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Formulation and Analysis of a Sequential Quadratic Programming Method for the Optimal Dirichlet Boundary Control of Navier–Stokes Flow*

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Abstract

The optimal boundary control of Navier–Stokes flow is formulated as a constrained optimization problem and a sequential quadratic programming (SQP) approach is studied for its solution. Since SQP methods treat states and controls as independent variables and do not insist on satisfying the constraints during the iterations, care must be taken to avoid a possible incompatibility of Dirichlet boundary conditions and incompressibility constraint. In this paper, compatibility is enforced by choosing appropriate function spaces. The resulting optimization problem is analyzed. Differentiability of the constraints and surjectivity of linearized constraints are verified and adjoints are computed. An SQP method is applied to the optimization problem and compared with other approaches.

Keywords: Optimal flow control, Navier-Stokes equations, sequential quadratic programming.

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1 Introduction

This paper is concerned with the application of a sequential quadratic programming algorithm for the numerical solution of a boundary control problem governed by the steady state Navier–Stokes equations. The optimal control problem is given by

$$\min \mathcal{J}(\mathbf{u}, \mathbf{g}) \tag{1}$$

subject to

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \qquad \text{in } \Omega, & (2)\\ \text{div } \mathbf{u} &= 0 \qquad \text{in } \Omega, & (3) \end{aligned}$$

with boundary conditions

$$\mathbf{u} = \mathbf{b} \qquad \text{in } \Gamma_{\mathbf{g}}, \qquad\qquad \mathbf{u} = \mathbf{b} + \mathbf{g} \qquad \text{in } \Gamma_c. \tag{4}$$

The domain $\Omega \subset I\!R^N, N = 2, 3$, is assumed to be Lipschitz and bounded. The functions $\mathbf{u}, p$ represent the vector of velocities and the pressure, respectively. These are the states. The function $\mathbf{g}$ plays the role of the control. The boundary part Γ_u is the uncontrolled part; controls are only applied on Γ_c.

A large class of objective functions considered in the literature for the control of Navier-Stokes flow can be written as

$$J(\mathbf{u}, \mathbf{g}) = \frac{1}{2}\|\mathbf{Z}\mathbf{u} - \mathbf{u}_d\|^p_{\mathbf{L}^p(D)} + \frac{\alpha}{2}\|\mathbf{g}\|^2_{\mathbf{G}}, \tag{5}$$

where $\mathbf{Z} \in \mathcal{L}(\mathbf{H}^1(\Omega), \mathbf{L}^p(D))$ and $D \subset \Omega$ or $D \subset \Gamma$. For two Banach spaces X, Y, $\mathcal{L}(X, Y)$ denotes the space of bounded linear operators mapping a X into Y. We set $\mathcal{L}(X) = \mathcal{L}(X, X)$. Special instances of (5) are

$$J(\mathbf{u}, \mathbf{g}) = \frac{1}{2}\|\mathbf{u} - \mathbf{u}_d\|^p_{\mathbf{L}^p(\Omega)} + \frac{\alpha}{2}\|\mathbf{g}\|^2_{\mathbf{G}}, \tag{6}$$

$$J(\mathbf{u}, \mathbf{g}) = \frac{1}{2}\int_D (u_2)^2 dx + \frac{\alpha}{2}\|\mathbf{g}\|^2_{\mathbf{G}}, \tag{7}$$

and

$$J(\mathbf{u}, \mathbf{g}) = \frac{1}{2}\int_D \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)^2 dx + \frac{\alpha}{2}\|\mathbf{g}\|^2_{\mathbf{G}}. \tag{8}$$

The objective (6) with $p = 4$ is used in [14], [16] for flow matching. Flow separation along a horizontal line D in two dimensions ($N = 2$) leads to (7),

see [10], and minimization of the vorticity in a subdomain $D \subset \Omega$ for $N = 2$ leads to the functional (8). For a discussion of other objective functions see, e.g., [16].

Optimal control of Navier–Stokes flow has been studied theoretically and numerically in [1], [2], [5], [10], [11], [14], [15], [16], [19], [21].

In most of the numerical work, the optimization problem resulting after discretization of (1)–(4) is solved using a gradient or a quasi–Newton method. In particular, the states $(\mathbf{u}, p)$ are viewed as functions of the controls $\mathbf{g}$. These functions $(\mathbf{u}(\mathbf{g}), p(\mathbf{g}))$ are implicitly defined through the Navier–Stokes equations (2)–(4). This results in an unconstrained minimization problem

$$\min \hat{\mathcal{J}}(\mathbf{g}) = \mathcal{J}(\mathbf{u}(\mathbf{g}), \mathbf{g}), \tag{9}$$

which is then solved using a gradient or a quasi–Newton method. This approach makes the assumption that the Navier–Stokes equations can be uniquely solved for the iterates $\mathbf{g}_k$ generated during the optimization. In addition to the possibly costly process of solving the nonlinear Navier–Stokes equations for every (trial) step, the strong coupling of states and controls may make the problem more nonlinear and may result in a higher number of optimization iterations. In the context of Navier–Stokes flow this is reported in [11].

A different approach treats states and controls as independent variables and views (1)–(4) as a constrained optimization problem. Optimization methods which have been used in this context are the augmented Lagrangian method [10] and the sequential quadratic programming (SQP) method [11]. The augmented Lagrangian method as applied in [10] requires the solution of a sequence of unconstrained nonlinear minimization problems. If the objective function $\mathcal{J}$ is quadratic, then the quadratic nature of the Navier–Stokes equations implies that the nonlinearity is of fourth order [10].

This work concentrates on SQP methods. It is motivated by the good theoretical convergence properties of SQP methods (the local convergence properties are superior to the ones of the augmented Lagrangian method applied in [10]) and by their successful application to optimal control problems governed by nonlinear partial differential equations. SQP methods find a solution of the constrained nonlinear minimization problem by solving a sequence of quadratic minimization problems. Although they have been successfully applied in [11], [17] to the solution of optimal control of Navier–Stokes flow, a rigorous theoretical justification is missing. This paper is a contribution towards that goal.

We formulate and analyze an SQP method for the solution of (1)–(4). One issue that has to be dealt with carefully is the compatibility of the boundary conditions (4) and the incompressibility condition (3). It holds that

$$\mathbf{u} = \mathbf{h} \text{ on } \Gamma \Rightarrow \int_\Omega \text{ div } \mathbf{u} = \int_\Gamma \mathbf{h} \cdot \mathbf{n}. \tag{10}$$

As a consequence, linearizations of (2)–(4) cannot be solved for arbitrary right hand sides. Since SQP methods treat states and controls as independent variables and do not insist on satisfying the constraints during the iterations, care must be taken to avoid a possible incompatibility of Dirichlet boundary conditions (corresponding to the computed step in the controls) and incompressibility constraint (to be satisfied by the corresponding step in the states). Therefore, the formulation of optimization problem is important. To ensure compatibility of (3) and (4) we assume that the functions $\mathbf{b}, \mathbf{g}$ satisfy the conditions

$$\int_\Gamma \mathbf{b} \cdot \mathbf{n} \, dx = 0, \quad \int_{\Gamma_c} \mathbf{g} \cdot \mathbf{n} \, dx = 0. \tag{11}$$

Moreover, if Γ_c has a boundary $\partial\Gamma_c$, then we also assume that

$$\mathbf{g} = 0 \qquad \text{on } \partial\Gamma_c. \tag{12}$$

In our optimization problem formulation we will eliminate the linear constraints (3) and (11) by absorbing them into the function spaces. This ensures that these constraints are always satisfied and that incompatibilities arising from (10) cannot occur. This formulation is in spirit of the approaches used in [12] for the analysis of Navier–Stokes equations and in [10] for the presentation of the augmented Lagrangian method for flow control. We then study the resulting optimization problem. In particular, we investigate differentiability of constraints, surjectivity of linearized constraints, and computation of adjoints. This study uses techniques already applied in some of the references above, in particular [10], [14]. However, our presentation is different because we use a different problem formulation and because we derive quantities like derivatives and their adjoints in the form in which they are used in the SQP method. For example, in [14], [16] the full system including (3) is considered and no particular optimization methods are discussed. In [10] it is assumed that the control is of the form $\mathbf{g}(x) = \sum_{i=1}^m f_i \chi_i(x)$, i.e. that the control $f \in I\!R^m$ is finite dimensional. Moreover, the application of the SQP method to (1)–(4) requires

information that is different from the information needed in the augmented Lagrangian method applied to the problem formulation in [10]. While we focus on the SQP method, the study in this paper is also applicable to other methods, e.g., the augmented Lagrangian–SQP method in [20], and in particular Lagrange–Newton methods [4], [27]. We comment more on this at the end of Section 2.

The paper is organized as follows. In § 2 we review the SQP method to establish notation and to provide the background for the presentation in the subsequent sections. In § 3 we formulate the optimal control problem as an equality constrained optimization problem. We discuss properties of the constraint functions, in particular differentiability and structure of the adjoints of the derivatives, and we formulate the optimality conditions. In § 4 we discuss the application of the SQP method to the optimal boundary control of Navier–Stokes flow. The purpose of this section is to outline the computations needed to implement the SQP method for the problem formulation established in this paper and to compare it with existing approaches.

2 Solution of Equality Constrained Problems by the SQP Method

In this section we sketch the SQP method for the solution of the abstract equality constrained problem

$$\begin{array}{ll} \min & J(\mathbf{u},\mathbf{g}) \\ s.t. & \mathbf{C}(\mathbf{u},\mathbf{g}) = \mathbf{0}\,. \end{array} \tag{13}$$

Here

$$J : \mathcal{U} \times \mathcal{G} \to I\!R, \qquad \mathbf{C} : \mathcal{U} \times \mathcal{G} \to \mathcal{C},$$

where $\mathcal{U}, \mathcal{G}$ are Hilbert spaces and $\mathcal{C}$ is a Banach space. We denote the dual space of $\mathcal{C}$ by $\mathcal{C}'$. We use the notation $\mathbf{x} = (\mathbf{u}, \mathbf{g})$. We assume that J and $\mathbf{C}$ are twice differentiable with Lipschitz continuous second derivatives.

