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Mixed Boundary Value Problems for the Stationary Navier–Stokes System in Polyhedral Domains

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Abstract

The authors consider boundary value problems for the Navier–Stokes system in a polyhedral domain, where different boundary conditions (in particular, Dirichlet, Neumann, slip conditions) are arbitrarily combined on the faces of the polyhedron. They prove existence and regularity theorems for weak solutions in weighted (and nonweighted) L_p Sobolev and Hölder spaces with sharp integrability and smoothness parameters.

1. Introduction

Steady-state flows of incompressible viscous Newtonian fluids are modeled by the Navier–Stokes equations

$$-\nu\,\Delta u + (u\cdot\nabla)\,u + \nabla p = f, \quad \nabla\cdot u = g \tag{1.1}$$

for the velocity u and the pressure p. For this system, one can consider different boundary conditions. For example on solid walls, we have the Dirichlet condition $u = 0$. On other parts of the boundary (an artificial boundary such as the exit of a channel, or a free surface) a no-friction condition (Neumann condition) $2\nu\varepsilon(u)\,n - pn = 0$ may be useful. Here $\varepsilon(u)$ denotes the matrix with the components $\frac{1}{2}(\partial_{x_i}u_j + \partial_{x_j}u_i)$, and n is the outward normal. Note that the Neumann problem naturally appears in the theory of hydrodynamic potentials (see [12]). It is also of interest to consider boundary conditions containing components of the velocity and of the friction. Frequently used combinations are the normal component of the velocity and the tangential component of the friction (slip condition for uncovered fluid surfaces) or the tangential component of the velocity and the normal component of the friction (condition for in/out-stream surfaces).

In the present paper, we consider mixed boundary value problems for the system (1.1) in a three-dimensional domain $\mathcal{G}$ of polyhedral type, where components of the velocity and/or the friction are given on the boundary. To be more precise, we have one of the following boundary conditions on each face Γ_j:

(i) $u = h,$
(ii) $u_\tau = h, \quad -p + 2\nu\varepsilon_{n,n}(u) = \phi,$
(iii) $u_n = h, \quad 2\nu\varepsilon_{n,\tau}(u) = \phi,$
(iv) $-pn + 2\nu\varepsilon_n(u) = \phi,$

where $u_n = u \cdot n$ denotes the normal and $u_\tau = u - u_n n$ the tangential component of u, $\varepsilon_n(u)$ is the vector $\varepsilon(u)\, n$, $\varepsilon_{n,n}(u)$ is the normal component and $\varepsilon_{n,\tau}(u)$ the tangential component of $\varepsilon_n(u)$.

Weak solutions, that is, variational solutions $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$, always exist if the data are sufficiently small. In the case when the boundary conditions (ii) and (iv) disappear, such solutions exist for arbitrary f (see the books by LADYZHENSKAYA [12], TEMAM [25], GIRAULT & RAVIART [5]). Our goal is to prove regularity assertions for weak solutions. As is well known, the local regularity result

$$(u, p) \in W^{l,s} \times W^{l-1,s}$$

is valid outside an arbitrarily small neighborhood of the edges and vertices if the data are sufficiently smooth. Here $W^{l,s}$ denotes the Sobolev space of functions which belong to L_s together with all derivatives up to order l. The same result holds for the Hölder space $C^{l,\sigma}$. Since solutions of elliptic boundary value problems in general have singularities near singular boundary points, the result cannot be globally true in $\mathcal{G}$ without any restrictions on l and s. Here we give a few particular regularity results which are consequences of more general theorems proved in the present paper. Suppose that the data belong to corresponding Sobolev or Hölder spaces and satisfy certain compatibility conditions on the edges. Then the following smoothness of the weak solution is guaranteed and is the best possible.

- If (u, p) is a solution of the Dirichlet problem in an arbitrary polyhedron or a solution of the Neumann problem in an arbitrary Lipschitz graph polyhedron, then

$$(u, p) \in W^{1,3+\varepsilon}(\mathcal{G})^3 \times L_{3+\varepsilon}(\mathcal{G}),$$
$$(u, p) \in W^{2,4/3+\varepsilon}(\mathcal{G})^3 \times W^{1,4/3+\varepsilon}(\mathcal{G}),$$
$$u \in C^{0,\varepsilon}(\mathcal{G})^3.$$

  Here ε is a positive number depending on the domain $\mathcal{G}$.
- If (u, p) is a solution of the Dirichlet problem in a convex polyhedron, then

$$(u, p) \in W^{1,s}(\mathcal{G})^3 \times L_s(\mathcal{G}) \quad \text{for all } s,\ 1 < s < \infty,$$
$$(u, p) \in W^{2,2+\varepsilon}(\mathcal{G})^3 \times W^{1,2+\varepsilon}(\mathcal{G}),$$
$$(u, p) \in C^{1,\varepsilon}(\mathcal{G})^3 \times C^{0,\varepsilon}(\mathcal{G}).$$

- If (u, p) is a solution of the mixed problem in an arbitrary polyhedron with the Dirichlet and Neumann boundary conditions prescribed arbitrarily on different faces, then

$$(u, p) \in W^{2,8/7+\varepsilon}(\mathcal{G})^3 \times W^{1,8/7+\varepsilon}(\mathcal{G}).$$

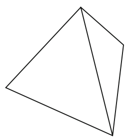
– Let (u, p) be a solution of the mixed boundary value problem with slip condition (iii) on one face Γ_1 and Dirichlet condition on the other faces. Then

$$(u, p) \in W^{1,3+\varepsilon}(\mathcal{G})^3 \times L_{3+\varepsilon}(\mathcal{G}) \quad \text{if } \theta < \tfrac{3}{2}\pi,$$
$$(u, p) \in W^{2,2+\varepsilon}(\mathcal{G})^3 \times W^{2,2+\varepsilon}(\mathcal{G}) \quad \text{if } \mathcal{G} \text{ is convex and } \theta < \pi/2,$$
$$(u, p) \in C^{1,\varepsilon}(\mathcal{G})^3 \times C^{0,\varepsilon}(\mathcal{G}) \quad \text{if } \mathcal{G} \text{ is convex and } \theta < \pi/2,$$

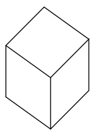
where θ is the maximal angle between Γ_1 and the adjoining faces.

General facts of this kind imply various precise regularity statements for special domains. Let us consider for example the flow outside a regular polyhedron G. On the boundary of G, the Dirichlet condition is prescribed. Then we obtain $(u, p) \in W^{2,s} \times W^{1,s}$ on every bounded subdomain of the complement of G, where the best possible value for s is (with all digits shown correct)

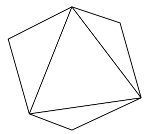
$s = 1.3516...$
for a regular tetrahedron,



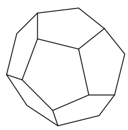
$s = 1.3740...$
for a cube,



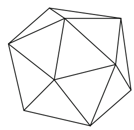
$s = 1.4133...$
for a regular octahedron,



$s = 1.4335...$ for a regular dodecahedron,



$s = 1.5248...$ for a regular icosahedron



Note that the above values for s are zeros of certain transcendental functions.

In the last decades, numerous papers appeared which treat boundary value problems for elliptic equations and systems in piecewise smooth domains. For the stationary linear Stokes system see for example, the references in the book [10]. The properties of solutions of the Dirichlet problem to the nonlinear Navier–Stokes system in two-dimensional polygonal domains were studied in papers by Kondrat'ev [8], Kellogg & Osborn [7], Kalex [6], Orlt & Sändig [21]. In particular, Kellogg and Osborn proved that the solution of the Dirichlet problem belongs to $W^{2,2}(\mathcal{G})^3 \times W^{1,2}(\mathcal{G})$ if $f \in L_2(\mathcal{G})$ and the polygon $\mathcal{G}$ is convex. Kalex, Orlt and Sändig considered solutions of mixed boundary value problems in polygonal domains. Mixed boundary value problems with boundary conditions (i) and (iii) in three-dimensional domains with smooth nonintersecting edges were handled by Solonnikov [23], Maz'ya et al. [14]. They proved in particular the solvability in weighted Sobolev and Hölder spaces. For the case of the Dirichlet problem and a polyhedral domain, solvability and regularity results in weighted Sobolev and Hölder spaces were proved by Maz'ya & Plamenevskiĭ [13]. Concerning regularity results in L_2 Sobolev spaces, we refer also to the papers of Nicaise [20] (Dirichlet problem), Ebmeyer & Frehse [4] [mixed problem with boundary conditions (i) and (iii)]. Ebmeyer and Frehse proved that $(u, p) \in W^{s,2}(\mathcal{G})^3 \times W^{s-1,2}(\mathcal{G})$ with arbitrary real $s < 3/2$ if the angle at the edge, where the boundary conditions change, is less than π. Finally, we mention the papers by Deuring & von Wahl [2], Dindos & Mitrea [3] dealing with the Navier–Stokes system in Lipschitz domains.

We give a short outline of the present paper. Section 2 concerns the existence of weak solutions in $W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$. In Section 3, we introduce and study weighted Sobolev and Hölder space. Here the weights are powers of the distances ρ_j and r_k to the vertices and edges of the domain $\mathcal{G}$, respectively. In particular, we establish imbedding theorems for these spaces. In contrast to the papers [13,14,23], we use weighted spaces with "nonhomogeneous" norms. The weighted Sobolev space $W^{l,s}_{\beta,\delta}(\mathcal{G})$ in our paper is defined as the set of all functions u in $\mathcal{G}$ such that

$$\rho_j^{\beta_j - l + |\alpha|} \prod_{k \in X_j} \left(\frac{r_k}{\rho_j}\right)^{\delta_k} \partial_x^\alpha u \in L_s(\mathcal{G} \cap \mathcal{U}_j)$$

for all $|\alpha| \leqq l$ and for all j. Here $\mathcal{U}_j$ is a neighborhood of the jth vertex, X_j is the set of the indices k such that this vertex is an end point of the edge M_k, β_j and δ_k are the components of β and δ, respectively. The norm in the weighted Hölder space $C^{l,\sigma}_{\beta,\delta}(\mathcal{G})$ has a similar structure. The use of weighted spaces with nonhomogeneous norms has several advantages. First, these spaces are applicable to a wider class of boundary value problems. For $\beta = 0$ and $\delta = 0$ they are closely related to the nonweighted spaces [see (4.12)]. Furthermore in some cases (for example, the Dirichlet problem when the edge angles are less than π), one can obtain higher regularity results when considering solutions in weighted spaces with nonhomogeneous norms. So we can partially improve the results in [13,14,23]. The main results of the paper are contained in Section 4. For the proofs, we use results from our previous papers [18,19], where we studied mixed boundary value problems for the linear Stokes system. We show in particular that the weak solution (u, p) belongs to the weighted space $W^{2,s}_{\beta,\delta}(\mathcal{G})^3 \times W^{1,s}_{\beta,\delta}(\mathcal{G})$ if the data are from corresponding spaces, satisfy certain compatibility conditions, and the numbers $\frac{5}{2} - \beta_j - \frac{3}{s}$ and $2 - \delta_k - \frac{2}{s}$ are positive and sufficiently small. The precise conditions on β and δ are given in terms of eigenvalues of certain operator pencils. The general results in Section 4 together with estimates for the eigenvalues of these pencils (see [10]) allow us in particular to deduce regularity assertions in nonweighted Sobolev and Hölder spaces. A number of examples are given at the end of Section 4.

The last section concerns the solvability in the space $W^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times W^{0,s}_{\beta,\delta}(\mathcal{G})$, where s may be less than two. Here, we assume that the Dirichlet condition is given on at least one of the adjoining faces of every edge M_k. One of our results is the following. Let $\mathcal{G}$ be an arbitrary polyhedron. Then the problem (1.1) with $g = 0$ and Dirichlet condition $u = 0$ on the boundary has a weak solution $(u, p) \in W^{1,s}(\mathcal{G})^3 \times L_s(\mathcal{G})$ for arbitrary $f \in W^{-1,s}(\mathcal{G})^3$, $3/2 < s < 3$, provided the norm of f is sufficiently small. The same result holds for the mixed problem with boundary conditions (i)–(iii) if we suppose that the angles at the edges where the boundary conditions change are less than or equal to $3\pi/2$.

2. Weak solutions of the boundary value problem

2.1. The domain

In the following, let $\mathcal{D}$ be the dihedron

$$\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < r < \infty, \ -\theta/2 < \varphi < \theta/2, \ x_3 \in \mathbb{R}\}, \tag{2.1}$$

where r, φ are the polar coordinates in the (x_1, x_2)-plane, $r = (x_1^2 + x_2^2)^{1/2}$, $\tan\varphi = x_2/x_1$, $\theta \in (0, 2\pi)$. Furthermore, let $\mathcal{K} = \{x \in \mathbb{R}^3 : x/|x| \in \Omega\}$ be a polyhedral cone with plane faces $\Gamma_1, \dots, \Gamma_N$ and edges $M_1, \dots, M_N$. Here, Ω is a domain of polygonal type on the unit sphere S^2 with the sides $\gamma_j = \Gamma_j \cap S^2$, $j = 1, \dots, N$.

The bounded domain $\mathcal{G} \subset \mathbb{R}^3$ is said to be a *domain of polyhedral type* if

(i) The boundary $\partial\mathcal{G}$ consists of smooth (of class C^∞) open two-dimensional manifolds Γ_j (the faces of $\mathcal{G}$), $j = 1, \dots, N$, smooth curves M_k (the edges), $k = 1, \dots, m$, and vertices $x^{(1)}, \dots, x^{(d)}$.

(ii) For every $\xi \in M_k$ there exist a neighborhood $\mathcal{U}_\xi$ and a diffeomorphism (a C^∞ mapping) κ_ξ which maps $\mathcal{G} \cap \mathcal{U}_\xi$ onto $\mathcal{D}_\xi \cap B_1$, where $\mathcal{D}_\xi$ is a dihedron of the form (2.1) and B_1 is the unit ball.

(iii) For every vertex $x^{(j)}$ there exist a neighborhood $\mathcal{U}_j$ and a diffeomorphism κ_j mapping $\mathcal{G} \cap \mathcal{U}_j$ onto $\mathcal{K}_j \cap B_1$, where $\mathcal{K}_j$ is a polyhedral cone with vertex at the origin.

The set $M_1 \cup \dots \cup M_m \cup \{x^{(1)}, \dots, x^{(d)}\}$ of the singular boundary points is denoted by $\mathcal{S}$.

2.2. Formulation of the problem

For every face Γ_j, $j = 1, \dots, N$, let a number $d_j \in \{0, 1, 2, 3\}$ be given. We consider the boundary value problem

$$-\nu\Delta u + \sum_{j=1}^{3} u_j\, \partial_{x_j} u + \nabla p = f, \quad -\nabla \cdot u = g \quad \text{in } \mathcal{G}, \tag{2.2}$$

$$S_j u = h_j, \quad N_j(u, p) = \phi_j \quad \text{on } \Gamma_j, \ j = 1, \dots, N, \tag{2.3}$$

where

$$S_j u = \begin{cases} u & \text{if } d_j = 0, \\ u_\tau & \text{if } d_j = 1, \\ u_n & \text{if } d_j = 2, \end{cases} \qquad N_j(u, p) = \begin{cases} -p + 2\nu\varepsilon_{n,n}(u) & \text{if } d_j = 1, \\ 2\nu\varepsilon_{n,\tau}(u) & \text{if } d_j = 2, \\ -pn + 2\nu\varepsilon_n(u) & \text{if } d_j = 3. \end{cases}$$

In the case $d_j = 0$ the condition $N_j(u, p) = \phi_j$ disappears, while the condition $S_j u = h_j$ does not appear if $d_j = 3$. By a weak solution of the problem (2.2), (2.3), we mean a vector function $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ satisfying

$$b(u, v) + \int_{\mathcal{G}} \sum_{j=1}^{3} u_j \frac{\partial u}{\partial x_j} \cdot v \, dx - \int_{\mathcal{G}} p \, \nabla \cdot v \, dx = F(v) \quad \text{for all } v \in V, \tag{2.4}$$

$$-\nabla \cdot u = g \ \text{ in } \mathcal{G}, \quad S_j u = h_j \ \text{ on } \Gamma_j, \ j = 1, \ldots, N, \tag{2.5}$$

where $V = \{u \in W^{1,2}(\mathcal{G})^3 : S_j u|_{\Gamma_j} = 0, \ j = 1, \ldots, N\}$,

$$b(u, v) = 2\nu \int_{\mathcal{G}} \sum_{i,j=1}^{3} \varepsilon_{i,j}(u) \, \varepsilon_{i,j}(v) \, dx, \tag{2.6}$$

$$F(v) = \int_{\mathcal{G}} (f + \nabla g) \cdot v \, dx + \sum_{j=1}^{n} \int_{\Gamma_j} \phi_j \cdot v \, dx. \tag{2.7}$$

Note that for arbitrary $u \in W^{1,2}(\mathcal{G})^3$, the functional $v \to \int_{\mathcal{G}} u_j \frac{\partial u}{\partial x_j} \cdot v \, dx$ is continuous on $W^{1,2}(\mathcal{G})^3$. This follows from the inequality

$$\left| \int_{\mathcal{G}} u_j \frac{\partial u}{\partial x_j} \cdot v \, dx \right| \leqq \|u_j\|_{L_4(\mathcal{G})} \, \|\partial_{x_j} u\|_{L_2(\mathcal{G})^3} \, \|v\|_{L_4(\mathcal{G})^3}$$

and the continuity of the imbedding $W^{1,2}(\mathcal{G}) \subset L_4(\mathcal{G})$.

2.3. Existence of solutions of the linearized problem

We consider the weak solution of the boundary value problem for the Stokes system

$$-\nu \Delta u + \nabla p = f, \quad \nabla \cdot u = g \quad \text{in } \mathcal{G}, \tag{2.8}$$

that is, a vector function $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ satisfying

$$b(u, v) - \int_{\mathcal{G}} p \, \nabla \cdot v \, dx = F(v) \ \text{ for all } v \in V, \tag{2.9}$$

$$-\nabla \cdot u = g \ \text{ in } \mathcal{G}, \quad S_j u = h_j \ \text{ on } \Gamma_j, \ j = 1, \ldots, N, \tag{2.10}$$

where F is given by (2.7). For the proof of the following theorem, we refer to [19, Theorem 5.1].

Theorem 2.1. *Let $g \in L_2(\mathcal{G})$ and $h_j \in W^{1/2,2}(\Gamma_j)^{3-d_j}$ be such that there exists a vector function $v \in W^{1,2}(\mathcal{G})^3$ satisfying the condition $S_j v = h_j$ on Γ_j, $j = 1, \ldots, N$. In the case when $d_j \in \{0, 2\}$ for all j, we assume, in addition, that*

$$\int_{\mathcal{G}} g \, dx + \sum_{j:\, d_j=0} \int_{\Gamma_j} h_j \cdot n \, dx + \sum_{j:\, d_j=2} \int_{\Gamma_j} h_j \, dx = 0. \tag{2.11}$$

Furthermore, let the functional $F \in V^*$ *satisfy the condition*

$$F(v) = 0 \quad \text{for all } v \in L_V, \tag{2.12}$$

where L_V *denotes the set of all* $v \in V$ *such that* $\varepsilon_{i,j}(v) = 0$ *for* $i, j = 1, 2, 3$. *Then there exists a solution* $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ *of the problem* (2.9), (2.10). *Here* p *is uniquely determined if* $d_j \in \{1, 3\}$ *for at least one* j *and unique up to constants if* $d_j \in \{0, 2\}$ *for all* j. *The vector function* u *is unique up to elements from* L_V.

Note that L_V contains only functions of the form $v = c + Ax$, where c is a constant vector and A is a constant matrix, $A = -A^t$ (rigid body motions). In particular, $\nabla \cdot v = 0$ for $v \in L_V$. In most cases (for example, if the Dirichlet condition is given on at least one face Γ_j), the set L_V contains only the function $v = 0$.

2.4. Existence of solutions of the nonlinear problem

Let the operator Q be defined by

$$Qu = (u \cdot \nabla)\, u.$$

Obviously, Q realizes a mapping $W^{1,2}(\mathcal{G}) \to V^*$. Furthermore, there exist constants c_1, c_2 such that

$$\|Qu\|_{V^*} \leqq c\, \|u\|^2_{W^{1,2}(\mathcal{G})^3} \quad \text{for all } u \in W^{1,2}(\mathcal{G}), \tag{2.13}$$

$$\|Qu - Qv\|_{V^*} \leqq c\, \big(\|u\|_{W^{1,2}(\mathcal{G})^3} + \|v\|_{W^{1,2}(\mathcal{G})^3}\big)\, \|u - v\|_{W^{1,2}(\mathcal{G})^3} \quad \text{for all } u, v \in W^{1,2}(\mathcal{G})^3. \tag{2.14}$$

Using the last two estimates together with Theorem 2.1, we can prove the following statement.

Theorem 2.2. *Let* g *and* h_j *be as in Theorem* 2.1. *Furthermore, we suppose that* $L_V = \{0\}$ *and*

$$\|F\|_{V^*} + \|g\|_{L_2(\mathcal{G})} + \sum_{j=1}^{N} \|h_j\|_{W^{1/2,2}(\Gamma_j)^{3-d_j}}$$

is sufficiently small. Then there exists a solution $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ *of the problem* (2.4), (2.5). *Here,* u *is unique on the set of all functions with norm less than a certain positive* ε, p *is unique if* $d_j \in \{1, 3\}$ *for at least one* j, *otherwise* p *is unique up to a constant.*

Proof. Let $(u^{(0)}, p^{(0)}) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ be the solution of the linear problem (2.9), (2.10). By our assumptions on F, g and h_j, we may assume that

$$\|u^{(0)}\|_{W^{1,2}(\mathcal{G})^3} \leqq \varepsilon_1,$$

where ε_1 is a small positive number. Let V_0 denote the set of all $v \in V$ such that $\nabla \cdot v = 0$. We put $w = u - u^{(0)}$ and $q = p - p^{(0)}$. Then (u, p) is a solution of the problem (2.4), (2.5) if and only if $(w, q) \in V_0 \times L_2(\mathcal{G})$ and

$$b(w, v) - \int_{\mathcal{G}} q \, \nabla \cdot v \, \mathrm{d}x = - \int_{\mathcal{G}} Q(w + u^{(0)}) \cdot v \, \mathrm{d}x \quad \text{for all } v \in V. \tag{2.15}$$

By Theorem 2.1, there exists a linear and continuous mapping

$$V^* \ni \Phi \to A\Phi = (w, q) \in V_0 \times L_2(\mathcal{G})$$

defined by

$$b(w, v) - \int_{\mathcal{G}} q \, \nabla \cdot v \, \mathrm{d}x = \Phi(v) \ \text{ for all } v \in V, \quad \int_{\mathcal{G}} q \, \mathrm{d}x = 0 \text{ if } d_j \in \{0, 2\} \text{ for all } j.$$

We write (2.15) as

$$(w, q) = T(w, q), \quad \text{where } T(w, q) = -AQ(w + u^{(0)}).$$

Due to (2.14), the operator T is contractive on the set of all $(w, q) \in V_0 \times L_2(\mathcal{G})$ with norm $\leqq \varepsilon_2$ if ε_1 and ε_2 are sufficiently small. Hence there exist $w \in W^{1,2}(\mathcal{G})^3$ and $q \in L_2(\mathcal{G})$ satisfying (2.15). The result follows. □

Remark 2.1. If $d_j \in \{0, 2\}$ for all j, then

$$\int_{\mathcal{G}} \sum_{j=1}^{3} u_j \frac{\partial v}{\partial x_j} \cdot v \, \mathrm{d}x = 0 \quad \text{for all } v \in W^{1,2}(\mathcal{G})^3, \ u \in V, \ \nabla \cdot u = 0 \tag{2.16}$$

(see [5, Lemma IV.2.2]). Thus, analogously to [5, Theorem IV.2.3] (see also [24]), the problem (2.4), (2.5) has at least one solution for arbitrary $F \in V^*$, $g = 0$, $h_j \in W^{1/2,2}(\Gamma_j)^{3-d_j}$ satisfying (2.11).

