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Variational Methods in Shape Optimization Problems

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Preface

The fascinating field of shape optimization problems has received a lot of attention in recent years, particularly in relation to a number of applications in physics and engineering that require a focus on shapes instead of parameters or functions. The goal of these applications is to deform and modify the admissible shapes in order to comply with a given cost function that needs to be optimized. In this respect the problems are both classical (as the isoperimetric problem and the Newton problem of the ideal aerodynamical shape show) and modern (reflecting the many results obtained in the last few decades).

The intriguing feature is that the competing objects are *shapes*, i.e., domains of $\mathbb{R}^N$, instead of functions, as it usually occurs in problems of the calculus of variations. This constraint often produces additional difficulties that lead to a lack of existence of a solution and to the introduction of suitable *relaxed* formulations of the problem. However, in certain limited cases an optimal solution exists, due to the special form of the cost functional and to the geometrical restrictions on the class of competing domains.

This volume started as a collection of the lecture notes from two courses given in the academic year 2000–2001 by the authors at the Dipartimento di Matematica Universit di Pisa and at Scuola Normale Superiore di Pisa respectively. The courses were mainly addressed to Ph.D. students and required as background the topics in functional analysis that are typically covered in undergraduate courses. Subsequently, more material has been added to the original base of lecture notes. However, the style of the work remains quite informal and follows, in large part, the lectures as given.

We decided to open the volume by presenting in Chapter 1 some relevant examples of shape optimization problems: the isoperimetric problem, the Newton problem of optimal aerodynamical profiles, the optimal distribution of two different media in a given region, and the optimal shape of a thin insulator around a given conductor. In Chapter 2 we consider the important case where the additional constraint of convexity is assumed on the competing domains: this situation often provides the extra compactness necessary to prove the existence of an optimal shape. A prototype for this class is the Newton problem, where the convexity of the competing bodies permits the existence of an optimal shape, together with some necessary conditions of optimality.

Many shape optimization problems can be seen in the larger framework of optimal control problems: indeed an admissible shape plays the role of an admissible control, and the corresponding state variable is usually the solution of a partial differential equation on the control domain. This point of view is developed in large generality in Chapter 3, together with the corresponding relaxation theory, which provides a general way to construct relaxed solutions through Γ-convergence methods. In Chapter 4 we study variational problems where the Dirichlet region is seen as one of the unknowns, and the corresponding optimization problems are considered. It must be pointed out that, due to the nature of the problem, in general an optimal Dirichlet region does not exist, and a relaxed formulation is needed to better understand the behavior of minimizing sequences.

Contrarily in Chapter 5 we present some particular cases where, due to the presence of suitable geometrical constraints and the monotonicity of the cost functional, a classical unrelaxed optimal solution does exist, admitting a solution in the family of classical admissible domains. Some relevant examples of problems that fulfill the required assumptions are also shown. Chapter 6 deals with the very special case of cost functionals that depend on the eigenvalues of an elliptic operator with Dirichlet conditions on the free boundary; we collected some classical and modern results together with several problems that are still open. Finally, we devote Chapter 7 to the case of shape optimization problems governed by elliptic equations with Neumann conditions on the free boundary. In this case several additional difficulties arise precluding the development of a complete theory; however, we made an effort to treat completely at least the so called problem of optimal cutting, where the existence of an optimal cut can be deduced in full generality.

The work also contains a substantial, yet hardly exhaustive, bibliography. The compilation of a more complete list of references would be prohibitive due to the rapid development of the field and the tremendous volume of associated papers that are regularly published on the subject.

This study can serve as an excellent text for a graduate course in variational methods for shape optimization problems, appealing to both students and instructors alike.

Dorin Bucur and Giuseppe Buttazzo
Metz and Pisa, March 31, 2005

1

Introduction to Shape Optimization Theory and Some Classical Problems

In this chapter we introduce a shape optimization problem in a very general way and we discuss some of the features that will be considered in the following chapters. We also present some classical problems like the isoperimetric problem and some of its variants, which can be viewed in the framework of shape optimization.

A shape optimization problem is a minimization problem where the unknown variable runs over a class of domains; then every shape optimization problem can be written in the form

$$\min\left\{F(A)\ :\ A \in \mathcal{A}\right\}$$

where $\mathcal{A}$ is the class of admissible domains and F is the cost function that one has to minimize over $\mathcal{A}$.

It must be noticed that the class $\mathcal{A}$ of admissible domains does not have any linear or convex structure, so in shape optimization problems it is meaningless to speak of convex functionals and similar notions. Moreover, even if several topologies on families of domains are available, in general there is not an a priori choice of a topology in order to apply the direct methods of the calculus of variations, for obtaining the existence of at least an optimal domain.

We want to stress that, as it also happens in other kinds of optimal control problems, in several situations an optimal domain does not exist; this is mainly due to the fact that in these cases the minimizing sequences are highly oscillating and converge to a limit object only in a *"relaxed"* sense. Then we may have, in these cases, only the existence of a *"relaxed solution"* that in general is not a domain, and whose characterization may change from problem to problem.

We shall introduce in the next chapters a general procedure to relax optimal control problems and in particular shape optimization problems. A case which will be considered in detail is when a Dirichlet condition is imposed on the free boundary: we shall see that in general one should not expect the existence of an optimal solution. However, the existence of an optimal domain occurs in the following cases:

i) when severe geometrical constraints on the class of admissible domains are imposed (see Section 5.1);
ii) when the cost functional fulfills some particular qualitative assumptions (see Section 5.4);
iii) when the problem is of a very special type, involving only the eigenvalues of the Laplace operator, where neither geometrical constraints nor monotonicity of the cost are required (see Section 6.4).

Far from being an exhaustive classification, this is simply the state of the art at present.

The case when Neumann conditions are considered on the free boundary is discussed in this volume only in Chapter 7. We refer the reader to books [4], [111], [140], [176], [186] or to the many available papers (see References) for some topics related to this subject.

In this chapter we present some problems of shape optimization that can be found in the classical literature. In all the cases we consider here, the existence of an optimal domain is due to the presence either of geometrical constraints in the class of admissible domains or of geometrical penalizations in the cost functional. The standard background of functional analysis and of function spaces (as Sobolev or BV spaces) is assumed to be known.

Among the classical questions which can be viewed as shape optimization problems we include the isoperimetric problem which will be presented in great generality and with several variants. In order to set the problem correctly, the notion of perimeter of a set is required; this will be introduced by means of the theory of BV functions. The main properties of BV functions will be recalled and summarized without entering into details; the reader interested in finer results and deeper discussions will be referred to one of the several books available in the field (see for instance [8], [12], [108], [131]), [161].

Another classical question which can be considered as a shape optimization problem is the determination of the best aerodynamical profile for a body in a fluid stream under some constraints on its size. The Newton model for the aerodynamical resistance will be considered and various kinds of constraints on the body will be discussed.

The Newton problem of optimal aerodynamical profiles gives us the opportunity to consider in the next chapter a larger class of shape optimization problems: indeed we shall take those whose admissible domains are convex. This geometrical constraint allows us in several cases to obtain the existence of an optimal solution.

In Section 1.4 we consider a problem of optimal interface between two given media; either a perimeter constraint on the interface, or a perimeter penalization, gives in this case enough compactness to guarantee the existence of an optimal classical solution. This problem gives us the opportunity to discuss some properties of Γ-convergence, which plays an important role in several shape optimization problems.

In Section 1.5 we deal with the problem of finding the optimal shape of a thin insulating layer around a thermally conducting body. The problem will be set as an optimal control problem, where the thickness function of the layer will be the control variable and the temperature will be the state variable.

1.1 General formulation of a shape optimization problem

As already said above, a shape optimization problem is a minimization problem of the form

$$\min\left\{F(A)\ :\ A\in\mathcal{A}\right\} \tag{1.1}$$

where $\mathcal{A}$ is the class of admissible domains and F is the cost functional. We shall see that, unless some geometrical constraints on the admissible sets are assumed or some very special cases for cost functionals are considered, in general the existence of an optimal domain may fail. In these situations the discussion will then be focused on the relaxed solutions, which always exist.

We shall see that, in order to give a qualitative description of the optimal solutions of a shape optimization problem, it is important to derive the so-called *necessary conditions of optimality*. These conditions, as it usually happens in all optimization problems, have to be derived from the comparison of the cost of an optimal solution A to the cost of other suitable admissible choices, close enough to A. This procedure is what is usually called a *variation* near the solution. The difficulty in obtaining necessary conditions of optimality for shape optimization problems consists in the fact that, being the unknown domain, the notion of neighbourhood is not a priori clear; the possibility of choosing a domain variation could then be rather wide. The same method can be applied, when no classical solution exists, to relaxed solutions, and this will provide qualitative information about the behaviour of minimizing sequences of the original problem.

Finally, for some particular problems presenting special behaviours or symmetries, one would like to exhibit explicit solutions (balls, ellipsoids, ...). This could be very difficult, even for simple problems, and often, instead of having established results, we can only give conjectures.

In general, since the explicit computations are difficult, one should develop efficient numerical schemes to produce approximated solutions; this is a challenging field we will not enter; we refer the interested reader to some recent books and papers (see for instance references [4], [128], [176], [185]).

1.2 The isoperimetric problem and some of its variants

The first and certainly most classical example of a shape optimization problem is the isoperimetric problem. It can be formulated in the following way: find, among all admissible domains with a given *perimeter* (this explains the term “isoperimetric”), the one whose Lebesgue measure is as large as possible. Equivalently, one could minimize the perimeter of a set among all admissible domains whose Lebesgue measure is prescribed.

The first difficulty consists in finding a definition of perimeter general enough to be applied to nonsmooth sets and to allow us to apply the direct method of the calculus of variations. The definition below goes back to De Giorgi (see [108]) and is now considered classical; we assume the reader is familiar with the spaces of functions with bounded variation and with their properties.

Given a set $A \subset \mathbb{R}^N$ we denote by 1_A the characteristic function of A, defined by

$$1_A(x) = \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ otherwise.} \end{cases} \tag{1.2}$$

Definition 1.2.1 *We say that a set A with finite Lebesgue measure is a set of finite perimeter in $\mathbb{R}^N$ if its characteristic function 1_A belongs to $BV(\mathbb{R}^N)$. This means that the distributional gradient $\nabla 1_A$ is a vector-valued measure with finite total variation. The total variation $|\nabla 1_A|$ is called the perimeter of A.*

The admissible domains A we consider are constrained to be contained in a given closed subset K of $\mathbb{R}^N$. Instead of fixing their Lebesgue measure, more generally we impose the constraint

$$\int_A f(x)\,dx = c$$

where c is a given constant and f is a given function in $L^1_{loc}(\mathbb{R}^N)$. Note that when f is a constant function, the class of admissible domains is simply the class of all subsets of K with a given volume.

With this notation the isoperimetric problem can be then formulated in the following way:

Given a closed subset K of $\mathbb{R}^N$ and a function $f \in L^1_{loc}(\mathbb{R}^N)$ find the subset of K whose perimeter is minimal, among all subsets A of K whose integral $\int_A f(x)\,dx$ is prescribed.

We then have a minimization problem of the form (1.1) with

$$F(A) = \mathrm{Per}(A) = \int |\nabla 1_A|,$$

$$\mathcal{A} = \left\{A \subset K \ : \ \int_A f(x)\,dx = c\right\}.$$

Note that all subsets of K with infinite perimeter are ruled out by the formulation above, because the cost functional evaluated on them takes the value $+\infty$.

Theorem 1.2.2 *With the notation above, if K is bounded and if the class of admissible sets is nonempty, then the minimization problem*

$$\min\left\{F(A)\ :\ A\in\mathcal{A}\right\} \tag{1.3}$$

admits at least a solution.

Proof The proof follows the usual scheme of the direct methods of the calculus of variations. Taking a minimizing sequence (A_n), the perimeters $\operatorname{Per}(A_n)$ are then equi-bounded; since $A_n \subset K$ and since K is bounded, the measures of A_n are equi-bounded as well. Therefore the sequence 1_{A_n} is bounded in $BV(Q)$ where Q is a large ball containing K; we may then extract a subsequence (which we still denote by the same indices) which converges weakly* to some function $u \in BV(Q)$ in the sense that

$$\begin{cases} 1_{A_n} \to u \text{ strongly in } L^1(Q),\\ \nabla 1_{A_n} \to \nabla u \text{ weakly* in the sense of measures.}\end{cases}$$

The function u has to be of the form 1_A for some set A with finite perimeter. Moreover, we obtain easily that $A \subset K$ (up to a set of measure zero) and $\int_A f(x)\,dx = c$, which shows that A is an admissible domain. This domain A achieves the minimum of the cost functional since (as it is well known) the perimeter is a weakly* lower semicontinuous function on BV. ■

Example 1.2.3 When K is not bounded, the existence of an optimal domain may fail. In fact, take $K = \mathbb{R}^N$ and $f(x) = |x|$. It is clear that a ball $B_{x_0,r}$ with $|x_0| \to +\infty$ and $r \to 0$, suitably chosen, may fulfill the integral constraint; on the other hand the perimeter of such a $B_{x_0,r}$ goes to zero. Then the infimum of the problem is zero, which is clearly not attained.

Example 1.2.4 If K is unbounded, the existence of an optimal domain for problem (1.3) may fail even if $f \equiv 1$. In fact, let c be the measure of the unit ball in $\mathbb{R}^N$ and let

$$K = \bigcup_{n\in\mathbb{N}} B_{x_n,r_n}$$

where (r_n) is a strictly increasing sequence of positive numbers converging to 1 (for instance $r_n = 1-1/n$) and (x_n) is a sequence of points in $\mathbb{R}^N$ such that $|x_n - x_m| \geq 2$ if $n \neq m$. Then it is easy to see that the infimum of problem (1.3) is given by the value $\operatorname{Per}\left(B_{0,1}\right)$ which is not attained, since the set K does not contain any ball of radius 1 (see Figure 1.1 below).

The case $K = \mathbb{R}^N$ and $f \equiv 1$ is the classical isoperimetric problem. It is well known that in this case the optimal domains are the balls of measure c, even if the proof of this fact is not trivial. In the case $N = 2$ the proof can be obtained in an elementary way by using the *Steiner symmetrization* method; in higher dimensions the proof is more involved. It is not our goal to enter into this kind of detail and we refer the interested reader to the wide literature on the subject.

A variant of the isoperimetric problem consists in counting in the cost functional only the part of the boundary of A which is interior to K. More precisely, we consider

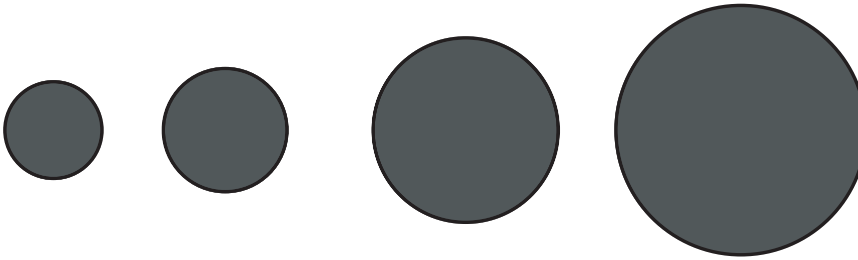

Figure 1.1. The set K.

an open subset D of $\mathbb{R}^N$ with a Lipschitz boundary and we define the perimeter relative to D of a subset A as

$$\mathrm{Per}_D(A) = |\nabla 1_A|(D).$$

In this way a set A will be of finite perimeter in D if the function 1_A belongs to the space $BV(D)$.

Again, we have a minimization problem of the form (1.1) with

$$F_D(A) = \mathrm{Per}_D(A) = \int_D |\nabla 1_A|,$$

$$\mathcal{A} = \left\{ A \subset D \ : \ \int_A f(x)\,dx = c \right\}.$$

Theorem 1.2.5 *With the notation above, if D is bounded and if the class of admissible sets is nonempty, then the minimization problem*

$$\min \left\{ F_D(A) \ : \ A \in \mathcal{A} \right\} \tag{1.4}$$

admits at least a solution.

Proof The proof can be obtained by repeating step by step the proof of Theorem 1.2.2. ■

Example 1.2.6 If the assumptions on the domain D are dropped, it is easy to construct counterexamples to the existence result above, even if the datum f is identically equal to 1. In fact, if we define the function

$$\begin{aligned}
&\phi(x) = -8x + 8 \qquad \text{if } x \in]1/2, 1], \\
&\phi(x) = -32x + 16 \qquad \text{if } x \in]1/4, 1/2], \\
&\ldots\ldots \\
&\phi(x) = -2^{2n+3}x + 2^{n+3} \qquad \text{if } x \in]2^{-n-1}, 2^{-n}], \\
&\ldots\ldots
\end{aligned}$$

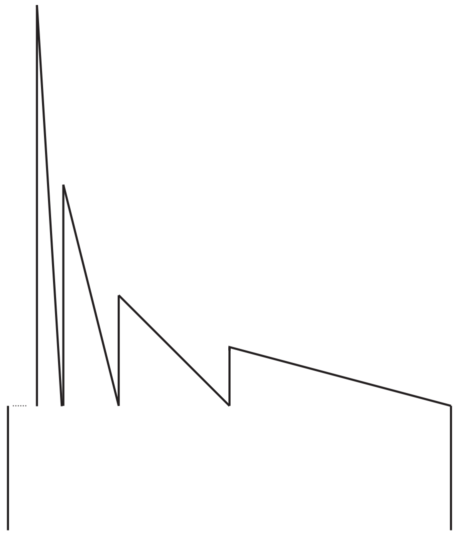

Figure 1.2. The constraint D.

and we take $c = 1$ and

$$D = \{(x, y) \in \mathbb{R}^2 \ : \ x \in]0, 1[, \ y < \phi(x)\},$$

an optimal domain for the constrained isoperimetric problem does not exist. To see this fact it is enough to consider the minimizing sequence

$$A_n = \{(x, y) \in \mathbb{R}^2 \ : \ x \in]2^{-n-1}, 2^{-n}[, \ 0 < y < \phi(x)\}.$$

All the sets A_n are admissible and their Lebesgue measure is equal to 1 for all n; however, we have $\mathrm{Per}_D(A_n) = 2^{-n-1} \to 0$, so that the infimum of the problem is zero. No optimal domain may then exist, because for every admissible set A we have $\mathrm{Per}_D(A) > 0$. A picture of the set D is in Figure 1.2 above.

The results above still hold for more general cost functionals. Instead of considering the cost given by the perimeter $|\nabla 1_A|$, take a function $j : \mathbb{R}^N \times \mathbb{R}^N \to \overline{\mathbb{R}}$ which satisfies the following properties:

i) j is lower semicontinuous on $\mathbb{R}^N \times \mathbb{R}^N$;

ii) for every $x \in \mathbb{R}^N$ the function $j(x, \cdot)$ is convex;

iii) there exists a constant $c_0 > 0$ such that

$$j(x, z) \geq c_0 |z| \qquad \forall (x, z) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Consider now the cost functional

$$F(A) = \int j(x, \nabla 1_A) \ .$$

The integral above must be intended in the sense of functionals over measures; more precisely, if μ is a measure and if $\mu = \mu^a\,dx + \mu^s$ is the Lebesgue–Nikodym decomposition of μ into absolutely continuous and singular parts (with respect to the Lebesgue measure), the integral $\int j(x,\mu)$ stands for

$$\int j\big(x,\mu^a(x)\big)\,dx + \int j^\infty\Big(x,\frac{d\mu^s}{d|\mu^s|}\Big)\,d\mu^s$$

where $|\mu^s|$ is the total variation of μ^s, $\frac{d\mu^s}{d|\mu^s|}$ is the Radon–Nikodym derivative of μ^s with respect to $|\mu^s|$, and j^∞ is the recession function of j defined by

$$j^\infty(x,z) = \lim_{t\to+\infty}\frac{j(x,tz)}{t}\;.$$

When $\mu = \nabla 1_A$ the expression above may be simplified; indeed, if A is a smooth domain it is easy to see that

$$\nabla 1_A = -\nu(x)\mathcal{H}^{N-1}\llcorner\partial A$$

with ν being the exterior unit normal vector to A and $\mathcal{H}^{N-1}$ the $N-1$ dimensional Hausdorff measure. When A is not smooth, the correct way to represent the measure $\nabla 1_A$ is to introduce the so-called *reduced boundary* ∂^*A.

Definition 1.2.7 *Let A be a set of finite perimeter; we say that $x \in \partial^*A$ if*

i) for every $r > 0$ we have $0 < \operatorname{meas}\big(A\cap B_{x,r}\big) < \operatorname{meas}\big(B_{x,r}\big)$;

ii) there exists the limit

$$\nu_A(x) = \lim_{r\to 0}\frac{-\nabla 1_A\big(B_{x,r}\big)}{|\nabla 1_A|\big(B_{x,r}\big)}$$

and $|\nu_A(x)| = 1$.

The vector $\nu_A(x)$ is called an exterior unit normal vector to A and the set ∂^*A is called the reduced boundary of A.

In this way, for every set A of finite perimeter we still have

$$\nabla 1_A = -\nu_A(x)\mathcal{H}^{N-1}\llcorner\partial^*A,$$

so that the cost functional above, dropping the constant term $\int_D j(x,0)\,dx$, can be written as

$$F(A) = \int_{\partial^*A} j^\infty(x,-\nu_A)\,d\mathcal{H}^{N-1}.$$

It has to be noticed that the integrand $j^\infty(x,z)$ is positively homogeneous of degree 1 with respect to z. In an analogous way we may consider the functional

$$F_D(A) = \int_{D\cap\partial^*A} j^\infty(x,-\nu_A)\,d\mathcal{H}^{N-1}.$$

Theorem 1.2.8 *With the notation above, if the classes of admissible sets are non-empty, then the minimization problems*

$$\min\left\{F(A)\ :\ A \subset K,\ \int_A f(x)\,dx = c\right\},$$

$$\min\left\{F_D(A)\ :\ A \subset D,\ \int_A f(x)\,dx = c\right\}$$

both admit at least a solution, provided K is a bounded set and D is a bounded open set.

Proof The proof in this more general framework is similar to the previous ones of Theorem 1.2.2 and Theorem 1.2.5. In fact, thanks to assumption iii) we still have the coercivity in the space BV, and thanks to assumptions i) and ii) the functionals F and F_D are lower semicontinuous with respect to the weak* convergence on BV (see for instance [62], [63]). ∎

The cases when K and D are unbounded can be treated by assuming that the function f is integrable. More precisely, we can prove the following result.

Proposition 1.2.9 *If the set K (respectively D) is unbounded, then the minimization problems of Theorem 1.2.8 still have a solution, provided $f \in L^1(K)$ (respectively $L^1(D)$).*

Proof By the same argument of Theorem 1.2.8 we can prove that a minimizing sequence (A_n) is such that the functions 1_{A_n} are bounded in $BV(B_{0,R})$ for every $R > 0$. Then by a diagonalization procedure we can extract a subsequence (still denoted by (A_n)) which converges in $L^1_{loc}(\mathbb{R}^N)$ to some function of the form 1_A. The L^1_{loc}-lower semicontinuity of the functional F (respectively F_D) concludes the proof, provided we can show that the set A is still admissible, that is $\int_A f(x)\,dx = c$. This last fact follows by the dominated convergence theorem using the a.e. convergence of 1_{A_n} to 1_A and the integrability of the function f. ∎

We can now see how the boundary variation method works in the isoperimetric problem and how this allows us to obtain necessary conditions of optimality. Assume A is a solution of the isoperimetric problem

$$\min\left\{\mathrm{Per}_D(A)\ :\ A \subset D,\ \mathrm{meas}(A) = c\right\} \tag{1.5}$$

and let $x_0 \in D \cap \partial A$; we assume that near x_0 the boundary ∂A is regular enough to perform all necessary operations. Actually, the regularity of ∂A does not need to be assumed as a hypothesis but is a consequence of some suitable conditions on the datum f; this is a quite delicate matter which goes under the name of *regularity theory*. We do not enter this field and we refer the interested reader to the various books available in the literature (see for instance references [8], [131], [161]).

We can then assume that in a small neighbourhood of x_0 the boundary ∂A can be written as the graph of a function $u(x)$, where x varies in an open subset ω of $\mathbb{R}^{N-1}$. The corresponding part of $\mathrm{Per}_D(A)$ can then be written in the Cartesian form as

$$\int_\omega \sqrt{1+|\nabla u|^2}\,dx.$$

The boundary variation method consists in perturbing ∂A, hence $u(x)$, by taking a comparison function of the form $u(x)+\varepsilon\phi(x)$, where $\varepsilon > 0$ and ϕ is a smooth function with support in ω. We also want the measure constraint to remain fulfilled, which turns out to require that the function ϕ satisfies the equality

$$\int_\omega \phi(x)\,dx = 0.$$

Since A is optimal we obtain the inequality

$$\int_\omega \sqrt{1+|\nabla u+\varepsilon\nabla\phi|^2}\,dx \geq \int_\omega \sqrt{1+|\nabla u|^2}\,dx. \tag{1.6}$$

The integrand on the left-hand side of (1.6) gives, as $\varepsilon \to 0$,

$$\sqrt{1+|\nabla u+\varepsilon\nabla\phi|^2} = \sqrt{1+|\nabla u|^2} + \varepsilon\frac{\nabla u\cdot\nabla\phi}{\sqrt{1+|\nabla u|^2}} + o(\varepsilon)$$

so that (1.6) becomes

$$\int_\omega \frac{\nabla u\cdot\nabla\phi}{\sqrt{1+|\nabla u|^2}}\,dx \geq 0.$$

Integrating by parts we obtain

$$-\int_\omega \operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big)\phi\,dx \geq 0,$$

and recalling that ϕ was arbitrary and with zero average in ω, we finally obtain that the function u must satisfy the partial differential equation

$$-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = \text{constant in } \omega. \tag{1.7}$$

The term $-\operatorname{div}\big(\nabla u/\sqrt{1+|\nabla u|^2}\big)$ represents the mean curvature of ∂A written in Cartesian coordinates; therefore we have found the following necessary condition of optimality for a regular solution A of the isoperimetric problem (1.5):

the mean curvature of $D\cap\partial A$ is locally constant.

A more careful inspection of the proof above actually shows that the constant is the same on all $D\cap\partial A$. Indeed, if x_1 and x_2 are two points with neighbourhoods ω_1 and ω_2, and $u(x)$ is a function whose graph is ∂A in $\omega_1\cup\omega_2$, the computation above gives

$$-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = c_1 \text{ in } \omega_1, \qquad -\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = c_2 \text{ in } \omega_2$$

with c_1 and c_2 constants. Take as a perturbation the function $u + \varepsilon(\phi_1 + \phi_2)$ where ϕ_1, ϕ_2 are smooth and with support in ω_1, ω_2 respectively. The measure constraint gives

$$\int_{\omega_1} \phi_1 \, dx + \int_{\omega_2} \phi_2 \, dx = 0. \tag{1.8}$$

By repeating the argument used above we obtain

$$\begin{aligned} 0 &\leq \int_{\omega_1} \frac{\nabla u \cdot \nabla \phi_1}{\sqrt{1 + |\nabla u|^2}} \, dx + \int_{\omega_2} \frac{\nabla u \cdot \nabla \phi_2}{\sqrt{1 + |\nabla u|^2}} \, dx \\ &= c_1 \int_{\omega_1} \phi_1 \, dx + c_2 \int_{\omega_2} \phi_2 \, dx. \end{aligned}$$

Since ϕ_1 and ϕ_2 are arbitrary, with the only constraint (1.8), we easily obtain that $c_1 = c_2$, and so the mean curvature of $D \cap \partial A$ is globally a constant.

When the measure constraint is replaced by the more general constraint $\int_A f(x) \, dx = c$, we may easily repeat all the previous steps and we obtain the partial differential equation

$$-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\Big) = \lambda f\big(x, u(x)\big) \tag{1.9}$$

where λ is a constant.

Finally, when the perimeter is replaced by the more general functional

$$\int_{D \cap \partial^* A} j^\infty(x, -\nu_A) \, d\mathcal{H}^{N-1},$$

then the exterior unit normal vector ν_A, when ∂A is the graph of a smooth function u, is given by

$$\nu_A = \Big(-\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \frac{1}{\sqrt{1 + |\nabla u|^2}}\Big),$$

so that the cost functional takes the form

$$\int_\omega j^\infty(x, u(x), \nabla u(x), -1) \, dx.$$

In this case the partial differential operator $-\operatorname{div}\big(\nabla u / \sqrt{1 + |\nabla u|^2}\big)$ has to be replaced by the new one obtained through the function $j^\infty(x, s, z, -1)$ that is

$$-\operatorname{div}\big(\partial_z j^\infty(x, u, \nabla u, -1)\big) + \partial_s j^\infty(x, u, \nabla u, -1).$$

1.3 The Newton problem of minimal aerodynamical resistance

The problem of finding the shape of a body which moves in a fluid with minimal resistance to motion is one of the first problems in the calculus of variations (see for instance Goldstine [134]). This can be again seen as a shape optimization problem, once the cost functional and the class of admissible shapes are defined.

In 1685 Newton studied this problem, proposing a very simple model to describe the resistance of a profile to the motion in an inviscid and incompressible medium. Here are his words (from *Principia Mathematica*):

If in a rare medium, consisting of equal particles freely disposed at equal distances from each other, a globe and a cylinder described on equal diameter move with equal velocities in the direction of the axis of the cylinder, (then) the resistance of the globe will be half as great as that of the cylinder. . . . I reckon that this proposition will be not without application in the building of ships.

The Newtonian pressure law states that the pressure coefficient is proportional to $\sin^2\theta$, with θ being the inclination of the body profile with respect to the stream direction. The deduction of this pressure law can be easily obtained from the assumption that the fluid consists of many independent particles with constant speed and velocity parallel to the stream direction, the interactions between the body and the particles obey the usual laws governing elastic shocks, and tangential friction and other effects are neglected (see Figure 1.3 below).

Suppose that the profile of the body is described by the graph of a nonnegative function u defined over the body cross section D (orthogonal to the fluid stream). A simple calculation gives that the effect due to the impact of a single particle, which slows the body down, that is the momentum in vertical direction, is proportional to the mass of the particle times $\sin^2\theta$. Since

$$\sin^2\theta = \frac{1}{1+\tan^2(\pi/2-\theta)} = \frac{1}{1+|\nabla u|^2}\,,$$

the total resistance of the body turns out to be proportional to the integral

$$F(u) = \int_D \frac{1}{1+|\nabla u|^2}\,dx. \tag{1.10}$$

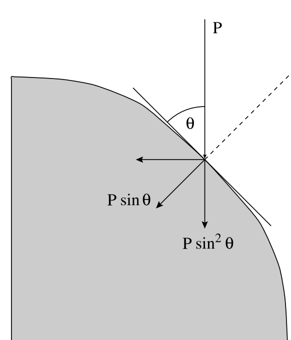

Figure 1.3. The Newtonian pressure law.

We may also define the relative resistance of a profile u, dividing the resistance $F(u)$ by the measure of D:

$$C_0(u) = \frac{F(u)}{|D|} \ .$$

In particular, if the body is a half-sphere of radius R we have $u(x) = \sqrt{R^2 - |x|^2}$ and an easy calculation gives the relative resistance

$$C_0(u) = \frac{F(u)}{\pi R^2} = 0.5$$

as predicted by Newton in 1685. Other bodies with the same value of C_0 are illustrated in Figures 1.4 and 1.5 below.

If we assume the total resistance to be our cost functional, it remains to determine the class of admissible shapes, that is the class of admissible functions u, over which the functional F has to be minimized.

Note that the integral functional F above is neither convex nor coercive. Therefore, obtaining an existence theorem for minimizers via the usual direct methods of the calculus of variations may fail.

If we do not impose any further constraint on the competing functions u, the infimum of the functional in (1.10) turns out to be zero, as it is immediate to see by taking for instance

$$u_n(x) = n \operatorname{dist}(x, \partial D)$$

for every $n \in \mathbb{N}$ and by letting $n \to +\infty$. Therefore, no function u can minimize the functional F, because $F(u) > 0$ for every function u.

One may think that the nonexistence of minimizers for F is due to the fact that the sequence $\{u_n\}$ above is unbounded in the L^∞ norm; however, even a constraint of the form

$$0 \le u \le M \tag{1.11}$$

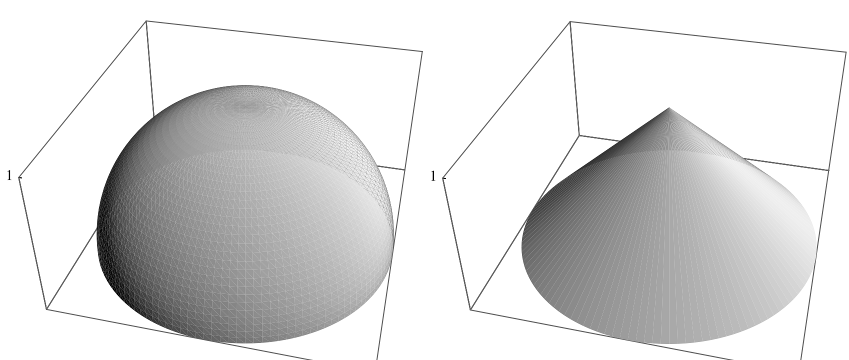

Figure 1.4. (a) half-sphere, (b) cone.

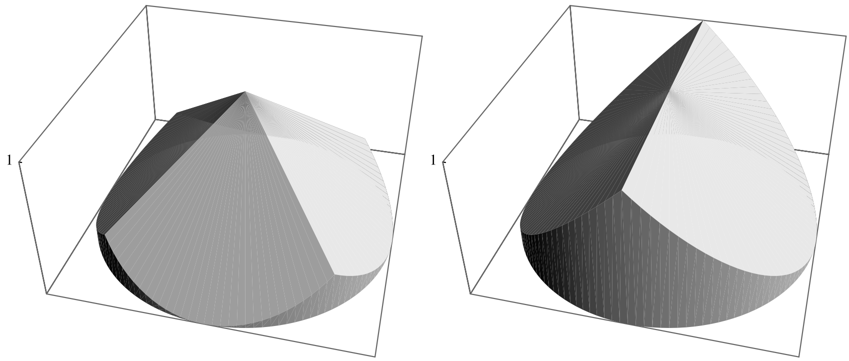

Figure 1.5. (c) pyramid 1, (d) pyramid 2.

does not help a lot for the existence of minimizers. Indeed, a sequence of functions like

$$u_n(x) = M \sin^2(n|x|)$$

satisfies the constraint (1.11) but we still have

$$\lim_{n\to+\infty} F(u_n) = 0,$$

and by the same argument used before we may conclude that again the cost functional F does not possess any minimizer, even in the more restricted class (1.11).

We shall take as admissible bodies only convex bounded domains, that is we restrict our analysis to functions u which are bounded and concave on D. More precisely, we study the minimization problem

$$\min\Big\{F(u)\ :\ 0 \le u \le M,\ u \text{ concave on } D\Big\}. \tag{1.12}$$

We shall see in Chapter 2 that the concavity constraint on u is strong enough to provide an extra compactness which implies the existence of a minimizer. On the other hand, from the physical point of view, a motivation for this constraint is that, thinking of the fluid as composed by many independent particles, each particle hits the body only once. If the body is not convex, it could happen that a particle hits the body more than once, but since $F(u)$ was constructed to measure only the resistance due to the first shock, it would no longer reflect the total resistance of the body.

Other kinds of constraints different from (1.11) can be imposed on the class of nonnegative concave functions: for instance, instead of (1.11) we may consider a bound on the surface area of the body, like

$$\int_D \sqrt{1+|\nabla u|^2}\,dx + \int_{\partial D} u\,dH^{n-1} \le c,$$

or on its volume, like

$$\int_D u\,dx \le c.$$

For a source of applications in aerodynamics, we refer for instance to Miele's book [163] and to some more recent papers ([27], [142], [192]).

Other classes of functions u, even if less motivated physically, can be considered from the mathematical point of view. A possibility could be the class of quasi-concave functions, that is of functions u whose upper level sets $\{x \in D \,:\, u(x) \geq t\}$ are all convex. Note that in the radially symmetric case a function $u = u(|x|)$ is quasi-concave if and only if it is decreasing as a function of $|x|$. Another class of admissible functions for which the problem can be studied is the class of superharmonic functions. Also the class of functions u with the property that the incoming particles hit the body only once deserves some interest. It is not the purpose of these notes to develop all details of these cases; thus we limit ourselves to the case of convex bodies, and we refer the interested reader to several papers where different situations are considered (see for instance [68], [72], [77], [87], [88], [153], [154]).

The most studied case of the Newton problem of a profile with minimal resistance is when the competing functions are supposed a priori with a radial symmetry, that is D is a (two-dimensional) disk and the functions u only depend on the radial variable $|x|$. In this case, after integration in polar coordinates, the functional F can be written in the form

$$F(u) = 2\pi \int_0^R \frac{r}{1 + |u'(r)|^2}\, dr$$

so that the resistance minimization problem becomes

$$\min\left\{ \int_0^R \frac{r}{1 + |u'(r)|^2}\, dr \;:\; u \text{ concave},\, 0 \leq u \leq M \right\}. \tag{1.13}$$

Several facts about the radial Newton problem can be shown; we simply list them by referring to [68], [70], [77] for all details.

- It is possible to show that the competing functions $u(r)$ must satisfy the conditions $u(0) = M$ and $u(R) = 0$; moreover the infimum does not change if we minimize over the larger class of decreasing functions. Therefore problem (1.13) can also be written as

$$\begin{aligned}\min\Bigg\{ \int_0^R \frac{r}{1 + |u'(r)|^2}\, dr \;:\; \\ u \text{ decreasing},\, u(0) = M,\; u(R) = 0 \Bigg\}.\end{aligned} \tag{1.14}$$

  Notice that, when the function u is not absolutely continuous, the symbol u' under the integral in (1.14) stands for the absolutely continuous part of u'.

- By using the functions $v(t) = u^{-1}(M - t)$, problem (1.14) can be rewritten in the more traditional form

$$\min\left\{ \int_0^M \frac{vv'^3}{1 + v'^2}\, dr \;:\; v \text{ increasing},\, v(0) = 0,\; v(M) = R \right\}. \tag{1.15}$$

  Again, when v is a general increasing function, v' is a nonnegative measure, and (1.14) has to be intended in the sense of BV functions, as

$$\int_0^M \frac{vv_a'^3}{1+v_a'^2}\,dt + \int_{[0,M]} vv_s' \tag{1.16}$$

where v_a' and v_s' are respectively the absolutely continuous and singular parts of the measure v' with respect to Lebesgue measure. The second integral in (1.16) has the product vv_s' which may have some ambiguity in its definition: it is then better to add and subtract vv_a' so that the functional in (1.16) can be written in a simpler way as

$$\frac{R^2}{2} - \int_0^M \frac{vv_a'}{1+v_a'^2}\,dt.$$

- The minimization problem (1.14) admits an Euler–Lagrange equation which is, in its integrated form,

$$ru' = C\big(1+u'^2\big)^2 \quad \text{on} \quad \{u' \neq 0\} \tag{1.17}$$

for a suitable constant $C < 0$. From (1.17) the solution u can actually be explicitly computed. Indeed, consider the function

$$f(t) = \frac{t}{(1+t^2)^2}\left(-\frac{7}{4} + \frac{3}{4}t^4 + t^2 - \ln t\right) \qquad \forall t \geq 1;$$

we can easily verify that f is strictly increasing so that the following quantities are well defined:

$$T = f^{-1}(M/R),$$
$$r_0 = \frac{4RT}{(1+T^2)^2}.$$

Then we obtain

$$u(r) = M \qquad \forall r \in [0, r_0]$$

and the solution u can be computed in the parametric form

$$\begin{cases} r(t) = \dfrac{r_0}{4t}(1+t^2)^2 \\ u(t) = M - \dfrac{r_0}{4}\Big(-\dfrac{7}{4} + \dfrac{3}{4}t^4 + t^2 - \ln t\Big) \end{cases} \qquad \forall t \in [1, T].$$

Notice that $|u'(r)| > 1$ for all $r > r_0$ and that $|u'(r_0^+)| = 1$; in particular, the derivative $|u'|$ never belongs to the interval $]0, 1[$.

- The optimal radial shape for $M = R$ is shown in Figure 1.6.
- It is possible to show that the optimal radial solution is unique.
- The optimal relative resistance C_0 of a radial body is then given by

$$C_0 = \frac{2}{R^2}\int_0^R \frac{r}{1+u'^2}\,dr$$

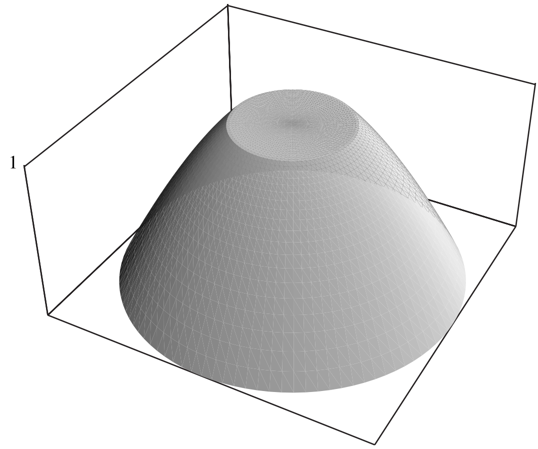

Figure 1.6. The optimal radial shape for $M = R$.

where u is the optimal solution above. We have $C_0 \in [0, 1]$ and it is easy to see that C_0 depends on M/R only. Some approximate calculations give

	$M/R = 1$	$M/R = 2$	$M/R = 3$	$M/R = 4$
r_0/R	0.35	0.12	0.048	0.023
C_0	0.37	0.16	0.082	0.049

- Moreover we obtain the following asymptotic estimates as $M/R \to +\infty$:

$$\begin{aligned} r_0/R &\approx \tfrac{27}{16}(M/R)^{-3} \qquad \text{as } M/R \to +\infty, \\ C_0 &\approx \tfrac{27}{32}(M/R)^{-2} \qquad \text{as } M/R \to +\infty. \end{aligned} \tag{1.18}$$

  Some more optimal radial shapes for different values of the ratio M/R are shown in Figure 1.7 below.
- It is interesting to notice that the optimal frustum cone, that is the frustum cone with height M, cross section radius R, and minimal resistance, is only slightly less performant than the optimal radial body computed above. Indeed, its top radius $\hat{r}_0$ and its relative resistance $\hat{C}_0$ can be easily computed, and we find

$$\hat{C}_0 = \frac{\hat{r}_0}{R} = 1 - \frac{(M/R)^2}{2}\left(\sqrt{1 + 4(M/R)^{-2}} - 1\right),$$

  with asymptotic behaviour

$$\hat{C}_0 \approx (M/R)^{-2} \qquad \text{as } M/R \to +\infty.$$

In the nonradial case, we shall see in the next chapter that it is still possible to show the existence of an optimal profile, even if little is known about its qualitative

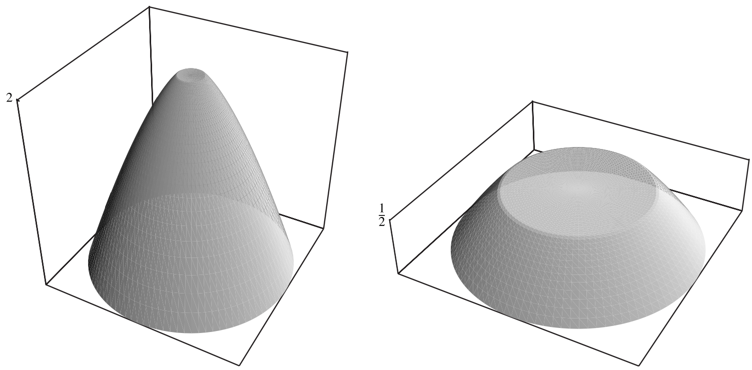

Figure 1.7. (a) the case $M = 2R$, (b) the case $M = R/2$.

behaviour. We shall see that a necessary condition of optimality is that the optimal profile must be flat, in the sense that det D^2u identically vanishes where u is of class C^2. In particular, when D is a disk, this excludes the radial Newton solution and so the optimal solution cannot be radial. This also shows that the solution is not unique in general. Up to now it is not known if optimal solutions always have a *flat nose* and if they always assume the value zero at the boundary.

1.4 Optimal interfaces between two media

In this section we study the problem of finding the minimal energy configuration for a mixture of two conducting materials when a constraint (or penalization) on the measure of the unknown interface between the two phases is added.

If D denotes a given bounded open subset of $\mathbb{R}^N$ (the prescribed container), denoting by α and β the conductivities of the two materials, the problem consists in filling D with the two materials in the most performant way according to some given cost functional. The volume of each material can also be prescribed. It is convenient to denote by A the domain where the conductivity is α and by $a_A(x)$ the conductivity coefficient

$$a_A(x) = \alpha 1_A(x) + \beta 1_{D\setminus A}(x).$$

In this way the state equation becomes

$$\begin{cases} -\operatorname{div}\big(a_A(x)\nabla u\big) = f & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{1.19}$$

where f is the (given) source density, and we denote by u_A its unique solution.

It is well known (see for instance Kohn and Strang [148], Murat and Tartar [169]) that if we take as a cost functional an integral of the form

$$\int_D j(x, 1_A, u_A, \nabla u_A)\, dx$$

in general an optimal configuration does not exist. However, the addition of a perimeter penalization is enough to imply the existence of classical optimizers. In other words, if we take as a cost the functional

$$J(u, A) = \int_D j(x, 1_A, u, \nabla u)\,dx + \sigma\,\mathrm{Per}_D(A)$$

where $\sigma > 0$, the problem can be written as an optimal control problem in the form

$$\min\big\{J(u, A) \;:\; A \subset D,\; u \text{ solves (1.19)}\big\}. \tag{1.20}$$

A volume constraint of the form $\mathrm{meas}(A) = m$ could also be present. The main ingredient for the proof of the existence of an optimal classical solution is the following result.

Theorem 1.4.1 *Let $a_n(x)$ be a sequence of $N \times N$ symmetric matrices with measurable coefficients such that the uniform ellipticity condition*

$$c_0|z|^2 \le a_n(x)z \cdot z \le c_1|z|^2 \qquad \forall x \in D,\; \forall z \in \mathbb{R}^N \tag{1.21}$$

holds with $0 < c_0 \le c_1$. Given $f \in H^{-1}(D)$ denote by u_n the unique solution of the problem

$$\begin{cases} -\,\mathrm{div}\,\big(a_n(x)\nabla u\big) = f,\\ u \in H_0^1(D). \end{cases} \tag{1.22}$$

If $a_n(x) \to a(x)$ a.e. in D then $u_n \to u$ weakly in $H_0^1(D)$, where u is the solution of (1.22) with a_n replaced by a.

Proof By the uniform ellipticity condition (1.21) we have

$$c_0 \int_D |\nabla u_n|^2\,dx \le \int_D f u_n\,dx,$$

and by the Poincaré inequality we have that u_n are bounded in $H_0^1(D)$ so that a subsequence (still denoted by the same indices) converges weakly in $H_0^1(D)$ to some v. All we have to show is that $v = u$ or equivalently that

$$-\,\mathrm{div}\,\big(a(x)\nabla v\big) = f. \tag{1.23}$$

This means that for every smooth test function ϕ we have

$$\int_D a(x)\nabla v \nabla\phi\,dx = \langle f, \phi\rangle.$$

Then it is enough to show that for every smooth test function ϕ we have

$$\lim_{n\to+\infty} \int_D a_n(x)\nabla u_n \nabla\phi\,dx = \int_D a(x)\nabla v \nabla\phi\,dx.$$

This is an immediate consequence of the fact that ϕ is smooth, $\nabla u_n \to \nabla v$ weakly in $L^2(D)$, and $a_n \to a$ a.e. in D remaining bounded.

Another way to show that (1.23) holds is to verify that v minimizes the functional

$$F(w) = \int_D a(x)\nabla w \nabla w\,dx - 2\langle f, w\rangle \qquad w \in H_0^1(D). \tag{1.24}$$

Since the function $\alpha(s, z) = sz \cdot z$, defined for $z \in \mathbb{R}^N$ and for s, a symmetric positive definite $N \times N$ matrix that is convex in z and lower semicontinuous in s, the functional

$$\Phi(a, \xi) = \int_D a(x)\xi \cdot \xi\,dx$$

is sequentially lower semicontinuous with respect to the strong L^1 convergence on a and the weak L^1 convergence on ξ (see for instance [62], [106],[143]). Therefore we have

$$F(v) = \Phi(a, \nabla v) - 2\langle f, v\rangle \le \liminf_{n\to+\infty} \Phi(a_n, \nabla u_n) - 2\langle f, u_n\rangle = \liminf_{n\to+\infty} F(u_n).$$

Since u_n minimizes the functional F_n defined as in (1.24) with a replaced by a_n, we also have for every $w \in H_0^1(D)$,

$$F_n(u_n) \le F_n(w) = \int_D a_n(x)\nabla w \nabla w\,dx - 2\langle f, w\rangle$$

so that taking the limit as $n \to +\infty$ and using the convergence $a_n \to a$ we obtain

$$\liminf_{n\to+\infty} F_n(u_n) \le \int_D a(x)\nabla w \nabla w\,dx - 2\langle f, w\rangle = F(w).$$

Thus $F(v) \le F(w)$ which shows what is required. ■

Remark 1.4.2 The result above can be rephrased in terms of G-convergence by saying that for uniformly elliptic operators of the form $-\operatorname{div}\big(a(x)\nabla u\big)$, the G-convergence is weaker than the L^1-convergence of coefficients. Analogously, we can say that the functionals

$$G_n(w) = \int_D a_n(x)\nabla w \nabla w\,dx$$

Γ-converge to the functional G defined in the same way with a in the place of a_n.

Corollary 1.4.3 *If $A_n \to A$ in $L^1(D)$, then $u_{A_n} \to u_A$ weakly in $H_0^1(D)$.*

A more careful inspection of the proof of Theorem 1.4.1 shows that the following stronger result holds.

Theorem 1.4.4 *Under the same assumptions of Theorem 1.4.1 the convergence of u_n is actually strong in $H_0^1(D)$.*

Proof We have already seen that $u_n \to u$ weakly in $H_0^1(D)$, which gives $\nabla u_n \to \nabla u$ weakly in $L^2(D)$. Denoting by $c_n(x)$ and $c(x)$ the square root matrices of $a_n(x)$ and $a(x)$ respectively, we have that $c_n \to c$ a.e. in D remaining equi-bounded. Then $c_n(x)\nabla u_n$ converge to $c(x)\nabla u$ weakly in $L^2(D)$. Multiplying equation (1.22) by u_n and integrating by parts we obtain

$$\begin{aligned}\int_D a(x)\nabla u \nabla u\,dx &= \langle f, u\rangle = \lim_{n\to+\infty}\langle f, u_n\rangle \\ &= \lim_{n\to+\infty}\int_D a_n(x)\nabla u_n \nabla u_n\,dx.\end{aligned}$$

This implies that

$$c_n(x)\nabla u_n \to c(x)\nabla u \quad \text{strongly in} \quad L^2(D).$$

Multiplying now by $\big(c_n(x)\big)^{-1}$ we finally obtain the strong convergence of ∇u_n to ∇u in $L^2(D)$. ■

We are now in a position to obtain an existence result for the optimization problem (1.20). On the function j we only assume that it is nonnegative, Borel measurable, and such that $j(x, s, z, w)$ is lower semicontinuous in (s, z, w) for a.e. $x \in D$.

Theorem 1.4.5 *Under the assumption above the minimum problem (1.20) admits at least a solution.*

Proof Let (A_n) be a minimizing sequence; then $\mathrm{Per}_D(A_n)$ are bounded, so that, up to extracting subsequences, we may assume (A_n) is strongly convergent in the L^1_{loc} sense to some set $A \subset D$. We claim that A is a solution of problem (1.20). Let us denote by u_n a solution of problem (1.19) associated to A_n; by Theorem 1.4.4 (u_n) converges strongly in $H_0^1(D)$ to some $u \in H_0^1(D)$. Then by the lower semicontinuity of the perimeter and by Fatou's lemma we have

$$J(u, A) \le \liminf_{n\to+\infty} J(u_n, A_n)$$

which proves the optimality of A. ■

Remark 1.4.6 The same proof works when volume constraints of the form $\mathrm{meas}(A) = m$ are present. Indeed this constraint passes to the limit when $A_n \to A$ strongly in $L^1(D)$.

The existence result above shows the existence of a classical solution for the optimization problem (1.20). This solution is simply a set with finite perimeter and additional assumptions have to be made in order to prove further regularity. For instance in [11] Ambrosio and Buttazzo considered the similar problem

$$\min\Big\{E(u, A) + \sigma\mathrm{Per}_D(A) \ :\ u \in H_0^1(D),\ A \subset D\Big\}$$

where $\sigma > 0$ and

$$E(u, A) = \int_D \left[a_A(x)|\nabla u|^2 + 1_A(x)g_1(x, u) + 1_{D\setminus A}g_2(x, u)\right] dx.$$

They showed that every solution A is actually an open set provided g_1 and g_2 are Borel measurable and satisfy the inequalities

$$g_i(x, s) \geq \gamma(x) - k|s|^2 \qquad i = 1, 2$$

where $\gamma \in L^1(D)$ and $k < \alpha\lambda_1$, with λ_1 being the first eigenvalue of $-\Delta$ on D.

1.5 The optimal shape of a thin insulating layer

In this section we study the optimization problem for a thin insulating layer around a conducting body; we have to put a given amount of insulating material on the boundary of a given domain in order to minimize a cost functional which describes the total heat dispersion of the domain. We consider the framework of a stationary heat equation, but the same model also applies to similar problems in electrostatics or in the case of elastic membranes.

Let D be a regular bounded open subset of $\mathbb{R}^N$ that, for simplicity, we suppose connected and let $f \in L^2(D)$ be a given function which represents the heat sources density. We assume that the boundary ∂D is surrounded by a thin layer of insulator, with thickness $d(\sigma)$, with σ being the variable which runs over ∂D. The limit problem, when the thickness of the layer goes to zero and simultaneously its insulating coefficient goes to infinity (i.e., the conductivity in the layer goes to zero too), has been studied in [38] through a PDE approach (called *reinforcement problem*) and in [1] through a Γ-limit approach, and the model obtained is the following. If u denotes the temperature of the system, then the family of approximating problems is

$$\min\left\{\int_D |\nabla u|^2\,dx + \varepsilon\int_{\Sigma_\varepsilon} |\nabla u|^2\,dx - 2\int_D fu\,dx \;:\; u \in H_0^1(D \cup \Sigma_\varepsilon)\right. \tag{1.25}$$

where Σ_ε is the thin layer of variable thickness $d(\sigma)$,

$$\Sigma_\varepsilon = \left\{\sigma + t\nu(\sigma) \;:\; \sigma \in \partial D,\; 0 \leq t < \varepsilon d(\sigma)\right\}. \tag{1.26}$$

In terms of PDE the Euler–Lagrange equation associated to problem (1.25) is an elliptic problem with a transmission condition along the interface ∂D

$$\begin{cases} -\Delta u = f & \text{in } \; D, \\ -\Delta u = 0 & \text{in } \; \Sigma_\varepsilon, \\ \frac{\partial u^-}{\partial \nu} = \varepsilon\frac{\partial u^+}{\partial \nu} & \text{on } \; \partial D, \\ u = 0 & \text{on } \; \partial(D \cup \Sigma_\varepsilon), \end{cases}$$

where u^- and u^+ respectively denote the traces of u in D and in Σ_ε.

Notice that the conductivity coefficient in the insulating layer Σ_ε has been taken equal to ε, as well as the size of the layer thickness. Passing to the limit as $\varepsilon \to 0$ (in the sense of Γ-convergence) in the sequences of energy functionals we obtain (see [1]) the limit energy which is given by

$$E(u,d) = \int_D |\nabla u|^2\,dx - 2\int_D f u\,dx + \int_{\partial D} \frac{u^2}{d}\,d\mathcal{H}^{N-1} \tag{1.27}$$

so that the temperature u solves the minimum problem

$$E(d) = \min\left\{E(u,d)\ :\ u \in H^1(D)\right\}. \tag{1.28}$$

Equivalently, problem (1.28) can be described through its Euler–Lagrange equation

$$\begin{cases} -\Delta u = f & \text{in}\quad D, \\ d\frac{\partial u}{\partial \nu} + u = 0 \text{ on} & \partial D. \end{cases} \tag{1.29}$$

We denote by u_d the unique solution of (1.28) or of (1.29). Equation (1.29) can be seen as the state equation of an optimal control problem whose state variable is the temperature of the system and whose control variable is the thickness function $d(\sigma)$. Given a fixed amount m of insulating material the control variables we consider are (measurable) thickness functions d defined on ∂D such that

$$d \geq 0 \quad \text{on} \quad \partial D, \qquad \int_{\partial D} d\,d\mathcal{H}^{N-1} = m.$$

We denote by Γ_m such a class of functions. Therefore, the optimization problem we are going to consider is

$$\min\left\{E(d)\ :\ d \in \Gamma_m\right\} = \min\left\{E(u,d)\ :\ u \in H^1(D),\ d \in \Gamma_m\right\}. \tag{1.30}$$

Remark 1.5.1 The energy $E(d)$ in (1.28) can be written in terms of the solution u_d; indeed, multiplying equation (1.29) by u_d and integrating by parts, we obtain

$$E(d) = E(u_d, d) = -\int_D f u_d\,dx. \tag{1.31}$$

Therefore, when the heat sources are uniformly distributed, that is f is (a positive) constant, the optimization problem (1.30) turns out to be equivalent to determining the function $d \in \Gamma_m$ for which the averaged temperature $\int_D u_d\,dx$ is maximal. Other criteria, different from the minimization of the energy $E(d)$, could be also investigated, as for instance obtaining a temperature as close as possible to a desired state $a(x)$,

$$\min\left\{\int_D |u_d - a(x)|^2\,dx\ :\ d \in \Gamma_m\right\}$$

or more generally

$$\min\Big\{\int_D f(x,u_d)\,dx+\int_{\partial D} g(x,d,u_d)\,d\mathcal{H}^{N-1}\ :\ d\in\Gamma_m\Big\}.$$

For further details we refer to the chapters of this volume where we consider the general theory of shape optimization for problems with Dirichlet condition on the free boundary.

Proposition 1.5.2 *For every $u\in L^2(\partial D)$ the minimum problem*

$$\min\Big\{\int_{\partial D}\frac{u^2}{d}\,d\mathcal{H}^{N-1}\ :\ d\in\Gamma_m\Big\}\tag{1.32}$$

admits a solution. This solution is unique if u is not identically zero.

Proof If $u=0$, then any function $d\in\Gamma_m$ solves the minimization problem (1.32). Assume that u is nonzero; then we claim that the function

$$d_u=m|u|\Big(\int_{\partial D}|u|\,d\mathcal{H}^{N-1}\Big)^{-1}$$

solves the minimization problem (1.32). In fact, by Hölder inequality we have, for every $d\in\Gamma_m$,

$$\Big(\int_{\partial D}|u|\,d\mathcal{H}^{N-1}\Big)^2\le\Big(\int_{\partial D}\frac{u^2}{d}\,d\mathcal{H}^{N-1}\Big)\Big(\int_{\partial D}d\,d\mathcal{H}^{N-1}\Big)=m\int_{\partial D}\frac{u^2}{d}\,d\mathcal{H}^{N-1}$$

so that

$$\int_{\partial D}\frac{u^2}{d_u}\,d\mathcal{H}^{N-1}=\frac{1}{m}\Big(\int_{\partial D}|u|\,d\mathcal{H}^{N-1}\Big)^2\le\int_{\partial D}\frac{u^2}{d}\,d\mathcal{H}^{N-1}$$

which proves the optimality of d_u. The uniqueness of the solution follows from the strict convexity of the mapping $d\mapsto 1/d$ and from the fact that every solution of (1.32) must vanish on the set $\{x\in\partial D\ :\ u(x)=0\}$. ■

Interchanging the order of the minimization in problem (1.30) we can perform first the minimization with respect to d, so that, thanks to the result of Proposition 1.5.2, problem (1.30) reduces to

$$\begin{aligned}&\min\Big\{\int_D|\nabla u|^2\,dx-2\int_D fu\,dx\\&+\frac{1}{m}\Big(\int_{\partial D}|u|\,d\mathcal{H}^{N-1}\Big)^2\ :\ u\in H^1(D)\Big\}.\end{aligned}\tag{1.33}$$

It is immediate to see that the variational problem above is convex; then it can equivalently be seen in terms of its Euler–Lagrange equation which has the form

$$\begin{cases}-\Delta u=f & \text{in}\ \ D,\\ 0\in m\frac{\partial u}{\partial\nu}+H(u)\int_{\partial D}|u|\,d\mathcal{H}^{N-1} & \text{on}\ \ \partial D,\end{cases}$$

where $H(t)$ denotes the multimapping

$$H(t) = \begin{cases} 1 & \text{if } \ t > 0, \\ -1 & \text{if } \ t < 0, \\ [-1, 1] & \text{if } \ t = 0. \end{cases}$$

The following Poincaré-type inequality will be useful.

Proposition 1.5.3 *There exists a constant C such that for every $u \in H^1(D)$,*

$$\int_D u^2 \, dx \le C\Big[\int_D |\nabla u|^2 \, dx + \Big(\int_{\partial D} |u| \, d\mathcal{H}^{N-1}\Big)^2\Big]. \tag{1.34}$$

Proof If we assume by contradiction that (1.34) is false we may find a sequence (u_n) in $H^1(D)$ such that

$$\int_D u_n^2 \, dx = 1, \qquad \int_D |\nabla u_n|^2 \, dx + \Big(\int_{\partial D} |u_n| \, d\mathcal{H}^{N-1}\Big)^2 \to 0.$$

Possibly passing to subsequences we may then assume that u_n converge weakly in $H^1(D)$ to some $u \in H^1(D)$ with $\int_D u^2 \, dx = 1$. Since $\int_D |\nabla u_n|^2 \, dx \to 0$ the convergence is actually strong in $H^1(D)$ and since $u_n \to u$ strongly in $L^2(\partial D)$ we have that

$$\nabla u \equiv 0 \text{ in } D, \qquad u \in H_0^1(D).$$

The proof is then concluded because this implies $u \equiv 0$ which is in contradiction with the fact that $\int_D u^2 \, dx = 1$. ■

Proposition 1.5.4 *For every $f \in L^2(D)$ the minimization problem (1.33) admits a unique solution.*

Proof Let (u_n) be a minimizing sequence of problem (1.33); by comparison with the null function we have

$$\int_D |\nabla u_n|^2 \, dx - 2\int_D f u_n \, dx + \frac{1}{m}\Big(\int_{\partial D} |u_n| \, d\mathcal{H}^{N-1}\Big)^2 \le 0.$$

Therefore, by using Hölder inequality, for every $\varepsilon > 0$ we have

$$\begin{aligned} \int_D |\nabla u_n|^2 \, dx + \frac{1}{m}\Big(\int_{\partial D} |u_n| \, d\mathcal{H}^{N-1}\Big)^2 &\le 2\int_D |f u_n| \, dx \\ \le 2\|f\|_{L^2(D)}\|u_n\|_{L^2(D)} &\le \frac{1}{\varepsilon}\int_D |f|^2 \, dx + \varepsilon \int_D |u_n|^2 \, dx \end{aligned}$$

for every $n \in \mathbb{N}$. By using the Poincaré-type inequality of Proposition 1.5.3 we obtain

$$\begin{aligned} \int_D |\nabla u_n|^2 \, dx + \frac{1}{m}\Big(\int_{\partial D} |u_n| \, d\mathcal{H}^{N-1}\Big)^2 &\le \frac{1}{\varepsilon}\int_D |f|^2 \, dx \\ +\varepsilon C\Big[\int_D |\nabla u_n|^2 \, dx &+ \frac{1}{m}\Big(\int_{\partial D} |u_n| \, d\mathcal{H}^{N-1}\Big)^2\Big], \end{aligned}$$

so that, by taking ε sufficiently small, (u_n) turns out to be bounded in $H^1(D)$. Possibly passing to subsequences, we may assume $u_n \to u$ weakly in $H^1(D)$ for some function $u \in H^1(D)$, and the weak $H^1(D)$-lower semicontinuity of the energy functional

$$G(u) = \int_D |\nabla u|^2 \, dx - 2 \int_D f u \, dx + \frac{1}{m}\Big(\int_{\partial D} |u| \, d\mathcal{H}^{N-1}\Big)^2$$

gives that u is a solution of problem (1.33).

In order to prove the uniqueness, assume u_1 and u_2 are two different solutions of problem (1.33); a simple computation shows that

$$\begin{aligned}
G\Big(\frac{u_1+u_2}{2}\Big) - \frac{G(u_1)+G(u_2)}{2} &= -\frac{1}{4}\int_D |\nabla u_1 - \nabla u_2|^2 \, dx \\
&+\frac{1}{4m}\Big(\int_{\partial D} |u_1+u_2| \, d\mathcal{H}^{N-1}\Big)^2 - \frac{1}{2m}\Big(\int_{\partial D} |u_1| \, d\mathcal{H}^{N-1}\Big)^2 \\
&-\frac{1}{2m}\Big(\int_{\partial D} |u_2| \, d\mathcal{H}^{N-1}\Big)^2.
\end{aligned}$$

Moreover, the right-hand side is strictly negative whenever $u_1 - u_2$ is nonconstant, which gives in this case a contradiction to the minimality of u_1 and u_2.

It remains to consider the case $u_1 - u_2 = c$ with c constant. If u_1 and u_2 have a different sign on a subset B of ∂D with $\mathcal{H}^{N-1}(B) > 0$, we have

$$|u_1 + u_2| < |u_1| + |u_2| \qquad \mathcal{H}^{N-1}\text{-a.e. on } B$$

which again contradicts the minimality of u_1 and u_2. If finally u_1 and u_2 have the same sign on ∂D we have

$$\begin{aligned}
\Big(\int_{\partial D} |u_1+u_2| \, d\mathcal{H}^{N-1}\Big)^2 - 2\Big(\int_{\partial D} |u_1| \, d\mathcal{H}^{N-1}\Big)^2 - 2\Big(\int_{\partial D} |u_2| \, d\mathcal{H}^{N-1}\Big)^2 \\
= -\Big(\int_{\partial D} |u_1-u_2| \, d\mathcal{H}^{N-1}\Big)^2 = -c^2 \mathcal{H}^{N-1}(\partial D)
\end{aligned}$$

which gives again a contradiction and concludes the proof. ∎

We are now in a position to prove an existence result for the optimization problem (1.30).

Theorem 1.5.5 *Let $f \in L^2(D)$ be fixed. Then the optimization problem (1.30) admits at least one solution d_{opt}. Moreover, denoting by u_{opt} the unique solution of (1.33), if u_{opt} does not vanish identically on ∂D we have that d_{opt} is unique and is given by*

$$d_{opt}(\sigma) = m|u_{opt}(\sigma)|\Big(\int_{\partial D} |u_{opt}| \, d\mathcal{H}^{N-1}\Big)^{-1} \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } \sigma \in \partial D.$$

Proof The proof follows straightforwardly from Proposition 1.5.2 and Proposition 1.5.4. ∎

Remark 1.5.6 It is clear that, when u_{opt} identically vanishes on ∂D, any function $d \in \Gamma_m$ can be taken as a solution of the optimization problem (1.30). However, this does not occur, at least if f is a nonnegative (and not identically zero) function, as it is easy to see by comparing the energy of the Dirichlet solution u_0 to the energy of the function $u_0 + \varepsilon\phi$ with $\phi > 0$ and $\varepsilon > 0$ small enough. Moreover, problem (1.30) does not change if we replace the constraint $\int_{\partial D} d\, d\mathcal{H}^{N-1} = m$ by the constraint $\int_{\partial D} d\, d\mathcal{H}^{N-1} \leq m$. Finally, all the previous analysis still holds if the heat sources density f is taken in the dual space $\big(H^1(D)\big)'$.

Even if u_{opt} cannot vanish identically on $\partial\Omega$ (if $f \geq 0$, $f \neq 0$) it may happen that u_{opt}, and so d_{opt}, vanishes somewhere on $\partial\Omega$. This is for instance the case when D is the annulus

$$D = \{x \in \mathbb{R}^2 : r < |x| < R\},$$

$f \equiv 1$, and m is small enough. In this case an explicit calculation (see [61]) gives that the most performant choice is to use all the insulator on the internal boundary (with a constant thickness) leaving the exterior boundary unprotected, as in Figure 1.8 below (where $r = 1$, $R = 2$)

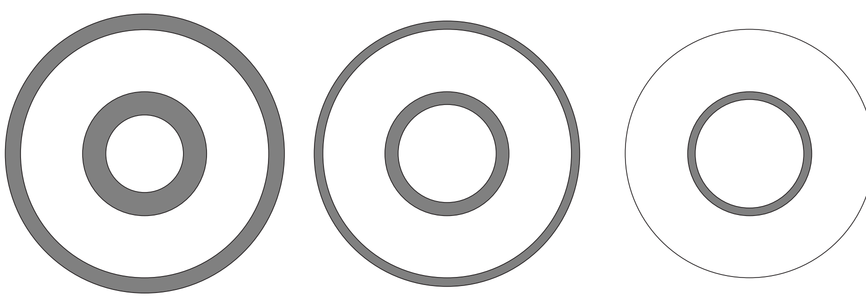

Figure 1.8. a) $m = 0.25|\partial D|$, b) $m = 0.15|\partial D|$, c) $m = 0.0379|\partial D|$.

It is then interesting to study the asymptotic behaviour of the optimal thickness $d_m(\sigma)$ as $m \to 0$. We denote by u_0 the solution of the Dirichlet problem

$$\begin{cases} -\Delta u_0 = f \text{ on } D, \\ u_0 = 0 \text{ on } \partial D, \end{cases}$$

and we assume for simplicity that D and f are regular enough to have $\frac{\partial u_0}{\partial \nu}$ continuous on ∂D (we refer to [114] for more details). It is convenient to use the rescaled variables

$$v(x) = \frac{u(x) - u_0(x)}{m}, \quad \delta(x) = \frac{d(\sigma)}{m} \tag{1.35}$$

so that the functional G can be written in the form

$$\begin{aligned}
G(u) &= \int_D |\nabla u_0 + m\nabla v|^2\,dx - 2\int_D f(u_0+mv)\,dx \\
&\quad + \frac{1}{m}\Big(\int_{\partial D} |u_0+mv|\,d\mathcal{H}^{N-1}\Big)^2 \\
&= -\int_D f u_0\,dx + 2m\int_D \nabla u_0 \nabla v\,dx + m^2\int_D |\nabla v|^2\,dx \\
&\quad -2m\int_D f v\,dx + m\Big(\int_{\partial D} |v|\,d\mathcal{H}^{N-1}\Big)^2 \\
&= -\int_D f u_0\,dx + m\Big[m\int_D |\nabla v|^2\,dx \\
&\quad +2\int_{\partial D} \frac{\partial u_0}{\partial \nu} v\,d\mathcal{H}^{N-1} + \Big(\int_{\partial D} |v|\,d\mathcal{H}^{N-1}\Big)^2\Big].
\end{aligned}$$

Denoting by u_m the optimal solutions of (1.33) and by d_m the optimal thickness

$$d_m = m|u_m(\sigma)|\Big(\int_{\partial D} |u_m|\,d\mathcal{H}^{N-1}\Big)^{-1},$$

we have that the rescaled solutions v_m and δ_m given by (1.35) are obtained by solving the minimum problems

$$\min\Big\{m\int\int_D |\nabla v|^2\,dx + 2\int_{\partial D} \frac{\partial u_0}{\partial \nu} v\,d\mathcal{H}^{N-1} + \Big(\int_{\partial D} |v|\,d\mathcal{H}^{N-1}\Big)^2 : v \in H^1(D)\Big\} \tag{1.36}$$

and by taking

$$\delta_m(\sigma) = |v_m(\sigma)|\Big(\int_{\partial D} |u_m|\,d\mathcal{H}^{N-1}\Big)^{-1}.$$

Since the functions v_m are involved only through their values on ∂D, it is convenient to denote, for every $\varphi \in H^{1/2}(\partial D)$, by w_φ the harmonic function on D having φ as boundary datum, and to write problem (1.36) as the minimization problem of the functional J_m defined on the space $\mathcal{M}(\partial D)$ of signed measures on ∂D by

$$J_m(\varphi) = \begin{cases} m\displaystyle\int_D |\nabla w_\varphi|^2 dx + 2\int_{\partial D} \frac{\partial u_0}{\partial \nu}\varphi\,d\mathcal{H}^{N-1} + \Big(\int_{\partial D} |\varphi|\,d\mathcal{H}^{N-1}\Big)^2 & \text{if } \varphi \in H^{1/2}(\partial D), \\ +\infty \quad \text{elsewhere.} \end{cases}$$

If φ_m is a minimum point of J_m we have

$$2\int_{\partial D} \frac{\partial u_0}{\partial \nu}\varphi_m\,d\mathcal{H}^{N-1} + \Big(\int_{\partial D} |\varphi_m|\,d\mathcal{H}^{N-1}\Big)^2 \le J(0) = 0$$

so that, setting

$$M = \max\Big\{\Big|\frac{\partial u_0}{\partial \nu}(\sigma)\Big| : \sigma \in \partial D\Big\},$$

we have

$$\left(\int_{\partial D} |\varphi_m| \, d\mathcal{H}^{N-1}\right)^2 \leq 2M \int_{\partial D} |\varphi_m| \, d\mathcal{H}^{N-1}$$

which implies that

$$\int_{\partial D} |\varphi_m| \, d\mathcal{H}^{N-1} \leq 2M.$$

The measures $\varphi_m \, d\mathcal{H}^{N-1} \llcorner \partial D$ are then bounded and converge (up to subsequences) to a measure μ on ∂D. It is also possible to show (see [114]) that the functional J_m converge in the sense of the Γ-convergence with respect to the weak* topology of $\mathcal{M}(\partial D)$ to the functional J defined on $\mathcal{M}(\partial D)$ by

$$J(\lambda) = \big(|\lambda|(\partial D)\big)^2 + 2 \int_{\partial D} \frac{\partial u_0}{\partial \nu} \, d\lambda.$$

By the general theory of the Γ-convergence (see [35], [91]) we have that the limit measure μ minimizes the functional J. It is now easy to show that (see [114]) $\mu = \mu^+ - \mu^-$ where

- μ^+ is nonnegative and supported by $K^- = \{x \in \partial D : \frac{\partial u_0}{\partial \nu}(x) = -M\}$;
- μ^- is nonnegative and supported by $K^+ = \{x \in \partial D : \frac{\partial u_0}{\partial \nu}(x) = +M\}$;
- $|\mu|(\partial D) = M$;
- $\int_{\partial D} \frac{\partial u_0}{\partial \nu} d\mu = -M^2$;
- the rescaled functions $\delta_m(\sigma)$ converge weakly* in $\mathcal{M}(\partial D)$ to $|\mu|/M$.

2

Optimization Problems over Classes of Convex Domains

In this chapter we deal with optimization problems whose class of admissible domains is made of convex sets. This geometrical constraint is rather strong and sufficient in many cases to guarantee the existence of an optimal solution.

In Section 2.1 the cost functional will be an integral functional of the form $\int_D f(x, u, \nabla u)\, dx$ where D is fixed and u varies in a class of convex (or concave, as in the case of the Newton problem) functions on D. We shall see that the convexity conditions provide an extra compactness which gives the existence of an optimal domain under very mild conditions on the integrand f.

