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Steady Navier-Stokes equations with mixed boundary value conditions in three-dimensional Lipschitzian domains

Carsten Ebmeyer · Jens Frehse

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1 Introduction

We treat the steady Navier-Stokes equations

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0 \qquad \text{in } \Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitzian domain, $\partial\Omega = \Gamma_1 \cup \Gamma_2$, $\mathbf{u}, \mathbf{f} \in \mathbb{R}^3$, and $p \in \mathbb{R}$.

We consider (1.1) under one of the two following mixed boundary value conditions:

$$\begin{aligned} \mathbf{u} &= 0 && \text{on } \Gamma_1, \\ \mathbf{u}\cdot\mathbf{n} &= 0 && \text{on } \Gamma_2, \\ \mathbf{n}\cdot T(\mathbf{u}, p)\cdot\mathbf{t} &= 0 && \text{on } \Gamma_2 \text{ for all } \mathbf{t} \in M_{\mathbf{n}}, \end{aligned} \tag{1.3}$$

or

$$\begin{aligned} \mathbf{u} &= 0 && \text{on } \Gamma_1, \\ \mathbf{u}\cdot\mathbf{n} &= 0 && \text{on } \Gamma_2, \\ (\nabla\times\mathbf{u})\times\mathbf{n} &= 0 && \text{on } \Gamma_2, \end{aligned} \tag{1.4}$$

where $\mathbf{n}$ is the outward normal of $\partial\Omega$, $M_{\mathbf{n}} = \{\mathbf{t} \in \mathbb{R}^3 : \mathbf{t}\cdot\mathbf{n} = 0\}$, and $T(\mathbf{u}, p)$ denotes the stress Tensor with the components

$$T_{ij}(\mathbf{u}, p) = -p\delta_{ij} + \nu(\partial_i u_j + \partial_j u_i)$$

C. Ebmeyer
Mathematisches Seminar, Universität Bonn, Nussallee 15, D–53115 Bonn, Germany

J. Frehse
Institut für Angewandte Mathematik, Universität Bonn, Beringstr. 6, D–53115 Bonn, Germany

for $1 \leq i, j \leq 3$, $\partial_i = \frac{\partial}{\partial x_i}$, and $\mathbf{u} = (u_1, u_2, u_3)^T$.

We refer to (1.1), (1.2), (1.3) as problem $(\mathcal{P}_1)$, and to (1.1), (1.2), (1.4) as problem $(\mathcal{P}_2)$.

The boundary conditions (1.3) are the classical no-slip and slip conditions (on Γ_1 and Γ_2 respectiveley), whereas the boundary conditions (1.4) are non standard. They are investigated in [5]. Details about the slip condition can be found in [2]. Other boundary conditions have been studied in [3,4,12,17,20].

We assume that $\Omega \subset \mathbb{R}^3$ is a bounded but not necessarily simply connected domain, and that the boundary of Ω is piecewise smooth. We treat polyhedral domains and a class of Lipschitzian domains.

We prove $W^{s,2}$-regularity of $\mathbf{u}$ ($s < \frac{3}{2}$) and of p ($s < \frac{1}{2}$). Our method of proof is developed in [9] and [8]. It uses a difference quotient technique and provides regularity results in Nikolskii spaces. The claim follows by the imbedding theorem of Nikolskii spaces into Sobolev spaces.

Other results, concerning the regularity of Navier-Stokes equations on non-smooth domains, can be found in [6,7,12–14,17,18], where different methods are used.

The outline of this paper is as follows. In the next section the assumptions on the domain are given and the main results are stated. In section 4 we consider problem $(\mathcal{P}_1)$ on polyhedral domains and prove the $W^{s,2}$-regularity results for $\mathbf{u}$ and p. In section 5 we investigate the problem $(\mathcal{P}_2)$ on polyhedral domains. In section 6 the problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ are treated on Lipschitzian domains.

2 The main results

We assume $\mathbf{f} \in L^2(\Omega; \mathbb{R}^3)$. Below we give the two weak formulations of $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$. In both cases it is known that there exist weak solutions $(\mathbf{u}, p)$ (as defined in (4.2) and (5.1) below) such that $\mathbf{u} \in W^{1,2}(\Omega; \mathbb{R}^3)$; cf. [19] and [5].

We consider polyhedral domains and a class of Lipschitzian domains.

2.1 Polyhedral domains

Let $\partial\Omega = \bigcup_{1\leq i\leq M} \overline{\Gamma^i}$, where each Γ^i is an open subset of a two-dimensional hyperplane. Further, let $\partial\Gamma^i$ be polygonal, let $\Gamma^i \cap \Gamma^j = \emptyset$ for $i \neq j$, let $\Gamma^1, \cdots, \Gamma^\sigma \subset \Gamma_1$ and $\Gamma^{\sigma+1}, \cdots, \Gamma^M \subset \Gamma_2$. We suppose

(1) $\overline{\Gamma^{i_1}} \cap \ldots \cap \overline{\Gamma^{i_k}} = \emptyset$, if $k > 3$ and $i_1 < \ldots < i_k$.

(2) If $\Gamma^i \subset \Gamma_2$, $\partial\Gamma^i \cap \partial\Gamma^j \neq \emptyset$, and $i \neq j$ then $\Gamma^j \subset \Gamma_1$ and $\text{angle}(\Gamma^i, \Gamma^j) \leq \pi$.

Remark 2.1 i) By $\text{angle}(\Gamma^i, \Gamma^j)$ we denote the inner angle between Γ^i and Γ^j where it is assumed that $\partial\Gamma^i \cap \partial\Gamma^j \neq \emptyset$.
ii) We do not need that Ω is simply connected. But we only consider classical

polyhedrons.
iii) Only in the case when $\Gamma^i, \Gamma^j \subset \Gamma_1$ we admit angle$(\Gamma^i, \Gamma^j) > \pi$.
iv) We do not treat the case that $\Gamma^i, \Gamma^j \subset \Gamma_2$, $i \neq j$, and $\partial\Gamma^i \cap \partial\Gamma^j \neq \emptyset$.
Let us note that there are weak solutions $(\mathbf{u}, p)$ (for the Stokes system) such that $\mathbf{u} \notin W^{\frac{3}{2}-\varepsilon,2}(\Omega; \mathrm{I\!R}^3)$, if angle$(\Gamma^i, \Gamma^j) > \frac{2}{3}\pi$, $\Gamma^i, \Gamma^j \subset \Gamma_2$, and $\partial\Gamma^i \cap \partial\Gamma^j \neq \emptyset$; cf. Orlt, Sändig [17] for problem $(\mathcal{P}_1)$ and Costabel, Dauge [6] for problem $(\mathcal{P}_2)$.

Now, we state the main results for polyhedral domains.

Theorem 2.2 *Let $(\mathbf{u}, p)$ be a weak solution of $(\mathcal{P}_1)$ and let Ω be a polyhedral domain satisfying the above conditions.*
a) There holds that

$$\mathbf{u} \in W^{s,2}(\Omega; \mathrm{I\!R}^3) \quad \textit{for all } s < \frac{3}{2}. \tag{2.1}$$

b) There holds that

$$p \in W^{s,2}(\Omega) \quad \textit{for all } s < \frac{1}{2}. \tag{2.2}$$

Theorem 2.3 *Let $(\mathbf{u}, p)$ be a weak solution of $(\mathcal{P}_2)$ and let Ω be a polyhedral domain satisfying the above conditions. Then there holds that*

$$\mathbf{u} \in W^{s,2}(\Omega; \mathrm{I\!R}^3) \quad \textit{for all } s < \frac{3}{2} \tag{2.3}$$

and

$$p \in W^{s,2}(\Omega) \quad \textit{for all } s < \frac{1}{2}. \tag{2.4}$$

Remark 2.4 i) The boundary conditions (1.3) and (1.4) are the same on polyhedral domains. But, due to the different variational formulations of $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$, the proofs of Theorems 2.2 and 2.3 are not the same. In the last section we modify these proofs in order to obtain our results for Lipschitzian domains.
ii) For convenience, we assume $\mathbf{f} \in L^s(\Omega, \mathrm{I\!R}^3)$ for $s = 2$. Let us note that we can prove our results for all $s \geq \frac{3}{2}$.
iii) In Kozlov, Maz'ya, Roßmann [12] similar results concerning the Stokes system under mixed boundary value conditions are proven. The authors use spectral analysis and obtain asymptotics of solutions on polyhedral cones.

2.2 Lipschitzian domains

Now, we state the results of Theorem 2.2 and Theorem 2.3 for more general Lipschitzian domains. We will consider domains where $\partial\Omega$ is $W^{2,\infty}$–piecewise.

Let $\partial\Omega = \bigcup_{1\le i\le M} \overline{\Gamma^i}$, where Γ^i are open 2-dimensional manifolds such that $\Gamma^i \cap \Gamma^j = \emptyset$ for $i \neq j$. Further, let $\partial\Gamma^i$ $(1 \le i \le M)$ be 1-dimensional Lipschitz continuous manifolds. We suppose that $\Gamma^1, \cdots, \Gamma^\sigma \subset \Gamma_1$ and $\Gamma^{\sigma+1}, \cdots, \Gamma^M \subset \Gamma_2$, and that $P \in \bigcap_{i\in\Lambda} \partial\Gamma^i$ implies that $|\Lambda| \le 3$. Next, we assume that to each point $P \in \partial\Omega$ there exists a mapping ϕ and a ball $B_R(\phi(P))$ such that
i) $B_R(\phi(P)) \cap \phi(\partial\Omega)$ is the intersection of $B_R(\phi(P))$ and a polyhedron,
ii) if $\Gamma^i \subset \Gamma_2$, $P \in \partial\Gamma^i \cap \partial\Gamma^j$, and $i \neq j$ then $\Gamma^j \subset \Gamma_1$ and angle $(\phi(\Gamma^i), \phi(\Gamma^j)) \le \pi$,
iii) $\phi, \phi^{-1} \in W^{2,\infty}_{loc}(\mathbb{R}^3)$,
iv) the Jacobian of ϕ is positive definite.

We state the main results for Lipschitzian domains.

Theorem 2.5 *Let* $(\mathbf{u}, p)$ *be a weak solution of* ($\mathcal{P}_1$) *and let* Ω *be a Lipschitzian domain satisfying the above conditions. Then there holds that*

$$\mathbf{u} \in W^{1+s,2}(\Omega;\mathbb{R}^3) \quad \textit{and} \quad p \in W^{s,2}(\Omega) \qquad \textit{for all } s < \frac{1}{2}. \tag{2.5}$$

Theorem 2.6 *Let* $(\mathbf{u}, p)$ *be a weak solution of* ($\mathcal{P}_2$) *and let* Ω *be a Lipschitzian domain satisfying the above conditions. Then there holds that*

$$\mathbf{u} \in W^{1+s,2}(\Omega;\mathbb{R}^3) \quad \textit{and} \quad p \in W^{s,2}(\Omega) \qquad \textit{for all } s < \frac{1}{2}. \tag{2.6}$$

Remark 2.7 In Kozlov, Maz'ya, Schwab [13] (and [11]) the Stokes system is investigated on a non-convex Lipschitzian domain under a Dirichlet boundary condition. There, Ω is a ball where a cone with top in the midpoint of the ball is cutted out. This domain Ω does not satisfy the above hyptheses (i.e., there is no smooth mapping ϕ fulfilling the conditions i)–v)). Let us note that we could try to apply our method to a domain with a curved boundary without using such a mapping ϕ. In principle, this is possible, if we treat, e.g., the Laplace equation. But in the case of the Stokes or Navier-Stokes equations this method of proof would fail, for our test functions would not be divergence free. (This is due to the reflections made in our proof.)

3 Notations

We use the Sobolev spaces $W^{s,p}(\Omega)$ and the Nikolskii spaces $\mathcal{H}^{s,p}(\Omega)$; cf. [1]. Let $m \geq 0$ be an integer, $0 < \sigma < 1$, $s = m + \sigma$, $z \in \mathbb{R}^3$, $\Omega_\eta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq \eta\}$, and $1 \leq p < \infty$. The spaces $W^{s,p}(\Omega)$ and $\mathcal{H}^{s,p}(\Omega)$ consist of all functions $f : \Omega \to \mathbb{R}$ for which the norms

$$\|f\|^p_{W^{s,p}(\Omega)} = \|f\|^p_{W^{m,p}(\Omega)} + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|^p}{|x-y|^{n+p\sigma}}\, dxdy$$

and

$$\|f\|^p_{\mathcal{H}^{s,p}(\Omega)} = \|f\|^p_{L^p(\Omega)} + \sum_{|\alpha|=m} \sup_{\substack{\eta>0\\ 0<|z|<\eta}} \int_{\Omega_\eta} \frac{|\partial^\alpha f(x+z) - \partial^\alpha f(x)|^p}{|z|^{\sigma p}}\, dx$$

are finite.

Let $B_R(x) = \{y \in \mathbb{R}^3 : |x-y| < R\}$. In what follows we consider a fixed point $P \in \partial\Omega$ and a radius $R_0 \in (0, 1]$ such that P is the only vertex of $B_{R_0}(P) \cap \partial\Omega$ or that there is no vertex of $\partial\Omega$ in $B_{R_0}(P)$. Further, let $B_{R_0}(P) \cap \partial\Omega$ be simply connected. (Let us note that the maximal size of the number R_0 depends only on the geometry of $\partial\Omega$.)

We need appropriate basis vectors $\{\zeta^1, \zeta^2, \zeta^3\}$ in $B_{R_0} \equiv B_{R_0}(P)$. Let us assume that $B_{R_0} \cap \partial\Omega$ is the intersection of B_{R_0} and a polyhedron. (In case that Ω is not a polyhedron we consider $B_{R_0}(\phi(P)) \cap \phi(\partial\Omega)$.)