3. Weighted Sobolev and Hölder spaces

Here, we introduce weighted SOBOLEV & HÖLDER spaces in polyhedral domains and prove imbeddings for these spaces which will be used in the next section. We start with the case of a polyhedral cone.

3.1. Weighted Sobolev spaces in a cone

Let $\mathcal{K} = \{x \in \mathbb{R}^3 : x/|x| \in \Omega\}$ be a polyhedral cone in $\mathbb{R}^3$ whose boundary consists of plane faces Γ_j and edges M_k, $j, k = 1, \ldots, N$. Here, Ω is a subdomain of polygonal type on the unit sphere. We denote by $\rho(x) = |x|$ the distance of x to the vertex of the cone, by $r_k(x)$ the distance to the edge M_k, and by $r(x)$ the distance to the set $\mathcal{S} = M_1 \cup \cdots \cup M_N \cup \{0\}$. Note that there exist positive constants c_1, c_2 such that

$$c_1\, r(x) \leqq \rho(x) \prod_{k=1}^{N} \frac{r_k(x)}{\rho(x)} \leqq c_2\, r(x) \quad \text{for all } x \in \mathcal{K}. \tag{3.1}$$

Let l be a nonnegative integer, $\beta \in \mathbb{R}$, $\delta = (\delta_1, \ldots, \delta_N) \in \mathbb{R}^N$, and $1 < s < \infty$. We define $V^{l,s}_{\beta,\delta}(\mathcal{K})$ as the closure of the set $C_0^\infty(\overline{\mathcal{K}}\backslash\mathcal{S})$ with respect to the norm

$$\|u\|_{V^{l,s}_{\beta,\delta}(\mathcal{K})} = \left(\int_{\mathcal{K}} \sum_{|\alpha| \leqq l} \rho^{s(\beta - l + |\alpha|)} \prod_{k=1}^{N} \left(\frac{r_k}{\rho}\right)^{s(\delta_k - l + |\alpha|)} |\partial_x^\alpha u|^s \, dx \right)^{1/s}.$$

The weighted Sobolev space $W^{l,s}_{\beta,\delta}(\mathcal{K})$, where $\delta_k > -2/s$ for $k = 1, \ldots, N$, is defined as the closure of the set $C_0^\infty(\overline{\mathcal{K}})$ with respect to the norm

$$\|u\|_{W^{l,s}_{\beta,\delta}(\mathcal{K})} = \left(\int_{\mathcal{K}} \sum_{|\alpha| \leqq l} \rho^{s(\beta - l + |\alpha|)} \prod_{k=1}^{N} \left(\frac{r_k}{\rho}\right)^{s\delta_k} |\partial_x^\alpha u|^s \, dx \right)^{1/s}.$$

If δ is a real number, then by $V^{l,s}_{\beta,\delta}(\mathcal{K})$ and $W^{l,s}_{\beta,\delta}(\mathcal{K})$, we mean the above introduced spaces with $\delta_1 = \cdots = \delta_N = \delta$. Note that the norm in $W^{l,s}_{\beta,\delta}(\mathcal{K})$ is equivalent to

$$\|u\| = \left(\sum_{k=-\infty}^{+\infty} \|\zeta_k u\|^s_{W^{l,s}_{\beta,\delta}(\mathcal{K})} \right)^{1/s},$$

where ζ_k are infinitely differentiable functions with support in $\{x : 2^{k-1} < |x| < 2^{k+1}\}$ such that

$$\left|\partial_x^\alpha \zeta_k(x)\right| < c\, 2^{-k|\alpha|} \quad \text{for } k \leqq l, \qquad \sum_{k=-\infty}^{+\infty} \zeta_k = 1. \tag{3.2}$$

For the proof of the following lemma, we refer to [16, Lemma 1].

Lemma 3.1. *Let* $1 < s \leqq t < \infty$, $l - 3/s \geqq l' - 3/t$, $\beta - l + 3/s = \beta' - l' + 3/t$, *and* $\delta_k - l + 3/s \leqq \delta'_k - l' + 3/t$ *for* $k = 1, \ldots, N$. *Then* $V^{l,s}_{\beta,\delta}(\mathcal{K})$ *is continuously imbedded into* $V^{l',t}_{\beta',\delta'}(\mathcal{K})$.

In particular, we have $V^{l,s}_{\beta,\delta}(\mathcal{K}) \subset V^{l',s}_{\beta',\delta'}(\mathcal{K})$ for $l \geqq l'$, $\beta - l = \beta' - l'$ and $\delta_k - l \leqq \delta'_k - l'$, $k = 1, \ldots, N$. If in addition $\delta_k > -2/s$ and $\delta'_k > -2/s$ for $k = 1, \ldots, N$, then also

$$W^{l,s}_{\beta,\delta}(\mathcal{K}) \subset W^{l',s}_{\beta',\delta'}(\mathcal{K}).$$

The spaces $V^{l,s}_{\beta,\delta}(\mathcal{K})$ and $W^{l,s}_{\beta,\delta}(\mathcal{K})$ coincide if $\delta_k > l - 2/s$ for all k (see [19]).

Lemma 3.2. *Let* $1 < s \leqq t < \infty$ *and* $l - 3/s \geqq \max(\delta_k, 0) - 3/t$. *Then* $W^{l,s}_{\beta,\delta}(\mathcal{K}) \subset W^{0,t}_{\beta-l+3/s-3/t,0}(\mathcal{K})$ *and*

$$\|\rho^{\beta-l+3/s-3/t} u\|_{L_t(\mathcal{K})} \leqq c\, \|u\|_{W^{l,s}_{\beta,\delta}(\mathcal{K})} \tag{3.3}$$

for all $u \in W^{l,s}_{\beta,\delta}(\mathcal{K})$ *with a constant* c *independent of* u. *Furthermore, for arbitrary* $u \in W^{l,s}_{\beta,\delta}(\mathcal{K})$, $l - 3/s > \max(\delta_k, 0)$, *the inequality*

$$\|\rho^{\beta-l+3/s} u\|_{L_\infty(\mathcal{K})} \leqq c\, \|u\|_{W^{l,s}_{\beta,\delta}(\mathcal{K})} \tag{3.4}$$

is valid with a constant c *independent of* u.

Proof. Let $u \in W^{l,s}_{\beta,\delta}(\mathcal{K})$, and let ζ_k be infinitely differentiable functions with support in the set $\{x : 2^{k-1} < |x| < 2^{k+1}\}$ satisfying (3.2). We define $v(x) = u(2^k x)$ and $\eta_k(x) = \zeta_k(2^k x)$. Then $\eta_k(x)$ vanishes for $|x| < 1/2$ and $|x| > 2$. Therefore for $l' = l - \max(\delta_k, 0)$, we have

$$\|\eta_k v\|_{W^{l',s}(\mathcal{K})} \leqq c\, \|\eta_k v\|_{W^{l,s}_{\beta,\delta}(\mathcal{K})}$$

(see [22, Theorem 3]). This inequality together with the continuity of the imbedding $W^{l',s} \subset L_t$ implies

$$\|\eta_k v\|_{L_t(\mathcal{K})} \leqq c\, \|\eta_k v\|_{W^{l,s}_{\beta,\delta}(\mathcal{K})},$$

where c is independent of u and k. Using the equalities

$$\|\eta_k v\|_{L_t(\mathcal{K})} = 2^{-3k/t} \|\zeta_k u\|_{L_t(\mathcal{K})} \quad \text{and} \quad \|\eta_k v\|_{W^{l,s}_{\beta,\delta}(\mathcal{K})} = 2^{-k(\beta-l)-3k/s} \|\zeta_k u\|_{W^{l,s}_{\beta,\delta}(\mathcal{K})},$$

we obtain

$$\|\rho^{\beta-l+3/s-3/t} \zeta_k u\|_{L_t(\mathcal{K})} \leqq c\, \|\zeta_k u\|_{W^{l,s}_{\beta,\delta}(\mathcal{K})}.$$

Consequently,

$$\begin{aligned}
\|\rho^{\beta-l+3/s-3/t} u\|_{L_t(\mathcal{K})} &\leqq c \left(\sum_{k=-\infty}^{+\infty} \|\rho^{\beta-l+3/s-3/t} \zeta_k u\|^t_{L_t(\mathcal{K})} \right)^{1/t} \\
&\leqq c \left(\sum_{k=-\infty}^{+\infty} \|\zeta_k u\|^t_{W^{l,s}_{\beta,\delta}(\mathcal{K})} \right)^{1/t} \\
&\leqq c \left(\sum_{k=-\infty}^{+\infty} \|\zeta_k u\|^s_{W^{l,s}_{\beta,\delta}(\mathcal{K})} \right)^{1/s} \leqq c\, \|u\|_{W^{l,s}_{\beta,\delta}(\mathcal{K})}.
\end{aligned}$$

This proves (3.3). The proof of (3.4) proceeds analogously. □

Lemma 3.3. *Let $u \in W^{l,s}_{\beta,\delta}(\mathcal{K})$, and let j be an integer, $j \geqq 1$. Then there exist functions $v \in V^{l,s}_{\beta,\delta}(\mathcal{K})$ and $w \in W^{l+j,s}_{\beta+j,\delta+j}(\mathcal{K})$ such that $u = v + w$ and*

$$\|v\|_{V^{l,s}_{\beta,\delta}(\mathcal{K})} + \|w\|_{W^{l+j,s}_{\beta+j,\delta+j}(\mathcal{K})} \leqq c\,\|u\|_{W^{l,s}_{\beta,\delta}(\mathcal{K})}\,.$$

Proof. Let ζ_k be infinitely differentiable functions with support in $\{x : 2^{k-1} < |x| < 2^{k+1}\}$ satisfying (3.2), and let u be an arbitrary function from $W^{l,s}_{\beta,\delta}(\mathcal{K})$. Obviously, the function $\tilde{u}_k$ defined by $\tilde{u}_k(x) = \zeta_k(2^k x)\,u(2^k x)$ belongs also to $W^{l,s}_{\beta,\delta}(\mathcal{K})$ and vanishes for $|x| < 1/2$ and $|x| > 2$. Consequently by [15, Lemma 1.3] (for integer $\beta + 2/s$ see [22, Theorem 5]), there exist functions $\tilde{v}_k \in V^{l,s}_{\beta,\delta}(\mathcal{K})$ and $\tilde{w}_k \in W^{l+j,s}_{\beta+j,\delta+j}(\mathcal{K})$ with supports in $\{x : 1/4 < |x| < 4\}$ such that $\tilde{u}_k = \tilde{v}_k + \tilde{w}_k$ and

$$\|\tilde{v}_k\|_{V^{l,s}_{\beta,\delta}(\mathcal{K})} + \|\tilde{w}_k\|_{W^{l+j,s}_{\beta+j,\delta+j}(\mathcal{K})} \leqq c\,\|\tilde{u}_k\|_{W^{l,s}_{\beta,\delta}(\mathcal{K})},$$

where c is independent of u and k. Let $v_k(x) = \tilde{v}_k(2^{-k}x)$ and $w_k(x) = \tilde{w}_k(2^{-k}x)$. Then the supports of v_k and w_k are contained in $\{x : 2^{k-2} < |x| < 2^{k+2}\}$, and we have $\zeta_k u = v_k + w_k$ for all k. Moreover,

$$\begin{aligned}\|v_k\|_{V^{l,s}_{\beta,\delta}(\mathcal{K})} + \|w_k\|_{W^{l+j,s}_{\beta+j,\delta+j}(\mathcal{K})} &= 2^{k(\beta-l+3/s)}\Big(\|\tilde{v}_k\|_{V^{l,s}_{\beta,\delta}(\mathcal{K})} + \|\tilde{w}_k\|_{W^{l+j,s}_{\beta+j,\delta+j}(\mathcal{K})}\Big)\\ &\leqq c\,2^{k(\beta-l+3/s)}\,\|\tilde{u}_k\|_{W^{l,s}_{\beta,\delta}(\mathcal{K})} = c\,\|\zeta_k u\|_{W^{l,s}_{\beta,\delta}(\mathcal{K})}\,.\end{aligned}$$

Let $v = \sum_{k=-\infty}^{+\infty} v_k$ and $w = \sum_{k=-\infty}^{+\infty} w_k$. Then $u = v + w$. Since v_k and v_m have disjoint supports for $|k-m| \geqq 4$, we have $v \in V^{l,s}_{\beta,\delta}(\mathcal{K})$ and

$$\|v\|^s_{V^{l,s}_{\beta,\delta}(\mathcal{K})} \leqq 7^{s-1}\sum_k \|v_k\|^s_{V^{l,s}_{\beta,\delta}(\mathcal{K})} \leqq c\sum_k \|\zeta_k u\|^s_{W^{l,s}_{\beta,\delta}(\mathcal{K})} \leqq c'\,\|u\|^s_{W^{l,s}_{\beta,\delta}(\mathcal{K})}.$$

Analogously, the norm of w in $W^{l+j,s}_{\beta+j,\delta+j}(\mathcal{K})$ can be estimated by the norm of u in $W^{l,s}_{\beta,\delta}(\mathcal{K})$. The lemma is proved. □

3.2. Weighted Hölder spaces in a polyhedral cone

Let l be a nonnegative integer, $\beta \in \mathbb{R}$, $\delta = (\delta_1, \dots, \delta_N) \in \mathbb{R}^N$, and $\sigma \in (0,1)$. We define the weighted Hölder space $\mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{K})$ as the set of all l times continuously differentiable functions on $\bar{\mathcal{K}}\backslash\mathcal{S}$ with finite norm

$$\begin{aligned}\|u\|_{\mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{K})} &= \sum_{|\alpha|\leqq l}\sup_{x\in\mathcal{K}} |x|^{\beta-l-\sigma+|\alpha|}\prod_{k=1}^N\Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k-l-\sigma+|\alpha|}\big|\partial_x^\alpha u(x)\big|\\ &\quad+\sum_{|\alpha|=l}\sup_{|x-y|<r(x)/2}|x|^\beta\prod_{k=1}^N\Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k}\frac{\big|\partial_x^\alpha u(x)-\partial_y^\alpha u(y)\big|}{|x-y|^\sigma}\,. \end{aligned}\quad (3.5)$$

Furthermore, the space $C^{l,\sigma}_{\beta,\delta}(\mathcal{K})$ is defined for nonnegative δ_k, $k = 1, \dots, N$, as the set of all l times continuously differentiable functions on $\bar{\mathcal{K}}\backslash\mathcal{S}$ with finite norm

$$\|u\|_{C^{l,\sigma}_{\beta,\delta}(\mathcal{K})} = \sum_{|\alpha|\leqq l} \sup_{x\in\mathcal{K}} |x|^{\beta-l-\sigma+|\alpha|} \prod_{k=1}^{N} \left(\frac{r_k(x)}{|x|}\right)^{\max(0,\delta_k-l-\sigma+|\alpha|)} \left|\partial_x^\alpha u(x)\right|$$
$$+ \sum_{k:\,\sigma_k\leqq l} \sum_{|\alpha|=l-\sigma_k} \sup_{\substack{x,y\in\mathcal{K}_k \\ |x-y|<|x|/2}} |x|^{\beta-\delta_k} \frac{\left|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)\right|}{|x-y|^{\sigma+\sigma_k-\delta_k}}$$
$$+ \sum_{|\alpha|=l} \sup_{\substack{x,y\in\mathcal{K} \\ |x-y|<r(x)/2}} |x|^{\beta} \prod_{k=1}^{N} \left(\frac{r_k(x)}{|x|}\right)^{\delta_k} \frac{\left|\partial_x^\alpha u(x)-\partial_y^\alpha u(y)\right|}{|x-y|^{\sigma}}, \quad (3.6)$$

where $\mathcal{K}_k = \{x \in \mathcal{K} : r_k(x) < 3r(x)/2\}$, $\sigma_k = [\delta_k - \sigma] + 1$, $[s]$ denotes the integral part of s. The trace spaces for $\mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{K})$ and $C^{l,\sigma}_{\beta,\delta}(\mathcal{K})$ on Γ_j are denoted by $\mathcal{N}^{l,\sigma}_{\beta,\delta}(\Gamma_j)$ and $C^{l,\sigma}_{\beta,\delta}(\Gamma_j)$, respectively.

Obviously, $\mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{K})$ is a subset of $C^{l,\sigma}_{\beta,\delta}(\mathcal{K})$. If $\delta_k \geqq l + \sigma$ for $k = 1, \dots, N$, then both spaces coincide. Furthermore, the following imbedding holds (and is continuous, see [18]).

$$\mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{K}) \subset \mathcal{N}^{l',\sigma'}_{\beta',\delta'}(\mathcal{K}) \ \text{ if } l + \sigma \geqq l' + \sigma',$$
$$\beta - l - \sigma = \beta' - l' - \sigma',\ \delta_k - l - \sigma \leqq \delta'_k - l' - \sigma'.$$

If in addition δ_k and δ'_k are nonnegative, then $C^{l,\sigma}_{\beta,\delta}(\mathcal{K})$ is continuously imbedded into $C^{l',\sigma'}_{\beta',\delta'}(\mathcal{K})$. Next, we prove a relation between the spaces $V^{l,s}_{\beta,\delta}$ and $\mathcal{N}^{l,\sigma}_{\beta,\delta}$.

Lemma 3.4. *Suppose that $l - 3/s > l' + \sigma$, $\beta - l + 3/s = \beta' - l' - \sigma$ and $\delta_k - l + 3/s \leqq \delta'_k - l' - \sigma$ for $k = 1, \dots, N$. Then $V^{l,s}_{\beta,\delta}(\mathcal{K})$ is continuously imbedded into $\mathcal{N}^{l',\sigma}_{\beta',\delta'}(\mathcal{K})$.*

Proof. It suffices to prove the lemma for $\delta_k - l + 3/s = \delta'_k - l' - \sigma, k = 1, \dots, N$. Let $u \in V^{l,s}_{\beta,\delta}(\mathcal{K})$. For an arbitrary point $x \in \mathcal{K}$, we denote by B_x the set $\{x' \in \mathcal{K} : |x - x'| < r(x)/2\}$. Note that

$$|x|/2 \leqq |x'| < 3|x|/2, \quad r_k(x)/2 \leqq r_k(x') < 3r_k(x)/2,$$
$$r(x)/2 \leqq r(x') \leqq 3r(x)/2 \quad \text{for } x' \in B_x. \quad (3.7)$$

First, let $r(x) = 1$. From the continuity of the imbedding $W^{l,s}(B_x) \subset C^{l',\sigma}(B_x)$ it follows that there exists a constant c independent of u and x such that

$$\left|(\partial^\alpha u)(x)\right| \leqq c\, \|u\|_{W^{l,s}(B_x)} \quad \text{for } |\alpha| \leqq l',$$
$$\frac{|(\partial^\alpha u)(x) - (\partial^\alpha u)(x')|}{|x - x'|^\sigma} \leqq c\, \|u\|_{W^{l,s}(B_x)} \quad \text{for } |\alpha| = l',\ x' \in B_x\,.$$

Due to (3.1) and (3.7), this implies

$$|x|^\beta \prod_k \Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k} \big|(\partial^\alpha u)(x)\big| \leqq c \sum_{|\gamma|\leqq l} \Big\| r^{|\gamma|-l}\, \rho^\beta \prod_k \Big(\frac{r_k}{\rho}\Big)^{\delta_k} \partial^\gamma u \Big\|_{L_s(B_x)} \leqq c\, \|u\|_{V^{l,s}_{\beta,\delta}(\mathcal{K})} \quad \text{for } |\alpha| \leqq l'$$

and analogously

$$|x|^\beta \prod_k \Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k} \frac{|(\partial^\alpha u)(x) - (\partial^\alpha u)(x')|}{|x-x'|^\sigma} \leqq c\, \|u\|_{V^{l,s}_{\beta,\delta}(\mathcal{K})} \quad \text{for } |\alpha| = l',\ x' \in B_x.$$

Now let x be an arbitrary point in $\mathcal{K}$ and $x' \in B_x$. We put $y = x/r(x)$, $y' = x'/r(x)$. Then $r(y) = 1$ and $y' \in B_y$. Consequently, the function $v(\xi) = u\,(r(x)\,\xi)$ satisfies the inequalities

$$|y|^\beta \prod_k \Big(\frac{r_k(y)}{|y|}\Big)^{\delta_k} \big|(\partial^\alpha v)(y)\big| \leqq c\, \|v\|_{V^{l,s}_{\beta,\delta}(\mathcal{K})} \leqq c\, r(x)^{l-\beta-3/s}\, \|u\|_{V^{l,s}_{\beta,\delta}(\mathcal{K})}$$

for $|\alpha| \leqq l'$ and

$$|y|^\beta \prod_k \Big(\frac{r_k(y)}{|y|}\Big)^{\delta_k} \frac{|(\partial^\alpha v)(y) - (\partial^\alpha v)(y')|}{|y-y'|^\sigma} \leqq c\, r(x)^{l-\beta-3/s}\, \|u\|_{V^{l,s}_{\beta,\delta}(\mathcal{K})}.$$

for $|\alpha| = l'$. Here, the constant c is independent of u, x and x'. Using (3.1), we obtain the inequalities

$$|x|^{\beta-l+|\alpha|+3/s} \prod_k \Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k-l+|\alpha|+3/s} \big|(\partial^\alpha u)(x)\big| \leqq c\, \|u\|_{V^{l,s}_{\beta,\delta}(\mathcal{K})} \quad \text{for } |\alpha| \leqq l',$$

$$|x|^{\beta'} \prod_k \Big(\frac{r_k(x)}{|x|}\Big)^{\delta_k-l+l'+\sigma+3/s} \frac{|(\partial^\alpha u)(x) - (\partial^\alpha u)(x')|}{|x-x'|^\sigma} \leqq c\, \|u\|_{V^{l,s}_{\beta,\delta}(\mathcal{K})} \text{ for } |\alpha| = l'.$$

The result follows. □

Corollary 3.1. *Let $u \in W^{l,s}_{\beta,\delta}(\mathcal{K})$, $l > 3/s$. Then*

$$\rho^{\beta-l+3/s} \prod_k \Big(\frac{r_k}{\rho}\Big)^{\sigma_k} u \in L_\infty(\mathcal{K}), \tag{3.8}$$

where $\sigma_k = 0$ for $\delta_k < l-3/s$, $\sigma_k = 1/s + \varepsilon$ for $l-3/s \leqq \delta_k \leqq l-2/s$, and $\sigma_k = \delta_k - l + 3/s$ for $\delta_k > l-2/s$ (ε is an arbitrarily small positive number).

