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Finite element approximation of incompressible Navier-Stokes equations with slip boundary condition II

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Summary. We consider mixed finite element approximations of the stationary, incompressible Navier-Stokes equations with slip boundary condition simultaneously approximating the velocity, pressure, and normal stress component. The stability of the schemes is achieved by adding suitable, consistent penalty terms corresponding to the normal stress component and to the pressure. A new method of proving the stability of the discretizations allows us to obtain optimal error estimates for the velocity, pressure, and normal stress component in natural norms without using duality arguments and without imposing uniformity conditions on the finite element partition. The schemes can easily be implemented into existing finite element codes for the Navier-Stokes equations with standard Dirichlet boundary conditions.

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1 Introduction

We consider the stationary, incompressible Navier-Stokes equations with slip boundary condition in a bounded, connected domain Ω in $\mathbb{R}^d$, $d = 2, 3$:

$$-\nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega \tag{1.1a}$$

$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega \tag{1.1b}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = g \quad \text{on } \Gamma \tag{1.1c}$$

$$\boldsymbol{T}(\nu\boldsymbol{u}, p)\cdot\boldsymbol{n} - \sigma(\nu\boldsymbol{u}, p)\boldsymbol{n} = 0 \quad \text{on } \Gamma\,. \tag{1.1d}$$

Here, $\nu > 0$, $\boldsymbol{u}$, p are the viscosity, velocity field, and pressure, resp. of the fluid, $\boldsymbol{n}$ is the unit exterior normal to Γ, and

$$\boldsymbol{T}(\boldsymbol{u}) := 2\boldsymbol{D}(\boldsymbol{u}) - p\boldsymbol{I}\,,$$

$$\sigma(\boldsymbol{u}, p) := \boldsymbol{n}\cdot\boldsymbol{T}(\boldsymbol{u}, p)\cdot\boldsymbol{n}$$

denote the stress tensor and normal stress component, resp., where $\boldsymbol{I}$ is the unit tensor and where

$$D(\boldsymbol{u})_{i,j} := \tfrac{1}{2}(\partial_i u_j + \partial_j u_i), \quad 1 \leqq i,j \leqq d\ ,$$

is the deformation tensor. Condition (1.1d) means that the tangential components of the vector $\boldsymbol{n} \cdot \boldsymbol{T}(v\boldsymbol{u}, p)$ vanish on the boundary. In order to ensure that problem (1.1) is well-posed and that its solutions have a certain amount of regularity (cf. (2.6), (2.7)), we assume that the function g has mean value 0 on Γ and is the trace of an H^2-function, that f is square integrable and orthogonal to all rigid body rotations of Ω (cf. (2.1)), and that the boundary Γ is of class C^3.

The slip boundary condition (1.1c, d) is the appropriate physical model for flows past chemically reacting walls [4], and for flows at high angles of attack and high Mach and Reynold's numbers (e.g., re-entry of a space orbiter), where the classical no-slip boundary condition $\boldsymbol{u} = 0$ is no longer valid. It is also part of the boundary conditions modelling flow problems with free boundaries (e.g., the coating problem [14, 15, 16]). These problems are usually solved by computing solutions to problem (1.1) on a sequence of domains iteratively approximating the unknown domain.

The slip and no-slip boundary conditions apparently describe different physical situations. This is also reflected in the mathematical properties. For example, the solutions of problem (1.1) are not stable with respect to polyhedral perturbations of the boundary. This effect is known as Babuška-paradox [3].

In order to avoid this difficulty, we have analyzed in [18] a saddle-point formulation of the slip-boundary condition and mixed finite element approximations thereof. The Lagrange-multiplier corresponding to the slip boundary condition is the normal stress component on the boundary. Thus, we obtain as a by-product additional information on an important physical parameter.

In the abstract framework for saddle-point problems [5], the primal variable has to balance the influence of the Lagrange-multiplier. In the present context this means, that the velocity has to weigh up the pressure and normal stress component. We have shown in [18] that this may be achieved by adding suitable "bubble" functions on the boundary to velocity spaces belonging to pairs of mixed finite element spaces, which are stable for the Navier-Stokes equations with no-slip boundary condition. The resulting mixed finite element method was implemented for two-dimensional problems in [19] using for the pressure and normal stress component continuous, piecewise linear functions and approximating the velocity by either continuous, piecewise linear functions on a refined mesh (*Taylor-Hood element*) or by continuous, piecewise linear functions augmented by piecewise cubic functions vanishing on the element boundaries (*mini-element*).

Here, we follow a different approach: the Lagrange-multiplier is stabilized by adding suitable, consistent penalty terms. We analyze two different methods:

(1) Use mixed finite element spaces for the velocity and pressure, which are stable for the Navier-Stokes equations with no-slip boundary condition, and stabilize the boundary terms on Γ by adding consistent, least-squares type penalty terms.

(2) Stabilize both the boundary terms on Γ and the pressure by adding consistent penalty terms thus allowing the use of arbitrary finite element spaces for the velocity, pressure, and normal stress component.

In both cases, the normal stress component can be eliminated on the element level. Compared with the approach of [18] we therefore end up with a smaller system of algebraic equations. Moreover, the methods considered here require fewer modifications when incorporating them into existng finite element codes for the Navier-Stokes equations with no-slip boundary condition.

A new method of establishing the stability of the discretizations allows us to obtain optimal error estimates for the velocity, pressure, and normal stress component in natural norms without imposing additional regularity assumptions and uniformity conditions on the solutions of problem (1.1) and the mesh, respectively. The idea of our stability proof is similar to the one used in [11, 17, 20] to verify the stability of the Taylor-Hood element and to analyze stabilized formulations of the Stokes equations with no-slip boundary condition. It combines a stability result with respect to a weaker norm, with an approximation result, and with the stability of the continuous saddle-point problem.

In contrast to standard stabilized formulations of the Stokes equations with no-slip boundary condition [6, 13] and as in [11, 20], we also allow the use of discontinuous pressure approximations. The stability of the discretizations is then maintained by additional pressure-jumps across inter-element boundaries. As in [11], the jump terms could be omitted when approximating the velocity by finite elements of sufficiently high degree.

The mixed finite element methods considered here lead to discrete problems having non-symmetric stiffness matrices. As in [10], they could be modified in order to obtain symmetric stiffness matrices. The methods also incorporate penalty parameters which must be chosen below a certain threshold, which is independent of the mesh-size h. As in [9], they could be modified in order to obtain discretizations which are stable independently of the penalty parameter. It should, however, be noted that, due to the saddle-point structure of the continuous problem, all these modifications cannot avoid the indefiniteness of the resulting discrete problems. In what follows, we will not analyze the above-mentioned modifications in order to not overload the presentation.

The outline of this paper is as follows. In Sect. 2, we introduce some function spaces and analyze the saddle-point formulation of the slip boundary condition. We then collect in Sect. 3 the general properties of the finite element spaces, which we will use. In Sects. 4 and 5 we present the two discretizations, prove their stability, and establish optimal order error estimates. In order to simplify the exposition, we consider in these sections only the linear problem, i.e., the Stokes equations, and assume that Ω is convex. In Sects. 6 and 7, we then show how the results for the linear problem naturally extend to the non-linear problem using the methods of [7] and how the previous analysis must be modified in order to treat non-convex domains and higher order approximations of the boundary. Finally, we comment in Sect. 8 on the implementation of the methods into existing finite element codes for Navier-Stokes equations.

2 Saddle-point formulation of the slip boundary condition

For any open subset ω of Ω with Lipschitz boundary γ, we denote by $L^2(\omega)$, $H^k(\omega)$, and $H^{k-1/2}(\gamma)$, $k \geqq 1$, the usual Lebesgue-, Sobolev-, and trace spaces equipped with the standard norms $\|\cdot\|_{0,\omega}$, $\|\cdot\|_{k,\omega}$, and $\|\cdot\|_{k-1/2,\gamma}$, respectively [1]. The inner products of $L^2(\omega)$ and $L^2(\gamma)$ are denoted by $(\cdot,\cdot)_\omega$ and $(\cdot,\cdot)_\gamma$, respectively. $H^{-1}(\omega)$

and $H^{-1/2}(\gamma)$ are the dual spaces of $H^1(\omega)$ and $H^{1/2}(\gamma)$, resp., and the corresponding duality pairings are denoted by $\langle\cdot,\cdot\rangle_\omega$ and $\langle\cdot,\cdot\rangle_\gamma$. Since no confusion can arise, we use the same notation for the corresponding norms and inner products on $L^2(\omega)^d$, $H^k(\omega)^d$, and $H^{k-1/2}(\gamma)^d$, respectively.

The rigid body rotations of Ω generate the space

$$\mathfrak{G} := \operatorname{span}\{\boldsymbol{u}(x) = \boldsymbol{b}\times\boldsymbol{x} : |\boldsymbol{b}| = 1,\ \boldsymbol{b} \text{ is an axis of symmetry of } \Omega\}\,, \tag{2.1}$$

where $\times$ denotes the vector product and where Ω has to be imbedded in the x–y-plane of $\mathbb{R}^3$, if $d = 2$. In order to simplify the notation, we introduce the following Hilbert- and Banach spaces

$$X := H^1(\Omega)^d/\mathfrak{G}\,, \tag{2.2a}$$

$$Y := \{p\in L^2(\Omega) : (p, 1)_\Omega = 0\}\,, \tag{2.2b}$$

$$Z := H^{-1/2}(\Gamma)\,. \tag{2.2c}$$

The division by $\mathfrak{G}$ in Eq. (2.2a) takes into account that the solutions of problem (1.1) are determined only up to rigid body rotations of Ω.

In what follows we will frequently use the following Green's formula

$$(-\Delta\boldsymbol{u} + \nabla p, \boldsymbol{v})_\Omega = 2(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_\Omega - (p, \nabla\cdot\boldsymbol{v})_\Omega - \langle\boldsymbol{n}\cdot\boldsymbol{T}(\boldsymbol{u}, p), \boldsymbol{v}\rangle_\Gamma \tag{2.3}$$

which holds for all $\boldsymbol{v}\in H^1(\Omega)^d$, $p\in H^1(\Omega)$, and $\boldsymbol{u}\in H^2(\Omega)^d$ with $\nabla\cdot\boldsymbol{u} = 0$.

The saddle-point formulation of problem (1.1) introduced in [18] is:

Find $\boldsymbol{u}\in X$, $p\in Y$, and $\sigma\in Z$ such that $\forall\boldsymbol{v}\in X$, $q\in Y$, $\tau\in Z$

$$2(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_\Omega + \frac{1}{\nu}((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \boldsymbol{v})_\Omega - (p, \nabla\cdot\boldsymbol{v})_\Omega - \langle\sigma, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma = \frac{1}{\nu}(\boldsymbol{f}, \boldsymbol{v})_\Omega \tag{2.4a}$$

$$(q, \nabla\cdot\boldsymbol{u})_\Omega = 0 \tag{2.4b}$$

$$\langle\tau, \boldsymbol{u}\cdot\boldsymbol{n}\rangle_\Gamma = \langle g, \tau\rangle_\Gamma\,. \tag{2.4c}$$

Note, that in problem (2.4)

$$\sigma = \sigma(\boldsymbol{u}, p) = \boldsymbol{n}\cdot\boldsymbol{T}(\boldsymbol{u}, p)\cdot\boldsymbol{n}$$

is the normal stress component on the boundary. Using Green's formula (2.3), one easily checks that problems (1.1) and (2.4) are equivalent in the usual sense for weak formulations modulo a scaling of the pressure by the viscosity ν.

Recall that the Stokes equations with slip boundary condition are formally obtained from problem (1.1) by setting $\nu = 1$ and omitting the convection term. The corresponding saddle-point problem is:

Find $\boldsymbol{u}\in X$, $p\in Y$, and $\sigma\in Z$ such that $\forall\boldsymbol{v}\in X$, $q\in Y$, $\tau\in Z$

$$2(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_\Omega - (p, \nabla\cdot\boldsymbol{v})_\Omega - \langle\sigma, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma = (\boldsymbol{f}, \boldsymbol{v})_\Omega \tag{2.5a}$$

$$(q, \nabla\cdot\boldsymbol{u})_\Omega = 0 \tag{2.5b}$$

$$\langle\tau, \boldsymbol{u}\cdot\boldsymbol{n}\rangle_\Gamma = \langle g, \tau\rangle_\Gamma\,. \tag{2.5c}$$

We have shown in [18] that problem (2.5) fits into the abstract framework for saddle-point problems and that it therefore has a unique solution. Moreover, the regularity estimate

$$\|\boldsymbol{u}\|_{2,\Omega} + \|p\|_{1,\Omega} + \|\sigma\|_{1/2,\Gamma} \leqq c(\Omega)\{\|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{3/2,\Gamma}\} \tag{2.6}$$

holds [16]. If the boundary Γ is sufficiently smooth, the estimate

$$\|\boldsymbol{u}\|_{k+2,\Omega} + \|p\|_{k+1,\Omega} + \|\sigma\|_{k+1/2,\Gamma} \leqq c_k(\Omega)\{\|\boldsymbol{f}\|_{k,\Omega} + \|g\|_{k+3/2,\Gamma}\} \tag{2.7}$$

can be derived from (2.6) using the general results of [2].

