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Structure of Shape Derivatives for Nonsmooth Domains*

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The object of this paper is to study the Shape gradient and the Shape Hessian by the Velocity (Speed) Method for arbitrary domains with or without constraints. It makes the connection between methods using a family of transformations such as first or second order Perturbations of the Identity Operator. New definitions for Shape derivatives are given. They naturally extend existing theories for C^k or Lipchitzian domains to arbitrary domains without any smoothness conditions on their geometric boundary. In this new framework extensions of the classical structure theorems are given for the Shape gradient and the Shape Hessian.

1. INTRODUCTION

Shape Analysis and *Optimization* deals with problems where the *control* or *optimization* variable is no longer a vector of parameters or functions but the shape of a geometric domain Ω contained in a fixed *hold-all* D of the Euclidean N-dimensional space $\mathbb{R}^N$. Here the space of all subsets $\mathscr{P}(D)$ of D is no longer a vector space and the traditional definitions of directional

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derivatives have to be adapted. To do that the domain Ω can be perturbed by a family of velocity fields $\{V(t): 0 \leqslant t \leqslant \tau\}$ on $\bar{D}$ or by introducing a family of transformations $\{T_t: 0 \leqslant t \leqslant \tau\}$ of $\bar{D}$. The first approach is known as the *Velocity (Speed) Method* (cf. J. Céa [7] and J. P. Zolésio [17]). The second approach is the classical viewpoint in Mechanics where it is convenient to work in both fixed and actual coordinates through an appropriate change of variable. A special case of this is known as the Method of *Perturbation of the Identity.*

In this paper we introduce a framework where the velocity and the transformation approaches are equivalent. We use velocities to define and study the *Shape gradient* and the *Shape Hessian* for nonsmooth constrained and unconstrained domains. We extend *Hadamard structure theorem* from C^k domains to nonsmooth domains. We also discuss their relationship to various methods based on perturbations of the identity operator.

In Section 2 we extend the Velocity (Speed) Method to nonsmooth domains Ω which are constrained to lie within a fixed domain D. This is done by a double application of the *Viability Theory* and the introduction of *Bouligand* contingent and *Clarke* tangent cones. We obtain natural extension of *Hadamard's structure theorem* for both the Shape gradient and the Shape Hessian (cf. Delfour and Zolésio [9(b)–9(e)] for a description of the smooth case) and recover known result in the smooth case. The *canonical structures* of the gradient and the Hessian are given for non-autonomous velocity fields. We show that Methods of Perturbation of the Identity Operator (first and second order) are special cases corresponding to non-autonomous velocity fields and indicate how to construct the associated velocity.

For the Shape gradient, the different methods yield expressions which may look different but are all equal. However, this is no longer true for the Shape Hessian. In fact we shall show in Section 4 that different perturbations of the identity yield final expressions which are not equal. It turns out that we can introduce an infinity of definitions based on perturbations of the identity. However, we shall show that they always contain a *canonical bilinear term* plus the Shape gradient of the functional acting in the direction of an *acceleration* field which is characteristic of the chosen perturbation. The canonical bilinear term exactly coincides with the second order Shape derivative obtained by the Velocity (Speed) Method for autonomous velocity fields. Moreover each expression obtained by a method of perturbation of the identity can be strictly recovered by adding to the canonical term the Shape gradient acting in the direction of an appropriate acceleration field. In view of this we propose to refer to this canonical term as the *Shape Hessian.*

A few papers have dealt with the second variation of a domain functional for partial differential equation models. To our knowledge the first one by

N. Fujii [10(a)] used a second order perturbation of the identity along the normal to the boundary for second order linear elliptic problems. An extremely interesting paper by Arumugam and Pironneau [2] used the Shape second variation to solve the *ribblet problem*. Finally J. Simon [16] presented a computation of the second variation using a first order perturbation of the identity. The first general approach to the computation of Shape Hessians can be found in Delfour and Zolésio [9(b)–9(e)]. It uses the Velocity (Speed) Method and includes simple illustrative examples for the Neumann and Dirichlet problems.

In conclusion, we would like to emphasize the importance of the equivalence between the velocity and the transformation viewpoints. It provides an essential link between many results which have been developped in parallel and direct constructions to compare them. In this paper we have used non-autonomous velocity fields but everything can be readily translated in the language of transformations.

2. Velocity (Speed) and Transformation Methods

In this section we review and extend the Velocity Method (cf. J. P. Zolésio [17]) and prove its equivalence with transformation methods. Under appropriate conditions we show how to construct a family of non-autonomous transformations of $\mathbb{R}^N$ (or the closure of a subset D of $\mathbb{R}^N$) from a family of non-autonomous velocity fields. Conversely we show how to construct the family of non-autonomous velocity fields from a family of non-autonomous transformations of $\mathbb{R}^N$ (or the closure of a subset D of $\mathbb{R}^N$). This construction is applied to various methods based on perturbations of the identity. In Subsections 2.1 and 2.2 we recall results from Delfour and Zolésio [9(e)] for unconstrained domains. In Subsections 2.3 and 2.4 we present new results for constrained domains which will be basic in this paper.

2.1. *Unconstrained Families of Domains*

Let the real number $\tau > 0$ and the map $V: [0, \tau] \times \mathbb{R}^N \to \mathbb{R}^N$ be given. Denote by V a family $\{V(t): 0 \leqslant t \leqslant \tau\}$ of non-autonomous velocity fields on $\mathbb{R}^N$ defined by

$$x \mapsto V(t)(x) \stackrel{\text{def}}{=} V(t, x): \mathbb{R}^N \mapsto \mathbb{R}^N. \tag{1}$$

Assume that

$$(V) \begin{cases} \forall x \in \mathbb{R}^N, \ V(\cdot, x) \in C^0([0, \tau]; \mathbb{R}^N) \\ \exists c > 0, \ \forall x, \ y \in \mathbb{R}^N, \ \|V(\cdot, y) - V(\cdot, x)\|_{C^0([0, \tau]; \mathbb{R}^N)} \leqslant c\, |y - x|, \end{cases}$$

where $V(\cdot, x)$ is the function $t \mapsto V(t, x)$. Associate with V the solution $x(t; V)$ of the ordinary differential equation

$$\frac{dx}{dt}(t) = V(t, x(t)), \qquad t \in [0, \tau],\ x(0) = X \in \mathbb{R}^N, \tag{2}$$

and introduce the homeomorphism

$$X \mapsto T_t(V)(X) \overset{\text{def}}{=} x(t; V): \mathbb{R}^N \to \mathbb{R}^N, \tag{3}$$

and the maps

$$(t, X) \mapsto T_V(t, X) \overset{\text{def}}{=} T_t(V)(X) : [0, \tau] \times \mathbb{R}^N \to \mathbb{R}^N, \tag{4}$$

$$(t, x) \mapsto T_V^{-1}(t, x) \overset{\text{def}}{=} T_t^{-1}(V)(x) : [0, \tau] \times \mathbb{R}^N \to \mathbb{R}^N. \tag{5}$$

Notation 2.1. In the sequel we shall drop the V in $T_V(t, X)$, $T_V^{-1}(t, x)$, and $T_t(V)$ whenever no confusion is possible.

THEOREM 2.1. (i) *Under hypothesis* (V) *the map* T *has the following properties*

$$(T1) \begin{cases} \forall X \in \mathbb{R}^N,\ T(\cdot, X) \in C^1([0, \tau]; \mathbb{R}^N) \\ \exists c > 0,\ \forall X,\ Y \in \mathbb{R}^N,\ \|T(\cdot, Y) - T(\cdot, X)\|_{C^1([0,\tau];\mathbb{R}^N)} \leqslant c\,|Y - X|, \end{cases}$$

$(T2)$ $\forall t \in [0, \tau]$, $X \mapsto T_t(X) = T(t, X): \mathbb{R}^N \to \mathbb{R}^N$ *is bijective,*

$$(T3) \begin{cases} \forall x \in \mathbb{R}^N,\ T^{-1}(\cdot, x) \in C^0([0, \tau]; \mathbb{R}^N) \\ \exists c > 0,\ \forall x,\ y \in \mathbb{R}^N,\ \|T^{-1}(\cdot, y) - T^{-1}(\cdot, x)\|_{C^0([0,\tau];\mathbb{R}^N)} \leqslant c\,|y - x|. \end{cases}$$

(ii) *Given a real* $\tau > 0$ *and a map* $T: [0, \tau] \times \mathbb{R}^N \to \mathbb{R}^N$ *verifying hypotheses* $(T1)$, $(T2)$, *and* $(T3)$, *then the map*

$$(t, x) \mapsto V(t, x) \overset{\text{def}}{=} \frac{\partial T}{\partial t}(t, T_t^{-1}(x)) : [0, \tau] \times \mathbb{R}^N \to \mathbb{R}^N, \tag{6}$$

verifies hypothesis (V), *where* T_t^{-1} *is the inverse of* $X \mapsto T_t(X) = T(t, X)$.

This first theorem is an equivalence result which says that we can either start from a family of velocity fields $\{V(t)\}$ on $\mathbb{R}^N$ or a family of transformations $\{T_t\}$ of $\mathbb{R}^N$ provided that the map V, $V(t, x) = V(t)(x)$, verifies (V) or the map T, $T(t, X) = T_t(X)$, verifies $(T1)$ to $(T3)$.

When we start from V, we obtain the *Velocity Method* and the perturbations of an initial domain Ω by the family of homeomorphisms $\{T_t(V)\}$ generates a new family of transformed domains

$$\Omega_t = T_t(V)(\Omega) = \{T_t(V)(X) : X \in \Omega\} \tag{7}$$

which will be used in Sections 3 and 4 to define shape derivatives. Note that interior (resp. boundary) points of Ω are mapped onto interior (resp. boundary) points of Ω_t.

2.2. *Perturbation of the Identity Operator*

In examples where we start from T, it is usually possible to verify hypotheses $(T1)$ to $(T3)$ and construct the corresponding velocity field V defined in (6). For instance perturbations of the identity to the first or second order fall in that category:

$$T_t(X) = X + tU(X) + \frac{t^2}{2} A(X)$$

$$(A = 0 \text{ for the first order}),\ t \geqslant 0,\ X \in \mathbb{R}^N, \tag{8}$$

where U and A are transformations of $\mathbb{R}^N$. It turns out that for Lipschitzian transformations U and A, hypotheses $(T1)$ to $(T3)$ are verified.

THEOREM 2.2. *Let U and A be two uniform Lipschitzian transformations of $\mathbb{R}^N$:*

$$\exists c > 0,\ \forall X,\ Y \in \mathbb{R}^N,\ |U(Y) - U(X)| \leqslant c\,|Y - X|,\ |A(Y) - A(X)| \leqslant c\,|Y - X|.$$

For $\tau = \min\{1, 1/4c\}$ and T given by (8), *the map T verifies hypotheses $(T1)$ to $(T3)$ on $[0, \tau]$. Moreover the associated velocity V is given by*

$$(t, x) \mapsto V(t, x) = U(T_t^{-1}(x)) + tA(T_t^{-1}(x)) : [0, \tau] \times \mathbb{R}^N \to \mathbb{R}^N, \tag{9}$$

and it verifies hypothesis (V) on $[0, \tau]$.

