1

Sobolev spaces

1.1 Motivation

Why do mathematicians use Sobolev spaces instead of the simpler looking spaces of continuously differentiable functions?

The most famous Sobolev space is $H^1(\Omega)$, the set of all functions u which are square integrable, together with all their first derivatives, in Ω, an open subset of $\mathbb{R}^n$, the usual n-dimensional Euclidian space. The derivatives are to be understood in the sense of distributions. It is not even true that any function in $H^1(\Omega)$ is continuous. For instance, the function

$$u(x, y) = |\ln \tfrac{1}{2}(x^2 + y^2)|^{1/3}$$

is in $H^1(\Omega_1)$, where Ω_1 is the unit circle in the plane:

$$\Omega_1 = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 < 1\}.$$

However, u is not continuous at $(0, 0)$ and not even bounded. Such spaces are obviously not easy to handle.

There are several reasons that lead us to use such spaces. The most significant is perhaps that they appear naturally in the solution of elliptic boundary value problems by the method of calculus of variations. The variational approach to the Dirichlet problem in Ω (with $n = 2$, say) is the following. Given a function f in Ω, we want to find a function u, also defined in Ω, a solution of

$$D_x^2 u(x, y) + D_y^2 u(x, y) = f(x, y) \qquad \text{for all } (x, y) \in \Omega, \tag{1,1,1}$$

with the boundary condition

$$u(x, y) = 0 \qquad \text{for all } (x, y) \in \partial\Omega. \tag{1,1,2}$$

We now try to view equation (1,1,1) as the equation of a critical point u for some functional. One possible functional is obviously

$$u \mapsto F(u) = \tfrac{1}{2}\int_\Omega [|D_x u|^2 + |D_y u|^2]\, dx\, dy + \int_\Omega fu\, dx\, dy. \tag{1,1,3}$$

If we assume that f is continuous, then F is a differentiable functional over V, the space of all functions which are continuous together with their first and second derivatives in $\bar{\Omega}$ and which vanish on the boundary $\partial\Omega$. The Frechet derivative of F at u is

$$v \mapsto \langle F'(u), v\rangle = \int_\Omega [D_x u D_x v + D_y u D_y v]\, dx\, dy + \int_\Omega fv\, dx\, dy,$$

or, after integrating by parts

$$v \mapsto \langle F'(u), v\rangle = \int_\Omega [-D_x^2 u - D_y^2 u + f]v\, dx\, dy. \tag{1,1,4}$$

Consequently, if u is a critical point for F, then u is solution of equation (1,1,1); u fulfils the boundary condition (1,1,2), simply because it is an element of V. Now our initial problem is converted into the problem of finding critical points for F. Obviously F is a convex quadratic functional on V; its minima are critical points, provided they exist. Usually a minimum is obtained by considering a minimizing sequence. This means a sequence u_n, $n = 1, 2, \ldots$ in V such that

$$F(u_n) \searrow m \tag{1,1,5}$$

where

$$m = \inf_{u \in V} F(u).$$

From (1,1,5), it follows that $D_x u_n$ and $D_y u_n$, $n = 1, 2, \ldots$ are bounded sequences in $L_2(\Omega)$, the space of all square integrable functions on Ω. Taking in account the boundary condition, an integration shows that u_n, $n = 1, 2, \ldots$ is also a bounded sequence in $L_2(\Omega)$.

We conclude, by using the property of bounded sequences in Hilbert space, that there exists a subsequence which is weakly convergent. Consequently, there exist

$$u, v_1, v_2 \in L_2(\Omega),$$

such that

$$\begin{cases} u_n \to u \\ D_x u_n \to v_1 \\ D_y u_n \to v_2 \end{cases}$$

weakly in $L_2(\Omega)$. The theory of distributions shows that $v_1 = D_x u$ and $v_2 = D_y u$, and therefore u is an element of the Sobolev space $H^1(\Omega)$.

Summing up, we have first replaced the original problem (1,1,1) (1,1,2) by the problem of finding a minimum for the functional F defined by (1,1,3). This was achieved in the space V, i.e. in the framework of spaces of twice continuously differentiable functions. Then the construction of a minimum for F leads to considering a sequence of functions in V (and, consequently, in $C^2(\bar{\Omega})$) which does not converge in $C^2(\bar{\Omega})$ but which is convergent in the weak topology of $H^1(\Omega)$. Its limit appears naturally as an element of $H^1(\Omega)$.

Actually, it can be proved that there exists a continuous f such that u, the solution of (1,1,1) (1,1,2), does not belong to $C^2(\Omega)$. Indeed, assume the contrary, then the mapping

$$f \mapsto D_x D_y u \tag{1,1,6}$$

would be a linear mapping from $C^0(\bar{\Omega})$ into itself; here we denote by $C^0(\bar{\Omega})$, the space of all continuous functions in $\bar{\Omega}$ equipped with the maximum norm. It follows from the closed graph theorem that (1,1,6) is a continuous mapping. Consequently, there exists a measure $d\mu$ on $\bar{\Omega}$ such that

$$D_x D_y u(0, 0) = \int_{\bar{\Omega}} f \, d\mu. \tag{1,1,7}$$

However, the solution u of problem (1,1,1) (1,1,2) is well known for some particular domains Ω. For instance, when Ω_1 is the unit circle, following Courant and Hilbert (1962) we have

$$u(x, y) = \iint_{\Omega_1} K(x, y; \xi, \eta) f(\xi, \eta) \, d\xi \, d\eta.$$

where

$$K(x, y; \xi, \eta) = +\frac{1}{2\pi} \log \frac{r_1}{r_2} - \frac{1}{2\pi} \log \rho$$

$$r_1 = \sqrt{[(x-\xi)^2 + (y-\eta)^2]}, \qquad \rho = \sqrt{(\xi^2 + \eta^2)}$$

$$r_2 = \sqrt{[(x-\xi/\rho^2)^2 + (y-\eta/\rho^2)^2]}.$$

It follows that

$$D_x D_y K(0, 0; \xi, \eta) = \frac{1}{\pi} \xi\eta \frac{1-\rho^4}{\rho^4}$$

and this is a singular kernel at the origin. Consequently, $D_x D_y u(0, 0)$ is given by the singular integral

$$D_xD_yu(0,0)=\lim_{\varepsilon\to 0}\frac{1}{\pi}\iint_{\varepsilon<\rho\leq 1}\xi\eta\frac{1-\rho^4}{\rho^4}f(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{1,1,8}$$

This is in contradiction with (1,1,7).

Now we have at least one good reason for using the space $H^1(\Omega)$; but, what about spaces of functions with more square integrable derivatives? And, what about spaces of functions of which certain derivatives have pth power integrable for some p, with $1\leq p<\infty$? The former appear in the variational method for solving equations of order higher than two, while the latter appear in the solution of nonlinear equations.

There are, of course, several other reasons for using Sobolev spaces in the solution of partial differential equations and boundary value problems. One of them is simply the property that the Fourier transform converts any partial differential equation with constant coefficients into a division problem. Plancherel's theorem allows one to handle functions with square integrable derivatives. Unfortunately, there is no counterpart of Plancherel's theorem for continuous functions. Consequently, the solutions are built in Sobolev spaces first and their differentiability properties in the classical sense are obtained through the so-called imbedding theorems (see Section 1.4.4).

To end this introductory section, let us define the scope of this chapter about Sobolev spaces. There is a tremendous amount of literature available concerning Sobolev spaces. Most of it is quoted in Avantaggiati (1975) and Triebel (1978), for instance. However, we shall mainly work with spaces defined on domains whose boundaries are polygons or polyhedras. On such domains, Sobolev spaces happen to have some strange properties, which are hard to find in the current literature. Consequently, the guideline that we shall follow throughout this chapter is to cite only those properties which are easy to find elsewhere and to give precise references for their proofs (most of them are to be found in Nečas (1967)). Meanwhile we shall give precise statements together with complete proofs for all those properties that we need and whose proofs are too scattered in the present literature. As far as only definitions and statements of properties are concerned, we attempt to make this chapter self-contained.

1.2 Boundaries

The properties of functions in a given Sobolev space, $H^1(\Omega)$ for instance, depend very strongly on the properties of the boundary Γ of the domain Ω. Several different points of view have been followed by mathematicians for specifying the properties of the boundary Γ. The purpose of the present section is to introduce the three main points of view and to compare them.

1.2.1 Graphs and manifolds

Many authors view (whenever possible) the boundary Γ of Ω as being locally the graph of a function φ. Then the properties of Γ are specified through the properties of φ, e.g. continuity, Lipschitz property, differentiability and so on. This is the point of view followed by Aronszjan and Smith (1961), Adams (1975), Ladyzenskaia and Uralc'eva (1968), Miranda (1970), Nečas (1967) for instance. This last author will be our usual reference in the present subsection.

Definition 1.2.1.1 *Let Ω be an open subset of $\mathbb{R}^n$. We say that its boundary Γ is continuous (respectively Lipschitz, continuously differentiable, of class $C^{k,1}$, m times continuously differentiable*†*) if for every $x \in \Gamma$ there exists a neighbourhood V of x in $\mathbb{R}^n$ and new orthogonal coordinates $\{y_1, \ldots, y_n\}$ such that*

(a) *V is an hypercube in the new coordinates:*

$$V = \{(y_1, \ldots, y_n) \mid -a_j < y_j < a_j, 1 \leq j \leq n\};$$

(b) *there exists a continuous*‡ *(resp. Lipschitz,*§ *continuously differentiable, of class $C^{k,1}$, m times continuously differentiable) function φ, defined in*

$$V' = \{(y_1, \ldots, y_{n-1}) \mid -a_j < y_j < a_j, 1 \leq j \leq n-1\}$$

and such that

$$\begin{aligned} &|\varphi(y')| \leq a_n/2 \text{ for every } y' = (y_1, \ldots, y_{n-1}) \in V', \\ &\Omega \cap V = \{y = (y', y_n) \in V \mid y_n < \varphi(y')\}, \\ &\Gamma \cap V = \{y = (y', y_n) \in V \mid y_n = \varphi(y')\}. \end{aligned} \tag{1.2.1.1}$$

In other words, in a neighbourhood of x, Ω is below the graph of φ and consequently the boundary Γ is the graph of φ. We recall that saying that φ belongs to the class $C^{k,1}$ means that it is k times continuously differentiable and its derivatives of order k are Lipschitz continuous.

If an open set Ω has a continuous boundary Γ, then Ω is not on both sides of Γ at any point of Γ. For instance, $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ has not a continuous

† m and k are integers ≥ 1, possibly equal to $+\infty$.

‡ Observe that the word continuous may be omitted there. Indeed, if a function fulfils the conditions (1.2.1.1), it is easily proved that φ has to be continuous.

§ By Lipschitz condition, we always mean uniform Lipschitz condition.

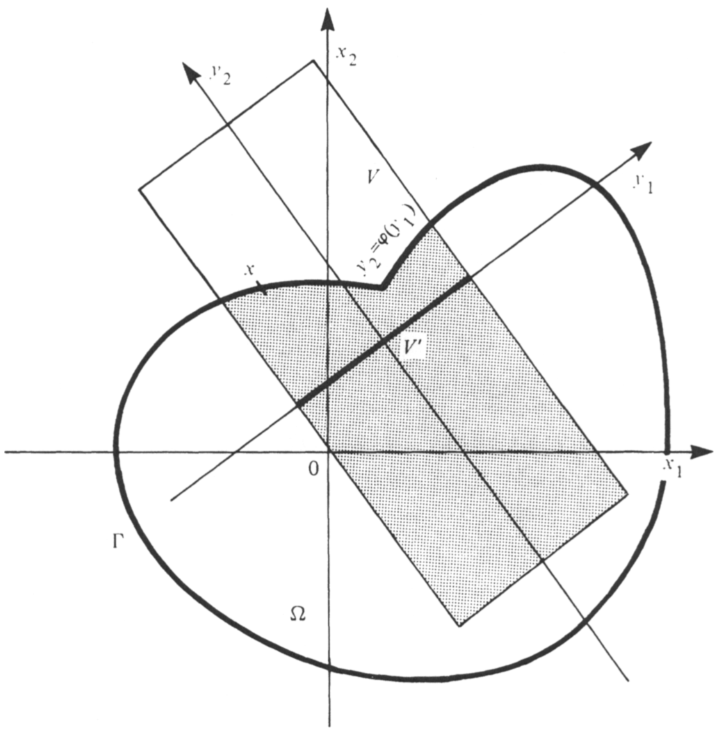

Figure 1.1

boundary in the sense of Definition 1.2.1.1. Likewise, a domain with a cut does not fulfil the conditions of Definition 1.2.1.1. However, this definition allows turning points.

The most important examples in the sequel are the following. A bounded open subset of $\mathbb{R}^2$, whose boundary Γ is a polygon, has a Lipschitz boundary and lacks a continuously differentiable boundary. Similarly, a bounded open subset of $\mathbb{R}^3$, whose boundary Γ is a polyhedron, has a Lipschitz boundary and lacks a continuously differentiable boundary.

Many other authors, such as Lions and Magenes (1968) and Hörmander (1963), prefer to consider (whenever possible) the closure $\bar{\Omega}$ of the domain Ω, as an n-dimensional manifold with boundary, imbedded in $\mathbb{R}^n$. They add various regularity assumptions on the manifold.

Definition 1.2.1.2 *Let Ω be an open subset of $\mathbb{R}^n$. We say that $\bar{\Omega}$ is an n-dimensional continuous (respectively Lipschitz, continuously differenti-*

able, of class $C^{k,1}$, m times continuously differentiable) submanifold† *with boundary in $\mathbb{R}^n$, if for every $x \in \Gamma$ there exists a neighbourhood V of x in $\mathbb{R}^n$ and a mapping ψ from V into $\mathbb{R}^n$ such that*

(a) *ψ is injective*
(b) *ψ together with ψ^{-1} (defined on $\psi(V)$) is continuous (respectively Lipschitz, continuously differentiable, of class $C^{k,1}$, m times continuously differentiable)*
(c) *$\Omega \cap V = \{y \in \Omega \mid \psi_n(y) < 0\}$ where $\psi_n(y)$ denotes the* nth *component of $\psi(y)$.*

As a consequence of condition (c), the boundary Γ of Ω is defined locally by the equation $\psi_n(y) = 0$.

In the notations of Definition 1.2.1.1, define ψ as follows:

$$\psi(y) = \{y_1, \ldots, y_{n-1}, y_n - \varphi(y')\} \tag{1,2,1,2}$$

It is easily seen that ψ fulfils all the conditions in Definition 1.2.1.2 with the same amount of differentiability for ψ and ψ^{-1} as for φ. In other words, Definition 1.2.1.1 implies Definition 1.2.1.2 and it is natural to ask whether the converse is also true. Unfortunately the converse statement is only partly true. It follows from the implicit functions theorem that Definition 1.2.1.2 implies Definition 1.2.1.1 provided everything is at least continuously differentiable. Indeed, we rebuild a function φ from a given ψ, by solving the equation

$$\psi_n(y) = 0$$

with respect to y_j where j is chosen in such a way that $D_j\psi_n$ does not vanish (locally). This is possible when ψ is continuously differentiable. Then the chain rule shows that φ is continuously differentiable (resp. of class $C^{k,1}$, m times continuously differentiable) when ψ is continuously differentiable (resp. of class $C^{k,1}$, m times continuously differentiable).

The implicit function theorem does not hold for Lipschitz functions and the following counterexample will show that Definition 1.2.1.2 does not imply Definition 1.2.1.1 under the single assumption that ψ together with ψ^{-1} is Lipschitz. This counterexample was shown to me by Zerner. We need some preliminary lemmas.

Lemma 1.2.1.3 *The Definition 1.2.1.2 of n-dimensional Lipschitz submanifolds with boundary in $\mathbb{R}^n$ is invariant under bi-Lipschitz homeomorphisms.*

A homeomorphism η of $\bar{\Omega}_1$ onto $\bar{\Omega}_2$ and of a neighbourhood W_1 of $\bar{\Omega}_1$

† A continuous manifold is more usually called a topological manifold.

onto a neighbourhood W_2 of $\bar{\Omega}_2$ will be called a bi-Lipschitz homeomorphism of $\bar{\Omega}_1$ onto $\bar{\Omega}_2$ if η and η^{-1} are uniformly Lipschitz-continuous. Lemma 1.2.1.3 is an easy consequence of the chain rule for the Lipschitz functions due to Rademacher (1919–20).

We now define a bi-Lipschitz homeomorphism from $\mathbb{R}^2$ onto $\mathbb{R}^2$ by

$$\eta(x) = \{x_1, \varphi(x_1) + x_2\}$$

where

$$\varphi(t) = \begin{cases} 3\,|t| - \dfrac{1}{2^{2k-1}} & \text{for } \dfrac{1}{2^{2k+1}} \leqslant |t| \leqslant \dfrac{1}{2^{2k}} \\ -3\,|t| + \dfrac{1}{2^{2k}} & \text{for } \dfrac{1}{2^{2k+2}} \leqslant |t| \leqslant \dfrac{1}{2^{2k+1}}. \end{cases}$$

The slope of φ is either 3 or -3. Consequently, φ is a uniformly Lipschitz function (with Lipschitz constant equal to 3). This implies that η together with η^{-1} are uniformly Lipschitz mappings.

Let Ω be defined as follows:

$$\Omega = \{(x_1, x_2) \mid 0 < x_1 < 1,\ 0 < x_2 < x_1\}.$$

It is clear that Ω has a Lipschitz boundary according to Definition 1.2.1.1.

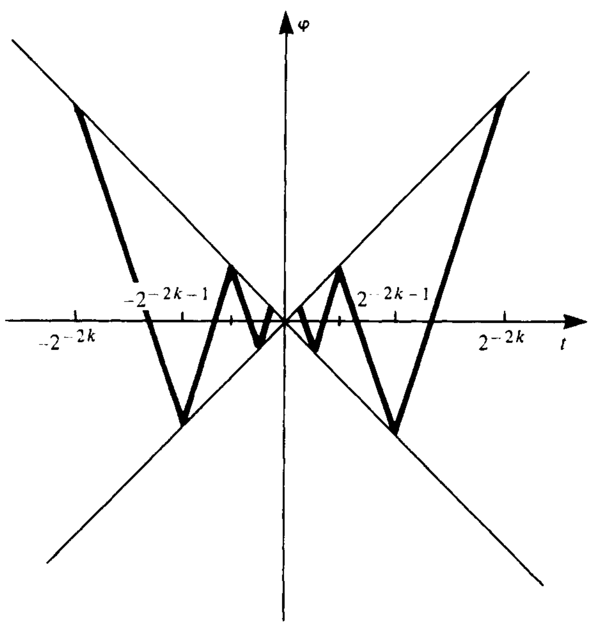

Figure 1.2

Therefore, $\bar{\Omega}$ is a two-dimensional Lipschitz submanifold with boundary, in $\mathbb{R}^2$, according to Definition 1.2.1.2, since Definition 1.2.1.1 implies Definition 1.2.1.2. Next, consider the new domain $\overline{\eta(\Omega)}$. This is also a two-dimensional Lipschitz submanifold with boundary in $\mathbb{R}^2$, owing to Lemma 1.2.1.3. Now we have the following result.

Lemma 1.2.1.4 *$\eta(\Omega)$ has not a continuous boundary according to Definition* 1.2.1.1.

Proof It is obvious from the geometry of $\eta(\Omega)$ (see Fig. 1.3) that every linear segment with origin at 0, which cuts Γ, actually cuts Γ at infinitely many points. This property is true without any restriction on the length of the segment under consideration. This prevents the existence of a neighbourhood V of 0, together with the existence of new coordinates such that $\Gamma \cap V$ should be the graph of a function as in Definition 1.2.1.1. Accordingly $\eta(\Omega)$ lacks a continuous boundary in the sense of Definition 1.2.1.1. ■

Summing up, the comparison between Definition 1.2.1.1 and Definition 1.2.1.2 is the following.

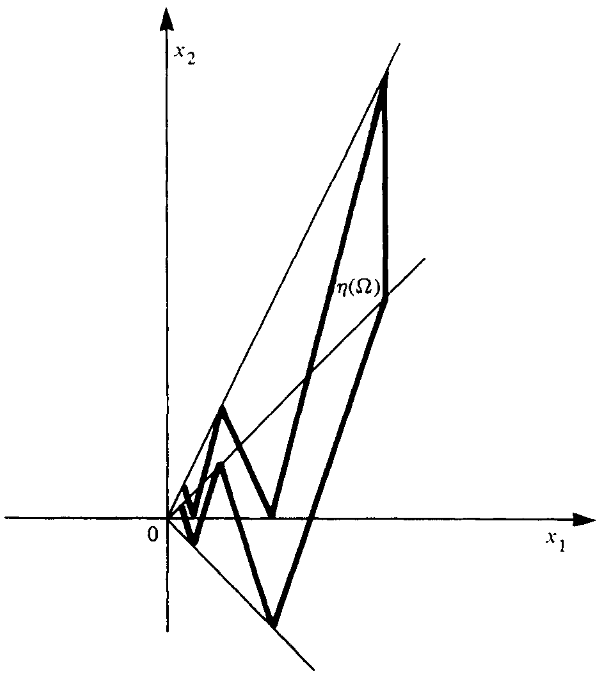

Figure 1.3

Theorem 1.2.1.5 *A bounded open subset Ω in $\mathbb{R}^n$ has a continuously differentiable (respectively of class $C^{k,1}$, m times continuously differentiable) boundary Γ if and only if $\bar{\Omega}$ is an n-dimensional continuously differentiable (respectively of class $C^{k,1}$, m times continuously differentiable) submanifold with boundary in $\mathbb{R}^n$. A bounded open subset Ω in $\mathbb{R}^n$ with continuous (respectively Lipschitz) boundary Γ has a closure $\bar{\Omega}$ which is an n-dimensional continuous (respectively Lipschitz) submanifold with boundary in $\mathbb{R}^n$. The converse statement is not true.*

In some special questions, for technical reasons, we shall need uniformly Lipschitz unbounded domains of the following kind.

Definition 1.2.1.6 *An open subset Ω of $\mathbb{R}^n$ is said to be a uniform Lipschitz epigraph if there exists new coordinates $\{y_1, \ldots, y_n\}$ and an uniformly Lipschitz continuous function φ of $n-1$ variables, such that*

$$\Omega = \{y = (y', y_n) \mid y_n > \varphi(y')\}. \tag{1,2,1,3}$$

Examples of such domains are infinite cones or plane sectors.

1.2.2 Segment property and cone property

In the early stages of the theory of Sobolev spaces, many authors preferred to describe the boundary properties of the possible domains Ω in a more straightforward fashion. Namely, they required that for each point x of the boundary Γ of Ω, there exists a linear segment C with origin at x or a cone C with vertex at x, such that $C\backslash\{x\}$ is contained in Ω. Usually a local uniformity assumption is added (cf. below). This point of view, adopted by Sobolev, has been followed by Agmon (1965) and Calderon (1961).

Definition 1.2.2.1 *Let Ω be an open subset of $\mathbb{R}^n$. We say that Ω has the uniform (or restricted) segment property (resp. cone property) if for every $x \in \Gamma$, there exists a neighbourhood V of x in $\mathbb{R}^n$ and new coordinates $\{y_1, \ldots, y_n\}$ such that*

(a) *V is a hypercube in the new coordinates:*

$$V = \{(y_1, \ldots, y_n) \mid -a_j < y_j < a_j,\ 1 \leq j \leq n\}$$

(b) *$y - z \in \Omega$ whenever $y \in \bar{\Omega} \cap V$ and $z \in C$, where C is the open segment $\{(0, \ldots, 0, z_n) \mid 0 < z_n < h\}$ (resp. C is the open cone $\{z = (z', z_n) \mid (\cot\theta)\,|z'| < z_n < h\}$ for some $\theta \in\,]0, \pi/2]$) for some $h > 0$.*

It is obvious that if Ω has a continuous boundary according to Defini-

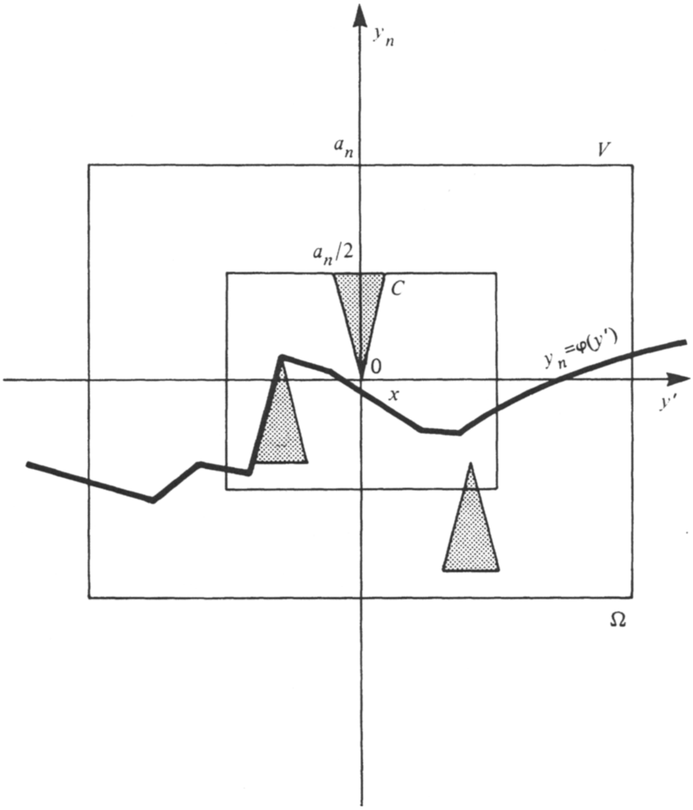

Figure 1.4

tion 1.2.1.1, then it has the uniform segment property (just choose $h < a_n/2$). The same way, if Ω has a Lipschitz boundary, then it has the uniform cone property. Indeed, this is seen by replacing all the a_j by $a_j/2$ in Definition 1.2.1.2 and by choosing $h < a_n/2$ together with

$$\theta \leq \inf\left(\arctan\frac{1}{K};\ \arctan\frac{a_1}{a_n};\ \cdots\ \arctan\frac{a_{n-1}}{a_n}\right),$$

where K is the Lipschitz constant of φ.

The converse statement has been known to be true for a long time by oral tradition. However, an actual proof has been published only recently by Chenais (1973). We shall give a transcript of the proof only for domains having the uniform cone property, because it is slightly simpler and it is the only one we need in the following sections.

Theorem 1.2.2.2 *A bounded open subset Ω of $\mathbb{R}^n$ has the uniform cone property if and only if its boundary Γ is Lipschitz.*

Proof We have already observed that the condition of having a Lipschitz boundary is sufficient. Thus, let us consider $x \in \Gamma$, assuming that Ω has the uniform cone property of Definition 1.2.2.1. We know that $\{x\}-C \subset \Omega$, but we can also observe that $\{x\}+C \subset C\Omega$, at least if the distance from x to CV is greater than $h/\cos\theta$; this last condition can always be achieved by choosing a smaller h. Indeed, if $\{x\}+C \cap \bar{\Omega}$ is not empty, let y be a point in the intersection; then $y \in \bar{\Omega} \cap V$ since $|y_n - x_n| < h$; consequently, $\{y\}-C \subset \Omega$, but this contradicts the fact that $\{y\}-C \ni x$.

From this remark it follows that if we translate the origin of the coordinates $\{y_1, \ldots, y_n\}$ at x and define a cylinder K by

$$K = \{(y', y_n) \mid -h < y_n < h, |y'| < h \tan\theta\},$$

then we have

$$\Gamma \cap K \subset \{(y', y_n) \mid |y_n| \tan\theta < |y'| < h \tan\theta\};$$

This means that Γ cannot 'escape' through the top of K.