In Section 2.2 we consider the case of cost functionals which are boundary integrals of the form $\int_{\partial A} f(x, \nu)\, d\mathcal{H}^{N-1}$. Again, the convexity hypothesis on the admissible domains A will enable us to obtain the existence of an optimal solution.

Section 2.3 deals with some optimization problems governed by partial differential equations of higher order; the situations considered are such that the convexity condition is strong enough to provide the existence of a solution.

In all these cases it would be interesting to enlarge the class of convex domains by imposing some weaker geometrical conditions but still strong enough to give the existence of an optimal solution.

2.1 A general existence result for variational integrals

Starting from the discussion about the Newton problem of an optimal aerodynamical profile made in Section 1.3, we consider in this section the general case of cost functionals of the form

$$F(u) = \int_D f(x, u, \nabla u)\, dx$$

where D is a given convex subset of $\mathbb{R}^N$ ($N = 2$ in the physical case) and the integrand f satisfies the very mild assumptions:

A1 the function $f : D \times \mathbb{R} \times \mathbb{R}^N \to \overline{\mathbb{R}}$ is nonnegative and measurable for the σ-algebra $\mathcal{L}_N \otimes \mathcal{B} \otimes \mathcal{B}_N$;

A2 for a.e. $x \in D$ the function $f(x, \cdot, \cdot)$ is lower semicontinuous on $\mathbb{R} \times \mathbb{R}^N$.

The case of a Newton resistance functional is described by the integrand

$$f(z) = \frac{1}{1 + |z|^2} \,.$$

Note that no convexity assumptions on the dependence of $f(x, s, z)$ on z are made. This lack of convexity in the integrand does not allow us to apply the direct methods of the calculus of variations in its usual form, with a functional defined on a Sobolev space endowed with a weak topology (see [62], [90]).

The class of admissible functions u we shall work with is, as in the Newton problem, the class

$$C_M = \left\{u \text{ concave on } D \ : \ 0 \le u \le M\right\}$$

where $M > 0$ is a given constant. Other kinds of classes are considered in the literature (see for instance [26], [68], [87], [88], [153], [154]).

The minimum problem we deal with is then

$$\min\left\{F(u) \ : \ u \in C_M\right\}. \tag{2.1}$$

Note that, since every bounded concave function is locally Lipschitz continuous in D, the functional F in (2.1) is well defined on C_M. Moreover, as a consequence of Fatou's lemma, conditions **A1** and **A2** imply that the functional F is lower semicontinuous with respect to the strong convergence of every Sobolev space $W^{1,p}(D)$ or also $W^{1,p}_{loc}(D)$.

The result we want to prove is the following.

Theorem 2.1.1 *Under assumptions* **A1** *and* **A2***, for every* $M > 0$ *the minimum problem*

$$\min\left\{F(u) \ : \ u \in C_M\right\} \tag{2.2}$$

admits at least a solution.

The proof of the existence Theorem 2.1.1 relies on the following compactness result for the class C_M (see [160]).

Lemma 2.1.2 *For every* $M > 0$ *and every* $p < +\infty$ *the class* C_M *is compact with respect to the strong topology of* $W^{1,p}_{loc}(D)$.

Proof Let (u_n) be a sequence of elements of C_M; since all u_n are concave, they are locally Lipschitz continuous on D, that is

$$\forall D' \subset\subset D \quad \forall x, y \in D' \quad |u_n(x) - u_n(y)| \le C_{n,D'}|x - y|$$

where $C_{n,D'}$ is a suitable constant. Moreover, from the fact that $0 \le u_n \le M$, the constants $C_{n,D'}$ can be chosen independent of n; in fact it is well known that we can take

$$C_{n,D'} = 2M/\operatorname{dist}(D', \partial D).$$

Therefore the sequence (u_n) is equi-bounded and equi-Lipschitz continuous on every subset D' which is relatively compact in D. Thus, by the Ascoli–Arzelà theorem, (u_n) is compact with respect to the uniform convergence in D' for every $D' \subset\subset D$. By a diagonal argument we may construct a subsequence of (u_n) (that we still denote by (u_n)) such that $u_n \to u$ uniformly on all compact subsets of D, for a suitable $u \in C_M$. Since the gradients ∇u_n are equi-bounded on every $D' \subset\subset D$, by the Lebesgue dominated convergence theorem, in order to conclude the proof it is enough to show that

$$\nabla u_n(x) \to \nabla u(x) \qquad \text{for a.e. } x \in D. \tag{2.3}$$

Let us fix an integer $k \in [1, n]$ and a point $x \in D$ where all u_n and u are differentiable (since all u_n and u are locally Lipschitz continuous, almost all points $x \in D$ are of this kind). Now, the functions $t \mapsto u_n(x + te_k)$ are concave, so that we get for every $\varepsilon > 0$,

$$\frac{u_n(x + \varepsilon e_k) - u_n(x)}{\varepsilon} \le \nabla_k u_n(x) \le \frac{u_n(x - \varepsilon e_k) - u_n(x)}{-\varepsilon}\,, \tag{2.4}$$

where we denoted by e_k the k-th vector of the canonical orthogonal basis of $\mathbb{R}^N$. Passing to the limit as $n \to +\infty$ in (2.4) we obtain for every $\varepsilon > 0$,

$$\begin{aligned}\frac{u(x + \varepsilon e_k) - u(x)}{\varepsilon} &\le \liminf_{n\to+\infty} \nabla_k u_n(x) \\ &\le \limsup_{n\to+\infty} \nabla_k u_n(x) \le \frac{u(x - \varepsilon e_k) - u(x)}{-\varepsilon}.\end{aligned} \tag{2.5}$$

Passing now to the limit as $\varepsilon \to 0$ we finally have

$$\nabla_k u(x) \le \liminf_{n\to+\infty} \nabla_k u_n(x) \le \limsup_{n\to+\infty} \nabla_k u_n(x) \le \nabla_k u(x),$$

that is (2.3), as required. ■

Proof of Theorem 2.1.1 The existence result follows from the direct methods of the calculus of variations. As we already noticed, thanks to assumptions **A1** and **A2** the functional F is lower semicontinuous with respect to the strong convergence of the Sobolev space $W^{1,p}_{loc}(D)$. By Lemma 2.1.2 the class C_M is also compact for the same convergence. This is enough to conclude that the minimum problem (2.2) admits at least a solution. ■

In particular, the problem of minimal Newtonian resistance

$$\min\Big\{\int_D \frac{1}{1 + |\nabla u|^2}\,dx \;:\; u \in C_M\Big\} \tag{2.6}$$

admits a solution for every $M \geq 0$.

A class larger than C_M that could be considered is the class of superharmonic functions

$$E_M = \{u \in H^1_{loc}(D) \ : \ 0 \leq u \leq M, \quad \Delta u \leq 0 \text{ in } D\}. \tag{2.7}$$

Here Δu is intended in the sense of distributions; then instead of requiring, as in the case C_M, that the $N \times N$ matrix $D^2 u$ is negative (as a measure), here we simply require that its trace Δu is negative. Nevertheless, we still have a compactness result as the following lemma shows.

Lemma 2.1.3 *Let (u_n) be a sequence of functions in E_M. Then for every $\alpha > 0$ there exists an open set $A_\alpha \subset D$ with* $\operatorname{meas}(A_\alpha) < \alpha$ *and a subsequence (u_{n_k}) such that ∇u_{n_k} converge strongly in $L^2_{loc}(D \setminus A_\alpha)$.*

Proof For every $\delta > 0$ let us denote by D_δ the set

$$D_\delta = \{x \in D \ : \ \operatorname{dist}(x, \partial D) > \delta\}.$$

Consider a smooth cut-off function η_δ with compact support in D and such that

$$0 \leq \eta_\delta \leq 1, \quad \eta_\delta = 1 \text{ on } D_\delta, \quad |\nabla \eta_\delta| \leq \frac{2}{\delta},$$

and set $\phi_{n,\delta} = \eta_\delta^2 (M - u_n)$. Since u_n are superharmonic we have

$$0 \leq \int_D \nabla u_n \nabla \phi_{n,\delta}\, dx = \int_D \left[2\eta_\delta (M - u_n) \nabla u_n \nabla \eta_\delta - \eta_\delta^2 |\nabla u_n|^2\right] dx$$

so that

$$\begin{aligned} \int_D \eta_\delta^2 |\nabla u_n|^2\, dx &\leq \int_D 2\eta_\delta (M - u_n) |\nabla u_n| |\nabla \eta_\delta|\, dx \\ &\leq \frac{1}{2} \int_D \eta_\delta^2 |\nabla u_n|^2\, dx + 2 \int_D (M - u_n)^2 |\nabla \eta_\delta|^2\, dx. \end{aligned} \tag{2.8}$$

Hence

$$\begin{aligned} \int_{D_\delta} |\nabla u_n|^2\, dx &\leq \int_D \eta_\delta^2 |\nabla u_n|^2\, dx \\ &\leq 4 \int_D (M - u_n)^2 |\nabla \eta_\delta|^2\, dx \\ &\leq \frac{16 M^2 \operatorname{meas}(D)}{\delta^2} = C(\delta). \end{aligned} \tag{2.9}$$

Therefore (u_n) is bounded in $H^1(D_\delta)$ and so it has a subsequence weakly convergent to some $u \in E_M$ in $H^1(D_\delta)$. Possibly passing to subsequences, and by using a diagonal argument, we may assume that (u_n) converges strongly in $L^2_{loc}(D)$.

Using Egorov's theorem, for every $\alpha > 0$ there exists an open set $A_\alpha \subset D$ with $\text{meas}(A_\alpha) < \alpha$ such that (u_n) converges uniformly on $D \setminus A_\alpha$.

Fix now $\varepsilon > 0$ and define

$$v_n = (\varepsilon + u - u_n)^+;$$

since $\Delta u_n \le 0$ we obtain

$$\begin{aligned} 0 &\le \int_D \nabla u_n \nabla(\eta_\delta^2 v_n)\,dx \\ &= \int_{\{u_n - u \le \varepsilon\}} \big[2\eta_\delta v_n \nabla u_n \nabla \eta_\delta + \eta_\delta^2 \nabla u_n \nabla v_n\big]\,dx \\ &= \int_{\{u_n - u \le \varepsilon\}} \big[2\eta_\delta(\varepsilon + u - u_n)\nabla u_n \nabla \eta_\delta - \eta_\delta^2 \nabla u_n \nabla(u_n - u)\big]\,dx \end{aligned} \tag{2.10}$$

so that

$$\int_{\{u_n - u \le \varepsilon\}} \eta_\delta^2 \nabla u_n \nabla(u_n - u)\,dx \le 2\int_{\{u_n - u \le \varepsilon\}} \eta_\delta(\varepsilon + u - u_n)\nabla u_n \nabla \eta_\delta. \tag{2.11}$$

Since $u_n - u \le \varepsilon$ on $D \setminus A_\alpha$, for n large enough, we have by (2.11)

$$\begin{aligned} \int_{D_\delta \setminus A_\alpha} |\nabla u_n - \nabla u|^2\,dx &\le \int_{D \setminus A_\alpha} \eta_\delta^2 |\nabla u_n - \nabla u|^2\,dx \\ &\le \int_{\{u_n - u \le \varepsilon\}} \big[\eta_\delta^2 \nabla u_n \nabla(u_n - u) - \eta_\delta^2 \nabla u \nabla(u_n - u)\big]\,dx \\ &\le 2\int_{\{u_n - u \le \varepsilon\}} \eta_\delta(\varepsilon + u - u_n)|\nabla u_n||\nabla \eta_\delta|\,dx \\ &\quad - \int_{\{u_n - u \le \varepsilon\}} \eta_\delta^2 \nabla u \nabla(u_n - u)\,dx. \end{aligned} \tag{2.12}$$

Since $\nabla u_n \to \nabla u$ weakly, the second integral in the last line tends to 0 as $n \to +\infty$, while

$$\begin{aligned} &\int_{\{u_n - u \le \varepsilon\}} \eta_\delta(\varepsilon + u - u_n)|\nabla u_n||\nabla \eta_\delta|\,dx \\ &\quad\le \Big(\int_D \eta_\delta^2 |\nabla u_n|^2\,dx\Big)^{1/2}\Big(\int_D |\varepsilon + u - u_n|^2 |\nabla \eta_\delta|^2\,dx\Big)^{1/2} \\ &\quad\le \frac{2C(\delta)^{1/2}}{\delta}\Big(\int_D |\varepsilon + u - u_n|^2\,dx\Big)^{1/2}. \end{aligned} \tag{2.13}$$

Passing to the limit as $n \to +\infty$ we get for every $\delta > 0$ and $\alpha > 0$

$$\lim_{n \to +\infty} \int_{D_\delta \setminus A_\alpha} |\nabla u_n - \nabla u|^2\,dx \le \frac{4\varepsilon}{\delta}\big[\text{meas}(D)C(\delta)\big]^{1/2},$$

and, since $\varepsilon > 0$ is arbitrary, the proof is concluded. ∎

The compactness result above allows us to obtain an existence result for optimization problems on the class E_M.

Theorem 2.1.4 *Let $f : D \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a bounded function which satisfies conditions* **A1**, **A2**. *Then the optimization problem*

$$\min\Big\{\int_D f(x,u,\nabla u)\,dx \;:\; u \in E_M\Big\} \tag{2.14}$$

admits a solution for every $M \geq 0$.

Proof Let (u_n) be a minimizing sequence for problem (2.14); by the argument used in the first part of Lemma 2.1.3, passing to subsequences we may assume that $u_n \to u$ weakly in $H^1(D_\delta)$, hence strongly in $L^2(D_\delta)$, for every $\delta > 0$, for a suitable $u \in E_M$. Moreover, always by Lemma 2.1.3, for every $\alpha > 0$ there exists an open set $A_\alpha \subset D$ with $\operatorname{meas}(A_\alpha) < \alpha$ and a subsequence (which we still denote by (u_n)) such that

$$\nabla u_n \to \nabla u \qquad \text{a.e. in } D \setminus A_\alpha.$$

Since f is bounded, possibly adding a constant we may reduce ourselves to the case $f \geq 0$. We may now apply Fatou's lemma to $f(x,u_n,\nabla u_n)$ on $D \setminus A_\alpha$ and we obtain

$$\begin{aligned}
&\int_D f(x,u,\nabla u)\,dx \\
&\quad = \int_{D\setminus A_\alpha} f(x,u,\nabla u)\,dx + \int_{A_\alpha} f(x,u,\nabla u)\,dx \\
&\quad \leq \liminf_{n\to+\infty} \int_{D\setminus A_\alpha} f(x,u_n,\nabla u_n)\,dx + \int_{A_\alpha} f(x,u,\nabla u)\,dx \\
&\quad \leq \liminf_{n\to+\infty} \int_D f(x,u_n,\nabla u_n)\,dx + C\alpha.
\end{aligned} \tag{2.15}$$

Finally, by letting $\alpha \to 0$ we get that u is a solution of problem (2.14). ∎

Remark 2.1.5 A more careful inspection of the proof above shows that the result of Theorem 2.1.4 still holds under the weaker growth assumption:

A3 there exist a constant C and a function $a(x,t)$ from $D \times \mathbb{R}$ into $\mathbb{R}$, increasing in t and with $a(\cdot,t) \in L^1_{loc}(D)$ such that

$$0 \leq f(x,s,z) \leq a(x,|s|) + C|z|^2 \qquad \forall (x,s,z) \in D \times \mathbb{R} \times \mathbb{R}^N.$$

Indeed, by repeating the proof above we have for every $\delta > 0$ and $\alpha > 0$,

$$\begin{aligned}
\int_{D_\delta\setminus A_\alpha} f(x,u,\nabla u)\,dx &\leq \liminf_{n\to\infty} \int_{D_\delta\setminus A_\alpha} f(x,u_n,\nabla u_n)\,dx \\
&\leq \liminf_{n\to\infty} \int_D f(x,u_n,\nabla u_n)\,dx,
\end{aligned} \tag{2.16}$$

$$\int_{D_\delta \cap A_\alpha} f(x,u,\nabla u)\,dx \le \int_{D_\delta \cap A_\alpha} [a(x,M) + C|\nabla u|^2]\,dx. \tag{2.17}$$

Summing (2.16) to (2.17) and passing to the limit as $\alpha \to 0$ we obtain

$$\int_{D_\delta} f(x,u,\nabla u)\,dx \le \liminf_{n\to\infty} \int_{D_\delta} f(x,u,\nabla u)\,dx$$

and the proof is achieved by taking $\delta \to 0$.

Other constraints than prescribing the maximal height M of the body are possible. For instance, in the case of convex bodies, we can prescribe a bound V on the volume of the body, so that we deal with the admissible class

$$C^V = \Big\{u : D \to \mathbb{R} \;:\; u \text{ concave },\; u \ge 0,\; \int_D u\,dx \le V\Big\}.$$

Alternatively, we can prescribe a bound S on the side surface of the body, so that the admissible class becomes

$$C(S) = \Big\{u : D \to \mathbb{R} \;:\; u \text{ concave },\; u \ge 0,\; \int_D \sqrt{1+|\nabla u|^2}\,dx \le S\Big\}.$$

In both cases we have an existence result similar to the one of Theorem 2.1.1. Indeed, if u is concave its sup-norm can be estimated in terms of its integral, as it is easily seen by comparing the body itself with the cone of equal height:

$$V \ge \int_D u\,dx \ge \frac{(\sup u)\operatorname{meas}(D)}{N+1}.$$

Then the volume class C^V is included in the height class C_M where $M = V(N+1)/\operatorname{meas}(D)$ and the corresponding compactness result follows from the one of Lemma 2.1.2.

The case of surface bound is similar: indeed, the sup-norm of a concave function can be estimated in terms of the surface of its graph, as it is easily seen by comparing again the body itself with the cone of equal height and by using Lemma 2.2.2:

$$S \ge \int_D \sqrt{1+|\nabla u|^2}\,dx \ge \frac{(\sup u)\mathcal{H}^{N-1}(\partial D)}{N}.$$

Then the surface class $C(S)$ is included in the height class C_M where $M = SN/\mathcal{H}^{N-1}(\partial D)$ and the corresponding compactness result again follows from the one of Lemma 2.1.2.

2.2 Some necessary conditions of optimality

Coming back to the Newton problem of minimal resistance, it is interesting to note that all solutions (we shall see that there is not uniqueness of the solution) of (2.6) verify a necessary condition of optimality, given by the following result.

Theorem 2.2.1 *Let u be a solution of problem (2.6). Then for a.e. $x \in D$ we have that $|\nabla u|(x) \notin]0, 1[$.*

In the proof of Theorem 2.2.1 we shall use the following lemma.

Lemma 2.2.2 *Let A, B be two N-dimensional closed convex subsets of $\mathbb{R}^N$ with $A \subset B$. Then $\mathcal{H}^{N-1}(\partial A) \le \mathcal{H}^{N-1}(\partial B)$ and equality holds if and only if $A = B$.*

Proof Let $P : \partial B \to \partial A$ be the projection on the closed convex set A, which maps every point of ∂B in the point of ∂A of least distance. It is well known (see for instance Brezis [37], Proposition V.3) that P is Lipschitz continuous with Lipschitz constant equal to 1. Therefore, by the general properties of Hausdorff measures (see for instance Rogers [182], Theorem 29), we obtain the inequality

$$\mathcal{H}^{N-1}(\partial A) = \mathcal{H}^{N-1}\big(P(\partial B)\big) \le \mathcal{H}^{N-1}(\partial B)$$

which proves the desired inequality.

In order to conclude the proof, if by contradiction $\mathcal{H}^{N-1}(\partial A) = \mathcal{H}^{N-1}(\partial B)$ and $A \neq B$, we can find a hyperplane S tangent to A such that, denoting by S^+ the half space bounded by S and containing A, it is

$$B \setminus S^+ \neq \emptyset.$$

It is easy to see that $B \setminus S^+$ contains an open set, so that

$$\begin{aligned} \mathcal{H}^{N-1}(\partial A) &\le \mathcal{H}^{N-1}\big(\partial(B \cap S^+)\big) \\ &= \mathcal{H}^{N-1}(\partial B) + \mathcal{H}^{N-1}(B \cap S) - \mathcal{H}^{N-1}(\partial B \setminus S^+) \\ &< \mathcal{H}^{N-1}(\partial B) \end{aligned} \tag{2.18}$$

which contradicts the assumption $H^{N-1}(\partial A) = H^{N-1}(\partial B)$ and achieves the proof. ■

Proof of Theorem 2.2.1 Let $u \in C_M$ be a solution of problem (2.6) and let v be defined as the infimum of M and of all tangent planes to the convex set $\big\{(x, y) \in D \times \mathbb{R} \ : \ 0 \le y \le u(x)\big\}$ having slope not belonging to $]0, 1[$. It is easy to see that $v \in C_M$, $v \ge u$ on D, $|\nabla v|(x) \notin]0, 1[$ for a.e. $x \in D$, and that on the set $\{v \neq u\}$ it is

$$|\nabla v| \in \{0, 1\} \quad \text{and} \quad |\nabla u| \in]0, 1[.$$

Consider now the function $\tilde{f} : \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$\tilde{f}(t) = \begin{cases} 1 - t/2 & \text{if } 0 \le t \le 1, \\ 1/(1 + t^2) & \text{if } t > 1 \end{cases} \tag{2.19}$$

and the functional

$$\tilde{F}(u) = \int_D \tilde{f}(|\nabla u|)\,dx.$$

The function $\tilde{f}$ is convex on $\mathbb{R}^+$ and we have

$$\tilde{f}(t) \le \frac{1}{1+t^2} \qquad \forall t \ge 0.$$

Therefore,

$$F(u) \ge \tilde{F}(u) = \int_{\{u=v\}} \tilde{f}(|\nabla u|)\,dx + \int_{\{u\ne v\}} \tilde{f}(|\nabla u|)\,dx.$$

Since $\nabla u = \nabla v$ a.e. on the set $\{u = v\}$, we get

$$\begin{aligned} F(u) &\ge \int_{\{u=v\}} \tilde{f}(|\nabla v|)\,dx + \int_{\{u\ne v\}} \tilde{f}(|\nabla u|)\,dx \\ &= \tilde{F}(v) + \int_{\{u\ne v\}} \big[\tilde{f}(|\nabla u|) - \tilde{f}(|\nabla v|)\big]\,dx. \end{aligned} \tag{2.20}$$

Since $|\nabla v| \notin]0, 1[$ on D we have

$$\tilde{f}(|\nabla v|) = f(|\nabla v|) \qquad \text{a.e. on } D;$$

moreover, since on $\{u \ne v\}$ it is $|\nabla v| \in \{0, 1\}$ and $|\nabla u| \in]0, 1[$, we have on $\{u \ne v\}$,

$$\tilde{f}(|\nabla u|) = 1 - \frac{|\nabla u|}{2}, \qquad \tilde{f}(|\nabla v|) = 1 - \frac{|\nabla v|}{2}.$$

Therefore,

$$\begin{aligned} F(u) &\ge F(v) + \frac{1}{2}\int_{\{u\ne v\}} \big[|\nabla v| - |\nabla u|\big]\,dx \\ &= F(v) + \frac{1}{2}\int_D \big[|\nabla v| - |\nabla u|\big]\,dx. \end{aligned} \tag{2.21}$$

By the coarea formula we obtain

$$\int_\Omega \big[|\nabla v| - |\nabla u|\big]\,dx = \int_0^M \big[\mathcal{H}^{N-1}(\{v = t\}) - \mathcal{H}^{N-1}(\{u = t\})\big]\,dt; \tag{2.22}$$

moreover, for every t the sets $\{u \ge t\}$ and $\{v \ge t\}$ are convex and

$$\{u \ge t\} \subset \{v \ge t\}.$$

Then, by Lemma 2.2.2 we get

$$\mathcal{H}^{N-1}(\{u = t\}) \le \mathcal{H}^{N-1}(\{v = t\})$$

so that, by (2.21) and (2.22)

$$F(v) \le F(u)$$

and equality holds if and only if $u = v$. Therefore, $|\nabla u|$ must be outside the interval $]0, 1[$ and the proof is achieved. ∎

For a problem of the form (2.2) let u be a solution; we assume that in an open set ω the function u

i) is of class C^2;
ii) does not attain the maximal value M;
iii) is strictly concave in the sense that its Hessian matrix is positive definite.

Moreover, we assume that the integrand f is smooth. Then it is easy to see that for every smooth function ϕ with compact support in ω we have $u + \varepsilon\phi \in C_M$ for ε small enough. Thus we can perform the usual first variation calculation which leads to the Euler–Lagrange equation

$$-\operatorname{div}\big(f_z(x,u,\nabla u)\big) + f_s(x,u,\nabla u) = 0 \ \text{ in } \ \omega.$$

In the case of the Newton functional this becomes

$$\operatorname{div}\Big(\frac{\nabla u}{(1+|\nabla u|^2)^2}\Big) = 0 \ \text{ in } \ \omega.$$

We can also perform the second variation; this gives for every ϕ,

$$\int_\omega \big[f_{zz}(x,u,\nabla u)\nabla\phi\nabla\phi + 2f_{sz}(x,u,\nabla u)\phi\nabla\phi + f_{ss}(x,u,\nabla u)\phi^2\big]\,dx \geq 0.$$

In particular, for the Newton functional we obtain for every ϕ,

$$\int_\omega \frac{2}{(1+|\nabla u|^2)^3}\big(4(\nabla u\nabla\phi)^2 - (1+|\nabla u|^2)|\nabla\phi|^2\big)\,dx \geq 0. \tag{2.23}$$

Condition (2.23) gives, as a consequence, the following result.

Theorem 2.2.3 *Let D be a circle. Then an optimal solution of the Newton problem*

$$\min\Big\{\int_D \frac{1}{1+|\nabla u|^2}\,dx \ : \ u \in C_M\Big\} \tag{2.24}$$

cannot be radial.

Proof We follow the proof given in [40], assuming for simplicity $N = 2$. Let u be the optimal radial solution of the Newton problem computed in Section 1.3; we have seen that, outside a circle of radius r_0 where $u \equiv M$, the function u is smooth, strictly concave, and does not attain the maximal value M. Then, using in (2.23) a function ϕ of the form $\eta(r)\psi(\theta)$ with $\operatorname{spt}\eta \subset]r_0, R[$, with R being the radius of D, we obtain

$$\int_{r_0}^R r\,dr\int_0^{2\pi}\Big[\frac{4|u'(r)\eta'(r)\psi(\theta)|^2}{\big(1+|u'(r)|^2\big)^3} - \frac{|\eta'(r)\psi(\theta)|^2 + |\eta(r)\psi'(\theta)|^2r^{-2}}{\big(1+|u'(r)|^2\big)^2}\Big]\,d\theta \geq 0.$$

Using $\psi(k\theta)$ instead of $\psi(\theta)$ the previous inequality becomes

$$\int_{r_0}^R r\,dr\int_0^{2\pi}\Big[\frac{4|u'(r)\eta'(r)\psi(\theta)|^2}{\big(1+|u'(r)|^2\big)^3} - \frac{|\eta'(r)\psi(\theta)|^2 + k^2|\eta(r)\psi'(\theta)|^2r^{-2}}{\big(1+|u'(r)|^2\big)^2}\Big]\,d\theta \geq 0$$

and the contradiction follows by taking $k \to +\infty$. ■

Remark 2.2.4 We may perform a similar computation for the integral

$$\int_D f\big(|\nabla u|\big)\,dx$$

and we find the second variation inequality

$$\int_\omega \frac{f'\big(|\nabla u|\big)}{|\nabla u|}|\nabla\phi|^2 + \Big(\frac{f''\big(|\nabla u|\big)}{|\nabla u|^2} - \frac{f'\big(|\nabla u|\big)}{|\nabla u|^3}\Big)(\nabla u\nabla\phi)^2\,dx \geq 0.$$

Assuming that the minimizer u is radial, the choice of ϕ as above leads to

$$\int_{r_1}^{r_2}\int_0^{2\pi} rf''(|u'|)|\eta'(r)\psi(\theta)|^2 + k^2\frac{f'(|u'|)}{r|u'|}|\eta(r)\psi'(\theta)|^2\,dr\,d\theta \geq 0,$$

with $]r_1, r_2[$ being an interval where u is smooth, strictly concave, and strictly less than M. Again, taking $k \to +\infty$ gives that the radial symmetry fails whenever $f'(|u'(r)|) < 0$ for some r, which implies the necessary condition of optimality for radial solutions

$$f'(|u'(r)|) \geq 0.$$

Remark 2.2.5 An immediate consequence of the nonradiality of the optimal Newton solutions is that problem (2.24) does not have a unique solution. In fact, rotating any nonradial solution u provides still another solution, as it is easy to verify, and therefore the number of solutions of problem (2.24) is infinite.

A more careful inspection of the proof of Theorem 2.2.3 allows us to obtain an additional necessary condition of optimality: all solutions of the Newton problem (2.24) must be "flat" in the sense specified by the following result (see [153]).

Theorem 2.2.6 *Let D be any convex domain and let u be a solution of the Newton problem (2.24). Assume that in an open set ω the function u is of class C^2 and does not touch the upper bound M. Then*

$$\det\nabla^2 u \equiv 0 \quad \text{in} \quad \omega. \tag{2.25}$$

Proof Let us fix a point $x_0 \in \omega$ and denote by a a unit vector orthogonal to $\nabla u(x_0)$. If (2.25) does not hold, then the second variation argument gives inequality (2.23) for every smooth function ϕ with support in a small neighbourhood of x_0. Take now

$$\phi(x) = \eta(x)\sin(ka\cdot x),$$

where the support of η is in a small neighbourhood of x_0 and k is large enough. We have

$$\nabla\phi(x) = \sin(ka\cdot x)\nabla\eta(x) + ka\cos(ka\cdot x)\eta(x)$$

so that, passing to the limit in (2.23) as $k \to +\infty$, we obtain

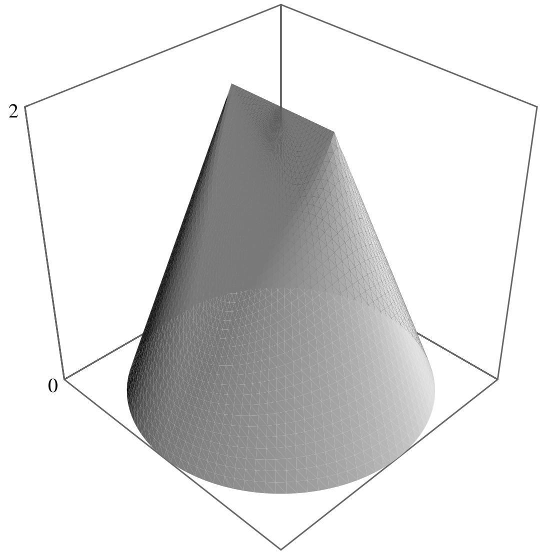

Figure 2.1. A “screwdriver” shape

$$\int_\omega \frac{2\eta^2(x)}{(1+|\nabla u|^2)^3}\big(4(a\cdot\nabla u)^2-(1+|\nabla u|^2)\big)\,dx \ge 0$$

for all η. As the support of the function η shrinks to x_0 this gives a contradiction, since $a\cdot\nabla u(x_0)=0$. ■

Remark 2.2.7 The result of Theorem 2.2.6 gives, in another way, that the solutions of the Newton problem for the case where a disc, D, cannot be radial. Moreover, the same argument can be repeated for functionals of the form $\int_D f(\nabla u)\,dx$. In this case we obtain that every minimizer u has to satisfy the condition

$$f_{zz}\big(\nabla u(x_0)\big)\ge 0 \quad \textit{whenever } u \textit{ is } C^2 \textit{ around } x_0\textit{, and} \quad \det\nabla^2 u(x_0)>0.$$

Finally, the flatness of solutions can be obtained also without assuming C^2 regularity, as it can be found in [153].

Remark 2.2.8 Another, more direct proof of the nonradiality of the solutions of the Newton problem when D is a disc, has been found by P. Guasoni in [135]. In fact, if S is the segment joining the points $(-a,0,M)$ and $(a,0,M)$, the convex hull of $S\cup(D\times\{0\})$ can be seen as the hypograph of a function $u_{a,M}\in C_M$ which is graphically represented in Figure 2.1 above.

If the number $a\in[0,R]$ is suitably chosen, the relative resistance of $u_{a,M}$ can be estimated and we obtain, after some calculations,

$$C_0(u_{a,M})=\frac{1}{\pi R^2}\int_{B_{0,R}}\frac{1}{1+|\nabla u_{a,M}|^2}\,dx\le C(M/R)^{-2}+o\big((M/R)^{-2}\big)$$

as $M/R \to +\infty$. The constant C can be computed and we find $C < 27/32$ which shows (at least for large values of M/R) that the radial function of Section 1.3 cannot be a minimizer.

The optimal solutions of the Newton problem have not yet been characterized, even if D is a disk in $\mathbb{R}^2$. Starting from the considerations made in Remark 2.2.8 concerning the Guasoni example shown in Figure 2.1, Lachand-Robert and Peletier introduced in [88] the subclass P_M of C_M made of all developable concave functions on D with values in $[0, M]$. These can be characterized as the functions whose hypograph coincides with the convex hull in $\mathbb{R}^{N+1}$ of the set

$$(K \times \{M\}) \cup (D \times \{0\})$$

where K varies among all closed convex subsets of $\overline{D}$. Therefore every function $u \in P_M$ can be identified with the closed convex set

$$K = \overline{\{x \in D : u(x) = M\}}.$$

By the compactness of C_M in $W^{1,p}_{loc}(D)$ for every $p < +\infty$ (see Lemma 2.1.2) it is easy to show that the class P_M is also compact for the same topologies. Then, under assumptions **A1** and **A2** on the integrand f, the minimization problem

$$\min\left\{\int_D f(x,u,\nabla u)\,dx : u \in P_M\right\} \tag{2.26}$$

admits a solution. In particular, if $f(z) = (1+|z|^2)^{-1}$ is the Newton integrand and D is a disk in $\mathbb{R}^2$, problem (2.26) above provides an optimal developable function w_M which we identify with the closed convex set

$$K_M = \overline{\{x \in D : w_M(x) = M\}}.$$

In [88] it is proved that all the functions w_M are more performant than the Newton radial function of the same height introduced in Section 1.3; moreover, all the sets K_M are regular polygons with n_M sides and centered in the disk D, where the number $n_m \geq 3$ of sides depends on M in a nonincreasing way.

Even if some analytical proof is not yet available, there is numerical evidence (see [152]) that the functions w_M are not optimal in the larger class C_M.

Below in Figure 2.2 are two numerical outputs from [152] which suggest that the optimal solutions are not in the developable class P_M.

2.3 Optimization for boundary integrals

In this section we consider shape optimization problems of the form

$$\min\left\{\int_{\partial A} f\big(x,\nu(x)\big)\,d\mathcal{H}^{N-1} \;:\; A \in \mathcal{A}\right\} \tag{2.27}$$

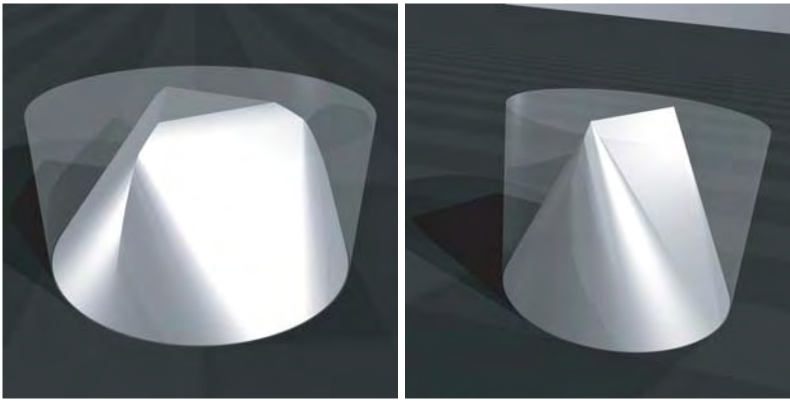

Figure 2.2. Two optimal nondevelopable Newton shapes.

where f is a nonnegative continuous function, ν is the normal unit vector exterior to A, and the class $\mathcal{A}$ of admissible domains is made of convex subsets of $\mathbb{R}^N$. This formulation allows us to consider convex bodies A which are not of Cartesian type, that is we do not need the admissible domains A to be the hypographs of concave functions u defined on a given convex set D.

The Newtonian resistance functional itself can be written in the form (2.27); in fact, for a Cartesian domain A given by the hypograph of a function u we have

$$\nu = \Big(\frac{-\nabla u}{\sqrt{1+|\nabla u|^2}}, \frac{1}{\sqrt{1+|\nabla u|^2}}\Big),$$

so that

$$\frac{1}{1+|\nabla u|^2} = (\nu_N)^2.$$

Therefore, since changing the integration on D into an integral on ∂A provides an additional factor $(1+|\nabla u|^2)^{-1/2} = \nu_N$, the Newtonian resistance functional takes the form

$$F(A) = \int_D \frac{1}{1+|\nabla u|^2}\,dx = \int_{\operatorname{graph} u} \nu_N^3(x)\,d\mathcal{H}^{N-1} = \int_{\partial A} \big(\nu_N^+(x)\big)^3\,d\mathcal{H}^{N-1},$$

where the positive part in $\nu_N^+(x)$ is due to the fact that we do not want to take into account the lower horizontal part $\partial A \setminus \operatorname{graph} u = D \times \{0\}$, on which $\nu_N < 0$. More generally, if a is the direction of the motion of the fluid stream, the Newtonian resistance has the form (2.27) with

$$f(x,\nu) = \big((a\cdot\nu)^+\big)^3.$$

The admissible class we consider is

$$C_{K,Q} = \big\{A \text{ convex subset of } \mathbb{R}^N \ :\ K \subset A \subset Q\big\} \tag{2.28}$$

where K and Q are two given compact subsets of $\mathbb{R}^N$. In the case of Newton's problem with prescribed height of Sections 1.3 and 2.1 we have

$$Q = \overline{D} \times [0, M], \qquad K = \overline{D} \times \{0\}.$$

The existence result we are going to prove is the following.

Theorem 2.3.1 *Let $f : \mathbb{R}^N \times S^{N-1} \to [0, +\infty]$ be a lower semicontinuous function and let K and Q be two given compact subsets of $\mathbb{R}^N$ such that the class $C_{K,Q}$ is nonempty. Then the minimum problem*

$$\min \left\{ \int_{\partial A} f\big(x, \nu(x)\big)\, d\mathcal{H}^{N-1} \ :\ A \in C_{K,Q} \right\} \tag{2.29}$$

admits at least one solution.

We shall use several notions about measures, collected in the following definition.

Definition 2.3.2 *For every Borel measure μ on $\mathbb{R}^N$ with values in $\mathbb{R}^N$ we define the variation of μ as the nonnegative measure $|\mu|$ defined for every Borel subset B of $\mathbb{R}^N$ by*

$$|\mu|(B) = \sup \left\{ \sum_n |\mu(B_n)| \ :\ \cup_n B_n = B \right\}.$$

We denote by $\mathcal{M}$ the class of all measures μ such that $|\mu|(\mathbb{R}^N) < +\infty$, and for each $\mu \in \mathcal{M}$ we set

$$\|\mu\| = |\mu|(\mathbb{R}^N).$$

If $\mu \in \mathcal{M}$ the symbol ν_μ will denote the Radon–Nikodym derivative $d\mu/d|\mu|$, which is a μ-measurable function from $\mathbb{R}^N$ into S^{N-1}.

Finally we say that a sequence (μ_h) of measures in $\mathcal{M}$ converges in variation to μ if

$$\mu_h \to \mu \quad \text{weakly}^* \quad \text{in } \mathcal{M} \quad \text{and} \quad \lim_{n\to+\infty} \|\mu_h\| = \|\mu\|.$$

The main tool we use in the proof of Theorem 2.3.1 is the following Reshetnyak result (see [179]) on functionals defined on measures.

Theorem 2.3.3 *Let $f : \mathbb{R}^N \times S^{N-1} \to \mathbb{R}$ be a bounded continuous function. Then the functional $F : \mathcal{M} \to \mathbb{R}$ defined by*

$$F(\mu) = \int_{\mathbb{R}^N} f(x, \nu_\mu)\, d|\mu| \tag{2.30}$$

is continuous with respect to the convergence in variation.

Corollary 2.3.4 *If $f : \mathbb{R}^N \times S^{N-1} \to [0, +\infty]$ is lower semicontinuous, then the functional defined in (2.30) turns out to be lower semicontinuous with respect to the convergence in variation.*

Proof It is enough to approximate the function f by an increasing sequence (f_n) of bounded continuous functions, to apply to every functional

$$F_n(\mu) = \int_{\mathbb{R}^N} f_n(x, \nu_\mu)\, d|\mu|$$

the result of Theorem 2.3.3, and to pass to the supremum as $n \to +\infty$ by using the monotone convergence theorem. ■

The following lemma will be also used.

Lemma 2.3.5 *Let A_n, A be bounded convex subsets of $\mathbb{R}^N$ with $A_n \to A$ in $L^1(\mathbb{R}^N)$. Then*

$$\lim_{n\to+\infty} \mathcal{H}^{N-1}(\partial A_n) = \mathcal{H}^{N-1}(\partial A).$$

Proof As A_n converges to A in $L^1(\mathbb{R}^N)$ it follows that

$$\forall \varepsilon > 0 \quad \exists n_\varepsilon \ : \ n > n_\varepsilon \quad \Rightarrow \quad A_n \subset A + B_{0,\varepsilon}.$$

Therefore, by Lemma 2.2.2, we obtain for $n > n_\varepsilon$,

$$\mathcal{H}^{N-1}(\partial A_n) \le \mathcal{H}^{N-1}\big(\partial(A + B_{0,\varepsilon}\big)$$

so that

$$\limsup_{n\to+\infty} \mathcal{H}^{N-1}(\partial A_n) \le \limsup_{\varepsilon\to0^+} \mathcal{H}^{N-1}\big(\partial(A + B_{0,\varepsilon}\big) = \mathcal{H}^{N-1}(\partial A).$$

On the other hand, by the L^1 lower semicontinuity of the perimeter,

$$\liminf_{n\to+\infty} \mathcal{H}^{N-1}(\partial A_n) \ge \mathcal{H}^{N-1}(\partial A)$$

and the proof is complete. ■

Proof of Theorem 2.3.1 It is convenient to restate the problem in terms of functionals depending on vector measures. To this aim, to every convex set $A \in \mathcal{C}_{K,Q}$ we associate its characteristic function 1_A defined by

$$1_A(x) = \begin{cases} 1 \text{ if } & x \in A, \\ 0 \text{ if } & x \notin A, \end{cases} \tag{2.31}$$

and the distributional gradient $\nabla 1_A$ which is a vector measure of the class $\mathcal{M}$. It is well known that, since A is convex, the measures $|\nabla 1_A|$ and $\mathcal{H}^{N-1} \llcorner \partial A$ coincide, so that the cost functional can be written in the form

$$\int_{\partial A} f\big(x, \nu(x)\big)\, d\mathcal{H}^{N-1} = \int_Q f\big(x, \nu_{\mu_A}(x)\big)\, d|\mu_A|$$

where we denote by μ_A the measure $\nabla 1_A$. By the Reshetnyak Theorem 2.3.3 and the related Corollary 2.3.4 the functional above is lower semicontinuous with respect to the convergence in variation of the measures μ_A, so in order to apply the direct methods of the calculus of variations it remains to show that the class

$$M_{K,Q} = \{\mu \in \mathcal{M} \ : \ \mu = \nabla 1_A, \ A \in C_{K,Q}\}$$

is compact for the same convergence. Let (A_n) be a sequence of convex domains of $C_{K,Q}$; by Lemma 2.2.2 we have

$$\|\nabla 1_{A_n}\| = \mathcal{H}^{N-1}(\partial A_n) \leq \mathcal{H}^{N-1}(\partial \tilde{Q})$$

where $\tilde{Q}$ denotes the convex envelope of Q. Hence the sequence (1_{A_n}) is bounded in BV, so that we may assume, up to extracting a subsequence, it converges weakly* in BV to some function of the form 1_A. In particular we have $A_n \to A$ strongly in L^1, which implies that A is a convex domain of $C_{K,Q}$, and by Lemma 2.3.5

$$\lim_{n\to+\infty} \|\nabla 1_{A_n}\| = \lim_{n\to+\infty} \mathcal{H}^{N-1}(\partial A_n) = \mathcal{H}^{N-1}(\partial A) = \|\nabla 1_A\|,$$

which gives the required convergence in variation and concludes the proof. ■

Remark 2.3.6 All the arguments above work in a similar way if instead of the class $C_{K,Q}$ we work with a volume constraint and so with one of the admissible classes

$$\mathcal{A}^{V,Q} = \{A \text{ convex subset of } \mathbb{R}^N \ : \ A \subset Q, \ \text{meas}(A) \geq V\},$$

$$\mathcal{A}_{K,V} = \{A \text{ convex subset of } \mathbb{R}^N \ : \ K \subset A, \ \text{meas}(A) \leq V\}.$$

Similarly, the optimization problem above can be considered with a surface constraint, in one of the admissible classes

$$\mathcal{B}^{S,Q} = \{A \text{ convex subset of } \mathbb{R}^N \ : \ A \subset Q, \ \mathcal{H}^{N-1}(A) \geq V\},$$

$$\mathcal{B}_{K,V} = \{A \text{ convex subset of } \mathbb{R}^N \ : \ K \subset A, \ \mathcal{H}^{N-1}(A) \leq V\}.$$

Another possible choice for the admissible class (see Buttazzo and Guasoni [71]) is obtained if also the section of the unknown domain A, with respect to a given hyperplane π, is involved in the optimization. We then have the class

$$S_{K,Q,m} = \{A \text{ convex subset of } \mathbb{R}^N \ : \ K \subset A \subset Q, \ \mathcal{H}^{N-1}(A \cap \pi) \geq m\}$$

for which all the previous analysis can be repeated.

2.4 Problems governed by PDE of higher order

In this section we deal with optimization problems on classes of convex domains, of a type different from the ones considered in Section 2.3. In particular, the class of admissible domains will be similar to the one of Section 2.3, that is

$$C_m(K, Q) = \Big\{A \text{ convex subset of } \mathbb{R}^N \ : \ K \subset A \subset Q, \ \operatorname{meas}(A) = m\Big\} \tag{2.32}$$

where K and Q are two given compact subsets of $\mathbb{R}^N$. The cost functional, however, is of a different type and may involve PDE of higher order as a state equation. Problems of this type have been studied for instance in [191].

Let us start by introducing some useful notions about convex sets and by studying their properties. A natural topology on the class of convex sets is given by the so-called Hausdorff distance.

Definition 2.4.1 *The Hausdorff distance between two closed sets A, B of $\mathbb{R}^N$ is defined by*

$$d(A, B) = \sup_{x\in A} d(x, B) \vee \sup_{x\in B} d(x, A)$$

where $d(x, E) = \inf\{|x - y| \ : \ y \in E\}$.

Remark 2.4.2 It is well known that the class of all closed subsets of a given compact set is compact with respect to the Hausdorff distance. Moreover, the convergence $A_n \to A$ induced by the Hausdorff distance is equivalent to the so-called uniform convergence, which occurs if for every $\varepsilon > 0$ there exists n_ε such that

$$A_n \subset A + B_{0,\varepsilon} \quad \text{and} \quad A \subset A_n + B_{0,\varepsilon} \qquad \forall n \geq n_\varepsilon$$

$B_{0,\varepsilon}$ being the ball in R^N centered at the origin and of radius ε.

We summarize here below some of the properties of convex sets.

Proposition 2.4.3 *The following facts hold for convex sets.*

i) *If $A \subset B$ then $\mathcal{H}^{N-1}(\partial A) \leq \mathcal{H}^{N-1}(\partial B)$;*

ii) *If $A_n \to A$ uniformly, then $A_n \to A$ in L^1, hence* $\operatorname{meas}(A_n) \to \operatorname{meas}(A)$ *and $\mathcal{H}^{N-1}(\partial A_n) \to \mathcal{H}^{N-1}(\partial A)$;*

iii) $\operatorname{meas}(A) < \rho\mathcal{H}^{N-1}(\partial A)$ *where ρ is the radius of the largest ball included in A.*

Proof Statement i) is proved in Lemma 2.2.2. To prove statement ii) it is enough to notice that, by the definition of uniform convergence we have for every $\varepsilon > 0$,

$$A_n \setminus A \subset \big(A + B_{0,\varepsilon}\big) \setminus A \quad \text{for all } n \text{ large enough,}$$

so that $\text{meas}(A_n \setminus A) \leq C\varepsilon$ for a suitable constant C. Analogously we have $\text{meas}(A \setminus A_n) \leq C\varepsilon$ which gives the L^1 convergence of A_n to A and the rest of the statement follows from Lemma 2.3.5. Finally, for the proof of statement iii) we refer to [172]. ∎

Proposition 2.4.4 *The class $C_m(K, Q)$ defined in (2.32) is compact for the uniform convergence.*

Proof Let (A_n) be a sequence in $C_m(K, Q)$; since all A_n are contained in the convex envelope $co(Q)$ of Q, by Proposition 2.4.3 i) we obtain

$$\mathcal{H}^{N-1}(\partial A_n) \leq \mathcal{H}^{N-1}(\partial co(Q))$$

so that by Proposition 2.4.3 iii) we have that the largest ball included in A_n has a radius

$$\rho_n > m/\mathcal{H}^{N-1}(\partial co(Q)) \ .$$

Therefore, possibly passing to a subsequence, that we still denote by (A_n), we may assume that there exists a ball $B_{x_0,\rho}$ with $\rho > 0$, which is contained in every A_n. Then the boundary ∂A_n can be described in the polar form

$$x - x_0 = r_n(\theta) \qquad x \in \partial A_n, \ \theta \in S^{N-1}.$$

Since $B_{x_0,\rho} \subset A_n \subset Q$ it is easy to see that the functions r_n have to be equi-Lipschitz continuous, so that by the Ascoli–Arzelà theorem we may assume they converge uniformly to some function $r(\theta)$. This function describes the boundary of the limit set A by the polar form

$$x - x_0 = r(\theta) \qquad x \in \partial A, \ \theta \in S^{N-1}.$$

Thus we have $A_n \to A$ uniformly; moreover it is easy to see that $A \in C_m(K, Q)$, which achieves the proof. ∎

Theorem 2.4.5 *Let $J : C_m(K, Q) \to [0, +\infty]$ be a cost functional which is lower semicontinuous with respect to the uniform convergence; then the optimization problem*

$$\min\left\{J(A) \ : \ A \in C_m(K, Q)\right\}$$

admits at least a solution.

Proof The proof is a straightforward consequence of Proposition 2.4.4 and of the direct methods of the calculus of variations. ∎

As an application of the previous result we present here two examples taken from [191] where the related optimization problems involve PDE of higher order.

In the first example we consider an elliptic operator L of order 2ℓ, of the form

$$Lu = \sum_{|\alpha|,|\beta|=\ell} (-1)^\ell D^\alpha \big(a_{\alpha,\beta}(x) D^\beta u\big), \tag{2.33}$$

where the coefficients $a_{\alpha,\beta}$ are bounded and satisfy the ellipticity condition

$$c_0 \sum_{|\alpha|=\ell} \xi^{2\alpha} \leq \sum_{|\alpha|,|\beta|=\ell} a_{\alpha,\beta}(x)\xi^\alpha \xi^\beta$$

for every $\xi \in \mathbb{R}^N$, where c_0 is a positive constant. For every $A \in C_m(K, Q)$ we denote by $\lambda_j(A)$ the j-th eigenvalue of L, counted with its multiplicity, on the Sobolev space $H_0^\ell(A)$, and by $e_{j,A}$ a corresponding eigenfunction which satisfies the equation

$$\begin{cases} Lu = \lambda_j(A)u \text{ in } A, \\ u \in H_0^\ell(A). \end{cases} \tag{2.34}$$

It is well known that $\lambda_j(A)$ admits the following variational characterization:

$$\lambda_j(A) = \min_{H\in\Sigma_j} \max \Big\{ \langle Lu, u\rangle \ : \ u \in H,\ \int_A u^2\,dx = 1 \Big\}$$

where Σ_j is the class of all linear subspaces of $H_0^\ell(A)$ of dimension j. Therefore it is easy to prove that all $\lambda_j(A)$ are monotone decreasing as functions of the domain A, with respect to the set inclusion. Moreover, in terms of eigenfunctions we also have

$$\lambda_j(A) = \min \Big\{ \langle Lu, u\rangle \ : \ u \in H_0^\ell(A),\ \int_A u^2\,dx = 1, \\ \int_A u e_{i,A}\,dx = 0 \text{ for } i < j \Big\}. \tag{2.35}$$

By the monotonicity of λ_j we have

$$\lambda_j\big(co(Q)\big) \leq \lambda_j(A) \leq \lambda_j\big(B_{x,\rho}\big)$$

where $B_{x,\rho}$ denotes the largest ball included in A, and by Proposition 2.4.3 iii) we have $\rho \geq m/\mathcal{H}^{N-1}(\partial co(Q))$. Since ρ is bounded from below, the previous inequality shows that for every integer j the quantity $\lambda_j(A)$ is bounded when A varies in $C_m(K, Q)$.

Proposition 2.4.6 *For every integer j, the mapping $\lambda_j : C_m(K, Q) \to \mathbb{R}$ is continuous for the uniform convergence.*

Proof Fix an integer j and take a sequence (A_n) in $C_m(K, Q)$ converging to A uniformly. Up to extracting a subsequence, thanks to Proposition 2.4.3, we may assume that all A_n and A contain a ball of radius ρ centered in a point that, without loss of generality, we may assume to be the origin. Moreover $A_n \to A$ in L^1. Then by

Remark 2.4.2 and by the monotonicity of λ_j, for every $\varepsilon > 0$ we have for n large enough

$$\begin{cases} \lambda_j(A) \geq \lambda_j\big(A_n + B_{0,\varepsilon}\big) \geq \lambda_j\big((1+c\varepsilon)A_n\big), \\ \lambda_j(A_n) \geq \lambda_k\big(A + B_{0,\varepsilon}\big) \geq \lambda_j\big((1+c\varepsilon)A\big), \end{cases} \tag{2.36}$$

where the constant $c > 0$ can be taken independent of n and ε. It is now easy, by repeating the arguments already seen in Section 1.4, and by using (2.35), to show that $\lambda_j(A_n) \to \lambda_j(A)$ and that the corresponding eigenfunctions $e_{j,A_n} \to e_{j,A}$ strongly in $H_0^\ell(\mathbb{R}^N)$. ■

Let us consider now a cost functional of the form

$$F(A) = \Phi\big(\Lambda(A)\big)$$

where $\Lambda(A)$ denotes the whole spectrum of the operator L over $H_0^\ell(A)$. We assume that the function Φ is lower semicontinuous, in the sense that

$$\Phi(\Lambda) \leq \liminf_{n\to+\infty} \Phi(\Lambda_n) \qquad \text{whenever } \Lambda_n \to \Lambda,$$

where the convergence $\Lambda_n \to \Lambda$ is defined by

$$\Lambda_n \to \Lambda \qquad \Longleftrightarrow \qquad \lambda_{j,n} \to \lambda_j \quad \forall j = 1, \ldots .$$

In particular, if Φ depends only on a finite number M of variables, then the lower semicontinuity above reduces to the usual lower semicontinuity in $\mathbb{R}^M$.

Theorem 2.4.7 *Let Φ be lower semicontinuous in the sense above. Then the optimization problem*

$$\min\big\{\Phi\big(\Lambda(A)\big) \;:\; A \in C_m(K, Q)\big\}$$

admits at least a solution.

Proof It is enough to apply the direct methods of the calculus of variations, taking into account the results previously obtained in Proposition 2.4.4 and in Proposition 2.4.6. ■

In the second example we consider again an operator of the form (2.33) and cost functionals

$$F(A) = \int_{\mathbb{R}^N} j(x, u_A, \nabla u_A, \ldots, D^\ell u_A)\, dx$$

where we denoted by u_A the solution of

$$\begin{cases} Lu = f & \text{in } A, \\ u \in H_0^\ell(A), \end{cases} \tag{2.37}$$

f being a given function in $L^2(\mathbb{R}^N)$, or more generally in $H^{-\ell}(\mathbb{R}^N)$.

Theorem 2.4.8 *Assume that j is a nonnegative Borel function such that $j(x, \cdot, \dots, \cdot)$ is lower semicontinuous. Then the optimization problem*

$$\min\left\{F(A)\ :\ A \in C_m(K, Q)\right\}$$

admits at least a solution.

Proof It is enough to repeat the arguments used in the proof of Theorem 2.4.7, noticing that, as before, we have $u_{A_n} \to u_A$ strongly in $H_0^\ell(\mathbb{R}^N)$ whenever $A_n \to A$ uniformly. ∎

3

Optimal Control Problems: A General Scheme

Optimal control problems are minimum problems which describe the behaviour of systems that can be modified by the action of an operator. Many problems in applied sciences can be modeled by means of optimal control problems. Two kinds of variables (or sets of variables) are then involved: one of them describes the state of the system and cannot be modified directly by the operator, it is called the *state variable*; the second one, on the contrary, is under the direct control of the operator that may choose its strategy among a given set of admissible ones, it is called the *control variable*.

The operator is allowed to modify the state of the system indirectly, acting directly on control variables; only these ones may act on the system, through a link control-state, usually called *state equation*. Finally, the operator, acting directly on controls and indirectly on states through the state equation, must achieve a goal usually written as a minimization of a functional, which depends on the control that has been chosen as well as on the corresponding state, the so-called *cost functional*.

Driving a car is a typical example of an optimal control problem: the driver may only act directly on controls which are in this case the accelerator, the brakes, and the steering-wheel; the state of the car is on the contrary described by its position and velocity which, of course, depend on the controls chosen by the driver, but are not directly controlled by him. The state equations are the usual equations of mechanics which, to a given choice of acceleration and steering angle, associate the position and velocity of the car, also taking into account the specifications of the engine (technological constraints, nonlinear behaviours, ...). Finally, the driver wants to achieve a goal, for instance to minimize the total fuel consumption to run along a given path. Then we have an optimal control problem, where the driver has to choose the best driving strategy to minimize the cost functional, which is in this case the total fuel consumption.

According to what was said above the ingredients of an optimal control problem are:

i) a space of states Y;

ii) a set of controls U;

iii) a the set $\mathcal{A}$ of *admissible pairs*, that is a subset of pairs $(u, y) \in U \times Y$ such that y is linked to u through the state equation;

iv) a cost functional $J : U \times Y \to \overline{\mathbb{R}}$.

The optimal control problem then takes the form of a minimization problem written as

$$\min\left\{J(u, y) \ : \ (u, y) \in \mathcal{A}\right\}.$$

We are specially interested in the study of shape optimization problems, where the control variable runs over classes of domains. For this reason we have to consider a framework general enough to include cases when the control variable does not belong to a space with a linear topological structure. On the contrary, taking the state variable as an element of a space of functions (a Sobolev space, a space of functions with bounded variation, ...) is the most studied case in the literature, and covers several important situations from the applications. Notice that in the list i) – iv) above we stressed the difference between the *space* Y and the *set* U.

The choice of a topology on Y and U is a very important matter when dealing with the question of existence of solutions to an optimal control problem. This is related to the use of direct methods of the calculus of variations, which require, for the problem under consideration, suitable lower semicontinuity and compactness assumptions.

In several cases of shape optimization problems it is known that an optimal solution does not exist; therefore minimizing sequences of domains cannot converge to an admissible domain, in any sense which preserves the lower semicontinuity of the cost functional. In order to study the asymptotic behaviour of minimizing sequences we shall endow U with an *ad hoc* topology, mainly depending on the state equation considered, and limits of minimizing sequences will be seen as optimal *relaxed* solutions which then turn out to belong to a larger space. In this chapter we give a rather general way of constructing this larger space of *relaxed controls*. Due to the great generality of our framework, the relaxed controls will be characterized simply as the elements of a Cauchy completion of a metric space; of course, when dealing with a more specific optimization problem, a more precise characterization will be needed: in the rest of these notes we shall see some relevant examples where this can be done.

3.1 A topological framework for general optimization problems

In this section we consider an abstract optimal control problem of the form

$$\min\left\{J(u, y) \ : \ (u, y) \in \mathcal{A}\right\} \tag{3.1}$$

where Y is the space of states, U is the set of controls, $J : U \times Y \to \overline{\mathbb{R}}$ is the cost functional, and $\mathcal{A} \subset U \times Y$ is the set of admissible pairs, determined in the applications by a state equation. We assume that Y is a separable metric space, while the controls vary in a set U with no topological structure a priori given. As already remarked in the introduction of Chapter 3 this happens in some quite important situations like shape optimization problems where the set of controls is given by suitable classes of admissible domains. To handle this situation it is convenient to write the set $\mathcal{A}$ of admissible pairs in the form

$$\mathcal{A} = \big\{(u, y) \in U \times Y \ : \ y \in \operatorname{argmin} G(u, \cdot)\big\} \tag{3.2}$$

where $G : U \times Y \to \overline{\mathbb{R}}$ is a given functional and where $\operatorname{argmin} G(u, \cdot)$ denotes the set of all minimum points of $G(u, \cdot)$. In the case $G(u, \cdot)$ is an integral functional of the calculus of variations whose integrand depends on the control u, its Euler–Lagrange equation provides the differential state equation. We shall call G the *state functional*. It is worth noticing that the set $\mathcal{A}$ can be always written in the form (3.2) by choosing

$$G(u, y) = \chi_{\mathcal{A}}(u, y) = \begin{cases} 0 & \text{if } (u, y) \in \mathcal{A}, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.3}$$

Therefore, the optimal control problem (3.1) can be written in the form

$$\min \big\{J(u, y) \ : \ y \in \operatorname{argmin} G(u, \cdot)\big\}. \tag{3.4}$$

For instance, a state equation like

$$\begin{cases} -\Delta y = f \ \text{ in } A, \\ y \in H_0^1(A) \end{cases} \tag{3.5}$$

is provided by the state functional

$$G(A, y) = \int_{\mathbb{R}^N} |\nabla y|^2 \, dx - \langle f, y \rangle + \chi_{H_0^1(A)}(y) \ ,$$

where the states vary in the Sobolev space $H^1(\mathbb{R}^N)$ and the control A varies in a class of domains.

Let us notice that in the applications the space Y of states is usually a separable reflexive Banach space of functions endowed with its weak topology (or the dual of a separable Banach space, endowed with its weak* topology), which is not, unless it is finite dimensional, metrizable. However, thanks to some growth assumptions on the cost functional J, we may often restrict ourselves to work on a bounded subset of Y which is, as it is well known, metrizable.

We shall endow U with a topology which is constructed by means of the functional G: the natural topology on U that takes into account the convergence of minimizers of G is the one related to the Γ-convergence of the mappings $G(u, \cdot)$ and will then be denoted by γ-convergence. Clearly, as soon as the convergence of controls implies the convergence of the associated states, it would be enough to have the compactness of minimizing sequences in U and the lower semicontinuity of the cost functional J in $U \times Y$ to obtain, always thanks to direct methods of the calculus of variation, the existence of an optimal pair (u, y). The lower semicontinuity of the cost functional J is not a very restrictive assumption: indeed in several cases J depends only on the state y in a continuous, or even more regular, way. On the contrary, the compactness of the set U, once endowed with the γ-convergence, is a rather severe requirement that in many cases does not occur: γ-limits of minimizing sequences may not belong to U. We will then construct a larger space of *relaxed controls* which is γ-compact so that the existence of an optimal relaxed solution will follow straightforward.

3.2 A quick survey on o-convergence theory

We recall here briefly the definition and the main properties of Γ-convergence. We do not want here to enter into the details of that theory, but only to use it in order to characterize the relaxed optimal control problem; we refer for all details to the book by Dal Maso [91] (see also [35]). In what follows Y denotes a separable metric space, endowed with a distance d.

Definition 3.2.1 *Given a sequence (G_n) of functionals from Y into $\overline{\mathbb{R}}$ we say that (G_n) Γ-converges to a functional G if for every $y \in Y$ we have:*

i) $\forall y_n \to y \quad G(y) \le \liminf\limits_{n\to+\infty} G_n(y_n)$;

ii) $\exists y_n \to y \quad G(y) \ge \limsup_{n\to+\infty} G_n(y_n)$.

We list here below the main properties of Γ-convergence.

• *Lower semicontinuity.* Every Γ-limit is lower semicontinuous on Y.

• *Convergence of minima.* If (G_n) Γ-converges to G and is equi-coercive on Y, that is for every $t \in \mathbb{R}$ there exists a compact set $K_t \subset Y$ such that

$$\{G_n \le t\} \subset K_t \qquad \forall n \in \mathbb{N},$$

then G is coercive too and so it attains its minimum on Y. We have

$$\min G = \lim_{n\to+\infty} \Big[\inf G_n\Big].$$

• *Convergence of minimizers.* Let (G_n) be an equi-coercive sequence of functionals on Y which Γ-converges to a functional G. If $y_n \in \operatorname{argmin} G_n$ is a sequence with $y_n \to y$ in Y, then we have $y \in \operatorname{argmin} G$. Moreover, if G is not identically $+\infty$ and if $y_n \in \operatorname{argmin} G_n$, then there exists a subsequence of (y_n) which converges to an element of $\operatorname{argmin} G$. In particular, if G has a unique minimum point y on Y, then every sequence $y_n \in \operatorname{argmin} G_n$ converges to y in Y.

It is interesting to notice (see Proposition 7.7 in [91]) that a sequence (G_n) of functionals is equi-coercive in Y if and only if there exists a lower semicontinuous coercive function $\Psi : Y \to \overline{\mathbb{R}}$ such that $G_n \ge \Psi$ for all $n \in \mathbb{N}$.

• *Compactness.* From every sequence (G_n) of functionals on Y it is possible to extract a subsequence Γ-converging to a functional G on Y.
• *Metrizability.* The Γ-convergence, considered on the family $S(Y)$ of all lower semicontinuous functions on Y, does not come from a topology, unless the space Y is locally compact, which never occurs in the infinite dimensional case. However, if instead of considering the whole family $S(Y)$, we take the smaller classes

$$S_\Psi(Y) = \left\{G : Y \to \overline{\mathbb{R}} \ : \ G \text{ l.s.c.}, G \geq \Psi\right\}$$

where $\Psi : Y \to \overline{\mathbb{R}}$ is lower semicontinuous and coercive (and nonnegative, for simplicity), then the Γ-convergence on $S_\Psi(Y)$ is metrizable. More precisely, it turns out to be equivalent to the convergence associated to the distance

$$d_\Gamma(F, G) = \sum_{i,j=1}^{\infty} 2^{-i-j} \left| \arctan\left(F_j(y_i)\right) - \arctan\left(G_j(y_i)\right)\right|$$

where (y_i) is a dense sequence in Y and H_j denotes the Moreau–Yosida transforms of a functional H, defined by:

$$H_j(y) = \inf\left\{H(x) + jd(x, y) \ : \ x \in Y\right\}.$$

According to the compactness property seen above, the family $S_\Psi(Y)$ endowed with the distance d_Γ turns out to be a compact metric space.

3.3 The topology of γ-convergence for control variables

We are now in a position to introduce a "natural" topology on the set U of control variables appearing in the general framework considered in Section 3.1 (see [25] and [69] for further details).

Definition 3.3.1 *We say that $u_n \to u$ in U if the associated state functionals $G(u_n, \cdot)$ Γ-converge to $G(u, \cdot)$ in Y. This convergence on U will be called γ-convergence.*

We shall always assume in the following that the state functional G satisfies the properties below:

- for every $u \in U$ the function $G(u, \cdot)$ is lower semicontinuous in the space Y;
- G is equi-coercive in the sense that there exists a coercive lower semicontinuous functional $\Psi : Y \to \overline{\mathbb{R}}$ such that

$$G(u, y) \geq \Psi(y) \qquad \forall u \in U, \ \forall y \in Y.$$

- the mapping $\Gamma_G : U \to S_\Psi(Y)$ defined by $\Gamma_G(u) = G(u, \cdot)$ is one-to-one. Otherwise, we may always reduce the space U to a smaller space which verifies this property.

Remark 3.3.2 By the assumptions above we have, in particular, that for every $u \in U$ the set argmin $G(u, \cdot)$ is nonempty. Moreover, according to the metrizability property of the Γ-convergence seen in Section 3.2, the γ-convergence on U is metrizable, and the mapping Γ_G is an isometry. However, even if $S_\Psi(Y)$ with the Γ-convergence is a compact metric space, in general U with the γ-convergence may be not compact. Indeed, a sequence $G(u_n, \cdot)$ of functionals may Γ-converge to a functional F, but this limit functional does not need to be of the form $G(u, \cdot)$ for some $u \in U$. This is why in many situations the existence of optimizers may fail and it is necessary to enlarge by relaxation the class of admissible controls U.

3.4 A general definition of relaxed controls

In this section we give the definition of relaxed controls in a rather general framework; the definition is given in the abstract scheme introduced in Section 3.1.

Definition 3.4.1 *The class $\hat{U}$ is defined as the completion of the metric space U endowed with the γ-convergence. The elements of $\hat{U}$ will be called relaxed controls and we still continue to denote by γ the convergence on $\hat{U}$.*

In order to define the relaxed optimal control problem associated to (3.1), (3.2) we have to introduce the relaxed cost functional $\hat{J}$ as well as the relaxed state functional $\hat{G}$. For every $\hat{u} \in \hat{U}$ we set

$$\hat{G}(\hat{u}, \cdot) = \Gamma \lim_{u \to \hat{u}} G(u, \cdot).$$

In other words, we define the mapping $\hat{\Gamma}_G : \hat{U} \to S_\psi(Y)$ as the unique isometry which extends Γ_G; more precisely,

$$\hat{\Gamma}_G(\hat{u}) = \Gamma \lim_{n \to +\infty} \Gamma_G(u_n),$$

where (u_n) is any sequence γ-converging to $\hat{u}$. Therefore we have $\hat{G} : \hat{U} \times Y \to \overline{\mathbb{R}}$ defined by

$$\hat{G}(\hat{u}, \cdot) = \hat{\Gamma}_G(\hat{u}) \qquad \forall \hat{u} \in \hat{U}$$

and we have

$$\hat{u}_n \to \hat{u} \text{ in } \hat{U} \qquad \Longleftrightarrow \qquad \Gamma \lim_{n \to +\infty} \hat{G}(\hat{u}_n, \cdot) = \hat{G}(\hat{u}, \cdot).$$

Proposition 3.4.2 *The metric space $\hat{U}$ is compact with respect to the γ-convergence.*

Proof Since $\hat{\Gamma}_G$ is an isometry and $\hat{U}$ is complete, $\hat{\Gamma}_G(\hat{U})$ is a complete subspace of the compact space $S_\psi(Y)$, so that $\hat{\Gamma}_G(\hat{U})$ is compact. Hence, using again the fact that $\hat{\Gamma}_G$ is an isometry, we get that $\hat{U}$ is compact too. ∎

The definition of the relaxed state functional allows us to define the relaxed state equation, linking a relaxed control $\hat{u} \in \hat{U}$ to a state $y \in Y$, which reads now

$$y \in \operatorname{argmin} \hat{G}(\hat{u}, \cdot).$$

The relaxed cost functional $\hat{J}$ is defined in a similar way. Take a pair $(\hat{u}, y)$ which verifies the state equation, i.e., such that $y \in \operatorname{argmin} \hat{G}(\hat{u}, \cdot)$; then we set

$$\hat{J}(\hat{u}, y) = \inf\Big\{ \liminf_{n\to+\infty} J(u_n, y_n) \ : \ u_n \to \hat{u} \text{ in } \hat{U},\ y_n \to y \text{ in } Y, \\ y_n \in \operatorname{argmin} G(u_n, \cdot)\Big\}.$$

Therefore the relaxed optimal control problem can be written in the form

$$\min\big\{\hat{J}(\hat{u}, y) \ : \ \hat{u} \in \hat{U},\ y \in Y,\ y \in \operatorname{argmin} \hat{G}(\hat{u}, \cdot)\big\}. \tag{3.6}$$

In several situations the cost functional J depends only on the state y and is continuous on Y; in this case it is easy to see that $\hat{J} = J$ so that the relaxed optimal control problem has the simpler form

$$\min\big\{J(y) \ : \ \hat{u} \in \hat{U},\ y \in Y,\ y \in \operatorname{argmin} \hat{G}(\hat{u}, \cdot)\big\}. \tag{3.7}$$

By the definition of relaxed control problem and by Proposition 3.4.2 we obtain immediately the following existence result.

Theorem 3.4.3 *Under the assumptions above the relaxed problem (3.6) admits at least a solution $(\hat{u}, y) \in \hat{U} \times Y$. Moreover, the infimum of the original problem given by (3.1) and (3.2) coincides with the minimum of the relaxed problem (3.6). Finally, if (u_n, y_n) is a minimizing sequence for the original problem, then there exists a subsequence converging in $\hat{U} \times Y$ to a solution $(\hat{u}, y)$ of the relaxed problem.*

Remark 3.4.4 On the one hand the result above gives the existence of an optimal pair $(\hat{u}, y)$ for a problem *"close"* to the original one; on the other hand the solution $\hat{u}$ belongs to a larger space and is only characterized as an element of an abstract topological completion, hence as an equivalence class of Cauchy sequences of the original control set U with respect to a quite involved distance function. In order to obtain further properties about the asymptotic behaviour of minimizing sequences it is then necessary, in concrete cases, to give a more explicit characterization of the space of relaxed controls $\hat{U}$.

3.5 Optimal control problems governed by ODE

In this section we consider optimal control problems where the control variable varies in a space of functions. For simplicity we consider the case of problems where the state and the control variables are functions of one real variable; therefore the state equation will be an ordinary differential equation.

Example 3.5.1 A car has to go from a point A to a point B (for simplicity assume along a straight line) in a given time T. Setting

$$y(t) : \text{the position of the car at the time } t,$$
$$u(t) : \text{the acceleration we give to the car at the time } t, \tag{3.8}$$

we have the state equation

$$y'' = u\ .$$

In this problem the position $y(t)$ plays the role of state variable and the acceleration $u(t)$ is the control variable; in this case we can control the acceleration but not the speed and the position: they are given indirectly by the state equation $y'' = u$. We can assume further constraints on the control, like $|u| \leq 1$ ($u = -1$ representing the maximum action of brakes, $u = 1$ the maximum acceleration).

If we take as a cost functional the total fuel consumption, we have to consider that this consumption may depend on several variables, as for instance:

- u (how much we push the accelerator),
- y (if we are going up or down on a hill),
- y' (the higher is the speed the higher is the consumption),
- t (on different hours of the day the consumption may be different).

Then the optimal control problem is given by the minimization of the functional

$$J(u, y) = \int_0^T f(t, y, y', u)\, dt$$

where the function f takes into account the variables above, with conditions

$$|u| \leq 1,\ \ y'' = u,\ \ y(0) = A,\ \ y(T) = B,\ \ y'(0) = 0.$$

Remark that an optimal solution is given by a pair (u, y).

In the following we want to derive some simple conditions for the existence of a solution.