Remark 2.1. Observe that from (8) and (9)

$$V(0) = U, \qquad \dot{V}(0)(x) \overset{\text{def}}{=} \frac{\partial V}{\partial t}(t, x)|_{t=0} = A - [DU]\,U, \tag{10}$$

where DU is the Jacobian matrix of U. The term $\dot{V}(0)$ is an *acceleration* at $t = 0$ which will always be present even when $A = 0$.

2.3. *Constrained Families of Domains*

In many applications the family of admissible domains Ω is constrained to subsets of a fixed larger domain or *hold-all D*. To reflect that constraint we consider transformations

$$T : [0, \tau] \times \bar{D} \to \mathbb{R}^N \tag{11}$$

with the following properties

$$(T1_D)\ \begin{cases}\forall X \in \bar{D},\ T(\cdot, X) \in C^1([0,\tau]; \mathbb{R}^N)\\ \exists c>0,\ \forall X,\ Y \in \bar{D},\ \|T(\cdot, Y) - T(\cdot, X)\|_{C^1([0,\tau];\mathbb{R}^N)} \leqslant c\,|Y-X|,\end{cases}$$

$$(T2_D)\ \forall t \in [0,\tau],\ X \mapsto T_t(X) \overset{\text{def}}{=} T(t,X) : \bar{D} \to \bar{D} \text{ is bijective,}$$

$$(T3_D)\ \begin{cases}\forall x \in \bar{D},\ T^{-1}(\cdot, x) \in C^0([0,\tau]; \mathbb{R}^N)\\ \exists c>0,\ \forall x,\ y \in \bar{D},\ \|T^{-1}(\cdot, y) - T^{-1}(\cdot, x)\|_{C^0([0,\tau];\mathbb{R}^N)} \leqslant c\,|y-x|,\end{cases}$$

where under hypothesis $(T2_D)$, T^{-1} is defined from the inverse of T_t as

$$(t,x) \mapsto T^{-1}(t,x) \overset{\text{def}}{=} T_t^{-1}(x) : [0,\tau] \times \bar{D} \to \mathbb{R}^N. \tag{12}$$

Those three properties are the analogue for $\bar{D}$ of the same three properties obtained for $\mathbb{R}^N$. In fact Theorem 2.1 extends from $\mathbb{R}^N$ to $\bar{D}$ by adding one hypothesis to (V). Specifically we shall consider for $\tau > 0$ velocities

$$V : [0,\tau] \times \bar{D} \to \mathbb{R}^N \tag{13}$$

such that

$$(V1_D)\ \begin{cases}\forall x \in \bar{D},\ V(\cdot, x) \in C^0([0,\tau]; \mathbb{R}^N)\\ \exists c>0,\ \forall x,\ y \in \bar{D},\ \|V(\cdot, y) - V(\cdot, x)\|_{C^0([0,\tau];\mathbb{R}^N)} \leqslant c\,|y-x|.\end{cases}$$

$$(V2_D)\ \forall x \in \bar{D}, \forall t \in [0,\tau],\ V(t,x) \in T_{\bar{D}}(x), \text{ and } -V(t,x) \in T_{\bar{D}}(x),$$

where $T_{\bar{D}}(x)$ is the Bouligand contingent cone to $\bar{D}$ at the point x in $\bar{D}$ (cf. Aubin and Cellina [3, p. 176]). Of course $(V2_D)$ need only be verified on the boundary ∂D of D.

The next theorem is a generalization of Theorem 2.1 from $\mathbb{R}^N$ to an arbitrary domain D which shows the equivalence between velocity and transformation viewpoints.

THEOREM 2.3. (i) *Let $\tau > 0$ and V be a family of velocity fields verifying hypotheses $(V1_D)$ and $(V2_D)$ and consider the family of transformations*

$$(t,X) \mapsto T(t,X) \overset{\text{def}}{=} x(t;X) : [0,\tau] \times \bar{D} \to \mathbb{R}^N, \tag{14}$$

where $x(\cdot, X)$ is the solution of

$$\frac{dx}{dt}(t) = V(t, x(t)), \qquad 0 \leqslant t \leqslant \tau, \qquad x(0) = X. \tag{15}$$

Then the family of transformations T verifies conditions $(T1_D)$ to $(T3_D)$.

(ii) *Conversely given a family of transformations T verifying hypotheses* $(T1_D)$ *to* $(T3_D)$, *the family of velocity fields*

$$(t, x) \mapsto V(t, x) \stackrel{\text{def}}{=} \frac{\partial T}{\partial t}(t, T_t^{-1}(x)) : [0, \tau] \times \bar{D} \to \mathbb{R}^N \tag{16}$$

verifies conditions $(V1_D)$ *and* $(V2_D)$ *and the transformations constructed from this V coincide with T.*

Remark 2.2. Under $(V1_D)$ and $(V2_D)$, $\{T_t : 0 \leqslant t \leqslant \tau\}$ is a family of homeomorphisms of $\bar{D}$ which map the interior $\mathring{D}$ (resp. the boundary ∂D) of D onto $\mathring{D}$ (resp. ∂D) (cf. J. Dugunji [18, pp. 87–88]).

Remark 2.3. Assumption $(V2_D)$ is a double viability condition. M. Nagumo's [13] usual viability condition

$$V(t, x) \in T_{\bar{D}}(x), \qquad \forall t \in [0, \tau], \forall x \in \bar{D} \tag{17}$$

is a necessary and sufficient condition for a *viable solution* to (15), that is,

$$\forall t \in [0, \tau], \forall X \in \bar{D}, x(t; X) \in \bar{D} \text{ (or } T_t(\bar{D}) \subset \bar{D}) \tag{18}$$

(cf. Aubin and Cellina [3, p. 174 and p. 180]). Condition $(V2_D)$

$$\forall t \in [0, \tau], \forall x \in \bar{D}, V(t, x) \in T_{\bar{D}}(x), \text{ and } -V(t, x) \in T_{\bar{D}}(x) \tag{19}$$

is a *strict viability condition* which not only says that T_t maps $\bar{D}$ into $\bar{D}$ but also that

$$\forall t \in [0, \tau], \qquad T_t : \bar{D} \to \bar{D} \text{ is a homeomorphism.} \tag{20}$$

In particular it maps interior points and boundary points onto boundary points.

Remark 2.4. Condition $(V2_D)$ is a generalization to arbitrary domains D of the following condition used by J. P. Zolésio [17(a)] in 1979: for all x in ∂D

$$\begin{aligned} &V(t, x) \cdot n(x) = 0, && \text{if the outward normal } n(x) \text{ exists} \\ &V(t, x) = 0, && \text{otherwise.} \end{aligned}$$

Proof of Theorem 2.3. (i) *Existence and uniqueness of viable solutions to* (15). We apply M. Nagumo's [13] theorem to the augmented system on $[0, \tau]$

$$\begin{aligned} &\frac{dx}{dt}(t) = V(t, x(t)), && x(0) = X \in \bar{D} \\ &\frac{dx_0}{dt}(t) = 1, && x_0(0) = 0, \end{aligned} \tag{21}$$

that is,

$$\frac{d\hat{x}}{dt}(t) = \hat{V}(\hat{x}(t)),$$
$$\hat{x}(0) = (0, X) \in \hat{D} \stackrel{\text{def}}{=} \mathbb{R}^+ \times \bar{D}, \tag{22}$$

where $\hat{x}(t) = (x_0(t), x(t)) \in \mathbb{R}^{N+1}$, $\hat{V}(\hat{x}) = (1, \tilde{V}(\hat{x}))$, and

$$\tilde{V}(x_0, x) = \begin{Bmatrix} V(x_0, x), & 0 \leqslant x_0 \leqslant \tau \\ V(\tau, x), & \tau < x_0 \end{Bmatrix}, \quad x \in \bar{D}. \tag{23}$$

It is easy to verify that systems (21) and (22) are equivalent on $[0, \tau]$ and that $\hat{x}(t) = (t, x(t))$. The new velocity field on $\hat{V} \subset \mathbb{R}^{N+1}$ is continuous at each point $\hat{x} \in \hat{D}$ by the first hypothesis $(V1_D)$ since

$$\hat{V}(\hat{y}) - \hat{V}(\hat{x}) = (0, \tilde{V}(y_0, y) - \tilde{V}(x_0, x))$$

and for $0 \leqslant x_0, y_0 \leqslant \tau$

$$\begin{aligned} |V(y_0, y) - V(x_0, x)| &\leqslant |V(y_0, y) - V(y_0, x)| + |V(y_0, x) - V(x_0, x)| \\ &\leqslant n\,|y - x| + |V(y_0, x) - V(x_0, x)|. \end{aligned}$$

In addition

$$T_{\hat{D}}(\hat{x}) = T_{\mathbb{R}^+}(x_0) \times T_{\bar{D}}(x)$$

and

$$\hat{V}(\hat{x}) = (1, V(\hat{x})) \in T_{\mathbb{R}^+}(x_0) \times T_{\bar{D}}(x).$$

Moreover $\hat{V}(\hat{D})$ is bounded and $\mathbb{R}^{N+1}$ is finite dimensional. So by using the version of Nagumo's theorem given in Aubin and Cellina [3, Theorem 3, Part (b), pp. 182–183] there exists a viable solution $\hat{x}$ to (22) for all $t \geqslant 0$. In particular

$$\forall t \in [0, \tau], \qquad \hat{x}(t) \in \hat{D} = \mathbb{R}^+ \times \bar{D}$$

which is necessarily of the form

$$\hat{x}(t) = (t, x(t)).$$

Hence there exists a viable solution $x, x(t) \in \bar{D}$ on $[0, \tau]$, to (15). The uniqueness now follows from the Lipschitz condition $(V1_D)$. The Lipschitzian continuity $(T1_D)$ can be established by a standard argument.

Condition $(T2_D)$. Associate with X in $\bar{D}$ the function

$$y(s) = T_{t-s}(X), \qquad 0 \leqslant s \leqslant t. \tag{24}$$

Then

$$\frac{dy}{ds}(s) = -V(t-s, y(s)), \qquad 0 \leqslant s \leqslant t,\ y(0) = T_t(X). \tag{25}$$

For each $x \in \bar{D}$, the differential equation

$$\frac{dy}{ds}(s) = -V(t-s, y(s)), \qquad 0 \leqslant s \leqslant t,\ y(0) = x \in \bar{D} \tag{26}$$

has a unique viable solution in $C^1([0, t]; \mathbb{R}^N)$:

$$\forall s \in [0, t], \qquad y(s) \in \bar{D} \tag{27}$$

since by hypothesis $(V2_D)$

$$\forall t \in [0, \tau], \forall x \in \bar{D}, \qquad -V(t, x) \in T_{\bar{D}}(x).$$

The proof is the same as above. The solutions of (26) define a Lipschitzian mapping

$$x \mapsto S_t(x) = y(t): \bar{D} \to \bar{D}$$

such that

$$\exists c > 0, \forall t \in [0, \tau], \forall x, y \in \bar{D}, \qquad |S_t(y) - S_t(x)| \leqslant c\,|y - x|. \tag{28}$$

Now in view of (25) and (26)

$$S_t(T_t(X)) = y(t) = T_{t-t}(X) = X \quad \Rightarrow \quad S_t \circ T_t = I \text{ on } \bar{D}.$$

To obtain the other identity, consider the function

$$z(r) = y(t - r; x),$$

where $y(\cdot, x)$ is the solution of Eq. (26). By definition

$$\frac{dz}{dr}(r) = V(r, z(r)), \qquad z(0) = y(t, x)$$

and necessarily

$$x = y(0; x) = z(t) = T_t(y(t; x)) = T_t(S_t(x))$$
$$\Rightarrow T_t \circ S_t = I \text{ on } \bar{D} \Rightarrow S_t = T_t^{-1}: \bar{D} \to \bar{D}.$$

Condition $(T3_D)$. The uniform Lipschitz continuity in $(T3_D)$ follows from (28) and we only need to show that

$$\forall x \in \bar{D}, \qquad T^{-1}(\cdot, x) \in C^0([0, \tau]; \mathbb{R}^N).$$

Given t in $[0, \tau]$ pick an arbitrary sequence $\{t_n\}$, $t_n \to t$. Then for each $x \in \bar{D}$ there exists $X \in \bar{D}$ such that

$$T_t(X) = x \qquad \text{and} \qquad T_{t_n}(X) \to T_t(X) = x.$$

But

$$\begin{aligned} T_{t_n}^{-1}(x) - T_t^{-1}(x) &= T_{t_n}^{-1}(T_t(X)) - T_t^{-1}(T_t(X)) \\ &= T_{t_n}^{-}(T_t(X)) - T_{t_n}^{-1}(T_{t_n}(X)). \end{aligned}$$

By the uniform Lipschitz continuity of T_t^{-1}

$$|T_{t_n}^{-1}(x) - T_t^{-1}(x)| = |T_{t_n}^{-1}(T_t(X)) - T_{t_n}^{-1}(T_{t_n}(X))| \leqslant c\, |T_t(X) - T_{t_n}(X)|$$

and the last term converges to zero as t_n goes to t.

