16, Canada

Published simultaneously in Canada by
D. VAN NOSTRAND COMPANY (Canada), LTD.

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

This book reproduces with few corrections notes of lectures given at the Summer Institute for Advanced Graduate Students held at the William Rice University from July 1, 1963, to August 24, 1963. The Summer Institute was sponsored by the National Science Foundation and was directed by Professor Jim Douglas, Jr., of Rice University.

The subject matter of these lectures is elliptic boundary valued problems. In recent years considerable advances have been made in developing a general theory for such problems. It is the purpose of these lectures to present some selected topics of this theory. We consider elliptic problems only in the framework of the L_2 theory. This approach is particularly simple and elegant. The hard core of the theory is certain fundamental L_2 differential inequalities.

The discussion of most topics, with the exception of that of eigenvalue problems, follows more or less along well-known lines. The treatment of eigenvalue problems is perhaps less standard and differs in some important details from that given in the literature. This approach yields a very general form of the theorem on the asymptotic distribution of eigenvalues of elliptic operators.

Only a few references are given throughout the text. The literature on elliptic differential equations is very extensive. A comprehensive bibliography on elliptic and other differential problems is to be found in the book by J. L. Lions, Equations différentielles opérationelles, Springer-Verlag, 1961.

These lectures were prepared for publication by Professor B. Frank Jones, Jr., with the assistance of Dr. George W.Batten, Jr. I am greatly indebted to them both. Professor Jones also took upon himself the trouble of inserting explanatory and complementary material in several places. I am particularly grateful to him. I would also like to thank Professor Jim Douglas for his active interest in the publication of these lectures.

Shmuel Agmon

Jerusalem

CONTENTS

Section

0. Notations and Conventions

The following notations and conventions will be used. E_n will denote real n-space. For any points $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$ $\in E_n$,

$$|x| = (x_1^2 + \ldots + x_n^2)^{1/2}, \quad x \cdot y = x_1 y_1 + \ldots + x_n y_n.$$

For an index or exponent $a = (a_1,\ldots,a_n)$, whose components are integers, $|a| = a_1 + \ldots + a_n$; from the context it will be clear whether this norm or the euclidean norm is intended. Also, $a! = a_1!\ldots a_n!$; for $\beta = (\beta_1,\ldots,\beta_n)$,

$$\binom{a}{\beta} = \frac{a!}{\beta!\,(a-\beta)!} = \frac{a_1!\ldots a_n!}{\beta_1!\ldots\beta_n!\,(a_1-\beta_1)!\ldots(a_n-\beta_n)!}$$

For any $x \in E_n$, $a = (a_1,\ldots,a_n)$,

$$x^\alpha = x_1^{\alpha_1}\ldots x_n^{\alpha_n}.$$

In particular, this will be used with the differentiation operator:

$$D^\alpha = D_1^{\alpha_1}\ldots D_n^{\alpha_n} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

Also $a \leq \beta \Leftrightarrow a_1 \leq \beta_1,\ldots,a_n \leq \beta_n$.

These conventions greatly simplify many expressions. For example, Taylor's series for a function $f(x)$ has the form

$$\sum_\alpha \frac{1}{a!} D^\alpha f(0)\, x^\alpha.$$

We will also find it convenient to have a special set inclusion: $\subset\subset$ We will writ $\Omega_1 \subset\subset \Omega_2$ if and only if Ω_1 and Ω_2 are open, $\overline{\Omega}_1$ is compact and $\overline{\Omega}_1 \subset \Omega_2$.

For any function u, the notation supp (u) will be used to denote the support of u; i.e., the closure of the set $\{x : u(x) \neq 0\}$.

1. Calculus of L_2 Derivatives–Local Properties

In this section Ω is an open set in n-dimensional Euclidean space E_n. For any non-negative integer m, $C^m(\Omega)$ is the class of m times continuously differentiable (complex-valued) functions on Ω and $C^\infty(\Omega) = \bigcap_{m=0}^{\infty} C^m(\Omega)$. Also $C_0^\infty(\Omega)$ is the subset of $C^\infty(\Omega)$ consisting of functions having compact support contained in Ω. The functions in $C_0^\infty(\Omega)$ will be called *test functions for* Ω. Now some norms and semi-norms will be defined.

Definition 1.1. *For* $u \in C^m(\Omega)$

$$||u||_{m,\Omega} = [\int_\Omega \sum_{|\alpha|\leq m} |D^\alpha u|^2 dx]^{1/2}.$$

If $||u||_{m\Omega}$, $||v||_{m,\Omega}$ *are both finite, then there exists*

$$(u, v)_{m,\Omega} = \int_\Omega \sum_{|\alpha|\leq m} \overline{D^\alpha u}\, D^\alpha v dx.$$

Likewise, for $u \in C^m(\Omega)$,

$$|u|_{m,\Omega} = [\int_\Omega \sum_{|\alpha|=m} |D^\alpha u|^2 dx]^{1/2}$$

This latter quantity is of course only a semi-norm: $|u|_{m,\Omega} = 0 \not\Rightarrow u = 0$. The notation $(u, v)_m = (u, v)_{m,\Omega}$, $||u||_m = ||u||_{m,\Omega}$ $|u|_m = |u|_{m,\Omega}$ may also be used if Ω is fixed during a discussion.

Definition 1.2. $C^{m*}(\Omega)$ *is the subset of* $C^m(\Omega)$ *consisting of functions u with* $||u||_{m,\Omega} < \infty$.

Definition 1.3. $H_m(\Omega)$ *is the completion of* $C^{m*}(\Omega)$ *with respect to the norm* $||\ \ ||_{m,\Omega}$.

Since $C^{m*}(\Omega)$ is obviously an inner product space with respect to the inner product $(u, v)_{m,\Omega}$, it follows that $H_m(\Omega)$ is itself a Hilbert space. According to the general concept of completion for a metric space, the members of $H_m(\Omega)$ can be assumed to be equivalence classes of Cauchy sequences of elements of $C^{m*}(\Omega)$. However, in the present case another characterization of $H_m(\Omega)$ is possible. Indeed, if $\{u_k\}$ is a Cauchy sequence in $C^{m*}(\Omega)$, then for any fixed α, $|\alpha| \leq m$, it follows from the inequality

$$[\int_\Omega |D^\alpha u_k - D^\alpha u_j|^2 dx]^{1/2} \leq ||u_k - u_j||_{m,\Omega}$$

that $\{D^\alpha u_k\}$ is a Cauchy sequence in $L_2(\Omega)$. As $L_2(\Omega)$ is complete, there exists $u^\alpha \in L_2(\Omega)$ such that $D^\alpha u_k \to u^\alpha$ in $L_2(\Omega)$. Therefore, every element of $H_m(\Omega)$ can be considered to be a function $u \in L_2(\Omega)$ with the property that there exists a sequence $\{u_k\} \subset C^{m*}(\Omega)$ such that $\{D^\alpha u_k\}$ is a Cauchy sequence in $L_2(\Omega)$, $|\alpha| \leq m$, and $u_k \to u$ in $L_2(\Omega)$. More will be said about this identification below. First, two kinds of derivatives will be defined.

Definition 1.4. *A function $u \in L_2(\Omega)$ has strong L_2 derivatives of order up to m if there exists a sequence $\{u_k\} \subset C^{m*}(\Omega)$ such that $\{D^\alpha u_k\}$ is a Cauchy sequence in $L_2(\Omega)$ for $|\alpha| \leq m$, and $u_k \to u$ in $L_2(\Omega)$.*

Suppose that u has strong L_2 derivatives of order up to m, and let $D^\alpha u_k \to u^\alpha$ in $L_2(\Omega)$, $|\alpha| \leq m$. If $\phi \in C_0^\infty(\Omega)$, then integrating by parts, using the fact that $\phi \equiv 0$ in a neighborhood of $\partial\Omega$,

$$\int_\Omega \phi D^\alpha u_k \, dx = (-1)^{|\alpha|} \int_\Omega u_k D^\alpha \phi dx.$$

Then, letting $k \to \infty$,

$$\int_\Omega \phi u^\alpha dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi dx, \; |\alpha| \leq m. \tag{1.1}$$

Taking this last relation on its own merit, the following definition is made.

Definition 1.5. *A locally integrable function u on Ω is said to have the weak derivative u^α if u^α is locally integrable on Ω and*

$$\int_\Omega \phi u^\alpha dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi dx, \text{ all } \phi \in C_0^\infty(\Omega). \tag{1.2}$$

A few results are immediate.

THEOREM 1.1. *If $u \in L_2(\Omega)$ has strong L_2 derivatives of order up to m, then u has weak L_2 derivatives of order up to m.*

This result is an immediate consequence of (1.1).

THEOREM 1.2. *Weak derivatives are unique. That is, if u has the weak derivative u^α and also the weak derivative v^α, then $u^\alpha = v^\alpha$ a.e.*

Proof. From (1.2) it follows that $\int_\Omega \phi(u^\alpha - v^\alpha) dx = 0$ for all $\phi \in C_0^\infty(\Omega)$. As $C_0^\infty(\Omega)$ is dense in $L_1(C)$ for any compact subset C of Ω, $u^\alpha - v^\alpha = 0$ a.e. Q. E. D.

Corollary. Strong derivatives are unique.

This theorem justifies the following notation for strong and weak derivatives. If u has the weak derivative u^α, write $u^\alpha = D^\alpha u$. Likewise, if u has strong L_2 derivatives, and if $\{u_k\} \subset C^{m*}(\Omega)$,

$u_k \to u$ in $L_2(\Omega)$, and $D^\alpha u_k \to u^\alpha$ in $L_2(\Omega)$, define $u^\alpha = D^\alpha u$. According to the above corollary, $D^\alpha u$ is independent of which sequence $\{u_k\}$ is chosen. Also, it is seen that if u has strong L_2 derivatives $D^\alpha u$, then u also has the weak L_2 derivatives $D^\alpha u$.

Definition 1.6. *$W_m(\Omega)$ is the class of functions in $L_2(\Omega)$ which have weak derivatives in $L_2(\Omega)$ of order up to m. If $u, v \in W_m(\Omega)$,*

$$(u, v)_{m,\Omega} = \int_\Omega \sum_{|\alpha| \leq m} D^\alpha u \overline{D^\alpha v}\, dx.$$

Since strong derivatives are unique, we will identify the class $H_m(\Omega)$ with the class of functions in $L_2(\Omega)$ which have strong L_2 derivatives of order up to m.

It is clear that if a function has continuous pointwise derivatives of certain order, these derivatives are also weak and strong derivatives.

Clearly, $W_m(\Omega)$ and $H_m(\Omega)$ are linear spaces of functions, and in either class

$$D^\alpha(u + v) = D^\alpha u + D^\alpha v,$$
$$D^\alpha(cu) = cD^\alpha u, \quad c \text{ constant}.$$

THEOREM 1.3. *$W_m(\Omega)$ is a Hilbert space.*

Proof. It suffices to show that if $\{u_k\} \subset W_m(\Omega)$ and if $u_k \to u$, $D^\alpha u_k \to u^\alpha$ in $L_2(\Omega)$, then $u \in W_m(\Omega)$ and $D^\alpha u = u^\alpha$, $|a| \leq m$. But this is immediate upon writing (1.2) for $u = u_k$ and letting $k \to \infty$. Q. E. D.

For reference we display the following fact, which has been established.

THEOREM 1.4. $H_m(\Omega) \subset W_m(\Omega)$.

In order to treat local properties we now introduce the idea of mollification. Let $j(x) \in C^\infty(E_n)$ satisfy

$$j(x) \geq 0,\ j(x) \equiv 0 \text{ for } |x| \geq 1,\ \int_{E_n} j(x)\, dx = 1.$$

For example, j can be the function

$$j(x) = c \exp\left(-\frac{1}{1 - |x|^2}\right) \text{ for } |x| < 1,\ j(x) \equiv 0 \text{ for } |x| \geq 1$$

where c is a suitable constant. Let

$$j_\epsilon(x) = \frac{1}{\epsilon^n} j\left(\frac{x}{\epsilon}\right).$$

Note that $j_\epsilon(x)$ vanishes for $|x| \geq \epsilon$ and that

$$\int_{E_n} j_\epsilon(x)\,dx = 1.$$

Definition 1.7. *The mollifier J_ϵ is defined by*

$$J_\epsilon u(x) = \int_\Omega j_\epsilon(x - y)u(y)dy$$

for any locally integrable function u in Ω.

One sees readily that $J_\epsilon u(x)$ is defined at all points x with dist $(x, \partial\Omega) > \epsilon$. If u is also integrable on bounded open subsets of Ω, then $J_\epsilon u(x)$ is defined for all x. The importance of J_ϵ arises in the fact that $J_\epsilon u$ behaves much like u, but it is very smooth. This is stated precisely in the following theorems.

THEOREM 1.5. *If u is locally integrable in Ω and also integrable on bounded open subsets of Ω, then $J_\epsilon u \in C^\infty(E_n)$. If, in addition, the support of u is contained in K, a compact subset of Ω, and if $\epsilon <$* dist $(K, \partial\Omega)$, then $J_\epsilon u \in C_0^\infty(\Omega)$.

Proof. Continuity of $J_\epsilon u$ follows from the continuity of j_ϵ. Differentiation can be carried under the integral sign, so that the differentiability properties of $J_\epsilon u$ follow from those of j_ϵ. The last statement is obvious. Q. E. D.

THEOREM 1.6 . *If $u \in L_2(\Omega)$, then $\|J_\epsilon u\|_{0,\Omega} \leq \|u\|_{0,\Omega}$.*

Proof. By the Cauchy-Schwarz inequality,

$$|J_\epsilon u|^2 = \left| \int_\Omega [j_\epsilon(x - y)]^{1/2} \{ [j_\epsilon(x - y)]^{1/2} u(y) \} dy \right|^2$$

$$\leq \int_\Omega j_\epsilon(x - y)dy \int_\Omega j_\epsilon(x - y) \, | u(y) |^2 dy$$

$$\leq \int_\Omega j_\epsilon(x - y) \, | u(y) |^2 dy.$$

Hence, by Fubini's theorem,

$$\|J_\epsilon u\|_{0,\Omega}^2 \leq \int_\Omega \left[\int_\Omega j_\epsilon(x - y) \, | u(y) |^2 dy \right] dx$$

$$= \int_\Omega \left[|u(y)|^2 \int_\Omega j_\epsilon(x - y)dx \right] dy \leq \|u\|_{0,\Omega}^2. \quad \text{Q. E. D.}$$

THEOREM 1.7. *If $u \in L_2(\Omega)$ then $J_\epsilon u \to u$ in $L_2(\Omega)$ as $\epsilon \to 0$. If u is continuous at a point x, then $(J_\epsilon u)(x) \to u(x)$, the convergence being uniform on any compact set of continuity points.*

Proof. Extending u to E_n, lettering $u \equiv 0$ outside Ω, we can assume without loss of generality that $\Omega = E_n$.

Clearly

$$u(x) = \int_{E_n} j_\epsilon (x-y)\, u(y)\, dy;$$

hence

$$|(J_\epsilon u)(x) - u(x)| \le \int_{E_n} j_\epsilon (x-y)\, |u(y) - u(x)|\, dy$$

$$\le \sup_{|y-x|<\epsilon} |u(y) - u(x)|.$$

Since the last term tends to zero with ϵ at each continuity point x, the convergence to zero being uniform for any compact set of continuity points, the last part of the theorem follows.

Let now η be an arbitrary positive number. Choose $v_\eta \in C_0^\infty(E_n)$ such that

$$||u - v_\eta||_{0,\, E_n} < \eta.$$

By the preceding theorem,

$$||J_\epsilon u - J_\epsilon v_\eta||_{0,\, E_n} \le ||u - v_\eta||_{0,\, E_n}.$$

Moreover, if the support of v_η is contained in the sphere $|x| < R$, then by the first part of the proof $(J_\epsilon v_\eta)(x) \to v_\eta(x)$ uniformly for $|x| \le 2R$, while $J_\epsilon v_\eta = v_\eta \equiv 0$ for $|x| > 2R$ and $\epsilon < R$. Thus for ϵ sufficiently small

$$||J_\epsilon v_\eta - v_\eta||_{0,\, E_n} < \eta,$$

whence

$$||J_\epsilon u - u||_{0,\, E_n} < 2\eta.$$

That is, $J_\epsilon u \to u$ in $L_2(E_n)$ as $\epsilon \to 0$. Q. E. D.

THEOREM 1.8 *If* $u \in W_m(\Omega)$, *and if* $|\alpha| \le m$, *then* $(D^\alpha J_\epsilon u)(x) = (J_\epsilon D^\alpha u)(x)$ *for* $x \in \Omega$, *provided dist* $(x, \partial\Omega) > \epsilon$.

Proof. Under the conditions of the theorem, we have

$$(D^\alpha J_\epsilon u)(x) = D^\alpha \int_\Omega j_\epsilon (x-y)\, u(y)\, dy$$

$$= \int_\Omega D_x^\alpha j_\epsilon (x-y)\, u(y)\, dy$$

$$= (-1)^{|\alpha|} \int_\Omega D_y^\alpha j_\epsilon (x-y)\, u(y)\, dy$$

$$= \int_\Omega j_\epsilon (x-y)\, D^\alpha u(y)\, dy$$

$$= (J_\epsilon D^\alpha u)(x),$$

the fourth equality following from the definition of the weak derivative. Q. E. D.

Now a theorem will be stated to guarantee the existence of a "partition of unity."

THEOREM 1.9. *Let $F \subset E_n$ be compact, and let $F \subset \bigcup_{i=1}^{\nu} 0_i$, where each 0_i is open. Then there exist functions $\xi_i \in C_0^\infty(0_i)$ such that $\sum_{i=1}^{\nu} \xi_i(x) = 1$ for $x \in F$.*

Proof. First we will choose $\{C_i\}$, a collection of compact sets such that $C_i \subset 0_i$ and $F \subset \cup\, C_i$. This can be done as follows. For $x \in 0_i$, let $S(x, 2r_x)$ be the sphere centered at x and having radius $2r_x > 0$ chosen so that $S(x, 2r_x) \subset 0_i$. Since F is compact and covered by $\{S(x, r_x) : x \in F\}$, it is covered by a finite number of these spheres, say by $\{S(x_k, r_{x_k}) : k = 1, \ldots, m\}$.
Let

$$C_i = \bigcup_{x_k \in 0_i} \overline{S(x_k, r_{x_k})} \;.$$

Then C_i has the desired properties.

Now let $\{C_{i*}\}$ be a collection of compact sets satisfying $C_i \subset \text{int}\, C_{i*} \subset C_{i*} \subset 0_i$. Let ψ_{i*} be the function which equals 1 on C_{i*} and vanishes elsewhere. Choose $\epsilon > 0$ less than dist $(C_i, \partial C_{i*})$ and dist $(C_{i*}, \partial 0_i)$. Let

$$\psi_i = J_\epsilon \psi_{i*}.$$

Then $\psi_i \in C_0^\infty(0_i)$ and $\psi_i(x) = 1$ for $x \in C_i$. Finally, let

$$\xi_1 = \psi_1$$

$$\xi_i = (1-\psi_1)\cdot(1-\psi_2)\ldots(1-\psi_{i-1})\psi_i,\; i > 1 \;.$$

Then $\xi_i \in C_0^\infty(0_i)$ and

$$\sum_{i=1}^{\nu} \xi_i = 1 - (1 - \psi_1)\ldots(1 - \psi_\nu)$$

$$\equiv 1 \text{ on } \bigcup_{i=1}^{\nu} C_i \supset F. \text{ Q. E. D.}$$

Terminology: The collection of functions ξ_i is called a *partition of unity subordinate to the open covering* $\{0_i\}$ *of* F.

Definition 1.8. $u \in W_m^{loc}(\Omega)$ $[H_m^{loc}(\Omega)]$ *if any* $x \in \Omega$ *has an open neighborhood* $0 \subset \Omega$ *such that* $u \in W_m(0)$ $[H_m(0)]$.

Note that $H_m^{loc}(\Omega) \subset W_m^{loc}(\Omega)$. For $m = 0$: $H_0^{loc}(\Omega) = W_0^{loc}(\Omega)$ $= L_2^{loc}(\Omega)$

THEOREM 1.10. *If* $u \in W_m(\Omega)$ $[H_m(\Omega)]$, *then for any* $\Omega_1 \subset\subset \Omega$ $J_\epsilon u \to u$ in $W_m(\Omega_1)$ $[H_m(\Omega_1)]$ *as* $\epsilon \to 0$. •

Proof. Clearly $J_\epsilon u$ is well defined in Ω_1 for $\epsilon > 0$ sufficiently small and it is enough to show that $D^\alpha J_\epsilon u \to D^\alpha u$ in $L_2(\Omega_1)$ for $0 \le |\alpha| \le m$. This however follows immediately from Theorem 1.8 and Theorem 1.7 since for ϵ sufficiently small

$$D^\alpha J_\epsilon u(x) = J_\epsilon D^\alpha u(x) \to D^\alpha u$$

in $L_2(\Omega_1)$. Q. E. D.

THEOREM 1.11. *If* $u \in W_m^{loc}(\Omega)$ $[H_m^{loc}(\Omega)]$, *then* $u \in W_m(\Omega_1)$ $[H_m(\Omega_1)]$ *for any* $\Omega_1 \subset\subset \Omega$.

Proof. Suppose $u \in W_m^{loc}(\Omega)$. Since $\overline{\Omega}_1$ is compact, it can be covered by a finite number if open subsets $0_1,\ldots,0_k$ of Ω, with $u \in W_m(0_i)$, $i = 1,\ldots,k$. Thus the restriction of u to 0_i admits weak derivatives $D^\alpha u \in L_2(0_i)$, $|\alpha| \le m$, which we denote by u_i^α. From the uniqueness of weak derivatives it follows further that $u_i^\alpha = u_j^\alpha$ almost everywhere in $0_i \cap 0_j$, $1 \le i, j \le k$. Thus after correction on null-sets we obtain functions $u^\alpha \in L_2(\bigcup_{i=1}^{k} 0_i)$ such that $u^\alpha =$ the weak derivative $D^\alpha u$ in $W_m(0_i)$, $|\alpha| \le m$ and $i = 1,\ldots,k$.

Let now $\{\xi_i\}$ be a partition of unity subordinate to $\{0_i\}$; $\sum_1^k \xi_i = 1$ on $\overline{\Omega}_1$. For any $\phi \in C_0^\infty(\Omega_1)$, let $\phi_i = \phi\,\xi_i$, $i = 1,\ldots,k$; then ϕ_i is a test function on 0_i and

$$\phi = \sum_{i=1}^{k} \phi_i \text{ on } \Omega_1.$$

Since $u \in W_m(0_i)$, and since the support of ϕ_i is contained in $0_i \cap \Omega_1$,

$$\int_{\Omega_1} u^\alpha \cdot \phi_i dx = (-1)^{|\alpha|} \int_{\Omega_1} uD^\alpha \phi_i dx$$

for $|\alpha| \leq m$, $i = 1,\ldots,k$. After summing on i, we have

$$\int_{\Omega_1} u^\alpha \cdot \phi dx = (-1)^{|\alpha|} \int_{\Omega_1} u \cdot D^\alpha \phi dx, \quad |\alpha| \leq m.$$

Thus, $u \in W_m(\Omega_1)$.

Suppose $u \in H_m^{\text{loc}}(\Omega) \subset W_m^{\text{loc}}(\Omega)$. Then by the first part of the proof, $u \in W_m(\Omega_2)$ with $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$. Then by the corollary to Theorem 1.10, $J_\epsilon u \to u$ in $W_m(\Omega_1)$. Since $J_\epsilon u \in C^\infty(\Omega_1)$, this implies $u \in H_m(\Omega_1)$. Q.E.D.

Corollary. If $\Omega_1 \subset\subset \Omega$, then the functions in $W_m^{\text{loc}}(\Omega)$, restricted to Ω_1, are in $H_m(\Omega_1)$; i.e., $u \in W_m^{\text{loc}}(\Omega)$ implies $u \in H_m(\Omega_1)$.

Proof. If $u \in W_m^{\text{loc}}(\Omega)$, then $J_\epsilon u \to u$ in $W_m(\Omega_1)$ as $\epsilon \to 0$ and $J_\epsilon u \in C^\infty$ by Theorem 1.5. Q.E.D.

THEOREM 1.12. $H_m^{\text{loc}}(\Omega) = W_m^{\text{loc}}(\Omega)$.

Proof. This follows immediately from the above corollary. Q.E.D.

THEOREM 1.13. (*Leibnitz's rule*). *If $u \in W_m(\Omega)$ $[H_m(\Omega)]$, and if $v \in C^m(\Omega)$ has bounded derivatives of all orders $\leq m$, then $uv \in W_m(\Omega)$ $[H_m(\Omega)]$ and*

$$(*) \quad D^\alpha(uv) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta} v.$$

Proof. If $u \in H_m(\Omega)$, then there is a sequence $\{u_k\} \subset C^{m*}(\Omega)$ such that $D^\alpha u_k \to D^\alpha u$ in $L_2(\Omega)$ for $0 \leq |\alpha| \leq m$. Since v has bounded derivatives in Ω,

$$D^\alpha(u_k v) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta u_k D^{\alpha-\beta} v$$

$$\to \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta} v$$

in $L_2(\Omega)$ for $0 \leq |\alpha| \leq m$, Hence the theorem is proved for $H_m(\Omega)$.

Now suppose $u \in W_m(\Omega)$. For any test function ϕ on Ω, let $\Omega_1 \subset\subset \Omega$ be a set containing the support of ϕ. By the corollary to Theorem 1.11, $u \in H_m(\Omega_1)$. Thus by the result for strong derivatives $uv \in H_m(\Omega_1) \subset W_m(\Omega_1)$ and (*) holds on Ω_1. Thus $uv \in W_m(\Omega_1)$ for every $\Omega_1 \subset\subset \Omega$ and (*) holds on each Ω_1. But since $D^{\alpha-\beta}v$ is bounded and $D^\beta u \in L_2(\Omega)$, $D^\alpha(uv) \in L_2(\Omega)$ by (*). But any function in $W_m^{loc}(\Omega)$ whose derivatives up to order m are square integrable over Ω is in $W_m(\Omega)$. Q.E.D.

Corollary. Functions $u \in W_m(\Omega)$ having compact support in E_n are dense in $W_m(\Omega)$.

Proof. Let $\zeta(x) \in C^\infty(E_n)$ *satisfy* $0 \leq \zeta(x) \leq 1$ *and*

$$\zeta(x) = \begin{matrix} 1 \text{ for } |x| \leq 1, \\ 0 \text{ for } |x| \geq 2. \end{matrix}$$

Let $u_N(x) = \zeta(\frac{x}{N})\, u(x)$. By Leibnitz's rule $u_N \in W_m(\Omega)$, and u_N has compact support in E_n. Moreover, for $|\alpha| \leq m$

$$D^\alpha u_N = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta[\zeta(\tfrac{x}{N})]\, D^{\alpha-\beta} u \to D^\alpha u$$

in $L_2(\Omega)$ as $N \to \infty$. For

$$D^\alpha u_N - D^\alpha u = \sum_{\substack{\beta \leq \alpha \\ \beta \neq 0}} \binom{\alpha}{\beta} \frac{1}{N^{|\beta|}} (D^\beta \zeta)(\tfrac{x}{N})\, D^{\alpha-\beta} u + [\zeta(\tfrac{x}{N}) - 1]\, D^\alpha u;$$

therefore by the triangle inequality

$$\|D^a u_N - D^a u\|_{0,\Omega} \le \sum_{\substack{\beta\le a\\ \beta\ne 0}} \binom{a}{\beta} \frac{1}{N} \max_x |D^\beta \zeta(x)|\, \|D^{a-\beta} u\|_{0,\Omega} + \Big[\int_{|x|>N} |D^a u|^2 dx\Big]^{1/2}$$

$$\to 0$$

as $N \to \infty$. Q.E.D.

2. Calculus of L_2 Derivatives–Global Properties

We wish to extend the local property that $H_m^{loc}(\Omega) = W_m^{loc}(\Omega)$ to a global property; i.e., to establish that $H_m(\Omega) = W_m(\Omega)$. This result which is true in general we shall prove only with some restrictions on Ω.

Definition 2.1. *Ω has the ordinary cone property if there is a cone C such that for each point $x \in \Omega$ there is a cone $C_i \subset \Omega$ with vertex x congruent to C. Ω has the restricted cone property if $\partial\Omega$ has a locally finite open covering $\{0_i\}$ and corresponding cones $\{C_i\}$ with vertices at the origin and the property that $x + C_i \subset \Omega$ for $x \in \Omega \cap 0_i$. Ω has the segment property if $\partial\Omega$ has a locally finite open covering $\{0_i\}$ and corresponding vectors $\{y^i\}$ such that for $0 < t < 1$, $x + ty^i \in \Omega$ for $x \in \bar{\Omega} \cap 0_i$.*

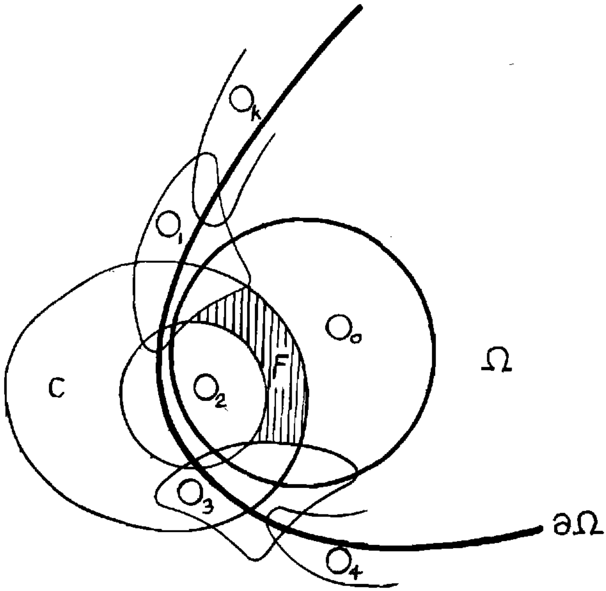
If Ω is bounded and has the restricted cone property, then Ω has the ordinary cone property. It is also easily seen that if Ω has the restricted cone property, then Ω has the segment property.

We will show that $H_m(\Omega) = W_m(\Omega)$ if Ω has the segment property. First, however, we prove the following approximation theorem, which is of interest by itself.

THEOREM 2.1. *If Ω has the segment property and if $u \in W_m(\Omega)$, then there is a sequence $\{u_k\} \subset C_0^\infty(E_n)$ such that $u_k \to u$ in $W_m(\Omega)$.*

Proof. Since by the corollary to Theorem 1.13 functions with compact support are dense in $W_m(\Omega)$, assume, without loss of generality, that u has its support in a compact set C. There are two cases: either i) $C \subset \Omega$, or ii) $C \cap \partial\Omega \ne 0$. In case i) we obviously have $J_\epsilon u \to u$ in $W_m(\Omega)$.

Case ii) is more difficult. Let $\{0_i\}$ be the locally finite open covering of $\partial\Omega$ guaranteed by the segment property. Let $F = C \cap (\bar{\Omega} - \cup\, 0_i)$. Then F is compact and $F \subset \Omega$. Choose 0_0 so that $F \subset 0_0 \subset\subset \Omega$. As



$C \cap \partial\Omega$ is compact, a finite collection $\{0_i\}_1^k$ covers $C \cap \partial\Omega$. Then $\{0_i\}_0^k$ is an open covering of $C \cap \bar{\Omega}$, hence of the support of u. As in the proof of Theorem 1.9, let $\{0_i'\}_0^k$ be a covering of $C \cap \bar{\Omega}$ such that $0_i' \subset\subset 0_i$ for $i = 0,\ldots,k$. Let $\{\xi_i\}$ be a partition of unity subordinate to $\{0_i'\}_0^k$, $\sum_{i=0}^{k} \xi_i = 1$ on $C \cap \bar{\Omega}$, and let $u_i = u\xi_i$. Then $u = \Sigma u_i$, the support of u_i is contained in $0_i'$, and $u_i \in W_m(\Omega)$. Thus it is sufficient to prove the theorem for the functions u_i individually. Since $0_0 \subset\subset \Omega$, case i) takes care of u_0.

Thus it is sufficient to prove the theorem under the assumption that the support of u is contained in $0_1' \subset\subset 0_1$. Extend u to E_n, defining u to be zero outside $0_1'$. Note that $\delta = \text{dist}\ (0_1', \partial\, 0_1) > 0$. Let $\Gamma = \overline{0_1'} \cap \partial\,\Omega$. Then $u \in W_m(E_n - \Gamma)$. For let ϕ be any test function on $E_n - \Gamma$. Then $K = \text{supp}\ (\phi) \cap \text{supp}\ (u) \subset \Omega$. Let $\zeta \in C_0^\infty(\Omega)$ be 1 in K. Then $\zeta\phi$ is a test function on Ω and we have

$$\int_{E_n-\Gamma} uD^\alpha\phi dx = \int_K uD^\alpha(\zeta\phi)dx$$

$$= \int_\Omega uD^\alpha(\zeta\phi)dx$$

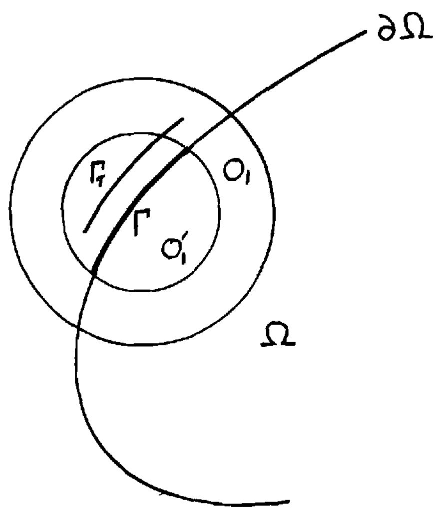
$$= (-1)^{|\alpha|}\int_\Omega D^\alpha u\,\zeta\phi dx$$

$$= (-1)^{|\alpha|}\int_K D^\alpha u\,\phi dx$$

$$= (-1)^{|\alpha|}\int_{E_n-\Gamma} D^\alpha u\,\phi dx,$$

where in the last integral we *define* $D^\alpha u$ as zero outside Ω. Hence $u \in W_m(E_n - \Gamma)$.

Let $\Gamma_r = \Gamma - ry^1$ where y^1 is the vector associated with 0_1 by the segment property and $0 < r < \min(1, \delta|y^1|^{-1})$. Then $\Gamma_r \subset 0_1$, for if $z \in \Gamma_r$ then dist $(z, 0_1') \le$ dist $(z, \Gamma) \le |ry^1| < \delta$. Also, $\Gamma_r \cap \bar\Omega = \emptyset$,



for if $z \in \Gamma_r \cap \bar\Omega$, then $z \in \bar\Omega \cap 0_1$ and $z + ry^1 \in \Gamma$, so that $z + ry^1 \in \Omega$, contradicting the segment property. Since Γ_r is compact, dist $(\Gamma_r, \Omega) >$

Let $u^\tau = u(x + \tau y^1)$. Then $u^\tau \in W_m(E_n - \Gamma_\tau)$; since $\Omega \subset E_n - \Gamma_\tau$, we have, *a fortiori*, $u^\tau \in W_m(\Omega)$. Since $(D^\alpha u^\tau)(x) = (D^\alpha u)(x + \tau y^1)$, $D^\alpha u^\tau \to D^\alpha u$ on $L_2(\Omega)$ by a familiar theorem on Lebesgue integration. Hence, $u^\tau \to u$ in $W_m(\Omega)$ as $\tau \to 0$. Thus it is sufficient to approximate u^τ by functions $u_k^\tau \in C_0^\infty(E_n)$. To obtain such a sequence we simply take the mollified sequence $u_k = J_{1/k}\, u$, $k = 1, 2, \ldots$. Clearly $u_k^\tau \in C_0^\infty(E_n)$ (Theorem 1.5). Since dist $(\Gamma_\tau, \Omega) > 0$ it follows from Theorem 1:8 that

$$(D^\alpha u_k^\tau)(x) = (J_{1/k} D^\alpha u^\tau)(x) \text{ for } x \in \Omega,\ |\alpha| \leq m$$

for all $k > [\text{dist}\,(\Gamma_\tau, \Omega)]^{-1}$. From this and from Theorem 1.7 we have: $D^\alpha u_k^\tau \to D^\alpha u^\tau$ in $L_2(\Omega)$ for $|\alpha| \leq m$. This shows that u_k^τ is the desired approximating sequence of u^τ. Q.E.D.

As a corollary we have

THEOREM 2.2. *If Ω has the segment property, then $H_m(\Omega) = W_m(\Omega)$.*

This theorem is important for obtaining properties of $H_m(\Omega)$, for frequently it is easier to obtain them for $W_m(\Omega)$. As an example, we prove

THEOREM 2.3. *If Ω has the segment property, if $u \in H_j(\Omega)$, and if $D^\alpha u \in H_k(\Omega)$ for $|\alpha| = j$, then $u \in H_{j+k}(\Omega)$.*

Proof. It is sufficient to prove the theorem for H replaced by W. This is trivial. Q.E.D.

Likewise, using the fact that $H_m^{\text{loc}}(\Omega) = W_m^{\text{loc}}(\Omega)$ for any Ω (Theorem 1.12), we have

THEOREM 2.4. *For any open set Ω, if $u \in H_j^{\text{loc}}(\Omega)$, and if $D^\alpha u \in H_k^{\text{loc}}(\Omega)$ for $|\alpha| = j$, then $u \in H_{j+k}^{\text{loc}}(\Omega)$.*

We will now extend the idea of generalized derivatives to differential operators. Let

$$A(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$$

be a linear differential operator with constant coefficients. Let

$$\tilde{A}(D) = \sum_{|\alpha| \leq m} a_\alpha (-1)^{|\alpha|} D^\alpha.$$

If $u \in C^m(\Omega)$, then integration by parts shows that

$$\int_\Omega u\tilde{A}\phi \, dx = \int_\Omega Au\phi \, dx$$

for any $\phi \in C_0^\infty(\Omega)$. More generally we make the following definition.

Definition 2.2. *Let u be a locally integrable function on Ω. If there exists a locally integrable function f on Ω such that*

$$\int_\Omega u\tilde{A}\phi \, dx = \int_\Omega f\phi \, dx$$

for all $\phi \in C_0^\infty(\Omega)$, then we will say that $Au = f$ exists weakly and that u is a weak solution of $Au = f$.

Note that the weak existence of Au *does not* imply the existence of any weak derivatives of u. However, weak existence does imply strong existence in the sense of the following theorem.

THEOREM 2.4. *If $u, f \in L_2^{\mathrm{loc}}(\Omega)$, and if u is a weak solution of $A_i u = f_i$, $i = 1,\dots,m$, on Ω, then there is a sequence $\{u_k\} \subset C^\infty(E_n)$ such that for any $\Omega_1 \subset\subset \Omega$, $u_k \to u$ and $A_i u_k \to f_i$ in $L_2(\Omega_2)$ as $k \to \infty$.*

Proof. Extend u to E_n, defining $u = 0$ outside Ω.
Let $u_k = J_{1/k} u$. As in Theorem 1.8, we see that

$$A_i J_{1/k} u(x) = J_{1/k} A_i u(x)$$

if dist $(x, \partial\Omega) > 1/k$. It follows from Theorem 1.7 that $A_i u_k =$

$A_i J_{1/k} u = J_{1/k} A_i u \to A_i u = f_i$ in $L_2(\Omega_1)$ as $k \to \infty$. Q.E.D.

Let $A^\alpha(\xi) = D_\xi^\alpha A(\xi)$. We will call $A^\alpha = A^\alpha(D)$ a *subordinate* of A.

We have the following analog of Leibnitz's rule.

THEOREM 2.5. *Let A be a linear differential operator of order m with constant coefficients. Suppose that $u \in L_2^{\mathrm{loc}}(\Omega)$ and that Au and $A^\alpha u$ exist weakly for $|\alpha| \le m$. If $\zeta \in C^m(\Omega)$, then $A(\zeta u)$ exists weakly and*

$$A(\zeta u) = \sum_{|\alpha| < m} \frac{D^\alpha \zeta}{\alpha!} A^\alpha u.$$

Proof. First assume that $u \in C^\infty(\Omega)$. Then by Leibnitz's rule we have

$$A(\zeta u) = \sum_{|\alpha| \le m} a_\alpha \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u$$

$$= \sum_{|\beta| \le m} D^\beta \zeta \sum_{\substack{\beta \le \alpha \\ |\alpha| \le m}} a_\alpha \binom{\alpha}{\beta} D^{\alpha-\beta} u$$

$$= \sum_{|\beta| \le m} \frac{D^\beta \zeta}{\beta!} A^\beta u,$$

since

$$A^\beta(\xi) = \sum_{|\alpha| \le m} a_\alpha D^\beta \xi^\alpha$$

$$= \sum_{\substack{\beta \le \alpha \\ |\alpha| \le m}} a_\alpha \frac{\alpha!}{(\alpha-\beta)!} \xi^{\alpha-\beta}.$$

The proof for arbitrary $u \in L_2^{\mathrm{loc}}(\Omega)$ is similar to that of Leibnitz's rule (Theorem 1.13), using Theorem 2.4. Q.E.D.

The following approximation theorem is analogous to Theorem 2.1.

THEOREM 2.6. *Let a be a collection of linear differential operators with constant coefficients, such that if $A \in a$ then for any α, $A^\alpha \in a$. Assume that Ω has the segment property, that $u \in L_2(\Omega)$, and that Au exists weakly for every $A \in a$. Then there exists a sequence $\{u_k\} \subset C_0^\infty(E_n)$ such that $u_k \to u$ and $Au_k \to Au$ in $L_2(\Omega)$ for every $A \in a$.*

Proof. Theorem 2.5 can be used to prove a density theorem analogous to the corollary to Theorem 1.13. The rest of the proof is just a repetition of that of Theorem 2.1. Q. E. D.

3. Some Inequalities

In this section we will derive some basic inequalities. We begin with some inequalities for functions on E_1.

THEOREM 3.1. *If $f \in C^2(a, b)$, then*

$$(3.1) \qquad \int_a^b |f'(x)|^2\,dx \le 54\left[\frac{1}{(b-a)^2}\int_a^b |f(x)|^2\,dx + (b-a)^2\int_a^b |f''(x)|^2\,dx\right].$$

If $f \in C^1(a, b)$, then

$$(3.2) \qquad \int_a^b |f(x)|^2\,dx \le 6\left[(b-a)^2\int_a^b |f'(x)|^2\,dx + \fint_a^b |f(x)|^2\,dx\right],$$

where $\fint_a^b = \int_{a+(b-a)/3}^{a+2(b-a)/3}$.

Proof. It is clearly sufficient to consider f a real function. Consider the case $a = 0$, $b = 1$. Let $0 < \alpha < 1/2$. For $x_1 < \alpha$, $1 - \alpha < x_2$, the mean value theorem gives

$$f'(\eta) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

for some $\eta \in (x_1, x_2)$; hence

$$|f'(\eta)| \le \frac{1}{(1-2\alpha)}\left(|f(x_2)| + |f(x_1)|\right).$$

Now

$$f'(x) = f'(\eta) + \int_\eta^x f''(\xi)\,d\xi,$$

so that

$$|f'(x)| \le \frac{1}{1-2\alpha}\left(|f(x_2)| + |f(x_1)|\right) + \int_0^1 |f''(\xi)|\,d\xi.$$

Integrate this inequality with respect to x_1 over $(0, a)$, and with respect to x_2 over $(1 - a, 1)$:

$$a^2 \, |f'(x)| \leq \frac{a}{1 - 2a} \left(\int_0^{\alpha} + \int_{1-\alpha}^{1}\right) |f(\xi)| \, d\xi + a^2 \int_0^1 |f''(\xi)| \, d\xi.$$

Squaring and using the Cauchy-Schwarz inequality twice, we obtain

$$|f'(x)|^2 \leq \frac{4}{a} \left(\frac{1}{1 - 2a}\right)^2 \left(\int_0^{\alpha} + \int_{1-\alpha}^{1}\right) |f(\xi)|^2 \, d\xi + 2 \int_0^1 |f''(\xi)|^2 \, d\xi.$$

Thus,

$$\int_0^1 |f'(x)|^2 \, dx \leq \frac{4}{a} \left(\frac{1}{1 - 2a}\right)^2 \int_0^1 |f(\xi)|^2 \, d\xi + 2 \int_0^1 |f''(\xi)|^2 \, d\xi.$$

The maximum of the first coefficient occurs at $a = 1/6$; hence

$$\int_0^1 |f'(x)|^2 \, dx \leq 54 \cdot \int_0^1 |f(x)|^2 \, dx + 2 \int_0^1 |f''(x)|^2 \, dx.$$

The transformation $\xi = x(b - a) + a$ yields (3.1).

Again consider $a = 0$, $b = 1$. For any $x \in (0, 1)$ and $x_1 \in (1/3, 2/3)$, we have

$$f(x) = f(x_1) + \int_{x_1}^{x} f'(\xi) \, d\xi,$$

whence, by the Cauchy-Schwarz inequality,

$$|f(x)|^2 \leq 2 \, |f(x_1)|^2 + 2 \, |x - x_1| \int_{x_1}^{x} |f'(\xi)|^2 d\xi$$

$$\leq 2 \, |f(x_1)|^2 + 4/3 \int_0^1 |f'(\xi)|^2 \, d\xi.$$

Integrate over $(1/3, 2/3)$ with respect to x_1:

$$1/3\ |f(x)|^2 \leq 2 \int_{1/3}^{2/3} |f(x_1)|^2\, dx_1 + 4/9 \int_0^1 |f'(\xi)|^2\, d\xi.$$

Thus,

$$\int_0^1 |f(x)|^2\, dx \leq 6 \int_{1/3}^{2/3} |f(x)|^2\, dx + 4/3 \int_0^1 |f'(\xi)|^2\, d\xi.$$

From this inequality (3.2) follows by the transformation $\xi = (b-a)\,x + a$ Q. E. D.

Corollary. If $f \in C^2(a, b)$, and if $0 < \epsilon \leq 0$, then

$$(3.3) \qquad \int_a^b |f'(x)|^2\, dx \leq \gamma \left[\, \epsilon \int_a^b |f''(x)|^2\, dx + \epsilon^{-1} \int_a^b |f(x)|^2\, dx \,\right],$$

where γ depends only on $b - a$. If $f \in C^1(a, b)$, and if $0 < \epsilon \leq 1$, then there is an interval $[a', b'] \subset (a, b)$ such that

$$(3.4) \qquad \int_a^b |f(x)|^2\, dx \leq \gamma \left[\, \epsilon \int_a^b |f'(x)|^2\, dx + \int_{a'}^{b'} |f(x)|^2\, dx \,\right],$$

where γ depends only on $b - a$, while $[a', b']$ depends only on (a, b) and

Proof. Let $a = a_0 < a_1 < \ldots < a_k = b$ be a partition of (a, b) such that $1/2\,\sqrt{\epsilon}\,(b-a) < a_i - a_{i-1} < \sqrt{\epsilon}\,(b-a)$ for $i = 1,\ldots,k$. If $f \in C^2(a, b)$, then by (3.1),

$$\int_{a_{i-1}}^{a_i} |f'(x)|^2\, dx \leq 54 \left[\, \epsilon\,(b-a)^2 \int_{a_{i-1}}^{a_i} |f''(x)|^2\, dx\right.$$

$$+ \frac{4}{\epsilon\,(b-a)^2} \int_{a_{i-1}}^{a_i} |\, f(x)\,|^2\, dx\ )$$

$$\leq 54 \max\, [(b-a)^2, \frac{4}{(b-a)^2}][\epsilon \int_{a_{i-1}}^{a_i} |f''(x)|^2\, dx$$

$$+ \epsilon^{-1} \int_{a_{i-1}}^{a_i} |f(x)|^2\, dx],$$

whence (3.3) follows by addition. If $f \in C^1\,(a, b)$, (3.2) yields

$$\int_{a_{i-1}}^{a_i} |f\,(x)\,|^2\, dx \leq 6\ [\ \epsilon\,(b-a)^2 \int_{a_{i-1}}^{a_i} |f'\,(x)\,|^2\, dx$$

$$+ \int_{a_{i-1}}^{a_i} |f\,(x)\,|^2\, dx\ ]$$

$$\leq 6 \max\ (\ (b-a)^2, 1\ )\ [\ \epsilon \int_{a_{i-1}}^{a_i} |f'\,(x)\,|^2\, dx$$

$$+ \int_{a_{i-1}}^{a_i} |f\,(x)\,|^2\, dx\ ],$$

and (3.4) follows, if we take $a' = a + (a_1 - a)\,/3$ and $b' = b - (b - a_{k-1})\,/3$. Q. E. D.

This theorem can be extended to functions on E_n; the following form will be convenient for our purposes.

THEOREM 3.2. *Let Ω be a bounded open set.*

1^0 *If Ω has the restricted cone property, then there is a constant γ, depending on only Ω, such that for any $u \in H_2(\Omega)$.*

$$(3.5) \qquad |u|^2_{1,\Omega} \leq \gamma \left(\epsilon\, |u|^2_{2,\Omega} + \epsilon^{-1} |u|^2_{0,\Omega}\right)$$

if $0 < \epsilon \leq 1$.

2^0 *If Ω has the segment property, then there is a constant γ, depending on only Ω, such that for $0 < \epsilon \leq 1$ there is a domain $\Omega_\epsilon \subset\subset \Omega$ for which*

$$(3.6) \qquad |u|^2_{0,\Omega} \leq \gamma \left(\epsilon\, |u|^2_{1,\Omega} + |u|^2_{0,\Omega_\epsilon}\right)$$

for any $u \in H_1(\Omega)$.

Proof. Since any function in $H_2(\Omega)$ $[H_1(\Omega)]$ can be approximated by functions in $C^2(\Omega)$ $[C^1(\Omega)]$, it is sufficient to prove the theorem for functions u in class $C^2(\Omega)$ for 1^0 and in class $C^1(\Omega)$ for 2^0.

For 1^0, let $\{0'_i\}$ and $\{C'_1\}$ be the covering of $\partial\,\Omega$ and the set of corresponding cones, respectively, as guaranteed by the restricted cone property. Since Ω is bounded $\{0'_1\}$ is finite. Let h_i denote the height of C'_i, and let $h = \min h'_1$. Let $\{0''_i\}$ be a finite open covering of $\partial\,\Omega$ with spheres $0''_i$ whose diameters are less than $h/2$. Let $\{0_i\}$ be the collection of all sets of the form $0'_j \cap 0''_k$; to any set $0_i = 0'_j \cap 0''_k$, assign the cone $C_i = C'_j$. Then $\{0_i\}$, $\{C_i\}$ is a covering of $\partial\,\Omega$, together with the set of corresponding cones as in the definition of the restricted cone property, and $\{0_i\}$ has the additional property that the diameter of 0_i is less than the height h_i of C_i. Let $\{Q_i\}$ be a finite collection of cubes such that $\Omega - \cup\, 0_i \subset \cup\, Q_i \subset\subset \Omega$, and the edges of each cube are parallel to the coordinate axes.

Let γ be a generic constant depending only on the Ω.

Consider a cube $Q_i = \{a_k < x_k < b_k\}$. For any $k = 1,\ldots n$, we have by the corollary to Theorem 3.1 that

$$\int_{a_k}^{b_k} |D_k u|^2 \, dx_k \leq \gamma \left[\epsilon \int_{a_k}^{b_k} |D_k^2 u|^2 \, dx_k + \epsilon^{-1} \int_{a_k}^{b_k} |u|^2 \, dx_k \right],$$

whence

$$(3.7) \qquad \int_{Q_i} |D_k u|^2 \, dx \leq \gamma \left[\epsilon \int_{Q_i} |D_k{}^2 u|^2 \, dx + \epsilon^{-1} \int_{Q_i} |u|^2 \, dx \right].$$

Summing over k we get

$$|u|^2_{1,\,Q_i} \leq \gamma \left[\epsilon \, |u|^2_{2,\,Q_i} + \epsilon^{-1} \, |u|^2_{2,\,Q_i} \right],$$

and summing this over i, we get

$$|u|^2_{1,\,\cup Q_i} \leq \gamma \left[\epsilon \, |u|^2_{2,\,\Omega} + \epsilon^{-1} \, |u|^2_{0,\,\Omega} \right].$$

Thus, it is sufficient to consider the boundary region $\Omega - \cup Q_i$. Since $\Omega - \cup Q_i \subset \cup 0_i$, it is sufficient to consider $0_i \cap \Omega$.

Let ξ be a unit vector which is a positive multiple of some vector in the cone C_i. We will establish an inequality for the directional derivative in the direction of ξ. However, we must be sure that the domain of integration is connected along each parallel to ξ. To this end introduce the set $\Omega_i^{\xi} = \{x\colon x = y + t\xi,\ y \in \Omega \cap 0_i,\ 0 \leq t < h_i\}$. Then $(\Omega \cap 0_i) \subset \Omega_i^{\xi} \subset \Omega$ (recall the cone property). One verifies easily that the intersection with Ω_i^{ξ} of any line parallel to ξ is either void or is a line segment of length d satisfying: $h_i \leq d \leq 2h_i$.

Let D_ξ be the operation of differentiation in the direction of ξ. If v is a function in Ω define $(v)_i$ by

$$(v)_i = v \text{ in } \Omega_i^{\xi};\ (v)_i \equiv 0 \text{ in } E_n - \Omega_i^{\xi}.$$

Denote by L_y a line in the direction of ξ passing through a point y. It follows from the observation concerning Ω_i^{ξ} and from the corollary to Theorem 3.1 that

$$\int_{L_y} |(D_\xi u)_i|^2\, ds \le \gamma\, [\epsilon \int_{L_y} |(D_\xi^2 u)_i|^2\, dx + \epsilon^{-1} \int_{L_y} |(u)_i|^2\, ds].$$

Letting y vary in a $n-1$ -space orthogonal to ξ, integrating the last inequality with respect to y, we find that

$$\int_{\Omega \cap 0_i} |D_\xi u|^2\, dx \le \gamma\, [\, \epsilon \int_{\Omega} |D_\xi^2 u|^2\, dx + \epsilon^{-1} \int_{\Omega} |u|^2\, dx\,].$$

Let $\{\xi^1,\ldots,\xi^n\}$ be a linearly independent set of vectors, each of which is a positive multiple of some vector in the cone C_i. Then the inequality above holds with ξ replaced by ξ^k for $k = 1,\ldots n$. Since any differentiation operator can be written as a linear combination of $D_{\xi^1},\ldots,D_{\xi^n}$, it follows on adding that

$$|u|^2_{1,\, \Omega \cap 0_i} \le \gamma\, [\epsilon\, |u|^2_{2,\, \Omega} + \epsilon^{-1}\, |u|^2_{0,\, \Omega}\,].$$

This completes the proof of part 1^0.

The proof of part 2^0 is similar, using (3.4) instead of (3.3), but needing only the segment property since (3.4) is applied in only one direction for each 0_i. In proving 2^0 the role of ξ is played by the segment guaranteed by the segment property. Q.E.D.

More generally, we have the *INTERPOLATION THEOREM:*

THEOREM 3.3. *Let Ω be a bounded open set with the restricted cone property, and assume $0 < \epsilon \leq 1$. If $u \in H_m(\Omega)$ for some $m \geq 2$, and if $1 \leq j \leq m - 1$, then*

$$(3.8) \qquad |u|_{j,\Omega}^2 \leq \gamma\,(\epsilon^{m-j}\,|u|_{0,\Omega}^2 + \epsilon^{-j}\,|u|_{0,\Omega}^2),$$

where $\gamma = \gamma(\Omega, m)$ depends only on Ω and m.

Proof. First we will show that (3.8) holds for $j = m - 1$. Observe that, by Theorem 3.2,

$$(3.9) \qquad |u|_{k-1,\Omega}^2 \leq \gamma\,(\epsilon\,|u|_{k,\Omega}^2 + \epsilon^{-(k-1)}\,|u|_{0,\Omega}^2)$$

holds for $k = 2$. Assume that (3.9) holds for $k < m$ and all $\epsilon \in (0, 1]$. Since $u \in H_m(\Omega)$, for $|a| \leq m - 2$, $D^\alpha u \in H_2(\Omega)$; hence, by Theorem 3.2

$$|D^\alpha u|_1^2 \leq \gamma_1\,(\epsilon\,|D^\alpha u|_2^2 + \epsilon^{-1}\,|D^\alpha u|_0^2).$$

and in particular, taking $|a| = k - 1$, we obtain

$$|u|_k^2 \leq \gamma_1\,(\epsilon\,|u|_{k+1}^2 + \epsilon^{-1}\,|u|_{k-1}^2).$$

Hence, using (3.9) with ϵ replaced by $\delta\,\epsilon$ for $0 < \delta \leq 1$,

$$|u|_k^2 \leq \gamma_1\,\epsilon\,|u|_{k+1}^2 + \gamma\,\gamma_1\,\delta\,|u_k|^2 + \gamma\,\gamma_1\,\delta^{-(k-1)}\,\epsilon^{-k}\,|u|_0^2.$$

Let $\delta = \min\left(1, \dfrac{1}{2\gamma\,\gamma_1}\right)$. Then

$$|u|_k^2 \leq \gamma_1 \, \epsilon \, |u|_{k+1}^2 + 1/2 \, |u_k|^2 + 1/2 \, \delta^{-k} \epsilon^{-k} \, |u|_0^2$$

or

$$|u|_k^2 \leq 2 \, \gamma_1 \epsilon \, |u|_{k+1}^2 + \delta^{-k} \epsilon^{-k} \, |u|_0^2$$

so that (3.9) holds for $k = 2,\ldots,m$.

Now we will show that if (3.8) holds for $j = k > 1$, then it holds for $j = k - 1$. Under the hypothesis that (3.8) holds for $j = k \leq m$ we have by (3.9) that

$$\begin{aligned} |u|_{k-1}^2 &\leq \gamma \, (\epsilon \, |u|_k^2 + \epsilon^{-(k-1)} \, |u|_0^2) \\ &\leq \gamma \, [\, \epsilon\gamma \, (\epsilon^{m-k} \, |u|_m^2 + \epsilon^{-k} \, |u|_0^2) + \epsilon^{-(k-1)} \, |u|_0^2 \,] \\ &\leq \gamma_2 \, (\epsilon^{m-(k-1)} \, |u|_m^2 + \epsilon^{-(k-1)} \, |u|_0^2) \, , \end{aligned}$$

and the theorem follows by induction. Q.E.D.

As a corollary we have

THEOREM 3.4. *If Ω is bounded and has the restricted cone property, if $\epsilon > 0$, and if $u \in H_m(\Omega)$ for some $m \geq 2$, then there exists a number $C_\epsilon = C_\epsilon(\Omega, m)$, which depends only on Ω, m, and ϵ, such that*

$$||u||_{m-1,\,\Omega} \leq \epsilon \, |u|_{m,\,\Omega} + C_\epsilon \, |u|_{0,\,\Omega}.$$

For convenience, we now collect some facts about Fourier series. We will use the notation $Q = \{x\colon |x_k| < 1/2\}$. The cube Q will serve as a fundamental domain for various classes of periodic functions.

Definition 3.1. *$C_{\#}^m$ [$C_{\#}^{\infty}$] is the class of functions $u(x)$ defined on E_n which are periodic of period 1 in each variable and which are m times continuously differentiable [infinitely differentiable]. Also, $H_m^{\#}$ is the completion of $C_{\#}^m$ (or $C_{\#}^{\infty}$) with respect to the norm $\| \ \|_{m,Q}$.*

We may obviously identify the elements of $H_m^{\#}$ with functions defined on Q, and, with this identification $H_m^{\#} \subset H_m(Q)$. Let $u \in C_{\#}^{\infty}$. Then u has a Fourier series expansion:

$$u(x) = \sum_{\xi} b_{\xi} e^{2\pi i \xi \cdot x}. \tag{3.10}$$

We use the notation $\sum_{\xi}$ to stand for summation over all $\xi = (\xi_1, \ldots, \xi_n)$, where $\xi_1, \ldots, \xi_n$ are integers.

This series and the derived series

$$D^{\alpha} u(x) = (2\pi i)^{|\alpha|} \sum_{\xi} b_{\xi} \xi^{\alpha} e^{2\pi i \xi \cdot x} \tag{3.11}$$

are uniformly convergent, and, by Parseval's identity

$$|D^{\alpha} u|_{0,Q}^2 = (2\pi)^{2|\alpha|} \sum_{\xi} |b_{\xi}|^2 \xi^{2\alpha}. \tag{3.12}$$

We will use the notation

$$\|u\|_m^{\#} = [\, |b_{0,\ldots,0}|^2 + \sum_{\xi} |b_{\xi}|^2 |\xi|^{2m} \,]^{1/2}. \tag{3.13}$$

($|\xi|$ Euclidean norm.)

THEOREM 3.5. *There is a constant* $C_m > 1$ *such that*

$$C_m^{-1} \|u\|_m^\# \leq \|u\|_{m,Q} \leq C_m \|u\|_m^\#$$

holds for all $u \in C_\#^\infty$.

Proof. Consider the ratio $|\xi|^{-2m} \cdot \sum_{|\alpha| \leq m} (2\pi)^{2|\alpha|} \xi^{2\alpha}$ for ξ an n-tuple of integers, $\xi \neq 0$. This ratio is bounded above for all such ξ. Therefore, since by (3.12)

$$\|u\|_{m,Q}^2 = \sum_{|\alpha| \leq m} |D^\alpha u|_{0,Q}^2 = \sum_\xi |b_\xi|^2 \cdot \sum_{|\alpha| \leq m} (2\pi)^{2|\alpha|} \xi^{2\alpha},$$

the result follows immediately. Q.E.D.

Consequently, the completion of $C_\#^\infty$ in the norm $\| \ \|_{m,Q}$ is the same as its competion in the norm $\| \ \|_m^\#$. Therefore, we may identi fy $H_m^\#$ with a class of formal Fourier series $\sum_\xi b_\xi e^{2\pi i x \cdot \xi}$ such that the expression (3.13) is finite. Now suppose that the formal series $\sum_\xi b_\xi e^{2\pi i x \cdot \xi}$ is such that the expression (3.13) is finite, and let $u_k(x) = \sum_{|\xi| \leq k} b_\xi e^{2\pi i x \cdot \xi}$. Then $D^\alpha u_k$ is a Cauchy sequence in $L_2(Q)$ for $|\alpha| \leq m$, so that, by the Riesz-Fischer theorem, $D^\alpha u_k$ converges in $L_2(Q)$, $|\alpha| \leq m$. Thus, $u_k \in C_\#^\infty$, $u_k \to u$ in $L_2(Q)$, and $D^\alpha u_k$ converges, $|\alpha| \leq m$. Hence, $u_k \to u$ in $H_m(Q)$, so that $u \in H_m^\#$. Therefore, we may identify $H_m^\#$ with the collection of all formal Fourier series

$$u = \sum_\xi b_\xi e^{2\pi i x \cdot \xi}$$

such that $\|u\|_m^\# < \infty$. Moreover, if $u \in H_m^\#$ has the Fourier series

$$u = \sum_{\xi} b_{\xi}\, e^{2\pi i x \cdot \xi},$$

then for $|\alpha| \le m$, $D^{\alpha} u$ has the Fourier series

$$D^{\alpha} u = \sum_{\xi} b_{\xi} (2\pi i \xi)^{\alpha}\, e^{2\pi i x \cdot \xi},$$

the series converging in $L_2(Q)$.

THEOREM 3.6. *If $\{u_j\}$ is a bounded sequence in $H^{\#}_{m}$ with the formal Fourier series*

$$u_j(x) = \sum_{\xi} b_{\xi,\, j} e^{2\pi i \xi \cdot x},$$

if $m \ge 1$, and if for fixed ξ, $\{b_{\xi, j}\}$ converges to b_{ξ} as $j \to \infty$, then $\{u_j\}$ converges in $H_{m-1}(Q)$ as $j \to \infty$.

Proof. Let

$$P_r u_j = \sum_{|\xi| \ge r} b_{\xi,\, j}\, e^{2\pi i \xi \cdot x},$$

$$S_r u_j = \sum_{|\xi| < r} b_{\xi,\, j}\, e^{2\pi i \xi \cdot x}.$$

By Theorem 3.5

$$\begin{aligned}
\|P_r u_j\|_{m-1,\,Q} &\le C_{m-1} \|P_r u_j\|^{\#}_{m-1} \\
&= C_{m-1} \Big(\sum_{|\xi| \ge r} |b_{\xi,\, j}|^2 |\xi|^{2(m-1)}\Big)^{1/2} \\
&\le \frac{C_{m-1}}{r} \Big(\sum_{|\xi| \ge r} |b_{\xi,\, j}|^2 |\xi|^{2m}\Big)^{1/2} \\
&\le \frac{C_{m-1}}{r} \Big(\sum_{\xi} |b_{\xi,\, j}|^2 |\xi|^{2m}\Big)^{1/2}
\end{aligned}$$

$$= \frac{C_{m-1}}{r} \, ||u_j||_m^{\#} .$$

Since $||u_j||_m^{\#}$ is bounded,

$$||P_r \, u_j||_{m-1, \, Q} \leq \frac{\text{const}}{r} \quad .$$

where by const we denote constants depending only on m and on the bound for $||u||_m^{\#}$. Thus

$$||u_j - u_k||_{m-1, \, Q} \leq ||S_r \, (u_r - u_k)||_{m-1, \, Q} + ||P_r \, u_j||_{m-1, \, Q}$$

$$+ ||P_r \, u_k||_{m-1, \, Q}$$

$$\leq C_{m-1} \, ||S_r \, (u_j - u_k)||_{m-1} + \frac{\text{const}}{r}$$

$$\leq C \, (r, \, m) \max \, |b_{\xi, \, j} - b_{\xi, \, k}| + \frac{\text{const}}{r} ,$$

where $C(r, \, m)$ depends only on r and m and the maximum is taken over $|\xi| < r$ only. Since $b_{\xi, \, j} - b_{\xi, \, k} \to 0$ as $j, \, k \to \infty$,

$$\limsup_{j, \, k \to \infty} \, ||u_j - u_k||_{m-1, \, Q} \leq \frac{\text{const}}{r} .$$

Since r is arbitrary,

$$\limsup_{j, \, k \to \infty} \, ||u_j - u_k||_{m-1, \, Q} = 0. \quad \text{Q.E.D.}$$

THEOREM 3.7. *Let* $\{u_j\}$ *be a bounded sequence in* $H_m^{\#}$. *If* $m \geq 1$, *then* $\{u_j\}$ *has a subsequence which converges in* H_{m-1} (Q).

Proof. Since $u_j \in H_m^{\#}$, u_j has a formal Fourier series expansion

$$u_j(x) = \sum_{\xi} b_{\xi_j} e^{2\pi i \xi \cdot x}.$$

Since $\{u_j\}$ is bounded, there is a number C such that for every j

$$\|u_j\|_m^{\#} < C.$$

Since

$$|b_{\xi_j}| \leq \|u_j\|_m^{\#} < C,$$

it follows that for fixed ξ, $\{b_{\xi_j}\}$ is a bounded set of complex numbers. Let $\xi_1, \xi_2, \ldots$ be some enumeration of the ξ's. We will use the diagonal process to extract a convergent subsequence form $\{u_j\}$. Since $\{b_{\xi_1, j}\}$ is bounded, it has a subsequence $\{b_{\xi_1, j_1}\}$ which converges as $j_1 \to \infty$. Since $\{b_{\xi_2 j_1}\}$ is bounded, it has a convergent subsequence $\{b_{\xi_2, j_2}\}$. Continuing in this manner, for each k we obtain a subsequence $\{b_{\xi_k, j_k}\}$ of $\{b_{\xi_k, j_{k-1}}\}$ which converges as $j_k \to \infty$. Then $\{b_{\xi_j, j_j}\}$ is a subsequence of $\{b_{\xi_j}\}$ converging for every ξ. By Theorem 3·6 the corresponding subsequence of $\{u_j\}$ converges in $H_{m-1}(Q)$. Q.E.D.

We can now prove the following compactness theorem which is due to *RELLICH*.

THEOREM 3.8. *Let Ω be a bounded domain having the segment property. Then every bounded sequence in $H_m(\Omega)$ has a subsequence which converges in $H_j(\Omega)$ if $j < m$.*

Proof. Since $\|\cdot\|_j \leq \|\cdot\|_{m-1}$ for $j \leq m-1$, it is sufficient to prove that if $\{u_i\} \subset H_m(\Omega)$, and if $\|u_i\|_{m,\Omega} \leq C$ for some constant $C > 0$, then there is a subsequence $\{u_{i_r}\}$ such that $\|u_{i_r} - u_{i_s}\|_{m-1,\Omega} \to 0$ as $r, s \to \infty$. We now perform several reductions to show that it is sufficient to prove the theorem for

$\{u_i\} \subset C_0^\infty(\Omega)$.

First, it is sufficient to show that some subsequence $\{u_{i_r}\}$ converges in $H_{m-1}(\Omega')$ for every $\Omega' \subset\subset \Omega$. For suppose that such a subsequence has been found. Since Ω has the segment property, for $0 < \epsilon \leq 1$ there is a $\Omega_\epsilon \subset\subset \Omega$ such that

$$\|u_{i_r} - u_{i_s}\|_{m-1,\,\Omega} \leq \gamma\epsilon \|u_{i_r} - u_{i_s}\|_{m,\,\Omega} + \gamma \|u_{i_r} - u_{i_s}\|_{m-1,\,\Omega_\epsilon};$$

this inequality is derived by applying Theorem 3.2 (2°) to derivatives D^α of the functions considered, for $|\alpha| \leq m-1$. Since u_{i_r} converges in $H_{m-1}(\Omega_\epsilon)$, it follows that

$$\limsup_{r,s\to\infty} \|u_{i_r} - u_{i_s}\|_{m-1,\,\Omega} \leq \epsilon \sup_{r,s} \|u_{i_r} - u_{i_s}\|_{m,\,\Omega} \leq 2\,C\epsilon.$$

Since this holds for every ϵ, $\|u_{i_r} - u_{i_s}\|_{m-1,\,\Omega} \to 0$. Note that only at this stage of the argument is the segment property needed.

Next, it is sufficient to prove the theorem in the case that there exists a $\Omega_1 \subset\subset \Omega$ such that each u_i has its support in Ω_1. For suppose the theorem to have been demonstrated in this case. Take an expanding sequence $\{\Omega_N\}$ such that $\lim_{N\to\infty} \Omega_N \subset\subset \Omega$; let $\zeta_N \in C_0^\infty(\Omega)$, $\zeta_N = 1$ on $\overline{\Omega_N}$. By Leibnitz's rule, for fixed N, $\{\zeta_N u_i\}$ is bounded in $H_m(\Omega)$ and supp $(\zeta_N u_i) \subset$ supp (ζ_N). Therefore, $\{\zeta_1 u_i\}$ has a subsequence $\{\zeta_1 u_{i_1}\}$ which converges in $H_{m-1}(\Omega)$. Similarly, $\{\zeta_2 u_{i_1}\}$ has a subsequence $\{\zeta_2 u_{i_2}\}$ which converges in $H_{m-1}(\Omega)$. Proceeding in this manner, for each N we obtain a subsequence $\{\zeta_N u_{i_N}\}$ of $\{\zeta_N u_{i_{N-1}}\}$ which converges in $H_{m-1}(\Omega)$. Now use the diagonal process: let $u_{i'_r} = u_{i_r}$. Then $\{\zeta_N u_{i'_r}\}$ converges in $H_{m-1}(\Omega)$, hence in $H_{m-1}(\Omega_N)$, for each N. Since $\zeta_N u_{i'_r} = u_{i'_r}$ on Ω_N, it follows that $\{u_{i'_r}\}$ converges in $H_{m-1}(\Omega')$ for every $\Omega' \subset\subset \Omega$. By the first reduction, $\{u_{i'_r}\}$ converges in $H_{m-1}(\Omega)$.

Finally, suppose the theorem has been proved for a sequence in $C_0^\infty(\Omega)$. If $\{u_i\}$ is a bounded sequence in $H_m(\Omega)$ and if supp $(u_i) \subset \Omega_1 \subset\subset \Omega$, then by Theorem 1.10 a sequence $\{v_i\} \subset C_0^\infty(\Omega)$ can be chosen such that $\|u_i - v_i\|_{m,\Omega} < i^{-1}$. Then $\{v_i\}$ is bounded in $H_m(\Omega)$, so that some subsequence $\{v_{i_r}\}$ converges in $H_{m-1}(\Omega)$. Thus $\{u_{i_r}\}$ converges in $H_{m-1}(\Omega)$ since $\|.\|_{m-1} \le \|\cdot\|_m$. Therefore, by the second reduction it is sufficient to prove the theorem for $\{u_i\} \subset C_0^\infty(\Omega)$.

Therefore, assume that $\{u_i\} \subset C_0^\infty(\Omega)$. By a suitable change of coordinates it may be assumed that $\overline{\Omega} \subset Q$. Extend u_i to $\overline{Q}$, defining $u_i = 0$ outside Ω, and then consider u_i to be extended by periodicity to E_n. Then $\{u_i\}$ is a bounded sequence in $H_m^\#$. By Theorem 3.7 $\{u_i\}$ has a subsequence which converges in $H_{m-1}(\Omega)$, hence, *a fortiori* in $H_{m-1}(\Omega)$. Q.E.D.

The following theorem is one of the versions of the important Sobolev inequality (reference: Ehrling, Math. Scand. vol. 2 (1954) p. 267).

THEOREM 3.9. (*Sobolev's Inequality*). *Let Ω be a domain having the ordinary cone property, let $u \in W_m(\Omega)$ $[H_m(\Omega)]$ with $m > n/2$, let ℓ be the largest integer less than $m - n/2$ ($\ell = m - [n/2] - 1$). Then u can be modified on a set of measure zero so that $u \in C^\ell(\Omega)$. Moreover, for any $r \ge 1$, $|\alpha| \le \ell$, $x \in \Omega$,*

$$|D^\alpha u(x)| \le \gamma\, r^{-(m - \frac{1}{2}n - |\alpha|)} \left(|u|_{m,\Omega} + r^m |u|_{0,\Omega}\right),$$

where γ is a constant that depends only on Ω and m. (Indeed, γ depends only on m and the dimensions of the cone associated with Ω.) If, in addition, Ω is bounded and has the segment property, then u can be modified on a set of measure zero so that $u \in C^\ell(\overline{\Omega})$.

Proof. First, assume that Ω contains a closed cone G_{h_0} of height h_0 and vertex at the origin. Let G_h denote the cone obtained by truncating G_{h_0} to height h. Assume that $u \in C^{m*}(\Omega)$. Let ξ be a unit vector which is a positive multiple of some vector in G_{h_0}. Let

$$f(t) = u(t\xi),\ 0 \le t \le h_0.$$

Then, by Taylor's formula we have

$$u(0) = \sum_{j=0}^{k-1} t^j f_j(t) + k \int_0^t \tau^{k-1} f_k(\tau)\, d\tau,$$

where

$$f_j(t) = \frac{(-1)^j}{j!} f^{(j)}(t).$$

We shall assume $n/2 < k \le m$. By the Cauchy-Schwarz inequality

$$|u(0)|^2 \le (k+1) \{ \sum_{j=0}^{k-1} t^{2j} |f_j(t)|^2 + k^2 \int_0^t \tau^{k-1} f_k(\tau)\, d\tau|^2 \}.$$

Now since $k > n/2$,

$$|\int_0^t \tau^{k-1} f_k(\tau)\, d\tau|^2 = |\int_0^t \tau^{k-\frac{1}{2}(n+1)} [\tau^{\frac{1}{2}(n-1)} f_k(\tau)]\, d\tau|^2$$

$$\le \int_0^t \tau^{2k-n-1}\, d\tau \cdot \int_0^t \tau^{n-1} |f_k(\tau)|^2\, d\tau$$

$$= \frac{1}{2k-n} t^{2k-n} \int_0^t \tau^{n-1} |f_k(\tau)|^2\, d\tau.$$

Thus

$$|u(0)|^2 \leq (k+1)\Big\{\sum_{j=0}^{k-1} t^{2j}\,|f_j(t)|^2 + \frac{k^2\, t^{2k-n}}{2k-n}\int_0^t \tau^{n-1}$$

$$\cdot\, |f_k(\tau)|^2\, d\tau\Big\}.$$

Now integrate this inequality over G_h for some $h < h_0$. The volume of G_h is ωh^n, where ω is a constant. Since $0 < t < h$, we have

$$\omega h^n\, |u(0)|^2 \leq (k+1)\Big\{\sum_{j=0}^{k-1} h^{2j}\int_{G_h} |f_j(t)|^2\, dx \tag{3.14}$$

$$+ \frac{k^2}{2k-n}\int_{G_h} t^{2k-n}\int_0^h \tau^{n-1}\,|f_k(\tau)|^2\, d\tau dx\Big\}.$$

Let γ be a generic constant depending only on ω, m, and h_0. Since f_j is a linear combination of derivatives of u, the first term on the right in (3.14) is bounded by

$$\gamma \sum_{j=0}^{k-1} h^{2j}\, |u|^2_{j,\,G_h}\ .$$

To estimate the second term, we introduce polar coordinates. Let σ be the ordinary surface measure on the surface of the unit sphere, and let Λ be that part of the surface of the unit sphere subtended by the cone G_{h_0}. We then have the formula for integration using the spherical measure σ

$$\int_{G_h} g dx = \int_\Lambda \int_0^h g t^{n-1}\, dt d\sigma\ . \tag{3.15}$$

Using (3.15), the second term on the right side of (3.14) becomes

$$\int_{G_h} t^{2k-n} \int_0^h \tau^{n-1} |f_k(\tau)^2 \, d\tau dx$$

$$= \int_\Lambda \int_0^h [t^{2k-n} \int_0^h \tau^{n-1} |f_k(\tau)|^2 \, d\tau] \, t^{n-1} dt d\sigma$$

$$= (\int_0^h t^{2k-1} \, dt) \, (\int_\Lambda \int_0^h \tau^{n-1} |f_k(\tau)|^2 \, d\tau d\sigma$$

$$= \frac{h^{2k}}{2k} \int_{G_h} |f_k(\tau)|^2 \, dx$$

$$\leq \gamma \frac{h^{2k}}{2k} |u|^2_{k, G_h} .$$

Hence, (3.14) implies

$$h^n |u(0)|^2 \leq \gamma \sum_{j=0}^{k} h^{2j} |u|^2_{j, G_h} . \tag{3.16}$$

By the interpolation inequality, Theorem 3.3, with $\epsilon = h^2$,

$$|u|^2_{j, G_h} \leq \gamma \, (h^{2(m-j)} |u|^2_{m, G_h} + h^{-2j} |u|^2_{0, G_h}), \; 0 \leq j \leq m,$$

so that (3.16) implies

$$|u(0)| \leq \gamma \, h^{-(\frac{1}{2}n - m)} (|u|_{m, G_h} + h^{-m} |u|_{0, G_h}).$$

For $h = h_0/r$, this gives

$$|u(0)| \leq \gamma \, r^{\frac{1}{2}n - m} (|u|_{m, G_h} + r^m |u|_{0, G_h}). \tag{3.17}$$

If $|\alpha| \le \ell$ and if $u \in C^{m^*}(\Omega)$, then $D^\alpha u \in C^{(m-|\alpha|)^*}(\Omega)$, and by (3.17),

$$|D^\alpha u(0)| \le \gamma r^{\frac{1}{2}n - m + |\alpha|}\left(|D^\alpha u|_{m-|\alpha|,\,G_h} + r^{m-|\alpha|}\,|D^\alpha u|_{0,\,G_h}\right)$$

$$\le \gamma r^{\frac{1}{2}n - m + |\alpha|}\left(|u|_{m,\,G_{h_0}} + r^{m-|\alpha|}\,|u|_{|\alpha|,\,G_{h_0}}\right).$$

By the interpolation inequality applied to the region G_{h_0} with $\epsilon = r^{-2}$,

$$|u|_{|\alpha|,\,G_{h_0}} \le \gamma\left(r^{-(m-|\alpha|)}\,|u|_{m,\,G_{h_0}} + r^{|\alpha|}\,|u|_{0,\,G_{h_0}}\right),$$

so that

$$|D^\alpha u(0)| \le \gamma r^{\frac{1}{2}n - m + |\alpha|}\left(|u|_{m,\,G_{h_0}} + r^m\,|u|_{0,\,G_{h_0}}\right),$$

for $|\alpha| \le \ell$.