Lemma 3.5.2 *Assume that for $n \in \mathbb{N} \cup \{\infty\}$ the functions $g_n : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$ are measurable in t and equi-Lipschitz continuous in s, i.e.,*

$$\exists L > 0 : |g_n(t, s_2) - g_n(t, s_1)| \leq L|s_2 - s_1|$$

for every $s_1, s_2 \in \mathbb{R}^N,\ t \in [0, T],\ n \in \mathbb{N} \cup \{\infty\}$. Assume further that

$$|g_n(t, 0)| \leq M$$

and fix initial data $\xi_n \in \mathbb{R}^N$. If for all $n \in \mathbb{N} \cup \{\infty\}$ we denote by y_n the unique solution of the differential equation

$$\begin{cases} y_n' = g_n(t, y_n) \text{ in } [0, T], \\ y_n(0) = \xi_n, \end{cases} \tag{3.9}$$

then the conditions $\xi_n \to \xi_\infty$ *and*

$$g_n(\cdot, s) \to g_\infty(\cdot, s) \qquad \textit{weakly in } L^1 \qquad \forall s \in \mathbb{R}^N$$

imply that $y_n \to y_\infty$ *uniformly as* $n \to +\infty$.

Proof It is convenient to write the differential equations in the integral form

$$(3.10) \qquad \begin{aligned} y_n(t) &= \xi_n + \textstyle\int_0^t g_n\big(\tau, y_n(\tau)\big)\,d\tau, \\ y_\infty(t) &= \xi_\infty + \textstyle\int_0^t g_\infty\big(\tau, y_\infty(\tau)\big)\,d\tau. \end{aligned}$$

Take now piecewise constant functions y_ε such that $\|y_\varepsilon - y_\infty\|_{L^\infty} < \varepsilon$. Then we have

$$\begin{aligned}
&|y_n(t) - y_\infty(t)| \le |\xi_n - \xi_\infty| + \Big|\int_0^t g_n(\tau, y_n)\,d\tau - \int_0^t g_\infty(\tau, y_\infty)\,d\tau\Big| \\
&\le |\xi_n - \xi_\infty| + \int_0^t |g_n(\tau, y_n) - g_n(\tau, y_\varepsilon)|\,d\tau \\
&\quad + \Big|\int_0^t g_n(\tau, y_\varepsilon) - g_\infty(\tau, y_\varepsilon)\,d\tau\Big| + \int_0^t |g_\infty(\tau, y_\varepsilon) - g_\infty(\tau, y_\infty)|\,d\tau \\
&\le |\xi_n - \xi_\infty| + \int_0^t L|y_n - y_\varepsilon|\,d\tau + \Big|\int_0^t g_n(\tau, y_\varepsilon) - g_\infty(\tau, y_\varepsilon)\,d\tau\Big| \\
&\quad + LT\|y_\varepsilon - y_\infty\| \\
&\le |\xi_n - \xi_\infty| + L\int_0^t |y_n - y_\infty|\,d\tau + \Big|\int_0^t g_n(\tau, y_\varepsilon) - g_\infty(\tau, y_\varepsilon)\,d\tau\Big| + C\varepsilon.
\end{aligned}$$

Since y_ε is piecewise constant we have

$$\Big|\int_0^t g_n(\tau, y_\varepsilon) - g_\infty(\tau, y_\varepsilon)\,d\tau\Big| \to 0 \quad \text{uniformly as } n \to +\infty$$

so that

$$|y_n(t) - y_\infty(t)| \le L\int_0^t |y_n(\tau) - y_\infty(\tau)|\,d\tau + \omega(n, \varepsilon)$$

where $\omega(n, \varepsilon) \to C\varepsilon$ as $n \to +\infty$. Applying now Gronwall's lemma we obtain

$$|y_n(t) - y_\infty(t)| \le \omega(n, \varepsilon)\exp\Big(\int_0^1 L(\tau)\,d\tau\Big).$$

Thus for a suitable constant C,

$$\|y_n - y_\infty\| \le C\omega(n, \varepsilon)$$

and, as ε was arbitrary, we get that $y_n \to y_\infty$ uniformly. ∎

Remark 3.5.3 The result of the lemma above holds as well if the constant L depends on t in an integrable way.

By using Lemma 3.5.2 we will prove an existence result for optimal control problems governed by equations of the form

$$y' = a(t, y) + b(t, y)u \ ,$$

with $a : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$ and $b : [0, T] \times \mathbb{R}^N \to \mathbb{R}^{Nm}$ measurable in t, Lipschitz continuous in y, and bounded at $y = 0$.

Let $f(t, s, z)$ be a Borel function such that

- $f \geq 0$;
- f is l.s.c in (s, z);
- f is convex in z.

Proposition 3.5.4 *Under the assumptions above, the functional*

$$F : L^1([0, T]; \mathbb{R}^m) \times W^{1,1}([0, T]; \mathbb{R}^N) \to [0, +\infty]$$

defined as

$$F(u, y) = \int_0^T f(t, y, u)\, dt + \chi_{\{y'=a(t,y)+b(t,y)u,\ y(0)=y_0\}}$$

is sequentially lower semicontinuous with respect to the $w - L^1 \times w - W^{1,1}$ topology.

Proof Assume $u_n \to u$ weakly in L^1 and $y_n \to y$ weakly in $W^{1,1}$. We can assume that for $n \in \mathbb{N}$,

$$y_n' = a(t, y_n) + b(t, y_n)u_n \ , \qquad y_n(0) = y_0.$$

Defining

$$\begin{aligned} g_n(t, s) &= a(t, s) + b(t, s)u_n(t), \\ g_\infty(t, s) &= a(t, s) + b(t, s)u(t) \ , \end{aligned} \tag{3.11}$$

the assumptions of Lemma 3.5.2 are fulfilled, hence we have

$$y' = a(t, y) + b(t, y)u \ , \qquad y(0) = y_0.$$

Therefore

$$F(u, y) = \int_0^T f(t, y, u)\, dt$$

and the lower semicontinuity follows from the general lower semicontinuity result for integral functionals (see for instance [62]). ■

It remains to show the coercivity of the functional F. For this we need to assume that there exist a superlinear function ϕ and $\gamma \in L^1$ such that

$$f(t, s, z) \geq \phi(|z|) - \gamma(t). \tag{3.12}$$

Proposition 3.5.5 *Under the assumptions above the functional F is coercive with respect to the $w - L^1 \times w - W^{1,1}$ topology.*

Proof Let $F(u_n, y_n) \leq c$. By the Dunford–Pettis weak L^1 compactness theorem for a subsequence we have $u_n \to u$ weakly in L^1 and

$$y_n' = a(t, y_n) + b(t, y_n)u_n, \quad y_n(0) = y_0.$$

It remains to show that $y_n \to y$ weakly in $W^{1,1}$, where y is the solution of

$$y' = a(t, y) + b(t, y)u, \quad y(0) = y = 0.$$

We have

$$\begin{aligned} |y_n'| &\leq |a(t, y_n)| + |b(t, y_n)||u_n| \\ &\leq |a(t, 0)| + A(t)|y_n| + |u_n|\big[|b(t, 0)| + B(t)|y_n|\big] \end{aligned}$$

where $A(t)$ and $B(t)$ are the Lipschitz constants of $a(t, \cdot)$ and $b(t, \cdot)$. By Gronwall's lemma it follows that

$$|y_n(t)| \leq \Big(|y_0| + \int_0^T (|a(t, 0)| + |b(t, 0)||u_n|)\, dt\Big) \exp\Big(\int_0^T \big(A(t) + B(t)|u_n|\big)\, dt\Big),$$

which implies that y_n are bounded in L^∞. From the relation $y_n' = a(t, y_n) + b(t, y_n)u_n$ we get that y_n' are equi-uniformly integrable. Therefore, by the Dunford–Pettis theorem again it follows that y_n' are weakly compact in L^1 and hence y_n are weakly compact in $W^{1,1}$. ■

Remark 3.5.6 Inspecting the proof of Proposition 3.5.5 we see that the growth assumption (3.12) requires that $a(t, 0)$, $b(t, 0)$, $A(t)$, $B(t)$ be bounded functions. If for $p \in]1, +\infty[$ we assume the stronger growth condition

$$f(t, s, z) \geq \alpha|z|^p - \gamma(t)$$

with $\alpha > 0$ and $\gamma \in L^1(0, T)$, it is then enough to require that $a(t, 0)$, $b(t, 0)$, $A(t)$, $B(t)$ be only in $L^{p'}(0, T)$. Finally, if we assume that

$$f(t, s, z) \geq \chi_{\{|z| \leq R\}} - \gamma(t)$$

with $R > 0$ and $\gamma \in L^1(0, T)$, then the proof above still works with the assumption that $a(t, 0)$, $b(t, 0)$, $A(t)$, $B(t)$ are in $L^1(0, T)$.

As an application of the results above, consider an optimal control problem governed by an ordinary differential equation (or system), and with integral cost functional, of the form

$$\min\Big\{\int_0^T j(t, y, u)dt \; : \; y' = g(t, y, u), \; y(0) = y_0\Big\}. \tag{3.13}$$

Here we have taken

- the space Y of states as the space $W^{1,1}(0, T; \mathbb{R}^N)$ of all absolutely continuous functions on $(0, T)$ with values in $\mathbb{R}^N$;
- the space U of controls as the space $L^1(0, T; \mathbb{R}^m)$ of all Lebesgue integrable functions on $(0, T)$ with values in $\mathbb{R}^m$;
- the set $\mathcal{A}$ of admissible pairs as the subset of $U \times Y$ of all pairs (u, y) which satisfy the state equation

$$y' = g(t, y, u) \qquad y(0) = y_0;$$

- the cost functional J as the integral functional

$$J(u, y) = \int_0^T j(t, y, u)\, dt.$$

In order to fulfill the conditions of Lemma 3.5.2, of Proposition 3.5.4, and of Proposition 3.5.5 we make the following assumptions on the data.

On the cost integrand j:

A1 the function $j : (0, T) \times \mathbb{R}^N \times \mathbb{R}^m \to [0, +\infty]$ is nonnegative and Borel measurable (or more generally measurable for the σ-algebra $\mathcal{L} \otimes \mathcal{B}_N \otimes \mathcal{B}_m$));

A2 the function $j(t, \cdot, \cdot)$ is lower semicontinuous on $\mathbb{R}^N \times \mathbb{R}^m$ for a. e. $t \in (0, T)$;

A3 the function $j(t, s, \cdot)$ is convex on $\mathbb{R}^m$ for a. e. $t \in (0, T)$ and for every $s \in \mathbb{R}^N$;

A4 there exist $\alpha \in L^1(0, T)$ and $\theta : \mathbb{R} \to \mathbb{R}$, with θ superlinear (that is, $\theta(r)/r \to +\infty$ as $r \to +\infty$) such that

$$\theta(|z|) - \alpha(t) \le j(t, s, z) \qquad \forall (t, s, z).$$

On the function g in the state equation we assume it is of the form

$$g(t, s, z) = a(t, s) + b(t, s)z,$$

where

A5 the function $a : (0, T) \times \mathbb{R}^N \to \mathbb{R}^N$ is measurable in t and continuous in s, and satisfies

$$\begin{gathered} |a(t, s_2) - a(t, s_1)| \le A(t)|s_2 - s_1| \qquad \text{with } A \in L^1(0, T), \\ |a(t, 0)| \le M(t) \qquad \text{with } M \in L^1(0, T); \end{gathered} \tag{3.14}$$

A6 the function $b : (0, T) \times \mathbb{R}^N \to \mathbb{R}^{mN}$ is measurable in t and continuous in s, and satisfies

$$\begin{gathered} |b(t, s_2) - b(t, s_1)| \le B|s_2 - s_1| \qquad \text{with } B \in \mathbb{R}, \\ |b(t, 0)| \le K \text{ with } K \in \mathbb{R}. \end{gathered} \tag{3.15}$$

The existence result is then the following.

Theorem 3.5.7 *Under assumptions* **A1–A6** *above the optimal control problem (3.13) admits at least one solution.*

Proof In order to apply the direct methods of the calculus of variations, we endow the space U of controls with the weak $L^1(0,T;\mathbb{R}^m)$ topology and the space Y of states with the topology of uniform convergence, and we make the following remarks.

• The cost functional J is sequentially lower semicontinuous on $U \times Y$; this follows from the De Giorgi–Ioffe lower semicontinuity theorem for integral functionals. The first proof has been given by De Giorgi in an unpublished paper [106]; another independent proof was given by Ioffe [143]; for a discussion about the lower semicontinuity of integral functionals we refer to the book by Buttazzo [62].

• The functional J is coercive with respect to the variable u; this is a consequence of the Dunford–Pettis weak compactness criterion.

• For every $u \in U$ the state equation

$$y' = a(t, y) + b(t, y)u, \qquad y(0) = y_0$$

has a unique solution $y \in Y$ defined on the whole interval $[0, T]$, thanks to the Lipschitz assumptions made on the coefficients $a(t, \cdot)$ and $b(t, \cdot)$.

• The set $\mathcal{A}$ of admissible pairs is sequentially closed in $U \times Y$ as it can be easily verified by writing the state equation in integral form

$$y(t) = y_0 + \int_0^t \big(a(s, y(s)) + b(s, y(s))u(s)\big)\,ds.$$

By the remarks above, it remains only to prove the coercivity of J on $\mathcal{A}$ with respect to y. In other words, if $u_n \to u$ weakly in $L^1(0,T;\mathbb{R}^m)$ and

$$y_n' = a(t, y_n) + b(t, y_n)u_n\,, \qquad y_n(0) = y_0,$$

we have to prove that (y_n), or a subsequence of it, converges uniformly. By Gronwall's lemma we obtain that (y_n) is uniformly bounded, so that by the state equations we obtain

$$|y_n'| \le c(t) + C|u_n| \tag{3.16}$$

for suitable $c \in L^1(0,T)$ and $C > 0$. Since (u_n) is weakly compact in $L^1(0,T;\mathbb{R}^m)$, by the Dunford–Pettis theorem again, it turns out to be equi-absolutely integrable on $(0, T)$, that is,

$$\forall \varepsilon > 0\ \exists \delta > 0\ :\ E \subset (0,T),\ |E| < \delta \quad \Rightarrow \quad \int_E |u_n|\,dt < \varepsilon \qquad \forall n \in \mathbb{N}.$$

Therefore by (3.16), also (y_n') is equi-absolutely integrable on $(0, T)$, which implies the weak compactness in $L^1(0,T;\mathbb{R}^N)$ of (y_n') and hence the compactness in $L^\infty(0,T;\mathbb{R}^N)$ of (y_n). ■

When the conditions of Theorem 3.5.7 are not fulfilled, we do not have, in general, the existence of a solution of the optimal control problem (3.13), and in order to study the asymptotic behaviour of minimizing sequences (u_n, y_n) we have to consider the associated relaxed formulation.

The simplest case is when we do not have to enlarge the class U of controls, which happens for instance when a coercivity assumption like **A4** is fulfilled. In this case it is enough to take the lower semicontinuous envelope in $U \times Y$ of the mapping

$$(u, y) \mapsto J(u, y) + \chi_{\mathcal{A}}(u, y).$$

In some cases, which often occur in applications to concrete problems, the lower semicontinuous envelope above can be easily computed in terms of the envelope $\overline{J}$ of the cost functional and of the closure $\overline{\mathcal{A}}$ of the state equation. More precisely, the following result can be proved.

Proposition 3.5.8 *Assume that*

i) $|J(u, y) - J(u, z)| \le \omega(y, z)\Phi(u)$ *for every* $u \in U$ *and* $y, z \in Y$ *with* Φ *locally bounded in* U *and*
$$\lim_{z \to y} \omega(y, z) = 0;$$

ii) if $(u, y) \in \overline{\mathcal{A}}$*, then for every* v *close to* u *there exists* y_v *such that* $(v, y_v) \in \mathcal{A}$ *and the mapping* $v \mapsto y_v$ *is continuous.*

Then the relaxed problem associated to

$$\min\left\{J(u, y) \;:\; (u, y) \in \mathcal{A}\right\}$$

can be written in the form

$$\min\left\{\overline{J}(u, y) \;:\; (u, y) \in \overline{\mathcal{A}}\right\}.$$

As an example let us consider again an optimal control problem governed by an ordinary differential equation:

$$(3.17) \qquad \begin{aligned} &J(u, y) = \int_0^T j(t, y, u)\, dt \\ &\mathcal{A} = \left\{(u, y) \in U \times Y \;:\; y' = a(t, y) + b(t, y)\beta(t, u),\;\; y(0) = y_0\right\} \end{aligned}$$

where the functions a and b satisfy conditions **A5** and **A6**, and β can be nonlinear and j nonconvex with respect to u. If the integrand j is bounded from below by

$$|u|^p - \alpha(t) \le j(t, y, u) \qquad \text{with } p > 1 \text{ and } \alpha \in L^1,$$

then we may take $U = L^p(0, T; \mathbb{R}^m)$ and $Y = W^{1,1}(0, T; \mathbb{R}^N)$ endowed with their weak topologies. Introducing the auxiliary variable $v = \beta(t, u)$ the new control space is $U \times V$ where V is an L^q space, provided

$$|\beta(t, u)| \le \beta_0(t) + c|u|^{p/q} \qquad \text{with } q > 1 \text{ and } \beta_0 \in L^q,$$

so that the problem can be written in an equivalent form with

$$
\begin{aligned}
\tilde{J}(u, v, y) &= \int_0^T \big(j(t, y, u) + \chi_{\{v=\beta(t,u)\}}\big)\, dt, \\
\tilde{\mathcal{A}} &= \big\{(u, v, y) \in U \times V \times Y : y' = a(t, y) + b(t, y)v,\ y(0) = y_0\big\}.
\end{aligned} \tag{3.18}
$$

In this form, we already know that the set $\tilde{\mathcal{A}}$ is closed, since the differential equation is now linear in the control. So it remains to relax the cost $\tilde{J}$ with respect to (u, v). If we assume the continuity condition on j,

$$
|j(t, y, u) - j(t, z, u)| \le \omega(y, z)(\alpha(t) + |u|^p)
$$

is satisfied with $\alpha \in L^1$ and ω such that

$$
\lim_{z \to y} \omega(y, z) = 0,
$$

then the relaxed form of $\tilde{J}$ is well known and is given by the integral functional

$$
\tilde{J}^{**}(u, v, y) = \int_0^T \Big(j(t, y, \xi) + \chi_{\{\eta=\beta(t,\xi)\}}\Big)^{**}(u, v)\, dt,
$$

where the convexification $\tilde{J}^{**}$ is intended with respect to the pair (u, v), and in the integrand with respect to the pair (ξ, η). Finally, eliminating the auxiliary variable v we obtain the relaxed form of the optimal control problem:

$$
\min \Big\{ \int_0^T \phi(t, y, u, y')\, dt : u \in L^p(0, T; \mathbb{R}^m),\ y \in W^{1,1}(0, T; \mathbb{R}^N),\ y(0) = y_0 \Big\},
$$

where the function ϕ takes into account cost and state equation at one time, and is defined by

$$
\phi(t, y, u, w) = \inf \Big\{ \Big(j(t, y, \xi) + \chi_{\{\eta=\beta(t,\xi)\}}\Big)^{**}(u, v) \ :\ w = a(t, y) + b(t, y)v \Big\}.
$$

A case in which the computation can be made explicitly is the following (see Example 5.3.7 of [62]):

$$
\begin{aligned}
J(u, y) &= \int_0^1 \Big(u^2 + \frac{1}{u^2} + |y - y_0|^2 + h(t)u\Big)\, dt, \\
\mathcal{A} &= \big\{(u, y) \in U \times Y \ :\ uy' = 1,\ 1/c \le u \le c,\ y(0) \in K\big\}.
\end{aligned} \tag{3.19}
$$

Here $y_0(t)$ and $h(t)$ are two functions in $L^2(0, 1)$, $c \ge 1$ is a constant, and K is a closed subset of $\mathbb{R}$. We obtain, after some elementary calculations, that the relaxed problem is the minimization problem for the functional

$$
\int_0^1 \big(u^2 + |y'|^2 + 2(uy' - 1) + |y - y_0(t)|^2 + h(t)u\big)\, dt
$$

with the constraints

$$\frac{1}{u} \le y' \le c + \frac{1}{c} - u, \qquad \frac{1}{c} \le u \le c, \qquad y(0) \in K.$$

Consider now the case of a control problem where the control occurs on the coefficient of a second order state equation. More precisely, given $\alpha > 0$ take

$$\begin{aligned} U &= \{u \in L^1(0,1) \ : \ u \ge \alpha \text{ a.e. on } (0,1)\}, \\ Y &= H_0^1(0,1) \text{ with the strong topology of } L^2(0,1) \end{aligned} \tag{3.20}$$

and consider the optimal control problem

$$\min\left\{ \int_0^1 \big(g(x,u) + \phi(x,y)\big)\,dx \ : \ u \in U,\ y \in Y,\ -(uy')' = f \right\}. \tag{3.21}$$

Here $f \in L^2(0,1)$, and g, ϕ are Borel functions from $(0,1) \times \mathbb{R}$ into $\overline{\mathbb{R}}$ with

B1 $\phi(x,\cdot)$ is continuous on $\mathbb{R}$ for a.e. $x \in (0,1)$,
B2 for a suitable function $\omega(x,t)$ integrable in x and increasing in t we have

$$|\phi(x,s)| \le \omega(x,|s|) \qquad \forall (x,s) \in (0,1) \times \mathbb{R}.$$

Setting for any $(u,y) \in U \times Y$,

$$\begin{aligned} J(u,y) &= \int_0^1 \big(g(x,u) + \phi(x,y)\big)\,dx, \\ G(u,y) &= \int_0^1 \big(uy'^2 - 2fy\big)\,dx, \end{aligned} \tag{3.22}$$

we obtain that problem (3.21) can be written in the form

$$\min\left\{ J(u,y) \ : \ u \in U,\ y \in Y,\ y \in \operatorname{argmin} G(u,\cdot) \right\}.$$

It is well known that

$$\Gamma \lim_{n\to+\infty} G(u_n,\cdot) = G(u,\cdot) \quad \iff \quad \frac{1}{u_n} \to \frac{1}{u} \text{ weakly* in } L^\infty(0,1);$$

therefore, by applying the framework of Section 3.5 we obtain $\hat{U} = U$, $\hat{G} = G$, and

$$\hat{J}(u,y) = \int_0^1 \big(\gamma(x,u) + \phi(x,y)\big)\,dx$$

where $\gamma(x,s) = \beta^{**}(x,1/s)$ with ** being the convexification operator (with respect to the second variable) and

$$\beta(x,t) = \begin{cases} g(x,1/t) \text{ if } t \in]0,1/\alpha], \\ +\infty \qquad \text{otherwise.} \end{cases} \tag{3.23}$$

For instance, if $\alpha < 1$ and $g(x,s) = |s-1|$ we have

$$\gamma(x,s) = \begin{cases} s-1 & \text{if } s \geq 1, \\ \alpha(1-s)/s & \text{if } \alpha \leq s < 1. \end{cases} \tag{3.24}$$

An analogous computation can be done in the case

$$U = \Big\{ u \in L^1(0,1) \ : \ u \geq 0, \ \int_0^1 \frac{1}{u}\, dx \leq c \Big\},$$

where $c > 0$. In this case, in order to satisfy the coercivity assumption required by the abstract framework, it is better to consider

$$\begin{aligned} &Y = BV(0,1) \text{ with the strong topology of } L^1(0,1), \\ &G(u,y) = \int_0^1 \big(uy'^2 - 2fy\big)\, dx + \chi_{\{y(0)=y_0,\ y(1)=y_1\}}(y) + \chi_{\{y' << dx\}}(y) \end{aligned}$$

where $y' << dx$ denotes the constraint that y' is a measure absolutely continuous with respect to the Lebesgue measure. Following Buttazzo and Freddi [69] we obtain that $\hat{U}$ coincides with the set of positive measures μ on $[0,1]$ such that $\mu([0,1]) \leq c$ and

$$\begin{aligned} \hat{G}(\mu, y) = &\int_{]0,1[} \Big|\frac{dy'}{d\mu}\Big|^2 d\mu - 2\int_{]0,1[} fy\, dx \\ &+ \frac{|y^+(0) - y_0|^2}{\mu(\{0\})} + \frac{|y^+(1) - y_1|^2}{\mu(\{1\})} + \chi_{\{y' << \mu\}}(y) \end{aligned}$$

where $dy'/d\mu$ is the Radon–Nikodym derivative of y' with respect to μ. It is not difficult to see that all assumptions required by the abstract framework are fulfilled, with

$$\psi(y) = a\|y\|_{BV}^2 - b$$

for suitable positive constants a, b. It remains to compute the functional $\hat{J}$. Assume for simplicity that $g(x,s) = g(s)$ and that **B1** and **B2** hold; then we obtain

$$\hat{J}(\mu, y) = \int_0^1 \beta^{**}\big(\mu^a(x)\big)\, dx + \int_{[0,1]} \big(\beta^{**}\big)^\infty(\mu^s) + \int_0^1 \phi(x,y)\, dx$$

where $\beta(t) = g(1/t)$, $\mu = \mu^a \cdot dx + \mu^s$ is the Lebesgue–Nikodym decomposition of μ, and $\big(\beta^{**}\big)^\infty$ is the recession function of β^{**}. For instance, if $g(s) = |s-1|^2$, the relaxed problem has the form

$$\min \Big\{ \int_{\{\mu^a \leq 1\}} \Big|\frac{\mu^a - 1}{\mu^a}\Big|^2 dx + \int_0^1 \phi(x,y)\, dx : \mu([0,1]) \leq c,\ y \in \operatorname{argmin} \hat{G}(\mu, \cdot) \Big\}.$$

3.6 Examples of relaxed shape optimization problems

In this section we give some applications of the abstract framework for relaxed controls introduced in the previous sections, and we characterize explicitly the set $\hat{U}$ of relaxed controls as well as the form of the relaxed control problems.

Example 3.6.1 The first example deals with a class of shape optimization problems with Dirichlet conditions on the free boundary, studied by Buttazzo and Dal Maso (see [64], [65], [66], [67] and references therein), in which the initial set U of controls is the class of all domains contained in a given open subset D of $\mathbb{R}^N$. We stress that this class has no linear or convex structure, and the usual topologies on families of domains are not suitable for the problems one would like to consider.

To set the problem more precisely, let D be a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$), let $f \in L^2(D)$ and let $j : D \times \mathbb{R} \to \mathbb{R}$ be a Borel function. Consider the shape optimization problem

$$\min\Big\{\int_D j\big(x, y_A(x)\big)\,dx \ :\ A \in \mathcal{A}(D)\Big\} \tag{3.25}$$

where $\mathcal{A}(D)$ is the family of all open subsets of D and where for every $A \in \mathcal{A}(D)$ we denoted by y_A the solution of the Dirichlet problem

$$\begin{cases} -\Delta y = f \quad \text{in} \quad A, \\ y \in H_0^1(A) \end{cases} \tag{3.26}$$

extended by zero to $D \setminus A$. In this way all states y belong to the Sobolev space $H_0^1(D)$, which will be taken as the space of states. The setting of the optimal control problem we consider is then:

- the space of states is $Y = H_0^1(D)$ with the strong topology of $L^2(D)$;
- the set of controls is $U = \mathcal{A}(D)$;
- the cost functional is taken of the integral form

$$J(A, y) = \int_D j\big(x, y(x)\big)\,dx \ ;$$

notice that in this case the cost does not explicitly depend on the control variable A;
- the state functional is

$$G(A, y) = \int_D \big(|\nabla y|^2 - 2fy\big)\,dx + \chi_{H_0^1(A)}(y) \tag{3.27}$$

which provides, via Euler–Lagrange equation, the state equation (3.26).

The shape optimization problem with Dirichlet conditions on the free boundary can then be written as an optimal control problem, in the form

$$\min\big\{J(y) \ :\ y \in Y,\ A \in U,\ y \in \operatorname{argmin} G(A, \cdot)\big\}. \tag{3.28}$$

It is easy to verify that, as a consequence of Poincaré inequality, the coercivity assumption required in Definition 3.3.1 ii) turns out to be fulfilled, with

$$\Psi(y) = C_1 \int_D |\nabla y|^2 dx - C_2$$

for suitable positive constants C_1 and C_2.

In order to identify the relaxed problem associated with (3.28) we have first to characterize the completion $\hat{U}$ of U with respect to the distance induced by the Γ-convergence on the functionals $G(A, \cdot)$. This has been done by Dal Maso and Mosco in [100], where it is shown that $\hat{\mathcal{U}}$ coincides with the space $\mathcal{M}_0(D)$ of all nonnegative Borel measures, possibly $+\infty$ valued, which vanish on all sets of capacity zero. The identification of the relaxed state functional $\hat{G}$ has also been given and, for every $\mu \in \mathcal{M}_0(D)$ and $y \in H_0^1(D)$ we have

$$\hat{G}(\mu, y) = \int_D \big(|\nabla y|^2 - 2fy\big)\, dx + \int_D y^2 \, d\mu.$$

The relation $y \in \operatorname{argmin} \hat{G}(\mu, \cdot)$ can also be written, via the Euler–Lagrange equation, in the form

$$\begin{cases} -\Delta y + \mu y = f \quad \text{in} \quad D, \\ y \in H_0^1(D) \end{cases} \tag{3.29}$$

which has to be intended in the following weak sense: $y \in H_0^1(D) \cap L^2(D, \mu)$ and

$$\int_D \nabla y \nabla \varphi \, dx + \int_D y\varphi \, d\mu = \int_D f\varphi \, dx \qquad \forall \varphi \in H_0^1(D) \cap L^2(D, \mu).$$

On the integrand j appearing in the cost functional J we make the following assumptions:

i) $j(x, \cdot)$ is continuous for a.e. $x \in D$;
ii) for suitable $a \in L^1(D)$ and $b \in \mathbb{R}$ we have $|j(x, s)| \le a(x) + b|s|^2$ for a.e. $x \in D$ and for every $s \in \mathbb{R}$.

In this way the functional J turns out to be continuous in the strong topology of L^2, so that the assumptions of the abstract scheme apply, and the corresponding relaxed problem can be written in the form

$$\min\Big\{ \int_D j\big(x, y(x)\big)\, dx \ : \ \mu \in \mathcal{M}_0(D),\ y \in H_0^1(D),\ -\Delta y + \mu y = f \Big\}.$$

Example 3.6.2 The second example we consider is the case of a control problem where the control occurs on the coefficient of the state equation, which is of partial differential type. More precisely, we consider a class of optimal control problems for two-phase conductors which has been studied by Cabib and Dal Maso (see [75],

[76]). As in the case seen in Section 3.5, also here the control occurs on the coefficients of the state equation which is actually an elliptic partial differential equation. More precisely, let D be a bounded open subset of $\mathbb{R}^N$, let α, β be two real positive numbers, $f \in L^2(D)$. We consider the class U of controls as the set of all functions $u : D \to \mathbb{R}$ with the property that there exists a Borel subset $A \subset D$ such that

$$u = \alpha 1_A + \beta 1_{D\setminus A}.$$

In this way we can identify the class U with the family of all Borel subsets of D. The space of states will be $Y = H_0^1(D)$ endowed with the strong topology of L^2.

Consider the optimal control problem

$$\min \Big\{ J(u, y) \ : \ -\operatorname{div}\big(u \nabla y\big) = f \text{ in } D,\ y = 0 \text{ on } \partial D \Big\}.$$

The cost functional J is still of the form

$$J(u, y) = \int_A g(x)\, dx + \int_D \varphi(x, y)\, dx,$$

where g is a given function in $L^1(D)$, and $\varphi : D \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory integrand which satisfies the growth condition

$$|\varphi(x, z)| \le c_1(x) + c_2 z^2 \quad \text{for suitable} \quad c_1 \in L^1(D),\ c_2 \ge 0.$$

The energy functional G is now given by

$$G(u, y) = \int_D \big(u |\nabla y|^2 - 2 f y\big)\, dx.$$

The completion $\hat{U}$ of U with respect to the G-convergence of the state equation or, equivalently, the Γ-convergence of the functionals $G(u, \cdot)$ has been characterized by Lurie and Cherkaev [158], [159] for the two-dimensional case and by Murat and Tartar [169], [189] for the general case. They proved that $\hat{U}$ is the space of all symmetric $N \times N$ matrices $a(x) = (a_{ij}(x))$ whose eigenvalues $\lambda_1(x) \le \lambda_2(x) \le \cdots \le \lambda_N(x)$ satisfy for a suitable $t \in [0, 1]$ (depending on x) the following $N + 2$ inequalities:

$$\begin{aligned}
&\sum_{i=1}^N \frac{1}{\lambda_i - \alpha} \le \frac{1}{\nu_t - \alpha} + \frac{N-1}{\mu_t - \alpha},\\
&\sum_{i=1}^N \frac{1}{\beta - \lambda_i} \le \frac{1}{\beta - \nu_t} + \frac{N-1}{\beta - \mu_t},\\
&\nu_t \le \lambda_i \le \mu_t, \qquad i = 1, \dots, N,
\end{aligned}$$

where μ_t and ν_t respectively denote the arithmetic and the harmonic mean of α and β, namely

$$\begin{aligned}
\mu_t &= t\alpha + (1-t)\beta,\\
\nu_t &= \Big(\frac{t}{\alpha} + \frac{1-t}{\beta}\Big)^{-1}.
\end{aligned}$$

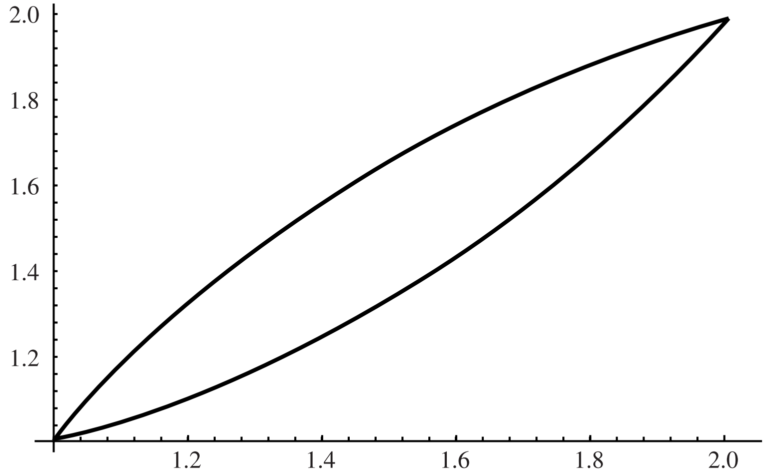

Figure 3.1. The set C in dimension 2.

For instance, when $N = 2$, then $\hat{U}$ consists of all symmetric 2×2 matrices $a(x)$ whose eigenvalues $\lambda_1(x)$, $\lambda_2(x)$ belong, for every $x \in D$, to the following convex domain C of $\mathbb{R}^2$:

$$C = \left\{(\lambda_1, \lambda_2) \in [\alpha, \beta] \times [\alpha, \beta] \ : \ \frac{\alpha\beta}{\beta + \alpha - \lambda_1} \leq \lambda_2 \leq \alpha + \beta - \frac{\alpha\beta}{\lambda_1}\right\}.$$

The picture of the set C in dimension 2 is given in Figure 3.1 above, with $\alpha = 1$ and $\beta = 2$.

The functional $\hat{G}$ can be computed and we have

$$\hat{G}(a, y) = \int_D \Big[a(x)DyDy - 2fy\Big]\,dx.$$

The computation of the functional $\hat{J}$ can be found in Cabib [75] where it is shown that

$$\hat{J}(a, y) = \int_D \big[\hat{g}(x, a) + \varphi(x, y)\big]\,dx,$$

with

$$\hat{g}(x, a) = \begin{cases} g(x)\dfrac{\beta - \underline{\mu}(a)}{\beta - \alpha} & \text{if } g(x) \leq 0, \\ g(x)\dfrac{\beta - \overline{\mu}(a)}{\beta - \alpha} & \text{if } g(x) \geq 0, \end{cases} \tag{3.30}$$

and

$$\underline{\mu}(a) = \max\left\{\lambda_N,\ \beta + \frac{(N-1)\beta + \alpha}{1 - \beta \sum_{i=1}^{N}\left(\beta - \lambda_i\right)^{-1}}\right\},$$
$$\overline{\mu}(a) = \alpha + \frac{(N-1)\alpha + \beta}{1 + \alpha \sum_{i=1}^{N}\left(\lambda_i - \alpha\right)^{-1}}\ .$$

4

Shape Optimization Problems with Dirichlet Condition on the Free Boundary

In this chapter we discuss shape optimization problems associated to elliptic operators of Dirichlet–Laplacian type. For simplicity, we concentrate our discussion on the Laplace operator. Nevertheless, we point out the fact that from *the shape optimization view point* this is the most important case. Further extensions to other operators are not so difficult. In order to give the reader a hint on how to deal with nonlinear problems, we discuss in Sections 4.8 and 4.9 the shape stability of the solution of a partial differential equation associated to the p-Laplace operator (with Dirichlet boundary conditions). Throughout all the chapter we assume that the dimension N is at least 2; in fact, in the one-dimensional case most of the results either are trivially true, or fail.

4.1 A short survey on capacities

Throughout the next chapters we shall often use the notion of Sobolev capacity of a subset E of $\mathbb{R}^N$, defined by

$$\operatorname{cap}(E) = \inf\Big\{\int_{\mathbb{R}^N} |\nabla u|^2 + u^2\,dx \;:\; u \in \mathcal{U}_E\Big\}\,,$$

where $\mathcal{U}_E$ is the set of all functions u of the Sobolev space $H^1(\mathbb{R}^N)$ such that $u \geq 1$ almost everywhere in a neighborhood of E.

Sometimes it is more convenient to work with a local capacity. Let D be a bounded open set. The capacity of a subset E in D is

$$\operatorname{cap}(E, D) = \inf\Big\{\int_D |\nabla u|^2\,dx \;:\; u \in \mathcal{U}_E\Big\}\,,$$

where $\mathcal{U}_E$ is the set of all functions u of the Sobolev space $H_0^1(D)$ such that $u \geq 1$ almost everywhere in a neighborhood of E. Since the two capacities are "locally equivalent", there is no important difference for many of our purposes. Nevertheless, in order to avoid any ambiguity, in all definitions given below we consider the first one. The second one is used for precise estimates of the oscillations of some harmonic functions.

If a property $P(x)$ holds for all $x \in E$ except for the elements of a set $Z \subseteq E$ with $\operatorname{cap}(Z) = 0$, we say that $P(x)$ holds *quasi-everywhere* on E (shortly *q.e.* on E). The expression *almost everywhere* (shortly *a.e.*) refers, as usual, to the Lebesgue measure.

A subset A of $\mathbb{R}^N$ is said to be *quasi-open* (resp. *quasi-closed*) if for every $\varepsilon > 0$ there exists an open (resp. closed) subset A_ε of $\mathbb{R}^n$, such that $\operatorname{cap}(A_\varepsilon \triangle A) < \varepsilon$, where $\triangle$ denotes the symmetric difference of sets. The class of all quasi-open subsets of D will be denoted by $\mathcal{A}$. In fact, in the definition of a quasi-open set we can additionally require that $A \subseteq A_\varepsilon$.

A function $f : D \to \mathbb{R}$ is said to be *quasi-continuous* (resp. *quasi-lower semi-continuous*) if for every $\varepsilon > 0$ there exists a continuous (resp. lower semicontinuous) function $f_\varepsilon : D \to \mathbb{R}$ such that $\operatorname{cap}(\{f \neq f_\varepsilon\}) < \varepsilon$, where $\{f \neq f_\varepsilon\} = \{x \in D : f(x) \neq f_\varepsilon(x)\}$. It is well known (see, e.g., Ziemer [196]) that every function u of the Sobolev space $H^1(D)$ has a quasi-continuous representative, which is uniquely defined up to a set of capacity zero. We shall always identify the function u with its quasi-continuous representative, so that a pointwise condition can be imposed on $u(x)$ for quasi-every $x \in D$. Notice that with this convention we have

$$\operatorname{cap}(E, D) = \min\Big\{\int_D |\nabla u|^2 dx \ : u \in H_0^1(D),\ u \geq 1 \text{ q.e. on } E\Big\}$$

for every subset E of D.

We recall the following theorems from [3].

Theorem 4.1.1 *Let $u \in H^1(\mathbb{R}^N)$. Then for q.e. $x \in \mathbb{R}^N$,*

$$\lim_{\varepsilon \to 0} \frac{\int_{B_{x,\varepsilon}} u(y)dy}{|B_{x,\varepsilon}|} = \tilde{u}(x),$$

where $\tilde{u}$ is a quasi-continuous representative of u.

Theorem 4.1.2 *Every strongly convergent sequence in $H^1(\mathbb{R}^N)$ has a subsequence converging q.e. in $\mathbb{R}^N$.*

For every $A \in \mathcal{A}$ we denote by $H_0^1(A)$ the space of all functions $u \in H_0^1(D)$ such that $u = 0$ q.e. on $D \setminus A$, with the Hilbert space structure inherited from $H_0^1(D)$. Note that $H_0^1(A)$ is a closed subspace of $H_0^1(D)$ as a consequence of well-known properties of quasi-continuous representatives of Sobolev functions (see, e.g., Ziemer [196]). If A is open, the previous definition of $H_0^1(A)$ is equivalent to the usual one (see Adams–Hedberg [3]). Indeed, we recall the following result.

Theorem 4.1.3 *Let $A \subseteq \mathbb{R}^N$ be an open set. A function $u \in H^1(\mathbb{R}^N)$ belongs to $H_0^1(A)$ if and only if $u = 0$ q.e. on A^c (here u is supposed to be quasi-continuous).*

In the statement above, the assertion u *belongs to* $H_0^1(A)$ has to be understood in the sense that u is the strong limit in $H^1(\mathbb{R}^N)$ of a sequence of $C_c^\infty(\mathbb{R}^N)$ functions with support in A.

Moreover, we have the following result (see [45]).

Lemma 4.1.4 *If C_1, C_2 are two quasi-open sets with* $\mathrm{cap}(C_1 \cap C_2) = 0$ *and* $u \in H_0^1(C_1 \cup C_2)$, *then* $u_{|C_1} \in H_0^1(C_1)$ *and* $u_{|C_2} \in H_0^1(C_2)$.

The fine topology on D is the coarsest topology making all super-harmonic functions continuous. The relation between quasi-open sets and the fine topology is studied in [3], [126], [145]. We recall the following theorem from [145].

Theorem 4.1.5 *Suppose $A \subseteq \mathbb{R}^N$. Then the following assertions are equivalent:*

i) A is quasi-open;

ii) A is the union of a finely open set and a set of zero capacity;

iii) $A = \{u > 0\}$ for some nonnegative quasi-continuous function $u \in H^1(\mathbb{R}^N)$.

Since the family of quasi-open sets of $\mathbb{R}^N$ is not a topology (only countable unions of quasi-open sets are quasi-open) when dealing with arbitrary unions of quasi-open sets, sometimes it is more interesting to work with the finely open sets given by the previous theorem at point ii).

To finish, we also recall the following result.

Theorem 4.1.6 *Suppose A is a quasi-open subset of $\mathbb{R}^N$ and u is a function on A. The following assertions are equivalent:*

i) u is quasi-l.s.c.;

ii) the sets $\{u > c\}$ are quasi-open for all $c \in \mathbb{R}$;

iii) u is finely l.s.c. up to a set of zero capacity.

Remark 4.1.7 All the definitions and results presented in this section have natural extension to the Sobolev spaces $W_0^{1,p}(\Omega)$ with $1 < p < +\infty$. We refer to [141] for a review of the main definitions and properties of the p-capacity. From the shape optimization point of view, the most interesting case is when $p \in (1, N]$, since for $p > N$ the p-capacity of a point is strictly positive and every $W^{1,p}$-function has a continuous representative. For this reason, a property which holds p-quasi-everywhere, with $p > N$, holds in fact everywhere, and this makes trivial several results concerning shape optimization problems.

4.2 Nonexistence of optimal solutions

In this section we give an explicit example where the existence of an optimal domain does not occur (see also Chapter 3). The shape optimization problem we consider is with Dirichlet conditions on the free boundary, of the form

$$\min\left\{J(u_A)\ :\ -\Delta u_A = f \text{ in } A,\ u_A \in H_0^1(A)\right\}. \tag{4.1}$$

Here the admissible domains A vary in the class of all open subsets of a given bounded open subset D of $\mathbb{R}^N$, $f \in L^2(D)$ is fixed, and the solutions u_A are considered extended by zero on $D \setminus A$.

The cost functional we consider is the $L^2(D)$ distance from a desired state $\overline{u}(x)$,

$$J(u) = \int_D |u - \overline{u}|^2\, dx. \tag{4.2}$$

In the thermostatic model the optimization problem (4.1) consists in finding an optimal distribution, inside D, of the Dirichlet region $D \setminus A$ in order to achieve a temperature which is as close as possible to the desired temperature $\overline{u}$, once the heat sources f are prescribed.

For simplicity, we consider a uniformly distributed heat source, that is we take $f \equiv 1$, and we take the desired temperature $\overline{u}$ constantly equal to $c > 0$. Therefore problem (4.1) becomes

$$\min\left\{\int_D |u_A - c|^2\, dx\ :\ -\Delta u_A = 1 \text{ in } A,\ u_A \in H_0^1(A)\right\}. \tag{4.3}$$

We will actually prove that for small values of the constant c no regular domain A can solve problem (4.3) above; the proof of nonexistence of any domain is slightly more delicate and requires additional tools like the capacitary form of necessary conditions of optimality (see for instance [64], [65], [82]).

Proposition 4.2.1 *If c is small enough, then no smooth domain A can solve the optimization problem (4.3).*

Proof Assume by contradiction that a regular domain A solves the optimization problem (4.3). Let us also assume first that A does not coincide with the whole set D, so that we can take a point x_0 in D which does not belong to the closure $\overline{A}$ and a small ball B_ε of radius ε, centered at x_0 and disjoint form A. If u_A denotes the solution of

$$\begin{cases} -\Delta u = 1 & \text{in } A, \\ u \in H_0^1(A), \end{cases} \tag{4.4}$$

then the solution $u_{A\cup B_\varepsilon}$, corresponding to the admissible choice $A \cup B_\varepsilon$, can be easily identified, and we find

$$u_{A\cup B_\varepsilon}(x)=\begin{cases} u_A(x) & \text{if } x\in A,\\ (\varepsilon^2-|x-x_0|^2)/2N & \text{if } x\in B_\varepsilon,\\ 0 & \text{otherwise.}\end{cases} \tag{4.5}$$

Therefore, we obtain

$$\begin{aligned} J(u_A) &= \int_A |u_A-c|^2\,dx+\int_{B_\varepsilon} c^2\,dx+\int_{D\setminus(A\cup B_\varepsilon)} c^2\,dx,\\ J(u_{A\cup B_\varepsilon}) &= \int_A |u_A-c|^2\,dx+\int_{B_\varepsilon}\left|\frac{\varepsilon^2-|x-x_0|^2}{2N}-c\right|^2 dx+\int_{D\setminus(A\cup B_\varepsilon)} c^2\,dx. \end{aligned}$$

Comparing the cost $J(u_A)$ to the cost $J(u_{A\cup B_\varepsilon})$ and using the minimality of A then gives

$$\begin{aligned} c^2\operatorname{meas}(B_\varepsilon) &\le \int_{B_\varepsilon}\left|\frac{\varepsilon^2-|x-x_0|^2}{2N}-c\right|^2 dx\\ &= N\varepsilon^{-N}\operatorname{meas}(B_\varepsilon)\int_0^\varepsilon\left|\frac{\varepsilon^2-r^2}{2N}-c\right|^2 r^{N-1}\,dr\\ &= c^2\operatorname{meas}(B_\varepsilon)+\frac{\operatorname{meas}(B_1)}{4N}\int_0^\varepsilon(\varepsilon^2-r^2)(\varepsilon^2-r^2-4Nc)r^{N-1}\,dr \end{aligned}$$

which, for a fixed $c>0$, turns out to be false if ε is small enough.

Thus all smooth domains $A\neq D$ are ruled out by the argument above. We can now exclude also the case $A=D$ if c is small, by comparing for instance the full domain D to the empty set. This gives, taking into account that $u_\emptyset\equiv 0$,

$$\begin{aligned} J(u_D) &= \int_D |u_D-c|^2\,dx,\\ J(u_\emptyset) &= \int_D c^2\,dx \end{aligned} \tag{4.6}$$

so that we have $J(u_\emptyset)<J(u_D)$ if c is sufficiently small. Hence all smooth subdomains of D are excluded, and the proof is complete. ■

Example 4.2.2 If we take into account the identification of the class of relaxed domains seen in Section 3.6, then we may produce, rather simply, other examples of nonexistence of optimal domains. Take indeed a smooth function f in (4.1) such that $f(x)>0$ in D and let w be the solution of the problem

$$\begin{cases} -\Delta w = f \text{ in } D,\\ w\in H_0^1(D). \end{cases} \tag{4.7}$$

It is well known, from the maximum principle, that $w(x)>0$ in D. Take now the desired state $\overline{u}(x)=w(x)/2$ and the cost density $j(x,s)=|s-\overline{u}(x)|^2$ like in (4.2). Then the optimization problem

$$\min\Big\{\int_D |u_A-\overline{u}|^2\,dx\ :\ -\Delta u_A=f\ \text{in}\ A,\ u_A\in H_0^1(A)\Big\}$$

admits the relaxed formulation

$$\min\Big\{\int_D |u_\mu-\overline{u}|^2\,dx\ :\ -\Delta u_\mu+u\mu=f\ \text{in}\ D,\ u_\mu\in H_0^1(D)\Big\},$$

where the measure μ varies now in the class of relaxed controls seen in Section 3.6. It is easy to see that the relaxed problem attains its minimum value 0 at the measure

$$\mu=(f/w)\cdot dx$$

which corresponds to the solution $u_\mu=w/2$ of the relaxed state equation

$$\begin{cases}-\Delta u_\mu+u\mu=f & \text{in}\quad D,\\ \qquad u_\mu\in H_0^1(D).\end{cases}\tag{4.8}$$

On the other hand, since $\overline{u}>0$ in D, it is clear that there are no domains $A\neq D$ such that $u_A=\overline{u}$ in D. The case $A=D$ has also to be excluded, because $u_D=w>w/2=\overline{u}$.

The assumption above that f is smooth can be weakened by simply requiring that $f(x)>0$ for a.e. $x\in D$.

4.3 The relaxed form of a Dirichlet problem

As already seen in Section 3.6 the relaxed form of a shape optimization problem with Dirichlet conditions on the free boundary involves relaxed controls which are measures. In this section we give more details about this topic; the reader may find a complete discussion in [65].

We know that the definition of relaxed controls only depends on the state equation, that we take for simplicity of the form

$$-\Delta u=f\ \text{in}\ A,\qquad u\in H_0^1(A).$$

Here the control variable A runs in the class of open subsets of a given bounded subset D of $\mathbb{R}^N$ and f is a given function in $L^2(D)$.

As already stated in Section 3.6, in order to discuss the relaxation of Dirichlet problems we denote by $\mathcal{M}_0(D)$ the set of all nonnegative Borel measures μ on D, possibly $+\infty$ valued, such that

i) $\mu(B)=0$ for every Borel set $B\subseteq D$ with $\operatorname{cap}(B)=0$,

ii) $\mu(B)=\inf\{\mu(U):U\ \text{quasi-open},\ B\subseteq U\}$ for every Borel set $B\subseteq D$.

We stress the fact that the measures $\mu\in\mathcal{M}_0(D)$ do not need to be finite, and may take the value $+\infty$ even on large parts of D.

For every measure $\mu \in \mathcal{M}_0(D)$ we denote by A_μ the *"set of finiteness"* of μ; more precisely A_μ is defined as the union of all finely open subsets A of D such that $\mu(A) < +\infty$; A_μ is called the regular set of the measure μ. By its definition, the set A_μ is finely open, hence quasi-open. We also denote by $S_\mu = D \setminus A_\mu$ the singular set of μ.

For example, if $N - 2 < \alpha \leq N$ the α-dimensional Hausdorff measure $\mathcal{H}^\alpha$ belongs to $\mathcal{M}_0(D)$ (and consequently every μ is absolutely continuous with respect to $\mathcal{H}^\alpha$ as well). In fact every Borel set with capacity zero has a Hausdorff dimension which is less than or equal to $N-2$. Another example of measure of the class $\mathcal{M}_0(D)$ is, for every $S \subseteq D$, the measure ∞_S defined by

$$\infty_S(B) = \begin{cases} 0 & \text{if } \operatorname{cap}(B \cap S) = 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.9}$$

In order to write correctly the relaxed form of the state equation we introduce the space $X_\mu(D)$ as the vector space of all functions $u \in H_0^1(D)$ such that $\int_D u^2 \, d\mu < \infty$. Note that, since μ vanishes on all sets with capacity zero and since Sobolev functions are defined up to sets of capacity zero, the definition of $X_\mu(D)$ is well posed. In other words we may think of $X_\mu(D)$ as $H_0^1(D) \cap L^2(D, \mu)$; moreover we can endow the space $X_\mu(D)$ with the norm

$$\|u\|_{X_\mu(D)} = \Big(\int_D |\nabla u|^2 \, dx + \int_D u^2 \, d\mu \Big)^{1/2}$$

which comes from the scalar product

$$(u, v)_{X_\mu(D)} = \int_D \nabla u \nabla v \, dx + \int_D uv \, d\mu.$$

It is possible to show (see [65]) that with the scalar product above the space $X_\mu(D)$ becomes a Hilbert space.

Since $X_\mu(D)$ can be embedded into $H_0^1(D)$ by the identity mapping $i(u) = u$, the dual space $H^{-1}(D)$ of $H_0^1(D)$ can be considered as a subspace of the dual space $X'_\mu(D)$. We then write for $f \in H^{-1}(D)$,

$$\langle f, v \rangle_{X'_\mu(D)} = \langle f, v \rangle_{H^{-1}(D)} \qquad \forall v \in X_\mu(D)$$

and so, when $f \in L^2(D)$,

$$\langle f, v \rangle_{X'_\mu(D)} = \int_D f v \, dx \qquad \forall v \in X_\mu(D).$$

Example 4.3.1 Take $\mu = a(x)\mathcal{H}^N$ where $a \in L^p(D)$ and

$$\begin{cases} N/2 \leq p \leq +\infty & \text{if } N \geq 3, \\ 1 < p \leq +\infty & \text{if } N = 2. \end{cases} \tag{4.10}$$

Then, by the Sobolev embedding theorem and Hölder inequality, we have that $X_\mu(D) = H_0^1(A)$ with equivalent norms.

Example 4.3.2 Let A be a finely open subset of D and let $S = D \setminus A$; take $\mu = \infty_S$ as defined in (4.9). Then, by the Poincaré inequality, we have that $X_\mu(D) = H_0^1(A)$ with equivalent norms. The same conclusion holds if $\mu = \infty_S + a(x)\mathcal{H}^N$ where $a \in L^p(D)$ with p satisfying the conditions of the previous example.

Consider now a measure $\mu \in \mathcal{M}_0(D)$. By the Riesz representation theorem, for every $f \in X'_\mu(D)$ there exists a unique $u \in X_\mu(D)$ such that

$$(4.11) \qquad (u, v)_{X_\mu(D)} = \langle f, v\rangle_{X'_\mu(D)} \qquad \forall v \in X_\mu(D).$$

By the definition of scalar product in $X_\mu(D)$ this turns out to be equivalent to

$$(4.12) \qquad \int_D \nabla u \nabla v \, dx + \int_D uv \, d\mu = \langle f, v\rangle_{X'_\mu(D)} \qquad \forall v \in X_\mu(D)$$

that we simply write in the form

$$u \in X_\mu(D), \qquad -\Delta u + \mu u = f \text{ in } X'_\mu(D).$$

This is the relaxed state equation of the optimal control problem we shall consider. In other words, the *resolvent operator* $R_\mu : X'_\mu(D) \to X_\mu(D)$ which associates to every $f \in X'_\mu(D)$ the unique solution u of (4.12) is well defined. Moreover it is easy to see that the operator R_μ is linear and continuous from $X'_\mu(D)$ onto $X_\mu(D)$, it is symmetric, that is

$$\langle g, R_\mu(f)\rangle_{X'_\mu(D)} = \langle f, R_\mu(g)\rangle_{X'_\mu(D)} \qquad \forall f, g \in X'_\mu(D),$$

and there exists a constant c, which depends only on D, such that

$$\|R_\mu(f)\|_{H^1(D)} \le c\|f\|_{H^{-1}(D)} \qquad \forall f \in H^{-1}(D).$$

Example 4.3.3 If we take $\mu = a(x)\mathcal{H}^N$ with $a \in L^p(D)$ and p satisfying the assumption of Example 4.3.1, and $f \in H^{-1}(D)$, then, according to what we saw in Example 4.3.1, the relaxed state equation simply becomes

$$u \in H_0^1(D), \qquad -\Delta u + au = f \text{ in } H^{-1}(D).$$

Notice that in this case we have $au \in H^{-1}(D)$.

Example 4.3.4 If we take $\mu = \infty_{D\setminus A}$ with A an open subset of D, and $f \in H^{-1}(D)$, then, according to what we saw in Example 4.3.2, the relaxed state equation simply becomes

$$u \in H_0^1(A), \qquad -\Delta u = f \llcorner A \text{ in } H^{-1}(A),$$

where the restriction $f \llcorner A$ is defined by

$$\langle f \llcorner A, v\rangle_{H^{-1}(A)} = \langle f, v\rangle_{H^{-1}(D)} \qquad \forall v \in H_0^1(A).$$

Example 4.3.5 If we take $\mu = \infty_{D\setminus A} + a(x)\mathcal{H}^N$ with a and A as in the examples above, then the relaxed state equation takes the form

$$u \in H_0^1(A), \qquad -\Delta u + au = f \llcorner A \text{ in } H^{-1}(A).$$

In Section 3.6 we have already stated the fact that the class $\mathcal{M}_0(D)$ is the class of relaxed controls obtained through the abstract relaxation procedure introduced in Section 3.5. In particular, $\mathcal{M}_0(D)$ can be endowed with the topology of γ-convergence (see Definition 3.3.1 in Chapter 3), which can be also defined through the resolvent operators.

Definition 4.3.6 *We say that a sequence (μ_n) of measures in $\mathcal{M}_0(D)$ γ-converges to a measure $\mu \in M_0(D)$ if and only if*

$$R_{\mu_n}(f) \to R_\mu(f) \text{ weakly in } H_0^1(D) \qquad \forall f \in H^{-1}(D).$$

The following compactness property follows from the abstract scheme introduced in Section 3.5 and the density follows from [100].

Proposition 4.3.7 *The space $\mathcal{M}_0(D)$, endowed with the topology of γ-convergence, is a compact metric space. Moreover, the class of measures of the form $\infty_{D\setminus A}$, with A an open (and smooth) subset of D, is dense in $\mathcal{M}_0(D)$.*

Remark 4.3.8 It is easy to see that also the class of measures of the form $a(x)\mathcal{H}^N$, where a is a nonnegative and smooth function in D, is dense in $\mathcal{M}_0(D)$.

Example 4.3.9 An explicit constructive way to approximate every Radon measure μ of $\mathcal{M}_0(D)$ by a sequence of measures of the form $\infty_{D\setminus A_n}$ is given in [99].

In the sequel we show (without proofs) how the relaxed form can be found in a direct way. For this approach we refer to [102]. We also refer the reader to the classical example of Cioranescu and Murat [84] which is briefly presented below.

Let $f \in L^2(D)$ and let (A_n) be a sequence of quasi-open subsets of the bounded design region D. We denote by u_n the solution of the following equation on A_n:

$$\begin{cases} -\Delta u_n = f \text{ in } A_n, \\ u_n \in H_0^1(A_n). \end{cases} \tag{4.13}$$

Suppose that w_n is the solution on A_n of the same equation, but for the right hand side $f \equiv 1$. Extracting a subsequence if necessary, we may suppose that $u_n \rightharpoonup u$ and $w_n \rightharpoonup w$ weakly in $H_0^1(D)$.

Let $\varphi \in C_0^\infty(D)$. Taking as a test function $w_n\varphi$ for (4.13) on A_n we have the following sequence of equalities:

$$\begin{aligned}
\int_D f w_n \varphi dx &= \int_D \nabla u_n \nabla (w_n \varphi) dx \\
&= \int_D \nabla u_n \nabla \varphi w_n dx + \int_D \nabla u_n \nabla w_n \varphi dx \\
&= \int_D \nabla u_n \nabla \varphi w_n dx - \int_D u_n \nabla w_n \nabla \varphi dx - \langle \Delta w_n, \varphi u_n \rangle_{H^{-1}(D) \times H_0^1(D)} \\
&= \int_D \nabla u_n \nabla \varphi w_n dx - \int_D u_n \nabla w_n \nabla \varphi dx + \int_D u_n \varphi dx.
\end{aligned}$$

Letting $n \to \infty$ we get

$$\int_D f \varphi w dx = \int_D \nabla u \nabla \varphi w dx - \int_D u \nabla w \nabla \varphi dx + \int_D u \varphi dx.$$

Since

$$-\int_D u \nabla w \nabla \varphi dx = \int_D \nabla u \nabla w \varphi dx + \langle \Delta w, u\varphi \rangle_{H^{-1}(D) \times H_0^1(D)},$$

we formally write

$$\int_D \nabla u \nabla (\varphi w) dx + \int_D u \varphi w d\mu = \int_D f \varphi w dx, \tag{4.14}$$

where μ is the Borel measure defined by

$$\mu(B) = \begin{cases} +\infty & \text{if } \operatorname{cap}(B \cap \{w = 0\}) > 0, \\ \displaystyle\int_B \frac{1}{w} d\nu & \text{if } \operatorname{cap}(B \cap \{w = 0\}) = 0. \end{cases} \tag{4.15}$$

Here $\nu = \Delta w + 1 \geq 0$ in $\mathcal{D}'(D)$ is a nonnegative Radon measure belonging to $H^{-1}(D)$.

This formal computation needs several rigorous proofs for which we refer the reader to [102]. We recall here the following facts:

i) u vanishes where w vanishes and $u \in H_0^1(D) \cap L^2(D, \mu)$;
ii) the set $\{\varphi w : \varphi \in C_0^\infty(D)\}$ is dense in $H_0^1(D) \cap L^2(D, \mu)$;
iii) $w \in \mathcal{K} := \{w \in H_0^1(D) : w \geq 0, -\Delta w \leq 1 \text{ in } D\}$;
iv) there exists a one to one mapping between $\mathcal{K}$ and $\mathcal{M}_0(D)$ given by $w \mapsto \mu$ where μ is defined by (4.15);
v) for every $w \in \mathcal{K}$ and every $\varepsilon > 0$, there exists an open set $A \subseteq D$ such that $\|w - w_A\|_{L^2(D)} \leq \varepsilon$.

Assertions i) and ii) give full sense to equation (4.14). Assertion v) proves that the family of open sets is dense in the family of relaxed domains which are identified with measures of $\mathcal{M}_0(D)$.

Remark 4.3.10 Note that for the γ-convergence of a sequence of measures, Definition 4.3.6 required the convergence of the resolvent operators for every $f \in H^{-1}(D)$. In fact, it is enough to have it only for $f \equiv 1$ (see [102]), and this implies the convergence for every $f \in H^{-1}(D)$. We will prove this fact for the special case of open sets in Proposition 4.5.3.

Remark 4.3.11 The same construction of relaxed domains can be performed for the p-Laplacian in $W_0^{1,p}(D)$ for $1 < p \le N$ (see [102]). Given $f \in L^q(D)$ and a sequence (A_n) of p-quasi-open subsets of D we denote by u_n the solution of the following equation on A_n:

$$\begin{cases} -\Delta_p u_n = f \text{ in } A_n, \\ u_n \in W_0^{1,p}(A_n), \end{cases} \tag{4.16}$$

which has to be understood in the sense

$$\forall v \in W_0^{1,p}(A_n) \qquad \int_{A_n} |\nabla u_n|^{p-2}\nabla u_n \nabla v dx = \int_{A_n} f v dx.$$

There exists a subsequence (still denoted by the same indices) of $(A_n)_n$ such that for every $f \in L^q(D)$, the sequence (u_n) weakly converges in $W_0^{1,p}(D)$ to the solution of the equation

$$\begin{cases} -\Delta_p u + \mu|u|^{p-2}u = f, \\ u \in W_0^{1,p}(D) \cap L^p_\mu(D), \end{cases} \tag{4.17}$$

μ being the Radon measure defined by

$$\mu(A) = \begin{cases} +\infty & \text{if } \operatorname{cap}_p(A \cap \{w = 0\}) > 0, \\ \displaystyle\int_A \frac{d\nu}{w^{p-1}} & \text{if } \operatorname{cap}_p(A \cap \{w = 0\}) = 0. \end{cases} \tag{4.18}$$

Here w is the weak limit in $W_0^{1,p}(D)$ of the solutions of (4.16) with $f = 1$ and $\nu = 1 + \Delta_p w$.

Example 4.3.12 (Cioranescu and Murat) In this example, we construct a sequence of open sets which are γ-convergent to an element of $\mathcal{M}_0(D)$ which is not a quasi-open set. Let D be an open set contained in the unit square of $\mathbb{R}^2$, $S =]0,1[\times]0,1[$.

We consider, for n large enough, the sequence of sets

$$C_n = \bigcup_{i,j=0}^{n} \overline{B}_{(i/n,j/n),r_n}, \qquad A_n = D \setminus C_n,$$

where $r_n = e^{-cn^2}$, $c > 0$ being a fixed positive constant. Let us denote by u_n the solution of (4.13) on A_n. For a subsequence, still denoted by the same indices, we can suppose that $u_n \rightharpoonup u$ weakly in $H_0^1(S)$.

Instead of working with the functions w_n used for finding the general form of a relaxed problem, in this particular case it is more convenient to introduce the following functions $z_n \in H^1(S)$:

$$z_n = \begin{cases} 0 & \text{on } \ C_n, \\ \dfrac{\ln\sqrt{(x-i/n)^2+(y-j/n)^2}+cn^2}{cn^2-\ln(2n)} & \text{on } \ \overline{B}_{(i/n,j/n),1/2n}\setminus C_n, \\ 1 & \text{on } \ S\setminus\bigcup_{i,j=0}^n \overline{B}_{(i/n,j/n),1/2n}. \end{cases}$$

We notice the following facts:

- $0 \le z_n \le 1$.
- $\nabla z_n \overset{L^2}{\rightharpoonup} 0$ as $n\to\infty$, hence z_n converges weakly in $H^1(S)$ to a constant function. Computing the limit of $\int_S z_n dx$ we find that this constant is equal to 1.

Let $\varphi \in C_0^\infty(D)$. Then $z_n\varphi \in H_0^1(A_n)$, thus we can take $z_n\varphi$ as a test function for equation (4.13) on A_n:

$$\int_D \nabla u_n \nabla z_n \varphi dx + \int_D \nabla u_n \nabla\varphi z_n dx = \int_D f\varphi z_n dx.$$

The second and third terms of this equality converge to $\int_D \nabla u \nabla\varphi dx$ and $\int_D f\varphi dx$, respectively. For the first term, the Green formula gives

$$\int_D \nabla u_n \nabla z_n \varphi dx = \sum_{i,j=0}^n \int_{\partial B_{(i/n,j/n),1/2n}} u_n \frac{\partial z_n}{\partial n}\varphi d\sigma - \int_D u_n \nabla z_n \nabla\varphi dx.$$

The boundary term on $\partial B_{(i/n,j/n),r_n}$ does not appear since u_n vanishes on it. The last term of this identity converges to 0 when $n\to\infty$.

We compute now the boundary integral. We have

$$\sum_{i,j=0}^n \int_{\partial B_{(i/n,j/n),1/2n}} u_n \frac{\partial z_n}{\partial n}\varphi d\sigma = \sum_{i,j=0}^n \int_{\partial B_{(i/n,j/n),1/2n}} \frac{2n}{cn^2-\ln(2n)} u_n\varphi d\sigma$$

$$= \frac{2n^2}{cn^2-\ln(2n)} \sum_{i,j=0}^n \int_{\partial B_{(i/n,j/n),1/2n}} \frac{1}{n} u_n \varphi d\sigma.$$

Let us denote by $\mu_n \in H^{-1}(S)$ the distribution defined by

$$\langle \mu_n, \psi\rangle_{H^{-1}(S)\times H_0^1(S)} = \sum_{i,j=0}^n \int_{\partial B_{(i/n,j/n),1/2n}} \frac{1}{n}\psi d\sigma.$$

We prove that μ_n converges strongly in $H^{-1}(S)$ to πdx. Indeed, we introduce the functions $v_n \in H^1(S)$ defined by

$$\begin{cases} \Delta v_n = 4 \ \text{ in } \ \bigcup B_{(i/n,j/n),1/2n}, \\ v_n = 0 \ \text{ on } \ \ S \setminus \bigcup B_{(i/n,j/n),1/2n}. \end{cases}$$

Therefore

$$\frac{\partial v_n}{\partial n} = \frac{1}{n} \ \text{ on } \bigcup \partial B_{(i/n,j/n),1/2n}.$$

We notice that $v_n \longrightarrow 0$ strongly in $H^1(S)$, therefore $\Delta v_n \longrightarrow 0$ strongly in $H^{-1}(S)$. But,

$$\langle -\Delta v_n, \psi \rangle_{H^{-1}(S)\times H_0^1(S)} = \sum_{i,j=0}^{n} \int_{B_{(i/n,j/n),1/2n}} \nabla v_n \nabla \psi dx$$

$$= \sum_{i,j=0}^{n} \int_{\partial B_{(i/n,j/n),1/2n}} \frac{1}{n} \psi d\sigma - \sum_{i,j=0}^{n} \int_{B_{(i/n,j/n),1/2n}} 4\psi dx.$$

Passing to the limit as $n \to \infty$ and using the fact that $1_{\bigcup B_{(i/n,j/n),1/2n}} \rightharpoonup \frac{\pi}{4} 1_S$ weakly in L^2 we get that

$$\mu_n \overset{H^{-1}(S)}{\longrightarrow} \pi dx.$$

Consequently, the equation satisfied by $u \in H_0^1(S)$ is

$$\forall \varphi \in C_0^\infty(D) \ \ \int_D \nabla u \nabla \varphi dx + \frac{2\pi}{c} \int_D u \varphi dx = \int_D f \varphi dx,$$

that is

$$-\Delta u + \frac{2\pi}{c} u = f.$$

The following results will be extensively used throughout the next chapters. For every quasi-open set A, we denote by w_A the solution of (4.13) for $f = 1$.

Lemma 4.3.13 *Let (A_n) be a sequence of quasi-open subsets of D and let $w \in H_0^1(D)$ be a function such that $w_{A_n} \rightharpoonup w$ weakly in $H_0^1(D)$. Let $u_n \in H_0^1(D)$ be such that $u_n = 0$ q.e. on $D \setminus A_n$ and suppose that $u_n \rightharpoonup u$ in $H_0^1(D)$. Then $u = 0 \quad$ q.e. on $\{w = 0\}$.*

Remark 4.3.14 A proof of this result, involving Γ-convergence tools, can be found in [66]. Let us sketch here the idea of the proof.

Let $f_n = -\Delta u_n \in H^{-1}(D)$. Then $f_n \rightharpoonup f := -\Delta u$ weakly in $H^{-1}(D)$. Consequently, if $v_n \in H_0^1(A_n)$ satisfies in $H_0^1(A_n)$ the equation $-\Delta v_n = f$, then $u_n - v_n \rightharpoonup 0$ weakly in $H_0^1(D)$, hence $v_n \rightharpoonup u$ weakly in $H_0^1(D)$. For every $\varepsilon > 0$, we consider $f_\varepsilon \in L^\infty(D)$ such that $|f_\varepsilon - f|_{H^{-1}(D)} \le \varepsilon$. If we denote by v_n^ε the solution in $H_0^1(A_n)$ of $-\Delta v_n^\varepsilon = f_\varepsilon$, then we get from the maximum principle

$$0 \leq |v_n^\varepsilon| \leq |f_\varepsilon|_\infty w_{A_n}.$$

Consequently, any weak limit of v_n^ε will vanish quasi-everywhere on $\{w = 0\}$. By a diagonal procedure, making $\varepsilon \to 0$ we get that $u \in H_0^1(\{w > 0\})$.

The nonlinear version of this lemma also holds true. A proof can be found in [102].

Lemma 4.3.15 *Let there be given a sequence of quasi-open sets (A_n) and another quasi-open set A such that $w_{A_n} \rightharpoonup w$ weakly in $H_0^1(D)$ and $w \in H_0^1(A)$. There exists a subsequence (still denoted using the same indices) and a sequence of open sets $G_n \subseteq D$ with $A_n \subseteq G_n$ and G_n γ-converges to A.*

Proof Following [102] we have $w \leq w_A$. For each $\varepsilon > 0$ we define the quasi-open set $A^\varepsilon = \{w_A > \varepsilon\}$. For a subsequence, still denoted by the same indices, we can suppose that

$$w_{A_n \cup A^\varepsilon} \overset{H_0^1(D)}{\rightharpoonup} w^\varepsilon$$

and by the comparison principle we have that $w^\varepsilon \geq w_{A^\varepsilon}$. But $w^\varepsilon \in H_0^1(A)$. Indeed, defining $v^\varepsilon = 1 - \frac{1}{\varepsilon}\min\{w_A, \varepsilon\}$ we get $0 \leq v^\varepsilon \leq 1$ and $v^\varepsilon = 0$ on A^ε, $v^\varepsilon = 1$ on $D \setminus A$. Taking $u_n = \min\{v^\varepsilon, w_{A_n \cup A^\varepsilon}\}$ we get $u_n = 0$ on $A^\varepsilon \cup (D \setminus (A_n \cup A^\varepsilon))$, and in particular on $D \setminus A_n$. Moreover $u_n \rightharpoonup \min\{v^\varepsilon, w^\varepsilon\}$ weakly in $H_0^1(D)$ and hence $\min\{v^\varepsilon, w^\varepsilon\}$ vanishes q.e. on $\{w = 0\}$. Since $v^\varepsilon = 1$ on $D \setminus A$ we get that $w^\varepsilon = 0$ q.e. on $D \setminus A$.

Using [102, Theorem 5.1], from the fact that $-\Delta w_{A_n \cup A^\varepsilon} \leq 1$ in D we get $-\Delta w^\varepsilon \leq 1$ and hence $w^\varepsilon \leq w_A$. Finally $w_{A^\varepsilon} \leq w^\varepsilon \leq w_A$, and by a diagonal extraction procedure we get that

$$w_{A_n \cup A^{\varepsilon_n}} \overset{H_0^1(D)}{\rightharpoonup} w_A.$$

■

Remark 4.3.16 The nonlinear version of this lemma is also true. We refer to [42] for the proof.

4.4 Necessary conditions of optimality

In this section we consider the shape optimization problem

$$\min\left\{\int_D j(x, u_A)\,dx \ : \ A \text{ open subset of } D\right\} \tag{4.19}$$

where we denote by u_A the unique solution of the Dirichlet problem

$$-\Delta u = f \text{ in } A, \qquad u \in H_0^1(A).$$

Here D is a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$), $f \in L^2(D)$, and the integrand $j(x, s)$ is supposed to be a Carathéodory function such that

$$|j(x,s)| \leq a(x) + c|s|^2 \tag{4.20}$$

for suitable $a \in L^1(D)$ and $c \in \mathbb{R}$.

As seen in the Sections 3.5, 3.6, 4.3 the relaxed form of the shape optimization problem above involves measures of $\mathcal{M}_0(D)$ as relaxed controls, and takes the form

$$\min\left\{ \int_D j(x, u_\mu)\,dx \ : \ \mu \in \mathcal{M}_0(D) \right\} \tag{4.21}$$

where we denoted by u_μ the unique solution of the relaxed Dirichlet problem

$$u \in X_\mu(D), \qquad -\Delta u + \mu u = f \quad \text{in} \quad X'_\mu(D).$$

Remark 4.4.1 By using the Sobolev embedding theorem, it is easy to see that it is possible to replace the growth condition (4.20) by the weaker one

$$|j(x,s)| \leq a(x) + c|s|^p \tag{4.22}$$

with $a \in L^1(D)$, $c \in \mathbb{R}$, and $p < 2N/(N-2)$.