We conclude by defining φ in the following way

$$\varphi(y') = \sup\{y_n \mid (y', y_n) \in \Gamma \cap K\};$$

φ is defined only for $|y'| < h \tan\theta$. Clearly, $(y', \varphi(y')) \in \Gamma$. Then the cone property shows that all points $(y', y_n) \in K$ with $y_n < \varphi(y')$ are in Ω; by contradiction, as we did previously for x, we show that all points $(y', y_n) \in K$ with $y_n > \varphi(y')$ are in $C\Omega$. Finally, if we consider two points $(y', \varphi(y'))$ and $(z', \varphi(z'))$ on the graph of φ, it follows from the cone property that

$$y_n - z_n > -|y' - z'| \cot\theta;$$

this implies that φ is a uniformly Lipshitz function with constant $K = \cot\theta$.

We conclude the proof by observing that all the conditions in Definition 1.2.1.1 are fulfilled when we choose the a_j small enough. ■

A useful consequence is the following:

Corollary 1.2.2.3 *Let Ω be a bounded open convex subset of $\mathbb{R}^n$, then Ω has a Lipschitz boundary.*

Proof Let x_0 be any point in Ω and let $r > 0$ be the radius of a ball B with centre x_0, contained in Ω. Since Ω is convex, all the points $ty + (1-t)z$ with $y \in \bar{\Omega}$, $z \in B$, $0 \leq t < 1$, are in Ω. This shows already that Ω has some kind of a cone property but we still need uniformity.

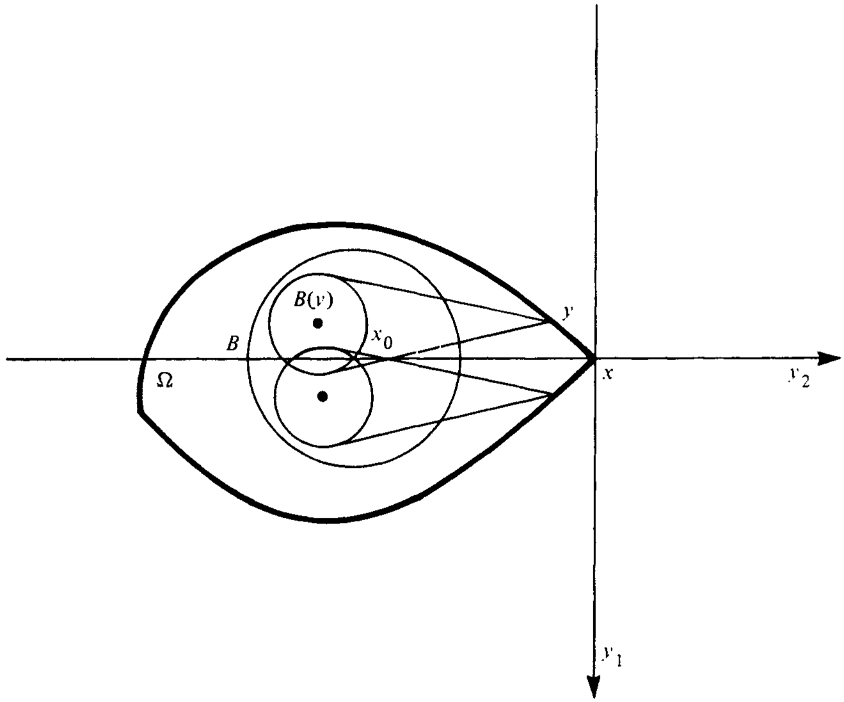

Figure 1.5

Now fix $x \in \Gamma$ and choose new coordinates $\{y_1, \ldots, y_n\}$ with, say, origin at x and such that x_0x is parallel to Oy_n. Denote by l the distance from x_0 to x. Then to each $y \in \bar{\Omega}$ at a distance less than $r/2$ from x, we associate a ball $B(y)$ centred at $(y', y_n - l)$ with radius $r/2$. Obviously $B(y) \subset B$, and therefore all the points $ty + (1-t)z$ with $z \in B(y)$, $0 \leqslant t < 1$ are in Ω. The property in Definition 1.2.2.1 is verified by choosing the a_j small enough, $h = l$ and $\sin\theta = r/2l$. ■

Remark 1.2.2.4 Unfortunately domains with cuts or with turning points are not well classified by the various previous definitions. Let us consider, for instance, the following domains in the plane:

$$\Omega_1 = \{(x_1, x_2) \mid -1 < x_1 < 1, -1 < x_2 < -|x_1|^{1/2}\}$$
$$\Omega_2 = \{(x_1, x_2) \mid 0 < x_1 < 2, -1 < x_2 < -(x_1/2)^{1/2}\}.$$

The domain Ω_1 is easily seen to have a continuous (and non-Lipschitz) boundary according to Definition 1.2.1.1. On the other hand, Ω_2 has not the segment property of Definition 1.2.2.1; consequently, it lacks a

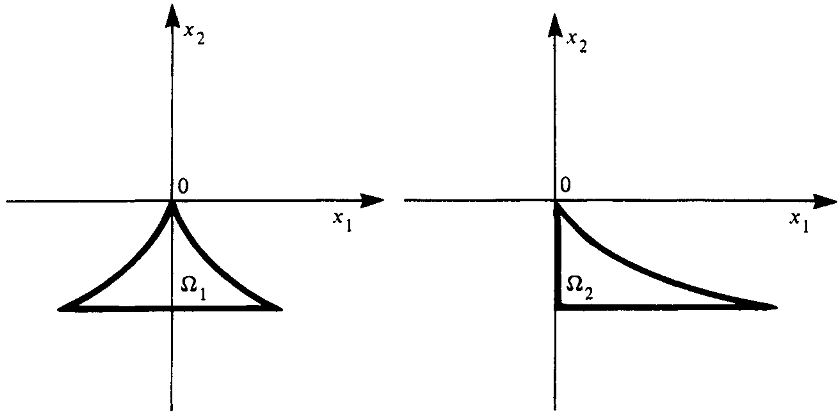

Figure 1.6

continuous boundary. However, Ω_2 is the image of Ω_1 through the mapping

$$\varphi : (x_1, x_2) \mapsto (x_1 + x_2^2, x_2)$$

which is a diffeomorphism of class C^∞ of $\mathbb{R}^2$ onto $\mathbb{R}^2$.

1.3 Spaces

This section is just a list of the definitions of the Sobolev spaces. We confine our attention strictly to those spaces which we really need in the following chapters. Consequently we will not consider any of those functional spaces such as Besov spaces and Nikolski spaces that are very closely related to Sobolev spaces and have better properties. The reader interested in those spaces is referred to Triebel (1978) for instance.

The Sobolev spaces are very easy to define on the whole Euclidean space. Then a possible definition of Sobolev spaces on a subdomain Ω of $\mathbb{R}^n$ with boundary uses restrictions to Ω. This is why we treat the spaces on $\mathbb{R}^n$ separately and first.

1.3.1 Euclidean space

In what follows, s is any real number and p is a real number such that $1<p<\infty$. We shall denote by m the integer part of s and by σ its fractional part; consequently, $s = m + \sigma$ and $0 \leqslant \sigma < 1$.

Definition 1.3.1.1 *We denote by $W_p^s(\mathbb{R}^n)$ the space of all distributions (all functions and distributions are complex valued unless otherwise specified) defined in $\mathbb{R}^n$, such that*

(a) *$D^\alpha u \in L_p(\mathbb{R}^n)$, for $|\alpha| \leqslant m$, when $s = m$ is a nonnegative integer,*

(b) $u \in W_p^m(\mathbb{R}^n)$ *and*

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+\sigma p}} \, dx \, dy < +\infty$$

for $|\alpha| = m$, *when* $s = m + \sigma$ *is nonnegative and is not an integer.*

As usual, $L_p(\mathbb{R}^n)$ is the space of all measurable functions u such that $|u|^p$ is integrable over $\mathbb{R}^n$. We define a Banach norm on $W_p^s(\mathbb{R}^n)$ by

$$\|u\|_{m,p,\mathbb{R}^n} = \left\{ \sum_{|\alpha| \leq m} \int_{\mathbb{R}^n} |D^\alpha u|^p \, dx \right\}^{1/p} \tag{1,3,1,1}$$

in case (a), and by

$$\|u\|_{s,p,\mathbb{R}^n} = \left\{ \|u\|^p_{m,p,\mathbb{R}^n} + \sum_{|\alpha|=m} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+\sigma p}} \, dx \, dy \right\}^{1/p} \tag{1,3,1,2}$$

in case (b).

The previous definition is extended to negative values of p by duality as follows:

Definition 1.3.1.2 *For* $s < 0$ *we denote by* $W_p^s(\mathbb{R}^n)$ *the dual space of* $W_q^{-s}(\mathbb{R}^n)$, *where* $p^{-1} + q^{-1} = 1$.

In the special case where $p = 2$, we shall use the more common notation $H^s(\mathbb{R}^n)$ instead of $W_2^s(\mathbb{R}^n)$. The norms defined in (1,3,1,1) and (1,3,1,2) are Hilbert norms when $p = 2$.

In some special questions where the use of Fourier transform cannot be avoided, it is useful to introduce a different kind of spaces as follows.

Definition 1.3.1.3 *We denote by* $H_p^s(\mathbb{R}^n)$ *the space of all distributions in* $\mathbb{R}^n$ *such that*

$$G_s * u \in L_p(\mathbb{R}^n),$$

where G_s *is the Bessel potential of order* s *defined by*

$$(FG_s)(\xi) = (1 + |\xi|^2)^{s/2}.$$

As usual, F is the Fourier transform operator defined by

$$(Fu)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi} u(x) \, dx$$

and the star $*$ denotes the convolution product.

It is known that $H_2^s(\mathbb{R}^n) = W_2^s(\mathbb{R}^n)$ (by Plancherel's theorem) for all real s, and that $H_p^m(\mathbb{R}^n) = W_p^m(\mathbb{R}^n)$ (by Mikhlin's theorem) for all integer m and $1 < p < \infty$. Furthermore, it is proved in Taibleson (1964) that

$$H_p^s(\mathbb{R}^n) \supseteq W_p^s(\mathbb{R}^n), \qquad 1 < p \leq 2,$$

while

$$W_p^s(\mathbb{R}^n) \supseteq H_p^s(\mathbb{R}^n), \qquad 2 \leq p < \infty,$$

for any real s.

It is also easily checked that $W_p^s(\mathbb{R}^n)$ and $H_p^s(\mathbb{R}^n)$ decrease when s increases and finally Lions and Peetre (1964) have proved that

$$W_p^{s'}(\mathbb{R}^n) \subset H_p^{s''}(\mathbb{R}^n) \subset W_p^{s'''}(\mathbb{R}^n)$$

for any real numbers s', s'', s''' such that

$$s' > s'' > s'''.$$

1.3.2 Open subsets of the Euclidean space

We now deal with Ω, an open subset of $\mathbb{R}^n$, possibly different from $\mathbb{R}^n$ itself. Our purpose is to extend the definitions given in Section 1.3.1, in order to define Sobolev spaces on Ω. In doing that, we can follow different schools. Here are the three main methods:

(a) We can reproduce Definition 1.3.1.1 by restricting the domain of integration (replacing $\mathbb{R}^n$ by Ω). This is the point of view in Lions and Magenes (1960–63) and Nečas (1967), for instance.
(b) We can define $W_p^s(\Omega)$ as being the set of restrictions to Ω of all functions in $W_p^s(\mathbb{R}^n)$. This is the point of view in Hörmander (1963).
(c) Finally, following Agmon (1965) and Miranda (1970), we can consider the completion of the space of smooth functions in $\bar{\Omega}$, with respect to the norm in (a).

It turns out that each of these three methods has its advantages. All three lead to the same spaces when Ω is smooth enough (we shall give a precise meaning to this sentence in the next sections). However, for general domains they may produce three different spaces, which we shall have to compare.

Definition 1.3.2.1 *We denote by $W_p^s(\Omega)$ the space of all distributions u defined in Ω, such that*

(a) *$D^\alpha u \in L_p(\Omega)$, for $|\alpha| \leq m$, when $s = m$ is a nonnegative integer,*

(b) $u \in W_p^m(\Omega)$ *and*

$$\iint_{\Omega\times\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+\sigma p}}\,dx\,dy < +\infty$$

for $|\alpha| = m$, *when* $s = m + \sigma$ *is nonnegative and is not an integer.*

We define a Banach norm on $W_p^s(\Omega)$ by

$$\|u\|_{m,p,\Omega} = \left\{ \sum_{|\alpha|\le m} \int_\Omega |D^\alpha u|^p\,dx \right\}^{1/p} \tag{1,3,2,1}$$

in the case (a), and by

$$\|u\|_{s,p,\Omega} = \left\{ \|u\|^p_{m,p,\Omega} + \sum_{|\alpha|=m} \iint_{\Omega\times\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+\sigma p}}\,dx\,dy \right\}^{1/p} \tag{1,3,2,2}$$

in the case (b).

We cannot directly reproduce Definition 1.3.1.2 since in general $\mathcal{D}(\Omega)$, the space of all C^∞ functions with compact support in Ω, is not dense in $W_p^s(\Omega)$. Consequently, the dual space of $W_p^s(\Omega)$ cannot be identified to a space of distributions in Ω. This is the reason for introducing another family of Sobolev spaces.

Definition 1.3.2.2 *For* $s > 0$, *we denote by* $\mathring{W}_p^s(\Omega)$ *the closure of* $\mathcal{D}(\Omega)$ *in* $W_p^s(\Omega)$.

Equivalently, it is the closure in $W_p^s(\Omega)$ of all distributions with compact support in Ω which belong to $W_p^s(\Omega)$.

Then the extension of Definition 1.3.1.2 is the following:

Definition 1.3.2.3 *For* $s < 0$, *we denote by* $W_p^s(\Omega)$ *the dual space of* $\mathring{W}_q^{-s}(\Omega)$, *where* $p^{-1} + q^{-1} = 1$.

In the special case when $p = 2$, we shall also use the common notation, namely $H^s(\Omega)$ instead of $W_2^s(\Omega)$ and $\mathring{H}^s(\Omega)$ instead of $\mathring{W}_2^s(\Omega)$. These are Hilbert spaces.

When s is a negative integer $-m$, $W_p^s(\Omega)$ is also the space of all distributions T in Ω such that

$$T = \sum_{|\alpha|\le m} D^\alpha f_\alpha \tag{1,3,2,3}$$

where $f_\alpha \in L_p(\Omega)$. The proof can be found in Magenes and Stampacchia (1958), for instance. An extension of (1,3,2,3) to non integer s is given in Lions (1961b).

For the sake of clarity in the following sections, it will be convenient to have a specific notation for the space defined by restriction.

Defininions 1.3.2.4 *For every $s>0$, we denote by $W_p^s(\bar{\Omega})$ the space of all distributions in Ω which are restrictions of elements of $W_p^s(\mathbb{R}^n)$.*

In other words, $W_p^s(\bar{\Omega})$ is the space of all $u|_\Omega$ where $u \in W_p^s(\mathbb{R}^n)$ and $u|_\Omega$ is defined by $\langle u|_\Omega; \varphi\rangle = \langle u, \tilde{\varphi}\rangle$ for all $\varphi \in \mathcal{D}(\Omega)$, where $\tilde{\varphi}$ is the continuation of φ by zero, outside Ω. We define a Banach norm on $W_p^s(\bar{\Omega})$ by

$$\|u\|_{s,p,\bar{\Omega}} = \inf_{\substack{U \in W_p^s(\mathbb{R}^n) \\ U|_\Omega = u}} \|U\|_{s,p,\mathbb{R}^n}. \tag{1,3,2,4}$$

As obvious consequences of the definition, we have the following inclusions:

$$W_p^s(\bar{\Omega}) \subseteq W_p^s(\Omega) \tag{1,3,2,5}$$

for every real $s>0$, and

$$\mathring{W}_p^m(\Omega) \subseteq W_p^m(\bar{\Omega}) \subseteq W_p^m(\Omega) \tag{1,3,2,6}$$

for every integer $m>0$.

Unfortunately we shall need one more kind of Sobolev space whose technical interest will appear much later.

Definition 1.3.2.5 *For every positive s, we denote by $\tilde{W}_p^s(\Omega)$, the space of all $u \in W_p^s(\Omega)$, where $\tilde{u} \in W_p^s(\mathbb{R}^n)$.*

$\tilde{W}_p^s(\Omega)$ is a Banach space for the norm

$$\|u\|_{\widetilde{s,p,\Omega}} = \|\tilde{u}\|_{s,p,\mathbb{R}^n} \tag{1,3,2,7}$$

The only obvious inclusions concerning $\tilde{W}_p^s(\Omega)$ are the following

$$\tilde{W}_p^s(\Omega) \subseteq W_p^s(\bar{\Omega}) \tag{1,3,2,8}$$

for all $s>0$ and

$$\mathring{W}_p^m(\Omega) \subseteq \tilde{W}_p^m(\Omega) \tag{1,3,2,9}$$

for m integer >0.

The norm defined in (1,3,2,7), although it is the natural one, is somewhat tricky, because it is the norm induced by $W_p^s(\Omega)$ only when s is an integer.

Lemma 1.3.2.6 *Let u belong to $\tilde{W}_p^s(\Omega)$; then*

$$\|u\|_{\widetilde{m,p,\Omega}} = \|u\|_{m,p,\Omega} \tag{1,3,2,10}$$

when $s = m$ is an integer, while

$$\|u\|^{\sim}_{s,p,\Omega} = \left\{ \|u\|^p_{s,p,\Omega} + \sum_{|\alpha|=m} \int_\Omega |D^\alpha u(x)|^p \rho_{\sigma,p}(x)\, dx \right\}^{1/p} \geq \|u\|_{s,p,\Omega}$$

when $s = m + \sigma$ is not an integer, where (1,3,2,11)

$$\rho_{\sigma,p}(x) = 2 \int_{C\Omega} \frac{dy}{|x-y|^{n+\sigma p}}.$$

It is not easy to describe the weight $\rho_{\sigma,p}$ in general. However, when Ω is bounded and has a Lipschitz boundary, there exist two constants C_1, C_2 with $0 < C_1 \leq C_2$, such that

$$C_1 d(x;\Gamma)^{-\sigma p} \leq \rho_{\sigma,p}(x) \leq C_2 d(x;\Gamma)^{-\sigma p} \tag{1,3,2,12}$$

where $d(x,\Gamma)$ denotes the distance from x to the boundary Γ of Ω. The same inequalities hold when Ω is a uniform Lipschitz epigraph (Definition 1.2.1.6).

1.3.3 Manifolds

In the following sections, we shall need Sobolev spaces on manifolds which are only (possibly part of) boundaries of open subsets of $\mathbb{R}^n$ fulfilling the assumptions in Definition 1.2.1.1. In other words, they will be $(n-1)$-dimensional hypersurfaces in $\mathbb{R}^n$. Consequently, keeping the same notations as in Definition 1.2.1.1, the boundary Γ of Ω is such that for every $x \in \Gamma$, there exists a neighbourhood V of x in $\mathbb{R}^n$, fulfilling the condition (a) in Definition 1.2.1.1 and a function fulfilling the condition (b) such that

$$\Gamma \cap V = \{y = (y', y_n) \in V \mid y_n = \varphi(y')\}.$$

Let us define Φ as follows:

$$\Phi(y) = \{y_1, \ldots, y_{n-1}, \varphi(y_1, \ldots, y_{n-1})\}, \tag{1,3,3,1}$$

then $(\Gamma \cap V, \Phi)$ is a map of Γ around x, where we now view Γ as a $(n-1)$-dimensional Lipschitz (respectively continuously differentiable, of class $C^{k,1}$, m times continuously differentiable) submanifold of $\mathbb{R}^n$.

The following statement expresses in a precise way the stability of Sobolev spaces under sufficiently smooth changes of variables. We assume that ψ is at least a bi-Lipschitz mapping from $\bar{\Omega}_1$ onto $\bar{\Omega}_2$ where Ω_1 and Ω_2 are bounded open subsets of $\mathbb{R}^n$. This hypothesis ensures that Lebesgue measure is mapped by ψ or ψ^{-1} to an equivalent image measure.

Lemma 1.3.3.1 *Let $u \in W^s_p(\Omega_2)$; assume that ψ and ψ^{-1} are of class $C^{k,1}$ where k is an integer $\geq s-1$; then $u \circ \psi \in W^s_p(\Omega_1)$.*

This property is easy to check with the help of the results in Rademacher (1919). It is a justification for the following definition.

Definition 1.3.3.2 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a boundary Γ of class $C^{k,1}$, where k is a nonnegative integer. Let Γ_0 be an open subset of Γ. A distribution u on Γ_0 belongs to $W_p^s(\Gamma_0)$ with $|s| \leq k+1$ if $u \circ \Phi \in W_p^s(V' \cap \Phi^{-1}(\Gamma_0 \cap V))$ for all possible V and φ fulfilling the assumptions in Definition 1.2.1.1.*

Usually distributions are defined only on C^∞ manifolds. When Γ_0 is only an open subset of a $C^{k,1}$ manifold we consider distributions whose order is less than or equal to $k+1$; those span the dual space of the space of all $C^{k,1}$ functions with compact support in Γ_0. Functions are identified with distributions by means of the usual injection $u \mapsto T_u$, defined by

$$\langle T_u ; v \rangle = \int_{\Gamma_0} uv \, \mathrm{d}\sigma$$

where $\mathrm{d}\sigma$ is the usual hypersurface measure on Γ (defined provided Γ is a Lipschitz hypersurface).

One possible Banach norm on $W_p^s(\Gamma)$ is

$$u \mapsto \sum_{j=1}^{J} \|u \circ \Phi_j\|^p_{s,p,V_j' \cap \Phi_j^{-1}(\Gamma_0 \cap V_j)} \tag{1,3,3,2}$$

where $(V_j, \Phi_j)_{j=1}^J$ is any atlas of Γ such that each couple (V_j, φ_j) fulfils the assumptions of Definition 1.3.3.1 (we recall that Φ_j is defined from φ_j by (1,3,3,1)).

In the particular case when $s \in]0, 1[$, one can easily check that any of the norms defined in (1,3,3,2) is equivalent to

$$u \mapsto \left\{ \int_{\Gamma_0} |u|^p \, \mathrm{d}\sigma + \iint_{\Gamma_0 \times \Gamma_0} \frac{|u(x) - u(y)|^p}{|x - y|^{n-1+sp}} \, \mathrm{d}\sigma(x) \, \mathrm{d}\sigma(y) \right\}^{1/p}. \tag{1,3,3,3}$$

1.4 Basic properties

This section is only a list of the main properties of the spaces defined above. We do not report any proof but just indicate easy references where all the details can be found.

1.4.1 Multiplication and differentiation

The question here is to know sufficient conditions on a function φ defined in Ω, ensuring that $u \to \varphi u$ is a continuous linear mapping in a given

$W_p^s(\Omega)$. We state here a very simple answer, which is just a straightforward consequence of the definitions given in Section 1.3. A more complete answer will be given in Section 1.4.4 as a consequence of the imbedding theorems (see Theorem 1.4.4.2).

We denote by $C_c^{k,\alpha}(\bar{\Omega})$ (k a nonnegative integer and $\alpha \in [0, 1]$) the space of all functions defined in $\bar{\Omega}$ which are restrictions to $\bar{\Omega}$ of functions of class $C^{k,\alpha}$ defined on the whole of $\mathbb{R}^n$ which have a compact support.

Theorem 1.4.1.1 *Let $\varphi \in C_c^{k,\alpha}(\bar{\Omega})$ with $k+\alpha \geq |s|$ when s is an integer and $k+\alpha > |s|$ when s is not an integer, then $\varphi u \in W_p^s(\Omega)$ for every $u \in W_p^s(\Omega)$, and there exists a constant $K = K(\varphi, s, p)$ such that*

$$\|\varphi u\|_{s,p,\Omega} \leq K \|u\|_{s,p,\Omega}. \tag{1,4,1,1}$$

An easy consequence is that under the same hypothesis on φ, $u \to \varphi u$ is a continuous linear mapping in $W_p^s(\bar{\Omega})$ and in $\tilde{W}_p^s(\Omega)$. The following result is also easy to check.

Theorem 1.4.1.2 *Let $\varphi \in C_c^{k,\alpha}(\bar{\Omega})$ with $k+\alpha \geq |s|$ when s is an integer and $k+\alpha > |s|$ when s is not an integer, then $\varphi u \in \mathring{W}_p^s(\Omega)$ for every $u \in \mathring{W}_p^s(\Omega)$.*

For a nonnegative integer m, the space $W_p^m(\Omega)$ is just the space of all functions defined in Ω, which are m times differentiable in $L_p(\Omega)$, so to say. The definition of $W_p^s(\Omega)$ for a noninteger s has been stated with the underlying idea that $W_p^s(\Omega)$ should be the space of all functions in Ω which are s times differentiable in some sense. Consequently, one could expect D^α to be a continuous linear operator from $W_p^s(\Omega)$ into $W_p^{s-|\alpha|}(\Omega)$. The extension of the definition of $W_p^s(\Omega)$ to negative values of s was devised with the hope that this rule should hold for every s. Unfortunately, this is not always true, as we shall begin to show now.

Firstly, D^α maps $W_p^s(\Omega)$ into $W_p^{s-|\alpha|}(\Omega)$ provided either $|\alpha| \leq s$ or $s \leq 0$. This follows from Definition 1.3.2.1 when $|\alpha| \leq s$. Then, from Definition 1.3.2.2, we see that D^α is also a continuous linear operator from $\mathring{W}_p^s(\Omega)$ into $\mathring{W}_p^{s-|\alpha|}(\Omega)$ when $|\alpha| \leq s$. Transposing this result and remembering Definition 1.3.2.3, we conclude that D^α maps $W_p^s(\Omega)$ into $W_p^{s-|\alpha|}(\Omega)$ when $s \leq 0$.

Now it remains to understand how differentiation operates from spaces with positive order to spaces with negative order. For this purpose it is enough to consider an elementary differentiation operator D_j with respect to x_j, with $1 \leq j \leq n$.

Lemma 1.4.1.3 *D_j is a continuous linear operator from $W_p^s(\mathbb{R}^n)$ into $W_p^{s-1}(\mathbb{R}^n)$.*

The only case we have to consider is when $0<s<1$. When $p=2$ and consequently $W^s_p(\mathbb{R}^n)=H^s(\mathbb{R}^n)=H^s_2(\mathbb{R}^n)$, the property is obvious from Definition 1.3.1.3. Unfortunately, we need another method of proof when p is not 2. We describe it now. Here R belongs to $\mathcal{D}(\mathbb{R}^n)$ and has its support in the unit ball and integral equal to one.

Lemma 1.4.1.4 *Let $u\in W^s_p(\mathbb{R}^n)$ and set*

$$U(x, x_{n+1})=\frac{1}{x^n_{n+1}}\int_{\mathbb{R}^n} R\left(\frac{y-x}{x_{n+1}}\right)u(y)\,dy, \qquad x\in\mathbb{R}^n, \quad x_{n+1}>0; \tag{1,4,1,2}$$

then $x_{n+1}^{1-s-1/p}D_kU\in L_p(\mathbb{R}^{n+1})$, $k=1,2,\ldots,n+1$.

Proof We follow the method used in Lemma 5.6, Chapter 2 in Nečas (1967), just adding a weight. We first consider D_kU where $1\leq k\leq n$. We have

$$\begin{aligned}D_kU(x, x_{n+1})&=-\frac{1}{x^{n+1}_{n+1}}\int_{\mathbb{R}^n} D_kR\left(\frac{y-x}{x_{n+1}}\right)u(y)\,dy\\ &=-\frac{1}{x^{n+1}_{n+1}}\int_{\mathbb{R}^n} D_kR\left(\frac{y-x}{x_{n+1}}\right)[u(y)-u(x)]\,dy\end{aligned}$$

since obviously

$$\int_{\mathbb{R}^n} D_kR\left(\frac{y-x}{x_{n+1}}\right)dy=0.$$

It follows that

$$D_kU(x, x_{n+1})=\int_{\mathbb{R}^n} D_kR(z)[u(x)-u(x+x_{n+1}z)]\frac{dz}{x_{n+1}}$$

and consequently

$$\begin{aligned}&\int_0^\infty\left(\int_{\mathbb{R}^n} x_{n+1}^{p-sp-1}\,|D_kU(x, x_{n+1})|^p\,dx\right)dx_{n+1}\\ &\leq c\int_0^\infty\left(\int_{\mathbb{R}^n} x_{n+1}^{p-sp-1}\left\{\int_{|z|\leq 1} |u(x)-u(x+x_{n+1}z)|^p\frac{dz}{x^p_{n+1}}\right\}dx\right)dx_{n+1}\\ &=c\int_0^\infty\left(\int_{\mathbb{R}^n} x_{n+1}^{-1-sp}\left\{\int_{|x-y|\leq x_{n+1}} |u(x)-u(y)|^p\frac{dy}{x^n_{n+1}}\right\}dx\right)dx_{n+1}\\ &=c\iint_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(y)|^p\left\{\int_{|x-y|}^\infty x_{n+1}^{-1-sp-n}\,dx_{n+1}\right\}dx\,dy\\ &=\frac{c}{sp+n}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\end{aligned}$$

where c is a constant depending on R.