Using standard techniques for Navier-Stokes equations, one can prove that problem (2.4) admits at least one solution and that the solution is unique provided $\nu^{-2}\|\boldsymbol{f}\|_{0,\Omega}$ is sufficiently small [12]. Moreover, standard bootstrapping arguments yield that the solutions of problem (2.4) also satisfy the regularity estimates (2.6) and (2.7).

3 Finite element spaces

Let Ω_h be a family of polyhedral domains with boundary Γ_h and unit exterior normal $\boldsymbol{n}_h$, which approximate Ω in the following sense: The vertices of Ω_h lie on Γ and

$$\operatorname{dist}(\Gamma, \Gamma_h) := \sup_{x\in\Gamma} \inf_{y\in\Gamma_h} |x-y| \leqq c_\kappa h^2 \tag{3.1}$$

where c_κ only depends on the curvature of Γ. Estimate (3.1) holds if, e.g., the edges of Ω_h are of length h.

Denote by $\mathfrak{T}_h$ a partition of Ω_h into polyhedral domains, which satisfies the usual compatibility conditions for finite elements. The partition $\mathfrak{T}_h$ must be shape regular, i.e., the ratio of the circumscribed ball for $T\in\mathfrak{T}_h$ to that of the inscribed ball is bounded independently of T and h. Note, that this allows the use of locally refined meshes.

Denote by $\mathfrak{G}_{h,\Gamma}$ the partition of Γ_h induced by $\mathfrak{T}_h$ and by $\mathfrak{G}_{h,\Omega}$ the set of all inter-element boundaries in $\mathfrak{T}_h$. Put $\mathfrak{G}_h := \mathfrak{G}_{h,\Gamma} \cup \mathfrak{G}_{h,\Omega}$. Note, that the shape regularity implies that the ratio h_E/h_T is bounded independently of h, T and $E \subset \partial T$, where h_T and h_E denote the diameter of T and E, respectively. For any $E\in\mathfrak{G}_{h,\Omega}$ with $E = T_1 \cap T_2$, $T_1, T_2 \in \mathfrak{T}_h$, and any $\varphi\in L^2(\Omega)$ with $\varphi_{|T_i}\in C(\bar{T}_i)$, $i=1,2$, we finally denote by $[\varphi]_E$ the jump of φ across E.

Let

$$X_h \subset X, \qquad Y_h \subset Y, \qquad Z_h \subset L^2(\Gamma_h)$$

be finite element spaces corresponding to $\mathfrak{T}_h$ and $\mathfrak{G}_{h,\Gamma}$. Put

$$\mathfrak{H}_h := X_h \times Y_h \times Z_h\,. \tag{3.2}$$

We assume that there are a number $\mathfrak{k} \geqq 1$ and three interpolation operators $\boldsymbol{I}_h : X \to X_h$, $I_h : Y \to Y_h$, and $\mathfrak{J}_h : Z \to Z_h$ such that the following approximation and inverse estimates hold for all $T \in \mathfrak{T}_h$ and $E \in \mathfrak{G}_h (E = T_1 \cap T_2$ with $T_1, T_2 \in \mathfrak{T}_h$ and $T_1 = T_2$ if $E \in \mathfrak{G}_{h,\Gamma})$

(3.3a) $\|u - I_h u\|_{m,T} \leqq c_1 h_T^{l-m} \|u\|_{l,T} \quad \forall u \in H^l(T), \quad 0 \leqq m \leqq 2, \quad m \leqq l \leqq \mathfrak{k} + 1$

(3.3b) $\|u - I_h u\|_{0,E} \leqq c_2 h_E^{l+1/2} \|u\|_{l+1, T_1 \cup T_2} \quad \forall u \in H^{l+1}(T_1 \cup T_2), \quad 0 \leqq l \leqq \mathfrak{k}$

(3.3c) $\|p - I_h p\|_{m,T} \leqq c_3 h_T^{l-m} \|p\|_{l,T} \quad \forall p \in H^l(T), \quad 0 \leqq m \leqq 1, \quad m \leqq l \leqq \mathfrak{k}$

(3.3d) $\|p - I_h p\|_{0,E} \leqq c_4 h_E^{l-1/2} \|p\|_{l, T_1 \cup T_2}, \quad \forall p \in H^l(T_1 \cup T_2), \quad 1 \leqq l \leqq \mathfrak{k}$

(3.3e) $\|\sigma - \mathfrak{J}_h \sigma\|_{-1/2,E} \leqq c_5 h_E^l \|\sigma\|_{l-1/2,E} \quad \forall \sigma \in H^{l-1/2}(E), \quad 0 \leqq l \leqq \mathfrak{k}$

(3.3f) $\|\sigma - \mathfrak{J}_h \sigma\|_{0,E} \leqq c_6 h_E^{l-1/2} \|\sigma\|_{l-1/2,E} \quad \forall \sigma \in H^{l-1/2}(E), \quad 1 \leqq l \leqq \mathfrak{k}$

(3.3g) $\|\nabla \cdot D(u_h)\|_{0,T} \leqq c_7 h_T^{-1} \|D(u_h)\|_{0,T} \quad \forall u_h \in X_h$

(3.3h) $\|D(u_h)\|_{0,E} \leqq c_8 h_E^{-1/2} \|D(u_h)\|_{0, T_1 \cup T_2} \quad \forall u_h \in X_h$

(3.3i) $\|p_h\|_{0,E} \leqq c_9 h_E^{-1/2} \|p_h\|_{0, T_1 \cup T_2} \quad \forall p_h \in Y_h .$

Here and in what follows $c, c_1, c_2, \ldots$ are constants which are independent of h. Conditions (3.3) are satisfied, if X_h and Y_h, Z_h contain functions which are piecewise polynomials of degree $\mathfrak{k}$ and $\mathfrak{k} - 1$, respectively [8].

For the first method we will also require that the spaces X_h and Y_h satisfy the well-known *Brezzi condition* for mixed finite element approximations of the Stokes equations with no-slip boundary condition, i.e., there is a constant β, *which does not depend on h*, such that

$$(3.4) \qquad \inf_{p_h \in Y_h \setminus \{0\}} \ \sup_{u_h \in X_h \setminus \{0\}, u_h = 0 \text{ on } \Gamma_h} \frac{(p_h, \nabla \cdot u_h)_{\Omega_h}}{\|p_h\|_{0,\Omega_h} \|u_h\|_{1,\Omega_h}} \geqq \beta .$$

Example 3.1. Let $\mathfrak{T}_h$ be a partition of Ω_h into triangles or tetrahedra.

(a) X_h, Y_h, and Z_h consist of continuous, piecewise quadratic functions on $\mathfrak{T}_h$, continuous, piecewise linear functions on $\mathfrak{T}_h$, and piecewise linear functions on $\mathfrak{G}_{h,\Gamma}$, respectively (*Taylor-Hood element*). The estimates (3.3) hold with $\mathfrak{k} = 2$; the stability condition (3.4) is satisfied.

(b) X_h, Y_h, and Z_h consist of continuous, piecewise linear functions on $\mathfrak{T}_h$ augmented by continuous functions, which are piecewise polynomials of degree $d + 1$ and which vanish on the element boundaries, continuous, piecewise linear functions on $\mathfrak{T}_h$, and piecewise constant functions on $\mathfrak{G}_{h,\Gamma}$, respectively (*mini element*). The estimates (3.3) hold with $\mathfrak{k} = 1$; the stability condition (3.4) is satisfied.

(c) X_h, Y_h, and Z_h consist of continuous, piecewise linear functions on $\mathfrak{T}_h$, piecewise constant functions on $\mathfrak{T}_h$, and piecewise constant functions on $\mathfrak{G}_{h,\Gamma}$, respectively (*linear/constant element*). The estimates (3.3) hold with $\mathfrak{k} = 1$; the stability condition (3.4) is not satisfied (*checker-board instability*).

4 Stabilization of the normal stress component

Throughout this section we assume that Ω is convex and that the Brezzi condition (3.4) is satisfied. We first construct on Γ_h an approximation g_h of the boundary data g as follows: Denote by $\tilde{g}_h$ the piecewise linear function on Γ_h interpolating g in the vertices of the partition $\mathfrak{G}_{h,\Gamma}$ and put

$$g_h := \tilde{g}_h - \frac{1}{|\Gamma_h|}(\tilde{g}_h, 1)_{\Gamma_h}$$

where $|\Gamma_h|$ is the measure of Γ_h.

The stabilized formulation of the slip boundary condition then is:

Find $\boldsymbol{u}_h \in X_h$, $p_h \in Y_h$, and $\sigma_h \in Z_h$ such that $\forall \boldsymbol{v}_h \in X_h$, $q_h \in Y_h$, $\tau_h \in Z_h$

$$2(\boldsymbol{D}(\boldsymbol{u}_h), \boldsymbol{D}(\boldsymbol{v}_h))_{\Omega_h} - (p_h, \nabla \cdot \boldsymbol{v}_h)_{\Omega_h} - \langle \sigma_h, \boldsymbol{v}_h \cdot \boldsymbol{n}_h \rangle_{\Gamma_h} = (\boldsymbol{f}, \boldsymbol{v}_h)_{\Omega_h} \tag{4.1a}$$

$$(q_h, \nabla \cdot \boldsymbol{u}_h)_{\Omega_h} = 0 \tag{4.1b}$$

$$\delta \sum_{E \in \mathfrak{G}_{h,\Gamma}} h_E(\sigma_h - \boldsymbol{n}_h \cdot \boldsymbol{T}(\boldsymbol{u}_h, p_h) \cdot \boldsymbol{n}_h, \tau_h)_E + \langle \tau_h, \boldsymbol{u}_h \cdot \boldsymbol{n}_h \rangle_{\Gamma_h} = \langle g_h, \tau_h \rangle_{\Gamma_h} . \tag{4.1c}$$

Here, δ is a small parameter *which is independent of h* and which will be determined later. Note. that the δ-terms in Eq. (4.1c) form a least-squares type penalty term which vanishes if $\boldsymbol{u}_h = \boldsymbol{u}$, $p_h = p$, $\sigma_h = \sigma$, and $\Omega_h = \Omega$.

In order to simplify the analysis of problem (4.1), we introduce a bilinear form $\mathfrak{B}_h$, a linear functional l_h, and a mesh-dependent norm $|||\,.\,|||_h$ on $\mathfrak{H}_h$ by

$$\begin{aligned} \mathfrak{B}_h([\boldsymbol{u}_h, p_h, \sigma_h], [\boldsymbol{v}_h, q_h, \tau_h]) := {} & 2(\boldsymbol{D}(\boldsymbol{u}_h), \boldsymbol{D}(\boldsymbol{v}_h))_{\Omega_h} - (p_h, \nabla \cdot \boldsymbol{v}_h)_{\Omega_h} \\ & - \langle \sigma_h, \boldsymbol{v}_h \cdot \boldsymbol{n}_h \rangle_{\Gamma_h} + (q_h, \nabla \cdot \boldsymbol{u}_h)_{\Omega_h} + \langle \tau_h, \boldsymbol{u}_h \cdot \boldsymbol{n}_h \rangle_{\Gamma_h} \\ & + \delta \sum_{E \in \mathfrak{G}_{h,\Gamma}} h_E(\sigma_h - \boldsymbol{n}_h \cdot \boldsymbol{T}(\boldsymbol{u}_h, p_h) \cdot \boldsymbol{n}_h, \tau_h)_E \end{aligned} \tag{4.2}$$

$$\mathrm{l}_h([\boldsymbol{v}_h, q_h, \tau_h]) := (\boldsymbol{f}, \boldsymbol{v}_h)_{\Omega_h} + \langle g_h, \tau_h \rangle_{\Gamma_h} \tag{4.3}$$

$$\begin{aligned} |||[\boldsymbol{u}_h, p_h, \sigma_h]|||_h := \Big\{ & \|\boldsymbol{D}(\boldsymbol{u}_h)\|^2_{0,\Omega_h} + \|p_h\|^2_{0,\Omega_h} + \|\sigma_h\|^2_{-1/2,\Gamma_h} \\ & + \delta \sum_{E \in \mathfrak{G}_{h,\Gamma}} h_E \|\sigma_h\|^2_{0,E} \Big\}^{1/2} . \end{aligned} \tag{4.4}$$

Proposition 4.1. *There are two constants $\delta_0 > 0$ and $\gamma > 0$, which do not depend on h, such that the stability estimate*

$$\inf_{[\boldsymbol{u}_h, p_h, \sigma_h] \in \mathfrak{H}_h \setminus \{0\}} \; \sup_{[\boldsymbol{v}_h, q_h, \tau_h] \in \mathfrak{H}_h \setminus \{0\}} \frac{\mathfrak{B}_h([\boldsymbol{u}_h, p_h, \sigma_h], [\boldsymbol{v}_h, q_h, \tau_h])}{|||[\boldsymbol{u}_h, p_h, \sigma_h]|||_h |||[\boldsymbol{v}_h, q_h, \tau_h]|||_h} \geqq \gamma \tag{4.5}$$

holds for all $0 < \delta \leqq \delta_0$.

