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Finite element discretization of the Navier–Stokes equations with a free capillary surface

Eberhard Bänsch

Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, 10117 Berlin, Germany; e-mail: baensch@wias-berlin.de

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Summary. The instationary Navier–Stokes equations with a free capillary boundary are considered in 2 and 3 space dimensions. A stable finite element discretization is presented. The key idea is the treatment of the curvature terms by a variational formulation. In the context of a discontinuous in time space–time element discretization stability in (weak) energy norms can be proved. Numerical examples in 2 and 3 space dimensions are given.

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1 Introduction

Many flow problems in physics and applied sciences lead to the incompressible Navier–Stokes equations with a free capillary surface. From the great variety of possible applications we mention

- coating flows, see for instance [24, 28],
- flow in semiconductor melts, see e.g. [9, 23, 25, 29, 31].

First applications of the method presented here to the simulation of semiconductor melts can be found in [5].

Physically speaking, a capillary boundary condition can be viewed as a balance of forces on the free surface: normal stresses of the fluid field are balanced by the surface tension. Mathematically we are dealing with the instationary Navier–Stokes equations, reading in non–dimensional form:

Let an initial velocity u_0 and an initial domain Ω_0 be given with $\partial\Omega_0 = \Gamma_N(0) \cup \Gamma_D$ and $\Gamma_N(0)$ be a closed surface without boundary. Find the ve-

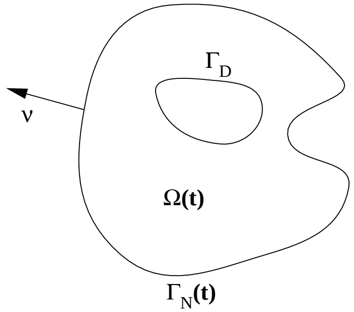

Fig. 1. Example of the geometric situation

locity u, the pressure p and the free capillary surface $\Gamma_N = \Gamma_N(t)$ (or equivalently $\Omega(t)$) such that for $0 < t < t^*$

$$
(1)\qquad
\begin{aligned}
\partial_t u \;+\; u\cdot\nabla u \;-\; \nabla\cdot\sigma = f &\quad \text{in } \Omega(t)\subseteq \mathbb{R}^d,\\
\operatorname{div} u = 0 &\quad \text{in } \Omega(t)\subseteq \mathbb{R}^d,\\
\\
u = 0 &\quad \text{on } \Gamma_D,\\
ReCa\,\nu\cdot\sigma\,\nu = \kappa &\quad \text{on } \Gamma_N(t),\\
\tau_i\cdot\sigma\,\nu = 0 &\quad \text{on } \Gamma_N(t), \quad i = 1,\ldots d-1,\\
u\cdot\nu = V_{\Gamma_N} &\quad \text{on } \Gamma_N(t),\\
\\
\Omega(0) = \Omega_0,&\\
u(0,\cdot) = u_0 &\quad \text{in } \Omega(0)
\end{aligned}
$$

with $\kappa = \sum_{i=1}^{d-1} \kappa_i$ the sum of the principal curvatures, i.e. $d-1$ times the mean curvature, and the convention that $\kappa < 0$ if Ω is convex. ν, τ_i are the unit outer normal and tangential vectors respectively and V_{Γ_N} is the normal velocity of the free boundary, see also Fig. 1 for notations.

$$
\sigma_{ij} = \sigma(u,p)_{ij} = \frac{1}{Re} D(u)_{ij} \;-\; p\,\delta_{ij}
$$

is the stress tensor and

$$
D(u)_{ij} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
$$

the deformation tensor. The Reynolds number $Re > 0$ and the capillary number $ReCa > 0$ are given by

$$
Re = \frac{UL}{\bar\nu}, \qquad Ca = \frac{\bar\nu \rho U}{\bar\gamma}
$$

with L a length scale, U a typical velocity, $\bar\nu$ the kinematic viscosity, ρ the density of the fluid and $\bar\gamma$ the surface tension.

There is a great deal of literature concerned with analysis for free boundary problems involving a capillary free boundary condition, see [6, 30, 32, 34] for a small selection. Usually existence of solutions for the time–dependent problem is shown by fix–point arguments in parabolic spaces. Since for a numerical treatment of the problem we are interested in a *time marching* procedure, the analytical methods do not give direct hints how to discretize problem (1). Discretizing the above problem, the free boundary conditions in (1) cause particular problems due to

Treatment of the curvature terms. It is not straightforward how to define a discrete curvature of the boundary of a triangulation, where the boundary is $C^{0,1}$ only and therefore a pointwise evaluation of the curvature is meaningless. Numerically this problem is even more involved in 3 space dimensions than in 2 dimensions.

Stable time discretization. In order to get an unconditionally stable discretization the curvature terms have to be incorporated in an implicit way.

"Efficiency" of the discretization. The time discretization should result for each time step in a quasi–stationary system of equations which may be solved efficiently. It is preferable that the structure of the discrete equations is more or less the same as in the case without free surface, so that one can extend standard Navier–Stokes solvers to problem (1).

We present a finite element discretization of problem (1) which is based on a variational formulation of the mean curvature as a functional, given by an integration by parts of the Laplace Beltrami operator on Γ_N, see also [13]. Due to the variational structure of the discretization the scheme works in 2D as well as in 3D.

The treatment of the curvature terms leads to an additional contribution to the left hand sides of the equations, which is symmetric and positive semi–definite, thus providing a nice structure of the algebraic systems to solve.

Stability in energy norms can be proved in the context of a first order in time space–time finite element discretization. Due to the variational formulation, the fundamental concept also works for other time–discretizations, which may be preferable from the point of view of efficiency and accuracy. The numerical examples presented in Sect. 7 were computed using the so called *fractional step θ–scheme*, see [3] for a more detailed discussion on algorithmical aspects.

Analyzing a discretization of problem (1) opens up additional mathematical questions:

Geometrical stability. The "natural" norms of the problems are too weak to control a sharp free surface. Furthermore situations may occur where for instance the topology of Ω changes, e.g. a splitting when forces are too large. Moreover, in cases with even low Reynolds numbers formations of (nearly) cusps etc. may occur, see [22]. Thus an a priori long time control of the geometry is delicate. This difficulty cannot be overcome —in our opinion— in the context of finite elements and simple energy estimates. Instead, we *assume* geometrical regularity *a priori* by (A), see Sect. 2 below, and we restrict ourselves to the case that the topology of Ω does not change.

Time–dependent domains. The motion of the domain causes a flux of energy at the boundary which is balanced by the nonlinearity. In order to get an energy estimate the discrete scheme should have a similar conservation property. One way to achieve this is the use of space–time finite elements.

Let us mention some related work:

Frederiksen and Watts [17] use a Galerkin approach for the bulk equations and couple them implicitly with a finite difference formulation for the curvature terms.

In [9, 10] the stationary problem is treated by a finite element method.

For a *linearized* problem, i.e. linear equations on a fixed domain, in [8, 10] a variational formulation is given. Stability and convergence of the approximate solutions are shown. The formulation used in this approach is similar to our formulation.

In [27] a finite element discretization for a stationary, 2D model problem, i.e. the Navier–Stokes equations in the bulk replaced by a scalar Laplace equation, is formulated and analyzed.

The rest of this paper is organized as follows:

First we introduce space–time finite elements on (discrete) time–dependent domains in Sect. 2. In Sect. 3 we review some simple differential geometry, which is needed to derive the variational formulation. The discretization of problem (1) is given in Sect. 4. In Sect. 5 we give a proof of unconditional stability of the proposed scheme, Theorem 1. Furthermore we prove the existence of solutions to the discrete problem in Theorem 2 and show that one may solve the equations by an iteration which decouples the geometry problem from the flow problem, Theorem 3. More general geometric situations, where Γ_N has a $d-2$ dimensional boundary, are treated in Sect. 6. Finally, in Sect. 7 we present some numerical examples in 2D and 3D. As mentioned above, the results were obtained using the fractional step θ–scheme instead of space–time elements. Although not covered by our theory, the incorporation of the curvature terms is done in the same way as outlined below. On the other hand, the fractional step θ–scheme provides an efficient way to solve the Navier–Stokes equations.

2 Space–time elements

A natural way to discretize problems on time–dependent domains is the use of space–time elements. We consider time–dependent domains $\Omega(t) \subseteq I\!R^d$, $d = 2$ or $d = 3$, of the following form:

There exists a reference domain $\hat{\Omega} \subseteq I\!R^d$ polygonally bounded with $\partial\hat{\Omega} = \hat{\Gamma}_N \cup \hat{\Gamma}_D$ and a conforming, regular triangulation $\mathcal{T}$ such that

$$\bar{\hat{\Omega}} = \bigcup_{T \in \mathcal{T}} T.$$

If not stated otherwise we assume that $\hat{\Gamma}_N$ is a closed surface without boundary. Let Σ be the $d-1$ dimensional conforming, regular triangulation of $\hat{\Gamma}_N$–faces of $\mathcal{T}$, i.e.

$$\hat{\Gamma}_N = \bigcup_{S \in \Sigma} S.$$

Let $S_h = S_h(\mathcal{T}) \subseteq C^0(\bar{\hat{\Omega}}, I\!R^d)$ and $U_h = U_h(\Sigma) \subseteq C^0(\hat{\Gamma}_N, I\!R^d)$ be finite element spaces with

$$\Lambda_{|\hat{\Gamma}_N} = \lambda \in U_h \qquad \text{for } \Lambda \in S_h.$$

Furthermore we assume that there exists a continuous extension operator

$$E : U_h \to S_h$$

with $(E\lambda)_{|\hat{\Gamma}_N} = \lambda$. That is, U_h is the space of traces (with respect to $\hat{\Gamma}_N$) of functions in S_h.

Let $0 = t_0 < t_1, \ldots < t_N = t^*$ be a partition of the time interval and $\tau_n := t_{n+1} - t_n$. For $\Lambda_n \in S_h$, $n = 0, \ldots N$, we define

$$\begin{aligned}
&\Phi_n : [t_n, t_{n+1}] \times \hat{\Omega} \to I\!R^d \\
&\Phi_n(t, \hat{x}) = (1-\hat{t})\Lambda_n(\hat{x}) + \hat{t}\Lambda_{n+1}(\hat{x}), \quad \hat{t} = \frac{t - t_n}{\tau_n} \\
&\Phi := \sum_{n=0}^{N-1} \Phi_n \chi_{(t_n, t_{n+1}]} \\
&\Omega(t) := \Phi_n\left(t, \hat{\Omega}\right) \qquad \text{for } t \in [t_n, t_{n+1}].
\end{aligned} \tag{2}$$

In order to have a proper space–time domain we pose the following assumption on the geometry:

(A) For $0 < t < t^*$ the transformation $\Phi(t, \cdot) : \hat{\Omega} \to \Omega(t)$ is globally injective and $\inf_{\hat{x} \in \hat{\Omega}} \det D\Phi(t, \hat{x}) > 0$.

