3

Second-order elliptic boundary value problems in convex domains

3.1 *A priori* estimates and the curvature of the boundary

One of our basic tools throughout Chapter 2 has been the *a priori* inequality (2,3,3,7) proved in Theorem 2.3.3.6. In the present chapter we propose an alternative proof of this inequality in the particular case when $p=2$ and when the function $u \in H^2(\Omega)$ under consideration fulfils the homogeneous boundary condition

$$\gamma Bu = 0.$$

We shall mainly consider boundary value problems for the Laplace operator (in order to avoid some extra technical difficulties). However, we shall allow some nonlinear boundary conditions.

The main idea of the forthcoming alternative proof is to bypass the use of local coordinates. These were used in Section 2.3 to reduce the problem to the case when the boundary Γ of the domain Ω is flat. Here we shall perform straightforward integration by parts to prove the inequality

$$\|u\|_{2,2,\Omega} \leq C(\Omega)\, \|Au\|_{0,2,\Omega} \tag{3,1,1}$$

for all $u \in H^2(\Omega)$ such that $\gamma Bu = 0$ on Γ. The constant $C(\Omega)$ takes into precise account the curvature of Γ.

This inequality has various applications, all of which are along the following lines. We shall consider very rough domains Ω, such as general convex domains or domains whose boundary has turning points. We shall approximate these domains by sequences of domains with a C^2 boundary for which the constant in inequality (3,1,1) can easily be controlled. Taking the limit will prove smoothness results for the solution of a boundary value problem in Ω, although Ω is far from having a C^2 boundary.

3.1.1 An identity based on integration by parts

In this section we shall consider a bounded open subset Ω of $\mathbb{R}^n$ with a C^2 boundary Γ together with its second fundamental quadratic form, denoted by $\mathcal{B}$. Let us recall briefly an elementary definition of $\mathcal{B}$. For that purpose, let P be any point on Γ. It is possible to find $n-1$ curves of class C^2 in a neighbourhood of P, passing through P, and being orthogonal there. Let us denote by $\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}$ those curves, by $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ the unit tangent vectors to $\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}$ respectively, and by $s_1, \ldots, s_{n-1}$ the arc lengths along $\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}$ respectively. We can assume that $\{\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}\}$ has the direct orientation at P.

Then, at P, $\mathcal{B}_P$ is the bilinear form (we shall often drop the subscript P and write $\mathcal{B}$ instead of $\mathcal{B}_P$, whenever this does not lead to any misunderstanding)

$$\boldsymbol{\xi}, \boldsymbol{\eta} \mapsto -\sum_{j,k} \frac{\partial \boldsymbol{\nu}}{\partial s_j} \cdot \boldsymbol{\tau}_k \xi_j \eta_k$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are the tangent vectors to Γ at P, whose components are $\{\xi_1, \ldots, \xi_{n-1}\}$ and $\{\eta_1, \ldots, \eta_{n-1}\}$, respectively, in the basis $\{\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}\}$. In other words, we have

$$\mathcal{B}(\boldsymbol{\xi}, \boldsymbol{\eta}) = -\frac{\partial \boldsymbol{\nu}}{\partial \xi} \cdot \boldsymbol{\eta}$$

where $\partial/\partial \xi$ denotes differentiation in the direction of $\boldsymbol{\xi}$. (Actually, we could also extend the definition of $\mathcal{B}$ to sets Ω with a $C^{1,1}$ boundary with just a little more extra work. All the subsequent results hold for domains with a $C^{1,1}$ boundary instead of C^2.)

Another point of view is this. Let us consider a point P of Γ and (according to Definition 1.2.1.1) related new coordinates $\{y_1, \ldots, y_n\}$ with origin at P as follows: there exists a hypercube

$$V = \{(y_1, \ldots, y_n) \mid -a_j < y_j < a_j, 1 \leq j \leq n\}$$

and a function φ of class C^2 in $\bar{V}'$, where

$$V' = \{(y_1, \ldots, y_{n-1}) \mid -a_j < y_j < a_j, 1 \leq j \leq n-1\}$$

such that

$$|\varphi(y')| \leq a_n/2 \text{ for every } y' \in \bar{V}'$$
$$\Omega \cap V = \{y = (y', y_n) \in V \mid y_n < \varphi(y')\}$$
$$\Gamma \cap V = \{y = (y', y_n) \in V \mid y_n = \varphi(y')\}.$$

Let us assume further that $\nabla\varphi(0) = 0$. This means that the new coordinates have been chosen in such a way that the hyperplane $y_n = 0$ is tangent to Γ at P. Then, it is easily checked that the form $\mathscr{B}$ is

$$\mathscr{B}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \sum_{j,k=1}^{n-1} \frac{\partial^2 \varphi}{\partial y_k\, \partial y_j}(0)\xi_k \eta_j,$$

where $\{\xi_1, \ldots, \xi_{n-1}\}$ and $\{\eta_1, \ldots, \eta_{n-1}\}$ are the components of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ respectively in the directions of $\{y_1, \ldots, y_{n-1}\}$. In particular, when Ω is convex, the function $-\varphi$ is also convex, and consequently the form $\mathscr{B}$ is nonpositive.

Let us observe, finally, that in the general case of a domain Ω with a C^2 boundary the form $\mathscr{B}$ is uniformly bounded on Γ. In other words, there exists K such that

$$|\mathscr{B}_P(\boldsymbol{\xi}, \boldsymbol{\eta})| \leq K\, |\boldsymbol{\xi}|\, |\boldsymbol{\eta}|$$

for all $P \in \Gamma$, where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are tangent vectors to Γ at P. Indeed, $\boldsymbol{\nu}$ is a C^1 vector field on Γ. This is why domains with a C^2 boundary (or even $C^{1,1}$) are said to have a boundary with bounded curvature.

Now we introduce some more notation. Let $\mathbf{v}$ be any vector field on Γ; we shall denote by v_ν the component of $\mathbf{v}$ in the direction of $\boldsymbol{\nu}$, while we shall denote by $\mathbf{v}_T$ the projection of $\mathbf{v}$ on the tangent hyperplane to Γ. In other words

$$v_\nu = \mathbf{v} \cdot \boldsymbol{\nu} \qquad \text{and} \qquad \mathbf{v}_T = \mathbf{v} - v_\nu \boldsymbol{\nu}.$$

In the same way, we shall denote by $\boldsymbol{\nabla}_T$ the projection of the gradient operator on the tangent hyperplane:

$$\boldsymbol{\nabla}_T u = \boldsymbol{\nabla} u - \frac{\partial u}{\partial \nu} \boldsymbol{\nu}.$$

We can now state the following.

Theorem 3.1.1.1 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a C^2 boundary, and let $\mathbf{v} \in H^1(\Omega)^n$. Then, we have*

$$\int_\Omega |\operatorname{div} \mathbf{v}|^2\, dx - \sum_{i,j=1}^{n} \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i}\, dx$$
$$= -2\langle (\gamma \mathbf{v})_T ; \boldsymbol{\nabla}_T[\gamma \mathbf{v} \cdot \boldsymbol{\nu}] \rangle - \int_\Gamma \{\mathscr{B}((\gamma \mathbf{v})_T ; (\gamma \mathbf{v})_T)$$
$$+ (\operatorname{tr} \mathscr{B})[(\gamma \mathbf{v}) \cdot \boldsymbol{\nu}]^2\}\, d\sigma. \qquad (3,1,1,1)$$

Here $\operatorname{tr} \mathscr{B}$ is the trace of the bilinear form $\mathscr{B}$, i.e.,

$$\operatorname{tr} \mathscr{B} = -\sum_{j=1}^{n-1} \frac{\partial \boldsymbol{\nu}}{\partial s_j} \cdot \boldsymbol{\tau}_j$$

in the above notation.

Proof First, we apply repeatedly the Green formula of Theorem 1.5.3.1 which holds as soon as Γ is Lipschitz. We thus get, for $\mathbf{v} \in C^2(\bar{\Omega})^n$,

$$\begin{aligned}\int_\Omega |\operatorname{div} \mathbf{v}|^2 \, dx &= \sum_{i,j=1}^n \int_\Omega \frac{\partial v_i}{\partial x_i} \frac{\partial v_j}{\partial x_j} \, dx \\ &= -\sum_{i,j=1}^n \int_\Omega v_i \frac{\partial^2 v_j}{\partial x_i \, \partial x_j} \, dx + \sum_{i,j=1}^n \int_\Gamma v_i \frac{\partial v_j}{\partial x_j} \nu_i \, d\sigma \\ &= \sum_{i,j=1}^n \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} \, dx + \sum_{i,j=1}^n \int_\Gamma v_i \frac{\partial v_j}{\partial x_j} \nu_i \, d\sigma \\ &\qquad - \sum_{i,j=1}^n \int_\Gamma v_i \frac{\partial v_j}{\partial x_i} \nu_j \, d\sigma.\end{aligned}$$

In other words, we have

$$\begin{aligned}I(v) &= \int_\Omega |\operatorname{div} \mathbf{v}|^2 \, dx - \sum_{i,j=1}^n \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} \, dx \\ &= \int_\Gamma v_\nu \operatorname{div} \mathbf{v} \, d\sigma - \int_\Gamma \{(\mathbf{v} \cdot \nabla)\mathbf{v}\} \cdot \boldsymbol{\nu} \, d\sigma. \end{aligned} \tag{3,1,1,2}$$

We shall now transform the integrand on the boundary. This can be done locally. Let us consider any point P on Γ and choose an open neighbourhood W of P in Γ small enough to allow the existence of $(n-1)$ families of C^2 curves on W with these properties: a curve of each family passes through every piont of W and the unit tangent vectors to these curves form an orthonormal system (which we assume to have the direct orientation) at every point of W. The lengths $s_1, \ldots, s_{n-1}$ along each family of curves, respectively, are a possible system of coordinates in W. We denote by $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ the unit tangent vectors to each family of curves, respectively.

With this notation, we have

$$\mathbf{v} = \mathbf{v}_T + v_\nu \boldsymbol{\nu}, \qquad \mathbf{v}_T = \sum_{k=1}^{n-1} v_k \boldsymbol{\tau}_k,$$

where $v_j = \mathbf{v} \cdot \boldsymbol{\tau}_j$. We also have for any $\varphi \in C^1(\bar{\Omega})$:

$$\nabla \varphi = \nabla_T \varphi + \frac{\partial \varphi}{\partial \nu} \boldsymbol{\nu}, \qquad \nabla_T \varphi = \sum_{j=1}^{n-1} \frac{\partial \varphi}{\partial s_j} \boldsymbol{\tau}_j$$

and consequently

$$\operatorname{div} \mathbf{v} = \sum_{j=1}^{n-1} \frac{\partial \mathbf{v}}{\partial s_j} \cdot \boldsymbol{\tau}_j + \frac{\partial \mathbf{v}}{\partial \nu} \cdot \boldsymbol{\nu}.$$

Consider now the first integrand on Γ in the right-hand side of

(3,1,1,2). We have

$$\begin{aligned}\operatorname{div} \mathbf{v} &= \sum_{j=1}^{n-1} \frac{\partial \mathbf{v}}{\partial s_j} \cdot \boldsymbol{\tau}_j + \frac{\partial \mathbf{v}}{\partial \nu} \cdot \boldsymbol{\nu} \\ &= \sum_{j,k=1}^{n-1} \left[\frac{\partial v_k}{\partial s_j} \boldsymbol{\tau}_k + v_k \frac{\partial \boldsymbol{\tau}_k}{\partial s_j}\right] \cdot \boldsymbol{\tau}_j + \sum_{j=1}^{n-1} \left[\frac{\partial v_\nu}{\partial s_j} \boldsymbol{\nu} + v_\nu \frac{\partial \boldsymbol{\nu}}{\partial s_j}\right] \cdot \boldsymbol{\tau}_j \\ &\quad + \sum_{k=1}^{n-1} \left[\frac{\partial v_k}{\partial \nu} \boldsymbol{\tau}_k + v_k \frac{\partial \boldsymbol{\tau}_k}{\partial \nu}\right] \cdot \boldsymbol{\nu} + \left[\frac{\partial v_\nu}{\partial \nu} \boldsymbol{\nu} + v_\nu \frac{\partial \boldsymbol{\nu}}{\partial \nu}\right] \cdot \boldsymbol{\nu}\end{aligned}$$

and thus

$$\begin{aligned}v_\nu \operatorname{div} \mathbf{v} = v_\nu \Bigg\{ &\sum_{j=1}^{n-1} \frac{\partial v_j}{\partial s_j} + \sum_{j,k=1}^{n-1} v_k \frac{\partial \boldsymbol{\tau}_k}{\partial s_j} \cdot \boldsymbol{\tau}_j + v_\nu \sum_{j=1}^{n-1} \frac{\partial \boldsymbol{\nu}}{\partial s_j} \cdot \boldsymbol{\tau}_j \\ &+ \sum_{k=1}^{n-1} v_k \frac{\partial \boldsymbol{\tau}_k}{\partial \nu} \cdot \boldsymbol{\nu} + \frac{\partial v_\nu}{\partial \nu} \Bigg\}\end{aligned} \tag{3,1,1,3}$$

since $(\partial \boldsymbol{\nu}/\partial \nu) \cdot \boldsymbol{\nu} = 0$.

We then consider the second integrand on Γ in the right-hand side of (3,1,1,2). We have

$$\begin{aligned}(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} &= \sum_{j=1}^{n-1} v_j \frac{\partial \mathbf{v}}{\partial s_j} + v_\nu \frac{\partial \mathbf{v}}{\partial \nu} \\ &= \sum_{j,k=1}^{n-1} v_j \left(\frac{\partial v_k}{\partial s_j} \boldsymbol{\tau}_k + v_k \frac{\partial \boldsymbol{\tau}_k}{\partial s_j}\right) + \sum_{j=1}^{n-1} v_j \left(\frac{\partial v_\nu}{\partial s_j} \boldsymbol{\nu} + v_\nu \frac{\partial \boldsymbol{\nu}}{\partial s_j}\right) \\ &\quad + \sum_{k=1}^{n-1} v_\nu \left(\frac{\partial v_k}{\partial \nu} \boldsymbol{\tau}_k + v_k \frac{\partial \boldsymbol{\tau}_k}{\partial \nu}\right) + v_\nu \left(\frac{\partial v_\nu}{\partial \nu} \boldsymbol{\nu} + v_\nu \frac{\partial \boldsymbol{\nu}}{\partial \nu}\right),\end{aligned}$$

and thus

$$\{(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}\} \cdot \boldsymbol{\nu} = \sum_{j,k=1}^{n-1} v_j v_k \frac{\partial \boldsymbol{\tau}_k}{\partial s_j} \cdot \boldsymbol{\nu} + \sum_{j=1}^{n-1} v_j \frac{\partial v_\nu}{\partial s_j} + \sum_{k=1}^{n-1} v_\nu v_k \frac{\partial \boldsymbol{\tau}_k}{\partial \nu} \cdot \boldsymbol{\nu} + v_\nu \frac{\partial v_\nu}{\partial \nu} \tag{3,1,1,4}$$

because $(\partial \boldsymbol{\nu}/\partial s_j) \cdot \boldsymbol{\nu} = 0$.