This inequality has been derived for the cone G_{h_0}, not for Ω. However, since Ω has the ordinary cone property, each point in Ω lies at the vertex of such a cone lying entirely in Ω. Thus, since the norm over a cone in Ω is no greater than the norm over Ω, for any $x \in \Omega$

$$\text{(3.18)} \qquad |D^\alpha u(x)| \le \gamma r^{\frac{1}{2}n - m + |\alpha|}\left(|u|_{m,\,\Omega} + r^m\,|u|_{0,\,\Omega}\right)$$

for $|\alpha| \le \ell$. Thus the theorem is proved for $u \in C^m(\Omega)$.

For any $u \in W_m(\Omega)$, let $x_0 \in \Omega$ and let $U \subset\subset \Omega$ be a neighborhood of x_0. Let $\Omega' \subset\subset \Omega$ be a domain such that the cones (of the ordinary cone property) with vertices in U are contained in Ω'. By Theorem 1.10, choose a sequence $\{u_k\} \subset C^\infty(E_n)$ such that $u_k \to u$

in $W_m(\Omega')$. Then by (3.17) applied to $u_k - u_j$, $\{u_k\}$ converges uniformly on U (to a continuous function u_0). As $u_k \to u$ in $L_2(\Omega)$, u_0 equals u almost everywhere on U. Moreover by (3.18) applied to $u_k - u_j$, $D^\alpha u_k$ converges uniformly on U for $|\alpha| \leq \ell$. It is easily seen then that $\lim D^\alpha u_k = D^\alpha u_0$. By the argument leading to (3.18), for $x \in U$

$$|D^\alpha u_k(x)| \leq \gamma r^{\frac{1}{2}n - m + |\alpha|} (|u_k|_{m,\Omega'} + r^m |u_k|_{0,\Omega'}).$$

In the limit as $k \to \infty$, for $|\alpha| \leq \ell$ we obtain .

$$|D^\alpha u_0(x)| \leq \gamma r^{\frac{1}{2}n - m + |\alpha|} (|u|_{m,\Omega'} + r^m |u|_{0,\Omega'}),$$

and the first part of the theorem is proved.

For the second part of the theorem, by Theorem 2.1 there is a sequence $\{u_k\} \subset C^\infty(E_n)$ such that $u_k \to u$ in $W_m(\Omega)$. By the first part of the proof, if $|\alpha| \leq \ell$,

$$|D^\alpha (u_k - u_j)(x)| \leq \gamma r^{-(m - \frac{1}{2}n - |\alpha|)} (|u_k - u_j|_{m,\Omega} + r^m |u_k - u_j|_{0,\Omega})$$

for $x \in \Omega$; by continuity, this inequality holds for all $x \in \bar{\Omega}$. Let $u_0 = \lim_{k\to\infty} u_k$; by the inequality above $D^\alpha u_k$ converges uniformly on $\bar{\Omega}$ for $|\alpha| \leq \ell$. Hence, $D^\alpha u_0$ is uniformly continuous on Ω and thus has a continuous extension to $\bar{\Omega}$, if $|\alpha| \leq \ell$. Q.E.D.

Corollary 1. If $u \in H_m^{\text{loc}}(\Omega)$ and if $m > n/2$, then there is a function $u_0 \in C(\Omega)$ such that $u_0 = u$ almost everywhere in Ω.

Corollary 2. If $u \in H_m^{\text{loc}}(\Omega)$ for all positive integers m, then there is a function $u_0 \in C^\infty(\Omega)$ such that $u_0 = u$ almost everywhere in Ω.

Let M be a smooth $(n-1)$-manifold in a domain Ω. If u is a function defined on Ω, then the restriction of u to M is the *trace* of u. For an arbitrary function in $L_2(\Omega)$, the trace defined in this manner has no significance since the measure of M is zero. If, however, $u \in H_1(\Omega)$, a more satisfactory definition can be given as follows: let $\{u_k\}$ be a sequence of funcetion, $u_k \in C^{1^*}(\Omega)$, such that $u_k \to u$ in $H_1(\Omega)$; if the restriction of u_k to M (that is, the trace of u_k) converges in $L_2(M)$, then we will call the limit function the *trace of u on M*. Of course, M could just as well be a lower dimensional manifold.

The following theorem (also a version of a Sobolev theorem) shows that the trace is well defined under suitable conditions.

E_ν will be used to denote a ν-dimensional subspace of E_n for $0 \le \nu \le n-1$.

THEOREM 3.10. *Let Ω be a bounded domain with the restricted cone property. Let $x^0 \in \Omega$, let $\Pi = x^0 + E_\nu$, and let $\Pi' = \Pi \cap \Omega$. Let σ be the ν-dimensional Lebesque measure on Π. If $u \in H_m(\Omega)$, and if $m > 1/2\,(n-\nu)$, then the trace u_0 of u on Π' is a well defined $L_2(\Pi')$ function. If, in addition $|\alpha| < m - 1/2\,(n-\nu)$, and $r \ge 1$, then the trace $D^\alpha u_0$ of $D^\alpha u$ on Π' exists, $u_0 \in H_{|\alpha|}(\Pi')$, and*

$$(3.19)\qquad \Big(\int_{\Pi'} |D^\alpha u_0(x)|^2\, d\sigma\Big)^{1/2} \le \gamma\, r^{-(m - \frac12 (n-\nu) - |\alpha|)}\cdot\big(|u|_{m,\Omega} + r^m |u|_{0,\Omega}\big),$$

where γ is a constant depending only on Ω.

Proof. First note that in Π there is a finite open covering $\{0_i\}$ of Π' such that to each 0_i there is associated an $(n-\nu)$ dimensional cone Σ_i which has the following properties: the vertex of Σ_i is at the origin, Σ_i spans a complementary space to E_ν, and $x + \Sigma_i \subset \Omega$ for each $x \in 0_i$. For let $\{0'_i\}_{i=1}^{N}$ be a finite covering of $\partial\Omega$ and $\{\Sigma'_i\}_{i=1}^{N}$ the set of corresponding cones guaranteed by the restricted cone property. Let $0_i'' \subset\subset 0_i'$ be chosen such that $\partial\Omega \subset \cup\, 0_i''$, and let $0_0' = \Omega - \overline{\cup\, 0_i''}$. If Σ_0 is any sufficiently small cone, then for $x \in 0_0'$ we have $x + \Sigma_0 \subset \Omega$. Let $0_i = 0_i' \cap \Pi$, $i = 0, 1, \ldots, N$, and let $\xi^{i1}, \ldots, \xi^{in}$ be linearly independent vectors in Σ_i'. By rearrangement (if necessary), assume that the projections of $\xi^{i1}, \ldots, \xi^{i,\, n-\nu}$ on the orthogonal complement of E_ν are linearly independent. Let Σ_i be the convex hull of $\{\xi^{i1}, \ldots, \xi^{i,\, n-\nu}\}$; that is,

$$\Sigma_i = \{x\colon x = \sum_{k=1}^{n-\nu} C_k\, \xi^{ik},\ C_k \geq 0,\ \sum_{k=1}^{n-\nu} C_k \leq 1\}.$$

Then $\Sigma_i \subset \Sigma_i'$, so that $x + \Sigma_i \subset \Omega$ for any $x \in 0_i$. Furthermore, Σ_i spans a complementary space to E_ν: this is evident from the fact that the projections span the orthogonal complement.

If $u \in C^m(\Omega)$, then for any point $x \in 0_i$, $u(x)$ can be estimated as in the proof of the Sobolev inequality, except that now the cone used is the n-ν dimensional cone $x + \Sigma_i$.

From relation (3.16), if h_i is the height of Σ_i and if $1/2\,(n-\nu) < k \leq m$, then for $x \in 0_i$

$$h_i^{n-\nu}\, |u(x)|^2 \leq \gamma \sum_{j=0}^{k} h_i^{2j}\, |u|_{j,\, x+\Sigma_i}^2$$

where γ is a generic constant depending only on Ω. On integrating over $0_i \cap \Pi'$, we have

$$h_i^{n-\nu} \int_{0_i \cap \Pi'} |u(x)|^2\, d\sigma \leq \gamma \sum_{j=0}^{k} h_i^{2j} \int_{0_i \cap \Pi'} |u|_{j,\, x+\Sigma_i}^2\, d\sigma$$

$$\leq \gamma \sum_{j=0}^{k} h_i^{2j}\, |u|_{j,\, \Omega}^2$$

the last inequality arising from the fact that the integration over $0_i \cap \Pi'$ is over an open set in E_ν, and $|u|_{j,\, x+\Sigma_i}^2$ contains an integration over a domain in $E_{n-\nu}$. Since the finite collection $\{0_i\}$ covers Π', a summation gives

$$h^{n-\nu} \int_{\Pi'} |u(x)|^2 \, d\sigma \leq \gamma \sum_{j=0}^{k} h^{2j} |u|_{j,\Omega}^2$$

where $h \leq h_0 = \min(h_i)$. Upon using the interpolation theorem (Theorem 3.3) and replacing h by h_0/r, we obtain as in the proof of Sobolev's inequality

$$\left[\int_{\Pi'} |u(x)|^2 \, d\sigma\right]^{1/2} \leq \gamma\, r^{1/2(n-\nu)-m} \left(|u|_{m,\Omega} + r^m |u|_{0,\Omega}\right).$$

For the derivatives of u, we have as in (3.18) that

$$(3.20) \qquad \int_{\Pi'} |D^\alpha u(x)|^2 \, d\sigma \leq \gamma\, r^{1/2(n-\nu)-m+|\alpha|} \left(|u|_{m,\Omega} + r^m |u|_{0,\Omega}\right)$$

for $0 \leq |a| \leq m\ y\ 1/2\,(n - \nu)$.

From (3.20) it follows that if $u_k \in C^m(\Omega)$, and if u_k converges in $H_m(\Omega)$, then u_k converges in $L_2(\Pi')$. Hence the trace is well defined.

For an arbitrary $u \in H_m(\Omega)$, u can be approximated by functions in $C^\infty(\overline{\Omega})$ as in the proof of the Sobolev inequality. Inequality (3.19) follows immediately. Q.E.D.

Now we will collect some simple lemmas which will be useful later. First, some definitions.

Definition 3.2. *Let E be a measurable subset of E_n. If $u \in L_2(E)$, a sequence $\{u_k\} \subset L_2(E)$ is said to converge weakly to u in $L_2(E)$ if and only if for every $v \in L_2(E)$*

$$\lim_{k\to\infty} \int_E u_k v\,dx = \int_E uv\,dx.$$

Weak convergence will be denoted $u_k \rightharpoonup u$.

The following *WEAK COMPACTNESS THEOREM* is well known.

THEOREM 3.11. *If $\{u_k\}$ is a sequence of functions in $L_2(E)$ such that $\|u_k\|_{0,E}$ is bounded, then $\{u_k\}$ has a subsequence which converges weakly in $L_2(E)$.*

Now we can prove

THEOREM 3.12. *Let Ω be a domain having the segment property, and let $u \in L_2(\Omega)$. If there is a sequence $\{u_k\} \subset H_m(\Omega)$ which is bounded in $H_m(\Omega)$ and converges weakly to u in $L_2(\Omega)$, then $u \in H_m(\Omega)$, and $D^\alpha u$ is the weak limit of $D^\alpha u_k$ in $L_2(\Omega)$ for $|\alpha| \le m$.*

Proof. Since $\{u_k\}$ is bounded in $H_m(\Omega)$, $\{D^\alpha u_k\}$ is bounded in $L_2(\Omega)$ if $|\alpha| \le m$. By the weak compactness theorem, $\{u_k\}$ has a subsequence $\{u_{k_i}\}$ such that $\{D^\alpha u_{k_i}\}$ converges weakly in $L_2(\Omega)$ if $|\alpha| \le m$, say

$$D^\alpha u_{k_i} \rightharpoonup u^\alpha$$

in $L_2(\Omega)$. Thus, for any $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega u^\alpha \phi dx = \lim_{k_i \to \infty} \int_\Omega D^\alpha u_{k_i} \phi dx$$

$$= \lim_{k_i \to \infty} (-1)^{|\alpha|} \int_\Omega u_{k_i} D^\alpha \phi dx$$

$$= (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi dx.$$

Thus u has the (uniquely determined) weak derivative $D^\alpha u = u^\alpha$; it follows that $u \in W_m(\Omega) = H_m(\Omega)$, and that

(3.21) $\quad D^\alpha u_{k_i} \rightharpoonup D^\alpha u$ in $L_2(\Omega)$ for $|\alpha| \le m$.

Since every subsequence of $\{u_k\}$ has a subsequence $\{u_{k_i}\}$ such that (3.21) holds, it follows that $D^\alpha u_k \longrightarrow D^\alpha u$ in $L_2(\Omega)$ for $|\alpha| \le m$. Q.E.D.

Definition 3.3. *If e^i is the unit vector $(\delta_{1i}, \delta_{2i}, \ldots, \delta_{ni})$ and if h is a real number, then the difference operator δ_h^i is defined by*

$$\delta_h^i u = h^{-1}[u(x + he^i) - u(x)]$$

for any function u.

THEOREM 3.13. *If $u \in H_m(\Omega)$, where $m \ge 1$, if $\bar{\Omega}' \subset \Omega$, and if dist $(\bar{\Omega}', \partial\Omega) > h > 0$, then*

$$\|\delta_h^i u\|_{m-1,\,\Omega'} \le \|u\|_{m,\,\Omega}.$$

Proof. For any function $f \in C^1(a, b + h)$, we have

$$f(x + h) - f(x) = \int_x^{x+h} f'(\xi)\, d\xi$$

whence, by the Cauchy-Schwarz inequality

$$|f(x + h) - f(x)|^2 \le h \int_x^{x+h} |f'(\xi)|^2\, d\xi .$$

Thus $\displaystyle\int_a^b |f(x + h) - f(x)|^2\, dx \le h \int_a^b \int_x^{x+h} |f'(\xi)|^2\, d\xi\, dx$

$$= h \int_a^{a+h} \int_a^{\xi} |f'(\xi)|^2\, dx d\xi$$

$$+ h \int_{a+h}^{b} \int_{\xi-h}^{\xi} |f'(\xi)|^2\, dx d\xi$$

$$+ h \int_b^{b+h} \int_{\xi-h}^{b} |f'(\xi)|^2\, dx d\xi$$

$$\leq h^2 \int_a^{b+h} |f'(\xi)|^2 \, d\xi .$$

Hence

$$(3.22) \qquad \int_a^b \left| \frac{f(x+h) - f(x)}{h} \right|^2 dx \leq \int_a^{b+h} |f'(\xi)|^2 \, d\xi .$$

Now if $u_k \in C^m(\Omega)$, then on integrating, using iterated integrals, and applying (3.22), we see that

$$\int_{\Omega'} |D^\alpha \delta_h^i u_k|^2 \, dx = \int_{\Omega'} |\delta_h^i D^\alpha u_k|^2 \, dx$$

$$\leq \int_{\Omega} |D^{e^i} D^\alpha u_k|^2 \, dx \text{ for } |a| \leq m-1,$$

so that

$$||\delta_h^i \; u_k||^2_{m-1,\, \Omega'} \leq ||u_k||^2_{m,\, \Omega} .$$

Finally, if $u_k \to u$ in $H_m(\Omega)$, then we see that

$$||\delta_h^i u||^2_{m-1,\, \Omega'} \leq ||u||^2_{m,\, \Omega} . \quad \text{Q.E.D.}$$

In the periodic case we can do even better.

THEOREM 3.14. *If* $u \in H_m^{\#}$, *then* $||\delta_h^i u||_{m-1,\, Q} \leq ||u||_{m,\, Q}$ for all $h > 0$.

Proof. Let h be fixed and let r be a fixed integer larger than $2h$. Now apply Theorem 3.13 to the case $\Omega = NQ = \{x: |x_k| < N/2\}$ *and* $\Omega' = (N-r)\, Q$, where $N > r$. For $x \in \partial\Omega$ and $y \in \bar{\Omega}'$,

$|x - y| \geq r/2 > h$, so by Theorem 3.13

$$\|\delta_h^i u\|^2_{m-1,\,(N-r)\,Q} \leq \|u\|^2_{m,\,NQ}.$$

By periodicity this inequality may be written

$$(N - r)^n \|\delta_h^i u\|^2_{m-1,\,Q} \leq N^n \|u\|^2_{m,\,Q}.$$

Now simply divide by N^n and let $N \to \infty$ to obtain the result. Q.E.D.

THEOREM 3.15. *Assume that Ω has the segment property. If $u \in H_m(\Omega)$, and if there is a number C such that for every $\Omega' \subset\subset \Omega$, $\|\delta_h^i u\|_{m,\,\Omega'} \leq C$ for all h sufficiently small, then $u \in H_{m+1}(\Omega)$ and $\|D_i u\|_{,\,\Omega} \leq C$ for $i = 1,\dots,n$.*

Proof. First we prove the theorem with $m = 0$. By the weak compactness theorem there is a sequence $\{h_k\}$ of real numbers with $h_k \to 0$ and functions $u_i \in L_2(\Omega')$ such that $\delta_{h_k}^i u \rightharpoonup u_i$ in $L_2(\Omega')$ as $k \to \infty$, $i = 1,\dots,n$. Clearly $\|u_i\|_{0,\,\Omega'} \leq C$. For any $\phi \in C_0^\infty(\Omega')$,

$$\int_{\Omega'} u_i \phi\, dx = \lim_{k\to\infty} \int_{\Omega'} \delta_{h_k}^i u\phi\, dx$$

$$= -\lim_{k\to\infty} \int_{\Omega'} u\delta_{-h_k}^i \phi\, dx$$

$$= -\int_{\Omega'} uD_i \phi\, dx.$$

As this relation holds for any $\Omega' \subset\subset \Omega$, u_i is uniquely determined in Ω and is equal to the weak derivative $D_i u$ there. Hence $u \in W_1^{\text{loc}}(\Omega) = H_1^{\text{loc}}(\Omega)$.

Now proceed by induction. Suppose the theorem is true if m is replaced by $m - 1$. As above, there is a sequence $\{h_k\}$ such that $\delta_{h_k}^i u \rightharpoonup D_i u$ in $L_2(\Omega')$, $i = 1,\dots,n$. By Theorem 3.12 $D_i u \in H_m(\Omega')$, $i = 1,\dots,n$. Hence $u \in W_{m+1}(\Omega')$ by Theorem 2.4. That $\|D_i u\|_{m,\,\Omega'} \leq C$ is immediate. Therefore $u \in W_{m+1}(\Omega)$ and Theorem 2.2 implies $u \in H_{m+1}(\Omega)$. Q.E.D.

THEOREM 3.16. *Let* $\Sigma_r = \{x: |x| < r, x_n > 0\}$, $r > 0$. *If* $u \in L_2(\Sigma_R)$, *and if there is a number C such that for every* $R' < R$ *and all h sufficiently small*

$$||\delta_h^i u||_{0, \Sigma_{R'}} \leq C \text{ for } i = 1,\dots,n-1,$$

then the weak derivatives $D_i u$, $i = 1,\dots,n-1$, *exist as functions in* $L_2(\Sigma_R)$ *and*

$$||D_i u||_{0, \Sigma_R} \leq C, \quad i = 1,\dots,n-1,$$

Proof. The proof is the same as that of Theorem 3.15. Q.E.D.

4. Elliptic Operators

Let $A(x, D)$ be a linear differential operator of order ℓ, that is

$$A(x, D) = \sum_{|\alpha| \leq \ell} a_\alpha(x) D^\alpha$$

where the coefficients $a_\alpha(x)$ are complex valued functions defined in some open set Ω in E_n. (It is assumed that not all coefficients a_α with $|\alpha| = \ell$ vanish indentically.) We associate with $A(x, D)$ the homogeneous polynomial in $\xi = (\xi_1,\dots,\xi_n)$ of degree ℓ:

$$A'(x, \xi) = \sum_{|\alpha| = \ell} a_\alpha(x) \xi^\alpha .$$

The corresponding differential operator $A'(x, D)$ is called the *principal part* of $A(x, D)$. We now introduce the notion of an elliptic operator.

Definition 4.1. $A = A(x, D)$ *is said to be* elliptic *at a point* x^0 *if and only if for any real* $\xi \neq 0$, $A'(x^0, \xi) \neq 0$. A *is* uniformly elliptic *in a domain* Ω *if and only if there is a constant C such that*

$$C^{-1} |\xi|^\ell \leq |A'(x, \xi)| \leq C |\xi|^\ell$$

for all real $\xi \neq 0$ *and all* $x \in \Omega$. *A is* strongly elliptic *if and only if there is a function* C (x) *such that*

$$\Re(C(x)\,A'(x, \xi)) > 0$$

for all real $\xi \neq 0$.

Note that uniform ellipticity and strong ellipticity are distinct properties of differential operators. In discussing strongly elliptic operators we will assume that a normalization has been made so that C (x) can be taken to be identically constant. Note that strongly elliptic operators are necessarily of even order.

The first result concerns the order of an elliptic operator.

THEOREM 4.1. *Let A be an elliptic operator of order* ℓ *at* x^0. *If*

1° *the coefficients of* A' *are real,*

or if

2° $n \geq 3$,

then ℓ *is even.*

Proof. Let $\xi' = (\xi_1,\ldots,\xi_{n-1})$, and let $P(\xi', \xi_n) = A'(x^0, \xi)$. Then

$$P(\xi', \xi_n) = b_0\,\xi_n^{\ell} + b_1(\xi')\,\xi_n^{\ell-1} + \ldots + b_{\ell}(\xi'),$$

where $b_i(\xi')$ is a homogeneous polynomial in ξ' of degree i, $i = 1,\ldots,n$. Since $A'(x^0, e^n) = P(0, 1) = b_0$ and A is elliptic at x^0, $b_0 \neq 0$.

Clearly, in case 1° it is sufficient to assume that $b_0 > 0$. Let ξ' be fixed, $\xi' \neq 0$. If ℓ is odd, $P(\xi', \xi_n)$ has the same sign as ξ_n for sufficiently large $|\xi_n|$. Since $P(\xi', \xi_n)$ is real and depends continuously on ξ_n, there is a real value of ξ_n for which $P(\xi', \xi_n) = 0$. This contradicts the ellipticity of A. Hence ℓ is even.

In case 2°, for $\xi' \neq 0$ let $N^+(\xi')$ $[N^-(\xi')]$ be the number of complex zeros ξ_n of $P(\xi', \xi_n) = 0$ having positive [negative] imaginary part. Since A is elliptic, there are no real zeros, so that $N^+(\xi') + N^-(\xi') \equiv \ell$. We will show that $N^+(\xi') \equiv N^-(\xi')$; from this it follows that ℓ is even. Note that $P(-\xi', -\xi_n) = (-1)^{\ell} P(\xi', \xi_n)$. Thus, any zero of $P(\xi', \xi_n)$ is also a zero of $P(-\xi', -\xi_n)$; hence $N^+(\xi') = N^-(-\xi')$.

Now we use Rouché's theorem to show that for $\xi'_0 \neq 0$, the number of zeros $N^+(\xi')$ in the upper half plane is constant for ξ' near ξ'_0. Let

Γ be a contour in the upper half plane containing all of the zeros of $P(\xi_0', \xi_n)$ which lie in the upper half plane. Then $P(\xi_0', \xi_n)$ does not vanish on Γ. Since $P(\xi', \xi_n)$ is a continuous function of ξ', for ξ' sufficiently near ξ_0'

$$|P(\xi_0', \xi_n) - P(\xi', \xi_n)| < |P(\xi_0', \xi_n)| \text{ on } \Gamma.$$

Hence, by Rouche's theorem, $P(\xi', \xi_n)$ and $P(\xi_0', \xi_n)$ have the same number of zeros within Γ. Applying the same technique to the lower half plane, it follows that $N^+(\xi_0') = N^+(\xi')$ for ξ' close to ξ_0'. Therefore, $N^+(\xi')$ is a continuous function of ξ' for $\xi' \neq 0$. Since $n \geq 3$, $\{\xi' \in E_{n-1}: \xi' \neq 0\}$ is a connected subset of E_{n-1}; $N^+(\xi')$ is a continuous, integer-valued function on a connected set, and therefore must be constant. Therefore, $N^+(\xi') = N^+(-\xi') = N^-(\xi')$; it follows that $\ell = N^+(\xi') + N^-(\xi') = 2N^+(\xi')$ must be even. Q.E.D.

Two classical examples of elliptic operators are the Laplacian $\Delta = D_1^2 + \ldots + D_n^2$ and, for $n = 2$, the Cauchy-Riemann operator $\frac{1}{2}(D_1 + iD_2)$.

5. Local Existence Theory

In this section it will be extablished that the elliptic equation $Au = f$ always has weak solutions in small neighborhoods, if $f \in L_2$ and mild assumptions are made on the coefficients in the differential operator A. First we treat the case which A has constant coefficients and coincides with its principal part. We shall also momentarily restrict our considerations to periodic f. Recall that Q is the cube $\{x: |x_k| < 1/2\}$.

THEOREM 5.1. *Let* $A(D) = \sum_{|\alpha| = \ell} a_\alpha D^\alpha$ *be an elliptic operator or order* ℓ *having constant coefficients. Then for every* $f \in L_2(Q)$ *such that* $\int_Q f dx = 0$ *there is a unique* u *satisfying*

$$u \in H_\ell^\#,$$

$$Au = f,$$
$$\int_Q u\,dx = 0.$$

Denote this uniquely determined u as $T^{\#} f$. Then there is a constant N depending only on A such that $\|T^{\#} f\|_{\ell, Q} \le N \|f\|_{0, Q}$. Moreover, if f has the Fourier series expansion

$$(5.1) \qquad f(x) = \sum_{\xi}{}' c_{\xi}\, e^{2\pi i x\cdot\xi},$$

then

$$(5.2) \qquad u(x) = (2\pi i)^{-\ell} \sum_{\xi}{}' \frac{c_{\xi}}{A(\xi)}\, e^{2\pi i x\cdot\xi},$$

where $\sum_{\xi}{}'$ stands for summation over all $\xi \ne 0$, $\xi_1, \ldots, \xi_n$ integers.

Proof. Note first that if $f \in L_2(Q)$ has the Fourier series expansion $\sum_{\xi} c_{\xi}\, e^{2\pi i x\cdot\xi}$, then $c_{(0,\ldots,0)} = 0$ if and only if $\int_Q f\,dx = 0$. Now suppose $u \in H^{\#}_{\ell}$, and $u = \sum_{\xi}{}' b_{\xi}\, e^{2\pi i x\cdot\xi}$. Then $Au = \sum_{\xi}{}' b_{\xi} A(2\pi i\xi)\, e^{2\pi i x\cdot\xi} = (2\pi i)^{\ell} \sum_{\xi}{}' A(\xi)\, b_{\xi}\, e^{2\pi i x\cdot\xi}$. Therefore, if f is given by (5.1) and $Au = f$, $A(\xi)\, b_{\xi} = c_{\xi}$. As A is elliptic, $A(\xi) \ne 0$ for $\xi \ne 0$, so that $b_{\xi} = c_{\xi}/A(\xi)$. Therefore, u is uniquely determined by the relation (5.2). Obviously, the function given by (5.2) is indeed a solution of $Au = f$. Finally, as $|A(\xi)| \ge c^{-1} |\xi|^{\ell}$,

$$\|u\|^{\#}_{\ell} = (2\pi)^{-\ell} \left[\sum_{\xi}{}' |c_{\xi}|^2 \, |A(\xi)|^{-2} \, |\xi|^{2\ell}\right]^{1/2}$$

$$\le (2\pi)^{-\ell} c \left[\sum_{\xi}{}' |c_{\xi}|^2\right]^{1/2}$$

$$= (2\pi)^{-\ell} c\, \|f\|^{\#}_{0}.$$

Since $\| \quad \|_m^{\#}$ and $\| \quad \|_{m,\,Q}$ are equivalent for all m, the estimate on $T^{\#} f$ is proved. Q.E.D.

It will also be necessary to solve the equation $Au = f$ for functions f which do not have mean value zero. Since in this section we wish only to state that *at least one* solution u exists, it will be sufficient here to construct a single particular solution of the equation $Au = 1$. This can be accomplished very simply in the present case of A having constant coefficients and $A = A'$. For since A is elliptic,

the coefficient in A of $D_1^{\ell} = \dfrac{\partial^{\ell}}{\partial x_1^{\ell}}$ cannot vanish: if $a = a_{(\ell,\,0,\dots,0)}$,

then $a \neq 0$. If $u_0(x) = x_1^{\ell}/a\ell!$, then $Au_0 = 1$.

Now if $f \in L_2(Q)$, let

$$(5.3) \qquad \mu(f) = \int_Q f dx.$$

Then the function $f - \mu(f)$ has mean value zero, and $u' = T^{\#}(f - \mu(f))$ is a solution of $Au' = f - \mu(f)$. Now define $u = Tf$, where

$$(5.4) \qquad Tf = T^{\#}(f - \mu(f)) + \mu(f)\, u_0.$$

Then $Au = [f - \mu(f)] + \mu(f) = f$, and we have the estimate

$$(5.5) \qquad \|Tf\|_{\ell,\,Q} \leq N\, \|f - \mu(f)\|_{0,\,Q} + |\mu(f)|\; \|u_0\|_{\ell,\,Q}$$

$$\leq N'\, \|f\|_{0,\,Q}$$

where we have used the estimate $|\mu(f)| \leq \|f\|$. Here $N' = N + 1 + \|u_0\|_{\ell,\,Q}$ depends only on A.

Now we pass to the general elliptic operator.

THEOREM 5.2. *Let $A(x, D) = \sum_{|\alpha| \leq \ell} a_\alpha(x) D^\alpha$ be an elliptic operator of order ℓ in a domain Ω. Let $a_\alpha(x)$ be continuous if $|a| = \ell$ and measurable and locally bounded if $|a| < \ell$. Then for any $x^0 \in \Omega$ there exists a neighborhood 0 of x^0 with the property that for any $f \in L_2(0)$ there exists $u \in H_\ell(0)$ satisfying $Au = f$ in 0.*

Proof. By a suitable change of coordinates it may be assumed that $x^0 = 0$ and $Q \subset \Omega$. Let Q_δ be the cube $\{x: |x_k| < \delta/2\}$, and introduce the coordinate transformation $x = \delta y$. Then if $D_x^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$, $D_x^\alpha = \delta^{-|\alpha|} D_y^\alpha$. If we let $v(y) = u(x) = u(\delta y)$, then the equation $A(x, D_x)\, u(x) = f(x)$ takes the form (after multiplying by δ^ℓ)

$$(5.6) \qquad \sum_{|\alpha| = \ell} a_\alpha(\delta y)\, D_y^\alpha v + \sum_{|\alpha| < \ell} a_\alpha(\delta y)\, \delta^{\ell - |\alpha|} D_y^\alpha v = \delta^\ell f(\delta y).$$

Letting $g(y) = \delta^\ell f(\delta y)$, (5.6) may be written for $y \in Q$

$$(5.7) \qquad A'(0, D_y)\, v - B_\delta(y, D_y)\, v = g(y).$$

where B_δ is a differential operator of order not exceeding ℓ, and B_δ has coefficients which tend to zero with δ. (We have used the assumptions on the a_α at this stage.) Thus, for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$(5.8) \qquad \|B_\delta v\|_{0,Q} \leq \epsilon \|v\|_{\ell,Q}$$

for all $v \in H_\ell(Q)$. Now let T be the solution operator for $A'(0, D_y)$; T has the form (5.4) and satisfies and estimate (5.5). Upon multiplying (5.7) by T, it will suffice to solve the equation

$$(5.9) \qquad v - TB_\delta\, v = Tg$$

for some $v \in H_\ell(Q)$. Let ϵ be chosen to satisfy $N'\epsilon < 1$ and let δ be chosen so that (5.8) is valid; then TB_δ is a bounded linear transformation of $H_\ell(Q)$ into $H_\ell(Q)$ satisfying

$$\|TB_\delta v\|_{\ell,Q} \leq N' \|B_\delta v\|_{0,Q} \leq N' \epsilon \|v\|_{\ell,Q}.$$

Since $N'\epsilon < 1$, the operator $1 - TB_\delta$ (where 1 is the identity on $H_\ell(Q)$) has an inverse, and the solution of (5.9) is $v = (1 - TB_\delta)^{-1} Tg$. Finally, $u(x) = v(\delta^{-1}x)$ is a solution of $Au = f$ for $|x| < \delta$. Q.E.D.

6. Local Regularity of Solutions of Elliptic Systems

In this section several results shall be derived showing regularity properties of weak solutions of elliptic systems. The gist of these results is that weak solutions of elliptic equations can be shown to have regularity properties not explicitly required of weak solutions of differential equations. Or, the smoother the data, the smoother the solution. In particular, we shall show that weak solutions of elliptic equations with infinitely differentiable coefficients and inhomogeneous terms are themselves infinitely differentiable.

First we introduce some convenient terminology.

Definition 6.1. *Let* $A(x, D) = \sum_{|\alpha| \leq \ell} a_\alpha(x) D^\alpha$ *be a differential operator of order* ℓ, *where* $a_\alpha(x)$ *is defined for* x *belonging to a domain* Ω. *We shall say that* $A(x, D)$ *is s-smooth in* Ω *if* $a_\alpha \in C^{|\alpha|-\ell+s}(\Omega)$ *for* $|\alpha| > \ell - s$, *and* a_α *is measurable and locally bounded for* $|\alpha| \leq \ell - s$.

Now weak solutions of differential equations with nonconstant coefficients are defined in the same fashion as weak solutions of equations with constant coefficients; cf. Definition 2.2. Thus, suppose that $A(x, D)$ is s-smooth in Ω, $s \geq \ell$, and suppose u is a C^ℓ solution of $Au = f$ in Ω. Then if $\phi \in C_0^\infty(\Omega)$, integration by parts shows that

$$(6.1) \qquad (Au, \phi)_{0,\Omega} = (u, A^* \phi)_{0,\Omega},$$

where

$$(6.2) \qquad A^* \phi = \sum_{|\alpha| \leq \ell} (-1)^{|\alpha|} D^\alpha (\overline{a_\alpha} \phi).$$

The operator defined by (6.2) is called the *formal adjoint* of A. Since $a_\alpha \in C^{|\alpha|}$, upon performing the indicated differentiations in (6.2) and using Leibnitz's rule, it is seen that A^* is differential operator of order ℓ with continuous coefficients. In fact, the coefficient of $D^\beta \phi$ in (6.2) is precisely

$$(6.3)\qquad \sum_{\substack{|\alpha|\leq \ell\\ \beta\leq\alpha}} (-1)^{|\alpha|}\binom{\alpha}{\beta} D^{\alpha-\beta}\,\overline{a_\alpha}\,;$$

as $a_\alpha \in C^{|\alpha|-\ell+s}$, the expression (6.3) belongs to $C^{|\beta|-\ell+s}$. Therefore, if A is s-smooth, $s \geq \ell$, then also A^* is s-smooth, and an easy computation shows that $A^{**} = (A^*)^* = A$.

Also, (6.3) shows that the coefficient of $D^\beta \phi$ in $A^*\phi$ for $|\beta| = \ell$ is just $(-1)^\ell\,\overline{a_\beta}$; consequently, the principal part of A^* is $(-1)^\ell\,\overline{A'}$. Thus, A is elliptic if and only if A^* is elliptic.

Now we make a definition of weak solution based upon (6.1). For convenience we will frequently write $(\ ,\ )$ in place of $(\ ,\ )_{0,\,\Omega}$.

Definition 6.2. *Let u and f be locally integrable in Ω and let $A(x, D)$ be ℓ-smooth and of order ℓ. Then u is a weak solution in Ω of $Au = f$ if for all $\phi \in C_0^\infty(\Omega)$*

$$(6.4)\qquad (f, \phi) = (u, A^*\phi).$$

Now suppose u is a weak solution in Ω of $Au = f$, and also that $f \in L_2(\Omega)$. Then by the Cauchy-Schwarz inequality

$$|(u, A^*\phi)| = |(f, \phi)| \leq ||f||_{0,\,\Omega}\, ||\phi||_{0,\,\Omega}.$$

Thus, there is a constant C such that

$$(6.5)\qquad |(u, A^*\phi)| \leq C\, ||\phi||_{0,\,\Omega},\ \text{all } \phi \in C_0^\infty(\Omega).$$

In proving the first regularity theorems of this section, we shall not even need to assume that u is a weak solution of $Au = f$, i.e., that (6.4) is satisfied, but only that (6.5) is satisfied.

Since for sufficiently smooth coefficients, $A = A^{**}$, and since A^* and A are simultaneously elliptic, in dealing with inequalities like (6.5) we shall replace A by A^*. This is more convenient notation and it gives more general results.

We shall also derive results using a generalization of (6.5). To motivate the considerations suppose that $u \in H_j^{\text{loc}}(\Omega)$, that $0 \leq j \leq \ell$, and that A is j-smooth. Then for $\phi \in C_0^\infty(\Omega)$, $(u.\ a_\alpha D^\alpha \phi) = (\overline{a_\alpha}\, u, D^\alpha \phi) = (D^\beta(\overline{a_\alpha}\, u), D^{\alpha-\beta}\phi)$, where D^β is any derivative such that $\beta \leq \alpha$ and $|\beta| = |\alpha| - \ell + j$, and then we obtain the estimate for $\phi \in C_0^\infty(\Omega)$ and $|\alpha| > \ell - j$

$$|(u, a_\alpha D^\alpha \phi)| \leq \text{const } ||D^{\alpha-\beta}\phi||_{0,\Omega}$$

$$\leq \text{const } |\phi|_{\ell-j,\Omega},$$

where the constant depends only on u, a_α, and supp (ϕ). For $|\alpha| \leq \ell - j$ we have by the Cauchy-Schwarz inequality that

$$|(u, a_\alpha D^\alpha \phi)| \leq \text{const } ||\phi||_{\ell-j,\Omega}.$$

Summing over all α, $|\alpha| \leq \ell$, we obtain

$$(6.6) \qquad |(u, A\phi)| \leq \text{const } ||\phi||_{\ell-j,\Omega}, \quad \phi \in C_0^\infty(\Omega).$$

Note that (6.5) is just the case $j = \ell$. It will be shown below that if (6.6) holds and A is elliptic and j-smooth, then $u \in H_j^{\,loc}(\Omega)$; cf. Theorem 6.3.

The key to all the following regularity results is now given in a lemma. We have to work hard to establish the lemma; having the lemma, all the other regularity results given in this section are proved with great ease.

LEMMA 6.1. *Let* $A(x, D) = \sum_{|\alpha|=\ell} a_\alpha(x) D^\alpha$ *be an elliptic operator of order* ℓ *in the cube* Q *and let* $a_\alpha \in C_\#^0$ *and satisfy a Lipschitz condition*

$$|a_\alpha(x) - a_\alpha(y)| \leq K\,|x - y|.$$

Let E_0 *be the largest constant such that*

$$E_0\,|\xi|^\ell \leq |A(0, \xi)|, \ \xi \text{ real.}$$

Then there exists a positive number ω_0 *depending on* E_0, n, *and* ℓ, *such that if* $|a_\alpha(x) - a_\alpha(0)| \leq \omega_0$, *then the following assertion is valid.*

If $u \in L_2(Q)$ *and if for all* $v \in C_\#^\infty$

(6.7) $$|(u, Av)| \leq C\,\|v\|_{\ell-1,\,Q},$$

then $u \in H_1(Q)$ *and*

$$\|u\|_{1,Q} \leq \gamma\,(C + \|u\|_{0,Q}),$$

where γ *depends only on* E_0, n, ℓ, *and* K.

Proof. We first prove the result under the assumption that u has mean value zero, i.e., that $\int_Q u\,dx = 0$. Then we extend u to a periodic function on E_n, so that we have $u \in H_0^\#$. Let i be fixed and consider the function $\delta_h^i u$, which we write $\delta_h u$ for the time being (cf. Definition 3.3). Note that $\delta_h u \in H_0^\#$ and $\delta_h u$ has mean value zero. Let $v_h \in H_\ell^\#$ be the unique solution of $A(0, D)\,v_h = \delta_h u$ having mean value zero; the existence of v_h is guaranteed by Theorem 5.1, where it is also shown that

(6.8) $$\|v_h\|_{\ell,Q} \leq N\,\|\delta_h u\|_{0,Q},$$

where N depends only on E_0, n, and ℓ.

Since $H_\ell^\#$ is the completion of $C_\#^\infty$ with respect to the norm $\|\ \|_{\ell,Q}$, it follows that (6.7) holds not only for $v \in C_\#^\infty$, but also for $v \in H_\ell^\#$. Now $\delta_{-h} v_h \in H_\ell^\#$, since $v_h \in H_\ell^\#$, and therefore we may in (6.7) insert for v the function $\delta_{-h} v_h$: We obtain

$$|(u, A\delta_{-h} v_h)| \leq C\,\|\delta_{-h} v_h\|_{\ell-1,Q}.$$

By Theorem 3.14 and by (6.8)

(6.9) $$|(u, A\delta_{-h} v_h)| \leq C\,\|v_h\|_{\ell,Q}$$

$$\leq CN\,\|\delta_h u\|_{0,Q}.$$

Now since $D^\alpha \delta_{-h} v_h = \delta_{-h} D^\alpha v_h$.

$$(6.10) \qquad (u, A\delta_{-h} v_h) = \sum_{|\alpha|=\ell} (u, a_\alpha D^\alpha (\delta_{-h} v_h))$$

$$= \sum_{|\alpha|=\ell} (u, a_\alpha \delta_{-h} D^\alpha v_h).$$

Now we need a formula analogous to Leibnitz's rule for derivatives: for two functions $f(x)$ and $g(x)$

$$\delta^i_{-h} (fg) = f(x)\, \delta_{-h} g + g(x - he^i)\, \delta_{-h} f.$$

Applying this formula in the case $f = a_\alpha$ and $g = D^\alpha v_h$, (6.10) becomes

$$(u, A\delta_{-h} v_h) = \sum_{|\alpha|=\ell} (u, \delta_{-h} (a_\alpha D^\alpha v_h))$$

$$- \sum_{|\alpha|=\ell} (u, D^\alpha v_h (x - he^i) \cdot \delta_{-h} a_\alpha)$$

$$(6.11) \qquad = -\sum_{|\alpha|=\ell} (\delta_h u, a_\alpha D^\alpha v_h)$$

$$- \sum_{|\alpha|=\ell} (u, D^\alpha v_h (x - he^i) \cdot \delta_{-h} a_\alpha)$$

$$= -(\delta_h u, A(x, D) v_h)$$

$$- \sum_{|\alpha|=\ell} (u, D^\alpha v_h (x - he^i) \cdot \delta_{-h} a_\alpha);$$

the second equality follows from the periodicity of a_α, v_h, and u. Next, since $A(0, D) v_h = \delta_h u$, (6.11) implies

$$(6.12) \qquad (u, A\delta_{-h} v_h) + (\delta_h u, \delta_h u) = (\delta_h u, [A(0, D) - A(x, D)] v_h)$$

$$- \sum_{|\alpha|=\ell} (u, D^\alpha v_h (x - he^i) \cdot \delta_{-h} a_\alpha).$$

Now let

$$\omega = \sup_{x \in Q, |\alpha| = \ell} |a_\alpha(x) - a_\alpha(0)|.$$

Let $p = p(n, \ell)$ be the number of derivatives D^α of order $|\alpha| = \ell$. Then the Cauchy-Schwarz inequality applied to (6.12) implies

$$|(u, A\delta_{-h} v_h)| + \|\delta_h u\|^2_{0,Q} \leq \|\delta_h u\|_{0,Q}\, \omega p\, \|v_h\|_{\ell,Q}$$

$$+ p\, \|u\|_{0,Q}\, K\, \|v_h\|_{\ell,Q}.$$

(We have used $|\delta_{-h} a_\alpha| \leq K$, a consequence of the Lipschitz condition assumed on a_α.) Combining this inequality and (6.9), (6.8), we obtain

$$\text{(6.13)} \qquad \|\delta_h u\|^2_{0,Q} \leq CN\, \|\delta_h u\|_{0,Q} + \omega pN\, \|\delta_h u\|^2_{0,Q}$$

$$+ pKN\, \|u\|_{0,Q}\, \|\delta_h u\|_{0,Q}.$$

We choose $\omega_0 = 1/2pN$. Then $\omega \leq \omega_0$, so that after dividing both sides of (6.13) by $\|\delta_h u\|_{0,Q}$,

$$\|\delta_h u\|_{0,Q} \leq CN + (1/2)\|\delta_h u\|_{0,Q} + pKN\, \|u\|_{0,Q};$$

thus,

$$\|\delta_h u\|_{0,Q} \leq 2CN + 2pKN\, \|u\|_{0,Q}.$$

Then, if $\gamma'_1 = \max(2N, 2pKN)$,

$$\|\delta^i_h u\|_{0,Q} \leq \gamma'_1 (C + \|u\|_{0,Q}),\ i = 1,\ldots,n.$$

This result holds for all positive h; as $u \in H_0(Q)$, Theorem 3.15 implies that $u \in H_1(Q)$ and

$$\|u\|_{1,Q} \leq \gamma_1 (C + \|u\|_{0,Q}).$$

This completes the proof in the case that u has mean value zero.

We now eliminate this assumption. This is relatively simple. In fact, ω_0 can be left unchanged. Let $u \in L_2(Q)$ and let $\mu(u) = \int_Q u dx$. We now show that $u' = u - \mu(u)$, which has mean value zero, satisfies all the requirements of the lemma.

Now

$$(6.14) \qquad (u', Av) = (u, Av) - \mu(u) \overline{\int_Q Avdx}.$$

The quantity $\int_Q Avdx$ is a sume of terms of the form $\int_Q a_\alpha D^\alpha vdx$ for $|\alpha| = \ell$. For each such α there is an index i such that $D^\alpha = D_i D^\beta$, $|\beta| = \ell - 1$. Thus, for $v \in C^\infty_\#$

$$\begin{aligned}
\int_Q a_\alpha D^\alpha vdx &= \int_Q a_\alpha D_i D^\beta vdx \\
&= \lim_{h \to 0} \int_Q a_\alpha \delta_h^i D^\beta vdx \\
&= \lim_{h \to 0} - \int_Q \delta_{-h}^i a_\alpha \cdot D^\beta vdx,
\end{aligned}$$

since a_α and v are periodic. Thus, the Lipschitz continuity of a_α implies

$$\begin{aligned}
\left|\int_Q a_\alpha D^\alpha vdx\right| &\leq K \int_Q |D^\beta v|\, dx \\
&\leq K \left[\int_Q |D^\beta v|^2 dx\right]^{1/2} \\
&\leq K\, ||v||_{\ell-1,Q}.
\end{aligned}$$

Hence,

$$\left|\overline{\int_Q Avdx}\right| \leq pK\, ||v||_{\ell-1,Q}.$$

Using this inequality in (6.14), together with the inequality $|\mu(u)| \leq ||u||_{0,Q}$, we find

$$|(u', Av)| \leq |(u, Av)| + pK\, ||u||_{0,Q}\, ||v||_{\ell-1,Q}.$$

Then applying (6.7), it follows that for all $v \in C^\infty_\#$

$$|(u', Av)| \leq [C + pK\, ||u||_{0,Q}]\, ||v||_{\ell-1,\,Q}.$$

Since u' has mean value zero and since we have already proved the lemma for such u', it follows that $u' \in H_1\,(Q)$ and

$$||u'||_{1,Q} \leq \gamma_1\,(C + pK\, ||u||_{0,Q} + ||u||_{0,Q}).$$

Since $||u||_{1,Q} = ||u' + \mu(u)||_{1,Q} \leq ||u'||_{1,Q} + |\mu(u)|\; ||1||_{1,Q}$ $\leq ||u'||_{1,Q} + ||u||_{0,Q}$, it follows that

$$||u||_{1,Q} \leq \gamma\,(C + ||u||_{0,Q}),$$

where $\gamma = \max\,[\gamma_1, \gamma_1\,(pK + 1) + 1]$. Q.E.D. (Whew)

We can easily generalize this result to obtain REGULARITY THEOREMS (Theorems 6.2–6.7).

THEOREM 6.2. *Let* $A(x, D) = \sum_{|\alpha| \leq \ell} a_\alpha\,(x)\, D^\alpha$ *be uniformly elliptic in* Ω*, and let* E *be the largest constant such that*

$$E\, |\xi|^\ell \leq |A\,(x, \xi)|,\ x \in \Omega,\ \xi \text{ real.}$$

Let a_α *be Lipschitz continuous for* $|\alpha| = \ell$*:*

$$|a_\alpha\,(x) - a_\alpha(y)| \leq K\, |x - y|;$$

and let a_α *be bounded and measurable for* $|a| \leq \ell$*:*

$$|a_\alpha\,(x)| \leq M.$$

Let $u \in L_2\,(\Omega)$ *and suppose that for all* $\phi \in C^\infty_0\,(\Omega)$

(6.15) $|(u, A\phi)| \le C\, ||\phi||_{\ell-1,\Omega}.$

Then $u \in H_1^{\mathrm{loc}}(\Omega)$, *and for every* $\Omega' \subset\subset \Omega$

(6.16) $||u||_{1,\Omega'} \le \gamma\,(C + ||u||_{0,\Omega}),$

where γ *depends only on* E, n, ℓ, K, M, *the diameter of* Ω', *and the distance from* $\overline{\Omega'}$ *to* $\partial\Omega$.

Proof. We make several reductions to simpler cases. First, we may assume A coincides with its principal part A'. For if $B = A - A'$, then

$$|(u, A'\phi)| \le |(u, A\phi)| + |(u, B\phi)|$$

$$\le C\,||\phi||_{\ell-1,\Omega} + ||u||_{0,\Omega}\, C_1\, ||\phi||_{\ell-1,\Omega},$$

where C_1 depends only on n, ℓ, and M. Thus, an estimate like (6.15) holds with A replaced by A' and C replaced by $C + C_1\, ||u||_{0,\Omega}$. Therefore, if (6.16) were proved in the case in which A has no lower order terms, we would have

$$||u||_{1,\Omega'} \le \gamma\,(C + C_1\, ||u||_{0,\Omega} + ||u||_{0,\Omega}).$$

and the result would hold also in the general case.

Also, since the result is of a local nature, it is sufficient to obtain the estimate (6.16) with Ω' replaced by a neighborhood 0 of a fixed x^0, as long as 0 depends only on the quantities E, n, ℓ, K and M. And then by a coordinate transformation it is obviously sufficient to treat the case $x^0 = 0$ and $\Omega = Q$.

Let ω_0 be the number whose existence is guaranteed by Lemma 6.1 and let δ be a fixed number so small that $|a_\alpha(x) - a_\alpha(0)| \le \omega_0$ for $|x| \le \delta$, and $0 < \delta < 1/4$; δ depends only on K and ω_0. Let ζ be a fixed real function in $C_0^\infty(Q)$ such that $\zeta(x) \equiv 1$ for $|x| \le \delta/2$, $\zeta(x) \equiv 0$ for $|x| \ge \delta$, and $0 \le \zeta \le 1$.

Then for any $v \in C_\#^\infty$, $\zeta v \in C_0^\infty(Q)$, and upon applying (6.15) with $\phi = \zeta v$,

$$|(u, A(\zeta v))| \leq C\, ||\zeta v||_{\ell-1,Q} \leq CC_2\, ||v||_{\ell-1,Q}.$$

where C_2 depends only on ζ in accordance with Leibnitz's rule. By Leibnitz's rule again $A(\zeta v) = \zeta A v + B_1 v$, where B_1 is a differential operator of order less than ℓ. Therefore.

$$\begin{aligned} |(\zeta u, Av)| &= |(u, \zeta A v)| \\ &\leq |(u, A(\zeta v))| + |(u, B_1 v)| \\ &\leq CC_2\, ||v||_{\ell-1,Q} + ||u||_{0,Q}\, C_3\, ||v||_{\ell-1,Q} \\ &= [CC_2 + C_3\, ||u||_{0,Q}]\, ||v||_{\ell-1,Q}, \end{aligned} \tag{6.17}$$

and C_3 depends only on ζ and M.

We could now apply the lemma, in view of (6.17), except that A does not have periodic coefficients. This situation is remedied by redefining A outside the sphere $S_\delta = \{x: |x| < \delta\}$. One way this can be done is to let

$$a_\alpha^\#(x) = \begin{cases} a_\alpha(x), & |x| \leq \delta, \\ a_\alpha([2\delta |x|^{-1} - 1]\, x), & \delta < |x| < 2\delta \\ a_\alpha(0), & |x| \geq 2\delta,\ x \in \bar{Q}. \end{cases}$$

As $a_\alpha^\#$ is constant on ∂Q, we may extend $a_\alpha^\#$ by periodicity to a function on E_n. Obviously, the extended $a_\alpha^\# \in C_\#^0$ and satisfies a Lipschitz condition with the same constant K that suffices for a_α. Also the operator $A^\#(x, D) = \sum_{|\alpha|=\ell} a_\alpha^\#(x) D^\alpha$ is elliptic and the same constant E suffices for $A^\#$. Since $\zeta \equiv 0$ for $|x| \geq \delta$, (6.17) may be written

$$|(\zeta u, A^\# v)| \leq [CC_2 + C_3\, ||u||_{0,Q}]\, ||v||_{\ell-1,Q}.$$

Since this holds for all $v \in C^{\infty}_{\#}$, Lemma 6.1 implies that $\zeta u \in H_1(Q)$ and

$$||\zeta u||_{1,Q} \leq \gamma\, [CC_2 + C_3\, ||u||_{0,Q} + ||\zeta u||_{0,Q}].$$

As $\zeta \equiv 1$ for $|x| \leq \delta/2$, we have $u \in H_1(S_{\delta/2})$ and

$$||u||_{1,S_{\delta/2}} \leq \gamma\, [CC_2 + (C_3 + 1)\, ||u||_{0,Q}]. \quad \text{Q.E.D.}$$

This result can be generalized to obtain stronger estimates on u if inequality (6.15) is strengthened. For simplicity we shall not be quite so precise in exhibiting the dependence of γ on the problem.

THEOREM 6.3. *Let $A(x, D)$ be a j-smooth elliptic operator of order ℓ in Ω, where $1 \leq j \leq \ell$. Let $u \in H_2(\Omega)$ satisfy the condition*

$$(6.18) \qquad |(u, A\phi)| \leq C\, ||\phi||_{\ell-j,\Omega}, \text{ all } \phi \in C_0^{\infty}(\Omega).$$

Then $u \in H_j^{\,loc}(\Omega)$ and for every $\Omega' \subset\subset \Omega$

$$(6.19) \qquad ||u||_{j,\Omega'} \leq \gamma\, (C + ||u||_{0,\Omega}),$$

where γ depends only on A, Ω', and Ω.

Proof. The proof is by induction. We have already proved the result in the case $j = 1$ (Theorem 6.2). Assume the result holds with j replaced by $j - 1$, where $2 \leq j \leq \ell$.

Let $A_0(x, D) = \sum_{\ell-j<|\alpha|\leq \ell} a_{\alpha}(x)\, D^{\alpha}$. It can be assumed that $A = A_0$. For, suppose that the result has been established in such cases, and let $B = A - A_0$. Then for $\phi \in C_0^{\infty}(\Omega)$

$$|(u, A_0\, \phi)| \leq |(u, A\phi)| + |(u, B\phi)|$$

$$\leq C\, ||\phi||_{\ell-j,\Omega} + ||u||_{0,\Omega}\, C_1\, ||\phi||_{\ell-j,\Omega},$$

so that $u \in H_j^{\,loc}(\Omega)$ and $||u||_{j,\Omega'} \leq \gamma\, (C + C_1\, ||u||_{0,\Omega} + ||u||_{0,\Omega})$, where C_1 depends only on A.

Thus, we assume $A = A_0$. Since $\|\phi\|_{\ell-j,\Omega} \leq \|\phi\|_{\ell-(j-1),\Omega}$, (6.18) together with the inductive hypothesis implies that $u \in H_{j-1,\Omega}^{loc}$. Since $u \in W_{1,\Omega}^{loc}$, we have for $\phi \in C_0^\infty(\Omega)$

$$(6.20) \qquad (D_i u, A\phi) = -(u, D_i A\phi).$$

Now

$$D_i A\phi = D_i \sum_\alpha a_\alpha D^\alpha \phi = \sum_\alpha a_\alpha D^\alpha D_i \phi + \sum_\alpha D_i a_\alpha \cdot D^\alpha \phi$$

$$= AD_i \phi + B_i \phi,$$

and (6.20) implies

$$(6.21) \qquad |(D_i u, A\phi)| \leq |(u, AD_i\phi)| + |(u, B_i \phi)|.$$

Now suppose $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. If $\phi \in C_0^\infty(\Omega'')$, (6.18) implies

$$(6.22) \qquad |(u, AD_i \phi)| \leq C \|D_i \phi\|_{\ell-j,\Omega''} \leq C \|\phi\|_{\ell-j+1,\Omega''}.$$

Also

$$(u, B_i \phi) = \sum_{\ell-j+1\leq|\alpha|\leq\ell} (u, D_i a_\alpha \cdot D^\alpha \phi)$$

$$= \sum_{\ell-j+1\leq|\alpha|\leq\ell} (uD_i \overline{a_\alpha}, D^\alpha \phi)$$

$$= \sum_{\ell-j+1\leq|\alpha|\leq\ell} (-1)^{|\beta|} (D^\beta (uD_i \overline{a_\alpha}), D^{\alpha-\beta} \phi),$$

where for each α we choose a single index β such that $|\beta| = |\alpha| - \ell + j - 1$ and $\alpha - \beta \geq 0$. Since $u \in H_{j-1}(\Omega'')$ and A is j-smooth, the Cauchy-Schwarz inequality implies

$$(6.23) \qquad |(u, B_i \phi)| \leq C_2 \|u\|_{j-1,\Omega''} \|\phi\|_{\ell-j+1,\Omega''},$$

where C_2 depends only on A and Ω''. Combining (6.21), (6.22), and (6.23),

$$|(D_i\ u,\ A\phi)| \leq (C + C_2\ ||u||_{j-1,\Omega''})\ ||\phi||_{\ell-(j-1),\Omega''},$$

all $\phi \in C_0^\infty(\Omega'')$. By the inductive hypothesis that the result holds for j replaced by $j-1$, we obtain that $D_i\ u \in H_{j-1}^{loc}(\Omega'')$; (6.19) with j replaced by $j-1$ and u replaced by $D_i\ u$ implies

$$(6.24)\qquad ||D_i\ u||_{j-1,\Omega'} \leq \gamma\ (C + C_2\ ||u||_{j-1,\Omega''} + ||D_i\ u||_{0,\Omega''})$$

$$\leq \gamma\ (C + C_2 + 1)\ ||u||_{j-1,\Omega''}).$$

(Recall that $j \geq 2$.) As Ω'' is arbitrary, we see that (using Theorem 2.4) $u \in H_j^{loc}(\Omega)$ and applying (6.24) for $i = 1,\ldots,n$,

$$||u||_{j,\Omega'} \leq \gamma\ (C + (C_2 + 1)\ ||u||_{j-1,\Omega''}).$$

The result now follows from the inductive hypothesis, since $||u||_{j-1,\Omega''} \leq \gamma\ (C + ||u||_{0,\Omega})$. Q.E.D.

We now wish to extend these results to systems of equations.

Definition 6.3. *Let* $A_i\ (x, D) = \sum_{|\alpha| \leq m_i} a_\alpha^i\ (x)\ D^\alpha$ *be differential operators of orders* m_i *in* Ω, $i = 1,\ldots,N$. *This system of operators is* an over-determined elliptic system at a point $x^0 \in \Omega$ *if there is no real* $\xi \neq 0$ *such that*

$$\sum_{|\alpha| = m_i} a_\alpha^i\ (x^0)\ \xi^\alpha = 0,\ i = 1,\ldots,N.$$

The system is an over-determined elliptic system *in* Ω *if it is an over-determined elliptic system at each point of* Ω.

An example of an over-determined elliptic system is the pairs of operators $\square \equiv D_1^2 + \ldots + D_{n-1}^2 - D_n^2,\ D_n$.

THEOREM 6.4. *Let $A_1(x, D),\ldots,A_N(x, D)$ be an over-determined elliptic system in Ω, where $A_i(x, D)$ is of order m_i, and each operator $A_i(x, D)$ is s-smooth in Ω. Let $u \in L_2(\Omega)$, $f_i \in L_2(\Omega)$, $i - 1,\ldots N$, and let u satisfy*

$$(6.25) \qquad (u, A_i \phi) = (f_i, \phi),\ i = 1,\ldots,N,$$

for all $\phi \in C_0^\infty(\Omega)$. Let $j = \min(s, m_1,\ldots,m_N)$. Then $u \in H_j^{loc}(\Omega)$ and for $\Omega' \subset\subset \Omega$

$$\|u\|_{j,\Omega'} \leq \gamma\, [\sum_{i=1}^{N} \|f_i\|_{0,\Omega} + \|u\|_{0,\Omega}],$$

where γ depends only on A_i, Ω', and Ω.

Proof. Let $m_0 = \min(m_1,\ldots,m_N)$, $m = \max(m_1,\ldots,m_N)$. Since the theorem is of a local nature, we need obtain the results only in a neighborhood of a point $x^0 \in \Omega$; we also take $x^0 = 0$ and let $S_\rho = \{x\colon |x| < \rho\}$.

Let Δ be the Laplacian operator: $\Delta = D_1^2+\ldots+D_n^2$, and set

$$A_0(x, D) = \sum_{i=1}^{N} A_i(x, D)\, \overline{A_i'(0, D)\, \Delta}^{\,m-m_i}.$$

Then A_0 is an operator of order $2m$ and for real $\xi \neq 0$

$$A_0'(0, \xi) = \sum_{i=1}^{N} |A_i'(0, \xi)|^2\, |\xi|^{2(m-m_i)} \neq 0$$

since the given system is over-determined elliptic at 0. Now if $s = 0$ the assertions of the theorem are trivial; thus we may assume $s \geq 1$, in which case the coefficients of $A_i'(x, D)$ are continuous, and hence the coefficients of $A_0'(x, D)$ are continuous. Thus, $A_0'(x, \xi) \neq 0$ for real $\xi \neq 0$ and for small $|x|$, so that A_0 is elliptic in the sphere S_ρ, for some sufficiently small ρ. For $\phi \in C_0^\infty(S_\rho)$ the function $\overline{A_i'(0, D)\, \Delta}^{\,m-m_i}\phi$ is again a test function on S_ρ, whence by (6.25)

$$(u, A_0\, \phi)_{0,S_\rho} = \sum_{i=1}^{N} (u, A_i\,(x, D)\, \overline{A_i'\,(0, D)}\, \Delta^{m-m_i}\, \phi)_{0,S_\rho}$$

$$= \sum_{i=1}^{N} (f_i, \overline{A_i'\,(0, D)}\, \Delta^{m-m_i}\, \phi)_{0,S_\rho}$$

Since $\overline{A_i'\,(0, D)}\, \Delta^{m-m_i}$ is an operator of order no greater than $m_i + 2\,(m-m_i) = 2m - m_i$,

$$|(u, A_0\, \phi)_{0,S_\rho}| \le C \sum_{i=1}^{N} ||f_i||_{0,S_\rho}\, ||\overline{A_i'\,(0, D)}\, \Delta^{m-m_i}\phi||_{0,S_\rho}$$

$$\le C_1 \sum_{i=1}^{N} ||f_i||_{0,S_\rho}\, ||\phi||_{2m-m_i,S_\rho}$$

$$\le [C_1 \sum_{i=1}^{N} ||f_i||_{0,S_\rho}]\, ||\phi||_{2m-j,S_\rho}$$

Since A_0 is elliptic of order $2m$, and since A_0 is also s-smooth, and, *a fortiori*, j-smooth, Theorem 6.3 can be invoked, yielding that $u \in H_j^{\,loc}\,(S_\rho)$ and for $\Omega' \subset\subset S_\rho$

$$||u||_{j,\Omega'} \le \gamma\, [C_1 \sum_{i=1}^{N} ||f_i||_{0,S_\rho} + ||u||_{0,S_\rho}]. \quad \text{Q.E.D.}$$

This result can be strengthened as follows.

THEOREM 6.5. *Let $A_1\,(x, D),\ldots A_N\,(x, D)$ be an over-determined elliptic system in Ω, where $A_i(x, D)$ is of order m_i, and each operator $A_i(x, D)$ is s-smooth in $\Omega(s \ge 1)$. Let $u \in L_2^{loc}(\Omega)$ satisfy* (6.25) *for all $\phi \in C_0^\infty(\Omega)$, where $f_i \in H_{k_i}^{loc}(\Omega)$. If $j = \min(s, m_1 + k_1,\ldots,m_N + k_N)$, then $u \in H_j^{loc}(\Omega)$, and if $\Omega' \subset\subset \Omega'' \subset\subset \Omega$,*

$$||u||_{j,\Omega'} \le \gamma\, [\sum_{i=1}^{N} |f_i|_{k_i,\,\Omega''} + ||u||_{0,\Omega''}].$$

where γ depends only on A_i, Ω', and Ω''.

Proof. Let $f_{i,\alpha} = (-1)^{k_i} D^\alpha f_i$ for $|\alpha| = k_i$, and let $A_{i,\alpha}(x, D) = A_i(x, D) D^\alpha$. Then the operators $A_{i,\alpha}$ for $i = 1,\ldots,N$ and $|\alpha| = k_i$ form an over-determined elliptic system in Ω. For if ξ is real non-zero, and if $x^0 \in \Omega$, then for some i $A_i(x^0, \xi) \neq 0$; some component (say ξ_r) of ξ is not zero, and so $A_i(x^0, \xi)\, \xi_r^{k_i} \neq 0$.

Since (6.25) holds, we have for any test function ϕ in Ω and for $|\alpha| = k_i$

$$
\begin{aligned}
(u, A_{i,\alpha}\phi) &= (u, A_i D^\alpha \phi) \\
&= (f_i, D^\alpha \phi) \\
&= (-1)^{|\alpha|} (D^\alpha f_i, \phi) \\
&= (f_{i,\alpha}, \phi).
\end{aligned}
$$

where we have used the fact that $D^\alpha \phi$ is again a test function. Thus, u is a weak solution of the system $A_{i,\alpha} u = f_{i,\alpha}$. Since $A_{i,\alpha}$ is of order $m_i + k_i$, all the assertions of the theorem are immediate consequences of Theorem 6.4. Q.E.D.

The following important corollary is now immediate.

THEOREM 6.6. *Let $A_1(x, D),\ldots,A_N(x, D)$ be an over-determined elliptic system in Ω, and suppose that each operator has infinitely differentiable coefficients. Let $u \in L_2^{\,loc}(\Omega)$ be a weak solution of $A_i u = f_i$, $i = 1,\ldots,N$, where $f_i \in C^\infty(\Omega)$. Then $u \in C^\infty(\Omega)$, if u is redefined on a set of measure zero.*

Remark. A differential operator $A(x, D)$ with infinitely differentiable coefficients is said to be *hypoelliptic* if every weak solution u of $Au = f$ for $f \in C^\infty(\Omega)$ must also be infinitely differentiable. Theorem 6.6 implies that elliptic operators with infinitely differentiable coefficients are hypoelliptic.

Proof. By definition of weak solution, we have for all test functions ϕ on Ω

$(u, A_i^* \phi) = (f_i, \phi)$.

As A_i^* has coefficients in $C^\infty(\Omega)$ and the system $A_1^*,\ldots,A_N^*$ is over-determined elliptic, and as $f_i \in C^\infty(\Omega)$, we may take s and $k_1,\ldots k_N$ in Theorem 6.5 to be any positive integers. Thus, $u \in H_j^{loc}(\Omega)$ for all $j = 1, 2, 3,$---. Thus, $u \in C^\infty$ by Corollary 2 of Theorem 3.9. Q.E.D.

Because of its importance we single out the following *A PRIORI ESTIMATE.*

THEOREM 6.7. *Let $A_1(x, D),\ldots,A_N(x, D)$ be an over-determined elliptic system in Ω, where $A_i(x, D)$ is m_i-smooth and of order m_i. Set $m_0 = \min(m_1,\ldots,m_N)$. Let $\Omega' \subset\subset \Omega$. Then for all $u \in C_0^\infty(\Omega')$*

$$||u||_{m_0, \Omega'} \leq \gamma \left[\sum_{i=1}^{N} ||A_i u||_{0,\Omega'} + ||u||_{0,\Omega'}\right],$$

where γ depends only on A_i, Ω', and Ω.

Proof. Since A_i is m_i-smooth, A_i has an adjoint A_i^* which is also m_i-smooth. Thus $(u, A_i^* \phi) = (A_i u, \phi)$ for all $\phi \in C_0^\infty(\Omega)$. Therefore Theorem 6.4 yields the estimate desired, since in that theorem $j = m_0$. Q.E.D.

Corollary. Let $A(x, D)$ be an m-smooth elliptic operator of order m in Ω and let $\Omega' \subset\subset \Omega$. Then for all $u \in C_0^\infty(\Omega')$

$$||u||_{m,\Omega'} \leq \gamma (||Au||_{0,\Omega'} + ||u||_{0,\Omega'}),$$

where γ depends only on A, Ω', and Ω.

To conclude this section Hans Lewy's example of a linear differential equation having no solution will be presented. If the data in the equation were analytic, then the Cauchy-Kowalewski theorem would imply that an analytic solution existed. Therefore, some of the date in the example must be non-analytic; moreover, all the data in the example will be infinitely differentiable.

The example is the equation

$$(6.26a) \qquad i\,(x_2 + ix_3)\frac{\partial u}{\partial x_1} + \frac{1}{2}\frac{\partial u}{\partial x_2} + \frac{1}{2}\,i\,\frac{\partial u}{\partial x_3} = f\,(x_1),$$

where f is a real-valued, infinitely differentiable function of x_1. Assume that $u\,(x_1, x_2, x_3)$ is a weak solution of this equation in a neighborhood of the origin. That is, assume that u is square integrable in a neighborhood of the origin and that

$$(6.26b) \qquad \int u\,[-i\,(x_2 + ix_3)\,D_1\,\phi - \tfrac{1}{2}D_2\,\phi - \tfrac{1}{2}iD_3\,\phi]\,dx = \int f\phi dx$$

for all infinitely differentiable ϕ vanishing outside a sufficiently small neighborhood of the origin. Now we introduce the coordinate transformation

$$(6.27) \qquad \begin{cases} x_1 = s, \\ x_2 = \sqrt{r}\,\cos\theta, \\ x_3 = \sqrt{r}\,\sin\theta, \end{cases}$$

and we shall consider only those test functions having the special form

$$\phi = \phi\,(r, s) = \phi\,(x_2^2 + x_3^2, x_1),$$

where $\phi\,(r, s)$ is any infinitely differentiable function vanishing for $|s| \geq \rho$ or $r \geq \rho$, where ρ is a suitable positive number depending on the neighborhood where u is defined. Since

$$(6.28) \qquad D_2\,\phi = 2x_2\frac{\partial\phi}{\partial r}, \quad D_3\,\phi = 2x_3\frac{\partial\phi}{\partial r},$$

while the Jacobian of the coordinate transformation (6.27) is

$$\frac{\partial\,(x_1, x_2, x_3)}{\partial\,(s, r, \theta)} \equiv \frac{1}{2},$$

(6.26b) combined with (6.27) and (6.28) implies

$$(6.29) \qquad \int_0^\rho \int_{-\rho}^\rho \int_0^{2\pi} u \, [- i \sqrt{r} \, e^{i\theta} \frac{\partial \phi}{\partial s} - \sqrt{r} \, e^{i\theta} \frac{\partial \phi}{\partial r} \,] \, d\theta ds dr$$

$$= 2\pi \int_0^\rho \int_{-\rho}^\rho f(s) \, \phi(r, s) \, ds dr.$$

The equality (6.29) holds for all infinitely differentiable functions $\phi(r, s)$ defined for $r \geq 0$ and having compact support in the set $R = \{(r, s): 0 \leq r < \rho, - \rho < s < \rho\}$; let $C_0^\infty(R)$ denote the class of all such functions. Note that the elements of $C_0^\infty(R)$ do not necessarily vanish on the portion of ∂R where $r = 0$.

Now let

$$(6.30) \qquad U(r, s) = \sqrt{r} \int_0^{2\pi} e^{i\theta} u(r, s, \theta) \, d\theta.$$

Note that

$$(6.31) \qquad |U(r, s)|^2 \leq 2\pi r \int_0^{2\pi} |u(r, s, \theta)|^2 \, d\theta,$$

so that

$$\int_R |U(r, s)|^2 \, ds dr \leq 2\pi\rho \int_0^\rho \int_{-\rho}^\rho \int_0^{2\pi} |u(r, s, \theta)|^2 \, d\theta ds dr < \infty,$$

and from Fubini's theorem it follows that the integral in (6.30) is absolutely convergent for almost all points $(r, s) \in R$, and that $U \in L_2(R)$. Also, (6.31) implies

$$(6.32) \qquad \int_0^\rho \int_{-\rho}^\rho r^{-1} |U(r, s)|^2 \, ds dr < \infty.$$

Integrating first with respect to θ in (6.29) and using (6.30),

$$(6.33) \qquad \int_R U(r, s) \, [- i \frac{\partial \phi}{\partial s} - \frac{\partial \phi}{\partial r}] \, ds dr = 2\pi \int_R f(s) \, \phi ds dr,$$

all $\phi \in C_0^\infty(R)$.

Now let $L = \{(r, s): -\rho < r \leq 0, -\rho < s < \rho\}$. We shall extend $U(r, s)$ to the region $L \cup R$, forgetting that r previously was allowed to have only non-negative values. The extension is given by the formula

$$U(r, s) = -\overline{U(-r, s)}, \quad r \geq 0.$$

Now for $\psi \in C_0^\infty(L)$ let $\phi(r, s) = -\overline{\psi(-r, s)}$, $r \geq 0$; then $\phi \in C_0^\infty(R)$. The relation (6.33) implies that

$$\int_L U(-r, s)\left[-i \frac{\partial\phi}{\partial s}(-r, s) - \frac{\partial\phi}{\partial r}(-r, s)\right] dsdr$$

$$-= 2\pi \int_L f(s)\, \phi(-r, s)\, dsdr.$$

Therefore,

$$-\int_L \overline{U(r, s)}\left[+i \overline{\frac{\partial\psi}{\partial s}(r, s)} - \overline{\frac{\partial\psi}{\partial r}(r, s)}\right] dsdr$$

$$= -2\pi \int_L f(s)\, \overline{\psi(r, s)}\, dsdr.$$

Conjugating both sides, it follows that since f is real

$$\int_L U(r, s)\left[-i \frac{\partial\psi}{\partial s} - \frac{\partial\psi}{\partial r}\right] dsdr = 2\pi \int_L f(s)\, \psi(r, s)\, dsdr. \tag{6.34}$$

Combining (6.33) and (6.34), we obtain the relation

$$\int_{L \cup R} U(r, s)\left[-i \frac{\partial\phi}{\partial s} - \frac{\partial\phi}{\partial r}\right] dsdr = 2\pi \int_{L \cup R} f(s)\, \phi dsdr, \tag{6.35}$$

which holds for all $\phi \in C_0^\infty(L \cup R)$.

But (6.35) just expresses the fact that U is a weak solution in $L \cup R$ of the equation

$$\frac{\partial U}{\partial r} + i \frac{\partial U}{\partial s} = 2\pi f(s). \tag{6.36}$$

Since the operator $\frac{\partial}{\partial r} + i \frac{\partial}{\partial s}$ is elliptic, the regularity result of Theorem 6.6 implies that $U \in C^\infty (L \cup R)$, if U is suitably modified on a set of measure zero. (Recall that $f(s)$ is assumed to be infinitely differentiable.) Now that U is known to be smooth, (6.32) implies that $U(0, s) \equiv 0$, $-\rho < s < \rho$.

Let $f(s)$ have the special form $f(s) = (2\pi)^{-1} g'(s)$, where g is a real-valued infinitely differentiable function. If $V(r, s) = U(r, s) + ig(s)$, then $V \in C^\infty (L \cup R)$ and (6.36) implies

$$\frac{\partial V}{\partial r} + i \frac{\partial V}{\partial s} \equiv 0. \tag{6.37}$$

But (6.37) is just the Cauchy-Riemann equation for the real and imaginary parts of V. Therefore, V is analytic in $L \cup R$. Since $U(0, s) \equiv 0$, it follows that $V(0, s) \equiv ig(s)$, and thus $g(s)$ is analytic, $-\rho < s < \rho$. Thus, if g is any non-analytic, infinitely differentiable, real-valued function, the equation (6.26a) has no weak solution in L_2.

7. Gårding's Inequality

The purpose of this chapter is to prove an important theorem of Gårding which will be used in the global existence theory in section 8. From the standpoint of motivation, this theorem should appear with the existence theory. Therefore, the reader who is meeting this material for the first time might well proceed to section 8, assume Theorem 7.6 when it is needed there, and return to the proof at aome convenient moment in the future.