Proof. Let ψ_k be a smooth function on the unit sphere S^2 such that $\psi_k = \delta_{j,k}$ in a neighborhood of the points $S^2 \cap M_j$ for $j = 1, \dots, N$. We extend ψ_k to $\mathbb{R}^3 \backslash \{0\}$ by $\psi_k(x) = \psi_k(x/|x|)$. Then $\psi_k u \in W^{l,s}_{\beta,\delta_k}(\mathcal{K})$. Obviously, it suffices to prove (3.8) for the function $\psi_k u$. If $\delta_k < l - 3/s$, then by Lemma 3.2, $\rho^{\beta-l+3/s}\psi_k u \in L_\infty(\mathcal{K})$. If $\delta_k > l - 2/s$, then $W^{l,s}_{\beta,\delta_k}(\mathcal{K}) = V^{l,s}_{\beta,\delta_k}(\mathcal{K})$. By Lemma 3.4, the last space is imbedded into $\mathcal{N}^{0,\sigma}_{\beta-l+\sigma+3/s,\delta_k-l+\sigma+3/s}(\mathcal{K})$ for arbitrary $\sigma < l - 3/s$. Therefore in particular,

$$\rho^{\beta-l+3/s} \prod_k \left(\frac{r_k}{\rho}\right)^{\delta_k-l+3/s} \psi_k u \in L_\infty(\mathcal{K}).$$

For $l - 3/s \leqq \delta_k < l - 2/s$, the assertion follows from the imbedding $W^{l,s}_{\beta,\delta_k}(\mathcal{K}) \subset W^{l,s}_{\beta,l-2/s+\varepsilon}(\mathcal{K})$. □

3.3. Weighted Sobolev and Hölder spaces in a bounded polyhedral domain

Let $\mathcal{G}$ be a domain of polyhedral type (see Section 2.1) with faces $\Gamma_1, \dots, \Gamma_N$, edges $M_1, \dots, M_m$ and vertices $x^{(1)}, \dots, x^{(d)}$. We denote the distance of x to the edge M_k by $r_k(x)$, the distance to the vertex $x^{(j)}$ by $\rho_j(x)$, and the distance to $\mathcal{S}$ (the set of all edge points and vertices) by $r(x)$. Furthermore, we denote by X_j the set of all indices k such that the vertex $x^{(j)}$ is an end point of the edge M_k. Let $\mathcal{U}_1, \dots, \mathcal{U}_d$ be domains in $\mathbb{R}^3$ such that

$$\mathcal{U}_1 \cup \dots \cup \mathcal{U}_d \supset \overline{\mathcal{G}} \quad \text{and} \quad \overline{\mathcal{U}}_j \cap \overline{M}_k = \emptyset \text{ if } k \notin X_j\,.$$

Then for arbitrary integer $l \geqq 0$, real $s > 1$ and real tuples $\beta = (\beta_1, \dots, \beta_d)$, $\delta = (\delta_1, \dots, \delta_m)$, we define $V^{l,s}_{\beta,\delta}(\mathcal{G})$ and $W^{l,s}_{\beta,\delta}(\mathcal{G})$ as the weighted Sobolev spaces with the norms

$$\|u\|_{V^{l,s}_{\beta,\delta}(\mathcal{G})} = \left(\sum_{j=1}^d \int_{\mathcal{G}\cap\mathcal{U}_j} \sum_{|\alpha|\leqq l} \rho_j^{s(\beta_j-l+|\alpha|)} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}\right)^{s(\delta_k-l+|\alpha|)} |\partial_x^\alpha u|^s \, \mathrm{d}x\right)^{1/s}$$

and

$$\|u\|_{W^{l,s}_{\beta,\delta}(\mathcal{G})} = \left(\sum_{j=1}^d \int_{\mathcal{G}\cap\mathcal{U}_j} \sum_{|\alpha|\leqq l} \rho_j^{s(\beta_j-l+|\alpha|)} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}\right)^{s\delta_k} |\partial_x^\alpha u|^s \, \mathrm{d}x\right)^{1/s},$$

respectively. In the case of the space $W^{l,s}_{\beta,\delta}(\mathcal{G})$, we suppose that $\delta_k > -2/s$ for $k = 1, \dots, m$. Obviously, the spaces $V^{l,s}_{\beta,\delta}(\mathcal{G})$ and $W^{l,s}_{\beta,\delta}(\mathcal{G})$ do not depend on the choice of the domains $\mathcal{U}_j$. The corresponding trace spaces on the faces Γ_j are denoted by $V^{l-1/s,s}_{\beta,\delta}(\Gamma_j)$ and $W^{l-1/s,s}_{\beta,\delta}(\Gamma_j)$, respectively. Furthermore, let $V^{-1,s}_{\beta,\delta}(\mathcal{G})$ denote the dual space of $V^{1,s'}_{-\beta,-\delta}(\mathcal{G})$, $s' = s/(s-1)$, with respect to the L_2 scalar product.

Lemma 3.5. *Let $l \geqq 0$, $1 < t < s < \infty$, $\beta_j + 3/s < \beta'_j + 3/t$ for $j = 1, \dots, d$, and $\delta_k + 2/s < \delta'_k + 2/t$ for $k = 1, \dots, N$. Then $V^{l,s}_{\beta,\delta}(\mathcal{G}) \subset V^{l,t}_{\beta',\delta'}(\mathcal{G})$ and $W^{l,s}_{\beta,\delta}(\mathcal{G}) \subset W^{l,t}_{\beta',\delta'}(\mathcal{G})$. These imbeddings are continuous.*

Proof. Let $q = st/(s-t)$. By Hölder's inequality,

$$\left\| \rho_j^{\beta'_j - l + |\alpha|} \prod_{k \in X_j} \left(\frac{r_k}{\rho_j}\right)^{\delta'_k} \partial_x^\alpha u \right\|_{L_t(\mathcal{G} \cap \mathcal{U}_j)} \leqq c \left\| \rho_j^{\beta_j - l + |\alpha|} \prod_{k \in X_j} \left(\frac{r_k}{\rho_j}\right)^{\delta_k} \partial_x^\alpha u \right\|_{L_s(\mathcal{G} \cap \mathcal{U}_j)},$$

where

$$c = \left\| \rho_j^{\beta'_j - \beta_j} \prod_{k \in X_j} \left(\frac{r_k}{\rho_j}\right)^{\delta'_k - \delta_k} \right\|_{L_q(\mathcal{G})} < \infty$$

if $\beta'_j - \beta_j > -3/q$ and $\delta'_k - \delta_k > -2/q$. This proves the imbedding $W^{l,s}_{\beta,\delta}(\mathcal{G}) \subset W^{l,t}_{\beta',\delta'}(\mathcal{G})$. Analogously, the imbedding $V^{l,s}_{\beta,\delta}(\mathcal{G}) \subset V^{l,t}_{\beta',\delta'}(\mathcal{G})$ holds. □

The following result can be directly deduced from Lemma 3.1.

Lemma 3.6. *Let $1 < s \leqq t < \infty$, $l - 3/s \geqq l' - 3/t$, $\beta_j - l + 3/s \leqq \beta'_j - l' + 3/t$ for $j = 1, \dots, d$, and $\delta_k - l + 3/s \leqq \delta'_k - l' + 3/t$ for $k = 1, \dots, m$. Then $V^{l,s}_{\beta,\delta}(\mathcal{G})$ is continuously imbedded into $V^{l',t}_{\beta',\delta'}(\mathcal{G})$.*

Corollary 3.2. *Let $1 < s \leqq t < \infty$, $3/s \leqq 1 + 3/t$, $\beta_j + 3/s \leqq \beta'_j + 1 + 3/t$ for $j = 1, \dots, d$, and $\delta_k + 3/s \leqq \delta'_k + 1 + 3/t$ for $k = 1, \dots, N$. Then $V^{0,s}_{\beta,\delta}(\mathcal{G})$ is continuously imbedded into $V^{-1,t}_{\beta',\delta'}(\mathcal{G})$.*

Proof. According to Lemma 3.6, we have $V^{1,t'}_{-\beta',-\delta'}(\mathcal{G}) \subset V^{0,s'}_{-\beta,-\delta}(\mathcal{G})$, where $s' = s/(s-1)$, $t' = t/(t-1)$. The result follows. □

Let us further note that (as in the case of a cone)

$$W^{l,s}_{\beta,\delta}(\mathcal{G}) \subset W^{l',s}_{\beta',\delta'}(\mathcal{G}) \quad \text{if } l \geqq l',\ \beta_j - l \leqq \beta'_j - l',$$
$$\delta_k - l \leqq \delta'_k - l',\ \delta_k > -2/s,\ \delta'_k > -2/s$$

for $j = 1, \dots, d$, $k = 1, \dots, m$. If $\delta_k > l - 2/s$ for $k = 1, \dots, m$, then $V^{l,s}_{\beta,\delta}(\mathcal{G}) = W^{l,s}_{\beta,\delta}(\mathcal{G})$.

We introduce the following weighted Hölder spaces in the domain $\mathcal{G}$. Let $\mathcal{U}_1, \dots, \mathcal{U}_d$ be the same domains as above. Furthermore, let

$$\mathcal{G}_j = \mathcal{G} \cap \mathcal{U}_j \quad \text{and} \quad \mathcal{G}_{j,k} = \{x \in \mathcal{G}_j : r_k(x) < 3r(x)/2\}.$$

Then $\mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{G})$ is defined as the space of all l times continuously differentiable functions on $\bar{\mathcal{G}}\backslash\mathcal{S}$ with finite norm

$$\|u\|_{\mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{G})} = \sum_{j=1}^{d}\sum_{|\alpha|\leqq l}\sup_{x\in\mathcal{G}_j}\rho_j(x)^{\beta_j-l-\sigma+|\alpha|}\prod_{k\in X_j}\left(\frac{r_k(x)}{\rho_j(x)}\right)^{\delta_k-l-\sigma+|\alpha|}\left|\partial_x^\alpha u(x)\right|$$
$$+\sum_{j=1}^{d}\sum_{|\alpha|=l}\sup_{\substack{x,y\in\mathcal{G}_j\\|x-y|<r(x)/2}}\rho_j(x)^{\beta_j}\prod_{k\in X_j}\left(\frac{r_k(x)}{\rho_j(x)}\right)^{\delta_k}\frac{\left|\partial_x^\alpha u(x)-\partial_y^\alpha u(y)\right|}{|x-y|^\sigma}.$$

Suppose that $\delta_k \geqq 0$ for $k = 1,\ldots,m$. Then $C^{l,\sigma}_{\beta,\delta}(\mathcal{G})$ is defined as the set of all l times continuously differentiable functions on $\bar{\mathcal{G}}\backslash\mathcal{S}$ with finite norm

$$\|u\|_{C^{l,\sigma}_{\beta,\delta}(\mathcal{G})} = \sum_{j=1}^{d}\sum_{|\alpha|\leqq l}\sup_{x\in\mathcal{G}_j}\rho_j(x)^{\beta_j-l-\sigma+|\alpha|}\prod_{k\in X_j}\left(\frac{r_k(x)}{\rho_j(x)}\right)^{\max(0,\delta_k-l-\sigma+|\alpha|)}\left|\partial_x^\alpha u(x)\right|$$
$$+\sum_{j=1}^{d}\sum_{\substack{k\in X_j\\\sigma_k\leqq l}}\sum_{|\alpha|=l-\sigma_k}\sup_{\substack{x,y\in\mathcal{G}_{j,k}\\|x-y|<\rho_j(x)/2}}\rho_j(x)^{\beta_j-\delta_k}\frac{\left|\partial_x^\alpha u(x)-\partial_y^\alpha u(y)\right|}{|x-y|^{\sigma+\sigma_k-\delta_k}}$$
$$+\sum_{j=1}^{d}\sum_{|\alpha|=l}\sup_{\substack{x,y\in\mathcal{G}_j\\|x-y|<r(x)/2}}\rho_j(x)^{\beta_j}\prod_{k\in X_j}\left(\frac{r_k(x)}{\rho_j(x)}\right)^{\delta_k}\frac{\left|\partial_x^\alpha u(x)-\partial_y^\alpha u(y)\right|}{|x-y|^\sigma},$$

where $\sigma_k = [\delta_k-\sigma]+1$. The trace space on Γ_j for $C^{l,\sigma}_{\beta,\delta}(\mathcal{G})$ is denoted by $C^{l,\sigma}_{\beta,\delta}(\Gamma_j)$.

Analogously to the case of a cone, we have

$$\mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{G})\subset\mathcal{N}^{l',\sigma'}_{\beta',\delta'}(\mathcal{G})\ \text{ if } l+\sigma\geqq l'+\sigma',$$
$$\beta_j-l-\sigma\leqq\beta_j'-l'-\sigma',\ \delta_k-l-\sigma\leqq\delta_k'-l'-\sigma'$$

for $j = 1,\ldots,d$, $k = 1,\ldots,m$. If in addition δ_k and δ_k' are nonnegative, then $C^{l,\sigma}_{\beta,\delta}(\mathcal{G})\subset C^{l',\sigma'}_{\beta',\delta'}(\mathcal{G})$. Furthermore, it follows from Lemma 3.4 that

$$V^{l,s}_{\beta,\delta}(\mathcal{G})\subset\mathcal{N}^{l',\sigma}_{\beta',\delta'}(\mathcal{G})\quad\text{if } l-3/s>l'+\sigma,$$
$$\beta_j-l+3/s\leqq\beta_j'-l'-\sigma,\ \delta_k-l+3/s\leqq\delta_k'-l'-\sigma$$

for $j = 1,\ldots,d$, $k = 1,\ldots,m$.

We introduce the following notation: If $\beta\in\mathbb{R}^d$, $\delta\in\mathbb{R}^m$ and $s,t\in\mathbb{R}$, then by $\mathcal{N}^{l,\sigma}_{\beta+s,\delta+t}(\mathcal{G})$ and $C^{l,\sigma}_{\beta+s,\delta+t}(\mathcal{G})$, we mean the spaces $\mathcal{N}^{l,\sigma}_{\beta',\delta'}(\mathcal{G})$ and $C^{l,\sigma}_{\beta',\delta'}(\mathcal{G})$ with $\beta' = (\beta_1+s,\ldots,\beta_d+s)$ and $\delta' = (\delta_1+t,\ldots,\delta_m+t)$. Analogous notation will be used for the weighted Sobolev spaces $V^{l,s}_{\beta,\delta}$ and $W^{l,s}_{\beta,\delta}$.

The next lemma follows immediately from the definition of the space $\mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{G})$.

Lemma 3.7. *If $f\in\mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{G})$ and $g\in\mathcal{N}^{l,\sigma}_{\beta',\delta'}(\mathcal{G})$, then $fg\in\mathcal{N}^{l,\sigma}_{\beta+\beta'-l-\sigma,\delta+\delta'-l-\sigma}(\mathcal{G})$.*

Finally, we define $C^{-1,\sigma}_{\beta,\delta}(\mathcal{G})$ as the space of all distributions of the form

$$f = f_0 + \sum_{j=1}^{3} \partial_{x_j} f_j, \quad \text{where} \quad f_0 \in C^{0,\sigma}_{\beta+1,\delta+1}(\mathcal{G}) \quad \text{and} \quad f_j \in C^{0,\sigma}_{\beta,\delta}(\mathcal{G}), \quad j = 1, 2, 3. \tag{3.9}$$

Note that every $f \in C^{-1,\sigma}(\mathcal{G})$, that is, every distribution of the form

$$f = f_0 + \sum_{j=1}^{3} \partial_{x_j} f_j, \quad \text{where} \quad f_j \in C^{0,\sigma}(\mathcal{G}), \quad j = 0, 1, 2, 3, \tag{3.10}$$

belongs to $C^{-1,\sigma}_{0,0}(\mathcal{G})$. Indeed, let χ_k be infinitely differentiable cut-off functions equal to one near $x^{(k)}$ and to zero near the vertices $x^{(l)}$, $l \neq k$. Then the distribution (3.10) can be written as

$$f = F_0 + \sum_{j=1}^{3} \partial_{x_j} F_j,$$

where

$$F_0(x) = f_0(x) + \sum_{k=1}^{d} \sum_{j=1}^{3} f_j(x^{(k)})\, \partial_{x_j} \chi_k(x), \; F_j(x) = f_j(x) - \sum_{k=1}^{d} \chi_k(x)\, f_j(x^{(k)}),$$

$j = 1, 2, 3$. Here, $F_0 \in C^{0,\sigma}(\mathcal{G}) \subset C^{0,\sigma}_{1,1}(\mathcal{G})$, and from $F_j \in C^{0,\sigma}(\mathcal{G})$, $F_j(x^{(k)}) = 0$ it follows that $F_j \in C^{0,\sigma}_{0,0}(\mathcal{G})$ for $j = 1, 2, 3$.

4. Regularity results for weak solutions

In this section, we establish regularity results for weak solutions in weighted Sobolev and Hölder spaces. The regularity assertions are formulated in terms of eigenvalues of operator pencils generated by the boundary value problem at the edge points and vertices of the domain.

4.1. Operator pencils generated by the boundary value problem

We introduce the operator pencils generated by the problem (2.2), (2.3) for the edge points and vertices of the domain $\mathcal{G}$.

(1) Let ξ be a point on an edge M_k, and let Γ_{k_+}, Γ_{k_-} be the faces of $\mathcal{G}$ adjacent to ξ. Then by $\mathcal{D}_\xi$ we denote the dihedron which is bounded by the half-planes $\Gamma^\circ_{k_\pm}$ tangent to $\Gamma_{k_\pm}$ at ξ and the edge $M^\circ_\xi = \bar{\Gamma}^\circ_{k_+} \cap \bar{\Gamma}^\circ_{k_-}$. The angle between the half-planes $\Gamma^\circ_{k_\pm}$ is denoted by θ_ξ. Furthermore, let r, φ be polar coordinates in the plane perpendicular to M°_ξ such that

$$\Gamma^\circ_{k_\pm} = \{x \in \mathbb{R}^3 : \; r > 0, \; \varphi = \pm\theta_\xi/2\}.$$

Then we define the operator $A_\xi(\lambda)$ as follows:

$$A_\xi(\lambda)\ (U(\varphi),P(\varphi))$$
$$= \Big(r^{2-\lambda}(-\Delta u+\nabla p), -r^{1-\lambda}\nabla\cdot u, r^{-\lambda}S_{k_\pm}u|_{\varphi=\pm\theta_\xi/2}, r^{1-\lambda}N_{k_\pm}(u,p)|_{\varphi=\pm\theta_\xi/2}\Big),$$

where $u(x)=r^\lambda U(\varphi)$, $p(x)=r^{\lambda-1}P(\varphi)$, $\lambda\in\mathbb{C}$. The operator $A_\xi(\lambda)$ depends quadratically on the parameter λ and realizes a continuous mapping

$$W^{2,2}(I_\xi)^3\times W^{1,2}(I_\xi)\to W^{1,2}(I_\xi)^3\times L_2(I_\xi)\times\mathbb{C}^3\times\mathbb{C}^3$$

for every $\lambda\in\mathbb{C}$, where I_ξ denotes the interval $(-\theta_\xi/2,+\theta_\xi/2)$. The spectrum of the pencil $A_\xi(\lambda)$ consists of eigenvalues with finite geometric and algebraic multiplicities. These eigenvalues are zeros of certain transcendental functions (see [17,21]). For example, in the cases $d_{k_+}=d_{k_-}=0$ (Dirichlet conditions on $\Gamma_{k_\pm}$) and $d_{k_+}=d_{k_-}=3$ (Neumann conditions on $\Gamma_{k_\pm}$), the spectrum of $A_\xi(\lambda)$ consists of the solutions of the equation

$$\sin(\lambda\theta_\xi)\,\Big(\lambda^2\sin^2\theta_\xi-\sin^2(\lambda\theta_\xi)\Big)=0,$$

$\lambda\neq 0$ for $d_{k_+}=d_{k_-}=0$.

Let $\lambda_1(\xi)$ be the eigenvalue with smallest positive real part of this pencil, and let $\lambda_2(\xi)$ be the eigenvalue with smallest real part greater than 1. We define

$$\mu(\xi)=\begin{cases}\operatorname{Re}\lambda_2(\xi) & \text{if } d_{k_+}+d_{k_-} \text{ is even and } \theta_\xi<\pi/m_k,\\ \operatorname{Re}\lambda_1(\xi) & \text{else,}\end{cases}$$

where $m_k=1$ if $d_{k_+}+d_{k_-}\in\{0,6\}$, $m_k=2$ if $d_{k_+}+d_{k_-}\in\{2,4\}$. Finally, let

$$\mu_k=\inf_{\xi\in M_k}\mu(\xi). \tag{4.1}$$

Note that in the case of even $d_{k_+}+d_{k_-}$, the number $\lambda=1$ belongs always to the spectrum of the pencil $A_\xi(\lambda)$.

