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The Couette–Taylor Problem

With 58 Illustrations

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I

Introduction

I.1 A paradigm

About one hundred years ago, Maurice Couette, a French physicist, designed an apparatus consisting of two coaxial cylinders, the space between the cylinders being filled with a viscous fluid and the outer cylinder being rotated at angular velocity Ω_2. The purpose of this experiment was, following an idea of the Austrian physicist Max Margules, to deduce the viscosity of the fluid from measurements of the torque exerted by the fluid on the inner cylinder (the fluid is assumed to adhere to the walls of the cylinders). At least when Ω_2 is not too large, the fluid flow is nearly laminar and the method of Couette is valuable because the torque is then proportional to $\nu\Omega_2$, where ν is the kinematic viscosity of the fluid. If, however, Ω_2 is increased to a very large value, the flow becomes eventually turbulent. A few years later, Arnulph Mallock designed a similar apparatus but allowed the inner cylinder to rotate with angular velocity Ω_1, while $\Omega_2 = 0$. The surprise was that the laminar flow, now known as the *Couette flow*, was not observable when Ω_1 exceeded a certain "low" critical value Ω_{1c}, even though, as we shall see in Chapter II, it is a solution of the model equations for any values of Ω_1 and Ω_2. In fact, Mallock did not find a value of Ω_1 at which the Couette flow would be stable. However, as shown by Geoffrey I. Taylor in a celebrated paper of 1923 [Tay], an instability occurs for the Couette flow when Ω_1 exceeds the value Ω_{1c}, which is lower than any of the values of Ω_1 used by Mallock. The mathematical meaning of Taylor's finding is that for certain initial conditions, even very close to the solution corresponding to the Couette flow for the model equation, the evolution of the system with time does not tend back to the Couette solution but instead relaxes to a new, more complicated state.

A flow such that any small perturbation tends back to it as time tends to infinity will be called *stable* in this book. Observable flows need to be stable in this sense. In order to perform the calculations, Taylor assumed that the apparatus has *infinite length*, which is of course an idealization of the real apparatus. He found that the first instability occurred via a stationary and axisymmetric disturbance of the Couette flow, but for an ideal apparatus and with given nonzero axial periodicity. Not only was the value of Ω_{1c} found by Taylor very close to its "experimental" value, but also the pattern of the solution associated with this type of disturbance was similar to that observed in the experiment. This flow is now called the *Taylor vortex flow*. It is organized as a superposition of horizontal vortices of length exactly equal to half of the axial period of the basic disturbance, as shown in Figure I.1. Taylor's work was an improvement on the work of Rayleigh [Ray], who dealt with *inviscid* rotating fluids and gave a criterion of stability for co-rotating cylinders. Since these pioneering works, the instability of Couette flow and Taylor vortex flow, now known as the "Couette-Taylor problem", has been greatly studied, both experimentally and theoretically. Why did such an apparently simple experimental device become so popular?

In fact, observers were astonished to see the rich variety of patterns that occur, for instance, when the rotation rate of the inner cylinder is increased. One immediate (and striking) feature of these patterns is their degree of "self-organization", that is, of spatial as well as temporal symmetries (e.g., the Taylor vortex flow is axisymmetric, periodic along the axis of the cylinders and stationary). However, when Ω_1 is increased, the pattern becomes

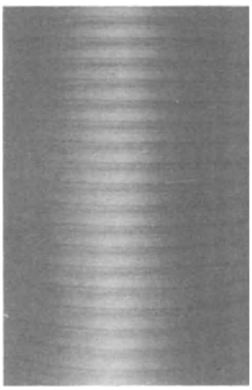


FIGURE I.1. The Taylor vortex flow.

more and more complicated, breaking more and more symmetries in space and time, leading at last to a *turbulent regime*, itself containing large-scale structures with, again, many symmetries. This model problem then appears as an example of a system that progressively approaches turbulence, which is still one of the biggest challenges for scientists and engineers in fluid mechanics. When both cylinders are rotated, but in opposite directions, even richer "routes" to turbulence are observed.

An attractive property of this experiment is the large number of different kinds of flows that can be observed, despite the simplicity of the apparatus. Many very good experimental results were obtained in the 1960s (see, for instance, Donald Coles [Col]). However, it is only by recent progress in measuring and analyzing velocities of fluid particles, about 15 years ago, that geometric and dynamic properties of many different flows can be clearly determined. Such results were achieved thanks to very systematic studies (Fourier spectrum, Poincaré maps, symmetries, ...). This experimental progress came together with new progress in the theory of dynamical systems, themselves greatly impulsed by the development of computer facilities. It appeared that this specific problem by itself could manage to motivate scientific meetings mixing mathematicians, mechanicians, and physicists (for instance, the "Taylor vortex working parties", which were held every two years in different parts of the world). It is a remarkable fact that comparisons between experiments and predictions from theoretical arguments (neglecting end effects) are reasonably good, provided the *height of cylinders is large enough*, even in such situations where complicated dynamics occur close to the points of onset of instability for Couette flow.

Our point of view emphazises *bifurcation and symmetry breaking* arguments. Under the word *bifurcation*, we understand a structural change in the observed flow when one or more parameters are progressively varied. For instance, when a parameter crosses a critical value, a flow with certain symmetries may lose some of these symmetries, or a steady flow may become time dependent (this is a symmetry breaking too, since the new flow is no longer invariant under shifts in time). The main point is that these changes are continuous—there is no jump in any measurable quantity when the parameter crosses the critical value, if the bifurcating flow is stable with respect to small perturbations when it appears. If it is not stable, this continuity cannot be observed experimentally, even though it is still valid for the unstable flow. In other words, a bifurcation occurs for instance when a branch of steady or time-periodic solutions (as function of a parameter) loses its *local* unicity—it intersects another branch of different solutions (it might be a manifold of dimension greater than 1), and the crossing point is called the bifurcation point. The analysis of such bifurcations is the main tool for the understanding of the mechanism of pattern formation and appearance of complicated dynamics in fluid flows.

Our study is of course *nonlinear* but *local*. This means that our analysis stays valid in a neighborhood of the linear threshold of instability of some basic flow (itself depending on parameters). This differs from studies using energy methods as developed for instance in Joseph's book [Jo], which gives *global* stability results but says nothing about the dynamics in case of instability. The method we use extensively rests on the "amplitude equations". It thus differs from the method used in the book of Golubitsky, Stewart, and Schaeffer [Go-St-Sc] (where a full chapter is concerned with the Couette-Taylor system), where the Lyapunov-Schmidt method is coupled with symmetry arguments (see Remark 3 in Chapter II). Our method has the advantage to *not a priori specialize the flow we are looking for* and to lead directly to the stability analysis. This technique appeared in the 1960s, generalizing the Landau equation [Lan]; it was for instance extensively used by the Stuart's school (see references in [St]). In fact, owing to the ignorance at that time of the full justification of the method, it was often limited to the computation of cubic terms for not too degenerate cases. The introduction of *center manifold theory* and of normalization technique (*normal forms*) allows us to write amplitude equations up to an arbitrary order and mathematically to prove the existence and determine the stability of many different interesting flows. One of the aims of this book is to show how these tools apply and to give fruitful qualitative results without explicitly numerically computing the coefficients of the amplitude equations. However, when these coefficients are computed, the dynamics given by this ordinary differential system fits surprisingly well for a large set of values of the parameters, even quantitatively, with the flow observed experimentally.

This book is an attempt to interest theoreticians as well as applied scientists, giving to the first an example of nice applications of sophisticated tools of analysis and to the second some ideas of techniques that can easily be used on such a problem to give predictions on flows we might expect to observe for some range of the parameters. The arguments we use apply to many nonlinear stability problems (dissipative and possessing some symmetries), so we can say that the Couette-Taylor problem is a *paradigm*. The Couette-Taylor problem has given rise to a huge number of publications since the pioneering works mentioned above. For those who are interested in further study of this problem, we mention the book of Koschmieder [Ko], which specializes more on the experimental side, and the very complete bibliography of Tagg [Ta] on this subject.

I.2 Experimental results

At this point of the introduction, the reader would probably (and hopefully!) like to know more about those flows discussed above. We now have to introduce the various parameters that appear in the Couette-Taylor problem. We have already defined Ω_1 and Ω_2. Let R_1 and R_2 be the inner and outer radii, respectively, and $\mathcal{R}_i, \mathcal{R}_o$ denote the inner and outer Reynolds numbers. More precisely, we set

$$\mathcal{R}_i = \frac{R_1(R_2 - R_1)\Omega_1}{\nu}, \qquad \mathcal{R}_o = \frac{R_2(R_2 - R_1)\Omega_2}{\nu}.$$

We shall discuss in more detail the numbers arising from a nondimensionalization of the equations in the first section of Chapter II. It is enough for the moment to know that the two Reynolds numbers are, with the length of the cylinders, the *characteristic* parameters of the problem (but an other choice of such parameters could be made). The easiest way to vary these numbers is of course by varying the angular velocities Ω_1 and Ω_2. In a recent work, Andereck, Liu, and Swinney [An-L-Sw] explored a large number of values of $\mathcal{R}_i$ and $\mathcal{R}_o$, with an apparatus of radius ratio 0.883. They obtained a diagram, which is reproduced in Figure I.2. This diagram shows regions in the $(\mathcal{R}_i, \mathcal{R}_o)$, plane, where the indicated flows have been observed. It assumes that the cylinders are long enough. In this experiment the length was 30 times the thickness $R_2 - R_1$. In this diagram, the only stationary flow apart from the Couette flow is the Taylor vortex flow, shown in Figure I.1. The spiral flow (Figure I.3) and wavy vortex flow (Figure I.4) are time periodic and assume the form of *rotating waves*, i.e., of an azimuthally periodic pattern rotating like a solid. The spiral flow is also a *traveling* wave in the direction of the axis of the cylinders (think of a rotating "barber's pole"). The "modulated waves" show a *modulation* of the waves in the

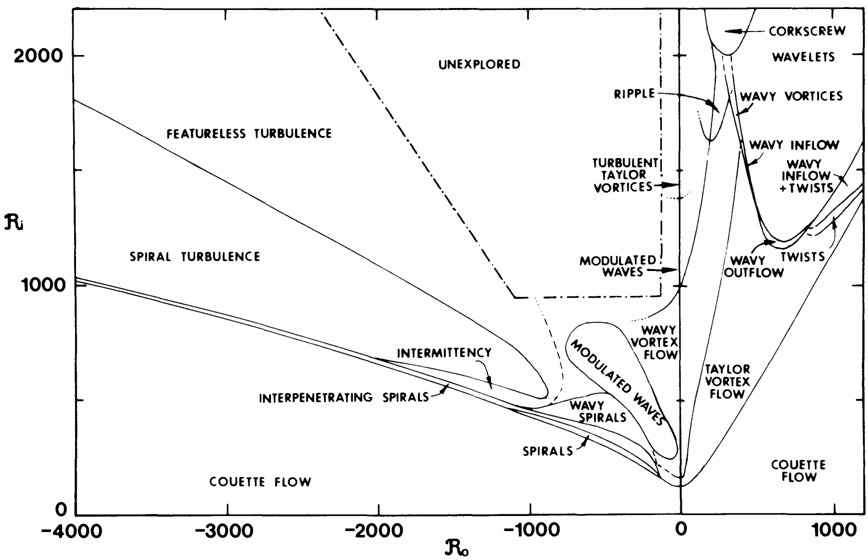


FIGURE I.2. Experimental stability diagram by [An-L-Sw].

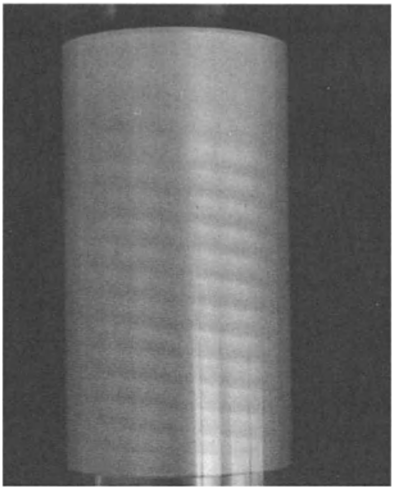


FIGURE I.3. Spiral flow.

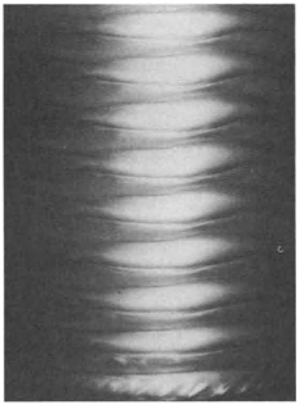


FIGURE I.4. Wavy vortex flow.

wavy vortex flow (quasi-periodic flow). Weak turbulence develops from the modulated waves (Figure I.5). We shall discuss in detail "interpenetrating spirals" in Chapter IV and the flows that occur after the loss of stability of the Taylor vortex flow, when $\mathcal{R}_o \geq 0$, in Chapter VI. We mention a recent review of experimental works on the subject made by Cognet [Cog]. In fact, some of the flows mentioned above are not immediately observed when one

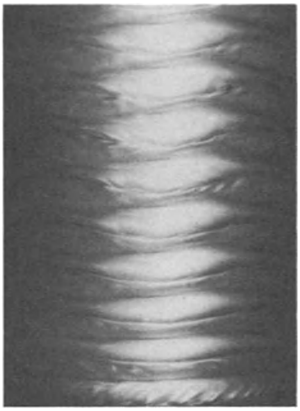


FIGURE I.5. Modulated waves.

performs the experiment. For instance, for values of the parameters where spiral waves are expected, one actually sees a flow with "defects," and one has to wait quite a long time before these defects disappear. The typical defects for the spiral regime are either sinks or sources; one of these is shown in Figure I.6. We show in Chapter VII how to obtain such solutions mathematically, when the axial periodicity assumption is relaxed. By now the reader should be convinced of our assertion that the Couette-Taylor problem shows an astonishing variety of flows with different spatio-temporal patterns, eventually leading to turbulence.

The diagram in Figure I.2 has several remarkable features, which we now enumerate: (i) Couette flow is stable when $\mathcal{R}_i$ is lower than a critical value which depends on $\mathcal{R}_o$; (ii) when $\mathcal{R}_o$ is positive or slightly negative, Couette flow loses stability and Taylor vortex flow develops, as mentioned in the previous section; (iii) when $\mathcal{R}_o$ is lower than a certain negative critical value, Couette flow loses stability and is replaced by the *nonstationary*, time-periodic spiral flow (or by the interpenetrating spiral flow). Moreover, when $\mathcal{R}_i$ is increased at a fixed value of $\mathcal{R}_o$, one observes a succession of transitions to flows that rapidly lose a simple spatio-temporal dependence. In particular, the spiral flow becomes unstable, leading to the regime of quasi-periodic "interpenetrating spirals", which themselves lead to aperiodic flows (e.g., intermittency). When $\mathcal{R}_o \geq 0$, the Taylor vortex flow becomes unstable and is replaced by various kinds of time-periodic flows, for example, "wavy vortex", "wavy outflow", "twist", etc.

In this book, it will appear that most of these patterns have a very general (one would be tempted to say a "universal") explanation in terms of *spontaneous* symmetry breaking of the initial symmetries of the problem. By

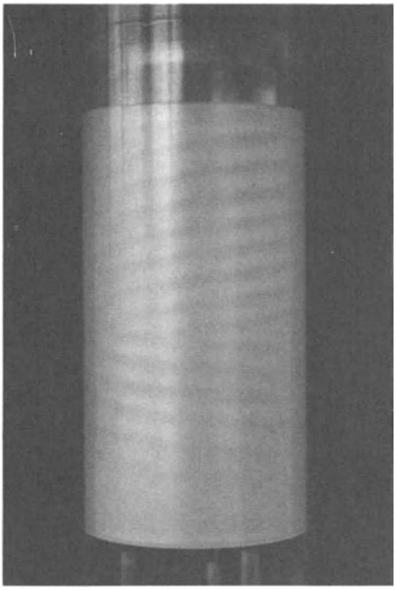


FIGURE I.6. Defect in the spiral regime.

"spontaneous" we mean that the mechanism responsible for the symmetry breaking depends on the conditions of bifurcation from the basic flow (e.g., Couette) but not on external perturbations to this flow. In other words, physical experiments with the same symmetries and the same kind of conditions for the loss of stability of some basic state would exhibit the same kind of patterns when this basic state becomes unstable. The next step is to build a model based as closely as possible on the experimental conditions and satisfying in addition the basic requirements needed for treating our problem. In the next section we make such requirements precise, from the point of view of symmetries.

I.3 Modeling for theoretical analysis

We observed in Section I.1 that in his analysis of the stability of the Couette flow, Taylor made the assumption that the cylinders had infinite length. Why did he make such an unrealistic hypothesis? In fact, keeping a bounded domain would make the mathematical analysis extremely complicated, even for computing the solution corresponding to Couette flow, because it would require solving the singularity on the ends of the cylinders. On the other hand, we shall see in the next chapter that under the infinite cylinder assumption, the Couette solution has a very simple, analytical form. Moreover, this assumption has the great advantage of allowing a Fourier analysis of the linear perturbation of the Couette solution along the axis of the cylinders. The reason is that with infinite cylinders the problem is *invariant* under translations along the z-axis. More precisely, let us write $U(r, \theta, z, t)$ for the perturbations of the Couette solution, where (r, θ) are polar coordinates in a plane perpendicular to the z-axis. Then we can write, formally at least,

$$U(r, \theta, z, t) = \sum_{m=-\infty}^{+\infty} e^{im\theta} \int_{-\infty}^{+\infty} e^{iz\alpha} \tilde{U}(r, m, \alpha, t)\, d\alpha. \tag{I.1}$$

If one looks for linear stability, i.e., if one retains only the *linear* terms in the equations for U, then, replacing U by expression (I.1) in these equations, the (θ, z) dependence is eliminated and one is led to a much simpler system for $\tilde{U}$. These statements will be made rigorous in Chapter II. Elementary solutions of the form $e^{i(m\theta+\alpha z)}\tilde{U}$ are called eigenmodes of the perturbation. They correspond to periodic perturbations in both azimuthal (i.e., θ) and axial (i.e., z) directions, with m azimuthal waves and axial periodicity $2\pi/\alpha$ (α is the *axial wave number*). There is, however, a counterpart to this simplification; because we assume infinite cylinders, *any* value of α may give an eigenmode. This means, in particular, that if an eigenmode is unstable, then, in general, a *continuum* of eigenmodes are unstable simultaneously. This complicates considerably the bifurcation analysis, because the "usual" techniques for dealing with such a problem assume that only a *finite* number of modes are unstable near threshold. We shall partially solve this difficulty in Chapter VII. One way to overcome it is by assuming a given periodicity h along the z-axis. This is not much more realistic than the simple infinite cylinder assumption, but it has the great advantage of assuming for the eigenmodes that $\alpha = n\alpha_0$, $n \in \mathbb{Z}$, $\alpha_0 = 2\pi/h$. Hence, only finitely many modes can be simultaneously unstable, but of course the theory says nothing about perturbations with other α's. This is nevertheless the point of view we have adopted in this book (except in Chapters VII and VIII). Taylor's analysis came back to assume that the system is periodic in z, with wave number α_c equal to the value of the critical α at threshold. We shall modify this by imposing the periodicity h, which will play an important role in Chapter IV. Now, the reader could ask whether this model has anything at all to do with the original experiment. We have already indicated that, in Taylor's case, this was quite successful. More generally, the hope is that in the case of an apparatus with large aspect ratio (length very large versus thickness of the domain), the flow near the middle of the cylinders is only weakly influenced by the effects due to the ends of the cylinders or, at least, influenced after a long enough time.

I.4 Arrangements of topics in the text

Chapter II presents the linear stability theory of the Couette flow and the basic tools for further nonlinear analysis, including the function spaces, the center manifold, and the normal form theorems. Even though our notations mostly follow a somewhat mathematical style, we try to present these tools in a comprehensive way for researchers not specially trained in this field; for instance, physicists may expect to recognize "adiabatic elimination" or "slaving principle" in what we call the "center manifold reduction".

Chapter III is the core of the book. It shows in detail the bifurcation analysis to Taylor vortices, spiral waves, and ribbons (the standing waves) as the first elementary bifurcating flows as the Couette flow becomes unstable. This chapter, by simply using group symmetry properties easily deduced from the structure of amplitude equations, gives the geometric structure of such flows. Numerical values are also given. Some simple degenerate (codimension two) situations due to nonlinear coefficients are also studied in this chapter.

Chapter IV studies the two main codimension two cases due to the linear operator. The first is the steady axisymmetric mode interacting with an oscillating nonaxisymmetric mode. The second is the interaction of two nonaxisymmetric oscillating modes. These cases are both very rich, giving rise to various interesting structures, and numerical values of the coefficients are given too. Here we reach analytically secondary and tertiary bifurcating solutions, like interpenetrating spiral waves solutions.

Chapter V deals with imperfections. The first imperfection is due to cylinders not being well centered. This leads to changes in the threshold for instability, and also may significantly change the type of flow (the azimuthal wave number for instance) that is observed in the counterrotating case! Another imperfection we study is when one adds a little mass flux through the cross section of the cylindrical domain, and new flows may be observed. Finally, the last imperfection we study is due to nonuniform rotation rates having a small time-periodic oscillation. These imperfections do not represent all possibilities, since one can imagine other situations, for instance, perturb the shape of the cylinders like in [Si]. However, we think that these examples are typical of what happens in a physical problem (with respect to symmetry breaking).

Chapter VI deals with secondary bifurcations in general. The problem is that primary bifurcating flows belong to group orbit of solutions. So, the theory developed in previous chapters does not apply and has to be adapted near group orbits (one- or two-dimensional here). By our analysis we easily reveal such regimes as wavy vortices, wavy inflow or outflow boundaries, twisted vortices, wavy spirals, modulated wavy vortices, and many other flows not necessarily observed yet. The idea is to present a tool general enough for the reader to be able to use in other circumstances. Here, the advantage is that we immediately have physical illustrations of the new computed solutions.

Chapter VII is an attempt to release the periodicity assumption in the axial coordinate. To be able to conduct an analysis (the difficulty being the occurrence of a continuous spectrum), we must specify the type of temporal behavior we wish to find. First, we look for all steady solutions in the case corresponding to the occurrence of Taylor vortices. We are able to show the existence of many other solutions not periodic in the axial coordinate z—for instance, quasi-periodic solutions—and solutions that are periodic at both infinities but with a phase shift between them and a little region in the middle where the flow is close to the Couette flow. Second, we look for all time-periodic bifurcating flows in the case when spirals might occur. Now, we find other very interesting solutions (no longer periodic in z); for instance, we can prove that a defect solution exists that connects two symmetric spiral waves regimes at both infinities (this is often observed in experiments when cylinders are counter-rotating, near threshold). We relate these mathematical results with heuristically obtained envelope equations of *Ginzburg-Landau* type which add time behavior (not necessarily steady or time-periodic).

Chapter VIII deals with the small gap case. Even though it seems to look like the plane Couette problem in the limit, different small gap limits are possible, depending on the rotation rates. We present two different cases and give some new results. In particular, we give numerical values of the coefficients of Ginzburg-Landau types of envelope equations describing the structure and the dynamics of the tridimensional flow near criticality.

To conclude, let us mention that this book is a result of several years of research and published works by a group of researchers comprised of Yves Demay, Patrice Laure, younger researchers such as Françoise Signoret and Rabah Raffaï working on their theses, and the authors of this book. Their works are, of course, referenced in the bibliography. In addition, it should be clearly said that Y. Demay provided the impetus for our first numerical computations (near 1983) of the coefficients in the amplitude equations and that these were greatly improved for high codimension problems by P. Laure, who was able to connect formal computation schemes with numerical ones. It is clear for us that this book benefited much from their contributions.

Finally, it is our pleasure to warmly acknowledge M. Golubitsky, E. L. Koschmieder, P. Manneville, H. Swinney, and R. Tagg for the help and interest they gave in the realization of this book. Special thanks to H. Swinney and R. Tagg, who kindly allowed us to use their pictures in Figures I.1, I.2, I.4, I.5, and IV.7, and to G. Cognet, who procured for us a Couette-Taylor apparatus with which Figures I.3 and I.6 were produced.

II

Statement of the Problem and Basic Tools

The aim of this chapter is, in Section II.1, to specify notation and parameters including the small gap case. In the literature, several different choices can be found, and quantitative comparisons of results are often difficult. Here we take the usual choice of nondimensionalization and parameters. In Section II.2 we describe the frame of all further analysis of our problem (except in Chapter VII where the spatial periodicity condition is released, and in Chapter VIII where a specific study is made in the small gap case). The idea is to be able to use ordinary differential equations techniques for the initial value problem. For this purpose, the simpler way is to fix the frame of the differential equation in some Hilbert space where the solution is differentiable in time. An element of this Hilbert space is then a velocity vector field of fluid particles. Therefore, to obtain the streamlines one has to find integral lines of this vector field. Section II.2 may be skipped by readers who are not very motivated by mathematical justifications of the analysis done in further chapters. In Section II.3, we give general results on the linear stability of the Couette flow, and in Section II.4 we introduce the general tools used in the following chapters, i.e., the Center Manifold Theorem combined with symmetry and normal forms arguments.

II.1 Nondimensionalization, parameters

II.1.1 Basic formulation

In the standard problem considered for the moment, we have a viscous incompressible fluid filling the domain $Q = \Sigma \times \mathbb{R}$ between two concentric rotating infinite cylinders where we denote by Σ the cross section (see Figure II.1). Equations describing the flow are the Navier-Stokes equations on Q:

$$\begin{cases} \dfrac{\partial V}{\partial t} + (V.\nabla)V + \dfrac{1}{\rho}\nabla_p &= \nu\Delta V + f, \\ \nabla.V &= 0. \end{cases} \tag{II.1}$$

Here ρ is the (constant) density, ν the kinematic viscosity, p the pressure, and V the velocity vector of fluid particles. In (II.1) we introduced f, which is a density of external forces per unit of mass. In the standard problem, there are *no external forces* since gravity can be incorporated into the pressure term p. The notation ∇ denotes the gradient operator, (∇.) the divergence, and Δ the Laplace operator, all acting on scalar fields as well as on vector fields. Functions V and p depend in general on (x, t), where $x \in Q$ is often written $x = (y, z)$ where $y \in \Sigma$, $z \in \mathbb{R}$. In cylindrical coordinates (r, θ), the cross section Σ is defined by $R_1 < r < R_2$, $\theta \in \mathbb{T}^1$ where $\mathbb{T}^1$ denotes the circle $\mathbb{R}/2\pi\mathbb{Z}$ and R_1 and R_2 denote the inner and outer radii of cylinders respectively. Components of $V(x, t)$ are written (v_r, v_θ, v_z) in cylindrical coordinates, so the no-slip boundary conditions for fluid particles on the cylinders can be expressed as

$$v_r = v_z = 0, \quad v_\theta = \Omega_j R_j \qquad \text{at } r = R_j,\ j = 1, 2, \tag{II.2}$$

where Ω_j, $j = 1, 2$, respectively, denote the angular velocities of the inner and outer cylinders.

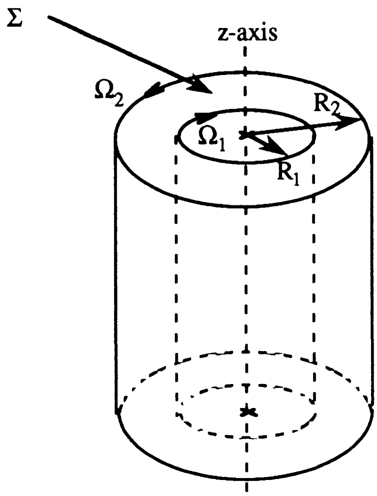


FIGURE II.1. The geometry of the problem.

II.1.2 Nondimensionalization

We reduce the number of parameters by choosing suitable scales. The fundamental assumption is that $\Omega_1 \neq 0$; we assume from now on $\Omega_1 > 0$; Ω_2 may be positive, zero, or negative. Several choices are possible. One possibility would be to choose scales such that dimensionless parameters occur in the simplest way, for instance, $\Omega_1^{-1}, R_1, R_1\Omega_1$ as time, length, and velocity scales, respectively. In fact, this is not the standard choice, because in general this does not lead to quantities and coefficients of order one in amplitude equations. So, we choose instead the following, better adapted, scales:

$$\begin{aligned} &d = R_2 - R_1 \text{ (gap)}, \quad R_1\Omega_1 \text{ (velocity of the inner cylinder)}, \\ &d^2/\nu \text{ (viscous time)}, \quad \text{and} \quad \rho\nu R_1\Omega_1/d, \end{aligned} \tag{II.3}$$

respectively for length, velocity, time, and pressure. Now the following three dimensionless parameters appear:

$$\Omega = \Omega_2/\Omega_1, \qquad \eta = R_1/R_2, \qquad \mathcal{R} = R_1\Omega_1 d/\nu, \tag{II.4}$$

where $\mathcal{R}$ is called the Reynolds number (or the inner cylinder Reynolds number). In what follows we again denote *the dimensionless quantities* by *the same symbols.* System (II.1), with no external forces, now reads

$$\left\{ \begin{aligned} \frac{\partial V}{\partial t} + \mathcal{R}(V.\nabla)V + \nabla p &= \Delta V, \\ \nabla . V &= 0, \end{aligned} \right. \tag{II.5}$$

and the boundary conditions become

$$\mathrm{v_r} = \mathrm{v_z} = 0, \ \mathrm{v}_\theta = 1, \ \text{at } \mathrm{r} = \eta/(1-\eta), \quad \text{and} \quad \mathrm{v}_\theta = \Omega/\eta \quad \text{at } \mathrm{r} = 1/(1-\eta). \tag{II.6}$$

Remark. It may be helpful to write (II.5) in cylindrical coordinates explicitly (see, for instance, Germain [Ge]):

$$\left\{ \begin{aligned} \frac{\partial v_r}{\partial t} &= \Delta v_r - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{v_r}{r^2} - \frac{\partial p}{\partial r} \\ &\quad -\mathcal{R}\left[v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right], \\ \frac{\partial v_\theta}{\partial t} &= \Delta v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2} - \frac{1}{r}\frac{\partial p}{\partial \theta} \\ &\quad -\mathcal{R}\left[v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right], \\ \frac{\partial v_z}{\partial t} &= \Delta v_z - \frac{\partial p}{\partial z} - \mathcal{R}\left[v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right], \end{aligned} \right. \tag{II.7}$$

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0, \tag{II.8}$$

where

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

II.1.3 Couette flow and the perturbation

The main fact is that we have an exact solution of (II.5–II.6), called the *Couette solution*, which has the form of a pure azimuthal flow (streamlines are concentric circles, see Figure II.2):

$$V^{(0)} = (0, v_\theta^{(0)}(r), 0), \qquad p^{(0)} = \mathcal{R}\int \frac{[v_\theta^{(0)}(r)]^2}{r}\,dr,$$

$$v_\theta^{(0)}(r) = Ar + \frac{B}{r}, \qquad A = \frac{\Omega - \eta^2}{\eta(1+\eta)}, \qquad B = \frac{\eta(1-\Omega)}{(1-\eta)(1-\eta^2)}. \tag{II.9}$$

We now define the perturbation (U, q) where $U = (u_r, u_\theta, u_z)$ by

$$V = V^{(0)} + U, \qquad p = p^{(0)} + q, \tag{II.10}$$

so the system (II.5–II.6) becomes

$$\left\{\begin{array}{rcl} \dfrac{\partial U}{\partial t} &=& \Delta U - \mathcal{R}[(V^{(0)}.\nabla)U + (U.\nabla)V^{(0)} + (U.\nabla)U] - \nabla q, \\ \nabla . U &=& 0, \end{array}\right. \tag{II.11}$$

$$u_r = u_z = u_\theta = 0, \quad \text{at } r = \eta/(1-\eta), r = 1/(1-\eta). \tag{II.12}$$

We observe that we have homogeneous boundary conditions on $\partial Q = \partial\Sigma \times \mathbb{R}$ and that three parameters enter in the problem, namely, $\mathcal{R}, \eta, \Omega$. When the outer cylinder is at rest, $\Omega = 0$.

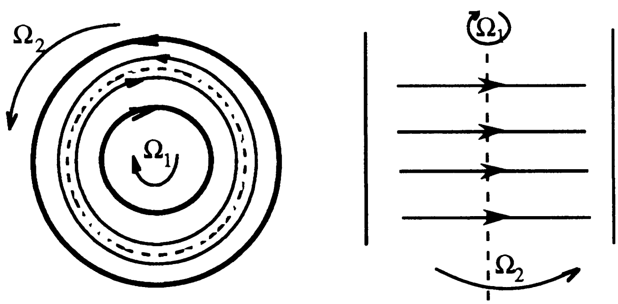


FIGURE II.2. Couette flow.

II.1.4 Symmetries

The standard problem under study has several fundamental invariance properties. It is well known that, owing to the independence of the laws of mechanics with respect to the reference frame, the Navier-Stokes equations are invariant under the group of rigid motions (a product of the orthogonal group O(3) by the group of translations in $\mathbb{R}^3$) (see, for instance, [Ge]). Let us define linear transformations $\mathbf{T}$ and τ as follows for any $g \in O(3)$, any x, and any a in $\mathbb{R}^3$:

$$\left\{\begin{array}{rcl} [\mathbf{T}(g)U](x) & = & gU(g^{-1}x), \\ [\tau(a)U](x) & = & U(x+a) \end{array}\right. \quad \text{for the velocity vector field } U, \tag{II.13}$$

$$\left\{\begin{array}{rcl} [\mathbf{T}(g)q](x) & = & q(g^{-1}x), \\ [\tau(a)q](x) & = & q(x+a) \end{array}\right. \quad \text{for the pressure scalar field } q. \tag{II.14}$$

Then the system (II.11) *commutes* with these linear operators. In particular, this means that we have

$$\left\{\begin{array}{rcl} \mathbf{T}(g)[(U.\nabla)V] & = & ([\mathbf{T}(g)U].\nabla)[\mathbf{T}(g)V], \\ \mathbf{T}(g)[\Delta U] & = & \Delta[\mathbf{T}(g)U], \\ \mathbf{T}(g)[\nabla.U] & = & \nabla.[\mathbf{T}(g)U], \\ \mathbf{T}(g)[\nabla q] & = & \nabla[\mathbf{T}(g)q] \end{array}\right. \tag{II.15}$$

and similar relations for τ.

The geometry of the problem (boundary conditions) and the symmetries of Couette flow imply that the system (II.11–II.12) is in fact invariant under the action of the subgroups of *translations along the z-axis, reflections $z \to -z$ and rotations about the z-axis.* In cylindrical coordinates, these actions on the velocity field are defined for a and $\varphi \in \mathbb{R}$ by

$$\left\{\begin{array}{rcl} [\tau_a U](r,\theta,z) & = & [u_r(r,\theta,z+a), u_\theta(r,\theta,z+a), u_z(r,\theta,z+a)], \\ [\mathbf{S}U](r,\theta,z) & = & [u_r(r,\theta,-z), u_\theta(r,\theta,-z), -u_z(r,\theta,-z)], \\ [\mathbf{R}_\varphi U](r,\theta,z) & = & [u_r(r,\theta+\varphi,z), u_\theta(r,\theta+\varphi,z), u_z(r,\theta+\varphi,z)], \end{array}\right. \tag{II.16}$$

where we note that the coordinate frame $(\vec{i}(\theta), \vec{j}(\theta), \vec{k})$ is invariant under these operators.

II.1.5 Small gap case

We want to consider the asymptotic limit of equations (II.6–II.8), when the gap $d = R_2 - R_1$ is small compared with R_1, and Ω_1 is large. We feel, of course, that at this limit we must obtain a flow between two parallel plates, i.e., a planar Couette flow problem. This is exactly what happens if we pass to the limit $\eta \to 1$, after changing the scale in r by setting

$$r = \bar{R}/d + x, \quad \text{where } \bar{R} = {}^1\!/\!_2(R_1 + R_2), \quad x \in (-{}^1\!/\!_2, {}^1\!/\!_2)\,. \tag{II.17}$$

In fact, this is a bad choice for studying instabilities since, as is well known, the planar Couette flow is linearly stable (see, for instance, Drazin and Reid [Dr-Re pp. 212–216]). The idea is now to catch an additional term at the limit. Several possible choices are possible, which lead to limits different from the planar Couette problem as $\eta \to 1$.

II.1.5.1 Case when the average rotation rate is very large versus the difference $\Omega_1 - \Omega_2$

Let us introduce the average rotation rate $\bar{\Omega} = 1/2(\Omega_1 + \Omega_2)$. The idea is first to choose a new frame, rotating with angular velocity $\bar{\Omega}$. This adds the Coriolis term $2\bar{\Omega} \times V$ in the left-hand side of the momentum equation (II.1) and changes the boundary conditions (II.2) into $v_\theta = (\Omega_j - \bar{\Omega})R_j$, at $r = R_j$, $j = 1, 2$. Notice that the entrainment inertial forces are incorporated into the gradient of the pressure. The chosen scales for length and time are the same as in (II.3), but we now choose ν/d and $\rho\nu^2/d^2$ for velocity and pressure scales, respectively. This choice of scales, which follows Nagata [Na 86], is good when $\Omega_2 \geq 0$, i.e., when both cylinders rotate in the same direction. If it is not the case, one makes another choice of scale like that in Tabeling [Tab] (see Section II.1.5.2).

The Couette flow is now such that $v_\theta^{(0)} = -\mathcal{R}'x + \mathrm{O}(1 - \eta)$ with the following new dimensionless parameters:

$$\mathcal{R}' = \frac{(\Omega_1 - \Omega_2)\bar{R}d}{\nu}, \qquad \Omega' = \frac{2\bar{\Omega}d^2}{\nu}. \tag{II.18}$$

If we set $y = \bar{R}\theta$ and use (II.17), in the rotating frame the system satisfied by the perturbation reads

$$\left\{\begin{aligned} \frac{\partial U}{\partial t} &= \Delta U - \Omega' \times U + \mathcal{R}'\left[x\frac{\partial U}{\partial y} + u_r\vec{j}\right] - (U.\nabla)U - \nabla q, \\ \nabla.U &= 0, \\ U|_{x=\pm 1/2} &= 0, \end{aligned}\right. \tag{II.19}$$

in the domain $-\frac{1}{2} < x < \frac{1}{2}$, $(y, z) \in \mathbb{R}^2$, where $\vec{j}$ is the unit vector along the y-axis. We observe that (II.19) is different from the planar Couette problem, due to the occurrence of the Coriolis term (see Figure II.3).

The symmetries of the new system (II.19) are represented by τ_a and $\mathbf{S}$ as in (II.16), but $\mathbf{R}_\varphi$ is now replaced by the translation along y, τ'_b defined by

$$[\tau'_b U](x, y, z) = [u_r(x, y + b, z),\ u_\theta(x, y + b, z),\ u_z(x, y + b, z)].$$

There is an additional symmetry due to the simple form of the Couette solution, which is invariant under the transformation $(x \to -x,\ v_\theta \to -v_\theta)$. The system (II.19) is now *symmetric with respect to the z-axis*, this symmetry being represented by $\mathbf{S}'$ defined by

$$[\mathbf{S}'U](x, y, z) = [-u_r(-x, -y, z),\ -u_\theta(-x, -y, z),\ u_z(-x, -y, z)]. \tag{II.20}$$

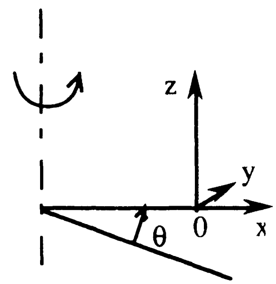


FIGURE II.3. System of coordinates in the small gap approximation.

Finally, in this small gap approximation the system commutes with the four linear representations $\tau_a, \mathbf{S}, \tau_b'$ and $\mathbf{S}'$.

II.1.5.2 Case when the rotation rate of the inner cylinder is very large

Let us now consider the case when the inner and possibly the outer Reynolds numbers $R_j\Omega_j d/\nu$, $j = 1, 2$, are very large when η is close to 1. The choice of scale we present follows Tabeling [Tab]. Contrary to the previous limiting case, there is no additional symmetry. Starting with (II.17) as above, we now take a different scale along the azimuthal coordinate y. For components of the velocity, we choose a different scale in such a way that $\nabla.V$ is unchanged. More precisely, we set

$$\left\{\begin{array}{l} \theta = y\sqrt{d/2R_1}, \qquad \text{scale of time } \dfrac{d^2}{\nu} \\ \text{scale of the } r\text{- and } z\text{-components of velocity } \nu/d \\ \text{scale of the } \theta\text{-component of velocity } (\nu/d)\sqrt{R_1/2d}. \end{array}\right. \tag{II.21}$$

After suppression of terms of order $O(1-\eta)$ in the Navier-Stokes equations (II.7), we have the Couette solution:

$$V^{(0)} = (0, v^{(0)}(x), 0), \quad \text{with } v^{(0)}(x) = (\mathfrak{T}_2 - \mathfrak{T}_1)x + (\mathfrak{T}_1 + \mathfrak{T}_2)/2,$$

where we define our new parameters $\mathfrak{T}_1, \mathfrak{T}_2$ by $\mathfrak{T}_j = (R_j\Omega_j d/\nu)\sqrt{2(1-\eta)}$, $j = 1, 2$ ($\mathfrak{T}_1$ is > 0, while $\mathfrak{T}_2$ may be ≤ 0). Now the perturbation U satisfies

$$\left\{\begin{array}{rcl} \dfrac{\partial U}{\partial t} & = & \tilde{\Delta} U - v^{(0)}(x)\dfrac{\partial U}{\partial y} + \begin{pmatrix} v^{(0)}(x)u_\theta + \dfrac{1}{2}u_\theta^2 \\ -v'^{(0)}(x)u_r \\ 0 \end{pmatrix} \\ & & -(U.\nabla)U - \tilde{\nabla} q + f(y)\vec{\mathbf{j}}, \\ \nabla.U & = & 0. \end{array}\right. \tag{II.22}$$

with the boundary conditions $U|_{x=\pm 1/2} = 0$.

Here $\tilde{\Delta}$ and $\tilde{\nabla}$ are the usual Laplace and gradient operators in the two coordinates x and z and f is an *arbitrary function of* y, which means that the gradient $\tilde{\nabla}$ gives nothing in the y-component of the momentum equation (see Chapter VIII for the details of the proof). It then appears that differentiations with respect to y do not occur in the two important terms $\tilde{\Delta}U$ and $\tilde{\nabla}q$. In fact, it could be seen that the problem is still well posed.

Remark. When $\Omega_2 = 0$, $\mathcal{T}_1^2 = 2\bar{R}\Omega_1^2 d^3/\nu^2$ is called the *Taylor number.*

II.2 Functional frame and basic properties

The reader who is unaware of basic definitions in functional analysis might jump to Section II.3 with a reasonable chance of still understanding the material.

For the moment, we concentrate our study on the classical problem, i.e., when the gap is neither small nor large. The small gap situation is studied in Chapter VIII. The large gap case, i.e., when we take the limit $R_1 \to 0$, while $\Omega_1 R_1^2$ stays constant, was studied by Ovchinnikova [Ov] and will not be considered here. As explained in Chapter I, we are looking for solutions that are h-periodic in $z \in \mathbb{R}$, where h is a length related to the physical height of the cylinders (dimensionalized with the scale d). Let us show how equations for the perturbation may be put into the form of a differential equation lying in a suitable function space $H(Q_h)$:

$$\frac{dU}{dt} = L_\mu U + N(\mu, U). \tag{II.23}$$

II.2.1 Projection on divergence-free vector fields

Let us denote by $I_h = \mathbb{R}/h\mathbb{Z}$ locally identified with the interval $[-h/2, h/2]$ and $L^2(Q_h)$ the closure with respect to the norm of $L^2(\Sigma \times I_h)$ of the set of continuous, h-periodic in z, functions on Q. By definition, we set

$$H(Q_h) = \{U \in [L^2(Q_h)]^3;\ \nabla.U = 0,\ U.n|_{\partial\Sigma\times I_h} = 0\}$$

with the scalar product of $[L^2(Q_h)]^3$. To give a meaning to the trace $U.n|_{\partial\Sigma\times I_h}$ and the divergence $\nabla.U$, see Témam [Té].

The space $H(Q_h)$ is the orthogonal supplement (in $[L^2(Q_h)]^3$) of the space of all ∇q with $q \in H^1(Q_h)$ the Sobolev space of functions belonging, with their first partial derivatives, to $L^2(Q_h)$. In $[L^2(Q_h)]^3$ we introduce the orthogonal projection Π_0 on $H(Q_h)$. This projection is used for eliminating the term ∇q in (II.11) by incorporating the condition $\nabla.U = 0$ and a part of boundary conditions in the space $H(Q_h)$.

The basic idea is to use the following identity for regular h-periodic functions in z:

$$\int_{Q_h} \nabla p.V\,dx + \int_{Q_h} p(\nabla.V)\,dx = \int_{\partial\Sigma\times I_h} p(V.n)\,ds,$$

where n is the exterior normal to Q. This identity shows that for $V \in [L^2(Q_h)]^3$ such that $\nabla.V \in L^2(Q_h)$, then $V.n|_{\partial\Sigma\times I_h}$ can be defined in the dual space of the traces $p|_{\partial\Sigma\times I_h}$ of $p \in H^1(Q_h)$ (see [Té] for details); hence, $V.n|_{\partial\Sigma\times I_h} \in H^{-1/2}(\partial\Sigma \times I_h)$. Now, if $V \in H(Q_h)$, we then see that V is orthogonal to ∇p for any p in $H^1(Q_h)$. A practical way to compute the projection $\Pi_0 V$ on $H(Q_h)$ for any V in $[H^k(Q_h)]^3$, $k \geq 1$, is to write the following decomposition of V:

$$V = U + \nabla\varphi,$$

where one looks for $U \in [H^k(Q_h)]^3$ such that $\nabla.U = 0$ and $U.n|_{\partial\Sigma\times I_h} = 0$. It is then clear that $U = \Pi_0 V$; i.e., this decomposition is orthogonal in $[L^2(Q_h)]^3$. It results that

$$\Delta\varphi = \nabla.V \in H^{k-1}(Q_h) \quad \text{and} \quad \frac{d\varphi}{dn}|_{\partial\Sigma\times I_h} = V.n|_{\partial\Sigma\times I_h} \in H^{k-1/2}(\partial\Sigma \times I_h);$$

hence, φ solves a Neumann problem and is uniquely determined in $H^{k+1}(Q_h)$ up to an additive constant.

Remark. For $k = 0$, one cannot define $V.n|_{\partial\Sigma\times I_h}$, so one has to decompose $\nabla\varphi$ in two parts $\nabla\varphi_1$ and $\nabla\varphi_2$ where $\varphi_1 \in H^1(Q_h)$ solves the Dirichlet problem $\Delta\varphi_1 = \nabla.V \in H^{-1}(Q_h)$, $\varphi_1|_{\partial\Sigma\times I_h} = 0$, and $\varphi_2 \in H^1(Q_h)$ solves the Neumann problem

$$\Delta\varphi_2 = 0, \qquad \frac{d\varphi_2}{dn}|_{\partial\Sigma\times I_h} = (V - \nabla\varphi_1).n|_{\partial\Sigma\times I_h} \in H^{-1/2}(\partial\Sigma \times I_h).$$

II.2.2 Alternative choice for the functional frame

The choice made above for the space $H(Q_h)$ is the simplest possible. However, it may happen that one needs to specify the mass flux of fluid across a section Σ. Owing to the divergence-free condition, this flux (here nondimensionalized)

$$D = \int_{\Sigma} U.n\,ds$$

is independent of z (it may depend on t). To control such a quantity, we notice that one only has to impose ∇p to be periodic in z and not necessarily p itself. As a consequence, we may now write

$$p = p' + az,$$

where p' is periodic as before and a is an undetermined constant. So, in leaving free this coefficient a, we are now able to impose the flux D.

Let us reconsider the following decomposition of V in $[H^k(Q_h)]^3$,

$$V = U + \nabla\varphi$$

with $U \in [H^k(Q_h)]^3$ such that $\nabla.U = 0$, $U.n|_{\partial\Sigma\times I_h} = 0$ and $\int\limits_\Sigma U.n\,ds = 0$.

It results that $\varphi = \varphi' + az$, with φ' h-periodic in z, satisfying the same equations as φ in the previous section. Hence, φ' is determined up to a constant and is in $H^{k+1}(Q_h)$. Now the coefficient a is determined by the following identity:

$$\int\limits_\Sigma V.n\,ds - \int\limits_\Sigma \frac{d\varphi'}{dn}\,ds = a\Sigma.$$

Let us show now that this decomposition of V is *still an orthogonal decomposition in* $[L^2(Q_h)]^3$, and let us define in this case $H(Q_h) = \{U \in [H^k(Q_h)]^3;\ \nabla.U = 0,\ U.n|_{\partial\Sigma\times I_h} = 0 \text{ and } \int\limits_\Sigma U.n\,ds = 0\}$. Indeed, for any U in $H(Q_h)$ and any p such that $\nabla p \in H^k(Q_h)$, we have the following identity (where $\Sigma = \tau_h\Sigma'$):

$$\int\limits_{Q_h} U.\nabla p\,dx = \int\limits_{\partial Q_h} p(U.n)\,ds = \int\limits_{\Sigma\cup\Sigma'} p(U.n)\,ds = \int\limits_\Sigma (U.n)(p|_\Sigma - p|_{\Sigma'})\,ds = 0$$

because $p|_\Sigma - p|_{\Sigma'} = ah$ is constant on Σ.

In all that follows, we use one or the other of the orthogonal projection Π_0 [associated with one or the other choice of space $H(Q_h)$]; this gives the same theoretical results in the view of functional analysis. We shall notice the differences in the structure of the bifurcating solutions and numerical results whenever this happens to be of interest.

II.2.3 Main results for the nonlinear evolution problem

The orthogonal projection Π_0 allows us to define the unbounded linear and quadratic operators, respectively, L_μ and $N(\mu, .)$, depending smoothly on a multiparameter $\mu = (\mathcal{R}, \eta, \Omega) \in \mathbb{R}^3$, as

$$\begin{cases} L_\mu U &= \Pi_0\{\Delta U - \mathcal{R}[(V^{(0)}.\nabla)U + (U.\nabla)V^{(0)}]\}, \\ N(\mu, U) &= -\mathcal{R}\,\Pi_0(U.\nabla)U. \end{cases} \tag{II.24}$$

If we define the domain of L_μ by

$$\mathcal{D}_h = \{U \in H(Q_h);\ U \in [H^2(Q_h)]^3,\ U|_{\partial\Sigma\times I_h} = 0\},$$

the following properties are well known [Lad][Iu 65][Io 71][Té]:

(i) The linear operator L_μ has a compact resolvent, and its (discrete) spectrum lies in a sector centered on the negative real half-axis (see Figure II.4). This property is related to the dissipative term due to viscosity. Moreover, the fact that the spectrum is only composed with isolated eigenvalues leads to a formulation very similar to the finite-dimensional one.

(ii) L_μ is the generator of a holomorphic and compact semigroup ([Ka]) in $H(Q_h)$, and the semigroup $\exp(L_\mu t)$ is analytic for t in a sector centered on the positive real axis, of vertex 0 and of angle independent of μ. Dependency in μ is analytic in the sense of Kato [Ka]. Property (ii) shows that the linear evolution operator has the same properties as for the heat equation.

(iii) The quadratic operator $N(\mu, .)$ is continuous from $\mathcal{D}_h$ to $\mathcal{K}_h$, where $\mathcal{K}_h$ is a space "between" $\mathcal{D}_h$ and $H(Q_h)$, defined by

$$\mathcal{K}_h = \{U \in H(Q_h);\ U \in [H^1(Q_h)]^3\}.$$

In fact, the Sobolev embedding theorem [Ag] shows that the identity operator is continuous from $H^2(Q_h)$ to $C^0(\bar{Q}_h)$ and from $H^1(Q_h)$ to $L^4(Q_h)$ when the dimension (as here) is ≤ 3. It then follows that, for bounded $\mathcal{R}$, there exists C only depending on Q_h such that

$$||N(\mu, U)||_{\mathcal{K}_h} \leq C||U||^2_{\mathcal{D}_h}.$$

To solve the Cauchy problem for initial data U_0 in $\mathcal{D}_h$, we want to use an integral formulation equivalent to (II.23):

$$U(t) = e^{L_\mu t}U_0 + \int_0^t e^{L_\mu (t-s)} N[\mu, U(s)]\, ds. \tag{II.25}$$

To give a meaning to the right-hand side we use the estimate:

$$||\exp(L_\mu t)||_{\mathcal{L}(\mathcal{K}_h, \mathcal{D}_h)} \leq c/t^\beta \quad \text{for } t \in]0, T], \tag{II.26}$$

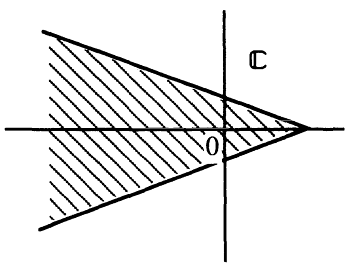


FIGURE II.4. Spectrum of L_μ.

where T is fixed, $\beta \in [0,1)$, and $\mathcal{L}(E_1, E_2)$ is the Banach space of linear bounded operators from E_1 to E_2 (both being Banach spaces). In the case of Navier-Stokes equations with rigid boundary conditions, one has $\beta = 3/4$ (see [Io 70] and [Bre] for a proof).

Now, using (II.25) as for a classical ordinary differential equation, one can prove the following:

Theorem. *(i) For any $T > 0$, $\exists \delta > 0$ such that if $||U_0||_{\mathcal{D}_h} \leq \delta$, there exists a unique solution of (II.11–II.12) with $U(0) = U_0$, $U(.)$ being continuous in $\mathcal{D}_h$ and differentiable in $H(Q_h)$ with respect to $t \in [0,T]$.*

(ii) For any data in $\mathcal{D}_h$, there exists $T > 0$ such that the solution U of (II.1) exists and is unique on $[0,T]$.

(iii) The dependence of the solution U in $\mathcal{D}_h$ with respect to the variable (t, μ, U_0) is analytic (see [Io77]).

The symmetry properties of the system pointed out in Subsection II.1.4, i.e., (i) the $(z \to z + a)$ and $(z \to -z)$ translational and reflectional invariance, and (ii) the $(\theta \to \theta + \varphi)$ rotational invariance, are expressed by the commutativity of L_μ and $N(\mu, .)$ with the two one-parameter groups of linear operators: $\boldsymbol{\tau}_a$ and $\mathbf{R}_\varphi (a, \varphi \in \mathbb{R})$, and with the symmetry operator $\mathbf{S}(\mathbf{S}^2 = \mathrm{Id})$ such that

$$\boldsymbol{\tau}_a \mathbf{S} = \mathbf{S} \boldsymbol{\tau}_{-a}, \tag{II.27}$$

all these operators being defined by (II.16). They are *unitary operators* in $\mathcal{D}_h$ and $H(Q_h)$.

II.3 Linear stability analysis

The stability of Couette flow, which corresponds to the solution $U = 0$ of (II.23), is governed by the position of the eigenvalues of the linear operator L_μ. The Couette flow is linearly stable if all eigenvalues have a negative real part and is linearly unstable if at least one eigenvalue has a positive real part.

Let U be an eigenvector belonging to eigenvalue σ. By definition, we have

$$L_\mu U = \sigma U, \qquad U \in \mathcal{D}_h. \tag{II.28}$$

This means that one can find q with $\nabla q \in [L^2(Q_h)]^3$ such that

$$\begin{cases} \Delta U - \mathcal{R}[(V^{(0)}.\nabla)U + (U.V)V^{(0)}] - \nabla q = \sigma U, & \nabla . U = 0, \\ U \in [H^2(Q_h)]^3, & U|_{\partial\Sigma \times I_h} = 0. \end{cases} \tag{II.29}$$

The above system (II.29) is valid for the choice of space $H(Q_h)$ made in Subsection II.2.1. If we make the choice of Subsection II.2.2, we add the flux condition

$$\int_\Sigma u_z(r, \theta, z) r \, dr \, d\theta = 0,$$

with an additional freedom on q to contain a term az not periodic in z.

Since U is h-periodic in z and 2π-periodic in θ, we can use the Fourier series

$$U = \sum_{(n,m)\in\mathbb{Z}^2} e^{i(2\pi nz/h)+im\theta}\hat{U}_{n,m}(r), \tag{II.30}$$

and $U \in \mathcal{D}_h$ leads to

$$\sum_{(n,m)\in\mathbb{Z}^2} (|n|+|m|)^4||\hat{U}_{n,m}||_0^2 + (|n|+|m|)^2||\hat{U}_{n,m}||_1^2 + ||\hat{U}_{n,m}||_2^2 < \infty,$$

where $||.||_j$ is the H^j norm for the vector functions of $r \in [\eta/(1-\eta), 1/(1-\eta)]$. Now, using (II.7–II.8) and equation (II.29) for eigenvalue σ leads to a set of uncoupled systems for each couple $(n,m) \in \mathbb{Z}^2$:

$$\begin{cases} [L_{\alpha m} - \sigma]\hat{U}_{n,m} - D_{\alpha m}\hat{q} = 0, \\ \nabla_{\alpha m}.\hat{U}_{n,m} = 0. \end{cases} \tag{II.31}$$

Here by definition we have for any vector function $\hat{V}(r)$ and scalar function $\hat{q}(r)$:

$$L_{\alpha m}\hat{V} = [\Delta_{\alpha m} - \frac{im}{r}\mathcal{R}v_\theta^{(0)}]\hat{V} + \mathcal{R}\begin{pmatrix} \frac{2}{r}v_\theta^{(0)}\hat{v}_\theta \\ -2A\hat{v}_r \\ 0 \end{pmatrix}, \quad D_{\alpha m}\hat{q} = \begin{pmatrix} D\hat{q} \\ im\hat{q}/r \\ i\alpha\hat{q} \end{pmatrix},$$

$$\nabla_{\alpha m}.\hat{V} = \frac{1}{r}D(r\hat{v}_r) + \frac{im}{r}\hat{v}_\theta + i\alpha\hat{v}_z, \quad \Delta_{\alpha m}\hat{V} = \begin{pmatrix} L\hat{v}_r - 2im\hat{v}_\theta/r^2 - \hat{v}_r/r^2 \\ L\hat{v}_\theta + 2im\hat{v}_r/r^2 - \hat{v}_\theta/r^2 \\ L\hat{v}_z \end{pmatrix},$$

where $\hat{V} = (\hat{v}_r, \hat{v}_\theta, \hat{v}_z)$, $v_\theta^{(0)} = Ar + B/r$ is defined in (II.9), D means d/dr, $\alpha = 2\pi n/h$, and, for any scalar function f, $Lf \equiv r^{-1}D(rDf) - [\alpha^2 + m^2/r^2]f$. For nonlinear terms that occur in the further analysis, we also need

$$\nabla_{\alpha m}\hat{V} = \begin{pmatrix} D\hat{v}_r & im\hat{v}_r/r - \hat{v}_\theta/r & i\alpha\hat{v}_r \\ D\hat{v}_\theta & im\hat{v}_\theta/r + \hat{v}_r/r & i\alpha\hat{v}_\theta \\ D\hat{v}_z & im\hat{v}_z/r & i\alpha\hat{v}_z \end{pmatrix}.$$

The boundary conditions are now for $\hat{U}_{n,m}$

$$\hat{u}_r = \hat{u}_\theta = \hat{u}_z = 0 \quad \text{for } r = \eta/(1-\eta) \text{ and } 1/(1-\eta). \tag{II.32}$$

Remark 1. In case of the choice of space $H(Q_h)$ made in Subsection II.2.2, we need to verify for the z-component of $\hat{U}_{n,m}$ the flux condition

$$\int_\Sigma \hat{u}_z(r)e^{im\theta}r\,dr\,d\theta = 0;$$

which is obviously satisfied for $m \neq 0$. It is also automatically satisfied for $m = 0$ and $n \neq 0$. Indeed, by the divergence free condition $(\text{II}.31)_2$ we have

$$-i\alpha \int_{\Sigma} \hat{u}_z(r) r \, dr \, d\theta = 2\pi \int_{\eta/(1-\eta)}^{1/(1-\eta)} D(r\hat{u}_r) \, dr = 0$$

because of the boundary conditions. The flux condition remains to be satisfied for $n = m = 0$. In this case, we redefine (II.31) for $n = m = 0$ by adding a term $a\vec{k}$ on the right-hand side of $(\text{II}.31)_1$ and the flux condition. This only influences the z-component $\hat{u}_z$ of $\hat{U}_{0,0}$, which now satisfies

$$(L - \sigma)\hat{u}_z = a, \qquad \int_{\eta/(1-\eta)}^{1/(1-\eta)} \hat{u}_z r \, dr = 0, \quad \hat{u}_z = 0 \text{ for } r = \frac{\eta}{1-\eta} \text{ and } \frac{1}{1-\eta}. \tag{II.31$_{00}$}$$

In fact, this modification does not affect the result on linear stability threshold since we may notice that the eigenvalue problem $(\text{II}.31)_{00}$ only gives negative σ's. To see this it suffices to multiply the first equation by the complex conjugate of $\hat{u}_z$ and to integrate by parts.

Remark 2. We observe that if $(\hat{u}_r, \hat{u}_\theta, \hat{u}_z, \hat{q})$ is a solution of (II.31–II.32) for (σ, α, m), then $(\hat{u}_r, \hat{u}_\theta, -\hat{u}_z, \hat{q})$ is a solution for $(\sigma, -\alpha, m)$ and $(\bar{\hat{u}}_r, \bar{\hat{u}}_\theta, \bar{\hat{u}}_z, \bar{\hat{q}})$ is a solution for $(\bar{\sigma}, -\alpha, -m)$. These properties are due to the equivariance of our system by the symmetry $\mathbf{S}$ and to the fact that the linear operator L_μ is real.