Roughly speaking, our basis vectors are parallel to Γ_1 or Γ_2 or across $\partial\Omega$. Let us suppose that Λ_1, Λ_2, and Λ_3 are disjoint index sets (some of them possibly empty) such that $\cup_{i=1}^3 \Lambda_i = \{1, 2, 3\}$. Let $\alpha^* > 0$, $|\zeta^i| = 1$ for $1 \leq i \leq 3$, and $\operatorname{angle}(\zeta^i, \zeta^j) \geq \alpha^*$ for $1 \leq i < j \leq 3$. Further, let $x + s\zeta^i \in \overline{\Omega}$ for $x \in \partial\Omega \cap B_{R_0}$, $1 \leq i \leq 3$, and $0 < s < R_0$. We assume that

(1) ζ^i ($i \in \Lambda_1$) is parallel to $\Gamma_1 \cap B_{R_0}$. (Parallel means parallel to all $\Gamma^i \subset \Gamma_1$ with $\Gamma^i \cap B_{R_0} \neq \emptyset$). If $\Gamma_1 \cap B_{R_0} = \emptyset$ then $\Lambda_1 = \{1, 2, 3\}$ (and all ζ^i must satisfy (2), (5), and (6)).

(2) If $i \in \Lambda_1$, $x \in \Gamma_1 \cap B_{R_0}$, $s > 0$ and $x + s\zeta^i \in B_{R_0}$ then $x + s\zeta^i \in \Gamma_1$.

(3) ζ^i ($i \in \Lambda_2$) is parallel to $\Gamma_2 \cap B_{R_0}$. If $\Gamma_2 \cap B_{R_0} = \emptyset$ then $\Lambda_2 = \{1, 2, 3\}$ (and all ζ^i must satisfy (4)–(6)).

(4) ζ^i ($i \in \Lambda_2$) satisfies $\operatorname{angle}(\zeta^i, \Gamma^k) \geq \alpha^*$, if $\Gamma^k \cap B_{R_0} \neq \emptyset$, $\Gamma^k \subset \Gamma_1$, and $\operatorname{angle}(\Gamma^k, \Gamma_2 \cap B_{R_0}) \neq \pi$.

(5) If ζ^i is not parallel to $\Gamma^k \cap B_{R_0}$ then $\operatorname{angle}(\zeta^i, \Gamma^k) \geq \alpha^*$.

(6) If $\operatorname{angle}(\zeta^i, \Gamma^k \cap B_{R_0}) \geq \alpha^*$ then $x - s\zeta^i \notin \overline{\Omega}$ for all $x \in \Gamma^k \cap B_{R_0}$ and $0 < s < R_0$.

(7) If $\operatorname{angle}(\Gamma^i, \Gamma^j) = \pi$ ($i \neq j$, $\partial\Gamma^i \cap \partial\Gamma^j \cap B_{R_0} \neq \emptyset$), then $\Lambda_3 = \{3\}$, otherwise $\Lambda_3 = \emptyset$.

(8) ζ^3 $(3 \in \Lambda_3)$ is parallel to $(\partial\Omega \cap B_{R_0}) \setminus (\Gamma^i \cup \Gamma^j)$ and angle$(\zeta^3, \Gamma^i \cup \Gamma^j) \geq \alpha^*$ (where i, j are given in (7)).

Remark 3.1 There exists a basis $\{\zeta^1, \zeta^2, \zeta^3\}$ of $\mathbb{R}^3$ satisfying the above conditions. (Some examples how to choose the basis can be found in [8].) Further, the constant α^* only depends on the geometry of Ω.

Let $h > 0$. We set

$$D_i^h f(x) = \frac{E_i^h f(x) - f(x)}{h} \quad \text{and} \quad D_i^{-h} f(x) = \frac{f(x) - E_i^{-h} f(x)}{h},$$

where $E_i^\sigma x = x + \sigma\zeta^i$ and $E_i^\sigma f(x) = f(x + \sigma\zeta^i)$. We will write $E_i^\sigma f(x)g(x)$ instead of $(E_i^\sigma f(x))g(x)$.

Let $R = \frac{R_0}{15}$, $B = B_R(P) \cap \Omega$, $B' = B_{6R}(P) \cap \Omega$, $B'' = B_{7R}(P) \cap \Omega$, and

$$\Omega_i^h = \{y \in B_{R_0} \cap \Omega : y \neq x + h\zeta^i,\ x \in B_{R_0} \cap \Omega\},$$

$$\Omega_i^{-h} = \{y \in B_{R_0} \setminus \Omega : y = x - h\zeta^i,\ x \in B_{R_0} \cap \Omega\}.$$

Further, let τ_0 be a cut-off function with $\tau_0 \equiv 1$ in B_{6R}, supp $\tau_0 = B_{7R}$ and $|\nabla\tau_0| \leq c$, where c depends only on R_0. By τ we denote the restriction of τ_0 onto $\overline{\Omega}$.

Next, we define an appropriate extension of $\mathbf{u}$ into Ω_i^{-h} for $i \in \Lambda_2$. Let $y \in \partial\Omega \cap \partial\Omega_i^{-h}$, $0 < \lambda \leq h$ and $y - \lambda\zeta^i \in \Omega_i^{-h}$. We set

$$\mathbf{u}(y - \lambda\zeta^i) = 0. \tag{3.1}$$

This provides an $W^{1,2}$-extension of $\mathbf{u}$, for $i \in \Lambda_2$ implies that $\mathbf{u} = 0$ on $\partial\Omega \cap \partial\Omega_i^{-h}$. Moreover, (3.1) is an $\mathcal{H}^{\frac{3}{2},2}$-extension for $\mathcal{H}_0^{\frac{3}{2},2}(\Omega)$-functions where $\mathcal{H}_0^{\frac{3}{2},2}(\Omega) = \{f \in \mathcal{H}^{\frac{3}{2},2}(\Omega) : f = 0 \text{ on } \partial\Omega\}$.

Below, we will write $\sum_i$ instead of $\sum_{i=1}^3$. Further, $\nabla\mathbf{u}$ is a (3×3)-matrix and $|\nabla\mathbf{u}|^2 = \sum_{i,j} |\partial_i u_j|^2$. The dot $\cdot$ denotes the Euclidean scalar product and c denotes a constant which will be allowed to vary from equation to equation.

4 The problem $(\mathcal{P}_1)$ on polyhedral domains

In this section the proof of Theorem 2.2 is given. Let

$$V = \{\mathbf{v} \in W^{1,2}(\Omega; \mathbb{R}^3) : \mathbf{v} = 0 \text{ on } \Gamma_1, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_2\} \tag{4.1}$$

and $\mathcal{D}_{ij}(\mathbf{u}) = \partial_i u_j + \partial_j u_i$. Notice that there holds the following Green's formula

$$\begin{aligned}
&\int_\Omega (-\nu\Delta\mathbf{u} + \nabla p) \cdot \mathbf{w} \\
&= \frac{\nu}{2} \sum_{i,j} \int_\Omega \mathcal{D}_{ij}(\mathbf{u})\mathcal{D}_{ij}(\mathbf{w}) - \int_\Omega p\nabla \cdot \mathbf{w} - \int_{\partial\Omega} \mathbf{n} \cdot T(\mathbf{u}, p) \cdot \mathbf{w},
\end{aligned}$$

where $\mathbf{w} \in W^{1,2}(\Omega; \mathbb{R}^3)$, $p \in W^{1,2}(\Omega)$, and $\mathbf{u} \in W^{2,2}(\Omega; \mathbb{R}^3)$. Thus, $(\mathbf{u}, p)$ is called a weak solution of problem $(\mathcal{P}_1)$ if

$$\frac{\nu}{2} \sum_{i,j} \int_\Omega \mathcal{D}_{ij}(\mathbf{u}) \mathcal{D}_{ij}(\mathbf{w}) + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{w} - \int_\Omega p \nabla \cdot \mathbf{w} = \int_\Omega \mathbf{f} \cdot \mathbf{w} \tag{4.2}$$

for all $\mathbf{w} \in V$.

It is known that there exists a weak solution $(\mathbf{u}, p)$ such that $\mathbf{u} \in W^{1,2}(\Omega; \mathbb{R}^3)$; cf. [19]. Thus, equation (1.1) yields $\nabla p \in H^{-1}(\Omega; \mathbb{R}^3)$. But if an H^{-1}–function p has all its first derivatives $\partial_i p$ $(1 \le i \le 3)$ in $H^{-1}(\Omega)$, then it follows that (cf. [15])

$$p \in L^2(\Omega). \tag{4.3}$$

Further, let us state the second Korn inequality.

Lemma 4.1 *There exist two constants $c_1, c_2 > 0$ depending only on Ω such that*

$$\sum_{i,j} \left\| \mathcal{D}_{ij}(\mathbf{w}) \right\|^2_{L^2(\Omega)} \ge c_1 \left\| \nabla \mathbf{w} \right\|^2_{L^2(\Omega)} - c_2 \left\| \mathbf{w} \right\|^2_{L^2(\Omega)} \tag{4.4}$$

for all $\mathbf{w} \in W^{1,2}(\Omega; \mathbb{R}^3)$.

A proof for Lipschitzian domains can be found in [16]. Next, we prove the following lemma.

Lemma 4.2 *Let $1 \le i \le 3$, $0 < \sigma < \frac{1}{2}$, and supp $z = B''$. Then there is a constant c depending on σ such that*

$$\sup_{0<h<R} \left\| h^\sigma D_i^h z \right\|_{L^3(B'')} \le c \sup_{0<h<R} \left\| h^{\frac{1}{2}+\sigma} D_i^h z \right\|_{W^{1,2}(B'')}. \tag{4.5}$$

Proof. We prove (4.5) for $i = 1$. Let $0 < h < R$ and $\mu = 4R$. We set $g = h^{\frac{1}{2}+\sigma} D_1^h z$ and $\Omega_\mu = \{y \in B_{R_0} : y = x + \lambda \zeta^j, x \in B'', 0 \le \lambda \le \mu, 1 \le j \le 3\}$. The chain of imbeddings

$$W^{1,2}(\Omega_\mu) \to W^{\frac{1}{2},3}(\Omega_\mu) \to \mathcal{H}^{\frac{1}{2},3}(\Omega_\mu)$$

implies that

$$\sup_{0<\bar{h}<\mu} \left\| \bar{h}^{\frac{1}{2}} D_j^{\bar{h}} g \right\|_{L^3(B'')} \le c \left\| g \right\|_{W^{1,2}(\Omega_\mu)} \equiv c \left\| g \right\|_{W^{1,2}(B'')}$$

for $1 \le j \le 3$, where we have used the fact that $g \equiv 0$ in $\Omega_\mu \setminus B''$. Let us choose $j = 1$ and $\bar{h} = h$. Replacing g by $h^{\frac{1}{2}+\sigma} D_1^h z$ we obtain

$$\left\| h^{1+\sigma} D_1^h D_1^h z \right\|_{L^3(B'')} \le c \left\| h^{\frac{1}{2}+\sigma} D_1^h z \right\|_{W^{1,2}(B'')}. \tag{4.6}$$

Now, the conclusion follows as in [1, p.223]. Let $0 \le j \le k$ then

$$D_1^h z = D_1^{h_k} z - \frac{1}{2} \sum_{j=0}^{k-1} h_j D_1^{h_j} D_1^{h_j} z,$$

where $h_j = 2^j h$ and $D_1^h D_1^h = h^{-2}(z(x + 2h\zeta^1) - 2z(x + h\zeta^1) + z(x))$. This yields

$$\begin{aligned}
&\left[\int_{B''} |h^\sigma D_1^h z|^3\right]^{\frac{1}{3}} \\
&\le \left[\int_{B''} |h^\sigma D_1^{h_k} z|^3\right]^{\frac{1}{3}} + \frac{1}{2} \sum_{j=0}^{k-1} \left[\int_{B''} |h^\sigma h_j D_1^{h_j} D_1^{h_j} z|^3\right]^{\frac{1}{3}} \\
&= \left[\frac{1}{2^k}\right]^\sigma \left[\int_{B''} |h_k^\sigma D_1^{h_k} z|^3\right]^{\frac{1}{3}} + \frac{1}{2} \sum_{j=0}^{k-1} \left[\frac{1}{2^j}\right]^\sigma \left[\int_{B''} |h_j^{1+\sigma} D_1^{h_j} D_1^{h_j} z|^3\right]^{\frac{1}{3}} \\
&= (I) + (II).
\end{aligned}$$

Taking on both sides $\sup_{0<h<R}$ and choosing $k = 2$ we can absorb (I) into the left-hand side. Further, (4.6) entails

$$|(II)| \le c \sum_{j=0}^{k-1} \left\| h_j^{\frac{1}{2}+\sigma} D_1^{h_j} z \right\|_{W^{1,2}(B'')} \le \sup_{0<h<2R} c \left\| h^{\frac{1}{2}+\sigma} D_1^h z \right\|_{W^{1,2}(B'')} .$$

This yields the assertion. □

Now, we start proving Theorem 2.2. First we investigate the regularity of $D_i^h \nabla \mathbf{u}$ for $i \in \Lambda_2$, Λ_1 and Λ_3 in the Propositions 4.4, 4.5 and 4.6 respectively. To begin with we prove the following lemma.

Lemma 4.3 *Let $0 < \delta < \frac{1}{2}$, $\Gamma_1 \cap B_{R_0} \ne \emptyset$, $\Gamma_2 \cap B_{R_0} \ne \emptyset$, and $\partial\Omega \cap B_{R_0} \subset E$, where $E \subset \mathbb{R}^3$ is a hyperplane. Then there exists a constant c depending only on R_0, δ and the data such that*

$$\sup_{0<h<R} \int_{B'} h^{1+\delta} \left|D_i^h \nabla \mathbf{u}\right|^2 \le c \qquad \textit{for } i \in \Lambda_1.$$

Proof. By assumption it holds that ζ^i $(i \in \Lambda_1)$ is parallel to $\partial\Omega \cap B_{R_0}$. We prove the assertion for some $i \in \Lambda_1$, say $i = 1$. Let $0 < h < R$ and $0 < \delta < \frac{1}{2}$. Note that $y \in \Gamma_1 \cap \partial B''$ implies that $y + h\zeta^1 \in \Gamma_1 \cap B_{R_0}$ due to the definition of Λ_1.

