5

More singular solutions

5.1 Behaviour of the derivatives of order higher than two

In this section, we look for $u \in W_p^{k+2}(\Omega)$, where k is a nonnegative integer, which are solutions of the same boundary value problems as in Chapter 4. However, we shall also consider non-homogeneous boundary conditions. In other words, we shall try to find necessary and sufficient conditions on the functions f and g_j, $1 \leqslant j \leqslant N$, ensuring that the following problem should have a solution u belonging to $W_p^{k+2}(\Omega)$:

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \gamma_j u = g_j & \text{on } \Gamma_j, \quad j \in \mathcal{D} \\ \gamma_j \dfrac{\partial u}{\partial \nu_j} + \beta_j \dfrac{\partial}{\partial \tau_j} \gamma_j u = g_j & \text{on } \Gamma_j, \quad j \in \mathcal{N}. \end{cases} \tag{5,1,1}$$

Here we keep the notation of Chapter 4.

Some necessary conditions are obvious. Indeed, if there exists $u \in W_p^{k+2}(\Omega)$ which is a solution of (5,1,1) we must have

$$f \in W_p^k(\Omega),$$

$$g_j \in W_p^{k+2-1/p}(\Gamma_j), \quad j \in \mathcal{D} \qquad \text{and} \qquad g_j \in W_p^{k+1-1/p}(\Gamma_j), \quad j \in \mathcal{N}.$$

In addition, we must have

(a) $g_j(S_j) = g_{j+1}(S_j)$, if j and $j+1 \in \mathcal{D}$ (5,1,2)

(b) $\dfrac{\partial g_j}{\partial \mu_{j+1}}(S_j) = g_{j+1}(S_j)$ if $j \in \mathcal{D}$, $j+1 \in \mathcal{N}$, $\boldsymbol{\mu}_{j+1}$ is parallel to $\boldsymbol{\tau}_j$ and $k+1 > (2/p)$

(c) $g_j(S_j) = \dfrac{\partial g_{j+1}}{\partial \mu_j}(S_j)$ if $j \in \mathcal{N}$, $j+1 \in \mathcal{D}$, $\boldsymbol{\mu}_j$ is parallel to $\boldsymbol{\tau}_{j+1}$ and $k+1 > (2/p)$

(d) $g_j(S_j) = \lambda^{-1} g_{j+1}(S_j)$ if j and $j+1 \in \mathcal{N}$, $\boldsymbol{\mu}_j = \lambda \boldsymbol{\mu}_{j+1}$ and $k+1 > (2/p)$.

(When $p=2$ and $k+1=1$, the pointwise conditions in (b)–(d) must be replaced by the corresponding conditions with integrals.)

As we saw in Chapter 4 in the particular case when $k=0$, some additional orthogonality conditions occur. They were the orthogonality of f to the space N_q. Here we shall find many more orthogonality conditions of the same nature when k is $\geqslant 1$. However, for several special values of the measure of the angles of Ω, we shall find additional conditions which generalize (5,1,2). This makes the study of higher derivatives of u a lot more difficult than the study of the second derivatives that we carried out in Chapter 4.

The technique that we shall use here is mainly based on the following idea. We shall extensively use the trace theorems of Chapter 1 to reduce the general problem (5,1,1) to the particular case when $g_j=0$ for every j and $f\in \mathring{W}_p^k(\Omega)$. In this particular case, the given function f fulfils a lot of unnatural homogeneous boundary conditions. However, due to these boundary conditions on f, it turns out that the derivatives up to the order k of the corresponding solution u are solutions of boundary value problems for the Laplace operator of the same kind as (4,1,1). Accordingly, we shall take advantage of the results proven in Chapter 4 to find the behaviour of these derivatives of u near the corners.

5.1.1 Special data

Let us look in a first step, at $u\in W_p^2(\Omega)$ which is a solution of (5,1,1) under the assumption that $g_j=0$ for every j and that $f\in \mathring{W}_p^1(\Omega)$. In other words we have

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \gamma_j u = 0 & \text{on } \Gamma_j,\quad j\in\mathcal{D} \\ \gamma_j \dfrac{\partial u}{\partial \nu_j} + \beta_j \dfrac{\partial}{\partial \tau_j}\gamma_j u = 0 & \text{on } \Gamma_j,\quad j\in\mathcal{N}. \end{cases} \tag{5,1,1,1}$$

We already know that u is smoother far from the corners.

Lemma 5.1.1.1 *Let $u\in W_p^2(\Omega)$ be the solution of problem* (5,1,1,1) *with f given in $W_p^1(\Omega)$; then we have*

$$u\in W_p^3(\Omega\setminus V)$$

for every closed neighbourhood V of the corners of Ω.

Proof First let $\eta\in\mathcal{D}(\Omega)$; we have

$$-\Delta(\widetilde{\eta u})+\widetilde{\eta u} = -\widetilde{\eta f}-\widetilde{[\Delta;\eta]}u+\widetilde{\eta u}\in W_p^1(\mathbb{R}^2)$$

and $\widetilde{\eta u} \in W_p^2(\mathbb{R}^2)$. Accordingly

$$\widetilde{\eta u} = E * \{-\widetilde{\eta f} - \widetilde{[\Delta; \eta]}u + \widetilde{\eta u}\}$$

where E is the elementary solution of $-\Delta + 1$ defined by

$$FE(\xi) = [1 + |\xi|^2]^{-1}, \qquad \xi \in \mathbb{R}^2.$$

Then Theorem 2.3.2.1 shows that $\eta u \in W_p^3(\Omega)$.

Now let $\eta \in \mathcal{D}(\bar{\Omega})$ be such that the support of η meets only the interior of one side, Γ_j say. Then let ω be any plane open set with smooth boundary such that $\bar{\omega}$ contains the support of η and $\Gamma \cap \partial\omega \subset \Gamma_j$. With the obvious modification when $j \in \mathcal{D}$, we have

$$\begin{cases} \Delta \widetilde{\eta u} = \widetilde{\eta f} + \widetilde{[\Delta; \eta]}u \in W_p^1(\omega) \\ \gamma \dfrac{\partial \widetilde{\eta u}}{\partial \nu} + \beta_j \dfrac{\partial}{\partial \tau} \gamma \widetilde{\eta u} = \left(\dfrac{\partial \eta}{\partial \nu} + \beta_j \dfrac{\partial \eta}{\partial \tau}\right) \gamma u \in W_p^{1-1/p}(\partial\omega) \end{cases}$$

and it follows from Theorem 2.5.1.1 that

$$\widetilde{\eta u} \in W_p^3(\omega).$$

The claim of Lemma 5.1.1.1 follows by partition of unity on $\Omega \setminus V$. ■

Now we consider any unit vector $\boldsymbol{\lambda} = (\alpha; \beta)$ and the corresponding derivative of u, i.e.

$$v = \frac{\partial u}{\partial \lambda} = \alpha D_x u + \beta D_y u.$$

This is obviously a function belonging to $W_p^1(\Omega)$ which is a solution of

$$\Delta v = \frac{\partial f}{\partial \lambda} \in L_p(\Omega).$$

The boundary conditions on v are the following.

Lemma 5.1.1.2 *Assume that $u \in W_p^2(\Omega)$ is the solution of problem (5,1,1,1) with f given in $\mathring{W}_p^1(\Omega)$. Let χ_j be the angle from $\boldsymbol{\tau}$ to $\boldsymbol{\lambda}$; then*

$$\gamma_j \frac{\partial v}{\partial \nu_j} + \tan(\Phi_j - \chi_j) \frac{\partial}{\partial \tau_j} \gamma_j v = 0 \qquad \text{on } \Gamma_j \tag{5,1,1,2}$$

when $(\Phi_j - \chi_j)/\pi - \frac{1}{2}$ is not an integer, and

$$\gamma_j v = 0 \qquad \text{on } \Gamma_j \tag{5,1,1,2'}$$

when $(\Phi_j - \chi_j)/\pi - \frac{1}{2}$ is an integer.

