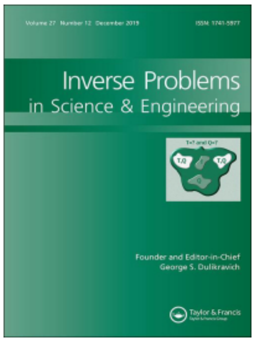

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Shape optimization for Navier–Stokes flow

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This article describes the optimal shape design of the newtonian viscous incompressible fluids driven by the stationary nonhomogeneous Navier–Stokes equations. We use three approaches to derive the structures of shape gradients for some given cost functionals. The first one is to use the Piola transformation and derive the state derivative and its associated adjoint state; the second one is to use the differentiability of a minimax formulation involving a Lagrangian functional with a function space parametrization technique; the last one is to employ the differentiability of a minimax formulation with a function space embedding technique. Finally, we apply a gradient type algorithm to our problem and numerical examples show that our theory is useful for practical purpose and the proposed algorithm is feasible.

Keywords: shape optimization; shape derivative; gradient algorithm; minimax formulation; material derivative; Navier–Stokes equations

AMS Subject Classifications: 35B37; 35Q30; 49K35; 49K40.

1. Introduction

This article deals with the optimal shape design for the stationary Navier–Stokes flow. This problem is of great practical importance in the design and control of many industrial devices such as aircraft wings, cars, turbines, boats and so on. The control variable is the shape of the fluid domain, the object is to minimize some cost functionals that may be given by the designer, and finally we can obtain the optimal shapes by numerical computation.

Optimal shape design has received considerable attention already. Early works concerning on existence of solutions and differentiability of the quantity (such as, state, cost functional, etc.) with respect to shape deformation occupied most of the 1980s [1–4]) the stabilization of structures using boundary variation technique has been fully addressed in [2–4]. For the optimal shape design for Stokes flow, many people are contributed to it, such as Pironneau [5], Simon [6], Gao *et al*. [7,8] and so on.

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In this article, in order to derive the structures of shape gradients for some given cost functionals in shape optimization problems for Navier–Stokes flow, we suggest the following three approaches:

(1) use the Piola transformation and derive the state derivative with respect to the shape of the fluid domain and its associated adjoint state;
(2) utilize the differentiability of a minimax formulation involving a Lagrangian functional with a function space parametrization technique;
(3) employ the differentiability of a minimax formulation involving a Lagrangian functional with a function space embedding technique;

In [9], we use the first approach to solve a shape optimization problem governed by a Robin problem, and in [8], we derive the expression of shape gradients for Stokes optimization problem by the first approach. In this article, we use this approach to study the optimal shape design for Navier–Stokes flow with small regularity data.

As we all known, many shape optimization problems can be expressed as a minimax of some suitable Lagrangian functional. Theorems on the differentiability of a saddle point (i.e. a minimax) of such Lagrangian functional with respect to a parameter provides very powerful tools to obtain shape gradients by function space parametrization or function space embedding without the usual study of the state derivative approach.

The function space parametrization technique and function space embedding technique are advocated by M.C. Delfour and J.-P. Zolésio to solve Poisson equation with Dirichlet and Nuemann condition [2]. In our article [7,10], we apply them to solve a Robin problem and a shape optimization problem for Stokes flow, respectively. However, in this article we extend them to study the optimal shape design for Navier–Stokes flow in spite of its lack of rigorous mathematical justification in case where the Lagragnian formulation is not convex. We shall show how this theorem allows, at least formally to bypass the study of the differentiability of the state and obtain the expression of shape gradients with respect to the shape of the variable domain for some given cost functionals.

We will find that the three approaches lead to the same expressions of the shape gradients for our given cost functionals. Hence, even if the last two approaches lacks a rigorous mathematical framework, they allow more flexible computations which can be very useful for practical purpose. On the numerical point of view, we give the implementation of our problem in two-dimensional case at the end of this article, and the numerical results show that the last two approaches provide big efficiency for the shape optimization problem.

This article is organized as follows. In Section 2, we briefly recall the velocity method, which is used for the characterization of the deformation of the shape of the domain and give the definitions of Eulerian derivative and shape derivative. We also give the description of the shape optimization problem for Navier–Stokes flow.

In Section 3, we prove the existence of the weak Piola material derivative, and give the description of the shape derivative. After that, we express the shape gradients of the typical cost functionals $J_i(\Omega)$, $(i=1, 2)$ by introducing the corresponding adjoint state systems.

Section 4 is devoted to the computation of the shape gradient of the Lagrangian functional, due to a minimax principle concerning the differentiability of the minimax formulation by function space parametrization technique and function space embedding technique.

Finally in the last section, we give a gradient-type algorithm with some numerical examples to prove that our theory could be very useful for the practical purpose and the proposed algorithm is feasible.

2. Preliminaries and statement of the problem

2.1. *Elements of the velocity method*

Domains Ω don't belong to a vector space and this requires the development of shape calculus to make sense of a derivative or a gradient. To realize it, there are about three types of techniques: Hadamard's [11] normal variation method, the perturbation of the identity method by Simon [12] and the velocity method, (Cea [1] and Zolesio [2,13]). We will use the velocity method, which contains the others. In that purpose, we choose an open set D in $\mathbb{R}^N$ with the boundary ∂D piecewise C^k, and a velocity space $\boldsymbol{V} \in E^k := \{\boldsymbol{V} \in C([0,\varepsilon]; \mathcal{D}^k(\bar{D}, \mathbb{R}^N)) : \boldsymbol{V} \cdot \boldsymbol{n}_{\partial D} = 0 \text{ on } \partial D\}$, where ε is a small positive real number and $\mathcal{D}^k(\bar{D}, \mathbb{R}^N)$ denotes the space of all k-times continuous differentiable functions with compact support contained in $\mathbb{R}^N$. The velocity field

$$\boldsymbol{V}(t)(x) = \boldsymbol{V}(t,x), \quad x \in D, \ t \geq 0$$

belongs to $\mathcal{D}^k(\bar{D}, \mathbb{R}^N)$ for each t. It can generate transformations

$$T_t(\boldsymbol{V})X = x(t,X), \quad t \geq 0, \ X \in D$$

through the following dynamical system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t}(t,X) = \boldsymbol{V}(t,x(t)) \\ x(0,X) = X \end{cases} \tag{1}$$

with the initial value X given. We denote the transformed domain $T_t(\boldsymbol{V})(\Omega)$ by $\Omega_t(\boldsymbol{V})$ at $t \geq 0$, and also set $\Gamma_t := T_t(\Gamma)$.

There exists an interval $I = [0, \delta)$, $0 < \delta \leq \varepsilon$, and a one-to-one map T_t from $\bar{D}$ onto $\bar{D}$ such that

(i) $T_0 = I$;
(ii) $(t,x) \mapsto T_t(x)$ belongs to $C^1(I; C^k(D; D))$ with $T_t(\partial D) = \partial D$;
(iii) $(t,x) \mapsto T_t^{-1}(x)$ belongs to $C(I; C^k(D; D))$.

Such transformation are well studied in [2].

Furthermore, for sufficiently small $t > 0$, the Jacobian J_t is strictly positive [2]:

$$J_t(x) := \det\left|\mathrm{D}T_t(x)\right| = \det \mathrm{D}T_t(x) > 0, \ (\mathrm{D}T_t)_{ij} = \partial_j(T_t)_i \tag{2}$$

where $\mathrm{D}T_t(x)$ denotes the Jacobian matrix of the transformation T_t evaluated at a point $x \in D$ associated with the velocity field $\boldsymbol{V}$. We will also use the following notation: $\mathrm{D}T_t^{-1}(x)$ is the inverse of the matrix $\mathrm{D}T_t(x)$, ${}^*\mathrm{D}T_t^{-1}(x)$ is the transpose of the matrix $\mathrm{D}T_t^{-1}(x)$. These quantities also satisfy the following lemma.

LEMMA 2.1 [2] *For any* $V \in E^k$, $\mathrm{D}T_t$ *and* J_t *are invertible. Moreover*, $\mathrm{D}T_t$, $\mathrm{D}T_t^{-1}$ *are in* $C^1([0,\varepsilon]; C^{k-1}(\bar{D}; \mathbb{R}^{N\times N}))$, *and* J_t, J_t^{-1} *are in* $C^1([0,\varepsilon]; C^{k-1}(\bar{D}; \mathbb{R}))$.

Now, let $J(\Omega)$ be a real valued functional associated with any regular domain Ω, we say that this functional has a 'Eulerian derivative' at Ω in the direction $\boldsymbol{V}$ if the limit

$$\lim_{t\searrow 0}\frac{J(\Omega_t)-J(\Omega)}{t} := \mathrm{d}J(\Omega;\, \boldsymbol{V})$$

exists.

Furthermore, if the map

$$\boldsymbol{V} \mapsto \mathrm{d}J(\Omega;\, \boldsymbol{V}) : E^k \to \mathbb{R}$$

is linear and continuous, we say that J is 'shape differentiable' at Ω. In the distributional sense we have

$$\mathrm{d}J(\Omega;\, \boldsymbol{V}) = \langle \nabla J, \boldsymbol{V}\rangle_{\mathcal{D}^k(\bar{D},\mathbb{R}^N)'\times\mathcal{D}^k(\bar{D},\mathbb{R}^N)}. \tag{3}$$

When J has a Eulerian derivative, we say that ∇J is the 'shape gradient' of J at Ω.

Before closing this subsection, we introduce the following functional spaces which will be used throughout this article:

$$\begin{aligned}
H(\mathrm{div},\Omega) &:= \{\boldsymbol{u}\in L^2(\Omega)^N : \mathrm{div}\,\boldsymbol{u}=0 \text{ in } \Omega, \boldsymbol{u}\cdot\boldsymbol{n}=0 \text{ on } \partial\Omega\},\\
H^1(\mathrm{div},\Omega) &:= \{\boldsymbol{u}\in H^1(\Omega)^N : \mathrm{div}\,\boldsymbol{u}=0 \text{ in } \Omega\},\\
H^1(\mathrm{div},\Omega) &:= \{\boldsymbol{u}\in H^1(\Omega)^N : \mathrm{div}\,\boldsymbol{u}=0 \text{ in } \Omega,\ \boldsymbol{u}|_{\partial\Omega}=0\}.
\end{aligned}$$

2.2. *Statement of the shape optimization problem*

Let Ω be the fluid domain in $\mathbb{R}^N (N = 2 \text{ or } 3)$, and the boundary $\Gamma := \partial\Omega$. The fluid is described by its velocity $\boldsymbol{y}$ and pressure p satisfying the stationary Navier–Stokes equations:

$$\begin{cases}
-\alpha\Delta\boldsymbol{y} + D\boldsymbol{y}\cdot\boldsymbol{y} + \nabla p = \boldsymbol{f} & \text{in } \Omega\\
\mathrm{div}\,\boldsymbol{y} = 0 & \text{in } \Omega\\
\boldsymbol{y} = \boldsymbol{g} & \text{in } \Gamma
\end{cases} \tag{4}$$

where α stands for the kinematic viscosity, $\boldsymbol{f}$ denotes the given body force per unit mass and $\boldsymbol{g}$ is the given velocity at the boundary Γ.

For the existence and uniqueness of the solution of the nonhomogeneous Navier–Stokes system (4), we have the following results [14].

THEOREM 2.2 *We suppose that* Ω *is of class* C^1. *For*

$$\boldsymbol{f} \in [L^2(\mathbb{R}^N)]^N \tag{5}$$

$$\boldsymbol{g} \in H^2(\mathrm{div},\mathbb{R}^N) := \left\{\boldsymbol{g} \in \left[H^2(\mathbb{R}^N)\right]^N : \mathrm{div}\,\boldsymbol{g} = 0\right\}, \tag{6}$$

there exists at least one $\boldsymbol{y} \in H^1(\mathrm{div},\Omega)$ *and a distribution* $p \in L^2(\Omega)$ *on* Ω *such that* (4) *holds. Moreover, if* α *is sufficiently large and* $\boldsymbol{g}$ *in* H^1*–norm, is sufficiently small, there* *exists a unique solution* $(\boldsymbol{y}, p) \in H^1(\mathrm{div}, \Omega) \times L_0^2(\Omega)$ *of* (4). *In addition, if* Ω *is of class* C^2, *we have* $(\boldsymbol{y}, p) \in (H^1(\mathrm{div}, \Omega) \cap H^2(\Omega)^N) \times H^1(\Omega)$.

We are interested in solving the following velocity-tracking type minimization problem

$$\min_{\Omega \in \mathcal{O}} J_1(\Omega) = \frac{1}{2} \int_\Omega |\boldsymbol{y} - \boldsymbol{y}_d|^2 \mathrm{d}x, \tag{7}$$

where $\vec{y}_d$ is the target velocity or the energy minimization problem

$$\min_{\Omega \in \mathcal{O}} J_2(\Omega) = \alpha \int_\Omega |\mathrm{D}\boldsymbol{y}|^2 \mathrm{d}x. \tag{8}$$

An example of the admissible set $\mathcal{O}$ is:

$$\mathcal{O} := \{\Omega \subset \mathbb{R}^N : |\tilde{\Gamma}| = 1\},$$

where $\tilde{\Gamma}$ is the domain inside the closed boundary Γ and $|\tilde{\Gamma}|$ is its volume or area in 2D.

3. State derivative approach

In this section, we use the Piola transformation to bypass the divergence free condition and then derive a weak material derivative by the weak implicit function theorem. Finally, we will derive the structure of the shape gradients of the cost functionals by introducing the adjoint state equations associated with the corresponding cost functionals.

3.1. *Piola material derivative*

In order to deal with the nonhomogeneous Dirichlet boundary condition on Γ, we take $\tilde{\boldsymbol{y}} = \boldsymbol{y} = \boldsymbol{g}$, where $\tilde{\boldsymbol{y}}$ satisfies the following homogeneous Navier–Stokes system

$$\begin{cases} -\alpha\Delta\tilde{\boldsymbol{y}} + \mathrm{D}\tilde{\boldsymbol{y}} \cdot \tilde{\boldsymbol{y}} + \mathrm{D}\tilde{\boldsymbol{y}} \cdot \boldsymbol{g} + \mathrm{D}\boldsymbol{g} \cdot \tilde{\boldsymbol{y}} + \nabla p = \boldsymbol{F} & \text{in } \Omega \\ \mathrm{div}\, \tilde{\boldsymbol{y}} = 0 & \text{in } \Omega \\ \tilde{\boldsymbol{y}} = 0 & \text{on } \Gamma \end{cases} \tag{9}$$

with $\boldsymbol{F} := \boldsymbol{f} + \alpha\Delta\boldsymbol{g} - \mathrm{D}\boldsymbol{g} \cdot \boldsymbol{g} \in [L^2(\mathbb{R}^N)]^N$.

We say that the function $\tilde{\boldsymbol{y}} \in H_0^1(\mathrm{div}, \Omega)$ is called a weak solution of problem (9) if it satisfies

$$\langle e(\tilde{\boldsymbol{y}}), w \rangle = 0, \quad \boldsymbol{w} \in H_0^1(\mathrm{div}, \Omega) \tag{10}$$

with

$$\langle e(\tilde{\boldsymbol{y}}), \boldsymbol{w} \rangle := \int_\Omega (\alpha \mathrm{D}\tilde{\boldsymbol{y}} : \mathrm{D}\boldsymbol{w} + \mathrm{D}\tilde{\boldsymbol{y}} \cdot \tilde{\boldsymbol{y}} \cdot \boldsymbol{w} + \mathrm{D}\tilde{\boldsymbol{y}} \cdot \boldsymbol{g} \cdot \boldsymbol{w} + \mathrm{D}\boldsymbol{g} \cdot \tilde{\boldsymbol{y}} \cdot \boldsymbol{w} - \boldsymbol{F} \cdot \boldsymbol{w}) \mathrm{d}x. \tag{11}$$

As we all known, the divergence free condition coming from the fact that the fluid has an homogeneous density and evolves as an incompressible flow is difficult to impose on the mathematical and numerical point of view. Therefore in order to work with the divergence free condition, we need to introduce the following lemma [15].

LEMMA 3.1 *The Piola transform*

$$\Psi_t : H^1(\mathrm{div}, \Omega) \mapsto H^1(\mathrm{div}, \Omega_t)$$
$$\boldsymbol{\varphi} \mapsto ((J_t)^{-1} \mathrm{D}T_t \cdot \varphi) \circ T_t^{-1}$$

is an isomorphism, where $\circ$ *denotes the composition of the two maps.*

Now by the transformation T_t, we consider the solution $\tilde{\boldsymbol{y}}_t$ defined on Ω_t of the perturbed weak formulation:

$$\int_{\Omega_t} (\alpha \mathrm{D}\tilde{\boldsymbol{y}} : \mathrm{D}\boldsymbol{w}_t + \mathrm{D}\tilde{\boldsymbol{y}} \cdot \tilde{\boldsymbol{y}} \cdot \boldsymbol{w}_t + \mathrm{D}\tilde{\boldsymbol{y}} \cdot \boldsymbol{g} \cdot \boldsymbol{w}_t + \mathrm{D}\boldsymbol{g} \cdot \tilde{\boldsymbol{y}} \cdot \boldsymbol{w}_t - \boldsymbol{F} \cdot \boldsymbol{w}_t) \mathrm{d}x = 0, \forall \boldsymbol{w}_t \in H_0^1(\mathrm{div}, \Omega_t) \quad (12)$$

and introduce $\tilde{\boldsymbol{y}}^t = \Psi_t^{-1}(\tilde{\boldsymbol{y}}_t), \boldsymbol{w}_t = \Psi_t^{-1}(\boldsymbol{w}_t)$ defined on Ω.