Proof. Let $[\boldsymbol{u}_h, p_h, \sigma_h] \in \mathfrak{H}_h \setminus \{0\}$ be arbitrary. Then we have

$$\begin{aligned} & \mathfrak{B}_h([\boldsymbol{u}_h, p_h, \sigma_h], [\boldsymbol{u}_h, p_h, \sigma_h]) \\ & = 2\|\boldsymbol{D}(\boldsymbol{u}_h)\|^2_{0,\Omega_h} + \delta \sum_{E \in \mathfrak{G}_{h,\Gamma}} h_E \|\sigma_h\|^2_{0,E} - \delta \sum_{E \in \mathfrak{G}_{h,\Gamma}} h_E(\boldsymbol{n}_h \cdot \boldsymbol{T}(\boldsymbol{u}_h, p_h) \cdot \boldsymbol{n}_h, \sigma_h)_E \\ & = 2\|\boldsymbol{D}(\boldsymbol{u}_h)\|^2_{0,\Omega_h} + \delta \sum_{E \in \mathfrak{G}_{h,\Gamma}} h_E \|\sigma_h\|^2_{0,E} - 2\delta \sum_{E \in \mathfrak{G}_{h,\Gamma}} h_E(\boldsymbol{n}_h \cdot \boldsymbol{D}(\boldsymbol{u}_h) \cdot \boldsymbol{n}_h, \sigma_h)_E \\ & \quad + \delta \sum_{E \in \mathfrak{G}_{h,\Gamma}} h_E(p_h, \sigma_h)_E . \end{aligned}$$

Using estimate (3.3h) and choosing δ sufficiently small, the third term in the above equation can be absorbed by the first and second one. In order to balance the fourth term, we note that the inf-sup condition (3.4) yields the existence of a $\boldsymbol{v}_h \in X_h$ with $\boldsymbol{v}_h = 0$ on Γ_h, $\|\boldsymbol{v}_h\|_{1,\Omega_h} = \|p_h\|_{0,\Omega_h}$, and $-(p_h, \nabla \cdot \boldsymbol{v}_h)_{\Omega_h} \geqq \beta \|p_h\|_{0,\Omega_h}^2$. This gives

$$\mathfrak{B}_h([\boldsymbol{u}_h, p_h, \sigma_h], [\boldsymbol{v}_h, 0, 0]) = 2(\boldsymbol{D}(\boldsymbol{u}_h), \boldsymbol{D}(\boldsymbol{v}_h))_{\Omega_h} - (p_h, \nabla \cdot \boldsymbol{v}_h)_{\Omega_h}$$

$$\geqq 2(\boldsymbol{D}(\boldsymbol{u}_h), \boldsymbol{D}(\boldsymbol{v}_h))_{\Omega_h} + \beta \|p_h\|_{0,\Omega_h}^2 .$$

The definition of $H^{-1/2}(\Gamma_h)$ and of $H^{1/2}(\Gamma_h)$, on the other hand, imply that there is a $\boldsymbol{w} \in H^1(\Omega_h)^d$ with $\|\boldsymbol{w}\|_{1,\Omega_h} = \|\sigma_h\|_{-1/2,\Gamma_h}$, and $-\langle \sigma_h, \boldsymbol{w} \cdot \boldsymbol{n}_h \rangle_{\Gamma_h} = \|\sigma_h\|_{-1/2,\Gamma_h}^2$. Let $\boldsymbol{w}_h$ be the H^1-projection of $\boldsymbol{w}$ onto X_h. This yields

$$\begin{aligned}\mathfrak{B}_h([\boldsymbol{u}_h, p_h, \sigma_h], [\boldsymbol{w}_h, 0, 0]) &= 2(\boldsymbol{D}(\boldsymbol{u}_h), \boldsymbol{D}(\boldsymbol{w}_h))_{\Omega_h} - (p_h, \nabla \cdot \boldsymbol{w}_h)_{\Omega_h} \\ &\quad - \langle \sigma_h, \boldsymbol{w} \cdot \boldsymbol{n}_h \rangle_{\Gamma_h} + \langle \sigma_h, (\boldsymbol{w} - \boldsymbol{w}_h) \cdot \boldsymbol{n}_h \rangle_{\Gamma_h} \\ &\geqq 2(\boldsymbol{D}(\boldsymbol{u}_h), \boldsymbol{D}(\boldsymbol{w}_h))_{\Omega_h} - (p_h, \nabla \cdot \boldsymbol{w}_h)_{\Omega_h} \\ &\quad + \|\sigma_h\|_{-1/2,\Gamma_h}^2 + \langle \sigma_h, (\boldsymbol{w} - \boldsymbol{w}_h) \cdot \boldsymbol{n}_h \rangle_{\Gamma_h} .\end{aligned}$$

Now, we take a suitable linear combination of the above estimates such that all mixed terms can be absorbed by the quadratic ones. To this end denote by a and b positive constants, which will be determined later. Using Hölder's inequality and the estimate $ab \leqq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$ with suitable $\varepsilon > 0$ several times, we conclude that

$$\begin{aligned}&\mathfrak{B}_h([\boldsymbol{u}_h, p_h, \sigma_h], [\boldsymbol{u}_h + a\boldsymbol{v}_h + b\delta \boldsymbol{w}_h, p_h, \sigma_h]) \\ &\quad = 2\|\boldsymbol{D}(\boldsymbol{u}_h)\|_{0,\Omega_h}^2 + \delta \sum_{E \in \mathfrak{G}_{h,\Gamma}} h_E \|\sigma_h\|_{0,E}^2 - \delta \sum_{E \in \mathfrak{G}_{h,\Gamma}} h_E (\boldsymbol{n}_h \cdot \boldsymbol{T}(\boldsymbol{u}_h, p_h) \cdot \boldsymbol{n}_h, \sigma_h)_E \\ &\qquad + 2a(\boldsymbol{D}(\boldsymbol{u}_h), \boldsymbol{D}(\boldsymbol{v}_h))_{\Omega_h} - a(p_h, \nabla \cdot \boldsymbol{v}_h)_{\Omega_h} \\ &\qquad + 2b\delta(\boldsymbol{D}(\boldsymbol{u}_h), \boldsymbol{D}(\boldsymbol{w}_h))_{\Omega_h} - b\delta(p_h, \nabla \cdot \boldsymbol{w}_h)_{\Omega_h} \\ &\qquad - b\delta\langle \sigma_h, \boldsymbol{w} \cdot \boldsymbol{n}_h \rangle_{\Gamma_h} + b\delta\langle \sigma_h, (\boldsymbol{w} - \boldsymbol{w}_h) \cdot \boldsymbol{n}_h \rangle_{\Gamma_h} \\ &\quad \geqq 2\|\boldsymbol{D}(\boldsymbol{u}_h)\|_{0,\Omega_h}^2 + \delta \sum_{E \in \mathfrak{G}_{h,\Gamma}} h_E \|\sigma_h\|_{0,E}^2 \\ &\qquad - 2\delta \sum_{E \in \mathfrak{G}_{h,\Gamma}, E \subset \partial T} c_8 h_E^{1/2} \|\boldsymbol{D}(\boldsymbol{u}_h)\|_{0,T} \|\sigma_h\|_{0,E} \\ &\qquad - \delta \sum_{E \in \mathfrak{G}_{h,\Gamma}, E \subset \partial T} c_9 h_E^{1/2} \|p_h\|_{0,T} \|\sigma_h\|_{0,E} \\ &\qquad - 4a\|\boldsymbol{D}(\boldsymbol{u}_h)\|_{0,\Omega_h} \|p_h\|_{0,\Omega_h} + a\beta\|p_h\|_{0,\Omega_h}^2 \\ &\qquad - 4b\delta\|\boldsymbol{D}(\boldsymbol{u}_h)\|_{0,\Omega_h} \|\sigma_h\|_{-1/2,\Gamma_h} - 2b\delta\|p_h\|_{0,\Omega_h} \|\sigma_h\|_{-1/2,\Gamma_h} \\ &\qquad + b\delta\|\sigma_h\|_{-1/2,\Gamma_h}^2 - b\delta \sum_{E \in \mathfrak{G}_{h,\Gamma}, E \subset \partial T} c_2 h_E^{1/2} \|\sigma_h\|_{0,E} \|\boldsymbol{w}\|_{1,T}\end{aligned}$$

$$\geqq \left(1 - \frac{16a}{\beta}\right) \| D(u_h) \|_{0,\Omega_h}^2$$

$$+ \frac{a\beta}{4} \| p_h \|_{0,\Omega_h}^2$$

$$+ \left(\frac{1}{2} - 2\delta c_8^2 - \frac{1}{a\beta} \delta c_9^2\right) \delta \sum_{E \in \mathfrak{G}_{h,\Gamma}} h_E \| \sigma_h \|_{0,E}^2$$

$$+ b\delta \left(1 - 8b\delta - \frac{4b\delta}{a\beta} - \frac{1}{2} bc_2^2\right) \| \sigma_h \|_{-1/2,\Gamma_h}^2 .$$

Put $C := \max\{1, c_2, c_8, c_9\}$. Assume without loss of generality that $\beta \leqq 1$ and choose

$$a = \frac{4\beta}{64 + \beta^2}, \qquad \delta_0 = \frac{\beta^2}{80C^2}, \qquad b = \frac{C^2}{2 + C^4} .$$

We then obtain for all $0 < \delta \leqq \delta_0$

$$\mathfrak{B}_h([u_h, p_h, \sigma_h], [u_h + av_h + b\delta w_h, p_h, \sigma_h]) \geqq \frac{\beta^2}{160(2 + C^4)} ||| [u_h, p_h, \sigma_h] |||_h^2$$

and

$$||| [u_h + av_h + b\delta w_h, p_h, \sigma_h] |||_h \leqq 2 ||| [u_h, p_h, \sigma_h] |||_h .$$

This proves the assertion. □

The estimates in the proof of Proposition 4.1 could be sharpened considerably. In the present context, however, we are not interested in computing the optimal value of the constant γ in (4.5). What is important for our purposes is that Proposition 4.1 implies the unique solvability of problem (4.1) and leads to optimal order error estimates. In what follows, we will always assume that δ is as in Proposition 4.1.

In order to compare the discrete normal stress component σ_h on Γ_h with the normal stress component σ on Γ, we denote by π the orthogonal projection of Γ onto Γ_h. Since Ω is convex, π is bijective. The same holds for non-convex domains provided $h \leqq c_\Gamma$, where the constant c_Γ only depends on the curvature of Γ. Using the mean-value theorem and Hölder's inequality, one easily checks that the estimate

$$\| v \circ \pi^{-1} - v \|_{0,\Gamma_h} \leqq ch \| v \|_{1,\Omega} \tag{4.6}$$

holds for all $v \in H^1(\Omega)$. Note that, due to inequ. (4.6), we can at best expect an $O(h)$-error estimate for the velocity, pressure, and normal stress component in the $||| \cdot |||_h$-norm.