Proof By possibly performing a rotation of the coordinate axes, we can assume that Γ_j is supported by the axis $\{y=0\}$ and that Ω is 'above' Γ_j. Accordingly we have

$$\boldsymbol{\tau}_j=(1,0), \qquad \boldsymbol{\nu}_j=(0,-1)$$

and the boundary condition for u on Γ_j is

$$-(\cos \Phi_j)\gamma_j D_y u+(\sin \Phi_j)D_x\gamma_j u=0. \tag{5,1,1,3}$$

Differentiating with respect to x, we get

$$-(\cos \Phi_j)D_x\gamma_j D_y u+(\sin \Phi_j)D_x^2\gamma_j u=0. \tag{5,1,1,4}$$

On the other hand, the assumption that $f\in \mathring{W}_p^1(\Omega)$ implies that

$$\gamma_j f=\gamma_j D_y^2 u+D_x^2\gamma_j u=0. \tag{5,1,1,5}$$

From the definition of χ_j it follows that

$$v=(\cos \chi_j)D_x u+(\sin \chi_j)D_y u.$$

Accordingly, we have

$$\gamma_j\frac{\partial v}{\partial \nu_j}=-(\cos \chi_j)D_x\gamma_j D_y u-(\sin \chi_j)\gamma_j D_y^2 u$$

$$\gamma_j\frac{\partial v}{\partial \tau_j}=(\cos \chi_j)D_x^2\gamma_j u+(\sin \chi_j)D_x\gamma_j D_y u$$

and finally

$$\begin{aligned}\cos(\Phi_j-\chi_j)\gamma_j\frac{\partial v}{\partial \nu_j}&+\sin(\Phi_j-\chi_j)\frac{\partial}{\partial \tau_j}\gamma_j v\\ &=-(\cos \Phi_j)D_x\gamma_j D_y u-\sin\chi_j\cos(\Phi_j-\chi_j)\gamma_j D_y^2 u\\ &\quad+\cos\chi_j\sin(\Phi_j-\chi_j)D_x^2\gamma_j u.\end{aligned}$$

Using (5,1,1,5), we deduce that

$$\begin{aligned}\cos(\Phi_j-\chi_j)\gamma_j\frac{\partial v}{\partial \nu_j}&+\sin(\Phi_j-\chi_j)\frac{\partial}{\partial \tau_j}\gamma_j v\\ &=-(\cos \Phi_j)D_x\gamma_j D_y u+(\sin \Phi_j)D_x^2\gamma_j u;\end{aligned}$$

this last expression is zero by (5,1,1,4). This is the desired result when $(\Phi_j-\chi_j)/\pi-\frac{1}{2}$ is not an integer. Otherwise we have

$$\cos \chi_j=\varepsilon \sin \Phi_j, \qquad \sin \chi_j=-\varepsilon\cos \Phi_j$$

where ε is either $+1$ or -1. Accordingly, we have

$$\gamma_j v=\varepsilon\{(\sin \Phi_j)D_x\gamma_j u-(\cos \Phi_j)\gamma_j D_y u\}$$

and this is zero due to (5,1,1,3). ∎

We are now able to apply the results of Chapter 4. However, a side difficulty arises here. We shall not be able to apply Theorem 4.4.4.13 because of the possible non-uniqueness in that statement. Accordingly, we shall try to make use of Theorem 4.4.3.7, which deals with variational problems only. This is why we first have to localize our problem.

For this purpose, we introduce a cut-off function related to S_j as follows: $\eta_j \in \mathcal{D}(\bar{\Omega})$, η_j is one near S_j, the support of η_j does not meet $\bar{\Gamma}_l$ for $l \neq j, j+1$ and

$$\frac{\partial \eta_j}{\partial \nu_{j+1}} + \tan\left(\Phi_{j+1} - \Phi_j - \frac{\pi}{2} - \omega_j\right) \frac{\partial \eta_j}{\partial \tau_{j+1}} = 0 \tag{5,1,1,6}$$

if $(\Phi_{j+1} - \Phi_j - \omega_j)/\pi$ is not an integer.

Then we look at

$$v_j = \frac{\partial u}{\partial \mu_j};$$

in other words, we choose $\boldsymbol{\lambda} = \boldsymbol{\mu}_j/|\boldsymbol{\mu}_j|$ in order to study the behaviour of u near S_j. By Lemma 5.1.1.2, we have (the boundary condition is plainly $\gamma_{j+1} v_j = 0$ on Γ_{j+1} when $(\Phi_{j+1} - \Phi_j - \omega_j)/\pi$ is an integer)

$$\begin{cases} \gamma_j v_j = 0 & \text{on } \Gamma_j \\ \gamma_{j+1} \dfrac{\partial v_j}{\partial \nu_{j+1}} + \tan\left(\Phi_{j+1} - \Phi_j - \dfrac{\pi}{2} - \omega_j\right) \dfrac{\partial}{\partial \tau_{j+1}} \gamma_{j+1} v_j = 0 & \text{on } \Gamma_{j+1}. \end{cases} \tag{5,1,1,7}$$

The boundary conditions on the other sides will not matter, since we finally introduce

$$w_j = \eta_j v_j.$$

By Lemma 5.1.1.1 we know that $w_j \in W_p^1(\Omega)$ and that

$$\Delta w_j = \eta_j \frac{\partial}{\partial \mu_j} f + [\eta_j, \Delta] v_j \in L_p(\Omega).$$

The boundary conditions on w_j follow from (5,1,1,6) and (5,1,1,7). Accordingly, we have (see the note preceding (5,1,1,7))

$$\begin{cases} \gamma_l w_j = 0 & \text{on } \Gamma_l, \quad l \neq j+1 \\ \gamma_{j+1} \dfrac{\partial w_j}{\partial \nu_{j+1}} + \tan\left(\Phi_{j+1} - \Phi_j - \dfrac{\pi}{2} - \omega_j\right) \dfrac{\partial}{\partial \tau_{j+1}} \gamma_{j+1} w_j = 0 & \text{on } \Gamma_{j+1}. \end{cases} \tag{5,1,1,8}$$

Assuming $p \geq 2$ so that $w_j \in H^1(\Omega)$, it follows from Theorem 4.4.3.7

that there exist numbers $c'_{j,m'}$ such that

$$w_j - \sum_{\substack{-2/q<\lambda'_{j,m'}<0\\ \lambda'_{j,m'}\neq -1}} c'_{j,m'} s'_{j,m'} \in W^2_p(V_j) \tag{5,1,1,9}$$

where V_j is a neighbourhood of S_j in Ω and

$$\lambda'_{j,m'} = \frac{\Phi_j - \Phi_{j+1} + m'\pi}{\omega_j} + 1,$$

$$S'_{j,m'} = \frac{r_j^{-\lambda'_{j,m'}}}{(\sqrt{\omega_j})\lambda'_{j,m'}} \cos\left(\lambda'_{j,m}\theta_j + \Phi_{j+1} - \Phi_j - \frac{\pi}{2} - \omega_j\right)\eta_j(r_j e^{i\theta_j}).$$

This holds provided $\lambda'_{j,m'} \neq -2/q$ for all m.

We can now easily derive the following result.

Proposition 5.1.1.3 *Let $u \in W^2_p(\Omega)$ be the solution of problem (5,1,1,1). Assume that $f \in \mathring{W}^1_p(\Omega)$ and $p \geq 2$. Then there exist real numbers $C_{j,m}$ such that*

$$u - \sum_{\substack{-1-2/q<\lambda_{j,m}<-1\\ \lambda_{j,m}\neq -2}} C_{j,m} S_{j,m} \in W^3_p(V_j) \tag{5,1,1,10}$$

where V_j is a neighbourhood of S_j and $\lambda_{j,m} = (\Phi_{j+1} - \Phi_j + m\pi)/\omega_j$ (m an integer),

$$S_{j,m} = \frac{r_j^{-\lambda_{j,m}}}{(\sqrt{\omega_j})\lambda_{j,m}} \cos(\lambda_{j,m}\theta_j + \Phi_{j+1})$$

provided $\lambda_{j,m} \neq -1 - 2/q$ for all m.

Proof We merely integrate (5,1,1,9). Indeed w_j coincides with $\partial u/\partial \mu_j$ near S_j. We can rewrite (5,1,1,9) as follows:

$$\frac{\partial}{\partial \mu_j}\Bigg(u - \sum_{\substack{-1-2/q<\lambda_{j,m}<-1\\ \lambda_{j,m}\neq -2}} C_{j,m} S_{j,m}\Bigg) \in W^2_p(V_j), \tag{5,1,1,11}$$

for some real numbers $C_{j,m}$. Then let us consider any vector $\boldsymbol{\xi}_j$ orthogonal to $\boldsymbol{\mu}_j$ and having the same length $l = |\boldsymbol{\mu}_j|$. We shall show that

$$\frac{\partial}{\partial \xi_j}\Bigg(u - \sum_{\substack{-1-2/q<\lambda_{j,m}<-1\\ \lambda_{j,m}\neq -2}} C_{j,m} S_{j,m}\Bigg) \in W^2_p(V_j). \tag{5,1,1,12}$$

Indeed, let us set

$$\psi = u - \sum_{\substack{-1-2/q<\lambda_{j,m}<-1\\ \lambda_{j,m}\neq -2}} C_{j,m} S_{j,m}.$$

We already know from (5,1,1,11), that

$$\frac{\partial^{\alpha+\beta}\psi}{\partial\mu_i^\alpha\,\partial\xi_i^\beta}\in L_p(V_j) \tag{5,1,1,13}$$

for $\alpha+\beta\leq 3$, provided $\alpha\geq 1$. We just need to check that

$$\frac{\partial^3\psi}{\partial\xi_i^3}\in L_p(V_j),$$

since it is clear from the assumptions on u and the $\lambda_{j,m}$ that $\psi\in W_p^2(V_j)$ and accordingly ψ, $\partial\psi/\partial\xi_i$, $\partial^2\psi/\partial\xi_i^2\in L_p(V_j)$.

For this purpose we observe that

$$\Delta\psi=f\in W_p^1(V_j)$$

since $S_{j,m}$ is harmonic near S_j. It follows that

$$\frac{\partial^3\psi}{\partial\xi_i^3}=\frac{1}{l^2}\frac{\partial f}{\partial\xi_i}-\frac{\partial^3\psi}{\partial\mu_i^2\partial\xi_i}\in L_p(V_j)$$

due to (5,1,1,13). The claim (5,1,1,10) is an obvious consequence of (5,1,1,11) and (5,1,1,12). ■

We are now able to reach the main purpose of this subsection.

