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DIRECTIONAL DERIVATIVE OF A MINIMAX FUNCTION

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1. INTRODUCTION

THE STUDY of the existence and characterization of the directional derivative of a minimax function has a great importance in optimization, from the theoretical viewpoint as well as from the purposes of devising algorithms. Given an extended real valued function L defined on $X \times U \times V$, where X is a locally convex space and U, V are two Hausdorff topological spaces, we study the existence and a characterization of the directional derivative $\mathrm{D}h(x_0; d)$ at $x_0 \in X$, in the direction $d \in X$, of the minimax function

$$h(x) = \operatorname*{Sup}_{v \in V_0} \operatorname*{Inf}_{u \in U_0} L(x, u, v)$$

with U_0, V_0 two nonempty sets in U and V respectively. The smooth case in $X = \mathbb{R}^n$, that is when the gradient $\nabla_x L(x, u, v)$ exists and is continuous in the variable (x, u, v) has been treated by Dem'janov [8, 9]. In our main result (theorem 2.1) of Section 2, we suppose some semicontinuity properties of the Dini directional derivatives $\underline{\mathrm{D}}_x L(x, u, v; d)$ and $\bar{\mathrm{D}}_x L(x, u, v; d)$ that guarantee the existence of $\mathrm{D}h(x_0; d)$, obtaining also a characterization of it. We show that a large class of nonsmooth functions verifies the hypotheses of this theorem. As a particular case, we study the directional derivative of the max function $h(x) = \operatorname{Sup}\{L(x, v)/v \in V_0\}$. We compare our result with one due to Clarke [5], who assumes that $L(\cdot, v)$ is a locally Lipschitz function defined over $X = \mathbb{R}^n$ verifying the so-called subdifferential regularity condition. Another interesting work in this topic was developed by Hiriart-Urruty [11], who gave some evaluations of the directional derivative of $h(x) = \operatorname{Sup}\{L(x, v)/v \in F(x)\}$ where F is an arbitrary multifunction with closed graph. A sharper estimation when L is independent of x and $F(x) = \{v \in \mathbb{R}^p/f_i(v) \leq x_i \text{ if } 1 \leq i \leq s \text{ and } f_i(v) = x_i \text{ if } s + 1 \leq i \leq n\}$, has been developped by Gauvin [10].

In Section 3, we give sufficient conditions to ensure the hypotheses of theorem 2.1 when $L(x, \cdot, \cdot)$ is a convex-concave function over $U_0 \times V_0$. This allows us, in Section 4, to treat the particular case $L(x, u, v) = f_0(x, u) + \sum_{i=1}^{m} v_i f_i(x, u)$ corresponding to the classical Lagrangian

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associated with the parametrized optimization problem $h(x) = \inf\{f_0(x, u)/ f_i(x, u) \leq 0, 1 \leq i \leq m\}$. In this way, we generalize a result due to Hogan [15], who assumes that the functions $f_i(\cdot, u)$ are continuously differentiable. Finally, we show how this generalization can be used to study the approximate second-order directional derivative of a nonsmooth convex function [12, 13].

2. DIRECTIONAL DERIVATIVE OF A MINIMAX FUNCTION

Let U_0 and V_0 be two nonempty subsets of the Hausdorff spaces U and V respectively. Let X be a Hausdorff locally convex real topological vector space (l.c.s.). Let us define the minimax functions

$$h_1(x) = \operatorname*{Sup}_{v \in V_0} \operatorname*{Inf}_{u \in U_0} L(x, u, v) \tag{2.1}$$

$$h_2(x) = \operatorname*{Inf}_{u \in U_0} \operatorname*{Sup}_{v \in V_0} L(x, u, v) \tag{2.2}$$

where L is an extended real valued function on $X \times U \times V$, and the solution sets

$$V(x) = \left\{ v \in V_0/h_1(x) = \operatorname*{Inf}_{u \in U_0} L(x, u, v) \right\} \tag{2.3}$$

$$U(x) = \left\{ u \in U_0/h_2(x) = \operatorname*{Sup}_{v \in V_0} L(x, u, v) \right\}. \tag{2.4}$$

The main result of this section is the existence and a characterization of the directional derivatives of the functions h_1 and h_2. For a given point $x_0 \in X$ and a direction $d \in X$, we introduce the following notation for the directional derivative and the Dini directional derivatives of $L(\cdot, u, v)$:

$$\mathrm{D}_x L(x_0, u, v; d) = \lim_{t \to 0^+} \{L(x_0 + td, u, v) - L(x_0, u, v)\}/t$$

$$\overline{\mathrm{D}}_x L(x_0, u, v; d) = \overline{\lim_{t \to 0^+}} \{L(x_0 + td, u, v) - L(x_0, u, v)\}/t$$

$$\underline{\mathrm{D}}_x L(x_0, u, v; d) = \underline{\lim}_{t \to 0^+} \{L(x_0 + td, u, v) - L(x_0, u, v)\}/t.$$

We start by introducing a new continuity notion for multifunctions.