Proposition 4.2. *Denote by $\boldsymbol{u}$, p, σ and by $\boldsymbol{u}_h$, p_h, σ_h the unique solution of problems* (2.5) *and* (4.1), *resp. and let δ be as in Proposition* 4.1. *Then the error estimate*

$$(4.7)\qquad \begin{aligned} &\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega_h}+\|p-p_h\|_{0,\Omega_h}+\|\sigma\circ\pi^{-1}-\sigma_h\|_{-1/2,\Gamma_h}\\ &\quad+\Big\{\delta\sum_{E\in\mathfrak{G}_{h,\Gamma}}h_E\|\sigma\circ\pi^{-1}-\sigma_h\|_{0,E}^2\Big\}^{1/2}\\ &\leqq ch\{\|\boldsymbol{u}\|_{2,\Omega}+\|p\|_{1,\Omega}+\|\sigma\|_{1/2,\Gamma}\}\end{aligned}$$

holds.

Proof. Proposition 4.1 implies that

$$\begin{aligned} &|||[\boldsymbol{u}-\boldsymbol{u}_h, p-p_h, \sigma\circ\pi^{-1}-\sigma_h]|||_h\\ &\leqq |||[\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u}, p-I_hp, \sigma\circ\pi^{-1}-\mathfrak{J}_h\sigma\circ\pi^{-1}]|||_h\\ &\quad+|||[\boldsymbol{I}_h\boldsymbol{u}-\boldsymbol{u}_h, I_hp-p_h, \mathfrak{J}_h\sigma\circ\pi^{-1}-\sigma_h]|||_h\\ &\leqq |||[\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u}, p-I_hp, \sigma\circ\pi^{-1}-\mathfrak{J}_h\sigma\circ\pi^{-1}]|||_h\\ &\quad+\frac{1}{\gamma}\sup_{|||[\boldsymbol{v}_h,q_h,\tau_h]|||_h=1}\mathfrak{B}_h(\boldsymbol{I}_h\boldsymbol{u}-\boldsymbol{u}_h, I_hp-p_h, \mathfrak{J}_h\sigma\circ\pi^{-1}-\sigma_h], [\boldsymbol{v}_h, q_h, \tau_h])\\ &\leqq |||[\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u}, p-I_hp, \sigma\circ\pi^{-1}-\mathfrak{J}_h\sigma\circ\pi^{-1}]|||_h\\ &\quad+\frac{1}{\gamma}\sup_{|||[\boldsymbol{v}_h,q_h,\tau_h]|||_h=1}\mathfrak{B}_h(\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u}, p-I_hp, \sigma\circ\pi^{-1}-\mathfrak{J}_h\sigma\circ\pi^{-1}], [\boldsymbol{v}_h, q_h, \tau_h])\\ &\quad+\frac{1}{\gamma}\sup_{|||[\boldsymbol{v}_h,q_h,\tau_h]|||_h=1}\mathfrak{B}_h(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h, \sigma\circ\pi^{-1}-\sigma_h], [\boldsymbol{v}_h, q_h, \tau_h])\\ &=: R_1+R_2+R_3\,.\end{aligned}$$

With arguments similar to those used in the proof of Theorem 5.1 in [18] and using the estimates (3.3), we conclude that the interpolation errors R_1 and R_2 can be bounded by

$$(4.8)\qquad \begin{aligned} R_1 &\leqq 4\|\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u}\|_{1,\Omega_h}+2\|p-I_hp\|_{0,\Omega_h}+2\|\sigma\circ\pi^{-1}-\mathfrak{J}_h\sigma\circ\pi^{-1}\|_{-1/2,\Gamma_h}\\ &\quad+2\Big\{\delta\sum_{E\in\mathfrak{G}_{h,\Gamma}}h_E\|\sigma\circ\pi^{-1}-\mathfrak{J}_h\sigma\circ\pi^{-1}\|_{0,E}^2\Big\}^{1/2}\\ &\leqq c_{10}h\{\|\boldsymbol{f}\|_{0,\Omega}+\|g\|_{3/2,\Gamma}\}\end{aligned}$$

$$(4.9)\qquad \begin{aligned} R_2 &\leqq c_{11}\Big\{\|\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u}\|_{1,\Omega_h}+\|p-I_hp\|_{0,\Omega_h}+\|\sigma\circ\pi^{-1}-\mathfrak{J}_h\sigma\circ\pi^{-1}\|_{-1/2,\Gamma_h}\\ &\quad+\Big\{\delta\sum_{E\in\mathfrak{G}_{h,\Gamma}}h_E\|\sigma\circ\pi^{-1}-\mathfrak{J}_h\sigma\circ\pi^{-1}\|_{0,E}^2\Big\}^{1/2}\Big\}\\ &\leqq c_{12}h\{\|\boldsymbol{f}\|_{0,\Omega}+\|g\|_{3/2,\Gamma}\}\,.\end{aligned}$$

Next, we consider the consistency error R_3. Using the Green's formula (2.3), we obtain

$$\begin{aligned}
&\mathfrak{B}_h([\boldsymbol{u}-\boldsymbol{u}_h, p-p_h, \sigma\circ\pi^{-1}-\sigma_h], [\boldsymbol{v}_h, q_h, \tau_h]) \\
&= \mathfrak{B}_h([\boldsymbol{u}, p, \sigma\circ\pi^{-1}], [\boldsymbol{v}_h, q_h, \tau_h]) - \mathfrak{l}_h([\boldsymbol{v}_h, q_h, \tau_h]) \\
&= \langle \boldsymbol{n}_h\cdot \boldsymbol{T}(\boldsymbol{u}, p), \boldsymbol{v}_h\rangle_{\Gamma_h} - \langle \sigma\circ\pi^{-1}, \boldsymbol{v}_h\cdot\boldsymbol{n}_h\rangle_{\Gamma_h} + \langle \boldsymbol{u}\cdot\boldsymbol{n}_h, \tau_h\rangle_{\Gamma_h} - \langle g_h, \tau_h\rangle_{\Gamma_h} \\
&\quad + \delta \sum_{E\in\mathfrak{G}_{h,\Gamma}} h_E(\sigma\circ\pi^{-1} - \boldsymbol{n}_h\cdot\boldsymbol{T}(\boldsymbol{u}, p)\cdot\boldsymbol{n}_h, \tau_h)_{0,E} \\
&= \langle \boldsymbol{n}_h\cdot\boldsymbol{T}(\boldsymbol{u}, p)\cdot\boldsymbol{n}_h - \sigma\circ\pi^{-1}, \boldsymbol{v}_h\cdot\boldsymbol{n}_h\rangle_{\Gamma_h} + \langle \boldsymbol{n}_h\cdot\boldsymbol{T}(\boldsymbol{u}, p) - [\boldsymbol{n}_h\cdot\boldsymbol{T}(\boldsymbol{u}, p)\cdot\boldsymbol{n}_h]\boldsymbol{n}_h, \boldsymbol{v}_h\rangle_{\Gamma_h} \\
&\quad + \langle \boldsymbol{u}\cdot\boldsymbol{n}_h - (\boldsymbol{u}\cdot\boldsymbol{n})\circ\pi^{-1}, \tau_h\rangle_{\Gamma_h} + \langle g\circ\pi^{-1} - g_h, \tau_h\rangle_{\Gamma_h} \\
&\quad + \delta \sum_{E\in\mathfrak{G}_{h,\Gamma}} h_E(\sigma\circ\pi^{-1} - \boldsymbol{n}_h\cdot\boldsymbol{T}(\boldsymbol{u}, p)\cdot\boldsymbol{n}_h, \tau_h)_{0,E} \\
&=: R_{3,1} + R_{3,2} + R_{3,3} + R_{3,4} + R_{3,5}\,.
\end{aligned}$$

Estimates (3.3) and the arguments used in the proof of Theorem 5.1 in [18] imply

$$R_{3,1} + R_{3,2} \leqq c_{13} h \|\boldsymbol{v}_h\|_{1,\Omega_h}\{\|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{3/2,\Gamma}\} \tag{4.10}$$

$$\begin{aligned} R_{3,3} + R_{3,4} \leqq c_{14} h &\left\{ \|\tau_h\|^2_{-1/2,\Gamma_h} + \delta \sum_{E\in\mathfrak{G}_{h,\Gamma}} h_E\|\tau_h\|^2_{0,E}\right\}^{1/2} \\ &\times\{\|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{3/2,\Gamma}\} \end{aligned} \tag{4.11}$$

$$\begin{aligned} R_{3,5} \leqq c_{15} h &\left\{ \|\tau_h\|^2_{-1/2,\Gamma_h} + \delta \sum_{E\in\mathfrak{G}_{h,\Gamma}} h_E\|\tau_h\|^2_{0,E}\right\}^{1/2} \\ &\times\{\|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{3/2,\Gamma}\}\,. \end{aligned} \tag{4.12}$$

Combining the estimates (4.8)–(4.12), we immediately obtain the error estimate (4.7). □

5 Stabilization of both the normal stress component and the pressure

Throughout this section we assume that Ω is convex. *However, we do not require that the Brezzi condition* (3.4) *is satisfied.* The notations are the same as in Sect. 4.

The stabilized formulation of both the slip boundary condition and the divergence constraint then is:

Find $\boldsymbol{u}_h\in X_h$, $p_h\in Y_h$, and $\sigma_h\in Z_h$ such that $\forall \boldsymbol{v}_h\in X_h$, $q_h\in Y_h$, $\tau_h\in Z_h$

$$2(\boldsymbol{D}(\boldsymbol{u}_h), \boldsymbol{D}(\boldsymbol{v}_h))_{\Omega_h} - (p_h, \nabla\cdot\boldsymbol{v}_h)_{\Omega_h} - \langle\sigma_h, \boldsymbol{v}_h\cdot\boldsymbol{n}_h\rangle_{\Gamma_h} = (\boldsymbol{f}, \boldsymbol{v}_h)_{\Omega_h} \tag{5.1a}$$

$$\begin{aligned} &(q_h, \nabla\cdot\boldsymbol{u}_h)_{\Omega_h} + \delta_2 \sum_{E\in\mathfrak{G}_{h,\Omega}} h_E([p_h]_E, [q_h]_E)_E \\ &\quad + \delta_2 \sum_{T\in\mathfrak{T}_h} h_T^2(\nabla p_h - 2\nabla\cdot\boldsymbol{D}(\boldsymbol{u}_h), \nabla q_h)_T = \delta_2 \sum_{T\in\mathfrak{T}_h} h_T^2(\boldsymbol{f}, \nabla q_h)_T \end{aligned} \tag{5.1b}$$

$$\delta_1 \sum_{E\in\mathfrak{G}_{h,\Gamma}} h_E(\sigma_h - \boldsymbol{n}_h\cdot\boldsymbol{T}(\boldsymbol{u}_h, p_h)\cdot\boldsymbol{n}_h, \tau_h)_E + \langle\tau_h, \boldsymbol{u}_h\cdot\boldsymbol{n}_h\rangle_{\Gamma_h} = \langle g_h, \tau_h\rangle_{\Gamma_h}\,. \tag{5.1c}$$

Here, δ_1 and δ_2 are small parameters *which are independent of h* and which will be determined later. As in the previous section, the δ-terms in Eqs. (5.1b, c) form least-squares type penalty terms which vanish if $\boldsymbol{u}_h = \boldsymbol{u}$, $p_h = p$, $\sigma_h = \sigma$, and $\Omega_h = \Omega$. Note, that the jump-terms in (5.1b) are needed to stabilize the divergence constraint when using discontinuous pressure approximations and that they vanish when using continuous pressure approximations. As in [11], they could be omitted when approximating the velocity by finite elements of sufficiently high degree.

