Mathematical Principles of Classical Fluid Mechanics.

By

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With 17 Figures.

A. Preface and introductory remarks.

1. Classical fluid mechanics is a branch of continuum mechanics; that is, it proceeds on the assumption that a fluid is practically continuous and homogeneous in structure. The fundamental property which distinguishes a fluid from other continuous media is that it cannot be in equilibrium in a state of stress such that the mutual action between two adjacent parts is oblique to the common surface. Though this property is the basis of hydrostatics and hydrodynamics, it is by itself insufficient for the description of fluid motion. In order to characterize the physical behavior of a fluid the property must be extended, given suitable analytical form, and introduced into the equations of motion of a general continuous medium, this leading ultimately to a system of differential equations which are to be satisfied by the velocity, density, pressure, etc. of an arbitrary fluid motion. In this article we shall consider these differential equations, their derivation from fundamental axioms, and the various forms which they take when more or less special assumptions concerning the fluid or the fluid motion are made.

Our intent, then, is to present in a mathematically correct way, in concise form, and with more than passing attention to the foundations, the principles of classical fluid mechanics. The work includes the body of exact theoretical knowledge which accompanies the fundamental equations, and at the same time excludes relativistic and quantum effects, most of the kinetic theory, special fields such as turbulence, and all numerical or approximate work. Other topics which have been omitted, but which properly come within the scope of the article, are hydrostatics, rotating fluid masses, one-dimensional gas flows, and stability theory; these subjects are treated elsewhere in this Encyclopedia. A basic knowledge of vector analysis and partial differential equations is expected of the reader, and some experience in hydrodynamics will prove helpful.

The paper proper begins with Division B, where the equations of motion are derived; we have attempted to give rigorous and complete discussions of the basic points, establishing the entire work on the concept of motion as a continuous point transformation. In the final part of this chapter we have discussed transformation of coordinates and variational principles. The material in Part C is to some extent standard in textbooks, but its omission would affect the unity of the article. Moreover, it is here that we first meet many of the ideas which are of importance in the more complex situations treated later. Part D returns to the foundations of the subject with a concise treatment of the thermodynamics of fluid motion, including a postulational summary of the relevant parts of classical thermodynamics. The presentation here may serve as a model for the discussion of multicomponent hydrodynamical systems.

C. Truesdell (ed.), *Fluid Dynamics I / Strömungsmechanik I*

In Part E we present the general theory of perfect (i.e. nonviscous) gases. We have attempted as much as possible to include results on non-isentropic motion and to avoid the ideal gas assumption $pV = RT$. Rather surprisingly, this point of view leads in many cases to a considerable economy of thought. Part F deals with the theory of shock waves in a perfect fluid. The treatment is based entirely on the postulates of motion (Parts B and D) and requires no new dynamical assumptions. The section on shock layers should be useful as an introduction to the specialized literature on the subject. The concluding chapter begins with a clearcut derivation of the constitutive equations of a viscous fluid and covers other theoretical work of recent years.

Some of the sections contain new material or improved treatment of known work. In particular we refer to the following items: the discussion of variational principles (Sects. 14, 15, 24, and 47), the theory of dynamical similarity (Sects. 36 and 66), the theory of the stress tensor (Sect. 59), the energy method (Sect. 73), an extension of the Helmholtz-Rayleigh theorem (Sect. 75), and several new formulas or equations, e.g. Eqs. (29.9), (40.6), (42.8), etc. An attempt has been made to cite original authorities whenever possible; on the other hand, complete references to a subject are seldom given, since they can usually be traced through the papers which are quoted. Finally, we must add that in a number of places proofs have been considerably modified and shortened from their original form.

This work owes much to the stimulating lectures and penetrating scholarship of my teachers DAVID GILBARG and CLIFFORD TRUESDELL. Although the responsibility for the material presented is solely mine, their influence is apparent in many places. Also to my wife BARBARA I owe sincerest thanks and gratitude, specifically for typing the entire manuscript and generally for smoothing the whole project to completion. Every work on fluid dynamics is the better for whatever degree of closeness it attains to the style, clarity, and thoroughness of Sir HORACE LAMB's *Hydrodynamics*. The author hopes he has stayed to the path there laid out.

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2. Vectors and tensors. The mathematical notation used in this article is that of ordinary Cartesian or Gibbsian vector analysis. This notation leads to the utmost conciseness of expression, and at the same time illuminates the physical meaning of the phenomena represented. Most of the vector operations which we use are standard, but occasionally an expression is needed which may appear unusual or ambiguous. For this reason it is convenient to define all operations in terms of vector components: then the meaning of an equation can always be made clear simply by rewriting it in component form. Another advantage accrues to this method, namely that any equation admits an immediate tensorial interpretation if so desired.

Except in a few special situations we shall use lower case bold face to denote vectors; in a fixed rectangular coordinate system, the components of vectors **b**, **c**, etc., will be denoted by b^i, c^i, etc., or equivalently b_i, c_i, etc., where $i = 1, 2, 3$. In this notation the scalar product $\mathbf{b} \cdot \mathbf{c}$ is defined by

$$\mathbf{b} \cdot \mathbf{c} = b^i c_i = b_i c^i,$$

with the usual convention that a repeated index is summed from 1 to 3 [1]. Similarly the vector product $\mathbf{b} \times \mathbf{c}$ is defined by its components

$$(\mathbf{b} \times \mathbf{c})^i = e^{ijk} b_j c_k,$$

[1] The simultaneous use of upper and lower indices has been adopted in order to conform with the standard notation of tensor analysis.

where e^{ijk} is the usual permutation symbol[1]. The magnitude of a vector $\mathbf{b}$ is denoted by the corresponding italic lower case letter, thus

$$b = |\mathbf{b}| = \sqrt{\mathbf{b} \cdot \mathbf{b}}.$$

(One important exception to this rule will be made: the magnitude of the velocity vector $\mathbf{v}$ will be denoted by q, the letter v being reserved to stand for a velocity component.)

The symbols grad φ, div $\mathbf{b}$, and curl $\mathbf{b}$ will be employed in their usual senses, thus

$$\operatorname{div} \mathbf{b} = b^i{}_{,i}$$

and

$$(\operatorname{curl} \mathbf{b})^i = e^{ijk} b_{k,j}, \quad (\operatorname{grad} \varphi)_i = \varphi_{,i}.$$

The comma in these formulas is a standard convention denoting differentiation. That is, if F is an arbitrary scalar or vector function of position we define

$$F_{,i} \equiv \frac{\partial F}{\partial x^i}, \qquad i = 1, 2, 3.$$

[This definition of $F_{,i}$ must be modified in case one wishes to consider curvilinear coordinate systems, as in Sect. 12. The modification need not concern us here, however, since except for a few instances the article is couched exclusively in the notation of Cartesian vector analysis.]

Second order tensors (dyadics) occur frequently in this work. They will be represented by *uppercase* bold face letters: $\mathbf{\Sigma}$, $\mathbf{T}$, etc. The components of a tensor $\mathbf{\Sigma}$ will be denoted by Σ^{ij}, and also, upon occasion, by $\Sigma^i{}_j$ and Σ_{ij}. By the equations

$$\mathbf{b} = \mathbf{c} \cdot \mathbf{\Sigma} \quad \text{and} \quad \mathbf{b} = \mathbf{\Sigma} \cdot \mathbf{c}$$

we mean, respectively

$$b^i = c_j \Sigma^{ji} \quad \text{and} \quad b^i = \Sigma^{ij} c_j.$$

Finally $\mathbf{\Sigma} : \mathbf{T}$ stands for the scalar product $\Sigma^{ij} T_{ij}$.

Several special notations are convenient. By $\mathbf{\Sigma}_\times$ we mean the *vector* with components $e^{ijk}\Sigma_{jk}$. By grad $\mathbf{b}$ we mean the *tensor* with components $b_{j,i}$, that is

$$(\operatorname{grad} \mathbf{b})_{ij} = b_{j,i}.$$

Finally, div $\mathbf{\Sigma}$ stands for the *vector* with components $\Sigma^{ji}{}_{,j}$. From these definitions it follows that

$$\operatorname{curl} \mathbf{b} = (\operatorname{grad} \mathbf{b})_\times \quad \text{and} \quad (\mathbf{c} \cdot \operatorname{grad} \mathbf{b})_i = c^j b_{i,j}.$$

The reader familiar with tensor analysis will observe that if $\mathbf{b}$ is regarded as a short name for the set of contravariant components b^i or covariant components b_i of a vector in a general curvilinear coordinate system, and if $\mathbf{\Sigma}$ is likewise regarded as a short name for the components of a tensor, then the above definitions are tensorially invariant. Thus the vector symbols we have introduced could equally well serve as a shorthand for writing tensor formulas.

A general transformation of volume integrals into surface integrals is embodied in the symbolic formula[2]

$$\int_{\mathscr{V}} F_{,i}\, dv = \oint_{\mathscr{S}} F\, n_i\, da. \tag{2.1}$$

Here F is any scalar, vector, or tensor, with or without an index i to be summed out; $\mathscr{V}$ is a volume in which F is continuously differentiable; $\mathscr{S}$ is the surface of

[1] That is, $e^{123} = e^{231} = e^{312} = 1$, $e^{213} = e^{132} = e^{321} = -1$, and all other components are 0.

[2] H. B. Phillips [*48*], formula (127).

this volume (assumed suitably smooth); and n_i are the components of the *outer* normal $\mathbf{n}$ to the surface $\mathfrak{s}$. Replacing F by b^i gives

$$\int_{\mathfrak{v}} \operatorname{div} \mathbf{b}\, dv = \oint_{\mathfrak{s}} \mathbf{b} \cdot \mathbf{n}\, da, \tag{2.2}$$

usually called the divergence theorem; replacing F by $e^{ijk} b_j$ gives

$$\int_{\mathfrak{v}} \operatorname{curl} \mathbf{b}\, dv = \oint_{\mathfrak{s}} \mathbf{n} \times \mathbf{b}\, da. \tag{2.3}$$

These formulas, and others like them, will be used frequently in this work.

List of frequently used symbols.

Within a single section sometimes these same symbols are defined and used in a different sense. Numbers refer to section where symbol is first used.

c sound speed, Sect. 35.
E internal energy, Sects. 30, 33.
F arbitrary function.
H total enthalpy, Sects. 18, 38.
I enthalpy, Sect. 38.
J Jacobian, Sect. 3.
M Mach number, Sect. 36.
n distance normal to streamline.
p pressure.
q speed.
Q mass flow, Sect. 37.
r radial distance.
s distance along streamline.
S entropy, Sects. 30, 33.
t time.
T absolute temperature.
u, v, w velocity components.

$\mathbf{a}$ acceleration vector.
$\mathbf{D}$ deformation tensor, Sect. 11.
$\mathbf{f}$ extraneous force vector.
$\mathbf{I}$ unit matrix.
$\mathbf{n}$ unit (outer) normal vector to a surface.
$\mathbf{t}$ stress vector, Sect. 6.

$\mathbf{T}$ stress tensor, Sect. 6.
$\mathbf{v}$ velocity vector.

ϱ density.
θ polar coordinate.
ϑ velocity inclination.
Θ expansion, Sect. 26.
φ velocity potential.
Φ dissipation function, Sects. 34, 61.
ψ stream function, Sects. 19, 42.
ω vorticity magnitude.
Ω extraneous force potential, Sect. 9.

$\boldsymbol{\omega}$ vorticity vector.
$\boldsymbol{\Omega}$ vorticity tensor, Sect. 11.

$\mathfrak{T}$ kinetic energy, Sect. 9.
$\mathfrak{W}$ vorticity measure, Sect. 27.

$\mathscr{C}, \mathscr{S}, \mathscr{V}$ curves, surfaces, volumes moving with the fluid.
$\mathfrak{v}, \mathfrak{s}$, fixed volume in space, and its bounding surface.

Other standard notations are introduced in Sects. 2 and 3.

B. The equation of motion.

I. Kinematics and dynamics of fluid motion.

3. Kinematical preliminaries. Fluid flow is an intuitive physical notion which is represented mathematically by a *continuous transformation* of three-dimensional Euclidean space into itself. The parameter t describing the transformation is identified with the time, and we may suppose its range to be $-\infty < t < +\infty$, where $t = 0$ is an arbitrary initial instant.

In order to describe the transformation analytically let us introduce a *fixed rectangular coordinate system* (x^1, x^2, x^3). We refer to the coordinate triple (x^1, x^2, x^3) as the *position* and denote it by $\mathbf{x}$. Now consider a typical point or particle P moving with the fluid. At time $t = 0$ let it occupy the position $\mathbf{X} = (X^1, X^2, X^3)$ and at time t suppose it has moved to the position $\mathbf{x} = (x^1, x^2, x^3)$. Then $\mathbf{x}$ is determined as a function of $\mathbf{X}$ and t, and the flow may be represented by the transformation

$$\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t) \quad \left(\text{or } x^i = \varphi^i(\mathbf{X}, t)\right). \tag{3.1}$$

If $\mathbf{X}$ is fixed while t varies, Eq. (3.1) specifies the *path* of the particle P initially at $\mathbf{X}$; on the other hand, for fixed t, Eq. (3.1) determines a transformation of the region initially occupied by the fluid into its position at time t.

We assume that initially distinct points remain distinct throughout the entire motion, or, in other words, that the transformation (3.1) possesses an inverse[1],

$$\mathbf{X} = \boldsymbol{\Phi}(\mathbf{x}, t) \qquad \left(\text{or } X^\alpha = \Phi^\alpha(\mathbf{x}, t)\right). \tag{3.2}$$

It is also assumed that φ^i and Φ^α possess continuous derivatives up to the third order in all variables, except possibly at certain singular surfaces, curves, or points. Unless otherwise specified, we shall be concerned only with those portions of a flow which *do not* contain singularities. Cases of exception (singular surfaces in particular) require a separate examination, and are dealt with in Sects. 51 and 54. Finally, notice that any closed surface whatever, which moves with the fluid, completely and permanently separates the matter on the two sides of it.

Although a flow is completely determined by the transformation (3.1), it is also important to consider the state of motion at a given point during the course of time. This is described by the functions

$$\varrho = \varrho(\mathbf{x}, t), \qquad \mathbf{v} = \mathbf{v}(\mathbf{x}, t), \quad \text{etc.} \tag{3.3}$$

which give the density and velocity, etc., of the particle which happens to be at the position $\mathbf{x}$ at the time t. It was D'ALEMBERT in 1749 and EULER in 1752 who first recognized the importance of the field description (3.3) in the study of fluid motion, and EULER who conceived the magnificent idea of studying the motion directly through partial differential equations relating the quantities (3.3)[2]. We must now develop the ideas just outlined.

The variables $(\mathbf{x}, t)$ used in the field description (3.3) of the flow will be called *spatial variables*; the variables $(\mathbf{X}, t)$, which single out individual particles will correspondingly be called *material variables*[3]. By means of Eq. (3.1) any quantity F which is a function of the spatial variables $(\mathbf{x}, t)$ is also a function of the material variables $(\mathbf{X}, t)$, and conversely. If we wish to indicate the dependence of F on a particular set of variables we write either

$$F = F(\mathbf{x}, t) \quad \text{or} \quad F = F(\mathbf{X}, t),$$

the functions $F(\mathbf{x}, t)$ and $F(\mathbf{X}, t)$ of course being related by the change of variables (3.1) and (3.2). Geometrically, $F(\mathbf{X}, t)$ is the value of F experienced at time t by the particle initially at $\mathbf{X}$, and $F(\mathbf{x}, t)$ is the value of F felt by the particle instantaneously at the position $\mathbf{x}$. We shall use the symbols

$$\frac{\partial F}{\partial t} \equiv \frac{\partial F(\mathbf{x}, t)}{\partial t} \quad \text{and} \quad \frac{dF}{dt} \equiv \frac{\partial F(\mathbf{X}, t)}{\partial t}$$

for the two possible time derivatives of F; obviously they are quite different quantities. dF/dt is called the *material derivative of* F. It measures the rate of

[1] Greek letters will be used as indices for particle coordinates.

[2] EULER's work on fluid mechanics will be found, for the most part, in volumes II **12**, **13** of his collected works (Opera Omnia, Zurich). Professor TRUESDELL's introductions to these volumes lucidly describe EULER's contributions to fluid mechanics in relation to those of his predecessors and contemporaries, and firmly establish EULER as the founder of rational fluid mechanics.

[3] The two sets of variables just introduced are usually called Eulerian and Lagrangian, respectively, though both are in fact due to EULER; cf. [*26*], § 14.

change of F following a particle, and it can of course be expressed in either material or spatial variables. $\partial F/\partial t$ on the other hand, gives the rate of change of F apparent to a viewer stationed at the position $\mathbf{x}$.

The *velocity* $\mathbf{v}$ of a particle is given by the definition

$$\mathbf{v} \equiv \frac{d\mathbf{x}}{dt}, \qquad \left(v^i \equiv \frac{dx^i}{dt} \equiv \frac{\partial \varphi^i(\mathbf{X}, t)}{\partial t}\right).$$

As defined, $\mathbf{v}$ is a function of the material variables; in practice, however, one usually deals with the spatial form

$$\mathbf{v} = \mathbf{v}(\mathbf{x}, t).$$

In most problems it is sufficient to know $\mathbf{v}(\mathbf{x}, t)$ rather than the actual motion (3.1).

We have introduced the velocity field in terms of the motion (3.1). It is naturally important to be able to proceed in the opposite direction, that is, to determine Eq. (3.1) from $\mathbf{v}(\mathbf{x}, t)$. This transition is effected by solving the system of ordinary differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t) \tag{3.4}$$

with the conditions $\mathbf{x}(0) = \mathbf{X}$. The integration of Eq. (3.4) should be carried out "in the large" and is therefore not always an easy problem[1].

Acceleration is the rate of change of velocity experienced by a moving particle. Denoting the acceleration vector by $\mathbf{a}$, we have then $\mathbf{a} = d\mathbf{v}/dt$. We observe that acceleration can be computed directly in terms of the velocity field $\mathbf{v}(\mathbf{x}, t)$, for we have

$$a^i = \frac{dv^i}{dt} = \frac{\partial v^i}{\partial t} + \frac{\partial v^i}{\partial x^j}\frac{dx^j}{dt},$$

or

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \operatorname{grad} \mathbf{v}. \tag{3.5}$$

Eq. (3.5) is a special case of the general formula

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \mathbf{v} \cdot \operatorname{grad} F \tag{3.6}$$

relating the material derivative to spatial derivatives. Eq. (3.6) may be interpreted as expressing, for an arbitrary quantity $F = F(\mathbf{x}, t)$, the time rate of change of F apparent to a viewer situated on the moving particle instantaneously at the position $\mathbf{x}$.

The Jacobian of the transformation (3.1), namely

$$J = \frac{\partial(x^1, x^2, x^3)}{\partial(X^1, X^2, X^3)} = \det\left(\frac{\partial x^i}{\partial X^\alpha}\right),$$

represents the dilatation of an infinitesimal volume as it follows the motion. From the assumption that Eq. (3.1) possesses a differentiable inverse it follows that

$$0 < J < \infty. \tag{3.7}$$

[1] In [*10*], § 9.21 there is a particularly interesting example of the integration of equation (3.4), due originally to MAXWELL, Proc. Lond. Math. Soc. **3**, 82 (1870). Other examples are discussed in [*10*], § 9.71 and [*8*], §§ 72, 159. The general problem of integration is considered by LICHTENSTEIN [*9*], pp. 159 to 170.

In the sequel we shall make use of the elegant formula

$$\frac{dJ}{dt} = J \operatorname{div} \mathbf{v}, \tag{3.8}$$

due originally to EULER. To prove this, let A_i^α be the cofactor of $\partial x^i/\partial X^\alpha$ in the expansion of the Jacobian determinant, so that

$$\frac{\partial x^i}{\partial X^\alpha} A_j^\alpha = J\,\delta_j^i.$$

Then clearly

$$\frac{dJ}{dt} = \frac{d}{dt}\left(\frac{\partial x^i}{\partial X^\alpha}\right) A_i^\alpha = \frac{\partial v^i}{\partial X^\alpha} A_i^\alpha = \frac{\partial v^i}{\partial x^j}\,\frac{\partial x^j}{\partial X^\alpha} A_i^\alpha = \frac{\partial v^i}{\partial x^i} J.$$

Incompressible fluids. If a fluid is assumed to be incompressible, that is, to move without change in volume, then by Eq. (3.8) we have

$$\operatorname{div} \mathbf{v} = 0. \tag{3.9}$$

Further study of incompressible fluid motion must involve dynamical considerations; in particular, the common assumption $\operatorname{curl} \mathbf{v} = 0$ needs dynamical justification whenever it is applied.

4. The transport theorem. Let $\mathscr{V} = \mathscr{V}(t)$ denote an arbitrary volume which is moving with the fluid[1], and let $F(\mathbf{x}, t)$ be a scalar or vector function of position. The volume integral

$$\int_{\mathscr{V}} F\,dv$$

is then a well-defined function of time. Its derivative is given by the important formula

$$\frac{d}{dt}\int_{\mathscr{V}} F\,dv = \int_{\mathscr{V}} \left(\frac{dF}{dt} + F \operatorname{div} \mathbf{v}\right) dv. \tag{4.1}$$

To prove Eq. (4.1), we introduce (X^1, X^2, X^3) as new variables of integration by means of Eq. (3.1). Then the moving region $\mathscr{V}(t)$ in the $\mathbf{x}$-variables is replaced by the fixed region $\mathscr{V}_0 = \mathscr{V}(0)$ in the $\mathbf{X}$-variables (recall that $\mathscr{V}$ is at all times composed of the same particles), and

$$\int_{\mathscr{V}} F\,dv = \int_{\mathscr{V}_0} F(\mathbf{X}, t)\, J\, dv_0,$$

where the formula $dv = J\,dv_0$ relates the element of volume dv in the $\mathbf{x}$-variables to the element of volume dv_0 in the $\mathbf{X}$-variables. The integral on the right in volves t only under the integral sign, hence

$$\frac{d}{dt}\int_{\mathscr{V}} F\,dv = \int_{\mathscr{V}_0} \left(J\,\frac{dF}{dt} + F\,\frac{dJ}{dt}\right) dv,$$

and Eq. (4.1) follows at once by transformation of the last integral using EULER's formula (3.8).

[1] We shall generally use script *capital* letters to denote volumes, surfaces, and curves which move with the particles of fluid. On the other hand, volumes, surfaces, and curves which are *fixed* in the physical space will be denoted by script *lower case* letters. This notation will prove to be a convenient one for the formulation of a number of the basic principles of hydrodynamics.

Eq. (4.1) can be expressed in an alternate way which brings out clearly its kinematical significance. Indeed, by virtue of Eq. (3.6) the integrand on the right of Eq. (4.1) can be written

$$\frac{\partial F}{\partial t} + \operatorname{div}(\mathbf{v}F),$$

and then by application of the divergence theorem (2.2) we find

$$\frac{d}{dt}\int_{\mathscr{V}} F\,dv = \frac{\partial}{\partial t}\int_{\mathscr{V}} F\,dv + \oint_{\mathscr{S}} F\,\mathbf{v}\cdot\mathbf{n}\,da. \tag{4.2}$$

Here $\mathscr{S}$ is the surface of $\mathscr{V}$, $\mathbf{v}\cdot\mathbf{n}$ is the component of $\mathbf{v}$ along the outward normal to $\mathscr{S}$, and $\partial/\partial t$ denotes differentiation with $\mathscr{V}$ held fixed. Eq. (4.2) expresses that the rate of change of the total F over a material volume $\mathscr{V}$ equals the rate of change of the total F over the fixed volume instantaneously coinciding with $\mathscr{V}$ plus the flux of F out of the bounding surface. It should be emphasized that Eqs. (4.1) and (4.2) express a *kinematical theorem*, independent of any meaning attached to F.

5. The equation of continuity. We suppose that the fluid possesses a density function $\varrho = \varrho(\mathbf{x}, t)$, which serves by means of the formula

$$\mathfrak{M} = \int_{\mathscr{V}} \varrho\,dv \tag{5.1}$$

to determine the mass $\mathfrak{M}$ of fluid occupying a region $\mathscr{V}$. We naturally assume $\varrho > 0$, and assign to ϱ the physical dimension "mass per unit volume".

Turning now to the physical significance of the concept of mass, we postulate the following *principle of conservation of mass: the mass of fluid in a material volume $\mathscr{V}$ does not change as $\mathscr{V}$ moves with the fluid.* The principle of conservation of mass is otherwise expressed by the statement

$$\frac{d}{dt}\int_{\mathscr{V}} \varrho\,dv = 0. \tag{5.2}$$

Now from Eqs. (4.1) and (5.2) it follows easily that

$$\int_{\mathscr{V}} \left(\frac{d\varrho}{dt} + \varrho\operatorname{div}\mathbf{v}\right) dv = 0,$$

and since $\mathscr{V}$ is arbitrary this implies

$$\frac{d\varrho}{dt} + \varrho\operatorname{div}\mathbf{v} = 0. \tag{5.3}$$

This is *the spatial, or Eulerian, form of the equation of continuity* and is a necessary and sufficient condition for a motion to conserve the mass of each moving volume. In virtue of Eq. (3.6) we can express the equation of continuity in the alternate form

$$\frac{\partial\varrho}{\partial t} + \operatorname{div}(\varrho\,\mathbf{v}) = 0. \tag{5.4}$$

The derivation just given is substantially due to EULER[1].

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[1] L. EULER: Principes généraux du mouvement des fluids. Hist. Acad. Berlin (1755) (Opera Omnia II **12**, pp. 54 to 92). As early as 1751 EULER had corresponding ideas for incompressible fluids, but this material did not appear in published form until 1761.

Multiplying Eq. (5.3) by J and using Eq. (3.8), we derive two forms of the *material, or Lagrangian, equation of continuity*:

$$\frac{d}{dt}(\varrho J) = 0, \qquad \varrho J = \varrho_0, \tag{5.5}$$

where $\varrho_0 = \varrho_0(\mathbf{X})$ is the initial density distribution.

The principle of conservation of mass is sometimes expressed in an equivalent form involving a fixed volume: the rate of change of mass in a fixed volume v is equal to the mass flux through its surface, i.e.

$$\frac{\partial}{\partial t}\int\limits_{v} \varrho\, dv = -\oint\limits_{s} \varrho\, \mathbf{v}\cdot\mathbf{n}\, da. \tag{5.6}$$

Applying the divergence theorem to the right hand side of Eq. (5.6) leads to

$$\int\limits_{v}\left(\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho\,\mathbf{v})\right) dv = 0,$$

from which Eq. (5.4) is easily obtained. It is essentially this derivation which is found in most texts, but with application of the divergence theorem disguised in a discussion of the variation of $\varrho\mathbf{v}$ over a small box. The only objection to this derivation is that the principle of conservation of mass in its first form is more convincing.

We conclude this section with an important formula, valid for an arbitrary function $F = F(\mathbf{x}, t)$, namely

$$\frac{d}{dt}\int\limits_{\mathscr{V}} \varrho F\, dv = \int\limits_{\mathscr{V}} \varrho \frac{dF}{dt}\, dv. \tag{5.7}$$

Eq. (5.7) is an easy consequence of Eqs. (4.1) and (5.3).

6. The equations of motion. We consider now the *dynamics* of fluid motion; our intention is to derive the equations which govern the action of forces, external and internal, upon the fluid. In this section we shall present what seems to be the most straight-forward and compelling treatment of this topic, stemming from the pioneer work of EULER and CAUCHY.

We adopt the *stress principle* of CAUCHY[1], which states that "upon any imagined closed surface $\mathscr{S}$ there exists a distribution of *stress vectors* $\mathbf{t}$ whose resultant and moment are equivalent to those of the actual forces of material continuity exerted by the material outside $\mathscr{S}$ upon that inside"[2]. It is assumed that $\mathbf{t}$ depends at any given time only on the position and the orientation of the surface element da; in other words, if $\mathbf{n}$ denotes the (outward) normal to $\mathscr{S}$, then $\mathbf{t} = \mathbf{t}(\mathbf{x}, t; \mathbf{n})$. As TRUESDELL remarks, the above principle "has the simplicity of genius. Its profound originality can be grasped only when one realizes that a whole century of brilliant geometers had treated very special elastic problems in very complicated and sometimes incorrect ways without ever hitting upon this basic idea, which immediately became the foundation of the mechanics of distributed matter"[3].

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[1] A.-L. CAUCHY: Ex. de Math. **2** (1827), (Oeuvres (2) **7**, pp. 79 to 81). A similar statement, but restricted to the case of perfect fluids, was given by EULER.

[2] This statement of CAUCHY's principle is due to TRUESDELL, J. Rational Mech. Anal. **1**, 125 (1952).

[3] C. TRUESDELL: Amer. Math. Monthly **60**, 445 (1953).

We now set forth the fundamental principle of the dynamics of fluid motion, the *principle of conservation of linear momentum: the rate of change of linear momentum of a material volume $\mathscr{V}$ equals the resultant force on the volume*[1]. This principle is otherwise expressed by the statement

$$\frac{d}{dt}\int_{\mathscr{V}} \varrho\,\mathbf{v}\,dv = \int_{\mathscr{V}} \varrho\,\mathbf{f}\,dv + \oint_{\mathscr{S}} \mathbf{t}\,da, \tag{6.1}$$

where $\mathbf{f}$ is the *extraneous force per unit mass*. In setting down axiom (6.1) it is tacitly assumed that the force $\mathbf{f}$ is a *known function* of position and time, and perhaps also of the state of motion of the fluid. This point of view bypasses one of the prime problems in the foundations of mechanics, namely the recognition, and even the existence, of a coordinate system in which $\mathbf{f}$ is known. Of course, in the situations to which fluid mechanics is usually applied, an inertial frame is generally evident beforehand, and the axiom (6.1) is patently applicable. By means of Eq. (5.7), Eq. (6.1) may be written in the form

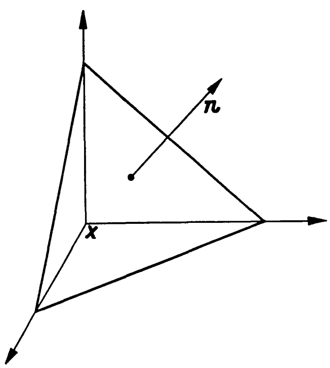

Fig. 1. Stress tetrahedron.

$$\int_{v} \varrho\,\frac{d\mathbf{v}}{dt}\,dv = \int_{v} \varrho\,\mathbf{f}\,dv + \oint_{s} \mathbf{t}\,da; \tag{6.2}$$

here integration over a moving volume can be replaced without loss of generality by integration over a fixed volume.

From the form alone of Eq. (6.2) follows a result of great importance. Let l^3 be the volume of v; dividing both sides of (6.2) by l^2, letting v tend to zero, and noting that the integrands are bounded, we obtain

$$\lim_{v\to 0} l^{-2} \oint_{s} \mathbf{t}\,da = 0, \tag{6.3}$$

that is, the stress forces are in local equilibrium. Consider the tetrahedron of Fig. 1, with vertex at an arbitrary point $\mathbf{x}$, and with three of its faces parallel to the coordinate planes. Let the slanted face have normal $\mathbf{n}$ and area Σ. The normals to the other faces are $-\mathbf{i}$, $-\mathbf{j}$, and $-\mathbf{k}$, and their areas are $n_1\Sigma$, $n_2\Sigma$, and $n_3\Sigma$. Now let us apply Eq. (6.3) to the family of tetrahedrons obtained by letting $\Sigma\to 0$. Since $\mathbf{t}$ is a continuous function of position, and $l^2 \sim \Sigma$, we obtain easily

$$\mathbf{t}(\mathbf{n}) + n_1\,\mathbf{t}(-\mathbf{i}) + n_2\,\mathbf{t}(-\mathbf{j}) + n_3\,\mathbf{t}(-\mathbf{k}) = 0, \tag{6.4}$$

where $\mathbf{t}(\mathbf{n})$ is an abbreviation for $\mathbf{t}(\mathbf{x}, t; \mathbf{n})$. This formula has been proved, of course, only for the case when all the components n_i are positive. To extend its validity, we first note that by continuity it holds if all the n_i are $\geqq 0$. Thus, in particular,

$$\mathbf{t}(\mathbf{i}) = -\,\mathbf{t}(-\mathbf{i}), \quad \mathbf{t}(\mathbf{j}) = -\,\mathbf{t}(-\mathbf{j}), \quad \mathbf{t}(\mathbf{k}) = -\,\mathbf{t}(-\mathbf{k}). \tag{6.5}$$

[1] The necessity for a clearcut statement of the postulates on which continuum mechanics rests was pointed out by FELIX KLEIN and DAVID HILBERT. The first axiomatic presentation is due to G. HAMEL, Math. Ann. **66**, 350 (1908); also [*38*], pp. 1 to 42. In a recent paper, W. NOLL has developed the foundations of continuum mechanics at a level of rigor comparable to that of advanced mathematical analysis.

It should be emphasized that the above postulate cannot be derived from classical mass-point mechanics by simple limiting processes; rather it is a plausible analogue of the basic equations of that subject.

Now applying the "tetrahedron" argument in the other octants, and using Eq. (6.5), we find that, in all cases,

$$\mathbf{t}(\mathbf{n}) = n_1\,\mathbf{t}(\mathbf{i}) + n_2\,\mathbf{t}(\mathbf{j}) + n_3\,\mathbf{t}(\mathbf{k}). \tag{6.6}$$

$\mathbf{t}$ *may therefore be expressed as a linear function of the components of* $\mathbf{n}$, that is

$$t^i = n_j\,T^{ji} \quad \text{where} \quad T^{ji} = T^{ji}(\mathbf{x}, t).$$

The matrix of coefficients T^{ij} obviously forms a tensor, called the *stress tensor* and here denoted by $\mathbf{T}$. Each component of $\mathbf{T}$ has a simple physical interpretation, namely, T^{ij} is the j-component of the force on the surface element with outer normal in the i-direction. The foregoing argument is due in principle to CAUCHY[1].

Replacing $\mathbf{t}$ by $\mathbf{n}\cdot\mathbf{T}$ in (6.2) and applying the divergence theorem, we find

$$\int_{\mathfrak{v}} \varrho\,\frac{d\mathbf{v}}{dt}\,dv = \int_{\mathfrak{v}} (\varrho\,\mathbf{f} + \operatorname{div}\mathbf{T})\,dv$$

and since $\mathfrak{v}$ is arbitrary it follows that

$$\varrho\,\frac{d\mathbf{v}}{dt} = \varrho\,\mathbf{f} + \operatorname{div}\mathbf{T}. \tag{6.7}$$

This is the simple and elegant *equation of motion* discovered by CAUCHY[2]. It is valid for any fluid, and indeed for any continuous medium, regardless of the form which the stress tensor may take.

Perfect fluids. All real fluids obviously can exert tangential stresses across surface elements, so that $\mathbf{t}$ generally will fail to be normal to the surface element on which it acts. The effect of the tangential stresses is small in many practical cases, however, and therefore it is not unreasonable to study the idealized situation in which the tangential stresses are neglected altogether. A *perfect fluid* is then by definition a material for which

$$\mathbf{t} = -p\,\mathbf{n}. \tag{6.8}$$

p is called the *pressure:* when $p > 0$, the vectors $\mathbf{t}$ acting on a closed surface tend to compress the fluid inside. Comparing Eqs. (6.6) and (6.8), we find $p(\mathbf{n}) = p(\mathbf{i}) = p(\mathbf{j}) = p(\mathbf{k})$. That is, p is independent of $\mathbf{n}$,

$$p = p(\mathbf{x}, t).$$

The equations of motion now take the simple form[3]

$$\varrho\,\frac{d\mathbf{v}}{dt} = \varrho\,\mathbf{f} - \operatorname{grad} p. \tag{6.9}$$

It is satisfying to note that we have obtained four equations, namely Eq. (5.3) and the three equations embodied in Eqs. (6.7) or (6.9), relating the four quantities ϱ and the components of $\mathbf{v}$. To be sure, further variables $\mathbf{T}$ or p enter, but one may reasonably expect to express them in terms of ϱ and $\mathbf{v}$ by direct mechanical or thermodynamical assumptions. The various possibilities for this form the material of the following chapters.

Material forms of the equations of motion. For the case of a perfect fluid it is relatively simple to find equations satisfied by $\mathbf{v}$, ϱ, and p as functions of the

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[1] A.-L. CAUCHY: Ex. de Math. **2** (1827), (Oeuvres (2) **7**, pp. 79 to 81).
[2] A.-L. CAUCHY: Ex. de Math. **3** (1828), (Oeuvres (2) **8**, pp. 195 to 226).
[3] L. EULER: Cf. footnote 1, p. 132.

variables X^α, t. Indeed, noting that $d\mathbf{v}/dt = d^2\mathbf{x}/dt^2$, and multiplying both sides of Eq. (6.9) by $x_{i,\alpha} \equiv x^i_{,\alpha}$, we obtain

$$\left(\frac{d^2 x^i}{dt^2} - f^i\right) x_{i,\alpha} = -\frac{1}{\varrho} p_{,\alpha}$$

which may be written vectorially as

$$\operatorname{Grad} \mathbf{x} \cdot \left(\frac{d^2\mathbf{x}}{dt^2} - \mathbf{f}\right) = -\frac{1}{\varrho} \operatorname{grad} p. \tag{6.10}$$

These equations are inconvenient to handle and infrequently used except for one dimensional flows. They are necessary, however, when one wishes to distinguish one particle from another, as in the case of a non-homogeneous fluid. The material equations for fluids sucseptible of tangential stresses are extremely cumbersome and never seem to be used[1].

7. Conservation of angular momentum. The principle of conservation of angular momentum is usually stated as a theorem in the classical dynamics of mass points or rigid bodies. Its proof, however, depends on certain axioms concerning the nature of the "inner forces" between the particles or bodies making up the dynamical system in question. The situation can be treated similarly in continuum mechanics[2]. Here, in order to guarantee the conservation of angular momentum it is necessary to make certain assumptions concerning the forces exerted across surface elements, or, in other words, concerning the stress tensor. Specifically, we postulate that the stress tensor is symmetric, i.e.

$$T^{ij} = T^{ji}. \tag{7.1}$$

(When extraneous *couples* are present this needs modification. However, we specifically exclude extraneous couples from this study, since they arise generally only for polarized media and thus are not important in fluid mechanics.) As a theorem, Eqs. (7.1) are due to CAUCHY[3]; that they can equally well serve as axioms was first recognized by BOLTZMANN[4]. As a consequence of Eqs. (7.1) the following result now holds:

Theorem (conservation of angular momentum). *For an arbitrary continuous medium satisfying the continuity equation (5.3), the dynamical equation (6.7), and the Boltzmann postulate (7.1), we have*

$$\frac{d}{dt}\int_{\mathscr{V}} \varrho(\mathbf{r}\times\mathbf{v})\,dv = \int_{\mathscr{V}} \varrho(\mathbf{r}\times\mathbf{f})\,dv + \oint_{\mathscr{S}} \mathbf{r}\times\mathbf{t}\,da, \tag{7.2}$$

where $\mathscr{V}$ is an arbitrary material volume.

Proof. From Eqs. (5.7) and (6.7) it is easy to show that

$$\begin{aligned}\frac{d}{dt}\int_{\mathscr{V}} \varrho(\mathbf{r}\times\mathbf{v})\,dv &= \int_{\mathscr{V}} \varrho\left(\mathbf{r}\times\frac{d\mathbf{v}}{dt}\right)dv \\ &= \int_{\mathscr{V}} \varrho(\mathbf{r}\times\mathbf{f})\,dv + \oint_{\mathscr{S}} \mathbf{r}\times\mathbf{t}\,da - \int_{\mathscr{V}} \mathbf{T}_{\mathrm{x}}\,dv.\end{aligned}$$

[1] In non-linear elasticity, on the other hand, great importance is attached to the material form of the equation of motion.

[2] The following presentation is similar to that of HAMEL, [*38*], p. 9. A different point of view is adopted by TRUESDELL and TOUPIN (this Encyclopedia, Vol. III, Part 1), who postulate a generalized law of conservation of angular momentum in which extraneous torques are admitted.

[3] A.-L. CAUCHY: Cf. footnote 1, p. 135.

[4] Cf. [*38*], p. 9.

Here $\mathbf{T}_x$ is the axial vector field defined by $(\mathbf{T}_x)^i = e^{ijk}\, \mathbf{T}_{jk}$. By virtue of Eq. (7.1) we have $\mathbf{T}_x = 0$, and Eq. (7.2) is proved. Conversely, if Eq. (7.2) holds for arbitrary volumes then $\mathbf{T}$ must be symmetric.

For certain types of fluids the stress tensor turns out to be symmetric on purely mechanical grounds, irrespective of any other considerations. We mention in particular perfect fluids, where $\mathbf{T} = -p\mathbf{I}$, and isotropic viscous fluids in which stress is a function of the rate of deformation (Sect. 59). For these important cases, then, the Boltzmann postulate is a tautology and Eq. (7.2) can be obtained directly from the equations of motion.

It is possible to imagine a mechanical system for which $\mathbf{T}$ is not symmetric, and HAMEL, in the reference already cited, gives several examples. In cases of this sort, which are not of interest in fluid mechanics, the principle of conservation of momentum as given in Eq. (7.2) no longer holds, but must be generalized to allow for "apparent" extraneous torques.

8. Surface conditions. If a surface in a moving fluid always consists of the same particles, it is clearly a possible bounding surface of the fluid. The converse proposition, namely that every bounding surface must be a material surface, is less obvious.

Suppose a fluid to be in continuous motion according to the conditions set down in Sect. 3, and let $F(\mathbf{x}, t) = 0$ be the equation of its boundary surface. Then F must satisfy the condition

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \mathbf{v} \cdot \operatorname{grad} F = 0 \quad \text{when} \quad F = 0, \tag{8.1}$$

(KELVIN[1]), and this condition in turn implies that the surface always consists of the same particles (LAGRANGE[2]).

Proof. It is well known that the normal velocity of a moving surface $F(\mathbf{x}, t) = 0$ is given by the formula

$$V = -\frac{\partial F}{\partial t} \Big/ |\operatorname{grad} F| .$$

But if $F = 0$ is a bounding surface, then

$$V = \mathbf{v} \cdot \mathbf{n} = \mathbf{v} \cdot (\operatorname{grad} F / |\operatorname{grad} F|) ,$$

and Eq. (8.1) follows at once. On the other hand, if Eq. (8.1) holds, we wish to show that $F = 0$ always consists of the same particles. Set

$$G(\mathbf{X}, t) = F\big(\boldsymbol{\varphi}(\mathbf{X}, t), t\big),$$

so that $G(\mathbf{X}, t) = 0$ describes the initial positions of particles which at time t are on the surface $F = 0$. Clearly

$$\frac{\partial G}{\partial t} = 0 \quad \text{when} \quad G = 0 .$$

Therefore the normal velocity of propagation of the surface $G = 0$ through the $\mathbf{X}$-space is zero. It follows that $G = 0$ is fixed in the $\mathbf{X}$-space, and hence always the same particles make up the moving surface $F = 0$.

At a fixed boundary we have the obvious condition $\mathbf{v} \cdot \mathbf{n} = 0$, independent of the preceding analysis.

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[1] W. THOMSON (Lord KELVIN): Cambridge and Dublin Math. J. (1848), (Papers **1**, p. 83).
[2] J.-L. LAGRANGE: Nouv. Mém. Acad. Sci. Berlin (1781), (Oeuvres **4**, p. 706).

II. Energy and momentum transfer.

9. The energy transfer equation. Let $\mathfrak{T}$ denote the kinetic energy of a volume $\mathscr{V}$,

$$\mathfrak{T} = \tfrac{1}{2} \int_{\mathscr{V}} \varrho q^2 \, dv,$$

and let $\mathbf{D}$ be the *deformation tensor*, $D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$. Then for an arbitrary material volume $\mathscr{V}$ we have

$$\frac{d\mathfrak{T}}{dt} = \int_{\mathscr{V}} \varrho \mathbf{f} \cdot \mathbf{v} \, dv + \oint_{\mathscr{S}} \mathbf{t} \cdot \mathbf{v} \, da - \int_{\mathscr{V}} \mathbf{T} : \mathbf{D} \, dv. \tag{9.1}$$

The proof is a simple exercise in use of Eqs. (5.7), (6.7), and the symmetry of $\mathbf{T}$. *Eq. (9.1) states that the rate of change of kinetic energy of a moving volume is equal to the rate at which work is being done on the volume by external forces, diminished by a "dissipation" term involving the interaction of stress and deformation.* This latter term must represent the rate at which work is being done in changing the volume and shape of fluid elements. Part of the power connected with this term may well be recoverable, but the rest must be accounted for as heat[1]. For a perfect fluid the energy equation takes the simpler form

$$\frac{d\mathfrak{T}}{dt} = \int_{\mathscr{V}} \varrho \mathbf{f} \cdot \mathbf{v} \, dv - \oint_{\mathscr{S}} p \, \mathbf{v} \cdot \mathbf{n} \, da + \int_{\mathscr{V}} p \operatorname{div} \mathbf{v} \, dv. \tag{9.2}$$

The last term is the rate at which work is done by the pressure in changing the volume of fluid elements.

A slight simplification of the energy equation may be effected if $\mathbf{f}$ is derivable from a time-independent potential; $\mathbf{f} = -\operatorname{grad} \Omega$, $\Omega = \Omega(\mathbf{x})$. In this case, setting $\mathfrak{U} = \int_{\mathscr{V}} \varrho \Omega \, dv$, Eq. (9.1) becomes

$$\frac{d}{dt}(\mathfrak{T} + \mathfrak{U}) = \oint_{\mathscr{S}} \mathbf{t} \cdot \mathbf{v} \, da - \int_{\mathscr{V}} \mathbf{T} : \mathbf{D} \, dv.$$

10. The momentum transfer equation. The principle of conservation of linear momentum, stated in Eq. (6.1), may be transformed by Eq. (4.2) into the form

$$\frac{\partial}{\partial t} \int_{\mathfrak{v}} \varrho \mathbf{v} \, dv = \int_{\mathfrak{v}} \varrho \mathbf{f} \, dv + \oint_{\mathfrak{s}} (\mathbf{t} - \varrho \mathbf{v} \, \mathbf{v} \cdot \mathbf{n}) \, da, \tag{10.1}$$

expressing the rate of change of momentum of a fixed volume $\mathfrak{v}$. Because of the physical interpretation of the final term, Eq. (10.1) is known as the *momentum transfer equation.* Eq. (10.1) is sometimes used instead of Eq. (6.1) as the basic expression of the law of conservation of linear momentum.

The momentum transfer equation is often used to determine the force on an obstacle immersed in a steady flow. To illustrate this with a single example, suppose that the fluid occupies the entire exterior of some obstacle, and that the external force field is zero. Then if $\mathfrak{s}$ denotes the surface of the obstacle and Σ denotes a "control surface" enclosing $\mathfrak{s}$, we have the following formula for the force $\mathbf{F}$ acting on the obstacle,

$$\mathbf{F} = -\int_{\mathfrak{s}} \mathbf{t} \, da = \int_{\Sigma} (\mathbf{t} - \varrho \mathbf{v} \, \mathbf{v} \cdot \mathbf{n}) \, da, \tag{10.2}$$

[1] See Sect. 34.

(note that $\mathbf{v}\cdot\mathbf{n}=0$ on $\mathcal{s}$). By an analogous argument proceeding from the Eq. (7.2) we find for the moment $\mathbf{L}$ on $\mathcal{s}$ the formula

$$\mathbf{L}=\int_{\Sigma}\mathbf{r}\times(\mathbf{t}-\varrho\,\mathbf{v}\,\mathbf{v}\cdot\mathbf{n})\,dv.$$

Another force formula of a different type can be derived from the energy equation (9.1). Consider a rigid body moving with rectilinear velocity $\mathbf{U}$ through a fluid, the fluid being bounded externally by fixed walls. Let $\mathcal{V}$ denote the flow region, $\mathcal{s}$ its external boundary, and $\mathcal{s}_0$ the surface of the moving body. Then

$$\int_{\mathcal{s}_0}\mathbf{t}\cdot\mathbf{v}\,da=\mathbf{U}\cdot\int_{\mathcal{s}_0}\mathbf{t}\,da \tag{10.3}$$

(for a perfect fluid this follows from the boundary condition $\mathbf{v}\cdot\mathbf{n}=\mathbf{U}\cdot\mathbf{n}$; for a viscous fluid it depends on the assumption $\mathbf{v}=\mathbf{U}$ on $\mathcal{s}_0$). Combining Eq. (10.3) with Eq. (9.1) gives

$$\mathbf{F}\cdot\mathbf{U}=\frac{d\mathfrak{T}}{dt}+\int_{\mathcal{V}}\mathbf{T}:\mathbf{D}\,dv \tag{10.4}$$

thus determining the component of $\mathbf{F}$ in the direction of motion. (The case where the flow region is infinite in extent can be handled similarly, given suitable asymptotic behavior of the flow at infinity. Further applications of the momentum principle will be found in [*23*], pp. 203 to 234, and in [*12*].)

11. Kinematics of deformation. The vorticity vector. This subject is based upon a simple decomposition of the tensor grad $\mathbf{v}$, namely

$$\operatorname{grad}\mathbf{v}=\mathbf{D}+\mathbf{\Omega} \tag{11.1}$$

where

$$D_{ij}=\tfrac{1}{2}(v_{i,j}+v_{j,i}),\qquad \Omega_{ij}=\tfrac{1}{2}(v_{j,i}-v_{i,j}).$$

The tensors $\mathbf{D}$ and $\mathbf{\Omega}$ are respectively the symmetric and skew-symmetric parts of grad $\mathbf{v}$. The discussion is conveniently divided into two parts.

1. The deformation tensor. Let $d\mathbf{x}$ denote a material element of arc. Its rate of change during the fluid motion is given by the formula

$$\frac{d}{dt}(dx^i)=\frac{d}{dt}\left(\frac{\partial x^i}{\partial X^\alpha}\,dX^\alpha\right)=\frac{\partial v^i}{\partial X^\alpha}\,dX^\alpha=\frac{\partial v^i}{\partial x^j}\,dx^j,$$

or simply

$$\frac{d}{dt}(d\mathbf{x})=d\mathbf{x}\cdot\operatorname{grad}\mathbf{v}. \tag{11.2}$$

From Eq. (11.2) we have easily

$$\frac{d}{dt}(ds^2)=2\,d\mathbf{x}\cdot\mathbf{D}\cdot d\mathbf{x},$$

where $ds=|d\mathbf{x}|$. The tensor $\mathbf{D}$ thus is a measure of the rate of change of the squared element of arc following a fluid motion. In a rigid motion $ds\equiv$ const, whence *a necessary and sufficient condition that a motion be locally and instantaneously rigid is that* $\mathbf{D}=0$. For this reason, $\mathbf{D}$ is called the *deformation tensor.* The tensor $\mathbf{D}-\frac{1}{3}$ (Trace $\mathbf{D}$) $\mathbf{I}$ is also of interest, for its vanishing is the necessary and sufficient condition that the motion locally and instantaneously preserves angles.

If $\mathbf{D}=0$ everywhere in the fluid, the motion is rigid and

$$\mathbf{v}=\tfrac{1}{2}\,\boldsymbol{\omega}\times\mathbf{r}+\text{const}, \tag{11.3}$$

where $\boldsymbol{\omega}$ is twice the (constant) angular velocity of the motion. Eq. (11.3) can also be derived analytically as the integral of the system of first order partial differential equations $\mathbf{D}=0$.

2. General motion of a fluid. Let us consider the velocity field in the neighborhood of a fixed point P. Denoting the evaluation of a quantity at the point P by a subscript, we have near P,

$$\mathbf{v} = \mathbf{v}_P + \mathbf{r} \cdot (\operatorname{grad} \mathbf{v})_P + O(r^2),$$

where $\mathbf{r}$ denotes the radius vector from P. Neglecting terms of order r^2 and using Eq. (11.1), we obtain

$$\mathbf{v} = \mathbf{v}_P + \mathbf{r} \cdot \mathbf{D}_P + \mathbf{r} \cdot \boldsymbol{\Omega}_P. \tag{11.4}$$

We must now interpret the various terms in this formula.

The first term on the right represents a uniform translation of velocity $\mathbf{v}_P$. If we set $D = \mathbf{r} \cdot \mathbf{D}_P \cdot \mathbf{r}$, then the second term can be written in the form

$$\operatorname{grad} \tfrac{1}{2} D. \tag{11.5}$$

This term represents a velocity field normal at each point to the quadric surface $D=\text{const}$ which passes through that point. In this velocity field there are three mutually perpendicular directions which are suffering no instantaneous rotation (the axes of strain). The principal (or eigen-) values of D measure the rates of extension per unit length of fluid elements in these directions.

The final term in Eq. (11.4) may be written

$$\tfrac{1}{2} \boldsymbol{\omega}_P \times \mathbf{r} \tag{11.6}$$

where $\boldsymbol{\omega} = \operatorname{curl} \mathbf{v}$ is the *vorticity vector*. [The simplest way to verify Eq. (11.6) is to note that

$$\boldsymbol{\omega} = (\operatorname{grad} \mathbf{v})_{\mathrm{x}} = \boldsymbol{\Omega}_{\mathrm{x}} = 2\,(\Omega_{23}, \Omega_{31}, \Omega_{12}),$$

whence the components of Eq. (11.6) are equal to those of $\mathbf{r} \cdot \boldsymbol{\Omega}_P$.] The vector form of Eq. (11.6) shows clearly that the final term $\mathbf{r} \cdot \boldsymbol{\Omega}_P$ represents a rigid rotation of angular velocity $\frac{1}{2}\boldsymbol{\omega}_P$.

By combining the results of the two previous paragraphs, the identity (11.1) can be fully interpreted. For an arbitrary motion, the velocity $\mathbf{v}$ in the neighborhood of a fixed point P is given, up to terms of order r^2, by

$$\mathbf{v} = \mathbf{v}_P + \operatorname{grad} \tfrac{1}{2} D + \tfrac{1}{2} \boldsymbol{\omega}_P \times \mathbf{r}, \tag{11.7}$$

where $D = \mathbf{r} \cdot \mathbf{D} \cdot \mathbf{r}$ is the rate of strain quadric and $\boldsymbol{\omega} = \operatorname{curl} \mathbf{v}$ is the vorticity vector: thus *an arbitrary instantaneous state of continuous motion is at each point the superposition of a uniform velocity of translation, a dilatation along three mutually perpendicular axes, and a rigid rotation of these axes*[1]. The angular velocity of the rotation is $\frac{1}{2}\boldsymbol{\omega}_P$. This result amply establishes that $\boldsymbol{\omega}$ *represents the local and instantaneous rate of rotation of the fluid.*

If $\mathbf{D}=0$ at a point it is apparent from Eq. (11.7) that the motion is locally and instantaneously a rotation, while if $\mathbf{D}=k\mathbf{I}$ the motion is a combination of pure expansion and rotation. These results provide a verification of the statements of paragraph 1. On the other hand, if throughout a finite portion of fluid we have $\boldsymbol{\omega} = \boldsymbol{\Omega} = 0$, the relative motion of any element of that portion consists of a pure deformation, and is called "irrotational". In this case it can be shown that $\mathbf{v}$ is everywhere derivable from a potential ($\mathbf{v} = \operatorname{grad} \varphi$), cf. [*48*], p. 101.

[1] A.-L. CAUCHY: Ex. d'Anal. Phys. Math. **2** (1841), [Oeuvres (2) **12**, pp. 343 to 377]. — G. STOKES: Trans. Cambridge Phil. Soc. **8** (1845), (Papers **1**, pp. 75 to 129).

III. Transformation of coordinates.

12. We shall here obtain the equations of continuity and motion in a general curvilinear coordinate system. For this purpose it is useful to employ the methods of elementary tensor analysis; the reader unfamiliar with this topic will find a lucid discussion in [*47*], or he may omit the entire section without serious detriment to the rest of the article. Let (x^1, x^2, x^3) be the coordinates of a point in a general curvilinear coordinate system. We set $\mathbf{x} = (x^1, x^2, x^3)$ as before, with the understanding however that $\mathbf{x}$ is not a vector. The motion is still represented by equations of the form (3.1), stating the position of the particles at time t; for example, in cylindrical polar coordinates motion is represented by the equations

$$r = \chi(\mathbf{X}, t), \qquad \theta = \varphi(\mathbf{X}, t), \qquad z = \psi(\mathbf{X}, t).$$

It is easy to see that the derivatives dx^i/dt of the functions (3.1) form the contravariant component of a vector, hence the velocity vector in curvilinear coordinates retains the form $v^i = dx^i/dt$. We define the *material derivative* of a scalar, vector, or tensor function F by the formula

$$\frac{\delta F}{\delta t} = \frac{\partial F}{\partial t} + v^i F_{,i} \tag{12.1}$$

where the subscript comma denotes *covariant* differentiation. This definition is clearly consistent with the previous formula (3.6), and furthermore makes the material derivative a tensor quantity. It should be observed that the definition of the material derivative given in Sect. 3 is not generally valid in a curvilinear coordinate system, since for vector or tensor quantities F the expression $dF/dt \equiv \partial F(\mathbf{X}, t)/\partial t$ does not transform as a tensor. To establish the correct form for the material derivative in *material coordinates*, one can proceed as follows. Writing the covariant derivative

$$F_{,i} = \frac{dF}{\partial x^i} + A_i,$$

where the A_i denote certain well known expressions involving the Christoffel symbols, we obtain from (12.1) the formula

$$\frac{\delta F}{\delta t} = \frac{\partial F}{\partial t} + v^i\left(\frac{\partial F}{\partial x_i} + A_i\right) = \frac{dF}{dt} + v^i A_i. \tag{12.1'}$$

Eq. (12.1'), which appears also in the theory of parallel translation in differential geometry, clearly shows the difference between $\delta F/\delta t$ and the more naive expression dF/dt. The reader should observe, however, that in rectangular coordinates $\delta F/\delta t \equiv dF/dt$; in other words, just as the covariant derivative is the tensor extension of the ordinary (Cartesian) derivative, so is $\delta F/\delta t$ an extension of dF/dt. Finally, it is evident that Eq. (12.1') could serve as the starting point for the discussion of material derivative, rather than Eq. (12.1). At this point it is convenient to introduce vector notation, the definitions of Sect. 2 being carried over in the obvious way. For example, $\mathbf{v}$ will now denote the set of contravariant or covariant components of the velocity vector, whichever is appropriate, and Eq. (12.1) will be written

$$\frac{\delta F}{\delta t} = \frac{\partial F}{\partial t} + \mathbf{v} \cdot \operatorname{grad} F.$$

With these preliminaries taken care of, we see that the equation of continuity can be written in either of the invariant forms,

$$\frac{\delta \varrho}{\delta t} + \varrho \operatorname{div} \mathbf{v} = 0 \quad \text{or} \quad \frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{v}) = 0, \tag{12.2}$$

where divergence has its usual tensorial meaning,

$$\operatorname{div}\mathbf{b} = b^i{}_{,i} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}\, b^i).$$

Let the stress tensor be defined in a curvilinear coordinate system by means of its components in rectangular coordinates. Then the relation between the stress vector $\mathbf{t}$ and the surface normal $\mathbf{n}$ retains the form $\mathbf{t}=\mathbf{n}\cdot\mathbf{T}$, even though the components of $\mathbf{T}$ are no longer equal to the magnitudes of forces acting upon surface elements. Finally, the equation of motion has the invariant form

$$\varrho\frac{\delta\mathbf{v}}{\delta t} = \varrho\mathbf{f} + \operatorname{div}\mathbf{T}, \tag{12.3}$$

where

$$(\operatorname{div}\mathbf{T})_i = T^k{}_{i,k} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g}\, T^k{}_i) - T^k{}_j\,\Gamma^j_{ik}. \tag{12.4}$$

It is useful to write out Eqs. (12.2) to (12.4) for an orthogonal coordinate system, where the line element has the special form

$$d s^2 = (h_1 dx^1)^2 + (h_2 dx^2)^2 + (h_3 dx^3)^2. \tag{12.5}$$

The equation of continuity becomes simply

$$\frac{\partial\varrho}{\partial t} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}\,\varrho\, v^i) = 0, \quad \sqrt{g} = h_1 h_2 h_3. \tag{12.6}$$

In order to write out Eq. (12.3) we first observe that

$$\mathbf{a} = \frac{\delta\mathbf{v}}{\delta t} = \frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\times\boldsymbol{\omega} + \operatorname{grad}\frac{1}{2}q^2, \tag{12.7}$$

[cf. Eq. (17.1)], whence the acceleration can easily be written down in terms of $\mathbf{v}$ and $\boldsymbol{\omega}$. The latter is given by the formula

$$\omega^i = \frac{e^{ijk}}{\sqrt{g}}\, v_{k,j} = \frac{e^{ijk}}{\sqrt{g}}\frac{\partial v_k}{\partial x^j}, \tag{12.8}$$

using the fact that $\Gamma^l_{jk} = \Gamma^l_{kj}$. The term div $\mathbf{T}$ requires more effort because of the fairly complicated form of Eq. (12.4). The Christoffel symbols corresponding to the metric (12.5) are given by

$$\Gamma^i_{ik} = \Gamma^i_{ki} = \frac{1}{h_i}\frac{\partial h_i}{\partial x^k}, \qquad \Gamma^k_{ii} = -\frac{h_i}{h_k^2}\frac{\partial h_i}{\partial x^k} \quad (i \neq k), \quad \text{all others zero,}$$

(i and k unsummed). Thus after a straightforward calculation,

$$(\operatorname{div}\mathbf{T})_i = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g}\, T^k{}_i) - T^k{}_k\frac{\partial\log h_k}{\partial x^i}, \tag{12.9}$$

(summed on k). The reader should note that this formula is not needed in the case of a perfect fluid, while for a viscous fluid obeying the Cauchy-Poisson law (Sect. 61) it is usually simpler to obtain the equations of motion without first determining div $\mathbf{T}$.