Now let us consider $D_{n+1}U$; we have

$$D_{n+1}U(x, x_{n+1}) = -\frac{n}{x_{n+1}^{n+1}}\int_{\mathbb{R}^n} R\left(\frac{y-x}{x_{n+1}}\right)u(y)\,dy - \frac{1}{x_{n+1}^{n+1}} \times \int_{\mathbb{R}^n} \sum_{j=1}^{n} D_j R\left(\frac{y-x}{x_{n+1}}\right)\frac{y_j - x_j}{x_{n+1}}\,u(y)\,dy$$

$$= -\frac{n}{x_{n+1}^{n+1}}\int_{\mathbb{R}^n} R\left(\frac{y-x}{x_{n+1}}\right)[u(y)-u(x)]\,dy - \frac{1}{x_{n+1}^{n+1}}\int_{\mathbb{R}^n} \sum_{j=1}^{n} D_j R\left(\frac{y-x}{x_{n+1}}\right)\frac{y_j - x_j}{x_{n+1}}[u(y)-u(x)]\,dy$$

since

$$\frac{n}{x_{n+1}^{n+1}}\int_{\mathbb{R}^n} R\left(\frac{y-x}{x_{n+1}}\right)dy + \frac{1}{x_{n+1}^{n+1}}\int_{\mathbb{R}^n} \sum_{j=1}^{n} D_j R\left(\frac{y-x}{x_{n+1}}\right)\frac{y_j - x_j}{x_{n+1}}\,dy = 0$$

by integration by parts. Then each integral in $D_{n+1}U$ is estimated as we did for $D_k U$. ■

Proof of Lemma 1.4.1.3 We consider the bilinear form

$$u, v \mapsto \int_{\mathbb{R}^n} D_j uv\,dx$$

and we prove that it is defined and continuous on $W_p^s(\mathbb{R}^n) \times W_q^{1-s}(\mathbb{R}^n)$, where $(1/p) + (1/q) = 1$. From u and v we construct U and V according to identity (1,4,1,2). We know from Lemma 1.4.1.4 that

$$\begin{cases} x_{n+1}^{1-s-1/p} D_k U \in L_p(\mathbb{R}^{n+1}), & 1 \leq k \leq n+1 \\ x_{n+1}^{s-1/q} D_k V \in L_q(\mathbb{R}^n), & 1 \leq k \leq n+1. \end{cases}$$

Furthermore, in the topology of $L_p(\mathbb{R}^n)$ we have

$$\lim_{x_{n+1}\to 0} U = u, \qquad \lim_{x_{n+1}\to\infty} U = 0$$

while in the topology of $L_q(\mathbb{R}^n)$ we have

$$\lim_{x_{n+1}\to 0} V = v, \qquad \lim_{x_{n+1}\to\infty} V = 0.$$

This implies that

$$\int_{\mathbb{R}^n} D_j U(x, x_{n+1}) V(x, x_{n+1})\,dx$$

$$= -\int_{x_{n+1}}^{+\infty}\int_{\mathbb{R}^n} [D_j U(x,t) D_{n+1} V(x,t) - D_{n+1} U(x,t) D_j V(x,t)]\,dx\,dt$$

and consequently

$$\left|\int_{\mathbb{R}^n} D_j U(x, x_{n+1}) V(x, x_{n+1})\, \mathrm{d}x\right|$$

$$\leqslant \sum_{k,l=1}^{n+1} \|x_{n+1}^{1-s-1/p} D_k U\|_{L_p(\mathbb{R}^{n+1})} \|x_{n+1}^{s-1/q} D_l V\|_{L_q(\mathbb{R}^{n+1})}$$

$$\leqslant K \|u\|_{W_p^s(\mathbb{R}^n)} \|v\|_{W_q^{1-s}(\mathbb{R}^n)},$$

where K is some constant produced by Lemma 1.4.1.4. Taking the limit when $x_{n+1} \to 0$, one gets

$$|\langle D_j u; v\rangle| \leqslant K \|u\|_{W_p^s(\mathbb{R}^n)} \|v\|_{W_q^{1-s}(\mathbb{R}^n)}.$$

This proves Lemma 1.4.1.3. ■

As a consequence of Lemma 1.4.1.3, it is clear that for $u \in W_p^s(\bar{\Omega})$, $D_j u$ is the restriction to Ω of a distribution belonging to $W_p^{s-1}(\mathbb{R}^n)$. Consequently, a complete answer to the question of whether or not D_j maps $W_p^s(\Omega)$ into $W_p^{s-1}(\Omega)$ will follow the study of continuation and restriction properties.

1.4.2 Density results

Here we quote only one basic result proved in Agmon (1959) and Nečas (1967) for instance. We denote by $C_c^\infty(\bar{\Omega})$ the space of all functions defined in Ω which are restrictions to Ω of C^∞ functions with compact support in $\mathbb{R}^n$.

Theorem 1.4.2.1 *Let Ω be an open subset of $\mathbb{R}^n$ with a continuous boundary; then $C_c^\infty(\bar{\Omega})$ is dense in $W_p^s(\Omega)$ for all $s > 0$.*

It follows easily from Definition 1.3.2.3 that $C^\infty(\bar{\Omega})$ is dense in $W_p^s(\Omega)$ for all $s < 0$, without any hypothesis on Ω.

Moreover, $\mathcal{D}(\mathbb{R}^n)$ is dense in $W_p^s(\mathbb{R}^n)$ for all s and consequently $C_c^\infty(\bar{\Omega})$ is dense in $W_p^s(\bar{\Omega})$ without any assumption on Ω.

Another result, closely related to Theorem 1.4.2.1, is the following:

Theorem 1.4.2.2 *Let Ω be an open subset of $\mathbb{R}^n$ with a continuous boundary, then $\mathcal{D}(\Omega)$ is dense in $\tilde{W}_p^s(\Omega)$ for all $s > 0$.*

Together with the identity (1,3,2,10), this shows that when $s = m$ is an integer and Ω is a domain with a continuous boundary, then

$$\tilde{W}_p^m(\Omega) = \mathring{W}_p^m(\Omega). \qquad (1,4,2,1)$$

An easy and useful consequence of Theorem 1.4.2.2 is the following:

Proposition 1.4.2.3 *Let Ω be an open subset of $\mathbb{R}^n$ with a continuous boundary and let T belong to $W_p^s(\mathbb{R}^n)$ with $s<0$. Then the restriction of T to Ω belongs to the dual space of $\tilde{W}_q^{-s}(\Omega)$.*

Finally we state an improvement of Theorem 1.4.2.1 in the particular case when $s \leqslant 1/p$.

Theorem 1.4.2.4 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary; then $\mathcal{D}(\Omega)$ is dense in $W_p^s(\Omega)$ for $0<s\leqslant 1/p$.*

The same is true when Ω is uniform Lipschitz epigraph (Definition 1.2.1.6).

In view of Theorem 1.4.2.1 we only have to approximate functions in $C^\infty(\bar{\Omega})$ by functions in $\mathcal{D}(\Omega)$ for the norm of $W_p^s(\Omega)$. This is easily achieved by means of a sequence of cut-off functions.

A direct consequence is that under the assumptions of Theorem 1.4.2.4, $\mathring{W}_p^s(\Omega)$ is the same space as $W_p^s(\Omega)$, when $0<s\leqslant 1/p$.

1.4.3 Continuation, compactness and convexity inequalities

Now we clarify partly the relation between $W_p^s(\Omega)$ and $W_p^s(\bar{\Omega})$. The following result is proved in Agmon (1965), Aronszajn and Smith (1961), Lions (1957), Nečas (1967), Stein (1970).

Theorem 1.4.3.1 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary; then for every $s>0$ there exists a continuous linear operator P_s from $W_p^s(\Omega)$ into $W_p^s(\mathbb{R}^n)$ such that*

$$P_s u\,|_\Omega = u. \tag{1,4,3,1}$$

The same results hold when Ω is an uniform Lipschitz epigraph or an infinite strip.

In other words each function $u \in W_p^s(\Omega)$ is the restriction of a function $P_s u \in W_p^s(\mathbb{R}^n)$. A counterexample in Lions (1957) shows that the property may not hold when Ω has not a Lipschitz boundary. Consequently we have $W_p^s(\Omega) = W_p^s(\bar{\Omega})$ when Ω is bounded and has a Lipschitz boundary.

In addition it has been shown in Seeley (1964) and Aronszajn and Smith (1961) that P_s can be chosen independently of s.

The continuation theorems are powerful tools for extending several results proved on $\mathbb{R}^n$ to similar results on a bounded domain with a Lipschitz boundary. We list some of them now.

Theorem 1.4.3.2 *Let $s' > s'' \geq 0$ and assume that Ω is a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary. Then the injection of $W_p^{s'}(\Omega)$ in $W_p^{s''}(\Omega)$ is compact.*

(For the sake of convenience here we define $W_p^0(\Omega)$ as being $L_p(\Omega)$.)

This result originally due to Kondrašov (1945) is proved in Nečas (1961) for the case in which s' and s'' are integers. The extension to possibly non-integer values of s' and s'' may be found in Lions and Peetre (1964).

The following inequality is closely related to the previous theorem, through a lemma of Lions (cf. Lemma 2,7, Chapter 1 in Magenes and Stampacchia (1958)).

Theorem 1.4.3.3 *Let $s' > s'' > s''' \geq 0$ and assume that Ω is a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary. Then there exists a constant C (depending on Ω, s', s'', s''' and p) such that*

$$\|u\|_{s'',p,\Omega} \leq \varepsilon \|u\|_{s',p,\Omega} + \cancel{K}\varepsilon^{-(s''-s''')/(s'-s'')} \|u\|_{s''',p,\Omega} \tag{1,4,3,2}$$

for all $u \in W_p^{s'}(\Omega)$. C

Such an inequality is also true when $\Omega = \mathbb{R}^n$ or Ω is any subset of $\mathbb{R}^n$ with the continuation property of Theorem 1.4.3.1.

This is an interpolation inequality which follows from the similar inequality on $\mathbb{R}^n$. See Lions and Magenes (1960–63) for a proof. In the case when s', s'' and s''' are integers, this is a particular case of more general inequalities by Gagliardo (1958) and Nirenberg (1959).

Let us also recall here a related inequality often referred to as Poincare's inequality (cf. Nečas (1967), for instance).

Theorem 1.4.3.4 *Assume that Ω is a bounded open subset of $\mathbb{R}^n$. Then there exists a constant $K(\Omega)$ which depends only on the diameter of Ω such that*

$$\|u\|_{0,p,\Omega} \leq K(\Omega) \left\{ \sum_{i=1}^{n} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^p dx \right\}^{1/p} \tag{1,4,3,3}$$

for all $u \in \mathring{W}_p^1(\Omega)$.

Closely related to the inequality in Theorem 1.4.3.3 is the interpolation theorem (cf. Lions and Peetre (1964)).

Theorem 1.4.3.5 *Let Π be a continuous linear operator in $W_p^s(\mathbb{R}^n)$, $1 < p < +\infty$, $s \in \mathbb{R}$. Assume that for some $t > s$ the restriction of Π to $W_p^t(\mathbb{R}^n)$*

is continuous in $W_p^t(\mathbb{R}^n)$. *Then for every* $u \in]s, t[$, *the restriction of* Π *to* $W_p^u(\mathbb{R}^n)$ *is continuous in* $W_p^u(\mathbb{R}^n)$.

Due to the continuation property a similar statement holds concerning the Sobolev spaces on Ω a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary.

1.4.4 Imbeddings

The most outstanding result about Sobolev spaces is the famous imbedding theorem, derived first by Sobolev himself. The main statement is this:

Theorem 1.4.4.1 *The following inclusions hold*:

$$W_p^s(\mathbb{R}^n) \subseteq W_q^t(\mathbb{R}^n) \qquad (1,4,4,1)$$

for $t \leq s$ *and* $q \geq p$ *such that* $s - n/p = t - n/q$† *and*

$$W_p^s(\mathbb{R}^n) \subset C^{k,\alpha}(\mathbb{R}^n) \qquad (1,4,4,2)$$

for $k < s - n/p < k+1$, *where* $\alpha = s - k - n/p$, k *a nonnegative integer.*

It is possible to state a weaker result in the limit cases when $s - n/p$ is an integer, as follows. We have

$$W_p^{n/p}(\mathbb{R}^n) \subset L_q(\mathbb{R}^n) \qquad (1,4,4,3)$$

for all $q \geq p$, and

$$W_p^{k+n/p}(\mathbb{R}^n) \subset C^{k-1,\alpha}(\mathbb{R}^n) \qquad (1,4,4,4)$$

for all $\alpha \in [0, 1[$, where k is an integer ≥ 1.

The proof may be found in any of the references quoted before about Sobolev spaces.

As a consequence we have the following inclusions

$$W_p^s(\bar{\Omega}) \subset W_q^t(\bar{\Omega}) \qquad (1,4,4,5)$$

for $t \leq s$, $q \geq p$ such that $s - n/p = t - n/q$ and

$$W_p^s(\bar{\Omega}) \subset C^{k,\alpha}(\bar{\Omega}) \qquad (1,4,4,6)$$

for $k < s - n/p < k+1$, $\alpha = s - k - n/p$, k a non-negative integer. These inclusions hold without any assumption on Ω. As a consequence of Theorem 1.4.3.1, similar inclusions hold for $W_p^s(\Omega)$, when Ω is a bounded open subset of $\mathbb{R}^n$, with a Lipschitz boundary.

† Negative values of t are admitted and $W_q^0(\mathbb{R}^n)$ means $L_q(\mathbb{R}^n)$.

The main interest of these results, in the subsequent sections, is the following. Assume we are able to build a solution to some given boundary value problem, which belongs to $W_p^s(\Omega)$, with s large enough; then we know that it is differentiable in the usual sense up to an order (strictly less) than $s-n/p$.

A by-product of Theorem 1.4.4.1 is that $W_p^s(\mathbb{R}^n)$ is an algebra for $s>n/p$. The more general result which follows has been proved by Zolesio (1977).

Theorem 1.4.4.2 *Let $s_1 \geq s$ and $s_2 \geq s$ be such that either*

$$s_1+s_2-s \geq n\left(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p}\right) \geq 0, \quad s_j-s>n\left(\frac{1}{p_j}-\frac{1}{p}\right), \quad j=1,2$$

or

$$s_1+s_2-s > n\left(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p}\right) \geq 0, \quad s_j-s \geq n\left(\frac{1}{p_j}-\frac{1}{p}\right), \quad j=1,2,$$

then $u, v \mapsto u \cdot v$ is a continuous bilinear map from $W_{p_1}^{s_1}(\mathbb{R}^n) \times W_{p_2}^{s_2}(\mathbb{R}^n)$ into $W_p^s(\mathbb{R}^n)$.

A similar statement holds for Sobolev spaces defined on a bounded open subset of $\mathbb{R}^n$, with a Lipschitz boundary. It is a complement to Theorem 1.4.1.1.

Imbedding results of a different sort deal with weighted spaces. They are consequences of the well-known Hardy inequality (more precisely Theorem 330 in *Hardy et al.* (1952)). Let us recall a convenient statement of the Hardy inequality. Here we denote by $L_{p,\alpha}(\mathbb{R}_+)$ the space of all measurable functions u defined in $\mathbb{R}_+$ such that

$$\|u\|_{L_{p,\alpha}}^p = \int_0^\infty |u(t)t^\alpha|^p \, dt < +\infty.$$

Then, we define two linear operators H and L by

$$(Hu)(t) = \frac{1}{t}\int_0^t u(s)\, ds$$

$$(Lu)(t) = \frac{1}{t}\int_t^\infty u(s)\, ds.$$

It turns out that H is linear continuous in $L_{p,\alpha}(\mathbb{R}_+)$ iff $\alpha+1/p<1$, while L is linear continuous in $L_{p,\alpha}(\mathbb{R}_+)$ iff $\alpha+1/p>1$. In both cases the norm of the operator is bounded by $|\alpha+1/p-1|^{-1}$.

Theorem 1.4.4.3 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz* *boundary Γ and denote by $\rho(x)$ the distance from a point x to Γ. Then when $0<s<1/p$, we have $u/\rho^s \in L_p(\Omega)$ for all $u \in W_p^s(\Omega)$ and when $1/p<s\leqslant 1$, we have $u/\rho^s \in L_p(\Omega)$ for all $u \in \mathring{W}_p^s(\Omega)$.*

The same result holds (with the same proof) when Ω is a uniform Lipschitz epigraph (Definition 1.2.1.6).

This result is proved in Grisvard (1963) for spaces defined on a half space $\mathbb{R}_+^n$. The result is extended to the case of a Lipschitz domain by bi-Lipschitz changes of coordinates (use Theorem 1.4.1.1 and Lemma 1.3.3.1).

Iteration of Theorem 1.4.4.3 provides a more complete result concerning the spaces $\mathring{W}_p^s(\Omega)$. Since this is not a result easy to find in the current literature on Sobolev spaces, we give the statement together with a detailed proof.

Theorem 1.4.4.4 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary Γ, then for all $u \in \mathring{W}_p^s(\Omega)$ such that $s-1/p$ is not an integer, the following property holds:*

$$\rho^{-s+|\alpha|}D^\alpha u \in L_p(\Omega) \tag{1,4,4,7}$$

for all $|\alpha|\leqslant s$.

Proof We observe first that by replacing s by $s-|\alpha|$ it is enough to prove the result when $|\alpha|=0$.

Let us consider first the case when $\Omega=\mathbb{R}_+$ the nonnegative real axis and $s=m$ is an integer. Then, for $u \in \mathcal{D}(\mathbb{R}_+)$ we have

$$u(x)=\int_0^x \frac{(x-y)^{m-1}}{(m-1)!}\,u^{(m)}(y)\,dy$$

and consequently

$$\frac{|u(x)|}{x^m}\leqslant\frac{1}{(m-1)!}\,\frac{1}{x}\int_0^x |u^{(m)}(y)|\,dy, \tag{1,4,4,8}$$

Hardy's inequality (mentioned above) implies that

$$\|x^{-m}u\|_{L_p(\mathbb{R}_+)}\leqslant\frac{p}{(m-1)!\,(p-1)}\,\|u^{(m)}\|_{L_p(\mathbb{R}_+)}.$$

By density, this implies the desired result for $\mathring{W}_p^m(\mathbb{R}_+)$.

Let us consider then the case when Ω is still $\mathbb{R}_+$ but $s=m+\sigma$ is no longer an integer. We consider now $v=u^{(m)}$, which belongs to $\mathring{W}_p^\sigma(\mathbb{R}_+)$. We make use of the following strange identity:

$$v(x)=-w(x)+\int_x^{+\infty}\frac{w(y)}{y}\,dy \tag{1,4,4,9}$$

where

$$w(x)=\frac{1}{x}\int_0^x [v(t)-v(x)]\,dt. \tag{1,4,4,10}$$

We first show that $x^{-\sigma}w \in L_p(\mathbb{R}_+)$. Indeed, we have

$$\begin{aligned}\int_0^\infty x^{-\sigma p}\Big(\frac{1}{x}\int_0^x [v(t)-v(x)]\,dt\Big)^p dx &\le \int_0^\infty x^{-\sigma p-1}\int_0^x |v(t)-v(x)|^p\,dt\,dx\\ &\le \int_0^\infty\int_0^\infty \frac{|v(t)-v(x)|^p}{|x-t|^{1+\sigma p}}\,dt\,dx \le \|v\|^p_{\sigma,p,\mathbb{R}_+}.\end{aligned}$$

Then Hardy's inequality shows that, when $\sigma<1/p$,

$$x^{-\sigma}\int_x^\infty \frac{w(y)}{y}\,dy \in L_p(\mathbb{R}_+)$$

and consequently $x^{-\sigma}v \in L_p(\mathbb{R}_+)$. Unfortunately, when $\sigma>1/p$, using formula (1,4,4,9) is inconclusive; we therefore use

$$v(x)=-w(x)-\int_0^x \frac{w(y)}{y}\,dy \tag{1,4,4,11}$$

with the same w. Now, Hardy's inequality shows that

$$x^{-\sigma}\int_0^x \frac{w(y)}{y}\,dy \in L_p(\mathbb{R}_+),$$

and consequently, again, $x^{-\sigma}v \in L_p(\mathbb{R}_+)$. Now inequality (1,4,4,8) and one more application of Hardy's inequality implies that

$$x^{-m-\sigma}u \in L_p(\mathbb{R}_+).$$

This is the desired result in $\mathring{W}^s_p(\mathbb{R}_+)$ provided $s-1/p$ is not an integer.

We conclude by extending this result to a general Ω. Let us use the same notation as in Definition 1.2.1.1 and consider a function u whose support is contained in V. One can always reduce the general case to this particular case, using a partition of unity. Now for $y'\in V'$ let us set

$$u_{y'}(t)=u(y',\varphi(y')-t).$$

For almost all $y'\in V'$, we have $u_{y'}\in \mathring{W}^s_p(\mathbb{R}_+)$ and consequently $t^{-s}u_{y'}\in L_p(\mathbb{R}_+)$ with

$$\|t^{-s}u_{y'}\|^p_{L_p(\mathbb{R}_+)} \le K^p\,\|u_{y'}\|^p_{s,p,\mathbb{R}_+}$$

where K does not depend on y'. Integrating this inequality in y' leads to

$$\|[\varphi(y')-y_n]^{-s}u\|_{L_p(\Omega)} \leq K\,\|u\|_{s,p,\Omega}.$$

Since φ is a Lipschitz function, the weight $\varphi(y')-y_n$ is equivalent to $\rho(y)$, the distance from y to Γ, throughout V. This completes the proof of Theorem 1.4.4.4. ■

Corollary 1.4.4.5 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary; then when $s-1/p$ is not an integer we have*

$$\tilde{W}^s_p(\Omega)=\mathring{W}^s_p(\Omega), \tag{1,4,4,12}$$

and furthermore, when $0<s<1/p$ we have

$$\tilde{W}^s_p(\Omega)=\mathring{W}^s_p(\Omega)=W^s_p(\Omega). \tag{1,4,4,13}$$

Proof From Lemma 1.3.2.6 and Theorem 1.4.4.4, we know that the norms of $W^s_p(\Omega)$ and of $\tilde{W}^s_p(\Omega)$ are equivalent at least on $\mathscr{D}(\Omega)$ when $s-1/p$ is not an integer. Then, from Definition 1.3.2.2 and Lemma 1.4.2.2 we know that $\mathscr{D}(\Omega)$ is dense in both spaces $\tilde{W}^s_p(\Omega)$ and $\mathring{W}^s_p(\Omega)$. Consequently, $\tilde{W}^s_p(\Omega)$ and $\mathring{W}^s_p(\Omega)$ are the completions on $\mathscr{D}(\Omega)$ for two equivalent norms. This proves identity (1,4,4,12).

We always have $\tilde{W}^s_p(\Omega)\subseteq W^s_p(\Omega)$. Then when $s<1/p$, it follows from Theorem 1.4.4.3 and Lemma 1.3.2.6 that $W^s_p(\Omega)=\tilde{W}^s_p(\Omega)$. This proves identity (1,4,4,13). ■

Another useful consequence of Theorem 1.4.4.4 is the extension of Lemma 1.4.1.3 to a bounded open domain Ω with a Lipschitz boundary.

Theorem 1.4.4.6 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary. Then D_j is a linear continuous operator from $W^s_p(\Omega)$ into $W^{s-1}_p(\Omega)$ unless $s=1/p$.*

Proof We have already seen in Subsection 1.4.1 that D_j maps $W^s_p(\Omega)$ into $W^{s-1}_p(\Omega)$ when either $s\geq 1$ or $s\leq 0$. Let us assume that $0<s<1$. We know from Theorem 1.4.3.1 that $W^s_p(\Omega)=W^s_p(\bar{\Omega})$. Consequently, for $u\in W^s_p(\Omega)$, D_ju is the restriction to Ω of a distribution $T\in W^{s-1}_p(\mathbb{R}^n)$. More precisely, we have

$$\langle D_ju, v\rangle=\langle T,\tilde{v}\rangle$$

for every $v\in\mathscr{D}(\Omega)$. Furthermore, we have

$$|\langle D_ju, v\rangle|\leq \|T\|_{s-1,p,\mathbb{R}^n}\,\|\tilde{v}\|_{1-s,q,\mathbb{R}^n}=\|T\|_{s-1,p,\mathbb{R}^n}\,\|V\|_{\widetilde{1-s},q,\Omega}$$

where $(1/p)+(1/q)=1$. This shows that T belongs to the dual space of

$\tilde{W}_q^{1-s}(\Omega)$. Due to Corollary 1.4.4.5, this last space coincides with $\mathring{W}_q^{1-s}(\Omega)$ provided $1-s \neq 1/q$; this means $s \neq 1/p$. Therefore $D_j u$ belongs to $W_p^{s-1}(\Omega)$ provided $s \neq 1/p$. ■

Remark 1.4.4.7 The preceding proof shows that $D_j u$ belongs to the dual space of $\tilde{W}_q^{1/q}(\Omega)$ when u belongs to $W_p^{1/p}(\Omega)$. This result cannot be improved as will be shown now. Here, for simplicity, we assume that $p=2$.

Proposition 1.4.4.8 *The bilinear form (defined for u and v smooth)*

$$u, v \mapsto \int_0^1 u'v \, dx \tag{1,4,4,14}$$

has no continuous extension to $H^{1/2}(]0,1[) \times H^{1/2}(]0,1[)$.

This obviously implies that for $u \in H^{1/2}(]0,1[)$, u' is not necessarily in $H^{-1/2}(]0,1[)$, the dual space of $H^{1/2}(]0,1[)$, since $H^{1/2}(]0,1[) = \mathring{H}^{1/2}(]0,1[)$ (see Theorem 1.4.2.4).

Proof Let us assume that (1,4,4,14) is continuous on $H^{1/2}(]0,1[) \times H^{1/2}(]0,1[)$; then there exists a constant K such that

$$\int_0^1 u'v \, dx \leq K \|u\|_{1/2,2,]0,1[} \|v\|_{1/2,2,]0,1[}$$

for all $u, v \in \mathcal{D}([0,1])$. Now let us assume that $v = \psi u$, where ψ is some cut-off function ($\psi \in \mathcal{D}([0,1])$, $\psi(0)=1$ and $\psi(1)=0$). We have

$$\int_0^1 u'v \, dx = \int_0^1 u'\psi u \, dx = -\tfrac{1}{2}u^2(0) - \tfrac{1}{2}\int_0^1 \psi' u^2 \, dx;$$

consequently, there exists a new constant C such that

$$|u(0)| \leq C \|u\|_{1/2,2,]0,1[}$$

for all $u \in \mathcal{D}([0,1])$. By translation we also have

$$\max_{x \in [0,1/2]} |u(x)| \leq C \|u\|_{1/2,2,]0,1[}.$$

By density, this last inequality implies that all the functions in $H^{1/2}(]0,1[)$ are continuous near zero. However, the particular function

$$u(x) = \log \left|\log \frac{x}{2}\right|$$

is an obvious counterexample to this property. Consequently, the form (1,4,4,14) cannot be continuous. ■

Remark 1.4.4.9 A by-product of the previous proof is that Sobolev's theorem (1.4.4.1) cannot be improved in the case where $s = n/p$ (here $n = 1$ and $p = 2$). Indeed, we have

$$H^{1/2}(]0, 1[) \not\subset L_\infty(]0, 1[);$$

this is the negation of the inclusion (1,4,4,1) in the limit case. The same way, we have

$$H^{1/2}(]0, 1[) \not\subset C^0([0, 1])$$

and this is the negation of (1,4,4,2) in the limit case.