We have already seen examples which show that the original problem (4.19) may have no solution; on the other hand, the relaxed optimization problem (4.21) always admits a solution, as shown in the abstract scheme of Section 3.5. Our goal is now to obtain some necessary conditions of optimality for the solutions μ of the relaxed optimization problem (4.21). They will be obtained by evaluating the cost functional on a family μ_ε of perturbations of μ and by computing the limit

$$\lim_{\varepsilon \to 0^+} \frac{J(u_{\mu_\varepsilon}) - J(u_\mu)}{\varepsilon}\,.$$

Some numerical computations for the relaxed solution can be found in [121] and [122].

In what follows we assume for simplicity that the function $j(x, \cdot)$ is continuously differentiable and that its differential verifies the growth condition

$$|j_s(x,s)| \leq a_1(x) + c_1|s|$$

for suitable $a_1 \in L^2(D)$ and $c_1 \in \mathbb{R}$.

The first perturbation we consider is of the form $\mu_\varepsilon = \mu + \varepsilon\phi\mathcal{H}^N$ where ϕ is a nonnegative function belonging to $L^\infty(D)$. If (u, μ) is an optimal pair of the relaxed optimization problem and $u_\varepsilon = R_{\mu_\varepsilon}(f)$, proceeding as in [65] we obtain

$$\frac{dJ(u_\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} = -\int_D j_s(x,u) R_\mu(\phi u)\,dx.$$

On the other hand, the optimality of μ gives that the derivative above has to be nonnegative, so that we obtain

$$\int_D j_s(x,u)R_\mu(\phi u)\,dx \le 0 \qquad \forall\phi \in L^\infty(D),\ \phi \ge 0.$$

By the symmetry of the resolvent operator R_μ we can also write

$$\int_D R_\mu\big(j_s(x,u)\big)\phi u\,dx \le 0 \qquad \forall\phi \in L^\infty(D),\ \phi \ge 0$$

which gives, since ϕ is arbitrary,

$$R_\mu\big(j_s(x,u)\big)u \le 0 \quad \text{a.e. in} \quad D.$$

It is now convenient to introduce the adjoint state equation

$$v \in X_\mu(D), \qquad -\Delta v + \mu v = j_s(x,u) \quad \text{in} \quad X'_\mu(D) \tag{4.23}$$

so that the optimality condition above reads

$$uv \le 0 \quad \text{a.e. in} \quad D.$$

Noticing that u and v are finely continuous q.e. in D, their product uv is still finely continuous q.e. in D, and since nonempty finely open sets have positive Lebesgue measure we obtain the following necessary condition of optimality.

Proposition 4.4.2 *If (u,μ) is an optimal pair of the relaxed optimization problem (4.21) and if v denotes the solution of the adjoint state equation (4.23), then we have*

$$uv \le 0 \quad \textit{q.e. in} \quad D. \tag{4.24}$$

We consider now another kind of perturbation of an optimal measure μ by taking the family of measures $\mu_\varepsilon = (1-\varepsilon)\mu$ that, for $\varepsilon < 1$, still belong to the class $\mathcal{M}_0(D)$. Again, denoting by $u_\varepsilon = R_{\mu_\varepsilon}(f)$ the optimal state related to μ_ε, proceeding as in [65] we obtain

$$\frac{dJ(u_\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} = \int_D j_s(x,u)R_\mu(\mu u)\,dx.$$

The optimality of μ gives that the derivative above has to be nonnegative, so that we obtain

$$\int_D j_s(x,u)R_\mu(\mu u)\,dx \ge 0$$

and, again by the symmetry of the resolvent operator R_μ we have

$$\int_D R_\mu\big(j_s(x,u)\big)u\,d\mu \ge 0.$$

On the other hand, the optimality condition obtained in Proposition 4.4.2 gives that the product $uR_\mu\big(j_s(x,u)\big)$ is less than or equal to zero q.e. in D, hence μ-a.e. in D, which implies the following second necessary condition of optimality.

Proposition 4.4.3 *If (u, μ) is an optimal pair of the relaxed optimization problem (4.21) and if v denotes the solution of the adjoint state equation (4.23), then we have*

$$uv = 0 \quad \mu\text{-a.e. in} \quad D. \tag{4.25}$$

In order to obtain further necessary conditions of optimality it is convenient to introduce, for every finely open subset A of $\mathbb{R}^N$, a boundary measure ν_A, carried by the fine boundary $\partial^* A$. If we denote by w_A the unique solution of the Dirichlet problem

$$-\Delta w_A = 1 \text{ in } A, \qquad w_A \in H_0^1(A), \tag{4.26}$$

then the following theorem gives the existence of ν_A.

Theorem 4.4.4 *There exists a unique nonnegative measure ν_A belonging to $H^{-1}(\mathbb{R}^N)$ and such that*

$$-\Delta w_A + \nu_A = 1_{cl^* A} \quad in \quad H^{-1}(\mathbb{R}^N), \tag{4.27}$$

where $cl^* A$ denotes the fine closure of A. Moreover, we have that ν_A is carried by $\partial^* A$ $\big($i.e., $\nu_A(\mathbb{R}^N \setminus \partial^* A) = 0\big)$, and

$$\nu_A(\partial^* A) = \mathcal{H}^N(cl^* A).$$

For the proof of the theorem above we refer to [65].

Example 4.4.5 If A is a smooth domain, then $\partial^* A = \partial A$ and the solution w_A is smooth up to the boundary. Using (4.27), an integration by parts gives

$$\int_{\mathbb{R}^N} v \, d\nu_A = -\int_{\partial A} v \frac{\partial w_A}{\partial n} \, d\mathcal{H}^{N-1} \qquad \forall v \in H^1(\mathbb{R}^N),$$

where n is the outer unit normal vector to A. Thus

$$\nu_A = -\frac{\partial w_A}{\partial n} \mathcal{H}^{N-1} \llcorner \partial A.$$

The measure ν_A above allows us to give a weak definition of the normal derivative for the solution u of a relaxed state equation

$$-\Delta u + \mu u = f \text{ in } X'_\mu(D), \qquad u \in X_\mu(D), \tag{4.28}$$

where $f \in L^2(D)$ and $\mu \in \mathcal{M}_0(D)$. We denote by $A = A_\mu$ the set of finiteness of μ, as defined in the previous section, and by ν_A the boundary measure defined above. The following result holds (see [65] for the proof).

Proposition 4.4.6 *There exists a unique $\alpha \in L^2(D, \nu_A)$ such that*

$$-\Delta u + \mu u + \alpha \nu_A = f 1_{cl^* A} \text{ in } H^{-1}(D). \tag{4.29}$$

Moreover we have

$$\int_D \alpha^2 \, d\nu_A \le \int_D f^2 \, dx$$

and $\alpha \ge 0$ ν_A-a.e. in D whenever $f \ge 0$ a.e. in D.

Example 4.4.7 Let A be a smooth domain and let $\mu = a(x)\mathcal{H}^N \llcorner A$, with $a \in L^\infty(A)$. Then equation (4.28) simply reads

$$-\Delta u + a(x)u = f \text{ in } H^{-1}(A) \qquad u \in H_0^1(A)$$

and $u \in H^2(A)$, so that $\partial u/\partial n \in L^2(\partial A, \mathcal{H}^{N-1})$. Using (4.29) and integrating by parts we obtain

$$\int_D v\alpha \, d\nu_A = -\int_{D\cap\partial A} v\frac{\partial u}{\partial n}\, d\mathcal{H}^{N-1} \qquad \forall v \in H^1(\mathbb{R}^N).$$

Therefore

$$\alpha\nu_A = -\frac{\partial u}{\partial n}\mathcal{H}^{N-1} \llcorner D \cap \partial A.$$

Since by Example 4.4.5 we have $\nu_A = -\frac{\partial w_A}{\partial n}\mathcal{H}^{N-1} \llcorner \partial A$, we finally deduce

$$\alpha = \frac{\partial u}{\partial n} \Big/ \frac{\partial w_A}{\partial n} \qquad \mathcal{H}^{N-1} \quad \text{a.e. on} \quad D \cap \partial A.$$

Note that by the Hopf maximum principle we have $\partial w_A/\partial n < 0$ on ∂A.

The last two necessary conditions of optimality for a solution μ of the relaxed problem (4.21) will be obtained by considering the perturbation

$$\mu_\varepsilon = \mu \llcorner A + \frac{1}{\varepsilon}\Big(\frac{1}{\phi}\mathcal{H}^N \llcorner \operatorname{int}^* S + \frac{1}{\psi}\nu_A\Big),$$

where $A = A_\mu$, $S = S_\mu$, and ϕ, ψ are two positive and continuous functions up to $\overline{D}$. We give a sketch of the proof by referring to [65] for all details. We denote by u_ε the corresponding solution of

$$u_\varepsilon \in X_{\mu_\varepsilon}(D), \qquad -\Delta u_\varepsilon + \mu_\varepsilon u_\varepsilon = f \quad \text{in} \quad X'_{\mu_\varepsilon}(D).$$

It is possible to show that $u_\varepsilon \to u$ strongly in $H_0^1(D)$. Moreover, by Proposition 4.4.6 there exists a unique $\alpha \in L^2(D, \nu_A)$ such that

$$-\Delta u + \mu u + \alpha\nu_A = f 1_{cl^* A} \quad \text{in} \quad H^{-1}(D);$$

analogously, if v is the solution of the adjoint equation

$$v \in X_\mu(D), \qquad -\Delta v + \mu v = j_s(x, u) \quad \text{in} \quad X'_\mu(D)$$

there exists a unique $\beta \in L^2(D, \nu_A)$ such that

$$-\Delta v + \mu v + \beta\nu_A = j_s(x, u) 1_{cl^* A} \quad \text{in} \quad H^{-1}(D).$$

Then, for every $g \in L^2(D)$ it is possible to compute the limit

$$\lim_{\varepsilon\to 0}\int_D \frac{u_\varepsilon - u}{\varepsilon} g\, dx$$

in terms of the function α, β introduced above, and, by using the fact that ϕ, ψ are arbitrary, we obtain the further necessary conditions of optimality:

$$\begin{cases} f(x)j_s(x,0) \geq 0 \qquad \text{for a.e. } x \in \text{int}^* S; \\ \alpha\beta \geq 0 \qquad \nu_A\text{-a.e. on } D. \end{cases} \tag{4.30}$$

Summarizing, the four optimality conditions we have obtained are:

$$\begin{cases} uv \leq 0 \qquad \text{q.e. in } D; \\ uv = 0 \qquad \mu\text{-a.e. in } D; \\ f(x)j_s(x,0) \geq 0 \qquad \text{for a.e. } x \in \text{int}^* S; \\ \alpha\beta \geq 0 \qquad \nu_A\text{-a.e. on } D. \end{cases} \tag{4.31}$$

Example 4.4.8 It is interesting to rewrite the conditions above in the case when the optimal measure μ has the form

$$\mu = a(x)\mathcal{H}^N \llcorner A + \infty_{D \setminus A}$$

with $a \in L^\infty(D)$, $a(x) \geq 0$ for a.e. $x \in D$, and A is an open subset of D with a smooth boundary. In this case the optimality conditions above become:

$$\begin{cases} uv \leq 0 \qquad \text{q.e. on } A; \\ uv = 0 \qquad \text{a.e. on } \{x \in A \;:\; a(x) > 0\}; \\ f(x)j_s(x,0) \geq 0 \qquad \text{for a.e. } x \in D \setminus A; \\ (\partial u/\partial n)(\partial v/\partial n) \geq 0 \qquad \mathcal{H}^{N-1}\text{-a.e. on } D \cap \partial A. \end{cases} \tag{4.32}$$

Since the boundary of A has been assumed smooth, the optimal state u and its adjoint state v both belong to the Sobolev space $H^2(A)$; hence the last condition can be written in the stronger form

$$(\partial u/\partial n)(\partial v/\partial n) = 0 \qquad \mathcal{H}^{N-1}\text{-a.e. on } D \cap \partial A.$$

Indeed, using the fact that $uv \leq 0$ on A, this follows by considering the one-dimensional functions $t \mapsto u\big(x + tn(x)\big)$ and $t \mapsto v\big(x + tn(x)\big)$ which are continuously differentiable in a neighborhood of $t = 0$ for $\mathcal{H}^{N-1}$-a.e. point $x \in D \cap \partial A$.

Specializing the conditions above to the particular case when $a \equiv 0$, which means that the original shape optimization problem has a classical solution, we obtain:

$$\begin{cases} uv \leq 0 \qquad \text{q.e. on } A; \\ f(x)j_s(x,0) \geq 0 \qquad \text{for a.e. } x \in D \setminus A; \\ (\partial u/\partial n)(\partial v/\partial n) = 0 \qquad \mathcal{H}^{N-1}\text{-a.e. on } D \cap \partial A. \end{cases} \tag{4.33}$$

Remark 4.4.9 In general, for a shape optimization problem of the form

$$\min_{\Omega \in \mathcal{U}_{ad}} J(\Omega),$$

which has a classical solution $\Omega^* \in \mathcal{U}_{ad}$, one can write two types of necessary optimality conditions that we briefly describe below.

Using the shape derivative. For an admissible vector field V one computes the shape derivative

$$dJ(\Omega^*; V) = \lim_{t \to 0} \frac{J((Id + tV)\Omega^*) - J(\Omega^*)}{t}.$$

Of course, the vector field V has to be chosen in such a way that $(Id+tV)\Omega^* \in \mathcal{U}_{ad}$, or we could use a Lagrange multiplier. The optimality condition is then written

$$dJ(\Omega^*; V) \geq 0.$$

Usually, the computation of the shape derivative requires that Ω^* is smooth enough (C^2 for example); the regularity of the optimum Ω^* is, in general, difficult to prove. Particular attention has to be given to the case when $\mathcal{U}_{ad}$ consists of convex sets, since the convexity constraint is "unstable" to small variations of the boundary. We refer the reader to [111], [140], [186] for detailed discussions of the shape derivative.

Using the topological derivative. For every $x_0 \in \Omega^*$, one computes the asymptotic development

$$J(\Omega^* \setminus \overline{B}_{x_0,\varepsilon}) = J(\Omega^*) + g(x_0) f(\varepsilon) + o(f(\varepsilon)),$$

where $f(\varepsilon) > 0$ is such that $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$. The optimality condition then writes $g(x_0) \geq 0$. We refer to [128], [185] for a detailed discussion of the topological derivative and for several applications to concrete problems.

4.5 Boundary variation

In this section we study the continuity of the mapping

$$\Omega \longrightarrow u_\Omega$$

where u_Ω is the weak variational solution of the following Dirichlet problem on Ω:

$$\begin{cases} -\Delta u_\Omega = f, \\ u_\Omega \in H_0^1(\Omega). \end{cases} \tag{4.34}$$

Here, $\Omega \subseteq D$ is an open subset of a bounded design region D of $\mathbb{R}^N$, with $N \geq 2$ and $f \in H^{-1}(D)$ is a fixed distribution. Since $u_\Omega \in H_0^1(\Omega)$, extending it by zero we can suppose that $u_\Omega \in H_0^1(D)$.

When speaking about "shape continuity" one has to endow the space of open sets with a topology. The γ-convergence is precisely the topology making continuous the shape functional $\Omega \to u_\Omega$. The difficulty is to relate this convergence of domains to a geometric one, which is easier to handle!

Assuming that we endow the family of open sets with a certain topology, the shape continuity holds if the topology is *strong*. A typical example is to consider a mapping $T \in C_c^1(\mathbb{R}^N, \mathbb{R}^N)$, to define $\Omega_t = (Id + tT)(\Omega)$ and wonder if u_{Ω_t} converges to u_Ω when $t \to 0$.

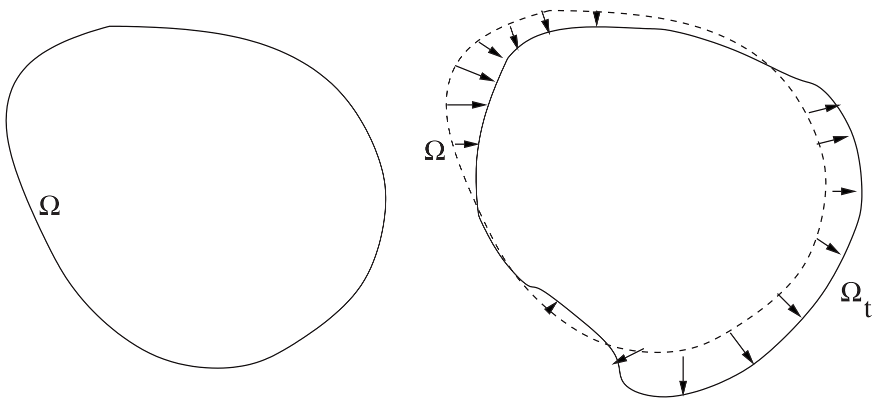

Figure 4.1. Perturbation given by a smooth vector field.

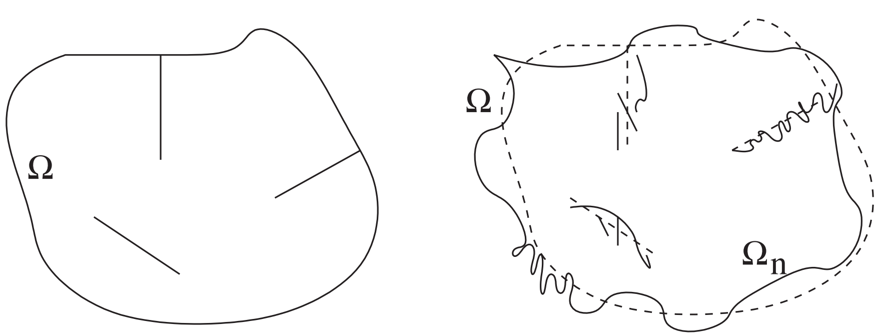

Figure 4.2. Nonsmooth perturbation.

On the contrary, if the topology on the space of open sets is *weak*, the continuity may not hold. Nevertheless, the interest to consider a weak topology is high when dealing with shape optimization problems. Indeed, in view of applying the direct methods of the calculus of variations, one needs some compactness result for a minimizing sequence (Ω_n), and this is easier to be obtained working with weak topologies.

Therefore one has to look for a kind of equilibrium: a topology weak enough in order to have compactness but strong enough in order to get continuity. This purpose is not easy to be attained; what we can do is to look for weak geometrical or topological constraints which would make a class of domains compact (in the chosen topology) and which are strong enough to give shape continuity.

The shape continuity of the solution of a PDE can be seen from two opposite points of view. First, we may suppose that an open set Ω is given and (Ω_n) is a perturbation of it. We are interested to see if the solution of the PDE on Ω is *stable* for this perturbation, i.e., if u_{Ω_n} converges to u_Ω (see Section 4.9). Second, we look for continuity-compactness results, i.e., a sequence (Ω_n) is given and we want to find a domain Ω such that (for a subsequence denoted using the same indices) we have that $u_{\Omega_n} \to u_\Omega$. This point of view is followed in Sections 4.2 to 4.8. In this case, one has to endow the family of open sets with a suitable topology τ, and the construction of Ω follows from compactness properties of this topology. On the other hand, the shape continuity of the solution is deduced from the geometrical constraints imposed on Ω_n. The role of these constraints is double: on one side they make the topological space $(\mathcal{A}, \tau)$ compact, and on the other side they provide the γ-convergence of any τ-convergent sequence (here $\mathcal{A}$ denotes the class of admissible domains). Notice that the choice of the topology is completely free, provided that the continuity-compactness occurs.

Notice that equation (4.34) is considered in a very simple case: homogeneous Dirichlet boundary conditions for the Laplace operator. We choose this easy linear setting only to simplify the proofs and to avoid heavy notation. At the end of the chapter we discuss how the results can be extended to non-linear elliptic operators and non-homogeneous boundary conditions. Roughly speaking, the shape continuity of the solution does not depend "so much" on the operator and on the right hand side f; on the other hand it "strongly" depends on the behaviour of the energy spaces (here H^1_0) on the moving domains.

How to prove shape continuity. The most abstract setting, which includes the relaxation, that we could consider for understanding the shape continuity, is to work into the frame of the Γ-convergence. When no relaxation occurs, then the Γ-convergence of the energy functionals can be seen through the Mosco convergence of the associated functional spaces. In the sequel, we follow this idea, with the only purpose to give the reader a more intuitive frame.

We start by giving a general definition for the convergence of spaces. Let X be a Banach space and $(G_n)_{n\in\mathbb{N}}$ a sequence of subsets of X. The weak upper and the strong lower limits in the sense of Kuratowski are defined as follows:

$$w - \limsup_{n\to\infty} G_n = \{u \in X : \exists (n_k)_k, \exists u_{n_k} \in G_{n_k} \text{ such that } u_{n_k} \overset{w-X}{\rightharpoonup} u\},$$

$$s - \liminf_{n\to\infty} G_n = \{u \in X : \exists u_n \in G_n \text{ such that } u_n \xrightarrow{s-X} u\}.$$

If $(G_n)_{n\in\mathbb{N}}$ are closed subspaces in X, it is said that G_n converges in the sense of Mosco to G if

M1) $G \subseteq s - \liminf\limits_{n\to\infty} G_n$,

M2) $w - \limsup\limits_{n\to\infty} G_n \subseteq G$.

Note that in general $s - \liminf_{n\to\infty} G_n \subseteq w - \limsup_{n\to\infty} G_n$. Therefore, if G_n converges in the sense of Mosco to G, then

$$s - \liminf_{n\to\infty} G_n = G = w - \limsup_{n\to\infty} G_n.$$

Coming back to the Dirichlet problem on varying domains, let us suppose that $(\Omega_n)_n$ is a sequence of quasi-open sets contained in a bounded design region D. The sequence of spaces $H_0^1(\Omega_n)$ converges in the sense of Mosco to the space $H_0^1(\Omega)$ if the following conditions are satisfied:

M1) For all $\phi \in H_0^1(\Omega)$ there exists a sequence $\phi_n \in H_0^1(\Omega_n)$ such that ϕ_n converges strongly in $H_0^1(D)$ to ϕ.

M2) For every sequence $\phi_{n_k} \in H_0^1(\Omega_{n_k})$ weakly convergent in $H_0^1(D)$ to a function ϕ we have $\phi \in H_0^1(\Omega)$.

For every open set $\Omega \subseteq D$ we denote by $P_{H_0^1(\Omega)}$ the orthogonal projection of $H_0^1(D)$ onto $H_0^1(\Omega)$ with respect to the norm $\left(\int_D |\nabla u|^2 dx\right)^{1/2}$. By R_Ω we denote the resolvent operator $R_\Omega : L^2(D) \to L^2(D)$ defined by $R_\Omega(f) = u_{\Omega,f}$.

Lemma 4.5.1 *Let A be a quasi-open subset of D. There exists a constant M depending only on $|f|_{H^{-1}(D)}$ and $|D|$ such that*

$$\|u_{A,f}\|_{H_0^1(D)} \leq M.$$

Proof Take $u_{A,f}$ as test function in the weak formulation of the equation and apply the Cauchy inequality together with the Poincaré inequality. ■

We also give the following estimate for $f \equiv 1$ which does not depend on the design region.

Lemma 4.5.2 *Let A be a quasi-open set with finite measure. There exist two constants M_1, M_2 which depend only on $|A|$ such that*

1. $\|u_{A,1}\|_{H^1(\mathbb{R}^N)} \leq M.$
2. $\|u_{A,1}\|_{L^\infty(\mathbb{R}^N)} \leq M.$

Proof Taking $u_{A,1}$ as test function gives $\int_A |\nabla u_{A,1}|^2\,dx = \int_A u_{A,1}\,dx$. Using the Poincaré inequality (for which the constant $\beta = \beta(|A|)$ depends only on $|A|$) we get

$$\|u_{A,1}\|^2_{H^1(\mathbb{R}^N)} \leq \beta^2 \int_A |\nabla u_{A,1}|^2\,dx = \beta^2 \int_A u_{A,1} dx \leq \beta^2 |A|^{\frac{1}{2}} \|u_{A,1}\|_{H^1(\mathbb{R}^N)}.$$

For the second assertion we refer the reader to [129, Theorem 8.16]. ■

Proposition 4.5.3 *Let $(\Omega_n)_n$ and Ω be open subsets of D. The following assertions are equivalent.*

1) *for every $f \in H^{-1}(D)$ we have $u_{\Omega_n,f} \longrightarrow u_{\Omega,f}$ strongly in $H_0^1(D)$ (i.e., Ω_n γ-converges to Ω);*
2) *for $f \equiv 1$ we have $u_{\Omega_n,1} \longrightarrow u_{\Omega,1}$ strongly in $H_0^1(D)$;*
3) *$H_0^1(\Omega_n)$ converges in the sense of Mosco to $H_0^1(\Omega)$;*
4) *$G(\Omega_n,\cdot) \xrightarrow{\Gamma} G(\Omega,\cdot)$ in $L^2(D)$, where $G(A,\cdot)$ are the associated energy functionals defined in (3.27) for $f \equiv 0$;*
5) *For every $u \in H_0^1(D)$ the sequence $(P_{H_0^1(\Omega_n)}u)$ converges strongly in $H_0^1(D)$ to $P_{H_0^1(\Omega)}u$;*
6) *R_{Ω_n} converges in the operator norm of $\mathcal{L}(L^2(D))$ to R_Ω.*

Proof 1) $\Longrightarrow$ 2) is obvious.

2) $\Longrightarrow$ 1) Let $f \in L^\infty(D)$, $f \geq 0$. By Lemma 4.5.1, for a subsequence we have

$$u_{\Omega_{n_k},f} \overset{H_0^1(D)}{\rightharpoonup} u.$$

From the maximum principle

$$0 \leq u_{\Omega_{n_k},f} \leq \|f\|_\infty u_{\Omega_{n_k},1},$$

hence passing to the limit as $k \to \infty$ we obtain

$$0 \leq u \leq \|f\|_\infty u_{\Omega,1},$$

therefore $u \in H_0^1(\Omega)$. Let us now take $\varphi \in \mathcal{D}(\Omega)$; there exists $\alpha > 0$ such that

$$0 \leq |\varphi| \leq \alpha u_{\Omega,1}.$$

We define the sequence $\varphi_n = \varphi_n^+ - \varphi_n^-$, where

$$\varphi_n^+ = \min\{\varphi^+, \alpha u_{\Omega_n,1}\}, \qquad \varphi_n^- = \min\{\varphi^-, \alpha u_{\Omega_n,1}\}.$$

On one side we have that $\varphi_n \in H_0^1(\Omega_n)$, and on the other side $\varphi_n \longrightarrow \varphi$ strongly in $H_0^1(D)$. Writing

$$\int_D \nabla u_{\Omega_{n_k},f} \nabla \varphi_{n_k} dx = \int_D f \varphi_{n_k} dx$$

and passing to the limit as $k \to \infty$ we get

$$\int_D \nabla u \nabla \varphi dx = \int_D f \varphi dx.$$

Consequently, $u = u_{\Omega,f}$. The convergence is strong in $H_0^1(D)$ since the norms also converge. Moreover, we get the convergence of the whole sequence from the uniqueness of the limit $u = u_{\Omega,f}$.

By the linearity of the equation and the density of $L^\infty(D)$ in $H^{-1}(D)$, the proof of point 1) is achieved.

1) $\Longleftrightarrow$ 5) Let $u \in H_0^1(D)$, and set $f := -\Delta u \in H^{-1}(D)$. Then it is easy to see that $P_{H_0^1(\Omega_n)}(u) = u_{\Omega_n,f}$, and so the equivalence between 1) and 5) follows straightforwardly.

5) $\Longrightarrow$ 4) Let $u_n \longrightarrow u$ strongly in $L^2(D)$. In order to prove that $G(\Omega, u) \leq \liminf_{n\to\infty} G(\Omega_n, u_n)$ we can suppose that $\liminf_{n\to\infty} G(\Omega_n, u_n) < +\infty$. The inequality would then follow as soon as $u \in H_0^1(\Omega)$. For that it is enough to prove that $u = P_{H_0^1(\Omega)}u$. For every $\varphi \in H_0^1(D)$ we have

$$\begin{aligned}(u, \varphi)_{H_0^1(D)\times H_0^1(D)} &= \lim_{n\to\infty} (u_n, \varphi)_{H_0^1(D)\times H_0^1(D)} \\ &= \lim_{n\to\infty} (P_{H_0^1(\Omega_n)}u_n, \varphi)_{H_0^1(D)\times H_0^1(D)} \\ &= \lim_{n\to\infty} (u_n, P_{H_0^1(\Omega_n)}\varphi)_{H_0^1(D)\times H_0^1(D)}.\end{aligned}$$

Using 5) and the pairing (weak,strong)-convergence we get

$$(u, \varphi)_{H_0^1(D)\times H_0^1(D)} = (u, P_{H_0^1(\Omega)}\varphi)_{H_0^1(D)\times H_0^1(D)} = (P_{H_0^1(\Omega)}u, \varphi)_{H_0^1(D)\times H_0^1(D)},$$

hence $u = P_{H_0^1(\Omega)}u$.

Let now $u \in H_0^1(\Omega)$. We define $u_n := P_{H_0^1(\Omega_n)}u \in H_0^1(\Omega_n)$ and get by 5) that $u_n \longrightarrow u$ strongly in $H_0^1(D)$. Then $G(\Omega, u) = \lim_{n\to\infty} G(\Omega_n, u_n)$.

4) $\Longrightarrow$ 3) Let $u \in H_0^1(\Omega)$. From the Γ-convergence, there exists $(u_n)_n$ such that $u_n \in H_0^1(\Omega_n)$ with $u_n \longrightarrow u$ strongly in $L^2(D)$ and $G(\Omega, u) = \lim_{n\to\infty} G(\Omega_n, u_n)$. This means that $u_n \in H_0^1(\Omega_n)$ and $u_n \longrightarrow u$ strongly in $H_0^1(D)$, hence the first Mosco condition is satisfied.

For the second Mosco condition, let $u_{n_k} \in H_0^1(\Omega_{n_k})$ such that $u_{n_k} \rightharpoonup u$ weakly in $H_0^1(D)$. Hypothesis 4) gives $G(\Omega, u) \leq \liminf_{n\to\infty} G(\Omega_{n_k}, u_{n_k}) < +\infty$, i.e., $u \in H_0^1(\Omega)$.

3) $\Longrightarrow$ 1) Let $\varphi \in H_0^1(\Omega)$. From the first Mosco condition there exists $\varphi_n \in H_0^1(\Omega_n)$ such that $\varphi_n \longrightarrow \varphi$ strongly in $H_0^1(D)$. On the other hand, for a subsequence we have

$$u_{\Omega_{n_k},f} \overset{H_0^1(D)}{\rightharpoonup} u,$$

and from the second Mosco condition $u \in H_0^1(\Omega)$. Writing the following chain of equalities we get $u = u_{\Omega,f}$:

$$\int_D \nabla u \nabla \varphi dx = \lim_{n\to\infty} \int_D \nabla u_{\Omega_n,f} \nabla \varphi_n dx$$

$$= \lim_{n\to\infty} \langle f, \varphi_n \rangle_{H^{-1}(D)\times H_0^1(D)} = \langle f, \varphi \rangle_{H^{-1}(D)\times H_0^1(D)}.$$

Classical arguments now give

$$u_{\Omega_n, f} \xrightarrow{H_0^1(D)} u_{\Omega, f}.$$

3) $\Longrightarrow$ 6) We have

$$|R_{\Omega_n} - R_\Omega|_{\mathcal{L}(L^2(\mathbb{R}^N))} = \sup_{\|f\|_{L^2(D)}\le 1} \|R_{\Omega_n}(f) - R_\Omega(f)\|_{L^2(D)}.$$

Supposing that $f_n \in L^2(D)$ is such that $\|f_n\|_{L^2(D)} \le 1$ and

$$|R_{\Omega_n} - R_\Omega|_{\mathcal{L}(L^2(\mathbb{R}^N))} \le \|R_{\Omega_n}(f_n) - R_\Omega(f_n)\|_{L^2(D)} + \frac{1}{n},$$

we can assume for a subsequence (still denoted using the same indices) that $f_n \rightharpoonup f$ weakly in $L^2(D)$. For proving 6), is enough to show that $R_{\Omega_n}(f_n)$ converges strongly in $L^2(D)$ to $R_\Omega(f)$. This is a consequence of 3) and of the compact embedding $H_0^1(D) \hookrightarrow L^2(D)$.

6) $\Longrightarrow$ 2) Let $f \equiv 1$. Then $R_{\Omega_n}(f) = u_{\Omega_n,1}$ and 2) follows. ■

Remark 4.5.4 The γ-convergence is local, i.e., $\Omega_n \xrightarrow{\gamma} \Omega$ if and only if there exists $\delta > 0$ such that for every $x \in D$ and for every $r \in (0, \delta)$ we have that $\Omega_n \cap B_{x,r} \xrightarrow{\gamma} \Omega \cap B_{x,r}$. This can be easily proved using the Mosco convergence and a partition of unity.

Remark 4.5.5 The results of Proposition 4.5.3 also hold for quasi-open sets. The only point which needs a more careful discussion is contained in the proof of the implication 2) $\Rightarrow$ 1). In that case, notice that the family $\mathcal{D}(\Omega)$ is not defined! One can use instead the following density result (see [102] and Section 4.8):

$$\{\phi u_{\Omega,1} : \phi \in C_0^\infty(D)\} \text{ is dense in } H_0^1(\Omega).$$

We end this section by a result that will be useful in the following (see [47] for its proof).

Proposition 4.5.6 *Let us consider two sequences of quasi-open sets* $A_n \xrightarrow{\gamma} A$ *and* $B_n \xrightarrow{\gamma} B$. *Then* $A_n \cap B_n \xrightarrow{\gamma} A \cap B$.

4.6 Continuity under geometric constraints

Let D be a bounded open set. In this section we set

$$\mathcal{A} = \{\Omega : \Omega \subseteq D, \ \Omega \text{ open}\}$$

and denote by τ the Hausdorff complementary topology on $\mathcal{A}$, given by the metric

$$d_{H^c}(\Omega_1, \Omega_2) = d(\Omega_1^c, \Omega_2^c).$$

Here d is the usual Hausdorff distance introduced in Definition 2.4.1.

Proposition 4.6.1 *The following properties of the Hausdorff convergence hold.*

1. *$(\mathcal{A}, d_{H^c})$ is a compact metric space.*
2. *If $\Omega_n \xrightarrow{H^c} \Omega$, then for every compact set $K \subseteq \Omega$, there exists $N_K \in \mathbb{N}$ such that for every $n \geq N_K$ we have $K \subseteq \Omega_n$.*
3. *The Lebesgue measure is lower semicontinuous in the H^c-topology.*
4. *The number of connected components of the complement of an open set is lower semicontinuous in the H^c-topology.*

Proof The proof of this proposition is quite simple; we refer to [140] for further details. For the convenience of the reader we only recall that property 1 is a consequence of the Ascoli–Arzelà theorem. ∎

Notice that the first Mosco condition is fulfilled for every sequence $\Omega_n \xrightarrow{H^c} \Omega$. Moreover, the space $(\mathcal{A}, \tau)$ is compact, even if it does not turn out to be γ-compact.

Proposition 4.6.2 *Suppose that $\Omega_n \xrightarrow{H^c} \Omega$ and let $f \in H^{-1}(D)$. There exists a subsequence of $(\Omega_n)_{n\in\mathbb{N}}$, still denoted using the same indices, such that*

$$u_{\Omega_n, f} \overset{H_0^1(D)}{\rightharpoonup} u$$

and

$$\int_\Omega \nabla u \nabla \phi dx = \langle f, \phi \rangle_{H^{-1}(D) \times H_0^1(D)} \tag{4.35}$$

for every $\phi \in H_0^1(\Omega)$, that is u verifies the equation $-\Delta u = f$ in $H^{-1}(\Omega)$.

Proof By Lemma 4.5.1 the sequence $(u_{\Omega_n, f})$ is bounded in $H_0^1(D)$ so that we may assume it converges weakly to some function $u \in H_0^1(D)$. It remains to prove equality (4.35). By a density argument, we may take $\phi \in C_c^\infty(\Omega)$. Since the support of ϕ is compact and Ω_n converges in H^c to Ω, equality (4.35) is valid for $u_{\Omega_n, f}$, when n is large enough. The proof is then achieved by passing to the limit as $n \to \infty$. ∎

In order to get $u = u_{\Omega, f}$ it remains to prove that $u \in H_0^1(\Omega)$. This is of course related to the second Mosco condition, which does not hold in general for sequences converging in H^c. The geometrical constraints play a crucial role for this case.

Indeed, the following counterexample shows that, in general, H^c-convergent sequences are not γ-convergent.

Example 4.6.3 Let $\{x_1, x_2, ...\}$ be an enumeration of points of rational coordinates of the square $D =]0,1[\times]0,1[$ in $\mathbb{R}^2$. Defining $\Omega_n = D \setminus \{x_1, x_2, ..., x_n\}$ we get that $\Omega_n \xrightarrow{H^c} \emptyset$ and $\Omega_n \xrightarrow{\gamma} D$ since $\text{cap}(D \setminus \Omega_n) = 0$.

A non-exhaustive list of classes of domains in which the γ-convergence is equivalent to the H^c-convergence is the following (from the strongest constraints to the weakest ones).

- The class $\mathcal{A}_{convex} \subseteq \mathcal{A}$ of convex sets contained in D.
- The class $\mathcal{A}_{unif\ cone} \subseteq \mathcal{A}$ of domains satisfying a uniform exterior cone property (see Chenais [80], [81]), i.e., such that for every point x_0 on the boundary of every $\Omega \in \mathcal{A}_{unif\ cone}$ there is a closed cone, with uniform height and opening, and with vertex in x_0, lying in the complement of Ω.
- The class $\mathcal{A}_{unif\ flat\ cone}$ of domains satisfying a uniform flat cone condition (see Bucur, Zolésio [56]), i.e., as above, but with the weaker requirement that the cone may be flat, that is of dimension $N - 1$.
- The class $\mathcal{A}_{cap\ density} \subseteq \mathcal{A}$ of domains satisfying a uniform capacity density condition (see [56]), i.e., such that there exist $c, r > 0$ such that for every $\Omega \in \mathcal{A}_{cap\ density}$, and for every $x \in \partial\Omega$, we have
$$\forall t \in (0, r) \quad \frac{\text{cap}(\Omega^c \cap B_{x,t}, B_{x,2t})}{\text{cap}(B_{x,t}, B_{x,2t})} \geq c,$$
where $B_{x,s}$ denotes the ball of radius s centered at x.
- The class $\mathcal{A}_{unif\ Wiener} \subseteq \mathcal{A}$ of domains satisfying a uniform Wiener condition (see [55]), i.e., domains satisfying for every $\Omega \in \mathcal{A}_{unif\ Wiener}$ and for every point $x \in \partial\Omega$,
$$\int_r^R \frac{\text{cap}(\Omega^c \cap B_{x,t}, B_{x,2t})}{\text{cap}(B_{x,t}, B_{x,2t})} \frac{dt}{t} \geq g(r, R, x) \quad \text{for every} \quad 0 < r < R < 1$$
where $g : (0,1) \times (0,1) \times D \to \mathbb{R}_+$ is fixed, such that for every $R \in (0,1)$ $\lim\limits_{r \to 0} g(r, R, x) = +\infty$ locally uniformly on x.

Another interesting class, which is only of topological type and is not contained in any of the previous ones, was given by Šverák [187] and consists in the following.

- For $N = 2$, the class of all open subsets Ω of D for which the number of connected components of $\overline{D} \setminus \Omega$ is uniformly bounded.

In fact, we shall see that this last constraint is strongly related to a capacity density type constraint: in two dimensions, any curve has a strictly positive capacity.

Roughly speaking, the following inclusions can be established:
$$\mathcal{A}_{convex} \subseteq \mathcal{A}_{unif\ cone} \subseteq \mathcal{A}_{unif\ flat\ cone} \subseteq \mathcal{A}_{cap\ density} \subseteq \mathcal{A}_{unif\ Wiener},$$
hence it would be enough to prove that the γ-convergence is equivalent to the H^c-convergence only in $\mathcal{A}_{unif\ Wiener}$. The shape continuity under a uniform Wiener

criterion was first observed by Frehse [125]. The proof of the continuity under the uniform Wiener criterion is slightly more technical than the continuity under capacity density condition. This is the main reason for which we prove in the sequel the continuity result only in $\mathcal{A}_{cap\ density}$ which is based on a uniform Holder estimate of the solutions (for the right-hand side $f \equiv 1$) on the moving domain. A uniform Wiener condition is in some sense the weakest reasonable constraint to obtain a continuity result in the Hausdorff complementary topology; it is based on a local equi-continuity-like property of the solutions on the moving domain.

The last part of this chapter is devoted to finding necessary and sufficient conditions for the shape continuity. In order to introduce the reader to nonlinear equations, the results of the last section are presented for the p-Laplacian (with $1 < p < +\infty$). A careful reading of Sections 4.8 and 4.9 will give the reader an idea of how to prove shape continuity under a uniform Wiener criterion.

Definition 4.6.4 *For $r, c > 0$ it is said that an open set Ω has the (r,c) capacity density condition if*

$$\forall x \in \partial\Omega, \forall t \in (0, r) \quad \frac{\text{cap}(\Omega^c \cap B_{x,t}, B_{x,2t})}{\text{cap}(B_{x,t}, B_{x,2t})} \geq c. \tag{4.36}$$

The class of open subsets of D having the (r,c) capacity density condition is denoted by $\mathcal{O}_{c,r}(D)$.

We recall the following result from [141] (the nonlinear version of this result will be used in Section 4.8).

Lemma 4.6.5 *Suppose that Ω is bounded. Let $\theta \in H^1(\Omega) \cap C(\overline{\Omega})$ and let h be the unique harmonic function in Ω with $\theta - h \in H_0^1(\Omega)$. If $x_0 \in \partial\Omega$, then for every $0 < r \leq R$ we have*

$$osc(h, \Omega \cap B_{x_0,r}) \leq osc(\theta, \partial\Omega \cap \overline{B}_{x_0,2R}) + osc(\theta, \partial\Omega) exp(-cw(\Omega, x_0, r, R))$$

where

$$w(\Omega, x_0, r, R) = \int_r^R \frac{\text{cap}(\Omega^c \cap B_{x_0,t}, B_{x_0,2t})}{\text{cap}(B_{x_0,t}, B_{x_0,2t})} \frac{dt}{t},$$
$$osc(h, \Omega) = |\sup_\Omega h(x) - \inf_\Omega h(x)|,$$

and c depends only on the dimension of the space.

Lemma 4.6.6 *Suppose that Ω belongs to $\mathcal{O}_{c,r}(D)$. If $\theta \in H^1(\Omega) \cap C(\overline{\Omega})$, and if h is the harmonic function in Ω with $h - \theta \in H_0^1(\Omega)$, then*

$$\lim_{x \to x_0} h(x) = \theta(x_0)$$

for any $x_0 \in \partial\Omega$.

The main continuity result can be expressed as follows:

Theorem 4.6.7 *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{O}_{c,r}(D)$, which converges in the H^c topology to an open set Ω. Then Ω_n γ-converges to Ω.*

Proof Let us fix $f \equiv 1$; it will be sufficient to prove the continuity for a subsequence of $(\Omega_n)_{n\in\mathbb{N}}$. Since $f \equiv 1$, we shortly write u_Ω instead of $u_{\Omega,1}$. From Proposition 4.6.2 there exists a subsequence of $(\Omega_n)_{n\in\mathbb{N}}$, which we still denote by $(\Omega_n)_{n\in\mathbb{N}}$, such that $u_{\Omega_n} \rightharpoonup u$ weakly in $H_0^1(D)$, and u satisfies the equation $-\Delta u = 1$ on Ω. We prove that $u \in H_0^1(\Omega)$, which will imply that $u = u_\Omega$. For that it is sufficient to prove $u = 0$ q.e. on $D \setminus \Omega$ where u is a quasi-continuous representative.

From the Banach–Saks theorem there exists a sequence of averages:

$$\psi_n = \sum_{k=n}^{N_n} \alpha_k^n u_{\Omega_n}$$

with

$$0 \le \alpha_k^n \le 1\ , \quad \sum_{k=n}^{N_n} \alpha_k^n = 1$$

such that

$$\psi_n \xrightarrow{H_0^1(D)} u.$$

From the strong convergence of ψ_n to u in $H_0^1(D)$, we have that

$$\psi_n(x) \longrightarrow u(x) \quad \text{q.e.} \quad on \quad D$$

for a subsequence of (ψ_n) which we still denote by (ψ_n).

Let G_0 be the set of zero capacity on which $\psi_n(x)$ does not converge to $u(x)$. Let $x \in D \setminus (\Omega \cup G_0)$, and $\varepsilon > 0$ arbitrary. We prove that $|u(x)| < \varepsilon$. Indeed, we have

$$|u(x)| \le |u(x) - \psi_n(x)| + |\psi_n(x)|.$$

We consider $n > N_{\varepsilon,x}$ such that

$$|u(x) - \psi_n(x)| < \frac{\varepsilon}{2}.$$

Let us consider the solution u_B on a large ball B containing D. From the smoothness of B and since $f \equiv 1$, we have that u_B is continuous on $\overline{B}$. Subtracting the corresponding equations, we obtain

$$\Delta(u_B - u_{\Omega_n}) = 0 \ \text{ in } \Omega_n. \tag{4.37}$$

Consequently, $u_B - u_{\Omega_n}$ is harmonic in Ω_n and continuous on Ω_n. We use Lemma 4.6.6 in the following way: θ is the restriction to Ω_n of the function u_B and $h_n = u_{B|\Omega_n} - u_{\Omega_n}$. Now $h_n - \theta = -u_{\Omega_n}$ which belongs to $H_0^1(\Omega_n)$. From the continuity of u_B we obtain that the continuous extension of $u_{B}\big|_{\Omega_n} - u_{\Omega_n}$ is equal to u_B on the boundary of Ω_n, and so the extension of u_{Ω_n} to the boundary of Ω_n is zero in the classical sense. Using Lemma 4.6.5, one obtains (see [141]) that if h_n is Lipschitz on $\partial\Omega_n$ then it is Hölderian on all Ω_n. Since u_B is Lipschitz on B and is equal to h_n on $\partial\Omega_n$ we have

$$\forall x, y \in \partial\Omega_n \quad |h_n(x) - h_n(y)| = |u_B(x) - u_B(y)| \leq M|x - y|.$$

There exist two constants $\delta_1 > 0$ and an M_1 given by Lemma 4.6.5 which depend only on c, r, diam B and the dimension of the space N such that

$$|h_n(x) - h_n(y)| \leq M_1|x - y|^{\delta_1} \quad \forall x, y \in \Omega_n.$$

This inequality holds obviously in B (changing the constant M_1 if necessary), since h_n is equal to u_B outside Ω_n. Therefore for every $x, y \in B$ we have

$$|u_{\Omega_n}(x) - u_{\Omega_n}(y)| \leq |h_n(x) - h_n(y)| + |u_B(x) - u_B(y)|$$

$$\leq M_1|x - y|^{\delta_1} + M|x - y| \leq M_2|x - y|^{\delta_2}.$$

Let us choose $R > 0$, such that $M_2 R^{\delta_2} < \varepsilon/2$. From the H^c convergence of Ω_n to Ω there exists $n_R \in \mathbb{N}$, such that for every $n \geq n_R$ we have $(B \setminus \Omega_n) \cap B_{x,R} \neq \emptyset$. Let us take $x_n \in (B \setminus \Omega_n) \cap B_{x,R}$. We have

$$|u_{\Omega_n}(x)| = |u_{\Omega_n}(x) - u_{\Omega_n}(x_n)| \leq M_2|x - x_n|^{\delta_2} \leq M_2 R^{\delta_2} \leq \frac{\varepsilon}{2}$$

because $u_{\Omega_n}(x_n) = 0$. Hence

$$|\psi_n(x)| = \Big|\sum_{k=n}^{N_n} \alpha_k^n u_{\Omega_n}(x)\Big| \leq \sum_{k=n}^{N_n} \alpha_k^n \frac{\varepsilon}{2} = \frac{\varepsilon}{2}, \qquad \forall n > n_R.$$

Finally we obtain $|u(x)| \leq \varepsilon$. Since ε was taken arbitrarily we have $u(x) = 0$ q.e. on $B \setminus \Omega$, which implies that $u = u_\Omega$. The strong convergence of u_{Ω_n} to u_Ω is now immediate, from the convergence of the norms of u_{Ω_n} to the norm of u_Ω. ∎

4.7 Continuity under topological constraints: Šverák's result

Let us denote by

$$\mathcal{O}_l(D) = \{\Omega \subseteq D : \sharp\Omega^c \leq l\}$$

the family of open subsets of D whose complements have at most l connected components. By $\sharp$ we denote the number of connected components.

A consequence of Theorem 4.6.7 is the following result due to Šverák [187].

Theorem 4.7.1 *Let $N = 2$. If $\Omega_n \in \mathcal{O}_l(D)$ and $\Omega_n \xrightarrow{H^c} \Omega$, then Ω_n γ-converges to Ω.*

Proof Let us fix $f \equiv 1$. Since the solution of equation (4.34) is unique, it is sufficient to prove the continuity result for a subsequence.

There exists a subsequence of $(\Omega_n)_{n\in\mathbb{N}}$ still denoted by $(\Omega_n)_{n\in\mathbb{N}}$, such that $u_{\Omega_n} \rightharpoonup u$ weakly in $H_0^1(D)$, and by Proposition 4.6.2, u satisfies the equation on Ω. To obtain that $u \in H_0^1(\Omega)$ we prove that $u = 0$ q.e. on Ω^c.

In general, one cannot find $c, r > 0$ such that $(\Omega_n)_{n\in\mathbb{N}} \subseteq \mathcal{O}_{c,r}(D)$, therefore a direct application of Theorem 4.6.7 is not possible. Let

$$\overline{D} \setminus \Omega_n = K_1^n \cup \cdots \cup K_l^n$$

be the decomposition of $\overline{D} \setminus \Omega_n$ in l connected components, which are compact and disjoint, possibly empty. According to Proposition 4.6.1 there exists a subsequence $(K_1^{k_n^1})_n$ of $(K_1^n)_n$ such that

$$K_1^{k_n^1} \xrightarrow{H} K_1.$$

By the same argument we can extract a subsequence of (k_n^1) which we denote by (k_n^2) such that

$$K_2^{k_n^2} \xrightarrow{H} K_2.$$

Finally, continuing this procedure, we obtain a subsequence of $(\Omega_n)_{n\in\mathbb{N}}$ (still denoted using the same indices) such that

$$K_j^n \xrightarrow{H} K_j \quad \forall j = 1, ..., l.$$

Obviously $\Omega = D \setminus (K_1 \cup \cdots \cup K_l)$. Since the K_j^n are connected, we have that K_j is connected. There are now three possibilities. Either K_j is the empty set, or it is a point, or it contains at least two points; in the latter case any connected open set which contains K_j, also contains a continuous curve which links the two points. If $K_j = \emptyset$ we ignore K_j and $(K_j^n)_n$ which are also empty (for n large enough). If K_j is a point, then it has zero capacity, so K_j can also be ignored. In that case we define the new sets $\Omega_n^+ = D \setminus \cup_{i\neq j} K_i^n$ which satisfy $\Omega_n \subseteq \Omega_n^+$. We continue this procedure for all $j = 1, \ldots, l$ and obtain that

$$\Omega_n^+ \xrightarrow{H^c} \Omega^+.$$

Of course $u_{\Omega^+} = u_\Omega$ because the difference between Ω^+ and Ω has zero capacity (it consists only of a finite number of points).

Let us prove that there exist $c, r > 0$ such that for n large enough $(\Omega_n^+)_{n\in\mathbb{N}} \subseteq \mathcal{O}_{c,r}$. There exists $\delta > 0$ such that diam $(K_i) \geq \delta$ for all remaining indices $i \in \{1, \ldots, l\}$. For n large enough, we get that diam $(K_i^n) \geq \delta/2$.

Let us set $r = \delta/4$ and

$$c = \frac{\text{cap}([0,1] \times \{0\}, B_{0,2})}{\text{cap}(\overline{B}_{0,1}, B_{0,2})} > 0.$$

Then we show that for n large enough $\Omega_n^+ \in \mathcal{O}_{c,r}$. In order to prove that for every $x \in \partial\Omega_n^+$ and $t \in (0, r)$ we have

$$\frac{\text{cap}((\Omega_n^+)^c \cap B_{x,t}, B_{x,2t})}{\text{cap}(B_{x,t}, B_{x,2t})} \geq c,$$

we simply remark that for every $x \in K_i^n$ and $t \in (0, r)$,

$$\text{cap}(K_i^n \cap B_{x,t}, B_{x,2t}) \geq \text{cap}([x, y], B_{x,2t}).$$

Here y is a point belonging on $\partial B_{x,t}$. This inequality follows straightforwardly by a Steiner symmetrization type argument. For details, we refer to [52], [39] and Section 6.3.

Using Theorem 4.6.7 we get that

$$u_{\Omega_n^+} \longrightarrow u_{\Omega^+} \text{ strongly in } H_0^1(D).$$

From the maximum principle we have $u_{\Omega_n^+} \geq u_{\Omega_n} \geq 0$, hence $u_{\Omega^+} \geq u \geq 0$, hence $u \in H_0^1(\Omega)$. ■

4.8 Nonlinear operators: Necessary and sufficient conditions for the γ_p-convergence

In order to make the reader familiar with the nonlinear framework, in this section we discuss the necessary and sufficient conditions for the γ_p-convergence in terms of the convergence of the local capacities.

The γ_p-convergence is defined similarly to the case $p = 2$ (see Definition 4.3.6).

Definition 4.8.1 *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of open subsets of a bounded smooth design region D, and $1 < p < \infty$. We say that Ω_n γ_p-converges to Ω if for every $f \in W^{-1,q}(D)$ and $g \in W_0^{1,p}(D)$ the solutions $u_{\Omega_n,f,g}$ of the equations*

$$\begin{cases} -\Delta_p u_{\Omega_n,f,g} = f & \text{in } \Omega_n, \\ u_{\Omega_n,f,g} = g & \text{on } \partial\Omega_n, \end{cases} \tag{4.38}$$

extended by g on $D \setminus \Omega_n$, converge weakly in $W_0^{1,p}(D)$ to the function $u_{\Omega,f,g}$ of the same equation with Ω_n replaced by Ω.

We consider the sequence of the associated Sobolev spaces $(W_0^{1,p}(\Omega_n))_{n\in\mathbb{N}}$ as subspaces in $W_0^{1,p}(D)$, and study the weak upper and strong lower limits in the sense of Kuratowski (see Section 4.5 for the definitions of the Kuratowski limits) of this sequence in terms of the behavior of the local capacity of the complements intersected with open and closed balls. We refer the reader to [42] for further details.

Given an open set $\Omega \subseteq D$ we establish necessary and sufficient conditions for each of the inclusions

$$W_0^{1,p}(\Omega) \subseteq s - \liminf_{n\to\infty} W_0^{1,p}(\Omega_n) \tag{4.39}$$

and

$$w - \limsup_{n\to\infty} W_0^{1,p}(\Omega_n) \subseteq W_0^{1,p}(\Omega), \tag{4.40}$$

without imposing a priori any *geometric convergence* of Ω_n to Ω.

It is easy to see (see Proposition 4.5.3 for the linear case) that the γ_p-convergence of Ω_n to Ω is equivalent to the Mosco convergence of the Sobolev spaces $W_0^{1,p}(\Omega_n)$ to $W_0^{1,p}(\Omega)$, i.e., to both relations (4.39) and (4.40).

We begin by two technical results.

Lemma 4.8.2 *Let $A_n, A \subseteq B_{0,1}$ be open sets and $0 < r_1 < r_2 < 1$. If*

$$W_0^{1,p}(A) \subseteq s - \liminf_{n\to\infty} W_0^{1,p}(A_n), \tag{4.41}$$

then

$$W_0^{1,p}(A \cup (B_{0,1} \setminus \overline{B}_{0,r_2})) \subseteq s - \liminf_{n\to\infty} W_0^{1,p}(A_n \cup (B_{0,1} \setminus \overline{B}_{0,r_1})). \tag{4.42}$$

If

$$w - \limsup_{n\to\infty} W_0^{1,p}(A_n) \subseteq W_0^{1,p}(A), \tag{4.43}$$

then

$$w - \limsup_{n\to\infty} W_0^{1,p}(A_n \cup (B_{0,1} \setminus \overline{B}_{0,r_2})) \subseteq W_0^{1,p}(A \cup (B_{0,1} \setminus \overline{B}_{0,r_1})). \tag{4.44}$$

Proof Let $u, v \in C^\infty(B_{0,1})$, $u, v \geq 0$, $u + v = 1$ be a partition of unity of $B_{0,1}$ such that $u = 1$ on B_{0,r_1} and $v = 1$ on $B_{0,1} \setminus B_{0,r_2}$.

Assume (4.41) and let $\varphi \in W_0^{1,p}(A \cup (B_{0,1} \setminus \overline{B}_{0,r_2}))$, $\varphi \geq 0$. We have $\varphi = u\varphi + v\varphi$ and $u\varphi \in W_0^{1,p}(A \cap (B_{0,1} \setminus \overline{B}_{0,r_2}))$. From (4.41), there exists $\varphi_n \in W_0^{1,p}(A_n)$ such that $\varphi_n \to u\varphi$ strongly in $W_0^{1,p}(B_{0,1})$. Then $\varphi_n^+ \wedge u\varphi \in W_0^{1,p}(A_n \cup (B_{0,1} \setminus \overline{B}_{0,r_1}))$ and $\varphi_n^+ \wedge u\varphi$ converges strongly to $u\varphi$. On the other hand, $v\varphi \in W_0^{1,p}(B_{0,1} \setminus \overline{B}_{0,r_1})$ hence $v\varphi + \varphi_n^+ \wedge u\varphi \in W_0^{1,p}(A_n \cup (B_{0,1} \setminus \overline{B}_{0,r_1}))$ and converges strongly to φ.

Assume now (4.43). Consider $\varphi_n \in W_0^{1,p}(A_{k_n} \cup (B_{0,1} \setminus \overline{B}_{0,r_2}))$ such that $\varphi_n \rightharpoonup \varphi$ weakly in $W_0^{1,p}(B_{0,1})$. Then $\varphi_n u \in W_0^{1,p}(A_{k_n})$, and from (4.43) any weak limit point belongs to $W_0^{1,p}(A)$. On the other hand, $\varphi_n v \in W_0^{1,p}(B_{0,1} \setminus \overline{B}_{0,r_1})$ hence any weak limit point belongs to $W_0^{1,p}(B_{0,1} \setminus \overline{B}_{0,r_1})$. ■

Lemma 4.8.3 *Under the hypotheses of Lemma 4.8.2, assume that (4.41) holds. Then*

$$\text{(4.45)} \qquad \mathrm{cap}_p(A^c \cap \overline{B}_{0,r_2}, B_{0,1}) \geq \limsup_{n\to\infty} \mathrm{cap}_p(A_n^c \cap \overline{B}_{0,r_1}, B_{0,1}).$$

If (4.43) holds, then

$$\text{(4.46)} \qquad \mathrm{cap}_p(A^c \cap \overline{B}_{0,r_1}, B_{0,1}) \leq \liminf_{n\to\infty} \mathrm{cap}_p(A_n^c \cap \overline{B}_{0,r_2}, B_{0,1}).$$

Proof Assume that (4.41) holds. Let $\varphi \in W_0^{1,p}(B_{0,1})$ be the function realizing the capacity $\mathrm{cap}_p(A^c \cap \overline{B}_{0,r_2}, B_{0,1})$. Using Lemma 4.8.2, we apply (4.41) to the function $(1-\varphi)u$ (u being chosen as in the proof of Lemma 4.8.2) and get a sequence $u_n \in W_0^{1,p}(A_n \cup (B_{0,1} \setminus \overline{B}_{0,r_1}))$ which strongly converges to $(1-\varphi)u$. We consider the functions $1-(u_n+(1-\varphi)v)$ which strongly converge to φ and are test functions for the capacities $\mathrm{cap}_p(A_n^c \cap \overline{B}_{0,r_1}, B_{0,1})$. Then (4.45) follows.

Assume that (4.43) holds, take the sequence of functions $\varphi_n \in W_0^{1,p}(B_{0,1})$ realizing the capacities $\mathrm{cap}_p(A_{n_k}^c \cap \overline{B}_{0,r_2}, B_{0,1})$ and assume that φ_n weakly converges to φ. We apply (4.44) to the functions $u(1-\varphi_n)$ and get that $u(1-\varphi) \in W_0^{1,p}(A \cup (B_{0,1} \setminus \overline{B}_{0,r_1})))$. On the other hand $v(1-\varphi) \in W_0^{1,p}(B_{0,1} \setminus \overline{B}_{0,r_1})$ hence $\varphi = 1-(u(1-\varphi)+v(1-\varphi))$ is a test function for the capacity $\mathrm{cap}_p(A^c \cap \overline{B}_{0,r_1}, B_{0,1})$ and (4.46) follows. ■

Study of the strong lower limit. We prove the following

Theorem 4.8.4 *Let* $(\Omega_n)_{n\in\mathbb{N}}, \Omega$ *be open subsets of* D. *Then*

$$W_0^{1,p}(\Omega) \subseteq s - \liminf_{n\to\infty} W_0^{1,p}(\Omega_n)$$

if and only if for every $x \in \mathbb{R}^N$ *and* $\delta > 0$,

$$\text{(4.47)} \qquad \mathrm{cap}_p(\Omega^c \cap \overline{B}_{x,\delta}, B_{x,2\delta}) \geq \limsup_{n\to\infty} \mathrm{cap}_p(\Omega_n^c \cap \overline{B}_{x,\delta}, B_{x,2\delta}).$$

Proof **(⇒)** Assume

$$W_0^{1,p}(\Omega) \subseteq s - \liminf_{n\to\infty} W_0^{1,p}(\Omega_n).$$

Then we also have

$$W_0^{1,p}(\Omega \cap B_{x,2\delta}) \subseteq s - \liminf_{n\to\infty} W_0^{1,p}(\Omega_n \cap B_{x,2\delta}).$$

Indeed, consider $\varphi \in W_0^{1,p}(\Omega \cap B_{x,2\delta})$, $\varphi \geq 0$. If $\varphi_n \in W_0^{1,p}(\Omega_n)$ converges strongly to φ, then $\varphi \wedge \varphi_n^+ \in W_0^{1,p}(\Omega_n \cap B_{x,2\delta})$ converges also strongly to φ.

We apply Lemmas 4.8.2 and 4.8.3 and get for $\varepsilon > 0$,

$$\mathrm{cap}_p(\Omega^c \cap \overline{B}_{x,\delta+\varepsilon}, B_{x,2\delta}) \geq \limsup_{n\to\infty} \mathrm{cap}_p(\Omega_n^c \cap \overline{B}_{x,\delta}, B_{x,2\delta}).$$

Making $\varepsilon \to 0$, relation (4.47) follows.

($\Leftarrow$) Let us consider $u \in C_c^\infty(\Omega)$, supp $u = K \subset\subset \Omega$ and $\varepsilon = d(K, \partial\Omega)$. There exists a finite family of k balls centered at points of K and of radius less than ε such that $K \subseteq \cup_{r=1}^k B_{x_r,\varepsilon/2}$. We also have

$$\mathrm{cap}_p(\Omega^c \cap \overline{B}_{x_r,\varepsilon/2}, B_{x_r,\varepsilon}) = 0$$

and thus we get from (4.47)

$$\lim_{n\to\infty} \mathrm{cap}_p(\Omega_n^c \cap \overline{B}_{x_r,\varepsilon/2}, B_{x_r,\varepsilon}) = 0.$$

Hence the capacity of $\Omega_n^c \cap \overline{B}_{x_r,\varepsilon/2}$ vanishes as $n \to \infty$ and we consider for $r = 1, ..., k$ smooth functions $\psi_n^r \in \mathcal{D}(B_{x,\varepsilon})$ equal to 1 on $\Omega_n^c \cap \overline{B}_{x_r,\varepsilon/2}$ and which approximate respectively the capacity of this set, namely

$$\|\psi_n^r\|_{W_0^{1,p}(D)} \leq \mathrm{cap}_p(\Omega_n^c \cap \overline{B}_{x_r,\varepsilon/2}, B_{x_r,\varepsilon}) + \frac{1}{n}.$$

Moreover the functions ψ_n^r can be chosen such that $0 \leq \psi_n^r \leq 1$. Therefore, since k is fixed, the sequence of functions defined by

$$u_n = u \prod_{r=1}^k (1 - \psi_n^r)$$

has the property

$$u_n \in W_0^{1,p}(\Omega_n) \text{ and } u_n \overset{W_0^{1,p}(D)}{\longrightarrow} u,$$

so (4.39) is satisfied.

The passage from $\mathcal{D}(\Omega)$ to $W_0^{1,p}(\Omega)$ is made by using the density $\overline{\mathcal{D}(\Omega)} = W_0^{1,p}(\Omega)$ and a standard diagonal procedure for extracting a convergent sequence. ∎

Remark 4.8.5 If one replaces in the previous theorem the sequence $(\Omega_n)_{n\in\mathbb{N}}$ of open sets by p-quasi-open sets, the only technical point which does not work as above deals with the necessity. The construction of the approximating sequence fails because the dense family in $W_0^{1,p}(\Omega)$, which replaces $\mathcal{D}(\Omega)$ for a p-quasi-open set Ω, does not have the same properties. In fact, even for a p-quasi-open set Ω, there exists a dense family of functions having compact support in Ω, but their supports cannot be covered by balls which do not intersect the complement of Ω (see [145]).

Study of the weak upper limit. We prove the following result.

Theorem 4.8.6 *Let $(\Omega_n)_{n\in\mathbb{N}}$, Ω be open subsets of D. Then*

$$w-\limsup_{n\to\infty} W_0^{1,p}(\Omega_n) \subseteq W_0^{1,p}(\Omega)$$

if and only if for every $x \in \mathbb{R}^N$ and every $\delta > 0$

$$\operatorname{cap}_p(\Omega^c \cap B_{x,\delta}, B_{x,2\delta}) \le \liminf_{n\to\infty} \operatorname{cap}_p(\Omega_n^c \cap B_{x,\delta}, B_{x,2\delta}). \tag{4.48}$$

Proof $(\Rightarrow)$ Assume

$$w-\limsup_{n\to\infty} W_0^{1,p}(\Omega_n) \subseteq W_0^{1,p}(\Omega).$$

Then, it is immediate to observe (using Hedberg's result [137]) that

$$w-\limsup_{n\to\infty} W_0^{1,p}(\Omega_n \cap B_{x,2\delta}) \subseteq W_0^{1,p}(\Omega \cap B_{x,2\delta}).$$

We apply Lemmas 4.8.2 and 4.8.3 and get for $\varepsilon > 0$,

$$\operatorname{cap}_p(\Omega^c \cap \overline{B}_{x,\delta-\varepsilon}, B_{x,2\delta}) \le \liminf_{n\to\infty} \operatorname{cap}_p(\Omega_n^c \cap \overline{B}_{x,\delta-\varepsilon/2}, B_{x,2\delta}).$$

Since

$$\operatorname{cap}_p(\Omega_n^c \cap \overline{B}_{x,\delta-\varepsilon/2}, B_{x,2\delta}) \le \operatorname{cap}_p(\Omega_n^c \cap B_{x,\delta}, B_{x,2\delta})$$

we have

$$\operatorname{cap}_p(\Omega^c \cap \overline{B}_{x,\delta-\varepsilon}, B_{x,2\delta}) \le \liminf_{n\to\infty} \operatorname{cap}_p(\Omega_n^c \cap B_{x,\delta}, B_{x,2\delta}).$$

Making $\varepsilon \to 0$ we get (4.48).

$(\Leftarrow)$ In order to reduce the study of arbitrary weak convergent sequences to particular sequences of solutions of equation (4.38) for $f \equiv 0$, we give the following lemma. We denote in the sequel $v_{\Omega_n,g} = u_{\Omega_n,0,g}$, and when no ambiguity occurs, $v_n = v_{\Omega_n,g}$.

Lemma 4.8.7 *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of open subsets of B, and Ω an open set such that for every $g \in \mathcal{D}(D)$ and for every $W_0^{1,p}(D)$-weak limit v of a sequence $(v_{\Omega_{n_k},g})$ we have $v - g \in W_0^{1,p}(\Omega)$. Then relation (4.40) holds.*

Proof Remark first that if the conclusion holds for any $g \in \mathcal{D}(D)$, then it holds for any $g \in W_0^{1,p}(D)$. Indeed, consider some $g \in W_0^{1,p}(D)$ and (with a re-notation of the indices) suppose that $v_{\Omega_n,g} \rightharpoonup v_g$ weakly in $W_0^{1,p}(D)$. There exists a sequence $g_k \in \mathcal{D}(D)$ such that $g_k \to g$ strongly in $W_0^{1,p}(D)$. Following [102], one can find a uniform bound for v_{Ω_n,g_k}, hence there exists a constant β such that

$$\|v_{\Omega_n,g_k} - v_{\Omega_n,g}\|_{W^{1,p}(D)} \le \beta\|g - g_k\|_{W_0^{1,p}(D)}$$

for all $k \in \mathbb{N}$. Then $\|v - v_k\|_{W^{1,p}(D)} \le \beta\|g - g_k\|_{W_0^{1,p}(D)}$ and if $v_k - g \in W_0^{1,p}(\Omega)$ we conclude that $v - g \in W_0^{1,p}(\Omega)$.

For proving (4.40) we consider a sequence $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(D)$, with $u_n \in W_0^{1,p}(\Omega_n)$. Following [102], for a subsequence still denoted using the same indices, Ω_n γ_p-converges to $(\mathbb{R}^N, \mu)$, μ being the measure defined by (4.18). Then we have $u \in W_0^{1,p}(D) \cap L^p(D, \mu)$. To prove that $u \in W_0^{1,p}(\Omega)$ it suffices to verify that

$$W_0^{1,p}(D) \cap L^p(D, \mu) \subseteq W_0^{1,p}(\Omega). \tag{4.49}$$

Consider some $g^* \in W_0^{1,p}(D)$. For a subsequence still denoted using the same indices, we have $v_{\Omega_n,g^*} \rightharpoonup v$ weakly in $W_0^{1,p}(D)$ and v satisfies the equation

$$\begin{cases} -\Delta_p v + \mu|v - g^*|^{p-2}(v - g^*) = 0 \ \text{ in } W_0^{1,p}(D) \cap L^p(D, \mu), \\ v - g^* \in W_0^{1,p}(D) \cap L^p(D, \mu). \end{cases} \tag{4.50}$$

Hence, on one side we obtain that $v - g^* \in W_0^{1,p}(D) \cap L^p(D, \mu)$, and on the other side the hypothesis we assumed gives $v - g^* \in W_0^{1,p}(\Omega)$. To obtain the conclusion it is sufficient to prove that the family of functions written in the form $v - g^*$ with the properties above is dense in $W_0^{1,p}(D) \cap L^p(D, \mu)$. This will provide inclusion (4.49). Set $v - g^* = z$. We have

$$\begin{cases} -\Delta_p(z + g^*) + \mu|z|^{p-2}z = 0 \ \text{ in } W_0^{1,p}(D) \cap L^p(D, \mu), \\ z \in W_0^{1,p}(D) \cap L^p(D, \mu). \end{cases} \tag{4.51}$$

The family $\{\varphi w\}_{\varphi \in \mathcal{D}(D)}$, where $\varphi \in C_c^\infty(D)$, and $w = u_{A_\mu,1,0}$, with A_μ being the regular set of the measure μ (i.e., the union of all finely open sets of finite μ-measure), is dense in $W_0^{1,p}(D) \cap L^p(D, \mu)$. Fix now some $z = \varphi w$. For this z we can prove the existence of some $g \in W_0^{1,p}(D)$ such that

$$-\Delta_p(z + g) + \mu|z|^{p-2}z = 0.$$

Indeed, the existence of such a g is trivial if $\mu|z|^{p-2}z \in W^{-1,q}(D)$. This follows immediately from the particular structure of z. Consider $\theta \in \mathcal{D}(D)$. Then

$$\begin{aligned} \langle \mu|z|^{p-2}z, \theta\rangle_{W^{-1,q}(D)\times W_0^{1,p}(D)} &= \int_{\{w>0\}} \theta|z|^{p-2}z\,d\mu \\ &= \int_{\{w>0\}} \theta|z|^{p-2}z\frac{d\nu}{w^{p-1}} \\ &= \langle \theta|\varphi|^{p-2}\varphi, 1 + \Delta_p w\rangle_{W^{-1,q}(D)\times W_0^{1,p}(D)}. \end{aligned}$$

Since $\Delta_p w \in W^{-1,q}(D)$ we get $\mu|z|^{p-2}z \in W^{-1,q}(D)$ and this concludes the proof. ∎

Proof of Theorem 4.8.6, continuation. According to Lemma 4.8.7, it suffices to study the behavior of $v_{\Omega_n,g}$. Therefore, let us consider some $g \in \mathcal{D}(D)$ and $v_{\Omega_n,g} \rightharpoonup v$ weakly in $W_0^{1,p}(D)$.

It is sufficient to prove $v = g$ p-q.e. on Ω^c. In fact we use an estimation of the oscillation of $v_{\Omega_n,g}$ near the boundary. The lower semicontinuity of the capacity will provide a uniform behavior. There exist convex combinations

$$\phi_n = \sum_{k=n}^{N_n} \alpha_k^n v_k \overset{W_0^{1,p}(D)}{\rightarrow} v$$

and for a subsequence (still denoted using the same indices) it converges p-q.e. (for p-quasi-continuous representatives). Let us denote by $D \setminus E$ the set of points where the convergence is pointwise, with $\operatorname{cap}_p(E) = 0$. So, for $x \in D \setminus E$, for every $\varepsilon > 0$ we have

$$|v(x) - g(x)| \le |v(x) - \phi_n(x)| + |\phi_n(x) - g(x)|$$

and for n large enough we also have $|v(x) - \phi_n(x)| < \varepsilon/2$. We must prove that we have $|\phi_n(x)| < \varepsilon/2$ or, even better, that $|v_n(x) - g(x)| < \varepsilon/2$ for n large enough.

One can apply directly Lemma 4.6.5 for $h = v_{\Omega,g}$ and $\theta = g$ and get for a point $x_0 \in \partial\Omega$, for all $x, y \in \Omega \cap B_{x_0,r}$,

$$|v_{\Omega,g}(x) - v_{\Omega,g}(y)| \le c_g M(4R)^\alpha + 2M \exp(-cw(\Omega, x_0, r, R)), \tag{4.52}$$

where α is the Hölder exponent of g. If $y \in \partial\Omega \cap B_{x_0,r}$ is a regular point in the sense of Wiener, i.e.,

$$\int_0^1 \frac{\operatorname{cap}(\Omega^c \cap B_{y,t}, B_{y,2t})}{\operatorname{cap}(B_{y,t}, B_{y,2t})} \frac{dt}{t} = +\infty, \tag{4.53}$$

then $v_{\Omega,g}(y) = g(y)$ (see also [2], [3]) and one can derive the inequality

$$|v_{\Omega,g}(x) - g(x)| \le 2c_g M(4R)^\alpha + 2M \exp(-cw(\Omega, x_0, r, R)) \tag{4.54}$$

for all $x \in \Omega \cap B_{x_0,r}$.