Another method for computing the acceleration may be had from the formula

$$a_i = \frac{\partial v_i}{\partial t} + v^k\left(\frac{\partial v_i}{\partial x^k} - v_k\frac{\partial\log h_k}{\partial x^i}\right), \tag{12.10}$$

proved by the same calculation which led to Eq. (12.9).

In practice, rather than using the covariant or contravariant components of a vector **b**, it is convenient to use its physical components β_i, defined by

$$\beta_i = h_i b^i = \frac{1}{h_i} b_i \qquad (i \text{ unsummed});$$

thus β_i is the magnitude of the projection of **b** on the i-curve through the point of action of **b**. The physical components of tensors are similarly defined, but they will not be needed here.

Example: *cylindrical polar coordinates.* We have in this case

$$d s^2 = d r^2 + (r\, d\theta)^2 + d z^2.$$

Letting v_r, v_θ, and v_z be the respective physical components of velocity, the equation of continuity (12.6) takes the form

$$\frac{\partial \varrho}{\partial t} + \frac{1}{r}\left[\frac{\partial}{\partial r}(\varrho\, r\, v_r) + \frac{\partial}{\partial \theta}(\varrho\, v_\theta) + \frac{\partial}{\partial z}(\varrho\, r\, v_z)\right] = 0. \tag{12.11}$$

The acceleration terms in the equation or motion are, from Eq. (12.7) or from Eq. (12.10)

$$a_r = D\, v_r - \frac{v_\theta^2}{r}, \qquad a_\theta = D\, v_\theta + \frac{v_r v_\theta}{r}, \qquad a_z = D\, v_z,$$

$$D \equiv \frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z}.$$

The physical components of div **T** are given in LOVE's treatise[1] and need not be reproduced here. Finally, the vorticity vector is given by

$$\omega_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}, \qquad \omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, \qquad \omega_z = \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\theta}{r}. \tag{12.12}$$

13. Riemannian space. It may be of interest to consider the nature of the hydrodynamical equations in a Riemannian space, given the line element

$$d s^2 = g_{ij}\, dx^i\, dx^j$$

in some coordinate system $\mathbf{x} = (x^1, \ldots, x^n)$. It is generally not possible to introduce a set of rectangular coordinates, so that one cannot derive suitable "equations of motion" merely by carrying out the steps of the previous work.

Motion in a Riemannian space is represented by a transformation of the form (3.1), although now i runs from 1 to n. We define the velocity vector by $v^i = dx^i/dt$, and the material derivative by

$$\frac{\delta F}{\delta t} = \frac{\partial F}{\partial t} + v^i F_{,i}.$$

(This definition is in analogy to the one used in Euclidean space, and also has the property that, should the space be embedded in a higher dimensional Euclidean space, as for example a surface in three space, then the material derivative is the surface component of the "natural" material derivative of Euclidean space.)

The equation of continuity is easily derived by the method of Sects. 4 and 5. In this procedure we must replace Eq. (4.2) with

$$\int_{\mathscr{V}} \varrho(x, t)\, dV = \int_{\mathscr{V}_0} \varrho(\mathbf{X}, t) \sqrt{g}\, J\, dv_0$$

[1] A. E. H. LOVE: A Treatise on the Mathematical Theory of Elasticity, 4th edit. Cambridge 1927. See p. 90.

and then make use of the formula

$$\frac{\delta}{\delta t}(\sqrt{g}\,J) = \sqrt{g}\,J \operatorname{div} \mathbf{v},$$

which follows readily from Eq. (3.7). In other respects the argument is exactly as before, the final result being

$$\frac{\delta \varrho}{\delta t} + \varrho \operatorname{div} \mathbf{v} = 0,$$

which is exactly the same as Eq. (12.2), but obtained now without recourse to rectangular coordinates.

Deriving appropriate equations of motion involves dynamical considerations which do not seem adapted to Riemannian space; in particular, it is not evident how to formulate the principle of conservation of momentum. On the other hand, there seems to be no valid objection to taking Eq. (12.3) as a postulate. This done, further considerations will closely parallel corresponding results of ordinary hydrodynamics.

IV. Variational principles.

The wide scope and great success of variational principles in classical dynamics have stimulated many efforts to formulate the laws of continuum mechanics in a similar way. In the following section we shall discuss some of these formulations; the work applies generally to all continuous media, though it is stated only for the motion of fluids. In Sect. 15 we consider some special variational principles which apply to perfect fluids.

14. General fluids. The variational principle appropriate to a given dissipative system takes a form exactly suited to and dependent on the particular mechanism of dissipation, and is generally not capable of extension in unchanged form to other problems. This fact makes it easy to formulate a variational principle for fluids, but also indicates something of the *a posteriori* nature of the undertaking. The reader will observe that the appropriate variational principle is little more than a reformulation of the equations of motion; it may, however, provide methods for handling constraints otherwise beyond the scope of the original equations.

Let $\delta \mathbf{x} = \boldsymbol{\eta}(\mathbf{x}, t)$ be a *virtual displacement* of the particles of fluid from their instantaneous position. The vector function $\boldsymbol{\eta}$ is assumed to be finite valued and continuously differentiable; moreover it should conform to any restrictions placed on the fluid position. This latter condition implies, in particular, that $\boldsymbol{\eta}$ should be tangent to any wall bounding the fluid. The *virtual work* corresponding to a virtual displacement is defined by

$$\delta \mathfrak{A} = \delta \mathfrak{A}_e - \int_{\mathscr{V}} \mathbf{T} : \operatorname{grad} \delta \mathbf{x}\, dv,$$

where $\mathscr{V}$ is the volume occupied by the fluid, $\mathbf{T}$ is a tensor function of position, and

$$\delta \mathfrak{A}_e = \int_{\mathscr{V}} \varrho \mathbf{f} \cdot \delta \mathbf{x}\, dv + \oint_{\mathscr{S}} \mathbf{t} \cdot \delta \mathbf{x}\, da \tag{14.1}$$

is the virtual work done against extraneous forces $\mathbf{f}$ and surface stresses $\mathbf{t}$. The second term in the definition of $\delta \mathfrak{A}$ is peculiar to continuum mechanics: it reflects the common observation that deformations of a fluid medium generally require the expenditure of work against stress forces. We need not assume that $\mathbf{T}$

is symmetric, but otherwise a rigid virtual displacement will produce virtual work of deformation. For this reason, it is usual to consider only symmetric stresses $\mathbf{T}$. We may now state the fundamental d'Alembert-Lagrange variational principle: *A fluid moves in such a way that*

$$\delta\mathfrak{A} - \int_{\mathscr{V}} \varrho\, \mathbf{a} \cdot \delta\mathbf{x}\, dv = 0, \tag{14.2}$$

for all virtual displacements which satisfy the given kinematical conditions[1]. If there are no constraints on the motion, except for wall conditions, it follows in a well known way that

$$\varrho\, \mathbf{a} = \varrho\, \mathbf{f} + \operatorname{div} \mathbf{T} \quad \text{and} \quad \mathbf{t} = \mathbf{n} \cdot \mathbf{T}. \tag{14.3}$$

The first equation holds at all interior points of the motion, the second at "free" surfaces. These are of course just the equations of motion already derived.

Fluid motions on surfaces, or subject to other sorts of constraints, can be handled by the usual techniques of the calculus of variations. The interested reader should consult HELLINGER's article in the Encyclopaedia of Mathematical Sciences, in particular §§ 3e, 4c, and 8b.

The d'Alembert-Lagrange principle may be expressed equivalently in the form of Hamilton's principle. This is obtained by letting the virtual displacements arise from variations in the paths of the particles. Thus let a set of varied paths be given by $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t; \varepsilon)$, where $-1 < \varepsilon < 1$, say, and the path $\varepsilon = 0$ is the one to be investigated. If

$$\delta \equiv \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0},$$

then the virtual displacement corresponding to a varied motion is defined by

$$\delta\mathbf{x} = \delta\boldsymbol{\varphi} = \frac{d\boldsymbol{\varphi}}{d\varepsilon}\bigg|_{\varepsilon=0}.$$

We have now the following identity

$$\mathbf{a} \cdot \delta\mathbf{x} = \frac{d}{dt}(\mathbf{v} \cdot \delta\mathbf{x}) - \mathbf{v} \cdot \frac{d\,\delta\mathbf{x}}{dt} = \frac{d}{dt}(\mathbf{v} \cdot \delta\mathbf{x}) - \delta\,\frac{1}{2}\,q^2, \tag{14.4}$$

since δ and d obviously commute. The density of the varied motions is determined by the condition that the mass of fluid corresponding to an arbitrary set of particles shall be the same wherever the particles may be. Mathematically this leads to the "continuity condition"

$$\delta\varrho = -\varrho \operatorname{div} \delta\mathbf{x} \tag{14.5}$$

governing the variation of density. To prove Eq. (14.5) we observe that $\delta\mathbf{x}$ is the initial velocity in a motion for which ε plays the role of time; thus to obtain Eq. (14.5) we simply replace d/dt and $\mathbf{v}$ in the equation of continuity by δ and $\delta\mathbf{x}$ respectively. The same reasoning also proves the formula

$$\delta \int \varrho F\, dv = \int \varrho\, \delta F\, dv. \tag{14.6}$$

[1] The statical equivalent of Eq. (14.2), namely that a continuous medium will be in equilibrium if and only if $\delta\mathfrak{A} = 0$ for all virtual displacements, is due to LAGRANGE (Mécan. Anal. **1** part. Sect. IV, § 1). The extension of this principle to dynamical systems was likewise given by LAGRANGE, the fundamental idea in his derivation being the application of D'ALEMBERT's principle to the equilibrium condition $\delta\mathfrak{A} = 0$ (Mécan. Anal. 2e parts. Sects. I, II). See the articles of P. VOSS (Ency. Math. Wiss. **4**[1], No. 1) and E. HELLINGER (Ency. Math. Wiss. **4**[4], No. 30).

Condition (14.5) is also the consequence of assuming, (i) that each varied motion satisfies the equation of continuity, and (ii) that the virtual displacement vanishes at some fixed time. If Eq. (14.4) is multiplied by ϱ and integrated over a *material volume* $\mathscr{V}$, application of formulas (5.7) and (14.6) yields

$$\int_{\mathscr{V}} \varrho\, \mathbf{a} \cdot \delta \mathbf{x}\, dv = \frac{d}{dt} \int_{\mathscr{V}} \varrho\, \mathbf{v} \cdot \delta \mathbf{x}\, dv - \delta \mathfrak{T}, \tag{14.7}$$

where

$$\mathfrak{T} = \tfrac{1}{2} \int_{\mathscr{V}} \varrho\, q^2\, dv = \text{kinetic energy}.$$

Finally, by virtue of the d'Alembert-Lagrange principle, Eq. (14.7) can be written in the form

$$\delta \mathfrak{T} + \delta \mathfrak{A} - \frac{d}{dt} \int_{\mathscr{V}} \varrho\, \mathbf{v} \cdot \delta \mathbf{x}\, dv = 0. \tag{14.8}$$

This equation holds under the condition that the varied motions satisfy the continuity condition (14.5) and conform to external constraints. If Eq. (14.8) is integrated from t_0 to t_1, and if $\delta \mathbf{x}$ vanishes at t_0 and t_1, we obtain the so-called Hamilton's principle[1]

$$\int_{t_0}^{t_1} (\delta \mathfrak{T} + \delta \mathfrak{A})\, dt = 0;$$

each varied motion must satisfy the equation of continuity and external constraints, as well as having $\delta \mathbf{x} = 0$ at t_0 and t_1.[2]

15. Perfect fluids. For an incompressible perfect fluid the d'Alembert-Lagrange principle can be formulated in a more elegant fashion, namely, *an incompressible perfect fluid moves in such a way that*

$$\delta \mathfrak{A}_e - \int_{\mathscr{V}} \mathbf{a} \cdot \delta \mathbf{x}\, dv = 0 \tag{15.1}$$

for all virtual displacements $\delta \mathbf{x}$ *which preserve the volume, or, in other words, satisfy* $\operatorname{div} \delta \mathbf{x} = 0$. *The virtual work* $\delta \mathfrak{A}_e$ *is defined by Eq.* (14.1).

According to the theory of Lagrange multipliers, this is equivalent to

$$\int_{\mathscr{V}} [\varrho (\mathbf{a} - \mathbf{f}) \cdot \delta \mathbf{x} - \lambda \operatorname{div} \delta \mathbf{x}]\, dv - \oint_{\mathscr{S}} \mathbf{t} \cdot \delta \mathbf{x}\, da = 0,$$

where λ is a Lagrange multiplier and $\delta \mathbf{x}$ is subjected to no side conditions. It follows from an integration by parts that

$$\varrho\, \mathbf{a} = \varrho\, \mathbf{f} - \operatorname{grad} \lambda \quad \text{and} \quad \mathbf{t} = -\lambda\, \mathbf{n}. \tag{15.2}$$

λ thus becomes the "pressure", one of the principal unknowns of the problem. Eqs. (15.2) together with the continuity condition $\operatorname{div} \mathbf{v} = 0$ constitute four equations for the four unknowns $\mathbf{v}$ and λ.

For the general case of a compressible perfect fluid, LAGRANGE took Eq. (15.2) to be the correct equation, where λ is to be considered a "reaction" against the volume changes which are, of course, now permitted[3]. This derivation of a

[1] Cf. E. HELLINGER: Ency. Math. Wiss. **4**[4], footnote 61.

[2] Other variational principles which may be mentioned are the principle of least time (HELLINGER, § 5c) and an interesting energy principle of J. W. HERIVEL [Proc. Roy. Irish Acad. **56**, 37, 67 (1954)]. Cf. also E. HÖLDER: Ber. sächs. Akad. Wiss. (Lpz.), Math-phys. Kl. **97** (1950).

[3] Cf. [*6*], pp. 173, 522. A similar method was used by G. PIOLA [Modena Mem. **24**, 1 (1848)] to derive the general equations of continuum mechanics.

general case from a particular one—by retaining the old equation, but considering the Lagrange multiplier as a new "force of reaction"—HAMEL calls the "Lagrange freeing principle". He notes further that the reaction is to depend precisely on the compressibility (i.e. the density) which was before not allowed to vary. This procedure, although interesting and leading to a correct result, is not entirely convincing—one difficulty becomes evident in the case of gas, where the pressure is a definite thermodynamical variable.

The variational principle (15.1) may be written in the form of Hamilton's principle by means of identity (14.5). Thus we have the result, *an incompressible perfect fluid moves in such a way that*

$$\int_{t_0}^{t_1} (\delta \mathfrak{T} + \delta \mathfrak{A}_e)\, dt = 0$$

for all variations $\delta \mathbf{x}$ *of the motion satisfying*

$$\operatorname{div} \delta \mathbf{x} = 0 \quad \text{and} \quad \delta \mathbf{x} = 0 \quad \text{at} \quad t = t_0, t_1.$$

LICHTENSTEIN[1] has obtained a similar variational principle for the motion of compressible perfect fluids. A certain artificiality in his formulation was noticed by TAUB[2], who substituted an alternative procedure; the most satisfying form of the principle is, however, due to HERIVEL[3], and in the following discussion we shall use the latter's formulation.

We begin with the remark that, for a mechanical system whose energy is completely known it should be possible to state Hamilton's principle in the form

$$\int_{t_0}^{t_1} (\delta \mathfrak{L} + \delta \mathfrak{A}_e)\, dt = 0, \tag{15.3}$$

where the Lagrangian function $\mathfrak{L}$ is the difference of the kinetic and potential energies. An essential difference between the principle (15.3) and those stated earlier is that (15.3) can be written without *a priori* knowledge of the equations of motion. Thus this principle provides a way of deriving the equations of motion by a method which is genuinely independent of momentum considerations. Let us apply this to the case of a gas.

We suppose the motion takes place without loss of energy through the generation of transfer of heat, or, more precisely, that the specific entropy S of each fluid particle remains constant during the motion[4],

$$\frac{dS}{dt} = 0. \tag{15.4}$$

In this type of motion the energy is completely known, having the form $\mathfrak{T} + \mathfrak{E}$, where $\mathfrak{T}$ is the kinetic energy and $\mathfrak{E}$ the internal energy of the volume of fluid considered,

$$\mathfrak{E} = \int_{\mathscr{V}} \varrho E\, dv, \qquad E = E(\varrho, S) = \text{specific internal energy}.$$

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[1] L. LICHTENSTEIN [*9*], Chap. 9.

[2] A. H. TAUB [*44*], p. 148.

[3] J. W. HERIVEL: Proc. Cambridge Phil. Soc. **51**, 344 (1955).

[4] The thermodynamical basis for the following work will be found in Sect. 30 and in the first paragraph of Sect. 33.

There seems only one reasonable choice for the Lagrangian function, namely $\mathfrak{L} = \mathfrak{T} - \mathfrak{E}$. For this $\mathfrak{L}$ we shall now show that Eq. (15.3) leads to the correct equations of motion for a compressible perfect fluid.

Let $\delta\mathbf{x} = \delta\mathbf{x}(\mathbf{X}, t)$ be a variation of the path, vanishing at t_0 and t_1. Assuming that the varied motions satisfy the equation of continuity, the variation of density is given by Eq. (14.5). By the same arguments, the variation of entropy must satisfy

$$\delta S = 0.$$

From Eqs. (14.6) and (14.5), and since $(\partial E/\partial\varrho)_S = p/\varrho^2$, there follows

$$\begin{aligned}\delta\mathfrak{E} &= \int_{\mathscr{V}} \varrho\,\delta E\,dv = -\int_{\mathscr{V}} p\,\mathrm{div}\,\delta\mathbf{x}\,dv\\ &= \int_{\mathscr{V}} \delta\mathbf{x}\cdot\mathrm{grad}\,p\,dv - \oint_{\mathscr{S}} p\,\mathbf{n}\cdot\delta\mathbf{x}\,da.\end{aligned}$$

$\delta\mathfrak{T}$ is evaluated by means of Eq. (14.7). We may now conclude in the usual way from Eq. (15.3) and the formulae for $\delta\mathfrak{T}$, $\delta\mathfrak{E}$, and $\delta\mathfrak{A}_e$, that

$$\varrho\,\mathbf{a} = \varrho\,\mathbf{f} - \mathrm{grad}\,p \quad\text{and}\quad \mathbf{t} = -p\,\mathbf{n}.$$

These are of course the correct equations[1]. We emphasize again that they have been derived from a principle whose statement involved no *a priori* knowledge of their form. This is in contrast to the earlier principle (14.2) and the derivation from it of Eqs. (14.3).

In theoretical mechanics the energy equation is a consequence of Hamilton's principle. It is interesting to see that this is also true in the present case. For since

$$\frac{d\mathfrak{E}}{dt} = \int_{\mathscr{V}} \varrho\,\frac{dE}{dt}\,dv = \int_{\mathscr{V}} p\,\mathrm{div}\,\mathbf{v}\,dv,$$

we have from Eq. (9.2),

$$\frac{d}{dt}(\mathfrak{T} + \mathfrak{E}) = \int_{\mathscr{V}} \varrho\,\mathbf{f}\cdot\mathbf{v}\,dv + \oint_{\mathscr{S}} \mathbf{t}\cdot\mathbf{v}\,dv,$$

which is the usual statement of conservation of energy for a non-heat-conducting media.

In the paper already referred to, HERIVEL attempted to found the equations of perfect fluids on a variational principle of spatial (Eulerian) type. He was not entirely successful, in that his principle yields as extremals only a subset of the class of flows satisfying the Euler equations. This difficulty was first pointed out by C.C. LIN, who then supplied a correct version of the principle[2]. Consider, in particular, the variational principle,

$$\delta \iint L(v, \varrho, S)\,dv\,dt = 0, \tag{15.5}$$

where L is the Lagrangian density

$$L = \tfrac{1}{2}\,\varrho\,q^2 - \varrho(E + \Omega)$$

[1] The preceding derivation is based on that in HERIVEL's paper, with, however, certain modifications in the formulation and proof.

[2] HERIVEL's principle included only the first pair of constraints in Eq. (15.6), the final constraint being due to C. C. LIN (unpublished). Without this additional constraint, isentropic flows could appear as extremals only if they were also irrotational [see Eq. $(15.7)_1$].

and the variations of the velocity, density, and entropy are subject to the following constraints,

$$\begin{aligned}
&\text{Conservation of mass:} && \frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho\,\mathbf{v}) = 0,\\
&\text{Conservation of energy:} && \frac{dS}{dt} = 0,\\
&\text{Conservation of the identity of particles:} && \frac{d\mathbf{X}}{dt} = 0,
\end{aligned} \tag{15.6}$$

where the vector field $\mathbf{X}(\mathbf{x}, t)$ establishes the initial position of the particle which occupies the position $\mathbf{x}$ at time t. We shall now verify that every extremal of the variational principle (15.5) is a flow (HERIVEL-LIN)[1].

Upon introduction of the Lagrange multipliers φ, β, $\boldsymbol{\gamma}$ the above principle becomes

$$\delta \iint \left\{L + \varphi\left(\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho\,\mathbf{v})\right) - \varrho\,\beta\,\frac{dS}{dt} - \varrho\,\boldsymbol{\gamma}\cdot\frac{d\mathbf{X}}{dt}\right\} dv\,dt = 0,$$

where v, ϱ, S and $\mathbf{X}$ are now to be varied without restrictions. The separate variations of these quantities now give the following equations

$$\left.\begin{aligned}
&\delta v: && \mathbf{v} = \operatorname{grad}\varphi + \beta\operatorname{grad}S + \operatorname{grad}\mathbf{X}\cdot\boldsymbol{\gamma},\\
&\delta\varrho: && \frac{d\varphi}{dt} = \frac{1}{2}q^2 - I - \Omega,\\
&\delta S: && \frac{d\beta}{dt} = \left(\frac{\partial E}{\partial S}\right)_{\varrho} = T,\\
&\delta\mathbf{X}: && \frac{d\boldsymbol{\gamma}}{dt} = 0.
\end{aligned}\right\} \tag{15.7}$$

With the help of Eqs. $(15.6)_2$ and $(15.6)_3$ these equations can be shown to imply Eq. (6.9). Indeed, if we write Eq. $(15.7)_1$ in the form $\mathbf{v} = \sum_{\varkappa} \xi_\varkappa \operatorname{grad}\eta_\varkappa$, then a straightforward calculation based on Eqs. (3.5) and (3.6) yields the acceleration formula

$$\mathbf{a} + \operatorname{grad}\frac{1}{2}q^2 = \sum_{\varkappa}\left(\xi_\varkappa \operatorname{grad}\frac{d\eta_\varkappa}{dt} + \frac{d\xi_\varkappa}{dt}\operatorname{grad}\eta_\varkappa\right). \tag{15.8}$$

But $dS/dt = d\mathbf{X}/dt = d\boldsymbol{\gamma}/dt = 0$, whence

$$\mathbf{a} = -\operatorname{grad}\frac{1}{2}q^2 + \operatorname{grad}\frac{d\varphi}{dt} + \frac{d\beta}{dt}\operatorname{grad}S = -\operatorname{grad}\Omega - \frac{1}{\varrho}\operatorname{grad}p$$

where we have used the simple thermodynamic identity $T\,dS = dI - \frac{1}{\varrho}\,dp$.

To complete the discussion, it must still be shown that every flow is an extremal for the Herivel-Lin principle Eq. (15.5) to (15.6). This has been done by the author of the present article (see Sect. 29A).

It is likely that one can derive the equations of motion for a viscous fluid by a variational argument similar to HERIVEL'S. The essential point to be observed is that the energy equation must be postulated as a *side condition* [in HERIVEL'S work, for example, this is reflected in the condition (15.4)]. Without this or some equivalent side condition, it does not appear possible to obtain the equations of motion of a viscous fluid from Hamilton's principle. Thus MILLIKAN[2] has shown that a principle of the type $\delta\int L\,dv = 0$ where L is a function

[1] Preliminary results of a similar kind are due to A. CLEBSCH, J. reine angew. Math. **54**, 293 (1857); **56**, 1 (1859); and to H. BATEMAN, Proc. Roy. Soc. Lond., Ser. A **125**, 598 (1929).

[2] C. MILLIKAN: Phil. Mag. (7) **7**, 641 (1929).

only of $\mathbf{v}$ and grad $\mathbf{v}$, cannot represent the steady motion of a viscous incompressible fluid except in certain special cases, namely those investigated in Sect. 75 of this article[1].

Other variational principles. In addition to the fundamental principles already discussed, there are numerous variational formulations of special problems in fluid dynamics. At the appropriate place we shall mention some of these special principles, eg. KELVIN's minimum energy theorem (Sect. 24), BATEMAN's principles (Sect. 47), the theorems of HELMHOLTZ and RAYLEIGH (Sect. 75), etc.

C. Incompressible and barotropic perfect fluids.

I. General principles.

16. Preliminary discussion. We shall begin our detailed considerations of fluid flow with the special but highly important case of perfect fluids. Here the stress vector has the simple form $\mathbf{t} = -p\,\mathbf{n}$, and we have the following equations governing the motion,

$$\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0, \tag{16.1}$$

$$\varrho \frac{d\mathbf{v}}{dt} = \varrho \mathbf{f} - \operatorname{grad} p. \tag{16.2}$$

In general, one may adjoin to these four equations a fifth (thermodynamical) relation

$$p = f(\varrho, T), \tag{16.3}$$

where T denotes the absolute temperature. Discussion of this situation is appropriately deferred to the following chapters, while here we consider the elegant theory arising when the pressure and density are *directly related:*

$$p = f(\varrho) \quad \text{or} \quad \varrho = g(p). \tag{16.4}$$

A flow in which density and pressure are thus related is called *barotropic*. We observe that Eq. (16.4) may arise from special circumstances in the flow considered, or it may be an inherent property of the fluid itself. In the latter case the fluid is called *piezotropic*; (the distinction between barotropic flow and piezotropic fluid is clarified if we note that every flow of a piezotropic fluid is barotropic, while the converse is not true, cf. the examples below). The special piezotropic fluids for which $\varrho \equiv$ const are called *incompressible.*

The following examples of barotropic flow may be noted:

1. Air in steady motion in the Mach number range 0 to 0.4. There is less than 8% overall variation of density in this range of Mach numbers, so that for many purposes the density can be supposed to have some appropriate constant value.

2. A gas in isentropic motion. For the case of an ideal gas with constant specific heats we have, in particular,

$$p = N\varrho^{\gamma}, \qquad N, \gamma = \text{const}.$$

We shall assume in this chapter that the extraneous force $\mathbf{f}$ is conservative, $\mathbf{f} = -\operatorname{grad} \Omega$, and all results will be stated subject to this condition. It is worth-

[1] Other negative results concerning variational principles yielding the Navier-Stokes equation are due to R. GERBER, Ann. Inst. Fourier (Grenoble) **1**, 157 (1950); J. Math. Pure Appl. **32**, 79 (1950). Cf. also H. BATEMAN: Phys. Rev. (2) **38**, 815 (1931).

while to point out that no further axioms of motion are necessary for the conclusions of this chapter.

The fundamental property which distinguishes barotropic motion is the simple formula of Euler,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\operatorname{grad}\left(\int \frac{dp}{\varrho} + \Omega\right), \tag{16.5}$$

which shows that acceleration is derivable from a potential. The results of this chapter are largely due to the simplifying effect of this single equation.

Plane motion. Axially-symmetric motion. Vector-lines. We conclude this section with a brief summary of these concepts, mainly in order to fix upon a standard terminology.

A motion is called a *plane* flow if, in some rectangular coordinate system $\mathbf{x} = (x, y, z)$, the velocities $u = v^1$, $v = v^2$ are functions of x, y only, while $v^3 = 0$. The motion takes place in a series of planes parallel to xy, and is the same in each one. For this reason our attention can be directed entirely at the single plane $z = 0$. A motion is said to be *axially-symmetric* if, in some cylindrical polar coordinate system $\mathbf{x} = (x, y, \theta)$[1] the velocities $u = v^1$, $v = v^2$ are functions of x, y only, while $v^3 = 0$. It is obvious that our attention can be confined to the meridian half-plane $\theta = 0$.

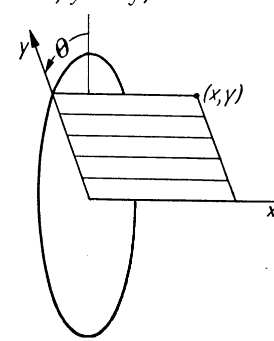

Fig. 2. Coordinates for axially-symmetric motion.

A curve every where tangent to a given continuous vector field is called a vector-line. In particular, the vector-lines of the velocity field are called *stream-lines*, and the vector-lines of the vorticity field are called *vortex-lines*. (It should be noted that streamlines and particle paths are identical in steady motion, but usually not otherwise.) Finally, a motion is said to be *irrotational* if its vorticity field is zero.

17. Convection of vorticity. One of the most important ways of gaining information about a fluid motion is to examine how its vorticity field changes with time. To this end, we shall derive a kinematical identity expressing the rate of change of vorticity in an arbitrary continuous motion. We begin with the well known vector identity

$$\mathbf{v} \cdot \operatorname{grad} \mathbf{v} = \boldsymbol{\omega} \times \mathbf{v} + \operatorname{grad} \tfrac{1}{2} q^2. \tag{17.1}$$

Taking the curl of Eq. (3.5) and using Eq. (17.1) yields

$$\operatorname{curl} \mathbf{a} = \frac{\partial \boldsymbol{\omega}}{\partial t} + \operatorname{curl}(\boldsymbol{\omega} \times \mathbf{v}) = \frac{d\boldsymbol{\omega}}{dt} - \boldsymbol{\omega} \cdot \operatorname{grad} \mathbf{v} + \boldsymbol{\omega} \operatorname{div} \mathbf{v}$$

whence by Eq. (5.3) follows the *diffusion equation* of BELTRAMI[2]:

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\varrho}\right) = \frac{\boldsymbol{\omega}}{\varrho} \cdot \operatorname{grad} \mathbf{v} + \frac{1}{\varrho} \operatorname{curl} \mathbf{a}. \tag{17.2}$$

Let us now apply this result to the barotropic flow of a perfect fluid. By virtue of Eq. (16.5) we have $\operatorname{curl} \mathbf{a} = 0$, so that Eq. (17.2) reduces to

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\varrho}\right) = \frac{\boldsymbol{\omega}}{\varrho} \cdot \operatorname{grad} \mathbf{v}. \tag{17.3}$$

[1] The orientation of coordinates is shown in Fig 2. Instead of the present notation, some authors (notably LAMB and MILNE-THOMPSON) use $(x, \tilde{\omega}, \theta)$. It may be observed that when polar coordinates (r, φ) are introduced into the meridian plane, the resulting spatial coordinates (r, φ, θ) become spherical polar coordinates.

[2] E. BELTRAMI: Mem. Acc. Sci. Bologna (1871 to 1873), (Opere **2**, pp. 202 to 379); especially § 6.

This celebrated equation[1] governs the convection of vorticity in a barotropic flow. It is a very remarkable fact that Eq. (17.3), considered as a differential equation for $\boldsymbol{\omega}$, can be integrated explicitly. For, introducing a new unknown $\mathbf{c}$ by means of[2]

$$\boldsymbol{\omega} = \varrho\,\mathbf{c}\cdot \operatorname{Grad}\mathbf{x}, \tag{17.4}$$

this being possible since $J \neq 0$, there results after a simple computation

$$\frac{d\mathbf{c}}{dt} = 0, \qquad \mathbf{c} = \mathbf{c}(\mathbf{X}).$$

Setting $t=0$ in Eq. (17.4) gives $\boldsymbol{\omega}_0 = \varrho_0\,\mathbf{c}$, and thus

$$\frac{\boldsymbol{\omega}}{\varrho} = \frac{\boldsymbol{\omega}_0}{\varrho_0}\cdot \operatorname{Grad}\mathbf{x}. \tag{17.5}$$

This beautiful result was obtained in 1815 by CAUCHY[3], by an entirely different method. Let us note three important consequences of Eq. (17.5).

1. Vortex-lines are material lines. This means simply that a set of particles which composes a vortex-line at one instant will continue to form a vortex-line at later instants (a quite surprising result!). The proof lies in the fact that a direction $d\mathbf{x}$ once tangent to a vortex-line is carried by the fluid so that it is always tangent to a vortex-line: if at $t=0$, $d\mathbf{x} = d\mathbf{X} = \boldsymbol{\omega}_0\,d\tau$, then at any other instant

$$d\mathbf{x} = d\mathbf{X}\cdot \operatorname{Grad}\mathbf{x} = \boldsymbol{\omega}_0\cdot \operatorname{Grad}\mathbf{x}\,d\tau = \frac{\varrho_0}{\varrho}\,\boldsymbol{\omega}\,d\tau, \tag{17.6}$$

and $d\mathbf{x}$ is tangent to a vortex-line. Incidentally, Eq. (17.6) shows that a material arc ds along a vortex-line varies during the motion according to the formula,

$$\frac{\varrho}{\omega}\,ds = \frac{\varrho_0}{\omega_0}\,ds_0. \tag{17.7}$$

This discussion will be amplified in Sect. 25.

2. The Lagrange-Cauchy theorem[4]. *If a fluid particle or a portion of fluid is initially in irrotational motion, then it will retain this property throughout its entire history.* This is an obvious consequence of CAUCHY's formula (17.5).

3. In plane flow we have

$$\omega_x = \omega_y = 0, \qquad \omega_z = \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{17.8}$$

$\boldsymbol{\omega}$ is thus normal to the flow plane, and $\boldsymbol{\omega}\cdot \operatorname{grad}\mathbf{v} = 0$. It follows from Eq. (17.3) that $\omega/\varrho = \text{const}$ following each particle, vividly illustrating the results of (1)

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[1] Due to E. NANSON: Mess. Math. **3**, 120 (1874). In a slightly different form the result was obtained by EULER, Novi. Comm. Acad. Sci. Petrop. (1762), (Opera Ommia II **12**, pp. 133 to 168). The special case valid for constant density is usually called Helmholtz's equation.

[2] Grad $\equiv \partial/\partial X^\alpha$.

[3] A.-L. CAUCHY: Mém. Divers Savants (2) **1**, (Oeuvres (1) **1**, pp. 5 to 318), especially 1st part, § 4.

[4] J. L. LAGRANGE: Nouv. Mém. Acad. Berlin (1781), (Oeuvres **4**, pp. 695 to 748). LAGRANGE's statement and proof were alike imperfect; a correct formulation and valid proof were given by CAUCHY, loc. cit. above. The proof is predicated on the assumption that the motion is continuous in the sense of Sect. 3. Shock waves of course introduce vorticity into an otherwise irrotational motion (Sect. 54), though singular surfaces across which the flow variables themselves are continuous do not (Sect. 51).

and (2) above. Similarly, in axially-symmetric flow

$$\omega_x = \omega_y = 0, \qquad \omega_\theta = \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{17.9}$$

(see Sect. 12). It follows from Eq. (17.7) that $\omega/y\varrho = \text{const}$ following each particle. This can also be obtained from Eq. (17.3), but not so easily.

18. Bernoullian theorems. By a Bernoulli equation is commonly meant a first integral of the equations of motion[1]. There are various forms which this can take, depending on the particular kinematical or dynamical assumptions made about the motion, though in all cases the basic expression

$$H \equiv \frac{1}{2} q^2 + \int \frac{dp}{\varrho} + \Omega$$

is present. In this section we shall consider the several Bernoulli equations holding for barotropic flow of a perfect fluid.

By means of Eq. (17.1) the fundamental Eq. (16.5) can be written

$$\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\omega} \times \mathbf{v} = -\operatorname{grad} H, \tag{18.1}$$

forming the starting point of the discussion. If it is now assumed that the flow is steady there results the following important

Bernoulli theorem: *Consider the steady barotropic flow of a perfect fluid. Then if* $\boldsymbol{\omega} \times \mathbf{v} \equiv 0$ *in the flow region we have*

$$\frac{1}{2} q^2 + \int \frac{dp}{\varrho} + \Omega \equiv \text{const}.$$

On the other hand, if $\boldsymbol{\omega} \times \mathbf{v} \neq 0$ *then there exists in the flow region a set of surfaces*

$$H = \frac{1}{2} q^2 + \int \frac{dp}{\varrho} + \Omega = \text{const},$$

each one covered by a network of vortex-lines and streamlines. In particular, H is constant on streamlines. The surfaces $H = \text{const}$ may be called Lamb surfaces, after their first investigator[2].

A very similar result may be proved under the weaker assumption that *merely the vorticity is steady*, rather than the entire flow[3]. Indeed, if $\boldsymbol{\omega} \times \mathbf{v} \equiv 0$, then by forming the curl of the relation $\mathbf{v} = k\boldsymbol{\omega}$ we find

$$\mathbf{v} = \frac{\omega^2}{\boldsymbol{\omega} \cdot \operatorname{curl} \boldsymbol{\omega}} \boldsymbol{\omega},$$

whence the velocity field as well as the vorticity field is steady. This is essentially the case already considered above. When $\boldsymbol{\omega} \times \mathbf{v} \neq 0$ the velocity field need not be steady, but instead we have

$$\operatorname{curl} \frac{\partial \mathbf{v}}{\partial t} = \frac{\partial \boldsymbol{\omega}}{\partial t} = 0, \quad \text{and so} \quad \frac{\partial \mathbf{v}}{\partial t} = \operatorname{grad} \chi$$

for some potential function χ. Thus, if $\boldsymbol{\omega} \times \mathbf{v} \neq 0$ there exists in the flow region a set of surfaces

$$\frac{1}{2} q^2 + \int \frac{dp}{\varrho} + \Omega + \chi = \text{const},$$

each one covered by a network of vortex-lines and streamlines.

---

[1] DANIEL BERNOULLI is the discoverer of the general type of theorem to be considered (Hydrodynamica, 1738). This does not mean that he was in possession of the various results which are today called Bernoulli's equation. See the Editor's Introduction to L. EULER, Opera Omnia II **12**.

[2] H. LAMB: Proc. Lond. Math. Soc. **9**, 91 (1878); also [8], § 165.

[3] This observation seems to be due to MASOTTI, Rend. Lincei (6) **6**, 224 (1927).

It is well known that an irrotational flow is characterized by the existence of a (possibly multiple-valued) *velocity potential* $\varphi = \varphi(\mathbf{x}, t)$, such that

$$\mathbf{v} = \operatorname{grad} \varphi. \tag{18.2}$$

(Many authors define the velocity potential by $\mathbf{v} = -\operatorname{grad} \varphi$, conforming with the minus sign found in force potentials. Modern usage tends to omit the minus sign as an unnecessary inconvenience.) The expression (18.2) for $\mathbf{v}$ allows an immediate integration of Eq. (18.1), and we thus obtain *the Bernoulli theorem for irrotational flow,*

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2} q^2 + \int \frac{dp}{\varrho} + \Omega = f(t). \tag{18.3}$$

For steady flow this reduces to

$$\frac{1}{2} q^2 + \int \frac{dp}{\varrho} + \Omega \equiv \text{const}. \tag{18.4}$$

Eqs. (18.3) and (18.4) constitute complete integrals of the equations of motion; their importance can hardly be overestimated.

19. The stream function. It is possible to introduce a stream function whenever the equation of continuity can be written as the sum of two derivatives. We consider in this section plane flow and axially-symmetric flow, though these are by no means the only cases which can be treated. Furthermore we shall assume the fluid to be incompressible, deferring the more elaborate treatment of compressible fluids until later (Sect. 42).

Plane flow. In this case $u = u(x, y)$, $v = v(x, y)$, $w = 0$, so that the equation of continuity takes the simple form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

It follows that the line integral $\int u\, dy - v\, dx$ from some fixed point (x_0, y_0) to the variable point (x, y) defines a (possibly multiple-valued) function $\psi = \psi(x, y, t)$. Obviously

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}, \tag{19.1}$$

so that knowledge of ψ determines the complete velocity field. Moreover, it is evident that the curves $\psi = \text{const}$ are the streamlines of the field. ψ is called the stream function.

From Eqs. (17.8) and (19.1) we obtain the following important equation satisfied by ψ, namely

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega. \tag{19.2}$$

Eq. (19.2) serves to determine ψ when the vorticity magnitude is known. Now for *steady plane flow* the equation of motion (18.1) is equivalent to

$$H = H(\psi), \qquad \omega = -\frac{dH}{d\psi}.$$

Thus any solution of the equation $\Delta^2 \psi = f(\psi)$ provides an example of steady two-dimensional flow; of course in a definite problem one must also take into account boundary conditions on ψ.

In irrotational motion a velocity potential φ exists and

$$u = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

The complex function $w = w(z, t) = \varphi + i\psi$, $z = z + iy$, is therefore analytic, a fact which makes plane irrotational motion of an incompressible fluid particularly amenable to exact solution. Concerning this subject to reader is referred to the works of LAMB [*8*] and MILNE-THOMSON [*10*], and to the articles of BERKER, WEHAUSEN, and GILBARG in this Encyclopedia.

Axially symmetric flow. The development here is entirely analogous to the preceding, and the results need only be sketched. The equation of continuity takes the form

$$\frac{\partial}{\partial x}(y u) + \frac{\partial}{\partial y}(y v) = 0,$$

[cf. formula (12.11)], hence we can define a stream function $\psi = \psi(x, y, t)$ such that

$$u = \frac{1}{y}\frac{\partial \psi}{\partial y}, \qquad v = -\frac{1}{y}\frac{\partial \psi}{\partial x}. \tag{19.3}$$

As equation relating ψ to the vorticity is obtained by eliminating u and v from Eqs. (17.9) and (19.3), with the result

$$E^2\psi \equiv \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - \frac{1}{y}\frac{\partial \psi}{\partial y} = -y\omega. \tag{19.4}$$

In *steady flow* H and ω are connected by

$$H = H(\psi), \qquad \omega = -y\frac{dH}{d\psi};$$

thus any solution of the equation $E^2\psi = y^2 f(\psi)$ provides an example of steady axially symmetric flow. An interesting case is furnished by HILL's spherical vortex, $\psi = \frac{1}{2} A y^2 (a^2 - r^2)$, where $r^2 = x^2 + y^2$. Here $\omega = 5Ay$ and $H = -5A\psi +$ const (cf. [*8*], p. 245).

20. Intrinsic equations of motion. Consider a steady plane motion in which s and n denote, respectively, arc length along streamlines and along their orthogonal trajectories. We wish to express the equations of motion in terms of intrinsic derivatives. To this end, it is convenient first to find an intrinsic expression for the divergence of the velocity vector, at the same time illustrating the general technique to be used throughout the section. Consider a rectangular coordinate system with origin at a fixed point P in the flow field and axes oriented along the streamline and orthogonal trajectory at P. Denoting the coordinates by (x', y'), we have

$$\operatorname{div}\mathbf{v} = \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = \frac{\partial}{\partial x'}[q\cos(\vartheta - \vartheta_P)] + \frac{\partial}{\partial y'}[q\sin(\vartheta - \vartheta_P)],$$

where ϑ is the inclination of the velocity vector to the original x-axis. Expanding the above formula and evaluating at P yields

$$\operatorname{div}\mathbf{v} = \frac{\partial q}{\partial s} + q\frac{\partial \vartheta}{\partial n}, \tag{20.1}$$

which is the required formula. Now introducing the curvatures $\varkappa$ and K of the streamlines and their orthogonal trajectories,

$$\varkappa = \frac{\partial \vartheta}{\partial s}, \qquad K = \frac{\partial \vartheta}{\partial n},$$

then by virtue of (20.1) we may write the equation of continuity in the form

$$\frac{\partial}{\partial s}(\varrho\, q) + K\, \varrho\, q = 0. \tag{20.2}$$

Similarly, using the formula

$$\frac{d}{dt} = q\,\frac{\partial}{\partial s}, \tag{20.3}$$

we obtain by resolution of Eq. (16.2) along streamlines and along normals,

$$\varrho\, q\,\frac{\partial q}{\partial s} = -\frac{\partial p}{\partial s}, \qquad \varrho\, q^2 \varkappa = -\frac{\partial p}{\partial n}. \tag{20.4}$$

Eqs. (20.2) and (20.4) constitute the *intrinsic equations of steady two-dimensional flow*. A calculation similar to that leading to Eq. (20.1) yields the vorticity formula

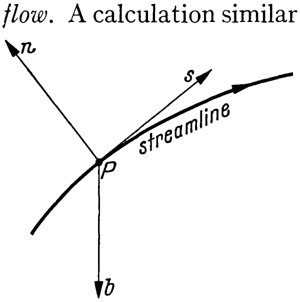

Fig. 3. Intrinsic directions in three-dimensional flow.

$$\omega = -\frac{\partial q}{\partial n} + \varkappa\, q. \tag{20.5}$$

Corresponding equations for axially symmetric flow are easily written down. In particular, we have

$$\frac{\partial}{\partial s}(y\, \varrho\, q) + K\, y\, \varrho\, q = 0 \tag{20.6}$$

in place of Eq. (20.2), while Eqs. (20.4) and (20.5) remain unchanged.

These equations are easily generalized to three-dimensional flows by using the Frenet formulas

$$\frac{\partial \mathbf{s}}{\partial s} = \varkappa\, \mathbf{n}, \qquad \frac{\partial \mathbf{b}}{\partial s} = -\tau\, \mathbf{n}, \tag{20.7}$$

where $\mathbf{s}$, $\mathbf{n}$, and $\mathbf{b}$ are respectively the tangent, normal, and binormal vectors to the congruence of streamlines. We have

$$\operatorname{div} \mathbf{v} = \operatorname{div}(q\, \mathbf{s}) = \frac{\partial q}{\partial s} + q \operatorname{div} \mathbf{s}.$$

Therefore, if we put $\mathfrak{M} = \operatorname{div} \mathbf{s}$, the equation of continuity can be written in the form

$$\frac{\partial}{\partial s}(\varrho\, q) + \mathfrak{M}\, \varrho\, q = 0. \tag{20.8}$$

Similarly, setting $\boldsymbol{v} = q\boldsymbol{s}$ in the equation of motion and using Eq. (20.3) yields

$$\varrho\, q\,\frac{\partial q}{\partial s}\,\mathbf{s} + \varrho\, q^2\,\frac{\partial \mathbf{s}}{\partial s} = -\operatorname{grad} p. \tag{20.9}$$

The second term on the left can be evaluated by one of Frenet's formulas, whence by resolving Eq. (20.9) along the directions $\mathbf{s}$, $\mathbf{n}$, and $\mathbf{b}$ we obtain

$$\varrho\, q\,\frac{\partial q}{\partial s} = -\frac{\partial p}{\partial s}, \qquad \varrho\, q^2\, x = -\frac{\partial p}{\partial n}, \qquad 0 = -\frac{\partial p}{\partial b}. \tag{20.10}$$

Eqs. (20.8) and (20.10) are the required *intrinsic equations of steady motion*.

A formula for the vorticity is obtained most simply as follows. We observe that

$$\boldsymbol{\omega} = \operatorname{curl} \mathbf{v} = \begin{vmatrix} \mathbf{s}_P & \mathbf{n}_P & \mathbf{b}_P \\ \dfrac{\partial}{\partial s} & \dfrac{\partial}{\partial n} & \dfrac{\partial}{\partial b} \\ \mathbf{v}\cdot\mathbf{s}_P & \mathbf{v}\cdot\mathbf{n}_P & \mathbf{v}\cdot\mathbf{b}_P \end{vmatrix}, \tag{20.11}$$

where P is a fixed point in the flow, and (s, n, b) are considered as fixed rectangular coordinates with origin at P (see Fig. 3). Setting $\mathbf{v} = q\,\mathbf{s}$ in Eq. (20.11), performing the indicated differentiations and evaluating at P yields the formula

$$\boldsymbol{\omega} = q\left(\mathbf{b}\cdot\frac{\partial \mathbf{s}}{\partial n} - \mathbf{n}\cdot\frac{\partial \mathbf{s}}{\partial b}\right)\mathbf{s} + \frac{\partial q}{\partial b}\,\mathbf{n} + \left(\varkappa q - \frac{\partial q}{\partial n}\right)\mathbf{b}. \tag{20.12}$$

In the case of irrotational flow, or, more generally, whenever the velocity field is normal to a one-parameter family of surfaces S, the factor $\mathfrak{M}$ in Eq. (20.8) can be interpreted as the sum of the principal curvatures (mean curvature) of the equipotential surfaces or the surfaces S, which ever the case may be.

The remainder of this chapter is divided into two major sections concerned respectively with results peculiar to irrotational motion and to rotational motion. We begin with the simpler and more fully investigated case.

II. Irrotational motion.

21. The assumption of irrotational motion. As is apparent from the Bernoulli equations (18.3) and (18.4), the equations of fluid motion are greatly simplified by the assumption of irrotational flow. The usual justification for this assumption comes from the Cauchy-Lagrange theorem (Sect. 17), which implies that *a barotropic motion of a perfect fluid is irrotational if each of its particles starts from a region of quiet.* This does not prove that a barotropic fluid will move *only* irrotationally, for, aside from the philosophical question whether every particle were sometime at rest, vorticity may well exist in a particular problem under study, introduced, for example, by the extraneous mechanisms of viscosity or shock waves. The theorem above nevertheless strongly indicates the importance of irrotational motion in the theory of perfect fluids.

There is an important related result in this connection which serves in many cases to guarantee irrotational flow. It is commonly expressed by the statement that a flow emanating from a quiet state or a uniform state must be irrotational. Although roughly correct, this assertion must be qualified somewhat to be entirely accurate, for consider the case when the uniform state is at ∞. In this case $\lim \mathbf{v}$, $\lim \varrho$, and $\lim p$ all exist as $\mathbf{x} \to \infty$, and $\lim \boldsymbol{\omega} = 0$. Now if the flow is plane or axially symmetric, then by Theorem 3 of Sect. 17 we do indeed have $\omega = 0$. Not so, however, in a genuine three-dimensional flow[1]. There the conclusion can only be drawn in the case of steady motion, and the argument goes as follows: From the Theorem of Sect. 18, we have

$$H = \frac{1}{2}q^2 + \int \frac{dp}{\varrho} + \Omega = \text{constant on streamlines};$$

evaluation of the constant at ∞ shows it to be the same for each streamline[2]. It follows then from Eq. (18.1) that $\boldsymbol{\omega}\times\mathbf{v} = 0$, which in turn implies

$$\boldsymbol{\omega} = k\,\mathbf{v}, \tag{21.1}$$

[1] Although no counter-example is known, a little reflection will show that irrotationality does not follow simply from Eq. (17.3) or the equivalent formula (17.7); cf. LECORNU: C. R. Acad. Sci., Paris **168**, 923 (1919). I am indebted to a remark of TRUESDELL [*26*], § 77 for this reference and for the germ of the following proof.

[2] It is assumed that Ω tends to a finite limit at ∞.

where k is a scalar. Taking the divergence of both sides of Eq. (21.1) gives

$$0 = \operatorname{div} k\,\mathbf{v} = \operatorname{div}\left(\frac{k}{\varrho}\,\varrho\,\mathbf{v}\right) = \varrho\,\mathbf{v}\cdot\operatorname{grad}\frac{k}{\varrho}.$$

But this is nothing more than the condition that k/ϱ be constant on streamlines. Thus Eq. (21.1) becomes

$$\frac{\boldsymbol{\omega}}{\varrho\,\mathbf{v}} = \text{constant on streamlines}. \tag{21.2}$$

If $\lim \mathbf{v} \neq 0$ we find by evaluation at ∞ that the constant is zero, hence $\boldsymbol{\omega} = 0$. On the other hand, if $\lim \mathbf{v} = 0$ (flow emanating from a rest at ∞) we must be content with the milder conclusion (21.2).

The results of the preceding paragraphs may be summarized in the statement: *Barotropic flow of a perfect fluid, subject to conservative extraneous force, is irrotational if each particle starts from a region of quiet or uniform state. Moreover, a plane flow, an axially-symmetric flow, or a steady flow with* $\lim \mathbf{v} \neq 0$, *is irrotational if it emanates from a uniform state at* ∞.

22. Principles of irrotational motion. Development of a potential at infinity. Irrotational motion is characterized by the existence of a velocity potential $\varphi = \varphi(\mathbf{x}, t)$ satisfying $\mathbf{v} = \operatorname{grad}\varphi$. If the fluid under consideration is incompressible we have also $\operatorname{div}\mathbf{v} = 0$, and φ therefore satisfies LAPLACE's equation

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0. \tag{22.1}$$

The problem of irrotational motion of an incompressible fluid thus reduces simply to the solution of (22.1) under suitable boundary conditions. We shall discuss some fundamental aspects of this problem in the present and following sections, deferring until Part E the corresponding treatment for compressible fluids.

To begin with, consider the important situation in which fluid occupies the entire exterior of one or more moving bodies, each one rigid and of finite size. Let us suppose that the flow at infinite distances is uniform and of velocity $\mathbf{U}$, that is, $\lim_{r\to\infty} \mathbf{v} = \mathbf{U}$. *Then the potential and velocity have the following asymptotic behavior as* $r \to \infty$

$$\left.\begin{aligned} \varphi &= \mathbf{U}\cdot\mathbf{x} + C + O(r^{-2}), \\ \mathbf{v} &= \mathbf{U} + O(r^{-3}). \end{aligned}\right\} \tag{22.2}$$

Here $\mathbf{x} = (x, y, z)$ *and* $r^2 = x^2 + y^2 + z^2$.

There are several proofs of this fundamental result, of which the following is probably the simplest. We observe first that it is sufficient to consider the case $\mathbf{U} = 0$, for the general case can be reduced to this one by superposing a uniform velocity field. Now under the assumption $\mathbf{U} = 0$ it can be shown that[1]

$$\varphi(\mathbf{x}) - C = \frac{1}{4\pi}\oint_{\Sigma}\left[\varphi\,\frac{\partial}{\partial n}\left(\frac{1}{r'}\right) - \frac{1}{r'}\,\frac{\partial\varphi}{\partial n}\right] da, \tag{22.3}$$

where C is an appropriate constant, Σ is a closed surface surrounding the obstacles, $\mathbf{x}$ is a point outside Σ, and r' denotes the distance from $\mathbf{x}$ to the point of integration. (Note that the velocity potential may be multiple-valued due to perforations in some of the obstacles, but that any branch may be considered single-

[1] L. M. MILNE-THOMSON [*10*], § 3.75.

valued for r sufficiently large.) The integral on the right-hand side of Eq. (22.3) obviously has the asymptotic behavior

$$\frac{\Delta}{4\pi r} + O(r^{-2}), \tag{22.4}$$

where

$$\Delta = \oint_{\Sigma} \frac{\partial \varphi}{\partial n} da$$

is the outflow integral. In the present case there is no outflow, hence Δ vanishes and (22.2) is proved. Differentiating (22.3) with respect to $\mathbf{x}$ similarly leads to

$$\mathbf{v} = O(r^{-3}), \tag{22.5}$$

(assuming always that $\Delta = 0$). This completes the proof of Eq. (22.2).

If desired, the asymptotic formulas (22.4) and (22.5) can be written out in terms of spherical harmonics, giving a complete development for the potential and velocity in the neighborhood of infinity[1].

An alternative proof is based on the representation of the velocity field as a Poisson integral

$$\mathbf{v}(\mathbf{x}) = \int_{r=R} \mathbf{v} \frac{|x|^2 - R^2}{4\pi r'^3} da, \tag{22.6}$$

where R is suitably large, and $|\mathbf{x}| > R$. The representation (22.6) is easily established on the basis of known properties of the Poisson integral[2]. From Eq. (22.6) and the fact that div $\mathbf{v} = 0$ it follows that

$$\mathbf{v} = O(r^{-2}). \tag{22.7}$$

Integration of (22.7) yields $\varphi - C = O(r^{-1})$, whence φ can also be represented by a Poisson integral. The remainder of the proof is similar to the one above.

The asymptotic behavior of a plane flow can be obtained from analytic function theory, or it can be derived directly from an argument of the type just given. (The first method given above fails since φ may be multiple-valued.) The result is that, as $r \to \infty$,

$$\left.\begin{aligned} \varphi &= \mathbf{U} \cdot \mathbf{x} + C + \frac{\Gamma}{2\pi} \theta + O(r^{-1}), \\ \mathbf{v} &= \mathbf{U} + \frac{\Gamma}{2\pi r^2} (-y, x) + O(r^{-2}). \end{aligned}\right\} \tag{22.8}$$

One easily verifies that $\Gamma = \oint \mathbf{v} \cdot d\mathbf{x}$, the integral being taken along any circuit surrounding the obstacles once in the counter-clockwise sense. Γ is called the *circulation of the flow*; cf. also the definition of circulation in a rotational flow (Sect. 25).

23. Principles of irrotational motion (continued). In this section we review the important standard theorems of the subject.

1. The velocity maximum occurs at the fluid boundary. A simple proof, due to KIRCHHOFF, will be found in LAMB [*8*], § 37; another is given in Sect. 28 of the present article.

2. Kinetic energy. For flow in a bounded simply-connected region $\mathscr{v}$ it is found easily from GREEN's theorem that

$$2\mathfrak{T} = \int_{\mathscr{v}} \varrho (\operatorname{grad} \varphi)^2 dv = \varrho \oint_{\mathscr{s}} \varphi \frac{\partial \varphi}{\partial n} da, \tag{23.1}$$

[1] O. D. KELLOGG [*46*], p. 143.
[2] O .D. KELLOGG [*46*], p. 243.

where $\mathscr{s}$ is the boundary of $\mathscr{v}$. If $\mathscr{v}$ is the exterior of a surface $\mathscr{s}$ and if the fluid is at rest at infinity, the kinetic energy is again given by Eq. (23.1). For if Σ denotes a large sphere enclosing $\mathscr{s}$, the kinetic energy $\mathfrak{T}^*$ of the fluid instantaneously between $\mathscr{s}$ and Σ is given by

$$2\mathfrak{T}^* = \varrho \oint_{\mathscr{s}} \varphi \frac{\partial \varphi}{\partial n} da + \varrho \oint_{\Sigma} \varphi \frac{\partial \varphi}{\partial n} da.$$

If outflow is not present the integrand of the second integral is $O(r^{-5})$, and otherwise $O(r^{-3})$, [it is assumed that φ is normalized so that $C=0$ in (22.2)]. In either case the second integral approaches zero as Σ becomes infinite, and Eq. (23.1) follows.

3. Uniqueness. Consider the problem discussed in the preceding section. We assert that in a simply-connected flow region the fluid motion is completely determined by the motion of the bodies and by the value of $\mathbf{U}$. Indeed if φ_1 and φ_2 are the potentials of two flows consistent with the prescribed conditions, then $\varphi = \varphi_1 - \varphi_2$ is the potential of a flow at rest at infinity and satisfying $\partial\varphi/\partial n = 0$ at the surfaces of the moving bodies. Assuming the flow region to be simply-connected, we then have $\mathfrak{T}=0$ from Eq. (23.1). It follows that $\operatorname{grad}\varphi = 0$ and $\varphi =$ constant, that is, the two flows must be identical. If the flow region is not simply-connected, simple examples show the result not to be true. For a discussion of the fluid motion in this case, the reader is referred again to LAMB, in particular §§ 47—55.