Definition 2.1. Let Z and T be Hausdorff spaces. The multifunction $A: T \rightrightarrows Z$ is sequentially semicontinuous (s.s.c.) at $t_0 \in T$ if for every sequence $\{t_k\}$ converging to t_0 there exist $z_0 \in A(t_0)$ and a sequence $\{z_k\}$ accumulating at z_0 such that $z_k \in A(t_k)$ for all k sufficiently large.

Remark 2.1. If the multifunction $A: T \rightrightarrows Z$ is nonempty valued in some neighbourhood of $t_0 \in T$, the sequential semicontinuity of A at t_0 is assured by some classical continuity notions. For example, when A is l.s.c. at t_0, that is when for every open set H in Z intersecting $A(t_0)$ there exists a neighbourhood N of t_0 such that $H \cap A(t)$ is nonempty for all $t \in N$. Similarly, A is s.s.c. at t_0 if A is u.s.c. at t_0, i.e. if $A(N) = U\{A(t)/t \in N\}$ is relatively compact for some neighbourhood N of t_0 and A is closed at t_0. The closedness of A at t_0 means that $z_0 \in A(t_0)$ for every sequence $\{t_k\}$ converging to t_0 and $\{z_k\}$ accumulating at z_0 with $z_k \in A(t_k)$.

With the above notions and denoting by $\mathbb{R}_+$ the set of nonnegative real numbers, we can state:

THEOREM 2.1. Assume that the multifunctions $t \in \mathbb{R}_+ \to U(x_0 + td)$ and $t \in \mathbb{R}_+ \to V(x_0 + td)$ are s.s.c. at 0 and that the following properties hold:

(P1) There exists $\delta > 0$ such that for every $(u, v) \in U_0 \times V_0$, the function $t \in \mathbb{R}_+ \to L(x_0 + td, u, v)$ is finite and continuous at $[0, \delta[$.

(P2) For all $u_0 \in U(x_0)$, the function $(t, v) \in \mathbb{R}_+ \times V_0 \to \underline{\mathrm{D}}_x L(x_0 + td, u_0, v; d)$ is finite and u.s.c. at $\{0\} \times V(x_0)$. For all $v_0 \in V(x_0)$, the function $(t, u) \in \mathbb{R}_+ \times U_0 \to \bar{\mathrm{D}}_x L(x_0 + td, u, v_0; d)$ is finite and l.s.c. at $\{0\} \times U(x_0)$.

(P3) There exists $\delta > 0$ such that $h_1(x_0 + td) = h_2(x_0 + td)$ for all $t \in [0, \delta[$; this value is denoted by $h(x_0 + td)$.

Then, the directional derivative $\mathrm{D}h(x_0; d) = \lim_{t \to 0^+} \{h(x_0 + td) - h(x_0)\}/t$ exists and is characterized by

$$\begin{aligned} \mathrm{D}h(x_0; d) &= \operatorname*{Sup}_{v \in V(x_0)} \operatorname*{Inf}_{u \in U(x_0)} \mathrm{D}_x L(x_0, u, v; d) \\ &= \operatorname*{Inf}_{u \in U(x_0)} \operatorname*{Sup}_{v \in V(x_0)} \mathrm{D}_x L(x_0, u, v; d). \end{aligned} \tag{2.5}$$

Proof. It is a direct consequence of the next two lemmas. ∎

LEMMA 2.1. Assume that the set $U(x_0)$ is nonempty and that the multifunction $t \in \mathbb{R}_+ \to V(x_0 + td)$ is s.s.c. at 0. Moreover

(P1) There exists $\delta > 0$ such that for every $(u, v) \in U_0 \times V_0$, the function $t \in \mathbb{R}_+ \to L(x_0 + td, u, v)$ is finite and l.s.c. at $[0, \delta[$.

