Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2009,**29B**(4):919–948
Acta Mathematica Scientia
数学物理学报
http://actams.wipm.ac.cn

THE NAVIER-STOKES EQUATIONS WITH THE KINEMATIC AND VORTICITY BOUNDARY CONDITIONS ON NON-FLAT BOUNDARIES*

Dedicated to Professor Wu Wenjun on the occasion of his 90th birthday

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Abstract We study the initial-boundary value problem of the Navier-Stokes equations for incompressible fluids in a general domain in $\mathbb{R}^n$ with compact and smooth boundary, subject to the kinematic and vorticity boundary conditions on the non-flat boundary. We observe that, under the nonhomogeneous boundary conditions, the pressure p can be still recovered by solving the Neumann problem for the Poisson equation. Then we establish the well-posedness of the unsteady Stokes equations and employ the solution to reduce our initial-boundary value problem into an initial-boundary value problem with absolute boundary conditions. Based on this, we first establish the well-posedness for an appropriate local linearized problem with the absolute boundary conditions and the initial condition (without the incompressibility condition), which establishes a velocity mapping. Then we develop *apriori* estimates for the velocity mapping, especially involving the Sobolev norm for the time-derivative of the mapping to deal with the complicated boundary conditions, which leads to the existence of the fixed point of the mapping and the existence of solutions to our initial-boundary value problem. Finally, we establish that, when the viscosity coefficient tends zero, the strong solutions of the initial-boundary value problem in $\mathbb{R}^n (n \geq 3)$ with nonhomogeneous vorticity boundary condition converge in L^2 to the corresponding Euler equations satisfying the kinematic condition.

*Received December 31, 2008. Gui-Qiang Chen's research was supported in part by the National Science Foundation under Grants DMS-0807551, DMS-0720925, and DMS-0505473, and the Natural Science Foundation of China (10728101). Zhongmin Qian's research was supported in part by EPSRC grant EP/F029578/1

Key words Navier-Stokes equations; incompressible; vorticity boundary condition; kinematic boundary condition; absolute boundary condition; non-flat boundary; general domain; Stokes operator; Neumann problem; Poisson equation; vorticity; strong solutions; inviscid limit; slip boundary condition

2000 MR Subject Classification 35Q30; 76D05; 76D03; 35A35; 35B35; 35P05; 76D07; 36M60; 58C35

1 Introduction

The motion of an incompressible viscous fluid in $\mathbb{R}^n, n \geq 2$, is described by the Navier-Stokes equations:

$$\partial_t u + u \cdot \nabla u + \nabla p = \mu \Delta u, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

with initial data

$$u|_{t=0} = u_0(x), \tag{1.3}$$

where $u(t,x) = (u^1, \cdots, u^n)(t,x)$ is the velocity vector field and $p(t,x)$ is the pressure that maintains the incompressibility of a fluid at (t,x). Equation (1.2), i.e., $\mathrm{div} u = 0$, is the incompressibility condition. As a nonlinear system of partial differential equations, u and p are regarded as unknown functions, and the initial velocity field $u_0(x)$ sets the fluid in motion. The constant $\mu > 0$ is the kinematic viscosity constant, $u \cdot \nabla$ denotes the covariant derivative along the flow trajectories, namely, the directional derivative in the direction u, Δu is the usual Laplacian on u, and $\mu \Delta u$ represents the stress applied to the fluid. As usual, we use $\nabla \cdot = \mathrm{div}$ to denote the divergence operator.

For inviscid flow, $\mu = 0$ and then the equations are referred to as the Euler equations for incompressible fluid flow:

$$\partial_t u + u \cdot \nabla u + \nabla p = 0, \tag{1.4}$$

$$\nabla \cdot u = 0. \tag{1.5}$$

When a fluid is confined in a bounded domain $\Omega \subset \mathbb{R}^n$ with non-empty boundary Γ, these equations must be supplied with proper boundary conditions in order to be well-posed. For concreteness, the bounded domain $\Omega \subset \mathbb{R}^n$ is assumed to have a compact, oriented, smooth surface boundary Γ. In this paper, we propose and study the initial-boundary value problem (1.1)–(1.3) with the following boundary conditions on the non-flat boundary Γ:

$$u^{\perp}\big|_{\Gamma} = 0 \qquad \text{(kinematic condition)}, \tag{1.6}$$

$$\omega^{\|}\big|_{\Gamma} = a \qquad \text{(vorticity condition)}, \tag{1.7}$$

where $\omega = \nabla \times u \in \mathbb{R}^n$ is the vorticity, the field $a = a(t,x)$ is defined on the non-flat boundary Γ, $u^{\perp} \in \mathbb{R}$ denotes the normal component and $u^{\|}$ the tangential part of $u \in \mathbb{R}^n$.

Various physical considerations for some flows observed in nature have led to the no-slip condition:

$$u^{\|}\big|_{\Gamma} = 0, \tag{1.8}$$

together with the kinematic condition (1.6). This is equivalent to the simple homogeneous Dirichlet boundary condition: $u|_\Gamma = 0$. Thus, it has been received intensive investigation for the Navier-Stokes equations.

On the other hand, if a vector field $u \in \mathbb{R}^n$ is identified with its corresponding differential 1-form on Ω, then various boundary problems for the linearized equations, the heat equations, have been studied in geometry under the Hodge theory on manifolds with boundaries; see Conner [9] and the references cited therein. In particular, three natural boundary conditions have been identified; since

$$\Delta u = -\nabla \times \omega + \nabla(\mathrm{div} u) \qquad \text{on } \Omega \subset \mathbb{R}^n,$$

boundary conditions may be imposed on the components $u^\perp$, $u^\parallel$, $\omega^\perp$, $\omega^\parallel$, and $\mathrm{div}u$. It has been revealed that the initial-boundary value problem with the boundary conditions (1.6)–(1.7) for the heat equations is well-posed. This leads to a natural question whether the same setup of initial-boundary value problem is well-posed for the Navier-Stokes equations. In the literature, the absolute boundary conditions mean our boundary conditions (1.6)–(1.7) with $a = 0$, i.e., homogeneous.

The initial-boundary value problem (1.1)–(1.3) and (1.6)–(1.7) for the Navier-Stokes equations has received less attention, partly due to the common agreement that the no-slip condition is a physical condition under ideal situations. However, recent experimental evidence (cf. Einzel-Panzer-Liu [11], Lauga-Brenner-Stone [23], and Zhu-Garnick [55] for a survey) demonstrates that the velocity field of a fluid does not satisfy the complete no-slip conditions in general. Indeed, it is revealed that the pressure does depend on how curved the surface boundary is; also see Bellout-Neustpua-Penel [6]. It is also evident that, for high speed flows, one can not expect to comply the complete no-slip conditions. Therefore, a careful study of incompressible fluids with various boundary conditions associated with the Navier-Stokes equations is in great demand.

Another motivation for our study in this paper is from the Navier slip boundary condition, which are widely accepted in many applications and numerical studies. It states that the slip velocity is proportional to the shear stress:

$$u^\parallel = -\zeta\big((\nabla u + (\nabla u)^\top)\nu\big)^\parallel, \tag{1.9}$$

where ν is the outward normal on Γ and $\zeta > 0$ is the slip length on Γ. Such boundary conditions can be induced by effects of free capillary boundaries, a rough boundary, a perforated boundary, or an exterior electric field (cf. [1, 4, 5, 7, 14, 15, 41, 43]). Take an asymptotic expansion of $u(t,x)$ with respect to ζ and the distance $d(x,\Gamma)$ to Γ such that $d(x,\Gamma) \ll \zeta$. Then, by a direct calculation, one finds that the Navier slip condition (1.9) yields (1.7) in which $a(t,x)$ is determined by the initial value problem of a family of linear heat equations on Γ, that is, $a(t,x)$ is determined solely by the initial data (1.3).

There is another appealing reason to consider other boundary problems associated with the Navier-Stokes equations. The Euler equations (1.4)–(1.5) subject to the kinematic condition (1.6), along with the initial condition (1.3), are often regarded as an ideal model in turbulence theory, where the Reynolds number is so big that the viscosity coefficient μ is negligible. Hence, a natural question is whether the flow determined by the Euler equations is, at best, a “singular limit" of the Navier-Stokes equations as the viscosity coefficient $\mu \downarrow 0$ (cf. Pope [39], page 18). However, it is very subtle and difficult in general, since physical boundary layers may be present. The classical no-slip boundary condition, $u|_\Gamma = 0$, gives arise to the phenomenon of strong boundary layers in general as formally derived by Prandtl [40]. For some sufficient conditions to ensure the convergence of viscous solutions to the ones of the Euler equations, see Kato [18, 52] and the references therein. However, as far as we know, such a claim has never been proved for general bounded domains with the Navier boundary condition. In contrary, we prove in this paper that the solutions of our problem (1.1)–(1.3) and (1.6)–(1.7) for the Navier-Stokes equations converge in L^2 to a solution of problem (1.4)–(1.6) for the Euler equations as the viscosity coefficient $\mu \downarrow 0$. To explain why the Euler equations, subject to the kinematic condition (1.6), are the limiting case of the solutions u_μ of our problem (1.1)–(1.3) and (1.6)–(1.7), we notice that the pressure p required in the Euler equations solves the Poisson equation:

$$\Delta p = -\nabla \cdot (u \cdot \nabla u)\,, \qquad \partial_\nu p\big|_\Gamma = \pi(u, u), \tag{1.10}$$

where $u(t, x)$ is a solution to the Euler equations (1.4)–(1.5), and π is the second fundamental form of the boundary Γ which describes the curvature of the boundary surface Γ. Since u does not necessarily comply the no-slip condition, the normal derivative of the pressure p does not vanish along the boundary in general. In particular, the pressure p along the boundary naturally depends on the curvature of the boundary. On the other hand, under the kinematic condition (1.6), the pressure p can be recovered from the Neumann problem of the Poisson equation:

$$\Delta p = -\nabla \cdot (u_\mu \cdot \nabla u_\mu), \tag{1.11}$$

$$\partial_\nu p\big|_\Gamma = \pi(u_\mu, u_\mu) - \mu \nabla^\Gamma \times \omega_\mu^{\|}. \tag{1.12}$$

As $\mu \downarrow 0$, the pressure p satisfies the same equation as that of the pressure required by the Euler equations. However, if we further enforce the no-slip condition, then the normal derivative of the pressure p vanishes, which does not in general coincide with that of the Euler equations. On the other hand, if we set a free condition on the tangent component of u but replace the boundary condition on the vorticity, we have the chance to recover the right boundary condition for the pressure in the Euler equations.

The rigorous mathematical analysis of the Navier-Stokes equations involving vorticity-type boundary conditions may date back the work by Solonnikov-Ščadilov [46] for the stationary linearized Navier-Stokes system under boundary conditions:

$$u^\perp\big|_\Gamma = 0, \qquad \big((\nabla u + (\nabla u)^\top)\nu\big)^{\|}\big|_\Gamma = 0. \tag{1.13}$$

For two-dimensional, simply connected, bounded domains, the vanishing viscosity problem has been rigorously justified by Yodovich [54]. We also refer to Lions [28, 29] for the boundary conditions (1.6)–(1.7) with $a = 0$, Clopeau, Mikelić, and Robert [8] and Lopes Filho, Nussenzveig Lopes and Planas [30] for the Navier boundary condition (1.13), and Mucha [35] under some geometrical constraints on the shape of the domains for the two-dimensional case. Also see [53] for an analysis for the complete slip boundary conditions as (1.14) below.

The main purpose of this paper is to develop an approach to deal with the well-posedness and the inviscid limit for the initial-boundary value problem (1.1)–(1.3) and (1.6)–(1.7) with nonhomogeneous boundary condition for the multidimensional Navier-Stokes equations ($n \geq 3$). One of the main difficulties for the higher dimension case, in comparison with the two-dimensional case, is that the maximum principle for the vorticity fails so that the two-dimensional techniques can not be directly extended to this case. Furthermore, the nonhomogeneous boundary condition causes another main difficulty in developing apriori estimates which require to be compatible with the nonlinear convection term. In Section 2, we introduce the local moving frames on the boundary Γ of a domain Ω and derive some basic identities for vector fields. Although we work with a domain in the Euclidean space, the boundary Γ is a curved surface, so that geometric tools have been brought in to carry out local computations. Since we can work only with the boundary coordinate system which may be not "normal", the computations we need to carry out are a little bit complicated and long. The results in Section 2 allow us to determine boundary values of several interesting quantities, which will be used to settle the local existence of a unique strong solution of problem (1.1)–(1.3) and (1.6)–(1.7) for the Navier-Stokes equations.