The uniqueness theorem proved above is important not only for its own sake, but also because it shows that one cannot in general prescribe more than the normal fluid velocity at a boundary surface. In particular, an *adherence condition* at a rigid surface will usually be incompatible with irrotational fluid motion[1].

4. The d'Alembert paradox. Consider the force acting on a solid body moving with uniform velocity through a fluid at rest at infinity, or, what comes to the same thing and is more easily computed, the action of a uniform stream on a fixed solid immersed in it. If $\mathbf{U}$ denotes the velocity of the uniform stream, then according to Eq. (22.2)

$$\mathbf{v} = \mathbf{U} + O(r^{-3}),$$

where, as in the rest of the paragraph, the order symbol refers to behavior as $r \to \infty$. It follows from BERNOULLI's theorem (18.4) that

$$p = p_0 + \tfrac{1}{2}\varrho(U^2 - q^2) = p_0 + O(r^{-3}),$$

assuming $\Omega = 0$. Thus by Eq. (10.2), with $\mathbf{t} = -p\mathbf{n}$ and Σ a large sphere of radius R,

$$\mathbf{F} = -\int_{\Sigma} (p\,\mathbf{n} + \varrho\,\mathbf{v}\,\mathbf{v}\cdot\mathbf{n})\,da = -\int_{\Sigma} (p_0\,\mathbf{n} + \varrho\,\mathbf{U}\,\mathbf{U}\cdot\mathbf{n})\,da + O(R^{-1}).$$

The integral on the right vanishes by virtue of the divergence theorem, and it follows easily that $\mathbf{F} = 0$.

This result, which at first glance appears unexpected, is known as d'Alembert's paradox[2]. The difficulty of course lies in the fact that the flow model is much too simplified[3].

[1] An extensive discussion of this point is given by TRUESDELL [*26*], § 37. The present argument, it should be noted, applies also to compressible irrotational motions once the appropriate uniqueness theorem is proved.

[2] J. L. D'ALEMBERT: Opuscules Mathematiques **5** (1768). The result is there proved only for symmetric obstacles, and by the trivial argument involving a symmetric pressure distribution. The same idea had occurred earlier in d'ALEMBERT's work, and the result was also given by EULER in his treatise on gunnery (1745): cf. the Editor's introduction to L. EULER, Opera Omnia (2) **12**.

[3] Cf. LAMB [*8*], §§ 370, 371.

The forces on an obstacle in plane flow are slightly more difficult to compute because of the presence of circulation. Let the coordinates be chosen with the positive x-axis in the direction of the uniform stream, so that $\mathbf{U} = (U, O)$. Then setting $\mathbf{v} = (u, v)$ we easily derive from Eq. (22.8) and BERNOULLI's equation that

$$p = p_0 + \varrho\, U(u - U) + O(r^{-2}).$$

Hence the force in the direction of the uniform stream is given by

$$\begin{aligned} X &= -\int_{\Sigma} (p \cos\theta + \varrho\, u\, \mathbf{v}\cdot\mathbf{n})\, ds \\ &= \varrho \int_{\Sigma} [U(u - U)\cos\theta - (u - U)\,\mathbf{v}\cdot\mathbf{n} - U\,\mathbf{v}\cdot\mathbf{n}]\, ds + O(R^{-1}), \end{aligned}$$

where $\mathbf{v}\cdot\mathbf{n} = u\cos\theta + v\sin\theta$. The sum of the first two terms of the integrand is $O(R^{-2})$ by (22.8). The third term represents outflow and correspondingly its integral vanishes. It follows then that $X = 0$, and again there is no resistance. A similar calculation yields, however, a non-zero value for the lift

$$Y = -\varrho\, \Gamma\, U. \tag{23.2}$$

Remarkably, Y depends on the circulation in a manner entirely independent of the shape and dimensions of the obstacle. Formula (23.2), obtained independently by KUTTA[1] and JOUKOWSKY[2], is the basis for the theory of lift of an airfoil.

We call attention to KIRCHHOFF's formulas for the force and moment on a solid moving *in an arbitrary way* through a fluid. Although there is not enough space here to consider this theory, the reader will find an interesting discussion in [*8*], Chap. 6.

24. KELVIN's minimum energy theorem. Consider flows occupying a bounded simply-connected region $\mathscr{v}$ of space, the normal mass-flux being prescribed at each point of the boundary of $\mathscr{v}$, that is,

$$\varrho\, \mathbf{v}\cdot\mathbf{n} = h \quad \text{prescribed on } \mathscr{s}. \tag{24.1}$$

The following criterion characterizes irrotational flow among the totality of incompressible flows satisfying Eq. (24.1).

Kelvin's principle: Among all motions of an incompressible fluid in $\mathscr{v}$ which satisfy Eq. (24.1), the irrotational motion has least kinetic energy.

The proof given by KELVIN and reproduced by LAMB, § 45, cannot be improved. KELVIN's theorem has a converse, which, for some reason, is seldom stated in works on hydrodynamics, this converse being, in fact, nothing more than a restatement of the classical Dirichlet principle of potential theory. The result can be formulated as follows.

Dirichlet's principle: Among all irrotational motions in $\mathscr{v}$, the one satisfying

$$\operatorname{div} \mathbf{v} = 0, \qquad \varrho\, \mathbf{v}\cdot\mathbf{n} = h \quad \text{on} \quad \mathscr{s} \tag{24.2}$$

gives the greatest value to the expression

$$\mathscr{J} = -\tfrac{1}{2}\varrho \int_{\mathscr{v}} q^2\, dv + \oint_{\mathscr{s}} \varphi\, h\, da; \tag{24.3}$$

here φ is the potential of a competing motion[3].

[1] W. M. KUTTA: Sitzgsber. bayr. Akad. Wiss. (Münch.) **40** (1910).

[2] N. E. JOUKOWSKI: Bull. Inst. Aero. Koutchino (St. Petersburg) (1906).

[3] A similar theorem has been given by A. R. PRATELLI, Rend. Ist. Lombardo (3) **17**, 484 (1953). PRATELLI also considers the possibility of a prescribed distribution of vorticity for the competing motions.

The factor ϱ in Eqs. (24.2) and (24.3) refers to the constant density of a given incompressible fluid under consideration. Thus the competing motions, because they need not satisfy $\operatorname{div} \mathbf{v} = 0$, may be impossible as far as the given fluid is concerned. To prove DIRICHLET'S principle, let φ be the potential of the irrotational motion satisfying Eq. (24.2). Obviously φ is harmonic and is uniquely determined up to an additive constant. Then by simple transformations we find

$$\mathscr{J}(\varphi) = \mathscr{J}(\varphi^*) + \tfrac{1}{2}\varrho \int_{\mathscr{v}} [\operatorname{grad}(\varphi - \varphi^*)]^2 dv,$$

where φ^* is any other potential. Thus $\mathscr{J}(\varphi) \geqq \mathscr{J}(\varphi^*)$, and the equality can hold if and only if $\varphi^* = \varphi + \text{const}$. This proves the assertion.

It is evident that the two principles above possess a common extremal motion. Moreover, *the minimized energy in the Kelvin principle is exactly equal to the maximized $\mathscr{J}$ in the Dirichlet principle:* This follows from the fact that, for the extremal flow,

$$\tfrac{1}{2}\varrho \int_{\mathscr{v}} q^2\, dv = \mathfrak{T}$$

and

$$\oint_{\mathscr{s}} \varphi\, h\, da = \varrho \int_{\mathscr{s}} \varphi \frac{\partial \varphi}{\partial n}\, da = 2\mathfrak{T}.$$

III. Rotational motion.

25. Kelvin's circulation theorem. The Helmholtz theorems. Kelvin in 1869 introduced the concept of circulation for the purpose of visualizing better the geometrical nature of a fluid motion, and in order to give new proofs for some remarkable theorems obtained just a little earlier by HELMHOLTZ[1].

The *circulation* around any closed curve (circuit) in the fluid is defined by the integral $\oint_{\mathscr{C}} \mathbf{v} \cdot d\mathbf{x} = \oint_{\mathscr{C}} v_i dx^i$. If now $\mathscr{C}$ moves with the fluid, $\mathscr{C} = \mathscr{C}(t)$, we may calculate the rate of change of the circulation around this moving circuit. The motion of the curve $\mathscr{C}$ is given parametrically by the equation

$$\mathbf{x} = \boldsymbol{\varphi}(s, t), \quad 0 \leqq s \leqq 1,$$

where s locates individual particles on $\mathscr{C}$, and t is the time. It follows easily that

$$\frac{d}{dt} \oint_{\mathscr{C}} \mathbf{v} \cdot d\mathbf{x} = \int_0^1 \frac{d}{dt}\left(\mathbf{v} \cdot \frac{d\mathbf{x}}{ds}\right) ds = \oint_{\mathscr{C}} \mathbf{a} \cdot d\mathbf{x}. \tag{25.1}$$

This result is purely kinematical, and thus valid for any motion whatsoever. Now for barotropic flow we recall from Eq. (16.5) that $\mathbf{a}$ is derivable from a potential; therefore the last integral in Eq. (25.1) vanishes, and

$$\frac{d}{dt} \oint_{\mathscr{C}} \mathbf{v} \cdot d\mathbf{x} = 0. \tag{25.2}$$

This is Kelvin's theorem; stated in words, *the barotropic flow of a perfect fluid under the action of a conservative extraneous force is circulation-preserving.* Conversely, if Eq. (25.2) holds for all circuits $\mathscr{C}$, then $\mathbf{a}$ is derivable from a potential. [From these results it is seen that a motion of an *incompressible fluid* is dynami-

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[1] W. THOMSON (Lord KELVIN): Trans. Roy. Soc. Edinb. **25**, 217 (1869), (Papers **4**, pp. 13 to 66). — H. HELMHOLTZ: J. reine angew. Math. **55**, 25 (1858).

cally possible if and only if Eq. (25.2) holds for all circuits $\mathscr{C}$; this is not generally true for compressible fluids.]

With the help of Kelvin's theorem it is now a simple matter to obtain the three theorems of HELMHOLTZ, which so vividly present the geometrical aspects of circulation-preserving motion. The first of the theorems is purely kinematical, the remaining ones are simple deductions from Kelvin's theorem and therefore apply in *any* situation where a motion can be assumed circulation-preserving, irrespective of the nature of the medium.

A *vortex-tube* is usually defined as the surface swept out by the vortex-lines passing through a given closed curve. This definition unfortunately embraces a number of configurations which would not ordinarily be called "tubes". In the future we shall therefore restrict consideration specifically to those vortex-tubes whose "cross sections" are simple closed curves, one of which is the defining curve.

Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be any two circuits drawn on the vortex tube, each encircling the tube in the same direction (more precisely, each circuit should be reducible on the tube to the closed curve defining the tube). *The first Helmholtz theorem states that the circulation about $\mathscr{C}_1$ is the same as the circulation about $\mathscr{C}_2$.* For the proof we refer the reader to nearly any work on hydrodynamics—particularly good is the discussion given by LAMB [*8*], § 145. There are several remarks to be made before we pass to the other Helmholtz theorems. First, by means of this theorem we are entitled to define the *strength* of a vortex-tube by the circulation around any curve $\mathscr{C}$ lying on it and encircling it. Thus

$$\text{Strength} = \int_{\mathscr{C}} \mathbf{v} \cdot d\mathbf{x} = \int_{s} \boldsymbol{\omega} \cdot \mathbf{n}\, da,$$

the last equality following by Stokes' theorem, where s is any (orientable) surface in the fluid spanning $\mathscr{C}$. Second, LAMB's proof shows that the theorem is true even if the vorticity field is only piecewise continuous, so long as the velocity field is continuous. And third, we wish to call attention to a statement commonly made on the basis of the first Helmholtz theorem, namely that *vortex-lines* are either closed curves, or else extend to the boundary of the fluid motion. KELLOGG[1] has pointed out an error in this statement, but notes that it becomes correct when *vortex-tube* is substituted for vortex-line.

The second and third Helmholtz theorems state respectively that *vortex-lines are material lines*, (equivalently: vortex-tubes move with the fluid), and that *the strength of a vortex-tube remains constant as the tube moves with the fluid.* The second Helmholtz theorem has already been noted in Sect. 17 of this article; it can also be proved from Kelvin's circulation theorem (see LAMB, § 146). The third Helmholtz theorem is obvious from Kelvin's theorem. We remark finally that these theorems remain true when the vorticity field is only piecewise-continuous.

Vorticity theorems for non-circulation-preserving motions will be found in Sects. 40 and 69.

26. General considerations of vortex motion. The subject of vortex motion is discussed at length by LAMB [*8*] and VILLAT [*28*]. Though these treatments are generally quite complete, nevertheless we wish to clarify a few basic results which are not adequately explained there. In particular, we consider the problem of determining a velocity field in terms of its vorticity and expansion, and some related results concerning the overall distribution of vorticity.

[1] O. D. KELLOGG [*46*], p. 41.

The expansion and vorticity of a velocity field are defined by

$$\Theta = \operatorname{div} \mathbf{v}, \quad \boldsymbol{\omega} = \operatorname{curl} \mathbf{v}. \tag{26.1}$$

It is easy to show that *corresponding to a given (piecewise continuous) distribution of expansion and vorticity in a finite region, there exists not more than one velocity field having a prescribed normal speed on the boundary* (see LAMB, § 147). The same result holds for an infinite region if $\mathbf{v}$ is prescribed at infinity. The *existence* of a velocity field with given vorticity and expansion is somewhat more difficult. We treat three cases.

1. Finite regions: $\boldsymbol{\omega} \cdot \mathbf{n} = 0$ *on boundary*. Assuming that the given vorticity and expansion are once differentiable in a finite region $\mathscr{v}$, then a corresponding velocity field can be constructed from the potentials

$$\varphi = \frac{1}{4\pi} \int_{\mathscr{v}} \frac{\Theta}{r'}\, dv, \qquad \boldsymbol{\pi} = \frac{1}{4\pi} \int_{\mathscr{v}} \frac{\boldsymbol{\omega}}{r'}\, dv, \tag{26.2}$$

where r' denotes the distance between the point $\mathbf{x}'$ of integration and the point $\mathbf{x}$ at which the values of φ and $\boldsymbol{\pi}$ are required. We define

$$\mathbf{w} = -\operatorname{grad} \varphi + \operatorname{curl} \boldsymbol{\pi},$$

and observe that

$$\operatorname{div} \mathbf{w} = -\nabla^2 \varphi = \Theta,$$

$$\operatorname{curl} \mathbf{w} = \operatorname{curl} \operatorname{curl} \boldsymbol{\pi} = \operatorname{grad} \operatorname{div} \boldsymbol{\pi} - \nabla^2 \boldsymbol{\pi} = \boldsymbol{\omega}.$$

In the final step we have used the fact that $\operatorname{div} \boldsymbol{\pi} = 0$, i.e.,

$$\operatorname{div} \boldsymbol{\pi} = \frac{1}{4\pi} \int_{\mathscr{v}} \boldsymbol{\omega} \cdot \operatorname{grad} \frac{1}{r'}\, dv = -\frac{1}{4\pi} \int_{\mathscr{v}} \operatorname{div}' \left(\frac{\boldsymbol{\omega}}{r'}\right) dv$$

$$= -\frac{1}{4\pi} \oint_{\mathscr{s}} \frac{\boldsymbol{\omega} \cdot \mathbf{n}}{r'}\, da = 0;$$

the change in sign in the second equality is due to the shift of differentiation from $\mathbf{x}$ to $\mathbf{x}'$. Since $\mathbf{w}$ has correct vorticity and expansion, the required velocity field $\mathbf{v}$ can be written in the form

$$\mathbf{v} = \mathbf{w} + \operatorname{grad} h = \operatorname{grad}(h - \varphi) + \operatorname{curl} \boldsymbol{\pi}, \tag{26.3}$$

where h is a harmonic function to be determined so that $\mathbf{v} \cdot \boldsymbol{n}$ takes prescribed values on s. (These values must of course be consistent with the total expansion.)

The velocity field $\mathbf{v}^* = \operatorname{curl} \boldsymbol{\pi}$ is continuous in all space, has vorticity $\boldsymbol{\omega}$ inside $\mathscr{v}$ and zero outside $\mathscr{v}$, and $\mathbf{v}^* = 0$ at infinity. The condition $\boldsymbol{\omega} \cdot \mathbf{n} = 0$ on $\mathscr{s}$ means simply that the vortex lines are closed curves in $\mathscr{v}$. $\mathbf{v}^*$ is thus the velocity field due to an isolated vortex system in an incompressible fluid.

A natural problem arising in connection with a given velocity field $\mathbf{v}^*$ concerns the existence and uniqueness of the subsequent motion of the fluid under the Euler equations. The difficult problem of existence has been treated by LICHTENSTEIN[1], HÖLDER[2], WOLBINER[3], SCHAEFFER[4], and MARUHN[5]. That the flow is uniquely determined by the initial velocity field can be proved by the methods of Sect. 72. Special cases where the subsequent motion can be determined explicitly will be found in the works of LAMB, VILLAT, LICHTENSTEIN and MILNE-THOMSON.

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[1] L. LICHTENSTEIN [*9*], Chap. 12; [*22*], Chaps. 11, 12.
[2] E. HÖLDER: Math. Z. **37**, 727 (1933).
[3] W. WOLBINER: Math. Z. **37**, 698 (1933).
[4] A. C. SCHAEFFER: Trans. Amer. Math. Soc. **42**, 497 (1937).
[5] K. MARUHN: Math. Z. **45**, 155 (1939).

2. *Infinite regions.* Assuming that Θ and $\boldsymbol{\omega}$ vanish outside some finite region and that $\mathbf{v}=0$ at infinity, we have

$$\mathbf{v} = -\operatorname{grad}\varphi + \operatorname{curl}\boldsymbol{\pi},$$

where φ and $\boldsymbol{\pi}$ are the potentials (26.1), taken over all space. When Θ and $\boldsymbol{\omega}$ vanish at infinity of order r^{-3}, the same formula can be shown to hold. Finally, if Θ and $\boldsymbol{\omega}$ vanish at infinity of order r^{-2}, then φ and $\boldsymbol{\pi}$ no longer exist, but the differentiated form of Eq. (26.2) remains valid,

$$\mathbf{v} = -\frac{1}{4\pi}\int \Theta \operatorname{grad}\left(\frac{1}{r'}\right) dv - \frac{1}{4\pi}\int \boldsymbol{\omega}\times\operatorname{grad}\left(\frac{1}{r'}\right) dv.$$

3. *Finite regions: general case.* The vector field (26.2) is no longer adequate for the description of $\mathbf{v}$ since $\operatorname{div}\boldsymbol{\pi}$ is not necessarily zero. However, if $\mathbf{v}$ is known on $\mathscr{s}$, then we can write

$$\mathbf{v} = -\operatorname{grad}\varphi^* + \operatorname{curl}\boldsymbol{\pi}^*,$$

where

$$\varphi^* = \frac{1}{4\pi}\int_{\mathscr{v}} \frac{\Theta}{r'}\,dv - \frac{1}{4\pi}\oint_{\mathscr{s}} \frac{\mathbf{n}\cdot\mathbf{v}}{r'}\,da, \tag{26.4}$$

$$\boldsymbol{\pi}^* = \frac{1}{4\pi}\int_{\mathscr{v}} \frac{\boldsymbol{\omega}}{r'}\,dv - \frac{1}{4\pi}\oint_{\mathscr{s}} \frac{\mathbf{n}\times\mathbf{v}}{r'}\,da, \tag{26.5}$$

([*48*], p. 190). It may be noticed that the surface integrals are harmonic in $\mathscr{v}$. If only $\mathbf{v}\cdot\mathbf{n}$ is given on $\mathscr{s}$, then the surface integral in Eq. (26.5) is not known, and a different procedure must be given.

Let us first extend $\boldsymbol{\omega}$ throughout all space by the following definition. Let φ' be harmonic in the exterior of $\mathscr{v}$ and satisfy $\partial\varphi'/\partial n = \boldsymbol{\omega}\cdot\mathbf{n}$ on $\mathscr{s}$. At points outside $\mathscr{v}$ we define $\boldsymbol{\omega} = \operatorname{grad}\varphi'$. This vector is divergenceless, and is $O(r^{-3})$ at infinity. With a revised meaning for $\boldsymbol{\pi}$, namely the integral being taken over all space, formula (26.2) can now be proved exactly as in paragraph 1.

Lamb includes two formulas for the kinetic energy of a vortex-system in an incompressible fluid. Suppose first that the fluid is at rest at infinity and $\boldsymbol{\omega}$ vanishes outside some finite region. Then

$$\mathfrak{T} = \frac{\varrho}{8\pi}\iint \frac{\boldsymbol{\omega}\cdot\boldsymbol{\omega}'}{r}\,dv\,dv',$$

where each integration extends over the whole space occupied by the vortices. Truesdell[1] has generalized this to the case of a finite region, allowing also for expansion. His result (corrected in one respect) is

$$\frac{1}{2}\int_{\mathscr{v}} q^2\,dv = \frac{1}{8\pi}\iint_{\mathscr{v}\,\mathscr{v}} \frac{\Theta\Theta' + \boldsymbol{\omega}\cdot\boldsymbol{\omega}'}{r}\,dv\,dv' - \frac{1}{2}\int_{\mathscr{v}} (\Theta f + \boldsymbol{\omega}\cdot\mathbf{g})\,dv - \\ - \frac{1}{2}\oint_{\mathscr{s}} (\varphi^*\mathbf{v} + \boldsymbol{\pi}^*\times\mathbf{v})\cdot\mathbf{n}\,da,$$

where f and $\mathbf{g}$ are the surface potentials in Eqs. (26.4) and (26.5). Lamb's second formula is given in Sect. 71 of this article.

In conclusion we mention two interesting vorticity formulas which hold for an arbitrary continuous motion. The first is due to Lamb[2]

$$\int_{\mathscr{v}} (\Theta\mathbf{v} + \boldsymbol{\omega}\times\mathbf{v})\,dv = \oint_{\mathscr{s}} [\mathbf{v}(\mathbf{v}\cdot\mathbf{n}) - \tfrac{1}{2}q^2\mathbf{n}]\,da, \tag{26.6}$$

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[1] C. Truesdell [*26*], § 35.

[2] H. Lamb [*8*], p. 218. Lamb considered only incompressible fluids, so that the term $\Theta\mathbf{v}$ does not appear in his formula.

and the second to TRUESDELL[1]

$$\frac{d}{dt}\int_{\mathscr{V}} \boldsymbol{\omega}\, dv = \oint_{\mathscr{S}} [\mathbf{v}(\boldsymbol{\omega}\cdot\mathbf{n}) - (\mathbf{a}\times\mathbf{n})]\, da. \tag{26.7}$$

(These formulas are simple applications of the divergence theorem and identities (17.1) and (17.2)) When $\mathbf{v}=0$ on s, or for an infinite region under conditions such that the surface integrals are negligible, these formulas become simply

$$\int_{v} (\boldsymbol{\omega}\times\mathbf{v})\, dv = 0, \qquad \frac{d}{dt}\int_{\mathscr{V}} \boldsymbol{\omega}\, dv = 0,$$

that is, the average of $\boldsymbol{\omega}\times\mathbf{v}$ is zero and the total vorticity is constant.

27. A vorticity measure. TRUESDELL[2] has pointed out that the angular velocity magnitude ω is not necessarily a good measure of the rotationally of a fluid motion. For ω, besides depending on the units chosen to measure it, certainly does not by itself indicate the relative importance of whatever rotation may be present in the motion. He has advanced good reasons for adopting as a vorticity measure the dimensionless invariant

$$\mathfrak{W} = \sqrt{\frac{\mathbf{\Omega}:\mathbf{\Omega}}{\mathbf{D}:\mathbf{D}}} = \frac{\omega}{\sqrt{2\mathbf{D}:\mathbf{D}}}. \tag{27.1}$$

Since in a non-rigid irrotational motion $\omega=0$, $\mathbf{D}\neq 0$, while in a rigid rotation $\mathbf{D}=0$, $\omega\neq 0$, we see that these cases are characterized respectively by $\mathfrak{W}=0$ and $\mathfrak{W}=\infty$. All possible motions with the exception of rigid translation are thus assigned a numerical degree of rotationality on a scale from 0 to ∞, a rigid rotation being the most rotational motion possible.

We shall now illustrate the concept of vorticity measure by computing $\mathfrak{W}$ for several well-known motions.

1. Generalized Poiseuille motion. This is given by

$$u = v = 0, \quad w = w(x, y), \tag{27.2}$$

in some rectangular coordinate system; Eq. (27.2) includes both pure shear flow, $w = ky$, and the laminar flow of viscous fluid down a straight pipe of uniform cross section. We find easily that $\mathfrak{W}=1$.

2. Flow of uniform vorticity. Consider a plane flow of incompressible fluid with uniform vorticity ω. The stream function satisfies

$$\nabla^2\psi = -\omega$$

and at fixed boundaries $\psi=$ const. If the fluid occupies a bounded vessel with fixed walls then clearly

$$\psi = \omega\,\psi^*(x, y),$$

and $\mathfrak{W}$ is independent of the vorticity of the motion, depending only on the shape of the vessel. For an elliptical vessel with semi-axes a and b we find in particular

$$\mathfrak{W} = \left|\frac{a^2+b^2}{a^2-b^2}\right|.$$

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[1] C. TRUESDELL: Phys. Rev. (2) **73**, 510 (1948). In another paper [Canad. J. Math. **3**, 69 (1951)], TRUESDELL obtained a whole series of vorticity formulas generalizing Eq. (26.7).
[2] C. TRUESDELL: J. Rational Mech. Anal. **2**, 173 (1953).

3. Gerstner's waves. As a less trivial example, we consider the famous wave motion due to GERSTNER and attempt to assess whether its degree of rotationality is serious enough to detract from physical interest[1].

The motion is represented in the material form (3.1) by

$$x = a + \frac{1}{k} e^{kb} \sin k(a + ct), \qquad y = b - \frac{1}{k} e^{kb} \cos k(a + ct),$$

where k and c are positive constants. Here a, b serve to distinguish the individual particles, though of course they are not the same as the initial positions X, Y. We consider the motion of those particles for which $b \leq b_0$, where $b_0 = \text{const} < 0$. Since

$$J' \equiv \frac{\partial(x, y)}{\partial(a, b)} = 1 - e^{2kb} > 0$$

if $b < 0$, it follows that these particles execute an admissable motion. The free surface $b = b_0$, and in fact any surface $b = \text{const}$, has the form of a trochoid of wave-length $2\pi/k$ moving with velocity c in the negative x-direction. Moreover

$$J = \frac{\partial(x, y)}{\partial(a, b)} \frac{\partial(a, b)}{\partial(X, Y)} = 1$$

so that the motion is possible for an incompressible fluid. The material Eqs. (6.10) can be integrated easily, giving the Bernoulli equation

$$\frac{p}{\varrho} = - g b + \frac{1}{2} c^2 e^{2kb} + \frac{1}{k} (g - k c^2) e^{kb} \cos k(a + ct) + \text{const}, \tag{27.3}$$

(we assume $\Omega = g y$). In order that $p = \text{const}$ on the free surface we must have

$$c^2 = g/k, \tag{27.4}$$

and Eq. (27.3) thereby reduces to

$$\frac{p}{\varrho} = - g b + \frac{1}{2} c^2 e^{2kb} + \text{const}.$$

This integration of Eq. (6.10) verifies that Gerstner's waves are dynamically possible *in a gravitational field.*

To compute $\mathfrak{W}$ it is necessary to determine the components of grad $\mathbf{v}$. To this end we have

$$u = \frac{dx}{dt} = c e^{kb} \cos \delta,$$

where $\delta = k(a + ct)$, so that

$$\frac{\partial u}{\partial x} = \frac{\partial(u, y)}{\partial(a, b)} \frac{\partial(a, b)}{\partial(x, y)} = - A \sin \delta, \quad (A = k c e^{kb} J'^{-1}).$$

In the same way it is found that

$$\frac{\partial u}{\partial y} = - A(\cos \delta + e^{kb}), \qquad \frac{\partial v}{\partial x} = - A(\cos \delta - e^{kb}), \qquad \frac{\partial v}{\partial y} = A \sin \delta.$$

Substitution into (27.1) then yields the remarkably simple expression

$$\mathfrak{W} = e^{kb} = 2\pi \frac{\text{Amplitude}}{\text{Wavelength}},$$

for the vorticity measure at any "depth" b.

[1] The following material is drawn from the paper cited above.

From this formula it is seen that $\mathfrak{W}$ is greatest at the free surface and decreases fairly rapidly with increasing depth. Also, if $\mathfrak{W}$ is to be less that 10%, say, we must have at the free surface

$$\text{Amplitude} < 1.5\,\%\ \text{Wavelength}.$$

That is, Gerstner's waves are strongly rotational except for very small amplitudes, amplitudes indeed small enough that the linearized theory of surface waves may be employed.

28. Acceleration identities. In this section we shall obtain several kinematical identities for div **a**. Their derivation and full exploitation are due to TRUESDELL[1], though special cases have long been known.

By taking the divergence of the acceleration formula (3.5) we find, with $\Theta = \operatorname{div} \mathbf{v}$,

$$\left.\begin{aligned} \operatorname{div} \mathbf{a} &= \frac{\partial \Theta}{\partial t} + \operatorname{div} (\mathbf{v} \cdot \operatorname{grad} \mathbf{v}) \\ &= \frac{d\Theta}{dt} + v^i{}_{,j} v^j{}_{,i} \\ &= \frac{d\Theta}{dt} + \mathbf{D} : \mathbf{D} - \frac{1}{2} \omega^2, \end{aligned}\right\} \tag{28.1}$$

or

$$\operatorname{div} \mathbf{a} = \frac{d\Theta}{dt} + (1 - \mathfrak{W}^2)\, \mathbf{D} : \mathbf{D}. \tag{28.2}$$

Alternately, from Eq. (17.1) follows

$$\operatorname{div} \mathbf{a} = \frac{\partial \Theta}{\partial t} + \operatorname{div} (\boldsymbol{\omega} \times \mathbf{v}) + \nabla^2 \frac{1}{2} q^2. \tag{28.3}$$

Eqs. (28.1) to (28.3) are the identities mentioned above.

Now it is known that any vector field may be written as the sum of a gradient and a curl ([*48*], p. 186), whence in particular

$$\mathbf{a} = -\operatorname{grad} \varphi^* + \operatorname{curl} \boldsymbol{\pi}^*. \tag{28.4}$$

For a barotropic flow, of course,

$$\varphi^* = \int \frac{dp}{\varrho} + \Omega, \qquad \boldsymbol{\pi}^* = 0, \tag{28.5}$$

while for an incompressible viscous fluid obeying the Navier-Stokes equation (68.2)

$$\varphi^* = \frac{p}{\varrho} + \Omega, \qquad \boldsymbol{\pi}^* = -\nu\, \boldsymbol{\omega}. \tag{28.6}$$

Regardless of the dynamical situation, however, inserting Eq. (28.4) into Eqs. (28.2) and (28.3) yields the following Poisson equations for the acceleration potential φ^*:

$$\nabla^2 \varphi^* = -\frac{d\Theta}{dt} + (\mathfrak{W}^2 - 1)\, \mathbf{D} : \mathbf{D}, \tag{28.7}$$

and

$$\nabla^2 \left(\varphi^* + \frac{1}{2} q^2\right) = -\frac{\partial \Theta}{\partial t} + \operatorname{div} (\mathbf{v} \times \boldsymbol{\omega}). \tag{28.8}$$

The second of these equations may be considered a Bernoulli equation, for it gives an expression for $\varphi^* + \frac{1}{2} q^2$ in terms of other flow quantities. Eqs. (28.7)

[1] C. TRUESDELL: J. Rational Mech. Anal. **2**, 173 (1953).

and (28.8) were first obtained by TRUESDELL, although special cases had early been given by BOBYLEFF[1] and ROWLAND[2].

For applications to fluid dynamics we confine our remarks to the case $\varrho = \text{const.}$ First, for irrotational motion from Eqs. (28.2) and (28.3) follows

$$\nabla^2 \tfrac{1}{2} q^2 = \mathbf{D} : \mathbf{D} \geqq 0,$$

whence $\mathbf{v}$ must take its maximum value on the boundary, a result derived in Sect. 23 by a different method. Secondly, consider the dynamical situations represented by Eqs. (28.5) and (28.6). In either of these cases (since $\Theta = 0$) we can write Eq. (28.7) in the following form, known as the *pressure equation*,

$$\nabla^2 \left(\frac{p}{\varrho} + \Omega\right) = (\mathfrak{W}^2 - 1)\, \mathbf{D} : \mathbf{D}. \tag{28.9}$$

It follows that *given a flow of an incompressible fluid, whether viscous or not, subject to conservative extraneous force, then*

1. In a region where $\mathfrak{W} \geqq 1$ the greatest value of $p + \varrho\Omega$ occurs on the boundary, while

2. In a region where $\mathfrak{W} \leqq 1$ the least value of $p + \varrho\Omega$ must occur on the boundary; in case $\Delta^2\Omega = 0$, as for example in any gravitational field, or when there is no extraneous force, $p + \varrho\Omega$ may be replaced in these statements by the pressure alone. Statement 2 includes, in particular, irrotational motion.

29. The transformations of WEBER and CLEBSCH. We conclude this chapter with a brief discussion of two celebrated transformations of the equations of motion for barotropic flow of a perfect fluid. Although rather little has come of these transformations, it is nevertheless of interest to set them down in a more modern style than that usually employed, and thus to make them more readily accessible to the reader.

We obtain first the transformation of Weber. To begin with we have the obvious identity

$$\operatorname{Grad} \mathbf{x} \cdot \mathbf{a} = \frac{d}{dt} (\operatorname{Grad} \mathbf{x} \cdot \mathbf{v}) - \operatorname{Grad} \frac{1}{2} q^2,$$

[note the similarity to the Lagrange formula (14.4)!]. Then from the dynamical expression (16.5) for acceleration follows

$$\frac{d}{dt} (\operatorname{Grad} \mathbf{x} \cdot \mathbf{v}) = - \operatorname{Grad} \left(\int \frac{dp}{\varrho} + \Omega - \frac{1}{2} q^2 \right).$$

Integrating this from 0 to t, setting

$$\Psi = \int_0^t \left[\int \frac{dp}{\varrho} + \Omega - \frac{1}{2} q^2 \right] dt,$$

and letting $\mathbf{v}_0$ denote the velocity field at $t = 0$, we obtain

$$\operatorname{Grad} \mathbf{x} \cdot \mathbf{v} - \mathbf{v}_0 = - \operatorname{Grad} \Psi, \tag{29.1}$$

which is *Weber's transformation*[3]. If desired, (29.1) can be used as an alternate form of the material equations of motion.

[1] D. BOBYLEFF: Math. Ann. **6**, 72 (1873).

[2] H. ROWLAND: Amer. J. Math. **3**, 226 (1880). Other writers who have introduced similar equations are LICHTENSTEIN [*9*], p. 409; HAMEL: Mh. Math. Phys. **43**, 345 (1936); and LAGALLY: Z. angew. Math. Mech. **17**, 80 (1937).

[3] H. WEBER: J. reine angew. Math. **68**, 286 (1868).

Building on the transformation of Weber it is not too difficult to obtain Clebsch's transformation[1]. Let us observe first that (locally, at least) the velocity field may always be expressed in the form[2]

$$\mathbf{v} = \operatorname{grad}\varphi + f\operatorname{grad} g. \tag{29.2}$$

But then $\boldsymbol{\omega} = \operatorname{grad} f \times \operatorname{grad} g$, so that the surfaces $f = \text{const}$ and $g = \text{const}$ are vortex-surfaces. Since vortex-surfaces are known to move with the fluid, this suggests that the representation (29.2) can be obtained in such a way that the surfaces $f = \text{const}$ and $g = \text{const}$ move with the fluid, that is, so that

$$\frac{df}{dt} = 0, \quad \frac{dg}{dt} = 0. \tag{29.3}$$

Clebsch's transformation consists in the assertion that this is indeed possible.

For let $\mathbf{v}_0 = \operatorname{Grad}\varphi_0 + f\operatorname{Grad} g$, where φ_0, f, and g are functions of $\mathbf{X}$ alone. If we insert this expression for $\mathbf{v}_0$ into Eq. (29.1) and multiply both sides by grad $\mathbf{X}$, there results

$$\mathbf{v} - \operatorname{grad}\varphi_0 = f\operatorname{grad} g = -\operatorname{grad}\Psi,$$

or

$$\mathbf{v} = \operatorname{grad}\varphi + f\operatorname{grad} g,$$

where $\varphi = \varphi_0 - \Psi$, and f and g satisfy Eq. (29.3).

CLEBSCH showed even more however, namely that the equation of motion (16.5) can be integrated when $\mathbf{v}$ is expressed in the form (29.2), (29.3). To see this, we have from a straightforward calculation beginning with the identity (15.8),

$$\begin{aligned}
\mathbf{a} &= \operatorname{grad}\left(\frac{d\varphi}{dt} - \frac{1}{2}q^2\right) + \frac{df}{dt}\operatorname{grad} g + f\operatorname{grad}\frac{dg}{dt} \\
&= \operatorname{grad}\left(\frac{d\varphi}{dt} - \frac{1}{2}q^2\right) = \operatorname{grad}\left(\frac{1}{2}q^2 + \frac{\partial\varphi}{\partial t} + f\frac{\partial g}{\partial t}\right).
\end{aligned}$$

Combining this with Eq. (16.5) gives at once

$$\frac{1}{2}q^2 + \int\frac{dp}{\varrho} + \Omega = -\frac{\partial\varphi}{\partial t} - f\frac{\partial g}{\partial t}, \tag{29.4}$$

where we have absorbed into φ an unimportant function of time. CLEBSCH'S Eq. (29.4) may be looked upon as a Bernoullian theorem, but unlike the earlier forms it depends neither on the assumption of steady motion nor on irrotationality. Finally, note that Eqs. (29.3) and (29.4) constitute three equations in the unknowns φ, f, g and ϱ, provided p and $\mathbf{v}$ are eliminated by means of Eqs. (16.4) and (29.2). A fourth relation is furnished by the equation of continuity[3].

A part of the apparent simplicity of Clebsch's transformation is illusory, since in steady motion the functions φ, f, and g will generally not be time independent.

If conditions (29.3) are not imposed on f and g then instead of (29.4) there arises

$$\frac{dg}{dt}\operatorname{grad} f - \frac{df}{dt}\operatorname{grad} g = \operatorname{grad}\mathfrak{H},$$

[1] A. CLEBSCH: J. reine angew. Math. **54**, 293 (1857); **56**, 1 (1859). These papers are not particularly easy reading: we follow here essentially the presentation of LAMB [*8*], § 167.

[2] H. B. PHILLIPS [*48*], § 49.

[3] CLEBSCH noted that this system of equations is equivalent to a certain variational problem. A more general form of the same variational problem was discovered independently by HERIVEL and LIN, cf. Sect. 15.

where

$$\mathfrak{H} = \frac{1}{2} q^2 + \int \frac{dp}{\varrho} + \Omega + \frac{\partial \varphi}{\partial t} + f \frac{\partial g}{\partial t}. \tag{29.5}$$

Hence $\mathfrak{H} = \mathfrak{H}(f, g, t)$, and

$$\frac{dg}{dt} = \frac{\partial \mathfrak{H}}{\partial f}, \qquad \frac{df}{dt} = -\frac{\partial \mathfrak{H}}{\partial g}, \tag{29.6}$$

formally reminiscent of HAMILTON's canonical equations. Clebsch's transformation is the special case $\mathfrak{H} = 0$ of Eqs. (29.5) and (29.6). It is interesting to consider Eqs. (29.5) and (29.6) when the motion is steady and when f and g satisfy not Eq. (29.3) but rather

$$\frac{\partial f}{\partial t} = \frac{\partial g}{\partial t} = 0.$$

Then $\mathfrak{H}$ no longer depends explicitly on t, and we find

$$\frac{d\mathfrak{H}}{dt} = \frac{\partial \mathfrak{H}}{\partial f}\frac{df}{dt} + \frac{\partial \mathfrak{H}}{\partial g}\frac{dg}{dt} = 0, \tag{29.7}$$

as in a well known proof of the conservation of energy in Hamiltonian dynamics. Evidently, Eq. (29.7) is equivalent to a Bernoulli theorem of Sect. 17. This surprising analogy between Clebsch's transformation and Hamiltonian dynamics may have its roots in the variational principles of Sect. 15.

29 A. Appendix: generalized Weber and Clebsch transformations. We consider here an extension of the results of Sect. 29 to apply to a general gas. The necessary thermodynamical background will be found in the following chapter, Sects. 30, 35.

To generalize the Weber transformation, we follow exactly the procedure leading to Eq. (29.1), observing, however, that Eq. (16.5) must now be replaced by

$$\mathbf{a} = T \operatorname{grad} S - \operatorname{grad}(I + \Omega), \qquad \frac{dS}{dt} = 0, \tag{29.8}$$

where I is the specific enthalpy, $I = E + p/\varrho$; (this follows in virtue of the identity $\varrho^{-1} \operatorname{grad} p = T \operatorname{grad} S - \operatorname{grad} I$). The end result is that Eq. (29.1) is replaced by

$$\operatorname{Grad} \mathbf{x} \cdot \mathbf{v} - \mathbf{v}_0 = \beta \operatorname{Grad} S - \operatorname{Grad} \Psi \tag{29.9}$$

where

$$\frac{d\beta}{dt} = T, \qquad \frac{d\Psi}{dt} = I + \Omega - \frac{1}{2} q^2. \tag{29.10}$$

Eq. (29.9) is a generalization of the Weber transformation and can be used as an alternate form of the material equations of motion.

Eqs. (29.9) and (29.10) have an immediate and important application to the Eulerian variational principle of Sect. 15. Specifically, if we multiply both sides of Eq. (29.9) by Grad $\mathbf{X}$ there results

$$\mathbf{v} = -\operatorname{grad} \Psi + \beta \operatorname{grad} S + \operatorname{grad} \mathbf{X} \cdot \mathbf{v}_0. \tag{29.11}$$

Eqs. (29.10) and (29.11), together with the obvious condition $d\mathbf{v}_0/dt = 0$, are (with minor changes in notation) just the Euler equations of the variational principle (15.5), (15.6). This proves that every flow is an extremal for this principle, as asserted in Sect. 15.

A generalization of the Clebsch transformation can also be obtained without difficulty, though we leave its derivation and formal statement to the reader.

D. Thermodynamics and the energy equation.

I. Thermodynamics of simple media.

Except in the simple situations considered in the previous chapter, the number of unknown quantities in the equations derived in Part B is greater than the number of equations, so that these equations are not in themselves sufficient for a complete description of fluid motion. This situation is remedied by the introduction of a *total energy equation* based on the principles of classical thermodynamics, and later by the use of certain constitutive equations. In the present chapter we take up the derivation of the total energy equation, beginning with a brief outline (Sects. 30 to 32) of those parts of classical thermodynamics which are relevant. We consider, in particular, the fundamental laws governing the behavior of a thermodynamic system whose phases have constant mass and are of fixed constitution. Before taking up the details we should point out that it is not our purpose, nor is it within the scope of this article, to present any physical justification for the logical structure of thermodynamics. Suffice it to say that such justification lies in laws of CLAUSIUS and KELVIN, in the results of kinetic theory and statistical mechanics, and perhaps most strongly in the success of the ideas in explaining observed fact[1].

30. The one-phase system. The simplest thermodynamic system which can be considered consists of a single homogeneous medium. The structure of such a one-phase system underlies all of the hydrodynamics of compressible fluids, and its mathematical description is therefore an appropriate starting point for our discussion. A single phase is described by certain *state variables*, the most important being the volume V, the entropy S, the internal energy E, the pressure p, and the absolute temperature T. The *structure* of the phase is specified by certain functional relations among the state variables.

A particularly elegant formulation of these relations is that of GIBBS, in which the fundamental state equation is some *definite* relation

$$E = E(S, V),$$

and the variables p, T are *defined* by

$$p = -\frac{\partial E}{\partial V}, \qquad T = \frac{\partial E}{\partial S};$$

it is assumed, of course, that both p and T are greater than zero. The various relations among E, S, V, T, and p which may be derived from these formulae are known as *equations of state*. It is evident that, in general, fixing any two

[1] It is hard to refer the reader to a perfectly rigorous treatment of the foundations of thermodynamics. One of the best elementary presentations is that of PHILLIPS, reproduced in KEENAN's *Thermodynamics* [33], Chap. 8. Chaps. 1–3, 6–7, and 19–24 of this book are also recommended. On a higher level, the method of CARATHÉODORY is extremely elegant [Math. Ann. **67**, 355 (1909); Sitzgsber. Akad. Wiss. Berlin, Math.-phys. Kl. 39 (1935); Handbuch der Physik, Vol. 9, Chap. 4, 1926]; but just as in the elementary treatments, CARATHÉODORY's method is so far not developed in sufficient generality to cover irreversible processes; see for example, B. LEAF, J. Chem. Phys. **12**, 89 (1944), especially p. 94. The modern viewpoint, in fact, seems to be that the thermodynamics of irreversible processes must be handled on an abstract postulated basis, independent of any derivation from presumably more basic assumptions; cf. S.R. DE GROOT: *Thermodynamics of Irreversible Processes.* New York 1951; K.G. DENBIGH: *The Thermodynamics of the Steady State.* London 1951.

The standard treatise on the kinetic theory of gases is that of CHAPMAN & COWLING [*30*].

of the state variables determines the remaining ones[1]. By forming the total differential of the relation $E = E(S, V)$ we obtain the basic formula

$$T\,dS = dE + p\,dV, \tag{30.1}$$

relating the state variables.

In the special case $E = E(S)$, the pressure can no longer be considered a thermodynamic variable, and the prime Eq. (30.1) must be altered to read simply $T\,dS = dE$. These fundamental differences require special treatment in the following work, though for brevity we shall leave the necessary changes to the reader: from here on, *we shall deal exclusively with the normal case* $(\partial E/\partial V)_S < 0$.

In a moment we shall consider the ideal gas as a specific example, but first we recall certain general results which apply to an arbitrary phase. A single phase system is said to undergo a *differentiable process* if its state variables are differentiable functions of time, e.g. $V = V(t)$, $T = T(t)$, etc. If the phase moves reversibly, that is, if it is in a state of equilibrium with its surroundings at each instant, then the heat $Q(t)$ supplied to the phase during the process is given by

$$dQ = dE + p\,dV;$$

(cf. Sect. 32). A phase has two important *heat capacities*[2]

$$c_v = \left(\frac{dQ}{dT}\right)_V \quad \text{and} \quad c_p = \left(\frac{dQ}{dT}\right)_p.$$

Since $c_v = T(\partial S/\partial T)_V$ and $c_p = T(\partial S/\partial T)_p$, it follows that both heat capacities are state variables. It is always found experimentally that

$$c_p > c_v > 0, \tag{30.2}$$

and we shall therefore take Eq. (30.2) as an additional postulate governing the makeup of a single phase.

Several theoretical relations will be needed later, including the well-known formulas

$$\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p, \tag{30.3}$$

$$c_p - c_v = T\left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_p, \tag{30.4}$$

proved in almost every work on thermodynamics. In addition, we have

$$\left(\frac{\partial p}{\partial T}\right)_V = \frac{\partial(p, V)}{\partial(T, V)} = \frac{\partial(p, V)}{\partial(p, T)}\,\frac{\partial(p, T)}{\partial(T, V)} = -\left(\frac{\partial V}{\partial T}\right)_p \left(\frac{\partial p}{\partial V}\right)_T, \tag{30.5}$$

and by a similar manipulation,

$$\frac{c_p}{c_v} = \left(\frac{\partial p}{\partial V}\right)_S \Big/ \left(\frac{\partial p}{\partial V}\right)_T. \tag{30.6}$$

From Eqs. (30.4) to (30.6) and the hypothesis (30.2) one easily deduces that

$$\left(\frac{\partial p}{\partial V}\right)_S < \left(\frac{\partial p}{\partial V}\right)_T < 0. \tag{30.7}$$

In Sect. 37 we shall introduce a final thermodynamic assumption, but until then the foregoing results will serve all our purposes.

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[1] We shall usually use either (S, V) or (p, V) as basic variables; in the latter case one must generally restrict consideration to a subset of states for which p and V uniquely determine the state.

[2] If the phase is of unit mass, the heat capacities are called specific heats.

Though it is not our intention to treat multi-component phases in this article (since they can hardly be considered a branch of classical fluid mechanics), nevertheless it will be of some values to point out the changes necessary in the above formulation when such situations are included. In particular, the fundamental equation of state takes the more general form

$$E = E(S, V, M_i), \tag{30.8}$$

where the state variable M_i denotes the amount of the i-th substance in the phase, $i = 1, \ldots, \beta$. Since increasing the amounts M_i in uniform proportion does not alter the constitution of the phase, it is clear that E must be homogeneous of degree 1 in the M_i. With temperature and pressure defined as before, the prime Eq. (30.1) becomes

$$T\,dS = dE + p\,dV + \mu_i\,dM_i$$

where $\mu_i = -\partial E/\partial M_i$. The definition of $Q(t)$ in a differentiable process is the same as before, and formulas (30.2) to (30.7) continue to hold, it being understood that the partial derivative are to be evaluated at fixed M_i.

31. The ideal gas. It is found that many real gases (including air) have an equation of state more or less closely approximated by

$$p\,V = R\,T, \tag{31.1}$$

where R is a positive constant. Simple considerations of kinetic theory indicate, moreover, that Eq. (31.1) is theoretically natural. Thus although no real material satisfies Eq. (31.1) exactly, it is extremely useful to study the behavior of a hypothetical material, called the ideal gas[1], which obeys Eq. (31.1).

Relations involving other state variables of an ideal gas are readily obtained by using Eqs. (30.3) and (30.4). In particular we find

$$\left(\frac{dE}{dV}\right)_T = 0, \qquad c_p - c_v = R.$$

It follows that $E = E(T)$, and since $c_v = (\partial E/\partial T)_V$, also $c_v = c_v(T)$. In the case of monatomic real gases, it is found that c_v is nearly constant over a large temperature range, although it is not safe to extrapolate to extremely low temperatures. In the case of polyatomic gases, c_v cannot be taken constant except for a limited range of temperatures on both sides of room temperature. This is because at low temperatures the rotational motions of the molecules require less energy, while at high temperatures, molecular vibrations come into play. In view of this situation, even for an ideal gas it is usual to retain the expression for internal energy in the form

$$E = \int c_v\,dT.$$

For a monatomic ideal gas, kinetic theory predicts $c_v = \frac{3}{2}R$, while for a diatomic gas, at temperatures where rotational but not vibrational effects are important, the result is $c_v = \frac{5}{2}R$. For these values of c_v the ratio $\gamma = c_p/c_v$ assumes the values $\frac{5}{3}$ and $\frac{7}{5}$, respectively. For air at moderate temperatures it is found that $\gamma = 1.40$, in agreement with the theoretical value for a diatomic gas. This value is presumably also accurate at high temperatures when they are not sustained long enough for the vibrational energy to be appreciably excited. It may be added finally that air at high pressures departs considerably from the type of behavior discussed here[2].

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[1] The terminology "perfect gas" is also in common use, but it seems preferable to reserve the word "perfect" to denote absence of viscosity.

[2] Concerning the behavior of real gases, and air in particular, the reader may consult BATEMAN [2], pp. 504–512; S. GOLDSTEIN: Aero, Res. Comm. R & M No. 2337 (1946); and Vol. XII of this Handbuch.

A formula for the entropy of an ideal gas may be obtained by integrating Eq. (30.1), thus

$$S = \int c_v \frac{dT}{T} + R \log V.$$

When c_v is constant this reduces to $S = c_v \log p V^\gamma + \text{const}$, which may in turn be solved for p, giving

$$p = e^{S/c_v} V^{-\gamma}, \quad \gamma = \text{const} > 1. \tag{31.2}$$

In Eq. (31.2) a constant of integration is avoided by choosing the zero point of entropy at $p = V = 1$.

32. The laws of thermodynamics. Consider a set of m phases of the type described in Sect. 30, which interact mechanically and thermally but which maintain their identity throughout any process. Let the phases of the system be indexed by a subscript α, $\alpha = 1, \dots, m$. Then the state of the system is described by the variables V_α, T_α, S_α, E_α and p_α. The entropy, volume, and internal energy of the system are defined by

$$\mathfrak{S} = \sum_\alpha S_\alpha, \quad \mathfrak{V} = \sum_\alpha V_\alpha, \quad \mathfrak{E} = \sum_\alpha E_\alpha.$$

Now imagine the system to undergo some process in which it passes from one state to another. During the process let the amount of work done by the system be $\mathfrak{W}$ and let an amount of heat Q_α^e be supplied to the phase α *from sources external to the system.* We define $\mathfrak{Q}^e = \sum_\alpha Q_\alpha^e$, that is, $\mathfrak{Q}^e$ is the total external heat supplied to the system. Then *the first law of thermodynamics* states that

$$\mathfrak{Q}^e = \Delta \mathfrak{E} + \mathfrak{W}, \tag{32.1}$$

where $\Delta \mathfrak{E}$ is the increment of internal energy of the system. Now consider a a differentiable process proceeding through a set of well-defined states, i.e. $V_\alpha = V_\alpha(t)$, $Q_\alpha^e = Q_\alpha^e(t)$, $\mathfrak{W} = \mathfrak{W}(t)$, etc. In this case Eq. (32.1) reduces to the special form

$$d\mathfrak{Q}^e = d\mathfrak{E} + d\mathfrak{W}. \tag{32.2}$$

For a single phase it is usual to write dQ instead of $d\mathfrak{Q}^e$. If the phase is in equilibrium with its surroundings, then the work done during a process is given by $d\mathfrak{W} = p\, dV$ ([*31*], Chap. 2). It follows that the heat supplied to a single phase during a reversible process is given by

$$dQ = dE + p\, dV.$$

This result has already been noted in Sect. 30.

The second law of thermodynamics has no complete formulation for arbitrary processes, but the following statement is sufficient for our purposes: *in a differentiable process the state functions and the heats supplied to the individual phases obey the inequality*

$$d\mathfrak{S} \geqq \sum_\alpha \frac{dQ_\alpha^e}{T_\alpha}. \tag{32.3}$$

In a process of any sort for which $Q_\alpha^e = 0$, *the entropy must increase.*

A process is called spontaneous if the inequality sign in Eq. (32.3) occurs; if the state variables are such that no spontaneous process is available, then the system is in equilibrium. As an example of the second law, consider a process conducted at a uniform temperature over all the phases, that is, $T_1 = T_2 = \cdots = T_m = T(t)$. Then Eq. (32.3) reduces simply to $T\, d\mathfrak{S} = d\mathfrak{Q}^e$, or, in integral form

$$\Delta\mathfrak{S} = \mathfrak{S}_2 - \mathfrak{S}_1 \geqq \int_1^2 \frac{d\mathfrak{Q}^e}{T}. \tag{32.4}$$

It is this equation which is usually given in textbooks.

Although we shall not need the result in the sequel, it is instructive to determine the equilibrium conditions for a thermally and mechanically isolated system of fixed volume $\mathfrak{V}$. For this case $d\,Q^e_\alpha = d\mathfrak{W} = 0$, hence by the first law $d\mathfrak{E} = 0$. Also since $\mathfrak{V}$ is fixed we have $d\mathfrak{V} = 0$. Now it is clear that spontaneous processes will be available to the system unless $d\mathfrak{S} = 0$ for all differential changes of state satisfying the constraints $d\mathfrak{E} = d\mathfrak{V} = 0$. A simple examination of the situation shows, however, that we can always make $d\mathfrak{S} \neq 0$ unless T_α and p_α are uniform over the system. We conclude that equilibrium is characterized by constant temperature and pressure throughout the system. This result, although certainly anticipated, is nevertheless important in showing that the second law is compatible with familiar observations. Notice finally that at equilibrium in this example we have

$$T\, d^2\mathfrak{S} = \sum_\alpha \left(\frac{\partial p}{\partial V}\right)_T (dV_\alpha)^2 - \left(\frac{\partial S}{\partial T}\right)_V (dT_\alpha)^2,$$

whence by Eqs. (30.2) and (30.7) every small change in state results in a decrease in total entropy; in other words, the equilibrium is stable. This result may be taken as the theoretical basis for the postulate (30.2).

Another point of contact with the standard presentation of thermodynamics may be made here. For an arbitrary process conducted between temperatures T_1 and T_2 the heat capacity c is defined as

$$c = \frac{\mathfrak{Q}^e}{\Delta T}.$$

When the process is locally reversible and conducted so that at each instant the temperature is uniform throughout the system, this becomes

$$c = \frac{d\mathfrak{Q}^e}{dT} \leqq T\frac{d\mathfrak{S}}{dT} = \sum_\alpha \frac{dQ_\alpha}{dT},$$

that is, *the total heat capacity of a multi-phase system can be no larger than the sum of the heat capacities of the individual phases.*

II. The energy equation.

33. Conservation of energy. As pointed out at the beginning of this chapter, it is necessary to augment the equations of motion with a total energy equation[1]. We define the *total energy* of a volume $\mathscr{V}$ as the sum of its kinetic energy $\mathfrak{T}$ and its internal energy $\mathfrak{E}$, where

$$\mathfrak{T} = \tfrac{1}{2}\int \varrho\, q^2\, dv, \quad \mathfrak{E} = \int \varrho\, E\, dv$$

[1] In the following discussion we are concerned only with homogeneous, chemically inert fluids. More general situations involving diffusion, chemical reactions, etc., are beyond the scope of this article. In this respect, the reader should consult the beautiful paper of C. TRUESDELL, *Sulle basi della termomeccanica* [Rend. Accad. naz. Lincei (8) **22**, 33–38, 158–166 (1957)], where for the first time these difficult topics are included in a rational and connected theory.

and E is the specific internal energy. The thermodynamical interpretation of E must naturally be somewhat different for compressible and incompressible fluids, and accordingly these cases are discussed separately. *For a compressible fluid we assume that E is a thermodynamic state variable satisfying the relation*

$$T\,dS = dE + p\,d\tau, \qquad \tau = 1/\varrho, \tag{33.1}$$

where T is the absolute temperature, S the specific entropy, p the pressure, and τ the specific volume (see Sect. 30). It should be noticed that this postulate *defines* fluid pressure as a thermodynamical variable; therefore, in general, $p = f(\varrho, S)$.

For an incompressible fluid we assume instead of Eq. (33.1) the simpler relation

$$T\,dS = dE, \qquad \varrho \equiv \text{const.} \tag{33.2}$$

It follows from Eqs. (33.2) that $E = E(S)$ and $T = dE/dS$. Consequently, pressure does not enter into the thermodynamical treatment of an incompressible fluid, it being, in fact, an entirely separate matter.

The principles stated above are by no means trivial or obvious when it is considered that they are intended to apply to fluids in all possible states of motion. In justification we offer firstly the relative simplicity and naturalness of Eq. (33.1), secondly, the kinetic theory, where it is shown that the same results are obtained, at least for simple enough molecular models, and, finally, the fact that a substantial body of experimental evidence exists which in no way contradicts the consequences which can be deduced from our assumptions. This last reason perhaps must be regarded as the ultimate justification. As GILBARG remarks[1], "equations have often been successful beyond the limits of their original derivation, and indeed this type of success is one of the hallmarks of a great theory".

With the fundamental assumptions of the preceding paragraphs understood, *we postulate that total energy is conserved*, i.e.

$$\frac{d}{dt}(\mathfrak{T} + \mathfrak{E}) = \int_{\mathscr{V}} \varrho\,\mathbf{f}\cdot\mathbf{v}\,dv + \oint_{\mathscr{S}} \mathbf{t}\cdot\mathbf{v}\,da - \oint_{\mathscr{S}} \mathbf{q}\cdot\mathbf{n}\,da. \tag{33.3}$$

This equation states that the rate of change of the total energy of a materia volume is equal to the rate at which work is being done on the volume plus the rate at which heat is conducted into the volume. The heat flux vector $\mathbf{q}$ has the dimensions energy per unit area per unit time $= \mathsf{MT}^{-3}$. Its relation to other mechanical and thermodynamical variables must later be specified according to the particular medium under consideration. By comparing Eqs. (9.1) and (33.3) and making use of standard arguments we obtain the *total energy equation*[2]

$$\varrho\frac{dE}{dt} = \mathbf{T}:\mathbf{D} - \operatorname{div}\mathbf{q}. \tag{33.4}$$

According to TRUESDELL[3] this equation should be attributed to C. NEUMANN.