(2) Let $x^{(j)}$ be a vertex of $\mathcal{G}$ and let I_j be the set of all indices k such that $x^{(j)}\in\overline{\Gamma}_k$. By our assumptions, there exist a neighborhood $\mathcal{U}$ of $x^{(j)}$ and a diffeomorphism κ mapping $\mathcal{G}\cap\mathcal{U}$ onto $\mathcal{K}_j\cap B_1$ and $\Gamma_k\cap\mathcal{U}$ onto $\Gamma_k^\circ\cap B_1$ for $k\in I_j$, where $\mathcal{K}_j=\{x: x/|x|\in\Omega_j\}$ is a polyhedral cone with vertex 0 and $\Gamma_k^\circ=\{x: x/|x|\in\gamma_k\}$ are the faces of this cone. By Ω_j we denoted again a domain of polygonal type on the unit sphere with the side γ_k. Without loss of generality, we may assume that the Jacobian matrix $\kappa'(x)$ is equal to the identity matrix I at the point $x^{(j)}$. We introduce spherical coordinates $\rho=|x|$, $\omega=x/|x|$ in $\mathcal{K}_j$ and define

$$V_{\Omega_j}=\{u\in W^{1,2}(\Omega_j)^3:\ S_ku=0 \text{ on } \gamma_k,\ k\in I_j\}.$$

On the space $V_{\Omega_j}\times L_2(\Omega_j)$, we define the bilinear form $a_j(\cdot,\cdot;\lambda)$ as

$$a_j\left(\begin{pmatrix}u\\p\end{pmatrix},\begin{pmatrix}v\\q\end{pmatrix};\lambda\right)$$
$$=\frac{1}{\log 2}\int\limits_{\substack{\mathcal{K}_j\\ 1<|x|<2}}\left(2\nu\sum_{i,j=1}^{3}\varepsilon_{i,j}(U)\cdot\varepsilon_{i,j}(V)-P\nabla\cdot V-(\nabla\cdot U)\,Q\right)dx,$$

where $U = \rho^\lambda u(\omega)$, $V = \rho^{-1-\lambda} v(\omega)$, $P = \rho^{\lambda-1} p(\omega)$, $Q = \rho^{-2-\lambda} q(\omega)$, u, $v \in V_{\Omega_j}$, $p, q \in L_2(\Omega_j)$, and $\lambda \in \mathbb{C}$. This bilinear form generates the linear and continuous operator

$$\mathfrak{A}_j(\lambda) : V_{\Omega_j} \times L_2(\Omega_j) \to V^*_{\Omega_j} \times L_2(\Omega_j)$$

by

$$\int_\Omega \mathfrak{A}_j(\lambda) \begin{pmatrix} u \\ p \end{pmatrix} \cdot \begin{pmatrix} v \\ q \end{pmatrix} \,\mathrm{d}\omega = a_j \left(\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix}; \lambda \right), \quad u, v \in V_{\Omega_j}, \ p, q \in L_2(\Omega_j).$$

The operator $\mathfrak{A}_j(\lambda)$ depends quadratically on the complex parameter λ. The spectrum of the pencil $\mathfrak{A}_j(\lambda)$ consists of isolated points, eigenvalues with finite geometric and algebraic multiplicities.

4.2. Regularity assertions for weak solutions of the linearized problem

The following two theorems are proved in [19].

Theorem 4.1. *Let $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ be a solution of the problem (2.9), (2.10). Suppose that the following conditions are satisfied.*

(i) $F \in V^* \cap V^{-1,s}_{\beta,\delta}(\mathcal{G})^3$, $g \in L_2(\mathcal{G}) \cap W^{0,s}_{\beta,\delta}(\mathcal{G})$, $h_j \in W^{1/2,2}(\Gamma_j) \cap W^{1-1/s,s}_{\beta,\delta}(\Gamma_j)$,
(ii) *there are no eigenvalues of the pencils $\mathfrak{A}_j(\lambda)$, $j = 1, \dots, d$, in the closed strip between the lines* $\mathrm{Re}\lambda = -1/2$ *and* $\mathrm{Re}\lambda = 1 - \beta_j - 3/s$,
(iii) *the components of δ satisfy the inequalities* $\max(1 - \mu_k, 0) < \delta_k + 2/s < 1$.

Then $u \in W^{1,s}_{\beta,\delta}(\mathcal{G})^3$ and $p \in W^{0,s}_{\beta,\delta}(\mathcal{G})$.

Theorem 4.2. *Let $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ be a solution of the problem (2.9), (2.10). Suppose that*

(i) $g \in W^{1,s}_{\beta,\delta}(\mathcal{G})$, $h_j \in W^{2-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, *and $F \in V^*$ has the representation* (2.7) *with* $f \in W^{0,s}_{\beta,\delta}(\mathcal{G})^3$, $\phi_j \in W^{1-1/s}_{\beta,\delta}(\Gamma_j)^{d_j}$,
(ii) *there are no eigenvalues of the pencils $\mathfrak{A}_j(\lambda)$, $j = 1, \dots, d$, in the closed strip between the lines* $\mathrm{Re}\,\lambda = -1/2$ *and* $\mathrm{Re}\,\lambda = 2 - \beta_j - 3/s$,
(iii) *the components of δ satisfy the inequalities* $\max(2 - \mu_k, 0) < \delta_k + 2/s < 2$,
(iv) *g, h_j and ϕ_j are such that there exist $w \in W^{2,s}_{\beta,\delta}(\mathcal{G})^3$ and $q \in W^{1,s}_{\beta,\delta}(\mathcal{G})$ satisfying*

$$S_j w = h_j, \quad N_j(w, q) = \phi_j \ \text{ on } \Gamma_j, \ j = 1, \dots, N, \quad \nabla \cdot w + g \in V^{1,s}_{\beta,\delta}(\mathcal{G}).$$

Then $u \in W^{2,s}_{\beta,\delta}(\mathcal{G})^3$ and $p \in W^{1,s}_{\beta,\delta}(\mathcal{G})$.

For the proof of the following two regularity assertions in weighted Hölder spaces, we refer to [18].

Theorem 4.3. *Let $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ be a weak solution of the problem (2.3), (2.8). Suppose that*

(i) $f \in C^{-1,\sigma}_{\beta,\delta}(\mathcal{G})^3$, $g \in C^{0,\sigma}_{\beta,\delta}(\mathcal{G})$, $h_j \in C^{1,\sigma}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, $\phi_j \in C^{0,\sigma}_{\beta,\delta}(\Gamma_j)^{d_j}$,

(ii) $\beta_j - \sigma < 3/2$ *for* $j = 1, \dots, d$, *and the strip* $-1/2 < \operatorname{Re}\lambda \leqq 1 + \sigma - \beta_j$ *is free of eigenvalues of the pencils* $\mathfrak{A}_j(\lambda)$, $j = 1, \dots, d$,

(iii) *the components of* δ *are nonnegative and satisfy the inequalities* $1 - \mu_k < \delta_k - \sigma < 1$, $\delta_k \neq \sigma$,

(iv) g, h_j *and* ϕ_j *are such that there exist* $w \in C^{1,\sigma}_{\beta,\delta}(\mathcal{G})^3$ *and* $q \in C^{1,\sigma}_{\beta,\delta}(\mathcal{G})$ *satisfying*

$$S_j w = h_j, \quad N_j(w,q) = \phi_j \ \text{on } \Gamma_j, \ j = 1, \dots, N, \quad \nabla \cdot w + g \in \mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{G}).$$

Then $(u, p) \in C^{1,\sigma}_{\beta,\delta}(\mathcal{G})^3 \times C^{0,\sigma}_{\beta,\delta}(\mathcal{G})$.

Note that under the conditions of Theorem 4.3 on β and δ, there are the imbeddings $C^{-1,\sigma}_{\beta,\delta}(\mathcal{G})^3 \subset V^*$, $C^{0,\sigma}_{\beta,\delta}(\mathcal{G}) \subset L_2(\mathcal{G})$, and $C^{1,\sigma}_{\beta,\delta}(\Gamma_j) \subset W^{1/2,2}(\mathcal{G})$.

Theorem 4.4. *Let* $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ *be a weak solution of the problem* (2.3), (2.8). *Suppose that*

(i) $f \in C^{0,\sigma}_{\beta,\delta}(\mathcal{G})^3$, $g \in C^{1,\sigma}_{\beta,\delta}(\mathcal{G})$, $h_j \in C^{2,\sigma}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, $\phi_j \in C^{1,\sigma}_{\beta,\delta}(\Gamma_j)^{d_j}$,

(ii) $\beta_j - \sigma < 5/2$ *for* $j = 1, \dots, d$, *the strip* $-1/2 < \operatorname{Re}\lambda \leqq 2 + \sigma - \beta_j$ *is free of eigenvalues of the pencil* $\mathfrak{A}_j(\lambda)$, $j = 1, \dots, d$,

(iii) *the components of* δ *are nonnegative, satisfy the inequalities* $2 - \mu_k < \delta_k - \sigma < 2$, $\delta_k \neq \sigma$, *and* $\delta_k \neq 1 + \sigma$,

(iv) g, h_j *and* ϕ_j *are such that there exist* $w \in C^{2,\sigma}_{\beta,\delta}(\mathcal{G})^3$ *and* $q \in C^{1,\sigma}_{\beta,\delta}(\mathcal{G})$, *satisfying*

$$S_j w = h_j, \quad N_j(w,q) = \phi_j \ \text{on } \Gamma_j, \ j = 1, \dots, N, \quad \nabla \cdot w + g \in \mathcal{N}^{1,\sigma}_{\beta,\delta}(\mathcal{G}).$$

Then $(u, p) \in C^{2,\sigma}_{\beta,\delta}(\mathcal{G})^3 \times C^{1,\sigma}_{\beta,\delta}(\mathcal{G})$.

Remark 4.1. For the validity of condition (iv) in Theorems 4.2–4.4, it is necessary and sufficient that the functions h_j and their derivatives, ϕ_j and g satisfy certain compatibility conditions on the edges of the domain $\mathcal{G}$ (see [18,19]).

4.3. Regularity results for solutions of the nonlinear problem in weighted Sobolev spaces

Our goal is to extend the results of Theorems 4.1–4.4 to the nonlinear problem (2.2), (2.3). We start with regularity results in weighted Sobolev spaces.

Lemma 4.1. *Let* $u \in L_6(\mathcal{G})$ *and* $v \in W^{1,s}_{\beta,\delta}(\mathcal{G})$, $s > 6/5$, $\beta'_j \geqq \beta_j - 1/2$ *for* $j = 1, \dots, d$, *and* $\delta'_k \geqq \delta_k - 1/2$. *Then* $u\,\partial_{x_i} v \in V^{-1,s}_{\beta',\delta'}(\mathcal{G})$ *for* $i = 1, 2, 3$.

Proof. Let $q = 6s/(s+6)$. By Hölder's inequality,

$$\|u\,\partial_{x_i} v\|_{V^{0,q}_{\beta,\delta}(\mathcal{G})} \leqq \|u\|_{L_6(\mathcal{G})}\, \|\partial_{x_i} v\|_{V^{0,s}_{\beta,\delta}(\mathcal{G})}.$$

Furthermore by Corollary 3.2, the space $V^{0,q}_{\beta,\delta}(\mathcal{G})$ is continuously imbedded into $V^{-1,s}_{\beta',\delta'}(\mathcal{G})$ if $\beta'_j \geqq \beta_j - 1/2$ and $\delta'_k \geqq \delta_k - 1/2$. The result follows. $\square$

Theorem 4.5. *Let $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ be a solution of the problem (2.4), (2.5). We suppose that $s > 6/5$ and that the conditions (i)–(iii) of Theorem 4.1 are satisfied. Then $u \in W^{1,s}_{\beta,\delta}(\mathcal{G})^3$ and $p \in W^{0,s}_{\beta,\delta}(\mathcal{G})$.*

Proof. (1) First, let $3/2 \leqq s \leqq 3$. From $u_j \in W^{1,2}(\mathcal{G}) \subset L_6(\mathcal{G})$ and $\partial_{x_j} u \in L_2(\mathcal{G})^3$ it follows that $u_j\, \partial_{x_j} u \in L_{3/2}(\mathcal{G})^3$. This together with Corollary 3.2 implies $(u \cdot \nabla)\, u \in V^{-1,s}_{1-3/s,1-3/s}(\mathcal{G})^3$. Hence, (u, p) is a solution of the problem

$$b(u, v) - \int_{\mathcal{G}} p\, \nabla \cdot v \, dx = \Phi(v) \;\text{ for all } v \in V, \tag{4.2}$$

$$-\nabla \cdot u = g \;\text{ in } \mathcal{G}, \quad S_j u = h_j \;\text{ on } \Gamma_j,\; j = 1, \dots, N, \tag{4.3}$$

where

$$\Phi = F - (u \cdot \nabla)u \in V^{-1,s}_{\beta',\delta'}(\mathcal{G})^3, \quad \beta'_j = \max(\beta_j, 1 - 3/s),\; \delta'_k = \max(\delta_k, 1 - 3/s).$$

We conclude from Theorem 4.1 that $(u, p) \in W^{1,s}_{\beta',\delta'}(\mathcal{G})^3 \times W^{0,s}_{\beta',\delta'}(\mathcal{G})$. Then by Lemma 4.1, we have $(u \cdot \nabla)\, u \in V^{-1,s}_{\beta'-1/2,\delta'-1/2}(\mathcal{G})^3$; therefore,

$$F - (u \cdot \nabla)u \in V^{-1,s}_{\beta'',\delta''}(\mathcal{G})^3, \quad \text{where } \beta''_j = \max(\beta_j, 1/2 - 3/s),$$
$$\delta''_k = \max(\delta_k, 1/2 - 3/s).$$

Consequently, Theorem 4.1 implies $(u, p) \in W^{1,s}_{\beta'',\delta''}(\mathcal{G})^3 \times W^{0,s}_{\beta'',\delta''}(\mathcal{G})$. Repeating the last consideration, we obtain $(u, p) \in W^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times W^{0,s}_{\beta,\delta}(\mathcal{G})$.

(2) Next, let $6/5 < s < 3/2$. Applying Lemma 3.5, we obtain

$$(u \cdot \nabla)u \in L_{3/2}(\mathcal{G})^3 = V^{0,3/2}_{0,0}(\mathcal{G})^3 \subset V^{0,s}_{\beta'+1,\delta'+1}(\mathcal{G})^3 \subset V^{-1,s}_{\beta',\delta'}(\mathcal{G})^3,$$

where $\beta'_j = \max(\beta_j, 1 - \frac{3}{s} + \varepsilon)$ and $\delta'_k = \max(\delta_k, \frac{1}{3} - \frac{2}{s} + \varepsilon)$ with an arbitrarily small positive ε. Hence, the vector function Φ in (4.2) belongs also to $V^{-1,s}_{\beta',\delta'}(\mathcal{G})^3$, and Theorem 4.1 implies that $(u, p) \in W^{1,s}_{\beta',\delta'}(\mathcal{G})^3 \times W^{0,s}_{\beta',\delta'}(\mathcal{G})$. From this and from Lemma 4.1, it follows that $\Phi \in V^{-1,s}_{\beta'',\delta}(\mathcal{G})^3$, where $\beta''_j = \max(\beta_j, \beta'_j - 1/2)$. Therefore by Theorem 4.1, we have $(u, p) \in W^{1,s}_{\beta'',\delta}(\mathcal{G})^3 \times W^{0,s}_{\beta'',\delta}(\mathcal{G})$. Repeating the last argument, we obtain $(u, p) \in W^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times W^{0,s}_{\beta,\delta}(\mathcal{G})$

(3) Now let $3 < s \leqq 6$. Then by Lemma 3.5, $V^{-1,s}_{\beta,\delta}(\mathcal{G}) \subset V^{-1,3}_{\beta',\delta'}(\mathcal{G})$, $W^{0,s}_{\beta,\delta}(\mathcal{G}) \subset W^{0,3}_{\beta',\delta'}(\mathcal{G})$, and $W^{1-1/s,s}_{\beta,\delta}(\Gamma_j) \subset W^{2/3,3}_{\beta',\delta'}(\Gamma_j)$, where $\beta'_j = \beta + 3/s - 1 + \varepsilon$, $\delta'_k = \delta_k + 2/s - 2/3 + \varepsilon$, ε is an arbitrarily small positive number. For sufficiently small ε, we conclude from part (1) that $u \in W^{1,3}_{\beta',\delta'}(\mathcal{G})^3$. By Hölder's inequality,

$$\|u_j\, \partial_{x_j} u\|_{W^{0,2}_{\beta',\delta'}(\mathcal{G})^3} \leqq \|u_j\|_{L_6(\mathcal{G})}\, \|\partial_{x_j} u\|_{W^{0,3}_{\beta',\delta'}(\mathcal{G})^3}\,.$$

Due to Corollary 3.2, we have $W^{0,2}_{\beta',\delta'}(\mathcal{G}) \subset V^{-1,s}_{\beta,\delta}(\mathcal{G})$ if $\varepsilon < 1/3$. Therefore, $\Phi = F - (u \cdot \nabla)\, u \in V^{-1,s}_{\beta,\delta}(\mathcal{G})^3$ and Theorem 4.1 implies $(u, p) \in W^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times W^{0,s}_{\beta,\delta}(\mathcal{G})$.

(4) Finally, let $s > 6$. Then again by Lemma 3.5, $V^{-1,s}_{\beta,\delta}(\mathcal{G}) \subset V^{-1,6}_{\beta',\delta'}(\mathcal{G})$, $W^{0,s}_{\beta,\delta}(\mathcal{G}) \subset W^{0,6}_{\beta',\delta'}(\mathcal{G})$, and $W^{1-1/s,s}_{\beta,\delta}(\Gamma_j) \subset W^{5/6,6}_{\beta',\delta'}(\Gamma_j)$, where $\beta'_j = \beta + 3/s - 1/2 + \varepsilon$, $\delta'_k = \delta_k + 2/s - 1/3 + \varepsilon$, ε is an arbitrarily small positive number. For sufficiently small ε, we conclude from part 4) that $u \in W^{1,6}_{\beta',\delta'}(\mathcal{G})^3$. Since $u \in L_6(\mathcal{G})^3$, it follows that $(u \cdot \nabla)\, u \in W^{0,3}_{\beta',\delta'}(\mathcal{G})^3$. The last space is imbedded into $V^{-1,s}_{\beta,\delta}(\mathcal{G})^3$ if $\varepsilon \leqq 1/3$ (see Corollary 3.2). Therefore, (u, p) is a solution of the problem (4.2), (4.3), where $\Phi \in V^{-1,s}_{\beta,\delta}(\mathcal{G})^3$. Applying Theorem 4.1, we obtain $(u, p) \in W^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times W^{0,s}_{\beta,\delta}(\mathcal{G})$. $\square$

For the proof of an analogous $W^{2,s}_{\beta,\delta}$ regularity result, we need the following lemma.

Lemma 4.2. *Let $u \in W^{2,s}_{\beta,\delta}(\mathcal{G}) \cap W^{1,2}(\mathcal{G})$, $1 < s < 6$, $\beta_j + 3/s \leqq 5/2$, $\delta_k + 2/s > 0$. Then $u\, \nabla u \in V^{0,s}_{\beta-1/2,\delta'}(\mathcal{G})^3$ for every δ', $\delta'_k \geqq \delta_k - 1/2$, $\delta'_k + 2/s > 0$.*

Proof. (1) Suppose that $\delta_k + 2/s > 1$ for $k = 1, \ldots, m$. Then by Lemma 3.6, $\nabla u \in W^{1,s}_{\beta,\delta}(\mathcal{G})^3 = V^{1,s}_{\beta,\delta}(\mathcal{G})^3 \subset V^{0,q}_{\beta-1/2,\delta-1/2}(\mathcal{G})^3$, $q = 6s/(6-s)$. From this, from the assumption $u \in W^{1,2}(\mathcal{G}) \subset L_6(\mathcal{G})$ and from Hölder's inequality, it follows that $u\, \nabla u \in V^{0,s}_{\beta-1/2,\delta-1/2}(\mathcal{G})^3$,

$$\|u\, \nabla u\|_{V^{0,s}_{\beta-1/2,\delta-1/2}(\mathcal{G})^3} \leqq c\, \|u\|_{L_6(\mathcal{G})}\, \|\nabla u\|_{V^{0,q}_{\beta-1/2,\delta-1/2}(\mathcal{G})^3}\,.$$

(2) We consider the case when $0 < \delta_k + 2/s \leqq 1$ for all k. Then u admits the decomposition (cf. Lemma 3.3)

$$u = v + w, \quad v \in V^{2,s}_{\beta,\delta}(\mathcal{G}), \quad w \in W^{4,s}_{\beta+2,\delta+2}(\mathcal{G}).$$

Let $\mathcal{U}_1, \ldots, \mathcal{U}_d$ be the same domains as in the definition of the spaces $V^{l,s}_{\beta,\delta}(\mathcal{G})$ and $W^{l,s}_{\beta,\delta}(\mathcal{G})$. Using Lemma 3.2, we obtain

$$\rho_j^{\beta_j-5/2+3/s} w \in L_6(\mathcal{G} \cap \mathcal{U}_j), \quad \rho_j^{\beta_j-2+3/s} w \in L_\infty(\mathcal{G} \cap \mathcal{U}_j)$$

and

$$\rho_j^{\beta_j-1+2/s}\, \nabla w \in L_{3s}(\mathcal{G} \cap \mathcal{U}_j)^3$$

for $j = 1, \ldots, d$. Since $\beta_j - 5/2 + 3/2 \leqq 0$, it follows in particular that $w \in L_6(\mathcal{G})$ and therefore also $v \in L_6(\mathcal{G})$. We estimate the norms of $v\nabla u$ and $w\nabla u$ in $V^{0,s}_{\beta-1/2,\delta-1/2}(\mathcal{G})^3$. Let $q = 6s/(6-s)$. Using Hölder's inequality and Lemma 3.6, we obtain

$$\|v\nabla v\|_{V^{0,s}_{\beta-1/2,\delta-1/2}(\mathcal{G})^3} \leqq \|v\|_{L_6(\mathcal{G})}\, \|\nabla v\|_{V^{0,q}_{\beta-1/2,\delta-1/2}(\mathcal{G})^3} \leqq c\, \|v\|_{L_6(\mathcal{G})}\, \|\nabla v\|_{V^{1,s}_{\beta,\delta}(\mathcal{G})^3}\,.$$

By Lemma 3.6, the space $V^{2,s}_{\beta,\delta}(\mathcal{G})$ is continuously imbedded into $V^{0,3s/2}_{\beta-2+1/s,\delta-2+1/s}(\mathcal{G})$. Consequently,

$$\|v\,\nabla w\|_{V^{0,s}_{\beta-1/2,\delta-1/2}(\mathcal{G})^3} \leqq c\,\|v\|_{V^{0,3s/2}_{\beta-2+1/s,\delta-2+1/s}(\mathcal{G})} \times \sum_{j=1}^{d} \Big\| \rho_j^{3/2-1/s} \prod_{k\in X_j} \Big(\frac{r_k}{\rho_j}\Big)^{3/2-1/s} \nabla w \Big\|_{L_{3s}(\mathcal{G}\cap\mathcal{U}_j)^3}$$

$$\leqq c\,\|v\|_{V^{2,s}_{\beta,\delta}(\mathcal{G})} \sum_{j=1}^{d} \Big\| \rho_j^{\beta_j-1+2/s}\,\nabla w \Big\|_{L_{3s}(\mathcal{G}\cap\mathcal{U}_j)^3}\,.$$

Thus, we have $v\nabla u \in V^{0,s}_{\beta-1/2,\delta-1/2}(\mathcal{G})^3 \subset V^{0,s}_{\beta-1/2,\delta'}(\mathcal{G})^3$. Furthermore, using the continuity of the imbedding $W^{2,s}_{\beta,\delta}(\mathcal{G}) \subset W^{1,s}_{3/2-3/s,\delta'}(\mathcal{G})$, we obtain

$$\|w\,\nabla u\|_{V^{0,s}_{\beta-1/2,\delta'}(\mathcal{G})^3} \leqq c \sum_{j=1}^{d} \Big\|\rho_j^{\beta_j-2+3/s}\, w\Big\|_{L_\infty(\mathcal{G}\cap\mathcal{U}_j)} \times \Big\| \rho_j^{3/2-3/s} \prod_{k\in X_j} \Big(\frac{r_k}{\rho_j}\Big)^{\delta_k'} \nabla u \Big\|_{L_s(\mathcal{G}\cap\mathcal{U}_j)^3}$$

$$\leqq c \sum_{j=1}^{d} \Big\|\rho_j^{\beta_j-2+3/s}\, w\Big\|_{L_\infty(\mathcal{G}\cap\mathcal{U}_j)} \|u\|_{W^{2,s}_{\beta,\delta}(\mathcal{G})^3}\,.$$

This proves the lemma for the case $\delta_k + 2/s < 1$, $k = 1, \dots, m$.