This last inequality proves that if one can handle the behavior of the sequence $\big(w(\Omega_n, x_0, r, R)\big)$, then the oscillations of $v_{\Omega_n,g}$ relatively to g are *uniform* in some neighborhoods of x_0.

We shall use the nonlinear version of Lemma 4.6.5 (see [141]).

Lemma 4.8.8 *Suppose that Ω is bounded. Let $\theta \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ and let h be the unique p-harmonic function in Ω with $\theta - h \in W_0^{1,p}(\Omega)$. If $x_0 \in \partial\Omega$, then for every $0 < r \le R$ we have*

$$osc(h, \Omega \cap B_{x_0,r}) \le osc(\theta, \partial\Omega \cap \overline{B}_{x_0,2R}) + osc(\theta, \partial\Omega) exp(-cw(\Omega, x_0, r, R)),$$

where

$$w(\Omega, x_0, r, R) = \int_r^R \Big(\frac{\mathrm{cap}_p(\Omega^c \cap B_{x_0,t}, B_{x_0,2t})}{\mathrm{cap}_p(B_{x_0,t}, B_{x_0,2t})}\Big)^{q-1} \frac{dt}{t},$$
$$osc(h, \Omega) = |\sup_\Omega h(x) - \inf_\Omega h(x)|,$$

and c depends only on the dimension of the space.

Proof of Theorem 4.8.6, continuation. We distinguish between two assertions: $x \in \partial\Omega$ and $x \in \mathrm{ext}\,(\Omega)$. Let us first consider the case $x \in \partial\Omega$. Since the set of non-regular points in the sense of Wiener (see (4.53) belonging to the boundary of Ω has zero capacity, without loss of generality one can suppose that x is regular. Let us give two lemmas.

Lemma 4.8.9 *For all $x \in \mathbb{R}^N$, and for all $0 < r < R$ we have*

$$\liminf_{n\to\infty} w(\Omega_n, x, r, R) \geq w(\Omega, x, r, R).$$

Proof The proof of this lemma is immediate from the lower semicontinuity of the local capacity, and the properties of the Lebesgue integral (Fatou's lemma and (4.48)).

■

Lemma 4.8.10 *There exists a positive constant $\overline{c}$ depending only on the dimension of the space and p, such that for all $R > r > 0$ and for all $x_1, x_2 \in \mathbb{R}^N$ with $|x_1 - x_2| = \delta \leq r/2$ we have*

$$w(\Omega, x_1, r, R) \geq \overline{c} w(\Omega, x_2, \frac{r}{2}, \frac{R}{2}).$$

Proof We have $\Omega^c \cap B_{x_2,\sigma} \subseteq \Omega^c \cap B_{x_1,\varepsilon}$ if $\sigma + \delta \leq \varepsilon$. Hence for any $R \geq t \geq r \geq 2\delta$ we have

$$\Omega^c \cap B_{x_2,t/2} \subseteq \Omega^c \cap B_{x_1,t}$$

since $t/2 + \delta \leq t$. So we get the inclusion $B_{x_1,2t} \subseteq B_{x_2,4t}$ and then we can write

$$\mathrm{cap}_p(\Omega^c \cap B_{x_2,t/2}, B_{x_2,4t}) \leq \mathrm{cap}_p(\Omega^c \cap B_{x_1,t}, B_{x_1,2t}).$$

According to [141] there exists a constant ξ depending only on the dimension N, such that

$$\begin{aligned}\xi\, \mathrm{cap}_p(\Omega^c \cap B_{x_2,t/2}, B_{x_2,t}) &\leq \mathrm{cap}_p(\Omega^c \cap B_{x_2,t/2}, B_{x_2,4t})\\ &\leq \mathrm{cap}_p(\Omega^c \cap B_{x_1,t}, B_{x_1,2t}).\end{aligned}$$

Hence

$$\begin{aligned}&\xi^{q-1} \int_r^R \Big(\frac{\mathrm{cap}_p(\Omega^c \cap B_{x_2,t/2}, B_{x_2,t})}{\mathrm{cap}_p(B_{x_2,t}, B_{x_2,2t})}\Big)^{q-1} \frac{dt}{t}\\ &\qquad\leq \int_r^R \Big(\frac{\mathrm{cap}_p(\Omega^c \cap B_{x_1,t}, B_{x_1,2t})}{\mathrm{cap}_p(B_{x_1,t}, B_{x_1,2t})}\Big)^{q-1} \frac{dt}{t}\end{aligned}$$

or, making a change of variables in the first integral, and using the behavior of the capacity on homothetic sets we get

$$\frac{\xi^{q-1}}{2^{(N-2)(q-1)}} \int_{r/2}^{R/2} \Big(\frac{\operatorname{cap}_p(\Omega^c \cap B_{x_2,t}, B_{x_2,2t})}{\operatorname{cap}_p(B_{x_2,t}, B_{x_2,2t})}\Big)^{q-1} \frac{dt}{t}$$

$$\leq \int_r^R \Big(\frac{\operatorname{cap}_p(\Omega^c \cap B_{x_1,t}, B_{x_1,2t})}{\operatorname{cap}_p(B_{x_1,t}, B_{x_1,2t})}\Big)^{q-1} \frac{dt}{t}.$$

Setting $(\xi 2^{2-N})^{q-1} = \overline{c}$ we get

$$w(\Omega, x_1, r, R) \geq \overline{c} w(\Omega, x_2, \frac{r}{2}, \frac{R}{2})$$

as soon as $|x_1 - x_2| \leq r/2 < R/2$. ■

Proof of Theorem 4.8.6, conclusion. Let us consider $x \in \partial\Omega$ a regular point in the sense of (4.53). We shall fix later $r, R > 0$ such that $w(\Omega, x, r/2, R/2) > \overline{M}$. The value of $\overline{M}$ will also be made precise. If $|x_n - x| \leq r/2$ we have from Lemma 4.8.10,

$$w(\Omega_n, x_n, r, R) \geq \overline{c} w(\Omega_n, x, \frac{r}{2}, \frac{R}{2}).$$

From Lemma 4.8.9, for n large enough one can write

$$w(\Omega_n, x, \frac{r}{2}, \frac{R}{2}) \geq \frac{1}{2} w(\Omega, x, \frac{r}{2}, \frac{R}{2})$$

which implies

$$w(\Omega_n, x_n, r, R) \geq \frac{\overline{c}\overline{M}}{2}$$

independently of the choice of x_n with $|x_n - x| \leq r/2$.

If $x \in \Omega_n^c$, then p-quasi-everywhere we have $v_n(x) = g(x)$. Let us suppose that $x \in \Omega_n$. Since $\operatorname{cap}_p(\Omega^c \cap B_{x,\delta}, B_{x,2\delta}) > 0$ for any $\delta > 0$ (the point x being regular) we have $\operatorname{cap}_p(\Omega_n^c \cap B_{x,\delta}, B_{x,2\delta}) > 0$ for n large enough. We fix $\delta = r/2$ and consider $x \in \Omega_n$, $x_n \in B_{x,\delta} \cap \partial\Omega_n$ and x_n regular. The existence of such a point follows from the fact that $\operatorname{cap}_p(\partial\Omega_n \cap B_{x,\delta}, B_{x,2\delta}) > 0$ (see [42]).

One can then write $v_n(x_n) = g(x_n)$, and using relation (4.52) we get

$$|v_n(x) - g(x)| \leq c_g M (4R)^\alpha + 2M \exp(-cw(\Omega_n, x_n, r, R)).$$

Now we fix $r, R, \overline{M}$ such that

$$R = \frac{1}{4}\Big(\frac{\varepsilon}{4c_g M}\Big)^{1/\alpha}, \qquad \overline{M} = -\frac{2}{c\overline{c}} \ln \frac{\varepsilon}{8M}$$

and $r < R/2$ such that $g(\Omega, x, r/2, R/2) > \overline{M}$. We have

$$c_g M(4R)^\alpha + 2M\exp\left(-c\overline{c}\frac{\overline{M}}{2}\right) \le \frac{\varepsilon}{2}$$

and for n large enough, such that $g(\Omega_n, x_n, r, R) \ge \overline{c}\overline{M}/2$ for $|x_n - x| \le r/2$, also $\mathrm{cap}_p(\Omega_n^c \cap B_{x,r/2}, B_{x,r}) > 0$. Hence $|v_n(x) - g(x)| < \varepsilon/2$ for n large enough, which finally implies $|v(x) - g(x)| \le \varepsilon$. Since ε was arbitrarily chosen we get $v(x) = g(x)$ for q.e. $x \in \partial\Omega$.

For the case $x \in \mathrm{ext}(\Omega)$, the same proof works with $w(\Omega, x, r, R) = \int_r^R 1/t dt = \ln(R/r)$. ∎

In Theorem 4.8.6 one can replace open sets by p-quasi-open sets. For the necessity the same proof follows. For the sufficiency, if $(A_n)_{n\in\mathbb{N}}$ and A are p-quasi-open subsets of D satisfying (4.48), we construct open sets Ω_n such that $A_n \subseteq \Omega_n$ and $\mathrm{cap}_p(\Omega_n \setminus A_n) \le 1/n$, and open sets Ω_ε such that $A \subseteq \Omega_\varepsilon$ and $\mathrm{cap}_p(\Omega_\varepsilon \setminus A) \le \varepsilon$. One sees that relation (4.48) still holds for Ω_n and for Ω_ε. Hence we apply Theorem 4.8.6 and get

$$w - \limsup_{n\to\infty} W_0^{1,p}(\Omega_n) \subseteq W_0^{1,p}(\Omega_\varepsilon).$$

Since $W_0^{1,p}(A_n)$ has the same Kuratowski limits as $W_0^{1,p}(\Omega_n)$ we get

$$w - \limsup_{n\to\infty} W_0^{1,p}(A_n) \subseteq W_0^{1,p}(\Omega_\varepsilon).$$

This inclusion holds for any $\varepsilon > 0$, therefore we can replace Ω_ε by A.

Remark 4.8.11 Note that for some fixed $x \in \mathbb{R}^N$, the family of $t \in \mathbb{R}_+$ such that the strict inequality

$$\mathrm{cap}_p(\Omega^c \cap \overline{B}_{x,t}, B_{x,2t}) > \mathrm{cap}_p(\Omega^c \cap B_{x,t}, B_{x,2t})$$

holds, is at most countable (see [42]). Hence, we can state that Ω_n *γ_p-converges to Ω if and only if for every $x \in \mathbb{R}^N$ there exists an at most countable family $T_x \subseteq \mathbb{R}_+$ such that for all $t \in \mathbb{R}_+ \setminus T_x$ we have*

$$\lim_{n\to\infty} \mathrm{cap}_p(\Omega_n^c \cap B_{x,t}, B_{x,2t}) = \mathrm{cap}_p(\Omega^c \cap B_{x,t}, B_{x,2t}).$$

Remark 4.8.12 The Mosco convergence of $W_0^{1,p}$-spaces was studied by Dal Maso [93], [94] and by Dal Maso-Defranceschi [95]. Using the frame of the relaxation theory, it was proved that Ω_n γ_p-converges to Ω if and only if there exists a family of sets $\mathcal{A} \subseteq \mathcal{P}(D)$ which is rich or dense (see the exact definitions in [94]) in $\mathcal{P}(D)$ such that

$$\mathrm{cap}_p(\Omega^c \cap A, D) = \lim_{n\to\infty} \mathrm{cap}_p(\Omega_n^c \cap A, D) \qquad \forall A \in \mathcal{A}. \tag{4.55}$$

Here $\mathcal{P}(D)$ denotes the family of all subsets of D.

By considering for example monotone sequences of the family $\mathcal{P}(D)$, in [94] it is shown that the convergence in the sense of Mosco is still equivalent to the following two relations which have to be satisfied for all p-quasi-open sets $A \subseteq D$ and p-quasi-compacts sets $F \subseteq A \subseteq D$:

$$\operatorname{cap}_p(\Omega^c \cap A, D) \geq \limsup_{n\to\infty} \operatorname{cap}_p(\Omega_n^c \cap F, D), \tag{4.56}$$

and

$$\operatorname{cap}_p(\Omega^c \cap A, D) \leq \liminf_{n\to\infty} \operatorname{cap}_p(\Omega_n^c \cap A, D). \tag{4.57}$$

In Theorem 4.8.4 and 4.8.6, we respectively proved that (4.39) is equivalent to a simpler version of (4.56) (where capacity is calculated by intersection with closed balls) and (4.40) is equivalent to a simpler version of (4.57) (where capacity is calculated on open balls).

4.9 Stability in the sense of Keldysh

In a large class of problems, the shape stability question for the solution of the elliptic equation

$$\begin{cases} -\Delta_p u_\Omega = f \ \text{ in } \Omega, \\ \quad u_\Omega \in W_0^{1,p}(\Omega) \end{cases} \tag{4.58}$$

has the following formulation: *let $(\Omega_n)_{n\in\mathbb{N}}$ be a perturbation of an open set Ω; the question is whether the solution u_{Ω_n} of equation (4.58) on Ω_n converges in $W_0^{1,p}(D)$ to u_Ω.*

Keldysh studied in [146] the so-called *compact convergence* (see also [139], [140], [180]).

Definition 4.9.1 *It is said that Ω_n compactly converges to Ω if for every compact $K \subseteq \Omega \cup \overline{\Omega}^c$ there exists $n_K \in \mathbb{N}$ such that for all $n \geq n_K$, $K \subseteq \Omega_n \cup \overline{\Omega}_n^c$.*

The compact convergence implicitly contains condition (4.39). Keldysh proved (in the linear setting) that the shape stability holds for this kind of perturbations provided that the limit set Ω is *stable*. By definition, Ω is p-stable, if every function $u \in W^{1,p}(\mathbb{R}^N)$ vanishing a.e on $\overline{\Omega}^c$ belongs to $W_0^{1,p}(\Omega)$. Using the result of Hedberg (see [137]), this is equivalent to

$$\forall u \in W^{1,p}(\mathbb{R}^N),\ u = 0 \text{ a.e. on } \overline{\Omega}^c \ \Rightarrow\ u = 0 \ p\text{-q.e. on } \Omega^c.$$

Roughly speaking, open sets with cracks are not stable. Notice that the stability of the solution depends only on Ω. No regularity assumption is made on the converging sequence (Ω_n). This is the main reason for which it is of interest to characterize the p-stable domains.

Using the results we obtained in Section 4.8, we are in a position to give a simple proof of the characterization of the p-stable domains.

Proposition 4.9.2 *A bounded open set Ω is p-stable if and only if for every $x \in \mathbb{R}^N$, $r > 0$ we have*

$$\operatorname{cap}_p(B_{x,r} \setminus \Omega, B_{x,2r}) = \operatorname{cap}_p(B_{x,r} \setminus \overline{\Omega}, B_{x,2r}). \tag{4.59}$$

Proof The proof is an immediate consequence of the fact that $\Omega_n = \cup_{x\in\Omega} B_{x,1/n}$ compactly converges to Ω. If Ω is p-stable, then Ω_n γ_p-converges to Ω. By Theorem 4.8.6 inequality (4.48) holds and we get

$$\operatorname{cap}_p(B_{x,r} \setminus \Omega, B_{x,2r}) \le \liminf_{n\to\infty} \operatorname{cap}_p(B_{x,r} \setminus \Omega_n, B_{x,2r}).$$

The behavior of the capacity on increasing sequences gives

$$\lim_{n\to\infty} \operatorname{cap}_p(B_{x,r} \setminus \Omega_n, B_{x,2r}) = \operatorname{cap}_p(B_{x,r} \setminus \overline{\Omega}, B_{x,2r}),$$

hence $\operatorname{cap}_p(B_{x,r} \setminus \Omega, B_{x,2r}) \le \operatorname{cap}_p(B_{x,r} \setminus \overline{\Omega}, B_{x,2r})$. The equality follows from the monotonicity of the capacity in the first argument.

Conversely, relation (4.59) yields that Ω_n γ_p-converges to Ω. Then any function $u \in W^{1,p}(\mathbb{R}^N)$ with $u = 0$ a.e. on $(\overline{\Omega})^c$ has the property that $u \in W_0^{1,p}(\Omega_n)$, since $u = 0$ p-q.e. on Ω_n^c. The γ_p-convergence of Ω_n to Ω gives $u \in W_0^{1,p}(\Omega)$, hence Ω is p-stable. ■

4.10 Further remarks and generalizations

Remark 4.10.1 Generalization of Šverák's result. In three or more dimensions a curve has zero capacity, hence an analogue of Šverák's result cannot be obtained for the Laplace operator. In [52] a Šverák type result is proved for the p-Laplacian for $p \in (N-1, N]$. For $p > N$ a trivial shape continuity result holds in the H^c-topology, since a point has strictly positive p-capacity.

Remark 4.10.2 Operators in divergence form. Instead of the Laplace operator, in equation (4.34) one could consider an elliptic operator of the form $-\operatorname{div}(A(x)\nabla u) + a(x)u$, where $A \in L^\infty(D, \mathbb{R}^{N\times N})$ is such that $\alpha I_d \le A \le \beta I_d$ and $a \in L^\infty(D, \mathbb{R}^+)$. Theorem 4.6.7 remains true with the same hypotheses; this is a consequence of the Mosco convergence of the Sobolev spaces. Nonhomogeneous boundary conditions can obviously be reduced to homogeneous boundary conditions by changing the right-hand side of the equation.

Remark 4.10.3 Nonlinear operators. The nonlinear case was treated in [52], where monotone operators similar to the p-Laplacian, of the form $-\operatorname{div}(A(x, \nabla u))$ are considered. All shape continuity results of Section 4.6 hold in similar classes of domains: convex, uniform cone, flat cone, p-capacity density condition, p-uniform Wiener criterion. Some extensions of these results can be found in [197].

Remark 4.10.4 Stronger convergence of solutions. If we limit ourselves to consider only domains Ω such that for $f \in L^\infty(D)$ the solutions $u_{\Omega,f}$ are continuous on $\overline{D}$, the question of studying the uniform convergence of solutions under geometric perturbations arises. We refer the interested reader to [13] (see also [104] and [105]) where this kind of problems are discussed.

Remark 4.10.5 Systems of equations. The case of elliptic systems, such as the elasticity equations, is treated in [51]; the Stokes equation is discussed in [187]. New difficulties appear when dealing with the convergence in the sense of Mosco of free divergence spaces, mainly because for nonsmooth open sets Ω it may happen that

$$\{u \in [H_0^1(\Omega)]^N : \operatorname{div} u = 0\} \neq \operatorname{cl}_{[H_0^1(\Omega)]^N}\{u \in [C_0^\infty(\Omega)]^N : \operatorname{div} u = 0\}.$$

Remark 4.10.6 Evolution equations. When studying the shape continuity of evolution equations, it is common to try to prove the following type of result: if shape continuity holds for the corresponding stationary equation, prove the shape continuity for the evolution equation. For the heat equation we refer to the book by Attouch [19], while for more general (degenerate) parabolic problems we refer to [184]. Hyperbolic equations were discussed by Toader in [190].

5

Existence of Classical Solutions

Let $\mathcal{A}$ be a class of admissible open (or, if specified, quasi-open) subsets of the design region D and $F : \mathcal{A} \to [0, +\infty]$ be a functional such that F is γ-lower semicontinuous. Our purpose is to look for the existence of a minimizer for the following problem.

$$(\wp) \qquad \min\{F(\Omega) : |\Omega| \leq m, \Omega \in \mathcal{A}\}.$$

We point out that the γ-convergence on the family of *all* open (or quasi-open) subsets of D is not compact if the dimension N is greater than 1; indeed several shape optimization problems of the form $(\wp)$ do not admit any solution, and the introduction of a relaxed formulation is needed in order to describe the behavior of minimizing sequences.

Even if in general problem $(\wp)$ does not admit a solution, some particular cases of existence results are available, provided that either the cost functional F is *regular* in some sense or the family of admissible domains $\mathcal{A}$ is smaller. This is for example the case when the cost functional F is monotone decreasing with respect to the set inclusion or if we search the minimizer in a class of admissible domains on which we impose some geometrical constraints.

5.1 Existence of optimal domains under geometrical constraints

In Chapter 4 we proved the continuity in the H^c topology of the solution of (4.34) in several classes of domains. In order to deduce that these classes are γ-compact, it would be sufficient to prove that they are closed in the H^c-topology and that the Lebesgue measure is lower semicontinuous in the H^c-topology. These below are easy exercises which use the geometric properties of the H^c-convergence and of the capacity.

Proposition 5.1.1 *The following classes of domains (defined in Section 4.6) are γ-compact: $\mathcal{A}_{convex}$, $\mathcal{A}_{unif\ cone}$, $\mathcal{A}_{unif\ flat\ cone}$, $\mathcal{A}_{cap\ density}$, $\mathcal{A}_{unif\ Wiener}$; in two dimensions the class $\mathcal{A}_l$.*

Exercise 1 Prove that if (Ω_n) is a sequence of convex sets converging in the H^c-topology to Ω, then Ω is also convex.

Exercise 2 Prove that the classes of domains satisfying a uniform exterior cone or flat cone condition are compact in the H^c-topology.

Hint: Suppose that $\Omega_n \overset{H^c}{\to} \Omega$. For every $x \in \partial\Omega$ there exists a subsequence of (Ω_n) (still denoted by the same indices) and $x_n \in \partial\Omega_n$ such that $x_n \to x$. The condition in x for Ω is satisfied by the cone obtained as the Hausdorff limit of the sequence of cones corresponding to the points x_n for Ω_n.

Exercise 3 Prove that the classes of domains satisfying a density capacity condition or a uniform Wiener criterion are compact in the H^c-topology.

Hint: Prove that if $\Omega_n \overset{H^c}{\to} \Omega$, then

$$\operatorname{cap}(\Omega^c \cap B_{x,t}, B_{x,2t}) \geq \limsup_{n\to\infty} \operatorname{cap}(\Omega_n^c \cap B_{x,t}, B_{x,2t}),$$

for every $x \in \mathbb{R}^N$ and every $t > 0$ (see the necessary and sufficient conditions for the γ-convergence in Section 4.8).

Exercise 4 Prove that if (K_n) is a sequence of compact connected sets converging in the Hausdorff topology to K, then K is also connected.

The direct methods of the calculus of variations and Proposition 5.1.1 give the following.

Theorem 5.1.2 *Let $j : D \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function. Then the shape optimization problem*

$$\min\left\{\int_\Omega j(x, u_{\Omega,f}, \nabla u_{\Omega,f})\, dx : \Omega \in \mathcal{U}_{ad}\right\}$$

has at least one solution for $\mathcal{U}_{ad} = \mathcal{A}_{convex}$, $\mathcal{A}_{unif\ cone}$, $\mathcal{A}_{unif\ flat\ cone}$, $\mathcal{A}_{cap\ density}$, $\mathcal{A}_{unif\ Wiener}$, $\mathcal{A}_l$ (for $N = 2$), respectively.

Remark 5.1.3 Let us consider again the optimization problem (4.3)

$$\min\left\{\int_D |u_A - c|^2\, dx \;:\; -\Delta u_A = 1 \text{ in } A,\; u_A \in H_0^1(A)\right\}.$$

We have seen in Section 4.2 that if c is sufficiently small, no regular optimal solution exists. The proof was obtained through a comparison argument between the cost of a smooth set A and the cost of $A \cup B_\varepsilon$, where B_ε is a ball of radius ε disjoint from A.

Consider now the same shape optimization problem, in the case of dimension two and with the additional constraint that admissible domains A only vary among simply connected open subsets of D, or more precisely in the class $\mathcal{O}_1(D)$, By Theorem 5.1.2 an optimal solution A_{opt} exists, even if the comparison argument between A_{opt} and $A_{opt} \cup B_\varepsilon$ (sometimes called *topological derivative*) still works. As a conclusion we obtain that A_{opt} must be dense in D and, if c is small enough, different from the whole D.

In particular, if D is a disk in $\mathbb{R}^2$, it is easy to see that A_{opt} cannot be radially symmetric, which gives a new and interesting example of break of symmetry.

5.2 A general abstract result for monotone costs

In this section we present a general framework in which the minimization problem of a monotone functional can be set.

Consider an ordered space $(X, \leq)$ and a functional $F : X \to \overline{\mathbb{R}}$. Suppose that X is endowed with two convergences denoted by γ and $w\gamma$, the last convergence being weaker than the first one and sequentially compact (it will be called *weak gamma* convergence). Moreover suppose that the functional F is γ lower semicontinuous. The relation we assume between γ and $w\gamma$ is the following one:

Assumption (A) *For every* $x_n \overset{w\gamma}{\rightharpoonup} x$ *there exists a sequence of integers* $\{n_k\}$ *and a sequence* $\{y_{n_k}\}$ *in* X *such that* $y_{n_k} \leq x_{n_k}$ *and* $y_{n_k} \overset{\gamma}{\to} x$.

The monotonicity of F becomes an important assumption because of the following result.

Proposition 5.2.1 *If* $F : X \to \overline{\mathbb{R}}$ *is monotone nondecreasing and* γ *lower semicontinuous, then* F *is* $w\gamma$ *lower semicontinuous.*

Proof Let us consider $x_n \overset{w\gamma}{\rightharpoonup} x$, and let $\left\{x_{n_k}\right\}$ be a subsequence such that

$$\lim_{k\to\infty} F(x_{n_k}) = \liminf_{n\to\infty} F(x_n).$$

Using assumption (A) above there exists a subsequence (which we still denote by $\{x_{n_k}\}$) and $y_{n_k} \leq x_{n_k}$ such that $y_{n_k} \overset{\gamma}{\to} x$. The γ lower semicontinuity of F gives

$$F(x) \leq \liminf_{k\to\infty} F(y_{n_k})$$

and the monotonicity of F gives $F(y_{n_k}) \leq F(x_{n_k})$. Therefore

$$F(x) \leq \liminf_{k\to\infty} F(y_{n_k}) \leq \liminf_{k\to\infty} F(x_{n_k}) = \liminf_{n\to\infty} F(x_n)$$

which concludes the proof. ∎

Consider now another functional $\Phi : X \to \overline{\mathbb{R}}$ and the minimization problem

$$\min\{F(x) : x \in X,\ \Phi(x) \leq 0\}. \tag{5.1}$$

Theorem 5.2.2 *Let* F *be a nondecreasing* γ *lower semicontinuous functional and let* Φ *be* $w\gamma$ *lower semicontinuous. Under the assumption (A) above, problem (5.1) admits at least one solution.*

Proof The proof follows straightforwardly by the direct methods of the calculus of variations, taking into account Proposition 5.2.1 and the fact that $w\gamma$ is supposed sequentially compact. ■

The general framework introduced above, even if quite trivial, applies very well in the case of shape optimization problems, and we shall apply it also in the case of obstacles. The main difficulty is to *"identify" the $w\gamma$-convergence, and to prove that assumption (A) is fulfilled.*

5.3 The weak γ-convergence for quasi-open domains

We use the general framework introduced in Section 5.2 for monotone functionals and introduce the *weak γ-convergence* for quasi-open sets.

Let us consider the admissible class

$$\mathcal{A} = \{A \subseteq D : A \text{ is quasi-open}\},$$

where D is a bounded open set. In the following definition we use the notation given in (4.26).

Definition 5.3.1 *We say that a sequence (A_n) of $\mathcal{A}$ weakly γ-converges to $A \in \mathcal{A}$ if w_{A_n} converges weakly in $H_0^1(D)$ to a function $w \in H_0^1(D)$ (that we may take quasi-continuous) such that $A = \{w > 0\}$.*

We point out that, in general, the function w in Definition 5.3.1 does not coincide with w_A (this happens only if A_n γ-converges to A). Moreover, if A_n weakly γ-converges to A, then the Sobolev space $H_0^1(A)$ contains all the weak limits of sequences of elements of $H_0^1(A_n)$. Indeed, by [66, Lemma 3.2], if $u_n \in H_0^1(A_n)$ converge to u weakly in $H_0^1(D)$, then $u = 0$ q.e. on $\{w = 0\}$, which gives $u \in H_0^1(A)$. Finally, notice that since w has been taken quasi-continuous (see Section 4.1), the set $A = \{w > 0\}$ is always quasi-open.

Lemma 5.3.2 *For every $A \in \mathcal{A}$ we have* $\operatorname{cap}\big\{A \Delta \{w_A > 0\}\big\} = 0$.

Proof Since $w_A = 0$ q.e. on $\overline{D} \setminus A$, the inclusion $\{w_A > 0\} \subseteq A$ q.e. is obvious. In order to show the inclusion $A \subseteq \{w_A > 0\}$ q.e., by using [92, Lemma 1.5] we may find an increasing sequence (v_n) of nonnegative functions in $H_0^1(D)$ such that $\sup v_n = 1_A$ q.e.; moreover, by [102, Proposition 5.5] for every v_n there exists a sequence $(\phi_{n,k})$ in $C_c^\infty(D)$ such that $\phi_{n,k} w_A$ tends to v_n strongly in $H_0^1(D)$ and q.e. too. Therefore, since $\phi_{n,k} w_A = 0$ on $\{w_A = 0\}$, we also have $v_n = 0$ q.e. on $\{w_A = 0\}$ and so $1_A = 0$ q.e. on $\{w_A = 0\}$, which shows the inclusion $A \subseteq \{w_A > 0\}$ q.e. ■

Proposition 5.3.3 *If (A_n) is a sequence in $\mathcal{A}$ which γ-converges to A, then (A_n) also weakly γ-converges to A.*

Proof It follows from the definitions of γ-convergence and weak γ-convergence, by using Lemma 5.3.2 ■

Proposition 5.3.4 *The weak γ-convergence on $\mathcal{A}$ is sequentially compact.*

Proof If (A_n) is a sequence in $\mathcal{A}$, by the boundedness of D we obtain that w_{A_n} is bounded in $H_0^1(D)$; hence we may extract a subsequence weakly converging in $H_0^1(D)$ to some function w. Defining $A = \{w > 0\}$ we get that a subsequence of A_n weakly γ-converges to A. ■

Assumption (A) for the γ and the $w\gamma$ convergences of quasi-open sets is contained in the following lemma.

Lemma 5.3.5 *Let (A_n), A, B in $\mathcal{A}$ be such that A_n weakly γ-converge to A and $A \subseteq B$. Then there exists a subsequence $\{A_{n_k}\}$ of (A_n) and a sequence $\{B_k\}$ in $\mathcal{A}$ such that $A_{n_k} \subseteq B_k$ and B_k γ-converge to B.*

Proof This is a consequence of Lemma 4.3.15. ■

Proposition 5.3.6 *The Lebesgue measure is weakly γ-lower semicontinuous on $\mathcal{A}$.*

Proof If $A_n \to A$ in the weak γ-sense, we have $w_{A_n} \to w$ weakly $H_0^1(\Omega)$, with $A = \{w > 0\}$, and for a subsequence the convergence is pointwise a.e. If $x \in A$ is such that $w_A(x) > 0$ and $w_{A_n}(x) \to w(x)$, then $w_{A_n}(x) > 0$ for n large enough, which implies that $x \in A_n$ for n large enough. Therefore, by Fatou's lemma,

$$|A| \leq \liminf_{n \to +\infty} |A_n|,$$

which concludes the proof. ■

5.4 Examples of monotone costs

Theorem 5.4.1 *Let $F : \mathcal{A} \to \overline{\mathbb{R}}$ be a function which is γ-lower semicontinuous and monotone decreasing with respect to the set inclusion. Then the optimization problem*

$$\min\{F(A) : |A| = m,\ A \in \mathcal{A}\}$$

admits at least a solution in $\mathcal{A}$.

Proof This is a consequence of Theorem 5.2.2 and Lemma 5.3.5. ■

Example 5.4.2 (Domains with minimal k^{th} eigenvalue) For every $A \in \mathcal{A}$ let $\lambda_k(A)$ be the k^{th} eigenvalue of the Dirichlet Laplacian on $H_0^1(A)$, with the convention $\lambda_k(A) = +\infty$ if $\text{cap}(A) = 0$. It is well known that the mappings $A \mapsto \lambda_k(A)$ are decreasing with respect to set inclusion (see, e.g., Courant & Hilbert [89]). They are moreover continuous with respect to γ-convergence (see Chapter 6), so that Theorem 5.4.1 applies and for every $k \in \mathbb{N}$ and $0 \le c \le |D|$ we obtain that the minimum

$$\min\big\{\lambda_k(A) : A \in \mathcal{A},\ |A| = c\big\}$$

is achieved. More generally, the minimum

$$\min\big\{\Phi\big(\lambda(A)\big) : A \in \mathcal{A},\ |A| = c\big\}$$

is achieved, where $\lambda(A)$ denotes the sequence $\big(\lambda_k(A)\big)$ and the function $\Phi\colon \overline{\mathbb{R}}^{\mathbb{N}} \to \overline{\mathbb{R}}$ is lower semicontinuous and nondecreasing, in the sense that

$$\lambda_k^h \to \lambda_k \quad \forall k \in \mathbb{N} \ \Rightarrow \ \Phi(\lambda) \le \liminf_{h\to\infty} \Phi(\lambda^h)\,,$$
$$\lambda_k \le \mu_k \quad \forall k \in \mathbb{N} \ \Rightarrow \ \Phi(\lambda) \le \Phi(\mu)\,.$$

Example 5.4.3 (Domains with minimal capacity) Since $\text{cap}(E)$ is nondecreasing, the mapping $A \mapsto F(A) := \text{cap}(D \setminus A)$ is decreasing with respect to the set inclusion. Since F is also γ-continuous, as we may easily verify, Theorem 5.4.1 applies, so that the minimum

$$\min\big\{F(A) : A \in \mathcal{A},\ |A| = c\big\}$$

is achieved. If $\mathcal{F}(D)$ denotes the class of all quasi-closed subsets of D, we see immediately that the problem

$$\min\big\{\text{cap}(E) : E \in \mathcal{F}(D),\ |E| = k\big\} \tag{5.2}$$

admits at least a solution E_0 (it is enough to take $c = |D| - k$ in the previous problem). Let us prove that

$$\text{cap}(E_0) = \min\big\{\text{cap}(E) : E \subseteq D,\ |E| = k\big\}. \tag{5.3}$$

For every subset E of D there exists a quasi-closed set E' such that $E \subseteq E'$ and $\text{cap}(E) = \text{cap}(E')$ (see, e.g., Fuglede [126, Section 2], or Dal Maso [94, Proposition 1.9]). If $|E| = k$, then $|E'| \ge k$, so that there exists $E'' \in \mathcal{F}(D)$ with $E'' \subseteq E'$ and $|E''| = k$. By (5.2) we have

$$\text{cap}(E_0) \le \text{cap}(E'') \le \text{cap}(E') = \text{cap}(E),$$

which proves (5.3).

Example 5.4.4 (Domains which minimize an integral functional) Let us take $f \in H^{-1}(D)$, with $f \ge 0$, and let $g\colon D \times \mathbb{R} \to \overline{\mathbb{R}}$ be a Borel function such that $g(x, \cdot)$ is lower semicontinuous and decreasing on $\mathbb{R}$ for a.e. $x \in D$, and $g(x,s) \ge$

$-\alpha(x) - \beta s^2$ for a suitable function $\alpha \in L^1(D)$ and for a suitable constant $\beta \in \mathbb{R}$. For every $A \in \mathcal{A}$ let $u_A = R_A(f)$ and let

$$F(A) = \int_D g\big(x, u_A(x)\big)\, dx\,.$$

Then F is lower semicontinuous with respect to γ-convergence and, since $f \geq 0$, the maximum principle and the monotonicity properties of g imply that F is decreasing with respect to set inclusion. Therefore, by Theorem 5.4.1 the minimum problem

$$\min\Big\{\int_D g\big(x, u_A(x)\big)\, dx \;:\; A \in \mathcal{A},\ |A| = c\Big\}$$

admits at least a solution.

5.5 The problem of optimal partitions

In this section we fix an integer k and we consider shape cost functionals $F : \mathcal{A}^k \to [0, +\infty]$; the optimization problems we deal with are of the form:

$$\min\Big\{F(A_1, \ldots, A_k) \;:\; A_i \in \mathcal{A},\ A_i \cap A_j = \emptyset \text{ for } i \neq j\Big\}. \tag{5.4}$$

In the following a family $\{A_1, \ldots, A_k\}$ of pairwise disjoint subsets of D will be called a *partition*.

We say that F is γ-lower semicontinuous if for all γ-convergent sequences $A_i^n \to A_i$ for $i = 1, \ldots, k$ we have

$$F(A_1, \ldots, A_k) \leq \liminf_{n\to+\infty} F(A_1^n, \ldots, A_k^n). \tag{5.5}$$

Analogously, we say that F is weakly γ-lower semicontinuous if (5.5) holds for all sequences A_i^n which weakly γ-converge to A_i for $i = 1, \ldots, k$.

It is clear that, without imposing extra assumptions on the cost functional F, we could not expect the existence of an optimal partition. In fact, even in the case $k = 1$, we have seen in Sections 4.2 and 5.2 that the existence of an optimal domain may fail and some monotonicity assumptions (or geometric constraints) are needed in order to obtain unrelaxed solutions; otherwise, only relaxed solutions in some suitable sense (see Section 4.3) can be obtained. Something similar happens for problems of optimal partitions of the form (5.4).

In order to characterize the expression of the relaxed problem associated to (5.4) we consider the case in which we have k sequences $\{A_n^1\}, \ldots, \{A_n^k\}$ of pairwise disjoint quasi-open subsets of D. If we consider the associated Dirichlet measures $\mu_n^i = \infty_{D\setminus A_n^i}$, $i = 1, \ldots, k$, of the class $\mathcal{M}_0(D)$ introduced in Chapter 4, from the sequential compactness property of $\mathcal{M}_0(D)$ we deduce that, up to a subsequence, there exist k measures $\mu_i \in \mathcal{M}_0(D)$, $i = 1, \ldots, k$, such that μ_n^i γ-converge to μ_i for any $i = 1, \ldots, k$.

Our goal is to characterize all k-tuples $(\mu_1, \dots, \mu_k)$ which are the γ-limits of $\mu_n^i = \infty_{D\setminus A_n^i}$, with A_n^i pairwise disjoint (we may call such a k-tuple an *admissible* or an *attainable* one). If we denote by A_{μ_i} the set of finiteness of the measure μ_i introduced in Section 4.3 we may prove (see [73]) the following result.

Theorem 5.5.1 *A k-tuple $(\mu_1, \dots, \mu_k)$ is admissible if and only if it satisfies the following property:*

$$\operatorname{cap}(A_{\mu_i} \cap A_{\mu_j}) = 0, \qquad \forall i, j = 1, \dots, k,\ i \neq j. \tag{5.6}$$

In particular, if the property above holds, it is possible to find k sequences of pairwise disjoint domains $\{A_n^i\}$ such that the corresponding Dirichlet measures $\mu_n^i = \infty_{D\setminus A_n^i}$ γ-converge to μ_i for any $i = 1, \dots, k$.

Remark 5.5.2 By the theorem above, the limit measures μ_i are not "independent". For example, it will not be possible to get an attainable k-tuple formed by k measures of the Lebesgue type on D.

A particular case of problem (5.4) occurs when we consider $f_1, \dots, f_k \in L^2(D)$ and we take an integrand $j : D \times \mathbb{R}^k \to \mathbb{R}$, satisfying the following conditions:

(i) the function $j(\cdot, s)$ is Lebesgue-measurable in D, for every $s \in \mathbb{R}^k$;
(ii) the function $j(x, \cdot)$ is continuous in $\mathbb{R}^k$, for a.e. $x \in D$;
(iii) there exist $a_0 \in L^1(D)$ and $c_0 \in \mathbb{R}$ such that, for a.e. $x \in D$ and for every $s \in \mathbb{R}^k$,

$$\big|j(x, s)\big| \le a_0(x) + c_0|s|^2.$$

For every k-tuple $(u_1, \dots, u_k) \in L^2(D)^k$ we define

$$J(u_1, \dots, u_k) = \int_D j\big(x, u_1(x), \dots, u_k(x)\big)\,dx. \tag{5.7}$$

If we denote by u_A^i the solution of the Dirichlet problem

$$u_A^i \in H_0^1(A), \qquad -\Delta u_A^i = f_i \quad \text{in} \quad H^{-1}(A) \tag{5.8}$$

we may then consider the cost functional

$$F(A_1, \dots, A_k) = J(u_{A_1}^1, \dots, u_{A_k}^k). \tag{5.9}$$

From Theorem 5.5.1 and from the assumptions made on the integrand j we obtain the following relaxation result.

Theorem 5.5.3 *The relaxed form of the optimization problem (5.4) with a cost functional F given by (5.9) is*

$$\min\Big\{ J(u^1_{\mu_1},\dots,u^k_{\mu_k}) \;:\; \mu_i \in \mathcal{M}_0(D),\\ \operatorname{cap}(A_{\mu_i}\cap A_{\mu_j}) = 0\ \forall i,j=1,\dots,k,\ i\neq j\Big\} \tag{5.10}$$

where u^i_μ are the solutions of the Dirichlet problems

$$u^i_\mu \in H^1_0(D)\cap L^2_\mu, \quad -\Delta u^i_\mu + \mu u^i_\mu = f_i, \tag{5.11}$$

and the partial differential equation is intended in the sense seen in Chapter 4. In particular, the relaxed optimization problem (5.10) admits an optimal solution $(\mu_1^{opt},\dots,\mu_k^{opt})$.

Proof The functional J turns out to be continuous in the strong topology of $L^2(D)^k$; therefore, the function $(\mu_1,\dots,\mu_k)\mapsto J(u_{\mu_1},\dots,u_{\mu_k})$ is continuous on $\mathcal{M}_0(D)^k$ with respect to the γ-convergence and the conclusion of the theorem follows immediately from Theorem 5.5.1. The existence of relaxed optimal solutions now follows by the sequential compactness of the class $\mathcal{M}_0(D)$ with respect to the γ-convergence. ■

Similarly to what we did in Section 4.4 for the case $k = 1$, some necessary conditions of optimality for the solutions of the relaxed optimization problem (5.10) can be obtained. The methods used to prove these conditions are quite similar to those used in Section 4.4 for the case of only one measure μ.

Let us suppose that, in addition to the conditions above, the function $j : D\times\mathbb{R}^k \to \mathbb{R}$ satisfies:

(iv) the function $j(x,\cdot)$ is of class C^1 on $\mathbb{R}^k$;

(v) the functions $j_{s_i}(\cdot,s)$ are Lebesgue-measurable on D for every $s\in\mathbb{R}^k$;

(vi) there exist $a_1\in L^2(D)$ and $c_1\in\mathbb{R}$ such that, for a.e. $x\in D$ and for every $s\in\mathbb{R}^k$

$$\sum_{i=1}^k \big|j_{s_i}(x,s)\big| \le a_1(x) + c_1|s|.$$

From the assumptions above it follows immediately that the map J defined by (5.7) is differentiable on $L^2(D)^k$ and its differential J' can be written as

$$\langle J'(u), v\rangle = \sum_{i=1}^k \int_D j_{s_i}(x,u)v_i\,dx,$$

for any $u,v\in L^2(D)^k$. Here, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $L^2(D)^k$ and its dual.

The necessary conditions of optimality we obtain are similar to the ones obtained in Section 4.4 and are described in the theorem below. We recall that if $u^i_{\mu_i}$ solve the state equations

$$u^i_{\mu_i} \in H^1_0(D) \cap L^2_{\mu_i}, \qquad -\Delta u^i_{\mu_i} + \mu_i u^i_{\mu_i} = f_i, \tag{5.12}$$

then the corresponding adjoint state equations are

$$v^i_{\mu_i} \in H^1_0(D) \cap L^2_{\mu_i}, \qquad -\Delta v^i_{\mu_i} + \mu_i v^i_{\mu_i} = j_{s_i}(x, u_{\mu_1}, \dots, u_{\mu_k}). \tag{5.13}$$

Notice that, since the measures $\{\mu_i\}$ satisfy the admissibility property of Theorem 5.5.1, then $u^i_{\mu_i}$ vanish q.e. on $D \setminus \bigcup_{j \neq i} A_{\mu_j}$. We also define in a way similar to the one of Section 4.4 the measures $\nu_{A_{\mu_i}}$ and the corresponding functions α_i and β_i.

Theorem 5.5.4 *Consider $f \in L^2(D)^k$ and an integrand $j : D \times \mathbb{R}^k \to \mathbb{R}$ satisfying the conditions (i)–(vi) above. Let $(\mu_1, \dots, \mu_k)$ be a solution of the relaxed optimization problem (5.10) and let $u^i_{\mu_i}$ and $v^i_{\mu_i}$ be the solutions of problems (5.12) and (5.13) respectively. Then, for every $i = 1, \dots, k$ we have*

(a) $u^i_{\mu_i} v^i_{\mu_i} \le 0$ *q.e. on D;*
(b) $u^i_{\mu_i} v^i_{\mu_i} = 0$ *μ_i-a.e. on D;*
(c) $f_i(x) j_{s_i}(x, 0, \dots, 0) \ge 0$ *for a.e. $x \in \mathrm{int}^*(D \setminus A_{\mu_i})$;*
(d) $\alpha_i \beta_i \ge 0$ *$\nu_{A_{\mu_i}}$-a.e. on $\partial^* A_{\mu_i} \setminus \big(\bigcup_{j \neq i} \partial^* A_{\mu_j}\big)$.*

Remark 5.5.5 Assume that the original optimization problem (5.4) admits a solution $(A_1, \dots, A_k)$ with A_i of class C^2. Then we have $\mu_i = \infty_{D \setminus A_i}$ and the conditions (a)–(d) of Theorem 5.5.4 take the form

(a′) $u^i_{A_i} v^i_{A_i} \le 0$ q.e. on D;
(c′) $f_i(x) j_{s_i}(x, 0, \dots, 0) \ge 0$ for a.e. $x \in D \setminus A_i$;
(d′) $\frac{\partial u^i_{A_i}}{\partial n} \frac{\partial v^i_{A_i}}{\partial n} = 0$ $\mathcal{H}^{N-1}$-a.e. on $\partial A_i \setminus \big(\bigcup_{j \neq i} \partial A_j\big)$;

while condition (b) is trivially satisfied in this case.

Similarly to what we have seen in the case $k = 1$ of shape optimization problems, also in the case of optimal partitions problems a monotonicity assumption on the cost functional leads to the existence of unrelaxed solutions in the original class $\mathcal{A}^k$.

Definition 5.5.6 *We say that $F : \mathcal{A}^k \to [0, +\infty]$ is monotonically decreasing (in the sense of the set inclusion) if for all $(A_1, \dots, A_k)$, $(B_1, \dots, B_k) \in \mathcal{A}^k$ such that $A_i \subseteq B_i$ for $i = 1, \dots, k$ in the sense of capacity, i.e., $\mathrm{cap}(A_i \setminus B_i) = 0$, then*

$$F(B_1, \dots, B_k) \le F(A_1, \dots, A_k).$$

We may formulate now our existence result for optimal partitions problems (see [46]).

Theorem 5.5.7 *Let $F : \mathcal{A}^k \to [0, +\infty]$ be a weak γ-lower semicontinuous shape functional. Then the following optimization problem admits a solution:*

$$\min \Big\{ F(A_1, \dots, A_k) \ : \ A_i \in \mathcal{A},\ \mathrm{cap}(A_i \cap A_j) = 0 \Big\}. \tag{5.14}$$

Proof Consider a minimizing sequence $(A_1^n, \dots, A_k^n)_{n\in\mathbb{N}}$. Since the weak γ-convergence is sequentially compact, there exists a subsequence (still denoted with the same indices) such that

$$A_i^n \to A_i \qquad (i = 1, \dots, k) \qquad \text{in the weak } \gamma\text{-sense.}$$

Since F is weakly γ-lower semicontinuous we have

$$F(A_1, \dots, A_k) \le \liminf_{n\to+\infty} F(A_1^n, \dots, A_k^n).$$

It remains only to prove that $(A_1, \dots, A_k)$ satisfies the constraint, that is $\operatorname{cap}(A_i \cap A_j) = 0$ for $i \neq j$. We have that $w_{A_i^n} \cdot w_{A_j^n} = 0$ a.e. on D and $w_{A_i^n} \to w_i$ strongly in L^2. Therefore, $w_i \cdot w_j = 0$ a.e. on D. Since w_i and w_j are quasi-continuous functions, their product $w_i \cdot w_j$ is quasi-continuous too. Following [137] a quasi-continuous function which vanishes almost everywhere on an open set vanishes quasi-everywhere. So $w_i \cdot w_j = 0$ q.e. on D and so $\operatorname{cap}(A_i \cap A_j) = 0$. ■

Corollary 5.5.8 *If $F : \mathcal{A}^k \to [0, +\infty]$ is monotonically decreasing and γ-lower semicontinuous, then the original optimization problem (5.14) admits a solution.*

As an example, we may consider a cost functional J of the form

$$J(A_1, \dots, A_k) = \phi\big(\lambda_{j_1}(A_1), \dots, \lambda_{j_k}(A_k)\big)$$

where $\lambda_j(A)$ are the eigenvalues of the Laplace operator $-\Delta$ on $H_0^1(A)$, $j_1, \dots, j_k$ are given positive integers, and $\phi(t_1, \dots, t_k)$ is lower semicontinuous and nondecreasing in each variable. Then J fulfills all the assumptions of Corollary 5.5.8, so that the minimization problem

$$\min\Big\{J(A_1, \dots, A_k) \;:\; A_i \in \mathcal{A},\;\; \operatorname{cap}(A_i \cap A_j) = 0\Big\}$$

has a solution. For instance this is the case of problem

$$\min\Big\{\lambda_1(A_1) + \lambda_1(A_2) \;:\; A_1, A_2 \in \mathcal{A},\; \operatorname{cap}(A_1 \cap A_2) = 0\Big\}.$$

Using the $w\gamma$-l.s.c. of the Lebesgue measure, we obtain existence results for shape optimization problems like

$$\min\Big\{J(A) + |A| \;:\; A \in \mathcal{A}\Big\}$$

with $J : \mathcal{A} \to [0, +\infty]$ weakly γ-semicontinuous, or more generally for

$$\min\Big\{J\big(A, |A|\big) \;:\; A \in \mathcal{A}\Big\}$$

with $J : \mathcal{A} \times \mathbb{R} \to [0, +\infty]$ lower semicontinuous with respect to the $\{\text{weak } \gamma\} \times \{\text{Euclidean}\}$-convergence and nondecreasing in the second variable.

5.6 Optimal obstacles

Only to simplify the comprehension of the topic, we discuss the obstacle problem in the linear setting. Nevertheless, we point out the fact that all results we present here hold for obstacle problems associated to p-Laplacian operators (see [47]).

We consider a bounded open set D of $\mathbb{R}^N$ (for $N \geq 2$), a function $\psi \in H_0^1(D)$, the family of admissible obstacles

$$X_\psi(D) = \{g : D \to \overline{\mathbb{R}} : g \leq \psi, \quad g \text{ quasi upper semicontinuous}\},$$

and a cost functional $F : X_\psi(D) \to \overline{\mathbb{R}}$ which is monotone nondecreasing, i.e., for all $g_1, g_2 \in X_\psi(D)$ with $g_1 \leq g_2$ we have $F(g_1) \leq F(g_2)$. Suppose that F is lower semicontinuous for the Γ-convergence of obstacle energy functionals (see the definition of the energy in relation (5.16) below).

The result we are going to prove is the following (see [47], for the nonlinear version of this theorem).

Theorem 5.6.1 *Under the assumptions above, for any constant $c \in \mathbb{R}$, the problem*

$$\min\{F(g) : g \in X_\psi(D), \int_D g\,dx = c\} \tag{5.15}$$

admits a solution.

In order to prove Theorem 5.6.1, we use the general framework of monotone functionals introduced in Section 5.2. Consequently we have to specify the γ and $w\gamma$-convergences and study their properties.

For a quasi-upper semicontinuous function $g : D \to \overline{\mathbb{R}}$ we define the set

$$K_g = \{u \in H_0^1(D) : u \geq g \quad \text{q.e.}\}$$

so that, for every $h \in L^2(D)$ the solution of the obstacle problem associated to h and g is given by minimizing the associated energy

$$\min\{\int_D \frac{1}{2}|\nabla u|^2 dx - \int_D hu\,dx : u \in K_g\}. \tag{5.16}$$

The choice of obstacles as quasi-upper semicontinuous functions is natural, since one can replace an arbitrary obstacle by a suitable upper quasi semicontinuous one (see [22], [93]) such that the solution of problem (5.15) does not change.

For some fixed h the study of the solution of problem (5.16) when the obstacle g varies is done by the classical tool of the Γ-convergence related to the energy functional (see [22], [98]).

If (g_n) is a sequence of admissible obstacles and if $\{K_{g_n}\}$ converges in the sense of Mosco to K_g in $H_0^1(D)$, then it easy to see that the solutions u_n of problem (5.16) associated to g_n converge weakly in $H_0^1(D)$ to the solution of (5.16) corresponding to g. It is also well known that the Mosco convergence of the convex sets K_{g_n} to K_g is equivalent to the Γ-convergence of the energy functionals associated to g_n and g.

Definition 5.6.2 *It is said that a sequence (g_n) of obstacles γ_o-converges to an obstacle g if the sequence of convex sets (K_{g_n}) converges to the convex set K_g in the sense of Mosco.*

In order to use the abstract framework of Section 5.2 we have to introduce a second convergence $w\gamma_o$ on the class of admissible obstacles, and to prove that assumption (A) is fulfilled. The definition of the $w\gamma_o$-convergence for the obstacles will be given by means of the $w\gamma$-convergence of the level sets introduced in Definition 5.3.1.

Definition 5.6.3 *We say that a sequence of obstacles $(g_n)_{n\in\mathbb{N}}$ weak γ_o-converges to g (and we write $g_n \overset{w\gamma_o}{\rightharpoonup} g$) if there exists a dense set $T \subseteq \mathbb{R}$ such that*

$$\{g_n < t\} \overset{w\gamma}{\rightharpoonup} \{g < t\} \quad \forall t \in T.$$

The relation between the γ_o-limit and the $w\gamma_o$-one, is not so simple to establish. Nevertheless, the γ_o-convergence of obstacles is stronger than the $w\gamma_o$-convergence (see Proposition 5.6.7 below), and the $w\gamma_o$-convergence is sequentially compact. Moreover, assumption (A) of the general framework is satisfied for the pair of topologies $(\gamma_o, w\gamma_o)$ and the classical order relation between functions in $X_\psi(D)$. Indeed, we split the proof into the following steps.

Step 1. Sequential compactness of the $w\gamma_o$-convergence. For every sequence $(g_n)_{n\in\mathbb{N}}$ of elements of $X_\psi(D)$ there exist a subsequence $\{g_{n_k}\}$ and an obstacle $g \in X_\psi(D)$ such that $g_{n_k} \overset{w\gamma_o}{\rightharpoonup} g$.

Step 2. Assumption (A) for obstacles. Consider a sequence of obstacles $(g_n)_{n\in\mathbb{N}} \in X_\psi(D)$ such that $g_n \overset{w\gamma_o}{\rightharpoonup} g$. There exist a subsequence $(g_{n_k})_k$ and a sequence $(f_k)_k$ with $f_k \le g_{n_k}$ such that $f_{n_k} \overset{\gamma_o}{\to} g$.

Step 3. Lower semicontinuity of the constraint. Let $g_n, g \in X_\psi(D)$, such that $g_n \overset{w\gamma_o}{\rightharpoonup} g$. Then

$$\int_D g\,dx \ge \limsup_{n\to\infty} \int_D g_n dx.$$

Assuming Steps 1,2 and 3 we can give the proof of the main result.

Proof of Theorem 5.6.1. Consider a minimizing sequence $(g_n)_{n\in\mathbb{N}}$ of admissible obstacles. According to Step 1. we may extract a subsequence (still denoted for simplicity by the same indices) which $w\gamma_o$-converges to some obstacle g in the sense of Definition 5.6.3. Since assumption (A) is fulfilled, by Proposition 5.2.1 we deduce

$$F(g) \le \liminf_{n\to\infty} F(g_n),$$

and by Step 3 on the upper semicontinuity of the constraint, we have $\int_D g\,dx \ge c$. If $\int_D g\,dx = c$, then the obstacle g is admissible and gives the minimum we are looking for. If $\int_D g\,dx > c$, then the new obstacle $\tilde{g}$ defined by

$$\tilde{g}(x) = g(x) - \frac{1}{|D|}\Big(\int_D g(y)dy - c\Big)$$

is admissible (i.e., $\tilde{g} \in X_\psi(D)$ and $\int_D \tilde{g}\,dx = c$) and using the monotonicity of F we get $F(\tilde{g}) \le F(g)$, which shows that $\tilde{g}$ is optimal. ■

In order to prove Steps 1, 2 and 3 we begin by the following result which is a very useful characterization of the γ_o-convergence of a sequence of obstacles in terms of the behavior of the level sets.

Lemma 5.6.4 *Let $g_n, g \in X_\psi(D)$. Then $g_n \xrightarrow{\gamma_o} g$ if and only if there exists a family $T \subseteq \mathbb{R}$ such that $\mathbb{R} \setminus T$ is at most countable and*

$$\{g_n < t\} \xrightarrow{\gamma} \{g < t\} \quad \forall t \in T.$$

Proof Following [93], if g_n and g are quasi-upper semicontinuous functions from D into $\overline{\mathbb{R}}$, a necessary and sufficient condition for having

$$K_g \neq \emptyset \quad \text{and} \quad g_n \xrightarrow{\gamma_o} g$$

is that the following assertions hold:

1. there exists $T \subseteq]0, +\infty[$ with $0 \in \overline{T}$ such that for every $t \in T$,
$$\lim_{n\to\infty} \operatorname{cap}(\{g_n > t\}) = \operatorname{cap}(\{g > t\});$$
2. there exist a dense set $T \subseteq \mathbb{R}$ and a family $\mathcal{B}$ of subsets of D such that for every $t \in T$, and every $B \in \mathcal{B}$,
$$\lim_{n\to\infty} \operatorname{cap}(\{g_n > t\} \cap B) = \operatorname{cap}(\{g > t\} \cap B);$$
the family $\mathcal{B}$ can be chosen dense in the sense of [94];
3. $\displaystyle\lim_{t\to\infty} \limsup_{n\to\infty} \int_t^{+\infty} \operatorname{cap}(\{g_n > s\})(s - t)ds = 0;$
4. $\displaystyle\lim_{t\to 0} \limsup_{n\to\infty} \int_0^t \operatorname{cap}(\{g_n > s\})s\,ds = 0.$

Following this result, if $g_n \xrightarrow{\gamma_o} g$, there exists a dense set $T \subseteq \overline{\mathbb{R}}$ with $\overline{\mathbb{R}} \setminus T$ at most countable, and a countable dense family $\mathcal{B}$ of subsets of D such that

$$\lim_{n\to\infty} \operatorname{cap}(\{g_n > t\} \cap B) = \operatorname{cap}(\{g > t\} \cap B)$$

for every $t \in T$ and every $B \in \mathcal{B}$.

At this point, we observe that for every $B \subseteq D$ and for every function g quasi-upper semicontinuous there exists an at most countable set $T(B)$ in $\mathbb{R}$ such that

$$\operatorname{cap}(\{g > t\} \cap B) = \operatorname{cap}(\{g \ge t\} \cap B) \qquad \forall t \in \mathbb{R} \setminus T(B).$$

Indeed, set for all $t \in \mathbb{R}$,

$$U_t = \{g > t\} \cap B, \quad \tilde{U}_t = \{g \geq t\} \cap B.$$

We have $U_t \subseteq \tilde{U}_t$, from which it follows that

$$\text{cap}(U_t) \leq \text{cap}(\tilde{U}_t) \qquad \forall t \in \mathbb{R}. \tag{5.17}$$

The function $t \mapsto \text{cap}(\tilde{U}_t)$ is decreasing in t, so it is continuous on $N = \mathbb{R} \setminus T(B)$, with $T(B) \subseteq \mathbb{R}$ at most countable. Let us fix $\tau \in N$. For each $t \in \mathbb{R}$ such that $\tau < t$, we have $\text{cap}(U_\tau) \geq \text{cap}(U_t)$. Making $t \to \tau$ we have

$$\text{cap}(U_\tau) \geq \text{cap}(\tilde{U}_\tau) \qquad \forall \tau \in N. \tag{5.18}$$

Now from (5.17) and (5.18) we deduce $\text{cap}(U_\tau) = \text{cap}(\tilde{U}_\tau)$.

Using this fact, for any $B_k \in \mathcal{B}$ and for any $n \in \mathbb{N}$ the family of $t \in \mathbb{R}$ such that

$$\text{cap}(\{g_n > t\} \cap B_k) \neq \text{cap}(\{g_n \geq t\} \cap B_k)$$

is at most countable. Therefore, eliminating all $t \in T$ for all $k, n \in \mathbb{N}$ such that the previous relation holds, one can find a set T' such that $\mathbb{R} \setminus T'$ is at most countable and such that

$$\lim_{n\to\infty} \text{cap}(\{g_n \geq t\} \cap B) = \text{cap}(\{g \geq t\} \cap B)$$

for every $t \in T'$ and every $B \in \mathcal{B}$. Using relation (4.55) we have that for all $t \in T'$ $\{g_n < t\} \xrightarrow{\gamma} \{g < t\}$. Since T' is dense in $\overline{\mathbb{R}}$ we conclude the proof of the necessity.

Suppose now that for a dense family $T \subseteq \overline{\mathbb{R}}$ we have $\{g_n < t\} \xrightarrow{\gamma} \{g < t\}$. We prove that conditions 1), 2), 3), 4) above are satisfied. From the fact that $g_n, g \leq \psi$ conditions 3) and 4) are satisfied. Following relation (4.55) condition 2) is also satisfied, by eliminating again an at most countable family of $t \in \mathbb{R}$ such that $\text{cap}(\{g_n > t\} \cap B_k) \neq \text{cap}(\{g_n \geq t\} \cap B_k)$. It remains to prove 1).

Let us fix some $t > 0$ and set $K_t = \{\psi \geq t\}$ which is a quasi-closed subset of D. Since $g_n \leq \psi$ we get

$$\text{cap}(\{g_n \geq t\} \cap K_t) = \text{cap}(\{g_n \geq t\}).$$

The idea is to find a set $B \in \mathcal{B}$ "between" K_t and D; this is not immediately possible since K_t is not closed but only quasi-closed. Nevertheless, for any $\varepsilon > 0$ there exists a closed set $K_\varepsilon \subseteq K_t$ such that $\text{cap}(K_t \setminus K_\varepsilon) < \varepsilon$. Then

$$|\,\text{cap}(\{g_n \geq t\} \cap K_t) - \text{cap}(\{g_n \geq t\} \cap K_\varepsilon)| \leq \text{cap}(K_t \setminus K_\varepsilon) < \varepsilon.$$

Choosing a set $B \in \mathcal{B}$ such that $K_\varepsilon \subseteq B \subseteq D$ and for which

$$\text{cap}(\{g_n > t\} \cap B) \to \text{cap}(\{g > t\} \cap B),$$

we have

$$\begin{aligned}
&|\operatorname{cap}(\{g>t\})-\operatorname{cap}(\{g_n>t\})|\\
&\qquad=|\operatorname{cap}(\{g>t\}\cap K_t)-\operatorname{cap}(\{g_n>t\}\cap K_t)|\\
&\qquad\le|\operatorname{cap}(\{g>t\}\cap K_t)-\operatorname{cap}(\{g>t\}\cap B)|\\
&\qquad\quad+|\operatorname{cap}(\{g>t\}\cap B)-\operatorname{cap}(\{g_n>t\}\cap B)|\\
&\qquad\quad+|\operatorname{cap}(\{g_n>t\}\cap K_t)-\operatorname{cap}(\{g_n>t\}\cap B)|.
\end{aligned}$$

The first and the last term of the right-hand side are less than ε by the choice of B, and the middle term vanishes as $n\to\infty$. Hence we get

$$\lim_{n\to+\infty}\operatorname{cap}(\{g_n>t\}=\operatorname{cap}(\{g>t\}).$$

Therefore, condition 1) of the lemma also holds, and so the proof is concluded. ■

Remark 5.6.5 The family T in Lemma 5.6.4 can be simply replaced by a a dense set in $\mathbb{R}$. Indeed, let $g_n, g\in X_\psi(D)$. Then $g_n\xrightarrow{\gamma_o} g$ if and only if there exists a countable dense family $\mathcal{T}_0\subseteq\mathbb{R}$ such that

$$\{g_n<t\}\xrightarrow{\gamma}\{g<t\}\qquad\forall t\in\mathcal{T}_0. \tag{5.19}$$

The necessity is like in the first step of Lemma 5.6.4. Conversely, suppose that (5.19) holds for a set $\mathcal{T}_0\subseteq\mathbb{R}$ countable and dense. We prove that (5.19) holds for $t\in T$ where $\mathbb{R}\setminus T$ is at most countable (which implies, by Lemma 5.6.4, that $g_n\xrightarrow{\gamma_o} g$).

For every $t\in\mathbb{R}$, possibly passing to subsequences, we have $w_{\{g_{n_k}<t\}}\rightharpoonup u_t$ weakly in $H^1_0(D)$ for a suitable function u_t. It will be enough to prove that $u_t=w_{\{g<t\}}$ up to an at most countable set. By assumption, $\mathcal{T}_0$ is dense in $\mathbb{R}$ and so we can find $t_1,t_2\in\mathcal{T}_0$ with $t_1\le t\le t_2$. Then

$$w_{\{g<t_1\}}\le u_t\le w_{\{g<t_2\}}. \tag{5.20}$$

Let us define

$$A_\tau=\{g<\tau\}\ \text{ for any }\ \tau\in\mathcal{T}_0,\quad \tau\le t,$$

$$B_\tau=\{g<\tau\}\ \text{ for any }\ \tau\in\mathcal{T}_0,\quad \tau\ge t.$$

The sets A_τ and B_τ are quasi-open and we have that $\{A_\tau\}_{\tau\in\mathcal{T}_0,\tau\le t}$ is nondecreasing and $\{B_\tau\}_{\tau\in\mathcal{T}_0,\tau\ge t}$ is decreasing with respect to the set inclusion. Now, from the theory of the γ-convergence (see [94], [145]), we have

$$A_\tau\xrightarrow{\gamma}\bigcup_{\tau\in\mathcal{T}_0,\tau\le t}A_\tau=A_t\qquad\text{as }\ \tau\to t \tag{5.21}$$

and

$$w_{\{g<t\}}=\sup\{w_{\{g<\tau\}}:\tau\in\mathcal{T}_0,\tau\le t\}. \tag{5.22}$$

One has to pay attention to the fact that an arbitrary union of quasi-open sets is not generally quasi-open. In this case, the monotonicity plays an essential role. In fact any nondecreasing sequence of quasi-open sets is γ-convergent to their union. Moreover

$$\{g < t\} \subseteq B_\tau \qquad \forall \tau \in \mathcal{T}_0,\ \tau \geq t, \tag{5.23}$$

so that

$$w_{\{g<t\}} \leq w_{\{g<\tau\}} \qquad \forall \tau \in \mathcal{T}_0,\ \tau \geq t, \tag{5.24}$$

from which it follows that

$$w_{\{g<t\}} \leq \inf\left\{w_{\{g<\tau\}} : \tau \in \mathcal{T}_0,\ \tau \geq t\right\}. \tag{5.25}$$

Now from (5.20), ... ,(5.25) we have

$$\begin{aligned} w_{\{g<t\}} &= \sup\left\{w_{\{g<\tau\}} : \tau \in \mathcal{T}_0,\ \tau \leq t\right\} \\ &\leq u_t \leq \inf\left\{w_{\{g<\tau\}} : \tau \in \mathcal{T}_0,\ \tau \geq t\right\}. \end{aligned} \tag{5.26}$$

Let us consider the mapping $t \mapsto \|w_{\{g<t\}}\|_{L^2(D)}$ which is nondecreasing, and hence it has an at most countable set of points of discontinuity. If t is a point where this mapping is continuous we have

$$\sup\left\{w_{\{g<\tau\}} : \tau \in \mathcal{T}_0,\ \tau \leq t\right\} = \inf\left\{w_{\{g<\tau\}} : \tau \in \mathcal{T}_0,\ \tau \geq t\right\}$$

and so from (5.26) we have $u_t = w_{\{g<t\}}$, which concludes the proof.

Proposition 5.6.6 *Let $g_n, g \in X_\psi(D)$. The following conditions are equivalent:*

(i) there exists a dense family $\mathcal{T} \in \mathbb{R}$ such that $\{g_n < t\} \overset{w\gamma}{\rightharpoonup} \{g < t\}$ for all $t \in \mathcal{T}$;
(ii) there exists an at most countable family $N \in \mathbb{R}$ such that $\{g_n < t\} \overset{w\gamma}{\rightharpoonup} \{g < t\}$ for all $t \in \mathbb{R} \setminus N$.

Proof If (ii) is true, obviously (i) follows. Conversely, let us suppose that (i) is true. For all $s \in \mathcal{T}$ we have $w_{\{g_n<t\}} \rightharpoonup w_s$ weakly in $H_0^1(D)$, $\{w_s > 0\} = \{g < s\}$ and, from the upper semicontinuity of g, for all $t \in \mathbb{R}$,

$$\{g < t\} = \bigcup_{s\in\mathcal{T}, s\leq t} \{g < s\}.$$

We define for any $t \in \mathbb{R}$ $w_t := \sup_{s\in\mathcal{T}, s\leq t} w_s$. From the monotonicity of the mapping $s \mapsto w_s$ the definition above is consistent and we have

$$\{w_t > 0\} = \bigcup_{s\in\mathcal{T}, s\leq t} \{w_s > 0\} = \bigcup_{s\in\mathcal{T}, s\leq t} \{g < s\} = \{g < t\}.$$

From the boundedness of $w_{\{g_n<t\}}$, possibly passing to a subsequence still denoted by the same indices, we have

$$w_{\{g_n<t\}} \rightharpoonup u_t.$$

It is sufficient to prove that there exists an at most countable set N such that $u_t(x) = w_t(x)$ q.e. for $t \in \mathbb{R} \setminus N$. We observe that, by monotonicity, for every $s \in \mathcal{T}$ with $s \leq t$ we have

$$w_{\{g_n<s\}} \leq w_{\{g_n<t\}}, \tag{5.27}$$

so passing to the limit in (5.27) as $n \to +\infty$ we obtain

$$w_s \leq u_t \qquad \forall s \in \mathcal{T}, s \leq t,$$

from which

$$u_t \geq \sup_{s\in\mathcal{T}, s\leq t} w_s = w_t. \tag{5.28}$$

On the other hand, again by the monotonicity, we also have

$$u_t \leq w_s \qquad \forall\, s \in \mathcal{T},\;\; s \geq t,$$

from which

$$u_t \leq \inf_{s\in\mathcal{T}, s\geq t} w_s. \tag{5.29}$$

Now, denoting by $\tilde{w}_t$ the right-hand side of (5.29), for $t \in \mathbb{R} \setminus N$, with N at most countable, we have $w_t = \tilde{w}_t$ since the mapping $t \mapsto \|w_t\|_{L^2(D)}$ is monotone nondecreasing and it only has at most countable points of discontinuity. So we obtain from (5.28) and (5.29) that $u_t = w_t$ for every $t \in \mathbb{R} \setminus N$ which ends the proof. ∎

Proposition 5.6.7 *Suppose that* $g_n \xrightarrow{\gamma_o} g$. *Then* $g_n \overset{w\gamma_o}{\rightharpoonup} g$.

Proof It is an immediate consequence of Lemma 5.6.4 and Definition 5.6.3. ∎

Proof of Step 1. Consider an enumeration $\{r_1, r_2, ...\}$ of the set $\mathbb{Q}$ of rational numbers. For the level r_1 there exists a subsequence of $(g_n)_{n\in\mathbb{N}}$ (still denoted by the same indices) such that $w_{\{g_n<r_1\}} \rightharpoonup w_{r_1}$ weakly in $H_0^1(D)$. For the level r_2 there exists a subsequence of the previous one such that $w_{\{g_n<r_2\}} \rightharpoonup w_{r_2}$ weakly in $H_0^1(D)$. In such a way for any $r_k \in \mathbb{Q}$ one can extract a subsequence (of the sequence established for r_{k-1}) such that $w_{\{g_n<r_k\}} \rightharpoonup w_{r_k}$ weakly in $H_0^1(D)$. By a diagonal procedure we can choose an element of the first sequence such that

$$\|w_{\{g_{n_1}<r_1\}} - w_{r_1}\|_{L^2(D)} < 1,$$

then a second element of the second sequence such that

$$\|w_{\{g_{n_2}<r_1\}} - w_{r_1}\|_{L^2(D)} < \frac{1}{2} \text{ and } \|w_{\{g_{n_2}<r_2\}} - w_{r_2}\|_{L^2(D)} < \frac{1}{2},$$

and continuing this procedure, one constructs a subsequence $(g_{n_i})_i$ such that

$$\forall k \in \mathbb{N} \qquad w_{\{g_{n_i}<r_k\}} \stackrel{L^2(D)}{\to} w_{r_k} \quad \text{as} \quad i \to \infty.$$

This means exactly that for all $t \in \mathbb{Q}$,

$$\{g_{n_i} < t\} \stackrel{w\gamma}{\rightharpoonup} \{w_t > 0\} \quad \text{for} \quad i \to \infty.$$

We define then the limit obstacle through its level sets

(5.30) $$\{g < t\} = \{w_t > 0\} \quad \text{for all} \quad t \in \mathbb{Q}.$$

If $t_1, t_2 \in \mathbb{Q}$ with $t_1 < t_2$, then obviously $\{g < t_1\} \subseteq \{g < t_2\}$. The function g is defined by

$$g(x) = \inf\{t \in \mathbb{Q} : x \in \{g < t\}\}.$$

Hence for every $t \in \mathbb{R} \setminus \mathbb{Q}$ we have

(5.31) $$\{g < t\} = \bigcup_{s \in \mathbb{Q}, s<t} \{g < s\}.$$

One can see that:

(i) g is correctly defined.

(ii) g is quasi-upper semicontinuous, because its level sets $\{g < t\}$ are quasi-open, being by definition, countable unions of quasi-open sets (see (5.30) and (5.31)).

(iii) $g \in X_\psi(D)$, because $\{\psi < t\} \supseteq \{g < t\}$ for every $t \in \mathbb{R}$. Indeed, let us consider first the case $t \in \mathbb{Q}$. By assumption we have $g_{n_i} \le \psi$, for all i, hence

$$\{\psi < t\} \supseteq \{g_{n_i} < t\} \qquad \forall i \in \mathbb{N},$$

from which

$$w_{\{g_{n_i}<t\}} \le w_{\{\psi<t\}}.$$

Passing to the limit as $i \to +\infty$, we get

$$w_t \le w_{\{\psi<t\}},$$

so that

$$\left\{w_{\{\psi<t\}} > 0\right\} = \{\psi < t\} \supseteq \{w_t > 0\} = \{g < t\}.$$

Fix now $t \in \mathbb{R} \setminus \mathbb{Q}$. We have

$$\{g < t\} = \bigcup_{s<t, s \in \mathbb{Q}} \{g < s\} \subseteq \bigcup_{s<t, s \in \mathbb{Q}} \{\psi < s\} \subseteq \{\psi < t\}$$

which shows that $g \in X_\psi(D)$.