Eq. (33.3) corresponds to the first law of thermodynamics. In analogy with the second law we postulate the inequality

$$\frac{d}{dt}\int_{\mathscr{V}} \varrho\,S\,dv \geqq -\oint_{\mathscr{S}} \frac{\mathbf{q}\cdot\mathbf{n}}{T}\,da. \tag{33.5}$$

[1] D. GILBARG and D. PAOLUCCI: J. Rational Mech. Anal. 2, 618 (1953).

[2] Occasionally a term R is added to the right of (33.3) to account for various other types of energy sources in the fluid, such as those resulting from chemical reaction, radiation, etc.

[3] C. TRUESDELL: J. Rational Mech. Anal. 1, 160 (1952).

It may be worthwhile to mention here that the results of this section complete the set of general dynamical principles on which continuum mechanics is based. Further progress is based on particular assumptions as to the form of the stress tensor, the heat flux vector, and the equation of state relating the thermodynamical variables.

34. Thermodynamics of deformation. For a perfect fluid, the pressure has already appeared as a dynamic variable in Eq. (6.9). Characteristic of the discipline of gas dynamics is the postulate that the thermodynamic pressure introduced in the preceding section is equal to this dynamic pressure. Thus for a perfect fluid we have the general stress formula $\mathbf{T} = -p\mathbf{I}$, in which p is the thermodynamic pressure in case the fluid is *compressible*, while p is simply an independent dynamical variable otherwise.

More generally, when tangential stresses may not be neglected, we write $\mathbf{T}$ in the form

$$\mathbf{T} = -p\,\mathbf{I} + \mathbf{V}, \quad \text{(defining } \mathbf{V}\text{)}, \tag{34.1}$$

where, for a compressible fluid, p is the thermodynamical pressure, and for an incompressible fluid, p is a primitive unknown which will later be specified in a convenient way (Sect. 60). So as to treat all cases on the same footing, we consider perfect fluids to be the special case $\mathbf{V} \equiv 0$ of Eq. (34.1). Then substituting (34.1) into Eq. (33.4) leads to

$$\varrho \frac{dE}{dt} + p \operatorname{div} \mathbf{v} = \Phi - \operatorname{div} \mathbf{q}, \tag{34.2}$$

where $\Phi = \mathbf{V} : \mathbf{D}$. By use of the equation of continuity and Eq. (33.1) or Eq. (33.2) depending on whether the fluid is compressible or not, Eq. (34.2) reduces to the elegant result

$$\varrho T \frac{dS}{dt} = \Phi - \operatorname{div} \mathbf{q}, \tag{34.3}$$

expressing the rate of change of entropy following a particle. Now the heat absorbed per unit mass of fluid is given by $dQ = T\,dS$, whence by Eq. (34.3) the rate per unit volume at which heat is absorbed is $\Phi - \operatorname{div} \mathbf{q}$. Since the second term represents the conduction of heat from neighboring fluid elements, it follows that Φ is the rate per unit volume at which heat is generated by the deformation of fluid elements. This heat must of course appear at the expense of mechanical energy, and Φ is accordingly called the *dissipation funktion*.

If we divide Eq. (34.3) by T and integrate over a volume moving with the fluid, there arises

$$\frac{d}{dt} \int_{\mathscr{V}} \varrho\, S\, dv = \int_{\mathscr{V}} \left[\frac{\Phi}{T} - \frac{\mathbf{q} \cdot \operatorname{grad} T}{T^2} \right] dv - \oint_{\mathscr{S}} \frac{\mathbf{q} \cdot \mathbf{n}}{T}\, da.$$

In order for this to be consistent with Eq. (33.5), it is necessary that

$$\Phi - \mathbf{q} \cdot \operatorname{grad} T / T \geqq 0.$$

This condition is most obviously satisfied if the stress and heat flux vectors satisfy the inequalities

$$\Phi \geqq 0, \qquad \mathbf{q} \cdot \operatorname{grad} T \leqq 0. \tag{34.4}$$

These conditions are the mathematical statements of two familiar facts: heat never flows against a temperature gradient, and, deformation absorbs energy (converting it to heat), but never releases it. Conversely, if we start with Ineqs. (34.4) we may *derive* Ineq. (33.5): thus the latter inequality is a direct consequence of simple physical observations.

In the subject of irreversible thermodynamics, one of the basic postulates is known as the "linear law"[1]. Its validity is certainly open to question, but at least it is worthwhile to examine its consequence in the present case. Stated roughly, it says that a thermodynamical system will tend to equilibrium at a rate linearly dependent on its displacement from equilibrium. In our case, this displacement is measured by $\mathbf{D}$ and grad T, hence $\mathbf{V}$ and $\mathbf{q}$ should be linear functions of $\mathbf{D}$ and grad T. By virtue of the tensorial difference of these various quantities it follows that

$$\mathbf{V} = \text{linear function of } \mathbf{D},$$

$$\mathbf{q} = \text{linear function of grad } T.$$

The first of these leads to the Cauchy-Poisson law of viscosity, while the second is the well-known law of Newton and Fourier. These considerations must be regarded as merely heuristic, however, and by no means provide an adequate derivation of the constitutive equations of a fluid.

E. The perfect gas.

I. General principles.

35. Preliminary discussion. The dynamical equations governing the motion of a perfect fluid have been derived in Part B; they are

$$\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0, \tag{35.1}$$

and

$$\varrho \frac{d\mathbf{v}}{dt} = \varrho \mathbf{f} - \operatorname{grad} p. \tag{35.2}$$

To these is to be added the energy equation (34.3) of Part D. Since tangential stresses are neglected in the definition of a perfect fluid, it is logical to suppose in addition that $\mathbf{q} = 0$. [Indeed, according to kinetic theory viscosity and heat conduction arise from similar mechanisms (molecular impact), and are of the same order of magnitude; thus if one is to be neglected, we should also neglect the other.] Accordingly we set $\Phi = 0$ and $\mathbf{q} = 0$ in Eq. (34.3), which then becomes simply

$$\frac{dS}{dt} = 0. \tag{35.3}$$

The system of equations (35.1) to (35.3) is completed by a thermodynamical equation of state relating p, ϱ and S, namely

$$p = f(\varrho, S). \tag{35.4}$$

When the fluid is an ideal gas with constant specific heats, Eq. (35.4) has the particular form

$$p = e^{S/c_v} \varrho^{\gamma}, \qquad \gamma = \text{const} > 1. \tag{35.5}$$

Ordinarily, however, we shall not specify the form of Eq. (35.4) beyond requiring that it be compatible with the general thermodynamic considerations of Sect. 30.

[1] Cf. L. ONSAGER: Phys. Rev. **37**, 405 (1931); **38**, 2265 (1931); an outline version of the results will be found in an article of C. F. CURTISS [*35*].

By Eq. (30.7) this involves the condition $(\partial p/\partial \varrho)_S > 0$, thus allowing us to *define* the thermodynamic variable c,

$$c = \sqrt{\left(\frac{\partial p}{\partial \varrho}\right)_S} = \sqrt{\frac{\partial f}{\partial \varrho}}. \tag{35.6}$$

In the sequel we shall call a fluid satisfying Eqs. (35.1) to (35.4) a *perfect gas*. Also, as is almost universal in dealing with the dynamics of gases, we shall neglect the effect of the external force $\mathbf{f}$ in Eq. (35.2).

Perhaps the most striking single feature of the system (35.1) to (35.4) is the propagation of pressure waves with finite velocity. This intuitive fact can be given quantitative form by the following procedure: Supposing that the pressure wave is of "small" amplitude, all flow quantities may be considered as perturbations from the rest state $\mathbf{v} = 0$, $p = p_0$, $\varrho = \varrho_0$, $S = S_0$. Upon neglect of squares of small quantities, Eqs. (35.1) and (35.2) become

$$\frac{\partial \varrho}{\partial t} + \varrho_0 \operatorname{div} \mathbf{v} = 0, \qquad \varrho_0 \frac{\partial \mathbf{v}}{\partial t} = -\operatorname{grad} p.$$

Elimination of $\mathbf{v}$ between these equations yields $\nabla^2 p = \partial^2 \varrho/\partial t^2$. On the other hand, from Eqs. (35.3) and (35.4) there follows $\partial^2 p/\partial t^2 = c_0^2 \partial^2 \varrho/\partial t^2$. The equation governing small pressure disturbances is then[1]

$$c_0^2 \nabla^2 p = \frac{\partial^2 p}{\partial t^2},$$

and according to well-known theory it follows that disturbances are propagated with the speed c_0. This semi-heuristic analysis justifies calling c the *speed of sound*[2]. Later (Sects. 51 and 54) we shall give a more rigorous approach to this topic.

It should be observed that c is not a constant, but a thermodynamic variable, depending on the state of the fluid. Indeed, for an ideal gas[3]

$$c = \sqrt{\frac{\gamma p}{\varrho}} = \sqrt{\gamma R T}, \tag{35.7}$$

where R is the gas constant for a unit mass of fluid ($= 2.87 \times 10^6$ for air, and $= 83.1 \times 10^6$/molecular weight, for a pure gas, in the C.G.S. system). The accuracy of formula (35.7) can be estimated by computing c for air at 0° Centigrade (273.16° absolute). We find, using $\gamma = 1.40$ and the above value of R, that $c = 331.3$ m/sec, in remarkable agreement with the observed value. This agreement

[1] This equation goes back to EULER, Mem. Acad. Sci. Berlin (1759), (Opera Omnia (3) **1**, pp. 428–507).

[2] The first theoretical formula for the speed of sound was found by NEWTON, Principia Mathematica, Lib. II, Sect. VIII, Props. 48, 49. EULER improved upon NEWTON's argument and derived the wave equation as an alternative approach to the subject. Since the concept of adiabatic changes was not then known, the final outcome of EULER's work, like that of NEWTON, was the formula $c^2 = p/\varrho$. The reconciliation of this theory with fact is due to LAPLACE, who pointed out that the temperature as well as the pressure rises in sudden compression. LAPLACE's remark appears to have been first published in a paper of BIOT, [Bull. Soc. Phil. Paris **3**, 116 (1802)]. In this paper the temperature rise is, however, still looked upon as an empirical fact. Some years later LAPLACE explained the rise as due to the adiabatic nature of sound transmission, and so found the formula $c^2 = \gamma p/\varrho$ with γ denoting the ratio of specific heats [Ann. Chim. Phys. **3**, 238 (1816)]. The general formula $c^2 = (\partial p/\partial \varrho)_S$ is, of course, the work of a later period.

[3] This formula holds whether or not the specific heats are constant, for by Eq. (30.6)

$$\left(\frac{\partial p}{\partial \varrho}\right)_S = \gamma \left(\frac{\partial p}{\partial \varrho}\right)_T = \frac{\gamma p}{\varrho}.$$

is, in fact, important among the reasons for accepting perfect fluid theory as an accurate account of the motion of gases.

From the concept of a finite speed of sound, one arrives directly at the well-known picture of subsonic and supersonic flows. Although the distinction will occur naturally in the subsequent analysis, nevertheless it is worthwhile to examine the situation here in an avowedly heuristic manner. Consider the case of subsonic steady motion—for example, the uniform level flight of an airplane. Here a pressure signal travels forward from the plane at sound velocity minus flight velocity, relative to the plane, whereas a signal travels backward at sound velocity plus flight velocity. Every point in space is therefore reaches by a signal, provided the flight has proceeded from an infinitely remote point. By contrast, in supersonic flight it is seen that all effect is restricted to a cone proceeding backward from the nose of the plane, the angle of the cone with respect to its axis being arcsin (c/q); (this picture will have to be amended when we consider disturbances of finite amplitude, i.e. shock waves)[1].

36. Dynamical similarity. In this section it is assumed that the reader knows the usual engineering treatment of dynamical similarity, and desires instead a discussion of the fundamental mathematical principles involved. The idea of dynamical similarity, it may be remarked, is due originally to STOKES. In his paper on the motion of a pendulum in a retarding fluid medium[2], not only is the notion of dynamical similarity formulated for the first time, but there even appears the combination of flow parameters now known as the Reynolds number.

Two perfect fluid motions are said to be *dynamically similar* if they are related by equations

$$\mathbf{v} = U\,\mathbf{v}', \quad \varrho = R\,\varrho', \quad p = P\,p', \tag{36.1}$$

and

$$\mathbf{x} = D\mathbf{x}', \quad t = T\,t' \tag{36.2}$$

where U, R, P, D, and T are similarity constants[3]. We shall show that these constants must be related in definite ways, or, in other words, that certain parameters must be the same for each flow[4]. Making the substitutions (36.1) and (36.2) in the equation of continuity we find

$$\frac{R}{T}\,\frac{\partial\varrho'}{\partial t'} + \frac{R\,U}{D}\,\mathrm{div}'\,(\varrho'\,\mathbf{v}') = 0.$$

Since the "primed" flow is also a solution of the equation of continuity, this implies

$$T = D/U, \tag{36.3}$$

[an exception occurs when the motion is steady, but then Eq. $(36.2)_2$ need not be considered]. With the help of Eq. (36.3) the remaining flow equations become

$$\varrho'\,\frac{d\mathbf{v}'}{dt} = -\,\frac{P}{R\,U^2}\,\mathrm{grad}'\,p' \quad \text{and} \quad \frac{dS}{dt'} = 0.$$

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[1] An excellent introduction to the theory of compressible perfect fluids will be found in the first chapter of [*43*]. General textbooks which can be recommended are [*17*], [*19*], [*21*] and [*23*].

[2] G. STOKES: Trans. Cambridge Phil. Soc. **9**, 8 (1850). (Papers **3**, pp. 1—141.)

[3] In the usual engineering treatments of dynamical similarity both motions are reduced to the same "dimensionless" flow. We find this procedure logically less appealing than the one adopted here.

[4] The conclusion is, of course, that mere geometric similarity of two flow regions does not guarantee dynamical similarity of the flows.

From the first of these we derive simply

$$\frac{R U^2}{P} = 1. \tag{36.4}$$

Now if the equation of state of the "primed" flow is $p' = g(\varrho', S')$, then

$$f(R\varrho', S) = P g(\varrho', S'). \tag{36.5}$$

Forming the material derivative of each side and using the fact that both S and S' are constant following particles,after cancellation of a factor there results

$$R c^2 = P c'^2. \tag{36.6}$$

It follows now from Eqs. (36.4) and (36.6) that *in order for two flows to be dynamically similar the local Mach*[1] *number* $M = q/c$ *must be the same at corresponding points of each flow.*

The condition just stated is not the only one necessary for dynamical similarity. Equally important, the equations of state must be such that Eq. (36.5) reduces to a relation not involving ϱ'. For example, in case of an ideal gas Eq. (36.5) reduces to

$$S - S' = c_v \log P R^{-\gamma},$$

which serves to determine the entropy of the "primed" flow.

In application of the above results (for example, in wind tunnel experiments) the external geometry of two flows is made similar and the reduced velocities $\mathbf{v}/c$ and $\mathbf{v}'/c$ are made to agree *at one point* P. Under these circumstances and provided that Eq. (36.5) can be satisfied, dynamically similar flows are mathematically possible. Whether dynamically similar flows actually occur or not is another question, one which rests squarely on the unique dependence of the flows on the conditions prescribed at P. At least in the case of subsonic flow past an obstacle, such uniqueness has been proved for conditions prescribed in the uniform stream (cf. Sect. 46). In actual wind tunnel tests, of course, so many factors intervene that the question whether dynamical similarity occurs must be answered at least partly on the basis of the particular experimental situation.

It has been shown that dynamically similar flows are possible for an ideal gas. Conversely, *if two gases allow dynamically similar flows for arbitrary values of* P *and* R *then each has an equation of state of the form*

$$p = \sigma(S)\, \varrho^m \tag{36.7}$$

with the same constant m.[2] This result shows how special a gas must be in order to possess useful properties of dynamical similarity. It is singularly fortunate that the common gases more or less accurately obey (36.7).

To prove the assertion, let us follow the path of a given particle, so that both S and S' in Eq. (36.5) will be constant. The unprimed flow being given once and for all, S is a fixed number which we can suppress for the moment in all relations; S', on the other hand, will be a function of the parameters P and R, or conversely

$$P = P(R, S'). \tag{36.8}$$

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[1] A survey of the original work of MACH in fluid mechanics and a discussion of the origin of the terms which bear his name will be found in an article by J. BLACK, J. Roy. Aero. Soc. **54**, 371 (1950).

[2] A similar result for isentropic flows is given by BIRKHOFF [*16*], p. 112.

Substituting Eq. (36.8) into Eq. (36.5) yields

$$g(\varrho', S') = \frac{f(R\varrho')}{P(R, S')} = \frac{f(\varrho')}{P(1, S')}, \tag{36.9}$$

since R may be varied independently of S'. The second equality in Eq. (36.9) can be written in the form

$$\frac{P(R, S')}{P(1, S')} = \frac{f(R\varrho')}{f(\varrho')} = \frac{f(R)}{f(1)},$$

the final equality following since the left hand side is independent of ϱ'. Hence

$$f(R\varrho') = \text{const}\, f(R)\, f(\varrho'), \tag{36.10}$$

and this in turn implies[1] that $f(\varrho) = \text{const}\ \varrho^m$. Therefore $f(\varrho, S) = \sigma(S)\,\varrho^m$, and our assertion is proved.

The above analysis does not provide a complete list of necessary conditions for dynamically similar motion if viscosity, thermal conductivity, extraneous forces, boundary conditions, etc., are important factors in determining the motion. The influence of some of these quantities is dealt with in Sect. 66.

Dimensional analysis and dynamical similarity. Some of the results described above can be arrived at by simple dimensional analysis. Thus, since the equations of motion are dimensionally consistent, it follows that the dimensionless ratio q/c must be the same at corresponding points of two dynamically similar flows. More generally, *if one assumes the existence of dynamically similar flows and their unique dependence on the flow variables at some reference point P*, then using standard procedures of dimensional analysis one sees that any dimensionless quantity connected with the flows must be a function solely of the Mach number at P. The italicized assumptions are, however, just the crux of the matter: without them the dimensional analysis cannot be logically defended[2]. It is at this point that the theory of dynamical similarity is of value, the analysis in the preceding paragraphs serving to determine *necessary and sufficient conditions for the existence of dynamically similar flows.* For reasons such as these GOLDSTEIN[3] says, in commenting on an investigation of dynamical similarity, "so we recover by a somewhat clearer and more logical method, the results obtained earlier by considerations of dimensions".

This is not to say that dimensional analysis is unimportant, for there are many instances where the above assumptions are verified, or where nothing else avails; the technique of dimensional analysis should, however, be supplemented whenever possible by a consideration of the full equations governing the phenomena in question.

II. Energy, entropy, and vorticity.

In the rest of this chapter it is assumed that the extraneous force $\boldsymbol{f} = 0$. Since we are dealing with gases this assumption involves no serious restriction of generality.

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[1] Differentiate (36.10) with respect to R and set $R = 1$.

[2] In ordinary applications of dimensional analysis the basic assumptions are, unfortunately, almost never stated. A critical discussion of these points, together with a number of examples, may be found in Chap. III of reference [*16*]; in particular, it is there emphasized (pp. 92–93) that the Pi Theorem, which is fundamental result of dimensional analysis, is never the point of objection, but rather that the difficulties lie in the assumptions upon which the dimensional analysis rests.

[3] [*42*], p. 112.

37. Bernoulli's equation. A gas flow is called *isentropic* (or *homentropic*) if the entropy is constant throughout the flow region. In this case Eq. (35.3) is identically satisfied, while Eq. (35.4) reduces simply to

$$p = f(\varrho). \tag{37.1}$$

We have therefore the case of barotropic flow already treated in some detail in Part C. In particular, for *isentropic, steady irrotational flow of a perfect gas subject to no extraneous force* the following Bernoulli equation holds,

$$\frac{1}{2} q^2 + \int \frac{dp}{\varrho} = \text{const}; \tag{37.2}$$

for definiteness it is customary to assign 0 as the lower limit of integration. The integral in Eq. (37.2) can be identified with the specific enthalpy of the fluid, but this need not concern us for the moment. Eq. (37.2) can be written in differential form

$$\varrho\, q\, dq + dp = 0, \tag{37.3}$$

or, equivalently,

$$\varrho\, q\, dq + c^2\, d\varrho = 0, \tag{37.4}$$

using formula (35.6) for the speed of sound.

For an ideal gas with constant specific heats the integral $\int dp/\varrho$ can be evaluated explicitly, and Eq. (37.2) becomes

$$\frac{q^2}{2} + \frac{c^2}{\gamma - 1} = \text{const}. \tag{37.5}$$

The various algebraic consequences of this equation are so familiar they need no comment. Instead we shall deal directly with the general equation (37.2). In so doing it is convenient to introduce the thermodynamic assumption[1]

$$\left(\frac{\partial^2 p}{\partial \tau^2}\right)_S > 0; \tag{37.6}$$

geometrically, Eqs. (30.7) and (37.6) imply that adiabatics in the (p, V)-plane are convex curves with negative slope. From inequality (37.6) it follows that

$$a \equiv \frac{1}{c} \frac{d}{d\varrho} (\varrho\, c) > 0. \tag{37.7}$$

It is the property $a > 0$ which we shall use in the subsequent analysis.

Now from Eqs. (37.4), (37.7), and the definition of the Mach number,

$$\frac{dM}{dq} = \frac{1}{c} \left[1 + (a-1) M^2\right]. \tag{37.8}$$

Thus $dM/dq > 0$ if $M \leqq 1$, and it follows that there is exactly one speed $q = q_*$ for which $M = 1$. When $q < q_*$ the flow is *subsonic* ($M < 1$), and when $q > q_*$ the flow is *supersonic* ($M > 1$). q_* is called the *critical speed*.

It is interesting to observe the behavior of the mass flow $Q = \varrho q$ at different speeds. In virtue of Eq. (37.4),

$$\frac{d}{dq} (\varrho\, q) = \varrho\,(1 - M^2). \tag{37.9}$$

---

[1] This condition is a recurrent one in gas dynamics; it is the condition that the velocity profile must steepen in an adiabatic compression wave ([*17*], § 41), it occurs in WEYL's theory of the shock transition (§ 56), and we shall note several other consequences in this section. Obviously, Eq. (37.6) is satisfied by an ideal gas with constant specific heats.

Thus at subsonic speeds ϱq increases with increasing speed, while at supersonic speeds ϱq decreases with increasing speed. It follows that there are two flow regimes, one subsonic and one supersonic, corresponding to a given mass-flow ϱq. These facts are illustrated by Fig. 4, which shows the dependence of ϱq on q for the case of an ideal gas with constant specific heats. Many phenomena in gas dynamics can be traced directly to this "reversed" behavior of the mass-flow at supersonic speeds.

The dependence of the density on the Mach number can be investigated very simply by means of the formula

$$\frac{d\varrho}{dM^2} = \frac{d\varrho}{dq}\,\frac{dq}{dM^2} = -\frac{\varrho}{2\,[1+(a-1)\,M^2]},$$

which is obtained directly from Eqs. (37.4) and (37.8). It follows by integration that

$$\log\frac{\varrho}{\varrho_0} = -\frac{1}{2}\int\limits_0^{M^2}\frac{du}{1+(a-1)\,u},$$
$$= -\frac{1}{2}\left[M^2 - \frac{a_0-1}{2}M^4 + \cdots\right];$$

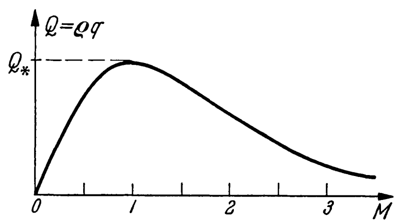

Fig. 4. Dependence of the mass flux $Q=\varrho q$ on the Mach number in steady, isentropic motion.

hence for an arbitrary gas

$$\varrho = \varrho_0\left(1 - \frac{1}{2}M^2 + \frac{2a_0-1}{8}M^4 + \cdots\right). \tag{37.10}$$

This equation shows how slowly the density changes with Mach number when M is relatively small. The accompanying table presents ϱ, ϱq, and $\varrho/\sqrt{1-M^2}$ as functions of M for an ideal gas. The latter quantity varies quite slowly for small Mach numbers.

Table 1. *Variables of state in steady, isentropic flow of an ideal gas* ($\gamma = 1.40$).

Subsonic table					Supersonic table				
M	$\frac{q}{q_*}$	$\frac{\varrho}{\varrho_0}$	$\frac{Q}{Q_*}$	$\varrho_0 k$	M	$\frac{q}{q_*}$	$\frac{\varrho}{\varrho_0}$	$\frac{Q}{Q_*}$	$\frac{\hat{p}_0}{p_0}$
0.1	0.109	0.995	0.172	1.000	1.1	1.082	0.582	0.992	0.999
0.2	0.218	0.980	0.337	1.000	1.2	1.158	0.531	0.970	0.993
0.3	0.326	0.956	0.491	0.998	1.3	1.231	0.483	0.938	0.979
0.4	0.431	0.924	0.629	0.992	1.4	1.300	0.437	0.897	0.958
0.5	0.535	0.885	0.746	0.978	1.5	1.365	0.395	0.850	0.930
0.6	0.635	0.840	0.842	0.952	1.7	1.483	0.320	0.748	0.856
0.7	0.732	0.792	0.914	0.903	2.0	1.633	0.230	0.593	0.721
0.8	0.825	0.740	0.963	0.810	3.5	2.064	0.045	0.147	0.213
0.9	0.915	0.687	0.991	0.635	5.0	2.236	0.011	0.040	0.062
1.0	1.000	0.634	1.000	0	∞	2.449	0	0	0

Abbreviations: $Q=\varrho q$, $k=\varrho^{-1}\sqrt{1-M^2}$,

$\hat{p}_0$ = Stagnation pressure behind normal shock wave.

Note: The final column in the table serves also to determine the ratios $\hat{\varrho}_0/\varrho_0$, $\hat{Q}_*/Q_*$, see Eq. (55.7). Moreover, the entropy jump across a shock is given by $\Delta S = R\log(\hat{p}_0/p_0)$.

38. The Crocco-Vazsonyi equation. In the present section we shall consider non-isentropic steady motion. If we neglect the extraneous force $\mathbf{f}$ in Eq. (35.2) and make use of Eq. (17.1), there results

$$\boldsymbol{\omega}\times\mathbf{v} = -\operatorname{grad}\frac{1}{2}q^2 - \frac{1}{\varrho}\operatorname{grad}p. \tag{38.1}$$

Now let I denote the specific enthalpy, $I = E + p/\varrho$; clearly

$$\operatorname{grad} I = T \operatorname{grad} S + \frac{1}{\varrho} \operatorname{grad} p.$$

Eliminating $\operatorname{grad} p$ from the two preceding equations now yields the important equation

$$\boldsymbol{\omega} \times \mathbf{v} = T \operatorname{grad} S - \operatorname{grad} H, \tag{38.2}$$

where

$$H = \tfrac{1}{2} q^2 + I.$$

Eq. (38.2) was given by CROCCO[1] for the special case of an ideal gas with $H \equiv$ const. The general result is due to VAZSONYI[2].

From Eq. (38.2) it follows that vorticity will generally be present in a non-isentropic flow. In particular, since the flow downstream of a curved shock is non-isentropic it will also be rotational.

Forming the scalar product of Eq. (38.2) with $\mathbf{v}$ and using the fact that $\mathbf{v} \cdot \operatorname{grad} S = 0$ in steady flow, there results $\mathbf{v} \cdot \operatorname{grad} H = 0$ or

$$H = \tfrac{1}{2} q^2 + I = \text{constant on streamlines.} \tag{38.3}$$

Now since $(\partial I/\partial p)_S = 1/\varrho$, we have

$$I = \int_0^p dp/\varrho$$

where it is assumed that $I = 0$ when $p = 0$ (the integration is of course undertaken at constant S). Eq. (38.3) is thus seen to be a generalization of the Bernoulli equation of the preceding section, to which it reduces when $\boldsymbol{\omega}$ and $\operatorname{grad} S$ are zero. If S and H are both known, then Eq. (38.3) determines density and pressure as functions of q.

For an ideal gas the enthalpy depends solely on the speed of sound. This follows from the fact that $I = E + RT = I(T)$ while also $c^2 = \gamma RT = c^2(T)$. Thus in any motion for which $H = $ const, whether or not it is isentropic, we have

$$\tfrac{1}{2} q^2 + I(c^2) = \text{const}.$$

For gases other than the ideal gas the enthalpy is a function of both c^2 and S.

It may be observed finally that the quantity H in steady flow is just the "stagnation" enthalpy; that is, according to Eq. (38.3) H is just the enthalpy I_0 which the particle would have if it ever should come to rest.

39. Isentropic, isoenergetic, and irrotational steady flow. We wish to examine the interrelations between steady flows of the following kinds:

$$\begin{aligned} \text{Isentropic:}\quad & S \equiv \text{const},\\ \text{Isoenergetic:}\quad & H \equiv \text{const},\\ \text{Irrotational:}\quad & \boldsymbol{\omega} \equiv 0. \end{aligned}$$

These types of flow are important because of their frequent occurence and their relative simplicity[3].

[1] L. CROCCO: Rend. Lincei (6) **23**, 115 (1936). — Z. angew. Math. Mech. **17**, 1 (1937).

[2] A. VAZSONYI: Quart. Appl. Math. **3**, 29 (1945).

[3] In some works, the words "homentropic" and "homenergic" are found in place of "isentropic" or "isoenergetic".

Consider first a flow in which all particles emanate from a uniform state. According to Eq. (35.3) the entropy will be constant over the flow region—assuming of course that the motion is continuous—and therefore the reasoning of Sect. 21 can be applied. Thus in general the motion will be isentropic, isoenergetic and irrotational. Even if shock waves intervene the flow will remain isoenergetic, though generally no longer isentropic. A related result is expressed in the following

Theorem 1. *A plane flow (or an axially symmetric flow) which is steady, isentropic, and isoenergetic will also be irrotational.*

Proof. By the Crocco-Vazsonyi equation, $\boldsymbol{\omega}\times\mathbf{v}=0$. Thus $\omega=0$ except at stagnation points. If the stagnation points are isolated, then $\omega=0$ everywhere, by continuity; otherwise, if the stagnation points fill up a region, then obviously $\omega=0$. There is a partial converse, which applies even to three-dimensional flow:

Theorem 2. *A steady irrotational flow is isentropic and isoenergetic, or it is of helicoidal type*[1].

Proof. If the given irrotational flow is isentropic, then it will also be isoenergetic, by the Crocco-Vazsonyi equation. Thus assume the flow is *non*-isentropic in some region; we must show that it is helicoidal in that region. This will be done in two steps. First, the speed and density must be constant along streamlines in the region. To see this, we have from Eq. (38.1) and the equation of state (35.4),

$$\frac{1}{2}\operatorname{grad} q^2 = -\frac{1}{\varrho}\operatorname{grad} p = -\frac{1}{\varrho}\left(f_\varrho \operatorname{grad}\varrho + f_S \operatorname{grad} S\right).$$

The first equality implies that $q=q(p)$ and also (wherever $\operatorname{grad} p\neq 0$) that $\varrho=\varrho(p)$. The second equality then gives, everywhere,

$$\operatorname{grad} p\times\operatorname{grad} S=0.$$

It follows that $p=p(S)$; hence $q=q(S)$ and $\varrho=\varrho(S)$. Since S is constant on streamlines, the first assertion is proved. The conclusion of the theorem is now obtained from the following fact:

If the speed and density are constant on streamlines in an irrotational flow, then the flow is of helicoidal type.

Proof. Since ϱ is constant on streamlines, Eq. (35.1) reduces to $\operatorname{div}\mathbf{v}=0$. The flow being irrotational, this establishes that the velocity potential φ is a harmonic function; moreover $|\operatorname{grad}\varphi|=\text{const}$ on streamlines. HAMEL[2] has shown that such a flow must be helicoidal. Q.E.D.

Since HAMEL's proof is exceedingly difficult[3], we add here a simple proof for the important case of plane flow (in this case helicoidal flow means vortex flow, of course). From Eq. (20.5), which is a kinematic result and therefore valid for any two-dimensional motion, the streamline curvature is given by

$$\varkappa=\frac{1}{q}\frac{\partial q}{\partial n}=\frac{1}{q}\frac{dq}{d\psi}\frac{\partial\psi}{\partial n}=\varrho\frac{dq}{d\psi}, \qquad q=q(\psi),$$

[1] A flow is said to be helicoidal if it can be obtained by the superposition of a two-dimensional vortex and a uniform translation along the axis of the vortex. The result stated is due to D. GILBARG, Amer. J. Math. **71**, 687 (1949).

[2] G. HAMEL: Sitzgsber. preuß. Akad. Wiss., Phys.-math. Kl. 5—20 (1937); see also R.C. PRIM: J. Rational Mech. Anal. **1**, 425 (1952), where HAMEL's result is extended to a larger class of flows.

[3] An estimate of the difficulty of HAMEL's proof can be gained from the thesis of L. HOWARD, Princeton 1952.

where ψ is the streamfunction. Thus $\varkappa$ is constant on streamlines, and the streamlines are circles. These circles are furthermore concentric, since

$$\frac{\partial\varkappa}{\partial n} = \frac{d\varkappa}{d\psi}\,\frac{\partial\psi}{\partial n} = \varrho q\,\frac{d\varkappa}{d\psi} = \text{function of } \psi \quad \text{Q.E.D.}$$

The hypotheses of theorems 1 and 2 can be lightened slightly if a corresponding weakening of the conclusions is allowed. Thus, one has the symmetric pair of theorems:

In a steady, isentropic, isoenergetic flow the vorticity vector satisfies $\boldsymbol{\omega}\times\mathbf{v}\equiv 0$.

If $\boldsymbol{\omega}\times\mathbf{v}\equiv 0$ *in a steady flow, then either the flow is isentropic and isoenergetic or it has constant speed on streamlines.*

Consider now the steady flow of a gas whose equation of state has the form,

$$\varrho = P(p)\,\Sigma(S). \tag{39.1}$$

We show that it is always possible to assume either S or H constant over the flow field. Indeed, consider the *substitute flow* defined by

$$\mathbf{v}^* = \frac{1}{m}\mathbf{v}, \qquad \varrho^* = m^2\varrho, \qquad S^* = \Sigma^{-1}[m^2\Sigma(S)], \tag{39.2}$$

where m is a scalar, *constant on streamlines*. It is easily verified that the substitute flow provides a genuine solution of the equations of motion. Moreover,

$$p^* = p, \qquad I^* = \int\frac{dp^*}{\varrho^*} = \frac{1}{m^2}I,$$

and Bernoulli's equation has the form

$$\tfrac{1}{2}q^{*2} + I^* = H^* = m^{-2}H.$$

We observe that the substitute flow has the same streamline pattern, pressure distribution, and Mach number as the original flow. Furthermore, if we set $m = 1/\sqrt{\Sigma(S)}$ then the substitute flow is isentropic, while if $m = \sqrt{H}$ it is isoenergetic (the latter conclusion remains valid even if the flow contains shock waves). That is, it is always possible to make either S or H constant over the flow field, as asserted[1].

In general, S and H cannot be made simultaneously constant. Indeed, we have the following criterion: *S and H can be made simultaneously constant if and only if the stagnation pressure p_0 is uniform over the flow region.*

[Stagnation pressure is obtained by setting $q=0$ in Bernoulli's equation and solving for p; that is, $I(p_0, S) = H$. Obviously p_0 is constant on streamlines.] To prove the theorem we observe that if p_0 is constant over the entire flow field, then $p_0^* = p_0$ is also. Choosing m so that S^* is constant, it follows that H^* must likewise be constant. The converse is proved in the same way.

It should be mentioned that if S^* and H^* are constant over the flow field, then also

$$\boldsymbol{\omega}^*\times\mathbf{v}^* = 0, \tag{39.3}$$

by the Crocco-Vazsonyi equation (which of course applies equally to the substitute flow). The class of flows for which Eq. (39.3) holds has been called *generalized Beltrami flows*; This class is generated from the class of Beltrami flows ($\boldsymbol{\omega}\times\mathbf{v} = 0$) by means of the substitution principle (39.2).

We emphasize in conclusion that the substitution principle is mainly of interest for rotational flow, and that it applies only to the steady flow of a gas whose equation of state has the form (39.1).

40. Diffusion of vorticity. The basic kinematical equations governing the distribution of vorticity in a fluid motion were derived in Sects. 17 and 25. They are BELTRAMI's equation

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\varrho}\right) = \frac{\boldsymbol{\omega}}{\varrho}\cdot\operatorname{grad}\mathbf{v} + \frac{1}{\varrho}\operatorname{curl}\mathbf{a}, \tag{40.1}$$

[1] M. MUNK and R. PRIM: Proc. Nat. Acad. Sci. U.S.A. **33**, 137 (1947).

and KELVIN'S equation
$$\frac{d}{dt}\oint_{\mathscr{C}} \mathbf{v}\cdot d\mathbf{x} = \oint_{\mathscr{C}} \mathbf{a}\cdot d\mathbf{x}. \tag{40.2}$$

When applied to the non-isentropic motion of a perfect gas, Eqs. (40.1) and (40.2) yield a number of important conclusions.

If we set $\tau = 1/\varrho$ the equation of motion may be written

$$\mathbf{a} = -\tau \operatorname{grad} p. \tag{40.3}$$

Taking the curl of both sides of Eq. (40.3) and using the thermodynamic identity

$$\tau\, dp = -T\, dS + dI \tag{40.4}$$

yields

$$\operatorname{curl} \mathbf{a} = \operatorname{grad} T \times \operatorname{grad} S.$$

Finally, substituting this expression for curl **a** into (40.1) gives the vorticity equation of VAZSONYI[1],

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\varrho}\right) = \frac{\boldsymbol{\omega}}{\varrho}\cdot \operatorname{grad} \mathbf{v} + \frac{1}{\varrho} \operatorname{grad} T \times \operatorname{grad} S. \tag{40.5}$$

But for the second term on the right the vorticity distribution in a perfect gas would follow the theorems of HELMHOLTZ. This term shows, however, that a non-uniform entropy field generally leads to a *diffusion* of vorticity, blurring out the sharp convective changes predicted by HELMHOLTZ's theorems.

In the case of *steady isoenergetic flow* we have by the Crocco-Vazsonyi equation

$$\begin{aligned} \operatorname{grad} T \times \operatorname{grad} S &= T^{-1}\left[\operatorname{grad} T \times (\boldsymbol{\omega} \times \mathbf{v})\right] \\ &= T\left[\mathbf{v}\left(\boldsymbol{\omega}\cdot \operatorname{grad}\frac{1}{T}\right) - \boldsymbol{\omega}\frac{d}{dt}\left(\frac{1}{T}\right)\right]. \end{aligned}$$

If this formula is used to eliminate grad $T \times$ grad S from Eq. (40.5), there follows after some simplification

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\varrho T}\right) = \frac{\boldsymbol{\omega}}{\varrho}\cdot \operatorname{grad}\left(\frac{\mathbf{v}}{T}\right). \tag{40.6}$$

For plane flow (40.6) reduces simply to

$$\frac{\omega}{\varrho T} = \text{constant on streamlines}, \tag{40.7}$$

a result due to CROCCO[2] for the special case of an ideal gas. Finally, if $\boldsymbol{\omega} = 0$ at a point P in steady isoenergetic flow, then from Eq. (40.6) follows $\boldsymbol{\omega} = 0$ along the entire streamline through P[3].

By a straightforward calculation based on Beltrami's equation one easily obtains the identity[4]

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\varrho}\cdot \operatorname{grad} F\right) = \frac{\boldsymbol{\omega}}{\varrho}\cdot \operatorname{grad}\frac{dF}{dt} + \frac{1}{\varrho}\operatorname{curl}\mathbf{a}\cdot \operatorname{grad} F, \tag{40.8}$$

where F is an arbitrary scalar, vector, or tensor function. Eq. (40.8) is the kinematical equivalent of an interesting vorticity formula discovered by ERTEL[5].

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[1] A. VAZSONYI: Quart. Appl. Math. **3**, 29 (1945).

[2] L. CROCCO: Z. angew. Math. Mech. **17**, 1 (1937). CROCCO wrote the result in the form $\omega/p =$ constant on streamlines.

[3] This should be compared with the Lagrange-Cauchy theorem of § 17.

[4] C. TRUESDELL: Z. angew. Math. Phys. **2**, 1 (1951); also [*26*], §§ 79, 98, 99.

[5] H. ERTEL: Naturwiss. **30**, 543 (1942). — Meteor. Z. **59**, 277, 385 (1942). — Phys. Z. **43**, 526 (1942).

If we take F to be the entropy of a perfect gas the right hand side of (40.8) reduces to zero, whence it follows that

$$\frac{\boldsymbol{\omega}}{\varrho} \cdot \operatorname{grad} S = \text{constant following particles}.$$

This interesting application of Eq. (40.8) is due to Truesdell; for other related results the reader is referred to the papers quoted above.

We turn now to the consequences of Kelvin's equation (40.2). Substituting for $\mathbf{a}$ from Eq. (40.3) yields

$$\frac{d}{dt} \oint_{\mathscr{C}} \mathbf{v} \cdot d\mathbf{x} = -\oint_{\mathscr{C}} \tau \, dp.$$

The last integral can be evaluated by plotting the curve $\mathscr{C}$ in the (p, τ)-plane, see Fig. 5. It follows that

$$\frac{d}{dt} \oint_{\mathscr{C}} \mathbf{v} \cdot d\mathbf{x} = \pm \left\{ \begin{array}{l} \text{Area surrounded in the } (p, \tau)\text{-plane} \\ \text{by the image of } \mathscr{C}. \end{array} \right\}.$$

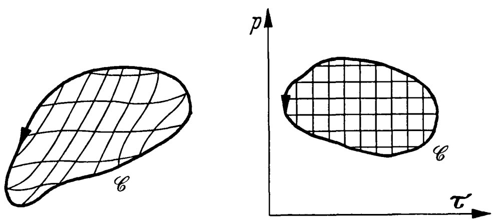

Fig. 5. Illustrating Bjerknes' circulation theorem.

In case of a barotropic flow the entire (p, τ)-image of $\mathscr{C}$ lies on a fixed curve. There is therefore no area, and we recover Kelvin's circulation theorem. Another interpretation of $\oint \tau \, dp$ may be had if we span $\mathscr{C}$ by an orientable surface $\mathscr{S}$ and plot on $\mathscr{S}$ the curves $p = 0, \pm 1, \pm 2, \ldots$ and $\tau = 0, \pm 1, \pm 2, \ldots$. Referring again to Fig. 5 we see that

$$\frac{d}{dt} \oint_{\mathscr{C}} \mathbf{v} \cdot d\mathbf{x} = \pm \text{ number of } (p, \tau)\text{-cells on } \mathscr{S}.$$

This is the celebrated theorem of Bjerknes[1]. It would be permissible to use (T, S)-cells instead of (p, τ)-cells because

$$\oint \tau \, dp = -\oint T \, dS,$$

by formula (40.4). Finally, it is easy to see that Bjerknes' theorem remains true even should first order singular surfaces (Sect. 51) intervene in the motion.

III. Special methods in two-dimensional flow.

41. Intrinsic equations. From Sect. 20 we carry over the intrinsic equations governing steady two-dimensional flow. They are

$$\varrho q \frac{\partial q}{\partial s} = -\frac{\partial p}{\partial s}, \qquad \varrho q^2 \varkappa = -\frac{\partial p}{\partial n}, \tag{41.1}$$

and the equation of continuity

$$\frac{\partial}{\partial s} (\varrho q) + K \varrho q = 0. \tag{41.2}$$

[1] V. Bjerknes: Vidensk. Skr. No. 5 (1898). — Kgl. svenska Vet. Handl. (2) **31**, No. 4 (1898).

Since $dp = c^2\, d\varrho$ along streamlines, Eq. (41.2) can be written with the help of Eq. (41.1), in the alternate form

$$\frac{\partial q}{\partial s} = \frac{K q}{M^2 - 1}. \tag{41.3}$$

In the special case of irrotational motion the above equations can be replaced by the pair

$$\frac{\partial q}{\partial s} = \frac{K q}{M^2 - 1}, \qquad \frac{\partial q}{\partial n} = \varkappa q, \tag{41.4}$$

together with the Bernoulli equation (37.2).

42. The stream function. In Sect. 19 a stream function was defined for certain incompressible flows. For compressible fluids it is necessary to assume *steady motion* in order to carry out the same ideas. The equation of continuity of a steady, two-dimensional flow has the form

$$\frac{\partial}{\partial x}(\varrho u) + \frac{\partial}{\partial y}(\varrho v) = 0.$$

Thus, as in Sect. 19, a stream function $\psi = \psi(x, y)$ can be defined such that

$$\varrho u = \frac{\partial \psi}{\partial y}, \qquad \varrho v = -\frac{\partial \psi}{\partial x}. \tag{42.1}$$

The curves $\psi = \text{const}$ are streamlines.

Now from Eqs. (17.8) and (42.1) we easily obtain

$$\psi_{xx} + \psi_{yy} - \mathbf{v}^* \cdot \operatorname{grad} \varrho = -\varrho\, \omega, \tag{42.2}$$

where $\mathbf{v}^*$ is the vector with components $(-v, u)$. This equation for ψ can be put in an elegant form by carrying out the indicated differentiation of ϱ. The calculation is routine, but since it has been given incorrectly in recent years we include it here for completeness. We begin with the relations

$$\left.\begin{aligned} \boldsymbol{\omega} \times \mathbf{v} &= \omega\, \mathbf{v}^*, \\ \operatorname{grad} q^2 &= \varrho^{-2}\left[\operatorname{grad}(\varrho^2 q^2) - 2\varrho q^2 \operatorname{grad} \varrho\right], \\ \operatorname{grad} p &= c^2 \operatorname{grad} \varrho + B \operatorname{grad} S, \end{aligned}\right\} \tag{42.3}$$

where $B = (\partial p/\partial S)_\varrho$. Substituting Eq. (42.3) into Eq. (38.1) and solving for $\operatorname{grad} \varrho$ gives

$$\operatorname{grad} \varrho = (\varrho q^2 - \varrho c^2)^{-1}\left[\varrho^2 \omega\, \mathbf{v}^* + \varrho B \operatorname{grad} S + \operatorname{grad} \tfrac{1}{2}(\varrho^2 q^2)\right].$$

Finally, inserting this into Eq. (42.2) and observing that

$$\operatorname{grad}(\varrho^2 q^2) = \operatorname{grad}(\psi_x^2 + \psi_y^2),$$
$$\operatorname{grad} S = \varrho\, \mathbf{v}^* S',$$

where $S' = dS/d\psi$, we find after some simplification that

$$(c^2 - u^2)\,\psi_{xx} - 2uv\,\psi_{xy} + (c^2 - v^2)\,\psi_{yy} = -\varrho\,(c^2 \omega + q^2 B S'). \tag{42.4}$$

This is the required equation for the stream function. (The derivation becomes somewhat simpler in the isentropic case, and even quite easy when the flow is both isentropic and irrotational.) The terms on the right hand side of Eq. (42.4) are connected by the Crocco-Vazsonyi equation, which can be written in the form

$$\omega = \varrho\,(T S' - H'), \tag{42.5}$$

where $S = S(\psi)$ and

$$H = H(\psi) = \tfrac{1}{2} q^2 + I(\varrho, S).$$

When $S(\psi)$ and $H(\psi)$ are known functions, Eq. (42.4) can be considered a second order partial differential equation for ψ.

The nature of Eq. (42.4) can be further clarified by considering its form in the special case of an ideal gas with constant specific heats. In this instance

$$B = \left(\frac{\partial p}{\partial S}\right)_\varrho = \frac{p}{c_v}$$

(see Sect. 35), whence after a short calculation we find

$$\left.\begin{aligned}(d^2 - \psi_y^2)\,\psi_{xx} + 2\psi_x \psi_y \psi_{xy} + (d^2 - \psi_x^2)\,\psi_{yy}\qquad&\\ = d^4\left\{\left(\frac{1}{\gamma-1} + \frac{1}{2} M^2\right)\frac{d \log H}{d\psi} - \left(\frac{1}{\gamma-1} + M^2\right)\frac{d\, S/c_p}{d\psi}\right\},&\end{aligned}\right\} \tag{42.6}$$

where $d = \varrho c$ is the acoustical impedance. Eq. (42.6) can be given a particularly simple form which at the same time is very convenient for the numerical computation of flows. To this end, consider the *substitute flow* (Sect. 39) defined by

$$\mathbf{v}^* = \varrho_0 c_0 \mathbf{v}, \qquad \varrho^* = \varrho/(\varrho_0 c_0)^2, \qquad p^* = p.$$

Its streamfunction Ψ satisfies $d\Psi = d\psi/\varrho_0 c_0$, hence

$$|\operatorname{grad} \Psi| = \varrho\, q/\varrho_0 c_0. \tag{42.7}$$

Consequently, once the streamfunction Ψ is determined, *one can use ordinary gas dynamic tables (or simple formulas) to obtain the various parameters M, p/p_0, ϱ/ϱ_0, etc. of the original flow.* Now the streamfunction Ψ obviously satisfies Eq. (42.6) with all flow quantities starred. But

$$d^* = d/d_0, \qquad M^* = M,$$

and

$$H^* = (\varrho_0 c_0)^2 H = \text{const}\, p_0^2,$$

$$\frac{S^*}{c_v} = \log \frac{p\,\varrho^{-\gamma}}{(\varrho_0 c_0)^{-2\gamma}} = \log p_0^{1+\gamma} + \text{const}.$$

Inserting these expressions into Eq. (42.6) leads at once to the result

$$\left.\begin{aligned}\left((d/d_0)^2 - \Psi_y^2\right)\Psi_{xx} + 2\Psi_x \Psi_y \Psi_{xy} + \left((d/d_0)^2 - \Psi_x^2\right)\Psi_{yy}&\\ = (1 - M^2)\,\frac{1}{\gamma}\left(\frac{d}{d_0}\right)^4 \frac{d \log p_0}{d\Psi}.&\end{aligned}\right. \tag{42.8}$$

Eq. (42.8) is the simplification mentioned earlier. It has the tremendous advantage over Eq. (42.6) that all its coefficients can be obtained easily and once and for all in terms of grad Ψ. In addition, the intervention of S' and H' into Eq. (42.8) is confined solely to the simple term $\log p_0$. In Table 2 the coefficients in Eq. (42.8) are tabulated numerically; it has been found convenient to list all quantities as functions of the Mach number, even though $|\operatorname{grad} \Psi|$ is the actual independent variable. It will be observed that each value of $|\operatorname{grad} \Psi|$ less than 0.579 appears twice, while no values greater than 0.579 appear. This is, of course, a reflection of the fact that $|\operatorname{grad} \Psi|$ is a mass-flux, and should cause no confusion; whether one uses the subsonic or supersonic portion of the table to determine the coefficients depends on the problem at hand.

For flows which are isoenergetic, an equation equivalent to (42.8) was discovered by CROCCO[1]. The present calculation may be considered an extension and simplification of CROCCO's original work. In conclusion, we remark that the substitute streamfunction Ψ is available only in the case of an ideal gas, and that Ψ generally suffers a discontinuity at a shock surface, while ψ itself remains continuous.

Isentropic irrotational flow. In this case Eq. (42.4) takes the simple and elegant form

$$(c^2 - u^2)\,\psi_{xx} - 2uv\,\psi_{xy} + (c^2 - v^2)\,\psi_{yy} = 0. \tag{42.9}$$

This result can of course be obtained directly from Eq. (42.2), if the reader so wishes. It is interesting to observe that the velocity potential satisfies an equation with the same coefficients (Sect. 45).

For an ideal gas the equation satisfied by the substitute streamfunction has some computational advantages over Eq. (42.9), and its use is recommended.

Table 2. *Coefficient table for Eq. (42.8).*

Subsonic flow				Supersonic flow			
M	$\lvert\mathrm{grad}\,\Psi\rvert$	$(d/d_0)^2$	$\mathscr{A}$	M	$\lvert\mathrm{grad}\,\Psi\rvert$	$(d/d_0)^2$	$\mathscr{A}$
0.0	0	1	0.714	1.1	0.574	0.272	−0.011
0.1	0.099	0.988	0.690	1.2	0.562	0.219	−0.015
0.2	0.195	0.953	0.623	1.3	0.543	0.174	−0.015
0.3	0.284	0.898	0.524	1.4	0.519	0.137	−0.013
0.4	0.364	0.828	0.411	1.5	0.492	0.108	−0.010
0.5	0.432	0.746	0.299	1.7	0.432	0.065	−0.006
0.6	0.487	0.659	0.199	2.0	0.343	0.029	−0.002
0.7	0.529	0.571	0.119	3.5	0.085	0.001	−0.000
0.8	0.557	0.485	0.061	5.0	0.023	0.000	−0.000
0.9	0.574	0.406	0.022	10.0	0.011	0.000	−0.000
1.0	0.579	0.335	0	∞	0	0	0

Note: The entries in this table are defined by

$$|\mathrm{grad}\,\Psi| = \varrho q/\varrho_0 c_0, \qquad d = \varrho c, \qquad \mathscr{A} = (1 - M^2)\,\frac{1}{\gamma}\left(\frac{d}{d_0}\right)^4,$$

and the computations are made for the specific heat ratio $\gamma = 1.40$.

Axially symmetric flow. The development here is entirely analogous to the preceding. The stream function may be defined by

$$\varrho\,u = \frac{1}{y}\,\frac{\partial\psi}{\partial y}, \qquad \varrho\,v = -\frac{1}{y}\,\frac{\partial\psi}{\partial x}.$$

After some calculation it is found that ψ satisfies the equation

$$(c^2 - u^2)\,\psi_{xx} - 2uv\,\psi_{xy} + (c^2 - v^2)\,\psi_{yy} - c^2\,\frac{\psi_y}{y} = -\,\varrho\,y\,(c^2\,\omega + y\,q^2\,B\,S'),$$

the terms on the right hand side being connected by the equation $\omega = y\varrho\,(T\,S' - H')$. When the flow is irrotational, the right hand side is zero. In this case the velocity potential satisfies a slightly different equation, namely

$$(c^2 - u^2)\,\varphi_{xx} - 2uv\,\varphi_{xy} + (c^2 - v^2)\,\varphi_{yy} + c^2\,\frac{\varphi_y}{y} = 0.$$

A pair of stream functions for three-dimensional flows has been introduced by GIESE[2].

[1] L. CROCCO: Z. angew. Math. Mech. **17**, 1 (1937).
[2] J. GIESE: J. Math. Phys. **30**, 31 (1951).

43. The hodograph method. The Eqs. (42.9) or (45.4) governing plane irrotational flow, although relatively simple in appearance, nevertheless are quite difficult to handle because they are non-linear. MOLENBROECK[1] discovered in 1890 that by transferance to the hodograph plane, i.e. by using the velocity components u, v as independent coordinates (or, equivalently, q and ϑ), the differential equations of motion became linear. This method was exploited by CHAPLYGIN[2] in a path-breaking paper on gas jets, and in recent years by many other writers[3]. The extensive work based on this transformation cannot be discussed here, although we can give an account of the method.

As we have seen, the velocity potential and stream function of a steady plane irrotational flow satisfy the equations

$$u = \varphi_x = \frac{1}{\varrho}\psi_y, \qquad v = \varphi_y = -\frac{1}{\varrho}\psi_x. \tag{43.1}$$

We are interested in the equations satisfied by φ and ψ when the speed q and the velocity inclination ϑ are considered as independent variables. The following is probably the most elegant derivation. We observe that

$$\begin{aligned} d\varphi + i\frac{d\psi}{\varrho} &= (u\,dx + v\,dy) + i(-v\,dx + u\,dy) \\ &= (u - iv)(dx + i\,dy) = q\,\mathrm{e}^{-i\vartheta}\,dz, \end{aligned}$$

where $z = x + iy$. Hence with q, ϑ as independent variables,

$$z_q = \mathrm{e}^{i\vartheta}\left(\frac{\varphi_q}{q} + i\frac{\psi_q}{\varrho q}\right), \qquad z_\vartheta = \mathrm{e}^{i\vartheta}\left(\frac{\varphi_\vartheta}{q} + i\frac{\psi_\vartheta}{\varrho q}\right).$$

Since $z_{q\vartheta} = z_{\vartheta q}$ we find after some cancellation

$$-\frac{\psi_q}{\varrho q} + i\frac{\varphi_q}{q} = -\frac{\varphi_\vartheta}{q^2} + i\psi_\vartheta\frac{d}{dq}\left(\frac{1}{\varrho q}\right).$$

Equating the real and imaginary parts in this formula and using the Bernoulli equation (37.9) leads at once to the *hodograph* equations

$$\varphi_q = -\frac{1 - M^2}{\varrho q}\psi_\vartheta, \qquad \varphi_\vartheta = \frac{q}{\varrho}\psi_q. \tag{43.2}$$

Solutions of Eq. (43.2) correspond to flows in the physical plane only when the Jacobian of the transformation from q, ϑ to x, y is non-zero. It is therefore of interest to express this Jacobian in the hodograph variables, i.e.

$$\frac{\partial(x, y)}{\partial(q, \vartheta)} = \frac{\partial(x, y)}{\partial(\varphi, \psi)}\,\frac{\partial(\varphi, \psi)}{\partial(q, \vartheta)} = -\frac{1}{\varrho^2 q^3}\left[\varrho^2\varphi_\vartheta^2 + (1 - M^2)\,\psi_\vartheta^2\right]. \tag{43.3}$$

We observe that this cannot vanish in subsonic flows, except at isolated points. Since the introduction of hodograph variables depends essentially on the assumption that q, ϑ rather than x, y can be considered as independent variables, it follows that solutions for which

$$\frac{\partial(q, \vartheta)}{\partial(x, y)} = 0$$

[1] P. MOLENBROECK: Arch. Math. Phys. **9**, 157 (1890).

[2] S.A. CHAPLYGIN: Sci. Memoirs, Imp. Univ. Moscow **21**, 1 (1902), [English translation, NACA Tech. Mem. No. 1063 (1944)].

[3] Extensive reference lists will be found in the following articles: L. BERS: Comm. Pure Appl. Math. **7**, 79 (1954); P. GERMAIN: Comm. Pure Appl. Math. **1** 117 (1954); M.J. LIGHTHILL [*35*], Chap. 7; M. SCHIFFER: This Handbuch, Vol. IX; see also T. CHERRY: Phil. Trans. Roy. Soc. Lond. A **245**, 583 (1953); M. SCHÄFER: J. Rational Mech. Anal. **2**, 383 (1953), and references quoted later in this and the following section.

cannot be obtained by the hodograph method: the image of the flow region of such a solution reduces to a curve in the (u, v) plane. For the theory of these solutions the reader is referred to the article of SCHIFFER in Vol. IX of this Encyclopedia. Further discussion of the singularities in the hodograph transformation may be found in COURANT and FRIEDRICHS ([*17*], §§ 30, 105).

Equations satisfied in the hodograph plane by φ or ψ alone can easily be obtained by eliminating the other variable from Eq. (43.2). We illustrate this remark by considering the elimination of φ. Thus, introducing the new variable

$$\sigma = \int \varrho q^{-1} dq,$$

we find easily

$$\psi_{\sigma\sigma} + K(\sigma)\,\psi_{\vartheta\vartheta} = 0, \qquad K(\sigma) = \varrho^{-2}(1 - M^2). \tag{43.4}$$

This equation has been the starting point for a number of investigations of subsonic and transonic flow. For subsonic flow another independent variable is sometimes used, namely

$$r = \int \frac{\sqrt{1-M^2}}{q}\, dq;$$

in terms of (r, ϑ) the equation for the stream function is[1]

$$\psi_{rr} + \psi_{\vartheta\vartheta} + (\log k)_r\,\psi_r = 0, \qquad k = \varrho^{-1}\sqrt{1 - M^2}.$$

Another hodograph equation, which we mention briefly, is that for the Legendre transforms

$$\Phi = u x + v y - \varphi, \qquad \Psi = \varrho u y - \varrho v x - \psi.$$

Since

$$x = \Phi_u = -\Psi_{\varrho v}, \qquad y = \Phi_v = \Psi_{\varrho u},$$

it is found that

$$\Psi_q = \frac{\varrho}{q}\,(1 - M^2)\,\Phi_\vartheta, \qquad \Psi_\vartheta = -\varrho q\,\Phi_q. \tag{43.5}$$

These equations are equivalent to the earlier obtained set (43.2).