In Section 3, we establish various L^2-estimates for vector fields that satisfy the absolute boundary conditions:

$$u^{\perp}\big|_{\Gamma} = 0, \qquad \omega^{\|}\big|_{\Gamma} = 0. \tag{1.14}$$

In particular, the main novel fact is that, if $u(t, x)$ satisfies the absolute boundary conditions (1.14), then

$$(\nabla \times \omega)^{\perp}\big|_{\Gamma} = 0.$$

The latter information on

$$\nabla \times \omega = -\Delta u + \nabla(\mathrm{div} u)$$

along the boundary is crucial in obtaining the necessary *apriori* estimates for the linearization of the absolute boundary problem.

In Section 4, we develop an old idea of Leray-Hopf [13, 24, 25, 26] for our initial-boundary value problem with nonhomogeneous vorticity boundary condition for the Navier-Stokes equations. Namely, under the nonhomogeneous boundary conditions, the pressure p may be recovered by solving the Neumann problem (1.11)–(1.12) of the Poisson equation. Then we establish the well-posedness of the unsteady Stokes equations and employ the solution w to reduce our initial-boundary value problem into an initial-boundary value problem with absolute boundary conditions for $v = u - w$. This leads to the following linearlization: One first considers the linear parabolic equations:

$$\partial_t v + (\beta + w) \cdot \nabla(v + w) + \nabla p_\beta = \mu \Delta v, \tag{1.15}$$

where p_β solves (1.10) replacing u by $\beta + w$ for a given $\beta = \beta(t, x), t \leq T$, satisfying the absolute conditions and the initial condition. Based on this, in Section 5, we demonstrate that the absolute boundary conditions are suitable boundary conditions, since they generally do not appear in the literature on parabolic equations, and conclude that (1.15) is well-posed, which thus establishes a velocity mapping $V :\ w \to v$. Note that we have dropped the incompressibility condition, so that (1.15) is local and a linear parabolic system. However, we will establish that any fixed point in a proper functional space is a strong solution to the Navier-Stokes equations, so that the incompressibility can be recovered.

We emphasize that the absolute boundary conditions in a general domain are a kind of boundary conditions which change from the Dirichlet to the Neumann boundary condition along the boundary. Therefore, the coercive estimates, also called the global L^p-estimates, for parabolic equations are not applicable in this setting. Indeed, we have to establish estimates quite close to the coercive estimates for the linear system (1.15) in a different functional space, which will be done in Section 6. In order to construct a local (in time) strong solution, we need to develop apriori estimates for the solution $v(t,x)$ of the linear parabolic equations (1.15). It would be enough to develop an estimate for the H^2-norm of $v(t,x)$ in terms of that of $(\beta+w)(t,x)$. In fact, it works for the Dirichlet boundary problem (which has been done by Solonnikov [45]) and also works for the periodic case (cf. [19]). Unfortunately, due to the complicated boundary conditions, we are unable to achieve such apriori estimates, instead, we have to involve the Sobolev norm for the time-derivative of $v(t,x)$.

Finally, in Sections 8–9, we establish that, when $\mu \to 0$, the strong solutions of the initial-boundary value problem (1.1)–(1.3) and (1.6)–(1.7) in $\mathbb{R}^n, n \geq 3$, with nonhomogeneous vorticity boundary condition converge in L^2 to the corresponding Euler equations satisfying the kinematic condition (1.6).

2 Local Moving Frames on the Boundaries and Basic Identities for Vector Fields

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a compact, smooth boundary $\Gamma = \partial\Omega$ with $n \geq 2$. We assume that Γ is an oriented surface (not necessarily connected) which carries an induced metric. Let ν denote the unit normal vector field along Γ pointing outwards. As usual, we use conventional notation that the repeated indices in a formula are understood to be summed up from 1 to n unless confusion may occur.

In order to reflect better the geometry of the boundary $\Gamma = \partial\Omega$ of Ω, it is convenient to work with local moving frames compatible to Γ (see Palais-Terng [37] for the details). More precisely, by a moving frame compatible to the boundary we mean a local orthonormal basis

$$\{e_1, \cdots, e_n\}$$

of the tangent bundle $T\Omega$ of Ω about a boundary point $x \in \Gamma$ such that $e_n = \nu$ when restricted to Γ. In terms of the Christoffel symbols Γ_{ij}^l:

$$\nabla_{e_i} e_j = \Gamma_{ij}^l e_l,$$

where ∇_X means the directional derivative in X. The torsion-free condition may be stated as $\Gamma_{ij}^k = -\Gamma_{ik}^j$ and, along the boundary $\Gamma = \partial\Omega$,

$$\Gamma_{ij}^n = \Gamma_{ji}^n \equiv -h_{ij}, \qquad \text{for any } 1 \leq i,j \leq n-1,$$

and $\pi = (h_{ij})$ is the second fundamental form, which is a symmetric tensor on Γ. By definition,

$$h_{ij} = \langle \nabla_{e_i} \nu, e_j \rangle.$$

If $u=\sum\limits_{i=1}^{n-1}u^ie_i$ and $w=\sum\limits_{i=1}^{n-1}w^ie_i$ are two vector fields tangent to Γ, then

$$\pi(u,w)=\sum_{i,j=1}^{n-1}h_{ij}u^iw^j.$$

The surface Γ carries an induced metric, its Levi-Civita connection is denoted by ∇^Γ. Then $\{e_1,\cdots,e_{n-1}\}$ restricted to Γ is a moving frame of $T\Gamma$. If u is a vector field on Ω, then the tangent part of u restricted to Γ, denoted by $u^{\|}$, is a section of $T\Gamma$, and its covariant derivative is denoted by $\nabla^\Gamma u^{\|}$. Let H denote the mean curvature of Γ, that is, H is the trace of the second fundamental form π:

$$H=\sum_{j=1}^{n-1}h_{jj}.$$

Lemma 2.1 Let u and w be two vector fields on Ω. Then

$$\begin{aligned}\langle w\cdot\nabla u,\,\nu\rangle=&-\pi(u^{\|},w^{\|})-H\langle w,\,\nu\rangle\langle u,\,\nu\rangle+\langle w,\,\nu\rangle(\nabla\cdot u)\\&+\langle w^{\|},\nabla^\Gamma\langle u,\nu\rangle\rangle+\langle u^{\|},\nabla^\Gamma\langle w,\nu\rangle\rangle-\nabla^\Gamma\cdot\big(\langle w,\,\nu\rangle\,u^{\|}\big).\end{aligned}\tag{2.1}$$

In particular, if $u^\perp=w^\perp=0$ on Γ, then

$$\langle w\cdot\nabla u,\,\nu\rangle=-\pi(u,w).\tag{2.2}$$

Proof Since $(w\cdot\nabla u)^i=w^j\big(e_j(u^i)+\Gamma^i_{jk}u^k\big)$, then

$$\langle w\cdot\nabla u,\,\nu\rangle=w^j\big(e_j(u^n)+\Gamma^n_{jk}u^k\big)=\langle w^{\|},\nabla^\Gamma\langle u,\,\nu\rangle\rangle-\pi(u^{\|},w^{\|})+w^n(\nabla_nu^n).\tag{2.3}$$

On the other hand, according to the Ricci equation,

$$\sum_{j=1}^{n-1}\nabla_ju^j=\sum_{j=1}^{n-1}\nabla^\Gamma_ju^j+\sum_{j=1}^{n-1}h_{jj}\langle u,\,\nu\rangle=\nabla^\Gamma\cdot u^{\|}+H\langle u,\,\nu\rangle,$$

so that

$$\nabla_nu^n=\nabla\cdot u-\sum_{j=1}^{n-1}\nabla_ju^j=\nabla\cdot u-\nabla^\Gamma\cdot u^{\|}-H\langle u,\nu\rangle.\tag{2.4}$$

Substitution (2.4) into (2.3) yields

$$\langle w\cdot\nabla u,\,\nu\rangle=\langle w^{\|},\nabla^\Gamma\langle u,\nu\rangle\rangle-\pi(u^{\|},w^{\|})+\langle w,\nu\rangle\big(\nabla\cdot u-\nabla^\Gamma\cdot u^{\|}-H\langle u,\nu\rangle\big),$$

which, together with the identity:

$$\langle w,\nu\rangle\nabla^\Gamma\cdot u^{\|}=\nabla^\Gamma\cdot\big(\langle w,\nu\rangle u^{\|}\big)-\langle u^{\|},\nabla^\Gamma\langle w,\nu\rangle\rangle,$$

yields (2.1).

Lemma 2.2 Let u be a vector field on Ω, $\omega=\nabla\times u$, and $d=\nabla\cdot u$. Then

(i) $\partial_\nu d|_\Gamma=\langle\Delta u,\,\nu\rangle+\nabla^\Gamma\times\omega^{\|}$, where

$$\nabla^\Gamma\times\omega^{\|}=\nabla^\Gamma_1\omega^2-\nabla^\Gamma_2\omega^1$$

is independent of the choice of a local moving frame, which can be identified with the exterior derivative of $\omega^{\parallel}$ on Γ;

(ii) $\frac{1}{2}\partial_\nu(|u|^2)|_\Gamma = \langle u \times \omega, \nu\rangle + \langle u, \nu\rangle d - \pi(u^{\parallel}, u^{\parallel}) - H\langle u, \nu\rangle^2 + 2\langle u^{\parallel}, \nabla^\Gamma\langle u, \nu\rangle\rangle$
$-\nabla^\Gamma \cdot \big(\langle u, \nu\rangle u^{\parallel}\big)$.

Proof (i) According to the vector identity:

$$\Delta u = -\nabla \times \omega + \nabla d,$$

one obtains

$$\partial_\nu d|_\Gamma = \langle \Delta u, \nu\rangle + \langle \nabla \times \omega, \nu\rangle.$$

It remains to verify that $\langle \nabla \times \omega, \nu\rangle$ coincides with $\nabla^\Gamma \times \omega^{\parallel}$ which depends only on the tangent part $\omega^{\parallel}\big|_\Gamma$. The last fact follows easily from a local computation.

(ii) This formula follows directly from the vector identity:

$$\frac{1}{2}\nabla(|u|^2) = u \times (\nabla \times u) + (u \cdot \nabla)\, u,$$

and Lemma 2.1.

3 L^2-Estimates for Vector Fields Satisfying the Absolute Boundary Conditions

In this section, we establish L^2-estimates for vector fields satisfying the absolute boundary conditions (1.14).

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a compact, smooth boundary Γ. The Sobolev spaces can be defined in terms of the total derivative ∇. If T is a (smooth) vector field on Ω and $\nabla^j T$ is the j-th covariant derivative, then, for a nonnegative integer k, the Sobolev norm $\|\cdot\|_{W^{k,p}}, p \geq 1$, is given by

$$\|T\|_{W^{k,p}} = \Big(\sum_{j=0}^{k} \|\nabla^j T\|_{L^p}^p\Big)^{1/p}, \tag{3.1}$$

where $\|\cdot\|_{L^p}$ denotes the L^p-norm of vector fields, that is, $\|T\|_{L^p} = (\int_\Omega |T|^p \mathrm{d}x)^{1/p}$ with $|T| = \sqrt{\langle T, T\rangle}$ the length of the vector field. When $p = 2$, we often use $H^k(\Omega) = W^{k,2}(\Omega)$ and $\|T\|_{H^k} = \|T\|_{W^{k,2}}$, since such spaces are Hilbert spaces. The most important in the theory of Sobolev spaces is the Sobolev imbedding theorem (see Theorem 4.12 of Adams-Fournier [2], page 85).

We will use the following integration by parts formula:

$$\int_\Omega \langle \nabla \times u, w\rangle \,\mathrm{d}x = \int_\Omega \langle u, \nabla \times w\rangle \,\mathrm{d}x + \int_\Gamma \langle u \times w, \nu\rangle \mathrm{d}S, \tag{3.2}$$

where $\mathrm{d}S$ is the surface measure on Γ. This formula follows easily from the Stokes theorem.

If u is a vector field on Ω, then three L^2-norms are related to the total derivative ∇u:

(i) The (squared) L^2-norm $\|\nabla u\|_2^2$;

(ii) $\|\nabla \cdot u\|_2^2 + \|\nabla \times u\|_2^2$, which is most useful in the study of the Navier-Stokes equations, since, if u is the velocity field of an incompressible fluid, then $\nabla \cdot u = 0$, and $\nabla \times u = \omega$ is the vorticity in physics;

(iii) The quantity $-\int_\Omega \langle \Delta u, u\rangle \mathrm{d}x$ which is related to the spectral gap of the (Hodge De Rham) Laplacian Δ which has a clear geometric meaning.

The relations among these three quantities are the following.