Subtracting identities (3,1,1,3) and (3,1,1,4), we finally obtain

$$\begin{aligned}v_\nu \operatorname{div} \mathbf{v} - \{(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}\} \cdot \boldsymbol{\nu} = v_\nu &\sum_{j=1}^{n-1} \frac{\partial v_j}{\partial s_j} + v_\nu \sum_{j,k=1}^{n-1} v_k \frac{\partial \boldsymbol{\tau}_k}{\partial s_j} \cdot \boldsymbol{\tau}_j \\ &+ v_\nu^2 \sum_{j=1}^{n-1} \frac{\partial \boldsymbol{\nu}}{\partial s_j} \cdot \boldsymbol{\tau}_j - \sum_{j,k=1}^{n-1} v_j v_k \frac{\partial \boldsymbol{\tau}_k}{\partial s_j} \cdot \boldsymbol{\nu} - \sum_{j=1}^{n-1} v_j \frac{\partial v_\nu}{\partial s_j}.\end{aligned} \tag{3,1,1,5}$$

On the other hand, using $s_1, \ldots, s_{n-1}$ as coordinates in W, we know that $\mathscr{B}$ is defined by

$$\mathscr{B}(\boldsymbol{\xi}, \boldsymbol{\eta}) = -\sum_{j,k=1}^{n-1} \frac{\partial \boldsymbol{\nu}}{\partial s_j} \cdot \boldsymbol{\tau}_k \xi_j \eta_k = +\sum_{j,k=1}^{n-1} \boldsymbol{\nu} \cdot \frac{\partial \boldsymbol{\tau}_k}{\partial s_j} \xi_j \eta_k$$

and consequently we have

$$\operatorname{tr} \mathscr{B} = -\sum_{j=1}^{n-1} \boldsymbol{\tau}_j \cdot \frac{\partial \boldsymbol{\nu}}{\partial s_j}.$$

It follows that (3,1,1,5) may be rewritten as:

$$v_\nu \operatorname{div} \mathbf{v} - \{(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}\} \cdot \boldsymbol{\nu} = v_\nu \sum_{j=1}^{n-1} \frac{\partial v_j}{\partial s_j} + v_\nu \sum_{j,k=1}^{n-1} v_k \frac{\partial \boldsymbol{\tau}_k}{\partial s_j} \cdot \boldsymbol{\tau}_j$$
$$-(\operatorname{tr} \mathscr{B}) v_\nu^2 - \mathscr{B}(\mathbf{v}_T; \mathbf{v}_T) - \sum_{j=1}^{n-1} v_j \frac{\partial v_\nu}{\partial s_j}. \qquad (3,1,1,6)$$

Let us finally calculate

$$\operatorname{div}_T (v_\nu \mathbf{v}_T) = \sum_{j=1}^{n-1} \frac{\partial (v_\nu \mathbf{v}_T)}{\partial s_j} \cdot \boldsymbol{\tau}_j = \sum_{j,k=1}^{n-1} \frac{\partial (v_\nu v_k \boldsymbol{\tau}_k)}{\partial s_j} \cdot \boldsymbol{\tau}_j.$$

Thus, we have

$$\operatorname{div}_T (v_\nu \mathbf{v}_T) = \sum_{j,k=1}^{n-1} \left\{ \frac{\partial v_\nu}{\partial s_j} v_k \boldsymbol{\tau}_k + v_\nu \frac{\partial v_k}{\partial s_j} \boldsymbol{\tau}_k + v_\nu v_k \frac{\partial \boldsymbol{\tau}_k}{\partial s_j} \right\} \cdot \boldsymbol{\tau}_j$$
$$= \sum_{j=1}^{n-1} \frac{\partial v_\nu}{\partial s_j} v_j + v_\nu \sum_{j=1}^{n-1} \frac{\partial v_j}{\partial s_j} + v_\nu \sum_{j,k=1}^{n-1} v_k \frac{\partial \boldsymbol{\tau}_k}{\partial s_j} \cdot \boldsymbol{\tau}_j. \qquad (3,1,1,7)$$

Then, from (3,1,1,6) and (3,1,1,7), we deduce

$$v_\nu \operatorname{div} \mathbf{v} - \{(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}\} \cdot \boldsymbol{\nu}$$
$$= \operatorname{div}_T (v_\nu \mathbf{v}_T) - (\operatorname{tr} \mathscr{B}) v_\nu^2 - \mathscr{B}(\mathbf{v}_T; \mathbf{v}_T) - 2 \sum_{j=1}^{n-1} v_j \frac{\partial v_\nu}{\partial s_j}$$
$$= \operatorname{div}_T (v_\nu \mathbf{v}_T) - (\operatorname{tr} \mathscr{B}) v_\nu^2 - \mathscr{B}(\mathbf{v}_T; \mathbf{v}_T) - 2\mathbf{v}_T \cdot \boldsymbol{\nabla}_T v_\nu. \qquad (3,1,1,8)$$

This expression of the integrand on Γ in (3,1,1,2) no longer involves the particular coordinates in W. Varying W, it is consequently true everywhere on Γ. Thus, we have

$$\int_\Omega |\operatorname{div} \mathbf{v}|^2 \, dx - \sum_{i,j=1}^{n} \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} \, dx$$
$$= -2 \int_\Omega \mathbf{v}_T \cdot \boldsymbol{\nabla}_T v_\nu \, d\sigma - \int_\Gamma \{(\operatorname{tr} \mathscr{B}) v_\nu^2 + \mathscr{B}(\mathbf{v}_T; \mathbf{v}_T)\} \, d\sigma. \qquad (3,1,1,9)$$

Indeed, the integral of $\operatorname{div}_T (v_\nu \mathbf{v}_T)$ is zero since the vector field $v_\nu \mathbf{v}_T$ is everywhere tangent to Γ.

We have proved (3,1,1,9) assuming that $\mathbf{v} \in C^2(\bar{\Omega})^n$. However, since the boundary of Ω is of class C^2, the space $C^2(\bar{\Omega})$ is dense in $H^1(\Omega)$. From (3,1,1,9) we deduce (3,1,1,1) by approximating $\mathbf{v} \in H^1(\Omega)^n$ by a sequence $\mathbf{v}_m$, $m = 1, 2, \ldots$ of elements of $C^2(\bar{\Omega})^n$ for which (3,1,1,9) holds. ∎

Theorem 3.1.1.1 is thus completely proved. Most of the previous proof can be carried out under weaker assumptions on Ω. We shall say that a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary Γ has a piecewise C^2 boundary if $\Gamma = \Gamma_0 \cup \Gamma_1$, where

(a) Γ_0 has zero measure (for the surface measure $d\sigma$).
(b) Γ_1 is open in Γ, and each point $x \in \Gamma_1$ has the property of Definition 1.2.1.1 with a function φ of class C^2.

Theorem 3.1.1.2 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary Γ. Assume, in addition, that Γ is piecewise C^2. Then for all $\mathbf{v} \in H^2(\Omega)^n$ we have*

$$\int_\Omega |\operatorname{div} \mathbf{v}|^2 \, dx - \sum_{i,j=1}^n \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} \, dx = \int_{\Gamma_1} \{\operatorname{div}_T (v_\nu \mathbf{v}_T) - 2\mathbf{v}_T \cdot \nabla_T v_\nu\} \, d\sigma$$
$$- \int_{\Gamma_1} \{(\operatorname{tr} \mathcal{B}) v_\nu^2 + \mathcal{B}(\mathbf{v}_T; \mathbf{v}_T)\} \, d\sigma. \tag{3,1,1,10}$$

Proof This is quite similar to the proof of Theorem 3.1.1.1. Indeed, (3,1,1,2) holds since Γ is Lipschitz. Then identity (3,1,1,8) holds at any point of Γ_1. Integrating (3,1,1,8), we obtain (3,1,1,10) since Γ_0 has measure zero. Finally, we extend (3,1,1,10) from $\mathbf{v} \in C^2(\bar{\Omega})^n$ to $\mathbf{v} \in H^2(\Omega)^n$ only, since it is now impossible to give a meaning to the bracket of $\mathbf{v}_T$ and $\nabla_T v_\nu$ when $\mathbf{v} \in H^1(\Omega)^n$ and Γ is only Lipschitz globally. ∎

3.1.2 *A priori* inequalities for the Laplace operator revisited

We now take advantage of the results of Section 3.1.1 to prove inequality (3,1,1). Such an inequality has been proved by Caccioppoli (1950–51) and Ladyzhenskaia and Ural'ceva (1968) in the case when Ω has a $C^{1,1}$ boundary. These latter authors call it 'the second fundamental *a priori* estimate'. All of them make use of local coordinates in order to flatten the boundary. The proof given below follows Grisvard and Iooss (1975); it allows better control of the constant $C(\Omega)$. It also allows one to consider some nonlinear boundary conditions. A slightly different point of view is developed in Lewis (1978) for two-dimensional domains.

For the sake of clarity, we shall first consider the particular case when the operator A is the Laplace operator Δ, or the modified Laplace operator $\Delta-\lambda$ with $\lambda>0$. The first inequality concerns a Dirichlet boundary condition.

Theorem 3.1.2.1 *Let Ω be a convex, bounded open subset of $\mathbb{R}^n$ with a C^2 boundary Γ. Then there exists a constant $C(\Omega)$, which depends only on the diameter of Ω, such that*

$$\|u\|_{2,2,\Omega}\leqslant C(\Omega)\,\|\Delta u\|_{0,2,\Omega} \tag{3,1,2,1}$$

for all $u\in H^2(\Omega)\cap \mathring{H}^1(\Omega)$.

Proof We first apply identity (3,1,1,1) to $\mathbf{v}=\nabla u$, observing that, since $\gamma u=0$ on Γ, we also have $(\gamma\mathbf{v})_T=\gamma\nabla_T u=0$ on Γ. Thus, we have

$$\int_\Omega |\Delta u|^2\,dx-\sum_{i,j=1}^n\int_\Omega\left|\frac{\partial^2 u}{\partial x_i\,\partial x_j}\right|^2 dx=-\int_\Gamma (\operatorname{tr}\mathcal{B})(\gamma\mathbf{v}\cdot\boldsymbol{\nu})^2\,d\sigma.$$

Due to the convexity of Ω, we have $\operatorname{tr}\mathcal{B}\leqslant 0$ and consequently

$$\sum_{i,j=1}^n\int_\Omega\left|\frac{\partial^2 u}{\partial x_i\,\partial x_j}\right|^2 dx\leqslant\int_\Omega |\Delta u|^2\,dx. \tag{3,1,2,2}$$

So far, we have estimated the second derivatives of u. The estimate for the first derivatives is well known to be obtainable by the straightforward integration by parts which follows. We have

$$\sum_{i=1}^n\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^2 dx=-\int_\Omega \Delta u\cdot u\,dx\leqslant\|\Delta u\|\,\|u\|$$

where the norm is the norm of $L_2(\Omega)$. On the other hand, the Poincaré inequality implies that

$$\|u\|^2\leqslant K(\Omega)^2\sum_{i=1}^n\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^2 dx$$

where $K(\Omega)$ depends only on the diameter of Ω (see Theorem 1.4.3.4). It follows that

$$\sum_{i=1}^n\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^2 dx\leqslant K(\Omega)^2\,\|\Delta u\|^2 \tag{3,1,2,3}$$

and that

$$\|u\|\leqslant K(\Omega)^2\,\|\Delta u\|. \tag{3,1,2,4}$$

Adding up inequalities (3,1,2,2) to (3,1,2,4), we obtain inequality (3,1,2,1) with $C(\Omega)^2\leqslant 1+K(\Omega)^2+K(\Omega)^4$.

Remark 3.1.2.2 In the case when we assume Ω to be only a bounded open subset of $\mathbb{R}^n$ with a C^2 boundary without the assumption of convexity, we again obtain inequality (3,1,2,1). This is achieved with the aid of identity (3,1,1,1), inequality (1,5,1,2) and by using an upper bound for $\operatorname{tr}\mathcal{B}$. Consequently, the constant $C(\Omega)$ depends not only on the diameter of Ω but also on an upper bound for $\operatorname{tr}\mathcal{B}$. In other words, the constant $C(\Omega)$ depends on the curvature of Γ. This is nothing but an alternative proof of the corresponding inequality in Section 2.3.

Let us now consider a Neumann boundary condition and even some nonlinear boundary conditions closely related to the 'third boundary problem'. Here we consider a real-valued, nondecreasing function β defined on the real line. In addition we assume $\beta(0)=0$, and we assume β to be uniformly Lipschitz continuous. We now deal with the following boundary problem for a function $u \in H^2(\Omega)$:

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } \Omega, \\ -\gamma \dfrac{\partial u}{\partial \nu} = \beta(\gamma u) & \text{on } \Gamma. \end{cases} \tag{3,1,2,5}$$

The corresponding estimate is the following:

Theorem 3.1.2.3 *Let Ω be a convex, bounded open subset of $\mathbb{R}^n$ with a C^2 boundary Γ, and let β be a uniformly Lipschitz, nondecreasing function such that $\beta(0)=0$. Then we have*

$$\|u\|_{2,2,\Omega} \le \left(\frac{1}{\lambda^2} + \frac{1}{\lambda} + 4\right)^{1/2} \|-\Delta u + \lambda u\|_{0,2,\Omega} \tag{3,1,2,6}$$

for all $u \in H^2(\Omega)$ such that $-\gamma\, \partial u/\partial \nu = \beta(\gamma u)$ on Γ and all $\lambda > 0$.

The particular case when the function β is identically zero is just a Neumann problem. Obviously the interest of inequality (3,1,2,6) is that the constant depends neither on Ω nor on β. This will allow us to extend widely the possible Ωs and the possible βs, in the following sections.

Proof We again apply identity (3,1,1,1) to $\mathbf{v} = \nabla u$. The boundary condition now means that $-(\gamma \mathbf{v}) \cdot \boldsymbol{\nu} = \beta(\gamma u)$ on Γ. Thus we get

$$\begin{aligned} \int_\Omega |\Delta u|^2\, dx - \sum_{i,j=1}^{n} \int_\Omega \left| \frac{\partial^2 u}{\partial x_i\, \partial x_j} \right|^2 dx \\ = +2\langle \nabla_T(\gamma u); \nabla_T \beta(\gamma u)\rangle \\ - \int_\Gamma \{\mathcal{B}((\gamma \mathbf{v})_T; (\gamma \mathbf{v})_T) + (\operatorname{tr}\mathcal{B})[(\gamma \mathbf{v}) \cdot \mathbf{v}]^2\}\, d\sigma. \end{aligned}$$

Due to the convexity of Ω, $\mathscr{B}$ is nonpositive. On the other hand, $\gamma u \in H^{3/2}(\Gamma)$, and since β is uniformly Lipschitz, we also have $\beta(\gamma u) \in H^1(\Gamma)$. This allows one to rewrite the bracket as an integral. Consequently, we have

$$\int_\Omega |\Delta u|^2 \, dx - \sum_{i,j=1}^{n} \int_\Omega \left|\frac{\partial^2 u}{\partial x_i \, \partial x_j}\right|^2 dx \geq +2\int_\Gamma \nabla_T(\gamma u) \cdot \nabla_T \beta(\gamma u) \, d\sigma.$$

The integrand on Γ is

$$2\nabla_T(\gamma u) \nabla_T \beta(\gamma u) = 2\beta'(\gamma u) \, |\nabla_T(\gamma u)|^2;$$

this is a nonnegative function since β is nondecreasing. We conclude that

$$\sum_{i,j=1}^{n} \int_\Omega \left|\frac{\partial^2 u}{\partial x_i \, \partial x_j}\right|^2 dx \leq \int_\Omega |\Delta u|^2 \, dx. \tag{3,1,2,7}$$

The estimate of the remaining terms in the $H^2(\Omega)$ norm of u is obtained, as usual, by integrating $(-\Delta u + \lambda u)u$. Indeed, we have

$$\int_\Omega (-\Delta u + \lambda u) u \, dx = \sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^2 dx + \lambda \int_\Omega |u|^2 \, dx - \int_\Gamma \gamma \frac{\partial u}{\partial \nu} \gamma u \, d\sigma.$$

Consequently, we have

$$\lambda \int_\Omega |u|^2 \, dx + \sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^2 dx \leq \|-\Delta u + \lambda u\| \, \|u\| - \int_\Omega \beta(\gamma u) \gamma u \, d\sigma.$$

Since we assume β to be nondecreasing and $\beta(0) = 0$, it follows that

$$\beta(\gamma u)\gamma u \geq 0 \qquad \text{a.e.}$$

on Γ. Then, we have

$$\lambda \int_\Omega |u|^2 \, dx + \sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^2 dx \leq \|-\Delta u + \lambda u\| \, \|u\|$$

and consequently

$$\|u\| \leq \frac{1}{\lambda} \|-\Delta u + \lambda u\| \tag{3,1,2,8}$$

and

$$\sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^2 dx \leq \frac{1}{\lambda} \|-\Delta u + \lambda u\|^2. \tag{3,1,2,9}$$

The conclusion follows from inequalities (3,1,2,7) to (3,1,2,9). ■

Remark 3.1.2.4 Again we can drop the convexity assumption on Ω and let Ω be any bounded open subset of $\mathbb{R}^n$ with a C^2 boundary. Then, we deduce from (1,5,1,2) and (3,1,1,1) the inequality

$$\|u\|_{2,2,\Omega} \leqslant C(\lambda; \Omega) \, \|-\Delta u + \lambda u\|_{0,2,\Omega} \tag{3,1,2,10}$$

for every $u \in H^2(\Omega)$ such that $-\gamma \, \partial u/\partial \nu = \beta(\gamma u)$ and every $\lambda > 0$. Here the constant $C(\lambda, \Omega)$ depends only on λ and on the curvature of Ω (or more precisely, on an upper bound for $\mathcal{B}$). It is important to observe that $C(\lambda, \Omega)$ does not depend on β.

Remark 3.1.2.5 *A priori* bounds in H^2 for solutions of the Laplace operator under oblique boundary conditions are also proved in Subsection 3.2.4 in the particular case $n = 2$.

3.1.3 *A priori* inequalities for more general operators

The purpose of this subsection is to extend the results of the previous subsection to the more general operators A that we introduced in Chapter 2. Here we shall no longer consider nonlinear boundary conditions. This is to avoid some very cumbersome calculations which can be found in Grisvard and Iooss (1975).

Accordingly, we consider an operator A defined by

$$Au = \sum_{i,j=1}^{n} D_i(a_{i,j} D_j u)$$

with $a_{i,j} = a_{j,i} \in C^{0,1}(\bar{\Omega})$. We assume again that $-A$ is strongly elliptic; i.e. there exists $\alpha > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leqslant -\alpha \, |\xi|^2 \tag{3,1,3,1}$$

for all $x \in \bar{\Omega}$ and $\xi \in \mathbb{R}^n$.

We only consider here Dirichlet and Neumann boundary conditions.

Theorem 3.1.3.1 *Let Ω be a convex, bounded open subset of $\mathbb{R}^n$ with a C^2 boundary. Then there exists a constant $C(\Omega; A)$, which depends only on the diameter of Ω and on the Lipschitz norms of the coefficients $a_{i,j}$, $1 \leqslant i, j \leqslant n$, such that*

$$\|u\|_{2,2,\Omega} \leqslant C(\Omega; A) \, \|Au\|_{0,2,\Omega} \tag{3,1,3,2}$$

for all $u \in H^2(\Omega) \cap \mathring{H}^1(\Omega)$.

Proof We could again use identity (3,1,1,1) with

$$\mathbf{v} = \mathcal{A} \, \nabla u$$

where $\mathcal{A}$ is the matrix of the $a_{i,j}$. However, the less natural method of proof that we shall follow here is simpler. Namely, we shall deduce inequality (3,1,3,2) directly from inequality (3,1,2,1) through the same perturbation procedure that we already used in Section 2.3.3. The main step is the following:

Lemma 3.1.3.2 *Each point $y \in \bar{\Omega}$ has a neighbourhood V_y such that*

$$\|u\|_{2,2,\Omega} \leq C_y\{\|Au\|_{0,2,\Omega} + \|u\|_{1,2,\Omega}\|\} \tag{3,1,3,3}$$

for all $u \in H^2(\Omega) \cap \mathring{H}^1(\Omega)$ whose support is contained in V_y. Furthermore, the constant C_y depends only on the diameter of Ω and on the Lipschitz norm of the $a_{i,j}$. The neighbourhood V_y depends only on the Lipschitz norm of the $a_{i,j}$.

