On the Navier-Stokes Initial Value Problem. I

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Dedicated to CHARLES LOEWNER on the occasion of his 70^{th} birthday

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1. Introduction and Summary

The present paper is concerned with the initial value problem for the non-stationary Navier-Stokes equation. The initial contribution to the mathematical study of this problem is contained in a series of remarkable papers of J. LERAY, which appeared about thirty years ago. Recently a number of authors have again become interested in this problem and have obtained some new results with the aid of various methods from modern functional analysis. For instance, the existence of unique and global (in time) solutions for 2-dimensional flows as well as the existence of unique and local (in time) solutions for 3-dimensional flows has been established. However, in most of these works the word "solution" has been interpreted in a more or less generalized sense[1]. The purpose of the present paper is to deduce such existence theorems for 3-dimensional flows through a Hilbert space approach, making use of the theory of fractional powers of operators and the theory of semi-groups of operators[2]. We shall reduce the initial value problem for the Navier-Stokes equations to an abstract initial value problem for an operator differential equation in a Hilbert space. After establishing the existence and uniqueness of solutions of this abstract initial value problem, we shall prove their regularity as functions of space-time variables, which will lead us to an existence theorem for the classical solutions.

Recently the writers published a paper [*10*] on this subject, in which we dealt exclusively with the existence and the uniqueness of the solutions of the abstract equation. This paper will be referred to as K-F. The basic idea in the present paper for constructing the solutions is the same as that of K-F. Some of the theorems given below are the same as those in K-F, although the proofs are independent and more elementary. As a rule, proofs in the present paper will be presented in more detail than is done in K-F.

With a view to clarifying the essential features of the method, we shall restrict our consideration to some simple cases. The flow is considered in a bounded domain D in R^3 with boundary ∂D of class C^3, and the boundary condition imposed on the velocity at ∂D is homogeneous. Extension of the method to more general cases, in particular, to the case of an unbounded domain, entails no essential difficulty, as will be explained in the last section.

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[1] In § 7 we refer to some existence theorems for classical solutions obtained by S. ITO and others.

[2] In this connection our method is quite close to that of SOBOLEVSKII [*23, 24*].

Let us now introduce some basic notions required for presenting a summary of the present paper. Without loss of generality we assume that the viscosity and the density of the fluid are both equal to 1. Then the initial value problem is written in its classical form as

$$\frac{\partial u}{\partial t}=\Delta u-\nabla p-(u\cdot\nabla)u+f, \qquad x\in D, t>0, \tag{1.1}$$

$$\nabla\cdot u\equiv \operatorname{div} u=0, \qquad x\in D, t>0, \tag{1.2}$$

$$u|_{\partial D}=0, \qquad t>0, \tag{1.3}$$

$$u|_{t=0}=a, \qquad x\in D, \tag{1.4}$$

with the usual notations. Here $u=u(t,x)=\big(u_1(t,x), u_2(t,x), u_3(t,x)\big)$ is the velocity field, $p=p(t,x)$ is the pressure, $a=a(x)$ is the initial velocity, $f=f(t,x)$ is the external force. By Pr. I we mean the initial value problem for these four equations with the unknown u, p and the given f, a.

We transform Pr. I into an abstract initial value problem. Let $\mathscr{H}$ be the Hilbert space of real vector functions in $L_2(D)$ with the inner product $(\,,\,)$ defined in the usual way. The set of real vector functions φ such that $\operatorname{div}\varphi=0$ and $\varphi\in C_0^\infty(D)$ is denoted by $C_{0,\sigma}^\infty(D)$. Let $\mathscr{H}_\sigma$ be the closure of $C_{0,\sigma}^\infty(D)$ in $\mathscr{H}$. If u is smooth in $\bar{D}$, $u\in\mathscr{H}_\sigma$ implies and is implied by

$$\operatorname{div} u=0 \quad\text{in}\quad D \quad\text{and}\quad u_n=0 \quad\text{on}\quad \partial D,$$

u_n being the normal component of u. In fact, if $\mathscr{M}_\pi$ is the subspace of $\mathscr{H}$ composed of all vector functions which are expressible as ∇h with scalar functions $h\in C^\infty(D)$, and if $\mathscr{H}_\pi$ denotes the closure of $\mathscr{M}_\pi$ in $\mathscr{H}$, then we have[1]

$$\mathscr{H}_\sigma=\mathscr{H}\ominus\mathscr{H}_\pi.$$

Let P be the orthogonal projection from $\mathscr{H}$ onto $\mathscr{H}_\sigma$. By A we denote the self-adjoint operator in $\mathscr{H}_\sigma$ formally given by $A=-P\Delta$. More precisely, A is the Friedrichs extension of the symmetric operator $-P\Delta$ in $\mathscr{H}_\sigma$ defined for every u such that $u\in C^2(D)\cap C^1(\bar{D})$, $\operatorname{div} u=0$ and $u|_{\partial D}=0$. Equivalently the relation $Au=w$ $(u\in\mathscr{D}(A)\subset H_{0,\sigma}^1(D),\ w\in\mathscr{H}_\sigma)$, is true if and only if

$$(\nabla u, \nabla v)=(w, v) \quad\text{for all}\quad v\in H_{0,\sigma}^1(D), \tag{1.5}$$

where $H_{0,\sigma}^1$ is the completion of the set $C_{0,\sigma}^1(D)$ of all solenoidal ($\operatorname{div} u=0$) functions in C_0^1 with the norm $|||u|||=(\|\nabla u\|^2+\|u\|^2)^{\frac12}$. From the definition it follows that $\mathscr{D}(A^{\frac12})=H_{0,\sigma}^1$ and

$$\|A^{\frac12}u\|=\|\nabla u\|. \tag{1.5)$'$}$$

We note that the class $H_{0,\sigma}^1$ of the test functions v in (1.5) can be equivalently replaced by $C_{0,\sigma}^1(D)$.

[1] The equality $\mathscr{H}_\sigma=\mathscr{H}\ominus\mathscr{H}_\pi$ can be shown as follows. $\mathscr{H}_\sigma\subset\mathscr{H}\ominus\mathscr{H}_\pi$ is obvious. Let w be such that $w\in\mathscr{H}\ominus\mathscr{H}_\pi$ and $w\perp\mathscr{H}_\sigma$. Then we have $(w,\nabla h)=0$ and $(w, \operatorname{rot}\varphi)=0$ for any $h\in C_0^\infty(D)$ and any $\varphi\in C_0^\infty(D)$. In other words, w satisfies $\operatorname{div} w=0$ and $\operatorname{rot} w=0$ weakly. From this we have $w\in C^\infty(D)$ by Weyl's lemma. The facts that $w\perp C_{0,\sigma}^\infty(D)$, $w\in C^\infty(D)$ and $w\in\mathscr{H}$ imply $w\in\mathscr{M}_\pi$ [5]. Hence $w=0$, and $\mathscr{H}_\sigma=\mathscr{H}\ominus\mathscr{H}_\pi$.

By applying P to (1.1) and taking account of the other equations, we are led to the following abstract initial value problem, Pr. II;

$$\frac{du}{dt} = -Au + Fu + Pf(t), \qquad t>0, \tag{1.6}$$

$$u(+0) = a, \tag{1.7}$$

where

$$Fu = -P(u \cdot \nabla)\, u. \tag{1.8}$$

Here $t \to u(t)$ and $t \to f(t)$ are regarded as functions on the real numbers to $\mathscr{H}_\sigma$ and $\mathscr{H}$ respectively. du/dt is the derivative of u in the strong topology of $\mathscr{H}_\sigma$. *In what follows we assume for simplicity that Pf is continuous for $t>0$.* Since we shall later show that $(u \cdot \nabla)\, u \in \mathscr{H}$ whenever $u \in \mathscr{D}(A)$, Fu is meaningful for a solution u, of which we now give a precise definition.

Definition 1.1. A function $u=u(t)$ is called a solution of (1.6) in an open interval I if du/dt and Au exist and are continuous in I and if (1.6) is satisfied there. If u is a solution of (1.6) in $(0, T)$ and also satisfies (1.7), then u is called a solution of the initial value problem Pr. II in $[0, T)$.

We introduce a class of functions, within which the solutions of Pr. II are unique.

Definition 1.2. Let I be a closed interval $[T_1, T_2]$ or a semiclosed interval $[T_1, T_2)$. By $\tilde{S}(I)$ we denote the set of all functions u, $(t \to u(t) \in \mathscr{H}_\sigma,\ t \in I)$, such that u is continuous on I, $A^{\frac{1}{2}} u$ is continuous in $I - \{T_1\}$ and

$$\|A^{\frac{1}{2}} u(t+T_1)\| = o(t^{-\frac{1}{4}}) \qquad (t \to +0).$$

Now our existence and uniqueness theorems read:

Theorem 1.1. *If $a \in \mathscr{H}_\sigma$ and $\|Pf\|$ is integrable on $[0, T]$, then the solution of* Pr. II *is unique within $\tilde{S}[0, T)$.*

Theorem 1.2. *In* Pr. II *assume that*

i) $a \in \mathscr{D}(A^{\frac{1}{4}})$,

ii) *Pf is Hölder continuous (strongly in $\mathscr{H}_\sigma$) for $t>0$ and*

$$\|Pf(t)\| = o(t^{-\frac{3}{4}}) \ \text{as} \ t \to +0.$$

Then there exists a solution $u \in \tilde{S}[0, T)$ of Pr. II, *where T is a positive number depending on a and Pf.*

Theorem 1.3. *In addition to the assumptions of Theorem* 1.2, *suppose that*

$$\|A^{\frac{1}{4}} a\| + B_1 M_\infty < \frac{1}{4 c_1 B_1}, \tag{1.9}$$

where

$$M_\infty = \sup_{0<t<\infty} t^{\frac{3}{4}} \|Pf(t)\|, \qquad B_1 = B(\tfrac{1}{2}, \tfrac{1}{4}),$$

$B(\,,\,)$ being the beta function, and where c_1 is an absolute constant to be introduced below in Lemma 1.1. *Then, there exists a solution $u \in \tilde{S}[0, \infty)$ of* Pr. II.

Theorem 1.4. *Let α_0, μ_0 be any positive numbers such that*

$$(1.10)\qquad 0<\alpha_0<\frac{1}{6c_1B_1},\qquad 0<108c'^3\mu_0\leqq\alpha_0,$$

where c_1, B_1 are as above and $c'=\|A^{-\frac{1}{4}}\|$. If a and Pf satisfy the conditions

$$\|A^{\frac{1}{2}}a\|\leqq\alpha_0,\qquad \|Pf(t)\|\leqq\mu_0\qquad (t\in(0,\infty)),$$

then there exists a solution $u\in\tilde{S}[0,\infty)$ of Pr. II.

The basic idea for constructing the solution is to reduce Pr. II to the following abstract integral equation which we designate as Pr. III, and then to proceed by iteration:

$$(1.11)\qquad u(t)=e^{-tA}a+\int_0^t e^{-(t-s)A}Fu(s)\,ds+\int_0^t e^{-(t-s)A}Pf(s)\,ds.$$

Unfortunately Fu does not make sense in general for $u\in\tilde{S}$. Therefore, in dealing with $u\in\tilde{S}$ we rewrite (1.11) as

$$(1.12)\qquad u(t)=e^{-tA}a+\int_0^t A^{\frac{1}{4}}e^{-(t-s)A}Hu(s)\,ds+\int_0^t e^{-(t-s)A}Pf(s)\,ds$$

with

$$(1.13)\qquad Hu=-A^{-\frac{1}{4}}P(u\cdot\nabla)u.$$

We note that Hu is well-defined whenever $u\in\mathscr{D}(A^{\frac{1}{2}})$, since we have the following lemma due to SOBOLEVSKII [*23*].

Lemma 1.1. There exists an absolute constant c_1 such that the inequality

$$(1.14)\qquad \|A^{-\frac{1}{4}}P(u\cdot\nabla)v\|\leqq c_1\|A^{\frac{1}{2}}u\|\cdot\|A^{\frac{1}{2}}v\|$$

holds for any u, $v\in C^1_{0,\sigma}(D)$.

Evidently this lemma allows us to define Hu for every $u\in\mathscr{D}(A^{\frac{1}{2}})$ and to obtain the following lemma.

Lemma 1.1′. If u, v belong to $\mathscr{D}(A^{\frac{1}{2}})=H^1_{0,\sigma}$, then the inequalities

$$(1.15)\qquad \|Hu\|\leqq c_1\|A^{\frac{1}{2}}u\|^2$$

and

$$(1.16)\qquad \|Hu-Hv\|\leqq c_1\|A^{\frac{1}{2}}(u-v)\|\left(\|A^{\frac{1}{2}}u\|+\|A^{\frac{1}{2}}v\|\right)$$

hold with the constant c_1 in the preceding lemma.

After obtaining a solution u of (1.12), by the iteration procedure we proceed to prove the differentiability of $u=u(t)$. This is accomplished, eventually, under the assumption that Pf is Hölder continuous. However, the direct proof of this fact seems troublesome, and we prefer to take a detour. Namely, we start again with constructing solutions of Pr. III within another class for which (1.11) is meaningful as it is, and the differentiability is easier to prove. The desired differentiability of the solutions in $\tilde{S}$ follows from that of the solutions of this class, since any solution $u\in\tilde{S}[0,T]$ is shown to belong to this class on the restricted interval $[\varepsilon,T]$ $(\varepsilon>0)$. The definition of this class is the following:

Definition 1.3 Let

$$(1.17)\qquad \tfrac{1}{4}<\beta<\tfrac{1}{2},\qquad \tfrac{3}{4}<\gamma<1,$$

and let I be as in Definition 1.2. We denote by $S_{\beta,\gamma}(I)$ the set of all u $(t \to u(t) \in \mathscr{H}_\sigma,\ t \in I)$ such that u is continuous on I, $A^\gamma u$ and $A^{\frac{1}{2}} u$ are continuous in $I - \{T_1\}$ and such that $(t-T_1)^{\gamma-\beta} \|A^\gamma u(t)\|$ and $(t-T_1)^{\frac{1}{2}-\beta} \|A^{\frac{1}{2}} u(t)\|$ are bounded as $t \to +0$.

We note that $\tilde{S}_{\beta,\gamma}(I) \subset S(I)$ and also that if $u \in S_{\beta,\gamma}(I)$, then Fu is well-defined in I according to the following lemma, which plays an important role throughout the present paper.

Lemma 1.2. Let γ be any real number larger than $\frac{3}{4}$. Then there exists a constant c_2, depending on γ and D, such that the following inequalities hold for any $u, v \in \mathscr{D}(A^\gamma)$:

(1.18)[1] $$\|u\|_\infty \leqq c_2 \|A^\gamma u\|,$$

(1.19) $$\|Fu\| \leqq c_2 \|A^{\frac{1}{2}} u\| \cdot \|A^\gamma u\|$$

and

(1.20) $$\|Fu - Fv\| \leqq c_2 \left(\|A^\gamma u\| \cdot \|A^{\frac{1}{2}} w\| + \|A^{\frac{1}{2}} v\| \cdot \|A^\gamma w\|\right),$$

where $w = u - v$.

As was noted in K-F, (1.19) and (1.20) are true even for $\gamma = \frac{3}{4}$, and this makes some of the proofs conceptually simpler. This result will not be employed in the present paper, however, for its derivation is not very elementary, being based on Lions' theory of interpolation of spaces and a theorem concerning $\mathscr{D}(A)$ due to Cattabriga *[1]* and to Ladyzhenskaia *[13]*.

The proofs of Lemmas 1.1 and 1.2 will be given in § 2 among other lemmas concerning the operator A. The existence and uniqueness of the solutions of Pr. III in $\tilde{S}$ will be established in § 3. The proof of the differentiability of these solutions and the proof of Theorems 1.1—1.4 will be completed at the end of § 4, which will be mainly devoted to consideration of the solution of Pr. III in $S_{\beta,\gamma}$.

We shall study the regularity of the solution of Pr. II in the remainder of the paper. In § 5 the regularity of the solution $u = u(t)$ of (1.6) considered as a $\mathscr{H}_\sigma$-valued function will be derived. From this regularity we can immediately obtain the regularity of u and $\nabla u \equiv \nabla_x u$ as functions of t and x, which is valid up to the boundary with respect to x. We state here some of the results:

Theorem 1.5. *Let u be a solution of* (1.6) *in an open interval I, where Pf is Hölder continuous in I as a $\mathscr{H}_\sigma$-valued function with the exponent ϑ $(0 < \vartheta < 1)$. Then the following propositions are true.*

i) *If $0 < \vartheta < 1$, $u = u(t, x)$ is Hölder continuous in $I \times \bar{D}$.*

ii) *If $\frac{1}{2} < \vartheta < 1$ and f is Hölder continuous as a $L_q(D)$-valued function for some $q > 3$, then $\nabla u = \nabla_x u(t, x)$ is Hölder continuous in $I \times \bar{D}$.*

Theorem 1.6. *Let u be as in the foregoing theorem. If f is of the class C^∞ as a $L_q(D)$-valued function for some $q > 3$, then u and ∇u are of the class C^∞ as $L_2(D)$-valued functions. Furthermore, $\partial^n u(t, x)/\partial t^n$ and $\partial^n \nabla u(t, x)/\partial t^n$ exist in the classical sense and are Hölder continuous in $I \times \bar{D}$ $(n = 0, 1, 2, \ldots)$.*

[1] For a vector function u defined in D, $|u(x)|$ means the Euclidean length of the vector $u(x)$, and $\|u\|_p \equiv \|u\|_{p,D}$ means
$$\|u\|_p = \Big(\int_D |u(x)|^p\, dx\Big)^{1/p}, \qquad 1 \leqq p \leqq \infty,$$
whenever the right side exists.

In consideration of SERRIN's remark [22] on the regularity of weak solutions of the Navier-Stokes equation, the C^∞-character of the solutions of (1.6) with respect to t seems to be quite interesting. § 6 will be devoted to the interior regularity of the solutions of (1.6). One of the main results is the following.

Theorem 1.7. *Let u be a solution of* (1.6) *in an open interval I, where Pf is Hölder continuous as a $\mathscr{H}_\sigma$-valued function. Assume, furthermore, that $f=f(t,x)$ is Hölder continuous in $I\times D$. Then $u(t,x)$, $\nabla u(t,x)$, $\nabla\nabla u(t,x)$, and $\partial u(t,x)/\partial t$ are Hölder continuous in $I\times D$.*

We shall conclude the present paper with some remarks given in § 7. There we shall refer to the pressure p associated with a solution u of Pr. II. After establishing the existence and the regularity of p, we shall point out that the solution u of Pr. II, together with p, satisfies all the equations of Pr. I if the given data are appropriately smooth. In this way we establish the existence of the classical solutions.

2. Some Properties of the Operator A

This section is composed of three parts I), II), and III). In Part I) we describe known properties of the Green function of the Stokes problem due to ODQVIST [19]. Part II) is devoted to fractional powers of the operator A. The final part contains several well-known lemmas concerning e^{-tA}, which we include for the sake of completeness. For convenience of reference, the lemmas in II) and III) are stated at the beginning of each part in advance of the proofs.

I. Odqvist's Green Function of the Stokes Problem

In the Stokes problem with the homogeneous boundary condition, one is required to find a vector function $v=v(x)$ and a scalar function $q=q(x)$ such that the Stokes system

$$\Delta v-\nabla q=-f,\qquad \nabla\cdot v=0\qquad (x\in D), \tag{2.1}$$

and the boundary condition

$$v|_{\partial D}=0 \tag{2.2}$$

are satisfied, where $f(x)$ is a given function. According to ODQVIST[1], the unique solution $\{v,q\}$ of this problem is given by

$$v(x)=\int_D G(x,y)\,f(y)\,dy\quad\text{and}\quad q(x)=\int_D g(x,y)\,f(y)\,dy. \tag{2.3}$$

Here $G(x,y)$ and $g(x,y)$ are a matrix function and a vector function defined in $D\times D$, respectively. The couple $\{G,g\}$, or sometimes G alone, is called the Green function of the Stokes problem. G satisfies the relations

$$G_{ij}(x,y)=G_{ji}(y,x), \tag{2.4}$$

$$\sum_i \frac{\partial}{\partial x_i}G_{ij}(x,y)=0\quad\text{and}\quad \sum_j \frac{\partial}{\partial y_j}G_{ij}(x,y)=0, \tag{2.5}$$

where $i,j=1,2,3$. Except for $x=y$, G and g are analytic in $D\times D$. G, $\nabla_x G$ and g are Hölder continuous in $\bar D\times\bar D$ except for $x=y$. $G(\xi,y)=0$ holds whenever

[1] ODQVIST's study has been retraced by LADYZHENSKAIA [13] with some improvements.

$\xi \in \partial D$ and $y \in D$. ODQVIST gave the following estimates regarding the behavior of G and g as $|x-y| \to 0$.