(P2) For all $u_0 \in U(x_0)$, the function $(t, v) \in \mathbb{R}_+ \times V_0 \to \underline{\mathrm{D}}_x L(x_0 + td, u_0, v; d)$ is finite and u.s.c. at $\{0\} \times V(x_0)$.

(P3) $h_1(x_0) = h_2(x_0)$.

Then,

$$\bar{\mathrm{D}}h_1(x_0; d) \leqslant \operatorname*{Sup}_{v \in V(x_0)} \operatorname*{Inf}_{u \in U(x_0)} \mathrm{D}_x L(x_0, u, v; d). \tag{2.6}$$

Proof. Let $\delta > 0$ be given by hypothesis (P1) and $\{t_k\}$ be a sequence in $]0, \delta[$ converging to 0 such that

$$\lim_{k \to \infty} \{h_1(x_0 + t_k d) - h_1(x_0)\}/t_k = \overline{\lim_{t \to 0^+}} \{h_1(x_0 + td) - h_1(x_0)\}/t.$$

Let us choose $u_0 \in U(x_0)$ and a sequence $\{v_k\}$ accumulating at some point $v_0 \in V(x_0)$ such that $v_k \in V(x_0 + t_k d)$ for all k sufficiently large. In this way, we obtain

$$\begin{aligned} h_1(x_0 + t_k d) - h_1(x_0) &= h_1(x_0 + t_k d) - h_2(x_0) \\ &= \operatorname*{Inf}_{u \in U_0} L(x_0 + t_k d, u, v_k) - \operatorname*{Sup}_{v \in V_0} L(x_0, u_0, v) \\ &\leqslant L(x_0 + t_k d, u_0, v_k) - L(x_0, u_0, v_k). \end{aligned}$$

Using (P1) and Dini's mean value theorem [14, proposition 2.1], we conclude that

$$L(x_0 + t_k d, u_0, v_k) - L(x_0, u_0, v_k) \leqslant t_k \underline{D}_x L(x_0 + \delta_k d, u_0, v_k; d)$$

for some $\delta_k \in [0, t_k]$ and therefore

$$\{h_1(x_0 + t_k d) - h_1(x_0)\}/t_k \leqslant \underline{D}_x L(x_0 + \delta_k d, u_0, v_k; d).$$

Taking into account (P2) and the fact that v_0 is an accumulation point of $\{v_k\}$ and that $\{\delta_k\}$ converges to 0, we obtain the inequality

$$\bar{D}h_1(x_0; d) \leqslant \underline{D}_x L(x_0, u_0, v_0; d).$$

On the other hand, using (P2) and Dini's mean value theorem, it can be shown that the directional derivative $D_x L(x_0, u, v; d)$ exists. Since $u_0 \in U(x_0)$ is arbitrary in the last inequality, the upper bound for the Dini directional derivative $\bar{D}h_1(x_0; d)$ is obtained. ■

LEMMA 2.2. Assume that the set $V(x_0)$ is nonempty and that the multifunction $t \in \mathbb{R}_+ \to U(x_0 + td)$ is s.s.c. at 0. Moreover

(P1) There exists $\delta > 0$ such that for every $(u, v) \in U_0 \times V_0$, the function $t \in \mathbb{R}_+ \to L(x_0 + td, u, v)$ is finite and u.s.c. at $[0, \delta[$.

(P2) For all $v_0 \in V(x_0)$, the function $(t, u) \in \mathbb{R}_+ \times U_0 \to \bar{D}_x L(x_0 + td, u, v_0; d)$ is finite and l.s.c. at $\{0\} \times U(x_0)$.

(P3) $h_1(x_0) = h_2(x_0)$.

Then,

$$\underline{D}h_2(x_0; d) \geqslant \operatorname*{Inf}_{u \in U(x_0)} \operatorname*{Sup}_{u \in U(x_0)} D_x L(x_0, u, v; d). \tag{2.7}$$

Proof. Analogous to lemma 2.1. ■

Sufficient conditions to guarantee the sequential semicontinuity property of the multifunctions in the last theorem and the stability hypothesis (P3) are given in the next section. To check (P2) it is usually sufficient to ensure the u.s.c. of $t \in \mathbb{R}_+ \to \underline{D}L(x_0 + td, u, v; d)$ and the l.s.c. of $t \in \mathbb{R}_+ \to \bar{D}_x L(x_0 + td, u, v; d)$. In the following proposition we exhibit a great class of locally Lipschitz functions satisfying this property. We assume that the reader is familiarized with the Clarke's directional derivative

$$D^\circ g(x_0; d) = \overline{\lim_{(x,t) \to (x_0, 0^+)}} \{g(x + td) - g(x)\}/t$$

and the subdifferential $\partial g(x_0) = \{z \in X^* / \forall d \in X, D^\circ g(x_0; d) \geqslant \langle d, z \rangle\}$ of a locally Lipschitz function g on X [5], where X and X^* are l.c.s. in duality by mean of $\langle \, , \rangle$.