(ii) The first condition $(V1_D)$ is verified since for each $x \in \bar{D}$ and t, s in $[0, \tau]$

$$\begin{aligned} |V(t, x) - V(s, x)| &\leqslant \left| \frac{\partial T}{\partial t}(t, T_t^{-1}(x)) - \frac{\partial T}{\partial t}(t, T_s^{-1}(x)) \right| \\ &\quad + \left| \frac{\partial T}{\partial t}(t, T_s^{-1}(x)) - \frac{\partial T}{\partial t}(s, T_s^{-1}(x)) \right| \\ &\leqslant c\, |T_t^{-1}(x) - T_s^{-1}(x)| \\ &\quad + \left| \frac{\partial T}{\partial t}(t, T_s^{-1}(x)) - \frac{\partial T}{\partial t}(s, T_s^{-1}(x)) \right|. \end{aligned}$$

The second condition $(V1_D)$ follows from $(T1_D)$ and $(T3_D)$ and the following inequality: for all x and y in $\bar{D}$

$$\begin{aligned} |V(t, y) - V(t, x)| &= \left| \frac{\partial T}{\partial t}(t, T_t^{-1}(y)) - \frac{\partial T}{\partial t}(t, T_t^{-1}(x)) \right| \\ &\leqslant c\, |T_t^{-1}(y) - T_t^{-1}(x)| \leqslant cc'\, |y - x|. \end{aligned}$$

To check condition $(V2_D)$, we go back to the definition of Bouligand contingent cone

$$T_{\bar{D}}(X) = \bigcap_{\varepsilon > 0} \bigcap_{\alpha > 0} \bigcup_{0 < h < \alpha} \left[\frac{1}{h}(\bar{D} - X) + \varepsilon B \right],$$

where B is the unit disk in $\mathbb{R}^N$. We first show that

$$V(t, x) = \frac{\partial T}{\partial t}(t, T_t^{-1}(x)) \in T_{\bar{D}}(x), \qquad \forall x \in \bar{D}.$$

By $(T2_D)$, T_t is bijective. So it is equivalent to show that

$$\frac{\partial T}{\partial t}(t, X) \in T_{\bar{D}}(T_t(X)), \qquad \forall X \in \bar{D}.$$

For simplicity we use the notation

$$x(t) = T_t(X) = T(t, X) \qquad \text{and} \qquad x'(t) = \frac{\partial T}{\partial t}(t, X). \tag{29}$$

By definition of $T_{\bar{D}}(x(t))$ we must prove that

$$\forall \varepsilon > 0, \forall \alpha > 0, \exists u \in x'(t) + \varepsilon B, \exists h \in]0, \alpha[, \text{ such that } x(t) + hu \in \bar{D}.$$

Choose δ, $0 < \delta < \alpha$, such that

$$\forall s, \qquad |s - t| < \delta \Rightarrow |x'(s) - s'(t)| < \varepsilon.$$

Then fix t', $0 < t' - t < \delta$,

$$x(t') - x(t) = \int_t^{t'} [x'(s) - x'(t)] \, ds + (t' - t) \, x'(t)$$

and

$$\left| \frac{x(t') - x(t)}{t' - t} - x'(t) \right| < \varepsilon.$$

Therefore choose $u = [x(t') - x(t)]/(t' - t)$. Now

$$x(t) + hu = x(t) + \frac{h}{t' - t} [x(t') - x(t)]$$

and choose $h = t' - t$ since $0 < t' - t < \delta < \alpha$:

$$T(t, X) + (t' - t)u = T(t, X) + [T(t', X) - T(t, X)] = T(t', X) \in \bar{D}$$

by hypothesis on T. This proves (29). The second part of $(V2_D)$ is

$$-V(t, x) = -\frac{\partial T}{\partial t}(t, T_t^{-1}(x)) \in T_{\bar{D}}(x), \qquad \forall x \in \bar{D}$$

which is equivalent to proving that

$$-\frac{\partial T}{\partial t}(t, X) \in T_{\bar{D}}(T_t(X)), \qquad \forall X \in \bar{D}$$

or with the simplified notation

$$-x'(t) \in T_{\bar{D}}(x(t)). \tag{30}$$

We proceed exactly as in the proof of (29) except that we choose t' such that $0 < t - t' < \delta$, $h = t - t'$, and $u = -[x(t) - x(t')]/(t - t')$. Then

$$|u + x'(t)| < \varepsilon$$

and

$$x(t) + hu = x(t) + (t - t')\left[-\left(\frac{x(t) - x(t')}{t - t'}\right)\right] = x(t') \in \bar{D}$$

and necessarily we get (30). This completes the proof of the theorem. ∎

2.4. *Transformation of Condition* $(V2_D)$ *into a Linear Constraint*

Condition $(V2_D)$ is equivalent to

$$\forall t \in [0, \tau], \forall x \in \bar{D} \text{ (resp } \partial D), \qquad V(t, x) \in \{-T_D(x)\} \cap \{T_D(x)\} \tag{31}$$

since $T_{\bar{D}}(x) = T_D(x)$. If $T_D(x)$ was convex, then the above intersection would be a closed linear subspace of $\mathbb{R}^N$. This is true when D is convex. In that case $T_D(x) = C_D(x) = \overline{co}T_D(x)$, where $C_D(x)$ is the (closed convex) Clarke tangent cone to $\bar{D}$ at x which is defined by

$$C_D(x) = \{v \in \mathbb{R}^N : \lim_{\substack{h \to 0 \\ y \to_{\bar{D}} x}} d_D(y + hv)/h = 0\},$$

$d_D(y)$ is the minimum distance from y to D, and $\to_{\bar{D}}$ denotes the convergence in $\bar{D}$. The sets

$$L_D(x) \stackrel{\text{def}}{=} \{-C_D(x)\} \cap \{C_D(x)\}$$

and (32)

$$L'_D(x) \stackrel{\text{def}}{=} \{-\overline{co}T_D(x)\} \cap \{\overline{co}T_D(x)\}$$

are closed linear subspaces of $\mathbb{R}^N$. This means that for convex domains D condition $(V2_D)$ is equivalent to

$$\forall t \in [0, \tau], \forall x \in \bar{D} \text{ (resp } \partial D), \qquad V(t, x) \in L_D(x) \tag{33}$$

or

$$\forall t \in [0, \tau], \forall x \in \bar{D} \text{ (resp } \partial D), \qquad V(t, x) \in L'_D(x). \tag{33'}$$

It turns out that for continuous vector fields $V(t, \cdot)$ the equivalence of $(V2_D)$, (33), and (33′) extends to arbitrary domains D.

THEOREM 2.4. *Given a velocity field V verifying $(V1_D)$, then condition $(V2_D)$ is equivalent to*

$$(V2_C) \ \forall t \in [0, \tau], \forall x \in \bar{D} \ (\textit{resp } \partial D),$$
$$V(t, x) \in L_D(x) = \{-C_D(x)\} \cap C_D(x),$$

or (34)

$$(V2_{\overline{co}T}) \ \forall t \in [0, \tau], \forall x \in \bar{D} \ (\textit{resp } \partial D),$$
$$V(t, x) \in L'_D(x) = \{-\overline{co}T_D(x)\} \cap \overline{co}T_D(x).$$

Moreover $L_D(x)$ and $L'_D(x)$ are closed linear subspaces of $\mathbb{R}^N$.

Proof. (i) The equivalence of $(V2_D)$, $(V2_C)$, and $(V2_{\overline{co}T})$ is a direct consequence of the following lemma which will be proved later.

LEMMA 2.1. *Given a vector field $W \in C^0(\bar{D}; \mathbb{R}^N)$, the following three conditions are equivalent:*

$$\forall x \in \bar{D} \ (\textit{resp } \partial D), \qquad W(x) \in C_D(x); \tag{35}$$

$$\forall x \in \bar{D} \ (\textit{resp } \partial D), \qquad W(x) \in T_D(x); \tag{36}$$

$$\forall x \in \bar{D} \ (\textit{resp } \partial D), \qquad W(x) \in \overline{co}T_D(x). \tag{37}$$

(ii) The set $L_D(x)$ is closed as the intersection of two closed sets. To show that it is linear we show that $\forall \alpha \in \mathbb{R}$, $\forall V \in L_D(x)$, $\alpha V \in L_D(x)$, and $\forall V, W \in L_D(x)$, $V + W \in L_D(x)$. Since $\pm C_D(x)$ are cones

$$\forall \alpha \in \mathbb{R}, \forall V \in L_D(x), \qquad \pm |\alpha| \, V \in C_D(x) \Rightarrow \pm \alpha V \in C_D(x) \Rightarrow \alpha V \in L_D(x).$$

By convexity of $\pm C_D(x)$

$$\forall V, W \in L_D(x), \qquad \pm (V + W) \in C_D(x) \Rightarrow V + W \in L_D(x).$$

The proof is analogous for $L'_D(x)$. This completes the proof of the theorem. ∎

Proof of Lemma 2.1. By definition $C_D(x) \subset T_D(x) \subset \overline{co} T_D$ and (35) $\Rightarrow$ (36) $\Rightarrow$ (37). Conversely we know that

$$\liminf_{y \to_{\bar{D}} x} \overline{co} T_D(y) = \liminf_{y \to_{\bar{D}} x} T_D(y) = C_D(x)$$

(cf. for instance Aubin and Frankowska [4(a), Theorem 4.1.10, Sect. 4.15, p. 130]). Since W is continuous in $\bar{D}$ and (35) is verified, then for each $x \in \bar{D}$

$$W(x) = \lim_{y \to_{\bar{D}} x} W(y) \in C_D(x)$$

and (36) $\Rightarrow$ (35) and (37) $\Rightarrow$ (35). ∎

Remark 2.5. Lemma 2.1 essentially says that for continuous vector fields we can relax the condition of M. Nagumo's [13] theorem from $(V2_D)$ involving the Bouligand contingent cone to $(V2_C)$ involving the smaller Clarke convex tangent cone or the bigger convex cone $\overline{co} T_D(x)$. In dimension $N = 3$, they are $\{0\}$, a line, a plane, or the whole space.