Further, $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega \cap B_{R_0}$ holds. Thus, the function $\mathbf{w} = \tau^2 h^\delta D_1^h \mathbf{u}$ is an admissible test function in (4.2). We get

$$\begin{aligned}
&\frac{\nu}{2} \sum_{i,j} \int_{B''} \mathcal{D}_{ij}(\mathbf{u}) \tau^2 h^\delta \mathcal{D}_{ij}(D_1^h \mathbf{u}) \\
&\quad + \frac{\nu}{2} \sum_{i,j} \int_{B''} \mathcal{D}_{ij}(\mathbf{u}) (\partial_i \tau^2 h^\delta D_1^h u_j + \partial_j \tau^2 h^\delta D_1^h u_i) \\
&\quad + \int_{B''} (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \tau^2 h^\delta D_1^h \mathbf{u} - \int_{B''} p \nabla \cdot (\tau^2 h^\delta D_1^h \mathbf{u}) \\
&= \int_{B''} \mathbf{f} \cdot \tau^2 h^\delta D_1^h \mathbf{u}.
\end{aligned} \tag{4.7}$$

Let M be the matrix with components m_{ij} $(1 \leq i, j \leq 3)$ and let $G(M) = \frac{1}{2} \sum_{i,j} m_{ij}^2$. Let us note that $\frac{\partial}{\partial m_{ij}} G(M) = m_{ij}$ and $\frac{\partial}{\partial m_{kl}} \frac{\partial}{\partial m_{ij}} G(M) = \delta_{ik}\delta_{jl}$, where δ_{ik} is the Kronecker symbol. The Taylor expansion yields

$$G(\widetilde{M}) - G(M) = \sum_{i,j} (\widetilde{M} - M)_{ij} m_{ij} + \frac{1}{2} \sum_{i,j,k,l} (\widetilde{M} - M)_{ij} (\widetilde{M} - M)_{kl} \delta_{ik} \delta_{jl}. \tag{4.8}$$

Let $m_{ij} = \mathcal{D}_{ij}(\mathbf{u})$ and $\widetilde{m}_{ij} = E_1^h m_{ij}$. Then $(\widetilde{M} - M)_{ij} = h D_1^h \mathcal{D}_{ij}(\mathbf{u}) = h \mathcal{D}_{ij}(D_1^h \mathbf{u})$ and

$$\frac{1}{2} \sum_{i,j} h D_1^h \left((\mathcal{D}_{ij}(\mathbf{u}))^2 \right) = \sum_{i,j} h \mathcal{D}_{ij}(D_1^h \mathbf{u}) \mathcal{D}_{ij}(\mathbf{u}) + \frac{1}{2} \sum_{i,j} \left(h \mathcal{D}_{ij}(D_1^h \mathbf{u}) \right)^2$$

thus

$$\begin{aligned}
&\int_{B''} \tau^2 \sum_{i,j} h^\delta \mathcal{D}_{ij}(D_1^h \mathbf{u}) \mathcal{D}_{ij}(\mathbf{u}) \\
&= \frac{1}{2} \int_{B''} \tau^2 \sum_{i,j} \left[h^\delta D_1^h \left((\mathcal{D}_{ij}(\mathbf{u}))^2 \right) - h^{1+\delta} \left(\mathcal{D}_{ij}(D_1^h \mathbf{u}) \right)^2 \right].
\end{aligned} \tag{4.9}$$

(4.7) and (4.9) entail

$$
\begin{aligned}
&\frac{\nu}{4}\sum_{i,j}\int_{B''}\tau^2 h^{1+\delta}\left(\mathcal{D}_{ij}(D_1^h\mathbf{u})\right)^2\\
&=\frac{\nu}{4}\sum_{i,j}\int_{B''}\tau^2 h^{\delta} D_1^h\left((\mathcal{D}_{ij}(\mathbf{u}))^2\right)\\
&\quad+\frac{\nu}{2}h^{\delta}\sum_{i,j}\int_{B''}\mathcal{D}_{ij}(\mathbf{u})\left(\partial_i\tau^2 D_1^h u_j+\partial_j\tau^2 D_1^h u_i\right)\\
&\quad+h^{\delta}\int_{B''}(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\tau^2 D_1^h\mathbf{u}\\
&\quad-h^{\delta}\int_{B''}p\nabla\cdot\left(\tau^2 D_1^h\mathbf{u}\right)-h^{\delta}\int_{B''}\mathbf{f}\tau^2 D_1^h\mathbf{u}\\
&=J_1+\ldots+J_5.
\end{aligned}
$$

First we estimate the integral on the left hand side from below. Using the second Korn inequality (4.4) we obtain

$$
\begin{aligned}
&\sum_{i,j}\int_{B''}h^{1+\delta}\left(\tau\mathcal{D}_{ij}(D_1^h\mathbf{u})\right)^2\\
&\geq\sum_{i,j}\int_{B''}h^{1+\delta}\left[c\left(\mathcal{D}_{ij}(\tau D_1^h\mathbf{u})\right)^2-c\left(\partial_i\tau D_1^h u_j+\partial_j\tau D_1^h u_i\right)^2\right]\\
&\geq c\int_{B''}h^{1+\delta}\left|\nabla(\tau D_1^h\mathbf{u})\right|^2-c\left(\|\tau\|^2_{W^{1,\infty}(B'')}\left\|D_1^h\mathbf{u}\right\|^2_{L^2(B'',\mathbb{R}^3)}\right).
\end{aligned}
$$

In view of

$$
\begin{aligned}
\int_{B''}h^{1+\delta}\left|\nabla(\tau D_1^h\mathbf{u})\right|^2 \geq c\int_{B''}h^{1+\delta}\tau^2\left|\nabla D_1^h\mathbf{u}\right|^2\\
-ch^{1+\delta}\|\nabla\tau\|^2_{L^\infty(B'')}\left\|D_1^h\mathbf{u}\right\|^2_{L^2(B'',\mathbb{R}^3)}
\end{aligned}
$$

we get

$$
\frac{\nu}{4}\sum_{i,j}\int_{B''}\tau^2 h^{1+\delta}\left(\mathcal{D}_{ij}(D_1^h\mathbf{u})\right)^2\geq c\int_{B''}\tau^2 h^{1+\delta}\left|D_1^h\nabla\mathbf{u}\right|^2-c.
$$

Next, let us estimate the integrals $J_1,\ldots,J_5$. Due to the identity $fD_1^h\tilde{f}=D_1^h(f\tilde{f})-D_1^h f E_1^h\tilde{f}$ we get

$$
\begin{aligned}
J_1&=\frac{\nu}{4}\sum_{i,j}\int_{B^*}\tau^2 h^{\delta}D_1^h\left((\mathcal{D}_{ij}(\mathbf{u}))^2\right)\\
&=\frac{\nu}{4}\sum_{i,j}\int_{B^*}h^{\delta}D_1^h\left(\tau^2(\mathcal{D}_{ij}(\mathbf{u}))^2\right)-\frac{\nu}{4}\sum_{i,j}\int_{B^*}h^{\delta}D_1^h\tau^2\left(E_1^h\mathcal{D}_{ij}(\mathbf{u})\right)^2\\
&=J_{11}+J_{12},
\end{aligned}
$$

where $B^* = \{x \in B_{R_0} : x = y \pm \lambda\zeta^1,\ y \in B'',\ 0 \le \lambda \le h\}$. Using supp $\tau = B''$ it follows that

$$
\begin{aligned}
J_{11} &= \frac{\nu}{4} h^{\delta-1} \sum_{i,j} \int_{B^*} E_1^h(\tau^2(\mathcal{D}_{ij}(\mathbf{u}))^2) - \frac{\nu}{4} h^{\delta-1} \sum_{i,j} \int_{B^*} \tau^2(\mathcal{D}_{ij}(\mathbf{u}))^2 \\
&= 0, \\
|J_{12}| &\le c h^\delta \int_{B^*} |\nabla \mathbf{u}|^2 \le c.
\end{aligned}
$$

Notice that $\tau \in W^{1,\infty}(\Omega)$. Hence,

$$
|J_2| \le c h^\delta \left(\|\nabla \mathbf{u}\|^2_{L^2(B'',\mathbb{R}^3)} + \left\| D_1^h \mathbf{u} \right\|^2_{L^2(B'',\mathbb{R}^3)} \right) \le c.
$$

Let $\varepsilon > 0$. The Hölder and Young inequalities yield

$$
|J_3| \le \frac{c h^\delta}{\varepsilon} \left(\|\mathbf{u}\|_{L^6(B'',\mathbb{R}^3)} \|\nabla \mathbf{u}\|_{L^2(B'',\mathbb{R}^3)} \right)^2 + \varepsilon \left\| h^{\frac{\delta}{2}} \tau D_1^h \mathbf{u} \right\|^2_{L^3(B'',\mathbb{R}^3)}.
$$

Due to $\mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^3)$ the Sobolev imbedding theorem provides $\mathbf{u} \in L^6(\Omega; \mathbb{R}^3)$. Further, using (4.5) (with $\sigma = \frac{\delta}{2}$) we obtain

$$
\left\| h^{\frac{\delta}{2}} D_1^h (\tau \mathbf{u}) \right\|^2_{L^3(B'',\mathbb{R}^3)} \le c \sup_{0<h<R} \left\| h^{\frac{1+\delta}{2}} D_1^h (\tau \mathbf{u}) \right\|^2_{W^{1,2}(B'',\mathbb{R}^3)}.
$$

Thus,

$$
|J_3| \le c + \varepsilon \sup_{0<h<R} \left\| h^{\frac{1+\delta}{2}} \tau D_1^h \nabla \mathbf{u} \right\|^2_{L^2(B'',\mathbb{R}^3)}.
$$

Next, $\nabla \cdot \mathbf{u} = 0$ and $p \in L^2(\Omega)$ imply that

$$
\begin{aligned}
|J_4| &= \left| -h^\delta \sum_i \int_{B''} p \partial_i \tau^2 D_1^h u_i \right| \\
&\le c h^\delta \left(\|p\|^2_{L^2(B'')} + \left\| D_1^h \mathbf{u} \right\|^2_{L^2(B'',\mathbb{R}^3)} \right) \le c.
\end{aligned}
$$

Further,

$$
|J_5| \le c h^\delta \left(\|\mathbf{f}\|^2_{L^2(B'',\mathbb{R}^3)} + \left\| D_1^h \mathbf{u} \right\|^2_{L^2(B'',\mathbb{R}^3)} \right) \le c.
$$

Altogether we obtain

$$
\sup_{0<h<R} \int_{B''} \tau^2 h^{1+\delta} \left| D_1^h \nabla \mathbf{u} \right|^2 \le c + \varepsilon \sup_{0<\bar{h}<R} \int_{B''} \tau^2 \bar{h}^{1+\delta} \left| D_1^{\bar{h}} \nabla \mathbf{u} \right|^2
$$

for a sufficiently small ε. Absorbing the integral on the right-hand side into the left-hand side and noting that $\tau \equiv 1$ in B' the assertion follows. □

Proposition 4.4 *For* $0 < \delta < \frac{1}{2}$ *there exists a constant* c *depending only on* R_0, δ, *and the data, such that*

$$\sup_{0<h<R} \int_{B'} h^{1+\delta} \left|D_i^h \nabla \mathbf{u}\right|^2 \leq c \qquad \text{for } i \in \Lambda_2. \tag{4.10}$$

Proof. We assume $1 \in \Lambda_2$. Let $0 < h < R$ and $0 < \delta < \frac{1}{2}$. Below, $\mathbf{u}$ and τ denote the extensions of these functions into Ω_1^{-h} defined by (3.1), and the definition of τ respectively.