PROPOSITION 2.1. Let $f: X \to \overline{\mathbb{R}}$ be such that $t \in \mathbb{R}_+ \to \bar{f}(t) = f(x_0 + td)$ is locally Lipschitz at 0. Moreover, assume that $\bar{f}$ is semismooth at 0, that is for every sequence $\{t_k\}$ in $\mathbb{R}_+$ converging to 0 and every sequence $\{g_k\}$ verifying $g_k \in \partial \bar{f}(t_k)$ it follows that $\{g_k\}$ converges to $\bar{f}'_+(0)$ (where $\bar{f}'_+(0) = D\bar{f}(0; 1)$).

Then, the functions $t \in \mathbb{R}_+ \to \bar{D}f(x_0 + td; d)$ and $t \in \mathbb{R}_+ \to \underline{D}f(x_0 + td; d)$ are continuous at 0.

Proof. Let $\{t_k\}$ be any sequence in $\mathbb{R}_-$ converging to 0. Using Lebourg's mean value theorem [18], it can be shown that $\bar{D}f(x_0 + t_k d; d) \in \partial \bar{f}(t_k)$ for all k sufficiently large. By the semismoothness of $\bar{f}$ it follows that $\bar{D}f(x_0 + t_k d; d)$ converges to $\bar{f}'_-(0) = Df(x_0; d)$. Analogously, the continuity at 0 of $t \in \mathbb{R}_- \rightarrow \underline{D}f(x_0 + td; d)$ can be proved. ■

Remark 2.2. The notion of semismoothness was introduced by Mifflin [19] and it can be demonstrated that convex, concave and continuously differentiable functions are semismooth. In a later publication, Spingarn [20] has proved that also the so-called lower-C^1 functions are semismooth, in particular the max of a finite number of continuously differentiable functions. Using the fact that the class of semismooth functions is a vector space, it follows that also the upper-C^1 functions are semismooth, in particular the min of a finite number of continously differentiable functions.

As a particular case of (2.1) we shall consider the max function

$$h(x) = \operatorname{Sup}\{L(x, v)/v \in V_0\} \tag{2.8}$$

and the solution set $V(x) = \{v \in V_0/h(x) = L(x, v)\}$, where L is an extended real valued function on $X \times V$.

COROLLARY 2.1. Assume that the multifunction $t \in \mathbb{R}_- \rightarrow V(x_0 + td)$ is s.s.c. at 0 and that L has the following properties:

(P1) There exists $\delta > 0$ such that for every $v \in V_0$, the function $t \in \mathbb{R}_+ \rightarrow L(x_0 + td, v)$ is finite and l.s.c. at $[0, \delta[$.

(P2) The function $(t, v) \in \mathbb{R}_- \times V_0 \rightarrow \overline{D}_x L(x_0 + td, v; d)$ is finite and u.s.c. at $\{0\} \times V(x_0)$.

Then, the directional derivative $Dh(x_0; d)$ exists and is characterized by

$$Dh(x_0; d) = \operatorname{Sup}\{D_x L(x_0, v; d)/v \in V(x_0)\}. \tag{2.9}$$

Proof. From lemma 2.1. we conclude the inequality

$$\bar{D}h(x_0; d) \leqslant \operatorname{Sup}\{D_x L(x_0, v; d)/v \in V(x_0)\}.$$

On the other hand

$$h(x_0 + td) - h(x_0) \geqslant L(x_0 + td, v) - L(x_0, v)$$

if $v \in V(x_0)$. Hence

$$Dh(x_0; d) \geqslant \operatorname{Sup}\{D_x L(x_0, v; d)/v \in V(x_0)\}$$

and (2.9) is proved. ■

In recent years, many articles have been devoted to obtain characterizations of the directional derivative of a max function such as (2.8). For L verifying some continuous differenciability properties, the most important results are due to Danskin [7], Dem'janov [8, 9], Hogan [15] and Contesse [6]. In the nonsmooth case, Clarke [5] obtained the same equality (2.9), using a different methodology and more restrictive hypotheses.