The Lagrange function for (13) is given by

$$\mathbf{L}(\mathbf{x}, \boldsymbol{\lambda}) = J(\mathbf{x}) + \langle \mathbf{C}(\mathbf{x}), \boldsymbol{\lambda} \rangle_{\mathcal{C}\times\mathcal{C}'}. \tag{14}$$

If $\mathbf{x}_*$ is a solution of (13) with $\mathcal{R}(\mathbf{C}_\mathbf{x}(\mathbf{x}_*)) = \mathcal{C}$, i.e. if $\mathbf{C}_\mathbf{x}(\mathbf{x}_*)$ is surjective, it is said that $\mathbf{x}_*$ is a *regular point*. See, e.g., [24]. If $\mathbf{x}_*$ is a local minimizer of (13) and is a regular point, then the following *first order necessary optimality conditions* hold: There exists $\boldsymbol{\lambda}_* \in \mathcal{C}'$ such that $\nabla_\mathbf{x}\mathbf{L}(\mathbf{x}_*, \boldsymbol{\lambda}_*) = \mathbf{0}$,

$\nabla_\lambda \mathbf{L}(\mathbf{x}_*, \boldsymbol{\lambda}_*) = \mathbf{0}$. Due to the partitioning of $\mathbf{x}$ this is equivalent to the following system:

$$\begin{aligned} \nabla_{\mathbf{u}} J(\mathbf{x}_*) + \mathbf{C}_{\mathbf{u}}(\mathbf{x}_*)^* \boldsymbol{\lambda}_* &= \mathbf{0}, \\ \nabla_{\mathbf{g}} J(\mathbf{x}_*) + \mathbf{C}_{\mathbf{g}}(\mathbf{x}_*)^* \boldsymbol{\lambda}_* &= \mathbf{0}, \\ \mathbf{C}(\mathbf{x}_*) &= \mathbf{0}. \end{aligned} \tag{15}$$

For optimal control problems the third equation in (15) represents the state equation, the first equation corresponds to the adjoint equation which determines the co–state $\boldsymbol{\lambda}$, and the left hand side of the second equation corresponds to the gradient. The *second order sufficient optimality conditions* (see, e.g., [24]) can be stated as follows: If $\mathbf{x}_*$ is a regular point and satisfies (15) and

$$\nabla_{\mathbf{xx}} \mathbf{L}(\mathbf{x}_*, \boldsymbol{\lambda}_*)[\mathbf{h}, \mathbf{h}] \geq \sigma \|\mathbf{h}\|^2 \quad \forall \mathbf{h} \in \{\mathbf{w} | \mathbf{C}_{\mathbf{x}}(\mathbf{x}_*)\mathbf{w} = \mathbf{0}\} \tag{16}$$

for some $\sigma > 0$, then $\mathbf{x}_*$ is a solution of (13).

We now give a brief derivation of the SQP method. This presentation is rather standard and follows the ones given in [9], [18], [22], [23]. See also the presentation in [11]. It is included to establish some notation and to motivate the use of the SQP method. In the following we assume that $\mathbf{C}_{\mathbf{u}}(\mathbf{x})$ is continuously invertible for all $\mathbf{x}$ under consideration. This implies that all points $\mathbf{x}$ are regular points.

If Newton's method is applied to the solution of the optimality system (15), then in each iteration the following system has to be solved:

$$\begin{pmatrix} \nabla_{\mathbf{xx}} \mathbf{L}(\mathbf{x}, \boldsymbol{\lambda}) & \mathbf{C}_{\mathbf{x}}(\mathbf{x})^* \\ \mathbf{C}_{\mathbf{x}}(\mathbf{x}) & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{s}_{\mathbf{x}} \\ \mathbf{s}_\lambda \end{pmatrix} = - \begin{pmatrix} \nabla_{\mathbf{x}} J(\mathbf{x}) + \mathbf{C}_{\mathbf{x}}(\mathbf{x})^* \boldsymbol{\lambda} \\ \mathbf{C}(\mathbf{x}) \end{pmatrix}. \tag{17}$$

The system (17) can also be interpreted as the optimality system for

$$\begin{aligned} \min \quad & \tfrac{1}{2} \langle \nabla_{\mathbf{xx}} \mathbf{L}(\mathbf{x}, \boldsymbol{\lambda}) \mathbf{s}_{\mathbf{x}}, \mathbf{s}_{\mathbf{x}} \rangle + \langle \nabla_{\mathbf{x}} \mathbf{L}(\mathbf{x}, \boldsymbol{\lambda}), \mathbf{s}_{\mathbf{x}} \rangle, \\ \text{s.t.} \quad & \mathbf{C}_{\mathbf{x}}(\mathbf{x}) \mathbf{s}_{\mathbf{x}} = -\mathbf{C}(\mathbf{x}), \end{aligned} \tag{18}$$

provided $\nabla_{\mathbf{xx}} \mathbf{L}(\mathbf{x}, \boldsymbol{\lambda})$ is positive on the null-space of $\mathbf{C}_{\mathbf{x}}(\mathbf{x})$. This is the case in a neighborhood of a point satisfying the second order sufficient condition (16).

Using $\mathbf{s}_{\mathbf{x}} = (\mathbf{s}_{\mathbf{u}}, \mathbf{s}_{\mathbf{g}})$ the linearized state equation $\mathbf{C}_{\mathbf{x}}(\mathbf{x}) \mathbf{s}_{\mathbf{x}} = -\mathbf{C}(\mathbf{x})$ can be written as

$$\mathbf{C}_{\mathbf{u}}(\mathbf{x}) \mathbf{s}_{\mathbf{u}} + \mathbf{C}_{\mathbf{g}}(\mathbf{x}) \mathbf{s}_{\mathbf{g}} = -\mathbf{C}(\mathbf{x}). \tag{19}$$

Thus, the set of solutions for the linearized constraint is given by

$$\mathbf{s_x} = \mathbf{r}(\mathbf{x}) + \mathbf{W}(\mathbf{x})\mathbf{s_g}, \qquad \mathbf{s_g} \in \mathcal{G}, \tag{20}$$

where

$$\mathbf{r}(\mathbf{x}) = \begin{pmatrix} -\mathbf{C_u}(\mathbf{x})^{-1}\mathbf{C}(\mathbf{x}) \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{W}(\mathbf{x}) = \begin{pmatrix} -\mathbf{C_u}(\mathbf{x})^{-1}\mathbf{C_g}(\mathbf{x}) \\ I \end{pmatrix}. \tag{21}$$

The operator $\mathbf{W}(\mathbf{x})$ characterizes the null space of $\mathbf{C_x}(\mathbf{x})$. It holds that $\mathcal{N}(\mathbf{C_x}(\mathbf{x})) = \mathcal{R}(\mathbf{W}(\mathbf{x}))$.

Using (20), (21), and the equalities $\mathbf{C_x}(\mathbf{x})\mathbf{W}(\mathbf{x}) = \mathbf{0}$, $\mathbf{C_x}(\mathbf{x})\mathbf{r}(\mathbf{x}) = -\mathbf{C}(\mathbf{x})$ we can rewrite (17) in the reduced form

$$\begin{aligned} & \left(\nabla_{\mathbf{xx}}\mathbf{L}(\mathbf{x},\boldsymbol{\lambda})\mathbf{W}(\mathbf{x}) \,|\, \mathbf{C_x}(\mathbf{x})\right) \begin{pmatrix} \mathbf{s_g} \\ \mathbf{s}_\lambda \end{pmatrix} \\ = \;& -(\nabla_{\mathbf{x}} J(\mathbf{x}) + \mathbf{C_x}(\mathbf{x})^*\boldsymbol{\lambda} + \nabla_{\mathbf{xx}}\mathbf{L}(\mathbf{x},\boldsymbol{\lambda})\mathbf{r}(\mathbf{x})). \end{aligned} \tag{22}$$

The systems (17), (22) are solvable if the reduced Hessian

$$\mathbf{H}(\mathbf{x},\boldsymbol{\lambda}) = \mathbf{W}(\mathbf{x})^*\nabla_{\mathbf{xx}}\mathbf{L}(\mathbf{x},\boldsymbol{\lambda})\,\mathbf{W}(\mathbf{x}) \tag{23}$$

is nonsingular. In fact, multiplying (22) with $\mathbf{W}(\mathbf{x})^*$ from the left and using $\mathbf{W}(\mathbf{x})^*\mathbf{C_x}(\mathbf{x})^* = \mathbf{0}$ yields

$$\mathbf{s_g} = -\mathbf{H}(\mathbf{x},\boldsymbol{\lambda})^{-1}\mathbf{W}(\mathbf{x})^*(\nabla_{\mathbf{x}} J(\mathbf{x}) + \nabla_{\mathbf{xx}}\mathbf{L}(\mathbf{x},\boldsymbol{\lambda})\mathbf{r}(\mathbf{x})). \tag{24}$$

Note that the reduced Hessian is positive definite at points near local minimizers for which the second order sufficient conditions are satisfied. The corresponding $\mathbf{u}$ component can be computed from (19).

At the new iterate $\mathbf{x}_+ = \mathbf{x} + \mathbf{s_x}$, the estimate for the Lagrange multiplier is computed from the adjoint equation

$$\mathbf{C_u}(\mathbf{x}_+)^*\boldsymbol{\lambda}(\mathbf{x}_+) = -\nabla_{\mathbf{u}} J(\mathbf{x}_+).$$

The vector $\mathbf{W}(\mathbf{x})^*\nabla_{\mathbf{x}} J(\mathbf{x})$ is called the reduced gradient. It can be written as

$$\begin{aligned} \mathbf{W}(\mathbf{x})^*\nabla_{\mathbf{x}} J(\mathbf{x}) &= -\mathbf{C_g}(\mathbf{x})^*(\mathbf{C_u}(\mathbf{x})^*)^{-1}\nabla_{\mathbf{u}} J(\mathbf{x}) + \nabla_{\mathbf{g}} J(\mathbf{x}) \\ &= \mathbf{C_g}(\mathbf{x})^*\boldsymbol{\lambda}(\mathbf{x}) + \nabla_{\mathbf{g}} J(\mathbf{x}). \end{aligned}$$

If exact second order information is used and if the new adjoint variable $\boldsymbol{\lambda}_+ = \boldsymbol{\lambda} + \mathbf{s}_\lambda$, where $\mathbf{s}_\lambda$ is the adjoint variable of the quadratic problem (18),

then one obtains the Lagrange–Newton method. Such methods in function spaces have been considered, e.g., in [4], [27].