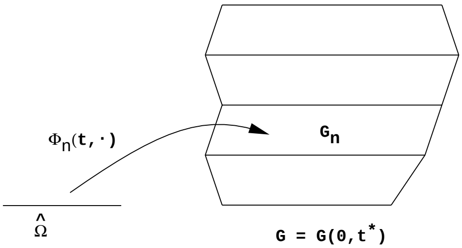

Fig. 2. Reference domain $\hat{\Omega}$ and the space–time domain G

Remark 1. i) If S_h is the finite element space of piecewise linears then $\hat{\Omega}$ can be chosen as $\Omega(0)$ and $\Lambda_0 = id$. More general choices of S_h allow for a higher order parameterization of the boundary. Usually, Λ_n will be piecewise linear in the interior of $\hat{\Omega}$ and of higher polynomial order at the boundary only.

ii) By the above construction of the space–time domain we restrict ourselves to the case that the topological types of Ω and Γ_N do not change.

For $0 \le r < s \le t^*$ define $G(r,s) := \{(t,x) \,|\, r < t < s,\, x \in \Omega(t)\}$, $G_n := G(t_n, t_{n+1})$ and $G := G(0,t^*)$. For $T \in \mathcal{T}$ and $0 \le n \le N-1$ we define the $d+1$ dimensional prism $Q_n(T) := \{(t, \Phi_n(t,x) \,|\, t \in [t_n, t_{n+1}], x \in T\}$. By the construction of G and assumption (A) we have that $Q := \{Q_n(T) \,|\, 0 \le n \le N-1,\, T \in \mathcal{T}\}$ is a conforming partition of G into $d+1$ dimensional space–time prisms, see also Fig. 2. For a function v given on G we define

$$\hat{v}(t,x) := v(t, \Phi(t,x)).$$

Let $V_h = V_h(\mathcal{T}) \subseteq \left(H^{1,2}(\hat{\Omega})\right)^d$ with $\hat{v}_{|\hat{\Gamma}_D} = 0$ for $\hat{v} \in V_h$ and $\hat{v}_{|\hat{\Gamma}_N} \in U_h$ for $\hat{v} \in V_h$, $W_h = W_h(\mathcal{T}) \subseteq L^2(\hat{\Omega})$ be finite element spaces for the velocity and the pressure respectively. We define the space–time element spaces

$$\mathcal{V} := \{v : G \to I\!R^d \,|\, \hat{v}_{|(t_n,t_{n+1}]} \in \mathcal{P}_0((t_n,t_{n+1}]) \times V_h,\ 0 \le n \le N-1\},$$
$$\mathcal{W} := \{q : G \to I\!R \,|\, \hat{q}_{|(t_n,t_{n+1}]} \in \mathcal{P}_0((t_n,t_{n+1}]) \times W_h,\ 0 \le n \le N-1\}.$$

By this definition $\mathcal{V}, \mathcal{W}$ are finite element spaces such that $\hat{v}, \hat{q}$ are piecewise constant in time and may be discontinuous at t_n. Note that in general v, q themselves are not constant in time. For $v \in \mathcal{V}$ (and analogously for $q \in \mathcal{W}$) we set

$$v^n := v(t_n,\cdot) = \lim_{\epsilon \searrow 0} v(t_n - \epsilon, \cdot)\,, \quad v^{n+0} := \lim_{\epsilon \searrow 0} v(t_n + \epsilon, \cdot)$$

and

$$[v](t_n) := v^{n+0} - v^n.$$

In the following a subscript corresponding to a domain in $I\!R^{d+1}$ indicates space–time integration, for instance

$$(u,v)_{G_n} := \int_{t_n}^{t_{n+1}} \int_{\Omega(t)} uv \, dx \, dt$$

and so on.

3 Some help from differential geometry

Let Γ be a closed orientable $(d-1)$–dimensional manifold without boundary embedded in $I\!R^d$ and χ a local parameterization of Γ. Denote by $(g)_{ij} = g_{ij} = \partial_{u_i}\chi \cdot \partial_{u_j}\chi$ the metric tensor of Γ and $g^{\mathrm{ij}} := (g^{-1})_{ij}$. For a function f on Γ define the tangential derivative $\underline{\nabla} f$ of f by

$$\underline{\nabla} f := g^{\mathrm{ij}} \partial_{u_i}(f \circ \chi) \, \partial_{u_j}\chi$$

and the Laplace–Beltrami operator

$$\underline{\Delta} f := \frac{1}{\sqrt{\det g}} \, \partial_{u_i}\Big(\sqrt{\det g} \; g^{\mathrm{ij}} \partial_{u_j}(f \circ \chi)\Big),$$

see for instance [11, 18]. We will make use of the identity

$$\underline{\Delta} id_\Gamma = \kappa \, \nu$$

with ν a unit normal vector field on Γ and $\kappa = \sum_{i=0}^{d-1} \kappa_i$ the sum of the principle curvatures (with an appropriate sign according to ν, see also the convention in (1)). If φ is a smooth vector valued function on Γ we can integrate by parts to get

$$\int_\Gamma \kappa \, \nu \cdot \varphi = \int_\Gamma (\underline{\Delta} id_\Gamma) \cdot \varphi = -\int_\Gamma \underline{\nabla} id_\Gamma \cdot \underline{\nabla} \varphi.$$

Note that there are no boundary terms in the above identity, because we assumed Γ to be a closed surface without boundary, compare Sect. 6 for a more general case. Now let $\Gamma = \Gamma_N$ be part of the boundary of $\Omega \subseteq I\!R^d$,

$\partial\Omega = \Gamma_N \cup \Gamma_D$. For smooth functions $u, \varphi : \Omega \to \mathbb{R}^d$ and $p : \Omega \to \mathbb{R}$ with $\operatorname{div} u = 0$ one computes

$$\begin{aligned}&\frac{1}{2Re}\int_\Omega D(u) : D(\varphi) \; - \; \int_\Omega p \operatorname{div} \varphi \\ &= \int_\Omega \left(-\frac{1}{Re}\Delta u + \nabla p\right) \cdot \varphi \; + \; \int_{\Gamma_N} \nu \cdot \sigma(u,p)\varphi\end{aligned}$$

for all φ with $\varphi = 0$ on Γ_D. If u, p are smooth functions fulfilling the boundary conditions in (1), i.e. $\nu \cdot \sigma\tau_i = 0$ for $i = 1, \ldots d-1$ and $ReCa\nu \cdot \sigma\nu = \kappa$ on Γ_N, then the above boundary integral can be transformed into

$$\begin{aligned}\int_{\Gamma_N} \nu \cdot \sigma(u,p)\varphi &= \int_{\Gamma_N} \nu \cdot \sigma(u,p)\nu \; \nu \cdot \varphi = \frac{1}{ReCa}\int_{\Gamma_N} \kappa \, \nu \cdot \varphi \\ = \; \frac{1}{ReCa}\int_{\Gamma_N} \underline{\Delta} id_{\Gamma_N} \cdot \varphi &= -\frac{1}{ReCa}\int_{\Gamma_N} \underline{\nabla} id_{\Gamma_N} \cdot \underline{\nabla}\varphi\end{aligned}$$

so that

$$\begin{aligned}&\frac{1}{2Re}\int_\Omega D(u) : D(\varphi) \; - \; \int_\Omega p \operatorname{div} \varphi \; + \; \frac{1}{ReCa}\int_{\Gamma_N} \underline{\nabla} id_{\Gamma_N} \cdot \underline{\nabla}\varphi \\ &= \int_\Omega \left(-\frac{1}{Re}\Delta u + \nabla p\right) \cdot \varphi\end{aligned} \tag{3}$$

for all φ with $\varphi = 0$ on Γ_D. On the other hand, if (3) holds for a pair of functions u, p with $\operatorname{div} u = 0$ for all such test functions φ, then the same calculation as above shows that the boundary conditions for the normal and tangential stresses in (1) are fulfilled. Thus (3) is the basis for the variational formulation of problem (1) in the next section.

We conclude this section by two lemmas. The first one is the basis to control the curvature terms in the stability estimate Theorem 1.

Lemma 1. *Let $m = 1$ or $m = 2$ and Γ a m–dimensional, closed, regular $C^{0,1}$–manifold embedded in $\mathbb{R}^d$, $d \in \mathbb{N}$. Moreover let $X : \Gamma \to \operatorname{rg}(\Gamma) \subseteq \mathbb{R}^d$ be a homeomorphism with $DX,\ (DX)^{-1} \in L^\infty$. Then it holds:*

$$\int_\Gamma \underline{\nabla} X \cdot \underline{\nabla}(X - Id) \; \geq \; |X(\Gamma)| \; - \; |\Gamma|.$$

Proof. Let $(\omega_k, \chi_k)_{k\in I}, \chi_k : \omega_k \subseteq I\!R^m \to \Gamma, (u_1, \dots, u_m) \mapsto \chi_k(u_1, \dots, u_m)$ be an atlas for Γ and $(\alpha_k)_{k\in I}$ a partition of unity subordinate to the covering $(\omega_k)_{k\in I}$. Then $(\omega_k, X \circ \chi_k)_{k\in I}$ is an atlas for $X(\Gamma)$. With the notation $(G(\chi_k))_{ij} = g(\chi_k)_{ij} = \partial_{u_i}\chi_k \cdot \partial_{u_j}\chi_k$, $g(\chi_k)^{\mathrm{ij}} = (G(\chi_k)^{-1})_{ij}$ one gets

$$\begin{aligned}
&\int_\Gamma \underline{\nabla} X \cdot \underline{\nabla}(X - Id) \\
&= \sum_{k\in I} \sum_{i,j=1}^{m} \int_{\omega_k} \alpha_k \partial_{u_i}(X \circ \chi_k) \cdot \partial_{u_j}(X \circ \chi_k - \chi_k) g(\chi_k)^{\mathrm{ij}} \sqrt{\det G(\chi_k)} \\
&\geq \sum_{k\in I} \left(\int_{\omega_k} \alpha_k \sqrt{\det G(X \circ \chi_k)} - \int_{\omega_k} \alpha_k \sqrt{\det G(\chi_k)} \right) \\
&= |X(\Gamma)| \; - \; |\Gamma|.
\end{aligned}$$

The last estimate is a consequence of the (pointwise) inequality

$$\sqrt{\det G(x)} - \sqrt{\det G(y)} \leq \sqrt{\det G(y)} \sum_{l=1}^{d} \sum_{i,j=1}^{m} x_i^l g^{\mathrm{ij}}(y) \, \left(x_j^l - y_j^l \right)$$

with $x, y \in I\!R^{m\times d}$, $x_i = \partial_{u_i}(X \circ \chi_k)$ and $y_i = \partial_{u_i}\chi_k$. This estimate is elementary for $m = 1$. To see this for $m = 2$ one can proceed as follows:

We may assume that $G(y)$ is diagonal. Otherwise there is an orthogonal matrix $S \in I\!R^{m\times m}$ with $D := S^{\mathrm{T}} G(y) S$, D diagonal. For $l \in \{1, \dots d\}$ we denote by $x^l := (x_1^l, \dots, x_m^l) \in I\!R^m$. Define $\tilde{x}^l := S^{\mathrm{T}} x^l, \tilde{y}^l := S^{\mathrm{T}} y^l \in I\!R^m$, $l = 1, \dots d$. It holds: $G(\tilde{x}) = S^{\mathrm{T}} G(x) S$, $G(\tilde{y}) = D$, i.e. $\tilde{y}_i \cdot \tilde{y}_j = |\tilde{y}_i|^2 \delta_{ij}$ and $G(y)^{-1} = S\, D^{-1} S^{\mathrm{T}}$.