Lemma 2.1. Let $k(x, y)$ be any one of $G_{ij}(x, y)$, and let $K(x, y)$ be any one of $\partial G_{ij}(x, y)/\partial x_m$ and $g_i(x, y)$ $(i, j, m = 1, 2, 3)$. Then,

i) there exists a constant C depending on D such that

$$(2.6) \qquad |k(x, y)| \leqq C\,|x-y|^{-1} \quad \text{and} \quad |K(x, y)| \leqq C\,|x-y|^{-2}$$

hold for any $x, y \in \bar{D}$.

ii) If ϑ and ε are arbitrary constants such that $0 < \vartheta < \vartheta + \varepsilon < 1$, then there exists a constant C depending on ϑ, ε and D such that

$$(2.7) \qquad |K(x_1, y) - K(x_2, y)| \leqq C d^{\vartheta} r^{-(2+\vartheta+\varepsilon)}$$

holds for every $x_1, x_2, y \in \bar{D}$, where $d = |x_1 - x_2|$ and $r = \min\{|x_1 - y|, |x_2 - y|\}$.

The estimate (2.7) can be obtained from

$$|K(x_1, y) - K(x_2, y)| \leqq C\{d\,|\log d|\, r^{-3} + d\,|\log d|^2\, r^{-2} + d^{\vartheta} r^{-1}\},$$

due to ODQVIST, with the aid of the obvious inequality

$$(2.8) \qquad |a-b| \leqq |a-b|^{\vartheta}\,(|a| + |b|)^{1-\vartheta} \qquad (0 < \vartheta < 1).$$

Now we introduce some notations to be used in most of this paper.

Notations for integral operators. Suppose that $H(x, y)$ is a function defined in $D_1 \times D_2$; it may be a scalar, vector, or matrix function. The formal integral operator with the kernel $H(x, y)$ which maps functions defined in D_2 into functions defined in D_1 is denoted by $\boldsymbol{H}$, while $\boldsymbol{H}^*$ means the formal adjoint of $\boldsymbol{H}$. We treat other letters similarly.

Notations for positive constants. Sometimes we use one and the same symbol C without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say λ, we write C_λ.

Let us return to consideration of the integral operator $\boldsymbol{G}$. In view of (2.4) and (2.6) $\boldsymbol{G}$ is a bounded and symmetric operator in $\mathscr{H} = L_2(D)$. The inclusion $\boldsymbol{G}\mathscr{H} \subset \mathscr{H}_\sigma$ follows from (2.5). Therefore the restriction $\boldsymbol{G}_\sigma$ of $\boldsymbol{G}$ to $\mathscr{H}_\sigma$ is a bounded symmetric operator in $\mathscr{H}_\sigma$. We note that $\boldsymbol{G}_\sigma P h = \boldsymbol{G} h$ holds for any $h \in \mathscr{H}$. In what follows we shall simply write $\boldsymbol{G}$ in place of $\boldsymbol{G}_\sigma$ if no confusion arises.

II. Lemmas on A^α

Here we establish some lemmas on A^α relevant to our use, including Lemmas 1.1 and 1.2 stated above. The proofs of these two lemmas are given after the proofs of Lemmas 2.7 and 2.5, respectively. Three of the lemmas involve the self-adjoint operator B in $\mathscr{H}$, which is the Friedrichs extension of $-\Delta$ defined for functions in $C_0^\infty(D)$. Similarly to the case of A, we notice the following. If H_0^1 means the completion of $C_0^\infty(D)$ with the norm $|||u||| = (\|\nabla u\|^2 + \|u\|^2)^{\frac{1}{2}}$, then we have $\mathscr{D}(B^{\frac{1}{2}}) = H_0^1$ and also

$$(2.9) \qquad \|\nabla u\| = \|B^{\frac{1}{2}} u\| \qquad (u \in \mathscr{D}(B^{\frac{1}{2}}) = H_0^1).$$

Lemma 2.2. A is strictly positive. In other words there exists a positive number δ such that

$$(2.10) \qquad (Au, u) \geqq \delta \|u\|^2 \quad (u \in \mathscr{D}(A)).$$

Corollary. $A^{-\alpha}$ is bounded for any $\alpha \geqq 0$.

Corollary. If λ is a nonnegative real number, then

$$(2.11) \qquad \|(A+\lambda)^{-1}\| \leqq (\delta+\lambda)^{-1}.$$

Lemma 2.3. $A^{-1} = G$, where G is considered an operator in $\mathscr{H}_\sigma$.

Lemma 2.4. If $u \in \mathscr{D}(A)$, then the inequality

$$(2.12) \qquad |u(x_1) - u(x_2)| \leqq C |x_1 - x_2|^{\frac{1}{2}} \|Au\|$$

holds for any $x_1, x_2 \in \bar{D}$, where C is a constant depending on D. Furthermore, we have $u(\xi) = 0$ for any $\xi \in \partial D$.

Remark. Strictly speaking, we mean by Lemma 2.4 that the element $u \in \mathscr{D}(A)$ of $\mathscr{H}$ is equivalent to a continuous function $u(x)$ satisfying (2.12) and $u(\xi) = 0$. In what follows we shall not state remarks of this sort explicitly.

Lemma 2.5. Let $v = v(x)$ satisfy

$$(2.13) \qquad |v(x_1) - v(x_2)| \leqq K d^{\vartheta} \quad (x_1, x_2 \in \bar{D}, \quad d = |x_1 - x_2|),$$

with some constants K and ϑ $(0 < \vartheta < 1)$. If v satisfies

$$(2.14) \qquad v|_{\partial D} = 0$$

in addition, then we have the inequality

$$(2.15) \qquad \|v\|_\infty^{2\vartheta+3} \leqq C K^3 \|v\|^{2\vartheta}$$

with a constant C depending on ϑ.

Lemma 2.6. Let γ and ϑ be constants such that

$$(2.16) \qquad \tfrac{3}{4} < \gamma < 1 \quad \text{and} \quad 0 < \vartheta < 2(\gamma - \tfrac{3}{4}).$$

If $u \in \mathscr{D}(A^\gamma)$, then

$$(2.17) \qquad |u(x_1) - u(x_2)| \leqq C |x_1 - x_2|^{\vartheta} \|A^\gamma u\|$$

for any $x_1, x_2 \in \bar{D}$, where C is a constant independent of u and x_j.

Lemma 2.7. $B^{-\alpha}$ $(0 < \alpha < 1)$ is an integral operator with the kernel $G_\alpha(x, y)$ satisfying

$$(2.18) \qquad |G_\alpha(x, y)| \leqq C_\alpha |x - y|^{-3+2\alpha},$$

where C_α is a constant depending on α.

Lemma 2.8. There exists an absolute constant c_0 such that

$$(2.19) \qquad \|u\|_6 \leqq c_0 \|\nabla u\| = c_0 \|A^{\frac{1}{2}} u\| \quad (u \in \mathscr{D}(A^{\frac{1}{2}}))$$

and

$$(2.20) \qquad \|u\|_6 \leqq c_0 \|\nabla u\| = c_0 \|B^{\frac{1}{2}} u\| \quad (u \in \mathscr{D}(B^{\frac{1}{2}})).$$

Lemma 2.9. Taking $\mathscr{D}(B^{\frac{1}{2}}) = \mathring{H}_0^1$ as its domain, define the operator $\mathring{S}_k$ by

$$(2.21) \qquad \mathring{S}_k u = A^{-\frac{1}{2}} P \partial_k u \quad (k = 1, 2, 3; \ \partial_k = \partial/\partial x_k).$$

Then the operator $\mathring{S}_k$ admits of the bounded extension S_k with $\|S_k\| \leqq 1$ as an operator from $\mathscr{H}$ into $\mathscr{H}_\sigma$.

Proof of Lemma 2.2. As is well known, $\|\nabla v\| \geqq \delta_1 \|v\|$ holds for every $v \in H^1_{0,\sigma} \subset H^1_0$ with a positive domain constant δ_1, whence follows (2.10) with $\delta = \delta_1^2$. Q.E.D.

Proof of Lemma 2.3. $C^\infty_{0,\sigma}$ is dense in $\mathscr{H}_\sigma$. A^{-1} and $\boldsymbol{G}$ are both bounded. Thus it suffices to show $A^{-1}\varphi = \boldsymbol{G}\varphi$ for $\varphi \in C^\infty_{0,\sigma}$. By means of the relation $\Delta \boldsymbol{G}\varphi - \nabla \boldsymbol{g}\varphi = -\varphi$, we can verify that $(\nabla \boldsymbol{G}\varphi, \nabla v) = -(\Delta \boldsymbol{G}\varphi, v) = (\varphi - \nabla \boldsymbol{g}\varphi, v) = (\varphi, v)$ for any $v \in C^1_{0,\sigma}(D)$, which implies $A\boldsymbol{G}\varphi = \varphi$ according to (1.5). Q.E.D.

Proof of Lemma 2.4. Putting $H(x_1, x_2, y) = G(x_1, y) - G(x_2, y)$, we have

$$u(x_1) - u(x_2) = \int_D H(x_1, x_2, y)\, A u(y)\, dy.$$

After applying Schwarz's inequality, we reduce the proof of the lemma to that of the inequality

$$\int_D |H(x_1, x_2, y)|^2\, dy \leqq C\, |x_1 - x_2|,$$

which can be easily verified with the aid of (2.6) by an argument familiar in potential theory. Q.E.D.

Proof of Lemma 2.5. (2.15) is trivial if $v \equiv 0$. Taking the case of $\|v\|_\infty \equiv L > 0$, we may assume $L = |v(0)|$ without loss of generality. Let σ be the open ball $\{x; |x| < R = (L/K)^{1/\vartheta}\}$. Then it follows from (2.13) that

$$|v(x)| \geqq |v(0)| - |v(x) - v(0)| \geqq L - K|x|^\vartheta > L - KR^\vartheta = 0$$

for any $x \in \sigma \cap \bar{D}$. By virtue of (2.14) this implies that $\sigma \subset D$ and also $|v(x)| \geqq L - K|x|^\vartheta > 0$ $(x \in \sigma)$. Therefore,

$$\|v\|^2 \geqq \int_\sigma (L - K|x|^\vartheta)^2\, dx = 4\pi L^2 R^3 \int_0^1 (1 - \eta^\vartheta)^2 \eta^2\, d\eta = C_\vartheta L^2 R^3.$$

From this we immediately obtain (2.15). Q.E.D.

Proof of Lemma 1.2. Obviously it is enough to deal with the case of $\frac{3}{4} < \gamma < 1$. Putting $A^\gamma u = v$, we reduce (1.18) to

$$\|A^{-\gamma} v\|_\infty \leqq c_2 \|v\| \qquad (v \in \mathscr{H}_\sigma). \tag{2.22}$$

In order to estimate $\|A^{-\gamma} v\|_\infty$ we resort to the formula [8]

$$A^{-\gamma} v = \frac{\sin \pi\gamma}{\pi} \int_0^\infty \lambda^{-\gamma} (A + \lambda)^{-1} v\, d\lambda. \tag{2.23}$$

Set $w = (A + \lambda)^{-1} v$, $\lambda \geqq 0$. Then we have

$$\|w\| \leqq (\delta + \lambda)^{-1} \|v\| \quad \text{and} \quad \|A w\| \leqq \|v\| \tag{2.24}$$

by (2.11). On the other hand, we have

$$\|w\|_\infty \leqq C \|A w\|^{\frac{3}{4}} \|w\|^{\frac{1}{4}}$$

by Lemmas 2.4 and 2.5. Hence we get

$$\|w\|_\infty \leqq C (\delta + \lambda)^{-\frac{1}{4}} \|v\|. \tag{2.25}$$

Substituting (2.25) into (2.23) and noting that

$$\int_0^\infty \lambda^{-\gamma}(\delta+\lambda)^{-\frac{1}{4}}\,d\lambda<+\infty \qquad (\tfrac{3}{4}<\gamma<1),$$

we arrive at (2.22). Q.E.D.

Proof of Lemma 2.6. We introduce the notation

$$\mu_\alpha(w)=\sup_{x,\,y\in D}\frac{|w(x)-w(y)|}{|x-y|^\alpha} \qquad (0<\alpha<1).$$

Our aim is to prove $\mu_\vartheta(u)\leqq C\|v\|$, where $v=A^\gamma u$. If $w=(A+\lambda)^{-1}v$ as above, then we have $\mu_{\frac{1}{2}}(w)\leqq C\|Aw\|$ by Lemma 2.4. On the other hand, it is clear that

$$\begin{aligned}\mu_\vartheta(w)&\leqq \sup\left\{\left(\frac{|w(x)-w(y)|}{|x-y|^{\frac{1}{2}}}\right)^{2\vartheta}\big(|w(x)|+|w(y)|\big)^{1-2\vartheta}\right\}\\&\leqq C_\vartheta\big(\mu_{\frac{1}{2}}(w)\big)^{2\vartheta}\cdot\|w\|_\infty^{1-2\vartheta}.\end{aligned}$$

Therefore we have $\mu_\vartheta(w)\leqq C_\vartheta\|Aw\|^{2\vartheta}\cdot\|w\|_\infty^{1-2\vartheta}$, and hence

$$\mu_\vartheta(w)\leqq C_\vartheta(\delta+\lambda)^{-(1-2\vartheta)/4}\|v\|$$

by (2.24) and (2.25). Thus we have by (2.23)

$$\begin{aligned}\mu_\vartheta(u)&\leqq C_\gamma\int_0^\infty\lambda^{-\gamma}\mu_\vartheta(w)\,d\lambda\\&\leqq C_{\vartheta,\gamma}\|v\|\int_0^\infty\lambda^{-\gamma}(\delta+\lambda)^{-(1-2\vartheta)/4}\,d\lambda=C\|v\|. \quad \text{Q.E.D.}\end{aligned}$$

Proof of Lemma 2.7. It is enough to deal with the scalar case. By means of the maximum principle for the Helmholtz equation we can easily show that $(B+\lambda)^{-1}$ is an integral operator with a kernel $H_\lambda(x,y)$ such that

$$0\leqq H_\lambda(x,y)\leqq\frac{1}{4\pi}\,\frac{e^{-\sqrt{\lambda}|x-y|}}{|x-y|} \qquad (\lambda\geqq 0). \tag{2.26}$$

The desired inequality (2.18) follows from (2.26) by virtue of (2.23) with A replaced by B. Q.E.D.

Proof of Lemma 1.1. As we noted above, A and B satisfy

$$\|A^{\frac{1}{2}}u\|=\|B^{\frac{1}{2}}Xu\|=\|\nabla u\| \qquad \big(u\in\mathscr{D}(A^{\frac{1}{2}})\big), \tag{2.27}$$

X being the injection from $\mathscr{H}_\sigma$ into $\mathscr{H}$. In virtue of the Heinz inequality [9] this gives

$$\|B^\alpha Xu\|\leqq\|A^\alpha u\| \qquad \big(u\in\mathscr{D}(A^\alpha)\big), \tag{2.28}$$

for any α in $0\leqq\alpha\leqq\frac{1}{2}$. Hence $B^\alpha XA^{-\alpha}$ is a bounded operator from $\mathscr{H}_\sigma$ into $\mathscr{H}$ with $\|B^\alpha XA^{-\alpha}\|\leqq 1$. Therefore $A^{-\alpha}PB^\alpha$ admits of the bounded extension $\overline{A^{-\alpha}PB^\alpha}$ with $\|\overline{A^{-\alpha}PB^\alpha}\|\leqq 1$ as an operator from $\mathscr{H}$ into $\mathscr{H}_\sigma$. It follows that (1.14) is implied by the inequality

$$\|B^{-\frac{1}{4}}w\|\leqq c_1\|\nabla u\|\cdot\|\nabla v\| \qquad (u,v\in C_0^1), \tag{2.29}$$

where $w=(u\cdot\nabla)\,v$. To prove (2.29), we use Schwarz's inequality and Lemma 2.7:

$$\|B^{-\frac{1}{2}}w\|^2=(B^{-1}w,w)\leqq C\int\limits_{D\times D}\frac{|w(x)|\cdot|w(y)|}{|x-y|^2}\,dx\,dy$$
$$\leqq C\Big(\int\limits_{D\times D}h(x,y)\,dx\,dy\Big)^{\frac{1}{2}}\cdot\Big(\int\limits_{D\times D}h(y,x)\,dx\,dy\Big)^{\frac{1}{2}},$$

where $h(x,y)=|u(x)|^2\,|\nabla v(y)|^2/|x-y|^2$.

On the other hand, the inequality

$$\int\limits_D\frac{|u(x)|^2}{|x-y|^2}\,dx\leqq 4\,\|\nabla u\|^2\qquad(u\in C_0^1)$$

is well known, whence follows

$$\int\limits_{D\times D}h(x,y)\,dx\,dy=\int\limits_{D\times D}h(y,x)\,dx\,dy\leqq 4\,\|\nabla u\|^2\cdot\|\nabla v\|^2.$$

Thus we have established (2.29). Q.E.D.

Proof of Lemma 2.8. The inequality $\|u\|_6\leqq c_0\|\nabla u\|$ is a special case of SOBOLEV's inequality. Actually we can take $c_0=(4/\pi)^{\frac{1}{3}}$ in the scalar case [3]. Q.E.D.

Proof of Lemma 2.9. Taking $\mathscr{D}(B^{\frac{1}{2}})$ as their domains, we define $\mathring{U}$, $\mathring{V}_k$ by

$$\mathring{U}u=A^{-\frac{1}{2}}PB^{\frac{1}{2}}u\quad\text{and}\quad\mathring{V}_k u=B^{-\frac{1}{2}}\partial_k u.$$

In view of $\mathring{S}_k=\mathring{U}\mathring{V}_k$, we notice that the lemma is implied by the fact that $\mathring{V}_k$ can be extended to a bounded operator V_k with $\|V_k\|\leqq 1$, since $\mathring{U}$ has been shown (see above) to admit a bounded extension U with $\|U\|\leqq 1$. Obviously, $\|\partial_k u\|\leqq\|\nabla u\|=\|B^{\frac{1}{2}}u\|$, which yields $\|\partial_k B^{-\frac{1}{2}}\|\leqq 1$. Consequently the required property of $\mathring{V}_k$ follows from the consideration of the adjoint operator.

III. Lemmas on e^{-tA}

Each of the lemmas given here is valid for any strictly positive, self-adjoint operator A in Hilbert space. First of all, the following basic properties of the semigroup e^{-tA} are well-known:

i) $\|e^{-tA}\|\leqq 1\quad(t\geqq 0)$,
$e^{-tA}\cdot e^{-sA}=e^{-(t+s)A}\quad(t,s\geqq 0)$,
$\|(e^{-tA}-I)\,u\|\to 0\quad(t\to+0,\ u\in\mathscr{H}_\sigma)$.

ii) The condition $u\in\mathscr{D}(A)$ is equivalent to the existence of

(2.30) $$\lim_{h\to+0}h^{-1}(e^{-hA}-I)\,u\qquad(\text{which is equal to }-Au).$$

iii) For any fixed $u\in\mathscr{H}_\sigma$ we have for $t>0$

$$\Big(\frac{d}{dt}\Big)^n e^{-tA}u=(-A)^n e^{-tA}u\qquad(n=1,2,\ldots).$$

Moreover, we can write

$$\left(\frac{d}{dt}\right)^n e^{-tA} u = (-1)^n A^{n-\alpha} e^{-tA} A^\alpha u,$$

provided that $u \in \mathscr{D}(A^\alpha)$ $(0 \leqq \alpha \leqq n)$. Finally, e^{-tA} is analytic in t $(t > 0)$.

We now introduce some notations in connection with continuous functions. These will be used not only in lemmas of this subsection but throughout the present paper.

Definition 2.1. Let I be a closed interval of t, and let $\mathscr{X}$ be a Banach space. Then $C(I; \mathscr{X})$ is the set of all continuous $\mathscr{X}$-valued functions defined on I. ϑ being any number in $0 < \vartheta < 1$, $C^\vartheta(I; \mathscr{X})$ means the set of all functions which are strongly (and uniformly) Hölder continuous with the exponent ϑ. More generally, $C^{n+\vartheta}(I; \mathscr{X})$ is the set of all functions v such that v is continuously differentiable up to the order n and $d^n v/dt^n \in C^\vartheta(I; \mathscr{X})$, where $n = 1, 2, \ldots$ and $0 \leqq \vartheta < 1$. If I is not closed, $v \in C^{n+\vartheta}(I; \mathscr{X})$ means that $v \in C^{n+\vartheta}(I_1; \mathscr{X})$ for any closed interval I_1 contained in I. We simply write $C^{n+\vartheta}(I)$ or $C^{n+\vartheta}$ instead of $C^{n+\vartheta}(I; \mathscr{X})$, when no confusion arises.

Lemma 2.10. Let α be a real number in $0 < \alpha \leqq e = 2.718\ldots$. Then,

$$\|A^\alpha e^{-tA}\| \leqq t^{-\alpha} \qquad (t > 0). \tag{2.31}$$

Furthermore,

$$t^\alpha \|A^\alpha e^{-tA} u\| \to 0 \qquad (t \to +0) \tag{2.32}$$

for every $u \in \mathscr{H}_\sigma$.