In applications where $\mathbf{C}(\mathbf{x}) = \mathbf{0}$ is given by a partial differential equation, the application of $\mathbf{W}(\mathbf{x})$ or $\mathbf{W}(\mathbf{x})^*$ to a vector requires the solution of a linear partial differential equation. Thus, the computation of $\mathbf{H}(\mathbf{x}, \boldsymbol{\lambda})\mathbf{v}$ for a given $\mathbf{v}$ is rather expensive and the explicit computation of $\mathbf{H}(\mathbf{x}, \boldsymbol{\lambda})$ is usually practically infeasible. In such cases, where second order information is too expensive to compute, the reduced Hessian $\mathbf{H}(\mathbf{x}, \boldsymbol{\lambda})$ is replaced by an operator $\mathbf{H}$ which is updated using some quasi–Newton formula. The vector $\mathbf{L}(\mathbf{x}, \boldsymbol{\lambda})\, \mathbf{r}(\mathbf{x})$ in (24) is also replaced by some vector $\mathbf{d}(\mathbf{x})$ using only lower order derivates. Usually the choice of $\mathbf{d}(\mathbf{x})$ depends on the quasi–Newton update used for $\mathbf{H}(\mathbf{x}, \boldsymbol{\lambda})$. It is beyond the scope of the paper to give an overview over the possible choices of $\mathbf{H}$ and $\mathbf{d}$ and their local convergence analysis. We refer to the literature, e.g. [6], [7], [8], [22], for a detailed discussion. To be specific, we state one version of a reduced SQP method below. It uses BFGS updates to approximate the reduced Hessian and $\mathbf{d} = \mathbf{0}$.

Algorithm 2.1 (Reduced SQP–BFGS Method (Local Version))

Initialization:

Given $\mathbf{x}_0 \in \mathcal{U} \times \mathcal{G}$ and $\mathbf{H}_0 \in \mathcal{L}(\mathcal{G})$, $\mathbf{H}_0$ positive definite.
Set $k = 0$.

Step k:

1. Compute the Lagrange multiplier estimate:
   Solve $\mathbf{C}_\mathbf{u}(\mathbf{x}_k)^* \boldsymbol{\lambda} = -\nabla_\mathbf{u} J(\mathbf{x}_k)$.
2. Compute the reduced gradient:
   $\mathbf{W}(\mathbf{x}_k)^* \nabla_\mathbf{x} J(\mathbf{x}_k) = \mathbf{C}_\mathbf{g}(\mathbf{x}_k)^* \boldsymbol{\lambda}(\mathbf{x}_k) + \nabla_\mathbf{g} J(\mathbf{x}_k)$.
3. Solve $\mathbf{H}_k \mathbf{s}_\mathbf{g} = -\mathbf{W}(\mathbf{x}_k)^* \nabla J(\mathbf{x}_k)$.
4. Solve $\mathbf{C}_\mathbf{u}(\mathbf{x}_k)\mathbf{s}_\mathbf{u} = -\mathbf{C}(\mathbf{x}_k) - \mathbf{C}_\mathbf{g}(\mathbf{x}_k)\mathbf{s}_\mathbf{g}$.
5. Set $\mathbf{u}_{k+1} = \mathbf{u}_k + \mathbf{s}_\mathbf{u}$, $\mathbf{g}_{k+1} = \mathbf{g}_k + \mathbf{s}_\mathbf{g}$.
6. Update $\mathbf{H}_k$ to obtain $\mathbf{H}_{k+1}$:
   $\mathbf{v} = \mathbf{W}(\mathbf{x}_k + \mathbf{W}(\mathbf{x}_k)\mathbf{s}_\mathbf{g})^* \nabla J(\mathbf{x}_k + \mathbf{W}(\mathbf{x}_k)\mathbf{s}_\mathbf{g}) - \mathbf{W}(\mathbf{x}_k)^* \nabla J(\mathbf{x}_k)$.
   $$\mathbf{H}_{k+1} = \mathbf{H}_k + \frac{\mathbf{v} \otimes \mathbf{v}}{\langle \mathbf{v}, \mathbf{s}_\mathbf{g} \rangle_\mathcal{G}} - \frac{(\mathbf{H}_k \mathbf{s}_\mathbf{g}) \otimes (\mathbf{H}_k \mathbf{s}_\mathbf{g})}{\langle \mathbf{s}_\mathbf{g}, \mathbf{H}_k \mathbf{s}_\mathbf{g} \rangle_\mathcal{G}}.$$

For given $\mathbf{v}, \mathbf{w}$, $\mathbf{v} \otimes \mathbf{w}$ denotes the linear rank-one operator $(\mathbf{v} \otimes \mathbf{w})\mathbf{h} = \langle \mathbf{w}, \mathbf{h} \rangle_\mathcal{G} \mathbf{v}$.

The following convergence theorem for the SQP Algorithm 2.1 is proven in [22].

Theorem 2.1 *Let $\mathcal{U}, \mathcal{G}$ be Hilbert spaces and let $\mathcal{C}$ be a Banach space. Furthermore, let $\mathbf{x}_* = (\mathbf{u}_*, \mathbf{g}_*)$ be a local minimizer of (13). Assume there exists a neighborhood D of $\mathbf{x}_*$ such that $J : \mathcal{U} \times \mathcal{G} \to I\!R$ and $\mathbf{C} : \mathcal{U} \times \mathcal{G} \to \mathcal{C}$ are twice Lipschitz continuously differentiable on D and $\mathbf{C_u}(\mathbf{u}, \mathbf{g})$ is continuously invertible on D.*

If the second order sufficient optimality conditions (16) are satisfied at $\mathbf{x}_*$, then there exist $\epsilon > 0$ and $\delta > 0$ such that for all $\mathbf{x}_0$ and all self-adjoint and positive $\mathbf{H}_0 \in \mathcal{L}(\mathcal{G})$ with

$$\|\mathbf{H}_0 - \mathbf{H}(\mathbf{x}_*, \boldsymbol{\lambda}_*)\| < \delta, \quad \|\mathbf{x}_0 - \mathbf{x}_*\| < \epsilon,$$

the SQP Algorithm 2.1 is well defined and the iterates $\mathbf{x}_k$ converge q–linearly to $\mathbf{x}_*$. If, in addition, $\mathbf{H}_0 - \mathbf{H}(\mathbf{x}_*, \boldsymbol{\lambda}_*)$ is a compact operator, then the convergence is 2–step q–superlinear, i.e.,

$$\lim_{k \to \infty} \frac{\|\mathbf{x}_{k+1} - \mathbf{x}_*\|_{\mathcal{U} \times \mathcal{G}}}{\|\mathbf{x}_{k-1} - \mathbf{x}_*\|_{\mathcal{U} \times \mathcal{G}}} = 0.$$

Proof. This result is a special case of [22, Thm. 3.12] in which the representation $\mathbf{T}(\mathbf{x})$ of the nullspace is chosen to be $\mathbf{W}(\mathbf{x})$, see (21), and a right inverse $\mathbf{R}(\mathbf{x})$ of $\mathbf{T}(\mathbf{x})$ is

$$\mathbf{R}(\mathbf{x}) = \begin{pmatrix} -\mathbf{C_u}(\mathbf{x})^{-1} \\ \mathbf{0} \end{pmatrix}.$$

The differentiability assumptions on $J, \mathbf{C}, \mathbf{W}$, and $\mathbf{R}$ follow from the assumptions on J and $\mathbf{C}$ stated in the theorem. See also [22, p. 156]. □

To apply this convergence result to the control of Navier-Stokes flow, we need to (i) verify differentiability of the objective and constraint function, (ii) establish continuous invertibility of $\mathbf{C_u}(\mathbf{x})$, and (iii) discuss the construction of starting approximations $\mathbf{H_0}$ for the reduced Hessian. Items (i) and (ii) are also the main tasks for applying local convergence results for the augmented Lagrangian–SQP method [20], or the Lagrange–Newton method, which is equivalent to applying Newton's method to (15) and was the starting point of our discussion of the SQP Algorithm 2.1. Therefore the presentation in the subsequent sections immediately applies to these methods.

Before we return to our specific application, we briefly contrast the SQP Algorithm 2.1 with gradient or quasi–Newton methods. Again, this discussion is standard, but it is included to support our choice of solution

algorithm over gradient or quasi–Newton methods, which are mostly used for these applications.

The SQP method moves towards optimality and feasibility at the same time. One step of the SQP Algorithm 2.1 can be compared to one step of a quasi–Newton method applied to the reduced problem

$$\min \hat{J}(\mathbf{g}) = J(\mathbf{u}(\mathbf{g}), \mathbf{g}),$$

where $\mathbf{u}(\mathbf{g})$ is the function implicitly defined as the solution of $\mathbf{C}(\mathbf{u}, \mathbf{g}) = \mathbf{0}$ (assuming that it exists). The SQP Algorithm 2.1 requires the solution of the linearized state equation $\mathbf{C}_\mathbf{u}(\mathbf{x}_k)\mathbf{s}_\mathbf{u} = -\mathbf{C}(\mathbf{x}_k) - \mathbf{C}_\mathbf{g}(\mathbf{x}_k)\mathbf{s}_\mathbf{g}$ and of the adjoint equation $\mathbf{C}_\mathbf{u}(\mathbf{x}_k)^* \boldsymbol{\lambda} = -\nabla_\mathbf{u} J(\mathbf{x}_k)$. The update of the Hessian approximation $\mathbf{H}_{k+1}$ uses the vector $\mathbf{W}(\mathbf{x}_k + \mathbf{W}(\mathbf{x}_k)\mathbf{s}_\mathbf{g})^* \nabla J(\mathbf{x}_k + \mathbf{W}(\mathbf{x}_k)\mathbf{s}_\mathbf{g})$, which require the solution of an additional linearized state and adjoint equation. A quasi–Newton method applied to the reduced problem requires the solution of the nonlinear equation $\mathbf{C}(\mathbf{u}, \mathbf{g}_k) = \mathbf{0}$ to compute $\mathbf{u}_k = \mathbf{u}(\mathbf{g}_k)$ and the solution of the adjoint equation to compute $\nabla \hat{J}(\mathbf{g}_k)$. If $(\mathbf{u}_k, \mathbf{g}_k)$ are feasible, i.e. if $\mathbf{C}(\mathbf{u}_k, \mathbf{g}_k) = \mathbf{0}$, then $\nabla \hat{J}(\mathbf{g}_k) = \mathbf{W}(\mathbf{x}_k)^* \nabla_\mathbf{x} J(\mathbf{x}_k)$. For more details see, e.g., the discussions in [9], [18]. Hence, if the solution of the nonlinear equation $\mathbf{C}(\mathbf{u}, \mathbf{g}_k) = \mathbf{0}$ is done using Newton's method and requires more than four Newton iterations, then the SQP iteration is computationally cheaper than the quasi–Newton iteration applied to the reduced problem. Moreover, the decoupling of states and controls often makes the optimization problem less nonlinear and might result in fewer SQP iterations to solve the problem compared to a quasi–Newton method applied to the reduced problem. In the context of optimal control of Navier–Stokes flow this has been reported in [11].