To check (P2) in the last corollary it is usually sufficient to ensure the u.s.c. of $t \in \mathbb{R}_- \to \underline{D}_x L(x_0 + td, v, d)$ at 0. Proposition 2.1. shows an important class of nonsmooth functions verifying this semicontinuity condition. This property is also verified when $L(\cdot, v)$ is locally Lipschitz and belongs to the class of subdifferentially regular functions, i.e. for all $x_0, d \in X$ $D_x L(x_0, v; d) = D_x^\circ L(x_0, v; d)$ where $D_x^\circ L(x_0, v; d)$ denotes the Clarke's directional derivatives of $L(\cdot, v)$.

As an illustration of corollary 2.1, consider the following example:

Example 2.1. Let $L: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ be defined by $L(x_1, x_2, v) = v^2(|x_1| - |x_2|)$. $V_0 = [-1, 1], x_0 = (0, 0)$ and $d \in \mathbb{R}_+ \times \mathbb{R}_-$. To calculate $Dh(x_0; d)$ applying the corollary 2.1. the sole nontrivial hypothesis to be checked is (P2). Since

$$D_x L(x, v; d) = \begin{cases} v^2(d_1 - d_2) & \text{if } x_1 \geqslant 0, x_2 \geqslant 0 \\ v^2(-d_1 - d_2) & \text{if } x_1 < 0, x_2 \geqslant 0 \\ v^2(-d_1 + d_2) & \text{if } x_1 < 0, x_2 < 0 \\ v^2(d_1 + d_2) & \text{if } x_1 \geqslant 0, x_2 < 0 \end{cases}$$

we obtain $D_x L(x_0 + td, v; d) = v^2(d_1 - d_2)$ for $(t, v) \in \mathbb{R}_- \times V_0$. Hence $Dh(x_0; d) = \operatorname{Sup}\{v^2(d_1 - d_2)/v \in V(x_0) = [-1, 1]\}$, i.e.

$$Dh(x_0; d) = \begin{cases} 0 & \text{if } d_1 \leqslant d_2 \\ d_1 - d_2 & \text{if } d_1 \geqslant d_2. \end{cases}$$

3. ANALYSIS OF THE CONVEX-CONCAVE CASE

In this section we consider a metrizable space T instead of the l.c.s. X and we define $h_1(t)$, $h_2(t)$, $V(t)$ and $U(t)$ by (2.1), (2.2), (2.3) and (2.4) respectively. The metrizability does not imply any loss of generality. In fact, in Section 2 we worked with the variable x only in a segment of the form $[x_0, x_0 + \delta d[$.

In order to give sufficient conditions to check the stability hypothesis (P3) and the topological properties of the multifunctions $U(\cdot)$ and $V(\cdot)$ in theorem 2.1, we assume in this section that $L(t, \cdot, \cdot)$ is a convex-concave function over $U_0 \times V_0$, where U_0 and V_0 are two nonempty closed and convex sets in the finite dimentional normed spaces U and V respectively.

PROPOSITION 3.1. Let N be a neighbourhood of $t_0 \in T$ such that:

(a) For all $t \in N$, $L(t, \cdot, \cdot)$ is convex–concave over $U_0 \times V_0$.

(b) $L: T \times U \times V \to \overline{\mathbb{R}}$ is finite and continuous at $N \times U_0 \times V_0$.

(c) For some $\bar{v} \in V_0$, $L(t_0, \cdot, \bar{v})$ is inf-compact over U_0, that is the level set $S_\lambda = \{u \in U_0 / L(t_0, u, \bar{v}) \leqslant \lambda\}$ is compact for each $\lambda \in \mathbb{R}$.

(d) For some $\bar{u} \in U_0$, $L(t_0, \bar{u}, \cdot)$ is sup-compact over V_0.

Then, there exists a neighbourhood of t_0 on which h_1 and h_2 are identical and the multifunctions $U(\cdot)$ and $V(\cdot)$ nonempty. Moreover, $U(\cdot)$ and $V(\cdot)$ are u.s.c. at t_0, in particular s.s.c. at t_0. (see remark 2.1.).