Now the assertion is equivalent to

$$\sqrt{\det G(\tilde{x})} - \sqrt{\det D} \;\leq\; \sqrt{\det D} \sum_{l,i} \tilde{x}_i^l d_{ii}^{-1} \left(\tilde{x}_i^l - \tilde{y}_i^l \right).$$

We may assume that $\det D = d_{11} d_{22} = |\tilde{y}_1|^2 |\tilde{y}_2|^2 = 1$, otherwise consider $\hat{x}^l := \dfrac{\tilde{x}^l}{\det(D)^{1/4}}$, $\hat{y}^l := \dfrac{\tilde{y}^l}{\det(D)^{1/4}}$. It remains to show

$$\sqrt{|\hat{x}_1|^2 |\hat{x}_2|^2 \; - \; |\hat{x}_1 \cdot \hat{x}_2|^2} \; - \; 1 \;\leq\; \frac{\hat{x}_1 \cdot (\hat{x}_1 - \hat{y}_1)}{|\hat{y}_1|^2} \; + \; \frac{\hat{x}_2 \cdot (\hat{x}_2 - \hat{y}_2)}{|\hat{y}_2|^2}$$

with $|\hat{y}_1|^2 |\hat{y}_2|^2 = 1$, which is again elementary. $\square$

Lemma 2. *Let $r < s$ be real numbers and $\Phi \in C^1([r,s]; C^{0,1}(\hat{\Omega}; I\!R^d))$ with $\hat{\Omega} \subseteq I\!R^d$ an open domain. Then the divergence of $\left[\mathrm{ad}(D\Phi)\dot{\Phi}\right]$ (in the distributional sense) is a function and the following identity holds:*

$$\operatorname{div}\left(\mathrm{ad}(D\Phi)\dot{\Phi}\right) = \frac{d}{dt}\det D\Phi$$

with $\mathrm{ad}(A) = \det(A^{ji})_{ij}$ *the matrix of cofactors of* A.

Proof. Let Φ be smooth. Then

$$\begin{aligned}\operatorname{div}\left(\mathrm{ad}(D\Phi)\dot{\Phi}\right) &= \underbrace{\operatorname{div}\left(\mathrm{ad}(D\Phi)\right)}_{=0}\cdot\dot{\Phi} + \operatorname{tr}\left(\mathrm{ad}(D\Phi)D\dot{\Phi}\right)\\ &= \operatorname{tr}\left(\mathrm{ad}(D\Phi)D\dot{\Phi}\right) = \frac{d}{dt}\det(D\Phi).\end{aligned}$$

The assertion now follows by approximating Φ by smooth Φ_n's. $\square$

4 Discretization

In order to discretize problem (1) we define the following bilinear and trilinear forms. For $0 \leq n \leq N-1$ and u, v, w, q such that $\hat{u}, \hat{v}, \hat{w} \in H^1(\hat{\Omega}) \cap C^0(\bar{G}_n)$ and $\hat{q} \in L^2(\hat{\Omega})$ we set:

$$\mathbf{a}(u,v)_{G_n} := \frac{1}{2Re}\int_{t_n}^{t_{n+1}}\int_{\Omega(t)} D(u) : D(v),$$

$$\mathbf{b}(u;v,w)_{G_n} := \frac{1}{2}\left\{\int_{t_n}^{t_{n+1}}\int_{\Omega(t)} (u\cdot\nabla v\, w - u\cdot\nabla w\, v) + \int_{t_n}^{t_{n+1}}\int_{\Gamma_N(t)} v\cdot w\, u^{n+0}\cdot\nu\right\},$$

$$\mathbf{c}(q,v)_{G_n} := \tau_n \int_{\Omega(t_n)} q^{n+0} \operatorname{div} v^{n+0},$$

$$\mathbf{d}(u,v)_{G_n} := \frac{\tau_n}{ReCa}\int_{\Gamma_N(t_n)} \underline{\nabla}\left(id_{\Gamma_N(t_n)} + \tau_n u^{n+0}\right)\cdot\underline{\nabla} v^{n+0}.$$

The term $u\cdot\nabla v\, w$ in the definition of $\mathbf{b}$ is understood as

$$u\cdot\nabla v\, w := \sum_{i,j=1}^{d} u^i \partial_i v^j w^j.$$

Remark 2. If $\Gamma_N(t) = \emptyset$ then the definition of $\mathbf{b}(\cdot;\cdot,\cdot)_{G_n}$ is the same as for instance in [19]. In case $\mathcal{H}^{d-1}(\Gamma_N(t)) > 0$, where $\mathcal{H}^{d-1}$ denotes the $(d-1)$–dimensional Hausdorff measure, we have for u with $\operatorname{div} u \equiv 0$:

$$\int_{t_n}^{t_{n+1}} \int_{\Omega(t)} u \cdot \nabla u v \; - \; \mathbf{b}(u; u, v)_{G_n} = \frac{1}{2} \int_{t_n}^{t_{n+1}} \int_{\Gamma_N(t)} u \cdot v \, (u - u^{n+0}) \cdot \nu .$$

This identity describes the consistency error introduced by the modification of the nonlinearity. Note that one expects $u - u^{n+0}$ to be of the order

$$u - u^{n+0} = \mathcal{O}(\tau_n)$$

for $t \in (t_n, t_{n+1}]$.

For $f \in L^2(G)$ we define f_n to be the L^2–projection of f such that $\hat{f}_n$ is piecewise constant in time. More precisely let $0 \leq n \leq N-1$, then f_n is characterized by the following conditions: $\hat{f}_n$ is constant in time on $(t_n, t_{n+1}]$ and

$$(f_n, w)_{G_n} = (f, w)_{G_n} \tag{4}$$

for all $w \in L^2(G_n)$ with $\hat{w}$ constant in time on $(t_n, t_{n+1}]$, or equivalently

$$\hat{f}_n = \frac{\int_{t_n}^{t_{n+1}} \hat{f} \det D\Phi}{\int_{t_n}^{t_{n+1}} \det D\Phi} . \tag{5}$$

Before defining the discrete problem we state the following auxiliary result.

Lemma 3. *Let $f \in L^2(G)$ and f_n given by (4), (5). Then*

$$\tau_n \, |f_n^{n+0}|^2_{\Omega(t_n)} \leq C \, |f|^2_{G_n} .$$

Proof. Since $\det D\Phi(\cdot, x)$ is a polynomial of degree d in time, there is a constant C such that

$$|\det D\Phi(x, t_n)| \leq C \int_{t_n}^{t_{n+1}} |\det D\Phi(x, s)| ds$$

for all $x \in \hat{\Omega}$. Using this fact and the positivity of $\det D\Phi$ we can conclude

$$\tau_n \, |f_n^{n+0}|^2_{\Omega(t_n)} = \tau_n \int\limits_{\hat{\Omega}} |\hat{f}_n|^2 \det D\Phi(t_n) \le C \int\limits_{\hat{\Omega}} |\hat{f}_n|^2 \int\limits_{t_n}^{t_{n+1}} \det D\Phi$$
$$= C \int\limits_{t_n}^{t_{n+1}} \int\limits_{\hat{\Omega}} |\hat{f}_n|^2 \det D\Phi = C \, |f_n|^2_{G_n}.$$

Since f_n is the L^2–projection of f we have $|f_n|^2_{G_n} \le |f|^2_{G_n}$. □

The discrete problem now reads: Let $f \in L^2(G)$, $\Lambda_0 \in S_h$ and $u_0 \in L^2(\Omega_0)$ be given. For $0 \le n \le N-1$ find $\Lambda_{n+1} \in S_h$ and $(u,p) \in \mathcal{V} \times \mathcal{W}$ such that

$$\boxed{\begin{aligned}
&(6) && (u_t, v)_{G_n} + ([u], v^{n+0})_{\Omega(t_n)} \;+\; \mathbf{b}(u; u, v)_{G_n} \\
& && \quad + \mathbf{a}(u,v)_{G_n} \;-\; \mathbf{c}(p,v)_{G_n} \;+\; \mathbf{d}(u,v)_{G_n} \\
& && \quad = \tau_n \, (f_n^{n+0}, v^{n+0})_{\Omega(t_n)} \qquad \text{for all } v \in \mathcal{V} \\
&(7) && \mathbf{c}(q,u)_{G_n} = 0 \qquad \text{for all } q \in \mathcal{W} \\
&(8) && X^{n+1} := id_{\Gamma_N(t_n)} \;+\; \tau_n u^{n+0} : \; \Gamma_N(t_n) \to I\!R^d \\
& && \Lambda_{n+1} = E(X^{n+1} \circ \Lambda_{n|\hat{\Gamma}_N})
\end{aligned}}$$

Remark 3. i) Since Λ_{n+1} is coupled with u^{n+0} by (8) the geometry and the velocity are linked in a nonlinear way.

ii) The discretization of the parabolic term by

$$(u_t, v)_{G_n} \;+\; ([u], v^{n+0})_{\Omega(t_n)}$$

is a generalization of the methods introduced in [21] and the usual discontinuous Galerkin method, see for instance [15,16]. In case $\Gamma_N(t) = \emptyset$ the discretization of the parabolic term is the same as in [21]. However, the definitions of the ansatz spaces are different.

iii) Since

$$\mathbf{d}(u,v)_{G_n} = \frac{\tau_n}{ReCa} \int\limits_{\Gamma_N(t_n)} \underline{\nabla} X^{n+1} \cdot \underline{\nabla} v^{n+0}$$

we have a semi–implicit discretization of the curvature terms: The tangential derivatives are evaluated for X^{n+1}, whereas the domain of integration is $\Gamma_N(t_n)$. This implies a linearization of the highly nonlinear curvature operator. Compare also the discussion in [14] on different time discretization schemes (explicit, implicit, semi–implicit) for the the mean curvature flow. Furthermore note that $\mathbf{d}(\cdot,\cdot)_{G_n}$ is symmetric in the unknowns and positive semi–definite.

iv) Defining $\mathbf{d}(\cdot,\cdot)_{G_n}$ by

$$\mathbf{d}(u,v)_{G_n} := \frac{\tau_n}{ReCa} \int\limits_{\Gamma_N(t_n)} \underline{\nabla} id_{\Gamma_N(t_n)} \cdot \underline{\nabla} v^{n+0}$$

one would discretize the curvature terms explicitly. Such a discretization is conditionally stable only, which is confirmed by numerical experiments, see [3].