Proof As in Lemma 2.3.3.3 we freeze the coefficients of A at y. Thus, we set $l_{i,j} = a_{i,j}(y)$. This defines a strictly negative symmetric matrix L, and consequently there exists a nonsingular matrix R such that $-RLR$ is the identity matrix (R is the inverse of the square root of $-L$.) If we set

$$v(x) = u(R^{-1}x), \qquad g(x) = f(R^{-1}x),$$

then the equation

$$\sum_{i,j=1}^{n} l_{i,j}D_iD_ju = f$$

is equivalent to

$$-\Delta v = g.$$

We can apply inequality (3,1,2,1) to v.

Precisely, our assumptions on u imply that

$$v \in H^2(R\Omega) \cap \mathring{H}^1(R\Omega).$$

On the other hand, $R\Omega$ is convex and has a $C^{1,1}$ boundary. Consequently, we have

$$\|v\|_{2,2,R\Omega} \leq C(R\Omega)\,\|g\|_{0,2,R\Omega}.$$

Going back to the original variables, we also have

$$\|u\|_{2,2,\Omega} \leq K(R,\Omega)\left\|\sum_{i,j=1}^{n} l_{i,j}D_iD_ju\right\|_{0,2,\Omega}, \tag{3,1,3,4}$$

where K is a continuous function of the matrix R and of the diameter of Ω.

We compare the right-hand side in (3,1,3,4) with the norm of Au in

$L_2(\Omega)$. Actually, we have

$$\sum_{i,j=1}^{n} l_{i,j} D_i D_j u - Au = \sum_{i,j=1}^{n} D_i (l_{i,j} - a_{ij}) D_j u$$
$$= \sum_{i,j=1}^{n} (l_{i,j} - a_{ij}) D_i D_j u - \sum_{i,j=1}^{n} (D_i a_{ij})(D_j u).$$

It follows that

$$\left\| \sum_{i,j=1}^{n} l_{i,j} D_i D_j u - Au \right\|$$
$$\leqslant n^2 \left\{ \max_{x \in V_y} |a_{ij}(y) - a_{ij}(x)| \, \|u\|_{2,2,\Omega} + M \, \|u\|_{1,2,\Omega} \right\}, \tag{3,1,3,5}$$

where M is a common bound for the Lipschitz norms of all the a_{ij}.

From (3,1,3,4) and (3,1,3,5), it follows that $\|u\|_{2,2,\Omega} \leqslant K(R, \Omega)$

$$\left(\|Au\|_{0,2,\Omega} + Mn^2 \left\{ \max_{x \in V_y} |x - y| \, \|u\|_{2,2,\Omega} + \|u\|_{1,2,\Omega} \right\} \right).$$

The inequality (3,1,3,3) follows by choosing the neighbourhood V_y of y small enough to ensure that

$$|x - y| \leqslant \frac{1}{2K(R, \Omega) M n^2}$$

for all $x \in V_y$.

The proof of Lemma 3.1.3.2 is complete. The claim in Theorem 3.1.3.1 follows easily with the aid of a partition of the unity on $\bar{\Omega}$. ■

We turn now to the Neumann problem.

Theorem 3.1.3.3 *Let Ω be a convex, bounded open subset of $\mathbb{R}^n$ with a C^2 boundary. Then there exists a constant $C(\lambda, A)$, which depends only on λ, α† and the $C^{0,1}$ norm of the a_{ij}, such that*

$$\|u\|_{2,2,\Omega} \leqslant C(\lambda, A) \, \|Au + \lambda u\|_{0,2,\Omega} \tag{3,1,3,6}$$

for all $u \in H^2(\Omega)$ such that $-\gamma \, \partial u / \partial \nu_A = 0$ on Γ and all $\lambda > 0$.

Proof We first apply identity (3,1,1,1) to

$$\mathbf{v} = \mathcal{A} \, \nabla u$$

where $\mathcal{A}$ is the matrix of the a_{ij}. We observe that

$$Au = \operatorname{div} \mathbf{v} \qquad \text{in } \Omega$$

† We recall that α is the ellipticity constant which occurs in (3,1,3,1).

and that

$$\gamma\frac{\partial u}{\partial \nu_A}=\sum_{i,j=1}^{n}\nu^i a_{ij}\gamma\frac{\partial u}{\partial x_j}=(\gamma\mathbf{v})\cdot\boldsymbol{\nu}\qquad\text{on }\Gamma.$$

Accordingly, we have

$$\int_\Omega |Au|^2\,dx-\sum_{i,j=1}^{n}\int_\Omega \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}\,dx=-\int_\Gamma \mathcal{B}((\gamma\mathbf{v})_T;(\gamma\mathbf{v})_T)\,d\sigma\geqslant 0 \tag{3,1,3,7}$$

since Ω is assumed to be convex.

We then use the following lemma, whose proof is postponed to the completion of the proof of Theorem 3.1.3.3 (cf. also Lemma 7.1, p. 152 in Ladyzhenskaia and Ural'ceva (1968)).

Lemma 3.1.3.4 *The following inequality holds for all $u\in H^2(\Omega)$:*

$$\alpha^2\sum_{i,j=1}^{n}\left|\frac{\partial^2 u}{\partial x_i\,\partial x_j}\right|^2\leqslant\sum_{i,j,k,l=1}^{n}a_{i,k}a_{j,l}\frac{\partial^2 u}{\partial x_j\,\partial x_k}\frac{\partial^2 u}{\partial x_i\,\partial x_l} \tag{3,1,3,8}$$

a.e. in Ω.

From (3,1,3,8) it follows that

$$\alpha^2\sum_{i,j=1}^{n}\left|\frac{\partial^2 u}{\partial x_i\,\partial x_j}\right|^2\leqslant\sum_{i,j=1}^{n}\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}+2\sum_{i,j,k,l=1}^{n}\left|a_{i,k}\frac{\partial^2 u}{\partial x_j\,\partial x_k}\frac{\partial a_{j,l}}{\partial x_i}\frac{\partial u}{\partial x_l}\right|$$

a.e. in Ω. Integrating, we have

$$\begin{aligned}\alpha^2\sum_{i,j=1}^{n}\int_\Omega\left|\frac{\partial^2 u}{\partial x_i\,\partial x_j}\right|^2 dx&\leqslant\sum_{i,j=1}^{n}\int_\Omega\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}\,dx\\&\quad+2n^4M^2\int_\Omega\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|\sum_{i,j=1}^{n}\left|\frac{\partial^2 u}{\partial x_i\,\partial x_j}\right|dx,\end{aligned}$$

where M is a common bound for the $C^{0,1}$ norms of all the a_{ij}. This, together with inequality (3,1,3,7), implies:

$$\begin{aligned}&\alpha^2\sum_{i,j=1}^{n}\int_\Omega\left|\frac{\partial^2 u}{\partial x_i\,\partial x_j}\right|^2 dx\\&\leqslant\int_\Omega|Au|^2\,dx+2n^4M^2\int_\Omega\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|\sum_{i,j=1}^{n}\left|\frac{\partial^2 u}{\partial x_i\,\partial x_j}\right|dx\\&\leqslant\int_\Omega|Au|^2\,dx+\frac{\alpha^2}{2}\sum_{i,j=1}^{n}\int_\Omega\left|\frac{\partial^2 u}{\partial x_i\,\partial x_j}\right|^2 dx+2\frac{n^8M^4}{\alpha^2}\sum_{i=1}^{n}\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^2 dx.\end{aligned}$$

Thus we get

$$\alpha^2 \sum_{i,j=1}^{n} \int_\Omega \left|\frac{\partial^2 u}{\partial x_i\, \partial x_j}\right|^2 \mathrm{d}x \leq 2\int_\Omega |Au|^2 \,\mathrm{d}x + 4\frac{n^2M^4}{\alpha^2} \sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^2 \mathrm{d}x. \tag{3,1,3,9}$$

The estimate of the remaining terms in the norm of u in $H^2(\Omega)$ is the classical one. Indeed, we have

$$\int_\Omega (Au+\lambda u)u \,\mathrm{d}x = \sum_{i,j=1}^{n} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,\mathrm{d}x + \lambda \int_\Omega |u|^2 \,\mathrm{d}x$$

and consequently

$$\lambda \|u\|^2 + \alpha \sum_{i=1}^{n} \|D_i u\|^2 \leq \|Au+\lambda u\|\, \|u\|.$$

It follows that

$$\|u\| \leq \frac{1}{\lambda} \|Au + \lambda u\| \tag{3,1,3,10}$$

and that

$$\sum_{i=1}^{n} \|D_i u\|^2 \leq \frac{1}{\alpha\lambda} \|Au+\lambda u\|^2. \tag{3,1,3,11}$$

Adding inequalities (3,1,3,9) to (3,1,3,11), we obtain (3,1,3,6). Indeed, we have

$$\begin{aligned} \|u\|^2_{2,2,\Omega} &= \|u\|^2 + \sum_{i=1}^{n} \|D_i u\|^2 + \sum_{i,j=1}^{n} \|D_i D_j u\|^2 \\ &\leq \left(\frac{1}{\lambda^2} + \frac{1}{\alpha\lambda}\right) \|Au+\lambda u\|^2 + \frac{2}{\alpha^2}\|Au\|^2 + \frac{4}{\alpha\lambda}\frac{n^2M^4}{\alpha^4}\|Au+\lambda u\|^2 \\ &\leq \left(\frac{1}{\lambda^2} + \frac{1}{\alpha\lambda} + \frac{4n^2M^4}{\lambda\alpha^5} + \frac{8}{\alpha^2}\right)\|Au+\lambda u\|^2. \quad \blacksquare \end{aligned}$$

Proof of Lemma 3.1.3.4 By density, it is enough to prove inequality (3,1,3,8) for $u \in C^2(\bar{\Omega})$. At a particular fixed point x, let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the matrix whose entries are the a_{ij}. Also, let $y_1, \ldots, y_n$ be a new system of orthogonal axes which diagonalize this matrix. The inequality (3,1,3,8) is equivalent to

$$\alpha^2 \sum_{i,j=1} \left|\frac{\partial^2 u}{\partial y_i\, \partial y_j}\right|^2 \leq \sum_{i,j=1}^{n} \lambda_i \lambda_j \left|\frac{\partial^2 u}{\partial y_i\, \partial y_j}\right|^2.$$

This is evident since all the eigenvalues are $\leq -\alpha$.

3.2 Boundary value problems in convex domains

For some of the boundary value problems introduced at the beginning of Chapter 2, we now have two kinds of results. First an existence and uniqueness result for a solution in $H^2(\Omega)$ provided Ω is bounded and has a $C^{1,1}$ boundary (see Section 2.4 mainly). On the other hand, we proved (in Section 3.1) *a priori* bounds for solutions in $H^2(\Omega)$, where the constants depend very weakly on Ω provided it is convex and has a C^2 boundary. In most of the inequalities the constants do not depend on the curvature of Γ, i.e. on the fact that Γ is $C^{1,1}$. This will allow us to take limits with respect to Ω, i.e. to let Ω vary among convex domains. Thus we shall extend our previous results to general bounded convex domains.

The first result of this kind is due to Kadlec (1964) and concerns the Dirichlet problem. The extension of this result to other boundary conditions has been achieved in Grisvard and Iooss (1975).

3.2.1 Linear boundary conditions

The possibility of approximating a general convex domain by domains with C^2 boundaries follows easily from the results in Eggleston (1958).

Lemma 3.2.1.1 *Let Ω be a convex, bounded and open subset of $\mathbb{R}^n$. Then for every $\varepsilon > 0$, there exist two convex open subsets Ω_1 and Ω_2 in $\mathbb{R}^n$ such that*

(a) $\Omega_1 \subset \Omega \subset \Omega_2$
(b) *Ω_j has a C^2 boundary Γ_j, $j = 1, 2$.*
(c) $d(\Gamma_1, \Gamma_2) \leq \varepsilon$,

where $d(\Gamma_1, \Gamma_2)$ denotes the distance from Γ_1 to Γ_2.

This lemma allows us to approximate a given Ω either from the inside or from the outside by a domain with a smoother boundary. The inside approximation is more convenient for studying the Dirichlet boundary condition while the outside approximation is more suitable for dealing with boundary conditions of the Neumann type. By the way, we recall that we already proved in Section 1.2 that a bounded convex open subset of $\mathbb{R}^n$ always has a Lipschitz boundary.

In the following results A denotes the same operator as in 3.1.3, fulfilling the assumption (3,1,3,1).

Theorem 3.2.1.2 *Let Ω be a convex, bounded and open subset of $\mathbb{R}^n$. Then for each $f \in L_2(\Omega)$, there exists a unique $u \in H^2(\Omega)$, the solution of*

$$\begin{cases} Au = f & \text{in } \Omega \\ \gamma u = 0 & \text{on } \Gamma. \end{cases} \tag{3,2,1,1}$$

Proof We choose a sequence Ω_m, $m = 1, 2, \ldots$ of convex open subsets of $\mathbb{R}^n$ with C^2 boundaries Γ_m such that $\Omega_m \subseteq \Omega$ and $d(\Gamma_m, \Gamma)$ tends to zero as $m \to +\infty$. We consider the solution $u_m \in H^2(\Omega_m)$ of the Dirichlet problem in Ω_m, i.e.

$$\begin{cases} Au_m = f & \text{in } \Omega_m \\ \gamma_m u_m = 0 & \text{on } \Gamma_m, \end{cases} \tag{3,2,1,2}$$

where γ_m denotes the trace operator on Γ_m, $m = 1, 2, \ldots$. Such a solution u_m exists by Theorem 2.2.2.3.

It follows from Theorem 1.5.1.5 that $u_m \in \mathring{H}^1(\Omega_m)$; in other words, we have $\tilde{u}_m \in H^1(\mathbb{R}^n)$. Then from Theorem 3.1.3.1 we know that there exists a constant C such that

$$\|u_m\|_{2,2,\Omega_m} \leqslant C. \tag{3,2,1,3}$$

This implies that $\tilde{u}_m$ is a bounded sequence in $H^1(\mathbb{R}^n)$, and in addition that

$$v_{m,i,j} = (D_i D_j u_m)^{\sim}, \qquad m = 1, 2, \ldots$$

are bounded sequences in $L_2(\mathbb{R}^n)$ for $1 \leqslant i, j \leqslant n$. Consequently there exist $U \in H^1(\mathbb{R}^n)$ and $V_{i,j} \in L_2(\mathbb{R}^n)$ and a suitable increasing sequence of integers m_k, $k = 1, 2, \ldots$ such that

$$\begin{cases} \tilde{u}_{m_k} \to U & \text{weakly in } H^1(\mathbb{R}^n), \quad k \to \infty \\ v_{m,i,j} \to V_{i,j} & \text{weakly in } L_2(\mathbb{R}^n), \quad k \to \infty. \end{cases}$$

First, we shall check that the restriction u of U to Ω is solution of the Dirichlet problem in Ω. Indeed we have $u \in H^1(\Omega)$. In addition, all the $\tilde{u}_m$ have their support in $\bar{\Omega}$; it follows that U also has its support in $\bar{\Omega}$, i.e. $U = \tilde{u}$. By Definition 1.3.2.5, this means that $u \in \tilde{H}^1(\Omega)$ and finally Corollary 1.5.1.5 implies that $\gamma u = 0$ on Γ (here the Corollary is applied with $k = 0$, which is possible owing to Corollary 1.2.2.3). Finally, let $\varphi \in \mathcal{D}(\Omega)$; then there exists $k(\varphi)$ such that the support of φ is contained in Ω_{m_k} for all $k \geqslant k(\varphi)$. Thus for $k \geqslant k(\varphi)$ we have

$$\begin{aligned} \int_\Omega f\varphi \, dx = \int_{\Omega_{m_k}} Au_{m_k}\varphi \, dx &= -\sum_{i,j=1}^n \int_{\Omega_{m_k}} a_{ij} D_i u_{m_k} D_j \varphi \, dx \\ &= -\sum_{i,j=1}^n \int_\Omega a_{ij} D_i \tilde{u}_{m_k} D_j \varphi \, dx. \end{aligned}$$

Taking the limit in k, we obtain

$$\int_\Omega f\varphi \, dx = -\sum_{i,j=1}^n \int_\Omega a_{ij} D_i u D_j \varphi \, dx.$$

This identity is valid for all $\varphi \in \mathcal{D}(\Omega)$; it means that

$$Au = f \qquad \text{in } \Omega$$

in the sense of distributions.

So far we proved the existence of $u \in H^1(\Omega)$, the solution of (3,2,1,1). The uniqueness of u is a classical result by the energy method (see for instance Nečas (1967)). To complete the proof we have to check that the second derivatives of u are square integrable. We again let φ belong to $\mathcal{D}(\Omega)$. Then for $k \geqslant k(\varphi)$, we have

$$\int_\Omega \tilde{u}_{m_k} D_i D_j \varphi \, dx = \int_{\Omega_{m_k}} u_{m_k} D_i D_j \varphi \, dx = \int_{\Omega_{m_k}} D_i D_j u_{m_k} \varphi \, dx = \int_\Omega v_{m,i,j} \varphi \, dx.$$

Taking the limit in k, we get

$$\int_\Omega u D_i D_j \varphi \, dx = \int_\Omega V_{i,j} \varphi \, dx.$$

In other words, the distributional derivative $D_i D_j u$ is the restriction of $V_{i,j}$ to Ω; this is a square integrable function for all $i, j = 1, \ldots, n$. ■

Theorem 3.2.1.3 *Let Ω be a convex, bounded and open subset of $\mathbb{R}^n$. Then for each $f \in L_2(\Omega)$ and for each $\lambda > 0$ there exists a unique $u \in H^2(\Omega)$ which is the solution of*

$$-\sum_{i,j=1}^{n} \int_\Omega a_{ij} D_i u D_j v \, dx + \lambda \int_\Omega uv \, dx = \int_\Omega fv \, dx \tag{3,2,1,4}$$

for all $v \in H^1(\Omega)$.