The set of eigenvalues (spectrum of the linear operator L_μ) is the union of all eigenvalues obtained for (II.31–II.32) for each couple $(n, m) \in \mathbb{Z}^2$. If $\mathcal{R}$ is close to 0, L_μ is close to the Stokes operator, and perturbation theory (see [Ka]) shows that the spectrum of L_μ is close to the spectrum of the Stokes operator, i.e., it is strictly on the left side of the complex plane (see [Lad]). Hence, for small enough $\mathcal{R}$, all eigenvalues have a negative real part, and numerical computations (see [Jo], Chapter V; [DP-S], [De-Io], [L-T-K-S-G] and references therein) show that when $\mathcal{R}$ is increased, a critical value $\mathcal{R}_c$ is reached where (at least) one eigenvalues lies on the imaginary axis and crosses this axis for $\mathcal{R} > \mathcal{R}_c$. Let us be more precise: if we denote the largest real part of eigenvalues σ obtained from (II.31–II.32) as λ_0, then λ_0 is a function of $(\alpha, m, \mathcal{R}, \eta, \Omega)$ where (α, m) defines the mode and $(\mathcal{R}, \eta, \Omega)$ defines the experimental conditions and, for a fixed value of (η, Ω), we now have a family (varying m in $\mathbb{N}$) of curves in the $(\alpha, \mathcal{R})$-plane where $\lambda_0(\alpha, m, \mathcal{R}, \eta, \Omega) = 0$. It was proved by Iudovich [Iu 66] that, except on a denumerable set of α's, the values $\mathcal{R}(\alpha, \eta, \Omega)$, given by $\lambda_0(\alpha, 0, \mathcal{R}, \eta, \Omega) = 0$, are *double eigenvalues* (we go back on this point later) of a family of linear operators directly derived from L_0. To prove this he uses the Krein theory of oscillatory kernels and the analyticity in α of operators such as $L_{\alpha 0}$. The curves $\lambda_0(\alpha, m, \mathcal{R}, \eta, \Omega) = 0$ are sketched in Figure II.5, where we must

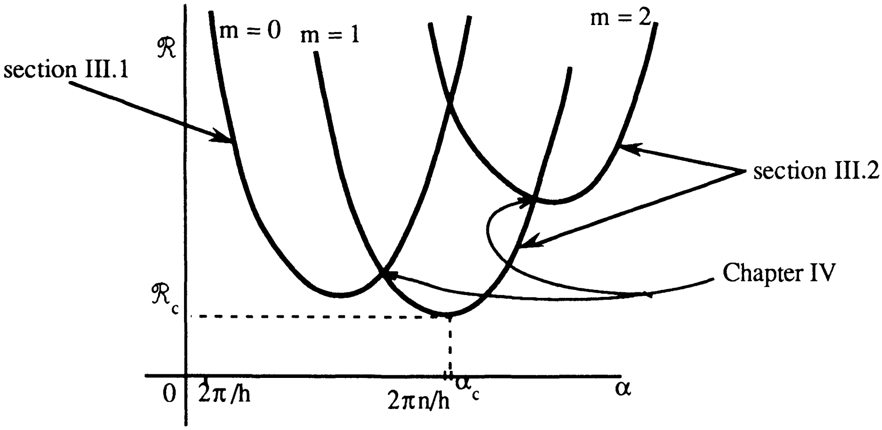


FIGURE II.5. Curves $\lambda_0(\alpha, m, \mathcal{R}, \eta, \Omega) = 0$ for fixed (η, Ω).

notice that the allowed values of α are multiples of $2\pi/h$. The minimal value of $\mathcal{R}$ on this set of curves is classically called the *critical Reynolds number* $\mathcal{R}_c$.

Remark 3. In the literature one sometimes finds curves in a plane $(\alpha, \mathcal{T})$ where $\mathcal{T}$ is the Taylor number given by $\mathcal{T} = -4A\mathcal{R}^2(1-\eta)\eta^{-1}$ where A is given in (II.9). Since we always have $A < 0$ at criticality, because of $\eta^2 > \Omega$ (see below the Rayleigh criterion), the curves look similar to those in Figure II.5. When $\Omega = 0$ and η is close to 1, we recover the Taylor number defined in Subsection II.1.5.2.

For $\mathcal{R} < \mathcal{R}_c$ all $\lambda_0(\alpha, m, \mathcal{R}, \eta, \Omega)$ are < 0, while for $\mathcal{R} > \mathcal{R}_c$ there is at least one m (and $-m$) and an interval for α's where $\lambda_0(\alpha, m, \mathcal{R}, \eta, \Omega) > 0$. This means that if some multiple $2\pi n/h$ lies in this interval, we then have an eigenvalue of positive real part $\lambda_0(2\pi n/h, m, \mathcal{R}, \eta, \Omega)$.

In the literature, instead of having a neutral stability surface under the form $\mathcal{R} = \mathcal{R}_c(\eta, \Omega)$, one finds curves $\mathcal{R} = \Omega_1 R_1 d\nu^{-1}$ as a function of $\mathcal{R}_2 = \Omega_2 R_2 d\nu^{-1}$ for a fixed η; it is clearly equivalent since this corresponds to writing $\mathcal{R}$ as a function of $\mathcal{R}\Omega\eta^{-1}$ for a fixed η. We give in Figure II.6 an idea of the form of these curves [it is heuristically shown for instance in [Li], [Cha], [Jo], [DP-S] that each curve is asymptotic to the line passing through the origin and of slope η^{-1} $(\eta^2 = \Omega)$ for $\mathcal{R}$ tending to infinity (Rayleigh criterion, see [Ray])]. However, in our case we have to take care of the fact that not all α are allowed. This modifies (slightly if the imposed wave number $2\pi/h$ is small, i.e., h large) the critical $\mathcal{R}$, which is reached for a certain value of $2\pi n/h$, where n may jump to another value when the parameters change, even with the same value for m. This results in the

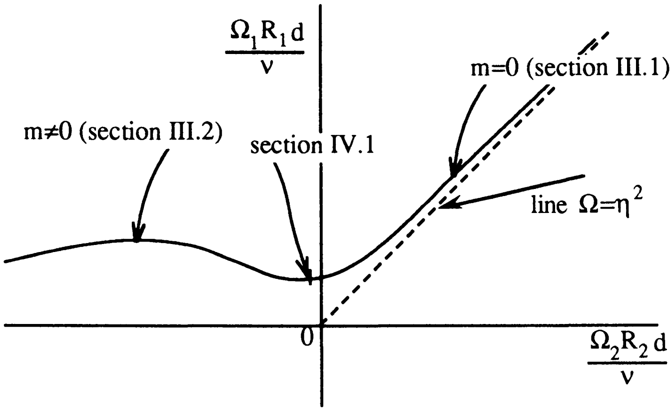


FIGURE II.6. Neutral stability curve for a fixed η.

existence of small discontinuities in the slope of the neutral surface each time n or m jumps when Ω and η varies. Numerical computations (see explicit results in following chapters) show that when $\Omega \geq 0$ or even for Ω slightly negative, the instability occurs via a real eigenvalue crossing zero, which corresponds to an axisymmetric mode (i.e., $m = 0$). For strongly counter-rotating cylinders (Ω sufficiently negative) the instability comes via a pair of pure imaginary eigenvalues corresponding to nonaxisymmetric modes ($m \neq 0$).

The last thing to be observed is that, due to the remark above, when σ is an eigenvalue for a nonzero α [in (II.31–II.32)], this eigenvalue is *at least double*, since the symmetric image of the eigenvector is an independent eigenvector too and belongs to the same eigenvalue σ. In fact, the symmetric eigenvector may be written as

$$\mathbf{S}[e^{i(2\pi nz/h+m\theta)}\hat{U}_{n,m}] = e^{i(-2\pi nz/h+m\theta)}\mathbf{S}[\hat{U}_{n,m}].$$

Section III.1 considers the case when $\mathcal{R}_c$ is reached for $m = 0$ and the corresponding critical eigenvalue is 0, i.e., *real and double*, with eigenvectors

$$\zeta = \hat{U}_0(r)e^{i\alpha z} \quad \text{and} \quad \mathbf{S}\zeta = [\mathbf{S}\hat{U}_0(r)]e^{-i\alpha z}. \tag{II.33$_1$}$$

In Section III.2, we consider cases when $\mathcal{R}_c$ is reached for $m \neq 0$ and the corresponding critical eigenvalues form a pair $\pm i\omega_0$ on the imaginary axis (each one being double), with eigenvectors denoted as

$$\zeta_1 = \hat{U}_1(r)e^{i(\alpha z+m\theta)} \quad \text{and} \quad \mathbf{S}\zeta_1 = \zeta_2 = [\mathbf{S}\hat{U}_1(r)]e^{i(-\alpha z+m\theta)}. \tag{II.33$_2$}$$

In Chapter IV, we consider codimension-two problems corresponding to the neighborhood of situations in which either 0 and one pair of pure imaginary eigenvalues, or two pairs are on the imaginary axis. At fixed η, these situations appear, when parameters $\mathcal{R}$ and Ω are varied through small slope discontinuities on the neutral curve of Figure II.2. Throughout this book we will not consider codimension-two situations generated by a jump of the value of the critical wave number $\alpha = 2\pi n/h$. This is due to the fact that we idealize the problem by assuming h-periodicity, so the definition of h is far from being obvious, a fortiori if h is very large, since a change by 1 of n would be equivalent, if we keep the same n, to a change of $2\pi/h$ by $2\pi/nh$. Finally, a study in this context of the eventual competition between different choices of numbers n for the critical wave number appears not to be justified here (aspect ratio not small). Let us mention that such a study on competing modes with different spatial wave numbers is made in works dealing with small aspect ratio systems; see, for instance, works of Mullin et al. [Mu, Cl, Pf], [Cl, Mu]. Instead of this, we shall consider in Chapter VII the large-scale effects due to the many excited modes for a very large h, when $\mathcal{R}$ is larger than $\mathcal{R}_c$.

II.4 Center Manifold Theorem

In this section, we give details on a very general tool that is used throughout this book. A reader who has difficulties understanding notation in the Taylor (not the same Taylor!) expansions (II.37), can forget them. In the next chapters, the concrete computations on specific cases are simpler, and do not in fact need general notations.

Let us consider parameter values near criticality, i.e., values of $\mathcal{R}$, η, Ω close to a specific value (say $\mu = 0$), lying on the neutral surface $\mathcal{R} = \mathcal{R}_c(\eta, \Omega)$. For $\mu = 0$, a part σ_0 of the spectrum (finite number of eigenvalues of finite multiplicities of L_0) lies on the imaginary axis, and the remaining part lies strictly in the left-hand side of the complex plane. Let us denote by σ_μ the part of the spectrum of L_μ formed by the eigenvalues close to the imaginary axis for μ close to 0. Then for the linearized problem a finite number of modes are weakly excited or weakly damped, while the remaining ones are strongly damped. For the nonlinear evolution problem (II.23), it is tempting to try to eliminate all the strongly damped modes, and, in fact, the correct way to do this follows from the *center manifold theory* (also called the slaving principle or adiabatic elimination by physicists). This theorem originates from [Pl] and [Ke] for ordinary differential equations; see [He] for a direct proof (not going through maps as in [M-M] or [Io 79] for evolution problems in partial differential equations using an estimate like (II.26) (with no part of the spectrum on the right of the imaginary axis). See also [Va 89a] and [Va-Io] for a modern proof.

Let us denote by P_μ the projection operator (of finite rank) that commutes with L_μ, associated with the part σ_μ of the spectrum (see [Ka]).

Then, thanks to the properties of the analytic semigroup $\exp(L_\mu t)$ indicated in Section II.2 and properties of the quadratic term $N(\mu, U)$, we may assert the following:

Center manifold Theorem. *For any $s > 0$, there exists a neighborhood $I \times \mathcal{O}$ of $(0,0)$ in $\mathbb{R}^3 \times \mathcal{D}_h$, on which a C^s map Φ: $I \times P_0\mathcal{O} \to (I - P_0)\mathcal{O}$ is defined, such that the graph of $Y = \Phi(\mu, X)$ represents a manifold $\mathcal{M}_\mu$ in $\mathcal{D}_h$ with the following properties:*

(i) $\mathcal{M}_0$ is tangent to the space $P_0\mathcal{D}_h$ at the origin $[\Phi(0,0) = 0$, *and* $D_X\Phi(0,0) = 0]$,

(ii) $\mathcal{M}_\mu$ is locally invariant under the flow induced by equation (II.23),

(iii) $\mathcal{M}_\mu$ is locally attracting under equation (II.23): *if U_0 is an initial data such that the solution $U(t)$ stays in $\mathcal{O}$ for all t, then dist$[U(t), \mathcal{M}_\mu] \to 0$ as $t \to \infty$,*

(iv) $\Phi(\mu, .)$ commutes with the group actions τ_a, $\mathbf{R}_\varphi$, and $\mathbf{S}$.

Remark 1. $\mu \in \mathbb{R}^3$ corresponds to the triplet $(\mathcal{R}, \Omega, \eta)$. Y is the "slave" part, function of the critical modes represented by X. We chose to represent the center manifold in $\mathcal{D}_h$, which is the domain of the linear operator acting in $H(Q_h)$. It is in fact possible to choose another less regular space with only first space derivatives that are square integrable.

Remark 2. The last property follows from a general result of Ruelle (see [Ru]) and from the fact that these transformations are unitary, in $\mathcal{D}_h$ as well as in $H(Q_h)$.

Let us denote the subspace $P_0\mathcal{D}_h$ as $\vee$. We may now notice that $\mathbf{R}_\varphi$ is a representation of the group SO(2) in the space $\vee$. It will be useful in the next chapters to express τ_a as an action of SO(2) by setting $a = \psi h/2\pi$ and $\mathbf{T}_\psi = \tau_{\psi h/2\pi}$, $0 \le \psi \le 2\pi$. Then, because of (II.27), $\mathbf{T}_\psi$ and $\mathbf{S}$ define a representation in $\vee$ of the group O(2). In the rest of this book we denote by Γ the group O(2) $\times$ SO(2). Γ acts in $\vee$ by $\mathbf{T}_\psi$, $\mathbf{S}$ and $\mathbf{R}_\varphi$.

It follows from the above theorem that the manifold $\mathcal{M}_\mu$ is finite dimensional and that the asymptotic dynamics of (II.23) lies on it. The trace of (II.23) on $\mathcal{M}_\mu$ is represented by an ordinary differential equation, which can be parameterized by $X \in P_0\mathcal{O}$:

$$\frac{dX}{dt} = F(\mu, X) \quad \text{in } \vee . \tag{II.34}$$

Here $F(\mu, .)$ [like $\Phi(\mu, .)$] commutes with the representations of $\mathbf{T}_\psi$, $\mathbf{S}$, and $\mathbf{R}_\varphi$ on $\vee$. We shall say that $F(\mu, .)$ is Γ-*equivariant.*

Remark 3. This reduction technique to an ODE should not be confused with the Lyapunov-Schmidt decomposition, which also allows the reduction of the bifurcation problem to a finite-dimensional equation. However the L-S reduction is only valid for the computation of branches of steady-state or time-periodic solutions, and is not suitable, in general, for the analysis of their stability. However, the resulting equations have a similar form in both methods, and in certain cases their lower-order terms are identical, as shown in [C-Go 87].

To conclude this chapter, let us indicate how to compute the Taylor expansions of F in (II.34) and that of Φ in the expression of the center manifold

$$U = X + \Phi(\mu, X) \text{ in } \mathcal{D}_h. \tag{II.35}$$

In fact, once we have a center manifold, we have to study the behavior of solutions of equation (II.34). This of course may be difficult, and it is often necessary to adapt X by a nonlinear change of variable to "improve" the form of (II.34) and make the study of solutions easier. This is precisely the aim of *normal form theory.* It is not the purpose of this book to present it, so we just give the main result below. Another important point is that, instead of computing (i) the center manifold and (ii) the normal form of (II.34) separately, we can do this *together.* This possibility and the observation that this is actually the most efficient way for studying high codimension bifurcation problems have been pointed out by Coullet and Spiegel [C-S]. In fact, it is a justification and a major extension of the very classical method of amplitude equations used in the 1960s (see the review paper by Stuart [St]) as initiated by Landau [Lan] in the context of nonlinear hydrodynamic stability problems near threshold.

Substituting (II.35) and (II.34) into (II.23), we obtain

$$[\,\mathrm{Id} + D_X\Phi(\mu, X)]F(\mu, X) = L_\mu[X + \Phi(\mu, X)] + N[\mu, X + \Phi(\mu, X)], \tag{II.36}$$

Now we expand L, N, Φ and F in Taylor expansion, use them in (II.36) and equate coefficients of like power. To do this we first define these Taylor expansions as follows:

$$\begin{aligned}
L_\mu &= \sum_{n\geq 0} L_n[\mu^{(n)}],\\
N(\mu, U) &= \sum_{p\neq 1} N_{np}[\mu^{(n)}; U^{(p)}],\\
\Phi(\mu, X) &= \sum_{(n,p)\neq(0,1)} \Phi_{np}[\mu^{(n)}; X^{(p)}],\\
F(\mu, X) &= \sum_{n+p\geq 1} F_{np}[\mu^{(n)}; X^{(p)}],
\end{aligned} \tag{II.37}$$

where L_n, Φ_{np}, F_{np}, N_{np} are n-linear symmetric in the first argument in $\mathbb{R}^k$ (here $k \leq 3$) and Φ_{np}, F_{np}, N_{np} are p-linear symmetric in the second argument in $P_0\mathcal{D}_h$ for Φ_{np} and F_{np} and in $\mathcal{D}_h$ for N_{np} (in the classical problem $N_{np} \equiv 0$ for $p \neq 2$, due to the form of Navier-Stokes equations). We write here $X^{(p)}$ for X repeated p times, and if $\mu \in \mathbb{R}$ (in the case when two of the three parameters are fixed), $\mu^{(n)}$ is in fact the scalar factor μ^n. The polynomial functions L_n, Φ_{np}, N_{np} take their values in the space $H(Q_h)$, while polynomials F_{np} take their values in the finite-dimensional space $P_0\mathfrak{D}_h$. A way to define the coefficients of the right-hand side of (II.37) is to write,

$$F_{np} = \frac{1}{n!p!}\frac{\partial^{n+p}F}{\partial\mu^n\partial X^p}(0,0);$$

when arguments are not all the same we set

$$F_{np} = F_{np}[\mu_1, \mu_2, \ldots, \mu_n; X_1, X_2, \ldots, X_p], \ \ \mu_j \in \mathbb{R}^k, \ \ X_j \in P_0\mathfrak{D}_h,$$

where F_{np} is linear with respect to each argument and separately symmetric in its n first arguments and in its p last arguments (hence the order for the X_j has no importance). We denote by $\tilde{L}_0 = F_{01}$ the linear operator which is the restriction of L_0 on the invariant finite dimensional subspace $P_0\mathfrak{D}_h$ (all its eigenvalues have a zero real part).

Equating order 1 terms in (μ, X) in equation (II.36) gives (for any X in $P_0\mathfrak{D}_h$)

$$\begin{aligned} \tilde{L}_0 X &= L_0 X, \\ -L_0\Phi_{10} &= N_{10} - F_{10}. \end{aligned} \tag{II.38}$$

Here $(\text{II.38})_1$ is just a definition and, if $N_{10} = 0$ as in the classical problem, we see that we can *choose* $\Phi_{10} = F_{10} = 0$ (this is of course the simplest choice).

Next, equating order 2 and some order 3 terms in $X^{(2)}$, $[\mu; X]$, $X^{(3)}$ leads to the following equations:

$$2\Phi_{02}[X, L_0X] - L_0\Phi_{02}[X^{(2)}] = N_{02}[X^{(2)}] - F_{02}[X^{(2)}], \tag{II.39}$$

$$\begin{aligned} \Phi_{11}[\mu; L_0X] - L_0\Phi_{11}[\mu; X] &= 2N_{02}[X, \Phi_{10}[\mu]] + L_1[\mu]X \\ &\quad -2\Phi_{02}[X, F_{10}[\mu]] - F_{11}[\mu; X], \end{aligned} \tag{II.40}$$

$$\begin{aligned} 3\Phi_{03}[X^{(2)}, L_0X] - L_0\Phi_{03}[X^{(3)}] &= 2N_{02}[X, \Phi_{02}[X^{(2)}]] + N_{03}[X^{(3)}] \\ &\quad -2\Phi_{02}[X, F_{02}[X^{(2)}]] \\ &\quad -F_{03}[X^{(3)}]. \end{aligned} \tag{II.41}$$

Now the general case of order $n+p$ yields:

$$\begin{aligned} &p\Phi_{np}[\mu^{(n)}; X^{(p-1)}, L_0X] - L_0\Phi_{np}[\mu^{(n)}; X^{(p)}] \\ &\quad = \mathcal{R}_{np}[\mu^{(n)}; X^{(p)}] - F_{np}[\mu^{(n)}; X^{(p)}], \end{aligned} \tag{II.42}$$

where $\mathcal{R}_{np}$ is determined by previous steps. At each step, the idea is to find Φ_{np} such that F_{np} is as simple as possible (0, for instance). We see that each order $n+p$ identity leads to a linear equation for Φ_{np} where the right-hand side is determined by lower-order terms of indices (n', p'), such that $n' = 0$, $p' \leq p-1$ if $n = 0$, and $n' \leq n-1$, $n'+p' \leq n+p$ if $n \geq 1$.

The problem is to study the range of the linear operator acting on $\Phi_{np}[\mu^{(n)}; .]$, called the "homological operator" in a vector space of homogeneous polynomials of degree p in the finite-dimensional variable X.

The compatibility condition for the right-hand side to belong to the image of this operator defines F_{np} (up to the addition of an element of this image). The choice of Φ_{np} and F_{np} is not unique, but the following is a simple, easy-to-use result.

Normal Form Theorem. *(i) We can choose the mapping Φ of the center manifold theorem such that the chosen parametrization on the center manifold gives an ordinary differential equation of the form*

$$\frac{dX}{dt} = \tilde{L}_0 X + \tilde{N}(\mu, X) + \mathrm{O}(||X||^P), \tag{II.43}$$

where P is fixed arbitrarily and $\tilde{N}$ is a polynomial in the variable X, with coefficients smooth in μ, taking values in V such that for any t in $\mathbb{R}$ and X in V, we have

$$\tilde{N}(\mu, \exp[\tilde{L}_0^* t]X) = \exp[\tilde{L}_0^* t]\tilde{N}(\mu, X), \tag{II.44}$$

where $\tilde{L}_0^*$ is the adjoint of $\tilde{L}_0$ in the finite-dimensional space V.

(ii) In addition, (II.43) *is equivariant under the representations of the symmetries of the original system* ($\mathbf{T}_\psi, \mathbf{R}_\varphi$, and $\mathbf{S}$).

The theorem in the present formulation is proved in [E-T-B-C-I], note that another version of (i) was previously shown in [Be]. The simplicity of the result is due to a suitable choice of a scalar product in the space of homogeneous polynomials, enabling the definition of a simple adjoint of the linear "homological" operator acting on Φ_{np}. In the following chapters, each time we have to use this theorem, we explicitly solve (II.44) (which is easy in most cases).

Remark 1. It is shown in [E-T-B-C-I] that the remaining higher-order terms have the form $\mathrm{O}((|\mu| + ||X||)^P)$ or, eventually, $\mathrm{O}(||X||^P)$ if 0 is not an eigenvalue of $\tilde{L}_0$ and if we do not require that $\tilde{N}$ depend polynomially on μ. In fact, a further result of Vanderbauwhede [Va 89b] shows that (II.43) hold even if 0 is an eigenvalue of $\tilde{L}_0$.

Remark 2. In Chapters III and IV we do not use this theorem, since it will be sufficient to use equivariance of equation (II.34) under $\mathbf{T}_\psi$, $\mathbf{R}_\varphi$, and $\mathbf{S}$. The system is in fact already into normal form, at least up to the fifth-order terms in (II.34). On the contrary, when we perturb the original symmetries of the problem, changing for instance the boundary conditions (see Chapter V), the above theorem is very useful.

III

Taylor Vortices, Spirals and Ribbons

We observed in Section II.3 that the Couette flow could lose stability in two different ways when the Reynolds number is increased, depending on the value of the other parameters, namely, the critical eigenvalue can be zero or purely imaginary. The crucial fact for bifurcation analysis is that the differential equation (II.34) is invariant under the representation of the group $\Gamma = O(2) \times SO(2)$ defined in Section II.4 and that, owing to this symmetry, the critical eigenvalues are double (see Remark 1 of Section II.3). These algebraic properties completely determine the number of branches and the pattern of the bifurcated solutions. The group-theoretic approach to bifurcation with symmetry can be avoided in our problem because the group action is quite simple. Readers interested in this algebraic approach should consult [Go-St-Sc], where they will also find many references. For the primary bifurcation analysis the azimuthal symmetry plays no role; the types and number of solutions would therefore be the same for a problem with eccentric cylinders, for example (see Chapter V). However, we will need this symmetry to explain the *spatial structure of the flows*. In this chapter we successively review the *steady-state* and *Hopf bifurcations*, describing first the theoretical analysis, then the actual bifurcation diagrams occurring for different values of the parameters, and finally the physical meaning of these solutions (structure of the flow). In equation (II.34), the parameter μ now designates $\mathcal{R} - \mathcal{R}_c$, the deviation from the critical value of the Reynolds number (see Figure II.1), the other parameters being kept fixed. In the differential equation (II.23) the operators have the form [(see II.24)]:

$$L_\mu = L_0 + \mu L_1, \qquad N(\mu, U) = N_0(U, U) + \mu N_1(U, U), \tag{III.1}$$

where

$$\begin{aligned} L_0 U &= \Pi_0\{\Delta U - \mathcal{R}_c[(V^{(0)}.\nabla)U + (U.\nabla)V^{(0)}]\}, \\ L_1 U &= -\Pi_0\{(V^{(0)}.\nabla)U + (U.\nabla)V^{(0)}\}, \\ N_0(U,V) &= -{}^1\!/_2\,\mathcal{R}_c\Pi_0[(U.\nabla)V + (V.\nabla)U], \\ N_1(U,V) &= -{}^1\!/_2\,\Pi_0[(U.\nabla)V + (V.\nabla)U]. \end{aligned}$$

III.1 Taylor vortex flow

III.1.1 Steady-state bifurcation with O(2)-symmetry

We suppose that the operator L_0 has an eigenvalue 0, as shown in Section II.3. Let us denote by ζ the eigenvector $\hat{U}_0(r)e^{i\alpha z}$, where $\alpha = 2\pi n/h$ as indicated in Figure II.1. We already noticed that the critical eigenvalue 0 is double, due to the other eigenvector $\mathbf{S}\zeta = \bar{\zeta}$; hence, the eigenspace $\mathbf{V}$ is *two dimensional.* For convenience we complexify $\mathbf{V}$, i.e., we write in (II.34)

$$X = A\zeta + \bar{A}\bar{\zeta}, \tag{III.2}$$

where $A \in \mathbb{C}$. We also notice here that the eigenvectors are *axisymmetric.* These properties come from the value of the azimuthal wave number $m = 0$ here and from the fact that one finds numerically only one critical eigenmode for one value of the axial wave number (see Section II.3), which is generic. It was already noticed by Taylor in 1923 (see [Tay]) that restricting the linear analysis to axisymmetric modes leads to stationary critical modes. The action of Γ on the eigenvectors defines the group representation of Γ acting in $\mathbf{V}$: by $(\text{II.33})_1$ we see that

$$\mathbf{T}_\psi\zeta = e^{in\psi}\zeta, \quad \mathbf{S}\zeta = \bar{\zeta}, \quad \mathbf{R}_\varphi\zeta = \zeta \qquad (\forall\psi \text{ and } \varphi \text{ in } \mathbb{R}). \tag{III.3}$$

Remark. We see from (III.3) that $\mathbf{R}_\varphi$ acts as the identity on $\mathbf{V}$; we say that "SO(2) acts trivially." On the contrary, the action of O(2), represented by rotations $\mathbf{T}_\psi$ and symmetry $\mathbf{S}$, is said to be an *"absolutely irreducible"* representation in $\mathbf{V}$. This means that the only linear maps commuting with this representation are the *scalar multiples* of the identity (the proof is immediate from formulas (III.5) applied to any linear vector field that commutes with the action of Γ in $\mathbf{V}$; see, for example, [Go-St-Sc] for more general considerations). This property is "generic" for one-parameter bifurcation problems with symmetry and plays a crucial role in the theory (see [Ru]); here it is simple enough so that a more general formalism is not needed in the subsequent analysis.

It follows from the decomposition (III.2) and the equivariance of $F(\mu, X)$ under Γ that, in these coordinates, (II.34) can be written as:

$$\frac{dA}{dt} = f(\mu, A, \bar{A}). \tag{III.4}$$

This equation is known as the *amplitude equation.* Here f is a complex function satisfying the relations

$$\begin{aligned} f(\mu, e^{\mathrm{i}n\psi}A, e^{-\mathrm{i}n\psi}\bar{A}) &= e^{\mathrm{i}n\psi}f(\mu, A, \bar{A}), \\ f(\mu, \bar{A}, A) &= \overline{f(\mu, A, \bar{A})}, \end{aligned} \tag{III.5}$$

for any ψ and A. Let us choose ψ such that $n\psi = -\arg A$ and $n\psi = \pi$ successively. This shows that

$$f(\mu, A, \bar{A}) = e^{\mathrm{i}n\psi}f(\mu, |A|, |A|) \quad \text{with } f \text{ odd in } |A|.$$

It is now clear that the Taylor expansion of the function f with respect to the variable A must be of the form $Ag(\mu, |A|^2)$, where g is a *real* function in its second argument, due to the property (III.5).

We may write the Taylor expansion of the amplitude equation as

$$\frac{dA}{dt} = Ag(\mu, |A|^2) = a\mu A + cA|A|^2 + \text{ h.o.t.} \tag{III.6}$$

where h.o.t. $= \mathrm{O}(|\mathrm{A}|^5 + |\mu||\mathrm{A}|^3 + |\mu|^2|\mathrm{A}|)$. The truncated equation is most often called the *Landau equation* since it was introduced by Landau in 1944 (see [Lan]). If we write $A = \rho e^{i\varphi}$ in polar coordinates, (III.6) reduces to

$$\frac{d\varphi}{dt} = 0 \quad \text{and} \quad \frac{d\rho}{dt} = \rho g(\mu, \rho^2).$$

The equation for ρ is a standard "pitchfork bifurcation equation", i.e., the equation obtained after the center manifold reduction for any steady-state bifurcation problem at a *simple eigenvalue* when the original equation is invariant under a symmetry acting nontrivially on the critical eigenvector. Here this symmetry is just $\mathbf{T}_{\pi/n}$, i.e., a transition by half the wavelength in the axial direction. Consequently, if $\partial g(0,0)/\partial\mu = a \neq 0$, there exists a one-sided branch of equilibria for equation (III.6) bifurcating from the trivial state at $\mu = 0$. Each steady solution A gives rise to a circle of solutions $Ae^{\mathrm{i}\psi}$, $\psi \in \mathbb{T}^1$. This circle (in the complex plane) of steady solutions corresponds to a *group orbit through the action of* $\mathbf{T}_\psi$. In fact, we have a solution in the form

$$U_0 = A\zeta + \bar{A}\bar{\zeta} + \Phi(\mu, A\zeta + \bar{A}\bar{\zeta}), \tag{III.7}$$

by simply using (III.2) in (II.35). Since we have $\mathbf{T}_\psi\Phi(\mu, X) = \Phi(\mu, \mathbf{T}_\psi X)$, it results that a group orbit is given by

$$\mathbf{T}_\psi U_0 = e^{ni\psi}A\zeta + e^{-ni\psi}\bar{A}\bar{\zeta} + \Phi(\mu, e^{ni\psi}A\zeta + e^{-ni\psi}\bar{A}\bar{\zeta}).$$

This group orbit simply means that the fluid flow is determined up to an arbitrary translation along the z-axis. We can assume in what follows that A in U_0 is built *real.*

By the same reasoning, it is shown that U_0 is *axisymmetric*, since X is invariant under the transformations $\mathbf{R}_\varphi$, $\forall\varphi \in \mathbb{T}^1$. We say that $\mathbf{R}_\varphi$ acts trivially on the bifurcated equilibria. In the same way, we see that for our choice of solution U_0 with a real A, we have $\mathbf{S}U_0 = U_0$; hence, the velocity vector field U_0 is symmetric with respect to the $z = 0$ plane (see in Section III.1.3. the geometrical description of the flow).

Moreover, the principle of "exchange of stability" holds. If the trivial state is stable for $\mu < 0$ (i.e., if $a > 0$) and if the bifurcation holds for $\mu > 0$, that is, if

$$\frac{\partial g}{\partial(\rho^2)}(0,0) = c < 0, \tag{III.8}$$

then the bifurcated solutions are orbitally stable. The adverb *orbitally* simply means that because the solutions form a *continuous* group orbit, this group orbit, rather than each individual equilibrium, is stable to any (small) perturbation. This statement is a direct consequence of the local attractivity of the center manifold and the phase-amplitude decoupling in (III.6). If $c = 0$, a general result applies provided there exists a nonvanishing higher-order derivative (see, for instance, [Io-Jo]).

This bifurcation was studied and solved, in the context of the Couette-Taylor problem, by Ovchinnikova and Iudovich [Ov-Iu 68] and by Kirchgässner and Sorger [Ki-So]. In their analysis they looked for solutions with an imposed symmetry (symmetry with respect to the plane $z = 0$, i.e., $\mathbf{S}$-invariant solutions) in order to reduce the study to a bifurcation at a simple eigenvalue. It follows from the foregoing discussion that in this way they found all the bifurcating solutions.... This approach can be formalized and gives a systematic way to compute many branches of solutions in bifurcation problems with symmetry, known more generally as the "equivalent branching lemma" [Go-St-Sc].

III.1.2 Identification of the coefficients in the amplitude equation

In order to determine the actual bifurcation diagram and the phase portrait for different values of the parameters, we have to compute the coefficients a and c defined in (III.6). We can derive these coefficients in terms of the operators (III.1) by following the method described in Section II.4. Using (III.2) and the fact that μ is one dimensional (codimension-one problem), we insert

$$\Phi(\mu, X) = \sum_{pqr} \mu^p A^q \bar{A}^r \Phi_{qr}^p \tag{III.9}$$

into (II.37). Using the notation introduced in Section II.4, this means that we have, for instance,

$$\Phi_{11}[\mu; X] = \mu(A\Phi_{10}^1 + \bar{A}\Phi_{01}^1)$$
$$2\Phi_{02}[X, X'] = 2AA'\Phi_{20}^0 + (A\bar{A}' + \bar{A}A')\Phi_{11}^0 + 2\bar{A}\bar{A}'\Phi_{02}^0.$$

Then formula (II.40) gives (for the coefficient of μA)

$$-L_0\Phi^1_{10} = L_1\zeta - a\zeta, \tag{III.10}$$

and (II.41) gives (for the coefficient of $A|A|^2$)

$$-L_0\Phi^0_{21} = 2N_0(\zeta, \Phi^0_{11}) + 2N_0(\bar{\zeta}, \Phi^0_{20}) - c\zeta, \tag{III.11}$$

where Φ^0_{11} and Φ^0_{20} satisfy the linear equations

$$\begin{aligned} -L_0\Phi^0_{11} &= 2N_0(\zeta, \bar{\zeta}), \\ -L_0\Phi^0_{20} &= N_0(\zeta, \zeta). \end{aligned} \tag{III.12}$$

Notice that the commutativity of $\Phi(\mu, .)$ with $\mathbf{T}_\psi$ (center manifold theorem, Section II.4) leads to the property that

$$\mathbf{T}_\psi \Phi^p_{qr} = e^{ni(q-r)\psi}\Phi^p_{qr},$$

and since $\Phi(\mu, .)$ is invariant under $\mathbf{R}_\varphi$, we have

$$\Phi^p_{qr} = e^{i(q-r)\alpha z}\hat{\Phi}^p_{qr}(r). \tag{III.13}$$

Using the notation introduced in (II.3) we rewrite (III.10)–(III.12) as

$$\begin{aligned} L_{\alpha 0}\hat{\Phi}^1_{10} - D_{\alpha 0}\hat{q}_3 &= a\hat{U}_0 + [V^{(0)}.\nabla_{\alpha 0}\hat{U}_0 + \hat{U}_0.\nabla_{00}V^{(0)}], \\ \nabla_{\alpha 0}.\hat{\Phi}^1_{10} &= 0; \end{aligned} \tag{III.14a}$$

$$\begin{aligned} L_{2\alpha,0}\hat{\Phi}^0_{20} - D_{2\alpha,0}\hat{q}_1 &= \mathcal{R}_c\hat{U}_0.\nabla_{\alpha 0}\hat{U}_0, \\ \nabla_{2\alpha,0}.\hat{\Phi}^0_{20} &= 0, \\ L_{00}\hat{\Phi}^0_{11} - D_{00}\hat{q}_2 &= \mathcal{R}_c[\hat{U}_0.\nabla_{-\alpha,0}\bar{\hat{U}}_0 + \bar{\hat{U}}_0.\nabla_{\alpha 0}\hat{U}_0], \\ \nabla_{00}.\hat{\Phi}^0_{11} &= 0; \end{aligned} \tag{III.14b}$$

$$\begin{aligned} L_{\alpha 0}\hat{\Phi}^0_{21} - D_{\alpha 0}\hat{q}_4 = {} & \mathrm{c}\,\hat{U}_0 + \mathcal{R}_c[\hat{U}_0.\nabla_{00}\hat{\Phi}^0_{11} + \hat{\Phi}^0_{11}.\nabla_{\alpha 0}\hat{U}_0 \\ & +\bar{\hat{U}}_0.\nabla_{2\alpha,0}\hat{\Phi}^0_{20} + \hat{\Phi}^0_{20}.\nabla_{-\alpha,0}\bar{\hat{U}}_0], \\ \nabla_{\alpha 0}.\hat{\Phi}^0_{21} = {} & 0, \end{aligned} \tag{III.14c}$$

with homogeneous boundary conditions.

The system (III.14a–c) is a set of linear differential equations over $r \in \left(\frac{\eta}{1-\eta}, \frac{1}{1-\eta}\right)$. We observe that $L_{2\alpha,0}$ and L_{00} are invertible; hence, $\hat{\Phi}^0_{20}$ and $\hat{\Phi}^0_{11}$ can be computed by a standard numerical scheme (see [De-Io] and [La-De] for details). Now, $L_{\alpha 0}$ is not invertible (by construction), and the Fredholm alternative applies because of the general results on linear operators with compact resolvant [Section II.2, property (i) of L_μ]. The compatibility condition allows us to compute a and c. If we define the eigenvector ζ^* for the adjoint operator L_0^* such that

$$L_0^*\zeta^* = 0 \quad \text{and} \quad (\zeta; \zeta^*) = 1,$$

where $(.;.)$ is the scalar product in $[L^2(Q_h)]^3$, then ζ^* has also the form $\hat{U}_0^*(r)e^{i\alpha z}$ and

$$2\pi h \int_{\eta/(1-\eta)}^{1/(1-\eta)} \hat{U}_0(r)\bar{\hat{U}}_0^*(r) r\, dr = 1.$$

Then we have by (III.14a–c)

$$a = (L_1\zeta;\zeta^*) \quad \text{and} \quad c = (2N_0(\bar{\zeta},\Phi_{20}^0) + 2N_0(\zeta,\Phi_{11}^0);\zeta^*).$$

In [De-Io] and [La-De] a and c are computed without explicit computing of the adjoint eigenvector ζ^* and tables of values for a and c are given for different values of η.

This method gives the same results as those obtained via the more classical method of asymptotic expansion in powers of the amplitude [St] or the Lyapunov-Schmidt method [Ov-Iu 68] and [Ki-So] (see also [Io 84]). Here we indicate the results obtained by [De-Io], up to a different choice of scales in the original equations. Details about the numerical procedure are given in [De-Io] and [La-De]. It applies to other situations reviewed in this book as well. There is no proof that the coefficient c is nonzero for (say) $\Omega = 0$. Computations shows however that $c \neq 0$. A proof exists in the small gap case (see [Ov-Iu 74]) because when η and Ω are both close to 1, the system for axisymmetric solutions is analogous to that obtained in the Rayleigh–Bénard convection problem with no-slip boundary conditions [Iu 67].

Results of calculations are given for two different values of the radius ratio: $\eta = 0.95$ and 0.75. In Figure III.1 we give the critical Reynolds numbers and critical wave numbers, while in Figure III.2 we plot the values of the coefficient $-c/a$ versus Ω. This coefficient gives the direction of the bifurcation and an idea of its "amplitude," since the solutions of the bifurcation equation $g(\mu,\rho^2) = 0$ have the form

$$\rho = \sqrt{-a\mu/c} + O(|\mu|^{3/2}). \tag{III.15}$$

The remarkable fact about Figure III.2 is that when Ω is smaller than a critical value located between -0.6 and -0.7 in the case $\eta = 0.95$ and between -0.7 and -0.8 in the case $\eta = 0.75$, the bifurcation is *subcritical*; therefore, the solution is unstable. In the case $\eta = 0.75$, however, this critical value belongs to the range of values for which the Couette flow loses stability by a Hopf bifurcation: the "critical" eigenvalue 0 is not first to cross the imaginary axis and, hence, the relevant bifurcation analysis in case $\eta = 0.75$, for Ω smaller than -0.57, is that of Section III.2. Let us mention that other interesting numerical computations, done in particular by Afendikov and Babenko [Af-Ba], show where the cubic coefficient c vanishes in function of the radius ratio η and rotation ratio Ω.

In the neighborhood of the case when c vanishes, it is necessary to compute the coefficient of ρ^4 in $g(\mu,\rho^2)$. Numerical evidence shows that

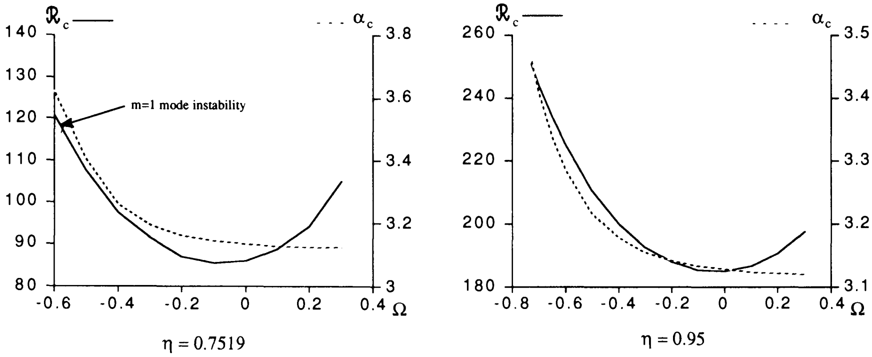


FIGURE III.1.

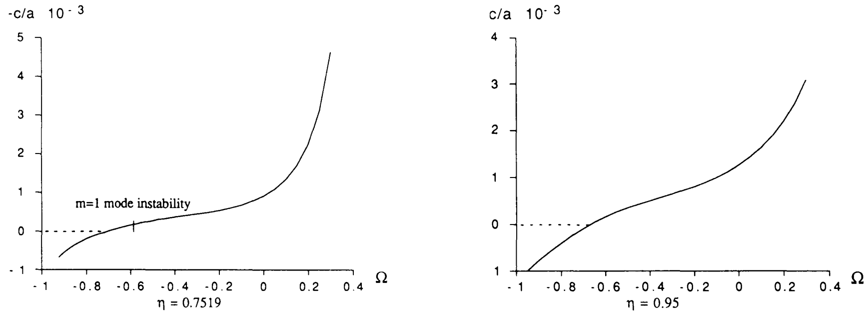


FIGURE III.2.

this coefficient is negative (see [La-De] and codimension-two problems in Chapter VI) which implies that one still can observe Taylor vortices in this case (see Subsection III.3.1).

III.1.3 Geometrical pattern of the Taylor cells

After Couette flow has lost stability to a stationary mode, the flow that is experimentally observed has a very remarkable structure. This flow is axisymmetric and is organized into a periodic system of horizontal cells, with boundaries not crossed by particles. These are called the Taylor cells, in honor of G. I. Taylor, who not only observed them in 1921 but was first

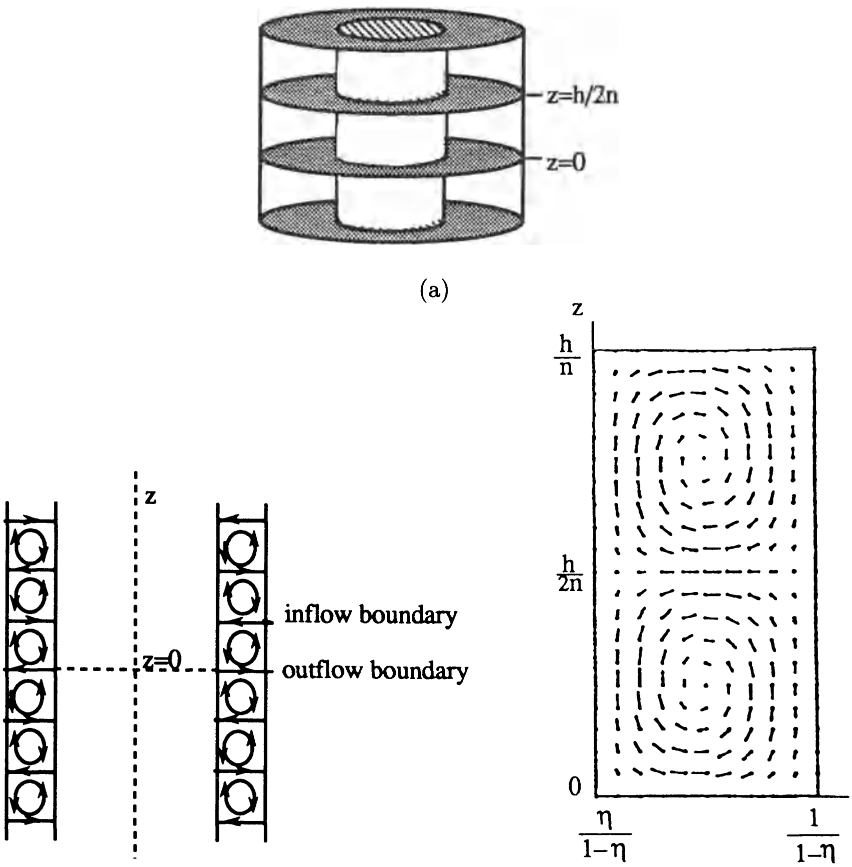


FIGURE III.3. (a) A sketch of the system of cells; (b) Projection of Taylor vortices in a meridian plane; and (c) View of the streamlines in the meridian plane for $\eta = 0.95$, and $\Omega = 0$.

to give the linear stability analysis of axisymmetric perturbations taking account of viscosity [Tay]. In this short paragraph we show that the pattern observed in experiments has a simple algebraic explanation. We note $V = (v_r, v_\theta, v_z) = V^{(0)} + U_0$, the velocity field representing the perturbation of the Couette flow by the stationary bifurcation. We fix the arbitrary phase of the amplitude A to be 0, so that for the bifurcated solution U_0 we have $\mathbf{S}X = X$ and therefore, $\mathbf{S}V = V$ since $\mathbf{S}\Phi(\mu, X) = \Phi(\mu, \mathbf{S}X)$ in (II.35). Moreover, the flow is periodic along the z-axis with period h/n, because $\mathbf{T}_{2\pi/n}X = X$ and hence $T_{2\pi/n}V = V$.

Claim. *Fluid particles are confined to horizontal cells of height $h/2n$. Moreover, the flow $V^{(0)} + U_0$ inside the cell $-h/2n < z < 0$ is the symmetric image cell under* $\mathbf{S}$ *of the flow in the symmetric cell $0 < z < h/2n$.*

Proof of the claim. Since $\mathbf{S}V = V$, we have the relation $v_z(r, \theta, -z) = -v_z(r, \theta, z)$. Therefore, $v_z(r, \theta, 0) = 0$. By periodicity we also have $v_z(r, \theta, -h/2n) = v_z(r, \theta, h/2n)$, hence, $v_z(r, \theta, h/2n) = 0$. These flat boundaries are of course repeated by periodicity. The same $\mathbf{S}$-invariance leads immediately to the second part of the claim.

In Figures III.3a-c we illustrate these properties. Figure III.3(a) is a sketch of the system of cells, while Figures III.3(b) and III.3(c) show a picture of the projection of the velocity field on a meridian plane.

Notice that $\mathbf{T}_{\pi/n}U_0$ is also a Taylor flow invariant under symmetry $\mathbf{S}$. The difference is that on the plane $z = 0$, the flow $V^{(0)} + U_0$ goes outward (one says that $z = 0$ in an outflow boundary), while the flow $V^{(0)} + \mathbf{T}_{\pi/n}U_0$ goes inward at $z = 0$, which is then an inflow boundary (see Figure III.3(b)).

III.2 Spirals and ribbons

III.2.1 The Hopf bifurcation with O(2)-symmetry

We suppose now that the operator L_0 has a pair of critical purely imaginary eigenvalues $\pm \mathrm{i}\omega_0$. We know from (II.33) that these eigenvalues are double, that the eigenvectors associated with $\mathrm{i}\omega_0$ have the form

$$\begin{aligned} \zeta_1(r, \theta, z) &= \hat{U}(r)e^{\mathrm{i}(m\theta + \alpha z)}, \\ \zeta_2(r, \theta, z) &= \mathbf{S}\zeta_1(r, \theta, z) = \mathbf{S}\hat{U}(r)e^{\mathrm{i}(m\theta - \alpha z)}, \end{aligned}$$

and that the eigenvectors associated with $-\mathrm{i}\omega_0$ are the conjugates of ζ_1 and ζ_2. Therefore, $\mathbf{V}$ has dimension four and we write

$$X = A_1\zeta_1 + A_2\zeta_2 + \bar{A}_1\bar{\zeta}_1 + \bar{A}_2\bar{\zeta}_2. \tag{III.16}$$

In the eigenspace belonging to $\mathrm{i}\omega_0$ and in the basis just defined, the group Γ acts by the following matrix representations:

$$\mathbf{R}_\varphi = e^{\mathrm{i}m\varphi}\begin{pmatrix} 1 0 \\ 0 1 \end{pmatrix}, \quad \mathbf{T}_\psi = \begin{pmatrix} e^{\mathrm{i}n\psi} & 0 \\ 0 & e^{-\mathrm{i}n\psi} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 0 1 \\ 1 0 \end{pmatrix}. \tag{III.17}$$

This is a Hopf bifurcation problem with O(2)-symmetry (see, for instance, [Go-St-Sc]). In fact, the additional circular symmetry (SO(2) acting via $\mathbf{R}_\varphi$) forces the full equation (II.34) to coincide with its normal form, because the group action induced by the normalization (see (II.43)–(II.44)) coincides with the action of $\mathbf{R}_\varphi$. As in the steady-state case, however, removing this circular symmetry would not change the qualitative features of the bifurcation diagrams and phase portraits. The Hopf bifurcation with O(2)-symmetry was studied by Ruelle in 1973 [Ru] and rediscovered about ten years after by several authors (see references in [Go-St-Sc]). In the following we present the most suitable version for our purpose (see also [Ch-Io]). The first "modern" treatment of the present situation appears to be due to Babenko, Afendikov, and Yur'ev [B-A-Y 81]; see also [B-A-Y 82] for the stability analysis and [Af] for a complete use of group invariances.

As before, the main step in our analysis is to compute the "equivariant structure" of the vector field $F(\mu, X)$. Let F_1 and F_2 respectively denote the components of F along ζ_1 and ζ_2.

Claim. $F_1(\mu, X) = A_1 f(\mu, |A_1|, |A_2|)$ and $F_2(\mu, X) = A_2 f(\mu, |A_2|, |A_1|)$ where f is complex valued and is even with respect to its second and third arguments.

Proof of the claim. (i) Let us write $X = (A_1, A_2, \bar{A}_1, \bar{A}_2)$ and in polar coordinates $A_j = \rho_j e^{\mathrm{i}\psi_j} (j = 1, 2)$. Owing to the equivariance under the group Γ, we know that for any θ, ψ, A_1, A_2

$$\begin{aligned} &F_1(\mu, e^{\mathrm{i}(m\theta+n\psi)} A_1, e^{\mathrm{i}(m\theta-n\psi)} A_2, e^{-\mathrm{i}(m\theta+n\psi)} \bar{A}_1, \\ &e^{\mathrm{i}(-m\theta+n\psi)} \bar{A}_2) = e^{\mathrm{i}(m\theta+n\psi)} F_1(\mu, A_1, A_2, \bar{A}_1, \bar{A}_2). \end{aligned} \tag{III.18}$$

We choose successively θ and ψ such that

(1) $m\theta + n\psi = -\psi_1, \; m\theta - n\psi = -\psi_2$,
(2) $m\theta + n\psi = \pi, \; m\theta - n\psi = 0$, and finally
(3) $m\theta + n\psi = 0, \; m\theta - n\psi = \pi$,

it is easy to show that F_1 is odd in $(A_1, \bar{A}_1)$ and even in $(A_2, \bar{A}_2)$ and that

$$F_1(\mu, A_1, A_2, \bar{A}_1, \bar{A}_2) = e^{\mathrm{i}\psi_1} F_1(\mu, \rho_1, \rho_2, \rho_1, \rho_2).$$

(ii) By applying the symmetry S we have

$$F_2(\mu, A_1, A_2, \bar{A}_1, \bar{A}_2) = F_1(\mu, A_2, A_1, \bar{A}_2, \bar{A}_1).$$

which completes the proof.

We can rewrite the amplitude equations in the form

$$\begin{aligned} \frac{dA_1}{dt} &= A_1 f(\mu, |A_1|, |A_2|) = A_1[\mathrm{i}\omega_0 + a\mu + b|A_1|^2 + c|A_2|^2 + \text{ h.o.t.}], \\ \frac{dA_2}{dt} &= A_2 f(\mu, |A_2|, |A_1|) = A_2[\mathrm{i}\omega_0 + a\mu + c|A_1|^2 + b|A_2|^2 + \text{ h.o.t.}], \end{aligned} \tag{III.19}$$

where h.o.t. $= O\{\mu^2 + \mu(|A_1|^2 + |A_2|^2) + (|A_1|^2 + |A_2|^2)^2\}$. Notice the terms $i\omega_0$ that come from the linear part of the vector field at $\mu = 0$. It follows that the phases and the moduli parts of the amplitudes are uncoupled. We have

$$\frac{d\rho_1}{dt} = \rho_1 f_r(\mu, \rho_1, \rho_2), \qquad \frac{d\rho_2}{dt} = \rho_2 f_r(\mu, \rho_2, \rho_1), \tag{III.20}$$

$$\frac{d\psi_1}{dt} = f_i(\mu, \rho_1, \rho_2), \qquad \frac{d\psi_2}{dt} = f_i(\mu, \rho_2, \rho_1), \tag{III.21}$$

where f_r and f_i are the real and imaginary parts of f.

Solving $\rho_1 f_r(\mu, \rho_1, \rho_2) = \rho_2 f_r(\mu, \rho_2, \rho_1) = 0$, we get

$$\text{either} \quad \rho_1\rho_2 = 0 \quad \text{or} \quad [\rho_1^2 - \rho_2^2]\{b_r - c_r + \text{ h.o.t.}\} = 0.$$

Hence, if $b_r \neq c_r$, and $a_r \neq 0$, we obtain two types of equilibria which bifurcate from the trivial state at $\mu = 0$ and correspond to two types of periodic solutions of (II.34):

(1) Rotating waves. $A_1 = \rho e^{i(\omega t + \psi)}$, $A_2 = 0$, where ρ satisfies

$$f_r(\mu, \rho, 0) = 0, \qquad \omega = f_i(\mu, \rho, 0),$$

and ψ is an arbitrary phase. Of course, the symmetric solutions $A_1 = 0$, $A_2 = \rho e^{i(\omega t + \psi)}$ exist as well. These solutions are waves rotating in two opposite directions. In fact, we can write

$$X(t) = \mathbf{R}_{\omega t/m}\mathbf{R}_{\psi/m}X_0 = \mathbf{T}_{\omega t/n}\mathbf{T}_{\psi/n}X_0, \tag{III.22}$$

where $X_0 = \rho(\zeta_1 + \bar{\zeta}_1)$. The group orbit of these solutions reduces to two circles (trajectories) that are exchanged by the symmetry $\mathbf{S}$.

(2) Standing waves. $A_1 = \rho e^{i(\omega t + \varphi_1)}$, $A_2 = \rho e^{i(\omega t + \varphi_2)}$, where ρ satisfies

$$f_r(\mu, \rho, \rho) = 0, \qquad \omega = f_i(\mu, \rho, \rho),$$

and φ_1, φ_2 are arbitrary phases. These solutions have an "up and down" symmetry since they satisfy:

$$\mathbf{T}_{(\varphi_1 - \varphi_2)/n}\mathbf{S}X = X. \tag{III.23}$$

The rotational part of the group Γ acts on these solutions by

$$\mathbf{R}_\varphi\mathbf{T}_\psi X(t) = \rho e^{i(\omega t + m\varphi)}[e^{i(n\psi + \varphi_1)}\zeta_1 + e^{-i(n\psi - \varphi_2)}\mathbf{S}\zeta_1] + \text{ complex conj.} \tag{III.24}$$

By shifting suitably the axial and azimuthal coordinates we can assume that φ_1 and φ_2 are 0. To obtain this, it is sufficient to act with $\mathbf{R}_\varphi\mathbf{T}_\psi$ on X, where $\varphi = -(\varphi_1 + \varphi_2)/2m$ and $\psi = (\varphi_2 - \varphi_1)/2n$. Then one has

$$X(t) = \mathbf{T}_{\omega t/n}X_0 + \mathbf{T}_{-\omega t/n}\mathbf{S}X_0 = \mathbf{R}_{\omega t/m}(X_0 + \mathbf{S}X_0), \tag{III.25}$$

where $X_0 = \rho(\zeta_1+\bar{\zeta}_1)$. This explains the *standing wave structure in the axial direction, and also the rotating wave structure in the azimuthal direction.* In fact, since Φ in the center manifold expression (II.35) commutes with the group action Γ, we also have, due to the propagation of these identities on $X(t)$, $\mathbf{R}_{\pi/m}\mathbf{T}_{\pi/n}U(t) = U(t)$ for the corresponding solution of Navier-Stokes equations and $U(t) = \mathbf{R}_{\omega t/m}U(0)$. The group orbit is now a 2-torus, parameterized by φ_1 and φ_2, and the cross product of a cycle (trajectory) by the O(2)-orbit.

The equations (III.20) have no other bounded solution close to 0, except $X = 0$. Their local phase portrait is determined by the asymptotic behavior of the equilibria, i.e., by the eigenvalues of the linearized vector field at each (local) equilibrium. The decoupling between phases and moduli of amplitudes implies that the (orbital) stability of the bifurcated periodic solutions only depends on the asymptotic behavior of the corresponding equilibrium solutions of equations (III.20).

The branch of rotating waves is given by

$$\mu = -b_r\rho^2/a_r + O(\rho^4), \tag{III.26}$$

while the branch of standing waves is given by

$$\mu = -(b_r + c_r)\rho^2/a_r + O(\rho^4), \tag{III.27}$$

From this and equations (III.20), an easy calculation shows that the eigenvalues that determine the stability of the *rotating waves* are

$$\sigma_0 = 2b_r\rho^2 + O(\rho^4) \quad \text{and} \quad \sigma_1 = (c_r - b_r)\rho^2 + O(\rho^4), \tag{III.28}$$

while the eigenvalues for the *standing waves* are

$$\sigma_0' = 2(b_r + c_r)\rho^2 + O(\rho^4) \quad \text{and} \quad \sigma_1' = 2(b_r - c_r)\rho^2 + O(\rho^4). \tag{III.29}$$

The phase portrait depends on the coefficients b_r and c_r only, except of course at the exceptional values at which the vector field is singular. In Figure III.4 we show the different possible phase diagrams depending on the (b_r, c_r)-plane, in the case $\mu > 0$ (supercritical bifurcation), which is the only case of interest in the Couette-Taylor problem.

Remark. If the eigenvectors were axisymmetric ($m = 0$).

In the standard physical problem, up to now both experiments and numerical computations have shown that when the critical modes are oscillating (Hopf bifurcation), they are nonaxisymmetric ($m \neq 0$). In fact, we could imagine, by changing the physical setting, that oscillating critical modes could be axisymmetric ($m = 0$) in (III.16)). All the discussions of this section would remain valid, provided the action of $\mathbf{R}_\varphi$ (now the identity) is replaced by the action generated by translations in time (Normal Form Theorem of Chapter II). This group action $(A, B) \to (Ae^{-i\omega_0 t}, Be^{-i\omega_0 t})$ is identical to the action of the group SO(2) when $m \neq 0$.

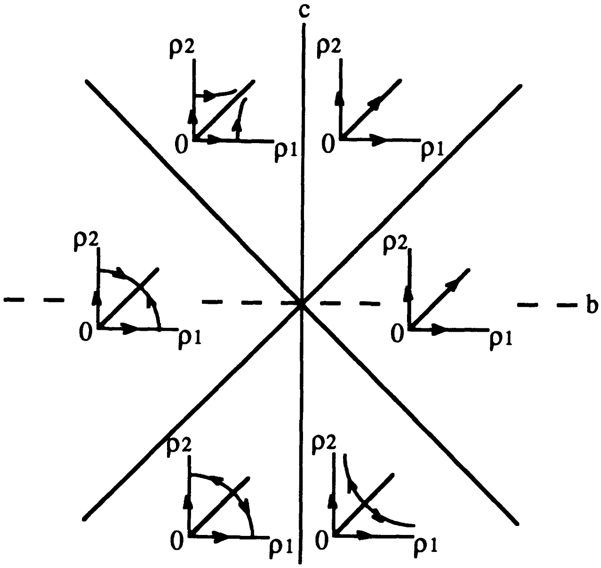


FIGURE III.4. ($\mu > 0$).

The rotating waves solutions would be axisymmetric in this case, traveling with constant velocity along the axis. The standing waves would also be axisymmetric with "flat" cells as above but "axially pulsating".