Lemma 2.11. Let α be a real number in $0 < \alpha < 1$. Then, the inequality

$$\|(e^{-hA} - I) u\| \leqq \frac{1}{\alpha} h^\alpha \|A^\alpha u\| \qquad (h > 0), \tag{2.33}$$

holds for any $u \in \mathscr{D}(A^\alpha)$.

The following three lemmas remain valid with the natural modifications of the statements, if we replace the interval $[0, T]$ by any closed interval.

Lemma 2.12. Consider

$$u(t) = \int_0^t e^{-(t-s)A} f(s)\, ds \qquad \big(t \in [0, T],\ T > 0\big), \tag{2.34}$$

where $f \in C(0, T] \equiv C\big((0, T]; \mathscr{H}_\sigma\big)$ is assumed to satisfy

$$\sup_{0 < s \leqq t} s^\lambda \|f(s)\| \leqq M(t) < +\infty \qquad (0 < t \leqq T), \tag{2.35}$$

for a constant $\lambda \in [0, 1)$ and a real-valued function M. If α is a number in $0 \leqq \alpha < 1$, then $A^\alpha u(t)$ exists for each $t \in (0, T]$ and satisfies the inequality

$$\|A^\alpha u(t)\| \leqq t^{1-\alpha-\lambda} M(t)\, B(1-\alpha, 1-\lambda), \tag{2.36}$$

where $B(\,,)$ represents the beta function. Moreover, $A^\alpha u \in C^\vartheta(0, T]$ for any $\vartheta \in (0, 1-\alpha)$. In particular, we have $A^\alpha u \in C^\vartheta[0, T]$ with $A^\alpha u(0) = 0$ if $0 < \vartheta \leqq 1 - \alpha - \lambda$.

Lemma 2.13. Let α, ϑ and μ be real numbers such that $0 \leqq \alpha < 1$ and $0 < \mu < \vartheta - \alpha$. Let

$$v(t) = \int_0^t e^{-(t-s)A} \{f(s) - f(t)\} ds \qquad (t \in [0, T], T > 0),$$

where $f \in C^\vartheta[0, T]$. Then $A^{1+\alpha} v(t)$ exists for each $t \in [0, T]$ and can be expressed as

$$A^{1+\alpha} v(t) = \int_0^t A^{1+\alpha} e^{-(t-s)A} \{f(s) - f(t)\} ds. \tag{2.37}$$

Moreover, $A^{1+\alpha} v \in C^\mu[0, T]$.

Lemma 2.14. Again consider u given by (2.34), assuming that $f \in C^\vartheta[0, T]$ for some $\vartheta \in (0, 1)$. Then

$$u \in C^{1+\nu}(0, T] \quad \text{and} \quad Au \in C^\nu(0, T] \tag{2.38}$$

for any ν subject to $0 < \nu < \vartheta$. Furthermore, $u' \equiv du/dt$ can be expressed as

$$u' = -Au + f \tag{2.39}$$

or as

$$u'(t) = e^{-tA} f(t) - \int_0^t A e^{-(t-s)A} \{f(s) - f(t)\} ds. \tag{2.40}$$

Proof of Lemma 2.10. (2.31) is an immediate consequence of the relation

$$\max_{\lambda \geqq 0} \lambda^\alpha e^{-t\lambda} = \alpha^\alpha e^{-\alpha} t^{-\alpha} \leqq t^{-\alpha} \qquad (t > 0),$$

in view of the spectral representation

$$A^\alpha e^{-tA} = \int_0^\infty \lambda^\alpha e^{-t\lambda} dE_\lambda \qquad \left(A = \int_0^\infty \lambda dE_\lambda\right).$$

On the other hand, (2.32) can be proved as follows. For an arbitrary fixed $\varepsilon \in (0, \alpha)$ it follows from (2.31) that

$$\begin{aligned} t^\alpha \|A^\alpha e^{-tA} u\| &\leqq t^\alpha \|A^\alpha e^{-tA}(u - v)\| + t^\alpha \|A^{\alpha-\varepsilon} e^{-tA} A^\varepsilon v\| \\ &\leqq \|u - v\| + t^\varepsilon \|A^\varepsilon v\| \end{aligned}$$

for any $v \in \mathscr{D}(A^\varepsilon)$. This implies (2.32), since $\mathscr{D}(A^\varepsilon)$ is dense in $\mathscr{H}_\sigma$. Q.E.D.

Proof of Lemma 2.11. In consideration of

$$(I - e^{-hA}) u = \int_0^h A^{1-\alpha} e^{-tA} A^\alpha u \, dt$$

we have

$$\begin{aligned} \|(I - e^{-hA}) u\| &\leqq \int_0^h \|A^{1-\alpha} e^{-tA}\| \cdot \|A^\alpha u\| dt \\ &\leqq \|A^\alpha u\| \int_0^h t^{\alpha-1} dt = h^\alpha \|A^\alpha u\| / \alpha \end{aligned}$$

with the aid of (2.31). Q.E.D.

Proof of Lemma 2.12. First of all we note that

$$\begin{aligned}\int_0^t \|A^\alpha e^{-(t-s)A} f(s)\| \, ds &\leqq \int_0^t \|A^\alpha e^{-(t-s)A}\| \cdot \|f(s)\| \, ds \\ &\leqq \int_0^t (t-s)^{-\alpha} M(t) \, s^{-\lambda} \, ds \\ &= t^{1-\alpha-\lambda} M(t) \, B(1-\alpha, 1-\lambda) < +\infty\end{aligned}$$

$(t \in (0, T])$. This implies that $u(t) \in \mathscr{D}(A^\alpha)$ and that (2.36) is true, since A^α is closed.

Assuming $0 < h \leqq 1$ without loss of generality, we put

$$\Delta_h A^\alpha u(t) = A^\alpha u(t+h) - A^\alpha u(t) \qquad (0 < t < t+h \leqq T).$$

Obviously we can write $\Delta_h A^\alpha u(t) = w_1 + w_2$ with

$$w_1 = A^\alpha \int_0^t \{e^{-(t+h-s)A} - e^{-(t-s)A}\} f(s) \, ds$$

and

$$w_2 = A^\alpha \int_t^{t+h} e^{-(t+h-s)A} f(s) \, ds.$$

Replacing α by $\alpha + \vartheta$ in (2.36), we obtain $A^\alpha u(t) \in \mathscr{D}(A^\vartheta)$ and also

$$\|A^{\vartheta+\alpha} u(t)\| \leqq t^{1-\alpha-\lambda-\vartheta} M(T) \, B(1-\alpha-\vartheta, 1-\lambda). \tag{2.41}$$

Since $w_1 = (e^{-hA} - I) A^\alpha u(t)$, we have

$$\|w_1\| \leqq h^\vartheta t^{1-\alpha-\lambda-\vartheta} M(T) \, B(1-\alpha-\vartheta, 1-\lambda)/\vartheta \tag{2.42}$$

in view of (2.33). On the other hand,

$$\begin{aligned}\|w_2\| &\leqq \int_t^{t+h} \|A^\alpha e^{-(t+h-s)A}\| \cdot \|f(s)\| \, ds \\ &\leqq \int_t^{t+h} (t+h-s)^{-\alpha} M(T) \, s^{-\lambda} \, ds = M(T) \int_0^h (h-s)^{-\alpha} (s+t)^{-\lambda} \, ds.\end{aligned}$$

From this we obtain

$$\|w_2\| \leqq h^{1-\alpha-\lambda} M(T) \, B(1-\alpha, 1-\lambda) \tag{2.43}$$

or

$$\|w_2\| \leqq t^{-\lambda} h^{1-\alpha} M(T)/(1-\alpha), \tag{2.44}$$

according as we make use of $(s+t)^{-\lambda} \leqq s^{-\lambda}$ or $(s+t)^{-\lambda} \leqq t^{-\lambda}$. Now suppose that $1-\alpha-\lambda \geqq \vartheta$. Then the factors $t^{1-\alpha-\lambda-\vartheta}$ in (2.42) and $h^{1-\alpha-\lambda}$ in (2.43) can be replaced respectively by $T^{1-\alpha-\lambda-\vartheta}$ and h^ϑ. Consequently, we have $\|\Delta_h A^\alpha u(t)\| \leqq C M(T) \, h^\vartheta$ with a constant C independent of $t \in (0, T]$, that is, $A^\alpha u \in C^\vartheta[0, T]$. In order to derive $A^\alpha u \in C^\vartheta(0, T]$ for any $\vartheta \in (0, 1-\alpha)$, it suffices to prove $A^\alpha u \in C^\vartheta[\varepsilon, T]$ for every small $\varepsilon > 0$. If t is restricted to $[\varepsilon, T]$, the factor $t^{1-\alpha-\lambda-\vartheta}$ in (2.42) can be replaced by $T^{1-\alpha-\lambda-\vartheta}$ or $\varepsilon^{1-\alpha-\lambda-\vartheta}$ according as $1-\alpha-\lambda-\vartheta \geqq 0$ or $1-\alpha-\lambda-\vartheta < 0$. Noting (2.44), we arrive at $\|\Delta_h A^\alpha u(t)\| \leqq C_\varepsilon M(T) \, h^\vartheta$ where C_ε is a constant independent of $t \in [\varepsilon, T]$. This completes the proof. Q.E.D.

Proof of Lemma 2.13. According to the assumption, there exists a constant K such that

$$\|f(t) - f(s)\| \leqq K |t-s|^\vartheta \qquad (t, s \in [0, T]). \tag{2.45}$$

Therefore we can estimate

$$\int_0^t \|A^{1+\alpha} e^{-(t-s)A} \{f(s) - f(t)\}\| ds \leqq \int_0^t \|A^{1+\alpha} e^{-(t-s)A}\| \cdot \|f(s) - f(t)\| ds$$
$$\leqq K \int_0^t (t-s)^{-1-\alpha+\vartheta} ds = K t^{\vartheta-\alpha}/(\vartheta-\alpha) < +\infty.$$

This ensures that $v(t) \in \mathscr{D}(A^{1+\alpha})$, and (2.37) is true. At the same time, we notice that

$$\|A^{1+\alpha} v(t)\| \leqq K T^{\vartheta-\alpha}/(\vartheta-\alpha).$$

Similarly we have $v(t) \in \mathscr{D}(A^{1+\alpha+\mu})$ and also

$$\|A^{1+\alpha+\mu} v(t)\| \leqq K T^{\vartheta-\alpha-\mu}/(\vartheta-\alpha-\mu). \tag{2.46}$$

Putting

$$\Delta_h A^{1+\alpha} v(t) = A^{1+\alpha} v(t+h) - A^{\alpha} v(t) \qquad (0 < t < t+h \leqq T),$$

we have

$$\Delta_h A^{1+\alpha} v(t) = w_1 + w_2 + w_3,$$

where

$$w_1 = A^{1+\alpha} \int_0^t (e^{-hA} - I) e^{-(t-s)A} \{f(s) - f(t)\} ds,$$

$$w_2 = A^{1+\alpha} \int_0^t e^{-(t+h-s)A} \{f(t) - f(t+h)\} ds$$

and

$$w_3 = A^{1+\alpha} \int_t^{t+h} e^{-(t+h-s)A} \{f(s) - f(t+h)\} ds.$$

By means of (2.46) we obtain $\|w_1\| \leqq C h^{\mu}$ with a constant C independent of t, since $w_1 = (e^{-hA} - I) A^{1+\alpha} v(t)$ (Lemma 2.11). Rewriting w_2 as

$$w_2 = A^{\vartheta-\mu} e^{-hA} \int_0^t A^{1+\alpha+\mu-\vartheta} e^{-(t-s)A} \{f(t) - f(t+h)\} ds,$$

we get

$$\|w_2\| \leqq \|A^{\vartheta-\mu} e^{-hA}\| \cdot \int_0^t \|A^{1+\alpha+\mu-\vartheta} e^{-(t-s)A}\| \cdot K h^{\vartheta} ds$$
$$\leqq h^{\mu-\vartheta} K h^{\vartheta} t^{\vartheta-\alpha-\mu}/(\vartheta-\alpha-\mu) \leqq C h^{\mu}.$$

Also we have

$$\|w_3\| \leqq \int_t^{t+h} \|A^{1+\alpha} e^{-(t+h-s)A}\| \cdot \|f(s) - f(t+h)\| ds$$
$$\leqq K \int_t^{t+h} (t+h-s)^{-1-\alpha+\vartheta} ds = K h^{\vartheta-\alpha}/(\vartheta-\alpha) \leqq C h^{\mu},$$

$(0 < h \leqq 1)$. Thus we are led to $\|\Delta_h A^{1+\alpha} v(t)\| \leqq C h^{\mu}$. Q.E.D.

Proof of Lemma 2.14. Using v in the preceding lemma, we can write

$$u(t) = A^{-1} f(t) - A^{-1} e^{-tA} f(t) + v(t). \tag{2.47}$$

From this it follows that $u(t) \in \mathscr{D}(A)$ and also

$$A u(t) = f(t) - e^{-tA} f(t) + A v(t). \tag{2.48}$$

Thus we have $Au \in C^{\nu}(0, T]$ according to Lemma 2.13. Putting $\Delta_h u(t) = u(t+h) - u(t)$ $(0 < t < t+h \leqq T)$, we write $\Delta_h u = w_1 + w_2 + w_3$, where

$$w_1 = \int_0^t (e^{-hA} - I)\, e^{-(t-s)A} f(s)\, ds,$$

$$w_2 = \int_t^{t+h} e^{-(t+h-s)A} \{f(s) - f(t)\}\, ds,$$

and

$$w_3 = \int_t^{t+h} e^{-(t+h-s)A} f(t)\, ds.$$

Since $w_1 = (e^{-hA} - I)\, u(t)$ and $u(t) \in \mathscr{D}(A)$, it follows from (2.30) that $w_1/h \to -Au$ as $h \to +0$. Similarly, $w_3/h \to AA^{-1}f(t) = f(t)$ as $h \to +0$, because $w_3 = (I - e^{-hA})\, A^{-1} f(t)$. On the other hand, we obtain

$$\|w_2\| \leqq K h^{1+\vartheta}/(1+\vartheta)$$

with the constant K in (2.45), whence follows that $w_2/h > 0$ as $h \to +0$. Thus we have established

$$\lim_{h \to +0} \frac{\Delta_h u(t)}{h} \equiv \left(\frac{d}{dt}\right)_+ u(t) = -Au + f. \tag{2.49}$$

In view of $Au \in C^{\nu}(0, T]$ we see that $-Au + f$ is continuous in $(0, T]$. Thus the right derivative of u exists and is equal to a continuous function. Therefore we have $du/dt = (d/dt)_+ u$. In other words, we see not only that $du/dt \in C^{\nu}(0, T]$ but also that (2.39) is true. (2.40) follows immediately from (2.39) and (2.48). Q.E.D.

3. Solutions of Pr. III in $\tilde{S}$

In this section we consider Pr. III in the form (1.12) within the class $\tilde{S}$. We begin with some preliminary remarks with respect to the integrals involved in (1.12).

Suppose that $u \in \tilde{S}[0, T]$ for a positive T. Then by the definition of $\tilde{S}$ there exists a nonnegative continuous function $K(t)$ such that $K(0) = 0$ and

$$\|A^{\frac{1}{2}} u(s)\| \leqq K(t)\, s^{-\frac{1}{4}} \qquad (0 < s \leqq t \leqq T). \tag{3.1}$$

Actually, one such K is given by

$$K(t) = \sup_{0 < s \leqq t} s^{\frac{1}{4}} \|A^{\frac{1}{2}} u(s)\|. \tag{3.2}$$

According to (1.15), (3.1) yields

$$\|Hu(s)\| \leqq c_1 K^2(t)\, s^{-\frac{1}{2}} \qquad (0 < s \leqq t \leqq T). \tag{3.3}$$

Put

$$\Phi(t) = \Phi(u; t) = \int_0^t A^{\frac{1}{4}} e^{-(t-s)A} Hu(s)\, ds. \tag{3.4}$$

Then the following facts are immediate consequences of Lemma 2.12 and (3.3). Φ exists and is continuous in $[0, T]$. $A^{\frac{1}{4}} \Phi$ also exists and is continuous in

$(0, T]$, with

$$\|A^{\frac{1}{4}}\Phi(t)\| \leqq c_1 B(\tfrac{1}{4}, \tfrac{1}{2}) K^2(t) t^{-\frac{1}{4}}. \tag{3.5}$$

Thus $\Phi \in \tilde{S}[0, T]$ whenever $u \in \tilde{S}[0, T]$. Supposing that the inequality

$$\|Pf(s)\| \leqq M(t) s^{-\frac{3}{4}} \qquad (0 < s \leqq t \leqq T) \tag{3.6}$$

is satisfied with a continuous function M, let us consider the other integral in (1.12):

$$\psi(t) = \int_0^t e^{-(t-s)A} Pf(s)\, ds. \tag{3.7}$$

It follows from Lemma 2.12 that ψ is continuous in $[0, T]$ and $A^{\frac{1}{4}}\psi$ is continuous in $(0, T]$ with

$$\|A^{\frac{1}{4}}\psi(t)\| \leqq B(\tfrac{1}{4}, \tfrac{1}{2}) M(t) t^{-\frac{1}{4}}. \tag{3.8}$$

If $\|Pf(t)\| = o(t^{-\frac{3}{4}})$ as $t \to +0$, we have $M(0) = 0$ by putting

$$M(t) = \sup_{0 < s \leqq t} s^{\frac{3}{4}} \|Pf(s)\|. \tag{3.9}$$

In this case (3.8) implies $A^{\frac{1}{4}}\psi = o(t^{-\frac{1}{4}})$ as $t \to +0$, and, therefore, $\psi \in \tilde{S}[0, T]$. Finally, it is obvious by Lemma 2.10 that $e^{-tA}a \in \tilde{S}[0, T]$ if $a \in \mathscr{D}(A^{\frac{1}{4}})$. Thus the right side of (1.12) belongs to $\tilde{S}[0, T]$, if $u \in \tilde{S}[0, T]$, $a \in \mathscr{D}(A^{\frac{1}{4}})$ and $\|Pf(t)\| = o(t^{-\frac{3}{4}})$ as $t \to +0$.

We are now going to deal with the uniqueness of the solutions of (1.12).

Theorem 3.1. *If $a \in \mathscr{H}_\sigma$ and $\|Pf\|$ is integrable on $[0, T]$ $(0 < T < \infty)$, then the solution of (1.12) is unique within the class $\tilde{S}[0, T]$.*

Remark 1. As is clear from the definition of $\tilde{S}$, the theorem remains valid if the interval $[0, T]$ is replaced by any $[0, T_1)$ $(0 < T_1 \leqq +\infty)$.

Remark 2. If we replace the initial moment $t = 0$ by any initial moment $t = t_1$, Theorem 3.1 as well as most of other theorems to be given below remain valid with the natural modifications of the statements. The integral equation of Pr.III with the initial moment t_1 and the initial value $u(t_1)$ is given by

$$u(t) = e^{-(t-t_1)A} u(t_1) + \int_{t_1}^t e^{-(t-s)A} Fu(s)\, ds + \int_{t_1}^t e^{-(t-s)A} Pf(s)\, ds \qquad (t \geqq t_1), \tag{3.10}$$

or by

$$u(t) = e^{-(t-t_1)A} u(t_1) + \int_{t_1}^t A^{\frac{1}{4}} e^{-(t-s)A} Hu(s)\, ds + \int_0^t e^{-(t-s)A} Pf(s)\, ds \quad (t \geqq t_1), \tag{3.11}$$

correspondingly to (1.11) or (1.12). Suppose that $\|Fu(s)\|$ and $\|Pf(s)\|$ are integrable in $[0, t]$, $(t > 0)$ and that u satisfies (1.11). Let t_1 be any constant in $(0, t)$. Then we can easily verify

$$\begin{aligned}\int_0^t e^{-(t-s)A} Fu(s)\, ds &= \int_0^{t_1} e^{-(t-s)A} Fu(s)\, ds + \int_{t_1}^t e^{-(t-s)A} Fu(s)\, ds \\ &= e^{(t-t_1)A} \int_0^{t_1} e^{-(t_1-s)A} Fu(s)\, ds + \int_{t_1}^t e^{-(t-s)A} Fu(s)\, ds.\end{aligned}$$

Similarly,

$$\int_0^t e^{-(t-s)A} Pf(s)\,ds = e^{-(t-t_1)A}\int_0^{t_1} e^{-(t_1-s)A} Pf(s)\,ds + \int_{t_1}^t e^{-(t-s)A} Pf(s)\,ds.$$

Noting $e^{-tA}a = e^{-(t-t_1)A}e^{-t_1A}a$, we thus have (3.10) from (1.11). In other words, if u is a solution of (1.11) in $[0, T]$, then u is a solution of (3.10) in $[t_1, T]$ for any $t_1 \in (0, T)$. This is the case also with respect to (1.12) and (3.11).