3 Existence and Characterization of Optimal Controls

In this section we formulate the optimal control problem as a constrained optimization problem. This optimization problem is then studied with the application of the SQP method in mind. As we have noted in the introduction, proof of existence of optimal controls and differentiability of the constraint function is standard. See, e.g., [10], [14]. However there are several differences between the presentations in [10], [14] and ours because we use a different problem formulation and because we derive quantities like derivatives and their adjoints in the form in which they are used in the

SQP method. The difference in problem formulation is partly due to our use of a method which views velocities, pressures, and controls as independent variables and the Navier–Stokes equations as constraints. Our problem formulation avoids incompatibility of (linearized) constraints, which is necessary to establish continuous invertibility of $\mathbf{C}_{\mathbf{x}}(\mathbf{x})$.

3.1 Weak Formulations and Existence of Optimal Controls

The state space is related to the function spaces

$$\mathbf{H}^1 = \mathbf{H}^1(\Omega) = \left\{ v_j \in L^2(\Omega) \mid \tfrac{\partial v_j}{\partial x_k} \in L^2(\Omega) \text{ for } j,k = 1,\ldots,N \right\},$$
$$\mathbf{H}_0^1 = \mathbf{H}_0^1(\Omega) = \left\{ \mathbf{v} \in \mathbf{H}^1 \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma \right\},$$
$$\mathbf{V} = \left\{ \mathbf{v} \in \mathbf{H}^1 \mid \text{ div } \mathbf{v} = 0 \right\},$$

and

$$L_0^2 = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \mid \int_\Omega q = 0 \right\}.$$

The control space is chosen to be $\mathbf{H}_0^1(\Gamma_c)$. (If Γ_c had no boundary, then $\mathbf{H}^1(\Gamma_c)$ would be chosen as the control space.) Moreover, we consider the closed subspace

$$\mathbf{G} = \left\{ \mathbf{g} \in \mathbf{H}_0^1(\Gamma_c) \mid \int_{\Gamma_c} \mathbf{g} \cdot \mathbf{n} \, dx = 0 \right\},$$

of $\mathbf{H}_0^1(\Gamma_c)$. For details on these and other function spaces used in this paper, we refer to [3], [12], [26]. Let $\gamma : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}(\Gamma)$ be the continuous trace operator.

If we multiply (2), (3) by sufficiently smooth test functions $\mathbf{v}$ and q, respectively, then we are lead to the bi– and tri–linear forms

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u} \cdot \nabla\mathbf{v} \, dx, \tag{25}$$
$$b(\mathbf{u};\mathbf{v},\mathbf{w}) = \int_\Omega ((\mathbf{u}\cdot\nabla)\mathbf{v}) \cdot \mathbf{w} \, dx, \tag{26}$$
$$c(\mathbf{v},q) = -\int_\Omega \text{div } \mathbf{v} \; q \, dx \tag{27}$$

and the following weak form of (2), (3), (4):

$$\nu\, a(\mathbf{u},\mathbf{v}) + b(\mathbf{u};\mathbf{u},\mathbf{v}) + c(\mathbf{v},p) = \langle \mathbf{f}, \mathbf{v} \rangle \qquad \forall \mathbf{v} \in \mathbf{H}_0^1, \tag{28}$$
$$c(\mathbf{u},q) = 0 \qquad \forall q \in L^2, \tag{29}$$
$$\mathbf{u} = \mathbf{b} \qquad \text{on } \Gamma_u, \tag{30}$$
$$\mathbf{u} = \mathbf{b} + \mathbf{g} \qquad \text{on } \Gamma_c. \tag{31}$$

The solvability of this system is investigated in, e.g., [12], [26]. In that analysis the following properties of the bi– and tri–linear forms are used. The bilinear form $a(\mathbf{u}, \mathbf{v})$ obeys

$$\begin{array}{rcll} a(\mathbf{u}, \mathbf{u}) & \geq & \|\nabla \mathbf{u}\|^2_{\mathbf{L}^2} & \forall \mathbf{u} \in \mathbf{H}^1, \\ a(\mathbf{u}, \mathbf{v}) & \leq & \|\nabla \mathbf{u}\|_{\mathbf{L}^2} \|\nabla \mathbf{v}\|_{\mathbf{L}^2} & \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}^1. \end{array} \tag{32}$$

See, e.g., [12, p. 86]. If $\mathbf{u} \in \mathbf{H}^1$ satisfies div $\mathbf{u} = 0$ and if $\mathbf{v} \in \mathbf{H}^1_0$, then

$$b(\mathbf{u}; \mathbf{v}, \mathbf{v}) = 0. \tag{33}$$

This equation is proven, e.g., in [12, p. 285], [26, p. 163]. For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1$ it holds that

$$b(\mathbf{u}; \mathbf{v}, \mathbf{w}) \leq \|\mathbf{u}\|_{\mathbf{L}^4} \|\nabla \mathbf{v}\|_{\mathbf{L}^2} \|\mathbf{w}\|_{\mathbf{L}^4} \leq \kappa \|\mathbf{u}\|_{\mathbf{H}^1} \|\mathbf{v}\|_{\mathbf{H}^1} \|\mathbf{w}\|_{\mathbf{H}^1}. \tag{34}$$

In (34) we have used that the imbeddings $H^1(\Omega) \subset L^p(\Omega), \quad p \in [1, \infty), N = 2$, $p \in [1, 6], N = 3$, are continuous. This and the compactness of the imbeddings $H^1(\Omega) \subset L^p(\Omega), \quad p \in [1, \infty), N = 2$, $p \in [1, 6), N = 3$, will be used again later.

For the solvability of (28)–(31) it is necessary that the boundary data are in $\mathbf{H}^{1/2}(\Gamma)$ and that the boundary data are compatible with (29), e.g., that the compatibility conditions (11) is satisfied. The first requirement is fulfilled by the assumption $\mathbf{b} \in \mathbf{H}^{1/2}(\Gamma)$ and the requirement that the controls satisfy $\mathbf{g} \in \mathbf{H}^1(\Gamma)$ and (12). We have chosen the control space $\mathbf{H}^1(\Gamma)$ over $\mathbf{H}^{1/2}(\Gamma)$ because terms resulting in the optimality conditions and the optimization algorithm can be implemented in a more straightforward way. The compatibility condition (11) represents a constraint on $\mathbf{g}$ that must be satisfied at least at the solution. However, care must also be taken in the formulation of the SQP method. In the SQP method we are required to solve linearized equations. If we consider (29), (30), (31) and (11) as constraints, then (10) implies that

$$\begin{array}{rcll} c(\hat{\mathbf{u}}, q) & = & r & \forall q \in L^2, \\ \hat{\mathbf{u}} & = & \mathbf{0} & \text{on } \Gamma_u, \\ \hat{\mathbf{u}} & = & \hat{\mathbf{g}} & \text{on } \Gamma_c, \\ \int_{\Gamma_c} \hat{\mathbf{g}} \cdot \mathbf{n} & = & \rho & \end{array} \tag{35}$$

can only be solved if $r = -\rho$. To avoid potential problems arising from incompatibilities, we enforce (29) and (11) strictly, i.e. we do not view these

as explicit constraints, but eliminate them by choosing appropriate function spaces.

Before we present the exact problem formulation, we remark on the weak formulation (28)–(31). Another weak form in which the Dirichlet boundary conditions are imposed weakly using a Lagrange multiplier is considered in [13] and is used for the optimal control of Navier–Stokes flow, e.g, in [14]. That weak form has advantages when deriving error estimates between solutions of the discrete and the infinite dimensional problem. In the infinite dimensional framework, however, the difficulty arising from incompatibility of right hand sides remain and the basic considerations applied in this section should carry over to the weak formulation used in [14].

As noted earlier, we do not view (29) and (11) as explicit constraints, but enforce them by essentially absorbing them into the function spaces. The constraint (11) on $\mathbf{g}$ is enforced using the projection $\mathbf{P}$ of $\mathbf{H}_0^1(\Gamma_c)$ onto $\mathbf{G}$. The function $\tilde{\mathbf{g}} = \mathbf{P}\mathbf{g}$ is the solution of

$$\min_{\text{s.t. } \tilde{\mathbf{g}} \in \mathbf{H}_0^1(\Gamma_c),\, \int_{\Gamma_c} \tilde{\mathbf{g}} \cdot \mathbf{n}\, dx = 0} \|\tilde{\mathbf{g}} - \mathbf{g}\|^2_{\mathbf{H}_0^1(\Gamma_c)}. \tag{36}$$

Let $\mathbf{S}$ satisfy

$$\begin{aligned} \mathbf{S} &\in \mathcal{L}(\{\mathbf{h} \in \mathbf{H}^{1/2}(\Gamma) \mid \textstyle\int_\Gamma \mathbf{h} \cdot \mathbf{n} = 0\}, \mathbf{H}^1), \\ \gamma \mathbf{S}\mathbf{h} = \mathbf{h}, \;\; \mathrm{div}\mathbf{S}\mathbf{h} = 0, \quad &\forall \mathbf{h} \in \{\mathbf{h} \in \mathbf{H}^{1/2}(\Gamma) \mid \textstyle\int_\Gamma \mathbf{h} \cdot \mathbf{n} = 0\}. \end{aligned} \tag{37}$$

Here $\mathcal{L}(X, Y)$ denotes the space of bounded linear operator from X into Y. Notice that in the definition of (37) the image is required to be divergence free. This requires appropriate restriction of the domain of definition. A concrete operator $\mathbf{S}$ that satisfies this is $\mathbf{u} = \mathbf{S}\mathbf{h}$ where $(\mathbf{u}, p)$ solves the Stokes equation

$$\begin{aligned} \nu\, a(\mathbf{u}, \mathbf{v}) + c(\mathbf{v}, p) &= \mathbf{0} \qquad \forall \mathbf{v} \in \mathbf{H}_0^1, & (38)\\ c(\mathbf{u}, q) &= 0 \qquad \forall q \in L^2, & (39)\\ \mathbf{u} &= \mathbf{h} \qquad \text{on } \Gamma. & (40) \end{aligned}$$

See, e.g., [12]. We apply $\mathbf{S}$ to a function $\mathbf{h} \in \mathbf{H}_0^1(\Gamma_c)$ by first extending $\mathbf{h}$ onto Γ by zero and then applying $\mathbf{S}$ to the extension.