III.2.2 Application to the Couette-Taylor problem

As mentioned in Subsection III.2.1, the above results are those generally obtained with the study of a Hopf bifurcation with O(2) symmetry, because the action of the additional SO(2) symmetry of the Couette-Taylor problem coincides with the group action induced by the normalization of the vector field (see II.43)–(II.44). The direction of bifurcation of the two types of solutions is given by formulas (III.26) and (III.27); hence, it depends on the values of the coefficients b_r and c_r (we know that $a_r > 0$ due to the stability of the Couette flow for $\mu < 0$). As for the Taylor vortices, we can compute these coefficients numerically for different values of η and Ω and plot the corresponding point in the (b_r, c_r)-plane of Fig. III.4. Let us first derive the expression of the coefficients in terms of the operators defined in (III.1). Using the form (III.16) $X = A_1\zeta_1 + A_2\zeta_2 + \bar{A}_1\bar{\zeta}_1 + \bar{A}_2\bar{\zeta}_2$, we expand $\Phi(\mu, X)$ as

$$\Phi(\mu, X) = \sum_{pqrst} \mu^p A_1^q \bar{A}_1^r A_2^s \bar{A}_2^t \Phi_{qrst}^p, \tag{III.30}$$

where, due to the commutativity with $\mathbf{S}$, we have $\Phi^p_{qrst} = \mathbf{S}\Phi^p_{stqr}$. Then, formula (II.40) gives, for the coefficient of μA_1:

$$(-L_0 + \mathrm{i}\omega_0)\Phi^1_{1000} = L_1\zeta_1 - a\zeta_1, \tag{III.31}$$

and (II.41) gives, for the coefficients of $A_1|A_1|^2$ and $A_1|A_2|^2$:

$$\begin{aligned}(-L_0 + \mathrm{i}\omega_0)\Phi^0_{2100} &= 2N_0(\zeta_1, \Phi^0_{1100}) + 2N_0(\bar{\zeta}_1, \Phi^0_{2000}) - b\zeta_1,\\ (-L_0 + \mathrm{i}\omega_0)\Phi^0_{1011} &= 2N_0(\zeta_1, \Phi^0_{0011}) + 2N_0(\zeta_2, \Phi^0_{1001})\\ &\quad +2N_0(\bar{\zeta}_2, \Phi^0_{1010}) - c\zeta_1,\end{aligned} \tag{III.32}$$

where Φ^0_{1100}, Φ^0_{2000}, Φ^0_{0011}, Φ^0_{1010}, Φ^0_{1001} satisfy the linear equations

$$\begin{aligned}(-L_0 + 2\mathrm{i}\omega_0)\Phi^0_{2000} &= N_0(\zeta_1, \zeta_1),\\ -L_0\Phi^0_{1100} &= 2N_0(\zeta_1, \bar{\zeta}_1),\\ (-L_0 + 2\mathrm{i}\omega_0)\Phi^0_{1010} &= 2N_0(\zeta_1, \zeta_2),\\ -L_0\Phi^0_{1001} &= 2N_0(\zeta_1, \bar{\zeta}_2),\\ \Phi^0_{0011} &= \mathbf{S}\Phi^0_{1100}.\end{aligned} \tag{III.33}$$

As in the case of Taylor cells, we reduce equations (III.31)–(III.33) to an ordinary system of differential equations in the radial coordinate r by noticing that

$$\Phi^p_{qrst} = \hat{\Phi}^p_{qrst} e^{\mathrm{i}[(q-r-s+t)\alpha z + (q-r+s-t)m\theta]}.$$

For the sake of clarity we define

$$w_1 = \hat{\Phi}^0_{2000}, \quad w_2 = \hat{\Phi}^0_{1100}, \quad w_3 = \hat{\Phi}^0_{0011}, \quad w_4 = \hat{\Phi}^0_{1001}, \quad w_5 = \hat{\Phi}^0_{1010}.$$

Then, using the notation introduced in Section II.3, we can rewrite (III.31)–(III.33) as follows (with homogeneous boundary conditions):

$$\begin{aligned}(L_{\alpha m} - \mathrm{i}\omega_0)\hat{\Phi}^1_{1000} - D_{\alpha m}\hat{q}_1 &= a\hat{U} + V^{(0)}.\nabla_{\alpha m}\hat{U} + \hat{U}.\nabla_{00}V^{(0)},\\ \nabla_{\alpha m}.\hat{\Phi}^1_{1000} &= 0;\end{aligned} \tag{III.34a}$$

$$\begin{aligned}(L_{\alpha m} - \mathrm{i}\omega_0)\hat{\Phi}^1_{2100} - D_{\alpha m}\hat{q}_2 &= b\hat{U} + \mathfrak{R}_c[\hat{U}.\nabla_{00}w_2 + w_2.\nabla_{\alpha m}\hat{U}\\ &\quad +\bar{\hat{U}}.\nabla_{2\alpha,2m}w_1 + w_1.\nabla_{-\alpha,-m}\bar{\hat{U}}],\\ \nabla_{\alpha m}.\hat{\Phi}^1_{2100} &= 0;\end{aligned} \tag{III.34b}$$

$$\begin{aligned}(L_{\alpha m} - \mathrm{i}\omega_0)\hat{\Phi}^1_{1011} - D_{\alpha m}\hat{q}_3 &= c\hat{U} + \mathfrak{R}_c[\hat{U}.\nabla_{00}w_3 + w_3.\nabla_{\alpha m}\hat{U}\\ &\quad +(\mathbf{S}\hat{U}).\nabla_{2\alpha,0}w_4 + w_4.\nabla_{-\alpha,m}(\mathbf{S}\hat{U})\\ &\quad +(\overline{\mathbf{S}\hat{U}}).\nabla_{0,2m}w_5 + w_5.\nabla_{\alpha,-m}(\overline{\mathbf{S}\hat{U}})],\\ \nabla_{\alpha m}.\hat{\Phi}^1_{1010} &= 0,\end{aligned} \tag{III.34c}$$

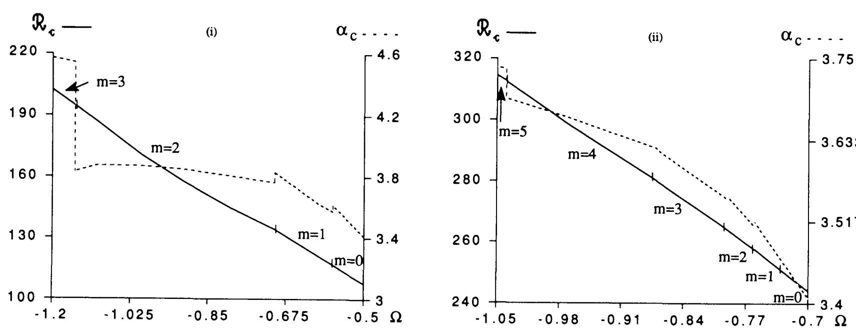
with the w_j's solving the following equations:

$$
\begin{aligned}
(L_{2\alpha,2m} - 2\mathrm{i}\omega_0)w_1 - D_{2\alpha,2m}\hat{q}_4 &= \mathcal{R}_c\hat{U}.\nabla_{\alpha m}.\hat{U},\\
\nabla_{2\alpha,2m}.w_1 &= 0,\\
L_{00}w_2 - D_{00}\hat{q}_5 &= \mathcal{R}_c[\hat{U}.\nabla_{-\alpha,-m}\bar{\hat{U}} + \bar{\hat{U}}.\nabla_{\alpha m}\hat{U}],\\
\nabla_{00}.w_2 &= 0,\\
L_{2\alpha,0}w_4 - D_{2\alpha,m}\hat{q}_6 &= \mathcal{R}_c[\hat{U}.\nabla_{\alpha,-m}(\overline{\mathbf{S}\hat{U}}) + (\overline{\mathbf{S}\hat{U}})\nabla_{\alpha m}.\hat{U}],\\
\nabla_{2\alpha,0}.w_4 &= 0,\\
(L_{0,2m} - 2\mathrm{i}\omega_0)w_5 - D_{0,2m}\hat{q}_7 &= \mathcal{R}_c[\hat{U}.\nabla_{-\alpha m}\mathbf{S}\hat{U} + \mathbf{S}\hat{U}.\nabla_{\alpha m}.\hat{U}],\\
\nabla_{0,2m}.w_5 &= 0,
\end{aligned}
\tag{III.35}
$$

and $w_3 = \mathbf{S}w_2$.

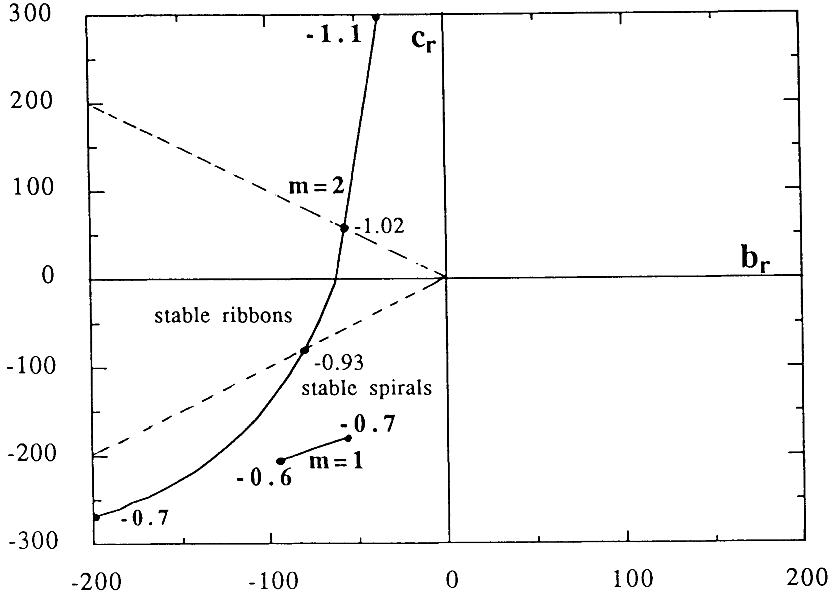
In (III.35) all the operators on the left-hand side are invertible, so these equations are all solved by using a standard numerical scheme (see [De-Io] and [La-De]). For solving (III.34a–c) one has to apply the Fredholm Alternative. The compatibility conditions give the values of the coefficients a, b, and c. In [De-Io] and [La-De] we show how to compute these compatibility conditions without the need of the adjoint operator L_0^*.

In [De-Io] these calculations have been done for two different types of apparatus: one with radius ratio $\eta = 0.7519$, the other with $\eta = 0.95$. In Figure III.5(a) we give the critical Reynolds numbers and critical wave numbers (see [L-T-K-S-G] for values of η different from those considered here), and in Figure III.5(b) we plot the points (b_r, c_r) when Ω is varied. The "graph" in Figure III.5(b) is composed of different segments, each corresponding to the relevant (different) critical azimuthal mode m. We see

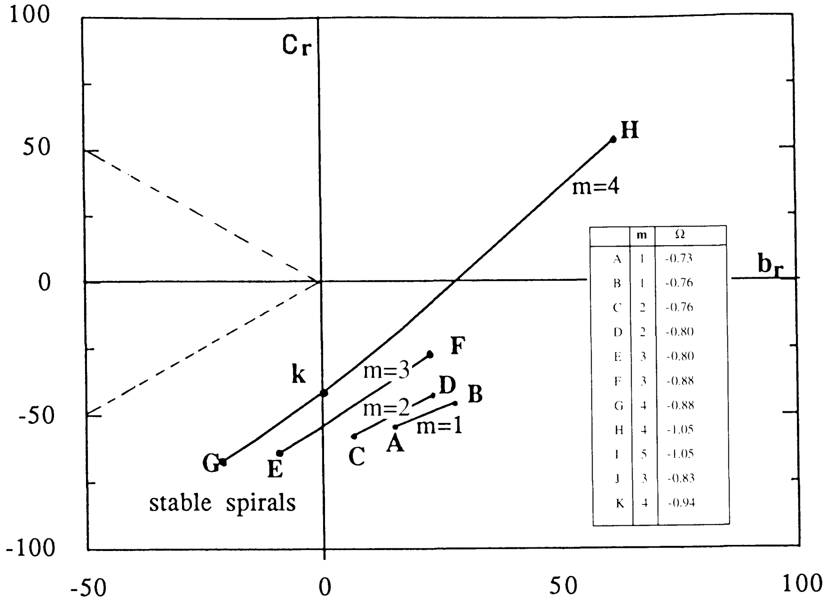


(a)

FIGURE III.5. (a) Critical Reynolds numbers and wave numbers; (b) preferred solutions for different values of the parameters; (i) $\eta = 0.75$, (ii) $\eta = 0.95$.



(i)



(ii)

(b)

FIGURE III.5. *Continued*

that in the case $\eta = 0.7519$ when $m = 2$, there exists a range of values of Ω for which the standing waves are stable (approximately $-1.02 < \Omega < -0.94$), while they seem never to be stable in the other case $\eta = 0.95$. Notice that, in both examples, spirals with different values of the azimuthal wave number m can be stable, depending on Ω. These results are in agreement with those announced by DiPrima and Grannick in 1971 [DP-G]. They used formal expansions in power of the amplitudes, with a basis different from ours.

III.2.3 Geometrical structure of the flows

III.2.3.1 Spirals

The geometrical structure of the rotating waves can be deduced by using the commutativity of Φ in (III.30) with the symmetries. In fact, we have (III.22) with

$$\mathbf{R}_{\varphi/m} X_0 = \mathbf{T}_{\varphi/n} X_0 \text{ for all } \varphi \text{ in } \mathbb{R}, \quad \text{and} \quad \mathbf{R}_{2\pi/m} X_0 = X_0, \tag{III.36}$$

and the action of the time shift σ_τ is such that

$$\sigma_\tau X(t) = X(t+\tau) = \mathbf{R}_{\omega\tau/m} X(t). \tag{III.37}$$

Now the bifurcating velocity field of our problem is given by

$$V(t) = \mathbf{R}_{(\omega t+\psi)/m} V_0 = \mathbf{T}_{(\omega t+\psi)/n} V_0, \tag{III.38}$$

where $V_0 = V^{(0)} + X_0 + \Phi(\mu, X_0)$ and $X_0 = \rho(\zeta_1 + \bar{\zeta}_1)$ (see (II.35)). This means that we have the following identities in cylindrical coordinates:

$$\begin{aligned} V(r,\theta,z,t) &= V_0(r,\theta+(\omega t+\psi)/m, z) = V_0(r,\theta,z+ \\ &\quad (\omega t+\psi)/\alpha), \\ V(r,\theta+\varphi/m, z+a, t+\tau) &= V_0(r,\theta+(\omega t+\psi)/m, z+ \\ &\quad (\omega\tau+\varphi)/\alpha + a); \end{aligned}$$

hence, in making $a = -z$, $\tau = -t$, $\varphi = \alpha z + \omega t$, we obtain

$$V(r,\theta,z,t) \equiv V(r, (m\theta + \alpha z + \omega t)/m, 0, 0), \tag{III.39}$$

i.e., the three variables (θ, z, t) only appear under the form $(m\theta + \alpha z + \omega t)$ and the velocity vector field is 2π periodic in this new variable. It is then clearly a *spiral structure*. In addition, we see by (III.39) that rotating θ by 2π is equivalent to shifting z by $2\pi m/\alpha = mh/n$. This gives a practical criterion for the determination of the number m when observing spirals in an experiment. In Figure III.6(a) we show two different views in projection of the velocity field for a radius ratio of 0.95: (a) projection (r, z), (b) projection (θ, z) at $r = (1+\eta)/2(1-\eta)$ (middle of the gap). In Figure III.6(b) we give a sketch of the spirals (see photographs in the introduction).

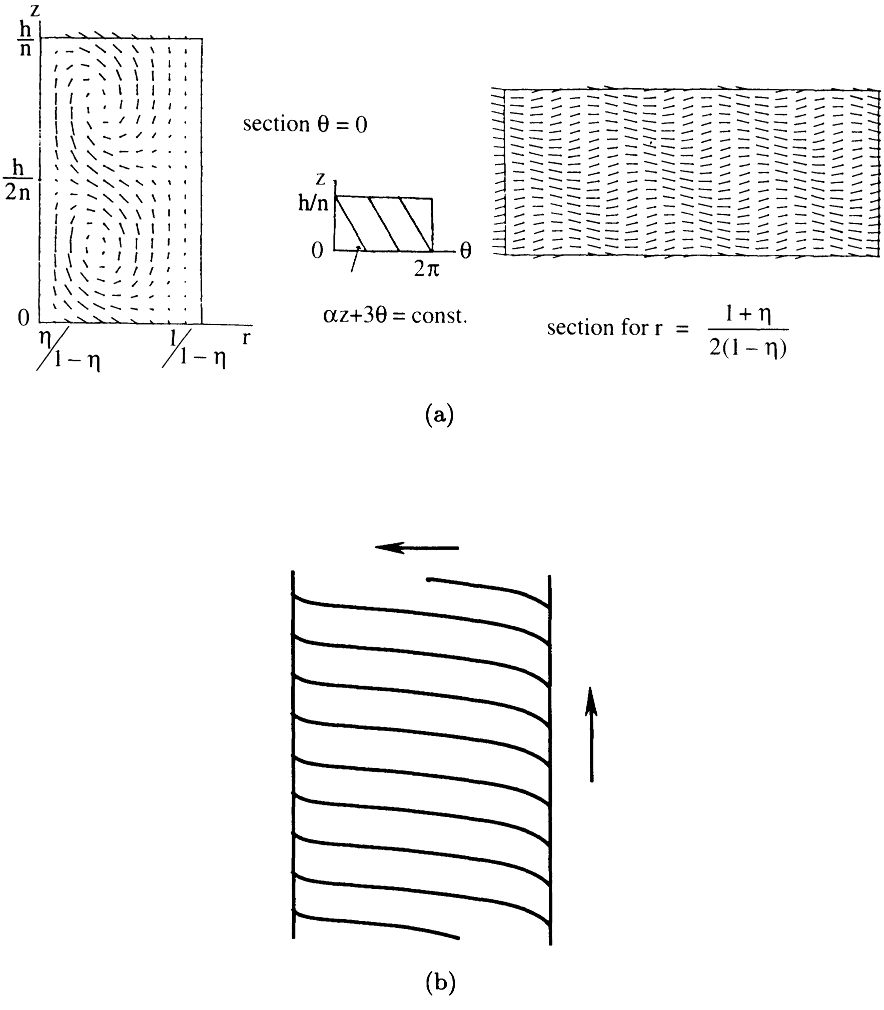


FIGURE III.6. (a) Velocity field for $\eta = 0.95$, $\Omega = -0.8$, $\mathcal{R}_c = 266.2$, $m = 3$. The vertical scale is exaggerated on the figure of the right-hand side. (b) Typical form of the spiral waves. Arrows show the drift along and the regular rotation around the axis.

Remark. We noticed in Subsection II.2.2 that the velocity flux through a cross section of the cylindrical domain is independent of z. Due to the spiral structure of the velocity field, it results that its flux is also independent of time t, and hence it is constant. It would be interesting to know what this velocity flux is through a horizontal cross section for such spiral-wave flows. Indeed, if we choose the functional frame described in Subsection II.2.1, the flux may very well be different from zero, despite the fact that the principal part in the space **V** has a zero flux (we thank V.N. Shtern from Novosibirsk for attracting this point to our attention). Hence the flux is $O(\mu)$ in such a case. Now, if we choose the functional frame described in Subsection II.2.2, the flux is zero by construction, and the pressure has a nonzero mean gradient along the z-axis. The difference between these two cases appears in the computation of w_2 and w_3 in Subsection III.2.2, where the operator L_{00} (with 0 on the right-hand side) is defined in (II.31)–(II.32) modified by $(\text{II.31})_{00}$. It appears from recent computations made by Raffaï (see [Raf]) that the numerical results for coefficients are not really affected by such a change in the functional frame. Notice that, in the theory developed in Chapter VII, we look for all time periodic solutions with a zero flux and do not impose a priori any spatial periodicity.

III.2.3.2 Ribbons

The geometrical structure of the standing waves given by (III.27) is such that, after a suitable shift along the z-axis, as in (III.25), these solutions are invariant under the symmetry **S**. Therefore, the claim of Subsection III.1.3 applies directly to show that the flow is confined into flat horizontal cells of height $h/2n$. The difference with the Taylor cells is that now the flow is not axisymmetric but *azimuthally periodic*. The full pattern rotates solidly at an angular velocity ω (rotating wave in the azimuthal direction). The wavy structure inside each cell can be specified by the existence of an additional symmetry for these solutions, which does not play any role in the mathematical analysis: by means of (III.16)–(III.17), we remark that combining a rotation of angle π/m with an axial translation of "angle" $\psi = \pi/n$ does not change the solution $X(t)$, hence the same holds for the velocity vector field $V(t)$. A sketch of an admissible pattern with these symmetries is drawn in Figure III.7. The general shape of this pattern led us [Ch-Io] to call this bifurcating flow "ribbons". The velocity field projected

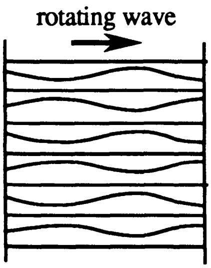


FIGURE III.7. Admissible pattern for ribbons.

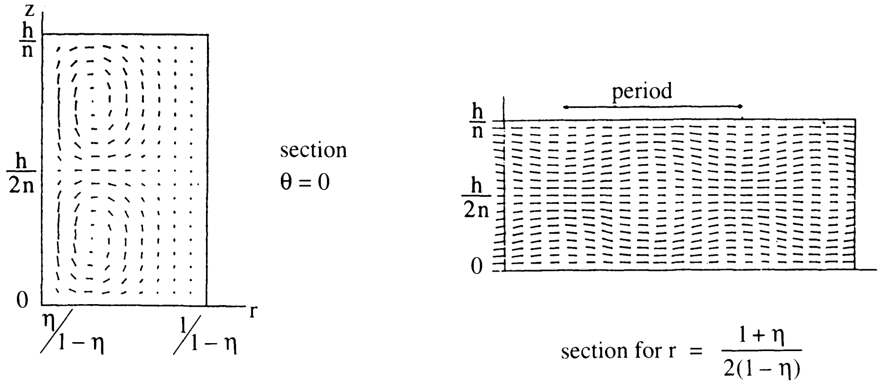


FIGURE III.8. Velocity field for $\eta = 0.75$, $\Omega = -0.95$, $\mathcal{R}_c = 163.6$, $m = 2$.

respectively on the meridian plane $(r,\ z)$ and on a cylindrical section (r fixed) is shown on Figure III.8 for a radius ratio of 0.75 in a specific case of stable flow.

III.3 Higher codimension bifurcations

III.3.1 Weakly subcritical Taylor vortices

In Subsection III.1.1 we derived the amplitude equation (III.6) when the Couette flow loses its stability by a real eigenvalue of the linear operator L_μ crossing 0 at $\mu = 0$ (i.e., $\mathcal{R} = \mathcal{R}_c$). Let us rewrite this equation for $A = \rho e^{i\psi}$ (polar coordinates) as

$$\frac{d\psi}{dt} = 0 \quad \text{and} \quad \frac{d\rho}{dt} = \rho g(\mu, \rho^2), \tag{III.40}$$

where g is a real function with Taylor expansion near (0, 0):

$$g(\mu, x) = a\mu + cx + dx^2 + e\mu x + f\mu^2 + \mathrm{O}\left((|x| + |\mu|)^3\right). \tag{III.41}$$

As shown on Figure III.2, the coefficient $-c/a$ is in general nonzero; hence, one easily deduces, in solving $g(\mu, x) = 0$ by the implicit function theorem with respect to $x = \rho^2$, the existence of a one-sided (pitch-fork) bifurcation of equilibria (III.15) for (III.40), which was shown to correspond to the Taylor vortices observed in experiments. However, for a radius ratio η close to 1, this coefficient can vanish at a value Ω^* of Ω for which the primary instability of the Couette flow still occurs via a steady-state bifurcation. This is notably the case for $\eta = 0.95$, as shown in Figure III.2.

When Ω is close to Ω^*, c is close to 0 in (III.41) and the bifurcation analysis of Section III.1 is no longer valid because higher-order terms in the Taylor expansion (III.41) are not negligible compared to cx. In order to study this phenomenon one has to consider $\Omega - \Omega^*$ as a small parameter; hence, this is in essence a *codimension-two bifurcation* problem. The implicit function theorem leads to the following expansion for the stationary solutions of (III.40):

$$\mu = -\frac{c}{a}\rho^2 + \left[-\frac{d}{a} + \frac{c}{a}\left(\frac{e}{a} - \frac{cf}{a^2}\right)\right]\rho^4 + \mathrm{O}(\rho^6). \tag{III.42}$$

The coefficient d in (III.42) has been computed numerically by Laure and Demay [La-De] for $\eta = 0.95$ and when $\Omega = \Omega^*$. This is the same kind of calculation used in Subsection III.1.2. However, since the algebra for the derivation of relations like (III.14) becomes very tedious at higher orders, these relations have been generated automatically by means of a symbolic computation code (Macsyma). Notice that such a calculation for $\eta = 0.95$ was performed by Eagles in 1971 [Ea] for the case when the outer cylinder is at rest, which is "far" from the degenerate situation we are considering here. The result is that $-c/a$ becomes negative when $\Omega < \Omega^*$, while the principal part $-d/a$ of the coefficient of ρ^4 is *positive*. Hence, when $\Omega < \Omega^*$, the bifurcation is first subcritical, but the branches *bend back* to the direction $\mu > 0$. If $\Omega > \Omega^*$, the bifurcation is supercritical. The stability analysis follows the classical "exchange of stability" principle, as already mentioned in Section III.1.1; hence, if $\Omega < \Omega^*$, the solutions bifurcate unstably and

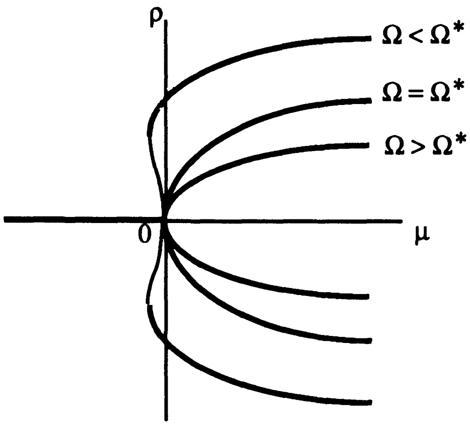


FIGURE III.9. Bifurcation diagrams for Ω close to Ω^* and η close to 1 (bold lines = stable solutions).

become stable after the turning point (see Figure III.9). Here, of course, "stability" means "orbital stability."

III.3.2 Competition between spirals and ribbons

We now turn to the case when the Couette flow loses stability by a pair of complex eigenvalues crossing the imaginary axis at $\mu = 0$. In Subsection III.2.1 we derived the amplitude equations on the center manifold. By writing the amplitudes in polar coordinates $A_j = \rho_j e^{i\psi_j}$ $(j = 1, 2)$, they take the form (III.20)–(III.21)

$$\frac{d\rho_1}{dt} = \rho_1 f_r(\mu, \rho_1^2, \rho_2^2), \qquad \frac{d\rho_2}{dt} = \rho_2 f_r(\mu, \rho_2^2, \rho_1^2), \tag{III.43}$$

$$\frac{d\psi_1}{dt} = f_i(\mu, \rho_1^2, \rho_2^2), \qquad \frac{d\psi_2}{dt} = f_i(\mu, \rho_2^2, \rho_1^2), \tag{III.44}$$

where f_r and f_i are the real and imaginary parts of a complex function f, the Taylor expansion of which we can write

$$f(\mu, x, y) = i\omega_0 + a\mu + bx + cy + dx^2 + exy + fy^2 + gx^3 + hx^2y + kxy^2 + ly^3 + \text{ h.o.t.} \tag{III.45}$$

By looking for the equilibria of the system (III.43), we showed that two kinds of time-periodic solutions bifurcated from the Couette flow, which we call spirals and ribbons. We also noticed that the direction of bifurcation and the stability of these solutions were given by the sign of the quantities b_r, $b_r + c_r$, and $b_r - c_r$, where b_r and c_r are the real parts of b and c (see Figure III.4). However, numerical computations of these coefficients have shown that when Ω is varied, stabilities can change, which means that one of the above quantities has changed sign. This is apparent in Figure III.5b, in which we notice, for example, that in the case $\eta = 0.7519$ an exchange of stabilities occurs between spirals and ribbons of azimuthal wave number 2 when $\Omega = \Omega^* \approx -0.93$. Let us now have a closer look at this situation. We assume that Ω is close to Ω^* and define (for commodity) $\nu = c_r - b_r$, which from the foregoing hypothesis is close to 0. Hence this is a *codimension-two* problem of bifurcation. The first-order dependence with respect to ρ of the eigenvalues that govern the stability of the spirals and ribbons is given, when $|\Omega - \Omega^*|$ is not small, by (III.28) and (III.29). Under the foregoing hypothesis, these eigenvalues depend on ρ and v, and their leading part is easy to compute. We get (using the notation of (III.28)–(III.29))

$$\begin{aligned} \sigma_0 &= 2b_r\rho^2 + \rho^2 O(\rho^2 + |\nu|), \\ \sigma_1 &= [\nu - (d_r - f_r)\rho^2]\rho^2 + \rho^2 O(\rho^2 + |\nu|)^2, \end{aligned} \tag{III.46}$$

$$\begin{aligned} \sigma_0' &= 2(b_r + c_r)\rho^2 + \rho^2 O(\rho^2 + |\nu|), \\ \sigma_1' &= 2[(d_r - f_r)\rho^2 - \nu]\rho^2 + \rho^2 O(\rho^2 + |\nu|)^2. \end{aligned} \tag{III.47}$$

The eigenvalues (III.46) give the stability of the spirals, while the eigenvalues (III.47) give the stability of the ribbons. A lengthy but straightforward computation shows the following:

(i) σ_1 vanishes when

$$\mu = \mu_1(\nu) = -\frac{b_r\nu}{a_r(d_r - f_r)} + \left[\frac{b_r(g_r - l_r)}{d_r - f_r} - d_r\right] \frac{\nu^2}{a_r(d_r - f_r)^2} + O(|\nu|^3);$$

(ii) σ_1' vanishes when

$$\mu = \mu_2(\nu) = -\frac{b_r\nu}{a_r(d_r - f_r)} + \left[\frac{b_r(3g_r + h_r - k_r - 3l_r)}{4(d_r - f_r)} - \frac{3d_r + e_r - f_r}{4}\right] \frac{\nu^2}{a_r(d_r - f_r)^2} + O(|\nu|^3).$$

Therefore, $\mu_1(\nu)$ is a bifurcation value for the branch of spirals and $\mu_2(v)$ is a bifurcation value for the branch of ribbons. Moreover, one can immediately infer from these formulas that in order to resolve these bifurcations, terms of order 7 are required in equations (III.43) since we have

$$\mu_1(\nu) - \mu_2(\nu) = \frac{-Q}{4a_r}\left(\frac{\nu}{d_r - f_r}\right)^2 + O(|\nu|^3), \tag{III.48}$$

where

$$Q = d_r - e_r + f_r - \frac{b_r(g_r - h_r + k_r - l_r)}{d_r - f_r}. \tag{III.49}$$

In order to study these bifurcations, we introduce the parameter ν in (III.43)–(III.44) and look again for branching solutions of this system.

Remark. The system (III.43)–(III.44) is the *normal form* for the Hopf bifurcation in a problem with O(2) symmetry (see Subsection III.2.1). The degenerate situation at which we are now looking was studied in a more general point of view (unfolding of singularities) in [Go-Ro] and [Na].

The spirals and the ribbons correspond, respectively, to equilibria of (III.43) with $(\rho_1, \rho_2) = (\rho, 0)$ [or $(0, \rho)$] and $(\rho_1, \rho_2) = (\rho, \rho)$. Introducing the small parameter ν in these equations allows us to find another kind of bifurcating equilibrium, with $\rho_1.\rho_2 \neq 0$ and $\rho_1 \neq \rho_2$. After the change of variables $s = \rho_1^2 + \rho_2^2$, $p = \rho_1^2\rho_2^2$, these solutions satisfy

$$\begin{aligned} s &= [(f_r - e_r + d_r)\nu + a_r(g_r - h_r + k_r - l_r)\mu]/(d_r - f_r)Q + O(\nu^2 + \mu^2), \\ p &= [b_r\nu + a_r(d_r - f_r)\mu]/(d_r - f_r)Q + O(\nu^2 + \mu^2). \end{aligned} \tag{III.50}$$

Then, solving (III.44), we find $\psi_j(t) = \omega_j t + \varphi_j$ $(j = 1, 2)$, with

$$\begin{aligned} \omega_1 + \omega_2 &= 2\omega_0 + [-2(b_r a_i/a_r) + b_i + c_i]\,(\rho_1^2 + \rho_2^2) + O(\rho_1^2 + \rho_2^2)^2, \\ \omega_1 - \omega_2 &= (b_i - c_i)(\rho_1^2 - \rho_2^2) + O(\rho_1^2 - \rho_2^2)^2. \end{aligned} \tag{III.51}$$

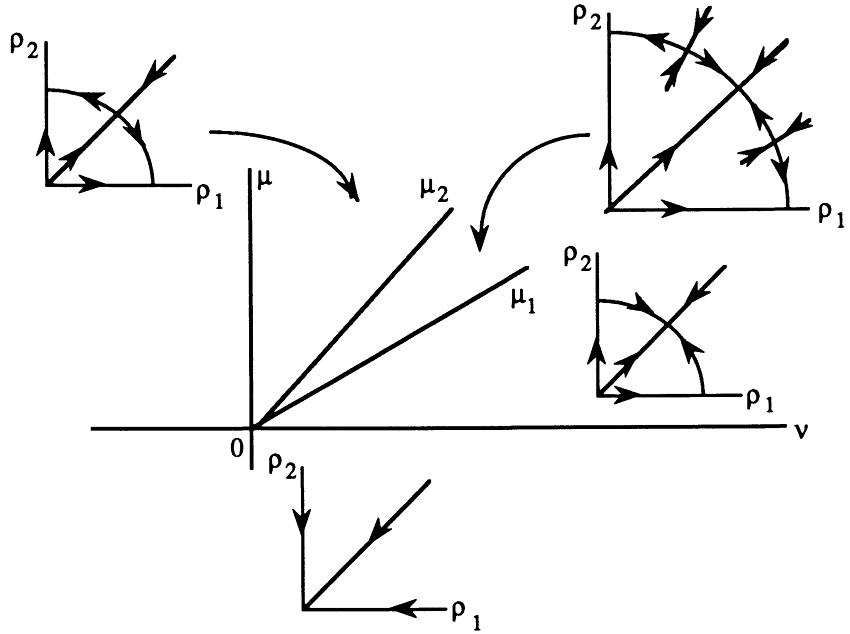


FIGURE III.10. Bifurcation diagram when $\eta = 0.7519$, $\Omega^* = -0.93$ $(\nu = b_r - c_r)$.

Therefore, these solutions are *quasi-periodic* with two frequencies. They exist when μ lies between $\mu_1(\nu)$ and $\mu_2(\nu)$, and they connect the branch of spirals to the branch of ribbons. Finally, a computation of the eigenvalues for the linearized equations (III.43) at these solutions leads to the stability conditions

$$Q(d_r - f_r) < 0 \quad \text{and} \quad b_r < 0. \tag{III.52}$$

The coefficients required for the complete determination of this bifurcation diagram were computed by Laure (see numerical values in [La]). He found that, in the above example, the spirals are stable for $\nu < 0$ and $\mu > 0$, and for $\nu > 0$ and $\mu > \mu_1(\nu)$, the ribbons are stable for $\nu > 0$ and $\mu < \mu_2(\nu)$, and therefore the quasi-periodic branch is unstable (see Figure III.10).

IV
Mode Interactions

Taylor vortices are commonly observed in experiments when the cylinders are co-rotating and Couette flow becomes unstable. Similarly, when the cylinders are counter-rotating, the instability that is usually seen first is the spiral flow. However, the analysis of Chapter III does not take into account the higher bifurcations, that are observed when the Reynolds number is increased and lead to more complicated spatio-temporal patterns. This is appeerent in Figure IV.1, which was established experimentally for an apparatus with radius ratio 0.883. This diagram shows the critical curves in the plane $(\mathcal{R}_o, \mathcal{R})$, where $\mathcal{R}_o$ is proportional to the outer angular velocity $\mathcal{R}_o = \mathcal{R}\Omega/\eta$. It is remarkable that some of these complicated regimes, for example, the wavy vortex flow or the interpenetrating spirals, can occur very close to the primary bifurcation curve. This suggests that we could reach such regimes by looking at higher codimension bifurcation points on the primary bifurcation curve. By codimension-two points, we mean situations where, for example, two or more different critical modes are competing, giving rise to secondary and higher-order bifurcations. In order to study these points we need to consider additional parameters, which we allow to vary. In Chapter II, when analyzing the linear stability of Couette flow, we saw that mode interaction could occur when varying the angular velocity Ω, i.e., when considering Ω as a *second* parameter (the first one being the Reynolds number $\mathcal{R}$). Let us be more precise: depending on Ω, Couette flow can lose stability by either an eigenvalue crossing the imaginary axis at 0 with axisymmetric eigenvectors (azimuthal wave number $m = 0$) or by a pair of purely imaginary eigenvalues with nonzero azimuthal wave number for the eigenmodes. The numerical evidence (see Figure IV.2) shows that

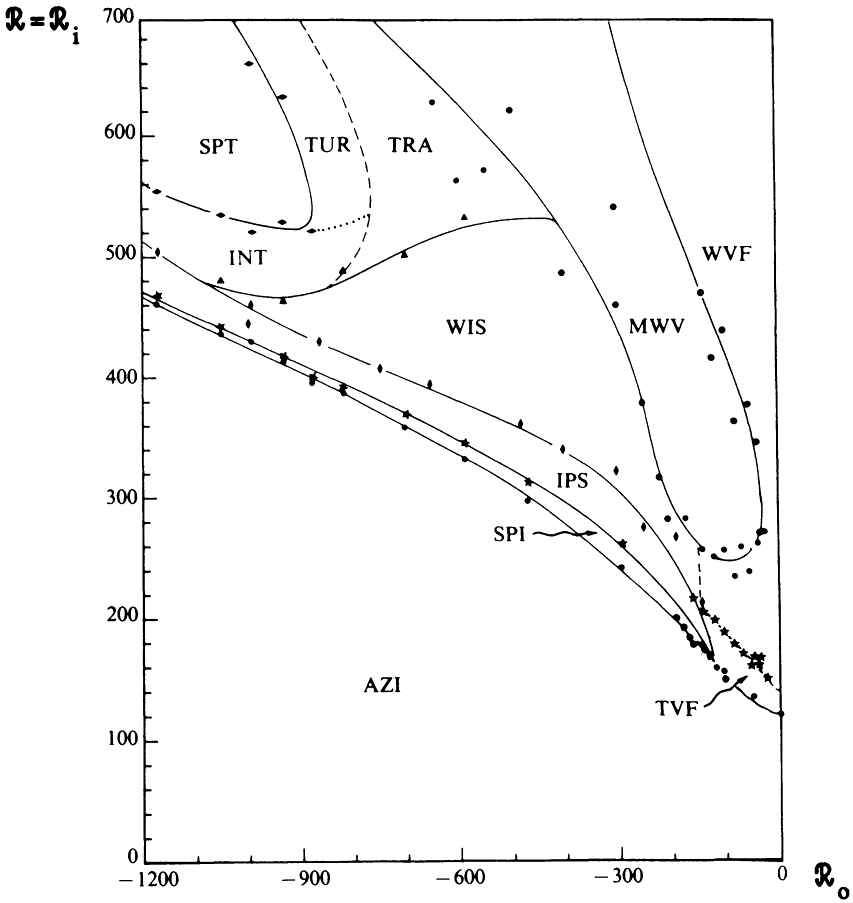


FIGURE IV.1. Experimental diagram (from [An-L-Sw]). AZI: Couette flow, TVF: Taylor vortex flow, SPI: Spiral flow, IPS: interpenetrating spirals, WVF: Wavy vortex flow, MWV: modulated vortex flow. The three first flows were analyzed in Chapter III. The next two flows (IPS and WVF) will be introduced in this chapter (WVF also in Chapter VI), and MWV will be analyzed in Chapter VI. Other regimes have a more turbulent structure, which is beyond the scope of this book.

there exists a sequence $\cdots < \Omega_m < \cdots < \Omega_1 < \Omega_0 < 0$, where Ω_0 is the value at which an eigenvalue with mode $m = 0$ and a pair of imaginary eigenvalues with mode $m = 1$ cross the imaginary axis at the same time and Ω_m $(m > 0)$ is the value at which two pairs of eigenvalues associated with modes m and $m + 1$ cross the imaginary axis at the same time. The first case corresponds to a stationary-Hopf interaction and the second one to a

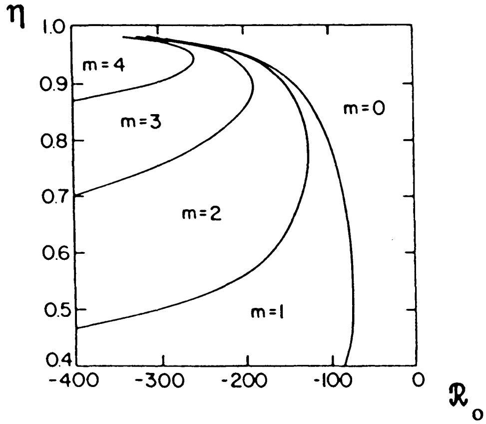


FIGURE IV.2. Mode interactions (from [Go-La]).

Hopf-Hopf interaction. We shall also assume that the *axial* wave number, defining the basic period in the z-direction, is the same for both (pairs of) eigenvalues. This is not obvious a priori, but it follows in fact from the numerical analysis. Indeed, it turns out that the neutral stability curves $\mathfrak{R}(m,\alpha)$ for the modes m and $m+1$ (see Figure II.1) intersect close to their minima. Therefore, by taking the value of α_c at this intersection point, we choose the length of the apparatus (i.e., the periodic boundary conditions) such that $\mathfrak{R}(m,\alpha_c)$ is the critical Reynolds number for both modes. The first case was analyzed in [Da-DP-S] (see also [St]), [DP-E-Si], [Go-Sta] and [Go-La] and the second case in [C-D-I]. We shall see that in both cases the interaction can lead to complicated spatio-temporal structures which, to some extent, give a good qualitative description (at least) of the experimental observations (see [Ok-Ta] for axial mode interaction).

IV.1 Interaction between an axisymmetric and a nonaxisymmetric mode

IV.1.1 The amplitude equations (6 dimensions)

We study the case when, at criticality, there is one eigenvalue at 0 and a pair $\pm i\omega_0$ on the imaginary axis. Because the eigenvalues are double as seen in (II.33), the center manifold is six dimensional in this case. Let us denote by ζ_0 and $\bar{\zeta}_0$ the eigenvectors for the eigenvalue 0 and by ζ_1 and ζ_2 the eigenvectors for the eigenvalue $i\omega_0$ such that $\zeta_2 = \mathbf{S}\zeta_1$. The group Γ

acts on ζ_0 as in (III.3) and on ζ_1 and ζ_2 as in (III.17) with $m = 1$. Let us write the elements in $\mathbf{V}$ as

$$X = \sum_{j=0}^{2} A_j \zeta_j + \bar{A}_j \bar{\zeta}_j.$$

The Γ-representation then acts on the coordinates as follows:

$$\begin{aligned} \mathbf{T}_\psi A_j &= e^{ni\psi} A_j \ (\mathrm{j} = 0,1) \quad \text{and} \quad \mathbf{T}_\psi A_2 = e^{-ni\psi} A_2, & \text{(IV.1a)} \\ \mathbf{R}_\varphi A_j &= e^{ji\varphi} A_j \ (\mathrm{j} = 0,1) \quad \text{and} \quad \mathbf{R}_\varphi A_2 = e^{i\varphi} A_2, & \text{(IV.1b)} \\ A_2 &= \mathbf{S} A_1 \quad \text{and} \quad \bar{A}_0 = \mathbf{S} A_0. & \text{(IV.1c)} \end{aligned}$$

In [Go-St] this problem was studied from a group-theoretical point of view. We have adapted this work for clarity. The bifurcation diagrams and stability results for given examples are essentially exposed in [Go-La].

The first step in our analysis is to compute the structure of Γ-equivariant maps in $\mathbf{V}$. We set $\mu = \mathcal{R} - \mathcal{R}_c$ and $\nu = \Omega - \Omega^{(0)}$, and

$$F(\mu,\nu,X) = \sum_{\mathrm{j}=0}^{2} F_j(\mu,\nu,X)\zeta_j + \overline{F_j(\mu,\nu,X)}\bar{\zeta}_j. \tag{IV.2}$$

The amplitude equations are therefore

$$\frac{dA_j}{dt} = F_j(\mu,\nu,X) \qquad (j = 0,1,2). \tag{IV.3}$$

By formulas (IV.1) we have the following relations for the F_j's for satisfying the group equivariance property, as indicated in Section II.4:

$$\begin{aligned} F_0(\mu,\nu,\mathbf{R}_\varphi X) &= F_0(\mu,\nu,X), \\ F_0(\mu,\nu,\mathbf{T}_\psi X) &= e^{ni\psi} F_0(\mu,\nu,X), \\ F_0(\mu,\nu,\mathbf{S}X) &= \overline{F_0(\mu,\nu,X)}; & \text{(IV.4a)} \\ F_1(\mu,\nu,\mathbf{R}_\varphi X) &= e^{i\varphi} F_1(\mu,\nu,X), \\ F_1(\mu,\nu,\mathbf{T}_\psi X) &= e^{ni\psi} F_1(\mu,\nu,X); & \text{(IV.4b)} \\ F_2(\mu,\nu,X) &= F_1(\mu,\nu,\mathbf{S}X). & \text{(IV.4c)} \end{aligned}$$

Remark. The vector field $F(\mu, v, .)$ is *already* in normal form. This is a consequence of the equivariance under $\mathbf{R}_\varphi$. Indeed, this representation of SO(2) can be written, in the former basis, as the matrix representation

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{mi\varphi} I_2 & 0 \\ 0 & 0 & e^{-mi\varphi} I_2 \end{pmatrix},$$

where I_2 is the 2×2 identity matrix. Since $m \neq 0$, this is, up to time rescaling, the same matrix as $\exp(\tilde{L}_0^* t)$, where

$$\tilde{L}_0^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\mathrm{i}\omega_0 I_2 & 0 \\ 0 & 0 & \mathrm{i}\omega_0 I_2 \end{pmatrix}$$

is the adjoint matrix to $\tilde{L}_0 = D_X F(0,0,0)$ (see the normal form theorem in Section II.4). The representations $\mathbf{R}_\varphi$ and $\exp(\tilde{L}_0^* t)$ are therefore equivalent, and they induce the same structure on equivariant vector fields.

Lemma 1. *Any polynomial map $F(X)$ satisfying the relations* (IV.4) *has the following structure: let us set $A_j = \rho_j e^{\mathrm{i}\psi_j}$, $u_j = \rho_j^2$, and $v = A_0^2 \bar{A}_1 A_2$. Then*

$$F_0 = A_0 g_0(u_0, u_1, u_2, v) + \bar{A}_0 A_1 \bar{A}_2 h_0(u_0, u_1, u_2, \bar{v}), \qquad \text{(IV.5a)}$$

$$F_1 = A_1 g_1(u_0, u_1, u_2, \bar{v}) + A_0^2 A_2 h_1(u_0, u_1, u_2, v), \qquad \text{(IV.5b)}$$

$$F_2 = A_2 g_1(u_0, u_2, u_1, v) + \bar{A}_0^2 A_1 h_1(u_0, u_2, u_1, \bar{v}), \qquad \text{(IV.5c)}$$

where g_j and h_j are complex polynomials and g_0, h_0 satisfy the relation $f(x,y,z,\bar{t}) = \overline{f(x,z,y,t)}$.

Proof. Let us prove the lemma for (IV.5a) (the proof is identical for the other formulas). Consider a monomial in F_0, of the form $P = A_0^{p_0} \bar{A}_0^{q_0} A_1^{p_1} \bar{A}_1^{q_1} A_2^{p_2} \bar{A}_2^{q_2}$. The first two relations (IV.4a) lead to

$$\begin{aligned} p_1 - q_1 + p_2 - q_2 &= 0, \\ p_0 - q_0 + p_1 - q_1 + q_2 - p_2 &= 1. \end{aligned} \qquad \text{(IV.6)}$$

The general solution of (IV.6) can be written as

$$\begin{aligned} p_0 - q_0 &= 1 + 2k, \\ p_1 - q_1 &= -k, \\ p_2 - q_2 &= k \end{aligned}$$

for any integer k. Then, if $k \geq 0$, one finds monomials of the form

$$A_0 |A_0|^{2q_0} |A_1|^{2p_1} |A_2|^{2q_2} (A_0^2 \bar{A}_1 A_2)^k,$$

while for $k' = -k > 0$ one finds

$$|A_0|^{2p_0} |A_1|^{2q_1} |A_2|^{2p_2} (\bar{A}_0^2 A_1 \bar{A}_2)^{k'-1} \bar{A}_0 A_1 \bar{A}_2.$$

Now (IV.5a) is obvious. The last property of polynomials g_0 and h_0 follows from the **S**-equivariance, and we leave this verification as an exercise for the reader (just write the **S**-equivariance relation for monomials by using (IV.4a)).

It follows from Lemma 1 that, expending $F(\mu,\nu,X)$ in Taylor series, the differential system in $\mathbf{V}$ can be written in terms of coordinates as follows:

$$\begin{aligned}
\frac{dA_0}{dt} &= A_0(\alpha_0\mu+\beta_0\nu+c_0\rho_0^2+d_0\rho_1^2+\bar{d}_0\rho_2^2)+f_0\bar{A}_0A_1\bar{A}_2+\text{ h.o.t.},\\
\frac{dA_1}{dt} &= A_1(\mathrm{i}\omega_0+\alpha_1\mu+\beta_1\nu+c_1\rho_0^2+d_1\rho_1^2+e_1\rho_2^2)+f_1A_0^2A_2+\text{ h.o.t.},\\
\frac{dA_2}{dt} &= A_2(\mathrm{i}\omega_0+\alpha_1\mu+\beta_1\nu+c_1\rho_0^2+e_1\rho_1^2+d_1\rho_2^2)+f_1\bar{A}_0^2A_1+\text{ h.o.t.},
\end{aligned} \tag{IV.7}$$

where α_0,β_0,c_0,f_0 are *real* coefficients.

Remark. The system (IV.7) was derived by Stuart in 1964 using a different basis (see [St] for a review paper and further references).

The amplitude equations with (IV.5) are not complete because we only show the Taylor expansion (to any order) of the vector field F with respect to X. We can in fact write these equations in a global way, taking account of all orders, as follows:

Lemma 1′. *The six-dimensional vector field satisfying relations (IV.4) can be written as*

$$\frac{dA_j}{dt}=e^{\mathrm{i}\psi_j}f_j(\mu,\nu,\rho_0,\rho_1,\rho_2,\theta),\qquad j=0,1,2, \tag{IV.8}$$

where $A_j=\rho_je^{\mathrm{i}\psi_j}$, f_j are 2π- periodic in $\theta=2\psi_0-\psi_1+\psi_2$, and f_0 is odd in ρ_0, even in (ρ_1,ρ_2), while f_1 and f_2 are even in ρ_0, odd in (ρ_1,ρ_2). Moreover,

$$\begin{aligned}
f_0(\mu,\nu,\rho_0,\rho_2,\rho_1,-\theta)&=\bar{f}_0(\mu,\nu,\rho_0,\rho_1,\rho_2,\theta),\\
f_2(\mu,\nu,\rho_0,\rho_2,\rho_1,-\theta)&=f_1(\mu,\nu,\rho_0,\rho_1,\rho_2,\theta),
\end{aligned} \tag{IV.9}$$

and if one of the ρ_j's is 0, then f_k is independent of θ and odd in ρ_k, even in ρ_l for $k\neq j$, $l\neq j$, $k\neq l$.

Proof. For the purpose of this proof let us simplify the notation by defining

$$F_j(\mu,\nu,X)=g_j(\psi_0,\psi_1,\psi_2).$$

Then (IV.4) leads, for any $(\psi,\varphi)\in\mathbb{T}^2$ (where $\pi=\mathbb{R}/2\pi\mathbb{Z}$) to the identities

$$g_j(\psi_0+n\psi,\psi_1+n\psi+\varphi,\psi_2-n\psi+\varphi)=g_j(\psi_0,\psi_1,\psi_2)\begin{cases} e^{\mathrm{i}n\psi} & (j=0),\\ e^{\mathrm{i}(n\psi+\varphi)} & (j=1),\\ e^{\mathrm{i}(-n\psi+\varphi)} & (j=2).\end{cases}$$

By choosing ψ and φ such that $n\psi=-\psi_0$, $n\psi+\varphi=-\psi_1$, we obtain the relations

$$g_j(\psi_0,\psi_1,\psi_2)=e^{\mathrm{i}\psi_j}g_j(0,0,\theta)\qquad(j=0,1)$$

and

$$g_2(\psi_0,\psi_1,\psi_2)=e^{\mathrm{i}\psi_2}g_2(0,-\theta,0),$$

which prove (IV.8). Properties (IV.9) follow directly from the equivariance under the symmetry **S**. Now, changing ρ_j into $-\rho_j$ *and* ψ_j into $\psi_j+\pi$ does not change A_j. For $j=0$, θ is not changed either; hence, we obtain the evenness of f_k $(k=1,2)$ and the oddness of f_0 with respect to ρ_0. For $j\neq 0$, we get for instance

$$\begin{aligned} f_1(\mu,\nu,\rho_0,\rho_1,\rho_2,\theta) &= -f_1(\mu,\nu,\rho_0,-\rho_1,\rho_2,\theta-\pi) \\ &= f_1(\mu,\nu,\rho_0,\rho_1,-\rho_2,\theta+\pi); \end{aligned}$$

hence, Lemma 1′ is proved, since the last assertion follows directly from the same proof. We leave as an exercise for the reader the verification of these global properties on the truncated form given in Lemma 1.

Remarks. (1) We observe that our system is in fact four dimensional in variables $(\rho_0,\rho_1,\rho_2,\theta)\in\mathbb{R}^3\times\mathbb{T}^1$ and that

$$\begin{aligned} \frac{d\theta}{dt} = \frac{2}{\rho_0}f_{0\mathrm{i}}(\mu,\nu,\rho_0,\rho_1,\rho_2,\theta) - \frac{1}{\rho_1}f_{1\mathrm{i}}(\mu,\nu,\rho_0,\rho_1,\rho_2,\theta) \\ +\frac{1}{\rho_2}f_{1\mathrm{i}}(\mu,\nu,\rho_0,\rho_2,\rho_1,-\theta). \end{aligned} \tag{IV.10}$$

Moreover, we might observe that for $\rho_1=\rho_2$, $f_{0\mathrm{i}}$, is odd in θ, hence, in this case we have

$$\frac{d\theta}{dt} = \tilde{f}(\mu,\nu,\rho_0,\rho_1,\theta),$$

which is odd in θ. This implies that

$$\tilde{f}(\mu,\nu,\rho_0,\rho_1,0) = \tilde{f}(\mu,\nu,\rho_0,\rho_1,\pi) = 0.$$

This remark leads to the possibility of looking for solutions with fixed $\theta=0$ or π, provided that $\rho_1(t)=\rho_2(t)$. In addition, in this case, we notice that $\frac{d\psi_0}{dt}=0$, therefore, the corresponding solutions $X(t)$ will be time periodic $\left(\frac{d\psi_1}{dt}=\frac{d\psi_2}{dt}\right)$.

(2) Another way to eliminate redundant variables is by writing the equations for the invariant functions u_j, v, and $\bar{v}$, i.e., considering these functions as new variables. The system is now defined in a manifold of $\mathbb{R}^5$; however, it can be shown that the new system thus obtained is equivalent to the original one. This idea has been recently applied by [Me] in this context. One advantage of this method compared to that of Lemma 1′ is that it avoids having a singularity at the origin, which is more convenient for the stability analysis.

IV.1.2 Restriction of the equations to flow-invariant subspaces

Although the problem has been significantly simplified by the computation of the equivariant structure of the differential system in **V**, we are still going to make use of a further reduction that is typical of bifurcation problems with symmetry. The idea is to look for subspaces that, thanks to the symmetry of the problem, are invariant under the vector field F and therefore under equations (IV.3). If such a subspace is transformed by the action of an element γ of the group Γ, then the resulting subspace is also invariant under the system (IV.3), thanks to the equivariance of F. Moreover, any trajectory is isometrically transformed by γ to another trajectory. Therefore, one only needs to know one representative of each equivalence class (for the Γ-action) of the flow-invariant subspaces. The next lemma summarizes these subspaces.

Lemma 2. *For any* $\psi \in \mathbb{T}^1$ *the following subspaces of* $\mathbf{V}$ *are invariant under the system* (IV.3):

(i) $e^{i\psi}A_0$ $(A_0 \in \mathbb{R})$, $A_1 = A_2 = 0$;
(ii) $A_0 = A_2 = 0$;
(iii) $A_0 = A_1 = 0$;
(iv) $A_0 = 0$, $A_2 = e^{i\psi}A_1$;
(v) $A_0 = 0$, $A_1A_2 \neq 0$;
(vi) $e^{i\psi}A_0$ $(A_0 \in \mathbb{R})$, $A_1 = -e^{2i\psi}A_2$;
(vii) $e^{i\psi}A_0$ $(A_0 \in \mathbb{R})$, $A_1 = e^{2i\psi}A_2$.

Remark. For pedagogic reasons, we mention here that (iii) is symmetric to (ii), even though they belong to the same group orbit.

Proof. The proof of this lemma is based on the following simple but useful property of equivariant vector fields. Let Σ be a subgroup of Γ; then the space Fix (Σ) of the elements in $\mathbf{V}$ that are fixed by the action of Σ is *invariant* under F. The proof of this claim is straightforward: let $X \in \mathrm{Fix}(\Sigma)$, and let σ denote any element of Σ; then $F(X) = F(\sigma X) = \sigma F(X)$ and, therefore $F(X) \in \mathrm{Fix}(\Sigma)$.

Let us first look for points in $\mathbf{V}$ that are fixed under some $\mathbf{R}_\varphi \mathbf{T}_\psi$ (pure rotations). Then we require

$$e^{ni\psi}A_0 = A_0, \qquad e^{i(\varphi+n\psi)}A_1 = A_1, \qquad e^{i(\varphi-n\psi)}A_2 = A_2.$$

Note that any element in $\mathbf{V}$ is invariant under the transformations of angles $\varphi = 0$ and $\psi = 2k\pi/n$, hence, we can just identify this action with the identity. Therefore, the fixed-point subspace for this action is $\mathbf{V}$ itself, and we shall forget this trivial case. Now the solutions to the above equations are

(a) $A_0 \neq 0$, $A_1 = A_2 = 0$, $\varphi \in \mathbb{T}^1$;
(b) $A_0 = 0$, $A_1 \neq 0$, $A_2 = 0$, $\varphi + n\psi = 2k\pi$;
(c) $A_0 = 0$, $A_1 = 0$, $A_2 \neq 0$, $\varphi - n\psi = 2k\pi$;
(d) $A_0 = 0$, $A_1A_2 \neq 0$, $\varphi = k\pi, \psi = l\pi/n, k = l \bmod 2$,

where k and l are integers. Solutions (b) and (c) lead to the invariant subspaces (ii) and (iii) of the lemma, and solution (d) to the subspace (v). We now look for those points that are fixed by a transformation containing the symmetry $\mathbf{S}$. They must satisfy

$$e^{ni\psi}\bar{A}_0 = A_0, \qquad e^{i(\varphi+n\psi)}A_2 = A_1, \qquad e^{i(\varphi-n\psi)}A_1 = A_2.$$

So, the solutions are

(e) $A_0 \in \mathbb{R},\ A_1 = A_2 = 0,\ \varphi \in \mathbb{T}^1$;

(f) $A_0 = 0,\ A_1 = e^{i\theta} A_2 \neq 0$ (θ fixed), $\varphi = k\pi,\ n\psi = \theta + l\pi,\ k = l \bmod 2$;

(g) $A_0 \in \mathbb{R},\ A_1 = -A_2,\ \varphi = (2k+1)\pi$;

(h) $A_0 \in \mathbb{R},\ A_1 = A_2$,

where k and l are integers. Note that (e) spans (a) by applying $\mathbf{T}_\psi$ and leads to subspaces (i) of the lemma. Solutions (f), (g), and (h) lead to the subspaces (iv), (vi), and (vii), respectively, which completes the proof.

Remarks. (1) The subspace (i) corresponds to the pure stationary (axisymmetric) modes as described in Section III.1, while (v) corresponds to the pure Hopf modes, described in Section III.2.

(2) We notice that the invariant subspaces (vi) and (vii) correspond to the cases when $\theta = 0$ or π in Remark 1, p. 65, while subspaces (i), (ii), (iii), (v) are trivially obtained by using the form of the vector field given in Lemma 1′ (notice that (iv) is a subspace of (v)).

In the next section we study the bifurcation problem for the equations restricted to the various subspaces listed in Lemma 2.

IV.1.3 Bifurcated solutions

IV.1.3.1 Primary branches

The lowest-dimensional (nontrivial) subspaces (i), (ii), and (iii) of Lemma 2 correspond to the situations studied in Chapter III, where only pure stationary or pure Hopf bifurcations were considered. Hence, in these spaces we obtain, respectively, the branches of axisymmetric stationary flow (Taylor vortices), spiral flow, and "ribbons" (with $m = 1$). It is often said that these are *pure mode* solutions. We shall of course need to know the leading part of each of these branches for the stability analysis, so we write the equations in these subspaces and specify the branching solutions.

1(a) Taylor vortices. The amplitude equation (IV.3) in the subspace (i) reduces to

$$\frac{dA_0}{dt} = F_0(\mu, \nu, A_0, \bar{A}_0, 0, 0, 0, 0) = e^{i\psi_0} f_0(\mu, \nu, \rho_0, 0, 0, 0),$$

where we set $A_0 = \rho_0 e^{i\psi_0}$ as in Lemma 1′. By the same analysis as in Section III.1, we obtain $\psi_0 =$ const, and the principal part of the bifurcation equation is given by (IV.7):

$$\alpha_0\mu + \beta_0\nu + c_0\rho_0^2 + \mathrm{O}(|\mu| + |\nu| + \rho_0^2)^2 = 0. \tag{IV.11}$$

As expected, we obtain a pitch-fork bifurcation of Taylor vortices, and

$$\rho_0 = \sqrt{(\alpha_0\mu + \beta_0\nu)/-c_0} + \mathrm{O}(|\mu| + |\nu|)^{3/2}.$$

All the discussion of Subsection III.1.3 about the geometrical pattern of these solutions is still valid here.

1(b) Spiral flow and ribbons. The equations in the pure Hopf mode subspace (v) are the following:

$$\frac{dA_1}{dt} = F_1(\mu,\nu,0,0,A_1,\bar{A}_1,A_2,\bar{A}_2) = e^{i\psi_1} f_1(\mu,\nu,0,\rho_1,\rho_2,0),$$
$$\frac{dA_2}{dt} = F_2(\mu,\nu,0,0,A_1,\bar{A}_1,A_2,\bar{A}_2) = e^{i\psi_2} f_1(\mu,\nu,0,\rho_2,\rho_1,0), \quad \text{(IV.12)}$$

where we set $A_1 = \rho_1 e^{i\psi_1}$ and $A_2 = \rho_2 e^{i\psi_2}$ as in Lemma 1′. We see, thanks to (III.7) and Lemma 1′, that the principal part of the above system can be written as follows (where we denote by indices r and i the real and imaginary parts):

$$\frac{d\rho_1}{dt} = \rho_1\left[\alpha_{1r}\mu + \beta_{1r}\nu + d_{1r}\rho_1^2 + e_{1r}\rho_2^2 + \mathrm{O}\left(|\mu| + |\nu| + \rho_1^2 + \rho_2^2\right)^2\right],$$
$$\frac{d\rho_2}{dt} = \rho_2\left[\alpha_{1r}\mu + \beta_{1r}\nu + e_{1r}\rho_1^2 + d_{1r}\rho_2^2 + \mathrm{O}\left(|\mu| + |\nu| + \rho_1^2 + \rho_2^2\right)^2\right]; \quad \text{(IV.13a)}$$
$$\frac{d\psi_1}{dt} = \omega_0 + \alpha_{1i}\mu + \beta_{1i}\nu + d_{1i}\rho_1^2 + e_{1i}\rho_2^2 + \mathrm{O}\left(|\mu| + |\nu| + \rho_1^2 + \rho_2^2\right)^2,$$
$$\frac{d\psi_2}{dt} = \omega_0 + \alpha_{1i}\mu + \beta_{1i}\nu + e_{1i}\rho_1^2 + d_{1i}\rho_2^2 + \mathrm{O}\left(|\mu| + |\nu| + \rho_1^2 + \rho_2^2\right)^2. \quad \text{(IV.13b)}$$

The bifurcation analysis is now *identical* to that of Section III.2. We obtain the following solutions:

(i) Spiral flow. $\rho_2 = 0$, $\alpha_{1r}\mu+\beta_{1r}\nu+d_{1r}\rho_1^2+\mathrm{O}\left(|\mu|+|\nu|+\rho_1^2\right)^2 = 0$, $\psi_1 = \Omega_1 t+\varphi$, with $\Omega_1 = \omega_0+\alpha_{1i}\mu+\beta_{1i}\nu+d_{1i}\rho_1^2+\mathrm{O}\left(|\mu|+|\nu|+\rho_1^2\right)^2$, and φ an arbitrary phase. These solutions (spirals) are described in Subsection III.2.3. Notice that these solutions occur in the subspace (ii) of Lemma 2. They are both rotating and traveling waves. The symmetric solutions (obtained by acting **S**) lie in the subspace (iii); their computations are left to the reader.