CROCCO[2] has noticed that the hodograph equations (43.2) and (43.5) can be written in an elegant symmetric form if the independent variable $Q = \varrho q$ is used simultaneously with q. In terms of the variables q, Q, and ϑ we have, in fact,

$$\varphi_\vartheta = -\frac{1}{Q}\,\psi_{q^{-1}}, \qquad \psi_\vartheta = \frac{1}{q}\,\varphi_{Q^{-1}}, \tag{43.6}$$

and similarly

$$\Phi_\vartheta = q\,\Psi_Q, \qquad \Psi_\vartheta = -Q\,\Phi_q. \tag{43.7}$$

44. Special solutions. It seems worthwhile to mention here several applications of the hodograph equations.

1. An approximation method. For subsonic flow Eq. (43.2) can be written in the form

$$\varphi_r = -k\psi_\vartheta, \qquad \varphi_\vartheta = k\psi_r, \tag{44.1}$$

where

$$r = \int_{q_0}^{q} \frac{\sqrt{1-M^2}}{q}\, dq, \qquad k = \frac{\sqrt{1-M^2}}{\varrho}, \tag{44.2}$$

[1] A similar equation can of course be obtained for supersonic flow. Further transformations of the hodograph equations can be made by means of the Bäcklund transformation, cf. C. LOEWNER: J. d'Anal. Math. **2**, 219 (1953).

[2] L. CROCCO: NACA Tech. Note 2432 (1951).

and q_0 is a reference speed in the flow. Now the function $k = k(r)$ is essentially constant over a relatively large range of Mach numbers, as can be seen from the expansion (37.10) or from Table 1. Indeed, for an ideal gas k varies less than 8% over the whole Mach number range $M = 0$ to $M = \frac{2}{3}$. For flows in this speed range, then, we can simplify Eq. (44.1) by setting $k = \text{const}$ and still not introduce serious error in the solutions. Similarly, for higher speed ranges we can again set $k = \text{const}$ provided the Mach number *variation* over the flow field is not too large.

Now with $k = \text{const}$ in Eq. (44.1) these equations reduce simply to the Cauchy-Riemann equations of analytic function theory, so that

$$\varphi + i k \psi = \text{analytic function of } r - i\vartheta.$$

For Mach number tending to zero we have, approximately,

$$r \approx \log \frac{q}{q_0}, \qquad r - i\vartheta \approx \log \frac{1}{q_0}(u - i v).$$

This suggests the following method for the generation of compressible flows from incompressible ones. Suppose $w(\zeta)$ is the complex velocity potential of an incompressible irrotational flow in the ζ-plane. Then the parametric equations

$$\varphi + i k \psi = \text{const}\, w(\zeta), \qquad r - i\vartheta = \log dw/d\zeta \tag{44.3}$$

clearly define a compressible flow (subject, of course, to the assumption $k = \text{const}$). For small Mach numbers, the flow thus defined will be very nearly the same as the incompressible flow; for larger Mach numbers there will be an appreciable difference, although it may be presumed that the new flow pattern will remain generally similar to that of the original incompressible one. The exact correspondence between the z and ζ-planes is determined from

$$z = \int^{\zeta} e^{i\vartheta}\left(\frac{d\varphi}{q} + i\,\frac{d\psi}{\varrho q}\right). \tag{44.4}$$

The quadratures involved in Eq. (44.4) are of course quite complicated, so that it is reasonable to keep the original flow pattern while at the same time changing the velocities according to the rule $(44.3)_2$, that is, according to

$$r(q) = \log q_{\text{inc}}.$$

The Karman-Tsien velocity correction formula[1], in particular, is based on this idea together with the choice $p = a/\varrho + b$ as equation of state [this equation of state makes $k = \text{const}$, and at the same time facilitates the integration (44.2)]. Using the correct equation of state may lead to a better velocity correction factor than that of KARMAN.

[The function $r(q)$ has not been tabulated for an ideal gas, so far as the author is aware, but the nearly equal integral

$$\sigma = \int \varrho q^{-1} dq$$

appears in the tables of GARRICK and KAPLAN[2] as a function of M. The velocity correction can then be read off from these tables by means of the formula

$$\sigma(M) = \sigma(M_0) + \log q_{\text{inc}},$$

it being assumed, as always, that the incompressible flow is normalized so that its speed is 1 at the reference point.]

Other velocity-correction formulas for subsonic compressible flow are due to GARRICK and KAPLAN, and to RINGLEB[3]. Though these formulas are derived from somewhat different arguments than the one presented above, nevertheless all of them predict approximately the same result, this being due in all likelihood to the close similarity of Eqs. (44.1) to the Cauchy-Riemann equations.

2. Spiral flow. There are two very simple solutions of Eq. (43.4), namely

$$\psi = \vartheta \quad \text{and} \quad \psi = \sigma = \int \varrho q^{-1} dq.$$

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[1] T. VON KARMAN: J. Aeronaut. Sci. **8**, 337 (1941).

[2] J. GARRICK and C. KAPLAN: NACA Rep. 789 (1944).

[3] See [*23*], p. 340.

The first represents radial flow and the second, vortex flow (see COURANT and FRIEDRICHS, [*17*], pp. 252—254). The linear combination $a\vartheta + b\sigma$ thus determines a *spiral flow*.

3. Ringleb's solution and Manwell's solution. We seek the most general solution of Eq. (43.2) which can be written in the form

$$\psi = f(q)\sin\vartheta, \qquad \varphi = h(q)\cos\vartheta.$$

This is readily found to be given by

$$f = q^{-1}\left[a + b\int_0^q \varrho\, q\, dq\right], \qquad h = -\varrho^{-1} q f',$$

where a and b are constants. The particular choice of constants $a=1$, $b=0$ leads to the well-known solution of RINGLEB[1]. MANWELL[2] pointed out another valuable example, namely

$$f = q^{-1}\int_a^q \varrho\, q\, dq, \qquad h = \varrho^{-1} f - q,$$

where a is constant. The transformation to the physical plane is obtained from

$$\left.\begin{aligned} z &= \int e^{i\vartheta}\left(\frac{d\varphi}{q} + i\,\frac{d\psi}{\varrho q}\right) \\ &= \frac{1}{4}\left(2i\vartheta - e^{2i\vartheta}\right) + e^{i\vartheta}\left[\cos\vartheta\int_a^q \frac{dh}{q} + i\sin\vartheta\int_a^q \frac{df}{\varrho q}\right]; \end{aligned}\right\} \tag{44.5}$$

this evaluation is greatly simplified by performing the ϑ integration at the value $q=a$, and the q integration for fixed ϑ. It is easy to see that the image of the hodograph region $0 \leqq q \leqq a$, $0 \leqq \vartheta \leqq \pi$, is the exterior of the semi-infinite body shown in Fig. 6. We note that $\psi = f\sin\vartheta$ vanishes on the boundary, which is therefore a streamline. Moreover, the boundary is independent of the parameter a. Thus, MANWELL's solution gives a class of flows of varying Mach number past a *fixed* semi-infinite body.

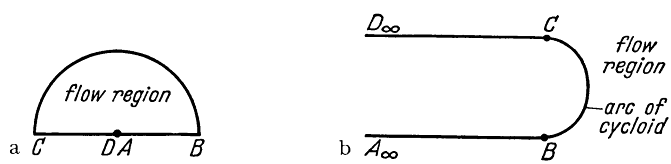

Fig. 6a and b. (a) Flow region in the hodograph plane for MANWELL's solution. (b) Flow region in the physical plane for MANWELL's solution.

For subsonic flow, that is, $a <$ critical speed, the Jacobian (43.3) is non-zero, and the transformation from (q, ϑ) to (x, y) is one-to-one. On the other hand, when a is greater than the critical speed, no physical flow exists. The simplest way to see this is to note that the boundary streamline $\psi = 0$ is doubled back in the physical plane. Indeed from Eq. (44.5),

$$\left.\frac{dx}{dq}\right|_{\vartheta=0} = \frac{1}{q}\,\frac{dh}{dq} = \frac{f}{\varrho q}\left(M^2 - 1\right),$$

[1] F. RINGLEB: Z. angew. Math. Mech. **20**, 185 (1940). See also G. TEMPLE and J. YARWOOD: Aero. Res. Comm. R & M 2077 (1942).

[2] A. MANWELL: J. Math. Phys. **34**, 113 (1955).

which changes sign at sonic speed. For $a <$ critical speed the flow can be continued across the boundary to exhibit a transonic flow exactly as in RINGLEB'S example.

The breakdown of MANWELL'S example at supersonic speeds seems to be connected with a theorem of NIKOLSKY and TAGANOV (Sect. 52) which asserts that no transonic flow can exist with a local supersonic zone adjacent to a straight arc of the boundary.

4. Chaplygin's solutions. We seek the most general solution of Eq. (43.2) of the form

$$\psi = f(q)\, F(\vartheta), \qquad \varphi = h(q)\, H(\vartheta).$$

Inserting the first of these into Eq. (43.4) we find that $F = e^{i n \vartheta}$ where n is any real number, and the equation satisfied by f is then

$$f_{\sigma\sigma} - n^2 K(\sigma)\, f = 0.$$

This equation is the starting point for much modern work on exact solutions[1].

5. *Crocco's solutions.* The complex velocity potential

$$w = (\log q - i\vartheta)^n = \sum_{k=0}^{n} c_{nk} (\log q)^k (-i\vartheta)^{n-k}$$

is obviously valid for motions of an incompressible fluid; CROCCO[2] has shown how to modify this solution so that it will satisfy the hodograph equations for flow of a compressible fluid. It is convenient in this process to think of these equations in the form (43.6). We set

$$\varphi = \sum_{k=0}^{n} c_{nk} L(q,k)\, (-i\vartheta)^{n-k}, \qquad \psi = i \sum_{k=0}^{n} c_{nk} \tilde{L}(q,k)\, (-i\vartheta)^{n-k}, \tag{44.6}$$

where the functions L and $\tilde{L}$ are defined by

$$L(q,k) = k!\, \underbrace{\int^{Q^{-1}} q\, dQ^{-1} \int^{q^{-1}} Q\, dq^{-1} \ldots}_{k \text{ times}}, \qquad L(q,0) = 1,$$

$$\tilde{L}(q,k) = k!\, \overbrace{\int^{q^{-1}} Q\, dq^{-1} \int^{Q^{-1}} q\, dQ^{-1} \ldots}, \qquad \tilde{L}(q,0) = 1;$$

it is understood that the iterated integrals are evaluated with a fixed reference state for the lower limit. It is obvious that

$$\frac{dL}{dQ^{-1}} = k\, q\, \tilde{L}(q, k-1), \qquad \frac{d\tilde{L}}{dq^{-1}} = k\, Q\, L(q, k-1).$$

This being verified, there is no trouble in showing that Eq. (44.6) yields a solution of Eq. (43.6). Real solutions are of course obtained by using only the real or imaginary parts of Eq. (44.6). In the paper referred to above, CROCCO has shown how the exact solutions (44.6) can be used to construct flows past obstacles in the physical plane.

IV. Subsonic potential flow.

In this subchapter we treat some of the important theoretical questions concerning steady subsonic potential flow of a perfect gas.

[1] M. LIGHTHILL [*35*], Chap. 7. — L. CROCCO: NACA Tech. Note 2432 (1951). — A. GHAFFARI: The Hodograph Method in Gas Dynamics. Univ. of Teheran Publ. No. 85 (1950).

[2] L. CROCCO: NACA Tech. Note 2432 (1951).

45. General principles. The basic equations of steady potential flow are BERNOULLI'S equation

$$\frac{1}{2} q^2 + \int \frac{dp}{\varrho} = \text{const}, \tag{45.1}$$

where $p = f(\varrho)$ is given, and the equation of continuity. Since $\mathbf{v} = \operatorname{grad} \varphi$ the latter can be written in the form $\operatorname{div}(\varrho \operatorname{grad} \varphi) = 0$, or in tensor notation,

$$(\varrho \varphi_{,i})_{,i} = 0, \qquad v^i = \varphi_{,i}. \tag{45.2}$$

If the differentiations in Eq. (45.2) are carried out, the result can be expressed in the form

$$c^2 \nabla^2 \varphi - v^i v^j \varphi_{,ij} = 0. \tag{45.3}$$

This celebrated differential equation is of course valid whether or not the flow is subsonic. For two-dimensional flow it takes the explicit form

$$(c^2 - u^2)\, \varphi_{xx} - 2uv\, \varphi_{xy} + (c^2 - v^2)\, \varphi_{yy} = 0. \tag{45.4}$$

Besides the two forms (45.2) and (45.3) of the potential equation, we have also $p_{v^i v^j}\, \varphi_{,ji} = 0$, this being connected with the fact that φ solves a certain variational problem (Sect. 47, Appendix).

Mathematically, Eq. (45.3) is a second order quasi-linear differential equation for the velocity potential; its type is determined by the nature of the quadratic form $c^2 \xi_i \xi_i - v^i v^j \xi_i \xi_j$ where ξ is an arbitrary real vector. Considering from here on only subsonic flow, this quadratic form is positive definite [$> (c^2 - q^2)\, \xi^2$, in fact], and accordingly Eq. (45.3) is an *elliptic* partial differential equation. Its mathematical treatment is not easy because the coefficients c^2 and $v^i v^j$ are functions of the velocity field. There are, nevertheless, a number of important results.

1. A velocity maximum cannot occur in the interior of the fluid. To see this, let us differentiate Eq. (45.3) with respect to x to obtain

$$c^2 \nabla^2 u - v^i v^j u_{,ij} + D^i u_{,i} = 0, \tag{45.5}$$

where u is the x-component of velocity and the D^i are certain coefficients whose exact form need not be given. For an everywhere subsonic flow Eq. (45.5) is an elliptic differential equation (see above) for u. According to a theorem of E. HOPF[1], non-constant solutions of an elliptic differential equation of the form (45.5) cannot take on interior maximum values. Therefore u does not take on an interior maximum, and, by the same reasoning, neither does v or w. The conclusion is now a consequence of KIRCHHOFF'S reasoning [*8*], § 37.

2. Flow past an obstacle; development at infinity. We consider for definiteness a plane flow, uniform at infinity, impinging on a fixed obstacle and in continuous motion around the obstacle. Let the velocity at infinity be $\mathbf{U}$, directed (for simplicity) along the x-axis. The flow being assumed subsonic, the Mach number M_∞ is less than 1. The asymptotic behavior of the flow as $\mathbf{x} \to \infty$ is given by the following formulas:

$$\left.\begin{aligned} \varphi &= \mathbf{U} \cdot \mathbf{x} + C + (\Gamma/2\pi) \arctan(\beta \tan \theta) + O(r^{-1+\varepsilon}), \\ \mathbf{v} &= \mathbf{U} + \frac{\varrho \Gamma}{2\pi} \frac{(-y, x)}{x^2 + \beta^2 y^2} + O(r^{-2+\varepsilon}). \end{aligned}\right\} \tag{45.6}$$

In these formulas (r, θ) are ordinary polar coordinates in the plane, $\beta = \sqrt{1 - M_\infty^2}$, Γ is the circulation, and ε is a *positive* number which can be taken as near zero

[1] E. HOPF: Sitzgsber. preuß. Akad. Wiss. 147 (1927).

as one cares. If there is an outflow of strength Δ from a source in the finite part of the plane, an additional term $(\Delta/4\pi\beta)\log(x^2+\beta^2 y^2)$ must be added to Eq. (45.6), and a corresponding term to Eq. $(45.6)_2$. A complete expansion for φ can be given when the pressure-density relation is analytic, namely,

$$\left.\begin{aligned}\varphi = \mathbf{U}\cdot\mathbf{x} + (\Delta/2\pi\beta)\log r + (\Gamma/2\pi)\arctan(\beta\tan\theta) + \\ + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{nm}(\theta)\left(\frac{\log r}{r}\right)^n\left(\frac{1}{r}\right)^m,\end{aligned}\right\} \tag{45.7}$$

unless both Δ and Γ are zero, in which case

$$\varphi = \mathbf{U}\cdot\mathbf{x} + \sum_{n=0}^{\infty} a_n(\theta)\, r^{-n}. \tag{45.7'}$$

These series are uniformly and absolutely convergent for large enough r, and can be differentiated term by term at will. The resulting series have similar convergence properties.

The history of these expansions is worth recounting. At an early stage the form of the initial terms was surmised from the fact that, for large r, the equation for φ becomes "nearly"

$$(1 - M_\infty^2)\,\varphi_{xx} + \varphi_{yy} = 0.$$

Making the transformation $x \to \sqrt{1-M_\infty^2}\,\xi$ gives LAPLACE's equation; we are thus led to Eq. (45.6) by analogy with the known expansions for the incompressible case. This treatment, although highly suggestive, is no proof that Eq. (45.6) really holds; moreover, the rest of the expansion remains unknown. BATEMAN[1] was the first to gues the true form of the complete expansion, though he included terms $(\log r^n)/r^p$ with $n > p$ which actually do not occur. Next, BERGMAN[2] obtained a development of φ in terms of the hodograph variables, and LUDFORD[3] performed the calculations necessary to transform BERGMAN's development into physical variables. The result of this work was formula (45.7). One step remained, however, namely, to show that the flow potential actually had the asymptotic behavior supposed by BERGMAN as the basis for his development. This was done by FINN and GILBARG[4], who gave an independent proof of the necessary formula (45.6). The reader will see from the above outline that formula (45.7) involves exceedingly deep considerations, beginning with BERGMAN's theory of singularities of solutions of linear analytic elliptic differential equations, and concluding with the (not simple) demonstration of Eq. (45.6).

One of the major applications of Eq. (45.6) is in the proof of the KUTTA-JOUKOWSKY lift formula for flows of compressible fluids. GILBARG and FINN[5] have recently pointed out that all that is really necessary for this proof is the estimate

$$\mathbf{v} = \mathbf{U} + o(r^{-\frac{1}{2}}). \tag{45.8}$$

Eq. (45.8) is relatively simpler than Eqs. (45.6) or (45.7), and we can outline its proof.

Following GILBARG and FINN, let us make the change of variables $x \to \sqrt{1-M_\infty^2}\,\xi$ in Eq. (45.4), yielding an equation of the form

$$A\,\varphi_{\xi\xi} + 2B\,\varphi_{\xi y} + C\,\varphi_{yy} = 0, \tag{45.9}$$

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[1] H. BATEMAN: Proc. Nat. Acad. Sci. U.S.A. **24**, 246 (1938).

[2] S. BERGMAN: Trans. Amer. Math. Soc. **62**, 452 (1947).

[3] G. LUDFORD: J. Math. Phys. **30**, 117 (1952). An explicit development for flow past a circle is given by I. IMAI, J. Phys. Soc. Japan **8**, 537 (1953).

[4] R. FINN and D. GILBARG: Comm. Pure Appl. Math. **10**, 23 (1957).

[5] R. FINN and D. GILBARG: Trans. Amer. Math. Soc. **88**, 375 (1958).

where $A, C \to 1$ and $B \to 0$ as (ξ, y) tends to infinity. From Eq. (45.9) the velocity "components" $\bar{u} = \varphi_\xi$ and $v = \varphi_y$ satisfy the equations

$$-C v_y = A \bar{u}_\xi + 2B \bar{u}_y, \quad v_\xi = \bar{u}_y.$$

Now, outside a circle Σ of suitably large radius A, B, C differ from their limiting values by less than ε. Hence

$$C(v_\xi \bar{u}_y - v_y \bar{u}_\xi) = A\bar{u}_\xi^2 + 2B\bar{u}_\xi \bar{u}_y + C\bar{u}_y^2 \geqq (1-2\varepsilon)(\bar{u}_\xi^2 + \bar{u}_y^2),$$

outside Σ. Similarly

$$A(v_\xi \bar{u}_y - v_y \bar{u}_\xi) \geqq (1-2\varepsilon)(v_\xi^2 + v_y^2),$$

and so

$$\bar{u}_\xi^2 + \bar{u}_y^2 + v_\xi^2 + v_y^2 \leqq 2K(v_\xi \bar{u}_y - v_y \bar{u}_\xi) \tag{45.10}$$

outside Σ, where $K = (1+\varepsilon)/(1-2\varepsilon)$. From inequality (45.10), which in mathematical terms means that $w = v + i\bar{u}$ is a quasi-conformal mapping, it can be shown that[1]

$$|v|, |\bar{u} - \beta U| \leqq \text{const} \left|\frac{R}{r'}\right|^\mu, \quad r' > 2R,$$

where $\mu = K - \sqrt{K^2 - 1}$, R is the radius of Σ, and $r' = \sqrt{\xi^2 + y^2}$. Returning to the original variables gives

$$|v|, |u - U| \leqq \text{const} \left|\frac{R}{r}\right|^\mu, \quad r > 2R/\beta,$$

and Eq. (45.8) is an immediate consequence provided ε is chosen so that $\mu > \frac{1}{2}$.

3. Flow past an obstacle: Force formulas. It is a remarkable fact that the familiar results for flow of an incompressible fluid,

$$X = 0, \quad Y = -\varrho \Gamma U,$$

hold also for subsonic flow past an obstacle. This may be inferred almost exactly as in the original proofs of the Kutta-Joukowsky theorem [*8*], § 370b. Indeed, from Bernoulli's equation we have

$$\begin{aligned} p &= p_\infty + \left(\frac{dp}{dq}\right)_\infty (q - q_\infty) + \cdots \\ &= p_\infty - \varrho_\infty U(u - U) + o(r^{-1}) \end{aligned}$$

using Eq. (45.8). Hence by the force formula (10.2),

$$X = -\oint (p\cos\theta + \varrho u \mathbf{v}\cdot\mathbf{n})\, ds = -U \oint \varrho \mathbf{v}\cdot\mathbf{n}\, ds + o(1),$$

since $\mathbf{v}\cdot\mathbf{n} = u\cos\theta + v\sin\theta$. Since the outflow integral vanishes, it follows in the usual way that $X = 0$. The treatment of the lift formula is the same. The above proof should be compared with the analogous one in Sect. 23, No. 4.

The d'Alembert paradox for three-dimensional flows has been proved by Finn and Gilbarg[2] using the asymptotic formula $\mathbf{v} = \mathbf{U} + o(r^{-2})$.

46. Existence and uniqueness theorems. As in the previous paragraphs, we consider plane steady flow past a profile. The basic problem is to determine the fluid motion when (1) the conditions in the uniform stream and the circulation are prescribed, or (2) the conditions in the uniform stream are prescribed and the circulation is determined by the Kutta-Joukowsky condition. In both cases a solution should be proved to exist, it should be unique, and a procedure for

[1] R. Finn and J. Serrin: Trans. Amer. Math. Soc. **89**, 1 (1958).

[2] R. Finn and D. Gilbarg: Acta math. **98**, 265 (1957).

computation should be given. All of these problems have been attacked more or less successfully in recent years. We shall outline the results concerning existence and uniqueness, but a discussion of the numerical computation of compressible flows is beyond the scope of this article. The reader interested in this field may consult recent textbooks in gas dynamics (e.g. [*23*], [*25*], [*40*], and [*43*]) from which further references may be obtained.

The best results on the problem of uniqueness have been obtained by FINN and GILBARG[1]. By using the asymptotic formulas (45.6) and certain integral identities somewhat analogous to the kinetic energy formula (23.1) they have proved the following theorem. *A plane subsonic potential flow past a smooth profile is uniquely determined by conditions in the uniform stream and by the circulation.*

A plane subsonic potential flow past a profile with a sharp trailing edge is uniquely determined by conditions in the uniform stream.

These results have the following significance in the theory of dynamical similarity. Suppose two ideal gases having the same ratio of specific heats are in continuous subsonic potential flow past geometrically similar profiles. Suppose also that both flows have the same Mach number M_∞ and the same circulation ratio Γ/U (the last condition can be dropped if the profiles have a sharp trailing edge). Then the flows are dynamically similar.

It would be of some theoretical interest to know if Γ/U is an increasing function of the Mach number in flows past a profile with a sharp trailing edge. The lift coefficient $C_L = Y/\frac{1}{2}\varrho U^2$ would then be an increasing function of M_∞.

The mathematical problem of existence, although extremely difficult, has been largely settled (for plane flows) by the work of FRANKL and KELDYSH[2], SHIFFMAN[3], and BERS[4]. For definiteness the problem is phrased for a *fixed* Bernoulli equation (45.1), a fixed profile, and a fixed velocity direction at ∞; the prescribed conditions are then the Mach number M_∞ and,in case of a smooth profile, the circulation Γ. The basic result is a follows:

For a given smooth profile and a given Bernoulli equation[5] *there is a region in the* (M_∞, Γ) *plane, including the origin, such that for any point in this region there exists a unique subsonic flow past the profile with these values of* M_∞ *and* Γ. *Moreover, as the point* (M_∞, Γ) *approaches the boundary of this region the maximum local Mach number of the corresponding flow approaches 1.*

For a given profile with a sharp trailing edge there is a number $\hat{M}$, *such that for every* M_∞ *in the interval* $0 \leqq M_\infty < \hat{M}$ *there exists a unique subsonic flow past the profile with this speed at infinity. Moreover, as* $M_\infty \to \hat{M}$, *the maximum local Mach number approaches 1.*

The existence proofs are too technical to be more than cursorily described here. FRANKL and KELDYSH use (essentially) an interation procedure, and establish the existence of flows only for "sufficiently small" M_∞. SHIFFMAN's proof uses direct methods in the calculus of variations (starting with the Bateman-

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[1] R. FINN and D. GILBARG: Trans. Amer. Math. Soc. **88**, 375 (1958). Earlier but less elegant attacks on the uniqueness problem are due to M. SHIFFMAN and L. BERS, see footnotes 3 and 4 below.

[2] F. FRANKL and M. KELDYSH: Bull. Acad. Sci. USSR. **12**, 561 (1934).

[3] M. SHIFFMAN: J. Rational Mech. Anal. **1**, 645 (1952).

[4] L. BERS: Comm. Pure Appl. Math. **7**, 441 (1954).

[5] For an ideal gas the condition of a fixed Bernoulli equation can be left out, provided that Γ is replaced by Γ/U. The flow is then unique only up to a similarity transformation, of course.

Kelvin principle), combined with an ingenious device which allows one always to conclude the existence of a minimizing extremal. BERS' proof may be described broadly as being function-theoretic (as in the treatment of plane flow of an incompressible fluid), but a whole arsenal of analysis comes into play. In addition to establishing the existence of subsonic flows, BERS proves their continuous dependence on M_∞, on the shape of the profile, and on the form of the speed-density relation. In spite of the highly difficult mathematical work in these papers, they fill a gap long felt by applied mathematicians and aerodynamicists[1].

The problem of existence and uniqueness of three-dimensional subsonic flows has been studied resently by GILBARG and FINN[2]. Their comprehensive results include a complete solution of the uniqueness problem, an asymptotic expansion of the potential at infinity, and the existence of subsonic flows when the local Mach number does not exceed 0.53. Moreover, a remarkable theorem of NASH[3] may lead to the solution of the existence problem in comparable generality to that for plane flows.

47. Variational principles in gas dynamics. In this section we consider some *maximum* and *minimum* principles in steady subsonic flow. The motivation for these principles is the desire to extend Kelvin's minimum energy theorem to gas dynamics, and also the need for practical methods to compute compressible flows. Our discussion does not include results such as Herivel's theorem (Sect. 15) because these are not true minimization principles. On the other hand, HERIVEL's discovery that $\mathfrak{T} - \mathfrak{E}$ is the appropriate Lagrangian function in Hamilton's principle may serve as a partial motivation for the choice of integrands below.

Consider now a finite region $\mathscr{v}$ with boundary $\mathscr{s}$. Let a normal mass-flux h be prescribed on $\mathscr{s}$ subject to the condition

$$\text{Outflow} = \oint_{\mathscr{s}} h\, d a = 0.$$

We are interest in setting up a variational problem which will characterize irrotational motion by the minimization of some function of the velocity field. For this purpose it is convenient to define for each velocity field a corresponding "density" field. This we do by means of the Bernoulli equation[4]

$$\frac{1}{2} q^2 + \int \frac{dp}{\varrho} = \text{const.} \tag{47.1}$$

The flow $(\varrho, \mathbf{v})$ thus determined will in general satisfy neither the equation of continuity nor the equation of motion. We shall say that $\mathbf{v}$ is subsonic if $q < c = \sqrt{dp/d\varrho}$. With these preliminaries understood we may proceed to the results in question.

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[1] Other theoretical work of an allied nature will be found in the following papers: L. BERS: Comm. Pure Appl. Math. **7**, 79 (1952); D. GILBARG: J. Rational Mech. Anal. **2**, 233 (1953); D. GILBARG and M. SHIFFMAN: J. Rational Mech. Anal. **3**, 209 (1954); C. LOEWNER: J. Rational Mech. Anal. **2**, 537 (1954) and an article on the critical Mach number in Studies in Mathematics and Mechanics, presented to R. VON MISES, New York, 1954; J. SERRIN: J. Math. Phys. **33**, 27 (1954).

[2] D. GILBARG and R. FINN: Acta math. **98**, 265 (1957).

[3] J. NASH: Proc. Nat. Acad. Sci. U.S.A. **43**, 754 (1957).

[4] If one wishes, (47.1) can be considered as a "side condition" for the following variational problems. The constant of integration and the constant on the right hand side of Eq. (47.1) are supposed fixed throughout the ensuing discussion.

The Bateman-Kelvin principle[1]. *Consider the variational problem, to minimize the integral*

$$\mathscr{I}(\mathbf{v}) = \int_{\mathfrak{v}} (p + \varrho q^2)\, dv$$

among all subsonic velocity fields which satisfy the equation of continuity and have prescribed mass-flux h on $\mathfrak{s}$. Then

$$\mathscr{I}(\mathbf{v}) = \text{minimum}$$

if and only if $\mathbf{v}$ *is irrotational.*

It is clear that the minimizing flow field provides a dynamically possible, isentropic, irrotational flow satisfying the prescribed boundary conditions. It is this fact which makes the Bateman principle especially valuable. There is a converse proposition, also due in essence to BATEMAN.

The Bateman-Dirichlet principle[2]. *Consider the variational problem, to maximize the integral*

$$\mathscr{J}(\mathbf{v}) = \int_{\mathfrak{v}} p\, dv + \oint_{\mathfrak{s}} \varphi\, h\, da$$

among all subsonic velocity fields $\mathbf{v} = \operatorname{grad} \varphi$. *Then*

$$\mathscr{J}(\mathbf{v}) = \text{maximum}$$

if and only if $\operatorname{div}(\varrho\, \mathbf{v}) = 0$ *and* $\varrho\, \mathbf{v} \cdot \mathbf{n} = h$ *on s.*

Now an extremal for one of these problems is also an extremal for the other. Moreover, as we shall see later, an extremal is unique if it exists at all. It follows therefore that these problems either possess a common extremal or none at all. In the former case, we see by application of the divergence theorem that

$$\mathscr{I}_{\min} = \mathscr{J}_{\max} = \int_{\mathfrak{v}} (p + \varrho\, q^2)\, dv,$$

(first noticed by LUSH and CHERRY). The possibility that there may be no extremal arises from the fact that there may be no irrotational subsonic flow satisfying the given boundary conditions. SHIFFMAN[3] has shown how to modify the variational problems so that they will always possess an extremal.

In order to prove the Bateman-Kelvin principle, we set $\mathbf{Q} = \varrho \mathbf{v}$ and note from the Bernoulli equation that $\mathbf{v}$, and hence also $\mathscr{I}$, may be considered a function of $\mathbf{Q}$. (There are, of course, two speeds corresponding to each value of Q, see Sect. 37. We naturally choose that one corresponding to subsonic flow.) The advantage of using $\mathbf{Q}$ instead of $\mathbf{v}$ as the fundamental variable is that $\mathbf{Q}$ is subject to simpler conditions than $\mathbf{v}$, namely just

$$\operatorname{div} \mathbf{Q} = 0.$$

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[1] H. BATEMAN: Proc. Nat. Acad. Sci. U.S.A. **16**, 816 (1930). The present formulation as a minimum problem is based on the results of a paper of P.E. LUSH and T.M. CHERRY, Quart. J. Mech. Appl. Math. **9**, 6 (1956). It is unlikely that any of these authors noted the strict analogy with KELVIN's theorem, for they were primarily concerned with the two-dimensional problem. Another formulation of the Bateman principle is given by E. HÖLDER, Math. Nachr. **4**, 366 (1950).

As a motivation for the choice of integrand, we observe the identity $p + \varrho\, q^2 = \frac{1}{2} \varrho\, q^2 - \varrho E + \varrho(\frac{1}{2} q^2 + I) = \frac{1}{2} \varrho\, q^2 - \varrho(E + \text{const})$; thus since E is defined only up to an arbitrary constant, $\mathscr{I} = \mathfrak{T} - \mathfrak{E}$.

[2] H. BATEMAN: Cf. footnote 1. Although BATEMAN discovered the integrand p, the additional surface integral necessary to make a true maximum principle was found only recently by LUSH and CHERRY (footnote 1). The present formulation is slightly more general than that of LUSH and CHERRY.

[3] M. SHIFFMAN: J. Rational Mech. Anal. **1**, 605 (1952).

Now consider a vector field $\mathbf{Q}$ which minimizes $\mathscr{I}$, and let $\mathbf{Q}^* = \mathbf{Q} + \varepsilon\boldsymbol{\zeta}$ be a competing field. Expanding $\mathscr{I}(\mathbf{Q}^*)$ in powers of ε yields

$$\mathscr{I}(\mathbf{Q}^*) = \mathscr{I}(\mathbf{Q}) + \varepsilon\,\delta\mathscr{I} + \varepsilon^2\mathscr{K},$$

where

$$\delta\mathscr{I} = \int_{v} \mathbf{v}\cdot\boldsymbol{\zeta}\,dv,$$

$$\mathscr{K} = \frac{1}{2}\int_{v} \frac{(\tilde{\mathbf{v}}\cdot\boldsymbol{\zeta})^2 + (\tilde{c}^2 - \tilde{q}^2)\,\zeta^2}{\tilde{\varrho}\,(\tilde{c}^2 - \tilde{q}^2)}\,dv;$$

here the tildes denote evaluations at some intermediate Q; the computation of $\delta\mathscr{I}$ and $\mathscr{K}$ is simplified by setting $p + \varrho q^2 = F(Q^2)$ and noting that $F' = (2\varrho)^{-1}$, $F'' = [4\varrho^3(c^2 - q^2)]^{-1}$.

Since $\mathscr{K} \geqq 0$ by the condition of subsonic flow, the extremal $\mathbf{Q}$ is characterized by $\delta\mathscr{I} = 0$. We must show that this condition is equivalent to $\operatorname{curl}\mathbf{v} = 0$.

First, if $\operatorname{curl}\mathbf{v} = 0$, then $\mathbf{v} = \operatorname{grad}\varphi$ and

$$\int_{v} \mathbf{v}\cdot\boldsymbol{\zeta}\,dv = \int_{v} \operatorname{div}(\varphi\boldsymbol{\zeta})\,dv = 0$$

where we have made use of the conditions $\operatorname{div}\boldsymbol{\zeta} = 0$ and $\boldsymbol{\zeta}\cdot\mathbf{n} = 0$ on s. On the other hand, if $\delta\mathscr{I} = 0$ for all admissible variations $\boldsymbol{\zeta}$ then $\mathbf{v}$ must be irrotational. For suppose $\operatorname{curl}\mathbf{v} \neq 0$ at some point P. Then we can find a vector $\mathbf{A}$ which vanishes outside the immediate neighborhood of P and has the property

$$\int_{v} \mathbf{A}\cdot\operatorname{curl}\mathbf{v}\,dv \neq 0.$$

But this implies $\delta\mathscr{I} \neq 0$ for the admissible variation $\boldsymbol{\zeta} = \operatorname{curl}\mathbf{A}$. This contradiction proves that $\operatorname{curl}\mathbf{v} = 0$.

Having proved the Bateman-Kelvin principle, it remains only to note the relation

$$\mathscr{I}(\mathbf{v}^*) = \mathscr{I}(\mathbf{v}) + \mathscr{K}$$

between an extremal flow $\mathbf{v}$ and any competing flow $\mathbf{v}^*$. It follows that $\mathscr{I}(v^*) > \mathscr{I}(\mathbf{v})$ unless $\mathbf{v}^* - \mathbf{v} = 0$, so the extremal is unique. The proof of the Bateman-Dirichlet principle is very similar, but simpler because $\mathscr{J}$ depends only on the single function φ.

In applying the Bateman-Kelvin principle to plane flow it is usual to introduce the stream function as dependent variable. The equation of continuity is then automatically satisfied while the boundary condition reduces to prescribing ψ on s. Similar schemes have been proposed for three-dimensional flows[1], but it is not apparent that they are any improvement over the use of the vector $\mathbf{Q}$ as dependent variable.

As we have already remarked, the above variational principles can be used in the construction of gas flows by actually attempting to minimize or maximize the integrals $\mathscr{I}$ and $\mathscr{J}$. For the important case of flow past a finite body, however, the integrals are divergent. Wang[2], Shiffman[3], and Lush and Cherry have shown that this difficulty is avoided if certain appropriate terms are added to the integrands. The modified integral for the Bateman-Kelvin principle (in the absence of circulation) can be written in the form

$$\int_{v} (p + \varrho q^2 - p_\infty - \varrho\,\mathbf{v}\cdot\mathbf{U})\,dv, \tag{47.2}$$

[1] Lush and Cherry (p. 13). — J. Giese: J. Math. Phys. **30**, 31 (1951).

[2] C. Wang: J. Aeronaut. Sci. **15**, 675 (1948). — Quart. Appl. Math. **9**, 99 (1951).

[3] M. Shiffman: Cf. footnote 3, p. 204.

where $\mathbf{U}$ is the velocity of the impinging uniform stream. It is easy to see that the integral (47.2) is convergent in case $\mathbf{v}-\mathbf{U}=O(r^{-1-\varepsilon})$ in plane flow and $\mathbf{v}-\mathbf{U}=O(r^{-\frac{3}{2}-\varepsilon})$ in threedimensional flow. These conditions are satisfied in particular for an actual motion (cf. Sect. 46, No. 2). Besides its use in the numerical computation of flows[1], minimization of the integral (47.2) is the starting point in SHIFFMAN's proof of the existence of subsonic flows past a prescribed obstacle.

Other variational principles in gas dynamics. The Bernoulli equation (47.1) may be modified by assuming $p=f(\varrho, S)$ where S is a given function of ψ. The Bateman-Kelvin principle remains valid, but the extremal flow is rotational and isoenergetic[2]. On the other hand, if Eq. (47.2) is modified to

$$\frac{1}{2}q^2+\int\frac{dp}{\varrho}=H(\psi),$$

then LUSH and CHERRY assert that the extremal gives rotational isentropic motion.

Finally, it is interesting to consider the variational principle arising from the integrand $\frac{1}{2}\varrho q^2-\varrho E$, where E is the internal energy, and ϱ and $\mathbf{v}$ are to be varied independently. With the side condition $\operatorname{div}(\varrho\,\mathbf{v})=0$ it can be shown that an extremal is irrotational, though it will not necessarily supply a minimum.

Appendix. The problems of maximizing $\mathscr{J}(\varphi)$, and in plane flow of minimizing $\mathscr{I}(\psi)$, are special cases of a variational problem which has been intensely studied, namely

$$\mathfrak{F}(\varphi)=\iint F(u,v)\,dx\,dy=\text{Min}, \tag{47.3}$$

where $u=\varphi_x$, $v=\varphi_y$. The Euler equation for the problem (47.3) is

$$(F_u)_x+(F_v)_y=0, \tag{47.4}$$

or

$$F_{uu}\varphi_{xx}+2F_{uv}\varphi_{xy}+F_{vv}\varphi_{yy}=0. \tag{47.5}$$

This is an elliptic equation provided $F_{uu}F_{vv}-F_{uv}^2>0$. The condition of ellipticity appears in another guise in the formula

$$\mathfrak{F}(\varphi+\zeta)=\mathfrak{F}(\varphi)+\tfrac{1}{2}\iint(\tilde{F}_{uu}\zeta_k^2+2\tilde{F}_{uv}\zeta_k\zeta_y+\tilde{F}_{vv}\zeta_y^2)\,dx\,dy,$$

which expresses the integral of an arbitrary function in terms of the integral of an extremal φ. The condition $F_{uu}F_{vv}-F_{uv}^2>0$ implies that $\mathfrak{F}(\varphi+\zeta)\geqq\mathfrak{F}(\varphi)$, thus showing that the extremal provides a unique minimum.

In the application to fluid dynamics, we have $F(u,v)=-p$, whence the potential φ satisfies the equation

$$p_{uu}\varphi_{xx}+2p_{uv}\varphi_{xy}+p_{vv}\varphi_{yy}=0. \tag{47.6}$$

A similar equations holds for the stream function, as the reader will easily see.

V. Supersonic flow and characteristics.

The theory of unsteady flow and supersonic flow of a compressible fluid is based largely on the existence of characteristic curves or surfaces in the flow field. Indeed, our insight into the behavior of gases at supersonic speeds is due largely to our understanding of the characteristic field, and most of the available solutions have been obtained from this knowledge.

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[1] See the interesting paper of H. GISPERT: Wiss. Z. Univ. Halle **6**, 14 (1956/57).

[2] C.C. LIN: Quart. Appl. Math. **9**, 421 (1952).

48. The nature of characteristics. Consider for the moment a general flow of a compressible fluid. Let Σ be a 3-dimensional manifold in the four-dimensional (x, t)-space occupied by the flow, or, in less abstract terms, let Σ be a *moving* surface in the three-dimensional physical space. Suppose now that the values of the flow variables $\mathbf{v}$, ϱ, p, and S are known on Σ. Then by means of Eqs. (35.1) to (35.4) we can, in general, uniquely determine their derivatives at points of Σ. The process is a familiar one: the derivatives tangential to Σ are known because the flow variables on Σ are known, while the remaining *normal* derivatives can be obtained by considering Eqs. (35.1) to (35.4) at points of Σ as a system of linear equations with these derivatives as the unknowns. It may happen, however, that Eqs. (35.1) to (35.4) do not thus determine the normal derivatives, and in this case Σ is called a characteristic manifold. The physical significance of a characteristic manifold lies in the fact that it is only along such a manifold that two solutions may be "tangent", or in other words, discontinuities in the derivative of a solution can only appear on characteristic manifolds. It may also be remarked that characteristic manifolds play an important role in the propagation of disturbances in a flow field, as is apparent from the above discussion.

In the following two sections we shall consider the special case of steady flow[1]. The characteristic manifolds then do not involve the variable t, and hence are curves in plane flow and surfaces in spatial flow. During the discussion it will become apparent that characteristics *do not* occur in steady subsonic flow, streamlines of the motion being excepted. Thus until Sect. 51 our considerations will be directed mainly to supersonic flows. An alternative treatment of characteristic manifolds is given in Sect. 51, serving to complement the present discussion.

The elegant theory of characteristic curves in one-dimensional unsteady flows is not covered in this article, since a number of excellent treatments are available.

49. Steady plane flow. Let C be a curve in a region of steady plane flow, defined parametrically by the equation $\mathbf{x} = \mathbf{x}(\sigma)$ where $\mathbf{x} = (x, y)$ and σ is arc-length on C. The values of the flow variables on C may be expressed in the form

$$\mathbf{v} = \mathbf{v}(\sigma), \quad p = p(\sigma), \quad \varrho = \varrho(\sigma), \quad S = S(\sigma). \tag{49.1}$$

We ask, under what conditions on C will the functions (49.1) not determine the derivatives of $\mathbf{v}, p, \varrho$, and S on C?

We have the following set of conditions for the determination of these derivatives

$$\mathbf{v} \cdot \operatorname{grad} \varrho + \varrho \operatorname{div} \mathbf{v} = 0,$$
$$\varrho \mathbf{v} \cdot \operatorname{grad} \mathbf{v} + \operatorname{grad} p = 0,$$
$$\mathbf{v} \cdot \operatorname{grad} S = 0,$$

where $p = f(\varrho, S)$ is the given equation of state. These equations can easily be reduced to the form

$$\left.\begin{aligned} \varrho \mathbf{v} \cdot \operatorname{grad} \mathbf{v} + \operatorname{grad} p &= 0, \\ \varrho c^2 \operatorname{div} \mathbf{v} + \mathbf{v} \cdot \operatorname{grad} p &= 0; \end{aligned}\right\} \tag{49.2}$$

it is this set of equations which we shall actually use. A simple artifice now suggests itself for determining the derivatives of $\mathbf{v}$ and p on C. We orient the

[1] Characteristic manifolds for a general three-dimensional nonsteady flow are treated in [*21*], pp. 112—116. This book also contains a useful discussion of characteristic manifolds for an arbitrary system of first order partial differential equations (pp. 103—112).

coordinates so that the y-axis is tangential to C at some point P; then the y-derivatives will be known at P, while it is only the x-derivatives which must be found. For these we have the system of linear equations

$$\left.\begin{aligned} \varrho\, u\, u_x \qquad\qquad + p_x &= -\varrho\, v\, u_y, \\ \varrho\, u\, v_x \qquad &= -\varrho\, v\, v_y - p_y, \\ \varrho\, c^2 u_x \qquad + u\, p_x &= -\varrho\, c^2 v_y - v\, p_y. \end{aligned}\right\} \tag{49.3}$$

The condition that C be a characteristic is that these equations *not* determine u_x, v_x and p_x.

Equating to zero the determinant of the coefficients of Eqs. (49.3) yields

$$\varrho^2 u (u^2 - c^2) = 0.$$

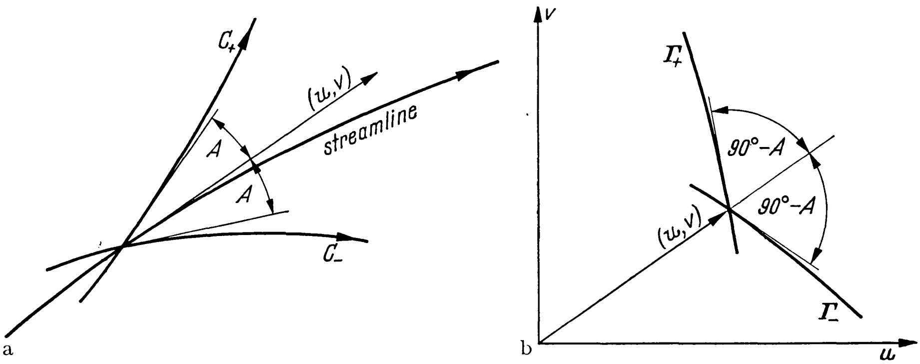

Fig. 7a and b. (a) Characteristic curves in the physical plane. (b) The corresponding curves in the hodograph plane.

That is, *a curve C is a characteristic if and only if at each point on C either (1) the normal velocity component is zero, or (2) the normal velocity component has magnitude equal to the local sound speed.* Characteristics of the first kind are obviously nothing more than the streamlines of the motion. The occurence of streamlines in this role is certainly not surprising and is relatively unimportant in the theory.

Characteristic curves satisfying the second condition evidently occur only in supersonic flow. In a fixed coordinate system (x, y) let us introduce the two direction fields

$$\frac{dy}{dx} = \tan(\vartheta + A), \qquad \frac{dy}{dx} = \tan(\vartheta - A), \tag{49.4}$$

where ϑ is the velocity inclination and A is the (local) *Mach angle,*

$$\sin A = \frac{c}{q} = \frac{1}{M}.$$

Then the curves C_+ satisfying the first of Eqs. (49.4), the curves C_- satisfying the second of Eqs. (49.4), together with the streamlines form the set of characteristics of two-dimensional steady flow. There are three characteristics, therefore, passing through each point of a supersonic flow region; we note that the C_+ and C_- characteristics are bisected by the streamline (Fig. 7a). One may interpret the $C_\pm$ characteristics as the paths of infinitesimal (steady-state) disturbances in the flow; equally important, however, is the interpretation of these characteristics as curves along which two solutions may be "tangent".

The flow variables cannot be entirely arbitrary along a characteristic curve. Indeed, on the streamline characteristics we have the evident conditions

$$S = \text{const}, \quad H = \text{const},$$

expressing constancy of entropy and "energy". On a Mach line (i.e., a $C_\pm$ characteristic) the vanishing of the determinant of coefficients in Eq. (49.3) similarly places a compatability condition on the right hand members, namely

$$\begin{vmatrix} -\varrho v u_y & & 1 \\ -\varrho v v_y - p_y & \varrho u & \\ -\varrho c^2 v_y - v p_y & & u \end{vmatrix} = 0,$$

or, after cancellation of a non-zero factor,

$$\varrho c^2 v_y - \varrho u v u_y + v p_y = 0. \tag{49.5}$$

This equation is of course expressed in the specially oriented coordinate system which was introduced at the beginning of the discussion. Using now the condition that $|u| = c$ on a Mach line, we may write Eq. (49.5) in the form

$$\varrho (u v_y - v u_y) \pm \cot A \, p_y = 0, \tag{49.6}$$

where the $+$ and $-$ signs refer respectively to the C_+ and C_- characteristics. Eq. (49.6) may in turn be written

$$\varrho q^2 \dot{\vartheta} \pm \cot A \, \dot{p} = 0, \tag{49.7}$$

the dot denoting differentiation with respect to "y" at a point on the characteristic, or with respect to σ along the length of the characteristic.

In irrotational isentropic flow the streamlines no longer appear as characteristics (the conditions $H \equiv$ const, $S \equiv$ const give additional information sufficient to determine flow derivatives on a streamline). Moreover, using the Bernoulli equation (37.3) we can eliminate p from Eq. (49.7), thus obtaining the important relation

$$\frac{d\vartheta}{dq} = \pm \frac{\cot A}{q} = \pm \frac{\sqrt{M^2 - 1}}{q}, \tag{49.8}$$

holding along a Mach line. The right hand side is of course a function of q alone. Consequently along any characteristic in the physical plane, we have the following simple relation between the speed and flow angle:

$$\vartheta = \pm \int \frac{\sqrt{M^2 - 1}}{q} \, dq = \pm \imath(q) + \text{const}. \tag{49.9}$$

Otherwise expressed, *the image of a $C_\pm$ characteristic in the hodograph plane belongs to a one-parameter family of fixed curves.* It is customary to call the image of a C_+ characteristic a Γ_+ curve, and the image of a C_- characteristic a Γ_- curve. (Since the $\Gamma_\pm$ curves are characteristics of the hodograph equations (43.2), they are also called *hodograph characteristics.*)

A considerable amount of information concerning the geometry of the $\Gamma_\pm$ curves can be deduced. To begin with, we observe that the function $\imath(q)$ is monotonically increasing for $q_* \leqq q < q_{\max}$. Moreover, for $q = q_*$ we have $M = 1$ and $d\vartheta/dq = 0$, while for $q = q_{\max}$ we have $M = \infty$ and $d\vartheta/dq = \infty$. The curvature of the Γ_+ curves is given by

$$\frac{q^2 + 2q'^2 - q q''}{(q^2 + q'^2)^{\frac{3}{2}}} = \frac{a}{c\sqrt{M^2 - 1}}, \qquad ' = \frac{d}{d\vartheta},$$

see Eq. (37.7), and is therefore always positive. Consequently, *the Γ_+ curves in the (u, v)-plane are counter-clockwise spirals between the fixed circles $q = q_*$ and*

$q = q_{\max}$; *all of them can be obtained by rotating any one about the origin* (Fig. 8). A similar result obviously holds for the Γ_- curves. Finally by elementary geometry Eq. (49.8) is seen to be equivalent to

$$\frac{dv}{du} = -\cot(\vartheta \mp A). \tag{49.10}$$

From Eqs. (49.4) and (49.10) it follows that *the directions of the C-characteristics of one kind are perpendicular to the Γ-characteristics of the other kind* (Fig. 7). The above results are valid for any gas, irrespective of its equation of state.

It can be shown that a discontinuity in grad $\mathbf{v}$ which appears across a characteristic in isentropic, irrotational flow will satisfy a Riccati equation along that characteristic[1]. It follows that such discontinuities are uniquely determined and are not zero along a whole characteristic if they are known to be different from zero at any point of the characteristic. It should be pointed out that these remarks on the propagation of discontinuities in grad $\mathbf{v}$ do not apply to discontinuities in $\mathbf{v}$ itself. Discontinuities in the functions themselves are propagated as "shocks" in quite a different manner (Part F).

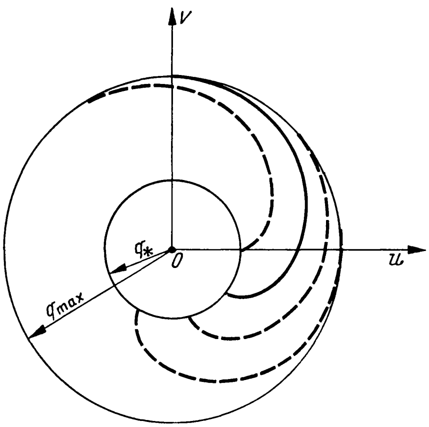

Fig. 8. Γ_+ curves in the hodograph plane.

In the special case of an ideal gas with constant specific heats, the Γ-characteristics are epicycloids generated by a circle of radius $\frac{1}{2}(q_{\max} - q_*)$ rolling on the circle $q = q_*$. One advantage of an ideal gas is that a single characteristic diagram will serve, whatever the reference state may be. For a general gas, on the contrary, the hodograph characteristics are essentially different for different entropy and energy levels of the reference state. This is no particular disadvantage theoretically, but for the purposes of numerical computation it is a decisive reason for employing the ideal gas law.

Axially-symmetric flow. The procedure applied above to deduce the characteristic equations for plane flow can be used similarly for axially-symmetric flow. Because of the changed form of div $\mathbf{v}$ we find that in place of Eqs. (49.3) there arises the slightly different set of equations,

$$\begin{aligned} \varrho u u_x \qquad\quad + p_x &= -\varrho v u_y, \\ \varrho u v_x \qquad\quad &= -\varrho v v_y - p_y, \\ \varrho c^2 u_x \qquad + u p_x &= -\varrho c^2 v_y - v p_y - \varrho q c^2 \frac{\sin\vartheta}{y}, \end{aligned}$$

where ϑ is the inclination of the velocity vector to the axis of revolution. It follows that the characteristic curves in the flow half-plane are given exactly as before. The characteristic condition (49.7) is, however, replaced by

$$\dot{\vartheta} \pm \left(\frac{\cot A}{\varrho q^2}\dot{p} + \frac{\sin A \sin\vartheta}{y}\right) = 0, \tag{49.11}$$

as is seen by following the preceding calculation.

[1] Joh. Nitsche: J. Rational Mech. Anal. 2, 291 (1953).

In isentropic irrotational flow Eq. (49.11) may be written

$$\frac{d}{d\sigma}(\imath \mp \vartheta) = \frac{\sin A \sin \vartheta}{y}. \tag{49.12}$$

The hodograph image of a Mach line is therefore not one of a fixed family of curves as in two-dimensional flow. This fact contributes to the difficulty of any exact treatment of axially-symmetric flow.

50. Steady irrotational flow in three dimensions. According to theorem 2 of Sect. 39, there is no loss of generality in assuming the motion to be isentropic and isoenergetic. Bernoulli's theorem therfore holds in its strong form (37.2).

The velocity potential φ of an irrotational flow in space satisfies the equation

$$c^2 \nabla^2 \varphi - v^i v^j \varphi_{,ij} = 0, \tag{50.1}$$

(see Sect. 45). In supersonic flow $q > c$, and Eq. (50.1) may be identified as a quasi-linear hyperbolic partial differential equation of the second order. The theory of such equations is well-known[1], and we shall merely recount here a few of the salient features.

A surface Σ is a characteristic manifold of Eq. (50.1) if and only if at each point of Σ,

$$\mathbf{n} \cdot \mathbf{A} \cdot \mathbf{n} = 0, \tag{50.2}$$

where $\mathbf{n}$ denotes the unit normal vector to Σ, and $A^{ij} = c^2 \delta^{ij} - v^i v^j$ is the matrix of coefficients of Eq. (50.1). We observe that Eq. (50.2) can be written in the simpler form

$$\mathbf{n} \cdot \mathbf{v} = \pm c, \tag{50.3}$$

i.e., the component of $\mathbf{v}$ normal to Σ has magnitude equal to the local speed of sound. Let P be any point in the (supersonic) flow region. We define the *Mach cone at* P to be the right circular cone having P as its vertex, the velocity vector $\mathbf{v}$ as its axis, and the Mach angle $A = \arcsin c/q$ as its angular opening. Then it is easily seen that Eq. (50.3) is equivalent to the condition that at each point P of Σ, Σ should be tangent to the Mach cone at P.

Consider the line element, or direction, on the surface Σ at P along which the Mach cone is tangent to Σ[2]. We call this the *characteristic direction* of Σ at P. The totality of characteristic directions on a characteristic manifold defines a direction field, the trajectories of which are called *characteristic rays.* Now according to the theory of characteristic surfaces, the characteristic rays may be obtained by solving the system of equations

$$\frac{dx^i}{d\sigma} = A^{ij} n_j, \qquad \frac{dn_i}{d\sigma} = -\frac{1}{2} \frac{\partial A^{jk}}{\partial x^i} n_j n_k, \tag{50.4}$$

subject to the initial conditions that $\mathbf{x}(0)$ is a point P on Σ, and $\mathbf{n}(0)$ is the normal to Σ at P. From this it is apparent that a characteristic ray is determined completely by its initial *surface element,* or, in other words, *two characteristic manifolds which are tangent at a point P are tangent on the entire characteristic ray through P.* This shows that characteristic rays play the same role as the characteristic curves of plane steady flow, in the propagation of infinitesimal disturbances.

A *strip* $\mathbf{x}(\sigma)$, $\mathbf{n}(\sigma)$ satisfying Eqs. (50.2) and (50.4) is called a *bicharacteristic.* From what has just been said, it is seen that all possible characteristic manifolds

[1] [*45*], Chap. 6.

[2] This line element can be obtained by projecting $\mathbf{v}$ onto Σ.

may be obtained by "piecing together" appropriate bicharacteristics. This can perhaps be seen more clearly if we introduce the *characteristic conoid*, the locus of all bicharacteristics passing through a given point. Then the effect of a disturbance emanating from an arbitrary point set is confined to the envelope of the characteristic conoids whose vertices range over that set of points. This is, of course, nothing more than the Huyghens construction for wave fronts.

A method of characteristics for dealing with certain problems of three-dimensional supersonic flow, somewhat analogous to the methods of characteristics in plane flow, has been prepared by COBURN and DOLPH[1]. More recently, HOLT[2] and COBURN[3] have considered the problem of characteristic manifolds for steady *rotational* supersonic motion. In this work, the equations of motion are expressed in intrinsic form with respect to differentiations along characteristic directions, much as in Sect. 49. COBURN has used his equations to determine some new exact spatial flows.

51. Singular surfaces and sound waves. The preceding work in this article has been concerned primarily with "continuous motion", i.e. twice differentiable velocity fields. Consider now a surface $\Sigma = \Sigma(t)$ in the flow region, such that the flow variables themselves are continuous across Σ, but certain derivatives of these quantities are discontinuous there; (it is assumed that on either side of Σ the flow variables are continuously differentiable and that the derivatives have definite limiting values on the two sides of Σ). Such a surface is called a *singular suface on order one*, but for brevity will be referred to here simply as a singular surface. It is the purpose of this section to determine the nature of singular surfaces and their laws of propagation in a general three-dimensional unsteady flow. One sees easily that a singular surface must be a characteristic manifold (Sect. 48), and we could therefore study these surfaces from that point of view. On the other hand, a direct attack has considerable appeal and leads quickly and elegantly to the required results.

