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ON THE INTERFACE BOUNDARY CONDITION OF BEAVERS, JOSEPH, AND SAFFMAN*

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Abstract. We consider the laminar viscous channel flow over a porous surface. It is supposed, as in the experiment by Beavers and Joseph, that a uniform pressure gradient is maintained in the longitudinal direction in both the channel and the porous medium. After studying the corresponding boundary layers, we obtain rigorously Saffman's modification of the interface condition observed by Beavers and Joseph. It is valid when the pore size of the porous medium tends to zero. Furthermore, the coefficient in the law is determined through an auxiliary boundary-layer type problem.

Key words. Beavers and Joseph's law, homogenization, effective interface law

AMS subject classifications. 76D05, 35B27, 76Mxx, 35Qxx

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1. Introduction.

It is well known that averaging of the equation describing an incompressible flow through a porous medium leads to Darcy's law. The physical velocity is replaced by the seepage velocity, being the velocity average over a representative pore volume. This approximation requires a homogeneity of the porous medium in all directions and, if the flow region contains two or more different media, the averaging procedure is not directly applicable to the whole region.

A very important example is finding effective boundary conditions at the interface between a porous medium and a free fluid. The homogeneity assumptions break down close to the interface and we have Darcy's law only in the interior of the porous medium. On the other hand, the flow of the free fluid is described by the Navier–Stokes system. Consequently, the flow is described in two adjacent domains by two quite different and incompatible PDEs.

The obvious boundary condition at such a permeable interface is the continuity of the normal velocity, which is a consequence of the incompressibility. In order to have a completely determined flow of the free fluid, we have to specify some condition on the tangential component of the free fluid velocity at the interface.

Classically, the vanishing of the tangential velocity of the free fluid at the porous interface was supposed. However, this condition, true for the permeable surface, is not satisfactory for a permeable interface. Then, in [2], Beavers and Joseph proposed a new condition postulating that the difference between the slip velocity of the free fluid and the tangential component of the seepage velocity is proportional to the shear rate of the free fluid. They have verified this law experimentally and found that the proportionality constant depends linearly on the square root of the permeability.

Their experimental boundary condition was further studied by Saffman in [11]. He pointed out that the seepage velocity was much smaller than other quantities in the law of Beavers and Joseph and in fact could be dropped. Then he made a derivation of the law at a physical level of rigor by ensemble averaging. His argument involves

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some ad hoc assumptions, e.g., the representation of the averaged interfacial forces as an integral functional of the seepage velocity with an unknown kernel.

Then the problem was reconsidered in the fundamental paper [6] and the pressure continuity was proposed as an alternative to the law of Beavers and Joseph. We note that this approach leads to some mathematical difficulties in solving the effective equations. On the contrary, the law of Beavers and Joseph leads to the well-posed boundary value problem in the free fluid part. The constants in the law are to be determined experimentally and determination of the seepage velocity in the porous part requires an additional boundary condition at the interface.

In this paper our principal objective is to find *if it is possible to justify mathematically Saffman's form of Beavers and Joseph's law*, which reads

$$\sqrt{k^\varepsilon}\frac{\partial u_\tau}{\partial \nu} = \alpha u_\tau + O(k^\varepsilon). \tag{1}$$

Here α is a dimensionless constant depending only on the structure of the porous medium, ε is the representative length of the porous matrix, and $k^\varepsilon = \varepsilon^2 k$ is the (scalar) permeability tensor. We note that k^ε is of order ε^2. ν is the outer unit normal vector and u denotes the effective velocity in the free fluid part, with u_τ denoting its tangential component at the interface. We are going to use the homogenization theory with matching at the boundary through an appropriate boundary layer. Decomposition of the problem to the dominant free flow effects and the next order effects of the boundary layer enables us to obtain the precise estimates and the mathematically rigorous proof. Furthermore, the construction of the boundary layer, using the technique developed in [9], gives the formulas for the coefficient in (1). Thus the boundary condition can be determined numerically by solving an auxiliary Stokes problem in an infinite strip, which proves advantageous when compared with the technique from [11].

We assume the situation of the experiment by Beavers and Joseph in [2] and, for simplicity, suppose a periodic porous part. More precisely, we consider the laminar viscous two-dimensional incompressible flow through a domain Ω consisting of the porous medium $\Omega_2 = (0,b)\times(-L,0)$, the channel $\Omega_1 = (0,b)\times(0,h)$, and the permeable interface $\Sigma = (0,b)\times\{0\}$ between them. We assume that the structure of the porous medium is periodic and generated by translations of a cell $Z^\varepsilon = \varepsilon Z$, where Z is the standard cell, $Z = (0,1)^2$, containing an open set Z^*, $\partial Z^* \in C^\infty$, strictly included in Z. Let $Y^* = Z \setminus \overline{Z}^*$ and let χ be the characteristic function of Y^*, extended by periodicity to $\mathbb{R}^2$. We set $\chi^\varepsilon(x) = \chi(\frac{x}{\varepsilon})$, $x \in \mathbb{R}^2$, and define Ω_2^ε by $\Omega_2^\varepsilon = \{x \mid x \in \Omega_2,\ \chi^\varepsilon(x) = 1\}$. Furthermore, $\Omega^\varepsilon = \Omega_1 \cup \Sigma \cup \Omega_2^\varepsilon$ is the fluid part of Ω. It is supposed that $(b/\varepsilon, L/\varepsilon) \in \mathbb{N}^2$.

Therefore, our porous medium is assumed to consist of a large number of periodically distributed channels of characteristic length ε, being small compared with a characteristic length of the macroscopic domain.

A uniform pressure gradient is maintained in the longitudinal direction in Ω^ε as in the experiment by Beavers and Joseph (see [2]). More precisely, for a fixed $\varepsilon > 0$ $\{u^\varepsilon, p^\varepsilon\}$ are defined by the equations of motion and mass conservation

$$-\mu\triangle u^\varepsilon + (u^\varepsilon\nabla)u^\varepsilon + \nabla p^\varepsilon = 0 \qquad \text{in } \Omega^\varepsilon, \tag{2}$$

$$\operatorname{div} u^\varepsilon = 0 \qquad \text{in } \Omega^\varepsilon, \tag{3}$$

$$u^\varepsilon = 0 \qquad \text{on } \partial\Omega^\varepsilon \setminus \partial\Omega, \tag{4}$$

$$u^\varepsilon = 0 \qquad \text{on } (0,b)\times(\{-L\}\cup\{h\}), \tag{5}$$

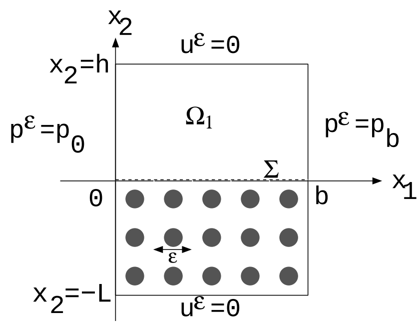

FIG. 1. *Flow regions and boundary conditions.*

$$u_2^\varepsilon = 0 \qquad \text{on } (\{0\} \cup \{b\}) \times (-L, h), \tag{6}$$

$$p^\varepsilon = p_0 \quad \text{on } \{0\} \times (-L, h) \quad \text{and} \quad p^\varepsilon = p_b \quad \text{on } \{b\} \times (-L, h), \tag{7}$$

where $\mu > 0$ is the viscosity and p_0 and p_b are given constants (see Fig. 1).

Now we would like to study the effective behavior of the velocities u^ε and pressures p^ε as $\varepsilon \to 0$, i.e., when the characteristic size of the pores tends to zero.

It is clear that in view of the classical homogenization results on the Navier–Stokes system in porous media we expect to have Darcy's law in Ω_2. In Ω_1 the flow continues to be governed by the Navier–Stokes system. Those two flows are to be coupled at the interface and the main goal of this paper is finding the effective behavior of $\{u^\varepsilon, p^\varepsilon\}$ on the interface Σ in the limit $\varepsilon \to 0$. Clearly, there will be complications arising from the incompatibility of the Navier–Stokes system in Ω_1 and the second order equation for the pressure in Ω_2.

Yet the main difficulty comes from the appearance of the boundary layers in the neighborhoods of the contact surface, where the gradient of the solution differs greatly from the behavior inside the interiors of the domains. The influence of the boundary layers on the effective behavior of the solution results in the particularity of the contact problems between a porous medium and a nonperforated domain under Dirichlet's conditions on the boundaries of the solid part.

Under the above assumptions on the geometry, the law (1) reads

$$\frac{\partial u_1}{\partial x_2} = \tilde{\alpha} u_1 + O(\varepsilon) \quad \text{on} \quad \Sigma, \tag{8}$$

and we are going to justify the form (8) of Beavers and Joseph's law.