(ii) Ribbons (standing waves). $\rho_1 = \rho_2$, $\alpha_{1r}\mu + \beta_{1r}\nu + (d_{1r} + e_{1r})\rho_1^2 + \mathrm{O}\left(|\mu|+|\nu|+\rho_1^2\right)^2 = 0$, $\psi_1 = \psi_2 + \psi = \Omega_2 t + \varphi$, with $\Omega_2 = \omega_0 + \alpha_{1i}\mu + \beta_{1i}\nu + (e_{1i} + d_{1i})\rho_1^2 + \mathrm{O}\left(|\mu|+|\nu|+\rho_1^2\right)^2$, and φ and ψ arbitrary phases. The geometric structure of these solutions (ribbons) is described in Subsection III.2.3. They are rotating waves and symmetric under **S**. Notice that they occur in the subspace (iv) of Lemma 2.

IV.1.3.2 Wavy vortices

Let us now study the differential system in the subspace (vi) of Lemma 2. By Lemma 1 or Lemma 1′ ($\theta = \pi$) and (IV.7) we have the following principal part for the system (IV.3):

$$\frac{dA_0}{dt} = A_0[\alpha_0\mu + \beta_0\nu + c_0\rho_0^2 + (2d_{0r} - f_0)\rho_1^2 + O(|\mu| + |\nu| + \rho_0^2 + \rho_1^2)^2], \tag{IV.14a}$$

$$\frac{dA_1}{dt} = A_1[i\omega_0 + \alpha_1\mu + \beta_1\nu + (c_1 - f_1)\rho_0^2 + (d_1 + e_1)\rho_1^2 + O(|\mu| + |\nu| + \rho_0^2 + \rho_1^2)^2], \tag{IV.14b}$$

an equation similar to (IV.14b) holds for A_2. This system *decouples* into phases and moduli equations. This property is well known for the codimension-two problem corresponding to an interaction between a pitch-fork and a Hopf bifurcation (see [L-Io] and [Gu-Ho]). Equations for moduli take the form

$$\begin{aligned}\frac{d\rho_0}{dt} &= \rho_0[\alpha_0\mu + \beta_0\nu + c_0\rho_0^2 + (2d_{0r} - f_0)\rho_1^2 \\ &\quad + O(|\mu| + |\nu| + \rho_0^2 + \rho_1^2)^2], \\ \frac{d\rho_1}{dt} &= \rho_1[\alpha_{1r}\mu + \beta_{1r}\nu + (c_{1r} - f_{1r})\rho_0^2 + (d_{1r} + e_{1r})\rho_1^2 \\ &\quad + O(|\mu| + |\nu| + \rho_0^2 + \rho_1^2)^2],\end{aligned} \tag{IV.15}$$

Looking for equilibria of the moduli equations leads to three cases: (1) $A_1 = 0$, which gives the branch of Taylor vortices; (2) $A_0 = 0$, which gives the ribbons; (3) $A_1A_0 \neq 0$, in which case the principal part of the system to solve becomes

$$\left\{\begin{aligned} \alpha_0\mu + \beta_0\nu + c_0\rho_0^2 + (2d_{0r} - f_0)\rho_1^2 &= 0, \\ \alpha_{1r}\mu + \beta_{1r}\nu + (c_{1r} - f_{1r})\rho_0^2 + (d_{1r} + e_{1r})\rho_1^2 &= 0. \end{aligned}\right. \tag{IV.16}$$

Define $D = c_0(d_{1r}+e_{1r}) - (2d_{0r} - f_0)(c_{1r} - f_{1r})$. Nonzero solutions to system (IV.16) do exist provided that $D \neq 0$, and the leading part of the branch is given by

$$\rho_0^2 = \frac{\alpha_{1r}(2d_{0r} - f_0) - \alpha_0(d_{1r} + e_{1r})}{D}\mu + \frac{\beta_{1r}(2d_{0r} - f_0) - \beta_0(d_{1r} + e_{1r})}{D}\nu, \tag{IV.17a}$$

$$\rho_1^2 = \frac{\alpha_0(c_{1r} - f_{1r}) - \alpha_{1r}c_0}{D}\mu + \frac{\beta_0(c_{1r} - f_{1r}) - \beta_{1r}c_0}{D}\nu. \tag{IV.17b}$$

The corresponding solutions of (IV.14) are time periodic (see Remark 1, p. 65). They mix the stationary and time-periodic modes and are rotating waves, as can be immediately seen from the imaginary part of (IV.14b):

$$\frac{d\psi_1}{dt} = \Omega_3 = \omega_0 + \alpha_{1i}\mu + \beta_{1i}\nu + (c_{1i} - f_{1i})\rho_0^2 + (d_{1i} + e_{1i})\rho_1^2 + O(|\mu| + |\nu| + \rho_0^2 + \rho_1^2)^2,$$

which gives the solutions in **V**:

$$X(\mu,\nu,t) = \rho_0(\mu,\nu)e^{in\psi}\zeta_0 + \rho_1(\mu,\nu)e^{i(\Omega_3 t+\varphi)}[e^{in\psi}\zeta_1 - e^{-in\psi}\zeta_2] + \text{ c.c.}, \tag{IV.18}$$

where φ and ψ are arbitrary phases and Ω_3 is the frequency. Notice that these solutions bifurcate from the branch of Taylor cells when

$$[\alpha_0(c_{1\mathrm{r}} - f_{1\mathrm{r}}) - \alpha_{1\mathrm{r}}c_0]\mu + [\beta_0(c_{1\mathrm{r}} - f_{1\mathrm{r}}) - \beta_{1\mathrm{r}}c_0]\nu = 0$$

and from the branch of ribbons when

$$[\alpha_{1\mathrm{r}}(2d_{0\mathrm{r}} - f_0) - \alpha_0(d_{1\mathrm{r}} + e_{1\mathrm{r}})]\mu + [\beta_{1\mathrm{r}}(2d_{0\mathrm{r}} - f_0) - \beta_0(d_{1\mathrm{r}} + e_{1\mathrm{r}})]\nu = 0.$$

These two equations represent two lines in the (μ,ν) plane, which are in fact tangent to the curves where these bifurcations occur. Since ρ_0^2 and ρ_1^2 must be positive, it follows from (IV.17) that the bifurcated solutions only exist in a sector in the parameter plane, bounded by these lines.

Physical interpretation of the solutions (IV.18). The solutions we just obtained in **V** are *rotating waves* and, as for the ribbons, the group orbit of these solutions is a 2-torus parametrized by φ and ψ. The phase φ determines a choice of time origin, and the phase ψ corresponds to a translation by $h\psi/2\pi$ along the z-axis. Contrary to the ribbons, however, these solutions are not invariant under the transformation $\psi \to \psi + \pi/n$, $\theta \to \theta + \pi$, but, in choosing the origin on the z-axis such that $\psi = 0$ in (IV.18), these solutions are invariant under the symmetry that consists of first rotating by π and then applying symmetry **S**. Therefore, they do not form flat

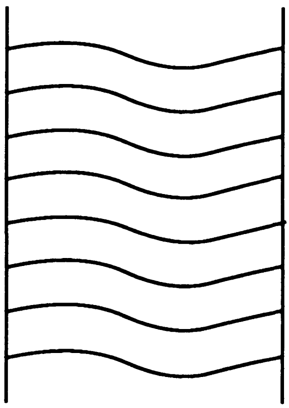


FIGURE IV.3. Wavy vortex flow.

cells, although the periodic azimuthal wave structure still exists. These symmetry properties that hold in **V** propagate on the full solution of the Navier-Stokes equation, as seen in Subsection III.2.3. Moreover, these symmetries are identical to the symmetries of the regime called *wavy vortex flow* that are observed in experiments, although the number of azimuthal waves is usually larger than 1. In fact, the region we are considering, in the parameter space, is very small, and the usually observed wavy vortex flow occurs when both cylinders are co-rotating (then it is not the result of a codimension-two singularity; this is studied in Chapter VI). In Figure IV.3 we sketch this pattern (assuming $m > 1$ for clarity of the picture).

IV.1.3.3 Twisted vortices

We now consider solutions in the subspace (vii) of Lemma 2. By Lemma 1′ (Remark 1, p. 65 with $\theta = 0$) and (IV.7) we have the principal part of the system under the following form:

$$\begin{aligned}\frac{dA_0}{dt} &= A_0[\alpha_0\mu + \beta_0\nu + c_0\rho_0^2 + (2d_{0\mathrm{r}} + f_0)\rho_1^2 \\ &\quad +\mathrm{O}(|\mu| + |\nu| + \rho_0^2 + \rho_1^2)^2], \qquad \text{(IV.19a)}\\ \frac{dA_1}{dt} &= A_1[i\omega_0 + \alpha_1\mu + \beta_1\nu + (c_1 + f_1)\rho_0^2 + (d_1 + e_1)\rho_1^2 \\ &\quad +\mathrm{O}(|\mu| + |\nu| + \rho_0^2 + \rho_1^2)^2], \qquad \text{(IV.19b)}\end{aligned}$$

and an equation similar to (IV.19b) holds for A_2. The only difference between equations (IV.14) and (IV.19) is a change of sign in front of the coefficients f_0 and f_1, so the analysis is pretty much the same! Once the primary branches have been eliminated [they are the same as for (IV.14), i.e., Taylor vortices and ribbons], the resolution of the amplitude equations for equilibria leads to the "mixed modes" branch

$$\begin{aligned}\rho_0'^2 &= \frac{\alpha_{1\mathrm{r}}(2d_{0\mathrm{r}} + f_0) - \alpha_0(d_{1\mathrm{r}} + e_{1\mathrm{r}})}{D'}\mu \\ &\quad + \frac{\beta_{1\mathrm{r}}(2d_{0\mathrm{r}} + f_0) - \beta_0(d_{1\mathrm{r}} + e_{1\mathrm{r}})}{D'}\nu, \qquad \text{(IV.20a)}\\ \rho_1'^2 &= \frac{\alpha_0(c_{1\mathrm{r}} + f_{1\mathrm{r}}) - \alpha_{1\mathrm{r}}c_0}{D'}\mu + \frac{\beta_0(c_{1\mathrm{r}} + f_{1\mathrm{r}}) - \beta_{1\mathrm{r}}c_0}{D'}\nu, \qquad \text{(IV.20b)}\end{aligned}$$

where $D' = c_0(d_{1\mathrm{r}} + e_{1\mathrm{r}}) - (2d_{0\mathrm{r}} + f_0)(c_{1\mathrm{r}} + f_{1\mathrm{r}})$. As in the previous case we can express the solutions in **V** as

$$X(\mu,\nu,t) = \rho_0'(\mu,\nu)e^{in\psi}\zeta_0 + \rho_1'(\mu,\nu)e^{\mathrm{i}(\Omega_3' t+\varphi)}[e^{in\psi}\zeta_1 + e^{-in\psi}\zeta_2] + \text{ c.c.}, \qquad \text{(IV.21)}$$

where φ and ψ are arbitrary phases and

$$\Omega_3' = \omega_0 + \alpha_{1\mathrm{i}}\mu + \beta_{1\mathrm{i}}\nu + (c_{1\mathrm{i}} + f_{1\mathrm{i}})\rho_0^2 + (d_{1\mathrm{i}} + e_{1\mathrm{i}})\rho_1^2 + \mathrm{O}(|\mu| + |\nu| + \rho_0^2 + \rho_1^2)^2$$

is the frequency. These solutions bifurcate from the branch of Taylor cells when one crosses the following line in the parameter plane:

$$[\alpha_0(c_{1r} + f_{1r}) - \alpha_{1r}c_0]\mu + [\beta_0(c_{1r} + f_{1r}) - \beta_{1r}c_0]\nu = 0$$

and from the branch of ribbons when one crosses the line:

$$[\alpha_{1r}(2d_{0r} + f_0) - \alpha_0(d_{1r} + e_{1r})]\mu + [\beta_{1r}(2d_{0r} + f_0) - \beta_0(d_{1r} + e_{1r})]\nu = 0.$$

The bifurcating solution (IV.20) takes place in the sector of the parameter space where $\rho_0'^2$ and $\rho_1'^2$ are both positive.

Physical interpretation of the solutions (IV.21). These solutions are again rotating waves and form a 2-torus group orbit by rotations (or time evolution) and translations. For $\psi = 0$ they are invariant under the symmetry **S**, and we can apply the same reasoning as in Chapter III for Taylor vortices and for ribbons to show that the fluid flow is confined to flat horizontal cells. So, what is the difference with the ribbons? The ribbons have an additional symmetry—they are invariant by rotations of angle π combined with translations along the z-axis of length $h/2n$. Lack of this symmetry implies that the flow associated with (IV.21) cannot be of the form symbolized in Figure III.7 but rather looks like the "twisted vortices" seen in the experiments. In fact, this flow is most often observed when cylinders are co-rotating (see Chapter VI). Then the azimuthal wave number is much larger than 1. In Figure IV.4 we sketch this pattern (assuming $m \gg 1$ for clarity of the picture). In the numerical result we present in Subsection IV.1.5, this flow is unstable and, hence, not expected to be seen experimentally.

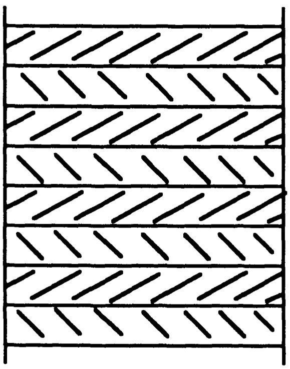


FIGURE IV.4. Twisted vortices.

IV.1.4 Stability of the bifurcated solutions

Preliminary remarks. (1) Since the main dynamics is reduced to the center manifold, the stability of a stationary solution X_0 is determined by the eigenvalues of the linearized operator $D_X F(\mu, \nu, X_0)$. For a T-periodic solution $X_0(t)$ one should in principle look for the eigenvalues (Floquet multipliers) of the monodromy matrix $\Phi(T)$, where $\Phi(t)$ is the fundamental matrix of the linearized equation $\dot{X} = D_X F(\mu, \nu, X_0(t))X$ such that $\Phi(0) = \mathrm{Id}$ (identity matrix). In our case, however, periodic solutions are *always* rotating waves because they break the rotational (azimuthal) symmetry as they do with time-translational symmetry. In other words they can be written as

$$X_0(t) = \mathbf{R}_{\omega t}\tilde{X}_0, \tag{IV.22}$$

where $\omega = 2\pi/mT$ and $\tilde{X}_0$ is a stationary solution of the equation

$$\frac{dX}{dt} + \omega \mathbf{J} X = F(\mu, \nu, X), \tag{IV.23}$$

where $\mathbf{J}$ is the infinitesimal generator of the group $\mathbf{R}_\varphi$ ($\mathbf{R}_\varphi = e^{\mathbf{J}\varphi}$). Therefore, the stability of the rotating waves is in fact determined by the eigenvalues of the operator

$$L_{\mu,\nu} = D_X F(\mu, \nu, \tilde{X}_0) - \omega \mathbf{J}. \tag{IV.24}$$

(2) Symmetry, although forcing multiple critical eigenvalues, usually simplifies the computation of the eigenvalues of $L_{\mu,\nu}$. The arguments we shall use for this purpose are the following:

(i) note $\mathbf{W} = \mathrm{Fix}(\Sigma)$ (see Subsection IV.1.2), Σ being the isotropy subgroup of the solution, and let $\mathbf{Z}$ be a complementary subspace to $\mathbf{W}$ in $\mathbf{V}$. Then the linearized operator takes the form

$$L_{\mu,\nu} = \begin{pmatrix} L_W & L' \\ 0 & L_Z \end{pmatrix},$$

where the operator $L_{\mu,\nu}$ is decomposed in $\mathbf{W} \oplus \mathbf{Z}$. Indeed, if we write $X = W \oplus Z$ and $F = F_W + F_Z$ according to this decomposition, then $F_Z(\mu, \nu, W) = 0\ \forall W \in \mathbf{W}$ and, hence, $D_W F_Z(\mu, \nu, W) = 0$. If in addition the group acts isometrically in $\mathbf{V}$ (which is the case in our problem) and we choose $\mathbf{Z}$ orthogonal to $\mathbf{W}$, then it can be shown that $L' = 0$ in the foregoing decomposition. This follows from a more general theorem about the decomposition of linear operators commuting with a group representation (the *isotypic* decomposition; see, for example, [Go-St-Sc]). In the following we do not need this theorem.

(ii) The tangent directions to the group-orbit at a steady solution are in the kernel of $L_{\mu,\nu}$. Indeed, let $g(\varphi)X_0$ be a curve in the group-orbit of X_0

such that $g(0)X_0 = X_0$. Then (forgetting the parameters),

$$F(g(\varphi)X_0) = g(\varphi)F(X_0) \quad \forall\varphi;$$

hence, by differentiating along φ at $\varphi = 0$, $DF(X_0)JX_0 = 0$, where J is here the generator of g.

These arguments work as well in the case of a rotating wave. In the simplest cases we can compute directly the eigenvalues at minimum expense by making use of the equivariant structure of F (Lemmas 1 and 1′) for computing the coefficients $\partial F_j/\partial A_k$ and $\partial F_j/\partial \bar{A}_k$. Then we apply the truncated equations (IV.7) for determining the leading part of the eigenvalues. In the following we use the same notation for $L_{\mu,\nu}$ and its matrix representation.

(iii) Computation of stability uses the form of the vector field given in Lemma 1. This is not the "exact" form of the field F, due to high-order terms (arbitrary order), but our argument is that we shall obtain a number of nonzero eigenvalues, which perturbation theory shows that the sign of their real part does not change by adding these high-order terms. For the zero eigenvalues, we just have to check that their number fits with the number predicted by the group action (see (ii)).

IV.1.4.1 Taylor vortices

These solutions are defined in the space $A_1 = A_2 = 0$. A straightforward computation shows that $L_{\mu,\nu}$ has the following eigenvalues: 0 (simple), $2|A_0|^2\partial g_0(\rho_0^2, 0, 0, 0)/\partial u_0$ (simple), $g_1(\rho_0^2, 0, 0, 0) \pm \rho_0^2 h_1(\rho_0^2, 0, 0, 0)$, and complex conjugates. Note that $g_0(\rho_0^2, 0, 0, 0) = 0$ along the branch. Using (IV.7) and the leading part of the branch of solutions namely, $\alpha_0\mu + \beta_0\nu + c_0\rho_0^2 = 0$, it follows that the nonzero eigenvalues are

$$\begin{aligned}
&2c_0\rho_0^2 + \mathrm{O}(\rho_0^4),\\
&\mathrm{i}\omega_0 + (\alpha_1 - \alpha_0(c_1 \pm f_1)/c_0)\mu + (\beta_1 - \beta_0(c_1 \pm f_1)/c_0)\nu + \mathrm{O}(|\mu| + |\nu|)^2,\\
&\text{and complex conjugates.}
\end{aligned} \qquad \text{(IV.25)}$$

Notice that when one of the two pairs of complex eigenvalues crosses the imaginary axis, this corresponds to the branching of wavy ["−" sign, see (IV.17b)] or twisted vortices ["+" sign, see (IV.20b)].

IV.1.4.2 Spirals

Consider the solutions with nonzero coordinate A_1. Again a straightforward computation using Lemma 1 shows that $L_{\mu,\nu} = D_X F(\mu, \nu, \tilde{X}_0) - \Omega_1\mathbf{J}$ has the eigenvalues 0 (simple), $g_0(0, \rho_1^2, 0, 0)$ (and complex conjugate), $|A_1|^2\partial g_{1\mathrm{r}}(0, \rho_1^2, 0, 0)/\partial u_1$ (simple), and $g_1(0, 0, \rho_1^2, 0)$ (and complex conjugate). The leading part of the solution branch is given by the equation $\alpha_{1\mathrm{r}}\mu + \beta_{1\mathrm{r}}\nu + d_{1\mathrm{r}}\rho_1^2 = 0$. Then using (IV.7) we can calculate the first-order expansion of the nonzero eigenvalues:

$$\begin{aligned}&(\alpha_0 - \alpha_{1\mathrm{r}}d_0/d_{1\mathrm{r}})\mu + (\beta_0 - \beta_{1\mathrm{r}}d_0/d_{1\mathrm{r}})\nu + \mathrm{O}(|\mu| + |\nu|)^2 \text{ and c.c.},\\ &2d_{1\mathrm{r}}\rho_1^2 + \mathrm{O}(\rho_1^4) \quad \text{(real)},\\ &(e_1 - d_1)\rho_1^2 + \mathrm{O}(\rho_1^4) \text{ and c.c.}\end{aligned} \tag{IV.26}$$

Notice that spirals can undergo a Hopf bifurcation when the first eigenvalues listed in (IV.26) cross the imaginary axis. This secondary bifurcation will be studied in Chapter VI in a more general way than in the present codimension-two singularity; this is a particular case of bifurcation from a "rotating wave".

IV.1.4.3 Ribbons

Consider such solutions with coordinates $A_1 = A_2$. We know that $\mathbf{T}_\psi$ generates an invariant 2-torus for system (IV.7) when applied to these periodic orbits. Going to system (IV.23) we now have a two-parameter group orbit of steady solutions. This implies that 0 is at least a *double* eigenvalue of $L_{\mu,\nu}$ defined by (IV.24) with $\omega = \Omega_2$. The other eigenvalues are those of the 2×2 matrix

$$\begin{pmatrix} g_0 & \rho_1^2 h_0 \\ \rho_1^2 \bar{h}_0 & \bar{g}_0 \end{pmatrix}$$

taken at the point $(0, \rho_1^2, \rho_1^2, 0)$, which occurs in the invariant space $\{A_0, \bar{A}_0\}$, and of a 4×4 matrix, which we do not need to identify because it occurs in the stability analysis of the ribbons in the "pure mode" case (see (III.20) in Subsection III.2.1). The leading part of the branch of solutions is given by $\alpha_{1\mathrm{r}}\mu + \beta_{1\mathrm{r}}\nu + (d_{1\mathrm{r}} + e_{1\mathrm{r}})\rho_1^2 = 0$. Using the form (IV.7) of the system, we get the first-order expansion of the nonzero eigenvalues (all real and simple):

$$\begin{aligned}&\left(\alpha_0 - \alpha_{1\mathrm{r}}\frac{2d_0 + f_0}{d_{1\mathrm{r}} + e_{1\mathrm{r}}}\right)\mu + \left(\beta_0 - \beta_{1\mathrm{r}}\frac{2d_0 + f_0}{d_{1\mathrm{r}} + e_{1\mathrm{r}}}\right)\nu + \mathrm{O}(|\mu| + |\nu|)^2,\\ &\left(\alpha_0 - \alpha_{1\mathrm{r}}\frac{2d_0 - f_0}{d_{1\mathrm{r}} + e_{1\mathrm{r}}}\right)\mu + \left(\beta_0 - \beta_{1\mathrm{r}}\frac{2d_0 - f_0}{d_{1\mathrm{r}} + e_{1\mathrm{r}}}\right)\nu + \mathrm{O}(|\mu| + |\nu|)^2,\\ &2(d_{1\mathrm{r}} + e_{1\mathrm{r}})\rho_1^2 + \mathrm{O}(\rho_1^4),\\ &2(d_{1\mathrm{r}} - e_{1\mathrm{r}})\rho_1^2 + \mathrm{O}(\rho_1^4).\end{aligned} \tag{IV.27}$$

We observe that the first two eigenvalues may change sign. For the first this leads to the secondary bifurcating twisted vortices [see (IV.20a)], while for the second this leads to wavy vortices [see (IV.17a)].

IV.1.4.4 Wavy vortices

Among the 2-torus family of bifurcating wavy vortices obtained in (IV.17–IV.18), we choose the solution such that $A_0 = \rho_0 \in \mathbb{R}$ and $A_1 = \rho_1 =$

$-A_2 \in \mathbb{R}$, i.e., $\psi = \varphi = 0$ in (IV.18). The eigenvectors tangent to the group orbit are obtained by differentiating the representations $\mathbf{R}_\varphi$ and $\mathbf{T}_\psi$ applied to the solution, at $\varphi = \psi = 0$. This gives the null eigenvectors $(A_0, A_1, A_2) = (\mathrm{i}\rho_0, \mathrm{i}\rho_1, \mathrm{i}\rho_1)$ and $(0, \mathrm{i}\rho_1, -\mathrm{i}\rho_1)$. The matrix $L_{\mu,\nu}$ (where $\omega = \Omega_3$) splits into two blocks: one 3×3 block in the flow-invariant subspace $\{A_0 \in \mathbb{R},\ A_1 = -A_2\}$ of elements having the same symmetry as wavy vortices (for $\psi = 0$) and one 3×3 block in the orthogonal complementary space $\{A_0 \in \mathrm{i}\mathbb{R},\ A_1 = A_2\}$ (orthogonality is understood in the 6-dimensional coordinates $(A_0, A_1, A_2, \bar{A}_0, \bar{A}_1, \bar{A}_2)$. Moreover, each of these subspaces contains one of the null eigendirections calculated above. The nonzero eigenvalues can therefore be computed as those of two 2×2 blocks.

In fact, a simpler computation can be performed by using the form of the vector field F given in Lemma 1$'$. It has been observed that wavy vortices correspond to *steady* solutions of the four-dimensional system in $(\rho_0, \rho_1, \rho_2, \theta)$ with $\rho_1 = \rho_2$ and $\rho_0\rho_1 \neq 0$. It is then clear that in this way we eliminate two additional zero eigenvalues, which correspond to two independent phase shifts. The principal part of the amplitude equations now reads as follows:

$$\frac{d\rho_0}{dt} = \rho_0[\alpha_0\mu + \beta_0\nu + c_0\rho_0^2 + d_{0\mathrm{r}}(\rho_1^2 + \rho_2^2) + f_0\rho_1\rho_2\cos\theta] \tag{IV.28a}$$

$$\frac{d\rho_1}{dt} = \rho_1[\alpha_{1\mathrm{r}}\mu + \beta_{1\mathrm{r}}\nu + c_{1\mathrm{r}}\rho_0^2 + d_{1\mathrm{r}}\rho_1^2 + e_{1\mathrm{r}}\rho_2^2] + \rho_0^2\rho_2(f_1e^{\mathrm{i}\theta})_\mathrm{r}, \tag{IV.28b}$$

$$\frac{d\rho_2}{dt} = \rho_2[\alpha_{1\mathrm{r}}\mu + \beta_{1\mathrm{r}}\nu + c_{1\mathrm{r}}\rho_0^2 + e_{1\mathrm{r}}\rho_1^2 + d_{1\mathrm{r}}\rho_2^2] + \rho_0^2\rho_1(f_1e^{-\mathrm{i}\theta})_\mathrm{r}, \tag{IV.28c}$$

$$\begin{aligned}\frac{d\theta}{dt} &= (2d_{0\mathrm{i}} + e_{1\mathrm{i}} - d_{1\mathrm{i}})(\rho_1^2 - \rho_2^2) - 2f_0\rho_1\rho_2\sin\theta \\ &\quad + \rho_0^2\left[\frac{\rho_1}{\rho_2}(f_1e^{-\mathrm{i}\theta})_\mathrm{i} - \frac{\rho_2}{\rho_1}(f_1e^{\mathrm{i}\theta})_\mathrm{i}\right].\end{aligned} \tag{IV.28d}$$

Now choosing coordinates $(\rho_0, \rho_1 + \rho_2, \rho_1 - \rho_2, \theta)$ to derive the 4×4 matrix of the linearized operator near the wavy vortices solution, we obtain

$$\begin{pmatrix} 2c_0\rho_0^2 & (2d_{0\mathrm{r}} - f_0)\rho_0\rho_1 & 0 & 0 \\ 4(c_{1\mathrm{r}} - f_{1\mathrm{r}})\rho_0\rho_1 & 2(d_{1\mathrm{r}} + e_{1\mathrm{r}})\rho_1^2 & 0 & 0 \\ 0 & 0 & \begin{matrix} 2(d_{1\mathrm{r}} - e_{1\mathrm{r}})\rho_1^2 \\ +2f_{1\mathrm{r}}\rho_0^2 \end{matrix} & 2f_{1\mathrm{i}}\rho_0^2\rho_1 \\ 0 & 0 & \begin{matrix} 2(2d_{0\mathrm{i}} + e_{1\mathrm{i}} - d_{1\mathrm{i}})\rho_1 \\ -2f_{1\mathrm{i}}\rho_0^2\rho_1^{-1} \end{matrix} & \begin{matrix} 2f_0\rho_1^2 \\ +2f_{1\mathrm{r}}\rho_0^2 \end{matrix} \end{pmatrix}. \tag{IV.29}$$

The two (2×2)-dimensional blocks give the four eigenvalues that determine the stability. We might observe that wavy vortices can lose their stability in two different ways. The upper block gives an eventual tertiary bifurcation "soft mode" quasi-periodic solution, as it is generic in codimension-two problems for interaction of pitch-fork and Hopf bifurcation, in the same invariant subspace ($\rho_1 = \rho_2,\ \theta = \pi$) as wavy vortices. The lower block leads to bifurcating solutions with less symmetry than wavy vortices. All these branches of solutions will be discussed in a more general framework in Chapter VII.

Remark. The above computation in fact excludes a neighborhood where ρ_1 is small compared to ρ_0^2, i.e., a neighborhood of the secondary bifurcation point of Taylor vortices into wavy vortices. This is only due to the choice of coordinates. Since these eigenvalues could be computed directly (heavier calculations) on the field F, our result stays valid up to this bifurcation point.

IV.1.4.5 Twisted vortices

Among the 2-torus family of bifurcating twisted vortices obtained in (IV.20–IV.21), we choose the solution such that $A_0 = \rho_0 \in \mathbb{R}$ and $A_1 = \rho_1 = A_2 \in \mathbb{R}$, i.e., $\psi = \varphi = 0$ in (IV.18). The eigenvectors tangent to the group orbit are now the null eigenvectors $(A_0, A_1, A_2) = (\mathrm{i}\rho_0, \mathrm{i}\rho_1, -\mathrm{i}\rho_1)$ and $(0, \mathrm{i}\rho_1, \mathrm{i}\rho_1)$. The same analysis as for the wavy vortices applies in the present case. The invariant subspaces are now $\{A_0 \in \mathbb{R},\ A_1 = A_2\}$ and the orthogonal complementary space $\{A_0 \in \mathrm{i}\mathbb{R},\ A_1 = A_2\}$. The same kind of calculations as in the wavy vortices case shows that the principal part of the nonzero eigenvalues of $L_{\mu,\nu}$ are eigenvalues of the following two matrices:

$$\begin{pmatrix} 2c_0\rho_0^2 & (2d_{0\mathrm{r}} + f_0)\rho_0\rho_1 \\ 4(c_{1\mathrm{r}} + f_{1\mathrm{r}})\rho_0\rho_1 & 2(d_{1\mathrm{r}} + e_{1\mathrm{r}})\rho_1^2 \end{pmatrix}, \tag{IV.30a}$$

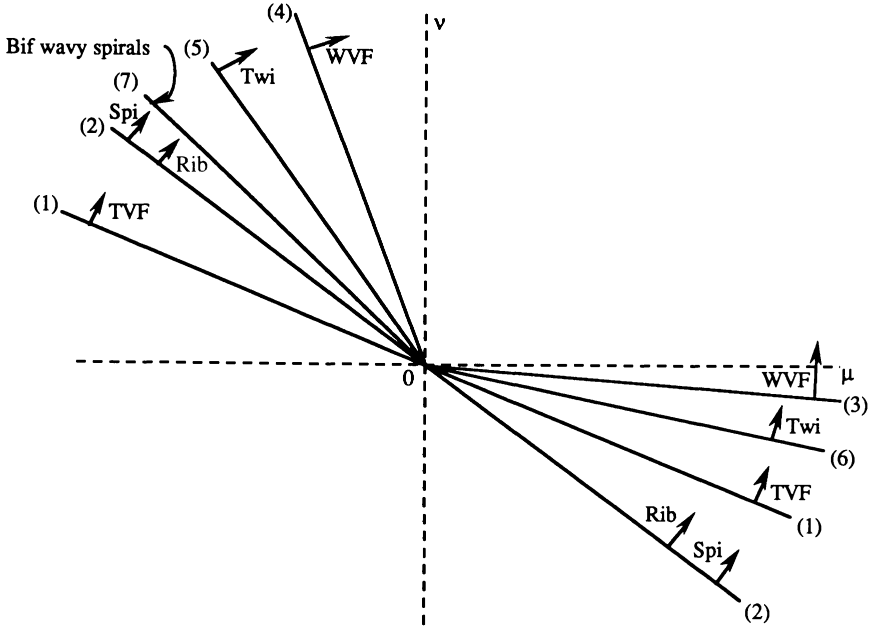
$$\begin{pmatrix} 2(d_{1\mathrm{r}} - e_{1\mathrm{r}})\rho_1^2 - 2f_{1\mathrm{r}}\rho_0^2 & -2f_{1\mathrm{i}}\rho_0^2\rho_1 \\ 2(2d_{0\mathrm{i}} + e_{1\mathrm{i}} - d_{1\mathrm{i}})\rho_1 + 2f_{1\mathrm{i}}\rho_0^2\rho_1^{-1} & -2f_0\rho_1^2 - 2f_{1\mathrm{r}}\rho_0^2 \end{pmatrix}. \tag{IV.30b}$$

The eventual loss of stability given by the upper block leads to a tertiary bifurcating "soft mode" quasi-periodic solution, as indicated in case 4, in the same invariant subspace ($\rho_1 = \rho_2,\ \theta = 0$) as twisted vortices. The lower block leads to bifurcating solutions with less symmetry than twisted vortices. All these branches of solutions will be discussed in a more general framework in Chapter VI.

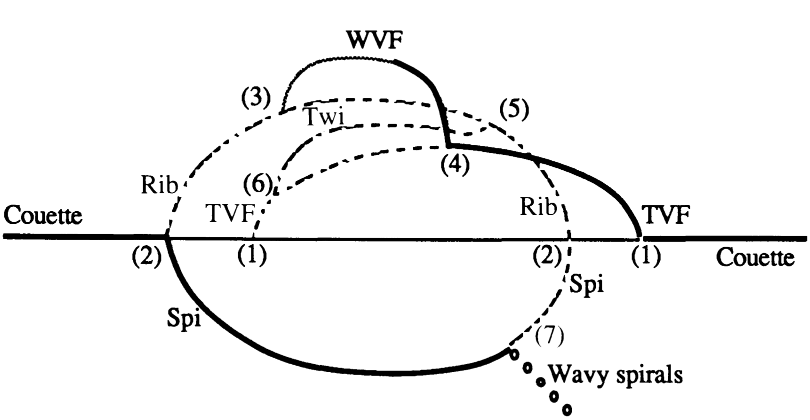
Remark. The same remark as in case 4 holds on the validity of this stability computation near the secondary bifurcation point or Taylor vortices into twisted vortices.

IV.1.5 A numerical example

It would be extremely long and of little interest to list all the possible bifurcation diagrams that follow from the foregoing analysis. Moreover, once the relevant coefficients of the amplitude equation (IV.7) have been computed (numerically), one can determine the actual diagram associated with a given set of physical parameter values. In [Go-La] several examples have been investigated in such a manner for different radius ratios. We


ν
μ
0
Bif wavy spirals
(1)
(2)
(3)
(4)
(5)
(6)
(7)
TVF
Rib
Spi
Twi
WVF
WVF
Twi
TVF
Rib
Spi
(3)
(6)
(1)
(2)

(a)


WVF
(3)
Twi
(5)
(4)
Rib
(6)
TVF
Couette
Rib
TVF
(2)
(1)
(2)
(1)
Couette
Spi
Spi
(7)
Wavy spirals

(b)

TABLE IV.1. Linear coefficient, following [Go-La].

η	$\mathfrak{R}_c$	$\Omega^{(0)}$	$\alpha_c^{(o)}$	$\alpha_c^{(1)}$	α	ω_0	α_0	β_0	α_{1r}	β_{1r}
0.8	129.6	−0.61	3.57	3.55	3.57	−0.345	0.46	40.2	0.43	34

TABLE IV.2. Nonlinear coefficients in (IV.7) (see [Go-La]).

c_0	d_0	f_0	c_{1r}	d_{1r}	e_{1r}	f_{1r}
−13.4	−84.4 + i 97.9	−99.2	−64.1	−27.9	−71.7	−56.5

reproduce one of these examples below, for radius ratio $\eta = 0.8$, for which real experiments have also been performed by Tagg et al. in a neighborhood of the critical point [T-H-S]. Comparison with experiments shows a good qualitative agreement with the computed diagrams.

Since we replace the physical boundary conditions by a period h condition, the two axial wave numbers must fit with this periodicity. Since the critical wave numbers $\alpha_c^{(0)}$ and $\alpha_c^{(1)}$ (corresponding respectively to $m = 0$ and $m = 1$) are very close, we choose one wave number α for the present analysis (then a multiple of $2\pi/h$), close to these critical values. The discussion made in Chapter II (see Figure II.1) shows that if h is not too large, other multiples of $2\pi/h$ will give larger critical Reynolds numbers on the neutral curves. The numerical values found in [Go-La] are transformed here to fit with our definition of coefficients in (IV.7). However, it should be noted that the coefficients of nonlinear terms depend on the choice of normalization for the eigenvectors $\zeta_0, \zeta_1, \zeta_2$ (not explicit in [Go-La]). (See Tables IV.1 and IV.2.)

We present two diagrams. Figure IV.5(a) represents in the parameter plane $[\mu = \mathfrak{R} - \mathfrak{R}_c,\ \nu = \Omega - \Omega^{(0)}]$, the lines where bifurcations occur, following the computations made with the principal part of the vector field F (in fact we qualitatively indicate the respective positions of these lines). We indicate the direction of the different bifurcations. By line (7) we know, following (IV.26), that a Hopf bifurcation occurs, leading to new

FIGURE IV.5. (a) Bifurcation lines in the parameter plane for $\eta = 0.80$. (b) Gyrant bifurcation diagram for $\eta = 0.80$. Stable branches are indicated with a full line, unstable branches with a dotted line. One does not know the stability of the WVF branch far from the bifurcation point (4). We observe that there are several flows which are stable together: spiral and Taylor vortices, as well as spirals with wavy vortex flow. A hysteresis phenomenon results when we move parameters in opposite directions.

quasi-periodic solutions (wavy spirals?). We do not know on what side they bifurcate because some necessary additional coefficients are unknown. In the present case, we can check that there are no Hopf bifurcations due to the 2×2 matrices (IV.30a) and the upper one in (IV.29). However, we do not know enough about coefficients to say whether or not there are other branches starting from wavy vortices or from the twisted vortices (due to the two other eigenvalues). Figure IV.5(b) is a "gyrant" bifurcation diagram. This means that one follows a clockwise path around the origin in the parameter plane, and we indicate qualitatively the bifurcating solutions. The left- and right-hand sides of this diagram must be identified.

IV.1.6 Bifurcation with higher codimension

The numerical calculation of the coefficients that govern the stability of the various branches of solutions described in this chapter show that for certain values of the radius ratio η the bifurcation is *degenerate*, i.e., the local shape of the branch of solutions is not determined by the cubic order terms in the normal form (IV.7) (this is analogous to the situation studied in Section III.3). In this case higher-order terms must be taken into account, and as η varies close to its critical value, the bifurcation can pass from

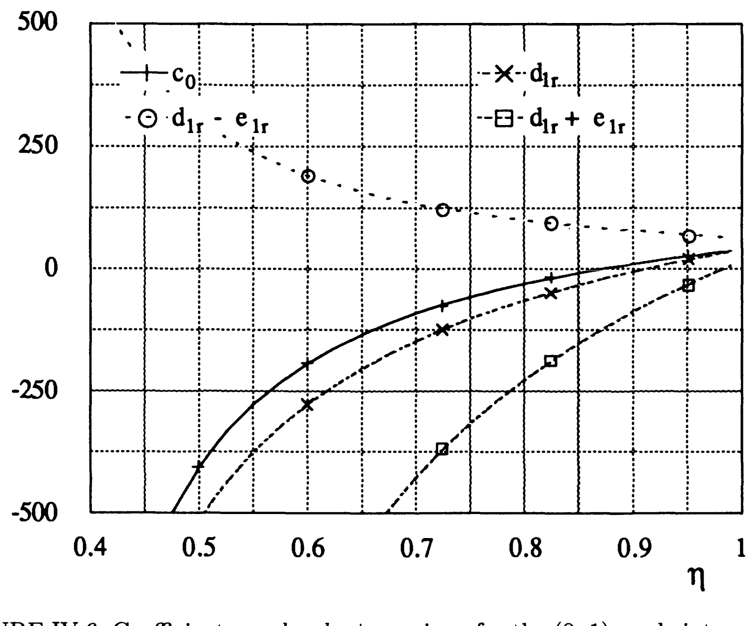


FIGURE IV.6. Coefficients $c_0, d_{1r}, d_{1r} \pm e_{1r}$ viz. η for the (0, 1) mode interaction.

supercritical to subcritical with a turning point. Near such a codimension-three point, a curve in the parameter space $\{\mathfrak{R}, \Omega, \eta\}$ can be found, along which the turning point is also a bifurcation point with interaction of two modes (axisymmetric and nonaxisymmetric). This new "codimension-two" bifurcation (there are now two free parameters, since the critical points form a curve in a 3-space) was studied in the case of weakly subcritical Taylor vortices by Signoret and Iooss [Si-Io]. Their main result is that, surprisingly, in some "large" regions of the critical point, ribbons or wavy vortices can be stable and that spirals can also be stable, so different kinds of solutions might be observable, depending on the initial data (*bistability*). Another case is that of weakly subcritical spirals. These bifurcations have been sought systematically by Laure [La 91], who also computed the coefficients of the normal form (IV.7). Figure IV.6 shows variation with respect to η of the coefficients $c_0, d_{1\mathrm{r}}, d_{1\mathrm{r}} \pm e_{1\mathrm{r}}$. We notice that c_0 and $d_{1\mathrm{r}}$, become positive at $\eta \approx 0.87$ and 0.92, respectively, meaning that branches of Taylor vortices and spirals can indeed become weakly subcritical. The second case has not been studied, to our knowledge. The coefficients $d_{1\mathrm{r}} \pm e_{1\mathrm{r}}$ govern the respective direction of branching of spirals and ribbons, so we observe that $d_{1\mathrm{r}} - e_{1\mathrm{r}}$ stays > 0 and $d_{1\mathrm{r}} + e_{1\mathrm{r}}$ stays < 0 except, maybe, for η extremely close to 1.

IV.2 Interaction between two nonaxisymmetric modes

IV.2.1 The amplitude equations (8 dimensions)

We now study the case when, at criticality, there are two pairs of eigenvalues $\pm \mathrm{i}\omega_1$ and $\pm \mathrm{i}\omega_2$ on the imaginary axis. This happens for many values of the parameters. In Figure IV.2, this situation corresponds to such cases given by the curves separating regions with different values of m ($\neq 0$). Again, criticality is defined by $\mu = \nu = 0$, where $\mu = \mathfrak{R} - \mathfrak{R}_c$, $\nu = \Omega - \Omega_m$, and Ω_m is as defined at the beginning of the chapter. Let us denote by ζ_0, ζ_1 the eigenvectors associated with the eigenvalue $\mathrm{i}\omega_1$ such that $\zeta_1 = \mathbf{S}\zeta_0$ and by ζ_2, ζ_3 the eigenvectors associated with the eigenvalue $\mathrm{i}\omega_2$ such that $\zeta_3 = \mathbf{S}\zeta_2$. Moreover, we assume that $\zeta_0(r,\theta,z) = \hat{U}_0(r)e^{\mathrm{i}(m\theta+\alpha z)}$ and $\zeta_2(r,\theta,z) = \hat{U}_2(r)e^{\mathrm{i}[(m+1)\theta+\alpha z]}$, $m > 0$ and $\alpha = 2\pi n/h$. This assumption essentially means that we only consider competition of modes such that the values of m differ by one (see [C-D-I] for the more general case of two azimuthal wave numbers $m_1 \neq m_2$) and such that the axial wave number can be taken as the same. These assumptions seem reasonable when we look at Figure IV.2 and for not too large gaps (see Figure III.5a). In fact, our periodicity assumption, which replaces the real boundary conditions, only allows that the two axial wave numbers be rationally related. The fact that their optimal values are very close gives us no other choice than to take them as equal. Then, writing the elements in **V** as

$$X = \sum_{j=0}^{3} A_j \zeta_j + \bar{A}_j \bar{\zeta}_j, \tag{IV.31}$$

the group Γ now acts on these coordinates as follows:

$$\begin{gathered}
\mathbf{T}_\psi A_0 = e^{ni\psi} A_0, \qquad \mathbf{T}_\psi A_1 = e^{-ni\psi} A_1, \qquad \mathbf{T}_\psi A_2 = e^{ni\psi} A_2, \\
\mathbf{T}_\psi A_3 = e^{-ni\psi} A_3, \\
\mathbf{R}_\varphi A_0 = e^{mi\varphi} A_0, \qquad \mathbf{R}_\varphi A_1 = e^{mi\varphi} A_1, \qquad \mathbf{R}_\varphi A_2 = e^{(m+1)i\varphi} A_2, \\
\mathbf{R}_\varphi A_3 = e^{(m+1)i\varphi} A_3, \\
A_1 = \mathbf{S} A_0, \qquad A_3 = \mathbf{S} A_2.
\end{gathered} \tag{IV.32}$$

We shall see that the Taylor expansion of the vector field F coincides with its normal form up to order three, so we do not need the extra symmetry introduced by the normalization of the vector field in our calculations.

Let us set

$$F(\mu, \nu, X) = \sum_{j=0}^{3} F_j(\mu, \nu, X) \zeta_j + \overline{F_j(\mu, \nu, X)} \bar{\zeta}_j.$$

The amplitude equations are therefore

$$\frac{dA_j}{dt} = F_j(\mu, \nu, X) \qquad (j = 0, 1, 2, 3). \tag{IV.33}$$

By formulas (IV.32) we have the relations

$$\begin{aligned}
F_0(\mu, \nu, \mathbf{R}_\varphi X) &= e^{im\varphi} F_0(\mu, \nu, X), \\
F_0(\mu, \nu, \mathbf{T}_\psi X) &= e^{ni\psi} F_0(\mu, \nu, X), \\
F_1(\mu, \nu, X) &= F_0(\mu, \nu, \mathbf{S} X);
\end{aligned} \tag{IV.34a}$$

$$\begin{aligned}
F_2(\mu, \nu, \mathbf{R}_\varphi X) &= e^{i(m+1)\varphi} F_2(\mu, \nu, X), \\
F_2(\mu, \nu, \mathbf{T}_\psi X) &= e^{ni\psi} F_2(\mu, \nu, X), \\
F_3(\mu, \nu, X) &= F_2(\mu, \nu, \mathbf{S} X).
\end{aligned} \tag{IV.34b}$$

The next remark is that the normal form of the vector field $F(\mu, \nu, .)$ (see Section II.4) is a polynomial vector field that commutes with the matrix $\exp(\tilde{L}_0^* t)$, where the operator $\tilde{L}_0^*$ is the adjoint matrix of $\tilde{L}_0 = D_X F(0,0,0)$, defined by

$$\tilde{L}_0^* = \begin{pmatrix} -i\omega_1 I_2 & 0 & 0 & 0 \\ 0 & -i\omega_2 I_2 & 0 & 0 \\ 0 & 0 & i\omega_1 I_2 & 0 \\ 0 & 0 & 0 & i\omega_2 I_2 \end{pmatrix},$$

where I_2 is the 2×2 identity matrix. If ω_1 and ω_2 are rationally dependent, i.e., if $\omega_1/\omega_2 = p/q \in \mathbb{Q}$, this is equivalent to saying that the normal form commutes with the S^1 representation in $\mathbf{V}$ defined by

$$\tilde{\mathbf{R}}_{\tilde{\varphi}} = \begin{pmatrix} e^{pi\tilde{\varphi}}I_2 & 0 & 0 & 0 \\ 0 & e^{qi\tilde{\varphi}}I_2 & 0 & 0 \\ 0 & 0 & e^{-pi\tilde{\varphi}}I_2 & 0 \\ 0 & 0 & 0 & e^{-qi\tilde{\varphi}}I_2 \end{pmatrix}.$$

Unless $mq = (m+1)p$, $\mathbf{R}_{\varphi}$ and $\tilde{\mathbf{R}}_{\tilde{\varphi}}$ define an action of the 2-torus $\mathbb{T}^2$ in $\mathbf{V}$, given by

$$\tilde{\mathbf{R}}_{\varphi_1,\varphi_2} = \begin{pmatrix} e^{i\varphi_1}I_2 & 0 & 0 & 0 \\ 0 & e^{i\varphi_2}I_2 & 0 & 0 \\ 0 & 0 & e^{-i\varphi_1}I_2 & 0 \\ 0 & 0 & 0 & e^{-i\varphi_2}I_2 \end{pmatrix}, \tag{IV.35}$$

where $\varphi_1 = m\varphi + p\tilde{\varphi}$ and $\varphi_2 = (m+1)\varphi + q\tilde{\varphi}$.

The same is clearly true if $\omega_1/\omega_2 \notin \mathbb{Q}$. However, the domain of validity of the transformation to a normal form up to some "high" order N can be very small if $m\omega_2 \approx (m+1)\omega_1$, and we have no interest in deriving the normal form as if it were $\neq$. Instead, it is better to work on a normal form given by the "worse case," which is when $m\omega_2 = (m+1)\omega_1$, considering an artificial "detuning parameter." This situation may well happen in examples (see Subsection IV.2.6). In fact, we shall be only interested in this problem through the equations up to third order, and since we show that the equations coincide at this order with their normal form (see Lemma 3), there is no real problem.

Lemma 3. (i) *Any polynomial vector field satisfying the relations (IV.34) has the following structure: let us set* $A_j = \rho_j e^{i\psi_j}$, $u_j = \rho_j^2$, *and* $v = A_0\bar{A}_1\bar{A}_2A_3$. *Then*

$$\begin{aligned} F_0 &= A_0 g_0(u_0,u_1,u_2,u_3,v) + A_1A_2\bar{A}_3 h_0(u_0,u_1,u_2,u_3,\bar{v}) \\ &\quad + O(||X||^{4m+1}), \end{aligned} \tag{IV.36a}$$

$$\begin{aligned} F_2 &= A_2 g_2(u_0,u_1,u_2,u_3,\bar{v}) + A_0\bar{A}_1A_3 h_2(u_0,u_1,u_2,u_3,v) \\ &\quad + O(||X||^{4m+1}), \end{aligned} \tag{IV.36b}$$

where g_j and h_j are complex polynomials (of arbitrary degree). The components F_1 and F_3 are deduced from F_0 and F_2 by permuting indices 0, 1 *on the one hand and* 2, 3 *on the other hand, in the arguments.*

(ii) *If in addition* $m\omega_2 \neq (m+1)\omega_1$ *the normal form of F is given by (IV.36) at any order.*

Proof. We first write a monomial in F_0 (for example) in its general form

$$P = A_0^{p_0}\bar{A}_0^{q_0}A_1^{p_1}\bar{A}_1^{q_1}A_2^{p_2}\bar{A}_2^{q_2}A_3^{p_3}\bar{A}_3^{q_3}.$$

Applying formulas (IV.34a) leads to the relations

$$m(p_0 - q_0 + p_1 - q_1) + (m+1)(p_2 - q_2 + p_3 - q_3) = m,$$
$$p_0 - q_0 - p_1 + q_1 + p_2 - q_2 - p_3 + q_3 = 1. \tag{IV.37}$$

It follows from these relations that

$$\begin{aligned} p_0 - q_0 + p_1 - q_1 - 1 &= (m+1)k, \\ p_2 - q_2 + p_3 - q_3 &= -mk, \\ 2(p_0 - q_0 + p_2 - q_2) &= k+2, \\ 2(p_1 - q_1 + p_3 - q_3) &= k, \qquad k \in \mathbb{Z}. \end{aligned}$$

Solving this linear system of equations leads to the structure F_0 without difficulty. The proof is the same for F_2. For the second part of the lemma, note that the use of the group representation (IV.35) replaces the first of relations (IV.37) by

$$\begin{aligned} p_0 - q_0 + p_1 - q_1 &= 1, \\ p_2 - q_2 + p_3 - q_3 &= 0. \end{aligned}$$

Then the rest of the reasoning is the same as for the first part.

Remark. The proof of Lemma 3 would allow a complete description of the equivariant structure of polynomials F (not only up to some finite order). This was made in [Ch 85], but we do not need this refinement here. There is in fact a "global" result, which gives the equivariant structure of F (not only of its Taylor expansion) and which is proved in [C-D-I] (Lemma 3):

Lemma 3′. *The eight-dimensional vector field satisfying relations (IV.34) can be written under the form*

$$\frac{dA_j}{dt} = e^{i\psi_j} f_j(\mu, \nu, \rho_0, \rho_1, \rho_2, \rho_3, \Theta_1, \Theta_2) \qquad (j = 0, 1, 2, 3), \tag{IV.38}$$

where $A_j = \rho_j e^{i\psi_j}$, *and* f_j *are* 2π*-periodic in* $\Theta_1 = \psi_1 - \psi_0 + \psi_2 - \psi_3$, $\Theta_2 = (m+1)(\psi_1 + \psi_0) - m(\psi_2 + \psi_3)$. *Moreover,* f_0 *and* f_1 *are odd in* (ρ_0, ρ_1) *and even in* (ρ_2, ρ_3), *while* f_2 *and* f_3 *are even in* (ρ_0, ρ_1) *and odd in* (ρ_2, ρ_3). *Finally, for* $(j, k) = (0, 1)$ *and (2, 3), we have*

$$f_j(\mu, \nu, \rho_0, \rho_1, \rho_2, \rho_3, \Theta_1, \Theta_2) = f_k(\mu, \nu, \rho_0, \rho_1, \rho_2, \rho_3, -\Theta_1, -\Theta_2), \tag{IV.39}$$

and if two of the ρ_j *are 0, then* f_k *is independent of* (Θ_1, Θ_2) *and odd in* ρ_k.

It follows from Lemma 3′ that the dynamics lies in a six-dimensional manifold. In what follows, we extensively use the following form of the principal part of the amplitude equations, which follows directly from Lemma 3:

$$\begin{aligned} \frac{dA_0}{dt} = {} & A_0[i\omega_1 + \alpha_0\mu + \beta_0\nu + b_0\rho_0^2 + c_0\rho_1^2 + d_0\rho_2^2 + e_0\rho_3^2] \\ & + f_0 A_1 A_2 \bar{A}_3 + \text{ h.o.t.}, \end{aligned}$$

$$\begin{aligned}\frac{dA_1}{dt} &= A_1[\mathrm{i}\omega_1 + \alpha_0\mu + \beta_0\nu + c_0\rho_0^2 + b_0\rho_1^2 + e_0\rho_2^2 + d_0\rho_3^2] \\ &\quad + f_0 A_0 \bar{A}_2 A_3 + \text{ h.o.t.}; \qquad \text{(IV.40a)} \\ \frac{dA_2}{dt} &= A_2[\mathrm{i}\omega_2 + \alpha_2\mu + \beta_2\nu + b_2\rho_0^2 + c_2\rho_1^2 + d_2\rho_2^2 + e_2\rho_3^2] \\ &\quad + f_2 A_0 \bar{A}_1 A_3 + \text{ h.o.t.}, \\ \frac{dA_3}{dt} &= A_3[\mathrm{i}\omega_2 + \alpha_2\mu + \beta_2\nu + c_2\rho_0^2 + b_2\rho_1^2 + e_2\rho_2^2 + d_2\rho_3^2] \\ &\quad + f_2 \bar{A}_0 A_1 A_2 + \text{ h.o.t.} \qquad \text{(IV.40b)}\end{aligned}$$

Remark. As noted in Subsection IV.1.1 (Remark 2, p. 65), we might use the invariant monomials u_j, v, and $\bar{v}$ as new variables. Equations (IV.40) then reduce to an equivalent system for these new unknowns. Here it means reducing to a six-dimensional real system, which is more convenient for the stability analysis (see [Me] for further details).

IV.2.2 Restriction of the equations to flow-invariant subspaces

We apply the method introduced in Subsection IV.1.2 to compute flow-invariant subspaces as the spaces of points that are fixed by the action of subgroups (called the isotropy subgroups) of Γ (Lemma 2). The following lemma identifies these subspaces.

Lemma 4. *The following subspaces of* $\mathbf{V}$ *are invariant for the system (IV.33):*

(i) $A_2 = A_3 = 0$,
(ii) $A_0 = A_1 = 0$,
(iii) $A_0 = A_3 = 0$,
(iv) $A_1 = A_2 = 0$,
(v) $A_0 = e^{\mathrm{i}\psi} A_1$ *and* $A_2 = e^{\mathrm{i}\psi} A_3$,
(vi) $A_0 = e^{\mathrm{i}\psi} A_1$ *and* $A_2 = -e^{-\mathrm{i}\psi} A_3$ *(for arbitrary* ψ*).*

Proof. We proceed as in the proof of Lemma 2, by looking first for the points in $\mathbf{V}$ that are fixed by nontrivial rotations $\mathbf{R}_\varphi \mathbf{T}_\psi$. Using relations (IV.32), one checks that only points with at least two vanishing coordinates A_j, A_k $(j \neq k)$ are fixed by such group elements. This leads to subspaces (i), (ii), (iii), (iv) of the lemma. Next we look for those points that are fixed by group elements including the symmetry $\mathbf{S}$. A simple reasoning then leads to subspaces (v) and (vi).

Remarks. (1) The subspace (iv) is simply the image by $\mathbf{S}$ of (iii). It will therefore be sufficient to look for solutions in (iii). In the same way it will be sufficient to look for solutions in the case $\psi = 0$ in subspaces (v) and (vi), the solutions in the subspaces with $\psi \neq 0$ being just translated from those by $\mathbf{T}_\psi$.

(2) In Lemma 4 we listed the subspaces of $\mathbf{V}$ that are flow-invariant by the vector field F, thanks to the fact that F commutes with the action

(IV.32) of the group Γ. If we deal with the *normal form* of F, instead of F itself, the group of symmetry may be larger than Γ. As we noticed in IV.2.1, if $m\omega_2 \not\approx (m+1)\omega_1$, the normal form is equivariant by the group $\mathrm{O}(2)\times\mathbb{T}^2$, with the action of $\mathbb{T}^2$ defined by (IV.35). In this case, the subspace $A_1 = A_3 = 0$ (and its symmetric $A_0 = A_2 = 0$) is flow-invariant for the normal form of F, in addition to the subspaces given in Lemma 4.

IV.2.3 Bifurcated solutions

The spaces (i) and (ii) of Lemma 4 correspond to *pure mode* bifurcations (no interaction of the azimuthal wave numbers m and $m+1$). Therefore, in these subspaces we essentially get the primary branches of spirals and ribbons (with azimuthal wave numbers m and $m+1$) studied in Chapter III.

IV.2.3.1 Primary branches

In Lemma 4 the spaces (i) and (ii) correspond to *pure mode* bifurcations (no interaction of different wave numbers). Therefore, in these subspaces we get the branches of spirals and ribbons (with azimuthal wave number m and $m+1$ respectively) studied in Chapter III. We call them m or $(m+1)$-spirals and ribbons.

1(a) m-Spirals and m-ribbons. Equations in the subspace (i) are as follows:

$$\frac{dA_0}{dt} = F_0(\mu, \nu, A_0, \bar{A}_0, A_1, \bar{A}_1, 0, 0, 0, 0),$$
$$\frac{dA_1}{dt} = F_1(\mu, \nu, A_0, \bar{A}_0, A_1, \bar{A}_1, 0, 0, 0, 0).$$

As in Lemmas 3 and 3′, we set $A_0 = \rho_0 e^{i\psi_0}$ and $A_1 = \rho_1 e^{i\psi_1}$. Then, thanks to (IV.40) and Lemma 3′, the above system can be written in the form

$$\left\{\begin{aligned} \frac{d\rho_0}{dt} &= \rho_0[\alpha_{0r}\mu + \beta_{0r}\nu + b_{0r}\rho_0^2 + c_{0r}\rho_1^2 + \mathrm{O}(|\mu| + |\nu| + \rho_0^2 + \rho_1^2)^2], \\ \frac{d\rho_1}{dt} &= \rho_1[\alpha_{0r}\mu + \beta_{0r}\nu + c_{0r}\rho_0^2 + b_{0r}\rho_1^2 + \mathrm{O}(|\mu| + |\nu| + \rho_0^2 + \rho_1^2)^2]; \end{aligned}\right. \tag{IV.41a}$$

$$\left\{\begin{aligned} \frac{d\psi_0}{dt} &= \omega_1 + \alpha_{0i}\mu + \beta_{0i}\nu + b_{0i}\rho_0^2 + c_{0i}\rho_1^2 + \mathrm{O}(|\mu| + |\nu| + \rho_0^2 + \rho_1^2)^2, \\ \frac{d\psi_1}{dt} &= \omega_1 + \alpha_{0i}\mu + \beta_{0i}\nu + c_{0i}\rho_0^2 + b_{0i}\rho_1^2 + \mathrm{O}(|\mu| + |\nu| + \rho_0^2 + \rho_1^2)^2. \end{aligned}\right. \tag{IV.41b}$$

The bifurcation analysis is now *identical* to that of Section III.2. We obtain the following solutions:

m-spiral flow.

$$\left\{\begin{array}{lcl}\rho_1 & = & 0, \qquad \alpha_{0\mathrm{r}}\mu+\beta_{0\mathrm{r}}\nu+b_{0\mathrm{r}}\rho_0^2+\mathrm{O}(|\mu|+|\nu|+\rho_0^2)^2=0,\\ \psi_0 & = & \Omega_0 t+\varphi_0 \qquad (\varphi_0 \text{ an arbitrary phase}),\\ \Omega_0 & = & \omega_1+\alpha_{0\mathrm{i}}\mu+\beta_{0\mathrm{i}}\nu+b_{0\mathrm{i}}\rho_0^2+\mathrm{O}(|\mu|+|\nu|)^2.\end{array}\right. \tag{IV.42}$$

m-ribbons.

$$\left\{\begin{array}{lcl}\rho_0 & = & \rho_1, \qquad \alpha_{0\mathrm{r}}\mu+\beta_{0\mathrm{r}}\nu+(b_{0\mathrm{r}}+c_{0\mathrm{r}})\rho_0^2+\mathrm{O}(|\mu|+|\nu|+\rho_0^2)^2=0,\\ \psi_0 & = & \psi_1+\varphi_1=\Omega_1 t+\varphi_1' \qquad (\varphi_1,\varphi_1' \text{ arbitrary phases}),\\ \Omega_1 & = & \omega_1+\alpha_{0\mathrm{i}}\mu+\beta_{0\mathrm{i}}\nu+(c_{0\mathrm{i}}+b_{0\mathrm{i}})\rho_0^2+\mathrm{O}(|\mu|+|\nu|)^2.\end{array}\right. \tag{IV.43}$$

Both spirals and ribbons are described in Section III.2. m-ribbons are rotating waves and invariant under symmetry **S** provided one chooses a suitable origin on the axis. Notice that they also occur in the subspaces (v) and (vi) of Lemma 4 while m-spirals also occur in the subspace (iv) [and (iii) for the symmetric ones].