We replace $\tilde{\boldsymbol{y}}$, $\boldsymbol{w}_{\mathrm{t}}$ by $\Psi_t(\tilde{\boldsymbol{y}}^t)$, $\Psi_{\mathrm{t}}(\boldsymbol{w}^t)$ in the weak formulation (12) and obtain that for all $\boldsymbol{w}^t \in H_0^1(\mathrm{div}, \Omega)$,

$$\begin{aligned} \int_{\Omega_t} &[\alpha \mathrm{D}(\Psi_t(\tilde{\boldsymbol{y}}^t)) : \mathrm{D}(\Psi_t(\boldsymbol{w}^t)) + \mathrm{D}(\Psi_t(\tilde{\boldsymbol{y}}^t)) \cdot \Psi_t(\tilde{\boldsymbol{y}}^t) \cdot \Psi_t(\boldsymbol{w}^t) \\ &+ \mathrm{D}(\Psi_t(\tilde{\boldsymbol{y}}^t)) \cdot \boldsymbol{g} \cdot \Psi_t(\boldsymbol{w}^t) + \mathrm{D}\boldsymbol{g} \cdot \Psi_t(\tilde{\boldsymbol{y}}^t) \cdot \Psi_t(\boldsymbol{w}^t) - \boldsymbol{F} \cdot \Psi_t(\boldsymbol{w}^t)] \mathrm{d}x = 0. \end{aligned}$$

By the transformation T_t and the following identities:

$$\begin{aligned} \mathrm{D}(T_t^{-1}) \circ T_t &= \mathrm{D}T_t^{-1}; \\ \mathrm{D}(\boldsymbol{\varphi} \circ T_t^{-1}) &= (\mathrm{D}\boldsymbol{\varphi} \cdot \mathrm{D}T_t^{-1}) \circ T_t^{-1}; \\ (\mathrm{D}\boldsymbol{g}) \circ T_t &= \mathrm{D}(\boldsymbol{g} \circ T_t) \cdot \mathrm{D}T_t^{-1}, \end{aligned}$$

we use a back transport in Ω and obtain the following weak formulation

$$\langle e(t, \tilde{\boldsymbol{y}}^t), \boldsymbol{w}^t \rangle = 0, \quad \forall \boldsymbol{w}^t \in H_0^1(\mathrm{div}, \Omega) \quad (13)$$

with the notations

$$\begin{aligned} \langle e(t, \boldsymbol{v}), \boldsymbol{w} \rangle := \alpha &\int_\Omega \mathrm{D}(B(t)\boldsymbol{v}) : [\mathrm{D}(B(t)\boldsymbol{w}) \cdot A(t)] \mathrm{d}x \\ &+ \int_\Omega \mathrm{D}(B(t)\boldsymbol{v}) \cdot \boldsymbol{v} \cdot (B(t)\boldsymbol{w}) \mathrm{d}x + \int_\Omega [\mathrm{D}(B(t)\boldsymbol{v}) \cdot \mathrm{D}T_t^{-1}] \cdot (\boldsymbol{g} \circ T_t) \cdot (\mathrm{D}T_t \boldsymbol{w}) \mathrm{d}x \\ &+ \int_\Omega \mathrm{D}(\boldsymbol{g} \circ T_t) \cdot \boldsymbol{v} \cdot (B(t)\boldsymbol{w}) \mathrm{d}x - \int_\Omega (\boldsymbol{F} \circ T_t) \cdot (\mathrm{D}T_t \boldsymbol{w}) \mathrm{d}x, \end{aligned} \quad (14)$$

and

$$A(t) := J_t \mathrm{D}T_t^{-1\,*} \mathrm{D}T_t^{-1}; \quad B(t)\tau := J_t^{-1} \mathrm{D}T_t \cdot \tau.$$

Now, we are interested in the derivability of the mapping

$$t \mapsto \tilde{\boldsymbol{y}}^t = \Psi_t^{-1}(\tilde{\boldsymbol{y}}_t) : [0, \varepsilon] \mapsto H_0^1(\mathrm{div}, \Omega),$$

where $\varepsilon > 0$ is sufficiently small and $\tilde{\boldsymbol{y}}^t \in H_0^1(\mathrm{div}, \Omega)$ is the solution of the state equation

$$\langle e(t, \boldsymbol{v}), \boldsymbol{w} \rangle = 0, \quad \forall \boldsymbol{w} \in H_0^1(\mathrm{div}, \Omega). \quad (15)$$

In order to prove the differentiability of $\tilde{\boldsymbol{y}}^t$ with respect to t in a neighbourhood of $t = 0$, there maybe two approaches:

(i) analysis of the differential quotient: $\lim\limits_{t \to 0} (\tilde{\boldsymbol{y}}^t - \tilde{\boldsymbol{y}})/t$;

(ii) derivation of the local differentiability of the solution $\tilde{\boldsymbol{y}}$ associated to the implicit Equation (10).

We use the second approach. However, we cannot use the classical implicit theorem, since it requires strong differentiability results in H^{-1} for our case. Then, we introduce the following weak implicit function theorem [13].

THEOREM 3.2 *Let X, Y' be two Banach spaces. I an open bounded set in $\mathbb{R}$, and consider the map $(t,x)\mapsto e(t,x) : I\times X\mapsto Y'$. If the following hypothesis hold*:

(i) *$t\mapsto\langle e(t,x),y\rangle$ is continuously differentiable for any $y\in Y$ and $(t,x)\mapsto\langle\partial_t e(t,x),y\rangle$ is continuous*;
(ii) *there exists $u\in X$ such that $u\in C^{0,1}(I;X)$ and $e(t,u(t))=0$, $\forall t\in I$*;
(iii) *$x\mapsto e(t,x)$ is differentiable and $(t,x)\mapsto\partial_x e(t,x)$ is continuous*;
(iv) *there exists $t_0\in I$ such that $\partial_x e(t,x)_{|(t_0,x(t_0))}$ is an isomorphism from X to Y'*,

the mapping $t\mapsto u(t) : I\mapsto X$ is differentiable at $t=t_0$ for the weak topology in X and its weak derivative $\dot{u}(t)$ is the solution of

$$\langle\partial_x e(t_0,u(t_0))\cdot\dot{u}(t_0),y\rangle+\langle\partial_t e(t_0,u(t_0)),y\rangle=0,\quad\forall y\in Y.$$

Now, we state the main theorem of this subsection concerning on the differentiability of $\tilde{\boldsymbol{y}}^t$ with respect to t.

THEOREM 3.3 *The weak 'Piola material derivative'*

$$\dot{\tilde{\boldsymbol{y}}}^P:=\lim_{t\to 0}\frac{\tilde{\boldsymbol{y}}^t-\tilde{\boldsymbol{y}}}{t}=\lim_{t\to 0}\frac{\Psi_t^{-1}(\tilde{\boldsymbol{y}}_t)-\tilde{\boldsymbol{y}}}{t}$$

exists and, is characterized, by the following weak formulation:

$$\langle\partial_{\boldsymbol{v}} e(0,\boldsymbol{v})\Big|_{\boldsymbol{v}=\tilde{\boldsymbol{y}}}\cdot\dot{\tilde{\boldsymbol{y}}}^P,w\rangle+\langle\partial_t e(0,\tilde{\boldsymbol{y}}),\boldsymbol{w}\rangle=0,\quad\forall\boldsymbol{w}\in H_0^1(\mathrm{div},\Omega),\tag{16}$$

i.e.

$$\begin{aligned}
&\alpha\int_\Omega \mathrm{D}\dot{\tilde{\boldsymbol{y}}}^P:\mathrm{D}\boldsymbol{w}\,\mathrm{d}x+\int_\Omega \mathrm{D}\dot{\tilde{\boldsymbol{y}}}^P\cdot\tilde{\boldsymbol{y}}\cdot\boldsymbol{w}+\mathrm{D}\tilde{\boldsymbol{y}}\cdot\dot{\tilde{\boldsymbol{y}}}^P\cdot\boldsymbol{w}+\mathrm{D}\dot{\tilde{\boldsymbol{y}}}^P\cdot\boldsymbol{g}\cdot\boldsymbol{w}+\mathrm{D}\boldsymbol{g}\cdot\dot{\tilde{\boldsymbol{y}}}^P\cdot\boldsymbol{w}\,\mathrm{d}x\\
&=\alpha\int_\Omega \mathrm{D}\tilde{\boldsymbol{y}}:\mathrm{D}\boldsymbol{w}\,\mathrm{div}\boldsymbol{V}\,\mathrm{d}x+\alpha\int_\Omega \mathrm{D}\tilde{\boldsymbol{y}}:[-\mathrm{D}(\mathrm{div}\boldsymbol{V}\boldsymbol{w})+\mathrm{D}(\mathrm{D}\boldsymbol{V}\boldsymbol{w})-\mathrm{D}\boldsymbol{w}\mathrm{D}\boldsymbol{V}]\mathrm{d}x\\
&\quad+\alpha\int_\Omega \mathrm{D}\boldsymbol{w}:[-\mathrm{D}(\mathrm{div}\boldsymbol{V}\tilde{\boldsymbol{y}})+\mathrm{D}(\mathrm{D}\boldsymbol{V}\tilde{\boldsymbol{y}})-\mathrm{D}\tilde{y}\mathrm{D}\boldsymbol{V}]\mathrm{d}x\\
&\quad+\int_\Omega[\mathrm{D}(\mathrm{D}\boldsymbol{V}\tilde{\boldsymbol{y}}-\mathrm{div}\boldsymbol{V}\tilde{\boldsymbol{y}})\cdot\tilde{\boldsymbol{y}}\cdot\boldsymbol{w}+\mathrm{D}\tilde{\boldsymbol{y}}\cdot\tilde{\boldsymbol{y}}\cdot(\mathrm{D}\boldsymbol{V}\boldsymbol{w}-\mathrm{div}\boldsymbol{V}\boldsymbol{w})]\mathrm{d}x\\
&\quad+\int_\Omega[\mathrm{D}(\mathrm{D}\boldsymbol{V}\tilde{\boldsymbol{y}}-\mathrm{div}\boldsymbol{V}\tilde{\boldsymbol{y}})-D\tilde{y}D\boldsymbol{V}]\cdot\boldsymbol{g}\cdot\boldsymbol{w}\,\mathrm{d}x+\int_\Omega \mathrm{D}\tilde{\boldsymbol{y}}\cdot(\mathrm{D}\boldsymbol{g}\boldsymbol{V}+{}^*\mathrm{D}\boldsymbol{V}\boldsymbol{g})\cdot\mathrm{d}x\\
&\quad+\int_\Omega[\mathrm{D}(\mathrm{D}\boldsymbol{g}\boldsymbol{V})-\mathrm{D}\boldsymbol{g}\,\mathrm{div}\boldsymbol{V}+{}^*\mathrm{D}\boldsymbol{V}\mathrm{D}\boldsymbol{g}]\cdot\tilde{\boldsymbol{y}}\cdot\boldsymbol{w}\,\mathrm{d}x\\
&\quad-\int_\Omega[\mathrm{D}(\boldsymbol{f}+\alpha\Delta\boldsymbol{g}+\mathrm{D}\boldsymbol{g}\cdot\boldsymbol{g})\boldsymbol{V}+{}^*\mathrm{D}\boldsymbol{V}(\boldsymbol{f}+\alpha\Delta\boldsymbol{g}-\mathrm{D}\boldsymbol{g}\cdot\boldsymbol{g})]\cdot\boldsymbol{w}\,\mathrm{d}x.
\end{aligned}\tag{17}$$

where $\tilde{\boldsymbol{y}}$ is the solution of the weak formulation (10).

Proof In order to apply Theorem 3.2, we need to verify the four hypothesis of Theorem 3.2 for the mapping

$$(t, \boldsymbol{v}) \mapsto e(t, \boldsymbol{v}) : [0, \varepsilon] \times H_0^1(\mathrm{div}, \Omega) \mapsto H_0^1(\mathrm{div}, \Omega)'.$$

First, since the flow map $T_t \in C^1([0, \varepsilon]; C^2(\bar{D}, \mathbb{R}^N))$ and Lemma 2.1, the mapping

$$t \mapsto \langle e(t, \boldsymbol{v})\boldsymbol{w}\rangle : [0, \varepsilon] \mapsto \mathbb{R}$$

is C^1 tor any $\boldsymbol{v}$, $\boldsymbol{w} \in H_0^1(\mathrm{div}, \Omega)$. On the other hand, since $\boldsymbol{F} \in [L^2(\mathbb{R}^N)]^N$, the mapping $t \mapsto \boldsymbol{F} \circ T_t$ is only weakly differentiable in the space $H^{-1}(\mathbb{R}^N)]^N$, thus the mapping $t \mapsto e(t, \boldsymbol{v})$ is weakly differentiable. We denote by $\partial_t e(t, \boldsymbol{v})$ its weak derivative. Since we have the following three identities,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}T_t = (\mathrm{D}\boldsymbol{V}(t) \circ T_t)\mathrm{D}T_t; \tag{18}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}J_t = (\mathrm{div}\,\boldsymbol{V}(t)) \circ T_t J_t; \tag{19}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{f} \circ T_t) = (\mathrm{D}\boldsymbol{f} \cdot \boldsymbol{V}(t)) \circ T_t, \tag{20}$$

the weak derivative $\partial_t e(t, \boldsymbol{v})$ can be expressed as follows,

$$\begin{aligned}
\langle \partial_t e(t, \boldsymbol{v}), \boldsymbol{w}\rangle &= \alpha \int_\Omega \mathrm{D}(B'(t)\boldsymbol{v}) : [\mathrm{D}(B(t)\boldsymbol{w}) \cdot A(t)]\mathrm{d}x \\
&\quad + \alpha \int_\Omega \mathrm{D}(B(t)\boldsymbol{v}) : [\mathrm{D}(B'(t)\boldsymbol{w}) \cdot A(t) + \mathrm{D}(B(t)\boldsymbol{w}) \cdot A'(t)]\mathrm{d}x \\
&\quad + \int_\Omega [\mathrm{D}(B'(t)\boldsymbol{v}) \cdot \boldsymbol{v} \cdot (B(t)\boldsymbol{w}) + \mathrm{D}(B(t)\boldsymbol{v}) \cdot \boldsymbol{v} \cdot (B'(t)\boldsymbol{w})]\mathrm{d}x \\
&\quad + \int_\Omega [\mathrm{D}(B'(t)\boldsymbol{v})\mathrm{D}T_t^{-1} - \mathrm{D}(B(t)\boldsymbol{v}) \cdot (\mathrm{D}T_t^{-1}(\mathrm{D}\boldsymbol{V}(t) \circ T_t)\mathrm{D}T_t^{-1})] \\
&\quad \cdot (\boldsymbol{g} \circ T_t) \cdot (\mathrm{D}T_t\boldsymbol{w})\mathrm{d}x \\
&\quad + \int_\Omega [\mathrm{D}(B(t)\boldsymbol{v})\mathrm{D}T_t^{-1}] \cdot \{[(\mathrm{D}\boldsymbol{g} \cdot \boldsymbol{V}(t)) \circ T_t) \cdot (\mathrm{D}T_t\boldsymbol{w}) + (\boldsymbol{g} \circ T_t) \\
&\quad \cdot [(\mathrm{D}\boldsymbol{V}(t) \circ T_t)\mathrm{D}T_t\boldsymbol{w}]\}\mathrm{d}x \\
&\quad + \int_\Omega [\mathrm{D}((\mathrm{D}\boldsymbol{g} \cdot \boldsymbol{V}(t)) \circ T_t) \cdot \boldsymbol{v} \cdot (B(t)\boldsymbol{w}) + \mathrm{D}(\boldsymbol{g} \circ T_t) \cdot \boldsymbol{v} \cdot (B'(t)\boldsymbol{w})]\mathrm{d}x \\
&\quad - \int_\Omega [{}^*\mathrm{D}\boldsymbol{V}(t)\boldsymbol{F} + \mathrm{D}\boldsymbol{F}\boldsymbol{V}(t)] \circ T_t\mathrm{D}T_t\boldsymbol{w}\,\mathrm{d}x,
\end{aligned} \tag{21}$$

where the notation

$$B'(t)\tau := \frac{\partial}{\partial t}[B(t)\tau] = [\mathrm{D}\boldsymbol{V}(t) \circ T_t - (\mathrm{div}\,\boldsymbol{V}(t) \circ T_t)\boldsymbol{I}]B(t)\boldsymbol{\tau}$$

and

$$A'(t) := \frac{\partial}{\partial t}A(t) = [\operatorname{div} \boldsymbol{V}(t) \circ T_t - \mathrm{D}T_t^{-1}\mathrm{D}\boldsymbol{V}(t) \circ T_t]A(t) - {}^*[\mathrm{D}T_t^{-1}\mathrm{D}\boldsymbol{V}(t) \circ T_t A(t)].$$

It is easy to check that the mapping $(t, \boldsymbol{v}) \mapsto \partial_t e(t, \boldsymbol{v})$ is weakly continuous from $[0, \varepsilon] \times H_0^1(\operatorname{div}, \Omega)$ to $H_0^1(\operatorname{div}, \Omega)'$.