Now, (11), (28)–(31) can be written as

$$\begin{aligned} \nu\, a(\mathbf{u}^b, \mathbf{v}) + b(\mathbf{u}^b; \mathbf{u}^b, \mathbf{v}) + c(\mathbf{v}, p) &= \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{H}_0^1, & (41)\\ c(\mathbf{u}, q) &= 0 \qquad \forall q \in L^2, & (42) \end{aligned}$$

where we have set

$$\mathbf{u}^b = \mathbf{u} + \mathbf{S}(\mathbf{b} + \mathbf{P}\mathbf{g}).$$

Note that the definition of $\mathbf{S}$ implies $c(\mathbf{u}, q) = c(\mathbf{u}^b, q)$ for all $q \in L^2$ and $\mathbf{u} \in \mathbf{H}_0^1$. It is shown in [12, IV §§1,2] that finding $\mathbf{u} \in \mathbf{H}_0^1$ and $p \in L_0^2$ with (41), (42) is equivalent to finding $\mathbf{u} \in \mathbf{V}$ with

$$\nu\, a(\mathbf{u}^b, \mathbf{v}) + b(\mathbf{u}^b; \mathbf{u}^b, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \qquad \forall \mathbf{v} \in \mathbf{V}. \tag{43}$$

This leads to the function

$$\mathbf{C} : \mathbf{V} \times \mathbf{H}^1(\Gamma_c) \to \mathbf{V}'$$

with

$$\begin{aligned} &\langle \mathbf{C}(\mathbf{u}, \mathbf{g}), \mathbf{v} \rangle_{\mathbf{V}' \times \mathbf{V}} \\ &\quad = \quad \nu\, a(\mathbf{u} + \mathbf{S}(\mathbf{b} + \mathbf{P}\mathbf{g}), \mathbf{v}) \\ &\qquad + b(\mathbf{u} + \mathbf{S}(\mathbf{b} + \mathbf{P}\mathbf{g}); \mathbf{u} + \mathbf{S}(\mathbf{b} + \mathbf{P}\mathbf{g}), \mathbf{v}) - \langle \mathbf{f}, \mathbf{v} \rangle. \end{aligned} \tag{44}$$

The abstract form of the optimal control problem (1)–(4), (11) is now given by

$$\min \qquad J(\mathbf{u}, \mathbf{g}) \tag{45}$$

$$\text{s.t.} \quad \mathbf{C}(\mathbf{u}, \mathbf{g}) = \mathbf{0}, \tag{46}$$

where

$$J(\mathbf{u}, \mathbf{g}) = \mathcal{J}(\mathbf{u} + \mathbf{S}(\mathbf{b} + \mathbf{P}\mathbf{g}), \mathbf{P}\mathbf{g}). \tag{47}$$

From the existence of solutions of the steady–state Navier–Stokes equations (28)–(30), see e.g. [12, Thm. IV.2.3], we obtain the following result (notice that the compatibility assumption (11) implies (2.14) in [12]):

Theorem 3.1 *For all $\mathbf{g} \in \mathbf{H}_0^1(\Gamma_c)$ with $\int_{\Gamma_c} \mathbf{g} \cdot \mathbf{n}\, dx = 0$ there exists $\mathbf{u} \in \mathbf{V}$ such that $\mathbf{C}(\mathbf{u}, \mathbf{g}) = 0$. In particular, there exist feasible points of (46).*

The following result on the existence of optimal controls is standard. See, e.g., [10], [14].

Theorem 3.2 *If J is weakly lower semicontinuous and if the level sets $\{(\mathbf{u}, \mathbf{g}) \mid J(\mathbf{u}, \mathbf{g}) \le J_0,\ (\mathbf{u}, \mathbf{g})$ satisfy (46)$\}$ are bounded, then there exists a solution $(\mathbf{u}_*, \mathbf{g}_*) \in \mathbf{V} \times \mathbf{H}^1(\Gamma_c)$ of the optimal control problem (45), (46).*

Weak lower semicontinuity of the objective function J is given if J is of tracking type (6) with $p \in [2,6)$ for $N = 3$ or $p \in [2,\infty)$ for $N = 2$, $D \subset \Omega$. This follows from the compactness of the imbedding $H^1 \subset L^p$ and the weak lower semicontinuity of the norm. The same arguments show the weak lower semicontinuity of (7). The objective function (8) is weakly lower semicontinuous as well since weak convergence of $\mathbf{u}$ in $\mathbf{H}^1$ implies weak convergence of the vorticity in L^2. The weak lower semicontinuity of the norm implies the desired result.

The boundedness of the level set for functionals (5) with $\alpha > 0$ follows with standard arguments. First, all $\mathbf{g}$ in the level set obey $\|\mathbf{g}\|_{\mathbf{H^1}} \leq \alpha^{-1} J_0$. If $\mathbf{u}$, $\mathbf{g}$ solve (46), then (44), (33) imply

$$\nu\, a(\mathbf{u} + \mathbf{S}(\mathbf{b} + \mathbf{P}\mathbf{g}), \mathbf{u} + \mathbf{S}(\mathbf{b} + \mathbf{P}\mathbf{g})) = \langle \mathbf{f}, \mathbf{u} + \mathbf{S}(\mathbf{b} + \mathbf{P}\mathbf{g})\rangle$$

and

$$\begin{aligned} c\nu\, \|\mathbf{u}\|^2_{\mathbf{H^1}} &\leq \nu\, \|\nabla \mathbf{u}\|^2_{\mathbf{L^2}} \leq \nu\, a(\mathbf{u} + \mathbf{S}(\mathbf{b} + \mathbf{P}\mathbf{g}), \mathbf{u} + \mathbf{S}(\mathbf{b} + \mathbf{P}\mathbf{g})) \\ &= (\|\mathbf{f}\|_{\mathbf{L^2}} + 2\|\mathbf{S}(\mathbf{b} + \mathbf{P}\mathbf{g})\|_{\mathbf{H^1}})\|\mathbf{u}\|^2_{\mathbf{H^1}} + \|\mathbf{S}(\mathbf{b} + \mathbf{P}\mathbf{g})\|^2_{\mathbf{H^1}}. \quad (48) \end{aligned}$$

With the boundedness of $\mathbf{g}$ and $\mathbf{S}$, this yields the boundedness of the level set.

3.2 Derivatives and Solvability of the Linearized Navier–Stokes Equation

Using the fact that the mappings $u \mapsto a(u,\cdot) \in (\mathbf{H}^1)'$, and $u \mapsto b(u;u,\cdot) \in (\mathbf{H}^1)'$ are linear and quadratic, respectively, and using the properties (32), (34) one can easily establish the Fréchet –differentiability of the constraint function $\mathbf{C}$. The partial derivatives of $\mathbf{C}$ are given by

$$\langle \mathbf{C_u}(\bar{\mathbf{u}}, \bar{\mathbf{g}})\mathbf{u}, \mathbf{v}\rangle_{\mathbf{V}'\times\mathbf{V}} = \nu\, a(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}; \bar{\mathbf{u}}^b, \mathbf{v}) + b(\bar{\mathbf{u}}^b; \mathbf{u}, \mathbf{v}) \quad (49)$$

and

$$\langle \mathbf{C_g}(\bar{\mathbf{u}}, \bar{\mathbf{g}})\mathbf{g}, \mathbf{v}\rangle_{\mathbf{V}'\times\mathbf{V}} = \nu\, a(\mathbf{S}\mathbf{P}\mathbf{g}, \mathbf{v}) + b(\mathbf{S}\mathbf{P}\mathbf{g}; \bar{\mathbf{u}}^b, \mathbf{v}) + b(\bar{\mathbf{u}}^b; \mathbf{S}\mathbf{P}\mathbf{g}, \mathbf{v}). \quad (50)$$

Here we have used the notation

$$\bar{\mathbf{u}}^b = \bar{\mathbf{u}} + \mathbf{S}(\mathbf{b} + \mathbf{P}\bar{\mathbf{g}}). \quad (51)$$

That (49) is the partial Fréchet –derivative of $\mathbf{C}$ can be seen from the estimate

$$\langle \mathbf{C}(\bar{\mathbf{u}} + \mathbf{u}, \bar{\mathbf{g}}) - \mathbf{C}(\bar{\mathbf{u}}, \bar{\mathbf{g}}) - \mathbf{C_u}(\bar{\mathbf{u}}, \bar{\mathbf{g}})\mathbf{u}, \mathbf{v}\rangle_{\mathbf{V}'\times\mathbf{V}} = b(\mathbf{u}; \mathbf{u}, \mathbf{v}) \leq \kappa \|\mathbf{u}\|^2_{\mathbf{H}^1} \|\mathbf{v}\|_{\mathbf{H}^1},$$

cf. (34), which implies that

$$\|\mathbf{C}(\bar{\mathbf{u}}+\mathbf{u},\bar{\mathbf{g}})-\mathbf{C}(\bar{\mathbf{u}},\bar{\mathbf{g}})-\mathbf{C}_{\mathbf{u}}(\bar{\mathbf{u}},\bar{\mathbf{g}})\mathbf{u}\|_{\mathbf{V}'}\leq\kappa\|\mathbf{u}\|^2_{\mathbf{H}^1}.$$

The partial Fréchet –derivative (50) can be established analogously.

The following result establishes the continuous invertibility of $\mathbf{C}_{\mathbf{u}}(\bar{\mathbf{u}},\bar{\mathbf{g}})$ and the surjectivity of the linearization of the constraints (46).