1(b) (m+1)-Spirals and (m+1)-Ribbons. Amplitude equations in the subspace (ii) are as follows:

$$\frac{dA_2}{dt}=F_2(\mu,\nu,0,0,0,0,A_2,\bar{A}_2,A_3,\bar{A}_3),$$
$$\frac{dA_3}{dt}=F_3(\mu,\nu,0,0,0,0,A_2,\bar{A}_2,A_3,\bar{A}_3).$$

As in Lemmas 3 and 3′, we set $A_2=\rho_2 e^{\mathrm{i}\psi_2}$ and $A_3=\rho_3 e^{\mathrm{i}\psi_3}$. Thanks to (IV.40) and Lemma 3′, the above system can be written

$$\left\{\begin{array}{lcl}\dfrac{d\rho_2}{dt} & = & \rho_2[\alpha_{2\mathrm{r}}\mu+\beta_{2\mathrm{r}}\nu+d_{2\mathrm{r}}\rho_2^2+e_{2\mathrm{r}}\rho_3^2+\mathrm{O}(|\mu|+|\nu|+\rho_2^2+\rho_3^2)^2],\\ \dfrac{d\rho_3}{dt} & = & \rho_3[\alpha_{2\mathrm{r}}\mu+\beta_{2\mathrm{r}}\nu+e_{2\mathrm{r}}\rho_2^2+d_{2\mathrm{r}}\rho_3^2+\mathrm{O}(|\mu|+|\nu|+\rho_2^2+\rho_3^2)^2];\end{array}\right. \tag{IV.44a}$$

$$\left\{\begin{array}{lcl}\dfrac{d\psi_2}{dt} & = & \omega_2+\alpha_{2\mathrm{i}}\mu+\beta_{2\mathrm{i}}\nu+d_{2\mathrm{i}}\rho_2^2+e_{2\mathrm{i}}\rho_3^2+\mathrm{O}(|\mu|+|\nu|+\rho_2^2+\rho_3^2)^2,\\ \dfrac{d\psi_3}{dt} & = & \omega_2+\alpha_{2\mathrm{i}}\mu+\beta_{2\mathrm{i}}\nu+e_{2\mathrm{i}}\rho_2^2+d_{2\mathrm{i}}\rho_3^2+\mathrm{O}(|\mu|+|\nu|+\rho_2^2+\rho_3^2)^2.\end{array}\right. \tag{IV.44b}$$

Accordingly to the analysis in Section III.2, we obtain the following solutions:

(m+1)-Spiral flow.

$$\left\{\begin{array}{lcl}\rho_3 & = & 0, \qquad \alpha_{2\mathrm{r}}\mu+\beta_{2\mathrm{r}}\nu+d_{2\mathrm{r}}\rho_2^2+\mathrm{O}(|\mu|+|\nu|+\rho_2^2)^2=0,\\ \psi_2 & = & \Omega_2 t+\varphi_2 \qquad (\varphi_2 \text{ arbitrary phase}),\\ \Omega_2 & = & \omega_2+\alpha_{2\mathrm{i}}\mu+\beta_{2\mathrm{i}}\nu+d_{2\mathrm{i}}\rho_2^2+\mathrm{O}(|\mu|+|\nu|)^2.\end{array}\right. \tag{IV.45}$$

$(m+1)$-Ribbons.

$$\begin{cases} \rho_2 &= \rho_3, \qquad \alpha_{2\mathrm{r}}\mu + \beta_{2\mathrm{r}}\nu + (d_{2\mathrm{r}} + e_{2\mathrm{r}})\rho_2^2 + \mathrm{O}(|\mu| + |\nu| + \rho_2^2)^2 = 0, \\ \psi_2 &= \psi_3 + \varphi_3 = \Omega_3 t + \varphi_3' \qquad (\varphi_3, \varphi_3' \text{ arbitrary phases}), \\ \Omega_3 &= \omega_2 + \alpha_{2\mathrm{i}}\mu + \beta_{2\mathrm{i}}\nu + (d_{2\mathrm{i}} + e_{2\mathrm{i}})\rho_2^2 + \mathrm{O}(|\mu| + |\nu|)^2. \end{cases} \tag{IV.46}$$

Notice that the $(m+1)$-ribbons also occur in the subspaces (v) and (vi) of Lemma 4 while $(m+1)$-spirals also occur in the subspace (iii) [and (iv) for the symmetric ones].

IV.2.3.2 Interpenetrating spirals (first kind)

Let us now consider solutions of (IV.33) in the subspace (iv) (or (iii) by applying **S**) of Lemma 4.

Proposition 5. *Setting $A_j = \rho_j e^{\mathrm{i}\psi_j}$ the equations in the space $\{A_1 = A_2 = 0\}$ are, following the notation of (IV.36),*

$$\begin{aligned} \frac{dA_0}{dt} &= A_0 g_0(\mu, \nu, \rho_0^2, 0, 0, \rho_3^2, 0), \\ \frac{dA_3}{dt} &= A_3 g_2(\mu, \nu, 0, \rho_0^2, \rho_3^2, 0, 0), \end{aligned} \tag{IV.47}$$

where the functions g_j (j = 1,2) are defined in Lemma 3.

Proof. This follows directly from Lemma 3$'$.

Equations (IV.47) obviously separate into phases and moduli equations, which with the notation introduced in (IV.40) gives the system

$$\begin{cases} \dfrac{d\rho_0}{dt} &= \rho_0[\alpha_{0\mathrm{r}}\mu + \beta_{0\mathrm{r}}\nu + b_{0\mathrm{r}}\rho_0^2 + e_{0\mathrm{r}}\rho_3^2 + \mathrm{O}(|\mu| + |\nu| + \rho_0^2 + \rho_3^2)^2], \\ \dfrac{d\rho_3}{dt} &= \rho_3[\alpha_{2\mathrm{r}}\mu + \beta_{2\mathrm{r}}\nu + c_{2\mathrm{r}}\rho_0^2 + d_{2\mathrm{r}}\rho_3^2 + \mathrm{O}(|\mu| + |\nu| + \rho_0^2 + \rho_3^2)^2]; \end{cases} \tag{IV.48a}$$

$$\begin{cases} \dfrac{d\psi_0}{dt} &= \omega_1 + \alpha_{0\mathrm{i}}\mu + \beta_{0\mathrm{i}}\nu + b_{0\mathrm{i}}\rho_0^2 + e_{0\mathrm{i}}\rho_3^2 + \mathrm{O}(|\mu| + |\nu| + \rho_0^2 + \rho_3^2)^2, \\ \dfrac{d\psi_3}{dt} &= \omega_2 + \alpha_{2\mathrm{i}}\mu + \beta_{2\mathrm{i}}\nu + c_{2\mathrm{i}}\rho_0^2 + d_{2\mathrm{i}}\rho_3^2 + \mathrm{O}(|\mu| + |\nu| + \rho_0^2 + \rho_3^2)^2. \end{cases} \tag{IV.48b}$$

If either ρ_0 or ρ_3 is set equal to 0, one gets the $(m+1)$- or m-spirals respectively. If $\rho_0\rho_3 \neq 0$, equilibria of (IV.48a) correspond to quasi-periodic solutions of (IV.47) with two frequencies given by integrating (IV.48b). For the further analysis, we assume that the following conditions are fulfilled:

$$\begin{aligned} \alpha_{0\mathrm{r}}\beta_{2\mathrm{r}} - \alpha_{2\mathrm{r}}\beta_{0\mathrm{r}} &\neq 0, \\ b_{0\mathrm{r}}d_{2\mathrm{r}} - c_{2\mathrm{r}}e_{0\mathrm{r}} &\neq 0. \end{aligned} \tag{IV.49}$$

Then indeed, the second condition allows the implicit function theorem to be applied on the stationary part of (IV.48a) in order to assert the existence of a branch of equilibria (ρ_0^2, ρ_3^2) function of (μ, ν), branching off the $(m+1)$-spirals when

$$(\alpha_{0\mathrm{r}} d_{2\mathrm{r}} - \alpha_{2\mathrm{r}} e_{0\mathrm{r}})\mu + (\beta_{0\mathrm{r}} d_{2\mathrm{r}} - \beta_{2\mathrm{r}} e_{0\mathrm{r}})\nu = 0 \tag{IV.50a}$$

and off the m-spirals when

$$(\alpha_{0\mathrm{r}} c_{2\mathrm{r}} - \alpha_{2\mathrm{r}} b_{0\mathrm{r}})\mu + (\beta_{0\mathrm{r}} c_{2\mathrm{r}} - \beta_{2\mathrm{r}} b_{0\mathrm{r}})\nu = 0. \tag{IV.50b}$$

In the space $\mathbf{V}$ these quasi-periodic solutions have the form

$$X(t) = \rho_0[e^{\mathrm{i}(\Omega_0 t + \varphi_0)}\zeta_0 + \text{ c.c.}] + \rho_3[e^{\mathrm{i}(\Omega_3 t + \varphi_3)}\zeta_3 + \text{ c.c.}], \tag{IV.51}$$

where

$$\Omega_0 = \omega_1 + \mathrm{O}(|\mu| + |\nu|) \quad \text{and} \quad \Omega_3 = \omega_2 + \mathrm{O}(|\mu| + |\nu|) \tag{IV.52}$$

and φ_0, φ_3 are arbitrary phases. In other words, the flow is linear on a 2-torus, and the two frequencies vary continuously and independently with the parameters. Frequency locking cannot happen in this case as is well known for rotationally invariant systems (generically), since this makes in fact only one frequency in a suitably rotating frame.

Physical interpretation. The solutions (IV.51) have exactly the form of a superposition of m-spirals and $(m+1)$-spirals in $\mathbf{V}$. One may ask whether this property extends to the solution $U = X + \Phi(\mu, \nu, X)$ of the Navier-Stokes equations (for the perturbation of the Couette flow). The answer is given in the following proposition.

Proposition 6. *The velocity vector field $U = X + \Phi(\mu, \nu, X)$ given by the center manifold theorem (see Section II.4) where X is the solution (IV.51) of system (IV.40) have the following form in cylindrical coordinates:*

$$U(r, \theta, z, t) = \tilde{U}(r, m\theta + \alpha z + \Omega_0 t + \varphi_0, (m+1)\theta - \alpha z + \Omega_3 t + \varphi_3), \tag{IV.53}$$

where φ_0 and φ_3 are arbitrary phases and $\tilde{U}$ is 2π-periodic in (φ_0, φ_3).

Notation. These solutions are called *interpenetrating spirals* and are denoted by $\mathrm{SI}^{(0,3)}$ (see Figure IV.7). The "mirror images" of $\mathrm{SI}^{(0,3)}$, i.e., their images by symmetry $\mathbf{S}$, are denoted by $\mathrm{SI}^{(1,2)}$.

Proof of Proposition 6. Notice that the solution (IV.51) takes the form

$$X(t) = \mathbf{R}_{\Omega_0 t/m}\tilde{X}_0 + \mathbf{R}_{\Omega_3 t/(m+1)}\tilde{X}_3,$$

where

$$\tilde{X}_0 = \rho_0 \hat{U}_0(r) e^{\mathrm{i}(m\theta + \alpha z + \varphi_0)} + \text{ c.c., and } \tilde{X}_3 = \rho_3 \mathbf{S}\hat{U}_2(r) e^{\mathrm{i}[(m+1)\theta - \alpha z + \varphi_3]} + \text{ c.c.}$$

Let us note for simplicity $Y(t) = \Phi(.,., X(t))$; then the equivariance of Φ for the symmetries Γ and time translation $\sigma_s : U(t) \to U(t+s)$ involves

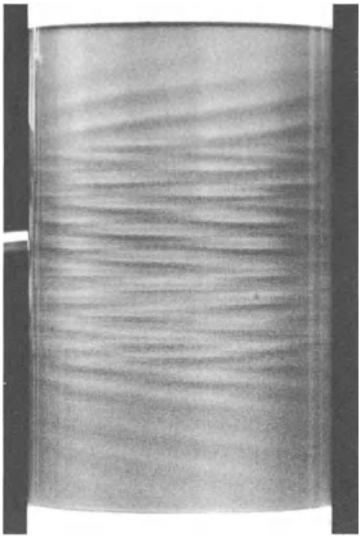


FIGURE IV.7. Interpenetrating spirals (from [An-L-Sw]).

the relations

$$\begin{aligned}
\mathbf{R}_\varphi Y(t) &= \Phi[\mu, \nu, \mathbf{R}_{\varphi+\Omega_0 t/m}\tilde{X}_0 + \mathbf{R}_{\varphi+\Omega_3 t/(m+1)}\tilde{X}_3], \\
\mathbf{T}_\psi Y(t) &= \Phi[\mu, \nu, \mathbf{R}_{n\psi+\Omega_0 t/m}\tilde{X}_0 + \mathbf{R}_{(-n\psi+\Omega_3 t)/(m+1)}\tilde{X}_3], \\
\sigma_s Y(t) &= \Phi[\mu, \nu, \mathbf{R}_{(s+t)\Omega_0/m}\tilde{X}_0 + \mathbf{R}_{(s+t)\Omega_3/(m+1)}\tilde{X}_3]. \qquad \text{(IV.54)}
\end{aligned}$$

Differentiating these relations with respect to φ, ψ, s at $\varphi = \psi = s = 0$, we get in cylindrical coordinates

$$\frac{\partial Y}{\partial t} = \left(\frac{\Omega_0 + \Omega_3}{2m+1}\right)\frac{\partial Y}{\partial \theta} + \frac{[(m+1)\Omega_0 - m\Omega_3]}{\alpha(2m+1)}\frac{\partial Y}{\partial z}.$$

Since $[m\theta + \alpha z + \Omega_0 t]$ and $[(m+1)\theta - \alpha z + \Omega_3 t]$ are both first integrals of the characteristic system, the result follows immediately.

Remark. The frequencies Ω_0 and Ω_3 have the sign of ω_0 and ω_2, respectively. If this is the same sign, the two trains of spirals rotate in the *same* direction but propagate along the axis in *opposite* directions, while if the signs are different they rotate in *opposite* directions but propagate in the *same* direction.

IV.2.3.3 Interpenetrating spirals (second kind)

All the calculations in the previous section transpose to the situation indicated in Remark (2) of this section, if we assume the vector field to be in normal form or just restrict it to orders less than $O(|X|^{4m+1})$, which we can do if there are no additional degeneracies. Then, as we saw, the subspace $A_1 = A_3 = 0$ (and its symmetric $A_2 = A_4 = 0$) is flow-invariant for the amplitude equations. Since spirals are rotating waves, bifurcation of quasi-periodic solutions from spirals can be transformed to Hopf bifurcation from stationary solutions by a simple change of frame [Io 84]. It follows that such quasi-periodic flows persist for the full equation and have the form of interpenetrating spirals up to order $O(|X|^{4m})$.

We now give some indications on these solutions. The differential system decomposed into phases and moduli takes the form

$$\left\{\begin{aligned} \frac{d\rho_0}{dt} &= \rho_0[\alpha_{0\mathrm{r}}\mu + \beta_{0\mathrm{r}}\nu + b_{0\mathrm{r}}\rho_0^2 + d_{0\mathrm{r}}\rho_2^2 + O(|\mu| + |\nu| + \rho_0^2 + \rho_2^2)^2], \\ \frac{d\rho_2}{dt} &= \rho_2[\alpha_{2\mathrm{r}}\mu + \beta_{2\mathrm{r}}\nu + b_{2\mathrm{r}}\rho_0^2 + d_{2\mathrm{r}}\rho_2^2 + O(|\mu| + |\nu| + \rho_0^2 + \rho_2^2)^2]; \end{aligned}\right. \tag{IV.55a}$$

$$\left\{\begin{aligned} \frac{d\psi_0}{dt} &= \omega_1 + \alpha_{0\mathrm{i}}\mu + \beta_{0\mathrm{i}}\nu + b_{0\mathrm{i}}\rho_0^2 + d_{0\mathrm{i}}\rho_2^2 + O(|\mu| + |\nu| + \rho_0^2 + \rho_2^2)^2, \\ \frac{d\psi_2}{dt} &= \omega_2 + \alpha_{2\mathrm{i}}\mu + \beta_{2\mathrm{i}}\nu + b_{2\mathrm{i}}\rho_0^2 + d_{2\mathrm{i}}\rho_2^2 + O(|\mu| + |\nu| + \rho_0^2 + \rho_2^2)^2. \end{aligned}\right. \tag{IV.55b}$$

For the further analysis, we assume that the following conditions are fulfilled:

$$\left\{\begin{aligned} \alpha_{0\mathrm{r}}\beta_{2\mathrm{r}} - \alpha_{2\mathrm{r}}\beta_{0\mathrm{r}} &\neq 0, \\ b_{0\mathrm{r}}d_{2\mathrm{r}} - b_{2\mathrm{r}}d_{0\mathrm{r}} &\neq 0, \end{aligned}\right. \tag{IV.56}$$

and the frequencies have the form

$$\Omega_0 = \omega_1 + O(|\mu| + |\nu|) \quad \text{and} \quad \Omega_2 = \omega_2 + O(|\mu| + |\nu|). \tag{IV.57}$$

Note that in this case too the frequencies are independent (no frequency locking).

The physical meaning of these solutions is better understood from the following proposition, the proof of which is identical to the proof of Proposition 6.

Proposition 7. *The velocity fields $U = X + \Phi(\mu, \nu, X)$ for the normal form (or for the equations expended up to order $O(|X|^4)$, where X is the quasi-periodic solution of (IV.55), have the following form in cylindrical coordinates:*

$$U(r, \theta, z, t) = \tilde{U}(r, m\theta + \alpha z + \Omega_0 t + \varphi_0, (m+1)\theta + \alpha z + \Omega_2 t + \varphi_2),$$

where φ_0 and φ_2 are arbitrary phases and $\tilde{U}$ is 2π-periodic in φ_0 and φ_2.

It follows that if m is "large" enough, these solutions will not be distinguishable from a "pure" superposition of spirals.

Notation. These solutions are called *interpenetrating spirals* and are denoted by $\mathrm{SI}^{(0,2)}$. Their images by $\mathbf{S}$ are denoted by $\mathrm{SI}^{(1,3)}$.

IV.2.3.4 Superposed ribbons (first kind)

We now look for solutions in the subspace (v) of Lemma 4. Proposition 5 (in the case of interpenetrating spirals) does not apply here; there exist terms of order $\geq 4m+1$ for which phases do not decouple with moduli in the equations restricted to this subspace. Setting the arbitrary phase equal to 0, the equations are now

$$\begin{aligned} \frac{dA_0}{dt} &= A_0 g_0(\mu, \nu, \rho_0^2, \rho_0^2, \rho_2^2, \rho_2^2, v) + \mathrm{O}(||X||^{4m+1}) \\ \frac{dA_2}{dt} &= A_2 g_2(\mu, \nu, \rho_0^2, \rho_0^2, \rho_2^2, \rho_2^2, v) + \mathrm{O}(||X||^{4m+1}) \end{aligned} \tag{IV.58}$$

and we have $A_0 = A_1$, $A_2 = A_3$, and v is defined by $v = \rho_0^2\rho_2^2$. If one of the amplitudes is set equal to 0, the corresponding equation is satisfied and the solutions of the remaining equation are the m- or $(m+1)$-ribbons. Suppose now that we forget about terms of order $\geq 4m+1$ in equations (IV.58); then by a now well-known technique we can decouple phases and moduli. Ribbons are equilibria for these moduli equations and if there exist other equilibria bifurcating from these, the corresponding solutions to the truncated equations (IV.58) are quasi-periodic. We claim that this property persists for the complete equations (IV.58). Indeed, ribbons are equilibria not only for the truncated moduli equations but for the full system too, after a suitable change of frame (see the preliminary remark 1 in Subsection IV.1.4). It follows that bifurcation of quasi-periodic solutions from the ribbons can essentially be reduced to Hopf bifurcation from the corresponding equilibria, which ensures the persistence of this bifurcation for the full system (IV.58). Therefore, we can now look for equilibria of the "decoupled" moduli equations

$$\left\{ \begin{aligned} \frac{d\rho_0}{dt} &= \rho_0[\alpha_{0\mathrm{r}}\mu + \beta_{0\mathrm{r}}\nu + (b_{0\mathrm{r}} + c_{0\mathrm{r}})\rho_0^2 + (d_{0\mathrm{r}} + e_{0\mathrm{r}} + f_{0\mathrm{r}})\rho_2^2 \\ &\quad +\mathrm{O}(|\mu| + |\nu| + \rho_0^2 + \rho_2^2)^2], \\ \frac{d\rho_2}{dt} &= \rho_2[\alpha_{2\mathrm{r}}\mu + \beta_{2\mathrm{r}}\nu + (b_{2\mathrm{r}} + c_{2\mathrm{r}} + f_{2\mathrm{r}})\rho_0^2 + (d_{2\mathrm{r}} + e_{2\mathrm{r}})\rho_2^2 \\ &\quad +\mathrm{O}(|\mu| + |\nu| + \rho_0^2 + \rho_2^2)^2]. \end{aligned} \right. \tag{IV.59}$$

Assuming in the following that the conditions

$$\begin{aligned} \alpha_{0\mathrm{r}}\beta_{2\mathrm{r}} - \beta_{0\mathrm{r}}\alpha_{2\mathrm{r}} &\neq 0, \\ (b_{0\mathrm{r}} + c_{0\mathrm{r}})(d_{2\mathrm{r}} + e_{2\mathrm{r}}) - (d_{0\mathrm{r}} + e_{0\mathrm{r}} + f_{0\mathrm{r}})(b_{2\mathrm{r}} + c_{2\mathrm{r}} + f_{2\mathrm{r}}) &\neq 0, \end{aligned} \tag{IV.60}$$

are satisfied, the implicit function theorem asserts the existence of a branch of equilibria of (IV.59) bifurcating from the $(m+1)$-ribbons when

$$[\alpha_{0\mathrm{r}}(d_{2\mathrm{r}}+e_{2\mathrm{r}})-\alpha_{2\mathrm{r}}(d_{0\mathrm{r}}+e_{0\mathrm{r}}+f_{0\mathrm{r}})]\mu+[\beta_{0\mathrm{r}}(d_{2\mathrm{r}}+e_{2\mathrm{r}})-\beta_{2\mathrm{r}}(b_{0\mathrm{r}}+e_{0\mathrm{r}}+f_{0\mathrm{r}})]\nu = 0 \tag{IV.61a}$$

and from the m-ribbons when

$$[\alpha_{0r}(b_{2r}+c_{2r}+f_{2r})-\alpha_{2r}(b_{0r}+c_{0r})]\mu+[\beta_{0r}(b_{2r}+c_{2r}+f_{2r})-\beta_{2r}(b_{0r}+c_{0r})]\nu = 0. \tag{IV.61b}$$

The corresponding solutions for (IV.40) *truncated* at order $4m+1$ have the form

$$\begin{aligned} A_0(t) &= \rho_0 e^{i(\Omega_0 t+\varphi_0)}, & A_1(t) &= \rho_0 e^{i(\Omega_0 t+\varphi_1)} \\ A_2(t) &= \rho_2 e^{i(\Omega_2 t+\varphi_2)}, & A_3(t) &= \rho_2 e^{i(\Omega_2 t+\varphi_3)}, \end{aligned} \tag{IV.62}$$

where $\Omega_j = \omega_j + O(|\mu| + |\nu|)$ and the phases φ_j satisfy the relation

$$\varphi_1 - \varphi_0 + \varphi_2 - \varphi_3 = 0 \qquad (\Theta_1 = 0 \text{ in lemma } 3').$$

There are three independent arbitrary phases. We have thus obtained a "circle" of invariant 2-tori on which trajectories are quasi-periodic.

Physical interpretation. Up to order $||X||^{4m+1}$ the solutions are a superposition of m- and $(m+1)$-ribbons, as appears from (IV.62). Notice that a suitable choice of the origin on the z-axis makes these solutions invariant under **S**. It follows that the fluid flow is confined into flat horizontal cells of height $h/2n$ by the same argument as in Chapter III for ribbons and Taylor vortices.
Notation. We call these solutions the *superposed ribbons* $\mathrm{RS}^{(0)}$.

IV.2.3.5 Superposed ribbons (second kind)

The discussion of the previous paragraph can be transposed exactly to the system restricted to the subspace (vi) of Lemma 4. The result is the existence of branches of quasi-periodic solutions that, up to order $||X||^{4m-1}$, have the form

$$\begin{aligned} A_0(t) &= \rho_0 e^{i(\Omega'_0 t+\varphi_0)}, & A_1(t) &= \rho_0 e^{i(\Omega'_0 t+\varphi_1)} \\ A_2(t) &= \rho_2 e^{i(\Omega'_2 t+\varphi_2)}, & A_3(t) &= -\rho_2 e^{i(\Omega'_2 t+\varphi_3)}, \end{aligned} \tag{IV.63}$$

where $\Omega_0' = \omega_1 + O(|\mu| + |\nu|)$, $\Omega_2' = \omega_2 + O(|\mu| + |\nu|)$, and the phases φ_j satisfy the relation

$$\varphi_1 - \varphi_0 + \varphi_2 - \varphi_3 = 0, \qquad (\Theta_1 = \pi \text{ in Lemma } 3').$$

Formulas (IV.60) and (IV.61) apply with only changing f_{0r} and f_{2r} by $-f_{0r}$ and $-f_{2r}$.

Physical interpretation. These solutions look "almost" like a superposition of m and $(m+1)$-ribbons. However, they do not form flat horizontal cells. If $\varphi_1 - \varphi_0 = 0$ and $\varphi_2 - \varphi_3 = 0$, these solutions are invariant under the composition of a rotation by π with the symmetry **S**, provided that m is even, while if m is odd one has to compose this transformation with a shift by π/α (half the axial period). One can also say that the two ribbon regimes are superposed but shifted in the axial direction by 1/4 of the axial period. Hence, the argument given for the ribbons and superposed ribbons $\mathrm{RS}^{(0)}$ does not work here. These solutions will be denoted by $\mathrm{RS}^{(\pi)}$.

Remark. In Chapter II B of [Si], a codimension-three situation is studied where b_{0r} is zero in (IV.40). This occurs for instance when $\eta \approx 0.94$, and either $\Omega_2/\Omega_1 \approx -0.76$ for (1,2) mode interaction, or $\Omega_2/\Omega_1 \approx -0.81$ for (2,3) mode interaction. The multiple stable states occur in an analogous way as mentioned in Subsection IV.1.6.

IV.2.4 Stability of the bifurcated solutions

The methodological remarks stated in the beginning of Subsection IV.1.4 are also valid in the present context and will help us to simplify the calculation of the eigenvalues for the rotating waves. For the quasi-periodic flows we need a little extra. We have seen that stability of a rotating wave reduces to stability of a stationary solution for the equation written in a suitable rotating frame. The same idea applies to the quasi-periodic solutions that were computed in Subsection IV.2.3, but now the equation is in normal form and is written in a frame moving quasi-periodically along a 2-torus. The idea is the following: for the *normal form*, or if one prefers for the truncated equations *at order* $||X||^{4m+1}$, the quasi-periodic solutions can be rewritten as

$$X(t) = \mathbf{R}^{(\alpha)}_{\Omega_\alpha t}\mathbf{R}^{(\beta)}_{\Omega_\beta t}\tilde{X}, \tag{IV.64}$$

where $\mathbf{R}^{(\alpha)}_{\varphi_\alpha}$ and $\mathbf{R}^{(\beta)}_{\varphi_\beta}$ are S^1-representations in $\mathbf{V}$ deduced from $\mathbf{T}_\psi$ and $\tilde{\mathbf{R}}_{\varphi_1,\varphi_2}$ (see (IV.35)) by a linear combination of the angles (we shall check this in each case). The normal form $\tilde{F}$ of the vector field F is equivariant under these S^1-representations. Therefore, if we write a perturbation of the quasi-periodic solution of the normal form as

$$X(t) = \mathbf{R}^{(\alpha)}_{\Omega_\alpha t}\mathbf{R}^{(\beta)}_{\Omega_\beta t}[\tilde{X} + \tilde{Y}(t)],$$

then the equation verified by $\tilde{Y}$ is

$$\frac{d\tilde{Y}}{dt} + (\Omega_\alpha J_\alpha + \Omega_\beta J_\beta)\tilde{Y} = \tilde{F}(\mu, \nu, \tilde{X} + \tilde{Y}) - \tilde{F}(\mu, \nu, \tilde{X}), \tag{IV.65}$$

where we denote by $J_\alpha = d\mathbf{R}^{(\alpha)}_\varphi(0)/d\varphi$ the generator of the group $\mathbf{R}^{(\alpha)}_\varphi$. It follows that the stability of the quasi-periodic solution of the normal form is determined by the eigenvalues of the *autonomous* operator

$$L_{\mu,\nu} = D_X\tilde{F}(\mu, \nu, \tilde{X}) - \Omega_\alpha J_\alpha - \Omega_\beta J_\beta \tag{IV.66}$$

This also determines the stability of the complete solutions.

Since F and $\tilde{F}$ coincide up to order $||X||^{4m}$, there is no difficulty in computing the coefficients of the matrix of $L_{\mu,\nu}$ to the order $||X||^{4m-1}$. In any case we define $L_{\mu,\nu}$ to be the part of the linearized operator when F is replaced by its Taylor expansion to order $4m-1$ in X. Note also that we shall evaluate the matrix coefficients of $L_{\mu,\nu}$ in the basis associated with the coordinates $\{A_k, \bar{A}_k,\ k=0,1,2,3\}$. For the full system the stability result obtained for the normal form remains valid, since it reduces, via perturbation theory, to the computation of Floquet exponents (in the rotating frame) very close to the previously obtained eigenvalues [provided that the right number (following Remark 2(ii) of Subsection IV.1.4) of nonzero eigenvalues is obtained].

IV.2.4.1 Stability of the m-spirals

Consider the m-spirals with $A_0 \neq 0,\ A_1 = 0$. A straightforward computation, using Lemma 3, shows that in the basis defined above,

$$L_{\mu,\nu} = \begin{pmatrix} L^{(0)} & 0 \\ 0 & L^{(1)} \end{pmatrix},$$

where

$$L^{(0)} = \begin{pmatrix} \bar{A}_0 \dfrac{\partial g_0}{\partial u_0} & A_0 \dfrac{\partial g_0}{\partial u_0} \\ \bar{A}_0 \dfrac{\partial \bar{g}_0}{\partial u_0} & A_0 \dfrac{\partial \bar{g}_0}{\partial u_0} \end{pmatrix}$$

and $L^{(1)}$ is diagonal. Knowing that the leading part of the branch is given by

$$\alpha_{0\mathrm{r}}\mu + \beta_{0\mathrm{r}}\nu + b_{0\mathrm{r}}\rho_0^2 = 0$$

and utilizing the order 3 expansion defined in (IV.40), we obtain the following:

(i) the eigenvalues of $L^{(0)}$ are 0 (due to the rotating wave structure) and

$$2b_{0\mathrm{r}}\rho_0^2 + \mathrm{O}(\rho_0^4) \quad \text{(real);}$$

(ii) the eigenvalues of $L^{(1)}$ are complex conjugate with the following real parts:

$$\begin{cases} (c_{0\mathrm{r}} - b_{0\mathrm{r}})\rho_0^2 + \mathrm{O}(\rho_0^4), \\ -b_{0\mathrm{r}}^{-1}[(\alpha_{0\mathrm{r}}b_{2\mathrm{r}} - \alpha_{2\mathrm{r}}b_{0\mathrm{r}})\mu + (\beta_{0\mathrm{r}}b_{2\mathrm{r}} - \beta_{2\mathrm{r}}b_{0\mathrm{r}})\nu + \mathrm{O}(|\mu| + |V|)^2], \\ -b_{0\mathrm{r}}^{-1}[(\alpha_{0\mathrm{r}}c_{2\mathrm{r}} - \alpha_{2\mathrm{r}}b_{0\mathrm{r}})\mu + (\beta_{0\mathrm{r}}c_{2\mathrm{r}} - \beta_{2\mathrm{r}}b_{0\mathrm{r}})\nu + \mathrm{O}(|\mu| + |\nu|)^2]. \end{cases} \tag{IV.67}$$

Note that the last two pairs of eigenvalues can cross the imaginary axis, which corresponds to the bifurcation of the two kinds of interpenetrating spirals [see (IV.50b) and a similar formula for the interpenetrating spirals of second kind].

IV.2.4.2 Stability of the $(m+1)$-spirals

By the same kind of calculation as for m-spirals, we get one real and six complex nonzero eigenvalues with real parts:

$$\begin{cases} 2d_{2\mathrm{r}}\rho_2^2 + \mathrm{O}(\rho_0^4) \text{ (real eigenvalue)}, \\ (e_{2\mathrm{r}} - d_{2\mathrm{r}})\rho_2^2 + \mathrm{O}(\rho_2^4), \\ -d_{2\mathrm{r}}^{-1}[(\alpha_{2\mathrm{r}}d_{0\mathrm{r}} - \alpha_{0\mathrm{r}}d_{2\mathrm{r}})\mu + (\beta_{2\mathrm{r}}d_{0\mathrm{r}} - \beta_{0\mathrm{r}}d_{2\mathrm{r}})\nu + \mathrm{O}(|\mu| + |\nu|)^2], \\ -d_{2\mathrm{r}}^{-1}[(\alpha_{2\mathrm{r}}e_{0\mathrm{r}} - \alpha_{0\mathrm{r}}d_{2\mathrm{r}})\mu + (\beta_{2\mathrm{r}}e_{0\mathrm{r}} - \beta_{0\mathrm{r}}d_{2\mathrm{r}})\nu + \mathrm{O}(|\mu| + |\nu|)^2]. \end{cases} \tag{IV.68}$$

The last two real parts become zero at points corresponding to the bifurcations of interpenetrating spirals [see (IV.50a) and a similar formula for the interpenetrating spirals of second kind].

IV.2.4.3 Stability of the m-ribbons

We consider the solution such that $A_0 = A_1 = \rho_0 e^{i\Omega_1 t}$ (see IV.43). Using Lemma 3 we see that

$$L_{\mu,\nu} = \begin{pmatrix} L^{(0)} & 0 \\ 0 & L^{(2)} \end{pmatrix},$$

where $L^{(0)}$ is the block on the coordinates $\{A_0, \bar{A}_0, A_1, \bar{A}_1\}$ and $L^{(2)}$ is the block on the coordinates $\{A_2, \bar{A}_2, A_3, \bar{A}_3\}$.

Remark. It follows from Remark 2(i) of Subsection IV.1.4 that this decomposition holds for the linearization of F itself and not only for its Taylor expansion to order $4m$.

Eigenvalues of $L^{(0)}$. This block is essentially the jacobian matrix for the m-ribbons when there is no mode interaction. It suffices therefore to look at the eigenvalues computed in Chapter III and adapt notation. We get 0 as a double eigenvalue (group-invariance property), and the two other eigenvalues are real and of the form

$$\begin{cases} 2(b_{0\mathrm{r}} + c_{0\mathrm{r}})\rho_0^2 + \mathrm{O}(\rho_0^4), \\ 2(b_{0\mathrm{r}} - c_{0\mathrm{r}})\rho_0^2 + \mathrm{O}(\rho_0^4). \end{cases} \tag{IV.69}$$

Eigenvalues of $L^{(2)}$. Making use of Lemma 3 one can check that

$$L^{(2)} = \begin{pmatrix} L_0^{(2)} & 0 \\ 0 & \bar{L}_0^{(2)} \end{pmatrix},$$

where $L_0^{(2)}$ is the block in coordinates $\{A_2, A_3\}$ and has the form

$$L_0^{(2)} = \begin{pmatrix} g_2 & \rho_0^2 h_2 \\ \rho_0^2 h_3 & g_3 \end{pmatrix}.$$

Remark. It can be shown that if $m > 1$, this decomposition holds at *any* order; see [C-D-I] for the proof.

Using the form (IV.40) of the equations, we obtain two pairs of complex conjugate eigenvalues with real parts

$$\begin{cases} [\alpha_{2r} - \dfrac{\alpha_{0r}}{b_{0r} + c_{0r}}(b_{2r} + c_{2r} - f_{2r})]\mu + [\beta_{2r} - \dfrac{\beta_{0r}}{b_{0r} + c_{0r}}(b_{2r} + c_{2r} - f_{2r})]\nu \\ \quad +\mathrm{O}(|\mu| + |\nu|)^2, \\ [\alpha_{2r} - \dfrac{\alpha_{0r}}{b_{0r} + c_{0r}}(b_{2r} + c_{2r} + f_{2r})]\mu + [\beta_{2r} - \dfrac{\beta_{0r}}{b_{0r} + c_{0r}}(b_{2r} + c_{2r} + f_{2r})]\nu \\ \quad +\mathrm{O}(|\mu| + |\nu|)^2. \end{cases} \tag{IV.70}$$

Both these pairs of eigenvalues can cross the imaginary axis, and this corresponds to the bifurcations of superposed ribbons $\mathrm{RS}^{(0)}$ and $\mathrm{RS}^{(\pi)}$ [see (IV.61b) and a similar formula for $\mathrm{RS}^{(\pi)}$].

IV.2.4.4 Stability of the $(m+1)$-ribbons

The calculations are identical to those for the m-ribbons. The nonzero eigenvalues are of two kinds: a pair of real eigenvalues

$$\begin{cases} 2(d_{2r} + e_{2r})\rho_2^2 + \mathrm{O}(\rho_2^4), \\ 2(d_{2r} - e_{2r})\rho_2^2 + \mathrm{O}(\rho_2^4), \end{cases} \tag{IV.71}$$

and two pairs of complex conjugate eigenvalues with real parts

$$\begin{cases} [\alpha_{0r} - \dfrac{\alpha_{2r}}{d_{2r} + e_{2r}}(d_{0r} + e_{0r} - f_{0r})]\mu + [\beta_{0r} - \dfrac{\beta_{2r}}{d_{2r} + e_{2r}}(d_{0r} + e_{0r} - f_{0r})]\nu \\ \quad +\mathrm{O}(|\mu| + |\nu|)^2, \\ [\alpha_{0r} - \dfrac{\alpha_{2r}}{d_{2r} + e_{2r}}(d_{0r} + e_{0r} + f_{0r})]\mu + [\beta_{0r} - \dfrac{\beta_{2r}}{d_{2r} + e_{2r}}(d_{0r} + e_{0r} + f_{0r})]\nu \\ \quad +\mathrm{O}(|\mu| + |\nu|)^2. \end{cases} \tag{IV.72}$$

These eigenvalues can cross the imaginary axis, which corresponds to the bifurcation of the superposed ribbons $\mathrm{RS}^{(0)}$ and $\mathrm{RS}^{(\pi)}$ [see (IV.61a) and a similar formula for $\mathrm{RS}^{(\pi)}$].

IV.2.4.5 Stability of the interpenetrating spirals $\mathrm{SI}^{(0,3)}$, and $\mathrm{SI}^{(1,2)}$

We now consider the solution $SI^{(0,3)}$, i.e., solutions found in Subsection IV.2.3.2) where $A_1 = A_2 = 0$. Let us set $\varphi_\alpha = m\varphi + n\psi$ and $\varphi_\beta = (m+1)\varphi - n\psi$. Then, we define the following linear operator $\mathbf{R}^{(\alpha)}_{\varphi_\alpha}\mathbf{R}^{(\beta)}_{\varphi_\beta}$ acting on the coordinates by

$$(A_0, A_1, A_2, A_3) \to (e^{\mathrm{i}\varphi_\alpha}A_0, e^{\mathrm{i}\frac{-\varphi_\alpha + 2m\varphi_\beta}{2m+1}}A_1, e^{\mathrm{i}\frac{2(m+1)\varphi_\alpha + \varphi_\beta}{2m+1}}A_2, e^{\mathrm{i}\varphi_\beta}A_3). \tag{IV.73}$$

Formula (IV.64) now holds with $\Omega_\alpha = \Omega_0$ and $\Omega_\beta = \Omega_3$, where Ω_0 and Ω_3 are defined in (IV.52). Note that the complete solution (not only the solution for the normal form equation) takes the form (IV.64) (see proof of Lemma 4). Now the linearized operator $L_{\mu,\nu}$ is defined by (IV.66). Remark 2(i) in Subsection IV.1.4 shows that

$$L_{\mu,\nu} = \begin{pmatrix} L^{(0,3)} & 0 \\ 0 & L^{(1,2)} \end{pmatrix},$$

where $L^{(j,k)}$ is the block on coordinates $\{A_j, \bar{A}_j, A_k, \bar{A}_k\}$.

Eigenvalues of $L^{(0,3)}$. This is the linearized operator for the system (IV.47), which decouples in polar form into phase equations (IV.48b) and moduli equations (IV.48a). It follows that 0 is a double eigenvalue (this is an effect of rotational symmetries), and the other eigenvalues are computed from (IV.48a). Since the solutions are zeros of this system, the jacobian matrix is easy to compute in terms of ρ_0 and ρ_3. Denoting the eigenvalues by σ_0' and σ_0'', we find

$$\begin{aligned} \boldsymbol{\sigma}_0' + \boldsymbol{\sigma}_0'' &= 2(b_{0\mathrm{r}}\rho_0^2 + d_{2\mathrm{r}}\rho_3^2) + \mathrm{O}(\rho_0^2 + \rho_3^2)^2, \\ \boldsymbol{\sigma}_0' \cdot \boldsymbol{\sigma}_0'' &= 4(b_{0\mathrm{r}}d_{2\mathrm{r}} - e_{0\mathrm{r}}c_{2\mathrm{r}})\rho_0^2\rho_3^2 + \mathrm{O}(\rho_0^2 + \rho_3^2)^3. \end{aligned} \tag{IV.74}$$

Note that these eigenvalues can be real or complex conjugate and that in the latter case a necessary condition for zero real part is $b_{0\mathrm{r}}d_{2\mathrm{r}} < 0$. If they are real, they keep their sign and take the value 0 only if ρ_0 or $\rho_3 = 0$.

Eigenvalues of $L^{(1,2)}$. Making use of Lemma 3 one can check that

$$L^{(1,2)} = \begin{pmatrix} L_0^{(1,2)} & 0 \\ 0 & \bar{L}_0^{(1,2)} \end{pmatrix},$$

where $L_0^{(1,2)}$ is the block in coordinates $\{A_1, \bar{A}_2\}$ and has the form

$$L_0^{(1,2)} = \begin{pmatrix} g_1 & A_0A_3h_1 \\ \bar{A}_0\bar{A}_3\bar{h}_2 & \bar{g}_2 \end{pmatrix}.$$

Using (IV.40) we get the leading part of $L_0^{(1,2)}$:

$$\hat{L}_0^{(1,2)} = \begin{pmatrix} \gamma & f_0\rho_0\rho_3 \\ \bar{f}_2\rho_0\rho_3 & \delta \end{pmatrix}, \tag{IV.75}$$

with $\gamma = (c_{0\mathrm{r}} - b_{0\mathrm{r}})\rho_0^2 + (d_{0\mathrm{r}} - e_{0\mathrm{r}})\rho_3^2 + \mathrm{i}\omega_3$, $\delta = (b_{2\mathrm{r}} - c_{2\mathrm{r}})\rho_0^2 + (e_{2\mathrm{r}} - d_{2\mathrm{r}})\rho_3^2 + \mathrm{i}\omega_3$, and

$$\omega_3 = \frac{2m+2}{2m+1}\Omega_0 - \frac{2m}{2m+1}\Omega_3. \tag{IV.76}$$

Note that the eigenvalues of $L_0^{(1,2)}$ are complex and may well cross the imaginary axis for some value of μ and ν. We shall come back to this point in Subsection IV.2.5.

IV.2.4.6 Stability of the interpenetrating spirals $\mathrm{SI}^{(1,3)}$ and $\mathrm{SI}^{(0,2)}$

The same reasoning as in the previous case applies here. We consider for example the solutions $\mathrm{SI}^{(0,2)}$, i.e., $A_1 = A_3 = 0$. Formula (IV.64) holds for the normal form, with $\varphi_\alpha = m\varphi + n\psi$ and $\varphi_\beta = (m+1)\varphi + n\psi$. Then $L_{\mu,\nu}$ has, apart from its double zero eigenvalue, two eigenvalues σ_0' and σ_0'' such that

$$\begin{aligned}\sigma_0' + \sigma_0'' &= 2(b_{0\mathrm{r}}\rho_0^2 + d_{2\mathrm{r}}\rho_2^2) + \mathrm{O}(\rho_0^2 + \rho_2^2)^2,\\ \sigma_0' \cdot \sigma_0'' &= 4(b_{0\mathrm{r}}d_{2\mathrm{r}} - d_{0\mathrm{r}}b_{2\mathrm{r}})\rho_0^2\rho_2^2 + \mathrm{O}(\rho_0^2 + \rho_2^2)^3.\end{aligned} \tag{IV.77}$$

and two pairs of complex conjugate eigenvalues, the leading parts of which are eigenvalues of

$$\hat{L}_0^{(1,3)} = \begin{pmatrix} \gamma' & f_0\rho_0\rho_2 \\ f_2\rho_0\rho_2 & \delta' \end{pmatrix} \tag{IV.78}$$

and its complex conjugate, with $\gamma' = (c_{0\mathrm{r}} - b_{0\mathrm{r}})\rho_0^2 + (e_{0\mathrm{r}} - d_{0\mathrm{r}})\rho_2^2 + \mathrm{i}\omega_4$, $\delta' = (c_{2\mathrm{r}} - b_{2\mathrm{r}})\rho_0^2 + (e_{2\mathrm{r}} - d_{2\mathrm{r}})\rho_2^2 + \mathrm{i}\omega_4$, and

$$\omega_4 = (2m+2)\Omega_0 - 2m\Omega_2. \tag{IV.79}$$

Again a pair of eigenvalues may cross the imaginary axis (off 0) at some value of μ and ν. This will be discussed in Subsection IV.2.5.

IV.2.4.7 Stability of the superposed ribbons $\mathrm{RS}^{(0)}$

Formula (IV.64) applies with $\mathbf{R}^{(\alpha)}_{\varphi_\alpha}\mathbf{R}^{(\beta)}_{\varphi_\beta} = \tilde{\mathbf{R}}_{\varphi_1,\varphi_2}$. This follows from Lemma 3 (see also the form (IV.62) of the solutions). Let us set the arbitrary phases equal to 0, and let us define the following change of basis in **V**:

$$\psi_\pm^{(0)} = 1/2(\zeta_0 \pm \zeta_1), \qquad \psi_\pm^{(2)} = 1/2(\zeta_2 \pm \zeta_3).$$

Then the invariant space in which the solution lives is spanned by $\{\psi_+^{(0)}, \bar{\psi}_+^{(0)}, \psi_+^{(2)}, \bar{\psi}_+^{(2)}\}$. By the preliminary remark 2(i) in Subsection IV.1.4, the operator $L_{\mu,\nu}$ can be written

$$L_{\mu,\nu} = \begin{pmatrix} L^{(+)} & 0 \\ 0 & L^{(-)} \end{pmatrix},$$

where each $L^{(\pm)}$ is a block on each of the subspaces $\{\psi_\pm^{(0)}, \bar{\psi}_\pm^{(0)}, \psi_\pm^{(2)}, \bar{\psi}_\pm^{(2)}\}$.

Eigenvalues of $L^{(+)}$. This matrix is just the linearized part arising from (IV.58). We know that 0 is a double eigenvalue, the other eigenvalues are easily computed from the moduli equations (IV.59). We find the eigenvalues σ' and σ'' such that

$$\begin{aligned}\sigma' + \sigma'' &= (b_{0\mathrm{r}} + c_{0\mathrm{r}})\rho_0^2 + (d_{2\mathrm{r}} + e_{2\mathrm{r}})\rho_2^2 + \mathrm{O}(\rho_0^2 + \rho_2^2)^2\\ \sigma' \cdot \sigma'' &= [(b_{0\mathrm{r}} + c_{0\mathrm{r}})(d_{2\mathrm{r}} + e_{2\mathrm{r}})\\ &\quad -(d_{0\mathrm{r}} + e_{0\mathrm{r}} + f_{0\mathrm{r}})(b_{2\mathrm{r}} + c_{2\mathrm{r}} + f_{2\mathrm{r}})]\rho_0^2\rho_2^2 + \mathrm{O}(\rho_0^2 + \rho_2^2)^3.\end{aligned} \tag{IV.80}$$

Eigenvalues of $L^{(-)}$. There is one zero eigenvalue in this subspace, thanks to the action of $\mathbf{T}_\psi$. Indeed $d\mathbf{T}_\psi(0)\tilde{X}/d\psi = \rho_0(\zeta_1 - \bar{\zeta}_1) + \rho_2(\zeta_3 - \bar{\zeta}_3)$; hence, there exists a null eigenvector in the direction

$$\rho_0(\psi_-^{(0)} - \bar{\psi}_-^{(0)}) + \rho_2(\psi_-^{(2)} - \bar{\psi}_-^{(2)}).$$

An elementary—but laborious—calculation shows that the remaining eigenvalues λ are the roots of the polynomial

$$\operatorname{Det}\begin{pmatrix} -\rho_0 & (b_0 - c_0)\rho_0^2 & (d_0 - e_0 + f_0)\rho_0\rho_2 & (d_0 - e_0 - f_0)\rho_0\rho_2 \\ \rho_0 & (\bar{b}_0 - \bar{c}_0)\rho_0^2 \\ -2\bar{f}_0\rho_2^2 - \lambda & (d_0 - \bar{e}_0 - \bar{f}_0)\rho_0\rho_2 & (\bar{d}_0 - \bar{e}_0 + f_0)\rho_0\rho_2 \\ -\rho_2 & (b_2 - c_2 - f_2)\rho_0\rho_2 & \begin{matrix}(d_2 - e_2)\rho_2^2 \\ -2f_2\rho_2^2 - \lambda\end{matrix} & (d_2 - e_2)\rho_2^2 \\ \rho_2 & (\bar{b}_2 - \bar{c}_2 + \bar{f}_2)\rho_0\rho_2 & (\bar{d}_2 - \bar{e}_2)\rho_2^2 & \begin{matrix}(\bar{d}_2 - \bar{e}_2)\rho_2^2 \\ -2\bar{f}_2\rho_2^2 - \lambda \\ +\mathrm{O}[(\rho_0^2 + \rho_2^2)]^4\end{matrix} \end{pmatrix}. \tag{IV.81}$$

IV.2.4.8 Stability of the superposed ribbons $\mathrm{RS}^{(\pi)}$

The calculations are the same as in the previous case, just by permuting the matrices $L^{(+)}$ and $L^{(-)}$ and changing in (IV.80) and (IV.81) the signs in front of f_0 and f_2.

IV.2.5 Further bifurcations

We have noticed that the branches of quasi-periodic solutions that were studied in Subsection IV.2.4 can undergo one or more secondary bifurcations. For example, the interpenetrating spirals $\mathrm{SI}^{(0,3)}$ can undergo a "Hopf-type" bifurcation, i.e., a pair of conjugate eigenvalues can cross the imaginary axis for the associated autonomous equation (IV.65), which in this case holds for the complete vector field F. The fact that $L_{\mu,\nu}$ possesses a double zero-eigenvalue whatever μ and ν, which is the consequence of the existence of an invariant 2-torus, can be treated as in [Io 84]. The bifurcated solutions would then be truly quasi-periodic with *three* frequencies: Ω_0/m, $\Omega_3/(m+1)$, and a frequency close to ω_3 (formula (IV.76)). For the interpenetrating spirals $\mathrm{SI}^{(0,2)}$ the situation is a little more involved. The equation for a perturbation does not reduce to an autonomous equation (except if restricted to the normal form). However, they reduce to an equation with *time-periodic* dependence in a suitable rotating frame. This is because interpenetrating spirals branch off by a Hopf bifurcation from the spirals, which are stationary in a rotating frame. In this case vanishing real parts for the eigenvalues of $L_{\mu,\nu}$ correspond to the presence of Floquet multipliers on the unit circle for the linear operator in the rotating frame. In this frame there would be a bifurcation to an invariant 2-torus.

The bifurcated solutions for the original equation would therefore lie on a 3-torus but would exhibit the expected behavior of frequency locking, i.e., they would show two or three true frequencies according to the presence or not of a weak resonance. The same sort of bifurcation would occur when a pair of eigenvalues cross the imaginary axis for the superposed ribbons. However, the region of validity of these phenomena is so small that they would be extremely difficult to detecte in experiments.

IV.2.6 Two numerical examples

The theoretical analysis of the azimuthal mode interaction was applied in [C-D-I] to two different examples with radius ratio $\eta = 0.7519$—one with critical azimuthal modes $m = 1$, 2; the other with critical azimuthal modes $m = 2$, 3. The relevant coefficients of the amplitude equations (IV.40) have been numerically computed. It turns out that these two examples give quite different bifurcation diagrams. In the first case [Figures IV.8(a),(b)] only spiral flows (of the two types) are stable in a neighborhood of the critical point, while in the second case [Figures IV.9(a),(b)] more complicated regimes are found stable (interpenetrating spirals, for instance). On the other hand, a series of experiments have been performed under physical conditions analogous to those chosen for these two numerical examples [St-C-H]. In these experiments the vicinity of the critical points has been systematically explored, by varying the velocity of the inner and outer cylinders. Computer control of these parameters allows description of circles (in both rotating directions) in the (μ, ν)-plane, hence allowing an easy comparison with the computed bifurcation diagrams of [C-D-I]. The comparison shows a good qualitative agreement, especially in example (1) $(m = 1, 2)$. In example (2) $(m = 2, 3)$, no stable flow was found by [C-D-I] in a whole wedge of the (μ, ν)-plane. In this region, however, multifrequency and seemingly chaotic flows were observed in [St-C-H]. Moreover, a direct numerical simulation of the amplitude equations (IV.40) (truncated at order 3) seems to indicate the presence of chaotic attractors for this differential system [Ste]. The question now posed is about the relevance of the amplitude equations for understanding the experimentally observed chaos.

In Tables IV.3 and IV.4, the numerical values of all relevant coefficients are indicated. The choice of the axial wave number is as indicated in Subsection IV.1.5. Linear coefficients of [C-D-I] have been transformed in order to keep coherence with our (different) choice of scales. The values of the nonlinear coefficients correspond to some nonspecified choice of normalization for the eigenvectors. Another choice would just change the eigenvectors by a factor and multiply all these values by some positive factor.

Next we present the bifurcation diagrams as in Subsection IV.1.5. Figures IV.8(a) and IV.9(a) represent the lines where bifurcation occurs in the parameter plane (remember that $\mu = \mathcal{R} - \mathcal{R}_c$ and $\nu = \Omega - \Omega_m$).

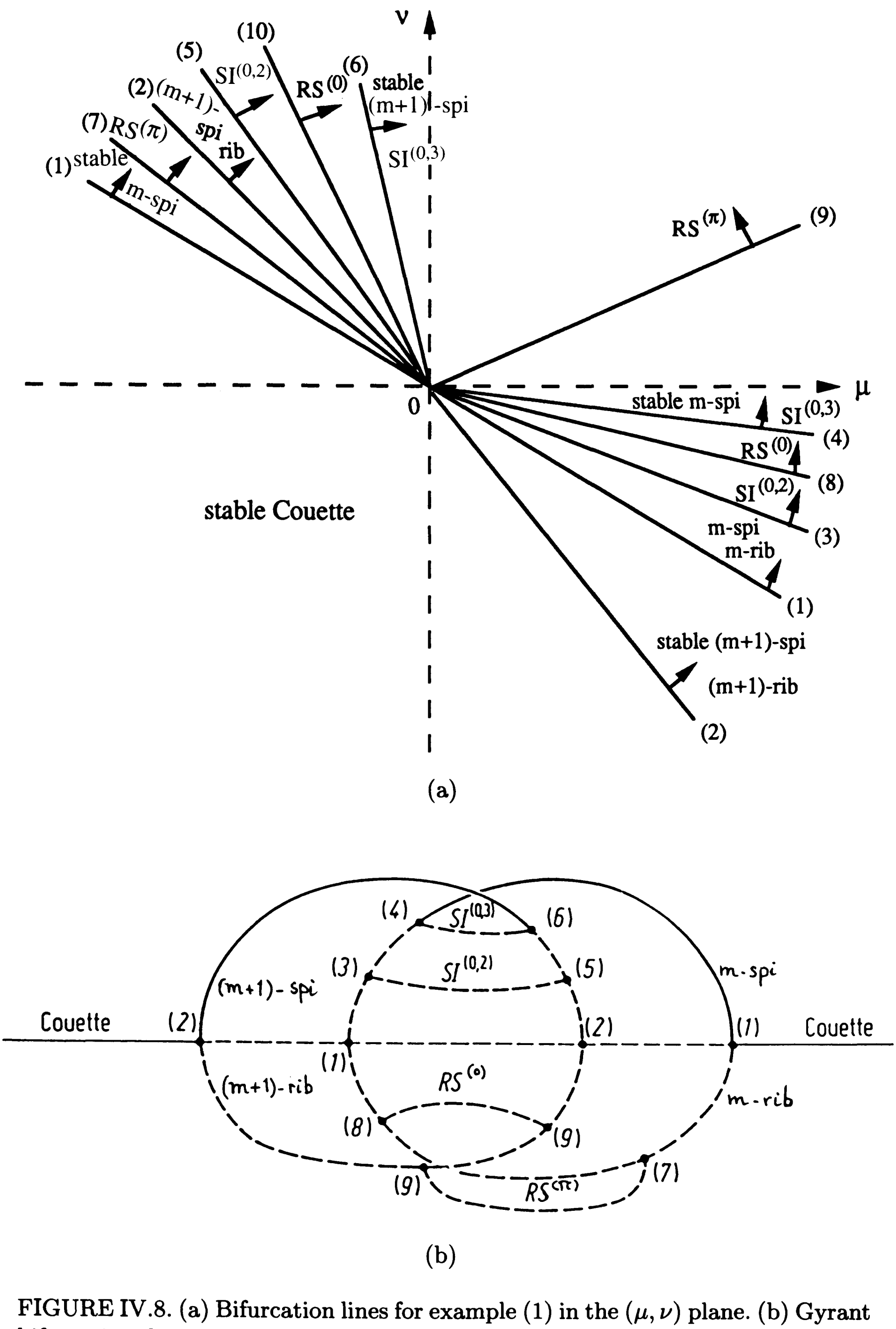


FIGURE IV.8. (a) Bifurcation lines for example (1) in the (μ, ν) plane. (b) Gyrant bifurcation diagram in example (1).

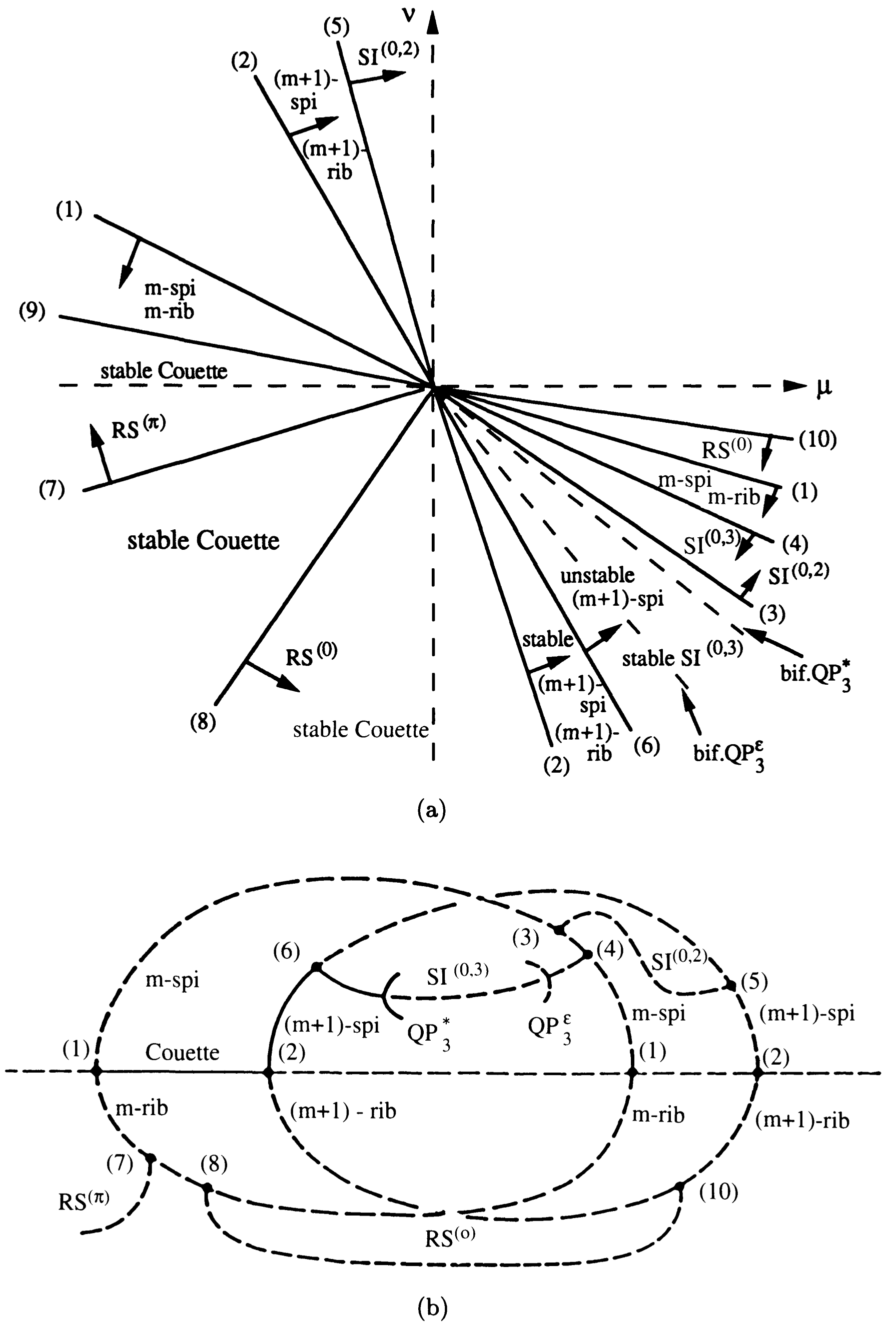


FIGURE IV.9. (a) Bifurcation lines for example (2) in the (μ, ν) plane. (b) Gyrant bifurcation diagram in example (2).

TABLE IV.3. Linear coefficients (from [C-D-I]).

EX.	η	$\mathcal{R}_c$	Ω_m	α	$m,\ m+1$	ω_1	ω_2	α_{0r}	β_{0r}	α_{2r}	β_{2r}
1	0.7519	134.3	−0.7	3.8	1, 2	−16.3	−28.9	0.308	0.022	0.288	0.014
2	0.7519	198.0	−1.15	4.3	2, 3	−48.9	−59.1	0.280	0.012	0.245	0.006

TABLE IV.4. Nonlinear coefficients in (IV.40) (from [C-D-I]).