The systematic mathematical study of the boundary condition at the permeable surface between a porous medium and a free fluid flow was undertaken in [9]. In that paper the Stokes flow in $(0, b) \times \mathbb{R}$ was considered with a given external force field depending on ε and with the periodic lateral boundary conditions. Various effective interface boundary conditions were found for various flow regimes. Also, a compatibility condition, somewhat similar to the condition (7), was observed. The flow determined by the system (2)–(7) qualitatively corresponds to the situation from Theorem 3 in [9]. Namely, the velocity field is of order $O(1)$ in $(0, b) \times (0, +\infty)$, it is of order $O(\varepsilon^2)$ in Ω_2^ε, and there is a boundary layer of order $O(\varepsilon)$ for the velocity at the interface. The pressure fields in both media were of order $O(1)$. The effective velocity field was described by the solution to the Stokes equation in Ω_1 with the forcing term of order $O(1)$, the no-slip condition at $(0, b) \times \{0\}$, and periodic boundary conditions on the lateral boundaries (the problem (1.85) from [9]), i.e., by the equations

$$\begin{cases} -\triangle u_0 + \nabla \pi_0 = f \quad \text{and} \quad \operatorname{div} u_0 = 0 \text{ in } (0,b)\times(0,+\infty); \\ u_0 = 0 \quad \text{on} \quad \Sigma, \qquad \{u_0,\pi_0\} \text{ is b-periodic in } x_1. \end{cases} \tag{9}$$

It was extended by zero to $(0,b)\times(-\infty,0)$. This was an L^2-approximation of order $O(\varepsilon)$ for the velocity u^ε. Then the higher order terms were necessary to determine the effective pressure field in the porous medium. The crucial role is played by the following auxiliary problem:

Find $\{\beta^{bl},\omega^{bl}\}$ that solve

$$-\triangle_y \beta^{bl} + \nabla_y \omega^{bl} = 0 \quad \text{in} \quad Z^+\cup Z^-, \tag{10}$$

$$\operatorname{div}_y \beta^{bl} = 0 \quad \text{in} \quad Z^+\cup Z^-, \tag{11}$$

$$\left[\beta^{bl}\right]_S(\cdot,0) = 0 \quad \text{on} \quad S, \tag{12}$$

$$\left[\{\nabla_y \beta^{bl} - \omega^{bl} I\}e_2\right]_S(\cdot,0) = e_1 \quad \text{on} \quad S, \tag{13}$$

$$\beta^{bl} = 0 \quad \text{on} \quad \cup_{k=1}^{\infty}\left(\partial Z^* - \{0,k\}\right), \qquad \{\beta^{bl},\omega^{bl}\} \text{ is } y_1\text{-periodic}, \tag{14}$$

where $S=(0,1)\times\{0\}$, $Z^+=(0,1)\times(0,+\infty)$, $Z^-=(0,1)\times(-\infty,0)\setminus\cup_{k=1}^{\infty}(Y^*-\{0,k\})$, and $Z_{BL}=Z^+\cup S\cup Z^-$.

Let $V=\{z\in L^2_{loc}(Z_{BL})^2 : \nabla_y z\in L^2(Z_{BL})^4;\ z\in L^2(Z^-)^2;\ z=0 \text{ on } \cup_{k=1}^{\infty}(\partial Z^*-\{0,k\});\ \operatorname{div}_y z=0 \text{ in } Z_{BL} \text{ and } z \text{ is } y_1\text{-periodic}\}$. Then Proposition 3.22 from section 3 in [9] implies the existence of a solution $\{\beta^{bl},\omega^{bl}\}\in V\cap C^\infty(Z^+\cup Z^-)^2\times C^\infty(Z^+\cup Z^-)$ to (10)–(14), where β^{bl} is unique and ω^{bl} is unique up to a constant. Furthermore, we are able to fix the constant in ω^{bl} and obtain the existence of constants $\gamma_0\in(0,1), C_1^{bl}$, and C_ω^{bl} such that

$$e^{\gamma_0|y_2|}\nabla_y\beta^{bl}\in L^2(Z_{BL})^4,\ e^{\gamma_0|y_2|}\beta^{bl}\in L^2(Z^-)^2, e^{\gamma_0|y_2|}\omega^{bl}\in L^2(Z^-)$$

and

$$\begin{cases} \mid \beta^{bl}(y_1,y_2)-(C_1^{bl},0)\mid \le Ce^{-\gamma_0 y_2}, \qquad y_2>y_*, \\ \mid \omega^{bl}(y_1,y_2)-C_\omega^{bl}\mid \le Ce^{-\gamma_0 y_2}, \qquad y_2>y_*. \end{cases} \tag{15}$$

In the neighborhood of S we have $\beta^{bl}-\big((y_2-y_2^2/2)e^{-y_2}H(y_2),0\big)\in W^{2,q}((0,1)^2\cup S\cup(Z-\{0,1\}))^2$ and $\omega^{bl}\in W^{1,q}((0,1)^2\cup S\cup(Z-\{0,1\}))\ \forall q\in[1,\infty)$.

In addition, constants C_1^{bl} and C_ω^{bl} are given by

$$\begin{cases} C_\omega^{bl}=\int_0^1 \omega^{bl}(y_1,a)\,dy_1 \quad \forall a\ge 0, \\ \int_0^1 \beta_1^{bl}(y_1,0)dy_1=\int_0^1\beta_1^{bl}(y_1,a)dy_1=C_1^{bl}<0 \quad \forall a\ge 0. \end{cases} \tag{16}$$

Then after determining the effective pressure field π_0 and the effective velocity field u_0 in the free fluid part, we determine the pressure field p^0 in the porous medium by solving the boundary value problem

$$\operatorname{div}\big(K(f-\nabla p^0)\big)=0 \qquad \text{in } (0,b)\times(-\infty,0), \tag{17}$$

$$p^0(x_1,-0)=\pi_0(x_1,+0)+C_\omega^{bl}\frac{\partial u_{01}}{\partial x_2}(x_1,+0) \qquad \text{on } \Sigma, \tag{18}$$

$$p^0 \text{ is } b\text{-periodic in } x_1, \tag{19}$$

where C_ω^{bl} is given by (15) and K is the permeability tensor. K is determined through the auxiliary problem

$$\left\{\begin{array}{l} -\triangle_y w^j + \nabla_y \pi^j = e_j \text{ in } Y^*; \\ \operatorname{div}_y w^j = 0 \quad \text{ in } Y^*, \ \int_{Y^*} \pi^j = 0; \\ w^j = 0 \quad \text{ on } \partial Z^*, \ \{w^j, \pi^j\} \text{ is Z-periodic,} \end{array}\right. \tag{20}$$

by $K_{ij} = \int_{Y^*} w_i^j(y)\, dy$. Then the pressure field $H(-x_2)p^0 + H(x_2)\pi_0$ is an L^2-approximation of the pressure of order $O(\sqrt{\varepsilon})$. We refer to Theorem 3 from [9] for the details.

The interface condition (18) connecting the pressures is not very intuitive since it involves the contribution from the geometry in the constant C_ω^{bl}. For a general periodic geometry the pressure continuity cannot be expected. In [6] the pressure continuity is proposed as an alternative to the law of Beavers and Joseph and the second term on the right-hand side of (18) does not appear. It should be noted that in [6] the boundary layer determining C_ω^{bl} was not constructed.

In view of the problem setting in [9] Beavers and Joseph's law corresponds to considering the next order corrections for the velocity. Let $H(x_2)$ denote the Heaviside function on $\mathbb{R}$. Using Theorem 3 from [9] we get

$$u^\varepsilon = u_0 H(x_2) - \varepsilon \beta^{bl}\left(\frac{x}{\varepsilon}\right)\frac{\partial u_{01}}{\partial x_2} + \varepsilon C_1^{bl}\left(\frac{\partial u_{01}}{\partial x_2}\vec{e}_1 + d^1\right) H(x_2) + O(\varepsilon^2), \tag{21}$$

where d^1 is the solution to the Stokes equation in $(0,b)\times(0,+\infty)$, with no external force, with the periodic lateral boundary conditions, and with the boundary condition $d^1 = -\frac{\partial u_{01}}{\partial x_2}\vec{e}_1$ on Σ. The asymptotic expansion of (21) at the interface Σ gives

$$\frac{\partial u_1^\varepsilon}{\partial x_2} = \frac{\partial u_{01}}{\partial x_2}\left(1 - \frac{\partial \beta_1^{bl}}{\partial y_2}\left(\frac{x}{\varepsilon}\right)\right) + O(\varepsilon) \ \text{ and } \ \frac{1}{\varepsilon}u_1^\varepsilon = -\beta_1^{bl}\left(\frac{x}{\varepsilon}\right)\frac{\partial u_{01}}{\partial x_2} + O(\varepsilon).$$

We note that the average of the product between a smooth function and an oscillatory function, of the period ε and the zero mean, is of order $O(\varepsilon)$. Then averaging formally leads to the familiar form of the Beavers and Joseph law

$$u_1^{eff} = -\varepsilon C_1^{bl}\frac{\partial u_1^{eff}}{\partial x_2} \quad \text{on} \quad \Sigma, \tag{22}$$

where u^{eff} is the average over the period of the right-hand side in (21) and C_1^{bl} is defined by (15). The higher order terms are neglected. This heuristic argument is going to be justified rigorously, for the system (2)–(7), in Theorem 7 and Proposition 8.

The rest of the paper contains the rigorous justification of the interface law (22). In order to cover the realistic flow from the experimental work of Beavers and Joseph, we give the proof for the Navier–Stokes equations (2)–(7) rather than for the simplified model from [9]. Solvability of the system (2)–(7) and the uniform a priori estimates are given in section 2. In section 3, we first construct an approximation for the pressure field in Ω_2^ε and the outer boundary layer for the velocity. Then we define a correction of order $O(\varepsilon^2|\log\varepsilon|)$ for the velocity. This result enables us not only to justify (22) but also to prove that the effective mass flow is an approximation of order almost $O(\varepsilon^{3/2})$ for the physical mass flow.

2. Solvability of the ε-problem and uniform a priori estimates. In this section we first address the existence of solutions for the ε-problem (2)–(7). The existence for fixed ε and a small pressure difference is presented in [3]. Similar results, but with the dynamic pressure $p^\varepsilon + \frac{1}{2}|u^\varepsilon|^2$ given instead of p^ε, were obtained in the exhaustive paper [5]. For (2)–(7) in general geometries global existence results for arbitrary data don't seem to be known. As pointed out in [1], the nonhomogeneous boundary conditions for the pressure lead to cubic terms in velocity which can't be easily estimated. Consequently, we can, in general, expect existence only for small $p_b - p_0$.