Putting $\widetilde{m}_{ij} = E_1^{-h} m_{ij}$ and $m_{ij} = \mathcal{D}_{ij}(\mathbf{u})$, the Taylor expansion (4.8) provides

$$\begin{aligned}\frac{1}{2}\sum_{i,j}\left((E_1^{-h}\mathcal{D}_{ij}(\mathbf{u}))^2 - (\mathcal{D}_{ij}(\mathbf{u}))^2\right) &= \sum_{i,j}\mathcal{D}_{ij}(E_1^{-h}\mathbf{u}-\mathbf{u})\mathcal{D}_{ij}(\mathbf{u}) \\ &\quad + \frac{1}{2}\sum_{i,j}\left(\mathcal{D}_{ij}(E_1^{-h}\mathbf{u}-\mathbf{u})\right)^2.\end{aligned}$$

Noting that $E_1^{-h}\mathbf{u} - \mathbf{u} = -hD_1^{-h}\mathbf{u}$, we get

$$\begin{aligned}-\sum_{i,j} h^\delta \mathcal{D}_{ij}(D_1^{-h}\mathbf{u})\mathcal{D}_{ij}(\mathbf{u}) &= -\frac{1}{2}\sum_{i,j} h^\delta D_1^{-h}\left((\mathcal{D}_{ij}(\mathbf{u}))^2\right) \\ &\quad - \frac{1}{2}\sum_{i,j} h^{1+\delta}\left(\mathcal{D}_{ij}(D_1^{-h}\mathbf{u})\right)^2. \end{aligned} \tag{4.11}$$

Next, $\Gamma_1 \cap B_{R_0} = \partial\Omega \cap \partial\Omega_1^{-h}$. The extension (3.1) implies that

$$-D_1^h\mathbf{u}(y) = h^{-1}(E_1^{-h}\mathbf{u}(y) - \mathbf{u}(y)) = 0 \qquad \text{for } y \in \partial\Omega \cap \partial\Omega_1^{-h}.$$

Further, ζ^1 is parallel to $\Gamma_2 \cap B_{R_0}$ and $\mathbf{u} = 0$ in Ω_1^{-h}, thus $(D_1^{-h}\mathbf{u}) \cdot \mathbf{n} = 0$ on Γ_2. Hence, $\mathbf{w} = -\tau^2 h^\delta D_1^{-h}\mathbf{u}$ is an admissible test function in (4.2). Using (4.2) and (4.11) we get

$$\begin{aligned} J_0 &= \frac{\nu}{4}\sum_{i,j}\int_{B''}\tau^2 h^{1+\delta}\left(\mathcal{D}_{ij}(D_1^{-h}\mathbf{u})\right)^2 \\ &= -\frac{\nu}{4}\sum_{i,j}\int_{B''}\tau^2 h^\delta D_1^{-h}\left((\mathcal{D}_{ij}(\mathbf{u}))^2\right) \\ &\quad -\frac{\nu}{2}h^\delta\sum_{i,j}\int_{B''}\mathcal{D}_{ij}(\mathbf{u})\left(\partial_i\tau^2 D_1^{-h}u_j + \partial_j\tau^2 D_1^{-h}u_i\right) \\ &\quad -h^\delta\int_{B''}(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\tau^2 D_1^{-h}\mathbf{u} \\ &\quad +h^\delta\int_{B''}p\nabla\cdot\left(\tau^2 D_1^{-h}\mathbf{u}\right) + h^\delta\int_{B''}\mathbf{f}\tau^2 D_1^{-h}\mathbf{u} \\ &= J_1 + \ldots + J_5. \end{aligned}$$

Applying Korn's inequality (4.4) we obtain

$$\sum_{i,j}\int_{B''} h^{1+\delta}\left(\mathcal{D}_{ij}(\tau D_1^{-h}\mathbf{u})\right)^2 \geq c\int_{B''} h^{1+\delta}\left|\nabla(\tau D_1^{-h}\mathbf{u})\right|^2 - ch^{1+\delta}\left\|\tau D_1^{-h}\mathbf{u}\right\|^2_{L^2(B'',\mathbb{R}^3)}.$$

Thus, we find

$$J_0 \geq c\int_{B''}\tau^2 h^{1+\delta}\left|D_1^{-h}\nabla\mathbf{u}\right|^2 - c'.$$

Next, let $B^* = \{x \in B_{R_0} : x = y + \lambda\zeta^1,\ y \in B'',\ 0 \leq \lambda \leq h\}$. Then it holds that

$$J_1 = -\frac{\nu}{4}\sum_{i,j}\int_{B^*} h^\delta D_1^{-h}\left(\tau^2(\mathcal{D}_{ij}(\mathbf{u}))^2\right) + \frac{\nu}{4}\sum_{i,j}\int_{B^*} h^\delta D_1^{-h}\tau^2\left(E_1^{-h}\mathcal{D}_{ij}(\mathbf{u})\right)^2 = J_{11} + J_{12}.$$

Notice that $\tau(\mathcal{D}_{ij}(\mathbf{u}))^2 = 0$ in Ω_1^{-h}. Consequently, we get

$$J_{11} = \frac{\nu}{4}h^{\delta-1}\sum_{i,j}\int_{\Omega_1^{-h}}\tau^2(\mathcal{D}_{ij}(\mathbf{u}))^2 = 0$$

and

$$|J_{12}| \leq ch^\delta\int_{B^*}|\nabla\mathbf{u}|^2 \leq c.$$

Estimating all other integrals as in the proof of Lemma 4.3 we obtain

$$\sup_{0<h<R}\int_{B''}\tau^2 h^{1+\delta}\left|D_1^{-h}\nabla\mathbf{u}\right|^2 \leq c + \varepsilon\sup_{0<\bar{h}<R}\int_{B''}\tau^2\bar{h}^{1+\delta}\left|D_1^{-\bar{h}}\nabla\mathbf{u}\right|^2.$$

Let $B_h = \{z \in B'' : z + h\zeta^1 \in B''\}$. Using

$$\sup_{0<h<R}\int_{B''} h^{1+\delta}\left|D_1^h\nabla(\tau\mathbf{u})\right|^2 \leq c\sup_{0<h<R}\int_{B_h} h^{1+\delta}\left|D_1^h\nabla(\tau\mathbf{u})\right|^2 \leq c\sup_{0<h<R}\int_{B''} h^{1+\delta}\left|D_1^{-h}\nabla(\tau\mathbf{u})\right|^2$$

the assertion follows. □

Proposition 4.5 *For* $0 < \delta < \frac{1}{2}$ *there exists a constant c depending only on* R_0, δ, *and the data, such that*

$$\sup_{0<h<R} \int_{B'} h^{1+\delta} \left| D_i^h \nabla \mathbf{u} \right|^2 \le c \qquad \textit{for } i \in \Lambda_1. \tag{4.12}$$

Proof. Let $1 \in \Lambda_1$ and let us assume

$$\zeta^1 \text{ is parallel to } \partial\Omega \cap B_{R_0}, \tag{4.13}$$

i.e., ζ^1 is parallel to all faces Γ^i with $\Gamma^i \cap B_{R_0} \neq \emptyset$. Then the assertion follows proceeding as in the proof of Lemma 4.3.

Now, let us consider the case when (4.13) does not hold. There is just one boundary manifold Γ^k satisfying $\Gamma^k \cap B_{R_0} = \Gamma_2 \cap B_{R_0}$. Let $\mathbf{n}$ be the outward normal of $\partial\Omega \cap \Gamma^k$. Let us note that ζ^1 is parallel to $\Gamma_1 \cap B_{R_0}$ but not to Γ^k. Thus, the function $\mathbf{w} = \tau^2 h^\delta D_1^h \mathbf{u}$ would be an admissible test function, if $\mathbf{w} \cdot \mathbf{n} = 0$ on Γ^k. In general, this is not satisfied.

Let us write $\mathbf{u} = \mathbf{u^n} + \mathbf{u^t}$, where $\mathbf{u^n}$ is the normal and $\mathbf{u^t}$ is the tangential component of $\mathbf{u}$, i.e., $\mathbf{u^n} = \langle \mathbf{u}, \mathbf{n} \rangle \mathbf{n}$ and $\mathbf{u^t} = \mathbf{u} - \mathbf{u^n}$.

Let Ω_*^{-h} be the reflection of Ω_1^h with respect to the hyperplane containing Γ^k. By ζ_*^1 we denote the reflection of ζ^1.

Let $y \in \Gamma^k \cap \partial\Omega_1^h$, $0 < \lambda \le h$, and $y + \lambda\zeta_*^1 \in \Omega_*^{-h}$. We extend the functions τ, $\mathbf{u^n}$, and $\mathbf{u^t}$ onto Ω_*^{-h} by setting

$$\mathbf{u^t}(y + \lambda\zeta_*^1) = \mathbf{u^t}(y + \lambda\zeta^1), \qquad \tau(y + \lambda\zeta_*^1) = \tau(y + \lambda\zeta^1), \tag{4.14}$$

and

$$\mathbf{u^n}(y + \lambda\zeta_*^1) = -\mathbf{u^n}(y + \lambda\zeta^1). \tag{4.15}$$

Thus, roughly speaking, we use an even extension of $\mathbf{u^t}$ and an odd extension of $\mathbf{u^n}$.

Next, let $y \in \Gamma^k \cap \partial\Omega_1^h$. We define

$$g(s) = \begin{cases} y + s\zeta^1 \text{ for } s \ge 0, \\ y + s\zeta_*^1 \text{ for } s < 0. \end{cases}$$

Let $s_0 > 0$ and $x = y + s_0\zeta^1 = g(s_0) \in \Omega$. We define $E_*^h f(x) \equiv E_*^h f(g(s_0)) = f(g(s_0+h))$, $E_*^{-h} f(x) \equiv E_*^{-h} f(g(s_0)) = f(g(s_0-h))$, $D_*^h f(x) = h^{-1}(E_*^h f(x) - f(x))$ and $D_*^{-h} f(x) = h^{-1}(f(x) - E_*^{-h} f(x))$.

Now, let us verify that the function $\mathbf{w} = \tau^2(D_*^h - D_*^{-h})\mathbf{u}$ is an admissible test function. On the one hand, ζ^1 is parallel to $\Gamma_1 \cap B_{R_0}$, thus, $x \in \Gamma_1 \cap B_{R_0}$ and $x + h\zeta^1 \in B_{R_0}$ imply that $x + h\zeta^1 \in \Gamma_1$. In view of (4.14) and (4.15) this yields $\mathbf{w} = 0$ on $\Gamma_1 \cap B_{R_0}$. On the other hand it holds that $(E_*^h\mathbf{u}(y) - 2\mathbf{u}(y) +$

$E_*^{-h}\mathbf{u}(y))\cdot\mathbf{n}=0$ for $y\in\Gamma^k\cap\partial\Omega_1^h$. Here we have used $\mathbf{u}\cdot\mathbf{n}=0$ on Γ^k and $E_*^h\mathbf{u}^{\mathbf{n}}(y)=-E_*^{-h}\mathbf{u}^{\mathbf{n}}(y)$.

Now, we proceed as above. After testing the equation and using the Taylor expansion, we obtain

$$\begin{aligned}
&\frac{\nu}{4}\sum_{i,j}\int_{B''}\tau^2h^{1+\delta}\left(\mathcal{D}_{ij}(D_*^h\mathbf{u})\right)^2+\frac{\nu}{4}\sum_{i,j}\int_{B''}\tau^2h^{1+\delta}\left(\mathcal{D}_{ij}(D_*^{-h}\mathbf{u})\right)^2\\
&=\frac{\nu}{4}\sum_{i,j}\int_{B''}\tau^2h^{\delta}(D_*^h-D_*^{-h})\left((\mathcal{D}_{ij}(\mathbf{u}))^2\right)\\
&\quad+\frac{\nu}{2}h^{\delta}\sum_{i,j}\int_{B''}\mathcal{D}_{ij}(\mathbf{u})\left[\partial_i\tau^2(D_*^h-D_*^{-h})u_j+\partial_j\tau^2(D_*^h-D_*^{-h})u_i\right]\\
&\quad+h^{\delta}\int_{B''}(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\tau^2(D_*^h-D_*^{-h})\mathbf{u}\\
&\quad-h^{\delta}\int_{B''}p\nabla\cdot\left(\tau^2(D_*^h-D_*^{-h})\mathbf{u}\right)-h^{\delta}\int_{B''}\mathbf{f}\tau^2(D_*^h-D_*^{-h})\mathbf{u}\\
&=J_1+\ldots+J_5.
\end{aligned}$$

Next, we estimate the integrals $J_1,\ldots,J_5$. Let $B^*=\{x\in B_{R_0}:x=y+\lambda\zeta^1,\ y\in B'',\ 0\le\lambda\le h\}$. Then

$$\begin{aligned}
J_1&=\frac{\nu}{4}\sum_{i,j}\int_{B^*}h^{\delta}(D_*^h-D_*^{-h})\left(\tau^2(\mathcal{D}_{ij}(\mathbf{u}))^2\right)\\
&\quad-\frac{\nu}{4}\sum_{i,j}\int_{B^*}h^{\delta}D_*^h\tau^2E_*^h\left((\mathcal{D}_{ij}(\mathbf{u}))^2\right)\\
&\quad+\frac{\nu}{4}\sum_{i,j}\int_{B^*}h^{\delta}D_*^{-h}\tau^2E_*^{-h}\left((\mathcal{D}_{ij}(\mathbf{u}))^2\right)\\
&=J_{11}+J_{12}+J_{13}.
\end{aligned}$$

Notice that $\mathbf{n}$ is a constant vector, thus, $\partial_j(\langle\mathbf{u},\mathbf{n}\rangle\mathbf{n})=\langle\partial_j\mathbf{u},\mathbf{n}\rangle\mathbf{n}$. Hence,

$$J_{11}=-\frac{\nu}{4}h^{\delta-1}\sum_{i,j}\int_{\Omega_1^h}\tau^2(\mathcal{D}_{ij}(\mathbf{u}))^2+\frac{\nu}{4}h^{\delta-1}\sum_{i,j}\int_{\Omega_*^{-h}}\tau^2(\mathcal{D}_{ij}(\mathbf{u}))^2.$$

Let A be an orthogonal matrix such that $A^T\mathbf{e_i}=\mathbf{t_i}$ for $1\le i\le 3$, and $\mathbf{t_1}=\mathbf{n}$. (Here $\mathbf{e_i}\in\mathbb{R}^3$ denotes the i-th unit vector.) Then it holds that $(\mathcal{D}_{ij}(\mathbf{u}))^2=(\partial_{\mathbf{t_i}}\mathbf{u}^{\mathbf{t_j}}+\partial_{\mathbf{t_j}}\mathbf{u}^{\mathbf{t_i}})^2$, where $\mathbf{u}^{\mathbf{t_i}}=\langle\mathbf{u},\mathbf{t_i}\rangle\mathbf{t_i}$. Due to the extensions (4.14) and (4.15), $(\mathcal{D}_{ij}(\mathbf{u}))^2$ is an even function with respect to the hyperplane containing Γ^k (i.e., $(\mathcal{D}_{ij}(\mathbf{u}(z+\lambda\mathbf{n})))^2=(\mathcal{D}_{ij}(\mathbf{u}(z-\lambda\mathbf{n})))^2$, if z is a point in the hyperplane, $z-\lambda\mathbf{n}\in\Omega$, and $z+\lambda\mathbf{n}\in\Omega_*^{-h}$. Altogether, it follows that

$$J_{11}=0.$$

Now, we estimate all other integrals as above. In order to estimate $|J_4|$, let us verify that $\operatorname{div}(D_*^h - D_*^{-h})\mathbf{u} = 0$. Let $y \in \partial\Omega \cap \partial\Omega_*^{-h}$, $y_1 = y + \lambda\zeta_*^1 \in \Omega_*^{-h}$, and $y_2 = y + \lambda\zeta^1 \in \Omega^h$. Using $\partial_{t_i}\mathbf{u}^{\mathbf{t_i}}(y_1) = \partial_{t_i}\mathbf{u}^{\mathbf{t_i}}(y_2)$ we get

$$\begin{aligned}\operatorname{div}\mathbf{u}(y_1) &= \langle\nabla, \mathbf{u}(y_1)\rangle = \langle A\nabla, A\mathbf{u}(y_1)\rangle = \sum_i \langle\partial_{t^i}\mathbf{u}(y_1), \mathbf{t_i}\rangle \\ &= \sum_i \langle\partial_{t^i}\mathbf{u}(y_2), \mathbf{t_i}\rangle = \operatorname{div}\mathbf{u}(y_2) = 0.\end{aligned}$$

Hence, we can estimate $|J_4|$ as above. Altogether, it follows that

$$\sup_{0<h<R} \int_{B''} \tau^2 h^{1+\delta} \left|D_*^h \nabla\mathbf{u}\right|^2 \le c.$$

Noting that $D_*^h g(x) = D_1^h g(x)$ for $x \in B''$ we get the assertion. □

In the next proposition we investigate the case $\Lambda_3 = \{3\}$. Let $k \neq j$ and $\operatorname{angle}(\Gamma^k, \Gamma^j) = \pi$. First, we assume that the basis vector ζ^3 satisfies $\zeta^3 = e_3$, where $e_3 = (0, 0, 1)^T$. Then it holds that $\Omega \cap B_{R_0} \subset \{x \in B_{R_0} : x_3 > 0\}$ and

$$\Gamma^k, \Gamma^j \subset \{x \in \mathbb{R}^3 : x_3 = 0\}. \tag{4.16}$$

Further, the basis vectors ζ^1, ζ^2 are parallel to the x_1-x_2-hyperplane.