Remark 4. Transforming the parabolic term in (6) to $\hat{\Omega}$ the momentum equation can be written in the form

$$\begin{aligned}(\hat{u}_t - \nabla\hat{u}\cdot D\Phi^{-1}\dot{\Phi}, \det D\Phi\,\hat{v})_{[t_n,t_{n+1}]\times\hat{\Omega}} \;+\; \mathbf{b}(u;u,v)_{G_n} \;+\; \mathbf{a}(u,v)_{G_n}& \\ -\,\mathbf{c}(p,v)_{G_n} \;+\; \mathbf{d}(u,v)_{G_n} \;=\; \tau_n\,(f_n^{n+0}, v^{n+0})_{\Omega(t_n)}.&\end{aligned} \tag{9}$$

Equation (9) may be the basis for applying other time discretization schemes to the above problem. The main difference to the standard Navier–Stokes equations on a fixed domain with, say, Dirichlet boundary data is the additional term $\mathbf{d}(u,v)_{G_n}$ in the bilinear form and the advection term $-\nabla\hat{u}\cdot D\Phi^{-1}\dot{\Phi}$ for the deformation of the domain. Thus, it is easy to extend standard Navier–Stokes solvers based on finite element discretization to the case of problem (1), see also [3].

The direct application of time discretization schemes to (9) instead of using space–time elements is also referred as ALE (*Arbitrary Lagrangian Eulerian Coordinates*), see for instance [20].

5 Stability estimate and existence of the discrete solutions

Since the corresponding meaning will be clear from the context, in the sequel we shall drop the arguments of $\dot{\Phi}$ and $D\Phi^{-1}$ for simplicity. This means that we do not distinguish explicitly by the notation between e.g. $\dot{\Phi}(t,\hat{x})$ and $\dot{\Phi}(t,\Phi^{-1}(t,x))$.

Lemma 4. *Let* $0 \le n \le N-1$ *and* $u, v \in \mathcal{V}$. *Then*

$$(u_t, v)_{G_n} = \frac{1}{2}\Bigg\{\left(u^{n+1}, v^{n+1}\right)_{\Omega(t_{n+1})} - \left(u^{n+0}, v^{n+0}\right)_{\Omega(t_n)} \tag{10}$$
$$- \int_{t_n}^{t_{n+1}} \int_{\Gamma_N(t)} u \cdot v\,\dot{\Phi} \cdot \nu + \int_{t_n}^{t_{n+1}} \int_{\Omega(t)} (u\nabla v - v\nabla u) \cdot \dot{\Phi}\Bigg\},$$

with $u\nabla v \cdot \dot{\Phi} := \sum_{i,j=1}^{d} u^i \partial_j v^i \dot{\Phi}_j$.

$$(u_t, u)_{G_n} = \frac{1}{2}\left\{|u^{n+1}|^2_{\Omega(t_{n+1})} - |u^{n+0}|^2_{\Omega(t_n)} - \int_{t_n}^{t_{n+1}} \int_{\Gamma_N(t)} |u|^2 \dot{\Phi} \cdot \nu\right\}. \tag{11}$$

Proof. Using the identity

$$u_t(t,x) = \frac{d}{dt}\hat{u}(t, \Phi^{-1}(t,x)) = \hat{u}_t\,(t,\hat{x}) - \nabla\hat{u}(t,\hat{x}) \cdot D\Phi^{-1}(t,x)\dot{\Phi}(t,\hat{x})$$

and the fact that $\hat{u}_t = 0$ for $u \in \mathcal{V}$ we have $u_t = \hat{u}_t - \nabla\hat{u} \cdot D\Phi^{-1}\dot{\Phi} = -\nabla\hat{u} \cdot D\Phi^{-1}\dot{\Phi}$. Then

$$\int_{t_n}^{t_{n+1}} \int_{\Omega(t)} u_t v$$
$$= - \int_{t_n}^{t_{n+1}} \int_{\hat{\Omega}} \hat{v}\nabla\hat{u} \cdot D\Phi^{-1}\dot{\Phi}\,\det D\Phi$$
$$= -\frac{1}{2} \int_{t_n}^{t_{n+1}} \int_{\hat{\Omega}} \operatorname{div}\left(\hat{v} \cdot \hat{u}\,\mathrm{ad}(D\Phi)\dot{\Phi}\right) - \frac{1}{2} \int_{t_n}^{t_{n+1}} \int_{\hat{\Omega}} \hat{v}\nabla\hat{u} \cdot \mathrm{ad}(D\Phi)\dot{\Phi}$$
$$+\frac{1}{2} \int_{t_n}^{t_{n+1}} \int_{\hat{\Omega}} \hat{u}\nabla\hat{v} \cdot \mathrm{ad}(D\Phi)\dot{\Phi} + \frac{1}{2} \int_{t_n}^{t_{n+1}} \int_{\hat{\Omega}} \hat{v} \cdot \hat{u}\,\operatorname{div}\left(\mathrm{ad}(D\Phi)\dot{\Phi}\right)$$
$$= -\frac{1}{2} \int_{t_n}^{t_{n+1}} \int_{\hat{\Gamma}_N} \left(\hat{v} \cdot \hat{u}\,\mathrm{ad}(D\Phi)\dot{\Phi}\right) \cdot \hat{\nu} + \frac{1}{2} \int_{t_n}^{t_{n+1}} \int_{\hat{\Omega}} (\hat{u}\nabla\hat{v} - \hat{v}\nabla\hat{u}) \cdot \mathrm{ad}(D\Phi)\dot{\Phi}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{t_n}^{t_{n+1}}\int_{\hat{\Omega}} \hat{v}\cdot\hat{u} \underbrace{\underbrace{\operatorname{div}\left(\operatorname{ad}(D\Phi)\dot{\Phi}\right)}_{\text{by Lemma 2}}}_{=\frac{d}{dt}\det D\Phi} \\
&= -\frac{1}{2}\int_{t_n}^{t_{n+1}}\int_{\hat{\Gamma}_N} \left(\hat{v}\cdot\hat{u}\operatorname{ad}(D\Phi)\dot{\Phi}\right)\cdot\hat{\nu} \\
&+\frac{1}{2}\int_{t_n}^{t_{n+1}}\int_{\hat{\Omega}} (\hat{u}\nabla\hat{v}-\hat{v}\nabla\hat{u})\cdot(D\Phi^{-1})\dot{\Phi}\det D\Phi \\
&+\frac{1}{2}\int_{t_n}^{t_{n+1}}\int_{\hat{\Omega}} \hat{v}\cdot\hat{u}\frac{d}{dt}\det D\Phi \\
&= \frac{1}{2}\Bigg(-\int_{t_n}^{t_{n+1}}\int_{\Gamma_N(t)} u\cdot v\,\dot{\Phi}\cdot\nu \;+\; \int_{t_n}^{t_{n+1}}\int_{\Omega(t)} (u\nabla v - v\nabla u)\cdot\dot{\Phi} \\
&\qquad + (u^{n+1},v^{n+1})_{\Omega(t_{n+1})} \;-\; (u^{n+0},v^{n+0})_{\Omega(t_n)}\Bigg)
\end{aligned}
$$

since $\hat{u},\hat{v}$ are constant in time on $(t_n,t_{n+1}]$. (11) is an immediate consequence of (10) by setting $v=u$. □

Next we show stability of the discretization (6)—(8) in “natural” norms. As already pointed out these norms are too weak to control the geometry. This is the reason why existence results are stated and proved in more regular spaces, usually in Hölder or K^r spaces, see [6,34]. Furthermore, in general the time interval of existence for a smooth solution cannot be estimated a priori. Thus we will *assume* that the geometry of the problem is regular, i.e. we will assume that (A) holds.

Theorem 1 (Stability estimate). *Let $f\in L^2(G)$ and $(u,p,\Lambda_n)\in\mathcal{V}\times\mathcal{W}\times S_h$, $0\le n\le N-1$ be the solution of problem (6)—(8). Furthermore assume that (A) holds. Then for $\tau_n\le C$ we have the following stability estimate:*

$$
\begin{aligned}
&\sup_{0\le n\le N}\left\{\frac{1}{2}|u^n|^2_{\Omega(t_n)}+\frac{1}{ReCa}\,|\Gamma_N(t_n)|\right\}+\frac{1}{Re}\int_0^{t^*}|D(u)|^2_{\Omega(t)} \\
&\quad\le C\left\{\frac{1}{2}|u^0|^2_{\Omega(t_0)}+\frac{1}{ReCa}\,|\Gamma_N(t_0)|+\|f\|^2_G\right\}
\end{aligned}
\tag{12}
$$

Proof. Set $v = u$ in (6) and get

$$\underbrace{(u_t, u)_{G_n} \;+\; ([u], u^{n+0})_{\Omega(t_n)} + \mathbf{b}(u; u, u)_{G_n}}_{(I)} + \underbrace{\mathbf{a}(u, u)_{G_n}}_{(II)}$$

$$- \underbrace{\mathbf{c}(p, u)_{G_n}}_{=0} \;+\; \underbrace{\mathbf{d}(u, u)_{G_n}}_{(III)} \;=\; \underbrace{\tau_n\, (f_n^{n+0}, u^{n+0})_{\Omega(t_n)}}_{(IV)}$$

With Lemma 4 one concludes

$$\begin{aligned}
(I) = \frac{1}{2}|u^{n+1}|^2_{\Omega(t_{n+1})} \;-\; \frac{1}{2}|u^{n+0}|^2_{\Omega(t_n)} \;-\; \frac{1}{2} \int\limits_{t_n}^{t_{n+1}} \int\limits_{\Gamma_N(t)} |u|^2 \dot{\Phi} \cdot \nu \\
+\frac{1}{2} \int\limits_{t_n}^{t_{n+1}} \int\limits_{\Gamma_N(t)} |u|^2 u \cdot \nu \;+\; (u^{n+0} - u^n, u^{n+0})_{\Omega(t_n)} \\
= \frac{1}{2}|u^{n+1}|^2_{\Omega(t_{n+1})} \;-\; \frac{1}{2}|u^{n}|^2_{\Omega(t_n)} \;-\; \frac{1}{2} \int\limits_{t_n}^{t_{n+1}} \int\limits_{\Gamma_N(t)} |u|^2 \Big(\dot{\Phi} - u\Big) \cdot \nu \\
+\frac{1}{2}|u^n - u^{n+0}|^2_{\Omega(t_n)}
\end{aligned}$$