Theorem 3.3 *Let* $\bar{\mathbf{u}}^b=\bar{\mathbf{u}}+\mathbf{S}(\mathbf{b}+\mathbf{P}\bar{\mathbf{g}})$ *satisfy* $\mathit{div}\ \bar{\mathbf{u}}^b=0$ *and*

$$\nu-\kappa^2\|\nabla\bar{\mathbf{u}}^b\|_{\mathbf{L}^2}>0, \tag{52}$$

where κ *is the imbedding constant with* $\|\mathbf{u}\|_{\mathbf{L}^4}\leq\kappa\|\nabla\mathbf{u}\|_{\mathbf{L}^2}$ *for all* $\mathbf{u}\in\mathbf{H}^1_0$, *then for all* $\mathbf{r}\in\mathbf{V}'$ *there exists a unique* $\mathbf{u}\in\mathbf{V}$ *such that* $\mathbf{C}_{\mathbf{u}}(\bar{\mathbf{u}},\bar{\mathbf{g}})\mathbf{u}=\mathbf{r}$. *The solution satisfies*

$$\|\mathbf{u}\|_{\mathbf{H}^1}\leq(\nu-\kappa^2\|\nabla\bar{\mathbf{u}}^b\|_{\mathbf{L}^2})^{-1}\|\mathbf{r}\|_{\mathbf{V}'}. \tag{53}$$

In particular, for all $\mathbf{r}\in\mathbf{V}'$ *there exists* $\mathbf{u}\in\mathbf{V}$ *and* $\mathbf{g}\in\mathbf{H}^1_0(\Gamma)$ *such that*

$$\mathbf{C}_{\mathbf{u}}(\bar{\mathbf{u}},\bar{\mathbf{g}})\mathbf{u}+\mathbf{C}_{\mathbf{g}}(\bar{\mathbf{u}},\bar{\mathbf{g}})\mathbf{g}=\mathbf{r}.$$

Proof. We have to show that

$$\sigma(\mathbf{u},\mathbf{v}):=\nu\,a(\mathbf{u},\mathbf{v})+b(\mathbf{u};\bar{\mathbf{u}}^b,\mathbf{v})+b(\bar{\mathbf{u}}^b;\mathbf{u},\mathbf{v})=\mathbf{r}(\mathbf{v})\qquad\forall\mathbf{v}\in\mathbf{V} \tag{54}$$

admits a unique solution. Using (33),(34) we find that

$$\begin{aligned}\sigma(\mathbf{u},\mathbf{u}) &= \nu\,a(\mathbf{u},\mathbf{u})+b(\mathbf{u};\bar{\mathbf{u}}^b,\mathbf{u})\\ &\geq (\nu-\kappa^2\|\nabla\bar{\mathbf{u}}^b\|_{\mathbf{L}^2})\,\|\nabla\mathbf{u}\|^2_{\mathbf{L}^2}.\end{aligned}$$

Hence, the bilinear form σ is $\mathbf{V}$–elliptic. From (32), (34) it is immediately seen that σ is also continuous. The Lax–Milgram theorem now yields the existence and uniqueness of a solution $\mathbf{u}$ of (54). □

Since $\nabla\bar{\mathbf{u}}^b$ reflects the change in velocity, the condition (52) on the flow velocity is a condition on the uniformity of the flow. If $\bar{\mathbf{u}}^b$ satisfies the Navier-Stokes equation, then one can use a-priori bounds similar to (48), to estimate $\|\bar{\mathbf{u}}^b\|_{\mathbf{H}^1}$ in terms of ν and $\bar{\mathbf{g}}$. If $(\bar{\mathbf{u}}^b,\bar{\mathbf{g}})$ satisfy the Navier-Stokes equation, (52) can be expected to hold for small ν. A different argument to establish solvability of the linearized Navier-Stokes equations is used in [14]. By premultiplication with the Stokes operator, the linearized Navier-Stokes equations are viewed as a compact perturbation of the identity and

an argument involving a perturbation of the domain is used to deduce that the compact perturbation does not have eigenvalues equal to minus one, i.e., the linearized Navier-Stokes equation is uniquely solvable. This argument does not require restrictions on $\bar{\mathbf{u}}^b$.

The differentiability properties of the objective function depend on the particular form of J. If J is of type (5) then it is infinitely often differentiable. In particular, the objective functions (6), (7), (8) are infinitely often differentiable.

3.3 Adjoints

The adjoint $\mathbf{C_u}(\bar{\mathbf{u}}, \bar{\mathbf{g}})^* \in \mathcal{L}(\mathbf{V}, \mathbf{V}')$ of $\mathbf{C_u}(\bar{\mathbf{u}}, \bar{\mathbf{g}})$ is given by

$$\langle \mathbf{C_u}(\bar{\mathbf{u}}, \bar{\mathbf{g}})^* \boldsymbol{\lambda}, \mathbf{v} \rangle_{\mathbf{V}' \times \mathbf{V}} = \nu\, a(\mathbf{v}, \boldsymbol{\lambda}) + b(\mathbf{v}; \bar{\mathbf{u}}^b, \boldsymbol{\lambda}) + b(\bar{\mathbf{u}}^b; \mathbf{v}, \boldsymbol{\lambda}). \tag{55}$$

This can be seen from

$$\begin{aligned} \langle \mathbf{C_u}(\bar{\mathbf{u}}, \bar{\mathbf{g}})^* \boldsymbol{\lambda}, \mathbf{u} \rangle_{\mathbf{V}' \times \mathbf{V}} &= \nu\, a(\mathbf{u}, \boldsymbol{\lambda}) + b(\mathbf{u}; \bar{\mathbf{u}}^b, \boldsymbol{\lambda}) + b(\bar{\mathbf{u}}^b; \mathbf{u}, \boldsymbol{\lambda}) \\ &= \langle \mathbf{C_u}(\bar{\mathbf{u}}, \bar{\mathbf{g}}) \mathbf{u}, \boldsymbol{\lambda} \rangle_{\mathbf{V}' \times \mathbf{V}}. \end{aligned}$$

Under the assumptions of Theorem 3.3 the adjoint $\mathbf{C_u}(\bar{\mathbf{u}}, \bar{\mathbf{g}})^*$ is continuously invertible and the inverse satisfies

$$\|(\mathbf{C_u}(\bar{\mathbf{u}}, \bar{\mathbf{g}})^*)^{-1}\| \leq (\nu - \kappa^2 \|\nabla \bar{\mathbf{u}}^b\|_{\mathbf{L}^2})^{-1}.$$

The adjoint $\mathbf{C_g}(\bar{\mathbf{u}}, \bar{\mathbf{g}})^* \in \mathcal{L}(\mathbf{V}, \mathbf{H}^{-1}(\Gamma_c))$ of $\mathbf{C_g}(\bar{\mathbf{u}}, p, \bar{\mathbf{g}})$ is given by

$$\mathbf{C_g}(\bar{\mathbf{u}}, \bar{\mathbf{g}})^* \boldsymbol{\lambda} = \langle \mathbf{z}, \cdot \rangle_{\mathbf{H}_0^1(\Gamma_c)}, \tag{56}$$

where $\mathbf{z} \in \mathbf{H}_0^1(\Gamma_c)$, the Riesz representation of $\mathbf{C_g}(\bar{\mathbf{u}}, \bar{\mathbf{g}})^* \boldsymbol{\lambda}$, solves

$$\langle \mathbf{z}, \mathbf{s} \rangle_{\mathbf{H}_0^1(\Gamma_c)} = \nu\, a(\mathbf{SPs}, \boldsymbol{\lambda}) + b(\mathbf{SPs}; \bar{\mathbf{u}}^b, \boldsymbol{\lambda}) + b(\bar{\mathbf{u}}^b; \mathbf{SPs}, \boldsymbol{\lambda}) \tag{57}$$

for all $\mathbf{s} \in \mathbf{H}_0^1(\Gamma_c)$. This can be seen from

$$\begin{aligned} \langle \mathbf{C_g}(\bar{\mathbf{u}}, \bar{\mathbf{g}})^* \boldsymbol{\lambda}, \mathbf{g} \rangle_{\mathbf{H}_0^1(\Gamma_c)} &= \langle \mathbf{z}, \mathbf{g} \rangle_{\mathbf{H}_0^1(\Gamma_c)} \\ &= \nu\, a(\mathbf{SPg}, \boldsymbol{\lambda}) + b(\mathbf{SPg}; \bar{\mathbf{u}}^b, \boldsymbol{\lambda}) + b(\bar{\mathbf{u}}^b; \mathbf{SPg}, \boldsymbol{\lambda}) \\ &= \langle \mathbf{C_g}(\bar{\mathbf{u}}, \bar{\mathbf{g}}) \mathbf{g}, \boldsymbol{\lambda} \rangle_{\mathbf{V}' \times \mathbf{V}}. \end{aligned}$$

3.4 Optimality Conditions

Let $(\mathbf{u}_*, \mathbf{g}_*)$ be a (local) solution of (45), (46) and let $\bar{\mathbf{u}}_*^b = \mathbf{u}_* + \mathbf{S}(\mathbf{b} + P\mathbf{g}_*)$ satisfy

$$\nu - \kappa^2 \|\nabla \mathbf{u}_*^b\|_{\mathbf{L}^2} > 0, \tag{58}$$

where κ is defined in Theorem 3.3. Theorem 3.3 ensures the surjectivity of the linearized constraints at $(\mathbf{u}_*, \mathbf{g}_*)$. Thus, $(\mathbf{u}_*, \mathbf{g}_*)$ is a regular point and the following necessary optimality condition holds true:
There exist $\boldsymbol{\lambda}_* \in \mathbf{V}$ such that $(\mathbf{u}_*, \mathbf{g}_*, \boldsymbol{\lambda}_*)$ satisfies the state equations $\mathbf{C}(\mathbf{u}_*, \mathbf{g}_*) = 0$, the adjoint equations

$$\mathbf{C}_{\mathbf{u}}(\mathbf{u}_*, \mathbf{g}_*)\boldsymbol{\lambda}_* = -\nabla_{\mathbf{u}} J(\mathbf{u}_*, \mathbf{g}_*), \tag{59}$$

and the (reduced gradient) equation

$$\nabla_{\mathbf{g}} J(\mathbf{u}_*, \mathbf{g}_*) + \mathbf{z}_* = 0, \tag{60}$$

where $\mathbf{z}_*$ solves (57) with $\bar{\mathbf{u}}, \bar{\mathbf{u}}^b, \bar{\mathbf{g}}, \boldsymbol{\lambda}$ replaced by $\mathbf{u}_*, \mathbf{u}_*^b = \mathbf{u}_* + \mathbf{S}(\mathbf{b} + \mathbf{P}\mathbf{g}_*)$, $\mathbf{g}_*, \boldsymbol{\lambda}_*$.