We set $t = 0$, use $T_t|_{t=0} = I$ and $\boldsymbol{V}(t)|_{t=0} = \boldsymbol{V}$, then obtain

$$\begin{aligned}\langle \partial_t e(0, \boldsymbol{v}), \boldsymbol{w}\rangle = {} & \alpha \int_\Omega \mathrm{D}\boldsymbol{v} : \mathrm{D}\boldsymbol{w} \operatorname{div} \boldsymbol{V} \mathrm{d}x + \alpha \int_\Omega \mathrm{D}\boldsymbol{v} : [-\mathrm{D}(\operatorname{div} \boldsymbol{V}\boldsymbol{w}) + \mathrm{D}(\mathrm{D}\boldsymbol{V}\boldsymbol{w}) - \mathrm{D}\boldsymbol{w}\mathrm{D}\boldsymbol{V}]\mathrm{d}x \\ & + \alpha \int_\Omega \mathrm{D}\boldsymbol{w} : [-\mathrm{D}(\operatorname{div} \boldsymbol{V}\boldsymbol{v}) + \mathrm{D}(\mathrm{D}\boldsymbol{V}v) - \mathrm{D}v\mathrm{D}\boldsymbol{V}]\mathrm{d}x \\ & + \int_\Omega [\mathrm{D}(\mathrm{D}\boldsymbol{V}\boldsymbol{v} - \operatorname{div} \boldsymbol{V}\boldsymbol{v}) \cdot \boldsymbol{v} \cdot \boldsymbol{w} + \mathrm{D}\boldsymbol{v} \cdot v \cdot (\mathrm{D}\boldsymbol{V}\boldsymbol{w} - \operatorname{div} \boldsymbol{V}\boldsymbol{w})]\mathrm{d}x \\ & + \int_\Omega [\mathrm{D}(\mathrm{D}\boldsymbol{V}\boldsymbol{v} - \operatorname{div} \boldsymbol{V}\boldsymbol{v}) - \mathrm{D}\boldsymbol{v}\mathrm{D}\boldsymbol{V}] \cdot \boldsymbol{g} \cdot \boldsymbol{w} \mathrm{d}x + \int_\Omega \mathrm{D}\boldsymbol{v} \cdot (\mathrm{D}\boldsymbol{g}\boldsymbol{V} + {}^*\mathrm{D}\boldsymbol{V}\boldsymbol{g}) \cdot \boldsymbol{w} \,\mathrm{d}x \\ & + \int_\Omega [\mathrm{D}(\mathrm{D}\boldsymbol{g}\boldsymbol{V}) - \mathrm{D}\boldsymbol{g} \operatorname{div} \boldsymbol{V} + {}^*\mathrm{D}\boldsymbol{V}\mathrm{D}\boldsymbol{g}] \cdot \tilde{\boldsymbol{y}} \cdot \boldsymbol{w} \mathrm{d}x - \int_\Omega (\mathrm{D}\boldsymbol{F}\boldsymbol{V} + {}^*\mathrm{D}\boldsymbol{V}\boldsymbol{F}) \cdot \boldsymbol{w} \,\mathrm{d}x.\end{aligned} \tag{22}$$

To verify (ii), we follow the same steps described in Dziri [16] to find an identity satisfied by $\tilde{\boldsymbol{y}}^{t_1} - \tilde{\boldsymbol{y}}^{t_2}$ and prove that the solution $\tilde{\boldsymbol{y}}^t \in H_0^1(\operatorname{div}, \Omega)$ of the weak formulation

$$\langle e(t, \boldsymbol{v}), \boldsymbol{w}\rangle = 0, \quad \forall \boldsymbol{w} \in H_0^1(\operatorname{div}, \Omega)$$

is Lipschitz with respect to t.

It is easy to find that $\boldsymbol{v} \mapsto e(t, \boldsymbol{v})$ is differentiable, and the derivative of $e(t, \mathbf{v})$ with respect to $\mathbf{v}$ in the direction $\delta\mathbf{v}$ is

$$\begin{aligned}\langle \partial_v e(t, \boldsymbol{v}) \cdot \delta\boldsymbol{v}, \boldsymbol{w}\rangle = {} & \int_\Omega \alpha \mathrm{D}(B(t)\delta\boldsymbol{v}) : [\mathrm{D}(B(t)\boldsymbol{w})A(t)]\mathrm{d}x \\ & + \int_\Omega [\mathrm{D}(B(t)\delta\boldsymbol{v}) \cdot \boldsymbol{v} \cdot (B(t)\boldsymbol{w}) + \mathrm{D}(B(t)\boldsymbol{v}) \cdot \delta\boldsymbol{v} \cdot (B(t)\boldsymbol{w})]\mathrm{d}x \\ & + \int_\Omega \{[\mathrm{D}(B(t)\delta\boldsymbol{v})\mathrm{D}T_t^{-1}] \cdot (\boldsymbol{g} \circ T_t) \cdot (\mathrm{D}T_t\boldsymbol{w}) + \mathrm{D}(\boldsymbol{g} \circ T_t) \cdot \delta\boldsymbol{v} \cdot (B(t)\boldsymbol{w})\}\mathrm{d}x.\end{aligned}$$

Obviously, $\partial_v e(t, \mathbf{v})$ is continuous, and when we take $t = 0$,

$$\begin{aligned}\langle \partial_v e(0, \boldsymbol{v}) \cdot \delta\boldsymbol{v}, \boldsymbol{w}\rangle = \alpha \int_\Omega \mathrm{D}\delta\boldsymbol{v} : \mathrm{D}\boldsymbol{w} \,\mathrm{d}x + \int_\Omega \mathrm{D}(\delta\boldsymbol{v}) \cdot \boldsymbol{v} \cdot \boldsymbol{w} + \mathrm{D}\boldsymbol{v} \cdot \delta\boldsymbol{v} \cdot \boldsymbol{w} \\ + \mathrm{D}(\delta\boldsymbol{v}) \cdot \boldsymbol{g} \cdot \boldsymbol{w} + \mathrm{D}\boldsymbol{g} \cdot \delta\boldsymbol{v} \cdot \boldsymbol{w} \,\mathrm{d}x.\end{aligned} \tag{23}$$

Furthermore, the mapping $\delta\boldsymbol{v} \mapsto \partial_v e(0, \boldsymbol{v}) \cdot \delta\boldsymbol{v}$ is an isomorphism from $H_0^1(\operatorname{div}, \Omega)$ to its dual. Indeed, this result follows from the uniqueness and existence of the Navier–Stokes system, i.e. Theorem 2.2.

Finally, all the hypothesis are satisfied by (14), we can apply Theorem 3.2 to (14) and then use (22) and (23) to obtain (17). ■

3.2. Shape derivative

In this subsection, we will characterize the shape derivative $\tilde{\boldsymbol{y}}'$, i.e. the derivative of the slate $\tilde{\boldsymbol{y}}$ with respect to the shape of the domain.

THEOREM 3.4 *Assume that Ω is of class C^2, $\tilde{\boldsymbol{y}} \in H_0^1(\mathrm{div}, \Omega)$ solves the weak formulation* (10) *in Ω and $\tilde{\boldsymbol{y}}_t \in H_0^1(\mathrm{div}, \Omega_t)$ solves the perturbed weak formulation* (12) *in Ω_t, then the 'shape derivative'*

$$\tilde{\boldsymbol{y}}' := \lim_{t \to 0} \frac{\tilde{\boldsymbol{y}}_t - \tilde{\boldsymbol{y}}}{t}$$

exists and is characterized as the solution of

$$\begin{cases} -\alpha \Delta \tilde{\boldsymbol{y}}' + \mathrm{D}\tilde{\boldsymbol{y}}' \cdot \tilde{\boldsymbol{y}} + \mathrm{D}\tilde{\boldsymbol{y}} \cdot \tilde{\boldsymbol{y}}' + \mathrm{D}\tilde{\boldsymbol{y}}' \cdot \boldsymbol{g} + \mathrm{D}\boldsymbol{g} \cdot \tilde{\boldsymbol{y}}' + \nabla \boldsymbol{p}' = 0 & in\, \Omega \\ \mathrm{div}\, \tilde{\boldsymbol{y}}' = 0 & in\, \Omega \\ \tilde{\boldsymbol{y}}' = -(\mathrm{D}\tilde{\boldsymbol{y}} \cdot n) \boldsymbol{V}_n & on\, \Gamma \end{cases} \tag{24}$$

Proof We recall that $\tilde{\boldsymbol{y}}_t \in H_0^1(\mathrm{div}, \Omega_t)$ satisfies the following weak formulation

$$\int_{\Omega_t} (\alpha \mathrm{D}\tilde{\boldsymbol{y}}_t : \mathrm{D}\boldsymbol{w} + \mathrm{D}\tilde{\boldsymbol{y}}_t \cdot \tilde{\boldsymbol{y}}_t \cdot \boldsymbol{w} + \mathrm{D}\tilde{\boldsymbol{y}}_t \cdot \boldsymbol{g} \cdot \boldsymbol{w} + \mathrm{D}\boldsymbol{g} \cdot \tilde{\boldsymbol{y}}_t \cdot \boldsymbol{w}) \mathrm{d}x - \int_{\Omega_t} \boldsymbol{F} \cdot \boldsymbol{w} \, \mathrm{d}x = 0 \tag{25}$$

for any $\boldsymbol{w} \in H_0^1(\mathrm{div}, \Omega_t)$.

At first, we introduce the following Hadamard formula [2,4]

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} \boldsymbol{F}(t, x) \mathrm{d}x = \int_{\Omega_t} \frac{\partial F}{\partial t}(t, x) \mathrm{d}x + \int_{\partial \Omega_t} \boldsymbol{F}(t, x) \boldsymbol{V} \cdot n_t \, \mathrm{d}\Gamma_t, \tag{26}$$

for a sufficiently smooth functional $F : [0, \tau] \times \mathbb{R}^N \to \mathbb{R}$.

Now, we set a function $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)^N$ and $\mathrm{div}\, \varphi = 0$ in Ω. Obviously when t is sufficiently small, $\boldsymbol{\varphi}$ belongs to the Sobolev space $H_0^1(\mathrm{div}, \Omega_t)$. Hence, we can use (26) to differentiate (25) with $\boldsymbol{w} = \boldsymbol{\varphi}$,

$$\begin{aligned} &\int_{\Omega} (\alpha \mathrm{D}\tilde{\boldsymbol{y}}' : \mathrm{D}\boldsymbol{\varphi} + \mathrm{D}\tilde{\boldsymbol{y}}' \cdot \tilde{\boldsymbol{y}} \cdot \boldsymbol{\varphi} + \mathrm{D}\tilde{\boldsymbol{y}} \cdot \tilde{\boldsymbol{y}}' \cdot \boldsymbol{\varphi} + \mathrm{D}\tilde{\boldsymbol{y}}' \cdot \boldsymbol{g} \cdot \boldsymbol{\varphi} + \mathrm{D}\boldsymbol{g} \cdot \tilde{\boldsymbol{y}}' \cdot \boldsymbol{\varphi}) \mathrm{d}x \\ &+ \int_{\Gamma} (\alpha \mathrm{D}\tilde{\boldsymbol{y}} : \mathrm{D}\boldsymbol{\varphi} + \mathrm{D}\tilde{\boldsymbol{y}} \cdot \tilde{\boldsymbol{y}} \cdot \boldsymbol{\varphi} + \mathrm{D}\tilde{\boldsymbol{y}} \cdot \boldsymbol{g} \cdot \varphi + \mathrm{D}\boldsymbol{g} \cdot \tilde{\boldsymbol{y}} \cdot \boldsymbol{\varphi} - \boldsymbol{F} \cdot \boldsymbol{\varphi}) \boldsymbol{V}_n \, \mathrm{d}s = 0. \end{aligned}$$

Since $\boldsymbol{\varphi}$ has a compact support, the boundary integral vanishes. Using integration by parts for the first term in the distributed integral, we obtain

$$\int_{\Omega} (-\alpha \Delta \tilde{\boldsymbol{y}}' + \mathrm{D}\tilde{\boldsymbol{y}}' \cdot \tilde{\boldsymbol{y}} + \mathrm{D}\tilde{\boldsymbol{y}} \cdot \tilde{\boldsymbol{y}}' + \mathrm{D}\tilde{\boldsymbol{y}}' \cdot \boldsymbol{g} + \mathrm{D}\boldsymbol{g} \cdot \tilde{\boldsymbol{y}}') \cdot \boldsymbol{\varphi} \, \mathrm{d}x = 0. \tag{27}$$

Then, the exists some distribution $p' \in L^2(\Omega)$ such that

$$-\alpha \Delta \tilde{\boldsymbol{y}}' + \mathrm{D}\tilde{\boldsymbol{y}}' \cdot \tilde{\boldsymbol{y}} + \mathrm{D}\tilde{\boldsymbol{y}} \cdot \tilde{\boldsymbol{y}}' + \mathrm{D}\tilde{\boldsymbol{y}}' \cdot \boldsymbol{g} + \mathrm{D}\boldsymbol{g} \cdot \tilde{\boldsymbol{y}}' = -\nabla p'$$

in the distributional sense in Ω.

Now we recall that for each t, $\Psi_t^{-1}(\tilde{\boldsymbol{y}}_t)$ belongs to the Sobolev space $H_0^1(\mathrm{div}, \Omega)$, then we can deduce that its material derivative vanishes on the boundary Γ. Thuswe obtain the shape derivative of $\tilde{\boldsymbol{y}}$ at the boundary,

$$\tilde{\boldsymbol{y}}' = -\mathrm{D}\tilde{\boldsymbol{y}} \cdot \boldsymbol{V}, \quad \text{on } \Gamma$$

Since $\tilde{\boldsymbol{y}}|_\Gamma = 0$, we have $\mathrm{D}\tilde{\boldsymbol{y}}\big|_\Gamma = \mathrm{D}\tilde{\boldsymbol{y}} \cdot \boldsymbol{n} * \boldsymbol{n}$, and then

$$\tilde{\boldsymbol{y}}' = -(\mathrm{D}\tilde{\boldsymbol{y}} \cdot \boldsymbol{n}) V_n \quad \text{on } \Gamma. \qquad \blacksquare$$

Remark 1 Notice that in Theorem 3.4, the pressure p' is the shape derivative of the pressure p_t which was defined on Ω_t.

The shape derivative $\boldsymbol{y}'$ of the solution $\boldsymbol{y}$ of the original Navier–Stokes system (4) is given by $\tilde{\boldsymbol{y}}' = \boldsymbol{y}'$, then we obtain the following corollary by substituting $\tilde{\boldsymbol{y}}' = \boldsymbol{y}'$ and $\tilde{\boldsymbol{y}} = \boldsymbol{y} - \boldsymbol{g}$ into (24).

COROLLARY 3.5 *The shape derivative $\boldsymbol{y}'$ of the solution $\boldsymbol{y}$ of* (4) *exists and satisfies the following system*

$$\begin{cases} -\alpha\Delta\boldsymbol{y}' + \mathrm{D}\boldsymbol{y}' \cdot \boldsymbol{y} + \mathrm{D}\boldsymbol{y} \cdot \boldsymbol{y}' + \nabla p' = 0 & \text{in } \Omega; \\ \mathrm{div}\, \boldsymbol{y}' = 0 & \text{in } \Omega; \\ \boldsymbol{y}' = (\mathrm{D}(\boldsymbol{g} - \boldsymbol{y}) \cdot n) V_n & \text{on } \Gamma. \end{cases} \tag{28}$$

Moreover, we have $\boldsymbol{y}' \in H^1(\mathrm{div}, \Omega)$.

3.3. *Adjoint state system and gradients of the cost functionals*

This subsection is devoted to the computation of the shape gradients for the cost functionals $J_1(\Omega)$ and $J_2(\Omega)$ by the adjoint method.

For the cost functional $J_1(\Omega) = \int_\Omega \frac{1}{2}\left|\boldsymbol{y} - \boldsymbol{y}_d\right|^2 \mathrm{d}x$, we have

THEOREM 3.6 *Let Ω be of class C^2 and the velocity $\boldsymbol{V} \in \mathrm{E}^2$, the shape gradient ∇J_1 of the cost functional $J_1(\Omega)$ can be expressed as*

$$\nabla J_1 = \left[\frac{1}{2}(\boldsymbol{y} - \boldsymbol{y}_d)^2 + \alpha(\mathrm{D}(\boldsymbol{y} - \boldsymbol{g}) \cdot \boldsymbol{n}) \cdot (\mathrm{D}\boldsymbol{v} \cdot \boldsymbol{n})\right]\boldsymbol{n}, \tag{29}$$

where the adjoint state $\boldsymbol{v} \in H_0^1(\mathrm{div}, \Omega)$ satisfies the following linear adjoint system

$$\begin{cases} -\alpha\Delta\boldsymbol{v} - \mathrm{D}\boldsymbol{v} \cdot \boldsymbol{y} + {}^*\mathrm{D}\boldsymbol{y} \cdot \boldsymbol{v} + \nabla \boldsymbol{q} = \boldsymbol{y} - \boldsymbol{y}_d, & \text{in } \Omega \\ \mathrm{div}\, \boldsymbol{v} = 0, & \text{in } \Omega \\ \boldsymbol{v} = 0, & \text{on } \Gamma. \end{cases} \tag{30}$$

Proof Since $J_1(\Omega)$ is differentiable with respect to $\boldsymbol{y}$, and the state $\boldsymbol{y}$ is shape differentiable with respect to t, i.e. the shape derivative $\boldsymbol{y}'$ exists, we obtain Eulerian derivative of $J_1(\Omega)$ with respect to t,

$$\mathrm{d}J_1(\Omega; \boldsymbol{V}) = \int_\Omega (\boldsymbol{y} - \boldsymbol{y}_d) \cdot \boldsymbol{y}' \mathrm{d}x + \int_\Gamma \frac{1}{2}\left|\boldsymbol{y} - \boldsymbol{y}_d\right|^2 V_n \,\mathrm{d}s. \tag{31}$$

by Hadamard formula (26).