Proof of Theorem 3.1. Supposing that u and v are solutions of (1.12) belonging to $\tilde{S}[0, T]$, we have for $w = u - v$

$$w(t) = \int_0^t A^{\frac{1}{4}} e^{-(t-s)A}\{Hu(s) - Hv(s)\}\,ds.$$

Introducing

$$K(t) = \max\Big\{\sup_{0<s\leq t} s^{\frac{1}{4}}\|A^{\frac{1}{2}}u(s)\|,\ \sup_{0<s\leq t} s^{\frac{1}{4}}\|A^{\frac{1}{2}}v(s)\|\Big\}$$

and

$$D(t) = \sup_{0<s\leq t} s^{\frac{1}{4}}\|A^{\frac{1}{2}}w(s)\|,$$

we have by (1.16)

$$\|Hu(s) - Hv(s)\| \leq 2c_1K(t)D(t)s^{-\frac{1}{2}} \qquad (0<s\leq t\leq T), \tag{3.12}$$

which implies by Lemma 2.12 that

$$t^{\frac{1}{4}}\|A^{\frac{1}{2}}w(t)\| \leq 2c_1B_1K(t)D(t) \qquad \big(B_1 = B(\tfrac{1}{2}, \tfrac{1}{4})\big).$$

This inequality gives

$$D(t) \leq 2c_1B_1K(t)D(t), \tag{3.13}$$

since $K(t)\cdot D(t)$ is increasing in t. Recalling that $K(0) = 0$, we can choose a small positive number t_1 such that $2c_1B_1K(t_1) < 1$. Then (3.13) implies $D(t_1) = 0$, that is,

$$w(t) \equiv 0 \qquad (0\leq t\leq t_1). \tag{3.14}$$

From now on, we restrict our consideration to $I_1 = [t_1, T]$. Since $A^{\frac{1}{2}}u$ as well as $A^{\frac{1}{2}}v$ is continuous on I_1, we can find a positive constant K_1 such that

$$\|A^{\frac{1}{2}}u(s)\| \leq K_1 \quad\text{and}\quad \|A^{\frac{1}{2}}v(s)\| \leq K_1 \qquad (s\in I_1).$$

According to (1.16) we then have

$$\|Hu(s) - Hv(s)\| \leq 2c_1K_1D_1(t) \qquad (t_1\leq s\leq t\leq T), \tag{3.15}$$

where

$$D_1(t) = \sup_{t_1\leq s\leq t}\|A^{\frac{1}{2}}w(s)\| = \sup_{0<s\leq t}\|A^{\frac{1}{2}}w(s)\|.$$

It is clear that the theorem is established by proving the following proposition:

Proposition (I). Let τ be any point in I_1, and let δ be given by

$$\delta = 1/(16c_1K_1)^4. \tag{3.16}$$

If $w(t) \equiv 0$ on $[0, \tau]$, then $w(t) \equiv 0$ on $[0, \tau+\delta]\cap I_1$.

We give the proof of this proposition only for the case $\tau+\delta\leqq T$. The other case can be dealt with similarly.

From the relation

$$w(t)=\int_{\tau}^{t} A^{\frac{1}{2}} e^{-(t-s)A}\{Hu(s)-Hv(s)\}\,ds$$

we obtain

$$\|A^{\frac{1}{2}}w(t)\|\leqq 8c_1K_1D_1(t)(t-\tau)^{\frac{1}{4}} \qquad (t\in[\tau, T]),$$

by (3.15) and Lemma 2.12. In particular, we have

$$\|A^{\frac{1}{2}}w(t)\|\leqq 8c_1K_1D_1(\tau+\delta)\delta^{\frac{1}{4}}=\tfrac{1}{2}D_1(\tau+\delta) \tag{3.17}$$

for any $t\in[\tau, \tau+\delta]$, whence follows $D_1(\tau+\delta)\leqq\frac{1}{2}D_1(\tau+\delta)$. Thus $D_1(\tau+\delta)=0$, and, consequently, $w\equiv 0$ on $[0, \tau+\delta]$. In this way we have established Proposition (I) and, eventually, the theorem. Q.E.D.

We proceed to existence theorems.

Theorem 3.2. *Assume that*

$$a\in\mathscr{D}(A^{\frac{1}{4}}), \tag{3.18}$$

$$\|Pf\|=o(t^{-\frac{3}{4}}) \tag{3.19}$$

as $t\to+0$. *Then there exists a solution* $u\in\tilde{S}[0, T]$ *of* (1.12) *for some* $T>0$ *depending on* a *and* Pf.

Proof. We attempt to construct the solution by the successive approximation:

$$\begin{aligned} u_0(t)&=e^{-tA}a+\psi(t),\\ u_{n+1}(t)&=u_0(t)+\Phi(u_n;t) \qquad (n=0, 1, \dots), \end{aligned} \tag{3.20}$$

where Φ and ψ are given by (3.4) and (3.7). From the preliminary consideration given above we see that this iteration can be continued indefinitely within the class $\tilde{S}[0, \infty)$. Therefore the functions

$$K_n(t)=\sup_{0<s\leqq t} s^{\frac{1}{4}}\|A^{\frac{1}{2}}u_n(s)\| \qquad (n=0, 1, \dots),$$

are continuous and increasing on $[0, \infty)$, with $K_n(0)=0$. Furthermore, $K_n(t)$ satisfies the recurrence inequalities

$$K_{n+1}(t)\leqq K_0(t)+c_1B_1K_n^2(t) \qquad (t>0,\ n=0, 1, \dots), \tag{3.21}$$

where $B_1=B(\frac{1}{4}, \frac{1}{2})$. In fact, by (3.5) we have

$$t^{\frac{1}{4}}\|A^{\frac{1}{2}}\Phi(u_n;t)\|\leqq c_1B_1K_n^2(t),$$

whence follows

$$\sup_{0<s\leqq t} s^{\frac{1}{4}}\|A^{\frac{1}{2}}\Phi(u_n;s)\|\leqq c_1B_1K_n^2(t)$$

owing to the monotonicity of $K_n(t)$ in t.

Because of $K_0(0)=0$ we can choose a T such that

$$4c_1B_1K_0(T)<1 \qquad (0<T). \tag{3.22}$$

Then an elementary consideration of (3.21) shows that the sequence $\{K_n(T)\}_{n=0}^{\infty}$ is bounded:

$$K_n(T) \leqq \chi(T) \qquad (n=0, 1, \ldots), \tag{3.23}$$

where

$$\chi(t) \equiv \left(1 - \sqrt{1 - 4c_1 B_1 K_0(t)}\right)/2c_1 B_1. \tag{3.24}$$

Similarly, $K_n(t) \leqq \chi(t)$ holds for any $t \in (0, T]$. Also we note that $\chi(t) \leqq 2K_0(t)$.

We now consider the equality

$$w_{n+1}(t) = \int_0^t A^{\frac{1}{4}} e^{-(t-s)A} \{H u_{n+1}(s) - H u_n(s)\} ds, \tag{3.25}$$

where $w_n = u_{n+1} - u_n$ $(n=0, 1, \ldots)$, and $t \in (0, T]$. Putting

$$D_n(t) = \sup_{0<s\leqq t} s^{\frac{1}{4}} \|A^{\frac{1}{4}} w_n(s)\| \qquad \left(n=0, 1, \ldots;\ t \in (0, T]\right),$$

we can derive

$$D_{n+1}(T) \leqq 2c_1 B_1 \chi(T) D_n(T) \leqq 4c_1 B_1 K_0(T) D_n(T) \tag{3.26}$$

in the same way as we derived (3.13). In view of (3.22) and (3.26) it is clear that the series $\sum D_n(T)$ converges. This implies that the series $\sum\limits_n t^{\frac{1}{4}} A^{\frac{1}{4}} w_n(t)$ converges uniformly (and strongly) in $(0, T]$ and, consequently, that the sequence $t^{\frac{1}{4}} A^{\frac{1}{4}} u_n(t)$ converges uniformly in $(0, T]$. Since $A^{-\frac{1}{4}}$ is bounded and $A^{\frac{1}{4}}$ is closed, it follows that $u_n(t)$ converges to a limit $u(t) \in \mathscr{D}(A^{\frac{1}{4}})$ and that $t^{\frac{1}{4}} A^{\frac{1}{4}} u_n(t)$ converges to $t^{\frac{1}{4}} A^{\frac{1}{4}} u(t)$ uniformly in $(0, T]$. The function

$$K(t) = \sup_{0<s\leqq t} s^{\frac{1}{4}} \|A^{\frac{1}{4}} u(s)\|$$

satisfies

$$K(t) \leqq \chi(t) \leqq 2K_0(t) \qquad \left(t \in (0, T]\right). \tag{3.27}$$

Finally,

$$\alpha_n \equiv \sup_{0<s\leqq T} s^{\frac{1}{2}} \|H u_n(s) - H u(s)\| \to 0 \tag{3.28}$$

as $n \to \infty$, as is seen from (1.16).

We now verify that u satisfies (1.12) in $[0, T]$. We have $\Phi(u_n; t) \to \Phi(u; t)$ since

$$\|\Phi(u_n; t) - \Phi(u; t)\| \leqq \int_0^t (t-s)^{-\frac{1}{4}} \alpha_n s^{-\frac{1}{2}} ds \to 0.$$

Hence we get

$$u(t) = u_0(t) + \Phi(u; t), \qquad \left(t \in (0, T]\right), \tag{3.29}$$

by making $n \to \infty$ in the second equation of (3.20). Setting $u(0) = a$, we see that (3.29) holds for any $t \in [0, T]$ and u is continuous on $[0, T]$. The continuity of $A^{\frac{1}{4}} u$ in $(0, T]$ is a consequence of the uniform convergence of $t^{\frac{1}{4}} A^{\frac{1}{4}} u_n(t)$ to $t^{\frac{1}{4}} A^{\frac{1}{4}} u(t)$. $\|A^{\frac{1}{4}} u(t)\| = o(t^{-\frac{1}{4}})$ is clear from (3.27) in consideration of $K_0(0) = 0$. Thus u satisfies not only (1.12) in $[0, T]$ but also the condition $u \in \tilde{S}[0, T]$. Q.E.D.

The following two theorems correspond to Theorems 1.3 and 1.4 respectively.

Theorem 3.3. *Let $a \in \mathscr{D}(A^{\frac{1}{4}})$, and $\|Pf(t)\| = o(t^{-\frac{3}{4}})$ as $t \to +0$. Let*

$$\|A^{\frac{1}{4}} a\| + B_1 M < \frac{1}{4\,c_1 B_1} \qquad \left(B_1 = B\left(\tfrac{1}{4}, \tfrac{1}{2}\right)\right), \tag{3.30}$$

where c_1 *is the absolute constant used in Lemma* 1.1 *and*

$$M = \sup_{0<s<\infty} s^{\frac{1}{4}}\|Pf(s)\|. \tag{3.31}$$

Then there exists a solution $u \in \tilde{S}[0, \infty)$ *of* (1.12).

Proof. On account of the uniqueness theorem it suffices to show the existence of a solution $\in \tilde{S}[0, T]$ for every positive T. By Lemma 2.10 we have

$$t^{\frac{1}{4}}\|A^{\frac{1}{2}}e^{-tA}a\| \leqq t^{\frac{1}{4}}\|A^{\frac{1}{4}}e^{-tA}\|\cdot\|A^{\frac{1}{4}}a\| \leqq \|A^{\frac{1}{4}}a\|. \tag{3.32}$$

Also $t^{\frac{1}{4}}\|A^{\frac{1}{2}}\psi(t)\| \leqq B_1 M$ by (3.8). Thus the function $K_0(t)$ used in the proof of the preceding theorem satisfies

$$K_0(T) \leqq \|A^{\frac{1}{4}}a\| + B_1 M \tag{3.33}$$

for any $T>0$. Combining (3.33) with (3.30), we see that (3.22) holds for any $T>0$. This guarantees the existence of a solution in $\tilde{S}[0, T]$, as is seen by examining the proof of the foregoing theorem. Q.E.D.

Theorem 3.4. *In the assumptions of Theorem* 3.3, *replace* (3.30) *by*

$$\|A^{\frac{1}{4}}a\| \leqq \alpha_0 \quad \textit{and} \quad \|Pf(t)\| \leqq \mu_0 \qquad (0<t<\infty), \tag{3.34}$$

where α_0 *and* μ_0 *are positive numbers such that*

$$0<6c_1B_1\alpha_0<1 \quad \textit{and} \quad 0<108c'^3\mu_0 \leqq \alpha_0, \tag{3.35}$$

$c' = \|A^{-\frac{1}{4}}\|$ *and* c_1, B_1 *being the same as above. Then there exists a solution* $u \in \tilde{S}[0, \infty)$ *of* (1.12).

Proof. First of all, we show that the solution in Theorem 3.2 exists in $[0, \tau]$ and satisfies

$$\|A^{\frac{1}{4}}u(\tau)\| \leqq \alpha_0, \tag{3.36}$$

where

$$\tau = \left(\frac{1}{4}\,\frac{\alpha_0}{\mu_0}\right)^{\frac{4}{3}}. \tag{3.37}$$

By means of (2.36) we have

$$t^{\frac{1}{4}}\|A^{\frac{1}{2}}\psi(t)\| \leqq 2\mu_0 t^{\frac{3}{4}} \leqq 2\mu_0\tau^{\frac{3}{4}} \qquad (t\in(0, \tau]).$$

Also we have $t^{\frac{1}{4}}\|A^{\frac{1}{2}}e^{-tA}a\| \leqq \alpha_0$ by (3.32). Hence

$$K_0(\tau) \leqq \alpha_0 + 2\mu_0\tau^{\frac{3}{4}} = \alpha_0 + 2\mu_0\left(\frac{1}{4}\,\frac{\alpha_0}{\mu_0}\right) = \frac{3}{2}\alpha_0 < \frac{1}{4c_1B_1}$$

by virtue of (3.35) and (3.37). Thus (3.22) holds with $T=\tau$, which implies that the solution u exists on $[0, \tau]$ and that (3.27) is true for any $t\in(0, \tau]$. Also we have

$$\tau^{\frac{1}{4}}\|A^{\frac{1}{2}}u(\tau)\| \leqq 2K_0(\tau) \leqq 3\alpha_0, \tag{3.38}$$

hence

$$\|A^{\frac{1}{2}}u(\tau)\| \leqq 3\alpha_0\tau^{-\frac{1}{4}} = 3\alpha_0\left(\frac{4\mu_0}{\alpha_0}\right)^{\frac{1}{3}} \leqq 3\alpha_0\left(\frac{4\mu_0}{108\alpha_0c'^3}\right)^{\frac{1}{3}} = \alpha_0/c'.$$

In this way we obtain (3.36) by $\|A^{\frac{1}{4}}u(\tau)\| \leqq \|A^{-\frac{1}{4}}\|\cdot\|A^{\frac{1}{2}}u(\tau)\| \leqq c'\|A^{\frac{1}{2}}u(\tau)\|$.

The second step of the proof is to extend the solution to a larger interval. It is clear that we can construct a solution v of Pr.III at the initial moment τ and with the initial value $u(\tau)$ (see the remark near (3.11)). From the foregoing argument it is also clear that v is defined in $[\tau, 2\tau]$ and $v\in\tilde{S}[\tau, 2\tau]$. Moreover the inequality

$$\|A^{\frac{1}{4}}v(2\tau)\|\leqq\alpha_0 \tag{3.39}$$

is satisfied. We can easily verify that the function w defined by

$$w(t)=\begin{cases}u(t), & 0\leqq t\leqq\tau\\ v(t), & \tau\leqq t\leqq 2\tau\end{cases}$$

is a solution of the original equation (1.12) in $[0, 2\tau]$. On the other hand, we can show $w\in\tilde{S}[0, 2\tau]$ as follows. All properties but the continuity of $A^{\frac{1}{4}}w$ at $t=\tau$, which are required for w to belong to $\tilde{S}[0, 2\tau]$, are obvious from $u\in\tilde{S}[0, \tau]$ and $v\in\tilde{S}[\tau, 2\tau]$. Since K_0 is continuous and $4c_1B_1K_0(\tau)<1$, we can find a number τ_1 such that

$$4c_1B_1K_0(\tau_1)<1 \quad \text{and} \quad \tau<\tau_1<2\tau.$$

If we reexamine the proof of Theorem 3.2, we see easily that a unique solution $u_1\in\tilde{S}[0, \tau_1]$ exists. By Theorem 3.1, u_1 is identical with u in $[0, \tau]$. On the other hand we have $u_1=v$ in $[\tau, \tau_1]$ by Theorem 3.1 and by the remarks to Theorem 3.1. Thus w and u_1 coincide with each other in $[0, \tau_1]$, whence follows the continuity of $A^{\frac{1}{4}}w$ at $t=\tau$.

By writing u instead of w, we have established that the solution $u\in\tilde{S}[0, \tau]$ can be extended from $[0, \tau]$ to $[0, 2\tau]$ with $\|A^{\frac{1}{4}}u(2\tau)\|\leqq\alpha_0$ satisfied. This process can be continued indefinitely, and, hence, u is extended to $[0, n\tau]$ within the class $\tilde{S}$ $(n=1, 2, \ldots)$. Again recalling the uniqueness, we have proved the theorem. Q.E.D.

4. Solutions of Pr. III in $S_{\beta,\gamma}$ and Solutions of Pr. II

The first half of this section is devoted to consideration of Pr.III within the class $S_{\beta,\gamma}$. Making use of the results, we shall later prove the differentiability in t of the solutions in the class $\tilde{S}$ constructed in the preceding section and, eventually, complete the proof of the theorems concerning the existence of the solutions of Pr.II stated in §1.

When considered in $S_{\beta,\gamma}$, Pr.III can be written in the form of (1.11). We begin with preliminary considerations of the terms that appear on the right side of (1.11). By the symbols Φ, ψ we mean the same as before, though Φ can now be expressed as

$$\Phi(t)\equiv\Phi(u; t)=\int_0^t e^{-(t-s)A}Fu(s)\,ds. \tag{4.1}$$

Suppose that $u\in S_{\beta,\gamma}[0, T]$ for some $T>0$. Set

$$N(w; t; \beta, \gamma)=\sup_{0<s\leqq t,\alpha=\frac{1}{2},\gamma} s^{\alpha-\beta}\|A^\alpha w(s)\|. \tag{4.2}$$

Then the function $K(t) = N(u; t; \beta, \gamma)$ is continuous and bounded in $(0, T]$. We note that N is increasing in t. According to Lemma 1.2 we have

$$\|Fu(s)\| \leqq c_2 K^2(t)\, s^{2\beta-\gamma-\frac{1}{2}} \qquad (0 < s \leqq t \leqq T). \tag{4.3}$$

Also the continuity of Fu in $(0, T]$ follows from Lemma 1.2. From (4.3) it follows by the Lemma 2.12 that $A^\alpha \Phi$ is continuous in $(0, T]$ for any α $(0 \leqq \alpha < 1)$ and that

$$\|A^\alpha \Phi(t)\| \leqq c_2 B(1-\alpha, 2\beta - \gamma + \tfrac{1}{2})\, K^2(t)\, t^{2\beta-\gamma-\alpha+\frac{1}{2}} \qquad \big(t \in (0, T]\big). \tag{4.4}$$

Φ itself is continuous on $[0, T]$ if we set $\Phi(0) = 0$. Suppose for the moment that

$$\varkappa \equiv \beta - \gamma + \tfrac{1}{2} \geqq 0. \tag{4.5}$$

Then the following inequality holds for $\alpha = \frac{1}{2}$ and $\alpha = \gamma$:

$$t^{\alpha-\beta} \|A^\alpha \Phi(t)\| \leqq c_2 B_2 K^2(t)\, t^\varkappa \leqq c_2 B_2 K^2(T)\, T^\varkappa \qquad \big(t \in (0, T]\big), \tag{4.6}$$

where

$$B_2 = B(1-\gamma, 2\beta - \gamma + \tfrac{1}{2}). \tag{4.7}$$

This means that $\Phi(u; t) \in S_{\beta,\gamma}[0, T]$ whenever $u \in S_{\beta,\gamma}[0, T]$. We have also

$$N\big(\Phi(u; t); t; \beta, \gamma\big) \leqq c_2 B_2 K^2(t)\, t^\varkappa \qquad \big(t \in (0, T]\big), \tag{4.8}$$

as is obvious from (4.6). Suppose, furthermore, that

$$\sup_{0 < s \leqq t} s^{1-\beta} \|Pf(s)\| \leqq M \qquad \big(t \in (0, T]\big), \tag{4.9}$$

with some constant M. Then an argument similar to that given above shows that $A^\alpha \psi$ is continuous in $(0, T]$ and satisfies the inequality

$$t^{\alpha-\beta} \|A^\alpha \psi(t)\| \leqq M B(1-\alpha, \beta) \qquad \big(t \in (0, T]\big)$$

for any α in $0 \leqq \alpha < 1$. In particular, ψ is continuous on $[0, T]$ if we set $\psi(0) = 0$. Thus $\psi \in S_{\beta,\gamma}[0, T]$, and

$$N(\psi; t; \beta, \gamma) \leqq M B(1-\gamma, \beta) \qquad \big(t \in (0, T]\big). \tag{4.10}$$

Finally we consider the function

$$h(t) = e^{-tA} a, \tag{4.11}$$

where a is assumed to satisfy

$$a \in \mathscr{D}(A^\beta). \tag{4.12}$$

Since we have

$$t^{\alpha-\beta} \|A^\alpha h(t)\| \leqq t^{\alpha-\beta} \|A^{\alpha-\beta} e^{-tA}\| \cdot \|A^\beta a\| \leqq \|A^\beta a\|, \qquad 0 \leqq \beta \leqq \alpha < 1$$

for any $t > 0$, it is clear that $h \in S_{\beta,\gamma}[0, \infty)$ and

$$N(h; t; \beta, \gamma) \leqq \|A^\beta a\| \qquad (t > 0). \tag{4.13}$$

In this way we see that, under the conditions (4.5), (4.9) and (4.12), the right side of (1.11) belongs to $S_{\beta,\gamma}[0, T]$ whenever $u \in S_{\beta,\gamma}[0, T]$. Since any solution of (1.11) in $S_{\beta,\gamma}[0, T]$ can be regarded as a solution of (1.12) in $\tilde{S}[0, T]$, we have by Theorem 3.1 the following theorem.