Theorem 5.1.1.4 *Let f be given in $\mathring{W}_p^1(\Omega)$ with $p\geq 2$ and $\int_\Omega f\,dx\,dy=0$ when $\mathscr{D}$ is empty. Assume that $(\Phi_j-\Phi_{j+1}+\omega_j+2\omega_j/q)/\pi$ is not an integer for any j. Then there exists a function u (possibly non-unique) and numbers $C_{j,m}$ such that*

$$u-\sum_{\substack{-1-2/q<\lambda_{j,m}<0\\ \lambda_{j,m}\neq -1,-2}} C_{j,m}S_{j,m}\in W_p^3(\Omega)$$

and u is solution of problem (5,1,1,1).

Proof We can choose $p_1\geq p$ such that $(\Phi_j-\Phi_{j+1}+2\omega_j/q_1)/\pi$ is not an integer for any j where q_1 is conjugate to p_1. Since $f\in L_{p_1}(\Omega)$, we can apply Theorem 4.4.4.13 with p replaced by p_1. Thus we know the existence of a function u and of numbers $C_{j,m}$ such that

$$\psi=u-\sum_{\substack{1\leq j\leq N\\ -2/q<\lambda_{j,m}<0\\ \lambda_{j,m}\neq -1}} C_{j,m}S_{j,m}\in W_{p_1}^2(\Omega)$$

and u is solution of problem (5,1,1,1). Since the functions $S_{j,m}$ are all solutions of the homogeneous problem corresponding to (5,1,1,1), we can apply Proposition 5.1.1.3 to ψ near each of the corners S_j. The smoothness of u far from the corners follows from Lemma 5.1.1.1. ■

Iterating the above procedure yields the following result.

Theorem 5.1.1.5 *Let f be given in $\mathring{W}_p^k(\Omega)$ with $p \geq 2$. Assume that*

$$\frac{\Phi_j - \Phi_{j+1} + k\omega_j + 2\omega_j/q}{\pi}$$

is not an integer for any j. Then there exists a function u (possibly non-unique) and numbers $C_{j,m}$ such that

$$u - \sum_{\substack{-k-2/q<\lambda_{j,m}<0\\ \lambda_{j,m}\neq -1,-2,\ldots,-k}} C_{j,m}S_{j,m} \in W_p^{k+2}(\Omega)$$

and u is solution of problem (5,1,1,1).

5.1.2 A trace theorem

We now want to solve the general non-homogeneous boundary value problem (5,1,1). This will be achieved by reducing the general case to the particular case which we solved in Section 5.1.1. For this purpose we need to find the necessary and sufficient conditions on the functions $f \in W_p^k(\Omega)$ and $g_j \in W_p^{k+2-1/p}(\Gamma_j)$, $j \in \mathscr{D}$, $g_j \in W_p^{k+1-1/p}(\Gamma_j)$, $j \in \mathscr{N}$ which ensure the existence of a function $v \in W_p^{k+2}(\Omega)$ s.t.

$$\begin{cases} \gamma_j v = g_j & \text{on } \Gamma_j, \quad j \in \mathscr{D} \\ \gamma_j \dfrac{\partial v}{\partial \nu_j} + \beta_j \dfrac{\partial}{\partial \tau_j} \gamma_j v = g_j & \text{on } \Gamma_j, \quad j \in \mathscr{N} \end{cases} \tag{5,1,2,1}$$

and

$$\Delta v - f \in \mathring{W}_p^k(\Omega).$$

In other words, we are looking for a function v which fulfils the boundary conditions (5,1,2,1) and

$$\gamma_j \frac{\partial^l}{\partial \nu_j^l} \Delta u = \gamma_j \frac{\partial^l f}{\partial \nu_j^l} \quad \text{on } \Gamma_j, \quad 0 \leq l \leq k-1 \tag{5,1,2,2}$$

(see Remark 1.5.2.11). This trace problem will be solved with the help of Theorems 1.6.1.4 and 1.6.1.5 (and Remark 1.6.1.8).

In a first step let us define the operators $B_{j,l}$, which we shall need to apply Theorem 1.6.1.4. The first operator $B_{j,1}$ will be either I when $j \in \mathscr{D}$ or $\partial/\partial\mu_j$ when $j \in \mathscr{N}$, while

$$B_{j,l} = \left(\frac{\partial}{\partial \nu_j}\right)^{l-2} \Delta, \qquad l = 2, \ldots, k+1.$$

Accordingly, the order of $B_{j,1}$ is either zero or one, while the order of $B_{j,l}$ is l, $l = 2, \ldots, k+1$. In order to be able to apply Theorem 1.6.1.4 we must find all the operators $P_{j,l}$ and $Q_{j+1,l}$ fulfilling condition (1,6,1,1).

Since we are dealing with a polygon (in the strict sense) and the operators $B_{j,l}$ are homogeneous with constant coefficients, we can restrict ourselves to looking for operators $P_{j,l}$ and $Q_{j+1,l}$, also homogeneous and with constant coefficients.

We shall first look for the operators $P_{j,1}$ and $Q_{j+1,1}$. We must have (by 1,6,1,1))

$$P_{j,1}B_{j,1}-Q_{j+1,1}B_{j+1,1}=\sum_{l=2}^{k+1}\left\{Q_{j+1,l}\left(\frac{\partial}{\partial\nu_{j+1}}\right)^{l-2}-P_{j,l}\left(\frac{\partial}{\partial\nu_j}\right)^{l-2}\right\}\Delta. \tag{5,1,2,3}$$

We observe that

$$\sum_{l=2}^{k+1}Q_{j+1,l}\left(\frac{\partial}{\partial\nu_{j+1}}\right)^{l-2}\quad\text{and}\quad\sum_{l=2}^{k+1}P_{j,l}\left(\frac{\partial}{\partial\nu_j}\right)^{l-2}$$

can be any homogeneous differential operators of order $d-2$ when d is the order of both sides of (5,1,2,3). Indeed $Q_{j+1,l}$ and $P_{j,l}$ are tangential operators to Γ_{j+1} and Γ_j respectively. Accordingly, the identity (5,1,2,3) means that the symbol of

$$P_{j,1}B_{j,1}-Q_{j+1,1}B_{j+1,1}$$

can be divided by the symbol of Δ.

Since $P_{j,1}$ and $Q_{j+1,1}$ are tangential to Γ_{j-1} and Γ_j, respectively, it turns out that

$$P_{j,1}=\begin{cases}a_j\left(\dfrac{\partial}{\partial\tau_j}\right)^{d} & \text{if } j\in\mathcal{D}\\[2ex] a_j\left(\dfrac{\partial}{\partial\tau_j}\right)^{d-1} & \text{if } j\in\mathcal{N}\end{cases} \tag{5,1,2,4}$$

$$Q_{j+1,1}=\begin{cases}b_j\left(\dfrac{\partial}{\partial\tau_{j+1}}\right)^{d} & \text{if } j+1\in\mathcal{D}\\[2ex] b_j\left(\dfrac{\partial}{\partial\tau_{j+1}}\right)^{d-1} & \text{if } j+1\in\mathcal{N}.\end{cases} \tag{5,1,2,5}$$

Lemma 5.1.2.1 *There exists real numbers a_j and b_j s.t. $P_{j,1}B_{j,1}-Q_{j+1,1}B_{j+1,1}$ can be divided by Δ where $P_{j,1}$ and $Q_{j+1,1}$ are defined by* (5,1,2,4) *and* (5,1,2,5) *and such that*

$$a_j^2+b_j^2\neq 0$$

iff $(\Phi_{j+1}-\Phi_j-\mathrm{d}\omega_j)/\pi$ is an integer.

Proof Let us look for instance at the particular case when both j and

$j+1$ belong to $\mathcal{N}$. Accordingly, we must be able to divide the symbol of

$$a_j\left(\frac{\partial}{\partial\tau_j}\right)^{d-1}\frac{\partial}{\partial\mu_j}-b_j\left(\frac{\partial}{\partial\tau_{j+1}}\right)^{d-1}\frac{\partial}{\partial\mu_{j+1}}$$

by the symbol of Δ. Equivalently we must be able to divide the polynomial

$$\begin{aligned}a_j(-1)^{d-1}(x\cos\omega_j+y\sin\omega_j)^{d-1}([-x\sin\omega_j+y\cos\omega_j]\\ -\tan\Phi_j[x\cos\omega_j+y\sin\omega_j])-b_jx^{d-1}(-y+\tan\Phi_{j+1}x)=p(x,y)\end{aligned}$$

by x^2+y^2. This means that $x=\pm iy$ are roots of the polynomial $p(x,y)$. Writing $p(\pm iy,y)=0$ leads to the following system of two equations in the two unknowns a_j and b_j:

$$\begin{aligned}a_j(-1)^{d-1}(\sin\omega_j\pm i\cos\omega_j)^{d-1}([\cos\omega_j\mp i\sin\omega_j]\\ -\tan\Phi_j[\sin\omega_j\pm i\cos\omega_j])-b_j(\pm i)^d(-1\pm i\tan\Phi_{j+1})=0.\end{aligned}$$

This system is equivalent to the following

$$a_j(-1)^{d-1}e^{\mp i\omega_jd}[1\mp i\tan\Phi_j]+b_j[1\mp i\tan\Phi_{j+1}]=0. \tag{5,1,2,6}$$