(3) The case when $\delta_k + 2/s < 1$ for some but not all k can be reduced to cases (1) and (2) using suitable cut-off functions. □

Theorem 4.6. *Let* $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ *be a solution of the problem* (2.4), (2.5). *We assume that the conditions* (i)–(iv) *of Theorem* 4.2 *are satisfied and that* $\beta_j + 3/s < 5/2$ *for* $j = 1, \dots, d$. *Then* $u \in W^{2,s}_{\beta,\delta}(\mathcal{G})^3$ *and* $p \in W^{1,s}_{\beta,\delta}(\mathcal{G})$.

Proof. (1) First, let $1 < s \leqq 3/2$. Let $\mathcal{U}_j$ and X_j be the same sets as in the definition of the spaces $V^{l,s}_{\beta,\delta}(\mathcal{G})$ and $W^{l,s}_{\beta,\delta}(\mathcal{G})$. We put $q = 3s/(3-2s)$ if $s < 3/2$, $q = \infty$ if $s = 3/2$. Since

$$\|u_i\,\partial_{x_i} u\|_{W^{0,s}_{\beta',\delta'}(\mathcal{G})^3} \leqq c\,\|u_i\|_{L_6(\mathcal{G})}\,\|\partial_{x_i} u\|_{L_2(\mathcal{G})^3} \sum_{j=1}^{d} \Big\| \rho_j^{\beta_j'} \prod_{k\in X_j} \Big(\frac{r_k}{\rho_j}\Big)^{\delta_k'} \Big\|_{L_q(\mathcal{G}\cap\mathcal{U}_j)}\,,$$

we obtain $(u\cdot\nabla)\,u \in W^{0,s}_{\beta',\delta'}(\mathcal{G})^3$ if $\beta_j' > -3/q$ and $\delta_k' > -2/q$ or, equivalently, if $\beta_j' + 3/s > 2$ for $j = 1, \dots, N$, $\delta_k' + 2/s > 4/3$ for $k = 1, \dots, m$. Let $\beta_j' = \max(\beta_j, 2-3/s+\varepsilon)$, $\delta_k' = \max(\delta_k, 4/3-2/s+\varepsilon)$, where ε is a sufficiently small positive number. Then (u, p) is a solution of the problem

$$-\nu\Delta u + \nabla p = f - (u\cdot\nabla)\,u, \quad -\nabla\cdot u = g \ \text{ in } \mathcal{G}, \tag{4.4}$$

$$S_j u = h_j, \quad N_j(u,p) = \phi_j \ \text{ on } \Gamma_j, \ j = 1, \dots, N, \tag{4.5}$$

where $f-(u\cdot\nabla)\,u \in W^{0,s}_{\beta',\delta'}(\mathcal{G})^3$, $g \in W^{1,s}_{\beta',\delta'}(\mathcal{G})$, $h_j \in W^{2-1/s,s}_{\beta',\delta'}(\Gamma_j)$ and $\phi_j \in W^{1-1/s,s}_{\beta',\delta'}(\Gamma_j)$. Consequently, by Theorem 4.2, we have $(u,p) \in W^{2,s}_{\beta',\delta'}(\mathcal{G})^3 \times W^{1,s}_{\beta',\delta'}(\mathcal{G})$. Applying Lemma 4.2, we obtain $f-(u\cdot\nabla)\,u \in W^{0,s}_{\beta'',\delta''}(\mathcal{G})^3$, where $\beta''_j = \max(\beta_j, 3/2-3/s+\varepsilon)$ and $\delta''_k = \max(\delta_k, 5/6-2/s+\varepsilon)$. Hence, Theorem 4.2 implies $(u,p) \in W^{2,s}_{\beta'',\delta''}(\mathcal{G})^3 \times W^{1,s}_{\beta'',\delta''}(\mathcal{G})$. Repeating this procedure, we obtain $(u,p) \in W^{2,s}_{\beta,\delta}(\mathcal{G})^3 \times W^{1,s}_{\beta,\delta}(\mathcal{G})$.

(2) Next, we consider the case $3/2 < s \leqq 2$. Let ε be a positive number less than 1/2 such that $\delta_k + 2/s < 2-\varepsilon$ for all k. Then by Lemma 3.5, $W^{l,s}_{\beta,\delta}(\mathcal{G}) \subset W^{l,3/2}_{\beta+3/s-2+\varepsilon,2/3-\varepsilon}(\mathcal{G})$, and from part (1) it follows that $(u,p) \in W^{2,3/2}_{\beta+3/s-2+\varepsilon,2/3-\varepsilon}(\mathcal{G})^3 \times W^{1,3/2}_{\beta+3/s-2+\varepsilon,2/3-\varepsilon}(\mathcal{G})$ if ε is sufficiently small. In particular,

$$\partial_{x_i} u \in W^{1,3/2}_{\beta+3/s-2+\varepsilon,2/3-\varepsilon}(\mathcal{G})^3 = V^{1,3/2}_{\beta+3/s-2+\varepsilon,2/3-\varepsilon}(\mathcal{G})^3 \subset V^{0,3}_{\beta+3/s-2+\varepsilon,2/3-\varepsilon}(\mathcal{G})^3$$

(see Lemma 3.6). Let $\delta'_k = \max(\delta_k, 5/3-2/s)$ for $k=1,\dots,m$. Then

$$\|u_i\,\partial_{x_i} u\|_{W^{0,s}_{\beta,\delta'}(\mathcal{G})^3} \leqq c\,\|u_i\|_{L_6(\mathcal{G})} \sum_{j=1}^{d} \left\| \rho_j^{\beta_j+3/s-2+\varepsilon} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}\right)^{2/3-\varepsilon} \partial_{x_i} u \right\|_{L_3(\mathcal{G}\cap\mathcal{U}_j)^3},$$

where

$$c = \sum_{j=1}^{d} \left\| \rho_j^{2-3/s-\varepsilon} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}\right)^{\delta'_k-2/3+\varepsilon} \right\|_{L_{2s/(2-s)}(\mathcal{G}\cap\mathcal{U}_j)} < \infty.$$

Consequently, $f-(u\cdot\nabla)\,u \in W^{0,s}_{\beta,\delta'}(\mathcal{G})^3$, and Theorem 4.2 implies $(u,p) \in W^{2,s}_{\beta,\delta'}(\mathcal{G})^3 \times W^{1,s}_{\beta,\delta'}(\mathcal{G})$. Using Lemma 4.2, we obtain $f-(u\cdot\nabla)\,u \in W^{0,s}_{\beta,\delta''}(\mathcal{G})^3$, where $\delta''_k = \max(\delta_k, \delta'_k - 1/2)$. Again Theorem 4.2 yields $(u,p) \in W^{2,s}_{\beta,\delta''}(\mathcal{G})^3 \times W^{1,s}_{\beta,\delta''}(\mathcal{G})$. Repeating this argument, we obtain $(u,p) \in W^{2,s}_{\beta,\delta}(\mathcal{G})^3 \times W^{1,s}_{\beta,\delta}(\mathcal{G})$.

(3) Let $2 < s \leqq 3$, and let ε be a positive number less than 1/2 such that $\delta_k + 2/s < 2-\varepsilon$ for all k. Then by Lemma 3.5, $W^{l,s}_{\beta,\delta}(\mathcal{G}) \subset W^{l,2}_{\beta+3/s-3/2+\varepsilon,1-\varepsilon}(\mathcal{G})$. Therefore by part (2), we have $(u,p) \in W^{2,2}_{\beta+3/s-3/2+\varepsilon,1-\varepsilon}(\mathcal{G})^3 \times W^{1,2}_{\beta+3/s-3/2+\varepsilon,1-\varepsilon}(\mathcal{G})$ provided ε is sufficiently small. Consequently,

$$\partial_{x_i} u \in W^{1,2}_{\beta+3/s-3/2+\varepsilon,1-\varepsilon}(\mathcal{G})^3 = V^{1,2}_{\beta+3/s-3/2+\varepsilon,1-\varepsilon}(\mathcal{G})^3 \subset V^{0,6}_{\beta+3/s-3/2+\varepsilon,1-\varepsilon}(\mathcal{G})^3.$$

Let $\delta'_k = \max(\delta_k, 5/3-2/s)$ for $k=1,\dots,m$. Then

$$\|u_i\,\partial_{x_i} u\|_{W^{0,s}_{\beta,\delta'}(\mathcal{G})^3} \leqq c\,\|u_i\|_{L_6(\mathcal{G})}\,\|\partial_{x_i} u\|_{V^{0,6}_{\beta+3/s-3/2+\varepsilon,1-\varepsilon}(\mathcal{G})^3},$$

where

$$c = \sum_{j=1}^{d} \left\| \rho_j^{3/2-3/s-\varepsilon} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}\right)^{\delta_k'-1+\varepsilon} \right\|_{L_{3s/(3-s)}(\mathcal{G}\cap\mathcal{U}_j)} < \infty.$$

Thus, we have $f - (u\cdot\nabla)\,u \in W^{0,s}_{\beta,\delta'}(\mathcal{G})^3$ and $(u,p) \in W^{2,s}_{\beta,\delta'}(\mathcal{G})^3 \times W^{1,s}_{\beta,\delta'}(\mathcal{G})$ (by Theorem 4.2). Hence by Lemma 4.2, $f - (u\cdot\nabla)\,u \in W^{0,s}_{\beta,\delta''}(\mathcal{G})^3$, where $\delta_k'' = \max(\delta_k, 7/6-2/s)$, and Theorem 4.2 implies $(u,p) \in W^{2,s}_{\beta,\delta''}(\mathcal{G})^3 \times W^{1,s}_{\beta,\delta''}(\mathcal{G})$. Repeating the last argument, we obtain $(u,p) \in W^{2,s}_{\beta,\delta}(\mathcal{G})^3 \times W^{1,s}_{\beta,\delta}(\mathcal{G})$.

(4) Finally, let $s > 3$. We define $\delta_k' = \max(\delta_k, 1-2/s+\varepsilon)$, where ε is a sufficiently small positive number. Then we have

$$g \in W^{1,s}_{\beta,\delta'}(\mathcal{G}) \subset W^{0,s}_{\beta-1,\delta'-1}(\mathcal{G}), \quad h_j \in W^{2-1/s,s}_{\beta,\delta'}(\Gamma_j)^{3-d_j} \subset W^{1-1/s,s}_{\beta-1,\delta'-1}(\Gamma_j)^{3-d_j}.$$

Furthermore, the functional (2.7) belongs to $V^{-1,s}_{\beta-1,\delta'-1}(\mathcal{G})^3$. Since $\max(1-\mu_k, 0) < \delta_k' - 1 + 2/s < 1$ it follows from Theorem 4.5 that $u \in W^{1,s}_{\beta-1,\delta'-1}(\mathcal{G})^3$. Then by Corollary 3.1,

$$\rho_j^{\beta_j-2+3/s} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}\right)^{\sigma_k} u \in L_\infty(\mathcal{G}\cap\mathcal{U}_j)^3,$$

where $\sigma_k = 0$ for $\delta_k < 2-3/s$, $\sigma_k = 1/s+\varepsilon$ for $2-3/s \leqq \delta_k < 2-2/s$. By Hölder's inequality, we have

$$\begin{aligned}
\|u_i\,\partial_{x_i} u\|_{W^{0,s}_{\beta,\delta}(\mathcal{G})} &\leqq c \sum_{j=1}^{d} \left\| \rho_j^{\beta_j-2+3/s} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}\right)^{\sigma_k} u_i \right\|_{L_\infty(\mathcal{G}\cap\mathcal{U}_j)} \\
&\quad \times \left\| \rho_j^{2-3/s} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}\right)^{\delta_k-\sigma_k} \partial_{x_i} u \right\|_{L_s(\mathcal{G}\cap\mathcal{U}_j)^3} \\
&\leqq c \sum_{j=1}^{d} \left\| \rho_j^{\beta_j-2+3/s} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}\right)^{\sigma_k} u_i \right\|_{L_\infty(\mathcal{G}\cap\mathcal{U}_j)} \|\partial_{x_i} u\|_{W^{0,s}_{\beta-1,\delta'-1}(\mathcal{G})^3}.
\end{aligned}$$

For the last inequality, we used the fact that $\beta - 1 \leqq 2-3/s$ and $\delta_k' - 1 \leqq \delta_k - \sigma_k$. Hence, (u,p) is a solution of the problem (4.2), (4.3), where $\Phi = F - (u\cdot\nabla)\,u$ is a functional of the form (2.7) with $f \in W^{0,s}_{\beta,\delta}(\mathcal{G})$, $\phi_j \in W^{1-1/s,s}_{\beta,\delta}(\mathcal{G})^{d_j}$. Applying Theorem 4.2, we obtain $u \in W^{2,s}_{\beta,\delta}(\mathcal{G})^3$ and $p \in W^{1,s}_{\beta,\delta}(\mathcal{G})$. $\square$

4.4. Regularity results in weighted Hölder spaces

In order to extend the results of Theorems 4.3 and 4.4 to problem (2.2), (2.3), we consider first the nonlinear term in the Navier–Stokes system.

Lemma 4.3. *Let $u \in C^{2,\sigma}_{\beta,\delta}(\mathcal{G})$, where $\beta_j \leqq 3+\sigma$ for $j = 1,\dots,d$, $0 \leqq \delta_k < 2+\sigma$, $\delta_k - \sigma$ is not an integer for $k = 1,\dots,m$. Then $u\,\partial_{x_i} u \in C^{0,\sigma}_{\beta,\delta'}(\mathcal{G})$ for every δ' such that $\delta'_k \geqq \max(0, \delta_k - 1)$, $k = 1,\dots,m$.*

Proof. Let $\mathcal{G}_j$, $\mathcal{G}_{j,k}$ and X_j be the same sets as in the definition of the spaces $\mathcal{N}^{l,\sigma}_{\beta,\delta}(\mathcal{G})$ and $C^{l,\sigma}_{\beta,\delta}(\mathcal{G})$. We have to show that there exists a finite constant C such that

$$\rho_j(x)^{\beta_j-\sigma} \prod_{k\in X_j} \Big(\frac{r_k}{\rho_j}\Big)^{\max(\delta'_k-\sigma,0)} \big|u(x)\partial_{x_i}u(x)\big| \leqq C \tag{4.6}$$

for $x \in \mathcal{G}_j$,

$$\rho_j(x)^{\beta_j} \prod_{k\in X_j} \Big(\frac{r_k(x)}{\rho_j(x)}\Big)^{\delta'_k} \frac{|u(x)\partial_{x_i}u(x) - u(y)\partial_{y_i}u(y)|}{|x-y|^\sigma} \leqq C \tag{4.7}$$

for $x, y \in \mathcal{G}_j$, $|x-y| < r(x)/2$ and

$$\rho_j(x)^{\beta_j-\delta'_k} \frac{|u(x)\partial_{x_i}u(x) - u(y)\partial_{y_i}u(y)|}{|x-y|^{\sigma-\delta'_k}} \leqq C \tag{4.8}$$

for $x, y \in \mathcal{G}_{j,k}$, $|x-y| < \rho_j(x)/2$, $\delta'_k < \sigma$. Inequality (4.6) follows immediately from the estimate

$$\rho_j(x)^{\beta_j-2-\sigma+|\alpha|} \prod_{k\in X_j} \Big(\frac{r_k}{\rho_j}\Big)^{\max(\delta_k-2-\sigma+|\alpha|,0)} |\partial_x^\alpha u(x)| \leqq \|u\|_{C^{2,\sigma}_{\beta,\delta}(\mathcal{G})} \tag{4.9}$$

($|\alpha| \leqq 2$) and from the inequalities $\beta_j \leqq 3+\sigma$, $\delta'_k \geqq \delta_k - 1$. Furthermore,

$$\frac{|u(x)-u(y)|}{|x-y|^\sigma} \leqq c\,|x-y|^{1-\sigma} \sup_z |\nabla u(z)|$$

for $x, y \in \mathcal{G}_j$, $|x-y| < r(x)/2$, where the supremum is taken over all $z \in \mathcal{G}_j$ such that $|x-z| < r(x)/2$. Using the inequalities $\rho_j(x)/2 < \rho_j(z) < 3\rho_j(x)/2$, $r_k(x)/2 < r_k(z) < 3r_k(x)/2$ and (4.9), we obtain

$$\frac{|u(x)-u(y)|}{|x-y|^\sigma} \leqq c\,r(x)^{1-\sigma}\,\rho_j(x)^{1+\sigma-\beta_j} \prod_{k\in X_j} \Big(\frac{r_k(x)}{\rho_j(x)}\Big)^{-\max(\delta_k-1-\sigma,0)}$$

for $x, y \in \mathcal{G}_j$, $|x-y| < r(x)/2$. Analogously,

$$\frac{\big|\partial_{x_i}u(x) - \partial_{y_i}u(y)\big|}{|x-y|^\sigma} \leqq c\,r(x)^{1-\sigma}\,\rho_j(x)^{\sigma-\beta_j} \prod_{k\in X_j} \Big(\frac{r_k(x)}{\rho_j(x)}\Big)^{-\max(\delta_k-\sigma,0)}$$

for $x, y \in \mathcal{G}_j$, $|x-y| < r(x)/2$. Hence,

$$\frac{|u(x)\partial_{x_i}u(x) - u(y)\partial_{y_i}u(y)|}{|x-y|^\sigma} \leqq \frac{|u(x)-u(y)|}{|x-y|^\sigma}\,\big|\partial_{x_i}u(x)\big| + \frac{\big|\partial_{x_i}u(x) - \partial_{y_i}u(y)\big|}{|x-y|^\sigma}\,|u(y)| \leqq c\,r(x)^{1-\sigma}\,\rho_j(x)^{2+2\sigma-2\beta_j} \times \left(\prod_{k\in X_j}\left(\frac{r_k(x)}{\rho_j(x)}\right)^{-2\max(\delta_k-1-\sigma,0)} + \prod_{k\in X_j}\left(\frac{r_k(x)}{\rho_j(x)}\right)^{-\max(\delta_k-\sigma,0)}\right)$$

for $x, y \in \mathcal{G}_j$, $|x-y| < r(x)/2$. From this estimate and from the inequalities

$$c_1\,r(x) \leqq \rho_j(x)\prod_k \frac{r_k(x)}{\rho_j(x)} \leqq c_2\,r(x),$$

$\delta'_k + 1 - \sigma \geqq 2\max(\delta_k - \sigma - 1, 0)$, $\delta'_k + 1 - \sigma \geqq \max(\delta_k - \sigma, 0)$, and $\beta_j \leqq 3+\sigma$, we obtain (4.7). Analogously, we obtain the estimate

$$\rho_j(x)^{\beta_j-\delta'_k}\frac{|u(x)-u(y)|}{|x-y|^{\sigma-\delta'_k}}|\partial_{x_i}u(x)| \leqq C \quad \text{for } \delta'_k < \sigma,\ x, y \in \mathcal{G}_{j,k},\ |x-y| < \rho_j(x)/2.$$

Since $\partial_{x_i}u(x) \in C^{1,\sigma}_{\beta,\delta}(\mathcal{G}) \subset C^{1,\sigma}_{\beta,\delta'+1}(\mathcal{G})$, there exists a constant C such that

$$\rho_j(x)^{\beta_j-1-\delta'_k}\frac{|\partial_{x_i}u(x)-\partial_{y_i}u(y)|}{|x-y|^{\sigma-\delta'_k}} \leqq C \quad \text{for } \delta'_k < \sigma,\ x, y \in \mathcal{G}_{j,k},\ |x-y| < \rho_j(x)/2.$$

This together with (4.9) implies

$$\rho_j(x)^{\beta_j-\delta'_k}\frac{|\partial_{x_i}u(x)-\partial_{y_i}u(y)|}{|x-y|^{\sigma-\delta'_k}}\,|u(x)| \leqq C \quad \text{for } \delta'_k < \sigma,\ x, y \in \mathcal{G}_{j,k},\ |x-y| < \rho_j(x)/2.$$

Thus, estimate (4.8) holds. The proof is complete. □

Theorem 4.7. *Let $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ be a weak solution of the problem (2.2), (2.3), and let the conditions (i)–(iv) of Theorem 4.4 are satisfied. Then $u \in C^{2,\sigma}_{\beta,\delta}(\mathcal{G})^3$ and $p \in C^{1,\sigma}_{\beta,\delta}(\mathcal{G})$.*

Proof. Suppose first that $\delta_k > \sigma$ for $k = 1, \dots, m$. Let ε be an arbitrarily small positive number and s an arbitrary real number greater than 1. We put $\beta'_j = \beta_j - \sigma - 3/s + \varepsilon$ for $j = 1, \dots, N$ and $\delta'_k = \delta_k - \sigma - 2/s + \varepsilon$. From our assumptions on f, g, h_j and ϕ_j it follows that

$$f \in W^{0,s}_{\beta',\delta'}(\mathcal{G})^3,\quad g \in W^{1,s}_{\beta',\delta'}(\mathcal{G}),\quad h_j \in W^{2-1/s,s}_{\beta',\delta'}(\Gamma_j)^{3-d_j},\quad \phi_j \in W^{1-1/s,s}_{\beta',\delta'}(\Gamma_j)^{d_j}.$$

Using Theorem 4.6, we obtain $u \in W^{2,s}_{\beta',\delta'}(\mathcal{G})^3$. Consequently,

$$u \in W^{0,s}_{\beta'-2,-2/s+\varepsilon}(\mathcal{G})^3,\quad \partial_{x_i}u \in W^{0,s}_{\beta'-1,\delta''}(\mathcal{G})^3,\quad \partial_{x_i}\partial_{x_j}u \in W^{0,s}_{\beta',\delta'}(\mathcal{G})^3$$

for $i, j = 1, 2, 3$, where $\delta''_k = \max(\delta_k - \sigma - 1, 0) - 2/s + \varepsilon$ for $k = 1, \dots, m$. This together with Hölder's inequality implies

$$u_j\, \partial_{x_j} u \in W^{0,s/2}_{2\beta'-3,\delta''-2/s+\varepsilon}(\mathcal{G})^3, \quad u_j\, \partial_{x_i} \partial_{x_j} u \in W^{0,s/2}_{2\beta'-2,\delta'-2/s+\varepsilon}(\mathcal{G})^3,$$
$$\times (\partial_{x_i} u_j)\, \partial_{x_j} u \in W^{0,s/2}_{2\beta'-2,2\delta''}(\mathcal{G})^3.$$

Therefore, $u_j\, \partial_{x_j} u \in W^{1,s/2}_{2\beta'-2,\delta'-2/s+\varepsilon}(\mathcal{G})^3$. The numbers ε and s can be chosen such that $\beta_j - \sigma \leqq 3 - 2\varepsilon$ for $j = 1, \dots, d$, $\delta_k - \sigma > 2/s$ for $k = 1, \dots, m$, $\varepsilon + 1/s \leqq 1/2$, and $1 - 6/s > \sigma$. Then $2\beta'_j - 2 \leqq \beta_j - \sigma + 1 - 6/s$, $\delta'_k - 2/s + \varepsilon \leqq \delta_k - \sigma + 1 - 6/s$, and $\delta_k - \sigma + 1 - 6/s > 1 - 4/s$. Consequently,

$$u_j\, \partial_{x_j} u \in W^{1,s/2}_{\beta-\sigma+1-6/s,\delta-\sigma+1-6/s}(\mathcal{G})^3 = V^{1,s/2}_{\beta-\sigma+1-6/s,\delta-\sigma+1-6/s}(\mathcal{G})^3 \subset \mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{G})^3$$

(see Lemma 3.4). Hence, (u, p) is a solution of the problem (4.4), (4.5), where $f - (u \cdot \nabla)\, u \in C^{0,\sigma}_{\beta,\delta}(\mathcal{G})^3$. Applying Theorem 4.4, we obtain $(u, p) \in C^{2,\sigma}_{\beta,\delta}(\mathcal{G})^3 \times C^{1,\sigma}_{\beta,\delta}(\mathcal{G})$.