As a last consequence of Theorem 1.4.4.4, we can investigate further the relations between $\tilde{W}_p^s(\Omega)$ and $\mathring{W}_p^s(\Omega)$ in the exceptional case when $s - 1/p$ is an integer.

Corollary 1.4.4.10 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary; then for all $s > 0$, we have*

$$\tilde{W}_p^s(\Omega) = \left\{u \mid u \in \mathring{W}_p^s(\Omega), \frac{D^\alpha u}{\rho^\sigma} \in L_p(\Omega), |\alpha| = m\right\} \tag{1,4,4,15}$$

where $\rho(x)$ is the distance from x to the boundary Γ of Ω and $s = m + \sigma$, m integer, $\sigma \in [0, 1[$.

Proof Let us denote by $Z(\Omega)$ the space on the right-hand side of (1,4,4,15). From Theorem 1.4.2.2 we know that $\mathscr{D}(\Omega)$ is dense in $\tilde{W}_p^s(\Omega)$ for the norm given by (1,3,2,11). This implies the inclusion

$$\tilde{W}_p^s(\Omega) \subseteq Z(\Omega).$$

To prove the converse inclusion, first we observe that

$$D^\alpha \tilde{u} = \widetilde{D^\alpha u}, \qquad |\alpha| \leq m \tag{1,4,4,16}$$

for all $u \in \mathring{W}_p^m(\Omega)$. This identity is obvious for $u \in \mathscr{D}(\Omega)$; it is extended to the whole of $\mathring{W}_p^m(\Omega)$ by density. Now let us start with $u \in Z(\Omega)$. From (1,4,4,16) we deduce that $\tilde{u} \in W_p^m(\mathbb{R}^n)$. To prove that $u \in \tilde{W}_p^s(\Omega)$, we just need to check that $D^\alpha \tilde{u} \in W_p^{s-m}(\mathbb{R}^n)$, for $|\alpha| = m$, according to Definitions 1.3.2.5 and 1.3.2.1. This means that

$$\|D^\alpha u\|^{\sim}_{s-m,p,\Omega}$$

has to be finite, in view of (1,3,2,11). This is obvious from the assumption that $u \in Z(\Omega)$. ■

1.4.5 Spaces defined on polygons

In most of the forthcoming sections, we shall deal with plane domains whose boundaries are (possibly curvilinear) polygons. First we shall make precise what we mean by curvilinear polygon. Then we shall review briefly the consequences of the results of the preceding sections, in the case when Ω has such a polygonal boundary.

Roughly speaking a curvilinear polygon is a manifold with corners. More precisely, let us state a definition similar in most respects to Definition 1.2.1.2.

Definition 1.4.5.1 *Let Ω be a bounded open subset of $\mathbb{R}^2$. We say that the boundary Γ is a curvilinear polygon of class C^m, m integer $\geqslant 1$ (respectively $C^{k,\alpha}$, k integer $\geqslant 1$, $0<\alpha\leqslant 1$) if for every $x\in\Gamma$ there exists a neighbourhood V of x in $\mathbb{R}^2$ and a mapping ψ from V in $\mathbb{R}^2$ such that*

(a) *ψ is injective,*
(b) *ψ together with ψ^{-1} (defined on $\psi(V)$) belongs to the class C^m (respectively $C^{k,\alpha}$),*
(c) *$\Omega\cap V$ is either*

$$\{y\in\Omega \mid \psi_2(y)<0\}, \{y\in\Omega \mid \psi_1(y)<0 \text{ and } \psi_2(y)<0\}$$

or

$$\{y\in\Omega \mid \psi_1(y)<0 \text{ or } \psi_2(y)<0\}$$

where $\psi_j(y)$ denotes the jth *component of ψ.*

Any domain Ω fulfilling the requirements in Definition 1.4.5.1 has a Lipschitz boundary according to Definition 1.2.1.1. Consequently, the Sobolev spaces on Ω will have all the properties already listed for Sobolev spaces on bounded domains with Lipschitz boundary. However, the actual advantages of these domains will appear clearly in the next section dedicated to the trace theorems.

Theorem 1.4.5.2 *Let Ω be a bounded open subset of $\mathbb{R}^2$ whose boundary Γ is a curvilinear polygon. Then we have the following inclusions and identities:*

(a) $\tilde{W}^s_p(\Omega)\subseteq \mathring{W}^s_p(\Omega)\subseteq W^s_p(\bar{\Omega})= W^s_p(\Omega)$ *for* $s>0$.
(b) $\tilde{W}^s_p(\Omega)=\mathring{W}^s_p(\Omega)$ *for* $s-1/p$ *non-integer,*
(c) $\mathring{W}^s_p(\Omega)=W^s_p(\Omega)$ *for* $s<1/p$,
(d) $$\tilde{W}^s_p(\Omega)=\left\{u\in \mathring{W}^s_p(\Omega) \,\middle|\, \frac{D^\alpha u}{\rho^\sigma}\in L_p(\Omega), |\alpha|=m\right\}$$

for $s=m+\sigma$, m a non-negative integer. Furthermore, $C^\infty(\bar{\Omega})$ is dense in $W^s_p(\Omega)$ and $\mathcal{D}(\Omega)$ is dense in $\tilde{W}^s_p(\Omega)$ for all $s>0$. We also have

(e) $W^s_p(\Omega)\subseteq W^t_q(\Omega)$, $s-\dfrac{2}{p}=t-\dfrac{2}{q}$, $t\leqslant s$

and

(f) $W_p^s(\Omega) \subseteq C^{k,\alpha}(\bar{\Omega}),\ k+\alpha = s - \dfrac{2}{p}$

for $s - 2/p > 0$, *not an integer.*

In practice, we shall often have to check whether or not some concrete functions belong to a given Sobolev space. For instance, we shall deal with functions which have an isolated singularity. A criterion for such functions is the following.

Theorem 1.4.5.3 *Let Ω be a bounded open subset of $\mathbb{R}^2$, whose boundary Γ is a curvilinear polygon. Assume that $0 \in \Gamma$. Let V be a neighbourhood of 0 such that*

$$V \cap \bar{\Omega} \subseteq \{(r\cos\theta, r\sin\theta) \mid r \geq 0,\ a \leq \theta \leq b\}$$

with $b - a < 2\pi$. Finally let u be a function which is smooth in $\bar{\Omega}\backslash\{0\}$ and which coincides with

$$r^\alpha \varphi(\theta)$$

in $V \cap \Omega$, where $\varphi \in C^\infty([a, b])$. Then

$$u \in W_p^s(\Omega) \text{ for } \operatorname{Re}\alpha > s - \frac{2}{p} \tag{1,4,5,1}$$

while

$$u \notin W_p^s(\Omega) \text{ for } \operatorname{Re}\alpha \leq s - \frac{2}{p} \tag{1,4,5,2}$$

when $\operatorname{Re}\alpha$ *is not an integer.*

It is very easy to check these inclusions when s is an integer. However, when s is not an integer the double integrals which appear in the norm (1,3,2,2) are so complicated that it is almost impossible to estimate them directly. The method of proof devised by Babuška consists in proving that $u \in W_r^m(\Omega)$ for m integer $>s$ and $r<p$ and then using the Sobolev imbeddings. We get thus all the desired results when $p \geq 2$. The general proof for $p<2$ makes use of weighted Sobolev spaces; we skip it since we shall mostly need inclusion (1,4,5,1) when $p \geq 2$.

Proof A derivative of order m of u behaves as a finite sum of functions $r^{\alpha-m}\psi(\theta)$, where $\psi \in C^\infty([a, b])$, in $V \cap \Omega$ (This is true unless α is an integer where cancellations can occur.) Consequently its rth power is integrable in Ω iff $\operatorname{Re}\alpha > m - 2/r$. In other words

$$\begin{cases} u \in W_r^m(\Omega) & \text{if } \operatorname{Re}\alpha > m - \dfrac{2}{r} \\ u \notin W_r^m(\Omega) & \text{if } \operatorname{Re}\alpha \leqslant m - \dfrac{2}{r}. \end{cases}$$

By Sobolev imbeddings it follows that

(a) $u \in W_p^s(\Omega)$
provided there exists an integer $m \geqslant s$ and an $r \in]1, p]$ such that $\operatorname{Re}\alpha > m - 2/r$ and $m - 2/r = s - 2/p$. This last condition is always fulfilled when $p \geqslant 2$.

(b) $u \notin W_p^s(\Omega)$
when there exists an integer $m \leqslant s$ and an $r \geqslant p$ such that $\operatorname{Re}\alpha \leqslant m - 2/r$ and $m - 2/r = s - 2/p$. This last condition is always fulfilled when $p \leqslant 2$.

So far, we have proved (1,4,5,1) when $p \geqslant 2$ and (1,4,5,2) when $p \leqslant 2$. We shall not attempt to extend (1,4,5,1) to all values of $p < 2$ since this requires the use of weighted spaces as we already mentioned it earlier. However, the extension of (1,4,5,2) to all $p > 2$ is simple at least when $s - 2/p$ is not an integer. Indeed, a derivative of order m of u is clearly Hölder continuous with exponent $\operatorname{Re}\alpha - m$ when $m < \operatorname{Re}\alpha \leqslant m + 1$, and it is not Hölder continuous with a larger exponent. Consequently, the second Sobolev imbedding implies (1,4,5,2) in the remaining cases when $\operatorname{Re}\alpha < s - 2/p$. ■

Remark 1.4.5.4 Similar results hold for the functions $r^\alpha (\ln r)\varphi(\theta)$.

1.5 Traces

Among the many consequences of Sobolev's imbeddings is the continuity of the functions belonging to $W_p^s(\Omega)$ when $s > n/p$. It is even continuity up to the boundary, which allows one to consider the values on the boundary, of such functions. This is obviously of the utmost importance in the study of boundary value problems. However, if we agree to consider boundary values of functions in a weaker sense, we can relax the condition on s. This is the purpose of the present section.

1.5.1 Hyperplanes

Here, for the sake of convenience, we denote by γ_n the operator defined by

$$(\gamma_n u)(x_1, \ldots, x_{n-1}) = u(x_1, \ldots, x_{n-1}, 0)$$

when u is a smooth function, continuous, say. In other words, we want to consider the restriction of u on the hyperplane $x_n = 0$. The basic fact about γ_n is that $\gamma_n u$ is well defined as soon as $u \in W_p^s(\mathbb{R}^n)$ when $s > 1/p$. We observe that this condition is less restrictive than the condition $s > n/p$ which is necessary for ensuring the continuity with respect to all variables.

The proof of the following result may be found in Agmon (1965) when $p = 2$, in Nečas (1967) when s is an integer and in Uspenskiĭ (1962) in the general case

Theorem 1.5.1.1 *Assume that $s - 1/p$ is not an integer and that $s - 1/p = k + \sigma$, $0 < \sigma < 1$, k an integer ≥ 0. Then the mapping*

$$u \mapsto \{\gamma_n u, \gamma_n D_n u, \ldots, \gamma_n D_n^k u\},$$

which is defined for $u \in \mathcal{D}(\mathbb{R}^n)$, has a unique continuous extension as an operator from

$$W_p^s(\mathbb{R}^n) \text{ onto } \prod_{j=0}^{k} W_p^{s-j-1/p}(\mathbb{R}^{n-1}).$$

This operator has a right continuous inverse which does not depend on p.

This result is easily extended to the case when $\mathbb{R}^{n-1}$ is replaced by an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$, which is smooth enough. This simply uses changes of variables. More precisely, when Γ is the Lipschitz boundary of a bounded open subset of $\mathbb{R}^n$, we define a normal vector field as follows. Let us keep the same notation as in Definition 1.2.1.1; then a unit outward normal vector $\boldsymbol{\nu}$ is defined a.e. (for the usual surface measure on Γ) by

$$\boldsymbol{\nu}(y', \varphi(y')) = \frac{\{-D_1\varphi(y'), \ldots, -D_{n-1}\varphi(y'), 1\}}{\sqrt{[1 + D_1\varphi(y')^2 + \cdots + D_{n-1}\varphi(y')^2]}}$$

for $y' \in V'$. This vector field is easily extended to the whole of V by defining it independently of x_n. Finally, by a partition of unity, we define an L^∞ vector field in a neighbourhood of $\bar{\Omega}$, such that $\boldsymbol{\nu}$ is the unit outward normal a.e. on Γ. Then we observe that when the boundary of Ω is of class $C^{k,1}$, the vector field $\boldsymbol{\nu}$ is only of class $C^{k-1,1}$. Now we denote by γ the operator defined by $(\gamma u) = u\,|_\Gamma$ when u is a smooth function.

Theorem 1.5.1.2 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a $C^{k,1}$ boundary Γ. Assume that $s - 1/p$ is not an integer, $s \leq k + 1$, $s - 1/p = l + \sigma$, $0 < \sigma < 1$, l an integer ≥ 0. Then the mapping*

$$u \mapsto \left\{\gamma u, \gamma\frac{\partial u}{\partial \nu}, \ldots, \gamma\frac{\partial^l u}{\partial \nu^l}\right\}$$

which is defined for $u \in C^{k,1}(\bar{\Omega})$, *has a unique continuous extension as an operator from*

$$W_p^s(\Omega) \text{ onto } \prod_{j=0}^{l} W_p^{s-j-1/p}(\Gamma).$$

This operator has a right continuous inverse which does not depend on p.

The particular case when $s = 1$ and $k = 0$ was proved a long time ago by Gagliardo (1957).

Theorem 1.5.1.3 *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *with a Lipschitz boundary* Γ. *Then the mapping* $u \to \gamma u$ *which is defined for* $u \in C^{0,1}(\bar{\Omega})$, *has a unique continuous extension as an operator from* $W_p^1(\Omega)$ *onto* $W_p^{1-1/p}(\Gamma)$. *This operator has a right continuous inverse independent of p.*

In the sequel we shall always denote by γ the extended operator defined on the whole of $W_p^1(\Omega)$ and we shall call it the trace operator.

In addition it is also possible to characterize the kernel of the trace operator γ and even of the mapping

$$u \mapsto \left\{\gamma u, \gamma \frac{\partial u}{\partial \nu}, \ldots, \gamma \frac{\partial^l u}{\partial \nu^l}\right\},$$

in several cases.

Theorem 1.5.1.4 *Assume that* $s - 1/p$ *is not an integer and that* $s - 1/p = k + \sigma$, $0 < \sigma < 1$, k *an integer* ≥ 0. *Then* $u \in \tilde{W}_p^s(\mathbb{R}_+^n)$ *if and only if* $u \in W_p^s(\mathbb{R}_+^n)$ *and*

$$\gamma_n u = \gamma_n D_n u = \cdots = \gamma_n D_n^k u = 0.$$

Here we denote by $\mathbb{R}_+^n$, the half space defined by $x_n > 0$. By changing variables, we deduce the following result.

Theorem 1.5.1.5 *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *with a* $C^{k,1}$ *boundary* Γ. *Assume that* $s - 1/p$ *is not an integer and that* $s - 1/p = l + \sigma$, $0 < \sigma < 1$, l *an integer* ≥ 0. *Then for* $s \leq k + 1$, $u \in \tilde{W}_p^s(\Omega)$ *if and only if* $u \in W_p^s(\Omega)$ *and*

$$\gamma u = \gamma \frac{\partial u}{\partial \nu} = \cdots = \gamma \frac{\partial^l u}{\partial \nu^l} = 0.$$

Remembering Corollary 1.4.4.5, we see that Theorem 1.5.1.5 implies also the following result.

Corollary 1.5.1.6 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a $C^{k,1}$ boundary Γ. Assume that $s \leq k+1$ and that $s - 1/p$ is not an integer. Let $s - 1/p = l + \sigma$, $0 < \sigma < 1$, l an integer ≥ 0. Then $u \in \mathring{W}_p^s(\Omega)$ if and only if $u \in W_p^s(\Omega)$ and*

$$\gamma u = \gamma \frac{\partial u}{\partial \nu} = \cdots = \gamma \frac{\partial^l u}{\partial \nu^l} = 0.$$

In some special problems related to the study of mixed boundary conditions on a regular boundary, it will be convenient to split the boundary Γ into pieces and correspondingly to split the trace operator γ. The related trace theorems follow. We first consider functions defined on $\mathbb{R}^n$ and define γ_+ and γ_- by

$$\begin{cases} \gamma_+ u(x_1, \ldots, x_{n-1}) = u(x_1, \ldots, x_{n-1}, 0), x_{n-1} > 0 \\ \gamma_- u(x_1, \ldots, x_{n-1}) = u(x_1, \ldots, x_{n-1}, 0), x_{n-1} < 0. \end{cases}$$

Theorem 1.5.1.7 *Assume that $s - 1/p$ is not an integer and that $s - 1/p = k + \sigma$, $0 < \sigma < 1$, k an integer ≥ 0. Then the mapping $u \mapsto (\{f_j^+\}_{j=0}^k, \{f_j^-\}_{j=0}^k)$ defined by*

$$f_j^{\pm} = \gamma_{\pm} D_n^j u, \qquad 0 \leq j \leq k$$

defined on $\mathcal{D}(\mathbb{R}^n)$ has a continuous extension as an operator from $W_p^s(\mathbb{R}^n)$ on the subspace of

$$T = \prod_{j=0}^{k} W_p^{s-j-1/p}(\mathbb{R}_+^{n-1}) \times \prod_{j=0}^{k} W_p^{s-j-1/p}(\mathbb{R}_-^{n-1})$$

defined by the conditions

(a) $\gamma_{n-1} D_{n-1}^l f_j^+ = \gamma_{n-1} D_{n-1}^l f_j^-, \qquad l < s - j - \dfrac{2}{p}$

and

(b) $$\int_0^{+\infty} \int_{\mathbb{R}^{n-2}} |D_{n-1}^l f_j^+(x_1, \ldots, x_{n-2}, t) - D_{n-1}^l f_j^-(x_1, \ldots, x_{n-2}, -t)|^p \, dx_1 \cdots dx_{n-1} \frac{dt}{t} < +\infty$$

for $l = s - j - 2/p$, when $s - 2/p$ is an integer.

The notation is self explanatory: we denote by $\mathbb{R}_{\pm}^{n-1}$ the subset of $\mathbb{R}^{n-1}$ defined by $x_{n-1} \gtrless 0$ respectively, γ_{n-1} is the trace operator on the hyperplane $x_{n-1} = 0$ defined in Theorem 1.5.1.1. This statement is a direct consequence of Theorem 1.5.1.1 through the following lemma.

Lemma 1.5.1.8 *Let $f^{\pm} \in W_p^r(\mathbb{R}_{\pm}^{n-1})$ and define f by*

$$f(x) = f^{\pm}(x) \text{ when } x_{n-1} \gtrless 0;$$

then $f \in W_p^r(\mathbb{R}^{n-1})$ if and only if

$$\text{(a)} \quad \gamma_{n-1} D_{n-1}^l f^+ = \gamma_{n-1} D_{n-1}^l f^-, \qquad l < r - \frac{1}{p},$$

$$\text{(b)} \quad \int_0^{+\infty} \int_{\mathbb{R}^{n-2}} |D_{n-1}^l f^+(x_1, \ldots, x_{n-2}, t) - D_{n-1}^l f^-(x_1, \ldots, x_{n-2}, t)|^2 \, dx_1 \cdots dx_{n-2} \frac{dt}{t} < +\infty$$

for $l = r - 1/p$ when $r - 1/p$ is an integer.

The corresponding results for a domain whose boundary is not a hyperplane will be detailed only in the case of a plane domain in the next subsection.

All the results that we have mentioned so far about trace properties are rather qualitative. It is often useful to have also quantitative results for traces. Here is a very elementary result in that direction. Before stating it, we need an auxiliary result about Lipschitz boundaries.

Lemma 1.5.1.9 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary Γ. Then there exist $\delta > 0$ and $\boldsymbol{\mu} \in C^\infty(\bar{\Omega})^n$ such that*

$$\boldsymbol{\mu} \cdot \boldsymbol{\nu} \geq \delta \text{ a.e. on } \Gamma. \tag{1,5,1,1}$$

Inequality (1,5,1,1) means that $\boldsymbol{\mu}$ is not very different from the normal $\boldsymbol{\nu}$ on Γ. However, $\boldsymbol{\mu}$ is much smoother than $\boldsymbol{\nu}$.

Proof It is very easy to define $\boldsymbol{\mu}$ locally. In the notation of Definition 1.2.1.1 we can choose $\boldsymbol{\mu}_V$ (in V) as the unit vector in the direction of y_n. Indeed, the component of $\boldsymbol{\nu}$ in the direction of y_n is $[1 + |\nabla\varphi(y')|^2]^{-1/2}$ and consequently we have

$$\boldsymbol{\mu}_V \cdot \boldsymbol{\nu} \geq [1 + L_V]^{-1/2}$$

where L_V^2 is the Lipschitz constant of φ in V'.

Then we cover the boundary Γ of Ω by the interiors of a finite number of hypercubes V_k, $1 \leq k \leq K$, each of which fulfils the conditions of Definition 1.2.1.1. To each V_k corresponds a vector $\boldsymbol{\mu}_{V_k}$ by the construction described above. Then we can define $\boldsymbol{\mu}$ as follows:

$$\boldsymbol{\mu} = \sum_{k=1}^{K} \theta_k \boldsymbol{\mu}_{V_k}$$

where θ_k, $1 \leqslant k \leqslant K$, is a partition of unity on Γ such that $\theta_k \in \mathcal{D}(\mathbb{R}^n)$, $\theta_k \geqslant 0$ and θ_k has its support in the interior of V_k.

Obviously $\boldsymbol{\mu}$ is a smooth vector field, and on Γ we have

$$\boldsymbol{\mu} \cdot \boldsymbol{\nu} = \sum_{k=1}^{K} \theta_k \boldsymbol{\mu}_{V_k} \cdot \boldsymbol{\nu} \geqslant \inf_{k=1}^{K} [1 + L_{V_k}^2]^{-1/2} = \delta. \quad \blacksquare$$

Theorem 1.5.1.10 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary Γ. Then there exists a constant K such that*

$$\int_\Gamma |\gamma u|^p \, d\sigma \leqslant K\left[\varepsilon^{1-1/p} \int_\Omega |\nabla u|^p \, dx + \varepsilon^{-1/p} \int_\Omega |u|^p \, dx\right] \tag{1,5,1,2}$$

for all $u \in W_p^1(\Omega)$ and $\varepsilon \in]0, 1[$. In addition K depends only on the norm of $\boldsymbol{\mu}$ in $C^1(\bar{\Omega})$ and on δ (defined in Lemma 1.5.1.9).

Proof In view of Theorem 1.4.2.1 it is sufficient to prove inequality (1,5,1,2) for all $u \in C^1(\bar{\Omega})$. For such a function, we have

$$\int_\Omega \nabla |u|^p \cdot \boldsymbol{\mu} \, dx = \sum_{j=1}^{n} \int_\Omega \frac{\partial |u|^p}{\partial x_j} \mu_j \, dx = \sum_{j=1}^{n} p \int_\Omega |u|^{p-2} u \frac{\partial u}{\partial x_j} \mu_j \, dx$$
$$= p \int_\Omega |u|^{p-2} u \, \nabla u \cdot \boldsymbol{\mu} \, dx.$$

On the other hand, applying the Green theorem (see also Theorem 1.1, Section 3.1 in Nečas (1967), or Theorem 1.5.3.1) we obtain

$$\int_\Omega \nabla |u|^p \cdot \boldsymbol{\mu} \, dx = \int_\Gamma |u|^p \, \boldsymbol{\mu} \cdot \boldsymbol{\nu} \, d\sigma - \int_\Omega |u|^p \operatorname{div} \boldsymbol{\mu} \, dx.$$

It follows that

$$\int_\Gamma |u|^p \, \boldsymbol{\mu} \cdot \boldsymbol{\nu} \, d\sigma = p \int_\Omega |u|^{p-2} u \nabla u \cdot \boldsymbol{\mu} \, dx + \int_\Omega |u|^p \operatorname{div} \boldsymbol{\mu} \, dx$$

and consequently that

$$\delta \int_\Gamma |u|^p \, d\sigma \leqslant p \max_{\bar{\Omega}} |\boldsymbol{\mu}| \int_\Omega |u|^{p-1} |\nabla u| \, dx + \max_{\bar{\Omega}} |\operatorname{div} \boldsymbol{\mu}| \int_\Omega |u|^p \, dx.$$

Then, applying Hölder's inequality, we get

$$\delta \int_\Gamma |u|^p \, d\sigma \leqslant \|\mu\|_{C^1(\bar{\Omega})} \left\{ p \left(\int_\Omega |u|^p \, dx \right)^{1/q} \left(\int_\Omega |\nabla u|^p \, dx \right)^{1/p} + \int_\Omega |u|^p \, dx \right\}$$

where $p^{-1}+q^{-1}=1$ and then

$$\delta\int_\Gamma |u|^p\,d\sigma \leq \|\mu\|_{C^1(\bar\Omega)}\left\{\varepsilon^{1-1/p}\int_\Omega |\nabla u|^p\,dx+\varepsilon^{-1/p}\frac{p}{q}\int_\Omega |u|^p\,dx\right.$$
$$\left.+\int_\Omega |u|^p\,dx\right\}.$$

This inequality, clearly implies (1,5,1,2) when $\varepsilon\in\,]0,1[$. ■

1.5.2 Polygons

The results stated in Theorem 1.5.1.2 and 1.5.1.3 are not sufficient for studying the Neumann problem in a domain whose boundary is a polygon. Indeed such a domain is never of class $C^{1,1}$. However, those theorems give us a hint of what happens.

First let us fix some notation. From now on, we consider a bounded open subset Ω of $\mathbb{R}^2$, whose boundary is a curvilinear polygon of class $C^{k,1}$. We denote each of the $C^{k,1}$ curves which constitute the boundary by $\bar\Gamma_j$ for some j ranging from 1 to N. The curve $\bar\Gamma_{j+1}$ follows $\bar\Gamma_j$ according to the positive orientation, on each connected component of Γ. We denote by S_j the vertex which is the end point of $\bar\Gamma_j$. Following the same method as in Section 1.5.1 we define a $C^{k-1,1}$ vector field $\boldsymbol{\nu}_j$ on a neighbourhood of $\bar\Omega$, which is the unit outward normal a.e. on Γ_j. (We observe that $\boldsymbol{\nu}_j=\boldsymbol{\nu}$ a.e. on Γ_j, but in general $\boldsymbol{\nu}_j\neq\boldsymbol{\nu}$ inside Ω.) Finally we denote by ω_j the measure of the angle at S_j (toward the interior of Ω). For a smooth function $u\in\mathcal{D}(\bar\Omega)$ we denote by $\gamma_j u$ its restriction to Γ_j. (Γ_j is the interior of $\bar\Gamma_j$, i.e., the set $\bar\Gamma_j$ without its endpoints S_{j-1} and S_j.)