Notation 2.2. In the sequel it will be convenient to introduce the following spaces and subspaces

$$\mathscr{L} = \{V : [0, \tau] \times \mathbb{R}^N \to \mathbb{R}^N \text{ such that } V \text{ verifies } (V) \text{ on } \mathbb{R}^N\} \tag{38}$$

and for an arbitrary domain D in $\mathbb{R}^N$

$$\mathscr{L}_D = \{V : [0, \tau] \times \bar{D} \to \mathbb{R}^N \text{ such that } V \text{ verifies } (V1_D) \text{ and } (V2_C) \text{ on } \partial D\}. \tag{39}$$

For any integers $k \geqslant 0$ and $m \geqslant 0$ and any compact subset K of $\mathbb{R}^N$ define the following subspaces of $\mathscr{L}$

$$\mathscr{V}_K^{m,k} = C^m([0, \tau], \mathscr{D}^k(K, \mathbb{R}^N)) \cap \mathscr{L}, \tag{40}$$

where $\mathscr{D}^k(K, \mathbb{R}^N)$ is the space of k-times continuously differentiable transformations of $\mathbb{R}^N$ with compact support in K. In all cases $\mathscr{V}_K^{m,k} \subset \mathscr{L}_K$. As usual $\mathscr{D}^\infty(K, \mathbb{R}^N)$ will be written $\mathscr{D}(K, \mathbb{R}^N)$.

Remark 2.6. The possible use of $\overline{co} T_D(x)$ instead of $C_D(x)$ has been pointed out to us by the referee. This follows from a specialization of a recent result by Aubin and Frankowska [4(b)] for convex compact valued upper semicontinuous set-valued functions to single-valued continuous functions. For completeness we have extended Lemma 2.1 to take this possibility into account. It is true that, in general,

$$C_D(x) \subset T_D(x) \subset \overline{co} T_D(x) \tag{41}$$

and that the three cones do not necessarily coincide. Since both $C_D(x)$ and $\overline{co}T_D(x)$ will generate linear spaces, the question of the choice between a small and a large candidate naturally arises. However, the linear space

$$\mathscr{L}'_D = \{V : [0, \tau] \times \bar{D} \to \mathbb{R}^N \text{ such that } V \text{ verifies } (V1_D) \text{ and } (V2_{\overline{co}T}) \text{ on } \partial D\} \tag{42}$$

constructed from $L'_D(x)$ coincides with the linear space $\mathscr{L}_D$ constructed from $L_D(x)$. Since all subsequent developments in this paper are based on the set $\mathscr{L}_D$ and not on the choice of any one of the tangent cones in (41), this question is purely a matter of taste at least for continuous vector fields. This may no longer be true for discontinuous fields and one tangent cone may be more fundamental than the others.

3. Shape Gradient

Consider the set $\mathscr{P}(D)$ of subsets Ω of a fixed domain D of $\mathbb{R}^N$ (possibly all of $\mathbb{R}^N$) which will play the role of a *hold-all*. Under the action of a velocity field V in $\mathscr{L}_D$, a domain Ω in an admissible family $\mathscr{A}$ of $\mathscr{P}(D)$ is transformed into a new domain

$$\Omega_t(V) = T_t(V)(\Omega) = \{T_t(V)(X) : X \in \Omega\}. \tag{1}$$

This provides our first notion of derivative for a domain functional, that is, a map

$$\Omega \mapsto J(\Omega) : \mathscr{A} \subset \mathscr{P}(D) \to \mathbb{R}. \tag{2}$$

Definition 3.1. Given a velocity field V in $\mathscr{L}_D$, J is said to have an *Eulerian semiderivative* at Ω in the direction V if the following limit exists and is finite

$$\lim_{t \searrow 0} [J(\Omega_t(V)) - J(\Omega)]/t. \tag{3}$$

Whenever it exists, the limit will be denoted $dJ(\Omega; V)$.

This definition is quite general and may include situations where $dJ(\Omega; V)$ is not only a function of $V(0)$ but also of $V(t)$ in a neighbourhood of $t = 0$. This will not occur under some appropriate continuity hypothesis on the map $V \mapsto dJ(\Omega; V)$. This immediately raises the question of the choice of topology and eventually the choice of gradient when we specialize to autonomous vector fields V. We choose to follow the classical framework of the Theory of Distributions (cf. L. Schwartz [15(a)]. Assume that D is an open domain in $\mathbb{R}^N$. Domains Ω in $\mathscr{P}(D)$ will be perturbed by velocity fields $V(t)$ with values in $\mathscr{D}^k(K, \mathbb{R}^N)$ for some compact subset K of D and integer $k \geqslant 0$. More precisely we shall consider velocity fields in

$$\vec{\mathscr{V}}_D^{m,k} = \varinjlim_K \{ V_K^{m,k} : \forall K \text{ compact in } D \}, \tag{4}$$

where $\varinjlim$ denotes the inductive limit set with respect to K endowed with its natural inductive limit topology. For autonomous fields, the above construction reduces to

$$\mathscr{V}_D^k = \left\{ \begin{array}{ll} \mathscr{D}^0(D, \mathbb{R}^N) \cap \operatorname{Lip}(\mathbb{R}^N, \mathbb{R}^N), & k=0 \\ \mathscr{D}^k(D, \mathbb{R}^N), & 1 \leqslant k \leqslant \infty \end{array} \right\}, \tag{5}$$

where $\operatorname{Lip}(\mathbb{R}^N, \mathbb{R}^N)$ denotes the space of uniformly Lipschitzian transformations of $\mathbb{R}^N$. In all cases hypotheses $(V1_D)$ and $(V2_D)$ are verified since for all $t \in [0, \tau]$, $V(t, x) = 0$ for all x in ∂D. When $D = \mathbb{R}^N$ we drop the index D in the above definitions and simply write $\vec{\mathscr{V}}^{m,k}$ and $\mathscr{V}^k$. In the sequel $\mathscr{D}^\infty(D, \mathbb{R}^N)$ will be written $\mathscr{D}(D, \mathbb{R}^N)$.

THEOREM 3.1. *Let Ω be a domain in the fixed open hold-all D. Assume that there exist integers $m \geqslant 0$ and $k \geqslant 0$ such that*

$$\forall V \in \vec{\mathscr{V}}_D^{m,k}, \qquad dJ(\Omega; V) \text{ exists}, \tag{6}$$

and that the map

$$V \mapsto dJ(\Omega; V) : \vec{\mathscr{V}}_D^{m,k} \to \mathbb{R} \tag{7}$$

is continuous. Then

$$\forall V \in \vec{\mathscr{V}}_D^{m,k}, \qquad dJ(\Omega; V) = dJ(\Omega; V(0)), \tag{8}$$

where $dJ(\Omega; V(0))$ is the Eulerian semiderivative for the autonomous vector field equal to $V(0)$.

Proof. It is sufficient to prove the theorem for any compact subset K of D. So given V in $\mathscr{V}_K^{m,k}$ construct the sequence

$$V_n(t) = V(t/n), \qquad 0 \leqslant t \leqslant \tau, \text{ for integers } n \geqslant 1.$$

By continuity of V, $\{V_n\}$ converges in $\mathscr{V}_K^{m,k}$ to the autonomous field $\tilde{V}(t) = V(0)$. Hence by continuity of (7)

$$dJ(\Omega; V_n) \to dJ(\Omega; V(0))$$

and by uniqueness of the limit we obtain (8). ∎

By virtue of this theorem we can now specialize to autonomous vector fields V to further study the properties and the structure of $dJ(\Omega; V)$.

DEFINITION 3.2. Let Ω be a domain in the open hold-all D of $\mathbb{R}^N$.

(i) The functional J is said to be *shape differentiable* at Ω, if the Eulerian semiderivative $dJ(\Omega; V)$ exists for all V in $\mathscr{D}(D, \mathbb{R}^N)$ and the map

$$V \mapsto dJ(\Omega; V) : \mathscr{D}(D, \mathbb{R}^N) \to \mathbb{R} \tag{9}$$

is linear and continuous.

(ii) The map (9) defines a vector distribution $G(\Omega)$ which will be referred to as the shape gradient of J at Ω.

(iii) When there exists some finite $k \geqslant 0$ such that $G(\Omega)$ is continuous for the $\mathscr{D}^k(D, \mathbb{R}^N)$-topology, we say that the shape gradient $G(\Omega)$ is of order k.

The next theorem gives additional properties of shape differentiable functionals.

Notation 3.1. Associate with a subset A of D and an integer $k \geqslant 0$ the set

$$L_A^k = \{ V \in \mathscr{D}^k(D, \mathbb{R}^N) : \forall x \in \partial A,\ V(x) \in L_A(x) \}. \tag{10}$$

THEOREM 3.2 (Generalized Hadamard's Structure Theorem). *Let Ω be a closed or open domain in $\mathbb{R}^N$ with boundary Γ in the open hold-all D of $\mathbb{R}^N$ and assume that J has a shape gradient $G(\Omega)$.*

(i) *The support of the shape gradient $G(\Omega)$ is contained in $\Gamma_D \stackrel{\text{def}}{=} \Gamma \cap D$.*

(ii) *If the shape gradient is of order k for some k, $0 \leqslant k \leqslant \infty$, then there exists $[G(\Omega)]$ in $(\mathscr{D}_D^k / L_\Omega^k)'$ such that for all V in $\mathscr{D}_D^k \stackrel{\text{def}}{=} \mathscr{D}^k(D, \mathbb{R}^N)$*

$$dJ(\Omega; V) = \langle [G(\Omega)], q_\Omega V \rangle_{\mathscr{D}_D^k / L_\Omega^k}, \tag{11}$$

where $q_\Omega : \mathscr{D}_D^k \to \mathscr{D}_D^k / L_\Omega^k$ is the canonical quotient surjection. Moreover

$$G(\Omega) = q_\Omega^* [G(\Omega)], \tag{12}$$

where q_Ω^* denotes the transpose of the linear map q_Ω.

Proof. (i) For any V in $\mathscr{D}_D$ such that $V = 0$ on $\Gamma \cap D$, we have $V = 0$ on Γ. Hence V verifies hypotheses $(V1_\Omega)$ and $(V2_\Omega)$ (with Ω in place of D) and $V \in L_\Omega^\infty$. Then by Theorem 2.2, $T_s : \bar{\Omega} \to \bar{\Omega}$ is a homeomorphism and

$(\bar{\Omega})_s = T_s(\bar{\Omega}) = \bar{\Omega}$ and $(\mathring{\Omega})_s = T_s(\mathring{\Omega}) = \mathring{\Omega}$. Thus when Ω is closed $(\Omega = \bar{\Omega})$ or open $(\Omega = \mathring{\Omega})$,

$$\forall s \geqslant 0, \qquad T_s(\Omega) = \Omega \Rightarrow J(\Omega_s) = J(\Omega) \Rightarrow dJ(\Omega; V) = 0.$$

(ii) It is sufficient to prove that $dJ(\Omega; V) = 0$ for all V in L^k_Ω. The other statements follow by standard arguments and the fact that L^k_Ω is a closed subspace of $\mathscr{D}^k_D$. From part (i) we know that the result is true for all V in L^∞_Ω and hence by a density argument for all V in L^k_Ω. ∎

Remark 3.1. When the boundary Γ of Ω is compact and J is shape differentiable at Ω, the distribution $G(\Omega)$ is of finite order. Once this is known, the conclusions of Theorem 3.2(ii) apply with k equal to the order of $G(\Omega)$.