Let us note that for a somewhat similar problem of the nonstationary incompressible flow through the filter, considered in [10], it was possible to prove the global existence, uniqueness, and regularity, regardless of the size of the pressure at the boundary. As in the case of the porous filter, here the porous medium Ω_2 is expected to considerably slow the flow and to help us obtain the global existence. But since in [2] the experiment gave a satisfactory conclusion only for Reynolds numbers corresponding to laminar flow, it is natural to consider the system (2)–(7) with $p_b - p_0$ not being too large and to avoid the nonessential technical difficulties connected with the nonstationary Navier–Stokes equations and their asymptotic behavior.

Since we need not only existence for a given ε but also the a priori estimates independent of ε, we give a direct proof of existence and uniqueness leading to uniform a priori estimates.

First, we observe that the classic Poiseuille flow in Ω_1, satisfying the no-slip conditions at Σ, is given by

$$\left\{ \begin{array}{ll} v^0 = \left(\dfrac{p_b - p_0}{2b\mu} x_2(x_2 - h), 0 \right) & \text{for } 0 \leq x_2 \leq h; \\ p^0 = \dfrac{p_b - p_0}{b} x_1 + p_0 & \text{for } 0 \leq x_1 \leq b. \end{array} \right. \tag{23}$$

We extend this solution to Ω_2 by setting $v^0 = 0$ for $-L \leq x_2 \leq 0$ and keeping the same form of p^0. Hence, if one considers the stationary incompressible Navier–Stokes system in the channel, satisfying the boundary condition (7) and the no-slip condition on $(0, b) \times (\{0\} \cup \{h\})$, then the Poiseuille flow (23) is always a solution. However, for a large pressure drop there can be several solutions. We are able to establish uniqueness only locally in a ball of L^4 of a radius that is not too large. More precisely, let the Reynolds number Re satisfy the bound

$$Re \overset{\text{def}}{=} \frac{|p_b - p_0|}{\mu^2} \frac{h^2}{8} \leq \frac{1}{16} \left(1 + \frac{h}{b\sqrt{2}} \right)^{-1/2} \max \left\{ \frac{1}{2} \sqrt{\frac{3}{2}}, \sqrt{\frac{b\sqrt{10}}{h}} \right\}.$$

Then, by a simple energy argument, we find that for the above boundary value problem for the Navier–Stokes system in the channel Ω_1 the Poiseuille flow (23) is the unique solution between all those lying in the ball

$$B = \left\{ z \in H^1(\Omega_1)^2 |\, \|z\|_{L^4(\Omega_1)^2} \leq \frac{\mu}{4\sqrt[4]{2bh}} \left(1 + \frac{h}{b\sqrt{2}} \right)^{-1/2} \right\}.$$

Now, the idea is to construct the solution to (2)–(7) as a small perturbation to the Poiseuille flow (23). We have the following nonlinear stability result.

Proposition 1. *Let us suppose the following bound on the Reynolds number Re:*

$$Re \overset{\text{def}}{=} \frac{|p_b - p_0|}{\mu^2}\frac{h^2}{8} \le \frac{3}{50}\sqrt{\frac{b}{h\sqrt{2}}}\left(1+\frac{h}{b\sqrt{2}}\right)^{-1/2}. \tag{24}$$

Then for

$$\varepsilon \le \varepsilon_0 = \max\left\{\frac{b}{\pi}\frac{25}{12\sqrt{3}}(1-|Y^*|),\ \frac{(1-|Y^*|)}{\sqrt{\pi}}\frac{h^4 L\sqrt{2}}{(\sqrt[4]{8}h^2+2bL)^2},\right. \tag{25}$$
$$\left.\frac{1}{3840}\frac{1-|Y^*|}{\sqrt{2\pi}}\frac{b^2}{h(Re)^2}\left(1+\frac{h}{b\sqrt{2}}\right)^{-2}\right\}$$

the problem (2)–(7) *has a solution* $\{u^\varepsilon, p^\varepsilon\} \in H^2(\Omega^\varepsilon)^2 \times H^1(\Omega^\varepsilon)$ *satisfying*

$$\|\nabla(u^\varepsilon - v^0)\|_{L^2(\Omega^\varepsilon)^4} \le \frac{8\sqrt[4]{2\pi}h^2}{\mu b\sqrt{b}\sqrt{1-|Y^*|}}|p_b - p_0|\sqrt{\varepsilon}. \tag{26}$$

Moreover, all solutions lying in the ball

$$B_0 = \left\{ z \in H^1(\Omega^\varepsilon)^2 |\ \|z\|_{L^4(\Omega^\varepsilon)^2} \le \frac{\mu}{15}\sqrt[4]{\frac{1}{2bh}}\left(1+\frac{h}{b\sqrt{2}}\right)^{-1/2}\right\}$$

are equal to $\{u^\varepsilon, p^\varepsilon\}$.

Proof. We search u^ε in the form $u^\varepsilon = v^0 + w^\varepsilon$. Let

$$\mathcal{Z}^\varepsilon = \left\{ z \in H^1(\Omega^\varepsilon)^2 : z = 0 \text{ on } \partial\Omega^\varepsilon\backslash\partial\Omega;\ z_2 = 0 \text{ on } \partial\Omega;\ z_1 = 0 \text{ on } (0,b)\times\big(\{-L\}\cup\{h\}\big)\right\}.$$

Then we are looking for $w^\varepsilon \in \mathcal{W}^\varepsilon = \{\varphi \in \mathcal{Z}^\varepsilon :\ \operatorname{div}\varphi = 0 \text{ in } \Omega^\varepsilon\}$ such that

$$\mu\int_{\Omega^\varepsilon}\nabla w^\varepsilon\nabla\varphi + \int_{\Omega^\varepsilon}(w^\varepsilon\nabla)w^\varepsilon\varphi + \int_{\Omega^\varepsilon} v_1^0\frac{\partial w^\varepsilon}{\partial x_1}\varphi + \int_{\Omega^\varepsilon} w_2^\varepsilon\frac{\partial v_1^0}{\partial x_2}\varphi_1 \tag{27}$$
$$= \mu\frac{\partial v_1^0}{\partial x_2}(0)\int_\Sigma\varphi_1 - \frac{p_b - p_0}{b}\int_{\Omega_2^\varepsilon}\varphi_1 \qquad \forall\varphi\in\mathcal{W}^\varepsilon.$$

Let $\mathcal{H} = \{\varphi \in H^1(\Omega_1)^2 :\ \varphi = 0 \text{ on } (0,b)\times\{h\}\}$ and let $\varpi(\psi,\varphi)$ be a bilinear form on $\mathcal{H}\times\mathcal{H}$ given by

$$\varpi(\psi,\varphi) = \mu\int_{\Omega_1}\nabla\psi\nabla\varphi + \int_{\Omega_1} v_1^0\frac{\partial\psi}{\partial x_1}\varphi + \int_{\Omega_1}\psi_2\frac{\partial v_1^0}{\partial x_2}\varphi_1.$$

Due to the assumption (24) we have

$$\varpi(\psi,\psi) \ge \frac{2\mu}{5}\int_{\Omega_1}|\nabla\psi|^2 \qquad \forall\psi\in\mathcal{H}.$$

Now let

$$w^k \in L^4(\Omega^\varepsilon)^2, \qquad \|w^k\|_{L^4(\Omega^\varepsilon)^2} \le R \overset{\text{def}}{=} \frac{\mu}{15}\sqrt[4]{\frac{1}{2bh}}\left(1+\frac{h}{b\sqrt{2}}\right)^{-1/2}.$$

We consider the problem

$$\varpi(w^{k+1},\varphi)+\mu\int_{\Omega_2^\varepsilon}\nabla w^{k+1}\nabla\varphi+\int_{\Omega^\varepsilon}(w^k\nabla)w^{k+1}\varphi \tag{28}$$
$$=\mu\frac{\partial v_1^0}{\partial x_2}(0)\int_\Sigma\varphi_1-\frac{p_b-p_0}{b}\int_{\Omega_2^\varepsilon}\varphi_1 \qquad \forall\varphi\in\mathcal{W}^\varepsilon.$$

Using Poincaré's inequality in Ω_2^ε and interpolation we obtain

$$\left|\int_{\Omega_2^\varepsilon}(w^k\nabla)\psi\varphi\right|\le\frac{3^{3/4}\sqrt{\pi}}{1-|Y^*|}R\sqrt{\varepsilon}\|\nabla\psi\|_{L^2(\Omega_2^\varepsilon)^4}\|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^4}. \tag{29}$$

Therefore, for $\varepsilon\le\varepsilon_0$ we have

$$\varpi(\varphi,\varphi)+\mu\int_{\Omega_2^\varepsilon}|\nabla\varphi|^2+\int_{\Omega^\varepsilon}(w^k\nabla)\varphi\varphi\ge\frac{\mu}{4}\int_{\Omega^\varepsilon}|\nabla\varphi|^2 \qquad \forall\varphi\in\mathcal{W}^\varepsilon, \tag{30}$$

and the problem (28) has a unique solution $w^{k+1}\in\mathcal{W}^\varepsilon$.

Now we define the nonlinear mapping $\mathcal{T}$ by

$$\mathcal{T}w^k=w^{k+1}.$$

It is easy to check that $\mathcal{T}$ is a continuous map: $\mathcal{T}:L^4(\Omega^\varepsilon)^2\to H^1(\Omega^\varepsilon)^2$.