Theorem 1.5.2.1 *Let Ω be a bounded open subset of $\mathbb{R}^2$, whose boundary is a curvilinear polygon of class $C^{k,1}$; then for each j, the mapping*

$$u\mapsto\left\{\gamma_j u,\ \gamma_j\frac{\partial u}{\partial\nu_j},\ldots,\gamma_j\frac{\partial^l u}{\partial\nu_j^l}\right\},\qquad l<\overset{m}{\cancel{s}}-\frac{1}{p},$$

which is defined for $u\in\mathcal{D}(\bar\Omega)$, has a unique continuous extension as an operator from

$$W_p^m(\Omega)\ \textit{onto}\ \prod_{j=0}^{l} W_p^{m-j-1/p}(\Gamma_j),\qquad l\leq m-1\leq k.$$

Lemma 1.5.2.2 *Let Ω be a bounded open subset of $\mathbb{R}^2$ whose boundary is a curvilinear polygon of class C^1. Let $f_j=\gamma_j u$; then we have*

$$\iint_{\Gamma_j\times\Gamma_k}\frac{|f_j(x)-f_k(y)|^p}{|x-y|^p}\,d\sigma(x)\,d\sigma(y)<+\infty. \tag{1,5,2,1}$$

This is just a consequence of the finiteness of the norm of γu in $W_p^{1-1/p}(\Gamma)$. From (1,3,3,3), splitting the domain of integration $\Gamma\times\Gamma$ in $\bigcup_{j,k}\Gamma_j\times\Gamma_k$, we get

$$\sum_{j=1}^{N}\iint_{\Gamma_j\times\Gamma_j}\frac{|f_j(x)-f_j(y)|^p}{|x-y|^p}\,d\sigma(x)\,d\sigma(y)$$
$$+2\sum_{j\neq k}\iint_{\Gamma_j\times\Gamma_k}\frac{|f_j(x)-f_k(y)|^p}{|x-y|^p}\,d\sigma(x)\,d\sigma(y)<+\infty.$$

Since we already knew from Theorem 1.5.2.1 that $f_j\in W_p^{1-1/p}(\Gamma_j)$, $1\leqslant j\leqslant N$ (and consequently $f_j\in L_p(\Gamma_j)$, $1\leqslant j\leqslant N$), the condition (1,5,2,1) is automatically fulfilled when the distance from Γ_j to Γ_k is strictly positive. In other words, (1,5,2,1) is an extra condition only when Γ_j and Γ_k have a common end point. By possibly exchanging j and k, we can assume that $k=j+1$. Then let σ be the distance along Γ, starting at S_j, and let $x_j(\sigma)$ be the point on Γ whose distance to S_j is σ. Consequently for $|\sigma|$ small enough, $|\sigma|\leqslant\delta_j$, say, we have $x_j(\sigma)\in\Gamma_j$ when $\sigma<0$ and $x_j(\sigma)\in\Gamma_{j+1}$ when $\sigma>0$. With these notations, condition (1,5,2,1) may be rewritten as

$$\int_0^{\delta_j}\int_0^{\delta_j}\frac{|f_{j+1}(x_j(\sigma))-f_j(x_j(-\tau))|^p}{|\sigma+\tau|^p}\,d\sigma\,d\tau<+\infty \tag{1,5,2,2}$$

since the angle at S_j is not allowed to be 0 or 2π (and therefore $|x_j(\sigma)-x_j(-\tau)|$ and $|\sigma+\tau|$ are equivalent functions). On the other hand, the fact that $f_j\in W_p^{1-1/p}(\Gamma_j)$ and $f_{j+1}\in W_p^{1-1/p}(\Gamma_{j+1})$ implies the convergence of the following integrals:

$$\int_0^{\delta_j}\int_0^{\delta_j}\frac{|f_j(x_j(-\sigma))-f_j(x_j(-\tau))|^p}{|\tau-\sigma|^p}\,d\sigma\,d\tau<+\infty \tag{1,5,2,3}$$

$$\int_0^{\delta_j}\int_0^{\delta_j}\frac{|f_{j+1}(x_j(\sigma))-f_{j+1}(x_j(\tau))|^p}{|\tau-\sigma|^p}\,d\sigma\,d\tau<+\infty. \tag{1,5,2,4}$$

From these inequalities, we shall deduce the following result, which is nothing but a rephrasing of Gagliardo's theorem. For simplicity we assume that Γ has only one connected component and agree that $\Gamma_{N+1}=\Gamma_1$; the extension of the forthcoming results to non-simply connected domains is obvious and just leads to complications in the notation.

Theorem 1.5.2.3 *Let Ω be a bounded open subset of $\mathbb{R}^2$ whose boundary Γ is a curvilinear polygon of class C^1. Then the mapping $u\mapsto\{f_j\}_{j=1}^N$, where $f_j=\gamma_j u$, is a linear continuous mapping from $W_p^1(\Omega)$ onto the subspace of*

$\prod_{j=1}^N W_p^{1-1/p}(\Gamma_j)$ *defined by*

(a) *no extra condition when* $1<p<2$,

(b) $f_j(S_j)=f_{j+1}(S_j)$, $1\leq j\leq N$ *when* $2<p<\infty$,

(c) $\displaystyle\int_0^{\delta_i} \frac{|f_{j+1}(x_j(\sigma))-f_j(x_j(-\sigma))|^2}{\sigma}\,d\sigma<\infty,\ 1\leq j\leq N$

when $p=2$.

We observe that condition (b) is meaningful since, for $p>2$, it follows from Sobolev's imbedding theorem that f_j and f_{j+1} are continuous on $\bar{\Gamma}_j$ and $\bar{\Gamma}_{j+1}$ respectively. Furthermore, if for some particular $u\in H^1(\Omega)$, f_j and f_{j+1} are Hölder continuous near S_j, then it is easily seen that condition (c) reduces to condition (b). Unfortunately, condition (b) is not always meaningful when $p=2$, since functions in $H^{1/2}(\Gamma_j)$ are not always continuous (see Remark 1.4.4.9). This is one of the few cases where Sobolev spaces related to $p=2$ are more complicated to handle than Sobolev spaces related to $p\neq 2$.

Proof We know from Theorem 1.5.2.1, that for $u\in W_p^1(\Omega)$,

$$\{f_j\}_{j=1}^N\in\prod_{j=1}^N W_p^{1-1/p}(\Gamma_j).$$

Furthermore, when $p>2$, u is continuous up to the boundary Γ and consequently we have $f_j=u|_{\Gamma_j}$ for all j; in particular, we have

$$u(S_j)=f_j(S_j)=f_{j+1}(S_j)$$

and this shows that condition (b) is necessary. Finally, in the limit case $p=2$, condition (c) will follow from (1,5,2,2). Indeed, we have

$$\begin{aligned}&\left[\int_0^{\delta_i}\int_0^{\delta_i}\frac{|f_{j+1}(x_j(\sigma))-f_j(x_j(-\sigma))|^2}{|\sigma+\tau|^2}\,d\sigma\,d\tau\right]^{1/2}\\ &\quad\leq\left[\int_0^{\delta_i}\int_0^{\delta_i}\frac{|f_{j+1}(x_j(\sigma))-f_j(x_j(-\tau))|^2}{|\sigma+\tau|^2}\,d\sigma\,d\tau\right]^{1/2}\\ &\quad+\left[\int_0^{\delta_i}\int_0^{\delta_i}\frac{|f_j(x_j(-\tau))-f_j(x_j(-\sigma))|^2}{|\sigma-\tau|^2}\,d\sigma\,d\tau\right]^{1/2}\end{aligned}$$

and this is finite in view of (1,5,2,2) and of the fact that f_j belongs to $H^{1/2}(\Gamma_j)$. We conclude by observing that

$$\lim_{\sigma\to 0}\sigma\int_0^{\delta_i}\frac{d\tau}{|\sigma+\tau|^2}=1$$

and consequently we have proved that

$$\int_0^{\delta_j} |f_{j+1}(x_j(\sigma)) - f_j(x_j(-\sigma))|^2 \frac{d\sigma}{\sigma} < +\infty.$$

This is why condition (b) is necessary.

We now turn to prove that those conditions are sufficient.

First case $1 < p < 2$. We want to prove that the mapping is onto; in other words, for every $\{f_j\}_{j=1}^N \in \prod_{j=1}^N W_p^{1-1/p}(\Gamma_j)$ we must show that there exists a $u \in W_p^1(\Omega)$ such that $\gamma_j u = f_j$. For that purpose, we take advantage of Theorem 1.5.1.3 and it is enough to build up a function f on Γ from all the given f_j and to check that $f \in W_p^{1-1/p}(\Gamma)$. Thus we set

$$f(x) = f_j(x), \qquad x \in \Gamma_j.$$

Since f_j is given in $W_p^{1-1/p}(\Gamma_j)$, we know that $f \in L_p(\Gamma)$ and that

$$\int_{\Gamma_j} \int_{\Gamma_k} \frac{|f_j(x) - f_k(y)|^p}{|x-y|^p} d\sigma(x)\, d\sigma(y) < +\infty$$

is finite when $j = k$ and when $k \neq j-1, j, j+1$ (so that the distance from Γ_j to Γ_k is strictly positive). Remembering identity (1,3,3,2), it remains to check that

$$I_j^p = \int_{\Gamma_j} \int_{\Gamma_{j+1}} \frac{|f_j(x) - f_{j+1}(y)|^p}{|x-y|^p} d\sigma(x)\, d\sigma(y) < +\infty. \tag{1,5,2,5}$$

Indeed, we have

$$I_j \leq \left[\int_{\Gamma_j} |f_j(x)|^p \left\{\int_{\Gamma_{j+1}} \frac{d\sigma(y)}{|x-y|^p}\right\} d\sigma(x)\right]^{1/p} + \left[\int_{\Gamma_{j+1}} |f_{j+1}(y)|^p \left\{\int_{\Gamma_j} \frac{d\sigma(x)}{|x-y|^p}\right\} d\sigma(y)\right]^{1/p}.$$

Since Γ_{j+1} is a C^1 curve, the function

$$\int_{\Gamma_{j+1}} \frac{d\sigma(y)}{|x-y|^p}$$

is equivalent to $d(x, \Gamma_{j+1})^{-p+1}$ and therefore

$$\begin{aligned} I_j &\leq K\left[\int_{\Gamma_j} |f_j(x)|^p \frac{d\sigma(x)}{d(x;\Gamma_{j+1})^{p-1}}\right]^{1/p} + K\left[\int_{\Gamma_{j+1}} |f_{j+1}(y)|^p \frac{d\sigma(y)}{d(y;\Gamma_j)^{p-1}}\right]^{1/p} \\ &\leq K'\left\{\|f\|_{L_p(\Gamma)} + \int_0^{\delta_j} |f_j(x_j(-\sigma))|^p \frac{d\sigma}{\sigma^{p-1}} + \int_0^{\delta_j} |f_{j+1}(x_j(\sigma))|^p \frac{d\sigma}{\sigma^{p-1}}\right\}^{1/p} \end{aligned}$$

where K and K' are some constants. The last two integrals are finite as is shown in Theorem 1.4.4.3, since

$$1-\frac{1}{p}<\frac{1}{p}$$

when $p<2$. Consequently, (1,5,2,5) is proved and $f\in W_p^{1-1/p}(\Gamma)$.

Second case $2\leqslant p<\infty$. We follow the same method and eventually we have to check the finiteness of I_j. This is equivalent to the finiteness of

$$I_j'=\left[\int_0^{\delta_j}\int_0^{\delta_j}\frac{|f_j(x_j(-\sigma))-f_{j+1}(x_j(\tau))|^p}{|\sigma+\tau|^p}\,d\sigma\,d\tau\right]^{1/p}.$$

This last integral is less than or equal to

$$\begin{aligned}&\left[\int_0^{\delta_j}\int_0^{\delta_j}\frac{|f_j(x_j(-\sigma))-f_{j+1}(x_j(\sigma))|^p}{|\sigma+\tau|^p}\,d\sigma\,d\tau\right]^{1/p}\\&\quad+\left[\int_0^{\delta_j}\int_0^{\delta_j}\frac{|f_{j+1}(x_j(\sigma))-f_{j+1}(x_j(\tau))|^p}{|\sigma-\tau|^p}\,d\sigma\,d\tau\right]^{1/p}\\&\leqslant K\left[\int_0^{\delta_j}|f_j(x_j(-\sigma))-f_{j+1}(x_j(\sigma))|^p\frac{d\sigma}{\sigma^{p-1}}\right]^{1/p}+K\,\|f_{j+1}\|_{1-1/p,p,\Gamma_{j+1}},\end{aligned}\tag{1,5,2,6}$$

where K is some constant. This is obviously finite when $p=2$ as a consequence of condition (c). Then when $p>2$ let us denote by h the function

$$\sigma\mapsto\psi(\sigma)[f_j(x_j(-\sigma))-f_{j+1}(x_j(\sigma))]$$

where ψ is some smooth cut-off function, which is identically equal to 1 near zero and which is zero for $\sigma\geqslant\frac{1}{2}\delta_j$. We know that h belongs to $W_p^{1-1/p}(]0,\delta_j[)$ and that $h(\delta_j)=0$ by construction and that $h(0)=0$ in view of condition (b). Consequently, we have $h\in\tilde{W}_p^{1-1/p}(]0,\delta_j[)$ by Corollary 1.5.1.5. Finally, it follows from Theorem 1.4.4.3 that

$$\frac{h}{\sigma^{1-(1/p)}}\in L_p(]0,\delta_j[)$$

and this shows the finiteness of the integral which appears in (1,5,2,6). Consequently we have shown the convergence of I_j' in all cases; this means that $f\in W_p^{1-1/p}(\Gamma)$ and the proof of Theorem 1.5.2.3 is complete. ■

The remainder of this section is devoted to the extension of Theorem 1.5.2.3 to the spaces $W_p^m(\Omega)$, when $m>1$. Essentially the method of proof is the same; however, for the sake of clarity we shall consider successively the cases where Ω is a quadrant, then a rectilinear polygon and finally a curvilinear polygon. First let us denote by $\mathbb{R}_+\times\mathbb{R}_+$ the first quadrant defined by $x>0$ and $y>0$.

Theorem 1.5.2.4 *The mapping* $u \mapsto \{\{f_k\}_{k=0}^{m-1}, \{g_l\}_{l=0}^{m-1}\}$ *defined by*

$$f_k = D_y^k u\,|_{y=0}, \qquad g_l = D_x^l u\,|_{x=0} \tag{1,5,2,7}$$

for $u \in \mathcal{D}(\overline{\mathbb{R}_+\times\mathbb{R}_+})$, *has a unique continuous extension as an operator from* $W_p^m(\mathbb{R}_+\times\mathbb{R}_+)$ *onto the subspace of*

$$T = \prod_{k=0}^{m-1} W_p^{m-k-1/p}(\mathbb{R}_+) \times \prod_{l=0}^{m-1} W_p^{m-l-1/p}(\mathbb{R}_+)$$

defined by

$$\begin{aligned} &\text{(a)}\quad D_x^l f_k(0) = D_y^k g_l(0),\ l+k<m-2/p \text{ for all } p, \text{ and}\\ &\text{(b)}\quad \int_0^1 |D_x^l f_k(t) - D_y^k g_l(t)|^2 \frac{dt}{t} < +\infty,\ l+k=m-1 \end{aligned} \tag{1,5,2,8}$$

for $p=2$.

(Here we shall denote by γ_1 the trace operator on $x=0$ and by γ_2 the trace operator on $y=0$; accordingly $f_k=\gamma_2 D_y^k u$ and $g_l=\gamma_1 D_x u$.)

The proof makes use of a simple continuation result which is proved by applying twice Nikolski's continuation method (once with respect to each variable). This method is explained in Section 3.6, §3, Chapter 2 of Nečas (1967). See also Theorem 1.4.3.1 since $\mathbb{R}_+\times\mathbb{R}_+$ is a uniform Lipschitz epigraph.)

Lemma 1.5.2.5 *We have* $W_p^m(\overline{\mathbb{R}_+\times\mathbb{R}_+}) = W_p^m(\mathbb{R}_+\times\mathbb{R}_+)$.

Proof of Theorem 1.5.2.4 Let U be any function in $W_p^m(\mathbb{R}^2)$ such that u is the restriction of U to $\mathbb{R}_+\times\mathbb{R}_+$. Applying Theorem 1.5.1.1, we see that the traces of u must be in T. Then for each k and l such that $k+l\leqslant m-1$, we consider

$$u_{k,l} = D_x^l D_y^k u, \quad \text{on} \quad]0,1[\times]0,1[.$$

It is obvious that $u_{k,l}$ belongs to $W_p^1(]0,1[\times]0,1[)$ and consequently conditions (a) and (b) in Theorem 1.5.2.4 follow from Theorem 1.5.2.3.

Now we are left with the problem of showing that the trace mapping is onto. We also need a continuation property on $\mathbb{R}_+$ for spaces of fractional order.

Lemma 1.5.2.6 *We have* $W_p^s(\overline{\mathbb{R}_+}) = W_p^s(\mathbb{R}_+)$ *for all* $s>0$.

Proof A short proof is the following. We can apply Theorem 1.4.3.1 to u_1, the restriction of a given $u \in W_p^s(\mathbb{R}_+)$ to $]0, 1[$. Let $U = P_s u_1$ and let φ be a cut-off function such that $\varphi(x) = 0$ for $x \geq \frac{2}{3}$ and $\varphi(x) = 1$ for $x \leq \frac{1}{3}$; then

$$\varphi U + (1-\varphi)\tilde{u}$$

is the desired continuation of u. ■

End of the proof of Theorem 1.5.2.4 Let $\{f_k\}_{k=0}^{m-1}$ and $\{g_l\}_{l=0}^{m-1}$ fulfil all the conditions in Theorem 1.5.2.4. We must find $u \in W_p^m(\mathbb{R}_+ \times \mathbb{R}_+)$ such that (1,5,2,7) holds. We first reduce our problem to the case when $g_l = 0$ for all l. For that purpose, let G_l be a continuation of g_l with $G_l \in W_p^{m-l-1/p}(\mathbb{R})$. From Theorem 1.5.1.1 we know that there exists $U \in W_p^m(\mathbb{R}^2)$, such that $\gamma_1 D_x^l U = G_l$, $0 \leq l \leq m-1$, where γ_1 refers here to the trace operator on the hyperplane $x = 0$.

Then, instead of looking for u, we shall look for $v = u - U|_{\mathbb{R}_+ \times \mathbb{R}_+} \in W_p^m(\mathbb{R}_+ \times \mathbb{R}_+)$ such that

$$\begin{cases} \gamma_2 D_y^k v = f_k - h_k, & 0 \leq k \leq m-1 \\ \gamma_1 D_x^l v = 0, & 0 \leq l \leq m-1 \end{cases}$$

where $h_k = \gamma_2 D_y^k(U|_{\mathbb{R}_+ \times \mathbb{R}_+})$. From the direct part of Theorem 1.5.2.4 which we have already proved, we know that

$$h_k \in W_p^{m-k-1/p}(\mathbb{R}_+)$$

and in addition

(c) $D_x^l h_k(0) = D_y^k g_l(0)$, $l + k < m - \dfrac{2}{p}$ for all p and

(d) $\displaystyle\int_0^1 |D_x^l h_k(t) - D_y^k g_l(t)|^2 \frac{dt}{t} < +\infty$, $l + k = m-1$

for $p = 2$. Let us denote by φ_k the difference $f_k - h_k$; then $\varphi_k \in W_p^{m-k-1/p}(\mathbb{R}_+)$ and from (a)–(d) it follows that

$$\varphi_k^{(l)}(0) = 0, \qquad l < m - k - \frac{2}{p} \text{ for all } p \tag{1,5,2,9}$$

and

$$\int_0^1 |\varphi_k^{(l)}(t)|^2 \frac{dt}{t} < +\infty, \qquad l = m-k-1 \text{ for } p = 2. \tag{1,5,2,10}$$

At this step our problem is the following. We are seeking $v \in$

$W_p^m(\mathbb{R}_+ \times \mathbb{R}_+)$ such that

$$\begin{cases} \gamma_2 D_y^k v = \varphi_k, & 0 \leqslant k \leqslant m-1 \\ \gamma_1 D_x^l v = 0, & 0 \leqslant l \leqslant m-1. \end{cases}$$

For the time being, let us accept the following result.

Lemma 1.5.2.7 *Under assumptions* (1,5,2,9) *and* (1,5,2,10), *we have* $\varphi_k \in \tilde{W}_p^{m-k-1/p}(\mathbb{R}_+)$, $0 \leqslant k \leqslant m-1$.

This means that $\tilde{\varphi}_k \in W_p^{m-k-1/p}(\mathbb{R})$, and applying Theorem 1.5.2.1 we know that there exists $w \in W_p^m(\mathbb{R}^2)$ such that

$$\gamma_2 D_y^k w = \tilde{\varphi}_k,$$

where γ_2 refers to the trace operator on the hyperplane $y = 0$. Then we obtain v as follows:

$$v(x, y) = w(x, y) - \sum_{j=1}^{m} \lambda_j w(-jx, y), \qquad x > 0, \quad y > 0$$

where the λ_j are real numbers such that

$$\sum_{j=1}^{m} (-j)^l \lambda_j = 1, \qquad 0 \leqslant l \leqslant m-1.$$

It is obvious that $v \in W_p^m(\mathbb{R}_+ \times \mathbb{R}_+)$. Then we have

$$(\gamma_1 D_y^k v)(x) = \tilde{\varphi}_k(x) - \sum_{j=1}^{m} \lambda_j \tilde{\varphi}_k(-jx) = \varphi_k(x)$$

for $0 \leqslant k \leqslant m-1$, since $x > 0$. We also have

$$(\gamma_2 D_x^l v)(y) = \left[1 - \sum_{j=1}^{m} \lambda_j (-j)^l\right] (\gamma_2 D_x^l w)(y) = 0$$

and consequently v is the desired function. The proof of Theorem 1.5.2.4 is complete provided we check Lemma 1.5.2.7. ■

Proof of Lemma 1.5.2.7 We just need to extend some of the previous results valid on a finite open interval to the case of $\mathbb{R}_+$. We again use a cut-off function ψ which is identically equal to 1 for $x \leqslant \frac{1}{3}$ and zero for $x \geqslant \frac{2}{3}$. Then $(1-\psi)\varphi_k \in W_p^{m-k-1/p}(\mathbb{R}_+)$ and its support is far from zero; it is readily seen from Definition 1.3.1.1 and Definition 1.3.2.5 that $(1-\psi)\varphi_k \in \tilde{W}_p^{m-k-1/p}(\mathbb{R}_+)$. On the other hand, we can consider $\psi\varphi_k$ as belonging to $W_p^{m-k-1/p}(]0, 1[)$. Applying Corollary 1.5.1.6 together with identity (1,4,4,12) when $p \neq 2$ and Corollary 1.4.4.10 when $p = 2$, we see that

(1,5,2,9) and (1,5,2,10) imply that $\psi\varphi_k \in \tilde{W}_p^{m-k-1/p}(]0,1[)$ and consequently that $\psi\varphi_k \in \tilde{W}_p^{m-k-1/p}(\mathbb{R}_+)$. The lemma is proved by addition, writing $\varphi_k = \psi\varphi_k + (1-\psi)\varphi_k$. ■

An extension of Theorem 1.5.2.4 to $W_p^s(\mathbb{R}_+ \times \mathbb{R}_+)$ with a noninteger s can be found in Grisvard (1966). The method of proof followed here is close to the method used in Nikolski (1956a,b, 1956–58, 1961) for studying the traces of some slightly different spaces.

The previous results are easily extended to an infinite sector with angle $\omega \in]0, \pi[$, by means of a linear change of coordinates. We also observe that the same results hold for the complement of the first quadrant owing to Lemma 1.5.2.5. Again, a linear change of coordinate allows one to extend those results to any infinite sector with angle $\omega \in]\pi, 2\pi[$. Eventually using a partition of unity, we obtain the corresponding results on a polygon; for simplicity we assume its boundary to be of class C^∞.

Theorem 1.5.2.8 *Let Ω be a bounded open subset of $\mathbb{R}^2$ whose boundary Γ is a curvilinear polygon of class C^∞. Then the mapping $u \to \{\gamma_j \partial^l u/\partial\nu_j^l\}$, $1 \leq j \leq N$, $0 \leq l \leq m-1$ is linear continuous from $W_p^m(\Omega)$ onto the subspace of*

$$T = \prod_{j=1}^{N} \prod_{l=0}^{m-1} W_p^{m-l-1/p}(\Gamma_j)$$

defined by the following condition: Let L be any linear differential operator with coefficients of class C^∞ and of order $d \leq m - 2/p$. Denote by $P_{j,l}$ the differential operator tangential to Γ_j such that $L = \sum_{l\geq 0} P_{j,l}\, \partial^l/\partial\nu_j^l$; then

(a) $\sum_{l\geq 0} (P_{j,l} f_{j,l})(S_j) = \sum_{l\geq 0} (P_{j+1,l} f_{j+1,l})(S_j)$ *for* $d < m - \dfrac{2}{p}$

(b) $$\int_0^{\delta_j} \left| \sum_{l\geq 0} (P_{j,l} f_{j,l})(x_j(-\sigma)) - \sum_{l\geq 0} (P_{j+1,l} f_{j+1,l})(x_j(\sigma)) \right|^2 \frac{d\sigma}{\sigma} < +\infty$$

for $d = m-1$ and $p = 2$.

Proof Using a partition of unity, we can restrict ourselves to the study of one vertex. Then a change of variables of class C^∞ replaces the corresponding vertex by zero, the angle by $\pi/2$ or $3\pi/2$ and the sides by the coordinate axis. Now the only difference between Theorems 1.5.2.4 and 1.5.2.8 is that in the former we only consider the operators $D_x^l D_y^k$, while in the latter we consider all the operators with coefficients in C^∞. However, in the case of a right angle with straight sides, this is equivalent. Indeed, let the f_k and the g_l fulfil condition (a) of Theorem 1.5.2.4 and let L be of order $d < m - 2/p$. We can write

$$L = \sum_{k=0}^{d} P_{2,k}(D_x)D_y^k = \sum_{l=0}^{d} P_{1,l}(D_y)D_x^l; \tag{1,5,2,11}$$

then

$$\begin{cases} P_{2,k}(D_x) = \sum_{l=0}^{d-k} a_{k,l}(x, y)D_x^l & (1,5,2,12) \\ P_{1,l}(D_y) = \sum_{k=0}^{d-l} a_{k,l}(x, y)D_y^k. & (1,5,2,13) \end{cases}$$

Consequently we have

$$\begin{aligned} \sum_{k=0}^{d} [P_{2,k}(D_x)f_k](0) &= \sum_{k=0}^{d} \left\{ \sum_{l=0}^{d-k} a_{k,l}(0, 0)D_x^l f_k(0) \right\} \\ &= \sum_{l=0}^{d} \left\{ \sum_{k=0}^{d-l} a_{k,l}(0, 0)D_y^k g_l(0) \right\} = \sum_{l=0}^{d} [P_{1,l}(D_y)g_l](0) \end{aligned}$$

and this is condition (a) in Theorem 1.5.2.8 (with the necessary change of notation).

Then in the case $p = 2$, let us assume further that the g_k and the f_l fulfil condition (b) of Theorem 1.5.2.4. We want to check condition (b) of Theorem 1.5.2.8. A preliminary remark is that we also have

$$\int_0^1 |D_x^l f_k(t) - D_y^k g_l(t)|^2 \frac{dt}{t} < +\infty \tag{1,5,2,14}$$

for $l + k < m - 1$. Indeed in that case we have

$$D_x^l f_k - D_y^k g_l \in H^{3/2}(]0, 1[).$$

From Sobolev's imbedding theorem, we know that $D_x^l f_k - D_y^k g_l$ is Hölder continuous of order α for every $\alpha \in]0, 1[$. Since this function also vanishes at zero by assumption, there exists a constant K such that

$$|D_x^l f_k(t) - D_y^k g_l(t)| \leq K t^\alpha, \qquad t \in]0, 1[.$$

This implies (1,5,2,14). Then using the same identities (1,5,2,11), (1,5,2,12) and (1,5,2,13), it is easy to check condition (b) in Theorem 1.5.2.8. ■

Remark 1.5.2.9 In some questions related to the solution of mixed boundary value problems, we have to admit the value π as possible value for the measure of the angles of Ω. In view of Theorem 1.5.1.7, the conditions (a) and (b) in Theorem 1.5.2.8 have to be replaced by the following, when the measure of the angle at S_j is π:

(a) $f_{j,l}(S_j) = f_{j+1,l}(S_j)$ for $l < m - \dfrac{2}{p}$

(b) $\displaystyle\int_0^{\delta_i} |f_{j,l}(x_j(-\sigma)) - f_{j+1,l}(x_j(\sigma))|^2 \frac{d\sigma}{\sigma} < +\infty$

for $l = m-1$ and $p = 2$.