Before giving the statement and proof of Gårding's inequality, it will be convenient to give the proof of an interpolation theorem similar to, but simpler than, Theorem 3.3. Also, Poincaré's inequality will be given.

LEMMA 7.1. *There is a constant γ_0 depending only on n and m such that for any $\epsilon > 0$ and any $\phi \in C_0^m (E_n)$*

$$|\phi|^2_{j, E_n} \leq \gamma_0 \left(\epsilon^{m-j} |\phi|^2_{m, E_n} + \epsilon^{-j} |\phi|^2_{0, E_n}\right), \; j = 0, \ldots, m. \tag{7.1}$$

Proof. We shall use the notation $\hat{\phi}$ for the Fourier transform of ϕ:

$$\hat{\phi}(\xi) = (2\pi)^{-n/2} \int_{E_n} \phi(x)\, e^{-ix\cdot\xi}\, dx.$$

Integration by parts shows that for $\phi \in C_0^1(E_n)$

$$\widehat{D_k \phi}(\xi) = (2\pi)^{-n/2} \int_{E_n} D_k \phi(x)\, e^{-ix\cdot\xi}\, dx$$

$$= (2\pi)^{-n/2} \int_{E_n} \phi(x)\, i\xi_k\, e^{-ix\cdot\xi}\, dx$$

$$= i\xi_k \hat{\phi}(\xi).$$

Therefore, if $\phi \in C_0^m(E_n)$ and $|a| = j \leq m$,

$$\widehat{D^\alpha \phi}(\xi) = (i\xi)^\alpha \hat{\phi}(\xi).$$

Then Parseval's identity gives

$$|D^\alpha \phi|_{0,E_n}^2 = \int_{E_n} \xi^{2\alpha} |\hat{\phi}(\xi)|^2\, d\xi$$

$$= \int_{|\xi|^2 \leq \epsilon^{-1}} \xi^{2\alpha} |\hat{\phi}(\xi)|^2\, d\xi$$

$$+ \int_{|\xi|^2 > \epsilon^{-1}} \xi^{2\alpha} |\hat{\phi}(\xi)|^2\, d\xi$$

$$\leq \epsilon^{-j} \int_{E_n} |\hat{\phi}|^2\, d\xi + \epsilon^{m-j} \int_{|\xi|^2 > \epsilon^{-1}} |\xi|^{2m} |\hat{\phi}|^2\, d\xi.$$

By another application of Parseval's identity,

$$|D^\alpha \phi|_{0,E_n}^2 \leq \epsilon^{-j} |\phi|_{0,E_n}^2 + \gamma\epsilon^{m-j} |\phi|_{m,E_n}^2,$$

whence the lemma follows. Q.E.D.

Corollary 1. *There is a constant* $\gamma_0 = \gamma_0(n, m)$ *such that for* $0 < \epsilon \leq 1$ *and for* $\phi \in C_0^m(E_n)$

$$||\phi||^2_{m-1,E_n} \leq \gamma_0 (\epsilon |\phi|^2_{m,E_n} + \epsilon^{1-m} |\phi|^2_{0,E_n}).$$

Lemma 7.1 is really a convexity theorem since the range on ϵ is not bounded above. Indeed, we have Corollary 2, as follows:

Corollary 2. *There is a constant* $\gamma_1 = \gamma_1(n, m)$ *such that for all* $\phi \in C_0^m(E_n)$

$$(7.2) \qquad |\phi|_{j,E_n} \leq \gamma_1 |\phi|^{j/m}_{m,E_n} |\phi|^{(m-j)/m}_{0,E_n}, \; j = 0, \ldots, m$$

Proof. Let

$$\epsilon = |\phi|^{2/m}_{0,E_n} |\phi|^{-2/m}_{m,E_n}.$$

Then the two terms on the right side of (7.1) are equal and (7.2) results. Q.E.D.

In fact, Corollary 2 implies Lemma 7.1: this is trivial if $j = 0$ or $j = m$; otherwise, use Young's inequality

$$|ab| \leq a^{-1} |a|^{\alpha} + \beta^{-1} |b|^{\beta}$$

for $a^{-1} + \beta^{-1} = 1$, $a, \beta > 0$, taking

$$a = |\phi|^{2j/m}_{m,E_n} \epsilon^{j(m-j)/m}, \; b = |\phi|^{2(m-j)/m}_{0,E_n} \epsilon^{-j(m-j)/m},$$

$$a = m/j, \; \beta = m/(m-j).$$

We shall say that Ω has *bounded width* $\leq d$ if and only if there is a line l such that each line parallel to l intersects Ω in a set whose diameter is no greater than d. The following lemma is the *POINCARE' INEQUALITY*.

LEMMA 7.3. *If* Ω *has bounded width* $\leq d$, *then* $|\phi|_{j,\Omega} \leq \gamma d^{m-j} |\phi|_{m,\Omega}$ *for all* $\phi \in C_0^m(\Omega)$, $0 \leq j \leq m-1$, *where* γ *is a constant depending only on* m *and* n.

Proof. Let l' be a line parallel to l, and assume that x^0 and $x^0 + q$ are points of $l' \cap \partial\Omega$ such that $l' \cap \Omega$ is contained in the segment between x^0 and $x^0 + q$. By defining ϕ to vanish outside Ω, we can assume $\phi \in C_0^m(E_n)$. Let

$$f(t) = \phi(x^0 + t\,|q|^{-1}\,q).$$

Then $f(0) = 0$, so that

$$f(t) = \int_0^t f'(\tau)\,d\tau.$$

By the Cauchy-Schwarz inequality,

$$(7.3) \qquad |f(t)|^2 \leq t \int_0^t f'(\tau)|^2\,d\tau \leq d \int_{-\infty}^{\infty} |f'(\tau)|^2\,d\tau.$$

Hence

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \int_0^d |f(t)|^2\,dt \leq d^2 \int_{-\infty}^{\infty} |f'(t)|^2\,dt.$$

Now express $|\phi|_{0,\Omega}^2$ as an iterated integral with one of the integrations taken in the direction of l. From the inequality above it follows that

$$|\phi|_{0,\Omega}^2 \leq d^2\,|\phi|_{1,\Omega}^2.$$

Applying this inequality to $D_i\,\phi$,

$$|D_i\,\phi|_{0,\Omega}^2 \leq d^2\,|D_i\,\phi|_{1,\Omega}^2.$$

Summing over all i, we obtain

$$|\phi|_{1,\Omega}^2 \leq 2d^2\,|\phi|_{2,\Omega}^2.$$

Proceeding in this manner,

$$|\phi|_{j,\Omega} \leq \gamma d \, |\phi|_{j+1,\Omega}$$

for $0 \leq j \leq m-1$. Q.E.D.

Although the Poincare′ inequality, as it stands, is completely sufficient for our purposes (indeed, we shall need the inequality only in the case that Ω is itself bounded), for the sake of completeness we present the following generalization:

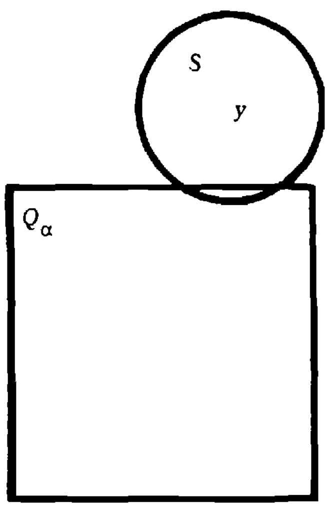
LEMMA 7.4. *Assume that there are positive constants δ and d such that for every $x \in \Omega$ there is a point y for which $|x-y| \leq d$ and $\{z\colon\ |z-y| < \delta\} \cap \Omega = \emptyset$. Then there is a constant c depending only on n and d/δ, such that*

$$|\phi|_{j,\Omega} \leq (dc)^{m-j} \, |\phi|_{m,\Omega}, \ 0 \leq j \leq m-1,$$

for any $\phi \in C_0^m(\Omega)$.

Proof. For any n-tuple $\alpha = (a_1,\ldots,a_n)$ of integers, let $Q_\alpha = \{x\colon n^{-1/2} a_k d < x_k \leq n^{-1/2}(a_k+1)d,\ k = 1,\ldots,n\}$. Then $E_n = \bigcup_\alpha Q_\alpha$. We shall prove something similar to the last estimate with Ω replaced by $Q_\alpha \cap \Omega$. Then, by summing over all α, the lemma will follow for Ω.

Assume that $\phi = 0$ outside Ω. Assume that $x' \in Q_\alpha \cap \Omega$, and let y be the corresponding point guaranteed by the hypothesis. Since for



$x \in Q_\alpha$, $|x - x'| < d$, the triangle inequality gives $|x - y| < 2d$. Let $S = \{z\colon\ |y - z| < \delta\}$. Now integrate $|\phi|^2$ over $Q_\alpha - S$; the integral can be expressed in polar coordinates about y as

$$\int_{Q_\alpha - S} |\phi|^2\, dx \le \int_\Sigma \int_\delta^{2d} |\phi|^2\, r^{n-1} dr d\sigma,$$

where σ is the spherical measure and Σ the surface of the unit sphere $r = 1$. By (7.3), we have for $\delta \le r \le 2d$

$$|\phi|^2\, r^{n-1} \le 2d \int_\delta^{2d} \left|\frac{\partial\phi}{\partial r_1}\right|^2 dr_1 \cdot r^{n-1}$$

$$\le 2d\, (2d)^{n-1} \int_\delta^{2d} \left|\frac{\partial\phi}{\partial r_1}\right|^2 dr_1$$

$$\le (2d)^n\, \delta^{1-n} \int_\delta^{2d} \left|\frac{\partial\phi}{\partial r_1}\right|^2 r_1^{n-1}\, dr_1,$$

where $\partial\phi/\partial r_1$ is the derivative of ϕ in the radial direction from y. Thus,

$$\int_\delta^{2d} |\phi|^2\, r^{n-1}\, dr \le (2d)^{n+1}\, \delta^{1-n} \int_\delta^{2d} \left|\frac{\partial\phi}{\partial r}\right|^2 r^{n-1}\, dr.$$

Since $\partial\phi/\partial r$ is a particular directional derivative of ϕ, it must be bounded in magnitude by the gradient of ϕ; that is,

$$\left|\frac{\partial\phi}{\partial r}\right|^2 \le \sum_{i=1}^{n} |D_i\, \phi|^2 .$$

It therefore follows from the preceding inequalities that

$$|\phi|_{0,Q_\alpha}^2 = \int_{Q_\alpha - S} |\phi|^2\, dx$$

$$\leq \int_{\Sigma} (2d)^{n+1} \delta^{1-n} \int_{\delta}^{2d} \sum_{i=1}^{n} |D_i \phi|^2 r^{n-1} \, dr d\sigma$$

$$= (2d)^{n+1} \delta^{1-n} \int_{\delta \leq |x-y| \leq 2d} \sum_{i=1}^{n} |D_i \phi(x)|^2 \, dx$$

$$\leq (2d)^{n+1} \delta^{1-n} |\phi|^2_{1,\tilde{Q}_\alpha},$$

where $\tilde{Q}_\alpha$ is the union of all cubes Q_β such that the distance from Q_β to the center of Q_α is less than $4d$. The number N of disjoint cubes Q_β which constitute $\tilde{Q}_\alpha$ depends only on n. Summing the above inequality over all a, we obtain

$$|\phi|^2_{0,E_n} \leq (2d)^{n+1} \delta^{1-n} N \, |\phi|^2_{1,E_n}.$$

This inequality can be applied to the derivatives of ϕ to show that

$$|\phi|_{j,E_n} \leq dc \, |\phi|_{j+1,E_n}$$

for $j = 0,\ldots,m-1$. Since $|\phi|_{j,E_n} = |\phi|_{j,\Omega}$ for all j, the result follows. Q.E.D.

We now pass to the statement and proof of the main result of this section. We shall be concerned with a quadratic form

$$B(\phi) = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} \int_{\Omega} a_{\alpha\beta}(x) \, D^\alpha \phi \, \overline{D^\beta \phi} \, dx.$$

We shall use the notation

$$B'(\phi) = \sum_{\substack{|\alpha| = m \\ |\beta| = m}} \int_{\Omega} a_{\alpha\beta}(X) \, D^\alpha \phi \, \overline{D^\beta \phi} \, dx$$

for the *principal part* of B.

Definition 7.1. *The quadratic form* $B(\phi)$ *is* uniformly strongly elliptic *in* Ω *if there exists a positive constant* E_0 *such that for all real* ξ *and all* $x \in \Omega$

$$\Re \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta}(x)\, \xi^{\alpha+\beta} \geq E_0\, |\xi|^{2m}. \tag{7.4}$$

The largest constant E_0 *for which* (7.4) *is valid is the* ellipticity constant *of* B.

One bit of notation must be introduced: $C_0^m(\Omega)$ is the class of functions in $C^m(\Omega)$ having compact support in Ω.

THEOREM 7.6. (GÅRDING'S INEQUALITY). *Let* Ω *be any open set, and let* $B(\phi)$ *be a uniformly strongly elliptic quadratic form on* Ω *with ellipticity constant* E_0. *Assume that*

1° $a_{\alpha\beta}(x)$ *is uniformly continuous on* Ω, $|\alpha| = |\beta| = m$;

2° $a_{\alpha\beta}(x)$ *is bounded and measurable,* $|\alpha| + |\beta| \leq 2m$.

Then there are constants $\gamma_0 > 0$ *and* $\lambda_0 \geq 0$ *such that*

$$\Re B(\phi) \geq \gamma_0\, E_0\, \|\phi\|^2_{m,\Omega} - \lambda_0\, \|\phi\|^2_{0,\Omega}$$

for all $\phi \in C_0^m(\Omega)$.

Here γ_0 *depends only on* n *and* m; *and* λ_0 *depends only on* n, m, E_0, *the modulus of continuity of* $a_{\alpha\beta}(x)$ *for* $|\alpha| = |\beta| = m$, *and* $\sup_{x\in\Omega} \sum_{|\alpha|+|\beta|\leq 2m} |a_{\alpha\beta}(x)|$.

The proof will be through a (terminating) sequence of lemmas. *In all of these lemmas it will be assumed at least that the hypothesis of Theorem* 7.6 *is satisfied by the form* B.

We shall in the remainder of this section use the notation γ for a positive generic constant depending only on n, m; and the notation K for a positive generic constant depending only on n, m, E_0, the modulus of continuity of $a_{\alpha\beta}(x)$ for $|\alpha| = |\beta| = m$, and

$\sup_{x \in \Omega} |a_{\sigma\beta}(x)|$ for $|\alpha| + |\beta| \leq 2m$.

LEMMA 7.7. *Assume that $B = B'$ and that B has constant coefficients. Then for all $\phi \in C_0^m(\Omega)$*

$$\Re B(\phi) \geq \gamma E_0 |\phi|^2_{m,\Omega}.$$

Proof. It is obviously sufficient to take $\Omega = E_n$ in this case. By Parseval's identity, for $|\alpha| = |\beta| = m$ we have (integrating on E_n)

$$\int D^\alpha \phi \overline{D^\beta \phi}\, dx = \int \widehat{D^\alpha \phi}\, \overline{\widehat{D^\beta \phi}}\, d\xi$$

$$= \int \xi^{\alpha+\beta} |\hat{\phi}(\xi)|^2\, d\xi.$$

Thus,

$$\Re B(\phi) = \Re \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \int \xi^{\alpha+\beta} |\hat{\phi}(\xi)|^2\, d\xi$$

$$= \int |\hat{\phi}(\xi)|^2\, \Re \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\, \xi^{\alpha+\beta}\, d\xi$$

By the ellipticity assumption on B (cf. (7.4)),

$$\Re \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\, \xi^{\alpha+\beta} \geq \gamma E_0 \sum_{|\alpha|=m} \xi^{2\alpha}.$$

Therefore,

$$\Re B(\phi) \geq \gamma E_0 \int |\hat{\phi}(\xi)|^2 \sum_{|\alpha|=m} \xi^{2\alpha}\, d\xi.$$

Finally, by Parseval's identity the last inequality is just

$$\Re B(\phi) \geq \gamma E_0 |\phi|^2_{m,E_n}. \quad \text{Q.E.D.}$$

Corollary. If $B = B'$ and B has constant coefficients, and if Ω satisfies the condition of Lemma 7.4, then there is a positive constant c depending only on n, m, d, and δ, such that for all $\phi \in C_0^m$ (Ω

$$\Re B(\phi) \geq cE_0 \, ||\phi||^2_{m,\Omega}.$$

Proof. From Lemma 7.4

$$||\phi||^2_{m,\Omega} = \sum_{j=0}^{m} |\phi|^2_{j,\Omega} \leq \sum_{j=0}^{m} (dc)^{2(m-j)} \cdot |\phi|^2_{m,\Omega}.$$

Thus, the corollary follows from Lemma 7.7. Q.E.D.

LEMMA 7.8. *Assume that B' has constant coefficients,. Then for all $\phi \in C_0^m$ (Ω),*

$$\Re B(\phi) \geq \gamma \, E_0 \, ||\phi||^2_{m,\Omega} - K \, ||\phi||^2_{0,\Omega}.$$

Proof. We have

$$B(\phi) = B'(\phi) + \sum_{|\alpha|+|\beta|\leq 2m-1} \int_{\Omega} a_{\alpha\beta}(x) \, D^\alpha \phi \, \overline{D^\beta \phi} \, dx.$$

By the Cauchy-Schwarz inequality, for $|a| + |\beta| \leq 2m - 1$

$$\left|\int_{\Omega} a_{\alpha\beta}(x) \, D^\alpha \phi \, \overline{D^\beta \phi} \, dx\right| \leq K \, |\phi|_{|\alpha|,\Omega} \, |\phi|_{|\beta|,\Omega}.$$

Thus,

$$\Re B(\phi) \geq \Re B'(\phi) - K \, ||\phi||_{m-1} \, |\phi|_m - K \, ||\phi||^2_{m-1}$$

$$\geq \gamma E_0 \, |\phi|^2_m - K \, ||\phi||_{m-1} \, |\phi|_m - K \, ||\phi||^2_{m-1}, \tag{7.5}$$

the latter inequality being a consequence of Lemma 7.7.

For any numbers a, b, ϵ, with $\epsilon > 0$,

$$2 \, |ab| \leq \epsilon \, |a|^2 + \epsilon^{-1} \, |b|^2.$$

Thus, (7.5) implies that

$$\Re B(\phi) \geq (\gamma E_0 - K\epsilon)|\phi|_m^2 - K(\epsilon^{-1} + 1)||\phi||_{m-1}^2.$$

Now choose $\epsilon = \gamma E_0/2K$. Then

$$\Re B(\phi) \geq \tfrac{1}{2}\gamma E_0|\phi|_m^2 - K||\phi||_{m-1}^2$$

$$= \tfrac{1}{2}\gamma E_0||\phi||_m^2 - (K + \tfrac{1}{2}\gamma E_0)||\phi||_{m-1}^2.$$

By Corollary 1 of Lemma 7.1, since ϕ can be considered to be in $C_0^m(E_n)$, we have for $0 < \eta \leq 1$

$$\Re B(\phi) \geq \tfrac{1}{2}\gamma E_0||\phi||_m^2 - K(\eta\, ||\phi||_m^2 + \eta^{1-m}||\phi||_0^2).$$

If we set $\eta = \gamma E_0/4K$, then

$$\Re B(\phi) \geq \tfrac{1}{4}\gamma E_0||\phi||_m^2 - K||\phi||_0^2.$$

Q.E.D.

This lemma completes the proof of Gårding's inequality in the case that the principal part of B has constant coefficients. For the general case, we first prove the theorem locally.

LEMMA 7.9. *Assume that $B = B'$. Then there exists a positive constant ρ such that*

$$\Re B(\phi) \geq \gamma E_0||\phi||_{m,\Omega}^2$$

for all $\phi \in C_0^m(\Omega)$ such that the diameter of supp (Ω) *is less than ρ. The constant ρ depends only on n, m, E_0, and the modulus of continuity of the coefficients $a_{\alpha\beta}$.*

Proof. Since the coefficients are uniformly continuous, for every positive ϵ there exists a positive number ρ such that $|a_{\alpha\beta}(x) - a_{\alpha\beta}(y)| < \epsilon$ if $|x - y| < \rho$, $x, y \in \Omega$. If $\phi \in C_0^m(\Omega)$ and the diameter of supp (ϕ) is less than ρ, let $x^0 \in$ supp (ϕ) and let

$$B_{x^0}(\phi) = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x^0) \int_{\mathrm{supp}(\phi)} D^\alpha\phi\overline{D^\beta\phi}\,dx.$$

Then by the Cauchy-Schwarz inequality

$$\Re B(\phi) = \Re B_{x^0}(\phi) + \Re \sum_{|\alpha|=|\beta|=m} \int_{\mathrm{supp}(\phi)} [a_{\alpha\beta}(x) - a_{\alpha\beta}(x^0)] \cdot D^\alpha\phi \overline{D^\beta\phi}dx$$

$$\geq \Re B_{x^0}(\phi) - \epsilon \sum_{|\alpha|=|\beta|=m} |D^\alpha\phi|_{0,\Omega}|D^\beta\phi|_{0,\Omega}$$

$$\geq \Re B_{x^0}(\phi) - K\epsilon||\phi||^2_{m,\Omega}.$$

Applying Lemma 7.8 to the form B_{x^0}, which has constant coefficients, (7.6) becomes

$$\Re B(\phi) \geq (\gamma E_0 - K\epsilon)||\phi||^2_{m,\Omega} - K||\phi||^2_{0,\Omega}.$$

Now choose $\epsilon = \gamma E_0/2K$, so that

$$(7.7) \qquad \Re B(\phi) \geq \tfrac{1}{2}\gamma E_0||\phi||^2_{m,\Omega} - K||\phi||^2_{0,\Omega}.$$

Finally, since the diameter of supp (ϕ) is less than ρ, Lemma 7.3 implies $||\phi||_0 \leq \gamma\rho^m||\phi||_m$, and then (7.7) becomes

$$\Re B(\phi) \geq (\tfrac{1}{2}\gamma E_0 - K\gamma^2\rho^{2m})||\phi||^2_{m,\Omega}.$$

Choosing ρ sufficiently small, the result follows. Q.E.D.

Corollary. Lemma 7.9 *remains valid if we omit the assumption that* $B = B'$.

Proof. Assume $\phi \in C_0^m(\Omega)$ and supp (ϕ) has diameter less than ρ, where ρ is the constant guaranteed for the form B' by Lemma 7.9. Then, by the lemma,

$$\Re B(\phi) = \Re B'(\phi) + \sum_{|\alpha|+|\beta|\leq 2m-1} \int_{\mathrm{supp}(\phi)} a_{\alpha\beta}(x)D^\alpha\phi\overline{D^\beta\phi}dx$$

$$(7.8) \quad \geq \gamma E_0||\phi||^2_{m,\Omega} - K||\phi||_{m,\mathrm{supp}(\phi)}\,||\phi||_{m-1,\mathrm{supp}(\phi)}.$$

Using the inequality $2ab \leq \epsilon a^2 + \epsilon^{-1}b^2$ for a, $b \geq 0$, (7.8) becomes

$$(7.9)\quad \Re B(\phi) \geq \gamma E_0 \|\phi\|^2_{m,\Omega} - K\epsilon\|\phi\|^2_{m,\Omega} - K\epsilon^{-1}\|\phi\|^2_{m-1,\mathrm{supp}(\phi)}.$$

By Lemma 7.3

$$\|\phi\|_{m-1,\mathrm{supp}(\phi)} \leq \gamma\rho\|\phi\|_{m,\mathrm{supp}(\phi)} = \gamma\rho\|\phi\|_{m,\Omega},$$

so that (7.9) becomes

$$\Re B(\phi) \geq (\gamma E_0 - K\epsilon - K\epsilon^{-1}\rho^2)\|\phi\|^2_{m,\Omega}.$$

Now simply choose $\epsilon = \gamma E_0/2K$ and then $\rho^2 \leq \epsilon\gamma E_0/4K$. Q.E.D.

The following lemma is similar to Theorem 1.9 in that it states the existence of a slightly modified partition of unity.

LEMMA 7.10. *Let F be a compact subset of E_n and let $F \subset \bigcup_{i=1}^{\nu} 0_i$, where each 0_i is open. Then there exist functions $\zeta_i \in C_0^\infty(0_i)$ such that $0 \leq \zeta_i \leq 1$ and $\sum_{i=1}^{\nu} \zeta_i(x)^2 \equiv 1$ for $x \in F$.*

Proof. Choose $0'_i$, $0''_i$ such that $0''_i \subset\subset 0'_i \subset\subset 0_i$, $i = 1,\ldots,\nu$, and $F \subset \bigcup_{i=1}^{\nu} 0''_i$. Let $\zeta'_i \in C_0^\infty(E_n)$, $i = 0, 1,\ldots,\nu$, such that: $0 \leq \zeta'_i \leq 1$ for all i; $\zeta'_0(x) \equiv 1$ on $\bigcup_{i=1}^{\nu} 0''_i$; supp $\zeta'_0 \subset \bigcup_{i=1}^{\nu} 0'_i$; and for $1 \leq i \leq \nu$, $\zeta'_i \equiv 1$ on $0'_i$, supp $\zeta'_i \subset 0_i$.

Then for $1 \leq i \leq \nu$ let

$$\zeta_i(x) = \zeta'_i(x)\Big[\sum_{j=1}^{\nu} \zeta'_j(x)^2 + (1 - \zeta'_0(x))^2\Big]^{-1/2}$$

Since the quantity in brackets never vanishes, $\zeta_i \in C_0^\infty(0_i)$; and since $1 - \zeta'_0(x) = 0$ for $x \in F$, $\sum_{i=1}^{\nu} \zeta_i(x)^2 = 1$ for $x \in F$. Q.E.D.

An easy modification of the above proof gives

LEMMA 7.11. *Let $d > 0$ and for each n-tuple $a = (a_1,\ldots,a_n)$ of integers let*

$$Q_\alpha = \{x:\ d(a_k - 1) < x_k < d(a_k + 1),\ k = 1,\ldots,n\}.$$

Then there exists a function $\zeta \in C_0^\infty(Q_0)$ such that $0 \leq \zeta \leq 1$ and

$$\sum_{\alpha} \zeta(x - a)^2 \equiv 1.$$

That is, if $\zeta_\alpha(x) \equiv \zeta(x - a)$, *then* $\zeta_\alpha \in C_0^\infty(Q_\alpha)$ *and*

$$\sum_{\alpha} \zeta_\alpha(x)^2 = 1.$$

Proof. Let $\zeta' \in C_0^\infty(Q_0)$ be chosen such that $\zeta(x) \equiv 1$ for $x \in 3/4Q_0$. Then set

$$\zeta(x) = \zeta'(x)[\sum_{\alpha} \zeta'(x - a)^2]^{-1/2}$$

As in the proof of Lemma 7.10, the functions $\zeta(x - a)$ satisfy the requirements of the lemma. Q.E.D.

We are now ready for the

Proof of Gårding's inequality. We shall apply Lemma 7.11, taking the cubes Q_α of side d, where d is chosen such that the diameter of Q_α is less than the number ρ of Lemma 7.9; thus $\sqrt{n}d < \rho$. Enumerate the cubes Q_α and the corresponding functions ζ_α in some order: $Q_1, Q_2, \ldots$, and $\zeta_1, \zeta_2, \ldots$. Since the functions ζ_i are just translates of ζ_1, we have the estimates

$$(7.10) \qquad |D^\alpha \zeta_i| \le K, \ |a| \le m.$$

Now let $\phi \in C_0^m(\Omega)$. Then by (7.8)

$$(7.11) \qquad \Re B(\phi) \ge \Re B'(\phi) - K||\phi||_{m,\Omega} \, ||\phi||_{m-1,\Omega}.$$

Now

$$B'(\phi) = \sum_{|a|=|\beta|=m} \int_\Omega \sum_{i=1}^{\infty} \zeta_i(x)^2 a_{a\beta}(x) D^a\phi \overline{D^\beta \phi} dx$$

$$(7.12) \qquad = \sum_{|a|=|\beta|=m} \sum_{i=1}^{\infty} \int_\Omega a_{a\beta} D^a(\zeta_i\phi) \overline{D^\beta(\zeta_i\phi)} dx + R(\phi),$$

where by Leibnitz's rule

$R(\phi) =$

$$- \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{i=1}^{\infty} \sum_{\substack{\gamma\leq\alpha \\ \gamma\neq\alpha}} \sum_{\substack{\delta\leq\beta \\ \delta\neq\beta}} \int_{\Omega} a_{\alpha\beta} \binom{\alpha}{\gamma} \binom{\beta}{\delta} D^{\alpha-\gamma}\zeta_i \cdot D^{\gamma}\phi \cdot D^{\beta-\delta}\zeta_i \cdot D^{\delta}\phi dx.$$

Thus, by (7.10)

$$(7.13) \qquad |R(\phi)| \leq K \sum_{|\gamma|+|\delta|\leq 2m-1} \sum_{i=1}^{\infty} \int_{Q_i} |D^{\gamma}\phi| \, |D^{\delta}\phi| dx.$$

Now the cubes Q_i overlap in such a fashion that any fixed point is contained in exactly 2^n distinct cubes, except for points on $\bigcup_{i=1}^{\infty} \partial Q_i$, a set of measure zero. Thus

$$\sum_{i=1}^{\infty} \int_{Q_i} |D^{\gamma}\phi| \, |D^{\delta}\phi| dx = 2^n \int_{\Omega} |D^{\gamma}\phi| \, |D^{\delta}\phi| dx;$$

using this relation in (7.13),

$$|R(\phi)| \leq K \sum_{|\gamma|+|\delta|\leq 2m-1} \int_{\Omega} |D^{\gamma}\phi| \, |D^{\delta}\phi| dx.$$

By the Cauchy-Schwarz inequality

$$|R(\phi)| \leq K \sum_{|\gamma|+|\delta|\leq 2m-1} ||D^{\gamma}\phi||_{0,\Omega} \, ||D^{\delta}\phi||_{0,\Omega}$$

$$(7.14) \qquad \leq K||\phi||_{m,\Omega} \, ||\phi||_{m-1,\Omega}.$$

Thus, (7.12) implies

$$(7.15) \qquad \Re B(\phi) \geq \sum_{i=1}^{\infty} \Re B'(\zeta_i\phi) - K||\phi||_{m,\Omega} \, ||\phi||_{m-1,\Omega}.$$

But the diameter of supp $(\zeta_i\phi)$ is no greater than the diameter of Q_i, which, by choice, is less than ρ. Thus, Lemma 7.9 implies that

$$\Re B'(\zeta_i\phi) \geq \gamma E_0 ||\zeta_i\phi||^2_{m,\Omega}.$$

Hence, (7.15) implies

$$(7.16) \qquad \Re B(\phi) \geq \gamma E_0 \sum_{i=1}^{\infty} ||\zeta_i\phi||^2_{m,\Omega} - K||\phi||_{m,\Omega}\, ||\phi||_{m-1,\Omega}\,.$$

The same type argument as was used to derive (7.15) from (7.12) shows that

$$\sum_{i=1}^{\infty} ||\zeta_i\phi||^2_{m,\Omega} = \sum_{|\alpha|\leq m} \sum_{i=1}^{\infty} \int_\Omega |D^\alpha \zeta_i\phi|^2 dx$$

$$= \sum_{|\alpha|\leq m} \sum_{i=1}^{\infty} \int_\Omega \zeta_i{}^2 |D^\alpha\phi|^2 dx + R_1(\phi)$$

$$= \sum_{|\alpha|\leq m} \int_\Omega |D^\alpha\phi|^2 dx + R_1(\phi)$$

$$= ||\phi||^2_{m,\Omega} + R_1(\phi),$$

where $R_1(\phi)$ is a quadratic form satisfying an estimate similar to (7.14):

$$|R_1(\phi)| \leq K||\phi||_{m,\Omega}\, ||\phi||_{m-1,\Omega}.$$

Combining these results with (7.16),

$$\Re B(\phi) \geq \gamma E_0 ||\phi||^2_{m,\Omega} - K||\phi||_{m,\Omega}\, ||\phi||_{m-1,\Omega}.$$

As in the proof of Lemma 7.8, this implies the required estimate in Theorem 7.6. Q.E.D.

We shall conclude this section with a proof that Gårding's inequality is very nearly equivalent to the ellipticity of the quadratic form B.

THEOREM 7.12. *Let $a_{\alpha\beta}(x)$ be bounded measurable functions in an open set Ω for $|\alpha| \leq m$, $|\beta| \leq m$, and let*

$$B(\phi) = \sum_{\substack{|\alpha|\le m\\|\beta|\le m}} \int_\Omega a_{\alpha\beta}(x) D^\alpha\phi\, \overline{D^\beta\phi}\, dx.$$

Let $c_0 > 0$ *and* $\lambda_0 \ge 0$ *be given constants.*

1° *If for all* $\phi \in C_0^\infty(\Omega)$

$$\Re B(\phi) \ge c_0 \|\phi\|^2_{m,\Omega} - \lambda_0 \|\phi\|^2_{0,\Omega}\,. \tag{7.17}$$

then for almost all $x \in \Omega$ *and for all real* ξ

$$\Re \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta} \ge c_0 \sum_{|\alpha|=m} \xi^{2\alpha}. \tag{7.18}$$

2° *If for all* $\phi \in C_0^\infty(\Omega)$

$$|B(\phi)| \ge c_0 \|\phi\|^2_{m,\Omega} - \lambda_0 \|\phi\|^2_{0,\Omega}, \tag{7.19}$$

then for almost all $x \in \Omega$ *and for all real* ξ

$$\Big| \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta}\Big| \ge c_0 \sum_{|\alpha|=m} \xi^{2\alpha}. \tag{7.20}$$

If moreover Ω *is connected and* $a_{\alpha\beta}(x) \in C^0(\Omega)$ *for* $|\alpha| = |\beta| = m$, *then there exists a real number* θ *such that for all* $x \in \Omega$ *and for all real* ξ

$$\Re\Big(e^{i\theta} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta}\Big) \ge c \sum_{|\alpha|=m} \xi^{2\alpha}, \tag{7.21}$$

where c *is some positive constant independent of* x *and* ξ.

Proof. Let ξ^1 and ξ^2 be any non-zero real vectors and let ψ_1, $\psi_2 \in C_0^\infty(\Omega)$ be such that supp (ψ_1) and supp (ψ_2) are disjoint. Set

$$\phi(x) = \psi_1(x)e^{i\tau\xi^1\cdot x} + \psi_2(x)e^{i\tau\xi^2\cdot x}.$$

We shall apply either (7.17) or (7.19) to the function ϕ and let $\tau \to \infty$.

So in computing $B(\phi)$, $||\phi||^2_{m,\Omega}$, and $||\phi||^2_{0,\Omega}$, it will be necessary to keep track only on those terms containing the factor τ^{2m}. Thus, since $\text{supp}\,(\psi_1) \cap \text{supp}\,(\psi_2) = \emptyset$, Leibnitz's rule implies

$$B(\phi) = \sum_{|\alpha|=|\beta|=m} \sum_{j=1}^{2} \int_\Omega a_{\alpha\beta}(x)(i\tau\xi^j)^\alpha \psi_j(x) \cdot \overline{(i\tau\xi^j)^\beta \psi_j(x)}\, dx + \ldots$$

$$= \tau^{2m} \sum_{j=1}^{2} \sum_{|\alpha|=|\beta|=m} (\xi^j)^{\alpha+\beta} \int_\Omega a_{\alpha\beta}(x)|\psi_j(x)|^2\, dx + \ldots,$$

where the other terms are of lower order in τ that $2m$. Likewise,

$$(7.23)\quad ||\phi||^2_{m,\Omega} = \tau^{2m} \sum_{j=1}^{2} (\xi^j)^{2\alpha} \int_\Omega |\psi_j(x)|^2\, dx + \ldots;\ ||\phi||^2_{0,\Omega} = 0 + \ldots$$

Assuming (7.19), if we use this special function ϕ, the estimates (7.22) and (7.23) imply, after letting $\tau \to \infty$, that

$$(7.24)\quad \left|\sum_{j=1}^{2} \sum_{|\alpha|=|\beta|=m} (\xi^j)^{\alpha+\beta} \int_\Omega a_{\alpha\beta}(x)|\psi_j(x)|^2\, dx\right|$$

$$\geq c_0 \sum_{j=1}^{2} (\xi^j)^{2\alpha} \int_\Omega |\psi_j(x)|^2\, dx.$$

This inequality has been derived for functions $\psi_1, \psi_2 \in C_0^\infty(\Omega)$. But since $C_0^\infty(\Omega)$ is dense in $L_2(\Omega)$, (7.24) is valid under the assumptions that $\psi_1, \psi_2 \in L_2(\Omega)$, $\text{supp}\,(\psi_1) \cap \text{supp}\,(\psi_2) = \emptyset$. Now let x^1 and x^2 be distinct points of Ω, and let for $j = 1, 2$

$$\psi_j(x) = \begin{cases} p_j\rho^{-n}, & |x - x^j| < \rho, \\ 0, & |x - x^j| \geq \rho, \end{cases}$$

where p_j is a non-negative number. If x^1 and x^2 are in the Lebesque sets of all the functions $a_{\alpha\beta}(x)$, $|\alpha| = |\beta| = m$, then applying (7.24) to the special functions ψ_j and letting $\rho \to 0$, we obtain

$$(7.25)\quad \left|\sum_{j=1}^{2} p_j^2 \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x^j)(\xi^j)^{\alpha+\beta}\right|$$

$$\geq c_0 \sum_{j=1}^{2} p_j^2 \sum_{|\alpha|=m} (\xi^j)^{2\alpha}.$$

To prove (7.20) let x^1 be any point in the Lebesque set of all the functions $a_{\alpha\beta}(x)$, $|\alpha| = |\beta| = m$, let $p_1 = 1$ and $p_2 = 0$. Then (7.25) becomes

$$\Big| \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x^1)(\xi^1)^{\alpha+\beta} \Big| \geq c_0 \sum_{|\alpha|=m} (\xi^1)^{\alpha+\beta},$$

which is just (7.20). Likewise, (7.18) can be proved by exactly the same type argument.

Finally, assume that $a_{\alpha\beta}(x)$ is continuous for $|\alpha| = |\beta| = m$ and assume (7.19). Then (7.25) holds for *all* x^1, $x^2 \in \Omega$, $x^1 \neq x^2$; and the continuity of $a_{\alpha\beta}$ also implies that (7.25) holds even if $x^1 = x^2$. Thus, if we let $p_1 = 1$, $p_2 = \sqrt{p}$ for $p \geq 0$, and if we define

$$H(x, \xi) = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta},$$

then (7.25) becomes

$$(7.26) \qquad |H(x^1, \xi^1) + pH(x^2, \xi^2)| \geq c_0'(|\xi^1|^{2m} + p|\xi^2|^{2m}),$$

an inequality which holds for all x^1, $x^2 \in \Omega$, all real vectors ξ^1, ξ^2, and all numbers $p \geq 0$.

Let Z be the set of numbers of the form $H(x, \xi)$ for $x \in \Omega$, ξ real. Since Ω is connected and H is continuous, Z is a connected subset of the complex plane. The set Z contains, together with any non-zero complex number z, all the numbers of the form rz, $0 < r < \infty$. Therefore, we need only show that the angle between the line from 0 to $H(x^1, \xi^1)$ ano the line from 0 to $H(x^2, \xi^2)$ is less than $\pi - \delta$ for some $\delta > 0$, δ independent of x^1, x^2, ξ^1, ξ^2. Obviously, we may assume $|\xi^1| = |\xi^2| = 1$. Then (7.26) becomes

$$|H(x^1 \xi^1) + pH(x^2, \xi^2)| \geq c_0'(1 + p).$$

Since the coefficients $a_{\alpha\beta}$ are bounded, we have $|H(x, \xi)| \leq K$ for

$|\xi| = 1$. Now let $H(x^j, \xi^j) = r_j e^{i\theta_j}$; then

$$|r_1 e^{i\theta_1} + p r_2 e^{i\theta_2}| \geq c_0'(1 + p).$$

Let $p = r_1/r_2$; then

$$|e^{i\theta_1} + e^{i\theta_2}| \geq c_0'(r_1^{-1} + r_2^{-1}) \geq 2c_0'/K.$$

But this means that $\theta_1 - \theta_2$ cannot be too near an odd mutiple of π. Q.E.D.

8. Global Existence

In this section we shall assume that A is a strongly elliptic operator of even order $\ell = 2m$ which has been normalized so that

$$(-1)^m \Re A'(x, \xi) > 0 \quad \text{for} \quad \xi \neq 0$$

(cf. Definition 4.1). Consider the problem of finding a function u such that

$$(8.1) \quad \begin{cases} Au = f & \text{in} \quad \Omega, \\ u = \phi_0 & \text{on} \quad \partial\Omega, \\ \dfrac{\partial u}{\partial n} = \phi_1 & \text{on} \quad \partial\Omega, \\ \quad \cdots\cdots\cdots, & \\ \dfrac{\partial^{m-1} u}{\partial n^{m-1}} = \phi_{m-1} & \text{on} \quad \partial\Omega, \end{cases}$$

where $\partial/\partial n$ indicates differentiation in the direction of the exterior normal to $\partial\Omega$. This is the *Dirichlet problem* for the elliptic operator A in Ω.

The existence theory developed in this section is for the Dirichlet problem. However, the formulation of the problem given above is fraught with difficulties. For instance, considerations involving normal differentiations at the boundary are quite complicated, and, indeed, we would like to consider domains whose boundaries may fail to have tangent planes. Therefore, we shall reformulate the Dirichlet boundary conditions in (8.1).

The reformulation can be motivated in the following way. Suppose that the boundary and the solution u are sufficiently smooth. Then the condition $u = \phi_0$ on $\partial\Omega$ automatically prescribes all derivatives of u in the directions tangent to $\partial\Omega$. Likewise, the remaining boundary conditions automatically prescribe all derivatives of u on $\partial\Omega$ in which enter at most $m - 1$ differentiations in the direction normal to $\partial\Omega$. Therefore, under sufficient smoothness conditions, the Dirichlet boundary conditions of u automatically prescribe at $\partial\Omega$ all derivatives $D^\alpha u$, $|\alpha| \leq m - 1$. Thus, instead of all *normal* derivatives of order less than m, we could prescribe for u all derivatives of order less than m on $\partial\Omega$.

However, we obviously cannot prescribe arbitrarily on $\partial\Omega$ all derivatives of u of order less than m, since these derivatives are not independent. One way out of this difficulty is to list a number of compatibility conditions these derivatives must satisfy. However, an easier procedure is simply to postulate the existence of a function $g(x)$ which is in $C^{m-1}(\bar{\Omega})$ and whose derivatives at $\partial\Omega$ of order less than m are precisely the prescribed derivatives for the solution of the Dirichlet problem. Thus, the Dirichlet boundary conditions for u now assume the form:

$$D^\alpha u = D^\alpha g \quad \text{on} \quad \partial\Omega, \quad |\alpha| \leq m - 1 .$$

There are many ways of proving the existence of functions u satisfying systems such as (8.1). One of the more important techniques was first used by Riemann to prove the existence of a solution of the Dirichlet problem for Laplace's equation. This method makes use of the Dirichlet principle: a solution u of Laplace's equation on Ω minimizes the *Dirichlet integral*,

$$\int_\Omega \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_1}\right)^2 dx,$$

among all smooth functions u which satisfy the given boundary conditions. As a generalization, the following program could be used to prove the existence of solution of (8.1).

In previous sections we have already discussed the meaning of the statement that a function u is a weak solution of an equation $Au = f$. In section 5 we proved the existence of weak solutions of elliptic equations, and in section 6 we established several regularity results, allowing us to give conditions on the smoothness of A and f which automatically insure that weak solutions are sufficiently regular to be classical smooth solutions. For boundary-value problems the procedure is similar. We shall first formulate a weak sense in which a function has certain boundary values, and shall then in sections 9 and 10 give conditions on the data which allow the conclusion that the weak solution of the boundary-value problem is indeed a classical solution.

We shall now give several formulations of the weak boundary conditions. The discussion will be motivated by considering the situation in which the functions treated and the boundary are sufficiently smooth. Suppose u is a smooth solution of $Au = f$. Let v be a smooth function, and assume that Ω has a smooth boundary. By Green's formula we have

$$(8.2) \qquad (A^*v, u) - (v, Au) = \sum_{j=0}^{2m-1} \int_{\partial\Omega} N_{2m-1-j}v \cdot \overline{\frac{\partial^j u}{\partial n^j}}\, d\sigma.$$

This formula is non-trivial; it will be derived and discussed in section 10. Here N_k is a linear differential operator of order k. If $D^\alpha v = 0$ on $\partial\Omega$ for $0 \leq |\alpha| \leq m - 1$, then v is a function with *zero Dirichlet data*. For such a function, and for u satisfying (8.1), the relation (8.2) becomes

$$(8.3) \qquad (A^*v, u) - (v, f) = \sum_{j=0}^{m-1} \int_{\partial\Omega} N_{2m-1-j}v \cdot \overline{\phi_j}\, d\sigma.$$

Conversely, suppose that u satisfies (8.3) for every function v having $2m$ continuous derivatives up to the boundary and zero Dirichlet data. Then u satisfies (8.1) in the following weak sense. If $v \in C_0^\infty(\Omega)$, then

$$(A^*v, u) = (v, f),$$

so that u is a weak solution of $Au = f$ in Ω. For the boundary conditions, let g be any smooth function such that $\partial^j g/\partial n^j = \phi_j$ on $\partial\Omega$, $0 \leq j \leq m - 1$. Note that (8.2) gives

$$(A^*v, g) - (v, Ag) = \sum_{j=0}^{m-1} \int_{\partial\Omega} N_{2m-1-j}v \cdot \overline{\phi_j}\, d\sigma,$$

where now v is assumed to have zero Dirichlet data. This subtracted from (8.3) gives

$$(8.4) \qquad (A^*v,\ u - g) - (v,\ Au - Ag) = 0.$$

Comparing this relation with (8.2) with u replaced by $u - g$, it follows that

$$(8.5) \qquad \sum_{j=0}^{m-1} \int_{\partial\Omega} N_{2m-1-j}v \cdot \overline{\frac{\partial^j(u-g)}{\partial n^j}}\, d\sigma = 0$$

for all sufficiently smooth v having zero Dirichlet data. But it is reasonable to expect that the functions $N_{2m-1-j}v$ for $0 \le j \le m-1$ can be essentially arbitrary for such v, so that (8.5) implies $\partial^j(u-g)/\partial n^j = 0$, $0 \le j \le m-1$. That is, $\partial^j u/\partial n^j = \phi_j$, $0 \le j \le m-1$. Thus, u satisfies (8.1).

Thus, we might make the definition that u is a generalized solution of (8.1) if (8.3) holds for all smooth v having zero Dirichlet data. However, this is somewhat inconvenient because of the presence of the boundary integrals in (8.3).

An alternate definition follows from (8.4). If g is any smooth function having the Dirichlet data $\partial^j g/\partial n^j = \phi_j$ on $\partial\Omega$, $0 \le j \le m-1$, then we might define u to be a generalized solution of (8.1) if

$$(A^*v,\ u - g) = (v,\ f) - (v,\ Ag)$$

for all smooth v having zero Dirichlet data. This is evidently a satisfactory definition, since no boundary integrals appear in this formulation. Indeed, this is almost our ultimate definition. It still needs modification so that something analogous to Dirichlet's integral for Laplace's equation appears.

We now proceed to the definition that will be used here. If both ϕ and ψ are smooth functions having zero Dirichlet data, then (8.2) shows

$$(A^*\phi,\ \psi) - (\phi,\ A\psi) = 0.$$

This result is obtained by transferring all the differentiations on ψ in the integral $(\phi,\ A\psi)$ to ϕ by integration by parts. However, we could transfer only some of the differentiations to obtain different formulas; since ϕ and ψ both have zero Dirichlet data, and since the boundary integrals contain only terms of the form $D^{\alpha}\phi \cdot \overline{D^{\beta}\psi}$ with $|\alpha| + |\beta| \le 2m - 1$, it is seen that all the boundary integrals vanish. In particular, suppose we integrate by parts the expression $(\phi, A\psi)$ in some manner so that no derivative for either ϕ or ψ of order greater than m results. Thus, we obtain some formula

$$(8.6)\qquad (\phi, A\psi) = \sum_{\substack{|\alpha|\le m\\ |\beta|\le m}} \int_{\Omega} c_{\alpha\beta}(x) D^{\alpha}\phi\, \overline{D^{\beta}\psi}\, dx.$$

(There is no reason to expect that there is a unique set of functions $c_{\alpha\beta}(x)$ with this property; we shall give an example below to show that there are in general many distinct possibilities.)

We shall set $a_{\alpha\beta}(x) \equiv \overline{c_{\alpha\beta}(x)}$; the bilinear form

$$(8.7)\qquad B[\phi, \psi] = \sum_{\substack{|\alpha|\le m\\ |\beta|\le m}} (D^{\alpha}\phi,\, a_{\alpha\beta} D^{\beta}\psi)$$

is called a *Dirichlet bilinear form* corresponding to the operator A. The relation (8.6) can be expressed

$$(8.8)\qquad (\phi, A\psi) = B[\phi, \psi],$$

all smooth ϕ, ψ having zero Dirichlet data.

Note that if in (8.6) we transfer all the differentiations of ϕ by integrating by parts, we have formally

$$(\phi, A\psi) = \sum_{\substack{|\alpha|\le m\\ |\beta|\le m}} (-1)^{|\alpha|} (\phi,\, D^{\alpha}(a_{\alpha\beta} D^{\beta}\psi)\,).$$

Thus, since even the functions ϕ in $C_0^{\infty}(\Omega)$ are dense in $L_2(\Omega)$, we have

$$(8.9)\qquad A\psi = \sum_{\substack{|\alpha|\le m\\ |\beta|\le m}} (-1)^{|\alpha|} D^{\alpha}(a_{\alpha\beta} D^{\beta}\psi).$$

Now we motivate our final choice for definition of weak solution. Assume all the functions and $\partial\Omega$ are sufficiently smooth, and let g be a smooth function having the Dirichlet data $\partial^j g/\partial n^j = \phi_j$, $0 \le j \le m-1$. If u is a solution of (8.1), then $u - g$ has zero Dirichlet data. Thus, if ϕ has zero Dirichlet data, (8.8) implies

$$(\phi,\ A(u-g)\ ) = B[\phi,\ u-g];$$

or setting $u_0 = u - g$, using $Au = f$,

$$(\phi,\ f - Ag) = B[\phi,\ u_0].$$

Thus, we shall say (approximately) that u is a generalized solution of (8.8) if $u = u_0 + g$, where u_0 is a function having zero Dirichlet data and satisfying

$$B[\phi,\ u_0] = (\phi,\ f) - (\phi,\ Ag),$$

all ϕ having zero Dirichlet data. Or, if we look only at the test functions $\phi \in C_0^\infty(\Omega)$, we require only that

$$B[\phi,\ u_0] = (\phi,\ f) - B[\phi,\ g], \quad \text{all} \quad \phi \in C_0^\infty(\Omega). \tag{8.10}$$

(Here we have used the relation (8.8) with $\psi = g$; the relation holds since all derivatives of ϕ are zero at $\partial\Omega$.)

After a few definitions, we shall make the statement (8.10) precise. First, we give some examples.

1. If
$$A = \sum_{\substack{|\alpha|\le m\\|\beta|\le m}} (-1)^{|\alpha|}\, D^\alpha a_{\alpha\beta} D^\beta,$$

as in (8.9), then the formal adjoint of A is given by

$$A^* = \sum_{\substack{|\alpha|\le m\\|\beta|\le m}} (-1)^{|\alpha|}\, D^\alpha \overline{a_{\beta\alpha}}\, D^\beta. \tag{8.11}$$

Thus, if $a_{\alpha\beta} = \overline{a_{\beta\alpha}}$, then $A = A^*$, a relation which is expressed by saying that A is *formally self-adjoint.* Note that in the case $\overline{a_{\alpha\beta}} = a_{\beta\alpha}$, the Dirichlet form $B[\phi, \psi]$ is Hermitian symmetric; that is,

$$B[\phi, \psi] = \overline{B[\psi, \phi]}.$$

Moreover, if $A = A^*$, then some Dirichlet form for A is symmetric in the sense that $\overline{a_{\alpha\beta}} = a_{\beta\alpha}$. For, suppose A has some Dirichlet form

$$B_1[\phi, \psi] = \sum_{\substack{|\alpha|\le m\\ |\beta|\le m}} (D^\alpha\phi, c_{\alpha\beta}D^\beta\psi).$$

Here A is assumed to be any linear differential operator, not necessarily of the form (8.9).
Then A^* has the Dirichlet form

$$(8.12)\qquad B_2[\phi, \psi] = \sum_{\substack{|\alpha|\le m\\ |\beta|\le m}} (D^\alpha\phi, \overline{c_{\beta\alpha}}\, D^\beta\psi),$$

as shown by a simpl e computation. But if $A = A^*$, then B_2 is also a Dirichlet form for A. Thus, $\frac{1}{2}\{B_1 + B_2\}$ is also a Dirichlet form for A, and

$$\tfrac{1}{2}\{B_1[\phi, \psi] + B_2[\phi, \psi]\} = \sum_{\substack{|\alpha|\le m\\ |\beta|\le m}} (D^\alpha\phi, a_{\alpha\beta}D^\beta\psi],$$

where

$$a_{\alpha\beta} = \tfrac{1}{2}(c_{\alpha\beta} + \overline{c_{\beta\alpha}}).$$

Thus, $a_{\alpha\beta} = \overline{a_{\beta\alpha}}$, as was desired.

2. The *biharmonic operator* in two-dimensional space is

$$\Delta^2 = (D_1{}^2 + D_2{}^2)^2.$$

For Δ^2 there are the two distinct Dirichlet forms:

$$B_1[\phi, \psi] = (\Delta\phi, \Delta\psi);$$

$$B_2[\phi, \psi] = ([D_1{}^2 - D_2{}^2]\phi, [D_1{}^2 - D_2{}^2]\psi) + 4\,(D_1D_2\phi, D_1D_2\psi).$$

The B_2 is a Dirichlet form corresponding to Δ^2 follows from the identity

$$(D_1{}^2 + D_2{}^2)^2 = (D_1{}^2 - D_2{}^2)^2 + 4(D_1D_2)^2.$$

Note that from B_1 and B_2 one obtains infinitely many distinct Dirichlet forms: $B = tB_1 + (1 - t)B_2$, t and arbitrary real number.

Now we give a rigorous definition of weak solution of (8.1).

Definition 8.1. $\overset{\circ}{H}_m(\Omega)$ *is the completion of* $C_0^\infty(\Omega)$ *under the norm* $\| \ \|_{m,\Omega}$.

Clearly, $\overset{\circ}{H}_m(\Omega)$ can be identified with the closure in $H_m(\Omega)$ of $C_0^\infty(\Omega)$; thus, $\overset{\circ}{H}_m(\Omega)$ is a closed subspace of $H_m(\Omega)$. The functions u in $\overset{\circ}{H}_m(\Omega)$ should be thought of as having zero Dirichlet data ($D^\alpha u = 0$ on $\partial\Omega$, $|\alpha| \leq m - 1$) in some vague sense.

We shall now even weaken the concept of differential operator. We shall assume that a bilinear form B is given:

$$(8.13) \qquad B[v, u] = \sum_{\substack{|\alpha|\leq m \\ |\beta|\leq m}} (D^\alpha v, a_{\alpha\beta} D^\beta u)_{0,\Omega},$$

and we assume only that the coefficients $a_{\alpha\beta}(x)$ are bounded and measurable in Ω, and that the leading coefficients $a_{\alpha\beta}(x)$, $|\alpha| = |\beta| = m$, are uniformly continuous in Ω. With such a form is associated a *formal* differential operator

$$(8.14) \qquad Au = \sum_{\substack{|\alpha|\leq m \\ |\beta|\leq m}} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta} D^\beta u),$$

in accordance with (8.9). It must be emphasized that A is just a symbol, and *may not be a differential operator at all,* since the coefficients $a_{\alpha\beta}$ are not assumed to be differentiable. Note that if we start with a genuine differential operator A, and if B is a Dirichlet form associated with

A, then the formal operator associated with B is just the differential operator A itself.

Finally, note that if A is a differential operator, then its principal part is

$$A' = (-1)^m \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta} D^{\alpha+\beta}.$$

Thus,

$$(-1)^m \Re A'(x, \xi) = \Re \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta}(x)\xi^{\alpha+\beta}.$$

Hence, the assumption of strong ellipticity of A is equivalent to the assumption of strong ellipticity of the quadratic form $B[\phi, \phi]$ (cf. Definition 7.1).

We can now define

THE GENERALIZED DIRICHLET PROBLEM: given $g \in H_m(\Omega)$ *and* $f \in L_2(\Omega)$, *find a function* $u \in H_m(\Omega)$ *such that*

1) $u - g \in \overset{\circ}{H}_m(\Omega)$,
2) $B[\phi, u] = (\phi, f)$ *for all* $\phi \in C_0^\infty(\Omega)$.

Alternatively, by setting $u_0 = u - g$ this can be stated in the equivalent form, "*find a function* $u_0 \in \overset{\circ}{H}_m(\Omega)$ *such that* $B[\phi, u_0] = (\phi, f) - B[\phi, g]$." Such a function u will be called a *generalized solution of* $Au = f$ with the same Dirichlet data as g. It will be convenient to abbreviate "generalized Direchlet problem" to GDP. The motivation for this definition of generalized solution is given in the discussion leading to (8.10).

Our first task is to show that in certain cases the GDP has a solution. The regularity theory in section 6 will then show that this solution is a solution in the classical sense, if enough assumptions are made on the regularity of A, f, g, and $\partial\Omega$.

The proof of existence makes use of the following *LAX-MILGRAM* theorem, which is derived and at the same time is a generalization of the Riesz-Fréchet representation theorem (see L. Nirenberg, On

Elliptic Partial Differential Equations, Annali della scuola Normale Superior di Pisa, v. 13 (1959 pp. 134-135).

THEOREM 8.1. *Let $B[u, v]$ be a bilinear form on a Hilbert space H with norm $\| \ \|$. If there are positive constants c_1 and c_2 such that*

$$|B[u, v]| \leq c_1 \|u\| \, \|v\|,$$

$$|B[u, u] \geq c_2 \|u\|^2,$$

for all u, $v \in H$, and if $F(x)$ is a bounded linear functional on H, then there exist unique v, $w \in H$ such that

$$F(x) = B[x, v] = \overline{B[w, x]}, \quad all \quad x \in H.$$

Observe that the Dirichlet bilinear form satisfies the first condition of the Lax-Milgram theorem. Indeed, since the coefficients $a_{\alpha\beta}$ are bounded, the Cauchy-Schwarz inequality implies $|B[v, u]| \leq K \|v\|_m \|u\|_m$. We now have

THEOREM 8.2. *If there is a constant $c > 0$ such that $|B[u, u]| \geq c\|u\|_m^2$ for all $u \in \overset{\circ}{H}_m(\Omega)$, then the GDP has a unique solution.*

Proof. We wish to find a function $u_0 \in \overset{\circ}{H}_m(\Omega)$ such that $B[\phi, u_0] = (\phi, f) - B[\phi, g]$ for all $\phi \in C_0^\infty(\Omega)$. Such a function is provided by the Lax-Milgram theorem with $F(\phi) = (\phi, f) - B[\phi, g]$. Note that $|F(\phi)| \leq \|\phi\|_0 \|f\|_0 + K\|\phi\|_m \|g\|_m \leq (\|f\|_0 + K\|g\|_m)\|\phi\|_m$, which implies F is a bounded linear functional on the Hilbert space $\overset{\circ}{H}_m(\Omega)$. Q.E.D.

Before proceeding further it will be convenient to prove the following compactness theorem which is a (simplified) version of *RELLICH'S THEOREM* (cf. Theorem 3.8):

THEOREM 8.3. *If Ω is bounded and if $j < m$, then the identity map of $\overset{\circ}{H}_m(\Omega)$ into $\overset{\circ}{H}_j(\Omega)$ is compact.*

Proof. We may clearly assume $j = m - 1$. Suppose that $\{u_k\} \subset \overset{\circ}{H}_m(\Omega)$ is a sequence such that $\|u_k\|_{m,\Omega} \leq c$. By the definition $\overset{\circ}{H}_m(\Omega)$, there is a sequence $\{\phi_k\} \in C_0^\infty(\Omega)$ such that $\|u_k - \phi_k\|_{m,\Omega} < 1/k$. As in the proof of Rellich's theorem, assume that $\overline{\Omega} \subset Q = \{x: |x_k| < 1/2\}$, and that ϕ_k has been extended to Q by defining $\phi_k = 0$ outside Ω. By Theorem 3.7 $\{\phi_k\}$ has a subsequence $\{\phi_{k_l}\}$ which converges in $\overset{\circ}{H}_{m-1}(\Omega)$. Since

$$\begin{aligned}\|u_{k_i}-u_{k_j}\|_{m-1,\Omega} &\le \|u_{k_i}-\phi_{k_i}\|_{m-1,\Omega}+\|\phi_{k_i}-\phi_{k_j}\|_{m-1,\Omega}\\ &\quad+\|\phi_{k_j}-u_{k_j}\|_{m-1,\Omega}\\ &< k_i^{-1}+\|\phi_{k_i}-\phi_{k_j}\|_{m-1,\Omega}+k_j^{-1}\\ &\to 0 \quad \text{as} \quad k_i,\ k_j\to\infty,\end{aligned}$$

it follows that $\{u_{k_i}\}$ converges in $\overset{\circ}{H}_{m-1}(\Omega)$. Q.E.D.

We shall call the form

$$B^*[u,\ v]=\overline{B[v,\ u]}$$

the *adjoint* of B. The formal differential operator associated with B^* is given in accordance with (8.14) by

$$A^*v=\sum_{\substack{|\alpha|\le m\\ |\beta|\le m}}(-1)^{|\alpha|}\,D^\alpha(\overline{a_{\beta\alpha}}D^\beta v).$$

In case both A and A^* are differential operators, then A^* is the formal adjoint of A in the usual sense; cf. (8.11).

Now assume that for some positive constant c_0,

$$(8.15)\qquad |B[u,\ u]|\ge c_0\|u\|_{m,\Omega}^2,\quad \text{all}\quad u\in\overset{\circ}{H}_m(\Omega).$$

By theorem 8.2, the GDP has a unique solution. In particular, if we take $g=0$, corresponding to zero Dirichlet data, there is for each $f\in L_2(\Omega)$ a unique element $u\in\overset{\circ}{H}_m(\Omega)$ such that

$$(8.16)\qquad B[\phi,\ u]=(\phi,\ f),\quad \text{all}\quad \phi\in C_0^\infty(\Omega).$$

Note that since both sides of (8.16) are continuous functions of ϕ for ϕ in $H_m(\Omega)$, we can even assert that (8.16) holds for all $\phi\in\overset{\circ}{H}_m(\Omega)$. Now we introduce the notation $u=Tf$. Thus, (8.16) becomes

(8.17) $B[\phi, Tf] = (\phi, f)$, all $\phi \in \overset{\circ}{H}_m(\Omega)$.

Now T maps $L_2(\Omega)$ into $\overset{\circ}{H}_m(\Omega)$, so in (8.17) we may take $\phi = Tf$; therefore,

$$\begin{aligned} c_0 \|Tf\|^2_{m,\Omega} &\leq |B[Tf, Tf]| \\ &= |(Tf, f)| \\ &\leq \|Tf\|_{0,\Omega} \, \|f\|_{0,\Omega} \\ &\leq \|Tf\|_{m,\Omega} \, \|f\|_{0,\Omega} \,. \end{aligned}$$

Hence,

$$\|Tf\|_{m,\Omega} \leq c_0^{-1} \|f\|_{0,\Omega} \,,$$

so that T is a bounded linear transformation of $L_2(\Omega)$ into $\overset{\circ}{H}_m(\Omega)$.

Now (8.15) implies also that $|B^*[u, u]| \geq c_0 \|u\|^2_{m,\Omega}$. Therefore, for each $h \in L_2(\Omega)$ there exists a unique $v = T_1 h \in \overset{\circ}{H}_m(\Omega)$ such that $B^*[\psi, T_1 h] = (\psi, h)$ for all $\psi \in \overset{\circ}{H}_m(\Omega)$. Or, by the definition of B^*,

(8.18) $B[T_1 h, \psi] = (h, \psi)$, all $\psi \in \overset{\circ}{H}_m(\Omega)$.

Moreover, T_1 is a bounded linear transformation of $L_2(\Omega)$ into $\overset{\circ}{H}_m(\Omega)$.

Now we can obviously think of T and T_1 as being linear transformations mapping $L_2(\Omega)$ into $L_2(\Omega)$. With this indentification we have

THEOREM 8.4. *In* $L_2(\Omega)$, $T_1 = T^*$. *That is,*

$(h, Tf) = (T_1 h, f)$, *all* $f, h \in L_2(\Omega)$.

Proof. By (8.17) with $\phi = T_1 h$,

$$B[T_1 h, Tf] = (T_1 h, f);$$

by (8.18) with $\psi = Tf$,

$$B[T_1 h, Tf] = (h, Tf).$$

Comparing these expressions, the result is immediate. Q.E.D.

The previous results were derived under the assumption that B satisfies (8.15). Now if A is a differential operator which is strongly and uniformly elliptic, then we have seen that the quadratic form $B[\phi, \phi]$ is uniformly strongly elliptic. Therefore, under the assumptions that $a_{\alpha\beta}$ is bounded and measurable, and $a_{\alpha\beta}$ is uniformly continuous in Ω for $|\alpha| = |\beta| = m$, we have at least the validity of Gårding's inequality; cf. Theorem 7.6. Moreover, a consequence of Theorem 8.2 and the corollary to Lemma 7.9 is that *in sufficiently small subsets of* Ω *the* GDP *for* $Au = f$ *is always uniquelv solvable.* More generally, we have

THEOREM 8.5. *Assume Gårding's inequality holds: for some constants* $c_0 > 0$ *and* $\lambda_0 \geq 0$, *and for all* $\phi \in C_0^\infty(\Omega)$

$$\Re B[\phi, \phi] \geq c_0\|\phi\|^2_{m,\Omega} - \lambda_0\|\phi\|^2_{0,\Omega}.$$

Then for any $\lambda \geq \lambda_0$ *the* GDP *for* $Au + \lambda u = f$ *has a unique solution. Moreover, if* Ω *is bounded, then* A *satisfies the* Fredholm alternative; *that is, if* $N(A)$ *is the* null space *of* A (*the set of functions* $u \in \overset{\circ}{H}_m$ *such that* $Au = 0$ *in the generalized sence) and if* $N(A^*)$ *is the null space of* A^*, *then* $N(A)$ *and* $N(A^*)$ *are finite demensional and have the same dimension. Also, the* GDP *for* $Au = f$ *and* u *having zero Dirichlet data has a solution if and only if* f *is orthogonal* (*in* $L_2(\Omega)$) *to* $N(A^*)$.

Proof. Let $B + \lambda$ be the bilinear form $B[v, u] + \lambda\, [v, u]_{0,\Omega}$. Then for $\lambda \geq \lambda_0$, $\Re(B + \lambda)[\phi, \phi] = \Re B[\phi, \phi] + \lambda\|\phi\|^2_{0,\Omega} \geq c_0\|\phi\|^2_{m,\Omega}$, so that Theorem 8.2 applies to $B + \lambda$. It remains to verify the Fredholm alternative.

Let T_0 be the operator associated with $B + \lambda_0$ according to the above analysis; that is, T_0 is a bounded linear transformation of $L_2(\Omega)$ into $\overset{\circ}{H}_m(\Omega)$, satisfying

$$(8.19)\qquad (B + \lambda_0)[\phi, T_0 f] = (\phi, f), \quad \text{all} \quad \phi \in \overset{\circ}{H}_m(\Omega)$$

(cf. 8.17). Now if $u \in \overset{\circ}{H}_m(\Omega)$, then u is a generalized solution of $Au = f$ precisely if $B[\phi, u] = (\phi, f)$, all $\phi \in \overset{\circ}{H}_m(\Omega)$. This is true if and only if $(B + \lambda_0)[\phi, u] = (\phi, f + \lambda_0 u)$, all $\phi \in \overset{\circ}{H}_m(\Omega)$. By (8.19), this is equivalent to the assertion that $u = T_0(f + \lambda_0 u)$. That is, for

$u \in \overset{\circ}{H}_m$ and $f \in L_2$,

$$(8.20) \quad B[\phi, u] = (\phi, f), \quad \text{all} \quad \phi \in \overset{\circ}{H}_m(\Omega) \Longleftrightarrow u - \lambda_0 T_0 u = f.$$

In particular, for $u \in \overset{\circ}{H}_m$, $u \in N(A) \Longleftrightarrow B[\phi, u] = 0$, all $\phi \in \overset{\circ}{H}_m(\Omega) \Longleftrightarrow u - \lambda_0 T_0 u = 0$. Thus, if $N(1 - \lambda_0 T_0)$ is the null space of $1 - \lambda_0 T_0$, considered as an operator in $L_2(\Omega)$, $N(A) \subset N(1 - \lambda_0 T_0)$. Next, suppose $u \in N(1 - \lambda_0 T_0)$; then $u \in L_2(\Omega)$ and $u = \lambda_0 T_0 u$, and, since T_0 maps $L_2(\Omega)$ into $\overset{\circ}{H}_m(\Omega)$, $T_0 u \in \overset{\circ}{H}_m(\Omega)$. But then $u = \lambda_0 T_0 u \in \overset{\circ}{H}_m(\Omega)$. By (8.20), $B[\phi, u] = 0$ for all $\phi \in \overset{\circ}{H}_m(\Omega)$. Thus, $u \in N(A)$. We conclude that $N(A) = N(1 - \lambda_0 T_0)$. Likewise, $N(A^*) = N(1 - \lambda_0 T_0^*)$.

Next, T_0, considered as a mapping of $L_2(\Omega)$ into $L_2(\Omega)$, is really a composition of T_0, the mapping of $L_2(\Omega)$ into $\overset{\circ}{H}_m(\Omega)$, followed by the identity mapping of $\overset{\circ}{H}_m(\Omega)$ into $H_0(\Omega) = L_2(\Omega)$. This identity mapping is compact, by Theorem 8.3. Thus, T_0 is a compact operator on $L_2(\Omega)$, and we can apply the Riesz-Schauder theory of compact operators. Thus, $N(1 - \lambda_0 T_0)$ and $N(1 - \lambda_0 T_0^*)$ have the same finite dimension. Also, $Au = f$ is solvable for $u \in \overset{\circ}{H}_m(\Omega) \Longleftrightarrow B[\phi, u] = (\phi, f)$, all $\phi \in \overset{\circ}{H}_m(\Omega) \Longleftrightarrow u - \lambda_0 T_0 u = f$, by (8.20). By the Riesz-Schauder theory, the range of $1 - \lambda_0 T_0$ is precisely the orthogonal complement of $N(1 - \lambda_0 T_0^*)$, which, as we have seen, is $N(A^*)$. Thus, $Au = f$ is solvable for $u \in \overset{\circ}{H}_m(\Omega)$ if and only if f is orthogonal to $N(A^*)$. Q.E.D.

9. Global Regularity of Solutions of Strongly Elliptic Equations

In section 6 we discussed local regularity results for elliptic systems. In the present section we are again confronted with the problem of regularity. This time, however, we shall obtain regularity properties up to the boundary of the domain; moreover, we shall in the process obtain some more local regularity theorems.

It should be noted that we have already given results which show that the solution of a generalized Dirichlet problem for an elliptic equation is regular in Ω. For suppose $u \in H_m(\Omega)$ satisfies $B[\phi, u] = (\phi, f)$ for all $\phi \in C_0^\infty(\Omega)$, where B is a Dirichlet form corresponding to an elliptic operator A. Suppose that A has coefficients which are smooth enough to guarantee that $B[\phi, u] = (A^*\phi, u)$. Then we have the result that u is actually a weak solution of the equation $Au = f$.

Thus, the local regularity results of section 6, especially Theorem 6.3, give conditions on A and f which allow the conclusion that u is as regular in Ω as we please. We shall not dwell on this point, however, since the regularity theorems of this section imply results both in the interior and at the boundary of Ω.

The method consists of using Gårding's inequality to show that if a solution u already has a certain regularity, say $u \in \overset{\circ}{H}_m(\Omega)$, then u actually has more regularity, say $u \in H_{m+j}(\Omega)$. As in section 6, the principal lemmas (Lemmas 9.2 and 9.3) require a considerable amount of work; the more precise regularity theorems then follow with relative ease.

We will consider only the Dirichlet problem with zero Dirichlet data. As a first result we have

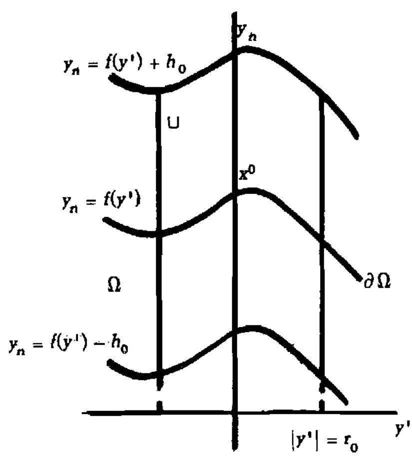
LEMMA 9.1. *Let $x^\circ \in \partial\Omega$ and assume that the coordinates can be rotated so that some neighborhood U of x° has the representation*

$$U = \{y\colon f(y') - h_0 < y_n < f(y') + h_0,\ |y'| < r_0\},$$

where $y' = (y_1, \ldots, y_{n-1})$, r_0 is a positive constant, f is a continuous function for $|y'| < r_0$, $x^\circ = (0, \ldots, 0, f(0))$, and

$$U \cap \partial\Omega = \{y\colon y_n = f(y'),\quad |y'| < r_0\}.$$

Let $u \in \overset{\circ}{H}_1(\Omega)$ and assume that u is defined and continuous in $U \cap \overline{\Omega}$. Then $u(x^\circ) = 0$.



Proof. Let $\phi_k \in C_0^\infty(\Omega)$. For fixed y' such that $|y'| < r_0$, and for $0 < h < h_0$,

$$\phi_k(y', f(y') - h) = \int_{f(y')}^{f(y')-h} D_n\phi_k dy_n,$$

since ϕ_k vanishes on $\partial\Omega$. By the Cauchy-Schwarz inequality,

$$|\phi_k(y', f(y') - h)|^2 \leq h \int_{f(y')-h}^{f(y')} |D_n\phi_k|^2\, dy_n,$$

so that, on integrating with respect to y', for $r < r_0$,

$$\int_{|y'|<r} |\phi_k(y', f(y') - h)|^2 dy' \leq h \int_{G_h} |D_n\phi_k|^2 dy,$$

where $G_h = \{y\colon f(y') - h < y_n < f(y'),\quad |y'| < r_0\}$. Thus,

$$\int_0^{h_1} \int_{|y'|<r} |\phi_k(y', f(y') - h)|^2\, dy' dh$$

$$\leq \int_0^{h_1} h \int_{G_h} |D_n\phi_k|^2 dy$$

$$\leq \tfrac{1}{2} h_1^2 \int_{G_{h_1}} |D_n\phi_k|^2 dy,$$

where $0 < h_1 < h_0$. Let $\phi_k \to u$ in $\overset{\circ}{H}_1(\Omega)$; we see that

$$h_1^{-1} \int_0^{h_1} \int_{|y'|<r} |u(y', f(y') - h)|^2 dy' dh$$

$$\leq \tfrac{1}{2} h_1 \int_{G_{h_1}} |D_n u|^2 dy$$

$$\leq \tfrac{1}{2} h_1 |u|_1^2.$$

Since u is continuous, on letting $h_1 \to 0$ we obtain

$$\int_{|y'|<r} |u(y', f(y'))|^2 dy' = 0,$$

so that $u(y', f(y')) = 0$ for $|y'| < r$. Q.E.D.

From this lemma it follows that if $u \in \overset{\circ}{H}_m(\Omega)$, if $x^\circ \in \partial\Omega$ satisfies the condition of the lemma, if $D^\alpha u$ is continuous in a neighborhood of x° in $\overline{\Omega}$, and if $|\alpha| \le m - 1$, then $D^\alpha u$ vanishes at x°. Consequently, if we succeed in proving the continuity of $D^\alpha u$ up to a portion of $\partial\Omega$, then it follows automatically that $D^\alpha u$ vanishes on that portion of $\partial\Omega$, $|\alpha| \le m - 1$.

As in section 6, it will not be necessary to assume that u satisfies certain identities in order to establish regularity results. Instead, we shall assume only that certain inequalities hold. If, for example, u satisfies

$$B[\phi, u] = (\phi, f)$$

for all $\phi \in C_0^\infty(\Omega)$, then

$$|B[\phi, u]| \le C||\phi||_{0,\Omega} = C||\phi||_{m-m,\Omega},$$

where C is some number depending on n, m, Ω, u. Even this inequality is more than is needed. Indeed, we shall assume only that

$$|B[\phi, u]| \le C||\phi||_{m-j,\Omega}$$

for some j: $1 \le j \le m$; and we shall show that if $u \in \overset{\circ}{H}_m(\Omega)$, this inequality implies that $u \in H_{m+j}(\Omega)$. This fact will first be proved for a neighborhood of a part of the boundary which is flat.

Since the principal lemma can be stated for either a sphere or a hemisphere, corresponding to interior or boundary estimates, respectively, the following notation will be convenient. By $G = G_R$, we shall denote either the sphere $\{x: |x| < R\}$ or the hemisphere $\{x: |x| < R, x_n > 0\}$; whenever a distinction between the two is necessary, we shall indicate which should be considered. We shall write $G' = G_{R'}$ and $G'' = G_{R''}$, assuming that $R' < R$ and $R'' = \frac{1}{2}(R' + R)$. The symbol ζ will denote a real function which is infinitely differentiable on E_n and has its support in $\{x: |x| < R\}$. Note that ζ need not vanish on the flat part of the boundary of the hemisphere.

Now we are ready to prove the *fundamental regularity lemma*. The essential tool will be Gårding's inequality.