Let us check that $\mathcal{T}(B_R)\subset B_R$. By variants of the trace inequality in the porous media (see, e.g., [9] or (48)) and Poincaré's inequality, we have

$$\left|\int_\Sigma\varphi_1\right|\le 2\sqrt{\frac{b\sqrt{2\pi}}{1-|Y^*|}}\left(1+\frac{4\pi\varepsilon^2}{L^2(1-|Y^*|)^2}\right)^{1/4}\sqrt{\varepsilon}\|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^4}\ \forall\varphi\in\mathcal{Z}^\varepsilon, \tag{31}$$

$$\left|\int_{\Omega_2^\varepsilon}\varphi_1\right|\le\frac{2\sqrt{bL\pi}}{1-|Y^*|}\varepsilon\|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^4}\ \forall\varphi\in\mathcal{Z}^\varepsilon. \tag{32}$$

Consequently, for $\varepsilon\le\varepsilon_0$, estimates (28)–(30) imply

$$\|\nabla w^{k+1}\|_{L^2(\Omega^\varepsilon)^4}\le\frac{8\sqrt[4]{2\pi}h^2}{\mu b\sqrt{b}\sqrt{1-|Y^*|}}|p_b-p_0|\sqrt{\varepsilon}$$

and $\mathcal{T}(B_R)\subset B_{C\sqrt{\varepsilon}}\subset B_R$.

Therefore, $\mathcal{T}$ has at least one fixed point $w^\varepsilon\in\mathcal{W}^\varepsilon$ and it satisfies the inequality (26). Uniqueness in B_0 and regularity are obvious. ∎

PROPOSITION 2. *For the solution to* (2)–(7)*, satisfying* (26)*, we have the following a priori estimates:*

$$\|u^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2}\le C\varepsilon\sqrt{\varepsilon}, \tag{33}$$
$$\|u^\varepsilon\|_{L^2(\Sigma)^2}\le C\varepsilon, \tag{34}$$
$$\|u^\varepsilon-v^0\|_{L^2(\Omega_1)^2}\le C\varepsilon, \tag{35}$$
$$\|p^\varepsilon-p^0\|_{L^2(\Omega_1)}\le C\sqrt{\varepsilon}. \tag{36}$$

Proof. The inequalities (33) and (34) are direct consequences of Poincaré's inequality and the trace inequality in Ω_2^ε, respectively.

In order to get the estimates (35)–(36) we note that $w^\varepsilon = u^\varepsilon - v^0$ and $\pi^\varepsilon = p^\varepsilon - p^0$ satisfy the system

$$\left\{ \begin{array}{ll} -\triangle w^\varepsilon + \nabla \pi^\varepsilon + v_1^0 \dfrac{\partial w^\varepsilon}{\partial x_1} + w_2^\varepsilon \dfrac{\partial v_1^0}{\partial x_2} e_1 = 0 & \text{in } \Omega_1, \\ \operatorname{div} \ w^\varepsilon = 0 & \text{in } \Omega_1, \\ w^\varepsilon = \xi^\varepsilon \text{ on } \Sigma, \qquad w^\varepsilon = 0 \text{ on } (0,b) \times \{h\}, & \\ w_2^\varepsilon = 0 \quad \text{and} \quad \pi^\varepsilon = 0 \quad \text{on} \quad (\{0\} \cup \{b\}) \times (0,h), & \end{array} \right. \tag{37}$$

where $\|\xi^\varepsilon\|_{L^2(\Sigma)^2} \le C\varepsilon$ by (34).

The theory of the very weak solutions for the Stokes system was developed in [4]. By using the analogous very weak variational formulation for Oseen's problem (37), we get (35).

The estimate (36) follows from the first equation in (37) and Nečas's inequality in Ω_1. ☐

Therefore, we have obtained the uniform a priori estimates for $\{u^\varepsilon, p^\varepsilon\}$. Moreover, we have found that Poiseuille's flow in Ω_1 is an $O(\varepsilon)$ L^2-approximation for u^ε. Following the formal asymptotic expansion from section 1, Beavers and Joseph's law should correspond to the next order velocity correction.

3. The next order velocity correction and Beavers and Joseph's law. The leading contribution for the estimate (26) was the interface integral term $\int_\Sigma \varphi_1$. Following the approach from [9], we eliminate it by using the boundary-layer type functions

$$\beta^{bl,\varepsilon}(x) = \varepsilon \beta^{bl}\left(\frac{x}{\varepsilon}\right) \qquad \text{and} \qquad \omega^{bl,\varepsilon}(x) = \omega^{bl}\left(\frac{x}{\varepsilon}\right), \quad x \in \Omega^\varepsilon. \tag{38}$$

We extend $\beta^{bl,\varepsilon}$ by zero to $\Omega \setminus \Omega^\varepsilon$. Then we have

$$\|\beta^{bl,\varepsilon} - \varepsilon(C_1^{bl}, 0) H(x_2)\|_{L^q(\Omega)^2} \le C\varepsilon^{1+1/q} \qquad \forall q \ge 1, \tag{39}$$

$$\|\omega^{bl,\varepsilon} - C_\omega^{bl} H(x_2)\|_{L^q(\Omega^\varepsilon)} + \|\nabla \beta^{bl,\varepsilon}\|_{L^q(\Omega_1 \cup \Sigma \cup \Omega_2)^4} \le C\varepsilon^{1/q} \quad \forall q \ge 1, \tag{40}$$

$$\left\{ \begin{array}{l} \|\omega^{bl,\varepsilon}(0,\cdot) - C_\omega^{bl} H(\cdot)\|_{H^{-1/2}(\mathbb{R})} + \sqrt{\varepsilon} \|\omega^{bl,\varepsilon}(0,\cdot) - C_\omega^{bl} H(\cdot)\|_{L^2(\mathbb{R})} \le C\varepsilon \ ; \\ \varepsilon^{-1/2} \|\beta^{bl,\varepsilon}(0,\cdot) - \varepsilon(C_1^{bl}, 0) H(\cdot)\|_{L^2(\mathbb{R})^2} + \left\| \dfrac{\partial \beta^{bl,\varepsilon}}{\partial x_2}(0,\cdot) \right\|_{H^{-1/2}(\mathbb{R})^2} \le C\varepsilon. \end{array} \right. \tag{41}$$

Finally,

$$-\triangle \beta^{bl,\varepsilon} + \nabla \omega^{bl,\varepsilon} = 0 \quad \text{in } \Omega_1 \cup \Omega_2^\varepsilon, \tag{42}$$

$$\operatorname{div} \ \beta^{bl,\varepsilon} = 0 \quad \text{in } \Omega^\varepsilon, \tag{43}$$

$$\left[\beta^{bl,\varepsilon}\right]_\Sigma(\cdot, 0) = 0 \quad \text{on } \Sigma, \tag{44}$$

$$\left[\{\nabla \beta^{bl,\varepsilon} - \omega^{bl,\varepsilon} I\} e_2\right]_\Sigma(\cdot, 0) = e_1 \quad \text{on } \Sigma. \tag{45}$$

As in [9], stabilization of $\beta^{bl,\varepsilon}$ towards a nonzero constant velocity $\varepsilon\big(C_1^{bl}, 0\big)$, at the upper boundary, generates a counterflow. It is given by the following Oseen system in Ω_1:

$$\left\{ \begin{array}{ll} -\mu \triangle d + \nabla g + v_1^0 \dfrac{\partial d}{\partial x_1} + d_2 \dfrac{\partial v_1^0}{\partial x_2} e_1 = 0 & \text{in } \Omega_1, \\ \operatorname{div} \ d = 0 & \text{in } \Omega_1, \\ d = e_1 \text{ on } \Sigma, & d = 0 \text{ on } (0,b) \times \{h\}, \\ d_2 = 0 \quad \text{and} \quad g = 0 & \text{on } (\{0\} \cup \{b\}) \times (0,h). \end{array} \right. \tag{46}$$

Under the assumption $\frac{|p_b-p_0|}{b\mu} \le C(b,h)\mu$ from Proposition 1, the problem (46) has a unique solution in the form of two-dimensional Couette flow $d = (1-\frac{x_2}{h})e_1$ and $g=0$.

Before defining an $o(\varepsilon)$ correction for the velocity field, we give the following simple auxiliary result.

LEMMA 3. *Let $\varphi \in H^1(\Omega_2^\varepsilon)$ be such that $\varphi = 0$ on $\partial\Omega_2^\varepsilon \setminus \partial\Omega_2$. Then we have*

$$\|\varphi\|_{L^2(\Omega_2^\varepsilon)} \le C\varepsilon \|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^2}, \tag{47}$$

$$\|\varphi\|_{L^2(\Sigma)} \le C\varepsilon^{1/2} \|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^2}, \tag{48}$$

$$\|\varphi(0,\cdot)\|_{H^{1/2}(-L,0)} \le C \|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^2}. \tag{49}$$

Proof. The estimate (47) is well known. For the estimate (48) we refer to [9].