Using (55) we see that solving the adjoint equation (59) requires solving the following problem: Find $\boldsymbol{\lambda}_* \in \mathbf{V}$ such that

$$\begin{aligned} & \nu\, a(\mathbf{v}, \boldsymbol{\lambda}_*) + b(\mathbf{v}; \mathbf{u}_*^b, \boldsymbol{\lambda}_*) + b(\mathbf{u}_*^b; \mathbf{v}, \boldsymbol{\lambda}_*) \\ = \;& -\langle \nabla_{\mathbf{u}} J(\mathbf{u}_*, \mathbf{g}_*), \mathbf{v} \rangle \qquad \forall \mathbf{v} \in \mathbf{V}. \end{aligned} \tag{61}$$

Here $\mathbf{u}_*^b = \mathbf{u}_* + \mathbf{S}(\mathbf{b} + \mathbf{P}\mathbf{g}_*)$. Using [12, IV §§1, 2] we see that (61) is equivalent to finding $\boldsymbol{\lambda}_* \in \mathbf{H}_0^1, \theta_* \in L_0^2$ with

$$\begin{aligned} \nu\, a(\mathbf{v}, \boldsymbol{\lambda}_*) + b(\mathbf{v}; \mathbf{u}_*^b, \boldsymbol{\lambda}_*) & & & \\ + b(\mathbf{u}_*^b; \mathbf{v}, \boldsymbol{\lambda}_*) + c(\mathbf{v}, \theta_*) &= -\langle \nabla_{\mathbf{u}} J(\mathbf{u}_*, \mathbf{g}_*), \mathbf{v} \rangle & \forall \mathbf{v} \in \mathbf{H}_0^1, & \quad (62) \\ c(\boldsymbol{\lambda}_*, q) &= 0 & \forall q \in L_0^2. & \quad (63) \end{aligned}$$

The system (62), (63) is the weak form of

$$\begin{aligned} -\nu \Delta \boldsymbol{\lambda}_* + (\nabla \mathbf{u}_*^b)^\top \boldsymbol{\lambda}_* - (\mathbf{u}_*^b \cdot \nabla)\boldsymbol{\lambda}_* + \nabla \theta_* &= -\nabla_{\mathbf{u}} \mathcal{J}(\mathbf{u}_*^b, \mathbf{g}_*) & \text{in } \Omega, \\ \text{div } \boldsymbol{\lambda}_* &= 0 & \text{in } \Omega, \\ \boldsymbol{\lambda}_* &= 0 & \text{on } \Gamma. \end{aligned}$$

Using the quadratic nature of the constraint function $\mathbf{C}$, one can easily show that it is twice Lipschitz–continuously Fréchet –differentiable. All higher derivatives are identical equal to zero.

If the objective function is also twice continuously differentiable, which is that case for (5), then the second order sufficient optimality conditions stated in § 2 for the abstract problem can be rewritten similar to the first order conditions using the specific form (44) of the constraint function. Since this is analogous to the presentation in [10, § 5], it is omitted here. We also remark that the second order conditions can often be verified for problems of tracking type. For nonlinear elliptic equations this is done in [20, § 4].

4 Application of the SQP Method

If the optimal control problem (45), (46) has a local solution $(\mathbf{u}_*, \mathbf{g}_*)$ which satsfies (58), then the SQP method can be applied for its computation. In a neighborhood of the local solution $(\mathbf{u}_*, \mathbf{g}_*)$ the SQP method is well–defined and under the conditions stated in §2 the iterates converge towards the solution fast. More precisely, application of Theorem 2.1 gives:

Theorem 4.1 *Let $\mathbf{x}_* = (\mathbf{u}_*, \mathbf{g}_*)$ be a local minimizer of (45), (46) satisfying the second order sufficient optimality conditions and (58). Assume there exists a neighborhood D of $\mathbf{x}_*$ such that $J : \mathcal{U} \times \mathcal{G} \to I\!R$ is twice Lipschitz continuously differentiable on D.*

There exist $\epsilon > 0$ and $\delta > 0$ such that for all $\mathbf{x}_0$ and all self-adjoint and positive $\mathbf{H}_0 \in \mathcal{L}(\mathcal{G})$ with $\|\mathbf{H}_0 - \mathbf{H}(\mathbf{x}_*, \boldsymbol{\lambda}_*)\| < \delta$ and $\|\mathbf{x}_0 - \mathbf{x}_*\| < \epsilon$ the SQP Algorithm 2.1 is well defined and the iterates $\mathbf{x}_k$ converge q–linearly to $\mathbf{x}_*$. If, in addition, $\mathbf{H}_0 - \mathbf{H}(\mathbf{x}_*, \boldsymbol{\lambda}_*)$ is a compact operator, then the convergence is 2–step q–superlinear.

If the penalty parameter α is not too small, $\mathbf{H}_0 = \alpha I$ is typically a good initial approximation.

The purpose of the remainder of this section is to combine the findings and presentations of the previous two sections and discuss the execution of the SQP method outlined in § 2. We assume that the operator $\mathbf{S}$ is implemented by solving the Stokes equations (38)–(40).

At the beginning of step k, we have given a new control $\mathbf{g}_k$ and a new 'state' $\mathbf{u}_k$. (Recall that $(\mathbf{u}_k, \mathbf{g}_k)$ does not satisfy the state equations and therefore the terminology 'state' is used somewhat loosely in this context.) Moreover, let a 'pressure' p_k be given. Initially p_0 can be chosen arbitrary, e.g., $p_0 = 0$. The discussion below will reveal how p_{k+1} is computed from $\mathbf{u}_k, \mathbf{g}_k$ and p_k.

For the evaluation of $\mathbf{C}(\mathbf{u}_k, \mathbf{g}_k)$ and its derivatives we have to compute $\mathbf{u}_k^b = \mathbf{u}_k + \mathbf{S}(\mathbf{b} + \mathbf{P}\mathbf{g}_k)$. In numerical implementations, the projection $\mathbf{P}\mathbf{g}_k$

can be computed easily from (36), since Γ_c contains relatively few mesh points and, therefore, (36) is a small dimensional problem. This problem is quadratic and strictly convex. It can be solved by solving the corresponding small dimensional linear Kuhn-Tucker system. The function $\mathbf{S}(\mathbf{b} + \mathbf{P}\mathbf{g}_k)$ is computed by solving the Stokes equations (38)–(40). Actually, we will show below that $\mathbf{u}_k^b$ can be updated from $\mathbf{u}_{k-1}^b$ using information already available.

For the computation of the Lagrange multiplier estimate $\boldsymbol{\lambda}_k$ in step 1 of Algorithm 2.1, we have to solve

$$\mathbf{C}_{(\mathbf{u},p)}(\mathbf{u}_k, \mathbf{g}_k)\boldsymbol{\lambda}_k = -\nabla_{\mathbf{u}} J(\mathbf{u}_k, \mathbf{g}_k). \tag{64}$$

We have seen in § 3.4, c.f. (59), (61)–(63), that this is equivalent to computing $\boldsymbol{\lambda}_k \in \mathbf{H}_0^1, \theta_k \in L_0^2$ with

$$\begin{aligned} \nu\, a(\mathbf{v}, \boldsymbol{\lambda}_k) + b(\mathbf{v}; \mathbf{u}_k^b, \boldsymbol{\lambda}_k) & & & \\ +b(\mathbf{u}_k^b; \mathbf{v}, \boldsymbol{\lambda}_k) + c(\mathbf{v}, \theta_k) &= & -\langle \nabla_{\mathbf{u}} J(\mathbf{u}_k, \mathbf{g}_k), \mathbf{v}\rangle & \forall \mathbf{v} \in \mathbf{H}_0^1, \\ c(\boldsymbol{\lambda}_k, q) &= & 0 & \forall q \in L_0^2. \end{aligned}$$

The latter system is the weak form of

$$\begin{aligned} -\nu\Delta\boldsymbol{\lambda}_k + (\nabla \mathbf{u}_k^b)^\top \boldsymbol{\lambda}_k - (\mathbf{u}_k^b \cdot \nabla)\boldsymbol{\lambda}_k + \nabla\theta_k &= -\nabla_{\mathbf{u}} \mathcal{J}(\mathbf{u}_k^b, \mathbf{g}_k) & \text{in } \Omega, \\ \text{div } \boldsymbol{\lambda}_k &= 0 & \text{in } \Omega, \\ \boldsymbol{\lambda}_k &= 0 & \text{on } \Gamma. \end{aligned} \tag{65}$$

Given the Lagrange multiplier estimate $\boldsymbol{\lambda}_k$, the reduced gradient in step 2 of Algorithm 2.1 is given by

$$\nabla_{\mathbf{g}} J(\mathbf{u}_k, \mathbf{g}_k) + \mathbf{z}_k \tag{66}$$

where $\mathbf{z}_k = \mathbf{C}_{\mathbf{g}}(\mathbf{u}_k, \mathbf{g}_k)^* \boldsymbol{\lambda}_k$ solves

$$\langle \mathbf{z}_k, \mathbf{s}\rangle_{\mathbf{H}_0^1(\Gamma_c)} = \nu\, a(\mathbf{S}\mathbf{P}\mathbf{s}, \boldsymbol{\lambda}_k) + b(\mathbf{S}\mathbf{P}\mathbf{s}; \mathbf{u}_k^b, \boldsymbol{\lambda}_k) + b(\mathbf{u}_k^b; \mathbf{S}\mathbf{P}\mathbf{s}, \boldsymbol{\lambda}_k). \tag{67}$$