We suppose that Σ has the equation $F(\mathbf{x}, t) = 0$, with the function F satisfying $|\operatorname{grad} F| > 0$. The normal vector $\mathbf{n}$ to Σ is given by

$$\mathbf{n} = \frac{\operatorname{grad} F}{|\operatorname{grad} F|},$$

and the *normal speed of advance* of Σ is

$$G = -\frac{\partial F/\partial t}{|\operatorname{grad} F|}.$$

Suppose now that a flow variable $f = f(\mathbf{x}, t)$ is continuous across Σ, but that at least one component of $\operatorname{grad} f$ is discontinuous there. We shall show that there is a multiplier $\alpha \neq 0$, defined over Σ and having the property that

$$[\operatorname{grad} f] = \alpha\,\mathbf{n}, \qquad \left[\frac{\partial f}{\partial t}\right] = -\alpha\, G, \tag{51.1}$$

where the square brackets about a quantity denote its *jump* across Σ (that is, the difference of its limiting values on either side of Σ)[4]. To prove (51.1), observe that for any direction $d\mathbf{x}$, dt lying on Σ we have

$$\left[d\mathbf{x}\cdot\operatorname{grad} f + \left(\frac{\partial f}{\partial t}\right) dt\right] = 0, \tag{51.2}$$

[1] N. COBURN and C. DOLPH [*36*], p. 55.
[2] M. HOLT: J. Fluid Mech. **1**, 409 (1956).
[3] N. COBURN: Quart. Appl. Math. **15**, 237 (1957).
[4] This theorem is due to J. C. MAXWELL: Electricity and Magnetism. Oxford 1881. Sect. 78a.

since f, and hence its surface gradient, is continuous across Σ. In other words, Eq. (51.2) holds for all $d\mathbf{x}$, dt such that

$$d\mathbf{x}\cdot\operatorname{grad}F+\left(\frac{\partial F}{\partial t}\right)dt=0.$$

It follows at once that

$$\frac{[\operatorname{grad}f]}{\operatorname{grad}F}=\frac{[\partial f/\partial t]}{\partial F/\partial t},$$

and setting this last ratio equal to $\alpha\,|\operatorname{grad}F|^{-1}$, we obtain Eq. (51.1). By virtue of Eq. (51.1),

$$\left[\frac{df}{dt}\right]=-\alpha G+\mathbf{v}\cdot\alpha\,\mathbf{n}=-\alpha\Pi$$

where $\Pi=G-\mathbf{v}\cdot\mathbf{n}$ is the speed of advance of Σ relative to the particles instantaneously situated on it.

Now on Σ, by assumption, certain derivatives of the flow variables p, ϱ, S, and $\mathbf{v}$ are discontinuous. The preceding argument therefore applies to these variables; that is, there exist multipliers α, β, γ and $\boldsymbol{\delta}$, defined over Σ and not all zero, such that

$$\begin{aligned}
[\operatorname{grad}p]&=\alpha\,\mathbf{n}, & \left[\frac{dp}{dt}\right]&=-\alpha\Pi,\\
[\operatorname{grad}\varrho]&=\beta\,\mathbf{n}, & \left[\frac{d\varrho}{dt}\right]&=-\beta\Pi,\\
[\operatorname{grad}S]&=\gamma\,\mathbf{n}, & \left[\frac{dS}{dt}\right]&=-\gamma\Pi,\\
[\operatorname{grad}\mathbf{v}]&=\boldsymbol{\delta}\,\mathbf{n}, & \left[\frac{d\mathbf{v}}{dt}\right]&=-\boldsymbol{\delta}\Pi.
\end{aligned}$$

Now from the Eqs. (35.1) to (35.4), which hold on *either side* of Σ, we find by subtraction

$$\left.\begin{aligned}
-\beta\Pi+\varrho\,\boldsymbol{\delta}\cdot\mathbf{n}&=0,\\
-\varrho\,\boldsymbol{\delta}\Pi+\alpha\,\mathbf{n}&=0,\\
\gamma\Pi=0,\quad \alpha&=c^2\beta+B\gamma.
\end{aligned}\right\}\tag{51.3}$$

Here $c^2=(\partial p/\partial\varrho)_S$ and $B=(\partial p/\partial S)_\varrho$ are thermodynamics variables. In treating Eqs. (51.3) it is convenient and useful to consider separately the two cases $\alpha\neq 0$ and $\alpha=0$.

A singular surface on which $\alpha\neq 0$ is called a *sound wave* since it carries a discontinuity in the pressure gradient. The relative speed of advance Π of a sound wave is then by definition the *speed of sound* (of course, Π will vary from point to point over Σ, so that it is more accurate to call Π the *local speed of sound*). We shall show that the speed of sound, as thus defined, has exactly the value c. In fact, under the assumption $\alpha\neq 0$ it follows from Eq. $(51.3)_2$ that $\Pi\neq 0$, and then from Eq. $(51.3)_3$ that $\gamma=0$. Now by forming the scalar product of Eq. $(51.3)_2$ with $\mathbf{n}$ and using Eq. $(51.3)_4$ there results

$$-\varrho\Pi\boldsymbol{\delta}\cdot\mathbf{n}+c^2\beta=0.\tag{51.4}$$

Elimination of $\varrho\,\boldsymbol{\delta}\cdot\mathbf{n}$ between Eqs. (51.4) and $(51.3)_1$ gives finally

$$\Pi=\pm c=\pm\sqrt{\left(\frac{\partial p}{\partial\varrho}\right)_S},\tag{51.5}$$

as asserted. This derivation of the speed of sound is due, in essence, to HUGONIOT[1]. It is relatively simple, mathematically rigorous, rests upon an unequivocal

[1] H. HUGONIOT: C. R. Acad. Sci., Paris **101**, 1118, 1229 (1885). — J. Math. Pures Appl. (4) **3**, 477 (1887); **4**, 153 (1888).

definition of the speed of sound, and applies to an entirely arbitrary motion of a perfect fluid. The earlier derivation of the speed of sound (Sect. 35) satisfies, at best, only the first of these criteria. We should point out, however, that even the present method is open to the criticism that it applies only to a perfect fluid. An alternate approach to the speed of sound, to some extent avoiding this criticism, will be given in Sect. 57.

There are several other points that should be made concerning sound waves. Since $\Pi = G - \mathbf{v}\cdot\mathbf{n}$ we have

$$|\mathbf{v}\cdot\mathbf{n} - G| = c \tag{51.6}$$

that is, the component of velocity normal to Σ, less the speed of advance of Σ, has magnitude equal to the local speed of sound c. This generalizes a result found earlier in connection with characteristic manifolds in steady flows. Also, across a sound wave we have $\gamma = 0$ and $[\boldsymbol{\omega}] = \boldsymbol{\delta}\times\mathbf{n} = 0$; *the entropy is continuously differentiable and the vorticity continuous across a sound wave.* Similarly, in view of Eq. $(51.3)_2$ *the discontinuity of the acceleration has no longitudinal component,* i.e.

$$[\mathbf{a}] = -\boldsymbol{\delta}\,\Pi = \frac{\alpha}{\varrho}\,\mathbf{n}.$$

Table 3. *Fundamental differences between the two types of singular surfaces of order one.*

Type	Sound wave	Material wave
Speed of propagation	$c = \sqrt{\left(\frac{\partial p}{\partial \varrho}\right)_s}$	0
Conserved quantities	$\mathbf{a}_t$ $\boldsymbol{\omega}$ grad S	$\mathbf{a}$ Θ, ω_n grad p

Consider next a singular surface on which $\alpha = 0$. From $(51.3)_2$ it follows that $\Pi = 0$ (otherwise $\alpha = \beta = \gamma = \boldsymbol{\delta} = 0$), and since $dF/dt = -\Pi\,|\mathrm{grad}\,F|$, we have by the theorem of Sect. 8 that Σ is a material surface. The pressure is continuously differentiable and the expansion and acceleration are continuous across such a singular surface.

For an incompressible fluid, Eqs. $(51.3)_1$ and $(51.3)_2$ remain valid, provided we set $\beta = 0$. Forming the scalar product of Eq. $(51.3)_2$ with $\boldsymbol{n}$ and using Eq. $(51.3)_1$ gives $\alpha = 0$, whence also $\Pi = 0$. This shows that *the only possible singular surfaces in an incompressible fluid motion are material surfaces.*

If a singular surface is considered as a fixed manifold in the four-dimensional $(\mathbf{x}, t)$-space, condition (51.6) may be interpreted by the following geometric criterion: *at each point P of a sound wave, the wave is tangent to the cone*

$$|\mathbf{x} - \mathbf{v}_P\,t|^2 = c_P^2\,t^2, \tag{51.7}$$

along the line (characteristic ray)

$$\mathbf{x} = (\mathbf{v}_P \pm c_P\,\mathbf{n})\,t. \tag{51.8}$$

The coordinates are chosen so that P is the origin. Formulae (51.7) and (51.8) admit an immediate interpretation in terms of the propagation of a sound wave in space (HUYGHEN's principle). In the same manner, *at every point of a material singular surface, the surface is tangent to the line*

$$\mathbf{x} = \mathbf{v}_P\,t. \tag{51.9}$$

We have seen that every singular surface is a characteristic manifold. The converse is certainly not true, but at least the geometric conditions defining a singular surface must also apply to a characteristic manifold. Thus, by the remarks of the preceding paragraph, there are two kinds of characteristic manifolds in a general flow of a compressible fluid: specifically, *a manifold Σ is a characteristic manifold if and only if at each point P it is tangent* *either to the cone (51.7) or to the line (51.9).* This result can also be obtained directly from the definition of a characteristic manifold. At the present time there are still some untouched problems in the theory of characteristic manifolds. The theory of bicharacteristics has been treated only sketchily, nor is it known what equation governs the propagation of discontinuities along characteristic rays.

VI. Special topics.

52. Transonic flow. A gas flow is called transonic if the motion is partly subsonic and partly supersonic. A number of transonic flow problems have been treated in the literature in recent years, but limitations of space make it impossible to deal here with more than one aspect of the situation. The problem chosen for discussion is of considerable physical interest, and also exhibits a remarkable mathematical behavior which has provoked much comment.

Consider the steady plane flow of a perfect gas past a fixed profile, the flow being uniform at infinite distances. We have noted the existence of a unique everywhere subsonic flow past the profile when the Mach number M_∞ lies in a certain range $0 \leqq M_\infty < \tilde{M}$; moreover, as M_∞ approaches $\tilde{M}$ the maximum local Mach number in the flow approaches 1. Now when M_∞ increases beyond $\tilde{M}$ it is observed experimentally that local supersonic zones develop on the sides of the profile, and, after further increase of M_∞, shock waves occur in the supersonic zones. The Mach number M_{sw} at which shock waves first appear is not too well-defined, but at all events satisfies $\tilde{M} < M_{sw} < 1$.

Now by using the hodograph method it is possible to construct exact transonic flows past profiles[1]. Although this result is of considerable importance, it does not guarantee that a solution exists for an arbitrary profile shape; furthermore no hodograph solution has ever provided a continuous transition from subsonic to transonic flow for a *given fixed* profile. These problems might be expected to yield to further analysis but for the difficulty that once a continuous transonic flow past a fixed profile is set up no mechanism is known which will explain the eventual breakdown of the flow as M_∞ is increased[2], the attractive limiting line hypothesis having been shown false[3]. A reason for these apparent anomalies may be gained from the following important theorem of NIKOLSKY and TAGANOV:

If in a continuous potential flow there is a local supersonic zone adjacent to an arc of the flow boundary, this arc must be strictly convex[4].

Leaving the proof until later, we see plainly from this result that, should a transonic flow exist, it could always be destroyed by the slightest variation of the profile, namely any variation putting a straight or concave arc into the supersonic region[5]. This result leads naturally to the following assertion, made in-

[1] M. J. LIGHTHILL [35], p. 251. — T.M. CHERRY: Phil. Trans. Roy. Soc. Lond., Ser. A **245**, 583 (1953).

[2] This statement refers only to breakdown mechanisms lying in the domain of perfect fluid theory; when viscosity is taken into account the situation may be somewhat different, see below.

[3] K.O. FRIEDRICHS: Comm. Pure Appl. Math. **1**, 287 (1948); a simpler proof is due to I. KOLODNER and C.S. MORAWETZ: Comm. Pure Appl. Math. **6**, 97 (1953); cf. also A. MANWELL: Quart. Appl. Math. **12**, 343; **13**, 337 (1955).

[4] A.A. NIKOLSKY and G.I. TAGANOV: Prikl. Mat. Mek., USSR. **10**, 481 (1946), [English translation, NACA Tech. Mem. 1213 (1949)].

[5] Using a method somewhat similar to that of NIKOLSKY and TAGANOV, H. JOHNSON (Masters thesis, U. of Minn.) has shown that an axially-symmetric transonic flow is likewise unstable to small variations of the boundary.

dependently by FRANKL, GUDERLEY, and BUSEMANN[1], and finally proved in 1957 by C. S. MORAWETZ[2]: *the problem of continuous transonic flow past a fixed profile is badly set in the theory of perfect fluids.*

The explanation for the observed phenomena seems therefore to involve the effect of viscosity in a boundary layer. Presumably the edge of the boundary layer makes the adjustments necessary to accommodate a non-viscous flow outside the boundary layer; as M_∞ increases, the edge of the boundary layer develops a very large curvature at some point and a shock line enters the flow. This surmise is supported by the result of Friedrichs quoted earlier. LIN has conjectured that the transonic flow problem can be solved for *analytic* convex profiles; if this is the case then the boundary layer argument just given provides a sort of existence theorem for transonic flows past a profile.

According to the discussion of the preceding paragraph we may expect continuous inviscid transonic flow to exist *outside* the boundary layer. Some information on the possible location of local supersonic zones can then be inferred from the Nikolsky-Taganov theorem: for example, the first appearance of sonic speed and the first appearance of a shock line must occur on convex portions of the profile boundary. The theorem applies also to local supersonic zones in a plane nozzle, and can be used to determine the location of the sonic point in transonic flow past a wedge.

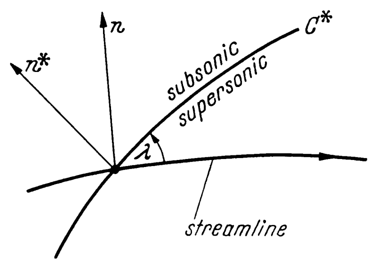

Fig. 9. Geometry of the sonic line C^*.

The proof of the Nikolsky-Taganov theorem follows in three parts.

1. Let C^* be a sonic arc in the flow region, separating a zone of subsonic flow from a zone of supersonic flow. Then the inclination of the velocity vector decreases monotonically as C^* is traversed so that the supersonic zone is to the right (Fig. 9).

For let s and s^* denote, respectively, arc length on the streamlines and on C^*. Then (see Fig. 9)

$$\frac{\partial\vartheta}{\partial s^*} = \frac{\partial\vartheta}{\partial s}\cos\lambda + \frac{\partial\vartheta}{\partial n}\sin\lambda. \tag{52.1}$$

Now on C^*, $M=1$ and $q=q_*=$ const. Therefore by Eq. (52.1) and the intrinsic Eqs. (41.4),

$$\frac{\partial\vartheta}{\partial s^*} = \frac{1}{q_*}\frac{\partial q}{\partial n}\cos\lambda = \frac{\cos^2\lambda}{q_*}\frac{\partial q}{\partial n^*}.$$

Since n^* is measured into the subsonic zone we have $\partial q/\partial n^* \leqq 0$; it follows that $\partial\vartheta/\partial s^* \leqq 0$, and our assertion is proved.

According to this lemma there cannot exist a bounded supersonic region in the interior of a flow: if this occured, then a complete traversal of the boundary would contradict the single-valuedness of ϑ.

2. A local supersonic zone $\mathfrak{D}$ adjacent to the flow boundary has a simply covered image in the hodograph plane.

For consider a C_- characteristic in the supersonic zone emanating from a point P on the profile. This characteristic ends at a point A on the sonic line, for it surely does not return to the profile. Let Q be a variable point on AP, and let R the sonic point on the C_+ characteristic through Q (see Fig. 10). In the hodograph plane, the image points A', R', B' have the order indicated since $\partial\vartheta/\partial s^* \leqq 0$. Thus as Q goes from A to P, R' moves from A' to B', and Q' moves from A' to P'. Now letting A vary the length of the sonic line, it is easily seen that the hodograph image of $\mathfrak{D}$ is simply covered.

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[1] F. I. FRANKL: Prikl. Mat. Mek., USSR. **11**, 192 (1947), [English translation, NACA Tech. Mem. 1251 (1949)]; G. GUDERLEY: Wright Field Rep. No. F-TR-1171-ND; and A. BUSEMANN: J. Aeronaut. Sci. **16**, 337 (1949).

[2] C. S. MORAWETZ: Comm. Pure Appl. Math. **10**, 107 (1957); **11**, 129 (1958).

3. The theorem of NIKOLSKY and TAGANOV can now be proved. Let Λ denote the curve in the hodograph plane which is the image of the boundary arc $P_1 P P_2$. As we follow Λ from P_1' to P_2' the angle ϑ must continually decrease, for otherwise Λ would intersect some characteristic twice. That characteristic would then, in the physical plane, begin and end on the boundary, which is impossible. It follows that $\partial\vartheta/\partial s \leqq 0$ on $P_1 P P_2$; that is, the boundary arc is necessarily convex. (A further argument shows that it must be *strictly* convex; this will be omitted, however, since it adds no basically new information.)

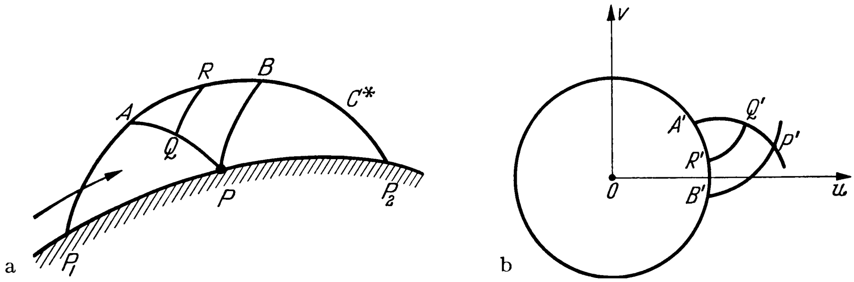

Fig. 10a and b. (a) Local supersonic zone. (b) The corresponding characteristic diagram.

The above argument also proves that the hodograph image of the supersonic zone must lie between the Γ_+ characteristic emanating from P_2' and the Γ_- characteristic emanating from P_1'. Thus the maximum speed $\bar{q}$ attainable in a local supersonic zone is given by

$$\imath(\bar{q}) = \frac{\vartheta_1 - \vartheta_2}{2},$$

where $\vartheta = \pm\,\imath(q)$ is the equation of the characteristic curves. For thin profiles the angles ϑ_1 and ϑ_2 are very nearly equal, so that high speeds in a local supersonic zone in continuous flow are impossible.

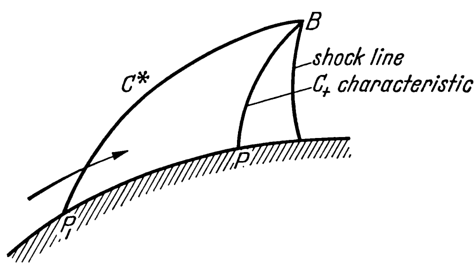

Fig. 11. Local supersonic zone bounded by a shock line.

The reader will note that the results of 1, 2, and 3 apply equally well to the supersonic region $P_1 B P$ in Fig. 11. Since such configurations are observed to be completely stable, we have another example of the predominant role which the boundary layer plays in smoothing out profile variations in transonic flows.

53. Elimination of the pressure and density from the equations of motion. It is of some interest to eliminate p, ϱ and S from Eqs. (35.1) to (35.4), thereby obtaining integrability conditions on the velocity field of any possible fluid motion. This was accomplished recently by BERKER[1], but since the matter is fairly complicated the reader must be referred to the original paper for details. As part of his investigation, BERKER finds a number of new exact solutions.

The situation is somewhat easier when the flow is restricted to being steady and two-dimensional. Considering an ideal gas with constant specific heats, ERICKSEN[2] finds the pair of equations

$$\left.\begin{aligned}\frac{\partial M^2}{\partial s} &= \frac{K M^2}{M^2 - 1}\,[2 + (\gamma - 1)\, M^2],\\ \frac{\partial M^2}{\partial n} &= \frac{M^2 (M^2 - 1)}{K}\left[\frac{\partial K}{\partial n} + \frac{\partial \varkappa}{\partial s} - 2K\varkappa - M^2\left(\frac{\partial \varkappa}{\partial s} + \gamma K \varkappa\right)\right],\end{aligned}\right\} \qquad (53.1)$$

[1] R. BERKER: C. R. Acad. Sci., Paris **242**, 342 (1956).

[2] J.L. ERICKSEN: Bull. Tech. Univ. Istanbul **6**, 1 (1953).

in which only the Mach number and flow pattern enter ($\varkappa$ and K are the curvatures of the streamlines and their orthogonal trajectories). By putting $f = M^2$ in the compatibility condition

$$\frac{\partial}{\partial n}\left(\frac{\partial f}{\partial s}\right) - \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial n}\right) = \varkappa \frac{\partial f}{\partial s} + K \frac{\partial f}{\partial n}, \tag{53.2}$$

he finds from Eq. (53.1) that

$$A_3 G^3 + A_2 G^2 + A_1 G + A_0 = 0, \tag{53.3}$$

where $G = M^2 - 1$ and the coefficients A_3, A_2, A_1 and A_0 depend only on $\varkappa$, K and their derivatives. This cubic equation is interesting for two applications:

1. *If the streamline pattern alone is known (say from a photograph) then Eq. (53.3) permits the determination of the Mach number and hence the speeds in the corresponding flow.*

2. *There cannot be more than three essentially different flows corresponding to a given streamline pattern*[1].

For irrotational motion in two dimensions there is of course no problem in eliminating the pressure and density, this being already accomplished by the intrinsic Eqs. (41.4). Applying condition (53.2) with $f = \log q$ to the intrinsic equations yields a quadratic equation in G, hence there cannot be more than two essentially different irrotational flows corresponding to a given flow pattern. By a different method ERICKSEN[2] has shown that actually there is only one. More precisely, his result is as follows:

If two plane irrotational flows of an ideal gas have the same streamline pattern, then the flows are dynamically similar.

F. Shock waves in perfect fluids.

It is a well-known experimental fact that abrupt changes in pressure and density can occur across surfaces in gas flow. The physical and mathematical reasons for the occurence of such transition surfaces or shock waves are also well-known and widely discussed in works on gas dynamics. This is not the place to present these arguments[3], but rather we shall assume that the reader is already familiar with the importance of shock waves. This chapter can then be devoted to the fundamental theoretical results of the subject. In particular we shall derive the shock relations, determine some elementary properties of shock waves, and finally consider the structure of shock waves.

54. Shock relations. Mathematically, a shock wave is a surface $\Sigma = \Sigma(t)$ in the flow region across which one or more of the flow variables $\mathbf{v}$, ϱ, p and S suffers a jump discontinuity. In order to describe the conditions which hold at a shock surface we begin by assigning the subscript 1 to denote quantities on one side of the shock and the subscript 2 to denote quantities on the other side, (eventually we shall interpret side 1 as the front of the shock and side 2 as the back, but for the moment no such distinction is made). Let $\mathbf{n}$ be the unit normal vector to the shock surface, directed towards side 2, and let G be the speed of advance of the surface in that direction. Then the following equations hold relat-

[1] See also T. Y. THOMAS and B. BERNSTEIN: J. Rational Mech. Anal. **1**, 703 (1955).

[2] J. L. ERICKSEN: J. Math. Phys. **31**, 63 (1952).

[3] See [*17*], §§ 50, 117, and the article of H. CABANNES in this Handbuch, vol. IX.

ing the flow variables on the two sides of Σ:

$$\left.\begin{aligned} [\varrho U] &= 0, \\ [\varrho U \mathbf{v} + p\,\mathbf{n}] &= 0, \\ [\varrho U (\tfrac{1}{2} q^2 + E) + p\,\mathbf{v}\cdot\mathbf{n}] &= 0, \\ [\varrho U S] &\geqq 0. \end{aligned}\right\} \tag{54.1}$$

In these equations $U = \mathbf{v}\cdot\mathbf{n} - G$ is the relative normal flow velocity on Σ, and the brackets denote the jump of a quantity across the discontinuity front, i.e.

$$[f] = f_2 - f_1.$$

The first three of conditions (54.1) express respectively, conservation of mass, momentum, and energy as the fluid crosses Σ [1]. The final condition is a consequence of postulate (33.5) governing the change of entropy of a material volume [2]. The above equations are usually obtained from considerations more or less independent of the basic postulates of Chaps. B and D, so that it is perhaps worthwhile to indicate here how they can be obtained directly from these postulates.

We assume in what follows that the shock surface Σ separates two states of continuous flow, that the flow variables have definite limiting values on either side of Σ, and that these limiting values are different for at least one of the flow variables. Consider now a volume $\mathcal{V}$ moving with the fluid particles, and let Σ divide $\mathcal{V}$ into two parts. According to Eq. (5.2)

$$\frac{d}{dt}\int_{\mathcal{V}} \varrho\, dv = 0. \tag{54.2}$$

In analogy with the derivations in Chap. B, we should like to use Eq. (4.1) to evaluate the left hand side of Eq. (54.2). Because of possible discontinuities of both the density and velocity across Σ, it is necessary to use Eq. (4.1) in a modified form, namely,

$$\frac{d}{dt}\int_{\mathcal{V}} f\, dv = \int_{\mathcal{V}} \left(\frac{df}{dt} + f \operatorname{div}\mathbf{v}\right) dv + \int_{\Sigma} [f\,U]\, da. \tag{54.3}$$

In this equation, which is easily inferred from Eq. (4.2), the designation Σ as the area of integration naturally refers only to the portion of Σ inside $\mathcal{V}$. From

[1] The concept of a discontinuity surface and the first two conditions above are due originally to G. Stokes, Phil. Mag. (3) **33**, 349 (1848). Stokes' remarks concerning these topics still make interesting reading: "These conclusions certainly seem sufficiently startling; yet a still more extraordinary result ..."; "the result, however, is so strange ..."; etc.

The fact that energy should be conserved across a shock wave is implicit in an investigation of Rankine [Phil. Trans. Roy. Soc. Lond. **160**, 277 (1870)], and was later given precise form by Hugoniot [J. Ecole Polytech. Cahier 57, 1 (1887); Cahier 58, 1 (1889)].

[2] That the entropy should increase across a shock was first pointed out by G. Zemplén, C. R. Acad. Sci., Paris **141**, 710 (1905). Zemplén's remark, which today seems obvious enough, is actually far from a triviality. At the time when his note was published, both Kelvin and Rayleigh were of the opinion that $[S] = 0$ at a shock surface. They therefore had serious doubts as to the validity of the shock wave hypothesis as a model for the behavior of real gases, since the first three of Eqs. (54.1) are incompatible with the presumed truth of $[S] = 0$. Even Stokes had joined this view, for in his Collected Papers the 1848 note is purposely abridged to omit the shock conditions! It should be added that the continental hydrodynamicists generally did not entertain the doubts of the English school, and that, in any case, the correct principles were more or less universally understood by 1915, (strangely enough, however, Lamb persists in the earlier misconception).

Eqs. (54.2) and (54.3), together with the equation of continuity, we obtain

$$\int_{\Sigma} [\varrho\, U]\, da = 0.$$

Since the portion of Σ over which this integration takes place may be arbitrarily small, it follows in the usual way that the integrand is zero, proving Eq. (54.1). The remaining equations of (54.1) are derived similarly from Eqs. (6.1), (33.3), and (33.5); of course, in these laws it is assumed that $\mathbf{t} = -p\mathbf{n}$ and $\mathbf{q} = 0$. (It is of related interest to study the shock relations which hold at a discontinuity surface in a viscous fluid motion, especially since certain of the results are quite remarkable[1]. Such a study must be mainly of theoretical importance, however, since there does not appear to be any experimental situation which would require viscous shocks for its explanation.)

The remainder of this section is concerned with certain elementary consequences of the shock relations. In order to facilitate the discussion it is convenient to consider separately the two cases

I. $U_1 = U_2 = 0$;

II. Neither U_1 nor U_2 is zero.

In the first case no fluid crosses Σ so that it can hardly be considered a true shock. Such a discontinuity surface in fact moves with the gas and separates two zones of different density (and temperature); but the pressure and normal velocity are the same on both sides. We shall exclude this simple and relatively unimportant case from the further discussion. The second case is that of a genuine shock wave. Here it can be assumed without loss of generality that

$$U_1 > 0, \qquad U_2 > 0, \tag{54.4}$$

for the other possibility can be converted into this one simply by relabeling the sides 1 and 2 of Σ (consequently reversing the direction of $\mathbf{n}$). The geometrical significance of Eq. (54.4) is that the fluid enters the shock surface on side 1 and emerges on side 2.

Now let $\mathbf{v}_t$ denote the (vector) projection of $\mathbf{v}$ onto the shock surface. By means of some simple reductions making use of the fact that U_1 and U_2 are positive we may rewrite Eq. (54.1) in the elegant form

$$\left.\begin{aligned} [\varrho\, U] &= 0, \\ [\varrho\, U^2 + p] &= 0, \quad [\mathbf{v}_t] = 0, \\ [\tfrac{1}{2} U^2 + I] &= 0, \\ [S] &\geqq 0, \end{aligned}\right\} \tag{54.5}$$

[1] The basic work on this subject is due to P. DUHEM, Recherches sur l'hydrodynamique, Ann. Toulouse (2) (1901—1903), reprinted Paris 1903—1904. Unfortunately, DUHEM's paper is quite difficult to read due to its shear bulk, its somewhat outdated notation, and an altogether remarkable profusion of symbols. In view of this fact, and since some of DUHEM's results (see below) have an inherent interest quite apart from possible applications, it seems desirable to reexamine DUHEM's work. This the author has done, with particular regard to the following two of DUHEM's results: (A) no singular surface of zero order (shock wave) can exist in a viscous fluid, and (B) singular surfaces of order one (Sect. 51) in a non-viscous heat-conducting fluid propagate with the Newtonian speed of sound. Our conclusions relative to these statements are as follows: (1) shock waves, across which all of the fundamental conservation laws hold, *are* possible in viscous fluids; (2) if, however, it is postulated that $[\mathbf{v}] = [T] = 0$ across any discontinuity surface in a viscous flow, then only contact discontinuities are possible (this is the true content of DUHEM's first result); (3) in a non-viscous, heat-conducting fluid it *is* possible for a singular surface to propagate at the Newtonian sound speed. A complete discussion of these results will be published in the Journal of Mathematics and Mechanics.

where I is the specific enthalpy, $I = E + p/\varrho$. When the motion is steady and the shock surface consequently stationary, we have from Eqs. $(54.5)_3$ and $(54.5)_4$,

$$[\tfrac{1}{2} q^2 + I] = 0. \tag{54.5a}$$

Thus BERNOULLI'S equation (38.3) holds for steady flow even when shock fronts intervene in the motion.

It is useful to exhibit Eqs. (54.5) in several other forms where the thermodynamic variables stand separated. To this end, let us introduce the mass flux m across the shock

$$m = \varrho_1 U_1 = \varrho_2 U_2. \tag{54.6}$$

Then from Eqs. $(54.5)_2$ and $(54.5)_1$,

$$\left.\begin{aligned} p_2 - p_1 &= \varrho_1 U_1^2 - \varrho_2 U_2^2 \\ &= m\,(U_1 - U_2) = m^2 (\tau_1 - \tau_2) = U_1 U_2 (\varrho_2 - \varrho_1), \end{aligned}\right\} \tag{54.7}$$

(τ denotes specific volume). From the first, second, and third forms respectively of the right hand side of Eq. (54.7) we derive

$$(p_2 - p_1)(\tau_2 + \tau_1) = U_1^2 - U_2^2, \tag{54.8}$$

and

$$\frac{p_2 - p_1}{\tau_2 - \tau_1} = - m^2, \tag{54.9}$$

$$\frac{p_2 - p_1}{\varrho_2 - \varrho_1} = U_1 U_2. \tag{54.10}$$

Finally from Eqs. (54.8) and $(54.5)_4$ follows

$$(p_2 - p_1)(\tau_2 + \tau_1) = 2\,(I_2 - I_1). \tag{54.11}$$

The important Eq. (54.11) was first obtained by HUGONIOT, although the result for ideal gases was already known to RANKINE. Eq. (54.11) determines all the possible thermodynamical states (p_2, τ_2) which may be reached across a shock wave from an initial state (p_1, τ_1). Another form of (54.11) is

$$(p_2 + p_1)(\tau_2 - \tau_1) = 2(E_2 - E_1).$$

An important property of shock waves is that they introduce vorticity into an otherwise irrotational flow. The usual reason advanced for this statement is that energy (more precisely, the stagnation enthalpy H) is conserved across a shock while entropy is introduced. This argument is restricted to steady motion. Recently, however, several papers have appeared in which the unsteady flow situation is analysed and in which the same conclusion is reached[1].

There is a remarkably simple formula for the vorticity introduced behind a shock when the flow in front is uniform, namely[2]

$$\boldsymbol{\omega} = \frac{(\tau_1 - \tau_2)^2}{\tau_2}\,\mathbf{n} \times \operatorname{grad}_\Sigma m, \tag{54.12}$$

where $\operatorname{grad}_\Sigma$ denotes the surface gradient operator, and $m = \varrho U$. The proof of Eq. (54.12) is too long to include here, but it is worthwhile to notice several

[1] W.D. HAYES: J. Fluid Mech. **2**, 595 (1957). — R.P. KANWAL: Arch. Rational Mech. Anal. **1**, 225 (1958).

[2] M.J. LIGHTHILL: J. Fluid Mech. **2**, 1 (1957). — W.D. HAYES: J. Fluid Mech. **2**, 595 (1957).

consequences. First, it is apparent that $\boldsymbol{\omega}$ is tangential to the shock surface. Also, in plane or axially-symmetric steady flow Eq. (54.12) reduces simply to

$$\omega = \frac{(\varrho_2 - \varrho_1)^2}{\varrho_1 \varrho_2} K v_t, \tag{54.13}$$

where K is the curvature of the shock line and v_t is the tangential component of velocity[1]. It is seen from Eq. (54.13) that vorticity is introduced at every point of a shock front where its curvature is non-zero and its inclination not normal to the uniform stream. Furthermore, although the entropy introduced by a shock front is of third order in shock strength, the vorticity is of second order.

In Sect. 56 we shall obtain several further properties of the shock transition in an arbitrary perfect fluid. In particular it will be shown that the entropy change across a shock front is of third order in the shock strength $\tau_1 - \tau_2$. Thus, considering a sequence of shock fronts whose strengths tend to zero, we have

$$\lim U_1^2 = \lim U_2^2 = \lim \frac{p_2 - p_1}{\varrho_2 - \varrho_1} = c^2.$$

In other words, *the speed of advance of an infinitely weak shock front relative to the fluid is precisely the speed of sound.* This result offers another justification for calling c the speed of sound.

55. Shock relations for an ideal gas. For an ideal gas with constant specific heats,

$$I = \frac{c^2}{\gamma - 1} = \frac{\gamma}{\gamma - 1} p \tau.$$

This allows the Hugoniot relation (54.11) to be written in the useful form[2]

$$\left(p_2 + \frac{\gamma - 1}{\gamma + 1} p_1\right)\left(\tau_2 - \frac{\gamma - 1}{\gamma + 1} \tau_1\right) = \frac{4\gamma}{(\gamma + 1)^2} p_1 \tau_1. \tag{55.1}$$

Eq. (55.1) determines all possible end states (p_2, τ_2) which can be reached across a shock wave from the initial state (p_1, τ_1). The locus of these end states in the (p, τ)-plane is a rectangular hyperbola with asymptotes

$$p = -\frac{\gamma - 1}{\gamma + 1} p_1, \qquad \tau = \frac{\gamma - 1}{\gamma + 1} \tau_1;$$

this curve is called the Hugoniot curve. The condition $S_2 \geqq S_1$ requires us to choose only that portion of the hyperbola above the point (p_1, τ_1), whence

$$\frac{\gamma - 1}{\gamma + 1} \tau_1 < \tau_2 < \tau_1 \quad \text{or} \quad \varrho_1 < \varrho_2 < \frac{\gamma + 1}{\gamma - 1} \varrho_1;$$

in other words the increase in density across a shock wave cannot be arbitrarily great. The adiabatic through (p_1, τ_1) and the Hugoniot curve through that point are shown in Fig. 12; these curves have a second order contact.

The fluid state in front of the shock wave, together with the shock speed G, suffices for the complete determination of the state behind the shock. Specifically, introducing the "Mach numbers"

$$\mathsf{M}_1 = U_1/c_1, \qquad \mathsf{M}_2 = U_2/c_2,$$

[1] C. TRUESDELL: J. Aeronaut. Sci. **19**, 826 (1952).

[2] Other forms of the Hugoniot relation for ideal gases are

$$\frac{p_2}{p_1} = \frac{(\gamma + 1)\varrho_2 - (\gamma - 1)\varrho_1}{(\gamma + 1)\varrho_1 - (\gamma - 1)\varrho_2} \quad \text{and} \quad \frac{p_2 - p_1}{\varrho_2 - \varrho_1} = \gamma \frac{p_2 + p_1}{\varrho_2 + \varrho_1}.$$

we have for the ideal gas,

$$\left.\begin{aligned}\frac{U_2-U_1}{U_1}&=\frac{\tau_2-\tau_1}{\tau_1}=\frac{2}{\gamma+1}\,\frac{1-\mathrm{M}_1^2}{\mathrm{M}_1^2}\\ \frac{p_2-p_1}{p_1}&=\frac{2\gamma}{\gamma+1}\,(\mathrm{M}_1^2-1)\\ \frac{T_2-T_1}{T_1}&=\frac{2(\gamma-1)}{(\gamma+1)^2}\,\frac{(\gamma\,\mathrm{M}_1^2+1)(\mathrm{M}_1^2-1)}{\mathrm{M}_1^2}\\ 1-\mathrm{M}_2^2&=\frac{\mathrm{M}_1^2-1}{1+\frac{2\gamma}{\gamma+1}\,(\mathrm{M}_1^2-1)}\,.\end{aligned}\right\}\tag{55.2}$$

It is sufficient to prove the first of these, for the others then follow readily. From Eqs. (54.9) and (54.6),

$$\left.\begin{aligned}p_2-p_1&=m^2(\tau_1-\tau_2)\\ &=\gamma p_1\mathrm{M}_1^2(1-\tau_2/\tau_1)\,.\end{aligned}\right\}\tag{55.3}$$

It is a simple matter to eliminate the quantity (p_2/p_1-1) between Eqs. (55.1) and (55.3), and thus to obtain Eq. $(55.2)_1$.

The entropy increase through a shock wave is given by

$$\frac{S_2-S_1}{c_v}=\log\left(\frac{p_2}{p_1}\right)\left(\frac{\tau_2}{\tau_1}\right)^{\gamma}.\tag{55.4}$$

This may be brought into a somewhat different form when the flow is steady. For by Eq. (54.5a) the stagnation enthalpy is the same in front of the shock as behind the shock, hence the same holds for the stagnation temperature. Thus, using the fact that entropy is constant along streamlines, we find that

$$\frac{p_2\tau_2^\gamma}{p_1\tau_1^\gamma}=\left(\frac{p_{02}}{p_{01}}\right)^{1-\gamma}\left(\frac{T_{02}}{T_{01}}\right)^{\gamma}=\left(\frac{p_{02}}{p_{01}}\right)^{1-\gamma},\tag{55.5}$$

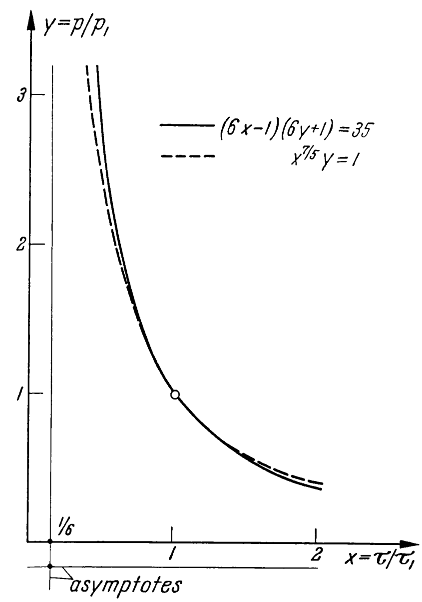

Fig. 12. Hugoniot curve (solid line) and adiabatic curve (broken line) for $\gamma=\frac{7}{5}$. The asymptotes of the Hugoniot curve are indicated.

where the subscript zeros denote stagnation quantities. With the help of Eq. (55.5) we can therefore write Eq. (55.4) in the alternate form

$$S_2-S_1=R\log\,(p_{01}/p_{02})\,.\tag{55.6}$$

Since the critical speed q_* and the critical enthalpy I_* are also unchanged across a shock front [because of Eq. (54.5a) and the fact that $I=c^2/\gamma-1$], the above reasoning also serves to prove the useful chain of equalities,

$$\frac{p_{01}}{p_{02}}=\frac{\varrho_{01}}{\varrho_{02}}=\frac{p_{*1}}{p_{*2}}=\frac{\varrho_{*1}}{\varrho_{*2}}=\frac{Q_{*1}}{Q_{*2}}\,.\tag{55.7}$$

The ratio (55.7) is tabulated as the final column in Table 1 of Sect. 37.

Finally, since $p_2/p_1>1$ we must have $\mathrm{M}_1>1$ according to Eq. $(55.2)_2$. This implies $U_1>c_1$; that is, *the relative normal flow velocity in front of a shock front is supersonic.* Conversely $\mathrm{M}_2<1$ and *the normal velocity behind a shock front is* *subsonic with respect to the shock front*[1]. These properties of a shock front are also true for a general gas, as we shall see in the following section. For steady flow, by Eq. $(55.2)_1$ and Bernoulli's equation,

$$\begin{aligned} v_{1n} v_{2n} = U_1 U_2 &= \frac{\gamma-1}{\gamma+1} U_1^2 + \frac{2}{\gamma+1} c_1^2 = \frac{\gamma-1}{\gamma+1} (q_{\max}^2 - v_t^2) \\ &= q_*^2 - \frac{\gamma-1}{\gamma+1} v_t^2 . \end{aligned}$$

This interesting formula is due to PRANDTL. For normal shocks it reduces to

$$q_1 q_2 = c_*^2 .$$

56. Basic properties of the shock transition. In this section we shall establish four important properties of the gas state on either side of a shock front, namely

I. The increase of entropy across a shock front is of third order in the shock strength $(\tau_1 - \tau_2)$.

II. Shocks are compressive, that is $p_2 > p_1$, $\tau_2 < \tau_1$.

III. The normal flow velocity relative to a shock front is supersonic at the front side, subsonic at the back side.

IV. The fluid state in front of a shock wave, together with the relative normal speed U_1, *completely determines the state behind the shock wave.*

For ideal gases with constant specific heats, properties (II) through (IV) were shown in the preceding section. It is remarkable that the same properties hold for arbitrary gases, if only very mild assumptions are made concerning their thermodynamic properties[2]. Specifically, we require of the gas that:

1. Its thermodynamical state Z is uniquely determined by the pressure and specific volume;

2. $$\left(\frac{\partial p}{\partial \tau}\right)_S < 0, \quad \left(\frac{\partial^2 p}{\partial \tau^2}\right)_S > 0\,^{3}.$$

[It is tacitly assumed that all points in the first quadrant of the (p, τ)-plane define possible thermodynamic states Z. If (1) and (2) do not hold over the whole quadrant, our argument still applies (with only slight changes) in any *convex* region where (1) and (2) are valid.] Before giving the formal proof of I to IV we observe that our assumptions imply that adiabatics in the (p, τ)-plane are convex curves with negative slope (Fig. 13). Now it can be shown that (1) and (2) imply furthermore that $(\partial S/\partial p)_\tau$ must be either everywhere positive or everywhere negative. We shall suppose

$$\left(\frac{\partial S}{\partial p}\right)_\tau > 0, \tag{56.1}$$

this being the case encountered in practice [actually only a few cases are known in which Eq. (56.1) is violated, the best example being liquid water below 4° centigrade; in any case, the alternative to Eq. (56.1) would make little difference]. Because of Eq. (56.1), adiabatics in the (p, τ)-plane corresponding to higher values of entropy lie above and to the right of adiabatics with lower entropy.

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[1] Some geometric consequences of this for the shock line-characteristic line configuration are given in [*17*], p. 305.

[2] First noticed by H. BETHE and H. WEYL; see H. WEYL: Comm. Pure Appl. Math. **2**, 103 (1949). Another treatment of this topic was recently given by R.D. COWAN, J. Fluid Mech. **3**, 531 (1958).

[3] These assumptions will be recognized as precisely those postulated in the earlier discussion of thermodynamical principles, see Eqs. (30.7) and (37.6).

Proof of I to IV. For *fixed* (p_1, τ_1) let us set

$$H(p, \tau) = 2(I - I_1) - (p - p_1)(\tau + \tau_1). \tag{56.2}$$

The locus $H = 0$ in the (p, τ)-plane is the Hugoniot curve for the state $Z_1 = (p_1, \tau_1)$; it will be denoted by $\mathfrak{H}$. Any state $Z_2 = (p_2, \tau_2)$ which can be reached across a shock wave from the front state Z_1 must lie on $\mathfrak{H}$. The differential of the function $H(p, \tau)$ is given by

$$dH = 2T\,dS - [(p - p_1)\,d\tau + (\tau_1 - \tau)\,dp], \tag{56.3}$$

which follows at once from $dI = T\,dS - \tau\,dp$. Along the Hugoniot curve $dH = 0$, hence

$$dS = 0 \quad \text{along } \mathfrak{H} \text{ at } Z_1.$$

Again, by forming the second and third differentials of H and evaluating at Z_1, we find

$$d^2S = 0, \quad 2T\,d^3S = dp\,d^2\tau - d\tau\,d^2p,$$

along $\mathfrak{H}$ at Z_1. Taking τ as independent variable now gives $dS = d^2S = 0$ and $d^3S > 0$ for $d\tau < 0$. This proves assertion I[1].

We now set

$$r = \frac{p - p_1}{\tau - \tau_1};$$

r is the slope of the straight line joining Z_1 to Z. In terms of r, Eq. (56.3) takes the form

$$dH = 2T\,dS + (\tau_1 - \tau)^2\,dr. \tag{56.4}$$

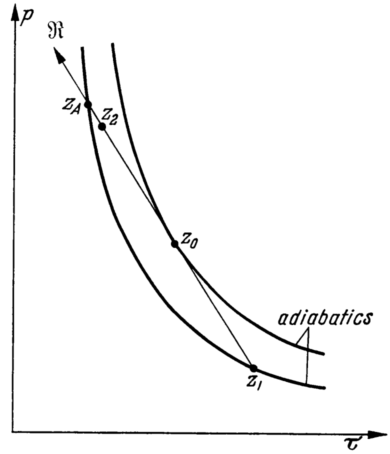

Fig. 13. (p, τ) diagram for the Hugoniot curve.

Now the adiabatic $\mathfrak{A}$ through Z_1 is convex, so that r increases as we trace $\mathfrak{A}$ from left to right. Thus by Eq. (56.4)

$$dH > 0 \quad \text{as we trace } \mathfrak{A} \text{ from left to right.}$$

Since $H = 0$ at Z_1, it follows that $H < 0$ on the upper branch of $\mathfrak{A}$ while $H > 0$ on the lower branch (see Fig. 13). If $\mathfrak{R}$ denotes a ray $r = \text{const}$ through Z_1, then $dH = 2T\,dS$ along $\mathfrak{R}$.

Consider a ray $\mathfrak{R}$ which is directed so that it intersects $\mathfrak{A}$ only at Z_1. Then by virtue of Eq. (56.1) dS is always positive or always negative as we follow $\mathfrak{R}$ away from Z_1. Since $dH = 2T\,dS$, such a ray cannot contain any points where $H = 0$. Next let $\mathfrak{R}$ intersect $\mathfrak{A}$ at a point Z_A on the upper branch of $\mathfrak{A}$. Because all adiabatic curves are convex it is geometrically evident that as we follow $\mathfrak{R}$ away from Z_1, S first increases from Z_1 to Z_0 (see Fig. 13), and decreases from then on. Since $dH = 2T\,dS$, H also increases and then decreases as we follow $\mathfrak{R}$ away from Z_1. But $H < 0$ when we arrive at Z_A, so that there must be exactly one point Z_2 on $\mathfrak{R}$ between Z_0 and Z_A where $H = 0$. The entropy at this point is obviously greater than the entropy at Z_1.

Finally, if $\mathfrak{R}$ intersects $\mathfrak{A}$ on the lower branch, the same procedure shows that $\mathfrak{R}$ can contain at most one point where $H = 0$. Such a point Z would have to

[1] In fact, we have the following development of $S - S_1$ along the Hugoniot curve,

$$S - S_1 = \frac{1}{12T_1}\left(\frac{\partial^2 p}{\partial \tau^2}\right)_{S,1}(\tau_1 - \tau)^3 + \cdots.$$

lie *below* $\mathfrak{A}$ (recall that $H > 0$ on the lower branch of $\mathfrak{A}$), and would therefore have $S < S_1$.

The Hugoniot curve is thus shown to be a simple line through Z_1 with $S > S_1$ on the upper branch and $S < S_1$ on the lower. According to the shock condition $(54.5)_5$ only states Z on the upper branch can be reached across a shock wave from the front state Z_1. Since $p > p_1$ and $\tau < \tau_1$ on this branch, assertion II is proved.

To prove III, note from Fig. 13 that for a shock front joining the front state Z_1 and the back state Z_2,

$$\left(\frac{\partial p}{\partial \tau}\right)_{S, Z_2} < r < \left(\frac{\partial p}{\partial \tau}\right)_{S, Z_1}. \tag{56.5}$$

Since $(\partial p/\partial \tau)_S = -\varrho^2 c^2$ and [see Eq. (54.6)]

$$r = -m^2 = -\varrho^2 U^2,$$

the two inequalities (56.5) are equivalent respectively to $c_2 > U_2$ and $U_1 > c_1$. That is, in front of the shock the relative normal speed is supersonic while in back it is subsonic.

Finally, if the front state Z_1 and the relative normal speed U_1 are prescribed, then to find the back state we need only intersect the upper branch of the Hugoniot contour with the ray $\mathfrak{R}$ of slope $r = -\varrho_1^2 U_1^2$. There is *exactly one* such intersection (provided of course $U_1 > c_1$: otherwise there is no shock possible). The value U_2 is then found from $U_2 = (\varrho_1 U_1)\, \tau_2$.

In conclusion note that r decreases monotonically as Z moves away from Z_1 along the upper branch of the Hugoniot contour. Moreover as r decreases, S must increase because of Eq. (56.4): that is, *for a given thermodynamic state in front of a shock wave, the greater the normal speed U_1, the greater the change in entropy across the shock.* The entropy increase across a detached bow wave in supersonic flight, for example, is greatest at the central streamline and monotonically decreases as one proceeds out along the wave.

57. The shock layer. In real gases, the passage of a particle through a shock front is not an instantaneous process in which the particle suddenly finds itself confronted with the new state behind the shock, but rather it involves a rapid transition from the front state to the back state through a narrow region, the shock layer. In this region the motion cannot be described adequately by perfect fluid theory, and therefore there is some question as to the validity of the preceding derivation of the Rankine-Hugoniot conditions. Considerable interest has thus been focused on the shock layer, and its structure has been widely studied. These investigations have yielded an increased understanding of the nature of shock waves, some information concerning the thickness of a shock layer, and an alternative justification of the Rankine-Hugoniot conditions. Moreover the comparison of the theoretical results with experiment provides a crucial testing ground for the validity of the Navier-Stokes equation. For mathematical reasons the problem has been considered mainly in the simplest case of one-dimensional steady motion, but this is in many respects the prototype of all shock phenomena.

The problem of shock structure in one-dimensional motion includes two essential phases: first, proving that the differential equations of viscous fluids actually admit a solution of the general type desired (the velocity profile should be of the form shown in Fig. 14), and second, describing the shock profile with particular emphasis on its thickness. The first of these problems, after being studied inconclusively by RAYLEIGH[1], was solved by VON MISES[2] and GILBARG[3]. The second problem—quantitative description of the profile—involves fairly difficult numerical computation and therefore lies mostly outside the scope of this article.

Since the work below is based on the equations of continuum mechanics, it is only fair to point out certain objections to the applicability of these equations. It is claimed, first, that since the thickness of a shock layer is of the order of a few molecular mean free paths, therefore any approach by continuum mechanics is *a priori* invalid[4]; and second, that continuum mechanics predicts too small values for shock thicknesses (bearing out the former criticism). The second objection has been completely negated by the work of GILBARG and PAOLUCCI[5], who have shown that if account is taken of the temperature dependence of viscosity and heat conductivity—effects only partially considered by most earlier investigators, then the Navier-Stokes equation provides at least as good values for shock thickness as does kinetic theory, values, moreover, which are in acceptable agreement with recent experiments[6]. Finally, the first objection, upon due reflection, can hardly be considered convincing[7]. For these reasons we definitely do not believe it outmoded to use continuum methods in studying the shock layer.

The mathematical theory of the shock layer, so far as it falls under the scope of continuum mechanics, is based on the equations of one-dimensional steady motion, namely

$$\left.\begin{aligned} &\frac{d}{dx}(\varrho u) = 0,\\ &\varrho u \frac{du}{dx} = \frac{dT_{xx}}{dx},\\ &\varrho u \frac{dE}{dx} = T_{xx}\frac{du}{dx} - \frac{dq}{dx}, \end{aligned}\right\} \quad (57.1)$$

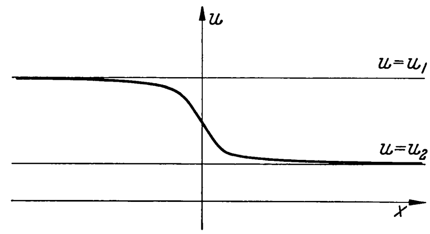

Fig. 14. Velocity profile in a shock layer.

[cf. Eqs. (5.4), (6.7), and (33.4)]. From the following chapter, Sects. 61 and 63, we draw the constitutive formulae

$$T_{xx} = -p + (\lambda + 2\mu)\frac{du}{dx}, \qquad q = -\varkappa \frac{dT}{dx}. \quad (57.2)$$

Eqs. (57.1) can be integrated once without difficulty, and with the help of Eq. (57.2) this integrated form may be written

$$\varrho u = m \quad (57.3)$$

and

$$\left.\begin{aligned} (\lambda + 2\mu)\frac{du}{dx} &= p + m(u-a),\\ \varkappa\frac{dT}{dx} &= m\left[E - \tfrac{1}{2}(u-a)^2 + b\right], \end{aligned}\right\} \quad (57.4)$$

[1] Lord RAYLEIGH: Proc. Roy. Soc. Lond., Ser. A **84**, 247 (1910); see also G.I. TAYLOR: Proc. Roy. Soc. Lond., Ser. A **84**, 371 (1910).

[2] R. VON MISES: J. Aeronaut. Sci. **17**, 551 (1950). The shock layer in three-dimensional flow is discussed by LUDFORD, Quart. Appl. Math. **10**, 1 (1952).

[3] D. GILBARG: Amer. J. Math. **73**, 256 (1951).

[4] The most recent statements to this effect will be found in [*43*], p. 126 and [*23*], p. 550.

[5] D. GILBARG and D. PAOLUCCI: J. Rational Mech. Anal. **2**, 617 (1953).

[6] F.S. SHERMAN: NACA Tech. Note 3298 (1955); see also the comprehensive review of G.N. PATTERSON [*34*], Chap. 4.

[7] A.E. PUCKETT and H.J. STEWART [Quart. Appl. Math. **7**, 457 (1950)] conclude that the Navier-Stokes equation should apply except in certain exceptional situations, e.g., where dissociation or condensation effects are important, in highly rarified gases, or when the stagnation temperature is very high. See also the penetrating remarks of C. TRUESDELL concerning the relative validity of methods based on continuum mechanics and on kinetic theory [J. Rational Mech. Anal. **2**, 678 (1954); **5**, 55 (1956)].

where a, b and m are constants. Here $p = f(\varrho, T)$ and $E = E(\varrho, T)$ can be considered known functions of u and T, by virtue of Eq. (57.3).

A solution $u = u(x)$, $T = T(x)$, of the system (57.4) is called a *shock layer* if, as x tends respectively to $-\infty$ and $+\infty$ the point $Z(u, T)$ tends to finite "endvalues" $Z_1(u_1, T_1)$ and $Z_2(u_2, T_2)$, with $u_2 < u_1$. It is easy to see that for a shock layer to exist the endvalues must reduce the right members of Eqs. (57.4) to zero, or in other words,

$$\left.\begin{aligned} \varrho_1 u_1 &= \varrho_2 u_2 = m\,, \\ p_1 + m u_1 &= p_2 + m u_2 = a m\,, \\ E_1 - \tfrac{1}{2}(u_1 - a)^2 &= E_2 - \tfrac{1}{2}(u_2 - a)^2 = b\,. \end{aligned}\right\} \tag{57.5}$$

These conditions are equivalent to the Rankine-Hugoniot conditions (54.5); therefore *a shock layer joining two states Z_1 and Z_2 can occur only if Z_1 and Z_2 are allowable initial and final states, respectively, of a normal shock of an ideal fluid having the same equations of state as the given fluid.* Conversely, if states Z_1 and Z_2 satisfy the Rankine-Hugoniot conditions, then to find a shock layer joining Z_1 and Z_2 one must solve the differential equations (57.4) where the values a, b and m are determined from Eq. (57.5).

A consequence of the above result may be mentioned in passing: *the speed of advance of an infinitely weak shock layer relative to the fluid is precisely the speed of sound c.* This is the last of our series of justifications for calling c the speed of sound.

In the sequel we shall specialize to ideal gases with constant specific heats[1]. Eqs. (57.4) then take the form

$$\left.\begin{aligned} (\lambda + 2\mu)\frac{du}{dx} &= m\,(RT/u + u - a)\,. \\ \varkappa\frac{dT}{dx} &= m\left[c_v T - \frac{1}{2}(u - a)^2 - b\right]. \end{aligned}\right\} \tag{57.6}$$

[These equations admit an exact solution when

$$\frac{c_p(\lambda + 2\mu)}{\varkappa} = 1\,.$$

Indeed in this case, if Eq. $(57.6)_1$ is multiplied by u and added to Eq. $(57.6)_2$ there results

$$(\lambda + 2\mu)\frac{d}{dt}\left(c_p T + \frac{1}{2}u^2\right) = m\left(c_p T + \frac{1}{2}u^2 - \frac{1}{2}a^2 - b\right),$$

whence

$$c_p T + \tfrac{1}{2}u^2 = \tfrac{1}{2}a^2 + b = \text{const}\,. \tag{57.7}$$

Using Eq. (57.7) to eliminate T from Eq. $(57.6)_1$ leads obviously to

$$(\lambda + 2\mu)\frac{du}{dx} = \text{const}\,\frac{(u - u_1)(u - u_2)}{u}$$

which is easily integrated if $\lambda + 2\mu$ is constant, and numerically integrable[2] if $\lambda + 2\mu = f(T)$.]

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[1] For a discussion of the case of a general gas, see D. Gilbarg, Amer. J. Math. **73**, 256 (1951).

[2] L.H. Thomas: J. Chem. Phys. **12**, 449 (1944). — M. Morduchow and P. Libby: J. Aeronaut. Sci. **16**, 674 (1949). — R. von Mises: J. Aeronaut. Sci. **17**, 551 (1950). — L. Meyerhoff: J. Aeronaut. Sci. **17**, 775 (1950). — A.E. Puckett and H.J. Stewart: Quart. Appl. Math. **7**, 457 (1953).

We now show, following GILBARG[1], that *there exists exactly one shock layer joining any two allowable endstates* Z_1 *and* Z_2. In order to do this, it is useful to visualize the direction field of Eqs. (57.6) in the $Z(u, T)$-plane. This has the form shown in Fig. 15, where the parabolic curves have the equations

$$L(u, T) = RT/u + u - a = 0,$$

$$M(u, T) = c_v T - \tfrac{1}{2}(u-a)^2 - b = 0,$$

necessarily intersecting at Z_1 and Z_2. The problem is to find an integral curve ("shock curve") joining Z_1 and Z_2. Now Z_1 and Z_2 are singular points of Eq. (57.6) so that to determine the nature of the direction field in the neighborhood of these points it is necessary to examine the roots of the "characteristic" equation

$$\begin{vmatrix} \frac{1}{\lambda+2\mu}\frac{\partial L}{\partial u} - \delta & \frac{1}{\lambda+2\mu}\frac{\partial L}{\partial T} \\ \frac{1}{\varkappa}\frac{\partial M}{\partial u} & \frac{1}{\varkappa}\frac{\partial M}{\partial T} - \delta \end{vmatrix} = 0. \tag{57.8}$$

It is easily found that the roots of Eq. (57.8) are real and of equal sign at Z_1, real and of opposite sign at Z_2. Hence the former point is a node and the latter a saddle[2], as indicated in Fig. 15. If we follow backwards along the integral curve entering Z_2 from the right, then we never cross the curves $L=0$ or $M=0$ (the arrows are directed so that this cannot occur). We must therefore approach the other singular point Z_1 as $x \to -\infty$. This proves the existence of a shock layer joining Z_1 and Z_2, and the uniqueness of this layer follows by a similar argument.