Lemma 3.1 If $u \in H^2(\Omega)$ is a vector field, then

$$\begin{aligned}\int_\Omega \langle \Delta u, u\rangle \,\mathrm{d}x = &-\int_\Omega |\omega|^2 \mathrm{d}x - \int_\Omega |\nabla\cdot u|^2 \mathrm{d}x \\ &+\int_\Gamma \langle u\times\omega, \nu\rangle \mathrm{d}S + \int_\Gamma (\nabla\cdot u)\,\langle u,\nu\rangle \mathrm{d}S,\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\int_\Omega |\nabla u|^2 \mathrm{d}x = &\int_\Omega |\omega|^2 \mathrm{d}x + \int_\Omega |\nabla\cdot u|^2 \mathrm{d}x - \int_\Gamma \pi\big(u^{\|}, u^{\|}\big)\mathrm{d}S \\ &- \int_\Gamma H|u^{\perp}|^2 \mathrm{d}S + 2\int_\Gamma \langle u^{\|}, \nabla^\Gamma \langle u,\nu\rangle\rangle \mathrm{d}S.\end{aligned} \tag{3.4}$$

Proof Formula (3.3) follows from the vector identity

$$\Delta u = -\nabla\times\omega + \nabla(\nabla\cdot u) \tag{3.5}$$

and integration by parts formula (3.2). To show (3.4), integrating the Bochner's identity:

$$\langle \Delta u, u\rangle = \frac{1}{2}\Delta |u|^2 - |\nabla u|^2, \tag{3.6}$$

together with integration by parts formulas, we obtain

$$\begin{aligned}\int_\Omega \langle \Delta u, u\rangle \mathrm{d}x &= -\int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{1}{2}\int_\Gamma \partial_\nu(|u|^2)\mathrm{d}S \\ &= -\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_\Gamma \langle u^{\|}\times\omega^{\|}, \nu\rangle \mathrm{d}S + \int_\Gamma \langle u,\nu\rangle \nabla\cdot u \mathrm{d}S \\ &\quad +2\int_\Gamma \langle u^{\|}, \nabla^\Gamma\langle u,\nu\rangle\rangle \mathrm{d}S - \int_\Gamma \pi\big(u^{\|}, u^{\|}\big)\mathrm{d}S - \int_\Gamma H\langle u,\nu\rangle^2 \mathrm{d}S,\end{aligned} \tag{3.7}$$

where we have used Lemma 2.2 (ii). Now (3.4) follows from (3.3) and (3.7).

Lemma 3.2 Suppose that $u \in H^1(\Omega)$ is a vector field satisfying the boundary conditions that $u^{\|}\big|_{\Gamma_1} = 0$ and $u^{\perp}\big|_{\Gamma_2} = 0$, where $\Gamma = \Gamma_1 \cup \Gamma_2$. Then

$$\|u\|_{H^1} \le C\|(\omega, \nabla\cdot u, u)\|_2 \tag{3.8}$$

for some constant C depending only on Ω.

Proof According to (3.4) and the boundary conditions, we have

$$\begin{aligned}\int_\Omega |\nabla u|^2 \,\mathrm{d}x &= \int_\Omega |(\omega, \nabla\cdot u)|^2 \,\mathrm{d}x - \int_{\Gamma_2} \pi(u^{\|}, u^{\|})\mathrm{d}S - \int_{\Gamma_1} H\langle u,\nu\rangle^2 \mathrm{d}S \\ &\le \int_\Omega |(\omega, \nabla\cdot u)|^2 \,\mathrm{d}x + C\int_\Gamma |u|^2 \mathrm{d}S.\end{aligned}$$

Now the conclusion follows from the trace imbedding theorem (Theorem 1.5.1.10 in Grisvard [12]): For any $\varepsilon \in (0,1]$,

$$\int_\Gamma |u|^2 \mathrm{d}S \le \varepsilon \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \frac{C}{\varepsilon}\int_\Omega |u|^2 \,\mathrm{d}x, \tag{3.9}$$

where C is a constant depending only on Ω.

Remark 3.1 The above lemma holds not only for the L^2-norm, but also for the L^p-norm for any $p \in [2, \infty)$. Indeed, under the same boundary conditions on u as in Lemma 3.2, as a special case of Theorem 10.5 of [3], we have

$$\|u\|_{W^{1,p}} \le C\big(\|(\omega, \nabla \cdot u)\|_p + \|u\|_2\big), \tag{3.10}$$

where C also depends on $p \ge 2$. Estimate (3.10) is the Agmon-Douglis-Nirenberg's estimate (cf. [3]).

Next we consider the second derivative $\nabla^2 u$ of u.

Lemma 3.3 There exists $C > 0$ depending only on Ω such that

$$\|\nabla^2 u\|_2^2 \le C\big(\|\Delta u\|_2^2 + \|\nabla u\|_2^2\big)$$

for any vector field $u \in H^2$ satisfying the absolute boundary conditions (1.14).

Proof Apply Lemma 2.2 (ii) to a gradient field ∇f for any scalar function f in Ω. Then

$$\begin{aligned}\frac{1}{2}\partial_\nu(|\nabla f|^2) = &-\pi\big((\nabla f)^\|, (\nabla f)^\|\big) - H\,|\partial_\nu f|^2 + \partial_\nu f \Delta f \\ &+2\langle(\nabla f)^\|, \nabla^\Gamma(\partial_\nu f)\rangle - \nabla^\Gamma \cdot \big(\partial_\nu f\,(\nabla f)^\|\,\big).\end{aligned} \tag{3.11}$$

Write $u = (u^1, \cdots, u^n)$ under an orthonormal basis of $T\Omega$. Then, according to the Bochner identity (3.6),

$$|\nabla^2 u|^2 = \sum_{k=1}^n |\nabla^2 u^k|^2 = \sum_{k=1}^n \Big(\frac{1}{2}\Delta|\nabla u^k|^2 - \langle \nabla \Delta u^k, \nabla u^k\rangle\Big),$$

and, after integration, we find

$$\begin{aligned}\|\nabla^2 u\|_2^2 &= \sum_{k=1}^n \Big(\frac{1}{2}\int_\Omega \Delta|\nabla u^k|^2 \mathrm{d}x - \int_\Omega \langle \nabla\Delta u^k, \nabla u^k\rangle \mathrm{d}x\Big) \\
&= \sum_{k=1}^n \Big(\int_\Omega (\Delta u^k)^2\,\mathrm{d}x + \frac{1}{2}\int_\Gamma \partial_\nu(|\nabla u^k|^2)\mathrm{d}S - \int_\Gamma (\Delta u^k)\partial_\nu u^k \mathrm{d}S\Big) \\
&= \sum_{k=1}^n \Big(\int_\Omega (\Delta u^k)^2 \mathrm{d}x - \int_\Gamma \sum_{i,j=1}^{n-1} h_{ij}(\nabla_i u^k)(\nabla_j u^k)\mathrm{d}S\Big) \\
&\quad - \int_\Gamma \sum_{k=1}^n |\partial_\nu u^k|^2 H \mathrm{d}S + 2\int_\Gamma \sum_{k=1}^n \langle \nabla^\Gamma u^k, \nabla^\Gamma(\partial_\nu u^k)\rangle \mathrm{d}S \\
&= -\int_\Gamma \sum_{i,j=1}^{n-1} h_{ij}\nabla_i u^k \nabla_j u^k \mathrm{d}S - \int_\Gamma \sum_{k=1}^n \langle \nabla u^k, \nu\rangle^2 H \mathrm{d}S \\
&\quad +2\sum_{k=1}^n \int_\Gamma \langle \nabla^\Gamma u^k, \nabla^\Gamma \langle \nabla u^k, \nu\rangle\rangle \mathrm{d}S + \|\Delta u\|_2^2,\end{aligned}$$

where the third equality follows from (3.11) applying to each u^k. We now handle the boundary integral:

$$\sum_{k=1}^n \int_\Gamma \langle \nabla^\Gamma u^k, \nabla^\Gamma(\partial_\nu u^k)\rangle \mathrm{d}S = \sum_{k=1}^{n-1} \int_\Gamma \langle \nabla^\Gamma u^k, \nabla^\Gamma(\partial_\nu u^k)\rangle \mathrm{d}S + \int_\Gamma \langle \nabla^\Gamma u^n, \nabla^\Gamma(\partial_\nu u^n)\rangle \mathrm{d}S$$

$$= \sum_{k=1}^{n-1} \int_\Gamma \langle \nabla^\Gamma u^k, \nabla^\Gamma(\partial_\nu u^k)\rangle \mathrm{d}S,$$

where we have used the fact that $u^n|_\Gamma = 0$ so that $\nabla^\Gamma u^n = 0$. For $1 \le k \le n-1$, we have $\nabla_n u^k = \nabla_k u^n = 0$ on Γ. However,

$$\partial_\nu u^k = \nabla_n u^k - \sum_{j=1}^{n-1} u^j \Gamma_{nj}^k = -\sum_{j=1}^{n-1} u^j \Gamma_{nj}^k,$$

so that

$$\sum_{k=1}^{n-1} \langle \nabla^\Gamma u^k, \nabla^\Gamma(\partial_\nu u^k)\rangle = -\sum_{k=1}^{n-1} \langle \nabla^\Gamma u^k, \nabla^\Gamma \sum_{j=1}^{n-1} u^j \Gamma_{nj}^k \rangle \le C\left(|u|^2 + |\nabla u|^2\right),$$

and the inequality follows from the trace imbedding theorem.

Corollary 3.1 Let $u \in H^2(\Omega)$ be a vector field which satisfies the absolute boundary condition (1.14) on Γ. Let $d = \nabla \cdot u$, $\omega = \nabla \times u$, and $\psi = \nabla \times \omega$. Then

(i) $\nabla \cdot \omega = \nabla \cdot \psi = 0$;

(ii) $\Delta u = \nabla d - \nabla \times \omega$;

(iii) $u^\perp\big|_\Gamma = 0$, $\omega^\parallel\big|_\Gamma = 0$, and $\psi^\perp\big|_\Gamma = 0$;

(iv) There exists $C > 0$ depending only on Ω such that

$$\|u\|_{H^2} \le C\|(\nabla d, \nabla \times \omega, u)\|_2.$$

That is, $\|(\nabla d, \nabla \times \omega, u)\|_2$ is an equivalent norm for any vector field $u \in H^2(\Omega)$ satisfying the absolute boundary condition (1.14).

Identity (i) is obvious, and (ii) follows from the vector identity:

$$\Delta u = -\nabla \times (\nabla \times u) + \nabla(\nabla \cdot u).$$

To show $\psi^\perp\big|_\Gamma = 0$, we use (ii) to obtain $(\Delta u)^\perp = (\nabla d)^\perp - \psi^\perp$, and the claim follows from Lemma 2.2. According to Lemma 3.3 and the Ehrling-Nirenberg-Gagliardo interpolation inequality, one has $\|u\|_{H^2} \le C\big(\|\Delta u\|_2 + \|u\|_2\big)$, and (iv) follows from (ii).

4 Navier-Stokes Equations

Consider the Navier-Stokes equations (1.1)–(1.2) in a bounded domain $\Omega \subset \mathbb{R}^3$, subject to the kinematic condition (1.6) and the nonhomogeneous vorticity condition (1.7).

Using the vector identity:

$$u \cdot \nabla u = \omega \times u + \frac{1}{2}\nabla(|u|^2), \tag{4.1}$$

one may rewrite (1.1) as

$$\partial_t u + \omega \times u = \mu \Delta u - \nabla\Big(p + \frac{1}{2}|u|^2\Big). \tag{4.2}$$

4.1 Neumann problem for the pressure p

Lemma 4.1 If $u(t,x)$ is a smooth solution of the Navier-Stokes equations (1.1)–(1.2) in $\Omega_T := \Omega \times (0,T]$ subject to the boundary conditions (1.6)–(1.7), then p solves the Neumann problem:

$$\Delta p = -\nabla \cdot (u \cdot \nabla u), \qquad \partial_\nu p|_\Gamma = \pi(u,u) - \mu \nabla^\Gamma \times a. \tag{4.3}$$

Proof Since $u(t,x)$ is a smooth solution in Ω_T, by taking the normal part of equations (1.1), we may determine the normal derivative of the pressure p. Indeed, since $u^\perp\big|_\Gamma = 0$, equations (1.1) imply that

$$((u \cdot \nabla)u)^\perp = \mu\,(\Delta u)^\perp - (\nabla p)^\perp \qquad \text{on the boundary } \Gamma.$$

We apply Lemmas 2.1–2.2 to obtain

$$\begin{aligned}\partial_\nu p|_\Gamma &= \mu\langle \Delta u,\, \nu\rangle - \langle u\cdot\nabla u,\, \nu\rangle \\ &= \mu\partial_\nu(\nabla\cdot u) - \mu\langle \nabla\times\omega,\, \nu\rangle + \pi(u,u) \\ &= \pi(u,u) - \mu\nabla^\Gamma \times a.\end{aligned}$$

Remark 4.1 Similarly, if $u(t,x)$ is a smooth solution of the Euler equations (1.4)–(1.5) in Ω_T satisfying the kinematic condition (1.6), then the pressure p is determined by the Neumann problem of the Poisson equation:

$$\Delta p = -\nabla \cdot (u \cdot \nabla u), \qquad \partial_\nu p|_\Gamma = \pi(u,u). \tag{4.4}$$

4.2 Reduction to an IBVP with absolute boundary conditions via the Stokes equations

We will employ the solution of the unsteady Stokes equations to reduce our initial-boundary value problem into an initial-boundary value problem with absolute boundary conditions. More precisely, let u be a solution of the initial-boundary value problem of the Navier-Stokes equations (1.1) with nonhomogeneous boundary conditions (1.6)–(1.7) and initial condition (1.3). Let w be the solution of the nonhomogeneous initial-boundary problem for the unsteady Stokes equations:

$$\begin{cases} \partial_t w = \mu\Delta w - \nabla q, \qquad \nabla\cdot w = 0, \\ w^\perp\big|_\Gamma = 0, \qquad (\nabla\times w)^\parallel\big|_\Gamma = a, \\ w|_{t=0} = u_0. \end{cases} \tag{4.5}$$

Then $v = u - w$ solves the following homogeneous initial-boundary value problem:

$$\partial_t v + (v+w)\cdot\nabla\,(v+w) = \mu\Delta v - \nabla p, \tag{4.6}$$

$$\nabla\cdot v = 0, \tag{4.7}$$

$$v|_{t=0} = 0, \tag{4.8}$$

subject to the absolute boundary condition:

$$v^\perp\big|_\Gamma = 0, \qquad (\nabla\times v)^\parallel\big|_\Gamma = 0. \tag{4.9}$$

Therefore, if the initial-boundary value problem (4.5) is well-posed with some appropriate estimates of its solution, then the well-posedness for the nonhomogeneous initial-boundary value problem (1.1)–(1.3) and (1.6)–(1.7) is equivalent to the well-posedness of the homogeneous initial-boundary value problem (4.6)–(4.9).