LEMMA 9.2. *Assume*

1° *that the bilinear form*

$$B[v, u] = \sum_{\substack{|\alpha| \le m \\ |\beta| \le m}} (D^{\alpha} v, a_{\alpha\beta} D^{\beta} u)_{0, G_R}$$

has bounded ($|a_{\alpha\beta}| \le M$) *measurable coefficients in* G_R, *and that, for* $|\alpha| = m$, $a_{\alpha\beta}$ *satisfies a uniform Lipschitz condition:*

$$|a_{\alpha\beta}(x) - a_{\alpha\beta}(y)| \le K|x - y|, \quad x, y \in G_R;$$

that the associated quadratic form is uniformly strongly elliptic: for ξ *real and* $x \in \Omega$

$$\Re \sum_{\substack{|\alpha| = m \\ |\beta| = m}} a_{\alpha\beta}(x)\xi^{\alpha+\beta} \ge E|\xi|^{2m};$$

2° *that* $u \in H_m(G_R)$ *and that* $\zeta u \in \overset{\circ}{H}_m(G_R)$ *for all* $\zeta \in C_0^{\infty}(\{x: |x| < R\})$ *(Note that this condition is needed only if* G_R *is a hemisphere; it is automatically satisfied if* G_R *is a sphere.); and*

3° *that there is a number* C *such that for all* $\phi \in C_0^{\infty}(G_R)$

$$|B[\phi, u]| \le C\|\phi\|_{m-1}.$$

Then $D_i u \in H_m(G_{R'})$ *and there is a constant* γ, *depending only on* m, n, E, M, K, R, *and* R', *such that*

$$\|D_i u\|_{m, G_{R'}} \le \gamma(C + \|u\|_{m, G_R}),$$

where either

(1) $i = 1,\ldots,n$, *if* G_R *is a sphere, or*

(2) $i = 1,\ldots,n-1$, *if* G_R *is a hemisphere.*

Proof. An argument, familiar by now, shows that without loss of generality we may assume that $a_{\alpha\beta} = 0$ for $|\alpha| < m$; for if hypothesis 3° of the lemma holds, then it certainly holds if $a_{\alpha\beta}$ is replaced by zero for $|\alpha| < m$, since from 3° and the inequality,

$$|B[\phi, u]| = \Big| \sum_{\substack{|\alpha|=m \\ |\beta|\le m}} (D^\alpha\phi, a_{\alpha\beta}D^\beta u)_{0,G} + \sum_{\substack{|\alpha|<m \\ |\beta|\le m}} (D^\alpha\phi, a_{\alpha\beta}D^\beta u)_{0,G}\Big|$$

$$\ge \Big| \sum_{\substack{|\alpha|=m \\ |\beta|\le m}} (D^\alpha\phi, a_{\alpha\beta}D^\beta u) - C_1||\phi||_{m-1,G}||u||_{m,G},$$

it follows that, for any $\phi \in C_0^\infty(G)$

$$\Big| \sum_{\substack{|\alpha|=m \\ |\beta|\le m}} (D^\alpha\phi, a_{\alpha\beta}D^\beta u)\Big| \le (C + C_1||u||_{m,G})||\phi||_{m-1,G},$$

where C_1 depends on m, n, and M.

The remainder of the proof consists of an estimation of difference quotients. Let ζ be a function as described above, such that $\zeta \equiv 1$ on G' and $\zeta \equiv 0$ outside G''. Throughout the proof the function ζ will be fixed. For $0 < h < R - R''$, let $v_h = \delta_h^i(\zeta u) = \delta_h(\zeta u)$; if G is a sphere [hemisphere], i can take on any of the values $1,\ldots,n$ $[1,\ldots,n-1]$. By 2° $\zeta u \in \overset{\circ}{H}_m(G_R)$. Since $i \neq n$ in the case of the hemisphere, it is not difficult to see that $v_h = \delta_h(\zeta u) \in \overset{\circ}{H}_m(G_R)$. Moreover, since we may assume $\zeta \equiv 0$ for $|x| \ge R''$ (and $x_n > 0$ in the case of the hemisphere), we may consider ζu to be defined ($\equiv 0$) for such x, and then we obtain a version of Leibnitz's rule:

$$D^\alpha(\zeta u) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^\beta\zeta \cdot D^{\alpha-\beta}u,$$

valid for $x \in G_r$, any $r > 0$. Here $D^{\alpha-\beta}u(x)$ can be defined to be anything (say zero) outside of G_R, and the formula remains valid.

By this convention we also have $D^{\beta}v_h = \delta_h D^{\beta}(\zeta u)$ in G_R. Therefore, for any $\phi \in C_0^{\infty}(G)$

$$(9.1) \quad B[\phi, v_h] = \sum_{\substack{|\alpha|=m \\ |\beta|\leq m}} (D^{\alpha}\phi, a_{\alpha\beta}\delta_h D^{\beta}(\zeta u))_{0,G}.$$

Let k be a generic constant depending only on m, n, M, K, E, R, and R' (k will also depend on ζ, but ζ is fixed for fixed R and R'). In (9.1) we shall use Leibnitz's rule; we then obtain terms of the form $\delta_h D^{\beta-\gamma}\zeta \cdot D^{\gamma}u$. The proof of Theorem 3.13 implies that for $|\gamma| \leq m-1$

$$(9.2) \quad ||\delta_h(D^{\beta-\gamma}\zeta \cdot D^{\gamma}u)_{0,G_R} \leq ||D^{\beta-\gamma}\zeta \cdot D^{\gamma}u||_{1,G_{R+h}}$$

$$= ||D^{\beta-\gamma}\zeta \cdot D^{\gamma}u||_{1,G_R}$$

$$\leq k||u||_{m,G}.$$

Thus, (9.1) implies

$$|B[\phi, v_h]| \leq |\sum_{\substack{|\alpha|=m \\ |\beta|\leq m}} (D^{\alpha}\phi, a_{\alpha\beta}\delta_h(\zeta D^{\beta}u))| + k||\phi||_{m,G}||u||_{m,G}.$$

Now the first term on the right can be estimated as follows:

$$|\sum_{\substack{|\alpha|=m \\ |\beta|\leq m}} (D^{\alpha}\phi, a_{\alpha\beta}\delta_h(\zeta D^{\beta}u))|$$

$$= |\sum_{\substack{|\alpha|=m \\ |\beta|\leq m}} (D^{\alpha}\phi, \delta_h(a_{\alpha\beta}\zeta D^{\beta}u))$$

$$- \sum_{\substack{|\alpha|=m \\ |\beta|\leq m}} (D^{\alpha}\phi, \delta_h a_{\alpha\beta} \cdot \zeta(x+he^i)D^{\beta}u(x+he^i)|$$

$$\leq \Big| \sum_{\substack{|\alpha|=m \\ |\beta|\leq m}} (D^\alpha\phi, \delta_h(a_{\alpha\beta}\zeta D^\beta u)\Big| + k\|\phi\|_{m,G}\|u\|_{m,G}.$$

where we have made use of the Lipschitz condition on $a_{\alpha\beta}$. Thus,

$$|B[\phi, v_h]| \leq \Big| \sum_{\substack{|\alpha|=m \\ |\beta|\leq m}} (D^\alpha\phi, \delta_h(a_{\alpha\beta}\zeta D^\beta u))\Big| + k\|\phi\|_{m,G}\|u\|_{m,G}$$

$$= \Big| -\sum_{\substack{|\alpha|=m \\ |\beta|\leq m}} (\zeta\delta_{-h}D^\alpha\phi, a_{\alpha\beta}D^\beta u)\Big| + k\|\phi\|_{m,G}\|u\|_{m,G}.$$

We have used the vanishing of ζ outside of G" to transfer δ_h from the right side of (,) to the left. Using Leibnitz's rule and the proof of Theorem 3.13 again, we see that

$$|B[\phi, v_h]| \leq \Big| \sum_{\substack{|\alpha|=m \\ |\beta|\leq m}} (D^\alpha(\zeta\delta_{-h}\phi), a_{\alpha\beta}D^\beta u)\Big| + k\|\phi\|_{m,G}\|u\|_{m,G}$$

$$= |B[\zeta\delta_{-h}\phi, u]| + k\|\phi\|_{m,G}\|u\|_{m,G}.$$

The function $\zeta\delta_{-h}\phi$ is a test function on G; here we use again the fact that for the case of a hemisphere, $i \neq n$. Therefore, by hypothesis 3° of the lemma,

$$|B[\zeta\delta_{-h}\phi, u]| \leq C\|\zeta\delta_{-h}\phi\|_{m-1,G}.$$

Thus

$$|B[\phi, v_h]| \leq C\|\zeta\delta_{-h}\phi\|_{m-1,G} + k\|\phi\|_{m,G}\|u\|_{m,G}$$

$$= k(C + \|u\|_{m,G})\|\phi\|_{m,G}.$$

Again we have used a modification of Theorem 3.13; cf. (9.2).

Since this inequality holds for every $\phi \in C_0^\infty(G)$, by continuity it holds for all $\phi \in \overset{\circ}{H}_m(G)$. In particular, it holds for $\phi = v_h$. Thus,

$$|B[v_h, v_h]| \le k\|v_h\|_{m,G}(C + \|u\|_{m,G}).$$

By Gårding's inequality, Theorem 7.6,

$$\begin{aligned}\|v_h\|^2_{m,G} &\le \gamma_0^{-1}E^{-1}[\Re B[v_h, v_h]+\lambda_0\|v_h\|^2_{0,G}] \\ &\le \gamma_0^{-1}E^{-1}[k\|v_h\|_{m,G}(C + \|u\|_{m,G}) + \lambda_0\|v_h\|^2_{0,G}] \\ &\le k\|v_h\|_{m,G}(C + \|u\|_{m,G} + \|v_h\|_{0,G}) \\ &\le k\|v_h\|_{m,G}(C + \|u\|_{m,G} + \|u\|_{1,G}),\end{aligned}$$

where we have utilized (9.2) with $\beta = \gamma = 0$ and m replaced by 1. Dividing by $\|v_h\|_{m,G}$,

$$\|\delta_h(\zeta u)\|_{m,G} = \|v_h\|_{m,G} \le k(C + \|u\|_{m,G}).$$

By Theorems 3.15 and 3.16, this implies that $D_i\zeta u \in H_m(G')$ and that

$$\|D_i\zeta u\|_{m,G'} \le k(C + \|u\|_{m,G}).$$

Since $\zeta \equiv 1$ on G', the conclusion of the lemma follows. Q.E.D.

Since this lemma can be applied to any sphere contained in a domain Ω, it implies interior regularity. More than this, it provides regularity properties up to a flat boundary. This is true even for the normal derivative $D_n u$, although the proof above certainly does not show this.

For $m = 1$, the situation is quite simple. If u satisfies the equation $B[\phi, u] = (\phi, f)_{0,\Omega}$ for all $\phi \in C_0^\infty(\Omega)$, then for any sphere $S \subset \Omega$ we have $|B[\phi, u]_S| \le \|f\|_{0,S}\|\phi\|_{1-1,S}$ for all $\phi \in C_0^\infty(S)$, and the lemma can be applied in S. It follows that $u \in H_2^{loc}(\Omega)$ and satisfies $Au = f$ weakly in Ω, if A is the elliptic operator with Dirichlet form B. If $G_R \subset \Omega$ is a hemisphere with the flat part on its boundary contained in $\partial\Omega$, and if the coordinate system is rotated so that the flat part of ∂G_R lies in the plane $x_n = 0$, then it follows from the lemma that, for $R' < R$, $D_i u \in H_1(G_{R'})$, $i = 1,\dots,n-1$, whence $D_iD_ju \in H_0(G_{R'})$,

$i = 1,\ldots,n-1$, $j = 1,\ldots,n$. Since B is uniformly strongly elliptic, the coefficient a_{nn} is bounded away from zero; indeed, $\Re a_{nn} \geq E$. Hence, assuming A is really a differential operator, the equation $Au = f$ can be written

$$D^2_n u = a_{nn}^{-1}\Big(f - \sum_{i=1}^{n-1} \sum_{j=1}^{n} a_{ij} D_i D_j u - \sum_{i=1}^{n} b_i D_i u - cu\Big).$$

Since all of the terms on the right are in $H_0(G_{R'})$, it follows that $D_n^2 u \in H_0(G_{R'})$ and that $u \in H_2(G_{R'})$. Moreover, by the estimates obtained in Lemma 9.2 for the derivatives $D_i D_j u$, $i \neq n$,

$$\|D_n u\|_{1,G_{R'}} \leq \gamma(C + \|u\|_{1,G}).$$

Thus,

$$\|u\|_{2,G_{R'}} \leq \gamma(C + \|u\|_{1,G}).$$

Even in case $\partial\Omega$ is not flat, but is sufficiently smooth, the same result holds. This will be seen later; cf. Theorem 9.7. Note further that if $n \leq 3$, then $2 > n/2$, so that Sobolev's inequality implies that, after modification on a set of measure zero, $u \in C^0(\overline{\Omega})$. Therefore, Lemma 9.1 implies that $u \equiv 0$ on $\partial\Omega$. Thus, for second-order elliptic equations $Au = f$, with only the assumption that $f \in L_2(\Omega)$ and weak assumptions on the smoothness of A, it follows that any solution u in $\overset{\circ}{H}_1(\Omega)$ is in $C^0(\overline{\Omega})$ and vanishes on $\partial\Omega$. Moreover, the regularity results of section 6 allow us easily to give smoothness conditions on A and f which show that u is even in $C^2(\Omega)$ and is thus a genuine solution of the entire Dirichlet problem. We shall later in this section obtain regularity results that allow the same conclusions for any n and any m.

If $m > 1$, this method will not work. Instead, we shall use a lemma which guarantees the existence of weak derivatives of a function known to have certain pure derivatives and known to satisfy a certain inequality:

LEMMA 9.3. *For any $r > 0$, let $\Sigma_r = \{x: |x| < r,\ x_n > 0\}$. Let R be a positive number and let j be a positive integer. Assume*

1° *that* $u \in L_2(\Sigma_R)$, *that* u *has weak derivatives* $D_i^j u \in L_2(\Sigma_R)$, $i = 1,\ldots,n-1$; *and*

2° *that there is a constant* C *and an integer* $m \geq j$ *such that*

$$|(D_n^m\phi, u)| \leq C\|\phi\|_{m-j,\Sigma_R} \tag{9.3}$$

for all $\phi \in C_0^\infty(\Sigma_R)$.

Then for every $R' < R$, $u \in H_j(\Sigma_{R'})$ *and*

$$\|u\|_{j,\Sigma_{R'}} \leq \gamma(C + \sum_{i=1}^{n-1} \|D_i^j u\|_{0,\Sigma_R} + \|u\|_{0,\Sigma_R}),$$

where $\gamma = \gamma(n, m, R, R')$.

Proof. The proof is by transforming the boundary problem into an interior problem by a type of reflection, and then using the local results of Theorem 6.3.

First, observe that inequality (9.3) holds for any function $\phi \in C_0^m(\Sigma_R)$ since $J_\epsilon\phi \to \phi$ in $H_m(\Sigma_R)$, by Theorems 1.7 and 1.8. Moreover, (9.3) holds for any function $\phi \in C^m(\Sigma_R)$ such that for some $\delta > 0$, $\phi(x) = 0$ for $R - \delta < |x| < R$ and $D_n^s\phi(x) = 0$ for $x_n = 0$, $s = 0,\ldots,m$ (note that it then follows that $D^\alpha\phi = 0$ for $x_n = 0$, $|\alpha| \leq m$): to see this, let

$$\phi_\epsilon(x) = \begin{cases} \phi(x - \epsilon e^n) & \text{for} \quad x \in \Sigma_{R-\delta},\ x_n \geq \epsilon, \\ 0 & \text{for all other} \quad x; \end{cases}$$

then $\phi_\epsilon(x) \in C_0^m(\Sigma_R)$ and $\phi_\epsilon \to \phi$ in $H_m(\Sigma_R)$ as $\epsilon \to 0$.

Now we extend u to E_n: let

$$u(x', x_n) = \begin{cases} 0 & \text{for} \quad |x| \geq R,\ x_n > 0, \\ \sum_{k=1}^{2m+1} \lambda_k u(x', -k^{-1}x_n) & \text{for} \quad x_n < 0, \end{cases} \tag{9.4}$$

where $x' = (x_1,\ldots,x_{n-1})$; here the λ_k's are constants which are chosen so that u is sufficiently differentiable (in the weak sense) for $x_n = 0$.

In order for the function and its tangential derivatives (that is, derivatives D^α with $\alpha_n = 0$) to match nicely, it is sufficient to assume that

$$\sum_{k=1}^{2m+1} \lambda_k = 1.$$

For the normal derivatives, we must have

$$D_n^s u(x', x_n)|_{x_n=0} = \sum_{k=1}^{2m+1} \lambda_k D_n^s[u(x', -k^{-1}x_n)]|_{x_n=0}$$

$$= D_n^s u(x', x_n)|_{x_n=0} \sum_{k=1}^{2m+1} \lambda_k(-k)^{-s};$$

i.e.,

$$\sum_{k=1}^{2m+1} \lambda_k(-k)^{-s} = 1. \tag{9.5}$$

Thus, in order to have derivatives up to order $2m - 1$, it is sufficient to assume that (9.5) holds for $s = 0,\ldots,2m - 1$. This imposes only $2m$ conditions on the $2m + 1$ quantities λ_k, $k = 1,\ldots,2m + 1$. Because we will need (9.5) to hold for $s = -1$ later, we take $\{\lambda_k\}$ to be the (unique) set of real numbers satisfying (9.5) for $s = -1, 0, 1,\ldots,2m - 1$. (The determinant of the coefficients in the system of equations (9.5) is a Vandermonde determinant for the quantities $-k^{-1}$, $k = 1,\ldots,2m + 1$, and thus is not zero.) Note that the argument $-k^{-1}x_n$ in (9.4) is chosen so that the value of u at any point in $S_R = \{x: |x| < R\}$ depends on only the values of u at a finite number of points in Σ_R. Thus, $u \in L_2(S_R)$ and

$$||u||_{0,S_R} \leq \gamma ||u||_{0,\Sigma_R}.$$

Moreover, if $|\alpha| \leq 2m - 1$, if $\alpha_n = 0$, and if $D^\alpha u$ exists weakly in Σ_R, then $D^\alpha u$ exists in S_R and it is the extension (by (9.4)) of $D^\alpha u$ on Σ_R. To see this, let $\phi \in C_0^\infty(S_R)$ and consider

$$\int_{S_R} D^\alpha\phi \cdot u dx = \int_{x_n>0} D^\alpha\phi \cdot u dx + \int_{x_n<0} D^\alpha\phi \cdot u dx$$

$$= \int_{x_n>0} D^\alpha\phi \cdot u\,dx + \sum_{k=1}^{2m+1} \lambda_k \int_{x_n<0} D^\alpha\phi(x) \cdot u(x', -k^{-1}x_n)\,dx$$

$$= \int_{x_n>0} D^\alpha\phi \cdot u\,dx + \sum_{k=1}^{2m+1} k\lambda_k \int_{x_n>0} D^\alpha\phi(x', -kx_n) \cdot u(x)\,dx,$$

where we have substituted $-kx_n$ for x_n. Recall that $\alpha_n = 0$.) Thus,

$$\int_{S_R} D^\alpha\phi \cdot u\,dx = \int_{\Sigma_R} D^\alpha\phi_0 \cdot u\,dx,$$

where

$$\phi_0(x) = \phi(x) + \sum_{k=1}^{2m+1} k\lambda_k\phi(x', -kx_n).$$

By (9.5) with $s = -1$, $\phi_0(x', 0) = 0$; however, ϕ_0 is not necessarily a test function on Σ_R, since its support may intersect $\{x\colon\ x_n = 0\}$. To circumvent this problem, let $\rho(\lambda)$ be a infinitely differentiable function on the real line with $\rho(\lambda) = 0$ for $\lambda \le 1$, $\rho(\lambda) = 1$ for $\lambda \ge 2$. Let

$$\zeta_\epsilon(x) = \rho(\epsilon^{-1}|x_n|).$$

Then, $D_i\zeta_\epsilon = 0$ if $i \ne n$, so that

$$\int_{\Sigma_R} \zeta_\epsilon D^\alpha\phi_0 \cdot u\,dx = \int_{\Sigma_R} D^\alpha(\zeta_\epsilon\phi_0) \cdot u\,dx.$$

Since $\zeta_\epsilon\phi_0$ is a test function on Σ_R, the existence of $D^\alpha u$ on Σ_R yields

$$\int_{\Sigma_R} \zeta_\epsilon D^\alpha\phi_0 \cdot u\,dx = (-1)^{|\alpha|} \int_{\Sigma_R} \zeta_\epsilon\phi_0 D^\alpha u\,dx.$$

Thus,

$$\int_{S_R} D^\alpha\phi \cdot u\,dx = \int_{\Sigma_R} \zeta_\epsilon D^\alpha\phi_0 \cdot u\,dx + \int_{\Sigma_R} (1-\zeta_\epsilon)D^\alpha\phi_0 \cdot u\,dx$$

$$= (-1)^{|\alpha|} \int_{\Sigma_R} \zeta_\epsilon \phi_0 D^\alpha u \, dx + \int_{\Sigma_R} (1 - \zeta_\epsilon) D^\alpha \phi_0 \cdot u dx.$$

As ϵ tends to zero, the second term on the right tends to zero, and the first term tends to be obvious limit, so that

$$\int_{S_R} D^\alpha \phi \cdot u \, dx = (-1)^{|\alpha|} \int_{\Sigma_R} \phi_0 D^\alpha u dx$$

$$= (-1)^{|\alpha|} \int_{\Sigma_R} (\phi(x) + \sum_{k=1}^{2m+1} k\lambda_k \phi(x', -kx_n)) D^\alpha u(x) dx$$

$$= (-1)^{|\alpha|} [\int_{x_n > 0} \phi(x) D^\alpha u(x) dx$$

$$+ \int_{x_n < 0} \phi(x) \sum_{k=1}^{2m+1} \lambda_k D^\alpha u(x', -k^{-1} x_n) dx],$$

where we have substituted $-k^{-1}x_n$ for x_n. This proves that the extension of $D^\alpha u$ is, indeed, the α-th derivative of the extension of u in S_R. Moreover, from (9.4) it follows that for $\alpha = je^i$

$$||D_i^j u||_{0,S_R} \leq \gamma ||D_i^j u||_{0,\Sigma_R}, \; i = 1,\ldots,n-1.$$

Let $\psi \in C_0^\infty(S_R)$. Then

$$(D_n^{2m}\psi, u)_{0,S_R} = \int_{x_n > 0} D_n^{2m}\psi \cdot \bar{u} dx + \int_{x_n < 0} D_n^{2m}\psi \cdot \bar{u} dx$$

$$= \int_{x_n} D_n^{2m}\psi \cdot \bar{u} dx + \sum_{k=1}^{2m+1} \lambda_k \int_{x_n < 0} D_n^{2m}\psi(x) \cdot \bar{u}(x', -k^{-1}x_n) dx$$

$$= \int_{x_n > 0} D_n^{2m}\psi \cdot \bar{u} dx + \sum_{k=1}^{2m+1} \lambda_k k \int_{x_n > 0} (D_n^{2m}\psi)(x', -kx_n) \cdot \overline{u(x)} \, dx$$

$$= \int_{x_n>0} D_n^{2m}\psi \cdot \bar{u}\,dx + \sum_{k=1}^{2m+1} \lambda_k k^{1-2m} \int_{x_n>0} D_n^{2m}[\psi(x',-kx_n)] \cdot \overline{u(x)}dx,$$

where we have substituted $-kx_n$ for x_n. Let, for $x_n > 0$,

$$\psi^*(x) = \psi(x', x_n) - \sum_{k=1}^{2m+1} \lambda_k(-k)^{1-2m}\psi(x', -kx_n).$$

Then

(9.7) $$(D_n^{2m}\psi, u)_{0,S_R} = (D_n^{2m}\psi^*, u)_{0,\Sigma_R};$$

moreover, by the assumption that (9.5) holds for $s = -1,\dots,2m-1$, we have

$$D_n^s\psi^* = 0 \text{ for } x_n = 0,\ s = 0,\dots,2m.$$

Note that also $\psi^* = 0$ for $R - \delta \le |x| \le R$, $x_n > 0$, some $\delta > 0$. Let $\phi = D_n^m\psi^*$; then, by the remark made at the beginning of the proof, (9.3) holds for ϕ:

$$\begin{aligned} |(D_n^{2m}\psi^*, u)_{0,\Sigma_R}| &= |(D_n^m\phi, u)_{0,\Sigma_R}| \\ &\le C\|\phi\|_{m-j,\Sigma_R} \\ &\le C\|\psi^*\|_{2m-j,\Sigma_R}. \end{aligned}$$

Thus, from (9.7),

(9.8) $$\begin{aligned} |(D_n^{2m}\psi, u)_{0,S_R}| &\le C\|\psi^*\|_{2m-j,\Sigma_R} \\ &\le \gamma C\|\psi\|_{2m-j,S_R} \end{aligned}$$

for any $\psi \in C_0^\infty(S_R)$. For $i \neq n$, we can use the fact that u has the weak derivatives D_i^j in S_R, together with the estimate (9.6), to obtain

$$(9.9) \qquad |(D_i^{2m}\psi, u)_{0,S_R}| = |(D_i^{2m-j}\psi, D_i^j u)_{0,S_R}|$$

$$\leq ||\psi||_{2m-j,S_R} \; ||D_i^j u||_{0,S_R}$$

$$\leq \gamma ||D_i^j u||_{0,\Sigma_R} ||\psi||_{2m-j,S_R}.$$

Now let A be the elliptic operator of order $2m$ given by

$$A = \sum_{i=1}^{n} D_i^{2m};$$

from (9.8) and (9.9) we see that

$$|(A\psi, u)_{0,S_R}| \leq \gamma(C + \sum_{i=1}^{n-1} ||D_i^j u||_{0,\Sigma_R}) \; ||\psi||_{2m-j,S_R}$$

for all $\psi \in C_0^\infty(S_R)$. Thus, the interior regularity theorm (Theorem 6.3) can be applied: for $R' < R$ we have $u \in H_j(S_{R'})$ so that, in particular, $u \in H_j(\Sigma_{R'})$, and

$$||u||_{j,\Sigma_{R'}} \leq ||u||_{j,S_{R'}} \leq \gamma(C + \sum_{i=1}^{n-1} ||D_i^j u||_{0,\Sigma_R} + ||u||_{0,\Sigma_R}).$$

Q.E.D.

One additional lemma is needed before proceeding again to the boundary regularity question.

LEMMA 9.4. *Let* $G = \{x: \; |x| < R, \; x_n > 0\}$. *If* $u \in H_{m+1}(G)$ *and if* $\zeta u \in \mathring{H}_m(G)$ *for all* $\zeta \in C_0^\infty(\{x: \; |x| < R\})$, *then* $\zeta D_i u \in \mathring{H}_m(G)$ *for all* $\zeta \in C_0^\infty(\{x: \; |x| < R\})$, $i = 1, 2, \ldots, n-1$.

Proof. Let $\zeta \in C_0^\infty(\{x: \; |x| < R\})$ be fixed, and let $i \neq n$. Defining u as zero outside G, we have, for $x \in G$,

$$(9.10) \qquad \zeta(x)\delta_h^i u(x) = \delta_h^i(\zeta u)(x) - \delta_h^i \delta(x) \cdot u(x + he^i).$$

Clearly, $\delta_h^i(\zeta u) \in \overset{\circ}{H}_m(G)$ for sufficiently small $|h|$. Also, $\delta_h^i \zeta \in C_0^\infty(\{x\colon\ |x| < R\})$ for sufficiently small $|h|$, so that $\delta_h^i \zeta(x) \cdot u(x + he^i) \in \overset{\circ}{H}_m(G)$ for sufficiently small $|h|$, by the assumption on the behavior of u. Therefore, $\zeta \delta_h^i u \in \overset{\circ}{H}_m(G)$ for small $|h|$.

For suitable $R' < R$, supp $(\zeta) \cap G \subset G'$. By the proof of Theorem 3.13, we have for small $|h|$

$$\begin{aligned}
||\delta_h^i(\zeta u)||_{m,G} &= ||\delta_h^i(\zeta u)||_{m,G''} \\
&\leq ||\zeta u||_{m+1,G} \\
&\leq \gamma ||\zeta||_{m+1,G} ||u||_{m+1,G'}
\end{aligned}$$

by Leibnitz's rule, Then (9.10) implies

$$(9.11) \qquad ||\zeta \delta_h^i u||_{m,G} \leq \gamma_1 ||u||_{m+1,G}$$

for small $|h|$, where γ_1 is independent of h. Since $\overset{\circ}{H}_m(G)$ is a Hilbert space, any closed sphere $\{v\colon\ ||v||_m \leq A\}$ in $\overset{\circ}{H}_m(G)$ is weakly sequentially compact. By (9.11), $\zeta \delta_h^i u$ is uniformly bounded in $\overset{\circ}{H}_m(G)$, and so a subsequence $\zeta \delta_{h_k}^i u$ converges weakly in $\overset{\circ}{H}_m(G)$. In particular, for any $\phi \in C_0^\infty(G)$

$$\begin{aligned}
\lim_{k\to\infty} \int_G \zeta \delta_{h_k}^i u \cdot \phi dx &= \lim_{k\to\infty} - \int_G u \delta_{-h_k}^i (\zeta\phi) dx \\
&= - \int_G u D_i(\zeta \phi) dx \\
&= \int_G D_i u \cdot \zeta \phi dx.
\end{aligned}$$

Hence, $\zeta \delta_{h_k}^i u$ converges weakly in $\overset{\circ}{H}_m(G)$ to $\zeta D_i u$, and so $\zeta D_i u \in \overset{\circ}{H}_m(G$
Q.E.D.

Now we can extend Lemma 9.2. We shall show that u and *all* of its derivatives of order $\leq j$ are smooth up to the flat part of the boundary of G. The theorem is stated precisely in Lemma 9.6.

We will use the notation $C_*^k(\Omega)$ for the class of functions which have bounded, continuous derivatives of all orders $\leq k$ on Ω.

Definition 9.1. *A Dirichlet bilinear form*

$$B[\phi, \psi] = \sum_{\substack{|\alpha|\leq m \\ |\beta|\leq m}} (D^{\alpha}\phi,\ a_{\alpha\beta}D^{\beta}\psi)_{0,\Omega}$$

is right j-smooth in Ω if the coefficients $a_{\alpha\beta}$ are bounded and measurable in Ω and $a_{\alpha\beta} \in C_*^{|\alpha|+j-m}(\Omega)$ for $|\alpha| + j - m > 0$.

LEMMA 9.5. *Assume that $j \leq m$ and*

1° *that B is uniformly strongly elliptic and right j-smooth in G;*

2° *that $u \in H_m(G)$, that $\zeta u \in \overset{\circ}{H}_m(G)$ for all $\zeta \in C_0^\infty(\{x\colon\ |x| < R\})$; and*

3° *that there is a positive number C such that*

$$|B[\phi, u]| \leq C||\phi||_{m-j,G}$$

for all $\phi \in C_0^\infty(G)$.

Then, for any $R' < R$, $u \in H_{m+j}(G')$ and there is a constant $\gamma = \gamma(m, n, B, R, R')$ such that

$$||u||_{m+j,G'} \leq \gamma(C + ||u||_{m,G}).$$

Proof. The proof is by induction on j. Observe that the lemma is trivial for $j = 0$. Thus, it suffices to establish an induction step.

As in the proof of Lemma 9.2, we assume, without loss of generality, that $a_{\alpha\beta} = 0$ for $|\alpha| + j - m \leq 0$. Assume that $1 \leq j \leq m$ and that the lemma holds if j is replaced by $j - 1$; we will show that it holds for j. Let $R''' = \frac{1}{2}(R + R'')$, and $G''' = G_{R'''}$. Since

$$|B[\phi, u]| \leq C||\phi||_{m-j,G} \leq C||\phi||_{m-(j-1),G},$$

the inductive hypothesis gives $u \in H_{m+j-1}(G''')$. For $j = 1$, $D_i u \in H_{m+j-1}(G'')$, $i = 1,\ldots,n - 1$, by Lemma 9.2. To establish a similar result for $j > 1$, we observe that $u \in H_{m+1}(G''')$; hence, by Lemma 9.4, $\zeta D_i u \in \overset{\circ}{H}_m(G''')$, $i = 1,\ldots,n - 1$, for any $\zeta \in C_0^\infty(\{x\colon\ |x| < R'''\})$.

Furthermore, for $\phi \in C_0^\infty(G''')$,

$$B[\phi, D_i u] = \sum_{\substack{m-j<|\alpha|\leq m \\ |\beta|\leq m}} (D^\alpha\phi, a_{\alpha\beta} D^\beta D_i u)_{0,G'''}$$

$$= \sum_{\substack{m-j<|\alpha|\leq m \\ |\beta|\leq m}} (D^\alpha\phi, D_i(a_{\alpha\beta} D^\beta u))_{0,G'''}$$

$$- \sum_{\substack{m-j<|\alpha|\leq m \\ |\beta|\leq m}} (D^\alpha\phi, D_i a_{\alpha\beta} \cdot D^\beta u)_{0,G'''}$$

$$= - \sum_{\substack{m-j<|\alpha|\leq m \\ |\beta|\leq m}} (D^\alpha D_i\phi, a_{\alpha\beta} D^\beta u)_{0,G'''}$$

$$- \sum_{\substack{m-j<|\alpha|\leq m \\ |\beta|\leq m}} (D^\alpha\phi, D_i a_{\alpha\beta} \cdot D^\beta u)_{0,G'''}$$

$$= - B[D_i\phi u] - \sum_{\substack{m-j<|\alpha|\leq m \\ |\beta|\leq m}} (D^\alpha\phi, D_i a_{\alpha\beta} \cdot D^\beta u)_{0,G'''}. \tag{9.12}$$

Consider any term in the sum on the right; integrate by parts $|\alpha| - m + j - 1$ times (which is possible, since, by the inductive hypothesis, $u \in H_{m+j-1}(G''')$), transfering $|\alpha| - m + j - 1$ differentiations from ϕ to the other side of $(\ ,\ )_{0,G'''}$. By the Cauchy-Schwarz inequality,

$$|(D^\alpha\phi, D_i a_{\alpha\beta} \cdot D^\beta u)_{0,G'''}| \leq \gamma ||\phi||_{m-j+1,G} ||u||_{m+j-1,G};$$

note that we have used the fact that the derivatives of $D_i a_{\alpha\beta}$ up to order $|\alpha| - m + j - 1$ are bounded since B is right j-smooth. Thus, by hypothesis 3° and (9.12),

$$|B[\phi, D_i u]| \leq C||D_i\phi||_{m-j,G'''} + \gamma||\phi||_{m-j+1,G'''}||u||_{m+j-1,G'''}$$

$$\leq \gamma(C + ||u||_{m+j-1,G'''})||\phi||_{m-j+1,G'''}.$$

Here γ is a generic constant depending only on m, n, B, R, R'. Therefore, $D_i u$ satisfies the conditions of the lemma with j replaced by $j - 1$, and G replaced by G'''. By the inductive hypothesis it follows that $D_i u \in H_{m+j-1}(G'')$ and that

$$(9.13) \quad ||D_i u||_{m+j-1,G''} \leq \gamma(C + ||u||_{m+j-1,G'''} + ||D_i u||_{m,G'''})$$

$$\leq \gamma(C + ||u||_{m+j-1,G'''})$$

(note that $m + 1 \leq m + j - 1$ since $j > 1$). In the case of the sphere, inequality (9.13) holds for $i = 1,\ldots,n$; this completes the proof in this case, since $u \in H_{m+j}(G'')$ and by (9.13) and the inductive hypothesis

$$||u||_{m+j,G''} \leq \gamma(C + ||u||_{m,G}).$$

In the case of the hemisphere, we still must show that these relations hold for $i = n$.

The proof is completed by using Lemma 9.3 to show that $D_n u \in H_{m+j-1}(G')$. For $\phi \in C_0^\infty(G'')$

$$(9.14) \quad B[\phi, u] = (D_n^m \phi, \sum_{|\beta| \leq m} a_{me^n,\beta} D^\beta u)_{0,G''} + \sum_{\substack{m-j<|\alpha|\leq m \\ \alpha \neq me^n \\ |\beta| \leq m}} (D^\alpha \phi, a_{\alpha\beta} D^\beta u)_{0,G''}.$$

For any term in the sum on the right, there is an $i \neq n$ such that $D^\alpha \phi = D^{e^i} D^{\alpha - e^i} \phi$. Thus

$$(9.15) \quad (D^\alpha \phi, a_{\alpha\beta} D^\beta u)_{0,G''} = -(D^{\alpha - e^i} \phi, D_i(a_{\alpha\beta} D^\beta u))_{0,G''}$$

$$= -(D^{\alpha - e^i} \phi, D_i a_{\alpha\beta} \cdot D^\beta u)_{0,G''}$$

$$- (D^{\alpha-e^i}\phi,\ a_{\alpha\beta}D^{\beta}D_i u)_{0,G''}.$$

Estimate the right side by transferring $|\alpha| - m + j - 1$ differentiations from $D^{\alpha-e^i}\phi$ to the other sides of (,) and applying the Cauchy-Schwarz inequality. Since B is right j-smooth, the derivatives of $a_{\alpha\beta}$ which occur will be bounded. For the first term we have

$$|(D^{\alpha-e^i}\phi,\ D_i a_{\alpha\beta}\cdot D^{\beta}u)_{0,G''}| \le \gamma||\phi||_{m-j,G''}||u||_{m+j-1,G''};$$

for the second, by (9.13) we have

$$|(D^{\alpha-e^i}\phi,\ a_{\alpha\beta}D^{\beta}D_i u)_{0,G''}| \le \gamma||\phi||_{m-j,G''}||D_i u||_{m+j-1,G''}$$

$$\le \gamma||\phi||_{m-j,G''}(C + ||u||_{m+j-1,G'''}).$$

Thus, (9.15) implies

$$|(D^{\alpha}\phi,\ a_{\alpha\beta}D^{\beta}u)_{0,G''}| \le \gamma||\phi||_{m-j,G''}(C + ||u||_{m+j-1,G'''}).$$

Therefore, from (9.14),

$$|(D_n^m\phi,\ \sum_{|\beta|\le m} a_{me^n,\beta}D^{\beta}u)_{0,G''}| \le |B[\phi,\ u]| + \gamma||\phi||_{m-j,G''}\cdot$$

$$\cdot\ (C + ||u||_{m+j-1,G'''}).$$

Let

$$v = \sum_{|\beta|\le m} a_{me^n,\beta}D^{\beta}u.$$

Then, by hypothesis 3°,

$$|(D_n^m\phi,\ v)_{0,G''}| \le C||\phi||_{m-j,G''} + \gamma||\phi||_{m-j,G''}(C + ||u||_{m+j-1,G'''})$$

$$\le \gamma(C + ||u||_{m+j-1,G'''})||\phi||_{m-j,G''}.$$

From the inductive hypothesis,

$$||u||_{m+j-1,G'''} \le \gamma(C + ||u||_{m,G}).$$

Therefore,

$$|(D_n^m\phi, v)_{0,G''}| \le \gamma(C + ||u||_{m,G})||\phi||_{m-j,G''},$$

so that, in G", v satisfies the conditions of Lemma 9.3. (Note that $D_i^j v \in L_2(G'')$ for $i \neq n$, since we have shown $D_i u \in H_{m+j-1}(G'')$, $i \neq n$.) Hence, Lemma 9.3 implies $v \in H_j(G')$ and

$$||v||_{j,G'} \le \gamma(C + ||u||_{m,G} + \sum_{i=1}^{n-1} ||D_i^j v||_{0,G''} + ||v||_{0,G''})$$

$$\le \gamma(c + ||u||_{m,G}).$$

(We have used (9.13) in this estimate.) Since, by the ellipticity of B, $a_{me^n,me^n} \neq 0$, it follows that

$$D_n^m u = (a_{me^n,me^n})^{-1}(v - \sum_{\substack{|\beta| \le m \\ \beta \neq me^n}} a_{me^n,\beta} D^\beta u) \in H_j(G'),$$

and that

$$||D_n^m u||_{j,G'} \le \gamma(C + ||u||_{m,G}). \quad \text{Q.E.D.}$$

From Lemma 9.5 we can easily get global regularity theorems for strongly elliptic equations. As a first result we have

THEOREM 9.6 *Assume*

1° *that for some* j, $1 \le j \le m$, $B[\phi, \psi]$ *is a right* j*-smooth, uniformly strongly elliptic Dirichlet bilinear form in* G;

2° *that* $u \in H_m(G)$, *that for every* $\zeta \in C_0^\infty(\{x\colon\ |x| < R\})$, $\zeta u \in \overset{\circ}{H}_m(G)$;

3° *that* $f \in L_2(G)$ *and that*

$$B[\phi,\ u] = (\phi,\ f)_{0,G}$$

for every $\phi \in C_0^\infty(G)$.

Then $u \in H_{m+j}(G')$ *and*

$$||u||_{m+j,G'} \leq \gamma(||f||_{0,G} + ||u||_{m,G}).$$

Proof. Hypothesis 3° of Lemma 9.5 is immediate, since

$$|B[\phi,\ u]| = |(\phi,\ f)_{0,G}| \leq ||f||_{0,G}||\phi||_{0,G} \leq ||f||_{0,G}||\phi||_{m-j,G}$$

Therefore, the lemma applies and the theorem is proved. Q.E.D.

As an extension, we have

THEOREM 9.7. *If hypotheses* 1°, 2°, *and* 3° *of Theorem* 9.6 *hold, and if for some* $k \geq 0$

4° $f \in H_k(G)$

5° B *is right* $(m + k)$*-smooth,*

then $u \in H_{2m+k}(G')$ *and*

$$||u||_{2m+k,G'} \leq \gamma(||f||_{k,G} + ||u||_{m,G}).$$

Proof. The proof is by induction on k. By Theorem 9.6 the conclusion holds for $k = 0$. Suppose that $k \geq 1$ and that the theorem is true if k is replaced by $k - 1$. Then $u \in H_{2m+k-1}(G'')$, and

$$(9.16) \qquad ||u||_{2m+k-1,G''} \leq \gamma(||f||_{k-1,G} + ||u||_{m,G})$$

$$\leq \gamma(||f||_{k,G} + ||u||_{m,G}).$$

Thus, for any $i = 1,\ldots,n - 1$, and for $\phi \in C_0^\infty(G'')$

$$B[D_i\phi,\ u] = \sum_{\substack{|\alpha|\leq m \\ |\beta|\leq m}} (D_i D^\alpha\phi,\ a_{\alpha\beta}D^\beta u)_{0,G''}$$

$$= - \sum_{\substack{|\alpha|\le m \\ |\beta|\le m}} (D^\alpha\phi,\ D_i(a_{\alpha\beta}D^\beta u))_{0,G''}$$

$$= - \sum_{\substack{|\alpha|\le m \\ |\beta|\le m}} (D^\alpha\phi,\ D_i a_{\alpha\beta} \cdot D^\beta u)_{0,G''}$$

$$- \sum_{\substack{|\alpha|\le m \\ |\beta|\le m}} (D^\alpha\phi,\ a_{\alpha\beta}D^\beta D_i u)_{0,G''}.$$

Let $a^i_{\alpha\beta} = D_i a_{\alpha\beta}$; in each term in the first sum on the right we transfer all the differentiations from ϕ to the other side of (,), obtaining

$$B[D_i\phi,\ u] = - \sum_{\substack{|\alpha|\le m \\ |\beta|\le m}} (-1)^{|\alpha|}(\phi,\ D^\alpha(a^i_{\alpha\beta}D^\beta u))_{0,G''} - B[\phi,\ D_i u]$$

$$= - (\phi,\ A_i u)_{0,G''} - B[\phi,\ D_i u],$$

where

$$A_i = \sum_{\substack{|\alpha|\le m \\ |\beta|\le m}} (-1)^{|\alpha|}D^\alpha a^i_{\alpha\beta}D^\beta.$$

Note that the smoothness of B implies $a_{\alpha\beta} \in C_*^{|\alpha|+k}(G) \subset C_*^{|\alpha|+1}(G)$, since $k \ge 1$. Thus, A_i is a differential operator of order at most $2m$, having bounded coefficients. Now

$$B[D_i\phi,\ u] = (D_i\phi,\ f)_{0,G''} = - (\phi,\ D_i f)_{0,G''};$$

thus,

$$B[\phi,\ D_i u] = (\phi,\ D_i f)_{0,G''} - (\phi,\ A_i u)_{0,G''}$$

$$= (\phi,\ f_i)_{0,G''},$$

where $f_i = D_i f - A_i u$. Clearly, $f_i \in H_{k-1}(G'')$. Moreover,

$$||f_i||_{k-1,G''} \leq ||D_i f||_{k-1,G''} + ||A_i u||_{k-1,G''}$$
$$\leq \gamma(||f||_{k,G''} + ||u||_{2m+k-1,G''});$$

by (9.16),

$$||f_i||_{k-1,G''} \leq \gamma(||f||_{k,G} + ||u||_{m,G}).$$

According to Lemma 9.4, $\zeta D_i u \in \overset{\circ}{H}_m(G'')$, so that, by the inductive hypothesis applied to $D_i u$, $D_i u \in H_{2m+k-1}(G')$ and

$$(9.17) \qquad ||D_i u||_{2m+k-1,G'} \leq \gamma(||f_i||_{k-1,G''} + ||D_i u||_{m,G''})$$
$$\leq \gamma(||f_i||_{k-1,G} + ||u||_{2m,G''})$$
$$\leq \gamma(||f||_{k,G} + ||u||_{m,G}),$$

where (9.16) has been used again. Thus, we have obtained the required estimates on $D_i u$ for $i \neq n$.

To treat the normal derivative, let A be the elliptic operator

$$A = \sum_{|\alpha|,|\beta| \leq m} (-1)^{|\alpha|} D^\alpha a_{\alpha\beta} D^\beta.$$

The operator A can be written as a genuine differential operator with bounded coefficients, since $a_{\alpha\beta} \in C_*^{|\alpha|+k}(G) \subset C_*^{|\alpha|}(G)$. By (8.8) and hypothesis 3°, $(\phi, Au) = (\phi, f)$, all $\phi \in C_0^\infty(G)$, since we know already that $u \in H_{2m}(G'')$. Thus, $Au = f$, and so, by Leibnitz's rule,

$$(9.18) \qquad (-1)^m a_{me^n,me^n} D_n^{2m} u = f + \sum_{|\alpha| \leq 2m} b_\alpha D^\alpha u,$$

where the coefficients b_α belong to $C_*^k(G)$, and $b_{2me^n} = 0$. Now a_{me^n,me^n} is bounded away from zero, since the Dirichlet form B is uniformly strongly elliptic on G. Since, by the results above, every term on the right in (9.18) is in $H_k(G')$ and satisfies bounds as given by (9.17), it follows that $D_n^{2m}u \in H_k(G')$ and

$$(9.19)\qquad \|D_n^{2m}u\|_{k,G'} \le \gamma(\|f\|_{k,G} + \|y\|_{m,G}).$$

Since $D_n^{2m}u \in H_k(G')$ and $D_i u \in H_{2m+k-1}(G')$ for $i \neq n$, it follows that $D^\alpha u \in W_k(G')$ for $|\alpha| = 2m$. Therefore, Theorem 2.2 and Theorem 2.3 imply that $u \in H_{2m+k}(G')$. The estimates we have derived, (9.16) and (9.19), then show that

$$\|u\|_{2m+k,G'} \le \gamma(\|f\|_{k,G} + \|u\|_{m,G}). \quad \text{Q.E.D.}$$

In order to extend the above results to arbitrary domains with smooth boundaries, we make

Definition 9.2. *For $k \ge 1$, an open set Ω is said to be of class C^k if*

1° *about every $x^\circ \in \partial\Omega$ there is an open neighborhood U such that for some i, $U \cap \partial\Omega$ has the representation*

$$x_i = g(x'),\ x' = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in U',$$

where U' is the projection of U on the hyperplane $x_i = 0$, and $g \in C^k(U')$; and if

2° *$U \cap \Omega$ is contained in the half cylinder $\{x\colon\ x_i > g(x'),\ x' \in U'\}$.*

Observe that such a region $U \cap \Omega$ can be mapped, by a mapping of class C^k, onto an open set with a flat part of its boundary corresponding to $U \cap \partial\Omega$: simply let

$$(9.20)\qquad \begin{aligned} y_j &= x_j - x_j^\circ,\ j \neq i, \\ y_i &= x_i - g(x'); \end{aligned}$$

then $U \cap \Omega$ is mapped into $\{y\colon\ y_i > 0\}$. In fact, we can assume that $U \cap \Omega$ is mapped onto a hemisphere: about x° there is an open neighborhood which is carried into a sphere centered at the image of x° having radius small enough so that the sphere is contained in the image of U; since $\partial\Omega$ is compact, a finite number of these neighborhoods cover it.

From these remarks, it is also seen that if Ω is bounded and of class C^k with $k \geq 1$, then Ω has the restricted cone property. For, the Jacobian of the transformation (9.20) is one.

THEOREM 9.8. *Assume*

1° *that* Ω *(bounded or not) is of class* C^{2m}, *that* $\Omega_R = \Omega \cap \{x: |x| < R\}$,

2° *that* $B[\phi, \psi]$ *is a Dirichlet bilinear form of order* m *satisfying Gårding's inequality in* Ω,

3° *that* $u \in \overset{\circ}{H}_m(\Omega)$,

4° *that* $f \in L_2(\Omega)$, *and that*

$$B[\phi, u] = (\phi, f)_{0,\Omega} \tag{9.21}$$

for all $\phi \in C_0^\infty(\Omega)$.

If for some j, $1 \leq j \leq m$, B *is right* j*-smooth in* Ω, *then for every* R, $u \in H_{m+j}(\Omega_R)$ *and*

$$||u||_{m+j,\Omega_R} \leq \gamma(||f||_{0,\Omega} + ||u||_{0,\Omega}). \tag{9.22}$$

If for some $k \geq 0$, Ω *is of class* C^{2m+k}, $f \in H_k(\Omega)$, *and* B *is right* $(m+k)$*-smooth, then* $u \in H_{2m+k}(\Omega_R)$ *and*

$$||u||_{2m+k,\Omega_R} \leq \gamma(||f||_{k,\Omega} + ||u||_{0,\Omega}). \tag{9.23}$$

Remark. If Ω is bounded, then the theorem holds with Ω_R replaced by Ω. This statement is also true if only $\partial\Omega$ is bounded or if Ω is ultimately cylindrical (i.e., it for sufficiently large R, $\Omega \cap \{x: |x| > R\}$ consists of a finite number of truncated cylinders).

Proof. By the definition of the class C^{2m}, every point in $\bar{\Omega}$ has a neighborhood which can be mapped into either a sphere or a hemisphere by a mapping of class $C^{2m}[C^{2m+k}$ if $\Omega \in C^{2m+k}]$. Only a finite number of such neighborhoods are needed to cover $\bar{\Omega}_R$. Theorem 9.6 and 9.7 can be applied in each of the mapped neighborhoods; since the mappings perserve the desired properties of u, we obtain (e.g., for the second part of the theorem) for each of the neighborhoods U that $u \in H_{2m+k}(U)$ and

$$||u||_{2m+k,U} \leq \gamma(||f||_{k,\Omega} + ||u||_{m,\Omega}).$$

Since a finite number of neighborhoods U cover Ω_R, we obtain $u \in W^{loc}_{2m+k}(\Omega)$ and

$$(9.24) \qquad ||u||_{2m+k,\Omega_R} \leq \gamma(||f||_{k,\Omega} + ||u||_{m,\Omega}).$$

Thus, indeed $u \in H_{2m+k}(\Omega_R)$. To obtain (9.23) we need only apply Gårding's inequality (Theorem 7.6). For, since $u \in \overset{\circ}{H}_m(\Omega)$, we may in (9.21) take $\phi = u$ to obtain

$$B[u, u] = (u, f)_{0,\Omega}.$$

Also, we may apply Gårding's inequality to u, obtaining

$$\begin{aligned} c_0||u||^2_{m,\Omega} &\leq \Re B[u, u] + \lambda_0||u||^2_{0,\Omega} \\ &\leq |(u, f)_{0,\Omega}| + \lambda_0||u||^2_{0,\Omega} \\ &\leq ||u||_{0,\Omega}||f||_{0,\Omega} + \lambda_0||u||^2_{0,\Omega} \\ &\leq (\tfrac{1}{2} + \lambda_0)||u||^2_{0,\Omega} + \tfrac{1}{2}||f||^2_{0,\Omega}. \end{aligned}$$

Thus,

$$||u||_{m,\Omega} \leq c_0^{-\frac{1}{2}}(\tfrac{1}{2} + \lambda_0)^{\frac{1}{2}}||u||_{0,\Omega} + c_0^{-\frac{1}{2}}2^{-\frac{1}{2}}||f||_{0,\Omega}.$$

Substituting this in (9.24) yields (9.23). Likewise, (9.22) is proved. Q.E.D.

As a corollary we have

THEOREM 9.9. *Assume*

1° *that* Ω *is of class* C^∞ *and is bounded,*

2° *that* A *is a uniformly strongly elliptic operator whose coefficients are in the class* $C^\infty(\bar{\Omega})$,

3° *that* $f \in C^\infty(\overline{\Omega})$, *and*

4° *that* $u \in H_m(\Omega)$ *is a solution of the* GDP *for* $Au = f$ *with infinitely differentiable Dirichlet data.*

Then u *can be modified on a set of measure zero so that* $u \in C^\infty(\overline{\Omega})$.

Proof. We are given f, $g \in C^\infty(\bar{\Omega})$, and $u_0 = u - g$ satisfies

$$B[\phi, u_0] = (\phi, f) - B[\phi, g]$$

for all $\phi \in C_0^\infty(\Omega)$, where B is some Dirichlet form corresponding to A. This is from the definition of GDP. Since the coefficients of A are smooth, we have $B[\phi, g] = (\phi, Ag)$, so that for all $\phi \in C_0^\infty(\Omega)$

$$B[\phi, u_0] = (\phi, f - Ag).$$

Now the second part of Theorem 9.8 applies, and k can be any positive integer. Thus, $u_0 \in H_{2m+k}(\Omega)$ for every positive k. By Theorem 3.9, u_0 is equivalent to a function in $C^j(\bar{\Omega})$ for every positive j (note that Ω has the restricted cone property). Since $u = u_0 + g$, the result follows. Q.E.D.

The results of this section can be used as *a priori* bounds. We shall first need

LEMMA 9.10. *If* Ω *is bounded and in class* C^m, *if* $u \in C^m(\bar{\Omega})$ *and if* $D^\alpha u = 0$ *for* $|\alpha| \le m - 1$, *then* $u \in \overset{\circ}{H}_m(\Omega)$.

Proof. First we prove the theorem locally. Let $x^\circ \in \partial\Omega$; x° has a neighborhood U such that $\Omega \cap U$ can be mapped onto the hemisphere $\Sigma_R = \{x\colon |x| < R,\ x_n > 0\}$ by a mapping of class $C^m(\overline{\Omega \cap U})$. Therefore, we can consider the restriction of u to $\Omega \cap U$ as a function in $C^m(\overline{\Sigma_R})$. Assume that $\operatorname{supp}(u) \subset \{x\colon |x| < R'\}$, $R' < R$, and let

$$u_\epsilon(x) = \begin{cases} u(x - \epsilon e^n) & \text{for} \quad x - \epsilon e^n \in \Sigma_R, \\ 0 & \text{otherwise.} \end{cases}$$

For $\epsilon < R - R'$, $u_\epsilon \in \overset{\circ}{C}{}^m_0(\Sigma_R) \subset \overset{\circ}{H}_m(\Sigma_R)$. Clearly, $u_\epsilon \to u$ in $\overset{\circ}{H}_m(\Sigma_R)$ as $\epsilon \to 0$. Hence $u \in \overset{\circ}{H}_m(\Sigma_R)$.

To prove the result globally, use a special partition of unity as in the proof of Gårding's inequality. Q.E.D.

THEOREM 9.11. *If*

$$A = \sum_{|\alpha| \leq 2m} a_\alpha D^\alpha$$

is a uniformly strongly elliptic operator on Ω, if for $k \geq 0$, Ω is bounded and in class C^{2m+k}, if $a_\alpha \in C^k(\bar{\Omega})$, and if $u \in C^{2m+k}(\bar{\Omega})$ satisfies $D^\alpha u = 0$ on $\partial\Omega$ for $|\alpha| \leq m - 1$, then

$$\text{(9.25)} \qquad ||u||_{2m+k,\Omega} \leq \gamma(||Au||_{k,\Omega} + ||u||_{0,\Omega}).$$

Proof. First, assume that $A = A'$ has constant coefficients. By Lemma 9.10, $u \in \overset{\circ}{H}_m(\Omega)$. Moreover, $f \equiv Au \in H_k(\Omega)$. Since, for $\phi \in C_0^\infty(\Omega)$,

$$B[\phi, u] = (\phi, Au)_{0,\Omega} = (\phi, f)_{0,\Omega},$$

where B is any Dirichlet bilinear form for A, it follows from Theorem 9.8 that

$$\text{(9.26)} \qquad ||u||_{2m+k,\Omega} \leq \gamma(||Au||_{k,\Omega} + ||u||_{0,\Omega}),$$

so that the theorem holds in this case.

Now assume that A' has constant coefficients. Then

$$A'u = Au - \sum_{|\alpha| \leq 2m-1} a_\alpha D^\alpha u,$$

so that

$$||A'u||_{k,\Omega} \leq ||Au||_{k,\Omega} + \gamma||u||_{2m+k-1}, \; k \geq 0.$$

Thus, by (9.26) applied to A' instead of A, we have

$$||u||_{2m+k,\Omega} \leq \gamma(||Au||_{k,\Omega} + ||u||_{2m+k-1,\Omega} + ||u||_{0,\Omega}).$$

By the interpolation inequality, Theorem 3.4, for any $\epsilon > 0$ there is a constant C_ϵ such that

$$||u||_{2m+k,\Omega} \leq \gamma(||Au||_{k,\Omega} + \epsilon||u||_{2m+k,\Omega} + C_\epsilon||u||_{0,\Omega}).$$

Choose ϵ so small that $\gamma\epsilon \leq \frac{1}{2}$. Then

$$||u||_{2m+k,\Omega} \leq 2\gamma(||Au||_{k,\Omega} + C_\epsilon||u||_{0,\Omega})$$

and (9.25) holds in this case.

The extension to the case in which A' need not have constant coefficients is similar to the proof of Gårding's inequality. For, consider the form

$$B(u) = ||Au||^2_{k,\Omega}$$

for $u \in C^{2m+k}(\bar{\Omega})$. Since the coefficients of A are in $C^k(\bar{\Omega})$, this is a Dirichlet form of the type we have considered. We wish to prove a version of Gårding's inequality:

$$(9.27) \quad B(u) \geq c_0||u||^2_{2m+k} - \lambda_0||u||^2_0,$$

for the subclass of functions satisfying: $D^\alpha u = 0$ on $\partial\Omega$ for $|\alpha| \leq m - 1$. We have succeeded in establishing this in the case that A' has constant coefficients. Exactly as in the proof of Lemma 7.9 and its corollary, we can establish (9.27) locally: that is, (9.27) holds if supp(u) has sufficiently small diameter. Of course, in Gårding's inequality we assumed that $u \in C_0^\infty(\Omega)$. But the proof showed that we really needed only two facts concerning the class of functions u: (1) the validity of an interpolation inequality, and (2) the fact that for

$\zeta \in C_0^\infty(E_n)$, ζu is in the same class of functions as u. These are both valid in our case. Thus, the interpolation inequality of Theorem 3.4 implies (9.27) holds locally. And the partition of unity argument of pp. 84-86 then implies the estimate (9.27) for all $u \in C^{2m+k}(\bar{\Omega})$ satisfying: $D^\alpha u = 0$ on $\partial\Omega$ for $|\alpha| \leq m - 1$. Q.E.D.

10. Coerciveness

This chapter contains a treatment of some boundary value problems other than the Dirichlet problem. We shall need the following special form of Green's formula, which we have already mentioned (vid. equation 8.2). For simplicity, we shall assume throughout this section that the differential operators and the bounded region Ω are sufficiently smooth.

THEOREM 10.1. *Let A be a strongly elliptic operator of order ℓ in Ω. Then there exist linear differential operators $N_k(x, D)$ for $x \in \partial\Omega$, such that N_k is of order k, the boundary of Ω is nowhere characteristic for each N_k, and for all $u, v \in C^\ell(\bar{\Omega})$*

$$(A^*v, u)_{0,\Omega} - (v, Au)_{0,\Omega} = \sum_{j=0}^{\ell-1} \int_{\partial\Omega} N_{\ell-1-j}v \cdot \overline{\frac{\partial^j u}{\partial n^j}}\, d\sigma,$$

where $\partial^j/\partial n^j$ is the exterior normal derivative of order j at $\partial\Omega$.

Proof. First, consider the case that Ω is the hemisphere $\{x\colon |x| < R,\ x_n > 0\}$ and v vanishes for $|x| \geq R'$, where $R' < R$. For a derivative $D^\alpha = D_i D^{\alpha - e^i}$ with $i \neq n$, an integration by parts gives

$$-(D_i v, D^{\alpha-e^i}u) - (v, D^\alpha u) = 0,$$

since the boundary terms vanish. Indeed, after $\alpha_1 + \ldots + \alpha_{n-1}$ partial integrations, we obtain

$$(-1)^{\alpha_1+\ldots+\alpha_{n-1}}(D^{\alpha'}v, D_n^{\alpha_n}u) - (v, D^\alpha u) = 0,$$

where $\alpha' = \alpha - \alpha_n e^n$. Further partial integrations contribute boundary terms:

$$(D^{\alpha'}v, D_n^{\alpha_n}u) = -(D^{\alpha'+e^n}v, D_n^{\alpha_n-1}u) - \int_{|x'|<R} D^{\alpha'}v\overline{D_n^{\alpha_n-1}u}\,dx',$$

where $x' = (x_1, \ldots, x_{n-1})$ and $dx' = dx_1 \ldots dx_{n-1}$. Therefore

$$(-1)^{\alpha_1+\cdots+\alpha_{n-1}+1}(D^{\alpha'+e^n}v, D_n^{\alpha_n-1}u) - (v, D^{\alpha}u)$$

$$= (-1)^{\alpha_1+\cdots+\alpha_{n-1}} \int_{|x'|<R} D^{\alpha'}v\overline{D_n^{\alpha_n-1}u}\,dx'.$$

On applying this argument $\alpha_n - 1$ more times, we obtain

$$(-1)^{|\alpha|}(D^{\alpha}v, u) - (v, D^{\alpha}u)$$

$$= \sum_{k=0}^{\alpha_n-1} (-1)^{|\alpha'|+k} \int_{|x'|<R} D^{\alpha'+ke^n}v \cdot \overline{D_n^{\alpha_n-k-1}u}\,dx'.$$

Since A is ℓ-smooth, $a_\alpha \in C^{|\alpha|}(\overline{\Omega})$, and the identity above applies to $\overline{a}_\alpha v$. Thus,

$$((-1)^{|\alpha|}D^{\alpha}(\overline{a}_\alpha v), u) - (v, a_\alpha u)$$

$$= \sum_{k=0}^{\alpha_n-1} (-1)^{|\alpha'|+k} \int_{|x'|<R} D^{\alpha'+ke^n}(\overline{a}_\alpha v) \cdot \overline{D_n^{\alpha_n-k-1}u}\,dx'.$$

On summing over α, we obtain

$$(A^* v, u) - (v, Au)$$

$$= \sum_{|\alpha|\leq\ell} \sum_{k=0}^{\alpha_n-1} (-1)^{|\alpha'|+k} \int_{|x'|<R} D^{\alpha'+ke^n}(\overline{a}_\alpha v) \cdot \overline{D_n^{\alpha_n-k-1}u}\,dx'$$

$$= \sum_{j=0}^{\ell-1} \sum_{\substack{\alpha_n\geq j+1\\ |\alpha|\leq\ell}} (-1)^{|\alpha|-j-1} \int_{|x'|<R} D^{\alpha-(j+1)e^n}(\overline{a}_\alpha v) \cdot \overline{D_n^j u}\,dx'.$$

Let

$$N_{\ell-1-j}v = \sum_{\substack{\alpha_n \geq j+1 \\ |\alpha| \leq \ell}} (-1)^{|\alpha|-1} D^{\alpha-(j+1)e^n}(\bar{a}_\alpha v).$$

Then the surface $x_n = 0$ is non characteristic for $N_{\ell-1-j}$, since $N_{\ell-1-j}$ contains the term

$$(10.1) \qquad (-1)^{\ell-1}\overline{a_{(0,\ldots,0,\ell)}}D_n^{\ell-j-1},$$

where $a_{(0,\ldots,0,\ell)}$ cannot vanish since A is elliptic. Also,

$$(10.2) \qquad (A^*v, u) - (v, Au) = \sum_{j=0}^{\ell-1} \int_{|x'|<R} N_{\ell-1-j}v \cdot \overline{(-D_n)^j u}dx',$$

so that the theorem is proved in the case of a hemisphere and v vanishing for $|x| \geq R'$.

In (10.2) the normal derivative $-D_n$ can be replaced by any non-tangential derivative, whose direction is a sufficiently smooth function of x'. Such an oblique derivative D_s can be represented as a linear combination of $D_1,\ldots,D_n$. Thus for some $d_j \neq 0$,

$$d_j D_s^j = (-D_n)^j + \sum_{\substack{|\alpha| \leq j \\ \alpha \neq je^n}} d_{\alpha,j} D^\alpha,$$

where d_j and $d_{\alpha,j}$ are smooth functions of x'. Since v vanishes for $|x'| > R'$, an integration by parts gives

$$\int_{|x'|<R} d_j N_{\ell-1-j}v \cdot \overline{D_s^j u}dx'$$

$$= \int_{|x'|<R} N_{\ell-1-j}v \cdot \overline{(-D_n)^j u}dx'$$

$$+ \sum_{\substack{|\alpha| \leq j \\ \alpha \neq je^n}} \int_{|x'|<R} N_{\ell-1-j}v \cdot \overline{d_{\alpha,j}D^\alpha u}dx'$$

$$(10.3) \quad = \int_{|x'|<R} N_{\ell-1-j}v \cdot \overline{(-D_n)^j u}\,dx'$$

$$+ \sum_{\substack{|\alpha|\le j \\ \alpha\ne je^n}} \int_{|x'|<R} (-1)^{|\alpha|} D^{\alpha'}(d_{\alpha,j}N_{\ell-1-j}v) \cdot \overline{(-D_n)^{\alpha_n} u}\,dx'$$

where $\alpha' = (\alpha_1,\ldots,\alpha_{n-1}, 0)$. Note that $\alpha_n < j$. Hence, the term with the oblique derivative of order j can be represented as a term with the normal derivative of order j plus terms in which the normal derivative is of order less than j. Moreover, the operator

$$N'_{j,\ell-1-\alpha_n} = \sum_{|\alpha'|\le j-\alpha_n} (-1)^{|\alpha|} D^{\alpha'}(d_{\alpha,j}N_{\ell-1-j})$$

associated with $(-D_n)^{\alpha_n}u$ is of order at most $\ell-1-\alpha_n$, and does not contain the pure normal derivative of order $\ell - 1 - \alpha_n$. The relation (10.3) can be written

$$(10.4) \quad (N_{\ell-1-j}v, (-D_n)^j u)' = (d_j N_{\ell-1-j}v, D_s^j u)'$$

$$- \sum_{\alpha_n=0}^{j-1} (N'_{j,\ell-1-\alpha_n}v, (-D_n)^{\alpha_n}u)',$$

where

$$(\phi, \psi)' = \int_{|x'|<R} \phi(x')\overline{\psi(x')}\,dx'.$$

Applying (10.4) with $j = \ell-1$, we obtain

$$\sum_{j=0}^{\ell-1} (N_{\ell-1-j}v, (-D_n)^j u)' = (d_{\ell-1}N_0 v, D_s^{\ell-1}u)'$$

$$- \sum_{\alpha_n=0}^{\ell-2} (N'_{\ell-1,\ell-1-\alpha_n}v, (-D_n)^{\alpha_n}u)'$$

$$+\sum_{j=0}^{\ell-2}(N_{\ell-1-j}v,\ (-D_n)^j u)'$$

$$(10.5)\qquad = (N_0^1 v,\ D_s^{\ell-1}u)' + \sum_{j=0}^{\ell-2}(N_{\ell-1-j}^1 v,\ (-D_n)^j u)',$$

where

$$N_0^1 = d_{\ell-1}N_0,$$

$$N_{\ell-1-j}^1 = N_{\ell-1-j} - N'_{\ell-1,\ell-1-j},\ 0 \le j \le \ell-2.$$

Since $N'_{\ell-1,\ell-1-j}$ does not contain the derivative $D_n^{\ell-1-j}$, but $N_{\ell-1-j}$ does contain this derivative, the surface $x_n = 0$ is non characteristic for $N_{\ell-1-j}^1$, $0 \le j \le \ell-2$. Since $d_{\ell-1} \neq 0$, this surface is non characteristic for N_0^1.

Now a relation similar to (10.4) holds with $N_{\ell-1-j}$ replaced by $N_{\ell-1-j}^1$, $0 \le j \le \ell-2$. Therefore, the argument just given shows that

$$\sum_{j=0}^{\ell-2}(N_{\ell-1-j}^1 v,\ (-D_n)^j u)' = (N_1^2 v,\ D_s^{\ell-2}u)' + \sum_{j=0}^{\ell-3}(N_{\ell-1-j}^2 v,\ (-D_n)^j u)',$$

where the surface $x_n = 0$ is non characteristic for $N_{\ell-1-j}^2$, $0 \le j \le \ell-2$. Therefore, (10.5) becomes

$$\sum_{j=0}^{\ell-1}(N_{\ell-1-j}v,\ (-D_n)^j u)' = (N_0^1 v,\ D_s^{\ell-1}u)' + (N_1^2 v,\ D_s^{\ell-2}u)'$$

$$+\sum_{j=0}^{\ell-3}(N_{\ell-1-j}^2 v,\ (-D_n)^j u)'.$$

An induction argument then shows that

$$\sum_{j=0}^{\ell-1}(N_{\ell-1-j}v,\ (-D_n)^j u)' = \sum_{j=0}^{\ell-1}(N_{\ell-1-j}^{\ell-j}v,\ D_s^j u)',$$

where the operator $N^{\ell-j}_{\ell-1-j}$ is of order j and the surface $x_n = 0$ is non-characteristic for this operator. Therefore, (10.2) becomes

$$(10.6)\quad (A^*v, u) - (v, Au) = \sum_{j=0}^{\ell-1} \int_{|x'|<R} \overset{\circ}{N}_{\ell-1-j}v \cdot \overline{D_s^j u}dx',$$

where $\overset{\circ}{N}_{\ell-1-j} = N^{\ell-j}_{\ell-1-j}$.

Now we continue with the proof of the theorem.

If v vanishes for $|x| \geq R'$, then the theorem holds trivially for the sphere $\{x: |x| < R\}$, since the boundary terms vanish.

Now consider an arbitrary bounded region $\Omega \in C^\ell$. Let $\{0_i\}$ be a finite open covering of $\partial\Omega$, of the type described in the remarks following Definition 9.2; i.e., there is a smooth mapping carrying $0_i \cap \Omega$ onto a hemisphere. Let $\{S_j\}$ be a finite collection of spheres such that $\overline{S}_j \subset \Omega$ and $\Omega - \cup 0_i \subset \cup S_j$. Let $\{0'_i\}$ and $\{S'_j\}$ be sets satisfying

$$0'_i \subset\subset 0_i, \qquad S'_j \subset\subset S_j,$$
$$\partial\Omega \subset \cup 0'_i, \qquad \Omega - \cup 0'_i \subset \cup S'_j.$$

Let $\{\zeta_i\}$ and $\{\zeta^j\}$ be collections of functions in $C^\infty(E_n)$ such that $\zeta_i \geq 0$, $\zeta^j \geq 0$, $\operatorname{supp}(\zeta_i) \subset 0'_i$, $\operatorname{supp}(\zeta^j) \subset S'_j$, and $\Sigma\zeta_i + \Sigma\zeta^j = 1$ on $\overline{\Omega}$. Let $v_i = \zeta_i v$, $v^j = \zeta^j v$. By the remarks above,

$$(10.7)\quad (A^*v^j, u) - (v^j, Au) = 0.$$

The boundary regions are more difficult to handle. Consider a particular boundary neighborhood 0_i and let τ_i be the mapping which carries $0_i \cap \Omega$ onto a hemisphere H. The mapping τ_i carries the operators A and A^* into smooth linear differential operators A_i and $\tilde{A}_i$, respectively, on the hemisphere. The operator $\tilde{A}_i$ is the formal adjoint of A: if $u, v \in C_0^\infty(0_i \cap \Omega)$, then τ_i carries $(Au, v)_{0,0_i\cap\Omega}$ $[(u, A^*v)_{0,0_i\cap\Omega}]$ into $(A_iu, v)_{0,H}[(u, \tilde{A}_iv)_{0,H}]$, since the Jacobian of the transformation is 1. If A_i^* is the formal adjoint of A_i, then we must have

$$(v, \tilde{A}_iv)_{0,H} = (A_iu, v)_{0,H} = (u, A_i^*v)_{0,H}$$

for all u, $v \in C_0^\infty(H)$; hence, $\tilde{A}_i = A_i^*$. Thus, by the first part of the proof, for A_i, A_i^*, we have an identity of the form (10.6) with the oblique derivative D_s chosen in such a direction that its image under r_i^{-1} is a normal derivative on $\partial\Omega$. Thus, after mapping back into $0_i \cap \Omega$, we obtain

$$(A^*v_i, u)_{0_i\cap\Omega} - (v_i, Au)_{0_i\cap\Omega}$$

$$= \sum_{j=0}^{\ell-1} \int_{\partial\Omega\cap 0_i} N_{\ell-1-j,i}v_i \cdot \overline{\frac{\partial^j u}{\partial n^j}}\, d\sigma,$$

where $N_{k,i}$ is a linear differential operator of order k, and the surface $\partial\Omega \cap 0_i$ is non characteristic for $N_{k,i}$.

Recalling (10.7), we sum over all i to obtain

$$(A^*v, u) - (v, Au)$$

$$= \Sigma_i \sum_{j=0}^{\ell-1} \int_{\partial\Omega\cap 0_i} N_{\ell-1-j,i}v_i \cdot \overline{\frac{\partial^j u}{\partial n^j}}\, d\sigma$$

$$= \sum_{j=0}^{\ell-1} \int_{\partial\Omega} (\Sigma_i N_{\ell-1-j,i}v_i) \overline{\frac{\partial^j u}{\partial n^j}}\, d\sigma,$$

since v_i vanishes outside 0_i. Therefore, we must show that

$$\Sigma_i N_{\ell-1-j,i}v_i = N_{\ell-1-j}v,$$

where $N_{\ell-1-j}$ is a linear differential operator of order $\ell - 1 - j$, for which $\partial\Omega$ is non characteristic. By Leibnitz's rule, we have

$$N_{\ell-1-j,i}v_i = \zeta_i\eta_i \frac{\partial^{\ell-1-j}v}{\partial n^{\ell-1-j}} + \ldots,$$

where η_i does not vanish and...represents terms in the tangential derivatives and lower order derivatives of v. Observe that the sign of $\Re\eta_i$ is $(-1)^{\ell-1-j}$, if A is suitably normalized (cf. (10.1)). Thus,

$$\Sigma_i N_{\ell-1-j,i} v_i = \Sigma_i \zeta_i \eta_i \frac{\partial^{\ell-1-j} v}{\partial n^{\ell-1-j}} + \dots,$$

where $\Sigma_i \zeta_i \eta_i$ cannot vanish since the numbers $\Re \eta_i$ all have the same sign and $\Sigma_i \zeta_i = 1$, $\zeta_i \geq 0$. Thus, the result is proved. Q.E.D.

As a corollary to the proof, we have

THEOREM 10.2. *Let*

$$B[u, v] = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} (D^\alpha u, a_{\alpha\beta} D^\beta v)_{0,\Omega}$$

be a bilinear form which is uniformly elliptic in Ω *(*Ω *bounded while* $a_{\alpha\beta}$ *are sufficiently smooth functions defined in* $\overline{\Omega}$*). Then there are linear differential operators* $N_k(x, D)$, $x \in \partial\Omega$ *of order* k, $k = 0, \dots, m-1$, *such that for* $u, v \in C^{2m}(\overline{\Omega})$,

$$B[u, v] = (u, Av)_{0,\Omega} + \sum_{j=0}^{m-1} \int_{\partial\Omega} \frac{\partial^j u}{\partial n^j} \overline{N_{2m-1-j} v}\, d\sigma,$$

where

$$A = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} (-1)^{|\alpha|} D^\alpha a_{\alpha\beta} D^\beta,$$

and the surface $\partial\Omega$ *is non characteristic for* N_k.

Definition 10.1. *Let* V *be a linear space,* $\mathring{H}_m(\Omega) \subset V \subset H_m(\Omega)$. *A bilinear form*

$$B[u, v] = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} (D^\alpha u, a_{\alpha\beta} D^\beta v)_{0,\Omega}$$

is coercive *over* V *if and only if there are constants* c_0 *and* λ_0 *such that* $c_0 > 0$ *and*

$$\Re B[u, u] \geq c_0 \|u\|_{m,\Omega}^2 - \lambda_0 \|u\|_{0,\Omega}^2 \tag{10.8}$$

for all $u \in V$. $B[u, v]$ *is* strongly coercive *over* V *if* (10.8) *holds with* $\lambda_0 = 0$.

From Theorem 7.12, we see that a coercive bilinear form must be strongly elliptic. Indeed, it would suffice to assume (10.8) with $\Re B[u, u]$ replaced by $|B[u, u]|$. Then Theorem 7.12 shows we could multiply B by a complex number to achieve strong ellipticity for B.

Assume that $B[u, v]$ is strongly coercive (a strongly coercive quadratic form can be obtained from one that is just coercive by adding $\lambda_0(u, v)_{0,\Omega}$). Then, if $f \in L_2(\Omega)$, by the Lax-Milgram theorem there is a unique $u \in V$ such that

$$(10.9) \qquad B[v, u] = (v, f)_{0,\Omega}$$

for all $v \in V$. For this we must assume V is closed in $H_m(\Omega)$.

Thus, we have solved the problem of finding for each $f \in L_2(\Omega)$ a function $u \in V$ such that (10.9) is satisfied for all $v \in V$. In case $V = \overset{\circ}{H}_m(\Omega)$, this is just the generalized Dirichlet problem already discussed. In case $\overset{\circ}{H}_m(\Omega)$ is a proper subspace of V, then we have no longer solved the GDP, but some other type of problem, which we can consider as an abstract boundary-value problem.