Theorem 4.1. *If $a \in \mathscr{H}_\sigma$ and $\|Pf\|$ is integrable in $[0, T]$, then the solution of* (1.11) *is unique within $S_{\beta,\gamma}[0, T]$.*

Concerning the existence of a solution we have

Theorem 4.2. *Assume that $a \in \mathscr{D}(A^\beta)$ and*

$$\|Pf(t)\| = O(t^{-1+\beta}) \qquad (t \to +0), \tag{4.14}$$

where β is a number in $\frac{1}{4} < \beta < \frac{1}{2}$. Then for any γ such that

$$\tfrac{3}{4} < \gamma < \beta + \tfrac{1}{2} \tag{4.15}$$

there exists a solution $u \in S_{\beta,\gamma}[0, T]$ of (1.11). *Here T is a positive number depending on β, γ, $\|A^\beta a\|$ and M to be given below by* (4.17).

Proof. Again we construct the solution by the iteration (3.20). Since the proof of convergence of this procedure is parallel to the previous one, we merely give the outline. The iteration can be continued indefinitely within $S_{\beta,\gamma}[0, \infty)$ according to the preliminary considerations given above. (Note that $\varkappa = \beta - \gamma + \frac{1}{2} > 0$ by (4.15).) The functions $K_n(t) \equiv N(u_n; t; \beta, \gamma)$ satisfy the recurrence inequalities

$$K_{n+1}(t) \leqq K_0(t) + c_2 B_2 K_n^2(t) t^\varkappa, \tag{4.16}$$

where c_2, B_2 and $\varkappa$ are as in (4.8). On the other hand, putting

$$M(t) = \sup_{0 < s \leqq t} s^{1-\beta} \|Pf(s)\| \qquad (t > 0), \tag{4.17}$$

we get

$$K_0(t) \leqq \|A^\beta a\| + M(t) B(1-\gamma, \beta) \qquad (t > 0). \tag{4.18}$$

Since $\varkappa$ is positive and $M(t)$ is bounded in $[0, T_1]$, $(T_1 > 0)$, we can choose a T such that

$$4c_2 B_2 (\|A^\beta a\| + M(T) B(1-\gamma, \beta)) T^\varkappa < 1 \qquad (T > 0). \tag{4.19}$$

By virtue of (4.18) T satisfies

$$4c_2 B_2 K_0(T) T^\varkappa < 1. \tag{4.20}$$

Making use of (4.20) instead of (3.22) we argue in the same way as before and establish that u_n converges in $[0, T]$ to a solution $u \in S_{\beta,\gamma}[0, T]$. Q.E.D.

Remark. Although we do not give them explicitly, we can prove existence theorems of global solutions $\in S_{\beta,\gamma}$ corresponding to Theorems 3.3 and 3.4, assuming that a and Pf are sufficiently small.

Before passing to the differentiability in t of the solutions in $S_{\beta,\gamma}$ we prepare the following lemma.

Lemma 4.1. Assume that $a \in \mathscr{H}_\sigma$ and $\|Pf(t)\| = O(t^{-1+\varepsilon})$ for some $\varepsilon > 0$ as $t \to +0$. Let u be a solution of (1.11) belonging to $S_{\beta,\gamma}[0, T]$, $(T > 0)$. Then $A^\alpha u \in C^\vartheta(0, T]$ for any α and ϑ such that $0 \leqq \alpha < 1$ and $0 < \vartheta < 1 - \alpha$.

Proof. Let the symbols h, ψ and $\Phi(u; t)$ be as above. Then $A^\alpha h \in C^\vartheta(0, T]$ ($A^\alpha h$ is even analytic in t). Also we have $A^\alpha \psi \in C^\vartheta(0, T]$ and $A^\alpha \Phi \in C^\vartheta(0, T]$, as is seen from $\|Pf\| = O(t^{-1+\varepsilon})$ and (4.3) with the aid of Lemma 2.12. Therefore we have $A^\alpha u \in C^\vartheta(0, T]$ by (1.11). Q.E.D.

Corollary to Lemma 4.1. Under the assumption of Lemma 4.1, we have

$$Fu \in C^{\vartheta}(0, T] \tag{4.21}$$

for any ϑ in $0<\vartheta<\frac{1}{4}$.

Proof. Lemma 4.1 implies that $A^{\frac{3}{4}}u$ and $A^{\lambda}u$ belong to $C^{\vartheta}(0, T]$, where $\frac{3}{4}<\lambda<1-\vartheta$. Hence (4.21) follows from Lemma 1.2. Q.E.D.

Theorem 4.3. *In addition to the assumptions of Lemma 4.1, suppose that*

$$Pf \in C^{\vartheta}(0, T] \qquad (0<\vartheta<1). \tag{4.22}$$

Then

$$u \in C^{1+\nu}(0, T] \quad and \quad Au \in C^{\nu}(0, T] \tag{4.23}$$

for any ν such that

$$0<\nu<\tfrac{1}{4} \quad and \quad 0<\nu<\vartheta. \tag{4.24}$$

Furthermore, u is a solution of (1.6) in $(0, T]$.

Proof. Let δ be an arbitrary small positive number. Then for any $t \in [\delta, T]$ u satisfies the relation (see Remark 2 on Theorem 3.1)

$$u(t) = h_\delta(t) + \psi_\delta(t) + \Phi_\delta(u; t),$$

where

$$\begin{aligned} h_\delta(t) &= e^{-(t-\delta)A} u(\delta), \\ \psi_\delta(t) &= \int_\delta^t e^{-(t-s)A} Pf(s)\, ds \qquad \text{and} \\ \Phi_\delta(t) &= \int_\delta^t e^{-(t-s)A} Fu(s)\, ds. \end{aligned} \tag{4.25}$$

It is obvious that $h_\delta \in C^\infty(\delta, T]$, $Ah_\delta \in C^\infty(\delta, T]$, and $dh_\delta/dt = -Ah_\delta$ in $(\delta, T]$. From $Pf \in C^{\vartheta}[\delta, T]$ it follows by Lemma 2.14 that $\psi_\delta \in C^{1+\nu}(\delta, T]$, $A\psi_\delta \in C^{\nu}(\delta, T]$ and $d\psi_\delta/dt = -A\psi_\delta + Pf$. If λ is chosen so that $\nu<\lambda<\frac{1}{4}$, we have $Fu \in C^{\lambda}(0, T]$ owing to (4.21) and, consequently, $Fu \in C^{\lambda}[\delta, T]$. Again according to Lemma 2.14 this yields that $\Phi_\delta \in C^{1+\nu}(\delta, T]$, $A\Phi_\delta \in C^{\nu}(\delta, T]$ and $d\Phi_\delta/dt = -A\Phi_\delta + Fu$. In this way we obtain $u \in C^{1+\nu}(\delta, T]$, $Au \in C^{\nu}(\delta, T]$ and $du/dt = -Au + Fu + Pf$ in $(\delta, T]$. Hence the theorem follows immediately, because δ is arbitrary. Q.E.D.

As an application of Theorem 4.3 we prove the differentiability of the solutions $\in \tilde{S}$ of (1.12).

Theorem 4.4. *In addition to (4.22), assume that $a \in \mathscr{H}_\sigma$, $Pf = O(t^{-1+\varepsilon})$ for some $\varepsilon>0$ as $t \to +0$. If u is a solution of (1.12) belonging to $\tilde{S}[0, T]$, $(T>0)$, then the conclusion of Theorem 4.3 is still true.*

Proof. Let t_1 be an arbitrary number in $(0, T)$. We choose β and γ so that (1.17) and (4.15) hold. Then $u(t_1) \in \mathscr{D}(A^{\beta})$ in view of $u(t_1) \in \mathscr{D}(A^{\frac{3}{4}})$, whereas $Pf \in C^{\vartheta}[t_1, T]$ is obvious. We now consider (3.10) with the initial moment t_1 and the initial value $u(t_1)$. According to Theorem 4.2 this problem has a solution v belonging to $S_{\beta,\gamma}[t_1, t_1+\delta]$ for some δ in $0<\delta \leqq T-t_1$. u and v coincide with each other in $[t_1, t_1+\delta]$ owing to the uniqueness. This implies that u satisfies the conclusion of the theorem with $(0, T]$ replaced by $(t_1, t_1+\delta]$, for we can apply Theorem 4.3 to v. Noting that δ is a certain positive number independent of t_1 as long as t_1 is bounded away from 0 by an arbitrary but fixed positive number, we conclude that the theorem is true.

Proof of Theorems 1.1.—1.4. If u is a solution of Pr.II belonging to the class $\tilde{S}[0, T)$, then u satisfies (1.12) in $[0, T)$ as is easily verified by

$$\frac{d}{ds}\{e^{-(t-s)A}u(s)\} = e^{-(t-s)A}\{du/ds + Au(s)\}.$$

Thus Theorem 1.1 follows from Theorem 3.1. Theorems 1.2, 1.3 and 1.4 follow with the aid of Theorem 4.4, from Theorems 3.2, 3.3 and 3.4, respectively. Q.E.D.

5. Regularity of Solutions of Pr.II

We are going to study the regularity of a solution u which satisfies the equation (1.6) in $(0, T)$ (in the sense of Definition 1.1) as well as the initial condition (1.7) at $t=0$. In this section we consider the regularity of u as a $\mathscr{H}_\sigma$-valued function and also some regularity properties of u with respect to t and $x \in \bar{D}$ obtainable as a consequence. The interior regularity of $u=u(t, x)$ will be dealt with in next section.

The first half of this section is devoted to the regularity for $t>0$; the behavior of u as $t \to +0$ is studied in the second half.

I. Regularity in $(0, T)$ *and* $(0, T)\times\bar{D}$

Throughout this part we assume that u is a solution of (1.6) in $I=(0, T)$ with Pf satisfying

$$Pf \in C^\vartheta(0, T) \equiv C^\vartheta(I; \mathscr{H}_\sigma) \tag{5.1}$$

for some ϑ $(0<\vartheta<1)$. If t_1 and T_1 are arbitrary numbers such that $0<t_1<T_1<T$, then $u \in \tilde{S}[t_1, T_1]$ and u is a solution of Pr.III in $[t_1, T_1]$ of the form given by (3.10). It follows from Theorem 4.4 that

$$u \in C^{1+\nu}(I; \mathscr{H}_\sigma) \quad \text{and} \quad Au \in C^\nu(I; \mathscr{H}_\sigma) \tag{5.2}$$

for any ν such that

$$0<\nu<\tfrac{1}{4} \quad \text{and} \quad 0<\nu<\vartheta. \tag{5.3}$$

Also we have by Lemma 4.1

$$A^\alpha u \in C^\mu(I; \mathscr{H}_\sigma) \tag{5.4}$$

if $0 \leqq \alpha < 1$ and $0<\mu<1-\alpha$.

With the aid of Lemma 1.2, (5.4) yields

$$u \in C^\mu\big(I; L_\infty(D)\big) \qquad (0<\mu<\tfrac{1}{4}). \tag{5.5}$$

Also (5.4) implies

$$\nabla u \in C^\lambda\big(I; L_2(D)\big) \qquad (0<\lambda<\tfrac{1}{2}), \tag{5.5$'$}$$

since $\|\nabla u\| = \|A^{\frac{1}{2}}u\|$. Thus we have

$$(u \cdot \nabla)u \in C^\mu\big(I; L_2(D)\big) \quad \text{and} \quad Fu \in C^\mu(I; \mathscr{H}_\sigma) \tag{5.6}$$

for any μ in $0<\mu<\frac{1}{4}$.

Without any further assumption the solution u is Hölder continuous with respect to t and x. Namely, we have

Theorem 5.1. *If* (5.1) *is satisfied, then* $u=u(t,x)$ *is Hölder continuous in* $I\times\bar{D}$. u *attains the prescribed boundary value* 0 *on* ∂D.

Proof. Taking an arbitrary closed subinterval I_1 of I, we can introduce constants M_1 and M_2 such that

$$\|Au(t)\|\leqq M_1 \qquad (t\in I_1), \tag{5.7}$$

and

$$\|Au(t_1)-Au(t_2)\|\leqq M_2|t_1-t_2|^\nu \qquad (t_1, t_2\in I_1), \tag{5.8}$$

since $Au\in C^\nu(I_1;\mathscr{H}_\sigma)$. According to Lemma 2.4, (5.7) implies

$$|u(t_1,x_1)-u(t_1,x_2)|\leqq CM_1 d^{\frac{1}{2}},$$

where $x_1, x_2\in\bar{D}$ and $d=|x_1-x_2|$. On the other hand, (5.8) gives

$$|u(t_1,x_2)-u(t_2,x_2)|\leqq CM_2\tau^\nu \qquad (\tau=|t_1-t_2|),$$

with the aid of Lemma 1.2. Thus we obtain

$$|u(t_1,x_1)-u(t_2,x_2)|\leqq C(d^{\frac{1}{2}}+\tau^\nu), \tag{5.9}$$

where C is a constant independent of $t_1, t_2\in I_1$ or $x_1, x_2\in\bar{D}$. (5.9) proves the first assertion of the lemma, since I_1 is arbitrary. The other one is implied simply by $u(t)\in\mathscr{D}(A)$. Q.E.D.

The theorem established above corresponds to the first statement in Theorem 1.5. The second statement will be proved below in Theorem 5.2 after the following lemmas.

Lemma 5.1. The integral representations

$$u(t,x)=\int_D G(x,y)w(t,y)\,dy \tag{5.10}$$

and

$$\partial_m u(t,x)=\partial u(t,x)/\partial x_m\equiv\int_D G_m(x,y)w(t,y)\,dy \qquad (m=1,2,3), \tag{5.11}$$

are true for the solution u in $I\times D$ with

$$w=-\frac{du}{dt}-(u\cdot\nabla)u+f, \tag{5.12}$$

where G is the Odqvist-Green function and $G_m(x,y)=\partial G(x,y)/\partial x_m$. If $Pf\equiv 0$, we can substitute $w=-du/dt-(u\cdot\nabla)u$ into (5.10) and (5.11).

Proof. From (1.6) we have

$$u=-A^{-1}u'+A^{-1}Fu+A^{-1}Pf \qquad (u'\equiv du/dt).$$

Since $A^{-1}=\boldsymbol{G}$ and $\boldsymbol{GP}=\boldsymbol{G}$ as noted in §2, this equation can be written as

$$u(t)=\boldsymbol{G}w(t) \qquad (w(t)=w(t,\cdot)\in L_2), \tag{5.13}$$

which is equivalent to (5.10). By considering the distribution derivatives (with respect to x-variables) of the two sides of (5.10) we get to

$$\partial_m u(t) = G_m w(t) \tag{5.14}$$

by means of Fubini's theorem which is applicable by virtue of (2.6). Q.E.D.

Lemma 5.2. If (5.1) is satisfied for some ϑ in

$$\tfrac{1}{2} < \vartheta < 1, \tag{5.15}$$

then the following propositions i), ii), iii) are true for any λ satisfying

$$0 < \lambda < \vartheta - \tfrac{1}{2} \quad \text{and} \quad \lambda < \tfrac{1}{4}: \tag{5.16}$$

$$\begin{aligned} &\text{i)} \quad (A^{\frac{1}{2}} u)' = A^{\frac{1}{2}} u' \in C^\lambda(I; \mathscr{H}_\sigma) \qquad (' = d/dt), \\ &\text{ii)} \quad u' \in C^\lambda(I; L_6(D)), \\ &\text{iii)} \quad \nabla(u') = (\nabla u)' \in C^\lambda(I; L_2(D)). \end{aligned} \tag{5.17}$$

Proof. With an arbitrary $\delta \in (0, T/2)$ we define h_δ, ψ_δ and $\Phi_\delta(u; t)$ by (4.25). Obviously $h'_\delta(t) \in \mathscr{D}(A^{\frac{1}{2}})$ for each $t \in I_\delta \equiv (\delta, T-\delta)$ and

$$A^{\frac{1}{2}} h'_\delta \in C^\lambda(I_\delta; \mathscr{H}_\sigma). \tag{5.18}$$

According to Lemma 2.14 we have

$$\psi'_\delta(t) = e^{-(t-\delta)A} P f(t) - \int_\delta^t A\, e^{-(t-s)A} \{P f(s) - P f(t)\} ds \tag{5.19}$$

for $t \in I_\delta$, whence follows

$$A^{\frac{1}{2}} \psi'_\delta \in C^\lambda(I_\delta; \mathscr{H}_\sigma) \tag{5.20}$$

by virtue of (5.16) and Lemma 2.13.

In order to deal with Φ_δ, we introduce a non-linear operator K by

$$K u = A^{-\frac{1}{2}} F u = -A^{-\frac{1}{2}} P(u \cdot \nabla) u. \tag{5.21}$$

Making use of the operators S_k in Lemma 2.9, we can write

$$-K u = S_1 v^{(1)} + S_2 v^{(2)} + S_3 v^{(3)}, \tag{5.22}$$

where $v^{(j)} = u_j u$, $(j = 1, 2, 3)$. In deriving[1] (5.22) we have made use of $\operatorname{div} u = 0$ and (5.5). In view of (5.2) and (5.5) we have $v^{(j)} \in C^{1+\nu}(I; L_2(D))$ for any $\nu \in (0, \frac{1}{4})$,

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[1] $w \in \mathscr{D}(A^{\frac{1}{2}})$ implies $w \in L_2(D)$, $\nabla w \in L_2(D)$ and $\operatorname{div} w = 0$, since $\mathscr{D}(A^{\frac{1}{2}}) = H^1_{0,\sigma}$. If u and v are any scalar functions belonging to H^1_0, which is the completion of $C_0^\infty(D)$ with the Dirichlet norm, then we have

$$\frac{\partial}{\partial x_k}(u v) = \frac{\partial u}{\partial x_k} v + u \frac{\partial v}{\partial x_k} \in L_1(D) \qquad (k = 1, 2, 3), \tag{$*$}$$

in the distribution sense. In fact, take smooth functions u_n and v_n $(n = 1, 2, \ldots)$ such that $u_n \to u$, $v_n \to v$, $\nabla u_n \to \nabla u$ and $\nabla v_n \to \nabla v$ in $L_2(D)$. Then for any $\varphi \in C_0^\infty(D)$ we have

$$-\left(u_n v_n, \frac{\partial \varphi}{d x_k}\right) = \left(\frac{\partial u_n}{\partial x_k} v_n + u_n \frac{\partial v_n}{\partial x_k}, \varphi\right).$$

From this we obtain $(*)$, since $u_n v_n \to u v$ and $\frac{\partial u_n}{\partial x_k} v_n + u_n \frac{\partial v_n}{\partial x_k} \to \frac{\partial u}{\partial x_k} v + u \frac{\partial v}{\partial x_k}$ in $L_1(D)$ as $n \to \infty$.

because we can easily verify[1] $(v^{(j)})' = u_j' u + u_j u'$. Since S_k is bounded, we thus obtain

$$Ku \in C^{1+\nu}(I; \mathscr{H}_\sigma) \tag{5.23}$$

which allows us to rewrite Φ_δ through integration by parts;

$$\Phi_\delta(t) = A^{-1}\{Fu(t) - e^{-(t-\delta)A} Fu(\delta)\} - A^{-\frac{1}{2}} \int_\delta^t e^{-(t-s)A} (Ku)'(s)\, ds. \tag{5.24}$$

Substituting $A\Phi_\delta(t)$ calculated from (5.24) into the equation $\Phi_\delta' = -A\Phi_\delta + Fu$, we get

$$\begin{aligned} \Phi_\delta'(t) &= e^{-(t-\delta)A} Fu(\delta) + A^{\frac{1}{2}} \int_\delta^t e^{-(t-s)A} (Ku)'(s)\, ds \\ &= w_1 + w_2 + w_3, \end{aligned} \tag{5.25}$$

where

$$w_1 = e^{-(t-\delta)A} Fu(\delta), \qquad w_2 = A^{-\frac{1}{2}}(1 - e^{-(t-\delta)A})(Ku)'(t)$$

and

$$w_3 = \int_\delta^t A^{\frac{1}{2}} e^{-(t-s)A} \{(Ku)'(s) - (Ku)'(t)\}\, ds.$$

Here $A^{\frac{1}{2}} w_1 \in C^\lambda(I_\delta; \mathscr{H}_\sigma)$ since $A^{\frac{1}{2}} w_1$ is analytic in t, $A^{\frac{1}{2}} w_2 \in C^\lambda(I_\delta; \mathscr{H}_\sigma)$ owing to (5.23) and $A^{\frac{1}{2}} w_3 \in C^\lambda(I_\delta; \mathscr{H}_\sigma)$ according to Lemma 2.13. Hence $A^{\frac{1}{2}} \Phi_\delta' \in C^\lambda(I_\delta; \mathscr{H}_\sigma)$.