4.3 Well-posedness of the Stokes equations and L^2-estimates for the solutions

The initial-boundary value problem (4.5) can be solved as follows. Taking the divergence in (4.5), we can decouple the scaler function q by solving the Neumann boundary problem:

$$\Delta q = 0, \qquad \partial_\nu q|_\Gamma = -\mu\nabla^\Gamma \times a, \tag{4.10}$$

where the normal derivative of q follows from the kinematic condition in (4.5) and $\nabla \cdot w = 0$, which implies from Lemma 2.2 that

$$\partial_\nu q|_\Gamma = \mu\langle \Delta u, \nu\rangle = -\mu\nabla^\Gamma \times (\nabla\times w)^\parallel = -\mu\nabla^\Gamma \times a.$$

Since $\int_\Gamma \nabla^\Gamma \times a\mathrm{d}S = 0$, there exists a unique solution q modulo a constant. We renormalize q so that $\int_\Omega q\,\mathrm{d}x = 0$. Therefore, it suffices to solve the linear parabolic equations:

$$\begin{cases} \partial_t w - \mu\Delta w = -\nabla q, \\ w^\perp\big|_\Gamma = 0, \qquad (\nabla\times w)^\parallel\big|_\Gamma = a, \\ w|_{t=0} = u_0, \end{cases} \tag{4.11}$$

where q is given by (4.10), which is well-posed. In fact, by taking any smooth vector field A such that $A^\perp\big|_\Gamma = 0$ and $(\nabla\times A)^\parallel\big|_\Gamma = a$, and setting $v := w - A$, then the linear parabolic problem can be written as

$$\begin{cases} \dfrac{\partial}{\partial t} v = \mu\Delta v + f\,, \\ v^\perp\big|_\Gamma = 0, \qquad (\nabla\times v)^\parallel\big|_\Gamma = 0, \end{cases}$$

where $f := \mu\Delta A - \nabla q - \frac{\partial}{\partial t}A$, which is well-posed; its proof directly follows the arguments in Section 5. Furthermore, all the required estimates on w and q are available.

We now make some necessary estimates for the solution w of (4.5).

Taking the curl operation $\nabla\times$ in the equations in (4.11), with $g = \nabla\times w$ and $h = \nabla\times g = -\Delta w$,

$$\partial_t g = \mu\Delta g; \qquad \partial_t h = \mu\Delta h,$$

together with the boundary conditions:

$$g^\parallel|_\Gamma = a, \qquad (\nabla\times g)^\perp|_\Gamma = \nabla^\Gamma\times a,$$

and

$$h^\perp|_\Gamma = \nabla^\Gamma \times a, \qquad (\nabla\times h)^\parallel|_\Gamma = -\frac{1}{\mu}\partial_t a.$$

We now make the L^2-estimates for g and h. First note the following identities:

$$\partial_t(|g|^2) = 2\mu\langle g, \Delta g\rangle, \qquad \partial_t(|h|^2) = 2\mu\langle h, \Delta h\rangle.$$

We integrate over Ω to obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|g\|_2^2 &= -2\mu\int_\Omega \langle g, \nabla\times(\nabla\times g)\rangle\,\mathrm{d}x \\ &= -2\mu\int_\Omega\langle\nabla\times g, \nabla\times g\rangle\,\mathrm{d}x - 2\mu\int_\Gamma\langle(\nabla\times g)^\parallel\times a, \nu\rangle\mathrm{d}S, \end{aligned}$$

which implies

$$\|g\|_2^2(t)+2\mu\int_0^t\|\nabla\times g\|_2^2(s)\mathrm{d}s=\|g\|_2^2(0)-2\mu\int_0^t\int_\Gamma\langle(\nabla\times g)^{\|}\times a,\,\nu\rangle\mathrm{d}S\mathrm{d}s\quad\text{for } 0<t\le T.\tag{4.12}$$

Similarly, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|h\|_2^2&=-2\mu\int_\Omega\langle h,\nabla\times(\nabla\times h)\rangle\,\mathrm{d}x\\&=-2\mu\int_\Omega\langle\nabla\times h,\nabla\times h\rangle\,\mathrm{d}x+2\int_\Gamma\langle(\partial_t a)\times h,\nu\rangle\mathrm{d}S,\end{aligned}$$

which implies

$$\|h\|_2^2(t)+2\mu\int_0^t\|\nabla\times h\|_2^2(s)\mathrm{d}s=\|h\|_2^2(0)+2\int_0^t\int_\Gamma\langle(\partial_t a)\times h,\,\nu\rangle\mathrm{d}S\mathrm{d}s\quad\text{for } 0<t\le T.\tag{4.13}$$

We now consider two different cases.

Case 1 $\partial_t a=0$ and $\sqrt{\mu}a\in L^2(\Gamma_T)$, for $\Gamma_T=\Gamma\times(0,T]$. In this case, we obtain from (4.13) that

$$\|h\|_2^2(t)+2\mu\int_0^t\|\nabla\times h\|_2^2(s)\mathrm{d}s=\|h\|_2^2(0)\le\|w(0,\cdot)\|_{H^2}=\|u_0\|_{H^2}.$$

Since $\nabla\cdot g=\nabla\cdot h=0$, we conclude

$$\|\nabla g\|^2_{L^2(\Omega)}(t)+\mu\|\nabla^2 g\|^2_{L^2(\Omega_T)}\le C\qquad\text{for } t\in[0,T],\tag{4.14}$$

where $C>0$ is independent of μ.

Then we have

$$\begin{aligned}2\mu\Big|\int_0^t\int_\Gamma\langle(\nabla\times g)^{\|}\times a,\nu\rangle\mathrm{d}S\mathrm{d}s\Big|&\le 2\mu\Big(\varepsilon\int_0^t\int_\Gamma|\nabla g|^2\mathrm{d}S\mathrm{d}s+C_\varepsilon\|a\|^2_{L^2(\Gamma_T)}\Big)\\&\le\mu\int_0^t\int_\Omega|\nabla\times g|^2\mathrm{d}x\mathrm{d}s+C\mu\|\nabla^2 g\|^2_{L^2(\Omega_T)}+\mu\|a\|^2_{L^2(\Gamma_T)}\\&\le\mu\int_0^t\int_\Omega|\nabla\times g|^2\mathrm{d}x\mathrm{d}s+C\|u_0\|^2_{H^2(\Omega)}+\mu\|a\|^2_{L^2(\Gamma_T)},\end{aligned}$$

where we have used the interpolation inequality in the second inequality above.

Substitution this into (4.12) yields

$$\|g\|^2_{L^2(\Omega)}(t)+\mu\|\nabla g\|^2_{L^2(\Omega_t)}\le C\|u_0\|^2_{H^2(\Omega)}+\mu\|a\|^2_{L^2(\Gamma_T)}.\tag{4.15}$$

Combining (4.14)–(4.15) with the fact $\nabla\cdot w=\nabla\cdot g=\nabla\cdot h=0$, we have

$$\|w\|^2_{H^2(\Omega)}(t)+\mu\|w\|^2_{H^3(\Omega_T)}\le C\|u_0\|^2_{H^2(\Omega)}+\mu\|a\|^2_{L^2(\Gamma_T)}\quad\text{for } t\in[0,T],\tag{4.16}$$

where $C>0$ is independent of μ.

Case 2 $\partial_t a\ne 0$ and $(a,\partial_t a)\in L^2(\Gamma_T)$. In this case, we have

$$\begin{aligned}2\Big|\int_0^t\int_\Gamma\langle(\partial_t a)\times h,\nu\rangle\mathrm{d}S\mathrm{d}s\Big|&\le\varepsilon\int_0^t\int_\Gamma|h|^2\mathrm{d}S\mathrm{d}s+C_\varepsilon\|\partial_t a\|^2_{L^2(\Gamma_T)}\\&\le\mu\int_0^t\int_\Omega|\nabla\times h|^2\mathrm{d}x\mathrm{d}s+\frac{C}{\mu}\int_0^t\int_\Omega|h|^2\mathrm{d}x\mathrm{d}s+C\|\partial_t a\|^2_{L^2(\Gamma_T)},\end{aligned}$$

where we have used $\nabla \cdot h = 0$ and the interpolation inequality in the second inequality above.

Substituting this into (4.13) yields

$$\|h\|_2^2(t) + \mu \int_0^t \|\nabla \times h\|_2^2(s)\mathrm{d}s \le \|u_0\|_{H^2(\Omega)}^2 + C\|\partial_t a\|_{L^2(\Gamma_T)} + \frac{C_0}{\mu}\int_0^t \|h\|_2^2(s)\mathrm{d}s.$$

Then the Gronwall inequality yields

$$\|h\|_2^2(t) \le M(\mu, T)(\|u_0\|_{H^2(\Omega)}^2 + \|\partial_t a\|_{L^2(\Gamma_T)}^2).$$

Combining this with (4.12), we conclude

$$\|w\|_{H^2(\Omega)}(t) + \|w\|_{H^3(\Omega_T)}^2 \le M(\mu, T)(\|u_0\|_{H^2(\Omega)}^2 + \|(a, \partial_t a)\|_{L^2(\Gamma_T)}^2), \tag{4.17}$$

where $M(\mu, T) > 0$ depends only on μ and T.

Proposition 4.1 Let $w \in H^3(\Omega)$ be a solution of problem (4.5). Then

(i) If $\partial_t a = 0$ and $\sqrt{\mu} a \in L^2(\Gamma_T)$, then there exists $C > 0$, independent of μ, such that (4.16) holds;

(ii) If $\partial_t a \neq 0$ and $(a, \partial_t a) \in L^2(\Gamma_T)$, there exists $M = M(\mu, T) > 0$ such that (4.17) holds.

5 A Linear Initial-Boundary Value Problem

In this section, we deal with the absolute boundary problem of a linearized parabolic system. Let $\beta(t, x), 0 \le t \le T$, be a smooth vector field on Ω satisfying the absolute boundary condition:

$$\beta^{\perp}\big|_\Gamma = (\nabla \times \beta)^{\|}\big|_\Gamma = 0.$$

Define the pressure function $p_\beta(t, x)$ for $0 \le t \le T$ by solving the Poisson equation:

$$\begin{cases} \Delta p_\beta = -\nabla \cdot \big((\beta + w) \cdot \nabla(\beta + w)\big), \\ \partial_\nu p_\beta|_\Gamma = \pi\left(\beta + w, \beta + w\right), \end{cases} \tag{5.1}$$

subject to the normalization $\int_\Omega p_\beta \,\mathrm{d}x = 0$. Here and hereafter, the time-variable t is omitted for simplicity in the equations if no confusion may arise. Thanks to Lemma 2.1,

$$\int_\Omega \nabla \cdot \big((\beta + w) \cdot \nabla(\beta + w)\big)\,\mathrm{d}x = \int_\Gamma \langle(\beta + w) \cdot \nabla(\beta + w), \nu\rangle \mathrm{d}S = -\int_\Gamma \pi\left(\beta + w, \beta + w\right)\mathrm{d}S,$$

hence the above boundary value problem (5.1) has a unique solution. The elliptic estimates yield that p_β is smooth both in t and x.