Identity (3,2,1,4) is the weak form of the Neumann problem for the equation

$$Au + \lambda u = f \qquad \text{in } \Omega. \tag{3,2,1,5}$$

As we saw in the proof of Theorem 2.2.2.5, identity (3,2,1,4) is equivalent to equation (3,2,1,5) together with the boundary condition

$$\sum_{i,j=1}^{n} \nu_i \gamma(a_{ij} D_j u) = 0 \qquad \text{a.e. on } \Gamma. \tag{3,2,1,6}$$

This makes sense since Γ is Lipschitz and $a_{ij} D_j u \in H^1(\Omega)$.

Proof This time, we choose a sequence Ω_m, $m = 1, 2, \ldots$, of bounded convex open subsets of $\mathbb{R}^n$ with C^2 boundaries Γ_m such that $\Omega \subseteq \Omega_m$ and $d(\Gamma_m, \Gamma)$ tends to zero as $m \to \infty$. We consider the solution $u_m \in H^2(\Omega_m)$

of the Neumann problem in Ω_m, i.e.

$$\begin{cases} Au_m + \lambda u_m = \tilde{f} & \text{in } \Omega_m \\ \gamma_m \dfrac{\partial u_m}{\partial \nu_A} = 0 & \text{on } \Gamma_m. \end{cases} \tag{3.2.1.7}$$

Obviously $\tilde{f} \in L_2(\Omega_m)$ and u_m exists by Corollary 2.2.2.6.

From Theorem 3.1.3.3, we know that there exists a constant C such that

$$\|u_m\|_{2,2,\Omega_m} \leq C. \tag{3.2.1.8}$$

Consequently, restricting the u_m to Ω we obtain a bounded sequence in $H^2(\Omega)$ and then a weakly convergent subsequence. In other words, there exists an increasing sequence of integers m_k and a function $u \in H^2(\Omega)$ such that

$$u_{m_k}|_\Omega \to u \qquad \text{weakly in } H^2(\Omega), \quad \text{as } k \to \infty.$$

We now complete the proof by checking that u is a solution of (3,2,1,4). Indeed let $v \in H^1(\Omega)$. Since Γ is Lipschitz, we can apply Theorem 1.4.3.1 to find a $V \in H^1(\mathbb{R}^n)$ such that $V|_\Omega = v$. It is clear that $V|_{\Omega_m} \in H^1(\Omega_m)$ and from (3,2,1,7) we deduce that

$$-\sum_{i,j,=1}^{n} \int_{\Omega_{m_k}} a_{i,j} D_i u_{m_k} D_j V \, dx + \lambda \int_{\Omega_{m_k}} u_{m_k} V \, dx = \int_\Omega f v \, dx. \tag{3,2,1,9}$$

We shall now consider the limit of (3,2,1,9) when $k \to \infty$. We have first

$$\int_{\Omega_{m_k}} u_{m_k} V \, dx - \int_\Omega uv \, dx = \int_{\Omega_{m_k} \setminus \Omega} u_{m_k} V \, dx + \int_\Omega (u_{m_k} - u) v \, dx$$

and consequently

$$\left| \int_{\Omega_{m_k}} u_{m_k} V \, dx - \int_\Omega uv \, dx \right|$$
$$\leq \|u_{m_k}\|_{0,2,\Omega_{m_k}} \left(\int_{\Omega_{m_k} \setminus \Omega} V^2 \, dx \right)^{1/2} + \|u_{m_k} - u\|_{0,2,\Omega} \|V\|_{0,2,\Omega}.$$

The right-hand side of this inequality converges to zero due to (3,2,1,8) and the compactness of the injection of $H^2(\Omega)$ in $L_2(\Omega)$ (see Theorem 1.4.3.2). In the same way, we prove that

$$\int_{\Omega_{m_k}} a_{i,j} D_i u_{m_k} D_j V \, dx \to \int_\Omega a_{i,j} D_i u D_j v \, dx$$

owing to the compactness of the injection of $H^2(\Omega)$ in $H^1(\Omega)$. Summing up we obtain identity (3,2,1,4) as the limit of (3,2,1,9) when $k \to \infty$. ■

Remark 3.2.1.4 One can prove results similar to those of Theorems 3.2.1.2 and 3.2.1.3 when Ω is a plane bounded domain with Lipschitz and piecewise C^2 boundary whose angles are all convex.

3.2.2 Nonlinear boundary conditions (review)

In the next subsection we shall take advantage of inequality (3,1,2,6), which concerns the nonlinear boundary condition.

$$-\gamma\frac{\partial u}{\partial \nu}=\beta(\gamma u) \qquad \text{on } \Gamma$$

where β is a uniformly Lipschitz continuous and nondecreasing function such that $\beta(0)=0$. We shall take limits with respect to Ω and β. In this subsection we review some known results about monotone operators. Here we follow Brezis (1971).

Let H be a Hilbert space and A a mapping from H into the family of all subsets of H. In other words, A is a (possibly multivalued) mapping from

$$D(A)=\{x\in H;\ Ax\neq\varnothing\}$$

into H. A is said to be monotone if

$$(y_1-y_2;\ x_1-x_2)\geqslant 0,$$

$\forall x_1, x_2\in D(A)$ and $y_1\in Ax_1$, $y_2\in Ax_2$. Then A is said to be maximal monotone if it is maximal in the sense of inclusions of graphs; i.e., it admits no proper monotone extension. For each $\lambda>0$ we define an inverse for the multivalued mapping $(\lambda A+I)$ as follows:

$$(\lambda A+I)^{-1}y=\left\{x\in D(A)\;\middle|\;\frac{1}{\lambda}(y-x)\in Ax\right\}.$$

It turns out that $(\lambda A+I)^{-1}$ is univalued and is a contraction in H provided A is monotone. It was shown by Minty (1962) that A is maximal if and only if $(\lambda A+I)$ is onto for $\lambda>0$, or equivalently $(\lambda A+I)^{-1}$ is defined everywhere.

In what follows we shall only consider monotone operators which are in some sense the gradient of a convex function. More precisely, let φ be a convex lower semicontinuous function from H into $]-\infty+\infty]$. We assume that φ is proper, i.e. that $\varphi\not\equiv+\infty$. Let

$$D(\varphi)=\{x\in H\mid \varphi(x)<+\infty\}.$$

For $x\in D(\varphi)$ the set

$$\partial\varphi(x)=\{y\in H\mid \varphi(z)-\varphi(x)\geqslant(y;\ z-x),\quad \forall z\in D(\varphi)\}$$

is called the subdifferential of φ at x. It was shown by Minty (1964) that the operator $x\mapsto\partial\varphi(x)$ is maximal monotone.

Following Moreau (1965), such a convex function can be approximated by smooth convex functions φ_λ defined for $\lambda > 0$, by

$$\varphi_\lambda(x) = \min_{z \in H} \left\{ \frac{1}{2\lambda} |x - z|^2 + \varphi(z) \right\}. \tag{3,2,2,1}$$

It turns out that φ_λ is convex and Frechet differentiable. Thus

$$\partial\varphi_\lambda(x) = \{\phi'_\lambda(x)\}$$

for every $x \in H$. In addition $\varphi_\lambda(x)$ is a decreasing function of λ and

$$\varphi_\lambda(x) \to \varphi(x)$$

for every $x \in H$ as $\lambda \to 0$. Finally

$$\varphi'_\lambda(x) = \frac{1}{\lambda} \{x - (\lambda\, \partial\varphi + I)^{-1} x\}. \tag{3,2,2,2}$$

This is simply the so-called Yosida approximation of $A = \partial\varphi$, which is monotone and Lipschitz continuous with Lipschitz constant $1/\lambda$.

Not all maximal monotone operators are subdifferentials of convex functions; however, in the particular case when H is just the real line $\mathbb{R}$, this does hold. Here are two typical examples of subdifferentials of convex functions in $\mathbb{R}$. First, if we assume that

$$\varphi(x) = \begin{cases} +\infty & x \neq 0 \\ 0 & x = 0 \end{cases}$$

then it is easy to check that

$$\partial\varphi(x) = \begin{cases} \varnothing & x \neq 0 \\ \mathbb{R} & x = 0. \end{cases}$$

This is a maximal monotone operator in $\mathbb{R}$. The Moreau approximation of φ is

$$\varphi_\lambda(x) = \frac{x^2}{2\lambda}$$

and accordingly the Yosida approximation of $\partial\varphi$ is

$$\varphi'_\lambda(x) = \frac{x}{\lambda}.$$

On the other hand, let

$$\varphi(x) = \begin{cases} +\infty & x < 0 \\ 0 & x \geqslant 0; \end{cases}$$

then it is easy to check that

$$\partial\varphi(x)=\begin{cases}\varnothing & x<0\\ [-\infty,0] & x=0\\ \{0\} & x>0\end{cases}$$

$$\varphi_\lambda(x)=\begin{cases}\dfrac{x^2}{2\lambda}, & x\leqslant 0\\ 0, & x\geqslant 0\end{cases}$$

$$\varphi_\lambda'(x)=\begin{cases}\dfrac{x}{\lambda}, & x\leqslant 0\\ 0, & x\geqslant 0.\end{cases}$$

Turning back to the general case, an important existence result is the following:

Lemma 3.2.2.1 *Let φ be a convex, lower semi-continuous and proper function on H. Assume that φ is coercive, i.e. that*

$$\varphi(x)\to+\infty \quad \textit{when} \quad \|x\|\to+\infty. \tag{3,2,2,3}$$

Then φ has a minimum in H. The minimum is unique when φ is strictly convex.

Accordingly, if x_0 is such a minimum, we have

$$0\in\partial\varphi(x_0).$$

This is an existence result for the subdifferential of φ.

Proof of Lemma 3.2.2.1 We denote by m the g.l.b. of φ. There exists a sequence x_n, $n=1,2,\ldots$ of elements of H such that

$$\varphi(x_n)\to m$$

when $n\to+\infty$. Since φ is proper, we have $m<+\infty$ and condition (3,2,2,3) implies that the sequence x_n, $n=1,2,\ldots$ is bounded in H.

Consequently, by possibly replacing the original sequence by a suitable subsequence, we can assume that x_n, $n=1,2,\ldots$ is weakly convergent to some limit $x\in H$. By the very definition of m we have

$$\varphi(x)\geqslant m.$$

On the other hand, since φ is lower semi-continuous (for either the strong topology or the weak topology on H), we have

$$\varphi(x) \leqslant \limsup_{n\to\infty} \varphi(x_n) \leqslant m.$$

Summing up we have proved that $\varphi(x) = m$ and x is the desired minimum.

The uniqueness of the minimum, when φ is strictly convex is obvious. ■

We shall use this existence result as follows. We consider a maximal operator β in $\mathbb{R}$ and the corresponding convex function j on $\mathbb{R}$ such that

$$\beta = \partial j.$$

Then we build a new convex function on $L_2(\Omega)$ by setting

$$\varphi(v) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla v|^2 \,\mathrm{d}x + \int_\Gamma j(\gamma v)\,\mathrm{d}\sigma & \\ \qquad\qquad \text{if } v \in H^1(\Omega) \text{ and } j(\gamma v) \in L_1(\Gamma) & \\ +\infty \qquad \text{otherwise} & \end{cases} \tag{3,2,2,4}$$

We are looking for solutions of the following boundary value problem where $c > 0$, $f \in L_2(\Omega)$ and Ω is, say, a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary Γ:

$$\begin{cases} -\Delta u + cu = f & \text{in } \Omega \\ -(\gamma \nabla u) \cdot \boldsymbol{\nu} \in \beta(\gamma u) & \text{a.e. on } \Gamma \end{cases} \tag{3,2,2,5}$$

The function φ allows a weak formulation for problem (3,2,2,5). Indeed we have this lemma.

Lemma 3.2.2.2 *Let $u \in H^2(\Omega)$ be a solution of* (3,2,2,5), *then we have*

$$\varphi(v) + \frac{c}{2}\|v\|^2 - \int_\Omega fv \,\mathrm{d}x \geqslant \varphi(u) + \frac{c}{2}\|u\|^2 - \int_\Omega fu \,\mathrm{d}x \tag{3,2,2,6}$$

for all $v \in L_2(\Omega)$.

Proof The boundary condition in (3,2,2,5) implies that for $v \in H^1(\Omega)$ such that $j(\gamma v) \in L_1(\Gamma)$ we have

$$j(\gamma v) - j(\gamma u) \geqslant -(\gamma \nabla u) \cdot \boldsymbol{\nu}(\gamma v - \gamma u)$$

and consequently that

$$\int_\Omega \{j(\gamma v) - j(\gamma u)\}\,\mathrm{d}\sigma \geqslant -\int_\Omega (\boldsymbol{\nu} \cdot \gamma \nabla u)(\gamma v - \gamma u)\,\mathrm{d}\sigma.$$

Then by the Green formula, we have

$$\begin{aligned}\int_\Omega f(v-u)\,\mathrm{d}x &= \int_\Omega (-\Delta u+cu)(v-u)\,\mathrm{d}x\\ &= -\int_\Gamma (\boldsymbol{\nu}\cdot\gamma\,\nabla u)(\gamma v-\gamma u)\,\mathrm{d}\sigma\\ &\quad+\int_\Omega \nabla u\cdot\nabla(v-u)\,\mathrm{d}x+c\int_\Omega u(v-u)\,\mathrm{d}x\end{aligned}$$

and consequently

$$\begin{aligned}\int_\Omega f(v-u)\,\mathrm{d}x &\leqslant \int_\Omega \{j(\gamma v)-j(\gamma u)\}\,\mathrm{d}\sigma+\int_\Omega \nabla u\cdot\nabla(v-u)\,\mathrm{d}x\\ &\quad+c\int_\Omega u(v-u)\,\mathrm{d}x.\end{aligned}$$

Finally, observing that

$$2u(v-u)\leqslant v^2-u^2$$

and that

$$2\,\nabla u\cdot\nabla(v-u)\leqslant|\nabla v|^2-|\nabla u|^2$$

we conclude that

$$\begin{aligned}&\tfrac{1}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x+\frac{c}{2}\int_\Omega |v|^2\,\mathrm{d}x+\int_\Gamma j(\gamma v)\,\mathrm{d}\sigma-\int_\Omega fv\,\mathrm{d}x\\ &\geqslant\tfrac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x+\frac{c}{2}\int_\Omega |u|^2\,\mathrm{d}x+\int_\Gamma j(\gamma u)\,\mathrm{d}\sigma-\int_\Omega fu\,\mathrm{d}x.\end{aligned}$$

This is exactly inequality (3,2,2,6) when $v\in H^1(\Omega)$ and $j(\gamma v)\in L_1(\Gamma)$. In the other cases, we have $\varphi(v)=+\infty$ and inequality (3,2,2,6) is obvious. ■

In other words, we have reduced the problem of solving (3,2,2,5) to that of minimizing the function

$$v\mapsto\varphi(v)+\frac{c}{2}\|v\|^2-\int_\Omega fv\,\mathrm{d}x. \tag{3,2,2,7}$$

This is easily achieved, owing to the following lemma.

Lemma 3.2.2.3 *For $c>0$ and $f\in L_2(\Omega)$ the function* (3,2,2,7) *is convex, lower semicontinuous, proper and coercive on $L_2(\Omega)$, provided $\beta(0)\ni 0$.*

Proof All but the coerciveness is obvious. To prove (3,2,2,3), we observe that $j(x)\geqslant 0$ everywhere. Indeed, the condition that $\partial j(0)=\beta(0)\ni 0$

means that the graph of j is contained in the upper half-plane. Then it follows that

$$\varphi(v)+\frac{c}{2}\|v\|^2-\int_\Omega fv\,dx \geq \tfrac{1}{2}\int_\Omega |\nabla v|^2\,dx+\frac{c}{4}\int_\Omega |v|^2\,dx-\frac{1}{c}\int_\Omega |f|^2 dx.$$

This lower bound obviously tends to $+\infty$ when $\|v\|\to\infty$. ■

3.2.3 Nonlinear boundary conditions (continued)

Let us first state the result which is the purpose of this subsection.

Theorem 3.2.3.1 *Let Ω be a bounded convex open subset of $\mathbb{R}^n$. Let β be a maximal monotone operator on $\mathbb{R}$ such that $\beta(0)\ni 0$. Then for each $f\in L^2(\Omega)$ and for each $c>0$, there exists a unique $u\in H^2(\Omega)$ which is the solution of* (3,2,2,5).

Before proving this theorem, let us take a look at some examples. Let us assume first that

$$j(x)=\begin{cases}+\infty & x\neq 0\\ 0 & x=0.\end{cases}$$

Then obviously we have

$$\varphi(v)=\begin{cases}\tfrac{1}{2}\int_\Omega |\nabla v|^2\,dx & \text{if } v\in \mathring{H}^1(\Omega)\\ +\infty & \text{otherwise.}\end{cases}$$

Then, surprisingly enough, problem (3,2,2,5) is just a Dirichlet problem. Indeed, the boundary condition means that a.e. on Γ,

$$(\gamma u, -\boldsymbol{\nu}\cdot\gamma\nabla u)$$

is a point of $\mathbb{R}^2$, which actually lies on the vertical axis. In other words, $\gamma u=0$ a.e. on Γ With this special choice of j, Theorem 3.2.3.1 is just a particular case of Theorem 3.2.1.2.