This proposition extends the results of Hogan [15, lemma 2] and Contesse [6, theorem 10] for the classical Lagrangian of parametrized convex optimization problems to the case of a general convex-concave function.

The proof of proposition 3.1 uses the following lemma:

LEMMA 3.1. Let $g: T \times U \to \overline{\mathbb{R}}$ be a finite and continuous function at $N \times U_0$ for some neighborhood N of $t_0 \in T$. Assume that $g(t_0, \cdot)$ is inf-compact over U_0 and that $g(t, \cdot)$ is quasiconvex over U_0, for all $t \in N$. Then, for every sequence $\{t_k\}$ converging to t_0 and every unbounded sequence $\{u_k\}$ in U_0 it follows that $\{g(t_k, u_k)\}$ is not bounded from above.

Proof. Assume that $g(t_k, u_k) \leq K$ for all k. Without loss of generality we suppose $\|u_k\| > k$. For $u_0 \in U_0$ and $\rho > 0$ let us define $z_k = (1 - \lambda_k)u_0 + \lambda_k u_k$ with $\lambda_k = \rho\|u_k\|^{-1}$. Then $g(t_k, z_k) \leq \operatorname{Max}\{g(t_k, u_k), g(t_k, u_0)\} \leq C$ for some constant C, so that $g(t_0, u_0 + \rho d) \leq C$, where d is an accumulation point of the sequence $\{u_k\|u_k\|^{-1}\}$. Since ρ is arbitrary, the last inequality contradicts the inf-compacity of $g(t_0, \cdot)$ over U_0. This methodology is used by Contesse [6, theorem 4]. ■

Proof of proposition 3.1. First, let us shown that $L(t, \cdot, \bar{v})$ is inf-compact over U_0 for all t in some neighborhood of t_0. For this purpose, given the convexity of $L(t, \cdot, \bar{v})$ it is sufficient to prove that the solution set $S(t) = \{u \in U_0/\alpha(t) = L(t, u, \bar{v})\}$ of $\alpha(t) = \operatorname{Inf}\{L(t, u, \bar{v})/u \in U_0\}$ is nonempty and bounded. Let $\{t_k\}$ be a sequence converging to t_0 such that $S(t_k)$ is empty. Since α is u.s.c. at t_0 and $\alpha(t_0)$ is finite, there is an unbounded sequence $\{u_k\}$ in U_0 such that $\{L(t_k, u_k, \bar{v})\}$ is bounded from above, which contradicts lemma 3.1. Thus, the nonemptiness of $S(t)$ for all t in some neighborhood of t_0 is proved. On the other hand, let $\{t_k\}$ be a sequence converging to t_0 such that $S(t_k)$ is unbounded for all k. Then, we can choose an unbounded sequence $\{u_k\}$ in U_0 verifying $\alpha(t_k) = L(t_k, u_k, \bar{v})$ and using the same previous argument, we obtain a contradiction. So, the boundedness of $S(t)$ for all t in some neighborhood of t_0 is proved. Similarly, the sup-compacity of $L(t, \bar{u}, \cdot)$ can also be proved. Using classical minimax theorems, see for example [1, theorem 4.2], the existence of a saddle point $(u_t, v_t) \in U(t) \times V(t)$ of $L(t, \cdot, \cdot)$ over $U_0 \times V_0$ can be assured and, in this way, it is shown that $U(t)$ and $V(t)$ are nonempty and $h_1(t) = h_2(t)$ for all t in some neighborhood of t_0. Now we shall prove the existence of a neighborhood M of t_0 such that the sets $U(M)$ and $V(M)$ are bounded. If it were not, there would be a sequence $\{t_k\}$ converging to t_0 and an unbounded sequence $\{u_k, v_k\}$ in $U_0 \times V_0$ such that $L(t_k, u_k, v_k) = \operatorname{Sup}\{L(t_k, u_k, v)/v \in V_0\} = \operatorname{Inf}\{L(t_k, u, v_k)/u \in U_0\}$. Since $\{u_k\}$ *or* $\{v_k\}$ is unbounded, let us assume that it is the first one. So, lemma 3.1 and the inequality $L(t_k, u_k, v_k) \geq L(t_k, u_k, \bar{v})$ show that $\{L(t_k, u_k, v_k)\}$ is not bounded from above, contradicting, in this way, the inequality $L(t_k, u_k, v_k) \leq L(t_k, \bar{u}, v_k)$, since $\{L(t_k, \bar{u}, v_k)\}$ is bounded from above. The proof of the closedness of the multifunctions $U(\cdot)$ and $V(\cdot)$ does not present any difficulty. ■