By the definition of Φ and (8) it follows that $\dot{\Phi}(t, \hat{x}) = u^{n+0} \circ \Lambda_n(\hat{x})$ on $\hat{\Gamma}_N$ and thus $\dot{\Phi}(t, \hat{x}) = u^{n+0} \circ \Lambda_n(\hat{x}) = u(t, x)$ with $x = \Phi(t, \hat{x})$ for $x \in \Gamma_N(t)$. Therefore:

$$(I) \;=\; \frac{1}{2}|u^{n+1}|^2_{\Omega(t_{n+1})} \;-\; \frac{1}{2}|u^n|^2_{\Omega(t_n)} \;+\; \frac{1}{2}|u^n - u^{n+0}|^2_{\Omega(t_n)}.$$

Obviously for (II) we have:

$$(II) = \frac{1}{2Re} \int\limits_{t_n}^{t_{n+1}} |D(u)|^2_{\Omega(t)}.$$

For (III) we have:

$$\begin{aligned}
\mathbf{d}(u, u)_{G_n} &= \frac{\tau_n}{ReCa} \int\limits_{\Gamma_N(t_n)} \underline{\nabla}(id_{\Gamma_N(t_n)} \;+\; \tau_n u^{n+0}) \cdot \underline{\nabla} u^{n+0} \\
&= \frac{\tau_n}{ReCa} \int\limits_{\Gamma_N(t_n)} \underline{\nabla} X^{n+1} \cdot \underline{\nabla} \frac{(X^{n+1} - id_{\Gamma_N(t_n)})}{\tau_n}
\end{aligned}$$

$$
\begin{aligned}
&\overset{\text{(Lemma 1)}}{\geq} \quad 1/(ReCa)\Big\{\, |X^{n+1}(\Gamma_N(t_n))| \;-\; |\Gamma_N(t_n)| \,\Big\} \\
&= \quad 1/(ReCa)\Big\{\, |\Gamma_N(t_{n+1})| \;-\; |\Gamma_N(t_n)| \,\Big\}
\end{aligned}
$$

Term (IV):

$$
\begin{aligned}
(IV) &= \tau_n \left(f_n^{n+0}, u^{n+0}\right)_{\Omega(t_n)} \\
&= \tau_n \left(f_n^{n+0}, u^{n+0} - u^n\right)_{\Omega(t_n)} + \tau_n \left(f_n^{n+0}, u^n\right)_{\Omega(t_n)} \\
&\leq \frac{\tau_n}{2}(1+\tau_n)|f_n^{n+0}|^2_{\Omega(t_n)} + \frac{1}{2}|u^n - u^{n+0}|^2_{\Omega(t_n)} + \frac{\tau_n}{2}|u^n|^2_{\Omega(t_n)}
\end{aligned}
$$

The first term on the right hand side is estimated by Lemma 3:

$$
\tau_n(1+\tau_n)|f_n^{n+0}|^2_{\Omega(t_n)} \leq C(1+\tau_n)\,|f|^2_{G_n}.
$$

We conclude the proof by summing up for $n = 0, \ldots N-1$ and using Gronwall's Lemma. □

Next we address the question of solvability of the discrete problem (6)—(8). To this end we transform (6) onto the reference domain $\hat{\Omega}$.

Lemma 5. *For* $0 \leq n \leq N-1$ *(6) is equivalent to*

$$
\begin{aligned}
&\frac{1}{2}\int\limits_{\hat{\Omega}} \hat{u}\cdot\hat{v}\Big(\det D\Phi(t_n) + \det D\Phi(t_{n+1})\Big) \\
&+\frac{1}{2}\int\limits_{t_n}^{t_{n+1}}\int\limits_{\hat{\Gamma}_N} \hat{u}\cdot\hat{v}\Big[\mathrm{ad}(D\Phi)\Big(\hat{u} - \dot{\Phi}\Big)\Big]\cdot\hat{\nu} \\
&+\frac{1}{2}\int\limits_{t_n}^{t_{n+1}}\int\limits_{\hat{\Omega}} (\hat{u}\nabla\hat{v} - \hat{v}\nabla\hat{u})\cdot\mathrm{ad}(D\Phi)\Big(\dot{\Phi} - \hat{u}\Big) \\
&+\frac{1}{2Re}\int\limits_{t_n}^{t_{n+1}}\int\limits_{\hat{\Omega}} \frac{\tilde{D}(\hat{u}):\tilde{D}(\hat{v})}{\det D\Phi} \\
&-\tau_n \int\limits_{\Omega(t_n)} p^{n+0}\mathrm{div}\, v^{n+0} + \frac{\tau_n}{ReCa}^2 \int\limits_{\Gamma_N(t_n)} \underline{\nabla} u^{n+0}\cdot\underline{\nabla} v^{n+0} \\
&= \tau_n\,(f_n^{n+0}, v^{n+0})_{\Omega(t_n)} + (u^n, v^{n+0})_{\Omega(t_n)} \\
&-\frac{\tau_n}{ReCa}\int\limits_{\Gamma_N(t_n)} \underline{\nabla} id_{\Gamma_N(t_n)}\cdot\underline{\nabla} v^{n+0}
\end{aligned}
$$

for all $\hat{v} \in V_h$ *with* $\tilde{D}(\hat{v}) := \nabla\hat{v}\,\mathrm{ad}(D\Phi) \; + \; \mathrm{ad}(D\Phi)^{\mathrm{T}}\nabla\hat{v}^{\mathrm{T}}$ *and where* $\hat{u}$ *is understood to be taken in the time interval* $(t_n, t_{n+1}]$.

Again, $(\hat{u}\nabla\hat{v})$ *is an abbreviation for* $(\hat{u}\nabla\hat{v})_j := \sum_{i=1}^{d} \hat{u}^i \partial_j \hat{v}^i$.

Proof. The proof follows the same line as the proof of Lemma 4.

Now let the solution be given up to time $t = t_n$. For the following considerations we assume that $\hat{f}$ is piecewise constant in time, i.e. $f \equiv f_n$. Define $\overset{\circ}{V}_h := \{\hat{v} \in V_h \,|\, \mathbf{c}(q,v)_{G_n} = 0 \;\forall \hat{q} \in W_h\}$ and for $0 \le \epsilon < 1$ an operator $F_\epsilon : \overset{\circ}{V}_h \to \overset{\circ}{V}{}_h'$ as follows. For $\hat{v} \in \overset{\circ}{V}_h$ set

$$
\begin{aligned}
&X := id_{\Gamma_N(t_n)} \; + \; \tau_n v^{n+0} : \; \Gamma_N(t_n) \to I\!R^d \\
&\Lambda \; = \; E(X \circ \Lambda_{n|\hat{\Gamma}_N}) \\
&\Phi \; = \; (1-\hat{t})\Lambda_n \; + \; \hat{t}\Lambda
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
\langle F_\epsilon(\hat{v}), \hat{\varphi} \rangle := {} & \frac{1}{2} \int_{\hat{\Omega}} \hat{v} \cdot \hat{\varphi} \Big(J_\epsilon(t_n) + J_\epsilon(t_{n+1}) \Big) \\
& + \frac{1}{2} \int_{t_n}^{t_{n+1}} \int_{\hat{\Omega}} (\hat{v}\nabla\hat{\varphi} - \hat{\varphi}\nabla\hat{v}) \cdot \mathrm{ad}(D\Phi) \Big(\dot{\Phi} - \hat{v} \Big) \\
& + \frac{1}{2Re} \int_{t_n}^{t_{n+1}} \int_{\hat{\Omega}} \frac{\tilde{D}(\hat{v}) : \tilde{D}(\hat{\varphi})}{J_\epsilon} \\
& + \frac{\tau_n^2}{ReCa} \int_{\Gamma_N(t_n)} \underline{\nabla} v^{n+0} \cdot \underline{\nabla} \varphi^{n+0} \; - \; l(\hat{\varphi})
\end{aligned}
$$

with $J_\epsilon(t,x) := \max\{\epsilon, \det D\Phi(t,x)\}$,

$$
\begin{aligned}
l(\hat{\varphi}) := {} & \tau_n \, (f^{n+0}, \varphi^{n+0})_{\Omega(t_n)} \; + \; (u^n, \varphi^{n+0})_{\Omega(t_n)} \\
& - \frac{\tau_n}{ReCa} \int_{\Gamma_N(t_n)} \underline{\nabla} id_{\Gamma_N(t_n)} \cdot \underline{\nabla} \varphi^{n+0}.
\end{aligned}
$$

That means J_ϵ is given by the formula in Lemma 5 except for that $\det D\Phi$ is replaced by J_ϵ. So, clearly $\hat{u}$ is a solution of (6), (8) for time step $n+1$ iff $\langle J_0(\hat{u}), \hat{\varphi} \rangle = 0$ for all $\hat{\varphi} \in \overset{\circ}{V}_h$ since (8) is fulfilled by the definition of Φ in (13) and thus $(\hat{u} - \dot{\Phi}) = 0$ on $\hat{\Gamma}_N$. Note that in general Φ given by (13) is *not* a transformation. This is the reason for introducing the regularization by J_ϵ. Note that even if $D\Phi$ is singular, the term $\mathrm{ad}(D\Phi)\dot{\Phi}$ makes sense. We are now looking for a $\hat{u}_\epsilon$ such that $\langle F_\epsilon(\hat{u}_\epsilon), \hat{v}\rangle = 0$ for all $\hat{v} \in \mathring{V}_h$. Such a $\hat{u}_\epsilon$ exists by Brouwer's fixpoint theorem for $\epsilon > 0$, since one concludes as in the stability estimate

$$\langle F_\epsilon(\hat{v}), \hat{v}\rangle \geq \frac{1}{2} \int_{\hat{\Omega}} |\hat{v}|^2 \Big(J_\epsilon(t_n) + J_\epsilon(t_{n+1}) \Big) \; + \; \frac{1}{2Re} \int_{t_n}^{t_{n+1}} \int_{\hat{\Omega}} \frac{|\tilde{D}(\hat{v})|^2}{J_\epsilon}$$
$$+ \; \frac{\tau_n^2}{ReCa} \int_{\Gamma_N(t_n)} |\underline{\nabla} v^{n+0}|^2 \; - \; l(\hat{v}) \; \geq \; 0,$$

if $|\hat{v}|_{\hat{\Omega}}$ is large enough. Clearly, since the above equation gives an identity on $\mathring{V}_h$, we also get the existence of $\hat{p}_\epsilon \in W_h$ such that

$$\langle F_\epsilon(\hat{u}_\epsilon), \hat{v}\rangle \; - \; \mathbf{c}(p_\epsilon, v)_{G_n} = 0 \quad \text{for all } \hat{v} \in V_h \tag{14}$$

In (A) it was assumed that a space–time domain was given. Here, we have to modify this assumption in order to derive the existence of the space–time domain. We define:

(A') Let u_ϵ, p_ϵ be solution of (14). Then there is an ϵ_0 (depending on h, n, τ_n, u^n, f) such that J_ϵ does not depend on ϵ for $0 < \epsilon \leq \epsilon_0$ and then Φ_n is also globally injective.