In order to simplify the analysis of problem (5.1) and in analogy to Sect. 4, we introduce a bilinear form $\mathfrak{B}'_h$, a linear functional $\mathfrak{l}'_h$, and a mesh-dependent norm $|||\cdot|||'_h$ on $\mathfrak{H}_h$ by

$$(5.2)\quad \begin{aligned} \mathfrak{B}'_h([\boldsymbol{u}_h, p_h, \sigma_h], [\boldsymbol{v}_h, q_h, \tau_h]) := {}& 2(\boldsymbol{D}(\boldsymbol{u}_h), \boldsymbol{D}(\boldsymbol{v}_h))_{\Omega_h} - (p_h, \nabla\cdot\boldsymbol{v}_h)_{\Omega_h} \\ & - \langle \sigma_h, \boldsymbol{v}_h\cdot\boldsymbol{n}_h\rangle_{\Gamma_h} + (q_h, \nabla\cdot\boldsymbol{u}_h)_{\Omega_h} \\ & + \langle \tau_h, \boldsymbol{u}_h\cdot\boldsymbol{n}_h\rangle_{\Gamma_h} \\ & + \delta_1 \sum_{E\in\mathfrak{G}_{h,\Gamma}} h_E(\sigma_h - \boldsymbol{n}_h\cdot\boldsymbol{T}(\boldsymbol{u}_h, p_h)\cdot\boldsymbol{n}_h, \tau_h)_E \\ & + \delta_2 \sum_{T\in\mathfrak{T}_h} h_T^2(\nabla p_h - 2\nabla\cdot\boldsymbol{D}(\boldsymbol{u}_h), \nabla q_h)_T \\ & + \delta_2 \sum_{E\in\mathfrak{G}_{h,\Omega}} h_E([p_h]_E, [q_h]_E)_E \end{aligned}$$

$$(5.3)\quad \mathfrak{l}'_h([\boldsymbol{v}_h, q_h, \tau_h]) := (\boldsymbol{f}, \boldsymbol{v}_h)_{\Omega_h} + \delta_2 \sum_{T\in\mathfrak{T}_h} h_T^2(\boldsymbol{f}, \nabla q_h)_T + \langle g_h, \tau_h\rangle_{\Gamma_h}$$

$$(5.4)\quad \begin{aligned} |||[\boldsymbol{u}_h, p_h, \sigma_h]|||'_h := \Big\{ & \|\boldsymbol{D}(\boldsymbol{u}_h)\|^2_{0,\Omega_h} + \|p_h\|^2_{0,\Omega_h} + \|\sigma_h\|^2_{-1/2,\Gamma_h} \\ & + \delta_1 \sum_{E\in\mathfrak{G}_{h,\Gamma}} h_E\|\sigma_h\|^2_{0,E} + \delta_2 \sum_{T\in\mathfrak{T}_h} h_T^2\|\nabla p_h\|^2_{0,T} \\ & + \delta_2 \sum_{E\in\mathfrak{G}_{h,\Omega}} h_E\|[p_h]_E\|^2_{0,E} \Big\}^{1/2} . \end{aligned}$$

Proposition 5.1. *There are three constants $\delta_1 > 0$, $\delta_2 > 0$, and $\gamma' > 0$, which do not depend on h, such that*

$$(5.5)\quad \inf_{[\boldsymbol{u}_h, p_h, \sigma_h]\in\mathfrak{H}_h\setminus\{0\}} \; \sup_{[\boldsymbol{v}_h, q_h, \tau_h]\in\mathfrak{H}_h\setminus\{0\}} \frac{\mathfrak{B}'_h([\boldsymbol{u}_h, p_h, \sigma_h], [\boldsymbol{v}_h, q_h, \tau_h])}{|||[\boldsymbol{u}_h, p_h, \sigma_h]|||'_h\, |||[\boldsymbol{v}_h, q_h, \tau_h]|||'_h} \geqq \gamma' .$$

Proof. Let $[\boldsymbol{u}_h, p_h, \sigma_h]\in\mathfrak{H}_h\setminus\{0\}$ and $\boldsymbol{w}$, $\boldsymbol{w}_h$ be as in the proof of Proposition 4.1. Now, we obtain

$$\begin{aligned} & \mathfrak{B}'_h([\boldsymbol{u}_h, p_h, \sigma_h], [\boldsymbol{u}_h, p_h, \sigma_h]) \\ & = 2\|\boldsymbol{D}(\boldsymbol{u}_h)\|_0^2 + \delta_1 \sum_{E\in\mathfrak{G}_{h,\Omega}} h_E\|\sigma_h\|^2_{0,E} - \delta_1 \sum_{E\in\mathfrak{G}_{h,\Omega}} h_E(\boldsymbol{n}_h\cdot\boldsymbol{T}(\boldsymbol{u}_h, p_h)\cdot\boldsymbol{n}_h, \sigma_h)_E \\ & \quad + \delta_2 \sum_{T\in\mathfrak{T}_h} h_T^2\|\nabla p_h\|^2_{0,T} + \delta_2 \sum_{E\in\mathfrak{G}_{h,\Omega}} h_E\|[p_h]_E\|^2_{0,E} \\ & \quad - 2\delta_2 \sum_{T\in\mathfrak{T}_h} h_T^2(\nabla\cdot\boldsymbol{D}(\boldsymbol{u}_h), \nabla p_h)_T \end{aligned}$$

and

$$\begin{aligned}
&\mathfrak{B}'_h([\boldsymbol{u}_h, p_h, \sigma_h], [\boldsymbol{w}_h, 0, 0]) \\
&\geqq 2(\boldsymbol{D}(\boldsymbol{u}_h), \boldsymbol{D}(\boldsymbol{w}_h))_{\Omega_h} - (p_h, \nabla \cdot \boldsymbol{w}_h)_{\Omega_h} + \|\sigma_h\|^2_{-1/2, \Gamma_h} + \langle \sigma_h, (\boldsymbol{w} - \boldsymbol{w}_h) \cdot \boldsymbol{n}_h \rangle_{\Gamma_h} .
\end{aligned}$$

In order to balance the pressure, we recall that there is a $\boldsymbol{v} \in H^1(\Omega_h)^d$ with $\boldsymbol{v} = 0$ on Γ_h, $\|\boldsymbol{v}\|_{1,\Omega_h} \leqq c_0 \|p_h\|_{0,\Omega_h}$, and $-\nabla \cdot \boldsymbol{v} = p_h$. Denote by $\boldsymbol{v}_h$ the H^1-projection of $\boldsymbol{v}$ onto X_h. This gives

$$\begin{aligned}
\mathfrak{B}'_h([\boldsymbol{u}_h, p_h, \sigma_h], [\boldsymbol{v}_h, 0, 0]) &= 2(\boldsymbol{D}(\boldsymbol{u}_h), \boldsymbol{D}(\boldsymbol{v}_h))_{\Omega_h} - (p_h, \nabla \cdot \boldsymbol{v})_{\Omega_h} + (p_h, \nabla \cdot (\boldsymbol{v} - \boldsymbol{v}_h))_{\Omega_h} \\
&= 2(\boldsymbol{D}(\boldsymbol{u}_h), \boldsymbol{D}(\boldsymbol{v}_h))_{\Omega_h} + \|p_h\|^2_{0,\Omega_h} + (p_h, \nabla \cdot (\boldsymbol{v} - \boldsymbol{v}_h))_{\Omega_h} .
\end{aligned}$$

With arguments similar to those used in the proof of Proposition 4.1 we conclude that

$$\begin{aligned}
&\mathfrak{B}'_h([\boldsymbol{u}_h, p_h, \sigma_h], [\boldsymbol{u}_h + a\delta_2 \boldsymbol{v}_h + b\delta_1 \boldsymbol{w}_h, p_h, \sigma_h]) \\
&= 2\|\boldsymbol{D}(\boldsymbol{u}_h)\|^2_0 + \delta_1 \sum_{E \in \mathfrak{G}_{h,\Gamma}} h_E \|\sigma_h\|^2_{0,E} - \delta_1 \sum_{E \in \mathfrak{G}_{h,\Gamma}} h_E (\boldsymbol{n}_h \cdot \boldsymbol{T}(\boldsymbol{u}_h, p_h) \cdot \boldsymbol{n}_h, \sigma_h)_E \\
&\quad + \delta_2 \sum_{T \in \mathfrak{T}_h} h_T^2 \|\nabla p_h\|^2_{0,T} + \delta_2 \sum_{E \in \mathfrak{G}_{h,\Omega}} h_E \|[p_h]_E\|^2_{0,E} \\
&\quad - 2\delta_2 \sum_{T \in \mathfrak{T}_h} h_T^2 (\nabla \cdot \boldsymbol{D}(\boldsymbol{u}_h), \nabla p_h)_T + 2a\delta_2 (\boldsymbol{D}(\boldsymbol{u}_h), \boldsymbol{D}(\boldsymbol{v}_h))_\Omega - a\delta_2 (p_h, \nabla \cdot \boldsymbol{v})_{\Omega_h} \\
&\quad + a\delta_2 (p_h, \nabla \cdot (\boldsymbol{v} - \boldsymbol{v}_h))_{\Omega_h} + 2b\delta_1 (\boldsymbol{D}(\boldsymbol{u}_h), \boldsymbol{D}(\boldsymbol{w}_h))_{\Omega_h} - b\delta_1 (p_h, \nabla \cdot \boldsymbol{w}_h)_{\Omega_h} \\
&\quad - b\delta_1 \langle \sigma_h, \boldsymbol{w} \cdot \boldsymbol{n} \rangle_{\Gamma_h} + b\delta_1 \langle \sigma_h, (\boldsymbol{w} - \boldsymbol{w}_h) \cdot \boldsymbol{n}_h \rangle_{\Gamma_h} \\
&\geqq 2\|\boldsymbol{D}(\boldsymbol{u}_h)\|^2_0 + \delta_1 \sum_{E \in \mathfrak{G}_{h,\Gamma}} h_E \|\sigma_h\|^2_{0,E} \\
&\quad - 2\delta_1 \sum_{E \in \mathfrak{G}_{h,\Gamma}, E \subset \partial T} c_8 h_E^{1/2} \|\boldsymbol{D}(\boldsymbol{u}_h)\|_{0,T} \|\sigma_h\|_{0,E} \\
&\quad - \delta_1 \sum_{E \in \mathfrak{G}_{h,\Gamma}, E \subset \partial T} c_9 h_E^{1/2} \|p_h\|_{0,T} \|\sigma_h\|_{0,E} \\
&\quad + \delta_2 \sum_{T \in \mathfrak{T}_h} h_T^2 \|\nabla p_h\|^2_{0,T} + \delta_2 \sum_{E \in \mathfrak{G}_{h,\Omega}} h_E \|[p_h]_E\|^2_{0,E} \\
&\quad - 2\delta_2 \sum_{T \in \mathfrak{T}_h} c_7 h_T \|\boldsymbol{D}(\boldsymbol{u}_h)\|_{0,T} \|\nabla p_h\|_{0,T} \\
&\quad - 4ac_0 \delta_2 \|\boldsymbol{D}(\boldsymbol{u}_h)\|_{0,\Omega_h} \|p_h\|_{0,\Omega_h} + a\delta_2 \|p_h\|^2_{0,\Omega_h} \\
&\quad - a\delta_2 \sum_{T \in \mathfrak{T}_h} c_1 h_T \|\nabla p_h\|_{0,T} \|\boldsymbol{v}\|_{1,T} \\
&\quad - a\delta_2 \sum_{E \in \mathfrak{G}_{h,\Omega}, E = T_1 \cap T_2} c_2 h_E^{1/2} \|[p_h]_E\|_{0,E} \|\boldsymbol{v}\|_{1, T_1 \cup T_2} \\
&\quad + 4b\delta_1 \|\boldsymbol{D}(\boldsymbol{u}_h)\|_{0,\Omega_h} \|\sigma_h\|_{-1/2,\Gamma_h} - 2b\delta_1 \|p_h\|_{0,\Omega_h} \|\sigma_h\|_{-1/2,\Gamma_h} \\
&\quad + b\delta_1 \|\sigma\|^2_{-1/2,\Gamma_h} - b\delta_1 \sum_{E \in \mathfrak{G}_{h,\Gamma}, E \subset \partial T} c_2 h_E^{1/2} \|\sigma_h\|_{0,E} \|\boldsymbol{w}\|_{1,T}
\end{aligned}$$

$$\geqq (1 - 2c_7^2\delta_2 - 24ac_0^2\delta_2)\|\boldsymbol{D}(\boldsymbol{u}_h)\|_0^2 + \tfrac{1}{6}a\delta_2\|p_h\|_0^2$$

$$+ (\tfrac{1}{2} - \tfrac{3}{2}ac_0^2c_1^2)\delta_2 \sum_{T\in\mathfrak{T}_h} h_T^2\|\nabla p_h\|_{0,T}^2$$

$$+ (1 - \tfrac{3}{2}ac_0^2c_2^2)\delta_2 \sum_{E\in\mathfrak{G}_{h,\Omega}} h_E\|[p_h]_E\|_{0,E}^2$$

$$+ \left(\frac{1}{2} - 2c_8^2\delta_1 - \frac{3}{2}\delta_1 c_9^2\frac{1}{a\delta_2}\right)\delta_1 \sum_{E\in\mathfrak{G}_{h,\Gamma}} h_E\|\sigma_h\|_{0,E}^2$$

$$+ b\delta_1\left(1 - 8b\delta_1 - 6b\delta_1\frac{1}{a\delta_2} - \frac{1}{2}bc_2^2\delta_1\right)\|\sigma_h\|_{-1/2,\Gamma_h}^2 .$$

Put $C' := \max\{2, c_0, c_1, c_2, c_7, c_8, c_9, c_0c_1, c_0c_2\}$ and choose

$$a = \frac{1}{6C'^2},\qquad b = \frac{17}{36},\qquad \delta_1 = \frac{1}{9C'^2(1+16C'^4)},\qquad \delta_2 = \frac{36C'^2}{1+144C'^4}.$$

We then obtain

$$\mathfrak{B}_h'([\boldsymbol{u}_h, p_h, \sigma_h], [\boldsymbol{u}_h + a\delta_2\boldsymbol{v}_h + b\delta_1\boldsymbol{w}_h, p_h, \sigma_h]) \geqq \frac{1}{1+144C'^6}|||[\boldsymbol{u}_h, p_h, \sigma_h]|||_h'^2$$

and

$$|||[\boldsymbol{u}_h + a\delta_2\boldsymbol{v}_h + b\delta_1\boldsymbol{w}_h, p_h, \sigma_h]|||_h' \leqq 3|||[\boldsymbol{u}_h, p_h, \sigma_h]|||_h' .$$

This proves the assertion. □

As in the proof of Proposition 4.1, the above estimates could be sharpened considerably. Proposition 5.1 implies that problem (5.1) has a unique solution. In what follows, we will always assume that δ_1 and δ_2 are as in Proposition 5.1. Note that, due to inequ. (4.6), we can at best expect an $O(h)$-error estimate for the velocity, pressure, and normal stress component in the $|||\,.\,|||_h'$-norm.