By Green formula, we have the following identity

$$\begin{aligned}&\int_\Omega [(-\alpha\Delta\boldsymbol{y}' + \mathrm{D}\boldsymbol{y}'\cdot\boldsymbol{y} + \mathrm{D}\boldsymbol{y}\cdot\boldsymbol{y}' + \nabla p')\cdot\boldsymbol{w} - \operatorname{div}\boldsymbol{y}'\pi]\mathrm{d}x\\ &= \int_\Omega [(-\alpha\Delta\boldsymbol{w} - \mathrm{D}\boldsymbol{w}\cdot\boldsymbol{y} + {}^*\mathrm{D}\boldsymbol{y}\cdot\boldsymbol{w} + \nabla\pi)\cdot\boldsymbol{y}' - p'\operatorname{div}\boldsymbol{w}]\mathrm{d}x\\ &\quad + \int_\Gamma (\boldsymbol{y}'\cdot\boldsymbol{w})(\boldsymbol{y}\cdot\boldsymbol{n})\mathrm{d}s + \int_\Gamma (\alpha\mathrm{D}\boldsymbol{w}\cdot\boldsymbol{n} - \pi\boldsymbol{n})\cdot\boldsymbol{y}'\mathrm{d}s + \int_\Gamma (p'\boldsymbol{n} - \alpha\mathrm{D}\boldsymbol{y}'\boldsymbol{n})\cdot\boldsymbol{w}\,\mathrm{d}s. \end{aligned} \tag{32}$$

Now we define $(\boldsymbol{v}, q) \in H_0^1(\operatorname{div}, \Omega) \times L^2(\Omega)$ to be the solution of (30), use (28) and set $(\boldsymbol{w}, \pi) = (\boldsymbol{v}, q)$ in (32) to obtain

$$\int_\Omega (\boldsymbol{y} - \boldsymbol{y}_d)\cdot\boldsymbol{y}'\mathrm{d}x = -\int_\Gamma (\alpha\mathrm{D}\boldsymbol{v}\cdot\boldsymbol{n} - q\boldsymbol{n})\cdot\boldsymbol{y}'\mathrm{d}s. \tag{33}$$

Since $\boldsymbol{y}' = (\mathrm{D}(\boldsymbol{g} - \boldsymbol{y})\cdot\boldsymbol{n})\boldsymbol{V}_n$ on the boundary Γ and $\operatorname{div}\boldsymbol{y}' = 0$ in Ω, we obtain the Eulerian derivative of $J_1(\Omega)$ from (31),

$$\mathrm{d}J_1(\Omega; \boldsymbol{V}) = \int_\Gamma \left[\frac{1}{2}\left|\boldsymbol{y} - \boldsymbol{y}_d\right|^2 + \alpha(\mathrm{D}(\boldsymbol{y} - \boldsymbol{g})\cdot\boldsymbol{n})\cdot(\mathrm{D}\boldsymbol{v}\cdot\boldsymbol{n})\right] \boldsymbol{V}_n\mathrm{d}s. \tag{34}$$

Since the mapping $\boldsymbol{V} \mapsto \mathrm{d}J_1(\Omega; \boldsymbol{V})$ is linear and continuous, we get the expression (29) for the shape gradient ∇J_1 by (3). ∎

For another typical cost functional $J_2(\Omega) = \alpha\int_\Omega \left|\mathrm{D}\boldsymbol{y}\right|^2\mathrm{d}x$, we have the following theorem.

THEOREM 3.7 *Let Ω be of class C^2 and the velocity $\boldsymbol{V} \in \mathrm{E}^2$, the cost functional $J_2(\Omega)$ possesses the shape gradient ∇J_2 which can be expressed as*

$$\nabla J_2 = [\alpha\left|\mathrm{D}\boldsymbol{y}\right|^2 - \alpha(\mathrm{D}(\boldsymbol{y} - \boldsymbol{g})\cdot\boldsymbol{n})\cdot(2\mathrm{D}\boldsymbol{y}\cdot\boldsymbol{n} - \mathrm{D}\boldsymbol{v}\cdot\boldsymbol{n})]\,\boldsymbol{n}, \tag{35}$$

where the adjoint state $\boldsymbol{v} \in H_0^1(\operatorname{div}, \Omega)$ satisfies the following linear adjoint system

$$\begin{cases} -\alpha\Delta\boldsymbol{v} - \mathrm{D}\boldsymbol{v}\cdot\boldsymbol{y} + {}^*\mathrm{D}\boldsymbol{y}\cdot\boldsymbol{v} + \nabla q = -2\alpha\Delta\boldsymbol{y}, & \text{in } \Omega\\ \operatorname{div}\boldsymbol{v} = 0, & \text{in } \Omega\\ \boldsymbol{v} = 0, & \text{on } \Gamma. \end{cases} \tag{36}$$

Proof The proof is similar to that of Theorem 3.6. Using Hadamard formula (26) for the cost functional J_2, we obtain the Eulerian derivative

$$\mathrm{d}J_2(\Omega; \boldsymbol{V}) = 2\alpha\int_\Omega \mathrm{D}\boldsymbol{y} : \mathrm{D}\boldsymbol{y}'\mathrm{d}x + \int_\Gamma \alpha\left|\mathrm{D}\boldsymbol{y}\right|^2 \boldsymbol{V}_n\,\mathrm{d}s \tag{37}$$

Then, we define $(\boldsymbol{v}, q) \in H_0^1(\operatorname{div}, \Omega) \times L^2(\Omega)$ to be the solution of (36), use (28) and set $(\boldsymbol{w}, \pi) = (\boldsymbol{v}, q)$ in (32) to obtain

$$2\alpha\int_\Omega \Delta\boldsymbol{y}\cdot\boldsymbol{y}'\mathrm{d}x = \int_\Gamma (\alpha\mathrm{D}\boldsymbol{v}\cdot\boldsymbol{n} - q\boldsymbol{n})\cdot\boldsymbol{y}'\mathrm{d}s. \tag{38}$$

Using Green formula for the domain integral, we obtain

$$2\alpha\int_\Omega \mathrm{D}\boldsymbol{y} : \mathrm{D}\boldsymbol{y}'\,\mathrm{d}x = 2\alpha\int_\Gamma (\mathrm{D}\boldsymbol{y}\cdot\boldsymbol{n})\cdot\boldsymbol{y}'\mathrm{d}s - \int_\Gamma (\alpha\mathrm{D}\boldsymbol{v}\cdot\boldsymbol{n} - q\boldsymbol{n})\cdot\boldsymbol{y}'\,\mathrm{d}s. \tag{39}$$

Combining (37), (38) with (39), we obtain the Eulerian derivative

$$\mathrm{d}J_2(\Omega;\, \boldsymbol{V}) = \int_\Gamma \alpha\Big[\big|\mathrm{D}\boldsymbol{y}\big|^2 - (\mathrm{D}(\boldsymbol{y}-\boldsymbol{g})\cdot\boldsymbol{n})\cdot(2\mathrm{D}\boldsymbol{y}\cdot\boldsymbol{n} - \mathrm{D}\boldsymbol{v}\cdot\boldsymbol{n})\Big]\boldsymbol{V}_n\,\mathrm{d}s. \tag{40}$$

Finally, we arrive at the expression (35) fur the shape gradient ∇J_2. ■

4. Function space pararmetrization and function space embedding

In this section, we restrict our study to the minimization problem (7), and problem (8) follows similarly. In section 3, we have used the local differentiability of the state with respect to the shape of the fluid domain and the associated adjoint system to derive the shape gradient of the given cost functional. However, we do not need to analyze the differentiability of the state in many cases. In this section we derive the structure of the shape gradient for the cost functional $J_1(\Omega) = \frac{1}{2}\int_\Omega \big|\boldsymbol{y}-\boldsymbol{y}_d\big|^2\mathrm{d}x$ by function space parametrization and function space embedding techniques in order to bypass the study of the state derivative.

4.1. *A saddle point formulation*

In this subsection, we shall describe how to build an appropriate Lagrange functional that the divergence condition and the nonhomogeneous Dirichlet boundary condition are taken into account.

We set $\boldsymbol{f} \in [H^1(\mathbb{R}^N)]^N$ and $\boldsymbol{g} \in \boldsymbol{H}^2(\mathrm{div}, \mathbb{R}^N)$, then introduce a Lagrange multiplier μ and a functional

$$L(\Omega, \boldsymbol{y}, p, \boldsymbol{v}, q, \boldsymbol{\mu}) = \int_\Omega [(\alpha\Delta\boldsymbol{y} - \mathrm{D}\boldsymbol{y}\cdot\boldsymbol{y} - \nabla p + \boldsymbol{f})\cdot\boldsymbol{v} + \mathrm{div}\,\boldsymbol{y}q]\mathrm{d}x + \int_\Gamma (\boldsymbol{y}-\boldsymbol{g})\cdot\boldsymbol{\mu}\,\mathrm{d}s \tag{41}$$

for $(\boldsymbol{y}, p) \in Y(\Omega)\times Q(\Omega)$, $(\boldsymbol{v}, q) \in P(\Omega)\times Q(\Omega)$, and $\boldsymbol{\mu} \in H^{-1/2}(\Gamma)^N$ with

$$Y(\Omega) := H^2(\Omega)^N; \quad P(\Omega) := H^2(\Omega)^N \cap H_0^1(\Omega)^N; \quad Q(\Omega) := H^1(\Omega).$$

Now we're interested in the following saddle point problem

$$\inf_{(\boldsymbol{y},p)\in Y(\Omega)\times Q(\Omega)} \quad \sup_{(\boldsymbol{v},q,\boldsymbol{\mu})\in P(\Omega)\times Q(\Omega)\times H^{-1/2}(\Gamma)^N} L(\Omega, \boldsymbol{y}, p, \boldsymbol{v}, q, \boldsymbol{\mu}).$$

The solution is characterized by the following systems:

(i) The state $(\boldsymbol{y}, p)$ is the solution of the problem

$$\begin{cases} -\alpha\Delta\boldsymbol{y} + \mathrm{D}\boldsymbol{y}\cdot\boldsymbol{y} + \nabla p = \boldsymbol{f} & \text{in } \Omega \\ \mathrm{div}\,\boldsymbol{y} = 0 & \text{in } \Omega \\ \boldsymbol{y} = \boldsymbol{g} & \text{on } \Gamma \end{cases} \tag{42}$$

(ii) The adjoint state $(\boldsymbol{v}, q)$ is the solution of the problem

$$\begin{cases} -\alpha\Delta\boldsymbol{v} - \mathrm{D}\boldsymbol{v}\cdot\boldsymbol{y} + {}^*\mathrm{D}\boldsymbol{y}\cdot\boldsymbol{u} + \nabla q = 0 & \text{in } \Omega \\ \mathrm{div}\,\boldsymbol{v} = 0 & \text{in } \Omega \\ \boldsymbol{v} = 0 & \text{on } \Gamma; \end{cases} \tag{43}$$

(iii) The multiplier: $\mu = \alpha\mathrm{D}\boldsymbol{v}\,\boldsymbol{n} - q\boldsymbol{n}$, on Γ.

Hence we obtain the following new functional,

$$L(\Omega, \boldsymbol{y}, p, \boldsymbol{v}, q) = \int_\Omega [(\alpha\Delta\boldsymbol{y} - \mathrm{D}\boldsymbol{y}\cdot\boldsymbol{y} - \nabla p + \boldsymbol{f})\cdot\boldsymbol{v} + \operatorname{div}\boldsymbol{y}q]\mathrm{d}x + \int_\Gamma (\boldsymbol{y}-\boldsymbol{g})\cdot(\alpha\mathrm{D}\boldsymbol{v}\cdot\boldsymbol{n} - q\boldsymbol{n})\mathrm{d}s.$$

To get rid of the boundary integral, the following identities are derived by Green formula,

$$\int_\Gamma (\boldsymbol{y}-\boldsymbol{g})\cdot(\mathrm{D}\boldsymbol{v}\cdot\boldsymbol{n})\mathrm{d}s = \int_\Omega [(\boldsymbol{y}-\boldsymbol{g})\cdot\Delta\boldsymbol{v} + \mathrm{D}(\boldsymbol{y}-\boldsymbol{g}) : \mathrm{D}\boldsymbol{v}]\mathrm{d}x;$$

$$\int_\Gamma (\boldsymbol{y}-\boldsymbol{g})\cdot(q\boldsymbol{n})\mathrm{d}s = \int_\Omega [\operatorname{div}(\boldsymbol{y}-\boldsymbol{g})q + (\boldsymbol{y}-\boldsymbol{g})\cdot\nabla q]\mathrm{d}x.$$

Thus we introduce the new Lagrangian associated with (4) and the cost functional $J_1(\Omega) = \frac{1}{2}\int_\Omega |\boldsymbol{y}-\boldsymbol{y}_d|^2\mathrm{d}x$:

$$G(\Omega, \boldsymbol{y}, p, \boldsymbol{v}, q) = \frac{1}{2}\int_\Omega |\boldsymbol{y}-\boldsymbol{y}_d|^2\mathrm{d}x + \int_\Omega [(\alpha\Delta\boldsymbol{y} - \mathrm{D}\boldsymbol{y}\cdot\boldsymbol{y} - \nabla p + \boldsymbol{f})\cdot\boldsymbol{v} + \operatorname{div}\boldsymbol{y}q]\mathrm{d}x$$
$$+ \alpha\int_\Omega [(\boldsymbol{y}-\boldsymbol{g})\cdot\Delta\boldsymbol{v} + \mathrm{D}(\boldsymbol{y}-\boldsymbol{g}) : \mathrm{D}\boldsymbol{v}]\mathrm{d}x - \int_\Omega [\operatorname{div}(\boldsymbol{y}-\boldsymbol{g})q + (\boldsymbol{y}-\boldsymbol{g})\cdot\nabla q]\mathrm{d}x.$$

Now, the minimization problem (7) can be expressed as the following form

$$\min_{\Omega\in\mathcal{O}} \inf_{(\boldsymbol{y},p)\in Y(\Omega)\times P(\Omega)} \sup_{(\boldsymbol{v},q)\in P(\Omega)\times Q(\Omega)} G(\Omega, \boldsymbol{y}, p, \boldsymbol{v}, q).$$

We can use the minimax framework to avoid the study of the state derivative with respect to the shape of the domain. The Karusch–Kuhn–Tucker (KKT) conditions will furnish the shape gradient of the cost functional $J_1(\Omega)$ by using the adjoint system. To begin with, we derive the formulation of the adjoint system that was satisfied by $(\boldsymbol{v}, q)$.