We conclude the above considerations in the following proposition.

Theorem 2 (Existence of the discrete solution). *Let u_0, Ω_0 and $f \in L^2(I\!R^+ \times I\!R^d)$ with $\hat{f}$ piecewise constant in time be given. Moreover assume (A'). Then there exist u^n, p^n solutions of (6)—(8) for $0 \leq n \leq N-1$.*

In order to solve the discrete equations on a time step $n+1$ it is natural to decouple the geometry problem and the flow problem by the following iteration.

Let $u_0 = u_0^{n+1}$ be an initial guess for u^{n+1} (usually $u_0 = u^n$). For $k \geq 1$ iterate:

- Step 1: Set
$$\Phi_{n,k} := (1 - \hat{t})\Lambda_n \; + \; \hat{t}E(X^{n+1} \circ \Lambda_{n|\hat{\Gamma}_N})$$
with
$$X^{n+1} = id_{\Gamma_N(t_n)} \; + \; \tau_n u_{k-1}^{n+0}.$$
- Step 2: Find $[\hat{u}_k, \hat{p}_k] \in \mathcal{V} \times \mathcal{W}$ fulfilling (6)—(7) but Φ_n replaced by $\Phi_{n,k}$ from step 1.

We show that this iteration converges locally in a neighborhood of the solution $[u^{n+1}, p^{n+1}]$.

Theorem 3. *Let $[u^{n+1}, p^{n+1}]$ be the solution of the discrete problem (6)—(8) on time step $n+1$. If τ_n is small enough (depending on h, n and data) and if u_0^{n+1} is close enough to u^{n+1}, then the above iteration is well defined and $u_k^{n+1} \to u^{n+1}$ for $k \to \infty$.*

Remark 5. The arguments to prove the above assertion make use of the equivalence of norms (say $L^2, L^\infty, H^{1,2}$–norms) in the discrete space V_h. Thus the convergence $u_k^{n+1} \to u^{n+1}$ holds for any norm in V_h.

Proof of Theorem 3. For a sufficiently small neighborhood $\mathcal{U}$ of the solution $\hat{u}$ we define

$$F : \mathcal{U} \times \mathring{V}_h \to \mathring{V}_h'$$

$$\begin{aligned}
\langle F(\hat{v}, \hat{w}), \hat{\varphi} \rangle := & \frac{1}{2} \int_{\hat{\Omega}} \hat{w} \cdot \hat{\varphi} \Big(\det D\Phi(t_n) + \det D\Phi(t_{n+1}) \Big) \\
& + \frac{1}{2} \int_{t_n}^{t_{n+1}} \int_{\hat{\Gamma}_N} \hat{w} \cdot \hat{\varphi} \, \mathrm{ad}(D\Phi) \Big(\hat{w} - \dot{\Phi} \Big) \cdot \hat{\nu} \\
& + \frac{1}{2} \int_{t_n}^{t_{n+1}} \int_{\hat{\Omega}} (\hat{w} \nabla \hat{\varphi} - \hat{\varphi} \nabla \hat{w}) \cdot \mathrm{ad}(D\Phi) \Big(\dot{\Phi} - \hat{w} \Big) \\
& + \frac{1}{2Re} \int_{t_n}^{t_{n+1}} \int_{\hat{\Omega}} \frac{\tilde{D}(\hat{w}) : \tilde{D}(\hat{\varphi})}{\det D\Phi} \\
& + \frac{\tau_n^2}{ReCa} \int_{\Gamma_N(t_n)} \underline{\nabla} w^{n+0} \cdot \underline{\nabla} \varphi^{n+0} - l(\hat{\varphi})
\end{aligned}$$

with Φ given by

$$\Phi := (1 - \hat{t}) \Lambda_n \; + \; \hat{t} E(X^{n+1} \circ \Lambda_{n|\hat{\Gamma}_N}) \tag{15}$$

and

$$X^{n+1} = id_{\Gamma_N(t_n)} \; + \; \tau_n v^{n+0}. \tag{16}$$

If $\mathcal{U}$ is sufficiently small, then $\det D\Phi > 0$ and F is well defined and continuously differentiable. Clearly the solution $\hat{u}$ fulfills $F(\hat{u}, \hat{u}) = 0$. If τ_n is small enough then the derivative of F with respect to the second variable

$\hat{w}$, $\partial_{\hat{w}} F(\hat{u}, \hat{u})$ is an isomorphism. To see this note that for $\hat{v}$ given, all terms in $\partial_{\hat{w}} F$ are of order $\mathcal{O}(\tau_n)$ except for the (definite) term of the mass matrix

$$\frac{1}{2} \int_{\hat{\Omega}} \hat{w} \hat{\varphi} \Big(\det D\Phi(t_n) + \det D\Phi(t_{n+1}) \Big).$$

By the implicit function theorem, see for instance [12], there is a neighborhood $\tilde{\mathcal{U}} \subseteq \mathcal{U}$ of $\hat{u}$ and a continuously differentiable function $K : \tilde{\mathcal{U}} \to \mathring{V}_h$ with

$$K(\hat{u}) = \hat{u} \quad \text{and } F(\hat{v}, K(\hat{v})) = 0 \quad \text{for } \hat{v} \in \tilde{\mathcal{U}}.$$

Moreover we observe that the derivative of F with respect to the first variable, $\partial_{\hat{v}} F$, is small for small τ_n. Recall that $\partial_{\hat{v}} \det D\Phi(t_n) = 0$ and $\partial_{\hat{v}} \det D\Phi(t_{n+1}) = \mathcal{O}(\tau_n)$ by the definition of Φ in (15), (16). Then

$$\|DK\| = \| - (\partial_{\hat{w}} F)^{-1} (\partial_{\hat{v}} F)\| \leq q < 1,$$

if τ_n is small enough, i.e. K is a contraction with unique fixpoint $\hat{u}$. Then the assertion is proved, because $F(\hat{u}_{k-1}, \hat{u}_k) = 0$, and therefore $K(\hat{u}_{k-1}) = \hat{u}_k$. □

Remark 6. i) Step 1 in the iteration above is just updating the geometry. Step 2 requires the solution of the quasi–stationary standard Navier–Stokes equations on a given geometry with a modified bilinear form. Note that the additional term in the bilinear form given by **d** is symmetric and positive definite.

ii) The arguments in the proof above strongly depend on the discretization parameters. However, numerical experiments show that it is sufficient to perform only *one* iteration step even for large time steps, see [3].

6 More general boundary conditions

We may also treat more general boundary conditions than in (1). Consider a geometrical situation as in Fig. 3. Here the liquid is in a container with "closed" bottom, open top and vertical side walls and cross section $\Omega' \subseteq \mathbb{R}^{d-1}$. $\Gamma_N(t)$ has now a $d-2$ dimensional boundary $\gamma(t)$, which is the intersection of $\Gamma_N(t)$ and the *slip boundary* $\Gamma_S(t)$: $\gamma(t) = \Gamma_N(t) \cap \Gamma_S(t)$. The slip boundary $\Gamma_S(t)$ is assumed to be the "wet" part of the side walls, that is $\Gamma_S(t) \subseteq \partial\Omega' \times \mathbb{R}_+$ for all $0 < t < t^*$.

On the *contact line* γ we have to prescribe a contact angle between $\Gamma_N(t)$ and $\Gamma_S(t)$ of for instance $\pi/2$.

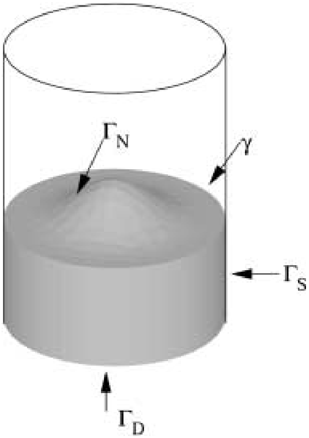

Fig. 3. Γ_N, Γ_S and Γ_D in 3D

To be more precise, additionally to (1) we pose the following boundary conditions

$$
\begin{aligned}
u &= 0 && \text{on } \Gamma_D,\\
u\cdot\nu_S &= 0 && \text{on } \Gamma_S(t),\\
\tau_{S,i}\cdot\sigma\,\nu_S &= 0 && \text{on } \Gamma_S(t), \quad i=1,\dots d-1,\\
ReCa\,\nu_N\cdot\sigma\,\nu_N &= \kappa && \text{on } \Gamma_N(t),\\
\tau_{N,i}\cdot\sigma\,\nu_N &= 0 && \text{on } \Gamma_N(t), \quad i=1,\dots d-1,\\
u\cdot\nu_N &= V_{\Gamma_N} && \text{on } \Gamma_N(t),\\
\nu_N\cdot\nu_S &= 0 && \text{on } \gamma(t)
\end{aligned}
\tag{17}
$$

with $\nu_N,\ \tau_{N,i}$ and $\nu_S,\ \tau_{S,i}$ corresponding to $\Gamma_N,\ \Gamma_S$ respectively.