It remains to prove that $g_{n_i} \overset{w\gamma_o}{\rightharpoonup} g$, or equivalently, that there exists a dense set $\mathcal{T} \subseteq \mathbb{R}$ such that

$$w_{\{g_{n_i}<t\}} \rightharpoonup w_t, \quad \{w_t > 0\} = \{g < t\}, \quad \forall t \in \mathcal{T}.$$

If $t \in \mathbb{Q}$ the above facts are true by the definition of g, which concludes the proof. ■

In order to prove assumption (A) for the $w\gamma_o$ and γ_o-convergences, we give a technical result for domains, necessary for the construction of the γ_o-convergent sequence deriving from the $w\gamma_o$-convergent one.

Proof of Step 2. This will be obtained in two steps. We first consider the particular case of obstacles whose ranges are in a finite set. Let us consider the numbers

$$-\infty = l_1 < l_2 < \cdots < l_q$$

and the special family of obstacles

$$\mathcal{A}[l_1, \dots, l_q] = \{g \in X_\psi(D) \ : \ g(x) \in \{l_1, \dots, l_q\}\}.$$

We construct the sets

$$A_1(g) = \{g < l_2\}, \quad A_2(g) = \{g < l_3\}, \quad \dots, \quad A_{q-1}(g) = \{g < l_q\}$$

which are quasi-open and such that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{q-1}$. Consider now a sequence $(g_n)_{n\in\mathbb{N}} \in \mathcal{A}[l_1, \dots, l_q]$ which weakly γ_o-converges to a function g. To every function g_n we associate as before the sets $A_1(g_n), \dots, A_{q-1}(g_n)$ and, using the compactness of the weak γ-convergence for sets, we can write (for a subsequence still denoted by the same indices)

$$A_i(g_n) \overset{w\gamma}{\rightharpoonup} A_i \qquad \forall i = 1, \dots, q-1.$$

By the definition of the weak γ_o-convergence we have

$$g = l_1 \ \text{ on } \ A_1, \quad g = l_2 \ \text{ on } \ A_2 \setminus A_1, \quad \dots, \quad g = l_q \ \text{ on } \ D \setminus (A_1 \cup \cdots \cup A_{q-1}).$$

Moreover, g is quasi-upper semicontinuous and $g \le \psi$. For this last inequality it is sufficient to prove for any $i = 2, \dots, q$ that $\{g < l_i\} \subseteq \{\psi < l_i\}$. This follows from the fact that $\{g_n < l_i\} \subseteq \{\psi < l_i\}$ and from the properties of the weak γ-convergence of sets.

We construct now the sequence f_n of admissible obstacles such that $f_n \overset{\gamma_o}{\to} g$ with $f_n \le g_n$. Using Lemma 4.3.15 for subsequences (still denoted by the same indices) there exist sets $G_i^n \supseteq A_i(g_n)$ such that $G_i^n \overset{\gamma}{\to} A_i$. For fixed n, the sets $G_1^n, \dots, G_{q-1}^n$ are not ordered. Therefore one applies Proposition 4.5.6 and considers

$$\begin{aligned}
\tilde{G}_1^n &= G_1^n \cap G_2^n \cap \cdots \cap G_{q-1}^n,\\
\tilde{G}_2^n &= G_2^n \cap G_3^n \cap \cdots \cap G_{q-1}^n,\\
&\dots\\
\tilde{G}_{q-1}^n &= G_{q-1}^n.
\end{aligned}$$

Then $\tilde{G}_1^n \subseteq \tilde{G}_2^n \subseteq \cdots \subseteq \tilde{G}_{q-1}^n$ and $\tilde{G}_i^n \xrightarrow{\gamma} A_i$. Moreover, $\tilde{G}_i^n \supseteq A_i(g_n)$. We define now the obstacle f_n by means of $\tilde{G}_1^n, \dots, \tilde{G}_{q-1}^n$ in the following way: $f_n = l_1$ on $\tilde{G}_1^n$, $f_n = l_2$ on $\tilde{G}_2^n \setminus \tilde{G}_1^n, \dots, f_n = l_q$ on $D \setminus \tilde{G}_{q-1}^n$. We get that f_n is quasi-upper semicontinuous, $f_n \le g_n$ and $f_n \xrightarrow{\gamma} g$.

In a second step of the proof, a general obstacle is approached by obstacles with a finite range. We define for any $k \in \mathbb{N}$ the family of levels

$$\mathcal{R}_k = \{l_1, \dots, l_k\} \cup -\infty$$

where $(l_k)_k$ is a dense set in $\overline{\mathbb{R}}$ with the property that the weak γ-convergence holds on levels l_i. We have $\mathcal{R}_{k_1} \subseteq \mathcal{R}_{k_2}$ if $k_1 \le k_2$. Set now $\mathcal{R} = \bigcup_{k \in \mathbb{N}} \mathcal{R}_k$.

For some obstacle $g \in X_\psi(D)$ we define the truncation on $\mathcal{R}_k$ of g in the following way:

$$T_k(g)(x) = \sup\{l \in \mathcal{R}_k : g(x) \ge l\}.$$

Obviously $T_k(g) \in \mathcal{A}[\mathcal{R}_k]$ and $T_k(g) \le g$. Moreover, if $l \in \mathcal{R}_k$, then for all $k' \ge k$ we have

$$\{T_{k'}(g) < l\} = \{g < l\}.$$

As in the first step, for $k = 1$ there exists a subsequence, still denoted by the same indices, such that

$$T_1(g_n) \overset{w\gamma_o}{\rightharpoonup} T_1(g);$$

we consider a subsequence and a sequence $f_n^1 \le T_1(g_n)$ with $f_n^1 \xrightarrow{\gamma_o} T_1(g)$. For $k = 2$ there exists (as in step one) a subsequence such that

$$T_2(g_n) \overset{w\gamma_o}{\rightharpoonup} T_2(g),$$

and again, we consider a subsequence and a sequence $f_n^2 \le T_2(g_n)$ with $f_n^2 \xrightarrow{\gamma_o} T_2(g)$. We continue this procedure for any $k \in \mathbb{N}$ and we choose a diagonal sequence $(f_{n_k}^k)_k$ with the property that $d_\gamma(f_{n_k}^k, T_k(g)) \le 1/k$. Here by d_γ one denotes the distance which generates the same topology of the γ_o-convergence (see [91] for the metrizability of the γ_o-convergence). On the other hand, using Lemma 5.6.4 we have $T_k(g) \xrightarrow{\gamma_o} g$ and therefore we have found a subsequence $(f_{n_k}^k)_k$ which satisfies the desired properties. ■

Since assumption (A) is fulfilled, the general framework presented in Section 5.2 and in particular Theorem 5.2.2 could be applied. Nevertheless, in the case of obstacles the integral constraint plays a very important role.

Proof of Step 3. Let $g_n, g \in X_\psi(D)$, with $g_n \overset{w\gamma_o}{\rightharpoonup} g$. Then

$$g(x) \ge \limsup_{n \to \infty} g_n(x) \quad \text{for a.e. } x \in D. \tag{5.32}$$

Indeed, consider some $x \in D$ and $l \in \mathbb{R}$ with $g(x) < l$. From the weak γ_o-convergence, there exists some l' between $g(x)$ and l, such that

$$\{g_n < l'\} \stackrel{w\gamma}{\rightharpoonup} \{g < l'\};$$

then $1_{\{g<l'\}} \leq \liminf_{n\to\infty} 1_{\{g_n<l'\}}$, where we denote by 1_C the function such that $1_C(x) = 1$ if $x \in C$ and and $1_C(x) = 0$ otherwise. In particular, this means that $g_n(x) < l'$ for n large enough, that is

$$\limsup_{n\to\infty} g_n(x) \leq l'.$$

Since l was arbitrary, and $g(x) < l' < l$ we get (5.32).

The conclusion now follows by Fatou's lemma. ■

Remark 5.6.8 The nonlinear frame. All the results of Section 5.6 hold in the nonlinear frame, i.e., for energy functionals associated to the p-Laplacian, into $W_0^{1,p}$-spaces. In this case, the γ_0 and $w\gamma_o$ convergences are defined with the help of the γ_p-convergence of p-quasi open sets. We refer to [47] for a detailed description of the problem.

Remark 5.6.9 Bilateral obstacles. By analogy, the previous results can be extended to the bilateral obstacle problem. In order to consider bilateral problems, we define the family

$$\tilde{X}_\psi(D) = \{g : D \to \overline{\mathbb{R}} : g \text{ quasi-l.s.c. } g \geq \psi\}$$

with ψ fixed in $W_0^{1,p}(D)$; the associated convex sets are

$$\tilde{K}_g = \{u \in W_0^{1,p}(D) : u \leq g \;\; p\text{-q.e.}\}.$$

In this case, the γ_o-convergence, respectively $w\gamma_o$-convergence, are defined as in Definitions 5.6.2 and 5.6.3 (naturally extended in $W_0^{1,p}$) by using the upper level sets: $\{g > t\}$. We denote them by $\tilde{\gamma}_o$, respectively $w\tilde{\gamma}_o$.

Then a nonlinear version of Theorem 5.6.1 can be formulated, under the assumptions:

1. F is lower semicontinuous with respect to the $\tilde{\gamma}_o$-convergence;
2. F is monotone decreasing with respect to the usual order of functions.

We can now consider the case of bilateral problems. Fix first a function $\psi \in W_0^{1,p}(D)$; the admissible set is now

$$\begin{aligned} Y_\psi(D) = \{(g_1, g_2) : g_i : D \to \overline{\mathbb{R}}, i = 1, 2,\\ g_1 \; p\text{-quasi-u.s.c.}, \; g_2 \; p\text{-quasi-l.s.c.}, g_1 \leq \psi \leq g_2\}. \end{aligned}$$

We define

$$K_{g_1,g_2} = \{u \in W_0^{1,p}(D) : g_1 \le u \le g_2 \;\; p\text{-q.e.}\}$$

and consider a functional $F : Y_\psi(D) \to \overline{\mathbb{R}}$. Then Theorem 5.6.1 still holds under the assumptions:

1. $F(\cdot, \cdot)$ is lower semicontinuous with respect to the $(\gamma_o, \tilde{\gamma}_o)$-convergence;
2. $F(\cdot, \cdot)$ is monotone nondecreasing with respect to the first variable and decreasing with respect to the second one.

In this case, the constraint is of the form

$$\int_D g_1 dx = c, \quad \int_D g_2 dx = \tilde{c},$$

with, of course, $c \le \tilde{c}$.

Remark 5.6.10 Nonlinear shape optimization problems. The shape optimization problem solved in Section 5.3 for $p = 2$ can be deduced as a particular case of the result mentioned in Remark 5.6.9. It follows by considering the family of obstacles that

$$\mathcal{O}(D) = \{(g_1, g_2) : \; g_1 = -\infty \cdot 1_A, \; g_2 = +\infty \cdot 1_A, \;\; A \subseteq D, \; A \; p\text{- quasi-open}\}.$$

It is sufficient to see that the family $\mathcal{O}(D)$ is closed with respect to the p-$(w\gamma_o, w\tilde{\gamma}_o)$-convergence. In fact, a bilateral obstacle $(-\infty \cdot 1_A, +\infty \cdot 1_A)$ is identified with the quasi-open set A. In this case, the bilateral obstacle problem becomes a Dirichlet problem with homogeneous boundary condition associated to the p-Laplacian on the quasi-open set A.

6

Optimization Problems for Functions of Eigenvalues

6.1 Stability of eigenvalues under geometric domain perturbation

For the convenience of the reader, we start by recalling some basic facts about eigenvalues of operators in Hilbert spaces. For simplicity, we limit ourselves to consider a Hilbert space H and a linear operator

$$R : H \to H$$

which is compact, self-adjoint and nonnegative. Further details, as well as the analysis of more general situations, can be found in [112].

Under the previous hypotheses, the spectrum of R consists only of eigenvalues, which can be ordered (counting their multiplicities) as

$$0 \leq \cdots \leq \Lambda_{n+1}(R) \leq \Lambda_n(R) \leq \cdots \leq \Lambda_1(R).$$

For every integer $n \geq 0$, the eigenvalue $\Lambda_{n+1}(R)$ is given by the formula

$$\Lambda_{n+1}(R) = \min_{\phi_1,\dots,\phi_n \in H} \max_{\substack{|\phi| = 1 \\ (\phi,\phi_1) = \cdots = (\phi,\phi_n) = 0}} |R\phi|,$$

where $|\cdot|$ is the norm and $(\cdot,\cdot)$ the scalar product of H.

Theorem 6.1.1 (Courant–Fischer) *For every $n \geq 1$ the following equality holds:*

$$\Lambda_n(R) = \max_{E \in S_n} \min_{\phi \in E, |\phi|=1} |R\phi|,$$

where S_n denotes the family of all subspaces of H of dimension n.

Theorem 6.1.2 (Rayleigh min-max formula) *For every $n \geq 0$ the following equality holds:*

$$\Lambda_{n+1}(R) = \min_{\phi_1,\dots,\phi_n \in H} \max_{\substack{|\phi| = 1 \\ (\phi,\phi_1) = \cdots = (\phi,\phi_n) = 0}} \frac{(R\phi,\phi)}{|\phi|^2}.$$

We recall the following general results from [112, Corollaries XI.9.3 and XI.9.4].

Theorem 6.1.3 *Let R_1, R_2 be compact, self-adjoint and nonnegative operators on H. For every $m, n \geq 1$ we have*

1. $\Lambda_{m+n-1}(R_1 + R_2) \leq \Lambda_m(R_1) + \Lambda_n(R_2)$,
2. $\Lambda_{m+n-1}(R_1 R_2) \leq \Lambda_m(R_1)\Lambda_n(R_2)$,
3. $|\Lambda_n(R_1) - \Lambda_n(R_2)| \leq |R_1 - R_2|_{\mathcal{L}(H)}$.

In Theorem 6.1.3, by $|R|_{\mathcal{L}(H)}$ we denoted the usual operator norm

$$|R|_{\mathcal{L}(H)} = \sup\{|R\phi| : |\phi| \leq 1\}.$$

We are interested in the study of optimization problems for functions of eigenvalues of some elliptic operators. Consider a measure μ belonging to the space $\mathcal{M}_0(\mathbb{R}^N)$ introduced in Chapter 4, and assume that the Lebesgue measure of its regular set A_μ (i.e., the union of all finely open sets of finite μ-measure, see Section 4.3) is finite. We define the resolvent operator associated to μ in the following way:

$$R_\mu : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N), \quad R_\mu(f) = u$$

where u is the solution of

$$\begin{cases} -\Delta u + \mu u = f, \\ u \in H^1(\mathbb{R}^N) \cap L^2_\mu(\mathbb{R}^N) \end{cases} \tag{6.1}$$

in the sense defined in Chapter 4.

Since $|A_\mu| < +\infty$, then $H^1_0(A_\mu)$ is compactly embedded in $L^2(A_\mu)$ (see [194]). Consequently, R_μ is well defined, compact, nonnegative and self-adjoint. Notice that we implicitly identify R_μ with $P_{L^2(A_\mu)} \circ R_\mu \circ P_{L^2(A_\mu)}$, where $P_{L^2(A_\mu)} : L^2(\mathbb{R}^N) \to L^2(A_\mu)$ is the orthogonal projector.

We define the eigenvalues λ_k associated to the measure μ as the eigenvalues of the elliptic operator $-\Delta + \mu I$ (in the sense of equation (6.1)), by setting

$$\lambda_k(\mu) = \frac{1}{\Lambda_k(R_\mu)},$$

and we obtain the following sequence (as soon as $R_\mu \neq 0$)

$$0 < \lambda_1(\mu) \leq \cdots \leq \lambda_n(\mu) \leq \lambda_{n+1}(\mu) \leq \cdots \to +\infty.$$

For every $n \geq 1$ there exists $u \in H^1(\mathbb{R}^N) \cap L^2_\mu(\mathbb{R}^N) \setminus \{0\}$ such that

$$-\Delta u + \mu u = \lambda_n(\mu)u,$$

in the sense defined in Chapter 4. Moreover, the Rayleigh formula can be used, and we obtain

$$\lambda_n(\mu) = \max_{\phi_1,\ldots,\phi_{n-1}\in L^2(A_\mu)} \min_{\substack{\phi \in L^2(A_\mu)\setminus\{0\} \\ (\phi,\phi_1)=\cdots=(\phi,\phi_{n-1})=0}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla\phi|^2\,dx + \int_{\mathbb{R}^N} \phi^2 d\mu}{\displaystyle\int_{\mathbb{R}^N} \phi^2 dx}$$

$$= \min_{E\in S_n} \max_{\phi\in E\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla\phi|^2 dx + \int_{\mathbb{R}^N} \phi^2 d\mu}{\displaystyle\int_{\mathbb{R}^N} \phi^2 dx}.$$

Here $(\cdot,\cdot)$ is the scalar product in $L^2(\mathbb{R}^N)$ and S_n denotes the family of subspaces of $H^1(\mathbb{R}^N) \cap L^2_\mu(\mathbb{R}^N)$, of dimension n.

Remark 6.1.4 If A is an open (or quasi-open) set of finite Lebesgue measure in $\mathbb{R}^N$, considering the measure $\mu = \infty_{\mathbb{R}^N\setminus A}$, we have

$$\lambda_n(\mu) = \lambda_n(A),$$

where $\lambda_n(A)$ is the usual n-th eigenvalue (counted with its multiplicity) of the Dirichlet–Laplacian on A.

Proposition 6.1.5 *Let $\mu_1, \mu_2 \in \mathcal{M}_0(\mathbb{R}^n)$ with $|A_{\mu_1}|, |A_{\mu_2}| < +\infty$. If $\mu_1 \le \mu_2$ in the usual sense of measures, then for every $n \ge 1$, $\lambda_n(\mu_1) \le \lambda_n(\mu_2)$.*

Proof It is a direct consequence of the Rayleigh formula. ■

Corollary 6.1.6 *Let A_1, A_2 be two open (or quasi-open) sets of finite Lebesgue measure. If $A_2 \subseteq A_1$, then for every $n \ge 1$, $\lambda_n(A_1) \le \lambda_n(A_2)$.*

Proof It follows from Proposition 6.1.5 by noticing that if $A_1 \subseteq A_2$, then $\infty_{\mathbb{R}^N\setminus A_1} \le \infty_{\mathbb{R}^N\setminus A_2}$. ■

Notice, that in Corollary 6.1.6, it is enough that the inclusion $A_2 \subseteq A_1$ holds up to a set of zero capacity.

Let us now fix a bounded open design region $D \subseteq \mathbb{R}^N$.

Proposition 6.1.7 *Let $\mu_n \in \mathcal{M}_0(D)$, and $\mu_n \xrightarrow{\gamma} \mu$. Then*

$$|R_{\mu_n} - R_\mu|_{\mathcal{L}(L^2(D))} \to 0.$$

Proof Let

$$|R_{\mu_n} - R_\mu|_{\mathcal{L}(L^2(D))} = |R_{\mu_n} f_n - R_\mu f_n|_{L^2(D)},$$

where $|f_n|_{L^2(D)} = 1$.

By weak compactness, we get that $f_n \rightharpoonup f$ weakly in $L^2(D)$. It is clear that $R_\mu f_n \rightharpoonup R_\mu f$ weakly in $H^1_0(D)$, since R_μ is continuous on $L^2(D)$ and

$|R_\mu f_n|_{H_0^1(D)} \le C$, where the constant C depends only on the measure of D. Consequently, $R_\mu f_n \to R_\mu f$ strongly in $L^2(D)$.

On the other hand, note that $R_{\mu_n} f_n \rightharpoonup R_\mu f$ weakly in $H_0^1(D)$. It is enough to observe that $R_{\mu_n} f_n \rightharpoonup R_\mu f$ weakly in $L^2(D)$; this is a consequence of the γ-convergence and the fact that R_{μ_n} and R_μ are self-adjoint. Indeed, for every $\phi \in L^2(D)$, we have

$$(R_{\mu_n} f_n, \phi)_{L^2(D)} = (f_n, R_{\mu_n}\phi)_{L^2(D)} \to (f, R_\mu \phi)_{L^2(D)} = (R_\mu f, \phi)_{L^2(D)}.$$

Therefore $R_{\mu_n} f_n \to R_\mu f$ strongly in $L^2(D)$ and we get that

$$|R_{\mu_n} f_n - R_\mu f_n|_{\mathcal{L}(L^2(D))} \to 0$$

as required. ■

Corollary 6.1.8 *Let $\mu_n \in \mathcal{M}_0(D)$, and $\mu_n \xrightarrow{\gamma} \mu$. Then, for every $k \ge 1$, $\lambda_k(\mu_n) \to \lambda_k(\mu)$ and the following uniform estimate holds:*

$$\left| \frac{1}{\lambda_k(\mu_n)} - \frac{1}{\lambda_k(\mu)} \right| \le |R_{\mu_n} - R_\mu|_{\mathcal{L}(L^2(D))}.$$

Proof We apply Theorem 6.1.3 and Proposition 6.1.7 to R_{μ_n} and R_μ. ■

Remark 6.1.9 Let A_n be a sequence of uniformly bounded open sets such that $A_n \xrightarrow{H^c} A$. If all A_n belong to one of the classes introduced in Section 4.5, i.e., $\mathcal{A}_{convex}$, $\mathcal{A}_{unif\ cone}$, $\mathcal{A}_{unif\ flat\ cone}$, $\mathcal{A}_{cap\ density}$, $\mathcal{A}_{unif\ Wiener}$ or, in two dimensions of space, the number of connected components of $\mathbb{R}^2 \setminus A_n$ is uniformly bounded, then for every $k \ge 1$ we have $\lambda_k(A_n) \to \lambda_k(A)$ and the uniform estimate of Corollary 6.1.8 holds. Indeed, in these cases the measures $\mu_n = \infty_{\mathbb{R}^N \setminus A_n}$ γ-converge to the measure $\mu = \infty_{\mathbb{R}^N \setminus A}$, as seen in Chapter 4.

Remark 6.1.10 Let A_n be a sequence of uniformly bounded open (or quasi-open) sets and A another open (or quasi-open) set.

If the first Mosco condition **M1** holds (see the definition in Chapter 4), then for every $k \ge 1$,

$$\limsup_{n\to\infty} \lambda_k(A_n) \le \lambda_k(A).$$

If the second Mosco condition **M2** holds, then for every $k \ge 1$,

$$\liminf_{n\to\infty} \lambda_k(A_n) \ge \lambda_k(A).$$

For open sets, if $A_n \xrightarrow{H^c} A$, then the first Mosco condition holds without any further assumption on the sequence $(A_n)_n$. In other words, the eigenvalues $\lambda_k(\cdot)$ are upper semicontinuous with respect to the Hausdorff complementary convergence H^c.

Remark 6.1.11 If the design region D is unbounded, for example $D = \mathbb{R}^N$, Proposition 6.1.7 fails to be true. The γ-convergence of measures implies only pointwise convergence of the resolvent operators and, in general, no convergence of eigenvalues. For example, take the sequence of open sets $A_n = B(x_n, 1)$ where $|x_n| \to +\infty$. Then R_{A_n} converges pointwise to 0, while the eigenvalues of A_n do not change with n. We refer to Section 6.5 of this chapter for a detailed discussion of the case $D = \mathbb{R}^N$.

Remark 6.1.12 To conclude this section, we recall that finer behavior of the eigenvalues can be observed as soon as the variation of the domain is more particular. For example, if A is smooth and A_t is the deformation of A by a smooth vector field V, i.e., $A_t = (\mathrm{Id} + tV)A$, one gets even differentiability of the mapping $t \mapsto \lambda_k(A_t)$, provided that $\lambda_k(A)$ is simple. We refer the reader to [85], [140], [186].

Moreover, the asymptotic behaviour of $\lambda_k(A_\varepsilon)$ as $\varepsilon \to 0$ can be obtained in the case $A_\varepsilon = A \setminus \overline{B}(x_0, \varepsilon)$, provided that $\lambda_k(A)$ is simple, and $x_0 \in A$ (see [174], [175]).

6.2 Setting the optimization problem

Minimization problems for eigenvalues are not new in the literature; since the first result by Krahn [150], [151] and Faber [119] concerning the minimality of a disk in $\mathbb{R}^2$ for the first eigenvalue of the Laplace operator $-\Delta$ with Dirichlet boundary conditions, among domains with equal area, many other results have been obtained.

Denoting by B_1 the ball of measure c, and by B_2 the union of two disjoint balls of measure $c/2$, for every bounded open set Ω of measure c the following inequalities hold:

- $\lambda_1(\Omega) \geq \lambda_1(B_1)$ (proved by Faber [119] and Krahn [150], [151]);
- $\lambda_2(\Omega) \geq \lambda_2(B_2) = \lambda_1(B_2)$ (we refer to Krahn [151]; see also [177] for a proof by P. Szegö);
- $\lambda_2(B_1)/\lambda_1(B_1) \geq \lambda_2(\Omega)/\lambda_1(\Omega) \geq 1$ (proved by Ashbaugh and Benguria [17]);

Notice that by density arguments and γ-continuity properties, in the previous inequalities the set Ω can be replaced by an arbitrary (not necessarily bounded) quasi-open set of measure less than or equal to c.

We are concerned with problems of the form

$$\min\{\Phi(\lambda(\Omega)) \ : \ |\Omega| \leq c,\ \Omega \in \mathcal{A}\} \tag{6.2}$$

where $\lambda(\Omega)$ denotes the sequence $(\lambda_j(\Omega))_j$ of all eigenvalues of the Laplace operator with Dirichlet boundary conditions on $\partial\Omega$, $\Phi : \mathbb{R}^{\mathbb{N}} \to [0, +\infty]$ is a given function, $c > 0$ is given and $\mathcal{A}$ is the class of admissible domains. A choice we can make is to take $\mathcal{A} = \{\Omega \subseteq D\}$ where the domain D (the so-called "*design region*") is either a given, bounded open subset of $\mathbb{R}^N$, or $D = \mathbb{R}^N$. We shall see that the two cases are quite different, and most of the results obtained for the first cannot be easily extended to the second.

Following Theorem 5.4.1, a general existence result can be adapted for functionals depending on eigenvalues. Let us note that the first assumption of Theorem 5.4.1 is verified for a large number of situations; for instance, by Proposition 6.1.7 and Corollary 6.1.8 we have (see also Buttazzo and Dal Maso [66])

$$\Omega_n \xrightarrow{\gamma} \Omega \qquad \Longrightarrow \qquad \lambda_j(\Omega_n) \to \lambda_j(\Omega) \quad \text{for all} \quad j \in \mathbb{N}.$$

Therefore the cost functions

$$F(\Omega) = \Phi(\lambda(\Omega))$$

are γ-lower semicontinuous as soon as the function $\Phi : \mathbb{R}^{\mathbb{N}} \to [0, +\infty]$ is lower semicontinuous, in the sense that

$$z_j^n \to z_j \quad \text{for all } j \in \mathbb{N} \quad \Longrightarrow \quad \Phi(z) \le \liminf_{n\to\infty} \Phi(z^n). \tag{6.3}$$

The monotonicity assumption is, on the contrary, much more restrictive. However, in the case of problems involving eigenvalues, of the form (6.2), Theorem 5.4.1 includes the case of functions Φ which are monotone increasing, that is

$$z_j^1 \le z_j^2 \ \text{ for all } \ j \in \mathbb{N} \quad \Longrightarrow \quad \Phi(z^1) \le \Phi(z^2). \tag{6.4}$$

This relies on the well-known fact that all the eigenvalues of an elliptic operator with Dirichlet boundary conditions are decreasing functions of the domain (see Proposition 6.1.5 and Corollary 6.1.6). Therefore, summarizing, from Theorem 5.4.1 we can deduce the following result.

Corollary 6.2.1 *Let $\Phi : \mathbb{R}^{\mathbb{N}} \to [0, +\infty]$ be a function which is lower semicontinuous in the sense of (6.3), and monotone increasing in the sense of (6.4). Assume also that the design region D is bounded and take $\mathcal{A} = \{\Omega \subseteq D,\ \Omega$ quasi-open$\}$. Then the optimization problem*

$$\min\{\Phi(\lambda(\Omega)) \ : \ |\Omega| = c,\ \Omega \in \mathcal{A}\}$$

admits at least a solution in $\mathcal{A}$.

In this chapter, we discuss two directions in which we weaken the hypotheses of the previous corollary. We consider *non-monotone functionals* Φ and *unbounded design regions*.

6.3 A short survey on continuous Steiner symmetrization

An important tool often used in shape optimization is the continuous Steiner symmetrization, often denoted by CSS (see Brock [39]).

For $N \geq 1$ denote by $M(\mathbb{R}^N)$ the class of Lebesgue measurable subsets of $\mathbb{R}^N$. Let us begin by the one-dimensional case. A family of mappings

$$E_t : M(\mathbb{R}) \to M(\mathbb{R}),\ 0 \leq t \leq +\infty,$$

is called a continuous symmetrization if it satisfies:

1. $m(E_t(M)) = m(M)$;
2. If $M \subseteq N$, then $E_t(M) \subseteq E_t(N)$;
3. $E_t(E_s(M)) = E_{s+t}(M)$;
4. If $I = [a, b]$ is a closed interval, then $E_t(I) = [a^t, b^t]$ where

$$a^t = \frac{1}{2}(a - b + e^{-t}(a+b)),$$
$$b^t = \frac{1}{2}(b - a + e^{-t}(a+b)).$$

In [39] the existence of such a family is proved. In the sequel, we will denote by $M^t = E_t(M)$, the continuous symmetrization of a set M. If M is a finite union of intervals, $M = (a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_k, b_k)$ such that $a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k$, then for t "small" we have

$$M^t = (a_1^t, b_1^t) \cup (a_2^t, b_2^t) \cup \cdots \cup (a_k^t, b_k^t),$$

where a^t, b^t are defined by the rule 4) above. There exists a first moment $t_0 > 0$ such that two intervals meet, i.e., $b_s^{t_0} = a_{s+1}^{t_0}$. In this case, we define the set

$$N = (a_1^{t_0}, b_1^{t_0}) \cup (a_2^{t_0}, b_2^{t_0}) \cup \cdots \cup (a_s^{t_0}, b_{s+1}^{t_0}) \cdots \cup (a_k^{t_0}, b_k^{t_0})$$

as union of $k-1$ intervals. For $t > 0$ the set M^{t_0+t} is defined as N^t up to the moment when two intervals of N^t meet. Then, the same procedure is continued. Since at each step the number of intervals diminishes by one, at some moment we get only one interval, and from this time on, the set M^t can be defined by the rule 4) above. An arbitrary open set is decomposed in a countable union of intervals, and the symmetrization is defined by interior approximation. Every measurable set is approached by open sets, the symmetrization being defined by exterior approximation.

For $N \geq 2$, the CSS is defined with respect to a hyperplane. For example, let us suppose that $\mathcal{H}$ is the hyperplane defined by $\mathcal{H} = \{x_N = 0\}$. If $M \subseteq \mathbb{R}^N$ is a polyhedron with all its faces parallel to the coordinate hyperplanes, then by definition

$$M^t = \left\{ \bigcup (M \cap l_{(x',0)})^t : x' \in \mathbb{R}^{N-1} \right\}.$$

Here $x = (x', 0)$ and l_x denotes the line orthogonal to $\mathcal{H}$ passing through the point x; the set $(M \cap l_{(x',0)})^t$ is the one-dimensional continuous symmetrization of $M \cap l_{(x',0)}$ introduced above. For an open set, the CSS is defined by interior approximation with a sequence of polyhedra and for a measurable set the CSS is defined by exterior approximation with open sets. It is possible to see that the symmetrized sets A^t are open whenever the set A is open.

For a bounded quasi-open set A, the previous construction only provides a measurable set defined up to a set of zero Lebesgue measure. On the contrary, for our purposes we need that the symmetrized sets be still quasi-open and defined quasi-everywhere. For this reason it is convenient, for a bounded quasi-open set A, to define (by an abuse of notation) the symmetrized set A^t in the following way: consider a decreasing sequence of bounded open sets $(A_n)_{n\in\mathbb{N}}$ with $\operatorname{cap}(A_n \setminus A) \to 0$ and $A \subseteq A_n$. For any $t \in [0, 1]$ the set A_n^t is well defined, and by monotonicity we define $A_n^t \supseteq A_{n+1}^t$. Then $(A_n^t)_{n\in\mathbb{N}}$ is γ-convergent and we define

$$A^t = \gamma - \lim_{n\to\infty} A_n^t.$$

In this way, the set A^t is quasi-open and we do not know if it is independent of the sequence A_n and if the Lebesgue measure is preserved. However, for our purposes this definition of A^t is convenient, as we shall see in Section 6.4.

For any positive measurable function u we define the continuous Steiner symmetrization of u by symmetrizing its level sets:

$$\forall s > 0 \qquad \{u^t > s\} := \{u > s\}^t.$$

In [39] the following is proved.

Theorem 6.3.1 *Let $u \in H^1(\mathbb{R}^N)$, $u \geq 0$. Then $u^t \in H^1(\mathbb{R}^N)$, $\|u\|_{L^2} = \|u^t\|_{L^2}$ and $\|u\|_{H^1} \geq \|u^t\|_{H^1}$. Moreover, if Ω is an open set and $u \in H_0^1(\Omega)$, $u \geq 0$, then $u^t \in H_0^1(\Omega^t)$.*

For other properties concerning the continuous Steiner symmetrization we refer to [39].

We recall from [50] some useful results (without proofs). Consider a measurable set A and a hyperplane $\mathcal{H} \subseteq \mathbb{R}^N$. For $t \in [0, 1]$ denote by A^t the Steiner symmetrization of A at time t in the orthogonal direction to $\mathcal{H}$.

Proposition 6.3.2 *For every bounded quasi-open set $A \subseteq \mathbb{R}^N$ and every positive integer i, the mapping $t \mapsto \lambda_i(A^t)$, is lower semicontinuous on the left and upper semicontinuous on the right.*

For a compact set $K \in \mathbb{R}^N$ the existence is known of a sequence of hyperplanes $(\mathcal{H}_n)_{n\in\mathbb{N}}$ such that denoting $K_0 = K$ and K_n the symmetrization of K_{n-1} with respect to $\mathcal{H}_n$ we have that $m(K_n \Delta K^\#) \to 0$, where by $C^\#$ we denote the closed ball of measure $m(C)$ (see [36]). If the convergence in measure is replaced by the Hausdorff convergence, a similar type of result can be found in the book of Federer [120].

For quasi-open sets we can formulate the following.

Proposition 6.3.3 *Let A be a bounded quasi-open set of $\mathbb{R}^N$. There exists a sequence of Steiner symmetrizations of A, denoted by $(A_n)_{n\in\mathbb{N}}$, such that $m(A_n \setminus A^\#) \to 0$ as $n \to \infty$.*

Proof This result appears to be weaker than the similar one for compact sets, but nevertheless it is still sufficient for our forthcoming purposes.

Suppose first that A is open. Consider $K_1 \subset\subset A$ such that $m(A \setminus K_1) \le \varepsilon_1/2$. We perform a finite number of Steiner symmetrizations given by the result of [36] for K_1 such that

$$m\Big((K_1)_{n_1} \Delta K_1^\#\Big) \le \frac{\varepsilon_1}{2}.$$

Then, by monotonicity,

$$m\Big(A_{n_1} \setminus A^\#\Big) \le \frac{\varepsilon_1}{2} + \frac{\varepsilon_1}{2} = \varepsilon_1.$$

Choosing now another set $K_2 \subset\subset A_{n_1}$ with $m(A_{n_1} \setminus K_2) \le \varepsilon_2/2$ we continue the process and obtain

$$m\Big((K_2)_{n_2} \setminus K_2^\#\Big) \le \frac{\varepsilon_2}{2}$$

and so on. Choosing a sequence $\varepsilon_n \to 0$ we conclude the proof in the case of open sets.

If A is quasi-open, consider a sequence of bounded open sets $(C_r)_{r\in\mathbb{N}}$ such that

$$A \subseteq C_{r+1} \subseteq C_r$$

and $\operatorname{cap}(C_r \setminus A) \to 0$, and we apply the previous result to C_r. We make a finite number of symmetrizations to C_1 such that

$$m\Big((C_1)_{n_1} \setminus C_1^\#\Big) \le \varepsilon_1.$$

Then $m(A_{n_1} \setminus C_1^\#) \le \varepsilon_1$. Making now a finite number of symmetrizations for C_2 we get $m((C_2)_{n_2} \setminus C_2^\#) \le \varepsilon_2$, and so on. Finally we get $m(A_n \setminus A^\#) \to 0$, since $m(C_n^\# \Delta A^\#) \to 0$. ■

Corollary 6.3.4 *For every bounded quasi-open set $A \subseteq \mathbb{R}^N$ there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of successive Steiner symmetrizations of A such that any weak γ-limit point of $(A_n)_{n\in\mathbb{N}}$ is contained in $A^\#$.*

Proof Indeed, from the previous proposition we have $m(A_n \setminus A^\#) \to 0$. If U is the weak γ-limit of $\{A_{n_k}\}$, then $w_{n_k} \rightharpoonup w$ weakly in $H_0^1(B)$ and $U = \{w > 0\}$. Since $m(A_n \setminus A^\#) \to 0$, and $w_{n_k} \to w$ in $L^2(B)$ we get $w = 0$ a.e. on $\mathbb{R}^N \setminus A^\#$, hence $w \in H_0^1(A^\#)$, which means $U \subseteq A^\#$. ■

Corollary 6.3.5 *For the sequence $(A_n)_{n\in\mathbb{N}}$ given by Corollary 6.3.4 we have*

$$\lambda_k(A^\#) \le \liminf_{n\to\infty} \lambda_k(A_n).$$

6.4 The case of the first two eigenvalues of the Laplace operator

Let $B \subseteq \mathbb{R}^N$ be a fixed ball and let $c > 0$ be a positive number. We denote by

$$\mathcal{A}_c(B) = \Big\{ A \subseteq B \ : \ A \text{ quasi-open}, \ m(A) \le c \Big\}$$

the family of all quasi-open subsets of B having Lebesgue measure less than or equal to c and by $s : \mathcal{A}_c(B) \to \mathbb{R}^2$ the *spot* function defined by

$$s(A) = (\lambda_1(A), \lambda_2(A)),$$

where $\lambda_1(A), \lambda_2(A)$ are the first two eigenvalues (counted with their multiplicity) of the Laplace operator $-\Delta$ on the Sobolev space $H_0^1(A)$.

The purpose of this section is to prove that the range of s is closed in $\mathbb{R}^2$, if for a given c, the ball B is large enough (for more details see [45]). This will immediately imply the existence of a solution for problems of the form

$$\min \Big\{ \Phi(\lambda_1(A), \lambda_2(A)) : \quad A \in \mathcal{A}_c(B) \Big\}$$

for a large class of cost functions Φ (in particular no monotonicity will be required).

Let us denote by $E = s(\mathcal{A}_c(B))$ the image of s in $\mathbb{R}^2$. The set E is conical with respect to the origin, that is $(tx, ty) \in E$ whenever $(x, y) \in E$ and $t \ge 1$. This is easily seen by considering the homothetical sets $A_t = A/\sqrt{t}$, where A is such that $s(A) = (x, y)$. Moreover, following the results of Faber and Krahn, and of Ashbaugh and Benguria, recalled at the beginning of Section 6.2, one has already an idea where the set E lies. Indeed, we have for all $(x, y) \in E$,

- $x \ge \lambda_1(B_c)$ with B_c being the ball of measure c;
- $y \ge \lambda_1(B_{c/2})$ with B_c being the ball of measure $c/2$;
- $1 \le y/x \le \lambda_2(B_c)/\lambda_1(B_c)$.

For a numerical study of the set E in the case $N = 2$ we refer the interested reader to the paper by Wolf and Keller [195] where the following picture for E is obtained.

Unfortunately, we are not able to prove the convexity of the set E, which Figure 6.1 seems to show; this would imply the closure result quite straightforward. However, we can prove that

- E is convex horizontally, that is for every $(x, y) \in E$ we have $\big((1-t)x + ty, y\big) \in E$ for every $t \in [0, 1]$,
- E is convex vertically, that is for every $(x, y) \in E$ we have $\big(x, (1-t)y + tx\lambda_2(B_c)/\lambda_1(B_c)\big) \in E$ for every $t \in [0, 1]$,

and this is enough to imply that the set E is closed.

Let $c > 0$ be given and let $B \subseteq \mathbb{R}^N$ be a ball large enough to contain two disjoint balls of mass $c/2$. We shall prove the following result.

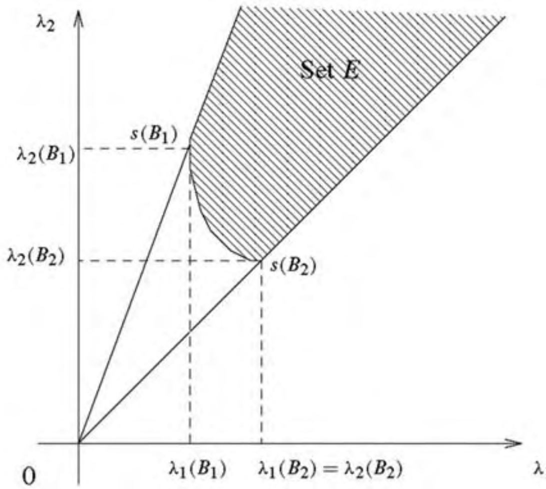

Figure 6.1. The set E for $N = 2$ and $c = 1$.

Theorem 6.4.1 *The set E is closed in $\mathbb{R}^2$.*

The proof of the theorem above is based on the following lemma.

Lemma 6.4.2 *If the set E is convex on the vertical and horizontal directions, then E is closed in $\mathbb{R}^2$.*

Proof Consider $(x, y) \in \bar{E}$. There exists a sequence of sets $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}_c(B)$ such that $s(A_n) \to (x, y)$. From the weak γ-compactness of the set $\mathcal{A}_c(B)$, for a subsequence still denoted by the same indices we can write $A_n \to A$ in the weak γ-sense. Then $A \in \mathcal{A}_c(B)$ and since the eigenvalues of the Laplacian are weakly γ-lower semicontinuous we get

$$\lambda_1(A) \le \liminf_{n\to\infty} \lambda_1(A_n) = x \quad \text{and} \quad \lambda_2(A) \le \liminf_{n\to\infty} \lambda_2(A_n) = y.$$

From the vertical convexity of E, the vertical segment joining $s(A)$ with the half line $\{ts(B_1) : t \ge 1\}$ is contained in E. If $y < \lambda_2(B_1)$ we can find the point $(\lambda_1(A), y)$ on this segment and using now the horizontal convexity, the segment joining $(\lambda_1(A), y)$ to $\{ts(B_2) : t \ge 1\}$ is in E. But this segment contains the point (x, y) since $\lambda_1(A) \le x$.

If $y \ge \lambda_2(B_1)$, then the horizontal convexity gives directly $(x, y) \in E$. ■

Following Lemma 6.4.2 it suffices to prove the convexity of E on vertical and horizontal directions. For this purpose, we split the proof in two steps:

Step 1 E is convex on horizontal lines, namely if $A \in \mathcal{A}_c(B)$, then the segment joining $(\lambda_1(A), \lambda_2(A))$ to $(\lambda_2(A), \lambda_2(A))$ is contained in E.

Step 2 E is convex on vertical lines, namely if $A \in \mathcal{A}_c(B)$, then the segment joining $(\lambda_1(A), \lambda_2(A))$ to $(\lambda_1(A), \frac{\lambda_2(B_1)}{\lambda_1(B_1)}\lambda_1(A))$ is contained in E.

In the proof of Step 2, the idea is to make a sequence of CSS to transform a given quasi-open set $A \in \mathcal{A}_c(B)$ into a ball. Here, one can see that the choice of B is important since if $A \in \mathcal{A}_c(B)$ for any hyperplane $\mathcal{H}$ we still have $A^t \in \mathcal{A}_c(B)$.

We proceed now with the proofs of Steps 1 and 2. It is convenient to indicate by d_1 the half line $\{ts(B_1) : t \geq 1\}$ and by d_2 the half line $\{ts(B_2) : t \geq 1\} = \{(x, x) \in \mathbb{R}^2 : x \geq \lambda_1(B_2)\}$.

We give first a general result which establishes the existence of a γ-continuous and decreasing homotopy between two quasi-open sets $A_1 \subseteq A_0$.

Proposition 6.4.3 *Let $A_1 \subseteq A_0$ be two quasi-open sets. There exists a γ-continuous mapping $h : [0, 1] \to \mathcal{A}(\mathbb{R}^N)$ such that for $t_1 < t_2$, $h(t_1) \supseteq h(t_2)$ and $h(0) = A_0$, $h(1) = A_1$.*

Proof Denote by K a closed cube containing A_0. We shall divide the cube in 2^N equal closed cubes $K_0, \dots, K_{2^N-1}$; each cube K_i is analogously divided in 2^N closed cubes $K_{i0}, \dots, K_{i2^N-1}$, and so on. Then to each real number $t \in [0, 1]$ written in the 2^N-basis by $0.\alpha_1\alpha_2\dots$ we associate the set

$$\Lambda_t = (A_0 \setminus F_t) \cup A_1,$$

where

$$F_t = \cup_{n=1}^{\infty} \cup_{i=0}^{\alpha_n-1} K_{\alpha_1\dots\alpha_{n-1}i}.$$

Remark first that Λ_t is quasi-open since F_t is quasi-closed. Indeed, let's denote by

$$F_{t,k} = \cup_{n=1}^{k} \cup_{i=0}^{\alpha_n-1} K_{\alpha_1\dots\alpha_{n-1}i}$$

the closed set consisting of the first k-blocks of F_t. Set also

$$\Lambda_{t,k} = (A_0 \setminus F_{t,k}) \cup A_1$$

which is obviously quasi-open, and remark that

$$\cap_{k\geq 1}\Lambda_{t,k} = \Lambda_t.$$

Since $\text{cap}(\Lambda_{t,k} \setminus \Lambda_t) \to 0$ for $k \to \infty$ we get that Λ_t is quasi-open. Moreover, the mapping $t \to \Lambda_t$ is continuous in capacity. Indeed, fix $t \in [0, 1]$ and consider $t_n \to t$. We have to distinguish two situations: either t has an infinite number of digits and is not finishing by $aa\dots aa\dots$, or t has a finite number of digits or it finishes by $aa\dots aa\dots$ (by a it is denoted the greatest digit in the basis 2^N, namely $a = 2^N - 1$). In the first case, if $t_n \to t$, then for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$

such that for every $n \geq n_k$ the numbers t_n and t have the same first k digits. In this case

$$\operatorname{cap}(A^t \Delta A^{t_n}) \leq \operatorname{cap}(K_{\alpha_1 \ldots \alpha_k})$$

and we derive the continuity in capacity.

If t has a finite number of digits, that is $t = 0.\alpha_1\alpha_2 \ldots \alpha_k$, then the number t written as

$$t = 0.\alpha_1\alpha_2 \ldots \alpha_k 0000 \ldots$$

is identified with

$$t' = 0.\alpha_1\alpha_2 \ldots (\alpha_k - 1)aaaa \ldots$$

The difference between A^t and $A^{t'}$ is a point, hence of zero capacity. Consider $t_n \to t$. If $t_n \geq t$, the first k digits of t_n and t coincide for $n \geq n_k$. If $t_n < t$, then the first k digits of t_n and t' coincide for $n \geq n_k$ and the conclusion follows.

Since the mapping $t \mapsto \Lambda_t$ is obviously decreasing and γ-continuous, taking

$$h(t) = \Lambda_t$$

achieves the proof. ■

Proof of Step 1. Let $A \in \mathcal{A}_c(B)$. If there exists a subset A^* of A, such that $\lambda_1(A^*) = \lambda_2(A^*) = \lambda_2(A)$, then one can apply directly Proposition 6.4.3, and Step 1 is proved since there exists a decreasing and γ-continuous homotopy from A to A^*. Since $\lambda_2(A) = \lambda_2(A^*)$, then by monotonicity $\lambda_2(\Lambda_t) = \lambda_2(A)$. Since the first eigenvalue is γ-continuous, for each $\alpha \in [\lambda_1(A), \lambda_2(A)]$ there exists some t_α such that $\lambda_1(\Lambda_{t_\alpha}) = \alpha$.

Let's prove now the existence of the set A^*. Denote by φ_1, φ_2, a first and second eigenfunctions, respectively. If $\lambda_1(A) = \lambda_2(A)$, there is nothing to prove. Hence we suppose $\lambda_1(A) < \lambda_2(A)$. We give then the following lemma.

Lemma 6.4.4 *Let A be a quasi-open set such that $\lambda_1(A) < \lambda_2(A)$. Then the fine interior of A_1 is finely connected and there are two possibilities: either $A_2 \subseteq A_1$ or* $\operatorname{cap}(A_1 \cap A_2) = 0$ *for a convenient second eigenfunction φ_2.*

Proof If A is open the result is immediate. If A is quasi-open, the proof is similar and based on Lemma 4.1.4 and the following assertion (see [126]): any nonnegative superharmonic function on a finely open and connected set is either strictly positive or equal to zero. In particular, this will be the case of the first eigenfunction.

Indeed, if A_1 is not finely connected (we denoted here by A_1 its fine interior) then it can be decomposed in a union of disjoint finely connected components $(C_i)_{i \in I}$ and since $\varphi_{1|C_i} \in H_0^1(C_i) \subseteq H_0^1(A)$ we have that

$$\forall i \in I \qquad \frac{\int_{C_i} |\nabla \varphi_{1|C_i}|^2 \, dx}{\int_{C_i} |\varphi_{1|C_i}|^2 \, dx} = \lambda_1(A) \; .$$

So, if I contains at least two indices this would mean that $\lambda_1(A)$ is at least double, since we have two independent eigenfunctions (defined by the restriction of φ_1 on each set). Therefore A_1 has only one finely connected component.

Suppose now that $\mathrm{cap}(A_1 \cap A_2) \neq 0$. Decomposing $A_2 = \cup_{i \in I} C'_i$, C'_i being finely connected, then for any component for which $\mathrm{cap}(A_1 \cap C'_i) \neq 0$ we have $C'_i \subseteq A_1$; indeed, otherwise $C'_i \cup A_1$ would be finely connected and u_1 could not vanish on $C'_i \setminus A_1$. So the finely connected components of A_2 are of two types: $C'_i \subseteq A_1$ or $\mathrm{cap}(C'_j \cap A_1) = 0$. In this case we can see that $\varphi_2\big|_{\cup C'_i}$ and $u_2\big|_{\cup C'_j}$ are both orthogonal to φ_1 and they are still second eigenfunctions. Then A_2 can be chosen as $\cup C'_i$ or $\cup C'_j$. ■

Proof of Theorem 6.4.1 (continuation). From Lemma 6.4.4, we have two possibilities: either $A_2 = \{\varphi_2 \neq 0\} \subseteq \{\varphi_1 > 0\} = A_1$ and in this case $A^* = A_2$, or $\mathrm{cap}(A_1 \cap A_2) = 0$ and denoting by P_t the open half space

$$P_t = \{(x_1, \dots, x_N) \in \mathbb{R}^N : t < x_1\}$$

there exists some $t_0 \in \mathbb{R}$ such that $\lambda_1(A_1 \cap P_{t_0}) = \lambda_2(A)$. Choosing

$$A^* = (A_1 \cap P_{t_0}) \cup A_2$$

the conclusion follows. ■

Proof of Step 2. Let's consider $A \in \mathcal{A}_c(B)$ and denote by $A^\#$ the closed ball of measure $m(A)$. The idea to prove the convexity in the vertical direction is to make a sequence of continuous Steiner symmetrizations transforming A, such that $m(A_n \setminus A^\#) \to 0$, and to use the horizontal convexity. If $\lambda_2(A) \geq \lambda_2(A^\#)$, then the segment

$$\Big\{(\lambda_1(A), \gamma) \ : \quad \gamma \in [\lambda_2(A), \lambda_1(A)\frac{\lambda_2(B_1)}{\lambda_1(B_1)}]\Big\}$$

is contained in E. This follows immediately from the convexity on the horizontal lines since all the half line supported by d_1 and having B_1 as extreme point is in E.

So let's suppose $\lambda_2(A) < \lambda_2(A^\#)$, and choose $\alpha \in]\lambda_2(A), \lambda_2(A^\#)[$. We intend to prove that $(\lambda_1(A), \alpha) \in E$. We use Corollary 6.3.4 and we find a sequence of continuous Steiner symmetrizations $(A_n)_{n \in \mathbb{N}}$ such that

$$\liminf_{n \to \infty} \lambda_2(A_n) \geq \lambda_2(A^\#).$$

In order to underline the evolution of the set A "toward" the ball, we say that the CSS from A_n to A_{n+1} is parametrized by $t \in [n, n+1]$, by simple translation of the interval [0, 1]. In this way we can define the set A^t for every $t \geq 0$, and the set A_n can also be written as A^n.

On the other hand, $\lambda_1(A_n) \leq \lambda_1(A)$. There exists some $n_0 \in \mathbb{N}$ such that $\lambda_2(A_{n_0}) \geq \alpha$ and denote

$$t^* = \sup\Big\{t \in [0, n_0] \ : \quad \lambda_2(A^t) \leq \alpha\Big\}.$$

From the upper semicontinuity on the right we have $\lambda_2(A^{t^*}) \geq \alpha$ and from the lower semicontinuity on the left we get $\lambda_2(A^{t^*}) \leq \alpha$ which give $\lambda_2(A^{t^*}) = \alpha$.

Using now the convexity on the horizontal lines, the segment joining $(\lambda_1(A^{t^*}), \alpha)$ with $(\alpha, \alpha) \in d_2$ is contained in E. But since $\lambda_1(A^{t^*}) \leq \lambda_1(A)$ the point $(\lambda_1(A), \alpha)$ belongs to E. ■

We remark that the previous result can be applied to prove the existence of solutions for some classes of shape optimization problems, for which the shape functional is not monotone with respect to the set inclusion (see [66]). We can consider problems of the form

$$\min\Big\{\Phi(\lambda_1(A), \lambda_2(A)) : \quad A \in \mathcal{A}_c(B)\Big\} \tag{6.5}$$

where $\Phi : E \to \overline{\mathbb{R}}$ is lower semi-continuous and goes to $+\infty$ at infinity. This is the case for instance of

$$\Phi(x, y) = (x - \alpha)^2 + (y - \beta)^2$$

where (α, β) is any element in $\mathbb{R}^2$. Therefore by Theorem 6.4.1 the minimization problem (6.5) admits at least a solution.

A typical example of functional Φ for which problem (6.5) does not have a solution is $\Phi(x, y) = x - y$.

Remark 6.4.5 Actually, the minimization problem (6.5) admits a solution under the sole assumption that $\Phi : E \to [0, +\infty]$ is lower semicontinuous. Indeed, take a minimizing sequence (x_n, y_n) in E, and assume that it is bounded; then the direct methods of the calculus of variations give the existence of an optimal domain, thanks to Theorem 6.4.1. Otherwise, one sequence between (x_n) and (y_n) goes to $+\infty$. Then, due to the shape of the set E, both (x_n) and (y_n) go to $+\infty$. In this case, it is easy to see that taking as A the empty set we obtain an element of $\mathcal{A}_c(B)$ which solves the optimization problem (6.5).

Remark 6.4.6 An interesting question is to study the boundary of the set E. We can provide only some information on it. For a set $A \in \mathcal{A}_c(B)$ we denote by $\mathcal{R}_{\inf}(A)$ the rectangle

$$\mathcal{R}_{\inf}(A) = \{(x, y) \in \mathbb{R}^2 \ : \ x \leq \lambda_1(A), \ \ y \leq \lambda_2(A)\}.$$

For every $A \in \mathcal{A}_c(B)$, there exists a set $\tilde{A} \in \mathcal{A}_c(B)$ which is either finely connected or coincides with two balls, such that $s(\tilde{A}) \in \mathcal{R}_{\inf}(A)$.

Indeed, let us fix $A \in \mathcal{A}_c(B)$ and set $A_1 = \{\varphi_1 > 0\}$ and $A_2 = \{\varphi_2 \neq 0\}$. If $\lambda_1(A) = \lambda_2(A)$ the assertion is obvious since $s(A) \in d_2$. If $\lambda_1(A) < \lambda_2(A)$ there are two possibilities. If $A_2 \subseteq A_1$, then A_1 is finely connected and $s(A_1) \in \mathcal{R}_{\inf}(A)$. If $\mathrm{cap}(A_2 \cap A_1) = 0$ we make the Schwarz rearrangements of A_1 and A_2 into the disjoint balls C_1 and C_2, and we get $s(C_1 \cup C_2) \in \mathcal{R}_{\inf}(A)$ and $C_1 \cup C_2 \in \mathcal{A}_c(B)$.

The argument above gives that any A whose $s(A)$ is on $\partial E \setminus (d_1 \cup d_2)$ is either finely connected or two balls. An open question is to study if these sets are simply connected and regular.

6.5 Unbounded design regions

The aim of this section is to give a variational method for proving global existence results for shape optimization problems depending on eigenvalues, if the design region is the entire space $\mathbb{R}^N$. In order to develop the method presented in Sections 6.2 and 6.4 to unbounded design regions, one has to relate the γ-convergence to the concentration-compactness principle, since the injection $H^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ fails to be compact, and the γ-convergence (which is still compact) does not yield the convergence of the eigenvalues. A detailed description of this subject can be found in [43]. To begin, let us give an example of a typical behavior in unbounded design regions.

Example 6.5.1 Let us consider the problem of minimizing $\lambda_1(\Omega)$ among all open subsets of $\mathbb{R}^N$ of measure c. Of course, the ball $B(0, R)$ is a solution (for the suitable value of R), but from a variational point of view one might search a minimizing sequence. An example of such minimizing sequence is

$$\Omega_n = B(x_n, R),$$

where $|x_n| \to +\infty$. Note that this sequence can not lead to a direct construction by means of *geometric convergence* of the optimal set, unless it is subject to some transformations (see also Remark 6.1.11).

For functions, this kind of behavior is understood via the concentration-compactness principle which describes the behavior in $L^2(\mathbb{R}^N)$ of a bounded sequence $(u_n)_{n\in\mathbb{N}}$ of $H^1(\mathbb{R}^N)$ (see [156]). More precisely, three situations may occur for a subsequence: compactness (possibly making some translations), vanishing or dichotomy.

Given a sequence of open (or quasi-open) sets $(A_n)_{n\in\mathbb{N}}$ of $\mathbb{R}^N$, with uniformly bounded measure ($|A_n| \le c$, for all $n \in \mathbb{N}$), a natural question is to see whether *all* bounded sequences $(u_n)_{n\in\mathbb{N}}$ of $H^1(\mathbb{R}^N)$, such that u_n belongs to $H_0^1(A_n)$ for every $n \in \mathbb{N}$, have the *same* behavior in $L^2(\mathbb{R}^N)$ with respect to the concentration-compactness principle. This is particularly important from the point of view of shape optimization problems. It is of interest to know whether for a suitable sequence $(y_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^N$ the injection

$$\bigcup_{n\in\mathbb{N}} H_0^1(y_n + A_n) \hookrightarrow L^2(\mathbb{R}^N) \tag{6.6}$$

is compact, i.e., a bounded subset of $\bigcup_{n\in\mathbb{N}} H_0^1(y_n + A_n)$ for the $H^1(\mathbb{R}^N)$-norm is relatively compact in $L^2(\mathbb{R}^N)$ (by $y_n + A_n$, one denotes the translation of A_n by the vector y_n). Notice that if A is a quasi-open set of finite measure, then $H_0^1(A)$ is compactly embedded in $L^2(A)$ (see [194]).

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of open (or quasi-open) subsets of $\mathbb{R}^N$. It is said that $(A_n)_{n\in\mathbb{N}}$ γ-converges to a measure μ if for any bounded open set Ω, the sequence of functionals

$$F_n(u,\Omega) = \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \chi_{H_0^1(A_n\cap\Omega)}(u)$$

Γ-converges to

$$F(u,\Omega) = \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} u^2\,d\mu + \chi_{H_0^1(\Omega)}(u)$$

in $L^2(\mathbb{R}^N)$. We will denote

$$F_n(u) = \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \chi_{H_0^1(A_n)}(u)$$

and

$$F(u) = \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} u^2\,d\mu.$$

Following [24], [100] the γ-convergence is sequentially compact in $\mathbb{R}^N$. Although several results are valid without any further assumption, in all this section we will work only with quasi-open sets with finite Lebesgue measure and with measures μ for which the regular set A_μ has finite Lebesgue measure.

Using our previous notation and denoting by R_{A_n} the resolvent operator of the Laplace equation with homogeneous Dirichlet boundary conditions and respectively by R_μ the resolvent operator associated to the measure μ (which are well defined since $|A_n|, |A_\mu| < +\infty$), a consequence of the γ-convergence is the following *pointwise* convergence of the resolvent operators:

$$\forall f \in L^2(\mathbb{R}^N) \qquad R_{A_n}(f) \overset{L^2(\mathbb{R}^N)}{\longrightarrow} R_\mu(f). \tag{6.7}$$

If the design region is $\mathbb{R}^N$, the γ-convergence does not involve the convergence of the resolvent operators in the operator norm; in bounded design regions the convergence in the operator norm is a consequence of the compact embedding $H_0^1(D) \hookrightarrow L^2(D)$ (see Proposition 6.1.7). We prove that the injection (6.6) is compact, if and only if the convergence (6.7) is uniform in the unit ball of $L^2(\mathbb{R}^N)$. Consequently, the convergence of the resolvent operators will hold in the operator norm.

Proposition 6.5.2 *Let us suppose that $(A_n)_{n\in\mathbb{N}}$ is a sequence of quasi-open sets of uniformly bounded measure which γ-converges to μ. Then F_n Γ-converges to F both in the $L^2(\mathbb{R}^N)$-strong convergence and in the $L^2(\mathbb{R}^N)$-weak convergence.*

Proof First, one has to prove that for every sequence $u_n \rightharpoonup u$ weakly in $L^2(\mathbb{R}^N)$ we have $F(u) \le \liminf_{n\to\infty} F_n(u_n)$. Without loss of generality we assume $\liminf_{n\to\infty} F_n(u_n) < +\infty$. From the structure of the functionals we get that $u_n \in H_0^1(A_n)$ and $\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx \le M$. Moreover, we have $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$.

Take a function $\rho_R \in C_0^\infty(\mathbb{R}^N)$, with $\rho_R = 1$ on $B_{0,R}$ and $\rho_R = 0$ on $\mathbb{R}^N \setminus B_{0,2R}$. Then $\rho_R u_n \rightharpoonup \rho_R u$ weakly in $H^1(\mathbb{R}^N)$, the convergence being strong in $L^2(\mathbb{R}^N)$ (from the compact injection $H_0^1(B_{0,2R}) \hookrightarrow L^2(\mathbb{R}^N)$).

From the Γ-convergence we get

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla(\rho_R u_n)|^2\, dx \geq \int_{\mathbb{R}^N} |\nabla(\rho_R u)|^2\, dx + \int_{\mathbb{R}^N} |\rho_R u|^2\, d\mu$$

so that

$$\begin{aligned}
&\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 \rho_R^2\, dx \\
&\quad = \liminf_{n\to\infty} \int_{\mathbb{R}^N} \left[|\nabla(\rho_R u_n)|^2 - 2\rho_R u_n \nabla\rho_R \nabla u_n - u_n^2 |\nabla\rho_R|^2\right] dx \\
&\quad \geq \int_{\mathbb{R}^N} |\nabla(\rho_R u)|^2\, dx + \int_{\mathbb{R}^N} |\rho_R u|^2 d\mu \\
&\qquad -2\int_{\mathbb{R}^N} \rho_R u \nabla\rho_R \nabla u\, dx - \int_{\mathbb{R}^N} u^2 |\nabla\rho_R|^2\, dx \\
&\quad = \int_{\mathbb{R}^N} |\nabla u|^2 \rho_R^2\, dx + \int_{\mathbb{R}^N} |\rho_R u|^2\, d\mu.
\end{aligned}$$

We get for $R \to +\infty$,

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \geq \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} u^2\, d\mu.$$

Second, for every $u \in L^2(\mathbb{R}^N)$ such that $F(u) < +\infty$, there exists a sequence $u_n \in L^2(\mathbb{R}^N)$ strongly convergent in $L^2(\mathbb{R}^N)$ to u such that $F_n(u_n) \to F(u)$.

Indeed, let us consider $u \in L^2(\mathbb{R}^N)$ with $F(u) < +\infty$. The sequence $\rho_R u$ converges in $L^2(\mathbb{R}^N)$ to u for $R \to +\infty$. Moreover $F(\rho_R u) \to F(u)$ since

$$\begin{aligned}
F(\rho_R u) &= \int_{\mathbb{R}^N} |\nabla(\rho_R u)|^2\, dx + \int_{\mathbb{R}^N} |\rho_R u|^2\, d\mu \\
&= \int_{\mathbb{R}^N} |\nabla u|^2 \rho_R^2\, dx + 2\int_{\mathbb{R}^N} \nabla u \nabla\rho_R u \rho_R\, dx \\
&\qquad + \int_{\mathbb{R}^N} |\nabla\rho_R|^2 u^2\, dx + \int_{\mathbb{R}^N} |\rho_R u|^2\, d\mu.
\end{aligned}$$

Making $R \to +\infty$, from the Lebesgue dominated convergence theorem, we have

$$\int_{\mathbb{R}^N} |\nabla u|^2 \rho_R^2 dx \to \int_{\mathbb{R}^N} |\nabla u|^2 dx \quad \text{and} \quad \int_{\mathbb{R}^N} |\rho_R u|^2 d\mu \to \int_{\mathbb{R}^N} u^2 d\mu.$$

The functions ρ_R can be chosen such that $\|\nabla\rho_R\|_{L^\infty(\mathbb{R}^N)} \to 0$ (for example $\rho_R(x) = \rho_1(x/R)$). Then

$$\int_{\mathbb{R}^N} \nabla u \nabla\rho_R u \rho_R\, dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} |\nabla\rho_R|^2 u^2\, dx \to 0 \quad \text{for} \quad R \to +\infty.$$

But, for all $R > 0$ there exists a sequence $u_n^R \longrightarrow \rho_R u$ strongly in $L^2(\mathbb{R}^N)$ such that $F_n(u_n^R) = F_n(u_n^R, B_{0,2R}) \to F(\rho_R u, B_{0,2R}) = F(\rho_R u)$. Then by a diagonal construction we find a sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n \longrightarrow u$ strongly in $L^2(\mathbb{R}^N)$ and $F_n(u_n) \longrightarrow F(u)$. ■

Proposition 6.5.3 *Assume that $(A_n)_{n\in\mathbb{N}}$ is a sequence of quasi-open sets of uniformly bounded measure which γ-converges to μ and $w_{A_n} \longrightarrow w$ strongly in $L^2(\mathbb{R}^N)$. Setting $A = \{w > 0\}$, the sequence of functionals*

$$G_n(u) = \int_{\mathbb{R}^N} |\nabla u|^2\, dx + \chi_{H_0^1(A_n)}(u) - 2\int_{A_n} u dx$$

Γ-converges in $L^2(\mathbb{R}^N)$ to

$$G(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} u^2 d\mu - 2\int_A u dx.$$

Moreover, w is a minimizer for G.

Proof Let $u_n \in H_0^1(A_n)$ such that $u_n \longrightarrow u$ strongly in $L^2(\mathbb{R}^N)$. It is enough to prove that $\int_{A_n} u_n\, dx \to \int_A u dx$.

First, one gets that $u \in H_0^1(A)$ and then

$$\begin{aligned}\Big|\int_{A_n} u_n\, dx - \int_A u\, dx\Big| &\le \int_{A_n\cup A} |u_n - u| dx \\ &\le |A_n \cup A|^{1/2}\Big(\int_{A_n\cup A} |u_n - u|^2\, dx\Big)^{1/2} \to 0.\end{aligned}$$

Since $w_{A_n} \longrightarrow w$ strongly in $L^2(\mathbb{R}^N)$, w is the minimizer of G as the limit of the minimizers, hence it satisfies the equation

$$-\Delta w + \mu w = 1 \quad \text{in } H^1(\mathbb{R}^N) \cap L^2_\mu(\mathbb{R}^N),$$

and the proof is achieved. ■

As usual, we denote in the sequel $R_{A_n}(1) = w_{A_n}$.

Theorem 6.5.4 *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of open (or quasi-open) sets with uniformly bounded measure. If $w_{A_n} \longrightarrow w$ strongly in $L^2(\mathbb{R}^N)$, then for any sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n \in H_0^1(A_n)$ and $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$ we have $u_n \longrightarrow u$ strongly in $L^2(\mathbb{R}^N)$, i.e., injection (6.6) is compact.*

Proof Let us suppose $u_n \in H_0^1(A_n)$ and $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$. We prove that $\|u_n - u\|_{L^2(\mathbb{R}^N)} \to 0$, as $n \to \infty$. Performing the Fourier transform, we have for every $R > 0$,

$$\begin{aligned}\|u_n - u\|^2_{L^2(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |u_n\widehat{(y) - u}(y)|^2\, dy \\ &= \int_{|y|>R} (1 + |y|^2)^{-1}(1 + |y|^2)|u_n\widehat{(y) - u}(y)|^2\, dy \\ &\quad + \int_{|y|<R} |u_n\widehat{(y) - u}(y)|^2\, dy \\ &\le \frac{1}{1 + R^2}\|u_n - u\|^2_{H^1(\mathbb{R}^N)} + \int_{|y|<R} |u_n\widehat{(y) - u}(y)|^2\, dy.\end{aligned}$$

Let us fix $\varepsilon > 0$. Since the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$, there exists $R > 0$, such that $\frac{1}{1+R^2}\|u_n - u\|^2_{H^1(\mathbb{R}^N)} \le \varepsilon/2$ for any $n \in \mathbb{N}$. With a fixed R, it remains to prove the existence of $n = n(R,\varepsilon) \in \mathbb{N}$ such that for all $n \ge n(R,\varepsilon)$ we have

$$\int_{|y|<R} |\widehat{u_n(y) - u}(y)|^2 dy \le \frac{\varepsilon}{2}.$$

In fact, it is sufficient to prove that $\int_{|y|<R} |\widehat{u_n(y) - u}(y)|^2\, dy \to 0$, for which we use the Lebesgue dominated convergence theorem. Fix $y \in B_{0,R}$ and consider the function $g_y(x) = e^{2i\pi\langle x,y\rangle}$. Since $u_n \in H^1_0(A_n)$ and $|A_n| \le c$ we have $u_n g_y \in H^1_0(A_n)$. By definition, we have

$$\hat{u}_n(y) = \int_{A_n} u_n(x) g_y(x) dx$$

and

$$\hat{u}(y) = \int_A u(x) g_y(x) dx.$$

Therefore $|\hat{u}_n(y) - \hat{u}(y)| \to 0$, if we prove that

$$\int_{A_n} u_n(x) g_y(x) dx \to \int_A u(x) g_y(x) dx. \tag{6.8}$$

On the other hand, we have $u_n g_y \rightharpoonup u g_y$ weakly in $H^1(\mathbb{R}^N)$. This does not imply relation (6.8) immediately, since $1_{\mathbb{R}^N} \notin L^2(\mathbb{R}^N)$, but it is a consequence of Lemma 6.5.5 below. Hence applying the Lebesgue dominated convergence theorem in $B_{0,R}$ we conclude the proof. ∎

Lemma 6.5.5 *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of quasi-open sets with uniformly bounded measure which γ-converges to a measure μ, and suppose that $w_{A_n} \longrightarrow w$ in $L^2(\mathbb{R}^N)$. Then for any sequence $v_n \in H^1_0(A_n)$ such that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$ we have*

$$\int_{\mathbb{R}^N} v_n dx \to \int_{\mathbb{R}^N} v dx.$$

Proof Let us denote by A the quasi-open set $\{w > 0\}$. Then

$$\int_{A_n} \nabla w_{A_n} \nabla v_n dx = \int_{A_n} v_n dx$$

and

$$\int_A \nabla w_A \nabla v dx + \int_A w v d\mu = \int_A v dx.$$

We have the estimate

$$
\begin{aligned}
\Big| \int_{\mathbb{R}^N} & \nabla w_{A_n} \nabla v_n \, dx - \int_A \nabla w_A \nabla v \, dx + \int_A wv \, d\mu \Big| \\
&\leq \Big| \int_{\mathbb{R}^N} (\nabla w_{A_n} - \nabla \tilde{w}_{A_n}) \nabla v_n dx \Big| \\
&\qquad + \Big| \int_{\mathbb{R}^N} \nabla \tilde{w}_{A_n} \nabla v_n \, dx - \int_A \nabla w_A \nabla v \, dx - \int_A wv \, d\mu \Big|
\end{aligned}
$$

where $\tilde{w}_{A_n}$ denotes the solution of

$$
\begin{cases} -\Delta \tilde{w}_{A_n} = 1_{A_n \cap A}, \\ \tilde{w}_{A_n} \in H_0^1(A_n). \end{cases} \tag{6.9}
$$

Usual Γ-convergence arguments give that $\tilde{w}_{A_n}$ converges weakly in $H^1(\mathbb{R}^N)$ to w. But

$$
\begin{aligned}
\Big| \int_{\mathbb{R}^N} & \nabla \tilde{w}_{A_n} \nabla v_n \, dx - \int_A \nabla w_A \nabla v \, dx - \int_A wv \, d\mu \Big| \\
&= \Big| \int_{\mathbb{R}^N} 1_{A_n \cap A} v_n \, dx - \int_{\mathbb{R}^N} 1_A v \, dx \Big| \\
&= \Big| \int_{\mathbb{R}^N} 1_A v_n \, dx - \int_{\mathbb{R}^N} 1_A v \, dx \Big| \to 0.
\end{aligned}
$$

On the other side

$$
\Big| \int_{\mathbb{R}^N} (\nabla w_{A_n} - \nabla \tilde{w}_{A_n}) \nabla v_n \, dx \Big|^2 \leq \Big(\int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx\Big)\Big(\int_{\mathbb{R}^N} |\nabla w_{A_n} - \nabla \tilde{w}_{A_n}|^2 \, dx\Big).
$$

Since the first term of the product in the right hand side is bounded, it remains to prove that $\int_{\mathbb{R}^N} |\nabla w_{A_n} - \nabla \tilde{w}_{A_n}|^2 dx \to 0$. But

$$
\begin{aligned}
\int_{\mathbb{R}^N} & |\nabla w_{A_n} - \nabla \tilde{w}_{A_n}|^2 dx \\
&= \int_{\mathbb{R}^N} |\nabla w_{A_n}|^2 dx - 2 \int_{\mathbb{R}^N} \nabla w_{A_n} \nabla \tilde{w}_{A_n} dx + \int_{\mathbb{R}^N} |\nabla \tilde{w}_{A_n}|^2 dx \\
&= \int_{\mathbb{R}^N} w_{A_n} \, dx - 2 \int_{\mathbb{R}^N} \tilde{w}_{A_n} \, dx + \int_{\mathbb{R}^N} \tilde{w}_{A_n} 1_A \, dx.
\end{aligned} \tag{6.10}
$$

Since w_{A_n} converges strongly in $L^2(\mathbb{R}^N)$ to w and since w_{A_n} are uniformly bounded (from Lemma 4.5.2), we get $\int_{\mathbb{R}^N} w_{A_n} dx \to \int_{\mathbb{R}^N} w dx$. Of course, we used the fact that $|A_n|, |A| \leq c$. If we prove that $\tilde{w}_{A_n}$ converges strongly in $L^2(\mathbb{R}^N)$ to w, we get that the expression in (6.10) goes to zero.