We now consider the following initial-boundary value problem in Ω:

$$\partial_t v + (\beta + w) \cdot \nabla(v + w) = \Delta v - \nabla p_\beta, \tag{5.2}$$

subject to the initial-boundary conditions:

$$v|_{t=0} = 0, \tag{5.3}$$

$$v^{\perp}\big|_\Gamma = \omega^{\|}\big|_\Gamma = 0. \tag{5.4}$$

In this section and Section 6, we denote $\omega = \nabla \times v$ again without confusion and always set $\mu = 1$ without loss of generality for the existence proof.

To solve problem (5.2)–(5.4), we recall that the Laplacian Δ acting on vector fields in

$$D(\Delta) = \left\{ v \in H^2(\Omega; \mathbb{R}^3) \,:\, v^{\perp}\big|_{\Gamma} = \omega^{\|}\big|_{\Gamma} = 0 \right\}$$

is self-adjoint on $L^2(\Omega)$ and is negative-definite:

$$(\Delta v, v) = -\int_{\Omega} |\nabla \cdot v|^2 \mathrm{d}x - \int_{\Omega} |\omega|^2 \mathrm{d}x \leq 0$$

for every $v \in D(\Delta)$. The Laplacian with domain $D(\Delta)$ will be still denoted by Δ for simplicity. According to Hille-Yosida's theorem, Δ is the infinitesimal generator of a strongly continuous semigroup of contractions on $L^2(\Omega)$, denoted by $\left(e^{-t\Delta}\right)_{t\geq 0}$. Indeed, Δ is the infinitesimal generator of an analytic semigroup, and the L^2-domain $D(\Delta)$ of Δ is invariant under $e^{-t\Delta}$. In addition, for each t, $e^{-t\Delta}$ commutes with the curl operation $\nabla\times$ and the divergence operator $\nabla\cdot$ (see [9] for the details).

For every $f \in D(\Delta)$, $\varphi(t) = e^{-t\Delta} f$ solves the following evolution equations:

$$\partial_t \varphi = \Delta \varphi, \qquad \varphi(0, \cdot) = f.$$

The regularity theory for parabolic equations yields that, if f is smooth, so is φ.

Our next aim is to solve the homogenous equations:

$$\partial_t \varphi = \big(\Delta - \theta \cdot \nabla\big)\varphi, \qquad \varphi|_{t=0} = f,$$

where $f \in D(\Delta)$, and θ is a smooth vector field on Ω (independent of t). To this end, we show that $A = \Delta - \theta \cdot \nabla$ with domain $D(A) = D(\Delta)$ is the generator of a strongly-continuous semigroup.

Lemma 5.1 Under the above notations, $(A, D(\Delta))$ is a densely defined, closed operator, which is indeed the infinitesimal generator of an analytic semigroup on $L^2(\Omega)$.

Proof If a vector field φ satisfies the absolute boundary conditions, then

$$\begin{aligned} \int_{\Omega} |\nabla \varphi|^2 \,\mathrm{d}x &= \frac{1}{2}\int_{\Omega} \Delta(|\varphi|^2)\,\mathrm{d}x - \int_{\Omega} \langle \Delta\varphi, \varphi\rangle \,\mathrm{d}x = \frac{1}{2}\int_{\Gamma} \partial_{\nu}(|\varphi|^2)\mathrm{d}S - \int_{\Omega} \langle \Delta\varphi, \varphi\rangle \,\mathrm{d}x \\ &= -\int_{\Gamma} \pi(\varphi,\varphi)\mathrm{d}S - \int_{\Omega} \langle \Delta\varphi, \varphi\rangle \,\mathrm{d}x \leq \int_{\Gamma} |\varphi|^2 \mathrm{d}S - \int_{\Omega} \langle \Delta\varphi, \varphi\rangle \,\mathrm{d}x \\ &\leq \varepsilon \|\nabla\varphi\|_2^2 + C\|\varphi\|_2^2 + \|\Delta\varphi\|_2 \|\varphi\|_2, \end{aligned}$$

that is,

$$\|\nabla\varphi\|_2^2 \leq C\big(\|\varphi\|_2^2 + \|\Delta\varphi\|_2\|\varphi\|_2\big).$$

Hence, we have

$$\|(\theta\cdot\nabla)\varphi\|^2 \leq \|\theta\|_{\infty}^2 \|\nabla\varphi\|^2 \leq C\|\theta\|_{\infty}^2 \big(\|\varphi\|_2^2 + \|\Delta\varphi\|_2\|\varphi\|_2\big) \leq C\|\theta\|_{\infty}^2\|\varphi\|_2^2 + \varepsilon\|\Delta\varphi\|_2^2,$$

for any $\varepsilon \in (0,1)$, where $C(\varepsilon, \|\theta\|_{\infty}^2, \Omega)$ is a constant depending only on ε, $\|\theta\|_{\infty}^2$, and Ω. According to Kato's perturbation theorem (for example, see Theorem 2.1, page 80 in [38]), all the conclusions follow.

The analytic semigroup with infinitesimal generator $\Delta-\theta\cdot\nabla$ with domain $D(\Delta)$ is denoted by $\big(e^{t(\Delta-\theta\cdot\nabla)}\big)_{t\geq 0}$.

Next, we want to solve the nonhomogeneous evolution equations

$$\partial_t\varphi=\big(\Delta-(\beta+w)\cdot\nabla\big)\varphi+h,\qquad \varphi(0,\cdot)=f,$$

where $\beta(t,x)$ is a smooth vector field and satisfies the absolute boundary conditions such that

$$\|\beta(t)-\beta(s)\|_\infty\leq C|t-s|^\alpha$$

for some constant C and $\alpha\in(0,1]$, $w(t,x)$ is the unique solution of (4.5), $h(t,\cdot)\in L^2(\Omega)$ for each $t\in[0,T]$, and $f\in D(\Delta)$.

Lemma 5.2 Let

$$\rho=\sup_{0\leq t\leq T}\left\{\int_\Omega\langle\big(\Delta-(\beta+w)(t,\cdot)\cdot\nabla\big)\psi,\psi\rangle\mathrm{d}x:\ \psi\in D(\Delta)\text{ and }\|\psi\|=1\right\}.$$

Then $\rho<\infty$. If $\psi\in D(\Delta)$ satisfies the Poisson equations:

$$\big(\Delta-(\beta+w)(t,\cdot)\cdot\nabla-\rho-1\big)\psi=\varphi,\qquad t\leq T,$$

and $\varphi\in L^2(\Omega)$, then $\|\psi\|\leq\|\varphi\|$ and $\|\nabla\psi\|\leq C\|\varphi\|$ for some constant $C>0$.

Proof It is easy to see that

$$\int_\Omega\langle\big(\Delta-(\beta+w)(t,\cdot)\cdot\nabla\big)\psi,\psi\rangle\,\mathrm{d}x\leq\|(\beta+w)(t,\cdot)\|_\infty\int_\Omega|\psi|^2\mathrm{d}x$$

so that $\rho<\infty$. Since

$$-\int_\Omega\langle\varphi,\psi\rangle\,\mathrm{d}x=\int_\Omega\langle\big(-\Delta+(\beta+w)(t,\cdot)\cdot\nabla+\rho+1\big)\psi,\psi\rangle\,\mathrm{d}x\geq\|\psi\|^2,$$

which gives the first estimate in the lemma. Using the Bochner identity (3.6), we have

$$\begin{aligned}\int_\Omega|\nabla\psi|^2\,\mathrm{d}x&=\frac{1}{2}\int_\Omega\Delta(|\psi|^2)\,\mathrm{d}x-\int_\Omega\langle\Delta\psi,\psi\rangle\,\mathrm{d}x=\frac{1}{2}\int_\Gamma\partial_\nu(|\psi|^2)\mathrm{d}S-\int_\Omega\langle\Delta\psi,\psi\rangle\,\mathrm{d}x\\&=-\int_\Gamma\pi(\psi,\psi)\mathrm{d}S-\int_\Omega\langle\Delta\psi,\psi\rangle\,\mathrm{d}x\\&=\int_\Gamma|\psi|^2\mathrm{d}S-\int_\Omega\langle\varphi+\big((\beta+w)(t,\cdot)\cdot\nabla+\rho+1\big)\psi,\psi\rangle\,\mathrm{d}x\\&\leq\varepsilon\|\nabla\psi\|_2^2+C\|\varphi\|_2^2+\|\varphi\|_2\|\psi\|_2+\|(\beta+w)(t,\cdot)\|_\infty\|\nabla\psi\|_2\|\psi\|_2+(\rho+1)\|\psi\|_2^2,\end{aligned}$$

which implies

$$\|\nabla\psi\|\leq C\|\varphi\|.$$

Lemma 5.3 Let $\beta(t,x),0\leq t\leq T$, be a smooth vector field on Ω satisfying the absolute boundary conditions (1.14). Suppose that

$$\|(\beta+w)(t,\cdot)\|_\infty\leq\rho\qquad\text{for every }t\in[0,T],$$

for some non-negative constant ρ. Let

$$A(t)=\Delta-(\beta+w)(t,\cdot)\cdot\nabla-(\rho+1)I$$

with domain $D(A(t)) = D(\Delta)$. Then, for each t, $A(t)$ is the infinitesimal generator of the strongly continuous semigroup $\{e^{-(\rho+1)s}e^{s(\Delta-(\beta+w)(t,\cdot)\cdot\nabla)}\}_{s\geq 0}$ of contractions on $L^2(\Omega)$. Moreover, for every s, t, and τ in $[0,T]$, we have

$$\|\big(A(t)-A(s)\big)A(\tau)^{-1}\| \leq C\|(\beta+w)(t,\cdot)-(\beta+w)(s,\cdot)\|_\infty$$

for some constant C.

Proof Let $\psi = A(\tau)^{-1}\varphi$. Then $\psi \in D(\Delta)$ and, according to Lemma 5.2, $\|\nabla\psi\| \leq C\|\varphi\|$ for some constant C. Since

$$\big(A(t)-A(s)\big)A(\tau)^{-1}\varphi = \big((\beta+w)(t,\cdot)-(\beta+w)(s,\cdot)\big)\cdot\nabla\psi$$

so that

$$\begin{aligned}\|\big(A(t)-A(s)\big)A(\tau)^{-1}\varphi\| &\leq \|(\beta+w)(t,\cdot)-(\beta+w)(s,\cdot)\|_\infty\|\nabla\psi\| \\ &\leq C\|(\beta+w)(t,\cdot)-(\beta+w)(s,\cdot)\|_\infty\|\varphi\|.\end{aligned}$$

Let us retain the notations as above. Assume that

$$\|(\beta+w)(t,\cdot)-(\beta+w)(s,\cdot)\|_\infty \leq C|t-s|^\alpha \qquad \text{for all } s,t\in[0,T].$$

We want to solve the nonhomogeneous evolution equations:

$$\partial_t\psi(t,\cdot)+(\beta+w)(t,\cdot)\cdot\nabla\psi(t,\cdot) = \Delta\psi(t,\cdot)+h(t),$$

where $\psi(t,\cdot)$ satisfies the absolute boundary conditions (1.14). The above evolution equations may be rewritten as

$$\partial_t\Psi(t) = A(t)\Psi(t)+e^{-(\rho+1)t}h(t),$$

where $A(t)$ is given in Lemma 5.3, $\psi(t) = e^{(\rho+1)t}\Psi(t)$, and $\Psi(t)$ satisfies the absolute boundary condition. According to Theorem 6.1 in [38], page 150, there is a unique evolution system $U(t,s)$ on $0\leq s\leq t\leq T$ such that the mild solution $\Psi(t)$ is given by

$$\Psi(t) = U(t,0)\psi(0,\cdot)+\int_0^t e^{-(\rho+1)s}U(t,s)h(s,\cdot)\mathrm{d}s$$

so that

$$\psi(t,\cdot) = e^{(\rho+1)t}U(t,0)\psi(0,\cdot)+e^{(\rho+1)t}\int_0^t e^{-(\rho+1)s}U(t,s)h(s,\cdot)\mathrm{d}s.$$

Therefore, we have the following theorem.

Theorem 5.1 Given the initial vector field $f\in D(\Delta)$, $h(t,\cdot)\in L^2(\Omega)$, and $\beta(t,\cdot)$ for $t\in[0,T]$ such that

$$\begin{cases}\beta(t,\cdot)\in D(\Delta) \quad \text{for each } t\in[0,T], \\ \sup\limits_{0\leq t\leq T}\|\beta(t,\cdot)\|_\infty<\infty, \\ \|(\beta+w)(t,\cdot)-(\beta+w)(s,\cdot)\|_\infty \leq C|t-s|^\alpha \qquad \forall s,t\in[0,T],\end{cases}$$

there exists a unique solution ψ which solves the absolute boundary problem on Ω of the linear parabolic equations:

$$\partial_t\psi+(\beta+w)\cdot\nabla\psi = \Delta\psi+h.$$

6 Construction of Local Solutions

The main goal of this section is to prove the existence of a unique strong solution, local in time, to the Navier-Stokes equations with the nonhomogeneous boundary conditions. To this end, we establish apriori estimate for solutions of the linear parabolic equations (5.2).