Suppose now that $\delta'_k < \sigma$ for at least one k. By the first part of the proof, we obtain $u \in C^{2,\sigma}_{\beta,\gamma}(\mathcal{G})^3$, where $\gamma_k = \max(\delta_k, \sigma + \varepsilon)$, ε is an arbitrarily small positive real number. Then Lemma 4.3 implies $f - (u \cdot \nabla)\, u \in C^{0,\sigma}_{\beta,\delta}(\mathcal{G})^3$, and from Theorem 4.4 it follows that $(u, p) \in C^{2,\sigma}_{\beta,\delta}(\mathcal{G})^3 \times C^{1,\sigma}_{\beta,\delta}(\mathcal{G})$. The proof is complete. □

Finally, we prove the analogous $C^{1,\sigma}_{\beta,\delta}$-regularity result.

Theorem 4.8. *Let $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ be a weak solution of the problem (2.2), (2.3). Suppose that conditions (i)–(iv) of Theorem 4.3 are satisfied. Then $u \in C^{1,\sigma}_{\beta,\delta}(\mathcal{G})^3$ and $p \in C^{0,\sigma}_{\beta,\delta}(\mathcal{G})$.*

Proof. (1) Let first $\delta_k > \sigma$ for $k = 1, \dots, m$. Then $g \in W^{0,s}_{\beta',\delta'}(\mathcal{G})$ and $h_j \in W^{1-1/s,s}_{\beta',\delta'}(\Gamma_j)^{3-d_j}$, where $\beta'_j = \beta_j - \sigma - 3/s + \varepsilon$, $\delta'_k = \delta_k - \sigma - 2/s + \varepsilon$, ε is an arbitrarily mall positive number, and $s > 1$. Furthermore, the functional (2.7) belongs to $V^{-1,s}_{\beta',\delta'}(\mathcal{G})^3$. Using Theorem 4.5, we obtain $(u, p) \in W^{1,s}_{\beta',\delta'}(\mathcal{G})^3 \times W^{1,s}_{\beta',\delta'}(\mathcal{G})^3$. We consider the term

$$(u \cdot \nabla)\, u = \sum_{j=1}^{3} \partial_{x_j}(u_j u) - \sum_{j=1}^{3} u\, \partial_{x_j} u_j = \sum_{j=1}^{3} \partial_{x_j}(u_j u) + gu.$$

Since $u_i \in W^{0,s}_{\beta'-1,-2/s+\varepsilon}(\mathcal{G})$ and $\partial_{x_k} u_i \in W^{0,s}_{\beta',\delta'}(\mathcal{G})$, we have

$$u_j u \in W^{1,s/2}_{2\beta'-1,\delta'-2/s+\varepsilon}(\mathcal{G})^3 \subset W^{1,s/2}_{\beta-\sigma+1-6/s,\delta-\sigma+1-6/s}(\mathcal{G})^3$$
$$= V^{1,s/2}_{\beta-\sigma+1-6/s,\delta-\sigma+1-6/s}(\mathcal{G})^3 \subset \mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{G})^3$$

if ε is sufficiently small and s is sufficiently large (see Lemma 3.4). Furthermore, from $g \in \mathcal{N}^{0,\sigma}_{\beta,\delta}(\mathcal{G})$,

$$u \in W^{1,s}_{\beta',1-2/s+\varepsilon}(\mathcal{G})^3 = V^{1,s}_{\beta',1-2/s+\varepsilon}(\mathcal{G})^3 \subset \mathcal{N}^{0,\sigma}_{\beta-1+\varepsilon,\sigma+\varepsilon+1/s}(\mathcal{G})^3$$

and Lemma 3.7 it follows that

$$gu \in \mathcal{N}^{0,\sigma}_{2\beta-1+\varepsilon-\sigma,\delta+\varepsilon+1/s}(\mathcal{G}) \subset \mathcal{N}^{0,\sigma}_{\beta+1,\delta+1}(\mathcal{G}).$$

Consequently, we have $(u \cdot \nabla)\, u \in C^{-1,\sigma}_{\beta,\delta}(\mathcal{G})^3$, and Theorem 4.3 implies $(u, p) \in C^{1,\sigma}_{\beta,\delta}(\mathcal{G})^3 \times C^{0,\sigma}_{\beta,\delta}(\mathcal{G})$.

(2) Suppose that $\delta_k < \sigma$ for some k. By the first part of the proof, we have $(u, p) \in C^{1,\sigma}_{\beta,\gamma}(\mathcal{G})^3 \times C^{0,\sigma}_{\beta,\gamma}(\mathcal{G})$, where $\gamma_k = \max(\delta_k, \sigma+\varepsilon)$ for $k = 1, \dots, m$, ε is an arbitrarily small positive number. In particular, $u_j \in \mathcal{N}^{0,\sigma}_{\beta-1,\gamma}(\mathcal{G})$, $\partial_{x_j} u \in \mathcal{N}^{0,\sigma}_{\beta,\gamma}(\mathcal{G})^3$, and therefore (by Lemma 3.7)

$$u_j \partial_{x_j} u \in \mathcal{N}^{0,\sigma}_{2\beta-1-\sigma,2\gamma-\sigma}(\mathcal{G})^3.$$

The last space is contained in $\mathcal{N}^{0,\sigma}_{\beta+1,\delta+1}(\mathcal{G})^3$ for sufficiently small ε. Applying Theorem 4.3, we obtain $(u, p) \in C^{1,\sigma}_{\beta,\delta}(\mathcal{G})^3 \times C^{0,\sigma}_{\beta,\delta}(\mathcal{G})$. □

4.5. Necessity of the conditions on β and δ

Let Λ_j be the eigenvalue of the pencil $\mathfrak{A}_j(\lambda)$ with smallest real part $> -1/2$. We show that the inequalities

$$\beta_j + 3/s > 2 - \mathrm{Re}\,\Lambda_j, \quad \delta_k + 2/s > 2 - \mu_k \tag{4.10}$$

in Theorem 4.6 cannot be weakened.

We assume first that $\beta_j + 3/s \leqq 2 - \mathrm{Re}\,\Lambda_j$ for some j and that δ satisfies the second condition of (4.10). By our assumptions on the domain, there exist a neighborhood $\mathcal{U}_j$ of $x^{(j)}$ and a diffeomorphism κ mapping $\mathcal{G} \cap \mathcal{U}_j$ onto the intersection of a cone $\mathcal{K}_j$ with the unit ball such that the Jacobian matrix $\kappa'(x^{(j)})$ at the point $x^{(j)}$ coincides with the identity matrix I. In the new coordinates $y = \kappa(x)$, the Navier–Stokes system takes the form

$$\sum_{i,j=1}^{3} a_{i,j}(y) \frac{\partial^2 u}{\partial y_i \partial y_j} + \sum_{i=1}^{3} a_i(y) \frac{\partial u}{\partial y_j} + (u \cdot \kappa' \nabla_y) u + \kappa' \nabla_y p = f, \quad \kappa' \nabla_y \cdot u = g,$$

where $a_{i,j}(0) = -\nu \delta_{i,j}$. Here by κ' we mean the matrix $\kappa'(\kappa^{-1}(y))$. We consider the functions

$$u = \zeta(y)\, |y|^{\Lambda_j}\, \Phi(y/|y|), \quad p = \zeta(y)\, |y|^{\Lambda_j - 1}\, \Psi(y/|y|),$$

where (Φ, Ψ) is an eigenvector of the pencil $\mathfrak{A}_j(\lambda)$ corresponding to the eigenvalue Λ_j and ζ is a smooth cut-off function equal to one near the origin. The eigenvector (Φ, Ψ) belongs to the space $W^{2,t}_{\gamma}(\Omega_j)^3 \times W^{1,t}_{\gamma}(\Omega_j)$ with arbitrary t and γ satisfying

$\max(2-\mu_k, 0) < \gamma_k + 2/t < 2$. Here, $W_\gamma^{l,t}(\Omega_j)$ is the closure of the set $C^\infty(\bar{\Omega}_j)$ with respect to the norm

$$\|u\|_{W_\gamma^{2,t}(\Omega_j)} = \left(\int\limits_{\substack{\mathcal{K}_j \\ 1/2<|x|<2}} \sum_{|\alpha|\leqq l} \prod_k r_k^{t\delta_k} \, |\partial_x^\alpha u(x)|^t \, dx \right)^{1/t},$$

where u is extended by $u(x) = u(x/|x|)$ to $\mathcal{K}_j$. In particular, $\Phi \in L_\infty(\Omega_j)^3$ and $\Phi_k \, \partial_{x_j} \Phi \in W_\delta^{0,s}(\Omega_j)^3$ for $k = 1, 2, 3$. Since (Φ, Ψ) is an eigenvector, the vector function $|y|^{\Lambda_j} \, (\Phi(y/|y|), |y|^{-1}\Psi(y/|y|))$ is a solution of the linear Stokes system with zero right-hand sides. From this and from the equalities $\kappa'(x^{(j)}) = I$ and $a_{i,j}(0) = -\nu\delta_{i,j}$, it follows that $f \in W_{\beta_j,\delta}^{0,t}(\mathcal{K}_j)^3$ and $g \in W_{\beta_j,\delta}^{1,t}(\mathcal{K}_j)$ if $\beta_j + 3/s > 1 - \operatorname{Re}\Lambda_j$, $\beta_j + 3/s > 1 - 2\operatorname{Re}\Lambda_j$. Analogously, the corresponding boundary data are from $W_{\beta_j,\delta}^{1-1/t,t}$ and $W_{\beta_j,\delta}^{2-1/t,t}$, respectively. However, $u \notin W_{\beta_j,\delta}^{2,t}(\mathcal{K}_j)^3$ for $\beta_j + 3/s \leqq 2 - \operatorname{Re}\Lambda_j$. This example shows that the inequality $\beta_j + 3/s > 2 - \operatorname{Re}\Lambda_j$ cannot be weakened.

Now we show that the inequality $\delta_k + 2/s > 2 - \mu_k$ cannot be weakened. We assume for the sake of simplicity that the edge M_k is a part of the x_3-axis and that the adjacent faces Γ_{k_+} and Γ_{k_-} are subsets of planes. Let $\delta_k + 2/s \leqq 2 - \mu_k$, and let λ_k be an eigenvalue of the pencil $A_k(\lambda)$ with the real part μ_k. Then we consider the functions

$$u(x) = \zeta(x) \, r^{\lambda_k} \, \Phi(\varphi), \quad p(x) = \zeta(x) r^{\lambda_k - 1} \, \Psi(\varphi),$$

where r, ϕ are the polar coordinates in the (x_1, x_2)-plane, (Φ, Ψ) is an eigenvector of the pencil $A_k(\lambda)$ corresponding to the eigenvalue λ_k, and ζ is a smooth cut-off function equal to one in a neighborhood of a certain point $x^{(0)} \in M_k$ and equal to zero near the other edges. Since the vector function $r^{\lambda_k} \, (\Phi(\varphi), r^{-1}\Psi(\varphi))$ is a solution of the linear Stokes system with zero right-hand sides, it follows that

$$-\nu\Delta u + (u \cdot \nabla)\, u + \nabla p \in W_{\beta,\delta}^{0,s}(\mathcal{G})^3, \quad \nabla \cdot u \in W_{\beta,\delta}^{1,s}(\mathcal{G})$$

if $\delta_k + 2/s > 1 - \mu_k$. However, $u \notin W_{\beta,\delta}^{2,s}(\mathcal{G})^3$ for $\delta_k + 2/s \leqq 2 - \mu_k$. This means, the result of Theorem 4.6 fails if $1 - \mu_k < \delta_k + 2/s \leqq 2 - \mu_k$.

Analogously, it can be shown that the inequalities for β_j and δ_k in Theorems 4.5, 4.7 and 4.8 cannot be weakened.

4.6. Examples

Here, we establish some regularity results for weak solutions in the class of the nonweighted spaces $W^{l,s}(\mathcal{G})$ and $C^{l,\sigma}(\mathcal{G})$. We assume that $\mathcal{G}$ is a polyhedron with faces Γ_j, $j = 1, ..., N$, and edges M_k, $k = 1, \ldots, m$. The angle at the edge M_k is denoted by θ_k. For the sake of simplicity, we restrict ourselves to the case $g = 0$ and to homogeneous boundary conditions

$$S_j u = 0, \quad N_j(u, p) = 0 \quad \text{on } \Gamma_j, \ j = 1, \ldots, N. \tag{4.11}$$

Analogous results are valid for inhomogeneous boundary conditions provided the boundary data satisfy certain compatibility conditions on the edges. Note that there are the following equalities

$$W^{1,s}(\mathcal{G}) = V^{1,s}_{0,0}(\mathcal{G}) \text{ if } s < 2, \quad W^{1,s}(\mathcal{G}) = W^{1,s}_{0,0}(\mathcal{G}) \text{ if } s < 3. \tag{4.12}$$

The Dirichlet problem. For arbitrary $f \in W^{-1,2}(\mathcal{G})^3$, there exists a solution $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ of the Dirichlet problem

$$-\nu\Delta u + (u \cdot \nabla)\, u + \nabla p = f,\ \nabla \cdot u = 0 \text{ in } \mathcal{G}, \quad u = 0 \text{ on } \Gamma_j,\ j = 1, \dots, N.$$

(see for example, [5, Theorem IV.2.1]).

The regularity results established below are based on the following properties of the operator pencils $\mathfrak{A}_j(\lambda)$ (see [10, Chap. 5] or [11]).

- The strip $-1/2 \leqq \operatorname{Re}\lambda \leqq 0$ is free of eigenvalues of the pencils $\mathfrak{A}_j(\lambda)$.
- If the cone $\mathcal{K}_j$ is contained in a half-space, then the strip $-1/2 \leqq \operatorname{Re}\lambda \leqq 1$ contains only the eigenvalue $\lambda = 1$ of the pencil $\mathfrak{A}_j(\lambda)$. This eigenvalue has only the eigenvector $(0, 0, 0, c)$, $c = const.$, and no generalized eigenvectors.
- The eigenvalues of the pencil $\mathfrak{A}_j(\lambda)$ in the strip $-1/2 \leqq \operatorname{Re}\lambda \leqq 1$ are real and monotonous with respect to the cone $\mathcal{K}_j$.

Moreover, the eigenvalues for a circular cone are solutions of a certain transcendental equation (see [10, Section 5.6] or [11]).

The numbers μ_k can be easily calculated. In the case $\theta_k < \pi$, we have $\mu_k = \pi/\theta_k$. If $\theta_k > \pi$, then μ_k is the smallest positive solution of the equation

$$\sin(\mu\theta_k) + \mu \sin\theta_k = 0. \tag{4.13}$$

Note that $\mu_k > 1/2$ for every $\theta_k < 2\pi$, $\mu_k > 2/3$ if $\theta_k < 3\arccos\frac{1}{4} \approx 1.2587\pi$, $\mu_k > 1$ if $\theta_k < \pi$, and $\mu_k > 4/3$ if $\theta_k < \frac{3}{4}\pi$. Using these facts together with Theorems 4.5 and 4.6, we obtain the following assertions.

- If $f \in (W^{1,s'}(\mathcal{G})^*)^3$, $2 < s \leqq 3$, $s' = s/(s-1)$, then $(u, p) \in W^{1,s}(\mathcal{G})^3 \times L_s(\mathcal{G})$. If the polyhedron $\mathcal{G}$ is convex, then this assertion is true for all $s > 2$.
- If $f \in W^{-1,2}(\mathcal{G})^3 \cap L_s(\mathcal{G})^3$, $1 < s \leqq 4/3$, then $(u, p) \in W^{2,s}(\mathcal{G})^3 \times W^{1,s}(\mathcal{G})$. If $\theta_k < 3\arccos\frac{1}{4} \approx 1.2587\pi$ for $k = 1, \dots, m$, then this result is true for $1 < s \leqq 3/2$. If $\mathcal{G}$ is convex, then this result is valid for $1 < s \leqq 2$. If, moreover, the angles at the edges are less than $\frac{3}{4}\pi$, then the result holds even for $1 < s < 3$.

Furthermore, the following assertion is valid.

- If $\mathcal{G}$ is convex, $f \in C^{-1,\sigma}(\mathcal{G})$, and σ is sufficiently small (such that $1+\sigma < \pi/\theta_k$ and there are no eigenvalues of the pencils $\mathfrak{A}_j$ in the strip $1 < \operatorname{Re}\lambda \leqq 1 + \sigma$), then $(u, p) \in C^{1,\sigma}(\mathcal{G})^3 \times C^{0,\sigma}(\mathcal{G})$.

We prove the last result. Let ε be a positive number, $\varepsilon < 1 - \sigma$. Since $C^{-1,\sigma}(\mathcal{G}) \subset C^{-1,\sigma}_{\sigma+\varepsilon,0}(\mathcal{G})$, it follows from Theorem 4.8 that $(u, p) \in C^{1,\sigma}_{\sigma+\varepsilon,0}(\mathcal{G}) \times C^{0,\sigma}_{\sigma+\varepsilon,0}(\mathcal{G})$ if $\sigma < \mu_k - 1$. In particular, we have $u \in C^{1,\sigma}_{\sigma+\varepsilon,0}(\mathcal{G}) \subset C^{0,\sigma}(\mathcal{G})$. This implies $u_j u \in$

$C^{0,\sigma}(\mathcal{G})^3$. Since $\nabla \cdot u = 0$, it follows that $(u \cdot \nabla)\, u = \sum_j \partial_{x_j}(u_j u) \in C^{-1,\sigma}(\mathcal{G})^3$. Thus, (u, p) satisfies the Stokes system

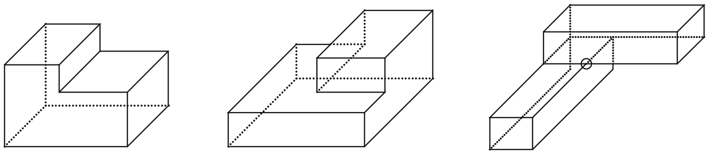
$$-\nu\, \Delta u + \nabla p = f - (u \cdot \nabla)u, \quad -\nabla \cdot u = 0,$$

where $f - (u \cdot \nabla)u \in C^{-1,\sigma}(\mathcal{G}) \subset C^{-1,\sigma}_{0,0}(\mathcal{G})$. Hence by [18, Theorem 4.3], the solution (u, p) admits the decomposition

$$\begin{pmatrix} u(x) \\ p(x) \end{pmatrix} = \begin{pmatrix} 0 \\ p(x^{(k)}) \end{pmatrix} + \begin{pmatrix} w(x) \\ q(x) \end{pmatrix}$$

in a neighborhood of the vertex $x^{(k)}$, where $(w, q) \in C^{1,\sigma}_{0,0}(\mathcal{G})^3 \times C^{0,\sigma}_{0,0}(\mathcal{G})$. This is true for every vertex $x^{(k)}$, $k = 1, \dots, d$. Consequently, $(u, p) \in C^{1,\sigma}(\mathcal{G})^3 \times C^{0,\sigma}(\mathcal{G})$.

For special domains, it is possible to obtain precise regularity results. Let for example $\mathcal{G}$ have the form of steps as in the first two pictures below with angles $\pi/2$ or $3\pi/2$ at every edge or the form of two beams, where one lies on the other as in the third picture. Note that the third polyhedron is not Lipschitz.



The greatest edge angle is $3\pi/2$, and we obtain $\min \mu_k = 0.54448373\ldots$. Moreover for every vertex, there exists a circular cone with the same vertex and aperture $3\pi/2$ which contains the polyhedron. For every vertex of the left polyhedron, there exists even a cone with aperture π (a half-space) containing the polyhedron. Consequently, the smallest positive eigenvalue of the pencils $\mathfrak{A}_j(\lambda)$ does not exceed the first eigenvalue for a circular cone with vertex $3\pi/2$. A numerical calculation (see [10, Section 5.6]) shows that this eigenvalue is greater than $(3 \min \mu_k - 1)/2$. This means that for $\beta = 0$ and $\delta = 0$, the condition (iii) in Theorems 4.5 and 4.6 is stronger than the condition (ii) in the same theorems. Thus, we obtain

$$\begin{aligned} &(u, p) \in W^{1,s}(\mathcal{G})^3 \times L_s(\mathcal{G}) \quad \text{if } f \in W^{-1,s}(\mathcal{G}),\ s < 2/(1 - \min \mu_k) = 4.3905\ldots, \\ &(u, p) \in W^{2,s}(\mathcal{G})^3 \times W^{1,s}(\mathcal{G}) \quad \text{if } f \in L_s(\mathcal{G}),\ s < 2/(2 - \min \mu_k) = 1.3740\ldots. \end{aligned}$$

Here the condition on s is sharp.