For smooth domains quotienting $\mathscr{D}^k_D$ by L^k_Ω intuitively means that we are dealing with vector fields in $\mathscr{D}^k_D$ which are normal to the boundary Γ_D. This notion can be made more precise and another interpretation of the dual of the quotient space $\mathscr{D}^k_D/L^k_\Omega$ can be given.

DEFINITION 3.3. Let Ω be a domain with boundary Γ in the open hold-all D of $\mathbb{R}^N$ and let k, $0 \leqslant k \leqslant \infty$, be an integer. We say that a distribution T in $(\mathscr{D}^k_D)'$ is *normal* to the boundary $\Gamma_D = \Gamma \cap D$ if

$$\forall V \in L^k_\Omega, \qquad \langle T, V \rangle_{\mathscr{D}^k_D} = 0.$$

The space of all distributions T in $(\mathscr{D}^k_D)'$ which are normal to the boundary Γ_D will be denoted $\mathscr{N}^k_D(\Gamma)$.

LEMMA 3.1. *Under the hypotheses of Definition* 3.2, *$\mathscr{N}^k_D(\Gamma)$ is a closed linear subspace of $(\mathscr{D}^k_D)'$ and the map*

$$q^*_\Omega : (\mathscr{D}^k_D/L^k_\Omega)' \to \mathscr{N}^k_D(\Gamma)$$

is an isomorphism.

Proof. By definition $\mathscr{N}^k_D(\Gamma)$ is a closed linear subspace of $(\mathscr{D}^k_D)'$. By construction the map q_Ω in Theorem 3.2(ii) is surjective and its transpose

$$q^*_\Omega : (\mathscr{D}^k_D/L^k_\Omega)' \to (\mathscr{D}^k_D)'$$

is linear continuous and injective. Its image in $(\mathscr{D}^k_D)'$ coincides with $\mathscr{N}^k_D(\Gamma)$: for all $[T]$ in $(\mathscr{D}^k_D/L^k_\Omega)'$ and V in L^k_Ω

$$\langle q^*_\Omega[T], V \rangle = \langle [T], q_\Omega V \rangle = 0.$$

Conversely given any $N \in \mathcal{N}_D^k(\Gamma_D)$, $\ker N \supset L_\Omega^k$ and the map

$$\langle [N], q_\Omega(V) \rangle = \langle N, V \rangle$$

is well-defined linear and continuous. Hence

$$\operatorname{Im} q_\Omega^* = \mathcal{N}_D^k(\Gamma). \quad \blacksquare$$

Theorem 3.2 can now be restated as follows.

COROLLARY 1. *Under the hypotheses of Theorem* 3.2, *for all integers* k, $0 \leqslant k \leqslant \infty$, $G(\Omega)$ *is a distribution in* $\mathcal{N}_D^k(\Gamma)$ (*that is, which is normal to* Γ_D) *with support in* Γ_D.

When the boundary Γ is defined by local maps, it is possible to introduce the corresponding spaces of traces on $\Gamma \cap D$ and define the surjective trace operator

$$\begin{gathered} \gamma_\Gamma : \mathscr{D}^k(D, \mathbb{R}^N) \to \mathscr{D}^k(\Gamma \cap D, \mathbb{R}^N) \\ (\text{resp. } \gamma_\Gamma : \mathscr{D}^k(D) \to \mathscr{D}^k(\Gamma \cap D)). \end{gathered} \tag{13}$$

Then a bijection p_L can be constructed such that the following diagram commutes

$$\begin{array}{ccc} \mathscr{D}^k(D, \mathbb{R}^N) & \xrightarrow{\gamma_\Gamma} & \mathscr{D}^k(\Gamma \cap D, \mathbb{R}^N) \\ {\scriptstyle q_\Omega}\downarrow & & \downarrow{\scriptstyle \nu} \\ \mathscr{D}_D^k / L_D^k & \xrightarrow{p_L} & \mathscr{D}^k(\Gamma \cap D) \end{array}$$

where

$$V \mapsto \nu(V) = V \cdot n : \mathscr{D}^k(\Gamma \cap D, \mathbb{R}^N) \to \mathscr{D}^k(\Gamma \cap D)$$

is continuous linear and surjective and n is the outward unit normal to Γ. It is easy to verify that the map p_L defined by

$$q_\Omega(V) \mapsto p_L(q_\Omega(V)) \stackrel{\text{def}}{=} \gamma_\Gamma(V) \cdot n \tag{14}$$

is well-defined and injective since the kernel of the map $V \mapsto \gamma_\Gamma(V) \cdot n$: $\mathscr{D}^k(D, \mathbb{R}^N) \to \mathscr{D}^k(\Gamma \cap D, \mathbb{R}^N)$ coincides with L_Ω^k. The surjectivity is a consequence of the fact that for a C^{k+1} boundary, $k \geqslant 0$, it is always possible to construct a C^k-extension N on D or even $\mathbb{R}^N$ of the unit normal n on Γ (cf. Agmon, Douglis, and Nirenberg [1]). Then for any v in $\mathscr{D}^k(\Gamma \cap D)$, there exists an extension $\tilde{v}$ in $\mathscr{D}^k(D)$ and the vector $V = \tilde{v}N$ belongs to $\mathscr{D}^k(D, \mathbb{R}^N)$ and coincides with vn on $\Gamma \cap D$.

The dual of the above diagram will also be commutative

$$\begin{array}{ccc} \mathscr{D}^k(\Gamma\cap D)' & \xrightarrow{p_L^*} & (\mathscr{D}_D^k/L_D^k)' \\ {\scriptstyle v^*}\big\downarrow & & \big\downarrow{\scriptstyle q_\Omega^*} \\ \mathscr{D}^k(\Gamma\cap D, \mathbb{R}^N)' & \xrightarrow{\gamma_L^*} & \mathscr{D}^k(D, \mathbb{R}^N)' \end{array}$$

and necessarily there exists $g(\Gamma)\in\mathscr{D}^k(\Gamma\cap D)'$ such that

$$G(\Omega)=q_\Omega^*[G(\Omega)]=q_\Omega^*\,p_L^*g(\Gamma)=\gamma_\Gamma^*v^*g(\Gamma).$$

Here the transpose of γ_Γ can be regarded as an extension operator by 0 outside of $\Gamma\cap D$, while the transpose of v associates with each element g in $\mathscr{D}^k(\Gamma\cap D)'$ the vector distribution v^*g which is normal to the boundary $\Gamma\cap D$. Following J. P. Zolésio [17(a), Theorem 3.1, pp. 53–54] it is legitimate to write this vector distribution gn (n, the outward normal to Γ).

COROLLARY 2. *Assume that the hypotheses of Theorem* 3.2 *are verified for an open domain* Ω, *that the order of* $G(\Omega)$ *is* $k\geqslant 0$ (*possibly infinite*), *and that the boundary* Γ *of* Ω *is* C^{k+1}. *Then for all* x *in* Γ, $L_\Omega(x)$ *is an* $(N-1)$-*dimensional hyperplane to* Ω *at* x *and there exists a unique outward unit normal* $n(x)$ *which belongs to* $C^k(\Gamma;\mathbb{R}^N)$. *as a result the spaces*

$$\mathscr{D}^k(\Gamma\cap D)'\xrightarrow{p_L^*}(\mathscr{D}_D^k/L_D^k)'\xrightarrow{q_\Omega^*}\mathscr{N}_D^k(\Gamma)$$

are isomorphic. In particular there exists a scalar distribution $g(\Gamma)$ *in* $\mathscr{D}^k(\Gamma\cap D)'$ *such that for all* V *in* $\mathscr{D}^k(D,\mathbb{R}^N)$

$$dJ(\Omega;V)=\langle g(\Gamma),\gamma_\Gamma(V)\cdot n\rangle_{\mathscr{D}^k(\Gamma\cap D)}=\langle\gamma_\Gamma^*g(\Gamma),(V\cdot n)n\rangle_{\mathscr{D}_D^k}\tag{15}$$

and

$$G(\Omega)=\gamma_\Gamma^*v^*g(\Gamma)=q_\Omega^*\,p_L^*g(\Gamma)=q_\Omega^*[G(\Omega)],\qquad [G(\Omega)]=p_L^*g(\Gamma).\tag{16}$$

Remark 3.2. In 1907, J. Hadamard [12] used velocity fields along the normal to the boundary Γ of a C^∞ domain to compute the derivative of the first eigenvalue of the plate. The generalization to open domains with a C^{k+1} boundary was done by J. P. Zolésio [17(a), Theorem 3.1, pp. 53–54] in 1979. Theorem 3.2 and Corollary 1 are generalizations for arbitrary shape functionals of that property to open or closed domains with an arbitrary boundary.

Remark 3.3. The space $\mathscr{D}^k(\Gamma\cap D)$ is not simple to characterize. However, when Γ is compact and $D=\mathbb{R}^N$, it coincides with $C^k(\Gamma)$.

EXAMPLE 3.1. For any measurable subset Ω of a measurable hold-all D of $\mathbb{R}^N$, consider the volume functional

$$J(\Omega) = \int_\Omega dx. \tag{17}$$

For Ω with finite volume and V in $\mathscr{D}^1(D, \mathbb{R}^N)$,

$$dJ(\Omega; V) = \int_\Omega \operatorname{div} V\, dx \tag{18}$$

but for a bounded open domain Ω with a C^1 boundary Γ

$$dJ(\Omega; V) = \int_\Gamma V \cdot n\, d\Gamma \tag{19}$$

which is continuous on $\mathscr{D}^0(D, \mathbb{R}^N)$. Here the smoothness of the boundary decreases the order of the distribution $G(\Omega)$. This raises the question of the characterization of the family of all domains Ω of D for which the map

$$V \mapsto \int_\Omega \operatorname{div} V\, dx : \mathscr{D}^1(D, \mathbb{R}^N) \to \mathbb{R} \tag{20}$$

can be continuously extended to $\mathscr{D}^0(D, \mathbb{R}^N)$. This is the family of *finite perimeter sets* or R. Caccioppoli [6] sets with respect to D (cf. E. De Giorgi [8]). It contains domains Ω whose characteristic function belongs to $BV(D)$, the space of L^1 functions on D with a distributional gradient in the space of (vectorial) Radon measures. They are the sets with finite volume and perimeter.

4. SHAPE HESSIAN

We first study the second order Eulerian semiderivative $d^2J(\Omega; V; W)$ of a functional $J(\Omega)$ for two non-autonomous vector fields V and W. A first theorem shows that under some natural continuity hypotheses, $d^2J(\Omega; V; W)$ is the sum of two terms: the *canonical term* $d^2J(\Omega; V(0); W(0))$ plus the first order Eulerian semiderivative $dJ(\Omega; \dot{V}(0))$ at Ω in the direction $\dot{V}(0)$ of the time-partial derivative $\partial_t V(t, x)$ at $t = 0$. As in the study of first order Eulerian semiderivatives, this first theorem reduces the study of second order Eulerian semiderivatives to the autonomous case. So we shall specialize to fields V and W in $\mathscr{D}^k(D, \mathbb{R}^N)$ and give the equivalent of Hadamard's structure theorem for the canonical term.