Let us assume now that

$$j(x)=\begin{cases}+\infty & x<0\\ 0 & x\geq 0.\end{cases}$$

Then we have

$$\varphi(v)=\begin{cases}\tfrac{1}{2}\int_\Omega |\nabla v|^2\,dx & \text{if } v\in H^1(\Omega) \quad\text{and}\quad \gamma v\geq 0 \text{ a.e. on } \Gamma\\ +\infty & \text{otherwise.}\end{cases}$$

The boundary condition in (3,2,2,5) means that a.e. on Γ

$$(\gamma u, -\boldsymbol{\nu}\cdot\gamma\,\nabla u)\in G$$

where G is the graph of $\beta=\partial j$, i.e.

$$G=\{(x,y)\in\mathbb{R}^2 \mid x\geq 0,\ y\leq 0,\ x\cdot y=0\}.$$

Accordingly we have

$$\gamma u\geq 0,\qquad \boldsymbol{\nu}\cdot\gamma\,\nabla u\geq 0,\qquad (\gamma u)(\boldsymbol{\nu}\cdot\gamma\,\nabla u)=0 \tag{3,2,3,1}$$

a.e. on Γ and this is the famous Signorini boundary condition.

Finally let us observe that if we assume that

$$j(x)=b\frac{x^2}{2}$$

where $b\geq 0$, then

$$\varphi(v)=\begin{cases}\frac{1}{2}\int_\Omega |\nabla v|^2\,dx+\frac{b}{2}\int_\Gamma |\gamma v|^2\,d\sigma & \text{if } v\in H^1(\Omega)\\ +\infty & \text{otherwise.}\end{cases}$$

Accordingly, we have $\beta(x)=j'(x)=bx$ and the boundary condition in (3,2,2,5) is just

$$-\boldsymbol{\nu}\cdot(\gamma\,\nabla u)=b\gamma u\qquad \text{a.e. on } \Gamma.$$

In particular, when $b=0$, this is a Neumann boundary condition and we have a particular case of Theorem 3.2.1.3.

Before proving Theorem 3.2.3.1, we need some preliminary results on the approximation of Ω.

Lemma 3.2.3.2 *Let Ω be a convex, bounded and open subset of $\mathbb{R}^n$ and let Ω_m, $m=1,2,\ldots$ be a sequence of convex, bounded and open subsets of $\mathbb{R}^n$ such that $\Omega\subseteq\Omega_m$, Ω_m has a C^2 boundary Γ_m and*

$$d(\Gamma_m,\Gamma)\to 0\quad\text{when}\quad m\to\infty.$$

Then, for large enough m, there exists a finite number of open subsets V_k, $k=1,2,\ldots,K$ in $\mathbb{R}^n$ with the following properties:

(a) *For each k there exist new coordinates $\{y_1^k,\ldots,y_n^k\}$ in which V_k is the hypercube*

$$\{(y_1^k,\ldots,y_n^k)\mid -a_j^k<y_j^k<a_j^k,\qquad 1\leq j\leq n\}$$

(b) *For each k there exist Lipschitz functions φ^k and φ_m^k defined in*

$$V_k'=\{(y_1^k,\ldots,y_{n-1}^k)\mid -a_j^k<y_j^k<a_j^k,\qquad 1\leq j\leq n-1\}$$

and such that

$$|\varphi^k(z^k)|, |\varphi_m^k(z^k)| \leq \frac{a_n^k}{2} \text{ for every } z^k \in V_k'$$

$$\Omega \cap V_k = \{y^k = (z^k, y_n^k) \mid y_n^k < \varphi^k(z^k)\}$$

$$\Omega_m \cap V_k = \{y^k = (z^k, y_n^k) \mid y_n^k < \varphi_m^k(z^k)\}$$

$$\Gamma \cap V_k = \{y^k = (z^k, y_n^k) \mid y_n^k = \varphi^k(z^k)\}$$

$$\Gamma_m \cap V_k = \{y^k = (z^k, y_n^k) \mid y_n^k = \varphi_m^k(z^k)\}$$

(c) $$\Gamma \subset \bigcup_{k=1}^{K} V_k, \qquad \Gamma_m \subset \bigcup_{k=1}^{K} V_k.$$

In addition $-\varphi^k$ *and* $-\varphi_m^k$ *are convex functions,* φ_m^k *is of class* C^2 *for all large enough* m *and*

(d) φ_m^k *converges uniformly to* φ^k *on* V_k' *and there exists* L *such that*

$$|\nabla\varphi^k(z^k)|, |\nabla\varphi_m^k(z^k)| \leq L \text{ for every } z^k \in V_k', \qquad 1 \leq k \leq K. \tag{3,2,3,2}$$

Finally, $\nabla\varphi_m^k \to \nabla\varphi^k$ *a.e. in* V_k'.

It follows from Corollary 1.2.2.3 that Ω and all the Ω_m have Lipschitz boundaries. Accordingly, properties (a) and (b) just refer to the corresponding properties in Definition 1.2.1.1. We actually just have to check that property (d) holds.

Proof of property (d) in Lemma 3.2.3.2 Since the distance from Γ to Γ_m converges to zero when $m \to \infty$, it follows that the distance from the graph of φ^k to the graph of φ_m^k converges to zero. This means that φ_m^k converges uniformly to φ^k.

Inequality (3,2,3,2) follows from the geometry. Indeed let us consider a fixed k and a fixed m. Let $y^k = (z^k, \varphi_m^k(z^k))$ be a point on $\Gamma_m \cap V^k$, with

$$-a_j^k + \varepsilon < y_j^k < a_j^k - \varepsilon, \qquad 1 \leq j \leq n-1$$

for a given $\varepsilon > 0$. The set

$$S = \left\{\left(z^k, -\frac{a_n^k}{2}\right) \,\middle|\, z^k \in V_k'\right\}$$

is included in Ω_m. So is the line segment from y^k to any point of ∂S. The slope of such a line has a modulus less than or equal to a_n^k/ε. This implies that

$$|\nabla\varphi_m^k(z^k)| \leq a_n^k/\varepsilon.$$

We conclude by replacing all the a_j^k, $1 \leq j \leq n-1$ by $a_j^k - \varepsilon$ with an

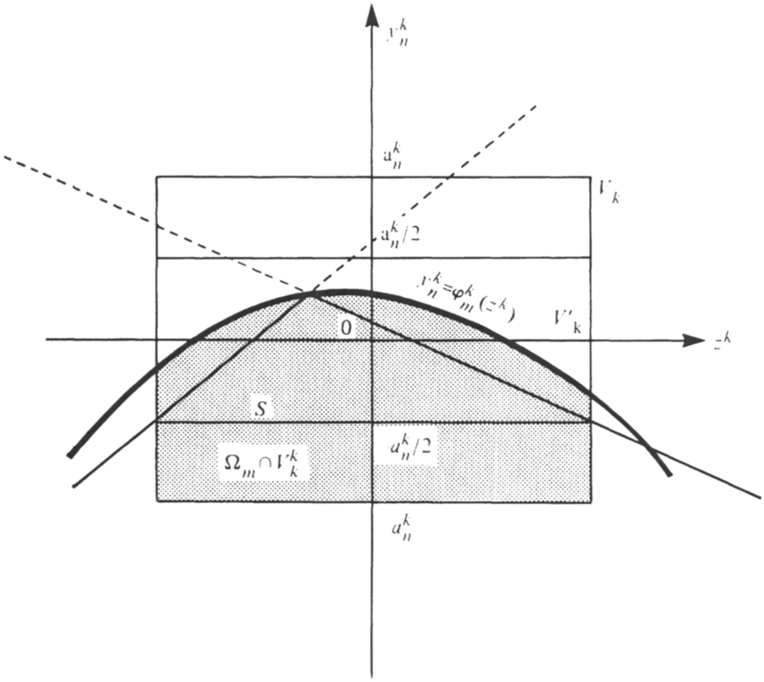

Figure 3.1

$\varepsilon > 0$ small enough to preserve the condition $\Gamma \subset \bigcup_{k=1}^{K} V_k$. Since $d(\Gamma_m, \Gamma) \to 0$ as $m \to +\infty$, the condition $\Gamma_m \subset \bigcup_{k=1}^{K} V_k$ is also preserved for m large enough. We can define L as follows:

$$L = \max_{k=1}^{K} a_n^k/\varepsilon.$$

Let us now complete the proof by looking at the convergence of $\nabla\varphi_m^k$ to $\nabla\varphi^k$. Since φ^k is Lipschitz continuous, it has a gradient a.e. in V_k'. Let us consider such a point $z \in V_k'$ such that $\nabla\varphi^k(z)$ exists. The tangent hyperplane at $(z, \varphi_m^k(z))$ to the graph of φ_m^k is above the graph of φ^k, since φ^k and φ_m^k are concave functions and $\varphi_m^k \geq \varphi^k$. In other words, we have

$$\varphi_m^k(z) + \nabla\varphi_m^k(z) \cdot (\boldsymbol{\xi} - \mathbf{z}) \geq \varphi^k(\xi)$$

for all $\xi \in V_k'$. Since φ^k has a gradient at z in the usual sense, it follows that

$$\varphi_m^k(z) + \nabla\varphi_m^k(z) \cdot (\boldsymbol{\xi} - \mathbf{z}) \geq \varphi^k(z) + \nabla\varphi^k(z) \cdot (\boldsymbol{\xi} - \mathbf{z}) - \mathcal{O}(|\boldsymbol{\xi} - \mathbf{z}|)$$

for all $\xi \in V'_k$. Then for each $j = 1, 2, \ldots, n-1$, we have

$$\varphi^k_m(z) + tD_j\varphi^k_m(z) \geq \varphi^k(z) + tD_j\varphi^k(z) + \mathcal{O}(|t|)$$

for $|t|$ small enough. If we denote by α_j and β_j the limits

$$\alpha_j = \liminf_{m\to\infty} D_j\varphi^k_m(z), \qquad \beta_j = \limsup_{m\to\infty} D_j\varphi^k_m(z),$$

we easily see that

$$\alpha_j \geq D_j\varphi^k(z) + \mathcal{O}(1), \qquad \beta_j \leq D_j\varphi^k(z) + \mathcal{O}(1).$$

This shows that

$$D_j\varphi^k_m(z) \to D_j\varphi^k(z)$$

when $m \to \infty$ and completes the proof. ■

Proof of Theorem 3.2.3.1 We shall approximate Ω by a sequence of convex open subsets Ω_m, $m = 1, 2, \ldots$ of $\mathbb{R}^n$ as in Lemma 3.2.3.2. We shall also approximate j by its Moreau approximation j_λ, or equivalently β by its Yosida approximation $\beta_\lambda = j'_\lambda$. Thus we start from $u_{\lambda,m} \in L_2(\Omega_m)$, which minimizes the functional

$$\psi_{m,\lambda}(v) = \begin{cases} \frac{1}{2}\int_{\Omega_m} |\nabla v|^2 \,dx + \frac{c}{2}\int_{\Omega_m} |v|^2 \,dx + \int_{\Omega_m} j_\lambda(\gamma_m v)\,d\sigma_m - \int_\Omega fv\,dx \\ \qquad\qquad \text{if } v \in H^1(\Omega_m) \\ +\infty \qquad \text{otherwise.} \end{cases}$$

We observe that since j'_λ is uniformly Lipschitz, its primitive j_λ does not grow faster than a quadratic function. Accordingly, when $v \in H^1(\Omega_m)$, we have $j_\lambda(\gamma_m v) \in L_1(\Gamma_m)$.

There are four main steps in the proof.

1st step We check that $u_{\lambda,m} \in H^2(\Omega_m)$.
2nd step We prove that $\|u_{\lambda,m}\|_{2,2,\Omega_m}$ remains bounded uniformly in m and λ.
3rd step We take the limit in m.
4th step We take the limit in λ.

In the *first step* we use the fact that $u_{\lambda,m}$ is the solution of

$$\begin{cases} -\Delta u_{\lambda,m} + cu_{\lambda,m} = \tilde{f} & \text{in } \Omega_m \\ -\gamma_m \dfrac{\partial u_{\lambda,m}}{\partial \nu_m} = \beta_\lambda(\gamma_m u_{\lambda,m}) & \text{on } \Gamma_m. \end{cases}$$

Indeed, we first observe that since $\psi_{m,\lambda}(u_{\lambda,m}) < +\infty$, $u_{\lambda,m}$ must belong to

$H^1(\Omega_m)$. Thus it is the minimum for $\psi_{m,\lambda}$ on $H^1(\Omega_m)$. It is easily checked that $\psi_{m,\lambda}$ is Frechet differentiable on $H^1(\Omega_m)$ and consequently we have

$$\psi'_{m,\lambda}(u_{\lambda,m}) = 0.$$

In other words we have

$$\int_{\Omega_m} \nabla u_{\lambda,m} \cdot \nabla v \, dx + c\int_{\Omega_m} u_{\lambda,m} v \, dx + \int_{\Gamma_m} \beta_\lambda(\gamma_m u_{\lambda,m})\gamma v \, d\sigma_m - \int_{\Omega_m} fv \, dx = 0 \quad (3,2,3,3)$$

for all $v \in H^1(\Omega_m)$. Making use of (3,2,3,3) with only $v \in \mathring{H}^1(\Omega_m)$, we readily see that

$$-\Delta u_{\lambda,m} + cu_{\lambda,m} = f \qquad \text{in } \Omega_m.$$

Then applying the Green formula (1,5,3,10), we rewrite (3,2,3,3) as follows:

$$-\left\langle \gamma_m \frac{\partial u_{\lambda,m}}{\partial \nu_m} ; \gamma_m v \right\rangle = \int_{\Gamma_m} \beta_\lambda(\gamma_m u_{\lambda,m})\gamma v \, d\sigma_m$$

for all $\gamma_m v \in H^{1/2}(\Gamma_m)$. This implies the boundary condition on $u_{\lambda,m}$, i.e. $-\gamma_m \, \partial u_{\lambda,m}/\partial \nu_m = \beta_\lambda(\gamma_m u_{\lambda,m})$ in the sense of $H^{-1/2}(\Gamma_m)$.

Let us now consider $\beta_\lambda(\gamma u_{\lambda,m})$ as the Neumann data for $u_{\lambda,m}$. Since $u_{\lambda,m} \in H^1(\Omega_m)$, we have $\gamma_m u_{\lambda,m} \in H^{1/2}(\Gamma_m)$. Then, taking advantage of the fact that β_λ is uniformly Lipschitz continuous, we conclude that

$$\beta_\lambda(\gamma_m u_{\lambda,m}) \in H^{1/2}(\Gamma_m).$$

Now it follows from Corollary 2.2.2.6 that $u_{\lambda,m} \in H^2(\Omega_m)$ since Γ_m is C^2.

The *second step* is just an application of Theorem 3.1.2.3. This theorem can be used here because Ω_m is convex and has a C^2 boundary, while β_λ is uniformly Lipschitz, non-decreasing and fulfils the condition

$$\beta_\lambda(0) = 0.$$

Indeed, we have $-\beta_\lambda(0) = (1/\lambda)(\lambda\beta + I)^{-1}0$ and $(\lambda\beta + I)^{-1}0 = 0$ since $0 \in (\lambda\beta + I)(0)$. Thus, we have

$$\|u_{\lambda,m}\|_{2,2,\Omega_m} \leq \left(\frac{1}{c^2} + \frac{1}{c} + 4\right)^{1/2} \|f\|_{0,2,\Omega}. \qquad (3,2,3,4)$$

$u_{\lambda,m}$, $m = 1, 2, \ldots$ is consequently a bounded sequence in $H^2(\Omega)$. By possibly considering a suitable subsequence we can therefore assume that there exists $u_\lambda \in H^2(\Omega)$ such that

$$u_{\lambda,m} \rightharpoonup u_\lambda$$

weakly in $H^2(\Omega)$ when $m \to +\infty$.