It remains to prove (49). We note that the $H^{1/2}$-norm on the boundary of Y is given by

$$\|v\|^2_{H^{1/2}(0,1)} = \int_0^1\int_0^1 \frac{|v(0,y_2')-v(0,z_2')|^2}{|y_2'-z_2'|^2}\, dy_2'\, dz_2' \le C\int_{Y^*} |\nabla_y v|^2\, dy.$$

Now, after the change of variables $x=\varepsilon y$, we obtain

$$\|\varphi(0,\cdot)\|^2_{H^{1/2}(\varepsilon((0,1)+k))} \le C\|\nabla\varphi\|^2_{L^2(\varepsilon(Y^*+k))^2} \quad \forall k\in\mathbb{N}.$$

After summation over k one obtains (49). ∎

Now, following (21), we would like to prove that the following quantities are $o(\varepsilon)$ for the velocity and $O(1)$ for the pressure:

$$\mathcal{U}_0^\varepsilon(x) = u^\varepsilon - v^0 + \left(\beta^{bl,\varepsilon} - \varepsilon\big(C_1^{bl},0\big)H(x_2)\right)\frac{\partial v_1^0}{\partial x_2}(0) + \varepsilon C_1^{bl}\frac{\partial v_1^0}{\partial x_2}(0)H(x_2)\left(1-\frac{x_2}{h}\right)e_1, \tag{50}$$

$$\mathcal{P}_0^\varepsilon = p^\varepsilon - p^0H(x_2) - p^{1,\varepsilon}H(-x_2) + \big(\omega^{bl,\varepsilon} - H(x_2)C_\omega^{bl}\big)\mu\frac{\partial v_1^0}{\partial x_2}(0), \tag{51}$$

where $p^{1,\varepsilon}\in H^1(\Omega_2)$.

We have the following result.

PROPOSITION 4. *Let $\mathcal{U}_0^\varepsilon$ be given by (50) and $\mathcal{P}_0^\varepsilon$ by (51). Then $\mathcal{U}_0^\varepsilon \in H^1(\Omega^\varepsilon)^2$, $\mathcal{U}_0^\varepsilon = 0$ on $\partial\Omega^\varepsilon\setminus\partial\Omega$, and $\operatorname{div}\mathcal{U}_0^\varepsilon = 0$ in Ω^ε. Furthermore, we have the following estimate:*

$$\left|\mu\int_{\Omega^\varepsilon}\nabla\mathcal{U}_0^\varepsilon\nabla\varphi - \int_{\Omega^\varepsilon}\mathcal{P}_0^\varepsilon\operatorname{div}\varphi + \int_{\Omega^\varepsilon} v_1^0\frac{\partial\mathcal{U}_0^\varepsilon}{\partial x_1}\varphi + \int_{\Omega^\varepsilon}(\mathcal{U}_0^\varepsilon)_2\frac{\partial v_1^0}{\partial x_2}\varphi_1\right| \le C\varepsilon^{3/2}\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^4} \tag{52}$$

$$+\left|-\mu C_\omega^{bl}\frac{\partial v_1^0}{\partial x_2}(0)\int_\Sigma\varphi_2 + \int_{\Omega_2^\varepsilon}\left(-\frac{p_b-p_0}{b}\varphi_1 + (p^{1,\varepsilon}-p^0)\operatorname{div}\varphi\right)\right| \ \forall\varphi\in\mathcal{Z}^\varepsilon.$$

Proof. First, we note that for $\varphi\in\mathcal{Z}^\varepsilon$ (27) reads

$$\mu\int_{\Omega^\varepsilon}\nabla(u^\varepsilon-v^0)\nabla\varphi - \int_{\Omega^\varepsilon}\big(p^\varepsilon - p^0H(x_2)\big)\operatorname{div}\varphi + \int_{\Omega_1} v_1^0\frac{\partial(u^\varepsilon-v^0)}{\partial x_1}\varphi + \int_{\Omega_1} u_2^\varepsilon\frac{\partial v_1^0}{\partial x_2}\varphi_1 \tag{53}$$
$$= -\int_{\Omega^\varepsilon}((u^\varepsilon-v^0)\nabla)(u^\varepsilon-v^0)\varphi + \mu\frac{\partial v_1^0}{\partial x_2}(0)\int_\Sigma\varphi_1 - \int_{\Omega_2^\varepsilon}\operatorname{div}\big(p^0(x_1)\varphi\big).$$

Next, the variational form of the problem (46) is

$$\int_{\Omega_1} \big(\mu \nabla d \nabla \varphi - g \operatorname{div} \varphi\big) + \int_{\Omega_1} v_1^0 \frac{\partial d}{\partial x_1} \varphi + \int_{\Omega_1} d_2 \frac{\partial v_1^0}{\partial x_2} \varphi_1 = -\mu \int_\Sigma \frac{\partial d_1}{\partial x_2} \varphi_1 + \int_\Sigma g \varphi_2 \qquad \forall \varphi \in \mathcal{Z}^\varepsilon \tag{54}$$

and, moreover, for $\{\beta^{bl,\varepsilon}, \omega^{bl,\varepsilon} - C_\omega^{bl} H(x_2)\}$ we have

$$\begin{aligned} &\int_{\Omega^\varepsilon} \big(\nabla(\beta^{bl,\varepsilon} - \varepsilon(C_1^{bl},0)H(x_2))\nabla\varphi - (\omega^{bl,\varepsilon} - C_\omega^{bl} H(x_2)) \operatorname{div} \varphi\big) + \int_{\Omega_1} v_1^0 \frac{\partial \beta^{bl,\varepsilon}}{\partial x_1} \varphi \\ &+ \int_{\Omega_1} \beta_2^{bl,\varepsilon} \frac{\partial v_1^0}{\partial x_2} \varphi_1 = \int_{-L}^h \big(\omega^{bl,\varepsilon} - C_\omega^{bl} H(x_2)\big)(0,x_2)\varphi_1 - \int_{-L}^h \big(\omega^{bl,\varepsilon} - C_\omega^{bl} H(x_2)\big)(b,x_2)\varphi_1 \\ &\qquad - \int_\Sigma \big(\varphi_1 + C_\omega^{bl} \varphi_2\big) + \int_{\Omega_1} \beta_2^{bl,\varepsilon} \frac{\partial v_1^0}{\partial x_2} \varphi_1 - \int_0^h \big(\beta_1^{bl,\varepsilon} - \varepsilon C_1^{bl}\big)(0,x_2) v_1^0 \varphi_1 \\ &\qquad + \int_0^h \big(\beta_1^{bl,\varepsilon} - \varepsilon C_1^{bl}\big)(b,x_2) v_1^0 \varphi_1 - \int_{\Omega_1} v_1^0 \big(\beta^{bl,\varepsilon} - \varepsilon(C_1^{bl},0)\big) \frac{\partial \varphi}{\partial x_1} \quad \forall \varphi \in \mathcal{Z}^\varepsilon. \end{aligned} \tag{55}$$

Because of the estimates (39)–(41), we have for $\varphi \in \mathcal{Z}^\varepsilon$

$$\left| \int_{-L}^h \big(\omega^{bl,\varepsilon} - C_\omega^{bl} H(x_2)\big)(0,x_2)\varphi_1 - \int_{-L}^h \big(\omega^{bl,\varepsilon} - C_\omega^{bl} H(x_2)\big)(b,x_2)\varphi_1 \right| \le C\varepsilon \|\nabla \varphi\|_{L^2(\Omega^\varepsilon)^4}, \tag{56}$$

$$\begin{aligned} &\left| \int_{\Omega_1} \beta_2^{bl,\varepsilon} \frac{\partial v_1^0}{\partial x_2} \varphi_1 - \int_0^h \big(\beta_1^{bl,\varepsilon} - \varepsilon C_1^{bl}\big)(0,x_2) v_1^0 \varphi_1 + \int_0^h \big(\beta_1^{bl,\varepsilon} - \varepsilon C_1^{bl}\big)(b,x_2) v_1^0 \varphi_1 \right| \\ &\qquad + \left| \int_{\Omega_1} v_1^0 \big(\beta^{bl,\varepsilon} - \varepsilon(C_1^{bl},0)\big) \frac{\partial \varphi}{\partial x_1} \right| \le C\varepsilon^{3/2} \|\nabla \varphi\|_{L^2(\Omega^\varepsilon)^4}, \end{aligned} \tag{57}$$

$$\left| \varepsilon C_1^{bl} \int_\Sigma \left(-\mu \frac{\partial d_1}{\partial x_2} \varphi_1 + g \varphi_2 \right) \right| \le C\varepsilon^{3/2} \|\nabla \varphi\|_{L^2(\Omega_2^\varepsilon)^4}. \tag{58}$$

Furthermore, a simple interpolation argument and the estimates (25) and (33)–(36) imply

$$\left| \int_{\Omega^\varepsilon} ((u^\varepsilon - v^0)\nabla)(u^\varepsilon - v^0)\varphi \right| \le C\varepsilon^{3/2} \|\nabla \varphi\|_{L^2(\Omega^\varepsilon)^4}. \tag{59}$$

Now the variational equations (53)–(55), the definitions of $\mathcal{U}_0^\varepsilon$ and $\mathcal{P}_0^\varepsilon$, and the estimates (56)–(59) give the estimate (52). □