4.1. *Non-autonomous Case*

The basic framework introduced in Sections 2 and 3 has reduced the computation of the Eulerian semiderivative of $J(\Omega)$ to the computation of the derivative

$$j(0) = dJ(\Omega; V(0)) \tag{1}$$

of the function

$$j(t) = J(\Omega_t(V)). \tag{2}$$

For $t \geqslant 0$, we naturally obtain

$$j'(t) = dJ(\Omega_t(V); V(t)). \tag{3}$$

This suggests the following definition.

DEFINITION 4.1. Let V and W belong to $\mathscr{L}_D$ and assume that for all $t \in [0, \tau]$, $dJ(\Omega_t(W); V(t))$ exists for $\Omega_t(W) = T_t(W)(\Omega)$. The functional J is said to have a second order Eulerian semiderivative at Ω in the directions (V, W) if the following limit exists

$$\lim_{t \searrow 0} [dJ(\Omega_t(W); V(t)) - dJ(\Omega; V(0))]/t. \tag{4}$$

When it exists, it is denoted $d^2J(\Omega; V; W)$.

Remark 4.1. This last definition is compatible with the second order expansion of $j(t)$ with respect to t around $t = 0$:

$$j(t) \cong j(0) + tj'(0) + \frac{t^2}{2} j''(0), \tag{5}$$

where

$$j''(0) = d^2J(\Omega; V; V). \tag{6}$$

Remark 4.2. It is easy to construct simple examples (see Example 4.2) with autonomous fields V and W showing that $d^2J(\Omega; V; W) \neq d^2J(\Omega; W; V)$ (cf. Delfour and Zolésio [9(b)].

The next theorem is the analogue of Theorem 3.1 and provides the canonical structure of the second order Eulerian semiderivative.

THEOREM 4.1. *Let Ω be a domain in the fixed open hold-all D of $\mathbb{R}^N$ and let $m \geqslant 0$ and $l \geqslant 0$ be integers. Assume that*

(i) $\forall V \in \vec{\mathscr{V}}_D^{m+1,l}$, $\forall W \in \vec{\mathscr{V}}_D^{m,l}$, *$d^2J(\Omega; V; W)$ exists,*

(ii) $\forall W \in \vec{\mathscr{V}}_D^{m,l}$, $\forall t \in [0, \tau]$, *J has a shape gradient at $\Omega_t(W)$ of order l,*

(iii) $\forall U \in \mathscr{V}_D^l$, *the map*

$$W \mapsto d^2J(\Omega; U; W): \vec{\mathscr{V}}_D^{m,l} \to \mathbb{R} \tag{7}$$

is continuous.

Then for all V in $\vec{\mathscr{V}}_D^{m+1,l}$ and all W in $\vec{\mathscr{V}}_D^{m,l}$

$$d^2J(\Omega; V; W) = d^2J(\Omega; V(0); W(0)) + dJ(\Omega; \dot{V}(0)), \tag{8}$$

where

$$\dot{V}(0)(x) = \lim_{t \searrow 0} [V(t, x) - V(0, x)]/t. \tag{9}$$

Proof. The differential quotient (4) can be split into the sum of two terms

$$\begin{aligned} &[dJ(\Omega_t(W); V(0)) - dJ(\Omega; V(0))]/t \\ &\quad + [dJ(\Omega_t(W); V(t)) - dJ(\Omega_t(W); V(0))]/t. \end{aligned} \tag{10}$$

In view of (i) and (iii), for all U in $\mathscr{V}_D^l$

$$d^2J(\Omega; U; W) = d^2J(\Omega; U; W(0))$$

by the same argument as in the proof of Theorem 3.1 for the gradient. Hence the first term converges to

$$d^2J(\Omega; V(0); W) = d^2J(\Omega; V(0); W(0)).$$

For the second term recall that V belongs to $\vec{\mathscr{V}}_D^{m+1,l}$ and observe that the vector field

$$\tilde{V}(t) = [V(t) - V(0)]/t$$

belongs to $\vec{\mathscr{V}}_D^{m,l}$ and that $\tilde{V}(0) = \dot{V}(0)$. Thus by linearity of $dJ(\Omega; V)$ the second term in (10) can be written as

$$dJ(\Omega_t(W); [V(t) - V(0)]/t) = dJ(\Omega_t(W); \tilde{V}(t)).$$

But for any V in $\vec{\mathscr{V}}_D^{m+2,l}$, $\tilde{V}$ belongs to $\vec{\mathscr{V}}_D^{m+1,l}$. Then by assumption (i)

$$\lim_{t \searrow 0} [dJ(\Omega_t(W); \tilde{V}(t)) - dJ(\Omega; \tilde{V}(0))]/t = d^2J(\Omega; \tilde{V}; W)$$

which implies that

$$\lim_{t \searrow 0} dJ(\Omega_t(W); \tilde{V}(t)) = dJ(\Omega; \tilde{V}(0)) = dJ(\Omega; \dot{V}(0)).$$

Now by hypothesis (ii), $U \mapsto dJ(\Omega; U)$ is linear and continuous on $\mathscr{D}^l(D, \mathbb{R}^N)$ and the map

$$V \mapsto \dot{V}(0) \mapsto dJ(\Omega; \dot{V}(0)) : \vec{\mathscr{V}}_D^{m+2,l} \to \mathscr{V}_D^l \to \mathbb{R}$$

is linear and continuous (hence uniformly continuous) for the topology $\vec{\mathscr{V}}_D^{m+1,l}$ for all V in the dense subspace $\vec{\mathscr{V}}_D^{m+2,l}$. Hence it uniquely and continuously extends to all elements of $\vec{\mathscr{V}}_D^{m+1,l}$. This completes the proof of the theorem. ∎

This important theorem gives the canonical structure of the second order Eulerian semiderivative: a first term which depends on $V(0)$ and $W(0)$ and a second term which is equal to $dJ(\Omega; \dot{V}(0))$. When V is autonomous the second term disappears and the semiderivative coincides with $d^2J(\Omega; V; W(0))$ which can be separately studied for autonomous vector fields in $\mathscr{V}_D^l$.

4.2. *Autonomous Case*

DEFINITION 4.2. Let Ω be a domain in the open hold-all D of $\mathbb{R}^N$.

(i) The functional $J(\Omega)$ is said to be *twice shape differentiable* at Ω if

$$\forall V, \forall W \text{ in } \mathscr{D}(D, \mathbb{R}^N), \qquad d^2J(\Omega; V; W) \text{ exists} \tag{11}$$

and the map

$$(V, W) \mapsto d^2J(\Omega; V; W) : \mathscr{D}(D, \mathbb{R}^N) \times \mathscr{D}(D, \mathbb{R}^N) \to \mathbb{R} \tag{12}$$

is bilinear and continuous. We denote by h the map (12).

(ii) Denote by $H(\Omega)$ the vector distribution in $(\mathscr{D}(D, \mathbb{R}^N) \otimes \mathscr{D}(D, \mathbb{R}^N))'$ associated with h,

$$d^2J(\Omega; V; W) = \langle H(\Omega), V \otimes W \rangle = h(V, W), \tag{13}$$

where $V \otimes W$ is the tensor product of V and W defined as

$$(V \otimes W)_{ij}(x, y) = V_i(x)\, W_j(y), \qquad 1 \leqslant i, j \leqslant N, \tag{14}$$

and $V_i(x)$ (resp. $W_j(y)$) is the ith (resp. jth) component of the vector V (resp. W) (cf. L. Schwartz's [15(b)] kernel theorem and Guelfand and Vilenkin [11]). $H(\Omega)$ will be called the *Shape Hessian* of J at Ω.

(iii) When there exists a finite integer $l \geqslant 0$ such that $H(\Omega)$ is continuous for the $\mathscr{D}^l(D, \mathbb{R}^N) \otimes \mathscr{D}^l(D, \mathbb{R}^N)$-topology, we say that $H(\Omega)$ is of order l.

THEOREM 4.2. *Let Ω be a domain with boundary Γ in the open hold-all D of $\mathbb{R}^N$ and assume that J is twice shape differentiable.*

(i) *The vector distribution $H(\Omega)$ has support in*

$$(\Gamma \cap D) \times (\Gamma \cap D).$$

(ii) *If Ω is an open or closed domain in D and $H(\Omega)$ is of order $l \geqslant 0$, then there exists a continuous bilinear form*

$$[h] : (\mathscr{D}^l_D / D^l_\Gamma) \times (\mathscr{D}^l_D / L^l_\Omega) \to \mathbb{R} \tag{15}$$

such that for all $[V]$ in $\mathscr{D}^l_D / D^l_\Gamma$ and $[W]$ in $\mathscr{D}^l_D / L^l_\Omega$

$$d^2 J(\Omega; V; W) = [h](q_D(V), q_\Omega(W)), \tag{16}$$

where $q_D : \mathscr{D}^l_D \to \mathscr{D}^l_D / D^l_\Gamma$ and $q_\Omega : \mathscr{D}^l_D \to \mathscr{D}^l_D / L^l_\Omega$ are the canonical quotient surjections and

$$D^l_\Gamma = \{ V \in \mathscr{D}^l(D, \mathbb{R}^N) : \partial^\alpha V = 0 \text{ on } \Gamma \cap D, \forall \alpha, |\alpha| \leqslant l \}. \tag{17}$$

Remark 4.3. Of course all the comments and constructions following Theorem 3.2 apply to the Shape Hessian.

Proof of Theorem 4.2. (i) It is sufficient to prove that

(a) $\forall V, W \in \mathscr{D}_D$ such that $W = 0$ in a neighbourhood of $\Gamma \cap D$, $d^2 J(\Omega; V; W) = 0$ and

(b) $\forall V, W \in \mathscr{D}_D$ such that $V = 0$ in a neighbourhood of $\Gamma \cap D$, $d^2 J(\Omega; V; W) = 0$.