The determinant is proportional to

$$\begin{aligned}&e^{-i\omega_jd}(1-i\tan\Phi_j)(1+i\tan\Phi_{j+1})-e^{i\omega_jd}(1+i\tan\Phi_j)(1-i\tan\Phi_{j+1})\\ &=e^{-i\omega_jd}(1+\tan\Phi_j\tan\Phi_{j+1}+i[\tan\Phi_{j+1}-\tan\Phi_j])\\ &\quad-e^{i\omega_jd}(1+\tan\Phi_j\tan\Phi_{j+1}-i[\tan\Phi_{j+1}-\tan\Phi_j])\\ &=(1+\tan\Phi_j\tan\Phi_{j+1})\{e^{-i\omega_jd}(1+i\tan[\Phi_{j+1}-\Phi_j])\\ &\quad-e^{+i\omega_jd}(1-i\tan[\Phi_{j+1}-\Phi_j])\}\\ &=(1+\tan\Phi_j\tan\Phi_{j+1})\frac{e^{-i\omega_jd}e^{i[\Phi_{j+1}-\Phi_j]}-e^{+i\omega_jd}e^{-i[\Phi_{j+1}-\Phi_j]}}{\cos[\Phi_{j+1}-\Phi_j]}\\ &=2i(1+\tan\Phi_j\tan\Phi_{j+1})\frac{\sin[\Phi_{j+1}-\Phi_j-\omega_jd]}{\cos[\Phi_{j+1}-\Phi_j]}.\end{aligned}$$

Obviously this determinant is zero iff $(\Phi_{j+1}-\Phi_j-\omega_jd)/\pi$ is an integer.

Similar calculations yield to the same result when j or $j+1$ belong to $\mathcal{D}$. The system (5,1,2,6) is replaced by

$$a_j(-1)^{d-1}e^{\mp id\omega_j}[\pm i]-b_j[1\mp i\tan\Phi_{j+1}]=0 \tag{5,1,2,7}$$

when $j\in\mathcal{D}$ and $j+1\in\mathcal{N}$,

$$a_j(-1)^{d-1}e^{\mp id\omega_j}[1\mp i\tan\Phi_j]\mp ib_j=0 \tag{5,1,2,8}$$

when $j\in\mathcal{N}$ and $j+1\in\mathcal{D}$ and

$$a_j(-1)^de^{\mp id\omega_j}-b_j=0 \tag{5,1,2,9}$$

when j and $j+1\in\mathcal{D}$. ■

Remark 5.1.2.2 When the determinant is zero, the space of the solutions of the systems (5,1,2,6) to (5,1,2,9) is one-dimensional. We shall completely determine a_j and b_j, by assuming in addition that $b_j = 1$. Accordingly, we have (setting $m = (\Phi_{j+1} - \Phi_j - d\omega_j)/\pi$):

$$a_j = (-1)^d e^{i\omega_j d} \frac{1 - i \tan \Phi_{j+1}}{1 - i \tan \Phi_j} = (-1)^{d+m} \frac{\cos \Phi_j}{\cos \Phi_{j+1}}$$

when j and $j+1 \in \mathcal{N}$,

$$a_j = (-1)^d e^{id\omega_j} i[1 - i \tan \Phi_{j+1}] = \frac{(-1)^{d+m}}{\cos \Phi_{j+1}}$$

when $j \in \mathcal{D}$ and $j+1 \in \mathcal{N}$,

$$a_j = (-1)^d e^{id\omega_j} \frac{-i}{[1 - i \tan \Phi_j]} = (-1)^{d+m} \cos \Phi_j$$

when $j \in \mathcal{N}$ and $j+1 \in \mathcal{D}$ and

$$a_j = (-1)^d e^{id\omega_j} = (-1)^{d+m}$$

when j and $j+1 \in \mathcal{D}$.

Let us now introduce one more notation. We denote by $R_{j,d}$ the differential operator (homogeneous and of order $d-2$: i.e., $R_{j,d} = 0$ if $d < 2$) s.t.

$$a_j \left(\frac{\partial}{\partial \tau_j}\right)^{d-1} \frac{\partial}{\partial \mu_j} - \left(\frac{\partial}{\partial \tau_{j+1}}\right)^{d-1} \frac{\partial}{\partial \mu_{j+1}} = R_{j,d} \Delta \tag{5,1,2,10}$$

when j and $j+1 \in \mathcal{N}$,

$$a_j \left(\frac{\partial}{\partial \tau_j}\right)^{d} - \left(\frac{\partial}{\partial \tau_{j+1}}\right)^{d-1} \frac{\partial}{\partial \mu_{j+1}} = R_{j,d} \Delta \tag{5,1,2,11}$$

when $j \in \mathcal{D}$ and $j+1 \in \mathcal{N}$,

$$a_j \left(\frac{\partial}{\partial \tau_j}\right)^{d-1} \frac{\partial}{\partial \mu_j} - \left(\frac{\partial}{\partial \tau_{j+1}}\right)^{d} = R_{j,d} \Delta \tag{5,1,2,12}$$

when $j \in \mathcal{N}$ and $j+1 \in \mathcal{D}$, and

$$a_j \left(\frac{\partial}{\partial \tau_j}\right)^{d} - \left(\frac{\partial}{\partial \tau_{j+1}}\right)^{d} = R_{j,d} \Delta \tag{5,1,2,13}$$

when j and $j+1 \in \mathcal{D}$. These identities are nothing but identity (1,6,1,1) in the particular case that we study here.

Accordingly, Theorem 1.6.1.4 implies the following result.

Theorem 5.1.2.3 *Assume* $p \neq 2$ *and let* $f \in W_p^k(\Omega)$ *and* $g_j \in$

$W^{k+2-1/p}(\Gamma_j)$, $j \in \mathcal{D}$, $g \in W^{k+1-1/p}(\Gamma_j)$, $j \in \mathcal{N}$ be given. Then there exists a solution $v \in W_p^{k+2}(\Omega)$ of the boundary conditions (5,1,2,1) *and* (5,1,2,2) *iff the following equalities hold*:

$$a_j \frac{\partial^{d-1} g_j}{\partial \tau_j^{d-1}}(S_j) - \frac{\partial^{d-1} g_{j+1}}{\partial \tau_{j+1}^{d-1}}(S_j) = R_{j,d} f(S_j) \tag{5,1,2,14}$$

when j and $j+1 \in \mathcal{N}$,

$$a_j \frac{\partial^{d} g_j}{\partial \tau_j^{d}}(S_j) - \frac{\partial^{d-1} g_{j+1}}{\partial \tau_{j+1}^{d-1}}(S_j) = R_{j,d} f(S_j) \tag{5,1,2,15}$$

when $j \in \mathcal{D}$ and $j+1 \in \mathcal{N}$,

$$a_j \frac{\partial^{d-1} g_j}{\partial \tau_j^{d-1}}(S_j) - \frac{\partial^{d} g_{j+1}}{\partial \tau_{j+1}^{d}}(S_j) = R_{j,d} f(S_j) \tag{5,1,2,16}$$

when $j \in \mathcal{N}$ and $j+1 \in \mathcal{D}$ and

$$a_j \frac{\partial^{d} g_j}{\partial \tau_j^{d}}(S_j) - \frac{\partial^{d} g_{j+1}}{\partial \tau_{j+1}^{d}}(S_j) = R_{j,d} f(S_j) \tag{5,1,2,17}$$

when j and $j+1 \in \mathcal{D}$, for all $d \in [0, k+2/q[$ and j s.t. $(\Phi_{j+1} - \Phi_j - d\omega_j)/\pi$ is an integer. ($d \geq 1$ when j or $j+1 \in \mathcal{N}$).

Proof We just have rewritten the identity

$$P_{j,1} g_j(S_j) - Q_{j+1,1} g_{j+1}(S_j) = \sum_{l=2}^{k+1} Q_{j+1,l} \frac{\partial^{l-2} f}{\partial \nu_{j+1}^{l-2}}(S_j) - \sum_{l=2}^{k-1} P_{j,l} \frac{\partial^{l-2} f}{\partial \nu_j^{l-2}}(S_j)$$

as

$$P_{j,1} g_j(S_j) - Q_{j+1,1} g_{j+1}(S_j) = R_{j,d} f(S_j)$$

for the corresponding value of d. ■

The similar result when $p = 2$ follows from Theorem 1.6.1.5.

Theorem 5.1.2.4 *Let $f \in H^k(\Omega)$ and $g_j \in H^{k+3/2}(\Gamma_j)$, $j \in \mathcal{D}$, $g_j \in H^{k+1/2}(\Gamma_j)$, $j \in \mathcal{N}$ be given. Then there exists a solution $v \in H^{k+2}(\Omega)$ of the boundary conditions* (5,1,2,1) *and* (5,1,2,2) *iff equalities* (5,1,2,14) *to* (5,1,2,17) *hold for all $d \in [0, k+1[$ and j s.t. $(\Phi_{j+1} - \Phi_j - d\omega_j)/\pi$ is an integer and provided*

$$\frac{\Phi_{j+1} - \Phi_j - (k+1)\omega_j}{\pi}$$

is not an integer for any j.