Remark 1.5.2.10 In the particular case when Ω is a rectilinear polygon, it is enough to consider only those operators L which are homogeneous and with constant coefficients in the corresponding statement of Theorem 1.5.2.8.

Remark 1.5.2.11 As in Corollary 1.5.1.6 we can characterize the kernel of the mapping

$$u \mapsto \left\{\gamma_j \frac{\partial^l u}{\partial \nu_j^l}\right\}, \qquad 1 \leq j \leq N, \quad 0 \leq l \leq m-1$$

as being $\mathring{W}_p^m(\Omega)$.

1.5.3 Maximal domains of elliptic operators

So far, we have defined the trace of a function belonging to some Sobolev space $W_p^s(\Omega)$, under the assumption that s is larger than $1/p$. However, it was shown in Lions and Magenes (1960–63) that when a function u is a solution, in Ω, of an elliptic equation, u has traces on the boundary provided it belongs to any Sobolev space, without any restriction on s and p. The purpose of the present subsection is just to extend part of this result to the case of a domain with a polygonal boundary. A different approach to this kind of result is presented in Goulaouic and Grisvard (1970).

The method of proof devised by Lions and Magenes uses Green's formula. First we recall that Green's formula is valid in any bounded Lipschitz domain, as is shown in Nečas (1967) (Theorem 1.1, §1, Chapter 3).

Theorem 1.5.3.1 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary Γ. Then for every $u \in W_p^1(\Omega)$ and $v \in W_q^1(\Omega)$, with $1/p + 1/q = 1$, we have*

$$\int_\Omega D_i u v \, dx + \int_\Omega u D_i v \, dx = \int_\Gamma \gamma u \gamma v \nu^i \, d\sigma \tag{1,5,3,1}$$

(ν^i denotes the ith *component of the vector field* $\boldsymbol{\nu}$ *which was defined in Section* 1.5.1.)

We shall apply this formula twice to derive the following, where A denotes a second-order elliptic operator with coefficients smooth enough.

$$Au = \sum_{i,k=1}^{n} D_i(a_{i,k} D_k u) + \sum_{i=1}^{n} a_i D_i u + a_0 u.$$

Precisely, we assume that $a_{i,k}$ and a_i are Lipschitz continuous and $a_0 \in L_\infty(\Omega)$. The adjoint operator will be denoted by A^*, i.e.,

$$A^* v = \sum_{i,k=1}^{n} D_k(a_{i,k} D_i v) - \sum_{i=1}^{n} D_i(a_i v) + a_0 v.$$

The corresponding 'conormal derivatives' are

$$\frac{\partial u}{\partial \nu_A} = \sum_{i,k=1}^{n} a_{i,k} \nu^i D_k u, \qquad \frac{\partial v}{\partial \nu_{A^*}} = \sum_{i,k=1}^{n} a_{i,k} \nu^k D_i v.$$

Lemma 1.5.3.2 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary Γ. Then for every $u \in W_p^2(\Omega)$ and $v \in W_q^2(\Omega)$ with $1/p + 1/q = 1$, we have*

$$\int_\Omega Auv \, dx - \int_\Omega uA^* v \, dx$$
$$= \int_\Gamma \gamma \frac{\partial u}{\partial \nu_A} \gamma v \, d\sigma - \int_\Gamma \gamma u \gamma \frac{\partial v}{\partial \nu_{A^*}} \, d\sigma + \int_\Gamma \left(\sum_{i=1}^{n} \nu^i a_i \right) \gamma u \gamma u \, d\sigma. \quad (1.5.3.2)$$

When Ω is a plane bounded domain, whose boundary Γ is a $C^{1,1}$ curvilinear polygon, we can restate this lemma. Using the same notation as in the previous subsection, we define, for each j, a Lipschitz vector field $\boldsymbol{\nu}_j$ on $\bar{\Omega}$, such that $\boldsymbol{\nu}_j$ is the unit outward normal a.e. on Γ_j. Accordingly, we define several 'conormal derivatives'

$$\frac{\partial u}{\partial \nu_{A,j}} = \sum_{i,k=1}^{2} a_{i,k} \nu_j^i D_k u, \qquad \frac{\partial v}{\partial \nu_{A^*,j}} = \sum_{i,k=1}^{2} a_{i,k} \nu_j^k D_i v.$$

For $u \in W_p^2(\Omega)$, $v \in W_q^2(\Omega)$, we have

$$\frac{\partial u}{\partial \nu_{A,j}} \in W_p^1(\Omega) \qquad \text{and} \qquad \frac{\partial v}{\partial \nu_{A^*,j}} \in W_q^1(\Omega)$$

since $a_{i,k}$ and ν_j^i are all Lipschitz functions. Consequently, $\gamma_j(\partial u/\partial \nu_{A,j})$ and $\gamma_j(\partial v/\partial \nu_{A^*,j})$ are well defined and coincide a.e. on Γ_j with $\gamma \partial u/\partial \nu_A$ and $\gamma \partial v/\partial \nu_{A^*}$ respectively, as defined previously.

Lemma 1.5.3.3 *Let Ω be a bounded open subset of $\mathbb{R}^2$ whose boundary is a curvilinear polygon of class $C^{1,1}$. Then for every $u \in W_p^2(\Omega)$ and $v \in W_q^2(\Omega)$ with $1/p+1/q=1$, we have*

$$\int_\Omega Auv\,dx - \int_\Omega uA^*v\,dx$$
$$= \sum_{j=1}^{N} \left(\int_{\Gamma_j} \gamma_j \frac{\partial u}{\partial \nu_{A,j}} \gamma_j v\,d\sigma - \int_{\Gamma_j} \gamma_j u \gamma_j \frac{\partial v}{\partial \nu_{A^*,j}}\,d\sigma + \int_{\Gamma_j} \boldsymbol{\nu}_j \cdot \mathbf{a}\gamma_j u \gamma_j v\,d\sigma \right\}, \tag{1,5,3,3}$$

where $\mathbf{a}$ denotes the vector with components a_1 and a_2.

The first consequence of this Green formula concerns the domain of the maximal extension of the operator A in $L_p(\Omega)$, which we denote by $D(A; L_p(\Omega))$. In other words†

$$D(A; L_p(\Omega)) = \{u \in L_p(\Omega);\ Au \in L_p(\Omega)\}.$$

This is a Banach space for the norm

$$u \mapsto \{\|u\|_{0,p,\Omega}^p + \|Au\|_{0,p,\Omega}^p\}^{1/p}.$$

Furthermore $\mathcal{D}(\bar{\Omega})$ is dense in $D(A; L_p(\Omega))$ when Ω has a $C^{1,1}$ boundary. The same proof as in Lions and Magenes (1960–63) works, although they only deal with C^∞ boundaries. Then these authors show that the mapping

$$u \mapsto \left\{\gamma u;\ \gamma \frac{\partial u}{\partial \nu}\right\}$$

has a continuous extension as an operator from $D(A; L_p(\Omega))$ into $W_p^{-1/p}(\Gamma) \times W_p^{-1-1/p}(\Gamma)$. Again, their method of proof allows one to handle domains with a $C^{1,1}$ boundary. However, the similar result for a domain with a polygonal boundary deserves a proof.

Theorem 1.5.3.4 *Let Ω be a bounded open subset of $\mathbb{R}^2$, whose boundary is a curvilinear polygon of class $C^{1,1}$. Then the mapping*

$$u \mapsto \left\{\gamma_j u,\ \gamma_j \frac{\partial u}{\partial \nu_{A,j}}\right\}$$

which is defined for $u \in W_p^2(\Omega)$ has a unique continuous extension as an operator from $D(A; L_p(\Omega))$ into

$$W_p^{-1/p}(\Gamma_j) \times W_p^{-1-1/p}(\Gamma_j)$$

† We observe that for $u \in L_p(\Omega)$, we have $D_j u \in W_p^{-1}(\Omega)$ and consequently $a_{ij} D_j u$ is well defined and belongs to $W_p^{-1}(\Omega)$ too, since a_{ij} is Lipschitz continuous. Therefore, Au is well defined as an element of $W_p^{-2}(\Omega)$.

when $p \neq 2$ and into

$$H^{-(1/2)-\varepsilon}(\Gamma_j) \times H^{-(3/2)-\varepsilon}(\Gamma_j)$$

for all $\varepsilon > 0$ when $p = 2$.

Actually when $p = 2$ we shall prove that the trace mapping defined above, maps $D(A, L_2(\Omega))$ into the dual space of

$$\tilde{H}^{1/2}(\Gamma_j) \times \tilde{H}^{3/2}(\Gamma_j).$$

It will also be clear from the proof that it is enough to assume that A is nowhere characteristic on the boundary Γ of Ω.

Finally, a different approach (as in Goulaouic and Grisvard (1970)) allows one to show that the trace mapping in Theorem 1.5.3.4 is onto. However, this is useless for the purpose of the next chapters.

Proof For $u \in W_p^2(\Omega)$ and $v \in W_q^2(\Omega)$, it follows from (1,5,3,3) that

$$\left|\sum_{j=1}^{N} \left\{\int_{\Gamma_j} \gamma_j \frac{\partial u}{\partial \nu_{A,j}} \gamma_j v \, d\sigma - \int_{\Gamma_j} \gamma_j u \left(\gamma_j \frac{\partial v}{\partial \nu_{A^*,j}} - \boldsymbol{\nu}_j \cdot \mathbf{a} \gamma_j v\right) d\sigma\right\}\right| \leq K \|v\|_{2,q,\Omega},$$

where K is some constant depending on u. In particular, for a fixed j, we consider those functions v which belong to

$$V = \left\{v \in W_q^2(\Omega) \mid \gamma_k v = \gamma_k \frac{\partial v}{\partial \nu_k} = 0 \quad \text{on} \quad \Gamma_k \quad \text{for} \quad k \neq j\right\}.$$

For $v \in V$ we also have $\gamma_k(\partial v/\partial \nu_{A^*,k}) = 0$ on Γ_k for $k \neq j$ and consequently

$$\left|\int_{\Gamma_j} \gamma_j \frac{\partial u}{\partial \nu_{A,j}} \gamma_j v \, d\sigma - \int_{\Gamma_j} \gamma_j u \left(\gamma_j \frac{\partial v}{\partial \nu_{A^*,j}} - \boldsymbol{\nu}_j \cdot \mathbf{a} \gamma_j v\right) d\sigma\right| \leq K \|v\|_{2,q,\Omega}. \tag{1,5,3,4}$$

On the other hand, we know from Theorem 1.5.2.8 that the mapping

$$v \mapsto \left\{f_{j,0} = \gamma_j v;\ f_{j,1} = \gamma_j \frac{\partial v}{\partial \nu_j}\right\}$$

maps V onto the subspace of $W_q^{2-1/q}(\Gamma_j) \times W_q^{1-1/q}(\Gamma_j)$ defined by the following conditions, where $\boldsymbol{\tau}_j$ is the unit tangent vector to Γ_j (following the positive orientation with respect to Ω).

(a) $f_{j,0}(S_j) = f_{j,0}(S_{j-1}) = 0$ for all q

(b) $\dfrac{\partial}{\partial \tau_j} f_{j,0}(S_j) = \dfrac{\partial}{\partial \tau_j} f_{j,0}(S_{j-1}) = 0$ and

$f_{j,1}(S_j) = f_{j,1}(S_{j-1}) = 0$ for $q > 2$

(c) $$\int_0^{\delta_j}\left|\left(\frac{\partial}{\partial\tau_j}f_{j,0}\right)(x_j(-\sigma))\right|^2\frac{d\sigma}{\sigma}<+\infty,$$

$$\int_0^{\delta_{j-1}}\left|\left(\frac{\partial}{\partial\tau_j}f_{j,0}\right)(x_{j-1}(\sigma))\right|^2\frac{d\sigma}{\sigma}<+\infty,$$

$$\int_0^{\delta_j}|f_{j,1}(x_j(-\sigma))|^2\frac{d\sigma}{\sigma}<+\infty \text{ and}$$

$$\int_0^{\delta_{j-1}}|f_{j,1}(x_{j-1}(\sigma))|^2\frac{d\sigma}{\sigma}<+\infty$$

for $q=2$.

These conditions show that $f_{j,0}\in \mathring{W}_q^{2-1/q}(\Gamma_j)$ and $f_{j,1}\in \mathring{W}_q^{1-1/q}(\Gamma_j)$ through Corollary 1.5.1.6, when $q\neq 2$; however, when $q=2$, these conditions show that $f_{j,0}\in \tilde{H}^{3/2}(\Gamma_j)$ and $f_{j,1}\in \tilde{H}^{1/2}(\Gamma_j)$ through Corollary 1.4.4.10. In other words, $v\mapsto\{f_{j,0}; f_{j,1}\}$ maps V onto

$$\mathring{W}_q^{2-1/q}(\Gamma_j)\times \mathring{W}_q^{1-1/q}(\Gamma_j)$$

when $q\neq 2$ (and consequently $p\neq 2$) and onto

$$\tilde{H}^{3/2}(\Gamma_j)\times \tilde{H}^{1/2}(\Gamma_j)$$

when $q=p=2$.

Now, since A is non-characteristic on Γ_j, we have

$$\gamma_j\frac{\partial v}{\partial\nu_{A^*,j}}=\alpha\gamma_j\frac{\partial v}{\partial\nu_j}+\beta_1\gamma_j v+\beta_0\frac{\partial}{\partial\tau_j}\gamma_j v$$

where α is strictly positive on Γ_j. Conversely, we have

$$\gamma_j\frac{\partial v}{\partial\nu_j}=\frac{1}{\alpha}\gamma_j\frac{\partial v}{\partial\nu_{A^*,j}}-\frac{1}{\alpha}\left\{\beta_1\gamma_j v+\beta_0\frac{\partial}{\partial\tau_j}\gamma_j v\right\}.$$

It follows that

$$v\mapsto\left\{\gamma_j v;\ \gamma_j\frac{\partial v}{\partial\nu_{A^*,j}}-\boldsymbol{\nu}_j\cdot\mathbf{a}\gamma_j v\right\}$$

maps V onto

$$\mathring{W}_q^{2-1/q}(\Gamma_j)\times \mathring{W}_q^{1-1/q}(\Gamma_j)$$

when $q\neq 2$, and onto

$$\tilde{H}^{3/2}(\Gamma_j)\times \tilde{H}^{1/2}(\Gamma_j).$$

This result, together with inequality (1,5,3,4), shows that

$$\{\varphi,\psi\}\mapsto\int_{\Gamma_j}\gamma_j\frac{\partial u}{\partial\nu_{A,j}}\varphi\, d\sigma-\int_{\Gamma_j}\gamma_j u\psi\, d\sigma$$

is a continuous bilinear form on

$$\mathring{W}_q^{2-1/q}(\Gamma_j) \times \mathring{W}_q^{1-1/q}(\Gamma_j)$$

when $q \neq 2$, and on

$$\tilde{H}^{3/2}(\Gamma_j) \times \tilde{H}^{1/2}(\Gamma_j)$$

when $q = 2$. This defines $\gamma_j\ \partial u/\partial \nu_{A,j}$ as an element of $W_p^{-1-1/p}(\Gamma_j)$ and $\gamma_j u$ as an element of $W_p^{-1/p}(\Gamma_j)$ when $p \neq 2$; while this defines $\gamma_j\ \partial u/\partial \nu_{A,j}$ as an element of the dual space of $\tilde{H}^{3/2}(\Gamma_j)$ and $\gamma_j u$ as an element of the dual space of $\tilde{H}^{1/2}(\Gamma_j)$ when $p = 2$. This proves Theorem 1.5.3.4. ■

Remark 1.5.3.5 Actually, since A is non characteristic on Γ, $\gamma_j\ \partial u/\partial \nu_j$ is also defined as an element of $W_p^{-1-1/p}(\Gamma_j)$ (respectively the dual of $\tilde{H}^{3/2}(\Gamma_j)$) when $p \neq 2$ (respectively $p = 2$).

We shall also need a Green's formula, extending (1,5,3,3) to $u \in D(A, L_p(\Omega))$. When Ω is a bounded open subset of $\mathbb{R}^n$ with a C^∞ boundary Γ, it is shown that (1,5,3,2) has a natural extension. Indeed, Lions and Magenes (1960–63) prove that

$$\int_\Omega Au\, dx - \int_\Omega uA^*v\, dx = \left\langle \gamma \frac{\partial u}{\partial \nu_A}; \gamma v \right\rangle - \left\langle \gamma u; \gamma \left[\frac{\partial v}{\partial \nu_{A^*}} - \left(\sum_{i=1}^n \nu_i a_i \right) \gamma v \right] \right\rangle \tag{1,5,3,5}$$

for all $u \in D(A; L_p(\Omega))$ and $v \in W_q^2(\Omega)$. Here the brackets denote the duality pairing between $W_p^{-1-1/p}(\Gamma)$ and $W_q^{1+1/p}(\Gamma)$ for the first and between $W_p^{-1/p}(\Gamma)$ and $W_q^{1/p}(\Gamma)$ for the second. The same result holds with the same proof, if we only assume that Ω has a $C^{1,1}$ boundary.

Unfortunately, the analogue of (1,5,3,5) no longer holds, if we consider a bounded plane open set Ω whose boundary Γ is a curvilinear polygon which actually has corners. The reason is that, in general, for $u \in D(A, L_p(\Omega))$ and $v \in W_q^2(\Omega)$, the traces $\gamma_j\ \partial u/\partial \nu_{A,j}$ and $\gamma_j v$ are in the spaces $W_p^{-1-1/p}(\Gamma_j)$ and $W_q^{1+1/p}(\Gamma_j)$ respectively and these spaces are not in duality. (This is for $p \neq 2$; the situation is even worse for $p = 2$.) Consequently, we shall prove only the following statement.

Theorem 1.5.3.6 *Let Ω be a bounded open subset of $\mathbb{R}^2$ whose boundary is a curvilinear polygon of class $C^{1,1}$. Then we have*

$$\int_\Omega Auv\, dx - \int_\Omega uA^*v\, dx$$
$$= \sum_{j=1}^N \left\{ \left\langle \gamma_j \frac{\partial u}{\partial \nu_{A,j}}; \gamma_j v \right\rangle - \left\langle \gamma_j u; \gamma_j \left[\frac{\partial v}{\partial \nu_{A^*,j}} - \boldsymbol{\nu}_j \cdot \mathbf{a} v \right] \right\rangle \right\} \tag{1,5,3,6}$$

for $u \in D(A, L_p(\Omega))$ *and* $v \in W_q^2(\Omega)$, $1/p + 1/q = 1$, *such that*

(a) $v(S_j) = 0$, $j = 1, 2, \ldots, N$ *when* $p > 2$
(b) $v(S_j) = 0$, *and* $\nabla v(S_j) = 0$, $j = 1, 2, \ldots, N$ *when* $p < 2$
(c) $v = 0$ *in a neighbourhood of* S_j, $j = 1, 2, \ldots, N$ *when* $p = 2$.

Proof We already know from Lemma 1.5.3.2 that (1,5,3,6) holds for $u \in W_p^2(\Omega)$ and $v \in W_q^2(\Omega)$. We also know that $W_p^2(\Omega)$ is dense in $D(A; L_p(\Omega))$. So we just have to prove that the right-hand side of (1,5,3,6) is continuous in u for the norm of $D(A; L_p(\Omega))$ for those particular v specified in the statement of Theorem 1.5.3.6.

Now, because of Theorem 1.5.3.4, we just have to check that

$$\gamma_j v \in \mathring{W}_q^{1+1/p}(\Gamma_j), \qquad \gamma_j \frac{\partial v}{\partial \nu_{A^*,j}} \in \mathring{W}_q^{1/p}(\Gamma_j),$$

at least when $p \neq 2$. It follows from Theorem 1.5.2.1 that

$$\gamma_j v \in W_q^{1+1/p}(\Gamma_j) \quad \text{and} \quad \gamma_j \frac{\partial v}{\partial \nu_{A^*,j}} \in W_q^{1/p}(\Gamma_j).$$

Then from the extra hypotheses (a) and (b), we have

$$\gamma_j v(S_j) = v(S_j) = 0, \qquad \gamma_j v(S_{j-1}) = v(S_{j-1}) = 0$$

for all p and

$$\begin{cases} \gamma_j \dfrac{\partial v}{\partial \nu_{A^*,j}}(S_j) = \dfrac{\partial v}{\partial \nu_{A^*,j}}(S_j) = 0 \\ \gamma_j \dfrac{\partial v}{\partial \nu_{A^*,j}}(S_{j-1}) = \dfrac{\partial v}{\partial \nu_{A^*,j}}(S_{j-1}) = 0 \end{cases}$$

for $p < 2$. By Corollary 1.5.1.6 we therefore know that

$$\gamma_j v \in \mathring{W}_q^{1+1/p}(\Gamma_j) \quad \text{and} \quad \gamma_j \frac{\partial v}{\partial \nu_{A^*,j}} \in \mathring{W}_q^{1/p}(\Gamma_j),$$

and this is enough to prove our Theorem for $p \neq 2$.

In the particular case when $p = 2$, $\gamma_j v$ and $\gamma_j\, \partial v/\partial \nu_{A^*,j}$ have closed supports inside Γ_j. Consequently, it follows from Corollary 1.4.4.10 that

$$\gamma_j v \in \tilde{H}^{3/2}(\Gamma_j) \quad \text{and} \quad \gamma_j \frac{\partial v}{\partial \nu_{A^*,j}} \in \tilde{H}^{1/2}(\Gamma_j).$$

This shows that the right-hand side terms in (1,5,3,6) depend continuously on $u \in D(A, L_2(\Omega))$ (in view of the first remark, just after the statement of Theorem 1.5.3.4). ■

In dealing with variational solutions of some boundary value problems, we shall often need similar results concerning a 'half' Green formula. Indeed, the following is an easy consequence of Theorem 1.5.3.1. (We restrict ourselves to the Laplace operator for simplicity, since we shall only need this result in the coming sections.)

Lemma 1.5.3.7 *let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary Γ. Then for every $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, we have*

$$\int_\Omega (\Delta u)v \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Gamma \gamma \frac{\partial u}{\partial \nu} \gamma v \, d\sigma. \tag{1,5,3,7}$$

The corresponding statement on polygons is (according to the notation introduced previously in this subsection) the following:

Lemma 1.5.3.8 *Let Ω be a bounded open subset of $\mathbb{R}^2$ whose boundary is a curvilinear polygon of class $C^{1,1}$. Then for every $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, we have*

$$\int_\Omega (\Delta u)v \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \sum_{j=1}^N \int_{\Gamma_j} \gamma_j \frac{\partial u}{\partial \nu_j} \gamma_j v \, d\sigma. \tag{1,5,3,8}$$

Again this can be extended to more functions u. Let us set

$$E(\Delta; L_p(\Omega)) = \{u \in H^1(\Omega); \Delta u \in L_p(\Omega)\}.$$

This is a Banach space for the obvious norm

$$u \mapsto \|u\|_{1,2,\Omega} + \|\Delta u\|_{0,p,\Omega}.$$

As before, $\mathcal{D}(\bar{\Omega})$ is dense in $E(\Delta; L_p(\Omega))$ when Ω has a Lipschitz boundary, but this now requires a proof.

Lemma 1.5.3.9 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary; then $\mathcal{D}(\bar{\Omega})$ is dense in $E(\Delta; L_p(\Omega))$.*

Proof Let P be any continuation operator defined on $H^1(\Omega)$. In other words, P is a continuous linear mapping from $H^1(\Omega)$ to $H^1(\mathbb{R}^n)$ such that

$$Pu\,|_\Omega = u,$$

for every $u \in H^1(\Omega)$ (see Theorem 1.4.3.1). With the help of P we can view $H^1(\Omega)$ as a closed subspace of $H^1(\mathbb{R}^n)$. Thus for every continuous linear form l on $E(\Delta, L_p(\Omega))$ there exists $f \in H^{-1}(\mathbb{R}^n)$ and $g \in L_q(\Omega)$ such that

$$l(u) = \langle f; Pu \rangle + \int_\Omega g \, \Delta u \, dx$$

for all $u \in E(\Delta, L_p(\Omega))$. In addition, since l depends only on u and not on $Pu|_{C\Omega}$, the support of f is contained in $\bar{\Omega}$. (See also Theorem 2.3 in Magenes and Stampacchia (1958), Chapter 1.)

Now, in order to prove the claim of Lemma 1.5.3.9, we just need to show that any l which vanishes on $\mathcal{D}(\bar{\Omega})$ is actually zero everywhere. Indeed for $U \in \mathcal{D}(\mathbb{R}^n)$, we have

$$\langle f; U\rangle + \langle \tilde{g}, \Delta U\rangle = 0,$$

since we have

$$\langle f; U\rangle = \langle f; Pu\rangle$$

where $u = U|_{\Omega}$, due to the properties of the support of f. It follows that

$$\Delta \tilde{g} = -f$$

in the sense of distributions.

The ellipticity of Δ implies that $\tilde{g} \in H^1(\mathbb{R}^n)$ and consequently that $g \in \mathring{H}^1(\Omega)$. Let us now consider a sequence g_m, $m = 1, 2, \ldots$ of functions belonging to $\mathcal{D}(\Omega)$ and such that

$$g_m \longrightarrow g, \qquad m \longrightarrow +\infty$$

in $\mathring{H}^1(\Omega)$. For every $u \in E(\Delta, L_p(\Omega))$, we have

$$\begin{aligned} l(u) &= \lim_{m\to\infty} \left\{ -\langle \Delta \tilde{g}_m, Pu\rangle + \int_\Omega g_m \Delta u \, dx \right\} \\ &= \lim_{m\to\infty} \left\{ -\int_\Omega \Delta g_m u \, dx + \int_\Omega g_m \Delta u \, dx \right\} \\ &= 0. \end{aligned}$$

Thus l is identically zero. ■

The inclusion

$$E(\Delta; L_p(\Omega)) \subseteq D(\Delta; L_p(\Omega))$$

shows that $\gamma_j \, \partial u/\partial \nu_j$ is well defined and

$$\gamma_j \frac{\partial u}{\partial \nu_j} \in W_p^{-1-1/p}(\Gamma_j)$$

when $p \neq 2$ and

$$\gamma_j \frac{\partial u}{\partial \nu_j} \in \tilde{H}^{3/2}(\Gamma_j)^*$$

when $p = 2$. However, this result can be improved.

Theorem 1.5.3.10 *Let Ω be a bounded open subset of $\mathbb{R}^2$, whose boundary is a curvilinear polygon of class $C^{1,1}$. Then the mapping*

$$u \mapsto \gamma_j \frac{\partial u}{\partial \nu_j},$$

which is defined on $\mathcal{D}(\bar{\Omega})$, has a unique continuous extension as an operator from $E(\Delta; L_p(\Omega))$ into

$$\tilde{H}^{1/2}(\Gamma_j)^*.$$

Proof Consider $v \in V$, where

$$V = \{v \in H^1(\Omega) \mid \gamma_k v = 0, k \neq j\}.$$

Then $\gamma_j v \in \tilde{H}^{1/2}(\Gamma_j)$ (see Subsection 1.5.2). Furthermore, for $u \in \mathcal{D}(\bar{\Omega})$, we have by Lemma 1.5.3.8

$$\int_{\Gamma_j} \gamma_j \frac{\partial u}{\partial \nu_j} \gamma_j v \, d\sigma = \int_\Omega (\Delta u) v \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx.$$

It follows that

$$\left| \int_{\Gamma_j} \gamma_j \frac{\partial u}{\partial \nu_j} \gamma_j v \, d\sigma \right| \leqslant \|u\|_{E(\Delta; L_p(\Omega))} \|v\|_{1,2,\Omega}$$

and consequently there exists a constant C such that

$$\left\| \gamma_j \frac{\partial u}{\partial \nu_j} \right\|_{\tilde{H}^{1/2}(\Gamma_j)^*} \leqslant K \|u\|_{E(\Delta; L_p(\Omega))}.$$

The result follows by density. ∎

We can now extend the Green formula (1,5,3,7) to $u \in E(\Delta; L_p(\Omega))$.