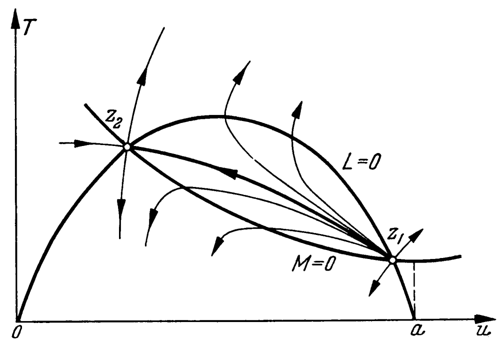

Fig. 15. Field of integral curves and the shock curve in the Z-plane.

(The method just outlined provides a practical procedure for the numerical determination of shock profiles, cf. GILBARG and PAOLUCCI.)

It remains only to verify that for small $\varkappa$ and $\lambda+2\mu$ the shock profile has qualitatively the form shown in Fig. 14, with the transition region arbitrarily narrow. Rather than give a formal proof of this fact, it will be sufficient to make it plausible. First of all, the monotone decrease of u is obvious from the fact that $du/dx < 0$ along the entire shock curve. Now suppose we wish 90%, say, of the change in u to occur in an interval of width less than ε. In other words, we wish the "shock curve" to cross the greater part of the distance between Z_1 and Z_2 with x-variation less than ε. A glance at Eq. (57.6) shows that this will indeed occur if $\lambda+2\mu$ and $\varkappa$ are sufficiently small whether they are variable or not.

Further discussion of Eq. (57.6) is facilitated by the substitution $v = u^2$. The resulting equation may be treated geometrically with some ease, and simple

[1] See footnote 3, p. 227.

[2] Cf. E. A. CODDINGTON and N. LEVINSON: Theory of Ordinary Differential Equations, Chap. 15. New York 1955.

and valuable inequalities for the shock thickness can be obtained (cf. the paper of VON MISES). A comprehensive treatment of the shock layer from a more physical point of view has recently been given by LIGHTHILL[1].

G. Viscous fluids.

I. The constitutive equations of a viscous fluid.

58. The stress tensor. The Cauchy stress vector $\mathbf{t}$ which describes the action of material forces across a surface was shown in Sect. 6 to have the decomposition

$$\mathbf{t} = \mathbf{n} \cdot \mathbf{T}$$

where $\mathbf{T} = \mathbf{T}(\mathbf{x}, t)$ is the stress tensor and $\mathbf{n}$ is the unit normal to the surface in question. Through its appearance in the equation of motion the stress tensor governs the dynamic response of the medium: thus by relating $\mathbf{T}$ to other kinematic and thermodynamic variables we effectively define or delimit the type of medium which we study, eg. fluid, elastic, plastic, and so on. Such a relation between $\mathbf{T}$ and other flow quantities is called a *constitutive equation*. A particular example of a constitutive equation has already been considered, namely that of a perfect fluid, $\mathbf{T} = -p\mathbf{I}$.

The purpose of the following sections is to derive a constitutive equation applicable to fluids which exert appreciable tangential stresses. The method goes back to STOKES[2] and BOUSSINESQ[3], and has been greatly simplified and extended by the work of REINER[4], RIVLIN[5], and TRUESDELL[6], until now constitutive equations of quite general type, including the classical Cauchy-Poisson law as a special case, are relatively easy to obtain. In the following section, a clearcut set of conditions is stated concerning the way the fluid should react in deformation. Once these conditions are stated the constitutive equations follow unavoidably; at the same time the structural simplicity of the derivation leads to increased insight into the mathematical nature of the fluid. The resulting theory includes non-linear viscosity terms which may be of importance in certain complex situations such as shock and boundary layer phenomena and high altitude flight.

The procedure outlined above belongs to the realm of continuum mechanics. It is completely independent of the molecular nature of liquids and gases, and accordingly has been criticized as inadequate in some situations. It is claimed, for example, that the Navier-Stokes equations cannot be applied to high-altitude flight or to shock layer phenomena, since the mean free path then becomes of appreciable size. This argument has considerable force, but ultimately it must stand or fall on the comparison of theory with experiment; and here the Navier-Stokes equation appears to give satisfactory results whenever it is properly used. In addition, the only alternative seems to be kinetic theory, and this has certainly not yet reached the simplicity or elegance of continuum mechanics.

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[1] M. J. LIGHTHILL: Surveys in Applied Mechanics, p. 250. Cambridge 1956.

[2] G. STOKES: Trans. Cambridge Phil. Soc. **8**, 287 (1845). (Papers **1**, pp. 75–129).

[3] J. BOUSSINESQ: J. Math. Pures Appl. (2) **13**, 377 (1868).

[4] M. REINER: Amer. J. Math. **67**, 350 (1945).

[5] R. S. RIVLIN: Nature, Lond. **160**, 611 (1947). — Proc. Roy. Soc. Lond., Ser. A **193**, 260 (1948).

[6] C. TRUESDELL: J. Math. Pures Appl. (9) **29**, 215 (1950); (9) **30**, 111 (1951). Cf. also J. Rational Mech. Anal. **1**, 125 (1952); **2**, 593 (1953).

59. STOKES' principle. In a remarkable paper published when he was 26 years old, Sir GEORGE STOKES stated the concept of fluidity in the following form[1]: "That the difference between the pressure on a plane in a given direction passing through any point P of a fluid in motion and the pressure which would exist in all directions about P if the fluid in its neighborhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about P; and that the relative motion due to any motion of rotation may be eliminated without affecting the differences of the pressures above mentioned." Although this may seem fairly vague, it nevertheless contains the essential idea of fluidity. In his actual computations Stokes makes his ideas more precise. Specifically, they can be stated in the following postulational form:

1. $\mathbf{T}$ *is a continuous function of the deformation tensor* $\mathbf{D}$, *and is independent of all other kinematic quantities.*

2. $\mathbf{T}$ *does not depend explicitly on the position* $\mathbf{x}$ *(spatial homogeneity).*

3. There is no preferred direction in space (isotropy).

4. When $\mathbf{D}=0$, $\mathbf{T}$ *reduces to* $-p\mathbf{I}$.

Other sets of postulates are course possible and in some cases desirable[2], but for the purposes of this article and for almost all present-day hydrodynamical applications, STOKES' assumptions are sufficient. A medium whose constitutive equation satisfies the above postulates will be called a *Stokesian fluid.*

The mathematical expression of (1) and (2) is simply

$$\mathbf{T}=f(\mathbf{D}), \tag{59.1}$$

while the requirement of isotropy is expressed by the condition[3]

$$\mathbf{S}\,\mathbf{T}\,\mathbf{S}^{-1}=f(\mathbf{S}\,\mathbf{D}\,\mathbf{S}^{-1}) \tag{59.2}$$

for all orthogonal transformation matrices $\mathbf{S}$. Eq. (59.2) implies that there is no preferred direction either in the fluid or in space, in other words, that regardless of its orientation a given deformation produces the same intrinsic response. Alternatively, Eq. (59.2) implies that Eq. (59.1) is invariant under all rectangular coordinate transformations.

It is tacitly assumed that $\mathbf{T}$ depends on the thermodynamic state, but for the moment this dependence is irrevelant and need not be shown in Eq. (59.1).

We shall now show that postulates (1) to (4) lead to the following simple formula for the stress tensor,

$$\mathbf{T}=\alpha\,\mathbf{I}+\beta\,\mathbf{D}+\gamma\,\mathbf{D}^2, \tag{59.3}$$

where α, β, γ are scalar functions of the principal invariants of $\mathbf{D}$, that is

$$\alpha=\alpha(\mathrm{I},\mathrm{II},\mathrm{III}),\ \text{etc.}$$

Note. Principal invariants may be defined as the coefficients of λ in the expansion of the determinant $D(\lambda)=\det(\lambda\mathbf{I}-\mathbf{D})$; thus

$$D(\lambda)=\lambda^3-\mathrm{I}\,\lambda^2+\mathrm{II}\,\lambda-\mathrm{III}. \tag{59.4}$$

It follows in particular that $\mathrm{I}=\operatorname{Trace}\mathbf{D}=\operatorname{div}\mathbf{v}=\Theta$. The principal values d_1, d_2, d_3 of $\mathbf{D}$ are the roots of the equation $D(\lambda)=0$; they are real since $\mathbf{D}$ is

[1] G. STOKES: Papers **1**, pp. 75–129.

[2] See the article of C. TRUESDELL: J. Rational Mech. Anal. **1**, 125 (1952). Also W. NOLL: J. Rational Mech. Anal. **4**, 1 (1955). An interesting point of NOLL's work is his careful treatment of the notion of isotropy.

[3] It is convenient at this point to use the notation and terminology of matrix algebra.

symmetric. Clearly the principal values of $\mathbf{D}$ are functions of the principal invariants.

An assignment of the functions α, β, and γ gives rise to a definite type of viscous response for the fluid. For example, if we choose α, β, and γ so that $\mathbf{T}$ is linear in $\mathbf{D}$ we obtain the classical Cauchy-Poisson law of viscosity. In Sect. 65 we shall present some flow examples retaining the full non-linearity of Eq. (59.3).

Proof of Eq. (59.3)[1]. It will first be shown that the principal directions of $\mathbf{T}$ coincide with the principal directions of $\mathbf{D}$, or, in other words, that any orthogonal transformation reducing $\mathbf{D}$ to diagonal form likewise reduces $\mathbf{T}$ to diagonal form. Indeed, suppose that $\mathbf{D}$ has been transformed into

$$\overline{\mathbf{D}} = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix},$$

and let $\overline{\mathbf{T}}$ be the corresponding transformation of $\mathbf{T}$. Then by Eqs. (59.1) and (59.2), $\overline{\mathbf{T}} = f(\overline{\mathbf{D}})$. Now the orthogonal transformation

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

leaves $\overline{\mathbf{D}}$ unaltered. Therefore, again by assumptions (59.1) and (59.2),

$$\mathbf{S}\,\overline{\mathbf{T}}\,\mathbf{S}^{-1} = f(\mathbf{S}\,\overline{\mathbf{D}}\,\mathbf{S}^{-1}) = f(\overline{\mathbf{D}}) = \overline{\mathbf{T}}. \tag{59.5}$$

Thus $\overline{\mathbf{T}}$ is also unaltered by $\mathbf{S}$. It follows that $\bar{t}_{12} = \bar{t}_{13} = \bar{t}_{21} = \bar{t}_{31} = 0$, and in the same way $\bar{t}_{23} = \bar{t}_{32} = 0$. Thus we have finally

$$\overline{\mathbf{T}} = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix}.$$

Geometrically, $\mathbf{S}$ represents a rotation of 180° about the x_1-axis, and Eq. (59.5) shows that $\overline{\mathbf{T}}$ is symmetric about this axis.

Since $\overline{\mathbf{T}}$ is diagonal, it follows that the principal values of $\overline{\mathbf{T}}$ are functions of d_1, d_2, and d_3, that is

$$t_i = f_i(d_1, d_2, d_3), \qquad i = 1, 2, 3. \tag{59.6}$$

Now assuming for the moment that the principal values of $\mathbf{D}$ are all different, we can determine multipliers α, β, and γ such that

$$t_i = \alpha + \beta\, d_i + \gamma\, d_i^2, \qquad i = 1, 2, 3. \tag{59.7}$$

Indeed, Eq. (59.7) can be considered as three linear equations in the unknowns α, β, γ with coefficient determinant

$$\Delta = \begin{vmatrix} 1 & d_1 & d_1^2 \\ 1 & d_2 & d_2^2 \\ 1 & d_3 & d_3^2 \end{vmatrix} = (d_1 - d_2)(d_2 - d_3)(d_3 - d_1) \neq 0.$$

[1] Except for the final part of the proof we follow the elegant method of J.L. ERICKSEN and R.S. RIVLIN, J. Rational Mech. Anal. **4**, 323 (1955), especially § 30. In another place ERICKSEN has examined certain restrictions which the physical situation might possibly impose on the coefficients α, β, γ [J. Washington Acad. Sci. **44**, 33 (1954)].

Thus by Cramer's rule,

$$\alpha = \frac{1}{\Delta} \begin{vmatrix} t_1 & d_1 & d_1^2 \\ t_2 & d_2 & d_2^2 \\ t_3 & d_3 & d_3^2 \end{vmatrix}, \tag{59.8}$$

with similar expressions for β and γ. The multipliers are continuous functions of t_i and d_i, and the t_i in turn are functions of d_i. Therefore, as long as the principal values of $\bar{\mathbf{D}}$ are unequal, α, β, γ can be expressed as continuous functions of these principal values.

We observe next that any permutation of the d_i leads to a corresponding permutation of the t_i, [this is because the permutation can be effected by an orthogonal transformation, and by Eq. (59.2) this has the same effect on $\bar{\mathbf{T}}$ as on $\bar{\mathbf{D}}$]. Using this fact, it then follows from Eq. (59.8) that a permutation of the d_i leaves α unchanged. Naturally, the same result is true of β and γ; that is, *α, β and γ are symmetric functions of the variables d_i*. But this implies that α, β, and γ depend solely on the principal invariants Θ, II, and III (this is because the d_i are roots of the polynomial (59.4) having Θ, II, and III as coefficients). We have thus shown that *α, β, and γ are single-valued, continuous functions of the principal invariants of* $\mathbf{D}$. Reverting to matrix notation, Eq. (59.7) reads

$$\bar{\mathbf{T}} = \alpha \mathbf{I} + \beta \bar{\mathbf{D}} + \gamma \bar{\mathbf{D}}^2,$$

and this, when transformed to the original matrices $\mathbf{T}$ and $\mathbf{D}$, is precisely Eq. (59.3)

When two or more principal values of $\mathbf{D}$ coincide, then by a permutation of these values one sees that likewise the corresponding principal values of $\mathbf{T}$ coincide. Therefore by the reasoning which led to Eq. (59.7) we can establish either

$$\bar{\mathbf{T}} = \alpha \mathbf{I} + \beta \bar{\mathbf{D}},$$

or, if all the principal values are equal, $\bar{\mathbf{T}} = \alpha \mathbf{I}$. Since these formulas are also of the type (59.3), our proof is completed.

It is a slight disappointment to find that α, β, γ need not be continuous at a coalescence of the principal value of $\mathbf{D}$. If, however, $\mathbf{T}$ is assumed to be *three times* continuously differentiable in the components of $\mathbf{D}$, then continuity holds even at a coalescence. Since this result is not used in this sequel, its (relatively simple) proof will be omitted.

If to postulates 1 to 4 is added the condition that the components of $\mathbf{T}$ be linear in the components of $\mathbf{D}$, then Eq. (59.3) must reduce to the form

$$\mathbf{T} = (-p + \lambda \Theta) \mathbf{I} + 2\mu \mathbf{D}. \tag{59.9}$$

Although this result will appear later as a special case of the theorem of Sect. 60, it is nevertheless of interest to present here an alternate and much simpler proof, which is independent of most of the preceeding argument.

We begin with formula (59.6), and observe that in virtue of postulate 4 and the hypothesis of linearity it must have the form

$$\begin{aligned} t_1 &= a_1 d_1 + a_2 d_2 + a_3 d_3 - p, \\ t_2 &= b_1 d_1 + b_2 d_2 + b_3 d_3 - p, \\ t_3 &= c_1 d_1 + c_2 d_2 + c_3 d_3 - p. \end{aligned}$$

Where the coefficients are independent of $\mathbf{D}$. Then since permutation of the d_i causes a corresponding permutation of the t_i, the cyclic permutation $\langle 231 \rangle$

leads to the conditions

$$a_1 = b_2 = c_3, \quad a_2 = b_3 = c_1, \quad a_3 = b_1 = c_2$$

while the acyclic permutation $\langle 213 \rangle$ gives in addition $a_2 = b_1$. These equalities lead at once to

$$a_2 = a_3 = b_1 = b_3 = c_1 = c_2 = \lambda, \quad a_1 = b_2 = c_3 = \lambda + 2\mu,$$

and Eq. (59.9) then follows by transformation to the original matrices $\mathbf{T}$ and $\mathbf{D}$.

In the next section the concept of fluid pressure will be discussed and clarified, following which we shall conclude our treatment of the Stokesian fluid by considering the interesting case where the stress components T_{ij} are assumed to be *polynomials* in the deformation components D_{ij}.

59A. Pressure. For a compressible fluid the pressure is a well defined thermodynamic variable. This fact allows us to write Eq. (59.3) in the form

$$\mathbf{T} = (-p + \alpha^*) I + \beta \mathbf{D} + \gamma \mathbf{D}^2, \tag{59.10}$$

where p is the thermodynamic pressure and $\alpha^* = \alpha + p$. According to postulate 4 on p. 231, α^* must vanish when $\mathbf{D} = 0$. Also, comparing Eq. (59.10) with Eq. (34.1) gives the formula

$$\mathbf{V} = \alpha^* \mathbf{I} + \beta \mathbf{D} + \gamma \mathbf{D}^2,$$

expressing the viscosity tensor in terms of the deformation tensor.

For an *incompressible* fluid, on the other hand, we have already noted (in Sect. 33) that pressure is *not* a thermodynamic variable, and, in fact, it has so far been left undefined[1]. We are at liberty, therefore, to introduce any definition of pressure whatsoever, so long as it is compatible with postulate 4 of the preceding section. No matter what choice is finally made, it should be borne in mind that what is called "pressure" in the hydrodynamical equations is not necessarily what is recorded by a pressure measuring device; in fact, the latter is usually a particular component of the stress tensor.

In view of the remarks above it seems reasonable to choose for pressure a quantity which will lead to the greatest simplicity in the equations of motion. We thus define p to be the negative of the coefficient α in Eq. (59.3). This gives the formula

$$\mathbf{T} = -p \mathbf{I} + \beta \mathbf{D} + \gamma \mathbf{D}^2 \tag{59.11}$$

for the stress tensor in an *incompressible* fluid. It should be noted that this definition of pressure is compatible with the condition that $\mathbf{T}$ reduce to $-p\mathbf{I}$ when $\mathbf{D} = 0$.

In some works (eg. LAMB [8]) fluid pressure is defined by $p = -\frac{1}{3}\mathrm{Trace}\,\mathbf{T}$. Although this is entirely permissible, it does not give as simple a formula for $\mathbf{T}$ as does the above definition. In the present article we shall call $-\frac{1}{3}\mathrm{Trace}\,\mathbf{T}$ the *mean pressure* and denote it by $\bar{p}$. In order to see the difference between pressure and mean pressure in an incompressible fluid, we have from Eq. (59.11),

$$3(p - \bar{p}) = \gamma \mathbf{D} : \mathbf{D}.$$

Thus $p = \bar{p}$ if and only if $\gamma = 0$; that is, general equality of pressure and mean pressure in an incompressible fluid is equivalent to a quasi-linear stress-deformation relation. In particular, the distinction between p and $\bar{p}$ vanishes in the

[1] This refers only to *viscous* fluids, of course.

classical linear viscosity theory: either definition of the pressure, $p = -\alpha$ or $p = -\frac{1}{3}\operatorname{Trace}\mathbf{T}$, leads to exactly the same result. We emphasize this point because in many places the somewhat arbitrary nature of the pressure in an incompressible fluid is entirely overlooked.

60. Polynomial dependence. It was shown in the previous section that Eqs. (59.10) and (59.11) are the most general stress-deformation equations consistent with postulates 1 to 4 on p. 231. The coefficients α^*, β, and γ in these equations are arbitrary functions of the principal invariants of $\mathbf{D}$ and of the thermodynamic state variables. In order to treat actual flow problems one must specify the particular forms of α^*, β, and γ, since otherwise, if they are allowed to remain arbitrary, prohibitive difficulties would arise in the solution of all but the simplest type of problem. For this reason, it is almost universal practice to choose α^*, β, γ so that the stress components T_{ij} are *polynomials* in the deformation components D_{ij}.

The most important and most frequently occuring case, namely $\mathbf{T}$ linear in $\mathbf{D}$, has already been treated in Sect. 59. Of next importance is quadratic dependence of $\mathbf{T}$ on $\mathbf{D}$. Since Θ, II, III are respectively of degrees 1, 2, and 3 in the components of $\mathbf{D}$, this can be achieved simply by specializing Eqs. (59.10) and (59.11) to the form

$$\mathbf{T} = \begin{cases} (-p + \lambda\Theta + \lambda'\Theta^2 + \lambda''\,\mathrm{II})\,\mathbf{I} + \\ \qquad + (2\mu + 2\mu'\Theta)\,\mathbf{D} + 4\nu\,\mathbf{D}^2 & \text{(compressible)}, \\ -p\,\mathbf{I} + 2\mu\,\mathbf{D} + 4\nu\,\mathbf{D}^2 & \text{(incompressible)}, \end{cases} \tag{60.1}$$

and higher order stress-deformations are likewise easily written down. The question remains, whether Eq. (60.1) and its higher order analogues are the *most general* relations possible when quadratic or higher order dependence is assumed. The crucial theorem is this:

The most general constitutive equation satisfying postulates 1 to 4 above, and such that the stress components T_{ij} are polynomials of degree N in the deformation components D_{ij}, is

$$\mathbf{T} = (-p + P_0)\,\mathbf{I} + P_1\mathbf{D} + P_2\mathbf{D}^2, \tag{60.2}$$

where the P_k, $k = 0, 1, 2$, are polynomials in the principal invariants of $\mathbf{D}$, with weight[1] *$P_k \leqq N - k$. For a compressible fluid, the constant term of P_0 must vanish, while for an incompressible fluid $P_0 \equiv 0$*[2].

Proof. It will be sufficient to demonstrate that α^* in Eq. (59.10) has the form P_0, the argument for β and γ being exactly similar. Now according to hypothesis the functions f_i in Eq. (59.6) must be polynomials of degree N at most (orthogonal transformations clearly do not affect the assumed polynomial dependence). Therefore the determinant in the expression Eq. (59.8) for α is a polynomial of degree $N+3$ at most in the variables d_i. It vanishes when $d_1 = d_2$ (since then

[1] By the weight of P_k is meant the maximum weight of any of its terms, a term $\Theta^k\,\mathrm{II}^l\,\mathrm{III}^m$ being counted of weight $k + 2l + 3m$.

[2] This theorem has been stated and proved in various degrees of generality by a number of writers. For the special case of linear dependence the result is due to Stokes. Interest in a general polynomial dependence of $\mathbf{T}$ on $\mathbf{D}$ has arisen more recently, the formula (60.2) appearing first in the work of Reiner and Rivlin (loc. cit. Sect. 58). The first rigorous proof, however, seems to be due to Rivlin and Ericksen, though their derivation (Sects. 39 and 41 of the paper cited above) is far from simple. The proof we give is different and quite straightforward; the author is indebted to E. Calabi for some of the ideas.

The theorem may be generalized to the case of analytic dependence of $\mathbf{T}$ on $\mathbf{D}$, the P_k in this case being analytic functions of the principal invariants.

$t_1 = t_2$), hence contains $(d_1 - d_2)$ as a factor; similarly it has $(d_2 - d_3)$ and $(d_3 - d_1)$ as factors. We may therefore divide out the factor Δ in Eq. (59.8), thus leaving for α a polynomial of at most degree N in d.

But we have shown earlier that α is a symmetric function of the d_i, so by a well known theorem of algebra[1] α is a polynomial of weight at most N in the principal invariants of $\mathbf{D}$. To satisfy postulate 4, the constant-term of α must be $-p$, hence $\alpha^* = \alpha + p$ is a polynomial P_0 with constant term zero. Q.E.D.

Using dimensional analysis, TRUESDELL[2] has shown how the coefficients in the polynomials P_k must depend on thermodynamic quantities.

61. Classical hydrodynamics. The Navier-Stokes equation. Because the deformation tensor is in general fairly small in comparison with, say, the ratio of some reference speed and reference length, it is a reasonable hypothesis to assume a linear relation between $\mathbf{T}$ and $\mathbf{D}$. The fact that this is a hypothesis should be clearly understood: it is not to be derived from experiments, nor can it be proved by abstract reasoning; if results obtained on the basis of this hypothesis agree with experiments, then of course so much the better for the hypothesis and our faith in its validity.

The hypothesis of linearity leads, by application of the final result of Sect. 59, to the classical constitutive equations

$$\mathbf{T} = \begin{cases} (-p + \lambda\Theta)\,\mathbf{I} + 2\mu\,\mathbf{D} & \text{(compressible)},\\ -p\,\mathbf{I} + 2\mu\,\mathbf{D} & \text{(incompressible)}. \end{cases} \tag{61.1}$$

Here, for a compressible fluid p is the thermodynamic pressure, $\Theta = \operatorname{div}\mathbf{v}$, and λ and μ are scalar functions of the thermodynamic state. For an incompressible fluid p is a fundamental dynamical variable and μ is a scalar function of temperature[3].

The dependence of the viscosity coefficients on the thermodynamic state is of some importance and is treated at length by CHAPMAN and COWLING[4]. TRUESDELL has clarified the nature of the kinetic theory argument used to support the result $\mu \sim T^{\frac{1}{2}}$, and has considered the problem from the point of view of dimensional analysis[5]. Except in extreme cases, both theory and experiment indicate that the viscosity coefficients are sensibly independent of pressure and that a formula $\mu \sim T^m$ is quite accurate. Further remarks on this subject will be found in the following section.

The dissipation function corresponding to the Cauchy-Poisson law (61.1) is

$$\Phi = \mathbf{V} : \mathbf{D} = \begin{cases} \lambda\,\Theta^2 + 2\mu\,\mathbf{D} : \mathbf{D} & \text{(compressible)},\\ 2\mu\,\mathbf{D} : \mathbf{D} & \text{(incompressible)}. \end{cases} \tag{61.2}$$

The condition $\Phi \geqq 0$, derived in Chap. D, Sect. 34, places some restrictions on the possible values of λ and μ. In particular, for an incompressible fluid we must

[1] B.L. VAN DER WAERDEN: Modern Algebra, Vol. 1, New York 1949, especially pp. 78 to 81.

[2] C. TRUESDELL: Cf. footnote 6, p. 230.

[3] Eq. (61.1) appeared first in the work of S.-D. POISSON, J. Ecole Polytech. **13**, Cahier 20, 1 (1831). CAUCHY had earlier found a similar constitutive equation, lacking only the term $-p\mathbf{I}$. Further references are given by H. BATEMAN [*2*], pp. 89—91 and by C. TRUESDELL, J. Rational Mech. Anal. **1**, 126 (1952).

[4] [*30*], Chaps. 9—12. COPE and HARTREE have examined the available experimental data for air [Phil. Trans. Roy. Soc. Lond. A **241**, 282 (1948)], and recent work of F.R. GILMORE [Rand. Corp. Memo. RM-1543 (Santa Monica, 1955)] may also be consulted in this respect.

[5] C. TRUESDELL: Z. Physik **131**, 273 (1952); cf. also footnote 6, p. 230.

have $\mu \geqq 0$. For a compressible fluid, a fairly straightforward computation shows that

$$3\,\Phi = (3\lambda + 2\mu)\,\Theta^2 + 2\mu\,[(d_1 - d_2)^2 + (d_2 - d_3)^2 + (d_3 - d_1)^2],$$

and hence Φ is positive for all $\mathbf{D}$ if and only if

$$\mu \geqq 0, \quad 3\lambda + 2\mu \geqq 0.$$

The quantity $(3\lambda + 2\mu)$ also arises in an expression for the difference between pressure and mean pressure in a compressible fluid, namely

$$3\,(p - \bar{p}) = (3\lambda + 2\mu)\,\Theta. \tag{61.3}$$

The Navier-Stokes equation[1]. The dynamical equation which results from inserting the Cauchy-Poisson law into (6.7) is known as the Navier-Stokes equation. It takes a slightly different form for compressible and for incompressible fluids, viz.

$$\text{Compressible:} \quad \varrho\,\frac{d\mathbf{v}}{dt} = \varrho\,\mathbf{f} - \operatorname{grad} p + \operatorname{grad}(\lambda\,\Theta) + \operatorname{div}(2\mu\,\mathbf{D}), \tag{61.4}$$

$$\text{Incompressible:} \quad \varrho\,\frac{d\mathbf{v}}{dt} = \varrho\,\mathbf{f} - \operatorname{grad} p + \operatorname{div}(2\mu\,\mathbf{D}). \tag{61.5}$$

If one wishes, Eq. (61.5) can be thought of as the special case of Eq. (61.4) which arises when $\Theta = \operatorname{div}\mathbf{v} = 0$. This view is somewhat superficial, however, because the pressure has fundamentally different meanings for compressible and incompressible fluids; we prefer therefore to write Eqs. (61.4) and (61.5) as separate equations. If λ and μ are constant Eqs. (61.4) and (61.5) simplify to the forms,

$$\text{Compressible:} \quad \varrho\,\frac{d\mathbf{v}}{dt} = \varrho\,\mathbf{f} - \operatorname{grad} p + (\lambda + \mu)\operatorname{grad}\Theta + \mu\,\nabla^2\mathbf{v}, \tag{61.6}$$

$$\text{Incompressible:} \quad \varrho\,\frac{d\mathbf{v}}{dt} = \varrho\,\mathbf{f} - \operatorname{grad} p + \mu\,\nabla^2\mathbf{v}. \tag{61.7}$$

In order to express the above equations in orthogonal coordinates it is necessary to have formulas for the acceleration, for Θ, and for $\nabla^2\mathbf{v}$. Formulas suitable for the evaluation of $\mathbf{a}$ and Θ will be found in Sect. 12, while the Laplacian is most easily obtained by using the identity

$$\nabla^2\mathbf{v} = \operatorname{grad}\Theta - \operatorname{curl}\boldsymbol{\omega};$$

even for incompressible fluids it is convenient to retain the term $\operatorname{grad}\Theta$, for it serves to cancel a number of terms from $\operatorname{curl}\boldsymbol{\omega}$. The components of the stress tensor (61.1) in general coordinates are given by

$$T^i_{\ k} = (-p + \lambda\,\Theta)\,\delta^i_k + 2\mu\,D^i_{\ k},$$

where

$$D^i_{\ i} = \frac{\partial v^i}{\partial x^i} + v^k\,\frac{\partial \log h_i}{\partial x^k}, \quad D^i_{\ k} = \frac{1}{2}\left[\frac{\partial v^i}{\partial x^k} + \left(\frac{h_k}{h_i}\right)^2 \frac{\partial v^k}{\partial x^i}\right] \quad (k \neq i).$$

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[1] C.-L.-M.-H. NAVIER: Mem. Acad. Sci. Inst. France (2) **6**, 289 (1827), the memoir is dated 1822. — G. STOKES: Papers **1**, pp. 75—129. NAVIER considered only incompressible fluids.

Formulas for $\mathbf{a}$, Θ, and ω in cylindrical polar coordinates are given at the end of Sect. 12. Also, by the method outlined in the previous paragraph we find

$$(\operatorname{grad}\Theta)_r = \frac{1}{r}\frac{\partial}{\partial r}(r\Theta) - \frac{\Theta}{r} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 v_\theta}{\partial r\,\partial\theta} + \frac{\partial^2 v_z}{\partial r\,\partial z} - \frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{v_r}{r^2}.$$

$$(\operatorname{grad}\Theta)_\theta = \frac{1}{r^2}\frac{\partial}{\partial\theta}(r\Theta) \quad = \frac{1}{r}\frac{\partial^2 v_r}{\partial r\,\partial\theta} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2} + \frac{1}{r}\frac{\partial^2 v_z}{\partial\theta\,\partial z} + \frac{1}{r^2}\frac{\partial v_r}{\partial\theta},$$

$$(\operatorname{grad}\Theta)_z = \frac{1}{r}\frac{\partial}{\partial z}(r\Theta) \quad = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial z}\right) + \frac{1}{r}\frac{\partial^2 v_\theta}{\partial\theta\,\partial z} + \frac{\partial^2 v_z}{\partial z^2},$$

and

$$(\nabla^2\mathbf{v})_r = \Delta v_r - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{v_r}{r^2},$$

$$(\nabla^2\mathbf{v})_\theta = \Delta v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r^2},$$

$$(\nabla^2\mathbf{v})_z = \Delta v_z, \qquad \Delta \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}.$$

Finally, letting $\widehat{rr}$, $\widehat{r\theta}$, etc., stand for the physical components of stress, we have

$$\widehat{rr} = -p + \lambda\Theta + 2\mu\frac{\partial v_r}{\partial r}, \qquad \widehat{zz} = -p + \lambda\Theta + 2\mu\frac{\partial v_z}{\partial z},$$

$$\widehat{\theta\theta} = -p + \lambda\Theta + 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r}\right),$$

$$\widehat{r\theta} = \mu\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right), \quad \widehat{\theta z} = \mu\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial\theta}\right), \quad \widehat{zr} = \mu\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right).$$

Explicit evaluation of Eqs. (61.6) and (61.7) for spherical coordinates will be found in reference [*42*], §§ 39 to 41.

62. The Stokes relation. In the paper referred to in Sect. 58 STOKES set forth two arguments in justification of the relation

$$3\lambda + 2\mu = 0, \tag{62.1}$$

between the viscosity coefficients of a compressible viscous fluid. Summarized briefly, they are that unless Eq. (62.1) holds we would find in a uniform spherical expansion both non-zero dissipation and a difference between pressure and mean pressure [see formula (61.3)]. Another and perhaps stronger argument in favor of Eq. (62.1) is that it arises in MAXWELL's kinetic theory of monatomic gases. Now STOKES' reasons are not very convincing, and indeed at a later time[1] STOKES said that he never put great faith in his relation. Moreover, MAXWELL's result has been shown by TRUESDELL[2] to be tantamount to assuming (62.1), though it nevertheless retains much force.

Another facet of the question arises if one adopts not the phenomenological viewpoint of Sect. 58 and 59, but considers instead the internal constitution of a fluid undergoing large stresses. Then already the simple thermodynamic theory of Chap. D may not be adequate for the explanation of observed behavior, and one may be inclined to use the bulk viscosity term in the stress tensor to make up in one way or another for the failure of the thermodynamic equation of state. Obviously if one adopts this point of view, Eq. (62.1) cannot be seriously maintained.

Turning to the experimental situation there are considerable complications, not the least because any experiment to determine λ must involve relatively

[1] G. STOKES: Papers **3**, p. 136—137.
[2] C. TRUESDELL: Z. Physik **131**, 273 (1952).

large values of the expansion, perhaps so large that a linear stress-deformation relation is already in doubt. From the basic papers on the subject[1] we draw the following conclusions: that Eq. (62.1) is reasonably correct for monatomic gases; that for liquids and polyatomic gases, on the other hand, λ is always positive and sometimes many times greater than μ. As for appropriate values to assign to λ and μ the reader is referred to the treatise of CHAPMAN and COWLING [*30*] and to the papers noted in the preceding footnotes.

Of course, the value of λ may not be very important in cases where compressibility effects are slight.

63. Heat conduction. In order to complete the system of equations of classical fluid mechanics it is necessary to express the heat conduction vector $\mathbf{q}$ in terms of mechanical and thermodynamical variables. We follow the commonly accepted formulation and postulate that $\mathbf{q}$ *is an isotropic function of the temperature gradient and thermodynamic state*. More general situations are easily conceived, in which deformations give rise to heat conduction (and vice versa)[2], but for present-day situations our assumption seems entirely adequate. Let us see where it leads.

From the condition of isotropy it is apparent that $\mathbf{q}$ must be parallel to grad T, whence follows the classical Newton-Fourier law

$$\mathbf{q} = -\varkappa \operatorname{grad} T, \tag{63.1}$$

where $\varkappa$ is a scalar function of $|\operatorname{grad} T|$ and of thermodynamical variables. The thermodynamical condition (34.4) implies that $\varkappa \geqq 0$. The particular functional form of $\varkappa$ to be adopted in a given problem is of course a matter for experiment and kinetic theory to decide. Here the evidence favors a relation[3]

$$\frac{\mu c_p}{\varkappa} = \text{const}. \tag{63.2}$$

This is consistent with the frequent assumption that $\varkappa$, μ and c_p are all constants, and in any case implies that $\varkappa$ should not depend on $|\operatorname{grad} T|$. For an ideal gas with constant specific heats, the ratio (63.2) is usually given values as follows:

$$\frac{\mu c_p}{\varkappa} = \frac{2}{3}, \quad \frac{3}{4}, \quad \text{and } 0.72,$$

respectively for monatomic gases, diatomic gases, and air. $\mu c_p/\varkappa$ is called the Prandtl number; it will appear again in considering the dynamical similarity of flows with non-zero heat conduction.

When Eq. (63.1) is substituted into the energy equation (34.3) there results

$$\varrho \frac{dS}{dt} = \Phi + \operatorname{div}(\varkappa \operatorname{grad} T). \tag{63.3}$$

This equation, the Navier-Stokes equation, the equation of continuity, and the thermodynamical equations of state constitute the general set of equations upon which classical hydrodynamics is based. Of course, these equations are hardly amenable to any general analysis, but rather one must be content with various special cases of the theory.

[1] L. TISZA: Phys. Rev. **61**, 531 (1942). — L. LIEBERMAN: Phys. Rev. **75**, 1415; **76**, 440 (1950). — S. KARIM and L. ROSENHEAD: Rev. Mod. Phys. **24**, 108 (1952). See also C. TRUESDELL: Proc. Roy. Soc. Lond., Ser. A **226**, 59 (1954).

[2] Cf. footnote 2, p. 231.

[3] CHAPMAN and COWLING [*30*], Chap. 13.

64. Boundary conditions. We must now inquire into the dynamical conditions to be satisfied at the boundaries of a fluid. At a free surface, or at an interface between two dissimilar fluids, the stress vector should be continuous, as remarked by LAMB. Whether this guarantees a continuous velocity distribution across the surface or interface seems to be an open question, although this appears likely from consideration of special examples.

The conditions to be satisfied at a solid boundary are more difficult to state, and subject to some controversy. STOKES argued[1] that the fluid must adhere to the solid, since the contrary assumption implies an infinitely greater resistance to the sliding of one portion of fluid past another than to the sliding of fluid over a solid. Another and perhaps stronger argument in favor of adherence, at least for the case of liquids at ordinary conditions, is found in experiments with tube viscometers where the fourth power of the diameter law is conclusively verified. Although these facts are quite convincing when moderate pressures and low surface stresses are involved, they do not apply in all cases; indeed in high altitude aerodynamics an adherence condition is no longer true[2].

To explain the observed phenomena, we have first of all the classical adherence condition

$$\Delta \mathbf{v} = 0 \quad \text{at solid boundaries,} \tag{64.1}$$

where $\Delta \mathbf{v}$ denotes the difference between the fluid and boundary velocities. Various slip conditions have been proposed in place of Eq. (64.1), the most important being

$$\Delta v_n = 0, \qquad \Delta \mathbf{v}_t = k\,\mathbf{t}_t, \tag{64.2}$$

(the subscripts n and t denote respectively normal and tangential components at the boundary). This law is phenomenologically reasonable and moreover has some justification in kinetic theory. If the coefficient k is a function of the thermodynamic variables (in particular, if k is zero except at low pressures), then Eq. (64.2) can even account for adherence. The present author would like to suggest a modification of Eq. (64.2) which includes a "coefficient of friction" k_1; this modification implies adherence at low tangential stresses and high pressures, and at the same time reduces to Eq. (64.2) for the opposite extremes. Specifically, we set

$$K = \begin{cases} 0 & \text{if } k\,|\mathbf{t}_t| \leqq k_1\,|t_n|, \\ k - k_1 \left|\dfrac{t_n}{\mathbf{t}_t}\right| & \text{otherwise,} \end{cases}$$

where k and k_1 are certain positive constants, and propose the boundary condition

$$\Delta v_n = 0, \qquad \Delta \mathbf{v}_t = K\,\mathbf{t}_t. \tag{64.3}$$

As yet no calculations have been made to see how well Eq. (64.3) agrees with observations. Other boundary conditions have been proposed by MAXWELL, DUHEM, KNUDSEN, and by CHANG and UHLENBECK, and the entire field has been reviewed by BATEMAN[3], and more recently by TRUESDELL[4] and PATTERSON[5].

In the remainder of this article we adopt Eq. (64.1) as the standard boundary condition. With this assumption it is easy to show that the vorticity vector is

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[1] G. STOKES: Cf. footnote 1 p. 231.
[2] H. TSIEN: J. Aeronaut. Sci. **13**, 653 (1946).
[3] H. BATEMAN [*2*], Part II, §§ 1.2, 1.7, 3.2.
[4] C. TRUESDELL: J. Rational Mech. Anal. **1**, 125 (1952), especially § 79.
[5] G.N. PATTERSON [*34*], Chap. 5.

tangential to a fixed boundary wall. Indeed by STOKES' theorem,

$$\int_{s} \boldsymbol{\omega} \cdot \mathbf{n}\, d a = \oint_{c} \mathbf{v} \cdot d\mathbf{x} = 0,$$

where s is an arbitrary area on the wall bounded by the circuit c. Thus necessarily $\boldsymbol{\omega} \cdot \mathbf{n} = 0$ on the wall. By a similar argument BERKER[1] has shown that the stress vector at a fixed wall reduces to

$$\mathbf{t} = \left\{ \begin{array}{ll} [-p + (\lambda + 2\mu)\,\Theta]\,\mathbf{n} + \mu\,(\boldsymbol{\omega} \times \mathbf{n}) & \text{(compressible)}, \\ \qquad\qquad -p\,\mathbf{n} + \mu\,(\boldsymbol{\omega} \times \mathbf{n}) & \text{(incompressible)}, \end{array} \right\} \tag{64.4}$$

provided the classical viscosity law (64.1) is used. This shows that tangential stresses depend only on the local vorticity, while normal stresses are due to the pressure and expansion. It may be noticed that the normal stress at the wall is *not* equal to the mean pressure $\bar{p}$ in the case of a compressible fluid; indeed, the following relation holds between the normal stress p_n the fluid pressure p, and the mean pressure $\bar{p}$,

$$p_n = p - (\lambda + 2\mu)\,\Theta = \bar{p} - \tfrac{4}{3}\mu\,\Theta. \tag{64.5}$$

65. Appendix: Special solutions with non-linear viscosity. The qualitative effect of the non-linear terms in formula (59.3) can be assessed most easily by examining special solutions of the equations of motion. We consider here two simple examples of motion of an incompressible fluid, first worked out in detail by RIVLIN[2]. They indicate a basic qualitative difference between linear and non-linear viscosity, namely that shearing may produce normal stresses.

1. Rectilinear shearing flow. Suppose that $u = k y$, $v = w = 0$, where k is constant. Then by Eq. (59.3) the stress tensor is

$$\mathbf{T} = -p\,\mathbf{I} + \tfrac{1}{2}\beta\,k \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \tfrac{1}{4}\gamma\,k^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{65.1}$$

where $\beta = \beta\,(\text{I, II, III}) = \beta\,(0, -\frac{1}{4}k^2, 0)$ and $\gamma = \gamma\,(0, -\frac{1}{4}k^2, 0)$. A particular consequence of Eq. (65.1) is that the specific resistance T_{xy} is necesarily an odd function of k, the rate of shearing. Since there is no acceleration in a shear flow, the dynamical equations (6.7) become

$$\left. \begin{aligned} 0 &= -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\frac{1}{4}\gamma\,k^2\right) + \frac{\partial}{\partial y}\left(\frac{1}{2}\beta\,k\right), \\ 0 &= -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\frac{1}{2}\beta\,k\right) + \frac{\partial}{\partial y}\left(\frac{1}{4}\gamma\,k^2\right). \end{aligned} \right\} \tag{65.2}$$

Now if β and γ do not depend on temperature, then Eq. (65.2) has the simple solution $p = \text{const}$, and

$$T_{yy} = -p + \frac{\gamma k^2}{2}.$$

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[1] R. BERKER: C. R. Acad. Sci., Paris **232**, 148 (1951). — BERKER's proof goes as follows: By [*48*], p. 72, we have

$$\oint \mathbf{v} \times d\mathbf{x} = -\int (\mathbf{n} \times \boldsymbol{\nabla}) \times \mathbf{v}\, d a = \int (\mathbf{n}\Theta + \tfrac{1}{2}\boldsymbol{\omega} \times \mathbf{n} - \mathbf{n} \cdot \mathbf{D})\, d a.$$

Since the integral on the left vanishes at a fixed wall, the integrand $\mathbf{n}\,\Theta + \frac{1}{2}\boldsymbol{\omega} \times \mathbf{n} - \mathbf{n} \cdot \mathbf{D}$ must likewise vanish there. Using this fact, the formula $\mathbf{t} = \mathbf{n} \cdot \mathbf{T} = (-p + \lambda\,\Theta)\mathbf{n} + 2\mu\,\mathbf{n} \cdot \mathbf{D}$ is easily transformed into Eq. (64.4).

[2] R. S. RIVLIN: Proc. Roy. Soc. Lond., Ser. A **193**, 260 (1948). — Proc. Cambridge Phil. Soc. **45**, 88 (1949).

This conclusion shows that to produce a shearing flow between two parallel infinite plates, one needs in addition to the usual shearing force and pressure, an additional stress normal to the plates and proportional to the *square* of the rate of shearing. This striking phenomenon has been called a Poynting effect. When temperature gradients are present, the simple solution $p=\text{const}$ is no longer adequate, and the specific dependence of β and γ on pressure and temperature must be given before a solution can be effected.

2. *Poiseuille flow*. In cylindrical polar coordinates r, θ, z, we suppose a motion of the form

$$v_r = v_\theta = 0, \qquad v_z \equiv w = w(r), \qquad 0 \leqq r \leqq a,$$

and $w(a)=0$. In this case $\mathbf{T}$ has the form (cf. Sect. 61)

$$\mathbf{T} = -p\mathbf{I} + \tfrac{1}{2}\beta w' \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \tfrac{1}{4}\gamma w'^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $\beta=\beta\ (0, -\frac{1}{4}w'^2, 0)$, $\gamma=\gamma\ (0, -\frac{1}{4}w'^2, 0)$. The equations of motion (12.3) reduce by means of Eq. (12.9) to

$$\left.\begin{aligned} \frac{\partial p}{\partial r} - \frac{1}{4r}\frac{\partial}{\partial r}(r\gamma w'^2) - \frac{w'}{2}\frac{\partial \beta}{\partial z} &= 0, \\ \frac{\partial p}{\partial z} - \frac{1}{2r}\frac{\partial}{\partial r}(r\beta w') - \frac{w'^2}{4}\frac{\partial \gamma}{\partial z} &= 0. \end{aligned}\right\} \tag{65.3}$$

If β, γ are independent of both pressure and temperature, then we can obtain a solution under the hypothesis

$$p = Cz + f(r), \qquad C = \text{const}. \tag{65.4}$$

Indeed, substituting this expression for p into Eq. (65.3) leads to the following ordinary differential equation for the velocity profile

$$\beta w' = Cr. \tag{65.5}$$

Then from Eq. (65.3), we have

$$f(r) = \int \frac{1}{4r}\frac{d}{dr}(r\gamma w'^2)\, dr. \tag{65.6}$$

The mass flux $\mathfrak{M}$ is given by

$$-\varrho \int w\, da = -\pi\varrho \int_0^a w\, d(r^2) = \pi\varrho C \int_0^a r^3 \beta^{-1}\, dr. \tag{65.7}$$

[With gravity present C is to be replaced by $C-\varrho q$ in Eq. (65.4).]

In the special case of quadratic dependence of $\mathbf{T}$ on $\mathbf{D}$ the coefficients β and γ are constant, since we are considering only incompressible fluids. Setting $\beta=2\mu$ and solving Eq. (65.5) yields the classical formula for velocity

$$w = \frac{C}{4\mu}(r^2 - a^2),$$

and mass flux,

$$\mathfrak{M} = \frac{\pi\varrho C}{8\mu} a^4. \tag{65.8}$$

On the other hand, the pressure distribution across a section of the pipe is no longer uniform. To investigate this effect in detail we may suppose the fluid to issue from the pipe into an atmosphere at pressure p_0, the latter exerting a force on the output cross section in the amount $\pi a^2 p_0$. Since

$$\widehat{zz} = \widehat{rr} = -Cz - \frac{1}{32}\frac{\gamma C^2}{\mu^2} r^2 - \text{const},$$

we have the following force balance at the exit section (taken to be $z=0$),

$$\pi a^2 p_0 = -\pi \int_0^a \widehat{zz}\, d(r^2) = \frac{\pi}{64}\frac{\gamma C^2}{\mu^2} a^4 + \pi a^2 \text{const}.$$

Thus the fluid exerts a force per unit area on the wall of the amount

$$P = -\widehat{rr}\Big|_{r=a} = Cz + \frac{1}{64}\frac{\gamma C^2}{\mu^2} a^2 + p_0.$$

If we introduce the quantity $\Gamma = \mathfrak{M}/\pi a^2 \varrho$ = volume of flow per second per unit cross section area, then from Eq. (65.8),

$$C = 8\mu \Gamma / a^2,$$

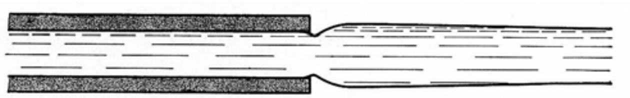

Fig. 16. Fluid stream issuing from a small diameter viscometer (after MERRINGTON).

and our preceding equation can be written in the interesting form

$$\frac{P - p_0}{(\Gamma/a)^2} = 8\mu \Gamma^{-1} z + \gamma. \tag{65.9}$$

Formula (65.9) may serve as the theoretical explanation of an interesting experimental phenomenon discovered by MERRINGTON[1], namely the tendency of a fluid stream to swell at the exit section of a viscometer (Fig. 16). In particular, according to Eq. (65.9) there is a pressure difference at the exit section ($z=0$) which is positive if $\gamma > 0$, so that MERRINGTON's experiment can be explained simply on the basis of a positive second coefficient of viscosity. It would be possible to check Eq. (65.9) by determining the variation of P along the pipe for various values of Γ; at the same time this would give a well verified experimental value for γ. In carrying out such an experiment, it may be noted from Eq. (65.9) that the effect of γ is emphasized by high flux and small pipe radius, and is *independent of pipe length* (MERRINGTON in fact observed that the swelling shown in Fig. 16 increased with increasing stress and concentration).

In the papers referred to earlier RIVLIN also treated the case of Couette flow, but this analysis will be omitted. GILBARG and PAOLUCCI[2] have considered the shock layer solution for a fluid with non-linear viscosity.

II. Dynamical similarity.

In the following sections we shall determine necessary conditions for two viscous fluid motions to be dynamically similar. We assume that the stress-deformation relation is linear, but do not require constant viscosity and conductivity[3]. It is convenient to consider compressible and incompressible fluids separately.

[1] A.C. MERRINGTON: Nature, Lond. **152**, 663 (1943).

[2] D. GILBARG and D. PAOLUCCI: J. Rational Mech. Anal. **2**, 617 (1953).

[3] In the usual treatment the viscosities are assumed either to be constant or to be proportional to a power of the absolute temperature. The following analysis shows that an assumption of this sort is *necessary*: without such behavior strict dynamical similarity is impossible.

66. Compressible viscous fluids. Two fluid motions are dynamically similar if they are related by equations

$$\mathbf{v} = U\,\mathbf{v}', \quad \varrho = R\,\varrho', \quad p = P\,p' \tag{66.1}$$

and

$$\mathbf{x} = D\,\mathbf{x}', \quad t = (D/U)\,t', \tag{66.2}$$

(cf. Sect. 36). We shall inquire into the conditions which these relations place on the two flows; since the analysis is somewhat lengthy, the reader interested only in the results may turn at once to a summarizing statement on p. 246.

The equation of continuity supplies no conditions, so we proceed to the Navier-Stokes equation (61.4), neglecting the extraneous force term. If Eqs. (66.1) and (66.2) are substituted into this equation there results

$$\left.\begin{aligned} R\,U\,\varrho'\,\frac{d\mathbf{v}'}{dt} &= -\frac{P}{U}\,\mathrm{grad}'\,p' + \frac{1}{D}\Big\{\lambda\,\mathrm{grad}'\,\Theta' + 2\mu\,\mathrm{div}'\,\mathbf{D}' + \\ &+ \Theta'\left(\frac{\partial\lambda}{\partial p'}\,\mathrm{grad}'\,p' + \frac{\partial\lambda}{\partial\varrho'}\,\mathrm{grad}'\,\varrho'\right) + 2\mathbf{D}'\cdot\left(\frac{\partial\mu}{\partial p'}\,\mathrm{grad}'\,p' + \frac{\partial\mu}{\partial\varrho'}\,\mathrm{grad}'\,\varrho'\right)\Big\}, \end{aligned}\right\} \tag{66.3}$$

it being assumed that λ and μ are functions of thermodynamic variables alone, and $\Theta = \mathrm{div}\,\mathbf{v}$. [In the special case of constant viscosities the terms on the second line of Eq. (66.3) vanish.] Now consider the congruence of curves C in the flow region determined by the equations $p' = \mathrm{const}$, $\varrho' = \mathrm{const}$. Along any one of these curves, Eq. (66.3) has the form

$$\sum_{i=1}^{8} a_i\,\boldsymbol{\xi}_i = 0,$$

where the a_i are constants and the $\boldsymbol{\xi}_i$ are vector variables; specifically $a_1 = R\,U$, $\boldsymbol{\xi}_1 = \varrho'\,d\boldsymbol{v}'/dt$, $a_2 = P/U$, $\boldsymbol{\xi}_2 = \mathrm{grad}'\,p'$, etc. Since the "primed" flow is a solution of the Navier-Stokes equation we have also

$$\sum_{i=1}^{8} b_i\,\boldsymbol{\xi}_i = 0$$

along the curves C, where $b_1 = b_2 = 1$, $b_3 = -\lambda'$, etc. It follows from the last two displayed equations that $a_i/b_i = \mathrm{const} = R\,U$, leading to the conditions

$$R\,U^2/P = 1, \tag{66.4}$$

and

$$R\,U\,D = \frac{\lambda}{\lambda'} = \frac{\mu}{\mu'}. \tag{66.5}$$

(This argument fails when the $\boldsymbol{\xi}_i$ are not independent along the curves C, as in plane flow. In such cases, however, we can assume constant viscosities, or more generally

$$\lambda, \mu = \mathrm{const}\,p^m\,\varrho^n,$$

and our results can be obtained without introducing the curves C.)

The interpretation of Eq. (66.4) is best deferred until certain other aspects of similarity have been derived. From Eq. (66.5) it follows that *the ratio λ/μ and the local Reynolds number $Re = \varrho q l/\mu$ must be the same at corresponding points of the two flows.* Here l is a representative distance in the (geometrically similar) flow fields. We emphasize that our assertion is proved even for *variable* viscosities, and is independent of the Stokes relation.

From Eq. (66.5) it can also be seen that the ratio μ/μ' is independent of p'. Therefore

$$\frac{\mu(P p')}{\mu'(p')} = \frac{\mu(P)}{\mu'(1)}.$$

If this is true for all values of P, as is reasonable to suppose, then

$$\frac{\mu'(p')}{\mu'(1)} = \frac{\mu(P p')}{\mu(P)} = \frac{\mu(p')}{\mu(1)},$$

since the left hand side is independent of P. Consequently

$$\mu(P\, p') = \text{const}\, \mu(P)\, \mu(p'),$$

and therefore $\mu \sim p^m$.[1] Similarly one has $\mu \sim \varrho^n$ so that finally $\mu = \text{const}\, p^m\, \varrho^n$. Using the relation $\mu/\mu' = \lambda/\lambda'$ now gives the theorem: *in order that two compressible fluids should allow dynamical similarity for arbitrary values of R and P it is necessary that the viscosities of each have the form*

$$\lambda, \mu = \text{const}\, p^m\, \varrho^n \tag{66.6}$$

with the same constants m and n. Moreover the ratio λ/μ must be the same for each.

In the subsequent work, the author has found it necessary to assume the ideal gas law[2]

$$p = \mathsf{R}\, \varrho\, T, \qquad E = E(T). \tag{66.7}$$

From Eqs. (66.1) and (66.7) we obtain

$$T = \frac{P}{R}\, \frac{\mathsf{R}'}{\mathsf{R}}\, T' \equiv T_0\, T', \tag{66.8}$$

the last equality serving to define the similarity constant T_0. The energy equation (63.3) becomes, upon making the substitutions (66.1), (66.2), (66.8) and (66.9)

$$\left.\begin{aligned} R\, T_0\, c_v\, \varrho'\, \frac{dT'}{dt'} + P\, p'\, \Theta' &= \frac{U}{D}\left(-\lambda\, \Theta'^2 + 2\mu\, \mathbf{D}\!:\!\mathbf{D}'\right) + \\ &+ \frac{T_0}{DU}\left\{\varkappa\, \nabla^{2\prime}\, T' + \text{grad}'\, T' \cdot \left(\frac{\partial \varkappa}{\partial p'}\, \text{grad}'\, p' + \frac{\partial \varkappa}{\partial \varrho'}\, \text{grad}'\, \varrho'\right)\right\}, \end{aligned}\right\} \tag{66.9}$$

where $c_v = dE/dT$. Now by introducing the congruence of curves C and using the earlier argument there results

$$R\, T_0 \left(\frac{c_v}{c_v'}\right) = P \quad \text{and} \quad R\, U\, D\left(\frac{c_v}{c_v'}\right) = \frac{\varkappa}{\varkappa'}. \tag{66.10}$$

Then from Eqs. (66.8) and (66.10), we have $\mathsf{R}' c_v = \mathsf{R} c_v'$. Since $\mathsf{R} = c_p - c_v$ this implies that $\gamma = \gamma'$; in other words the ratio of specific heats for each gas must be the same at corresponding points in the flows. Using this fact, we obtain from Eqs. (66.5) and $(66.10)_2$ that the Prandtl number

$$\sigma = \mu\, c_p/\varkappa$$

is also the same at corresponding points in the two flows. Finally

$$\frac{c^2}{c'^2} = \frac{\gamma\, p/\varrho}{\gamma'\, p'/\varrho'} = \frac{R}{P}. \tag{66.11}$$

Returning to condition (66.4) and combining it with Eq. (66.11) shows that the local Mach number must be the same at corresponding points of the two flows.

[1] See footnote 1, p. 183.

[2] But note that the specific heats are *not* assumed to be constant.

When the gases admit dynamical similarity for all values of R and P, then γ and γ' must be constant. But this implies c_v and c_v' are constant, and finally $\varkappa = \text{const}\, p^m \varrho^n$. The foregoing work may be summarized as follows:

If two ideal gases with non-zero viscosities and non-zero heat conductivities are in dynamically similar motion, then the local Reynolds number and the local Mach number must be the same at corresponding points in the two flows. Moreover, the ratio of specific heats γ, the ratio of viscosities λ/μ, and the Prandtl number σ must be the same at corresponding points.

If the gases allow dynamically similar motions for arbitrary values of R and P, then for each gas

$$\mu, \lambda, \varkappa = \text{const}\, p^m \varrho^n,$$

with the same constants m and n. Moreover the ratio of specific heats and the Prandtl number of each gas must be constant and equal.

The conditions enumerated in the above theorem are relatively restrictive, and it is fortunate that the common gases obey them more or less well.

Other factors which must be considered in determining dynamical similarity arise from the given boundary conditions. These may introduce complicated problems more readily treated by dimensional analysis than by a full-scale investigation of the type just concluded. In all cases where a complete investigation is not made, however, it is well to keep in mind the earlier discussion (Sect. 36) concerning the possible inadequacy of dimensional analysis or incomplete similarity arguments.

67. Dynamical similarity; Incompressible viscous fluids. The situation here is actually somewhat more complicated, because the thermodynamical behavior of liquids is less well understood than that of gases. Since pressure is only defined up to an additive constant, condition (66.1) should be replaced by

$$p = P p' + \text{const}.$$

Then in case the viscosity is constant we derive easily that *the local Reynolds number must be the same at corresponding points in each flow*. Further progress must be based on assumptions such as

$$T = T_0 T' \quad \text{and} \quad E = kT,$$

somewhat difficult to justify. These assumptions being made, however, then it follows as in the preceding work that the viscosity and heat conductivity must have the form

$$\mu, \varkappa = \text{const}\, T^m,$$

and the "Prandtl number" $\mu k/\varkappa$ must be the same for each fluid. For a discussion of the influence of gravity, see BIRKHOFF [*16*].