In order to prove the strong L^2-convergence of $\tilde{w}_{A_n}$ to w, let us denote by $\tilde{w}^r_{A_n}$ the weak solution of the problem

$$
\begin{cases} -\Delta \tilde{w}^r_{A_n} = 1_{A_n \cap A \cap B_{0,r}}, \\ \tilde{w}^r_{A_n} \in H_0^1(A_n \cap B_{0,r}). \end{cases} \tag{6.11}
$$

The maximum principle yields $\tilde{w}_{A_n} \geq \tilde{w}^r_{A_n}$. But

$$\int_{\mathbb{R}^N} |\tilde{w}^r_{A_n}|^2 dx \longrightarrow \int_{\mathbb{R}^N} |w^r|^2 dx,$$

where by w_r we denoted the solution of

$$\begin{cases} -\Delta w^r + \mu w^r = 1_{A \cap B_{0,r}}, \\ \tilde{w}^r \in H^1_0(B_{0,r}) \cap L^2_\mu(B_{0,r}). \end{cases} \tag{6.12}$$

Hence

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\tilde{w}_{A_n}|^2 dx \geq \int_{\mathbb{R}^N} |w^r|^2 dx.$$

Making $r \to +\infty$ and using the fact that $w^r \to w$ strongly in $L^2(\mathbb{R}^N)$ we get

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |\tilde{w}_{A_n}|^2 dx \geq \int_{\mathbb{R}^N} w^2 dx.$$

On the other side $w_{A_n} \geq \tilde{w}_{A_n} \geq 0$, and $w_{A_n} \longrightarrow w$ strongly in $L^2(\mathbb{R}^N)$, hence

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\tilde{w}_{A_n}|^2 dx = \int_{\mathbb{R}^N} w^2 dx,$$

which gives $\tilde{w}_{A_n} \longrightarrow w$ strongly in $L^2(\mathbb{R}^N)$. ■

The following theorem is a sort of "uniform" concentration-compactness result, which is described in terms of the resolvent operators.

Theorem 6.5.6 *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of open (or quasi-open) sets with uniformly bounded measure. Then there exists a subsequence (still denoted by the same indices) such that one of the following situations occurs.*

Compactness: *there exists a sequence of vectors $(y_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^N$ and a positive Borel measure μ, vanishing on sets of zero capacity, such that $y_n + A_n$ γ-converges to the measure μ and $R_{y_n+A_n}$ converges in the uniform operator topology of $L^2(\mathbb{R}^N)$ to R_μ. Let us denote further by $\|R\|_2$ the operator norm of R.*

Dichotomy: *there exists a sequence of subsets $\tilde{A}_n \subseteq A_n$, such that*

$$\|R_{A_n} - R_{\tilde{A}_n}\|_2 \to 0, \quad \textit{and} \quad \tilde{A}_n = A^1_n \cup A^2_n$$

with $d(A^1_n, A^2_n) \to +\infty$ and $\liminf\limits_{n\to\infty} |A^i_n| > 0$ for $i = 1, 2$.

For the convenience of the reader, we also recall the concentration-compactness principle. (See [115], [144], [156].)

The concentration-compactness principle. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} u_n^2 dx \to \lambda > 0$. There exists a subsequence $(n_k)_{k\in\mathbb{N}}$ satisfying one of the following three possibilities:

i) *(compactness)* there exists $y_k \in \mathbb{R}^N$ such that

$$\forall \varepsilon > 0,\ \exists R < +\infty,\ \int_{y_k+B_{0,R}} u_{n_k}^2\, dx \geq \lambda - \varepsilon; \tag{6.13}$$

ii) *(vanishing)*

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{y+B_{0,R}} u_{n_k}^2\, dx = 0, \quad \text{for all} \quad R < +\infty; \tag{6.14}$$

iii) *(dichotomy)* there exist $\alpha \in (0, \lambda)$, $k_0 \geq 1$, u_k^1, u_k^2 bounded in $H^1(\mathbb{R}^N)$ satisfying for $k \geq k_0$:

$$\begin{aligned}
&\|u_{n_k} - (u_k^1 + u_k^2)\|_{L^2(\mathbb{R}^N)} \leq \delta(\varepsilon) \to 0 \quad \text{for} \quad \varepsilon \to 0^+,\\
&\left|\int_{\mathbb{R}^N} (u_k^1)^2\, dx - \alpha\right| \leq \varepsilon \quad \text{and} \quad \left|\int_{\mathbb{R}^N} (u_k^2)^2\, dx - (\lambda - \alpha)\right| \leq \varepsilon,\\
&\operatorname{dist}\left(\operatorname{supp} u_k^1, \operatorname{supp} u_k^2\right) \to +\infty \quad \text{as} \quad k \to \infty,\\
&\liminf_{k\to\infty} \int_{\mathbb{R}^N} \left[|\nabla u_{n_k}|^2 - |\nabla u_k^1|^2 - |\nabla u_k^2|^2\right] dx \geq 0.
\end{aligned} \tag{6.15}$$

Notice that the vanishing situation does not appear in Theorem 6.5.6. Vanishing is, roughly speaking, covered by compactness, since if vanishing occurs for the sequence $(R_{A_n}(1))$ we prove that $R_{A_n} \longrightarrow 0$ in the norm operator of $\mathcal{L}(L^2(\mathbb{R}^N))$.

Proof of Theorem 6.5.6. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of open (or quasi-open and not necessarily bounded) sets in $\mathbb{R}^N$ of uniformly bounded measure (say $|A_n| \leq c$). We apply the concentration-compactness principle to the sequence $(w_{A_n})_{n\in\mathbb{N}}$ which from Lemma 4.5.2 is bounded in $H^1(\mathbb{R}^N)$. Without loss of generality, we can suppose that $\int_{\mathbb{R}^N} w_{A_n}^2\, dx \to \lambda \geq 0$. We study separately each situation. The compactness and the vanishing cases will give uniform convergence for the sequence of operators $(R_{A_n})_{n\in\mathbb{N}}$, while dichotomy of $(w_{A_n})_{n\in\mathbb{N}}$ will give a dichotomy behavior for $(R_{A_n})_{n\in\mathbb{N}}$.

Compactness Let us suppose that for a subsequence (still denoted by the same indices) and some translations (again we re-note $y_n + A_n$ by A_n) we have the L^2-strong convergence of the sequence $(w_{A_n})_{n\in\mathbb{N}}$. Following γ-convergence arguments, if $w_{A_n} \longrightarrow w$ in $L^2(\mathbb{R}^N)$, then A_n γ-converges to a measure μ and $w = w_\mu$. In order to conclude the compactness case, we give the following version of Proposition 6.1.7.

Lemma 6.5.7 *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of quasi-open sets of uniformly bounded measure and suppose that $w_{A_n} \longrightarrow w$ in $L^2(\mathbb{R}^N)$. Then R_{A_n} converges in $\mathcal{L}(L^2(\mathbb{R}^N))$ to R_μ.*

Proof We have to prove that

$$\lim_{n\to\infty}\Big[\sup_{\|f\|_{L^2(\mathbb{R}^N)}\le 1}\|R_{A_n}(f)-R_\mu(f)\|_{L^2(\mathbb{R}^N)}\Big]=0.$$

This is equivalent to

$$\lim_{n\to\infty}\|R_{A_n}(f_n)-R_\mu(f_n)\|_{L^2(\mathbb{R}^N)}=0,$$

where $\|f_n\|_{L^2(\mathbb{R}^N)}\le 1$. It is enough to consider a subsequence which weakly converges in $L^2(\mathbb{R}^N)$ to f. Hence

$$\begin{aligned}
&\limsup_{n\to\infty}\|R_{A_n}(f_n)-R_\mu(f_n)\|_{L^2(\mathbb{R}^N)}\\
&\quad=\limsup_{n\to\infty}\|R_{A_n}(f_n)-R_\mu(f)+R_\mu(f)-R_\mu(f_n)\|_{L^2(\mathbb{R}^N)}\\
&\quad\le\limsup_{n\to\infty}\|R_{A_n}(f_n)-R_\mu(f)\|_{L^2(\mathbb{R}^N)}\\
&\qquad+\limsup_{n\to\infty}\|R_\mu(f)-R_\mu f_n\|_{L^2(\mathbb{R}^N)}.
\end{aligned}$$

We have $R_{A_n}(f_n)\rightharpoonup R_\mu(f)$ weakly in $H^1(\mathbb{R}^N)$, hence applying Theorem 6.5.4, this convergence is strong in $L^2(\mathbb{R}^N)$. On the other side, $R_\mu(f_n)\rightharpoonup R_\mu(f)$ weakly in $H^1(\mathbb{R}^N)$. Denoting again by A the regular set of the measure μ (we have $|A|\le c$), the compact injection $H^1_0(A)\hookrightarrow L^2(A)$ proves that this convergence is also strong in $L^2(\mathbb{R}^N)$. ■

Proof of Theorem 6.5.6 (continuation).
Vanishing Let us suppose that $(w_{A_n})_{n\in\mathbb{N}}$ is in the vanishing case, i.e., for all $R>0$,

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_{y,R}}w^2_{A_n}\,dx=0. \tag{6.16}$$

Assume that $\lambda_1(A_n)\to+\infty$. The inequality $\|u_n\|_{L^2(A_n)}\le\frac{1}{\lambda_1(A_n)}\|\nabla u_n\|_{L^2(A_n,\mathbb{R}^N)}$ yields that every $H^1(\mathbb{R}^N)$-bounded sequence of elements $u_n\in H^1_0(A_n)$ converges strongly in $L^2(\mathbb{R}^N)$ to 0. In particular this will be the case of w_{A_n} and, following Lemma 6.5.7, R_{A_n} converges to 0 in $\mathcal{L}(L^2(\mathbb{R}^N))$.

In order to prove that $\lambda_1(A_n)\to+\infty$, we use a result of Lieb from [155], namely that for any $\varepsilon>0$, there exists some $R>0$ and $y_n\in\mathbb{R}^N$ such that

$$\lambda_1(A_n\cap B_{y_n,R})\le\lambda_1(A_n)+\varepsilon. \tag{6.17}$$

The maximum principle yields $w_{A_n}\ge w_{A_n\cap B_{y_n,R}}\ge 0$, hence relation (6.16) gives

$$\lim_{n\to\infty}\int_{A_n\cap B_{y_n,R}}w^2_{A_n\cap B_{y_n,R}}\,dx=0.$$

Translating A_n by the vector $-y_n$, we can suppose (possibly extracting a subsequence, still denoted by the same indices), that the sequence of sets $(-y_n+A_n)\cap B_{0,R}$ γ-converges to the empty set, which implies

$$\lambda_1((-y_n + A_n) \cap B_{0,R}) \to +\infty.$$

Hence, relation (6.17) gives that $\lambda_1(A_n) \to +\infty$ (see for example [66]). Then $w_{A_n} \longrightarrow 0$ in $L^2(\mathbb{R}^N)$ and from Lemma 6.5.7 we get $R_{A_n} \longrightarrow 0$ in $\mathcal{L}(L^2(\mathbb{R}^N))$.

Dichotomy Supposing that $(w_{A_n})_{n\in\mathbb{N}}$ is in the dichotomy case, by a diagonal procedure we find a subsequence (still denoted by the same indices) such that there exists $\alpha > 0$ and $u_n^1, u_n^2 \in H^1(\mathbb{R}^N)$ with

$$\|w_{A_n} - (u_n^1 + u_n^2)\|_{L^2(\mathbb{R}^N)} \to 0, \tag{6.18}$$

$$\int_{\mathbb{R}^N} (u_n^1)^2\, dx \to \alpha \quad \text{and} \quad \int_{\mathbb{R}^N} (u_n^2)^2 dx \to \lambda - \alpha, \tag{6.19}$$

$$\operatorname{dist}(\operatorname{supp} u_n^1, \operatorname{supp} u_n^2) \to +\infty, \tag{6.20}$$

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} \left[|\nabla w_{A_n}|^2 - |\nabla u_n^1|^2 - |\nabla u_n^2|^2\right] dx \geq 0. \tag{6.21}$$

It is easy to see that u_n^1, u_n^2 can be chosen positive, and belonging to $H_0^1(A_n)$ (see the construction of u_n^1, u_n^2 in [156]). We define

$$\tilde{A}_n = A_n^1 \cup A_n^2 \quad \text{where} \quad A_n^1 = \{u_n^1 > 0\} \quad \text{and} \quad A_n^2 = \{u_n^2 > 0\},$$

which is a quasi-open set contained in A_n. Let us assume that

$$\|w_{A_n} - w_{\tilde{A}_n}\|_{H^1(\mathbb{R}^N)} \to 0. \tag{6.22}$$

In order to conclude, one can use [43, Lemma 3.6] and the fact that there exist two constants K, α depending only on the measure of A_n and the dimension of the space, such that

$$\|R_{A_n} - R_{\tilde{A}_n}\|_2 \leq K \|w_{A_n} - w_{\tilde{A}_n}\|^{\alpha}_{L^2(\mathbb{R}^N)}. \tag{6.23}$$

The proof of this inequality is trivial in 2 and 3 dimensions, and needs an interpolation argument related to the Riesz–Thorin theorem for $N \geq 4$.

In order to prove (6.22) notice that $w_{\tilde{A}_n} = P_{H_0^1(\tilde{A}_n)} w_{A_n}$, where $P_{H_0^1(\tilde{A}_n)}$ denotes the orthogonal projection from $H_0^1(A_n)$ onto $H_0^1(\tilde{A}_n)$. Then

$$
\begin{aligned}
\int_{\mathbb{R}^N} |\nabla w_{A_n} - \nabla w_{\tilde{A}_n}|^2\,dx &\le \int_{\mathbb{R}^N} |\nabla w_{A_n} - \nabla u_n^1 - \nabla u_n^2|^2\,dx \\
&= \int_{\mathbb{R}^N} |\nabla w_{A_n}|^2\,dx - 2\int_{\mathbb{R}^N} \nabla w_{A_n} \nabla(u_n^1 + u_n^2)\,dx \\
&\quad + \int_{\mathbb{R}^N} |\nabla(u_n^1 + u_n^2)|^2\,dx \\
&= \int_{\mathbb{R}^N} w_{A_n}\,dx - 2\int_{\mathbb{R}^N} (u_n^1 + u_n^2)\,dx \\
&\quad + \int_{\mathbb{R}^N} |\nabla(u_n^1 + u_n^2)|^2\,dx \\
&= 2\Big(\int_{\mathbb{R}^N} w_{A_n}\,dx - \int_{\mathbb{R}^N} (u_n^1 + u_n^2)\,dx\Big) \\
&\quad + \int_{\mathbb{R}^N} |\nabla(u_n^1 + u_n^2)|^2\,dx - \int_{\mathbb{R}^N} |\nabla w_{A_n}|^2\,dx.
\end{aligned}
$$

But

$$
\begin{aligned}
0 &\le \lim_{n\to\infty} \Big| \int_{\mathbb{R}^N} w_{A_n}\,dx - \int_{\mathbb{R}^N} (u_n^1 + u_n^2)\,dx \Big| \\
&\le \lim_{n\to\infty} |A_n|^{1/2} \|w_{A_n} - (u_n^1 + u_n^2)\|_{L^2(A_n)} = 0
\end{aligned}
$$

and

$$
\limsup_{n\to\infty} \Big[\int_{\mathbb{R}^N} |\nabla(u_n^1 + u_n^2)|^2\,dx - \int_{\mathbb{R}^N} |\nabla w_{A_n}|^2\,dx \Big] \le 0,
$$

hence relation (6.22) follows by the Poincaré inequality. ∎

Example 6.5.8 The first two eigenvalues of the Dirichlet Laplacian.

Let us consider, as in Section 6.4, the problem of minimizing functionals depending on the first and the second eigenvalues, taking as the design region the entire space $\mathbb{R}^N$. We define the set

$$
E = \{(\lambda_1(A), \lambda_2(A)) : A \in \mathcal{A}_c(\mathbb{R}^N)\} \subseteq \mathbb{R}^2, \tag{6.24}
$$

where $\mathcal{A}_c(\mathbb{R}^N)$ is the family of all quasi-open sets of $\mathbb{R}^N$ of measure less than or equal to c. Using the same arguments as in Section 6.4 and the concentration-compactness principle for the γ-convergence, we prove the following result.

Theorem 6.5.9 *The set E is closed in $\mathbb{R}^2$.*

Proof Let $(x, y) \in \mathbb{R}^2$ be such that

$$
x = \lim_{n\to\infty} \lambda_1(A_n), \quad y = \lim_{n\to\infty} \lambda_2(A_n), \quad \text{with } (A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}_c(\mathbb{R}^N).
$$

In order to prove the closedness of the set E, we have to prove the existence of a set $A \in \mathcal{A}_c(\mathbb{R}^N)$ such that $x = \lambda_1(A)$, $y = \lambda_2(A)$. Using Lemma 6.5.6 we distinguish between two situations. We begin with the dichotomy.

There exists a sequence $\tilde{A}_n = A_n^1 \cup A_n^2$ given by Lemma 6.5.6, which in particular satisfies $\lambda_1(A_n^1 \cup A_n^2) \to x$ and $\lambda_2(A_n^1 \cup A_n^2) \to y$. There are two possibilities (up to a re-notation of the indices).

1. $\lambda_1(A_n^1) \to x$ and $\lambda_2(A_n^1) \to y$;
2. $\lambda_1(A_n^1) \to x$ and $\lambda_1(A_n^2) \to y$.

If the first situation occurs, there exists $\varepsilon > 0$ such that for every $n \geq n_\varepsilon$, we have $|A_n^1| \leq c - \varepsilon$. For every $\delta > 0$, there exists $n_\delta \in \mathbb{N}$ such that for every $n \geq n_\delta$ we have $|\lambda_1(A_n^1) - x| + |\lambda_2(A_n^1) - y| \leq \delta$. For every $\delta' > 0$, there exists $r > 0$ large enough such that

$$\begin{aligned}
&\lambda_1(A_n^1 \cap B_r) - \lambda_1(A_n^1) \leq \delta, \\
&\lambda_2(A_n^1 \cap B_r) - \lambda_2(A_n^1) \leq \delta, \\
&|A_n^1 \setminus (A_n^1 \cap B_r)| \leq \delta'.
\end{aligned}$$

Choosing $\delta > 0$ and $\delta' > 0$ such that

$$\Big(\frac{c - \varepsilon + \delta'}{c}\Big)^{2/n} < \frac{y}{y + 2\delta}$$

we make a homothety of ratio $\big(\frac{c}{c-\varepsilon+\delta'}\big)^{1/n}$ of $A_n^1 \cap B_r$ for some $n \geq \max\{n_\varepsilon, n_\delta\}$ and find a bounded quasi-open set $A^* = \big(\frac{c}{c-\varepsilon+\delta'}\big)^{1/n}(A_n^1 \cap B_r)$ such that $|A^*| \leq c$ and $\lambda_1(A^*) \leq x$, $\lambda_2(A^*) \leq y$.

If the second situation occurs, we replace A_n^1 by the ball of mass $|A_n^1|$ denoted B_n^1 and A_n^2 by the ball of mass $|A_n^2|$ denoted B_n^2. For a subsequence (still denoted by the same indices) we find two balls B_1, B_2, such that $|B_1| + |B_2| \leq c$, and the γ-limits of $(B_n^1)_{n\in\mathbb{N}}, (B_n^2)_{n\in\mathbb{N}}$ are respectively B_1 and B_2. Then $\lambda_1(B_1 \cup B_2) \leq x$, $\lambda_2(B_1 \cup B_2) \leq y$ and the same arguments as in the bounded case can be applied.

If compactness occurs when applying Lemma 6.5.6, there are two possibilities: either A_μ is quasi-connected, or not. If A_μ is not quasi-connected, then we write $A_\mu = A_1 \cup A_2$ and repeat the same arguments as in the dichotomy situation. If A_μ is quasi-connected, two more possibilities may occur.

1. $\lambda_1(A_\mu) = \lambda_1(\mu)$,
2. $\lambda_1(A_\mu) < \lambda_1(\mu)$.

If the first situation occurs, we get $\mu = \infty_{\mathbb{R}^N \setminus A_\mu}$. Indeed, let us denote by u_μ a first eigenvector associated to $\lambda_1(\mu)$. Since $u_\mu \in H_0^1(A_\mu)$ we get

$$\lambda_1(\mu) = \frac{\int_{\mathbb{R}^N} |\nabla u_\mu|^2\,dx + \int_{\mathbb{R}^N} u_\mu^2\,d\mu}{\int_{\mathbb{R}^N} u_\mu^2 dx} = \frac{\int_{\mathbb{R}^N} |\nabla u_\mu|^2\,dx}{\int_{\mathbb{R}^N} u_\mu^2\,dx} = \lambda_1(A_\mu).$$

Consequently, $\int_{\mathbb{R}^N} u_\mu^2 \, d\mu = 0$ and u_μ is a first eigenvector for A_μ. Since A_μ is quasi-connected, we get that $u_\mu(x) > 0$ for q.e. $x \in A_\mu$. Then from $\int_{\mathbb{R}^N} u_\mu^2 \, d\mu = 0$ we get $\mu(A_\mu) = 0$, hence $\mu = \infty_{\mathbb{R}^N \setminus A_\mu}$. Thus, $\lambda_1(A_\mu) = x$, $\lambda_2(A_\mu) = y$.

If the second situation occurs, there are still two possibilities: either $\lambda_2(A_\mu) < y$, or $\lambda_2(A_\mu) = y$. If $\lambda_2(A_\mu) < y$ we can consider a ball B_r large enough such that $\lambda_1(A_\mu \cap B_r) < x$, $\lambda_2(A_\mu \cap B_r) < y$ and follow the same arguments as in the bounded case.

If $\lambda_2(A_\mu) = y$, let us denote by u_2 an eigenvector associated to the second eigenvalue, and $\tilde{A}_\mu = \{u \neq 0\}$. It follows, as in the bounded case, that

$$\lambda_1(\tilde{A}_\mu) = \lambda_2(\tilde{A}_\mu) = y.$$

There exists a mapping, as in the bounded case (see Section 6.4)

$$[0, +\infty] \ni t \mapsto A(t) \in \mathcal{A}_c(\mathbb{R}^N)$$

with the properties $A(0) = A_\mu$, $A(+\infty) = \tilde{A}_\mu$, for every $t_1 < t_2$, $\operatorname{cap}(A(t_2) \setminus A(t_1)) = 0$, the mapping $t \mapsto \lambda_1(A(t))$ is continuous and increasing, and the mapping $t \mapsto \lambda_2(A(t))$ is constant. The idea to prove this assertion is to "delete" continuously in capacity the nodal line of u_2. We do not give the proof here, since the passage from bounded to unbounded sets can be done by classical arguments.

Then, there exists some $t \in (0, +\infty)$ such that $\lambda_1(A(t)) = x$, $\lambda_2(A(t)) = y$ and $A(t) \in \mathcal{A}_c(\mathbb{R}^N)$. ■

Example 6.5.10 The buckling load of a clamped plate.

Let $\Omega \subseteq \mathbb{R}^2$ be an open set (bounded or unbounded) with $|\Omega| < +\infty$. The buckling load of the clamped plate Ω is defined as:

$$\Lambda_1(\Omega) = \min_{u \in W_0^{2,2}(\Omega), u \neq 0} \frac{\int_\Omega |\Delta u|^2 dx}{\int_\Omega |\nabla u|^2 dx}. \tag{6.25}$$

It is an open question to prove that the disk minimizes the buckling load among all open sets of given measure. Willms and Weinberger proved that the conjecture could be solved provided there exists a **smooth** bounded simply connected open set Ω of class C^2 which minimizes Λ_1 among all open domains of measure equal to c in $\mathbb{R}^2$ (see [18] for a detailed description of the problem). Concentration-compactness arguments, similar to the one presented into this chapter, were used in [18] to prove the existence of a simply connected minimizer. Up to our knowledge, the conjecture is still open, since the regularity of the minimizer was not yet proved.

6.6 Some open questions

- Concerning Theorem 6.4.1, there are many other questions which can be raised. Is the set E convex? Is E still closed if the pair (λ_1, λ_2) is replaced by (λ_i, λ_j), or more generally if we consider the set

$$E_K = \left\{(\lambda_i(A))_{i\in K} : \quad A \in \mathcal{A}_c(B)\right\}$$

where K is a given subset of positive integers? Are the sets A on the boundary of E smooth? When are they convex? If the design region is an open set D, is the set $s(\mathcal{A}_c(D))$ still closed? Or if the Laplace operator is replaced by another elliptic operator of the form

$$L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c \quad ?$$

- In [49] it was proved that the problem

$$\min\{\lambda_k(\Omega) : \Omega \subseteq \mathbb{R}^N, |\Omega| = c\} \tag{6.26}$$

  has a solution for $k = 3$. In two dimensions, the disk is suspected to be the solution (see [16]); up to now, as far as we know, this is still a conjecture. The existence for (6.26) in the case $k \geq 4$ is not solved. Roughly speaking, if one proves the existence of bounded minimizers for $\lambda_3, \dots, \lambda_k$ (under the volume constraint) then the existence of a minimizer (bounded or unbounded) for λ_{k+1} follows.
- An important question deals with the regularity of the optimal shape. Is there a smooth solution for Problem (6.26) (even for $k = 3$)? For the first eigenvalue only, it was proved by variational methods that the minimizer among all quasi-open subsets $A \subseteq D$ of prescribed measure is in fact an open set (see [136]).
  We refer the reader to [44] for a C^1-regularity result for functionals depending on eigenvalues, provided that the class of admissible domains consists only of open convex sets.
- For every $k \geq 2$, prove that if A^* is a solution of

$$\min\{\lambda_k(A) : A \subseteq \mathbb{R}^N, |A| \leq c\},$$

  then $\lambda_k(A^*) = \lambda_{k-1}(A^*)$, i.e., on the optimal set, the k-th eigenvalue is not simple, and equals the one of lower order. This happens for $k = 2$ and in view of the conjectured optimum for λ_3, this should also hold for $k = 3$. Moreover, the numerical computations of Oudet [173] for several values of k support the conjecture.

For a more detailed list of open problems related to eigenvalues we refer the reader to [16].

7

Shape Optimization Problems with Neumann Condition on the Free Boundary

In this chapter we are concerned with shape optimization problems with Neumann boundary conditions on the unknown part of the boundary. We will consider only the case of homogeneous boundary conditions: on the one hand, this is the situation which is very often encountered in different physical models (cracks, free parts of structures, image segmentation, etc.); on the other hand, nonhomogeneous Neumann boundary conditions are unnatural on free boundaries and not well defined on irregular boundaries (even not in a weak sense).

In order to study the existence question for shape optimization problems with Neumann conditions on the free boundary, the first step is to understand the stability of the solution of a partial differential equation for nonsmooth perturbations of the geometric domain.

As for the Dirichlet problem, it is quite clear that the stability of the solution is strictly related to the convergence in the sense of Mosco of the corresponding variational spaces. The choice of the differential operator and the fact that the problem is scalar or vectorial is also of interest, since, via the Mosco convergence, a lot of problems may be reduced to the scalar case and to the Laplace operator. There are two spaces which are of interest for the scalar problems: the usual Sobolev space $H^1(\Omega)$ and the Dirichlet space $L^{1,2}(\Omega)$ (see the precise definition below). For other problems, e.g., arising in elasticity, the variational spaces are related to the previous ones.

As for the Dirichlet problem, the difficult part of the question is to exhibit situations when the Mosco convergence holds; to precisely answer this question, we restrict ourselves to the case of dimension 2, even if most of the general results are presented in N dimensions, $N \geq 2$. For $N \geq 3$, up to our knowledge, all shape stability results in the literature involve (too) strong geometrical constraints. Either one assumes a certain regularity of the boundary (uniform Lipschitz condition [80] or equicontinuity [157]) or the boundaries are allowed to move into a fixed manifold and satisfy a capacity condition [57].

It is worth noticing that for Neumann boundary conditions, the general form of a relaxed problem is not known. By relaxed problem we mean the problem solved by a weak limit of a sequence of solutions on arbitrarily varying domains.

7.1 Some examples

Example 7.1.1 Optimal cutting in a membrane. This can be seen as the scalar version of the well-known cantilever problem. In a simplified form, the question is to find one or more “optimal cuts or holes” in a membrane which leave it as strong as possible (see [48]). Typical constraints are that the cut connects two or more points, or that the holes have a minimal measure.

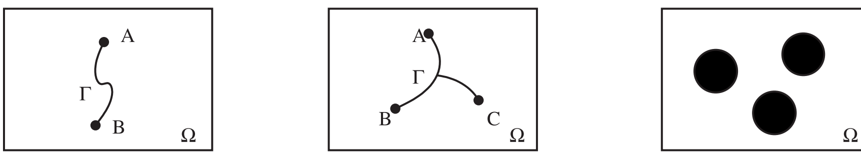

Figure 7.1. Three admissible cuts Γ in the membrane D.

Let D be a two-dimensional bounded open set with a smooth boundary (say the rectangle in Figure 7.1), $u_0 \in H^1(D)$, K a compact subset of D. An admissible cut in D will simply be a compact subset Γ of $\overline{D}$ containing K and satisfying some connectedness assumptions. We denote by $\mathcal{U}_{ad}$ the class of admissible cuts, that is

$$\mathcal{U}_{ad} = \Big\{\Gamma \subseteq \overline{D} \;:\; K \subseteq \Gamma,\ \Gamma \quad \text{is compact,} \quad \sharp\Gamma \le l,\ \ |\Gamma| \ge c\Big\},$$

where $\sharp\Gamma$ denotes the number of connected components of Γ, and $|\Gamma|$ is the Lebesgue measure of Γ. In Figure 7.1 we took $K_1 = \{A, B\}$, $l_1 = 1$, $c_1 = 0$, $K_2 = \{A, B, C\}$, $l_2 = 1$, $c_2 = 0$, $K_3 = \emptyset$, $l_3 = 3$, $c_3 = 10\%$, respectively.

For every $\Gamma \in \mathcal{U}_{ad}$ the energy $\mathcal{E}(\Gamma)$ associated to Γ will be

$$\mathcal{E}(\Gamma) = \min\Big\{\int_{D\setminus\Gamma} |\nabla u|^2\,dx \;:\; u \in H^1_{loc}(D\setminus\Gamma),\ u = u_0 \quad \text{on} \quad \partial D\setminus\Gamma\Big\},$$

so that the optimization problem we deal with can be written as

$$\max\big\{\mathcal{E}(\Gamma) \;:\; \Gamma \in \mathcal{U}_{ad}\big\}. \tag{7.1}$$

For fixed $\Gamma \in \mathcal{U}_{ad}$, the function $u_\Gamma \in H^1_{loc}(D\setminus\Gamma)$ minimizing the energy of the membrane is the weak variational solution of the problem

$$\begin{cases} -\Delta u_\Gamma = 0 \text{ in } D\setminus\Gamma, \\ \frac{\partial u_\Gamma}{\partial n} = 0 \quad \text{on} \quad \partial\Gamma, \quad u_\Gamma = u_0 \quad \text{on} \quad \partial D\setminus\Gamma. \end{cases} \tag{7.2}$$

We prove the existence of a solution to problem (7.1) in Section 7.3.

In Figure 7.1 there are some examples of admissible cuts or holes. Notice that, in general, even if $c = 0$ admissible cuts Γ do not need to be curves; for instance if we want to connect three points we have to expect the optimal cut has a triple junction shape.

Example 7.1.2 The image segmentation problem. A similar situation occurs in the so-called image segmentation problem, where the question is to transform an image g by introducing a family of suitable contours Γ. Given a function $g \in L^2(D)$, the energy of a segmentation Γ is

$$\mathcal{E}(\Gamma) = \min\left\{ \int_{D\setminus\Gamma} |\nabla u|^2\,dx + \int_{D\setminus\Gamma} (u-g)^2\,dx \ : \ u \in H^1(D\setminus\Gamma)\right\}.$$

The optimal segmentation of the image g is then obtained by minimizing the Mumford–Shah functional, i.e., by solving the optimization problem

$$\min\left\{\mathcal{E}(\Gamma) + \mathcal{H}^1(\Gamma) \ : \ \Gamma \in \mathcal{U}_{ad}\right\} \tag{7.3}$$

where $\mathcal{U}_{ad}$ is the family of compact sets contained in $\overline{D}$ and $\mathcal{H}^1(\Gamma)$ is the one-dimensional Hausdorff measure of Γ. For a given Γ, the minimizer u_Γ of the energy satisfies the equation

$$\begin{cases} -\Delta u_\Gamma + u_\Gamma = g \text{ in } D\setminus\Gamma, \\ \frac{\partial u_\Gamma}{\partial n} = 0 \text{ on } \Gamma \cup \partial D. \end{cases} \tag{7.4}$$

Let us observe that problems (7.1) and (7.3) are somehow similar, in the sense that for a given Γ the minimizer of the energy solves an elliptic equation with *homogeneous Neumann boundary conditions* on Γ.

Problems (7.1) and (7.3) are nevertheless deeply different. First of all, in problem (7.1) one has to *maximize* the energy, while in problem (7.3) one has to *minimize* it. This is the main reason for which problem (7.3) can be seen as a minimum problem in the space SBV (see [7], [21]), the "crack" Γ being seen as the "jump set" of the SBV function u.

A second difference is that the elliptic equation (7.4) has a zero order term, therefore the solution belongs to the Sobolev space $H^1(D\setminus\Gamma)$. The solution of problem (7.2) belongs only to the Dirichlet space (see [141]) $L^{1,2}(D\setminus\Gamma)$. By definition, for every open set $A \subseteq \mathbb{R}^N$,

$$L^{1,2}(A) = \left\{u \in L^2_{loc}(A) \ : \ \nabla u \in L^2(A,\mathbb{R}^N)\right\}, \tag{7.5}$$

which coincides with $H^1(A)$ only if A is smooth enough, for instance if ∂A is Lipschitz continuous (see [130]).

Another main difference between problems (7.1) and (7.3) is the presence of the penalty term given by the Hausdorff measure. Without this term the minimizer of the Mumford–Shah functional would not exist in general, the infimum being equal to zero. The presence of this term in functional (7.1) is not necessary for the existence of a solution. From an intuitive point of view, since one looks for the strongest membrane, the length of the crack should be not "too big," hence the constraint on the Hausdorff measure is somehow redundant. One could add in problem (7.1) a penalty term given by the Hausdorff measure, by considering

$$\max\{\mathcal{E}(\Gamma) - \alpha\mathcal{H}^1(\Gamma) : \Gamma \in \mathcal{U}_{ad}\}, \tag{7.6}$$

where $\mathcal{E}(\Gamma)$ is the energy appearing in (7.1) and $\alpha > 0$ is fixed. In this case, the existence of an optimal crack could be derived as a consequence of the result of Chambolle and Doveri [79].

Example 7.1.3 The quasi-static growth of brittle fracture. An approach by duality, close to the one we will present below, was followed by Dal Maso and Toader in [103], where they studied a model for the quasi-static growth of a brittle fracture proposed by Francfort and Marigo [123]. Given a crack Γ_0 in an open set D, at the next time step when a new condition u_0 on the boundary ∂D is imposed, they deal with an optimization problem of the form

$$\min_{\Gamma\in\mathcal{U}_{ad},\ \Gamma_0\subseteq\Gamma} \min\Big\{\int_{D\setminus\Gamma} |\nabla u|^2\,dx + \alpha\mathcal{H}^1(\Gamma)\ :\ u \in H^1_{loc}(D\setminus\Gamma),$$
$$u = u_0 \text{ on } \partial D\setminus\Gamma\Big\},$$

for which they proved the existence of an optimum. It is clear that the minimizer u is the solution of an elliptic equation and, on the unknown crack Γ, the natural conditions are of homogeneous Neumann type.

Example 7.1.4 The cantilever problem. A celebrated and very classical shape optimization problem arising in elasticity is the so-called cantilever problem (see [5], [6], [28] and [78]). This is similar to the optimal cutting problem but it is formulated within the elasticity framework.

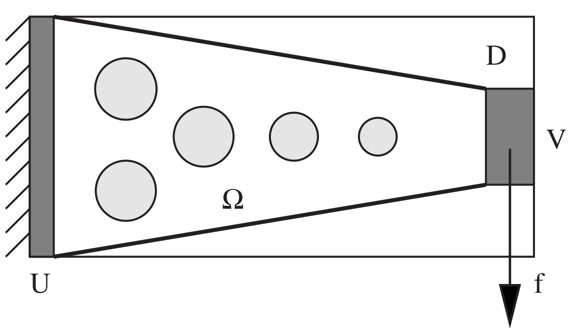

Figure 7.2. The Cantilever problem.

Let D be a rectangle and let the rectangles U and V be as in Figure 7.3. The force f is supposed supported by V. The admissible cantilevers are

$$\mathcal{U}_{ad} = \big\{\Omega \subseteq D\ :\ U\cup V\subseteq\Omega,\ \ \Omega \text{ is open},\ \ \sharp\Omega^c \le l,\ |\Omega| \le c\big\}.$$

For every $\Omega\in\mathcal{U}_{ad}$ the energy $\mathcal{E}(\Omega)$ associated to Ω is

$$\mathcal{E}(\Omega) = \inf \left\{ \frac{1}{2} \int_\Omega \varepsilon(u) : \varepsilon(u)\, dx - \int_V fu dx \ : \right.$$
$$\left. u \in L^2_{loc}(\Omega, \mathbb{R}^2),\ u = 0 \text{ on } U \right\},$$

where the strain tensor $\varepsilon(u)$ is defined as the symmetrized gradient $({}^t\nabla u + \nabla u)/2$.

The optimization problem we deal with can be written as

$$\max \left\{ \mathcal{E}(\Omega) \ : \ \Omega \in \mathcal{U}_{ad} \right\}. \tag{7.7}$$

The proof of the existence result relies on the first condition of the Mosco convergence for the "natural" elasticity spaces

$$\{u \in L^2_{loc}(\Omega, \mathbb{R}^2) \ : \ \varepsilon(u) \in L^2(\Omega, \mathbb{R}^4)\}. \tag{7.8}$$

Chambolle proved in [78] that $L^{1,2}(\Omega, \mathbb{R}^2)$ is dense into the elasticity space (7.8) for every $\Omega \in \mathcal{U}_{ad}$ and consequently the proof is reduced to the first condition of the Mosco convergence for the scalar Dirichlet spaces $L^{1,2}(\Omega, \mathbb{R})$ (which is discussed in Section 7.2 below).

Remark 7.1.5 The admissibility condition $\sharp\Omega^c \leq l$ is crucial in the formulation above. Indeed, this assumption provides the necessary compactness property to obtain the existence of an optimal solution in Problem (7.7).

On the other hand, removing this admissibility condition would lead to non-existence phenomena and some suitable relaxed formulation would be necessary to have a well posed minimization problem. We refer to the pioneering works by Murat and Tartar [168], [169] where the homogenization method to shape optimization has been applied, and to the book by Allaire [4, Section 4.1]. We do not enter into this very interesting field, also important for many industrial applications, and refer the reader further reading to one of the several books available on the subject [4], [29], [83].

In Figure 7.3 there is a numerical solution for a the Cantilever problem with $f = -1$ and $c = |D|/2$ (see Bendsøe and Sigmund [29], and also Allaire [4] for other numerical computations of optimal shapes). The figure has been obtained by using the TOPOPT source code available at `http://www.topopt.dtu.dk` (see also the paper [183]).

Figure 7.3. Numerical solution to the Cantilever problem.

7.2 Boundary variation for Neumann problems

The purpose of this section is to give a quite general method, based on duality, for the study of the shape stability of the weak solution of a linear elliptic problem with homogeneous Neumann boundary conditions in two dimensions of the space. Shape stability is influenced by the presence of the zero order term (compare equations (7.2) and (7.4)). For example, equation (7.4) is related to the Mosco convergence of H^1-spaces, while equation (7.2) is related to the Mosco convergence of $L^{1,2}$-spaces.

We concentrate our discussion only on purely homogeneous boundary conditions (over all the boundary), but we keep in mind that shape stability for mixed boundary conditions or non-homogeneous Neumann boundary conditions on a fixed part of the boundary (like $\partial\Omega$ in equation (7.2)) is a straightforward consequence of the results we present here.

Let us fix a bounded open set $D \subseteq \mathbb{R}^N$, and $h \in L^2(D)$. For every open set $\Omega \subseteq D$, we consider the problem

$$(7.9) \qquad \begin{cases} - \triangle\, u_{\Omega,h} + u_{\Omega,h} = h \text{ in } \Omega, \\ \frac{\partial u_{\Omega,h}}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases}$$

We study the stability of the solution $u_{\Omega,h}$ for perturbations of the geometric domain Ω inside D, i.e., the "continuity" of the mapping $\Omega \mapsto u_{\Omega,h}$. The family of domains is endowed as in Chapter 4 with the Hausdorff complementary topology. In order to compare two solutions on two different domains we extend $u_{\Omega,h}$ and $\nabla u_{\Omega,h}$ by zero on $D \setminus \Omega$. More precisely, we embed $L^{1,2}(\Omega)$ and $H^1(\Omega)$ into two fixed space as follows:

$$(7.10) \qquad L^{1,2}(\Omega) \hookrightarrow L^2(D, \mathbb{R}^N),$$

$$(7.11) \qquad H^1(\Omega) \hookrightarrow L^2(D) \times L^2(D, \mathbb{R}^N)$$

respectively, by means of the mappings

$$(7.12) \qquad u \mapsto 1_\Omega \nabla u,$$

$$(7.13) \qquad u \mapsto (1_\Omega u, 1_\Omega \nabla u).$$

According to these conventions and to the definition of the Mosco convergence (see Section 4.5), the Mosco convergence of H^1-spaces is seen in $L^2(D) \times L^2(D, \mathbb{R}^N)$ while the Mosco convergence of $L^{1,2}$-spaces is seen in $L^2(D, \mathbb{R}^N)$.

Roughly speaking, shape stability for Neumann problems can be split in two situations. The easy case is when the "limit" domain Ω is smooth enough such that $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$. The difficult case is when the "limit" domain Ω is not smooth, and this density property fails. In this case, as far as we know, the only available results are those of [53], [54], [79] which hold in two dimensions of the space and require a uniformly finite number of holes for the perturbation. Recently, the results of [54] were extended in [96] to nonlinear operators of p-Laplacian type (still in two dimensions of the space).

Concerning equation (7.9), we recall the result obtained in [80], where continuity is obtained under geometric constraints on the variable domains (uniform Lipschitz boundary), which in particular imply the existence of uniformly bounded extension operators from $H^1(\Omega)$ to $H^1(\mathbb{R}^2)$; the existence of extension operators across the boundary is the key result for the shape continuity. In [171] the shape continuity is established for the same equation in a class of domains satisfying weaker geometrical constraints which still provide the existence of a dense set of functions for which the extension property holds (hence the first Mosco condition).

Here are the main results (see Bucur and Varchon [53], [54]), where we denote by $\mathcal{O}_l(D)$ the class of all open subsets Ω of D with $\sharp\Omega^c \le l$. We prefer to express the main results in terms of Mosco convergence, which can be easily adapted to every concrete example in which shape stability is investigated.

Theorem 7.2.1 *Let $N = 2$ and $\{\Omega_n\}_{n\in\mathbb{N}} \in \mathcal{O}_l(D)$ be such that $\Omega_n \xrightarrow{H^c} \Omega$. Then $H^1(\Omega_n)$ converges in the sense of Mosco to $H^1(\Omega)$ if and only if $|\Omega_n| \to |\Omega|$.*

In particular, the Mosco convergence holds if the number of the connected components and the Hausdorff measure of $\partial\Omega_n$ are uniformly bounded (Chambolle and Doveri [79]).

As an example of direct application of Theorem 7.2.1, we give a shape stability result for the solutions of (7.9).

Proposition 7.2.2 *Let $N = 2$ and $\{\Omega_n\}_{n\in\mathbb{N}} \in \mathcal{O}_l(D)$ be such that $\Omega_n \xrightarrow{H^c} \Omega$. For every $h \in L^2(D)$ we have that $1_{\Omega_n} u_{\Omega_n,h}$ converges to $1_\Omega u_{\Omega,h}$ in $L^2(D)$ if and only if $|\Omega_n| \to |\Omega|$.*

The shape stability of the solution of equations with zero right-hand side, like (7.2), is related to the following result which does not require, as Theorem 7.2.1, the stability of the Lebesgue measure. More details will be given in the next paragraph, but for a full comprehension of the role played by the stability of the Lebesgue measure, we refer the reader to [54].

Theorem 7.2.3 *Let $N = 2$ and $\{\Omega_n\}_{n\in\mathbb{N}} \in \mathcal{O}_l(D)$ be such that $\Omega_n \xrightarrow{H^c} \Omega$. Then the first Mosco condition holds for $\{L^{1,2}(\Omega_n)\}_n$ and $L^{1,2}(\Omega)$.*

7.2.1 General facts in $\mathbb{R}^N$

Proposition 7.2.4 *Let D be a bounded design region in $\mathbb{R}^N$ and let $\Omega_n, \Omega \subseteq D$. Suppose that $H^1(\Omega_n)$ converges in the sense of Mosco to $H^1(\Omega)$. Then, for every $h \in L^2(D)$,*

$$(1_{\Omega_n} u_{\Omega_n,h}, 1_{\Omega_n} \nabla u_{\Omega_n,h}) \xrightarrow{L^2(D)\times L^2(D,\mathbb{R}^2)} (1_\Omega u_{\Omega,h}, 1_\Omega \nabla u_{\Omega,h}).$$

Proof It is clear that $(1_{\Omega_n} u_{\Omega_n,h}, 1_{\Omega_n} \nabla u_{\Omega_n,h})$ is bounded in $L^2(D) \times L^2(D, \mathbb{R}^N)$ by $\|h\|_{L^2(D)}$. For a subsequence, still denoted using the same indices, we can write

$$(1_{\Omega_n} u_{\Omega_n,h}, 1_{\Omega_n} \nabla u_{\Omega_n,h}) \overset{L^2(D)\times L^2(D,\mathbb{R}^2)}{\rightharpoonup} (u, u_1, \dots, u_N).$$

From the second Mosco condition we get $u = u_1 = \cdots = u_N = 0$ a.e. on Ω^c and $\nabla u = (u_1, \dots, u_N)$ in the sense of distributions on Ω. In this way, $u_{|\Omega} \in H^1(\Omega)$. To prove that $u_{|\Omega} = u_{\Omega,h}$, we have to prove for every $\phi \in H^1(\Omega)$,

$$\int_\Omega \nabla u \nabla \phi dx + \int_\Omega u \phi dx = \int_\Omega h \phi dx.$$

From the first Mosco condition, there exists $\phi_n \in H^1(\Omega_n)$ such that

$$(1_{\Omega_n} \phi_n, 1_{\Omega_n} \nabla \phi_n) \overset{L^2(D)\times L^2(D,\mathbb{R}^N)}{\longrightarrow} (1_\Omega \phi, 1_\Omega \nabla \phi).$$

We conclude by passing to the limit the equality

$$\int_D 1_{\Omega_n} \nabla u_{\Omega_n,h} \nabla \phi_n dx + \int_D 1_{\Omega_n} u_{\Omega_n,h} 1_{\Omega_n} \phi_n dx = \int_D h 1_{\Omega_n} \phi_n dx,$$

and observing that the $L^2(D) \times L^2(D, \mathbb{R}^N)$-norms of $u_{\Omega_n,h}$ also converge.

Since the solution of (7.9) is unique, the whole sequence $(u_{\Omega_n,h})_n$ converges strongly to $u_{\Omega,h}$ (in the sense of extensions). ■

Corollary 7.2.5 *If $H^1(\Omega_n)$ converges in the sense of Mosco to $H^1(\Omega)$, then 1_{Ω_n} converges in L^1 to 1_Ω.*

Proof Take $h = 1$ and apply Proposition 7.2.4. ■

Corollary 7.2.6 *Let $\Omega_n \overset{H^c}{\longrightarrow} \Omega$. Then $H^1(\Omega_n)$ converges in the sense of Mosco to $H^1(\Omega)$, if and only if, for every $h \in L^2(D)$,*

$$(1_{\Omega_n} u_{\Omega_n,h}, 1_{\Omega_n} \nabla u_{\Omega_n,h}) \overset{L^2(D)\times L^2(D,\mathbb{R}^2)}{\longrightarrow} (1_\Omega u_{\Omega,h}, 1_\Omega \nabla u_{\Omega,h}).$$

Proof The necessity follows by Proposition 7.2.4. Let us prove now the sufficiency. Condition M_2 follows from Corollary 7.2.5 and the properties of the Hausdorff convergence (see Proposition 4.6.1). For proving M_1, it is enough to observe that

$$Y = \{u_{\Omega,h} : h \in L^2(D)\} \subseteq H^1(\Omega)$$

is dense in $H^1(\Omega)$. Indeed, suppose by contradiction that $u \in H^1(\Omega)$ is orthogonal to Y, i.e.,

$$\forall h \in L^2(D) \qquad \int_\Omega \nabla u \nabla u_{\Omega,h} + u u_{\Omega,h} dx = 0.$$

Consequently

$$\int_{\Omega} uh dx = 0,$$

hence, $u \equiv 0$. ∎

Theorem 7.2.7 *Let D be a bounded design region in $\mathbb{R}^N$ and assume that Ω_n, $\Omega \subseteq D$ satisfy a uniform cone condition. If $\Omega_n \xrightarrow{H^c} \Omega$, then $H^1(\Omega_n)$ converges in the sense of Mosco to $H^1(\Omega)$.*

Proof First, as all Ω_n satisfy a uniform cone condition, if $\Omega_n \xrightarrow{H^c} \Omega$ we also have that 1_{Ω_n} converges in L^1 to 1_Ω (see [140]). The first Mosco condition follows by taking $u_n = (Eu)_{|\Omega_n}$, where $Eu \in H^1(D)$ is an extension of u on $D \setminus \Omega$. The second Mosco condition follows using the properties of the H^c-convergence and the convergence of the characteristic functions. ∎

For technical purposes, we introduce the equation

$$\begin{cases} -\triangle u_{\Omega,g} = g \text{ in } \Omega, \\ \frac{\partial u_{\Omega,g}}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{7.14}$$

Here, $g \in L^2(\Omega)$ has a compact support in Ω and $\int_C g dx = 0$ for every connected component C of Ω. The solution $u_{\Omega,g}$ then belongs to $L^{1,2}(\Omega)$ and is obtained by the minimization of the functional

$$L^{1,2}(\Omega) \ni u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega ug dx.$$

One of the main ideas introduced in [54] is to consider a second equation which is easier to study from the point of view of the shape stability, but which carries most of the information concerning the shape stability of (7.9). Let $B = B(0, r)$ be such that $B(0, r + \delta) \subseteq \Omega \subseteq D$ for some $\delta > 0$ and let $\gamma \in H^{1/2}(\partial B)$ be such that $\int_{\partial B} \gamma d\sigma = 0$. Note that, under this last assumption, γ is also an element of the dual of $H^{1/2}(\partial B)/\mathbb{R}$. We consider the equation

$$\begin{cases} -\triangle v_{\Omega,\gamma} = 0 \text{ in } \Omega \setminus \overline{B}, \\ \frac{\partial v_{\Omega,\gamma}}{\partial n} = 0 \text{ on } \partial\Omega, \\ \frac{\partial v_{\Omega,\gamma}}{\partial n} = \gamma \text{ on } \partial B. \end{cases} \tag{7.15}$$

Equation (7.15) has a unique variational solution in $L^{1,2}(\Omega \setminus \overline{B})$ obtained by the minimization of the energy functional

$$L^{1,2}(\Omega \setminus \overline{B}) \ni v \mapsto F(v) = \frac{1}{2}\int_{\Omega\setminus\overline{B}} |\nabla v|^2 dx - \int_{\partial B} \gamma v d\sigma. \tag{7.16}$$

This is a consequence of the Lax–Milgram theorem. Note that in $L^{1,2}(\Omega \setminus \overline{B})$ we implicitly assumed the equivalence relation $u \equiv v$ if $\nabla u = \nabla v$ a.e.

The main interest to relate the shape stability of the solution of problem (7.9) to the shape stability of the solution of problem (7.15) relies on the fact that all solutions of problem (7.15) (even in open sets with non-smooth boundaries) have, in two dimensions of the space, harmonic conjugates which satisfy Dirichlet boundary conditions (which are easier to handle on varying domains using the results of Chapter 4). Observe that a new difficulty (of different type) appears, since the traces of the conjugate functions on the boundary are locally constant, but, globally, the constants may vary. Nevertheless, in concrete examples, this seems easier to handle, rather than investigating directly the stability of the original problem.

We give first a result which relates the shape stability of (7.14) to the shape stability of (7.15). We recall that for a bounded open set D, we denote by $\mathcal{O}(D)$ the family of all open subsets of D. Given a sequence $\Omega_n \xrightarrow{H^c} \Omega$, we say that g is admissible, if supp g is contained in a ball B, such that $\overline{B} \subseteq \Omega$ and $\int_B g dx = 0$. This implies for n large enough that supp $g \subseteq \Omega_n$.

Proposition 7.2.8 *Let $\Omega_n, \Omega \in \mathcal{O}(D)$ such that $\Omega_n \xrightarrow{H^c} \Omega$. The following assertions are equivalent:*

1. *(Behavior of the solutions of (7.14)). For every admissible g, we have $1_{\Omega_n} \nabla u_{\Omega_n, g} \xrightarrow{L^2(D, \mathbb{R}^N)} 1_\Omega \nabla u_{\Omega, g}$.*
2. *(Behavior of the solutions of (7.15)). For every ball B such that $\overline{B} \subseteq \Omega$ and for every $\gamma \in H^{1/2}(\partial B)$ with $\int_{\partial B} \gamma d\sigma = 0$ we have $1_{\Omega_n \setminus B} \nabla v_{\Omega_n, \gamma} \longrightarrow 1_{\Omega \setminus B} \nabla v_{\Omega, \gamma}$ strongly in $L^2(D, \mathbb{R}^N)$,*
3. *(The first Mosco condition for the spaces $L^{1,2}(\Omega_n)$). We have*

$$L^{1,2}(\Omega) \subseteq s - \liminf_{n \to \infty} L^{1,2}(\Omega_n).$$

In assertion 3 above, we used the notation introduced in Section 4.5. Using embedding (7.10), this condition reads: for every $u \in L^{1,2}(\Omega)$ there exists $u_n \in L^{1,2}(\Omega_n)$ such that $1_{\Omega_n} \nabla u_n \longrightarrow 1_\Omega \nabla u$ strongly in $L^2(D, \mathbb{R}^N)$.

Proof *1.* $\Rightarrow$ *3.* Let us set

$$Y = \{\psi \in L^{1,2}(\Omega) : \exists \psi_n \in L^{1,2}(\Omega_n) \text{ such that } 1_{\Omega_n} \nabla \psi_n \xrightarrow{L^2(D, \mathbb{R}^2)} 1_\Omega \nabla \psi\}.$$

It is sufficient to prove that Y is dense in $L^{1,2}(\Omega)$; then 3 follows straightforwardly by an usual diagonal procedure. Let $\Psi \in L^{1,2}(\Omega)$ such that

$$\Psi \perp_{L^{1,2}(\Omega)} Y,$$

i.e., $\int_\Omega \nabla \Psi \nabla v dx = 0$ for all $v \in Y$, and fix one of its representatives in $L^{1,2}(\Omega)$. According to Proposition 4.6.1 on the Hausdorff convergence, the equivalence class generated by $C_0^\infty(\Omega)$ in $L^{1,2}(\Omega)$ is contained in Y. Hence, for all $v \in C_0^\infty(\Omega)$ we have $\int_\Omega \nabla\Psi\nabla v dx = 0$, therefore $-\Delta\Psi = 0$ in $\mathcal{D}'(\Omega)$. Let now B be a ball such that $\bar{B} \subset \Omega$. For every $g \in L^2(D)$, with supp $g \subset \bar{B}$ and $\int_B g dx = 0$ we have, by assertion *1*, $\int_B g\Psi dx = 0$, so Ψ is constant in B, hence $\nabla\Psi = 0$ in the connected component of Ω which contains B. Applying this argument to every connected component of Ω, we deduce that $\nabla\Psi = 0$ in Ω i.e., $\Psi \equiv 0$ in $L^{1,2}(\Omega)$.

3 $\Rightarrow$ *1*. Let $g \in L^2(\Omega)$ admissible. Taking $u_{\Omega_n,g}$ as test function in (7.14) and applying the Poincaré inequality in $H^1(B)$, we obtain that the sequence $\|1_{\Omega_n}\nabla u_{\Omega_n,g}\|_{L^2(D,\mathbb{R}^N)}$ is bounded. Up to a subsequence denoted by the same indices we have $1_{\Omega_n}\nabla u_{\Omega_n,g} \rightharpoonup w$ weakly in $L^2(D)$. From the H^c-convergence, we get that for every $q \in C_0^\infty(\Omega, \mathbb{R}^N)$, div $\overline{q} = 0$,

$$\langle w|_\Omega, \overline{q}\rangle_{H^{-1}(\Omega,\mathbb{R}^N)\times H_0^1(\Omega,\mathbb{R}^N)} = 0.$$

Applying successively De Rham's theorem [130, Theorem 2.3] on an increasing sequence of smooth sets covering Ω, there exists $u \in L^2_{loc}(\Omega)$ such that $w|_\Omega = \nabla u$ in the distributional sense in Ω. Moreover, from the compact injection $H^1(B) \hookrightarrow L^2(B)$ we have $u_{\Omega_n,g} \longrightarrow u$ strongly in $L^2(U)$. By assertion *3*, for every $v \in L^{1,2}(\Omega)$ we have

$$\int_\Omega \nabla u \nabla v dx = \int_D \langle w, 1_\Omega \nabla v\rangle dx = \lim_{n\to\infty}\int_D 1_{\Omega_n}\nabla u_n 1_{\Omega_n}\nabla v_n dx$$

$$= \lim_{n\to\infty}\int_{\Omega_n} g v_n dx = \int_\Omega g v dx.$$

Hence $u|_\Omega = u_{\Omega,g}$ and moreover

$$\begin{aligned}\lim_{n\to\infty}\int_D |1_{\Omega_n}\nabla u_{\Omega_n,g}|^2 dx &= \lim_{n\to\infty}\int_U g u_{\Omega_n,g} dx \\ &= \int_U g u_{\Omega,g} dx = \int_D |1_\Omega \nabla u_{\Omega,g}|^2 dx.\end{aligned} \tag{7.17}$$

By the uniqueness of the solution of (7.14), the whole sequence $1_{\Omega_n}\nabla u_{\Omega_n,g}$ converges to $1_\Omega\nabla u_{\Omega,g}$ in $L^2(D,\mathbb{R}^N)$.

2 $\Rightarrow$ *3*. Let C be a connected component of Ω and denote by Y the subspace of $L^{1,2}(C \setminus \overline{B})$ given by

$$Y = \Big\{\psi \in L^{1,2}(C\setminus\overline{B}) : \exists\psi_n \in L^{1,2}(\Omega_n\setminus\overline{B}) \quad \text{such that} \quad 1_{\Omega_n}\nabla\psi_n \overset{L^2(D,\mathbb{R}^N)}{\longrightarrow} 1_{C\setminus\overline{B}}\nabla\psi\Big\}.$$

Let $\Psi \in L^{1,2}(C\setminus\overline{B})$, $\Psi \perp Y$ i.e., $\int_{C\setminus\overline{B}} \nabla\Psi\nabla v dx = 0$ for all $v \in Y$; let us fix a representative of Ψ in $L^{1,2}(C\setminus\overline{B})$. Using Proposition 4.6.1, we deduce, as above, that $-\Delta\Psi = 0$ in $\mathcal{D}'(C\setminus\overline{B})$. Since every solution $v_{\Omega,\gamma}$ belongs to Y, writing the orthogonality property we get

$$0 = \int_{C \setminus \overline{B}} \nabla \Psi \nabla v_{\Omega,\gamma} dx = \int_{\partial B} \gamma \Psi d\sigma.$$

This relation holds for every $\gamma \in H^{1/2}(\partial B)$ such that $\int_{\partial B} \gamma d\sigma = 0$. Since $H^{1/2}(\partial B)$ is dense in $L^2(\partial B)$ we get that Ψ is constant on ∂B. Let now $\bar{\Psi} \in L^{1,2}(C)$ such that $\bar{\Psi} = \Psi$ in $C \setminus \bar{B}$ and $\bar{\Psi} = c$ a.e. on B. Since $\Omega_n \xrightarrow{H^c} \Omega$, for every function $\varphi \in C_0^\infty(C)$ the restriction $\varphi_{|\Omega \setminus \bar{B}}$ belongs to Y, hence we have

$$\int_\Omega \nabla \bar{\Psi} \nabla \varphi dx = 0.$$

Therefore the extension of Ψ by the same constant on B gives a harmonic function, constant on a set of strictly positive measure, hence $\nabla \Psi = 0$ on $\Omega \setminus \overline{B}$. We conclude that Y is dense in $L^{1,2}(C \setminus \bar{B})$.

To prove that for every $u \in L^{1,2}(C)$ there exists $u_n \in L^{1,2}(\Omega_n)$ such that $1_{\Omega_n} \nabla u_n \longrightarrow 1_C \nabla u$ strongly in $L^2(D, \mathbb{R}^N)$, we use an argument based on the partition of unity of D. Let $\varphi \in C_0^\infty(C)$ such that $\varphi = 1$ on B. Let $u_n = 1_C u\varphi + (1-\varphi) 1_{\Omega \setminus \overline{B}} v_n$ where $v_n \in L^{1,2}(\Omega \setminus \bar{B})$ and $1_{\Omega \setminus \overline{B}} \nabla v_n \longrightarrow 1_{C \setminus \overline{B}} \nabla u$ strongly in $L^2(D, \mathbb{R}^N)$. So $u_n \in L^{1,2}(\Omega_n)$ and $1_{\Omega_n} \nabla u_n \longrightarrow 1_\Omega \nabla u$ strongly $L^2(D, \mathbb{R}^N)$.

Let now $(C_i)_{i \in \mathbb{N}}$ be the family of all connected components of Ω. Since the set

$$\{u \in L^{1,2}(\Omega) \text{ such that } \nabla u = 0 \text{ on } C_i \text{ except for a finite number of } i\}$$

is dense in $L^{1,2}(\Omega)$ assertion *3* follows.

3 $\Rightarrow$ *2*. The proof follows the same arguments as in the implication *3* $\Rightarrow$ *1*. with the remark that every function of $L^{1,2}(\Omega \setminus \overline{B})$ has an extension on $L^{1,2}(\Omega)$. ∎

Corollary 7.2.9 *Let D be a bounded design region of $\mathbb{R}^N$ and let $\{\Omega_n\}_{n \in \mathbb{N}}$ be a sequence of open subsets of D converging in the Hausdorff complementary topology to Ω. Then assertions A) and B) below are equivalent.*
A) $H^1(\Omega_n)$ converges in the sense of Mosco to $H^1(\Omega)$.
B) The following three conditions hold:

B.1. $s - \liminf_{n \to \infty} L^{1,2}(\Omega_n) \subseteq L^{1,2}(\Omega)$;

B.2. for every $u \in H^1(\Omega)$ such that $\nabla u = 0$ there exist $u_n \in H^1(\Omega_n)$ such that $(1_{\Omega_n} u_n, 1_{\Omega_n} \nabla u_n) \xrightarrow{L^2(D) \times L^2(D, \mathbb{R}^N)} (1_\Omega u, 0)$;

B.3. $|\Omega| = \lim_{n \to \infty} |\Omega_n|$.

Proof A) $\Rightarrow$ B) For proving $B.1$ take $u \in L^{1,2}(\Omega)$ and define for every $M > 0$,

$$u_M := (u^* \wedge M) \vee (-M),$$

where u^* is a representative of u in $L^{1,2}(\Omega)$. Then u_M converges in $L^{1,2}(\Omega)$ to u when $M \to +\infty$ and moreover u_M belongs to $H^1(\Omega)$, hence from A) $u_M \in s - \liminf_{n \to \infty} H^1(\Omega_n)$.

$B.2$ is a direct consequence of A).

In order to prove $B.3$ take $u_{n,} = 1_{\Omega_n}$.

B) $\Rightarrow$ A) It is enough to prove that the set

$$
(7.18) \qquad \begin{aligned} Y = \{\phi \in H^1(\Omega) : \exists \phi_n \in H^1(\Omega_n) \quad \text{such that} \\ (1_{\Omega_n}\phi_n, 1_{\Omega_n}\nabla\phi_n) \to (1_\Omega\phi, 1_\Omega\nabla\phi) \text{ in } L^2(D) \times L^2(D, \mathbb{R}^N)\} \end{aligned}
$$

is dense in $H^1(\Omega)$.

By linearity and a truncation argument, we can fix $\phi \in H^1(\Omega)$ such that $\phi \in L^\infty(\Omega)$ and $\phi = 0$ on $\Omega \setminus C$, where C is a connected component of Ω. According to $B.1$ there exists $u_n \in L^{1,2}(\Omega_n)$ such that

$$1_{\Omega_n}\nabla u_n \overset{L^2(D,\mathbb{R}^N)}{\longrightarrow} 1_\Omega\nabla\phi.$$

Let us fix a ball B such that $\overline{B} \subseteq C$ and choose the representative of u_n in $L^{1,2}(\Omega_n)$ by adding a suitable constant, so that we can assume that $\int_B u_n dx = \int_B \phi dx$. Let M be a positive constant such that $\|\phi\|_\infty < M$ and define

$$u_n^M = (u_n \wedge M) \vee (-M).$$

We notice that $u_n^M \in H^1(\Omega_n)$ and

$$1_{\Omega_n}\nabla u_n^M \overset{L^2(D,\mathbb{R}^N)}{\longrightarrow} 1_\Omega\nabla\phi.$$

Moreover, since $\{1_{\Omega_n}u_n^M\}_n$ is uniformly bounded in $L^\infty(D)$, we can write (for a subsequence)

$$1_{\Omega_{n_k}}u_{n_k}^M \overset{L^2(D)}{\rightharpoonup} v,$$

where $\nabla v = \nabla\phi$ on Ω and $v = \phi$ on C. Using the Poincaré inequality on smooth open subsets compactly contained in C we have that the convergence above is actually strong in $L^2_{loc}(C)$.

According to $B.2$, there exists $v_{n_k} \in H^1(\Omega_{n_k})$ such that

$$(1_{\Omega_{n_k}}v_{n_k}, 1_{\Omega_{n_k}}\nabla v_{n_k}) \overset{L^2(D)\times L^2(D,\mathbb{R}^N)}{\longrightarrow} (1_\Omega v - 1_\Omega\phi, 0).$$

It is obvious that v_{n_k} can be chosen such that $\|v_{n_k}\|_\infty \le 2M$. Let us define $\phi_{n_k} := u_{n_k}^M - v_{n_k} \in H^1(\Omega_{n_k})$. We have

$$1_{\Omega_{n_k}}\nabla\phi_{n_k} \overset{L^2(D,\mathbb{R}^N)}{\longrightarrow} 1_\Omega\nabla\phi.$$

Let us prove that $\int_D (1_{\Omega_{n_k}}\phi_{n_k} - 1_\Omega\phi)^2 dx \to 0$. First, we have

$$\int_\Omega (1_{\Omega_{n_k}}\phi_{n_k} - 1_\Omega\phi)^2 dx \to 0$$

since on every compact set $\omega \subseteq \Omega$ the sequence $1_{\Omega_{n_k}}\phi_{n_k} - 1_\Omega\phi$ weakly converges to 0 in $L^2(\omega)$, the gradients converge to zero strongly and the sequence is uniformly bounded in $L^\infty(D)$. Second,

$$\int_{D\setminus\Omega} (1_{\Omega_{n_k}}\phi_{n_k} - 1_\Omega\phi)^2 dx \le 4M^2 \int_{D\setminus\Omega} 1_{\Omega_{n_k}} dx,$$

the last term converging to zero from $B.3$.

Notice that we found a subsequence $\{\phi_{n_k}\}$ and not a sequence converging to ϕ. Suppose by contradiction that a sequence $\{\phi_n\}$ strongly converging to ϕ does not exist. For a subsequence, we would have that the distance in $L^2(D) \times L^2(D, \mathbb{R}^N)$ from ϕ to $H^1(\Omega_{n_k})$ would be bounded below by a positive number. This cannot occur, since using the same arguments as above, we would find a sub-subsequence which gives the contradiction. ■

7.2.2 Topological constraints for shape stability

The main result of this section consists in proving that the solution of equation (7.15) is stable in the H^c-topology, provided that the number of the connected components of Ω^c is uniformly bounded. In this section we assume that the dimension N of the space is 2. For a bounded design region $D \subseteq \mathbb{R}^2$ and for $l \in \mathbb{N}$, we denote as in Chapter 4,

$$\mathcal{O}_l(D) = \{\Omega \subseteq D : \Omega \text{ open}, \ \sharp\Omega^c \le l\}.$$

Relying on the results of the previous section, we begin with the proof of Theorem 7.2.3.

Proof of Theorem 7.2.3. According to Proposition 7.2.8, it is enough to prove that for every ball B such that $\overline{B} \subseteq \Omega$ and for every $\gamma \in H^{1/2}(\partial B)$ with $\int_{\partial B} \gamma d\sigma = 0$ we have $1_{\Omega_n\setminus B}\nabla v_{\Omega_n,\gamma} \longrightarrow 1_{\Omega\setminus B}\nabla v_{\Omega,\gamma}$ strongly in $L^2(D, \mathbb{R}^N)$.

We use a duality argument to transform the Neumann problem into a Dirichlet problem, then use a Šveràk type result, and then return to the Neumann problem, again by duality.

Let $\Omega \in \mathcal{O}_l(D)$ such that $\overline{B} \subseteq \Omega$, and denote by $K_1, \ldots, K_l$ the connected components of Ω^c. Consider problem (7.15) on $\Omega \setminus \overline{B}$. If Ω is not connected, in every connected component which does not contain B, the solution is set to be 0.

For the existence of a conjugate function of $v_{\Omega,\gamma}$ into a smooth domain with a finite number of (smooth) holes we refer to [130, Theorem 3.1]. By approaching the non-smooth holes with smooth ones and applying [130, Theorem 3.1], in [54] is proved the following result.

Lemma 7.2.10 *There exists a function $\phi \in H_0^1(D)$ and constants $c_1, \ldots, c_l \in \mathbb{R}$ such that $\nabla v_{\Omega,g} = \mathbf{curl}\,\phi$ in $\Omega \setminus \overline{B}$ and*

$$\begin{cases} -\Delta\phi = 0 \text{ in } \Omega \setminus \overline{B}, \\ \quad \phi = c_i \text{ q.e. on } K_i \quad i = 1, \ldots, l, \\ \quad \phi = G \text{ on } \partial B, \end{cases} \tag{7.19}$$

where $G \in H^{3/2}(\partial B)$ is such that $G' = \gamma$ in the sense of distributions on ∂B with respect to the arc length parametrization.

The equality $\phi = c_i$ q.e. on K_i means that the usual restriction of a quasi-continuous representative of ϕ in $H_0^1(D)$ is equal to c_i on K_i.

We recall two technical lemmas. The first one is an immediate consequence of [41], [56] while the second one can be proved using circular rearrangements (see [86]) and noticing that in one dimension the step functions are not in $H^{1/2}(\mathbb{R})$ (see [54] for more details).

Lemma 7.2.11 *Let $\{\phi_n\}_{n\in\mathbb{N}} \subseteq H_0^1(D)$, $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of compact connected sets in D and $\{c_n\}_{n\in\mathbb{N}}$ be a sequence of constants such that $\phi_n(x) = c_n$ q.e. on K_n. If $K_n \xrightarrow{H} K$ and $\phi_n \rightharpoonup \phi$ weakly in $H_0^1(D)$, there exists a constant $c \in \mathbb{R}$ such that $c_n \longrightarrow c$ and $\phi(x) = c$ q.e. on K.*

Lemma 7.2.12 *Let $\phi \in H_0^1(D)$ and K_1, K_2 two compact connected sets in D with positive diameter. If there exist two constants $c_1, c_2 \in \mathbb{R}$ such that $\phi(x) = c_1$ q.e. on K_1 and $\phi(x) = c_2$ q.e. on K_2, then $K_1 \cap K_2 = \emptyset$.*

Let us assume that $\{\Omega_n\}_{n\in\mathbb{N}}$ is a sequence satisfying the hypotheses of Theorem 7.2.3. As in the previous step, we denote by ϕ_n, ϕ the corresponding functions found by Lemma 7.2.10 applied to $v_{\Omega_n,\gamma}$ on Ω_n and $v_{\Omega,\gamma}$ on Ω, respectively. We denote the connected components of $\overline{D} \setminus \Omega_n$ by $K_1^n, \dots, K_l^n$, some of them being possibly empty.

Lemma 7.2.13 *There exists a subsequence $\{\phi_{n_k}\}_{k\in\mathbb{N}}$ such that $\phi_{n_k} \rightharpoonup \phi$ weakly in $H_0^1(D)$, and a function $v \in L^{1,2}(\Omega \setminus \overline{B})$ such that $\mathbf{curl}\, \phi = \nabla v$ in $\Omega \setminus \overline{B}$.*

Proof Since the extension by constants of ϕ_n does not increase the norm of the gradient and since we have $\int_{\Omega_n \setminus \overline{B}} |\nabla \phi_n|^2 dx = \int_{\Omega_n \setminus \overline{B}} |\nabla u_n|^2 dx$, we get that $\{1_{\Omega_n \setminus \overline{B}} \nabla \phi_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(D, \mathbb{R}^2)$. Hence for a subsequence we have $\phi_{n_k} \rightharpoonup \phi$ weakly in $H_0^1(D)$. From the Hausdorff convergence we get $-\Delta\phi = 0$ in $\Omega \setminus \overline{B}$.

Without loss of generality, we can suppose that for a subsequence (still denoted by the same indices) and for all $i = 1, \dots, l$ we have $K_i^{n_k} \xrightarrow{H} K_i$. Using Lemma 7.2.11 we also get $c_{n_k,i} \to c_i$ and $\phi = c_i$ q.e. on K_i. If there exists two compact sets with positive diameter K_{i_1} and K_{i_2} and nonempty intersection, then from Lemma 7.2.12 we get that $c_{i_1} = c_{i_2}$.

Since $\overline{D} \setminus \Omega = \cup_{i=1}^l K_i$ we get that ϕ is constant q.e. on every connected component of $\overline{D} \setminus \Omega$. From the H^c-convergence, there exists $v \in L^{1,2}(\Omega)$ such that $1_{\Omega_n \setminus \overline{B}} \nabla v_{\Omega_n,\gamma} \rightharpoonup (v_1, v_2)$ weakly in $L^2(D, \mathbb{R}^2)$ and $\nabla v = (v_1, v_2)$ in Ω. The relation $\nabla v_{\Omega_n,\gamma} = \mathbf{curl}\, \phi_n$ in $\Omega_n \setminus \overline{B}$ gives that $\nabla v = \mathbf{curl}\, \phi$ in $\Omega \setminus \overline{B}$. ■

The result above asserts that the weak limit ϕ is such that $-\Delta\phi = 0$ in $\Omega \setminus \overline{B}$ and ϕ is q.e. constant on each connected component of $\overline{D} \setminus \Omega$. In the sequel we prove that ϕ is exactly the function obtained by applying Lemma 7.2.10 to $v_{\Omega,\gamma}$ on $\Omega \setminus \overline{B}$. We also recall from [54] the following result without proof.