4. APPLICATIONS TO CONVEX OPTIMIZATION

Let X and U be two l.c.s. and U_0 be a nonempty convex and closed subset of U. Let $f_0, f_1, \ldots, f_m$ be $(m + 1)$ extended real valued functions defined over $X \times U$ such that, for $x_0, d \in X$ and $\delta > 0$, they are convex, l.s.c. and finite in the variable $u \in U_0$ for each x fixed in $[x_0, x_0 + \delta d[$.

In this section, we shall study the directional derivative of the optimal value function

$$h_2(x) = \inf\{f_0(x, u)/u \in M(x)\} \tag{4.1}$$

where $M(x) = \{u \in U_0/f_i(x, u) \leq 0, 1 \leq i \leq m\}$ is supposed nonempty for all x in $[x_0, x_0 + \delta d[$.

Defining the Lagrangian

$$L(x, u, v) = f_0(x, u) + \sum_{i=1}^{m} v_i f_i(x, u),$$

the function h_2 takes the form (2.2) with $V_0 = \mathbb{R}^m_+$ and the solution set (2.4) is reduced to $U(x) = \{u \in M(x)/h_2(x) = f_0(x, u)\}$. The function h_1 defined by (2.1) corresponds to the so-called dual problem of the convex optimization problem given in (4.1).

As a direct consequence of theorem 2.1, we can state the following result.

THEOREM 4.1. Assume that the multifunctions $t \in \mathbb{R}_+ \to U(x_0 + td)$ and $t \in \mathbb{R}_+ \to V(x_0 + td)$ are s.s.c. at 0 and that the stability condition

(P3) $h_1(x_0 + td) = h_2(x_0 + td)$ for all $t \in [0, \delta[$ holds. Moreover, if for all $0 \leqslant i \leqslant m$, the next conditions are verified:

(a) For all $u \in U_0$, the function $t \in \mathbb{R}_+ \to f_i(x_0 + td, u)$ is continuous and has right derivative at $[0, \delta[$.

(b) The function $(t, u) \in \mathbb{R}_+ \times U_0 \to \mathrm{D}_x f_i(x_0 + td, u; d)$ is l.s.c. at $\{0\} \times U(x_0)$.

(c) For all $u_0 \in U(x_0)$, the function $t \in \mathbb{R}_+ \to \mathrm{D}_x f_i(x_0 + td, u_0; d)$ is u.s.c. at 0.

then, the directional derivative $\mathrm{D}h(x_0; d)$ (with h defined by (P3) over $[x_0, x_0 + \delta d[$) exists and is characterized by

$$\begin{aligned} \mathrm{D}h(x_0; d) &= \operatorname*{Inf}_{u \in V(x_0)} \operatorname*{Sup}_{v \in V(x_0)} \mathrm{D}_x L(x_0, u, v; d) \\ &= \operatorname*{Sup}_{v \in V(x_0)} \operatorname*{Inf}_{u \in U(x_0)} \mathrm{D}_x L(x_0, u, v; d) \end{aligned} \tag{4.2}$$

where

$$\mathrm{D}_x L(x_0, u, v; d) = \mathrm{D}_x f_0(x_0, u; d) + \sum_{i=1}^{m} v_i \mathrm{D}_x f_i(x_0, u; d).$$

The s.s.c. of the multifunctions and the stability condition in the last theorem, is assured in the next proposition when U is a finite dimensional normed space. This result was first obtained by Hogan [15, lemma 2] and recently generalized by Contesse [6, theorem 10].

PROPOSITION 4.1. Assume that U is a finite dimensional normed space and that the following hypotheses hold:

(i) For all $0 \leqslant i \leqslant m$, the function $(t, u) \in \mathbb{R}_+ \times U_0 \to f_i(x_0 + td, u)$ is continuous at $[0, \delta[\times U_0$.

(ii) There exists $\bar{u} \in U_0$ such that, for all $1 \leqslant i \leqslant m$, $f_i(x_0, \bar{u}) < 0$

(iii) $U(x_0)$ is nonempty and bounded.

Then, the multifunctions $t \in \mathbb{R}_+ \to U(x_0 + td)$ and $t \in \mathbb{R}_+ \to V(x_0 + td)$ are s.s.c. at 0 and the stability condition (P3) holds.