In case (a) the proof is similar to the one in Theorem 3.2 for the gradient and we prove the stronger result that for W such that $W = 0$ on $\Gamma \cap D$

$$\Omega_t(W) = \Omega, \forall t \geqslant 0 \Rightarrow dJ(\Omega_t(W); V) = dJ(\Omega; V)$$
$$\Rightarrow d^2 J(\Omega; V; W) = 0.$$

In case (b), $V = 0$ in a neighbourhood N of $\Gamma \cap D$ and in $\mathbb{R}^N \backslash K$, the complement of the compact support K of V. But $\Omega \subset D$ and $\Gamma = (\Gamma \cap D) \cup (\Gamma \cap \partial D)$. So $U = (\mathbb{R}^N \backslash K)$ is a neighbourhood of Γ where $V = 0$. By construction $U \cap K = \varnothing$ and there exists a bounded neighbourhood $\mathscr{U}$ of K

such that $\bar{\mathscr{U}} \cap \Gamma = \varnothing$. Since $\bar{\mathscr{U}}$ is compct and Γ is closed, the minimum distance d from $\bar{\mathscr{U}}$ is finite and non-zero. Let

$$N(\Gamma) = \{ y \in \mathbb{R}^N : d_\Gamma(y) < d/2 \},$$

where

$$d_\Gamma(y) = \inf\{ |y - x| : x \in \Gamma \}.$$

For all X in Γ

$$T_t(X) - X = \int_0^t W(T_s(X))\, ds = tW(X) + \int_0^t [W(T_s(X)) - W(X)]\, ds$$

and by hypotheses $(V2_D)$ on W

$$|T_t(X) - X| \leqslant t\, |W(X)| + ct \max_{[0,t]} |T_s(X) - X|$$

and it can easily be shown that for $t < 1/c$

$$\max_{[0,t]} |T_s(X) - X| < \frac{t}{1 - ct} |W(X)|.$$

Thus

$$\sup_{X \in \Gamma} \max_{[0,t]} |T_s(X) - X| \leqslant \frac{t}{1 - ct} \sup_{X \in \Gamma} |W(X)|.$$

But W is continuous with compact support. Therefore

$$\sup_{X \in \Gamma} |W(X)| \leqslant \sup_{X \in \operatorname{supp} W} |W(X)| = \|W\|_{C^0(\bar{D}; \mathbb{R}^N)} < \infty$$

and there exist $\tau > 0$ such that

$$\forall s \in [0, \tau], \qquad \frac{s}{1 - cs} \|W\|_{C^0} < \frac{d}{2}.$$

By definition and the previous inequalities

$$d_\Gamma(T_s(X)) = \inf_{Y \in \Gamma} |T_s(X) - Y| \leqslant |T_s(X) - X| < \frac{d}{2}$$

for all s in $[0, \tau]$ and all $X \in \Gamma$. This implies that

$$\forall s \in [0, \tau], \forall X \in \Gamma, \qquad \Gamma_s(W) = T_s(W)(\Gamma) \subset N(\Gamma).$$

By construction $V=0$ in $N(\Gamma)$ since the distance from K to Γ is greater or equal to d. Therefore

$$\forall s \in [0, \tau], \qquad V \in L^{\infty}_{\Omega_s(W)}$$

and as in the proof of Theorem 3.2, $dJ(\Omega_s(W); V)=0$ and necessarily $d^2J(\Omega; V; W)=0$.

(ii) We have already established in (i) that the bilinear form

$$(V, W) \mapsto h(V, W) : \mathscr{D}_D \times \mathscr{D}_D \to \mathbb{R}$$

is zero for all $V \in \mathscr{D}_D$ and $W \in \mathscr{D}_D$ such that $W=0$ on Γ and also zero for all $W \in \mathscr{D}_D$ and $V \in \mathscr{D}_D$ for which $V=0$ in a neighbourhood of Γ. By density all this is still true in $\mathscr{D}^l_D$ and now by the same argument as in the proof of Theorem 3.2 for all V in $\mathscr{D}^l_D$

$$[W] \mapsto h(V, W) : \mathscr{D}^l_D / L^l_\Omega \to \mathbb{R}$$

is well-defined, linear, and continuous. For the first component it is necessary to show that

$$D^l_\Gamma = \{ V \in \mathscr{D}^l(D, \mathbb{R}^N) : \partial^\alpha V = 0 \text{ on } \Gamma \cap D, \forall\, |\alpha| \leqslant l \}$$

the bilinear form $h(V, W)=0$. We first prove the result for the subspace

$$A = \mathscr{D}(\Omega; \mathbb{R}^N) \oplus \mathscr{D}(D \backslash \bar{\Omega}; \mathbb{R}^N).$$

Then by density and continuity the result holds for the $\mathscr{D}^l(D, \mathbb{R}^N)$-closure $\bar{A}$ of A. Finally we prove that $\bar{A} = D^l_\Gamma$. For any V in A, there exist $V_1 \in \mathscr{D}(\Omega; \mathbb{R}^N)$ and $V_2 \in \mathscr{D}(D \backslash \bar{\Omega}; \mathbb{R}^N)$ such that $V = V_1 + V_2$. Moreover

$$K_1 = \operatorname{supp} V_1 \subset \Omega \qquad \text{and} \qquad K_2 = \operatorname{supp} V_2 \subset D \backslash \bar{\Omega}$$

are compact subsets of the open sets Ω and $D \backslash \bar{\Omega}$, respectively. Hence $V_1 = 0$ (resp. $V_2 = 0$) in the open neighbourhood $\mathbb{R}^N \backslash K_1$ (resp. $\mathbb{R}^N \backslash K_2$) of Γ and necessarily $V = V_1 + V_2 = 0$ in the neighbourhood $U = \mathbb{R}^N \backslash (K_1 \cup K_2)$ of $D \cap \Gamma$. Hence from part (i), $h(V, W) = 0$. By definition of D^l_Γ, $D^l_\Gamma \subset \mathscr{D}^l(\bar{\Omega}; \mathbb{R}^N) \oplus \mathscr{D}^l(\mathbb{R}^N \backslash \Omega; \mathbb{R}^N)$. Now $A \subset D^l_\Gamma$,

$$\bar{A} = \overline{\mathscr{D}(\Omega; \mathbb{R}^N)} \oplus \overline{\mathscr{D}(D \backslash \bar{\Omega}; \mathbb{R}^N)}$$

and

$$\overline{\mathscr{D}(\Omega; \mathbb{R}^N)} = \mathscr{D}^l(D \cap \bar{\Omega}; \mathbb{R}^N), \qquad \overline{\mathscr{D}(D \backslash \bar{\Omega}; \mathbb{R}^N)} = \mathscr{D}^l(D \backslash \Omega; \mathbb{R}^N).$$

By construction each V in $\bar{A}$ is of the form $V = V_1 + V_2$ for

$$\begin{aligned} &V_1 \in \mathscr{D}^l(D \cap \bar{\Omega}; \mathbb{R}^N), && K_1 = \operatorname{supp} V_1 \text{ compact in } D \cap \bar{\Omega} \\ & && \partial^\alpha V_1 = 0 \text{ on } D \cap \Gamma, \forall\, |\alpha| \leqslant l \\ &V_2 \in \mathscr{D}^l(D \backslash \Omega; \mathbb{R}^N), && K_2 = \operatorname{supp} V_2 \text{ compact in } D \backslash \Omega \\ & && \partial^\alpha V_2 = 0 \text{ on } D \cap \Gamma, \forall\, |\alpha| \leqslant l. \end{aligned}$$

Hence

$$\operatorname{supp} V = K_1 \cup K_2 \text{ compact in } [D \cap \bar{\Omega}] \cup [D \backslash \Omega] = D$$

and $V \in \mathscr{D}^l(D, \mathbb{R}^N)$. Moreover

$$\partial^\alpha V = \partial^\alpha V_1 + \partial^\alpha V_2 = 0 \qquad \text{on} \quad D \cap \Gamma, \forall\, |\alpha| \leqslant l.$$

This proves that $\bar{A} \subset D^l_\Gamma$ and hence $\bar{A} = D^l_\Gamma$. To complete the proof notice that by continuity of $V \mapsto h(V, W)$, for all W in $\mathscr{D}^l_D$ the map

$$[V] \mapsto h(V, W) \colon \mathscr{D}^l_D / D^l_\Gamma \to \mathbb{R}$$

is well-defined, linear, and continuous. Finally the map

$$([V], [W]) \mapsto h(V, W) : (\mathscr{D}^l_D / L^l_\Omega) \times (\mathscr{D}^l_D / D^l_\Gamma) \to \mathbb{R}$$

is well-defined, bilinear, and continuous. ∎

The next and last result is the extension of Hadamard's structure theorem to second order Eulerian semiderivatives. We need the result established in the Corollary to Theorem 3.2. For a domain Ω with a boundary Γ which is C^{l+1}, $l \geqslant 0$, the map

$$q_\Omega(W) \mapsto p_L(q_\Omega(W)) = \gamma_\Gamma(W) \cdot n : \mathscr{D}^l_\Omega / L^l_\Omega \to \mathscr{D}^l(\Gamma \cap D) \tag{18}$$

is a well-defined isomorphism. This will be used for the V-component. For the W-component we need the following lemma.

LEMMA 4.1. *Assume that the boundary Γ of Ω is C^{l+1}, $l \geqslant 0$. Then the map*

$$q_D(V) \mapsto p_D(q_D(V)) = \gamma_\Gamma(V) : \mathscr{D}^l_D / D^l_\Gamma \to \mathscr{D}^l(\Gamma \cap D, \mathbb{R}^N) \tag{19}$$

is a well-defined isomorphism where

$$p_D \colon \mathscr{D}^l_D \to \mathscr{D}^l_D / D^l_\Gamma \tag{20}$$

is the canonical surjection.

Proof. This is by standard arguments. ∎

Remark 4.4. When $D=\mathbb{R}^N$ and Γ is compact, $\mathscr{D}^l(\Gamma\cap D, \mathbb{R}^N)=\mathscr{D}^l(\Gamma, \mathbb{R}^N)$ coincides with the space of l-times continuously differentiable maps from Γ to $\mathbb{R}^N$.

THEOREM 4.3. *Assume that the hypotheses of Theorem* 4.2(ii) *hold and that the boundary Γ of the open domain Ω is C^{l+1} for $l\geqslant 0$.*

(i) *The map*

$$(v, w)\mapsto h_{D\times L}(v, w)=[h](p_D^{-1}v, p_L^{-1}w):\mathscr{D}^l(\Gamma_D, \mathbb{R}^N)\times\mathscr{D}^l(\Gamma_D)\to\mathbb{R} \tag{21}$$

is bilinear and continuous and for all V and W in $\mathscr{D}^l(D, \mathbb{R}^N)$

$$d^2J(\Omega; V; W)=h_{D\times L}(\gamma_\Gamma V, ((\gamma_\Gamma W)\cdot n)), \tag{22}$$

where $\Gamma_D=\Gamma\cap D$.

(ii) *This induces a vector distribution $h(\Gamma_D\otimes\Gamma_D)$ on $\mathscr{D}^l(\Gamma_D, \mathbb{R}^N)\otimes\mathscr{D}^l(\Gamma_D)$ of order l*

$$h(\Gamma_D\otimes\Gamma_D):\mathscr{D}^l(\Gamma_D, \mathbb{R}^N)\otimes\mathscr{D}^l(\Gamma_D)\to\mathbb{R} \tag{23}$$

such that for all V and W in $\mathscr{D}^l(D, \mathbb{R}^N)$

$$\langle h(\Gamma_D\otimes\Gamma_D), (\gamma_\Gamma V)\otimes((\gamma_\Gamma W)\cdot n)\rangle=d^2J(\Omega; V; W), \tag{24}$$

where $(\gamma_\Gamma V)\otimes((\gamma_\Gamma W)\cdot n)$ is defined as the tensor product

$$((\gamma_\Gamma V)\otimes((\gamma_\Gamma W)\cdot n))_i(x, y)=(\gamma_\Gamma V_i)(x)((\gamma_\Gamma W)\cdot n)(y), \qquad x, y\in\Gamma_D \tag{25}$$

$V_i(x)$ is the ith component of $V(x)$, and

$$(\gamma_\Gamma(W)\cdot n)(y)=(\gamma_\Gamma W)(y)\cdot n(y), \qquad \forall y\in\Gamma_D. \tag{26}$$

Remark 4.5. Again we can introduce constructions similar to the ones preceding Corollary 2 to Theorem 3.2.

Remark 4.6. Finally under the hypotheses of Theorems 4.1 and 4.3

$$\begin{aligned} d^2J(\Omega; V; W)=&\langle h(\Gamma_D\otimes\Gamma_D), (\gamma_\Gamma V(0))\otimes((\gamma_\Gamma W(0))\cdot n)\rangle\\ &+\langle(g(\Gamma_D), (\gamma_\Gamma \dot{V}(0))\cdot n\rangle \end{aligned} \tag{27}$$

for all V in $\vec{V}_D^{m+1,l}$ and W in $\vec{V}_D^{m,l}$.