We give some explanations concerning the examples in the introduction (the flow outside a regular polyhedron G). By Theorem 4.6, the regularity result $(u, p) \in W^{2,s} \times W^{1,s}$ in an arbitrary bounded subdomain of the complement of G holds if $s < 2/(2-\mu_k)$ and there are no eigenvalues of the pencils $\mathfrak{A}_j(\lambda)$ in the strip $-1/2 < \operatorname{Re} \lambda < 2 - 3/s$. Here, μ_k is the smallest positive solution of the equation (4.13), where $\theta_k = \theta$ is the edge angle in the exterior of G, $\sin\theta$ is equal to $-\frac{2}{3}\sqrt{2}$ if G is a regular tetrahedron or octahedron, -1 if G is a cube, $-\frac{2}{5}\sqrt{5}$ if G is a regular dodecahedron, and $-2/3$ if G is a regular icosahedron. The smallest positive solutions of (4.13) are $\mu_k = 0.52033360\ldots$ for the regular tetrahedron, $\mu_k = 0.54448373\ldots$ for a cube, $\mu_k = 0.58489758\ldots$ for the regular octahedron, $\mu_k = 0.60487306\ldots$ for the regular dodecahedron, and $\mu_k = 0.68835272\ldots$ for the regular icosahedron. In the case of a regular tetrahedron, cube, octahedron or dodecahedron, the inequality $s < 2/(2-\mu_k)$ implies $2 - 3/s < 3\mu_k/2 - 1 < 0$. Then the absence of eigenvalues of the pencil $\mathfrak{A}_j(\lambda)$ in the strip $-1/2 < \mathrm{Re}\,\lambda < 2 - 3/s$ follows from [10, Theorem 5.5.6]. The exterior of a regular icosahedron is contained in a right circular cone with aperture less than 255°. Numerical results for right circular cones together with the monotonicity of the eigenvalues of the pencils $\mathfrak{A}_j(\lambda)$ in the interval $[-1/2, 1)$ show that also for this polyhedron, the strip $-1/2 < \mathrm{Re}\,\lambda < 3\mu_k/2 - 1$ is free of eigenvalues of the pencil $\mathfrak{A}_j(\lambda)$. Thus, the above mentioned regularity result holds for all $s < 2/(2-\mu_k)$. This inequality cannot be weakened.

The Neumann problem for the Navier–Stokes system. We consider a weak solution $u \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ of the Neumann problem

$$-\Delta u + (u \cdot \nabla)\, u + \nabla p = f, \ -\nabla u = 0 \text{ in } \mathcal{G},$$
$$-pn + 2\nu\varepsilon_n(u) = 0 \text{ on } \Gamma_j, \ j = 1, \ldots, N.$$

For this problem it is known that the strip $-1 \leqq \mathrm{Re}\,\lambda \leqq 0$ contains only the eigenvalues $\lambda = 0$ and $\lambda = 1$ of the operator pencils $\mathfrak{A}_j(\lambda)$ (see [10, Theorem 6.3.2]) if $\mathcal{G}$ is a Lipschitz polyhedron. The numbers μ_k are the same as for the Dirichlet problem. Therefore, the following assertions are valid.

- If $f \in (W^{1,s'}(\mathcal{G})^*)^3$, $s' = s/(s-1)$, $2 < s < 3$, then $(u, p) \in W^{1,s}(\mathcal{G})^3 \times L_s(\mathcal{G})$.
- If $f \in (W^{1,2}(\mathcal{G})^*)^3 \cap L_s(\mathcal{G})^3$, $1 < s \leqq 4/3$, then $(u, p) \in W^{2,s}(\mathcal{G})^3 \times W^{1,s}(\mathcal{G})$. If the angles θ_k are less than $3\arccos\frac{1}{4}$, then this result is true for $1 < s < 3/2$.

The mixed problem with Dirichlet and Neumann boundary conditions. We assume that on each face Γ_j either the Dirichlet condition $u = 0$ or the Neumann condition $\frac{\partial u}{\partial n} = 0$ is given. If on both adjoining faces of the edge M_k the same boundary conditions are given, then $\mu_k > 1/2$. If on one of the adjoining faces the Dirichlet condition and on the other face the Neumann condition is given, then $\mu_k > 1/4$. This implies the following result.

- If $f \in (W^{1,2}(\mathcal{G})^*)^3 \cap L_s(\mathcal{G})^3$, $1 < s \leqq 8/7$, then every weak solution (u, p) belongs to $W^{2,s}(\mathcal{G})^3 \times W^{1,s}(\mathcal{G})$.

The mixed problem with boundary conditions (i)–(iii). Let $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ be a weak solution of the problem (2.2), (2.3), where $g = 0$, $h_j = 0$, $\phi_j = 0$ for $j = 1, \ldots, N$, and $d_k \leqq 2$ for all k (that is, the Neumann condition does not appear in the boundary conditions). We assume that the Dirichlet condition is given on at least one of the adjoining faces of every edge. Then, by [10, Theorem 6.1.5], the strip $-1 \leqq \mathrm{Re}\,\lambda \leqq 0$ is free of eigenvalues of the pencils $\mathfrak{A}_j(\lambda)$. Furthermore, we have $\mu_k > 1/2$ if the Dirichlet condition is given on both adjoining faces of the edge M_k. For the other indices k, we have $\mu_k > 1/4$ and $\mu_k > 1/3$ if $\theta_k < \frac{3}{2}\pi$.

- If $f \in (W^{1,s'}(\mathcal{G})^*)^3$, $2 < s \leqq 8/3$, then $(u, p) \in W^{1,s}(\mathcal{G})^3 \times L_s(\mathcal{G})$. Suppose that $\theta_k < \frac{3}{2}\pi$ if the boundary condition (ii) or (iii) is given on one of the adjoining faces of the edge M_k. Then this result is even true for $2 < s \leqq 3$.
- If $f \in (W^{1,2}(\mathcal{G})^*)^3 \cap L_s(\mathcal{G})^3$, $1 < s \leqq 8/7$, then $(u, p) \in W^{2,s}(\mathcal{G})^3 \times W^{1,s}(\mathcal{G})$. Suppose that $\theta_k < 3 \arccos\frac{1}{4}$ if the Dirichlet condition is given on both adjoining faces of M_k, $\theta_k < \frac{3}{2}\arccos\frac{1}{4}$ if the boundary condition (ii) is given on one of the adjoining faces of M_k, and $\theta_k < \frac{3}{4}\pi$ if the boundary condition (iii) is given on one of the adjoining faces of M_k. Then the last result is true for $1 < s \leqq 3/2$.

Note that in the last case we have $\mu_k > 2/3$ for $k = 1, \dots, m$.

Finally, we assume that the homogeneous Dirichlet condition $u = 0$ is given on the faces $\Gamma_1, \dots, \Gamma_{N-1}$, while the homogeneous boundary condition (iii) is given on Γ_N. Let I be the set of all k such that $M_k \subset \bar{\Gamma}_N$ and $I' = \{1, \dots, m\}\backslash I$. We suppose that the polyhedron $\mathcal{G}$ is convex and $\theta_k < \pi/2$ for $k \in I$. Then $\mu_k > 1$ for all k, and the strip $-1/2 \leqq \operatorname{Re}\lambda \leqq 1$ contains only the simple eigenvalue $\lambda = 1$ of the pencils $\mathfrak{A}_j(\lambda)$ (see [10, Theorem 6.2.7]). If $\theta_k < \frac{3}{8}\pi$ for $k \in I$ and $\theta_k < \frac{3}{4}\pi$ for $k \in I'$, then even $\mu_k > 4/3$. This implies the following result.

- Let $f \in (W^{1,2}(\mathcal{G})^*)^3 \cap L_s(\mathcal{G})^3$, $1 < s \leqq 2$. Then any weak solution belongs to $W^{2,s}(\mathcal{G})^3 \times W^{1,s}(\mathcal{G})$. If $\theta_k < \frac{3}{8}\pi$ for $k \in I$ and $\theta_k < \frac{3}{4}\pi$ for $k \in I'$, then the result holds even for $1 < s < 3$.

Furthermore, analogously to the Dirichlet problem, the following assertion holds.

- Let $f \in (W^{1,2}(\mathcal{G})^*)^3 \cap C^{-1,\sigma}(\mathcal{G})$. Then for sufficiently small σ, we have $(u, p) \in C^{1,\sigma}(\mathcal{G}) \times C^{0,\sigma}(\mathcal{G})$.

5. Existence of weak solutions in $W^{1,s}(\mathcal{G}) \times L_s(\mathcal{G})$, $s < 2$

In Section 2, we proved the existence of weak solutions of the boundary value problem in $W^{1,2}(\mathcal{G}) \times L_2(\mathcal{G})$. Using the regularity result of Theorem 4.5, we obtain also the existence of weak solutions in $W^{1,s}(\mathcal{G}) \times L_s(\mathcal{G})$ for sufficiently small $s > 2$. In this section, we will prove that weak solutions exist also in the space $W^{1,s}(\mathcal{G}) \times L_s(\mathcal{G})$ with $s < 2$ provided the norms of the right-hand sides of (2.4), (2.5) in the corresponding Sobolev spaces are sufficiently small. Throughout this section, we suppose that the Dirichlet condition is given on at least one of the adjoining faces of every edge M_k.

5.1. Solvability of the linearized problem in a cone

Let $\mathcal{K}$ be the same polyhedral cone as in Section 3.1. We consider weak solutions $(u, p) \in V^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times V^{0,s}_{\beta,\delta}(\mathcal{G})$ of the linear Stokes system in $\mathcal{K}$ with boundary conditions (i)–(iv) on the faces Γ_j. This means that (u, p) satisfies the equations

$$b_{\mathcal{K}}(u, v) - \int_{\mathcal{K}} p\, \nabla \cdot v \, dx = F(v) \quad \text{for all } v \in V^{1,s'}_{-\beta,-\delta}(\mathcal{K})^3,\ S_j v|_{\Gamma_j} = 0, \tag{5.1}$$

$$-\nabla \cdot u = g \ \text{ in } \mathcal{K}, \quad S_j u = h_j \ \text{ on } \Gamma_j,\ j = 1, \dots, N. \tag{5.2}$$

Here $b_{\mathcal{K}}$ denotes the bilinear form (2.6), where $\mathcal{G}$ has to be replaced by $\mathcal{K}$. We define the space $V^{-1,s}_{\beta,\delta}(\mathcal{K};S)$ as the set of all linear and continuous functionals on the space $\{v \in V^{1,s'}_{-\beta,-\delta}(\mathcal{K})^3 : S_j v|_{\Gamma_j} = 0\}$, where $s' = s/(s-1)$. Furthermore, the pencils $A_k(\lambda)$ for the edges M_k and $\mathfrak{A}(\lambda)$ for the vertex of the cone $\mathcal{K}$ are defined as in Section 4.1. If the Dirichlet condition (i) is given on at least one of the adjoining faces of the edge M_k, then $\lambda = 0$ is not an eigenvalue of the pencil $A_k(\lambda)$.

The following lemma is proved in [19] under the condition $\max(1-\operatorname{Re}\lambda_1^{(k)}, 0) < \delta_k + 2/s < 1$. Using the sharper estimates of Green's matrix given in [17] for the case when $\lambda = 0$ is not an eigenvalue of the pencils $A_k(\lambda)$, this theorem can be proved in the same way if

$$1 - \operatorname{Re}\lambda_1^{(k)} < \delta_k + 2/s < 1 + \operatorname{Re}\lambda_1^{(k)} \tag{5.3}$$

for $k = 1, \dots, N$. For the Dirichlet problem, we refer to [13, Theorem 4.1].

Lemma 5.1. *Suppose that $F \in V^{-1,s}_{\beta,\delta}(\mathcal{K};S)$, $g \in V^{0,s}_{\beta,\delta}(\mathcal{K})$, $h_j \in V^{1-1/s,s}_{\beta,\delta}(\Gamma_j)$, there are no eigenvalues of the pencil $\mathfrak{A}(\lambda)$ on the line* $\operatorname{Re}\lambda = 1-\beta_j-3/s$*, and the components of δ satisfy the inequalities (5.3). Then there exists a unique solution $(u,p) \in V^{1,s}_{\beta,\delta}(\mathcal{K})^3 \times V^{0,s}_{\beta,\delta}(\mathcal{K})$ of problem (5.1), (5.2).*

Moreover, the following regularity results hold analogously to [19, Le.4.4 and Th.4.4].

Lemma 5.2. (1) *Suppose that in addition to the assumptions of Lemma 5.1, we have $F \in V^{-1,t}_{\beta',\delta'}(\mathcal{K};S)$, $g \in V^{0,t}_{\beta',\delta'}(\mathcal{K})$, $h_j \in V^{1-1/t,t}_{\beta',\delta'}(\Gamma_j)$, where* $1-\operatorname{Re}\lambda_1^{(k)} < \delta'_k + 2/t < 1 + \operatorname{Re}\lambda_1^{(k)}$ *for $k = 1, \dots, N$ and β' is such that there are no eigenvalues of the pencil $\mathfrak{A}(\lambda)$ in the closed strip between the lines* $\operatorname{Re}\lambda = 1-\beta-3/s$ *and* $\operatorname{Re}\lambda = 1-\beta'-3/t$*. Then the solution $(u,p) \in V^{1,s}_{\beta,\delta}(\mathcal{K})^3 \times V^{0,s}_{\beta,\delta}(\mathcal{K})$ belongs to $V^{1,t}_{\beta',\delta'}(\mathcal{K})^3 \times V^{0,t}_{\beta',\delta'}(\mathcal{K})$.*

(2) *Suppose that in addition to the assumptions of Lemma 5.1, $g \in V^{1,t}_{\beta',\delta'}(\mathcal{K})$, $h_j \in V^{2-1/t,t}_{\beta',\delta'}(\Gamma_j)$, and the functional F has the form*

$$F(v) = \int_{\mathcal{K}} f \cdot v \, dx + \sum_{j=1}^{N} \Phi_j \cdot v \, dx$$

with vector function $f \in V^{0,t}_{\beta',\delta'}(\mathcal{K})$, $\Phi_j \in V^{1-1/t,t}_{\beta',\delta'}(\Gamma_j)$, where β' is such that there are no eigenvalues of the pencil $\mathfrak{A}(\lambda)$ in the closed strip between the lines $\operatorname{Re}\lambda = 1-\beta-3/s$ *and* $\operatorname{Re}\lambda = 2-\beta'-3/t$*, and the components of δ' satisfy the inequalities* $2-\operatorname{Re}\lambda_1^{(k)} < \delta'_k + 2/t < 2 + \operatorname{Re}\lambda_1^{(k)}$*. Then the solution $(u,p) \in V^{1,s}_{\beta,\delta}(\mathcal{K})^3 \times V^{0,s}_{\beta,\delta}(\mathcal{K})$ belongs to $V^{2,t}_{\beta',\delta'}(\mathcal{K})^3 \times V^{1,t}_{\beta',\delta'}(\mathcal{K})$.*

5.2. Solvability of the linearized problem in $\mathcal{G}$

We consider the operator

$$V^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times V^{0,s}_{\beta,\delta}(\mathcal{G}) \ni (u,p) \to (F,g,h) \in V^{-1,s}_{\beta,\delta}(\mathcal{G};S) \\ \times V^{0,s}_{\beta,\delta}(\mathcal{G}) \times \prod_j V^{1-1/s,s}_{\beta,\delta}(\Gamma_j) \tag{5.4}$$

of problem (2.9), (2.10) and denote this operator by $\mathcal{A}_{s,\beta,\delta}$. Here again $V^{-1,s}_{\beta,\delta}(\mathcal{G};S)$ is defined as the dual space of $\{v \in V^{1,s'}_{-\beta,-\delta}(\mathcal{G})^3 : S_j v|_{\Gamma_j} = 0\}$. We show that this operator is Fredholm if there are no eigenvalues of the pencil $\mathfrak{A}_j(\lambda)$ on the line $\mathrm{Re}\lambda = 1-\beta_j-3/s$, $j=1,\dots,d$, and the components of δ satisfy the inequalities

$$1 - \inf_{\xi\in M_k} \mathrm{Re}\,\lambda_1(\xi) < \delta_k + 2/s < 1 + \inf_{\xi\in M_k} \mathrm{Re}\,\lambda_1(\xi) \tag{5.5}$$

for $k=1,\dots,m$. To this end, we construct a left and right regularizer for the operator $\mathcal{A}_{s,\beta,\delta}$.

Lemma 5.3. *Let $\mathcal{U}$ be a sufficiently small open subset of $\mathcal{G}$, and let φ be a smooth function with support in $\overline{\mathcal{U}}$. Suppose that there are no eigenvalues of the pencil $\mathfrak{A}_j(\lambda)$ on the line $\mathrm{Re}\lambda = 1-\beta_j-3/s$, $j=1,\dots,d$, and the components of δ satisfy (5.5). Then there exists an operator $\mathcal{R}$ continuously mapping the space of all $(F,g,h) \in V^{-1,s}_{\beta,\delta}(\mathcal{G};S) \times V^{0,s}_{\beta,\delta}(\mathcal{K}) \times \prod_j V^{1-1/s,s}_{\beta,\delta}(\Gamma_j)$ with support in $\overline{\mathcal{U}}$ into $V^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times V^{0,s}_{\beta,\delta}(\mathcal{G})$ such that $\varphi\mathcal{A}_{s,\beta,\delta}\mathcal{R}(F,g,h) = \varphi(F,g,h)$ for all (F,g,h) with support in $\overline{\mathcal{U}}$ and $\varphi\mathcal{R}\mathcal{A}_{s,\beta,\delta}(u,p) = \varphi\,(u,p)$ for all (u,p) with support in $\overline{\mathcal{U}}$.*

Proof. Suppose first that $\overline{\mathcal{U}}$ contains the vertex $x^{(1)}$ of $\mathcal{G}$. Then there exists a diffeomorphism κ mapping $\mathcal{U}$ onto a subset $\mathcal{V}$ of a polyhedral cone $\mathcal{K}$ with vertex at the origin such that $\kappa(x^{(1)}) = 0$ and the Jacobian matrix κ' coincides with the identity matrix I at $x^{(1)}$. We assume that the supports of u and p are contained in $\overline{\mathcal{U}}$. Then the coordinate change $y = \kappa(x)$ transforms (2.9), (2.10) into

$$\tilde b(\tilde u,\tilde v) - \int_{\mathcal{K}} \tilde p\, \tilde{\mathcal{D}}\tilde v\, |\det\kappa'|^{-1}\, dy = \tilde F(\tilde v) \ \text{ for all } \tilde v \in V^{1,s'}_{\beta,\delta}(\mathcal{K})^3,\ S_j\tilde v = 0 \text{ on } \Gamma_j^\circ, \tag{5.6}$$

$$-\tilde{\mathcal{D}}\tilde u = \tilde g \ \text{ in } \mathcal{K}, \quad S_j\tilde u = \tilde h_j \ \text{ on } \Gamma_j^\circ, \tag{5.7}$$

where $\tilde u = u\circ\kappa^{-1}$, $\tilde F(\tilde v) = F(\tilde v\circ\kappa)$, Γ_j° are the faces of $\mathcal{K}$, $\tilde{\mathcal{D}}$ is a first order differential operator of the form

$$\tilde{\mathcal{D}}\tilde u = \big(D(y)\nabla_y\big)\cdot\tilde u,$$

and $\tilde b$ is a bilinear form having the representation

$$\tilde b(\tilde u,\tilde v) = 2\nu \int_{\mathcal{K}} \sum_{i,j=1}^{3} B_{i,j}(y)\,\partial_{y_i}\tilde u\cdot\partial_{y_j}\tilde v\, dy.$$

Here $D(y)$ and $B_{i,j}(y)$ are quadratic matrices such that $D(0) = I$ and

$$\sum_{i,j=1}^{3} B_{i,j}(0)\,\partial_{y_i}\tilde{u}\cdot\partial_{y_j}\tilde{v} = \sum_{i,j=1}^{3} \varepsilon_{i,j}(\tilde{u})\,\varepsilon_{i,j}(\tilde{v}).$$

Let ζ be an infinitely differentiable cut-off function on $[0,\infty)$ equal to 1 in $[0,1)$ and to zero in $(2,\infty)$. For arbitrary positive ε, we define $\zeta_\varepsilon(y) = \zeta(|y|/\varepsilon)$, Moreover, we put $\zeta_0 = 0$ and $\eta_\varepsilon = 1 - \zeta_\varepsilon$ for $\varepsilon \geqq 0$. We consider the operator

$$\begin{aligned} V^{1,s}_{\beta,\delta}(\mathcal{K})^3 \times V^{0,s}_{\beta,\delta}(\mathcal{K}) \ni (\tilde{u},\tilde{p}) \to (\tilde{F},\tilde{g},\tilde{h}) \in V^{-1,s}_{\beta,\delta}(\mathcal{K};S) \\ \times V^{0,s}_{\beta,\delta}(\mathcal{K}) \times \prod_j V^{1-1/s,s}_{\beta,\delta}(\Gamma_j^\circ) \end{aligned} \tag{5.8}$$

defined by

$$\begin{aligned} &b_\varepsilon(\tilde{u},\tilde{v}) - \int_{\mathcal{K}} \tilde{p}\,\Big(\zeta_\varepsilon\,\tilde{\mathcal{D}}\tilde{v}\,|\det\kappa'|^{-1} + \eta_\varepsilon\,\nabla_y\cdot\tilde{v}\Big)\,dy \\ &= \tilde{F}(\tilde{v}) \quad \text{for all } \tilde{v}\in V^{1,s'}_{\beta,\delta}(\mathcal{K})^3,\ S_j\tilde{v} = 0 \text{ on } \Gamma_j^\circ, \\ &\ -\big(\zeta_\varepsilon D(y)\nabla_y + \eta_\varepsilon\nabla_y\big)\cdot\tilde{u} = \tilde{g} \ \text{ in } \mathcal{K}, \quad S_j\tilde{u} = \tilde{h}_j \ \text{ on } \Gamma_j^\circ, \end{aligned}$$

where

$$b_\varepsilon(\tilde{u},\tilde{v}) = 2\nu\int_{\mathcal{K}} \sum_{i,j=1}^{3} \big(\zeta_\varepsilon B_{i,j}(y) + \eta_\varepsilon B_{i,j}(0)\big)\,\partial_{y_i}\tilde{u}\cdot\partial_{y_j}\tilde{v}\,dy.$$

We denote the operator (5.8) by $\tilde{\mathcal{A}}_\varepsilon$. According to Lemma 5.1, the operator $\tilde{\mathcal{A}}_0$ is an isomorphism. Since the norm of $\tilde{\mathcal{A}}_0 - \tilde{\mathcal{A}}_\varepsilon$ is small for small ε, the operator $\tilde{\mathcal{A}}_\varepsilon$ is an isomorphism if $\varepsilon \leqq \varepsilon_0$ and ε_0 is sufficiently small. We may assume that $\zeta_\varepsilon = 1$ on $\mathcal{V}$ for $\varepsilon = \varepsilon_0$. Then problem (5.6), (5.7) can be written as $\tilde{\mathcal{A}}_{\varepsilon_0}(\tilde{u},\tilde{p}) = (\tilde{F},\tilde{g},\tilde{h})$ if the supports of $\tilde{u}$ and $\tilde{p}$ are contained in $\bar{\mathcal{V}}$. Let

$$u(x) = \tilde{u}(\kappa(x)),\ p(x) = \tilde{p}(\kappa(x)) \ \text{ for } x\in\mathcal{U}, \quad \text{where } (\tilde{u},\tilde{p}) = \tilde{\mathcal{A}}_{\varepsilon_0}^{-1}\,(\tilde{F},\tilde{g},\tilde{h}). \tag{5.9}$$

Outside $\mathcal{U}$, let (u,p) be continuously extended to a vector function from $V^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times V^{0,s}_{\beta,\delta}(\mathcal{G})$. The so defined mapping $(F,g,h)\to(u,p)$ is denoted by $\mathcal{R}$ and has the desired properties.