The question now arises concerning the regularity of the solution u. The *interior* regularity is no different from the theory we presented in section 9, or, alternatively, the interior regularity theory of section 6. Thus, in case f and B are sufficiently regular, then we can give precise statements concerning the regularity of u. Then, under sufficient assumptions, $u \in H_{2m}^{loc}(\Omega)$, and since (10.9) is satisfied *a fortiori* for all $v \in C_0^\infty(\Omega)$, u is a weak solution of $Au = f$, where

$$A = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} (-1)^{|\alpha|} D^\alpha a_{\alpha\beta} D^\beta.$$

As usual, the question of boundary regularity is more complicated. We can reduce the considerations to the case of a hemisphere $\{x\colon\ |x| < R,\ x_n > 0\}$, by making sufficient smoothness assumptions on Ω. Then we need to consider functions $\zeta \in C_0^\infty(\{x\colon\ |x| < R\})$, as in the results of section 9. If one examines the proofs in section 9, it is evident that the same arguments will work in our case if we make two assumptions of the nature of V. First, if $v \in V$, then $\zeta v \in V$. Second, if $v \in V$, then sufficiently small translations of ζv tangent to the surface $x_n = 0$ also give elements of V. Thus, if both these assumptions hold, then one can establish the boundary regularity of u.

As an example, consider the case in which $V = H_m(\Omega)$. Both the above conditions are obviously satisfied, so that u is highly regular in $\overline{\Omega}$ in case f, B, and Ω are sufficiently regular. Let us then assume that everything is sufficiently regular for the following manipulations. We shall show that u is actually a solution of a certain boundary-value problem. For, according to Theorem 10.2,

$$B[v, u] = (v, Au)_{0,\Omega} + \sum_{j=0}^{m-1} \int_{\partial\Omega} \frac{\partial^j v}{\partial n^j} \overline{N_{2m-1-j}u}\, d\sigma$$

for all $v \in C^{2m}(\overline{\Omega})$. But (10.9) implies $B[v, u] = (v, Au)_{0,\Omega}$, so we obtain

$$\sum_{j=0}^{m-1} \int_{\partial\Omega} \frac{\partial^j v}{\partial n^j} \overline{N_{2m-1-j}u}\, d\sigma = 0.$$

Since the functions $\partial^j v/\partial n^j$ are essentially arbitrary, this implies that on $\partial\Omega$

$$N_{2m-1-j}u = 0, \quad j = 0,\ldots,m-1. \tag{10.10}$$

Recall that N_{2m-1-j} is order $2m-1-j$, and $N_{2m-1-j}u$ contains the term $\partial^{2m-1-j}u/\partial n^{2m-1-j}$ with a non-vanishing coefficient. Thus, the boundary conditions for u are specified by m differential operators on $\partial\Omega$, having orders $m, m+1,\ldots,2m-1$. Note that in this case the boundary conditions actually depend on the choice of the Dirichlet form B. These are so-called *natural* boundary conditions.

The idea can be generalized. Let $B_0,\ldots,B_{k-1}$ be sufficiently smooth linear differential operators on $\partial\Omega$, where $k \leq m$ and the order of B_i is less than m. Assume that the orders of $B_0,\ldots B_{k-1}$ are distinct, and that $\partial\Omega$ is non characteristic for these operators. In this case V shall be the closure in $H_m(\Omega)$ of the set of functions $\phi \in C^m(\overline{\Omega})$, such that $B_i\phi = 0$ on $\partial\Omega$, $i = 0,\ldots,k-1$. If $k < m$, let $B_k,\ldots,B_{m-1}$ be differential operators on $\partial\Omega$, such that $\partial\Omega$ is non characteristic for these operators, and if $0 \le j \le m-1$, then, for some i, B_i has order j. Then, by the proof of Theorem 10.2, we obtain a Green's formula

$$B[v, u] = (v, Au)_{0,\Omega} + \sum_{j=0}^{m-1} \int_{\partial\Omega} B_j v \cdot \overline{M_{2m-1-j} u} d\sigma,$$

where M_{2m-1-j} is an operator for which $\partial\Omega$ is non characteristic. If $u \in V$ and (10.9) holds for all $v \in V$, then we obtain

$$\sum_{j=0}^{m-1} \int_{\partial\Omega} B_j v \cdot \overline{M_{2m-1-j} u} d\sigma = 0.$$

$v \in C^m(\bar{\Omega})$, $B_i v = 0$ on $\partial\Omega$, $i = 0,\dots,k-1$. Thus, since $B_k v,\dots,B_{m-1} v$ can be chosen essentially arbitrarily, we obtain

$$(10.11) \qquad M_{2m-1-j} u = 0 \quad \text{on} \quad \partial\Omega,\ j = k,\dots,m-1.$$

Thus, in addition to the given boundary conditions, $B_0 u = \dots = B_{k-1} u = 0$ on $\partial\Omega$, the function u also satisfies the *natural* boundary conditions (10.11). Note that if $k = m$, then no natural boundary conditions a appear.

It is therefore seen that the critical fact needed for treatment of general boundary-value problems is the coerciveness of a quadratic form. For an example of the usefulness of this criterion, let us consider the case of second-order problems ($m = 1$), and a strongly elliptic operator

$$(10.12) \qquad A = -\sum_{i,j=1}^{n} a_{ij} D_i D_j + \sum_{i=1}^{n} a_i D_i + a,$$

where a_{ij} is real, and for real ξ,

$$(10.13) \qquad \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \ge E_0 |\xi|^2.$$

By the procedure given in section 8 we can associate with A a Dirichlet bilinear form,

$$(10.14)\qquad B[v,\ u] = \sum_{i,j=1}^{n} (D_i v,\ a_{ij} D_j u) + \sum_{i=1}^{n} (D_i v,\ b_i u)$$

$$+ \sum_{i=1}^{n} (v,\ c_i D_i u) + (v,\ du),$$

in a non-unique fashion. The form B comes from integrating by parts so that

$$(10.15)\qquad B[v,\ u] = (v,\ Au)$$

is satisfied in case $v,\ u \in C_0^\infty(\Omega)$. Now one possible selection for B has the principal part

$$(10.16)\qquad \sum_{i,j=1}^{n} \int_\Omega a_{ij} D_i u \overline{D_j u} dx,$$

so that the condition (10.13) implies the ellipticity of this particular form, and, hence, by (10.15) the ellipticity of B. Note that Gårding's inequality is *trivial* for this second-order case, since the *integrand* in (10.16) is already bounded below by

$$E_0 \sum_{i=1}^{n} |D_i u|^2,$$

so that the inequality

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij} D_i u \overline{D_j u} dx \geq E_0 |u|_{1,\Omega}^2$$

is immediate. In higher order problems it is no longer the case that the integrand in $B[u,\ u]$ is positive. Since the lower order terms in (10.14) can be easily estimated in terms of $|u|_{1,\Omega}^2$ with a small coefficient, and $|u|_{0,\Omega}^2$, it follows that B is coercive over $H_1(\Omega)$ if we assume only (10.13).

As we have remarked, the boundary conditions are connected with the choice of B. In the second-order case, the computations are fairly simple. Let us write A in the form

$$A = -\sum_{i,j=1}^{n} a_{ij} D_i D_j + \sum_{i=1}^{n} \alpha_i D_i + \sum_{i=1}^{n} \beta_i D_i + a,$$

where, of course, by comparison with (10.12) we must have

$$\alpha_{ij} + \alpha_{ji} = a_{ij} + a_{ji},$$

$$\alpha_i + \beta_i = a_i.$$

Then, if n_i is the i^{th} component of the exterior normal to $\partial\Omega$, Green's formula gives

$$(v, \alpha_i D_i u) = -(D_i(\overline{\alpha_i} v), u) + \int_{\partial\Omega} v\overline{\alpha_i u} n_i d\sigma,$$

$$-(v, \alpha_{ij} D_i D_j u) = (D_i(\alpha_{ij} v), D_j u) - \int_{\partial\Omega} v\alpha_{ij} D_j \overline{u} n_i d\sigma,$$

assuming α_{ij} to be real. Thus, summing these expressions gives

$$(v, Au) = B[v, u] + \int_{\partial\Omega} [-\Sigma_{i,j}\, \alpha_{ij} n_i D_j \overline{u} + \Sigma_i \overline{\alpha}_i n_i \overline{u}] v d\sigma,$$

for a certain bilinear form B. Thus, the associated boundary condition for u is

$$(10.17) \qquad -\Sigma_{i,j} \alpha_{ij} n_i D_j u + \Sigma_i \alpha_i n_i u = 0.$$

In case $\alpha_{ij} = a_{ij} = a_{ji}$, the expression $\Sigma_{i,j} \alpha_{ij} n_i D_j u$ is called the *conormal derivative* of u corresponding to the operator A; if $A = \Delta$ (the Laplacian), the conormal derivative becomes $\Sigma_j n_j D_j u = \partial u/\partial n$, the *normal* derivative.

Usually, of course, the boundary condition is given in advance. Thus, it is important to be able to pick the form B in such a manner as to obtain the given boundary condition. That is, α_{ij} and α_i must be chosen in such a fashion that (10.17) coincides with the given boundary condition. We shall now demonstrate how this can be accomplished in the case that $A = \Delta$. Suppose then that the given boundary condition is

$$\frac{\partial u}{\partial n} + g\frac{\partial u}{\partial \tau} + au = 0,$$

where $\tau = (\tau_1,\dots,\tau_n)$ is a given smoothly varying unit tangent vector. Since the problem of choosing the quantities a_i is trivial, we concentrate on the choice of a_{ij}. Now the given boundary condition can be written

$$\frac{\partial u}{\partial n} + \Sigma_j g\tau_j D_j u + au = 0.$$

Since $A = \Delta$, the coefficients a_{ij} must satisfy

(10.18) $$a_{ii} = 1,$$

$$a_{ij} + a_{ji} = 0, \quad i \neq j.$$

we shall show that we may indeed choose

$$a_{ij} = \delta_{ij} + (n_i\tau_j - n_j\tau_i)g.$$

Here δ_{ij} is the Kronecker δ-function while the functions n_i, τ_i and g which a-priori are defined only on the boundary are supposed to possess smooth extensions into $\bar{\Omega}$. Using these extensions, a_{ij} are smooth functions in $\bar{\Omega}$ which obviously satisfy (10.18). Moreover, the boundary condition (10.17) becomes

$$0 = \Sigma_{i,j}[\delta_{ij} + (n_i\tau_j - n_j\tau_i)g]n_iD_ju - \Sigma_i a_i n_i u$$

$$= \Sigma_j n_j D_j u + g\Sigma_j \tau_j D_j u \Sigma_i n_i^2 - g\Sigma_j n_j D_j u \Sigma_i n_i \tau_i - \Sigma_i a_i n_i u$$

$$= \frac{\partial u}{\partial n} + g\,\frac{\partial u}{\partial \tau} + 0 - \Sigma_i a_i n_i u,$$

since on $\partial\Omega$ $\Sigma n_i^2 = 1$ and $\Sigma n_i\tau_i = 0$. Therefore, this boundary condition is equivalent to the given one, if the quantities a_i are chosen correctly.

We can also treat mixed boundary-value problems using this procedure. For instance, suppose we wish to obtain a Dirichlet form giving the boundary conditions

$$\frac{\partial u}{\partial \ell} + au = 0 \text{ on } \partial_1\Omega,$$

$$u = 0 \text{ on } \partial_2\Omega,$$

where $\partial_1\Omega$ and $\partial_2\Omega$ are disjoint; and their union is $\partial\Omega$. Then we take for V the closure in $H_1(\Omega)$ of the class of functions in $C_1^1(\bar{\Omega})$ which vanish vanish on $\partial_2\Omega$. And then the Dirichlet form must be chosen in such a manner that the boundary condition on $\partial_1\Omega$ emerges as a natural boundary condition. It should be noted that our regularity theorems hold except in neighborhoods of points in $\overline{\partial_1\Omega} \cap \overline{\partial_2\Omega}$. Our theory fails there because at such points tangential translations of functions in V do not remain in V. And, indeed, singularities actually occur at such points.

To conclude this section, we shall briefly mention some *a priori* estimates. Let A be a sufficiently smooth, uniformly strongly elliptic operator on Ω, let $\Omega \in C^2$, and suppose that $u \in H_2(\Omega) \cap C^1(\bar{\Omega})$ satisfies

$$\frac{\partial u}{\partial \ell} + au = 0 \quad \text{on } \partial\Omega,$$

where $\partial/\partial\ell$ is a smoothly varying directional derivative that is not tangent to $\partial\Omega$. From the regularity theorems, we obtain the *a priori estimate*

$$||u||_{2,\Omega} \leq C(||Au||_{0,\Omega} + ||u||_{0,\Omega}).$$

We can obtain such results for strongly elliptic second order problems which are not real, such as problems for the operator $D_1^2 + D_2^2 + e^{i\theta}D_3^2$, where θ is a constant, $|\theta| < \pi$.

$$B'[v, u] = \sum_{i,j=1}^{n} (D_i v, a_{ij} D_j u)_{0,\Omega}$$

corresponding to the principal part A' of a strongly elliptic operator,

$$A' = -\sum_{i,j=1}^{n} a_{ij} D_i D_j.$$

Now

$$\Re B'[u, u] = ½ \sum_{i,j=1}^{n} [(D_i u, a_{ij} D_j u)_{0,\Omega} + (a_{ij} D_j u, D_i u)_{0,\Omega}]$$

$$= \sum_{i,j=1}^{n} (D_i u, a^*_{ij} D_j u)_{0,\Omega},$$

where $a^*_{ij} = ½(a_{ij} + \overline{a_{ji}})$. For real ξ,

$$\sum_{i,j=1}^{n} a^*_{ij} \xi_i \xi_j = \Re \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j > 0.$$

Hence, for some positive constatn c_0,

$$\sum_{i,j=1}^{n} a^*_{ij} \xi_i \xi_j \geq c_0 |\xi|^2.$$

It follows that

$$\Re B'[u, u] = \int_\Omega \sum_{i,j=1}^{n} D_i u \overline{a^*_{ij}} \overline{D_j u} dx$$

$$\geq c_0 \sum_{i=1}^{n} |D_i u|^2_{0,\Omega}$$

$$= c_0 |u|^2_{1,\Omega}.$$

Therefore, $B'[u, u]$ is coercive over $H_1(\Omega)$. Hence, we obtain the regularity results and the *a priori* estimates.

As an application, we present the following trick which will be useful later. Let A be a uniformly strongly elliptic second order operator over Ω, normalized in the usual fashion, such that A has real coefficients. For $x = (x_1, \ldots, x_n)$, let (x, τ) be a vector in E_{n+1}. For some constant $\theta, |\theta| < \pi$, let

$$L = A - e^{i\theta} D_\tau^2$$

on the cylindrical domain $\Gamma = \Omega \times E_1$. Then L is uniformly strongly elliptic on Γ, as is easily verified. Suppose that $u \in H_2(\Gamma) \cap C^1(\bar{\Gamma})$ satisfies

$$\frac{\partial u}{\partial \ell} + a(x) u = 0 \quad \text{on} \quad \partial\Gamma,$$

where ℓ is a smoothly varying vector parallel to the Ω plane, but not tangent to $\partial\Omega$. Then we have the *a priori* estimate

$$||u||_{2,\Gamma} \leq C(||Lu||_{0,\Gamma} + ||u||_{0,\Gamma}).$$

We note that if $B[v, u]$ is a Dirichlet bilinear form corresponding to A, then

$$B_t[v, u] = \int_{-\infty}^{\infty} B[v(x, t), u(x, t)]\, dt + e^{i\theta}(D_t v, D_t u)_{0,\Gamma}$$

is a Dirichlet form corresponding to L.

Finally, we present a brief remark concerning the biharmonic operator in two variables. As mentioned on p. 96, there are the following two distinct Dirichlet forms corresponding to Δ^2. The first, $B_1[\phi, \psi] = (\Delta\phi, \Delta\psi)$, comes from the Green's formula

$$(v, \Delta^2 u) = (\Delta v, \Delta u) + \int_{\partial\Omega} (v \frac{\partial \Delta \bar{u}}{\partial n} - \Delta \bar{u} \frac{\partial v}{\partial n}) d\sigma,$$

and gives the associated boundary conditions

$$\Delta u = 0,$$

$$\frac{\partial}{\partial n}\Delta u = 0.$$

This is not a regular boundary-value problem; the form B_1 is not coercive over $H_2(\Omega)$ since the space of harmonic functions in $\bar{\Omega}$ has infinite dimension. However, the second,

$$B_2[\phi, \psi] = ([D_1^2 - D_2^2]\phi, [D_1^2 - D_2^2]\psi) + 4(D_1D_2\phi, D_1D_2\psi),$$

leads to a regular boundary problem. This form *is* coercive over $H_2(\Omega)$, since the polynomials $\xi_1^2 - \xi_2^2$ and $2\xi_1\xi_2$ have no common complex zero, except $\xi = 0$. Cf. Theorem 11.9.

More generally, it can be shown that if one considers the Dirichlet form $tB_1 + (1-t)B_2$, t a real parameter, then this form is coercive over $H_2(\Omega)$ if and only if $-3 < t < 1$. For a proof see S. Agmon, *Proceedings of the International Symposium on Linear Spaces*, Pergamon Press, Jerusalem 1961, pp. 1-13.

11. Coerciveness Results of Aronszajn and Smith

In this section we shall give some very precise conditions for a special type of quadratic form to be coercive over $H_m(\Omega)$. Before obtaining these results, we need two auxiliaries: some results of Calderón and Zygmund, and the Sobolev representation formula.

Definition 11.1. *A* Calderón-Zygmund kernel *is a measurable function on $E_n - \{0\}$ satisfying the conditions:*

1° *k is (positively)* homogeneous of degree $-n$; *that is,*

$$k(ax) = a^{-n}k(x), \quad a > 0, \quad x \neq 0; \quad \textit{and}$$

2° *for $|\sigma| = 1$ $k(\sigma)$ is a bounded measurable function on the unit sphere Σ, and k has* mean value zero; *that is,*

$$\int_{\Sigma} k(\sigma)d\sigma = 0.$$

In general, a function $h(x)$ is said to be (positively) homogeneous of degree r if $h(ax) = a^r h(x)$ for $a > 0$ and $x \neq 0$. Note that if h is differentiable, then $D_i h$ is homogeneous of degree $r - 1$.

The following result exhibits a wide class of Calderón-Zygmund kernels.

LEMMA 11.1. *Let $h \in C^1(E_n - \{0\})$ and let h be homogeneous of degree $1 - n$. Then for $i = 1,\dots,n$, $D_i h$ is a Calderón-Zygmund kernel.*

Proof. We have remarked that $D_i h$ is homogeneous of degree $-n$. Thus, we need only exhibit that $D_i h$ has mean value zero. Let $\rho(t)$ be a non-negative function of a real variable such that $\rho \in C(E_1)$, supp $(\rho) \subset [1, 2]$, and $\int_0^\infty \frac{\rho(t)}{t}\, dt = 1$. Now an integration by parts shows that

$$\int_{E_n} D_i h(x) \cdot \rho(|x|) dx = -\int_{E_n} h(x)\rho'(|x|)\frac{x_i}{|x|}\, dx,$$

where $\rho'(t) = d\rho/dt$. Next we introduce polar coordinates $x = r\sigma$, $r = |x|$, in each of these integrals to obtain

$$\int_0^\infty \int_\Sigma \frac{(D_i h)(\sigma)}{r^n} \rho(r) r^{n-1} d\sigma dr = -\int_0^\infty \int_\Sigma \frac{h(\sigma)}{r^{n-1}} \rho'(r)\sigma_i r^{n-1} d\sigma dr,$$

which simplifies to

$$\text{(11.1)} \qquad \int_0^\infty \frac{\rho(r)}{r}\, dr \int_\Sigma (D_i h)(\sigma) d\sigma = -\int_0^\infty \rho'(r) dr \int_\Sigma h(\sigma)\sigma_i d\sigma.$$

Since $\int_0^\infty r^{-1}\rho(r) dr = 1$ and since $\int_0^\infty \rho'(r) dr = 0$, (11.1) becomes

$$\int_\Sigma (D_i h)(\sigma) d\sigma = 0.$$

Q.E.D.

We shall need also the following basic lemma of Calderón and Zygmund, which we state without proof. For an elementary proof see A. Zygmund, on singular integrals, *Rendiconti di Matematica* 16 (1957), pp. 479-481.

LEMMA 11.2. *Let k be a Calderón-Zygmund kernel, let $0 < \epsilon < R$, and let*

$$(11.2) \qquad k_{\epsilon,R}(x) = \begin{cases} k(x), & \epsilon < |x| < R, \\ 0, & |x| \leq \epsilon,\ |x| \geq R. \end{cases}$$

Then if $\hat{k}_{\epsilon,R}$ is the Fourier transform of $k_{\epsilon,R}$, there exists a constant c, independent of ϵ and R, such that for all $\xi \in E_n$

$$|\hat{k}_{\epsilon,R}(\xi)| \leq c.$$

Moreover, for all $\xi \in E_n$ there exists

$$\hat{k}(\xi) \equiv \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \hat{k}_{\epsilon,R}(\xi).$$

Remark. The notation $\hat{k}$ is used here since this function behaves somewhat like a classical Fourier transform, even though k itself does not have a Fourier transform in the usual sense. In the distribution sense, $\hat{k}$ is actually the Fourier transform of k, though we shall not need this fact.

As a corollary of this lemma we have

LEMMA 11.3. *Let k be a Calderón-Zygmund kernel and let $f \in L_2(E_n)$. Let $k_{\epsilon,R}*f$ be the* convolution

$$(k_{\epsilon,R}*f)(x) = \int_{E_n} k_{\epsilon,R}(x-y)f(y)dy.$$

Then, $k_{\epsilon,R}*f$ converges in $L_2(E_n)$ as $\epsilon \to 0$, $R \to \infty$. If $k*f$ denotes the limit in $L_2(E)$, then the following estimate holds:

$$(11.3) \qquad ||k*f||_{0,E_n} \leq (2\pi)^{n/2}c||f||_{0,E_n},$$

where c is the constant appearing in Lemma 11.2.

Proof. Using a familiar fact about the Fourier transform of a convolution,

$$k_{\epsilon,R}*f = (2\pi)^{n/2}\hat{k}_{\epsilon,R}\hat{f}. \tag{11.4}$$

Thus, Parseval's relation shows

$$\begin{aligned} \|k_{\epsilon,R}*f - k_{\epsilon',R'}*f\|^2_{0,E_n} &= (2\pi)^n\|[\hat{k}_{\epsilon,R} - \hat{k}_{\epsilon',R'}]\hat{f}\|^2_{0,E_n} \\ &= (2\pi)^n \int_{E_n} |\hat{k}_{\epsilon,R}(x) - \hat{k}_{\epsilon',R'}(x)|^2 \cdot \\ &\qquad \cdot |\hat{f}(x)|^2 dx. \end{aligned}$$

Using the previous lemma, the integrand in the last expression is bounded by the integrable function $4c^2|\hat{f}(x)|^2$, and the integrand tends to zero a.e. as $\epsilon, \epsilon' \to 0$, $R, R' \to \infty$. By Lebesgue's dominated convergence theorem, the integral tends to zero, so that by the completeness of $L_2(E)$, $k_{\epsilon,R}*f$ converges to a function $k*f$ in $L_2(E)$. By (11.4) and Lemma 11.2,

$$|\widehat{k_{\epsilon,R}*f}(x)|^2 \le (2\pi)^n c^2 |\hat{f}(x)|^2.$$

If we integrate this expression over E_n and use Parseval's relation, (11.3) follows. Q.E.D.

Now we state a result we shall frequently use in the remainder of this section.

THEOREM 11.4. *Let $h \in C^1(E_n - \{0\})$, let h be homogeneous of degree $1 - n$, and let k^i be the Calderón-Zygmund kernel given by $k^i = D_i h$. Let*

$$c_i = \int_\Sigma h(\sigma)\sigma_i d\sigma. \tag{11.5}$$

Let f be a bounded measurable function having compact support in E_n. Then $h*f \in H_1^{loc}(E_n)$, and $h*f$ has weak derivatives given by the formula

$$D_i(h*f) = k^i*f + c_i f.$$

Moreover, for a certain constant c depending only on h,

$$||D_i(h*f)||_{0,E_n} \le c||f||_{0,E_n}.$$

Remark. Although the last estimate shows that really $D_i(h*f) \in L_2(E_n)$, still we must write $h*f \in H_1^{loc}(E_n)$, since $h*f$ itself may not be in $L_2(E_n)$.

Proof. We have

$$\int_{E_n} |h(x-y)|\, |f(y)|dy \le \max_{|\sigma|=1} |h(\sigma)| \sup_{y\in E_n} |f(y)|. \int_{\mathrm{supp}(f)} |x-y|^{1-n}dy.$$

Since f is bounded and has compact support, it follows that $h*f$ exists as an absolutely convergent integral, and is uniformly bounded.

Let $\phi \in C_0^\infty(E_n)$. Then

$$(11.6) \qquad -\int_{E_n} (h*f)(x)D_i\phi(x)dx = \lim_{\substack{\epsilon\to 0\\ R\to\infty}} - \iint_{\epsilon<|x-y|<R} h(x-y)f(y)D_i\phi(x)dydx.$$

By Green's formula, we have for fixed y

$$(11.7) \qquad -\int_{\epsilon<|x-y|<R} h(x-y)D_i\phi(x)dx = -\int_{|x-y|=R} h(x-y)\phi(x)\frac{x_i-y_i}{R}\, d\sigma_x$$
$$-\int_{|x-y|=\epsilon} h(x-y)\phi(x)\frac{y_i-x_i}{\epsilon}\, d\sigma_x$$
$$+\int_{\epsilon<|x-y|<R} D_ih(x-y)\phi(x)dx.$$

Suppose that $f(y) \equiv 0$ for $|y| \ge A$, and suppose $\phi(x) \equiv 0$ for $|x| \ge B$. Then if $|y| < A$ and $R > A + B$, it follows that for $|x-y| = R$ we have $|x| > R - A > B$, so that $\phi(x) = 0$. Thus, if $y \in \mathrm{supp}\,(f)$ and $R > A + B$, then the integral in (11.7) taken over the sphere $|x-y| = R$ vanishes. Also, if in the integral over the sphere $|x-y| = \epsilon$ we set $x = y + \epsilon\sigma$, then we have

$$-\int_{|x-y|=\epsilon} h(x-y)\phi(x)\frac{y_i-x_i}{\epsilon}\,d\sigma_x = \int_\Sigma \frac{h(\sigma)}{\epsilon^{n-1}}\phi(y+\epsilon\sigma)\sigma_i\epsilon^{n-1}d\sigma$$

$$= \int_\Sigma h(\sigma)\sigma_i\phi(y+\epsilon\sigma)d\sigma$$

$$= c_i\phi(y)$$

$$+ \int_\Sigma h(\sigma)\sigma_i[\phi(y+\epsilon\sigma)-\phi(y)]d\sigma,$$

by (11.5). The last integral tends to zero with ϵ uniformly for $y \in E_n$, since $D_i\phi$ is bounded. Thus, from (11.7) we have for R sufficiently large and for $y \in \operatorname{supp}(f)$

$$-\int_{\epsilon<|x-y|<R} h(x-y)D_i\phi(x)dx = \int_{E_n} k^i_{\epsilon,R}(x-y)\phi(x)dx + c_i\phi(y) + J_\epsilon(y),$$

where $\lim_{\epsilon\to 0} J_\epsilon(y) = 0$, uniformly for all y. Here $k^i_{\epsilon,R}$ is defined as in (11.2). Substituting this is (11.6),

$$-\int_{E_n}(h*f)(x)D_i\phi(x)dx = \lim_{\substack{\epsilon\to 0\\ R\to\infty}}\int_{E_n} f(y)dy\int_{E_n} k^i_{\epsilon,R}(x-y)\phi(x)dx + c_i\int_{E_n} f(y)\phi(y)dy$$

$$= \lim_{\substack{\epsilon\to 0\\ R\to\infty}}\int_{E_n}\phi(x)dx\int_{E_n} k^i_{\epsilon,R}(x-y)f(y)dy + c_i\int_{E_n} f(y)\phi(y)dy$$

$$= \lim_{\substack{\epsilon\to 0\\ R\to\infty}}\int_{E_n}(k^i_{\epsilon,R}*f)(x)\phi(x)dx + c_i\int_{E_n} f(x)\phi(x)dx.$$

By Lemma 11.3, $k^i_{\epsilon,R}*f$ converges in $L_2(E_n)$ to k^i*f. Thus, we have

$$-\int_{E_n}(h*f)(x)D_i\phi(x)dx = \int_{E_n}[(k^i*f)(x) + c_if(x)]\phi(x)dx. \qquad \text{Q.E.D.}$$

We now pass to the proof of the *SOBOLEV REPRESENTATION FORMULA*:

THEOREM 11.5. *Let C be an open cone of height h and opening θ with vertex at the origin. That is, for some unit vector ξ^0, the axis of the cone, $C = \{\xi: \ |\xi| < h, \ \xi \cdot \xi^0 > |\xi| \cos\theta\}$. Let Σ be the surface of the unit sphere, and let Λ be the portion of Σ subtended by C. Let $\phi \in C^\infty(\Sigma)$,* supp $(\phi) \subset \Lambda$, *and suppose*

$$\int_\Sigma \phi(\sigma)d\sigma = \frac{(-1)^m}{(m-1)!},$$

where $d\sigma$ is the surface element on Σ. Let $u \in C^m(\overline{C})$, where $u \equiv 0$ for $|x| \geq h - \delta$, some $\delta > 0$. Then

$$u(0) = \sum_{|\alpha|=m} \int_C \frac{m!}{\alpha!} \phi\left(\frac{y}{|y|}\right) \frac{y^\alpha}{|y|^n} D^\alpha u(y)dy.$$

Proof. Let σ be any unit vector such that $\sigma \cdot \xi^0 > \cos\theta$, and consider the function of a single real variable, $u(t\sigma)$, $0 \leq t \leq h$. Expand this function in its Taylor series about the point $t = h$. Then, since $u(t\sigma)$ vanishes for $t \geq h - \delta$,

$$u(0) = \frac{(-1)^m}{(m-1)!} \int_0^h t^{m-1} \frac{d^m}{dt^m} u(t\sigma)dt.$$

Multiply this relation by $\phi(\sigma)$ and integrate over Λ to obtain

$$u(0) = \int_\Lambda \phi(\sigma)d\sigma \int_0^h t^{m-1} \frac{d^m}{dt^m} u(t\sigma)dt. \tag{11.8}$$

Now

$$\frac{d^m}{dt^m} u(t\sigma) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \sigma^\alpha (D^\alpha u)(t\sigma). \tag{11.9}$$

This formula can be proved by induction on m. For, if it is valid for a given value of m and we differentiate once more, we obtain

$$\frac{d^{m+1}}{dt^{m+1}} u(t\sigma) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \sigma^\alpha \sum_{i=1}^n \sigma_i (D^{\alpha+e^i} u)(t\sigma), \tag{11.10}$$

so that, for any β, $|\beta| = m + 1$, the coefficient in (11.10) of $\sigma^\beta D^\beta u$ is

$$\sum_{i=1}^{n} \frac{m!}{(\beta - e^i)!} = \sum_{i=1}^{n} \frac{m!}{\beta_1!\ldots(\beta_i - 1)!\ldots\beta_n!}$$

$$= \frac{m!}{\beta!} \sum_{i=1}^{n} \beta_i$$

$$= \frac{(m+1)!}{\beta!}.$$

Thus, (11.9) is established by induction, as it is obvious for the case $m = 0$.

Substituting (11.9) into (11.8),

$$u(0) = \sum_{|\alpha|=m} \int_\Lambda \int_0^h t^{m-1} \frac{m!}{\alpha!} \phi(\sigma)\sigma^\alpha (D^\alpha u)(t\sigma)dtd\sigma.$$

Setting $y = t\sigma$*

$$u(0) = \sum_{|\alpha|=m} \int_\Lambda \int_0^h \frac{m!}{\alpha!} \phi(\frac{y}{|y|}) \, |y|^{m-1} (\frac{y}{|y|})^\alpha (D^\alpha u)(y)dtd\sigma.$$

Since the formula for volume integration in polar coordinates is $dy = t^{n-1}dtd\sigma$, we obtain

$$u(0) = \sum_{|\alpha|=m} \int_C \frac{m!}{\alpha!} \phi(\frac{y}{|y|}) \, y^\alpha |y|^{-n} D^\alpha u(y)dy.$$

Q.E.D.

The preliminaries having been completed, we shall now discuss some important coerciveness results due originally to Aronszajn; we shall give proofs of K. T. Smith.

First, two lemmas are needed.

LEMMA 11.6. *Let Ω be a bounded domain having the segment property. Let $B[u, u]$ be a coercive quadratic form over a subspace*

L *of* $H_m(\Omega)$. *If for all* $u \in L$

$$(11.11) \qquad |\Re B[u, u]| \le C\|u\|^2_{m-1,\Omega},$$

then L *is finite-dimensional.*

Proof. Since B is coercive over L, (11.1) and Definition 10.1 imply

$$(11.12) \qquad c_0\|u\|^2_m \le (C + \lambda_0)\|u\|^2_{m-1} \quad \text{for} \quad u \in L.$$

Although $L \subset H_m(\Omega)$, we shall consider L as a subspace of $H_{m-1}(\Omega)$; let $\bar{L}$ be the closure of L in the space $H_{m-1}(\Omega)$. It follows from (11.12) that $\bar{L} \subset H_m(\Omega)$ and that (11.12) holds for $u \in \bar{L}$. Let $\{u_k\} \subset \bar{L}$, $\|u_k\|_{m-1} \le 1$. By (11.12), the norms $\|u_k\|_m$ are bounded. Thus, Rellich's theorem, Theorem 3.8, implies that a subsequence of $\{u_k\}$ converges in $H_{m-1}(\Omega)$. Therefore, the unit ball in $\bar{L}$ is compact. But if the unit ball in a normed linear space is compact, the space must have finite dimension. Thus $\bar{L}$ is finite-dimensional. Q.E.D.

LEMMA 11.7. *Let* Ω *be a bounded domain having the segment property. Let* $B[u, u]$ *be a coercive quadratic form over a subspace* V *of* $H_m(\Omega)$, *and suppose* $\Re B[u, u] \ge 0$ *for* $u \in V$. *Then the set*

$$L = \{u\colon\ u \in V,\ \Re B[u, u] = 0\}$$

is linear and of finite dimension.

Proof. By Lemma 11.6, we need only show that L is linear. For $u, v \in V$, let

$$B_1[u, v] = \tfrac{1}{2}B[u, v] + \tfrac{1}{2}\overline{B[v, u]}.$$

Then $B_1[u, u[= \Re B[u, u] \ge 0$ and $B_1[u, v] = \overline{B_1[v, u]}$. Thus B_1 is a Hermitian symmetric, non-negative definite, bilinear form on V, and the Cauchy-Schwarz inequality must be valid for B_1:

$$|B_1[u, v]| \le B_1[u, u]^{1/2}\, B_1[v, v]^{1/2}$$

$$= \{\Re B[u, u] \cdot \Re B[v, v]\}^{1/2}.$$

Thus, if $u, v \in L$, then $B_1[u, v] = 0$, and

$$B_1[u+v, u+v] = B_1[u, u] + 2\Re B_1[u, v] + B_1[v, v]$$
$$= 0,$$

or, $\Re B[u+v, u+v] = 0$, so that $u + v \in L$. Obviously, L is closed under scalar multiplication. Thus, L is linear. Q.E.D.

We now turn to the results of Aronszajn and Smith. The first result is almost trivial, using Gårding's inequality and the previous lemma.

THEOREM 11.8. *Let $P_1(\xi),\dots,P_N(\xi)$ be homogeneous polynomials of degree m, having constant coefficients. For $u, v \in H_m(\Omega)$ let*

$$B[u, v] = \sum_{k=1}^{n} \int_\Omega P_k(D)u\overline{P_k(D)v}dx, \tag{11.13}$$

where Ω is a bounded domain.

1° *A necessary and sufficient condition that B be coercive over $\overset{\circ}{H}_m(\Omega)$ is that there is no common non-zero real zero of $P_1(\xi),\dots,P_N(\xi)$.*

2° *If Ω has the segment property, a necessary condition that B be coercive over $H_m(\Omega)$ is that there is no common non-zero complex zero of $P_1(\xi),\dots,P_N(\xi)$.*

Proof. Note that the non-vanishing of $\sum_{k=1}^{n} |P_k(\xi)|^2$ for all non-zero ξ is equivalent to the ellipticity of the form $B[u, u]$. Thus, in 1° the sufficiency is an immediate result of Gårding's inequality, and the necessity follows from the converse of Gårding's inequality, Theorem 7.12. Thus, 1° is merely a restatement of previously derived results.

To prove 2°, suppose $z = (z_1,\dots,z_n)$ is a common non-zero complex zero of $P_1,\dots,P_N$. Let $u_\lambda(x) = e^{\lambda z\cdot x}$; $u \in H_m(\Omega)$ since Ω is bounded. Now

$$P_k(D)u_\lambda = P_k(\lambda z)e^{\lambda z\cdot x}$$

$$= \lambda^m P_k(z) e^{\lambda z \cdot x}$$

$$= 0,$$

so that $B[u_\lambda, u_\lambda] = 0$. Obviously, $\Re B[v, v] \geq 0$ for any v. Since the set of functions u_λ of the above form is not of finite dimension, when λ takes on all complex values, Lemma 11.7 implies that B cannot be coercive over $H_m(\Omega)$. Q.E.D.

Now a partial converse of the second part of this theorem will be given. The only additional assumption is a requirement on the domain. The proof is taken from K. T. Smith, Inequalities for formally positive integro-differential forms, *Bull. Amer. Math. Soc.* 67(1961), 368-370.

THEOREM 11.9. *Let $P_1(\xi),\ldots,P_N(\xi)$ be homogeneous polynomials of degree m, having constant coefficients, and let $P_1(\xi),\ldots,P_N(\xi)$ have no common non-zero complex zero. Let Ω be a bounded open set having the restricted cone property. Then the quadratic form in* (11.13) *is coercive over $H_m(\Omega)$; that is, there exists a constant C such that for all $u \in H_m(\Omega)$*

$$(11.14)\qquad \|u\|_{m,\Omega} \leq C\Big[\sum_{k=1}^{N} \|P_k(D)u\|_{0,\Omega} + \|u\|_{0,\Omega}\Big].$$

Proof. According to Hilbert's Nullstellensatz, if $P(\xi)$ is any polynomial which vanishes at all common complex zeros of $P_1(\xi),\ldots,P_N(\xi)$, then some power ρ of P is a linear combination of $P_1,\ldots,P_N$, in the sense that there exist polynomials $A_1(\xi),\ldots,A_N(\xi)$, such that $P^\rho = \sum_{k=1}^{N} A_k P_k$. Since the only common zero of the polynomials $P_1,\ldots,P_N$ is zero, the requirement on P is simply that $P(0) = 0$. Notice that if the above formula holds for P^ρ, then a similar expression is obviously valid for $P^{\rho'}$, any $\rho' > \rho$. For a proof of the Nullstellensatz, cf. Van der Waerden, *Modern Algebra*, II, Ungar Pub. Co., pp. 5-6.

Now we apply this result to the polynomial $P(\xi) = \xi_j$. We then obtain for some sufficiently large ρ and for some polynomials $A_k^{(j)}(\xi)$

$$(11.15)\qquad \xi_j^{\rho} = \sum_{k=1}^{N} A_k^{(j)}(\xi)P_k(\xi),\ j = 1,\ldots,N.$$

Therefore, if $m' \geq \rho n$ and if $|\alpha| = m'$, then $\alpha_j \geq \rho$ for some j, so that (11.15) implies that for some polynomials $A'_{\alpha k}(\xi)$

$$(11.16)\qquad \xi^{\alpha} = \sum_{k=1}^{N} A'_{\alpha k}(\xi)P_k(\xi),\ |\alpha| = m'.$$

We can certainly assume $m' \geq m$. (This is actually implied by (11. 16), though we need not use this fact.) Now if $A_{\alpha k}(\xi)$ is the part of the polynomial $A'_{\alpha k}(\xi)$ containing terms of degree precisely $m' - m$, then the homogeneity of $P_k(\xi)$ implies

$$(11.17)\qquad \xi^{\alpha} = \sum_{k=1}^{N} A_{\alpha k}(\xi)P_k(\xi),\ |\alpha| = m'.$$

If we interpret (11,17) as a formula for derivatives, it follows that

$$(11.18)\qquad D^{\alpha} = \sum_{k=1}^{N} A_{\alpha k}(D)P_k(D),\ |\alpha| = m',$$

where $A_{\alpha k}$ is a homogeneous polynomial of degree $m' - m$.

Obviously, it is sufficient in proving (11.14) to assume $u \in C^{\infty}(E_n)$, by the approximation theorem, Theorem 2.1. Let $\{0_i\}$ be a finite open covering of $\bar{\Omega}$ and let C_i be the cones guaranteed by assumption that Ω has the restricted cone property. If h is the minimum height of the cones C_i, we may refine the covering $\{0_i\}$, if necessary, to insure that the diameter of 0_i is less than h. We have for any $x \in \Omega \cap 0_i$ that $x + C_i \subset \Omega$.

Now suppose first that u has its support in 0_i. Since the diameter of 0_i is less than the height of C_i, it follows that if $x \in 0_i$ the cone $x + C_i$ has the spherical part of its boundary outside 0_i, and therefore outside the support of u. Thus, we may apply the Sobolev representation formula (Theorem 11.5) to obtain for each fixed $x \in \Omega \cap 0_i$

$$(11.19)\qquad u(x) = \sum_{|\alpha| = m'} \int_{C_i} \phi_{\alpha}(y)D^{\alpha}u(x+y)dy,$$

where $\phi_\alpha(y) \in C^\infty(E_n - \{0\})$, $\phi_\alpha(y)$ is homogeneous of degree $m' - n$, and supp $[\phi_\alpha(y/|y|)] \subset \Lambda_i$, where Λ_i is the portion of the unit sphere Σ subtended by C_i. Now utilizing the formula (11.18) for D^α, the representation (11.19) becomes

$$(11.20) \quad u(x) = \sum_{|\alpha|=m'} \sum_{k=1}^{N} \int_{C_i} \phi_\alpha(y) A_{\alpha k}(D) P_k(D) u(x+y)dy.$$

We wish to integrate by parts in (11.20), moving all the differentiations in $A_{\alpha k}(D)$ so that they apply to ϕ_α. Note that there are no boundary contributions, since $u(x + y)$ vanishes for y near the spherical part of ∂C_i and $\phi_\alpha(y)$ vanishes for y near the lateral part of ∂C_i. Thus, since $A_{\alpha k}(D)$ has constant coefficients and is homogeneous of degree $m' - m$, (11.20) becomes

$$(11.21) \quad u(x) = \sum_{|\alpha|=m'} \sum_{k=1}^{N} (-1)^{m'-m} \int_{C_i} A_{\alpha k}(D)\phi_\alpha(y) \cdot P_k(D)u(x+y)dy.$$

Let

$$\psi_k(y) = \sum_{|\alpha|=m'} (-1)^{m'-m} A_{\alpha k}(D)\phi_\alpha(y).$$

Then ψ_k is homogeneous of degree $m' - n - (m' - m) = m - n$; in particular ψ_k is integrable over C_i, so the integration by parts leading to (11.21) is fully justified, and we have

$$(11.22) \quad u(x) = \sum_{k=1}^{N} \int_{C_i} \psi_k(y) P_k(D) u(x+y)dy, \quad x \in \Omega \cap 0_i.$$

Also $\psi_k \in C^\infty(E_n - \{0\})$ and supp $[\psi_k(y/|y|)] \subset \Lambda_i$.

We now need to apply Theorem 11.4. To do so, it is necessary that the formula (11.22) be modified to contain integrals over all of E_n, not just over C_i. To accomplish this, let

$$w_k(y) = \begin{cases} P_k(D)u(y), & y \in \Omega, \\ 0, & y \notin \Omega. \end{cases}$$

Also let C_l' be the infinite cone which subtends the protion Λ_l of Σ. Since supp $(u) \subset 0_l$, $u(x + y)$ vanishes for $y \in C_l' - C_l$, and the integration in (11.22) may be taken over all of C_l' with no change in the value of the integrals. Next, the integration may be performed over E_n, since $\psi_k(y)$ vanishes outside C_l'. Thus, (11.22) becomes

$$u(x) = \sum_{k=1}^{N} \int_{E_n} \psi_k(y) w_k(x + y)dy, \ x \in \Omega \cap 0_l.$$

Note that we must restrict x to be in $\Omega \cap 0_l$ for the validity of this formula. Now replace $x + y$ by y in this formula to obtain

$$(11.23) \qquad u(x) = \sum_{k=1}^{N} \int_{E_n} \psi_k'(x - y) w_k(y)dy, \ x \in 0_l \cap \Omega,$$

where $\psi_k'(y) = \psi_k(-y)$. Since $\psi_k' \in C^\infty(E_n - \{0\})$ and ψ_k' is homogeneous of degree $m - n$, we may differentiate formally the relation (11.23) a total of $m - 1$ times to obtain for $|\beta| = m - 1$

$$(11.24) \qquad D^\beta u(x) = \sum_{k=1}^{N} \int_{E_n} \psi_{\beta k}(x - y) w_k(y)dy, \ x \in 0_l \cap \Omega,$$

where $\psi_{\beta k} = D^\beta \psi_k'$ is homogeneous of degree $m - n - (m - 1) = - n +$ and is in $C^\infty(E_n - \{0\})$. This formula is valid because $\psi_{\beta k}(y)$ is integrable near $y = 0$.

By Theorem 11.4 it now follows that each of the functions $\psi_{\beta k} * w_k(x) = \int_{E_n} \psi_{\beta k}(x - y) w_k(y)dy$ has weak derivatives of first order which satisfy

$$||D_l(\psi_{\beta k} * w_k)_{0,E_n} \le c||w_k||_{0,E_n}.$$

Thus, (11.24) implies

$$||D_i D^\beta u||_{0,\Omega} \le \sum_{k=1}^{N} ||D_i(\psi_{\beta k} * w_k)||_{0,E_n}$$

$$\le \sum_{k=1}^{N} c||w_k||_{0,E_n}$$

$$= c \sum_{k=1}^{N} ||P_k(D)u||_{0,\Omega}.$$

Since this holds for all β, $|\beta| = m$, and all $i = 1,\ldots,N$, we obtain

$$|u|_{m,\Omega} \le c_1 \sum_{k=1}^{N} ||P_k(D)u||_{0,\Omega}.$$

By the interpolation result of Theorem 3.4, we have

$$(11.25) \qquad ||u||_{m,\Omega} \le c_2[\sum_{k=1}^{N} ||P_k(D)u||_{0,\Omega} + ||u||_{0,\Omega}],$$

which is just (11.14) for the case in which supp $(u) \subset 0_l$.

Thus, we have established (11.14) locally. The procedure for obtaining the global result is precisely the same as for obtaining Gårding's inequality from the local result in Lemma 7.9; cf. pp. 81-83. Therefore, (11.14) is established. Q.E.D.

We now extend this result to show that the existence of certain weak derivatives of u implies the existence of all weak derivatives up to a certain order.

THEOREM 11.10. *Let $P_1(\xi),\ldots,P_N(\xi)$ be homogeneous polynomials of degree m, having constant coefficients, and having no common non-zero complex zero. Let Ω be a bounded open set having the restricted cone property. Let $u \in L_2(\Omega)$ and let $P_k(D)u$ exist weakly and belong to $L_2(\Omega)$, $k = 1,\ldots,N$. Then $u \in H_m(\Omega)$, and*

$$||u||_{m,\Omega} \le C[\sum_{k=1}^{N} ||P_k(D)u||_{0,\Omega} + ||u||_{0,\Omega}].$$

Proof. Obviously, once we have shown that $u \in H_m(\Omega)$, the previous theorem implies the estimate on $||u||_{m,\Omega}$. Thus, we need only show that $u \in H_m(\Omega)$. It is easily seen that $u \in H_m^{\mathrm{loc}}(\Omega)$. For, extend u to vanish outside Ω, and let $u_\epsilon = J_\epsilon u$, where J_ϵ is the mollifier introduced in Definition 1.7. Then $u_\epsilon \in C_0^\infty(E_n)$ and, by Theorem 2.4, $u_\epsilon \to u$ in $L_2^{\mathrm{loc}}(\Omega)$, $P_k(D)u_\epsilon \to P_k(D)u$ in $L_2^{\mathrm{loc}}(\Omega)$, $k = 1,\ldots,N$. If S is a sphere such that $S \subset\subset \Omega$, then (11.14) implies

$$||u_\epsilon||_{m,S} \leq C[\sum_{k=1}^{N} ||P_k(D)u_\epsilon||_{0,S} + ||u_\epsilon||_{0,S}].$$

Therefore, it follows that $u_\epsilon \to u$ in $L_2(S)$ and $||u_\epsilon||_{m,S}$ is bounded as $\epsilon \to 0$. By Theorem 3.12 it follows that $u \in H_m(S)$. Since S is any sphere whose closure is contained in Ω, it follows that $u \in H_m^{\mathrm{loc}}(\Omega)$.

Now let $\{0_i'\}$ be a finite open covering of $\partial\Omega$ and let $\{C_i\}$ be the associated cones such that $x + C_i \subset \Omega$ for $x \in \Omega \cap 0_i'$. By approximating each open set $0_i'$ by finite unions of spheres contained in $0_i'$, we may actually assume that there is a finite open covering $\{0_i\}$ of $\partial\Omega$ such that each set 0_i is a sphere whose diameter is less than the height of C_i. Let Ω_i be the union of all the cones $x + C_i$ for $x \in \Omega \cap 0_i$. Obviously, Ω_i itself has the restricted cone property.

By decreasing the size of the sets 0_i, if necessary, we can also assume that for any fixed unit vector ξ in the cone C_i, the translated set $\Omega_{i,\epsilon} = \epsilon\xi + \Omega_i$ satisfies $\Omega_{i,\epsilon} \subset\subset \Omega$, if ϵ is sufficiently small. Let $u^\epsilon(x) = u(x + \epsilon\xi)$ for $x \in \Omega_i$. Since $x + \epsilon\xi \in \Omega_{i,\epsilon}$ for $x \in \Omega_i$, and since we have shown already that $u \in H_m^{\mathrm{loc}}(\Omega) \subset H_m(\Omega_{i,\epsilon})$, it follows that $u^\epsilon \in H_m(\Omega_i)$. Moreover, we have the estimate

$$||u^\epsilon||_{m,\Omega_i} \leq C[\sum_{k=1}^{N} ||P_k(D)u^\epsilon||_{0,\Omega_i} + ||u^\epsilon||_{0,\Omega_i}],$$

which follows from (11.14) since Ω_i has the restricted cone property. But then it follows, since u^ϵ is just a translation of u, that

$$||u^\epsilon||_{m,\Omega_i} \leq C[\sum_{k=1}^{N} ||P_k(D)u||_{0,\Omega} + ||u||_{0,\Omega}].$$

Therefore, when $\epsilon \to 0$, $u^{\epsilon} \to u$ in $L_2(\Omega_i)$ and $||u^{\epsilon}||_{m,\Omega_i}$ is bounded. Applying Theorem 3.12 again it follows that $u \in H_m(\Omega_i) \subset H_m(\Omega \cap 0_i)$.

Since this holds for each i, it follows that $D^{\alpha}u \in L_2(\Omega)$ for $|\alpha| \leq m$. As mentioned on page 10, any function in $W_m^{loc}(\Omega)$ whose $D^{\alpha}u$ belong to $L_2(\Omega)$, $|\alpha| \leq m$, is in $W_m(\Omega)$. By Theorem 2.2, it follows that $u \in H_m(\Omega)$. Q.E.D.

This result will prove extremely useful in section 13. In that application, it will prove essential that no assumption is necessary on the smoothness of $\partial\Omega$. This emphasizes the importance of Smith's proof, since Aronszajn's original proof required a smooth boundary. Specifically, we shall need the foll owing corollary of the theorem.

Corollary. Let Ω be a bounded open set having the restricted cone property and let $u \in L_2(\Omega)$. Suppose that $D_k^m u$ exists weakly and belongs to $L_2(\Omega)$, $k = 1,\ldots,n$. Then $u \in H_m(\Omega)$ and for some constant C depending only on Ω and m

$$||u||_{m,\Omega} \leq C[\sum_{k=1}^{n} ||D_k^m u||_{0,\Omega} + ||u||_{0,\Omega}].$$

For the case of variable coefficients we have

THEOREM 11.11. *Let Ω be a bounded open set having the restricted cone property. Let $P_1(x, D),\ldots,P_N(x, D)$ be differential operators of order m whose coefficients are bounded in Ω and for which the coefficients of the principal parts $P_k'(x, D)$ are continuous in $\bar{\Omega}$. Assume that*

1° *for each $x^{\circ} \in \Omega$ the polynomials $P_k'(x^{\circ}, D)$ possess no common non-zero real zero; and*

2° *for each $x^{\circ} \in \partial\Omega$ the polynomials $P_k'(x^{\circ}, D)$ posses no common non-zero complex zero.*

Then there exists a constant C such that for all $u \in H_m(\Omega)$

$$||u||_{m,\Omega} \leq C[\sum_{k=1}^{N} ||P_k u||_{0m\Omega} + ||u||_{0,\Omega}]. \tag{11.26}$$

Proof. First we will prove the theorem for a neighborhood of the boundary. Let $x^{\circ} \in \partial\Omega$. Then the polynomials $P'_k(x^{\circ}, \xi)$, $k = 1,\ldots,N$, have no common non-zero complex zeros. Let 0 be a small neighborhood of x°; since Ω has the restricted cone property, we may assume that $U = 0 \cap \Omega$ has this property. By Theorem 11.9 there is a constant C_1 such that for $u \in H_m(U)$

$$||u||_{m,U} \leq C_1 \Big(\sum_{k=1}^{N} ||P'_k(x^{\circ}, D)u||_{0,U} + ||u||_{0,U} \Big).$$

Now

$$\begin{aligned} P_k(x, D)u &= \sum_{|\alpha| \leq m} P_{k,\alpha}(x) D^{\alpha} u \\ &= P'_k(x^{\circ}, D)u + \sum_{|\alpha| = m} [P_{k,\alpha}(x) - P_{k,\alpha}(x^{\circ})] D^{\alpha} u \\ &\qquad + \sum_{|\alpha| \leq m-1} P_{k,\alpha}(x) D^{\alpha} u. \end{aligned}$$

Hence, for some constant C_2,

$$\begin{aligned} ||P'_k(x^{\circ}, D)u||_{0,U} &\leq ||P_k(x, D)u||_{0,U} \\ &\quad + C_2 [\sup_{\substack{x \in U \\ |\alpha| = m}} |P_{k,\alpha}(x) - P_{k,\alpha}(x^{\circ})| \; ||u||_{m,U} \\ &\quad + ||u||_{m-1,U}]. \end{aligned}$$

Thus,

$$||u||_{m,U} \leq C_1 \sum_{k=1}^{N} ||P_k(x, D)u||_{0,U}$$

$$+ NC_1C_2 \sup_{\substack{x\in U \\ |\alpha|=m \\ 1\le k\le N}} |p_{k,\alpha}(x) - p_{k,\alpha}(x^\circ)| \, ||u||_{m,U}$$

$$+ NC_1C_2 \, ||u||_{m-1} + C_1||u||_{0,U}.$$

By the interpolation inequality, there is for $\epsilon > 0$ a constant C_ϵ, such that

$$NC_1C_2||u||_{m-1,U} \le \epsilon||u||_{m,U} + C_\epsilon||u||_{0,U}.$$

Furthermore, since, for $|\alpha| = m$, $p_{k,\alpha}(x)$ is continuous on $\bar{\Omega}$, U can be taken small enough that

$$NC_1C_2 \sup_{\substack{x\in U \\ |\alpha|=m \\ 1\le k\le N}} |p_{k,\alpha}(x) - p_{k,\alpha}(x^\circ)| < \epsilon.$$

Thus,

$$||u||_{m,U} \le C_1 \sum_{k=1}^{N} ||P_k(x, D)u||_{0,U} + 2\epsilon||u||_{m,U} + (C_1 + C_\epsilon)||u||_{0,U}.$$

Choose $\epsilon < 1/4$. Then we have

$$||u||_{m,U} \le C[\sum_{k=1}^{N} ||P_k(x, D)u||_{0,U} + ||u||_{0,U}],$$

where $C = 2(C_1 + C_\epsilon)$. Now let $0_1,\ldots,0_\nu$ be a finite covering of $\partial\Omega$ by such sets 0; thus, for $u \in H_m(\Omega \cap 0_i)$,

$$(11.27) \qquad ||u||_{m,\Omega\cap 0_i} \le C[\sum_{k=1}^{N} ||P_k(x, D)u||_{0,\Omega\cap 0_i} + ||u||_{0,\Omega\cap 0_i}] .$$

Let $0_0 = \bigcup_{i=1}^{\nu} 0_i$.

Let $U \subset\subset \Omega' \subset\subset \Omega$, where U is chosen such that $\Omega - 0_0 \subset U$. Let

$$B(u) = \sum_{k=1}^{N} ||P'_k(x, D)u||^2_{0,U}.$$

Then $B(u)$ is a uniformly strongly elliptic quadratic form on U. For, since $P'_1(x, \xi),\dots,P'_N(x, \xi)$ possess no common non-zero real zero ξ for $x \in \Omega$,

$$\sum_{k=1}^{N} |P'_k(x, \xi)|^2 \neq 0, \quad x \in \Omega', \quad |\xi| = 1$$

Hence, for $x \in U$ and $|\xi| = 1$ this quantity is bounded away from zero. It follows that there is a constant E_0 such that for real ξ and $x \in U$,

$$\sum_{k=1}^{N} |P'_k(x, \xi)|^2 \geq E_0|\xi|^{2m}.$$

Furthermore, the coefficients of B are uniformly continuous on U. By Gårding's inequality there are constants $\gamma_0 > 0$ and $\lambda_0 \geq 0$ such that for $u \in \overset{\circ}{H}_m(U)$

$$\Re B(u) \geq \gamma_0||u||^2_{m,U} - \lambda_0||u||^2_{0,U};$$

in other words,

$$||u||^2_{m,U} \leq \gamma_0^{-1} \sum_{k=1}^{N} ||P'_k(x, D)u||^2_{0,U} + \lambda_0\gamma_0^{-1}||u||^2_{0,U}.$$

By the Cauchy-Schwarz inequality, it follows that there is a constant C such that

$$||u||_{m,U} \leq C[\sum_{k=1}^{N} ||P'_k(x, D)u||_{0,U} + ||u||_{0,U}].$$

As above, the interpolation theorem yields

$$||u||_{m,U} \leq C[\sum_{k=1}^{N} ||P_k(x, D)u||_{0,U} + ||u||_{0,U}].$$

On combining this inequality with (11.27), it is seen that we have actually proved a local version of (11.26). As in the proof of Gårding's inequality, the global version now follows from a familiar argument utilizing a partition of unity. Q.E.D.

The original result of Aronszajn can be expressed as follows. Suppose Ω is of class C^m and consider the condition

3° *for each $x^\circ \in \partial\Omega$ let n be the unit normal vector to $\partial\Omega$ at x°. For each real tangent vector ξ at x° the polynomials in τ given by $P'_k(x^\circ, \xi + \tau n)$ possess no common non-zero complex zero.*

The theorem of Aronszajn states that if Ω is of class C^m and Ω is bounded, then the conditions 1° and 3° are necessary and sufficient for the validity of (11.26). We shall not give the proof of this result.

Remark. The estimates guaranteed by the preceding theorems can be extended to more general classes of norms: for instance, L_p norms and norms involving the Hölder continuity of derivatives of u.

We conclude this section with an application of Sobolev's representation formula and the Calderón-Zygmund theorem already mentioned. The theorem will not be needed in the following but is of considerable interest in itself.

THEOREM 11.12. (*CALDERÓN'S EXTENSION THEOREM*). *Let Ω be a bounded domain having the restricted cone property. Then there exists a bounded linear transformation $\mathfrak{E}$ of $H_m(\Omega)$ into $H_m(E_n)$ such that for every $u \in H_m(\Omega)$ the restriction of $\mathfrak{E}u$ to Ω coincides with u.*

Proof. We prove the theorem first for $u \in C^m(\Omega) \cap H_m(\Omega)$.

Let $\{0_i\}_{i=1}^{\nu}$ be a finite open covering of $\partial\Omega$ with associated cones $\{C_i\}$ such that $x + C_i \subset \Omega$ for $x \in \Omega \cap 0_i$. As in the proof of Theorem 3.10 we add to this covering an open set $0_0 \subset\subset \Omega$ such that $\{0_i\}_{i=0}^{\nu}$ is an open covering of $\bar{\Omega}$; let C_0 be a sufficiently small cone such that $x + C_0 \subset \Omega$ for $x \in 0_0$. We can arrange this covering such that the diameter of 0_l is less than the height of C_l. Next let $0'_l \subset\subset 0_l$ be such that $\{0'_l\}_{l=0}^{\nu}$ is still a covering of $\bar{\Omega}$. Let $\{\zeta_l\}_{l=0}^{\nu}$ be a partition of unity subordinate to $\{0'_l\}$, such that $\sum_{l=0}^{\nu} \zeta_l \equiv 1$ on Ω. Then $u = \sum_{l=0}^{\nu} u_l$, where $u_l = u\zeta_l$ is in $C^m(\Omega)$ and $\operatorname{supp}(u_l) \subset 0_l$. To extend u to E_n it is clearly sufficient to extend each u_l.

By the Sobolev representation formula we have for $x \in \Omega \cap 0_l$

$$(11.28) \qquad u_l(x) = \sum_{|\alpha|=m} \int_{C_l} \phi_{\alpha l}(y) D^{\alpha} u_l(x+y)\,dy,$$

where $\phi_{\alpha l} \in C^{\infty}(E_n - \{0\})$, $\phi_{\alpha l}$ is homogeneous of degree $m-n$, and $\operatorname{supp}\,[\phi_{\alpha l}(y/|y|)] \subset \Lambda_l$, the portion of Σ subtended by C_l.

As in the proof of Theorem 11.9, we shall arrange things so the integration in (11.28) is taken over all of E_n. First, let

$$(11.29) \qquad w_{\alpha l}(y) = \begin{cases} D^{\alpha} u_l(y), & y \in \Omega, \\ 0, & y \in \Omega. \end{cases}$$

Then (11.28) can be written

$$(11.30) \qquad u_l(x) = \sum_{|\alpha|=m} \int_{E_n} \phi_{\alpha l}(y) w_{\alpha l}(x+y)\,dy, \quad x \in \Omega\ 0_l.$$

Now let $\psi_l \in C_0^{\infty}(E_n)$ satisfy $\psi_l(x) = 1$ for $x \in 0'_l$, $\psi_l(x) = 0$ for $x \notin 0_l$. Since $u_l(x) = 0$ for $x \notin 0'_l$, (11.30) implies

$$(11.31) \qquad u_l(x) = \psi_l(x) \sum_{|\alpha|=m} \int_{E_n} \phi'_{\alpha l}(x-y) w_{\alpha l}(y)\,dy, \quad x \in \Omega.$$

Here we have set $\phi'_{\alpha l}(y) = \phi_{\alpha l}(-y)$. Now let

$$(11.32) \qquad v_l(x) = \psi_l(x) \sum_{|\alpha|=m} \int_{E_n} \phi'_{\alpha l}(x-y) w_{\alpha l}(y)\,dy, \quad x \in E_n.$$

Then we set

(11.33) $\quad v(x) = \sum_{i=0}^{\nu} v_i(x), \quad x \in E_n.$

By the very construction, $v(x) = u(x)$ on Ω.

Now as in the proof of Theorem 11.9, and by Leibnitz's rule,

(11.34) $\quad D^\beta v_i(x) = \sum_{\gamma \le \beta} \binom{\beta}{\gamma} D^{\beta-\gamma}\psi_i(x) \cdot \int_\Omega D^\gamma \phi'_{\alpha i}(x-y) \cdot w_{\alpha i}(y)dy,$

if $|\beta| \le m-1$. Here $D^\gamma \phi'_{\alpha i}$ is homogeneous of degree $m-n-|\gamma|$. We now need to estimate the L_2 norm of $D^\beta v_i$. In this computation K will stand for a generic constant depending only on Ω, the functions $\phi_{\alpha i}$ and ψ_i, and m. Suppose $\Omega \subset \bigcup_{i=0}^{\nu} 0'_i \subset \{x\colon\ |x| < a\}$. Then the L_2 norm of any term on the right in (11.34) is no greater than

$$K[\int_{0'_i} |D^{\beta-\gamma}\psi_i(x)|^2 dx |\int_{E_n} D^\gamma\phi'_{\alpha i}(y) \cdot w_{\alpha i}(x-y)dy|^2]^{1/2}$$

(11.35) $\quad \le K[\int_{0'_i} dx\{\int_{E_n} |y|^{m-n|\gamma|}|w_{\alpha i}(x-y)|dy\}^2]^{1/2},$

where we have again replaced $x-y$ by y in the inner integral, and have used the homogeneity of $D^\gamma\phi'_{\alpha i}$. By the Minkowski inequality, (11.35) is no greater than

$$K\int_{E_n} dy[\int_{0'_i} |y|^{2(m-n-|\gamma|)}|w_{\alpha i}(x-y)|^2 dx]^{1/2}$$

$$= K\int_{E_n} |y|^{m-n-|\gamma|}dy[\int_{0'_i} |w_{\alpha i}(x-y)|^2 dx]^{1/2}$$

(11.36) $\quad = K\int_{E_n} |y|^{m-n-|\gamma|}dy[\int_{\substack{x\in 0'_i\\ x-y\in\Omega}} |D^\alpha u_i(x-y)|^2 dx]^{1/2}.$

For $x \in 0'_i$ and $x - y \in \Omega$, we have $|x| < a$ and $|x-y| < a$, so that $|y| < 2a$. Thus, since $|\gamma| \le m-1$, (11.36) is no greater than

$$K(2a)^{m-|\gamma|-1}\int_{|y|<2a}|y|^{-n+1}dy\,|u_i|_{m,\Omega} \le K|u_i|_{m,\Omega}.$$

Therefore, summing over γ in (11.34), we have

$$(11.37) \qquad ||v_i||_{m-1,\Omega} \le K|u|_{m,\Omega}.$$

Now we need to estimate $D_jD^\beta v_i$ for $j = 1,\dots,n$ and $|\beta| = m - 1$. In computing $D_jD^\beta v_i$ from (11.34), terms arise which can be estimated exactly as above, and we also have a term of the form

$$(11.38) \qquad \psi_i(x)D_j\int_\Omega D^\beta\phi'_{ai}(x-y)\cdot w_{ai}(y)dy.$$

Since $D^\beta\phi'_{ai}$ is homogeneous of degree $1 - n$, the inequality of Calderón and Zygmund (Theorem 11.4) implies that the L_2 norm of (11.38) is no greater than $K||w_{\alpha i}||_{0,E_n} \le K|u_i|_{m,\Omega}$. Thus we have

$$|v_i|_{m,\Omega} \le K|u_i|_{m,\Omega}.$$

Combining this with (11.37) yields

$$(11.39) \qquad ||v_i||_{m,E_n} \le K|u_i|_{m,\Omega}.$$

Now (11.33) implies that

$$(11.40) \qquad ||v||_{m,E_n} \le K\sum_{i=0}^{\nu}|u_i|_{m,\Omega}.$$

Next, Leibnitz's rule implies

$$\sum_{i=0}^{\nu}|u_i|^2_{m,E_n} \le K||u||^2_{m,\Omega}.$$

Therefore, (11.40) implies $||v||_{m,E_n} \le K||u||_{m,\Omega}$.

Therefore, the correspondence $u \to v$ is a linear transformation of $C^m(\Omega) \cap H_m(\Omega)$ into $H_m(E_n)$ with the properties

$$||v||_{m,E_n} \leq K||u||_{m,\Omega}.$$

$$v = u \text{ on } \Omega.$$

Since this transformation is a bounded transformation from a dense subset of $H_m(\Omega)$ into $H_m(E_n)$, it may be extended in a unique fashion to $H_m(\Omega)$. Moreover, it follows that the extended mapping still satisfies the requirement $v = u$ almost everywhere on Ω. Q.E.D.

Remarks. The same argument shows that we can use L_p norms $(1 < p < \infty)$ everywhere instead of L_2 norms. The only additional tool is the Calderón-Zygmund estimate for L_p instead of L_2. We could also obtain estimates for extensions of u preserving Hölder continuity of derivatives of u. Here a slight change appears, since the extension of $D^\alpha u_i$ outside Ω can no longer be $w_{\alpha i}$ (cf. (11.29)), but must be an extension perserving the Hölder continuity of $D^\alpha u_i$.

12. Some Results on Linear Transformations on a Hilbert Space

This section is devoted to preliminary material needed in the discussion of eigenvalue problems in section 13. It is divided into three parts.

Part 1. ELEMENTARY SPECTRAL THEORY

Definition 12.1. *Let A be a linear transformation defined on a (not necessarily closed) subspace $D(A)$ of a complex Hilbert space X; $D(A)$ is the* domain *of A. Let $R(A)$ be the* range *of A; that is, $R(A) = \{Af\colon f \in X\}$. The* resolvent set *of A, denoted $\rho(A)$, is the set of all complex numbers λ such that $\lambda - A$ is one-to-one, $R(\lambda - A) = X$, and the inverse $(\lambda - A)^{-1}$ is a bounded linear transformation on X. Here the notation λ stands for the transformation mapping f into λf. all $f \in X$. The* spectrum *of A, denoted $\sigma(A)$, is the set of all complex numbers not in $\rho(A)$.*

THEOREM 12.1. *The resolvent set of A is open, and $(\lambda - A)^{-1}$ is an analytic function on $\rho(A)$. If A is bounded, then $\rho(A) \supset \{\lambda\colon |\lambda| > ||A||\}$.*

Proof. If $\lambda_0 \in \rho(A)$, then

$$\lambda - A = (\lambda_0 - A) + (\lambda - \lambda_0)$$

$$= [1 - (\lambda_0 - \lambda)(\lambda_0 - A)^{-1}](\lambda_0 - A). \tag{12.1}$$

The operator $1 - (\lambda_0 - \lambda)(\lambda_0 - A)^{-1}$ is invertible if $||(\lambda_0 - \lambda)(\lambda_0 - A)^{-1}|| < 1$, or, if

$$|\lambda_0 - \lambda| < ||(\lambda_0 - A)^{-1}||^{-1}.$$

Moreover, we have the Neumann expansion

$$[1 - (\lambda_0 - \lambda)(\lambda_0 - A)^{-1}]^{-1} = \sum_{k=0}^{\infty} (\lambda_0 - \lambda)^k(\lambda_0 - A)^{-k}$$

where $(\lambda_0 - A)^{-k} = [(\lambda_0 - A)^{-1}]^k$. Therefore, (12.1) implies that for $|\lambda_0 - \lambda| < ||(\lambda_0 - A)^{-1}||^{-1}$, $\lambda \in \rho(A)$, and the continuity of $(\lambda_0 - A)^{-1}$ implies that

$$(\lambda - A)^{-1} = \sum_{k=0}^{\infty} (\lambda_0 - \lambda)^k(\lambda_0 - A)^{-k-1}. \tag{12.2}$$

Thus, $(\lambda - A)^{-1}$ is expressed in terms of a power series about any point $\lambda_0 \in \rho(A)$; that is to say, $(\lambda - A)^{-1}$ is analytic.

Finally, if A is bounded and $|\lambda| > ||A||$, then

$$\lambda - A = \lambda(1 - \lambda^{-1}A).$$

and $||\lambda^{-1}A|| < 1$. Thus, the above argument shows that $\lambda \in \rho(A)$. Q.E.D.

Definition 12.2. *The function* $(\lambda - A)^{-1}$, *defined on* $\rho(A)$, *is the* resolvent *of* A.

Note that (12.2) implies

$$||(\lambda - A)^{-1}|| \leq \sum_{k=0}^{\infty} |\lambda_0 - \lambda|^k ||(\lambda_0 - A)^{-1}||^{k+1}.$$

Summing this geometric series, we obtain a very useful estimate

$$(12.3) \quad ||(\lambda - A)^{-1}|| \leq \frac{||(\lambda_0 - A)^{-1}||}{1 - |\lambda_0 - \lambda| \, ||(\lambda_0 - A)^{-1}||}, \quad |\lambda_0 - \lambda| < ||(\lambda_0 - A)^{-1}||^{-1}.$$

Frequently in dealing with integral equations, we shall consider relations of the form

$$f = g + \lambda A f,$$

in which it is desired to solve for f in terms of g. In case $1 - \lambda A$ is invertible, i.e., $\lambda^{-1} \in \rho(A)$, the solution will be given by $f = (1 - \lambda A)^{-1} g$ (the case $\lambda = 0$ is obviously without interest). It will prove to be very fruitful, however, not to analyze the operator $(1 - \lambda A)^{-1}$ itself, but, instead, $A(1 - \lambda A)^{-1}$. One reason for this is that the *range* of $A(1 - \lambda A)^{-1}$ is the same as the range of A, and we shall frequently have some very precise information about the range of A. Note that

$$\begin{aligned} 1 &= (1 - \lambda A)(1 - \lambda A)^{-1} \\ &= (1 - \lambda A)^{-1} - \lambda A(1 - \lambda A)^{-1}, \end{aligned}$$

so that

$$(12.4) \quad (1 - \lambda A)^{-1} = 1 + \lambda A(1 - \lambda A)^{-1}.$$

This notion is in the following definition.

Definition 12.3. *The* modified resolvent set $\rho_m(A)$ *of* A *is the set of all non-zero complex numbers* λ *such that* $1 - \lambda A$ *is one-to-one,* $R(1 - \lambda A) = X$, *and* $(1 - \lambda A)^{-1}$ *is a bounded linear transformation on* X. *If* $\lambda \in \rho_m(A)$, *then the transformation* $A_\lambda \equiv A(1 - \lambda A)^{-1}$ *is the* modified resolvent *of* A.

Note that (12.4) can be expressed in the form

$$(12.5) \quad (1 - \lambda A)^{-1} = 1 + \lambda A_\lambda.$$

Also note that

$$(12.6)\qquad \lambda \in \rho_m(A) \Longleftrightarrow \lambda^{-1} \in \rho(A).$$

It should be remarked that if the linear transformation is *closed*, then the condition in Definition 12.1 that $(\lambda - A)^{-1}$ be bounded is superfluous. For, if $\lambda - A$ is one-to-one and onto, then $(\lambda - A)^{-1}$ is closed (since A is closed) and defined everywhere on X. Then the closed graph theorem implies that $(\lambda - A)^{-1}$ is bounded. Likewise, if A is closed, then the condition in Definition 12.3 that $(1 - \lambda A)^{-1}$ be bounded is superfluous.

It should also be remarked that if $D(A) = X$, and if $\lambda \in \rho(A)$, then A and $(\lambda - A)^{-1}$ commute. For, in any case, $(\lambda - A)(\lambda - A)^{-1} = 1$ and $(\lambda - A)^{-1}(\lambda - A)$ is the identity *on* $D(A)$. Thus, if $D(A) = X$, then $(\lambda - A)^{-1}(\lambda - A) = 1$. Likewise, if $D(A) = X$ and $\lambda \in \rho_m(A)$, then A and A_λ commute.

We now state the analog of Theorem 12.1 for modified resolvents.

THEOREM 12.2. *The modified resolvent set of A is open, and A_λ is an analytic function on $\rho_m(A)$. If $\lambda_0 \in \rho_m(A)$, then for $|\lambda - \lambda_0| < \|A_{\lambda_0}\|^{-1}$, $\lambda \neq 0$, $\lambda \in \rho_m(A)$ and*

$$(12.7)\qquad A_\lambda = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k A_{\lambda_0}^{k+1},$$

in the sense of convergence in the operator norm. Also,

$$(12.8)\qquad \|A_\lambda\| \le \frac{\|A_{\lambda_0}\|}{1 - |\lambda - \lambda_0|\,\|A_{\lambda_0}\|}, \qquad |\lambda - \lambda_0| < \|A_{\lambda_0}\|^{-1}.$$

If A is bounded, $\rho_m(A) \supset \{\lambda\colon\ 0 < |\lambda| < \|A\|^{-1}\}$.

Proof. We have

$$1 - \lambda A = 1 - \lambda_0 A + (\lambda_0 - \lambda)A$$

$$= [1 - (\lambda - \lambda_0)A(1 - \lambda_0 A)^{-1}](1 - \lambda_0 A)$$

$$= [1 - (\lambda - \lambda_0)A_{\lambda_0}](1 - \lambda_0 A).$$

Thus, if $|\lambda - \lambda_0| < ||A_{\lambda_0}||^{-1}$, we have, as before,

$$(1 - \lambda A)^{-1} = (1 - \lambda_0 A)^{-1} \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k A_{\lambda_0}^k.$$

Multiplying by A, we obtain (12.7). The other assertions are then immediate. Q.E.D.

In the situation of interest to us, A will be a differential operator, and there will exist some number λ in the resolvent set of A. For example, if we are interested in the generalized Dirichlet problem with zero Dirichlet data, $X = L_2(\Omega)$ and A is defined on $\overset{\circ}{H}_m(\Omega)$. By Theorem 8.5, if Gårding's inequality holds, then $-\lambda \in \rho(A)$ for all sufficiently large real λ. Also, if Ω is bounded, then $(A + \lambda)^{-1}$ is compact. If we consider the differential operator $A + \lambda$ instead of A, then this differential operator is itself invertible and has a compact inverse. The abstract case of this situation is a special case of what we shall consider.

A simple result shall now be given.

THEOREM 12.3. *Let $0 \in \rho(A)$ and let $T = A^{-1}$. Then for $\lambda \neq 0$, $\lambda \in \rho_m(T)$ if and only if $\lambda \in \rho(A)$; if $\lambda \in \rho_m(T)$, then*

$$T_\lambda = (\lambda - A)^{-1}. \tag{12.9}$$

Proof. We have

$$(\lambda - A)T = \lambda T - AT$$

$$= \lambda T - 1.$$

The assertions about $\rho_m(T)$ and $\rho(A)$ are now easily checked, and then (12.9) is immediate. Q.E.D.

Definition 12.4. *A complex number λ is an* eigenvalue *of A if there exists a non-zero vector f such that $Af = \lambda f$. A complex number λ is a* characteristic value *of A if there exists a non-zero vector g such that $\lambda A g = g$.*

Note that every characteristic value of A is non-zero and that λ is a characteristic value of A if and only if λ^{-1} is an eigenvalue of A.