Recalling (5.18) and (5.20), we thus obtain

$$A^{\frac{1}{2}} u' \in C^\lambda(I_\delta; \mathscr{H}_\sigma) \tag{5.26}$$

because $u = h_\delta + \psi_\delta + \Phi_\delta$. (5.26) yields

$$A^{\frac{1}{2}} u' \in C^\lambda(I; \mathscr{H}_\sigma) \tag{5.27}$$

owing to the arbitrariness of δ. Since $A^{\frac{1}{2}}$ is closed and u' and $A^{\frac{1}{2}} u'$ are continuous, (5.27) implies i). ii) follows immediately from i) by (2.19).

Finally, $\nabla(u') \in C^\lambda(I; L_2)$ follows from i) on account of (2.19), and so does $(\nabla u)' = \nabla(u')$. This proves iii). Q.E.D.

Lemma 5.3. If (5.1) is satisfied for some ϑ in $\frac{1}{2} < \vartheta < 1$, and if

$$f \in C^\varkappa(I; L_q(D)) \tag{5.28}$$

is satisfied for some $\varkappa$ and q such that $0 < \varkappa < 1$ and $3 < q$, then

$$\nabla u \in C^\nu(I; L_\infty(D)) \tag{5.29}$$

for any ν such that

$$0 < \nu < \min\{\vartheta - \tfrac{1}{2}, \tfrac{1}{4}, \varkappa\}. \tag{5.30}$$

If $Pf \equiv 0$, (5.28) can be disregarded.

---

[1] Put $g = u_j' u + u_j u'$. Let $t \in I$, and let σ be a sufficiently small number. Then

$$\|\sigma^{-1}(v^{(j)}(t+\sigma) - v^{(j)}(t)) - g(t)\|_2 \leq \left\| \frac{u_j(t+\sigma) - u_j(t)}{\sigma} - u_j'(t) \right\|_2 \|u(t+\sigma)\|_\infty + $$
$$+ \|u_j'(t)\|_2 \|u(t+\sigma) - u(t)\|_\infty + \|u_j(t)\|_\infty \left\| \frac{u(t+\sigma) - u(t)}{\sigma} - u'(t) \right\|_2.$$

Hence we have $(v^{(j)})' = g$ by (5.2) and (5.5).

Proof. We make use of (5.11). The integral operator G_m is bounded if it is considered either as

$$G_m: L_2(D) \to L_p(D) \qquad (1 \leqq p < 6),$$

or as

$$G_m: L_q(D) \to L_\infty(D), \qquad (3 < q).$$

This can be verified with the aid of Young's inequality and Hölder's inequality by virtue of (2.6). Hence we have $G_m f \in C^\varkappa(I; L_\infty(D))$ by (5.28), and $G_m u' \in C^\nu(I; L_\infty(D))$ by ii) of Lemma 5.2. In view of (5.6) we have also $G_m(u \cdot \nabla)u \in C^\nu(I; L_5(D))$. This implies $\nabla u \in C^\nu(I; L_5(D))$ by virtue of (5.11). Recalling (5.5), we now have $(u \cdot \nabla)u \in C^\nu(I; L_5(D))$, whence follows $G_m(u \cdot \nabla)u \in C^\nu(I; L_\infty(D))$. This gives $\partial_m u \in C^\nu(I; L_\infty(D))$ owing to (5.11). Q.E.D.

Theorem 5.2. *Under the assumptions of the foregoing Lemma* 5.3, $\nabla u = \nabla_x u(t, x)$ *is Hölder continuous in* $I \times \bar{D}$.

Proof. If v is any function in $L_p(D)$ with $p > 3$ and α is any number $\in \left(0, 1 - \frac{3}{p}\right)$, then

$$|(G_m v)(x_1) - (G_m v)(x_2)| \leqq C |x_1 - x_2|^\alpha \|v\|_p \tag{5.31}$$

holds for any $x_1, x_2 \in \bar{D}$. This can be proved easily from (2.7). We have, in the same way,

$$|(G_m v)(t_1, x_1) - (G_m v)(t_2, x_2)| \leqq C\left(|x_1 - x_2|^\alpha + |t_1 - t_2|^\nu\right) \tag{5.32}$$

for any $x_1, x_2 \in \bar{D}$ and for any t_1, t_2 contained in an arbitrary, fixed, closed subinterval of I, provided that $v = v(t) = v(t, \cdot)$ is a $L_p(D)$-valued function of t and $v \in C^\nu(I; L_p(D))$.

Taking account of these facts, we see that $G_m f$ and $G_m u'$ are Hölder continuous in $I \times \bar{D}$ because $f \in C^\varkappa(I; L_q(D))$, and $u' \in C^\nu(I; L_6(D))$ (see Lemma 5.2, ii)). Also $G_m(u \cdot \nabla)u$ is Hölder continuous in $I \times \bar{D}$, since (5.5) and (5.29) imply $(u \cdot \nabla)u \in C^\nu(I; L_\infty(D))$.

In view of (5.11) we thus see that $\partial_m u$ is Hölder continuous in $I \times \bar{D}$ for each m. Q.E.D.

Proof of Theorem 1.5. Theorem 1.5 follows from Theorems 5.1 and 5.2. Q.E.D.

Theorem 5.3. *Assume that* (5.1) *is satisfied for some* $\vartheta \in (\frac{3}{4}, 1)$ *and that* (5.28) *holds. Then* $u'(t, x) = (du/dt)(t, x)$ *is Hölder continuous in* $I \times \bar{D}$ *and coincides with the ordinary derivative* $\partial u(t, x)/\partial t$.

Proof. According to Lemmas 1.2 and 2.6, the desired Hölder continuity of u' is implied by the fact that, for any α and μ such that

$$\tfrac{3}{4} < \alpha < \vartheta \quad \text{and} \quad 0 < \mu < \vartheta - \alpha, \tag{5.33}$$

the relation

$$A^\alpha u' = (A^\alpha u)' \in C^\mu(I; \mathscr{H}_\sigma) \tag{5.34}$$

is true. To prove (5.34) we note, using the symbols h_δ, ψ_δ and Φ_δ employed in the proof of Lemma 5.2, that $A^\alpha h'_\delta \in C^\mu(I_\delta; \mathscr{H}_\sigma)$, $A^\alpha \psi'_\delta \in C^\mu(I_\delta; \mathscr{H}_\sigma)$. By means of Lemmas 5.2 and 5.3, we have[1]

$$\frac{d}{dt}(u\cdot\nabla)u=\left(\frac{du}{dt}\cdot\nabla\right)u+(u\cdot\nabla)\frac{du}{dt}, \tag{5.35}$$

and, hence,

$$(u\cdot\nabla)u\in C^{1+\nu}(I;L_2(D)) \tag{5.36}$$

for any ν in $0<\nu<\min\{\frac{1}{4},\varkappa\}$. Therefore,

$$Fu\in C^{1+\nu}(I;\mathscr{H}_\sigma). \tag{5.37}$$

This enables us to rewrite (5.25) as

$$\Phi_\delta'(t)=e^{-(t-\delta)A}Fu(\delta)+\int_\delta^t e^{-(t-s)A}(Fu)'(s)\,ds, \tag{5.38}$$

whence we have $A^\alpha\Phi_\delta'\in C^\mu(I_\delta;\mathscr{H}_\sigma)$ by Lemma 2.12. Thus we get to $A^\alpha u'\in C^\mu(I_\delta;\mathscr{H}_\sigma)$ and, finally, to $A^\alpha u'\in C^\mu(I;\mathscr{H}_\sigma)$. The equality $A^\alpha u'=(A^\alpha u)'$ can be obtained in the same way as for i) of Lemma 5.2. Q.E.D.

We are now going to show that the C^∞-character of f is inherited by u. Since A^α is closed with $A^{-\alpha}$ bounded $(0\leqq\alpha<1)$, the condition $A^\alpha u\in C^n(I;\mathscr{H}_\sigma)$ is equivalent to the condition that $u\in C^n(I;\mathscr{H}_\sigma)$ and $A^\alpha u^{(k)}\in C(I;\mathscr{H}_\sigma)$ for $k=0,1,\dots,n$. In this case we have $A^\alpha u^{(k)}=(A^\alpha u)^{(k)}$. Similar relations hold for the condition $A^\alpha u\in C^{n+h}$ $(n=0,1,\dots,0\leqq h<1)$.

Lemma 5.4. If $Pf\in C^\infty(I;\mathscr{H}_\sigma)$ and (5.28) is satisfied, then $A^\alpha u\in C^\infty(I;\mathscr{H}_\sigma)$ for any α in $0\leqq\alpha<1$. In the case of $Pf\equiv 0$, (5.28) can be disregarded.

Proof. n being any non-negative integer, we denote the following proposition by $(P)_n$.

$(P)_n$: For any α and ϑ such that $0\leqq\alpha<1$ and $0<\vartheta<1-\alpha$, we have $A^\alpha u\in C^{n+\vartheta}(I;\mathscr{H}_\sigma)$.

We prove by induction that $(P)_n$ is true for any n. $(P)_0$ is true in view of (5.4). Let us show that $(P)_{n+1}$ follows from $(P)_n$. Assuming $(P)_n$, we have

$$\nabla u\in C^{n+\nu}(I;L_2(D)) \tag{5.39}$$

and

$$u\in C^{n+\nu}(I;L_\infty(D)), \tag{5.40}$$

where ν is arbitrary in $0<\nu<\frac{1}{4}$. Here use has been made of (1.5)′ and Lemma 1.2. (5.39) and (5.40) yield $(u\cdot\nabla)u\in C^{n+\nu}(I;L_2(D))$ and, consequently,

$$Fu\in C^{n+\nu}(I;L_2(D)). \tag{5.41}$$

We shall again use the notations employed in the proof of Lemma 5.2. By virtue of (5.41) we can express $\Phi_\delta^{(n)}$ as

$$\Phi_\delta^{(n)}(t)=S_n(t)+W_n(t)\qquad(t\in I_\delta),$$

where

$$S_n(t)=\sum_{k=0}^{n-1}(-A)^{n-k-1}e^{-(t-\delta)A}(Fu)^{(k)}(\delta)$$

[1] See the second footnote given to the proof of Lemma 5.2.

and

$$W_n(t)=\int_\delta^t e^{-(t-s)A}(Fu)^{(n)}(s)\,ds.$$

Obviously $S_n\in C^\infty(I_\delta;\mathscr{H}_\sigma)$. According to Lemma 2.14, $W_n\in C^{1+\mu}(I_\delta;\mathscr{H}_\sigma)$ for any $\mu\in(0,\nu)$. Thus we have $\Phi_\delta^{(n)}\in C^{1+\mu}(I_\delta;\mathscr{H}_\sigma)$. On the other hand, $h_\delta\in C^\infty(I;\mathscr{H}_\sigma)$ and $\psi_\delta\in C^\infty(I;\mathscr{H}_\sigma)$. Therefore we get $u^{(n)}\in C^{1+\mu}(I_\delta;\mathscr{H}_\sigma)$, which yields

(5.42) $$u\in C^{n+1+\mu}(I;\mathscr{H}_\sigma)$$

on account of the arbitrariness of δ. Furthermore, we get

(5.43) $$u\in C^{n+1+\nu}(I;\mathscr{H}_\sigma)\qquad(0<\nu<\tfrac{1}{4}),$$

considering that μ is arbitrary in $0<\mu<\nu$ and ν is arbitrary in $0<\nu<\frac{1}{4}$. Combining (5.40) with (5.43), we obtain

$$u_j u\in C^{n+1+\nu}\big(I;L_2(D)\big)\qquad(j=1,2,3),$$

whence follows $Ku\in C^{n+1+\nu}(I;\mathscr{H}_\sigma)$, K being the operator given by (5.22). By the same argument as in the proof of Lemma 5.2 we can then deduce that

(5.44) $$A^{\frac{1}{2}}\Phi_\delta^{(n+1)}\in C^{\mu}(I_\delta;\mathscr{H}_\sigma)\qquad(0<\mu<\nu).$$

Since $A^{\frac{1}{2}}h_\delta^{(n+1)}\in C^\infty(I_\delta;\mathscr{H}_\sigma)$ and $A^{\frac{1}{2}}\psi_\delta\in C^\infty(I;\mathscr{H}_\sigma)$, (5.44) implies

(5.45) $$A^{\frac{1}{2}}u^{(n+1)}\in C^{\mu}(I_\delta;\mathscr{H}_\sigma),$$

and, finally,

(5.45)′ $$A^{\frac{1}{2}}u\in C^{n+1+\nu}(I;\mathscr{H}_\sigma)\qquad(0<\nu<\tfrac{1}{4}).$$

Consequently,

(5.46) $$\nabla u\in C^{n+1+\nu}\big(I;L_2(D)\big).$$

With the aid of (5.40), (5.46) implies

$$(u^{(k)}\cdot\nabla)\,u^{(n+1-k)}\in C^{\nu}\big(I;L_2(D)\big)\qquad(k=0,1,\dots,n).$$

On the other hand, we have

$$(u^{(n+1)}\cdot\nabla)\,u\in C^{\mu}\big(I;L_2(D)\big)$$

for some positive μ, by means of (5.43) and Lemma 5.3. Thus we have

(5.47) $$(u\cdot\nabla)\,u\in C^{(n+1)}\big(I;L_2(D)\big),$$

and, consequently, $Fu\in C^{(n+1)}(I;\mathscr{H}_\sigma)$. This allows us to write

(5.48) $$\Phi_\delta^{(n+1)}=S_{n+1}+W_{n+1}.$$

According to Lemma 2.12 we have

$$A^\alpha W_{n+1}\in C^{\vartheta}(I_\delta;\mathscr{H}_\sigma)\qquad(0\leqq\alpha<1-\vartheta<1),$$

for $\|(Fu)^{(n+1)}\|$ is bounded in I_δ. Since $A^\alpha S_{n+1}$, $A^\alpha h_\delta^{(n+1)}$ and $A^\alpha\psi_\delta^{(n+1)}$ behave nicely in I_δ, this leads to $A^\alpha u^{(n+1)}\in C^{\vartheta}(I_\delta;\mathscr{H}_\sigma)$ and, finally, to $A^\alpha u\in C^{n+1+\vartheta}(I;\mathscr{H}_\sigma)$. Thus we have proved $(P)_{n+1}$ and therefore established the lemma. Q.E.D.

Theorem 5.4. *Under the assumptions of the Lemma 5.4, $u=u(t, x)$ is infinitely differentiable with respect to t and $\partial^n u(t, x)/\partial t^n$ is Hölder continuous in $I\times\bar{D}$ $(n=0, 1, \ldots)$.*

Proof. Let γ be any number in $\frac{3}{4}<\gamma<1$. Then $A^\gamma u\in C^\infty(I; \mathscr{H}_\sigma)$ according to Lemma 5.4. Hence we can derive the Hölder continuity of $u^{(n)}(t, x)$ in $I\times\bar{D}$ with the aid of Lemma 1.2 and Lemma 2.6 for any non-negative integer n. The identity of $u^{(n)}(t, x)$ with $\partial^n u(t, x)/\partial t^n$ is obvious in this case. Q.E.D.

Theorem 5.5. *If $f\in C^\infty(I; L_q(D))$ for some $q>3$, then $\nabla u=\nabla_x u(t, x)$ is infinitely differentiable with respect to t and $\partial^n \nabla u(t, x)/\partial t^n$ is Hölder continuous in $I\times\bar{D}$ $(n=0, 1, \ldots)$. If $Pf\equiv 0$, then the same result holds.*

Proof. We give the proof only for the case $f\in C^\infty(I; L_q(D))$. We note that $Pf\in C^\infty(I; \mathscr{H}_\sigma)$ follows from $f\in C^\infty(I; L_q(D))$, since D is bounded. Thus we can apply Lemma 5.4. Putting $\alpha=\frac{1}{2}$ and $\alpha=\gamma>\frac{3}{4}$ in Lemma 5.4, we get (see Lemma 1.2)

$$\nabla u\in C^\infty(I; L_2(D)) \tag{5.49}$$

and

$$u\in C^\infty(I; L_\infty(D)) \tag{5.50}$$

respectively. Therefore, $(u\cdot\nabla)u\in C^\infty(I; L_2(D))$, and hence $w\in C^\infty(I; L_2(D))$, w being as in Lemma 5.1. Then we can easily verify the relation

$$(\partial_m u)^{(n)}(t, x)=\int_D G_m(x, y)\, w^{(n)}(t, y)\, dy \qquad (n=0, 1, \ldots). \tag{5.51}$$

By means of (5.51) and by an argument similar to the proof of Lemma 5.3, we can prove that $\nabla u\in C^\infty(I; L_\infty(D))$. Consequently we have $(u\cdot\nabla)u\in C^\infty(I; L_\infty(D))$. Then we can derive the Hölder continuity of $(\nabla u)^{(n)}(t, x)$ in $I\times\bar{D}$ by the same arguments as in the proof of Theorem 5.2. Finally, the identity of $(\nabla u)^{(n)}(t, x)$ with $\partial^n\nabla u(t, x)/\partial t^n$ is obvious in this case. Q.E.D.

Proof of Theorem 1.6. Theorem 1.6 follows from Theorems 5.4 and 5.5. Q.E.D.

II. Regularity at $t=+0$ and in $[0, T)\times\bar{D}$

Here we study the behavior at $t=+0$ of a solution u of Pr.II which satisfies (1.6) in $(0, T)$ as well as (1.7) at $t=0$. First of all we introduce the following notion.

Definition 5.1. Let $0<\mu<1$. A function $v=v(t)$ with its values in $\mathscr{H}_\sigma$ is said to belong to the class $(R)_\mu$ if the following conditions are satisfied:

i) v is defined on $[0, \varepsilon)$ for some $\varepsilon>0$ and $A^\alpha v$ is Hölder continuous in $(0, \varepsilon)$ for any α in $0<\alpha<1$.

ii) If $0\leqq\alpha<\mu$, then $A^\alpha v$ is Hölder continuous in $[0, \varepsilon)$ and $A^\mu v$ is continuous in $[0, \varepsilon)$. Moreover, $\|A^\alpha v(t)\|=o(t^{\mu-\alpha})$ as $t\to+0$, if $\mu<\alpha<1$.

We give some examples of functions $\in(R)_\mu$ in the following lemmas, the verification of which is immediate according to Lemmas 2.10 and 2.12.

Lemma 5.5. $e^{-tA}a\in(R)_\mu$, if $a\in\mathscr{D}(A^\mu)$ $(0<\mu<1)$.

Lemma 5.6. If $w \in C(0, T)$ for some $T>0$ and $\|w(t)\|=o(t^{-1+\mu})$ as $t \to +0$, then

$$\int_0^t e^{-(t-s)A} w(s)\, ds \in (R)_\mu \qquad (0<\mu<1).$$

As to the solution u mentioned above we have the following

Theorem 5.6. *Assume that* $a \in \mathscr{D}(A^\mu)$ *and*

$$\|Pf(t)\| = o(t^{-1+\mu}) \quad as \quad t \to +0 \tag{5.52}$$

for some μ *in* $\frac{1}{4}<\mu<1$. *If the solution* u *mentioned above belongs to* $\tilde{S}[0, T)$, *then* $u \in (R)_\mu$.

Remark. This theorem is true even for $\mu=\frac{1}{4}$. The proof then depends on the inequality

$$\|Fu\| \leqq \text{const.} \|A^{\frac{1}{4}} u\| \cdot \|A^{\frac{3}{4}} u\| \tag{5.53}$$

established in K-F (*cf.* the remark after Lemma 1.2).