This last provision is made to avoid possible identities (5,1,2,10) to

(5,1,2,13) corresponding to the order $d = k+1$. Such an identity would yield a condition with an integral similar to (1,6,1,3).

Remark 5.1.2.5 The identities (5,1,2,10) to (5,1,2,13) corresponding to either $d=0$ or $d=1$ are just (5,1,2).

5.1.3 More singular solutions

We first infer the consequences of Theorem 5.1.1.5 and Theorem 5.1.2.3 or 5.1.2.4.

Theorem 5.1.3.1 *Let $f \in W_p^k(\Omega)$ and $g_j \in W_p^{k+2-1/p}(\Gamma_j)$, $j \in \mathcal{D}$, $g_j \in W_p^{k+1-1/p}(\Gamma_j)$, $j \in \mathcal{N}$, be given. Assume that $(\Phi_{j+1} - \Phi_j - k\omega_j - 2\omega_j/q)/\pi$ is not an integer for any j and that the identities* (5,1,2,14) *to* (5,1,2,17) *hold for all $d \in [0, k+1[$ and all j such that $(\Phi_{j+1} - \Phi_j - d\omega_j)/\pi$ is an integer. Then there exists a function u (possibly non-unique) and numbers $c_{j,m}$ such that*

$$u - \sum_{\substack{-k-2/q<\lambda_{j,m}<0\\ \lambda_{j,m}\neq -1,-2,\ldots,-k}} C_{j,m} S_{j,m} \in W_p^{k+2}(\Omega)$$

and u is solution of problem (5,1,1,1).

Indeed, let v be given by Theorem 5.1.2.3 or Theorem 5.1.2.4; we just have to apply Theorem 5.1.1.5 with f replaced by $f - \Delta v$, since this function belongs to $\mathring{W}_p^k(\Omega)$.

Remark 5.1.3.2 The number of extra conditions (5,1,2,14) to (5.1.2.17) on the data of f and g_j, $1 \leq j \leq N$, is exactly the number of eigenvalues $\lambda_{j,m}$ which are excluded from the sum which describes the singular behaviour of u, by the condition

$$\lambda_{j,m} \neq -1, -2, \ldots, -k.$$

For practical purposes, the identities (5,1,2,14) to (5,1,2,17) are not easy to check on functions given explicitly. This is due, in particular, to the fact that we did not attempt to find the operators $R_{j,d}$. We shall rather try to understand the particularly singular behaviour of the solution u which occurs when one of these identities is not fulfilled. We need a preliminary result.

Lemma 5.1.3.3 *Let s be the function*

$$s(r;\theta) = r^{-\lambda_{j,m}}[\ln r \cos(\lambda_{j,m}\theta + \Phi_{j+1}) + \theta \sin(\lambda_{j,m}\theta + \Phi_{j+1})]$$

where $\lambda_{j,m} = (\Phi_j - \Phi_{j+1} + m\pi)/\omega_j$ is assumed to be a negative integer. Then *s is harmonic for* $r>0$ *and* $\theta\in\,]0,\omega_j[$. *Furthermore, we have*

$$\begin{cases} -\dfrac{1}{r}\dfrac{\partial s}{\partial\theta}+\tan\Phi_{j+1}\dfrac{\partial s}{\partial r}=0 & \textit{for } r>0,\quad \theta=0\\ s=0\quad \textit{for}\quad r>0,\quad \theta=0 \textit{ if } \cos\Phi_{j+1}=0 \end{cases}$$

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial s}{\partial\theta}-\tan\Phi_j\dfrac{\partial s}{\partial r}=(-1)^m\dfrac{\lambda_{j,m}\omega_j}{\cos\Phi_j}r^{-\lambda_{j,m}-1} & \textit{for } r>0,\quad \theta=\omega_j\\ s=(-1)^m\omega_j\sin\Phi_j r^{-\lambda_{j,m}} \quad \textit{for } r>0,\quad \theta=\omega_j & \textit{if } \cos\Phi_j=0. \end{cases}$$

This can be verified directly. Now let us consider a cut-off function η_j similar to the one we have used in Chapter 4. In other words, we have $\eta_j\in\mathcal{D}(\bar{\Omega})$, $\eta_j=1$ near S_j; the support of η_j does not meet $\bar{\Gamma}_l$ for $l\neq j, j+1$, and

$$\begin{cases} \dfrac{\partial\eta_j}{\partial\mu_j}=0 & \text{on } \Gamma_j \quad \text{if } j\in\mathcal{N}\\ \dfrac{\partial\eta_j}{\partial\mu_{j+1}}=0 & \text{on } \Gamma_{j+1} \quad \text{if } j+1\in\mathcal{N}. \end{cases}$$

Let us then set

$$\begin{aligned} \mathfrak{S}_{j,m}(r_je^{i\theta_j}) = r_j^{-\lambda_{j,m}}[&\ln r_j\cos(\lambda_{j,m}\theta_j+\Phi_{j+1})\\ &+\theta_j\sin(\lambda_{j,m}\theta_j+\Phi_{j+1})]\eta_j(r_je^{i\theta_j}). \end{aligned} \tag{5,1,3,1}$$

This function has the following properties:

$$\Delta\mathfrak{S}_{j,m}=f_{j,m}\in C^\infty(\bar{\Omega})$$

where $f_{j,m}$ is zero near all the corners and

$$\gamma_k\frac{\partial\mathfrak{S}_{j,m}}{\partial\nu_k}+\tan\Phi_k\frac{\partial}{\partial\gamma_k}\mathfrak{S}_{j,m}=g_{j,m,k}\in C^\infty(\bar{\Gamma}_k)$$

for $k\in\mathcal{N}$ and

$$\gamma_k\mathfrak{S}_{j,m}=g_{j,m,k}\in C^\infty(\bar{\Gamma}_k)$$

for $k\in\mathcal{D}$. In addition $g_{j,m,k}$ is zero for all k, but $k=j$ and $k=j+1$, and we have

$$g_{j,m,j}=\begin{cases} (-1)^m\dfrac{\lambda_{j,m}\omega_j}{\cos\Phi_j}r_j^{-\lambda_{j,m}-1} & \text{if } j\in\mathcal{N}\\ (-1)^m\omega_j\sin\Phi_j r_j^{-\lambda_{j,m}} & \text{if } j\in\mathcal{D} \end{cases}$$

near S_j, while

$$g_{j,m,j+1} = 0$$

near S_j.

Accordingly, the necessary condition at S_j in Theorem 5.1.2.3 is not fulfilled. On the other hand, it is easy to check that

$$\mathfrak{S}_{j,m} \in H^s(\Omega)$$

iff $s < -\lambda_{j,m} + 1$, while $\mathfrak{S}_{j,m} \notin H^{-\lambda_{j,m}+1}(\Omega)$.

The conclusion of these preliminaries is the following, where for convenience, we introduce a new definition.

Definition 5.1.3.4 *We define the function* $\mathfrak{S}_{j,m}$ *as follows:*

$$\mathfrak{S}_{j,m}(r_j e^{i\theta_j}) = r_j^{-\lambda_{j,m}} \cos(\lambda_{j,m}\theta_j + \Phi_{j+1})\eta_j(r_j e^{i\theta_j})$$

when $\lambda_{j,m}$ *is not an integer*† *and*

$$\mathfrak{S}_{j,m}(r_j e^{i\theta_j}) = r_j^{-\lambda_{j,m}}\{\ln r_j \cos(\lambda_{j,m}\theta_j + \Phi_{j+1}) + \theta_j \sin(\lambda_{j,m}\theta_j + \Phi_{j+1})\}\eta_j(r_j e^{i\theta_j})$$

when $\lambda_{j,m}$ *is an integer.*

Theorem 5.1.3.5 *Let* $f \in W_p^k(\Omega)$ *and*

$$g_j \in W_p^{k+2-1/p}(\Gamma_j), \quad j \in \mathcal{D}, \qquad g_j \in W_p^{k+1-1/p}(\Gamma_j), \quad j \in \mathcal{N},$$

be given. Assume that $(\Phi_{j+1} - \Phi_j - k\omega_j - 2\omega_j/q)/\pi$ *is not an integer for any* j *and that* $g_j(S_j) = g_{j+1}(S_j)$ *whenever* j *and* $j+1 \in \mathcal{D}$*. Then there exists a function* u *(possibly non-unique) and numbers* $k_{j,m}$*, such that*

$$u - \sum_{-k-2/q < \lambda_{j,m} < 0} k_{j,m}\mathfrak{S}_{j,m} \in W_p^{k+2}(\Omega)$$

and u *is solution of problem* (5,1,1,1).

Proof We shall apply Theorem 5.1.3.1. For each j,m such that $\lambda_{j,m}$ is an integer belonging to the interval $]-k-2/q, 0[$, let us define $k_{j,m}$ as follows:

$$k_{j,m}\left[a_j \frac{\partial^{d-1} g_{j,m,j}}{\partial \tau_j^{d-1}}(S_j) - \frac{\partial^{d-1} g_{j,m,j+1}}{\partial \tau_{j+1}^{d-1}}(S_j)\right]$$
$$= \left[a_j \frac{\partial^{d-1} g_j}{\partial \tau_j^{d-1}}(S_j) - \frac{\partial^{d-1} g_{j+1}}{\partial \tau_{j+1}^{d-1}}(S_j) - R_{j,d} f(S_j)\right]$$

† Observe that here $\mathfrak{S}_{j,m}$ is just a relabelling for $(\sqrt{\omega_j})\lambda_{j,m} S_{j,m}$.

when j and $j+1 \in \mathcal{N}$, $d = (\Phi_{j+1} - \Phi_j - m\pi)/\omega_j$. We define $k_{j,m}$ in a similar fashion (*mutatis mutandis*) when j or $j+1 \in \mathcal{D}$.