Ex.	b_{0r}	c_{0r}	d_{0r}	e_{0r}	f_{0r}	f_{0i}	b_{2r}	c_{2r}	d_{2r}	e_{2r}	f_{2r}	f_{2i}
1	−0.242	−0.750	−2.307	−1.330	−1.381	−1.238	−1.214	−0.729	−0.766	−1.337	−2.185	−1.856
2	0.056	1.262	−4.080	−2.104	−0.718	−6.363	−0.757	−2.209	−0.931	−2.647	−4.688	−7.533

The direction of the different bifurcations is indicated. Notice that in example (2), when one turns counterclockwise in the (μ, ν)-plane, the interpenetrating spirals $\mathrm{SI}^{(0,3)}$ become unstable by a "Hopf-type" bifurcation, which would lead to a flow named QP_3^ϵ with a third frequency close to ω_3 [see Subsection IV.2.5 and formula (IV.76)]. However, no attempt was made to compute the direction of branching and the stability of this tertiary bifurcation. Another 3-frequencies flow, named QP_3^*, bifurcates from $\mathrm{SI}^{(0,3)}$, with a third frequency close to 0 (this one comes from the eigenvalues (IV.74) crossing the imaginary axis). However, this flow is unstable. In Figures IV.8(b) and IV.9(b) we show the gyrant bifurcation diagrams, obtained by following a clockwise path in the parameter plane. Left- and right-hand sides of the diagram have to be identified.

V
Imperfections on Primary Bifurcations

In this chapter we are interested in the perturbations introduced by an imperfection in the apparatus of the Couette-Taylor experiment, when we are close to a primary bifurcation. This imperfection is supposed to come from the boundary conditions of the problem (for instance, the geometry), except that we still assume here that cylinders are infinitely long. We concentrate on three types of imperfections, which could be superposed. The first type is when the two cylinders are not exactly well centered, which corresponds to breaking the SO(2) azimuthal symmetry of the perfect problem. The second case is when there is a mass flux through the cross section of the cylindrical domain, which corresponds to breaking the reflectional symmetry. Another simple way to break this symmetry would be to heat the bottom of the cylinders. In such a case one might use the Boussinesq approximation, as for the Rayleigh-Bénard convection problem. The last imperfection is when the rotation rates of the cylinders are no longer constant but oscillate periodically in time, with small oscillations, around a mean value. This last imperfection breaks time translations invariance—the equations of the perturbed problem are no longer autonomous. Other imperfections could also be treated; for example, the case of a periodic modulation of the shape of cylinders in the axial direction, these "bumps" being axisymmetric. This breaks the translational invariance of the perfect problem in shifts parallel to the z-axis. We only give a summary of the results in such a case (see the thesis of Signoret [Si]). All these types of imperfections have the property that the domain of the flow is time invariant. The more general case when the domain is no longer time invariant leads to additional (tractable) complications, but it is not our aim here to describe all possible imperfections of this problem. We think that the three types considered below are representative enough for the presentation of the general technique, as always, using amplitude equations.

V.1 General setting when the geometry of boundaries is perturbed

First, we set the theory for obtaining the amplitude equations when one perturbs the geometry of the boundary. The idea is to arrive first at a formulation like (II.23), i.e., with a differential equation in a suitable function space, with an additional parameter ϵ measuring the imperfection which breaks some of the basic symmetries. Then we can use, near the critical value of the parameter μ and near $\epsilon = 0$, the center manifold and normal form theorems as stated in Section II.4.

V.1.1 Reduction to an equation in $H(Q_h)$

Let us denote by Q_ϵ the perturbed domain of the flow that is close to Q. We mean that when ϵ tends toward 0, Q_ϵ tends toward Q. We again basically assume that h-periodicity is preserved. As in previous chapters, this assumption is suitable to avoid mathematical complications and is justified when cylinders are long enough but not too long.

There exists a diffeomorphism $H_\epsilon = \mathrm{Id} + \mathrm{O}(\epsilon)$ that transforms Q_ϵ into Q, respecting the h-periodicity, and we can choose Q and the diffeomorphism H_ϵ in such a way that there is *local conservation of volumes.* If we choose another type of H_ϵ, the divergence-free condition on the velocity vector field U no longer holds, and this leads to additional theoretical complications. In any case, as we shall see, we do *not need to give explicitly* H_ϵ neither for deriving the theory nor for the computation of the amplitude equation, since we directly identify monomials whose existence is known a priori.

However, to give an idea of how such diffeomorphisms H_ϵ, close to identity, could be built, we first transform each section by a planar area-preserving diffeomorphism (close to Id) to obtain a region bounded by two concentric circles, the inner one being fixed. The next step is to make all sections equal, in conserving the volume, extending or contracting in the axial direction depending on whether the radius is contracted or extended, in such a way that the element of volume $(2\pi r\, dr\, dz)$ is locally conserved. We can choose the outer radius in such a way that the axial period h is preserved.

Let us verify that the divergence of a vector field is conserved for the vector field transported by the tangent mapping DH_ϵ.
We have the mapping

$$x \in Q_\epsilon \to x' = H_\epsilon(x) \in Q,$$

and the transported velocity vector field defined by

$$U(x) \to U'(x') = DH_\epsilon(x).U(x),$$

which means that for components we have:

$$u'_j = \sum_i \frac{\partial H_\epsilon^{(j)}}{\partial x_i} u_i.$$

Then, using $x = H_\epsilon^{-1}(x')$, we obtain by the chain rule

$$\nabla'.U' = \sum_j \frac{\partial}{\partial x'_j}\left(\sum_i \frac{\partial H_\epsilon^{(j)}}{\partial x_i} u_i\right) = \sum_{ijk} \frac{\partial H_\epsilon^{-1(k)}}{\partial x'_j} \frac{\partial}{\partial x_k}\left[\frac{\partial H_\epsilon^{(j)}}{\partial x_i} u_i\right].$$

Using now the identity $\sum_j \partial H_\epsilon^{-1(k)}/\partial x'_j . \partial H_\epsilon^{(j)}/\partial x_i = \delta_{ki}$, this leads to

$$\nabla'.U' = \nabla.U + \sum_{ijk} \frac{\partial H_\epsilon^{-1(k)}}{\partial x'_j} \frac{\partial^2 H_\epsilon^{(j)}}{\partial x_i \partial x_k} u_i.$$

Let us show that the second term on the right-hand side is zero, because of the volume-preserving property. Recalling that $\det(DH_\epsilon) = 1$ and denoting by A_{jk} the cofactor of $\partial H_\epsilon^{(j)}/\partial x_k$, we have $\frac{\partial H_\epsilon^{-1(j)}}{\partial x'_k} = A_{kj}$ by construction. Then we have (by differentiating successively each row of the determinant)

$$0 = \frac{\partial}{\partial x_i}[\det(DH_\epsilon)] = \sum_{jk} \frac{\partial^2 H_\epsilon^{(j)}}{\partial x_i \partial x_k} A_{jk} = \sum_{jk} \frac{\partial H_\epsilon^{-1(k)}}{\partial x'_j} \frac{\partial^2 H_\epsilon^{(j)}}{\partial x_i \partial x_k} \quad \text{for any } i;$$

hence, $\nabla'.U' = \nabla.U$.

After this diffeomorphism, the Navier-Stokes equations (II.5) in Q_ϵ are now written as a system in Q of the form (V' now being the transported vector field by the tangent mapping DH_ϵ)

$$\left\{\begin{array}{rcl} \dfrac{\partial V'}{\partial t} + \mathcal{R}(V'.\nabla)V' + \nabla p - \Delta V' &=& \mathcal{L}_\epsilon V' + \mathcal{B}_\epsilon(V', V') + \mathcal{Q}_\epsilon \nabla p, \\ \nabla.V' &=& 0, \qquad V' = W_\epsilon \text{ on } \partial Q, \end{array}\right. \tag{V.1}$$

where $\mathcal{L}_\epsilon$ is a linear differential operator of order 2, $\mathcal{B}_\epsilon$ is quadratic in V'—its terms have the form of products of first derivative of some component of V' by some other component of V', and $\mathcal{Q}_\epsilon$ is linear acting on ∇p. All these operators have coefficients of order ϵ. Since the transported boundary conditions W_ϵ on ∂Q are a perturbation of the usual Couette condition (II.6), there always exists a *divergence-free vector field* $V_\epsilon^{(0)}$ *satisfying the boundary conditions* (see [Lad]), which is ϵ close to the Couette flow $V^{(0)}$

but is *not necessarily a solution of our problem for* $\epsilon \neq 0$. Now the vector field $U = V' - V_\epsilon^{(0)}$ in Q is a solution of the new equation:

$$\begin{aligned}\frac{dU}{dt} &= L_\mu U + N(\mu, U) + \Pi_0\{\Delta V_\epsilon^{(0)} - \mathcal{R}(V_\epsilon^{(0)}.\nabla)V_\epsilon^{(0)}\} \\ &+\mathcal{R}\Pi_0\{[(V^{(0)} - V_\epsilon^{(0)}).\nabla]U + (U.\nabla)(V^{(0)} - V_\epsilon^{(0)})\} \\ &+\Pi_0\{\mathcal{L}_\epsilon(V_\epsilon^{(0)} + U) + \mathcal{B}_\epsilon(V_\epsilon^{(0)} + U, V_\epsilon^{(0)} + U)\} + \Pi_0\{\mathcal{Q}_\epsilon \nabla p\}, \quad \text{(V.2)}\end{aligned}$$

and we have in the orthogonal complement $H(Q_h)^\perp$ of $H(Q_h)$ in $[L^2(Q_h)]^3$:

$$\begin{aligned}[\mathrm{Id} - (\mathrm{Id} - \Pi_0)\mathcal{Q}_\epsilon]\nabla p = (\mathrm{Id} - \Pi_0)\{&\Delta(V_\epsilon^{(0)} + U) \\ &-\mathcal{R}[(V_\epsilon^{(0)} + U).\nabla](V_\epsilon^{(0)} + U) + \mathcal{L}_\epsilon(V_\epsilon^{(0)} + U) \\ &+\mathcal{B}_\epsilon(V_\epsilon^{(0)} + U, V_\epsilon^{(0)} + U)\}. \qquad \text{(V.3)}\end{aligned}$$

The operator on the left-hand side of (V.3), acting on ∇p, is invertible in $H(Q_h)^\perp$ since $(\mathrm{Id} - \Pi_0)\mathcal{Q}_\epsilon$ has a small norm, of order ϵ. Now replacing in (V.2) ∇p by its expression given by solving (V.3), we finally obtain a differential equation of the form

$$\frac{dU}{dt} = L_{\mu,\epsilon}U + N(\mu, \epsilon, U) + G_{\mu,\epsilon}, \qquad \text{(V.4)}$$

where $L_{\mu,\epsilon}$ is linear, $N(\mu, \epsilon, .)$ is quadratic in U, and $G_{\mu,\epsilon}$ is independent of U. Moreover, we have the following estimates (see Subsection II.2.3):

$$\begin{aligned}||L_{\mu,\epsilon} - L_\mu||_{\mathcal{L}(\mathcal{D}_h, H(Q_h))} &= \mathrm{O}(\epsilon), \qquad ||G_{\mu,\epsilon}||_{\mathcal{X}_h} = \mathrm{O}(\epsilon), \\ ||N_{(\mu,\epsilon,U)} - N(\mu, U)||_{\mathcal{X}_h} &\leq C|\epsilon|\,||U||^2_{\mathcal{D}_h},\end{aligned}$$

which allow us to use again an integral formulation as (II.25) and estimates (II.26) to prove a theorem on the Cauchy problem as in Section II.3. For instance, estimates (II.26) can be obtained using the perturbation theory developed in [Ka].

Remark. Notice that the original velocity vector field V defined on Q_ϵ satisfies

$$V = (DH_\epsilon)^{-1}[V_\epsilon^{(0)} + U] = V^{(0)} + U + \mathrm{O}(\epsilon),$$

where U is *now defined on* Q, *with homogeneous boundary conditions.*

V.1.2 Amplitude equations

The assumption now is that for μ and $\epsilon = 0$ we have a critical situation as described in Section II.4 for $\mu = 0$. We want to know what happens for the dynamics when $\epsilon \neq 0$. We are in a position to apply the center manifold theorem, with an additional parameter ϵ. The center manifold has the form

$$U = X + \Phi(\mu, \epsilon, X) \quad \text{in } \mathcal{D}_h \qquad \text{(V.5)}$$

for X in $\mathbf{V}$ satisfying an ordinary differential equation

$$\frac{dX}{dt} = F(\mu, \epsilon, X) \quad \text{in } \mathbf{V}. \tag{V.6}$$

For $\epsilon = 0$, $F(\mu, 0, .)$ is Γ-equivariant as we observed in Section II.4, and if $\epsilon \neq 0$, this is in general no longer true and, hence we obtain in general additional non-Γ-equivariant terms in F, with respect to the case $\epsilon = 0$.

We might be intimidated by all the technical tools needed to arrive at equation (V.4) in $H(Q_h)$ and amplitude equation (V.6). The idea now is to show how to escape from all these technicalities, using the fact that we know the type of equation for which we are looking.

Following the remark at the end of Subsection V.1.1, we finally get, for the velocity field, a solution in Q_ϵ of the form

$$V = DH_\epsilon^{-1}[V_\epsilon^{(0)} + X + \Phi(\mu, \epsilon, X)] = V^{(0)} + X + \tilde{\Phi}(\mu, \epsilon, X), \tag{V.7}$$

where $V^{(0)}$ is the Couette solution defined in Subsection II.1.3, $X \in \mathbf{V}$, and $\tilde{\Phi}$ has the same type of expansion as Φ.

Important Remark. We should note that it is a priori *useless to try to compute the perturbation of the Couette flow*, which might take place for $\epsilon \neq 0$. If $X = 0$ is a solution of amplitude equation (V.6), we automatically get this perturbed Couette flow by setting $X = 0$ in (V.7). If $X = 0$ is not a solution of (V.6), the search for a perturbed Couette flow for $\epsilon \neq 0$ is *just a mistake.*

This way of writing the expansion in (V.7) leads at each order to vector fields in Q, which satisfy in general nonzero boundary conditions, expressed on ∂Q. In fact, once we replace (V.7) and (V.6) in Navier-Stokes equation (II.5), the only new thing that appears with respect to the computations made in Chapters III and IV is that we have to expand the boundary condition $DH_\epsilon^{-1}W_\epsilon$ on ∂Q_ϵ in powers of ϵ and μ. Since we have $DH_\epsilon^{-1}W_\epsilon - V^{(0)}|_{\partial Q_\epsilon} = O(\epsilon)$ and since $X|_{\partial Q_\epsilon} = O(\epsilon)$ (because $X|_{\partial Q} = 0$), we finally observe that

$$\tilde{\Phi}(\mu, \epsilon, X)|_{\partial Q_\epsilon} = DH_\epsilon^{-1}W_\epsilon - V^{(0)}|_{\partial Q_\epsilon} - X|_{\partial Q_\epsilon} = O(\epsilon) \tag{V.8}$$

should be expanded in powers of ϵ and μ. Each term of the expansion has a now determined boundary value on ∂Q (expansion in the neighborhood of ∂Q), and these boundary conditions are no longer homogeneous. We shall see in the examples treated below what type of identification this leads to for coefficients of the expansion of $\tilde{\Phi}$.

V.2 Eccentric cylinders

In this section we consider the case when the distance between the axis of the two cylinders ϵR_2 is nonzero, small with respect to the gap between the cylinders (i.e., $\epsilon \ll 1 - \eta$).

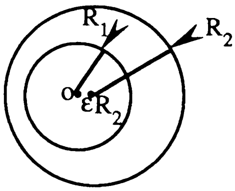


FIGURE V.1.

Let us take the origin of coordinates on the axis of the inner cylinder and use the same scales as usual, as defined in Subsection II.1.2. Then, the boundary ∂Q_ϵ, represented by the cylinders, is defined in cylindrical coordinates by (see Figure V.1)

$$r = \frac{\eta}{1-\eta}, \quad \text{and} \quad r = r_2(\theta) = \frac{1}{1-\eta}\{\sqrt{1-\epsilon^2\sin^2\theta} + \epsilon\cos\theta\}, \qquad (V.9)$$

respectively, for the inner and the outer cylinders. Since we assume here again that both cylinders rotate with constant rotation rates Ω_1 and Ω_2, the boundary conditions for the nondimensionalized velocity vector field are as follows:

$$v_r = v_z = 0, \quad v_\theta = 1 \qquad \text{at } r = \eta/(1-\eta), \qquad (V.10)$$

$$v_z = 0, \quad v_r = -\epsilon\frac{\Omega}{\eta}\sin\theta, \quad v_\theta = \frac{\Omega}{\eta}\sqrt{1-\epsilon^2\sin^2\theta} \quad \text{at } r = r_2(\theta). \quad (V.11)$$

Note that for $\epsilon = 0$ we recover the basic problem studied in Chapters III and IV.

To use the technique explained in Subsection V.1.2 for computing the amplitude equations, we also need to look at $V^{(0)}|_{\partial Q_\epsilon}$. A little calculation leads to

$$V^{(0)} = (0, v_\theta^{(0)}(r), 0), \quad \text{with } v_\theta^{(0)}|_{r=\eta/(1-\eta)} = 1, \qquad (V.12)_1$$

and

$$v_\theta^{(0)}|_{r=r_2(\theta)} = \frac{\Omega}{\eta}\sqrt{1-\epsilon^2\sin^2\theta} + \epsilon\cos\theta\frac{\Omega(1+\eta^2)-2\eta^2}{\eta(1-\eta^2)}$$
$$+\epsilon^2\frac{\eta(1-\Omega)}{1-\eta^2}\{\sqrt{1-\epsilon^2\sin^2\theta} + \epsilon\cos\theta\}^{-1}. \qquad (V.12)_2$$

V.2.1 Effect on Taylor vortices

Let us consider the critical situation studied in Section III.1, i.e., the case when we have a double 0 eigenvalue for $\mu = \epsilon = 0$ and the eigenspace $\mathbf{V}$ is spanned by ζ and $\mathbf{S}\zeta = \bar{\zeta}$, such that

$$T_\psi \zeta = e^{ni\psi}\zeta, \qquad \mathbf{R}_\varphi \zeta = \zeta \quad (\forall \psi \text{ and } \varphi \text{ in } \mathbb{R}). \tag{V.13}$$

We observe that when $\epsilon \neq 0$ the rotational invariance of the problem is broken, but since $\mathbf{R}_\varphi$ acts trivially on $\mathbf{V}$, breaking this symmetry of the system does not change the form of the amplitude equation, which therefore satisfies (III.3)–(III.4). We observe now that a change of ϵ into $-\epsilon$ is equivalent to a change of θ into $\theta+\pi$. Since this symmetry does not play on the space $\mathbf{V}$, g should be even in ϵ. Hence, we have an equation of the form

$$\frac{dA}{dt} = Ag(\mu, \epsilon^2, |A|^2) = (a\mu + b\epsilon^2)A + cA|A|^2 + \text{ h.o.t.} \tag{V.14}$$

with a real function g. We now observe that $A = 0$ is a steady solution that leads to a solution for the velocity vector field of the form

$$V = V^{(0)} + \tilde{\Phi}(\mu, \epsilon, 0), \tag{V.15}$$

which corresponds to the perturbation of the Couette solution. We may observe that this solution is invariant under T_ψ and $\mathbf{S}$ (not invariant under $\mathbf{R}_\varphi$). This results from $A = 0$, from the commutativity of $\tilde{\Phi}(\mu, \epsilon, \cdot)$ with T_ψ and $\mathbf{S}$, and from the full invariance of $V^{(0)}$. It follows that the perturbed Couette flow is still invariant under translations along the z-axis and is symmetric with respect to any orthogonal section of the cylinders.

The bifurcation now takes place for $\mu = -b\epsilon^2/a + O(\epsilon^4)$; i.e., *there is a little shift on the critical Reynolds number* independent of the sign of ϵ. The structure of the bifurcated steady solution may be studied exactly as for the Taylor vortices, i.e., we find that the flow is cellular, the size of cells being $\pi/\alpha = h/2n$, but it should be clear here that the modified Taylor vortices and the modified Couette flow (V.15) are no longer axisymmetric. The only new interesting thing here to be computed is the coefficient b, since a and c are known from Subsection III.1.2.

V.2.2 Computation of the coefficient b

To compute explicitly b and the Taylor expansion of $\tilde{\Phi}$, let us set

$$\tilde{\Phi}(\mu, \epsilon, X) = \sum_{p,q,r,s} \mu^p \epsilon^q A^r \bar{A}^s \tilde{\Phi}^{pq}_{rs}. \tag{V.16}$$

Due to the equivariances of the problem, we have

$$\tilde{\Phi}^{pq}_{rs} = \tilde{\Phi}^{pq}_{rs}(r, \theta) e^{i(r-s)\alpha z}, \tag{V.17}$$

where $\tilde{\Phi}^{pq}_{rs}(r, \theta + \pi) = (-1)^q \tilde{\Phi}^{pq}_{rs}(r, \theta)$ and $\tilde{\Phi}^{p0}_{rs}$ is independent of θ and already known from Subsection III.1.3. Since at order ϵ the azimuthal dependence only occurs through the boundary conditions, via $\cos\theta$ or $\sin\theta$, identification at order ϵ leads to

$$\tilde{\Phi}_{00}^{01}(r,\theta) = e^{\mathrm{i}\theta}\Phi_{00}^{01(1)}(r) + e^{-\mathrm{i}\theta}\bar{\Phi}_{00}^{01(1)}(r) \tag{V.18}$$

with (see definition of notation in Section II.3):

$$\begin{cases} L_{01}\Phi_{00}^{01(1)} - D_{01}\hat{q} = 0, \qquad \nabla_{01}.\Phi_{00}^{01(1)} = 0, \\ \Phi_{00}^{01(1)}|_{r=\eta/(1-\eta)} = 0, \\ \Phi_{00}^{01(1)}|_{r=1/(1-\eta)} = \left(\dfrac{\mathrm{i}\Omega}{2\eta}, \dfrac{2\eta^2 - \Omega(1+\eta^2)}{2\eta(1-\eta^2)}, 0\right). \end{cases} \tag{V.19}$$

At order ϵ^2 we obtain

$$\tilde{\Phi}_{00}^{02}(r,\theta) = e^{2\mathrm{i}\theta}\Phi_{00}^{02(2)}(r) + e^{-2\mathrm{i}\theta}\bar{\Phi}_{00}^{02(2)}(r) + \Phi_{00}^{02(0)}(r), \tag{V.20}$$

and, for the calculation of b, we need $\Phi_{00}^{02(0)}(r)$ defined by

$$\begin{cases} L_{00}\Phi_{00}^{02(0)} - D_{00}\hat{q} = \mathcal{R}_c[\Phi_{00}^{01(1)}.\nabla_{0,-1}\bar{\Phi}_{00}^{01(1)} + \bar{\Phi}_{00}^{01(1)}.\nabla_{01}\Phi_{00}^{01(1)}], \\ \nabla_{00}.\Phi_{00}^{02(0)} = 0, \qquad \Phi_{00}^{02(0)}|_{r=\eta/(1-\eta)} = 0, \\ \Phi_{00}^{02(0)}|_{r=1/(1-\eta)} + \dfrac{1}{2(1-\eta)}\dfrac{d}{dr}[\Phi_{00}^{01(1)} + \bar{\Phi}_{00}^{01(1)}]|_{r=1/(1-\eta)} \\ \quad = \left(0, \frac{\eta(\Omega-1)}{(1-\eta^2)}, 0\right). \end{cases} \tag{V.21}$$

Order ϵA leads to

$$\tilde{\Phi}_{10}^{01}(r,\theta) = e^{\mathrm{i}\theta}\Phi_{10}^{01(1)}(r) + e^{-\mathrm{i}\theta}\Phi_{10}^{01(-1)}(r), \tag{V.22}$$

with

$$\begin{cases} L_{\alpha 1}\Phi_{10}^{01(1)} - D_{\alpha 1}\hat{q} = \mathcal{R}_c[\Phi_{00}^{01(1)}.\nabla_{\alpha 0}\hat{U}_0 + \hat{U}_0.\nabla_{01}\Phi_{00}^{01(1)}], \\ \nabla_{\alpha 1}.\Phi_{10}^{01(1)} = 0, \qquad \Phi_{10}^{01(1)}|_{r=\eta/(1-\eta)} = 0, \\ \Phi_{10}^{01(1)}|_{r=1/(1-\eta)} + \dfrac{1}{2(1-\eta)}\dfrac{d}{dr}\hat{U}_0|_{r=1/(1-\eta)} = 0 \end{cases} \tag{V.23}$$

and

$$\begin{cases} L_{\alpha,-1}\Phi_{10}^{01(-1)} - D_{\alpha,-1}\hat{q} = \mathcal{R}_c[\bar{\Phi}_{00}^{01(1)}.\nabla_{\alpha 0}\hat{U}_0 + \hat{U}_0.\nabla_{0,-1}\bar{\Phi}_{00}^{01(1)}], \\ \nabla_{\alpha,-1}.\Phi_{10}^{01(-1)} = 0, \qquad \Phi_{10}^{01(-1)}|_{r=\eta/(1-\eta)} = 0, \\ \Phi_{10}^{01(-1)}|_{r=1/(1-\eta)} + \dfrac{1}{2(1-\eta)}\dfrac{d}{dr}\hat{U}_0|_{r=1/(1-\eta)} = 0, \end{cases} \tag{V.24}$$

where we recall that $\zeta = e^{\mathrm{i}\alpha z}\hat{U}_0$. Order $\epsilon^2 A$ now gives

$$\tilde{\Phi}_{10}^{02}(r,\theta) = e^{2\mathrm{i}\theta}\Phi_{10}^{02(2)}(r) + e^{-2\mathrm{i}\theta}\Phi_{10}^{02(-2)}(r) + \Phi_{10}^{02(0)}(r), \tag{V.25}$$

and for the calculation of b we only need to solve

$$\begin{cases} L_{\alpha 0}\Phi_{10}^{02(0)} - D_{\alpha 0}\hat{q} = b\hat{U}_0 + \mathcal{R}_c[\Phi_{00}^{01(1)}.\nabla_{\alpha,-1}\Phi_{10}^{01(-1)} \\ \qquad + \bar{\Phi}_{00}^{01(1)}.\nabla_{\alpha 1}\Phi_{10}^{01(1)} + \Phi_{10}^{01(-1)}.\nabla_{01}\Phi_{00}^{01(1)} \\ \qquad + \Phi_{10}^{01(1)}.\nabla_{0,-1}\bar{\Phi}_{00}^{01(1)} + \Phi_{00}^{02(0)}.\nabla_{\alpha 0}\hat{U}_0 \\ \qquad + \hat{U}_0.\nabla_{00}\Phi_{00}^{02(0)}], \\ \nabla_{\alpha 0}.\Phi_{10}^{02(0)} = 0, \end{cases} \tag{V.26$_1$}$$

with the boundary conditions

$$\Phi_{10}^{02(0)}|_{r=\eta/(1-\eta)} = 0,$$

$$\Phi_{10}^{02(0)}|_{r=1/(1-\eta)} = \frac{-1}{2(1-\eta)}\frac{d}{dr}\left(\Phi_{10}^{01(1)} + \Phi_{10}^{01(-1)} - \frac{1}{2}\hat{U}_0 + \frac{1}{2(1-\eta)}\frac{d\hat{U}_0}{dr}\right)|_{r=1/(1-\eta)}. \tag{V.26$_2$}$$

In all systems (V.19,21,23,24,26) the functions $\hat{q}$ are different (we abusively use the same notation for all $\hat{q}$). In these systems, we might subtract a divergence-free vector field satisfying the boundary conditions; hence, we observe that the vector fields $\Phi_{00}^{01(1)}(r)$, $\Phi_{00}^{02(0)}(r)$, $\Phi_{10}^{01(\pm 1)}(r)$ can be computed in a standard way, since $L_{p\alpha,m}$ is invertible for homogeneous boundary conditions when $(p,m) \neq (\pm 1, 0)$, as we saw in Subsection III.1.3. Now $\hat{U}_0$ is in the kernel of $L_{\alpha 0}$; hence, b and $\Phi_{10}^{02(0)}(r)$ are determined via a Fredholm alternative, as in Subsection III.1.3 for other coefficients of the amplitude equation. Numerical results due to Laure and Raffaï (see [Raf] and [Ra-La 91]) are presented at Figure V.2. and Table V.1. In Figure V.2 it is shown that the sign of b changes in the (η, Ω) plane, for positive $\Omega_0(\eta)$. This means that the critical Reynolds number is exactly the unperturbed one for $\Omega = \Omega_0(\eta)$ and that the new critical Reynolds number is larger (or smaller) than the unperturbed one, depending on $\Omega < \Omega_0(\eta)$ (or>). In Table V.1 we give typical values of the coefficient $-b/a$ as a function of

TABLE V.1. (see Figure III.1 for $\mathcal{R}_c$).

Ω	.5	.45	.3	.2	0	$-.1$
$\mathfrak{R}_c(\eta = .95)$	224.30	215.30	197.50	190.70	185.0	185.40
$\mathfrak{R}_c(\eta = .7519)$	193.20	148.60	105.10	94.44	86.60	85.58
$-\frac{b}{a}10^{-4}(\eta = .95)$	8.94	9.15	9.31	9.33	9.50	9.71
$-\frac{b}{a}10^{-4}(\eta = .7519)$	$-.417$	$-.0922$	.109	.136	.162	.174

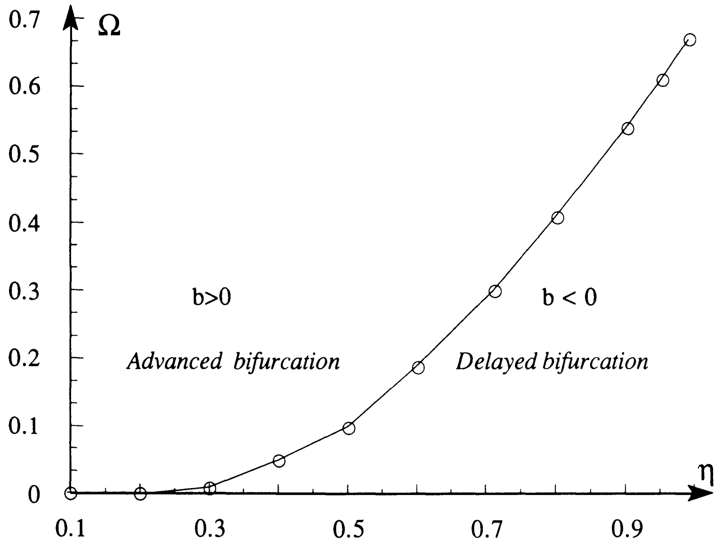


FIGURE V.2.

η and Ω. It should be understood that the expansion of the new critical Reynolds number $\mathcal{R}_c(\epsilon) = \mathcal{R}_c - b\epsilon^2/a + O(\epsilon^4)$ is only valid for $\epsilon \ll 1 - \eta$.

V.2.3 Effect on spirals and ribbons

Let us now consider the critical situation studied in Section III.2, i.e., the case when we have two pure imaginary eigenvalues $\pm i\omega_0$ for $\mu = \epsilon = 0$ and the critical space $\mathbf{V}$ is spanned by ζ_1 and $\zeta_2 = \mathbf{S}\zeta_1$ and their complex conjugate $\bar{\zeta}_1$ and $\bar{\zeta}_2$, which satisfy:

$$\mathbf{T}_\psi \zeta_1 = e^{in\psi}\zeta_1, \qquad \mathbf{T}_\psi \zeta_2 = e^{-in\psi}\zeta_2 \qquad (\text{V.27})_1$$

$$\mathbf{R}_\varphi \zeta_j = e^{im\varphi}\zeta_j, \qquad j = 1, 2, \text{ for all } \varphi,\ \psi \text{ in } \mathbb{R} \qquad (\text{V.27})_2$$

Contrary to the previous case studied in Subsection V.2.1, the operator $\mathbf{R}_\varphi$ has a nontrivial action on the critical space $\mathbf{V}$. When $\epsilon \neq 0$ we lose one of the two properties used in the identity (III.18) for the amplitude equation in A and B. Indeed, it is now necessary to use normal form theory as presented in Section II.4. The corresponding group action $(A_1, A_2) \to (A_1 e^{i\omega_0 t}, A_2 e^{i\omega_0 t})$ replaces the action of $\mathbf{R}_\varphi$ (with $\varphi = \omega_0 t/m$). As a consequence, the normal form looks exactly like (III.19), with the additional parameter ϵ. Let us observe as above that we have in addition the fact that changing ϵ into $-\epsilon$ is equivalent to changing θ in $\theta + \pi$, i.e., changing A_j into $(-1)^m A_j$ in amplitude equations. For instance, we now have in the normal form part of dA_1/dt the following identity [see (III.19)]:

$$(-1)^m A_1 f(\mu, -\epsilon, (-1)^m A_1, (-1)^m A_2) = (-1)^m A_1 f(\mu, \epsilon, A_1, A_2),$$

hence, f *is even in* ϵ. Finally, the perturbed amplitude equations become

$$\begin{aligned} \frac{dA_1}{dt} &= A_1[\mathrm{i}\omega_0 + a\mu + e\epsilon^2 + b|A_1|^2 + c|A_2|^2] + \text{ h.o.t.}, \\ \frac{dA_2}{dt} &= A_2[\mathrm{i}\omega_0 + a\mu + e\epsilon^2 + c|A_1|^2 + b|A_2|^2] + \text{ h.o.t.}, \end{aligned} \tag{V.28}$$

where the computation of coefficient e follows the same process as in the previous section. These computations were made by Laure and Raffaï (see [Raf] and [Ra-La 91]), and results are presented in Figure V.3, where the coefficient $-e_r/a_r$ is plotted for two different values of η. The new threshold is given here by

$$\mathcal{R}_c(\epsilon) = \mathcal{R}_c - \frac{e_r}{a_r}\epsilon^2 + O(\epsilon^4),$$

where $\mathcal{R}_c$ is given at Figure III.5a.

A consequence of the numerical results is that the threshold is changed differently following the different modes and, for a not so large ϵ, this can, in particular, modify significantly the codimension-two points, studied in Chapter IV. Some point might even disappear! (See the discussion in [Raf].)

V.3 Little additional flux

This section is based on a paper by Raffaï and Laure [Ra-La 92]. Let us consider the case when a small axial flux is imposed. Then we shall consider it as an imperfection of the basic problem that breaks the reflection symmetry $z \to -z$. In what follows, we study the effect of this perturbation on the primary bifurcating flows.

In the present case, the geometry of the domain of the flow is unchanged, even though the perturbation comes from the boundary conditions. We now have for any section Σ of the flow

$$\int_\Sigma V.n\, ds = \epsilon, \tag{V.29}$$

where ϵ is the imposed little flux. The theory developed in Section V.1 is simplified here, since it is enough to subtract a velocity vector such that div $V = 0$ and satisfying adherence conditions at $r = \eta/1-\eta$ and $r = 1/1-\eta$ in addition to condition (V.29). In fact, here we have the additional luck of being able to use a solution of the full equations, which takes the form of

$$V_\epsilon^0 = V^{(0)} + \epsilon V_P \tag{V.30}$$

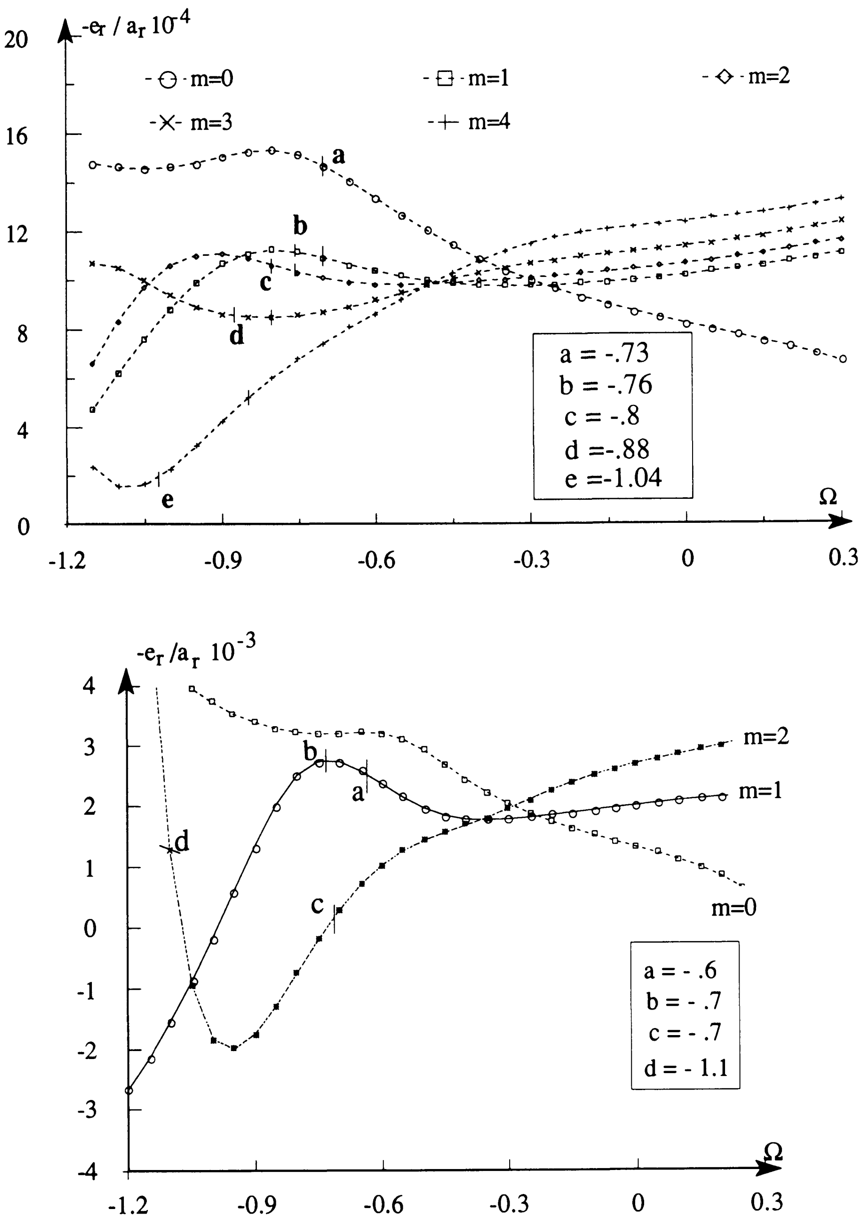


FIGURE V.3. Coefficient $-e_r/a_r$ in function of Ω, for $\eta = 0.95$ and 0.7519. We mention, for the different m-modes, the most unstable ones (for $\epsilon = 0$) in full line.

where $V^{(0)}$ is the Couette flow (II.9) and V_{P} is the Poiseuille flow between the cylinders at rest, such that the volume flux is 1. Indeed, in cylindrical coordinates we have

$$V_\epsilon^0 = \left(0, Ar + B/r, \epsilon(A_{\mathrm{P}} r^2 + B_{\mathrm{P}} \ln r + C_{\mathrm{P}})\right),$$

where A and B are given below (II.9) and

$$A_{\mathrm{P}} = -\frac{2(1-\eta)^3 \ln(\eta)}{\pi(1+\eta)[1+\ln(\eta)+\eta^2(\ln(\eta)-1)]},$$
$$B_{\mathrm{P}} = \frac{A_{\mathrm{P}}(1+\eta)}{(1-\eta)\ln(\eta)}, \qquad C_{\mathrm{P}} = A_{\mathrm{P}} \frac{(1-\eta^2)\ln(1-\eta) - \ln(\eta)}{(1-\eta)^2 \ln(\eta)}$$

Remark. In theory it is not necessary to have a steady solution of the full equations for making the explicit analysis, since it might very well happen that there is no such solution for the perturbed problem!

Now, if we set $V = V_\epsilon^0 + U$, the vector field U satisfies an equation of the form (V.4) with $G_{\mu,\epsilon} = 0$ since V_ϵ^0 is a solution of the original equation. This shows that one can apply the center manifold reduction theorem as indicated in V.1.2 in the simplest way.
About the symmetry breaking, one has the property:

the system is *invariant under the transformation* $(\epsilon, U) \to (-\epsilon, \mathbf{S}U)$. (V.31)

V.3.1 Perturbed Taylor vortices lead to traveling waves

In this case, a center manifold reads

$$U = X + \Phi(\mu, \epsilon, X), \tag{V.32}$$

where $X = A\zeta + \bar{A}\bar{\zeta}$, $\Phi(\mu, \epsilon, 0) = 0$, and $D_X\Phi(0,0,0) = 0$ (see Section III.1). Moreover, we have $\mathbf{S}\Phi(\mu, \epsilon, X) = \Phi(\mu, -\epsilon, \mathbf{S}X)$.
The amplitude equation may be written as

$$\frac{dA}{dt} = Ag(\mu, \epsilon, |A|^2) \tag{V.33}$$

since we can use the translational invariance (SO(2) part of O(2) symmetry), even though $\epsilon \neq 0$, as shown in Section III.1. Now $g(\mu, 0, |A|^2)$ is real as indicated in Section III.1, and due to the invariance (V.31), we have equivariance of the field (V.33) under the transformation $(\epsilon, A) \to (-\epsilon, \bar{A})$. So we get

$$g(\mu, -\epsilon, |A|^2) = \overline{g(\mu, \epsilon, |A|^2)}$$

Finally, equation (V.33) written at leading orders becomes

$$\frac{dA}{dt} = A(a\mu + b|A|^2 + id\epsilon + e\epsilon^2 + \cdots), \tag{V.34}$$

where a, b, d, e are *real* coefficients.
Let us define again $A = \rho e^{\mathrm{i}\varphi}$; then (V.34) reads

$$\frac{d\rho}{dt} = \rho(a\mu + e\epsilon^2 + b\rho^2 + \cdots),$$
$$\frac{d\varphi}{dt} = d\epsilon + \cdots. \tag{V.35}$$

This system shows that the bifurcation point has changed (no longer at $\mu = 0$):

$$\mu_c = -\frac{e\epsilon^2}{a} + \mathrm{O}(\epsilon^4), \tag{V.36}$$

and the bifurcation is again of pitchfork type for the amplitude ρ. The constant amplitude regime $\rho_0 \neq 0$ leads to a phase satisfying $\varphi = \omega t + \varphi_0$, where ω is of order ϵ.
The bifurcating solution takes the form

$$X(t) = \rho_0 e^{\mathrm{i}(\omega t + \varphi_0)}\zeta + \text{ c.c.},$$

i.e., it is a traveling wave for the velocity vector field U, which, in polar coordinates, can be written as $U(r, \omega t + \alpha z)$ (the proof follows the lines of Subsection III.2.3). The velocity $-\omega/\alpha$ of the waves is of order ϵ. The stability of this traveling wave is the same as that of Taylor vortices for the shifted values of the parameter μ.

V.3.2 Identification of coefficients d and e

In the present case, equation (V.4) takes the form

$$\frac{dU}{dt} = L_0 U + \mu L_1 U + \epsilon L_2 U + N_0(U,U) + \mu N_1(U,U), \tag{V.37}$$

where all operators except L_2 are defined in Chapter III. We have

$$L_2 U = -\Pi_0[(V_\mathrm{P}.\nabla)U + (U.\nabla)V_\mathrm{P}], \tag{V.38}$$

where V_P is defined in (V.30). Let us observe that

$$\mathbf{S}L_2 = -L_2\mathbf{S}, \tag{V.39}$$

because $\mathbf{S}V_\mathrm{P} = -V_\mathrm{P}$.
Now the center manifold reads

$$U = A\zeta + \bar{A}\bar{\zeta} + \Sigma \mu^p \epsilon^q A^r \bar{A}^s \Phi_{rs}^{pq}, \tag{V.40}$$

where $\Phi_{00}^{pq} = 0$ because V_ϵ^0 is a solution of the full equations and by construction $\Phi_{rs}^{00} = 0$ for $r + s = 1$. Identifying orders ϵA and $\epsilon^2 A$ in (V.37), using (V.40) and (V.34), leads to

$$\mathrm{i}d\zeta = L_0\Phi_{10}^{01} + L_2\zeta,$$
$$e\zeta = L_0\Phi_{10}^{02} + L_2\Phi_{10}^{01}. \tag{V.41}$$

Hence we get

$$\begin{aligned} id &= (\mathrm{L}_2\zeta;\zeta^*), & \mathbf{S}\Phi^{01}_{10} &= -\overline{\Phi^{01}_{10}},\\ e &= (\mathrm{L}_2\Phi^{01}_{10};\zeta^*), & \mathbf{S}\Phi^{02}_{10} &= \overline{\Phi^{02}_{10}}. \end{aligned}$$

More specifically, we have $\Phi^{pq}_{rs} = \hat{\Phi}^{pq}_{rs}(r)e^{\mathrm{i}(r-s)\alpha z}$, and we must solve the following system with homogeneous boundary conditions, as in Chapter III:

$$\left\{\begin{aligned} L_{\alpha 0}\hat{\Phi}^{01}_{10} - D_{\alpha 0}\hat{q}_1 &= \mathrm{i}d\hat{U}_0 + [V_{\mathrm{P}}.\nabla_{\alpha 0}\hat{U}_0 + \hat{U}_0.\nabla_{00}V_{\mathrm{P}}],\\ L_{\alpha 0}\hat{\Phi}^{02}_{10} - D_{\alpha 0}\hat{q}_2 &= e\hat{U}_0 + [V_{\mathrm{P}}.\nabla_{\alpha 0}\hat{\Phi}^{01}_{10} + \hat{\Phi}^{01}_{10}.\nabla_{00}V_{\mathrm{P}}],\\ \nabla_{\alpha 0}.\hat{\Phi}^{01}_{10} &= \nabla_{\alpha 0}.\hat{\Phi}^{02}_{10} = 0, \end{aligned}\right. \tag{V.42}$$

where

$$\mathbf{S}\hat{\Phi}^{01}_{10} = -\overline{\hat{\Phi}^{01}_{10}}, \qquad \mathbf{S}\hat{\Phi}^{02}_{10} = \overline{\hat{\Phi}^{02}_{10}}.$$

Values of d and e are given by Raffaï and Laure in [Ra-La 92]. We reproduce their main results in Table V.2.

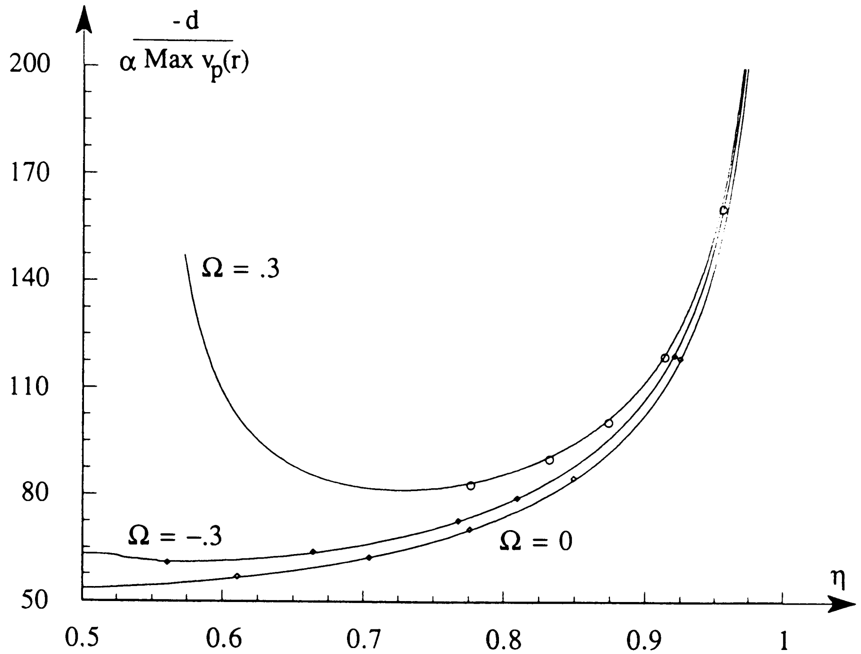


FIGURE V.4. Comparison between the axial velocity and the velocity of traveling waves (see [Ra-La 92]).

TABLE V.2.

η	Ω	$\Re_c$	a	b	d	e
0.95	0.3	197.5	0.1319	−406.9	−5.885	−0.131
	0	185	0.1416	−178.2	−5.524	−0.1115
	−0.3	192.5	0.1386	−89.91	−5.794	−0.1157
0.7519	0.3	105.1	0.2487	−1138	−17.35	−1.19
	0	86.06	0.3061	−279.9	−14.25	−0.767
	−0.3	90.94	0.3013	−134.7	−15.31	−0.8047

Since $e < 0$, bifurcation is *always delayed* (critical Reynolds number larger than with no flux). The traveling wave of velocity $\approx -\epsilon d/\alpha$ has the same direction as the axial flow, since $d < 0$; moreover, its velocity is about 100 times the maximum velocity ϵv_P, as shown in Figure V.4.

V.3.3 Effects on spirals and ribbons

We consider now the case studied in Section III.2, when we have two pure imaginary eigenvalues $\pm i\omega_0$ when $\mu = \epsilon = 0$ and the critical space $\mathbf{V}$ is spanned by ζ_1 and $\zeta_2 = \mathbf{S}\zeta_1$ and their complex conjugates. We have again a center manifold of (V.32), now with

$$X = A_1\zeta_1 + A_2\zeta_2 + \bar{A}_1\bar{\zeta}_1 + \bar{A}_2\bar{\zeta}_2, \tag{V.43}$$

and the amplitude equations must commute with the transformations $(A_1, A_2, \epsilon) \to (A_2, A_1, -\epsilon)$ and $(A_1, A_2) \to (e^{i(n\psi+m\varphi)}A_1,\ e^{i(-n\psi+m\varphi)}A_2)$ as in Section III.2. Finally, we obtain (see Subsection III.2.1 for the main point of the proof)

$$\left\{\begin{aligned} \frac{dA_1}{dt} &= A_1 f(\mu, \epsilon, |A_1|^2, |A_2|^2), \\ \frac{dA_2}{dt} &= A_2 f(\mu, -\epsilon, |A_2|^2, |A_1|^2), \end{aligned}\right. \tag{V.44}$$

hence, the leading part of this system reads

$$\left\{\begin{aligned} \frac{dA_1}{dt} &= A_1(i\omega_0 + a\mu + d\epsilon + b|A_1|^2 + c|A_2|^2 + \cdots), \\ \frac{dA_2}{dt} &= A_2(i\omega_0 + a\mu - d\epsilon + b|A_2|^2 + c|A_1|^2 + \cdots). \end{aligned}\right. \tag{V.45}$$

Identification of coefficients may be performed as in previous section. The numerical results are as indicated in Table V.3 [Ra-La 92].

Remark. Notice in Table V.3 that numerical values for b_r and c_r differ from those given in Chapter III, Figure III.5(b). Indeed, the present ones are obtained with the flux condition on the velocity instead of the periodicity condition on the pressure.

TABLE V.3. (see [Ra-La 92]).

η	Ω	$\Re_c$	m	α	ω_c	a	b	c	d
0.95	-0.73	251.09	1	3.42	-4.76	0.130 - 0.024 i	15.51 + 35.84 i	-54.58 - 42.74 i	0.116 - 8.167 i
	-0.76	257.96	1	3.52	-4.99	0.131 - 0.024 i	27.37 + 38.71 i	-45.53 - 45.77 i	0.125 - 8.495 i
		257.86	2	3.51	-9.81	0.131 - 0.047 i	8.32 + 69.63 i	-58.45 - 89.26 i	0.230 - 8.442 i
	-0.80	266.53	2	3.57	-10.37	0.132 - 0.047 i	22.85 + 75.98 i	-42.53 - 98.17 i	0.243 - 8.821 i
		266.37	3	3.56	-15.11	0.131 - 0.068 i	-4.42 + 95.88 i	-68.13 - 141.20 i	0.326 - 8.755 i
	-0.88	282.31	3	3.63	-16.51	0.131 - 0.067 i	22.12 + 113.57 i	-26.80 -171.34 i	0.340 - 9.357 i
		282.18	4	3.63	-21.06	0.129 - 0.087 i	-11.08 + 121. 40 i	-78.07 - 216.06 i	0.411 - 9.331 i
	-1.04	321.76	4	3.70	-24.22	0.126 - 0.085 i	41.18 + 169.62 i	22.78 - 324.69 i	0.397 - 10.31 i
0.7519	-0.6	120.92	1	3.62	-13.70	0.300 - 0.133 i	-89.41 + 111.30 i	-205.09 - 116.24 i	0.841 - 22.38 i
	-0.7	134.29	1	3.83	-16.28	0.308 - 0.128 i	-49.56 + 145.87 i	-168.09 -183.70 i	0.386 - 25.72 i
		134.28	2	3.76	-28.82	0.284 - 0.249 i	-151.05 + 201. 18 i	-301.99 - 269.99 i	1.500 - 25.30 i
	-1.06	180.34	2	3.88	-42.88	0.258 - 0.241 i	-57.50 + 543. 53 i	196.86 - 1510.6 i	1.016 - 33.14 i
0.5	-0.38	94.37	1	3.80	-26.45	0.413 - 0.325 i	-477.22 + 294. 27 i	-817.16 - 103.80 i	2.92 - 43.00 i
	-0.4	96.80	1	3.83	-27.56	0.414 - 0.324 i	-487.00 + 310.53 i	-829.10 - 150.08 i	2.96 - 44.22 i
	-0.5	111.31	1	4.074	-34.41	0.424 - 0.321 i	-596.64 + 479. 19 i	-967.95 - 539.20 i	2.88 - 52.25 i

Since $d_r \neq 0$, we obtain two different traveling wave regimes (spirals) S^+ ($A \neq 0$, $B = 0$) and S^- ($A = 0$, $B \neq 0$), which are no longer symmetric under the reflection **S**. The standing waves (ribbons) here disappear since $|A| \neq |B|$, and we obtain a (true) *quasi-periodic regime* denoted below by QP.

More precisely, we have

$$S^+ : \quad A = \rho_+ e^{\mathrm{i}(\omega_+ t + \delta_+)}, \ B = 0, \qquad S^- : \quad A = 0, \ B = \rho_- e^{\mathrm{i}(\omega_- t + \delta_-)},$$

with

$$a_\mathrm{r}\mu + b_\mathrm{r}\rho_\pm^2 \pm d_\mathrm{r}\epsilon + \cdots = 0, \qquad \omega_\pm = \omega_0 + a_\mathrm{i}\mu + b_\mathrm{i}\rho_\pm^2 \pm d_\mathrm{i}\epsilon + \cdots . \quad \text{(V.46)}$$

We notice that at first order the existence of traveling waves S^+ (or S^-) is on one side of the line $d_+ : a_\mathrm{r}\mu + d_\mathrm{r}\epsilon = 0$ (or $d_- : a_\mathrm{r}\mu - d_\mathrm{r}\epsilon = 0$) in the parameter plane (μ, ϵ), this side depending on the sign of b_r. Assume $\epsilon > 0$, then $d_\mathrm{r} > 0$ means that the bifurcation is advanced for S^+ and delayed for S^-. These solutions are stable only if

$$b_\mathrm{r} < 0 \quad \text{and} \quad k_1\mu \pm k_2\epsilon > 0 \qquad \text{for } S^\pm,$$

where $k_1 = b_\mathrm{r} - c_\mathrm{r}$ and $k_2 = -d_\mathrm{r}(b_\mathrm{r} + c_\mathrm{r})/a_\mathrm{r}$. Defining the lines $D_\pm$ by $k_1\mu \pm k_2\epsilon = 0$, we refer to Figure V.5 [Ra-La], which gives in the (μ, ϵ) plane the regions of existence and stability for $S^\pm$.

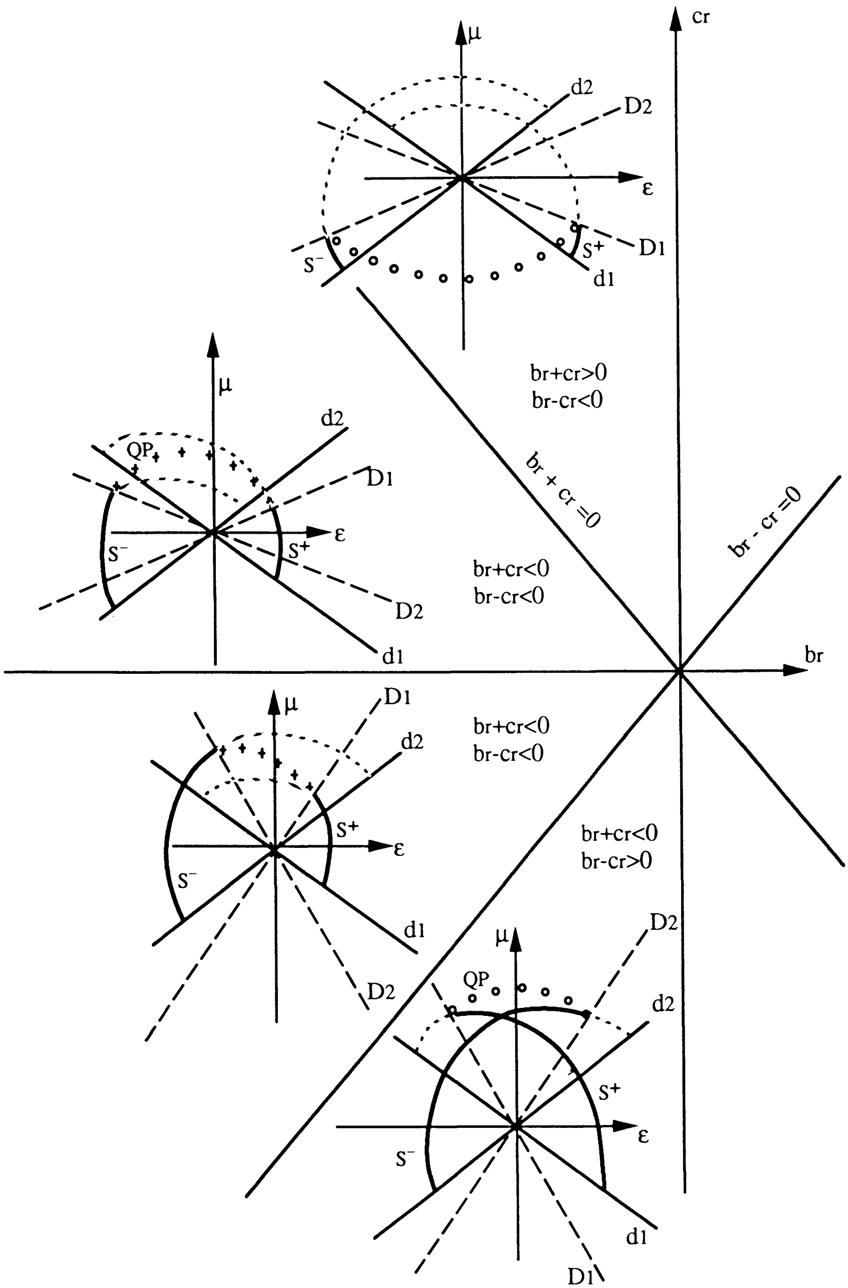


FIGURE V.5. [Ra-La 92]. Dotted and plain lines mean unstable (and stable) traveling waves respectively. Crosses and circles mean stable (and unstable) quasi-periodic solutions respectively.

For the quasi-periodic solution QP we find

$$A = \rho_1 e^{\mathrm{i}\omega_{\mathrm{QP}}^{(1)} t + \delta_1}, \qquad B = \rho_2 e^{\mathrm{i}\omega_{\mathrm{QP}}^{(2)} t + \delta_2}, \tag{V.47}$$

where

$$\begin{aligned}
\rho_1^2 - \rho_2^2 &= \frac{2d_{\mathrm{r}}\epsilon}{c_{\mathrm{r}} - b_{\mathrm{r}}} + \cdots, \qquad \rho_1^2 + \rho_2^2 = -\frac{2a_{\mathrm{r}}\mu}{c_{\mathrm{r}} + b_{\mathrm{r}}} + \cdots, \\
\omega_{\mathrm{QP}}^{(1)} &= \omega_0 + a_{\mathrm{i}}\mu + b_{\mathrm{i}}\rho_1^2 + c_{\mathrm{i}}\rho_2^2 + d_{\mathrm{i}}\epsilon + \cdots, \\
\omega_{\mathrm{QP}}^{(2)} &= \omega_0 + a_{\mathrm{i}}\mu + c_{\mathrm{i}}\rho_1^2 + b_{\mathrm{i}}\rho_2^2 - d_{\mathrm{i}}\epsilon + \cdots.
\end{aligned}$$

We notice that this solution exists only if $b_{\mathrm{r}} + c_{\mathrm{r}}$ has a sign opposite to $a_{\mathrm{r}}\mu$ and if (for the principal part)

$$\left|\frac{d_{\mathrm{r}}\epsilon}{c_{\mathrm{r}} - b_{\mathrm{r}}}\right| < -\frac{a_{\mathrm{r}}\mu}{b_{\mathrm{r}} + c_{\mathrm{r}}}, \tag{V.48}$$

Equality in (V.48) again gives equations of $D_{\pm}$, and we obtain at the limit for QP the traveling waves S^+ or S^-. It then appears that QP may be considered as a *secondary bifurcating branch*, emanating from the branches of traveling waves.

The stability of the solution QP appears to be exactly as for the standing waves in Section III.2. Figure V.5 shows the regions of existence and stability of the QP solution in the plane (μ, ϵ), depending on the region in which the coefficients b_{r} and c_{r} sit. Only the case $b_{\mathrm{r}} < 0$ is shown, because in the other case all solutions are found unstable. We observe for instance that we have bistability of S^+ and S^- for certain values of μ, ϵ, when $b_{\mathrm{r}} < 0$ and $b_{\mathrm{r}} - c_{\mathrm{r}} > 0$. It might then occur in such a case that the spiral waves travel in a direction opposite to the mean flux.

V.4 Periodic modulation of the shape of cylinders in the axial direction

This case was considered, in particular, by Signoret in her thesis [Si]. We only present a summary of the results. We now have the following perturbed domain of the flow

$$\frac{\eta}{1-\eta} < r < r_2(z) = \frac{1}{1-\eta}(1 + \epsilon \cos \beta z), \tag{V.49}$$

which means that the exterior cylinder has periodic axisymmetric bumps (harmonic ones). As a consequence, the boundary conditions become

$$v_r = v_z = 0, \quad v_\theta = 1 \qquad \text{at } r = \eta/(1-\eta), \tag{V.50}$$

$$v_r = v_z = 0, \quad v_\theta = \Omega(1 + \epsilon \cos \beta z)/\eta \qquad \text{at } r = r_2(z). \tag{V.51}$$

In the purpose of using the technique explained in Section V.1, we need to assume that the wave number β of the bumps respects the basic h-periodicity of the problem. Since this is also true for the "natural wave number" α, it results that we have integers m and p, with no common divisor, such that

$$m\beta = p\alpha. \tag{V.52}$$

The process is analogous to that described in Section V.2. We need to compute the trace of the Couette flow on the new boundary $V^{(0)}|\partial Q_\epsilon$:

$$v_\theta^{(0)}|_{r=r_2(z)} = \frac{\Omega}{\eta}(1+\epsilon\cos\beta z) + \frac{\eta(1-\Omega)}{1-\eta^2}(-2\epsilon\cos\beta z + \epsilon^2\cos^2\beta z) + \mathrm{O}(\epsilon^3). \tag{V.53}$$

V.4.1 *Effect on Taylor vortices*

As in Subsection V.2.1, let us assume that for $\mu = \epsilon = 0$, 0 is a double eigenvalue, the eigenspace $\mathbf{V}$ being spanned by ζ and $\mathbf{S}\zeta = \bar{\zeta}$. Here again, a center manifold reads

$$V = V^{(0)} + A\zeta + \bar{A}\bar{\zeta} + \tilde{\Phi}(\mu, \epsilon, A, \bar{A}), \tag{V.54}$$

where $\tilde{\Phi}$ satisfies *nonhomogeneous* boundary conditions on the *unperturbed cylinders*, which are expended in powers of ϵ.