In case that $\zeta^3 \neq e_3$ we make a rotation of the domain. This will be discussed later, see Remark 4.9 below.

Proposition 4.6 *Let $\Lambda_3 = \{3\}$, $\zeta^3 = e_3$, $0 < \delta < \frac{1}{2}$, and $\bar{R} = \frac{R}{4}$. There exists a constant c depending only on R_0, δ, and the data, such that*

$$\sup_{0<h<\bar{R}} \int_B h^{1+\delta} \left|D_3^h \nabla\mathbf{u}\right|^2 \le c. \tag{4.17}$$

We give the proof of (4.17) using Fourier series. Therefore we proceed in several steps. To begin with, we prove the following two lemmas.

Lemma 4.7 *Let $\Lambda_3 = \{3\}$, $\zeta^3 = e_3$, and $0 < \delta < \frac{1}{2}$. Then there exists a constant c depending only on R_0, δ, and the data, such that*

$$\sup_{0<h<R} \int_B h^{1+\delta} |D_3^h \partial_k \mathbf{u}|^2 \le c \qquad \textit{for k=1,2}. \tag{4.18}$$

Proof. Let P be the origin, i.e., $B = B_R(0) \cap \Omega$ etc. Let $\varepsilon > 0$ be small and $Q_\varepsilon = \{x \in \mathbb{R}^3 : -2R < x_1, x_2 < 2R,\ \varepsilon < x_3 < 4R + \varepsilon\}$. Further, we assume that $\partial\Omega \cap B_{R_0}$ consists of just two boundary manifolds, thus,

$$\partial\Omega \cap B_{R_0} = \{x \in B_{R_0} : x_3 = 0\}.$$

Otherwise, $\partial\Omega \cap B_{R_0}$ may consist of three boundary manifolds and $\partial\Omega \cap B_{R_0} \not\subset \{x \in B_{R_0} : x_3 = 0\}$. Then $Q_\varepsilon \not\subset \Omega$. Thus, instead of a cube Q_ε we consider a parallelogram (for details see [9]).

Let $1 \le k \le 3$, $1 \le i \le 2$, and

$$H(x) = \partial_i u_k(x) \qquad \text{for } x \in Q_\varepsilon. \tag{4.19}$$

By successive reflections of $\overline{Q}_\varepsilon$ at the hyperplanes $\{x_3 = \varepsilon\}$, $\{x_2 = 2R\}$, and $\{x_1 = 2R\}$, we get an extension of H into the cube

$$Q = \{x \in \mathbb{R}^3 : -2R < x_1, x_2 < 6R,\ -4R + \varepsilon < x_3 < 4R + \varepsilon\},$$

and H has periodic boundary values on ∂Q. Next, we extend H periodically into $\mathbb{R}^3$ such that $H(x + 8Re_j) = H(x)$ for each $j \in \{1, 2, 3\}$ and $x \in \mathbb{R}^3$. Let us note that this is an $W^{1,2}$-extension. We denote the Fourier series of $H(x)$ by

$$H(x) = \sum_{m_1,m_2,m_3} \mu_{m_1m_2m_3} e^{im_1\rho(x_1-2R)} e^{im_2\rho(x_2-2R)} e^{im_3\rho(x_3-\varepsilon)}, \tag{4.20}$$

where $\rho = \frac{\pi}{4R}$, and $\mu_{m_1m_2m_3}$ are the Fourier coefficients with respect to Q. Let $I_j^h = [x_j, x_j + h]$ for $1 \le j \le 3$. Note that

$$h^{-\frac{1}{2}} \int_{I_i^h} \partial_3 H\, dx_i = h^{-\frac{1}{2}} \int_{I_i^h} \partial_i \partial_3 u_k\, dx_i = \frac{\partial_3 u_k(x + he_i) - \partial_3 u_k(x)}{h^{\frac{1}{2}}}.$$

(4.16) implies that the basis vectors ζ^1 and ζ^2 are linear dependent on $e_1 = (1, 0, 0)^T$ and $e_2 = (0, 1, 0)^T$, and that $\zeta^3 = e_3 = (0, 0, 1)^T$. Using (4.12) and (4.10) we obtain

$$\sup_{0<h<R} \int_Q \left| h^{-\frac{3}{2}+\frac{\delta}{4}} \int_{I_2^h \times I_1^h} \partial_3 H(x)\, dx_1 dx_2 \right|^2 dx \le c, \tag{4.21}$$

where $0 < \delta < \frac{1}{2}$. In view of (4.20) this yields

$$\sup_{0<h<R} \left(h^{-3+\frac{\delta}{2}} \sum_{m_1,m_2,m_3} \mu^2_{m_1m_2m_3} \rho^2 |m_3|^2 \prod_{k=1}^{2} \psi_k \right) \le c, \tag{4.22}$$

where

$$\psi_k = \begin{cases} \frac{|e^{im_k\rho h}-1|^2}{\rho^2|m_k|^2} & \text{for } |m_k| \ne 0, \\ h^2 & \text{for } |m_k| = 0. \end{cases}$$

Let $a = \frac{\pi}{2\rho}$, $S_j = (2^{j-1}a, 2^j a]$, $h_j = 2^{-j}$, and $j \geq j_0$ for an appropriate j_0. Notice that $\sin x \geq \frac{x}{2}$ for $x \in [0, \frac{\pi}{4}]$, and that $|m_k|\rho\frac{h_j}{2} \leq \frac{\pi}{4}$ for $|m_k| \in S_j$. It follows for $|m_1| \in S_j$ and $|m_1| \geq |m_2| > 0$ that

$$\begin{aligned}
h_j^{-3+\delta} \prod_{k=1}^{2} \frac{|e^{im_k\rho h_j} - 1|^2}{\rho^2|m_k|^2} &= h_j^{-3+\delta} \prod_{k=1}^{2} \frac{|e^{-im_k\rho\frac{h_j}{2}}(e^{im_k\rho h_j} - 1)|^2}{\rho^2|m_k|^2} \\
&= h_j^{-3+\delta} \prod_{k=1}^{2} \frac{\left(2\sin\left(m_k\rho\frac{h_j}{2}\right)\right)^2}{\rho^2|m_k|^2} \geq h_j^{-3+\delta} \prod_{k=1}^{2} \frac{|m_k\rho h_j|^2}{4\rho^2|m_k|^2} \\
&= ch_j^{1+\delta} \geq ca^{1+\delta}|m_1|^{-1-\delta}.
\end{aligned}$$

Let

$$\begin{aligned}
M_j &= \{(m_1, m_2, m_3) : |m_1| \geq |m_2| > 0,\ |m_1| \in S_j\}, \\
M_0 &= \{(m_1, m_2, m_3) : |m_1| \geq |m_2| > 0,\ |m_1| \geq 2^{j_0-1}a\}.
\end{aligned}$$

In view of (4.22) we get

$$\begin{aligned}
c &\geq \sum_{j=j_0}^{\infty} \frac{1}{j^2} \left(h_j^{-3+\frac{\delta}{2}} \sum_{M_j} \mu^2_{m_1m_2m_3}\rho^2|m_3|^2 \prod_{k=1}^{2} \frac{|e^{im_k\rho h_j} - 1|^2}{\rho^2|m_k|^2} \right) \\
&\geq c \sum_{j=j_0}^{\infty} \frac{(2^j)^{\frac{\delta}{2}}}{j^2} \sum_{M_j} \mu^2_{m_1m_2m_3}|m_3|^2|m_1|^{-1-\delta} \\
&\geq c \sum_{M_0} \mu^2_{m_1m_2m_3}|m_3|^2|m_1|^{-1-\delta} \\
&\geq c \sum_{M_0} \mu^2_{m_1m_2m_3}|m_3|^2|m_0|^{-1-\delta},
\end{aligned}$$

where $m_0 = \max\{1, |m_1|, |m_2|, |m_3|\}$. Now, it is not hard to estimate the last sum on the right-hand side over the index sets $\{(m_1, m_2, m_3) : |m_1| \geq |m_2| > 0\}$ and $\{(m_1, m_2, m_3) : m_1 = m_2 = 0\}$. For example, in the latter case we use

$$h_j^{1+\delta}|m_3|^{\,2} \geq c|m_3|^{1-\delta} \qquad \text{for } |m_3| \in S_j.$$

Next, we interchange m_1 and m_2, and proceed as above. Altogether we get

$$\sum_{m_1,m_2,m_3} \mu^2_{m_1m_2m_3}|m_3|^2|m_0|^{-1-\delta} \leq c. \tag{4.23}$$

The Young inequality entails (with $q_1 = \frac{2}{1+\sigma},\ q_2 = \frac{2}{1-\sigma}$ and $\sigma \in (0, \frac{1}{2})$)

$$\begin{aligned}
&\sum_{m_1,m_2,m_3} \mu^2_{m_1m_2m_3}|m_0|^{\frac{1}{2}}|m_0|^{-\frac{1}{2}}|m_3|^{1-\sigma} \\
&\le c \sum_{m_1,m_2,m_3} \mu^2_{m_1m_2m_3}\left(|m_0|^{\frac{1}{1+\sigma}} + |m_3|^2|m_0|^{-\frac{1}{1-\sigma}}\right). \qquad (4.24)
\end{aligned}$$

From (4.12) and (4.10) we conclude

$$\begin{aligned}
c &\ge \sup_{0<h<R} \int_Q h^{1+\frac{\delta}{2}} \left|\frac{H(x+he_k) - H(x)}{h}\right|^2 \\
&= \sup_{0<h<R} \sum_{m_1,m_2,m_3} \mu^2_{m_1m_2m_3} h^{\frac{\delta}{2}-1} \left|e^{im_k\rho h} - 1\right|^2 \qquad (4.25)
\end{aligned}$$

for $1 \le k \le 2$. Let $|m_k| \in S_j$ then

$$\begin{aligned}
&h_j^{\frac{\delta}{2}-1}\left|e^{im_k\rho h_j} - 1\right|^2 = h_j^{\frac{\delta}{2}-1}\left(2\sin(m_k\rho\frac{h_j}{2})\right)^2 \ge ch_j^{\frac{\delta}{2}-1}|m_kh_j|^2 \\
&= ch_j^{\frac{\delta}{2}-1}|m_kh_j|^{1+\delta}|m_kh_j|^{1-\delta} \ge ch_j^{-\frac{\delta}{2}}|m_k|^{1-\delta}. \qquad (4.26)
\end{aligned}$$

(4.25) and (4.26) imply that

$$\begin{aligned}
c &\ge \sum_{j=j_0}^{\infty} \frac{1}{j^2} \sum_{\substack{m_1,m_2,m_3 \\ |m_k|\in S_j}} \mu^2_{m_1m_2m_3} h_j^{\frac{\delta}{2}-1}\left|e^{im_k\rho h_j} - 1\right|^2 \\
&\ge \sum_{j=j_0}^{\infty} \frac{(2^j)^{\frac{\delta}{2}}}{j^2} \sum_{\substack{m_1,m_2,m_3 \\ |m_k|\in S_j}} \mu^2_{m_1m_2m_3}|m_k|^{1-\delta},
\end{aligned}$$

thus,

$$\sum_{m_1,m_2,m_3} \mu^2_{m_1m_2m_3}|m_k|^{1-\delta} \le c \qquad \text{for } 1 \le k \le 2. \qquad (4.27)$$

Altogether, (4.24), (4.23), and (4.27) entail

$$\sum_{m_1,m_2,m_3} \mu^2_{m_1m_2m_3}|m_3|^{1-\delta} \le c. \qquad (4.28)$$

Using $|\sin(x)| \le |x|$ we have

$$\begin{aligned}
\int_Q h^{1+\delta}|D_3^hH|^2 &= \sum_{m_1,m_2,m_3} \mu^2_{m_1m_2m_3} h^{\delta-1}\left(2\sin(m_3\rho\frac{h}{2})\right)^2 \\
&\le c \sum_{m_1,m_2,m_3} \mu^2_{m_1m_2m_3}\left|2\sin(m_3\rho\frac{h}{2})\right|^{1+\delta} |m_3\rho|^{1-\delta} \\
&\le c \sum_{m_1,m_2,m_3} \mu^2_{m_1m_2m_3}|m_3|^{1-\delta},
\end{aligned}$$

thus, by (4.28) we get

$$\sup_{0<h<R} \int_{Q_\varepsilon} h^{1+\delta} \left| D_3^h H(x) \right|^2 \, dx \leq c. \tag{4.29}$$

Now, we take $\lim_{\varepsilon \to 0}$. Let $Q_0 := \lim_{\varepsilon \to 0} Q_\varepsilon$. Notice that

$$\{y \in B_{R_0} : y = x + \lambda e_3, \; x \in B, \; 0 \leq \lambda < R\} \subset Q_0 \tag{4.30}$$

and that $H(x) = \partial_i u_k(x)$ for $x \in Q_0$. Thus, $D_3^h H(x) = D_3^h \partial_i u_k(x)$ holds for $x \in B$, $1 \leq i \leq 2$, and $1 \leq k \leq 3$. Hence, we have proven the assertion. □

Lemma 4.8 *Let $\Lambda_3 = \{3\}$, $\zeta^3 = e_3$, and $0 < \delta < \frac{1}{2}$. There are two constants c_1 and c_2 depending only on R_0, δ, and the data such that*

$$\sup_{0<h<R} \int_B h^{1+\delta} \left| D_3^h \partial_3 u_3 \right|^2 \leq c_1 \tag{4.31}$$

and

$$\sup_{0<h<R} \int_{B^*} \left(h^{-\frac{5}{2}+\delta} \int_{I_3^h \times I_2^h \times I_1^h} \partial_3 p \right)^2 \leq c_2, \tag{4.32}$$

where $B^* = B_{5R}(P) \cap \Omega$.