It is clear that

$$f' = f - \sum_{\substack{-k-2/q<\lambda_{j,m}<0 \\ \lambda_{j,m} \text{ integer}}} k_{j,m} f_{j,m}$$

and

$$g_j' = g_j - \sum_{\substack{-k-2/q<\lambda_{l,m}<0 \\ \lambda_{l,m} \text{ integer}}} k_{l,m} g_{l,m,j}, \qquad 1 \leq j \leq N,$$

fulfil the assumptions of Theorem 5.1.3.1. Consequently, there exist a function u' and numbers $c_{j,m}$ such that

$$u' - \sum_{\substack{-k-2/q<\lambda_{j,m}<0 \\ \lambda_{j,m} \neq -1,-2,\ldots,-k}} c_{j,m} S_{j,m} \in W_p^{k+2}(\Omega)$$

and u' is a solution of

$$\begin{cases} \Delta u' = f' & \text{in } \Omega \\ \gamma_j u' = g_j' & \text{on } \Gamma_j, \quad j \in \mathcal{D} \\ \gamma_j \dfrac{\partial u'}{\partial \nu_j} + \beta_j \dfrac{\partial}{\partial \tau_j} \gamma_j u' = g_j' & \text{on } \Gamma_j, \quad j \in \mathcal{N} \end{cases}$$

We conclude the proof by setting

$$u = u' + \sum_{\substack{-k-2/q<\lambda_{j,m}<0 \\ \lambda_{j,m} \text{ integer}}} k_{j,m} \mathfrak{S}_{j,m}$$

and $k_{j,m} = c_{j,m}/(\sqrt{\omega_j})\lambda_{j,m}$.

Remark 5.1.3.6 The solution u in Theorem 5.1.3.5 is unique when there is uniqueness in the space $H^1(\Omega)$ (see Section 4.4.3) or in the space $W_r^1(\Omega)$ with $r > 2$ (see Section 4.4.4).

Remark 5.1.3.7 Again one can also handle the domains with cuts by applying the same techniques (see Remark 1.7.4 in connection with the trace results of Subsection 5.1.2). Therefore the results in Theorem 5.1.3.5 still hold if we allow $\omega_j = 2\pi$ for some j. For instance in the case of a Dirichlet problem on both sides of a cut (i.e. $\omega_j = 2\pi$ and $j \in \mathcal{D}$, $j+1 \in \mathcal{D}$) the singular solutions are the following:

$$\mathfrak{S}_{j,m} = r_j^{m/2} \sin(m\theta_j/2)\eta_j(r_j e^{i\theta_j})$$

when m is odd and

$$\mathfrak{S}_{j,m} = r_j^{m/2}\{\ln r_j \sin(m\theta_j/2) + \theta_j \cos(m\theta_j/2)\}\eta_j(r_j e^{i\theta_j})$$

when m is even.

5.2 Operators with variable coefficients

A natural continuation of the study carried out in the previous sections would be to investigate boundary value problems with variable coefficients in a plane domain whose boundary is a curvilinear polygon. The simplest idea is to apply the well-known perturbation method to reduce such a problem to similar problems involving only homogeneous operators with constant coefficients. This method will enable us to extend only part of the preceding results to problems with variable coefficients.

Here just to illustrate such a method, we shall restrict ourselves to the study of a Dirichlet problem. Thus we will also avoid a lot of side difficulties which have nothing to do with the specific problem of singular behaviour of solutions near the corners.

The data are the following. We consider a plane bounded open domain Ω whose boundary Γ is a curvilinear polygon of class $C^{1,1}$ (see Definition 1.4.5.1). Thus $\Gamma = \bigcup_{j=1}^{N} \Gamma_j$, where Γ_j is an open arc of curve of class $C^{1,1}$ and $\bar{\Gamma}_j$ meets $\bar{\Gamma}_{j+1}$ at S_j. The measure of the angle of the tangent vectors to $\bar{\Gamma}_j$ and $\bar{\Gamma}_{j+1}$ at S_j (toward the interior of Ω), will again by denoted by ω_j, $1 \leq j \leq N$. Next, we consider the elliptic operator A defined by

$$Au = \sum_{i,j=1}^{2} D_i(a_{ij}D_j u) + \sum_{i=1}^{2} a_i D_i u + a_0 u,$$

where $a_{i,j} = a_{j,i} \in C^{0,1}(\bar{\Omega})$ (it is sufficient throughout this section to assume that $a_{i,j} \in W_p^1(\Omega)$ for some $p > 2$) $a_i \in L^\infty(\bar{\Omega})$, $0 \leq i \leq 2$. The ellipticity of A means the existence of $\alpha > 0$ such that

$$\sum_{i,j=1}^{2} a_{i,j}(x)\xi_i\xi_j \leq -\alpha\, |\xi|^2 \tag{5,2,1}$$

for all $x \in \bar{\Omega}$ and $\xi \in \mathbb{R}^2$.

Our first purpose is to calculate the index of the operator A from $W_p^2(\Omega) \cap \mathring{W}_p^1(\Omega)$ into $L_p(\Omega)$. For simplicity, we assume that the corresponding boundary value problem has a unique variational solution in $\mathring{H}^1(\Omega)$. This can be achieved by assuming for instance that

$$\min_{x \in \bar{\Omega}} a_0(x) > \frac{1}{4\alpha} \max_{\substack{x \in \bar{\Omega} \\ i=1,2}} |a_i(x)|^2. \tag{5,2,2}$$

Consequently, applying Lemma 2.2.1.1, it is easily seen that for any given $f \in L_p(\Omega)$, there exists a unique solution $u \in \mathring{H}^1(\Omega)$ of the equation

$$Au = f \qquad \text{in } \Omega. \tag{5,2,3}$$

In addition it follows from the results in Subsection 2.5.1, that

$$u \in W_p^2(\Omega \setminus V)$$

for every closed neighbourhood V of the corners. Thus we just have to investigate the behaviour of u near the corners.

Before stating our main result, we need to introduce one more notation. We consider the operator A_j obtained from A by freezing the coefficients of its principal part at S_j:

$$A_j u = \sum_{k,l=1}^{2} a_{k,l}(S_j) D_k D_l u.$$

Then let us consider one matrix $\mathcal{T}_j$ such that

$$-\mathcal{T}_j \mathcal{A}_j \mathcal{T}_j = I$$

where $\mathcal{A}_j$ is the symmetrix matrix whose entries are the $a_{k,l}(S_j)$, $1 \leq k, l \leq 2$. We clearly have

$$(A_j u)(\mathcal{T}_j^{-1} x) = -(\Delta v_j)(x)$$

where $v_j(x) = u(\mathcal{T}_j^{-1} x)$.

Definition 5.2.1 *We denote by $\omega_j(A)$ the measure of the angle at $\mathcal{T}_j S_j$ of $\mathcal{T}_j \Omega$.*

We are now able to calculate the index of A.

Theorem 5.2.2 *Assume that $2\omega_j(A)/\pi q$ is not an integer for any j; then the image of $W_p^2(\Omega)$ through the mapping*

$$T: u \mapsto \{Au; \gamma_j u, 1 \leq j \leq N\}$$

is a closed subspace of codimension

$$\sum_{j=1}^{N} \operatorname{card}\left\{m \;\middle|\; -\frac{2}{q} < \frac{m\Pi}{\omega_j(A)} < 0\right\}$$

in the space

$$Z = \{(f; g_j, 1 \leq j \leq N) \mid f \in L_p(\Omega), g_j \in W_p^{2-1/p}(\Gamma_j), 1 \leq j \leq N, g_j(S_j) \\ = g_{j+1}(S_j), 1 \leq j \leq N\}.$$

This general result will be deduced from a sequence of lemmas of technical character. In these we are going to deal with the particular case when Ω has only one corner. This is possible since we now consider curvilinear polygons. One can think of the cross section of a wing, for instance. We shall refer to this particular case as the case when $N = 1$.

Lemma 5.2.3 *Assume that $N = 1$; then there exists a constant C s.t.*

$$\|u\|_{2,p,\Omega} \leq C \|Au\|_{0,p,\Omega} \tag{5,2,4}$$

for all $u \in W_p^2(\Omega) \cap \mathring{W}_p^1(\Omega)$, provided $2\omega_1(A)/\pi q$ is not an integer.

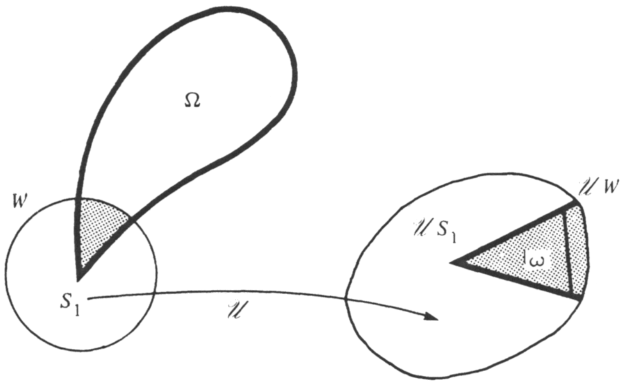

Figure 5.1

Proof This will be derived from inequalities (2,3,3,1) and (4,1,2), with the help of the same technique that we have already used in Subsection 2.3.3.