Definition 12.5. *Let λ be an eigenvalue of A. A non-zero vector f is a* generalized eigenvector of A corresponding to λ *if for some positive integer k, $(\lambda - A)^k f = 0$. The set of all generalized eignenvectors of A corresponding to λ, together with the origin in X, forms a subspace of X, whose dimension is the* multiplicity *of the eigenvalue λ. Likewise, if λ is a characteristic value of A, then a non-zero vector f is a* generalized characteristic vector of A corresponding to λ *if for some positive integer k, $(1 - \lambda A)^k f = 0$.*

If we use the notation $N(A)$ for the null space of a transformation A, then f is a generalized eigenvector of T corresponding to λ if and only if $f \neq 0$ and f is in the set

$$M(\lambda) = \bigcup_{k=1}^{\infty} N((\lambda - T)^k).$$

Thus, the multiplicity of λ is the dimension of this set. Note that this set is a subspace of X, since $N((\lambda - T)^k) \subset N((\lambda - T)^{k+1})$, $k \geq 1$. We shall call $M(\lambda)$ the *generalized eigenspace* of the operator T corresponding to λ.

THEOREM 12.4. *Let T be defined on X, and let $\lambda_0 \in \rho_m(T)$. If $\lambda\lambda_0 = 1$, then $\lambda \in \rho(T)$. If $\lambda\lambda_0 \neq 1$, then λ is an eigenvalue of T if and only if $\frac{\lambda}{1-\lambda\lambda_0}$ is an eigenvalue of T_{λ_0}. Furthermore, the multiplicity of λ as an eigevalue of T is the same as the multiplicity of $\frac{\lambda}{1-\lambda\lambda_0}$ as an eigenvalue of T_{λ_0}. Indeed, the set of all generalized eigenvectors of T corresponding to λ is the same as the set of all generalized eigenvectors of T_{λ_0} corresponding to $\frac{\lambda}{1-\lambda\lambda_0}$.*

Proof. The first assertion follows from (12.6). Suppose $\lambda\lambda_0 \neq 1$. Then by definition of T_{λ_0} (Definition 12.3),

$$T_{\lambda_0} - \frac{\lambda}{1-\lambda\lambda_0} = [T - \frac{\lambda}{1-\lambda\lambda_0}(1 - \lambda_0 T)](1 - \lambda_0 T)^{-1}$$

$$= [(1 + \frac{\lambda\lambda_0}{1-\lambda\lambda_0})T - \frac{\lambda}{1-\lambda\lambda_0}](1 - \lambda_0 T)^{-1}$$

$$= \frac{1}{1-\lambda\lambda_0}(T - \lambda)(1 - \lambda_0 T)^{-1}.$$

Therefore, since $T - \lambda$ and $(1 - \lambda_0 T)^{-1}$ commute,

$$(12.10) \quad (T_{\lambda_0} - \frac{\lambda}{1-\lambda\lambda_0})^k = \frac{1}{(1-\lambda\lambda_0)^k}(T - \lambda)^k(1 - \lambda_0 T)^{-k}.$$

The relation (12.10) for $k = 1$ implies that $T - \lambda$ is one-to-one if and only if $T_{\lambda_0} - \frac{\lambda}{1-\lambda\lambda_0}$ is one-to-one. Thus, λ is an eigenvalue of $T \Leftrightarrow \frac{\lambda}{1-\lambda\lambda_0}$ is an eigenvalue for T_{λ_0}. Moreover, (12.10) implies that $(T - \lambda)^k f = 0 \Leftrightarrow (T_{\lambda_0} - \frac{\lambda}{1-\lambda\lambda_0})^k f = 0$. Q.E.D.

Before giving the next definition, some notation will be convenient. For θ real and $a \geq 0$, let

$$\Xi(\theta, a) = \{re^{i\theta}: r > a\}.$$

Definition 12.6. *Let A be a linear transformation defined on a subspace $D(A)$ of a Hilbert space X. A vector $e^{i\theta}$ in the complex plane is a* direction of minimal growth of the resolvent of A *if for some positive a, $\Xi(\theta, a) \subset \rho(A)$ and $\|(\lambda - A)^{-1}\| = 0(|\lambda|^{-1})$ for $\lambda \in \Xi(\theta, a)$, $|\lambda| \to \infty$. Likewise, a vector $e^{i\theta}$ is a* direction of minimal growth of the modified resolvent of A *if for some positive a, $\Xi(\theta, a) \subset \rho_m(A)$ and* $\|A_\lambda\| = 0(|\lambda|^{-1})$ for $\lambda \in \Xi(\theta, a)$, $|\lambda| \to \infty$.

In general, we cannot expect the resolvent, or modified resolvent, to decay faster along any ray than $0(|\lambda|^{-1})$. This is shown by the following theorem.

THEOREM 12.5. *Suppose that for some positive constant a and for some sequence of complex numbers λ_n in $\rho(A)$ such that $|\lambda_n| \to \infty$, the following estimate holds:*

$$||(\lambda_n - A)^{-1}|| \leq a|\lambda_n|^{-1}.$$

Then $a \geq 1$. Correspondingly, if for some sequence of complex numbers in $\rho_{\mathfrak{m}}(A)$ such that $|\lambda_n| \to \infty$, the estimate

$$||A_{\lambda_n}|| \leq a|\lambda_n|^{-1}$$

holds, then either $A = 0$ or $a \geq 1$.

Proof. By (12.3) it follows that $\lambda \in \rho(A)$ if $|\lambda - \lambda_n| < a^{-1}|\lambda_n|$, and we have an estimate

$$||(\lambda - A)^{-1}|| \leq \frac{a|\lambda_n|^{-1}}{1-|\lambda-\lambda_n|a|\lambda_n|^{-1}}$$

$$= \frac{1}{a^{-1}|\lambda_n|-|\lambda-\lambda_n|}. \tag{12.11}$$

Now assume $a < 1$. Then $a = (1 + 2\epsilon)^{-1}$ for some positive ϵ. Also all of the circles $\{\lambda\colon\ |\lambda - \lambda_n| < (1 + \epsilon)\ |\lambda_n|\}$ are contained in $\rho(A)$, and for λ in such a circle (12.11) implies

$$||(\lambda - A)^{-1}|| < \frac{1}{(1 + 2\epsilon)|\lambda_n|-(1 + \epsilon)|\lambda_n|} = \frac{1}{\epsilon|\lambda_n|}. \tag{12.12}$$

Thus, since such circles cover the complex plane, (12.12) implies $(\lambda - A)^{-1}$ exists and is uniformly bounded in the complex plane. As $(\lambda - A)^{-1}$ is also analytic on $\rho(A)$ = complex plane (by Theorem 12.1), Liouville's theorem implies $(\lambda - A)^{-1}$ is constant. But then $\lambda - A$ is constant, an absurdity.

For the proof of the second part, the same procedure as given above shows that if $a < 1$, then A_λ is constant. But since $||A_{\lambda_n}|| \to 0$, this implies $A_\lambda = 0$ for all λ. That is, $A(1 - \lambda A)^{-1} = 0$ for all λ. Multiplying this relation by $1 - \lambda A$ implies $A = 0$. Q.E.D.

THEOREM 12.6. *The set of numbers θ such that $e^{i\theta}$ is a direction of minimal growth of the resolvent [modified resolvent] of A is open.*

Proof. Suppose that

$$||(\lambda - A)^{-1}|| \le \frac{K}{|\lambda|}$$

for $\lambda \in \Xi(\theta, a)$. By (12.3), $\mu \in \rho(A)$ if $|\mu - \lambda| < K^{-1}|\lambda|$ and $\lambda \in \Xi(\theta, a)$, and we have an estimate

$$||(\mu - A)^{-1}|| \le \frac{K|\lambda|^{-1}}{1-|\mu-\lambda|K|\lambda|^{-1}}.$$

Thus, if $\lambda \in \Xi(\theta, a)$ and if $|\mu - \lambda| < \frac{1}{2}K^{-1}|\lambda|$, then

$$||(\mu - A)^{-1}|| \le 2K|\lambda|^{-1}.$$

Since

$$\begin{aligned}|\mu| &\le |\lambda| + |\mu - \lambda| \\ &< (1 + \frac{1}{2K})|\lambda|,\end{aligned}$$

we have

$$||(\mu - A)^{-1}|| < \frac{2K + 1}{|\mu|},$$

an estimate which holds for all μ such that $|\mu - \lambda| < \frac{1}{2}K^{-1}|\lambda|$ for some $\lambda \in \Xi(\theta, a)$. An easy computation then shows that ψ is a direction of minimal growth of the resolvent of A if $|\psi - \theta| < \sin^{-1} 1/2K)$. A similar argument gives the result for the directions of minimal growth of the modified resolvent. Q.E.D.

We now give some examples.

THEOREM 12.7. *Let A be a self-adjoint operator on the Hilbert space X. Then any non-real direction $e^{i\theta}$ is a direction of minimal growth of both the resolvent and the modified resolvent of A. Furthermore, for* $\arg \lambda = \theta$

(12.13) $\quad ||(\lambda - A)^{-1}|| \leq |\csc \theta| \; |\lambda|^{-1},$

(12.14) $\quad ||A_\lambda|| \leq |\csc \theta| \; |\lambda|^{-1}.$

Proof. Since A is self-adjoint, (Af, f) is real for $f \in D(A)$. Thus, if $\lambda = |\lambda| e^{i\theta}$, then

$$\mathcal{I}[((\lambda - A)f, f)] = |\lambda| \sin\theta (f, f).$$

Therefore, the Cauchy-Schwarz inequality implies

$$||f||^2 \leq |\lambda|^{-1}| \csc \theta| \; ||(\lambda - A)f|| \; ||f||;$$

dividing by $||f||$,

(12.15) $\quad ||f|| \leq |\csc \theta| \; |\lambda|^{-1}||(\lambda - A)||f||.$

This inequality shows that $N(\lambda - A) = \{0\}$ for all non-real λ. Thus, for non-real λ, $N((\lambda - A)^*) = N(\bar{\lambda} - A) = \{0\}$. But the orthogonal complement of the range of $\lambda - A$ is precisely $N((\lambda - A)^*)$. Therefore, the range of $\lambda - A$ is dense in X. Since (12.15) implies that the range of $\lambda - A$ is closed (here we use the fact that a self-adjoint transformation is *closed*), it follows that $\lambda - A$ is one-to-one and onto; and another application of (12.15) implies

$$||(\lambda - A)^{-1}|| \leq |\csc \theta| \; |\lambda|^{-1}.$$

To obtain the result for the modified resolvent of A, note that for non-real λ and $f \in D(A)$

$$\mathcal{I}[((1 - \lambda A)f, Af)] = |\lambda| \sin \theta ||Af||^2.$$

Therefore, applying the Cauchy-Schwartz inequality, dividing by $||f||$,

$$||Af|| \leq |\csc\theta|\ |\lambda^{-1}|\ ||(1-\lambda A)f||.$$

Putting in the last inequality $f = (1-\lambda A)^{-1}g$, g an arbitrary element in X, (12.14) follows. Q.E.D.

We also have the following result.

THEOREM 12.8. *Let A be a linear transformation with domain $D(A) \subset X$, such that for all $f \in D(A)$*

$$\Re(Af, f) \geq \lambda_0 ||f||^2,$$

*for some real number λ_0. Suppose that some number λ_1 is in the resolvent set of A, and that $\Re\lambda_1 < \lambda_0$. Then the half-*plane $\{\lambda$: $\Re\lambda < \lambda_0\}$ *is contained in the resolvent set of A, and every direction $e^{i\theta}$ with $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ is a direction of minimal growth of the resolvent of A.*

Proof. We have for $f \in D(A)$

$$\begin{aligned}\Re((\lambda - A)f, f) &= \Re\lambda ||f||^2 - \Re(Af, f)\\ &\leq \Re\lambda||f||^2 - \lambda_0||f||^2\\ &= (\Re\lambda - \lambda_0)||f||^2.\end{aligned}$$

Therefore, if $\Re\lambda < \lambda_0$, the Cauchy-Schwarz inequality implies

$$\begin{aligned}||f||^2 &= \frac{-1}{\lambda_0 - \Re\lambda}\Re((\lambda - A)f, f)\\ &\leq \frac{1}{\lambda_0 - \Re\lambda}\ ||(\lambda - A)f||\ ||f||^2\end{aligned}$$

or,

(12.16) $$||f|| \leq \frac{1}{\lambda_0 - \Re\lambda}\ ||(\lambda - A)f||.$$

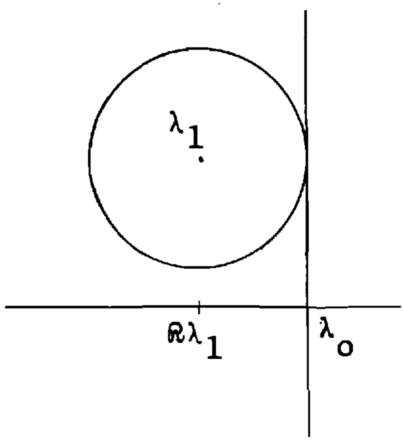
Since $\lambda_1 \in \rho(A)$, (12.16) implies that for $f = (\lambda_1 - A)^{-1}g$

$$||(\lambda_1 - A)^{-1}g|| \leq \frac{1}{\lambda_0 - \Re\lambda_1} ||g||.$$

Thus,

$$||(\lambda_1 - A)^{-1}|| \leq \frac{1}{\lambda_0 - \Re\lambda_1}.$$

But (12.3) implies that $\lambda \in \rho(A)$ if $|\lambda_1 - \lambda| < ||(\lambda_1 - A)^{-1}||^{-1}$; therefore, $\lambda \in \rho(A)$ if $|\lambda_1 - \lambda| < \lambda_0 - \Re\lambda_1$. We can then extend this process (see figure), to show that $\lambda \in \rho(A)$ if $\Re\lambda < \lambda_0$. And then (12.16)



yields the estimate

$$||(\lambda - A)^{-1}|| \leq \frac{1}{\lambda_0 - \Re\lambda}, \; \Re\lambda < \lambda_0.$$

From this estimate the assertion of the theorem is obvious.

Part 2. OPERATORS OF FINITE DOUBLE-NORM ON AN ABSTRACT HILBERT SPACE

THEOREM 12·9. *Let T be a bounded linear transformation on a Hilbert space X. If $\{\phi_1, \phi_2, \dots\}$ and $\{\psi_1, \psi_2, \dots\}$ are orthonormal bases in X, then*

$$\sum_{i=1}^{\infty} ||T\phi_i||^2 = \sum_{i=1}^{\infty} ||T\psi_i||^2$$

$$= \sum_{i=1}^{\infty} ||T^*\phi_i||^2$$

$$= \sum_{i,j=1}^{\infty} |(T\phi_i, \psi_j)|^2;$$

it is to be understood that these sums may be either finite or infinite.

Proof. By Parseval's relation,

$$||T\phi_i||^2 = \sum_{j=1}^{\infty} |(T\phi_i, \psi_j)|^2.$$

Summing over i gives

$$\sum_{i=1}^{\infty} ||T\phi_i||^2 = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |(T\phi_i, \psi_j)|^2$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |(\phi_i, T^*\psi_j)|^2$$

$$= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |(T^*\psi_j, \phi_i)|^2$$

$$= \sum_{j=1}^{\infty} ||T^*\psi_j||^2,$$

where we have again used Parseval's identity. Applying this argument to T^* instead of T gives

$$\sum_{i=1}^{\infty} ||T^*\phi_i||^2 = \sum_{j=1}^{\infty} ||T^{**}\psi_j||^2$$

$$= \sum_{j=1}^{\infty} ||T\psi_j||^2.$$

Therefore all the assertions of the theorem follow. Q.E.D.

Definition 12.7. *A bounded linear transformation T on a Hilbert space has* finite double-norm *if the sums in Theorem 12.9 are finite. In this case, for any orthonormal sequence $\{\phi_1, \phi_2, \ldots\}$ the double-norm of T is given by*

$$|||T||| = [\sum_{i=1}^{\infty} ||T\phi_i||^2]^{1/2}.$$

The next theorem gives some elementary properties.

THEOREM 12.10. *Let T_1 and T_2 each have finite double-norm, and let A be a bounded linear transformation. Then T_1^*, $T_1 + T_2$, AT_1, and T_1A each have finite double-norm, and*

$$|||T_1||| = |||T_1^*|||;$$

$|||aT_1||| = |a|\ |||T_1|||$ for any complex number a;

$$|||T_1 + T_2||| \le |||T_1||| + |||T_2|||;$$

$$|||AT_1||| \le ||A||\ |||T_1|||;$$

$$|||T_1A||| \le ||A||\ |||T_1|||;$$

$$||T_1|| \le |||T_1|||.$$

Proof. The first relation follows from Theorem 12.9, the second is trivial, and the third follows from Minkowki's inequality. The fourth follows easily:

$$\sum_{i=1}^{\infty} ||AT_1\phi_i||^2 \leq \sum_{i=1}^{\infty} ||A||^2||T_i\phi_i||^2 = ||A||^2|||T|||^2.$$

And the fifth follows from the computation

$$\begin{aligned} |||T_1A||| &= |||(T_1A)^*||| \\ &= |||A^*T^*_1||| \\ &\leq ||A^*||\ |||T^*_1||| \\ &= ||A||\ |||T_1|||. \end{aligned}$$

Finally, suppose $f \in X$. Then

$$f = \sum_{i=1}^{\infty} (f, \phi_i)\phi_i;$$

$$Tf = \sum_{i=1}^{\infty} (f, \phi_i)T\phi_i.$$

The Cauchy-Schwarz inequality shows that

$$\begin{aligned} ||Tf|| &\leq \sum_{i=1}^{\infty} |(f, \phi_i)|\ ||T\phi_i|| \\ &\leq [\sum_{i=1}^{\infty} |(f, \phi_i)|^2]^{1/2}[\sum_{i=1}^{\infty} ||T\phi_i||^2]^{1/2} \\ &= ||f||\ |||T|||. \quad \text{Q.E.D.} \end{aligned}$$

THEOREM 12.11 *A transformation having finite double-norm is compact.*

Proof. Let $\{\phi_1, \phi_2 \ldots\}$ be an orthonormal basis for X. Then there exist numbers $t^{ij} (i, j = 1, 2, \ldots)$ such that

$$T\phi_j = \sum_{i=1}^{\infty} t^{ij} \phi_i.$$

Moreover, Parseval's relation shows that

$$\|T\phi_j\|^2 = \sum_{i=1}^{\infty} |t^{ij}|^2.$$

Therefore, summing on i, we obtain

$$(12.17) \qquad |||T|||^2 = \sum_{i,j=1}^{\infty} |t^{ij}|^2.$$

If N is a positive integer, let

$$t_N^{ij} = \begin{cases} t^{ij}, & i, j \leq N, \\ 0, & i > N \quad \text{or} \quad j > N. \end{cases}$$

Then let T_N be the unique linear transformation satisfying

$$T_N \phi_j = \sum_{i=1}^{\infty} t_N^{ij} \phi_i.$$

Thus,

$$(12.18) \qquad T_N \phi_j = \begin{cases} \sum_{i=1}^{N} t^{ij} \phi_i, & 1 \leq j \leq N, \\ 0, & j > N. \end{cases}$$

Then it is obvious that T_N has finite double-norm and

$$|||T - T_N|||^2 = \sum_{i,j=1}^{\infty} |t^{ij}|^2 - \sum_{i,j=1}^{N} |t^{ij}|^2.$$

Therefore, $|||T - T_N||| \to 0$, so that Theorem 12.10 implies $||T - T_N|| \to 0$. Since the range of T_N is finite-dimensional, it is obvious that T_N is compact. But then the fact that $||T - T_N|| \to 0$ implies T is compact. Q.E.D.

THEOREM 12.12. *Let T and S be transformations each having finite double-norm, and let $\{\phi_1, \phi_2, \ldots\}$ be an orthonormal basis in X. Then the series*

$$\sum_{i,j=1}^{\infty} (T\phi_j, \phi_i)(S\phi_i, \phi_j)$$

is absolutely convergent and is independent of the particular orthonormal basis $\{\phi_i\}$. If the sum of the series is designated $tr(TS)$, then

$$|tr(TS)| \leq |||T|||\ |||S|||. \tag{12.19}$$

Proof. The absolute convergence of the series and the estimate (12.19) are immediate consequences of the Cauchy-Schwarz inequality, the definition of double-norm, and Theorem 12.9. Note that

$$TS\phi_i = T \sum_{j=1}^{\infty} (S\phi_i, \phi_j)\phi_j$$

$$= \sum_{j=1}^{\infty} (S\phi_i, \phi_j)T\phi_j$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (S\phi_i, \phi_j)(T\phi_j, \phi_k)\phi_k.$$

Thus,

$$(TS\phi_i, \phi_i) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (S\phi_i, \phi_j)(T\phi_j, \phi_k)(\phi_k, \phi_i)$$

$$= \sum_{j=1}^{\infty} (S\phi_i, \phi_j)(T\phi_j, \phi_i).$$

Summing on i,

$$(12.20) \qquad \sum_{i=1}^{\infty} (TS\phi_i, \phi_i) = \sum_{i,j=1}^{\infty} (S\phi_i, \phi_j)(T\phi_j, \phi_i).$$

Now let $\{\psi_i\}$ be another orthonormal basis. Then

$$\sum_{i,j=1}^{\infty} (T\phi_j, \phi_i)(S\phi_i, \phi_j) = \sum_{i,j=1}^{\infty} (S\phi_j, \phi_i)(T\phi_i, \phi_j)$$

$$= \sum_{i=1}^{\infty} (ST\phi_i, \phi_i)$$

$$= \sum_{i=1}^{\infty} (T\phi_i, S^*\phi_i)$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (T\phi_i, \psi_j)\overline{(S^*\phi_i, \psi_j)},$$

by Parseval's relation. Thus, by the absolute convergence of the double series and Parseval's relation again,

$$\sum_{i,j=1}^{\infty} (T\phi_j, \phi_i)(S\phi_i, \phi_j) = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} (S\psi_j, \phi_i)\overline{(T^*\psi_j, \phi_i)}$$

$$= \sum_{j=1}^{\infty} (S\psi_j, T^*\psi_j)$$

$$= \sum_{j=1}^{\infty} (TS\psi_j, \psi_j)$$

$$= \sum_{i,j=1}^{\infty} (S\psi_i, \psi_j)(T\psi_j, \psi_i) ,$$

by (12.20). Q.E.D.

Notation. The quantity $tr(TS)$ is called the *trace* of the operator TS. Note that (12.20) can be written

$$tr(TS) = \sum_{i=1}^{\infty} (TS\phi_i, \phi_i).$$

Since any transformation T having finite double-norm is compact, the Riesz-Schauder theory of compact operators applies to T. In particular, a non-zero complex number λ is either in the resolvent set of T or is an eigenvalue of T; the non-zero eigenvalues of T have finite multiplicity; and the only limit point of the set of eigenvalues of T is the complex number zero.

The next result of immediate interest is Theorem 12.14. But first a lemma is needed.

LEMMA 12.13. *Let T be a linear transformation on a linear space X, and let $M(\lambda)$ be the generalized eigenspace of T corresponding to the eigenvalue λ. The subspaces $M(\lambda)$ are independent; that is, if $\lambda_1,\ldots,\lambda_\nu$ are distinct, if $f_i \in M(\lambda_i)$, $i = 1,\ldots,\nu$, and if $\sum_{i=1}^{\nu} f_i = 0$, then $f_i = 0$, $1 \le i \le \nu$.*

Proof. The proof is by induction on ν. The theorem is trivial for $\nu = 1$, but this fact does not help in the proof. We first establish the lemma for $\nu = 2$. Thus, since $f_1 + f_2 = 0$, we may simply assume that for some $f \in X$,

$$(12.21) \quad (\lambda_1 - T)^{k_1} f = 0,$$

$$(\lambda_2 - T)^{k_2} f = 0.$$

If $k_1 = 0$, then it is trivial that $f = 0$. If $k_1 > 0$, let $g = (\lambda_1 - T)^{k_1 - 1} f$. Then the two relations imply

$$(\lambda_1 - T)g = 0,$$

$$(\lambda_2 - T)^{k_2} g = 0.$$

The first follows from the definition of g and the second by multiplying the second equality in (12.21) by $(\lambda_1 - T)^{k_1 - 1}$. But now it is seen that $Tg = \lambda_1 g$, so that $(\lambda_2 - T)^{k_2} g = (\lambda_2 - \lambda_1)^{k_2} g = 0$. Since $\lambda_1 \neq \lambda_2$, $g = 0$, or, $(\lambda_1 - T)^{k_1 - 1} f = 0$. If $k_1 - 1 = 0$, then $f = 0$; if $k_1 - 1 > 0$, then we obtain $(\lambda_1 - T)^{k_1 - 2} f = 0$. Proceeding in this manner, it follows that $f = 0$, and the lemma is proved for $\nu = 2$.

Now suppose the lemma to be valid for $\nu - 1$, $\nu \geq 3$; we then establish the result for ν. We have

$$(12.22) \qquad (\lambda_i - T)^{k_i} f_i = 0, \; 1 \leq i \leq \nu;$$

$$(12.23) \qquad \sum_{i=1}^{\nu} f_i = 0.$$

Multiplying (12.23) by $(\lambda_\nu - T)^{k_\nu}$, setting $g_i = (\lambda_\nu - T)^{k_\nu} f_i$, and taking (12.23) into account, we obtain

$$(\lambda_i - T)^{k_i} g_i = 0, \; 1 \leq i \leq \nu - 1;$$

$$\sum_{i=1}^{\nu-1} g_i = 0.$$

By the induction hypothesis, $g_i = 0$; that is, $(\lambda_\nu - T)^{k_\nu} f_i = 0$, $1 \leq i \leq \nu - 1$. But since $(\lambda_\nu - T)^{k_\nu} f_i = (\lambda_i - T)^{k_i} f_i = 0$, the result for $\nu = 2$ implies $f_i = 0$, $1 \leq i \leq \nu - 1$. And then (12.23) implies also $f_\nu = 0$. Q.E.D.

THEOREM 12.14. *Let T be a linear transformation on a separable Hilbert space X, and let T have finite double-norm. Then each non-zero eigenvalue of T has finite multiplicity and the eigenvalues of T can have only 0 as a limit point. If $\{\lambda_j\}$ is the set of non-zero eigenvalues of T arranged in some order, where each eigenvalue is repeated the number of times equal to its multiplicity, then*

$$(12.24) \qquad \Sigma_j |\lambda_j|^2 \leq |||T|||^2.$$

Proof. As we have already remarked, the non-zero eigenvalues have finite multiplicity and have 0 as the only possible limit point. Arrange the non-zero eigenvalues $\{\lambda_j\}$ such that the repetitions of a given eigenvalue appear in succession. Suppose that λ is a non-zero eigenvalue, and that $\lambda_{j_0+1},\ldots,\lambda_{j_0+m}$ are the members of $\{\lambda_j\}$ equal to λ. Then $M(\lambda)$, the generalized eigenspace of T corresponding to λ, has dimension m. Obviously, $\lambda - T$ maps $M(\lambda)$ into itself, so we can consider the transformation A which is the restriction of $\lambda - T$ to $M(\lambda)$; A is then a linear transformation on $M(\lambda)$. Obviously, the only eigenvalue of A is 0. A simple theorem on matrices shows that there is a basis $\{f_{j_0+1},\ldots,f_{j_0+m}\}$ for $M(\lambda)$ such that the matrix of A with respect to this basis is triangular. That is,

$$(12.25)\qquad Af_j = \sum_{i=j_0+1}^{j} a^{ij}f_i,\ j_0+1 \le j \le j_0+m,$$

for certain constants a^{ij}. Moreover, $a^{ii}=0$, since a^{ii} must be an eigenvalue of A. Since $Af_j = (\lambda - T)f_j$, (12.25) implies (since also $\lambda = \lambda_j$)

$$(12.26)\qquad Tf_j = \lambda_j f_j - \sum_{i=j_0+1}^{j-1} a^{ij}f_i,\ j_0+1 \le j \le j_0+m.$$

If this is done for each eigenvalue, then the independence of the subspaces $M(\lambda)$ (cf. Lemma 12.13) implies that there is a linearly independent set $\{f_1, f_2,\ldots\}$ such that

$$(12.27)\qquad Tf_j = \lambda_j f_j + \sum_{i=1}^{j-1} a_1^{ij}f_i,$$

for certain constants a_1^{ij}.

By the Gram-Schmidt orthogonalization process, there is an orthonormal sequence $\{g_1, g_2,\ldots\}$ such that

$$g_j = \sum_{l=1}^{j} b^{lj} f_l, \quad b^{jj} \neq 0,$$

$$f_j = \sum_{l=1}^{j} b_1^{lj} g_l, \quad b_1^{jj} = (b^{jj})^{-1}.$$

Therefore, (12.27) implies

$$Tg_j = \sum_{l=1}^{j} b^{lj}[\lambda_l f_l + \sum_{k=1}^{l-1} a_1^{kl} f_k]$$

$$= b^{jj}\lambda_j f_j + \sum_{l=1}^{j-1} a_2^{lj} f_l$$

$$= b^{jj}\lambda_j [b_1^{jj} g_j + \sum_{l=1}^{j-1} b_1^{lj} g_l] + \sum_{l=1}^{j-1} a_2^{lj} \sum_{k=1}^{l} b_1^{kl} g_k$$

$$= \lambda_j g_j + \sum_{l=1}^{j-1} a_3^{lj} g_l,$$

for certain constants a_3^{lj}. The orthonormality of the sequence $\{g_1, g_2, \ldots\}$ implies

$$||Tg_j||^2 = |\lambda_j|^2 + \sum_{l=1}^{j-1} |a_3^{lj}|^2$$

$$\geq |\lambda_j|^2.$$

Summing over all j, we obtain

$$|||T|||^2 \geq \Sigma_j ||Tg_j||^2 \geq \Sigma_j |\lambda_j|^2.$$

Q.E.D.

To finish the main results of this section, an important formula is needed, which will be given in Theorem 12.17. But first a lemma is needed. Recall that a *projection* on a linear space is a linear transformation P satisfying the equation $P^2 = P$.

LEMMA 12.15. *Let P and Q be continuous non-zero projections on a Hilbert space X, such that $||P - Q|| < ||P||^{-1}$. Then the dimension of $R(P)$ is not greater than the dimension of $R(Q)$.*

Proof. Suppose that $R(P)$ has larger dimension than $R(Q)$. Now $R(Q)$ has dimension at least that of $R(PQ)$, so it follows that $R(PQ)$ is a proper subspace of $R(P)$. Since $R(P)$ and $R(PQ)$ are both closed subspaces, there is a unit vector f in $R(P)$ which is orthogonal to $R(PQ)$. In particular, f is orthogonal to PQf, so that

$$(12.28) \qquad ||f - PQf||^2 = 1 + ||PQf|^2 \geq 1.$$

But since $f \in R(P)$, it follows that $f = P^2 f$, and we have from (12.28)

$$1 \leq ||P(Pf - Qf)|| \leq ||P||\ ||(P - Q)f||$$

$$\leq ||P||\ ||P - Q||.$$

This contradicts the hypothesis. Therefore, $R(P)$ has dimension not greater than that of $R(Q)$. Q.E.D.

As a corollary we have

LEMMA 12.16. *Let P be a non-zero projection on a Hilbert space, and let $\{P_k\}$ be a sequence of projections such that $\lim_{k\to\infty} ||P_k - P|| = 0$. Then for all sufficiently large k, the ranges of P_k have the same dimension as the range of P.*

Proof. For sufficiently large k we have $||P_k|| < 2||P||$ and also $||P_k - P|| < \frac{1}{2}||P||^{-1}$. Therefore also $||P_k - P|| < ||P_k||^{-1}$ and the result follows immediately from the lemma. Q.E.D.

THEOREM 12.17. *Let T be a transformation having finite double-norm on a Hilbert space X, and let $\{\lambda_j\}$ be the sequence of characteristic values of T, each repeated a number of times equal to its multiplicity. Then for all $\lambda \in \rho_m(T)$, the modified resolvent T_λ has finite double norm, and*

$$(12.29) \qquad Tr(TT_\lambda) = \lambda^{-1}\Sigma_j \left(\frac{1}{\lambda_j - \lambda} - \frac{1}{\lambda_j}\right) + tr(T^2) - \Sigma_j \frac{1}{\lambda_j^2}.$$

Proof. First note that since $T_\lambda = T(1 - \lambda T)^{-1}$, Theorem 12.10 implies that T_λ has finite double-norm. Let $\{\phi_1, \phi_2, \ldots\}$ be an orthonormal basis in X, and let $t^{ij} = (T\phi_j, \phi_i)$. Let T_N be the transformation defined by (12.18). As shown in the proof of Theorem 12.11, $|||T - T_N||| \to 0$.

Now T_N is essentially a transformation on a Hilbert space of dimension N, and so T_N has characteristic values $\lambda_{j,N}$, of which there are at most N, counted according to their multiplicity. Moreover, using a triangular representation of the matrix of T_N, the formula (12.29) for T_N is obvious; thus,

$$\lambda tr(T_N(T_N)_\lambda) = \Sigma_j \left(\frac{1}{\lambda_{j,N} - \lambda} - \frac{1}{\lambda_{j,N}}\right). \tag{12.30}$$

Note that in this finite-dimensional case, $tr(T_N^2) = \Sigma_j \lambda_{j,N}^{-2}$.

Since the trace function is linear, we have

$$tr(TT_\lambda) - tr(T_N(T_N)_\lambda) = tr[(T - T_N)T_\lambda] + tr[T_N(T_\lambda - (T_N)_\lambda)]$$

Applying (12.19), we obtain

$$|tr(TT_\lambda) - tr(T_N(T_N)_\lambda)| \leq |||T - T_N|||\ |||T_\lambda||| + |||T_N|||\ |||T_\lambda - (T_N)_\lambda|||. \tag{12.31}$$

Now

$$\begin{aligned} T_\lambda - (T_N)_\lambda &= T(1 - \lambda T)^{-1} - T_N(1 - \lambda T_N)^{-1} \\ &= (T - T_N)(1 - \lambda T)^{-1} + T_N[(1 - \lambda T)^{-1} - (1 - \lambda T_N)^{-1}] \\ &= (T - T_N)(1 - \lambda T)^{-1} + T_N(1 - \lambda T)^{-1}[1 - \lambda T_N - (1 - \lambda T)] \cdot \\ &\qquad \cdot (1 - \lambda T_N)^{-1} \\ &= (T - T_N)(1 - \lambda T)^{-1} + \lambda T_N(1 - \lambda T)^{-1}(T - T_N)(1 - \lambda T_N)^{-} \end{aligned}$$

Therefore, Theorem 12.10 implies

$$|||T_\lambda - (T_N)_\lambda||| \le [||(1-\lambda T)^{-1}|| + |\lambda|\,|||T_N|||\,||(1-\lambda T)^{-1}|| \cdot$$

$$\cdot\, ||(1-\lambda T_N)^{-1}||]\,|||T - T_N|||.$$

Therefore, by (12.3), $|||T_\lambda - (T_N)_\lambda||| \to 0$ uniformly for λ bounded and a fixed positive distance away from characteristic values of T and T_N. Hence, (12.31) implies that $tr(T_N(T_N)_\lambda) \to tr(TT_\lambda)$, and (12.30) implies

$$(12.32) \qquad \lambda tr(TT_\lambda) = \lim_{N\to\infty} \Sigma_j \left(\frac{1}{\lambda_{j,N} - \lambda} - \frac{1}{\lambda_{j,N}}\right),$$

the limit being uniform for λ bounded and a fixed positive distance from $\{\lambda_j\}$ and $\{\lambda_{j,N}\}$.

Now some results of operational calculus are needed. See, for instance, Taylor, *Introduction to Functional Analysis*, Wiley, 1958, pp. 287-307. Let λ_j be fixed. Since the characteristic values of T are isolated, a circle of sufficiently small radius 2ϵ with center λ_j^{-1} contains only the eigenvalue λ_j^{-1} of T. The result of operational calculus we shall need is that the operator

$$(12.33) \quad \frac{1}{2\pi i}\int_{|\mu-\lambda_j^{-1}|=\epsilon} (\mu - T)^{-1} d\mu$$

is a continuous projection onto the generalized eigenspace $M(\lambda_j)$. Likewise, the operator

$$(12.34) \qquad \frac{1}{2\pi i}\int_{|\mu-\lambda_j^{-1}|=\epsilon} (\mu - T_N)^{-1} d\mu$$

is a continuous projection onto the direct sum of the generalized eigenspaces of T_N corresponding to eigenvalues of T_N within a distance of ϵ to λ_j^{-1}. Now since $(\mu - T)^{-1}$ exists for $|\mu - \lambda_j^{-1}| = \epsilon$, also, for all sufficiently large N, $(\mu - T_N)^{-1}$ exists for $|\mu - \lambda_j^{-1}| = \epsilon$. Moreover, $(\mu - T_N)^{-1}$ converges uniformly to $(\mu - T)^{-1}$ on the circle

$|\mu - \lambda_j^{-1}| = \epsilon$. All this follows since $||T_N - T|| \to 0$.

But this implies that the projection (12.34) converges uniformly to the projection (12.33). By the corollary to Lemma 12.15, it follows that the total multiplicity of all eigenvalues of T_N within ϵ of λ_j^{-1} is the same as the multiplicity m of λ_j^{-1}. Moreover, by taking ϵ smaller and smaller, it follows that among the eigenvalues of T_N there are precisely m (counted according to multiplicity) which converge to λ_j^{-1}. Therefore, the characteristic values $\{\lambda_{j,N}\}$ can be arranged in such a manner that $\lambda_{j,N} \to \lambda_j$, for all λ_j. In case T has only a finite number of characteristic values, the remaining characteristic values of T_N must tend to infinity. For, if they had a finite limit point, that limit point would necessarily be a characteristic val ue of T. This follows since if $(\mu - T)^{-1}$ exists, then also $(\lambda - T)^{-1}$ exists for $|\lambda - \mu| < 2\epsilon$ (some $\epsilon > 0$), whence also $(\lambda - T_N)^{-1}$ exists for $|\lambda - \mu| < \epsilon$, all N sufficiently large.

Now let

$$(12.35) \qquad F(\lambda) = \Sigma_j \left(\frac{1}{\lambda_j - \lambda} - \frac{1}{\lambda_j}\right) - \lambda tr(TT_\lambda).$$

Note that since

$$(12.36) \qquad \frac{1}{\lambda_j - \lambda} - \frac{1}{\lambda_j} = \frac{\lambda}{(\lambda_j - \lambda)\lambda_j},$$

and since

$$\Sigma_j \frac{1}{|\lambda_j|^2} \leq |||T|||^2$$

(cf. Theorem 12.14), it follows that the series in the definition of $F(\lambda)$ converges uniformly for λ in a compact subset of $\rho_m(T)$. By (12.32) and the previous discussion, we have for any integer n

$$F(\lambda) = \Sigma_{j \geq n} \left(\frac{1}{\lambda_j - \lambda} - \frac{1}{\lambda_j}\right) - \lim_{N \to \infty} \Sigma_{j \geq n} \left(\frac{1}{\lambda_{j,N} - \lambda} - \frac{1}{\lambda_{j,N}}\right),$$

the convergences being uniform in any compact subset of $\rho_m(T)$. Therefore, $F(\lambda)$ is an entire function of λ. We shall estimate $F(\lambda)$ for $|\lambda| \leq r$, taking n so large that $|\lambda_j| \geq 2r$, $|\lambda_{j,N}| \geq 2r$ for $j \geq n$. Thus, by (12.36)

$$|F(\lambda)| \leq \Sigma_{j\geq n} \frac{|\lambda|}{|\lambda_j - \lambda|\,|\lambda_j|} + \limsup_{N\to\infty} \Sigma_{j\geq n} \frac{|\lambda|}{|\lambda_{j,N} - \lambda|\,|\lambda_{j,N}|}$$

$$\leq \Sigma_{j\geq n} \frac{|\lambda|}{\frac{1}{2}|\lambda_j|^2} + \limsup_{N\to\infty} \Sigma_{j\geq n} \frac{|\lambda|}{\frac{1}{2}|\lambda_{j,N}|^2}$$

$$\leq 2|\lambda|(|||T|||^2 + \limsup_{N\to\infty} |||T_N|||^2)$$

$$= 4|||T|||^2|\lambda|,$$

by Theorem 12.14. Therefore, $F(\lambda)$ is an entire function satisfying $|F(\lambda)| \leq \text{const}\ |\lambda|$ for *all* λ. Hence, F is linear: $F(\lambda) = a\lambda + b$, some constants a, b. Letting $\lambda \to 0$ in (12.35), it follows that $b = 0$. Thus, dividing (12.35) by λ gives

$$\Sigma_j \frac{1}{(\lambda_j - \lambda)\lambda_j} - tr(TT_\lambda) = a$$

Letting $\lambda \to 0$ in this relation, we obtain

$$\Sigma_j \frac{1}{\lambda_j^2} - tr(T^2) = a.$$

Q.E.D.

Part 3. HILBERT-SCHMIDT KERNELS

We shall now give two results identifying the class of operators of finite double-norm in case the Hilbert space is $L_2(\Omega)$, for some measurable subset Ω of E_n.

THEOREM 12.18. *Let $K(x, y)$ be a measurable function on $\Omega \times \Omega$, square integrable on $\Omega \times \Omega$; i.e., $K \in L_2(\Omega \times \Omega)$. If $f \in L_2(\Omega)$, then*

$$\int_\Omega K(x, y)f(y)dy$$

converges absolutely for almost all x in Ω, and represents an element Tf in $L_2(\Omega)$. The transformation T is a linear transformation having finite double-norm on $L_2(\Omega)$, and

$$|||T|||^2 = \int_{\Omega\times\Omega} |K(x, y)|^2 dxdy. \tag{12.37}$$

Proof. Fubini's theorem shows that, for almost all x, the function $K(x, y)$ is measurable for $y \in \Omega$, and

$$\int_\Omega |K(x, y)|^2 dy < \infty.$$

Thereofre, $\int_\Omega K(x, y)f(y)dy$ exists as an absolutely convergent integral for such x, as a result of the Cauchy-Schwarz inequality; furthermore,

$$|(Tf)(x)|^2 = \left|\int_\Omega K(x, y)f(y)dy\right|^2$$

$$\leq \int_\Omega |K(x, y)|^2 dy\,||f||^2.$$

Integrating over Ω,

$$||Tf||^2 \leq \int_{\Omega\times\Omega} |K(x, y)|^2 dydx\,||f||^2$$

Thus, T is a bounded linear transformation on $L_2(\Omega)$.

To derive (12.37), let $\{\phi_1, \phi_2, \ldots\}$ be any orthonormal basis in $L_2(\Omega)$. Then for almost all x

$$(T\phi_i)(x) = \int_\Omega K(x, y)\phi_i(y)dy.$$

Squaring and summing on i,

$$(12.38) \quad \sum_{i=1}^{\infty} |(T\phi_i)(x)|^2 = \sum_{i=1}^{\infty} |\int_{\Omega} \overline{K(x, y)\phi_i(y)}\, dy|^2.$$

As remarked above, for fixed x the function $\overline{K(x, y)}$ is in $L_2(\Omega)$; and $\int_{\Omega} \overline{K(x, y)}\phi_i(y)dy$ is just the inner product of this function with $\phi_i(y)$. Therefore, Parseval's relation implies that (12.38) may be written for almost all x

$$\sum_{i=1}^{\infty} |(T\phi_i)(x)|^2 = \int_{\Omega} |\overline{K(x, y)}|^2 dy.$$

Integrating with respect to x,

$$\sum_{i=1}^{\infty} ||T\phi_i||^2 = \int_{\Omega\times\Omega} |K(x, y)|^2 dydx.$$

Q.E.D.

Definition 12.8. *An operator of the type given in Theorem 12.17 is said to be an* integral operator with Hilbert-Schmidt kernel $K(x, y)$.

We now give a converse to Theorem 12.18.

THEOREM 12.19. *Let T be a transformation having finite double-norm on the Hilbert space $L_2(\Omega)$. Then T is an integral operator with a Hilbert-Schmidt kernel.*

Proof. Let $\{\phi_1, \phi_2, \ldots\}$ be a fixed orthonormal basis in $L_2(\Omega)$. Let $t^{ji} = (T\phi_i, \phi_j)$, and let

$$K_N(x, y) = \sum_{i,j=1}^{N} t^{ij}\phi_i(x)\overline{\phi_j(y)}.$$

Then for $N_1 < N_2$,

$$\int_{\Omega\times\Omega} |K_{N_2}(x, y) - K_{N_1}(x, y)|^2 dxdy$$

$$= \int_{\Omega\times\Omega} \Big| \sum_{\substack{i,j\le N_2 \\ i>N_1 \text{ or } j>N_1}} t^{ij}\phi_i(x)\overline{\phi_j(y)}\Big|^2 dxdy$$

$$= \int_{\Omega} \sum t^{ij}\overline{t^{i'j'}}\phi_i(x)\overline{\phi_{i'}(x)\phi_j(y)}\phi_{j'}(y)dxdy$$

$$= \sum_{\substack{i,j\le N_2 \\ i>N_1 \text{ or } j>N_1}} |t^{ij}|^2$$

$$\le \sum_{i=N_1+1}^{\infty} \sum_{j=1}^{\infty} |t^{ij}|^2 + \sum_{j=N_1+1}^{\infty} \sum_{i=1}^{\infty} |t^{ij}|^2.$$

Since $\sum_{i,j=1}^{\infty} |t^{ij}|^2 = |||T|||^2 < \infty$, it follows that as $N_1, N_2 \to \infty$,

$$\int_{\Omega\times\Omega} |K_{N_2}(x, y) - K_{N_1}(x, y)|^2 dxdy \to 0.$$

As $L_2(\Omega \times \Omega)$ is complete, there exists $K(x, y) \in L_2(\Omega \times \Omega)$ such that

$$\int_{\Omega\times\Omega} |K_N(x, y) - K(x, y)|^2 dxdy \to 0.$$

Let T_1 be the integral operator associated with Hilbert-Schmidt kernel K; we then must show $T = T_1$.

To see this, note that

$$(T_1\phi_i, \phi_j) = \int_{\Omega\times\Omega} K(x, y)\phi_i(y)\overline{\phi_j(x)}dydx$$

$$= \lim_{N\to\infty} \int_{\Omega\times\Omega} K_N(x, y)\phi_i(y)\overline{\phi_j(x)}dydx$$

$$= \lim_{N\to\infty} \sum_{i',j'=1}^{N} t^{i'j'} \int_{\Omega} \phi_{i'}(x)\overline{\phi_j(x)}dx \cdot \int_{\Omega} \overline{\phi_{j'}(y)}\phi_i(y)dy$$

$$= t^{ji}$$

$$= (T\phi_i, \phi_j).$$

Thus, $T_1\phi_i = T\phi_i$ for all i. Since $\{\phi_1, \phi_2, \ldots\}$ is a basis, $T_1 = T$. Q.E.D.

To complete this section, we give two results showing how certain operations with integral operators can be carried out by performing corresponding operations on the kernels.

THEOREM 12.20. *Let T be an integral operator on $L_2(\Omega)$ having Hilbert-Schmidt kernel $K(x, y)$. Then the adjoint T^* is an integral operator on $L_2(\Omega)$ having the Hilbert-Schmidt kernel $K^*(x, y) = \overline{K(y, x)}$.*

Proof. If $f, g \in L_2(\Omega)$, then

$$\begin{aligned}
(Tf, g) &= \int_\Omega (Tf)(x)\overline{g(x)}dx \\
&= \int_{\Omega\times\Omega} K(x, y)f(y)\overline{g(x)}dxdy \\
&= \int_{\Omega\times\Omega} K(y, x)f(x)\overline{g(y)}dydx \\
&= \int_\Omega f(x)[\int_\Omega K(y, x)\overline{g(y)}dy]dx \\
&= (f, T^*g).
\end{aligned}$$

Since this holds for all $f \in L_2(\Omega)$, we have almost all $x \in \Omega$

$$(T^*g)(x) = \int_\Omega \overline{K(y, x)}g(y)dy.$$

Q.E.D.

THEOREM 12.21. *Let T_i be an integral operator on $L_2(\Omega)$ having Hilbert-Schmidt kernel $K_i(x, y)$, $i = 1, 2$. Then the integral*

$$K(x, y) = \int_\Omega K_1(x, z)K_2(z, y)dz \tag{12.39}$$

is absolutely convergent for almost all $(x, y) \in \Omega \times \Omega$, *and* $T_1 T_2$ *is an integral operator whose Hilbert-Schmidt kernel is* $K(x, y)$. *Moreover,* $K(x, x)$ *as defined by* (12.39) *exists for almost all* $x \in \Omega$, *and*

$$(12.40) \qquad tr(T_1 T_2) = \int_\Omega K(x, x)dx,$$

the integral being absolutely convergent.

Proof. We have seen that for almost all x, $K_1(x, z)$ is in $L_2(\Omega)$ as a function of z; likewise, for almost all y, $K_2(z, y)$ is in $L_2(\Omega)$ as a function of z. Thus, (12.39) is convergent for almost all $(x, y) \in \Omega \times \Omega$. Also, we see that $K(x, x)$ exists for almost all $x \in \Omega$. We next establish the measurability of $K(x, y)$.

Let $f, g \in L_2(\Omega)$. As we have seen $K_2(z, y)f(y)$ is an integrable function of y for almost all z. Moreover, $\int_\Omega |K_2(z, y)f(y)|dy$ is then in $L_2(\Omega)$ as a function of z. Therefore, it follows that for almost all x, $|K_1(x, z)|\int_\Omega |K_2(z, y)f(y)|dy$ is integrable, and that

$$\int_\Omega |K_1(x, z)|dz \int_\Omega |K_2(z, y)f(y)|dy$$

is in $L_2(\Omega)$. But then $g(x)$ times this function is integrable. Since the function $K_1(x, z)K_2(z, y)f(y)g(x)$ is measurable in $\Omega \times \Omega \times \Omega$, it is integrable over $\Omega \times \Omega \times \Omega$. But then Fubini's theorem shows that its integral with respect to z, which is $K(x, y)f(y)g(x)$, is measurable in $\Omega \times \Omega$. Taking f and g to be characteristic functions of measurable sets, it then follows that $K(x, y)$ is a measurable function on $\Omega \times \Omega$.

By the Cauchy-Schwarz inequality,

$$|K(x, y)|^2 \le \int_\Omega |K_1(x, z)|^2 dz \cdot \int_\Omega |K_2(z, y)|^2 dz.$$

Integrating this over $\Omega \times \Omega$, it is seen that $K \in L_2(\Omega \times \Omega)$.

It remains to identify K with the operator $T_1 T_2$ and to prove (12.40). Now for $f, g \in L_2(\Omega)$.

$$(T_1T_2f, g) = \int (T_1T_2f)(x)\overline{g(x)}dx$$

$$= \int [\int K_1(x, z)(T_2f)(z)dz]\overline{g(x)}dx$$

$$= \int [\int K_1(x, z)\{\int K_2(z, y)f(y)dy\}dz]\overline{g(x)}dx$$

$$= \int \{[\int K_1(x, z)K_2(z, y)dz]f(y)dy\}\overline{g(x)}dx,$$

since the integrand is integrable over $\Omega \times \Omega \times \Omega$. Thus,

$$(T_1T_2f, g) = \int [K(x, y)f(y)dy]\overline{g(x)}dx.$$

Since this holds for all g, we must have all almost all x

$$(T_1T_2f)(x) = \int K(x, y)f(y)dy.$$

Therefore, T_1T_2 has kernel K.

Finally, we must prove (12.40). First, we have by the Cauchy-Schwarz inequality

$$\int_\Omega |K(x, x)|dx \leq \int_\Omega \int_\Omega |K_1(x, y)|\ |K_2(y, x)|dydx$$

$$\leq [\int_{\Omega\times\Omega} |K_1(x, y)|^2 dydx]^{1/2}[\int_{\Omega\times\Omega} |K_2(y, x)^2 dydx]^{1/2}$$

$$= |||T_1|||\ |||T_2||| < \infty.$$

Thus, $K(x, x)$ is integrable over Ω. Moreover, by (12.20), if $\{\phi_1, \phi_2,\ldots\}$ is an orthonormal basis in $L_2(\Omega)$,

$$tr(T_1T_2) = \sum_{i,j=1}^{\infty} (T_2\phi_i, \phi_j)(T_1\phi_j, \phi_i)$$

$$(12.41) \qquad = \sum_{i,j=1}^{\infty} \int_{\Omega\times\Omega} K_2(x, y)\phi_i(y)\overline{\phi_j(x)}dxdy \cdot$$

$$\cdot \int_{\Omega\times\Omega} \overline{\overline{K_1(y, x)\phi_j(x)}\phi_i(y)dxdy}.$$

Since the functions $\overline{\phi_i(y)}\phi_j(x)$, $i, j = 1, 2, \ldots$, form a complete orthonormal sequence in $L_2(\Omega \times \Omega)$, (12.41) becomes, after applying Parseval's relation,

$$tr(T_1 T_2) = \int_{\Omega\times\Omega} K_2(x, y)K_1(y, x)dxdy$$

$$= \int_{\Omega} K(y, y)dy.$$

Q.E.D.

13. Spectral Theory of Abstract Operators

Before giving results for elliptic equations, we can give several abstract theorems in which no mention need be made of differential equations. In these abstract results and in the results of section 14, essential use will be made of Sobolev's inequality (Theorem 3.9) and Rellich's theorem (Theorem 3.8). In order that both these theorems be valid, it is sufficient to assume that Ω is a bounded open set having both the segment property and the ordinary cone property. Actually, as the proofs of these theorems show, it is enough to assume Ω is a finite union of disjoint open sets of such type. This is obvious in the case of Sobolev's theorem, since a finite union of open sets which have the ordinary cone property still has the ordinary cone property. In the case of Rellich's theorem, note that the only use of the segment property for Ω was in the application of the second inequality of Theorem 3.2. But it is obvious from the proof of Theorem 3.2 that the theorem is valid for the case that Ω is a finite disjoint union of bounded open sets having the segment property. We shall be somewhat more restrictive than this, since we shall also need the restricted cone property in Theorem 13.9. *We require everywhere in this section and the section following that Ω be a finite disjoint union of bounded open sets, each of which has the restricted cone property. We shall use this assumption throughout these sections without mentioning it explicity.*

Of the various constants appearing in estimates in this section, we wish to isolate the one appearing in the statement of Sobolev's inequality. First a lemma is needed, showing that Theorem 3.9 is really a sort of convexity result.

LEMMA 13.1. *Let $m > n/2$ and let $u \in H_m(\Omega)$. Then u can be modified on a set of measure zero so that $u \in C^0(\bar{\Omega})$. Moreover, there exists a constant γ, depending only on Ω and m, such that for all* $x \in \bar{\Omega}$

$$(13.1) \qquad |u(x)| \le \gamma(|u|_0^{1-(n/2m)}|u|_m^{n/2m} + |u|_0).$$

Proof. By Theorem 3.9 it is enough to prove the estimate. According to Theorem 3.9,

$$(13.2) \qquad |u(x)| \le \gamma_1 r^{-(m-\frac{1}{2}n)}(|u|_m + r^m|u|_0), \quad r \ge 1.$$

If $|u|_m \le |u|_0$, then, taking $r = 1$, we obtain from (13.2)

$$|u(x)| \le \gamma_1(|u|_0^{1-(n/2m)}|u|_m^{n/2m} + |u|_0).$$

If $|u|_m > |u|_0$, then take $r^m = |u|_m|u|_0^{-1}$, so that (13.2) implies

$$|u(x)| \le \gamma_1(|u|_m|u|_0^{-1})^{-1+(n/2m)}(|u|_m + |u|_m)$$

$$= 2\gamma_1|u|_0^{1-(n/2m)}|u|_m^{n/2m}$$

$$= 2\gamma_1(|u|_0^{1-(n/2m)}|u|_m^{n/2m} + |u|_0).$$

Q.E.D.

Remark. An easy argument based on Young's inequality shows also that (13.1) implies (13.2). The argument is the same as was given on page 73 to show that the inequality (7.2) implies (7.1).

As an immediate corollary we have

LEMMA 13.2. *Let $m > n/2$ and let $u \in H_m(\Omega)$. Then there exists a constant γ_S, depending only on Ω and m, such that, after modification of u on a set of measure zero,*

$$|u(x)| \le \gamma_S ||u||_m^{n/2m} ||u||_0^{1-(n/2m)}, \quad x \in \bar{\Omega}. \tag{13.3}$$

Definition 13.1. *The smallest constant* γ_S *such that* (13.3) *is valid for all* $u \in H_m(\Omega)$ *is the* Sobolev constant *of* Ω.

Likewise, we can prove a similar result concerning interpolation inequalities. This result will not actually be used until the next section.

LEMMA 13.3. *There exists a constant* γ, *depending only on* Ω *and* m, *such that for* $u \in H_m(\Omega)$ *and* $0 \le j \le m$,

$$|u|_{j,\Omega} \le \gamma[|u|_0^{1-(j/m)}|u|_m^{j/m} + |u|_0]$$

$$\le 2\gamma|u|_0^{1-(j/m)}||u||_m^{j/m}.$$

Proof. The proof follows from Theorem 3.3 by choosing the parameter ϵ appropriately; the procedure is just the same as given in the proof of Lemma 13.1. Q.E.D.

Now we turn to the consideration of linear transformations which will later play the role of inverses to differential operators.

LEMMA 13.4. *Let* T *be a bounded linear transformation on* $L_2(\Omega)$, *such that the range of* T *is contained in* $H_m(\Omega)$, *some* $m \ge 1$. *Then* T *is a bounded linear transformation of* $L_2(\Omega)$ *into* $H_m(\Omega)$.

Proof. By the closed graph theorem, the result will follow if we show that T is closed, considered as a mapping of $L_2(\Omega)$ into $H_m(\Omega)$. Suppose then that $\{u_k\} \subset L_2(\Omega)$, $u_k \to u$ in $L_2(\Omega)$, and $Tu_k \to v$ in $H_m(\Omega)$. Then, since T is continuous on $L_2(\Omega)$, $Tu_k \to Tu$ in $L_2(\Omega)$. But *a fortiori* $Tu_k \to v$ in $L_2(\Omega)$. Therefore $Tu = v$. Q.E.D.

Definition 13.2. *Let* T *be a bounded linear transformation on* $L_2(\Omega)$, *such that* $R(T) \subset H_m(\Omega)$. *For* $0 \le k \le m$ *let*

$$||T||_k = \sup_{f \in L_2(\Omega), ||f||_{0,\Omega}=1} ||Tf||_{k,\Omega};$$

$$|T|_k = \sup_{f \in L_2(\Omega), ||f||_{0,\Omega}=1} |Tf|_{k,\Omega}.$$

By Lemma 13.4 it follows that these norms all are finite; of course, $|T|_k$ is not a norm, but a semi-norm, if $k > 0$.

THEOREM 13.5. *Let T be a bounded linear transformation on $L_2(\Omega)$, such that $R(T) \subset H_m(\Omega)$, where $m > n/2$. Then T has finite double-norm, and the following inequalities hold:*

$$(13.4) \qquad |||T||| \leq \gamma|\Omega|^{1/2}(|T|_m^{n/2m}|T|_0^{1-(n/2m)} + |T|_0^{1-(n/2m)});$$

$$(13.5) \qquad |||T||| \leq \gamma_S|\Omega|^{1/2}||T||_m^{n/2m}||T||_0^{1-(n/2m)}.$$

Here $|\Omega|$ is the Lebesgue measure of Ω, γ_S is the Sobolev constant of Ω, and γ is the constant appearing in (13.1).

Proof. Let $\{\phi_1, \phi_2, \ldots\}$ be an orthonormal basis in $L_2(\Omega)$, and let $u_j = T\phi_j$, $j = 1, 2, \ldots$. By Lemma 13.1 it may be assumed that $u_j \in C^\infty(\Omega)$. Let $a_1, \ldots a_N$ be any complex numbers, and let $f = \sum_{j=1}^N a_j\phi_j$. Then, the continuity of the function u_j and of $u = Tf$ implies that

$$u(x) = \sum_{j=1}^N a_j u_j(x), \quad \text{all} \quad x \in \Omega.$$

By Lemma 13.1 we have for all $x \in \Omega$

$$|u(x)| \leq \gamma(|Tf|_0^{1-(n/2m)}|Tf|_m^{n/2m} + |Tf|_0)$$

$$\leq \gamma(|T|_0^{1-(n/2m)}|T|_m^{n/2m} + |T|_0)\,||f||_{0,\Omega}.$$

That is,

$$(13.6) \qquad |\sum_{j=1}^N a_j u_j(x)| \leq \gamma(|T|_0^{1-(n/2m)}|T|_m^{n/2m} + |T|_0)[\sum_{j=1}^N |a_j|^2]^{1/2}.$$

This inequality holds for all $x \in \Omega$ and for all complex numbers $a_1, \ldots, a_N$. If for fixed x we take $a_j = u_j(x)$, then we obtain from (13.6)

$$[\sum_{j=1}^{N} |u_j(x)|^2]^{1/2} \leq \gamma(|T|_0^{1-(n/2m)}|T|_m^{n/2m} + |T|_0).$$

Since this holds for all $x \in \Omega$, we may square and integrate over Ω to obtain

$$\sum_{j=1}^{N} ||T\phi_j||_{0,\Omega}^2 \leq \gamma^2|\Omega|(|T|_0^{1-(n/2m)}|T|_m^{n/2m} + |T|_0)^2.$$

Since this holds for all N, we have

$$\sum_{j=1}^{\infty} ||T\phi_j||_{0,\Omega}^2 \leq \gamma^2|\Omega|(|T|_0^{1-(n/2m)}|T|_m^{n/2m} + |T|_0)^2.$$

Thus, (13.4) is proved. The proof of (13.5) is made in the same manner, using (13.3) in place of (13.1). Q.E.D.

As an application of this result we have

THEOREM 13.6. *Let T be a bounded linear transformation on $L_2(\Omega)$, such that $R(T) \subset H_m(\Omega)$, where $m > n/2$. Let there exist at least one direction of minimal growth of the modified resolvent of T. Let $\{\lambda_j\}$ be the characteristic values of T, repeated according to multiplicity. Let $N(t)$ be the number of characteristic values of modulus less than or equal to t: $N(t) = \sum_{|\lambda_j| \leq t} 1$. Then $N(t) = 0(t^{n/m})$ as $t \to \infty$,*

and for all $\lambda \in \rho_m(T)$

$$(13.7) \qquad tr(TT_\lambda) = \Sigma_j \frac{1}{(\lambda_j - \lambda)\lambda_j}.$$

Indeed, if $||T_\lambda||_0 \leq C|\lambda|^{-1}$ for λ on the ray $\Xi(\theta, a)$, then

$$N(t) \leq 4\gamma_S^2|\Omega| \; ||T||_m^{n/m}(1 + C)^2 t^{n/m}, \quad t \geq a.$$

Proof. We have by (12.5)

$$(1 - \lambda T)^{-1} = 1 + \lambda T_\lambda, \quad \lambda \in \Xi(\theta, a).$$

Therefore,

$$||(1-\lambda T)^{-1}||_0 \le 1 + |\lambda|\ ||T_\lambda||_0 \le 1 + C, \quad \lambda \in \Xi(\theta, a).$$

Also,

$$(13.8) \quad ||T_\lambda||_m = ||T(1-\lambda T)^{-1}||_m \le ||T||_m ||(1-\lambda T)^{-1}||_0$$
$$\le ||T||_m (1 + C).$$

Therefore, Theorem 13.5 implies T_λ has finite double-norm, and we have an estimate

$$(13.9) \quad |||T_\lambda||| \le \gamma_S |\Omega|^{1/2} ||T_\lambda||_m^{n/2m} ||T_\lambda||_0^{1-(n/2m)}$$
$$\le \gamma_S |\Omega|^{1/2} ||T||_m^{n/2m} (1 + C)^{n/2m} C^{1-(n/2m)} |\lambda|^{-1+(n/2m)}$$

By Theorem 12.4, the characteristic values of T_λ are precisely $\dfrac{\lambda_j^{-1}}{1-\lambda_j^{-1}\lambda} = \dfrac{1}{\lambda_j - \lambda}$, and the multiplicity of $\dfrac{1}{\lambda_j - \lambda}$ is the same as the multiplicity of the characteristic value λ_j of T. Therefore, applying Theorem 12.14 to T_λ, we obtain the estimate valid for all $\lambda \in \rho_m(T)$

$$(13.10) \quad \Sigma_j \frac{1}{|\lambda_j - \lambda|^2} \le |||T_\lambda|||^2.$$

In case $|\lambda_j| \le t$ and $|\lambda| = t$, then $|\lambda_j - \lambda| \le 2t$. Therefore, if in (13.10) we sum only over those j for which $|\lambda_j| \le t$, we obtain for $|\lambda| = t$, $\lambda \in \Xi(\theta, a)$,

$$\sum_{|\lambda_j| \le t} \frac{1}{4t^2} \le |||T_\lambda|||^2$$
$$\le \gamma_S^2 |\Omega|\ ||T||_m^{n/m} (1 + C)^{n/m} C^{2-(n/m)} t^{-2+(n/m)}$$

taking (13.9) into account. Multiplying by $4t^2$, we obtain

$$N(t) \leq 4\gamma_S^2|\Omega| \; ||T||_m^{n/m}(1 + C)^{n/m}C^{2-(n/m)}t^{n/m}.$$

Finally, we must prove (13.7). By Theorem 12.17 we have

$$tr(TT_\lambda) = \Sigma_j \frac{1}{(\lambda_j - \lambda)\lambda_j} + c.$$

for all $\lambda \in \rho_m(T)$ and for a certain constant c. For $\lambda \in \Xi(\theta, a)$, we obtain from (12.19) and (13.9)

$$|tr(TT_\lambda)| \leq |||T||| \; |||T_\lambda|||$$

$$\leq \text{const } |\lambda|^{-1+(n/2m)}.$$

Thus, $tr(TT_\lambda) \to 0$ as $|\lambda| \to \infty$, $\lambda \in \Xi(\theta, a)$. But also $\Sigma_j \dfrac{1}{(\lambda_j - \lambda)\lambda_j} \to 0$ as $|\lambda| \to \infty$. Therefore, c must vanish. Q.E.D.

As a result of this estimate on $N(t)$, we can sometimes obtain lower bounds for individual characteristic values, as we now show.

Corollary. In addition to the hypothesis of Theorem 13. 6, suppose also that T_λ exists for λ on the ray $\Xi(\theta, 0)$, and that for such λ the estimate $||T_\lambda||_0 \leq C|\lambda|^{-1}$ is satisfied. If the characteristic values $\{\lambda_j\}$ of T are arranged according to increasing modulus, $|\lambda_1| \leq |\lambda_2| \leq \ldots$, then for $j = 1, 2, \ldots$,

$$|\lambda_j| \geq [4\gamma_S^2|\Omega|(1 + C)^2]^{-m/n}||T||_m^{-1}j^{m/n}.$$

Proof. This is an immediate consequence of the theorem, since $j \leq N(|\lambda_j|)$. Q.E.D.

In a sense, this is better than an asymptotic estimate, since the estimate for λ_j is valid for *all j*. However, it gives only a lower bound for $|\lambda_j|$. Under more assumptions we shall obtain asymptotic formulas for $N(t)$ in the next section.

Let us now give an example.

THEOREM 13.7. *Let $B[v, u]$ be a bilinear form over a closed subspace V of $H_m(\Omega)$, such that for certain positive constants K and c_0 and for all $u, v \in V$*

$$|B[v, u]| \leq K||v||_{m,\Omega}||u||_{m,\Omega},$$

$$\Re B[u, u] \geq c_0||u||^2_{m,\Omega}.$$

Let $m > n/2$. Let T be the unique linear transformation of $L_2(\Omega)$ into V satisfying

$$B[v, Tf] = (v, f)_{0,\Omega} \tag{13.11}$$

for all $f \in L_2(\Omega)$, $v \in V$. Then the half plane $\{\lambda: \Re\lambda < 0\}$ is contained in $\rho_m(T)$, and the negative real axis is a direction of minimal growth of the modified resolvent of T. If the characteristic values $\{\lambda_j\}$ of T are arranged in order of increasing modulus, then

$$|\lambda_j| \geq \frac{c_0}{(16\gamma_S^2|\Omega|)^{m/n}} j^{m/n}.$$

Proof. The existence of T is a consequence of the Lax-Milgram theorem (Theorem 8.1). If in (13.11) we substitute $v = Tf$, we obtain

$$\begin{aligned} \Re(Tf, f) &= \Re B[Tf, Tf] \\ &\geq c_0||Tf||^2_{m,\Omega} \\ &\geq 0. \end{aligned} \tag{13.12}$$

Since T is bounded as a mapping of $L_2(\Omega)$ into $L_2(\Omega)$, it follows that for sufficiently large $|\lambda|$, λ is in the resolvent set of T. Therefore, Theorem 12.8 shows that each λ with negative real part is in the resolvent set of T. Therefore, $(1 - \lambda T)^{-1}$ exists for all λ such that $R\lambda < 0$, since $1 - \lambda T = \lambda(\lambda^{-1} - T)$, and $\Re\lambda^{-1} < 0$. To obtain the estimate, note that if $u = T_\lambda f = T(1 - \lambda T)^{-1}f$, then, since T and $(1 - \lambda T)^{-1}$ commute, $(1 - \lambda T)u = Tf$, so that $u = T(\lambda u + f)$. Therefore, (13.11) implies that for all $v \in V$.

$$B[v, u] = (v, \lambda u + f);$$

i.e.,

$$B[v, T_\lambda f] = (v, \lambda T_\lambda f + f), \quad \Re\lambda < 0.$$

Now, if we set $v = T_\lambda f$, then

$$B[T_\lambda f, T_\lambda f] = \bar{\lambda}||T_\lambda f||_0^2 + (T_\lambda f, f).$$

Therefore,

$$\begin{aligned} c_0||T_\lambda f||_0^2 &\leq \Re B[T_\lambda f, T_\lambda f] \\ &= \Re\lambda||T_\lambda f||_0^2 + \Re(T_\lambda f, f) \\ &\leq \Re\lambda||T_\lambda f||_0^2 + ||T_\lambda f||_0||f||_0. \end{aligned}$$

Dividing by $||T_\lambda f||_0$, we obtain

$$||T_\lambda||_0 \leq \frac{1}{c_0 - \Re\lambda}, \Re\lambda < 0.$$

Therefore, if λ is a negative real number,

$$||T_\lambda||_0 \leq \frac{1}{c_0 - \lambda} \leq \frac{1}{|\lambda|} \ .$$

Thus, it is seen that the hypothesis of the corollary to Theorem 13.6 is satisfied with $C = 1$. Since it follows from (13.12) that $||T||_m \leq c_0^{-1}$, the estimate for $|\lambda_j|$ is a consequence of the estimate in the corollary. Q.E.D.

In case the operator T in Theorem 13.5 is self-adjoint, the result can be improved to obtain the "correct" order of growth of λ_j. We have *Theorem 13.7. bis. Suppose that the conditions of Theorem 13.7 hold and that in addition the bilinear form $B[u, v]$ is Hermitian symmetric so that the operator T defined by* (13.11) *is a self-adjoint compact operator. If the characteristic values $\{\lambda_j\}$ of T (all of which are positive) are arranged in an increasing order, then*

$$\lambda_j \geq \frac{c_0}{(16\gamma_S^2|\Omega|)^{2m/n}} j^{2m/n}, \quad j = 1, 2, \ldots.$$

Proof. From (13.12) we have $(Tf, f) \geq 0$ for all f so that T is a *positive* compact self-adjoint operator. We now use a well-known result that a positive self-adjoint operator admits a positive self-adjoint square root. More precisely, since T is compact there exists a (unique) self-adjoint positive and compact operator S such that $S^2 = T$. Also the sequence of characteristic values of S is given by the sequence $\{\lambda_j^{1/2}\}$ (positive square root). We observe that since $Tf \neq 0$ for $f \neq 0$ (an immediate consequence of (13.11)) also $Sf \neq 0$ for $f \neq 0$. From this it follows that the range of S is dense in $L_2(\Omega)$. Using now (13.11) with $v = Tf$ we have:

$$B[Tf, Tf] = (Tf, f) = (S^2f, f) = ||Sf||_0^2.$$

Putting $Sf = g$, we then get

$$c_0||Sg||_m^2 = c_0||Tf||_m^2 \leq B[Tf, Tf] = ||g||_0^2.$$

Since by the previous remark elements $g = Sf$ are dense in $L_2(\Omega)$ it follows from the last inequality that the range of S is contained in $H_m(\Omega)$ and that

$$||S||_m \leq c_0^{-1/2}.$$

Consider now the modified resolvent S_λ along the positive imaginary axis. By (12.14) we have: $||S_\lambda||_0 \leq |\lambda|^{-1}$. Thus it is seen that the hypothesis of the corollary to Theorem 13.6 (for S_λ) is satisfied with $C = 1$. Using this and the estimate for $||S||_m$, recalling that $\{\lambda_j^{1/2}\}$ is the sequence of characteristic values of S, we obtain our theorem as a consequence of the estimate in the corollary.

In Theorem 13.5 it is reasonable to expect that if m is much bigger than $n/2$, then more can be expected of the transformation T. This is in fact the case, as will now be shown. First, a lemma is needed.

LEMMA 13.8. *The open set $\Omega \times \Omega$ is a finite union of disjoint open sets, each of which has the restricted cone property.*

Proof. Since Ω is a finite union of disjoint open sets having the restricted cone property, it is obviously sufficient to verify that the Cartesian product of two cones contains a cone. By shrinking the cones, we may assume that the two cones in question have the same height and same opening. Thus, let ξ_0 and η_0 be unit vectors in E_n, let $0 < \theta < \pi/2$, let $h > 0$, and let

$$C_1 = \{\xi:\ \xi \cdot \xi_0 \geq |\xi| \cos \theta,\ |\xi| \leq h\},$$

$$C_2 = \{\eta:\ \eta \cdot \eta_0 \geq |\eta| \cos \theta,\ |\eta| \leq h\}.$$

Vectors in $E_n \times E_n$ will be denoted (ξ, η). Choose the angle θ_1 such that $\cos 2\theta_1 = \cos^2\theta$, $0 < \theta_1 < \pi/2$, let v_0 be the unit vector $v_0 = 2^{-1/2}(\xi_0, \eta_0)$, and let C be the cone

$$C = \{(\xi, \eta):\ (\xi, \eta) \cdot v_0 \geq |(\xi, \eta)| \cos \theta_1,\ |(\xi, \eta)| \leq h\}.$$

We shall show that $C \subset C_1 \times C_2$ and the proof will then be complete. So let $(\xi, \eta) \in C$; then

$$(\xi \cdot \xi_1 + \eta \cdot \eta_0)2^{-1/2} \geq (|\xi|^2 + |\eta|^2)^{1/2} \cos \theta_1.$$

If $\xi \neq 0$, then, using the Cauchy-Schwarz inequality,

$$\frac{\xi}{|\xi|} \cdot \xi_0 \geq \left(1 + \frac{|\eta|^2}{|\xi|^2}\right)^{1/2} \sqrt{2} \cos \theta_1 - \frac{|\eta|}{|\xi|}.$$

If a is any real number greater than one, then

$$(1 + t^2)^{1/2} a - t \geq \sqrt{a^2 - 1}, \quad 0 \leq t \leq \infty.$$

Applying this with $a = \sqrt{2} \cos \theta_1$ and $t = |\eta|/|\xi|$, we obtain

$$\frac{\xi}{|\xi|} \cdot \xi_0 \geq \sqrt{2 \cos^2 \theta_1 - 1} = \sqrt{\cos 2\theta_1} = \cos \theta,$$

so that $\xi \cdot \xi_0 \geq |\xi| \cos \theta$. This relation obviously holds also if

$\xi = 0$. Moreover, $|(\xi, \eta)| \leq h$ obviously implies $|\xi| \leq h$. Therefore, $(\xi, \eta) \in C \Rightarrow \xi \in C_1$. Likewise, $(\xi, \eta) \in C \Rightarrow \eta \in C_2$, so that $C \subset C_1 \times C_2$. Q.E.D.

THEOREM 13.9. *Let T be a bounded linear transformation on $L_2(\Omega)$, such that $R(T) \subset H_m(\Omega)$, where $m > n$ if n is odd, $m > n + 1$ if n is even (equivalently, $m > 2[n/2] + 1$). Suppose also that $R(T^*) \subset H_m(\Omega)$. Let $\ell = m - [n/2] - 1$. Then T is an integral operator with Hilbert-Schmidt kernel $K(x, y)$. The function K belongs to $H_\ell(\Omega \times \Omega)$. Moreover, the trace of K on the diagonal of $\Omega \times \Omega$ exists in the sense of Theorem 3.10, and*

$$(13.13) \quad [\int_\Omega |K(x, x)|^2 dx]^{1/2} \leq \gamma[(|T|_m^{n/m} + |T^*|_m^{n/m})|T|_0^{1-(n/m)} + |T|_0],$$

where γ is a constant depending only on Ω and m.

Remark: We state here without proof that under the conditions of the theorem the conclusion (13.13) could be replaced by a much stronger result. Namely, it could be shown that $K(x, y)$ is actually continuous and bounded in $\Omega \times \Omega$ and that

$$(13.13)' \quad |K(x, y)| \leq \gamma[(|T|_m^{n/m} + |T^*|_m^{n/m}) \, |T|_0^{1-(n/m)} + |T|_0].$$

where γ is a constant depending only on Ω and m. Also for the continuity of K and for the last estimate to hold it is enough to assume that Ω possesses the *ordinary* cone property and that $m > n$ (also for n even). On the other hand from the proof of the theorem that follows one can deduce easily (using Sobolev's inequality and a dimensional argument) that K is continuous and satisfies (13.13)' if $\ell > n$.

Proof. It follows from Theorem 13.5 that T has finite double-norm. From Theorem 12.19 it is then seen that T is an integral operator with a Hilbert-Schmidt kernel $K(x, y)$. It remains to verify that $K \in H_\ell(\Omega \times \Omega)$ and that the estimate (13.13) holds.

Let

$$r = \max \{|T|_m^{1/m}|T|_0^{-1/m}, \ |T^*|_m^{1/m}|T|_0^{-1/m}, \ 1\}.$$

Then we have

$$(13.14)\qquad |T|_m \le r^m |T|_0,$$

$$|T^*|_m \le r^m |T|_0, \qquad (\text{note: } |T|_0 = |T^*|_0)$$

$$1 \le r.$$

By Theorem 12.20, T^* has the Hilbert-Schmidt kernel $K^*(x, y) = \overline{K(y, x)}$.