Theorem 1.5.3.11 *Let Ω be a bounded open subset of $\mathbb{R}^2$, whose boundary is a curvilinear polygon of class $C^{1,1}$. Then we have*

$$\int_\Omega \Delta u v \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \sum_{j=1}^{N} \left\langle \gamma_j \frac{\partial u}{\partial \nu_j}; \gamma_j v \right\rangle \tag{1,5,3,9}$$

for $u \in E(\Delta; L_p(\Omega))$ and $v \in W_r^1(\Omega)$, $r > 2$ such that

$$v(S_j) = 0, \qquad 1 \leqslant j \leqslant N.$$

Proof The identity (1,5,3,9) holds by Lemma 1.5.3.8 for every $u \in \mathcal{D}(\bar{\Omega})$ and $v \in W_r^1(\Omega)$, $r > 2$ (since this last space is included in $H^1(\Omega)$). Then assuming that v is zero at the corners, implies that

$$\gamma_j v \in \mathring{W}_r^{1-1/r}(\Gamma_j) \subseteq \tilde{H}^{1/2}(\Gamma_j).$$

Consequently, all the terms involved in identity (1,5,3,9) are continuous in u for the norm of $E(\Delta; L_p(\Omega))$. Again, the result follows for density. ■

Finally, let us recall for later reference, the corresponding result on domains with smooth boundary, due to Lions and Magenes (1960–63). Here Ω is a bounded open subset of $\mathbb{R}^n$ with a $C^{1,1}$ boundary. Then $\mathscr{D}(\bar{\Omega})$ is dense in $E(\Delta; L_p(\Omega))$, the trace operator

$$u \mapsto \gamma \frac{\partial u}{\partial \nu}$$

is linear continuous from $E(\Omega; L_p(\Omega))$ to $H^{-1/2}(\Gamma)$ and the Green formula

$$\int_\Omega \Delta uv \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \left\langle \gamma \frac{\partial u}{\partial \nu}; \gamma v \right\rangle \tag{1,5,3,10}$$

holds for every $u \in E(\Omega; L_p(\Omega))$ and $v \in H^1(\Omega)$.

1.6 Boundary conditions

So far, we have studied the traces on the boundary of a function u, together with its derivatives in the direction of $\boldsymbol{\nu}$ up to a certain order. For the purpose of studying boundary value problems, it is convenient to replace the powers of $\partial/\partial\nu$ by a more general set of differential operators. This is the main goal of this section.

1.6.1 Normal systems

From now on, we consider a set of given differential operators

$$B_k(x, D_x) = \sum_{|\alpha| \le d_k} a_\alpha(x) D_x^\alpha, \qquad k = 1, \ldots, K$$

with C^∞ coefficients defined in $\bar{\Omega}$. For convenience, we assume that these operators are numbered according to the increasing orders of their degrees; in other words, we assume that $k \to d_k$ is a nondecreasing function of k.

Furthermore, we make the very restrictive assumption that the system $\{B_k\}_{k=1}^K$ is 'normal'. This means the following

Definition 1.6.1.1 *Let Ω be an open subset of $\mathbb{R}^n$ with a Lipschitz boundary Γ. The system $\{B_k\}_{k=1}^K$ is said to be normal on a subset Γ' of Γ if*

(a) *the degrees d_k are all different*

(b) *the B_k are all uniformly noncharacteristic on Γ: i.e. there exists m and M such that $0<m\leq M$ and*

$$m\leq\left|\sum_{|\alpha|=d_k} a_\alpha(x)\boldsymbol{\nu}^\alpha\right|\leq M$$

a.e. on Γ'. (As usual $\boldsymbol{\nu}^\alpha$ means $(\nu^1)^{\alpha_1}\cdots(\nu^n)^{\alpha_n}$.)

This definition agrees with the usual one given for a bounded Ω with C^∞ boundary. We observe that $k\to d_k$ is now a strictly increasing function.

We shall now investigate the mapping

$$u\mapsto\{\gamma B_k u\}_{k=1}^K$$

when u varies in some Sobolev space. We first quote the classical results of Lions and Magenes (1960–63).

Theorem 1.6.1.2 *Let $\{B_k\}_{k=1}^K$ be a system of homogeneous differential operators with constant coefficients in $\mathbb{R}^n$, which is normal on the hyperplane $x_n=0$. Then for $s-1/p$ non-integer and $>d_K$, the mapping*

$$u\mapsto\{\gamma_n B_k u\}_{k=1}^K$$

from $W_p^s(\mathbb{R}^n)$ into $\prod_{k=1}^K W_p^{s-d_k-1/p}(\mathbb{R}^{n-1})$ is onto.

This is a consequence of Theorem 1.5.1.1. The following is a consequence of Theorem 1.5.1.2.

Theorem 1.6.1.3 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a boundary of class $C^{l,1}$. Let also $\{B_k\}_{k=1}^K$ be a system of differential operators in Ω with coefficients belonging to $C^\infty(\bar\Omega)$, which is normal on the boundary Γ of Ω. Then for $s-1/p$ non-integer, $s-1/p>d_K$ and $s\leq l+1$, the mapping*

$$u\mapsto\{\gamma B_k u\}_{k=1}^K$$

from $W_p^s(\Omega)$ into $\prod_{k=1}^K W_p^{s-d_k-1/p}(\Gamma)$ is onto.

We now restrict our purpose to plane domains whose boundaries are curvilinear polygons of class C^∞, for simplicity. We also assume that $s=m$ is an integer. We use the same notation as in Section 1.5.2. With each of the curves Γ_j we consider a set of differential operators

$$B_{j,k}(x,D_x)=\sum_{|\alpha|\leq d_{j,k}} a_{j,\alpha}(x)D_x^\alpha,\qquad k=1,2,\ldots,K_j.$$

We assume that the coefficients $a_{j,\alpha}$ belong to $C^\infty(\bar\Omega)$ and that the set $\{B_{j,k}\}_{k=1}^K$ is normal on Γ_j for each $j=1,2,\ldots,N$. It follows from

Theorem 1.5.2.1 that for $u \in W_p^m(\Omega)$ we have

$$f_{j,k} = \gamma_j B_{j,k} u \in W_p^{m-d_{j,k}-1/p}(\Gamma_j).$$

Then, let us consider for each j, all the possible sets of differential operators $P_{j,k}(x, D_x)$, $k = 1, 2, \ldots, K_j$ and $Q_{j+1,k}(x, D_x)$, $k = 1, 2, \ldots, K_{j+1}$ such that $P_{j,k}$ is tangential to Γ_j for all k and $Q_{j+1,k}$ is tangential to Γ_{j+1} for all k. In addition, we assume that

$$\sum_{k=1}^{K_j} P_{j,k} B_{j,k} = \sum_{k=1}^{K_{j+1}} Q_{j+1,k} B_{j+1,k} \tag{1,6,1,1}$$

at S_j and that the degree of $P_{j,k}$ is $\leqslant d - d_{j,k}$ and the degree of $Q_{j+1,k}$ is $\leqslant d - d_{j+1,k}$. Consequently, the degree of the operator L in (1,6,1,1) is $\leqslant d$ and for $u \in W_p^m(\Omega)$ we have

$$Lu \in W_p^{m-d}(\Omega).$$

Then, Theorem 1.5.2.8 shows that

$$\sum_{k=1}^{K_j} (P_{j,k} f_{j,k})(S_j) = \sum_{k=1}^{K_{j+1}} (Q_{j+1,k} f_{j+1,k})(S_j) \tag{1,6,1,2}$$

when $d < m - 2/p$, while

$$\int_0^{\delta_j} \left| \sum_{k=1}^{K_j} (P_{j,k} f_{j,k})(x_j(-\sigma)) - \sum_{k=1}^{K_{j+1}} (Q_{j+1,k} f_{j+1,k})(x_j(\sigma)) \right|^2 \frac{d\sigma}{\sigma} < +\infty \tag{1,6,1,3}$$

when $d = m - 1$ and $p = 2$.

Conditions (1,6,1,2) are obviously necessary conditions on the traces $f_{j,k}$. It turns out that they are also sufficient conditions.

Theorem 1.6.1.4 *Let Ω be a bounded open subset of $\mathbb{R}^2$, whose boundary is a curvilinear polygon of class C^∞. Let also $\{B_{j,k}\}_{k=1}^{K_j}$ be, for each j, a system of differential operators in Ω, with coefficients belonging to $C^\infty(\bar{\Omega})$, which is normal on Γ_j. Then for $p \neq 2$, the mapping*

$$u \mapsto \{f_{j,k} = \gamma_j B_{j,k} u\}, \qquad j = 1, \ldots, N, \quad k = 1, \ldots, K_j,$$

maps $W_p^m(\Omega)$ onto the subspace of

$$T = \prod_{j=1}^{N} \prod_{k=1}^{K_j} W_p^{m-d_{j,k}-1/p}(\Gamma_j)$$

defined by the conditions (1,6,1,2) *for all possible systems of differential operators $\{P_{j,k}\}_{k=1}^{K_j}$ tangential to Γ_j and $\{Q_{j+1,k}\}_{k=1}^{K_{j+1}}$ tangential to Γ_{j+1}, such that identity* (1,6,1,1) *holds.*

Proof The conditions (1,6,1,2) have a local character. This allows one, through partition of the unity and changes of variables, to reduce the proof of the sufficiency of (1,6,1,2) to the case when Ω is replaced by $\mathbb{R}_+ \times \mathbb{R}_+$ and the functions $f_{j,k}$ have bounded supports. (Indeed, when the angle at S_j is less than π the problem is thus reduced to the case when Ω is replaced by $\mathbb{R}_+ \times \mathbb{R}_+$. On the other hand, when the angle at S_j is more than π, Ω is replaced by the complement of $\mathbb{R}_+ \times \mathbb{R}_+$. However, owing to the continuation theorem, it is equivalent to prove sufficiency in $\mathbb{R}_+ \times \mathbb{R}_+$.)

Since we consider a domain with only one corner, it will be convenient, throughout the proof, to adopt some slightly different notation. We replace $P_{j,k}$ by P_k (setting $K_j = K'$), $Q_{j+1,k}$ by Q_k (setting $K_{j+1} = K''$), $B_{j,k}$ by B_k (setting $d_{j,k} = m'_k$) and finally $B_{j+1,k}$ by C_k (setting $d_{j+1,k} = m''_k$). Thus, we start from

$$\varphi_k \in W_p^{m-m'_k-1/p}(\mathbb{R}_+), \qquad \psi_l \in W_p^{m-m''_l-1/p}(\mathbb{R}_+)$$

$1 \leq k \leq K'$ and $1 \leq l \leq K''$, such that

$$\sum_{k=1}^{K'} (P_k(D_y)\varphi_k)(0) = \sum_{l=1}^{K''} (Q_l(D_x)\psi_l)(0) \tag{1,6,1,4}$$

for all possible systems of operators P_k and Q_l such that

$$\sum_{k=1}^{K'} P_k(D_y)B_k(D_x, D_y) = \sum_{l=1}^{K''} Q_l(D_x)C_l(D_x, D_y) \tag{1,6,1,5}$$

and this sum is an operator of degree $d < m - 2/p$. With this data, we look for a function $u \in W_p^m(\mathbb{R}_+ \times \mathbb{R}_+)$ such that

$$\gamma_1 B_k u = \varphi_k \qquad \text{and} \qquad \gamma_2 C_l u = \psi_l. \tag{1,6,1,6}$$

Instead of building directly a function u, we shall only look for those functions f_k and g_l which, through the application of Theorem 1.5.2.4, allow us to find a function $u \in W_p^m(\mathbb{R}_+ \times \mathbb{R}_+)$, such that

$$\gamma_1 D_x^l u = g_l \qquad \text{and} \qquad \gamma_2 D_y^k u = f_k$$

for $k, l = 1, 2, \ldots, m-1$. In other words, we have to solve the following problem: Find

$$f_k \in W_p^{m-k-1/p}(\mathbb{R}_+), \quad g_l \in W_p^{m-l-1/p}(\mathbb{R}_+), \qquad k, l = 0, \ldots, m-1$$

such that $D_x^l f_k(0) = D_y^k g_l(0)$, $l + k < m - 2/p$ and that

$$\left.\begin{aligned} &\sum_{l=0}^{m'_k} B_{k,l}(D_y)g_l = \varphi_k, \qquad 1 \leq k \leq K' \\ &\sum_{k=0}^{m''_l} C_{l,k}(D_x)f_k = \psi_l, \qquad 1 \leq l \leq K'' \end{aligned}\right. \tag{1,6,1,7}$$

where the operators $B_{k,l}$ and $C_{l,k}$ are defined by

$$B_k(D_x, D_y) = \sum_{l=0}^{m'_k} B_{k,l}(D_y)D_x^l$$
$$C_l(D_x, D_y) = \sum_{k=0}^{m''_l} C_{l,k}(D_x)D_y^k.$$

Since we have assumed that the systems of operators $\{B_k\}_{k=1}^{K'}$ and $\{C_l\}_{l=1}^{K''}$ are normal on $x=0$ and $y=0$ respectively, B_{k,m'_k} and C_{l,m''_l} are just nonvanishing functions. Consequently (1,6,1,7) may be rewritten as

$$\begin{cases} g_{m'_k} = \dfrac{1}{B_{k,m'_k}}\left\{\varphi_k - \displaystyle\sum_{l=0}^{m'_k-1} B_{k,l}(D_y)g_l\right\} \\ f_{m''_l} = \dfrac{1}{C_{l,m''_l}}\left\{\psi_l - \displaystyle\sum_{k=0}^{m''_l-1} C_{l,k}(D_x)f_k\right\}. \end{cases} \tag{1,6,1,8}$$

In other words, we know $g_{m'_k}$ and $f_{m''_l}$ as soon as we know g_l for $l \leq m'_k - 1$ and f_k for $k \leq m''_l - 1$. However, these identities do not define all the g_l and the f_k since we never assumed that $m'_k = k-1$ and that $m''_l = l-1$.

As a first step in the construction of the missing g_l and f_k, we first look for the numbers

$$a_{k,l} = D_x^l f_k(0) = D_y^k g_l(0), \qquad k + l < m - \frac{2}{p}.$$

These numbers are solution of the linear system of equations which we obtain by differentiating the first equation in (1,6,1,7) with respect to y and the second with respect to x and then writing the corresponding equations at 0 when this makes sense. Namely, we have

$$\begin{aligned} &\sum_{l=0}^{m'_k} D_y^\lambda B_{k,l}(D_y)g_l = \varphi_k^{(\lambda)}, \qquad 1 \leq k \leq K', \quad \lambda \geq 0 \\ &\sum_{k=0}^{m''_l} D_x^\mu C_{l,k}(D_x)f_k = \psi_l^{(\mu)}, \qquad 1 \leq l \leq K'', \quad \mu \geq 0. \end{aligned} \tag{1,6,1,9}$$

We now adopt the following notation:

$$\begin{cases} D_y^\lambda B_{k,l}(D_y) = \displaystyle\sum_{\alpha=0}^{m'_k+\lambda-l} b_{k,l}^{\lambda,\alpha} D_y^\alpha \\ D_x^\mu C_{l,k}(D_x) = \displaystyle\sum_{\beta=0}^{m''_l+\mu-k} c_{l,k}^{\mu,\beta} D_x^\beta, \end{cases}$$

where $b_{k,l}^{\lambda,\alpha}$ and $c_{l,k}^{\lambda,\beta}$ are functions. Then at 0, (1,6,1,9) implies

$$\begin{cases} \sum_{l=0}^{m_k'} \sum_{\alpha=0}^{m_k'+\lambda-l} b_{k,l}^{\lambda,\alpha}(0)a_{\alpha,l} = \varphi_k^{(\lambda)}(0), & m_k'+\lambda < m - \dfrac{2}{p} \\ \sum_{k=0}^{m_l''} \sum_{\beta=0}^{m_l''+\mu-k} c_{l,k}^{\mu,\beta}(0)a_{k,\beta} = \psi_l^{(\mu)}(0), & m_l''+\mu < m - \dfrac{2}{p}. \end{cases} \tag{1,6,1,10}$$

This is the system of linear equations in $a_{k,l}$. This system is possibly overdetermined. We must therefore check that it has a solution.

The simplest way to prove that (1,6,1,10) has actually a solution is to prove that the data are in the kernel of the transposed matrix. In other words, the data have to annihilate all the linear forms which are zero after composition with the matrix of the system. We must therefore check that

$$\sum_{k,\lambda} p_{k,\lambda}\varphi_k^{(\lambda)}(0) = \sum_{l,\mu} q_{l,\mu}\psi_l^{(\mu)}(0) \tag{1,6,1,11}$$

for all possible numbers $p_{k,\lambda}$, $1 \leq k \leq K'$, $\lambda = 0, 1, \ldots, r - m_k'$ and $q_{l,\mu}$, $1 \leq l \leq K''$, $\mu = 0, 1, \ldots, r - m_l''$ (r is the integral part of $m - 2/p$), such that

$$\sum_{k,\lambda} p_{k,\lambda} b_{k,\hat{l}}^{\lambda,\hat{\alpha}}(0) = \sum_{l,\mu} q_{l,\mu} c_{l,\hat{k}}^{\mu,\hat{\beta}}(0) \tag{1,6,1,12}$$

for all possible values of $\hat{\alpha} = \hat{k}$ and $\hat{\beta} = \hat{l}$. For that purpose, let us consider the operators

$$P_k(D_y) = \sum_{\lambda=0}^{r-m_k'} p_{k,\lambda}(x, y)D_y^\lambda, \qquad Q_l(D_x) = \sum_{\mu=0}^{r-m_l''} q_{l,\mu}(x, y)D_x^\mu,$$

where the numbers $p_{k,\lambda} = p_{k,\lambda}(0, 0)$ and $q_{l,\mu} = q_{l,\mu}(0, 0)$ fulfil (1,6,1,12). We then have

$$\begin{aligned} \sum_{k=1}^{K'} P_k(D_y)B_k(D_x, D_y) &= \sum_{k=1}^{K'} \sum_{\lambda=0}^{r-m_k'} p_{k,\lambda}D_y^\lambda \sum_{l=0}^{m_k'} B_{k,l}(D_y)D_x^l \\ &= \sum_{k=1}^{K'} \sum_{\lambda=0}^{r-m_k'} \sum_{l=0}^{m_k'} \sum_{\alpha=0}^{m_k'+\lambda-l} p_{k,\lambda} b_{k,l}^{\lambda,\alpha} D_x^l D_y^\alpha. \end{aligned}$$

As a consequence of (1,6,1,12), this operator is also

$$\begin{aligned} \sum_{l=1}^{K''} \sum_{\mu=0}^{r-m_l''} \sum_{k=0}^{m_l''} \sum_{\beta=0}^{m_l''+\mu-k} q_{l,\mu} C_{l,k}^{\mu,\beta} D_x^\beta D_y^k &= \sum_{l=1}^{K''} \sum_{\mu=0}^{r-m_l''} q_{l,\mu} D_x^\mu \sum_{k=0}^{m_l''} C_{l,k}(D_x)D_y^k \\ &= \sum_{l=1}^{K''} Q_l(D_x)C_l(D_x, D_y). \end{aligned}$$

This is exactly identity (1,6,1,5). It follows that (1,6,1,4) holds and this can be rewritten as

$$\sum_{k,\mu} p_{k,\lambda}\varphi_k^{(\lambda)}(0)=\sum_{l,\mu} q_{l,\mu}\psi_l^{(\mu)}(0).$$

This proves identity (1,6,1,11). Summing up, we have shown that the linear system (1,6,1,10) has a solution.

From now on, let us consider any solution of (1,6,1,10). We must find

$$f_k\in W_p^{m-k-1/p}(\mathbb{R}_+),\qquad g_l\in W_p^{m-l-1/p}(\mathbb{R}_+) \tag{1,6,1,13}$$

such that

$$f_k^{(l)}(0)=a_{k,l},\qquad 0\leqslant l<m-k-\frac{2}{p} \tag{1,6,1,14}$$

$$g_l^{(k)}(0)=a_{k,l},\qquad 0\leqslant k<m-l-\frac{2}{p} \tag{1,6,1,15}$$

and that (1,6,1,8) holds. We obtain the functions f_k such that $k\neq m_l''$ ($l=1,2,\ldots,K''$) and the functions g_l such that $l\neq m_k'$ ($k=1,2,\ldots,K'$) from the one-dimensional version of Theorem 1.5.1.1. Indeed this theorem implies that the mapping

$$h\mapsto\{h(0),\ldots,h^{(k)}(0)\}$$

from $W_p^s(\mathbb{R})$ into $\mathbb{R}^{k+1}$, is onto, when $k<s-1/p$. We then obtain the functions $f_{m_l''}$ ($l=1,2,\ldots,K''$) and $g_{m_k'}$ ($k=1,2,\ldots,K'$) from (1,6,1,8).

The functions which have been thus constructed satisfy (1,6,1,13) and (1,6,1,8). They also satisfy (1.6.1.14) for $k\neq m_l''$ ($l=1,2,\ldots,K''$) and (1,6,1,15) for $l\neq m_k'$ ($k=1,2,\ldots,K'$). The last step of the proof consists in checking (1,6,1,14) and (1,6,1,15) for the remaining k and l. We do this by induction on m_l'' and m_k' separately.

Let us assume that we already know that

$$f_k^{(\mu)}(0)=a_{k,\mu}$$

for $k\leqslant m_l''-1$ ($0\leqslant\mu<m-k-2/p$) and for $k=m_l''$ but only for $0\leqslant\mu\leqslant\hat\mu-1$ (possibly with $\hat\mu=0$; this means no information about $f_{m_l''}$). We shall then show that

$$f_{m_l''}^{(\hat\mu)}(0)=a_{m_l'',\hat\mu}.$$

Indeed we have (1,6,1,8) which is equivalent to (1,6,1,7). Thus

$$\begin{aligned}\psi_l^{(\hat\mu)}&=D_x^{\hat\mu}\sum_{k=0}^{m_l''}C_{l,k}(D_x)f_k=\sum_{k=0}^{m_l''}\sum_{\beta=0}^{m_l''+\hat\mu-k}c_{l,k}^{\hat\mu,\beta}f_k^{(\beta)}\\&=c_{l,m_l''}^{\hat\mu,\hat\mu}f_{m_l''}^{(\hat\mu)}+\sum_{\beta=0}^{\hat\mu-1}c_{l,m_l''}^{\hat\mu,\beta}f_{m_l''}^{(\beta)}+\sum_{k=0}^{m_l''-1}\sum_{\beta=0}^{m_l''+\hat\mu-k}c_{l,k}^{\hat\mu,\beta}f_k^{(\beta)}.\end{aligned}$$

At zero this implies

$$c_{l,m_l''}^{\hat\mu,\hat\mu}(0)f_{m_l''}^{(\hat\mu)}(0)+\sum_{\beta=0}^{\hat\mu-1}c_{l,m_l''}^{\hat\mu,\beta}(0)a_{m_l'',\beta}+\sum_{k=0}^{m_l''-1}\sum_{\beta=0}^{m_l''+\hat\mu-k}c_{l,k}^{\hat\mu,\beta}(0)a_{k,\beta}=\psi_l^{(\hat\mu)}(0).$$

This is one of the equations of the system (1,6,1,10) with $a_{m_l'',\hat\mu}$ replaced by $f_{m_l''}^{(\hat\mu)}$. Fortunately $c_{l,m_l''}^{\hat\mu,\hat\mu}(0)$ is not zero because $c_{l,m_l''}^{\hat\mu,\hat\mu}$ is just the coefficient of $D_y^{m_l''}$ in $C_l(D_x, D_y)$ and the axis $\{y=0\}$ is not characteristic for this operator, by assumption. This shows that

$$f_{m_l''}^{(\hat\mu)}(0)=a_{m_l'',\hat\mu}.$$

Consequently we check (1,6,1,14) by induction. We do the same for (1,6,1,15). The proof of Theorem 1.6.1.4 is now complete. ■

In the case when $p=2$, which we have excluded so far, conditions (1,6,1,2) and (1,6,1,3) turn out to be also sufficient conditions on the traces $f_{j,k}$. However, this is really a result which is more easily proved by using interpolation methods. For this reason we shall only prove the sufficiency of (1,6,1,2) and (1,6,1,3) in some particular cases that we need in the forthcoming chapters.

Theorem 1.6.1.5 *Let Ω be a bounded open subset of $\mathbb{R}^2$, whose boundary is a polygon. Let also $\{B_{j,k}\}_{k=1}^{K_j}$ be for each j, a system of homogeneous linear differential operators, with constant coefficients, which is normal on Γ_j. Then the mapping*

$$u\mapsto\{f_{j,k}=\gamma_j B_{j,k}u\},\qquad j=1,\ldots,N,\quad k=1,2,\ldots,K_j$$

maps $H^m(\Omega)$ onto the subspace of

$$T=\prod_{j=1}^{N}\prod_{k=1}^{K_j}H^{m-d_{j,k}-1/2}(\Gamma_j)$$

defined by the conditions (1,6,1,2) *and* (1,6,1,3) *for all possible systems of homogeneous differential operators with constant coefficients $\{P_{j,k}\}_{k=1}^{K_j}$ tangential to Γ_j and $\{Q_{j+1,k}\}_{k=1}^{K_{j+1}}$ such that* (1,6,1,1) *holds.*

Proof The beginning of the proof is quite similar to that of Theorem 1.6.1.4. Using a partition of unity and affine changes of coordinates, we reduce the proof to the case when Ω is replaced by $\mathbb{R}_+\times\mathbb{R}_+$. After this reduction, we still deal with homogeneous operators with constant coefficients.