First we select a neighbourhood W of S_1 and a change of variable $\mathcal{U}: W \to \mathcal{U}W$ with the following properties:

(a) $\mathcal{U}$ is of class $C^{1,1}$,
(b) the Frechet derivative of $\mathcal{U}$ at S_1 is the linear operator defined by the matrix $\mathcal{T}_1$,
(c) $\mathcal{U}(\Gamma \cap W)$ is the union of two straight segments with origin at $\mathcal{U}S_1$.

Then we choose a cut-off function η with support in W s.t. η is identically equal to one near S_1. We shall look separately at ηu and $(1-\eta)u$. Set

$$v(x) = (\eta u)(\mathcal{U}^{-1}x), \qquad x \in \mathcal{U}W$$

and select any plane open domain ω with a polygonal boundary such that

(a) $\bar{\omega} \subset \mathcal{U}W$,
(b) $\bar{\omega}$ contains the support of v,
(c) $\partial\omega$ coincides with $\mathcal{U}\,\partial\Omega$ near $\mathcal{U}S_1$.

It is clear that $v \in W_p^2(\omega) \cap \mathring{W}_p^1(\omega)$ and that

$$-\Delta v + \sum_{i,j=1}^{2} b_{i,j} D_i D_j v + \sum_{i=1}^{2} b_i D_i v + b_0 v = g$$

in ω, where

$$g = (A\eta u) \circ \mathcal{U}^{-1}$$

and where $b_{i,j} \in C^{0,1}(\bar{\omega})$, $b_i \in L^\infty(\omega)$, $0 \leq i \leq 2$ and in addition:

$b_{i,j}(\mathcal{U}S_1) = 0.$

We apply inequality (4,1,2) to v. This is possible since the angle of ω at $\mathcal{U}S_1$ is $\omega_1(A)$ and we have assumed that $2\omega_1(A)/\pi q$ is not an integer. It is always possible to choose the other angles of ω so as to avoid the exceptional cases for inequality (4,1,2). Accordingly, we have

$$\|v\|_{2,p,\omega} \leq C \left\| g - \sum_{i,j=1}^{2} b_{i,j} D_i D_j v + \sum_{i=1}^{2} b_i D_i v + b_0 v \right\|_{0,p,\omega}.$$

It follows that

$$\|v\|_{2,p,\omega} \leq C\left\{\|g\|_{0,p,\omega} + 4 \max_{\substack{i,j=1,2 \\ x \in \operatorname{supp} \eta \circ \mathcal{U}^{-1}}} |b_{i,j}(x)| \, \|v\|_{2,p,\omega} + \|v\|_{1,p,\omega}\right\}.$$

Now since $b_{i,j}$ vanishes at $\mathcal{U}S_1$, we can choose the support of η small enough so that

$$\max_{\substack{i,j=1,2 \\ x \in \operatorname{supp} \eta \circ \mathcal{U}^{-1}}} |b_{i,j}(x)| \leq 1/8C.$$

Accordingly, we have

$$\|v\|_{2,p,\omega} \leq 2C\{\|g\|_{0,p,\omega} + \|v\|_{1,p,\omega}\}.$$

Going back to u, this implies that

$$\|\eta u\|_{2,p,\Omega} \leq C(\|A\eta u\|_{0,p,\Omega} + \|\eta u\|_{1,p,\Omega}\} \tag{5,2,5}$$

with possibly another value for C.

Then we can choose another plane open domain, Ω' with a $C^{1,1}$ boundary, such that the boundary of Ω' coincides with the boundary of Ω out of the set $\{\eta = 1\}$ where η is equal to one. Accordingly, we have $(1-\eta)u \in W_p^2(\Omega') \cap \mathring{W}_p^1(\Omega')$ and applying inequality (2,3,3,1), we have

$$\|(1-\eta)u\|_{2,p,\Omega} \leq C\{\|A(1-\eta)u\|_{0,p,\Omega} + \|(1-\eta)u\|_{1,p,\Omega}. \tag{5,2,6}$$

Adding inequalities (5,2,5) and (5,2,6), we obtain the estimate

$$\|u\|_{2,p,\Omega} \leq C\{\|Au\|_{0,p,\Omega} + \|u\|_{1,p,\Omega}\}. \tag{5,2,7}$$

On the other hand, a direct integration by parts shows that

$$\|u\|_{1,2,\Omega} \leq C \|Au\|_{0,p,\Omega}. \tag{5,2,8}$$

The inequality (5,2,4) follows from (5,2,7) and (5,2,8) with the help of inequality (1,4,3,2). ∎

The next step is the following.

Lemma 5.2.4 *Assume that $N=1$ and that the boundary of Ω is rectilinear near S_1. Then the image of $W_p^2(\Omega)\cap \mathring{W}_p^1(\Omega)$ through Δ is a closed subspace of codimension $[2\omega_1/\pi q]$*† *in $L_p(\Omega)$, provided $2\omega_1/\pi q$ is not an integer.*

Proof Let $f\in L_p(\Omega)$ be given and let $u\in \mathring{H}^1(\Omega)$ be the solution of

$$-\Delta u=f \qquad \text{in } \Omega. \tag{5,2,9}$$

It is clear from the results in Chapter 2 that

$$u\in W_p^2(\Omega\setminus W)$$

where W is any closed neighbourhood of S_1. Again let η be a cut-off function which is equal to 1 near S_1. We have

$$(1-\eta)u\in W_p^2(\Omega),$$

and if we choose the support of η small enough, ηu is solution of the Dirichlet problem for the equation

$$-\Delta\eta u=\eta f-[\Delta\,;\eta]u=f_1$$

in a plane open domain ω, whose boundary $\partial\omega$ is a polygon which coincides with $\partial\Omega$ near S_1. It is clear that $f_1\in L_p(\omega)$ and, applying Theorem 4.4.3.7, we know that there exists numbers C_m such that

$$\eta u-\sum_{0<m\pi/\omega_1<2/q} C_m r_1^{m\pi/\omega_1}\sin\frac{m\pi\theta_1}{\omega_1}\in W_p^2(W),$$

where W is a neighbourhood of S_1 in ω.

Adding these results, we have

$$u-\sum_{0<m\pi/\omega_1<2/q} C_m r_1^{m\pi/\omega_1}\sin\frac{m\pi\theta_1}{\omega_1}\eta_1\in W_p^2(\Omega).$$

The numbers C_m are continuous linear functionals of f_1 and consequently also of f through the Green operator

$$\Sigma: f\to u$$

defined by (5,2,9). Accordingly, we have $u\in W_p^2(\Omega)\cap\mathring{W}_p^1(\Omega)$ iff f annihilates $[2\omega_1/\pi q]$ linear functionals on $L_p(\Omega)$. ■

Lemma 5.2.5 *Assume that $N=1$ and that the boundary of Ω is rectilinear near S_1. Then the image of $W_p^2(\Omega)\cap\mathring{W}_p^1(\Omega)$ through A is a closed subspace whose codimension is $[2\omega_1(A)/\pi q]$ provided $2\omega_1(A)/\pi q$ is not an integer.*

† $[S]$ denotes the integral part of S.

Proof Let us first look at the particular case when $A_1 = -\Delta$ or equivalently

$$a_{k,l}(S_1) = \delta_{k,l}, \qquad 1 \leq k, \qquad l \leq 2.$$

Then we shall derive the result by homotopy from $-\Delta$ to A. Let us set

$$A(t) = tA - (1-t)\Delta, \qquad t \in [0, 1].$$

Applying Lemma 5.2.3, we know that for each $t \in [0, 1]$ there exists a constant C_t such that

$$\|u\|_{2,p,\Omega} \leq C_t \, \|A(t)u\|_{0,p,\Omega}$$

for all $u \in W_p^2(\Omega) \cap \mathring{W}_p^1(\Omega)$. Accordingly, $A(t)$ is a semi-Fredholm operator from $W_p^2(\Omega) \cap \mathring{W}_p^1(\Omega)$ into $L_p(\Omega)$ for every $t \in [0, 1]$.

The operator $A(t)$ depends continuously on t. Thus by a theorem in Kato (1966), the index of $A(t)$ does not depend on t. Consequently, the index of $A(1) = A$ is equal to the index of $A(0) = -\Delta$, which is $-[2\omega_1/\Pi q]$, by Lemma 5.2.1.4. This completes the proof of Lemma 5.2.5 when $A_1 = -\Delta$, since A is one to one (inequality (5,2,4)).

The general case is reduced to the particular case when $A_1 = -\Delta$ by composition with the matrix $\mathcal{T}_1$. ∎

Proof of Theorem 5.2.2 Let us start from $f \in L_p(\Omega)$. There exists a solution $u \in \mathring{H}^1(\Omega)$ of

$$Au = f \qquad \text{in } \Omega \tag{5,2,10}$$

and in addition $u \in W_p^2(\Omega \setminus V)$, where V is any closed neighbourhood of the corners. This was observed earlier. For further convenience we denote by Σ the continuous linear operator from $L_p(\Omega)$ into $\mathring{H}^1(\Omega)$ defined by (5,2,10).