We must now study the reduced symmetries of the problem when $\epsilon \neq 0$. The perturbed problem is still axisymmetric (so, all our new steady solutions are axisymmetric), but it is now invariant under the translation $z \to z + 2\pi/\beta$, instead of any translation in the z-direction. This translation multiplies $e^{i\alpha z}$ by $e^{2i\pi m/p}$. Since p and m are prime, we have the Bezout identity: $pq_1 + mq_2 = 1$ for some integers q_1, q_2. Hence the q_2th iterate of the translation above gives a multiplication by $e^{2i\pi/p}$. Finally, the new amplitude equations are equivariant under the following representation of the dihedral group D_p:

$$\begin{aligned} \mathbf{S} &: (A, \bar{A}) \to (\bar{\mathrm{A}}, \mathrm{A}), \\ \mathbf{T}_{2\pi/pn=\tau_{2\pi/m\beta}} &: (A, \bar{A}) \to (Ae^{2i\pi/p}, \bar{A}e^{-2i\pi/p}). \end{aligned} \tag{V.55}$$

Moreover, shifting half the period π/β is equivalent to changing ϵ into $-\epsilon$; hence, we have an additional equivariance under the transformation:

$$(A, \epsilon) \to (Ae^{i\pi m/p}, -\epsilon). \tag{V.56}$$

In particular, it results for (V.56) that *if m is even, the amplitude equation is then even in ϵ*. The reader may now check that equivariant monomials $A^r\bar{A}^s$ lead to a system of the following form (up to an arbitrary order):

$$\frac{dA}{dt} = AP(\mu, \epsilon, |A|^2, A^p) + \bar{A}^{p-1}Q(\mu, \epsilon, |A|^2, \bar{A}^p), \tag{V.57}$$

where polynomials P and Q have real coefficients and satisfy

$$\begin{aligned} P(\mu, 0, |A|^2, A^p) &\equiv g(\mu, |A|^2) \quad \text{[see (III.6)]}, \\ Q(\mu, 0, |A|^2, \bar{A}^p) &\equiv 0. \end{aligned}$$

Case $p = 1$: $\beta = \alpha/m$ (the period of the bumps is a multiple of the natural period). In such a case, the transformation $(V.55)_2$ is trivial and the principal part of (V.57) reads

$$\frac{dA}{dt} = A(a\mu + c|A|^2) + \begin{cases} b\epsilon & \text{if } m \text{ is odd,} \\ b\epsilon^2 & \text{if } m \text{ is even.} \end{cases} \tag{V.58}$$

Defining again polar coordinates $A = \rho e^{i\theta}$, we see that if $\epsilon \neq 0$ then the steady solutions correspond to a *phase* θ *locked at* 0 *or* π and ρ satisfying

$$\rho(a\mu + c\rho^2) \pm \begin{cases} b\epsilon \ (m \text{ odd}) \\ b\epsilon^2 \ (m \text{ even}) \end{cases} = 0.$$

Numerical values for the coefficient b are given in [Si]. We obtain two pairs of branches of symmetric steady solutions, exchanged by a shift of π/α (half of the period). If m is odd, they are also exchanged by $\epsilon \to -\epsilon$. These solutions look like the Taylor vortices for μ near 0, but their phase is locked by the bumps, and there is no solution like a "perturbed Couette flow" for μ near 0. Notice that these solutions have axial period m times the period for Taylor vortices, since $2\pi/\beta = m2\pi/\alpha$. In Figure V.6 we sum up the result in this case, giving in addition the stability of the solutions.

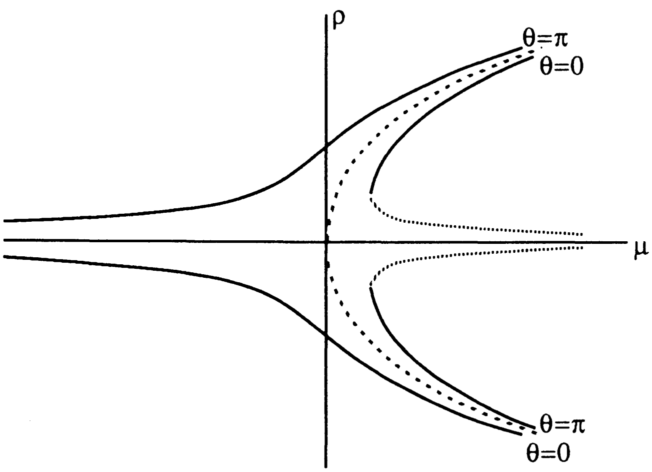


FIGURE V.6. Solutions for $\epsilon \neq 0$ in the case $\beta = \alpha/m$ (here $a > 0$, $c < 0$, $b < 0$, and $\epsilon > 0$ for m odd).

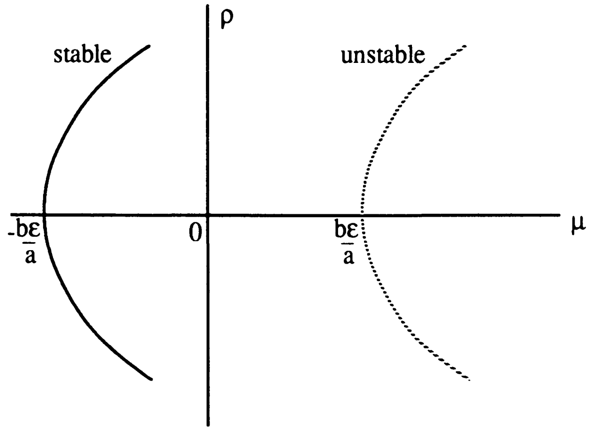


FIGURE V.7. Solutions for $\epsilon \neq 0$ in the case $\beta = 2\alpha/m$ (here $a > 0$, $c < 0$, $b > 0$, $\epsilon > 0$).

Case $p = 2$: $\beta = 2\alpha/m$ (the period of bumps is an odd multiple of half of the natural period). In such a case, the leading part of (V.57) reads

$$\frac{dA}{dt} = A(a\mu + c|A|^2) + b\epsilon\bar{A}. \tag{V.59}$$

Defining again polar coordinates $A = \rho e^{i\theta}$, we see that if $\epsilon \neq 0$ then the steady solutions are given by

(i) $\rho = 0$,

(ii) $\theta = 0 \pmod{\pi}$ or $\theta = \pi/2 \pmod{\pi}$ with $a\mu + c\rho^2 \pm b\epsilon = 0$.

Hence there exists a "perturbed Couette flow" losing its stability at the first bifurcation point, and there are two bifurcating branches (solutions looking like locked Taylor vortices), the first bifurcating one being stable ($c < 0$), the second being unstable. Numerical values of b are given in [Si]. Notice that a shift by π/α in the z direction is equivalent to changing ρ into $-\rho$ (or a change θ into $\theta + \pi$) on the same bifurcating branch, and notice that, here again, the axial period of the bifurcating flow is m times the period of Taylor vortices—$2.2\pi/\beta = m.2\pi/\alpha$, while the period of the perturbed Couette flow is half $(2\pi/\beta)$, so the bifurcation has here the effect of doubling the axial period of the flow. We sum up the situation in Figure V.7.

V.4.2 Effects on spirals and ribbons

Let us consider the case of oscillatory instability as in Subsection V.2.3. We again have (V.27). When $\epsilon \neq 0$, the amplitude equations are now equivariant under the transformations:

$$\left\{\begin{aligned} \mathbf{S}: \ (A_1, A_2) &\to \ (A_2, A_1), \\ \mathbf{T}_{2\pi/\mathrm{pn}} = \tau_{2\pi/m\beta}: \ (A_1, A_2) &\to \ (A_1 e^{2\mathrm{i}\pi/p}, A_2 e^{-2\mathrm{i}\pi/p}), \\ \mathbf{R}_\varphi: \ (A_1, A_2) &\to \ (A_1 e^{\mathrm{i}m\varphi}, A_2 e^{\mathrm{i}m\varphi}) \quad \forall \varphi \in \mathbb{R}. \end{aligned}\right. \tag{V.60}$$

Moreover, shifting half the period π/β is equivalent to changing ϵ into $-\epsilon$; hence, we have an additional equivariance under the transformation

$$(A_1, A_2, \epsilon) \to (A_1 e^{\mathrm{i}\pi m/p}, A_2 e^{-\mathrm{i}\pi m/p}, -\epsilon).$$

The general form for the amplitude equations is given in [Si]. Let us just give the principal part associated with the cases when $p = 1$ or 2 (as in previous section):

$$\begin{aligned} \frac{dA_1}{dt} &= A_1[\mathrm{i}\omega_0 + a\mu + e\epsilon^2 + b|A_1|^2 + c|A_2|^2] \\ &\quad + \left\{\begin{array}{l} d\epsilon^2 A_2 \ (p=1) \\ d\epsilon A_2 \ (p=2) \end{array}\right. + \text{ h.o.t.}, \\ \frac{dA_2}{dt} &= A_2[\mathrm{i}\omega_0 + a\mu + e\epsilon^2 + c|A_1|^2 + b|A_2|^2] \\ &\quad + \left\{\begin{array}{l} d\epsilon^2 A_1 \ (p=1) \\ d\epsilon A_1 \ (p=2) \end{array}\right. + \text{ h.o.t.}, \end{aligned} \tag{V.61}$$

where coefficients e and d are computed in [Si] for $p = 1$, $m = 1$ and $p = 2$, $m = 1$.

Let us rewrite the system in polar coordinates $A_j = \rho_j e^{\mathrm{i}\psi_j}$ $(j = 1, 2)$. Then we notice that it reduces to a system in three variables $(\rho_1, \rho_2, \theta = \psi_2 - \psi_1)$. Now looking for equilibria in these three variables leads to different solutions perturbing the traveling and standing waves found in Chapter III. First, for $\rho_1 = \rho_2$ we get $\theta = 0$ or π, and we obtain an advanced or delayed bifurcation to standing waves of ribbons type (exercise left to the reader). Now if $\rho_1 \neq \rho_2$, we find, for the other equilibria of the three-dimensional system (for instance, for $p = 1$)

$$\epsilon^2 \cos\theta = \rho_1\rho_2 |d|^{-2}[(b-c)\bar{d}]_{\mathrm{r}}, \qquad \epsilon^2 \sin\theta = \rho_1\rho_2 \frac{\rho_1^2 - \rho_2^2}{\rho_1^2 + \rho_2^2} |d|^{-2}[(b-c)\bar{d}]_{\mathrm{i}}$$

(change ϵ^2 into ϵ in the case $p = 2$) and eliminating θ, we have the following system in $X = \rho_1^2 + \rho_2^2$ and $Y = \rho_1^2 - \rho_2^2$:

$$\begin{aligned} (b_{\mathrm{r}} + c_{\mathrm{r}} + d_{\mathrm{r}}|d|^{-2}[(b-c)\bar{d}]_{\mathrm{r}})X^2 - d_{\mathrm{i}}|d|^{-2}[(b-c)\bar{d}]_{\mathrm{i}} Y^2 & \\ +2(a_{\mathrm{r}}\mu + b_{\mathrm{r}}\epsilon^2)X &= 0, \end{aligned} \tag{V.62}$$

$$\{[(b-c)\bar{d}]_{\mathrm{r}}{}^2 X^2 + [(b-c)\bar{d}]_{\mathrm{i}}{}^2 Y^2\}(X^2 - Y^2) - 4\epsilon^4|d|^4 X^2 = 0. \tag{V.63}$$

In the (X, Y)-plane, the first equation (V.62) represents a conic (ellipse or hyperbola in general) passing through 0 and symmetric with respect to $Y = 0$. The second represents a quartic symmetric with respect to both axes. When ϵ is small enough they intersect, on the side $X > 0$, in two symmetric points, which lead to two symmetric traveling wave regimes. Values of the coefficients e and d are given in [Si] for $p = 1$ and $p = 2$ (modifications of (V.62)–(V.63) for this last case are left to the reader).

Now for the stability of all these perturbed solutions, we should notice that the two eigenvalues found in the study made in Chapter III are regularly perturbed, but an additional eigenvalue appears near 0 (no longer 0 when $\epsilon \neq 0$), the sign of which depends on the new coefficients. Following the results of [Si], the standing waves should be unstable in most cases. For the traveling waves, X is close to Y, and the stability results obtained at Section III.2 do not change for ϵ close to 0 (one Floquet multiplier is 1 while the three other main ones stay off the unit circle for small ϵ).

V.5 Time-periodic perturbation

In this section, we consider the case when the cylinders are not rotated uniformly. More precisely, we assume that

$$\Omega_1(t) = \Omega_1(1 + \epsilon \cos \omega_f t), \tag{V.64}$$

while Ω_2 is constant. We can proceed as in Section V.1 to transform the problem into a nonautonomous one, with homogeneous boundary conditions, in subtracting from the velocity V a divergence-free vector field $V_\epsilon^{(0)}(t)$ satisfying the oscillating boundary conditions. This shows that one can use the center manifold reduction argument again. The additional feature is that the function Φ, which gives the graph of the center manifold, depends periodically on t, when $\epsilon \neq 0$. So, we have now

$$V = V^{(0)} + X + \Phi(t, \mu, \epsilon, X), \tag{V.65}$$

where $X \in \mathbf{V}$ as in previous sections and where Φ now satisfies nonhomogeneous oscillatory boundary conditions at $r = \eta/1 - \eta$ (inner cylinder):

$$\Phi(t, \mu, \epsilon, X)|_{r=\eta/1-\eta} = (0, \epsilon \cos \omega_f t, 0) \quad \text{in cylindrical coordinates.} \tag{V.66}$$

This additional time dependency modifies the basic equation (II.36) for Φ and the normal form F by adding the term $\partial\Phi/\partial t$ on its left-hand side. Now, identifying powers of (X, μ, ϵ) does lead to equations of the following type (modifying (II.42)):

$$p\Phi_{np}(t)[(\mu, \epsilon)^{(n)}; X^{(p-1)}, L_0 X] - \mathbf{L}_0 \Phi_{np}(t)[(\mu, \epsilon)^{(n)}; X^{(p)}]$$

$$+\frac{\partial \Phi_{np}(t)}{\partial t}[(\mu,\epsilon)^{(n)};X^{(p)}] = \mathcal{R}_{np}(t)[(\mu,\epsilon)^{(n)};X^{(p)}]$$
$$-F_{np}(t)[(\mu,\epsilon)^{(n)};X^{(p)}], \tag{V.67}$$

where at each step $\mathcal{R}_{np}(t)$ is periodic in t and is known from resolutions at lower orders. Since we are looking for time periodic Φ_{np}, we can look for the Fourier coefficients of (V.67). This gives the series of equations

$$p\Phi_{npq}[(\mu,\epsilon)^{(n)};X^{(p-1)},L_0X] - L_0\Phi_{npq}[(\mu,\epsilon)^{(n)};X^{(p)}]$$
$$+\mathrm{i}q\omega_f\Phi_{npq}[(\mu,\epsilon)^{(n)};X^{(p)}] = \mathcal{R}_{npq}[(\mu,\epsilon)^{(n)};X^{(p)}]$$
$$-F_{npq}[(\mu,\epsilon)^{(n)};X^{(p)}], \tag{V.68}$$

which have the same type as usual! If $-\mathrm{i}\omega_f q$ is not an eigenvalue of the homological operator defined in Section II, then one can take $F_{npq} = 0$. In particular, when 0 is the only eigenvalue for L_0, it is also the only eigenvalue for the homological operator. It results in such a case that $F_{npq} = 0$ for all $q \neq 0$, i.e., F_{np} is independent of t. In the general case, the set of eigenvalues of the homological operator has the form $\Sigma\alpha_j\lambda_j - \lambda_k$ and one has to look at all possible combinations of eigenvalues λ_n of L_0 such that $\Sigma\alpha_j = p$, α_j being nonnegative integers. One can show that the normal form can be chosen such that it solves the adjoint equation built with L_0^* and $\partial/\partial t$ replaced by $-\partial/\partial t$ (see, for instance, [E-T-I] for results on periodically forced bifurcations and [Io 88] and [Io-Ad] for detailed and more general results).

V.5.1 Perturbed Taylor vortices

In this case, the only eigenvalue of L_0 is 0 (double). So the new normal form is *again autonomous*. As indicated above we can find a $2\pi/\omega_f$- periodic Φ such that the center manifold reads

$$U = A\zeta + \bar{A}\bar{\zeta} + \Phi(t,\mu,\epsilon,A\zeta + \bar{A}\bar{\zeta}) \tag{V.69}$$

and, since we do not break any geometrical symmetry, the amplitude equation becomes

$$\frac{dA}{dt} = Ag_\epsilon(\mu,|A|^2) = a\mu A + cA|A|^2 + d\epsilon^2 A + \text{ h.o.t.}, \tag{V.70}$$

where the order ϵ^2 is due to the invariance of the problem under the transformation $(t,\epsilon) \to (t+\pi/\omega_f, -\epsilon)$, which propagates on Φ and g_ϵ. Coefficient d may be computed as other coefficients, by a straightforward identification of powers of $(A,\bar{A},\mu,\epsilon)$ in the basic equations, taking care of the fact that the pure coefficient in ϵ of Φ satisfies a nonhomogeneous boundary condition at the inner cylinder. As can be seen immediately, the perturbation of the Taylor vortices gives rise again to Taylor vortices, indeed oscillating periodically, but with the cell boundaries fixed. The bifurcation is then delayed or advanced, depending on the sign of coefficient d.

V.5.2 Perturbation of spirals and ribbons

In this case, the eigenvalues of L_0 are $\pm i\omega_0$ (double). So, for the remaining terms of the normal form in the A_1 or A_2 component, we have for the coefficient of $\exp(ni\omega_f t)$ in the Fourier expansion of a monomial $A_1^{p_1} A_2^{p_2} \bar{A}_1^{q_1} \bar{A}_2^{q_2}$

$$(p_1 - q_1 + p_2 - q_2 - 1)i\omega_0 + in\omega_f = 0. \tag{V.71}$$

In fact, none of the geometrical symmetries are broken by our perturbation. Hence the analysis made in Section III.2 is still valid, and this leads to $n = 0$ in (V.71). This means that the *perturbed amplitude equations are again autonomous*, just introducing a shift of order ϵ^2 (as above) in the threshold for bifurcation of perturbed spirals or ribbons, both now quasi-periodic, due to the additional frequency.

A much more drastic modification would occur if we consider the case of eccentric cylinders. Indeed, in such a case, one can use neither the SO(2) symmetry nor the symmetry due to Hopf bifurcation, which is equivalent, since the first is broken by geometry and the second broken by the nonautonomy of the system. We are then faced with the problem of an O(2) symmetric system having a Hopf bifurcation, periodically forced. This leads to many works in the literature. Let us establish the leading terms of the amplitude equations. We have two small parameters: ϵ_1 from the eccentricity, and ϵ_2 from the time forcing. For $\epsilon_2 = 0$ the system is written in Section V.2. For $\epsilon_2 \neq 0$ and $\epsilon_1 = 0$ we also have the same type of system for the amplitudes, and for $\epsilon_2 \neq 0$ and $\epsilon_1 \neq 0$ one has the following equivariances:

$$\begin{aligned} \mathbf{T}_\psi : \ (A_1, A_2) &\rightarrow \ (e^{in\psi} A_1, e^{-in\psi} A_2), \\ \mathbf{S} : \ (A_1, A_2) &\rightarrow \ (A_2, A_1). \end{aligned}$$

We must not forget that the system also must be equivariant under the transformations

$$(A_1, A_2, \epsilon_1) \rightarrow ((-1)^m A_1, (-1)^m A_2, -\epsilon_1), \qquad (\epsilon_2, t) \rightarrow (-\epsilon_2, t + \pi/\omega_f), \tag{V.72}$$

and the relationship (V.71) for resonant terms has to be exploited following the value of the ratio

$$\omega_0/\omega_f = p/q + \gamma, \tag{V.73}$$

where p and q have no common divisor and γ is the *detuning* with respect to p/q. The detuning plays the role of an additional (small) parameter. It should be understood that in the interval of values of γ one has to avoid the occurrence of other rational numbers for ω_0/ω_f with a smaller denominator than q.

Since our analysis wants to be valid for $|\gamma|$ small, it is necessary to keep all coefficients such that (V.71) is satisfied with $\omega_0/\omega_f = p/q$. This property, with the remaining geometric equivariance, leads to the following conditions for the A_1 component of the amplitude equation:

$$(p_1 - q_1 + p_2 - q_2 - 1)p + nq = 0, \qquad p_1 - q_1 - p_2 + q_2 - 1 = 0. \quad \text{(V.74)}$$

Notice that our problem is now equivalent to saying that we perturb the SO(2) part of the symmetry of the original problem by S_q (invariance under rotations of angle $2\pi/q$). This case is then different from that studied in Subsection V.4.2. Moreover, this case is also different from the case of periodically forced Hopf bifurcations, appearing many times in the literature (see the most detailed analysis made by Gambaudo [Gam]), since in those cases one only has one complex mode.

Equation (V.74) gives $p_1 - q_1 + p_2 - q_2 = 1 - lq$, $n = lp$, $2(p_1 - q_1) = 2 - lq$, $2(p_2 - q_2) = -lq$, for relative integers l, such that lq is even. Then we have two different cases depending on the parity of q.

Case 1. q odd. We can set $l = 2l'$, where l' is any integer, and it can easily be shown that the normal form reads

$$\begin{aligned}\frac{dA_1}{dt} &= A_1 f(|A_1|^2, |A_2|^2, (A_1A_2)^q e^{-2i\omega_f pt}) \\ &\quad + e^{2i\omega_f pt}\bar{A}_1^{q-1}\bar{A}_2^{q} g(|A_1|^2, |A_2|^2, (\bar{A}_1\bar{A}_2)^q e^{2i\omega_f pt}) \qquad \text{(V.75)}\end{aligned}$$

and dA_2/dt is obtained by exchanging A_1 and A_2 in the above equation. In addition, f and g are even in ϵ_1 and in ϵ_2, due to (V.72). The lowest-order term that is exotic with respect to normal forms (III.19) or (V.28) has the form

$$\epsilon_1^2\epsilon_2^2 e^{2i\omega_f pt}\bar{A}_1^{q-1}\bar{A}_2^{q}. \qquad \text{(V.76)}$$

Case 2. q even. We can set $q = 2q_0$, and p is necessarily odd. Then the normal form reads

$$\begin{aligned}\frac{dA_1}{dt} &= A_1 f(|A_1|^2, |A_2|^2, (A_1A_2)^{q_0} e^{-i\omega_f pt}) \\ &\quad + e^{i\omega_f pt}\bar{A}_1^{q_0-1}\bar{A}_2^{q_0} g(|A_1|^2, |A_2|^2, (\bar{A}_1\bar{A}_2)^{q_0} e^{i\omega_f pt}) \qquad \text{(V.77)}\end{aligned}$$

and dA_2/dt is obtained by exchanging A_1 and A_2 in the above equation. Due to (V.72), f and g are even in ϵ_1 and f is even while g is odd in $(\epsilon_2, (A_1A_2)^{q_0})$. The lowest-order term that is exotic with respect to normal forms (III.19) or (V.28) has the form

$$\epsilon_1^2\epsilon_2 e^{i\omega_f pt}\bar{A}_1^{q_0-1}\bar{A}_2^{q_0}. \qquad \text{(V.78)}$$

As a result, the strongest resonances are obtained by $q = 1$ and 2, where a new type of term of degree 1 appears. For $q = 4$, a term of degree 3 appears, while for $q = 3$ as well as for $q = 6$ it is of degree 5, hence, the resonating terms are more "soft" than those we had already for the unperturbed problem.

A general feature of all these normal forms is that one can choose *new variables such that the system becomes autonomous.* Let us set

$$A_j = A'_j \exp(p/q)\omega_f t, \qquad j = 1,2. \tag{V.79}$$

Then the principal part of the amplitude equations is

$$\begin{aligned}\frac{dA'_1}{dt} &= A'_1[\mathrm{i}\gamma\omega_f + a\mu + e_1\epsilon_1^2 + e_2\epsilon_2^2 + b|A'_1|^2 + c|A'_2|^2] \\ &\quad + \begin{cases} d\epsilon_1^2\epsilon_2^2\bar{A}'^{q-1}_1\bar{A}'^q_2 & (q \text{ odd}), \\ d\epsilon_1^2\epsilon_2\bar{A}'^{q_0-1}_1\bar{A}'^{q_0}_2 & (q \text{ even}), \end{cases}\end{aligned} \tag{V.80}$$

where the second equation is obtained by exchanging A_1 and A_2 and where the detuning parameter γ appears in the linear part.

Now, because of the invariances of the normal form one can look for equilibria in $|A_1|$, $|A_2|$, θ, where $\theta = q(\arg A'_1 + \arg A'_2)$, and discuss the possibility of having such equilibria in suitable regions of the parameter plane (μ, γ). For q odd ≥ 3 and q even ≥ 6 one finds "horns" in this plane where this occurs and where the perturbed bifurcating solutions (same symmetries as in Section V.2) are periodic as for the unperturbed problem but "locked" to the forcing frequency because of (V.79) and of the $2\pi/\omega_f$-periodicity of Φ in the expression of the center manifold. In such horns, the period of the new flows is then q times that of the forcing one; hence, close to p times the natural one given by the Hopf bifurcation. Now for $q = 1, 2, 4$ a more specific study is necessary because the above "horns" look different and are a little harder to get. This is left to a courageous reader, who might proceed along the same lines as indicated at the end of Section V.4.

VI

Bifurcation from Group Orbits of Solutions

In Chapter III we studied the primary bifurcations that occur when the basic Couette flow becomes unstable. In Chapter IV secondary bifurcations were encountered in the local analysis of bifurcation with mode interaction, when the cylinders are counterrotating. In this case two parameters were required to allow mode interaction. However, it is an experimental fact that the primary bifurcated branches can also lose their stability "far" from these critical points. In this case, the analysis of Chapter IV fails to explain the various regimes that are observed, and this is especially true in the situation of co-rotating cylinders. For example, Figure VI.1 shows relatively "simple" flows (by this we mean nonchaotic flows) like the wavy inflow-(WIB) and wavy outflow-boundaries flows (WOB), which are not accessible to a local analysis from the instability of the Couette flow. The question is therefore the following: How much can be said about these further bifurcations, or, more precisely, can we understand these bifurcations from purely theoretical considerations? In answering this question we shall make use of two tools. First, note that the flows we now consider as the basic solutions have a residual symmetry inherited from the original symmetry of the problem, which in turn allows a symmetry-breaking classification of the possible bifurcations. The second tool is the center manifold reduction from a group orbit of solutions. Indeed, the standard center manifold procedure is not appropriate here because the basic solutions are not isolated, contrary to the Couette flow. The reason is that every solution is transformed to another by any element of the symmetry group that does not belong to the isotropy subgroup of that solution. For example, any translation along the z-axis transforms the Taylor vortex

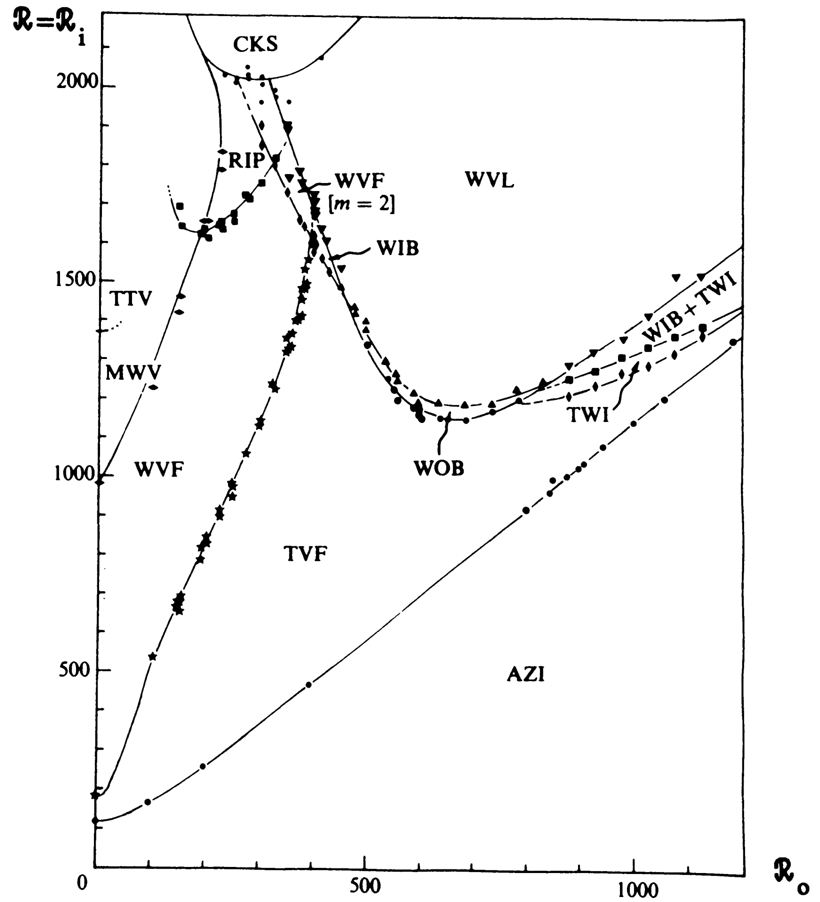


FIGURE VI.1. (see Andereck, Liu, and Swinney [An-L-Sw]).

flow to another solution, which is a shifted "copy" of this flow. Hence, as it was noticed in Subsection IV.1.4 (Preliminary remark 2), the linearized operator around this solution has a zero eigenvalue with a corresponding eigenvector tangent to the group orbit, whatever the location of the solution on its branch. In order to handle this additional degeneracy, one should consider the bifurcation from the group orbit itself, instead of looking for a bifurcation from individual solutions. This idea, applied together with the Lyapunov-Schmidt decomposition, was used by Iooss [Io 84] for the Hopf bifurcation from a rotating wave and by Chossat [Ch 86] for a bifurcation analysis from standing waves in a problem with O(2) symmetry. In the context of the Couette-Taylor problem, Iooss [Io 86] has derived a center manifold reduction from a group orbit and in this chapter we essentially follow this method. Note that a geometrical derivation of this idea is given by Krupa [Kru].

The chapter is organized as follows: in the first section we expose the center manifold reduction from group orbits of equilibria (Taylor vortices) and traveling as well as standing waves (spirals and ribbons). The bifurcation from these three "basic" regimes is then studied.

VI.1 Center manifold for group orbits

VI.1.1 Group orbits of first bifurcating solutions

VI.1.1.1 Taylor vortex flow

It was shown in Subsection III.1.3 that the perturbation U of the Couette flow that satisfies the Navier-Stokes equations (II.23) has the form

$$U = \mathbf{T}_\psi U_0, \tag{VI.1}$$

where ψ is an arbitrary phase and U_0 is steady and axisymmetric and satisfies in addition the relations

$$\mathbf{T}_{2\pi/n} U_0 = \mathbf{S} U_0 = U_0, \tag{VI.2}$$

which express spatial periodicity and reflexion symmetry.
The group orbit of solutions (VI.1) is therefore a circle of equilibria parametrized by ψ, which we denote by

$$\mathcal{O}_T = \{\mathbf{T}_\psi U_0;\ 0 \leq \psi < 2\pi/n\}. \tag{VI.3}$$

Each individual solution is invariant under the action of $\mathbf{T}_{2\pi/n}$ (axial periodicity) and $\mathbf{R}_\varphi$ for any $\varphi \in \mathbb{R}$ (axisymmetry), and there are two elements on $\mathcal{O}_T$ that are invariant under $\mathbf{S}$, namely, U_0 and $\mathbf{T}_{\pi/n} U_0$. Indeed,

$$\mathbf{S}\mathbf{T}_{\pi/n} U_0 = \mathbf{T}_{-\pi/n} \mathbf{S} U_0 = \mathbf{T}_{-\pi/n} U_0 = \mathbf{T}_{\pi/n} U_0.$$

Note that U_0 and $\mathbf{T}_{\pi/n} U_0$ are therefore invariant under the action of the group D_n generated by $\psi = 2\pi/n$ and S.

VI.1.1.2 Spirals

These solutions satisfy the relations (III.38), which we rewrite as

$$U(t) = \mathbf{R}_{(\omega t+\psi)/m} U_0 = \mathbf{T}_{(\omega t+\psi)/n} U_0, \tag{VI.4}$$

where U_0 is stationary and $\omega \neq 0$. The action of rotations as well as translations is identical to the time phase shift along a trajectory. Note that U_0 is invariant under the transformations $\mathbf{R}_{n\varphi}\mathbf{T}_{-m\varphi}$. In other words, there is only one arbitrary phase, and the group orbit reduces to the two limit cycles $U(t)$ and $\mathbf{S}U(t)$. We then define

$$\mathcal{O}_S = \{\mathbf{R}_\varphi U_0;\ 0 \leq \varphi < 2\pi/m\} = \{\mathbf{T}_\psi U_0;\ 0 \leq \psi < 2\pi/n\}. \tag{VI.5}$$

VI.1.1.3 Ribbons

It follows from (III.25) and the fact that the symmetries of the solutions on the center manifold propagate to the corresponding solutions of the Navier-Stokes equations that the ribbons can be written in the form

$$U(t) = \mathbf{R}_{(\omega t+\varphi)/m}\mathbf{T}_{\psi}U_0, \tag{VI.6}$$

where ψ and φ are arbitrary phases and U_0 satisfies the invariance relations

$$\mathbf{S}U_0 = \mathbf{R}_{\pi/m}\mathbf{T}_{\pi/n}U_0 = U_0. \tag{VI.7}$$

These solutions are rotating waves in the azimuthal direction and standing waves in the axial direction. The phase ψ defines a one-parameter orbit of rotating waves. Therefore, in this case the group orbit is a 2-torus parameterized by φ and ψ, which we denote by

$$\mathcal{O}_R = \{\mathbf{R}_{\varphi}\mathbf{T}_{\psi}U_0;\ 0 \le \varphi < 2\pi/m,\ 0 \le \psi < 2\pi/n\}. \tag{VI.8}$$

VI.1.2 The center manifold reduction for a group-orbit of steady solutions

Let us first define the infinitesimal generators of the groups $\mathbf{R}_{\varphi}$ and $\mathbf{T}_{\psi}$ acting in the space $H(Q_h)$ in which equation (II.23) for the perturbation of the Couette flow was defined (see Section II.2):

$$\mathfrak{r} = d\mathbf{R}_{\varphi}(0)/d\varphi \quad \text{and} \quad \mathfrak{t} = d\mathbf{T}_{\psi}(0)/d\psi. \tag{VI.9}$$

These operators are differential operators and, hence, unbounded in $H(Q_h)$. For instance $\mathfrak{t}$ and $\mathfrak{r}$ represent differentiation with respect to the axial variable αz and to the aximuthal variable θ, respectively. We can also write $\mathbf{R}_{\varphi} = \exp(\mathfrak{r}\varphi)$ and $\mathbf{T}_{\psi} = \exp(\mathfrak{t}\psi)$, in the sense of strongly continuous groups [Ka].

For the sake of clarity we discuss first the method *in the case of Taylor vortex flow.* Given the orbit $\mathcal{O}_T$, the idea is to consider a perturbation of this orbit, of the form

$$U(t) = \mathbf{T}_{\psi(t)}[U_0 + W(t)], \tag{VI.10}$$

where W is orthogonal to the tangent direction to $\mathcal{O}_T$, at U_0 defined by the vector

$$\xi_0 = \mathfrak{t}U_0. \tag{VI.11}$$

The angle ψ therefore plays the role of a coordinate "along" the orbit. We now replace U by (VI.10) in (II.23), which, thanks to the equivariance of the operators under $\mathbf{T}_{\psi}$, gives the equation

$$(\xi_0 + \mathfrak{t}W)\frac{d\psi}{dt} + \frac{dW}{dt} = L_{\mu}(U_0 + W) + N(\mu, U_0 + W) = \mathcal{F}(\mu, W). \tag{VI.12}$$

Let $P_\perp$ be the orthogonal projection in $H(Q_h)$ onto the line $\{\xi_0\}$: if $\hat{\xi}_0 = \xi_0/||\xi_0||^2$, we have

$$P_\perp U = (U; \hat{\xi}_0)\xi_0, \tag{VI.13}$$

where (.;.) is the scalar product in the space $H(Q_h)$ defined in Chapter II, and which is such that the group actions $\mathbf{R}_\varphi$, $\mathbf{T}_\psi$, $\mathbf{S}$, are unitary. Then $P_\perp$ commutes with these operators. Applying $P_\perp$ and $1-P_\perp$ to (VI.12) we get the following two equations:

$$[1 + (\mathfrak{t}W; \hat{\xi}_0)]\frac{d\psi}{dt} = (\mathcal{F}(\mu, W); \hat{\xi}_0), \tag{VI.14}$$

$$(1 - P_\perp)\mathfrak{t}W\frac{d\psi}{dt} + \frac{dW}{dt} = (1 - P_\perp)\mathcal{F}(\mu, W). \tag{VI.15}$$

Since we look for W such that $||W|| \ll 1$, the scalar equation (VI.14) reduces to an equation of the form

$$\frac{d\psi}{dt} = \mathfrak{n}(\mu, W), \tag{VI.16}$$

where $\mathfrak{n}(\mu, 0) = 0$, and (VI.15) becomes an equation for W only:

$$\frac{dW}{dt} = (1 - P_\perp)[\mathcal{F}(\mu, W) - \mathfrak{n}(\mu, W)\mathfrak{t}W]. \tag{VI.17}$$

Observe that the linear part in (VI.17) only comes from the projection of $\mathcal{F}$. Let us set $\mathcal{L}(\mu) = D_W\mathcal{F}(\mu, 0)$. If for some value μ_0 of the parameter the operator $(1 - P_\perp)\mathcal{L}(\mu_0)$ in $(1 - P_\perp)H(Q_h)$ has eigenvalues on the imaginary axis with corresponding invariant subspace $\mathbf{V}$, a classical center manifold reduction can be performed on (VI.17) according to the method shown in Section II.4. Finally, setting $W = X + \Phi(\mu, X)$, where X is a coordinate of W on $\mathbf{V}$ and Φ is the graph map of the center manifold as defined in Section II.4, the problem reduces to an ordinary differential system

$$\frac{d\psi}{dt} = \mathfrak{n}(\mu, X + \Phi(\mu, X)) = h(\mu, X), \tag{VI.18}$$

$$\frac{dX}{dt} = F(\mu, X). \tag{VI.19}$$

The remarkable fact about this system is that the phase equation (VI.18) is uncoupled from the amplitude equation (VI.19). If X is an equilibrium of (VI.19), then ψ represents a slow uniform drift along the group orbit (unless $h(\mu, X) = 0$). Notice that the equivariance properties of (II.23) propagate to this system. Indeed, the group transformations $\mathbf{R}_\varphi$, $\mathbf{T}_\psi$, and $\mathbf{S}$ are unitary and, hence, $P_\perp$ commutes with them; then the result follows from the center manifold theorem. The consequences of this fact are analyzed in Section VI.2.

For the spiral flow the procedure is essentially the same, but we have the choice to parametrize the group orbit $\mathcal{O}_S$ by the angle φ or by the angle ψ (see (VI.4)). Let us define

$$U(t) = \mathbf{T}_{\Omega t+\psi(t)}(U_0 + W(t)) \tag{VI.20}$$

where $\Omega = \omega/n$. As before, we set $\xi_0 = \mathfrak{t}U_0$ and define $P_\perp$ to be the orthogonal projection on the line $\{\xi_0\}$, so W is orthogonal to ξ_0. Observing that

$$\Omega\xi_0 - L_\mu U_0 - N(\mu, U_0) = 0,$$

equation (II.23) is then transformed into

$$(\xi_0 + \mathfrak{t}W)\frac{d\psi}{dt} + \frac{dW}{dt} = \mathcal{F}(\mu, W), \tag{VI.21}$$

where we set $\mathcal{F}(\mu, W) = L_\mu W + N(\mu, U_0 + W) - N(\mu, U_0) - \Omega\mathfrak{t}W$, and $\mathcal{L}(\mu) = D_W\mathcal{F}(\mu, 0)$. Suppose that for some value μ_0 of the parameter the operator $(1 - P_\perp)\mathcal{L}(\mu_0)$ in $(1 - P_\perp)H(Q_h)$ has eigenvalues on the imaginary axis with corresponding invariant subspace $\mathbf{V}$. By the same reasoning as in the case of the bifurcation from the Taylor vortex flow, this equation is decomposed into two *autonomous* equations, which, after a center manifold reduction, have exactly the same form as (VI.18)–(VI.19). Moreover, these equations inherit the symmetry of the spiral solutions.

For the ribbons the group orbit $\mathcal{O}_R$ is two-dimensional (see (VI.8)). In this case we define

$$U(t) = \mathbf{R}_{\Omega t+\varphi(t)}\mathbf{T}_{\psi(t)}(U_0 + W(t)), \tag{VI.22}$$

where $\Omega = \omega/m$. Now the eigenvectors spanning the tangent plane to $\mathcal{O}_R$ at U_0 are

$$\xi_0 = \mathfrak{t}U_0 \quad \text{and} \quad \chi_0 = \mathfrak{r}U_0, \tag{VI.23}$$

so W is taken orthogonal to the space $[\xi_0, \chi_0]$ spanned by ξ_0 and χ_0.
The transformations $\mathbf{R}_\varphi$ and $\mathbf{T}_\psi$ commute; hence, replacing U by (VI.22) in (II.23) gives the equation

$$(\chi_0+\mathfrak{r}W)\left(\Omega + \frac{d\varphi}{dt}\right) + (\xi_0+\mathfrak{t}W)\frac{d\psi}{dt} + \frac{dW}{dt} = L_\mu(U_0+W) + N(\mu, U_0+W), \tag{VI.24}$$

where we know that $\chi_0\Omega = L_\mu(U_0) + N(\mu, U_0)$.
Let $P_\perp$ now denote the orthogonal projection onto the plane $[\xi_0, \chi_0]$ in $H(Q_h)$:

$$P_\perp U = (U; \hat{\xi}_0)\xi_0 + (U; \hat{\chi}_0)\chi_0, \tag{VI.25}$$

where $\{\hat{\xi}_0, \hat{\chi}_0\}$ is now the dual basis to $\{\xi_0, \chi_0\}$ in this plane. Let us project (VI.24) on $[\xi_0, \chi_0]$, then we obtain a linear system in $(d\psi/dt, d\varphi/dt)$ that we can solve (matrix close to Id). Finally the system (VI.24) takes the new form:

$$\frac{d\psi}{dt} = \mathfrak{n}(\mu, W), \tag{VI.26}$$

$$\frac{d\varphi}{dt} = \mathfrak{k}(\mu, W), \tag{VI.27}$$

$$\frac{dW}{dt} = (1 - P_\perp)[\mathfrak{F}(\mu, W) - \mathfrak{k}(\mu, W)\mathfrak{r}W - \mathfrak{n}(\mu, W)\mathfrak{t}W], \tag{VI.28}$$

where $\mathfrak{F}(\mu, W) = L_\mu W + N(\mu, U_0 + W) - N(\mu, U_0) - \Omega\mathfrak{r}W$. Observe that

$$\mathfrak{n}(\mu, 0) = \mathfrak{k}(\mu, 0) = 0;$$

hence, the linear part of the right-hand side in (VI.28) is $(1 - P_\perp)\mathfrak{L}(\mu)$, where $\mathfrak{L}(\mu) = D_W\mathfrak{F}(\mu, 0)$.

If $(1 - P_\perp)\mathfrak{L}(\mu_0)$ has eigenvalues on the imaginary axis, the center manifold theorem can be applied to (VI.28). We set $W = X + \Phi(\mu, X)$, where X is a coordinate of W on V and Φ is the graph map of the center manifold. Then equation (VI.24) reduces to the differential system

$$\frac{d\psi}{dt} = h(\mu, X), \tag{VI.29}$$

$$\frac{d\varphi}{dt} = k(\mu, X), \tag{VI.30}$$

$$\frac{dX}{dt} = F(\mu, X). \tag{VI.31}$$

Once (VI.31) is solved, the phases φ and ψ are readily obtained from (VI.29) and (VI.30). Note that the equivariance properties of (II.23) propagate through this reduction to the differential system (VI.29)–(VI.31), which will be of fundamental importance in the subsequent analysis.

VI.2 Bifurcation from the Taylor vortex flow

When μ is varied, the Taylor vortex flow, which we shall also denote by TVF, can lose its stability either by an eigenvalue of the linear operator $(1 - P_\perp)\mathfrak{L}(\mu)$ defined from (VI.17) crossing the imaginary axis at 0 or by a pair of eigenvalues crossing the imaginary axis away from 0. We study successively these two cases. We take criticality at $\mu = 0$ without loss of generality, set $\mathfrak{L}(0) = D_W\mathfrak{F}(0, 0) = \mathfrak{L}$ (see VI.12), and define the linearized operator at $\mu = 0$ in the orthogonal complement of $\{\xi_0\}$ by

$$L = (1 - P_\perp)D_W\mathfrak{F}(0, 0) = (1 - P_\perp)\mathfrak{L}. \tag{VI.32}$$

The operator $\mathfrak{F}$ is supposed to have a Taylor expansion at the origin of the form

$$\mathfrak{F}(\mu, W) = \sum_{p,q>0} \mu^p \mathfrak{F}_{pq}[W^{(q)}], \tag{VI.33}$$

where $\mathcal{F}_{pq}$ is q-linear symmetric (with respect to W). Note that for any integer p, $\mathcal{F}_{p0}$ is invariant under $\mathbf{S}$, as a straightforward consequence of the equivariance of $\mathcal{F}$.

VI.2.1 The stationary case

We assume that there exists ζ_0 in $(1 - P_\perp)\mathcal{D}_h$ such that

$$L\zeta_0 = 0. \tag{VI.34}$$

It follows from the definition (VI.12) that $\mathcal{F}$ commutes with $\mathbf{R}_\varphi, \mathbf{T}_{2\pi/n}$, and $\mathbf{S}$, which are the symmetries of U_0. Since $1 - P_\perp$ commutes also with these operators (thanks to the fact that they leave invariant the scalar product), it results that L commutes with $\mathbf{R}_\varphi, \mathbf{T}_{2\pi/n}$, and $\mathbf{S}$. We also assume that $\mathbf{R}_\varphi\zeta_0 = \zeta_0$ for all φ, i.e., ζ_0 does not break the rotational invariance of the Taylor vortices.

Remark. It is not generic that ζ_0 breaks the rotational invariance of U_0. By this we mean that this breaking would imply the corresponding eigenvalues to be pure imaginary, and to have them at zero would follow from an additional condition on L (for example, an additional symmetry or an additional parameter in the equations). We do not prove this statement. Let us only note that rotational symmetry breaking would imply having a double eigenvalue (eigenvectors $\zeta_0 = \hat{\zeta}_0(r, z)e^{im\theta}$, $m \neq 0$, and its complex conjugate), but a 2×2 matrix commuting with $\mathbf{R}_\varphi$ has complex conjugate eigenvalues in general, and at criticality these eigenvalues are expected to be pure imaginary.

Therefore, in the stationary case, $m = 0$ and only the D_n invariance is expected to break (D_n is the group generated by $\mathbf{T}_{2\pi/n}$ and $\mathbf{S}$). This can happen in two possible ways: either at a simple or at a double eigenvalue of L. The case of a simple eigenvalue occurs when there is no symmetry breaking or when the only nontrivial effect of the group action is to transform ζ_0 into $-\zeta_0$. In the case of a double eigenvalue, one can find an eigenvector that transforms as follows: $\mathbf{T}_{2\pi/n}\zeta_0 = e^{2ik\pi/n}\zeta_0$, $k \neq 0, \pm n/2$. This kind of "translational" symmetry breaking has not been reported in experiments and we shall not consider it here.

The next important point is that ξ_0 satisfies the relation

$$\mathbf{S}\xi_0 = -\xi_0. \tag{VI.35}$$

Indeed, $\mathbf{ST}_\psi = \mathbf{T}_{-\psi}\mathbf{S}$; hence, by differentiation at $\psi = 0$, $\mathbf{S}\mathrm{t}U_0 = -\mathrm{t}\mathbf{S}U_0 = -\mathrm{t}U_0$.

VI.2.1.1 No symmetry breaking

We assume now that ζ_0 is still invariant under the action of the symmetry group of U_0, i.e.,

$$\mathbf{T}_{2\pi/n}\zeta_0 = \mathbf{S}\zeta_0 = \mathbf{R}_\varphi\zeta_0 = \zeta_0. \tag{VI.36}$$

Lemma 1. *If* $\mathbf{S}\zeta_0 = \zeta_0$, *then* $\mathcal{L}\zeta_0 = 0$. *Moreover, the phase equation* (VI.18) *reduces to* $d\psi/dt = 0$.

Proof. By (VI.34) and the definition of L we have $\mathcal{L}\zeta_0 = (\mathcal{L}\zeta_0; \hat{\xi}_0)\xi_0$. Applying $\mathbf{S}$ to this equation and observing that $\mathbf{S}\mathcal{L}\zeta_0 = \mathcal{L}\zeta_0$, we see, thanks to (VI.35), that $\mathcal{L}\zeta_0 = 0$. Let us consider $\mathbf{S}U$ in (VI.10). We have $\mathbf{S}U = \mathbf{S}\mathbf{T}_\psi(U_0 + W) = \mathbf{T}_{-\psi}(U_0 + \mathbf{S}W)$, which shows that the system in (ψ, W) commutes with the action $(\psi, W) \longrightarrow (-\psi, \mathbf{S}W)$. In our case, $\mathbf{S}W = W$ on the center manifold. This implies $h(\mu, X) = h(\mu, \mathbf{S}X) = -h(\mu, X)$, hence $h \equiv 0$.

Let us set

$$W = A\zeta_0 + \Phi(\mu, A), \tag{VI.37}$$

i.e., $X = A\zeta_0$, so that the system (VI.18)–(VI.19) can now be written as

$$\left\{ \begin{aligned} \frac{d\psi}{dt} &= 0, \\ \frac{dA}{dt} &= f(\mu, A). \end{aligned} \right. \tag{VI.38}$$

The scalar function f is the component of F along ζ_0. We already know that $f(0,0) = 0$ and $\partial f(0,0)/\partial A = 0$, so that we have

$$f(\mu, A) = a_{10}\mu + a_{02}A^2 + a_{11}\mu A + a_{20}\mu^2 + \text{ h.o.t.}$$

In general, $a_{10}a_{02} \neq 0$, so that (0, 0) is in fact a turning point of the branch of Taylor vortex flows (saddle-node bifurcation). This turning point corresponds to an exchange of stability between the upper and lower branches of solutions. However, it appears that no experiment so far refers to such a phenomenon, which would lead to a jump to another type of flow, not reachable by the present analysis. The coefficients in the Taylor expansion of f can be computed in the following way. We set

$$f(\mu, A) = \sum_{(p,q)\neq(0,1)} a_{pq}\mu^p A^q \quad \text{and} \quad \Phi(\mu, A) = \sum_{(p,q)\neq(0,1)} \mu^p A^q \Phi_{pq}, \tag{VI.39}$$

$$\frac{dW}{dt} = \frac{dA}{dt}\left(\zeta_0 + \frac{\partial \Phi}{\partial A}\right). \tag{VI.40}$$

By identification of the terms of same order in (VI.12) and since we know that $d\psi/dt = 0$, we obtain

$$\text{at order } A : 0 = \mathcal{L}\zeta_0 \text{ (which was expected!)}; \tag{VI.41a}$$

$$\text{at order } \mu : a_{10}\zeta_0 = \mathcal{L}\Phi_{10} + \mathcal{F}_{10}; \tag{VI.41b}$$

$$\text{at order } A^2 : a_{02}\zeta_0 = \mathcal{L}\Phi_{02} + \mathcal{F}_{02}[\zeta_0, \zeta_0]. \tag{VI.41c}$$

These equations have the form $\mathcal{L}u = v$, which we can decompose into

$$Lu = (1 - P_\perp)v, \tag{VI.42a}$$

$$(u; \mathcal{L}^*\xi_0) = (v; \xi_0), \tag{VI.42b}$$

where $\mathcal{L}^*$ is the adjoint operator to $\mathcal{L}$. Note that in (VI.41), $\mathbf{S}v = v$; hence, we can find u such that $\mathbf{S}u = u$ and (VI.42b) is automatically satisfied. There exists a unique solution of (VI.42) in $(1 - P_\perp)H(Q_h)$ if $(1 - P_\perp)v$ is orthogonal to the kernel of the adjoint operator L^* (Fredholm alternative). Let ζ_0^* be such that

$$(\zeta_0; \zeta_0^*) = 1, \qquad L^*\zeta_0^* = 0, \qquad \text{and} \qquad (\xi_0; \zeta_0^*) = 0.$$

Then the solvability condition of (VI.42) is

$$(v; \zeta_0^*) = 0. \tag{VI.43}$$

This follows from the identity $((1 - P_\perp)v; \zeta_0^*) = (v; \zeta_0^*) - (v; \hat{\xi}_0)(\xi_0; \zeta_0^*)$ and the fact that $(\xi_0; \zeta_0^*) = 0$.

Lemma 2. $\forall v \perp \xi_0$, $L^*v = \mathcal{L}^*v$.

Proof. Let u and v be orthogonal to ξ_0. We have

$$(u; L^*v) = (Lu; v) = (\mathcal{L}u; v) - (\mathcal{L}u; \hat{\xi}_0)(\xi_0; v).$$

Since $(\xi_0; v) = 0$, it follows that $(u; L^*v) = (\mathcal{L}u; v) = (u; \mathcal{L}^*v)$, from which we deduce that

$$\mathcal{L}^*v = L^*v + \alpha\xi_0,$$

but $(\xi_0; \mathcal{L}^*v) = 0$, hence $\alpha = 0$.

It follows in particular that ζ_0^* satisfies $\mathcal{L}^*\zeta_0^* = 0$. Now the equations (VI.41b), (VI.41c) have solutions Φ_{10} and Φ_{02} orthogonal to ξ_0 and ζ_0^* and unique iff

$$a_{10} = (\mathcal{F}_{10}; \zeta_0^*) \qquad \text{and} \qquad a_{02} = (\mathcal{F}_{02}[\zeta_0, \zeta_0]; \zeta_0^*). \tag{VI.44}$$

If U_0 and the vectors ζ_0, ζ_0^* were known (numerically, for example, by means of a path following method), (VI.44) would allow a numerical computation of these coefficients of (VI.38).

VI.2.1.2 Breaking reflectional symmetry creates a traveling wave

We now assume that ζ_0 only breaks the reflectional symmetry, i.e.,

$$\mathbf{T}_{2\pi/n}\zeta_0 = \mathbf{R}_\varphi\zeta_0 = \zeta_0 \qquad \text{and} \qquad \mathbf{S}\zeta_0 = -\zeta_0. \tag{VI.45}$$

Lemma 3. *If $\mathbf{S}\zeta_0 = -\zeta_0$, then generically ζ_0 can be chosen so that $\mathcal{L}\zeta_0 = \xi_0$, realizing a two-dimensional Jordan block. Moreover, a suitable rescaling of the amplitude A allows us to set $h(\mu, A) \equiv A$ in the phase equation.*

Proof. From the definition of L, we have $\mathcal{L}\zeta_0 = (\mathcal{L}\zeta_0; \hat{\xi}_0)\xi_0$. But $\mathcal{L}\zeta_0$ and $\hat{\xi}_0$ are both antisymmetric, so their scalar product need not vanish (in general); hence, the first part of the lemma follows. Now, by setting $W = A\zeta_0 + \Phi(\mu, A)$, system (VI.18)–(VI.19) becomes

$$\left\{ \begin{array}{rcl} \dfrac{d\psi}{dt} & = & h(\mu, A), \\ \dfrac{dA}{dt} & = & \tilde{f}(\mu, A). \end{array} \right. \tag{VI.46}$$

Now the action of $\mathbf{S}$ is such that $(\psi, A) \longrightarrow (-\psi, -A)$; hence,

$$\Phi(\mu, -A) = \mathbf{S}\Phi(\mu, A),$$

and the equivariance property of (VI.46) leads to h and $\tilde{f}$ odd in A. Now, (VI.14) shows that $h(\mu, A) = A + o(|A|)$, and it follows by setting

$$A' = h(\mu, A)$$

that we obtain a new system of the form (suppressing the primes)

$$\begin{cases} \dfrac{d\psi}{dt} &= A, \\ \dfrac{dA}{dt} &= f(\mu, A). \end{cases} \tag{VI.47}$$

where f is still odd in A. The bifurcation in this case is of pitchfork type. This means that the bifurcated branches of (VI.47) are one-sided and are exchanged by the transformation $\mathbf{S}$. Moreover, there is an exchange of stability between the basic solution $W = 0$ and the bifurcated solutions W, which now take the form

$$W = A\zeta_0 + \Phi(\mu(A), A).$$

The bifurcated flow $U(t)$ is a *traveling wave*, with a frequency $\mathrm{O}(|A|)$. It is *axisymmetric* but no longer confined into flat horizontal cells since the symmetry $\mathbf{S}$ is broken. More precisely, this solution takes the form

$$U(t) = \mathbf{T}_{At}[U_0 + A\zeta_0 + \Phi(\mu(A), A)],$$

and

$$\mathbf{S}U(t) = \mathbf{T}_{-At}[U_0 - A\zeta_0 + \Phi(\mu(A), -A)]$$

represents the symmetric traveling wave regime (μ is an even function of A).

In order to compute the coefficients in the Taylor expansion of f, we proceed as in Subsection VI.2.1.1. Applying (VI.39) and (VI.40) in equation (VI.17), we get the following linear relations:

$$\text{order } A : \xi_0 = \mathcal{L}\zeta_0, \tag{VI.48a}$$

$$\text{order } \mu : 0 = \mathcal{L}\Phi_{10} + \mathcal{F}_{10}, \tag{VI.48b}$$

$$\text{order } A^2 : \mathrm{t}\zeta_0 = \mathcal{L}\Phi_{02}, + \mathcal{F}_{02}[\zeta_0, \zeta_0], \tag{VI.48c}$$

$$\text{order } \mu A : \mathrm{t}\Phi_{10} + a_{11}\zeta_0 = \mathcal{L}\Phi_{11} + \mathcal{F}_{11}[\zeta_0] + 2\mathcal{F}_{02}[\zeta_0, \Phi_{10}], \tag{VI.48d}$$

$$\text{order } A^3 : \mathrm{t}\Phi_{02} + a_{03}\zeta_0 = \mathcal{L}\Phi_{03} + \mathcal{F}_{03}[\zeta_0^{(3)}] + 2\mathcal{F}_{02}[\zeta_0, \Phi_{02}]. \tag{VI.48e}$$

Thanks to Lemma 2 we can find ζ_0^* such that

$$(\zeta_0; \zeta_0^*) = 1, \quad \mathcal{L}^*\zeta_0^* = 0 \qquad \text{and} \qquad (\xi_0; \zeta_0^*) = 0, \quad \mathbf{S}\zeta_0^* = -\zeta_0^*.$$

The method shown in Subsection VI.2.1.1 applies here too. The equations (VI.48b, c) have a unique solution because $(\mathcal{F}_{p0};\zeta_0^*) = (\mathcal{F}_{p0};\hat{\xi}_0) = 0$, thanks to the fact that $\mathcal{F}_{p0}$ and $\mathcal{F}_{02}[\zeta_0,\zeta_0]$ are $\mathbf{S}$ invariant. Observe that $\mathrm{t}\zeta_0, \Phi_{10}$, and Φ_{02} are also invariant by $\mathbf{S}$. The conditions of solvability for (VI.48d, e) lead to the determination of a_{11} and a_{03}:

$$\begin{cases} a_{11} &= (\mathcal{F}_{11}[\zeta_0] + 2\mathcal{F}_{02}[\zeta_0,\Phi_{10}] - \mathrm{t}\Phi_{10};\zeta_0^*), \\ a_{03} &= (\mathcal{F}_{03}[\zeta_0^{(3)}] + 2\mathcal{F}_{02}[\zeta_0,\Phi_{02}] - \mathrm{t}\Phi_{02};\zeta_0^*). \end{cases} \tag{VI.49}$$

Observe that if we write equations (VI.48) in the form $\mathcal{L}u = v$, then in (VI.48d, e) v is antisymmetric: $\mathbf{S}v = -v$. In this case Φ_{11} and Φ_{03} have a nonvanishing component on ζ_0, determined by the projection of (VI.48d, e) on ξ_0. This results from the observation that by writing $u = \bar{u} + \alpha\zeta_0$ with $\bar{u} \perp \zeta_0^*$, we get from (VI.48a): $\alpha = (v;\hat{\xi}_0) - (\mathcal{L}\bar{u};\hat{\xi}_0)$, where $\bar{u}$ is determined in $[\hat{\xi}_0,\zeta_0^*]^\perp$ by $L\bar{u} = (1-P_\perp)v$.

VI.2.1.3 Doubling the axial wave length

Let us now assume the following:

$$\mathbf{T}_{2\pi/n}\zeta_0 = -\zeta_0, \qquad \mathbf{S}\zeta_0 = \zeta_0 \quad \text{and} \quad \mathbf{R}_\varphi\zeta_0 = \zeta_0. \tag{VI.50}$$

In this case Lemma 1 applies, hence $d\psi/dt = 0$. However, the function f is *odd* with respect to A, because now ζ_0 is antisymmetric for the symmetry $\mathbf{T}_{2\pi/n}$ and the system in (ψ, A) has to commute with the action $(\psi, A) \to (\psi, -A)$, which now represents $\mathbf{T}_{2\pi/n}$. Therefore, the bifurcation is of pitchfork type. The amplitude equation takes the form

$$\frac{dA}{dt} = f(\mu, A) = A(a_{11}\mu + a_{03}A^2 + \text{ h.o.t.}), \tag{VI.51}$$

and the classical stability argument applies.

Moreover, the structure of the flow undergoes a period doubling in the axial direction, since $\mathbf{T}_{4\pi/n}U = U$ and $\mathbf{T}_{2\pi/n}U \neq U$, but the $\mathbf{S}$ invariance implies that horizontal cells (of length h/n) persist for the bifurcated flows. If we take the convention that U_0 is the Taylor vortex flow with $z = 0$ as an outflow boundary (see Figure III.3(b)), then the outflow boundaries of these solutions stay flat, while the inflow boundaries ($z = h/2n$, for instance) are destroyed. We leave the calculation of the coefficients in the Taylor expansion of $f(\mu, A)$ as an exercise for the reader (same calculation as in the previous cases).