Lemma 7.2.14 *Let O be a smooth open connected set and K a compact connected subset of O not reduced to a point. Let us denote by θ the capacitary potential of K in O, i.e., the function $\theta \in H_0^1(O)$ such that*

$$\begin{cases} -\Delta\theta = 0 \text{ in } O \setminus K, \\ \theta = 0 \text{ on } \partial O, \\ \theta = 1 \text{ q.e. on } \partial K. \end{cases} \tag{7.20}$$

Then, for every function $\xi \in L^{1,2}(O \setminus K)$ we have $\mathbf{curl}\,\theta \neq \nabla\xi$.

Lemma 7.2.15 *Let $\Omega \in \mathcal{O}_l(D)$ such that $\overline{B} \subseteq \Omega$. Suppose that there exists a function $\phi \in H_0^1(D)$ and a function $u \in L^{1,2}(\Omega \setminus \overline{B})$ such that $\nabla u = \mathbf{curl}\,\phi$ in $\Omega \setminus \overline{B}$ and*

$$\begin{cases} -\Delta\phi = 0 \text{ in } \Omega \setminus \overline{B}, \\ \phi = c_i \text{ q.e. on } K_i \quad i = 1, \dots, l, \\ \phi = G + c \text{ on } \partial B. \end{cases} \tag{7.21}$$

Then u is the weak solution of (7.15) on $\Omega \setminus \overline{B}$.

Proof Since $u \in L^{1,2}(\Omega \setminus \overline{B})$ it suffices to prove that for any $\xi \in L^{1,2}(\Omega \setminus \overline{B})$ we have

$$\int_{\Omega\setminus\overline{B}} \nabla u \nabla \xi dx = \int_{\partial B} \gamma \xi d\sigma.$$

Considering smooth neighbourhoods O_i of K_i, by an argument of partition of unity, it suffices to prove that for any function $\xi \in H^1(O_i \setminus K_i)$ vanishing q.e. on ∂O_i we have

$$\int_{O_i\setminus K_i} \nabla u \nabla \xi dx = 0.$$

It suffices actually to prove that u solves the following problem on $O_i \setminus K_i$:

$$\begin{cases} -\Delta u = 0 \text{ in } O_i \setminus K_i, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial K_i, \\ \frac{\partial u}{\partial n} = \frac{\partial \phi}{\partial t} \text{ on } \partial O_i. \end{cases} \tag{7.22}$$

To the solution u^* of this equation we associate the function ϕ^* given by Lemma 7.2.10. We have that $-\Delta\phi^* = 0$ in $O_i \setminus K_i$, $\phi^* = \phi$ on ∂O_i, $\phi^* = c^*$ on K_i. Denoting $\theta = \phi - \phi^*$, we get that $\nabla\theta = \mathbf{curl}\,(u - u^*)$, $-\Delta\theta = 0$ in $O_i \setminus K_i$, $\theta = 0$ on ∂O_i, $\theta = c - c^*$ on K_i. According to Lemma 7.2.14, since $\text{diam}(K_i) > 0$, we get $c = c^*$, hence $u = u^*$. ∎

Proof of Theorem 7.2.8 (conclusion). According to Lemma 7.2.15 the conjugate function of ϕ obtained in Lemma 7.2.13 is the solution of equation (7.15) on $\Omega \setminus \overline{B}$, hence

$$1_{\Omega_n \setminus \overline{B}} \nabla v_{\Omega_n,\gamma} \stackrel{L^2(D,\mathbb{R}^2)}{\rightharpoonup} 1_{\Omega \setminus \overline{B}} \nabla v_{\Omega,\gamma}.$$

The strong convergence is a consequence of the convergence of the L^2-norms of the gradients, which follows as usual by taking $v_{\Omega_n,\gamma}$ as test function in the equation and passing to the limit as $n \to \infty$. ■

Remark 7.2.16 Let $\Omega_n \in \mathcal{O}_l(D)$ such that $\Omega_n \xrightarrow{H^c} \Omega$. Then in general $L^{1,2}(\Omega_n)$ does not converge in the sense of Mosco to $L^{1,2}(\Omega)$. For example, consider a situation when $|\Omega| < \liminf_{n\to\infty} |\Omega_n|$ and take $u_n(x, y) = x$. The second Mosco condition is not satisfied in general. By Proposition 7.2.8 the first Mosco condition is automatically satisfied.

As a consequence of Theorem 7.2.3 and Corollary 7.2.9 we can prove now Proposition 7.2.2.

Proof of Proposition 7.2.2

Necessity By Corollary 7.2.9 condition $B.3$ holds.

Sufficiency Let us prove that $B.1$, $B.2$ and $B.3$ hold. Condition $B.1$ is a consequence of Theorem 7.2.3 and condition $B.3$ is assumed by hypothesis. One has only to verify condition $B.2$ of Corollary 7.2.9. If Ω is connected, this is trivial, since every function with zero gradient in Ω is constant, say $c1_\Omega$. Therefore, the sequence $c1_{\Omega_n}$ solves $B.2$. If Ω is not connected, then condition $B.2$ is a consequence of the more involved geometric argument relating the Hausdorff convergence to the capacity. We recall this result from [53].

Lemma 7.2.17 *If $\{\Omega_n\}_{n\in\mathbb{N}}$ is a sequence of simply connected open sets in a bounded design region $D \subseteq \mathbb{R}^2$ such that $\Omega_n \xrightarrow{H^c} \Omega_a \cup \Omega_b$, where $\Omega_a \cap \Omega_b = \emptyset$, then there exists a subsequence (still denoted by the same indices) of $(\Omega_n)_n$, and two sequences of simply connected open sets $\{\Omega_n^a\}_{n\in\mathbb{N}}$, $\{\Omega_n^b\}_{n\in\mathbb{N}}$, such that $\Omega_n^a \cap \Omega_n^b = \emptyset$, $\Omega_n^a \cup \Omega_n^b \subseteq \Omega_n$, $\mathrm{cap}(\Omega_n \setminus (\Omega_n^a \cup \Omega_n^b)) \to 0$ and $\Omega_n^a \xrightarrow{H^c} \Omega_a$, $\Omega_n^b \xrightarrow{H^c} \Omega_b$.*

Using this lemma, condition $B.2$ can be proved using a partition of the unity and localizing around the boundary of $\partial\Omega$, as in [79]. ■

Proof of Theorem 7.2.1 For the necessity, use Corollary 7.2.5. For the sufficiency, use Proposition 7.2.2 together with Corollary 7.2.9 to obtain the first Mosco condition (relation (7.18) in the proof of Corollary 7.2.9). To prove the second Mosco condition, we observe that if $\phi_n \in H^1(\Omega_n)$ is such that

$$(1_{\Omega_n}\phi_n, 1_{\Omega_n}\nabla\phi_n) \stackrel{L^2(D)\times L^2(D,\mathbb{R}^2)}{\rightharpoonup} (\phi, \phi_1, \phi_2),$$

we have directly from the H^c-convergence and the convergence of the Lebesgue measures that $\phi = \phi_1 = \phi_2 = 0$ a.e. on Ω^c. To prove that on Ω we have

$\nabla \phi = (\phi_1, \phi_2)$ in the sense of distributions, we simply use Proposition 4.6.1 on the Hausdorff convergence. ∎

Remark 7.2.18 We observe the following facts.

- For other operators in divergence form (e.g., $u \mapsto -\operatorname{div}(A(x)\nabla u)$) the result of Proposition 7.2.2 holds true. As well, those results can be directly adapted to vector problems, where the variational spaces are of the form $H^1(\Omega, \mathbb{R}^d)$ or $L^{1,2}(\Omega, \mathbb{R}^d)$.
- For the Cantilever problem, the natural space is

$$\{u \in L^2_{loc}(\Omega, \mathbb{R}^2) : \varepsilon(u) \in L^2(\Omega, \mathbb{R}^4)\},$$

endowed with the norm $|\varepsilon(u)|_{L^2}$; in concrete examples, Dirichlet boundary conditions can be imposed on some regions. When dealing with shape stability or existence of optimal shapes, the difficult condition to prove is the first Mosco condition for the spaces defined above. According to Theorem 7.2.3, if $\Omega_n \xrightarrow{H^c} \Omega$ is such that the number of connected components of Ω_n^c is uniformly bounded, the first Mosco condition holds true for functions belonging to

$$\{u \in L^2_{loc}(\Omega, \mathbb{R}^2) : \nabla u \in L^2(\Omega, \mathbb{R}^4)\}.$$

Chambolle proved in [78] that this set is dense in the elasticity space, hence the first Mosco condition holds for the elasticity problem, as well.
- Nonlinear problems in $\mathbb{R}^2$ were discussed in [96]. The main idea is to adapt the duality argument of Proposition 7.2.8 into a nonlinear setting. For operators of p-Laplacian type, with $1 < p \leq 2$, the result of Proposition 7.2.2 is true.

7.3 The optimal cutting problem

In this section we treat the optimal cutting problem in detail, and show how the continuity results presented in the previous section can be adapted in order to prove existence of solutions for the shape optimization problem.

Let D be a two-dimensional bounded open connected set. For simplicity, we suppose that the boundary of D is Lipschitz (see [118]). Consequently the number of connected components of D^c is finite.

For $i = 1, \ldots, l$ let K_i be l compact sets contained in $\overline{D}$ and $K \subseteq \overline{D}$ be a compact set such that $\cup_{i=1}^l K_i \subseteq K$. Let $f \in L^2(D)$ such that $\operatorname{supp} f \cap K = \emptyset$.

Remark 7.3.1 The assumption that $\operatorname{supp} f \cap K = \emptyset$ is made for technical reasons that will be clear in the proof of Theorem 7.3.2. However, we want to stress the fact that the most interesting case is when $f \equiv 0$, so that the only datum of the problem is the boundary condition u_0.

We also notice that, when the datum K is regular enough (for instance a set with a Lipschitz boundary), then, thanks to the equality $L^{1,2}(D \setminus K) = H^1(D \setminus K)$, the assumption $\operatorname{supp} f \cap K = \emptyset$ can be relaxed into the weaker one $f = 0$ a.e. on K.

Remark also that the optimization criterion (7.1) rules out the admissible Γ with $\mathcal{E}(\Gamma) = -\infty$. This automatically implies that the optimization (7.1) is performed on the class of cuts Γ such that the integral of f vanishes on every connected component of Γ which does not touch the boundary ∂D on a set of positive capacity.

In the sequel, we denote by $\mathcal{U}_{ad}$ the following admissible class of "cuts" which is supposed to be nonempty. Let $c \geq 0$, $l \in \mathbb{N}$, and $K_1, \dots, K_l$ pairwise disjoint compact subsets of D. We set

$$\begin{aligned}\mathcal{U}_{ad} = \big\{\Gamma \ :\ & \Gamma = \cup_{i=1}^{l} \Gamma^i, \\ & \forall i = 1, \dots, l \ \ K_i \subseteq \Gamma^i \subseteq K,\ \Gamma^i \text{ compact connected, } |\Gamma| \geq c\big\},\end{aligned}$$

and for every $\Gamma \in \mathcal{U}_{ad}$ we consider the energy

$$\mathcal{E}(\Gamma) = \min\big\{E(u, \Gamma)\ :\ u \in H^1_{loc}(D \setminus \Gamma),\ u = u_0 \ \text{ on } \ \partial D\big\} \tag{7.23}$$

where

$$E(u, \Gamma) = \frac{1}{2}\int_{D\setminus\Gamma} \langle A\nabla u \cdot \nabla u\rangle\, dx - \int_D f u\, dx.$$

Here $u_0 \in H^1(D)$ is a given function and $A \in L^\infty(D, \mathbb{R}^4)$ is a given symmetric matrix satisfying for some $\alpha > 0$ the ellipticity condition

$$\langle A\xi.\xi\rangle \geq \alpha|\xi|^2 \quad \text{for every } \ \xi \in \mathbb{R}^2.$$

If $u \in H^1_{loc}(D \setminus \Gamma)$, the trace of u on ∂D does not exist in general, even if ∂D is smooth. Nevertheless, in our case $\nabla u \in L^2(D \setminus \Gamma, \mathbb{R}^2)$, hence u belongs to the Dirichlet space $L^{1,2}(D \setminus \Gamma)$. In that case, the trace of u on $\partial D \setminus \Gamma$ is well defined, since ∂D is supposed to be Lipschitz continuous. A second equivalent way to give a meaning to the equality $u = u_0$ on $\partial D \setminus \Gamma$ is as follows. Let us fix an extension u_0^* of u_0 outside D, say in $D^* \setminus D$, where D^* is a Lipschitz bounded open set such that $\overline{D} \subseteq D^*$. The trace of u is equal to u_0 on $\partial D \setminus \Gamma$ if and only if the function

$$u^* = \begin{cases} u(x) & \text{if } \ x \in D \setminus \Gamma, \\ u_0^*(x) & \text{if } \ x \in D^* \setminus (D \cup \Gamma) \end{cases} \tag{7.24}$$

belongs to $L^{1,2}(D^* \setminus \Gamma)$.

For every fixed $\Gamma \in \mathcal{U}_{ad}$ problem (7.23) has a solution. This is an immediate consequence of the fact that the support of f is compactly embedded in $\overline{D} \setminus K$ and that, thanks to Remark 7.3.1, the integral of f vanishes on the connected sets of $\overline{D} \setminus K$ not touching ∂D on a set of positive capacity. In fact, if a connected component of $\overline{D} \setminus \Gamma$ contains a part of the support of f and does not touch ∂D on a set of positive capacity, in this region the solution is defined up to a constant, the gradient being fixed. With this remark, the solution is unique (more precisely its gradient is unique) and belongs to the Dirichlet space $L^{1,2}(D \setminus \Gamma)$.

The main result of this section is contained in the following theorem.

Theorem 7.3.2 *The optimization problem*

$$\max\left\{\mathcal{E}(\Gamma)\ :\ \Gamma\in\mathcal{U}_{ad}\right\} \tag{7.25}$$

has at least one solution.

Proof In order to prove the existence of a solution for problem (7.25), we follow the direct methods of the calculus of variations. Let $\{\Gamma_n\}_n\subseteq\mathcal{U}_{ad}$ be a maximizing sequence for (7.25). Without loss of generality, we can suppose that for every $i=1,\dots,l$,

$$\Gamma_n^i\overset{H}{\to}\Gamma^i,$$

the convergence being understood in the Hausdorff metric (see for instance [79], [187]). We denote $\Gamma=\cup_{i=1}^l\Gamma^i$, the Hausdorff limit of Γ_n. Our purpose is to prove that Γ is a solution for problem (7.25). Notice that for every $i=1,\dots,l$ the set Γ^i is compact, connected, $K_i\subseteq\Gamma_i\subseteq K$ and $|\Gamma|\geq c$, hence $\Gamma\in\mathcal{U}_{ad}$.

It remains to prove that for every $u\in L^{1,2}(D\setminus\Gamma)$ with $u=u_0$ on ∂D there exists a sequence $\{u_n\}_n$ such that $u_n\in L^{1,2}(D\setminus\Gamma_n)$ with $u_n=u_0$ on $\partial D\setminus\Gamma_n$ and

$$E(\Gamma,u)\geq\limsup_{n\to\infty}E(\Gamma_n,u_n). \tag{7.26}$$

The construction of the sequence $\{u_n\}_n$ is strongly related to the Mosco convergence of the spaces $L^{1,2}(D\setminus\Gamma_n)$. We observe that if $u\notin L^{1,2}(D\setminus\Gamma)$, then $E(u,\Gamma)=+\infty$ and inequality (7.26) holds trivially. For $u\in L^{1,2}(D\setminus\Gamma)$ we construct a sequence $u_n\in L^{1,2}(D\setminus\Gamma_n)$ with $u=u_0$ on $\partial D\setminus\Gamma_n$ such that

$$\widetilde{\nabla}u_n\to\widetilde{\nabla}u\quad\text{strongly in}\quad L^2(D) \tag{7.27}$$

and

$$\int_D u_n f\,dx\to\int_D uf\,dx.$$

In relation (7.27) we denoted by $\widetilde{\nabla}u_n=1_{D\setminus\Gamma_n}\nabla u_n$ the extension by zero of ∇u_n on Γ_n, since ∇u_n is only defined on $D\setminus\Gamma_n$. Of course, the function $\widetilde{\nabla}u_n$ is not anymore a gradient on D.

In order to construct the sequence $\{u_n\}_n$ we rely on condition 3. of Proposition 7.2.8 and Theorem 7.2.3.

Proposition 7.3.3 *Let $\Gamma_n,\Gamma\in\mathcal{U}_{ad}$ be such that $\Gamma_n\overset{H}{\to}\Gamma$. Then for every $u\in L^{1,2}(D\setminus\Gamma)$ such that $u\big|_{\partial D\setminus\Gamma}=u_0$ there exists a sequence $u_n\in L^{1,2}(D\setminus\Gamma_n)$ such that $\widetilde{\nabla}u_n\longrightarrow\widetilde{\nabla}u$ strongly in $L^2(D)$ and $u_n\big|_{\partial D\setminus\Gamma_n}=u_0$.*

Proof Let us denote by u^* the extension of u by u_0^* on $D^*\setminus D$. Then we apply condition 3 of Proposition 7.2.8 to $D^*\setminus G_n$ and $D^*\setminus\Gamma$ and we find a sequence $u_n^*\in L^{1,2}(D^*\setminus\Gamma_n)$ such that $\widetilde{\nabla}u_n^*\longrightarrow\widetilde{\nabla}u^*$ strongly in $L^2(D^*)$.

For every $n \in \mathbb{N}$, let us denote by u_n the solution of the minimization problem

$$\min\{\int_{D^*\setminus\Gamma_n} |\nabla\phi - \nabla u_n^*|^2\, dx : \phi \in L^{1,2}(D^* \setminus \Gamma_n), \phi = u_0^* \text{ on } D^* \setminus D\}. \tag{7.28}$$

Since $u_n - u_n^* \in L^{1,2}(D^* \setminus \Gamma_n)$ and since $D^* \setminus D$ is Lipschitz, we get that $u_n - u_n^* \in H^1(D^* \setminus D)$. Moreover, there exists a bounded continuous linear extension operator T from $H^1(D^* \setminus D)$ to $H^1(D^*)$. Taking as a test function in (7.28) the function $\phi = u_n^* + T(u_n - u_n^*)$ we get

$$\min\{\int_{D^*\setminus\Gamma_n} |\nabla\phi - \nabla u_n^*|^2\, dx : \phi \in L^{1,2}(D^* \setminus \Gamma_n), \phi = u_0^* \text{ on } D^* \setminus D\}$$

$$\leq \int_{D^*\setminus\Gamma_n} |\nabla T(u_n - u_n^*)|^2 dx.$$

Using the Poincaré inequality on the space $\{u \in H^1(D^*) : \int_{\partial D^*} u dx = 0\}$ and the boundedness of the extension operator T we get

$$\begin{aligned}\int_{D^*\setminus\Gamma_n} |\nabla T((u_n - u_n^*)_{|D^*\setminus D})|^2 dx &\leq C \int_{D^*\setminus D} |\nabla(u_n - u_n^*)|^2 dx \\ &= C \int_{D^*\setminus D} |\nabla(u_0^* - u_n^*)|^2 dx.\end{aligned}$$

This last term converges to zero.

Taking the restrictions of u_n to $D \setminus \Gamma_n$, all the requirements are satisfied and the proof is concluded. ■

Proof of Theorem 7.3.2 (continuation). Back to the proof of Theorem 7.3.2, we observe that the sequence $\{u_n\}_n$ defined in Proposition 7.3.3 satisfies relation (7.26). Indeed, the gradients extended by zero converge strongly in L^2 by construction, hence using the boundedness of A we have

$$\int_D \langle A\widetilde{\nabla} u_n, \widetilde{\nabla} u_n\rangle dx \to \int_D \langle A\widetilde{\nabla} u, \widetilde{\nabla} u\rangle dx.$$

It remains to prove that

$$\int_D u_n f dx \to \int_D u f dx.$$

Fix a connected component U of $\overline{D} \setminus K$ containing a part of the support of f. Two possibilities may occur.

Suppose first that $\text{cap}(U \cap \partial D) > 0$. Since ∂D is Lipschitz and Γ is closed, the set $\partial D \setminus \Gamma$ is relatively open, hence there exists an open Lipschitz set V such that

$\operatorname{supp} f \subseteq V \subseteq U$ and $\operatorname{cap}(\overline{V} \cap \partial D) > 0$. Then, the Poincaré inequality holds true in $H^1((D^* \setminus \overline{D}) \cup V)$, so that $u_n \longrightarrow u$ strongly in $L^2(V)$, which implies

$$\int_V u_n f dx \to \int_V u f dx.$$

Suppose now that $\operatorname{cap}(U \cap \partial D) = 0$. In this case there exists an open Lipschitz set V such that $\operatorname{supp} f \subseteq V \subseteq U$ and $\overline{V} \cap \partial D = \emptyset$. By hypothesis, we have that $\int_V f dx = 0$, hence the Poincaré inequality holds in $H^1(V)/\mathbb{R}$. Consequently

$$\int_V u_n f dx \to \int_V u f dx.$$

The support of f being compactly contained in $\overline{D} \setminus K$, the proof is concluded. ■

If K does not touch ∂D, one could drop the hypothesis on the regularity of D, by simply imposing a constraint of the type $(u - u_0)\varphi \in H_0^1(D \setminus \Gamma)$, where $\varphi \in C^\infty(\mathbb{R}^2)$ is a fixed function such that $\varphi = 0$ on Γ and use a partition of unity.

Remark 7.3.4 The uniqueness of the optimal cut does not hold in general. Trivially, let $u_0 \equiv 0$, $f \equiv 0$, $K_1 = \{A, B\}$, $K = \overline{D}$, $c = 0$. Then any compact connected set containing A and B solves problem (7.25).

In some particular situations one can make explicit at least one solution of the problem. In a symmetric setting, there exists an optimal cut which is also symmetric. Indeed, let $f \equiv 0$ and D be a rectangle; let d be a symmetry line of the rectangle. Suppose that $K_1 = \{A, B\}$ are two points on d and that u_0 is also symmetric with respect to d. It can be easily seen that a solution of problem (7.25) (with $K = \overline{D}$) is the segment AB.

7.4 Eigenvalues of the Neumann Laplacian

Contrary to the case of Dirichlet boundary conditions, the behavior of the eigenvalues of the Neumann Laplacian for nonsmooth variations of the boundary of the geometric domain is (almost) uncontrollable. Several facts can explain this phenomenon, like the following ones.

- For a nonsmooth domain Ω, the injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ may not be compact, hence the spectrum of the Neumann–Laplacian is not necessarily discrete, and may not consist only on eigenvalues. This means that a small geometric perturbation of a smooth boundary may produce essential spectrum. Figure 7.4 shows an example of a set for which the injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is not compact.

  Due to the lack of compactness of the resolvent operators, in the Neumann case the framework introduced in Chapter 6 has to be made precise with more details.

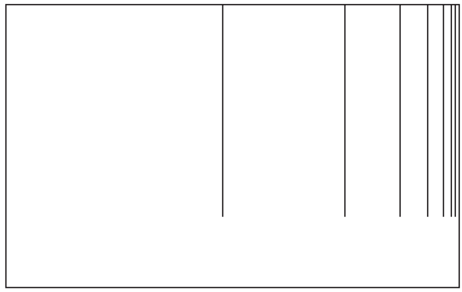

Figure 7.4. A rectangle with an infinite number of cracks collapsing on the right edge.

- The resolvent operators R_{Ω_n} may converge pointwise, where $R_{\Omega_n} : L^2(D) \to L^2(D)$ is naturally defined by $R_{\Omega_n}(f) = 1_{\Omega_n} u_{\Omega_n,f}$ (see Proposition 7.2.2). Here $u_{\Omega_n,f}$ is the weak solution in $H^1(\Omega_n)$ of the equation

$$(7.29)\qquad \begin{cases} -\triangle u + u = f \text{ in } \Omega_n, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega_n. \end{cases}$$

  This convergence is (contrary to the Dirichlet Laplacian), in general not in the operator norm, hence there is no *a priori* convergence of eigenvalues even if all Ω_n and Ω are smooth (see Example 7.4.3 and Figure 7.5).
- A "small" geometric perturbation of the boundary may produce low eigenvalues which highly perturb the spectrum (Example 7.4.3 and Figure 7.5). In a similar situation, for the Dirichlet–Laplacian the "new" eigenvalues produced by small perturbations are large and do not perturb the low part of the spectrum.
- Except particular cases (like monotone cracks, for example), there is no monotonicity of eigenvalues with respect to the domain inclusion.

Let Ω be a bounded Lipschitz domain. The injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is then compact, and the spectrum of the Neumann–Laplacian consists only on eigenvalues:

$$0 = \mu_1(\Omega) \le \mu_2(\Omega) \le \cdots \le \mu_k(\Omega) \le \cdots \to +\infty.$$

For every $k \in \mathbb{N}$, there exists $u_k \in H^1(\Omega) \setminus \{0\}$ such that, in the usual weak sense,

$$(7.30)\qquad \begin{cases} -\triangle u_k = \mu_k(\Omega) u_k \text{ in } \Omega, \\ \frac{\partial u_k}{\partial n} = 0 \text{ on } \partial\Omega, \end{cases}$$

i.e., for every $\phi \in H^1(\Omega)$,

$$\int_\Omega \nabla u_k \nabla \phi dx = \mu_k(\Omega) \int_\Omega u\phi dx.$$

Let us observe that the resolvent operator R_Ω is positive, self-adjoint and compact. Denoting by $\Lambda_k(\Omega)$ its k-th eigenvalue, we have $\Lambda_k(\Omega) = (1 + \mu_k(\Omega))^{-1}$.

In view of the result of Chapter 6, if Ω_n is a perturbation of Ω, in order to get the convergence of the spectrum one can try to prove the norm-convergence of the resolvent operators. The Mosco convergence of H^1-spaces gives straightforwardly the pointwise convergence of the resolvent operators (via extensions by zero outside Ω_n). As Example 7.4.3 shows, in general this convergence is not in the operator norm. We have the following.

Theorem 7.4.1 *Let $H^1(\Omega_n)$ converge in the sense of Mosco to $H^1(\Omega)$. A sufficient condition for*

(7.31)
$$|R_{\Omega_n} - R_\Omega|_{\mathcal{L}(L^2(D))} \to 0$$

is that the following injection is compact:

(7.32)
$$\cup_{n\in\mathbb{N}} H^1(\Omega_n) \hookrightarrow L^2(D).$$

In (7.32), $H^1(\Omega_n)$ is supposed embedded in $L^2(D)$ by the composition of the projection mapping $P : L^2(D) \times L^2(D, \mathbb{R}^N) \mapsto L^2(D)$ with mapping (7.13).

Also note that (7.32) implies that the injection $H^1(\Omega_n) \hookrightarrow L^2(\Omega_n)$ is compact for every $n \in \mathbb{N}$.

Proof Let $|R_{\Omega_n} - R_\Omega|_{\mathcal{L}(L^2(D)} \le |(R_{\Omega_n} - R_\Omega) f_n|_{(L^2(D)} + 1/n$, where $|f_n|_{(L^2(D)} \le 1$. We can assume $f_n \rightharpoonup f$ weakly in $L^2(D)$. Then $R_{\Omega_n} f \to R_\Omega f$ strongly in $L^2(D)$, from Proposition 7.2.4. The sequence $R_{\Omega_n}(f_n - f)$ is bounded in $L^2(D) \times L^2(D, \mathbb{R}^N)$ and converges weakly to 0 in $L^2(D)$. Indeed, for every $\phi \in L^2(D)$, we have

$$\langle R_{\Omega_n}(f_n - f), \phi\rangle_{L^2(D)} = \langle f_n - f, R_{\Omega_n}\phi\rangle_{L^2(D)} \to 0.$$

Using the compact injection (7.32), we get that $R_{\Omega_n}(f_n - f)$ converges strongly to zero in $L^2(D)$. ■

Corollary 7.4.2 *Let D be a bounded design region in $\mathbb{R}^N$ and $\Omega_n, \Omega \subseteq D$ satisfy a uniform cone condition. If $\Omega_n \xrightarrow{H^c} \Omega$, then $\mu_k(\Omega_n) \to \mu_k(\Omega)$.*

Proof The pointwise convergence of the resolvent operators follows from Theorem 7.2.7. To prove that the convergence is in norm, one uses Theorem 7.4.1 relying on the existence of uniformly bounded extension operators $E_n : H^1(\Omega_n) \mapsto H^1(D)$. ■

If Ω_n, Ω are Lipschitz (but not uniformly Lipschitz) such that $\Omega_n \xrightarrow{H^c} \Omega$, then the convergence of the spectrum does not hold in general. Either particular cases of domains linked by channels or pieces of domain disconnecting from a fixed domain were considered in [14], or particular situations where the uniform Lipschitz constraint is weakened, were discussed in [60] (see also [15]).

In fact, in [138] it is proved that a small geometric perturbation of a smooth set may produce “a wild perturbation” of the spectrum. More precisely (see [138]), for every closed set $S \subseteq [0, +\infty)$ and for every $\varepsilon > 0$, there exists an open connected set $\Omega \subseteq B(0, \varepsilon)$ such that the essential spectrum (i.e., the part of the spectrum which does not consist of eigenvalues of finite multiplicity) of the Laplacian coincides with the set S.

From the shape optimization point of view, when Ω is *not smooth* it is very convenient to introduce the *relaxed values* which coincide with the usual eigenvalues as soon as Ω is smooth, and inherit *some* properties of the eigenvalues. A typical example of such relaxed values may be the singular values defined in [112].

Several choices can be made for the definition of the *relaxed values*. For our purpose, it is more suitable to consider the following, which consists in the relaxation of the Rayleigh formula. We set for a bounded open set Ω,

$$\mu_k(\Omega) = \inf_{E \in S_k(\Omega)} \sup_{\phi \in E \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla \phi|^2 dx}{\displaystyle\int_{\mathbb{R}^N} \phi^2 dx}, \tag{7.33}$$

where $S_k(\Omega)$ is the family of all linear spaces of $H^1(\Omega)$ of dimension k.

The scheme of a shape optimization problem for eigenvalues would then be the following:

1. consider the initial problem of eigenvalues on smooth domains;
2. relax the problem for non-smooth domains and replace the eigenvalues by the *relaxed values*;
3. prove the existence of the optimal shape (which is a priori nonsmooth);
4. prove the regularity of an optimal shape, and recover true eigenvalues at the optimum.

In general, this last step is the most difficult one. We will not be able to afford it here, but we will give below an example supporting the introduction of the *relaxed values*.

Example 7.4.3 The example of Courant–Hilbert [89]. Let Ω be a fixed rectangle as in Figure 7.5. By a thin channel of fixed length l and thickness ε, we join to Ω another rectangle of size ν. If we take $\varepsilon = \nu^3$ and make $\varepsilon \to 0$, one can readily observe by taking test functions which are constant on each rectangle and affine on the channel, that the second eigenvalue of $\Omega_{\varepsilon,\nu}$ converges to zero. Since the second eigenvalue of Ω is not vanishing, we have an example of non-convergence of eigenvalues, despite the pointwise convergence of resolvents.

Example 7.4.4 Examples of shape optimization problems for the eigenvalues of the Neumann Laplacian.

We refer to Weinberger [193] and Szegö [188] for the following results:

1. The ball is the unique solution of

$$\max\{\mu_2(\Omega) : \Omega \subseteq \mathbb{R}^N,\ \Omega \text{ smooth },\ |\Omega| = c\}.$$

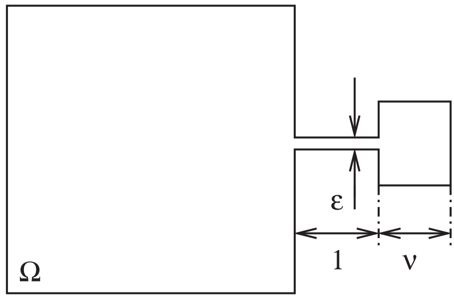

Figure 7.5. The example of Courant and Hilbert [89]: l is fixed and $\varepsilon = \nu^3$.

2. The ball is the unique solution of

$$\min\{\frac{1}{\mu_2(\Omega)} + \frac{1}{\mu_3(\Omega)} : \Omega \subseteq \mathbb{R}^2,\ \Omega \text{ simply connected and smooth },\ |\Omega| = c\}.$$

We refer to [16] for a list of open problems involving the eigenvalues of the Neumann–Laplacian.

Remark 7.4.5 We note that in the first example, the *smoothness* assumption can be eliminated, simply replacing for a bounded open set Ω the eigenvalue by the *relaxed value* introduced in relation (7.33). We still have that the maximizer is the ball. In order to apply the Weinberger idea, one has only to check that for a nonsmooth Ω, we have that

$$\mu_2(\Omega) = \inf_{u\in H^1(\Omega), \int_\Omega u dx=0} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}.$$

This fact supports the idea of replacing the true eigenvalues by the *relaxed values* introduced in (7.33) for more general shape optimization problems (see Remark 7.4.10).

Following Remark 6.1.10, for the Dirichlet boundary conditions the Mosco convergence of the H_0^1-spaces implies the convergence of the full spectrum. For Neumann boundary conditions this is not anymore true; we can only establish the following result.

Theorem 7.4.6 *Let D be a bounded design region in $\mathbb{R}^N$ and let $\Omega_n, \Omega \subseteq D$ such that $H^1(\Omega_n)$ converges in the sense of Mosco to $H^1(\Omega)$ (in the sense of extensions in $L^2(D) \times L^2(D, \mathbb{R}^N)$).*

Then, for every $k \in \mathbb{N}^*$, we have

$$\mu_k(\Omega) \geq \limsup_{n\to\infty} \mu_k(\Omega_n). \tag{7.34}$$

Proof Let $\varepsilon > 0$ and let S_k be a space of dimension k in $H^1(\Omega)$ such that

$$\mu_k(\Omega) \geq \sup_{u \in S_k} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx} - \varepsilon. \tag{7.35}$$

Let $u_1, \dots, u_k \in S_k$ be a basis of the space S_k, such that $\int_\Omega u_i u_j dx = \delta_{ij}$.

From the first Mosco condition, there exist sequences

$$(u_n^1)_n, \dots, (u_n^k)_n$$

such that $u_n^i \in H^1(\Omega_n)$ for every $i = 1, \dots, k$ and $n \in \mathbb{N}$ and

$$u_n^i \longrightarrow u_i \text{ strongly in } L^2(D) \times L^2(D, \mathbb{R}^N).$$

For n large enough, the vectors $u_n^1, \dots, u_n^k$ are independent in $L^2(\Omega_n)$. Indeed, suppose by contradiction that (up to a subsequence)

$$u_n^k = a_n^1 u_n^1 + \dots + a_n^{k-1} u_n^{k-1}.$$

Dividing by $\alpha_n = \max\{|a_n^1|, \dots, |a_n^{k-1}|, 1\}$ and passing to the limit for a subsequence, we get that $u_1, \dots, u_k$ are linearly dependent, which contradicts our assumption.

We correct the sequences $(u_n^1)_n, \dots, (u_n^k)_n$ and transform them into an orthonormal basis of k-dimensional spaces in $L^2(\Omega_n)$. We take, as usual

$$\begin{aligned}
\tilde{u}_n^1 &= u_n^1, \\
\tilde{u}_n^2 &= u_n^2 - \tilde{u}_n^1 \frac{\int_{\Omega_n} \tilde{u}_n^1 u_n^2 dx}{\int_{\Omega_n} \tilde{|}u_n^1|^2 dx}, \\
&\dots \\
\tilde{u}_n^k &= u_n^k - \tilde{u}_n^1 \frac{\int_{\Omega_n} \tilde{u}_n^1 u_n^k dx}{\int_{\Omega_n} \tilde{|}u_n^1|^2 dx} - \dots - \tilde{u}_n^{k-1} \frac{\int_{\Omega_n} \tilde{u}_n^{k-1} u_n^k dx}{\int_{\Omega_n} \tilde{|}u_n^{k-1}|^2 dx}.
\end{aligned}$$

We normalize all these functions in $L^2(\Omega_n)$, and by abuse of notation we still call them $u_n^1, \dots, u_n^k$.

Let us denote by $S_{k,n}$ the space of dimension k generated by $u_n^1, \dots, u_n^k$ in $L^2(\Omega_n)$. There exists a function $u_n \in S_{k,n}$ with $\int_{\Omega_n} u_n^2 dx = 1$ such that

$$\frac{\int_{\Omega_n} |\nabla u_n|^2 dx}{\int_{\Omega_n} u_n^2 dx} = \max_{u \in S_{k,n}} \frac{\int_{\Omega_n} |\nabla u|^2 dx}{\int_{\Omega_n} u^2 dx}.$$

Writing $u_n = a_n^1 u_n^1 + \dots + a_n^k u_n^k$, we get $\sum_1^k |a_n^i|^2 = 1$. Therefore, for a subsequence we have for every $i = 1, \dots, k$ that $a_n^i \to a_i$, $\sum_1^k |a^i|^2 = 1$ and

$$u_n \longrightarrow a^1 u^1 + \dots + a^k u^k := u \in H^1(\Omega),$$

the convergence being strong in the sense of extensions in $L^2(D) \times L^2(D, \mathbb{R}^N)$. Hence

$$\frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx} = \lim_{n\to\infty} \frac{\int_{\Omega_n} |\nabla u_n|^2 dx}{\int_{\Omega_n} u_n^2 dx} = \lim_{n\to\infty} \sup_{\phi\in S_{k,n}} \frac{\int_{\Omega_n} |\nabla\phi|^2 dx}{\int_{\Omega_n} \phi^2 dx}$$

$$\geq \limsup_{n\to\infty} \inf_{S\in S_k(\Omega_n)} \sup_{\phi\in S} \frac{\int_{\Omega_n} |\nabla\phi|^2 dx}{\int_{\Omega_n} \phi^2 dx}.$$

According to (7.35) we get

$$\mu_k(\Omega) + \varepsilon \geq \limsup_{n\to\infty} \mu_k(\Omega_n).$$

Taking $\varepsilon \to 0$ we conclude the proof. ■

A somehow similar, but much weaker result than the one for Dirichlet problems in Corollary 6.2.1, is given below.

Theorem 7.4.7 *Let $D \subseteq \mathbb{R}^2$ be a bounded design region, c, l, M positive constants, and let us denote*

$$\mathcal{U}_{ad} = \{\Omega \subseteq D : \Omega \text{ open},\ |\Omega| = c,\ \sharp\Omega^c \leq l,\ \mathcal{H}^1(\partial\Omega) \leq M\}.$$

Let $F : \mathbb{R}^k_+ \mapsto \mathbb{R}$ be an upper semicontinuous function which is nondecreasing in each variable. Then, the problem

$$\max_{\Omega\in\mathcal{U}_{ad}} F(\mu_1(\Omega), \ldots, \mu_k(\Omega))$$

has at least one solution.

Proof Note that if $\sharp\Omega^c \leq l$, this does not imply that $\sharp\partial\Omega$ is finite. Nevertheless, if the number of the connected components of Ω is less than or equal to k, then the number of the connected components of $\partial\Omega$ is less than or equal to $k + l - 1$. Note that, unless the functional F is trivial, it is enough to search the maximum only among domains Ω which have less than k connected components. Indeed, if $\sharp\Omega = k$, then $\mu_1(\Omega) = \cdots = \mu_k(\Omega) = 0$, hence F is minimal on such a set.

We use the direct methods of the calculus of variations and consider a maximizing sequence for F, say $(\Omega_n)_n$. Up to a subsequence we can assume that $\Omega_n \xrightarrow{H^c} \Omega$. Since $\sharp(\partial\Omega_n) \leq k + l - 1$ we get that $\mathcal{H}^1(\partial\Omega) \leq M$. The properties of the H^c-convergence for sets with uniformly bounded perimeter give (see [59]) $\Omega \in \mathcal{U}_{ad}$. Theorem 7.4.6 together with Proposition 7.2.2 give that $\mu_i(\Omega) \geq \limsup_{n\to\infty} \mu_i(\Omega_n)$. The upper semicontinuity and the monotonicity of F give that Ω is a maximizer. ■

Remark 7.4.8 A way to replace the Hausdorff measure in shape optimization problems involving Neumann boundary conditions is to use the *density perimeter* introduced in [58] and developed in [59]. The reason to replace the Hausdorff measure is related to its bad continuity properties for the Hausdorff convergence.

Let $H : [0, \infty) \to \mathbb{R}$ be a given continuous function with $H(0) = 0$ (this is a "corrector" of the perimeter) and $\gamma > 0$ a fixed number (which plays the role of a scale in the problem).

Definition 7.4.9 *Let* $\gamma > 0$. *The* (γ, H)*-density perimeter of the set* A *is*

$$P_{\gamma,H}(A) = \sup_{\varepsilon \in (0,\gamma)} \left[\frac{m(A^\varepsilon)}{2\varepsilon} + H(\varepsilon)\right], \tag{7.36}$$

where $A^\varepsilon = \cup_{x \in A} B(x, \varepsilon)$.

The family

$$\{\Omega \subseteq D : \Omega \text{ open}, \quad P_{\gamma,H}(\partial\Omega) \leq k\} \tag{7.37}$$

is compact in the H^c-topology, and if $\Omega_n \xrightarrow{H^c} \Omega$, then $1_{\Omega_n} \xrightarrow{L^1} 1_\Omega$ and $P_{\gamma,H}(\partial\Omega) \leq \liminf_{n\to\infty} P_{\gamma,H}(\partial\Omega_n)$.

It is worth noticing that in two-dimensional space, for every compact set A with at most l connected components and for a suitable function H (e.g., $H(x) = -l\pi x/2$) we have $P_{1,H}(A) = \mathcal{H}^1(A)$.

Remark 7.4.10 We end this section by pointing out an open problem. Given a bounded open set D, and indicating by $\mu_k(\Omega)$ the k-th relaxed value on an open subset $\Omega \subseteq D$, prove, or disprove, the existence of a solution for the maximization problem

$$\max\{\mu_k(\Omega) : |\Omega| = c, \Omega \subseteq D\}.$$

Bibliography

1. E. ACERBI, G. BUTTAZZO: *Reinforcement problems in the calculus of variations. (English.* Ann. Inst. H. Poincaré Anal. Non Linéaire, **3** (1986), no. 4, 273–284.
2. R. A. ADAMS: *Sobolev Spaces.* Academic Press, New York (1975).
3. D. R. ADAMS, L.I. HEDBERG: *Function Spaces and Potential Theory*. Springer-Verlag, Berlin (1996).
4. G. ALLAIRE: *Shape optimization by the homogenization method*, Applied Mathematical Sciences **146**. Springer-Verlag, New York (2002).
5. G. ALLAIRE, E. BONNETIER, G. FRANCFORT, F.JOUVE: *Shape optimization by the homogenization method.* Numer. Math., **76** (1997), no. 1, 27–68.
6. G. ALLAIRE, R. V. KOHN: *Optimal design for minimum weight and compliance in plane stress using extremal microstructures.* Europ. J. Mech. A/Solids, **12** (1993), no. 6, 839–878.
7. L. AMBROSIO: *Existence theory for a new class of variational problems.* Arch. Ration. Mech. Anal., **111**(4) (1990), 291–322.
8. L. AMBROSIO: *Corso introduttivo alla Teoria Geometrica della Misura ed alle Superfici Minime*. Scuola Normale Superiore, Pisa (1997).
9. L. AMBROSIO, A. BRAIDES: *Functionals defined on partitions of sets of finite perimeter, I: integral representation and Γ-convergence.* J. Math. Pures Appl., **69** (1990), 285–305.
10. L. AMBROSIO, A. BRAIDES: *Functionals defined on partitions of sets of finite perimeter, II: semicontinuity, relaxation and homogenization.* J. Math. Pures Appl., **69** (1990), 307–333.
11. L. AMBROSIO, G. BUTTAZZO: *An optimal design problem with perimeter penalization.* Calc. Var., **1** (1993), 55–69.
12. L. AMBROSIO, N. FUSCO, D. PALLARA: *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs, Clarendon Press, Oxford (2000).
13. W. ARENDT, D. DANERS: *Uniform convergence for elliptic problems on varying domains*. Math. Nachr., (to appear).
14. J. M. ARRIETA, J. K. HALE, Q. HAN: *Eigenvalue problems for non-smoothly perturbed domains.* J. Differential Equations **91** (1991), no. 1, 24–52.
15. J. M. ARRIETA, A.N. CARVALHO: *Spectral convergence and nonlinear dynamics of reaction-diffusion equations under perturbations of the domain.* J. Differential Equations **199** (2004), no. 1, 143–178.

16. M. S. ASHBAUGH: *Open problems on eigenvalues of the Laplacian.* In “Analytic and geometric inequalities and applications”, Math. Appl. **478**, Kluwer Acad. Publ., Dordrecht (1999), 13–28.
17. M. S. ASHBAUGH, R. D. BENGURIA: *Proof of the Payne-Pólya-Weinberger conjecture.* Bull. Amer. Math. Soc., **25** (1991), 19–29.
18. M. S. ASHBAUGH, D. BUCUR: *On the isoperimetric inequality for the buckling of a clamped plate.* Special issue dedicated to Lawrence E. Payne., Z. Angew. Math. Phys. **54** (2003), no. 5, 756–770.
19. H. ATTOUCH: *Variational Convergence for Functions and Operators.* Pitman, Boston, (1984).
20. H. ATTOUCH, G. BUTTAZZO: *Homogenization of reinforced periodic one-codimensional structures.* Ann. Scuola Norm. Sup. Pisa Cl. Sci., **14** (1987), 465–484.
21. H. ATTOUCH, G. BUTTAZZO, G. MICHAILLE: *Variational Analysis in Sobolev and BV spaces, Applications to PDE and Optimization.* Book in preparation.
22. H. ATTOUCH, C. PICARD: *Variational inequalities with varying obstacles: The general form of the limit problem.* J. Funct. Anal., **50** (1983), 329–386.
23. D. AZE, G. BUTTAZZO: *Some remarks on the optimal design of periodically reinforced structures.* RAIRO Modél. Math. Anal. Numér., **23** (1989), 53–61.
24. J. BAXTER, G. DAL MASO: *Stopping times and Γ-convergence*, Trans. American Math. Soc., **303** (1987), 1–38.
25. M. BELLONI, G. BUTTAZZO, L. FREDDI: *Completion by Gamma-convergence for optimal control problems.* Ann. Fac. Sci. Toulouse Math., **2** (1993), 149–162.
26. M. BELLONI, B. KAWOHL: *A paper of Legendre revisited.* Forum Math. 9 (1997), no. 5, 655–667.
27. M. BELLONI, A. WAGNER: *Newton’s problem of minimal resistance in the class of bodies with prescribed volume.* J. Convex Anal., **10**(2) (2003), 491–500.
28. M. BENDSØE: *Optimal shape design as a material distribution problem.* Struct. Optim., **1** (1989), 193–202.
29. M. BENDSØE, O. SIGMUND: *Topology optimization. Theory, methods and applications.* Springer-Verlag, Berlin (2003).
30. G. BOUCHITTÉ, G. BUTTAZZO: *Characterization of optimal shapes and masses through Monge-Kantorovich equation.* J. Eur. Math. Soc., (JEMS) **3**(2) (2001), 139–168.
31. G. BOUCHITTÉ, G. BUTTAZZO, I. FRAGALÀ: *Convergence of Sobolev spaces on varying manifolds.* J. Geom. Anal., **11**(3) (2001), 399–422.
32. G. BOUCHITTÉ, G. BUTTAZZO, P. SEPPECHER: *Energies with respect to a measure and applications to low dimensional structures.* Calc. Var., **5** (1997), 37–54.
33. G. BOUCHITTÉ, G. BUTTAZZO, P. SEPPECHER: *Shape optimization solutions via Monge-Kantorovich equation.* C. R. Acad. Sci. Paris, **324-I** (1997), 1185–1191.
34. G. BOUCHITTÉ, G. DAL MASO: *Integral representation and relaxation of convex local functionals on $BV(\Omega)$.* Ann. Scuola Norm. Sup. Pisa Cl. Sci., **20**(4) (1993), 483–533.
35. A. BRAIDES: *Γ-convergence for beginners.* Oxford Lecture Series in Mathematics and its Applications, **22**. Oxford University Press, Oxford (2002).
36. H.J. BRASCAMP, E. LIEB, J.M. LUTTINGER: *A general rearrangement inequality for mulptiple integrals*, J. Funct. Anal., **17** (1974), 227–237.
37. H. BREZIS: *Analyse Fonctionelle.* Masson, Paris (1983).
38. H. BREZIS, L. CAFFARELLI, A. FRIEDMAN: *Reinforcement problems for elliptic equations and variational inequalities*, Ann. Mat. Pura Appl., **123**(4) (1980), 219–246.
39. F. BROCK: *Continuous Steiner Symmetrization.* Math. Nachr., **172** (1995), 25–48.

40. F. BROCK, V. FERONE, B. KAWOHL: *A symmetry problem in the calculus of variations.* Calc. Var. **4**(6) (1996), 593–599.
41. D. BUCUR: *Shape analysis for Nonsmooth Elliptic Operators.* Appl. Math. Lett., **9**(3) (1996), 11–16.
42. D. BUCUR: *Characterization for the Kuratowski Limits of a Sequence of Sobolev Spaces*, J. Differential Equations, **151** (1999), 1–19.
43. D. BUCUR: *Uniform concentration-compactness for Sobolev spaces on variable domains*, J. Differential Equations, **162** (2) (2000), 427–450.
44. D. BUCUR: *Regularity of optimal convex shapes.* J. Convex Anal. **10** (2003), no. 2, 501–516.
45. D. BUCUR, G. BUTTAZZO, I. FIGUEIREDO: *On the attainable eigenvalues of the Laplace operator.* SIAM J. Math. Anal., **30** (1999), 527–536.
46. D. BUCUR, G. BUTTAZZO, A. HENROT: *Existence results for some optimal partition problems.* Adv. Math. Sci. Appl., **8** (1998), 571–579.
47. D. BUCUR, G. BUTTAZZO, P. TREBESCHI: *An existence result for optimal obstacles.* J. Funct. Anal., **162** (1999), 96–119.
48. D. BUCUR, G. BUTTAZZO, N. VARCHON: *On the problem of optimal cutting*, SIAM J. Optimization **13**(1) (2002), 157–167.
49. D. BUCUR, A. HENROT: *Minimization of the third eigenvlaue of the Dirichlet Laplacian.* Proc. Roy. Soc. London, Ser. A, **456** (2000), 985–996.
50. D. BUCUR, A. HENROT: *Stability for the Dirichlet problem under continuous Steiner symmetrization.* Potential Anal., **13** (2000), 127–145.
51. D. BUCUR, A. HENROT, J. SOKOLOWSKI, A. ZOCHOWSKI: *Continuity of the elasticity system solutions with respect to boundary variations.* Adv. Math. Sci. Appl., **11**(1) (2001), 57–73.
52. D. BUCUR, P. TREBESCHI: *Shape optimization problem governed by nonlinear state equation.* Proc. Roy. Soc. Edinburgh, **128 A** (1998), 945–963.
53. D. BUCUR, N. VARCHON: *Boundary variation for the Neumann problem* , Ann. Scuola Norm. Sup. Pisa Cl. Sci., **24**(4) (2000), 807–821.
54. D. BUCUR, N. VARCHON: *A duality approach for the boundary variations of Neumann problems*, SIAM J. Math. Anal., **34**(2) (2002), 460–477.
55. D. BUCUR, J. P. ZOLÉSIO: *Wiener's criterion and shape continuity for the Dirichlet problem.* Boll. Un. Mat. Ital., **B 11**(4) (1997), 757–771.
56. D. BUCUR, J. P. ZOLÉSIO: *N-Dimensional Shape Optimization under Capacitary Constraints.* J. Differential Equations, **123**(2) (1995), 504–522.
57. D. BUCUR, J. P. ZOLÉSIO: *Shape continuity for Dirichlet-Neumann problems.* In "Progress in Partial Differential Equations: the Metz Surveys", Pitman Res. Notes Math. Ser. **345**, Longman, Harlow (1996), 53–65.
58. D. BUCUR, J. P. ZOLÉSIO: *Free boundary problems and density perimeter.* J. Differential Equations **126** (1996), no. 2, 224–243.
59. D. BUCUR, J. P. ZOLÉSIO: *Boundary optimization under pseudo curvature constraint.* Ann. Scuola Norm. Sup. Pisa Cl. Sci., **23**(4) (1996), 681–699.
60. V. I. BURENKOV, E. B. DAVIES: *Spectral stability of the Neumann Laplacian.* J. Differential Equations **186** (2002), no. 2, 485–508.
61. G. BUTTAZZO: *Thin insulating layers: the optimization point of view.* Proceedings of "Material Instabilities in Continuum Mechanics and Related Mathematical Problems", Edinburgh 1985–1986, edited by J. M. Ball, Oxford University Press, Oxford (1988), 11–19.
62. G. BUTTAZZO: *Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations.* Pitman Res. Notes Math. Ser. **207**, Longman, Harlow (1989).

63. G. BUTTAZZO: *Semicontinuità inferiore di funzionali definiti su BV.* Lecture notes of a series of lectures held in "Scuola Internazionale di Equazioni Differenziali e Calcolo delle Variazioni", Pisa 14–26 September 1992, Pitagora Editrice, Bologna (1995), 197–235.
64. G. BUTTAZZO, G. DAL MASO: *Shape optimization for Dirichlet problems: relaxed solutions and optimality conditions.* Bull. Amer. Math. Soc., **23** (1990), 531–535.
65. G. BUTTAZZO, G. DAL MASO: *Shape optimization for Dirichlet problems: relaxed formulation and optimality conditions.* Appl. Math. Optim., **23** (1991), 17–49.
66. G. BUTTAZZO, G. DAL MASO: *An existence result for a class of shape optimization problems.* Arch. Rational Mech. Anal., **122** (1993), 183–195.
67. G. BUTTAZZO, G. DAL MASO, A. GARRONI, A. MALUSA: *On the relaxed formulation of Some Shape Optimization Problems.* Adv. Math. Sci. Appl., **7** (1997), 1–24.
68. G. BUTTAZZO, V. FERONE, B. KAWOHL: *Minimum problems over sets of concave functions and related questions.* Math. Nachr., **173** (1995), 71–89.
69. G. BUTTAZZO, L. FREDDI: *Relaxed optimal control problems and applications to shape optimization.* Lecture notes of a course held at the NATO-ASI Summer School "Nonlinear Analysis, Differential Equations and Control", Montreal, July 27 – August 7, 1998, Kluwer, Dordrecht (1999), 159–206.
70. G. BUTTAZZO, M. GIAQUINTA, S. HILDEBRANDT: *One-dimensional Calculus of Variations: an Introduction.* Oxford University Press, Oxford (1998).
71. G. BUTTAZZO, P. GUASONI: *Shape optimization problems over classes of convex domains.* J. Convex Anal., **4** (1997), 343–351.
72. G. BUTTAZZO, B. KAWOHL: *On Newton's problem of minimal resistance.* Math. Intelligencer, **15** (1993), 7–12.
73. G. BUTTAZZO, C. TIMOFTE: *On the relaxation of some optimal partition problems.* Adv. Math. Sci. Appl. **12**(2) (2002), 509–520.
74. G. BUTTAZZO, A. WAGNER: *On the optimal shape of a rigid body supported by an elastic membrane.* Nonlinear Anal., **39** (2000), 47–63.
75. E. CABIB: *A relaxed control problem for two-phase conductors.* Ann. Univ. Ferrara - Sez. VII - Sc. Mat., **33** (1987), 207–218.
76. E. CABIB, G. DAL MASO: *On a class of optimum problems in structural design.* J. Optimization Theory Appl., **56** (1988), 39–65.
77. G. CARLIER, T. LACHAND-ROBERT: *Regularity of solutions for some variational problems subject to a convexity constraint.* Comm. Pure Appl. Math., **54** (2001), no. 5, 583–594.
78. A. CHAMBOLLE: *A density result in two-dimensional linearized elasticity and applications*, Arch. Ration. Mech. Anal. **167**(3) (2003), 211–233.
79. A. CHAMBOLLE, F. DOVERI: *Continuity of Neumann linear elliptic problems on varying two-dimensional bounded open sets.*, Commun. Partial Differ. Equations, **22**(5-6) (1997), 811–840.
80. D. CHENAIS: *On the existence of a solution in a domain identification problem.* J. Math. Anal. Appl., **52** (1975), 189–219.
81. D. CHENAIS: *Homéomorphisme entre ouverts lipschitziens.* Ann. Mat. Pura Appl. **118**(4) (1978), 343–398.
82. M. CHIPOT, G. DAL MASO: *Relaxed shape optimization: the case of nonnegative data for the Dirichlet problem.* Adv. Math. Sci. Appl., **1** (1992), 47–81.
83. A. CHERKAEV: *Variational methods for structural optimization.* Applied Mathematical Sciences, 140. Springer-Verlag, New York (2000).

84. D. CIORANESCU, F. MURAT: *Un terme étrange venu d'ailleurs.* Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. II (Paris, 1979/1980), pp. 98–138, 389–390, Res. Notes in Math., 60, Pitman, Boston, Mass.-London (1982).
85. S. COX: *The generalized gradient at a multiple eigenvalue.* J. Funct. Anal., **133**(1) (1995), 30–40.
86. S. COX, B. KAWOHL: *Circular symmetrization and extremal Robin conditions,* Z. Angew. Math. Phys., **50** (1999), 301–311 .
87. M. COMTE, T. LACHAND-ROBERT: *Existence of minimizers for Newton's problem of the body of minimal resistance under a single impact assumption.* J. Anal. Math. **83** (2001), 313–335.
88. M. COMTE, T. LACHAND-ROBERT: *Newton's problem of the body of minimal resistance under a single-impact assumption.* Calc. Var. Partial Differential Equations **12** (2001), no. 2, 173–211.
89. R. COURANT, D. HILBERT: *Methods of mathematical physics.* Vol. I. Interscience Publishers, Inc., New York, N.Y. (1953).
90. B. DACOROGNA: *Direct Methods in the Calculus of Variations.* Appl. Math. Sciences **78**, Springer-Verlag, Berlin (1989).
91. G. DAL MASO: *An Introduction to* Γ*-convergence.* Birkhäuser Boston (1993).
92. G. DAL MASO: *On the integral representation of certain local functionals.* Ricerche Mat., **32** (1983), 85–113.
93. G. DAL MASO: *Some necessary and sufficient conditions for convergence of sequences of unilateral convex sets.* J. Funct. Anal, **62** (1985), 119–159.
94. G. DAL MASO: Γ*-convergence and* μ*-capacities.* Ann. Scuola Norm. Sup. Pisa, **14** (1988), 423–464.
95. G. DAL MASO, A. DE FRANCESCHI: *Limits of nonlinear Dirichlet problems in varying domains.* Manuscripta Math., **61** (1988), 251–268.
96. G. DAL MASO, F. EBOBISSE, M. PONSIGLIONE: *A stability result for nonlinear Neumann problems under boundary variations.* J. Math. Pures Appl., **82**(9) (2003), no. 5, 503–532.
97. G. DAL MASO, A. GARRONI: *New results on the asymptotic behaviour of Dirichlet problems in perforated domains.* Math. Mod. Meth. Appl. Sci., **3** (1994), 373–407.
98. G. DAL MASO, P. LONGO: Γ*-limits of obstacles.* Ann. Mat. Pura Appl., **128** (1980), 1–50.
99. G. DAL MASO, A. MALUSA: *Approximation of relaxed Dirichlet problems by boundary value problems in perforated domains.* Proc. Roy. Soc. Edinburgh Sect. **A 125** (1995), no. 1, 99–114.
100. G. DAL MASO, U. MOSCO: *Wiener's criterion and* Γ*-convergence.* Appl. Math. Optim., **15** (1987), 15–63.
101. G. DAL MASO, U. MOSCO: *Wiener criteria and energy decay for relaxed Dirichlet problems.* Arch. Rational Mech. Anal., **95** (1986), 345–387.
102. G. DAL MASO, F. MURAT: *Asymptotic behavior and correctors for Dirichlet problems in perforated domains with homogeneous monotone operators.* Ann. Scuola Norm. Sup. Pisa, **24** (1997), 239–290.
103. G. DAL MASO, R. TOADER: *A model for the quasi-static growth of a brittle fracture: existence and approximation results*. Arch. Ration. Mech. Anal. **162** (2) (2002), 101–135.
104. E.N. DANCER, D. DANERS: *Domain perturbation for elliptic equations subject to Robin boundary conditions*. J. Differential Equations **138** (1997), 86–132.

105. D. DANERS: *Dirichlet problems on varying domains*. J. Differential Equations **188** (2003), 591–624.
106. E. DE GIORGI: *Teoremi di semicontinuità nel calcolo delle variazioni.* Notes of a course given at the Instituto Nazionale de Alta Matematica, Rome (1968).
107. E. DE GIORGI: *Γ-convergenza e G-convergenza.* Boll. Un. Mat. Ital., (5) **14-A** (1977), 213–224.
108. E. DE GIORGI, F. COLOMBINI, L.C. PICCININI: *Frontiere orientate di misura minima e questioni collegate*. Quaderni della Scuola Normale Superiore, Pisa, (1972).
109. E. DE GIORGI, G. DAL MASO, P. LONGO: *Γ-limiti di ostacoli.* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., (8) **68** (1980), 481–487.
110. E. DE GIORGI, T. FRANZONI: *Su un tipo di convergenza variazionale.* Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur., (8) **58** (1975), 842–850.
111. M. DELFOUR, J.-P. ZOLÉSIO: *Shapes and geometries. Analysis, differential calculus, and optimization*, Advances in Design and Control (SIAM), Philadelphia, PA (2001).
112. N. DUNFORD, J.T. SCHWARTZ: *Linear Operators, Part II: Spectral Theory*, Interscience Publishers, New York, London, (1963).
113. I. EKELAND, R. TEMAM: *Convex Analysis and Variational Problems.* Studies in Mathematics and its Applications **1**, North-Holland, Amsterdam (1976).
114. P. ESPOSITO, G. RIEY: *Asymptotic behaviour of a thin insulation problem*. J. Convex Anal. **10** (2003), no. 2, 379–388.
115. M.J. ESTEBAN, P.L. LIONS: *Γ-convergence and the concentration-compactness method for some variational problems with lack of compactness.* Ricerche di Matematica, **36** (1), (1987), 73–101.
116. L. C. EVANS: *Partial differential equations and Monge-Kantorovich mass transfer.* Current Developments in Mathematics, Cambridge MA (1997), 65–126, Int. Press, Boston (1999).
117. L. C. EVANS, W. GANGBO: *Differential Equations Methods for the Monge-Kantorovich Mass Transfer Problem.* Mem. Amer. Math. Soc. **137**, Providence (1999).
118. L. C. EVANS, R. F. GARIEPY: *Measure Theory and Fine Properties of Functions.* Studies in Advanced Math., CRC Press, Ann Harbor (1992).
119. G. FABER: *Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt.* Sitz. Ber. Bayer. Akad. Wiss., (1923),169–172.
120. H. FEDERER: *Geometric Measure Theory.* Springer-Verlag, Berlin (1969).
121. S. FINZI VITA: *Numerical shape optimization for relaxed Dirichlet problems.* Math. Models Methods Appl. Sci. **3** (1) (1993), 19–34.
122. S. FINZI VITA: *Constrained shape optimization for Dirichlet problems: discretization via relaxation.* Adv. Math. Sci. Appl. **9** (2) (1999), 581–596.
123. G. A. FRANCFORT, J.-J. MARIGO: *Revisiting brittle fracture as an energy minimization problem.* J. Mech. Phys. Solids **46** (1998), no. 8, 1319–1342.
124. G. A. FRANCFORT, F. MURAT: *Homogenization and optimal bounds in linear elasticity.* Arch. Rational Mech. Anal., **94** (1986), 307–334.
125. J. FREHSE: *Capacity methods in the theory of partial differential equations.* Jahresber. Deutsch. Math.-Verein. **84** (1) (1982), 1–44.
126. B. FUGLEDE: *Finely harmonic functions*, Lecture notes in Math., **289**, Springer, Berlin, Heidelberg, New York (1972).
127. W. GANGBO, R. J. McCANN: *The geometry of optimal transportation.* Acta Math., **177** (1996), 113–161.
128. S. GARREAU, P. GUILLAUME, M. MASMOUDI: *The topological asymptotic for PDE systems: the elasticity case.* SIAM J. Control Optim. **39** (2001), no. 6, 1756–1778.

129. D. GILBARG, N.S. TRUDINGER: *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, Second Edition, Revised Third Printing (1998).
130. V. GIRAULT, P.A. RAVIART: *Finite element methods for Navier-Stokes Equations*, Springer-Verlag, Berlin 1986.
131. E. GIUSTI: *Minimal Surfaces and Functions of Bounded Variation.* Birkäuser, Boston (1984).
132. E. GIUSTI: *Metodi diretti nel calcolo delle variazioni.* Unione Matematica Italiana, Bologna (1994).
133. F. GOLAY, P. SEPPECHER: *Locking materials and topology of optimal shapes.* Eur. J. Mech. A Solids **20** (4) (2001), 631–644.
134. H. H. GOLDSTINE: *A History of the Calculus of Variations from the 17th through the 19th Century.* Springer-Verlag, Heidelberg (1980).
135. P. GUASONI: *Problemi di ottimizzazione di forma su classi di insiemi convessi.* Tesi di Laurea, Università di Pisa, 1995-1996.
136. M. HAYOUNI: *Sur la minimisation de la première valeur propre du laplacien.* C. R. Acad. Sci. Paris Série. I Math. **330** (7) (2000), 551–556.
137. L.I. HEDBERG: *Spectral Synthesis in Sobolev Spaces and Uniqueness of Solutions of Dirichlet Problems*. Acta Math., **147** (1981), no. 3-4, 237–263.
138. R. HEMPEL, L. A. SECO, B. SIMON: *The essential spectrum of Neumann Laplacians on some bounded singular domains,* J. Funct. Anal. **102** (1991), no. 2, 448–483.
139. A. HENROT: *Continuity with respect to the domain for the Laplacian: a survey*. Control and Cybernetics, **23** (1994), no.3, 427–443.
140. A. HENROT, M. PIERRE: *Optimisation de forme* (book in preparation).
141. J. HEINONEN, T. KILPELAINEN, O. MARTIO: *Nonlinear potential theory of degenerate elliptic equations.* Clarendon Press, Oxford (1993).
142. D. HORSTMANN, B. KAWOHL, P. VILLAGGIO: *Newton's aerodynamic problem in the presence of friction.* NoDEA Nonlinear Differential Equations Appl. **9** (3) (2002), 295–307.
143. A. D. IOFFE: *On lower semicontinuity of integral functionals I. II.* SIAM J. Control Optim., **15** (1977), 521–538 and 991–1000.
144. O. KAVIAN: *Introduction à la théorie des points critiques et applications aux problèmes elliptiques.* Mathématiques et applications, **13**, Springer-Verlag, (1993).
145. T. KILPELAINEN, J. MALY: *Supersolutions to degenerate elliptic equations on quasi open sets*, Commun. in Partial Diff. Eq., **17** (1992), 371–405.
146. M.V. KELDYSH: *On the Solvability and Stability of the Dirichlet Problem.* Amer. Math. Soc. Translations, **51-2** (1966), 1–73.
147. D. KINDERLEHRER, G. STAMPACCHIA: *An Introduction to Variational Inequalities and their Applications.* Academic Press, New York (1980).
148. R. V. KOHN, G. STRANG: *Optimal design and relaxation of variational problems, I,II,III.* Comm. Pure Appl. Math., **39** (1986), 113–137, 139–182, 353–377.
149. R. V. KOHN, M. VOGELIUS: *Relaxation of a variational method for impedance computed tomography.* Comm. Pure Appl. Math., **40** (1987), 745–777.
150. E. KRAHN: *Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises.* Math. Ann., **94** (1924), 97–100.
151. E. KRAHN: *Über Minimaleigenschaften der Kugel in drei un mehr Dimensionen.* Acta Comm. Univ. Dorpat. **A9** (1926), 1–44.
152. T. LACHAND-ROBERT, E. OUDET: *Minimizing within convex bodies using a convex hull method.* SIAM J. Optim. (to appear).

153. T. LACHAND-ROBERT, M.A. PELETIER: *An example of non-convex minimization and an application to Newton's problem of the body of least resistance.* Ann. Inst. H. Poincaré Anal. Non Linéaire **18** (2) (2001), 179–198
154. T. LACHAND-ROBERT, M.A. PELETIER: *Newton's problem of the body of minimal resistance in the class of convex developable functions.* Math. Nachr. **226** (2001), 153–176.
155. E.H. LIEB: *On the lowest eigenvalue of the Laplacian for the intersection of two domains.* Invent. Math., **74** (1983), 441–448.
156. P.L. LIONS: *The concentration-compactness principle in the Calculus of Variations. The locally compact case, part 1.* Ann. Inst. Poincaré, **1** (2) (1984), 109–145.
157. W. LIU, P. NEITTAANMAKI, D. TIBA: *Sur les problèmes d'optimisation structurelle.* C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), no. 1, 101–106.
158. K. A. LURIE, A. V. CHERKAEV: *G-closure of a Set of Anisotropically Conductivity Media in the Two-Dimensional Case.* J. Optimization Theory Appl., **42** (1984), 283–304.
159. K. A. LURIE, A. V. CHERKAEV: *G-closure of Some Particular Sets of Admissible Material Characteristics for the Problem of Bending of Thin Elastic Plates.* J. Optimization Theory Appl., **42** (1984), 305–316.
160. P. MARCELLINI: *Nonconvex integrals of the calculus of variations.* Methods of nonconvex analysis (Varenna, 1989), 16–57, Lecture Notes in Math., 1446, Springer, Berlin, (1990).
161. U. MASSARI, M. MIRANDA: *Minimal surfaces of codimension one*, North-Holland, Amsterdam (1984).
162. V. G. MAZ'JA: *Sobolev Spaces.* Springer-Verlag, Berlin (1985).
163. A. MIELE: *Theory of Optimum Aerodynamic Shapes.* Academic Press, New York (1965).
164. C. B. MORREY: *Multiple integrals in the calculus of variations.* Springer, Berlin, (1966).
165. U. MOSCO: *Convergence of convex sets and of solutions of variational inequalities.* Adv. in Math., **3** (1969), 510–585.
166. U. MOSCO: *Composite media and asymptotic Dirichlet forms.* J. Funct. Anal., **123** (1994), 368–421.
167. F. MURAT, J. SIMON: *Sur le contrôle par un domaine géometrique.* Preprint 76015, Univ. Paris VI, (1976).
168. F. MURAT, L. TARTAR: *Calcul des variations et homogénéisation.* Proceedings of "Les Méthodes de l'homogénéisation: Théorie et applications en physique", Ecole d'Eté d'Analyse Numérique C.E.A.-E.D.F.-INRIA, Bréau-sans-Nappe 1983, Collection de la direction des études et recherches d'electricité de France **57**, Eyrolles, Paris, (1985), 319–369.
169. F. MURAT, L. TARTAR: *Optimality conditions and homogenization.* Proceedings of "Nonlinear variational problems", Isola d'Elba 1983, Res. Notes in Math. **127**, Pitman, London, (1985), 1–8.
170. S. A. NAZAROV, J. SOKOLOWSKI *Asymptotic analysis of shape functionals*, Journal de Mathématiques pures et appliquées. **82**(2003), 125–196.
171. P. NEITTAANMAKI, D. TIBA: *Shape optimization in free boundary systems. Free boundary problems: theory and applications*, II (Chiba, 1999), 334–343, GAKUTO Internat. Ser. Math. Sci. Appl., **14**, Gakkōtosho, Tokyo (2000).
172. R. OSSERMAN: *Bonnesen-style isoperimetric inequalities*. Amer. Math. Monthly, **86** (1979), 1–29.

173. E. OUDET: *Numerical Minimization of eigenmodes of a membrane with respect to the domain*, ESAIM COCV **10**(2004), 315–330 .
174. S. OZAWA: *An asymptotic formula for the eigenvalues of the Laplacian in a three-dimensional domain with a small hole.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. **30** (1983), no. 2, 243–257.
175. S. OZAWA: *A shape of eigenfunction of the Laplacian under singular variation of domains. II. The Neumann boundary condition.* Proc. Japan Acad. Ser. A Math. Sci. **60** (1984), no. 4, 125–126
176. O. PIRONNEAU: *Optimal Shape Design for Elliptic Systems.* Springer-Verlag, Berlin, 1984.
177. G. POLYA: *On the characteristic frequencies of a symmetric membrane*, Math. Zeit., **63** (1955), 331–337.
178. S. T. RACHEV, L. RÜSCHENDORF: *Mass transportation problems. Vol. I Theory, Vol. II Applications.* Probability and its Applications, Springer-Verlag, Berlin (1998).
179. Y.G. RESHETNYAK: *Weak convergence of completely additive vector measures on a set.* Sibirskii Math. Zh., **9** (1968), 1039–1045, 1386–1394.
180. J. RAUCH, M. TAYLOR: *Potential and Scattering Theory on Wildly Perturbed Domains*. J. Funct. Analysis, **18** (1975), 27–59.
181. R. T. ROCKAFELLAR: *Convex Analysis.* Princeton University Press, Princeton (1972).
182. C. A. ROGERS: *Hausdorff Measures*. Cambridge University Press, Princeton (1972).
183. O. SIGMUND: *A 99 line topology optimization code written in Matlab*, Struct. Multidisc. Optim. **21** (2001), 120–127.
184. F. SIMONDON: *Domain perturbation for parabolic quasilinear problems.* Commun. Appl. Anal. **4** (1) (2000), 1–12.
185. J. SOKOLOWSKI, A. ZOCHOVSKI: *On the topological derivative in shape optimization.* SIAM J. Control Optim. **37** (1999), no. 4, 1251–1272
186. J. SOKOLOWSKI, J.-P. ZOLÉSIO: *Introduction to shape optimization. Shape sensitivity analysis.* Springer Series in Computational Mathematics, 16. Springer-Verlag, Berlin (1992).
187. V. ŠVERÁK: *On optimal shape design.* J. Math. Pures Appl., **72** (1993), 537–551.
188. G. SZEGO: *Inequalities for certain eigenvalues of a membrane of given area.* J. Rational Mech. Anal. **3**, (1954), 343–356.
189. L. TARTAR: *Estimations Fines des Coefficients Homogénéises.* Ennio De Giorgi Colloqium, Edited by P.Krée, Res. Notes in Math. **125** Pitman, London, (1985) 168–187.
190. R. TOADER: *Wave equation in domains with many small obstacles.* Asymptot. Anal. **23** (2000), no. 3-4, 273–290.
191. N. VAN GOETHEM: *Variational problems on classes of convex domains.* Comm. Appl. Anal. **8**, (2004), 353–372.
192. A. WAGNER: *A remark on Newton's resistance formula.* ZAMM Z. Angew. Math. Mech. 79 (6) (1999), 423–427.
193. H. WEINBERGER: *An isoperimetric inequality for the N-dimensional free membrane problem.* J. Rat. Mech. Anal., **5** (1956), 633–636.
194. M. WILLEM: *Analyse harmonique réelle.* Hermann, Paris (1995).
195. S. A. WOLF, J. B. KELLER: *Range of the first two eigenvalues of the Laplacian.* Proc. Roy. Soc. Lond., **A 447** (1994), 397–412.
196. W.P. ZIEMER: *Weakly Differentiable Functions.* Springer-Verlag, Berlin, 1989.
197. X. ZHONG: *On nonhomogeneous quasilinear elliptic equations.* Ann. Acad. Sci. Fenn. Math. Diss. No. 117, (1998).

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