Let η_j be a cut-off function whose support is small near S_j and such that $\eta_j = 1$ in a neighbourhood of S_j. Obviously, we have

$$A\eta_j u = \eta_j f + [A\,;\,\eta_j]\Sigma f = f_j \in L_p(\omega_j)$$

and $\eta_j u \in W_p^2(\omega_j) \cap \mathring{W}_p^1(\omega_j)$, where ω_j is an open plane domain whose boundary is a curvilinear polygon of class $C^{1,1}$ with only one corner at S_j, which contains the support of η_j and such that the boundary of ω_j coincides with the boundary of Ω near S_j.

Finally, we select a change of variable $\mathcal{U}_j$ defined in a neighbourhood W_1 of S_j such that

(a) $\mathcal{U}_j$ is of class $C^{1,1}$
(b) the Frechet derivative of $\mathcal{U}_j$ at S_j is the identity operator $\mathcal{U}$
(c) $\mathcal{U}_j(\Gamma \cap W_j)$ is the union of two straight segments with origin at $\mathcal{U}_j S_j$.

If the support of η_j is small enough, we can also choose ω_j small enough to be contained in W_j. Under these assumptions, we can apply Lemma 5.2.1.5 in the domain $\mathcal{U}_j\omega_j$.

The conclusion is that $\eta_j u \in W_p^2(\omega_j)$ iff f_j annihilates $[2\omega_j(A)/\pi q]$ continuous linear forms on $L_p(\omega_j)$, $1 \leq j \leq N$. Accordingly, $u \in W_p^2(\Omega)$ iff f annihilates

$$\nu = \sum_{j=1}^{N} \left[\frac{2\omega_j(A)}{\pi q}\right]$$

continuous linear forms on $L_p(\Omega)$, since

$$f_j = \eta_j f + [A\,;\,\eta_j]\Sigma f.$$

Thus we have shown that A is a one-to-one mapping from $W_p^2(\Omega) \cap \mathring{W}_p^1(\Omega)$ onto a closed subspace of $L_p(\Omega)$, whose codimension is ν. This, together with a trace theorem (Subsection 1.6) implies the claim of Theorem 5.2.2. ∎

Remark 5.2.6 The general principle underlying Theorem 5.2.2 is that we can easily extend the index property of our boundary value problems from the case when all the operators have constant coefficients to the general case. However, for practical purposes, one also needs to know which singular functions must be added to $W_p^2(\Omega)$ in order to get surjectivity. The perturbation method used here will allow us to conclude only in some particular cases when the singular functions remain the same when passing from the constant coefficient case to the general case.

Theorem 5.2.7 *Assume that $2\omega_j(A)/\pi q$ is not an integer and that*

$$p < \omega_j(A)/(\omega_j(A) - \pi) \tag{5,2,11}$$

for all j s.t. $\omega_j(A) > \pi$. Then for every $(f;\, g_j, 1 \leq j \leq N)$ given in Z, (as defined in the statement of Theorem 5.2.2), there exists a unique function u and unique numbers $C_{j,m}$ such that

$$u - \sum_{\substack{0<m<2\omega_j(A)/\pi q \\ j=1,2,\ldots,N}} C_{j,m}\hat{S}_{j,m} \in W_p^2(\Omega),$$

and

$$\begin{cases} Au = f & \text{in } \Omega \\ \gamma_j u = g_j & \text{on } \Gamma_j, \quad 1 \leq j \leq N, \end{cases}$$

where

$$\hat{S}_{j,m}(\mathcal{T}_j^{-1}x) = r_j^{m\pi/\omega_j(A)} \sin \frac{m\pi\theta_j}{\omega_j(A)}\, \eta_j(r_j e^{i\theta_j}).\dagger$$

† We identify x with $r_j e^{i\theta_j}$ in $\mathcal{T}_j\Omega$, where r_j, θ_j are the polar coordinates of the corner $\mathcal{T}_j S_j$ of $\mathcal{T}_j\Omega$; i.e., $\mathcal{T}_j S_j$ is the point $r_j = 0$, while the lines tangent to $\mathcal{T}_j\Gamma$ at $\mathcal{T}_j S_j$ correspond to $\theta_j = 0$ and $\theta_j = \omega_j(A)$ respectively.

Proof Clearly the functions $\hat{S}_{j,m}$ belong to $H^1(\Omega)\backslash W_p^2(\Omega)$ when $0 < m\pi/\omega_j(A) < 2/q$. We shall show that

$$A\hat{S}_{j,m} \in L_p(\Omega), \qquad \gamma_k \hat{S}_{j,m} \in W_p^{2-1/p}(\Gamma_k), \qquad 1 \leq k \leq N,$$

provided condition (5,2,11) is fulfilled.

Indeed, we have

$$(A\hat{S}_{j,m}) \circ \mathcal{T}_j^{-1} = \left(-\Delta + \sum_{k,l=1}^{2} b_{k,l} D_k D_l + \sum_{k=1}^{2} b_k D_k + b_0\right) \hat{S}_{j,m} \circ \mathcal{T}_j^{-1}$$

where $b_{k,l} \in C^{0,1}(\mathcal{T}_j \bar{\Omega})$, $b_k \in L_\infty(\mathcal{T}_j \bar{\Omega})$, $0 \leq k \leq 2$ and

$$b_{k,l}(\mathcal{T}_j S_j) = 0.$$

From the above definition of $\hat{S}_{j,m}$, it follows that

$$(A\hat{S}_{j,m}) \circ \mathcal{T}_j^{-1}(x) = 0(r_j^{m\pi/\omega_j(A)-1})$$

near $\mathcal{T}_j S_j$, while $A\hat{S}_{j,m}$ is smooth in $\mathcal{T}_j(\bar{\Omega}\backslash S_j)$. This shows that

$$A\hat{S}_{j,m} \in L_p(\Omega).$$

On the other hand, we have $\gamma_k \hat{S}_{j,m} \in W_p^{2-1/p}(\Gamma_k)$ for all k, because $\gamma_k \hat{S}_{j,m} = 0$ for $k \neq j, j+1$, while when k is j or $j+1$ we can apply the following Lemma which follows easily from the definition of the space $W_p^{2-1/p}(]0, a[)$.

Lemma 5.2.8 *Assume that $\varphi \in C^{1,1}([0, a])$ with $a > 0$ and that $\varphi(0) = \varphi'(0) = 0$. Then the function $u = \varphi^\alpha$ belongs to $W_p^{2-1/p}(]0, a[)$ provided $\alpha > 1 - 1/p$.*

Here, choosing the coordinate axes suitably, we can assume that $\mathcal{T}_j \Gamma_{j+1}$ is the graph of a function φ fulfilling the assumption of Lemma 5.2.8. Then we have

$$\gamma_{j+1} \hat{S}_{j,m} \mathcal{T}_j^{-1}(x, y) = \sqrt{(x^2 + \varphi(x)^2)}^{m\pi/\omega_j(A)} \sin\left(\frac{m\pi}{\omega_j(A)} \arctan \frac{\varphi(x)}{x}\right)$$

near the origin. Accordingly, $\gamma_{j+1}\hat{S}_{j,m}$ belongs to $W_p^{2-1/p}(\Gamma_{j+1})$ iff $p < \omega_j(A)/(\omega_j(A) - \pi)$ when $\omega_j(A) > \pi$, $m = 1, 2, \ldots$.

A similar proof shows that $\gamma_j \hat{S}_{j,m} \in W_p^{2-1/p}(\Gamma_j)$.

Summing up, we have shown that the mapping T (defined in the statement of Theorem 5.2.2) maps the space E spanned by $W_p^2(\Omega)$ and the functions $\hat{S}_{j,m}$ $(1 \leq j \leq N, 0 < m < 2\omega_j(A)/\pi q)$ into Z. The codimension of $W_p^2(\Omega)$ in E is obviously

$$\nu = \sum_{j=1}^{N} \left[\frac{2\omega_j(A)}{\pi q}\right].$$

Consequenty if follows from Theorem 5.2.2 that T is an isomorphism from E onto Z, since T is one to one on $E \subseteq H^1(\Omega)$.

Remark 5.2.9 In the particular case when Ω is a strict polygon (but A still has variable coefficients), we can replace the condition (5,2,11) by the weaker one

$$P < \frac{2\omega_j(A)}{\pi - \omega_j(A)}.$$

Indeed we still have $A\hat{S}_{j,m} \in L_p(\Omega)$, while $\gamma_k \hat{S}_{j,m} = 0$ for all k.

Remark 5.2.10 Starting from the results explained in Remark 4.4.1.14, one can also apply the techniques of this section when $\omega_j(A) = 2\pi$. This takes care of domains with turning points toward the interior of Ω (see Section 3.3 for the treatment of turning points toward the exterior of Ω).

Remark 5.2.11 A technique for calculating singular solutions of other boundary value problems has been worked out in Mghazli (1983). This technique allows one to handle second-order elliptic boundary value problems for nonhomogeneous operators (i.e. including lower-order terms).