Proposition 5.2. *Denote by $\boldsymbol{u}$, p, σ and by $\boldsymbol{u}_h$, p_h, σ_h the unique solution of problems* (2.5) *and* (5.1), *resp. and let δ_1 and δ_2 be as in Proposition* 5.1. *Then the error estimate*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega_h} + \|p - p_h\|_{0,\Omega_h} + \|\sigma\circ\pi^{-1} - \sigma_h\|_{-1/2,\Gamma_h} \tag{5.6}$$

$$+ \left\{\delta_1 \sum_{E\in\mathfrak{G}_{h,\Gamma}} h_E\|\sigma\circ\pi^{-1} - \sigma_h\|_{0,E}^2\right\}^{1/2}$$

$$+ \left\{\delta_2 \sum_{T\in\mathfrak{T}_h} h_T^2\|\nabla(p - p_h)\|_{0,T}^2 + \delta_2 \sum_{E\in\mathfrak{G}_{h,\Omega}} h_E\|[p - p_h]_E\|_{0,E}^2\right\}^{1/2}$$

$$\leqq ch\{\|\boldsymbol{u}\|_{2,\Omega} + \|p\|_{1,\Omega} + \|\sigma\|_{1/2,\Gamma}$$

holds. □

Proof. The proof follows along the same lines as the one of Proposition 4.2. Proposition 5.1 implies that

$$
\begin{aligned}
&|||[\boldsymbol{u}-\boldsymbol{u}_h, p-p_h, \sigma\circ\pi^{-1}-\sigma_h]|||_h' \\
&\leqq |||[\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u}, p-I_hp, \sigma\circ\pi^{-1}-\mathfrak{J}_h\sigma\circ\pi^{-1}]|||_h' \\
&\quad + |||[\boldsymbol{I}_h\boldsymbol{u}-\boldsymbol{u}_h, I_hp-p_h, \mathfrak{J}_h\sigma\circ\pi^{-1}-\sigma_h]|||_h' \\
&\leqq |||[\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u}, p-I_hp, \sigma\circ\pi^{-1}-\mathfrak{J}_h\sigma\circ\pi^{-1}]|||_h' \\
&\quad + \frac{1}{\gamma'}\sup_{|||[\boldsymbol{v}_h,q_h,\tau_h]|||_h'=1}\mathfrak{B}_h'([\boldsymbol{I}_h\boldsymbol{u}-\boldsymbol{u}_h, I_hp-p_h, \mathfrak{J}_h\sigma\circ\pi^{-1}-\sigma_h],[\boldsymbol{v}_h,q_h,\tau_h]) \\
&\leqq |||[\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u}, p-I_hp, \sigma\circ\pi^{-1}-\mathfrak{J}_h\sigma\circ\pi^{-1}]|||_h' \\
&\quad + \frac{1}{\gamma'}\sup_{|||[\boldsymbol{v}_h,q_h,\tau_h]|||_h'=1}\mathfrak{B}_h'([\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u}, p-I_hp, \sigma\circ\pi^{-1}-\mathfrak{J}_h\sigma\circ\pi^{-1}],[\boldsymbol{v}_h,q_h,\tau_h]) \\
&\quad + \frac{1}{\gamma'}\sup_{|||[\boldsymbol{v}_h,q_h,\tau_h]|||_h'=1}\mathfrak{B}_h'([\boldsymbol{u}-\boldsymbol{u}_h, p-p_h, \sigma\circ\pi^{-1}-\sigma_h],[\boldsymbol{v}_h,q_h,\tau_h]) \\
&=: R_1'+R_2'+R_3'\,.
\end{aligned}
$$

As in the proof of Proposition 4.2, we obtain for the interpolation errors R_1' and R_2'

$$
\begin{aligned}
R_1' &\leqq c_{16}h\{\|\boldsymbol{f}\|_{0,\Omega}+\|g\|_{3/2,\Gamma}\} \\
R_2' &\leqq c_{17}h\{\|\boldsymbol{f}\|_{0,\Omega}+\|g\|_{3/2,\Gamma}\}\,.
\end{aligned}
$$

Using the Green's formula (2.3) and the continuity of p, we now get

$$
\begin{aligned}
&\mathfrak{B}_h'([\boldsymbol{u}-\boldsymbol{u}_h, p-p_h, \sigma\circ\pi^{-1}-\sigma_h],[\boldsymbol{v}_h,q_h,\tau_h]) \\
&= \mathfrak{B}_h'([\boldsymbol{u}, p, \sigma\circ\pi^{-1}],[\boldsymbol{v}_h,q_h,\tau_h]) - \mathfrak{l}_h'([\boldsymbol{v}_h,q_h,\tau_h]) \\
&= \langle \boldsymbol{n}_h\cdot\boldsymbol{T}(\boldsymbol{u},p)\cdot\boldsymbol{n}_h-\sigma\circ\pi^{-1}, \boldsymbol{v}_h\cdot\boldsymbol{n}_h\rangle_{\Gamma_h} \\
&\quad + \langle \boldsymbol{n}_h\cdot\boldsymbol{T}(\boldsymbol{u},p)-[\boldsymbol{n}_h\cdot\boldsymbol{T}(\boldsymbol{u},p)\cdot\boldsymbol{n}_h]\boldsymbol{n}_h, \boldsymbol{v}_h\rangle_{\Gamma_h} \\
&\quad + \langle \boldsymbol{u}\cdot\boldsymbol{n}_h-(\boldsymbol{u}\cdot\boldsymbol{n})\circ\pi^{-1}, \tau_h\rangle_{\Gamma_h} + \langle g\circ\pi^{-1}-g_h, \tau_h\rangle_{\Gamma_h} \\
&\quad + \delta_1\sum_{E\in\mathfrak{G}_{h,\Gamma}} h_E(\sigma\circ\pi^{-1}-\boldsymbol{n}_h\cdot\boldsymbol{T}(\boldsymbol{u},p)\cdot\boldsymbol{n}_h, \tau_h)_{0,E} \\
&\quad + \delta_2\sum_{E\in\mathfrak{G}_{h,\Omega}} h_E([p-p_h]_E, \boldsymbol{v}_h\cdot\boldsymbol{n}_h)_E\,.
\end{aligned}
$$

With the same arguments as in the proof of Proposition 4.2, we therefore obtain for the consistency error R_3'

$$
R_3' \leqq c_{18}h\{\|\boldsymbol{f}\|_{0,\Omega}+\|g\|_{3/2,\Gamma}\}\,.
$$

This immediately implies the error estimate (5.7). □

6 Analysis of the non-linear problem

In this section, we extend the methods of Sects. 4 and 5 to the Navier-Stokes equations (1.1) and establish error estimates similar to those of Propositions 4.2 and 5.2. In order to avoid technical difficulties, we still assume that Ω is convex. We refer to the next section for the general case.

The stabilized formulations of the Navier-Stokes equations corresponding to problems (4.1) and (5.1) are given by

Find $\boldsymbol{u}_h \in X_h$, $p_h \in Y_h$, and $\sigma_h \in Z_h$ such that $\forall \boldsymbol{v}_h \in X_h$, $q_h \in Y_h$, $\tau_h \in Z_h$

$$(6.1a)\quad 2(\boldsymbol{D}(\boldsymbol{u}_h), \boldsymbol{D}(\boldsymbol{v}_h))_{\Omega_h} + \frac{1}{\nu}((\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}, \boldsymbol{v})_{\Omega_h} - (p_h, \nabla\cdot\boldsymbol{v}_h)_{\Omega_h} - \langle\sigma_h, \boldsymbol{v}_h\cdot\boldsymbol{n}_h\rangle_{\Gamma_h} = \frac{1}{\nu}(\boldsymbol{f}, \boldsymbol{v}_h)_{\Omega_h}$$

$$(6.1b)\quad (q_h, \nabla\cdot\boldsymbol{u}_h)_{\Omega_h} = 0$$

$$(6.1c)\quad \delta \sum_{E\in\mathfrak{G}_{h,\Gamma}} h_E(\sigma_h - \boldsymbol{n}_h\cdot\boldsymbol{T}(\boldsymbol{u}_h, p_h)\cdot\boldsymbol{n}_h, \tau_h)_E + \langle\tau_h, \boldsymbol{u}_h\cdot\boldsymbol{n}_h\rangle_{\Gamma_h} = \langle g_h, \tau_h\rangle_{\Gamma_h}\,.$$

and

Find $\boldsymbol{u}_h \in X_h$, $p_h \in Y_h$, and $\sigma_h \in Z_h$ such that $\forall \boldsymbol{v}_h \in X_h$, $q_h \in Y_h$, $\tau_h \in Z_h$

$$(6.2a)\quad 2(\boldsymbol{D}(\boldsymbol{u}_h), \boldsymbol{D}(\boldsymbol{v}_h))_{\Omega_h} + \frac{1}{\nu}((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \boldsymbol{v})_{\Omega_h} - (p_h, \nabla\cdot\boldsymbol{v}_h)_{\Omega_h} - \langle\sigma_h, \boldsymbol{v}_h\cdot\boldsymbol{n}_h\rangle_{\Gamma_h} = \frac{1}{\nu}(\boldsymbol{f}, \boldsymbol{v}_h)_{\Omega_h}$$

$$(6.2b)\quad \delta_2 \sum_{T\in\mathfrak{T}_h} h_T^2(\nabla p_h - 2\nabla\cdot\boldsymbol{D}(\boldsymbol{u}_h) + \frac{1}{\nu}(\boldsymbol{u}_h\cdot\nabla)\,\boldsymbol{u}_h, \nabla q_h)_T + \delta_2 \sum_{E\in\mathfrak{G}_{h,\Omega}} h_E([p_h]_E, [q_h]_E)_E + (q_h, \nabla\cdot\boldsymbol{u}_h)_{\Omega_h} = \frac{\delta_2}{\nu}\sum_{T\in\mathfrak{T}_h} h_T^2(\boldsymbol{f}, \nabla q_h)_T$$

$$(6.2c)\quad \delta_1 \sum_{E\in\mathfrak{G}_{h,\Gamma}} h_E(\sigma_h - \boldsymbol{n}_h\cdot\boldsymbol{T}(\boldsymbol{u}_h, p_h)\cdot\boldsymbol{n}_h, \tau_h)_E + \langle\tau_h, \boldsymbol{u}_h\cdot\boldsymbol{n}_h\rangle_{\Gamma_h} = \langle g_h, \tau_h\rangle_{\Gamma_h}\,.$$