In order to state our results, we use the following norm: For a vector field $v(t,x), 0 \le t \le T, x \in \Omega$,

$$\|v(t,\cdot)\|_{\mathbf{N}} = \sqrt{\|v(t,\cdot)\|^2_{H^2} + \|v_t(t,\cdot)\|^2_{H^1}}.$$

6.1 Main estimate

Let $T > 0$ be a fixed but arbitrary constant. Let $\beta(t,x), 0 \le t \le T$, be a given smooth vector field satisfying the absolute boundary conditions (1.14). Let $v = V(\beta)$ is the unique solution to the linear parabolic equations (5.2)–(5.4), with p_β the unique solution to problem (5.1) and $w(t,x)$ the unique solution of the initial-boundary value problem (4.5) for the Stokes equations. Then the main *apriori* estimate is given in the following theorem.

Theorem 6.1 Let $\beta(t,x), 0 \le t \le T$, be a vector field which satisfies the following conditions:

(i) For every $0 \le t \le T$, $\beta(t,\cdot) \in H^2(\Omega)$ and $\partial_t \beta(t,\cdot) \in H^1(\Omega)$;

(ii) For each $0 \le t \le T$, $\beta^\perp\big|_\Gamma = (\nabla \times \beta)^\parallel\big|_\Gamma = 0$;

(iii) $t \to \|\beta(t,\cdot)\|^2_{H^2} + \|\partial_t \beta(t,\cdot)\|^2_{H^1}$ is continuous;

(iv) $\beta(0,\cdot) = 0$.

Let $v = V(\beta)$ defined by (5.1)–(5.4). Then there exists a constant C depending only on the domain Ω such that the following inequality holds:

$$\|v(t,\cdot)\|^2_{\mathbf{N}} \le C\|u_0\|^4_{H^2} e^{CQ(t)} + C\int_0^t e^{C(Q(t)-Q(s))}\big(1 + \|(\beta,w)(s,\cdot)\|^2_{\mathbf{N}}\big)^2 \mathrm{d}s, \tag{6.1}$$

where

$$Q(t) = \int_0^t \big(1 + \|(\beta,w)(s,\cdot)\|^2_{\mathbf{N}}\big)\mathrm{d}s. \tag{6.2}$$

To establish estimate (6.1), we will frequently use an elementary L^2-estimate for elliptic equations, which can be stated as follows: If ϕ is a solution to the Neumann boundary problem:

$$\begin{cases} \Delta\phi = \nabla \cdot f, \\ \partial_\nu \phi|_\Gamma = \langle f, \nu\rangle, \end{cases} \tag{6.3}$$

then we have the Solonnikov estimate:

$$\|\nabla\phi\|_2 \le \|f\|_2. \tag{6.4}$$

Estimate (6.4) is a special case of Theorem 2.2 in Solonnikov [45]. In fact, estimate (6.4) follows easily from an integration by parts argument. Since

$$\begin{aligned} \|\nabla\phi\|^2_2 &= -\int_\Omega \langle \phi, \Delta\phi\rangle \,\mathrm{d}x + \int_\Gamma \phi\, \partial_\nu \phi \mathrm{d}S = -\int_\Omega \langle \phi, \nabla\cdot f\rangle \,\mathrm{d}x + \int_\Gamma \phi\,\partial_\nu\phi \mathrm{d}S \\ &= \int_\Omega \langle \nabla\phi, f\rangle \,\mathrm{d}x - \int_\Gamma \phi\langle f,\nu\rangle \mathrm{d}S + \int_\Gamma \phi\,\partial_\nu\phi\mathrm{d}S = \int_\Omega \langle\nabla\phi, f\rangle \mathrm{d}S, \end{aligned}$$

then (6.4) follows from the Schwartz inequality directly.

We now present the proof of Theorem 6.1. Throughout the proof, we will use C to denote a constant depending only on the domain Ω, which may be different at each occurrence. We also omit the lowerscript β for simplicity: For example, we write p for p_β.

Let $d = \nabla \cdot v$, $\omega = \nabla \times v$, and $\psi = \nabla \times \omega$ for simplicity. Let f_t denote the time-derivative $\partial_t f$ for a vector field f. Observe that v_t again satisfies the absolute boundary conditions (1.14). Thus, according to Corollary 3.1,

$$\|v_t\|_{H^1} \le C\|(v_t, d_t, \omega_t)\|_2, \qquad \|v\|_{H^2} \le C\|(v, \nabla d, \psi)\|_2. \tag{6.5}$$

It is therefore natural to bound the time-derivative of each term appearing on the right-hand sides of (6.5). However, as a matter of fact, we are unable to bound $\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla d\|_2^2$. A different approach to handle $\|\nabla d\|_2$ is required.

Taking the divergence $\nabla\cdot$ on both sides of the linear parabolic equations (5.2), one obtains

$$\partial_t d = \Delta d + \nabla \cdot \left((\beta + w) \cdot \nabla(\beta - v)\right). \tag{6.6}$$

From (5.2)–(5.4), we have

$$\langle \Delta v, \nu\rangle = -\pi(\beta + w, v + w) + \partial_\nu p_\beta = \pi\left(\beta + w, \beta - v\right)$$

so that, according to Lemma 2.2, d satisfies the Neumann boundary condition:

$$\partial_\nu d|_\Gamma = \pi\left(\beta + w, \beta - v\right), \tag{6.7}$$

which is nonhomogeneous. We define, for every $0 \le t \le T$, a function $q(t, \cdot)$ by solving the Poisson equation:

$$\begin{cases} \Delta q = -\nabla \cdot \left((\beta + w) \cdot \nabla(\beta - v)\right), \\ \partial_\nu q|_\Gamma = \pi\left(\beta + w, \beta - v\right), \end{cases} \tag{6.8}$$

subject to $\int_\Omega q\, \mathrm{d}x = 0$.

Let $g = d - q$. Then g solves the linear parabolic equation:

$$\begin{cases} \partial_t g = \Delta g - q_t, \\ \partial_\nu g|_\Gamma = 0. \end{cases} \tag{6.9}$$

Since $\|\nabla d\|_2 \le \|(\nabla g, \nabla q)\|_2$, we have

$$\|v\|_{H^2} \le C\|(v, \nabla g, \nabla q, \psi)\|_2.$$

We thus consider the following function:

$$F(t) = \|(v, \psi)(t, \cdot)\|_2^2 + \|(\nabla g, \nabla q)(t, \cdot)\|_2^2 + \|(v_t, d_t, \omega_t)(t, \cdot)\|_2^2.$$

Then $\|v(t, \cdot)\|_{\mathbf{N}}^2 \le F(t)$, and

$$F(0) = \|(u_0 \cdot \nabla)u_0 + \nabla p_0\|_2^2 + \|\nabla \times (u_0 \cdot \nabla)u_0\|_2^2,$$

where p_0 is the solution to

$$\Delta p_0 = -\nabla \cdot ((u_0 \cdot \nabla) u_0), \qquad \partial_\nu p_0|_\Gamma = \pi(u_0, u_0).$$

Hence

$$F(0) \le C\|u_0\|_{H^2}^4,$$

and (6.1) follows from the following lemma.

Lemma 6.1 Under the assumptions in Theorem 6.1 and notations introduced above, we have the following differential inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t} F \le C\left(\|(\beta, w)\|_{\mathbf{N}}^2 + 1\right)\left(F + 1 + \|(\beta, w)\|_{\mathbf{N}}^2\right). \tag{6.10}$$

Proof Using (5.2)–(5.4) and integration by parts, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_2^2 &= 2\int_\Omega \langle v, \Delta v\rangle \,\mathrm{d}x - 2\int_\Omega \langle v, \nabla p\rangle \,\mathrm{d}x - 2\int_\Omega \langle v, (\beta + w)\cdot \nabla (v + w)\rangle \,\mathrm{d}x \\ &\le 2\|v\|_{H^2}^2 + 2\|v\|_2 \|\nabla p\|_2 + 2\|v\|_2 \|\beta + w\|_\infty \|\nabla (v + w)\|_2.\end{aligned}$$

Using the Sobolev embedding, together with the L^2-estimate for p, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_2^2 \le C\|\beta + w\|_{H^2}(F + 1)\|\nabla w\|_2. \tag{6.11}$$

Next we show the following inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v_t\|^2 \le C\left(1 + \|\beta + w\|_{\mathbf{N}}^2\right)\left(F + 1 + \|w_t\|_2^2 + \|\nabla w\|_2^2\right). \tag{6.12}$$

By differentiating (5.2), we obtain

$$\partial_t v_t = \Delta v_t - (\beta + w)\cdot \nabla (v_t + w_t) - (\beta_t + w_t)\cdot \nabla (v + w) - \nabla p_t, \tag{6.13}$$

and v_t also satisfies the absolute boundary condition (1.14). Since p_t is the time-derivative of p, it solves the Neumann problem:

$$\begin{cases} \Delta p_t = -\nabla \cdot \big((\beta_t + w_t)\cdot \nabla (\beta + w)\big) - \nabla \cdot \big((\beta + w)\cdot \nabla (\beta_t + w_t)\big), \\ \partial_\nu p_t|_\Gamma = 2\,\pi\,(\beta_t + w_t, \beta + w). \end{cases} \tag{6.14}$$

According to the Solonnikov estimate (6.4),

$$\begin{aligned}\|\nabla p_t\| &\le \|(\beta_t + w_t)\cdot \nabla (\beta + w) + (\beta + w)\cdot \nabla (\beta_t + w_t)\|_2 \\ &\le \|\beta_t + w_t\|_4 \|\nabla (\beta + w)\|_4 + \|\beta + w\|_\infty \|\nabla \beta_t + \nabla w_t\|_2 \\ &\le C\|\beta_t + w_t\|_{H^1}\|\beta + w\|_{H^2},\end{aligned} \tag{6.15}$$

where the second inequality follows from the Hölder inequality, and the last one follows from the Sobolev imbedding: $\|T\|_4 \le C\|T\|_{H^1}$ and $\|T\|_\infty \le C\|T\|_{H^2}$ in $\mathbb{R}^3$. Using (6.13) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|v_t\|_2^2 = {}& 2\int_\Omega \langle v_t, \Delta v_t\rangle \,\mathrm{d}x - 2\int_\Omega \langle v_t, (\beta + w)\cdot \nabla (v_t + w_t)\rangle \,\mathrm{d}x \\ & -2\int_\Omega \langle v_t, \nabla p_t\rangle \,\mathrm{d}x - 2\int_\Omega \langle v_t, (\beta_t + w_t)\cdot \nabla (v + w)\rangle \,\mathrm{d}x.\end{aligned}$$

Then, using the Hölder inequality, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|v_t\|_2^2 \le& -2\|\nabla v_t\|_2^2 - 2\int_\Gamma \pi(v_t, v_t)\mathrm{d}S + \|\beta + w\|_\infty \|v_t\|_2 \|\nabla v_t + \nabla w_t\|_2\\ &+2\|v_t\|_2\|\nabla p_t\|_2 + \|\beta_t + w_t\|_4 \|v_t\|_4 \|\nabla v + \nabla w\|_2,\end{aligned}$$

where we have used (3.3) and (3.4). According to the trace imbedding theorem (Theorem 1.5.1.10 of Grisvard [12], page 41),

$$\Big|\int_\Gamma \pi(v_t, v_t)\mathrm{d}S\Big| \le C\|v_t\|_{H^1}^2.$$

Thus we establish

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|v_t\|^2 \le& C\|v_t\|^2 + 2\|v_t\|_2\|\nabla p_t\|_2 + C\|\beta + w\|_{H^2}^2 \|v_t + w_t\|_2^2\\ &+C\|\beta_t + w_t\|_{H^1}\|v_t\|_{H^1}\|\nabla v + \nabla w\|_2,\end{aligned} \tag{6.16}$$

which implies (6.12).