For $(p, \boldsymbol{v}, q) \in Q(\Omega)\times P(\Omega)\times Q(\Omega)$, $G(\Omega, \boldsymbol{y}, p, \boldsymbol{v}, q)$ is differentiable with respect to $\boldsymbol{y}\in Y(\Omega)$ and we get

$$\partial \boldsymbol{y} G(\Omega, \boldsymbol{y}, p, \boldsymbol{v}, q)\cdot\delta\boldsymbol{y} = \int_\Omega (\boldsymbol{y}-\boldsymbol{y}_d)\cdot\delta\boldsymbol{y}\,\mathrm{d}x + \int_\Omega [\alpha\Delta(\delta\boldsymbol{y}) - \mathrm{D}(\delta\boldsymbol{y})\cdot\boldsymbol{y} - \mathrm{D}\boldsymbol{y}\cdot\delta\boldsymbol{y}]\cdot\boldsymbol{v}\,\mathrm{d}x$$
$$+ \alpha\int_\Omega [\delta\boldsymbol{y}\cdot\Delta\boldsymbol{v} + \mathrm{D}(\delta\boldsymbol{y}) : \mathrm{D}\boldsymbol{v}]\mathrm{d}x - \int_\Omega \delta\boldsymbol{y}\cdot\nabla q\,\mathrm{d}x, \quad \forall\delta\boldsymbol{y}\in Y(\Omega).$$

Integrating by parts, we obtain

$$\partial \boldsymbol{y} G(\Omega, \boldsymbol{y}, p, \boldsymbol{v}, q)\cdot\delta\boldsymbol{y} = \int_\Omega (\alpha\Delta\boldsymbol{v} + \mathrm{D}\boldsymbol{v}\cdot\boldsymbol{y} -{}^{*}\mathrm{D}\boldsymbol{y}\cdot\boldsymbol{v} - \nabla q + \boldsymbol{y} - \boldsymbol{y}_d)\cdot\delta\boldsymbol{y}\,\mathrm{d}x. \tag{44}$$

Similarly for $(\boldsymbol{y}, \boldsymbol{v}, q)\in Y(\Omega)\times P(\Omega)\times Q(\Omega)$, $G(\Omega, \boldsymbol{y}, p, \boldsymbol{v}, q)$ is differentiable with respect to $p\in Q(\Omega)$, and we have

$$\partial_p G(\Omega, \boldsymbol{y}, p, \boldsymbol{v}, q)\cdot\delta p = \int_\Omega -\nabla(\delta p)\cdot\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \delta p\operatorname{div}\boldsymbol{v}\,\mathrm{d}x, \quad \forall\delta p\in Q(\Omega). \tag{45}$$

Hence, (44) and (45) lead to the following linear adjoint system

$$\begin{cases} -\alpha\Delta \boldsymbol{v} - \mathrm{D}\boldsymbol{v}\cdot\boldsymbol{y} + {}^*\mathrm{D}\boldsymbol{y}\cdot\boldsymbol{v} + \nabla q = \boldsymbol{y} - \boldsymbol{y}_d & \text{in } \Omega \\ \operatorname{div} \boldsymbol{v} = 0 & \text{in } \Omega \\ \boldsymbol{v} = 0 & \text{on } \Gamma; \end{cases} \tag{46}$$

Given a velocity field $\boldsymbol{V} \in E^2$ and transformed domain $\Omega_t := T_t(\Omega)$, our main task of this section is to get the limit

$$\lim_{t\searrow 0}\frac{j(t)-j(0)}{t} \tag{47}$$

with

$$j(t) = \inf_{(\boldsymbol{y}_t, p_t)\in Y(\Omega_t)\times Q(\Omega_t)} \sup_{(\boldsymbol{v}_t, q_t)\in P(\Omega_t)\times Q(\Omega_t)} G(\Omega_t, \boldsymbol{y}_t, p_t, \boldsymbol{v}_t, q_t), \tag{48}$$

where $(\boldsymbol{y}_t, p_t)$ satisfies

$$\begin{cases} -\alpha\Delta \boldsymbol{y}_t + \mathrm{D}\boldsymbol{y}_t\cdot\boldsymbol{y}_t + \nabla p_t = \boldsymbol{f} & \text{in } \Omega_t \\ \operatorname{div} \boldsymbol{y}_t = 0 & \text{in } \Omega_t \\ \boldsymbol{y}_t = \boldsymbol{g} & \text{on } \Gamma_t \end{cases} \tag{49}$$

and $(\mathbf{v}_t, q_t)$ satisfies

$$\begin{cases} -\alpha\Delta \boldsymbol{v}_t - \mathrm{D}\boldsymbol{v}_t\cdot\boldsymbol{y}_t + {}^*\mathrm{D}\boldsymbol{y}_t\cdot\boldsymbol{v}_t + \nabla q_t = \boldsymbol{y}_t - \boldsymbol{y}_d & \text{in } \Omega_t \\ \operatorname{div} \boldsymbol{v}_t = 0 & \text{in } \Omega_t \\ \boldsymbol{v}_t = 0 & \text{on } \Gamma_t. \end{cases} \tag{50}$$

Unfortunately, the Sobolev space $Y(\Omega_t)$, $Q(\Omega_t)$ and $P(\Omega_t)$ depend on the parameter t, so we need a theorem to differentiate a saddle point with respect to the parameter t, and there are two techniques to get rid of it:

- Function space parametrization technique;
- Function space embedding technique.

4.2. *Function space parametrization*

This subsection is devoted to the function space parametrization, which consists in transporting the different quantities (such as, a cost functional) defined on the variable domain Ω_t back into the reference domain Ω, which does not depend on the perturbation parameter t. Thus, we can use differential calculus since the functionals involved are defined in a fixed domain Ω with respect to the parameter t.

We parameterize the functions in $H^m(\Omega)^d$ by elements of $H^m(\Omega)^d$ through the transformation:

$$\boldsymbol{\varphi} \mapsto \boldsymbol{\varphi}\circ T_t^{-1}: \quad H^m(\Omega)^d \to H^m(\Omega_t)^d, \quad \text{integer } m \geq 0,$$

where d is the dimension of the function φ.

Note that since T_t and T_t^{-1} are diffeomorphisms, reference domain Ω_t (respectively, the boundary Γ) into the new domain Ω_t (respectively, the boundary Γ_t of Ω_t).

This parameterization cannot change the value of the saddle point. We can rewrite (48) as

$$j(t) = \inf_{(\boldsymbol{y},p)\in Y(\Omega)\times Q(\Omega)} \sup_{(\boldsymbol{v},q)\in P(\Omega)\times Q(\Omega)} G(\Omega_t, \boldsymbol{y}\circ T_t^{-1}, p\circ T_t^{-1}, \boldsymbol{v}\circ T_t^{-1}, q\circ T_t^{-1}). \tag{51}$$

It amounts to introducing the new Lagrangian for $(\boldsymbol{y}, p, \boldsymbol{v}, q) \in Y(\Omega)\times Q(\Omega)\times P(\Omega)\times Q(\Omega)$:

$$\tilde{G}(t, \boldsymbol{y}, p, \boldsymbol{v}, q) := G(\Omega_t, \boldsymbol{y}\circ T_t^{-1}, p\circ T_t^{-1}\boldsymbol{v}\circ T_t^{-1}, q\circ T_t^{-1}).$$

The expression for $\tilde{G}(t, \boldsymbol{y}, p, \boldsymbol{v}, q)$ is given by

$$\tilde{G}(t, \boldsymbol{y}, p, \boldsymbol{v}, q) := I_1(t) + I_2(t) + I_3(t) + I_4(t), \tag{52}$$

where

$$\begin{aligned}
I_1(t) &:= \frac{1}{2}\int_{\Omega_t} \left|\boldsymbol{y}\circ T_t^{-1} - \boldsymbol{y}_d\right|^2 \mathrm{d}x;\\
I_2(t) &:= \int_{\Omega_t} [(\alpha\Delta(\boldsymbol{y}\circ T_t^{-1}) - \mathrm{D}(\boldsymbol{y}\circ T_t^{-1})\cdot(\boldsymbol{y}\circ T_t^{-1}) - \nabla(p\circ T_t^{-1}) + \boldsymbol{f}]\cdot(\boldsymbol{v}\circ T_t^{-1})\\
&\qquad + \mathrm{div}(\boldsymbol{y}\circ T_t^{-1})(q\circ T_t^{-1})\mathrm{d}x;\\
I_3(t) &:= \alpha\int_{\Omega_t} [(\boldsymbol{y}\circ T_t^{-1} - \boldsymbol{g})\cdot\Delta(\boldsymbol{v}\circ T_t^{-1}) + \mathrm{D}(\boldsymbol{y}\circ T_t^{-1} - \boldsymbol{g}) :\ \mathrm{D}(\boldsymbol{v}\circ T_t^{-1})]\mathrm{d}x;\\
I_4(t) &:= -\int_{\Omega_t} [\mathrm{div}(\boldsymbol{y}\circ T_t^{-1} - \boldsymbol{g})(q\circ T_t^{-1}) + (\boldsymbol{y}\circ T_t^{-1} - \boldsymbol{g})\cdot\nabla(q\circ T_t^{-1})]\mathrm{d}x.
\end{aligned}$$

Now, we introduce the theorem concerning on the differentiability of a saddle point (or a minimax). First, some notations are given as follows.

Define a functional $\mathcal{G} : [0, \tau]\times X\times Y \to \mathbb{R}$ with $\tau > 0$, and X, Y are the two topological spaces. For any $t \in [0, \tau]$, define, $g(t) = \inf_{x\in X}\sup_{y\in Y}\mathcal{G}(t, x, y)$ and the sets

$$X(t) = \{x^t \in X \colon g(t) = \sup_{y\in Y}\mathcal{G}(t, x^t, y)\}$$

$$Y(t, x) = \{y^t \in Y \colon \mathcal{G}(t, x, y^t) = \sup_{y\in Y}\mathcal{G}(t, x, y)\}$$

Similarly, we can define dual functionals

$$h(t) = \sup_{y\in Y}\inf_{x\in X}\mathcal{G}(t, x, y)$$

and the corresponding sets

$$Y(t) = \{y^t \in Y : h(t) = \inf\nolimits_{x\in X}\mathcal{G}(t, x, y^t)\}$$
$$X(t, y) = \{x^t \in X : \mathcal{G}(t, x^t, y) = \inf\nolimits_{x\in X}\mathcal{G}(t, x, y)\}$$

Furthermore, we introduce the set of saddle points

$$S(t) = \{(x, y) \in X\times Y : g(t) = \mathcal{G}(t, x, y) = h(t)\}$$

Now we can introduce the following theorem (see [17] or page 427 of [2]):

THEOREM 4.1 *Assume that the following hypothesis hold*:

($H1$) $S(t) \neq \emptyset,\ t \in [0, \tau]$;
($H2$) *The partial derivative* $\partial_t \mathcal{G}(t, x, y)$ *exists in* $[0, \tau]$ *for all*

$$(x, y) \in \left[\bigcup_{t \in [0, \tau]} X(t) \times Y(0) \right] \bigcup \left[X(0) \times \bigcup_{t \in [0, \tau]} Y(t) \right];$$

($H3$) *There exists a topology* $\mathcal{T}_X$ *on* X *such that for any sequence* $\{t_n \colon t_n \in [0, \tau]\}$ *with* $\lim_{n \nearrow \infty} t_n = 0$, *there exists* $x^0 \in X(0)$ *and a subsequence* $\{t_{n_k}\}$, *and for each* $k \geq 1$, *there exists* $x_{n_k} \in X(t_{n_k})$ *such that*

(i) $\lim_{n \nearrow \infty} x_{n_k} = x^0$ *in the* $\mathcal{T}_X$ *topology*,
(ii) $\liminf_{\substack{t \searrow 0 \\ k \nearrow \infty}} \partial_t \mathcal{G}(t, x_{n_k}, y) \geq \partial_t \mathcal{G}(0, x^0, y),\ \forall y \in Y(0)$;

($H4$) *There exists a topology* $\mathcal{T}_Y$ *on* Y *such that for any sequence* $\{t_n \colon t_n \in [0, \tau]\}$ *with* $\lim_{n \nearrow \infty} t_n = 0$, *there exists* $y^0 \in Y(0)$ *and a subsequence* $\{t_{n_k}\}$, *and for each* $k \geq 1$, *there exists* $y_{n_k} \in Y(t_{n_k})$ *such that*

(i) $\lim_{n \nearrow \infty} y_{n_k} = y^0$ *in the* $\mathcal{T}_Y$ *topology*,
(ii) $\limsup_{\substack{t \searrow 0 \\ k \nearrow \infty}} \partial_t \mathcal{G}(t, x, y_{n_k}) \leq \partial_t \mathcal{G}(0, x, y^0),\ \forall x \in X(0)$.

Then, there exists $(x^0, y^0) \in X(0) \times Y(0)$ *such that*

$$\begin{aligned} \mathrm{d}g(0) &= \lim_{t \searrow 0} \frac{g(t) - g(0)}{t} \\ &= \inf_{x \in X(0)} \sup_{y \in Y(0)} \partial_t \mathcal{G}(0, x, y) = \partial_t \mathcal{G}(0, x^0 y^0) = \sup_{y \in Y(0)} \inf_{x \in X(0)} \partial_t \mathcal{G}(0, x, y). \end{aligned} \tag{53}$$

This means that $(x^0, y^0) \in X(0) \times Y(0)$ *is a saddle point of* $\partial_t \mathcal{G}(0, x, y)$.

Following Theorem 4.1, we need to differentiate the perturbed Lagrange functional $\tilde{G}(t, \boldsymbol{y}, p, \boldsymbol{v}, \boldsymbol{q})$. Since $(\boldsymbol{y}, p, \boldsymbol{v}, q) \in H^3(\Omega)^N \times H^2(\Omega) \times H^3(\Omega)^N \times H^2(\Omega)$ provided that Γ is at least C^3 [14], we can use Hadamard formula (26) to differentiate $\tilde{G}(t, \boldsymbol{y}, p, \boldsymbol{v}, q)$ with respect to the parameter $t > 0$,

$$\partial_t \tilde{G}(t, \boldsymbol{y}, p, \boldsymbol{v}, q) = I_1'(0) + I_2'(0) + I_3'(0) + I_4'(0),$$

where

$$I_1'(0) := \frac{\partial}{\partial t} \{I_1(t)\}|_{t=0} = \int_\Omega (\boldsymbol{y} - \boldsymbol{y}_d) \cdot (-D\boldsymbol{y}\boldsymbol{V}) \mathrm{d}x + \frac{1}{2} \int_\Gamma |\boldsymbol{y} - \boldsymbol{y}_d|^2 \boldsymbol{V}_n \, \mathrm{d}s; \tag{54}$$

$$\begin{aligned} I_2'(0) := \frac{\partial}{\partial t} \{I_2(t)\}|_{t=0} &= \int_\Omega (\alpha \Delta \boldsymbol{y} - \mathrm{D}\boldsymbol{y} \cdot \boldsymbol{y} - \nabla p + \boldsymbol{f}) \cdot (-\mathrm{D}\boldsymbol{v}\boldsymbol{V}) \mathrm{d}x \\ &+ \int_\Omega [\alpha \Delta(-\mathrm{D}\boldsymbol{y}\boldsymbol{V}) - \mathrm{D}(-\mathrm{D}\boldsymbol{y}\boldsymbol{V}) \cdot \boldsymbol{y} - \mathrm{D}\boldsymbol{y} \cdot (-\mathrm{D}\boldsymbol{y}\boldsymbol{V}) + \nabla(\nabla p \cdot \boldsymbol{V})] \cdot \boldsymbol{v} \, \mathrm{d}x \\ &+ \int_\Omega [\mathrm{div}(-\mathrm{D}\boldsymbol{y}\boldsymbol{V}) q + \mathrm{div}\, \boldsymbol{y}(-\nabla q \cdot \boldsymbol{V})] \mathrm{d}x \\ &+ \int_\Gamma [\alpha \Delta \boldsymbol{y} - \mathrm{D}\boldsymbol{y} \cdot \boldsymbol{y} - \nabla p + \boldsymbol{f}) \cdot \boldsymbol{v} + \mathrm{div}\, \boldsymbol{y}\, q] \boldsymbol{V}_n \, \mathrm{d}s; \end{aligned} \tag{55}$$

$$I_3'(0) := \frac{\partial}{\partial t}\{I_3(t)\}|_{t=0} = \alpha \int_\Omega \{(-\mathrm{D}\boldsymbol{y}\boldsymbol{V}) \cdot \Delta \boldsymbol{v} + (\boldsymbol{y}-\boldsymbol{g}) \cdot \Delta(-\mathrm{D}\boldsymbol{v}\boldsymbol{V}) + \mathrm{D}(-\mathrm{D}\boldsymbol{y}\boldsymbol{V}) : \mathrm{D}\boldsymbol{v} + \mathrm{D}(\boldsymbol{y}-\boldsymbol{g}) : \mathrm{D}(-\mathrm{D}\boldsymbol{v}\boldsymbol{V})\}\,\mathrm{d}x + \alpha \int_\Gamma [(\boldsymbol{y}-\boldsymbol{g}) \cdot \Delta \boldsymbol{v}\rangle + \mathrm{D}(\boldsymbol{y}-\boldsymbol{g}) : \mathrm{D}\boldsymbol{v}]\, V_n\,\mathrm{d}s; \tag{56}$$

$$I_4'(0) := \frac{\partial}{\partial t}\{I_4(t)\}|_{t=0} = -\int_\Omega [\mathrm{div}(-\mathrm{D}\boldsymbol{y}\boldsymbol{V})q + \mathrm{div}(\boldsymbol{y}-\boldsymbol{g})(-\nabla q \cdot \boldsymbol{V}) + (-\mathrm{D}\boldsymbol{y}\boldsymbol{V}) \cdot \nabla q - (\boldsymbol{y}-\boldsymbol{g}) \cdot \nabla(\nabla q \cdot \boldsymbol{V})]\mathrm{d}x - \int_\Gamma [\mathrm{div}(\boldsymbol{y}-\boldsymbol{g})q + (\boldsymbol{y}-\boldsymbol{g}) \cdot \nabla q]\boldsymbol{V}_n\,\mathrm{d}s. \tag{57}$$

Since $(\boldsymbol{y}, p)$ satisfies (4), $\mathrm{div}\,\boldsymbol{v}=0$ and $\boldsymbol{v}|_\Gamma = 0$, also by Green formula we can simplify (55) to

$$I_2'(0) = \int_\Omega [\alpha\Delta(-\mathrm{D}\boldsymbol{y}\boldsymbol{V}) - \mathrm{D}(-\mathrm{D}\boldsymbol{y}\boldsymbol{V}) \cdot \boldsymbol{y} - \mathrm{D}\boldsymbol{y} \cdot (-\mathrm{D}\boldsymbol{y}\boldsymbol{V})] \cdot \boldsymbol{v}\,\mathrm{d}x + \int_\Omega \mathrm{div}(-\mathrm{D}\boldsymbol{y}\boldsymbol{V})q\,\mathrm{d}x. \tag{58}$$

By Green formula, $\boldsymbol{y}\big|_\Gamma = \boldsymbol{g}$ and $\boldsymbol{v}|_\Gamma = 0$, we can simplify (56) to

$$I_3'(0) = \alpha \int_\Omega (-\mathrm{D}\boldsymbol{y}\boldsymbol{V}) \cdot \Delta \boldsymbol{v}\,\mathrm{d}x - \alpha \int_\Omega \Delta(-\mathrm{D}\boldsymbol{y}\boldsymbol{V}) \cdot \boldsymbol{v}\,\mathrm{d}x + \alpha \int_\Gamma [\mathrm{D}(\boldsymbol{y}-\boldsymbol{g}) : \mathrm{D}\boldsymbol{v}]\boldsymbol{V}_n\,\mathrm{d}s. \tag{59}$$