Proof. Notice that $\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2$. Using (4.18) we obtain

$$\int_B h^{1+\delta} |D_3^h \partial_3 u_3|^2 \leq c \sum_{i=1}^{2} \int_B h^{1+\delta} |D_3^h \partial_i u_i|^2 \leq c.$$

Thus, (4.31) is proven.

Next, we prove (4.32). Let $⨍$ denote the mean value, i.e., $⨍_{I_1^h} = h^{-1} \int_{I_1^h}$, $⨍_{I_1^h \times I_2^h} = h^{-2} \int_{I_1^h \times I_2^h}$, etc. We set $g_3 = f_3 - (\mathbf{u} \cdot \nabla) u_3$, where $\mathbf{f} = (f_1, f_2, f_3)^T$ is the right-hand side of (1.1). Using $\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2$ the equation (1.1) entails

$$\partial_3 p = \nu \Delta u_3 + g_3 = \nu(\partial_1 \partial_1 u_3 + \partial_2 \partial_2 u_3 - \partial_3(\partial_1 u_1 + \partial_2 u_2)) + g_3,$$

thus,

$$\begin{aligned} &\int_{B^*} \left(h^{\frac{1}{2}+\delta} ⨍_{I_3^h \times I_2^h \times I_1^h} \partial_3 p \right)^2 \\ &\leq c \int_{B^*} \left(h^{\frac{1}{2}+\delta} ⨍_{I_3^h \times I_2^h \times I_1^h} \sum_{i=1}^{2} \partial_i (\partial_i u_3 - \partial_3 u_i) \right)^2 \\ &\quad + c \int_{B^*} \left(h^{\frac{1}{2}+\delta} ⨍_{I_3^h \times I_2^h \times I_1^h} g_3 \right)^2 \\ &= J_1 + J_2. \end{aligned} \tag{4.33}$$

If we assume $\zeta^1 = e_1$, (4.12) and (4.10) yield

$$\int_{B^*} \left(h^{\frac{1}{2}+\delta} \fint_{I_3^h \times I_2^h \times I_1^h} \partial_1(\partial_1 u_3 - \partial_3 u_1) \right)^2 = \int_{B^*} \left(\fint_{I_3^h \times I_2^h} h^{\frac{1}{2}+\delta} D_1^h(\partial_1 u_3 - \partial_3 u_1) \right)^2 \le c.$$

In general, $\zeta^1 = \alpha_1 e_1 + \alpha_2 e_2$ holds for some $\alpha_1, \alpha_2 \in \mathbb{R}$. Notice that (4.12) and (4.10) yield $|h^{\frac{1+\delta}{2}} D_i^h \nabla \mathbf{u}| \in L^2(B', \mathbb{R}^3)$ for $i = 1, 2$. Thus, we conclude

$$J_1 \le c. \tag{4.34}$$

Further, (1.2) implies that $g_3 = f_3 - \sum_{i=1}^3 \partial_i(u_i u_3)$. Due to $\mathbf{u} \in W^{1,2}(\Omega; \mathbb{R}^3)$ and the Sobolev imbedding theorem we obtain $\mathbf{u} \in L^6(\Omega; \mathbb{R}^3) \cap W^{\frac{1}{2},3}(\Omega; \mathbb{R}^3)$. Using the imbedding theorem of Sobolev spaces into Nikolskii spaces

$$W^{\frac{1}{2},3}(\Omega; \mathbb{R}^3) \to \mathcal{H}^{\frac{1}{2},3}(\Omega; \mathbb{R}^3)$$

it follows that $h^{\frac{1}{2}} D_i^h \mathbf{u} \in L^3(\Omega; \mathbb{R}^3)$ for $1 \le i \le 3$. Notice that

$$h^{-\frac{1}{2}} \int_{I_1^h} \partial_1(u_1 u_3) = h^{\frac{1}{2}} D_1^h(u_1 u_3) = h^{\frac{1}{2}} (D_1^h u_1 E_1^h u_3 + u_1 D_1^h u_3)$$

for $\zeta^1 = e_1$. Thus, we conclude

$$J_2 \le c. \tag{4.35}$$

Altogether, (4.33), (4.34), and (4.35) yield (4.32). □

Next, we give some notations. Let $g_k = f_k - (\mathbf{u} \cdot \nabla) u_k$ for $1 \le k \le 3$. As in the proof of Lemma 4.7 we define the cubes Q_ε, Q, and the functions

$$H_1(x) = \partial_1 u_2(x), \quad H_2(x) = \partial_2 u_2(x), \quad H_3(x) = \partial_3 u_2(x),$$
$$H_4(x) = p(x), \quad \text{and} \quad H_5(x) = g_2(x) \qquad \text{for } x \in Q_\varepsilon.$$

By successive reflections we extend these functions into Q and then periodically into $\mathbb{R}^3$.

Proof of Proposition 4.6: Let $\bar{R} = \frac{R}{4}$ and $0 < \delta < \frac{1}{2}$. In view of (4.18) and (4.31) it remains to show

$$\sup_{0<h<\bar{R}} \int_B h^{1+\delta} \left| D_3^h \partial_3 u_k \right|^2 \le c \qquad \text{for } k = 1, 2, \tag{4.36}$$

where the constant c only depends on R_0, δ, and the data.

In the sequel let $0 < h < \bar{R}$. As above, we suppose $\zeta^3 = e_3$. Then the basis vectors ζ^1 and ζ^2 are parallel to the x_1-x_2-hyperplane. For simplicity we may suppose $\zeta^2 = e_2$. Then it holds $h^{-1} \int_{I_j^h} \partial_j g = D_j^h g$ for $j = 2, 3$.

We will prove (4.36) for some k, say $k = 2$. The equation (1.1) yields

$$\nu \hat{\partial}_3 H_3 = -H_5 - \nu(\hat{\partial}_1 H_1 + \hat{\partial}_2 H_2) + \hat{\partial}_2 H_4, \tag{4.37}$$

where $\hat{\partial}_k = \mu_k(x)\partial_k$, and $\mu_k(x) \in \{\pm 1\}$ is defined appropriately. For example, let χ_2 be the characterisitic function of the strip $2R < x_2 < 6R$, then it holds that $\mu_2(x) = 1 - 2\chi_2(x)$ for $x \in Q$.

Let τ_2 be a smooth cut-off function depending only on x_2 such that for $x \in Q$ there holds i) $\tau_2 \equiv 1$ for $|x_2| \leq R$, and ii) $\tau_2 \equiv 0$ for $|x_2| \geq \frac{3R}{2}$. Further, let τ_2 be a periodic function, i. e., $\tau_2(x + 8Re_2) = \tau_2(x)$. We multiply the equation (4.37) with $\tau_2\mu_3$ and integrate it over $I_3^h \times I_2^h \times I_2^h \times I_1^h$. Testing with a function ϕ entails

$$\begin{aligned} J_0 = &\int_Q \left[\fint_{I_3^h \times I_2^h \times I_2^h \times I_1^h} \nu \tau_2 \partial_3 H_3 \right] \phi \\ = &-\int_Q \left[\fint_{I_3^h \times I_2^h \times I_2^h \times I_1^h} \tau_2 \mu_3 H_5 \right] \phi \\ &-\nu \int_Q \left[\fint_{I_3^h \times I_2^h \times I_2^h \times I_1^h} \tau_2 \mu_3 (\hat{\partial}_1 H_1 + \hat{\partial}_2 H_2) \right] \phi \\ &+\int_Q \left[\fint_{I_3^h \times I_2^h \times I_2^h \times I_1^h} \tau_2 \mu_3 \hat{\partial}_2 H_4 \right] \phi \\ = &\, J_1 + J_2 + J_3. \end{aligned} \tag{4.38}$$

We choose

$$\phi = h^{1+\sigma} \fint_{I_3^h \times I_2^h \times I_2^h \times I_1^h} \tau_2 \partial_3 H_3.$$

Now, we start estimating $J_0, \ldots, J_3$. The main difficulty is to estimate J_3 from above.

For $\eta > 0$ the Hölder inequality yields

$$\begin{aligned} |J_1| \leq &\frac{c}{\eta} \int_Q h^{1+\sigma} \left| \fint_{I_3^h \times I_2^h \times I_2^h \times I_1^h} \tau_2 \mu_3 H_5 \right|^2 \\ &+\eta \int_Q h^{1+\sigma} \left| \fint_{I_3^h \times I_2^h \times I_2^h \times I_1^h} \tau_2 \partial_3 H_3 \right|^2 = J_{11} + J_{12}. \end{aligned}$$

Estimating as above (cf. (4.35)) and using some easy calculations like $\tau_2\partial_i(u_iu_2) = \partial_i(\tau_2u_iu_2) - \partial_i\tau_2u_iu_2$ we obtain

$$J_{11} \le c. \tag{4.39}$$

Further,

$$\begin{aligned} |J_2| \le \frac{c}{\eta}\int_Q h^{1+\sigma}\left|\fint_{I_3^h\times I_2^h\times I_2^h\times I_1^h}\tau_2\mu_3(\hat{\partial}_1H_1+\hat{\partial}_2H_2)\right|^2 \\ +\eta\int_Q h^{1+\sigma}\left|\fint_{I_3^h\times I_2^h\times I_2^h\times I_1^h}\tau_2\partial_3H_3\right|^2. \end{aligned}\tag{4.40}$$

It holds that $\hat{\partial}_2 = \partial_2 - 2\chi_2\partial_2$ and $\tau_2\chi_2 = 0$, thus, $\tau_2\hat{\partial}_2H_2 = \tau_2\partial_2H_2$ on $\{y \in Q : y = x + 2\lambda\zeta^2,\ x \in \operatorname{supp}\tau_2,\ 0 \le \lambda \le h\}$. Notice that $\zeta^2 = e_2$. Using (4.12), (4.10), and calculations like $\tau_2\mu_3\partial_2H_2 = \partial_2(\tau_2\mu_3H_2) - \partial_2\tau_2\mu_3H_2$ we get

$$\begin{aligned} &\int_Q h^{1+\sigma}\left|\fint_{I_3^h\times I_2^h\times I_2^h\times I_1^h}\tau_2\mu_3\partial_2H_2\right|^2 \\ &\le \int_Q\left|\fint_{I_3^h\times I_2^h\times I_1^h}\tau_2\mu_3h^{\frac{1+\sigma}{2}}D_2^hH_2\right|^2 + c \le c'. \end{aligned}$$

Moreover, in order to estimate $\hat{\partial}_1(\tau_2\mu_3H_1)$, we employ the following one-dimensional argument. Let $\chi(z)$ be the characteristic function of the intervall $(2R, 6R)$ and let $h_j = (\frac{1}{2})^j(h + z - 2R)$ and $\bar{h}_j = h_j + 2R - z$. Then it holds for some smooth function $G(z)$ that

$$\begin{aligned} &\int_{2R-h}^{2R} h^{-1}\left|\int_z^{z+h}\chi(s)G'(s)\,ds\right|^2 dz \\ &= \int_{2R-h}^{2R} h^{-1}\left|\int_{2R}^{z+h}G'(s)\,ds\right|^2 dz \\ &= \int_{2R-h}^{2R} h^{-1}\,|G(z+h) - G(2R)|^2\,dz \\ &= \int_{2R-h}^{2R}\left|\sum_{j=0}^{\infty}\frac{G(z+\bar{h}_j) - G(z+\bar{h}_{j+1})}{h^{\frac{1}{2}}}\right|^2 \\ &\le \int_{2R-h}^{2R}\sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^j\left|h_j^{\frac{1}{2}}D^{h_j}G(z+\bar{h}_j)\right|^2 \\ &\le \sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^j\sup_{0<h<R}\int_R^{3R}\left|h^{\frac{1}{2}}D^hG\right|^2. \end{aligned}$$