Proof of Theorem 5.6. We first consider the case $\frac{1}{4}<\mu<\frac{1}{2}$. We put

$$u_0(t) = e^{-tA} a + \int_0^t e^{-(t-s)A} Pf(s)\, ds$$

and

$$\Phi(u;t) = \int_0^t e^{-(t-s)A} Fu(s)\, ds = \int_0^t A^{\frac{1}{4}} e^{-(t-s)A} Hu(s)\, ds$$

as before. Then $u_0 \in (R)_\mu$ by the foregoing lemmas. Put $\beta=\mu$, and choose γ so that (4.15) holds. Then Theorem 4.2 guarantees that there exists a solution $v \in S_{\beta,\gamma}[0, \varepsilon]$ of Pr.II for some small $\varepsilon>0$. v must coincide with u owing to the uniqueness of the solutions within the class $\tilde{S}$. Consequently $u \in S_{\beta,\gamma}[0, T)$, which implies the existence of a constant K such that

$$t^{\frac{1}{4}-\beta} \|A^{\frac{1}{4}} u(t)\| \leqq K \quad \text{and} \quad t^{\gamma-\beta} \|A^\gamma u(t)\| \leqq K$$

hold for any $t \in (0, T_1]$, T_1 being an arbitrary, fixed number in $0<T_1<T$. Therefore, by means of (4.3) we have

$$\|Fu(t)\| = O(t^{-1+\beta+\varkappa}) = o(t^{-1+\mu}) \qquad (t \to +0), \tag{5.54}$$

because $\mu=\beta$ and $\varkappa=\beta-\gamma+\frac{1}{2}>0$. According to Lemma 5.6 this gives $\Phi(u;t) \in (R)_\mu$, and hence $u \in (R)_\mu$ by virtue of $u=u_0+\Phi$.

Now we consider the remaining case $\frac{1}{2} \leqq \mu<1$. We choose γ so that $\frac{3}{4}<\gamma<1$ and $\mu<\gamma$. Retracing the proof of Theorem 4.2 with slight modifications, we can construct a solution v in some interval $[0, T_1]$ $(0<T_1<T)$ such that the inequalities

$$\sup_{0<t\leqq T_1} \|A^{\frac{1}{4}} v(t)\| \leqq L \quad \text{and} \quad \sup_{0<t \leqq T_1} t^{\gamma-\mu} \|A^\gamma v(t)\| \leqq L \tag{5.55}$$

hold for some constant L. On account of the uniqueness we notice that u itself satisfies the same conditions (5.55). Hence $\|Fu(t)\| = O(t^{-\gamma+\mu}) = o(t^{-1+\mu})$, $t \to +0$, according to Lemma 1.2. Thus we get $\Phi \in (R)_\mu$ and, $u \in (R)_\mu$. Q.E.D.

Using Lemmas 1.2 and 2.6, we obtain the following theorem as a result of the preceding theorem.

Theorem 5.7. *If the assumption of Theorem* 5.6 *is satisfied for some* μ *larger than* $\frac{3}{4}$, *then* $u=u(t, x)$ *is Hölder continuous in* $[0, T)\times\bar{D}$.

6. Regularity of Solutions of Pr. II (continued)

In this section we consider the interior regularity of solutions of (1.6). Namely, we study how the smoothness of $f(t, x)$ in $I\times D$ is inherited by $u(t, x)$ in $I\times D$, where $u(t, \cdot)$ is a solution of (1.6) in an open interval I. To this end we make use of a local (with respect to t and x) integral representation of u with kernels obtained by truncation from the fundamental solution of the time-dependent Stokes equation. The first half of this section is devoted to preliminary study of the properties of these kernels, and we derive the interior regularity of u in the second half by using these properties. The interior regularity of u with respect to x at $t=+0$ will be considered in the next section.

I. Preliminaries

According to OSEEN [*21*] the fundamental solution $E(t-s, x-y)=\{E_{ij}(t-s, x-y)\}_{i,j=1,2,3}$ of the time-dependent Stokes system,

$$\partial u/\partial t-\Delta u+\nabla p=0, \qquad \operatorname{div} u=0, \tag{6.1}$$

is given by

$$E_{ij}(t-s, x-y)=(-\delta_{ij}\Delta_x+\partial^2/\partial x_i\partial x_j)\,\Phi(t-s, x-y), \tag{6.2}$$

where

$$\Phi(t, x)=\frac{1}{4\pi\sqrt{\pi}}\,\frac{1}{|x|\sqrt{t}}\int_0^{|x|}\exp(-\alpha^2/4t)\,d\alpha \qquad (t>0,\ x\in R^3). \tag{6.3}$$

$\Phi(t-s, x-y)$ is regular in $t>s$ and there satisfies

$$\Delta\Phi=-h \quad\text{and}\quad (\Delta-\partial/\partial t)\,\Phi=0, \tag{6.4}$$

h being the ordinary Green function of the heat equation:

$$h(t, x)=\frac{1}{(\sqrt{4\pi t})^3}\exp\Bigl(-\frac{|x|^2}{4t}\Bigr). \tag{6.4$'$}$$

Therefore we can write

$$E_{ij}=\delta_{ij}h+\partial^2\Phi/\partial x_i\partial x_j. \tag{6.5}$$

From (6.4) it is easily seen that each column vector of E, considered as a function of t, x, satisfies (6.1) with $p\equiv 0$. On the other hand, each row vector of E, considered as a function of s, y, satisfies

$$\partial v/\partial s+\Delta v=0 \quad\text{and}\quad \operatorname{div} v=0.$$

If we put

$$\varphi(t, x)=\int_D E(t, x-y)\,\psi(y)\,dy \qquad (t>0), \tag{6.6}$$

with any $\psi \in \mathscr{H}_\sigma$, then we have by virtue of (6.5)

$$\varphi(t, x) = \int_D h(t, x-y)\,\psi(y)\,dy - \nabla_x \int_D \nabla_y \Phi(t, x-y)\,\psi(y)\,dy$$
$$= \int_D h(t, x-y)\,\psi(y)\,dy.$$

Hence follows that

$$\lim_{t\to+0} \varphi(t, \cdot) = \psi \tag{6.7}$$

strongly in $L_2(D)$. In particular,

$$\varphi(t, x) \to \psi(x) \quad \text{as} \quad t \to +0 \tag{6.8}$$

in the sense of locally uniform convergence, if ψ is continuous and bounded in D.

In order to derive a local integral representation of the solutions, we truncate E in the following way. We fix a real-valued function $\eta = \eta(\sigma)$ defined in $[0, \infty)$ such that $\eta \in C^\infty[0, \infty)$ and

$$\eta(\sigma) = \begin{cases} 1 & (0 \leqq \sigma < 1), \\ 0 & (2 \leqq \sigma). \end{cases} \tag{6.9}$$

Introducing a positive parameter δ, we put

$$\eta^\delta(t, x) = \eta(2t/\delta)\,\eta(2|x|/\delta) \qquad (t > 0,\ x \in R^3), \tag{6.10}$$

$$\Phi^\delta(t, x) = \eta^\delta(t, x)\,\Phi(t, x) \tag{6.11}$$

and

$$h^\delta(t, x) = \eta^\delta(t, x)\,h(t, x). \tag{6.12}$$

Putting

$$\begin{aligned} &\Sigma = [0, \infty) \times R^3, \qquad \Sigma_0 = \Sigma - (0, 0) \quad \text{and} \\ &Q^\delta = \{(t, x);\ 0 \leqq t \leqq \delta,\ |x| \leqq \delta\}, \end{aligned} \tag{6.13}$$

we notice that $\Phi(t, x)$ as well as $\Phi^\delta(t, x)$ is of the class $C^\infty(\Sigma_0)$ and that the support of $\Phi^\delta(t, x)$ is contained in Q^δ. By the truncated fundamental solution we mean $E^\delta = \{E^\delta_{ij}\}_{i,j=1,2,3}$ and $e^\delta = \{e^\delta_j\}_{j=1,2,3}$, where

$$\begin{aligned} E^\delta_{ij}(t-s, x-y) &= (-\delta_{ij}\Delta + \partial^2/\partial x_i\,\partial x_j)\,\Phi^\delta(t-s, x-y), \\ e^\delta_j(t-s, x-y) &= (\Delta\,\partial/\partial x_j - \partial^2/\partial x_j\,\partial t)\,\Phi^\delta(t-s, x-y). \end{aligned} \tag{6.14}$$

Obviously $E^\delta = E^\delta(t, x) \in C^\infty(\Sigma_0)$ and its singularity at $(0, 0)$ is identical with that of E. On the other hand, e^δ is of the class $C^\infty(\Sigma)$ by virtue of (6.4). Both of E^δ and e^δ are supported by Q^δ. Furthermore, we put

$$R^\delta_{ij}(t, x) = -\delta_{ij}\Delta(\Delta - \partial/\partial t)\,\Phi^\delta(t, x), \tag{6.15}$$

and set $R^\delta(t, x) = \{R^\delta_{ij}(t, x)\}$. Then we have $R^\delta \in C^\infty(\Sigma)$ and

$$\begin{aligned} &(\Delta_x - \partial/\partial t)\,E^\delta_{ij} - \partial e^\delta_j/\partial x_i = R^\delta_{ij}, \\ &\sum_i (\partial/\partial x_i)\,E^\delta_{ij} = 0, \end{aligned} \tag{6.16}$$

$$\begin{aligned} &(\Delta y + \partial/\partial s)\,E^\delta_{ij} + \partial e^\delta_i/\partial y_j = R^\delta_{ij}, \\ &\sum (\partial/\partial y_j)\,E^\delta_{ij} = 0, \end{aligned} \tag{6.17}$$

where the arguments of E^δ, e^δ and R^δ are $t-s$ and $x-y$. If we put

(6.18) $$\varphi(t, x)=\int_D E^\delta(t, x-y)\,\psi(y)\,dy$$

with an arbitrary $\psi\in\mathscr{H}_\sigma$, then we have

$$\varphi(t, x) = -\int_D (\Delta\,\Phi^\delta)\,\psi(y)\,dy$$
$$=\int_D h^\delta(t, x-y)\,\psi(y)\,dy+\int_D K^\delta(t, x-y)\,\psi(y)\,dy.$$

Here $K^\delta(t, x)$ is a function in $C^\infty(\Sigma)$ and is supported by Q^δ. Moreover, the function

(6.19) $$S^\delta(x)=K^\delta(0, x)$$

belongs to $C^\infty(R^3)$ and is supported by the sphere $\{x; |x|\leqq\delta\}$. Making use of this function we can obtain an anologue of (6.7), namely,

(6.20) $$\lim_{t\to+0}\varphi(t, \cdot)=\psi(\cdot)+\int_D S^\delta(\cdot, -y)\,\psi(y)\,dy$$

in the L_2-convergence. If ψ is continuous and bounded in D, (6.20) holds in the sense of local uniform convergence.

Concerning the singularity of E^δ, we state the following lemmas which can be verified by straightforward calculations. There ∂_t means $\partial/\partial t$, and ∂_x^σ means $\partial^{|\sigma|}/\partial x_1^{\sigma_1}\,\partial x_2^{\sigma_2}\,\partial x_3^{\sigma_3}$, σ being a triple of non-negative integers $\sigma_1, \sigma_2, \sigma_3$ with $|\sigma|=\sigma_1+\sigma_2+\sigma_3$.

Lemma 6.1. If $K(t, x)$ denotes $(\partial_t)^n\,\partial_x^\sigma E^\delta(t, x)$, then the following propositions are true:

i) For any $\varrho>0$ the inequality

(6.21) $$|K(t, x)|\leqq C\,t^{-\frac{3}{2}+\varrho-n}|x|^{-2\varrho-|\sigma|}\qquad ((t, x)\in\Sigma),$$

holds with a constant C depending on δ, n, σ and ϱ.

ii) For any $\varrho>0$ and $\vartheta\in(0, 1)$ the inequality

(6.22) $$|K(t, x_1)-K(t, x_2)|\leqq C|x_1-x_2|^\vartheta t^{-\frac{3}{2}+\varrho-n}|x_1|^{-2\varrho-|\sigma|-\vartheta}$$

holds with a constant C depending on $\delta, n, \sigma, \vartheta$ and ϱ, where

$$(t, x_1)\in\Sigma,\qquad (t, x_2)\in\Sigma\ \text{ and }\ |x_1|\leqq|x_2|.$$

iii) For any $\varrho\in(0, \frac{3}{2}+n)$ and $\vartheta\in(0, 1)$ the inequality

(6.23) $$|K(t_1, x)-K(t_2, x)|\leqq C|t_1-t_2|^\vartheta t_1^{-\frac{3}{2}+\varrho-n-\vartheta}|x|^{-2\varrho-|\sigma|}$$

holds with a constant C depending on $\delta, n, \sigma, \vartheta$ and ϱ, where

$$(t_1, x)\in\Sigma,\qquad (t_2, x)\in\Sigma\ \text{ and }\ t_1\leqq t_2.$$

We now consider integral operators with the kernel E^δ. In what follows V is an arbitrary bounded domain in R^3 and J is an arbitrary open interval (T_0, T_1). Furthermore we put

$$\Omega=J\times V,\qquad V(\delta)=\{x;\ x\in V \text{ and dis.}(x, \partial V)>\delta\},$$
$$J(\delta)=(T_0+\delta, T_1)\ \text{ and }\ \Omega(\delta)=J(\delta)\times V(\delta)$$

for any sufficiently small $\delta>0$.

Lemma 6.2. Let

$$w(t, x) = \int_{T_0}^{t} ds \int_{V} E^{\delta}(t-s, x-y)\, v(s, y)\, dy \qquad \big((t, x) \in \Omega(\delta)\big). \tag{6.24}$$

If $v(t, x)$ is bounded in Ω and satisfies

$$|v(t_1, x) - v(t_2, x)| \leqq C|t_1 - t_2|^{\vartheta} \qquad ((t_j, x) \in \Omega), \tag{6.25}$$

for some constants C and ϑ $(0<\vartheta<1)$ independent of (t_j, x), then $\partial w(t, x)/\partial t$ exists and is Hölder continuous in $\Omega(\delta)$.

Proof. As is obvious from Lemma 6.1, $w(t, x)$ is continuous. Hence it suffices to show that the distribution derivative of w with respect to t is Hölder continuous. In order to obtain the distribution derivative we calculate with an arbitrary $\varphi \in C_0^{\infty}(\Omega(\delta))$:

$$-\iint_{\Omega(\delta)} \frac{\partial \varphi}{\partial t} w\, dt\, dx = -\lim_{\varepsilon \downarrow 0} \iint_{\Omega(\delta)} \frac{\partial \varphi}{\partial t} dt\, dx \int_{T_0}^{t-\varepsilon} ds \int_{V} E^{\delta}(t-s, x-y)\, v(s, y)\, dy$$

$$= \lim_{\varepsilon \downarrow 0} \iint_{\Omega(\delta)} \varphi\, dt\, dx \int_{V} E^{\delta}(\varepsilon, x-y)\, \{v(t-\varepsilon, y) - v(t, y)\}\, dy +$$

$$+ \lim_{\varepsilon \downarrow 0} \iint_{\Omega(\delta)} \varphi\, dt\, dx \int_{T_0}^{t-\varepsilon} ds \int_{V} \frac{\partial E^{\delta}}{\partial t}(t-s, x-y)\, \{v(s, y) - v(t, y)\}\, dy$$

$$= \iint_{\Omega(\delta)} \varphi\, W_1\, dx\, dt,$$

where

$$W_1(t, x) = \int_{T_0}^{t} ds \int_{V} \frac{\partial E^{\delta}}{\partial t}(t-s, x-y)\, \{v(s, y) - v(t, y)\}\, dy.$$

Here use has been made of the fact that

$$\Big| \int_{V} E^{\delta}(\varepsilon, x-y)\, \{v(t-\varepsilon, y) - v(t, y)\}\, dy \Big| \leqq C\, \varepsilon^{\mu} \qquad (0<\mu<\vartheta),$$

which follows from (6.21) and (6.25). The Hölder continuity of the distribution derivative W_1 of w is easily verified with the aid of Lemma 6.1 by an argument familiar in potential theory. Q.E.D.

We proceed to consideration of the derivatives of w with respect to x. In the following lemma, ∇ means the gradient in x-variables, which is interpreted in the distribution sense if necessary.

Lemma 6.3. Let v and w be as in the preceding lemma.

i) If $\|v(t, \cdot)\|_{2,V}$ is bounded in J, then $\|\nabla w(t, \cdot)\|_{5,V(\delta)}$ is bounded in $J(\delta)$. (See the footnote to Lemma 1.2.)

ii) If $\|v(t, \cdot)\|_{5,V}$ is bounded in J, then $\|\nabla w(t, \cdot)\|_{\infty,V(\delta)}$ is bounded in $J(\delta)$, that is, ∇w is bounded in $\Omega(\delta)$.

iii) If $v(t, x)$ is bounded in Ω, then for any α and β with $0<\alpha<1$ and $0<\beta<\frac{1}{2}$ we have the inequality

$$|\nabla w(t_1, x_1) - \nabla w(t_2, x_2)| \leqq C\,(|x_1 - x_2|^{\alpha} + |t_1 - t_2|^{\beta}) \tag{6.26}$$

with a constant C independent of $(t_j, x_j) \in \Omega(\delta)$. Thus ∇w is Hölder continuous in $\Omega(\delta)$.

iv) If $v(t, x)$ is bounded in Ω and satisfies

$$|v(t, x_1) - v(t, x_2)| \leqq C|x_1 - x_2|^\vartheta \qquad (0<\vartheta<1),$$

with constants C and ϑ independent of $(t, x_j) \in \Omega$, then $\nabla \nabla w(t, x)$ exists and is Hölder continuous in $\Omega(\delta)$.

Proof. First of all we note that

$$\nabla w(t, \cdot) = \int_{t-\delta}^{t} \boldsymbol{E}^1(t-s)\, v(s, \cdot)\, ds,$$

where $\boldsymbol{E}^1$ means the formal integral operator defined by

$$\left(\boldsymbol{E}^1(t-s)\, h\right)(x) = \int_V \nabla_x E^\delta(t-s, x-y)\, h(y)\, dy \qquad \left(x \in V(\delta)\right).$$

Proof of i). We put $\nu = \frac{10}{7}$ and choose ϱ so that $\frac{1}{2} < \varrho < \frac{11}{20}$. Then we have by (6.21)

(6.27) $$\left(\int |\nabla E^\delta(\tau, \xi)|^\nu d\xi\right)^{1/\nu} \leqq C \tau^{\varrho - \frac{8}{5}},$$

whence follows

$$\|\boldsymbol{E}^1(t-s)\, v(s)\|_{5, V(\delta)} \leqq C(t-s)^{\varrho - \frac{8}{5}} \|v(s)\|_{2, V} \leqq C(t-s)^{\varrho - \frac{8}{5}}$$

by Young's inequality. This immediately yields

$$\|\nabla w(t, \cdot)\|_{5, V(\delta)} \leqq C \int_{t-\delta}^{t} (t-s)^{\varrho - \frac{8}{5}}\, ds \leqq C.$$

Proof of ii). Putting $\nu = \frac{5}{4}$ and choosing $\varrho \in (\frac{1}{2}, \frac{7}{10})$, we note that (6.27) is still true. By means of Hölder's inequality we therefore have

$$\|\boldsymbol{E}^1(t-s)\, v(s)\|_{\infty, V(\delta)} \leqq C(t-s)^{\varrho - \frac{8}{5}} \|v(s)\|_{5, V}.$$

This leads immediately to the required boundedness of ∇w.

Proof of iii). Applying (6.22) with $\varrho \in (\frac{1}{2}, 1-\alpha/2)$, we obtain

$$|\nabla w(t, x_1) - \nabla w(t, x_2)| \leqq C M d^\alpha \int_{t-\delta}^{t} (t-s)^{\varrho - \frac{8}{5}}\, ds \leqq C d^\alpha,$$

where $d = |x_1 - x_2|$ and $M = \sup |v(t, x)|$. On the other hand, we have

$$|\nabla w(t_2, x) - \nabla w(t_1, x)| \leqq I_1 + I_2$$

with

$$I_1 = \left| \int_{T_0}^{t} \{\boldsymbol{E}^1(t_2 - s) - \boldsymbol{E}^1(t_1 - s)\}\, v(s)\, ds \right|$$

and

$$I_2 = \left| \int_{t_1}^{t_2} \boldsymbol{E}^1(t_2 - s)\, v(s)\, ds \right|.$$

Assuming that $\sigma \equiv t_2 - t_1$ is sufficiently small and positive, and applying (6.23) with $\varrho \in (\frac{1}{2} + \beta, 1)$, we obtain

$$I_1 \leqq C M \sigma^\beta \int_{T_0}^{t_1} (t_1 - s)^{\varrho - \frac{8}{5} - \beta}\, ds \leqq C \sigma^\beta$$

and also

$$I_2 \leqq C M \int_{t_1}^{t_2} (t_2 - s)^{\varrho - \frac{3}{2}} \leqq C \sigma^{\varrho - \frac{1}{2}} \leqq C \sigma^{\beta}.$$

Thus we have

$$|\nabla w(t_2, x) - \nabla w(t_1, x)| \leqq C \sigma^{\beta}.$$

The required Hölder continuity of ∇w follows from these inequalities.