In the *third step* we take the limit in m in the equation which expresses that $\psi_{m,\lambda}$ has a minimum at $u_{\lambda,m}$. Actually we have

$$\begin{aligned}\tfrac{1}{2}\int_{\Omega_m} |\nabla u_{\lambda,m}|^2\,dx+\frac{c}{2}\int_{\Omega_m} |u_{\lambda,m}|^2\,dx+\int_{\Gamma_m} j_\lambda(\gamma_m u_{\lambda,m})\,d\sigma_m-\int_\Omega u_{\lambda,m}f\,dx\\ \leqslant \tfrac{1}{2}\int_{\Omega_m} |\nabla V|^2\,dx+\frac{c}{2}\int_{\Omega_m} |V|^2\,dx+\int_{\Gamma_m} j_\lambda(\gamma_m V)\,d\sigma_m-\int_\Omega Vf\,dx\end{aligned} \tag{3,2,3,5}$$

for all $V\in H^1(\mathbb{R}^n)$. It is easy to check that

$$\tfrac{1}{2}\int_{\Omega_m} |\nabla V|^2\,dx+\frac{c}{2}\int_{\Omega_m} |V|^2\,dx \to \tfrac{1}{2}\int_{\Omega} |\nabla V|^2\,dx+\frac{c}{2}\int_{\Omega} |V|^2\,dx.$$

Then obviously we have

$$\begin{aligned}\liminf_{m\to\infty}\int_{\Omega_m} |\nabla u_{\lambda,m}|^2\,dx \geqslant \lim_{m\to\infty}\int_\Omega |\nabla u_{\lambda,m}|^2\,dx=\int_\Omega |\nabla u_\lambda|^2\,dx\\ \liminf_{m\to\infty}\int_{\Omega_m} |u_{\lambda,m}|^2\,dx \geqslant \lim_{m\to\infty}\int_\Omega |u_{\lambda,m}|^2\,dx=\int_\Omega |u_\lambda|^2\,dx\end{aligned}$$

since $\Omega_m \supseteq \Omega$. Also, we have

$$\int_\Omega fu_{m,\lambda}\,dx \to \int_\Omega fu_\lambda\,dx.$$

Thus the only difficult point in taking the limit in (3,2,3,5), is to prove that

$$\int_{\Gamma_m} j_\lambda(\gamma_m V)\,d\sigma_m \to \int_\Gamma j_\lambda(\gamma V)\,d\sigma \tag{3,2,3,6}$$

$$\liminf_{m\to\infty}\int_{\Gamma_m} j_\lambda(\gamma_m u_{\lambda,m})\,d\sigma_m \geqslant \int_\Gamma j_\lambda(\gamma u_\lambda)\,d\sigma. \tag{3,2,3,7}$$

For this purpose, we fix a partition of unity on Γ and Γ_m corresponding to the covering V_k, $1\leqslant k\leqslant K$, introduced in Lemma 3.2.3.2, i.e. we consider $\theta_k\in\mathcal{D}(\mathbb{R}^n)$, $1\leqslant k\leqslant K$, such that θ_k has its support in V_k and

$$1=\sum_{k=1}^K \theta_k(x)$$

for all $x\in\Gamma$ and all $x\in\Gamma_m$ (for m large enough). We have to prove that

$$\int_{\Gamma_m} \theta_k j_\lambda(\gamma_m V)\,d\sigma_m \to \int_\Gamma \theta_k j_\lambda(\gamma V)\,d\sigma. \tag{3,2,3,8}$$

We drop the index k and set $\eta = \theta j_\lambda(V)$. It follows that

$$\int_{\Gamma_m} \gamma_m \eta \, d\sigma_m = \int_{V'} (\gamma_m \eta)(z, \varphi_m(z))[1 + |\nabla \varphi_m(z)|^2]^{1/2} \, dz.$$

Clearly we have

$$(\gamma_m \eta)(z, \varphi_m(z))[1 + |\nabla \varphi_m(z)|^2]^{1/2} \to (\gamma \eta)(z, \varphi(z))[1 + |\nabla \varphi(z)|^2]^{1/2}$$

a.e. in V'. In addition, we have

$$\begin{aligned} &|(\gamma_m \eta)(z, \varphi_m(z))| \, [1 + |\nabla \varphi_m(z)|^2]^{1/2} \\ &\quad \le [1 + L^2]^{1/2} \Big\{ |(\gamma \eta)(z, \varphi(z))| + [\varphi_m(z) - \varphi(z)]^{1/2} \\ &\quad \times \Big[\int_{-a_n/2}^{a_n/2} |D_n \eta(z, y)|^2 \, dy \Big]^{1/2} \Big\}. \end{aligned}$$

Thus we have a fixed square integrable bound, and applying Lebesgue's dominated convergence theorem, we conclude that (3,2,3,8) holds.

To prove (3,2,3,7) we introduce $U_\lambda \in H^2(\mathbb{R}^n)$ such that $U_\lambda|_\Omega = u_\lambda$. Then, we have

$$\begin{aligned} &\int_{\Gamma_m} j_\lambda(\gamma_m u_{\lambda,m}) \, d\sigma_m - \int_\Gamma j_\lambda(\gamma u_\lambda) \, d\sigma \\ &\quad = \int_{\Gamma_m} \{ j_\lambda(\gamma_m u_{\lambda,m}) - j_\lambda(\gamma_m U_\lambda) \} \, d\sigma_m \\ &\quad + \int_{\Gamma_m} j_\lambda(\gamma_m U_\lambda) \, d\sigma_m - \int_\Gamma j_\lambda(\gamma u_\lambda) \, d\sigma. \end{aligned}$$

It follows from (3,2,3,6) that

$$\int_{\Gamma_m} j_\lambda(\gamma_m U_\lambda) \, d\sigma_m \to \int_\Gamma j_\lambda(\gamma u_\lambda) \, d\sigma.$$

Then we observe that for $x, y \in \mathbb{R}$ we have

$$j_\lambda(x) - j_\lambda(y) \ge \beta_\lambda(y)(x - y)$$

and thus

$$\int_{\Gamma_m} \{ j_\lambda(\gamma_m u_{\lambda,m}) - j_\lambda(\gamma_m U_\lambda) \} \, d\sigma_m \ge \int_{\Gamma_m} \beta_\lambda(\gamma_m U_\lambda) \{ \gamma_m u_{\lambda,m} - \gamma_m U_\lambda \} \, d\sigma_m. \tag{3,2,3,9}$$

We shall show that the right-hand side of this inequality converges to zero. Indeed, we have

$$\int_{\Gamma_m} |\beta_\lambda(\gamma_m U_\lambda)|^2 \, d\sigma_m \le \frac{1}{\lambda^2} \int_{\Gamma_m} |\gamma_m U_\lambda|^2 \, d\sigma_m \le \frac{K}{\lambda^2} \int_{\Omega_m} [|\nabla U_\lambda|^2 + |U_\lambda|^2] \, dx$$

with a constant K which does not depend on m, due to Lemma 3.2.3.3 below and Theorem 1.5.1.10.

Lemma 3.2.3.3 *Under the assumptions of Lemma 3.2.3.2, there exists a Lipschitz vector field* $\boldsymbol{\mu}$ *defined on* $\mathbb{R}^n$ *and a constant* $\delta > 0$ *such that* $\boldsymbol{\mu} \cdot \boldsymbol{\nu}_m \geq \delta$ *on* Γ_m *for all* m.

We postpone the proof of this lemma until the proof of Theorem 3.2.3.1 is completed.

This shows that $\|\beta_\lambda(\gamma_m U_\lambda)\|_{0,2,\Gamma_m}$ remains bounded when $m \to \infty$. Then let us set

$$\eta_m = \theta_k(u_{\lambda,m} - U_\lambda).$$

Use local coordinates and drop k. We have

$$\int_{\Gamma_m} |\theta\{\gamma_m u_{\lambda,m} - \gamma_m U_\lambda\}|^2 \, d\sigma_m = \int_{V'} |\eta_m(z, \varphi_m(z))|^2 [1 + |\nabla \varphi_m(z)|^2]^{1/2} \, dz.$$

We write

$$\eta_m(z, \varphi_m(z)) = \eta_m(z, \varphi(z)) + \int_{\varphi(z)}^{\varphi_m(z)} D_n \eta_m(z, y) \, dy$$

and consequently

$$|\eta_m(z, \varphi_m(z))| \leq |\eta_m(z, \varphi(z))| + [\varphi_m(z) - \varphi(z)]^{1/2} \left(\int_{\varphi(z)}^{\varphi_m(z)} |D_n \eta_m(z, y)|^2 \, dy \right)^{1/2}.$$

Thus we have

$$\begin{aligned} &\left\{ \int_{V'} |\eta_m(z, \varphi_m(z))|^2 [1 + |\nabla \varphi_m(z)|^2]^{1/2} \, dz \right\}^{1/2} \\ &\quad \leq [1 + L^2]^{1/4} \left\{ \int_{V'} |\eta_m(z, \varphi(z))|^2 [1 + |\nabla \varphi(z)|^2]^{1/2} \, dz \right\}^{1/2} \\ &\quad\quad + [1 + L^2]^{1/4} \max_{z \in V'} [\varphi_m(z) - \varphi(z)]^{1/2} \\ &\quad\quad \times \left\{ \int_{V'} \int_{\varphi(z)}^{\varphi_m(z)} |D_n \eta_m(z, y)|^2 \, dz \, dy \right\}^{1/2}. \end{aligned}$$

In other words

$$\left\{\int_{\Gamma_m} |\theta\gamma_m(u_{\lambda,m} - U_\lambda)|^2 \, d\sigma_m\right\}^{1/2}$$
$$\leqslant [1+L^2]^{1/4}\left(\left\{\int_\Gamma |\theta\gamma(u_{\lambda,m} - U_\lambda)|^2 \, d\sigma\right\}^{1/2}\right.$$
$$\left. + \max_{z\in V'} [\varphi_m(z) - \varphi(z)]^{1/2} \|\theta(u_{\lambda,m} - U_\lambda)\|_{1,2,\Omega_m}\right).$$

This shows that

$$\int_{\Gamma_m} |\gamma_m(u_{\lambda,m} - U_\lambda)|^2 \, d\sigma_m \to 0$$

when $m \to \infty$ since $u_{\lambda,m} \to U_\lambda$ in $H^1(\Omega)$ and $\|u_{\lambda,m}\|_{1,2,\Omega_m}$ remains bounded. Summing up we have proved that

$$\int_{\Gamma_m} \beta_\lambda(\gamma_m U_\lambda)\{\gamma_m u_{\lambda,m} - \gamma_m U_\lambda\} \, d\sigma_m \to 0$$

and remembering (3,3,2,9), this implies (3,2,3,7).

Taking the infimum limit in m, of inequality (3,2,3,5), we eventually obtain

$$\psi_\lambda(u_\lambda) \leqslant \psi_\lambda(v) \tag{3,2,3,10}$$

for all $v \in H^1(\Omega)$, where ψ_λ is defined by

$$\psi_\lambda(v) = \tfrac{1}{2}\int_\Omega |\nabla v|^2 \, dx + \frac{c}{2}\int_\Omega |v|^2 \, dx + \int_\Gamma j_\lambda(\gamma v) \, d\sigma - \int_\Omega v f \, dx.$$

In addition, taking the limit in (3,2,3,4) we also have

$$\|u_\lambda\|_{2,2,\Omega} \leqslant \left(\frac{1}{c^2} + \frac{1}{c} + 4\right)^{1/2} \|f\|_{0,2,\Omega}. \tag{3,2,3,11}$$

We can now perform the *last step* of the proof. Due to (3,2,3,11) we can find a sequence λ_j, $j = 1, 2, \ldots$ such that

$$\lambda_j \to 0, \qquad j \to \infty$$

and there exists $u \in H^2(\Omega)$ such that

$$u_{\lambda_j} \to u, \qquad j \to \infty$$

weakly in $H^2(\Omega)$ and consequently strongly in $H^{2-\varepsilon}(\Omega)$ for $\varepsilon > 0$. In addition, by the Lebesgue theorem on subsequences, we can also assume that

$$\begin{cases} \gamma u_{\lambda_j} \to \gamma u, & j \to \infty \\ \boldsymbol{\nu} \cdot \gamma \nabla u_{\lambda_j} \to \boldsymbol{\nu} \cdot \gamma \nabla u, & j \to \infty \end{cases}$$

a.e. on Γ.

On the other hand, as in the first step of the proof, inequality (3,2,3,10) implies that

$$\begin{cases} -\Delta u_\lambda + c u_\lambda = f & \text{in } \Omega \\ -\boldsymbol{\nu} \cdot \gamma \nabla u_\lambda = \beta_\lambda(\gamma u_\lambda) & \text{a.e. on } \Gamma. \end{cases} \tag{3,2,3,12}$$

It is easy to take the limit in this equation. We thus obviously obtain

$$-\Delta u + cu = f \qquad \text{in } \Omega.$$

To take the limit in the boundary condition, we use the following trick.

Lemma 3.2.3.4 *Let β be a maximal monotone operator in $\mathbb{R}$ and let β_λ be its Yosida approximation. Let x_m, y_m, λ_m be three sequences of real numbers such that*

$$\begin{cases} x_m \to x, \; y_m \to y \\ \lambda_m \to 0 \\ y_m = \beta_{\lambda_m}(x_m). \end{cases}$$

Then

$$y \in \beta(x).$$

We consequently obtain

$$-\boldsymbol{\nu} \cdot \gamma \nabla u \in \beta(\gamma u)$$

a.e. on Γ. Summing up, $u \in H^2(\Omega)$ is the solution of problem (3,2,2,5). The uniqueness of u follows from Lemma 3.2.2.2 since the functional which is minimized there is strictly convex (see also Lemma 3.2.2.3). This completes the proof of Theorem 3.2.3.1. ■

Proof of Lemma 3.2.3.3 We fix a point $x_0 \in \Omega$ and a ball B of radius $\rho > 0$ and centre at x_0 such that $B \subseteq \Omega$. Then we fix a function $\theta \in \mathcal{D}(\mathbb{R}^n)$ such that $\theta \equiv 1$ on all Ω_m for large enough m. Then we can define $\boldsymbol{\mu}$ by

$$\boldsymbol{\mu}(x) = \theta(x)(\mathbf{x} - \mathbf{x}_0).$$

It is clear that the angle of $\boldsymbol{\mu}$ with $\boldsymbol{\nu}_m$ is less than or equal to

$$\left\{\frac{\pi}{2} - \arcsin \frac{\rho}{|x - x_0|}\right\}$$

at $x \in \Gamma_m$. Consequently we have

$$\boldsymbol{\mu} \cdot \boldsymbol{\nu}_m \geqslant |\boldsymbol{\mu}| \frac{\rho}{|x - x_0|}.$$

Thus a $\delta > 0$ obviously exists and on the other hand, it is easy to check that $\boldsymbol{\mu}$ is Lipschitz continuous. ■

Proof of Lemma 3.2.3.4 Using the definition of β_λ it is easy to check that

$$y_m \in \beta(x_m - \lambda_m y_m).$$

Since $y_m \to y$ and $x_m - \lambda_m y_m \to x$, it follows that

$$y \in \beta(x)$$

since the graph of β is closed (by maximality). ■

3.2.4 Oblique boundary conditions

Here, for the sake of simplicity, we shall restrict our purpose to boundary value problems in a plane domain Ω. Thus let us assume that Ω is a bounded convex open subset of $\mathbb{R}^2$; its boundary is a closed Lipschitz curve Γ along which the arc-length s is well defined. We assume, in addition, that c is a given Lipschitz function in $\bar{\Omega}$. We shall solve the following problem: for a given $f \in L_2(\Omega)$ and $\lambda > 0$, find $u \in H^2(\Omega)$ which is a solution of

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } \Omega \\ -\gamma \dfrac{\partial u}{\partial \nu} = c \dfrac{\partial}{\partial s} \gamma u & \text{on } \Gamma \end{cases} \tag{3,2,4,1}$$

The main result below is due to Moussaoui (1974).

Theorem 3.2.4.1 *Let Ω be a bounded convex open subset of $\mathbb{R}^2$ and let $c \in C^{0,1}(\bar{\Omega})$. Then there exists λ_0 such that for each $\lambda > \lambda_0$ and for each $f \in L_2(\Omega)$ there exists a unique $u \in H^2(\Omega)$ which is a solution of*

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \lambda \int_\Omega uv \, dx = \int_\Omega fv \, dx - \left\langle c \frac{\partial}{\partial s} \gamma u; \gamma v \right\rangle \tag{3,2,4,2}$$

for all $v \in H^1(\Omega)$.

Of course, identity (3,2,4,2) is a weak form for problem (3,2,4,1). Indeed, applying the Green formula of Theorem 1.5.3.1, it is easy to check that for $u \in H^2(\Omega)$, (3,2,4,1) and (3,2,4,2) are equivalent.

Exactly as we did in the proof of Theorem 3.2.3.1, we shall approximate Ω by a decreasing sequence of smoother convex domains Ω_m, $m = 1, 2, \ldots$ (see Lemma 3.2.3.2). In each of these domains, we shall solve a problem similar to (3,2,4,1). Then it will be possible to take the limit in m with the help of an *a priori* estimate. Let us first prove this estimate.

Theorem 3.2.4.2 *Let Ω be a bounded convex open subset of $\mathbb{R}^2$ with a C^2 boundary. Let $c \in C^{0,1}(\bar{\Omega})$ and $\boldsymbol{\mu} \in C^{0,1}(\bar{\Omega})^2$ be such that $\boldsymbol{\mu} \cdot \boldsymbol{\nu} \geqslant \delta > 0$ everywhere on Γ; then there exists λ_0 and k such that*

$$\|u\|_{2,2,\Omega} \leqslant k \, \|-\Delta u + \lambda u\|_{0,2,\Omega} \tag{3,2,4,3}$$

for all $u \in H^2(\Omega)$, such that $-\gamma \, \partial u/\partial \nu = c \, \partial(\gamma u)/\partial s$ on Γ. Moreover, k and λ_0 depend only on δ, the Lipschitz norm of $\boldsymbol{\mu}$ and $\max_\Gamma |\partial c/\partial s|$.

Note that the existence of $\boldsymbol{\mu}$ follows from Lemma 1.5.1.9.