Suppose now that $\overline{\mathcal{U}}$ contains an edge point $\xi\in M_1$ but no points of other edges and no vertices of $\mathcal{G}$. Then again there exists a diffeomorphism mapping $\mathcal{U}$ onto a subset of a cone $\mathcal{K}$. We assume that the point $\kappa(\xi)$ lies on the edge M_1° of $\mathcal{K}$ and coincides with the origin (in contrast to the first part of the proof, the vertex of the cone is not the origin). Let $\tilde{\mathcal{A}}_\varepsilon$ be the same operator as above. Then there exist a number β_0 and a tuple δ', $\delta_1' = \delta_1$, such that $\tilde{\mathcal{A}}_0$ and for sufficiently small ε also $\tilde{\mathcal{A}}_\varepsilon$ are isomorphisms

$$V^{1,s}_{\beta_0,\delta'}(\mathcal{K})^3 \times V^{0,s}_{\beta_0,\delta'}(\mathcal{K}) \to V^{-1,s}_{\beta_0,\delta'}(\mathcal{K};S) \times V^{0,s}_{\beta_0,\delta'}(\mathcal{K}) \times \prod_j V^{1-1/s,s}_{\beta_0,\delta'}(\Gamma_j^\circ).$$

Since $\overline{\mathcal{U}}$ does not contain points of the edges M_k, $k \neq 1$, the vector function (5.9) can be continuously extended to a vector function from $V^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times V^{0,s}_{\beta,\delta}(\mathcal{G})$. The so defined mapping $(F, g, h) \to (u, p)$ defines the desired operator $\mathcal{R}$. Analogously, the lemma can be proved for the case when $\bar{\mathcal{U}}$ contains no edge points of $\mathcal{G}$. □

Remark 5.1. Suppose that

$$2 - \inf_{\xi \in M_k} \operatorname{Re} \lambda_1(\xi) < \delta'_k + 2/t < 2 + \inf_{\xi \in M_k} \operatorname{Re} \lambda_1(\xi) \quad \text{for } k = 1, \dots, m$$

and there are no eigenvalues of the pencil $\mathfrak{A}_j(\lambda)$ in the closed strip between the lines $\operatorname{Re}\lambda = 1 - \beta_j - 3/s$ and $\operatorname{Re}\lambda = 2 - \beta'_j - 3/t$. Then it follows from Lemma 5.2 that the operator $\mathcal{R}$ constructed in the proof of Lemma 5.3 continuously maps the subspace of all (F, g, h), where

$$g \in V^{0,s}_{\beta,\delta}(\mathcal{G}) \cap V^{1,t}_{\beta',\delta'}(\mathcal{G}), \quad h_j \in V^{1-1/s,s}_{\beta,\delta}(\Gamma_j) \cap V^{2-1/t,t}_{\beta',\delta'}(\Gamma_j)$$

and the functional $F \in V^{-1,s}_{\beta,\delta}(\mathcal{G}; S)$ has the form

$$F(v) = \int_{\mathcal{G}} f \cdot v \, dx + \sum_j \int_{\Gamma_j} \Phi_j \cdot v \, dx \quad \text{with vector functions}$$
$$f \in V^{0,t}_{\beta',\delta'}(\mathcal{G})^3, \ \Phi_j \in V^{1-1/t,t}_{\beta',\delta'}(\Gamma_j)^3,$$

into $V^{2,t}_{\beta',\delta'}(\mathcal{G})^3 \times V^{1,t}_{\beta',\delta'}(\mathcal{G})$.

Lemma 5.4. *Suppose that there are no eigenvalues of the pencil $\mathfrak{A}_j(\lambda)$ on the line* $\operatorname{Re}\lambda = 1 - \beta_j - 3/s$, *$j = 1, \dots, d$, and the components of δ satisfy* (5.5). *Then there exists a continuous operator*

$$\mathcal{R} : V^{-1,s}_{\beta,\delta}(\mathcal{G}; S) \times V^{0,s}_{\beta,\delta}(\mathcal{K}) \times \prod_j V^{1-1/s,s}_{\beta,\delta}(\Gamma_j) \to V^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times V^{0,s}_{\beta,\delta}(\mathcal{G})$$

such that $\mathcal{R}\mathcal{A}_{s,\beta,\delta} - I$ and $\mathcal{A}_{s,\beta,\delta}\mathcal{R} - I$ are compact operators in $V^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times V^{0,s}_{\beta,\delta}(\mathcal{G})$ and $V^{-1,s}_{\beta,\delta}(\mathcal{G}; S) \times V^{0,s}_{\beta,\delta}(\mathcal{G}) \times \prod_j V^{1-1/s,s}_{\beta,\delta}(\Gamma_j)$, respectively.

Proof. For the sake of brevity, we write $\mathcal{A}$ instead of $\mathcal{A}_{s,\beta,\delta}$. Let $\{\mathcal{U}_j\}$ be a sufficiently fine open covering of $\mathcal{G}$, and let φ_j, ψ_j be infinitely differentiable functions such that

$$\operatorname{supp} \varphi_j \subset \operatorname{supp} \psi_j \subset \mathcal{U}_j, \quad \varphi_j \, \psi_j = \varphi_j, \quad \text{and} \quad \sum_j \varphi_j = 1.$$

For every j, there exists an operator $\mathcal{R}_j$ having the properties of Lemma 5.2 for $\mathcal{U} = \mathcal{U}_j \cap \mathcal{G}$. We consider the operator $\mathcal{R}$ defined by

$$\mathcal{R}(F, g, h) = \sum_j \varphi_j \, \mathcal{R}_j \, \psi_j \, (F, g, h).$$

Obviously,

$$\begin{aligned}\mathcal{R}\,\mathcal{A}(u,p) &= \sum_j \varphi_j \mathcal{R}_j\,\big(\mathcal{A}\psi_j(u,p) - [\mathcal{A},\psi_j]\,(u,p)\big)\\ &= (u,p) - \sum_j \varphi_j\,\mathcal{R}_j\,[\mathcal{A},\psi_j]\,(u,p),\end{aligned}$$

where $[\mathcal{A},\psi_j]$ is the commutator of $\mathcal{A}$ and ψ_j. Here, the mapping $(u,p)\to[\mathcal{A},\psi_j]\,(u,p)$ is continuous from $V^{1,s}_{\beta,\delta}(\mathcal{G})^3\times V^{0,s}_{\beta,\delta}(\mathcal{G})$ into the set of all (F,g,h), where $g\in V^{1,s}_{\beta',\delta'}(\mathcal{G})$, $h=0$, and F is a functional of the form

$$F(v)=\int_{\mathcal{G}} f\cdot v\,\mathrm{d}x+\sum_j\int_{\Gamma_j}\Phi_j\cdot v\,\mathrm{d}x,\ \text{ where } f\in V^{0,s}_{\beta',\delta'}(\mathcal{G})^3,\ \Phi_j\in V^{1-1/s,s}_{\beta',\delta'}(\Gamma_j)^3,$$

with arbitrary $\beta'\geqq\beta$, $\delta'\geqq\delta$. We can choose β' and δ' such that $\beta_j\leqq\beta'_j<\beta_j+1$ for $j=1,\dots,d$, $\delta_k\leqq\delta'_k<\delta_k+1$ for $k=1,\dots,m$, and β', δ' satisfy the conditions of Remark 5.1 with $t=s$. Then the mapping $(u,p)\to\mathcal{R}_j\,[\mathcal{A},\psi_j]\,(u,p)$ is continuous from $V^{1,s}_{\beta,\delta}(\mathcal{G})^3\times V^{0,s}_{\beta,\delta}(\mathcal{G})$ into $V^{2,s}_{\beta',\delta'}(\mathcal{G})^3\times V^{1,s}_{\beta',\delta'}(\mathcal{G})$. Since the last space is compactly imbedded into $V^{1,s}_{\beta,\delta}(\mathcal{G})^3\times V^{0,s}_{\beta,\delta}(\mathcal{G})$ for $\beta'_j<\beta_j+1$, $\delta'_k<\delta_k+1$ (cf. [9, Lemma 6.2.1]), it follows that $\mathcal{R}\mathcal{A}-I$ is compact. Analogously, the compactness of $\mathcal{A}\mathcal{R}-I$ holds. $\square$

Due to Lemma 5.4, the operator $\mathcal{A}_{s,\beta,\delta}$ is Fredholm if the line $\mathrm{Re}\lambda=1-\beta_j-3/s$ if free of eigenvalues of the pencil $\mathfrak{A}_j(\lambda)$ for $j=1,\dots,d$ and the components of δ satisfy the condition (5.5). Using Theorem 2.1 and Lemma 5.2, we can prove the following existence and uniqueness theorem.

Theorem 5.1. *Let $F\in V^{-1,s}_{\beta,\delta}(\mathcal{G};S)$, $g\in V^{0,s}_{\beta,\delta}(\mathcal{G})$, and $h_j\in V^{1-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}$. We suppose that the components of δ satisfy condition (5.3) and that there are no eigenvalues of the pencils $\mathfrak{A}_j(\lambda)$ in the closed strip between the lines* $\mathrm{Re}\,\lambda=-1/2$ *and* $\mathrm{Re}\,\lambda=1-\beta-3/s$. *In the case when $d_j\in\{0,2\}$ for all j, we assume in addition that g and h_j satisfy condition (2.11). Then there exists a solution $(u,p)\in V^{1,s}_{\beta,\delta}(\mathcal{G})^3\times V^{0,s}_{\beta,\delta}(\mathcal{G})$ of problem (2.9), (2.10). (Here V has to be replaced by the set of all $v\in V^{1,s'}_{-\beta,-\delta}(\mathcal{G})$ satisfying $S_jv=0$ on Γ_j.) The vector function u is unique, p is unique if $d_j\notin\{0,2\}$ for at least one j, and p is unique up to a constant if $d_j\in\{0,2\}$ for all j.*

Proof. First note that in the case when $d_j\in\{0,2\}$ for all j, the spectra both of $\mathfrak{A}_j(\lambda)$ and $A_k(\lambda)$ contain the eigenvalue $\lambda=1$. An eigenvector corresponding to this eigenvalue is $(U,P)=(0,1)$. Furthermore, the spectra of the pencils $\mathfrak{A}_j(\lambda)$ contain the eigenvalue $\lambda=-2$. Therefore from the conditions of the theorem on β and δ it follows in particular that $0<\beta_j+3/s<3$ and $0<\delta_k+2/s<2$. Then $L_\infty(\mathcal{G})\subset V^{0,s}_{\beta,\delta}(\mathcal{G})\subset L_1(\mathcal{G})$. In particular, any constant belongs to the space $V^{0,s}_{\beta,\delta}(\mathcal{G})$, and the integrals in (2.11) exist for $g\in V^{0,s}_{\beta,\delta}(\mathcal{G})$, $h_j\in V^{1-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}$.

We prove the uniqueness of u and p. Let $(u, p) \in V^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times V^{0,s}_{\beta,\delta}(\mathcal{G})$ be a solution of problem (2.9), (2.10) with $F = 0$, $g = 0$, $h_j = 0$. Then according to Lemma 5.2, we obtain $u \in V^{1,2}_{0,0}(\mathcal{G})^3 \subset W^{1,2}(\mathcal{G})^3$ and $p \in L_2(\mathcal{G})$. Hence by Theorem 2.1, we have $u = 0$ and p is constant, $p = 0$ if $d_j \in \{1, 3\}$ for at least one j. Since the operator $\mathcal{A}_{s,\beta,\delta}$ is Fredholm, it follows that every solution $(u, p) \in V^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times V^{0,s}_{\beta,\delta}(\mathcal{G})$ of problem (2.9), (2.10) ($\int_{\mathcal{G}} p \, dx = 0$ if $d_j \in \{0, 2\}$ for all j) satisfies the inequality

$$\|u\|_{V^{1,s}_{\beta,\delta}(\mathcal{G})^3} + \|p\|_{V^{0,s}_{\beta,\delta}(\mathcal{G})} \leqq c \left(\|F\|_{V^{-1,s}_{\beta,\delta}(\mathcal{G};S))} + \|g\|_{V^{0,s}_{\beta,\delta}(\mathcal{G})} + \sum_j \|h_j\|_{V^{1-1/s,s}_{\beta,\delta}(\Gamma_j)} \right) \tag{5.10}$$

with a constant c independent of u and p.

We prove the existence of solutions. Let $d_j \in \{0, 2\}$ for all j, and $F \in V^{-1,s}_{\beta,\delta}(\mathcal{G}; S)$, Furthermore, let $g \in V^{0,s}_{\beta,\delta}(\mathcal{G})$ and $h_j \in V^{1-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}$ be arbitrary functions satisfying (2.11). Then there exist sequences $\{F^{(n)}\} \subset V^* \cap V^{-1,s}_{\beta,\delta}(\mathcal{G}; S)$, $\{g^{(n)}\} \subset L_2(\mathcal{G}) \cap V^{0,s}_{\beta,\delta}(\mathcal{G})$ and $\{h^{(n)}_j\} \subset W^{1/2,2}(\Gamma_j)^{3-d_j} \cap V^{1-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}$ converging to F, g, and h_j, respectively. From (2.11) it follows that the sequence of the numbers

$$a_n = \int_{\mathcal{G}} g^{(n)} \, dx + \sum_{j:\, d_j=0} \int_{\Gamma_j} h^{(n)}_j \cdot n \, dx + \sum_{j:\, d_j=2} \int_{\Gamma_j} h^{(n)}_j \, dx$$

converges to zero. Therefore, the sequence of functions $\tilde{g}^{(n)} = g^{(n)} - \frac{1}{|\mathcal{G}|} a_n$ converges also to g. Moreover, $\tilde{g}^{(n)}$ and $h^{(n)}_j$ satisfy condition (2.11). By Theorem 2.1, there exist solutions $(u^{(n)}, p^{(n)}) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ of problem (2.9), (2.10) with right-hand sides $F^{(n)}$, $\tilde{g}^{(n)}$ and $h^{(n)}_j$, $\int_{\mathcal{G}} p^{(n)} \, dx = 0$. From Lemma 5.2 and from the imbedding $W^{1,2}(\mathcal{G}) \subset V^{1,2}_{0,\varepsilon}(\mathcal{G})$ with arbitrary positive ε, it follows that $(u^{(n)}, p^{(n)}) \in V^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times V^{0,s}_{\beta,\delta}(\mathcal{G})$. Due to (5.10), the vector functions $(u^{(n)}, p^{(n)})$ form a Cauchy sequence in $V^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times V^{0,s}_{\beta,\delta}(\mathcal{G})$. The limit of this sequence solves problem (2.9), (2.10). The proof of the existence of solutions in the case when $d_j \in \{1, 3\}$ for at least one j proceeds analogously. □

5.3. Solvability of the nonlinear problem

We consider the nonlinear problem (2.2), (2.3). For the proof of the existence of solutions in $V^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times V^{0,s}_{\beta,\delta}(\mathcal{G})$, we have to show inequalities analogous to (2.13), (2.14) for the operator

$$V^{1,s}_{\beta,\delta}(\mathcal{G})^3 \ni u \to Qu = (u \cdot \nabla)u \in V^{-1,s}_{\beta,\delta}(\mathcal{G}; S).$$

Lemma 5.5. *Suppose that $s > 3/2$, $\beta_j + 3/s \leqq 2$ for $j = 1, \ldots, N$, and $\delta_k + 3/s \leqq 2$ for $k = 1, \ldots, m$. Then*

$$\|u \, \partial_{x_j} v\|_{V^{-1,s}_{\beta,\delta}(\mathcal{G})^3} \leqq c \, \|u\|_{V^{1,s}_{\beta,\delta}(\mathcal{G})} \, \|v\|_{V^{1,s}_{\beta,\delta}(\mathcal{G})}$$

for all $u, v \in V^{1,s}_{\beta,\delta}(\mathcal{G})$, $j = 1, 2, 3$.

Proof. Let t be a real number such that

$$t \geqq 3, \; t \geqq s, \quad \text{and} \quad \frac{3}{s} - 1 \leqq \frac{3}{t} < 3 - \frac{3}{s}.$$

For example, we can put $t = \max(s, 3)$. Then according to Lemma 3.6, the space $V^{1,s}_{\beta,\delta}(\mathcal{G})$ is continuously imbedded into $V^{0,t}_{\beta',\delta'}(\mathcal{G})$, where $\beta'_j = \beta - 1 + 3s^{-1} - 3t^{-1}$, $\delta'_k = \delta_k - 1 + 3s^{-1} - 3t^{-1}$. Let $q^{-1} = s^{-1} + t^{-1}$. From the conditions on t it follows that $q > 1$. By Hölder's inequality, we have

$$\|u\partial_{x_j} v\|_{V^{0,q}_{\beta+\beta',\delta+\delta'}(\mathcal{G})} \leqq c\, \|u\|_{V^{0,t}_{\beta',\delta'}(\mathcal{G})}\, \|\partial_{x_j} v\|_{V^{0,s}_{\beta,\delta}(\mathcal{G})}$$

for arbitrary $u, v \in V^{1,s}_{\beta,\delta}(\mathcal{G})$. Using the continuity of the imbedding $V^{0,q}_{\beta+\beta',\delta+\delta'}(\mathcal{G}) \subset V^{-1,s}_{\beta,\delta}(\mathcal{G})$ which follows from Corollary 3.2, we obtain the desired inequality. □

As a consequence of Lemma 5.5, the following inequalities hold for arbitrary $u, v \in V^{1,s}_{\beta,\delta}(\mathcal{G})^3$:

$$\|Qu\|_{V^{-1,s}_{\beta,\delta}(\mathcal{G};S)} \leqq c\, \|u\|^2_{V^{1,s}_{\beta,\delta}(\mathcal{G})^3}, \tag{5.11}$$

$$\|Qu - Qv\|_{V^{-1,s}_{\beta,\delta}(\mathcal{G};S)} \leqq c \left(\|u\|_{V^{1,s}_{\beta,\delta}(\mathcal{G})^3} + \|v\|_{V^{1,s}_{\beta,\delta}(\mathcal{G})^3} \right) \|u - v\|_{V^{1,s}_{\beta,\delta}(\mathcal{G}))^3}. \tag{5.12}$$

Using these estimates, one can prove the following theorem analogously to Theorem 2.2.

Theorem 5.2. *Let the conditions of Theorem 5.1 on F, g, h_j, β and δ be satisfied. In addition, we assume that $s > 3/2$, $\beta_j + 3/s \leqq 2$, $\delta_k + 3/s \leqq 2$ and that*

$$\|F\|_{V^{-1,s}_{\beta,\delta}(\mathcal{G};S)} + \|g\|_{V^{0,s}_{\beta,\delta}(\mathcal{G})} + \sum_j \|h_j\|_{V^{1-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}}$$

is sufficiently small. Then there exists a solution $(u, p) \in V^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times V^{0,s}_{\beta,\delta}(\mathcal{G})$ of problem (2.4), (2.5)*, where V has to be replaced by the set of all $v \in V^{1,s'}_{-\beta,-\delta}(\mathcal{G})^3$ such that $S_j v = 0$ on Γ_j, $j = 1, \dots, N$. The function u is unique on the set of all functions with norm less than a certain positive ε, p is unique if $d_j \in \{1, 3\}$ for at least one j, otherwise p is unique up to a constant.*

Proof. Let $(u^{(0)}, p^{(0)}) \in V^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times V^{0,s}_{\beta,\delta}(\mathcal{G})$. By Theorem 5.1, the norm of $u^{(0)}$ can be assumed to be small. Obviously, (u, p) is a solution of problem (2.4), (2.5) if and only if $(w, q) = (u - u^{(0)}, p - p^{(0)})$ is a solution of the problem

$$b(w, v) - \int_{\mathcal{G}} q \nabla \cdot v \, dx = - \int_{\mathcal{G}} Q(w + u^{(0)}) \cdot v \, dx \;\text{ for all } v \in v \in V^{1,s'}_{-\beta,-\delta}(\mathcal{G})^3,$$
$$S_j v|_{\Gamma_j} = 0,$$
$$\nabla \cdot w = 0 \;\text{ in } \mathcal{G}, \quad S_j u = 0 \;\text{ on } \Gamma_j, \; j = 1, \dots, N.$$

This problem can be written as

$$(w, q) = -AQ(w + u^{(0)})$$

where A is the inverse to the operator $F \to \mathcal{A}_{s,\beta,\delta}(F, 0, 0)$ considered in the preceding subsection. Due to (5.12), the operator $(w, q) \to -AQ(w + u^{(0)})$ is contractive on the set of all $(w, q) \in V^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times V^{0,s}_{\beta,\delta}(\mathcal{G})$ with norm $\leqq \varepsilon$ if ε and the norm of $u^{(0)}$ are sufficiently small. Hence this operator has a fixed point. This proves the theorem. □

Finally, we give a result in nonweighted Sobolev spaces which follows immediately from the last theorem.

Let $\mathcal{G}$ be a polyhedron. We assume that one of the boundary conditions (i)–(iii) *is given on every face Γ_j, that the Dirichlet condition is given on at least one of the adjoining faces of every edge M_k, and that $\theta_k \leqq 3\pi/2$ if the boundary conditions* (ii) *or* (iii) *are given on one of the adjoining faces of M_k. Here, θ_k denotes again the angle at the edge M_k. Then the problem*

$$\begin{aligned} b(u, v) + \int_{\mathcal{G}} \sum_{j=1}^{3} u_j \frac{\partial u}{\partial x_j} \cdot v \, dx - \int_{\mathcal{G}} p \, \nabla \cdot v \, dx \\ = F(v) \ \textit{for all } v \in W^{1,s'}(\mathcal{G})^3, \ S_j v = 0 \textit{ on } \Gamma_j, \\ -\nabla \cdot u = 0 \ \textit{in } \mathcal{G}, \quad S_j u = 0 \ \textit{on } \Gamma_j, \ j = 1, \ldots, N, \end{aligned}$$

with $F \in W^{-1,s}(\mathcal{G}; S)$, $3/2 < s < 3$, has a solution $(u, p) \in W^{1,s}(\mathcal{G})^3 \times L_s(\mathcal{G})$ if the norm of F is sufficiently small.

For the proof it suffices to note that the strip $-1 \leqq \operatorname{Re} \lambda \leqq 0$ does not contain eigenvalues of the pencils $\mathfrak{A}_j(\lambda)$ and that $\operatorname{Re} \lambda_1^{(k)} \geqq 1/3$ for $k = 1, \ldots, m$. Therefore, the conditions of Theorem 5.2 are satisfied for $\beta = 0$, $\delta = 0$, $3/2 < s < 3$.

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