Similarly, $\omega = \nabla \times v$ evolves according to the vorticity system:

$$\begin{cases}\partial_t \omega = \Delta\omega - \nabla\times((\beta + w)\cdot\nabla(v + w)),\\ \omega^{\|}\big|_\Gamma = 0,\end{cases} \tag{6.17}$$

and ω_t satisfies the evolution system:

$$\begin{cases}\partial_t \omega_t = \Delta\omega_t - \nabla\times((\beta_t + w_t)\cdot\nabla(v + w) + (\beta + w)\cdot\nabla(v_t + w_t))\,,\\ \omega_t^{\|}\big|_\Gamma = 0.\end{cases} \tag{6.18}$$

Moreover, $\nabla\cdot\omega_t = 0$. Hence, integration by parts yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_t\|_2^2\\ &= 2\int_\Omega \langle\omega_t, \Delta\omega_t\rangle\,\mathrm{d}x - 2\int_\Omega \langle\omega_t, \nabla\times\big((\beta_t + w_t)\cdot\nabla(v + w) + (\beta + w)\cdot\nabla(v_t + w_t)\big)\rangle\,\mathrm{d}x\\ &= -2\int_\Omega |\nabla\times\omega_t|^2\mathrm{d}x - 2\int_\Omega \langle\nabla\times\omega_t, (\beta_t + w_t)\cdot\nabla(v + w) + (\beta + w)\cdot\nabla(v_t + w_t)\rangle\,\mathrm{d}x\\ &\le \frac{1}{2}\|(\beta_t + w_t)\cdot\nabla(v + w) + (\beta + w)\cdot\nabla(v_t + w_t)\|_2^2,\end{aligned} \tag{6.19}$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_t\|_2^2 \le C\|\beta + w\|_{\mathbf{N}}^2\left(F + \|(\nabla w, \nabla w_t)\|_2^2\right). \tag{6.20}$$

Next we consider $\|d_t\|_2^2$ to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|d_t\|_2^2 \le C\|\beta + w\|_{\mathbf{N}}^2\left(F + \|\beta\|_{\mathbf{N}}^2\right). \tag{6.21}$$

Indeed, differentiating equation (6.6) for d to obtain

$$\partial_t d_t = \Delta d_t + \nabla\cdot\big((\beta_t + w_t)\cdot\nabla(\beta - v) + (\beta + w)\cdot\nabla(\beta_t - v_t)\big), \tag{6.22}$$

subject to the Neumann boundary condition:

$$\partial_\nu d_t|_\Gamma = \pi(\beta_t + w_t, \beta - v) + \pi(\beta + w, \beta_t - v_t). \tag{6.23}$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|d_t\|_2^2 = 2\int_\Omega \langle d_t, \Delta d_t\rangle \,\mathrm{d}x + 2\int_\Omega \langle d_t, \nabla\cdot((\beta_t + w_t)\cdot\nabla(\beta - v) + (\beta + w)\cdot\nabla(\beta_t - v_t))\rangle \,\mathrm{d}x.$$

Performing integrating by parts for the two integrals on the right-hand side and noting that the boundary integrals cancel out, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|d_t\|_2^2 &= -2\int_\Omega |\nabla d_t|^2 \mathrm{d}x + 2\int_\Omega \langle \nabla d_t, (\beta_t + w_t)\cdot\nabla(\beta - v)\rangle \,\mathrm{d}x \\ &\quad +2\int_\Omega \langle \nabla d_t, (\beta + w)\cdot\nabla(\beta_t - v_t)\rangle \,\mathrm{d}x \\ &\le C\|(\beta_t + w_t)\cdot\nabla(\beta - v)\|_2^2 + \|(\beta + w)\cdot\nabla(\beta_t - v_t)\|_2^2 \\ &\le C\|\beta_t + w_t\|_4^2\|\nabla\beta - \nabla v\|_4^2 + \|\beta + w\|_\infty^2\|\nabla\beta_t - \nabla v_t\|_2^2 \\ &\le C\|\beta + w\|_{\mathbf{N}}^2\big(\|\beta\|_{H^2}^2 + \|v\|_{H^2}^2 + \|\nabla\beta_t - \nabla v_t\|_2^2\big),\end{aligned}$$

and (6.21) follows.

Next we handle the second-order derivative. That is, we need to bound $\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla g(t,\cdot)\|_2^2$, $\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla q(t,\cdot)\|_2^2$, and $\frac{\mathrm{d}}{\mathrm{d}t}\|\psi(t,\cdot)\|_{L^2}^2$. We handle them one by one.

Since g satisfies (6.9), integration by parts yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla g\|_2^2 = -2\int_\Omega (\Delta g)^2 \mathrm{d}x + 2\int_\Omega \langle \nabla q_t, \nabla g\rangle \,\mathrm{d}x \le \|\nabla q_t\|_2\|\nabla g\|_2.$$

Notice that q_t solves the Poisson equation:

$$\Delta q_t = -\nabla\cdot\big((\beta_t + w_t)\cdot\nabla(\beta - v) + (\beta + w)\cdot\nabla(\beta_t - v_t)\big),$$

subject to

$$\partial_\nu q_t|_\Gamma = \pi(\beta_t + w_t, \beta - v) + \pi(\beta + w, \beta_t - v_t) = -\langle \nabla\cdot\big((\beta + w)\cdot\nabla(\beta - v)\big), \nu\rangle.$$

Thus, according to the Solonnikov estimate (6.4),

$$\begin{aligned}\|\nabla q_t\|_2 &\le \|(\beta_t + w_t)\cdot\nabla(\beta - u) + (\beta + w)\cdot\nabla(\beta_t - v_t)\|_2 \\ &\le \|\beta + w\|_{\mathbf{N}}\big(\|\beta\|_{H^2} + \|v\|_2 + \|\nabla(\beta_t - v_t)\|_2\big) \\ &\le \|\beta + w\|_{\mathbf{N}}\big(\|\beta\|_{\mathbf{N}} + \sqrt{F}\big),\end{aligned} \tag{6.24}$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla g(t,\cdot)\|_2^2 \le 2\|\beta + w\|_{\mathbf{N}}\big(\|\beta\|_{\mathbf{N}} + \sqrt{F}\big)\sqrt{F}.$$

To estimate $\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla q(t,\cdot)\|_2^2$, we begin with

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla q(t,\cdot)\|_2^2 = 2\int_\Omega \langle \nabla q_t, \nabla q\rangle \,\mathrm{d}x = -2\int_\Omega q\Delta q_t \mathrm{d}x + 2\int_\Gamma q\,\partial_\nu q_t \mathrm{d}S.$$

Using the boundary condition for q_t and integrating by parts again, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla q(t,\cdot)\|_2^2 &= 2\int_\Omega q\nabla\cdot\big((\beta_t+w_t)\cdot\nabla(\beta-v)+(\beta+w)\cdot\nabla(\beta_t-v_t)\big)\,\mathrm{d}x\\ &\quad +2\int_\Gamma q\big(\pi(\beta_t+w_t,\beta-v)+\pi(\beta+w,\beta_t-v_t)\big)\mathrm{d}S\\ &= -2\int_\Omega\langle\nabla q,(\beta_t+w_t)\cdot\nabla(\beta-v)+(\beta+w)\cdot\nabla(\beta_t-v_t)\rangle\mathrm{d}S.\end{aligned}$$

Hence, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla q\|_2^2\le C\|\beta+w\|_{\mathbf{N}}\big(\|\beta\|_{\mathbf{N}}+\sqrt{F}\big)\sqrt{F}. \tag{6.25}$$

Finally, we establish a differential inequality for $\|\psi\|_2^2$. Recall that the vorticity ω evolves according to the parabolic system (6.17). Taking the curl operation $\nabla\times$ both sides of system (6.17), we obtain the evolution system for ψ:

$$\partial_t\psi=\Delta\psi-\nabla\times\nabla\times\left((\beta+w)\cdot\nabla(v+w)\right) \tag{6.26}$$

so that

$$\partial_t(|\psi|^2)=2\langle\Delta\psi,\psi\rangle-2\langle\nabla\times\nabla\times\left((\beta+w)\cdot\nabla(v+w)\right),\psi\rangle.$$

Integrating over Ω and performing integration by parts, one then obtains

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi(t,\cdot)\|_2^2 &= 2\int_\Omega\langle\Delta\psi,\psi\rangle\,\mathrm{d}x-2\int_\Omega\langle\nabla\times\nabla\times\big((\beta+w)\cdot\nabla(v+w)\big),\psi\rangle\,\mathrm{d}x\\ &= -2\int_\Omega|\nabla\times\psi|^2\mathrm{d}x-2\int_\Omega\langle\nabla\times\big((\beta+w)\cdot\nabla(v+w)\big),\nabla\times\psi\rangle\,\mathrm{d}x\\ &\quad +2\int_\Gamma\langle\psi\times(\nabla\times\psi),\nu\rangle\mathrm{d}S-2\int_\Gamma\langle\nabla\times\big((\beta+w)\cdot\nabla(v+w)\big)\times\psi,\nu\rangle\mathrm{d}S,\end{aligned} \tag{6.27}$$

where we have used the fact that $\nabla\cdot\psi=0$. Now we have to handle the last two boundary integrals. The vector identity:

$$\nabla\times\psi=\nabla\times(\nabla\times\omega)=-\Delta\omega$$

yields

$$\langle\psi\times(\nabla\times\psi),\nu\rangle=\langle\Delta\omega\times\psi,\nu\rangle=\langle(\Delta\omega)^{\|}\times\psi,\nu\rangle.$$

However, since $\omega^{\|}\big|_\Gamma=0$, it follows from the vorticity system (6.17) that

$$(\Delta\omega)^{\|}\big|_\Gamma=\big(\nabla\times\big((\beta+w)\cdot\nabla(v+w)\big)\big)^{\|}. \tag{6.28}$$

Therefore, the two boundary integrals sum up to zero. Hence, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_2^2=-2\int_\Omega|\nabla\times\psi|^2\mathrm{d}x-2\int_\Omega\langle\nabla\times\big((\beta+w)\cdot\nabla(v+w)\big),\nabla\times\psi\rangle\,\mathrm{d}x. \tag{6.29}$$

To estimate the second integral, we consider $\alpha=\nabla\times(X\cdot\nabla Y)$, where X and Y are two vector fields. Since

$$(X\cdot\nabla Y)^k=X^i\nabla_iY^k,$$

then

$$\begin{aligned}
\alpha^j &= \frac{1}{2}\varepsilon_{jkl}\left(\nabla_k(X^a\nabla_aY^l)-\nabla_l(X^a\nabla_aY^k)\right)\\
&=\varepsilon_{jkl}(\nabla_kX^a)(\nabla_aY^l)+\frac{1}{2}\varepsilon_{jkl}X^a\left(\nabla_k\nabla_aY^l-\nabla_l\nabla_aY^k\right)\\
&=\varepsilon_{jkl}(\nabla_kX^a)(\nabla_aY^l)+X^a\nabla_a\left(\nabla\times Y\right)^j,
\end{aligned}$$

where ε_{jkl} are the Kronecker symbols. That is,

$$\nabla\times(X\cdot\nabla Y)=\varepsilon_{jkl}(\nabla_kX^a)(\nabla_aY^l)+X\cdot\nabla\left(\nabla\times Y\right),$$

so that

$$|\nabla\times(X\cdot\nabla Y)|\le|\nabla X||\nabla Y|+|X||\nabla\left(\nabla\times Y\right)|.$$

Using this inequality in (6.29), one deduces that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\psi(t,\cdot)\|_2^2 &\le -2\int_\Omega|\nabla\times\psi|^2\,\mathrm{d}x+2\int_\Omega|\nabla\times\psi|\,|\nabla(\beta+w)|\,|\nabla(v+w)|\,\mathrm{d}x\\
&\quad+2\int_\Omega|\nabla\times\psi|\,|\beta+w|\,|\nabla(\omega+\nabla\times w)|\,\mathrm{d}x\\
&\le -2\|\nabla\times\psi\|_2^2+2\|\nabla\times\psi\|_2\|\nabla(\beta+w)\|_4\|\nabla(v+w)\|_4\\
&\quad+2\|\nabla\times\psi\|_2\|\beta+w\|_\infty\|\nabla\omega+\nabla(\nabla\times w)\|_2\\
&\le -2\|\nabla\times\psi\|_2^2+C\|\nabla\times\psi\|_2\|\beta+w\|_{H^2}\|v+w\|_{H^2},
\end{aligned}$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_2^2\le C\|\beta+w\|_{\mathbf{N}}^2\left(F+\|w\|_{H^2}^2\right). \tag{6.30}$$

Combining all the estimates yields (6.10). We thus complete the proof of Theorem 6.1.

6.2 Local strong solutions

For $T>0$, denote $\mathbf{W}_T$ the space of all vector fields $\beta(t,x)$ that satisfy conditions (i)–(iii) in Theorem 6.1 with the uniform norm:

$$\|\beta\|_{\mathbf{W}_T}=\sup_{0\le t\le T}\sqrt{\|\beta(t,\cdot)\|_{H^2}^2+\|\partial_t\beta(t,\cdot)\|_{H^1}^2}.$$

Then $\mathbf{W}_T$ is a Banach space. According to (6.1), there is a constant $C>0$ depending only on Ω such that

$$\|V(\beta)\|_{\mathbf{W}_T}\le C\sqrt{C_0}\,e^{C(1+\|(\beta,w)\|_{\mathbf{W}_T}^2)T}+C\sqrt{T}\,e^{C(1+\|(\beta,w)\|_{\mathbf{W}_T}^2)T}\left(1+\|(\beta,w)\|_{\mathbf{W}_T}^2\right). \tag{6.31}$$

Equipped with the apriori estimate (6.1), we are now in a position to establish the existence of a local (in time) strong solution of the initial-boundary value problem with nonhomogeneous boundary conditions.