Obviously, the estimate (52) is useful only if the pressure field $p^{1,\varepsilon}$ is chosen in an appropriate way. By analogy to the problem (17)–(19), one can try to take as $p^{1,\varepsilon}$ the solution p for the problem

$$\operatorname{div}\big(K \nabla p\big) = 0 \qquad \text{in } \Omega_2, \tag{60}$$

$$p(x_1, -0) = p^0(x_1) + \mu C_\omega^{bl} \frac{\partial v_1^0}{\partial x_2}(0) \qquad \text{on } \Sigma, \tag{61}$$

$$K \nabla p \; e_2 = 0 \qquad \text{on } (0,b) \times \{-L\}, \tag{62}$$

$$p = p_0 \quad \text{on} \quad \{0\} \times (-L,0) \quad \text{and} \quad p = p_b \qquad \text{on } \{b\} \times (-L,0), \tag{63}$$

where the permeability tensor K is defined through the problem (20). The pressure field p is an element of $W^{1,q}(\Omega_2)\,\forall q \in (1,2)$, and it is a C^∞-function outside the corners. However, due to the discontinuities of the traces at $(0,0)$ and at $(b,0)$, it is not an element of $H^1(\Omega_2)$. We have to regularize the values at the upper corners. Let δ^ε be a Lipschitzian function defined in a neighborhood of $(0,0)$ by

$$\begin{cases} \dfrac{x_1}{\varepsilon} & \text{if } x_1^2 + x_2^2 \le \varepsilon^2, \\ \dfrac{x_1}{\sqrt{x_1^2 + x_2^2}} & \text{if } x_1^2 + x_2^2 > \varepsilon^2 \end{cases} \tag{64}$$

analogously in a neighborhood of $(b,0)$ and by a smooth extension elsewhere in Ω_2. Then we introduce $p^{1,\varepsilon}$ as the solution to the problem

$$\operatorname{div}\left(K\nabla p^{1,\varepsilon}\right) = 0 \qquad \text{in } \Omega_2, \tag{65}$$

$$p^{1,\varepsilon}(x_1,-0) = p^0(x_1) + C_\omega^{bl}\frac{\partial v_1^0}{\partial x_2}(0)\delta^\varepsilon(x_1,-0) \qquad \text{on } \Sigma, \tag{66}$$

$$K\nabla p^{1,\varepsilon}\ e_2 = 0 \qquad \text{on } (0,b)\times\{-L\}, \tag{67}$$

$$p^{1,\varepsilon} = p_0 \quad \text{on} \quad \{0\}\times(-L,0) \quad \text{and} \quad p^{1,\varepsilon} = p_b \quad \text{on } \{b\}\times(-L,0). \tag{68}$$

Then

$$\left\| p^{1,\varepsilon} - p^0 - C_\omega^{bl}\frac{\partial v_1^0}{\partial x_2}(0)\right\|_{L^2(\Sigma)} \le C\sqrt{\varepsilon}, \tag{69}$$

$$\|\nabla p^{1,\varepsilon}\|_{L^2(\Omega_2^\varepsilon)^2} \le C|\log\varepsilon| \tag{70}$$

and, consequently, for every $\varphi \in \mathcal{Z}^\varepsilon$ we have

$$\left| -\mu C_\omega^{bl}\frac{\partial v_1^0}{\partial x_2}(0)\int_\Sigma \varphi_2 + \int_{\Omega_2^\varepsilon}\left(-\frac{p_b - p_0}{b}\varphi_1 + (p^{1,\varepsilon} - p^0)\ \operatorname{div}\ \varphi\right)\right| \le C\varepsilon|\log\varepsilon|\|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^4}. \tag{71}$$

Thus, we have obtained the following result.

Corollary 5. *Let $p^{1,\varepsilon}$ be defined by* (65)–(68)*. Then for $\{\mathcal{U}_0^\varepsilon, \mathcal{P}_0^\varepsilon\}$ given by* (50)–(51)*, we have*

$$\left|\mu\int_{\Omega^\varepsilon}\nabla\mathcal{U}_0^\varepsilon\nabla\varphi - \int_{\Omega^\varepsilon}\mathcal{P}_0^\varepsilon\operatorname{div}\varphi + \int_{\Omega^\varepsilon}v_1^0\frac{\partial\mathcal{U}_0^\varepsilon}{\partial x_1}\varphi + \int_{\Omega^\varepsilon}(\mathcal{U}_0^\varepsilon)_2\frac{\partial v_1^0}{\partial x_2}\varphi_1\right|$$
$$\le C\varepsilon|\log\varepsilon|\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^4} \quad \forall\varphi\in\mathcal{Z}^\varepsilon.$$

At this stage we would like to follow the ideas from [9], take $\varphi = \mathcal{U}_0^\varepsilon$ as the test function, and get the required higher order a priori estimate. Nevertheless, here we are in the presence of the physical outer boundaries and $\mathcal{U}_0^\varepsilon \notin \mathcal{Z}^\varepsilon$. At $(0,b)\times(\{-L\}\cup\{h\})$ the velocity field $\mathcal{U}_0^\varepsilon$ is exponentially small and we can suppose it is zero without loss of generality.

At the inflow/outflow boundaries $(\{0\}\cup\{b\})\times(-L,h)$ the situation is different. We are going to correct the values of $\mathcal{U}_0^\varepsilon$ there. For this purpose we introduce the following outer boundary layer:

$$-\triangle_y s^{bl} + \nabla_y\vartheta^{bl} = 0 \quad \text{in } (0,\ell)\times\mathbb{R}, \tag{72}$$

$$\operatorname{div}_y s^{bl} = 0 \quad \text{in } (0,\ell)\times\mathbb{R}, \tag{73}$$

$$s^{bl} = 0 \quad \text{on } \{\ell\}\times\mathbb{R}, \tag{74}$$

$$s_2^{bl} = \beta_2^{bl} \text{ and } \vartheta^{bl} = \omega^{bl} - C_\omega^{bl}H(y_2) \quad \text{on} \quad \{0\}\times\mathbb{R}, \tag{75}$$

where $0 < \ell < dist\big(Z^*, \{x_1 = 0\}\big)$. Then the properties of $\{\beta^{bl}, \omega^{bl}\}$ imply the existence of a unique solution $\{s^{bl}, \vartheta^{bl}\} \in H^1((0,\ell)\times\mathbb{R})^2 \times L^2((0,\ell)\times\mathbb{R})$. Furthermore, by the Saint–Venant principle for the Dirichlet problem for the Stokes system in the unbounded strips (see [7] or [8]), there is a positive constant $\gamma_0 > 0$ such that

$$e^{\gamma_0|y_2|}\nabla_y s^{bl} \in L^2((0,\ell)\times\mathbb{R})^4,\ e^{\gamma_0|y_2|}s^{bl} \in L^2((0,\ell)\times\mathbb{R})^2, e^{\gamma_0|y_2|}\vartheta^{bl} \in L^2((0,\ell)\times\mathbb{R}).$$

We have $s^{bl} \in W^{2,q}((0,\ell)\times\mathbb{R})^2$ and $\vartheta^{bl} \in W^{1,q}((0,\ell)\times\mathbb{R})$ $\forall q \in (1,2)$. Both functions are C^∞ outside a neighborhood of the point $(0,0)$. We set

$$\left\{ \begin{array}{ll} s^\varepsilon(x) = \varepsilon s^{bl}(\frac{x}{\varepsilon}),\ x_1 \in [0,\varepsilon\ell); & s^\varepsilon(x) = 0 \text{ for } x_1 \in [\varepsilon\ell, b-\varepsilon\ell], \\ \vartheta^\varepsilon(x) = \vartheta^{bl}(\frac{x}{\varepsilon}),\ x_1 \in [0,\varepsilon\ell); & \vartheta^\varepsilon(x) = 0 \text{ for } x_1 \in [\varepsilon\ell, b-\varepsilon\ell] \end{array} \right. \tag{76}$$

and analogously for $x_1 \in (b-\varepsilon\ell, b]$. Because of the symmetry, we shall systematically neglect the right lateral boundary $\{b\}\times(-L,h)$ and present the calculations only for the left one $\{0\}\times(-L,h)$.

Then we have

$$\|\vartheta^\varepsilon(\varepsilon\ell,\cdot)\|_{H^t(\mathbb{R})} + \left\|\frac{\partial s_2^\varepsilon}{\partial x_1}(\varepsilon\ell,\cdot)\right\|_{H^t(\mathbb{R})} \le C\varepsilon^{1/2-t} \quad \forall t \in [-1,1]. \tag{77}$$

After introducing all auxiliary functions we are in position to prove our main result.

THEOREM 6. *Let*

$$\mathcal{U}^\varepsilon(x) = u^\varepsilon - v^0 + \left(\beta^{bl,\varepsilon} - s^\varepsilon\right)\frac{\partial v_1^0}{\partial x_2}(0) - \varepsilon C_1^{bl}\frac{\partial v_1^0}{\partial x_2}(0)H(x_2)\frac{x_2}{h}e_1, \tag{78}$$

$$\mathcal{P}^\varepsilon = p^\varepsilon - p^0 H(x_2) - p^{1,\varepsilon}H(-x_2) + \left(\omega^{bl,\varepsilon} - \vartheta^\varepsilon - H(x_2)C_\omega^{bl}\right)\mu\frac{\partial v_1^0}{\partial x_2}(0), \tag{79}$$

where $\{v^0, p^0\}$ *is defined by* (23), $p^{1,\varepsilon}$ *by* (65)–(68), $\{\beta^{bl,\varepsilon}, \omega^{bl,\varepsilon}\}$ *by* (38), *and* $\{s^\varepsilon, \vartheta^\varepsilon\}$ *by* (76).