Proof We apply again identity (3,1,1,1) to $\mathbf{v} = \nabla u$. The two-dimensional version of this identity and the convexity of Ω, imply that†

$$\int_\Omega |\operatorname{div} \mathbf{v}|^2 \, dx - \sum_{i,j=1}^{2} \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} \, dx \geqslant -2 \int_\Gamma (\gamma \mathbf{v})_T \, d(\gamma \mathbf{v})_\nu.$$

In other words, we have

$$\int_\Omega |\Delta u|^2 \, dx \quad \sum_{i,j=1}^{2} \int_\Omega \left| \frac{\partial^2 u}{\partial x_i \, \partial x_j} \right|^2 dx \geqslant -2 \int_\Gamma \frac{\partial}{\partial s} (\gamma u) \, d\left(\gamma \frac{\partial u}{\partial \nu}\right).$$

Due to the boundary condition, we have

$$-2 \int_\Gamma \frac{\partial}{\partial s} (\gamma u) \, d\left(\gamma \frac{\partial u}{\partial \nu}\right) = 2 \int_\Gamma \frac{\partial}{\partial s} (\gamma u) \, d\left(c \frac{\partial}{\partial s} \gamma u\right) = \int_\Gamma \left| \frac{\partial}{\partial s} \gamma u \right|^2 dc.$$

Denoting by M an upper bound for $|\partial c/\partial s|$ on Γ, we have

$$\sum_{i,j=1}^{2} \int_\Omega \left| \frac{\partial^2 u}{\partial x_i \, \partial x_j} \right|^2 dx \leqslant \int_\Omega |\Delta u|^2 \, dx + M \int_\Gamma \left| \frac{\partial}{\partial s} \gamma u \right|^2 ds. \tag{3,2,4,4}$$

Besides this, we have as usual

$$\int_\Omega (-\Delta u + \lambda u) u \, dx = \int_\Omega |\nabla u|^2 \, dx + \lambda \int_\Omega |u|^2 \, dx - \int_\Gamma \gamma \frac{\partial u}{\partial \nu} \gamma u \, dx.$$

It follows from the boundary condition that

$$-\int_\Gamma \gamma \frac{\partial u}{\partial \nu} \gamma u \, ds = \int_\Gamma c \left(\frac{\partial}{\partial s} \gamma u \right) \gamma u \, ds = -\tfrac{1}{2} \int_\Gamma \frac{\partial c}{\partial s} |\gamma u|^2 \, ds$$

† For any vector field $\mathbf{a}$, a_ν and a_T are the normal and tangential components of $\mathbf{a}$ on Γ.

and consequently

$$\begin{aligned}
&\int_\Omega |\nabla u|^2\,dx + \lambda\int_\Omega |u|^2\,dx \\
&= \int_\Omega (-\Delta u + \lambda u)u\,dx + \tfrac{1}{2}\int_\Gamma \frac{\partial c}{\partial s}|\gamma u|^2\,ds \\
&\leq \|-\Delta u + \lambda u\|\,\|u\| + M\int_\Gamma |\gamma u|^2\,d\sigma \\
&\leq \|-\Delta u + \lambda u\|\,\|u\| + MK\sqrt{\varepsilon}\int_\Omega |\nabla u|^2\,dx + \frac{MK}{\sqrt{\varepsilon}}\int_\Omega |u|^2\,dx
\end{aligned}$$

owing to inequality (1,5,1,2) of Lemma 1.5.1.10. Choosing ε small enough (e.g. such that $Mk\sqrt{\varepsilon} \leq \frac{1}{2}$ and $\lambda_0 \geq (MK/\sqrt{\varepsilon}) + 1$, we finally obtain

$$\tfrac{1}{2}\|\nabla u\|^2 + \|u\|^2 \leq \|-\Delta u + \lambda u\|^2. \tag{3,2,4,5}$$

On the other hand, we have

$$\left|\frac{\partial}{\partial s}\gamma u\right| \leq |\gamma\,\nabla u|$$

consequently it follows from inequality (3,2,4,4) that

$$\begin{aligned}
\sum_{i,j=1}^{2}\int_\Omega \left|\frac{\partial^2 u}{\partial x_i\,\partial x_j}\right|^2 dx &\leq \int_\Omega |\Delta u|^2\,dx + M\int_\Gamma |\gamma\,\nabla u|^2\,ds \\
&\leq \int_\Omega |\Delta u|^2\,dx + MK\sqrt{\varepsilon}\sum_{i,j=1}^{2}\int_\Omega \left|\frac{\partial^2 u}{\partial x_i\,\partial x_j}\right|^2 dx + \frac{MK}{\sqrt{\varepsilon}}\|\nabla u\|^2
\end{aligned}$$

by a new application of Theorem 1.5.1.10. Then we have

$$\sum_{i,j=1}^{2}\int_\Omega \left|\frac{\partial^2 u}{\partial x_i\,\partial x_j}\right|^2 dx \leq 2\int_\Omega |\Delta u|^2\,dx + L\,\|\nabla u\|^2 \tag{3,2,4,6}$$

where $L = MK/\sqrt{\varepsilon}$, $MK\sqrt{\varepsilon} = \frac{1}{2}$.

Combining inequalities (3,2,4,5) and (3,2,4,6), we plainly obtain the desired result. ■

Remark 3.2.4.3 The same proof can be worked out, without the convexity assumption on Ω; of course in that case, the constants k and λ_0 will also depend on the curvature of Γ.

Remark 3.2.4.4 We can combine the two kinds of computations that we did in the proof of Theorems 3.1.2.3 and 3.2.4.2 to deal with the more complicated boundary condition

$$-\gamma\frac{\partial u}{\partial \nu} = c\frac{\partial}{\partial s}\gamma u + \beta(\gamma u)$$

assuming that β is a nondecreasing Lipschitz function. Unfortunately, we shall not be able to take advantage of such a result in what follows.

We now turn to the

Proof of Theorem 3.2.4.1 We use a sequence of plane convex domains Ω_m, $m = 1, 2, \ldots$, as in Lemma 3.2.3.2. We extend c to the whole plane in such a fashion that

$$c \in C^{0,1}(\mathbb{R}^2).$$

Consequently c is defined on Γ_m. Then, in each Ω_m, we consider $u_m \in H^2(\Omega_m)$, a solution of

$$\begin{cases} -\Delta u_m + \lambda u_m = \tilde{f} & \text{in } \Omega_m \\ -\gamma_m \dfrac{\partial u_m}{\partial \nu_m} = c \dfrac{\partial}{\partial s_m} \gamma_m u_m & \text{on } \Gamma_m. \end{cases} \tag{3,2,4,7}$$

The existence of such a function u_m follows from Theorem 2.4.2.7 for $\lambda > 0$. We shall now use inequality (3,2,4,3) to show that $\|u_m\|_{2,2,\Omega}$ remains bounded when $m \to \infty$.

Here, we take advantage of Lemma 3.2.3.3 again. Obviously $\max_{\Gamma_m} |\partial c/\partial s_m|$ remains bounded uniformly in m. This implies the existence of constants k and λ_0, both independent of m, such that

$$\|u_m\|_{2,2,\Omega_m} \leq k \, \|f\|_{0,2,\Omega_m} \tag{3,2,4,8}$$

provided $\lambda \geq \lambda_0$.

Now we proceed as in the proof of Theorem 3.2.3.1. We first observe that (3,2,4,7) implies this:

$$\int_{\Omega_m} \nabla u_m \cdot \nabla V \, dx + \lambda \int_{\Omega_m} u_m V \, dx = \int_\Omega f V \, dx - \int_{\Gamma_m} c \frac{\partial}{\partial s_m} \gamma_m u_m \gamma_m V \, ds_m \tag{3,2,4,9}$$

for all $V \in H^1(\Omega_m)$.

On the other hand by inequality (3,2,4,8), there exists a subsequence of the sequence u_m, $m = 1, 2, \ldots$ which is weakly convergent in $H^2(\Omega)$ to some $u \in H^2(\Omega)$. Let us again denote this subsequence u_m, $m = 1, 2, \ldots$ for the sake of avoiding further complications in the notation. We shall show that u is the desired solution of (3,2,4,2); this will be achieved by taking the limit in identity (3,2,4,9).

First let $V \in C^1(\mathbb{R}^2)$ be fixed. We shall set $v = V|_\Omega$. Exactly as in the proof of Theorem 3.2.1.3 we show that

$$\int_{\Omega_m} \nabla u_m \cdot \nabla V \, dx \to \int_\Omega \nabla u \cdot \nabla v \, dx \tag{3,2,4,10}$$

and that

$$\int_{\Omega_m} u_m V \, dx \to \int_\Omega uv \, dx. \tag{3,2,4,11}$$

To complete the proof, we shall show that

$$\int_{\Gamma_m} c \frac{\partial}{\partial s_m} \gamma_m u_m \gamma_m V \, ds_m = \int_{\Gamma_m} c(\gamma_m \nabla u_m)_{T_m} \gamma_m V \, ds_m$$
$$\to \int_\Gamma c(\gamma \nabla u)_T \gamma v \, ds = \left\langle c \frac{\partial}{\partial s} \gamma u; \gamma v \right\rangle. \tag{3,2,4,12}$$

This requires much more care. First we fix a partition of unity (θ_k, $1 \leq k \leq K$) on Γ and Γ_m corresponding to the covering V_k, $1 \leq k \leq K$ in Lemma 3.2.3.2. The limit in (3,2,4,12) will follow by adding these limits:

$$\int_{\Gamma_m} \theta_k c(\gamma_m \nabla u_m)_{T_m} V \, ds_m \to \int_\Gamma \theta_k c(\gamma \nabla u)_T v \, ds. \tag{3,2,4,13}$$

We shall use the local coordinates of Lemma 3.2.3.2 in each V_k. Let us first introduce some auxiliary notation:

$$\mathbf{w}_m = \nabla u_m, \qquad \eta = \theta_k c V.$$

From now on, we shall drop the subscript k to simplify the notation. In addition, we consider a function $U \in H^2(\mathbb{R}^2)$ such that $U|_\Omega = u$ (see Theorem 1.4.3.1) we shall consider separately

$$\int_{\Gamma_m} \eta[\gamma_m(\mathbf{w}_m - \mathbf{W})]_{T_m} \, ds_m$$

and

$$\int_{\Gamma_m} \eta[\gamma_m \mathbf{W}]_{T_m} \, ds_m - \int_\Gamma \eta[\gamma \mathbf{W}]_T \, ds$$

where $\mathbf{W} = \nabla U$.

According to the notation in Lemma 3.2.3.2, we have (after dropping the k):

$$\left| \int_{\Gamma_m} \eta[\gamma_m(\mathbf{w}_m - \mathbf{W})]_{T_m} \, ds_m \right| \leq a \int_{\Gamma_m \cap V} |\gamma_m(\mathbf{w}_m - \mathbf{W})| \, ds_m$$
$$= a \int_{V'} |\gamma_m(\mathbf{w}_m - \mathbf{W})(z, \varphi_m(z))| \times \sqrt{(1 + \varphi_m'(z)^2)} \, dz$$

where a does not depend on m. Owing to (3,2,3,2) we have

$$\left|\int_{\Gamma_m} \eta[\gamma_m(\mathbf{w}_m-\mathbf{W})]_{T_m}\, ds_m\right|$$
$$\leqslant a\sqrt{(1+L^2)}\left\{\int_{V'} |\gamma(\mathbf{w}_m-\mathbf{W})(z,\varphi(z))|\, dz\right.$$
$$\left.+\int_{V'} |\gamma_m(\mathbf{w}_m-\mathbf{W})(z,\varphi_m(z))-\gamma(\mathbf{w}_m-\mathbf{W})(z,\varphi(z))|\, dz\right\}$$
$$\leqslant b\,\|\gamma(\nabla u_m-\nabla u)\|_{0,2,\Gamma}+a\sqrt{(1+L^2)}$$
$$\times\int_{V'}\left\{\int_{\varphi(z)}^{\varphi_m(z)} |D_y(\mathbf{w}_m-\mathbf{W})(z,y)|\, dy\right\}dz$$
$$\leqslant b\,\|\gamma(\nabla u_m-\nabla u)\|_{0,2,\Gamma}+\max_{z\in V'}|\varphi_m(z)-\varphi(z)|^{1/2}\, b\,\|u_m-U\|_{2,2,\Omega_m}.$$

This implies that

$$\int_{\Gamma_m} \eta[\gamma_m(\mathbf{w}_m-\mathbf{W})]_{T_m}\, ds_m \to 0 \tag{3,2,4,14}$$

since $u_m\to u$ weakly in $H^2(\Omega)$ and $\varphi_m\to\varphi$ uniformly in V'.

Next we have

$$\int_{\Gamma_m} \eta(\gamma_m\mathbf{W})_{T_m} ds_m=\int_{\Gamma_m} \eta\boldsymbol{\tau}_m\cdot\gamma_m\mathbf{W}\, ds_m$$
$$=\int_{V'}[\eta\boldsymbol{\tau}_m\cdot\gamma_m\mathbf{W}](z,\varphi_m(z))\sqrt{[1+\varphi_m'(z)^2]}\, dz$$
$$=\int_{V'}\eta(z,\varphi_m(z))[\gamma_m D_1U(z,\varphi_m(z))$$
$$+\varphi_m'(z)\gamma_m D_2U(z,\varphi_m(z))]\, dz. \tag{3,2,4,15}$$

Obviously $\eta(z,\varphi_m(z))$ converges uniformly to $\eta(z,\varphi(z))$ in V'. On the other hand, we have

$$|\gamma_m\mathbf{W}(z,\varphi_m(z))-\gamma\mathbf{W}(z,\varphi(z))|\leqslant\int_{\varphi(z)}^{\varphi_m(z)}|D_2\mathbf{W}(z,y)|\, dy$$
$$\leqslant|\varphi_m(z)-\varphi(z)|^{1/2}$$
$$\times\left\{\int_{-a_n/2}^{a_n/2}|D_2\mathbf{W}(z,y)|^2\, dy\right\}^{1/2}$$

and consequently $\gamma_m\mathbf{W}(z,\varphi_m(z))$ converges to $\gamma\mathbf{W}(z,\varphi(z))$ almost everywhere in V'. Summing up this shows that the integrand in (3,2,4,15) has a limit almost everywhere in V'.

In addition, we have

$$|\gamma_m \mathbf{W}(z, \varphi_m(z))| \leq |\gamma \mathbf{W}(z, \varphi(z))| + |\varphi_m(z) - \varphi(z)|^{1/2} \times \left\{ \int_{-a_n/2}^{a_n/2} |D_2 \mathbf{W}(z, y)|^2 \, dy \right\}^{1/2}.$$

Thus the integrand in (3,2,4,15) is bounded by a fixed integrable function on V'. One of Lebesgue's theorems implies, therefore, that

$$\int_{\Gamma_m} \eta(\gamma_m \mathbf{W})_{T_m} \, ds_m \to \int_\Gamma \eta(\gamma \mathbf{W})_T \, ds. \tag{3,2,4,16}$$

Now (3,2,4,13) follows from (3,2,4,14) and (3,2,4,16). Accordingly, we have shown that (3,2,4,2) holds for all $v \in C^1(\bar{\Omega})$. This identity is extended to all $v \in H^1(\Omega)$ by density. Finally, the uniqueness of u follows from (3,2,4,2) by substituting u for v and assuming λ large enough. ■

Remark 3.2.4.5 We can make the lower bound λ_0 more precise in the statement of Theorem 3.2.4.1. Indeed, let us set $v = u$ in (3,2,4,2); thus we obtain

$$\int_\Omega |\nabla u|^2 \, dx + \lambda \int_\Omega |u|^2 \, dx = \int_\Omega fu \, dx + \tfrac{1}{2} \int_\Gamma \frac{\partial c}{\partial s} |\gamma u|^2 \, ds.$$

Then let K be the best possible constant in inequality (1,5,1,2) and set

$$M = \max_\Gamma \frac{\partial c}{\partial s};$$

it follows that

$$\int_\Omega |\nabla u|^2 \, dx + \lambda \int_\Omega |u|^2 \, dx \leq \int_\Omega fu \, dx + \frac{KM}{2} \times \left\{ \sqrt{\varepsilon} \int_\Omega |\nabla u|^2 \, dx + \frac{1}{\sqrt{\varepsilon}} \int_\Omega |u|^2 \, dx \right\}.$$

Choosing $\sqrt{\varepsilon} = 2/KM$, we obtain

$$\left\{ \lambda - \left(\frac{KM}{2} \right)^2 \right\} \int_\Omega |u|^2 \, dx \leq \int_\Omega fu \, dx$$

accordingly we have

$$\left\{ \lambda - \left(\frac{KM}{2} \right)^2 \right\} \|u\| \leq \|f\|$$

and this shows that $(KM/2)^2$ is a possible value for λ_0.

Remark 3.2.4.6 In all the previous results the convexity of Ω can be replaced by the weaker assumptions that there exists a sequence Ω_m, $m = 1, 2, \ldots$ of bounded open subsets of $\mathbb{R}^n$, with C^2 boundaries Γ_m such that $d(\Gamma_m, \Gamma) \to 0$ as $u \to +\infty$, the sequence of the corresponding second fundamental forms $\mathcal{B}_m$, $m = 1, 2, \ldots$ is uniformly bounded from above independently of m, and $\Omega_m \subseteq \Omega$ to solve the Dirichlet problem or $\Omega_m \supseteq \Omega$ to solve the other boundary value problems. This assumption is obviously fulfilled when Ω is a bounded open subset of $\mathbb{R}^2$ whose boundary is a curvilinear polygon of class $C^{1,1}$, provided all the angles are strictly convex.

3.3 Boundary value problems in domains with turning points

As we observed in Remark 3.2.4.6, the good domains Ω for the regularity in $H^2(\Omega)$ are those which are piecewise C^2 with convex corners. This is an upper bound on the measure of the possible angles and this leads naturally to the question whether turning points (i.e. angles with measure zero) allow the solution of an elliptic boundary value problem to belong to $H^2(\Omega)$. The answer is yes for several boundary conditions as it is shown in Khelif (1978).

We shall consider here only the simplest problem, namely the Dirichlet problem for the Laplace equation in a plane domain Ω with a boundary Γ which is C^2 everywhere except in one point, which is a turning point. To be more precise we assume that this turning point is at O and that there exists $\rho > 0$ such that, denoting by V the disc with centre at zero and radius ρ, we have

$$V \cap \Omega = \{(x, y) \in V; x > 0, \varphi_1(x) < y < \varphi_2(x)\}$$

where φ_1 and φ_2 are a pair of C^2 functions such that

$$\begin{cases} \varphi_1(0) = \varphi_2(0) = 0 \\ \varphi_1'(0) = \varphi_2'(0) = 0 \\ \varphi_1'(x) < \varphi_2'(x), \; 0 < x < \rho. \end{cases}$$

Thus, near the origin, the boundary of Ω is a pair of C^2 curves which meet at zero and which are tangent there to the positive half x-axis.

We are going to prove the following result of Ibuki (1974) applying the method of Khelif (1978) which is simpler and more general.