Theorem 6.2 Given $C_0>1$, there exists $T>0$ depending only on C_0, the viscosity coefficient $\mu>0$, and the domain Ω such that, for any given $u_0\in H^2(\Omega)$ which satisfies the boundary conditions (1.6)–(1.7), $\|u_0\|_{H^2}\le C_0$, and $\nabla\cdot u_0=0$, there exists a strong solution $u(t,x)$ of (1.1)–(1.3) and (1.6)–(1.7) with the form $u(t,x)=v(t,x)+w(t,x), 0\le t\le T$, such that

(i) For every $0 \le t \le T$, $v(t,\cdot) \in H^2(\Omega)$ and $\partial_t v(t,\cdot) \in H^1(\Omega)$;

(ii) For every $0 \le t \le T$, $v^{\perp}\big|_{\Gamma} = 0$ and $(\nabla \times v)^{\|}\big|_{\Gamma} = 0$;

(iii) $t \to \|v(t,\cdot)\|_{H^2}^2 + \|\partial_t v(t,\cdot)\|_{H^1}^2$ is continuous;

(iv) $v|_{t=0} = 0$, and v satisfies the Navier-Stokes equations

$$\partial_t v + (v+w)\cdot\nabla(v+w) = \Delta v - \nabla p, \qquad \nabla\cdot v = 0,$$

where p solves the Poisson equation for each $t \in (0,T]$:

$$\Delta p = -\nabla\cdot\left((v+w)\cdot\nabla(v+w)\right), \qquad \partial_\nu p|_{\Gamma} = \pi(v+w, v+w),$$

and w solves the initial-boundary value problem (4.5) for the unsteady Stokes equations.

Proof Step 1. If $\beta_1(t,x)$ and $\beta_2(t,x)$, $0 \le t \le T$, are two vector fields which satisfy conditions (i)-(iv) in Theorem 6.1, then $U := v_1 - v_2$ with $v_k = V(\beta_k), k = 1, 2$, satisfies the linear parabolic equations:

$$\partial_t U = \Delta U - (\beta_1 + w)\cdot\nabla U - \nabla P - (\beta_1 - \beta_2)\cdot\nabla(v_2 + w), \qquad U|_{t=0} = 0,$$

subject to $U^{\perp}\big|_{\Gamma} = 0$ and $(\nabla \times U)^{\|}\big|_{\Gamma} = 0$, where $P = p_{\beta_1} - p_{\beta_2}$ is the unique solution to

$$\Delta P = -\nabla\cdot\big((\beta_1 + w)\cdot\nabla(\beta_1 + w) - (\beta_2 + w)\cdot\nabla(\beta_2 + w)\big)$$

such that

$$\partial_\nu P\big|_{\Gamma} = \pi\big(\beta_1 + w, \beta_1 + w\big) - \pi\big(\beta_2 + w, \beta_2 + w\big).$$

Applying the same arguments used in the proof of Theorem 6.1, we obtain that there exists $C > 0$ depending only on the domain Ω such that the following inequality holds:

$$\begin{aligned} &\|V(\beta_1) - V(\beta_2)\|_{\mathbf{W}_T} \\ &\le C\sqrt{T}e^{C(1+\|(\beta_1,\beta_2,w)\|^2_{\mathbf{W}_T})T}\big(1 + \|(V(\beta_1), \beta_1, \beta_2, w)\|_{\mathbf{W}_T}\big)\|\beta_1 - \beta_2\|_{\mathbf{W}_T}. \end{aligned} \tag{6.32}$$

It follows from (6.31)–(6.32) that, for every $K > \sqrt{CC_0}$, there exists $T > 0$ depending only K, C, and $\|w\|_{\mathbf{N}}$ such that, if $v \in \mathbf{W}_T$ with $\|v\|_{\mathbf{W}_T} \le K$, then $\|V(v)\|_{\mathbf{W}_T} \le K$ and

$$\|V(v_1) - V(v_2)\|_{\mathbf{W}_T} \le \frac{2}{3}\|v_1 - v_2\|_{\mathbf{W}_T}$$

for any $v_1, v_2 \in \mathbf{W}_T$ such that $v_1(0,x) = v_2(0,x) = 0$, $\|v_i\|_{\mathbf{W}_T} \le K$, $i = 1, 2$. Therefore, there is a unique fixed point $v \in \mathbf{W}_T$ such that $V(v) = v$. Then $u = v + w$, according to Theorem 6.1, a solution of the nonhomogeneous initial-boundary value problem (1.1)–(1.3) and (1.6)–(1.7) for the Navier-Stokes equations.

Step 2. Incompressibility. If $v(t,\cdot), 0 \le t \le T$, is a fixed point of the velocity map V, then v is a solution to

$$\nabla\cdot v = 0. \tag{6.33}$$

Setting $\beta = v$ in (5.1)–(5.2) and taking the divergence to obtain

$$\partial_t(\nabla\cdot v) + \nabla\cdot\big((v+w)\cdot\nabla(v+w)\big) = \Delta(\nabla\cdot v) - \Delta p_v$$

so that $d = \nabla \cdot v$ solves the heat equation:

$$\partial_t d = \Delta d, \qquad d(0, \cdot) = 0.$$

According to our assumptions:

$$\partial_t v + (v + w) \cdot \nabla (v + w) = \Delta v - \nabla p_v. \tag{6.34}$$

Identifying the normal part of each term of equations (6.34), together with the boundary condition: $v^\perp\big|_\Gamma = w^\perp\big|_\Gamma = 0$, we conclude

$$(\Delta v)^\perp = (\nabla p_v)^\perp + ((v + w) \cdot \nabla (v + w))^\perp = 0.$$

On the other hand, by Lemma 2.2, together with the boundary condition: $\omega^\parallel\big|_\Gamma = 0$, we have

$$\partial_\nu d\big|_\Gamma = \langle \Delta v, \nu \rangle = 0.$$

By the uniqueness of the Neumann problem, $d(t, \cdot) = 0$ for all t.

7 Inviscid Limit in $\Omega \subset \mathbb{R}^n, n \geq 3$

In this section we study the inviscid limit of the solutions to the nonhomogeneous initial-boundary value problem (1.1)–(1.3) and (1.6)–(1.7) for the Navier-Stokes equations.

Let u_μ be the solution to the initial-boundary value problems (1.1)–(1.3) and (1.6)–(1.7) for the Navier-Stokes equations for each $\mu > 0$. Let u be the solution to the initial-boundary value problem (1.4)–(1.6) for the Euler equations. Notice that all solutions u_μ subject to the same boundary conditions: $u_\mu^\perp\big|_\Gamma = 0$ and $(\nabla \times u_\mu)^\parallel\big|_\Gamma = \nabla^\Gamma \times a$ for the given smooth vector field a, while the solution u satisfies only the kinematic condition $u^\perp\big|_\Gamma = 0$ which is independent of the viscosity constant μ.

Theorem 7.1 Let $a \in L^2([0, T]; L^2(\Gamma))$ be a smooth vector field on Γ. Suppose that, for all $\mu \in (0, \mu_0]$, a unique strong solution u_μ of problem (1.1)–(1.3) and (1.6)–(1.7) and a unique strong solution $u \in H^2(\Omega)$ of problem (1.4)–(1.6) exist up to time $T^* > 0$. Then there exists $C > 0$ depending on $\mu_0, T, \|a\|_{L^2([0,T];L^2(\Gamma))}$, and $\|u\|_{H^2 \cap W^{1,\infty}(\Omega)}$, independent of μ, such that, for any $T \in [0, T^*]$,

$$\sup_{0 \leq t \leq T} \|u_\mu(t, \cdot) - u(t, \cdot)\|_2 \leq C(T)\mu \to 0 \quad \text{as } \mu \downarrow 0, \tag{7.1}$$

and

$$\int_0^T ||\nabla (u_\mu - u)(s, \cdot)||_2^2 \mathrm{d}s \leq C.$$

It follows that the solutions u_μ of problem (1.1)–(1.3) and (1.6)–(1.7) for the Navier-Stokes equations converge to the unique solution $u(t, x)$ of problem (1.4)–(1.6) in $L^\infty([0, T]; L^2(\Omega))$.

Proof Let $v_\mu = u_\mu - u$. Then v_μ satisfies the following equations:

$$\begin{cases} \partial_t v_\mu = \mu \Delta v_\mu - (v_\mu + u) \cdot \nabla v_\mu - \nabla P_\mu - v_\mu \cdot \nabla u + \mu \Delta u, \\ \nabla \cdot v_\mu = 0, \end{cases} \tag{7.2}$$

and the initial condition:

$$v_\mu(0,\cdot)=0, \tag{7.3}$$

where $P_\mu = p_\mu - p$.

Since both u_μ and u satisfy the kinematic condition (1.6), so does v_μ. Thus, by means of the energy method, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|v_\mu\|_2^2 = 2\mu\int_\Omega v_\mu\cdot\Delta v_\mu \mathrm{d}x - \int_\Omega (v_\mu+u)\cdot\nabla\big(|v_\mu|^2\big)\,\mathrm{d}x - 2\int_\Omega\langle\nabla P_\mu, v_\mu\rangle\,\mathrm{d}x \\ -2\int_\Omega\langle v_\mu\cdot\nabla u, v_\mu\rangle\,\mathrm{d}x + 2\mu\int_\Omega\langle\Delta u, v_\mu\rangle\,\mathrm{d}x.\end{aligned}$$

Integrating by parts in the first three integrals, one then deduces that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|v_\mu\|_2^2 &= -2\mu\int_\Omega|\nabla\times v_\mu|^2\mathrm{d}x + 2\mu\int_\Gamma\langle v_\mu\times(\nabla\times v_\mu)\,,\nu\rangle\mathrm{d}S \\ &\quad -2\int_\Omega\langle v_\mu\cdot\nabla u, v_\mu\rangle\,\mathrm{d}x + 2\mu\int_\Omega\langle\Delta u, v_\mu\rangle\,\mathrm{d}x \\ &= -2\mu\|\nabla v_\mu\|_2^2 - 2\mu\int_\Gamma\pi(v_\mu,v_\mu)\mathrm{d}S + 2\mu\int_\Gamma\langle v_\mu\times b,\nu\rangle\mathrm{d}S \\ &\quad -2\int_\Omega\langle v_\mu\cdot\nabla u, v_\mu\rangle\,\mathrm{d}x + 2\mu\int_\Omega\langle\Delta u, v_\mu\rangle\,\mathrm{d}x,\end{aligned}$$

where $b = a - \nabla\times u$. Furthermore, we use the following estimate:

$$\int_\Gamma\langle v_\mu\times b,\nu\rangle\mathrm{d}S \le \|b\|_{L^2(\Gamma)}\|v_\mu\|_{L^2(\Gamma)} \le C\big(\|v_\mu\|^2_{L^2(\Gamma)} + \|a-\nabla\times u\|^2_{L^2(\Gamma)}\big)$$

to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|v_\mu\|_2^2 \le -2\mu\|\nabla v_\mu\|_2^2 + 2\mu C\big(\|v_\mu\|^2_{L^2(\Gamma)} + \|a-\nabla\times u\|^2_{L^2(\Gamma)}\big) \\ +2\|\nabla u\|_\infty\|v_\mu\|_2^2 + 2\mu\|\Delta u\|_2\|v_\mu\|_2\ .\end{aligned} \tag{7.4}$$

Finally, we use the Sobolev trace theorem:

$$2C\|v_\mu\|^2_{L^2(\Gamma)} \le \|\nabla v_\mu\|_2^2 + \tilde{C}\|v_\mu\|_2^2, \qquad \|\nabla\times u\|^2_{L^2(\Gamma)} \le \|u\|^2_{H^2(\Omega)}$$

to establish the differential inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v_\mu\|_2^2 + \mu\|\nabla v_\mu\|_2^2 \le C\big((\|\nabla u\|_\infty + \mu_0)\|v_\mu\|_2^2 + \mu(\|a\|^2_{L^2(\Gamma)} + \|u\|^2_{H^2(\Omega)})\big). \tag{7.5}$$

The Gronwall inequality yields that, for $t\in[0,T^*]$,

$$\|v_\mu(t,\cdot)\|_2^2 \le \mu C\int_0^t e^{C\int_s^t(\|\nabla u(\tau,\cdot)\|_\infty + \mu(t-s))}\big(\|b\|^2_{L^2(\Gamma)} + \|u(s,\cdot)\|^2_{H^2(\Omega)}\big)\mathrm{d}s := C\mu. \tag{7.6}$$

Hence

$$\int_0^t\|\nabla v_\mu(s,\cdot)\|_2^2\mathrm{d}s \le C,$$

which implies the conclusions of the theorem.

In order to ensure the convergence of u_μ to u in the strong sense (say, in $H^2(\Omega)$) up to the boundary, a necessary condition is that u must match with the boundary data $(\nabla\times u)^\perp\big|_\Gamma = a$.

Acknowledgments This paper was written as part of the international research program on Nonlinear Partial Differential Equations at the Centre for Advanced Study at the Norwegian Academy of Science and Letters in Oslo during the academic year 2008–09.

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