Then we have the following estimates:

$$\|\nabla\mathcal{U}^\varepsilon\|_{L^2(\Omega^\varepsilon)^4} \le C\varepsilon|\log\varepsilon|, \tag{80}$$

$$\|\mathcal{U}^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2} \le C\varepsilon^2|\log\varepsilon|, \tag{81}$$

$$\|\mathcal{U}^\varepsilon\|_{L^2(\Sigma)^2} \le C\varepsilon^{3/2}|\log\varepsilon|, \tag{82}$$

$$\|\mathcal{U}^\varepsilon\|_{L^2(\Omega_1)^2} \le C\varepsilon^{3/2}|\log\varepsilon|, \tag{83}$$

$$\|\mathcal{P}^\varepsilon\|_{L^2(\Omega_1)} \le C\varepsilon|\log\varepsilon|. \tag{84}$$

Proof. By analogy to Proposition 4 and Corollary 5 we have

$$\left|\mu\int_{\Omega^\varepsilon}\nabla\mathcal{U}^\varepsilon\nabla\varphi - \int_{\Omega^\varepsilon}\mathcal{P}^\varepsilon\text{div}\varphi + \int_{\Omega_1}v_1^0\frac{\partial\mathcal{U}^\varepsilon}{\partial x_1}\varphi + \int_{\Omega_1}\mathcal{U}_2^\varepsilon\frac{\partial v_1^0}{\partial x_2}\varphi_1\right|$$

$$\le C\varepsilon|\log\varepsilon|\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^4} + \mu\left|\frac{\partial v_1^0}{\partial x_2}(0)\right|\left|\int_{\Omega^\varepsilon}\big(\nabla s^\varepsilon\nabla\varphi - \vartheta^\varepsilon \text{ div } \varphi\big)\right| \quad \forall\varphi \in \mathcal{Z}^\varepsilon.$$

By definition of $\{s^\varepsilon, \vartheta^\varepsilon\}$, (41), and (49)

$$\left|\int_{\Omega^\varepsilon}\big(\nabla s^\varepsilon\nabla\varphi - \vartheta^\varepsilon\text{div}\varphi\big)\right| \le \left|\int_{\mathbb{R}}\vartheta^\varepsilon(0,x_2)\varphi_1(0,x_2)\,dx_2\right| + \left|\int_{\mathbb{R}}\frac{\partial s_2^\varepsilon}{\partial x_1}(\varepsilon\ell,x_2)\varphi_2(\varepsilon\ell,x_2)\,dx_2\right|$$

$$+ \left|\int_{\mathbb{R}}\vartheta^\varepsilon(\varepsilon\ell,x_2)\varphi_1(\varepsilon\ell,x_2)\;dx_2\right| \le C\varepsilon\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^4} \quad \forall\varphi \in \mathcal{Z}^\varepsilon.$$

Therefore, we obtain the estimate

$$\left|\mu\int_{\Omega^\varepsilon}\nabla\mathcal{U}^\varepsilon\nabla\varphi-\int_{\Omega^\varepsilon}\mathcal{P}^\varepsilon\mathrm{div}\varphi+\int_{\Omega_1}v_1^0\frac{\partial\mathcal{U}^\varepsilon}{\partial x_1}\varphi+\int_{\Omega_1}\mathcal{U}_2^\varepsilon\frac{\partial v_1^0}{\partial x_2}\varphi_1\right| \le C\varepsilon|\log\varepsilon|\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^4}\quad\forall\varphi\in\mathcal{Z}^\varepsilon. \tag{85}$$

Now let us note that $\mathcal{U}^\varepsilon\in\mathcal{Z}^\varepsilon$ and $\mathrm{div}\,\mathcal{U}^\varepsilon=0$. Hence it is possible to take $\varphi=\mathcal{U}^\varepsilon$. With this choice we get the estimate (80). Poincaré's inequality (47) applied to (80) gives (81). Equation (82) is a consequence of (48) and (84) follows from (80).

It remains to prove (83). We note that $\{\mathcal{U}^\varepsilon,\mathcal{P}^\varepsilon\}$ satisfies the following Oseen system in Ω_1:

$$\begin{cases} -\mu\triangle\mathcal{U}^\varepsilon+\nabla\mathcal{P}^\varepsilon+v_1^0\dfrac{\partial\mathcal{U}^\varepsilon}{\partial x_1}+(\mathcal{U}^\varepsilon)_2\dfrac{\partial v_1^0}{\partial x_2}e_1=G^\varepsilon & \text{in }\Omega_1,\\ \mathrm{div}\,\mathcal{U}^\varepsilon=0 & \text{in }\Omega_1,\\ \mathcal{U}^\varepsilon=\xi^\varepsilon\ \text{ on }\Sigma, & \|\xi^\varepsilon\|_{L^2(\Sigma)^2}\le C\varepsilon^{3/2}|\log\varepsilon|,\\ \mathcal{U}^\varepsilon=0 & \text{on }(0,b)\times\{h\},\\ \mathcal{U}_2^\varepsilon=0\ \text{ and }\ \mathcal{P}^\varepsilon=0 & \text{on }(\{0\}\cup\{b\})\times(0,h), \end{cases} \tag{86}$$

where

$$G^\varepsilon=v_1^0\frac{\partial}{\partial x_1}\big(\beta^{bl,\varepsilon}-s^\varepsilon-\varepsilon(C_1^{bl},0)\big)+(\beta_2^{bl,\varepsilon}-s_2^\varepsilon)\frac{\partial v_1^0}{\partial x_2}e_1-\left(\vartheta^\varepsilon e_1+\frac{\partial s_2^\varepsilon}{\partial x_1}e_2\right)\delta_{\{\varepsilon\ell\}\times(0,h)}. \tag{87}$$

The adjoint problem for (86) reads

$$\begin{cases} -\mu\triangle\Phi+\nabla\eta-v_1^0\dfrac{\partial\Phi}{\partial x_1}+\Phi_1\dfrac{\partial v_1^0}{\partial x_2}e_2=\bar g & \text{in }\Omega_1,\\ \mathrm{div}\,\Phi=z & \text{in }\Omega_1,\\ \Phi=0 & \text{on }\Sigma\cup(0,b)\times\{h\},\\ \Phi_2=0\ \text{ and }\ \eta-v_1^0\Phi_1=\mu z & \text{on }(\{0\}\cup\{b\})\times(0,h). \end{cases} \tag{88}$$

It is easily seen that for $\bar g\in L^2(\Omega_1)^2$ and $z\in H^1(\Omega_1)$ the problem (88) has a unique solution $\{\Phi,\eta\}\in H^2(\Omega_1)^2\times H^1(\Omega_1)$, which depends continuously on the data.

Following [4] we write the very weak formulation corresponding to (86) as

$$\begin{aligned}\int_{\Omega_1}\mathcal{U}^\varepsilon\bar g-\langle\mathcal{P}^\varepsilon,z\rangle_{\Omega_1}&=\int_\Sigma(\mu\nabla\Phi-\eta I)e_2\xi^\varepsilon+\int_{\Omega_1}(\beta_2^{bl,\varepsilon}-s_2^\varepsilon)\frac{\partial v_1^0}{\partial x_2}\Phi_1\\ &-\int_{\Omega_1}v_1^0\big(\beta^{bl,\varepsilon}-s^\varepsilon-\varepsilon(C_1^{bl},0)\big)\frac{\partial\Phi}{\partial x_1}-\int_{\{\varepsilon\ell\}\times(0,h)}\left(\vartheta^\varepsilon\Phi_1+\frac{\partial s_2^\varepsilon}{\partial x_1}\Phi_2\right)\\ &-\int_{\{0\}\times(0,h)}\big(\beta_1^{bl,\varepsilon}-s_1^\varepsilon-\varepsilon C_1^{bl}\big)v_1^0\Phi_1\quad\forall\bar g\in L^2(\Omega_1)^2,\ \forall z\in H^1(\Omega_1).\end{aligned} \tag{89}$$

Thus we have obtained the estimate

$$\begin{aligned}\|\mathcal{U}^\varepsilon\|_{L^2(\Omega_1)^2}\le C\bigg\{&\|\xi^\varepsilon\|_{L^2(\Sigma)^2}+\|\beta^{bl,\varepsilon}-s^\varepsilon-\varepsilon(C_1^{bl},0)\|_{L^2(\Omega_1)^2}+\|\vartheta^\varepsilon\|_{H^{-1}(\{\varepsilon\ell\}\times(0,h))}\\ &+\left\|\frac{\partial s_2^\varepsilon}{\partial x_1}\right\|_{H^{-1}(\{\varepsilon\ell\}\times(0,h))}+\|\beta_1^{bl,\varepsilon}-s_1^\varepsilon-\varepsilon C_1^{bl}\|_{H^{-1}(\{\varepsilon\ell\}\times(0,h))}\bigg\}.\end{aligned} \tag{90}$$

Now (90) and (77) imply (83). □

The estimates (80)–(84) allow us to justify Saffman's modification of the Beavers and Joseph law.

We start with a result related to the behavior of the velocity field u^ε at the interface Σ with the porous body.

THEOREM 7. *Let u^ε be the velocity field determined in Proposition* 1 *and let the boundary layer tangential velocity at infinity C_1^{bl} be given by* (15)–(16).

Then we have

$$\left\| u_1^\varepsilon + \varepsilon C_1^{bl} \frac{\partial u_1^\varepsilon}{\partial x_2} \right\|_{\left(H_{00}^{1/2}(\Sigma)\right)'} \le C\varepsilon^{3/2} |\log \varepsilon|. \tag{91}$$

Proof. Using the definition of the correction $\mathcal{U}_0^\varepsilon$, we get

$$\begin{aligned}\left\| u_1^\varepsilon + \varepsilon C_1^{bl} \frac{\partial u_1^\varepsilon}{\partial x_2} \right\|_{\left(H_{00}^{1/2}(\Sigma)\right)'} &\le C\varepsilon^2 + \left\| \mathcal{U}_{01}^\varepsilon + \varepsilon C_1^{bl} \frac{\partial \mathcal{U}_{01}^\varepsilon}{\partial x_2} \right\|_{\left(H_{00}^{1/2}(\Sigma)\right)'} \\ &+ C\|\beta_1^{bl,\varepsilon}(0,\cdot) - \varepsilon C_1^{bl}\|_{\left(H_{00}^{1/2}(\Sigma)\right)'} + \ C\varepsilon \left\| \frac{\partial \beta_1^{bl,\varepsilon}}{\partial x_2}(0,\cdot) \right\|_{\left(H_{00}^{1/2}(\Sigma)\right)'}.\end{aligned} \tag{92}$$

It should be noted that $\left(H_{00}^{1/2}(\Sigma)\right)' = \left[L^2(\Sigma), H^{-1}(\Sigma)\right]_{1/2}$ and that $\beta_1^{bl,\varepsilon}(0,\cdot) - \varepsilon C_1^{bl}$ and $\frac{\partial \beta_1^{bl,\varepsilon}}{\partial x_2}(0,\cdot)$ are ε-periodic functions with zero mean.