Thus, by (4.12) and (4.10), we conclude that the first integral on the right-hand side of (4.40) is bounded. Next, using again $\tau_2 \hat{\partial}_2 H_4 = \tau_2 \partial_2 H_4$ we get

$$\begin{aligned}
J_3 = & \int_Q h^{\frac{1+\sigma}{2}} \fint_{I_3^h \times I_2^h \times I_2^h \times I_1^h} \partial_2(\tau_2 \mu_3 H_4) h^{\frac{1+\sigma}{2}} \fint_{I_3^h \times I_2^h \times I_2^h \times I_1^h} \partial_3(\tau_2 H_3) \\
& - \int_Q h^{\frac{1+\sigma}{2}} \fint_{I_3^h \times I_2^h \times I_2^h \times I_1^h} \partial_2 \tau_2 \mu_3 H_4 h^{\frac{1+\sigma}{2}} \fint_{I_3^h \times I_2^h \times I_2^h \times I_1^h} \partial_3(\tau_2 H_3) \\
= & J_{31} + J_{32}.
\end{aligned}$$

Now, we are dealing with periodic functions, thus, it holds that

$$\begin{aligned}
J_{31} &= \int_Q h^{\frac{1+\sigma}{2}} D_2^h \left[\fint_{I_3^h \times I_2^h \times I_1^h} \tau_2 \mu_3 H_4 \right] h^{\frac{1+\sigma}{2}} D_3^h \left[\fint_{I_2^h \times I_2^h \times I_1^h} \tau_2 H_3 \right] dx \\
&= \int_Q h^{\frac{1+\sigma}{2}} D_3^h \left[\fint_{I_3^h \times I_2^h \times I_1^h} \tau_2 \mu_3 H_4 \right] h^{\frac{1+\sigma}{2}} D_2^h \left[\fint_{I_2^h \times I_2^h \times I_1^h} \tau_2 H_3 \right] dx.
\end{aligned}$$

Next, we apply the Hölder inequality. Then (4.12) and (4.10) yield

$$\int_Q h^{1+\sigma} \left| D_2^h \fint_{I_2^h \times I_2^h \times I_1^h} \tau_2 H_3 \right|^2 \leq c.$$

Using (4.32) and some easy calculations as above we also get

$$\int_Q h^{1+\sigma} \left| D_3^h \fint_{I_3^h \times I_2^h \times I_1^h} \tau_2 \mu_3 H_4 \right|^2 \leq c.$$

Thus, it holds that

$$|J_{31}| \leq c.$$

Next, for $\eta > 0$ the Hölder inequality entails

$$\begin{aligned}
|J_{32}| \leq \frac{c}{\eta} \int_Q h^{1+\sigma} & \left| \fint_{I_3^h \times I_2^h \times I_2^h \times I_1^h} \partial_2 \tau_2 \mu_3 H_4 \right|^2 \\
& + \eta \int_Q h^{1+\sigma} \left| \fint_{I_3^h \times I_2^h \times I_2^h \times I_1^h} \tau_2 \partial_3 H_3 \right|^2 dx.
\end{aligned}$$

The first integral on the right-hand side is bounded. Altogether, for a sufficiently small $\eta > 0$, we get

$$\begin{aligned}
J_0 &= \nu \int_Q h^{1+\sigma} \left| \fint_{I_3^h \times I_2^h \times I_2^h \times I_1^h} \tau_2 \partial_3 H_3 \right|^2 \\
&= \nu \int_Q h^{1+\sigma} \left| \fint_{I_3^h \times I_2^h \times I_2^h \times I_1^h} \partial_3(\tau_2 H_3) \right|^2 \leq c. \qquad (4.41)
\end{aligned}$$

This yields

$$\sup_{0<h<\bar{R}} \int_Q h^{-7+\sigma} \left| \int_{I_3^h \times I_2^h \times I_2^h \times I_1^h} \partial_3(\tau_2 H_3) \right|^2 \leq c. \tag{4.42}$$

Next, we get rid of the mean value $h^{-4} \int_{I_3^h \times I_2^h \times I_2^h \times I_1^h}$. We set $H(x) = \tau_2(x) H_3(x)$ and consider the Fourier series

$$H(x) = \sum_{m_1,m_2,m_3} \beta_{m_1 m_2 m_3} e^{im_1\rho(x_1-2R)} e^{im_2\rho(x_2-2R)} e^{im_3\rho(x_3-\varepsilon)},$$

where $\rho = \frac{\pi}{4R}$, and the Fourier coefficients are defined with respect to Q. Thus, in view of (4.42), we have

$$\sup_{0<h<\bar{R}} \sum_{m_1,m_2,m_3} \beta^2_{m_1 m_2 m_3} \rho^2 |m_3|^2 h^{-7+\sigma} \psi_3 \psi_2 \psi_2 \psi_1 \leq c,$$

where

$$\psi_k = \begin{cases} \frac{|e^{im_k\rho h}-1|^2}{\rho^2|m_k|^2} & \text{for } |m_k| \neq 0, \\ h^2 & \text{for } |m_k| = 0. \end{cases}$$

Let $m_0 = \max\{1, |m_1|, |m_2|, |m_3|\}$ and $\sigma = \frac{\delta}{2}$. Now, we proceed as in (4.22)-(4.29). We conclude that (cf. (4.23))

$$\sum_{m_1,m_2,m_3} \beta^2_{m_1 m_2 m_3} |m_3|^2 |m_0|^{-1-\delta} \leq c. \tag{4.43}$$

The Young inequality yields (with $q_1 = \frac{2}{1-\delta}$ and $q_2 = \frac{2}{1+\delta}$)

$$\begin{aligned} &\sum_{m_1,m_2,m_3} \beta^2_{m_1 m_2 m_3} |m_3|^{1-\delta} |m_0|^{-\frac{1}{2}} |m_0|^{\frac{1}{2}} \\ &\leq c \sum_{m_1,m_2,m_3} \beta^2_{m_1 m_2 m_3} \left(|m_3|^2 |m_0|^{-\frac{1}{1-\delta}} + |m_0|^{\frac{1}{1+\delta}} \right). \end{aligned} \tag{4.44}$$

By (4.43), an analogue of (4.27), and (4.44) we get

$$\sum_{m_1,m_2,m_3} \beta^2_{m_1 m_2 m_3} |m_3|^{1-\delta} \leq c.$$

This yields (cf. (4.29))

$$\sup_{0<h<\bar{R}} \int_{Q_\varepsilon} h^{1+\delta} \left| D_3^h H \right|^2 \leq c. \tag{4.45}$$

Next, we take $\lim_{\varepsilon\to 0}$. In view of (4.30) we obtain

$$\sup_{0<h<\bar{R}} \int_B h^{1+\delta} \left| D_3^h(\tau_2 H_3) \right|^2 \le c.$$

Let us note that $\tau_2 \equiv 1$ and $H_3 \equiv \partial_3 u_2$ on $B + \lambda e_3$ for $0 \le \lambda \le \bar{R}$. Thus, we have proven (4.36) for $k = 2$. □

Remark 4.9 We have proven Proposition 4.6 under the assumption (4.16), i.e., $\zeta^3 = e_3$. In case when $\zeta^3 \neq e_3$, we rotate the domain Ω such that the outward normal of $\Gamma^k \cup \Gamma^j$ is mapped onto $-e_3$.

Let M be the matrix describing the rotation. Let $\hat{x} = Mx$, $\hat{\mathbf{u}}(\hat{x}) = \mathbf{u}(M^{-1}\hat{x})$ etc, and $\widetilde{\partial}_i = \sum_k m_{ik}\partial_k$. Then we use

$$-\nu\widetilde{\nabla}\cdot\widetilde{\nabla}\hat{\mathbf{u}} + (\hat{\mathbf{u}}\cdot\widetilde{\nabla})\hat{\mathbf{u}} + \widetilde{\nabla}\hat{p} = \hat{\mathbf{f}} \quad \text{in } \hat{\Omega}, \tag{4.46}$$

$$\widetilde{\nabla}\cdot\hat{\mathbf{u}} = 0 \quad \text{in } \hat{\Omega}. \tag{4.47}$$

Let us note that $\widetilde{\partial}_1$ and $\widetilde{\partial}_2$ are derivatives in directions tangential to the boundary, and that $\widetilde{\partial}_3$ is the normal derivative. Thus, (4.12) and (4.10) yield regularity in the $\hat{x}_1$- and $\hat{x}_2$-directions. Proceeding as above we obtain (cf. (4.31))

$$\sup_{0<h<R} \int_{\hat{B}} h^{1+\delta} \left| D_3^h \widetilde{\partial}_3 \hat{u}_3 \right|^2 \le c.$$

The third equation of (4.46) implies that (cf. (4.32))

$$\sup_{0<h<R} \int_{\hat{B}^*} \left(h^{-\frac{5}{2}+\delta} \int_{I_3^h\times I_2^h\times I_1^h} \widetilde{\partial}_3\hat{p} \right)^2 \le c.$$

Then the proof of Proposition 4.6 yields

$$\sup_{0<h<\bar{R}} \int_{\hat{B}} h^{1+\delta} \left| D_3^h \widetilde{\nabla}\hat{\mathbf{u}} \right|^2 \le c.$$

In view of

$$\left| D_3^h \widetilde{\nabla}\hat{\mathbf{u}} \right|^2 = \left| M D_3^h \nabla\hat{\mathbf{u}} \right|^2 = \left| D_3^h \nabla\hat{\mathbf{u}} \right|^2,$$

the substitution rule for integrals yields the desired result

$$\sup_{0<h<\bar{R}} \int_B h^{1+\delta} \left| D_3^h \nabla\mathbf{u} \right|^2 \le c. \tag{4.48}$$

Now, we are able to investigate the regularity of the pressure p.

Proposition 4.10 *Let* $0 < \delta < \frac{1}{2}$, $\bar{R} = \frac{R}{8}$, *and* $B_0 = \{x \in \Omega : |x - P| < \frac{R}{8}\}$. *Then there is a constant* c *depending only on* R_0, δ, *and the data such that*

$$\sup_{0<h<\bar{R}} \int_{B_0} h^{1+\delta} \left|D_k^h p\right|^2 \le c \qquad \textit{for } 1 \le k \le 3. \tag{4.49}$$

Proof. First, we suppose $\Lambda_3 = \{3\}$, $\zeta^3 = e_3$, $\Omega \cap B_{R_0} = \{x \in B_{R_0} : x_3 > 0\}$, and

$$\partial\Omega \cap B_{R_0} = \{x \in B_{R_0} : x_3 = 0\}. \tag{4.50}$$

We define a sufficiently small cube Q_ε such that $Q_\varepsilon \subset B$. We set $Q_\varepsilon = \{x \in \mathbb{R}^3 : -\frac{R}{4} < x_1, x_2 < \frac{R}{4},\ \varepsilon < x_3 < \frac{R}{2} + \varepsilon\}$. Then we define, as above, the cube Q and the functions $H_1, \ldots, H_5$ by reflections. Multiplying equation (4.37) by μ_2 yields

$$\partial_2 H_4 = \nu \sum_{j=1}^{3} \mu_2 \hat{\partial}_j H_j + \mu_2 H_5.$$

We get

$$\int_Q h^{1+\delta} \left| \fint_{I_3^h \times I_2^h \times I_1^h} \partial_2 H_4 \right|^2 \le c \int_Q h^{1+\delta} \left| \fint_{I_3^h \times I_2^h \times I_1^h} \sum_{j=1}^{3} \mu_2 \hat{\partial}_j H_j \right|^2 + c \int_Q h^{1+\delta} \left| \fint_{I_3^h \times I_2^h \times I_1^h} \mu_2 H_5 \right|^2.$$

Using (4.12), (4.10), (4.17), (4.35), and calculations as above, we conclude

$$\int_Q h^{1+\delta} \left| \fint_{I_3^h \times I_2^h \times I_1^h} \partial_2 H_4 \right|^2 \le c.$$

In the same way, we can estimate $\partial_1 H_4$. Hence, in view of (4.32), it follows that

$$\int_Q h^{1+\delta} \left| \fint_{I_3^h \times I_2^h \times I_1^h} \partial_k H_4 \right|^2 \le c \qquad \text{for } 1 \le k \le 3. \tag{4.51}$$

Next, we consider the Fourier series of H_4. Let us denote the Fourier coefficients by $\beta_{m_1 m_2 m_3}$. Estimate (4.51) implies that

$$\sum_{m_1, m_2, m_3} h^{-5+\delta} \beta_{m_1 m_2 m_3}^2 \rho^2 |m_k|^2 \prod_{j=1}^{3} \psi_j \le c \qquad \text{for } 1 \le k \le 3.$$

Let $m_0 = \max\{1, |m_1|, |m_2|, |m_3|\}$. Proceeding as above (cf. (4.22)-(4.23)) we obtain

$$\sum_{m_1,m_2,m_3} \beta^2_{m_1m_2m_3}|m_k|^2|m_0|^{-1-\delta} \le c \qquad \text{for } 1 \le k \le 3, \tag{4.52}$$

thus,

$$\sum_{\substack{m_1,m_2,m_3\\|m_k|=m_0}} \beta^2_{m_1m_2m_3}|m_0|^{1-\delta} \le c \qquad \text{for } 1 \le k \le 3$$

and

$$\sum_{m_1,m_2,m_3} \beta^2_{m_1m_2m_3}|m_k|^{1-\delta} \le \sum_{m_1,m_2,m_3} \beta^2_{m_1m_2m_3}|m_0|^{1-\delta} \le c$$

for $1 \le k \le 3$. This implies that (cf. (4.28)-(4.29))

$$\sup_{0<h<\bar{R}} \int_{Q_\varepsilon} h^{1+\delta} \left|D_k^h H_4\right|^2 \le c \qquad \text{for } 1 \le k \le 3.$$

Now, we take $\lim_{\varepsilon\to 0}$. For $x \in B_0$ it holds that $x + h\zeta^k \in Q_0$, thus $D_k^h H_4(x) \equiv D_k^h p(x)$. Hence, we have proven the assertion under the additional assumption (4.50).