Now consider the operator T^α which maps $f \in L_2(\Omega)$ into $D^\alpha Tf$. If $|\alpha| \le \ell$, then since $Tf \in H_m(\Omega)$, it follows that $D^\alpha Tf$ is in $H_{m-|\alpha|}(\Omega)$, and $m - |\alpha| \ge m - \ell = [n/2] + 1 > n/2$. Therefore, T^α is a bounded linear transformation of $L_2(\Omega)$ into $H_{m-\ell}(\Omega)$, since T is bounded as a transformation of $L_2(\Omega)$ into $H_m(\Omega)$; cf. Lemma 13·4. Thus, we may apply Theorem 13.5 to T^α, if $|\alpha| \le \ell$. It follows then that T^α is an integral operator with a Hilbert-Schmidt kernel $K^\alpha(x, y)$.

We could certainly estimate $|||T^\alpha|||$ by using (13.4) and (13.5), but we shall find it better to repeat the proof of Theorem 13.5 in the present situation. Suppose then that $\{\phi_1, \phi_2, \ldots\}$ is an orthonormal basis in $L_2(\Omega)$, and let $u_j = T\phi_j$.

Let $f = \sum_{j=1}^N a_j\phi_j$, $u = Tf = \sum_{j=1}^N a_j u_j$, where $a_1, \ldots, a_N$ are complex numbers. By Sobolev's inequality it follows that, after modification on a set of measure zero, $u_j \in C^\ell(\Omega)$ and $u \in C^\ell(\Omega)$. Therefore, if α is a fixed index with $|\alpha| \le \ell$, we have for all $x \in \Omega$

$$u(x) = \sum_{j=1}^N a_j u_j(x),$$

$$(13.15)\qquad D^\alpha u(x) = \sum_{j=1}^N a_j D^\alpha u_j(x).$$

By Sobolev's inequality we also have the estimate for all $x \in \Omega$,

$$|D^\alpha u(x)| \le \gamma r^{-(m-\frac{1}{2}n-|\alpha|)}(|u|_{m,\Omega} + r^m |u|_{0,\Omega}),$$

since $r \ge 1$. But by Definition 13.2 this inequality implies since $u = Tf$

$$|D^{\alpha}u(x)| \leq \gamma r^{-(m-\frac{1}{2}n-|\alpha|)}(|T|_m + r^m|T|_0)||f||_0.$$

By definition of f and by (13.15), this may be written

$$\left|\sum_{j=1}^{N} a_j D^{\alpha}u_j(x)\right| \leq \gamma r^{-(m-\frac{1}{2}n-|\alpha|)}(|T|_m + r^m|T|_0)\left(\sum_{j=1}^{N} |a_j|^2\right)^{\frac{1}{2}}.$$

This inequality is valid for all constants $a_1,\dots,a_N$, and for all $x \in \Omega$. Choosing $a_j = \overline{D^{\alpha}u_j(x)}$, we obtain

$$\left[\sum_{j=1}^{N} |D^{\alpha}u_j(x)|^2\right]^{\frac{1}{2}} \leq \gamma r^{-(m-\frac{1}{2}n-|\alpha|)}(|T|_m + r^m|T|_0).$$

After squaring and integrating over Ω, and using the relation $D^{\alpha}u_j = D^{\alpha}T\phi_j = T^{\alpha}\phi_j$, this becomes

$$\sum_{j=1}^{N} ||T^{\alpha}\phi_j||_0^2 \leq \gamma^2|\Omega|r^{-2(m-\frac{1}{2}n-|\alpha|)}(|T|_m + r^m|T|_0)^2.$$

Therefore, as this holds for all positive integers N, we find

$$|||T^{\alpha}||| \leq \gamma|\Omega|^{\frac{1}{2}}r^{-(m-\frac{1}{2}n-|\alpha|)}(|T|_m + r^m|T|_0).$$

By (13.14) this implies

$$(13.16) \qquad |||D^{\alpha}T||| = |||T^{\alpha}|| \leq 2\gamma|\Omega|^{\frac{1}{2}}r^{\frac{1}{2}n+|\alpha|}|T|_0.$$

Since $T^{\alpha} = D^{\alpha}T$, it should be expected that the kernel $K^{\alpha}(x, y)$ of T^{α} is given by the formula $K^{\alpha}(x, y) = D_x^{\alpha}K(x, y)$. This is indeed true, if this derivative is interpreted as a weak derivative in $L_2(\Omega \times \Omega)$, as we now show.

To see this, take first for a test function in $\Omega \times \Omega$ any function $\phi(x, y) = a(x)b(y)$, where $a \in C_0^{\infty}(\Omega)$, $b \in C_0^{\infty}(\Omega)$. Then

$$\int_{\Omega\times\Omega} K^{\alpha}(x, y)\phi(x, y)dxdy = \int_{\Omega} a(x)dx \int_{\Omega} K^{\alpha}(x, y)b(y)dy.$$

Since K^α is the kernel corresponding to $T^\alpha = D^\alpha T$, it follows that for almost all $x \in \Omega$

$$\int_\Omega K^\alpha(x, y)b(y)dy = (D^\alpha Tb)(x).$$

Therefore,

$$\int_{\Omega\times\Omega} K^\alpha \phi dxdy = \int_\Omega aD^\alpha(Tb)dx$$

$$= (-1)^{|\alpha|}\int_\Omega D^\alpha a \cdot Tbdx.$$

But T has the kernel $K(x, y)$, so this relation can be expressed

$$\int_{\Omega\times\Omega} K^\alpha \phi dxdy = (-1)^{|\alpha|} \int_\Omega D^\alpha a(x)dx \int_\Omega K(x, y)b(y)dy$$

$$= (-1)^{|\alpha|} \int_{\Omega\times\Omega} K(x, y)D_x^\alpha(a(x)b(y))dxdy$$

$$= (-1)^{|\alpha|} \int_{\Omega\times\Omega} KD_x^\alpha \phi dxdy.$$

Therefore, it follows immediately that

$$(13.17) \qquad \int_{\Omega\times\Omega} K^\alpha \phi dxdy = (-1)^{|\alpha|} \int_{\Omega\times\Omega} KD_x^\alpha \phi dxdy$$

for any test function ϕ of the form

$$(13.18) \qquad \phi(x, y) = \sum_{i=1}^{N} a_i(x)b_i(y),$$

where $a_i, b_i \in C_0^\infty(\Omega)$. One can then show that (13.17) holds for *all* test functions ϕ on $\Omega \times \Omega$. In order not to impair the continuity of the proof, this will be shown as the final stage of the proof.

Thus, it follows that $K(x, y)$ has all the weak derivatives $D_x^\alpha K(x, y)$, $|a| \leq \ell$. We also wish to show that $K(x, y)$ has all the weak derivatives $D_y^\alpha K(x, y)$, $|a| \leq \ell$. It is precisely for this point that we need the assumptions of the theorem on the adjoint T^* of T. For, we can obviously carry out all the preceding lines of the argument with T replaced everywhere by T^*. Since T^* has the kernel $K^*(x, y) = \overline{K(y, x)}$, the results already obtained show that $\overline{K(y, x)}$ has all the weak derivatives $D_x^\alpha \overline{K(y, x)}$, $|\alpha| \le \ell$, and we have an estimate analogous to (13.16),

$$(13.19) \quad |||D^\alpha T^*||| \le 2\gamma\, |\Omega|^{\frac{1}{2}} r^{\frac{1}{2}n+|\alpha|} |T|_0.$$

Here, the kernel of $D^\alpha T^*$ is precisely $D_x^\alpha \overline{K(y, x)}$. But this means that $K(x, y)$ has weak derivatives $D_y^\alpha K(x, y)$, $|\alpha| \le \ell$. Moreover, (13.16) and (13.19), when combined with the formula (12.37) for the double-norm of an integral operator, show that for $|\alpha| \le \ell$

$$||D_x^\alpha K||_{0,\Omega\times\Omega} \le 2\gamma |\Omega|^{\frac{1}{2}} r^{\frac{1}{2}n+|\alpha|} |T|_0,$$

$$||D_y^\alpha K||_{0,\Omega\times\Omega} \le 2\gamma |\Omega|^{\frac{1}{2}} r^{\frac{1}{2}n+|\alpha|} |T|_0.$$

Therefore, *a fortiori* all *pure* derivatives of $K(x, y)$ of order $\le \ell$ exist weakly and satisfy such estimates. But then since by Lemma 13·8, $\Omega \times \Omega$ is a disjoint union of open sets having the restricted cone property, and since the corollary to Theorem 11.10 obviously holds in such open sets, we obtain the result that $K \in H_\ell(\Omega \times \Omega)$, and that

$$(13.20) \quad |K|_{p,\Omega\times\Omega} \le \gamma_1 r^{\frac{1}{2}n+p} |T|_0, \quad p \le \ell.$$

Now we are in a position to apply the trace theorem (Theorem 3.10). In our case the basic open set is $\Omega \times \Omega$, of dimension $2n$, and we wish to consider the trace on the intersection of this set with the manifold $\Pi = \{(x, x)\colon\ x \in E_n\}$, of dimension n. Thus, since $K \in H_\ell(\Omega \times \Omega)$, in order to apply the theorem we need that $\ell > \frac{1}{2}(2n - n) = n/2$, or, that $m - [n/2] - 1 > n/2$, or, that $m > n/2 + [n/2] + 1$. But this is precisely the assumption made on m. Therefore, Theorem 3.10 applies. Thus, $K(x, x)$ is uniquely determined for $x \in \Omega$, and we have an estimate corresponding to (3.19). Since $d\sigma$, the Lebesgue measure on the diagonal of $E_n \times E_n$, in our case is just $2^{\frac{1}{2}n} dx$, (3.19) (with $\alpha = 0$) becomes in our case

$$[\int_\Omega |K(x, x)|^2 dx]^{1/2} \leq \gamma_2 r^{-(\ell - 1/2(2n-n))}(|K|_{\ell,\Omega\times\Omega} + r^\ell |K|_{0,\Omega\times\Omega})$$

$$\leq \gamma_3 r^{-(\ell - 1/2 n)}(r^{1/2 n+\ell}|T|_0 + r^\ell r^{1/2 n}|T|_0),$$

by (13.20). Therefore,

$$(13.21) \qquad [\int_\Omega |K(x, x)|^2 dx]^{1/2} \leq 2\gamma_3 r^n |T|_0.$$

To finish the proof we simply need to use the definition of r. If $r = 1$, then (13.13) is immediate. If $r > 1$, then r is equal to one of the numbers $|T|_m^{1/m}|T|_0^{-1/m}$, $|T^*|_m^{1/m}|T|_0^{-1/m}$; thus,

$$r^n \leq |T|_m^{n/m}|T|_0^{-n/m} + |T^*|_m^{n/m}|T|_0^{-n/m}.$$

Applying this to (13.21), it follows that (13.13) is valid also in this case.

Finally, we have to fulfil the promise made earlier to establish (13.17) for every $\phi \in C_0^\infty(\Omega \times \Omega)$. Suppose that ϕ is such a function. Then there exists an open set $\Omega' \subset\subset \Omega$ such that supp $(\phi) \subset \Omega' \times \Omega'$. Let $\zeta \in C_0^\infty(\Omega)$ be chosen such that $\zeta \equiv 1$ on Ω'. By rescaling, if necessary, we can obviously assume that $\Omega \times \Omega$ is contained in a cube $Q = \{(x, y)\colon\ |x_k| < 1/2,\ |y_k| < 1/2,\ k = 1,\dots,n\}$. Suppose $\phi(x, y)$ to be extended to Q by setting $\phi \equiv 0$ in $Q \sim$ supp (ϕ). Extend ϕ by periodicity to all of $E_n \times E_n$. Then ϕ is periodic and in $C^\infty(E_n \times E_n)$. Therefore, the Fourier expansion of ϕ converges uniformly to ϕ:

$$\phi(x, y) = \Sigma_{\xi,\eta} a_{\xi,\eta} e^{2\pi i(x\cdot\xi + y\cdot\eta)},$$

where $\Sigma_{\xi,\eta}$ is taken over all ξ, η with integral components. Also,

$$D_x^\alpha \phi(x, y) = \Sigma_{\xi,\eta}(2\pi i\xi)^\alpha a_{\xi,\eta} e^{2\pi i(x\cdot\xi + y\cdot\eta)},$$

Now consider the function

$$\phi_N(x, y) = \zeta(x)\zeta(y) \sum_{|\xi|+|\eta|\leq N} a_{\xi,\eta} e^{2\pi i(x\cdot\xi + y\cdot\eta)}$$

$$= \sum_{|\xi|+|\eta|\leq N} a_{\xi,\eta}\zeta(x)e^{2\pi i x\cdot\xi}\zeta(y)e^{2\pi i y\cdot\eta}.$$

By choice of ζ, $\zeta(x)e^{2\pi i x\cdot\xi}$ and $\zeta(y)e^{2\pi i y\cdot\eta}$ are each in $C_0^\infty(\Omega)$. Thus, $\phi_N(x, y)$ has the form (13.18), and it follows that (13.17) holds with ϕ replaced by ϕ_N:

$$\text{(13.22)}\quad \int_{\Omega\times\Omega} K^\alpha\phi_N dxdy = (-1)^{|\alpha|}\int_{\Omega\times\Omega} KD_x^\alpha\phi_N dxdy.$$

Now it follows from Leibnitz's rule that as $N \to \infty$ we have the following limits in $L_2(\Omega\times\Omega)$:

$$\phi_N \to \zeta(x)\zeta(y)\phi(x, y),$$

$$D_x^\alpha\phi_N \to D_x^\alpha(\zeta(x)\zeta(y)\phi(x, y)).$$

Therefore, (13.22) becomes

$$\int_{\Omega\times\Omega} K^\alpha\zeta(x)\zeta(y)\phi dxdy = (-1)^{|\alpha|}\int_{\Omega\times\Omega} KD_x^\alpha(\zeta(x)\zeta(y)\phi)dxdy;$$

or,

$$\text{(13.23)}\quad \int_{\text{supp}(\phi)} K^\alpha\zeta(x)\zeta(y)\phi dxdy = (-1)^{|\alpha|}\int_{\text{supp}(\phi)} KD_x^\alpha(\zeta(x)\zeta(y)\phi)\,dxdy.$$

If $(x, y) \in \text{supp}(\phi) \subset \Omega' \times \Omega'$, then on a neighborhood of (x, y), $\zeta(x) \equiv 1$ and $\zeta(y) \equiv 1$. Therefore, (13.23) is the same formula as (13.17). Q.E.D.

We shall later need an expression relating the dependence of the constant γ in (13.13) on the dimensions of Ω. More precisely, we have the

Corollary. Under the assumptions of Theorem 13.9, let γ be the constant in (13.13). *If $a > 0$ and if the assumptions of the theorem hold for the open set $a\Omega$ and an operator $T_{(a)}$ on $L_2(a\Omega)$, and if $K_{(a)}$*

is the kernel corresponding to $T_{(a)}$, *then*

$$a^{n/2}\Big[\int_{a\Omega}|K_{(a)}(x,x)|^2dx\Big]^{1/2} \le \gamma\Big[a^n\Big(|T_{(a)}|_m^{n/m} + |T^*_{(a)}|_m^{n/m}\Big)|T_{(a)}|_0^{1-(n/m)} + |T_{(a)}|_0\Big].$$

Proof. For a function $u(y)$ on Ω let

$$u'(x) = u(a^{-1}x), \quad x \in a\Omega;$$

thus,

$$u'(ax) = u(x), \quad x \in \Omega.$$

Clearly, $u \in H_k(\Omega) \Leftrightarrow u' \in H_k(a\Omega)$; and

$$|u'|_{k,a\Omega} = a^{\frac{1}{2}n-k}|u|_{k,\Omega}. \tag{13.24}$$

Now we shall define an operator on $L_2(\Omega)$ which corresponds to $T_{(a)}$. Precisely, for $f \in L_2(\Omega)$ let Tf be given by the expression

$$(Tf)' = T_{(a)}f'.$$

Since $R(T_{(a)}) \subset H_m(a\Omega)$, it follows that $Tf \in H_m(\Omega)$. Furthermore, it can be easily checked that

$$(T^*f)' = T^*_{(a)}f'.$$

Now if $k = 0$ or m, and if $f \in L_2(\Omega)$, we have from (13.24)

$$\frac{|Tf|_{k,\Omega}}{|f|_{0,\Omega}} = \frac{a^{k-\frac{1}{2}n}|(Tf)'|_{k,a\Omega}}{a^{-\frac{1}{2}n}|f'|_{0,a\Omega}}$$

$$= a^k\,\frac{|T_{(a)}f'|_{k,a\Omega}}{|f'|_{0,a\Omega}}.$$

Therefore, $|T|_k = a^k|T_{(a)}|_k$. Likewise, $|T^*|_k = a^k|T^*_{(a)}|_k$. Substituting these relations in (13.13), it follows that if $K(x, y)$ is the kernel of T,

(13.25)
$$[\int_\Omega |K(x, x)|^2 dx]^{1/2} \leq \gamma[a^n(|T_{(a)}|_m^{n/m} + T^*_{(a)}|_m^{n/m})|T_{(a)}|_0^{1-n/m} + |T_{(a)}|_0].$$

Finally, suppose $K_{(a)}(x, y)$ is the kernel of $T_{(a)}$. Then for $f \in L_2(\Omega)$ and $y = a\xi$

$$\begin{aligned}(Tf)'(x) &= (Tf)(a^{-1}x) \\ &= \int_\Omega K(a^{-1}x, \xi)f(\xi)d\xi \\ &= \int_\Omega K(a^{-1}x, \xi)f'(a\xi)d\xi \\ &= \int_{a\Omega} K(a^{-1}x, a^{-1}y)f'(y)a^{-n}dy \\ &= (T_{(a)}f')(x) \\ &= \int_{a\Omega} K_{(a)}(x, y)f'(y)dy.\end{aligned}$$

Therefore, $K_{(a)}(x, y) = a^{-n}K(a^{-1}x, a^{-1}y)$. Thus,

$$\begin{aligned}\int_{a\Omega} |K_{(a)}(x, x)|^2 dx &= \int_{a\Omega} a^{-2n}|K(a^{-1}x, a^{-1}x)|^2 dx \\ &= a^{-n}\int_\Omega |K(y, y)|^2 dy.\end{aligned}$$

Compare this with (13.25), and the result follows. Q.E.D.

THEOREM 13.10. *In addition to the hypothesis of Theorem 13.9, assume that there exists a direction $e^{i\theta}$ of minimal growth of the modified resolvent of T. Let $\{\lambda_j\}$ be the sequence of characteristic values of T, repeated according to multiplicity. Then for any λ which is not a characteristic value of T, the modified resolvent T_λ*

is an integral operator having Hilbert-Schmidt kernel $K_\lambda(x, y)$, and $K_\lambda \in H_\ell(\Omega \times \Omega)$. Also K_λ has a trace on the diagonal of $\Omega \times \Omega$, and

(13.26) $$[\int_\Omega |K_\lambda(x, x)|^2 dx]^{1/2} = 0(|\lambda|^{-1+(n/m)})$$

for $|\lambda| \to \infty$, $\arg \lambda = \theta$. Also, for λ not a characteristic value of T,

(13.27) $$\int_\Omega K_\lambda(x, x)dx = \Sigma_j \frac{1}{\lambda_j - \lambda} .$$

Proof. If $\lambda \in \rho_m(T)$, then $T_\lambda = T(1 - \lambda T)^{-1}$ satisfies the same conditions as T. For, obviously $R(T_\lambda) \subset R(T) \subset H_m(\Omega)$. Since T and $(1 - \lambda T)^{-1}$ commute, $R((T_\lambda)^*) = R(T^*(1 - \lambda T)^{-1*}) \subset R(T^*) \subset H_m(\Omega)$. Therefore, by Theorem 13.9, T_λ has a Hilbert-Schmidt kernel $K_\lambda \in H_\ell(\Omega \times \Omega)$, and we have the estimate

(13.28) $$[\int_\Omega |K_\lambda(x, x)|^2 dx]^{1/2} \le \gamma[(|T_\lambda|_m^{n/m} + |(T_\lambda)^*|_m^{n/m})|T_\lambda|_0^{1-(n/m)} + |T_\lambda|_0].$$

Now on a ray $\Xi(\theta, a)$ we have an estimate $|T_\lambda|_0 \le C|\lambda|^{-1}$. By (13.8), $||T_\lambda||_m \le ||T||_m(1 + C)$ for such λ. Also, $(T_\lambda)^* = T^*(1 - \lambda T)^{-1*}$, so that for λ on $\Xi(\theta, a)$

$$\begin{aligned} ||(T_\lambda)^*||_m &\le ||T^*||_m ||(1 - \lambda T)^{-1*}||_0 \\ &= ||T^*||_m ||(1 - \lambda T)^{-1}||_0 \\ &\le ||T^*||_m (1 + C), \end{aligned}$$

since $(1 - \lambda T)^{-1} = 1 + \lambda T_\lambda$. Thus, (13.28) implies for $\lambda \in \Xi(\theta, a)$

$$\begin{aligned} [\int_\Omega |K_\lambda(x, x)|^2 dx]^{1/2} &\le \gamma[(1 + C)^{n/m}(||T||_m^{n/m} + ||T^*||_m^{n/m}) \cdot C^{1-(n/m)} \\ &\qquad |\lambda|^{-1+(n/m)} + C|\lambda|^{-1}] \\ &\le \text{const } |\lambda|^{-1+(n/m)}. \end{aligned}$$

Thus, it remains to prove the formula (13.27).

For this, note first that the corollary to Theorem 13.6 implies that $|\lambda_j| \geq \text{const}\ j^{m/n}$ for sufficiently large j, since at most a finite number of characteristic values have modulus no greater than a. Since $m > n$, this implies that the series $\Sigma_j |\lambda_j|^{-1}$ converges. By Theorem 13.6, we then have for $\lambda \in \rho_m(T)$

$$tr(\lambda T T_\lambda) = \Sigma_j \frac{\lambda}{(\lambda_j - \lambda)\lambda_j}$$

$$= \Sigma_j \left(\frac{1}{\lambda_j - \lambda} - \frac{1}{\lambda_j}\right)$$

$$= \Sigma_j \frac{1}{\lambda_j - \lambda} - \Sigma_j \frac{1}{\lambda_j}. \tag{13.29}$$

Now

$$(1 - \lambda T)T_\lambda = (1 - \lambda T)T(1 - \lambda T)^{-1}$$

$$= (1 - \lambda T)(1 - \lambda T)^{-1} T$$

$$= T,$$

so that

$$\lambda T T_\lambda = T_\lambda - T. \tag{13.30}$$

By Theorem 13.9, there exists $\int_\Omega K(x, x)dx$. Since Theorem 12.21 implies that the trace of $\lambda T T_\lambda$ is the integral of its kernel over the diagonal of $\Omega \times \Omega$, (13.30) shows that

$$tr(\lambda T T_\lambda) = \int_\Omega [K_\lambda(x, x) - K(x, x)]dx.$$

Therefore, (13. 29) implies that for all $\lambda \in \rho_m(T)$

$$\int_\Omega K_\lambda(x, x)dx = \Sigma_j \frac{1}{\lambda_j - \lambda} + c,$$

where c is independent of λ. Finally, we show that $c = 0$.

First, the estimate (13.26) shows that $\int_\Omega K_\lambda(x, x)dx \to 0$ as $|\lambda| \to \infty$, $\arg \lambda = \theta$. Next, we show the same result for $\Sigma_j(\lambda_j - \lambda)^{-1}$. By Theorem 12.6 there are no characteristic values of T in an entire angle about $\Xi(\theta, a)$; thus, if $|\lambda_j| > a$, then $|\arg \lambda_j - \theta| > \delta$ for some positive number δ. Now, if $z = re^{i\theta}$ and $w = se^{i\phi}$ are the polar forms of two complex numbers, and if $\delta \le |\theta - \phi| \le \frac{1}{2}\pi$, then $|z - w|^2 = r^2 - 2rs \cos(\theta - \phi) + s^2 = r^2 \sin^2(\theta - \phi) + [r \cos(\theta - \phi) - s]^2 \ge r^2 \sin^2 \delta$, or, $|z - w| \ge |z| \sin \delta$. If $\frac{1}{2}\pi \le |\theta - \phi| \le \pi$, then $|z - w|^2 \ge r^2 + s^2 \ge r^2$, or $|z - w| \ge |z|$. Thus in any case $|z - w| \ge |z| \sin \delta$ if $\delta \le |\theta - \phi| \le \pi$. Applying this inequality to λ_j and λ, we have

$$(13.31) \qquad |\lambda_j - \lambda| \ge \max(|\lambda_j|, |\lambda|) \sin \delta.$$

Therefore, if λ_j is outside this angle for $j \ge j_0$, then for $\lambda \in \Xi(\theta, a)$

$$\left|\Sigma_j \frac{1}{\lambda_j - \lambda}\right| \le \frac{\text{const}}{|\lambda|} + \csc \delta \sum_{j \ge j_0} \frac{1}{\max(|\lambda_j|, |\lambda|)}.$$

From this it follows easily that $\Sigma_j(\lambda_j - \lambda)^{-1} \to 0$ as $|\lambda| \to \infty$, $\lambda \in \Xi(\theta, a)$, since we know that $\Sigma_j|\lambda_j|^{-1}$ converges. Therefore $c = 0$. Q.E.D.

14. Eigenvalue Problems for Elliptic Equations; The Self-Adjoint Case

Part 1. PRELIMINARY RESULTS ON FUNDAMENTAL SOLUTIONS

The results of the preceding section allow considerable sharpening in the case of operators T associated with elliptic differential operators. Before considering such cases, we need a preliminary discussion of the concept of fundamental solution. Before beginning this discussion, we need the following characterization of $H_\ell(E_n)$.

LEMMA 14.1. *Let $u \in L_2(E_n)$ and let $\hat{u}$ be the Fourier transform of u. Then $u \in H_\ell(E_n)$ if and only if the integral*

$$(14.1)\qquad \int_{E_n} (1+|\xi|)^{2\ell}|\hat{u}(\xi)|^2 d\xi$$

converges. Moreover,

$$(14.2)\qquad ||u||^2_{\ell, E_n} = \int_{E_n} \sum_{|\alpha|\le\ell} \xi^{2\alpha}|\hat{u}(\xi)|^2 d\xi.$$

Proof. If $u \in H_\ell(E_n)$, then $\widehat{D^\alpha u}(\xi) = (i\xi)^\alpha \hat{u}(\xi)$so that Parseval's relation implies

$$||D^\alpha u||^2_{0,E_n} = \int_{E_n} \xi^{2\alpha}|\hat{u}(\xi)|^2 d\xi.$$

From this relation, (14.2) follows upon addition over all a, $|a| \le \ell$.

Conversely, suppose (14.1) is satisfied; notice then that $\xi^\alpha \hat{u}(\xi)$ is square integrable over E_n if $|a| \le \ell$. Therefore, $(i\xi)^\alpha \hat{u}(\xi)$ is the Fourier transform of a certain function u^α in $L_2(E_n)$. Then Parseval's relation implies that if $\phi \in C_0^\infty(E_n)$,

$$\int_{E_n} u\overline{D^\alpha \phi}\,dx = \int_{E_n} \hat{u}(\xi)\overline{\widehat{D^\alpha \phi}(\xi)}\,d\xi$$

$$= \int_{E_n} \hat{u}(\xi)\overline{(i\xi)^\alpha \hat{\phi}(\xi)}\,d\xi$$

$$= (-1)^{|\alpha|}\int_{E_n} (i\xi)^\alpha \hat{u}(\xi)\overline{\hat{\phi}(\xi)}\,d\xi$$

$$= (-1)^{|\alpha|}\int_{E_n} \widehat{u^\alpha}(\xi)\overline{\hat{\phi}(\xi)}\,d\xi$$

$$= (-1)^{|\alpha|}\int_{E_n} u^\alpha(x)\overline{\phi(x)}\,dx.$$

Thus, u has the weak derivative $D^\alpha u = u^\alpha$ in $L_2(E_n)$, $|a| \le \ell$. Therefore, $u \in W_\ell(E_n) = H_\ell(E_n)$. Q.E.D.

Let $A(D)$ be a homogeneous elliptic operator of order ℓ having constant coefficients; thus,

$$A(D) = \sum_{|\alpha| = \ell} a_\alpha D^\alpha,$$

and for some positive E_0,

$$|A(i\xi| = |A(\xi)| \geq E_0 |\xi|^\ell$$

for real ξ. Assume that $\ell > n$ and that λ is a complex number such that the equation $A(i\xi) = \lambda$ has no solution for real ξ. Our aim is to solve the equation

$$(14.3) \quad (A(D) - \lambda)u = f$$

in case $f \in L_2(E_n)$. We shall indeed find a function $u \in H_\ell(E_n)$ such that (14.3) is satisfied in all of E_n. The technique is similar to that employed in Theorem 5.1, in which f and u were periodic. There we could use techniques of Fourier series. Since now f is not periodic, we shall have to apply Fourier transform techniques.

Suppose first that $u \in H_\ell(E_n)$ is a solution. Since the Fourier transform of $D^\alpha u$ is

$$\widehat{D^\alpha u}(\xi) = (i\xi)^\alpha \hat{u}(\xi),$$

it follows upon taking the Fourier transform of both sides of (14.3) that

$$(A(i\xi) - \lambda)\hat{u}(\xi) = \hat{f}(\xi).$$

Therefore,

$$(14.4) \quad \hat{u}(\xi) = \frac{1}{A(i\xi) - \lambda} \hat{f}(\xi)$$

Since $A(i\xi) - \lambda \neq 0$ for real ξ, it follows from the ellipticity of A that for all real ξ

$$(14.5) \quad \frac{1}{|A(i\xi) - \lambda|} \leq \frac{\text{const}}{(1 + |\xi|)^\ell}.$$

Therefore, since $\ell > n$ it follows that $(A(i\xi) - \lambda)^{-1}$ is in $L_2(E_n)$ and even in $L_1(E_n)$. Therefore, $(A(i\xi) - \lambda)^{-1}$ is the Fourier transform of a function in $L_2(E_n)$, say $(2\pi)^{n/2}F_\lambda(x)$. By the Fourier inversion formula,

$$(14.6) \quad F_\lambda(x) = (2\pi)^{-n} \int_{E_n} \frac{e^{ix\cdot\xi}}{A(i\xi) - \lambda}\, d\xi.$$

Since $(A(i\xi) - \lambda)^{-1}$ is integrable, it follows not only that $F_\lambda(x)$ is in $L_2(E_n)$, but also that $F_\lambda(x)$ is a continuous function on E_n. Moreover, $F_\lambda(x)$ decreases rapidly as $|x| \to \infty$. For, since

$$\widehat{(-ix)^\alpha F_\lambda}(\xi) = D_\xi^\alpha \hat{F}_\lambda(\xi),$$

it follows from the Fourier inversion formula that for all α

$$(14.7) \quad (-ix)^\alpha F_\lambda(x) = (2\pi)^{-n} \int_{E_n} e^{ix\cdot\xi} D_\xi^\alpha (A(i\xi) - \lambda)^{-1} d\xi.$$

The justification of this formula comes from the fact that $D_\xi^\alpha(A(i\xi) - \lambda)^{-1}$ is square integrabel for every α. Since $D_\xi^\alpha(A(i\xi) - \lambda)^{-1}$ is integrable, the relation (14,7) implies that for all positive N and for some positive constant C_N,

$$|F_\lambda(x)| \le \frac{C_N}{1 + |x|^N}, \qquad x \in E_N.$$

By the definition of F_λ, (14.4) can be written

$$(14.8) \quad \hat{u} = (2\pi)^{n/2}\hat{F}_\lambda \hat{f}.$$

Now the rapid decrease of F_λ implies that the convolution $F_\lambda * f$ has the Fourier transform

$$\widehat{F_\lambda * f} = (2\pi)^{n/2}\hat{F}_\lambda \hat{f}.$$

Therefore, (14.8) implies that u and $F_\lambda * f$ have the same Fourier transform, and must therefore coincide. That is

$$(14.9) \quad u(x) = \int_{E_n} F_\lambda(x-y)f(y)dy.$$

What we have shown is that if $u \in H_\ell(E_n)$ and if u satisfies (14.3), then u is given by the formula (14.9). Conversely, suppose that $f \in L_2(E_n)$ and that u is defined by (14.9). By retracing the above argument in the opposite direction, it follows that $u \in L_2(E_n)$ and that its Fourier transform satisfies (14.4). Therefore, the estimate (14.5) implies that $(1 + |\xi|)^\ell |\hat{u}(\xi)| \leq \text{const } |\hat{f}(\xi)|$, so that

$$\int_{E_n} (1 + |\xi|)^{2\ell} |\hat{u}(\xi)|^2 d\xi \leq \text{const} \int_{E_n} |\hat{f}(\xi)|^2 d\xi.$$

Thus, Lemma 14.1 implies that $u \in H_\ell(E_n)$. Then it follows that

$$\widehat{(A(D) - \lambda)u} = (A(i\xi) - \lambda)\hat{u} = \hat{f},$$

so that u is a solution of (14.3). The formula (14.2) also implies that

$$(14.10) \quad ||u||_{\ell, E_n} \leq \text{const } ||f||_{0, E_n}.$$

The function $F_\lambda(x)$ will be called the *fundamental solution* for the operator $A(D) - \lambda$.

Now we shall investigate the formula (14.9) for the solution of $(A(D) - \lambda)u = f$ in the case in which $f \in L_2(\Omega)$ for some bounded open set Ω. Regarding f as being extended to E_n by setting $f = 0$ outside Ω, then we see from the previous results that the function

$$(14.11) \quad u(x) = \int_\Omega F_\lambda(x-y)f(y)dy, \quad x \in \Omega,$$

is in $H_\ell(\Omega)$ and satisfies

$$(A(D) - \lambda)u = f \text{ in } \Omega.$$

The relation (14.11) represents an integral transformation of $L_2(\Omega)$ into $H_\ell(\Omega)$, having a Hilbert-Schmidt kernel $F_\lambda(x-y)$, since Ω is bounded. Since $F_\lambda(x)$ is continuous, the kernel $F_\lambda(x-y)$ certainly has a trace on the diagonal of $\Omega \times \Omega$, and this function is obviously the constant $F_\lambda(0)$, since $x = y$ on the diagonal.

Recall that we have assumed that $A(i\xi) - \lambda \neq 0$ for real ξ. If λ has the polar form

$$\lambda = r^\ell e^{i\theta},\ r = |\lambda|^{1/\ell} > 0,$$

this condition is obviously equivalent to the condition that $A(i\xi) \neq e^{i\theta}$ for real ξ; here we use again the homogeneity of A.

If we set $\xi = ry$ in the formula (14.6) with $x = 0$, then

$$\begin{aligned}(14.12)\qquad F_\lambda(0) &= (2\pi)^{-n} \int_{E_n} \frac{1}{r^\ell A(iy) - r^\ell e^{i\theta}}\, r^n dy \\ &= (2\pi)^{-n} r^{n-\ell} \int_{E_n} (A(iy) - e^{i\theta})^{-1} dy \\ &= (2\pi)^{-n} |\lambda|^{(n/\ell)-1} \int_{E_n} (A(iy) - e^{i\theta})^{-1} dy.\end{aligned}$$

This formula will be needed in some later computations. Note that, taking the integral over Ω of the trace of $F_\lambda(x-y)$, we obtain

$$\int_\Omega F_\lambda(0) dx = |\Omega|\ |\lambda|^{(n/\ell)-1} (2\pi)^{-n} \int_{E_n} \frac{1}{A(iy) - e^{i\theta}}\, dy$$

We shall obtain analogous expressions in Part 2 of this section.

Two more preliminary results are needed before giving the main results of this section. Note first that the condition, $A(i\xi) \neq \lambda$ for ξ real, actually implies that the values $A(i\xi)$ for real ξ omit an entire angle in the complex plane. Thus, for some $\delta > 0$ it follows that for real ξ, $A(i\xi) \neq \mu$ if $|\arg \mu - \arg \lambda| \leq \delta$.

LEMMA 14.2. *Let $A(D)$ be a homogeneous elliptic operator of order ℓ having constant coefficients, and let E_0 be a positive constant such that*

$$|A(\xi)| \geq E_0 |\xi|^\ell, \quad \xi \textit{ real.}$$

Let δ be a positive number such that

$$A(i\xi) \neq e^{i\theta}$$

for $|\theta - \theta_0| < 2\delta$ and ξ real. Let $\Omega' \subset\subset \Omega \subset E_n$. Then there exist positive constants $\omega = \omega(E_0, \delta, \ell, n)$ and $C = C(E_0, \delta, \ell, n, \Omega, \Omega')$ such that for all $u \in H_\ell(\Omega)$, for $0 \le k \le \ell$, and for $|\arg \lambda - \theta_0| < \delta$

$$|u|_{k,\Omega'} \le |\lambda|^{(k/\ell)-1}[\omega||(A - \lambda)u||_{0,\Omega} + C||u||_{\ell-1,\Omega}].$$

Proof. By (13.31) we have for real ξ and for $|\arg \lambda - \theta_0| < \delta$

$$(14.13) \qquad |A(i\xi) - \lambda| \ge \sin\delta \; \max\,(|A(i\xi)|, |\lambda|).$$

Suppose first that $u \in H_\ell(E_n)$, and let $f = (A(D) - \lambda)u$. Then $f \in L_2(E_n)$, and it follows from Parseval's relation and (14.4) that

$$\begin{aligned}(14.14) \qquad ||D^\alpha u||_{0,E_n} &= ||\widehat{D^\alpha u}||_{0,E_n} \\ &= ||(i\xi)^\alpha \hat{u}||_{0,E_n} \\ &= ||(i\xi)^\alpha (A(i\xi) - \lambda)^{-1}\hat{f}||_{0,E_n}.\end{aligned}$$

Now (14.13) implies for $|\alpha| = k$

$$\begin{aligned}\frac{(i\xi)^\alpha}{A(i\xi) - \lambda} &\le \frac{|\xi|^k}{|A(i\xi) - \lambda|^{k/\ell}|A(i\xi) - \lambda|^{1-(k/\ell)}} \\ &\le \frac{|\xi|^k}{\sin\delta\;|A(i\xi)|^{k/\ell}|\lambda|^{1-(k/\ell)}} \\ &\le \csc\delta\; E_0^{-k/\ell}|\lambda|^{(k/\ell)-1}.\end{aligned}$$

Thus, (14.14) implies

$$(14.15) \qquad ||D^\alpha u||_{0,E_n} \le \csc\delta\; E_0^{-k/\ell}|\lambda|^{(k/\ell)-1}||\hat{f}||_{0,E_n}$$

$$= \csc\delta \ E_0^{-k/\ell}|\lambda|^{(k/\ell)-1}||(A-\lambda)u||_{0,E_n}.$$

Now suppose only that $u \in H_\ell(\Omega)$. Then let $\zeta \in C_0^\infty(\Omega)$ satisfy $|\zeta| \leq 1$ on Ω and $\zeta \equiv 1$ on Ω'. Let $v = \zeta u$. Then $v \in H_\ell(E_n)$, and (14.15) may be applied to the function v. Note that by Leibnitz's rule

$$(A-\lambda)v = A(\zeta u) - \lambda\zeta u$$

$$= \zeta Au + Bu - \lambda\zeta u,$$

where Bu involves derivatives of ζ, and derivatives of u of order less than ℓ. Thus, by (14.15) with $|\alpha| = k$ and with u replaced by ζu,

$$||D^\alpha(\zeta u)||_{0,E_n} \leq \csc\delta \ E_0^{-k/\ell}|\lambda|^{(k/\ell)-1}[||\zeta(A-\lambda)u||_{0,E_n} + ||Bu||_{0,E_n}]$$

$$\leq \csc\delta \ E_0^{-k/\ell}|\lambda|^{(k/\ell)-1}[||(A-\lambda)u||_{0,\Omega} + C_1||u||_{\ell-1,\Omega}].$$

The lemma then follows from the relation

$$||D^\alpha u||_{0,\Omega'} = ||D^\alpha(\zeta u)||_{0,\Omega'} \leq ||D^\alpha(\zeta u)||_{0,\Omega}.$$

Q.E.D.

We can give a simple condition that $A'(x, i\xi)$ omit a certain complex number when x varies over Ω and ξ takes on all real values.

LEMMA 14.3. *Let $A(x, D)$ be a differential operator of order ℓ in Ω, having continuous leading coefficients and bounded, measurable lower order coefficients. Assume that for a certain θ and positive numbers C, a,*

$$||\phi||_{0,\Omega} \leq \frac{C}{|\lambda|} \ ||(A-\lambda)\phi||_{0,\Omega},$$

for all $\phi \in C_0^\infty(\Omega)$, $\lambda \in \Xi(\theta, a)$. Then $A'(x, i\xi) - e^{i\theta} \neq 0$ for all $x \in \Omega$, ξ real.

Proof. Let $\psi \in C_0^\infty(\Omega)$ and let

$$\phi(x) = e^{it\xi\cdot x}\psi(x), \quad t > 0.$$

For $\lambda = t^{\ell}e^{i\theta}$, Leibnitz's rule implies that as $t \to \infty$,

$$\begin{aligned}(A - \lambda)\phi &= (A'(x, D)e^{it\xi\cdot x} - \lambda e^{it\xi\cdot x}) \cdot \psi(x) + 0(t^{\ell-1}) \\ &= (A'(x, it\xi) - \lambda)e^{it\xi\cdot x}\psi(x) + 0(t^{\ell-1}) \\ &= t^{\ell}(A'(x, i\xi) - e^{i\theta})e^{it\xi\cdot x}\psi(x) + 0(t^{\ell-1}).\end{aligned}$$

Thus,

$$\begin{aligned}||\phi||^2_{0,\Omega} &= \int_\Omega |\psi(x)|^2 dx \\ &\leq \frac{C^2}{|\lambda|^2}\, ||(A - \lambda)\phi||^2_{0,\Omega} \\ &= \frac{C^2}{t^{2\ell}}\,[t^{2\ell}\int_\Omega |A'(x, i\xi) - e^{i\theta}|^2|\psi(x)|^2 dx + 0(t^{2\ell-1})] \\ &= C^2\int_\Omega |A'(x, i\xi) - e^{i\theta}|^2|\psi(x)|^2 dx + 0(t^{-1}).\end{aligned}$$

Letting $t \to \infty$, we obtain

$$\int_\Omega [C^2|A'(x, i\xi) - e^{i\theta}|^2 - 1]\, |\psi(x)|^2 dx \geq 0.$$

Since this holds for all $\psi \in C_0^\infty(\Omega)$, it follows that it must also hold hold for all $\psi \in L_2(\Omega)$. Therefore, since $A'(x, i\xi)$ is a continuous function of x,

$$C^2|A'(x, i\xi) - e^{i\theta}|^2 - 1 \geq 0,$$

so that for all $x \in \Omega$ and all real ξ,

$$|A'(x, i\xi) - e^{i\theta}| \geq C^{-1}.$$

Q.E.D.

Part 2. EIGENVALUE PROBLEMS FOR ELLIPTIC EQUATIONS

Now we are ready for the fundamental result of this section. Because of the generality of the situations to which the theorem applies, the theorem requires quite a long statement. As in section 13, it will be assumed throughout the remainder of section 14 that Ω is a finite union of disjoint open sets, each of which has the restricted cone property.

THEOREM 14.4. Hypothesis. *T is a bounded linear transformation on $L_2(\Omega)$ such that both the range of T and the range of T^* are contained in $H_{m'}(\Omega)$, where $m' > n$ if n is odd, $m' > n+1$ if n is even. The direction $e^{i\theta}$ is a direction of minimal growth of the modified resolvent of T. There exists an open set Ω_0 contained in Ω and an elliptic operator $P(x, D)$ of order m' in Ω_0 of the form*

$$P(x, D) = \sum_{|\alpha|=m'} a_\alpha(x) D^\alpha$$

where $a_\alpha \in C^0(\Omega_0)$.

1° *For $x \in \Omega_0$, for all real ξ, and for all complex $\lambda \neq 0$ such that $\arg \lambda = \theta$, $P(x, i\xi) \neq \lambda$.*

2° *For any $x^\circ \in \Omega_0$ and any positive ϵ, there exists a neighborhood U of x°, $U \subset \Omega_0$, and a constant C_ϵ, such that if*

$$P_0(D) = \sum_{|\lambda|=m'} a_\alpha(x^\circ) D^\alpha,$$

then for all $f \in L_2(\Omega)$

$$\|P_0 Tf - f\|_{0,U} \le \epsilon \|f\|_{0,\Omega} + C_\epsilon \|Tf\|_{m'-1,\Omega} \tag{14.16}$$

$$\|P_0^* T^* f - f\|_{0,U} \le \epsilon \|f\|_{0,\Omega} + C_\epsilon \|T^* f\|_{m'-1,\Omega}$$

where P_0^* is the formal adjoint of P_0.

Conclbsion. *For* $\lambda \in \rho_m(T)$, T_λ *is an integral operator with a Hilbert-Schmidt kernel* $K_\lambda \in H_\ell(\Omega \times \Omega)$, *where* $\ell = m' - [n/2] - 1$. *The kernel* K_λ *has a trace* $K_\lambda(x, x) \in L_2(\Omega)$ *on the diagonal of* $\Omega \times \Omega$. *For* $\lambda \in \Xi(\theta, 0)$, $|\lambda| \to \infty$,

$$[\int_\Omega |K_\lambda(x, x)|^2 dx]^{\frac{1}{2}} = 0(|\lambda|^{(n/m')-1}),$$

$$\int_{\Omega_0} K_\lambda(x, x)dx = c|\lambda|^{(n/m')-1} + o(|\lambda|^{(n/m')-1}),$$

where c *is a constant depending only on* P, θ, *and* Ω_0. *Indeed, if*

$$\rho_\theta(x) = (2\pi)^{-n} \int_{E_n} \frac{d\xi}{P(x, i\xi) - e^{i\theta}},$$

and if Ω_δ *is the subset of* Ω_0 *consisting of points whose distance from* $\partial\Omega_0$ *is greater than* δ, *and if* $\Omega_\delta \subset \Omega'_\delta \subset\subset \Omega_0$, *then*

$$c = \lim_{\delta \to 0} \int_{\Omega'_\delta} \rho_\theta(x)dx.$$

Remark: Some care was taken in the definition of c, since the following proof does not show that $\rho_\theta(x)$ is absolutely convergent over Ω_0. In reality, however, the function $\rho_\theta(x)$ is bounded in Ω_0. This follows from a much stronger version of Theorem 14.4 (compare remark at the end of Theorem 13.9) which yields *pointwise* asymptotic results. Thus, under the conditions of the theorem, one can show that $K_\lambda(x, y)$ is continuous and bounded in $\Omega \times \Omega$ for $\lambda \in \rho_m(T)$. Moreover, if $x \in \Omega_0$, $y \in \Omega_0$, $x \neq y$, then as $|\lambda| \to \infty$, $\lambda \in \Xi(\theta, 0)$,

$$K_\lambda(x, x) = \rho_\theta(x)|\lambda|^{(n/m')-1} + o(|\lambda|^{(n/m')-1}),$$

while

$$K_\lambda(x, y) = o(|\lambda|^{(n/m')-1}).$$

Proof. We first show that Hypothesis 2° implies that we also have the following. For any positive ϵ, there exists a neighborhood U' of x°, $U \subset \Omega_0$ and a constant C'_ϵ, such that for all $f \in L_2(\Omega)$ and for all $|\lambda|$ sufficiently large, $\arg \lambda = \theta$,

$$(14.16)' \qquad ||(P_0 - \lambda)T_\lambda f - f||_{0,U} \leq \epsilon||f||_{0,\Omega} + C'_\epsilon||T_\lambda f||_{m-1,\Omega}$$

$$||(P_0^* - \lambda)(T_\lambda)^* f - f||_{0,U} \leq \epsilon||f||_{0,\Omega} + C'_\epsilon||(T_\lambda)^* f||_{m-1,\Omega}.$$

Indeed, observe that

$$(P_0 - \lambda)T_\lambda f - f = P_0 T_\lambda f - (1 - \lambda T)^{-1} f = P_0 T(1 - \lambda T)^{-1} f - (1 - \lambda T)^{-1} f.$$

Given $\epsilon > 0$ let U be the neighborhood of x° for which (14.16) holds. Then, applying this inequality to $(1 - \lambda T)^{-1} f$, using the above relation, we obtain

$$||(P_0 - \lambda)T_\lambda f - f||_{0,U} \leq \epsilon||(1 - \lambda T)^{-1} f||_{0,\Omega} + C_\epsilon||T_\lambda f||_{m'-1,\Omega}$$

$$\leq \epsilon||1 - \lambda T)^{-1}||_0||f||_{0,\Omega} + C_\epsilon||T_\lambda f||_{m-1,\Omega}.$$

Using (12.5) we have for $|\lambda|$ sufficiently large, $\arg \lambda = \theta$,

$$||(1 - \lambda T)^{-1}||_0 \leq 1 + |\lambda|\ ||T_\lambda||_0 \leq 1 + K$$

where K is some constant. This combined with the preceding inequality yield the first part of (14.16)' (one replaces ϵ by $\epsilon' = \epsilon(1 + K)^{-1}$ choosing U' as the neighborhood which corresponds to ϵ' in Hypothesis 2°). The proof of the second inequality in (14.16)' is similar.

Let now Ω'_δ be fixed and let $\Omega' \subset\subset \Omega'_\delta$. Let $\epsilon > 0$. Then there exists a collection of congruent non-overlapping cubes $Q'_i, \ldots, Q'_N$, such that if $\mathcal{E} = \bigcup_{i=1}^{N} Q'_i$, then $\Omega' \subset \mathcal{E} \subset \Omega'_\delta$, and the estimates (14.16)' are valid in Q_i, the cube concentric to Q'_i having twice the side length of Q'_i. Let x^i be the center of Q'_i, and let b be the length of the side of Q'_i.

Let $\arg \lambda = \theta$. By the results of Part 1, there exists a fundamental solution $F^i_\lambda(x)$ for the elliptic operator with constant coefficients, $P(x^i, D) - \lambda$. For $\arg \lambda = \theta$, $|\lambda|$ sufficiently large, the modified resolvent T_λ exists. Suppose $f \in L_2(Q_i')$; extend f to be zero outside Q_i'. Define an operator S_λ^i on $L_2(Q_i')$ by the formula

$$S_\lambda^i f = T_\lambda f - F_\lambda^i * f.$$

Now $R(T_\lambda) \subset R(T) \subset H_{m'}(\Omega)$, and, as we have seen in Part 1, $F_\lambda^i * f \in H_{m'}(E_n)$. Thus, if we restrict our attention to Q_i', $R(S_\lambda^i) \subset H_{m'}(Q_i')$. By Lemma 13.4 and by the estimate (14.10), it follows that S_λ^i is a bounded linear transformation from $L_2(Q_i')$ into $H_{m'}(Q_i')$.

The assumed relation between m' and n shows that the hypothesis of Theorem 13.10 is satisfied, so it follows that the Hilbert-Schmidt kernel $K_\lambda(x, y)$ of T_λ exists, and that $K_\lambda(x, x)$ exists as an L_2 function in Ω. The same remarks apply to the kernel $G_\lambda^i(x, y)$ of the operator S_λ^i, by Theorem 13.9. It should be noted in this case that the adjoint of S_λ^i is given by the formula

$$S_\lambda^{i*} f = T_\lambda^* f - \int_{Q_i} \overline{F_\lambda^i(y - x)} f(y)dy,$$

and $F_\lambda^{i*}(x) = \overline{F_\lambda^i(- x)}$ is the fundamental solution for $P^*(x^i, D) - \bar{\lambda}$. Thus, $R(S_\lambda^{i*}) \subset H_{m'}(Q_i')$, and Theorem 13.9 can be applied.

Therefore, corresponding to the operator definition of S_λ^i is the equation for the kernels:

$$G_\lambda^i(x, y) = K_\lambda(x, y) - F_\lambda^i(x - y)\,, \tag{14.17}$$

$$G_\lambda^{i*}(x, y) = K_\lambda^*(x, y) - F_\lambda^{i*}(x - y)\,,$$

where G_λ^{i*} and K_λ^* are the Hilbert-Schmidt kernels for the operators $(S_\lambda^i)^*$ and $(T_\lambda)^*$, respectively. Let γ be the constant of (13.13) which applies to a fixed cube of side δ_0. Then for $b < \delta_0$, we have by the corollary to Theorem 13.9

$$(b/\delta_0)^{n/2}\, [\textstyle\int_{Q_i'} |G_\lambda^i(x, x)|^2 dx]^{1/2} \tag{14.18}$$

$$\leq \gamma[(b/\delta_0)^n (|S_\lambda^i|_{m'}^{n/m'} + |S_\lambda^{i*}|_{m'}^{n/m'})|S_\lambda^i|_0^{1-(n/m')} + |S_\lambda^i|_0].$$

We now need to estimate the semi-norms of S_λ appearing in this expression. First we shall obtain some rough estimates.

By the triangle inequality and the definition of S_λ^i, for $f \in L_2(Q_i')$ we have

$$(14.19) \qquad |S_\lambda^i f|_k \le |T_\lambda f|_k + |F_\lambda^{i*} f|_k.$$

For large $|\lambda|$, are $\lambda = \theta$, we have by hypothesis

$$\|T_\lambda\|_0 \le K|\lambda|^{-1}.$$

Therefore, by (13.8) it follows that for such λ

$$\|T_\lambda\|_{m'} \le \|T\|_{m'}(1 + K).$$

Combining this with (14.10) implies

$$\|S_\lambda^i f\|_{m', Q_i} \le [\|T\|_{m'}(1 + K) + \text{const}]\, \|f\|_{0, Q_i}.$$

Thus, $\|S_\lambda^i\|_{m'}$ is bounded independently of λ. Therefore, Lemma 13.3 implies

$$|S_\lambda^i|_k \le \text{const}\ |S_\lambda|_0^{1-(k/m')} \|S_\lambda\|_{m'}^{k/m'}$$

$$(14.20) \qquad \le \text{const}\ |S_\lambda^i|_0^{1-(k/m')}.$$

By Lemma 14.2, we obtain for $f \in L_2(Q_i')$,

$$|F_\lambda^{i*} f|_{0, Q_i'} \le |\lambda|^{-1} [\omega \|f\|_{0, E_n} + \text{const}\ \|F_\lambda^{i*} f\|_{m', E_n}]$$

$$\le \text{const}\ |\lambda|^{-1} \|f\|_{0, Q_i'},$$

where we have used (14.10). Thus, (14.19) implies

$$|S_\lambda^i|_0 \le |T_\lambda|_0 + \text{const}\ |\lambda|^{-1}$$

$$\leq \text{const } |\lambda|^{-1}.$$

Thus, by (14.20) we have for $0 \leq k \leq m' - 1$

$$|S_\lambda^i|_k \leq \text{const } |\lambda|^{(k/m')-1}.$$

Therefore, for sufficiently large $|\lambda|$ it follows that

$$(14.21) \qquad \|S_\lambda^i\|_{m'-1} \leq \text{const } |\lambda|^{-1/m'}.$$

Now we shall obtain some sharp estimates, using the inequalities (14.16)'. We have for $f \in L_2(Q_i')$

$$\|(P(x^i, D) - \lambda)T_\lambda f - f\|_{0,Q_i}$$

$$= \|(P(x^i, D) - \lambda)(T_\lambda f - K_\lambda^i * f)\|_{0,Q_i}$$

$$= \|(P(x^i, D) - \lambda)S_\lambda^i f\|_{0,Q_i}$$

$$\leq \epsilon\|f\|_{0,Q_i} + C_\epsilon'\|T_\lambda f\|_{m'-1,\Omega}$$

$$\leq [\epsilon + C_\epsilon'|T_\lambda|_0^{1/m'}\|T_\lambda\|_{m',\Omega}]\|f\|_{0,Q_i'}$$

$$\leq [\epsilon + C_\epsilon|\lambda|^{-1/m'}]\|f\|_{0,Q_i'},$$

where we have used again Lemma 13.3; we are now using C_ϵ as a generic constant. It follows that for sufficiently large $|\lambda|$, $\arg \lambda = \theta$,

$$(14.22) \qquad \|(P(x^i, D) - \lambda)S_\lambda^i f\|_{0,Q_i} \leq 2\epsilon\|f\|_{0,Q_i'}.$$

Now we shall again apply the estimate in Lemma 14.2, taking $\Omega = Q_i$, $\Omega' = Q_i'$, $u = S_\lambda^i f$. We obtain, using (14.22),

$$|S_\lambda^i f|_{k,Q_i'} \leq |\lambda|^{(k/m')-1}[\omega 2\epsilon\|f\|_{0,Q_i'} + C_\epsilon\|S_\lambda^i f\|_{m'-1,Q_i}].$$

By the rough estimate (14.21), this implies

$$|S_\lambda^i f|_{k,Q_i'} \le |\lambda|^{(k/m')-1}[2\omega\epsilon\|f\|_{0,Q_i'} + C_\epsilon|\lambda|^{-1/m'}\|f\|_{0,Q_i'}].$$

Therefore, we have for $|\lambda|$ sufficiently large, $\arg\lambda = \theta$,

$$|S_\lambda^i|_k \le 3\omega\epsilon|\lambda|^{(k/m')-1}.$$

The same argument shows also that for such λ,

$$|(S_\lambda^i)^*|_k \le 3\omega\epsilon|\lambda|^{(k/m')-1}.$$

The proof is now almost completed, except for some easy, but lengthy, computations. Substituting the estimates just obtained into (14.18), we find, using (14.17),

$$(b/\delta_0)^{n/2}\,[\textstyle\int_{Q_i'}|K_\lambda(x,\,x) - F_\lambda^i(0)|^2dx]^{1/2}$$

$$\le \gamma[(b/\delta_0)^n 3\omega\epsilon 2|\lambda|^{(n/m')-1} + 3\omega\epsilon|\lambda|^{-1}];$$

or,

$$[\textstyle\int_{Q_i'}|K_\lambda(x,\,x) - F_\lambda^i(0)|^2dx]^{1/2}$$

$$\le 6\gamma\omega\epsilon(b/\delta_0)^{n/2}|\lambda|^{(n/m')-1} + 3\gamma\omega\epsilon(b/\delta_0)^{-n/2}|\lambda|^{-1}.$$

Therefore, we obtain for sufficiently large $|\lambda|$, after squaring,

$$\int_{Q_i'}|K_\lambda(x,\,x) - F_\lambda^i(0)|^2dx \le (12\gamma\omega)^2\epsilon^2(b/\delta_0)^n|\lambda|^{(2n/m')-2}.$$

Next, summing over all the cubes Q_i', we obtain since $b^n = |Q_i'|$,

$$(14.23)\qquad \Sigma_i\int_{Q_i'}|K_\lambda(x,\,x) - F_\lambda^i(0)|^2dx \le (12\gamma\omega)^2\epsilon^2\delta_0^{-n}\Big|\bigcup_{i=1}^{N} Q_i'\Big|\;|\lambda|^{(2n/m')-2}$$

$$\leq C\epsilon^2|\lambda|^{(2n/m')-2}.$$

We shall use C as a generic constant for the remainder of the proof.

Now we utilize the formula (14.12) for $F^i_\lambda(0)$:

$$F^i_\lambda(0) = |\lambda|^{(n/m')-1}(2\pi)^{-n} \int_{E_n} (P(x^i, i\xi) - e^{i\theta})^{-1} d\xi$$

$$= |\lambda|^{(n/m')-1}\rho_\theta(x^i).$$

Thus, (14.23) becomes, after multiplying by $|\lambda|^{2-(2n/m')}$,

$$(14.24) \qquad \Sigma_i \int_{Q'_i} \big|\, |\lambda|^{1-(n/m')}K_\lambda(x, x) - \rho_\theta(x^i)\big|^2 dx \leq C\epsilon^2.$$

Now let $\tilde{\rho}_\theta(x)$ be the step function on Ω_0, equal to $\rho_\theta(x^i)$ on Q'_i and equal to zero on $\Omega_0 - \cup Q'_i = \Omega_0 - \mathfrak{S}$. Then (14.24) can be written ($\mathfrak{S} = \cup Q'_i$)

$$[\int_{\mathfrak{S}} \big|\, |\lambda|^{1-(n/m')}K_\lambda(x, x) - \tilde{\rho}_\theta(x)\big|^2 dx]^{1/2} \leq C\epsilon.$$

By the Cauchy-Schwarz inequality,

$$\Big|\int_{\mathfrak{S}} |\lambda|^{1-(n/m')}K_\lambda(x, x)dx - \int_{\mathfrak{S}} \tilde{\rho}_\theta(x)dx\Big|$$

$$\leq \int_{\mathfrak{S}} \big|\, |\lambda|^{1-(n/m')}K_\lambda(x, x) - \tilde{\rho}_\theta(x)\big| dx$$

$$\leq |\mathfrak{S}|^{1/2}[\int_{\mathfrak{S}} \big|\, |\lambda|^{1-(n/m')}K_\lambda(x, x) - \tilde{\rho}_\theta(x)\big|^2 dx]^{1/2}$$

$$(14.25) \leq C|\Omega_0|^{1/2}\epsilon.$$

Applying Theorem 13.10, it follows that $K_\lambda \in H\ell(\Omega \times \Omega)$, and for $|\lambda|$ large, $\arg \lambda = \theta$,

$$|\lambda|^{1-(n/m')}\,[\int_\Omega |K_\lambda(x, x)|^2 dx]^{1/2} \leq C.$$

Therefore, the Cauchy-Schwarz inequality implies

$$|\lambda|^{1-(n/m')} \int_{\Omega_0-\mathcal{E}} |K_\lambda(x, x)|dx \leq C|\Omega_0-\mathcal{E}|^{1/2}.$$

Therefore, the triangle inequality applied to (14.25) shows that

$$\begin{aligned}\limsup_{\substack{|\lambda|\to\infty \\ \arg\lambda=\theta}} \Big| |\lambda|^{1-(n/m')} \int_{\Omega_0} K_\lambda(x, x)dx - \int_{\Omega'_\delta} \rho_\theta(x)dx\Big| &\leq C|\Omega_0|^{1/2}\epsilon + C|\Omega_0 - \mathcal{E}|^{1/2} \\ &+ \Big|\int_{\mathcal{E}}\rho_\theta(x)dx - \int_{\mathcal{E}}\tilde{\rho}_\theta(x)dx\Big| \\ &+ \int_{\Omega'_\delta-\mathcal{E}} |\rho_\theta(x)|dx.\end{aligned}$$

Now the integral $\int_{\Omega'_\delta} \rho_\theta(x)dx$ exists, and $\int_{\mathcal{E}}\tilde{\rho}_\theta(x)dx$ is an approximating Riemann sum for the integral $\int_{\mathcal{E}}\rho_\theta(x)dx$. By choosing the cubes Q'_i smaller and smaller, the above inequality becomes

$$\begin{aligned}\limsup_{\substack{|\lambda|\to\infty \\ \arg\lambda=\theta}} \Big| |\lambda|^{1-(n/m')} \int_{\Omega_0} K_\lambda(x, x)dx - \int_{\Omega'_\delta} \rho_\theta(x)dx\Big| &\leq C|\Omega_0|^{1/2}\epsilon + C|\Omega_0 - \Omega'|^{1/2} \\ &+ \int_{\Omega'_\delta-\Omega'} |\rho_\theta(x)|dx.\end{aligned}$$

Since Ω' was any set satisfying $\Omega' \subset\subset \Omega'_\delta$, and since ϵ was any positive number, the last inequality implies

$$\begin{aligned}\limsup_{\substack{|\lambda|\to\infty \\ \arg\lambda=\theta}} \Big| |\lambda|^{1-(n/m')} \int_{\Omega_0} K_\lambda(x, x)dx - \int_{\Omega'_\delta} \rho_\theta(x)dx\Big| &\leq C|\Omega_0 - \Omega'_\delta|^{1/2} \\ &\leq C|\Omega_0 - \Omega_\delta|^{1/2}.\end{aligned}$$

Since $|\lambda|^{1-(n/m')}\int_{\Omega_0} K_\lambda(x, x)dx$ is independent of δ, and $\int_{\Omega'_\delta} \rho_\theta(x)dx$

is independent of λ, it must be the case that the two limits

$$\lim_{\substack{|\lambda|\to\infty \\ \arg\lambda=\theta}} \int_{\Omega_0} K_\lambda(x, x)dx,$$

and

$$\lim_{\delta\to 0} \int_{\Omega'_\delta} \rho_\theta(x)dx,$$

exist and are equal. Q.E.D.

Before giving the proof of our main application of Theorem 14.4, we need a Tauberian theorem of Hardy and Littlewood, which we state without proof. For a proof see Karamata, Neuer Beweis und Verallgemeinerung der Tauberschen Sätze, welch die Laplacesche und Stieltjessche Transformation betreffen, *Journal für die reine und angewandte Mathematik 164* (1931); pp. 27-39.

THEOREM 14.5. *Let $\sigma(\lambda)$ be a non decreasing function for $\lambda > 0$, let $0 < a < 1$, let P be a non-negative number, and suppose that as t $t \to \infty$*

$$\int_0^\infty \frac{d\sigma(\lambda)}{\lambda + t} = Pt^{a-1} + o(t^{a-1}).$$

Then as $\lambda \to \infty$

$$\sigma(\lambda) = P\,\frac{\sin \pi a}{\pi a}\;\lambda^a + o(\lambda^a). \tag{14.26}$$

The converse ("Abelian theorem") of this result is valid, even if we omit the assumption of monotonicity of $\sigma(\lambda)$. Because its proof is elementary, we include it here. Thus, we assume the asymptotic behavior (14.26) of the function $\sigma(\lambda)$. For every positive number b, we obtain by integration by parts the formula

$$\int_0^\infty \frac{d\sigma(\lambda)}{\lambda + t} = \int_0^b \frac{d\sigma(\lambda)}{\lambda + t} - \frac{\sigma(b)}{b + t} + \int_b^\infty \frac{\sigma(\lambda)}{(\lambda + t)^2}\;d\lambda. \tag{14.27}$$

Another integration by parts shows that

$$\int_b^\infty \frac{\lambda^a}{(\lambda+t)^2}\,d\lambda = \frac{b^a}{b+t} + a\int_b^\infty \frac{\lambda^{a-1}}{\lambda+t}\,d\lambda.$$

Setting $\lambda = t\mu$ in the last integral, we obtain

$$(14.28)\qquad \int_b^\infty \frac{\lambda^a}{(\lambda+t)^2}\,d\lambda = \frac{b^a}{b+t} + at^{a-1}\int_{b/t}^\infty \frac{\mu^{a-1}}{\mu+1}\,d\mu.$$

Now a familiar formula concerning beta functions is

$$(14.29)\qquad \int_0^\infty \frac{\mu^{a-1}}{\mu+1}\,d\mu = \frac{\pi}{\sin \pi a}\,.$$

Therefore, (14.28) becomes

$$(14.30)\qquad \int_b^\infty \frac{\lambda^a}{(\lambda+t)^2}\,d\lambda = \frac{b^a}{b+t} + \frac{\pi a}{\sin \pi a}\,t^{a-1} - at^{a-1}\int_0^{b/t} \frac{\mu^{a-1}}{\mu+1}\,d\mu.$$

Multiplying this relation by $P\dfrac{\sin \pi a}{\pi a}$ and combining with (14.27) yields

$$\begin{aligned}\int_0^\infty \frac{d\sigma(\lambda)}{\lambda+t} - Pt^{a-1} &= \int_b^\infty \frac{1}{(\lambda+t)^2}\left[\sigma(\lambda) - P\frac{\sin \pi a}{\pi a}\lambda^a\right]d\lambda \\ &\quad - \frac{1}{b+t}\left[\sigma(b) - P\frac{\sin \pi a}{\pi a}b^a\right] + \int_0^b \frac{d\sigma(\lambda)}{\lambda+t} \\ &\quad - P\frac{\sin \pi a}{\pi a}t^{a-1}\int_0^{b/t}\frac{\mu^{a-1}}{\mu+1}\,d\mu.\end{aligned}$$

Given $\epsilon > 0$, there exists a positive number b such that for $\lambda \geq b$

$$\left|\sigma(\lambda) - P\frac{\sin \pi a}{\pi a}\lambda^a\right| < \epsilon\lambda^a.$$

Thus, we obtain

$$\begin{aligned}\left|\int_0^\infty \frac{d\sigma(\lambda)}{\lambda+t} - Pt^{a-1}\right| &< \epsilon\int_b^\infty \frac{\lambda^a}{(\lambda+t)^2}\,d\lambda + \epsilon\frac{b^a}{b+t} + \int_0^b \frac{d\sigma(\lambda)}{\lambda+t} \\ &\quad + \left|P\frac{\sin \pi a}{\pi a}\right| t^{a-1}\int_0^{b/t}\frac{\mu^{a-1}}{\mu+1}\,d\mu.\end{aligned}$$

Therefore, using (14.30) again,

$$\limsup_{t\to\infty} t^{1-a} \left|\int_0^\infty \frac{d\sigma(\lambda)}{\lambda + t} - Pt^{a-1}\right| \le \epsilon \limsup_{t\to\infty} t^{1-a} \int_b^\infty \frac{\lambda^a}{(\lambda + t)^2}\, d\lambda$$

$$= \epsilon \frac{\pi a}{\sin \pi a}\ .$$

Since ϵ is arbitrary, the converse of Theorem 14.5 is proved.

One further comment should be included. The Tauberian theorem is usually stated in the case that $P > 0$. But, having the result for $P > 0$, it is trivial to derive the result for the case $P = 0$. For suppose $\sigma(\lambda)$ is non-decreasing, and

$$\int_0^\infty \frac{d\sigma(\lambda)}{\lambda + t} = o(t^{a-1}).$$

Then $\sigma(\lambda) + \lambda^a$ is non-decreasing, and the converse proposition just proved shows that

$$\int_0^\infty \frac{d[\sigma(\lambda) + \lambda^a]}{\lambda + t} = \frac{\pi a}{\sin \pi a}\ t^{a-1} + o(t^{a-1}).$$

Therefore, the Tauberian theorem shows directly that

$$\sigma(\lambda) + \lambda^a = \lambda^a + o(\lambda^a),$$

or,

$$\sigma(\lambda) = o(\lambda^a).$$

Therefore, the result of Theorem 14.5 is also demonstrated in the case $P = 0$.

The spectrum of a linear transformation on a Hilbert space is *discrete* if every complex number in the spectrum is an eigenvalue.

THEOREM 14.6. *Let $A(x, D)$ be an elliptic operator of order m' in Ω, having continuous leading coefficients and bounded, measurable lower order coefficients. Let A be symmetric over $C_0^\infty(\Omega)$ in the sense that for all ϕ, $\psi \in C_0^\infty(\Omega)$, $(A\phi, \psi)_{0,\Omega} = (\phi, A\psi)_{0,\Omega}$. (This holds, in particular, if the coefficients of A are sufficiently smooth, and A coincides with its formal adjoint.) Suppose that there exists an unbounded self-adjoint transformation $\mathfrak{A}$ on $L_2(\Omega)$, such that $C_0^\infty(\Omega) \subset D(\mathfrak{A}) \subset H_{m'}(\Omega)$ and $\mathfrak{A}u = Au$ for $u \in D(\mathfrak{A})$. Let $n' = n$ if*

n is odd, $n' = n+1$ if n is even. In case $m' \le n'$, suppose that there exists an odd positive integer k such that $k > n'/m'$, the coefficients of A are in $C^{(k-1)m'*}(\Omega)$, and $D(\mathcal{A}^k) \subset H_{km'}(\Omega)$.

Then the spectrum of $\mathcal{A}$ is discrete, and the eigenvalues of $\mathcal{A}$ have finite multiplicity. Let $\{\lambda_j\}$ be the sequence of eigenvalues of $\mathcal{A}$ counted according to multiplicity. For $\lambda > 0$ let $N_+(\lambda)$ be the number of non-negative eigenvalues $\lambda_j \le \lambda$, and let $N_-(\lambda)$ be the number of negative eigenvalues $\lambda_j \ge -\lambda$. Then

$$N_\pm(\lambda) = c_\pm \lambda^{n/m'} + o(\lambda^{n/m'}) \text{ as } \lambda \to \infty,$$

where

$$c_\pm = (2\pi)^{-n} \int_\Omega \omega_\pm(x)dx,$$

and

$$\omega_\pm(x) = |\{\xi\colon\ 0 < \pm A'(x, i\xi) < 1\}|.$$

Remark. Since $\mathcal{A}$ is self adjoint, its spectrum is real, by Theorem 12.7. Also, all powers $\mathcal{A}^k$ of $\mathcal{A}$ are again self-adjoint transformations on $L_2(\Omega)$, a well-known fact about self-adjoint transformations.

Proof. By Theorem 12.7, $(\lambda - \mathcal{A})^{-1}$ exists for all non-real λ, and all non-real directions $e^{i\theta}$ are directions of minimal growth of the resolvent of $\mathcal{A}$. Moreover, since $D(\mathcal{A}) \subset H_{m'}(\Omega)$, it follows that $(\lambda - \mathcal{A})^{-1}$ maps $L_2(\Omega)$ into $H_{m'}(\Omega)$. Therefore, $(\lambda - \mathcal{A})^{-1}$ is *compact*, by Rellich's theorem. Therefore, the Riesz-Schauder theory of compact operators implies that the spectrum of $(\lambda - \mathcal{A})^{-1}$ consists only of eigenvalues of finite multiplicity, whose only possible limit point is the number 0, which is also in the spectrum of $(\lambda - \mathcal{A})^{-1}$. By Theorem 12.4, this implies that the spectrum of $\mathcal{A}$ is itself discrete, and the eigenvalues of $\mathcal{A}$ have finite multiplicity.

Therefore, there exists a real number λ_0 such that $\lambda_0 \in \rho(\mathcal{A})$. By considering $\mathcal{A} - \lambda_0$ instead of $\mathcal{A}$, we may now assume that $T = \mathcal{A}^{-1}$ exists as a bounded linear transformation on $L_2(\Omega)$. By Theorem 12.3, $(\lambda - \mathcal{A})^{-1} = -T_\lambda$, so it follows that all non-real directions are directions of minimal growth of the modified resolvent of T. This also follows immediately, as noted above, just from the fact that T is self-adjoint. Note that $R(T) = D(\mathfrak{A}) \subset H_{m'}(\Omega)$.

Let us now examine carefully the case in which $m' \leq n'$. Then, if k is the integer guaranteed by the hypothesis, $T^k = (\mathfrak{A}^k)^{-1}$, so that $R(T^k) = D(\mathfrak{A}^k) \subset H_{km'}(\Omega)$. By the smoothness assumptions on the coefficients of A, the operator A^k exists as a differential operator, and the coefficients of A^k are continuous and bounded in Ω. Moreover, the principal part of A^k is just

$$(A^k)'(x, \xi) = (A'(x, \xi))^k.$$

Therefore, A^k is an elliptic operator of order km'. Our aim is to apply Theorem 14.4 to the operator T^k. We shall set $k = 1$ in case $m' > n'$. Since in any case $R(T^k) \subset H_{km'}(\Omega)$ and $km' > n'$, and since T^k is self-adjoint, it remains to verify the hypotheses 1°, 2° of that theorem, as we now do.

Since $T^k = (\mathfrak{A}^k)^{-1}$, it follows that for any $f \in L_2(\Omega)$,

$$A^kT^kf = f. \tag{14.31}$$

Now let $\Omega_0 = \Omega$, and let the operator P of Theorem 14.4 be the principal part of A^k. Thus, $A^k = P + B$, where B is an operator of order less than km' having bounded coefficients. By (14.31),

$$\begin{aligned} \|PT^kf - f\|_{0,\Omega} &\leq \|BT^kf\|_{0,\Omega} \\ &\leq C\|T^kf\|_{km'-1,\Omega}. \end{aligned} \tag{14.32}$$

Now let $x^0 \in \Omega$, let U be a small neighborhood of x^0, such that the coefficients of P vary by less than ϵ in U. If P_0 is the operator P with coefficients evaluated and frozen at x^0, then (14.32) implies

$$\|P_0T^kf - f\|_{0,U} \leq \|(P_0 - P)T^kf\|_{0,U} + C\|T^kf\|_{km'-1,U}$$

$$\leq \gamma\epsilon \| T^k f \|_{km',\Omega} + C \| T^k f \|_{km'-1,\Omega}$$

$$\leq \gamma\epsilon \| T^k \|_{km'} \| f \|_{0,\Omega} + C \| T^k f \|_{km'-1,\Omega}, \tag{14.33}$$

where γ depends only on m and n. But this is precisely the first estimate of (14.16), with m' replaced by km', the order of P. Likewise, the estimate for the adjoints of the operators can be derived.

Next, if $e^{i\theta}$ is not real, then $e^{i\theta}$ is a direction of minimal growth of the resolvent of the self adjoint operator $\mathfrak{A}$, as we have remarked. Therefore, for all $u \in D(\mathfrak{A})$ and for $\arg \lambda = \theta$,

$$\| u \|_{0,\Omega} \leq \frac{\text{const}}{|\lambda|} \| (\mathfrak{A} - \lambda) u \|_{0,\Omega}.$$

In particular, since $D(\mathfrak{A}) \supset C_0^\infty(\Omega)$, we have for all $\phi \in C_0^\infty(\Omega)$

$$\| \phi \|_{0,\Omega} \leq \frac{\text{const}}{|\lambda|} \| (\mathfrak{A} - \lambda) \phi \|_{0,\Omega}.$$

Therefore, Lemma 14.3 implies $A'(x, i\xi) - e^{i\theta} \neq 0$. This also implies that $(A'(x, i\xi))^k - e^{i\theta} \neq 0$ for real ξ, so that condition 1° of Theorem 14.4 is verified. Note that in our case it even follows that $A'(x, i\xi)$ is *real* for all real ξ.