See (56), (57). In numerical implementations, (67) corresponds to a small dimensional system. Its left hand side is the discretization of the Laplace operator on the boundary Γ_c. The evaluation of the right hand side, however, is potentially costly. For each basis function ϕ_i of the control space one has to compute $\mathbf{S}\mathbf{P}\phi_i$, which involves the solution of the Stokes equations, and then one has to form $\nu\, a(\mathbf{S}\mathbf{P}\phi_i, \boldsymbol{\lambda}_k) + b(\mathbf{S}\mathbf{P}\phi_i; \mathbf{u}_k^b, \boldsymbol{\lambda}_k) + b(\mathbf{u}_k^b; \mathbf{S}\mathbf{P}\phi_i, \boldsymbol{\lambda}_k)$. Note that

$\mathbf{SP}\phi_i$ can be reused throughout the SQP algorithm and, if possible, should be stored. The definition (47) of J also requires some attention. It holds that

$$\nabla_{\mathbf{g}} J(\mathbf{u}_k, \mathbf{g}_k) = \mathbf{P}^* \mathbf{S}^* \nabla_{\mathbf{u}} \mathcal{J}(\mathbf{u}_k^b, \mathbf{g}_k) + \mathbf{P}^* \nabla_{\mathbf{g}} \mathcal{J}(\mathbf{u}_k^b, \mathbf{g}_k).$$

Therefore, it is favorable to define

$$\tilde{\mathbf{z}}_k = \mathbf{P}^* \mathbf{S}^* \nabla_{\mathbf{u}} \mathcal{J}(\mathbf{u}_k^b, \mathbf{g}_k) + \mathbf{z}_k.$$

This vector can be computed by solving

$$\begin{aligned} \langle \tilde{\mathbf{z}}_k, \mathbf{s} \rangle_{\mathbf{H}_0^1(\Gamma_c)} &= \langle \mathbf{P}^* \mathbf{S}^* \nabla_{\mathbf{u}} \mathcal{J}(\mathbf{u}_k^b, \mathbf{g}_k), \mathbf{s} \rangle_{\mathbf{H}_0^1(\Gamma_c)} + \langle \mathbf{z}_k, \mathbf{s} \rangle_{\mathbf{H}_0^1(\Gamma_c)} \\ &= \langle \nabla_{\mathbf{u}} \mathcal{J}(\mathbf{u}_k^b, \mathbf{g}_k), \mathbf{SPs} \rangle_{\mathbf{H}^1} \\ &\quad + \nu\, a(\mathbf{SPs}, \boldsymbol{\lambda}_k) + b(\mathbf{SPs}; \mathbf{u}_k^b, \boldsymbol{\lambda}_k) + b(\mathbf{u}_k^b; \mathbf{SPs}, \boldsymbol{\lambda}_k) \quad (68) \end{aligned}$$

for all $\mathbf{s} \in \mathbf{H}_0^1(\Gamma_c)$. The numerical expense of solving (68) is essentially equal to the one of solving (67).

If $\mathbf{H}_k$ is a quasi–Newton approximation, step 4 can be implemented easily. In the context of infinite dimensional problems, quasi–Newton updates have to be computed with the appropriate norm of the control space, in our case $\mathbf{H}_0^1(\Gamma_c)$. Therefore, it might be advantageous to solve the system in step 4 with the conjugate gradient method. The reason for this is that the conjugate gradient method requires operator vector products of the form $\mathbf{H}_k\mathbf{h}$. If the updates that generate $\mathbf{H}_k$ are stored in vector format, then operator vector products $\mathbf{H}_k\mathbf{h}$ can be computed nicely using the function space scalar products, while forming the operator explicitly (like it is often done in finite dimensional optimization) often cannot be done easily. The computation of $\mathbf{H}_k\mathbf{h}$ in the infinite dimensional context can proceed similar to the implementation of limited memory quasi–Newton updates, see e.g., [25]. Only the Euclidean scalar product has to be replaced by the infinite dimensional one.

The system in step 5 is equivalent to computing $\mathbf{s_u} \in \mathbf{V}$ with

$$\begin{aligned} &\nu\, a(\mathbf{s_u}, \mathbf{v}) + b(\mathbf{s_u}; \mathbf{u}_k^b, \mathbf{v}) + b(\mathbf{u}_k^b; \mathbf{s_u}, \mathbf{v}) \\ &\quad = -\nu\, a(\mathbf{SPs_g}, \mathbf{v}) - b(\mathbf{SPs_g}; \mathbf{u}_k^b, \mathbf{v}) - b(\mathbf{u}_k^b; \mathbf{SPs_g}, \mathbf{v}) \\ &\qquad -\nu\, a(\mathbf{u}_k^b, \mathbf{v}) - b(\mathbf{u}_k^b; \mathbf{u}_k^b, \mathbf{v}) \end{aligned}$$

for all $\mathbf{v} \in \mathbf{V}$. This is equivalent to computing $\mathbf{s_u} \in \mathbf{H}_0^1$, $s_p \in L_0^2$ with

$$\nu\, a(\mathbf{s_u} + \mathbf{SPs_g}, \mathbf{v}) + b(\mathbf{s_u} + \mathbf{SPs_g}; \mathbf{u}_k^b, \mathbf{v})$$

$$
\begin{aligned}
+b(\mathbf{u}_k^b; \mathbf{s_u} + \mathbf{SPs_g}, \mathbf{v}) + c(s_p, \mathbf{v}) &= -\nu\, a(\mathbf{u}_k^b, \mathbf{v}) - b(\mathbf{u}_k^b; \mathbf{u}_k^b, \mathbf{v}) \\
&\quad -c(\mathbf{v}, p_k) \qquad \forall \mathbf{v} \in \mathbf{H}_0^1, \\
c(\mathbf{s_u} + \mathbf{SPs_g}, q) &= 0 \qquad\qquad \forall q \in L_0^2.
\end{aligned}
$$

One can instead solve for $\mathbf{w}_k = \mathbf{s_u} + \mathbf{SPs_g}$ at the same cost. If one does this, then the previous system is the weak form of

$$
\begin{aligned}
-\nu\Delta\mathbf{w}_k + (\mathbf{w}_k \cdot \nabla)\mathbf{u}_k^b & \\
+(\mathbf{u}_k^b \cdot \nabla)\mathbf{w}_k + \nabla(s_p)_k &= \nu\Delta\mathbf{u}_k^b - (\mathbf{u}_k^b \cdot \nabla)\mathbf{u}_k^b - \nabla p_k \quad \text{in } \Omega, \\
\text{div } \mathbf{w}_k &= 0 \qquad\qquad\qquad\qquad\quad \text{in } \Omega, \\
\mathbf{w}_k &= 0 \qquad\qquad\qquad\qquad\quad \text{on } \Gamma_u, \\
\mathbf{w}_k &= \mathbf{s_g} \qquad\qquad\qquad\qquad\quad \text{on } \Gamma_c.
\end{aligned}
\tag{69}
$$

Here p_k is the known pressure. Notice that $\mathbf{u}_{k+1}^b = \mathbf{u}_k^b + \mathbf{w}_k$. Therefore, solving for $\mathbf{w}_k$ saves the computation of $\mathbf{u}_{k+1}^b$ mentioned at the beginning of this section.

The new 'pressure' is computed to be $p_{k+1} = p_k + s_p$.

As noted earlier similar steps may have to be performed for the update of $\mathbf{H}_k$ in step 6 of the SQP Algorithm 2.1.

We point out that (69) with $\mathbf{s_g}$ replaced by $\mathbf{g}_k$ is equal to the linear system one has to solve if one wants to compute the solution of the Navier-Stokes equations using Newton's method. Moreover, an adjoint system equal to (65) also has to be solved to compute the gradient $\nabla\hat{\mathcal{J}}(\mathbf{g})$ of the reduced function (9). See also the discussion at the end of § 2. Hence solution of these two systems is also required in every gradient or quasi–Newton method applied to (9). The computation of the gradient for the reduced function $\hat{\mathcal{J}}$ in (9) may differ from $\nabla_{\mathbf{g}}\mathcal{J}(\mathbf{u}_k^b, \mathbf{g}_k) + \tilde{\mathbf{z}}_k$, depending on how the boundary conditions in (2)–(4) are handled. If a function $\hat{\mathbf{u}}$ satisfying the boundary conditions $\mathbf{g}$ is computed using $\mathbf{S}$, then $\nabla_{\mathbf{g}}\hat{\mathcal{J}}(\mathbf{g}_k) = \nabla_{\mathbf{g}}\mathcal{J}(\mathbf{u}_k^b, \mathbf{g}_k) + \tilde{\mathbf{z}}_k$.

This SQP method is different than the SQP methods used in [11], [17]. In [11], the Navier–Stokes equations are discretized using finite elements with Dirichlet boundary conditions implemented via boundary interpolation. The pressure is eliminated using a penalty method. The SQP method is then applied to the discrete system. In [17] the SQP method is applied to the discretization of (1)–(4). The infinite dimensional problem structure is incorporated by using the discrete $\mathbf{H}^1(\Gamma_c)$ scalar product in the control space. (This enters in the quasi–Newton updates and for the computation of the reduced gradient and has shown to be important.) Adjoint equation and

linearized state equation correspond to the systems (65) and (69), respectively. The equation for computing the reduced gradient differs from (66), (67) essentially in that the operator $\mathbf{S}$ is replaced by the right inverse of the trace operator (boundary interpolation in the discretized case). The test problem used in [17] is the problem of flow separation in a two–dimensional cavity from [10]. Due to the structure of the boundary control in that example, all controls automatically satisfy (11). The potentially expensive term $\mathbf{SPg}$, which is introduced in the present paper to ensure compatibility of constraints, is not needed in that example. A rigorous analysis of the approaches in [11], [17], which seem to be able to avoid the use of the operator $\mathbf{S}$, and their comparison with the one presented in this paper is current research.

5 Conclusions

We have investigated the rigorous application of an SQP method for the solution of optimal Dirichlet boundary control problems governed by the steady state Navier–Stokes equations in infinite dimensions. This required a careful handling of compatibility requirements between Dirichlet boundary conditions and incompressibility constraint. Since SQP methods treat states and controls as independent variables and do not insist on satisfying the constraints during the iterations, (trial) boundary controls and (trial) states have to be chosen in way that guarantees solvability of the subproblems in the SQP method. In this paper, compatibility is enforced by choosing appropriate function spaces. In the finite dimensional framework this corresponds to the elimination of variables form parts of the (in this case) linear constraints. Differentiability of the constraints and surjectivity of linearized constraints were verified and adjoints were computed. The SQP method was formulated and compared with other approaches.

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