Proof. It is sufficient to apply proposition 3.1 making the adequate identifications. We shall verify the hypotheses of such proposition. Clearly, (a) and (b) hold. The convexity of $f_0(x_0, \cdot)$ together with (i) and (iii) ensure the inf-compacity of this function over U_0, then (c) holds with $\bar{v} = 0 \in V_0$. The Slater's condition (ii) guarantees (d). ∎

Remark 4.1. When U is not finite dimensional, sufficient conditions to check the stability condition and the nonemptiness of the solution sets of the primal and dual problems, are given in [16, chapter 7].

5. APPLICATIONS IN ε-SUBDIFFERENTIAL ANALYSIS

As an interesting application of theorem 4.1, we can obtain the characterization of the approximate second-order directional derivative of a convex function. An extensive analysis of this notion and the so-called ε-subdifferential calculus has been developed by Hiriart-Urruty (see for example [12, 13]).

Let X be a l.c.s. in duality with its topological dual X^* by means of the canonical bilinear form $\langle\,.\,\rangle$. In what follows, we consider a proper, l.s.c. and convex function $f: X \to \bar{\bar{\mathbb{R}}}$ and $x_0 \in \operatorname{dom} f = \{x \in X/f(x) < \infty\}$ a point at which f is continuous. For $\varepsilon > 0$, the ε-subdifferential of f at x_0 is defined as the nonempty convex w^*-compact set

$$\partial_\varepsilon f(x_0) = \{x^* \in X^*/f(x_0) + f^*(v) - \langle x_0, x^*\rangle \leqslant \varepsilon\}$$

where $f^*: X^* \to \overline{\mathbb{R}}$ denotes the conjugate of f. The ε-directional derivative of f at x in the direction $p \in X$ is defined by $f'_\varepsilon(x, p) = \operatorname{Sup}\{\langle p, x^*\rangle/x^* \in \partial_\varepsilon f(x)\}$, that is as a convex optimization problem whose classical Lagrangian formulation takes the form

$$f'_\varepsilon(x; p) = \operatorname*{Sup}_{x^* \in \operatorname{dom} f^*} \inf_{u \geqslant 0} \{\langle p, x^*\rangle - u(f(x) + f^*(x^*) - \langle x, x^*\rangle - \varepsilon)\}.$$

In this way, the existence and a characterization of the approximate second order directional derivative

$$f''_\varepsilon(x_0; p, d) = \lim_{t \to 0^-} \{f'_\varepsilon(x_0 + td; p) - f'_\varepsilon(x_0; p)\}/t$$

can be obtained using theorem 4.1:

$$\begin{aligned} f''_\varepsilon(x_0; p, d) &= \operatorname*{Min}_{u \in M(x_0)} \operatorname*{Max}_{x^* \in \partial_\varepsilon f(x_0)_p} \mu\{\langle d, x^*\rangle - f'(x_0; d)\} \\ &= \operatorname*{Max}_{x^* \in \partial_\varepsilon f(x_0)_p} \operatorname*{Min}_{\mu \in M(x_0)} \mu\{\langle d, x^*\rangle - f'(x_0; d)\} \end{aligned}$$

where $\partial_\varepsilon f(x_0)_p = \{x^* \in \partial_\varepsilon f(x_0)/f'_\varepsilon(x_0; p) = \langle p, x^*\rangle\}$ and $M(x_0) = \{\mu \geqslant 0/f'_\varepsilon(x_0; p) = r(\mu)\}$ with $r(\mu) = \mu\{f(x_0 + \mu^{-1}p) - f(x_0) + \varepsilon\}$ if $\mu > 0$ and $r(0) = \operatorname{Sup}\{\langle p, x^*\rangle/x^* \in \operatorname{dom} f^*\}$.

This formula was given by Lemarechal and Nurminski [17] assuming that f and f^* are finite. Recently, Auslender [3] has obtained the same result under the assumption that f is a finite function defined on $\mathbb{R}^n$.

With the same methodology, that is using theorem 4.1, the right derivative at $\varepsilon_0 > 0$ of the function $\varepsilon \to f'_\varepsilon(x_0; p)$ and the directional derivative at $p_0 \in X$ of the function $p \to f'_\varepsilon(x_0; p)$ can be easily obtained. At present, the authors are developing new applications in this area.

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