Theorem 3.3.1 *Given $f \in L_2(\Omega)$ there exists a unique $u \in H^2(\Omega) \cap \mathring{H}^1(\Omega)$ such that*

$$\Delta u = f$$

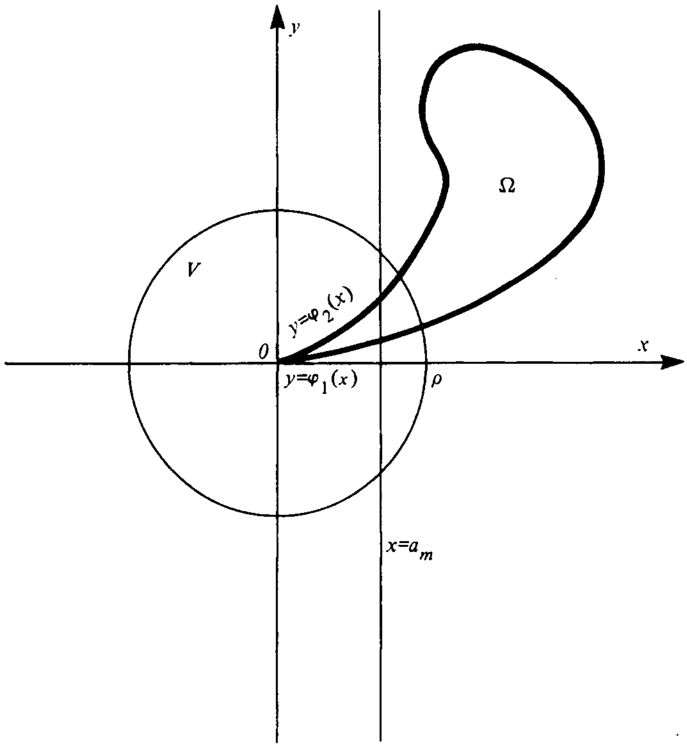

Figure 3.2

in Ω provided

$$\limsup_{x\to 0} \frac{(|\varphi_1''(x)|+|\varphi_2''(x)|)[\varphi_2(x)-\varphi_1(x)]}{[\varphi_2'(x)-\varphi_1'(x)]^2}<2$$

(*One can observe that those conditions are fulfilled in the following example:* $\varphi_1(x)=0$ *and* $\varphi_2(x)=x^a$, *where a is any real number* >1.)

Exactly as in the previous section the method consists in approximating Ω by a sequence Ω_m, $m=1,2,\ldots$, of 'better' domains. For this purpose we consider a decreasing sequence a_m, $m=1,2,\ldots$ of positive real numbers and we set

$$\Omega_m=\Omega\cap\{(x,y);\,x>a_m\}.$$

Clearly, we have

$$\Omega=\bigcup_{m=1}^{\infty}\Omega_m,$$

and each Ω_m has a piecewise C^2 boundary with two convex angles.

Consequently there exists a unique

$$u_m \in H^2(\Omega_m) \cap \mathring{H}^1(\Omega_m)$$

which is a solution of

$$\Delta u_m = f|_{\Omega_m}$$

in Ω_m. We are going to show that the sequence

$$\|u_m\|_{2,2,\Omega_m}$$

is bounded as $m \to +\infty$.

Lemma 3.3.2 *Under the assumptions of Theorem 3.3.1 there exists a constant K such that*

$$\|u_m\|_{2,2,\Omega_m} \leq K \|f\|_{0,2,\Omega} \tag{3,3,1}$$

for every m and every $f \in L_2(\Omega)$.

Proof Integration by parts of $(\Delta u_m)u_m$ and the Poincaré inequality (Theorem 1.4.3.4) yield a constant C_1 such that

$$\|u_m\|_{1,2,\Omega_m} \leq C_1 \|f\|_{0,2,\Omega}. \tag{3,3,2}$$

Then we apply Theorem 3.1.1.2 in order to bound the second derivatives of u_m. For this purpose, we assume that m is fixed and we set

$$v = D_x u_m, \qquad w = D_y u_m. \tag{3,3,3}$$

The functions v and w only belong to $H^1(\Omega_m)$ and we approximate them by functions belonging to $H^2(\Omega_m)$.

We observe that v and w fulfil the following boundary conditions:

$$\begin{cases} \gamma w(a_m, y) = 0, & \varphi_1(a_m) < y < \varphi_2(a_m) \\ \lambda\gamma v + \mu\gamma w = 0 & \text{on} \quad \Gamma \cap \{(x, y) \mid x > a_m\}, \end{cases}$$

where λ and μ are the components of the unit tangent vector to Γ. Accordingly, there exists a couple of sequences of functions v_k, w_k, $k = 1, 2, \ldots$, such that

$$v_k \to v, \qquad w_k \to w$$

in $H^1(\Omega_m)$ as $k \to +\infty$ and such that

$$\begin{cases} v_k, w_k \in H^2(\Omega_m) \\ \gamma w_k(a_m, y) = 0, & \varphi_1(a_m) < y < \varphi_2(a_m) \\ \lambda\gamma v_k + \mu\gamma w_k = 0 & \text{on} \quad \Gamma \cap \{(x, y) \mid x > a_m\}. \end{cases}$$

(We skip the proof of this density result due to its similarity to Lemma 4.3.1.3.)

Applying identity (3,1,1,10) to the vector function $\{v_k, w_k\}$, we obtain

$$\int_{\Omega_m} |D_x v_k + D_y w_k|^2 \, dx\, dy = \int_{\Omega_m} [|D_x v_k|^2 + |D_y w_k|^2 + 2D_x w_k D_y v_k] \, dx\, dy - \int_{\Gamma_m} (\operatorname{tr} \mathcal{B}) \, |\nu_1 \gamma v_k + \nu_2 \gamma w_k|^2 \, d\sigma.$$

Then, taking the limit in k, we have

$$\int_{\Omega_m} |D_x v + D_y w|^2 \, dx\, dy = \int_{\Omega_m} [|D_x v|^2 + |D_y w|^2 + 2D_x w D_y v] \, dx\, dy - \int_{\Gamma_m} (\operatorname{tr} \mathcal{B}) \, |\nu_1 \gamma_v + \nu_2 \gamma_w|^2 \, d\sigma$$

and consequently, using (3,3,3), we have

$$\int_{\Omega_m} |f|^2 \, dx\, dy = \int_{\Omega_m} [|D_x^2 u_m|^2 + |D_y^2 u_m|^2 + 2\,|D_x D_y u_m|^2] \, dx\, dy - \int_{\Gamma_m} (\operatorname{tr} \mathcal{B}) \left|\gamma \frac{\partial u_m}{\partial \nu}\right|^2 d\sigma. \tag{3,3,4}$$

Let us now consider the boundary integral in (3,3,4). The second fundamental form $\mathcal{B}$ vanishes on the segment

$$\{(a_m, y), \varphi_1(x) < y < \varphi_2(x)\}$$

which is curvature free. On the other hand the form $\mathcal{B}$ is bounded on each C^2 curve; in addition $\mathcal{B}$ is bounded by $|\varphi_j''(x)|$ at $(x, \varphi_j(x))$. Thus we shall consider differently the points of the boundary Γ_m according to their distance to the origin. For this purpose let $\delta > 0$ be small enough to ensure that the points $(\delta, \varphi_1(\delta))$ and $(\delta, \varphi_2(\delta))$ lie in V. Assuming that m is large enough to ensure $a_m < \delta$, there exists a constant M_2 such that

$$\int_{\Gamma_m} (\operatorname{tr} \mathcal{B}) \left|\gamma \frac{\partial u_m}{\partial \nu}\right|^2 d\sigma \le M_2 \int_{\Gamma_m \cap \{(x,y);x\ge\delta\}} \left|\gamma \frac{\partial u_m}{\partial \nu}\right|^2 d\sigma + \int_{a_m}^{\delta} \left|\gamma \frac{\partial u_m}{\partial \nu}(x, \varphi_1(x))\right|^2 |\varphi_1''(x)| \, dx + \int_{a_m}^{\delta} \left|\gamma \frac{\partial u_m}{\partial \nu}(x, \varphi_2(x))\right|^2 |\varphi_2''(x)| \, dx. \tag{3,3,5}$$

Since

$$\Omega_\delta = \Omega \cap \{(x, y) : x > \delta\}$$

has a Lipschitz boundary, Theorem 1.5.1.10 implies the existence of a constant K_δ such that

$$\int_{\Gamma_m\cap\{(x,y);x\geq\delta\}} \left|\gamma\frac{\partial u_m}{\partial\nu}\right|^2 d\sigma$$

$$\leq K_\delta\left\{\varepsilon^{1/2}\int_{\Omega_\delta}[|D_x^2u_m|^2+|D_y^2u_m|^2+2\,|D_xD_yu_m|^2]\,dx\,dy\right.$$

$$\left.+\varepsilon^{-1/2}\int_{\Omega_\delta}|\nabla u_m|^2\,dx\,dy\right\} \tag{3,3,6}$$

for every $\varepsilon>0$.

Next on the graph of φ_j $(j=1,2)$ we have

$$\gamma u_m(x,\varphi_j(x))=0 \tag{3,3,7}$$

and consequently

$$\gamma(D_xu_m)(x,\varphi_j(x))+\varphi_j'(x)\gamma(D_yu_m)(x,\varphi_j(x))=0$$

for a.e. x. On the other hand we have

$$\gamma\frac{\partial u_m}{\partial\nu}=\frac{-\varphi_2'\gamma D_xu_m+\gamma D_yu_m}{\sqrt{(1+\varphi_2'^2)}}$$

at $(x,\varphi_2(x))$. It follows that

$$\gamma\frac{\partial u_m}{\partial\nu}=\sqrt{(\varphi_2'^2+1)}\,\frac{\gamma D_xu_m+\varphi_1'\gamma D_yu_m}{\varphi_1'-\varphi_2'}$$

and there exists a constant N_δ such that

$$\left|\gamma\frac{\partial u_m}{\partial\gamma}\right|\leq N_\delta\frac{1}{(\varphi_2'-\varphi_1')}\,|\gamma D_xu_m+\varphi_1'\gamma D_yu_m|$$

for $a_m<x\leq\delta$ (and $N_\delta\to1$ as $\delta\to0$).

The boundary integral corresponding to the graph of φ_2 in (3,3,5) is bounded by

$$I=N_\delta\int_{a_m}^{\delta}|\gamma D_xu_m(x,\varphi_2(x))+\varphi_1'(x)D_yu_m(x,\varphi_2(x))|^2\frac{|\varphi_2''(x)|\,dx}{|\varphi_1'(x)-\varphi_2'(x)|^2}. \tag{3,3,8}$$

Since we have

$$\gamma D_xu_m(x,\varphi_1(x))+\varphi_1'(x)\gamma D_yu_m(x,\varphi_1(x))=0,$$

by (3,3,7), we can write

$$\gamma D_xu_m(x,\varphi_2(x))+\varphi_1'(x)\gamma D_yu_m(x,\varphi_2(x))$$

$$=\int_{\varphi_1(x)}^{\varphi_2(x)}D_y\{D_xu_m+\varphi_1'(x)D_yu_m\}\,dy.$$

Hence

$$
\begin{aligned}
&|\gamma D_x u_m(x, \varphi_2(x)) + \varphi_1'(x)\gamma D_y u_m(x, \varphi_2(x))| \\
&\quad \leq [\varphi_2(x) - \varphi_1(x)]^{1/2}\left[\int_{\varphi_1(x)}^{\varphi_2(x)} |D_y D_x u_m|^2 \, dy\right]^{1/2} \\
&\quad\quad + |\varphi_1'(x)|\,[\varphi_2(x) - \varphi_1(x)]^{1/2}\left[\int_{\varphi_1(x)}^{\varphi_2(x)} |D_y^2 u_m|^2 \, dy\right]^{1/2}
\end{aligned}
$$

Finally this implies that

$$
\begin{aligned}
I^{1/2} &\leq N_\delta^{1/2} \max_{0<x\leq\delta} \frac{(|\varphi_2''(x)|\,[\varphi_2(x) - \varphi_1(x)])^{1/2}}{|\varphi_1'(x) - \varphi_2'(x)|} \\
&\quad \times \left[\left\{\int_{a_m}^{\delta}\int_{\varphi_1(x)}^{\varphi_2(x)} |D_y D_x u_m|^2 \, dx\, dy\right\}^{1/2}\right. \\
&\quad \left. + \max_{0<x\leq\delta} |\varphi_1'(x)| \int_{a_m}^{\delta}\int_{\varphi_1(x)}^{\varphi_2(x)} |D_y^2 u_m|^2 \, dx\, dy\right\}^{1/2}\Bigg]. \qquad (3,3,9)
\end{aligned}
$$

We have a similar inequality for the boundary integral corresponding to φ_1. Summing up, there exists a constant M_2 such that

$$
\begin{aligned}
&\left|\int_{\Gamma_m} (\operatorname{tr} \mathscr{B}) \left|\gamma \frac{\partial u_m}{\partial \gamma}\right|^2 d\sigma\right| \\
&\quad \leq M_2 K_\delta \left\{\varepsilon^{1/2} \int_{\Omega_\delta} [|D_x^2 u_m|^2 + |D_y^2 u_m|^2 + 2\,|D_x D_y u_m|^2]\, dx\, dy\right. \\
&\quad\quad \left. + \varepsilon^{-1/2} \int_{\Omega_\delta} |\nabla u_m|^2 \, dx\, dy\right\} \\
&\quad\quad + N_\delta \max_{0<x\leq\delta} \frac{[|\varphi_1''(x)| + |\varphi_2''(x)|][\varphi_2(x) - \varphi_1(x)]}{|\varphi_1'(x) - \varphi_2'(x)|^2} \\
&\quad\quad \times \left\{(1+\eta)\int_{\Omega_m \setminus \Omega_\delta} |D_y D_x u_m|^2 \, dx\, dy\right. \\
&\quad\quad \left. + \left(1 + \frac{1}{\eta}\right) \max_{0<x\leq\delta} [|\varphi_1'(x)|^2 + |\varphi_2'(x)|^2] \int_{\Omega_m \setminus \Omega_\delta} |D_y^2 u_m|^2 \, dx\, dy\right\}
\end{aligned} \qquad (3,3,10)
$$

for every $\eta > 0$.

Now we can make precise the choice of δ. We choose δ small enough so that

$$
A = \max_{0<x\leq\delta} \frac{[|\varphi_1''(x)| + |\varphi_2''(x)][\varphi_2(x) - \varphi_1(x)]}{[\varphi_2'(x) - \varphi_2'(x)]^2} < 2.
$$

Then we choose η in such a way that again

$$A(1+\eta)<2.$$

Then we can replace δ by a smaller one to ensure that for this value of η we have

$$N_\delta A(1+\eta)<2$$

and

$$N_\delta A\left(1+\frac{1}{\eta}\right)\max_{0<x\leq\delta}[|\varphi_1'(x)|^2+|\varphi_2'(x)|^2]<1.$$

This is clearly possible since $\varphi_1'(0)=\varphi_2'(0)=0$ and $N_\delta\to 1$ as $\delta\to 0$. Finally we choose ε small enough so that

$$M_2K_\delta\varepsilon^{1/2}<1.$$

Then the inequality (3,3,10) may be rewritten merely as

$$\begin{aligned}\left|\int_{\Gamma_m}(\operatorname{tr}\mathcal{B})\left|\gamma\frac{\partial u_m}{\partial\nu}\right|^2 d\sigma\right| &\leq \alpha\int_{\Omega_m}[|D_xu_m|^2+|D_yu_m|^2+2\,|D_xD_yu_m|^2]\,dx\,dy\\ &\quad+\beta\int_{\Omega_m}|\nabla u_m|^2\,dx\,dy \qquad (3,3,11)\end{aligned}$$

where $\alpha<1$ and neither α nor β depend on m.

Let us go back to the identity (3,3,4). Together with (3,3,11) it yields

$$\begin{aligned}\int_{\Omega_m}&[|D_x^2u_m|^2+|D_yu_m|^2+2\,|D_xD_yu_m|^2]\,dx\,dy\\ &\leq\frac{1}{1-\alpha}\left[\int_{\Omega_m}|f|^2\,dx\,dy+\beta\int_{\Omega_m}|\nabla u_m|^2\,dx\,dy\right]. \qquad (3,3,12)\end{aligned}$$

This last inequality combined with (3,3,2) implies (3,3,1).

Proof of Theorem 3.3.1 This is very similar to the proof of Theorem 3.2.1.2. By the same technique we find an increasing sequence of integers m_k, $k=1,2,\ldots$ and a function u such that

$$\tilde{u}_{m_k}\to u$$

weakly in $H^1(\Omega)$,

$$\widetilde{D_iD_ju_{m_k}}\to D_iD_ju$$

weakly in $L_2(\Omega)$ such that

$$\Delta u = f$$

in Ω.

Obviously u belongs to $H^2(\Omega)$ and we just have to check that u belongs to $\mathring{H}^1(\Omega)$. Here Ω has no Lipschitz boundary and $\mathring{H}^1(\Omega)$ must be understood as the closure of $\mathscr{D}(\Omega)$ in $H^1(\Omega)$ (according to Definition 1.3.2.2). Indeed u_{m_k} belongs to the closure of $\mathscr{D}(\Omega_{m_k})$ for the norm of $H^1(\Omega_{m_k})$. Therefore $\tilde{u}_{m_k}$ belongs to the closure of $\mathscr{D}(\Omega)$ for the norm of $H^1(\Omega)$ since Ω_{m_k} is a subset of Ω. By taking the limit in k it is clear that u belongs to the closure of $\mathscr{D}(\Omega)$ in $H^1(\Omega)$. ■

Remark 3.3.3 In the work of Khelif (1978) conditions on φ_1 and φ_2 are given for the smoothness of the solution of the Laplace equation under other boundary conditions. For instance the conditions corresponding to the Neumann problem are

$$\limsup_{x\to 0} \frac{|\varphi_j''(x)|\,[\varphi_2(x)-\varphi_1(x)]}{[\varphi_2'(x)-\varphi_1'(x)]^2} < 1.$$

Mixed boundary conditions are also considered.