III. Incompressible viscous fluids.

The remaining sections of the article concern incompressible viscous fluids for which the stress-deformation relation is linear,

$$\mathbf{T} = -p\,\mathbf{I} + 2\mu\,\mathbf{D}.$$

It will be assumed that the viscosity coefficient μ is constant, since otherwise considerable mathematical difficulties intervene. In any case, experimental evidence indicates that the assumption is a good one, provided the overall temperature variation in the fluid is not great. Finally, it is assumed that the external force is derivable from a potential, $\mathbf{f} = -\operatorname{grad} \Omega$.

The theory of compressible viscous fluids has not been appreciably developed except in the acoustic approximation (this Encyclopedia, Vol. XI), and in boundary layer theory (this Encyclopedia, Vol. VIII), and so will not be reported here. There are, however, some known exact solutions which should be mentioned, in particular the shock layer solution (Sect. 57), the simple shear flows of ILLINGWORTH[1], and the discussion of Rayleigh's problem by HOWARTH[2].

68. The equations of motion. The motion of an incompressible viscous fluid is governed by the equation of continuity

$$\operatorname{div} \mathbf{v} = 0, \tag{68.1}$$

and by the Navier-Stokes equation

$$\varrho \frac{d\mathbf{v}}{dt} = -\operatorname{grad}(p + \varrho\Omega) + \mu \nabla^2 \mathbf{v}. \tag{68.2}$$

Now by virtue of the equation of continuity

$$\nabla^2 \mathbf{v} = -\operatorname{curl} \boldsymbol{\omega}, \tag{68.3}$$

whence by using the identity (17.1) the Navier-Stokes equation can be written in the alternate form

$$\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\omega} \times \mathbf{v} = -\operatorname{grad} H - \nu \operatorname{curl} \boldsymbol{\omega} \tag{68.4}$$

where $H = \frac{1}{2} q^2 + p/\varrho + \Omega$ and $\nu = \mu/\varrho$. The energy transfer equation (9.2) will be useful in later considerations. Under the present assumptions it is easy to see that $\mathbf{T}:\mathbf{D} = 2\mu \mathbf{D}:\mathbf{D} = \Phi$, where Φ is the dissipation function defined in Sect. 61. Thus the energy transfer equation for an incompressible viscous fluid may be written

$$\frac{d}{dt}(\mathfrak{T} + \mathfrak{U}) = \oint_{\mathscr{S}} \mathbf{t} \cdot \mathbf{v}\, da - \int_{\mathscr{V}} \Phi\, dv; \tag{68.5}$$

the physical interpretation of the various terms is obvious. Finally, Eq. (63.3) governing temperature distribution in the fluid takes the form

$$\varrho c_v \frac{dT}{dt} = \Phi + \varkappa \nabla^2 T, \tag{68.6}$$

where it is assumed that c_v and $\varkappa$ are constants. Since Eqs. (68.1) and (68.2) do not involve the temperature of the fluid (assuming always that μ is constant), there is no need to consider Eq. (68.6) unless one needs the actual temperature distribution in the fluid. The temperature distribution problem has hardly been touched in the literature, except in engineering applications[3], and we shall not consider it further in this article.

The reader will have no trouble expressing the above equations in curvilinear coordinates if he uses the general methods indicated in Sect. 61.

69. Vorticity. A salient feature of the motion of viscous fluids is the presence of vorticity. The reason for the appearance of vorticity, and the equations governing its distribution, form the content of this section.

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[1] C.R. ILLINGWORTH: Proc. Cambridge Phil. Soc. **46**, 469 (1950). The case of plane Couette flow is further elaborated in [*19*], pp. 306—313.

[2] L. HOWARTH: Quart. J. Mech. Appl. Math. **4**, 157 (1951).

[3] See the article of H.B. SQUIRE in [*43*], Vol. 2. Also N.A. HALL: The Thermodynamics of Fluid Flow. New York 1951.

Impossibility of irrotational motion. The velocity field $\mathbf{v} = \operatorname{grad} \varphi$ is easily seen to satisfy Eqs. (68.1) and (68.2) if φ is harmonic, and it follows that irrotational motion is dynamically allowable for an incompressible viscous fluid. In spite of this fact, it is virtually impossible for irrotational flow to exist. The reason is that viscous fluids adhere to bounding surfaces (Sect. 64), while irrotational motions generally cannot so adhere (Sect. 23, No. 3). (The above result is in no sense contradictory to boundary layer theory where the flow outside the boundary layer is assumed to be irrotational: the explanation is simply that the flow outside the boundary layer, although definitely rotational, has so little vorticity that for all practical purposes it may be calculated as if it were irrotational.)

A special case where viscous irrotational flow can exist occurs when fluid occupies the exterior of a rotating circular cylinder; specifically, a vortex flow with potential $\varphi = A/r$ can be found such that the fluid adheres to the wall of the cylinder. No exhaustive list of such special cases seems to be available, although there are certainly not many.

Vorticity distribution. The basic equation of vorticity distribution in an incompressible viscous fluid is

$$\frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\omega} \cdot \operatorname{grad} \mathbf{v} + \nu \nabla^2 \boldsymbol{\omega}, \tag{69.1}$$

which may be obtained most simply by taking the curl of Eq. (68.4). But for the second term on the right, the vorticity distribution in a viscous fluid would follow the theorems of HELMHOLTZ. This term shows, however, that vorticity variation in the flow field generally leads to a *diffusion* of that same vorticity. A general result of some importance may be drawn from these considerations, namely that *vorticity cannot be generated in the interior of a viscous incompressible fluid, but is necessarily diffused inward from the boundaries.* In actual fluids, appreciable vorticity exists only in those parts of the fluid which have passed near to rigid boundaries, as is strikingly exhibited in the case of the wake behind a ship which arises solely in water which has passed near to the ship's hull. The same observation shows that the eddy disturbance in the wake is damped out by friction. The above conclusions can also be reached by examining the rate of change of circulation around a closed curve $\mathscr{C}$. Using Eqs. (25.1) and (68.2) we have

$$\frac{d}{dt} \oint_{\mathscr{C}} \mathbf{v} \cdot d\mathbf{x} = \oint_{\mathscr{C}} \mathbf{a} \cdot d\mathbf{x} = \nu \int_{\mathscr{S}} \nabla^2 \mathbf{v}\, da,$$

where $\mathscr{S}$ denotes any surface in the fluid spanning the curve $\mathscr{C}$.

In plane flow Eq. (69.1) reduces to the elegant form

$$\frac{d\omega}{dt} = \nu \nabla^2 \omega, \tag{69.2}$$

which is similar to the heat equation. The resulting analogy with heat flow constitutes additional heuristic evidence for the diffusion of vorticity inwards from bounding surfaces.

Examples. It is of interest to consider some special solutions of the Navier-Stokes equations which illustrate the birth and decay of vorticity in a viscous fluid. Consider first a motion in concentric circles about the z-axis, where the speed q is a function of the radial distance from that axis. The vorticity is found from formula (12.12), namely

$$\omega = \frac{1}{r} \frac{\partial}{\partial r} (r q). \tag{69.3}$$

By use of the fact that ω is a function of r alone, Eq. (69.2) becomes simply

$$\frac{\partial\omega}{\partial t} = \nu\left(\frac{\partial^2\omega}{\partial r^2} + \frac{1}{r}\,\frac{\partial\omega}{\partial r}\right).$$

The solution

$$\omega = \frac{A}{4\pi\nu t}\,\mathrm{e}^{-\chi} \quad \text{where} \quad \chi = \frac{r^2}{4\nu t}, \tag{69.4}$$

corresponds to the decay of an *irrotational* vortex of strength A. For this solution we find from Eq. (69.3) that

$$q = \frac{A}{2\pi r}\left(1 - \mathrm{e}^{-\chi}\right),$$

where an additive term $f(t)/r$ has been omitted in order that q should be finite along the z-axis. The motion is easily seen to have initial circulation A about the z-axis, and to have constant total vorticity. At a fixed place ($r \neq 0$) the vorticity is zero at $t=0$ and $t=\infty$ and takes a maximum value at some inter mediate time[1]. On the other hand, the kinetic energy, the angular momentum, and the energy dissipation are infinite so that the motion is not physically attainable in an unbounded region.

A more realistic motion is obtained from the vorticity distribution[2]

$$\omega = \frac{A}{2\pi\nu t^2}\,(1-\chi)\,\mathrm{e}^{-\chi};$$

[this solution can be derived from Eq. (69.4) by differentiating with respect to t]. The speed is given by

$$q = \frac{rA}{4\pi\nu t^2}\,\mathrm{e}^{-\chi}.$$

Here the circulation is zero at $t=0$; but the total energy and dissipation of energy are finite, and the angular momentum constant:

$$\text{Ang. Mom.} = \int\limits_0^\infty 2\pi\cdot\varrho r q\cdot r\,dr = \varrho A.$$

At a fixed time $t>0$ the speed is zero at the origin and at infinity and has a maximum value $A/2\pi\sqrt{2\nu\,\mathrm{e}\,t^3}$ at the radius $r=\sqrt{2\nu t}$. The value

$$r_0 = \sqrt{2\nu t_0}$$

is called by Taylor the "radius" of the eddy at time t_0. Since the maximum speed in the eddy decreases according to a $t^{-\frac{3}{2}}$ law, the time for an eddy of radius r_0 and maximum speed q_0 to die down to an eddy of maximum speed $\frac{1}{2}\,q_0$ is

$$t - t_0 = \left(2^{\frac{2}{3}} - 1\right)t_0 = 0.296\,r_0^2/\nu.$$

Taylor adopts this figure as a measure of the rate of decay of the eddy[3].

A final example which may be mentioned is that of the dissolution of a discontinuity surface of gliding contact, discussed by Lamb [8], p. 590. The result indicates how rapidly such a surface would be obliterated, if indeed it could

[1] The vorticity measure of this motion is $\mathfrak{W} = \chi/(e^\chi - 1 - \chi)$, which increases steadily with time from $\mathfrak{W}=0$ at $t=0$ to $\mathfrak{W}=\infty$ at $t=\infty$. That is, the motion becomes steadily more rotational even while the vorticity itself decays to zero.

[2] G.I. Taylor: Aero. Res. Comm. R & M No. 598 (1918).

[3] Here, also, the vorticity measure $\mathfrak{W} = |\chi^{-1} - 1|$ eventually increases beyond all bounds, in spite of the fact that the vorticity itself decays to zero.

ever be formed. The same example also may be interpreted as giving the subsequent motion of an infinite mass of fluid after a plane edge is suddenly set in motion (Rayleigh's problem).

70. Intrinsic equations of steady motion. For plane flow, resolution of Eq. (68.2) along streamlines and their normals yields

$$\varrho q \frac{\partial q}{\partial s} + \frac{\partial p}{\partial s} = -\mu \frac{\partial \omega}{\partial n}, \quad \varrho q^2 \varkappa + \frac{\partial p}{\partial n} = \mu \frac{\partial \omega}{\partial s}.$$

Combining these with

$$\operatorname{div} \mathbf{v} = \frac{\partial q}{\partial s} + K q = 0, \quad \omega = -\frac{\partial q}{\partial n} + \varkappa q,$$

(cf. Sect. 20) gives a complete set of intrinsic equations.

For axially-symmetric flow we have similarly

$$\varrho q \frac{\partial q}{\partial s} + \frac{\partial p}{\partial s} = -\frac{\mu}{y} \frac{\partial}{\partial n} (y \omega), \quad \varrho q^2 \varkappa + \frac{\partial p}{\partial n} = \frac{\mu}{y} \frac{\partial}{\partial s} (y \omega),$$

and

$$\frac{1}{y} \frac{\partial}{\partial s} (y q) + K q = 0, \quad \omega = -\frac{\partial q}{\partial n} + \varkappa q.$$

Equations for the stream function in plane and axially-symmetric flow are given in [*42*].

71. Energy formulas. There are two identities which deserve special mention here because of their frequent usefulness, namely the Lamb-Thomson formula[1] for kinetic energy,

$$\mathfrak{T} = \varrho \int_{\mathscr{V}} (\mathbf{r} \cdot \boldsymbol{\omega} \times \mathbf{v})\, dv + \varrho \oint_{\mathscr{S}} [\tfrac{1}{2} q^2 (\mathbf{r} \cdot \mathbf{n}) - (\mathbf{r} \cdot \mathbf{v})(\mathbf{v} \cdot \mathbf{n})]\, da, \tag{71.1}$$

and the Bobyleff-Forsythe formula[2] for energy dissipation,

$$\int_{\mathscr{V}} \Phi\, dv = \mu \int_{\mathscr{V}} \omega^2\, dv + 2\mu \oint_{\mathscr{S}} \mathbf{a} \cdot \mathbf{n}\, da. \tag{71.2}$$

In these formulas it is assumed that $\mathscr{V}$ is a finite region with bounding surface $\mathscr{S}$. It may be noted that the only appeal to mechanics in the proof of Eqs. (71.1) and (71.2) is in the use of the equation of continuity, $\operatorname{div} \mathbf{v} = 0$, and the dissipation formula $\Phi = 2\mu\, \mathbf{D}:\mathbf{D}$. When the velocity vanishes on $\mathscr{S}$ they reduce simply to

$$\mathfrak{T} = \varrho \int_{\mathscr{V}} (\mathbf{r} \cdot \boldsymbol{\omega} \times \mathbf{v})\, dv, \tag{71.3}$$

and

$$\int_{\mathscr{V}} \Phi\, dv = \mu \int_{\mathscr{V}} \omega^2\, dv. \tag{71.4}$$

In order to prove Eq. (71.1) we note the simple identities

$$\operatorname{div}(q^2 \mathbf{r}) = 3 q^2 + \mathbf{r} \cdot \operatorname{grad} q^2,$$
$$\operatorname{div}[(\mathbf{r} \cdot \mathbf{v})\, \mathbf{v}] = q^2 + \mathbf{v} \cdot \mathbf{r} \times \boldsymbol{\omega} + \tfrac{1}{2} \mathbf{r} \cdot \operatorname{grad} q^2;$$

where, in proving the second, we have used $\operatorname{div} \mathbf{v} = 0$. Combining these identities and applying the divergence theorem yields

$$\oint_{s} [\tfrac{1}{2} q^2 \mathbf{r} - (\mathbf{r} \cdot \mathbf{v})\, \mathbf{v}] \cdot \mathbf{n}\, da = \int_{v} (\tfrac{1}{2} q^2 - \mathbf{v} \cdot \mathbf{r} \times \boldsymbol{\omega}]\, dv,$$

[1] H. LAMB: A Treatise on the Mathematical Theory of the Motion of Fluids. Cambridge 1879 (1st edition of [*8*]). J. J. THOMSON: A Treatise on the Motion of Vortex Rings. London 1883.

[2] D. BOBYLEFF: Math. Ann. **6**, 72 (1873). — A.R. FORSYTHE: Mess. Math. **9**, 134 (1880).

which leads at once to Eq. (71.1). Eq. (71.1) is easily generalized to allow for compressibility, when it becomes

$$\int_{\mathfrak{D}} \tfrac{1}{2} q^2\, dv = \int_{\mathfrak{D}} [(\mathbf{r}\cdot\mathbf{v})\,\Theta + \mathbf{r}\cdot\boldsymbol{\omega}\times\mathbf{v}]\, dv + \oint_{\mathfrak{S}} [q^2(\mathbf{r}\cdot\mathbf{n}) - (\mathbf{r}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{n})]\, da. \tag{71.5}$$

LAMB has pointed out that the closely similar Eq. (26.6) is the condition that the right hand side of Eq. (71.5) be independent of the origin of coordinates. Finally, let us observe that the surface integral in Eq. (71.1) or (71.5) can be written in the more symmetric form $\frac{1}{2}[(\mathbf{r}\times\mathbf{v})\cdot(\mathbf{v}\times\mathbf{n}) + (\mathbf{r}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{n})]$.

Next, to prove Eq. (71.2), we have from (28.1)

$$\mathbf{D}:\mathbf{D} = \tfrac{1}{2}\omega^2 + \operatorname{div}\mathbf{a},$$

since $\Theta = \operatorname{div}\mathbf{v} = 0$. Eq. (71.2) follows at once by application of the divergence theorem. The proofs just given should be compared for simplicity with those given by LAMB ([8], p. 218 and 580). Several other integral identities will be found in Sect. 26 of this article.

72. Uniqueness of viscous flows. Consider a viscous incompressible fluid filling a bounded region $\mathscr{V} = \mathscr{V}(t)$ of space, whose boundary $\mathscr{S}$ consists of finitely many closed rigid surfaces moving in some prescribed manner (rigid bodies moving in a bounded container). The velocity of the fluid on $\mathscr{S}$ is the same as the velocity of $\mathscr{S}$ itself, by virtue of the adherence condition of Sect. 64. In these circumstances, it is natural to ask whether the complete fluid motion is uniquely determined by the velocity distribution at some initial instant $t=0$. The answer is yes; in fact, more generally, *if two flows in a bounded region $\mathscr{V} = \mathscr{V}(t)$ have the same velocity distribution at $t=0$ and on the boundary of $\mathscr{V}$, then they must be identical*[1].

The proof of this result is based upon a simple identity for the kinetic energy of the difference motion. Thus let $\mathbf{v}$ and $\mathbf{v}^*$ be the velocity fields of the two flows in question, and define

$$\mathbf{u} = \mathbf{v}^* - \mathbf{v}, \qquad \mathfrak{K} = \tfrac{1}{2}\int_{\mathscr{V}} u^2\, dv.$$

Then we have[2]

$$\frac{d\mathfrak{K}}{dt} = -\int_{\mathscr{V}} (\nu \operatorname{grad}\mathbf{u} : \operatorname{grad}\mathbf{u} + \mathbf{u}\cdot\mathbf{D}\cdot\mathbf{u})\, dv, \tag{72.1}$$

where $\mathbf{D}$ is the deformation tensor of the motion $\mathbf{v}$. In order to prove Eq. (72.1), we use the fact that both $\mathbf{v}$ and $\mathbf{v}^*$ are solutions of the Navier-Stokes equation (68.2), thus obtaining by subtraction

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{v}^*\cdot\operatorname{grad}\mathbf{u} + \mathbf{u}\cdot\operatorname{grad}\mathbf{v} = -\operatorname{grad}\frac{p^*-p}{\varrho} + \nu\nabla^2\mathbf{u}.$$

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[1] This theorem is due to E. FOÁ, L'Industria (Milan) **43**, 426 (1929). FOÁ's theorem was rediscovered in 1954 by D.E. DOLIDZE, Dokl. Akad. Nauk SSSR. **96**, 437, and an extension to the case of compressible fluids was given by D. GRAFFI, J. Rational Mech. Anal. **2**, 99 (1953). Cf. also a paper of the present writer, to appear in the Archive for Rational Mechanics and Analysis, 1959.

Other uniqueness theorems for the Navier-Stokes equations, having particular reference to so-called "weak" solutions (cf. below), were proved by LERAY and by KISELEV and LADYZHENSKAYA in the papers referred to at the close of this section.

[2] Eq. (72.1) may be traced to the work of OSBOURNE REYNOLDS, Phil. Trans. Roy. Soc. Lond. A **186**, 123 (1894), and W. McF. ORR, Proc. Roy. Irish Acad. A **27**, 69–138 (1907).

Several alternate and occasionally useful forms of Eq. (72.1) arise from the simple identity

$$\int \operatorname{grad}\mathbf{u}:\operatorname{grad}\mathbf{u}\, dv = \int |\operatorname{curl}\mathbf{u}|^2\, dv = 2\int \mathbf{D}':\mathbf{D}'\, dv, \tag{72.1$'$}$$

which holds in virtue of the vanishing of $\mathbf{u}$ on S, and the fact that $\operatorname{div}\mathbf{u} = 0$. A proof of Eq. (72.1)$'$ is easily given by the methods of Sect 28.

Forming the scalar product of Eq. (72.1) with $\mathbf{u}$ and using the incompressibility conditions $\operatorname{div}\mathbf{u} = \operatorname{div}\mathbf{v} = 0$ then leads easily to

$$\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) = -\nu \operatorname{grad}\mathbf{u} : \operatorname{grad}\mathbf{u} - \mathbf{u}\cdot\mathbf{D}\cdot\mathbf{u} + \operatorname{div}\left(\nu \operatorname{grad}\frac{1}{2}u^2 - \frac{p^* - p}{\varrho}\mathbf{u} - \frac{1}{2}u^2\mathbf{v}^*\right),$$

from which Eq. (72.1) follows by integration over $\mathscr{V}$ and application of the boundary condition $\mathbf{u} = 0$ on $\mathscr{S}$.

Now let $-m$ be a lower bound for the characteristic values of the tensor $\mathbf{D}$ in the time interval $0 < t < T$; since Trace $\mathbf{D} = \operatorname{div}\mathbf{v} = 0$, we observe that $m \geqq 0$. From the definition of m and the properties of characteristic values it follows that at each point of $\mathscr{V}$ and for all t, $0 < t < T$,

$$\mathbf{u}\cdot\mathbf{D}\cdot\mathbf{u} \geqq -m u^2. \tag{72.2}$$

Therefore from Eq. (72.1),

$$\frac{d\mathfrak{K}}{dt} \leqq m \int_{\mathscr{V}} u^2\, dv = 2m\,\mathfrak{K};$$

writing this in the form

$$\frac{d}{dt}\left(\mathfrak{K}\, e^{-2mt}\right) \leqq 0$$

and integrating from $t = 0$ to $t = T$ yields finally

$$\mathfrak{K}(T)\, e^{-2mT} \leqq 0.$$

Since T was an arbitrary instant it follows that $\mathfrak{K}$ must be identically zero. Hence $\mathbf{u} = 0$, and the two flows $\mathbf{v}$ and $\mathbf{v}^*$ are identical. Under appropriate hypotheses concerning the asymptotic behavior of the flow as $r \to \infty$ the above result can be carried over to infinite regions bounded internally by surfaces $\mathscr{S}$. In particular, if both $\mathbf{v}$ and $\mathbf{v}^*$ satisfy

$$\mathbf{v} = \mathbf{U} + O(r^{-k}), \qquad \operatorname{grad}\mathbf{v} = O(r^{-k+1}), \qquad p = p_\infty + O(r^{-k+1}),$$

where $k > \frac{3}{2}$, then the integral $\mathfrak{K}$ exists, the formal procedures leading to Eq. (72.1) can be justified, and the proof remains valid. (These are fairly strict conditions and possibly not satisfied in the wake behind an obstacle; other conditions could easily be framed, however, which would allow for special behavior in the wake.)

We can no more than mention here the corresponding problem of existence of solutions of the Navier-Stokes equations for preassigned initial and boundary data. This extremely difficult subject is still not completely settled, although a number of formidable papers have dealt with the problem. While a complete bibliography would be beyond our limits, the following fundamental monographs form the core of present-day knowledge:

C. W. OSEEN, Neuere Methoden und Ergebnisse in der Hydrodynamik, Leipzig, 1927.

J. LERAY, Étude de diverses équations intégrales nonlinéaires et de quelques problèmes que pose l'hydrodynamique, J. Math. Pures Appl. (9) **12**, 1–82 (1933).

J. LERAY, Essai sur les mouvements plans d'un liquide visqueux que limitent des parois, J. Math. Pures Appl. (9) **13**, 331–418 (1934).

J. LERAY, Sur le mouvement d'un liquide visqueux emplissant l'espace, Acta Math. **63**, 193–248 (1934).

D. E. DOLIDZE, A non-linear boundary value problem for unsteady motion of a viscous fluid, Prikl. Mat. Meh. (Akad. Nauk SSSR), **12**, 165–180 (1948).

E. HOPF, Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen, Math. Nachr. **4**, 213–231 (1951).

A. A. KISELEV and O. A. LADYSHENSKAYA, On the existence and uniqueness of solutions of the initial value problem for viscous incompressible fluids, Izvestia Akad. Nauk SSSR. **21**, 655–680 (1957).

The major difficulty in the problem is that smooth initial data apparently in some cases give rise to solutions not continuously differentiable for more than a finite time interval. This has necessitated the introduction of various types of "weak" solutions of the Navier-Stokes equations (see the references above), with the result that penetration of the problem's defenses has called for the most refined of modern mathematical methods.

73. Stability of viscous flows. Closely related to the uniqueness problem is the much more difficult question of hydrodynamic stability. We consider a definite flow filling a region $\mathscr{V}$, and subject to prescribed velocity distribution on the boundary of $\mathscr{V}$. In the cases of greatest interest $\mathscr{V}$ is bounded by material walls and the boundary conditions arise from the motion of these walls (as, for example, in Couette flow). Now suppose the velocity field of the flow under consideration is varied slightly at the instant $t=0$; it is natural to ask whether the subsequent motion, subject to the same boundary conditions, will alter only slightly from what it was, or whether it will radically change character. There are two different methods available for treating this problem, the first involving standard linear perturbation procedures, the second depending on the energy formula (72.1). We can consider only the second method here[1]. It is based on the observation that if $\mathfrak{K}$ tends to zero, then $\mathbf{u}$ must likewise tend to zero almost everywhere. Thus if $\mathbf{v}$ is the velocity field of the basic flow and if $\mathbf{v}^*$ is the velocity field of a perturbed motion, then the basic flow will be stable (stable in the mean) provided the energy of the disturbance $\mathbf{u}=\mathbf{v}^*-\mathbf{v}$ tends to zero as t increases. To apply the method, one seeks to determine the sign of the right hand side of Eq. (72.1); if it is negative for arbitrary $\mathbf{u}$ satisfying $\operatorname{div}=\mathbf{u}=0$ then there is stability. Since the first term on the right of Eq. (72.1) is always negative, the viscosity tends to damp out any disturbance, while on the other hand a high rate of shear in the basic flow can cause the second term to be highly positive, thus fostering the growth of disturbances. The relative importance of these terms determines the stability of the flow.

Another criterion of the same sort arises when the right hand side of Eq. (72.1) is written in a slightly different form. Indeed, since $\operatorname{div}\mathbf{u}=0$ we have

$$\mathbf{u}\cdot\mathbf{D}\cdot\mathbf{u}=\operatorname{div}\left[(\mathbf{u}\cdot\mathbf{v})\,\mathbf{u}\right]-\mathbf{u}\cdot\operatorname{grad}\mathbf{u}\cdot\mathbf{v}$$

whence, by application of the divergence theorem, Eq. (72.1) can be written in the form

$$\frac{d\mathfrak{K}}{dt}=\int\limits_{\mathscr{V}}(\mathbf{u}\cdot\operatorname{grad}\mathbf{u}\cdot\mathbf{v}-\nu\operatorname{grad}\mathbf{u}:\operatorname{grad}\mathbf{u})\,dv. \tag{73.1}$$

If the first term on the right of Eq. (73.1) is less than the second term for all admissable functions $\mathbf{u}$, then clearly the basic flow $\mathbf{v}$ will be stable. Since the size of the first term is governed by the magnitude of $\mathbf{v}$, it follows that high speeds in the basic flow, as well as high rates of shear, tend to cause instability. The qualitative nature of these effects will be investigated in the next several paragraphs.

It is important to note that the energy method cannot provide accurate knowledge of the limits of stability, such as can be gained from the perturbation method, for in the energy method one considers the sign of the right hand side of Eq. (72.1) or Eq. (73.1) for arbitrary fields $\mathbf{u}$, not just those which are hydrodynamically possible. Nevertheless, the investigation of these equations leads to a number of interesting and valuable results.

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[1] For a discussion of the first approach, see the article of J.L. SYNGE in Semi-Centennial Publications of the Amer. Math. Soc. **2** (Addresses), pp. 227–269. Also C.C. LIN: The Theory of Hydrodynamic Stability, Cambridge 1954; H. BATEMAN [2], pp. 335–384.

Let $\mathscr{V}$ have a finite diameter d. Then the kinetic energy of any disturbance $\mathbf{u} = \mathbf{v}^* - \mathbf{v}$ *satisfies both the inequalities*

$$\mathfrak{K} \leqq \mathfrak{K}_0 \, e^{(2m - 6\pi^2 \nu/d^2)t}, \tag{73.2}$$

and

$$\mathfrak{K} \leqq \mathfrak{K}_0 \, e^{(V^2 - 3\pi^2 \nu^2/d^2)t/\nu}, \tag{73.3}$$

where $\mathfrak{K}_0$ is the initial energy of the disturbance, $-m$ is a lower bound for the characteristic values of the deformation tensor of the basic flow in the time interval 0 *to t, and V is the maximum speed of the basic flow in the same time interval. If $m < 3\pi^2\nu/d^2$ or if $V < \sqrt{3}\,\pi\nu/d$ for all t, then $\mathfrak{K} \to 0$ as $t \to \infty$ and the basic motion is stable*[1].

Proof. Let $\mathbf{h}$ be an arbitrary differentiable vector field in $\mathscr{V}$, and let $\mathscr{S}$ denote the boundary of $\mathscr{V}$. Then we have

$$0 \leqq (\operatorname{grad}\mathbf{u} + \mathbf{h}\mathbf{u}) : (\operatorname{grad}\mathbf{u} + \mathbf{h}\mathbf{u}) = \operatorname{grad}\mathbf{u} : \operatorname{grad}\mathbf{u} + \mathbf{h} \cdot \operatorname{grad} u^2 + h^2 u^2,$$

whence by integration over $\mathscr{V}$ and use of the condition $\mathbf{u} = 0$ on $\mathscr{S}$ follows

$$\int_{\mathscr{V}} \operatorname{grad}\mathbf{u} : \operatorname{grad}\mathbf{u}\, dv \geqq \int_{\mathscr{V}} (\operatorname{div}\mathbf{h} - h^2)\, u^2\, dv. \tag{73.4}$$

Now the particular field $\mathbf{h} = C \tan C r \cdot (\mathbf{r}/r)$ is differentiable in a sphere of radius $\pi/2C$ about the origin. If we set $C = \pi/d$ and suitably locate the origin in $\mathscr{V}$, this vector can be substituted into (72.6). A simple computation then gives

$$\int_{\mathscr{V}} \operatorname{grad}\mathbf{u} : \operatorname{grad}\mathbf{u}\, dv \geqq 3C^2 \int_{\mathscr{V}} u^2\, dv = (6\pi^2/d^2)\, \mathfrak{K}. \tag{73.5}$$

Combining Eqs. (72.1), (72.2), and (73.5) yields

$$\frac{d\mathfrak{K}}{dt} \leqq (2m - 6\pi^2 \nu/d^2)\, \mathfrak{K},$$

and Eq. (73.2) follows by integration, as in the earlier proof of uniqueness.

The proof of Eq. (73.3) follows similar lines, beginning with the observation that, for any dyadic $\mathbf{A}$,

$$\mathbf{A} : \mathbf{A} - 2\mathbf{u} \cdot \mathbf{A} \cdot \mathbf{v} + u^2 v^2 = (\mathbf{A} - \mathbf{u}\mathbf{v}) : (\mathbf{A} - \mathbf{u}\mathbf{v}) \geqq 0,$$

hence

$$\mathbf{u} \cdot \mathbf{A} \cdot \mathbf{v} \leqq \tfrac{1}{2} (\mathbf{A} : \mathbf{A} + u^2 v^2). \tag{73.6}$$

Putting $\mathbf{A} = \nu \operatorname{grad}\mathbf{u}$ in Eq. (73.6) and using the result to eliminate $\mathbf{u} \cdot \operatorname{grad}\mathbf{u} \cdot \mathbf{v}$ from the right side of (73.1) yields

$$\begin{aligned} \frac{d\mathfrak{K}}{dt} &\leqq \frac{1}{2\nu} \int_{\mathscr{V}} (u^2 v^2 - \nu^2 \operatorname{grad}\mathbf{u} : \operatorname{grad}\mathbf{u})\, dv \\ &\leqq \frac{1}{\nu} (V^2 - 3\pi^2 \nu^2/d^2)\, \mathfrak{K}, \end{aligned}$$

and Eq. (73.3) then follows by integration.

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[1] This theorem and the proof below are due essentially to T.Y. THOMAS, Proc. Nat. Acad. Sci. U.S.A. **29**, 243 (1943), and to E. HOPF, Math. Ann. **117**, 769 (1941). Similar results are true when the region $\mathscr{V}$ can be enclosed in an infinite cylinder, or between two planes, or if the disturbances can be assumed periodic.

It is interesting to observe that the stability criterion $V \leqq \sqrt{3}\,\pi\nu/d$ can be rephased in terms of a "Reynolds number": *if* $\mathrm{Re} = \varrho V d/\mu \leqq \sqrt{3}\,\pi = 5.44$ *then the motion will be stable.*

Although actual numerical results based on Eqs. (73.2) and (73.3) are somewhat disappointing[1], there are still some interesting corollaries to be gained. For example, *disturbances of small enough wave length d will be damped out no matter what the basic motion may be.* In other words, even in a turbulent motion small size "eddies" will not appear; macroscopically, the flow may have the appearance of an intricate chance movement, but, if observed with sufficient magnifying power, the regularity of the flow would never be doubted.

At the other extreme, suppose the boundaries of $\mathscr{V}$ consist of rigid *fixed* walls, so that $\mathbf{v}=0$ on $\mathscr{S}$. Then taking $\mathbf{v}\equiv 0$ as the basic flow, we see that the kinetic energy of any motion whatever in $\mathscr{V}$ must tend to zero according to the law

$$\mathfrak{T} \leqq \mathfrak{T}_0 \, e^{-6\pi^2 \nu t/d^2}, \tag{73.7}$$

where $\mathfrak{T}_0$ is the kinetic energy at $t=0$[2]. It is still an open question whether the velocity itself must tend to zero as $t\to\infty$. As a final application, we have the following uniqueness theorem concerning *steady* motions in a fixed bounded region $\mathscr{v}$.

Let $\mathbf{v}$ *and* $\mathbf{v}^*$ *be two steady flows in* $\mathscr{v}$, *subject to a prescribed time-independent velocity distribution on the boundary of* $\mathscr{v}$. *Let* $-m$ *be a lower bound for the characteristic values of the deformation tensor of the motion* $\mathbf{v}$, *let* $V=\max|\mathbf{v}|$, and *suppose that* either

$$m < 3\nu\pi^2/d^2 \quad \text{or} \quad V < \sqrt{3}\,\pi\nu/d. \tag{73.8}$$

Then the two flows must be identical.

The proof is immediate: the kinetic energy of the difference motion must be constant and at the same time it must satisfy Eqs. (73.2) and (73.3). In view of Eq. (73.8) this can happen only if $\mathfrak{K}_0=\mathfrak{K}=0$, and this in turn implies $\mathbf{v}=\mathbf{v}^*$. The above theorem depends strongly on the assumption (73.8), but without some such condition it is extremely unlikely that the conclusion is true.

For flow under prescribed boundary conditions, the preceding work shows that provided the viscosity is large enough every flow settles down in the long run to a single definite pattern. On the other hand, for smaller viscosity (or what comes to the same thing, larger Reynolds' numbers) the observed flows do not tend to a single limit flow. This is readily illustrated by the simple examples of Couette and Poiseuille flow, where steady laminar motion occurs only at low Reynolds' numbers. In view of these experimental results, E. Hopf[3] has conjectured that there exist certain solutions of the Navier-Stokes equation which correspond to flows observed in the long run after the influence of initial conditions has died down. For high viscosity there is but one such solution, but as the viscosity decreases, more and more possibilities appear, these solutions constituting for fixed ν a "stable" manifold in the phase space of all solutions. In the paper referred to, the conjecture is given more precise form and is supported by an interesting "mathematical model" of the Navier-Stokes equation whose solutions can be studied in exact form.

In line with Hopf's conjecture, it is to be expected that the kinetic energy of any flow must eventually fall beneath some value which depends only on the

[1] See, for example, [2], p. 336 and p. 377, and the paper of T.Y. Thomas cited in the preceding section.

[2] Inequality (73.7) is due to J. Kampé de Fériet, Ann. Soc. Sci. Bruxelles (1) **63**, 36 (1949); this paper also contains references to earlier work of Leray on a similar inequality for plane flows.

[3] E. Hopf: Comm. Pure Appl. Math. **1**, 303 (1948) and Proc. of the Conference on Differential Equations, Univ. of Maryland 1956, p. 49.

viscosity and upon the boundary conditions. This problem has been treated by E. HOPF[1] for the situation described at the opening of the section. Although the results are essentially simple, the mathematical work is somewhat involved and we must refer the reader to the original papers for further details.

Treatment of flows in an infinite region is no different in principle from the work already carried out, so that the statement of further results can be left to the reader.

74. Variational techniques associated with the stability problem. In the preceding section we have found stability criteria based on "arbitrary" perturbations $\mathbf{u}$, whereas in reality not all vector fields $\mathbf{u}$ are allowable. To investigate this situation more fully, consider the maximum value of the right hand side of Eq. (72.1) under the additional condition that $\operatorname{div}\mathbf{u}=0$. After a suitable normalization, we are led to the following variational problem for studying stability:

$$\int_{\mathscr{V}} \mathbf{u}\cdot\mathbf{D}\cdot\mathbf{u}\,dv = \text{Minimum}, \tag{74.1}$$

under the side conditions

$$\operatorname{div}\mathbf{u} = 0, \quad \int_{\mathscr{V}} \operatorname{grad}\mathbf{u} : \operatorname{grad}\mathbf{u}\,dv = 1, \quad \mathbf{u} = 0 \text{ on } \mathscr{S}. \tag{74.2}$$

If the minimum value is $-\tilde{\nu}$, and if $\tilde{\nu}<\nu$, then the right hand side of Eq. (72.1) will be negative for all "hydrodynamically allowable" perturbations $\mathbf{u}$, and the basic motion will be stable (in the mean).

The problem (74.1), (74.2) can be reformulated as a partial differential equation for the extremal function $\mathbf{u}$, according to known procedures of the calculus of variations. Thus, introducing Lagrange multipliers ν^* and $\lambda=\lambda(\mathbf{x})$, the problem (74.1), (74.2) may be stated (formally) as

$$\delta\int_{\mathscr{V}} (\mathbf{u}\cdot\mathbf{D}\cdot\mathbf{u} + \nu^*\operatorname{grad}\mathbf{u} : \operatorname{grad}\mathbf{u} - \lambda\operatorname{div}\mathbf{u})\,dv = 0. \tag{74.3}$$

The differential equations which correspond to Eq. (74.3) are

$$\mathbf{u}\cdot\mathbf{D} = -\operatorname{grad}\lambda + \nu^*\nabla^2\mathbf{u}, \quad \operatorname{div}\mathbf{u} = 0, \tag{74.4}$$

and these are to be solved subject to the side condition $\mathbf{u}=0$ on $\mathscr{S}$. The linear Eqs. (74.4) presumably have non-trivial solutions only for a bounded set of eigenvalues ν^*. If $\tilde{\nu}$ is the greatest of these eigenvalues, then standard techniques of variational calculus yield the inequality

$$-\int_{\mathscr{V}} \mathbf{u}\cdot\mathbf{D}\cdot\mathbf{u}\,dv \leqq \tilde{\nu},$$

for all $\mathbf{u}$ satisfying Eq. (74.2). This (formally) proves the following result.

Let $\tilde{\nu}$ be the greatest eigenvalue of the system (72.14) *in $\mathscr{V}$. Then the basic flow $\mathbf{v}$ in $\mathscr{V}$ is stable provided that*

$$\tilde{\nu} < \nu.$$

The reader will observe a remarkable similarity between equations (74.4) and the equations of hydrodynamics.

Unfortunately, the solution of Eq. (74.4) is not an easy matter even for relatively simple basic flows. For the case of plane shear flows $\mathbf{v}=\{u(y), 0, 0\}$

[1] E. HOPF: Math. Ann. **117**, 764 (1941).

the equations corresponding to Eq. (74.4) were found by ORR[1] and HAMEL[2]. In this case **D** has only two non-vanishing components $D_{xy} = D_{yx} = \frac{1}{2} U'$, and for plane disturbances one can introduce a stream function ψ. This reduces (74.4) to the fourth order equation of ORR and HAMEL,

$$\nu^* \nabla^4 \psi = \tfrac{1}{2} \{U' \psi_{xy} + U'' \psi_x\}. \tag{74.5}$$

Eq. (74.5) was carefully studied by ORR. As one might expect, the numerical results do not give very accurate criteria of stability, though of course they indicate correct trend. In particular, for the distribution $U = ky$ in a channel $0 \leqq y \leqq d$ ORR found the critical REYNOLDS number

$$\widetilde{Re} = k\, d^2 \tilde{\nu}^{-1} = 177.$$

In his book[3] LIN comments that the flow in question is probably stable, irrespective of the Reynolds number. The seeming paradox is resolved if we remember that LIN refers only to infinitesimal disturbances; it is quite possible that finite size disturbances once present in a flow would not damp out, even though infinitesimal ones would never grow important. Clearly ORR's critical value applies to the former type. The reader will find further remarks on the interpretation of ORR's theory in [2], pp. 335—336 and 377.

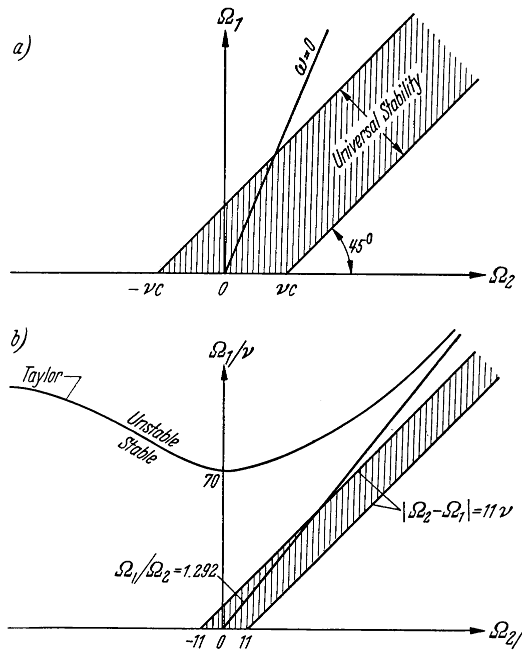

Fig. 17a and b. (a) Stability zones in Couette flow. (b) Stability zones in Couette flow for $R_1 = 3.55$, $R_2 = 4.03$.

Similar considerations apply to the flow between rotating cylinders, the socalled Couette motion. If the inner and outer cylinders have radii R_1 and R_2, respectively, and rotate with angular speeds $\Omega_1 (>0)$ and Ω_2, one finds the velocity field

$$v_r = 0, \quad v_\theta = A\, r^{-1} + B\, r,$$

where

$$A = -\frac{(R_1 R_2)^2 (\Omega_2 - \Omega_1)}{R_2^2 - R_1^2}, \quad B = \frac{R_2^2 \Omega_2 - R_1^2 \Omega_1}{R_2^2 - R_1^2},$$

and

$$\omega = 2B, \quad \mathbf{D} = -\frac{A}{r^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since **D** depends only on the coefficient A, we see immediately that the solution of Eq. (74.4) will yield stability criteria for flows with suitably small $|A|$. More precisely, one finds complete stability to all disturbances whenever

$$\left| \frac{(\Omega_2 - \Omega_1)}{\nu} \right| < C(R_1, R_2).$$

The form of the function $C(R_1, R_2)$ is at present unknown.

[1] W. McF. ORR: Proc. Roy. Irish Acad. A **27**, 69 (1906).
[2] G. HAMEL: Nachr. Ges. Wiss. Göttingen, pp. 261—270 (1911).
[3] C.C. LIN: The Theory of Hydrodynamic Stability, p. 11. Cambridge 1954.

Previously, SYNGE[1] had shown stability to infinitesimal disturbances when $\omega \geqq 0$, i.e. $\Omega_1/\Omega_2 \leqq (R_2/R_1)^2$. The relative zones of stability in the (Ω_1, Ω_2) plane are shown in Fig. 17a. For the case $R_1 \sim R_2$, TAYLOR's theory[2] applies; the situation for the particular pair of radii $R_1 = 3.55$, $R_2 = 4.03$ is illustrated in Fig. 17b. Calculations indicate that C is about 11, and this value has been used in preparing the figure[3].

75. The Helmholtz-Rayleigh dissipation theorem. Some interesting general theorems, relating to the dissipation of energy in the motion of a viscous fluid, have been given by HELMHOLTZ[4] and RAYLEIGH[5]. They involve the basic assumption that the vector curl $\boldsymbol{\omega}$ is derivable from a potential

$$\operatorname{curl} \boldsymbol{\omega} = \operatorname{grad} \sigma. \tag{75.1}$$

According to the Navier-Stokes equation, Eq. (75.1) is equivalent to the condition that the acceleration be derivable from a potential. Condition (75.1) is satisfied in particular for plane shear flow, Poiseuille and Couette flow, steady Beltrami flow, and generally in any flow where inertia terms may be neglected. Although these are admittedly quite special situations, the fact that they can be handled by rigorous mathematical demonstration is of great importance.

Consider a given motion of fluid satisfying Eq. (75.1) in a finite region $\mathscr{v}$. Let $\mathbf{v}$ be the velocity vector of this motion, and let $\mathbf{v}^* = \mathbf{v} + \mathbf{v}'$ be the velocity vector of *any other* incompressible motion. Finally let $\mathfrak{F}$, $\mathfrak{F}^*$, and $\mathfrak{F}'$ denote the dissipation in these respective motions (whether or not $\mathbf{v}^*$ and $\mathbf{v}'$ are dynamically possible). Then we have the formula

$$\mathfrak{F}^* = \mathfrak{F} + \mathfrak{F}' + 2\mu \oint_{\mathscr{s}} [\sigma \mathbf{v}' \cdot \mathbf{n} + 2\mathbf{v}' \cdot \mathbf{D} \cdot \mathbf{n}] \, da, \tag{75.2}$$

slightly generalizing the result of RAYLEIGH and HELMHOLTZ. To prove Eq. (75.2), we observe that

$$\mathfrak{F}^* = \int_{\mathscr{v}} \Phi^* dv = \mathfrak{F} + \mathfrak{F}' + 4\mu \int_{\mathscr{v}} \mathbf{D} : \mathbf{D}' dv,$$

and (in tensor notation)

$$\int_{\mathscr{v}} \mathbf{D} : \mathbf{D}' dv = \int_{\mathscr{v}} \left[(D^{ik} v_i')_{,k} - D^{ik}{}_{,k} v_i' \right] dv = \oint_{\mathscr{s}} \mathbf{v}' \cdot \mathbf{D} \cdot \mathbf{n} \, da - \tfrac{1}{2} \int_{\mathscr{v}} \nabla^2 \mathbf{v} \cdot \mathbf{v}' dv,$$

(using $\operatorname{div} \mathbf{v} = 0$). Now $\nabla^2 \mathbf{v} = -\operatorname{curl} \boldsymbol{\omega} = -\operatorname{grad} \sigma$, so that the last integral transforms into $-\oint_{\mathscr{s}} \sigma \mathbf{v}' \cdot \mathbf{n} \, da$. This proves Eq. (75.2).

Let us see what can be learned from the Helmholtz-Rayleigh formula. Suppose first that the altered motion has the same velocity distribution on $\mathscr{s}$ as the original motion, that is, suppose $\mathbf{v}' = 0$ on $\mathscr{s}$. Then the surface integral in Eq. (75.2) vanishes, and, since $\mathfrak{F}' \geqq 0$, we have

Helmholtz's Theorem: A motion of an incompressible viscous fluid satisfying Eq. (75.1) is characterized by the property that the dissipation in any region is less than in any other motion consistent with the same values of $\mathbf{v}$ *on the boundary.*

Next, consider the case of *generalized Poiseuille flow* (laminar flow) in a straight pipe of arbitrary cross section R[6]. The velocity components are $u = v = 0$,

[1] J.L. SYNGE: Proc. 5th Int. Congr. Appl. Math. Cambridge, USA, pp. 326–332.
[2] G.I. TAYLOR: Phil. Trans. Roy. Soc. Lond. A **223**, 289 (1923).
[3] J. SERRIN: Arch. Rational Mech. Anal. **2**, 1 (1959).
[4] H. HELMHOLTZ: Verh. naturhist.-med. Ver. (1868), (Wiss. Abh. **1**, p. 223).
[5] Lord RAYLEIGH: Phil. Mag. (6) **26**, 776 (1913).
[6] This application was suggested by some work of T.Y. THOMAS: Amer. J. Math. **64**, 754 (1942).

$w = w(x, y)$, and the Navier-Stokes equation becomes

$$0 = -\operatorname{grad} p + \mu \nabla^2 \mathbf{v}.$$

It follows that Eq. (75.1) is satisfied with

$$\sigma = -p/\mu; \tag{75.3}$$

in addition, it should be observed that the pressure must be a linear function of z. In the Helmholtz-Rayleigh formula take $\mathscr{v}$ to be a section of the pipe of length l. Also, let us assume that the altered flow is laminar, or more generally that w' is spatially periodic of period l along the length of the pipe. Then, since $v' = 0$ on the pipe walls, we have in formula (75.2),

$$\mu \oint_{\mathscr{s}} \sigma \mathbf{v}' \cdot \mathbf{n}\, da = \Delta p\,(Q^* - Q), \qquad \oint_{\mathscr{s}} \mathbf{v}' \cdot \mathbf{D} \cdot \mathbf{n}\, da = 0;$$

here $\Delta p > 0$ is the pressure drop along the section of pipe in question, and Q and Q^* are, respectively, the fluxes in the original and altered flows. The Helmholtz-Rayleigh formula then takes the form

$$\mathfrak{F}^* = \mathfrak{F} + \mathfrak{F}' + 2\Delta p\,(Q^* - Q), \tag{75.4}$$

and this proves the following

Theorem: The steady laminar flow of an incompressible viscous fluid down a straight pipe of arbitrary cross section is characterized by the property that its energy dissipation is least among all laminar (or spatially periodic) flows down the pipe which have the same total flux.

Now suppose the flow $\mathbf{v}^*$ has the same pressure drop $\Delta p > 0$ as the laminar flow $\mathbf{v}$. Then it can be shown that

$$\frac{\partial \mathfrak{T}^*}{\partial t} = -\mathfrak{F}' - \Delta p\,(Q^* - Q), \tag{75.5}$$

where $\mathfrak{T}^*$ refers to the kinetic energy of $\mathbf{v}^*$ over the section of pipe in question. To prove Eq. (75.5), we have first from Eq. (68.5),

$$\frac{d\mathfrak{T}}{dt} = \oint_{\mathscr{S}} \mathbf{t} \cdot \mathbf{v}\, da - \mathfrak{F}, \qquad \frac{d\mathfrak{T}^*}{dt} = \oint_{\mathscr{S}} \mathbf{t}^* \cdot \mathbf{v}^*\, da - \mathfrak{F}^*.$$

In these equations d/dt can be replaced by $\partial/\partial t$ because of the assumed periodicity, and clearly $\partial \mathfrak{T}/\partial t = 0$. Thus using Eq. (75.4)

$$\begin{aligned} \frac{\partial \mathfrak{T}^*}{\partial t} &= \oint_{\mathscr{S}} \mathbf{t}^* \cdot \mathbf{v}^*\, da - \{\mathfrak{F} + \mathfrak{F}' + 2\Delta p\,(Q^* - Q)\} \\ &= -\mathfrak{F}' - 2\Delta p\,(Q^* - Q) + \oint_{\mathscr{S}} (\mathbf{t}^* \cdot \mathbf{v}^* - \mathbf{t} \cdot \mathbf{v})\, da. \end{aligned}$$

To evaluate the last integral, we observe that the integrand is zero on the bounding walls, while (by virtue of the spatial periodicity) $\mathbf{t}^*$ and $\mathbf{t}$ at the ends of the section may be replaced respectively by $-p^* \mathbf{n}$ and $-p\,\mathbf{n}$. Evaluation of the resulting integral leads at once to Eq. (75.5).

As an application of Eq. (75.5), suppose that $Q^* \geqq Q$. Then $\partial \mathfrak{T}^*/\partial t < 0$, and $\mathfrak{T}^*$ decreases. Moreover, $\mathfrak{T}^*$ obviously continues to decrease so long as $Q^* \geqq Q$. This proves the following.

Instability Theorem[1]*: Spatially periodic flow in a straight pipe with fixed pressure drop is unstable when its total flux exceeds that in laminar flow with the same pressure drop.*

[1] T. Y. THOMAS: Amer. J. Math. **64**, 754 (1942). THOMAS considered only pipes of circular cross section. See also Proc. Nat. Acad. Sci. U.S.A. **29**, 243 (1943).

This result can also be stated in somewhat more physical terms: *the pressure drop required to produce a given discharge in a pipe is greater for turbulent flow than for laminar flow.* This is in strict agreement with observation and naturally expresses a fact of considerable practical importance. (The condition on the turbulent flow that it be spatially periodic at each instant can be replaced by the less stringent condition that the perturbing flow be independent of z *in the average*; in this case the conclusion of course involves *averaged* dissipation rather than instantaneous dissipation.)

76. Bernoulli theorems. The condition $\operatorname{curl}\boldsymbol{\omega} = \operatorname{grad}\sigma$ introduced in the previous section has several further implications. In particular, for flows possessing this property the acceleration is derivable from a potential,

$$\mathbf{a} = -\operatorname{grad}\left(\frac{p}{\varrho} + \Omega + \nu\sigma\right),$$

hence almost all results of Part C may be carried over. For example, the vorticity convection formulas (17.3) and (17.5) remain valid, together with Kelvin's circulation theorem and the Helmholtz vorticity theorems. Kelvin's theorem, in particular, shows that *a necessary and sufficient condition for* (75.1)

$$\operatorname{curl}\boldsymbol{\omega} = \operatorname{grad}\sigma$$

to hold in the flow of an incompressible viscous fluid is that the flow be circulation-preserving.

Finally, the derivation of the Bernoullian theorem of Sect. 18 depends only on the existence of an acceleration potential, so that the results of that section retain their validity if we but add the term $\nu\sigma$. We thus obtain the following theorem: *In a steady, circulation-preserving motion of a viscous incompressible fluid the function*

$$\frac{1}{2}q^2 + \frac{p}{\varrho} + \Omega + \nu\sigma$$

is constant along streamlines and vortex lines. A similar result holds if only the vorticity is steady (cf. Sect. 18).

The special types of flow which satisfy Eq. (75.1), although rare among the totality of all viscous flows, may nevertheless be useful in determining new exact solutions of the Navier-Stokes equations by means of semi-inverse methods[1]. This field is as yet relatively unexplored.

When the flow is not circulation-preserving [equivalently, when Eq. (75.1) does not hold], there is in general no Bernoulli theorem of the usual type. We can show, however, that[2] *in steady motion the classical Bernoulli expression*

$$H = \frac{1}{2}q^2 + \frac{p}{\varrho} + \Omega$$

is constant along each trajectory of the direction field

$$\boldsymbol{\lambda} = (\boldsymbol{\omega}\times\mathbf{v})\times\operatorname{curl}\boldsymbol{\omega}. \tag{76.1}$$

This follows at once from Eq. (68.4). Degenerate cases of Eq. (76.1) occur when $\operatorname{curl}\boldsymbol{\omega} = 0$, when $\boldsymbol{\omega}\times\mathbf{v} = 0$, and when $\operatorname{curl}\boldsymbol{\omega}$ is parallel to $\boldsymbol{\omega}\times\mathbf{v}$. In the first two of these cases the motion already admits a potential for $\operatorname{curl}\boldsymbol{\omega}$ so that the preceding Bernoulli theorem applies. Finally if $\operatorname{curl}\boldsymbol{\omega}$ is parallel to $\boldsymbol{\omega}\times\mathbf{v}$ it is clear from Eq. (68.4) that the classical Bernoulli theorem holds, that is, H is constant on streamlines and vortex lines.

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[1] P.F. NEMÉNYI [*29*], Vol. 2, pp. 123–151.

[2] C. TRUESDELL: Phys. Rev. **77**, 535 (1950).

Another theorem concerning the Bernoulli expression H is the following: *in the steady motion of a viscous fluid H takes its maximum value on the boundary of the flow region.* To prove this, we observe that from Eq. (68.4) follows both

$$\nabla^2 H = \operatorname{div} \operatorname{grad} H = \operatorname{div} (\mathbf{v} \times \boldsymbol{\omega}) \tag{76.2}$$

and

$$\mathbf{v} \cdot \operatorname{grad} H = -\nu \mathbf{v} \cdot \operatorname{curl} \boldsymbol{\omega}. \tag{76.3}$$

On the other hand, we have the simple identity

$$\operatorname{div} (\mathbf{v} \times \boldsymbol{\omega}) = \omega^2 - \mathbf{v} \cdot \operatorname{curl} \boldsymbol{\omega},$$

hence using Eqs. (76.2) and (76.3) one easily verifies that

$$\nu \nabla^2 H - \mathbf{v} \cdot \operatorname{grad} H = \nu \omega^2. \tag{76.4}$$

The theorem now follows from the maximum principle for elliptic partial differential equations (footnote 1, p. 199), or else from the fact that Eq. (76.4) could not possible hold at a point interior to the flow region where H attained a local maximum value. [The above theorem, and the auxillary equations (76.2) and (76.4), should be compared with the similar results of Sect. 28.]

77. Asymptotic behavior of viscous flows. Both theoretically and practically there is a need for asymptotic formulas applicable to the motion of a viscous fluid at large distances. For the Stokes and Oseen approximations we have, in fact[1],

$$\mathbf{v} = \mathbf{U} + O(r^{-1}) \quad \text{as} \quad r \to \infty, \tag{77.1}$$

but for the exact Navier-Stokes equation no formula of the type (77.1) seems to be known. In lieu of a precise result, UDESCHINI[2] and BERKER[3] have approached the problem from a slightly different point of view. They assume for a steady flow past a fixed obstacle that the asymptotic behavior is of the type

$$\mathbf{v} = \mathbf{U} + O(r^{-k}), \qquad D\mathbf{v}, D^2\mathbf{v} = O(r^{-k-1}), \tag{77.2}$$

where D denotes any partial derivative of the first order; it is then proved that only values $k \leqq 2$ can occur.

BERKER's proof begins with the identity

$$\operatorname{div} (\mathbf{v} \times \boldsymbol{\omega}) = \omega^2 - \mathbf{v} \cdot \operatorname{curl} \boldsymbol{\omega}.$$

Now from Eq. (68.4) and the fact that the motion is assumed to be steady,

$$\mathbf{v} \cdot \operatorname{curl} \boldsymbol{\omega} = -\frac{1}{\nu} \mathbf{v} \cdot \operatorname{grad} H = -\frac{1}{\nu} \operatorname{div} (H \mathbf{v}).$$

Combining the previous two formulas shows that

$$\omega^2 = \operatorname{div}\left(\mathbf{v} \times \boldsymbol{\omega} - \frac{1}{\nu} H \mathbf{v}\right);$$

if this is integrated over a sphere of (large) radius R containing the obstacle and use is made of the divergence theorem and the boundary condition $\mathbf{v} = 0$, there results

$$\int_{\mathscr{V}} \omega^2 \, dv = \oint_{\Sigma} \left(\left(\mathbf{v} \times \boldsymbol{\omega} - \frac{1}{\nu} H \mathbf{v}\right) \cdot \mathbf{n} \, da, \right. \tag{77.3}$$

[1] [8], §§ 336, 342; cf. also R. FINN and W. NOLL: Arch. Rational Mech. Anal. **1**, 97 (1957).
[2] P. UDESCHINI: Rend. Accad. Italia Cl. Sci.-Fis. (7) **2**, 957 (1941).
[3] R. BERKER: Rend. Circ. Mat. Palermo (2) **1**, 1 (1952).

where Σ denotes the surface of the sphere. Now by assumption

$$\mathbf{v}\times\boldsymbol{\omega} = O(r^{-k-1}), \quad H = H_0 + O(r^{-k}),$$

(the second of these is not quite obvious, and the reader is referred to BERKER's paper for a proof). Since Eq. (77.3) holds equally with H replaced by $H - H_0$ it follows that

$$\int_v \omega^2 dv = O(R^{-k+2}) \quad \text{as} \quad R \to \infty.$$

Therefore if $k > 2$ the motion must be irrotational. This, however, is impossible as we have seen in Sect. 69.

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