In the sequel we will make use of the following identities, which can be found for instance in [18]: For smooth functions f_1, f_2 and Γ_N, γ smooth it holds

$$\int\limits_{\Gamma_N} \underline{\Delta} f_1 f_2\, d\mathcal{H}^{d-1} = -\int\limits_{\Gamma_N} \underline{\nabla} f_1\cdot\underline{\nabla} f_2\, d\mathcal{H}^{d-1} + \int\limits_{\gamma} \underline{\partial}_{\nu_\gamma} f_1\, f_2 d\,\mathcal{H}^{d-2}$$

with $\underline{\partial}_{\nu_\gamma} f := \nu_\gamma\cdot\underline{\nabla} f$. Here ν_γ denotes the intrinsic outer unit normal of Γ_N at $\gamma=\partial\Gamma_N$, in 3 space dimensions given by

$$\nu_\gamma = \nu_{\Gamma_N}\ \wedge\ \tau_\gamma,$$

where τ_γ is the tangential unit vector of γ with appropriate sign. Now proceeding as in Sect. 3 we compute for smooth functions φ

$$\int\limits_{\Gamma_N} \kappa\,\nu_N\cdot\varphi = \int\limits_{\Gamma_N} \underline{\Delta} id_{\Gamma_N}\cdot\varphi = -\int\limits_{\Gamma_N} \underline{\nabla} id_{\Gamma_N}\cdot\underline{\nabla}\varphi\ +\ \int\limits_{\gamma} \underline{\partial}_{\nu_\gamma} id_{\Gamma_N}\cdot\varphi$$

$$= -\int_{\Gamma_N} \underline{\nabla} id_{\Gamma_N} \cdot \underline{\nabla}\varphi \; + \; \int_\gamma \nu_\gamma \cdot \varphi,$$

since $\underline{\partial}_{\nu_\gamma} id_{\Gamma_N} = \nu_\gamma$. We decompose φ as

$$\varphi = \varphi \cdot \nu_S \, \nu_S \; + \; \sum_{i=1}^{d-1} \varphi \cdot \tau_{S,i} \, \tau_{S,i} \quad \text{on } \Gamma_S.$$

Thus, if we test with functions φ fulfilling $\varphi \cdot \nu_S = 0$ on Γ_S we get

$$\int_\gamma \nu_\gamma \cdot \varphi = \sum_{i=1}^{d-1} \int_\gamma \left(\nu_\gamma \cdot \tau_{S,i}\right) \left(\tau_{S,i} \cdot \varphi\right) \quad .$$

Now observe that

$$\nu_\gamma \cdot \tau_{S,i} = 0 \quad \text{for all } i = 1, \ldots d-1 \quad \text{on } \gamma \qquad \Leftrightarrow \qquad \nu_N \cdot \nu_S = 0.$$

Analogously as in (3) we conclude by these computations that a pair (u, p) of smooth functions with $\operatorname{div} u = 0$ fulfills

$$\begin{aligned} &\frac{1}{2Re} \int_\Omega D(u) : D(\varphi) \; - \; \int_\Omega p \operatorname{div} \varphi \; + \; \frac{1}{ReCa} \int_{\Gamma_N} \underline{\nabla} id_{\Gamma_N} \cdot \underline{\nabla}\varphi \\ &= \int_\Omega \left(-\frac{1}{Re} \Delta u + \nabla p \right) \cdot \varphi \end{aligned} \tag{18}$$

for all φ with $\varphi = 0$ on Γ_D and $\varphi \cdot \nu_S = 0$ on Γ_S, iff (17) is satisfied.

This means (18) is a weak formulation of (1) together with (17). Thus we may take the same bilinear forms as in Sect. 4 to define the discrete problem. We just have to modify the definition of the space $V_h = V_h(\hat{\Omega})$ in order to incorporate the slip boundary condition $u \cdot \nu_S = 0$ on Γ_S. Let us denote this modified ansatz space by V_h^S. If V_h consists of finite element functions of Lagrange type defined in nodal points $\mathrm{p} \in \mathcal{N}(\mathcal{T})$, then V_h^S may be defined by

$$V_h^S := \{ v_h \in V_h \mid v_h(\mathrm{p}) \cdot \nu_S(\mathrm{p}) = 0 \; \forall \mathrm{p} \in \mathcal{N}(\mathcal{T}) \}.$$

Thus we enforce the slip boundary condition pointwise. Note that $\nu_S = (\nu_{\partial\Omega'}, 0)$ is time independent, which implies that also V_h^S is time independent.

Error estimates for finite element solutions using such a type of discretization in the case of the stationary Navier–Stokes equations and $\Gamma_S = \partial\Omega$ can be found e.g. in [2]. In [2,4] one can also find a simple and efficient way how to deal with such a boundary condition from an algorithmical point of view.

Note that a similar computation shows that the above considerations also hold for $d = 2$.

Let us finally remark that with the above settings Theorems 1–3 remain valid also for problem (1) with the additional boundary conditions (17).

7 Numerical examples

In this section we present numerical examples to demonstrate the behavior of the proposed method. As already mentioned in the Introduction, we do not use the formulation (6)–(8) based on space–time elements. The use of space–time elements was necessary to control the bulk energy (term (I) in the proof of Theorem 1). In practice, such terms do not cause problems. Numerically the most crucial point is the treatment of the curvature terms. Due to the variational structure of our approach, it is algorithmically possible to use other time discretization schemes, see Remark 4. Thus we use the so called fractional step θ–method in a variant as an operator splitting, see for instance [7,33]. Although not completely covered by our theory, numerical examples show that also this scheme is stable.

For the space discretization the Taylor–Hood element, i.e. globally continuous, piecewise quadratics for the velocity space V_h and globally continuous, piecewise linears for the pressure W_h are used. The free boundary is also parameterized by piecewise quadratics.

A detailed description of the scheme can be found in [3].

All computations were done using grids obtained by a refinement procedure: Starting from an initial coarse grid, successive refinement was used to obtain the final grid. In particular we used the so called *bisection method* (see for instance [1]), which implies that d refinement steps yield a triangulation with halved grid size. For the visualization of velocity and temperature fields in the subsequent examples the GRAPE visualization package was used, see [26].

7.1 Oscillating liquid drop

As a first example we consider a fluid motion solely driven by capillary forces, cp. also [30]. As initial domain we take an ellipse in 2D and an ellipsoid in 3D, deformations of the unit ball in $I\!R^2$ and $I\!R^3$ respectively, having different radii of the main axes, which point in the directions of the coordinate axes. The fluid is assumed to be at rest for $t = 0$, i.e. $u_0 = 0$. The mean curvature of $\Gamma_N := \partial\Omega$ is larger at the tips corresponding to large radii and smaller at tips corresponding to smaller radii. By this imbalance of forces there is an onset of motion and the “drop” starts to oscillate. For Reynolds numbers Re not too small we expect a periodic or quasiperiodic behavior

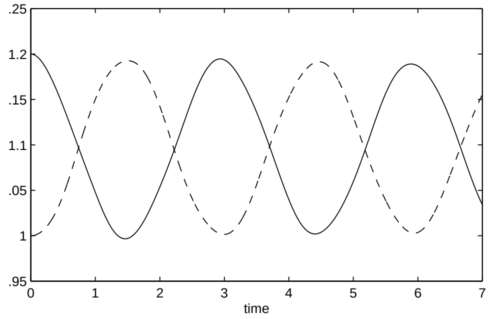

Fig. 4. Trajectories of the tips for the 2D drop, level=10, $\triangle t = 0.0025$, $Re = 300$, $ReCa = 1.0$

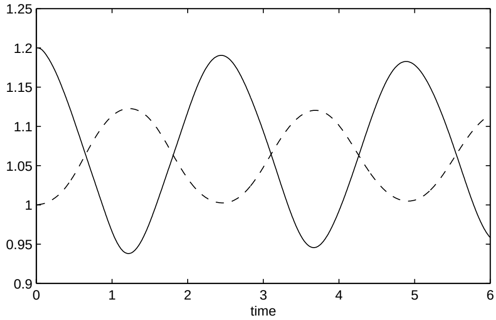

Fig. 5. Trajectories of the tips for the 3D drop with initial radii $r_1 = r_2 = 1.0$, $r_3 = 1.2$; level=9, $\triangle t = 0.008$, $Re = 300$, $ReCa = 1.0$

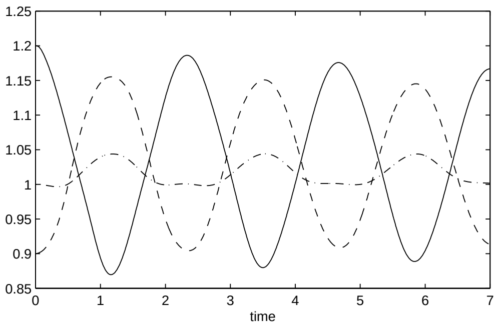

Fig. 6. Trajectories of the tips for the 3D drop with initial radii $r_1 = 0.9, r_2 = 1.0, r_3 = 1.2$; level=9, $\triangle t = 0.005$, $Re = 200$, $ReCa = 1.0$

with some damping depending on Re. This can be seen from Figs. 4–5, where the trajectories of the tips are plotted. Note that "level" is the number of refinement steps, $\mathcal{N}$ and $\mathcal{N}_N$ denote the number of velocity points and free boundary points respectively, i.e. $d \times \mathcal{N} = \dim V_h$, $d \times \mathcal{N}_N = \dim U_h$.

To study this numerical experiment in a more quantitative way we compute the mean frequency f after n periods and the "damping factor" δ,

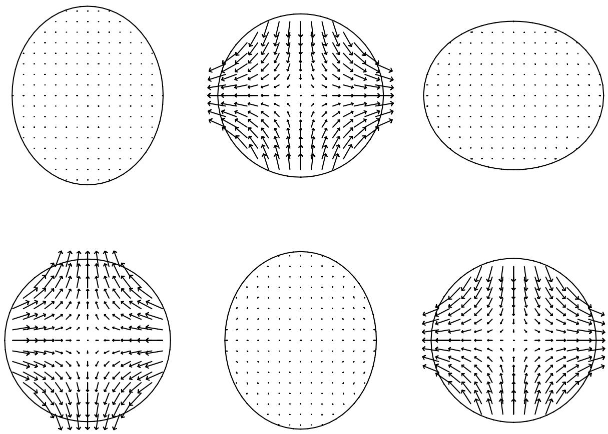

Fig. 7. Solution for $d = 2$ at $t_0 = 0.0$, $t_1 = 0.735$, $t_2 = 1.47$, $t_3 = 2.205$, $t_4 = 2.94$ and $t_5 = 3.675$, $Re = 300$, $ReCa = 1.0$, level=8, $\triangle t = 0.01$

Fig. 8. Clip into the triangulation at $t = 0$, level=9

defined by

$$f := \frac{n}{t_n},$$

$$\delta := \sqrt[n]{\frac{r_{\max}(t_n) - r(t_\infty)}{r_{\max}(t_0) - r(t_\infty)}},$$

where t_n is the time for n periods, $r_{\max}(\cdot)$ is the trajectory of the tip with largest initial radius and $r(t_\infty)$ is the radius of the ball with the same volume as $\Omega(0)$: $|B_{r(t_\infty)}| = |\Omega(0)|$. In 2D we set $n = 5$ and in 3D, because of the much larger CPU–time required, $n = 2$.

The damping factor δ is an increasing function of the Reynolds number Re. This is illustrated in Fig. 13. There, the damping factor is plotted as a function of Re. For values of Re less than about 1.5 no oscillatory behavior could be observed.

Tables 2–3 report the numerical values of f and δ for different discretization parameters in 2D and 3D.