Then we adopt the same simplified notation as in the previous proof. Thus we are given

$$\varphi_k\in H^{m-m_k'-1/2}(\mathbb{R}_+),\qquad \psi_l\in H^{m-m_l''-1/2}(\mathbb{R}_+)$$

$1 \leqslant k \leqslant K'$, $1 \leqslant l \leqslant K''$ such that

$$\sum_{k=1}^{K'} (P_k(D_y)\varphi_k)(0) = \sum_{l=1}^{K''} (Q_l(D_x)\psi_l)(0) \tag{1,6,1,16}$$

when $d < m-1$, while

$$\int_0^1 \left| \sum_{k=1}^{K'} (P_k(D_y)\varphi_k)(t) - \sum_{l=1}^{K''} (Q_l(D_x)\psi_l)(t) \right|^2 \frac{dt}{t} < \infty \tag{1,6,1,17}$$

when $d = m-1$, for all possible systems of operators P_k and Q_l such that

$$\sum_{k=1}^{K'} P_k(D_y)B_k(D_x, D_y) = \sum_{l=1}^{K''} Q_l(D_x)C_l(D_x, D_y) \tag{1,6,1,18}$$

and this sum is an operator of degree $d \leqslant m-1$. We look for a function $u \in H^m(\mathbb{R}_+ \times \mathbb{R}_+)$ such that

$$\gamma_1 B_k u = \varphi_k \qquad \text{and} \qquad \gamma_2 C_l u = \psi_l. \tag{1,6,1,19}$$

Equivalently we look for functions f_k and g_l such that

$$f_k \in H^{m-k-1/2}(\mathbb{R}_+), \qquad g_l \in H^{m-l-1/2}(\mathbb{R}_+)$$

for k, $l = 1, 2, \ldots, m-1$, with

$$D_x^l f_k(0) = D_y^k g_l(0), \qquad l+k < m-1$$

$$\int_0^1 |D_x^l f_k(t) - D_y^k g_l(t)|^2 \frac{dt}{t} < +\infty, \qquad l+k = m-1$$

and such that

$$\begin{cases} \displaystyle\sum_{l=0}^{m'_k} b_{k,l} D_y^{m'_k-l} g_l = \varphi_k, & 1 \leqslant k \leqslant K' \\ \displaystyle\sum_{k=0}^{m''_l} c_{l,k} D_x^{m''_l-k} f_k = \psi_l, & 1 \leqslant l \leqslant K'' \end{cases} \tag{1,6,1,20}$$

where the numbers $b_{k,l}$ and $c_{l,k}$ are defined by

$$B_k(D_x, D_y) = \sum_{l=0}^{m'_k} b_{k,l} D_y^{m'_k-l} D_x^l$$

$$C_l(D_x, D_y) = \sum_{k=0}^{m''_l} c_{l,k} D_x^{m''_l-k} D_y^k.$$

The first step will be to find the numbers

$$a_{k,l} = D_x^l f_k(0) = D_y^k g_l(0), \qquad k+l < m-1$$

together with those functions $a_{k,l} \in H^{1/2}(\mathbb{R}_+)$ such that

$$\begin{cases} \displaystyle\int_0^1 |D_x^l f_k(t) - a_{k,l}(t)|^2 \frac{dt}{t} < +\infty \\ \displaystyle\int_0^1 |D_y^k g_l(t) - a_{k,l}(t)|^2 \frac{dt}{t} < +\infty \end{cases} \qquad k + l = m - 1.$$

The necessary conditions on the $a_{k,l}$ are obtained by differentiating (1,6,1,20) and then considering the behaviour of $\varphi_k^{(\lambda)}$ and $\psi_l^{(\mu)}$ near zero. Namely, we have

$$\begin{cases} \displaystyle\sum_{l=0}^{m_k'} b_{k,l} a_{m_k'-l+\lambda,l} = \varphi_k^{(\lambda)}(0), & m_k' + \lambda < m - 1 \\ \displaystyle\sum_{k=0}^{m_l''} c_{l,k} a_{k,m_l''-k+\mu} = \psi_l^{(\mu)}(0), & m_l'' + \mu < m - 1 \end{cases} \tag{1,6,1,21}$$

and in addition

$$\begin{cases} \displaystyle\int_0^1 \left| \sum_{l=0}^{m_k'} b_{k,l} a_{m-l-1,l}(t) - \varphi_k^{(m-1-m_k')}(t) \right|^2 \frac{dt}{t} < +\infty, & k = 1, \ldots, K' \\ \displaystyle\int_0^1 \left| \sum_{k=0}^{m_l''} c_{l,k} a_{k,m-k-1}(t) - \psi_l^{(m-1-m_l'')}(t) \right|^2 \frac{dt}{t} < +\infty, & l = 1, \ldots, K''. \end{cases} \tag{1,6,1,22}$$

Now the interest of the assumption that all the involved operators are homogeneous is that the systems (1,6,1,21) and (1,6,1,22) are not coupled. In other words, the unknowns in (1,6,1,21) are only the $a_{k,l}$ with $k + l < m - 1$, while the unknowns in (1,6,1,22) are only the $a_{k,l}$ with $k + l = m - 1$. This allows one to solve the two systems separately. The system (1,6,1,21) is the same as (1,6,1,10) and it has a solution since we assumed (1,6,1,16) which is identical with (1,6,1,4).

We are left with the problem of solving (1,6,1,22). We can consider the set of functions $\{a_{k,l}\}_{k+l=m-1}$ as a vector valued function $\mathbf{a}$ in $\mathbb{R}^m$ and consequently (1,6,1,22) can be rewritten as

$$\int_0^1 \|A\mathbf{a}(t) - \mathbf{b}(t)\|^2 \frac{dt}{t} < +\infty \tag{1,6,1,23}$$

where A is some matrix from $\mathbb{R}^m$ into $\mathbb{R}^N$ and $\mathbf{b}$ is some given function of class $H^{1/2}(\mathbb{R}_+)$ with values in $\mathbb{R}^N$. Here $N = K' + K''$ and $\mathbf{b}$ is the function whose components are the corresponding $\varphi_k^{(m-1-m_k')}$ and $\psi_l^{(m-1-m_l'')}$. We shall use the following auxiliary result

Lemma 1.6.1.6 *Let* $\mathbf{b} \in H^{1/2}(\mathbb{R}_+; \mathbb{R}^N)$; *then there exists* $\mathbf{a} \in H^{1/2}(\mathbb{R}_+; \mathbb{R}^m)$ *which is a solution of* (1,6,1,23) *if and only if* $\int_0^1 |\varphi(\mathbf{b}(t))|^2 \, dt/t < +\infty$ *for all linear forms* φ *on* $\mathbb{R}^N$ *such that* $\varphi \circ A = 0$.

Applying this to (1,6,1,22), we find a solution if and only if

$$\int_0^1 \left| \sum_{k=1}^{K'} p_k \varphi_k^{(m-1-m_k')}(t) - \sum_{l=1}^{K''} q_l \psi_l^{(m-1-m_l'')}(t) \right|^2 \frac{dt}{t} < +\infty \tag{1,6,1,24}$$

for all numbers p_k and q_l such that

$$\sum_{k=1}^{K'} p_k b_{k,\hat{l}} = \sum_{l=1}^{K''} q_l c_{l,\hat{k}} \tag{1,6,1,25}$$

for all $\hat{k}$ and $\hat{l}$ such that $\hat{k} + \hat{l} = m - 1$. Let us now introduce the operators

$$P_k(D_y) = p_k D_y^{m-1-m_k'}, \qquad Q_l(D_x) = q_l D_x^{m-1-m_l''};$$

then we have

$$\sum_{k=1}^{K'} P_k(D_y) B_k(D_x; D_y) = \sum_{k=1}^{K'} p_k D_y^{m-1-m_k'} \sum_{l=0}^{m_k'} b_{k,l} D_y^{m_k'-l} D_x^l$$
$$= \sum_{k=1}^{K'} \sum_{l=0}^{m_k'} p_k b_{k,l} D_y^{m-1-l} D_x^l.$$

As a consequence of (1,6,1,25), this operator is also

$$\sum_{l=1}^{K''} \sum_{k=0}^{m_l''} q_l c_{l,k} D_x^{m-1-k} D_y^k = \sum_{l=1}^{K''} q_l D_x^{m-1-m_l''} \sum_{k=0}^{m_l''} c_{k,l} D_x^{m_l''-k} D_y^k$$
$$= \sum_{l=1}^{K''} Q_l(D_y) C_l(D_x, D_y).$$

This is exactly identity (1,6,1,18), from which (1,6,1,17) follows. This last inequality is exactly inequality (1,6,1,24), which we wanted to check. Summing up, we have proved the existence of the numbers $a_{k,l}$ and the functions $a_{k,l}$ which are solutions of (1,6,1,21) and (1,6,1,22).

Now we are left with the problem of finding the functions f_k and g_l. We recall that (1,6,1,20) implies identities similar to (1,6,1,8), namely

$$\begin{cases} g_{m_k'} = \dfrac{1}{b_{k,m_k'}} \left\{ \varphi_k - \displaystyle\sum_{l=0}^{m_k'-1} b_{k,l} g_l^{(m_k'-l)} \right\}, & 1 \leq k \leq K' \\ f_{m_l''} = \dfrac{1}{c_{l,m_l''}} \left\{ \psi_l - \displaystyle\sum_{k=0}^{m_l''-1} c_{l,k} f_k^{(m_l''-k)} \right\}, & 1 \leq l \leq K'' \end{cases} \tag{1,6,1,26}$$

We obtain the functions f_k such that $k \neq m_l''$ for all l and the functions g_l such that $l \neq m_k'$ for all k by applying the following lemma.

Lemma 1.6.1.7 *Let $a_0, \ldots, a_{m-2} \in \mathbb{R}$ be given together with $a_{m-1} \in H^{1/2}(\mathbb{R}_+)$; then there exists $f \in H^{m-1/2}(\mathbb{R}_+)$ such that*

$$\begin{cases} f^{(k)}(0) = a_k, & 0 \leq k \leq m-2 \\ \displaystyle\int_0^1 |f^{(m-1)}(t) - a_{m-1}(t)|^2 \frac{dt}{t} < +\infty. \end{cases}$$

The remainder of the proof is similar to that of Theorem 1.6.1.4. Indeed, so far, we have built $f_k \in H^{m-k-1/2}(\mathbb{R}_+)$ and $g_l \in H^{m-l-1/2}(\mathbb{R}_+)$ such that (1,6,1,20) holds and such that

$$\begin{cases} f_k^{(l)}(0) = a_{k,l}, & 0 \leqslant l \leqslant m-k-1 \\ \displaystyle\int_0^1 |f_k^{(m-k-1)}(t) - a_{k,m-k-1}(t)|^2 \frac{\mathrm{d}t}{t} < +\infty \end{cases} \tag{1,6,1,27}$$

for $k \neq m_l''$ ($l = 1, 2, \ldots, K''$), and such that

$$\begin{cases} g_l^{(k)}(0) = a_{k,l}, & 0 \leqslant k \leqslant m-l-1 \\ \displaystyle\int_0^1 |g^{(m-l-1)}(t) - a_{m-l-1,l}(t)|^2 \frac{\mathrm{d}t}{t} < +\infty \end{cases} \tag{1,6,1,28}$$

for $l \neq m_k'$ ($k = 1, \ldots, K'$). We still have to check the similar property for the remaining indexes $k = m_l''$ and $l = m_k'$. Proving that $f_k^{(l)}(0) = a_{k,l} = g_l^{(k)}(0)$ for $k + l < m - 1$ is exactly the last step of the proof of Theorem 1.6.1.4. Consequently, let us only check that

$$\int_0^1 |f_{m_l''}^{(m-1-m_l'')}(t) - a_{m_l'',m-m_l''-1}(t)|^2 \frac{\mathrm{d}t}{t} < +\infty$$

for one $l = \hat{l}$ provided we already know that the property holds for $l \leqslant \hat{l} - 1$. Indeed we have

$$f_{m_l''}^{(m-m_l''-1)} = \frac{1}{c_{l,m_l''}} \left\{ \psi_l^{(m-m_l''-1)} - \sum_{k=0}^{m_l''-1} c_{l,k} f_k^{(m-k-1)} \right\}.$$

Near zero this implies that

$$\int_0^1 \left| c_{\hat{l},m_{\hat{l}}''} f_{m_{\hat{l}}''}^{(m-m_{\hat{l}}''-1)}(t) - \psi_{\hat{l}}^{(m-m_{\hat{l}}''-1)}(t) - \sum_{k=0}^{m_{\hat{l}}''-1} c_{\hat{l},k} a_{k,m-k-1}(t) \right|^2 \frac{\mathrm{d}t}{t} < +\infty$$

because of our induction hypothesis. On the other hand, it follows from (1,6,1,22) that

$$\int_0^1 \left| c_{\hat{l},m_{\hat{l}}''} a_{m_{\hat{l}}'',m-m_{\hat{l}}''-1}(t) - \psi_{\hat{l}}^{(m-m_{\hat{l}}''-1)}(t) - \sum_{k=0}^{m_{\hat{l}}''-1} c_{\hat{l},k} a_{k,m-k-1}(t) \right|^2 \frac{\mathrm{d}t}{t} < +\infty$$

and since $c_{\hat{l},m_{\hat{l}}''} \neq 0$, this shows that

$$\int_0^1 \left| f_{m_{\hat{l}}''}^{(m-m_{\hat{l}}''-1)}(t) - a_{m_{\hat{l}}'',m-m_{\hat{l}}''-1}(t) \right|^2 \frac{\mathrm{d}t}{t} < +\infty.$$

This is the desired result. Consequently (1,6,1,27) is proved by induction for $k = m_l''$ for all l; (1,6,1,28) is proved the same way.

The existence of u solving our trace problem follows from that of f_k and g_l, through the application of Theorem 1.5.2.4. This completes the proof of Theorem 1.6.1.5. ■

Proof of Lemma 1.6.1.6 We just write that

$$\mathbf{a} = M\mathbf{b},$$

where M is any matrix such that $AM = P$ and P is a projection operator on the image of A. Fredholm's alternative theorem implies that the image of A is the orthogonal of all linear forms φ such that $\varphi \circ A = {}^tA\varphi = 0$. ■

Proof of Lemma 1.6.1.7 We set

$$f(t) = \left\{\sum_{k=0}^{m-2} a_k \frac{t^k}{k!} + \int_0^t \frac{(t-s)^{m-2}}{(m-2)!} a_{m-1}(s)\, ds\right\}\xi(t),$$

where ξ is any smooth cut-off function which is zero for $t \geqslant \frac{2}{3}$ and identically equal to 1 for $t < \frac{1}{3}$. ■

Remark 1.6.1.8 Assume in Theorem 1.6.1.4 that Ω has a (strictly) polygonal boundary and the $B_{j,k}$ are homogeneous and have constant coefficients (as in Theorem 1.6.1.5); then the necessary and sufficient conditions (1,6,1,2) in Theorem 1.6.1.4 involve only operators $P_{j,k}$ and $Q_{j+1,k}$ homogeneous and with constant coefficients. This is easily checked by inspecting the proof of Theorem 1.6.1.4.

1.7 A model domain with a cut

Domains with cuts sometimes occur in practice (in fracture mechanics for instance). We shall not undertake here a comprehensive study of the properties of Sobolev spaces on such domains. We shall only illustrate, on the simplest possible example, the basic trick which reduces most of the proofs for domains with cuts to the more classical proofs of the previous sections. This relies on the trace theorems.

Our model domain is

$$\Omega = \{(x, y) \mid x^2 + y^2 < 1, x < 0 \quad \text{when} \quad y = 0\}.$$

In other words Ω is obtained by removing the right half x-axis from the unit disc. Such a domain does not fulfil the assumptions of any of the definitions in Section 1.2.

The space $W_p^m(\Omega)$ has been defined in Section 1.3, but no property has been obtained in Sections 1.4–1.6 for this space. However, the trace theorems in Section 1.5.2 imply many properties. The trick consists in splitting Ω into two pieces: let us denote by $\Omega_\pm$ the domains

$$\Omega_+ = \Omega \cap \{y > 0\}, \qquad \Omega_- = \Omega \cap \{y < 0\}.$$

Then $\Omega_\pm$ are plane open domains whose boundaries are curvilinear

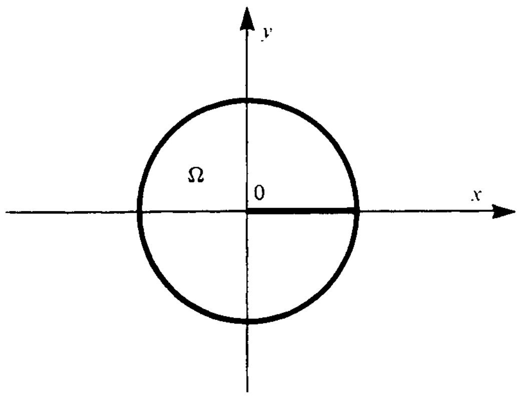

Figure 1.7

polygons of class C^∞. For $u \in W_p^m(\Omega)$ we set

$$u_+ = u\,|_{\Omega_+}, \qquad u_- = u\,|_{\Omega_-}.$$

It is obvious that $u_\pm \in W_p^m(\Omega_\pm)$ and consequently $u_\pm$ have well-defined traces, up to the order $m-1$, on $\{y=0\}$. To make things more precise we denote by $\gamma_\pm$ the trace operators from $W_p^1(\Omega_\pm)$ onto $W_p^{1-1/p}(]-1, +1[)$ which are defined by

$$(\gamma_\pm u)(x) = u(x, 0)$$

for $u \in \mathscr{D}(\bar{\Omega}_\pm)$ respectively (see Theorem 1.5.2.1). We can reconstruct u from u_+ and u_- by the following result.

Theorem 1.7.1 *Let u belong to $L_p(\Omega)$ and denote by $u_\pm$ its restrictions to $\Omega_\pm$ respectively; then $u \in W_p^m(\Omega)$ iff $u_\pm \in W_p^m(\Omega_\pm)$ and*

$$(\gamma_+ D_y^k u_+)(x) = (\gamma_- D_y^k u_-)(x), \qquad 0 \leqslant k \leqslant m-1, \tag{1,7,1}$$

for almost every $x \in]-1, 0[$.

Proof We prove the necessity of (1,7,1) by approximating u by $u_\nu \in \mathscr{D}(\bar{\Omega}_g)$, $\nu = 1, 2, \ldots$ in the norm of $W_p^m(\Omega_g)$, where

$$\Omega_g = \Omega \cap \{x < 0\}.$$

Since Ω_g has a continuous boundary we can apply Theorem 1.4.2.1. Now, on $]-1, 0[$, we have

$$(\gamma_+ D_y^k u_\nu)(x) = (\gamma_- D_y^k u_\nu)(x) = (D_y^k u_\nu)(x, 0).$$

By continuity the first identity is extended to u.

We prove the sufficiency as follows. We can approximate u_+ by $u_+^\nu \in \mathcal{D}(\bar{\Omega}_+)$ and u_- by $u_-^\nu \in \mathcal{D}(\bar{\Omega}_-)$, $\nu = 1, 2, \ldots$ in the norm of $W_p^m(\Omega_+)$ and $W_p^m(\Omega_-)$ respectively. We define a distribution u^ν by setting

$$\langle u^\nu; \varphi\rangle = \int_{\Omega_+} u_+^\nu \varphi \, dx\, dy + \int_{\Omega_-} u_-^\nu \varphi \, dx\, dy$$

for all test functions $\varphi \in \mathcal{D}(\Omega)$. It is a classical result that

$$\begin{aligned}\langle D_x^k D_y^l u^\nu; \varphi\rangle = &\int_{\Omega_+} D_x^k D_y^l u_+^\nu \varphi \, dx\, dy + \int_{\Omega_-} D_x^k D_y^l u_-^\nu \varphi \, dx\, dy \\ &+ \sum_{j=0}^{l-1} (-1)^j \int_{-1}^{0} [(D_x^k \gamma_+ D_y^{l-j-1} u_+^\nu)(x, 0) \\ &\qquad - (D_x^k \gamma_- D_y^{l-j-1} u_-^\nu)(x, 0)] D_y^j \varphi(x, 0)\, dx.\end{aligned}$$

By continuity, we obtain

$$\begin{aligned}\langle D_x^k D_y^l u; \varphi\rangle = &\int_{\Omega_+} D_x^k D_y^l u_+ \varphi \, dx\, dy + \int_{\Omega_-} D_x^k D_y^l u_- \varphi \, dx\, dy \\ &+ \sum_{j=0}^{l-1} (-1)^j \int_{-1}^{0} [(D_x^k \gamma_+ D_y^{l-j-1} u_+)(x, 0) \\ &\qquad - (D_x^k \gamma_- D_y^{l-j-1} u_-)(x, 0)] D_y^j \varphi(x, 0)\, dx\end{aligned}$$

provided $k + l \leq m - 1$. This, together with (1,7,1), proves that

$$\langle D_x^k D_y^l u; \varphi\rangle = \int_{\Omega_+} D_x^k D_y^l u_+ \varphi \, dx\, dy + \int_{\Omega_-} D_x^k D_y^l u_- \varphi \, dx\, dy \qquad (1,7,2)$$

for $k + l \leq m - 1$. Consequently $D_x^k D_y^l u$ is a function and belongs to $L_p(\Omega)$. This completes the proof. ■

We shall now draw some conclusions from Theorem 1.7.1. First, in general, there is no reason why $\gamma_+ u_+$ and $\gamma_- u_-$ should coincide on $]0, 1[$. This implies that $C^\infty(\bar{\Omega})$ is not dense in $W_p^m(\Omega)$. Indeed $\bar{\Omega}$ is the closed unit disc. For all $u \in C^\infty(\bar{\Omega})$ we have $\gamma_+ u_+ = \gamma_- u_-$ on $]-1, +1[$; by continuity this identity is also valid for all u belonging to the closure of $C^\infty(\bar{\Omega})$ in $W_p^m(\Omega)$. It is also obvious that the norms of $W_p^m(\Omega)$ and of $W_p^m(\Omega_1)$, where Ω_1 is the open unit disc, coincide on $W_p^m(\Omega_1)$. Consequently the closure of $C^\infty(\bar{\Omega})$ in $W_p^m(\Omega)$ is the space of the restrictions to Ω of all the functions in $W_p^m(\Omega_1)$. Since $W_p^m(\bar{\Omega}_1) = W_p^m(\Omega_1)$, this shows that $W_p^m(\bar{\Omega}) \neq W_p^m(\Omega)$; in other words $W_p^m(\Omega)$ has no extension property similar to that in Theorem 1.4.3.1.

In order to obtain some convenient density and imbedding results we must introduce some other spaces.

Definition 1.7.2 *We denote by $C^{k,\alpha}(\tilde{\Omega})$ the space of all functions u*

defined in Ω, which are uniformly continuous together with their derivatives up to the order k and such that their derivatives of order k satisfy a uniform Hölder condition with exponent α, in Ω.

It is easily seen that $u \in C^{k,\alpha}(\tilde{\Omega})$ iff

$$u_\pm \in C^{k,\alpha}(\Omega_\pm)$$

and

$$D_y^l u_+(x, 0) = D_y^l u_-(x, 0), \qquad -1 < x < 0$$

for all l such that $0 \leqslant l \leqslant k$.

An easy consequence of Theorems 1.7.1 and 1.4.5.2 is that

$$W_p^s(\Omega) \subseteq W_q^t(\Omega) \tag{1.7.3}$$

provided $s - 2/p \geqslant t - 2/q$, $t \leqslant s$ and

$$W_p^s(\Omega) \subseteq C^{k,\alpha}(\tilde{\Omega}) \tag{1.7.4}$$

provided $k + \alpha \leqslant s - 2/p$ and $s - 2/p$ is not an integer.

The main consequence of Theorem 1.7.1 is a trace theorem for the space $W_p^m(\Omega)$. We shall state it and derive it carefully since it is fundamental for studying boundary value problems in a domain like Ω.

For this purpose, besides the trace operators $\gamma_\pm$ already defined, we introduce the trace operator γ_c on the unit circle. We consider the subdomains Ω_ε defined by

$$\Omega_\varepsilon = \{(r \cos \theta; r \sin \theta) \mid 0 < r < 1, \varepsilon < \theta < 2\pi - \varepsilon\}$$

for $\varepsilon > 0$. Clearly $\Omega = \Omega_0 = \bigcup_{\varepsilon > 0} \Omega_\varepsilon$ and Ω_ε has a Lipschitz boundary. We denote by C_ε the interior of the intersection of the unit circle with Γ_ε the boundary of Ω_ε. The function

$$\gamma u \,|_{C_\varepsilon}$$

is well defined by Theorem 1.5.2.1 applied to Ω_ε. In addition it belongs to $W_p^{1-1/p}(C_\varepsilon)$ when u belongs to $W_p^1(\Omega)$. We define a function $\gamma_c u$ a.e. on C_0 by setting

$$\gamma_C u \,|_{C_\varepsilon} = \gamma u \,|_{C_\varepsilon}$$

for every $\varepsilon > 0$. (This definition may seem artificial, but it saves the proof of a density theorem which is not easy.)

From now on we set

$$\gamma_C \frac{\partial^l u}{\partial r^l} = g_l, \qquad l = 0, 1, \ldots, m-1 \tag{1.7.5}$$

and

$$\gamma_\pm D_y^l u = f_{\pm,l}, \qquad l = 0, 1, \ldots, m-1$$

for $u \in W_p^m(\Omega)$. We consider g_l as a function of $\theta \in]0, 2\pi[$, and $f_{\pm,l}$ as a function of $x \in]0, 1[$.

Theorem 1.7.3 *The mapping*

$$u \to \{g_l, f_{+,l}, f_{-,l}\}_{l=0}^{m-1}$$

is linear continuous from $W_p^m(\Omega)$ onto the subspace of

$$\prod_{l=0}^{m-1} W_p^{m-l-1/p}(]0, 2\pi[) \times W_p^{m-l-1/p}(]0, 1[) \times W_p^{m-l-1/p}(]0, 1[)$$

defined by the following conditions

(a) $g_l^{(k)}(0) = f_{+,k}^{(l)}(1), \quad g_l^{(k)}(2\pi) = f_{-,k}^{(l)}(1), \quad f_{+,k}^{(l)}(0) = f_{-,k}^{(l)}(0)$

for $k + 1 < m - 2/p$

(b) $\displaystyle\int_0^1 |g_l^{(k)}(t) - f_{+,k}^{(l)}(1-t)|^2 \frac{dt}{t} < +\infty,$

$$\int_0^1 |g_l^{(k)}(2\pi - t) - f_{-,k}^{(l)}(1-t)|^2 \frac{dt}{t} < +\infty,$$

$$\int_0^1 |f_{+,k}^{(l)}(t) - f_{-,k}^{(l)}(t)|^2 \frac{dt}{t} < +\infty$$

for $k + l = m - 1$, when $p = 2$.

Proof The necessity of the compatibility conditions between the g_l and the $f_{+,k}$ follows from Theorem 1.5.2.8 on Ω_+. In the same way Theorem 1.5.2.8 on Ω_- implies the compatibility conditions between the g_l and the $f_{-,k}$.

On the other hand the functions $D_y^l u_\pm$ have traces $\gamma_\pm D_y^l u_\pm$ which belong to

$$W_p^{m-l-1/p}(]-1, +1[)$$

and which coincide for $x \in]-1, 0[$ by Definition 1.7.1. Consequently, we have

$$(f_{+,l} - f_{-,l}) \in \tilde{W}_p^{m-l-1/p}(]0, 1[).$$

The compatibility conditions between $f_{+,l}$ and $f_{-,l}$ follow from Theorem 1.5.1.5 for $k + l < m - 2/p$ and from Lemma 1.3.2.6 for $k + l = m - 1$ (and $p = 2$).

In order to prove that the above conditions are sufficient, we start from given functions g_l and $f_{\pm,k}$ fulfilling those conditions. Then instead of looking for $u \in W_p^m(\Omega)$, having such traces, we first look for functions F_k, $k=0,1,\ldots,m-1$, the traces of u on the segment $\{(x,y); -1<x<0\}$. We claim that there exists

$$F_k \in W_p^{m-k-1/p}(]-1,0[)$$

such that

$$F_k^{(l)}(0)=f_{+,k}^{(l)}(0) \tag{1,7,6}$$

for $k+l<m-2/p$,

$$\int_0^1 |F_k^l(-t)-f_{+,k}^l(t)|^2 \frac{dt}{t}<+\infty \tag{1,7,7}$$

for $k+l=m-1$ $(p=2)$ and

$$F_k^{(l)}(-1)=(-1)^{k+l}g_l^{(k)}(\pi) \tag{1,7,8}$$

for $k+l<m-2/p$

$$\int_0^1 |F_k^{(l)}(-1+t)-(-1)^{k+l}g_l^{(k)}(\pi-t)|^2 \frac{dt}{t}<+\infty \tag{1,7,9}$$

for $k+l=m-1$ $(p=2)$.

The construction of such functions F_k is easy when $p\neq 2$, while their existence follows from Lemma 1.6.1.7 when $p=2$. Then we set

$$F_{\pm,k}(x)=\begin{cases} F_k(x) & \text{for } -1<x<0 \\ f_{\pm,k}(x) & \text{for } 0<x<1. \end{cases}$$

Clearly $F_{\pm,k}\in W_p^{m-k-1/p}(]-1,+1[)$ and applying Theorem 1.5.2.8 to Ω_+ and Ω_- we check that there exists

$$u_\pm \in W_p^m(\Omega_\pm)$$

such that

$$\gamma_\pm D_y^k u_\pm = F_{\pm,k}, \qquad 0\leq k\leq m-1$$

and that

$$\gamma_c \frac{\partial^l u_\pm}{\partial r^l}=g_l, \qquad 0\leq l\leq m-1$$

on $]0,\pi[$ and $]\pi,2\pi[$ respectively.

Since we have obviously

$$\gamma_+ D_y^k u_+ = \gamma_- D_y^k u_- \qquad \text{on} \quad]-1,0[$$

for $0\leq k\leq m-1$, it follows that the function on Ω built up from u_+ and u_- belongs to $W_p^m(\Omega)$. In addition it has the required traces. ■

Remark 1.7.4 It is easy to combine the results of Theorems 1.7.3 and 1.5.2.8 to obtain the description of the traces for more general domains with cuts. It turns out that the statement of Theorem 1.5.2.8 remains valid if we admit domains with cuts provided we consider both sides of the cut as two different sides of Ω. In the same way the Theorems 1.6.1.4 and 1.6.1.5 remain valid for domains with cuts. Also Theorem 1.4.5.3 holds for such domains.