EXAMPLE 4.1. Consider Example 3.1. Recall that for V in $\mathscr{D}^1(D, \mathbb{R}^N)$

$$dJ(\Omega; V)=\int_\Omega \operatorname{div} V\, dx. \tag{28}$$

Now for V in $\mathscr{D}^2(D, \mathbb{R}^N)$ and W in $\mathscr{D}^1(D, \mathbb{R}^N)$

$$d^2J(\Omega; V; W) = \int_\Omega \operatorname{div}[(\operatorname{div} V) W] \, dx \tag{29}$$

and if Γ is C^1

$$d^2J(\Omega; V; W) = \int_\Gamma \operatorname{div} V \, W \cdot n \, d\Gamma \tag{30}$$

which is continuous for pairs $(V, W) \in \mathscr{D}^1(D, \mathbb{R}^N) \times \mathscr{D}^0(D, \mathbb{R}^N)$ or $\mathscr{D}^1(\Gamma, \mathbb{R}^N) \times \mathscr{D}^0(\Gamma, \mathbb{R}^N)$.

Another interesting observation is that the shape Hessian is, in general, not symmetrical as can be seen from the following example in Delfour and Zolésio [9(b)].

EXAMPLE 4.2. We use the functional (28) and expression (30) in Example 4.1. Choose the following two vector fields

$$V(x, y) = (1, 0) \qquad \text{and} \qquad W(x, y) = (x^2/2, 0).$$

Then

$$\operatorname{div} V = 0, \qquad \text{and} \qquad W|_\Gamma = x = \cos\theta$$

and

$$V \cdot n = n_x = \cos\theta \qquad \text{on } \Gamma.$$

As a result $d^2J(\Omega; V; W) = 0$ and

$$d^2J(\Omega; W; V) = \int_\Gamma \operatorname{div} W (V \cdot n) \, d\Gamma = \int_0^{2\pi} \cos^2\theta \, d\theta > 0.$$

4.3. *Comparison with Methods of Perturbation of the Identity*

At this juncture it is instructive to compare first and second order Eulerian semiderivatives obtained by the Velocity (Speed) Method with those obtained by first and second order perturbations of the identity: that is, when the transformations T_t are specified a priori by

$$T_t(X) = X + tU(X) + \frac{t^2}{2} A(X), \qquad X \in \mathbb{R}^N, \tag{31}$$

where U and A are transformations of $\mathbb{R}^N$ verifying the hypotheses of Theorem 2.2. The transformation T_t in (31) is a *second order* perturbation when $A \neq 0$ and a *first order* perturbation when $A = 0$. According to Theorem 2.2, first and second order Eulerian semiderivatives associated with (31) can be equivalently obtained by applying the Velocity (Speed) Method to the non-autonomous velocity fields V_{UA} given by (2.9) and

$$dJ(\Omega; V_{UA}) = dJ(\Omega; V_{UA}(0)) = dJ(\Omega; U), \tag{32}$$

where we have used Remark 2.1 which says that

$$V_{UA}(0) = U \qquad \text{and} \qquad \dot{V}_{UA}(0) = A - [DU]\, U. \tag{33}$$

Similarly if V_{WB} is another velocity field corresponding to

$$T_t(X) = X + tW(X) + \frac{t^2}{2} B(X), \qquad X \in \mathbb{R}^N, \tag{34}$$

where W and B verify the hypotheses of Theorem 2.2, then

$$d^2J(\Omega; V_{UA}; V_{WB}) = d^2J(\Omega; V_{UA}(0); V_{WB}(0)) + dJ(\Omega; \dot{V}_{UA}(0)) \tag{35}$$

and

$$d^2J(\Omega; V_{UA}; V_{WB}) = d^2J(\Omega; U; W) + dJ(\Omega; A - [DU]\, U). \tag{36}$$

Expressions (32) and (35) are to be compared with the following expressions obtained by the Velocity (Speed) Method for two autonomous vector fields U and W

$$dJ(\Omega; U) \qquad \text{and} \qquad d^2J(\Omega; U; W). \tag{37}$$

For the Shape gradient the two expressions coincide; for the Shape Hessian we recognize the bilinear term in (36) and (37) but the two expressions differ by the term

$$dJ(\Omega; A - [DU]\, U). \tag{38}$$

Even for a first order perturbation ($A = 0$), we have a quadratic term in U. This situation is analogous to the classical problem of defining second order derivatives on a manifold. The term (38) would correspond to the connexion while the bilinear term $d2J(\Omega; V; W)$ would be the candidate for the *canonical* second order shape derivative. In this context we shall refer to the corresponding distribution $H(\Omega)$ as the *canonical Shape Hessian.* All other second order shape derivatives will be obtained from $H(\Omega)$ by adding the gradient term $G(\Omega)$ acting as the appropriate acceleration field connexion).

Remark 4.7. The method of perturbation of the identity can be made *more canonical* by using the following family of transformations

$$T_t(X) = X + tU(X) + \frac{t^2}{2}(A + [DU]U) \tag{39}$$

which yields

$$dJ(\Omega; U) \qquad \text{for the gradient} \tag{40}$$

and

$$d^2J(\Omega; U; W) + dJ(\Omega; A) \qquad \text{for the Hessian,} \tag{41}$$

where for a first order perturbation ($A = 0$) the second term disappears.

Remark 4.8. When Ω^* is an appropriately smooth domain which minimizes a twice shape differentiable functional $J(\Omega)$ without constraints on Ω, the classical necessary conditions would be (at least formally)

$$dJ(\Omega^*; V) = 0, \qquad \forall V, \tag{42}$$

$$d^2J(\Omega^*; W; W) \geqslant 0, \qquad \forall W, \tag{43}$$

or equivalently for "smooth velocity fields V and W"

$$dJ(\Omega^*; V(0)) = 0, \qquad \forall V \tag{44}$$

$$d^2J(\Omega^*; W(0); W(0)) + dJ(\Omega^*; \dot{W}(0)) \geqslant 0, \qquad \forall W. \tag{45}$$

But in view of (44), condition (45) reduces to the following condition on the canonical Shape Hessian

$$d^2J(\Omega^*; W(0); W(0)) \geqslant 0, \qquad \forall W. \tag{46}$$

References

1. S. Agmon, A. Douglis, and L. Nirenberg, (a) Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I, *Comm. Pure Appl. Math.* **12** (1959), 623–727; (b) Estimates near the boundary for slutions of elliptic partial differential equations satisfying general boundary conditions, II, *Comm. Pure Appl. Math.* **17** (1964), 35–92.
2. G. Arumugam and O. Pironneau, On the problems of riblets as a drag reduction device, *Optimal Control Appl. Methods* **10** (1989), 93–112.
3. J. P. Aubin and A. Cellina, "Differential Inclusions," Springer-Verlag, Berlin, 1984.
4. J. P. Aubin and H. Frankowska, (a) "Set-Valued Analysis," Birkäuser, Basel/Berlin, 1990; (b) "Feedback Controls as Solutions to Systems of Partial Differential Inclusions," Cahiers de Mathématiques de la décision, CEREMADE, Université Paris-Dauphine, Summer 1990.

5. V. M. BABIČ, Sur le prolongement des fonctions, *Uspehi Mat. Nauk* **8** (1953), 111–113. [In Russian]
6. R. CACCIOPPOLI, Misura e integrazione sugli insiemi dimensionalmenti orientati, *Rend. Accad. Naz. Lincei Cl. Sci. Fis. Mat. Nat.* **12** (1952), 3–11, 137–146.
7. J. CÉA, (a) Problems of shape optimal design, *in* "Optimization of Distributed Parameter Structures, Vol. II" (E. J. Haug and J. Céa, Eds.), pp. 1005–1048, Sijhoff and Noordhoff, Alphen aan den Rijn, The Netherlands, 1981; (b) Numerical methods of shape optimal design, *in* "Optimization of Distributed Parameter Structures, Vol. II" (E. J. Haug and J. Céa, Eds.), pp. 1049–1087, Sijhoff and Noordhoff, Alphen aan den Rijn, The Netherlands, 1981.
8. E. DE GIORGI, "Frontiere orientati di misura minima, Sem. Mat. Scuola Norm. Sup. Pisa 1960-61," Editrice Technico Scientifica, Pisa, 1961.
9. M. C. DELFOUR AND J. P. ZOLÉSIO, (a) Shape sensitivity analysis via min max differentiability, *SIAM J. Control Optim.* **26** (1988), 834–862; (b) Anatomy of the shape Hessian, *Ann. Mat. Pura Appl.* **153** (1989); (c) Computation of the shape Hessian by a Lagrangian method, *in* "Fifth Symp. on Control of Distributed Parameter Systems," (A. El Jai and M. Amouroux, Eds.), pp. 85–90, Pergamon, Elmsford, NY, in press; (d) Shape Hessian by the velocity method: A Lagrangian approach, *in* "Stabilization of Flexible Structures" (Proc. CONCOM Conference, Montpellier, France, January 1989) (J. P. Zolésio, Ed.), Springer-Verlag, New York/Berlin, in press; (e) Velocity method and Lagrangian formulation for the computation of the shape Hessian, *SIAM J. Control Optim.*, in press.
10. N. FUJII, (a) Domain optimization problems with a boundary value problem as a constraint, *in* "Control of Distributed Parameter Systems," pp. 5–9, Pergamon, Oxford/New York, 1986; (b) Second variation and its application in a domain optimization problem, *in* "Control of Distributed Parameter Systems," pp. 431–436, Pergamon, Oxford/New York, 1986.
11. M. GUELFAND AND N. Y. VILENKIN, "Les distributions, Applications de l'analyse harmonique" (translated in French by G. Rideau), Dunod, Paris, 1967.
12. J. HADAMARD, Mémoire sur le problème d'analyse relatif à l'équilibre des plaques élastiques encastrées, *in* "Œuvres de J. Hadamard, Vol. II" (original reference, *Mem. Sav. Etrang.* **33** (1907), mémoire couronné par l'Académie des Sciences), pp. 515–641, C.N.R.S., Paris, 1968.
13. M. NAGUMO, Über die Loge der Integralkurven gewöhnlicher Differentialgleichungen, *Proc. Phys. Math. Soc. Japan* **24** (1942), 551–559.
14. J. NEČAS, "Les méthodes directes en théorie des équations elliptiques," Masson, Paris, and Academia, Pragues, 1967.
15. L. SCHWARTZ, (a) "Théorie des distributions," Hermann, Paris, 1966; (b) Théorie des noyaux, *in* "Proceedings of the International Congress of Mathematicians, Vol. I," pp. 220–230, 1950.
16. J. SIMON, Second variations for domain optimization problems, *in* "Control of Distributed Parameter Systems (Proc. 4th Int. Conf. in Vorau)," Birkhäuser-Verlag, Basel, in press.
17. J. P. ZOLÉSIO, (a) "Identification de domaines par déformation," Thèse de doctorat d'état, Université de Nice, France, 1979; (b) The material derivative (or speed) method for shape optimization, *in* "Optimization of Distributed Parameter Structures, Vol. II" (E. J. Haug and J. Céa, Eds.), pp. 1089–1151, Sijhoff and Nordhoff, Alphen aan den Rijn, 1981.
18. J. DUGUNJI, "Topology," Allyn and Bacon, Inc., Boston, second printing, August 1966.