By Green formula, $\boldsymbol{y}\big|_\Gamma = \boldsymbol{g}$ and div $\boldsymbol{y} = \mathrm{div}\,\boldsymbol{g} = 0$, (57) can be simplified to

$$I_4'(0) = -\int_\Omega [(-\mathrm{D}\boldsymbol{y}\boldsymbol{V}) \cdot \nabla q + \mathrm{div}(-\mathrm{D}\boldsymbol{y}\boldsymbol{V})q]\mathrm{d}x. \tag{60}$$

Adding (54), (58), (59) and (60) together,

$$\sum_{i=1}^{4} I_i'(0) = \int_\Gamma \left\{\frac{1}{2}|\boldsymbol{y}-\boldsymbol{y}_d|^2 + \alpha D(\boldsymbol{y}-\boldsymbol{g}) : D\boldsymbol{v}\right\} \boldsymbol{V} \cdot \boldsymbol{n}\,\mathrm{d}s - \int_\Omega [\alpha\Delta \boldsymbol{v} \cdot (D\boldsymbol{y}\boldsymbol{V}) - \mathrm{D}(\mathrm{D}\boldsymbol{y}\boldsymbol{V}) \cdot \boldsymbol{y} - \mathrm{D}\boldsymbol{y} \cdot (\mathrm{D}\boldsymbol{y}\boldsymbol{V}) - (\mathrm{D}\boldsymbol{y}\boldsymbol{V}) \cdot \nabla q + (\boldsymbol{y}-\boldsymbol{y}_d) \cdot (\mathrm{D}\boldsymbol{y}\boldsymbol{V})]\mathrm{d}x. \tag{61}$$

Since $(\boldsymbol{v}, q)$ are characterized by (46), we multiply the first equation of (46) by $(\mathrm{D}\boldsymbol{y}\boldsymbol{V})$ and integrate over Ω, then the distributional integral in (61) vanishes, finally we obtain the boundary expression for the Eulerian derivative

$$\mathrm{d}J_1(\Omega; \boldsymbol{V}) = \int_\Gamma \left\{\frac{1}{2}|\boldsymbol{y}-\boldsymbol{y}_d|^2 + \alpha\mathrm{D}(\boldsymbol{y}-\boldsymbol{g}) : \mathrm{D}\boldsymbol{v}\right\} \boldsymbol{V} \cdot \boldsymbol{n}\,\mathrm{d}s. \tag{62}$$

Since $\boldsymbol{y}\big|_\Gamma = \boldsymbol{g}$ and $\boldsymbol{v}|_\Gamma = 0$, we have $\mathrm{D}(\boldsymbol{y}-\boldsymbol{g})\big|_\Gamma = \mathrm{D}(\boldsymbol{y}-\boldsymbol{g})\ \boldsymbol{n} * \boldsymbol{n}$ and $\mathrm{D}\boldsymbol{v}|_\Gamma = \mathrm{D}\boldsymbol{v}\cdot\boldsymbol{n}*\boldsymbol{n}$, thus we obtain an expression for the shape gradient

$$\nabla J_1 = \left\{\frac{1}{2}\left|\boldsymbol{y}-\boldsymbol{y}_d\right|^2 + \alpha[\mathrm{D}(\boldsymbol{y}-\boldsymbol{g})\cdot\boldsymbol{n}]\cdot(\mathrm{D}\boldsymbol{v}\cdot\boldsymbol{n})\right\}\boldsymbol{n}, \tag{63}$$

which is the same as (29) in the previous section.

4.3. *Function space embedding*

In the previous subsection, we have used the technique of function space parametrization in order to get the derivative of $j(t)$, i.e.

$$j(t) = \inf_{(\boldsymbol{y}_t,p_t)\in Y(\Omega_t)\times Q(\Omega_t)}\ \sup_{(\boldsymbol{v}_t,q_t)\in P(\Omega_t)\times Q(\Omega_t)} G(\Omega_t,\boldsymbol{y}_t,p_t,\boldsymbol{v}_t,q_t), \tag{64}$$

with respect to the parameter $t > 0$. This subsection is devoted to a different method based on function space embedding technique. It means that the state and adjoint state are defined on a large enough domain D (called a hold-all [2]), which contains all the transformations $\{\Omega_t\colon 0 \le t \le \varepsilon\}$ of thp reference domain Ω for some small $\varepsilon > 0$.

For convenience, let $D = \mathbb{R}^N$. Use the function space embedding technique,

$$j(t) = \inf_{(\mathcal{Y},\mathcal{P})\in Y(\mathbb{R}^N)\times Q(\mathbb{R}^N)}\ \sup_{(\mathcal{V},\mathcal{Q})\in P(\mathbb{R}^N)\times Q(\mathbb{R}^N)} G(\Omega_t,\mathcal{Y},\mathcal{P},\mathcal{V},\mathcal{Q}), \tag{65}$$

where the new Lagrangian

$$\begin{aligned}G(\Omega_t,\mathcal{Y},\mathcal{P},V,\mathcal{Q}) = {} & \frac{1}{2}\int_{\Omega_t}\left|\mathcal{Y}-\boldsymbol{y}_d\right|^2\mathrm{d}x\\ & + \int_{\Omega_t}[(\alpha\Delta\mathcal{Y}-\mathrm{D}\mathcal{Y}\cdot\mathcal{Y}-\nabla\mathcal{P}+\boldsymbol{f})\cdot V + \operatorname{div}\mathcal{Y}\mathcal{Q}]\mathrm{d}x\\ & + \alpha\int_{\Omega_t}[(\mathcal{Y}-\boldsymbol{g})\cdot\Delta V + \mathrm{D}(\mathcal{Y}-\boldsymbol{g}) : \mathrm{D}V]\mathrm{d}x\\ & - \int_{\Omega_t}[\operatorname{div}(\mathcal{Y}-\boldsymbol{g})\mathcal{Q} + (\mathcal{Y}-\boldsymbol{g})\cdot\nabla\mathcal{Q}]\mathrm{d}x.\end{aligned} \tag{66}$$

Since $\boldsymbol{f}$, $\boldsymbol{y}_d \in H^1(\mathbb{R}^N)^N$, $\boldsymbol{g}\in H^2(\mathbb{R}^N)^N$, and Ω_t is of class C^3, the solution $(\boldsymbol{y}_t, p_t, v_t, q_t)$ of belongs to $H^3(\Omega_t)^N \times (H^2(\Omega_t)\cap L^2_0(\Omega_t)) \times (H^3(\Omega_t)^N \cap H^1_0(\Omega_t)^N) \times (H^2(\Omega_t)\cap L^2_0(\Omega_t))$ instead of $Y(\Omega_t)\times Q(\Omega_t)\times P(\Omega_t)\times Q(\Omega_t)$. Therefore, the sets

$$X = Y = H^3(\mathbb{R}^N)^N \times H^2(\mathbb{R}^N),$$

and the saddle points $S(t) = X(t)\times Y(t)$ are given by

$$X(t) = \left\{(\mathcal{Y},\mathcal{P})\in X : \mathcal{Y}\big|_{\Omega_t} = \boldsymbol{y}_t, \mathcal{P}\big|_{\Omega_t} = p_t\right\} \tag{67}$$

$$Y(t) = \left\{(\mathcal{V},\mathcal{Q})\in Y : \mathcal{V}\big|_{\Omega_t} = \boldsymbol{v}_t, \mathcal{Q}\big|_{\Omega_t} = q_t\right\} \tag{68}$$

Using Theorem 4,1, we may make the conjecture that we can bypass the inf–sup and state

$$\mathrm{d}J(\Omega;\mathcal{V}) = \inf_{(\mathcal{Y},\mathcal{P})\in X(0)}\ \sup_{(\mathcal{V},\mathcal{Q})\in Y(0)} \partial_t G(\Omega_t,\mathcal{Y},\mathcal{P},\mathcal{V},\mathcal{Q})\big|_{t=0}. \tag{69}$$

Now we compute the partial derivative of the expression (66) by Hadamard formula (26),

$$\partial_t G(\Omega_t, \mathcal{Y}, \mathcal{P}, \mathcal{V}, \mathcal{Q}) = \int_{\Gamma_t} [\mathcal{W}_1(\mathcal{Y}, \mathcal{V}) + \mathcal{W}_2(\mathcal{Y}, \mathcal{P}, \mathcal{V}, \mathcal{Q})]\mathcal{V} \cdot \boldsymbol{n}_t \, \mathrm{d}s \tag{70}$$

where

$$\mathcal{W}_1(\mathcal{Y}, \mathcal{V}) := \frac{1}{2}|\mathcal{Y} - \boldsymbol{y}_d|^2 + \alpha \mathrm{D}(\boldsymbol{Y} - \boldsymbol{g}) : \mathrm{D}\mathcal{V},$$

and

$$\begin{aligned}\mathcal{W}_2(\boldsymbol{Y}, \mathcal{P}, \mathcal{V}, \mathcal{Q}) := {} & (\alpha \Delta \mathcal{Y} - \mathrm{D}\mathcal{Y} \cdot \mathcal{Y} - \nabla \mathcal{P} + \boldsymbol{f}) \cdot \mathcal{V} \\ & + (\mathcal{Y} - \boldsymbol{g}) \cdot (\alpha \Delta \mathcal{V} - \nabla \mathcal{Q}) + \mathcal{Q} \operatorname{div} \boldsymbol{g}.\end{aligned}$$

and $\boldsymbol{n}_t$ denotes the outward unit normal to be boundary Γ_t.

We note that the expression (70) is a boundary integral on Γ_t, which will not depend on $(\mathcal{Y}, \mathcal{P})$ and $(\mathcal{V}, \mathcal{Q})$ outside of $\bar{\Omega}_t$ so the inf and the sup in (69) can be dropped, we then get

$$\mathrm{d}J_1(\Omega; \boldsymbol{V}) = \int_{\Gamma} [\mathcal{W}_1(\boldsymbol{y}, \boldsymbol{v}) + \mathcal{W}_2(\boldsymbol{y}, p, \boldsymbol{v}, q)] \boldsymbol{V}_n \, \mathrm{d}s$$

However, $(\boldsymbol{y} - \boldsymbol{g})|_\Gamma = \boldsymbol{v}|_\Gamma = 0$ and $\operatorname{div} \boldsymbol{g} = 0$ imply $\mathcal{W}_2(\boldsymbol{y}, p, \boldsymbol{v}, q) = 0$ on the boundary Γ. Finally we have

$$\mathrm{d}J_1(\Omega; \boldsymbol{V}) = \int_{\Gamma} \left\{ \frac{1}{2}|\boldsymbol{y} - \boldsymbol{y}_d|^2 + \alpha \mathrm{D}(\boldsymbol{y} - \boldsymbol{g}) : \mathrm{D}\boldsymbol{v} \right\} \boldsymbol{V}_n \mathrm{d}s$$

As in the previous subsection, we also have the shape gradient of the functional $\mathrm{J}_1(\Omega)$,

$$\nabla J_1 = \left\{ \frac{1}{2}|\boldsymbol{y} - \boldsymbol{y}_d|^2 + \alpha [\mathrm{D}(\boldsymbol{y} - \boldsymbol{g}) \cdot \boldsymbol{n}] \cdot (\mathrm{D}\boldsymbol{v} \cdot \boldsymbol{n}) \right\} \boldsymbol{n}, \tag{71}$$

which is the same as the expressions (29) and (63).

Before closing this section, we give the following remark about the previous three approaches.

Remark 1

(1) Three approaches can lead to the same expression of shape gradient.
(2) The state derivative approach is more classical and general, but it requires the differentiability of the state, i.e. the existence of the material derivative and shape derivative. But in many practical cases, we cannot obtain this differentiability. This is the key drawback of this approach. However, function space parametrization and function space embedding approaches can bypass the differentiability of the state, and we only to analyse the differentiability of the saddle point of the associated Lagrange formulation.
(3) Function space parametrization approach is very complicated to implement. We need the new shape calculus technique [2] to simplify the expression of shape gradient with very rigorous mathematical deduction.

(4) Function space embedding approach is the quickest and easiest one to obtain shape gradient of the cost functional. However, the associated Lagrange formulation in this approach must only consist of domain integral. As we all know, the cost functional or the variational formulation of the state system may contain a boundary integral. So, we cannot use function space embedding approach in this case.

5. Gradient algorithm and numerical simulation

In this section, we will give a gradient-type algorithm and some numerical examples in two dimensions to prove that our previous methods could be very useful and efficient for the numerical implementation of the shape optimization problems for Navier–Stokes flow.

5.1. *A gradient type algorithm*

In this subsection, we will describe a gradient type algorithm for the minimization of cost functional $J_i(\Omega)$ ($i = 1$ or 2). As we have just seen, the general form of its Eulerian derivative is

$$\mathrm{d}J_1(\Omega; \boldsymbol{V}) = \int_\Gamma \nabla J \cdot \boldsymbol{V}\,\mathrm{d}s,$$

where ∇J denotes the shape gradient of the cost functional J. Ignoring regularization, a descent direction is found by defining

$$\boldsymbol{V} = -h_k \nabla J, \tag{72}$$

and then we can update the shape Ω as

$$\Omega_k = (I + h_k \boldsymbol{V})\Omega, \tag{73}$$

where h_k is a descent step at k-th iteration.

There are also other choices for the definition of the descent direction. Since the gradient of the functional has necessarily less regularity than the parameter, an iterative scheme like the method of descent deteriortates the regularity of the optimized parameter. We need to project or smooth the variation into $H^1(\Omega)^2$. Hence, the method used in this article is to change the scalar product with respect to which we compute a descent direction, for instance, $H^1(\Omega)^2$. In this case, the descent direction is the unique element $\boldsymbol{d} \in H^1(\Omega)^2$ such that for every $\boldsymbol{V} \in H^1(\Omega)^2$,

$$\int_\Omega \mathrm{D}\boldsymbol{d} : \mathrm{D}\boldsymbol{V}\,\mathrm{d}x = -\mathrm{d}J(\Omega; \boldsymbol{V}), \tag{74}$$

or equivalently the following vectorial Poisson equation,

$$\begin{cases} -\Delta \boldsymbol{d} = 0 & \text{in } \Omega \\ \mathrm{D}\boldsymbol{d} \cdot \boldsymbol{n} = -\nabla J & \text{on } \Gamma \end{cases} \tag{75}$$

The computation of $\boldsymbol{d}$ can also be interpreted as a regularization of the shape gradient, and the choice of $\mathrm{H}^1(\Omega)^2$ as space of variations is more dictated by technical considerations rather than theoretical ones.

The resulting algorithm can be summarized as follows:

(1) Choose an initial shape Ω_0;
(2) Compute the state system and adjoint state system, then we can evaluate the descent direction $\boldsymbol{d}_k$ by using (74) with $\Omega = \Omega_k$;
(3) Set $\Omega_{k+1} = (\mathrm{Id} + h_k \boldsymbol{d}_k)\ \Omega_k$, where h_k is a small positive real number.

The choice of the descent step h_k is not an easy task. If it is very large, the algorithm is unstable; if it is too small, the rate of convergence is insignificant. In order to refresh h_k, we compare h_k with h_{k-1}. If $(\boldsymbol{d}_k, \boldsymbol{d}_{k-1})_{H^1}$ is negative, we should reduce the step; on the other hand, if $\boldsymbol{d}_k$ and $\boldsymbol{d}_{k-1}$ are very close, we increase the step. In addition, if reversed triangles are appeared when moving the mesh, we also need to reduce the step.

In our algorithm, we do not choose any stopping criterion. A classical stopping criterion is to find that whether the shape gradients ∇J in some suitable norm is small enough. However, since we use the continuous shape gradients, it's hopeless for us to expect very small gradient norm because of numerical discretization errors. Instead, we fix the number of iterations. If it is too small, we can restart it with the previous final shape as the initial shape.

5.2. *Numerical examples*

5.2.1. Numerical test 1: a shape reconstruction problem. To illustrate the theory, we want to solve a simple academic problem

$$\min_{\Omega} J_1(\Omega) = \frac{1}{2}\int_{\Omega} |\boldsymbol{y} - \boldsymbol{y}_d|^2 \mathrm{d}x \tag{76}$$

subject to

$$\begin{cases} -\alpha\Delta \boldsymbol{y} + \mathrm{D}\boldsymbol{y}\cdot\boldsymbol{y} + \nabla p = \boldsymbol{f} & \text{in } \Omega \\ \operatorname{div} \boldsymbol{y} = 0 & \text{in } \Omega \\ \boldsymbol{y} = 0 & \text{on } \Gamma := \Gamma_{\text{out}} \cup \Gamma_{\text{in}}; \end{cases} \tag{77}$$

The domain Ω is an annuli, and its boundary has two parts: the outer boundary Γ_{out} is an unit circle which is fixed; the inner boundary Γ_{in} that is to be optimized. We choose the body force $\boldsymbol{f} = (f_1, f_2)$ as follows:

$$\begin{aligned} f_1 &= -x^3 + \frac{\alpha y(15x^2 + 15y^2 - 1)}{5(x^2+y^2)^{3/2}} \\ &\quad - x\left[y^2 + \frac{1}{775}\left(1426 + \frac{31}{x^2+y^2} + \frac{753}{\sqrt{x^2+y^2}} - 1860\sqrt{x^2+y^2}\right)\right]; \\ f_2 &= -y^3 - \frac{\alpha y(15x^2 + 15y^2 - 1)}{5(x^2+y^2)^{3/2}} \\ &\quad - y\left[x^2 + \frac{1}{775}\left(1426 + \frac{31}{x^2+y^2} + \frac{753}{\sqrt{x^2+y^2}} - 1860\sqrt{x^2+y^2}\right)\right]. \end{aligned}$$

The target velocity $\boldsymbol{y}_d$ is determined by the data $\boldsymbol{f}$ and the target shape of the domain Ω.