Proposition 6.1. *Let $\Lambda \subset (0, \infty)$ be a compact interval. Consider a continuous branch $\lambda \to \boldsymbol{u}_\lambda$, $\lambda = \frac{1}{\nu}$, of solutions of the Navier-Stokes equations* (1.1), *which are non-singular in the sense of* [7]. *Denote by p_λ and σ_λ the corresponding pressure and normal stress component. Then there are an $h_0 > 0$ and an $\alpha = \alpha(\lambda, h) > 0$ such that problems* (6.1) *and* (6.2) *have, for all $h \leqq h_0$ and $\lambda \in \Lambda$, a unique solution $\boldsymbol{u}_{\lambda,h}$ and $\boldsymbol{u}'_{\lambda,h}$, resp. in the ball $B_{\lambda,h}(\alpha) := \{\boldsymbol{v}\in H^1(\Omega_h)^d : \|\boldsymbol{v} - \boldsymbol{I}_h\boldsymbol{u}_\lambda\|_{1,\Omega_h} \leqq \alpha\}$. Denote by $p_{\lambda,h}$, $\sigma_{\lambda,h}$ and $p'_{\lambda,h}$, and $\sigma'_{\lambda,h}$, the pressure and normal stress component corresponding to $\boldsymbol{u}_{\lambda,h}$ and $\boldsymbol{u}'_{\lambda,h}$, respectively. Then the error estimates*

$$(6.3)\quad \|\boldsymbol{u}_\lambda - \boldsymbol{u}_{\lambda,h}\|_{1,\Omega_h} + \|p_\lambda - p_{\lambda,h}\|_{0,\Omega_h} + \|\sigma_\lambda\circ\pi^{-1} - \sigma_{\lambda,h}\|_{-1/2,\Gamma_h} + \left\{\delta \sum_{E\in\mathfrak{G}_{h,\Gamma}} h_E\|\sigma_\lambda\circ\pi^{-1} - \sigma_{\lambda,h}\|_{0,E}^2\right\}^{1/2} \leqq ch \quad \forall 0 < h \leqq h_0,\quad \lambda\in\Lambda$$

and

$$
\begin{aligned}
(6.4)\quad & \|u_\lambda - u'_{\lambda,h}\|_{1,\Omega_h} + \|p_\lambda - p'_{\lambda,h}\|_{0,\Omega_h} + \|\sigma_\lambda \circ \pi^{-1} - \sigma'_{\lambda,h}\|_{-1/2,\Gamma_h} \\
& + \left\{ \delta_1 \sum_{E \in \mathfrak{G}_{h,\Gamma}} h_E \|\sigma_\lambda \circ \pi^{-1} - \sigma'_{\lambda,h}\|_{0,E}^2 \right\}^{1/2} \\
& + \left\{ \delta_2 \sum_{T \in \mathfrak{T}_h} h_T^2 \|\nabla(p_\lambda - p'_{\lambda,h})\|_{0,T}^2 + \delta_2 \sum_{E \in \mathfrak{G}_{h,\Omega}} h_E \|[p_\lambda - p'_{\lambda,h}]_E\|_{0,E}^2 \right\}^{1/2} \\
& \leqq c'h \quad \forall 0 < h \leqq h_0, \lambda \in \Lambda
\end{aligned}
$$

hold with constants c and c' which are independent of h and λ.

Proof. The proof closely follows the approach of [7] to non-linear problems and the proof of Theorem 6.1 in [18]. Therefore, we only sketch the essential steps.

Denote by $T: H^{-1}(\Omega)^d \to H^1(\Omega)^d$ the Stokes operator which associates with each right-hand side $w \in H^{-1}(\Omega)^d$ the unique solution $u \in H^1(\Omega)^d$ of the Stokes problem

$$
\begin{aligned}
& \text{Find } u \in X,\ p \in Y, \text{ and } \sigma \in Z \text{ such that } \forall v \in X,\ q \in Y,\ \tau \in Z \\
& 2(D(u), D(v))_\Omega - (p, \nabla \cdot v)_\Omega - \langle \sigma, v \cdot n \rangle_\Gamma = \langle w, v \rangle_\Omega \\
& \qquad (q, \nabla \cdot u)_\Omega = 0 \\
& \qquad \langle \tau, u \cdot n \rangle_\Gamma = \langle g, \tau \rangle_\Gamma .
\end{aligned}
$$

The operator T is affine. Moreover, T is continuous, since by Lemma 3.1 in [18] we have

$$
(6.5)\qquad \|T(u) - T(v)\|_{1,\Omega} \leqq c_T \|u - v\|_{-1,\Omega} \quad \forall u, v \in H^{-1}(\Omega)^d .
$$

The non-linearity is taken into account by the mapping $G: H^1(\Omega)^d \to L^{3/2}(\Omega)^d$ with

$$
G(u) := (u \cdot \nabla) u - f .
$$

Put

$$
F(\lambda, u) := u + \lambda T(G(u)) .
$$

Since $\lambda \to u_\lambda$ is a non-singular branch of solutions of the Navier-Stokes Eq. (1.1), we know that

$$
(6.6)\qquad F(\lambda, u_\lambda) = 0, \qquad \lambda = \frac{1}{\nu} .
$$

and that $D_u F(\lambda, u_\lambda)$ is a homeomorphism of $H^1(\Omega)^d$ onto itself.

Put

$$
(6.7)\qquad \gamma(\lambda) := \sup_{0 \neq v \in H^1(\Omega)^d} \frac{\|D_u F(\lambda, u_\lambda)^{-1} v\|_{1,\Omega}}{\|v\|_{1,\Omega}} .
$$

The continuity of the mapping $\lambda \to u_\lambda$, the compactness of Λ, and the regularity estimate (2.6) imply

$$
(6.8)\qquad \tilde{\gamma} := \sup_{\lambda \in \Lambda} \gamma(\lambda) < \infty
$$

$$
(6.9)\qquad K := \max\left\{ \|f\|_{0,\Omega}, \|g\|_{3/2,\Gamma}, \sup_{\lambda \in \Lambda} \|u_\lambda\|_{2,\Omega} \right\} < \infty .
$$

Note, that, in what follows, all constants will be independent of h and λ.

In order to take into account the approximation of the domain Ω, we define the mapping $G_h: H^1(\Omega)^d \cup H^1(\Omega_h)^d \to L^{3/2}(\Omega)^d \cup L^{3/2}(\Omega_h)^d$ by

$$G_h(\boldsymbol{u}) := \chi_{\Omega_h}(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \boldsymbol{f},$$

where χ_{Ω_h} denotes the characteristic function of Ω_h. Let R_h be the operator which associates with each function $\varphi: \Omega \to \mathbb{R}$ its restriction to Ω_h and put

$$\tilde{F}_h(\lambda, \boldsymbol{u}) := R_h[\boldsymbol{u} + \lambda T(G_h(\boldsymbol{u}))] .$$

The Sobolev imbedding theorem, inequality (6.9), and Banach's fixed point theorem imply that there is an $h_1 > 0$ such that, for all $0 < h \leqq h_1$ and $\lambda \in \Lambda$, $D_u\tilde{F}_h(\lambda, \boldsymbol{u}_\lambda)$ is a homeomorphism of $H^1(\Omega_h)^d$ onto itself which satisfies the estimates

$$\|\tilde{F}_h(\lambda, \boldsymbol{u}_\lambda)\|_{1,\Omega_h} \leqq c_{19} h^{4/3} K^2 \tag{6.10}$$

$$\sup_{0 \neq \boldsymbol{v} \in H^1(\Omega_h)^d} \frac{\|D_u\tilde{F}_h(\lambda, \boldsymbol{u}_\lambda)^{-1}\boldsymbol{v}\|_{1,\Omega_h}}{\|\boldsymbol{v}\|_{1,\Omega_h}} \leqq 2\gamma(\lambda) . \tag{6.11}$$

Denote by $T_h: H^{-1}(\Omega)^d \cup H^{-1}(\Omega_h)^d \to X_h$ the discrete Stokes operator which associates with each right-hand side $\boldsymbol{w} \in H^{-1}(\Omega)^d \cup H^{-1}(\Omega_h)^d$ the unique solution $\boldsymbol{u}_h \in X_h$ of

$$\text{Find } \boldsymbol{u}_h \in X_h,\ p_h \in Y_h, \text{ and } \sigma_h \in Z_h \text{ such that } \forall \boldsymbol{v}_h \in X_h,\ q_h \in Y_h,\ \tau_h \in Z_h$$

$$2(\boldsymbol{D}(\boldsymbol{u}_h), \boldsymbol{D}(\boldsymbol{v}_h))_{\Omega_h} - (p_h, \nabla\cdot\boldsymbol{v}_h)_{\Omega_h} - \langle \sigma_h, \boldsymbol{v}_h\cdot\boldsymbol{n}_h\rangle_{\Gamma_h} = \langle \boldsymbol{w}, \boldsymbol{v}_h\rangle_{\Omega_h}$$

$$(q_h, \nabla\cdot\boldsymbol{u}_h)_{\Omega_h} = 0$$

$$\delta \sum_{E \in \mathfrak{E}_{h,\Gamma}} h_E(\sigma_h - \boldsymbol{n}_h\cdot\boldsymbol{T}(\boldsymbol{u}_h, p_h)\cdot\boldsymbol{n}_h, \tau_h)_E + \langle \tau_h, \boldsymbol{u}_h\cdot\boldsymbol{n}_h\rangle_h = \langle g_h, \tau_h\rangle_{\Gamma_h},$$

where δ is as in Proposition 4.1. The operator T_h is affine. Proposition 4.1 implies that

$$\|T_h(\boldsymbol{u}) - T_h(\boldsymbol{v})\|_{1,\Omega_h} \leqq \tilde{c}_T \|\boldsymbol{u} - \boldsymbol{v}\|_{-1,\Omega_h} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in H^{-1}(\Omega_h)^d,\ h > 0 \tag{6.12}$$

and Proposition 4.2 yields the error estimate

$$\|T(\boldsymbol{w}) - T_h(\boldsymbol{w})\|_{1,\Omega_h} \leqq ch\{\|\boldsymbol{w}\|_{0,\Omega} + \|g\|_{3/2,\Omega}\} \quad \forall \boldsymbol{w} \in L^2(\Omega)^d . \tag{6.13}$$

Put

$$F_h(\lambda, \boldsymbol{u}) := \boldsymbol{u} + \lambda T_h(G_h(\boldsymbol{u})) .$$

Then, $\boldsymbol{u}_h$ is a solution of problem (6.1) if and only if it is a solution of

$$F_h(\lambda, \boldsymbol{u}_h) = 0, \quad \lambda = \frac{1}{\nu} . \tag{6.14}$$

The error estimates (3.3), inequalities (6.10)–(6.13), and Banach's fixed point theorem imply that there is an $0 < h_2 \leqq h_1$ such that, for all $0 < h \leqq h_2$ and $\lambda \in \Lambda$,

$D_u F_h(\lambda, I_h u_\lambda)$ is a homeomorphism of $H^1(\Omega_h)^d$ onto itself which satisfies the estimates

$$\varepsilon_h(\lambda) := \| F_h(\lambda, I_h u_\lambda) \|_{1,\Omega_h} \leqq c_{20} h K^2 \tag{6.16}$$

$$\sup_{0 \neq w \in H^1(\Omega_h)^d} \frac{\| D_u F_h(\lambda, I_h u_\lambda) w - D_u \tilde{F}_h(\lambda, u_\lambda) w \|_{1,\Omega_h}}{\| w \|_{1,\Omega_h}} \leqq c_{21} h K \tag{6.17}$$

$$\sup_{0 \neq w \in H^1(\Omega_h)^d} \frac{\| D_u F_h(\lambda, v_1) w - D_u F_h(\lambda, v_2) w \|_{1,\Omega_h}}{\| w \|_{1,\Omega_h}} \leqq c_{22} \| v_1 - v_2 \|_{1,\Omega_h} \tag{6.18}$$

$$\sup_{0 \neq w \in H^1(\Omega_h)^d} \frac{\| D_u F_h(\lambda, I_h u_\lambda)^{-1} w \|_{1,\Omega_h}}{\| w \|_{1,\Omega_h}} \leqq 4\gamma(\lambda) \tag{6.19}$$

for all $v_1, v_2 \in H^1(\Omega_h)^d$. Hence, we can define a mapping $\Phi : H^1(\Omega_h)^d \to H^1(\Omega_h)^d$ by

$$\Phi(v) := v - D_u F_h(\lambda, I_h u_\lambda)^{-1} F_h(\lambda, v) .$$

Every fixed point of Φ is a solution of problem (6.14) and, therefore, of problem (6.1). Inequalities (6.16)–(6.19) imply that there is an $0 < h_3 \leqq h_2$ such that, for all $0 < h \leqq h_3$ and $\lambda \in \Lambda$, Φ is a contraction of the ball

$$B_{\lambda,h}(8\gamma(\lambda)\varepsilon_h(\lambda)) := \{ v \in H^1(\Omega_h)^d : \| v - I_h u_\lambda \|_{1,\Omega_h} \leqq 8\gamma(\lambda)\varepsilon_h(\lambda) \}$$

into itself. Hence, Φ has a unique fixed point in the ball $B_{\lambda,h}(8\gamma(\lambda)\varepsilon_h(\lambda))$. This proves the unique solvability of problem (6.1) in the neighbourhood of the solution branch $\lambda \to u_\lambda$ and the error estimate (6.3) for the velocity.