Thus, we can apply Theorem 14.4 in the case of the operator T^k. In particular, we wish to apply the asymptotic formula for the kernel K_λ of the operator $(T^k)_\lambda$; we have from Theorem 14.4

$$\int_\Omega K_\lambda(x, x) dx = c|\lambda|^{(n/km')-1} + o(|\lambda|^{(n/km')-1}). \tag{14.34}$$

where c is given in accordance with the formulas given in the statement of the theorem. Now since the characteristic values of T are just the eigenvalues $\{\lambda_j\}$ of $\mathfrak{A}$, the characteristic values of T^k, counted according to multiplicity, are $\{\lambda_j^k\}$. By Theorem 13.10, we then have

$$\int_\Omega K_\lambda(x, x)\, dx = \Sigma_j \frac{1}{\lambda_j^k - \lambda}.$$

Comparing this with the asymptotic formula above, we have for $\arg \lambda = \theta$ fixed, $e^{i\theta}$ not real,

$$\Sigma_j \frac{1}{\lambda_j^k - \lambda} = c|\lambda|^{(n/km')-1} + o(|\lambda|^{(n/km')-1}) \text{ as } |\lambda| \to \infty.$$

In particular, let us take for simplicity $\theta = \frac{\pi}{2}$. Thus, letting $\lambda = it$, $t > 0$, we have

$$(14.35)\qquad \Sigma_j \frac{1}{\lambda_j^k - it} = ct^{(n/km')-1} + o(t^{(n/km')-1}).$$

Using the formulas derived in Theorem 14.4,

$$(14.36)\qquad c = \int_\Omega \rho(x)dx,$$

where

$$(14.37)\qquad \rho(x) = (2\pi)^{-n} \int_{E_n} \frac{d\xi}{[A'(x, i\xi)]^k - i}.$$

The problem is now to use this information to find expressions for $N_+(\lambda)$ and $N_-(\lambda)$. Let $q = n/m'$; $0 < q < k$. Then, if c_1 and c_2 are the real and imaginary parts of c, it follows from (14.35) that

$$(14.38)\qquad \Sigma_j \frac{\lambda_j^k}{\lambda_j^{2k} + t^2} = c_1 t^{(q/k)-1} + o(t^{(q/k)-1}),$$

$$\Sigma_j \frac{t}{\lambda_j^{2k} + t^2} = c_2 t^{(q/k)-1} + o(t^{(q/k)-1}).$$

Divide the latter relation by t, and replace t^2 by t, to obtain

$$(14.39)\qquad \Sigma_j \frac{1}{\lambda_j^{2k} + t} = c_2 t^{(q/2k)-1} + o(t^{(q/2k)-1}).$$

Let $N(\lambda) = N_+(\lambda) + N_-(\lambda)$, the number of λ_j such that $|\lambda_j| \leq \lambda$. Then

$$\Sigma_j \frac{1}{\lambda_j^{2k} + t} = \int_0^\infty \frac{dN(\lambda^{1/2k})}{\lambda + t},$$

so that (14.39) becomes

$$\int_0^\infty \frac{dN(\lambda^{1/2k})}{\lambda + t} = c_2 t^{(q/2k)-1} + o(t^{(q/2k)-1}).$$

Applying the Tauberian theorem of Hardy and Littlewood (Theorem 14.5), it follows that

$$(14.40)\quad N(\lambda^{1/2k}) = c_2 \frac{\sin(\pi q/2k)}{\pi q/2k} \lambda^{q/2k} + o(\lambda^{q/2k}),$$

or, replacing λ by λ^{2k},

$$(14.41)\qquad N(\lambda) = 2kc_2 \frac{\sin(\pi q/2k)}{\pi q} \lambda^q + o(\lambda^q).$$

We now must separate the parts $N_+(\lambda)$ and $N_-(\lambda)$ out of their sum $N(\lambda)$. This is just the reason we had to assume k is odd: the sign of λ_j^k is the same as that of λ_j in this case. Now

$$\Sigma_j \frac{\lambda_j^k}{\lambda_j^{2k} + t^2} = \sum_{\lambda_j > 0} \frac{\lambda_j^k}{\lambda_j^{2k} + t^2} + \sum_{\lambda_j < 0} \frac{\lambda_j^k}{\lambda_j^{2k} + t^2}$$

$$= \int_0^\infty \frac{\sqrt{\lambda}}{\lambda + t^2}\, dN_+(\lambda^{1/2k}) - \int_0^\infty \frac{\sqrt{\lambda}}{\lambda + t^2}\, dN_-(\lambda^{1/2k})$$

$$= \int_0^\infty \frac{\sqrt{\lambda}}{\lambda + t^2}\, dN(\lambda^{1/2k}) - 2\int_0^\infty \frac{\sqrt{\lambda}}{\lambda + t^2}\, dN_-(\lambda^{1/2k})$$

$$(14.42)\qquad = c_1 t^{(q/k)-1} + o(t^{(q/k)-1}),$$

by (14.38). Let

$$\tilde{N}(\lambda) = \int_0^\lambda \sqrt{\mu}\, dN(\mu^{1/2k}).$$

Then, by an integration by parts and the asymptotic formula (14.40),

$$\tilde{N}(\lambda) = \sqrt{\lambda}N(\lambda^{1/2k}) - \tfrac{1}{2}\int_0^\lambda \mu^{-1/2}N(\mu^{1/2k})d\mu$$

$$= 2kc_2\frac{\sin(\pi q/2k)}{\pi q}\,[\lambda^{\frac{q}{2k}+\frac{1}{2}} - \tfrac{1}{2}\int_0^\lambda \mu^{-\frac{1}{2}+\frac{q}{2k}}\,d\mu] + o(\lambda^{\frac{q}{2k}+\frac{1}{2}})$$

$$= 2kc_2\frac{\sin(\pi q/2k)}{\pi q}\int_0^\lambda \mu^{\frac{1}{2}}d(\mu^{q/2k}) + o(\lambda^{\frac{q}{2k}+\frac{1}{2}})$$

$$= c_2\,\frac{\sin(\pi q/2k)}{\pi}\,\frac{\lambda^{\frac{q}{2k}+\frac{1}{2}}}{\frac{q}{2k}+\frac{1}{2}} + o(\lambda^{\frac{q}{2k}+\frac{1}{2}}).$$

Thus, the converse of the Tauberian theorem implies

$$\int_0^\infty \frac{\sqrt{\lambda}}{\lambda + t^2}\,dN(\lambda^{1/2k}) = \int_0^\infty \frac{d\tilde{N}(\lambda)}{\lambda + t^2}$$

$$= c_2\,\frac{\sin(\pi q/2k)}{\sin\left(\frac{\pi q}{2k}+\frac{\pi}{2}\right)}\,t^{2(\frac{q}{2k}-\frac{1}{2})} + o(t^{2(\frac{q}{2k}-\frac{1}{2})})$$

$$= c_2\,\tan(\pi q/2k)t^{(q/k)-1} + o(t^{(q/k)-1}).$$

Therefore, (14.42) implies

$$2\int_0^\infty \frac{\sqrt{\lambda}}{\lambda + t^2}\,dN_-(\lambda^{1/2k}) = [c_2\,\tan\,(\pi q/2k) - c_1]t^{(q/k)-1} + o(t^{(q/k)-1}).$$

In order to apply the Tauberian theorem, let

$$(14.43)\qquad \tilde{N}_-(\lambda) = \int_0^\lambda \sqrt{\mu}dN_-(\mu^{1/2k}).$$

The previous asymptotic formula then shows that, after replacing t^2 by t,

$$\int_0^\infty \frac{1}{\lambda + t}\,d\tilde{N}_-(\lambda) = \tfrac{1}{2}[c_2\,\tan(\pi q/2k) - c_1]t^{\frac{q}{2k}-\frac{1}{2}}) + o(t^{\frac{q}{2k}-\frac{1}{2}}).$$

Therefore, Theorem 14.5 implies

$$\tilde{N}_-(\lambda) = \tfrac{1}{2}[c_2 \tan(\pi q/2k) - c_1] \frac{\sin\left(\frac{\pi q}{2k} + \frac{\pi}{2}\right)}{\pi\left(\frac{q}{2k} + \frac{1}{2}\right)} \lambda^{\frac{q}{2k}+\frac{1}{2}} + o(\lambda^{\frac{q}{2k}+\frac{1}{2}})$$

$$= \tfrac{1}{2}[c_2 \sin(\pi q/2k) - c_1 \cos(\pi q/2k)] \frac{1}{\pi\left(\frac{q}{2k}+\frac{1}{2}\right)} \lambda^{\frac{q}{2k}+\frac{1}{2}} + o(\lambda^{\frac{q}{2k}+\frac{1}{2}}).$$

But from (14.43) it then follows that

$$N_-(\lambda^{\frac{1}{2k}}) = \text{const} + \int_0^\lambda \mu^{-\frac{1}{2}} d\tilde{N}_-(\mu)$$

$$= \tfrac{1}{2}[c_2 \sin(\pi q/2k) - c_1 \cos(\pi q/2k)] \frac{1}{\pi\left(\frac{q}{2k}+\frac{1}{2}\right)} \cdot \int_0^\lambda \mu^{-\frac{1}{2}} d\mu^{\frac{q}{2k}+\frac{1}{2}} + o(\lambda^{q/2k})$$

$$= [c_2 \sin(\pi q/2k) - c_1 \cos(\pi q/2k)] \frac{k}{\pi q} \lambda^{q/2k} + o(\lambda^{q/2k}).$$

Replacing λ by λ^{2k}, this relation, becomes

$$N_-(\lambda) = [c_2 \sin(\pi q/2k) - c_1 \cos(\pi q/2k)] \frac{k}{\pi q} \lambda^q + o(\lambda^q). \tag{14.44}$$

Since $N = N_+ + N_-$, it follows from (14.44) and (14.41) that

$$N_+(\lambda) = [c_2 \sin(\pi q/2k) + c_1 \cos(\pi q/2k)] \frac{k}{\pi q} \lambda^q + o(\lambda^q). \tag{14.45}$$

Since $q = n/m'$, the theorem will be proved if we now compute $c_1 + ic_2 = c$. This we proceed to do.

As we have remarked, $A'(x, i\xi)$ is real for real ξ. Let x be a fixed point of Ω and let $p(\xi) = (A'(x, i\xi))^k$. Then (14.37) can be written in terms of its real and imaginary parts as

$$(2\pi)^n\rho(x) = \int_{E_n} \frac{d\xi}{p(\xi) - i}$$

$$= \int_{E_n} \frac{p(\xi)}{p(\xi)^2 + 1}\, d\xi + i \int_{E_n} \frac{1}{p(\xi)^2 + 1}\, d\xi.$$

For $0 < t < \infty$ let $V(t)$ be the region $\{\xi\colon\ |p(\xi)| < t\}$, and let $\nu(t)$ be the measure of $V(t)$: $\nu(t) = |V(t)|$. Then the homogeneity of A' shows that $V(t) = t^{1/km'}V(1)$, so that

$$\nu(t) = t^{n/km'}\nu(1) = t^{q/k}\,\nu(1).$$

Therefore,

$$(2\pi)^n\mathcal{J}\rho(x) = \int_0^\infty \frac{1}{t^2+1}\, d\nu(t)$$

$$= \frac{q}{k}\,\nu(1)\int_0^\infty \frac{t^{(q/k)-1}}{t^2+1}\, dt$$

$$= \frac{q}{k}\,\nu(1)\,\frac{1}{2}\int_0^\infty \frac{s^{(q/2k)-1}}{s+1}\, ds$$

$$= \frac{q}{2k}\,\nu(1)\,\frac{\pi}{\sin\frac{\pi q}{2k}}, \tag{14.46}$$

by (14.29) (we have substituted $s = t^2$).

It will be convenient to treat two cases separately.

Case 1. *m' is odd.* In this case $p(\xi)$ is odd, so that $\mathcal{R}\rho(x) = 0$. Also

$$\nu(1) = |\{\xi\colon\ |p(\xi)| < 1\}|$$

$$= 2|\{\xi\colon\ 0 < p(\xi) < 1\}|$$

$$= 2|\{\xi\colon\ 0 < A'(x, i\xi) < 1\}|$$

$$= 2\omega_+(x) = 2\omega_-(x).$$

Thus, (14.46) implies

$$\rho(x) = i\mathfrak{I}\rho(x)$$

$$= i(2\pi)^{-n} 2\omega_{\pm}(x) \frac{\pi q}{2k} \frac{1}{\sin(\pi q/2k)}.$$

Therefore, it follows from (14.36) that

$$c_1 + ic_2 = i(2\pi)^{-n} \frac{\pi q}{k} \frac{1}{\sin(\pi q/2k)} \int_{\Omega} \omega_{\pm}(x)dx$$

$$= \frac{\pi q}{k} \frac{1}{\sin(\pi q/2k)} ic_{\pm}.$$

Therefore, $c_1 = 0$ and $\sin(\pi q/2k)c_2 = (\pi q/k)c_{\pm}$. Hence, (14.44) and (14.45) imply

$$N_{\pm}(\lambda) = c_{\pm}\lambda^q + o(\lambda^q),$$

so that the theorem is proved in case m' is odd.

Case 2. m' is even. In this case $p(\xi) = (A'(x, i\xi))^k$ is even, and the ellipticity of A implies $p(\xi)$ does not change sign. Let z be the common sign of $p(\xi)$ and $A'(x, i\xi)$. Then

$$(2\pi)^{-n}\mathfrak{R}\rho(x) = \int_{E_n} \frac{p(\xi)}{p(\xi)^2 + 1} d\xi$$

$$= z\int_0^{\infty} \frac{t}{t^2 + 1} d\nu(t)$$

$$= z\frac{q}{k}\nu(1)\int_0^{\infty} \frac{t^{q/k}}{t^2 + 1} dt$$

$$= z\frac{q}{2k}\nu(1)\frac{\pi}{\sin\left(\frac{\pi q}{2k} + \frac{\pi}{2}\right)}$$

$$= z\frac{q}{2k}\nu(1)\frac{\pi}{\cos(\pi q/2k)},$$

as in (14.46). Also,

$$\nu(1) = |\{\xi\colon\ |p(\xi)| < 1\}|$$

$$= |\{\xi:\ 0 < zp(\xi) < 1\}|$$

$$= |\{\xi:\ 0 < zA'(x, i\xi) < 1\}|$$

$$= \omega_z(x);$$

also,

$$\omega_{-z}(x) = |\{\xi:\ -1 < zA'(x, i\xi) < 0\}| = 0.$$

Thus, (14.36) implies

$$c_1 + ic_2 = (2\pi)^{-n}[z \frac{q}{2k} \frac{\pi}{\cos(\pi q/2k)} \int_\Omega \omega_z(x)dx$$

$$+ i \frac{q}{2k} \frac{\pi}{\sin(\pi q/2k)} \int_\Omega \omega_z(x)dx]$$

$$= z \frac{q}{2k} \frac{\pi}{\cos(\pi q/2k)} c_z + i \frac{q}{2k} \frac{\pi}{\sin(\pi q/2k)} c_z.$$

Hence, $\cos(\pi q/2k)c_1 = z(\pi q/2k)c_z$ and $\sin(\pi q/2k)c_2 = (\pi q/2k)c_z$, so that (14.44) and (14.45) imply

$$N_-(\lambda) = (\tfrac{1}{2} - \tfrac{1}{2}z)c_z\lambda^q + o(\lambda^q),$$

$$N_+(\lambda) = (\tfrac{1}{2} + \tfrac{1}{2}z)c_z\lambda^q + o(\lambda^q).$$

As $c_{-z} = 0$, we find for $z = +$

$$N_-(\lambda) = c_-\lambda^q + o(\lambda^q),$$

$$N_+(\lambda) = c_+\lambda^q + o(\lambda^q),$$

and likewise for $z = -$. Q.E.D.

Some comments are in order concerning the theorem just proved. In case $n \geq 2$, $A'(x, i\xi)$ has constant sign for each fixed x. The proof of this is like that of Theorem 4.1, but simpler. If $A'(x, i\xi^1) > 0$ and

$A'(x, i\xi^2) < 0$, then there is a curve connecting ξ^1 and ξ^2 which does not pass through $\xi = 0$. Since $A'(x, i\xi)$ is continuous, it must happen that $A'(x, i\xi) = 0$ for some ξ on the curve, a contradiction of the ellipticity of A. Likewise, in case $n = 1$ and m' is even, $A'(x, i\xi)$ has constant sign for each fixed x. (Note that our theory certainly is valid in the case $n = 1$, that is for *ordinary* differential equations.) Therefore, in either of these cases it follows that if Ω is *connected*, either $\omega_+(x)$ or $\omega_-(x)$ vanishes identically. Therefore, either the positive or negative eigenvalues predominate in number. This is not the case for the exceptional situation of an ordinary differential equation of odd order. For example, the eigenvalues of the problem

$$\frac{1}{i}\frac{du}{dx} = \lambda u,$$

$$u(0) = u(\pi),$$

are all the even integers.

This brings up another question. Even in the cases $n \geq 2$ or m' even, it can actually happen that there are infinitely many eigenvalues of each sign. For second-order problems this indeed cannot happen, but there are examples involving the biharmonic operator in which there are infinitely many eigenvalues of each sign.

15. Non-Self-Adjoint Eigenvalue Problems

We still shall assume Ω to be a finite union of disjoint bounded open sets, each of which has the restricted cone property.

THEOREM 15.1. *Let* $A(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha$ *be an elliptic operator or order* $2m$ *in* Ω, *having real leading coefficients; let* A *be normalized so that* $A'(x, i\xi) > 0$ *for* $x \in \Omega$, ξ *real*, $\xi \neq 0$. *Let* $a_\alpha \in C^{|\alpha|*}(\Omega)$, *and let* A^* *be the formal adjoint of* A. *Let* $\mathfrak{A}$ *be a closed, densely defined operator with domain in* $L_2(\Omega)$, *such that*

1° $D(\mathfrak{A}) \cup D(\mathfrak{A}^*) \subset H_{2m}(\Omega)$;

2° *for* $u \in D(\mathfrak{A})$, $\mathfrak{A}u = Au$;
for $u \in D(\mathfrak{A}^*)$, $\mathfrak{A}^*u = A^*u$;

3° *all directions except the positive real axis are directions of minimal growth of the resolvent of* $\mathfrak{A}$;

4° *in case* $2m < n + 1$, *there exists an integer* k *such that* $k > (n+1)/2m$, *the coefficients of* A *are* $C^{(k-1)2m*}(\Omega)$, *and* $D(\mathfrak{A}^k) \cup D((\mathfrak{A}^*)^k) \subset H_{2mk}(\Omega)$.

Then the spectrum of $\mathfrak{A}$ *is discrete and the eigenvalues of* $\mathfrak{A}$ *have finite multiplicity. There are only a finite number of eigenvalues outside the angle* $|\arg \lambda| < \epsilon$, *for any positive* ϵ. *If* $\{\lambda_j\}$ *is the sequence of eigenvalues of* $\mathfrak{A}$, *counted according to multiplicity, and if for* $\lambda > 0$, $N(\lambda)$ *is the number of* λ_j *such that* $\Re\lambda_j \leq \lambda$, *then*

$$N(\lambda) = c_0\lambda^{n/2m} + o(\lambda^{n/2m}),$$

where

$$c_0 = (2\pi)^{-n} \int_\Omega |\{\xi\colon \; A'(x, i\xi) < 1\}| dx.$$

Proof. The proof is quite similar to that of Theorem 14.6 except that the condition of self adjointness is replaced by other restrictions. Let $\lambda_0 \in \rho(\mathfrak{A})$. Then $R((\lambda_0 - \mathfrak{A})^{-1}) = D(\mathfrak{A}) \subset H_{2m}(\Omega)$, so that Rellich's theorem implies that $(\lambda_0 - \mathfrak{A})^{-1}$ is a compact transformation of $L_2(\Omega)$ into $L_2(\Omega)$. Therefore, it follows that $(\lambda_0 - \mathfrak{A})^{-1}$ has a discrete spectrum, with eigenvalues of finite multiplicity. Hence, $\mathfrak{A}$ has the same properties. Note that this property of the spectrum of $\mathfrak{A}$ requires only that the resolvent set of $\mathfrak{A}$ be non empty. By considering the operator $\mathfrak{A} - \lambda_0$ instead of $\mathfrak{A}$ itself, it is seen that we may assume that $0 \in \rho(\mathfrak{A})$, and we shall thus consider the compact operator $T = \mathfrak{A}^{-1}$. The assertion on the location of the eigenvalues follows immediately from Theorem 12.2, since, for any $\epsilon > 0$, there exists a number λ_ϵ such that the set $\{\lambda\colon \; |\lambda| > \lambda_\epsilon, |\arg \lambda| > \epsilon\}$ is contained in the resolvent set of $\mathfrak{A}$.

Note that the conditions on m, n, and k are no different than those made in the statement of Theorem 14.6, concerning m', n', and k, since now $m' = 2m$ is even. As before, if $2m \geq n + 1$, set $k = 1$.

We need to verify the hypothesis of Theorem 14.4, applied to the operator T^k. Note that our assumptions 1° and 4° imply $R(T^k) = D(\mathfrak{A}^k) \subset H_{2mk}(\Omega)$, and $R((T^k)^*) = R((T^*)^k) = R((\mathfrak{A}^*)^{-k}) = D((\mathfrak{A}^*)^k) \subset H_{2mk}(\Omega)$, and $2mk > n$ if n is odd, $2mk > n + 1$ if n is even. Also, our assumption 3°, together with the relation

$$(T^k)_\lambda = -(\lambda - \mathfrak{A}^k)^{-1},$$

implies that except for the positive real axis, all directions are directions of minimal growth of the modified resolvent of T^k. We take $\Omega_0 = \Omega$, and for the elliptic operator $P(x, D)$ we take that operator whose characteristic polynomial is $(A'(x, i\xi))^k$. Our assumption that $A'(x, i\xi)$ is positive shows that 1° of Theorem 14.4 is verified trivially. Finally, 2° of Theorem 14.4 is checked in exactly the same manner as in the proof of Theorem 14.6. Therefore, we may now apply the conclusion of Theorem 14.4 to our case.

Combining the estimate of Theorem 14.4 for the trace of the operator $(T^k)_\lambda$ with Theorem 13.10, we obtain just as in the proof of Theorem 14.6 the asymptotic relation for fixed arg λ,

$$\Sigma_j \frac{1}{\lambda_j^k - \lambda} = c|\lambda|^{(n/k2m)-1} + o(|\lambda|^{(n/k2m)-1}),$$

for $|\lambda| \to \infty$, arg $\lambda \neq 0$. Now let $q = n/2m$ and let $\lambda = -t$, $t > 0$, to obtain for $t \to \infty$

$$(15.1) \quad \Sigma_j \frac{1}{\lambda_j^k + t} = ct^{(q/k)-1} + o(t^{(q/k)-1}).$$

Now let $\lambda_j = \mu_j + i\nu_j$ be the Cartesian representation of the complex number λ_j. The result that the eigenvalues are eventually located in any angle about the real axis implies that $\mu_j \to \infty$ and $\nu_j/\mu_j \to 0$. Therefore, it follows that $\lambda_j^k = \mu_j^k(1 + i\nu_j/\mu_j)^k = \mu_j^k + \mu_j^k 0(\nu_j|/\mu_j)$, and so if $|\nu_j/\mu_j| < \epsilon$ for $j > j_0$, then

$$\left| \sum_{j>j_0} \frac{1}{\lambda_j^k + t} - \sum_{j>j_0} \frac{1}{\mu_j^k + t} \right| \leq \sum_{j>j_0} \frac{|\mu_j^k - \lambda_j^k|}{(\lambda_j^k + t)(\mu_j^k + t)}$$

$$\leq \gamma\epsilon \sum_{j > j_0} \frac{\mu_j^k}{(\lambda_j^k + t)(\mu_j^k + t)}$$

$$\leq \gamma\epsilon \Sigma_j \frac{1}{\lambda_j^k + t} .$$

From this it is a consequence of (15.1) that as $t \to \infty$

$$\Sigma_j \frac{1}{\mu_j^k + t} = ct^{(q/k)-1} + o(t^{(q/k)-1}).$$

By definition of $N(\lambda)$, this may be written

$$\int_0^\infty \frac{1}{\mu + t} \, dN(\mu^{1/k}) = ct^{(q/k)-1} + o(t^{(q/k)-1}).$$

Now we may apply the Tauberian theorem, Theorem 14.5. The result is that

$$N(\mu^{1/k}) = c \frac{\sin (\pi q/k)}{\pi q/k} \mu^{q/k} + o(\mu^{q/k}).$$

Replacing $\mu^{1/k}$ by λ, this becomes

$$(15.2) \qquad N(\lambda) = \frac{ck}{\pi q} \sin (\pi q/k)\lambda^q + o(\lambda^q), \ \lambda \to \infty.$$

Thus, all that is left is the computation of c.

According to Theorem 14.4, with $\theta = \pi$,

$$c = \int_\Omega \rho(x)dx,$$

where

$$\rho(x) = (2\pi)^{-n} \int_{E_n} \frac{d\xi}{(A'(x, i\xi))^k + 1} .$$

If we let $p(\xi) = (A'(x, i\xi))^k$, and let

$$\nu(t) = |\{\xi:\ p(\xi) < t\}|,$$

then we obtain, as in the proof of Theorem 14.6,

$$\rho(x) = (2\pi)^{-n} \int_{E_n} \frac{d\xi}{p(\xi) + 1}$$

$$= (2)^{-n} \int_0^\infty \frac{1}{t+1}\, d\nu(t).$$

Since

$$\nu(t) = t^{n/2mk}\nu(1) = t^{q/k}\nu(1),$$

this becomes by (14.29)

$$\rho(x) = (2\pi)^{-n}\, \frac{q}{k}\nu(1) \int_0^\infty \frac{t^{(q/k)-1}}{t+1}\, dt$$

$$= (2\pi)^{-n} \frac{\pi q}{k \sin(\pi q/k)}\, \nu(1).$$

Therefore,

$$\frac{ck}{\pi q} \sin(\pi q/k) = (2\pi)^{-n} \int_\Omega \nu(1)dx$$

$$= (2\pi)^{-n} \int_\Omega |\{\xi:\ A'(x, i\xi) < 1\}|dx.$$

When this expression is compared with (15.2), the theorem is proved. Q.E.D.

Remark. To obtain results similar to those of Theorem 14.6, we need only assume in Theorem 15.1 that all directions except the positive real axis and negative real axis are directions of minimal growth of the resolvent of $\mathfrak{A}$. Then we get asymptotic expressions for the number of eigenvalues λ_j such that $0 < \Re\lambda_j < \lambda$ (or such that $-\lambda < \Re\lambda_j < 0$).

The question now arises concerning the hypothesis 3° of Theorem 15.1. Note that Lemma 14.3 implies that a direction $e^{i\theta}$ is of minimal growth of the resolvent of $\mathfrak{A}$ only if $A'(x, i\xi)$ omits the value $e^{i\theta}$. In case $\mathfrak{A}$ is self adjoint, then the remark made above allows the result of the theorem. Of course, this is not important, since we already had obtained Theorem 14.6 for the self-adjoint case. What is important, however, is that Theorem 15.1 applies to perturbations of self-adjoint operators. This concept shall now be discussed. First, a preliminary result is needed.

THEOREM 15.2. *Let $\mathfrak{A}_0$ be a closed, densely defined linear operator in the space $L_2(\Omega)$, such that $D(\mathfrak{A}_0) \subset H_{m'}(\Omega)$. Then there exists a positive number C such that for all $u \in D(\mathfrak{A}_0)$*

$$(15.3) \qquad ||u||_{m',\Omega} \leq C(||\mathfrak{A}_0 u||_{0,\Omega} + ||u||_{0,\Omega}).$$

Proof. Let $G \subset L_2(\Omega) \times L_2(\Omega)$ be the *graph* of $\mathfrak{A}_9$; that is,

$$G = \{(u, \mathfrak{A}_0 u): \ u \in D(\mathfrak{A}_0)\}.$$

Since $\mathfrak{A}_0$ is closed, G is a closed subspace of $L_2(\Omega) \times L_2(\Omega)$. Consider the mapping of G into $H_{m'}(\Omega)$ given by $(u, \mathfrak{A}_0 u) \to u$. This is a closed linear transformation of the Hilbert space G into the Hilbert space $H_{m'}(\Omega)$. For if

$$(u_j, \mathfrak{A}_0 u_j) \to (u, \mathfrak{A}_0 u) \text{ in } L_2(\Omega) \times L_2(\Omega),$$

$$u_j \to v \text{ in } H_{m'}(\Omega),$$

then certainly $u_j \to v$ in $L_2(\Omega)$, so that $v = u$. By the closed graph theorem it follows that the mapping $(u, \mathfrak{A}_0 u) \to u$ is a continuous mapping of G into $H_{m'}(\Omega)$. But this is precisely the statement of the theorem. Q.E.D.

This theorem is of considerable interest in other areas than the one at hand, and we shall have more to say about it later. For the time being, however, we wish to apply this result to perturbation of self-adjoint operators.

First, suppose the transformation $\mathfrak{A}_0$ of Theorem 15.2 is self-adjoint. Theorem 12.7 implies that for non-real λ,

$$\|u\|_{0,\Omega} \le |\mathfrak{I}\lambda|^{-1}\|(\lambda - \mathfrak{A}_0)u\|_{0,\Omega},\ u \in D(\mathfrak{A}_0);$$

or, equivalently,

$$(15.4)\qquad \|(\lambda - \mathfrak{A}_0)^{-1}\|_0 \le |\mathfrak{I}\lambda|^{-1}.$$

Therefore, for $u \in D(\mathfrak{A}_0)$ it follows from (15.3) that

$$\begin{aligned}\|u\|_{m',\Omega} &\le C[\|\mathfrak{A}_0 u\|_{0,\Omega} + \|u\|_{0,\Omega}]\\ &\le C[\|(\lambda - \mathfrak{A}_0)u\|_{0,\Omega} + |\lambda|\,\|u\|_{0,\Omega} + \|u\|_{0,\Omega}]\\ &\le C[\|(\lambda - \mathfrak{A}_0)u\|_{0,\Omega} + (|\lambda| + 1)|\mathfrak{I}\lambda|^{-1}\|(\lambda - \mathfrak{A}_0)u\|_{0,\Omega}]\\ &= C[1 + \frac{|\lambda| + 1}{|\mathfrak{I}\lambda|}]\,\|(\lambda - \mathfrak{A}_0)u\|_{0,\Omega}.\end{aligned}$$

Therefore,

$$\|(\lambda - \mathfrak{A}_0)^{-1}\|_{m'} \le C\,\frac{1 + 2|\lambda|}{|\mathfrak{I}\lambda|}\ .$$

By the convexity result of Lemma 13.3, this estimate together with (15.4) implies

$$(15.5)\qquad \|(\lambda - \mathfrak{A}_0)^{-1}\|_{m'-1} \le C_1\,\frac{(1 + |\lambda|)^{1-(1/m')}}{|\mathfrak{I}\lambda|}\ ,$$

with a different constant C_1.

Now suppose that $\mathfrak{A} = \mathfrak{A}_0 + \mathfrak{B}$, where $\mathfrak{A}_0$ is a self-adjoint transformation in $L_2(\Omega)$, with $D(\mathfrak{A}_0) \subset H_{m'}(\Omega)$, and $\mathfrak{B}$ is an operator of lower order. Precisely, assume $D(\mathfrak{B}) \supset D(\mathfrak{A}_0)$ and that $\mathfrak{B}$ satisfies an estimate

(15.6) $\quad ||\mathcal{B}u||_{0,\Omega} \leq C_2 ||u||_{m'-1,\Omega}.$

This inequality allows us to extend $\mathcal{B}$, if necessary, so that $D(\mathcal{B}) \supset H_{m'-1}(\Omega)$. For λ not real,

$$\lambda - \mathcal{A} = \lambda - \mathcal{A}_0 - \mathcal{B},$$

and, multiplying by $(\lambda - \mathcal{A}_0)^{-1}$ gives

(15.7) $\quad (\lambda - \mathcal{A}_0)^{-1}(\lambda - \mathcal{A}) = 1 - (\lambda - \mathcal{A}_0)^{-1}\mathcal{B};$

this equation must be interpreted as being valid on $D(\mathcal{A}_0) = D(\mathcal{A})$. Now $\mathcal{B}$ maps $H_{m'-1}(\Omega)$ into $L_2(\Omega)$, and $(\lambda - \mathcal{A}_0)^{-1}$ maps $L_2(\Omega)$ into $D(\mathcal{A}_0) \subset H_{m'-1}(\Omega)$. Therefore, $(\lambda - \mathcal{A}_0)^{-1}\mathcal{B}$ is a mapping of $H_{m'-1}(\Omega)$ into $H_{m'-1}(\Omega)$; the estimates (15.6) and (15.5) imply that the norm of $(\lambda - \mathcal{A}_0)^{-1}\mathcal{B}$, considered as a mapping on $H_{m'-1}(\Omega)$, is bounded by

(15.8) $\quad C_1 C_2 \dfrac{(1 + |\lambda|)^{1-(1/m')}}{|\mathcal{I}\lambda|}.$

If this quantity is less than, say, 1/2, then $1 - (\lambda - \mathcal{A}_0)^{-1}\mathcal{B}$ is an invertible mapping on $H_{m'-1}(\Omega)$, and its inverse has norm less than 2. Therefore, since

$$\lambda - \mathcal{A} = (\lambda - \mathcal{A}_0)[1 - (\lambda - \mathcal{A}_0)^{-1}\mathcal{B}]$$

on $D(\mathcal{A})$, then there exists

$$(\lambda - \mathcal{A})^{-1} = [1 - (\lambda - \mathcal{A}_0)^{-1}\mathcal{B}]^{-1}(\lambda - \mathcal{A}_0)^{-1}.$$

Moreover, the estimate of 2 for the norm of $[1 - (\lambda - \mathcal{A}_0)^{-1}\mathcal{B}]^{-1}$, as an operator on $H_{m'-1}(\Omega)$, shows

(15.9) $\quad ||(\lambda - \mathcal{A})^{-1}||_{m'-1} \leq 2||(\lambda - \mathcal{A}_0)^{-1}||_{m'-1}$

Multiplying (15.7) by $(\lambda - \mathcal{A})^{-1}$ yields the relation

$$(\lambda - \mathcal{A})^{-1} = (\lambda - \mathcal{A}_0)^{-1} + (\lambda - \mathcal{A}_0)^{-1}\mathcal{B}(\lambda - \mathcal{A})^{-1}.$$

Hence, by (15.6) and (15.9)

$$\begin{aligned} ||(\lambda - \mathcal{A})^{-1}||_0 &\le ||(\lambda - \mathcal{A}_0)^{-1}||_0[1 + ||\mathcal{B}(\lambda - \mathcal{A})^{-1}||_0] \\ &\le ||(\lambda - \mathcal{A}_0)^{-1}||_0[1 + C_2||(\lambda - \mathcal{A})^{-1}||_{m'-1}] \\ &\le ||(\lambda - \mathcal{A}_0)^{-1}||_0[1 + 2C_2||(\lambda - \mathcal{A}_0)^{-1}||_{m'-1}]. \end{aligned}$$

Since the expression (15. 8) is less than 1/2, (15.5) implies

$$2C_2||(\lambda - \mathcal{A}_0)^{-1}||_{m'-1} \le 1,$$

so that we obtain

$$\begin{aligned} ||(\lambda - \mathcal{A})^{-1}||_0 &\le 2||\lambda - \mathcal{A}_0)^{-1}||_0 \\ &\le \frac{2}{|\mathcal{I}\lambda|} . \end{aligned}$$

That is, if

(15.10) $\quad |\mathcal{I}\lambda| > 2C_1C_2(1 + |\lambda|)^{1-(1/m')},$

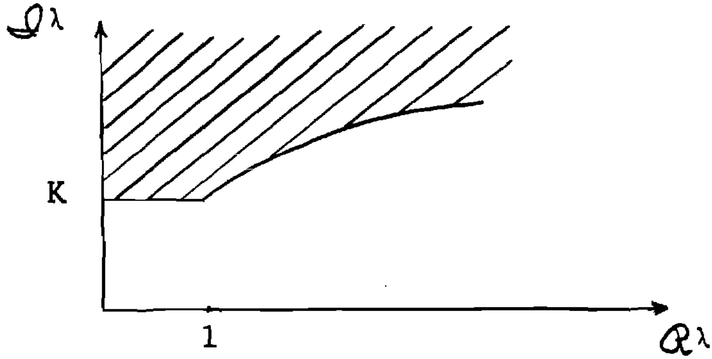
then $\lambda \in \rho(\mathcal{A})$ and we have the estimate

(15.11) $\quad ||(\lambda - \mathcal{A})^{-1}||_0 \le 2/|\mathcal{I}\lambda|.$

One can easily see that there exists a constant K such that if

(15.12) $\quad |\mathcal{I}\lambda| \ge K \max(|\mathcal{R}\lambda|^{1-(1/m')}, 1),$

then (15.10) is satisfied. A pictorial description of the region described by (15.12) is given in the figure; we have shown on the first quadrant.



Obviously, our results imply that any non-real direction is a direction of minimal growth of the resolvent of $\mathfrak{A}$. We have actually done much better, becuase our estimate (15.11) holds in a region of the λ plane which includes all rays $\Xi(\theta, a)$ with $e^{i\theta}$ not real, and a sufficiently large, depending on θ. And since the region (15.12) is in $\rho(\mathfrak{A})$, it follows that all the sufficiently large eigenvalues of $\mathfrak{A}$ lie in the "parabolic" type region. This is of course a much more restricted behavior of the eigenvalues than the conclusion of Theorem 15.1 can give.

Remark. If also $\mathfrak{A}_0$ satisfies an estimate

$$(\mathfrak{A}_0 u, u)_{0,\Omega} \geq \lambda_0 \|u\|^2_{0,\Omega}, \quad u \in D(\mathfrak{A}_0),$$

then the argument given above allows one to show that also the negative real axis is a direction of minimal growth of the resolvent of $\mathfrak{A}$. Cf. also Theorem 12.8.

We now wish to examine the abstract Theorem 15.2 and consider its consequences.

THEOREM 15.3. *Let $A(x, D)$ be a differential operator of order m' on Ω having continuous leading coefficients and measurable, locally bounded lower order coefficients. Let $\mathfrak{A}$ be a closed, densely defined linear operator in the space $L_2(\Omega)$, such that $C_0^\infty(\Omega) \subset D(\mathfrak{A}) \subset H_{m'}(\Omega)$, and $\mathfrak{A}u = Au$ for $u \in D(\mathfrak{A})$. Then A is elliptic.*

Proof. By Theorem 15.2, there exists a positive C such that

$$||u||^2_{m',\Omega} \le C[||Au||^2_{0,\Omega} + ||u||^2_{0,\Omega}], \quad u \in C_0^\infty(\Omega).$$

Thus, the quadratic form

$$||Au||^2_{0,\Omega} = \int_\Omega \Big| \sum_{|\alpha| \le m'} a_\alpha(x) D^\alpha u(x)\Big|^2 dx$$

$$= \sum_{\substack{|\alpha| \le m' \\ |\beta| \le m'}} \int_\Omega a_\alpha(x) \overline{a_\beta(x)} D^\alpha u(x) \cdot \overline{D^\beta u(x)} dx$$

satisfies Gårding's inequality. Therefore, the necessity of Gårding's inequality, Theorem 7.12, implies that for real non-zero ξ,

$$\sum_{\substack{|\alpha| = m' \\ |\beta| = m'}} a_\alpha(x) \overline{a_\beta(x)} \xi^\alpha \xi^\beta > 0,$$

or,

$$\Big| \sum_{|\alpha| = m'} a_\alpha(x) \xi^\alpha \Big|^2 > 0.$$

That is, $A'(x, \xi) \neq 0$. Q.E.D.

A consequence of Theorem 15.3 is that if $A(x, D)$ is *not* elliptic, and we wish to close the operator defined by applying A to functions in $C_0^\infty(\Omega)$, then the extended operator $\mathfrak{A}$ will have a domain which includes functions *not* in $H_{m'}(\Omega)$. For $u \in D(\mathfrak{A})$, the function Au still exists weakly, in the sense of Definition 2.2, but now Au can no longer be computed by taking the individual strong derivatives of u and forming the appropriate linear combination of them.

In the discussion above of operators of the form $\mathfrak{A} = \mathfrak{A}_0 + \mathfrak{B}$, where $\mathfrak{A}_0$ is self-adjoint and $\mathfrak{B}$ is of lower order, we obtained estimates for $\lambda - \mathfrak{A}$ just having an estimate (15.3) involving $\mathfrak{A}_0$. Of course, we also had to use the estimate $||(\lambda - \mathfrak{A}_0)^{-1}||_0 \le |\Im\lambda|^{-1}$, valid for arbitrary self adjoint $\mathfrak{A}_0$. We shall now show a general procedure for obtaining estimates containing a parameter λ, using only estimates not involving parameters. These latter estimates will be for an operator in a space of dimension one higher.

THEOREM 15.4. *Let $A(x, D)$ be an elliptic operator of order $2m$ in Ω, having leading coefficients in $C^0(\bar{\Omega})$ and lower order coefficients bounded and measurable in Ω. Let $\mathfrak{A}$ be a closed, densely defined operator in the space $L_2(\Omega)$ such that $D(\mathfrak{A})$ is the closure in $H_{2m}(\Omega)$ of a linear subspace M of $C^{2m}(\bar{\Omega})$, and $\mathfrak{A}u = Au$ for $u \in D(\mathfrak{A})$. Suppose that for some real θ, and for all real ξ and all $x \in \Omega$, $A'(x, i\xi) \neq e^{i\theta}$. Let $D_x^\alpha = D^\alpha$ and $D_t = \partial/\partial t$, and set*

$$L(x, D_x, D_t) = A_1(x, D_x) - (-1)^m e^{i\theta} D_t^{2m},$$

where A_1 is an operator having the same principal part as A, and having bounded measurable coefficients in Ω. Then L is elliptic in $\Gamma = \Omega \times E_1$.

Suppose that for all $v \in C^{2m}(\bar{\Gamma})$ such that $v(x, t_0) \in M$ for each fixed t_0 and $v(x, t) \equiv 0$ for $|t| \geq 1$,

$$\text{(15.13)} \qquad ||v||_{2m,\Gamma} \leq C(||Lv||_{0,\Gamma} + ||v||_{0,\Gamma}),$$

for some positive C. Then for $u \in D(\mathfrak{A})$, $\arg \lambda = \theta$, and $|\lambda|$ sufficiently large,

$$||u||_{0,\Omega} \leq \frac{2C}{|\lambda|} ||(\lambda - \mathfrak{A})u||_{0,\Omega}.$$

Moreover, if the resolvent set of $\mathfrak{A}$ is not empty, then $e^{i\theta}$ is a direction of minimal growth of the resolvent of $\mathfrak{A}$.

Proof. Let ζ be a function in $C^\infty(E_1)$ satisfying $|\zeta(t)| \leq 1$ for all t, $\zeta(t) \equiv 1$, $|t| \leq ½$, and $\zeta(t) \equiv 0$, $|t| \geq 1$. First, let $u \in M$, and let $v(x, t) = \zeta(t)e^{irt}u(x)$, where r is a positive parameter. According to the hypothesis, (15.13) is satisfied by the function v. Thus, for r $r \to \infty$ we have by Leibnitz's rule

$$|\zeta(t)e^{irt}u(x)|_{2m,\Gamma} \leq c[||\zeta(t)e^{irt}A_1 u - (-1)^m e^{i\theta} u D_t^{2m}(\zeta e^{irt})||_{0,\Gamma}$$

$$+ ||\zeta e^{irt} u||_{0,\Gamma}]$$

$$(15.14) \qquad \leq C\|\zeta e^{irt}Au - r^{2m}e^{i\theta}\zeta e^{irt}u\|_{0,\Gamma}$$

$$+ \gamma\|u\|_{2m-1,\Omega} + \gamma r^{2m-1}\|u\|_{0,\Omega},$$

for a certain constant γ. Now also

$$|\zeta e^{irt}u(x)|^2_{2m,\Gamma} = \sum_{|\alpha|+a=2m} \int_\Gamma |D_x^\alpha D_t^a(\zeta(t)e^{irt}u(x))|^2 dxdt$$

$$\geq \sum_{|\alpha|+a=2m} \int_{-1/2}^{1/2} \int_\Omega |D_x^\alpha D_t^a(e^{irt}u(x))|^2 dxdt$$

$$= \sum_{|\alpha|+a=2m} r^{2a} \int_\Omega |D_x^\alpha u(x)|^2 dx$$

$$= \sum_{a=0}^{2m} r^{2a}|u|^2_{2m-a,\Omega}.$$

Thus, if $r \geq 1$,

$$|\zeta e^{irt}u(x)|^2_{2m,\Gamma} \geq r^{4m}\|u\|^2_{0,\Omega} + r^2 \sum_{a=1}^{2m} |u|^2_{2m-a,\Omega}$$

$$= r^{4m}\|u\|^2_{0,\Omega} + r^2\|u\|^2_{2m-1,\Omega}.$$

Therefore, (15. 14) implies

$$[r^{4m}\|u\|^2_{0,\Omega} + r^2\|u\|^2_{2m-1,\Omega}]^{1/2} \leq C\| Au - r^{2m}r^{i\theta}u\|_{0,\Gamma}$$

$$+ \gamma\sqrt{2}[r^{4m-2}\|u\|^2_{0,\Omega} + \|u\|^2_{2m-1,\Omega}]^{1/2}$$

$$\leq C\sqrt{2}\|Au - r^{2m}e^{i\theta}u\|_{0,\Omega}$$

$$+ \gamma\sqrt{2}r^{-1}[r^{4m}\|u\|^2_{0,\Omega} + r^2\|u\|^2_{2m-1,\Omega}]^{1/2}.$$

Thus, if r is sufficiently large,

$$[r^{4m}\|u\|^2_{0,\Omega} + r^2\|u\|^2_{2m-1,\Omega}]^{1/2} \leq 2C\|Au - r^{2m}e^{i\theta}u\|_{0,\Omega}.$$

Hence,

$$||u||_{0,\Omega} \le 2Cr^{-2m}||Au - r^{2m}e^{i\theta}u||_{0,\Omega}.$$

Thus, if $\lambda = r^{2m}e^{i\theta}$, this becomes

$$||u||_{0,\Omega} \le 2C|\lambda|^{-1}||(A-\lambda)u||_{0,\Omega}, \quad |\lambda| \text{ large},$$

which is the required estimate.

In case u is an arbitrary element of $D(\mathfrak{A})$, let $\{u_k\} \subset M$ be chosen such that $u_k \to u$ in $H_{2m}(\Omega)$. Then $\mathfrak{A}u_k = Au_k \to Au = \mathfrak{A}u$ in $L_2(\Omega)$, so that the inequality for u follows from the proved inequality for u_k.

Finally, suppose that there exists a complex number λ_0 in $\rho(\mathfrak{A})$. Then $(\lambda_0 - \mathfrak{A})^{-1}$ is a mapping of $L_2(\Omega)$ into $D(\mathfrak{A}) \subset H_{2m}(\Omega)$, and so is compact by Rellich's theorem. Thus, the spectrum of $(\lambda_0 - \mathfrak{A})^{-1}$ is discrete. Hence, the same holds for the spectrum of $\mathfrak{A}$. For arg $\lambda = \theta$, $|\lambda|$ large, it follows from the estimate obtained above that $\lambda - \mathfrak{A}$ is one-to-one, so that λ is not an eigenvalue of $\mathfrak{A}$. Since the spectrum of $\mathfrak{A}$ is discrete, $\lambda \in \rho(\mathfrak{A})$. This estimate then implies

$$||(\lambda - \mathfrak{A})^{-1}||_0 \le 2C|\lambda|^{-1}.$$

Thus, $e^{i\theta}$ is a direction of minimal growth of the resolvent of $\mathfrak{A}$. Q.E.D.

We now shall give some indications of how the abstract theorems on eigenvalues can be applied to specific problems. First, we discuss the Dirichlet problem. For convenience, we shall assume that the operator has infinitely differentiable coefficients in $\bar{\Omega}$, and that Ω is of class C^∞. Of course, these restrictions are far more than is needed, and we could easily keep track of the regularity actually needed. But for simplicity, let us assume the infinite regularity. Assuming zero Dirichlet data, the Dirichlet problem can be stated:

$$\text{(15.15)} \quad \begin{cases} Au = f & \text{in} \quad \Omega, \\ D^\alpha u = 0 & \text{on } \partial\Omega, \ |a| \le m-1. \end{cases}$$

Here $A(x, D)$ shall be a strongly elliptic operator of order $2m$ in $\bar{\Omega}$.

By Gårding's inequality (Theorem 7.6) and Theorem 8.2, it follows that the GDP for $A + \lambda$ has a unique solution, for certain complex numbers λ. If A is suitably normalized, λ can be any sufficiently large real number. We shall replace A by $A + \lambda$, and thus assume (15.15) has a unique solution $u \in \overset{\circ}{H}_m(\Omega)$ for every f in $L_2(\Omega)$. Let $u = Tf$. By Theorem 9.8 it follows that u is actually in $H_{2m}(\Omega)$. Thus, $R(T) \subset H_{2m}(\Omega)$. Since the same results hold for the formal adjoint A^* of A, and since the solution operator associated with A^* is just the adjoint of T^*, it follows that $R(T^*) \subset H_{2m}(\Omega)$. For the operator $\mathfrak{A}$ we can take T^{-1}. Then $D(\mathfrak{A}) = H_{2m}(\Omega) \cap \overset{\circ}{H}_m(\Omega)$, and $\mathfrak{A}u = Au$ for $u \in D(\mathfrak{A})$. For, clearly $D(\mathfrak{A}) = R(T) \subset H_{2m}(\Omega) \cap \overset{\circ}{H}_m(\Omega)$. Conversely, suppose $u \in H_{2m}(\Omega) \cap \overset{\circ}{H}_m(\Omega)$. Let $f = Au$. Then $f \in L_2(\Omega)$, and, trivially, $u = Tf$. Thus, $H_{2m}(\Omega) \cap \overset{\circ}{H}_m(\Omega) \subset R(T)$, and $\mathfrak{A}u = \mathfrak{A}Tf = f = Au$. Likewise, $D(\mathfrak{A}^*) = H_{2m}(\Omega) \cap \overset{\circ}{H}_m(\Omega)$.

We also need to check the behavior of powers of T. Again, we shall appeal to the regularity theory of section 9, and prove that $R(T^k) \subset H_{2mk}(\Omega)$. For, suppose this has been proved for k, and let $u = T^{k+1}f$. Then $u = TT^kf$, so that $u \in H_{2m}(\Omega) \cap \overset{\circ}{H}_m(\Omega)$ and $Au = T^kf$. Again, Theorem 9.8 implies that $u \in H_{2m+2mk}(\Omega)$, since $T^kf \in H_{2mk}(\Omega)$. Now there are three cases we shall discuss.

1. In case A is formally self-adjoint, then $T = T^*$, so that $\mathfrak{A}$ is self-adjoint. Theorem 14.6 then applies to our case immediately.

2. Suppose next only that $A'(x, i\xi)$ is real. By passing to some Dirichlet form for A, it follows that A can be written in the form

$$A(x, D) = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} D^\alpha a_{\alpha\beta}(x) D^\beta;$$

cf. (8.9). Now let

$$a^0_{\alpha\beta}(x) = \tfrac{1}{2}[a_{\alpha\beta}(x) + a_{\beta\alpha}(x)],$$

and

$$A_0(x, D) = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} D^\alpha a^0_{\alpha\beta} D^\beta.$$

Then A_0 is formally self-adjoint (cf. (8.11)), since $a^0_{\alpha\beta} = a^0_{\beta\alpha}$, and A and A_0 have the same principal part. For, Leibnitz's rule implies

$$\begin{aligned} A_0' &= \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a^0_{\alpha\beta} D^{\alpha+\beta} \\ &= \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \tfrac{1}{2}[a_{\alpha\beta} + a_{\beta\alpha}] D^{\alpha+\beta} \\ &= \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta} D^{\alpha+\beta} \\ &= A'. \end{aligned}$$

Thus, $A = A_0 + B$, where A_0 is formally self-adjoint, $A_0'(x, i\xi)$ is real, and B is an operator of order less than $2m$. Moreover, the strong ellipticity of A implies by means of Gårding's inequality that, with suitable normalization,

$$\begin{aligned} (A_0 u, u)_{0,\Omega} &\geq \gamma_0 E_0 \|u\|^2_{m,\Omega} - \lambda_0 \|u\|^2_{0,\Omega} \\ &\geq -\lambda_0 \|u\|^2_{0,\Omega}. \end{aligned}$$

Therefore, we can apply all the results dealing with operators of the form $\mathfrak{A}_0 + \mathfrak{B}$, and obtain, in particular, that all directions except the positive real axis are directions of minimal growth of the resolvent of $\mathfrak{A}$. Compare the remark preceding Theorem 15.3. Also, we have "parabolic" condensation of eigenvalues along the positive real axis, as discussed above.

3. Finally, consider the case in which we assume only that A is strongly elliptic. By suitable normalization it may be assumed that $\{A'(x, i\xi)\colon\ x \in \bar{\Omega},\ \xi \text{ real}\}$ fills out precisely the region $\{\lambda\colon\ |\arg \lambda| \leq \theta_0\}$, where $\theta_0 < \tfrac{1}{2}\pi$. In this case, all directions not in this sector are directions of minimal growth of the resolvent of $\mathfrak{A}$. To prove this, we shall apply Theorem 15.4. Let θ be any number satisfying $\theta_0 < \theta < 2\pi - \theta_0$, and let

$$L(x, \xi, \tau) = A'(x, \xi) - (-1)^m e^{i\theta}\tau^{2m}.$$

Then

$$(-1)^m L(x, \xi, \tau) = A'(x, i\xi) - e^{i\theta}\tau^{2m}.$$

It is clear that for such θ, the values $(-1)^m L(x, \xi, \tau)$ lie in a cone properly contained in an open half space of the complex plane, so that $L(x, D_x, D_t)$ is actually strongly elliptic. Therefore, Gårding's inequality holds for L in a cylindrical domain $\Omega \times E_1$, so that (15.13) is verified. Moreover, since $\mathfrak{A} = T^{-1}$ and T is compact, it follows that the spectrum of $\mathfrak{A}$ is discrete; in particular, the resolvent set of $\mathfrak{A}$ is not empty. Thus, Theorem 15.4 applies to our case.

We conclude this section with a brief discussion of real second-order problems. We shall assume that

$$A = \sum_{i,j=1}^{n} a_{ij} D_i D_j + \sum_{i=1}^{n} a_i D_i + a$$

and that $-A'(x, \xi) \geq c_0 |\xi|^2$ for real ξ. In the case of the Neumann problem, the boundary condition has the form $\partial u/\partial \nu = 0$, in which $\partial u/\partial \nu$ is the conormal derivative corresponding to A. In case A is formally self-adjoint, we obtain immediately asymptotic expressions for the number of eigenvalues less than t. But in any case, this problem is almost self-adjoint, as in 2. above. Therefore, all but a finite number of eigenvalues lie in a parabolic region $\{\lambda\colon \Re\lambda > 0, |\lambda| \leq C|\Re\lambda|^{1/2}\}$. This is a result first obtained by Carleman.

In the case of an oblique derivative problem, we can show again that all directions except the positive real axis are directions of minimal growth. For this we consider the operator $L = A + e^{i\theta} D_t^2$ and apply Theorem 15.4. In this case the estimate (15.13) follows from remarks made at the end of section 10. It should be noted that it is not known in this case whether the eigenvalues condense in a parabolic fashion along the positive real axis.

16. Completeness of the Eigenfunctions

In this section we shall state conditions under which the generalized eigenvectors of an eigenvalue problem span the Hilbert space, or, at least, a certain subspace of the Hilbert space. First we shall state some theorems on the growth of analytic functions.

THEOREM 16.1. *Let* $F(\lambda)$ *be an entire complex-valued function, and let*

$$|F(\lambda)| \leq C_1 |\lambda|^{-1}$$

on two rays making an angle π/a *at the origin. Let* $r_1 < r_2 <\dots, r_k \to \infty$, *and let*

$$\max_{|\lambda| = r_k} |F(\lambda)| \leq C_2 e^{r_k^\beta},$$

where $\beta < a$. *Then* $|F(\lambda)| \leq C_1 |\lambda|^{-1}$ *everywhere between the two lines.*

THEOREM 16.2. *Let* $F(\lambda)$ *be an entire complex-valued function of finite order* ρ. *Then for any positive* ϵ *there exists a sequence* $r_1 < r_2 <\dots, r_k \to \infty$ *such that*

$$\min_{|\lambda| = r_k} |F(\lambda)| > e^{-r_k^{\rho+\epsilon}}.$$

The second of these theorems is a direct statement of a theorem on p. 273 in Titchmarsh, *The Theory of Functions*, 2nd ed., Oxford University Press (1939); the first is a Phragmen-Leindelöf theorem, and can be found in a slightly different form on p. 177 of the same book.

Now we give an estimate due to Carleman.

LEMMA 16.3. *Let* T *be an operator of finite double-norm on a Hilbert space, and let* ϵ *be a positive number. Then there exists a sequence* $r_1 < r_2 <\dots, r_k \to \infty$, *such that the modified resolvent* T_λ *exists on the circles* $\{\lambda: |\lambda| = r_k\}$, and for $|\lambda| = r_k$,

$$|||T_\lambda||| < e^{r_k^{2+\epsilon}}.$$

Proof. By a modification of the classical Fredholm theory of integral equations, we have a formula,

$$T_\lambda = \frac{\Delta_\lambda}{\delta(\lambda)}, \ \lambda \in \rho_m(T),$$

where $\delta(\lambda) = \sum_{m=0}^{\infty} \delta_m \lambda^m$ is a complex entire function, and $\Delta_\lambda = \sum_{m=0}^{\infty} \Delta_m \lambda^m$ is an operator-valued entire function, the convergence of the latter series being in the double-norm sense. Moreover, the following estimates are valid:

$$\delta_0 = 1,$$

$$|\delta_m| \leq (e/m)^{m/2}|||\,T|||^m;$$

$$\Delta_0 = T,$$

$$|||\Delta_m||| \leq \ e^{1/2}(e/m)^{m/2}|||T|||^{m+1}.$$

Also, T_λ exists precisely when $\delta(\lambda) \neq 0$. For a derivation of these formulas, see Zaanen, *Linear Analysis*, 1st ed. (1953), Amsterdam-Groningen, pp. 261-274. From these estimates we obtain

$$|\delta(\lambda)| \leq 1 + \sum_{m=1}^{\infty} (e/m)^{m/2}|||T|||^m|\lambda|^m.$$

Now let $\mu = \frac{1}{2}e|||T|||^2|\lambda|^2$, and split this sum into the odd and even terms, to obtain

$$|\delta(\lambda)| \leq 1 + \sum_{k=1}^{\infty} \frac{\mu^k}{k^k} + \sum_{k=0}^{\infty} \frac{\mu^{k+1/2}}{(k+1/2)^{k+1/2}}$$

$$\leq 1 + (2\mu)^{1/2} + \sum_{k=1}^{\infty} \frac{\mu^k}{k^k} + \mu^{1/2} \sum_{k=1}^{\infty} \frac{\mu^k}{k^k}$$

$$\leq 1 + (2\mu)^{1/2} + (1 + \mu^{1/2}) \sum_{k=1}^{\infty} \frac{\mu^k}{k!}$$

$$= 1 + (2\mu)^{1/2} - 1 - \mu^{1/2} + (1 + \mu^{1/2})e^{\mu}$$

$$\leq (1 + (2\mu)^{1/2})e^{\mu}.$$

Therefore,

$$|\delta(\lambda)| \leq (1 + e^{1/2}|||T|||\ |\lambda|)e^{\frac{1}{2}e|||T|||^2|\lambda|^2}.$$

Likewise, one sees that

$$|||\Delta_\lambda||| \leq |||T||| + \sum_{m=1}^{\infty} e^{1/2}(e/m)^{m/2}|||T|||^{m+1}|\lambda|^m$$

$$\leq e^{1/2}|||T|||[1 + \sum_{m=1}^{\infty} (e/m)^{m/2}|||T|||^m|\lambda|^m]$$

$$\leq (e^{1/2}|||T||| + e|||T|||^2|\lambda|)e^{\frac{1}{2}e|||T|||^2|\lambda|^2}.$$

Hence, both $\delta(\lambda)$ and Δ_λ are entire functions of order not exceeding 2.

Applying Theorem 16.2 to $\delta(\lambda)$, we obtain $r_1 < r_2 <\ldots, r_k \to \infty$, such that

$$\min_{|\lambda| = r_k} |\delta(\lambda)| > e^{-r_k^{2+\frac{1}{2}\epsilon}}.$$

Therefore, for $|\lambda| = r_k$ we have

$$|||T_\lambda||| \leq (e^{1/2}|||T||| + e|||T|||^2 r_k)e^{\frac{1}{2}e|||T|||^2 r_k^2} e^{r_k^{2+\frac{1}{2}\epsilon}}$$

$$< e^{r_k^{2+\epsilon}}$$

for r_k sufficiently large. Q.E.D.

If T is a linear transformation on a Hilbert space, let $sp(T)$ be the closed subspace spanned by the generalized eigenvectors of T, and let $sp'(T)$ be the closed subspace spanned by the generalized eigenvectors of T which correspond to non-zero eigenvalues. In case $Tf = \lambda f$ and $\lambda \neq 0$, then it is clear that $f \in R(T)$. Thus, we have the trivial relation $sp'(T) \subset \overline{R(T)}$. Therefore, if $R(T)$ is not dense in the Hilbert space, it cannot be expected that $sp'(T)$ is dense. The most that could ever be expected is that $sp' = \overline{R(T)}$. We shall now give an abstract theorem containing a sufficient condition for this to occur. This theorem is found in Dunford and Schwarz, *Linear Operators*, II, Interscience, 1963.

THEOREM 16.4. *Let T be an operator of finite double-norm on a Hilbert space X, such that there exist N directions of minimal growth of the modified resolvent of T, where the angle between any two adjacent rays is less than $\frac{1}{2}\pi$. (Thus, $N \geq 5$.) Then $sp'(T) = \overline{R(T)}$.*

Proof. The Riesz-Schauder theory of compact operators has as a consequence that the analytic function $(\lambda - T)^{-1}$ has poles at the non-zero eigenvalues of T. Thus, if λ_0 is a non-zero eigenvalue of T, the Laurent expansion of $(\lambda - T)^{-1}$ about λ_0 has the form

$$(\lambda - T)^{-1} = \sum_{\nu=1}^{k} B_\nu(\lambda - \lambda_0)^{-\nu} + \sum_{\nu=0}^{\infty} A_\nu(\lambda - \lambda_0)^{\nu}, \tag{16.1}$$

where all the operators B_ν, A_ν, commute with T. Therefore,

$$1 = (\lambda - T)^{-1}[(\lambda - \lambda_0) + (\lambda_0 - T)]$$

$$= \sum_{\nu=1}^{k} B_\nu(\lambda - \lambda_0)^{-\nu+1} + \sum_{\nu=0}^{\infty} A_\nu(\lambda - \lambda_0)^{\nu+1}$$

$$+ \sum_{\nu=1}^{k} B_\nu(\lambda_0 - T)(\lambda - \lambda_0)^{-\nu} + \sum_{\nu=0}^{\infty} A_\nu(\lambda_0 - T)(\lambda - \lambda_0)^{\nu}$$

$$= B_k(\lambda_0 - T)(\lambda - \lambda_0)^{-k} + \sum_{\nu=1}^{k-1} [B_\nu(\lambda_0 - T) + B_{\nu+1}](\lambda - \lambda_0)^{-\nu} + B_1 + A_0(\lambda_0 - T$$

$$+ \sum_{\nu=1}^{\infty} [A_\nu(\lambda_0 - T) + A_{\nu-1}](\lambda - \lambda_0)^\nu.$$

From this it follows that

$$B_k(\lambda_0 - T) = 0,$$

$$B_\nu(\lambda_0 - T) = -B_{\nu+1}, \quad \nu = 1,\dots,k-1.$$

Therefore,

$$(16.2) \qquad (\lambda_0 - T)^{k+1-\nu} B_\nu = 0, \quad \nu = 1,\dots,k.$$

Now suppose $f \in X$. By (16.1),

$$(\lambda - T)^{-1} f = \sum_{\nu=1}^{k} (\lambda - \lambda_0)^{-\nu} B_\nu f + \sum_{\nu=0}^{\infty} (\lambda - \lambda_0)^\nu A_\nu f.$$

The relation (16.2) implies that

$$(\lambda_0 - T) B_k f = 0,$$

$$(\lambda_0 - T)^2 B_{k-1} f = 0,$$

$$\vdots$$

$$(\lambda_0 - T)^k B_1 f = 0.$$

Therefore, the non-zero vectors among $B_k f, B_{k-1} f, \dots, B_1 f$ are all generalized eigenvectors of T corresponding to the eigenvalue λ_0.

Next, suppose λ_0 is a characteristic value of T. Since λ_0^{-1} is then an eigenvalue of T, for λ near λ_0 we obtain from (16.1)

$$T_\lambda = T(1 - \lambda T)^{-1}$$

$$= \lambda^{-1} T(\lambda^{-1} - T)^{-1}$$

$$(16.3) \qquad = T\Big[\sum_{\nu=1}^{k} B_\nu \lambda^{-1}(\lambda^{-1} - \lambda_0^{-1})^{-\nu} + \sum_{\nu=1}^{\infty} A_\nu \lambda^{-1}(\lambda^{-1} - \lambda_0^{-1})^{\nu}\Big].$$

Now

$$\lambda^{-1}(\lambda^{-1} - \lambda_0^{-1})^{-\nu} = \lambda^{-1}\Big(\frac{\lambda_0 - \lambda}{\lambda\lambda_0}\Big)^{-\nu}$$

$$= \lambda_0^{\nu}\lambda^{\nu-1}(-1)^{\nu}(\lambda - \lambda_0)^{-\nu}$$

$$= (-1)^{\nu}\lambda_0^{\nu}[\lambda_0 + (\lambda - \lambda_0)]^{\nu-1}(\lambda - \lambda_0)^{-\nu}$$

$$= (-1)^{\nu}\lambda_0^{\nu} \sum_{\mu=0}^{\nu-1} \binom{\nu-1}{\mu} \lambda_0^{\mu}(\lambda - \lambda_0)^{\nu-1-\mu}(\lambda - \lambda_0)^{-\nu}$$

$$= \sum_{\mu=0}^{\nu-1} (-1)^{\nu}\binom{\nu-1}{\mu} \lambda_0^{\mu+\nu}(\lambda - \lambda_0)^{-\mu-1}$$

$$= \sum_{\mu=1}^{\nu} (-1)^{\nu}\binom{\nu-1}{\mu-1} \lambda_0^{\mu+\nu-1}(\lambda - \lambda_0)^{-\mu}.$$

Therefore, (16.3) may be written

$$T_\lambda = \sum_{\nu=1}^{k}\sum_{\mu=1}^{\nu} (-1)^{\nu}\binom{\nu-1}{\mu-1}\lambda_0^{\mu+\nu-1} T B_\nu (\lambda - \lambda_0)^{-\mu} + T_1(\lambda),$$

where $T_1(\lambda)$ is analytic at λ_0. Rearranging this series, we obtain

$$T_\lambda = \sum_{\mu=1}^{k} (\lambda - \lambda_0)^{-\mu}\Big(\sum_{\nu=\mu}^{k} (-1)^{\nu}\binom{\nu-1}{\mu-1}\lambda_0^{\mu+\nu-1} T B_\nu\Big) + T_1(\lambda)$$

$$(16.4) \qquad = \sum_{\mu=1}^{k} C_\mu (\lambda - \lambda_0)^{-\mu} + T_1(\lambda)\,.$$

Here, as above, it holds that for any $f \in X$, $C_\mu f$ is either zero or a generalized eigenvector of T corresponding to the eigenvalue λ_0^{-1}. This follows from the corresponding fact for $B_\nu f$ and the formula for C_μ.

Now we shall prove that $sp'(T) = \overline{R(T)}$. Since we already know $sp'(T) \subset \overline{R(T)}$, it is enough to show that $\overline{R(T)} \subset sp'(T)$. This is equivalent to showing that the orthogonal complement of $R(T)$ contains that of $sp'(T)$. Thus, assume that g is orthogonal to $sp'(T)$ and that $f \in X$. The theorem will be proved once we show that $(Tf, g) = 0$.

Consider the function

$$F(\lambda) = (T_\lambda f, g).$$

This function is analytic on $\rho_m(T)$, and, as we have seen, has only poles at the characteristic values of T. Let λ_0 be any characteristic value. By (16.4), it follows that for λ near λ_0,

$$F(\lambda) = \sum_{\mu=1}^{k} (C_\mu f, g)(\lambda - \lambda_0)^{-\mu} + (T_1(\lambda)f, g).$$

As we have shown, $C_\mu f$ is either zero or is an element of $sp'(T)$. Since g is orthogonal to $sp'(T)$, in any case we have $(C_\mu f, g) = 0$. Therefore, $F(\lambda)$ has a removable singularity at λ_0. Since this holds for any λ_0 not in $\rho_m(T)$, $F(\lambda)$ is *entire*.

On any of the N rays of minimal growth of the modified resolvent of T,

$$|F(\lambda)| \le ||T_\lambda|| \; ||f|| \; ||g|| = 0(|\lambda|^{-1}).$$

Now let ϵ be a positive number such that the angle between any two such adjacent rays is less than $\pi/(2+\epsilon)$. By Lemma 16.3 there exist numbers r_k, $r_1 < r_2 < \ldots, r_k \to \infty$, such that for $|\lambda| = r_k$

$$||T_\lambda|| \le |||T_\lambda||| < e^{r_k^{2+\epsilon}}.$$

But now the Phragmen-Lindelöf theorem, Theorem 16.1, implies that $|F(\lambda)| \le C|\lambda|^{-1}$ between two such rays. Therefore, since such angular regions fill the complex plane, $|F(\lambda)| \le C_1|\lambda|^{-1}$ for *all* λ. Therefore, $F(\lambda)$ is bounded in the plane, and must be constant, by Liouville's theorem. Since $F(\lambda) \to 0$ as $|\lambda| \to \infty$, $F(\lambda) \equiv 0$; *i.e.*, $(T_\lambda f, g) \equiv 0$. Letting $\lambda \to 0$, we obtain $T_\lambda \to T$, so $(Tf, g) \equiv 0$. Q.E.D.

For applications of this result to elliptic problems, the following theorem is useful.

THEOREM 16.5. *Let $\mathfrak{A}$ be a closed, densely defined operator in the space $L_2(\Omega)$, such that $D(\mathfrak{A}) \subset H_{m'}(\Omega)$. Suppose that all but a finite number of directions are directions of minimal growth of the resolvent of $\mathfrak{A}$. In case $m' \leq \frac{1}{2}n$, suppose there exists an integer $k > n/2m'$ such that $D(\mathfrak{A}^k) \subset H_{km'}(\Omega)$. Then $sp(\mathfrak{A}) = L_2(\Omega)$.*

Proof. Let $\lambda_0 \in \rho(\mathfrak{A})$. Then $(\mathfrak{A} - \lambda_0)^{-1}$ maps $L_2(\Omega)$ into $D(\mathfrak{A}) \subset H_{m'}(\Omega)$, and is therefore compact, by Rellich's theorem. By considering $\mathfrak{A} - \lambda_0$ instead of $\mathfrak{A}$, it follows that it is sufficient to suppose $0 \in \rho(\mathfrak{A})$. Let $T = \mathfrak{A}^{-1}$. Clearly, $sp'(T) = sp(\mathfrak{A})$. In case $m' > \frac{1}{2}n$, then Theorem 13.5 implies T has finite double-norm. Since $T_\lambda = -(\lambda - \mathfrak{A})^{-1}$, the directions of minimal growth of $(\lambda - \mathfrak{A})^{-1}$ are also directions of minimal growth of T_λ. Therefore, Theorem 16.4 implies $sp'(T) = \overline{R(T)}$. But $R(T) = D(\mathfrak{A})$, so $\overline{R(T)} = L_2(\Omega)$. Thus, $sp(\mathfrak{A}) = sp'(T) = L_2(\Omega)$, and the result follows in this case.

Now suppose $m' \leq \frac{1}{2}n$, but $k > n/2m'$. Then $R(T^k) = D(\mathfrak{A}^k) \subset H_{km'}(\Omega)$, so that again Theorem 13.5 implies T^k has finite double-norm. Also, $R(T^k)$ is dense in $L_2(\Omega)$, since $R(T)$ is dense and T is continuous. That is, $\overline{R(T^k)} = L_2(\Omega)$.

Let $\omega_1,\ldots,\omega_k$ be the k roots of unity, and let z be any complex number such that $|z| < 1$. Then

$$z \sum_{i=1}^{k} \omega_i(1 - \omega_i z)^{-1} = z \sum_{i=1}^{k} \omega_i \sum_{j=0}^{\infty} \omega_i^j z^j$$

$$= \sum_{j=0}^{\infty} \left(\sum_{i=1}^{k} \omega_i^{j+1} \right) z^{j+1}.$$

Now

$$\sum_{i=1}^{k} \omega_i^{j+1} = \begin{cases} k, & \text{if } j+1 \text{ divides } k, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$z \sum_{i=1}^{k} \omega_i(1 - \omega_i z)^{-1} = \sum_{\nu=1}^{\infty} k z^{\nu k} = k z^k (1 - z^k)^{-1}.$$

Multiplying this relation by $1 - z^k = \prod_{j=1}^{k} (1 - \omega_j z)$, and dividing by z.

$$(16.5) \qquad \sum_{i=1}^{k} \omega_i \prod_{\substack{j=1 \\ j \neq i}}^{k} (1 - \omega_j z) = kz^{k-1}.$$

Although (16.5) has been derived under the assumption $0 < |z| < 1$, it must actually hold for *all* complex z, since both sides are just polynomials in z. Therefore, if we replace z by the *operator* zT, a corresponding relation is obtained:

$$(16.6) \qquad \sum_{i=1}^{k} \omega_i \prod_{\substack{j=1 \\ j \neq i}}^{k} (1 - \omega_j zT) = kz^{k-1}T^{k-1}.$$

Since also

$$\prod_{j=1}^{k} (1 - \omega_j zT) = 1 - z^k T^k,$$

it follows that $z^k \in \rho_m(T^k) \Leftrightarrow \omega_j z \in \rho_m(T)$ for all j. Assuming that $z^k \in \rho_m(T^k)$, multiply (16.6) by

$$(1 - z^k T^k)^{-1} = \prod_{j=1}^{k} (1 - \omega_j zT)^{-1}$$

to obtain

$$\sum_{i=1}^{k} \omega_i (1 - \omega_i zT)^{-1} = kz^{k-1}T^{k-1}(1 - z^k T^k)^{-1}.$$

Finally, multiply both sides of this by T to obtain

$$(16.7) \qquad \sum_{i=1}^{k} \omega_i T_{\omega_i z} = kz^{k-1}(T^k)_{z^k}.$$

This relation holds for all complex z such that $z^k \in \rho_m(T^k)$.

Let $e^{i\theta}$ be a complex number all of whose kth roots are on rays of minimal growth of the modified resolvent of T. The assumptions of the theorem imply that all but a finite set of the numbers $e^{i\theta}$ possess this property. Indeed, we didn't need to assume so much, but only enough to insure that such numbers $e^{i\theta}$ determine at least five lines such that any two adjacent lines make an angle less than $\frac{1}{2}\pi$. Letting $z = t^{1/k}e^{i\theta/k}$ in (16.7), we obtain

$$(T^k)_{te^{i\theta}} = k^{-1}t^{(1/k)-1}e^{i\theta(1-k)/k} \sum_{j=1}^{k} \omega_j T_{\omega_j t^{1/k}e^{i\theta/k}}.$$

Therefore, if $\lambda = te^{i\theta}$,

$$\begin{aligned}\|(T^k)_\lambda\| &\le k^{-1}|\lambda|^{(1/k)-1} \sum_{j=1}^{k} \|T_{\omega_j t^{1/k}e^{i\theta/k}}\| \\ &\le k^{-1}|\lambda|^{(1/k)-1} \sum_{j=1}^{k} C|\omega_j t^{1/k}e^{i\theta/k}|^{-1} \\ &\le C|\lambda|^{(1/k)-1}|\lambda|^{-1/k} \\ &\le C|\lambda|^{-1}.\end{aligned}$$

Therefore, $e^{i\theta}$ is a direction of minimal growth of the modified resolvent of T^k. Applying Theorem 16.4 to T^k, it is seen that $sp'(T^k) = \overline{R(T^k)} = L_2(\Omega)$.

Now we shall show that $sp'(T^k) \subset sp'(T)$. From this it will follow that $sp(\mathfrak{A}) = sp'(T) = L_2(\Omega)$, and the proof will be completed. First, if a is any positive integer, the rational function $\prod_{j=1}^{k} (z - \omega_j)^{-\alpha}$ can be decomposed into its partial fraction expansion. Thus, for certain complex numbers $c_{i\nu}$ we have

$$\prod_{j=1}^{k} (z - \omega_j)^{-\alpha} = \sum_{i=1}^{k} \sum_{\nu=1}^{\alpha} c_{i\nu}(z - \omega_i)^{-\nu}.$$

Multiplying this relation by $\prod_{j=1}^{k}(z-\omega_j)^{\alpha}$, we have an identity for all complex z,

$$\sum_{i=1}^{k}\sum_{\nu=1}^{\alpha} c_{i\nu}(z-\omega_i)^{\alpha-\nu}\prod_{\substack{j=1\\ j\neq i}}^{k}(z-\omega_j)^{\alpha}\equiv 1.$$

Since the left side is a polynomial in z, we obtain a corresponding identity if we replace z by the operator $z^{-1}T$. And then, multiplying the resulting expression by $z^{\alpha k}$, we obtain

$$(16.8)\qquad \sum_{i=1}^{k}\sum_{\nu=1}^{\alpha} c_{i\nu}z^{\nu}(T-\omega_i z)^{\alpha-\nu}\prod_{\substack{j=1\\ j\neq i}}^{k}(T-\omega_j z)^{\alpha}=z^{\alpha k},$$

a relation which holds for all complex z.

Now suppose f is a generalized eigenvector of T^k corresponding to a non-zero eigenvalue λ. Thus, for some positive integer a, $(T^k-\lambda)^{\alpha}f=0$. Let z be any complex number such that $z^k=\lambda$. Then $T^k-\lambda=T^k-z^k=\prod_{j=1}^{k}(T-\omega_j z)$.
Thus,

$$\prod_{j=1}^{k}(T-\omega_j z)^{\alpha}f=0.$$

This implies that for any i, $i=1,\ldots,k$, and any ν, $\nu=1,\ldots,a$,

$$(T-\omega_i z)^{\nu}[(T-\omega_i z)^{\alpha-\nu}\prod_{\substack{j=1\\ j\neq i}}^{k}(T-\omega_j z)^{\alpha}f]=0,$$

so that the vector

$$(T-\omega_i z)^{\alpha-\nu}\prod_{\substack{j=1\\ j\neq i}}^{k}(T-\omega_j z)^{\alpha}f$$

is either zero or is a generalized eigenvector of T corresponding to

$\omega_l z$. Hence, this vector belongs to $sp'(T)$. But then (16.8) implies $z^{\alpha k} f \in sp'(T)$, and, since $z \neq 0$, $f \in sp'(T)$. Thus, $sp'(T^k) \subset sp'(T)$. Q.E.D.

This result may be applied to the Dirichlet problem for elliptic operators of order $2m$. For example, if the principal part A' satisfies either the condition that $A'(x, i\xi)$ is real, or the condition that $|\arg A'(x, i\xi)| \leq a$, where $a < \frac{1}{4}\pi$ and $2m > \frac{1}{2}n$, then the generalized eigenvectors span $L_2(\Omega)$.

INDEX