Since for smaller values of $ReCa$ the influence of the mean curvature on the flow field is stronger and in this sense the system is “stiffer”, one expects a higher frequency f for smaller values of $ReCa$. This is confirmed

Table 1. Geometry data for the 3D drop

level	# tets	# $\mathcal{N}$	# $\mathcal{N}_N$
6	384	729	386
9	3072	4913	1538
12	24576	15937	6146

Table 2. Frequencies f (left) and damping factors δ (right) for the oscillating drop in 2D, $Re = 300$, $ReCa = 1.0$

	$\triangle t$		
level	0.04	0.01	0.0025
6	0.338	0.339	0.340
8	0.336	0.338	0.338
10	0.335	0.338	0.338

	$\triangle t$		
level	0.04	0.01	0.0025
6	0.8840	0.9222	0.9311
8	0.9068	0.9459	0.9564
10	0.9088	0.9484	0.9593

Table 3. Frequencies f and damping factors δ for the oscillating drop in 3D with initial radii $r_1 = r_2 = 1.0$, $r_3 = 1.2$; $Re = 300$, $ReCa = 1.0$

level	$\triangle t$	f	δ
6	0.032	0.411	0.8406
9	0.008	0.409	0.9350
12	0.002	0.408	0.9664

by numerical results reported in Fig. 12, where the frequency f is plotted versus $ReCa$. Numerically we get a relation

$$f \sim ReCa^{-1/2}.$$

The corresponding plot δ versus $ReCa$ can be seen in Fig. 11.

7.2 Sloshing liquid in a container

In this example we consider a situation as in Sect. 6. The liquid occupies part of a container with open top. Initially the liquid is at rest but the (upper) free surface is not in equilibrium. For times $t > 0$ there will be a motion, which eventually tends due to the influence of viscosity to the equilibrium state $u = 0$ and a planar surface Γ_N.

The boundary conditions are given by (17). In particular we impose a contact angle of $\pi/2$. With the notations of Sect. 6 we set $\Omega' = B_1(0) \subseteq I\!R^2$. The results are shown in Figs. 14–16.

For another example we choose $\Omega' =]0,1[\times]0,1[$. Fig. 17 shows the geometry of this example for different times.

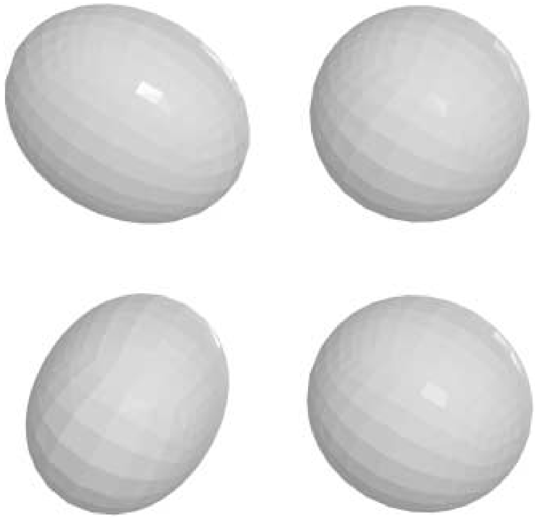

Fig. 9. $\Omega(t)$ for $d = 3$ with initial radii $r_1 = 0.9$, $r_2 = 1.0$ and $r_3 = 1.2$ at $t = 0.0$, $t = 0.6$, $t = 1.2$ and $t = 1.8$, $Re = 200$, $ReCa = 1.0$

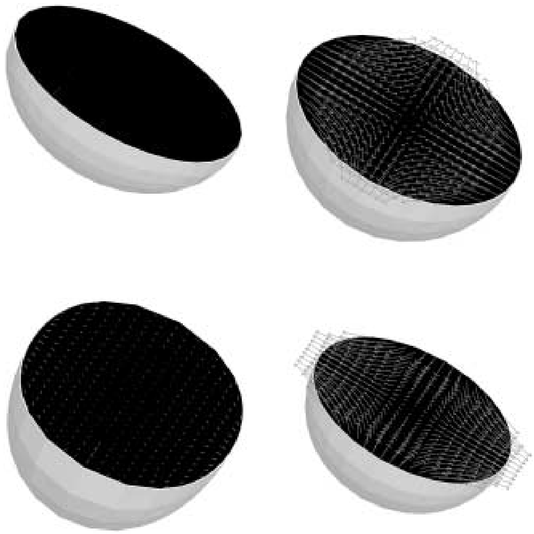

Fig. 10. Velocity field on a clipping plane at $t = 0.0$, $t = 0.6$, $t = 1.2$ and $t = 1.8$; initial radii $r_1 = 0.9$, $r_2 = 1.0$ and $r_3 = 1.2$, $Re = 200$, $ReCa = 1.0$

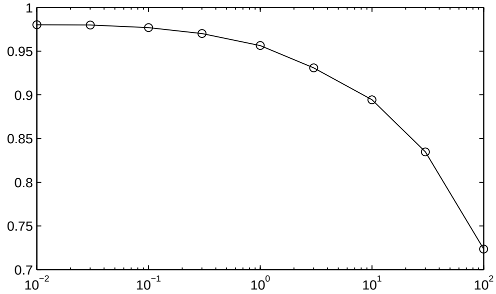

Fig. 11. Damping factor δ versus $ReCa$ for the 2D drop, level=8, $Re = 300$

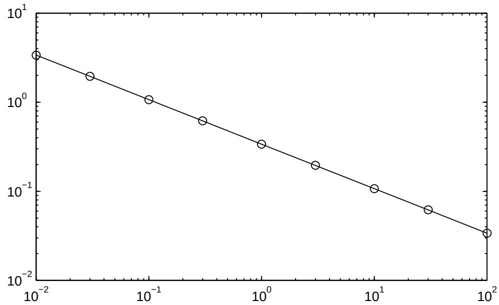

Fig. 12. Frequency f versus $ReCa$ for the 2D drop, level=8, $Re = 300$

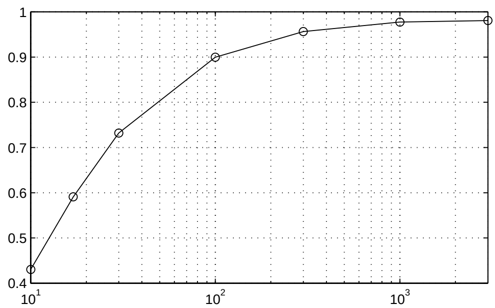

Fig. 13. Damping factor δ versus Re for the 2D drop, level=8, $ReCa = 1$

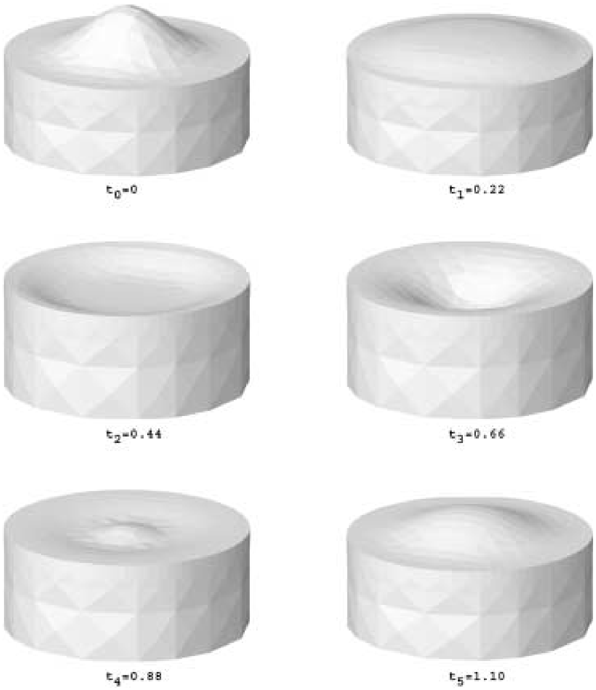

Fig. 14. $\Omega(t)$ for $\Omega' = B_1(0)$ at different times, $Re = 500$, $ReCa = 2$

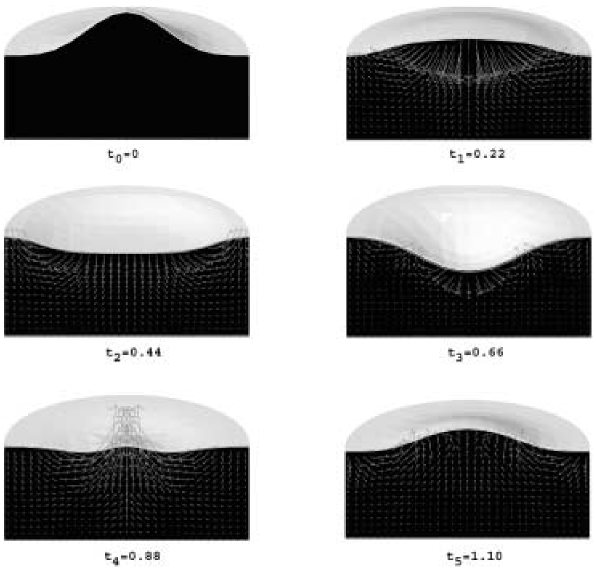

Fig. 15. Velocity on a clipping plane at different times, $Re = 500$, $ReCa = 2$

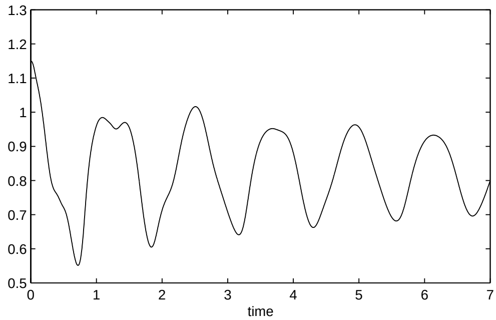

Fig. 16. Height of the free boundary at $(x_1, x_2) = (0, 0)$ as a function of time for the example from Figs. 14–15

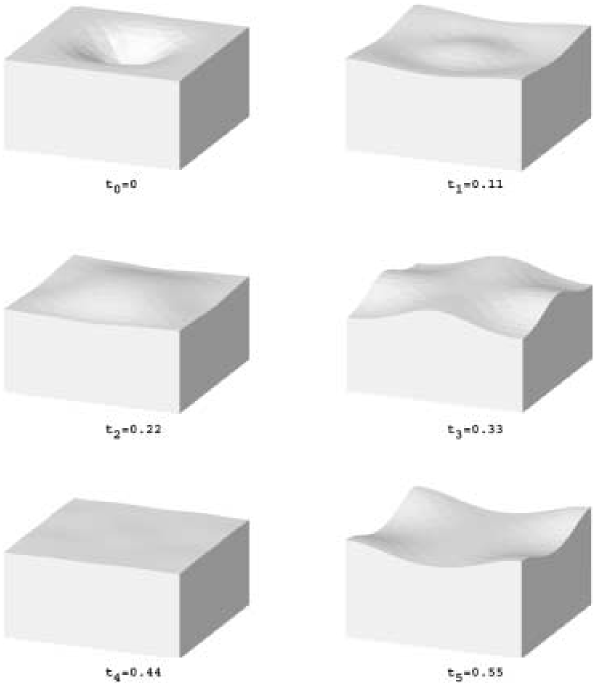

Fig. 17. $\Omega(t)$ for $\Omega' =]0,1[\times]0,1[$ at different times, $Re = 500$, $ReCa = 2$

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