In this model problem, we have the following Eulerian derivative:

$$\mathrm{d}J_1(\Omega; \boldsymbol{V}) = \int_{\Gamma_{\text{in}}} \left\{ \frac{1}{2} \left| \boldsymbol{y} - \boldsymbol{y}_d \right|^2 + \alpha \mathrm{D}\boldsymbol{y} : \mathrm{D}\boldsymbol{v} \right\} \boldsymbol{V}_n \,\mathrm{d}s.$$

We will solve this shape problem with two different target shapes:

Case 1 A circle: $\Gamma_{\text{in}} = \left\{ (x, y) \middle| x^2 + y^2 = 0.2^2 \right\}$.

Case 2 An ellipse: $\Gamma_{\text{in}} = \left\{ (x, y) \middle| \frac{x^2}{4} + \frac{4y^2}{9} = \frac{1}{25} \right\}$.

Our numerical solutions are obtained under FreeFem++ [18] and we run the program on a home PC.

In Case 1, we choose the initial shape to be elliptic: $\left\{ (x, y) \middle| (x^2/9) + (y^2/4) = 1/25 \right\}$, and the initial mesh is shown in Figure 1. In Case 2, we take the initial shape to be a circle whose center is at origin with radius 0.6, and the initial mesh is shown in Figure 2.

In Case 1, Figures 3–6 give the comparison between the target shape with iterated shape for the viscosity coefficients $\alpha = 0.1$, 0.01 and 0.001, respectively.

For $\alpha = 0.1$ and $\alpha = 0.01$, we choose the initial step $h = 20$ and in Figures 3 and 4, we give the final shape at iteration 10 with CPU times. However for $\alpha = 0.001$, one cannot obtain a good result when $h = 20$ (Figure 5). Thus we should reduce the initial descent step, and then in Figure 6, we give the final shape at iteration 40 with the initial step $h = 5$. By comparison with Figure 5, we find that though we need much more CPU time for $h = 5$, but it have a nicer reconstruction.

Figure 7 represents the fast convergence of the cost functional for the various viscosities in Case 1.

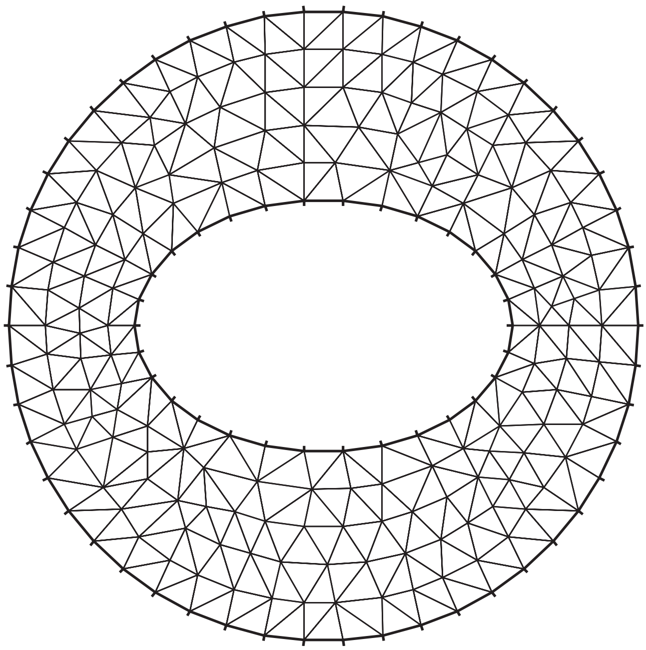

Figure 1. Initial mesh in Case 1 with 226 nodes.

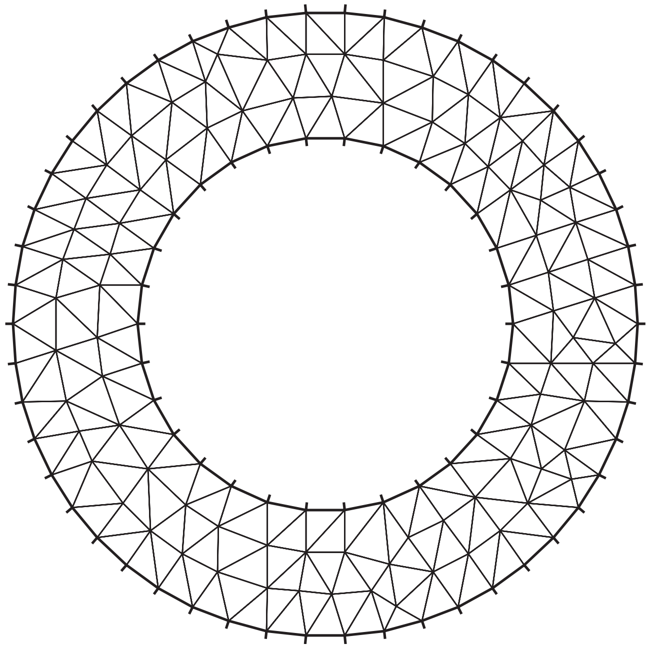

Figure 2. Initial mesh in Case 2 with 161 nodes.

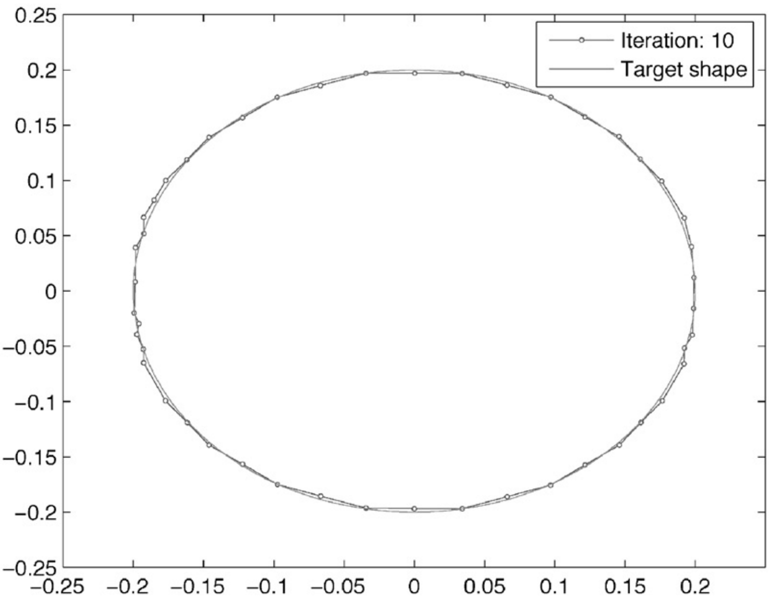

Figure 3. Case 1: $\alpha = 0.1$, CPU time: 111.984 s.

In Case 2, Figures 8 and 9 show the comparison between the target shape with iterated shape at iteration 15 for the viscosity $\alpha = 0.1$ and $\alpha = 0.01$, respectively. At this time, we choose the initial step $h = 20$. We also give the CPU run times at the 15 iterations for $\alpha = 0.1$ and $\alpha = 0.01$. Unfortunately, we cannot get a good reconstruction

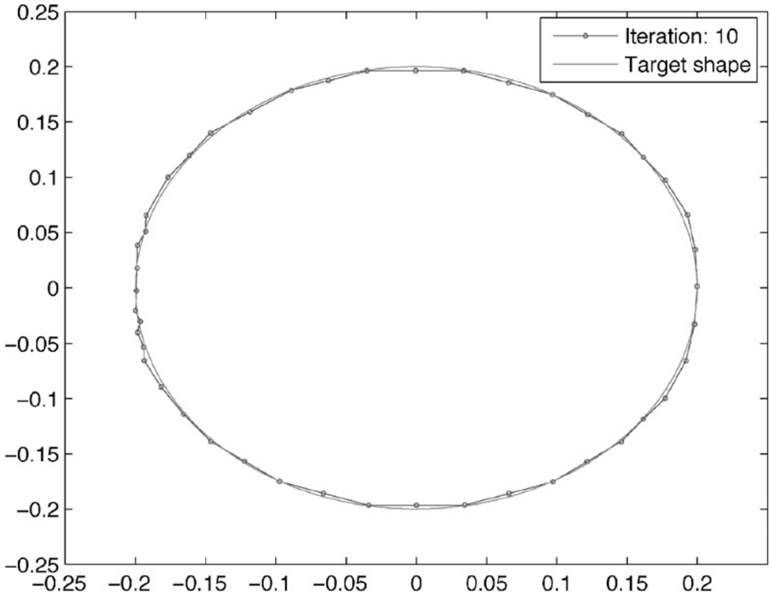

Figure 4. Case 1: $\alpha = 0.01$, CPU time: 125.969 s.

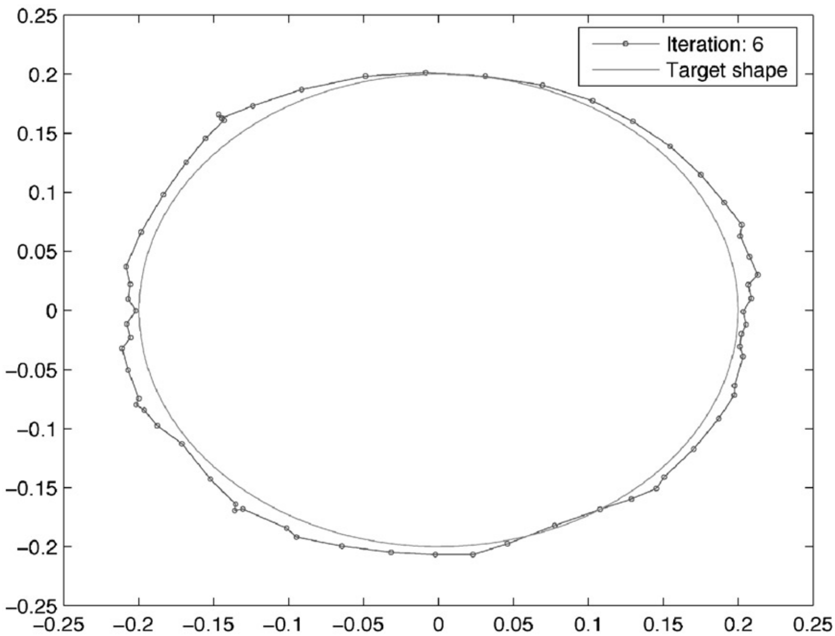

Figure 5. Case 1: $\alpha = 0.001$ with $h = 20$, CPU time: 185.578 s.

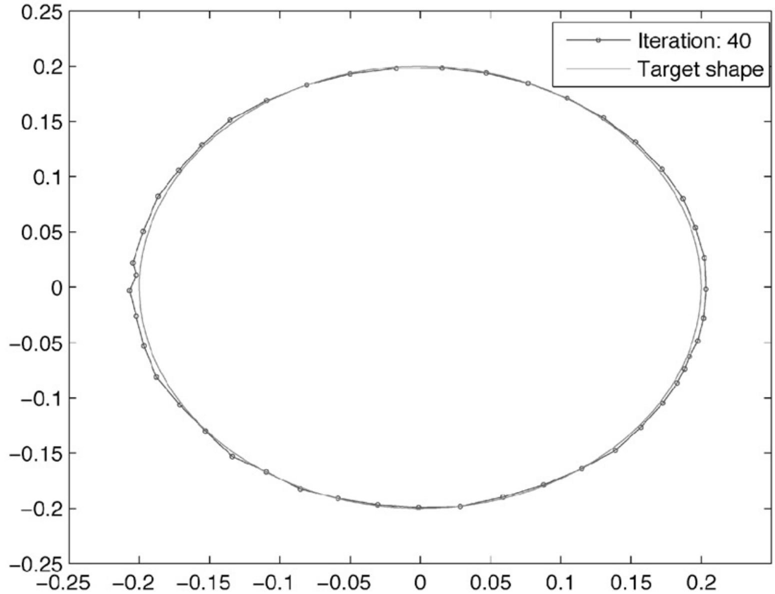

Figure 6. Case 1: $\alpha = 0.001$ with $h = 5$, CPU time: 700.016 s.

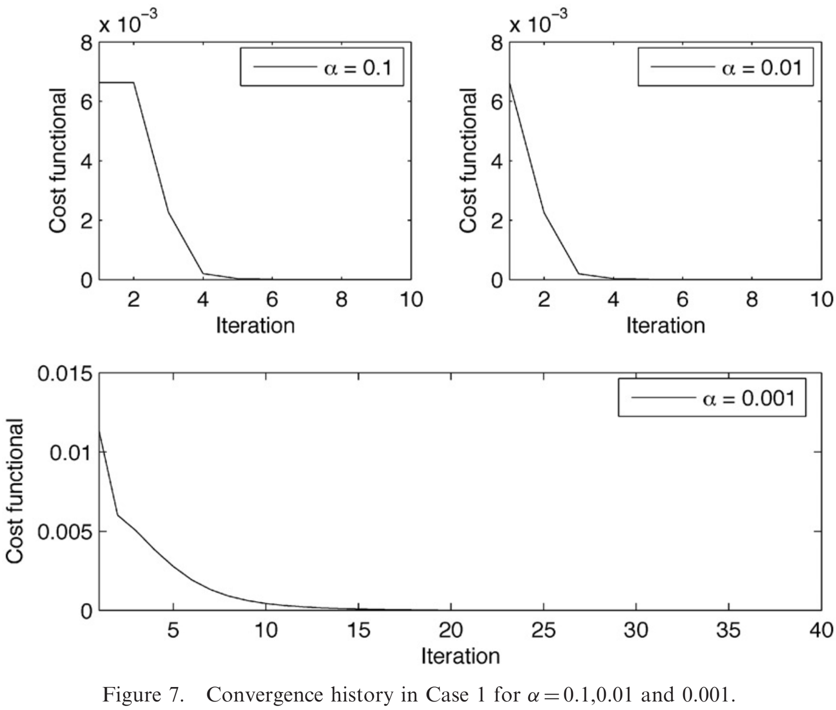

Figure 7. Convergence history in Case 1 for $\alpha = 0.1$,0.01 and 0.001.

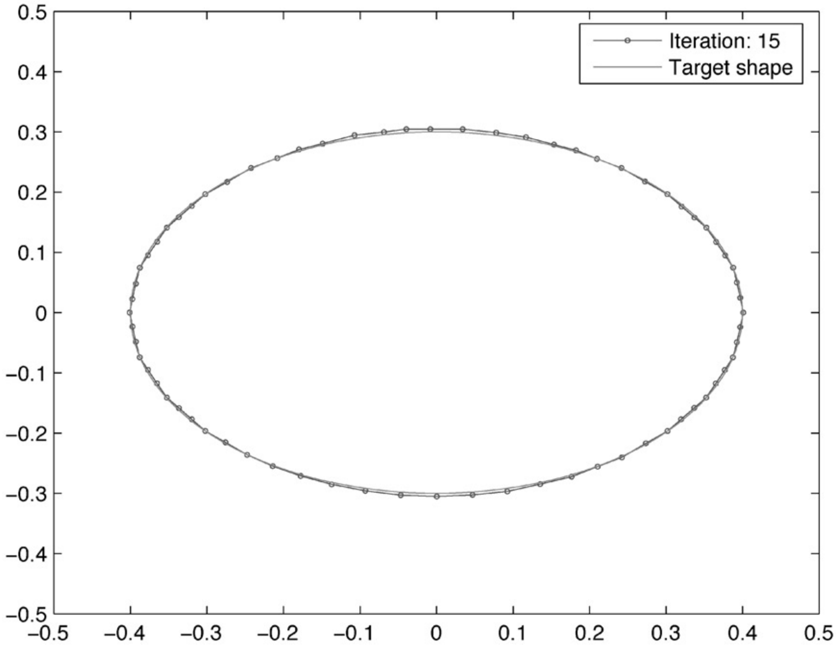

Figure 8. Case 2: $\alpha = 0.1$, CPU time: 196.609 s.

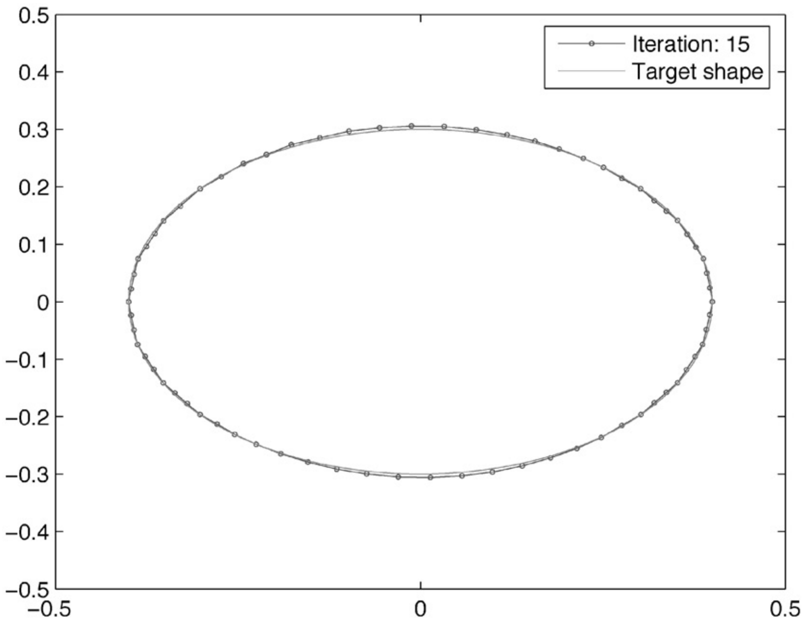

Figure 9. Case 2: $\alpha = 0.01$, CPU time: 207.469 s.