Proof of iv). We can argue in the same manner as in the proof of Lemma 6.2. In fact, the second derivatives can be expressed as

$$\frac{\partial^2 w(t, x)}{\partial x_i \partial x_j} = \int_{T_0}^{t} ds \int_{V} \frac{\partial^2}{\partial x_i \partial x_j} E^{\delta}(t - s, x - y) \{v(s, y) - v(s, x)\} dy,$$

of which the Hölder continuity is derived by means of Lemma 6.1. Q.E.D.

II. Theorems on the Interior Regularity

The primary objective of this part is to prove the interior regularity of the solutions of (1.6) in the form of Theorem 1.7. Throughout this part u means a solution of (1.6) in $I = (0, T)$ with $Pf \in C^{\vartheta}(I; \mathscr{H}_{\sigma})$, $(0 < \vartheta < 1)$.

Lemma 6.4. Let a subdomain V and a subinterval J be such that $\bar{V} \subset D$ and $\bar{J} \subset I$. Define Ω, $V(\delta)$, $J(\delta)$ and $\Omega(\delta)$ as given in Lemma 6.2. Then at any point $(t, x) \in \Omega(\delta)$ we can write

$$u(t, x) = W_1(t, x) + W_2(t, x) + W_3(t, x), \tag{6.28}$$

where

$$W_1(t, x) = \int_{T_0}^{t} ds \int_{V} dy E^{\delta}(t - s, x - y) \big(f - (u \nabla) u\big)(s, y),$$

$$W_2(t, x) = \int_{T_0}^{t} ds \int_{V} dy R^{\delta}(t - s, x - y) u(s, y),$$

and

$$W_3(t, x) = -\int_{V} S^{\delta}(x - y) u(t, y) dy.$$

Here E^{δ}, R^{δ} and S^{δ} are the kernels introduced in the first part of this section.

Proof. (1.6) may be written

$$du(s)/ds = -A u(s) + P\hat{f}(s) \qquad (0 < s < T), \tag{6.29}$$

where $\hat{f} = f - (u \cdot \nabla) u$. Taking an arbitrary $\psi \in C_0^{\infty}(V(\delta))$, we put

$$\begin{aligned} \varphi(t - s, y) &= \int_{V(\delta)} \psi(x) E^{\delta}(t - s, x - y) dx, \\ q(t - s, y) &= \int_{V(\delta)} \psi(x) e^{\delta}(t - s, x - y) dx. \end{aligned} \tag{6.30}$$

Then we have $\varphi(t - s, \cdot) \in C_0^{\infty}(V)$, $q(t - s, \cdot) \in C_0^{\infty}(V)$ and

$$\begin{aligned} &\frac{\partial \varphi}{\partial s} + \Delta \varphi + \nabla q = \boldsymbol{R}^{\delta}(t - s) \psi, \qquad (s < t) \\ &\operatorname{div} \varphi = 0 \end{aligned} \tag{6.31}$$

owing to (6.17). Here $\boldsymbol{R}^\delta(t-s)$ is the integral operator with the kernel $R^\delta(t-s, x-y)$, see the convention concerning integral operators and their kernels stated in § 2. In view of (6.30) and (6.31) we have

$$\begin{aligned}\frac{d}{ds}\big(\varphi(t-s), u(s)\big) &= (\partial\varphi/\partial s, u) + (\varphi, -Au + P\hat{f}) \\ &= (\partial\varphi/\partial s, u) + (\Delta\varphi, u) + (\varphi, \hat{f}),\end{aligned}$$

since

$$\begin{aligned}(\varphi, -Au) &= (-A\varphi, u) = (P\Delta\varphi, u) = (\Delta\varphi, Pu) \\ &= (\Delta\varphi, u)\end{aligned}$$

and

$$(\varphi, P\hat{f}) = (P\varphi, \hat{f}) = (\varphi, \hat{f}).$$

Hence, taking account of $(\nabla q, u) = 0$, we have

$$\begin{aligned}\frac{d}{ds}\big(\varphi(t-s), u(s)\big) &= \left(\frac{\partial\varphi}{\partial s} + \Delta\varphi + \nabla q, u\right) + (\varphi, f) \\ &= \big(\boldsymbol{R}^\delta(t-s)\,\psi, u\big) + \big(\boldsymbol{E}^\delta(t-s)\,\psi, \hat{f}\big) \\ &= \big(\psi, \boldsymbol{R}^\delta(t-s)\,u + \boldsymbol{E}^\delta(t-s)\,\hat{f}\big)\end{aligned} \tag{6.32}$$

by virtue of (6.30), (6.31) and the symmetry of $\boldsymbol{R}^\delta$ and $\boldsymbol{E}^\delta$. Assuming $t\in J(\delta)$, we now integrate both sides of (6.32) over $(T_0, t-\varepsilon)$ with respect to s, where ε is a sufficiently small positive number, and then make $\varepsilon\to+0$. By virtue of (6.20) we are thus led to

$$\big(\psi, u(t)\big) + \big(\boldsymbol{S}^\delta\psi, u(t)\big) = (\psi, W_1 + W_2), \tag{6.33}$$

which implies

$$\big(\psi, u(t) - (W_1 + W_2 + W_3)(t)\big) = 0 \qquad \big(t\in J(\delta)\big),$$

owing to the symmetry of $\boldsymbol{S}^\delta$. Noting that ψ is arbitrary, we see that (6.28) is true for every $t\in J(\delta)$ and almost every $x\in V(\delta)$. Q.E.D.

Lemma 6.5. Let $f = f(t, x)$ be Hölder continuous in $\overline{J\times V}$. Then, with J, V, Ω, $J(\delta)$, $V(\delta)$ and $\Omega(\delta)$ being the same as in the preceding lemma,

i) If $\|\nabla u\|_{2,V}$ is bounded for $t\in J$, then $\|\nabla u\|_{5,V(\delta)}$ is bounded for $t\in J(\delta)$.

ii) If $\|\nabla u\|_{5,V}$ is bounded for $t\in J$, then $\|\nabla u\|_{\infty,V(\delta)}$ is bounded for $t\in J(\delta)$, that is, $|\nabla u(t, x)|$ is bounded in $\Omega(\delta)$.

iii) If $|\nabla u(t, x)|$ is bounded in Ω, then $\nabla u(t, x)$ is Hölder continuous in $\Omega(\delta)$.

iv) If $\nabla u(t, x)$ is Hölder continuous in $\overline{\Omega}$, then u, ∇u, $\nabla\nabla u$ and $\partial u/\partial t$ are Hölder continuous in $\Omega(\delta)$.

Proof. Let us recall that $Pf\in C^\vartheta(I;\mathscr{H}_\sigma)$ $(0<\vartheta<1)$, implies $u\in C^{1+\nu}(I;\mathscr{H}_\sigma)\subset C^{1+\nu}\big(I; L_2(D)\big)$ for sufficiently small $\nu>0$ (see (5.2)) and that $u(t, x)$ is Hölder continuous in $I\times\overline{D}$ (see Theorem 5.1).

We now consider each term on the right side of (6.28). Let D_x^α be any differentiation with respect to x-variables. Then $D_x^\alpha W_3$ and $(\partial/\partial t)D_x^\alpha W_3$ are Hölder continuous in $\Omega(\delta)$ since $u\in C^{1+\nu}\big(I; L_2(D)\big)$ and $S^\delta\in C_0^\infty$. Similarly, $D_x^\alpha W_2$ and

$(\partial/\partial t)D_x^\alpha W_2$ are Hölder continuous in $\Omega(\delta)$ in virtue of the smoothness of $R^\delta(t-s, x-y)$. In order to prove the propositions of the lemma, it suffices to prove i) $\sim$ iv) for u replaced by W_1. i) is proved as follows. Since we know that u is bounded in Ω, the assumption implies that $\|(u\cdot\nabla)u\|_{2,V}$ is bounded for $t\in J$. On the other hand, $\|f\|_{2,V}$ is obviously bounded for $t\in J$. Therefore, $\|\hat{f}\|_{2,V}$ is bounded for $t\in J$, and i) follows with the aid of the first proposition of Lemma 6.3. Similarly, ii) and iii) can be easily established by means of the corresponding propositions of Lemma 6.3. Finally, iv) follows from Lemma 6.2 and the last proposition of Lemma 6.3. Q.E.D.

Proof of Theorem 1.7. It suffices to apply the propositions i) $\sim$ iv) in the preceding lemma successively. Q.E.D.

In the same way we can establish the following theorem, making use of Theorem 5.5.

Theorem 6.1. *Let u be a solution of* (1.6) *in an open interval I. If f satisfies the assumption of Theorem* 5.5 *and is of the class $C^\infty(I\times D)$, then $u\in C^\infty(I\times D)$.*

7. Concluding Remarks

In this section we first consider the interior regularity of solutions of Pr.II at $t=+0$ and the existence and the regularity of the pressure p associated with the solutions u. This consideration is naturally required in order to show that the solutions of Pr.II are classical solutions of the initial value problem Pr.I. Thereafter we give some remarks concerning the relations between other authors' results and ours. The last part of this section is devoted to a brief discussion of the possible generalizations of our method to more general cases including the two dimensional problems.

I. Interior Regularity at $t=+0$

Suppose that u is a solution of the initial value problem Pr.II in $[0, T)$. As to the behavior of u near $t=0$, we have established Theorems 5.6 and 5.7. However, more detailed results can be derived if we confine our attention to the interior of D. For instance, we have

Theorem 7.1. *Let*

$$a\in\mathcal{D}(A^\beta) \quad \text{and} \quad \|Pf(t)\|=O(t^{-1+\beta}) \qquad (t\to+0) \tag{7.1}$$

for some $\beta>\frac{1}{4}$. Let V be an open subset of D such that $a(x)$ is continuous in V and $\|f(t)\|_{2,V}=O(t^{-\mu})$ for some $\mu<\frac{1}{4}$. Then the solution $u(t, x)$ tends to $a(x)$ locally uniformly in V as $t\to+0$.

The proof of this theorem is nearly parallel to that of Theorem 1.7. In this case, however, we make use of an integral representation of $u(t, x)$ with the kernel $\tilde{E}^\delta$ which we construct from E by truncating only with respect to x-variables. This integral representation is essentially of the same structure as (6.28), though one new term $\tilde{W}_0(t, x)$ appears on the right side:

$$\tilde{W}_0(t, x)=\int_V \tilde{h}^\delta(t, x-y)\,a(y)\,dy,$$

where $\tilde{h}^{\delta}(t, x)$ is the ordinary Green function of the heat equation truncated with respect to x. Moreover, this integral representation is valid in $[0, T) \times V(\delta)$, whereas Lemma 6.1 remains true if we replace E^{δ} by $\tilde{E}^{\delta}$. Thus we can improve the regularity of u in $[0, T) \times V(\delta)$ in an iterative manner as before. For instance, $\|u(t)\|_{\infty, V(\delta)} = O(1 + t^{\alpha+\beta+\varrho-1})$ $(t \to +0)$, follws from $\|u(t)\|_{\infty, V} = O(1+t^{\alpha})$ and $\|\nabla u(t)\| = O(t^{\beta-\frac{1}{2}})$ for any $\varrho \in (\frac{1}{2}, \frac{3}{4})$ if $\alpha + \beta + \frac{1}{2} > 0$.

Furthermore, we can prove the following theorem:

Theorem 7.2. *In the preceding theorem, assume in addition that $a \in C^1(V)$ and $\|f(t)\|_{p, V} = O(1)$, $t \to 0$, for some $p > 3$. Then $\nabla u(t, x)$ tends to $\nabla a(x)$ locally uniformly in V as $t \to +0$.*

II. Pressure Associated with the Solutions of (1.6)

Suppose that u is a solution of (1.6) in an open interval I. Then a scalar function $p \in L_1^{\text{loc}}(I \times D)$ is called the pressure associated with u, if p satisfies the Navier-Stokes equation (1.1) together with u in the distribution sense. Obviously, ∇p is uniquely determined by u, so that p is unique apart from an arbitrary additive function $p_0(t)$ depending only on t. The existence of the pressure p is given by the following

Theorem 7.3. *Let u be a solution of* (1.6) *in an interval I, where $f \in C(I; L_2(D))$. Define*

$$p(t, x) = \int_D g(x, y) w(t, y) dy \qquad ((t, x) \in I \times D), \tag{7.2}$$

where $w = -du/dt - (u \cdot \nabla)u + f$ and g is the pressure part of Odqvist's Green function introduced in § 2. *Then p is the pressure associated with u.*

The proof of this theorem can be carried out easily by means of Lemma 5.1 and the basic properties of G and g.

According to Lemma 2.1 the singularity of g is similar to that of ∇G. Therefore we can derive the following theorem from Lemma 5.3 in the same way as for Theorem 5.2.

Theorem 7.4. *If $Pf \in C^{\vartheta}(I; \mathscr{H}_{\sigma})$ $(\frac{1}{2} < \vartheta < 1)$, and if $f \in C^{\varkappa}(I; L_q(D))$ $(0 < \varkappa < 1$, $3 < q)$, then the pressure p given by* (7.2) *is Hölder continuous in $I \times \bar{D}$.*

The interior regularity of p follows immediately from that of u, because p satisfies

$$\Delta p = -\operatorname{div}(u \cdot \nabla) u + \operatorname{div} f$$

in the distribution sense.

Theorem 7.5. *If $f \in C^{\vartheta}(I; L_2(D))$, $(0 < \vartheta < 1)$, and $f(t, x)$ is Hölder continuous in $I \times D$, then the pressure p given by* (7.2) *as well as ∇p is Hölder continuous in $I \times D$. Moreover, the equation* (1.1) *is satisfied in the classical sense by u and p.*

Summing up the results so far obtained, we obtain the following theorem which states that the solutions of Pr.II are actually classical solutions of Pr.I.

Theorem 7.6. *Suppose that the assumptions of Theorems* 7.1 *and* 7.5 *are satisfied with $V = D$. Then any solution u of* Pr.II *and the associated pressure p satisfy all equations of* Pr.I.

Recalling the existence theorems of the solutions of Pr.II stated in § 1, we now arrive at the following existence theorem of the classical solutions.

Theorem 7.7. *Let*

$$a \in \mathscr{D}(A^{\beta}) \cap C(D), \tag{7.3}$$

and let $t^{1-\beta} f(t, x)$ be Hölder continuous in $[0, \infty) \times \bar{D}$ for some $\beta > \frac{1}{4}$. Then there exists a unique (except for an arbitrary function $p_0(t)$ added to p) solution $\{u, p\}$ of Pr.I *in $0 \leqq t < T$, where T is a positive number depending on a and f. In particular, we can put $T = \infty$ if a and f are so weak as to satisfy the conditions stated in Theorem* 1.3 *or Theorem* 1.4.

III. Remarks on Other Results[1]

Restricting our consideration to the three dimensional case, we here refer to some relevant results due to other authors.

In [*11*] KISELEV & LADYZHENSKAIA established the existence of a unique weak solution of Pr.I, which is, in general, local in time, under the assumption that

$$a \in \mathscr{D}(A) \tag{7.4}$$

and

$$\int_0^T \int_D \left(|f|^2 + \left| \frac{\partial f}{\partial t} \right|^2 \right) dx\, dt < \infty. \tag{7.5}$$

In their work, they did not make use of the operator A. Nevertheless, their assumption on a can be written in the form (7.4) in consideration of a theorem due to CATTABRIGA [*1*] and to LADYZHENSKAIA [*13*]. It is easy to see that (7.5) implies $f \in C^{\frac{1}{2}}([0, T]; L_2(D))$. Thus the assumptions (7.4) and (7.5) are more stringent than our assumptions for the existence of solutions. On the other hand, our regularity theorems for the solutions of (1.6) are applicable to their weak solutions on account of uniqueness[2]. Incidentally, this remark establishes the regularity of their weak solutions, of which the differentiability with respect to t seems to have been unsettled.

As we mentioned in § 1, SOBOLEVSKII [*23, 24*] attacked the Navier-Stokes initial value problem through an approach quite close to ours. However, he

[1] After completion of this manuscript our attention was called to the following book by D. E. DOLIDZE which contains his work published in Prikl. Matem. Mech. **2** (1948): DOLIDZE, D. E., Some problems for non-stationary flows of viscous fluids. IAN Gruz. SSR (1960), Tbilisi. There he states some theorems concerning smooth solutions of the initial-boundary value problem, which are based on the potential theory for the time-dependent Stokes equation and apparently include known results on smooth solutions. However, his arguments for the differentiability of the solutions and for some other propositions seem to be incomplete.

[2] Let $v = v(t, x)$ be a weak solution in the sense of [*11*] defined in $0 \leqq t \leqq T$. $\|\nabla v(t, \cdot)\| = \|A^{\frac{1}{2}} v\|$ is uniformly bounded in the interval $[0, T]$ according to [*11*]. Let τ be any number in $0 < \tau < T$. Then we can construct a solution u of Pr. II with the initial moment τ and initial value $u(\tau) = v(\tau, \cdot)$ in an interval $[\tau, \tau + \delta]$, $(\delta > 0)$. $\|A^{\frac{1}{2}} u(t)\|$ is bounded in $[\tau, \tau + \delta]$ by Theorem 5.6. It is easy to see that u is a weak solution in the sense of [*11*]. By the uniqueness theorem contained in [*11*] v and u coincide with each other in $[\tau, \tau + \delta] \cap [0, T]$. Since τ is arbitrary, we see that $v(t, \cdot)$ is a solution of (1.6) in $(0, T)$.

did not use the inequality (1.18) and, consequently, was led to somewhat weaker results. For instance, he needed $\vartheta > \frac{3}{4}$ in order to prove the differentiability of his solution, ϑ being the Hölder exponent of f.

S. ITO [6] was the first to publish a complete proof of the existence of the classical solutions. Moreover, his work is interesting in that it admits non-homogeneous boundary conditions and in that the solution is constructed chiefly with elementary and classical methods. When the boundary condition is homogeneous, however, his assumptions are stronger than ours, for he assumes that

$$a \in C(\bar{D}) \cap C^2(D), \qquad \Delta a \in L_2(D) \cap C^1(D), \qquad \operatorname{div} a = 0$$
$$a \mid \partial D = 0 \quad \text{and} \quad f \in C^1([0, \infty) \times \bar{D}).$$

IV. Remarks on Further Generalizations

Here we indicate briefly how our method could be extended to more general cases.

a) Inhomogeneous Boundary Conditions. Suppose that we are given an inhomogeneous boundary condition

$$u(t, \xi) = \beta(t, \xi) \qquad (\xi \in \partial D,\ t > 0) \tag{7.6}$$

instead of (1.3), where β is the boundary value of a given solenoidal function $b(t, x)$. Then we can seek the solution u in the form $u = v + b$, $\big(v(t) \in \mathscr{H}_\sigma\big)$. If b is not too singular, the additional linear terms appearing in the equation for v do not entail any essential difficulty when we apply our method to v. When β alone is given, we have only to extend β to a suitable b using ITO's technique [6].

b) Unbounded Domains. Suppose that the flow is considered in an unbounded domain D with a bounded complement. Then we first reduce the problem for u to that of $v = e^{-t}u$. We can deal with the problem for v in the same way as above, if we replace A by $A_1 = A + 1$ which possesses the bounded inverse A_1^{-1}. The details will be given in a subsequent paper.

c) Two-dimensional Case. In the previous paper K-F [10] we have established the unique and global existence of the solutions of Pr.II in the 2-dimensional problem. There the only assumption on a is

$$a \in \mathscr{H}_\sigma. \tag{7.7}$$

The regularity of the solution can be proved with the aid of the estimate

$$\|u\|_\infty \leqq C_\gamma \|A^\gamma u\| \qquad (\tfrac{1}{2} < \gamma), \tag{7.8}$$

which takes the place of (1.18) in the 3-dimensional problem.

d) L_p-theory. It should be remarked that (7.7) does not involve the fractional power of A, whose domain is in general difficult to describe explicitly, whereas the assumption on a involves a fractional power of A in the 3-dimensional problem. In principle, it is desirable to have existence theorems in which the assumption on the initial velocity is not only sufficiently weak but easy to verify. It appears that this is possible if we release ourselves from L_2-theory to general L_p-theory. In a subsequent paper we shall show that, under certain circumstances, it suffices to assume

$$\|a\|_p < +\infty \quad \text{for some } p > m \tag{7.9}$$

in the m-dimensional problem ($m \geqq 2$) in addition to the assumptions $\operatorname{div} a = 0$, $a|_{\partial D} = 0$ in a certain generalized sense.

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