In order to prove the error estimates for the pressure and the normal stress component, denote by $\tilde{u}_{\lambda,h} \in X_h$, $\tilde{p}_{\lambda,h} \in Y_h$, $\tilde{\sigma}_{\lambda,h} \in Z_h$ the unique solution of

$$2(D(\tilde{u}_{\lambda,h}), D(v_h))_{\Omega_h} - (\tilde{p}_{\lambda,h}, \nabla \cdot v_h)_{\Omega_h} - \langle \tilde{\sigma}_{\lambda,h}, v_h \cdot n_h \rangle_{\Gamma_h} = \lambda \langle f - (u_\lambda \cdot \nabla) u_\lambda, v_h \rangle_{\Omega_h} \quad \forall v_h \in X_h$$

$$(q_h, \nabla \cdot \tilde{u}_{\lambda,h})_{\Omega_h} = 0 \quad \forall q_h \in Y_h$$

$$\delta \sum_{E \in \mathfrak{G}_{h,\Gamma}} h_E (\tilde{\sigma}_{\lambda,h} - n_h \cdot T(\tilde{u}_{\lambda,h}, \tilde{p}_{\lambda,h}) \cdot n_h, \tau_h)_E + \langle \tau_h, \tilde{u}_{\lambda,h} \cdot n_h \rangle_{\Gamma_h} = \langle g_h, \tau_h \rangle_{\Gamma_h} \quad \forall \tau_h \in Z_h .$$

Proposition 4.1 yields the stability estimate

$$\begin{aligned} &\| u_{\lambda,h} - \tilde{u}_{\lambda,h} \|_{1,\Omega_h} + \| p_{\lambda,h} - \tilde{p}_{\lambda,h} \|_{0,\Omega_h} + \| \sigma_{\lambda,h} - \tilde{\sigma}_{\lambda,h} \|_{-1/2,\Gamma_h} \\ &\quad + \left\{ \delta \sum_{E \in \mathfrak{G}_{h,\Gamma}} h_E \| \sigma_{\lambda,h} - \tilde{\sigma}_{\lambda,h} \|_{0,E}^2 \right\}^{1/2} \\ &\leqq c_{23} \lambda \{ \| u_\lambda \|_{1,\Omega} + \| u_{\lambda,h} \|_{1,\Omega_h} \} \| u_\lambda - u_{\lambda,h} \|_{1,\Omega_h} \\ &\leqq c_{24} h K^2 . \end{aligned} \tag{6.20}$$

Since $(\boldsymbol{u}_\lambda \cdot \nabla)\boldsymbol{u}_\lambda \in L^2(\Omega)^d$, we conclude from Proposition 4.2 that

$$
\begin{aligned}
(6.21)\qquad & \|\boldsymbol{u}_\lambda - \tilde{\boldsymbol{u}}_{\lambda,h}\|_{1,\Omega_h} + \|p_\lambda - \tilde{p}_{\lambda,h}\|_{0,\Omega_h} + \|\sigma_\lambda \circ \pi^{-1} - \tilde{\sigma}_{\lambda,h}\|_{-1/2,\Gamma_h} \\
& + \left\{\delta \sum_{E \in \mathfrak{G}_{h,\Gamma}} h_E \|\sigma_\lambda \circ \pi^{-1} - \tilde{\sigma}_{\lambda,h}\|_{0,E}^2\right\}^{1/2} \\
& \leqq c_{25} h \{\|(\boldsymbol{u}_\lambda \cdot \nabla)\boldsymbol{u}_\lambda\|_{0,\Omega} + \|g\|_{3/2,\Gamma}\} \\
& \leqq c_{26} h K^2 .
\end{aligned}
$$

Combining inequalities (6.20) and (6.21), we immediately obtain the error estimate (6.3).

The analysis of problem (6.2) is identical to the preceding one. One only has to replace the operator T_h by the operator $T_h': H^{-1}(\Omega)^d \cup H^{-1}(\Omega_h)^d \to X_h$ which associates with each right-hand side $\boldsymbol{w} \in H^{-1}(\Omega_h)^d$ the unique solution $\boldsymbol{u}_h \in X_h$ of

$$
\begin{aligned}
&\text{Find } \boldsymbol{u}_h \in X_h,\ p_h \in Y_h, \text{ and } \sigma_h \in Z_h \text{ such that } \forall \boldsymbol{v}_h \in X_h,\ q_h \in Y_h,\ \tau_h \in Z_h \\
&2(\boldsymbol{D}(\boldsymbol{u}_h), \boldsymbol{D}(\boldsymbol{v}_h))_{\Omega_h} - (p_h, \nabla \cdot \boldsymbol{v}_h)_{\Omega_h} - \langle \sigma_h, \boldsymbol{v}_h \cdot \boldsymbol{n}_h \rangle_{\Gamma_h} = \langle \boldsymbol{w}, \boldsymbol{v}_h \rangle_{\Omega_h} \\
&(q_h, \nabla \cdot \boldsymbol{u}_h)_{\Omega_h} + \delta_2 \sum_{E \in \mathfrak{G}_{h,\Omega}} h_E([p_h]_E, [q_h]_E)_E \\
&+ \delta_2 \sum_{T \in \mathfrak{T}_h} h_h^2 (\nabla p_h - 2\nabla \cdot \boldsymbol{D}(\boldsymbol{u}_h), \nabla q_h)_T = \delta_2 \sum_{T \in \mathfrak{T}_h} h_T^2 \langle \boldsymbol{w}, \nabla q_h \rangle_T \\
&\delta_1 \sum_{E \in \mathfrak{G}_{h,\Gamma}} h_E(\sigma_h - \boldsymbol{n}_h \cdot \boldsymbol{T}(\boldsymbol{u}_h, p_h) \cdot \boldsymbol{n}_h, \tau_h)_E + \langle \tau_h, \boldsymbol{u}_h \cdot \boldsymbol{n}_h \rangle_{\Gamma_h} = \langle g_h, \tau_h \rangle_{\Gamma_h} ,
\end{aligned}
$$

where δ_1 and δ_2 are as in Proposition 5.1. □

7 Treatment of non-convex domains and of higher order approximations of the boundary

If Ω is non-convex or if we approximate Γ by functions which are piecewise polynomials of order s, $s \geqq 2$, the set $\Omega_h \backslash \Omega$ is in general not empty. Hence, we cannot integrate equation (1.1) over Ω_h. To overcome this difficulty we consider the larger domain

$$
(7.1)\qquad \Omega_\varepsilon := \Omega \cup \left\{ x \in \mathbb{R}^d : \inf_{y \in \Gamma} |x - y| < \varepsilon \right\} .
$$

Note, that for any $\varepsilon > 0$ there is an $h_\varepsilon > 0$ such that $\Omega_h \subset \Omega_\varepsilon$ for all $0 < h \leqq h_\varepsilon$.

In [18], we have shown that there are an $\varepsilon_0 > 0$ and continuations $\tilde{\boldsymbol{u}} \in H^2(\Omega_{\varepsilon_0})^d$, $\tilde{p} \in H^1(\Omega_{\varepsilon_0})$, and $\tilde{\boldsymbol{f}} \in L^2(\Omega_{\varepsilon_0})^d$ such that

$$
(7.2)\qquad \|\tilde{\boldsymbol{u}}\|_{2,\Omega_{\varepsilon_0}} \leqq c\|\boldsymbol{u}\|_{2,\Omega}, \qquad \|\tilde{p}\|_{1,\Omega_{\varepsilon_0}} \leqq c\|p\|_{1,\Omega}, \qquad \|\tilde{\boldsymbol{f}}\|_{0,\Omega_{\varepsilon_0}} \leqq c\|\boldsymbol{f}\|_{0,\Omega}
$$

and

$$
\begin{aligned}
(7.3)\qquad & \|-\Delta\tilde{\boldsymbol{u}} + \nabla\tilde{p} - \tilde{\boldsymbol{f}}\|_{0,\Omega_h\backslash\Omega} + \|\nabla \cdot \tilde{\boldsymbol{u}}\|_{1,\Omega_h\backslash\Omega} \\
& \leqq c \operatorname{meas}(\Omega_h\backslash\Omega)^{1/3}\{\|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{3/2,\Gamma}\} .
\end{aligned}
$$

Note for practical computations, that $\tilde{f}$ can easily be computed by a reflection of f with respect to Γ. More precisely, let $x \in \Omega_{\varepsilon_0} \setminus \Omega$ and denote by x_0 the projection of x onto Γ and by $x' = 2x_0 - x$ its reflection across Γ. Then, $\tilde{f}$ is given by

$$\tilde{f}(x) = f(x') - 2[n(x_0) \cdot f(x')]n(x_0)\,. \tag{7.4}$$

With the same arguments as in [18], we conclude that the error estimates of Propositions 4.2, 5.2, and 6.1 remain valid for non-convex domains, provided f is replaced by $\tilde{f}$ in problems (4.1), (5.1), (6.1), and (6.2) and Ω_h is replaced by $\Omega \cap \Omega_h$ in the correspondng error estimates.

Similarly, we obtain $O(h^{\mathfrak{k}})$-error estimates for any value of $\mathfrak{k} \geqq 1$ in (3.3), provided the solutions of the Navier-Stokes equations (1.1) are sufficiently smooth and the boundary Γ is approximated by functions which are piecewise polynomials of sufficiently high degree.

8 Numerical implementation

The linear system of equations resulting from problem (4.1) has the structure

$$\begin{pmatrix} A & B^{\mathrm{T}} & C^{\mathrm{T}} \\ B & 0 & 0 \\ C + \delta E & \delta F & \delta D \end{pmatrix} \begin{pmatrix} u \\ p \\ \sigma \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}.$$

The matrix D is block-diagonal with blocks of size $\dim(Z_{h|E})$, $E \subset \mathfrak{G}_{h,\Gamma}$, and can be inverted locally on each $E \in \mathfrak{G}_{h,\Gamma}$. Therefore, the unknown σ_h can be eliminated on the element level and one obtains the reduced linear system of equations

$$\begin{pmatrix} A - C^{\mathrm{T}}D^{-1}E - \delta^{-1}C^{\mathrm{T}}D^{-1}C & B^{\mathrm{T}} - C^{\mathrm{T}}D^{-1}F \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \tag{8.1}$$

involving only the velocity and pressure. System (8.1) is only a slight modification near the boundary Γ_h of the corresponding system for the Stokes equations with no-slip boundary condition and may easily be implemented into standard finite element codes for the Navier-Stokes equations.

Similarly, the linear system of equations resulting from problem (5.1) has the structure

$$\begin{pmatrix} A & B^{\mathrm{T}} & C^{\mathrm{T}} \\ B + \delta_2 G & \delta_2 H & 0 \\ C + \delta_1 E & \delta_1 F & \delta_1 D \end{pmatrix} \begin{pmatrix} u \\ p \\ \sigma \end{pmatrix} = \begin{pmatrix} f \\ \delta_2 \tilde{f} \\ 0 \end{pmatrix}$$

where $\tilde{f}$ can easily be computed from f and where the matrix D is as above. The local elimination of the unknown σ_h now yields the reduced linear system of equations

$$\begin{pmatrix} A - C^{\mathrm{T}}D^{-1}E - \delta_1^{-1}C^{\mathrm{T}}D^{-1}C & B^{\mathrm{T}} - C^{\mathrm{T}}D^{-1}F \\ B + \delta_2 G & \delta_2 H \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ \delta_2 \tilde{f} \end{pmatrix} \tag{8.2}$$

involving once more only the velocity and pressure. System (8.2) can also be implemented easily into standard finite element codes for the Navier-Stokes equations. Note, that the matrix H vanishes when using continuous pressure approximations. When using discontinuous pressure approximations, the matrix H, however, cannot be inverted on the element level due to the jump-terms in (5.1). As in [11], the jump-terms could be omitted when approximating the velocity by finite elements of sufficiently high degree. The well-known example of the linear-constant element, however, shows that these terms are necessary in order to obtain a stable discretization when using low order approximations of the velocity.

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