In general, (4.50) may not be satisfied. Then we consider an appropriate parallelogram instead of the cube Q_ε; cf. [9]. □

Proof of Theorem 2.2: a) Let $B_\eta = \{x \in B : \mathrm{dist}(x, \partial\Omega) \ge \eta\}$. The basis vectors ζ^i satisfy $\mathrm{angle}(\zeta^i, \zeta^j) \ge \alpha^*$ for $1 \le i < j \le 3$, where the constant α^* depends only on the geometry of $\partial\Omega$. Thus, from (4.10), (4.12), (4.17), and (4.48), it follows for all $\delta \in (0, \frac{1}{2})$ that

$$\sup_{\substack{\eta>0\\0<|z|<\eta}} \int_{B_\eta} \frac{|\nabla\mathbf{u}(x+z) - \nabla\mathbf{u}(x)|^2}{|z|^{1-\delta}}\, dx \le c, \tag{4.53}$$

where the constant c depends on δ, the data, and R_0.

We can find finite sets of numbers $\{R_1, \ldots, R_N\}$ and of points $\{P_1, \ldots, P_N\}$ depending only on the geometry of $\partial\Omega$ such that $\overline{\Omega} \subset \bigcup_{i=1}^N B_{R_i}(P_i)$, and each ball $B_{R_i}(P_i)$ satisfies: (i) if $B_{R_i}(P_i) \cap \partial\Omega \ne \emptyset$ then $B_{R_i}(P_i) \cap \partial\Omega$ is simply connected; (ii) P_i is the only vertex of $B_{R_i}(P_i) \cap \partial\Omega$ or there is no vertex of $\partial\Omega$ in $B_{R_i}(P_i)$.

This implies that

$$\mathbf{u} \in \mathcal{H}^{\frac{3}{2}-\frac{\delta}{2},2}(\Omega; \mathbb{R}^3) \qquad \text{for } \delta \in \left(0, \frac{1}{2}\right).$$

The imbedding theorem of Nikolskii spaces into Sobolev spaces (cf. [1])

$$\mathcal{H}^{s,p}(\Omega) \to W^{s-\varepsilon,p}(\Omega) \qquad \text{for all } \varepsilon > 0$$

entails $\mathbf{u} \in W^{s,2}(\Omega; \mathbb{R}^3)$ for all $s < \frac{3}{2}$. This yields the assertion (2.1).
b) Next, we investigate the regularity of the pressure p and prove (2.2). Let $0 < \delta < \frac{1}{2}$. By (4.49) there exists a constant c depending only on R_0, δ, and the data, such that

$$\sup_{0<h<\bar{R}} \int_{B_0} h^{1+\delta} \left|D_k^h p\right|^2 \, dy \le c \qquad \text{for } 1 \le k \le 3.$$

The conclusion follows using the same covering argument as above and the imbedding theorem of Nikolskii spaces into Sobolev spaces. □

5 The problem ($\mathcal{P}_2$) on polyhedral domains

In this section we prove Theorem 2.3. We consider the variational formulation of ($\mathcal{P}_2$) given in [5]. Let V be the space defined in (4.1). We call $(\mathbf{u}, p)$ a weak solution of ($\mathcal{P}_2$) if

$$\nu \int_\Omega (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{w}) + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{w} - \int_\Omega p \nabla \cdot \mathbf{w} = \int_\Omega \mathbf{f} \cdot \mathbf{w} \qquad (5.1)$$

for all $\mathbf{w} \in V$.

In [5] it is shown that there exists a weak solution $(\mathbf{u}, p)$ such that $\mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^3)$. Multiplying (1.1) by $\mathbf{w}$ and using the identity

$$-\Delta \mathbf{u} = \nabla \times (\nabla \times \mathbf{u}) - \nabla(\nabla \cdot \mathbf{u})$$

the equivalence between (5.1) and ($\mathcal{P}_2$) follows.

Instead of the second Korn inequality we use the following Lemma.

Lemma 5.1 *There exists a constant $c > 0$ such that*

$$\int_\Omega (\nabla \times \mathbf{w}) \cdot (\nabla \times \mathbf{w}) \ge c \, \|\mathbf{w}\|^2_{W^{1,2}(\Omega, \mathbb{R}^3)} \qquad \textit{for all } \mathbf{w} \in V_0, \qquad (5.2)$$

where $V_0 = \{\mathbf{w} \in V : \nabla \cdot \mathbf{w} = 0\}$.

Proof. The proof is given in [5]. Let us sketch the idea. In case when Ω is convex, the claim follows applying the imbedding Theorems 3.7-3.9 in [10].

Next, we consider the case when Ω is not convex. Let $\widetilde{\Omega}$ be the convex hull of Ω and let $\mathbf{w} \in V_0$. Note that $\mathbf{w} = 0$ on Γ_1. Thus, the function $\widetilde{\mathbf{w}}$, defined by $\widetilde{\mathbf{w}} = \mathbf{w}$ in Ω and $\widetilde{\mathbf{w}} = 0$ in $\widetilde{\Omega} \setminus \Omega$, belongs to the space

$$\widetilde{V}_0(\widetilde{\Omega}) = \{\widetilde{\mathbf{w}} \in V_0(\widetilde{\Omega}) : \widetilde{\mathbf{w}} = 0 \text{ in } \widetilde{\Omega} \setminus \Omega\}.$$

Here we have used $(\partial\Omega \setminus \partial\widetilde{\Omega}) \subset \Gamma_1$.

Notice that $\widetilde{\Omega}$ is convex and $\widetilde{V}_0(\widetilde{\Omega})$ is a subspace of $V_0(\widetilde{\Omega})$. Thus, (5.2) follows as above for all functions $\widetilde{\mathbf{w}} \in \widetilde{V}_0(\widetilde{\Omega})$. This yields the assertion. □

Next, we investigate the regularity of $\mathbf{u}$ proceeding as in section 4.

Proposition 5.2 *Let* $0 < \delta < \frac{1}{2}$ *and* $\bar{R} = \frac{R}{4}$. *There exists a constant* c *depending only on* R_0, δ, *and the data, such that*

$$\sup_{0<h<\bar{R}} \int_B h^{1+\delta} \left|D_i^h \nabla \mathbf{u}\right|^2 \, dy \le c \qquad \textit{for } 1 \le i \le 3. \tag{5.3}$$

Proof. First, we suppose that the assumptions of Lemma 4.3 are satisfied. We prove (5.3) for some $i \in \Lambda_1$, say $i = 1$. Let us proceed as in the proof of Lemma 4.3. Using the test function $\mathbf{w} = \tau^2 h^\delta D_1^h \mathbf{u}$ in (5.1) yields

$$\begin{aligned}
&\nu \int_{B''} (\nabla \times \mathbf{u}) \cdot \tau^2 h^\delta D_1^h (\nabla \times \mathbf{u}) + \nu \int_{B''} (\nabla \times \mathbf{u}) \cdot (\nabla \tau^2 \times h^\delta D_1^h \mathbf{u}) \\
&+ \int_{B''} (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \tau^2 h^\delta D_1^h \mathbf{u} - \int_{B''} p \nabla \cdot (\tau^2 h^\delta D_1^h \mathbf{u}) \\
&= \int_{B''} \mathbf{f} \cdot \tau^2 h^\delta D_1^h \mathbf{u}.
\end{aligned} \tag{5.4}$$

Let $\mathbf{m} \in \mathbb{R}^3$ have the components $m_i = (\nabla \times \mathbf{u})_i$ $(1 \le i \le 3)$. We put $\widetilde{m}_i = E_1^h m_i$, $G(\mathbf{m}) = \frac{1}{2} \sum_i m_i^2$, and use the Taylor expansion

$$G(\widetilde{\mathbf{m}}) - G(\mathbf{m}) = \sum_i (\widetilde{\mathbf{m}} - \mathbf{m})_i m_i + \frac{1}{2} \sum_{i,j} (\widetilde{\mathbf{m}} - \mathbf{m})_i (\widetilde{\mathbf{m}} - \mathbf{m})_j \delta_{ij}.$$

Then we obtain

$$h^\delta D_1^h (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{u}) = \frac{1}{2} h^\delta D_1^h |\nabla \times \mathbf{u}|^2 - \frac{1}{2} h^{1+\delta} \left(D_1^h (\nabla \times \mathbf{u})\right)^2. \tag{5.5}$$

(5.4) and (5.5) yield

$$\begin{aligned}
&\frac{\nu}{2} \int_{B''} \tau^2 h^{1+\delta} \left|D_1^h (\nabla \times \mathbf{u})\right|^2 \\
&= \frac{\nu}{2} \int_{B''} \tau^2 h^\delta D_1^h \, |\nabla \times \mathbf{u}|^2 + \nu h^\delta \int_{B''} (\nabla \times \mathbf{u}) \cdot \left(\nabla \tau^2 \times D_1^h \mathbf{u}\right) \\
&\quad + h^\delta \int_{B''} (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \tau^2 D_1^h \mathbf{u} - h^\delta \int_{B''} p \nabla \cdot \left(\tau^2 D_1^h \mathbf{u}\right) - h^\delta \int_{B''} \mathbf{f} \tau^2 D_1^h \mathbf{u} \\
&= J_1 + \ldots + J_5.
\end{aligned}$$

The integrals $J_1, \dots, J_5$ on the right-hand side can be estimated as in the proof of Lemma 4.3. Next, applying Lemma 5.1 we obtain

$$\frac{\nu}{2}\int_{B''}\tau^2 h^{1+\delta}\left|D_1^h(\nabla\times\mathbf{u})\right|^2 \geq \frac{\nu}{2}\int_{B'} h^{1+\delta}\left|\nabla\times D_1^h\mathbf{u}\right|^2$$
$$\geq c\int_{B'} h^{1+\delta}\left|\nabla D_1^h\mathbf{u}\right|^2.$$

Thus, we conclude

$$\sup_{0<h<R}\int_{B''}\tau^2 h^{1+\delta}\left|D_i^h\nabla\mathbf{u}\right|^2 \leq c+\varepsilon\sup_{0<h<R}\int_{B''}\tau^2 h^{1+\delta}\left|D_i^h\nabla u\right|^2 \tag{5.6}$$

for $i\in\Lambda_1$ and a sufficiently small $\varepsilon>0$.

Clearly, following the proofs of Proposition 4.4 and 4.5, we obtain (5.6) for $i\in\Lambda_1\cup\Lambda_2$. Next, the proof of Proposition 4.6 uses the equation (1.1) but not the variational formulation (5.1). Hence, no changes in the proof of Proposition 4.6 must be made. Thus, we get the assertion. □

Proof of Theorem 2.3: Using the same cover argument as in the proof of Theorem 2.2 yields the $W^{s,2}$-regularity of u (2.3). Next, following the proof of Proposition 4.10 we obtain the $W^{s,2}$-regularity of the pressure p stated in (2.4). □

6 Lipschitzian domains

In this section we sketch the proofs of Theorem 2.5 and Theorem 2.6. (Further details can be found in [9] and [8] where the same method is used.)

Let $P\in\partial\Omega$ and $\hat{x}=\phi(x)$. By the assumptions given in section 2.2 there is a radius $R_0>0$ and a $W^{2,\infty}$-mapping ϕ such that $B_{R_0}(\phi(P))\cap\phi(\partial\Omega)$ is the intersection of $B_{R_0}(\phi(P))$ and a polyhedron.

By ϕ_k $(1\leq k\leq 3)$ we denote the k-th component of ϕ. Let A be the matrix whose elements are defined by $\hat{a}_{ik}(\hat{x})$, where $\hat{a}_{ik}(\hat{x})=a_{ik}(\phi^{-1}(\hat{x}))$ and $a_{ik}(x)=\frac{\partial}{\partial x_i}\phi_k(x)$. Then A is positive definite, the eigenvalues of A only depend on the geometry of $\partial\Omega$, and it holds $a_{ik}\in W^{1,\infty}(\Omega)$.

Let $\hat{\mathbf{u}}(\hat{x})=\mathbf{u}(\phi^{-1}(\hat{x}))$ etc. Then

$$-\nu\widetilde{\nabla}\cdot\widetilde{\nabla}\hat{\mathbf{u}}+(\hat{\mathbf{u}}\cdot\widetilde{\nabla})\hat{\mathbf{u}}+\widetilde{\nabla}\hat{p}=\hat{\mathbf{f}}\qquad\text{in }\phi(\Omega), \tag{6.1}$$
$$\widetilde{\nabla}\cdot\hat{\mathbf{u}}=0\qquad\text{in }\phi(\Omega), \tag{6.2}$$

where $\widetilde{\partial}_i=\sum_k\hat{a}_{ik}\partial_k$.

In order to prove Theorem 2.5 we follow the proofs of Proposition 4.4 and 4.5. We obtain for $i\in\Lambda_1,\Lambda_2$

$$\sup_{0<h<R}\int_{\hat{B}} h^{1+\delta}\left|D_i^h\widetilde{\nabla}\hat{\mathbf{u}}\right|^2\leq c. \tag{6.3}$$

Next, proceeding as in the proof of Proposition 4.6, we get (6.3) for $i \in \Lambda_3$.

Let us note that

$$D_i^h \widetilde{\nabla} \hat{u}_k = D_i^h (A \nabla \hat{u}_k) = D_i^h A \nabla E_i^h \hat{u}_k + A D_i^h \nabla \hat{u}_k$$

and that $|A D_i^h \nabla \hat{u}_k|^2 \geq \lambda^2 |D_i^h \nabla \hat{u}_k|^2$, where $\lambda > 0$ is the smallest eigenvalue of A. We conclude

$$\sup_{\substack{\eta>0 \\ 0<|z|<\eta}} \int_{((\phi)^{-1}(B))_\eta} \frac{|\nabla \mathbf{u}(x+z) - \nabla \mathbf{u}(x)|^2}{|z|^{1-\delta}} \, dx \leq c,$$

where $\Omega_\eta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \geq \eta\}$. We complete the proof of Theorem 2.5 by using the same cover argument as in the proof of Theorem 2.2.

Obviously, the proof of Theorem 2.6 follows in the same way.

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