Consequently, by a simple duality argument we obtain

$$\|\beta_1^{bl,\varepsilon}(0,\cdot) - \varepsilon C_1^{bl}\|_{\left(H_{00}^{1/2}(\Sigma)\right)'} + \varepsilon \left\| \frac{\partial \beta_1^{bl,\varepsilon}}{\partial x_2}(0,\cdot) \right\|_{\left(H_{00}^{1/2}(\Sigma)\right)'} \le C\varepsilon^{3/2}. \tag{93}$$

It remains to estimate the first term on the right-hand side of the inequality (92). The difficulty comes from the derivative of $\mathcal{U}_{01}^\varepsilon$. Since we have no information on the H^2-norm of $\mathcal{U}_0^\varepsilon$, the only possibility is to use the generalized Green formula.

First, we have

$$\operatorname{div}\left(\mu \nabla \mathcal{U}_{01}^\varepsilon - \mathcal{P}_0^\varepsilon e_1\right) = v_1^0 \frac{\partial \mathcal{U}_{01}^\varepsilon}{\partial x_1} + \mathcal{U}_{02}^\varepsilon \frac{\partial v_1^0}{\partial x_2} + v_1^0 \frac{\partial \beta_1^{bl,\varepsilon}}{\partial x_1} + \beta_2^{bl,\varepsilon} \frac{\partial v_1^0}{\partial x_2} \quad \text{in } \Omega_1. \tag{94}$$

Hence

$$\| \operatorname{div}\left(\mu \nabla \mathcal{U}_{01}^\varepsilon - \mathcal{P}_0^\varepsilon e_1\right)\|_{L^2(\Omega_1)} \le C\sqrt{\varepsilon}. \tag{95}$$

Now, by the generalized Green formula,

$$\left\| \frac{\partial \mathcal{U}_{01}^\varepsilon}{\partial x_2} \right\|_{\left(H_{00}^{1/2}(\Sigma)\right)'} \le C\left\{ \|\mu\nabla \mathcal{U}_{01}^\varepsilon - \mathcal{P}_0^\varepsilon e_1\|_{L^2(\Omega_1)^2} + \| \operatorname{div}\left(\mu\nabla \mathcal{U}_{01}^\varepsilon - \mathcal{P}_0^\varepsilon e_1\right)\|_{L^2(\Omega_1)} \right\} \le C\sqrt{\varepsilon}. \tag{96}$$

Since (82) implies $\|\mathcal{U}_{01}^\varepsilon\|_{(H_{00}^{1/2}(\Sigma))'} \le C\varepsilon^{3/2}|\log\varepsilon|$, after inserting (93) and (96) into (92), we obtain the estimate (91). □

Now we introduce the effective flow equations in Ω_1 through the following boundary value problem:

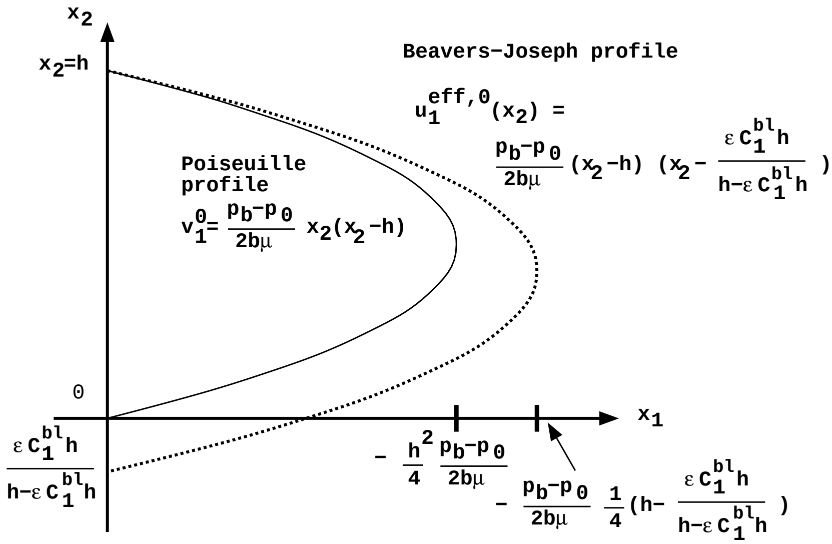

FIG. 2. *Classical Poiseuille velocity profile versus Poiseuille flow with Beavers and Joseph's slip boundary condition.*

Find a velocity field u^{eff} and a pressure field p^{eff} such that

$$-\mu\triangle u^{eff} + (u^{eff}\nabla)u^{eff} + \nabla p^{eff} = 0 \quad \text{in}\,\Omega_1, \tag{97}$$

$$\text{div } u^{eff} = 0 \quad \text{in}\,\Omega_1, \tag{98}$$

$$u^{eff} = 0 \quad \text{on}\,(0,b)\times\{h\}, \tag{99}$$

$$u_2^{eff} = 0 \quad \text{on}\,(\{0\}\cup\{b\})\times(0,h), \tag{100}$$

$$p^\varepsilon = p_0 \text{ on } \{0\}\times(0,h) \quad \text{and} \quad p^\varepsilon = p_b \quad \text{on}\,\{b\}\times(0,h), \tag{101}$$

$$u_2^{eff} = 0 \quad \text{and} \quad u_1^{eff} + \varepsilon C_1^{bl}\frac{\partial u_1^{eff}}{\partial x_2} = 0 \quad \text{on} \quad \Sigma. \tag{102}$$

Under the assumptions of Proposition 1, the problem (97)–(102) has a unique solution

$$\begin{cases} u^{eff} = \left(\dfrac{p_b-p_0}{2b\mu}\left(x_2 - \dfrac{\varepsilon C_1^{bl}h}{h-\varepsilon C_1^{bl}}\right)(x_2-h), 0\right) & \text{for } 0\le x_2\le h; \\ p^{eff} = p^0 = \dfrac{p_b-p_0}{b}x_1 + p_0 & \text{for } 0\le x_1\le b. \end{cases} \tag{103}$$

(See Fig. 2.) The effective mass flow rate through the channel is then

$$M^{eff} = b\int_0^h u_1^{eff}(x_2)\,dx_2 = -\frac{p_b-p_0}{12\mu}h^3\frac{h-4\varepsilon C_1^{bl}}{h-\varepsilon C_1^{bl}}, \tag{104}$$

where $C_1^{bl}<0$.

Let us estimate the error made when replacing $\{u^\varepsilon, p^\varepsilon, M^\varepsilon\}$ with $\{u^{eff}, p^{eff}, M^{eff}\}$. We have the following.

PROPOSITION 8. *Under the assumptions of Proposition* 1 *we have*

$$\|\nabla(u^\varepsilon - u^{eff})\|_{L^1(\Omega_1)^4} \le C\varepsilon|\log\varepsilon|, \tag{105}$$

$$\|u^\varepsilon - u^{eff}\|_{H^{1/2-\gamma}(\Omega_1)^2} \le C\varepsilon^{3/2}|\log\varepsilon|, \qquad 1/2>\gamma>0, \tag{106}$$

$$|M^\varepsilon - M^{eff}| \le C\varepsilon^{3/2}|\log\varepsilon|. \tag{107}$$

Proof. We have

$$u^\varepsilon - u^{eff} = \mathcal{U}^\varepsilon + v^0 - u^{eff} - \varepsilon C_1^{bl} \frac{\partial v_1^0}{\partial x_2}(0)\left(1 - \frac{x_2}{h}\right) e_1 - \left(\beta^{bl,\varepsilon} - s^\varepsilon - \varepsilon C_1^{bl} e_1\right) \frac{\partial v_1^0}{\partial x_2}(0) \text{ in } \Omega_1.$$

After a simple calculation we find the identity

$$v^0 - u^{eff} - \varepsilon C_1^{bl} \frac{\partial v_1^0}{\partial x_2}(0)\left(1 - \frac{x_2}{h}\right) e_1 = \frac{p_b - p_0}{2b\mu}(x_2 - h)\frac{(\varepsilon C_1^{bl})^2}{h - \varepsilon C_1^{bl}}.$$

Now (105) and (106) follow from the theory of very weak solutions for the Oseen system (89) and from the estimates on $\mathcal{U}^\varepsilon$, $\beta^{bl,\varepsilon}$, and s^ε. The estimate (107) is a consequence of the incompressibility and the estimate (82). ◻

Concluding remark. Therefore, we have found that the velocity field satisfies approximately the interface condition (22) of Beavers and Joseph in the sense of the estimate (91). It is important to point out that the parameter $\tilde{\alpha}$ from the expression (8) is determined through the auxiliary problems (10)–(14) and (20) by $\tilde{\alpha} = -(1/\varepsilon C_1^{bl}) > 0$.

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