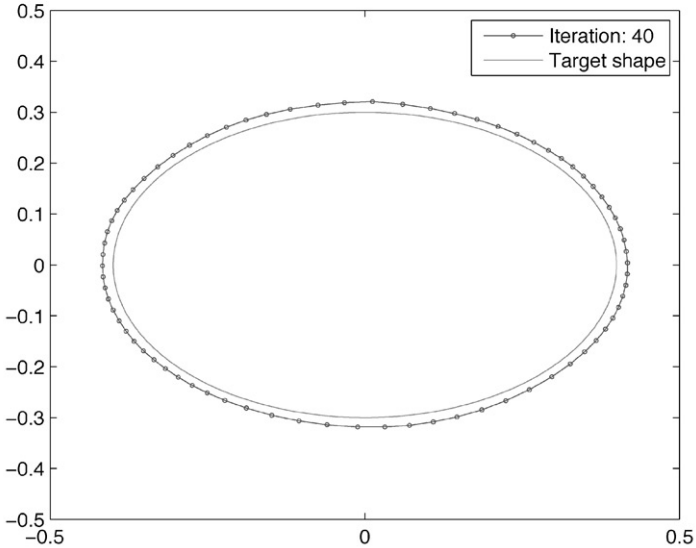

Figure 10. Case 2: $\alpha = 0.001$, CPU time: 707.469 s.

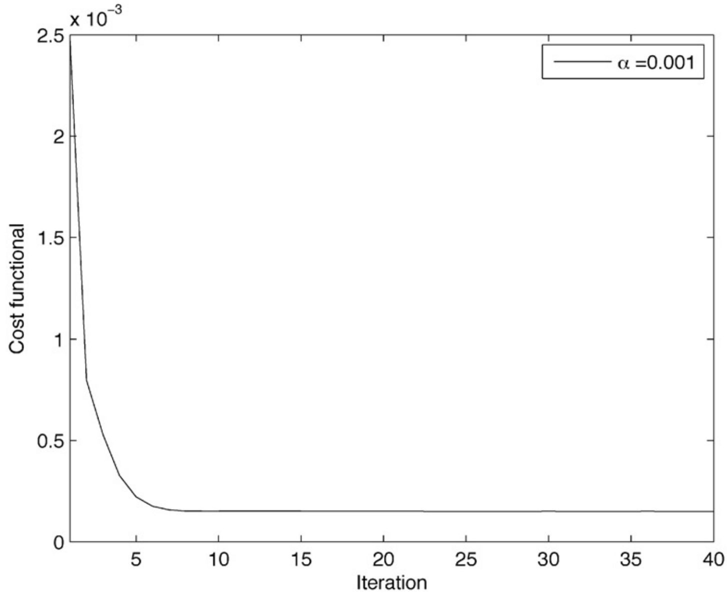

Figure 11. Convergence history in Case 2 for $\alpha = 0.001$.

for $\alpha = 0.001$ in this case (Figure 10), but as shown in Figure 11 the corresponding functional converges so fast. Figure 12 gives the convergence history of the cost functional $J(\Omega)$ for $\alpha = 0.1$, 0.01.

Remark 1 In this numerical example, we take this common definition for Reynolds number in annuli shape:

$$\mathrm{Re} = \frac{2R(1 - R/r)v\rho}{\alpha},$$

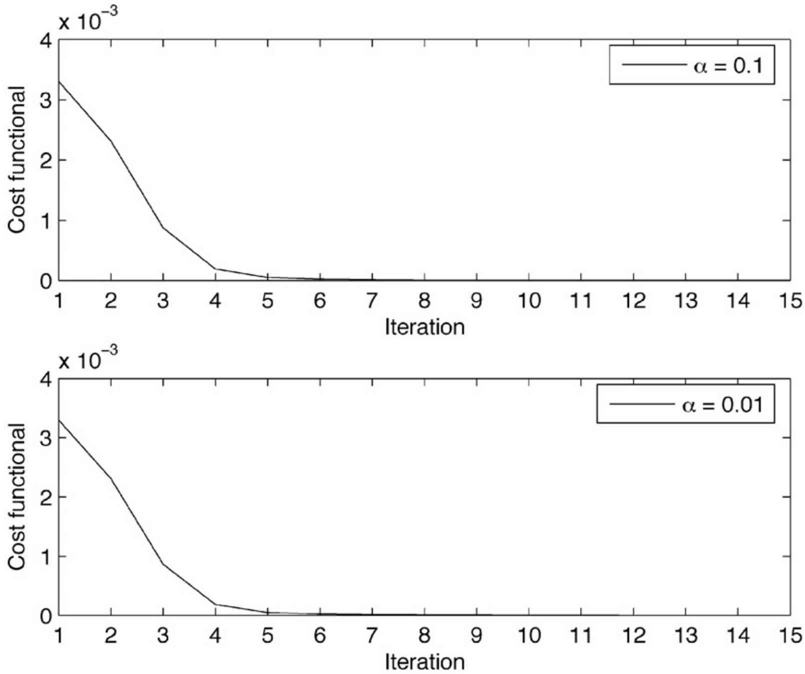

Figure 12. Convergence history in Case 2 for $\alpha = 0.1$, 0.01.

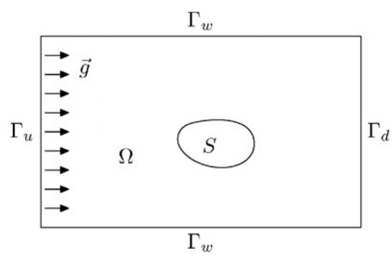

Figure 13. External flow around a solid body S.

where the outer radius $R = 1$, the density $\rho = 1$ and the inner radius $r = 0.6$. Since the velocity v does not appear in our version of the Reynolds number, if we wish to translate our results in terms of this experimentalist Reynolds number, we must assume that a velocity on the order of 1 is a typical velocity. Hence in our problem for $\alpha = 0.1$, 0.01, 0.001, the corresponding Reynolds numbers are $\mathrm{Re} = 8$, 80, 800, respectively.

5.2.2. Numerical test 2: Optimization of a solid body in Navier-Stokes flow. As a second test case, we consider the optimization of a isolated body located in a viscous incompressible flow. The schematic geometry of the fluid domain is described in Figure 13, corresponding to an external flow around a solid body S. We reduce the problem to a bounded domain D by introducing an artificial boundary

$\partial D := \Gamma_u \cup \Gamma_d \cup \Gamma_w$ which has to be taken sufficiently far from S so that the corresponding flow is a good approximation of the unbounded external flow around S and $\Omega := D \backslash \bar{S}$ is the effective domain. In addition, the boundary $\Gamma_s := \partial S$ is to be optimized.

Finally, we want to solve the following optimization problem

$$\min_{\Omega \in \mathcal{O}} J_2(\Omega) = \alpha \int_\Omega |\mathrm{D}\boldsymbol{y}|^2 \mathrm{d}x,$$

where $\boldsymbol{y}$ satisfies

$$\begin{cases} -\alpha\Delta\boldsymbol{y} + \mathrm{D}\boldsymbol{y} \cdot \boldsymbol{y} - \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{y} = 0 & \text{in } \Omega \\ \boldsymbol{y} = \boldsymbol{g} & \text{on } \Gamma_u \\ \boldsymbol{y} = 0 & \text{on } \Gamma_s \cup \Gamma_w \cup \Gamma_d \end{cases}$$

and the admissible set

$$\mathcal{O} = \{\Omega \in \mathbb{R}^2 : \Gamma_u, \Gamma_w \text{ and } \Gamma_d \text{ are fixed}, V_{\text{target}}(\Omega) = 1.9\}.$$

We choose D to be a rectangle $(-0.5,\ 1.5) \times (-0.5,\ 1.5)$ and S is to be determined in our simulations. The inflow velocity is assumed to be parabolic with a profile $\boldsymbol{g}(-0.5,\ \mathrm{y}) = ((0.5 - y)(y + 0.5),\ 0)^T$. In this problem we have the following shape gradient

$$\nabla J_2 = \alpha\Big[(\mathrm{D}\boldsymbol{y} \cdot \boldsymbol{n}) \cdot (\mathrm{D}\boldsymbol{v} \cdot \boldsymbol{n}) - |\mathrm{D}\boldsymbol{y} \cdot \boldsymbol{n}|^2\Big]\boldsymbol{n}.$$

We choose the initial shape of the body S to be a circle of center (0, 0) with radius $r = 0.3$. We present results for Reynolds numbers Re = 40,200 defined by $\mathrm{Re} = 2r|\boldsymbol{y}_m|/\alpha$, where $\boldsymbol{y}_m$ is the maximum velocity at the inflow Γ_u. The finite element mesh used of the initial domain at Re = 200 has been shown in Figure 14 with 3208 elements with 1724 vertices.

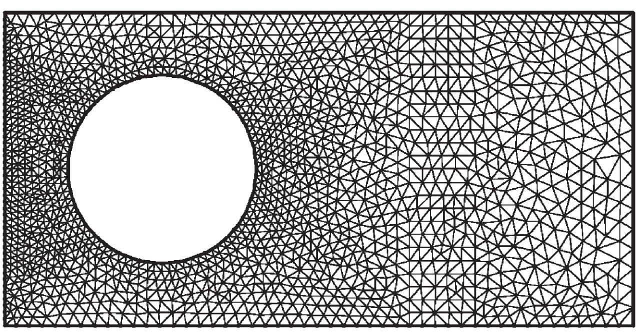

Figure 14. Finite element mesh of the initial shape (Re = 200).

The distributions of the horizontal velocity for the initial and optimal shapes with Re = 40,200 are shown in Figures 15 and 16. We also give the computational cost and the area of optimal shape in both cases.

In Figures 17 and 18, we run many iterations in order to show the good convergence and stability properties of our algorithm. For the Reynolds numbers 40 and 200, the total dissipated energy reduced about 79.76% and 79.04% respectively.

6. Conclusion

Shape optimization in the two-dimensional Navier–Stokes flow has been presented. We derived the structures of shape gradients for two kinds of cost functionals by three different approaches: state derivative approach, function space parametrization approach and function space embedding approach. Finally, a gradient-type algorithm is effectively used for the minimization problem for various Reynolds numbers. Further research is necessary on efficient implementations for time-dependent Navier–Stokes flow and much more real problems in the industry.

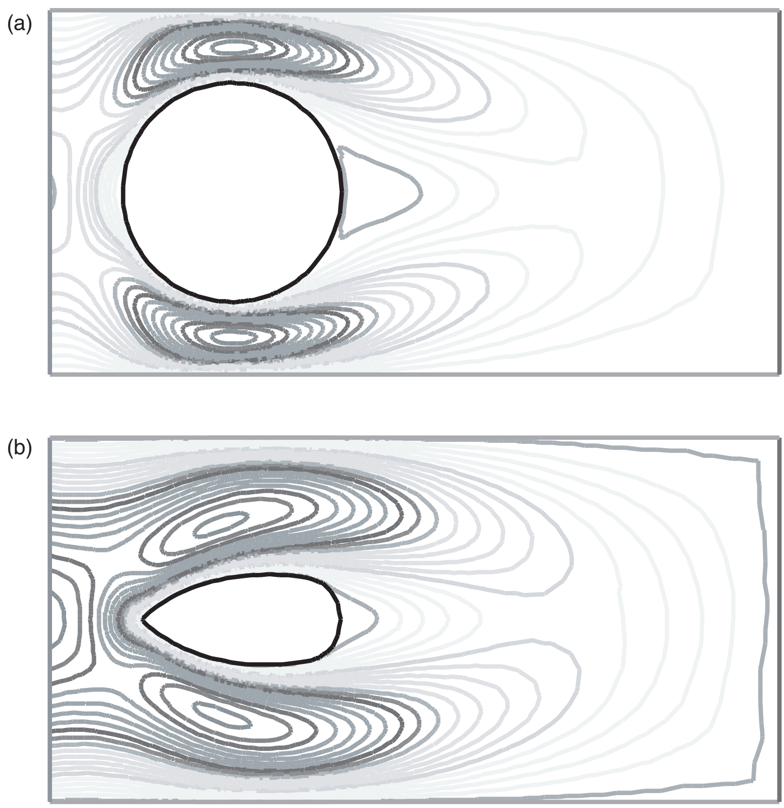

Figure 15. Distribution of horizontal velocity of initial shape (top) and optimal shape (bottom). CPU time: 176.422s. Area of optimal shape: 1.89773 (Re = 40).

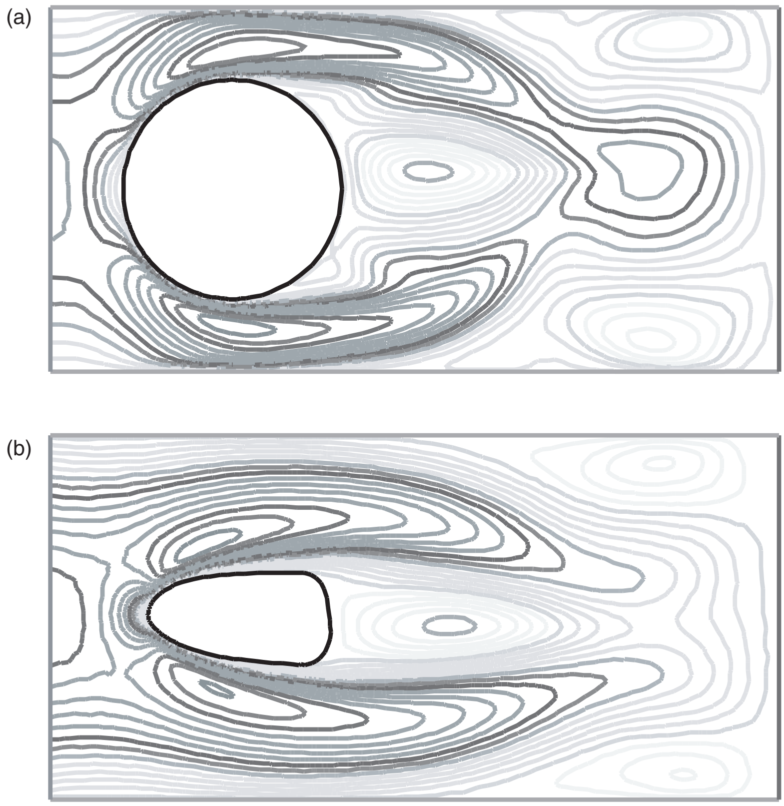

Figure 16. Distribution of horizontal velocity of initial shape (top) and optimal shape (bottom). cpu time: 438.05s. Area of optimal shape: 1.89762 (Re = 200).

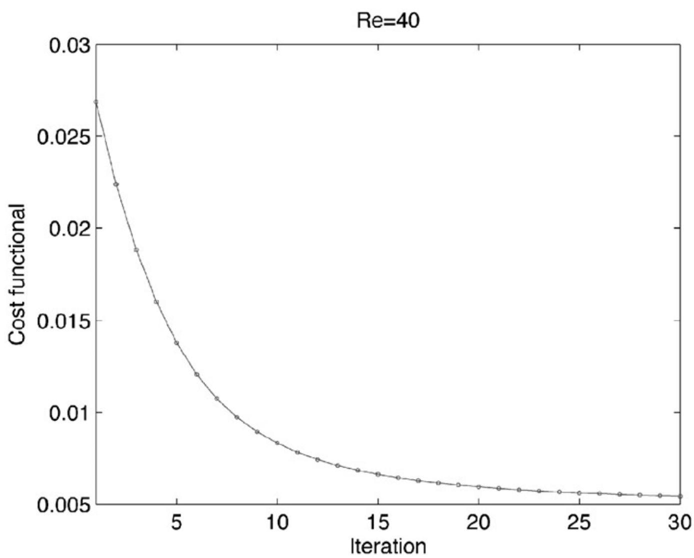

Figure 17. Convergence history of the cost functional (Re = 40).

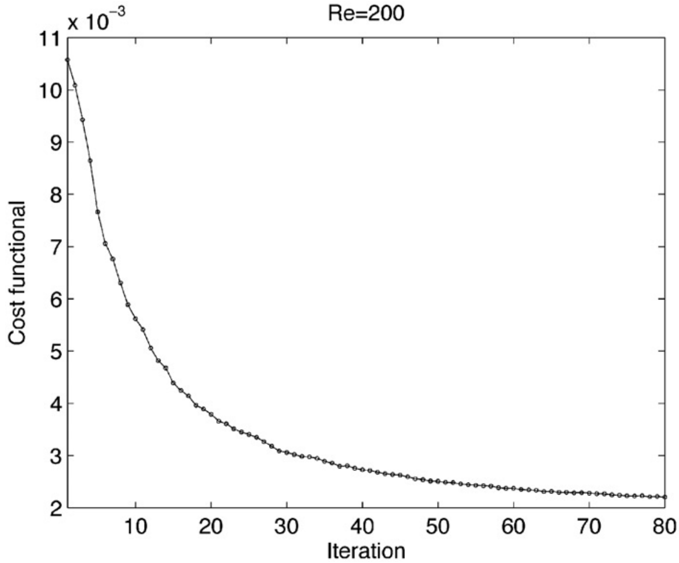

Figure 18. Convergence history of the cost functional (Re = 200).

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