By the same argument as in Lemma 1, one has $\mathcal{L}\zeta_0 = 0$. The Fredholm alternative exposed in Subsection VI.2.1.1 applies and enables one to get the coefficients in the Taylor expansion (VI.51) of $f(\mu, A)$ (notice that the coefficients such as computed in (VI.44) are 0 because of the $\mathbf{T}_{2\pi/n}$-action):

$$a_{11} = (\mathcal{F}_{11}[\zeta_0] + 2\mathcal{F}_{02}[\zeta_0,\Phi_{10}];\zeta_0^*), \tag{VI.52}$$

$$a_{03} = (\mathcal{F}_{03}[\zeta_0^{(3)}] + 2\mathcal{F}_{02}[\zeta_0,\Phi_{02}];\zeta_0^*) \tag{VI.53}$$

We observe that we again obtain a steady axisymmetric bifurcating solution. This differs from the case with Taylor vortices in that one boundary over two of the cells is destroyed (particules of fluid can cross these planes in the new flow), but since the perturbation of the Taylor flow is small close to the bifurcation, these "boundaries" will still be physically visible like a blurred separation.

VI.2.1.4 Doubling the axial wave length and breaking reflectional symmetry

Let us now assume the following:

$$\mathbf{T}_{2\pi/n}\zeta_0 = -\zeta_0, \qquad \mathbf{S}\zeta_0 = -\zeta_0 \quad \text{and} \quad \mathbf{R}_\varphi\zeta_0 = \zeta_0. \tag{VI.54}$$

In this case again $d\psi/dt = 0$, because the first relation in (VI.54) and the equivariance properties imply that $h(\mu, A)$ must be even with respect to A, while the second relation in (VI.54) implies that it must be odd. From the same relations we deduce that $f(\mu, A)$ is odd with respect to A, hence, the bifurcation is a pitchfork. As above, the bifurcated solutions correspond to a flow with doubled axial periodicity. Moreover, if one takes the same convention on the TVF as in Subsection VI.2.1.3, then the inflow boundaries stay flat while the outflow boundaries are destroyed. Indeed, the flow is now invariant under the transformation $\mathbf{ST}_{2\pi/n}$ (since this property propagates first from $X = A\zeta_0$ to $W = X + \Phi(\mu, X)$ and then to U), and it is easy to verify that in this case the vertical component of the velocity still vanishes at $z = \pi/\alpha$.

Exercise left to the reader. Show that $\mathbf{ST}_{2\pi/n}U = U$ implies $\mathbf{ST}_{\pi/n}U = \mathbf{T}_{\pi/n}U$; then show that $u_z = 0$ at $z = \pm\pi/\alpha$.

In this case, $\mathcal{L}\zeta_0 = 0$. The calculation of the coefficients in the Taylor expansion of $f(\mu, A)$ (see (VI.39)) gives exactly the same expression as (VI.52)–(VI.53). Physically, the bifurcating flow looks like that of the previous section. However, it is really different since flat boundaries now have an inward flow instead of outward.

VI.2.2 Hopf bifurcation from Taylor vortices

Let us now assume that there exists ζ such that

$$L\zeta = \mathrm{i}\omega\zeta, \qquad \omega \neq 0. \tag{VI.55}$$

Now, according to the experimental reports, we assume that the rotational invariance of the Taylor vortex flow is broken, which generically means that there is an integer m such that

$$\zeta = \hat{\zeta}(r, z)e^{\mathrm{i}m\theta}, \qquad m \neq 0. \tag{VI.56}$$

However, as in the stationary case, we suppose that either the D_n invariance is not broken, or it breaks through the reflectional and/or axial period doubling symmetries. Hence the eigenvalue $\mathrm{i}\omega$ is simple and the eigenvector satisfies relations like

$$\mathbf{T}_{2\pi/n}\zeta = \pm\zeta, \qquad \mathbf{S}\zeta = \pm\zeta. \tag{VI.57}$$

Note that (VI.35), i.e., $\mathbf{S}\xi_0 = -\xi_0$, is still valid. We set the following expression for the center manifold:

$$W = A\zeta + \bar{A}\bar{\zeta} + \Phi(\mu, A, \bar{A}). \tag{VI.58}$$

The following lemma is a straightforward consequence of relations (VI.57) and of the equivariance of the right-hand side $h(\mu, .)$ of the phase equation under the following actions of $\mathbf{R}_\varphi$ and $\mathbf{S}$:

$$\begin{aligned} \mathbf{S} :\ & (\psi, A, \bar{A}) \ \rightarrow (-\psi, \pm A, \pm\bar{A}) \\ \mathbf{R}_\varphi :\ & (\psi, A, \bar{A}) \ \rightarrow (\psi, Ae^{m\mathrm{i}\varphi}, \bar{A}e^{-m\mathrm{i}\varphi}). \end{aligned}$$

Lemma 4. *If the eigenvector ζ transforms like (VI.57), then $h(\mu, X) \equiv 0$, i.e., the phase ψ is constant in (VI.18).*

Let us now study the bifurcation equation (VI.19). Thanks to (VI.56), (VI.19) reduces to

$$\frac{dA}{dt} = Af(\mu, |A|), \tag{VI.59}$$

where f is even in $|A|$ and

$$f(\mu, |A|) = \mathrm{i}\omega + a_{11}\mu + a_{03}|A|^2 + \text{ h.o.t.}. \tag{VI.60}$$

Note that if $m = 0$ in (VI.56), the action of $\mathbf{R}_\varphi$ is trivial. But we recover up to an arbitrary order an equation of the same form as (VI.59) by using normalization (see Section II.4). Here the normalized system (VI.60) commutes with

$$(A, \bar{A}) \rightarrow (e^{\mathrm{i}\omega t}A, e^{-\mathrm{i}\omega t}\bar{A}).$$

If $m \neq 0$, from (VI.59) we deduce that the bifurcated solutions have the form

$$A(t) = \rho e^{\mathrm{i}(\Omega t + \varphi)}, \tag{VI.61}$$

with $\rho = \sqrt{-\mathrm{Re}(a_{03})\mu/\mathrm{Re}(a_{11})} + \mathrm{O}(|\mu|^{3/2})$, $\Omega = \omega + \mathrm{O}(|\mu|)$ and φ an arbitrary phase. These solutions are *rotating waves*, since we can write $X(t) = \mathbf{R}_{(\Omega t+\varphi)/m}X_0$. Depending on which of relations (VI.52) is true, the symmetry type of these solutions is different. In all cases, the classical stability argument for Hopf bifurcation holds.

Case 1. $\mathbf{T}_{2\pi/n}\zeta = \zeta$, $\mathbf{S}\zeta = \zeta$. In this case, the flow is still confined in flat horizontal cells of length $h/2n$ (as for the Taylor vortex flow) but with a wavy azimuthal structure inside each cell, which is invariant by the reflection $\mathbf{S}$. This is precisely the structure of the *twisted vortices* already described in Subsection IV.1.3. (see Figure IV.4).

Case 2. $\mathbf{T}_{2\pi/n}\zeta = \zeta$, $\mathbf{S}\zeta = -\zeta$. Now the flow is no longer confined into horizontal cells, but the azimuthal periodicity corresponds to the uniform undulation observed in the *wavy vortices* (see Subsection IV.1.3, Figure IV.3). Notice that the characterization of the symmetries of this flow is such that in addition to the $2\pi/n$ axial periodicity and $2\pi/m$ azimuthal periodicity, it is *invariant* under the transformation $\mathbf{R}_{\pi/m}\mathbf{S}$.

Case 3. $\mathbf{T}_{2\pi/n}\zeta = -\zeta$, $\mathbf{S}\zeta = \zeta$. In this case the flow is confined into flat horizontal cells of height h/n (axial period doubling), as seen in Subsection VI.2.1.3. If we take the convention that U_0 is the TVF with $z = 0$ as an outflow boundary, then the *outflow boundaries of these solutions stay flat* while the inflow boundaries become wavy (see Figure VI.2). The bifurcation of flows with such a structure was described by [An-L-Sw] as the WIB (*wavy inflow boundaries regime*), see the diagram in Figure VI.1.

Case 4. $\mathbf{T}_{2\pi/n}\zeta = -\zeta$, $\mathbf{S}\zeta = -\zeta$. Under the convention on the TVF defined in Case 3, the *inflow boundaries now stay flat* while the outflow boundaries become wavy as seen in Subsection VI.2.1.4 (see Figure VI.3). This regime was described by [An-L-Sw] as the WOB (*wavy outflow boundaries regime*), see the diagram in Figure VI.1. It seems that the first time such a flow was referred to is in a paper by Snyder in 1970 [Sn], who named this flow the "$\theta - T$ mode." He observed this flow for $\eta = 0.2$.

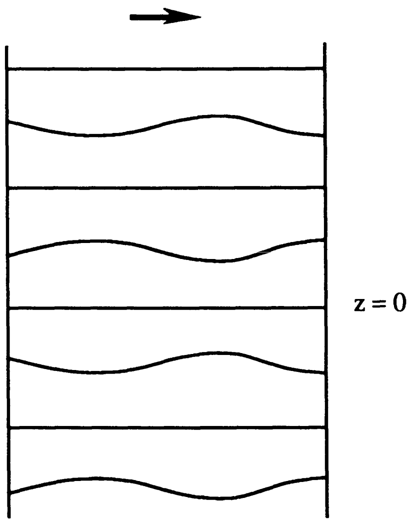


FIGURE VI.2. The WIB.

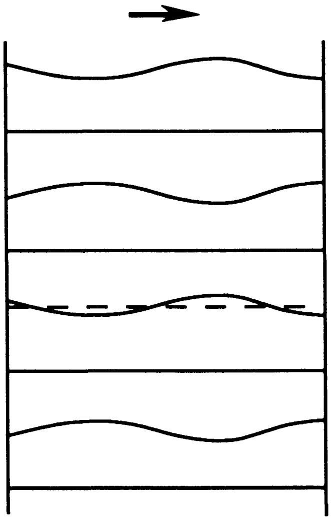


FIGURE VI.3. The WOB.

In order to calculate the coefficients in (VI.60), let us define the following expansions:

$$f(\mu, A) = \sum_{n,p,q} a_{npq}\mu^n A^p \bar{A}^q \quad \text{and} \quad \Phi(\mu, A) = \sum_{n,p,q} \mu^n A^p \bar{A}^q \Phi_{npq}, \tag{VI.62}$$

where A is defined in (VI.58). Replacing this expansion in (VI.12) and identifying the terms of same power in each argument, we get

$$\text{order } A : \mathrm{i}\omega\zeta = \mathfrak{L}\zeta, \tag{VI.63a}$$
$$\text{order } \mu : 0 = \mathfrak{L}\Phi_{100} + \mathfrak{F}_{10}, \tag{VI.63b}$$
$$\text{order } A^2 : 0 = (-2\mathrm{i}\omega + \mathfrak{L})\Phi_{020} + \mathfrak{F}_{02}[\zeta, \zeta], \tag{VI.63c}$$
$$\text{order } A\bar{A} : 0 = \mathfrak{L}\Phi_{011} + 2\mathfrak{F}_{02}[\zeta, \bar{\zeta}], \tag{VI.63d}$$
$$\text{order } \mu A : a_{11}\zeta = (-\mathrm{i}\omega + \mathfrak{L})\Phi_{110} + \mathfrak{F}_{11}[\zeta] + 2\mathfrak{F}_{02}[\zeta, \Phi_{100}], \tag{VI.63e}$$
$$\begin{aligned}\text{order } A^2\bar{A} : a_{03}\zeta = {} & (-\mathrm{i}\omega + \mathfrak{L})\Phi_{021} + 3\mathfrak{F}_{03}[\zeta, \zeta, \bar{\zeta}] \\ & + 2\mathfrak{F}_{02}[\bar{\zeta}, \Phi_{020}] + 2\mathfrak{F}_{02}[\zeta, \Phi_{011}].\end{aligned} \tag{VI.63f}$$

Equation (VI.63c) has a unique solution Φ_{020}, since $2\mathrm{i}\omega - \mathfrak{L}$ is invertible. Equations (VI.63b, d) have unique solutions Φ_{100}, Φ_{011} invariant under **S**, since their right-hand sides are symmetric in all cases [see the resolution of (VI.42)]. In order to apply the Fredholm alternative to (VI.63e, f), notice that there exists ζ^* such that $L^*\zeta^* = -\mathrm{i}\omega\zeta^*$, $(\zeta; \zeta^*) = 1$, and $(\xi_0; \zeta^*) = 0$. In particular, we have $((1-P_\perp)v;\zeta^*) = (v;\zeta^*)$. Then the solvability condition for these two equations leads to the expressions

$$a_{11} = (\mathfrak{F}_{11}[\zeta] + 2\mathfrak{F}_{02}[\zeta,\Phi_{100}];\zeta^*), \tag{VI.64}$$
$$a_{03} = (3\mathfrak{F}_{03}[\zeta,\zeta,\bar{\zeta}] + 2\mathfrak{F}_{02}[\bar{\zeta},\Phi_{020}] + 2\mathfrak{F}_{02}[\zeta,\Phi_{011}];\zeta^*), \tag{VI.65}$$

and Φ_{110} and Φ_{021} are uniquely determined being orthogonal to ζ^*.

VI.3 Bifurcation from the spirals

We have seen in Subsection VI.1.1 that the solution in spirals U_0 is invariant under the transformations $\mathbf{R}_{n\varphi}\mathbf{T}_{-m\varphi}$, and therefore the system (VI.18)–(VI.19) possesses the same symmetry. Let us define the corresponding representation of the rotation group SO(2) by

$$\tilde{\mathbf{T}}_\varphi = \mathbf{R}_{n\varphi}\mathbf{T}_{-m\varphi}, \qquad 0 \le \varphi < 2\pi. \tag{VI.66}$$

We now set the field $\mathfrak{F}$ occurring in (VI.21):

$$\mathfrak{F}(\mu, W) = L_\mu W + [N(\mu, U_0 + W) - N(\mu, U_0)] - \Omega \mathrm{t} W,$$

and define the linear operators

$$\left\{\begin{array}{rcl} \mathfrak{L}(\mu) & = & D_W\mathfrak{F}(\mu,0), \qquad \mathfrak{L}(0) = \mathfrak{L}, \\ L & = & (1-P_\perp)\mathfrak{L}. \end{array}\right. \tag{VI.67}$$

Assume that L has a pair of pure imaginary simple eigenvalues $\pm i\eta_0$ (we shall not consider here the steady-state bifurcation). Let ζ and $\bar{\zeta}$ denote the corresponding eigenvectors. The invariance by $\tilde{\mathbf{T}}_\varphi$ of U_0 leads to $\tilde{\mathbf{T}}_\varphi L = L\tilde{\mathbf{T}}_\varphi$ and $\tilde{\mathbf{T}}_\varphi\mathfrak{L} = \mathfrak{L}\tilde{\mathbf{T}}_\varphi$. This implies that one should have

$$\tilde{\mathbf{T}}_\varphi\zeta = e^{ki\varphi}\zeta \quad \text{for some integer } k. \tag{VI.68}$$

In what follows we assume $k \neq 0$. The center manifold now reads

$$W = A\zeta + \bar{A}\bar{\zeta} + \Phi(\mu, A, \bar{A}), \tag{VI.69}$$

and the system (VI.18)–(VI.19), which is equivariant under $\tilde{\mathbf{T}}_\varphi$, commutes with the action

$$(\psi, A) \to (\psi, Ae^{ki\varphi}).$$

It results that this system takes the form

$$\frac{d\psi}{dt} = h(\mu, A, \bar{A}) = h_1(\mu, |A|), \tag{VI.70}$$
$$\frac{dA}{dt} = Af(\mu, |A|), \tag{VI.71}$$

where h_1 and f are even in $|A|$. Let us define more precisely

$$f(\mu, |A|) = \mathrm{i}\eta_0 + a_{11}\mu + a_{03}|A|^2 + \text{ h.o.t.}. \tag{VI.72}$$

The solutions of (VI.70)–(VI.71) have the form

$$\psi = h_1(\mu, \rho)t + \psi_0, \tag{VI.73}$$

$$A(t) = \rho e^{\mathrm{i}(\eta t + \varphi)}, \tag{VI.74}$$

with $\rho = \sqrt{-\mathrm{Re}(a_{03})\mu/\mathrm{Re}(a_{11})} + \mathrm{O}(|\mu|^{3/2})$, $\eta = \eta_0 + \mathrm{O}(|\mu|)$, and ψ_0 an arbitrary phase. Moreover, W is a rotating wave along $\tilde{\mathbf{T}}_\varphi$, since time shift is the same as the action $\tilde{\mathbf{T}}_\varphi$ with a suitable angle, and U is a quasi-periodic flow with two independent frequencies: the renormalized frequency of the basic spiral flow $\Omega' = \Omega + h_1(\mu, \rho)$ and η. The corresponding solution (VI.20) of (II.23) can now be rewritten

$$U(t) = \mathbf{T}_{\Omega' t + \varphi_1}\tilde{\mathbf{T}}_{\eta t/k + \varphi_2}(U_0 + W_0), \tag{VI.75}$$

where $W_0 = \rho(\zeta + \bar{\zeta}) + \Phi(\mu, \rho, \rho)$ and φ_j, $j = 1, 2$, are arbitrary phases.

This can be interpreted as a uniform modulation superimposed to the spirals (see Fig. VI.4), and we call these solutions "wavy spirals."

Remark. In Chapter IV, several types of quasi-periodic solutions bifurcating from the spirals have been found in the local analysis of azimuthal mode interactions (Subsection IV.1.4 in the case of $m = 0$, $m = 1$ mode interaction, Subsection IV.2.3 in the case of m, $m + 1$ mode interaction with $m \neq 0$). Do they fit in the foregoing analysis? We must check that the eigenvectors corresponding to the critical eigenvalues at the bifurcation from the branch of spirals satisfy relation (VI.68) with some nonzero integer k. Let us first examine the case of $m = 0$, $m = 1$ interaction. It was shown in the stability analysis of the spiral flow (Subsection IV.1.4) that a pair of complex eigenvalues cross the imaginary axis for the same linear operator $\mathcal{L}(\mu)$ (denoted by $L_{\mu,\nu}$), leading to a Hopf bifurcation from the spirals (first eigenvalues listed in (IV.26)). The corresponding eigendirections have components in coordinates A_0 and $\bar{A}_0$. Since $\mathbf{T}_\psi A_0 = e^{ni\psi}A_0$ and $\mathbf{R}_\varphi A_0 = A_0$ (see relations (IV.1)), A_0 is transformed by $\tilde{\mathbf{T}}_\varphi$ to $e^{-ni\varphi}A_0$, as it results from the definition (VI.66) of $\tilde{\mathbf{T}}_\varphi$, where we set $m = 1$ in this case. Hence in this case, we have $k = n$. A similar reasoning, left to the reader as an exercise, shows that the interpenetrating spirals $\mathrm{SI}^{(0,3)}$ analyzed in Section IV.2 correspond to a Hopf bifurcation from the m-spirals with $k = (2m + 1)n$ while the interpenetrating spirals $\mathrm{SI}^{(0,2)}$ correspond to a Hopf bifurcation from the m-spirals with $k = mn$. Note that the interpenetrating spirals are quasi-periodic with two frequencies as expected. Notice also that this flow looks physically different from the interpenetrating spirals shown in Figure IV.4, because we consider in general this regime with nearly equal influence of the two modes, instead of looking close to the bifurcation from one mode (as in Fig. VI.4).

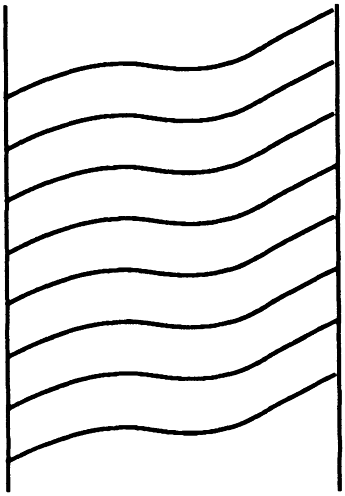


FIGURE VI.4. Sketch of the wavy spirals.

VI.4 Bifurcation from ribbons

For a family of ribbons of the form (VI.6) we define

$$\mathcal{F}(\mu, W) = L_\mu W + [N(\mu, U_0 + W) - N(\mu, U_0)] - \Omega \mathfrak{r} W, \tag{VI.76}$$

and

$$\begin{cases} \mathcal{L}(\mu) = D_W \mathcal{F}(\mu, 0), & \mathcal{L}(0) = \mathcal{L}, \\ L = (1 - P_\perp)\mathcal{L}. \end{cases} \tag{VI.77}$$

Notice that $\mathcal{F}$ and L commute with the symmetry group of U_0, i.e., with the operators $\mathbf{S}$ and $\mathbf{R}_{\pi/m}\mathbf{T}_{\pi/n}$ (accordingly to (VI.7)) and $\mathbf{R}_{2\pi/m}$, $\mathbf{T}_{2\pi/n}$. This symmetry group is generated by $\mathbf{R}_{\pi/m}\mathbf{T}_{\pi/n}$, $\mathbf{T}_{2\pi/n}$, and $\mathbf{S}$. We could also replace $\mathbf{T}_{2\pi/n}$ by $\mathbf{R}_{2\pi/m}$, thanks to the relation

$$\mathbf{T}_{2\pi/n} = (\mathbf{R}_{\pi/m}\mathbf{T}_{\pi/n})^2 \mathbf{R}_{-2\pi/m}.$$

In the following we shall refer to $\mathbf{R}_{\pi/m}\mathbf{T}_{\pi/n}$ as the *twist symmetry*, since it consists in performing successively a translation by half a period and a rotation by half a period. The condition for bifurcation from the ribbons is that either L has a 0 eigenvalue or a pair $\pm i\eta_0$ of pure imaginary eigenvalues. Both situations are equally likely to occur, and we consider them successively in this section. We recall that the kernel of $\mathcal{L}$ always contains two vectors tangent to the group orbit (which for the ribbons is a 2-torus [see (VI.6)]:

$$\xi_0 = \mathfrak{t} U_0 \quad \text{and} \quad \chi_0 = \mathfrak{r} U_0. \tag{VI.78}$$

Hence ξ_0 is tangent to the orbit of axial translations and χ_0 is tangent to the orbit along azimuthal rotations. Moreover, formula (VI.35) holds again, i.e.,

$$\mathbf{S}\xi_0 = -\xi_0, \tag{VI.79}$$

while χ_0 is $\mathbf{S}$-invariant.

VI.4.1 The stationary case

We assume the existence of ζ_0 in $(1 - P_\perp)\mathfrak{D}_h$ such that

$$L\zeta_0 = 0. \tag{VI.80}$$

As for the bifurcation from Taylor vortex flows, several cases are to be envisaged, depending on which symmetry is broken by ζ_0. We shall assume in this chapter that

$$\mathbf{K}\zeta_0 = \pm\zeta_0, \tag{VI.81}$$

where $\mathbf{K}$ is any of the generators of the symmetry group of U_0. This makes eight cases to study, but we reduce this number to four, thanks to the following lemma.

Lemma 5. *The case $\mathbf{T}_{2\pi/n}\zeta_0 = -\zeta_0$ cannot occur in the set of transformations (VI.81).*

Proof. Let us set $\mathbf{R}_{\pi/m}\mathbf{T}_{\pi/n}\zeta_0 = \varepsilon_1\zeta_0$, $\mathbf{T}_{2\pi/n}\zeta_0 = \varepsilon_2\zeta_0$, and $\mathbf{S}\zeta_0 = \varepsilon_3\zeta_0$. By multiplying the first of these equalities by $\mathbf{T}_{\pi/n}$, we get

$$\varepsilon_2\mathbf{R}_{\pi/m}\zeta_0 = \varepsilon_1\mathbf{T}_{\pi/n}\zeta_0.$$

By applying $\mathbf{S}$ to both sides of the latter identity we get

$$\varepsilon_2\varepsilon_3\mathbf{R}_{\pi/m}\zeta_0 = \varepsilon_1\varepsilon_3\mathbf{T}_{-\pi/n}\zeta_0$$

and, hence,

$$\mathbf{T}_{\pi/n}\zeta_0 = \mathbf{T}_{-\pi/n}\zeta_0,$$

which implies $\varepsilon_2 = 1$.

We shall not describe in this section the rules that facilitate the computation of the bifurcated branches of solutions in terms of the operators appearing in (VI.24). Such rules closely follow those explained in Subsection VI.2.1 for the bifurcation from the Taylor vortex flow, and we leave them as exercises for the interested reader.

Finally, notice that since the operator L has a simple zero eigenvalue, the classical "exchange of stability" between the basic flow (ribbons) and the bifurcated ones applies: if the bifurcation is supercritical the bifurcated solution is stable.

VI.4.1.1 No symmetry breaking

Let us assume that

$$\mathbf{R}_{\pi/m}\mathbf{T}_{\pi/n}\zeta_0 = \mathbf{T}_{2\pi/n}\zeta_0 = \mathbf{S}\zeta_0 = \zeta_0. \tag{VI.82}$$

Lemma 6. *If* $\mathbf{S}\zeta_0 = \zeta_0$, $d\psi/dt = 0$.

The proof is identical to that of Lemma 1 in Subsection VI.2.1.1.

Let us now set the center manifold under the form

$$W = A\zeta_0 + \Phi(\mu, A). \tag{VI.83}$$

The system (VI.29)–(VI.31) can now be rewritten as

$$\begin{cases} \dfrac{d\psi}{dt} &= 0, \\ \dfrac{d\varphi}{dt} &= k(\mu, A), \\ \dfrac{dA}{dt} &= f(\mu, A). \end{cases} \tag{VI.84}$$

The function f is the component of F along ζ_0. We already know that $f(0,0) = 0$ and $\partial f(0,0)/\partial A = 0$, and in general $\partial f(0,0)/\partial\mu \neq 0$, so that (0, 0) is in fact a turning point on the branch of ribbons. The function k is nonzero in general, but it only has the action of rescaling the frequency of the solutions along the "turned-back" branch of ribbons (viewed as rotating waves). The coefficients in the Taylor expansion of f and k may be computed by the same rule as that described in Subsection VI.2.1.1. We leave this computation to the reader.

VI.4.1.2 Breaking the twist symmetry

We now assume that

$$\mathbf{R}_{\pi/m}\mathbf{T}_{\pi/n}\zeta_0 = -\zeta_0 \quad \text{and} \quad \mathbf{T}_{2\pi/n}\zeta_0 = \mathbf{S}\zeta_0 = \zeta_0. \tag{VI.85}$$

Lemma 6 applies in this case, hence, the system (VI.29)–(VI.31) can again be rewritten as (VI.84). However, thanks to the first relation in (VI.85), the function $f(\mu, A)$ is now odd in A, while $k(\mu, A)$ is even in A (indeed, $\mathbf{R}_{\pi/m}\mathbf{T}_{\pi/n}$ acts trivially on φ). The bifurcation is therefore of *pitchfork* type (one-sided with branches exchanged by the twist symmetry) and the bifurcated solutions are *rotating waves*, since they take the form

$$U(t) = \mathbf{R}_{(\Omega+\Omega')t+\varphi}[U_0 + W], \tag{VI.86}$$

where $\Omega' = k(\mu, A)$ and $W = A\zeta_0 + \mathrm{O}(|A|^2)$. Moreover, the classical stability analysis for pitchfork bifurcation holds.

Notice that the symmetry of this flow is identical to that of the flow described in Subsection VI.2.2 as the twisted vortices. Since the $\mathbf{S}$-symmetry is preserved, the flow corresponding to these solutions is confined into flat horizontal cells. Moreover, the twist symmetry of the ribbons being replaced by a twist "antisymmetry," the geometrical pattern realized by this flow, should look like the picture drawn in Figure VI.4. In fact, the flow described in Chapter IV, Subsection IV.1.3, as the twisted vortices, belongs to this case. In order to check this claim, one has to show that the linear perturbation that is responsible for the bifurcation of twisted vortices from the branch of ribbons satisfies relations (VI.85). It follows from the stability analysis of ribbons in Subsection IV.1.4 that this perturbation sits along the eigenvector $\zeta_0 + \bar{\zeta}_0$ (here ζ_0 is the vector defined in Chapter IV and must not be mistaken with ζ_0 defined in the present section). This vector obviously satisfies (VI.85) thanks to relations (VI.1).

VI.4.1.3 Breaking the reflectional symmetry (stationary bifurcation creating a traveling wave)

We now assume that

$$\mathbf{R}_{\pi/m}\mathbf{T}_{\pi/n}\zeta_0 = \mathbf{T}_{2\pi/n}\zeta_0 = \zeta_0 \quad \text{and} \quad \mathbf{S}\zeta_0 = -\zeta_0. \tag{VI.87}$$

Lemma 6 does not apply any longer, and the system of equations to be solved is

$$\begin{cases} \dfrac{d\psi}{dt} &= h(\mu, A), \\ \dfrac{d\varphi}{dt} &= k(\mu, A), \\ \dfrac{dA}{dt} &= f(\mu, A). \end{cases} \tag{VI.88}$$

Notice that k is an even function of A in (VI.88) while f and h are odd functions of A. This follows from the now familiar argument about the way the vectors ξ_0, χ_0, and ζ_0 are transformed by $\mathbf{S}$. As in Subsection VI.2.1.2, we can find Φ in the center manifold expressions such that $h(\mu, A) \equiv A$. Hence the bifurcation is of pitchfork type, the two bifurcated branches being exchanged by the symmetry $\mathbf{S}$. Moreover, we see that now there are two low-frequency drifts: A and $k(\mu, A)$. The drift φ modifies the velocity of rotation around the z-axis of the wave bifurcated from the ribbons, whereas the drift ψ induces a slow uniform translation of the flow along the z-axis. Therefore, the flow is now *quasi-periodic* and can be written in the form

$$U(t) = \mathbf{R}_{(\Omega+\Omega')t+\varphi}\mathbf{T}_{\Omega''t+\psi}[U_0 + W], \tag{VI.89}$$

where $\Omega' = k(\mu, A)$, $\Omega'' = A$, and $W = A\zeta_0 + O(|A|^2)$. Notice that the symmetric flow corresponds to $-A$, which gives the same Ω' but an opposite Ω''.

This flow is no longer confined into a horizontal cell, but it keeps the twist symmetry. Its geometrical pattern should therefore look like the picture drawn in Figure VI.5. We call this flow the *wavy ribbons*. The stability analysis, is once again as in classical pitchfork bifurcation.

VI.4.1.4 Breaking the reflectional and twist symmetries

Let us now assume that

$$\mathbf{T}_{2\pi/n}\zeta_0 = \zeta_0, \qquad \mathbf{R}_{\pi/m}\mathbf{T}_{\pi/n}\zeta_0 = -\zeta_0, \qquad \text{and} \qquad \mathbf{S}\zeta_0 = -\zeta_0. \tag{VI.90}$$

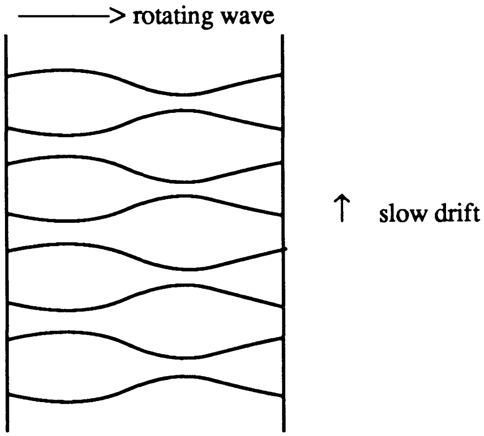


FIGURE VI.5. Sketch of the "wavy ribbons".

Lemma 7. *If* (VI.90) *hold, then* $d\psi/dt = 0$ (*no drift along the axis*).
Proof. Thanks to the second relation in (VI.90) and the fact that ψ is invariant by the twist symmetry, $h(\mu, A)$ must be even in A. On the other hand, $h(\mu, A)$ must be odd in A, thanks to the third relation in (VI.90) and the fact that $\mathbf{S}$ transforms ψ into $-\psi$. Hence $h \equiv 0$.

The system (VI.29)–(VI.31) can now be rewritten as (VI.84), where $k(\mu, A)$ is even in A and $f(\mu, A)$ is odd in A. Therefore, the bifurcation is of pitchfork type and the flow is time periodic with the structure of a rotating wave of the form (VI.86). The stability result is as in classical pitchfork bifurcations.

Since the $\mathbf{S}$ symmetry is broken, this regime should exhibit a wavy pattern. In addition, the twist symmetry is broken. Figure IV.3 (Subsection IV.1.3) shows an example of such a pattern, the so-called *wavy vortices*. In fact, these wavy vortices fall exactly in the present case. Indeed, the linear perturbation that is responsible for the bifurcation of wavy vortices from the branch of ribbons sits along $\zeta_0 - \bar{\zeta}_0$, where ζ_0 is now the eigenvector defined in Section IV.1. This is easily seen from the stability analysis of the ribbons in Subsection IV.1.4. This vector obviously satisfies (VI.90) thanks to relations (IV.1). Let us show that this flow has the symmetry characterizing the wavy vortices defined in Case 2 of Subsection VI.2.2: this flow is invariant under $\mathbf{R}_{\pi/m}\mathbf{S}$ (see also Subsection IV.1.3). This property is not directly verifiable on our formulation (VI.22). Let us consider instead the representation

$$U = \mathbf{R}_{\Omega t+\varphi}\mathbf{T}_{\psi}[U_1 + W],$$

where $U_1 = \mathbf{R}_{-\pi/2m}\mathbf{T}_{\pi/2n}U_0$ and W is now orthogonal to $\{\xi_1, \chi_1\}$ obtained from $\{\xi_0, \chi_0\}$ by acting $\mathbf{R}_{-\pi/2m}\mathbf{T}_{\pi/2n}$. The flow U_1 is still invariant under $\mathbf{R}_{\pi/m}\mathbf{T}_{\pi/n}$ but no longer invariant by $\mathbf{S}$. We have now $\mathbf{R}_{\pi/m}\mathbf{S}U_1 = U_1$. In addition, if we define $\zeta_1 = \mathbf{R}_{-\pi/2m}\mathbf{T}_{\pi/2n}\zeta_0$, then it is easy to show by using (VI.90) that ζ_1 *is also invariant under* $\mathbf{R}_{\pi/m}\mathbf{S}$. An exercise left to the reader shows that the amplitude equation for (A, φ, ψ) has the same form as (VI.84) with f odd and k even in A. Now the bifurcating rotating wave is invariant under the action of $\mathbf{R}_{\pi/m}\mathbf{S}$. This property, added to the axial periodicity and azimuthal wavelength m, shows that it is exactly the "wavy vortex flow."

VI.4.2 Hopf bifurcation from ribbons

Let us now assume that there exists ζ such that

$$L\zeta = i\eta_0\zeta, \qquad \eta_0 \neq 0. \tag{VI.91}$$

This can be associated with a variety of symmetry-breaking situations, among which we select only a few "typical" cases, notably cases including situations already encountered in Chapter IV.

Recall that the symmetry group we are considering in this problem is generated by $\mathbf{R}_{\pi/m}\mathbf{T}_{\pi/n}$ (which generates a subgroup isomorphic to $\mathbf{Z}_{2mn}$) and $\mathbf{T}_{2\pi/n}$ and $\mathbf{S}$ (which generate a subgroup of order $2n$ isomorphic to D_n, the symmetry group of the regular n-polygon). We shall only consider the breaking of the twist symmetry by a complex transformation, i.e.,

$$\mathbf{R}_{\pi/m}\mathbf{T}_{\pi/n}\zeta = e^{ki\pi/mn}\zeta, \qquad k \neq 0, mn. \tag{VI.92}$$

In addition, we assume that the eigenvector ζ keeps axial $2\pi/n$-periodicity and that it is either symmetric or antisymmetric by $\mathbf{S}$. Since ζ does not break the axial periodicity, (VI.92) results from a breaking of the azimuthal periodicity. An easy calculation shows that we can write

$$\mathbf{R}_{2\pi/m}\zeta = e^{2li\pi/m}\zeta, \qquad l = k/n. \tag{VI.93}$$

Then, in this case, k is a multiple of n.

A more complicated situation would be that of breaking the D_n symmetry by a two-dimensional transformation, i.e.,

$$\mathbf{T}_{2\pi/n}\zeta = e^{2ki\pi/n}\zeta, \quad k \neq 0, n, \qquad \text{and} \qquad \mathbf{S}\zeta = \zeta',$$

where $\zeta' \neq \zeta$. In this case the eigenvalues $\pm i\eta_0$ are double. The problem reduces then to the study of a Hopf bifurcation with D_n symmetry and eigenvectors transforming accordingly to the previous formulas. Several types of solution branches are possible. For a detailed study of this bifurcation, we refer the reader to [Go-St b] and [C-Go 88].

VI.4.2.1 Breaking the twist symmetry

Let us assume that (VI.92) holds, with

$$\mathbf{T}_{2\pi/n}\zeta = \mathbf{S}\zeta = \zeta. \tag{VI.94}$$

We set the center manifold under the form

$$W = A\zeta + \bar{A}\bar{\zeta} + \Phi(\mu, A, \bar{A}). \tag{VI.95}$$

Since we assume that ζ is $\mathbf{S}$ -invariant, $d\psi/dt = 0$ in equation (VI.29) (same proof as in Lemma 1). Hence the system (VI.29)–(VI.31) reduces to

$$\begin{cases} \dfrac{d\psi}{dt} &= 0, \\ \dfrac{d\varphi}{dt} &= k(\mu, A, \bar{A}), \\ \dfrac{dA}{dt} &= f(\mu, A, \bar{A}) \end{cases} \tag{VI.96}$$

where k is a real and f a complex function, satisfying the following relations that reflect the equivariance of the system by the action of $\mathbf{R}_{\pi/m}\mathbf{T}_{\pi/n}$:

$$\begin{aligned} k(\mu, e^{li\pi/m}A, e^{-li\pi/m}\bar{A}) &= k(\mu, A, \bar{A}), \\ f(\mu, e^{li\pi/m}A, e^{2li\pi/m}\bar{A}) &= e^{li\pi/m}f(\mu, A, \bar{A}). \end{aligned} \tag{VI.97}$$

By applying the normal form theorem (Section II.4) to the system, we can have more: We know from (II.44) that the normal form of (VI.29)–(VI.31) commutes with the action

$$A \to e^{i\eta t}A. \tag{VI.98}$$

Hence the normal form satisfies relations (VI.97) where the phase is taken to be *arbitrary*. It follows that up to higher order (as high as we wish), the system (VI.96) can be replaced by

$$\begin{cases} \dfrac{d\varphi}{dt} &= k_0(\mu, |A|^2), \\ \dfrac{dA}{dt} &= Af_0(\mu, |A|^2), \end{cases} \tag{VI.99}$$

where k_0 and f_0 are polynomial functions and

$$f_0(\mu, |A|^2) = i\eta_0 + a_{11}\mu + a_{03}|A|^2 + \text{ h.o.t.}. \tag{VI.100}$$

The system (VI.99) undergoes a bifurcation to solutions of the form

$$A(t) = \rho e^{i(\eta t + \kappa_0)}, \qquad \varphi(t) = \Omega' t + \varphi_0, \tag{VI.101}$$

with $\rho = \sqrt{-\mathrm{Re}(a_{03})/\mathrm{Re}(a_{11})\mu} + \mathrm{O}(|\mu|^{3/2})$, $\eta = \eta_0 + \mathrm{O}(|\mu|)$, $\Omega' = k_0(\mu, \rho^2)$, and κ_0, φ_0 are arbitrary phases. It can be shown that the limit cycle persists for the system (VI.96), with amplitude and frequency of the same leading part as in (VI.101). This comes from a classical argument using the Lyapunov-Schmidt method [Io 84].

The corresponding fluid flow is quasi-periodic with two independent frequencies η and $\Omega + \Omega'$, and can be written under the form

$$U(t) = \mathbf{R}_{(\Omega+\Omega')t+\varphi_0}[U_0 + W(t)], \tag{VI.102}$$

where W has the form (VI.95) and is periodic with frequency η. The stability result is as in usual Hopf bifurcations. The symmetry $\mathbf{S}$ is preserved; hence, the flow is confined into flat horizontal cells. Notice that the flow described in Subsection IV.2.3 as *superposed ribbons* $\mathbf{RS}^{(0)}$ fits exactly in this case. Indeed, the corresponding eigendirections responsible for this bifurcation are $\zeta_2 + \zeta_3$ and complex conjugate (notation of Section IV.2). Hence they are $\mathbf{S}$-invariant and satisfy

$$\mathbf{R}_\varphi \mathbf{T}_\psi(\zeta_2 + \zeta_3) = e^{\mathrm{i}(m+1)\varphi}(e^{in\psi}\zeta_2 + e^{-in\psi}\zeta_3).$$

By replacing φ by π/m and ψ by π/n, we see that in this case, $k = n$ in (VI.92).

VI.4.2.2 Breaking the twist and reflectional symmetries

Let us assume now that in addition to (VI.92), we have

$$\mathbf{T}_{2\pi/n}\zeta = \zeta \quad \text{and} \quad \mathbf{S}\zeta = -\zeta. \tag{VI.103}$$

Lemma 8. *Let us set $k/mn = p/q$ with p prime with q. If p is odd, and if (VI.92) and (VI.103) hold, we can set $d\psi/dt = 0$ in equation (VI.29).*
Proof. Thanks to the second relation in (VI.103), $h(\mu, -X) = -h(\mu, X)$. On the other hand,

$$\mathbf{R}_{\pi/m}\mathbf{T}_{\pi/n}\zeta = e^{p\mathrm{i}\pi/q}\zeta. \tag{VI.104}$$

If p is odd, it follows that $(\mathbf{R}_{\pi/m}\mathbf{T}_{\pi/n})^q X = -X$. Since h is an invariant function by $\mathbf{R}_{\pi/m}\mathbf{T}_{\pi/n}$, it follows that $h(\mu, -X) = h(\mu, X)$. Hence $h(\mu, X) = -h(\mu, X)$, which proves the lemma.

We now set for the center manifold:

$$W = A\zeta + \bar{A}\bar{\zeta} + \Phi(\mu, A, \bar{A}). \tag{VI.105}$$

The system (VI.29)–(VI.31) can be rewritten as

$$\begin{cases} \dfrac{d\psi}{dt} = h(\mu, A, \bar{A}) \\ \dfrac{d\varphi}{dt} = k(\mu, A, \bar{A}), \\ \dfrac{dA}{dt} = f(\mu, A, \bar{A}) \end{cases} \tag{VI.106}$$

where $h \equiv 0$ if p is odd by Lemma 8. We know that the normal form of this system commutes with the action defined by (VI.98). It follows that, up to higher order terms in $|A|$, (VI.106) can be replaced by

$$\begin{cases} \dfrac{d\psi}{dt} &= 0, \\ \dfrac{d\varphi}{dt} &= k_0(\mu, |A|^2), \\ \dfrac{dA}{dt} &= Af_0(\mu, |A|^2), \end{cases} \tag{VI.107}$$

where k_0 and f_0 are polynomial functions and the right-hand side of the first phase equation is zero (up to higher order terms) because it is odd in X. Let us set more precisely

$$f_0(\mu, |A|^2) = \mathrm{i}\eta_0 + a_{11}\mu + a_{03}|A|^2 + \text{ h.o.t.}. \tag{VI.108}$$

The system (VI.107) undergoes a bifurcation to solutions of the form

$$A(t) = \rho e^{\mathrm{i}(\eta t + \kappa_0)}, \qquad \varphi(t) = \Omega' t + \varphi_0, \qquad \psi(t) = \psi_0 \tag{VI.109}$$

with $\rho = \sqrt{-\mathrm{Re}(a_{03})/\mathrm{Re}(a_{11})\mu} + \mathrm{O}(|\mu|^{3/2})$, $\eta = \eta_0 + \mathrm{O}(|\mu|)$, $\Omega' = k_0(\mu, \rho^2)$, and $\kappa_0, \varphi_0, \psi_0$ are arbitrary phases. It could be shown that the limit cycle persists for the system (VI.106), with amplitude and frequencies of the same leading part as in (VI.109). Note that it is not obvious that there is no additional small frequency, but a Liapunov-Schmidt method leads to this result (as used in [Io 84]).

The corresponding fluid flow is quasi-periodic with two independent frequencies η and $\Omega + \Omega'$. It takes the form

$$U(t) = \mathbf{R}_{(\Omega+\Omega')t+\varphi_0}\mathbf{T}_{\Omega'' t+\psi_0}[U_0 + W(t)], \tag{VI.110}$$

where $W(t)$ is periodic with frequency η. Moreover, the stability analysis is as in classical Hopf bifurcations.

The symmetry $\mathbf{S}$ is not preserved, and the flow is no longer confined into flat horizontal cells. Notice that the flow described in Subsection IV.2.3 as *superposed ribbons* $\mathbf{RS}^{(\pi)}$ fits exactly in this case. Indeed, the corresponding eigendirections responsible for this bifurcation are $\zeta_2 - \zeta_3$ and complex conjugate. Hence, they are antisymmetric by $\mathbf{S}$ and satisfy

$$\mathbf{R}_\varphi \mathbf{T}_\psi(\zeta_2 - \zeta_3) = e^{\mathrm{i}(m+1)\varphi}(e^{\mathrm{i}n\psi}\zeta_2 - e^{-\mathrm{i}n\psi}\zeta_3).$$

It follows that $k = n$ in (VI.92), and therefore $p = 1$, $q = m$ in Lemma 8.

VI.5 Bifurcation from wavy vortices, modulated wavy vortices

In the previous sections we studied bifurcations from group-orbits of solutions that themselves result from an instability of the basic Couette flow.

However, the method applies as well to further bifurcations. One interesting case is the bifurcation from wavy vortices. Indeed, this flow is commonly seen in experiments, and its instability leads to the so-called *modulated wavy vortices* (MWV, for short), which are doubly periodic with a second frequency appearing spatially as a modulation of the wavy azimuthal structure. A group-theoretic analysis of this phenomenon was undertaken by Rand [Ra], as well as a comparison with experiments. In this section we study the occurrence of MWV by means of the method introduced in this chapter.

Recall that the wavy vortex regime corresponds to a group-orbit of solutions of the system (II.23) (Navier-Stokes equations for a perturbation of the Couette flow), which can be written as

$$U(t) = \mathbf{T}_{\psi_0}\mathbf{R}_{\omega t/m+\varphi_0}U_0, \tag{VI.111}$$

where ψ_0, φ_0 are arbitrary phases (parameters of the group orbit), ω is a nonzero frequency, and U_0 satisfies the following symmetries (see Subsection IV.1.3, Figure IV.3):

$$\mathbf{T}_{2\pi/n}U_0 = U_0, \qquad \mathbf{R}_{2\pi/m}U_0 = U_0, \qquad \text{and} \qquad \mathbf{R}_{\pi/m}\mathbf{S}U_0 = U_0. \tag{VI.112}$$

Notice that $(\mathbf{R}_{\pi/m}\mathbf{S})^2 = \mathbf{R}_{2\pi/m}$ and we have the symmetry $(\mathbf{R}_{\pi/m}\mathbf{S})^m = \mathbf{R}_\pi$ if m even, $\mathbf{R}_\pi\mathbf{S}$ if m odd.

We set perturbations of this flow in the form

$$U(t) = \mathbf{R}_{\Omega t+\varphi(t)}\mathbf{T}_{\psi(t)}(U_0 + W(t)), \tag{VI.113}$$

where $\Omega = \omega/m$. Note that this is exactly the form (IV.22) of the perturbation of the ribbons introduced in Subsection VI.1.2. The derivation of the center manifold is therefore identical to that for ribbons, and the differential system to which the problem reduces is (VI.29)–(VI.31). As before, we define

$$\begin{cases} \mathcal{L}(\mu) = D_W\mathcal{F}(\mu,0), & \mathcal{L}(0) = \mathcal{L}, \\ L = (1-P_\perp)\mathcal{L}. \end{cases} \tag{VI.114}$$

VI.5.1 Hopf bifurcation of wavy vortices into modulated wavy vortices

In view of the experimental reports, let us assume that the branch of wavy vortices undergoes a "Hopf" bifurcation at $\mu = 0$ (after a suitable change of origin in the bifurcation parameter), i.e., there exists a vector ζ and a positive real η_0 such that $L\zeta = i\eta_0\zeta$. Moreover, we assume that ζ does not break the axial periodicity of the wavy vortices. Then, in general, the eigenvalues $\pm i\eta_0$ are simple and ζ must satisfy relations of the form

$$\mathbf{R}_{2\pi/m}\zeta = e^{2li\pi/m}\zeta, \qquad -m < l < m, \tag{VI.115}$$

and

$$\mathbf{R}_{\pi/m}\mathbf{S}\zeta = \pm e^{li\pi/m}\zeta, \tag{VI.116}$$

Remark. We can write $l/m = p/q$ in irreducible form. Then q divides m : $m = qr$, and the new bifurcated flow is invariant under $\mathbf{R}_{2\pi q/m} = \mathbf{R}_{2\pi/r}$.

We now set the center manifold under the form

$$W = A\zeta + \bar{A}\bar{\zeta} + \Phi(\mu, A, \bar{A}), \tag{VI.117}$$

which allows us to rewrite (VI.29)–(VI.31) as

$$\begin{cases} \dfrac{d\psi}{dt} &= h(\mu, A, \bar{A}), \\ \dfrac{d\varphi}{dt} &= k(\mu, A, \bar{A}), \\ \dfrac{dA}{dt} &= f(\mu, A, \bar{A}). \end{cases} \tag{VI.118}$$

These equations are invariant under the group actions defined by (VI.115)–(VI.116). Moreover, by the same arguments as in Subsection VI.4.2, the normal form for this system is

$$\begin{cases} \dfrac{d\psi}{dt} &= 0, \\ \dfrac{d\varphi}{dt} &= k_0(\mu, |A|^2), \\ \dfrac{dA}{dt} &= Af_0(\mu, |A|^2), \end{cases} \tag{VI.119}$$

where k_0 and f_0 are polynomial functions. Notice that the right-hand side in the first equation is 0 because $\mathbf{R}_{\pi/m}\mathbf{S}$ acts on the arguments by $\psi \to -\psi$, $A \to \pm e^{\mathrm{i}p\pi/q}A$ and because of the group invariance of the normal form. The third equation (amplitude equation) undergoes a Hopf bifurcation, exactly as in Subsection VI.4.2, to solutions of the form

$$A(t) = \rho e^{\mathrm{i}(\eta t + \kappa_0)}, \qquad \varphi(t) = \Omega' t + \varphi_0, \qquad \psi(t) = \psi_0, \tag{VI.120}$$

with $\rho = \sqrt{-\mathrm{Re}(a_{03})/\mathrm{Re}(a_{11})\mu} + \mathrm{O}(|\mu|^{3/2})$, $\eta = \eta_0 + \mathrm{O}(|\mu|)$, $\Omega' = k_0(\mu, \rho^2)$, and $\kappa_0, \varphi_0, \psi_0$ are arbitrary phases. By a classical argument, it can be shown that the limit cycle persists for the third equation in (VI.118). Then the stability analysis proceeds as in the classical Hopf bifurcation. We finally ask about the persistence of the phase $\psi(t) =$ const for the full system. Indeed, the identity $d\psi/dt = 0$ in (VI.119) persists for (VI.118) *up to "flat terms."* In fact, if (VI.116) occurs with l even, it is easy to check in the first equation in (VI.118) that ψ is a constant phase for the full solution. If (VI.116) occurs with an odd l, it can be shown, now by a Lyapunov-Schmidt procedure, that indeed the phase ψ is constant (see [Io 84]).

The solution resulting from this bifurcation can in all cases be written as

$$U(t) = \mathbf{R}_{\Omega' t+\varphi_0}\mathbf{T}_{\psi_0}[U_0 + \rho(e^{\mathrm{i}(\eta t+\kappa_0)}\zeta + e^{-\mathrm{i}(\eta t+\kappa_0)}\bar{\zeta}) + \Phi]. \qquad \text{(VI.121)}$$

This flow looks like the superposition of a flow with m waves rotating around the axis, with an oscillating modulation with r waves (r dividing m). We interpret this flow as a modulation of the wavy vortex structure of the basic flow. Figure VI.6 shows a sketch of various modulated wavy patterns corresponding, for example, to the evolution in time of a single vortex outflow boundary in the case of four waves ($m = 4$). We assume a

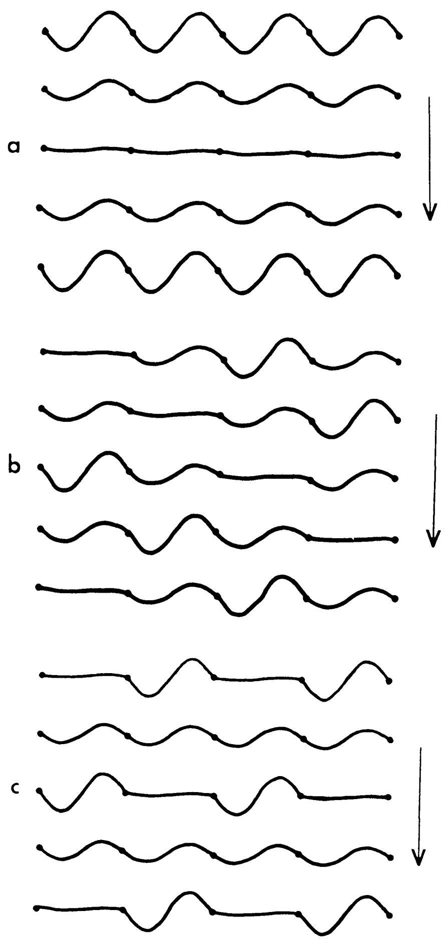


FIGURE VI.6. Arrows indicate time evolution. (see [Go-Re-Sw]).

reference frame rotating with the wavy pattern. This picture is taken from experimental observations of [Go-Re-Sw]. In (a) $r = 4$ and $q = 1$, so the waves are modulated in phase; in (b) $r = 1$ and $q = 4$ (azimuthal symmetry is completely broken), and successive waves differ in phase by about one-fourth of the modulation cycle; in (c) $r = q = 2$, and alternate waves are modulated in phase while adjacent waves differ in phase by about half of the modulation cycle.

Remark. Randy Tagg reported to us in July 1992 that in some cases the bifurcating modulated wavy vortices have a double axial period. This means that in such cases, ζ does break the axial periodicity, as in Subsection VI.2.2. (see Cases 3 and 4). This, however, does not change the foregoing analysis, except for the resulting symmetry of the flow (VI.121).

VI.5.2 Steady bifurcation of the wavy vortices into a quasi-periodic flow with a slow drift

When one makes a systematic numerical study of all possibilities, it appears that there are cases where the wavy vortices bifurcate via a zero eigenvalue for the linear operator L defined by (VI.114). This situation happens, for instance, when one analyzes the codimension-two problem studied in Section IV.1 (interaction of the steady axisymmetric mode, with $m = 1$ azimuthal mode). Then we showed the possibility of bifurcation to wavy vortices, as a secondary bifurcation (see Subsection IV.1.3). Numerical studies (see Laure [La 91]) show that wavy vortices may lose their stability through a "steady-state" bifurcation. Let us show briefly that this leads again to the *slow drift phenomenon*, provided a simple condition is realized.

Let us assume 0 is a simple eigenvalue of L and denote by ζ the eigenvector satisfying

$$L\zeta = 0. \tag{VI.122}$$

We assume that there is no breaking of the axial nor of the azimuthal periodicity. Now, due to the fact that $(\mathbf{R}_{\pi/m}\mathbf{S})^2\zeta = \mathbf{R}_{2\pi/m}\zeta = \zeta$, we necessarily have $\mathbf{R}_{\pi/m}\mathbf{S}\zeta = \pm\zeta$.

Let us assume that we indeed have

$$\mathbf{R}_{\pi/m}\mathbf{S}\zeta = -\zeta. \tag{VI.123}$$

If we define, as it is now classical, a real amplitude A, the amplitude equations in (φ, ψ, A) are invariant under the transformation $(\varphi, \psi, A) \longrightarrow (\varphi, -\psi, -A)$, and we find again a system of the form

$$\begin{cases} \dfrac{d\psi}{dt} &= A, \\ \dfrac{d\varphi}{dt} &= k_0(\mu, A^2), \\ \dfrac{dA}{dt} &= Af_0(\mu, A^2), \end{cases} \tag{VI.124}$$

which shows that we have a pitchfork bifurcation to a flow with constant amplitude A, a renormalization of the frequency Ω by Ω', and an *additional* frequency A corresponding to a slow drift along the axis. The action of $\mathbf{R}_{\pi/m}\mathbf{S}$ gives the symmetric solution, propagating the opposite direction. The flow velocity field takes the form

$$U_{\pm}(t) = \mathbf{R}_{\Omega' t+\varphi_0}\mathbf{T}_{\pm At+\psi_0}[U_0 + A\zeta + \Phi(\mu, A)], \tag{VI.125}$$

with $\Omega' = \Omega + k_0(\mu, A^2)$. This flow would look like wavy vortices with a slow axial drift. To our knowledge, no such solution has been reported yet in the experiments.

VI.6 Codimension-two bifurcations from Taylor vortex flow

We studied in Section VI.2 the generic bifurcations from Taylor vortex flow, and we showed in Subsection VI.2.2 that a Hopf bifurcation from this solution might lead to azimuthally periodic flow patterns like the wavy inflow (or outflow) boundary flow (denoted by WIB and WOB respectively). We now assume that, by allowing two parameters to vary (typically the Reynolds number $\mathcal{R}$ and the angular velocity ratio Ω, as in Chapter IV), two different azimuthal modes with waves numbers m_+ and m_- interact, the axial wave number being the same. Then, the reduction to a system of amplitude equations can be done by following rigorously the same method as in VI.2.2. This system takes the form

$$\begin{cases} \dfrac{d\psi}{dt} &= h(\mu, \nu, A, \bar{A}, B, \bar{B}), \\ \dfrac{dA}{dt} &= f(\mu, \nu, A, \bar{A}, B, \bar{B}), \\ \dfrac{dB}{dt} &= g(\mu, \nu, A, \bar{A}, B, \bar{B}), \end{cases} \tag{VI.126}$$

where μ and ν are the two parameters (see IV.1), A and B are the amplitudes associated with the two azimuthal modes, and ψ is the translational mode. The "equivariant structure" of functions f, g, and h depends on additional hypotheses regarding the way the critical modes break the symmetry along the axis of the cylinders (same type of discussion as in VI.2.2). By setting $A = 0$ or $B = 0$ in (VI.126), the solutions described in VI.2.2 are recovered. Additional solutions, which mix both modes and are quasi-periodic with, maybe, an additional slow axial drift $\psi(t)$ can be found. The interesting observation here is that the so-called *wavelets* observed by Andereck et al. [An-L-Sw] can be understood as a special case of such solutions, when the two interacting modes are the WIB and the WOB with azimuthal wave numbers m_+ and m_-, respectively. Up to a higher order, there are m_- waves on the inflow boundaries with a certain azimuthal velocity and m_+ waves on the outflow boundaries with another azimuthal velocity. A complete description of the amplitude equations (VI.126) in the case of WIB-WOB interaction can be found in [Io 86]. Other cases may be treated in a similar way.

VII
Large-scale Effects

In this chapter we remove the assumption made up to now—that the solutions are spatially periodic along the z-axis. We saw in Section II.3 that, for any fixed Reynolds number above criticality, there is an interval of axial wave numbers (mainly denoted by k in this chapter, instead of α) such that the perturbations of the linearized system grow exponentially (see Figure VII.1(b)). This is a major source of difficulty for our problem, since the method we used until now, crucially depended on the discreteness of the set of eigenvalues of the linearized operator, which precisely results from the h-periodicity in z (as shown in Figure II.5). For about 20 years, physicists have overcome this difficulty by considering slow modulations in space of the amplitudes of critical modes. The envelope equation they obtain is usually called the Ginzburg-Landau (G-L) equation (see Newell and Whitehead [New-Wh], Segel [Seg], and DiPrima et al. [DP-Ec-Seg]). We show in Section VII.3 how to obtain this complex partial differential equation in the present context.

In Section VII.1, based on the paper by Iooss, Mielke, and Demay [Io-Mi-De], we only consider *steady solutions*, now in the unbounded space variable z. This situation corresponds to the case when the usual theory, as developed in Section III.1, leads to steady bifurcating solutions. The variable z will play the role of an "evolution variable," even though the problem is ill posed as an initial value problem, because of the ellipticity of the Navier-Stokes system. This method of studying elliptic problems was initiated by Kirchgässner [Ki 82] [Ki 84] [Ki 88] and is now extensively used for water wave problems in a moving frame (see also Mielke [Mi 86b], Amick and Kirchgässner [Am-Ki], and Iooss and Kirchgässner

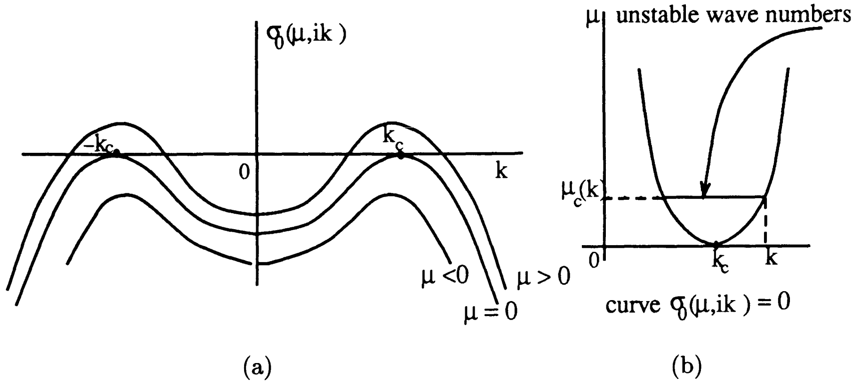


FIGURE VII.1.

[Io-Ki]), and for elasticity problems (long beams) [Mi 88b]. Following this idea, the usual techniques—center manifold and normal form theories—apply near criticality. The center manifold theorem is used here for finding solutions that are bounded at both infinities in z and are close to the basic fully symmetric solution. Using results of Mielke [Mi 88a], this method is shown to be applicable to steady Navier-Stokes equations in a cylindrical domain once they are written as an evolution problem in the x-variable. We then obtain a *reversible* four-dimensional system that is written in normal form and whose relationship with the steady (G-L) equation is emphasized in Section VII.3. The idea of using z as an evolution variable for obtaining steady bifurcating solutions in hydrodynamical nonlinear stability problems is due to Coullet and Repaux [Co-Re], where they give heuristic arguments leading at first order to the steady (G-L) equation. In fact, we present a way to compute the coefficients of our new system and give the relationship between these coefficients and those of the (G-L) equation.

The study of our normal form allows us to recover all bifurcating solutions of Section III.1 and many other types of solutions. In particular, we recover the known solutions of the (G-L) equation truncated at lowest orders (see [New-Wh] and [Kr-Zim]). We obtain spatially quasi-periodic solutions and some "front" types of solutions. In addition, we are able to answer the following questions: What is the meaning of (for instance) a third-order spatial derivative in the (G-L) equation? Has a solution of the (G-L) equation a meaning, even though the amplitude is zero while its gradient does not vanish?

In Section VII.2, based on the paper by Iooss and Mielke [Io-Mi], we consider time-periodic solutions, again suppressing the spatial periodicity in z, contrary to the works developed, for instance, by Iudovich [Iu 71], Sattinger [Sa 71], and Iooss [Io 72]. The corresponding situation is that of Section III.2 where the critical modes are oscillatory in time. The unknown period appears as an additional parameter. By the way, we recover the already-known bifurcating time- and space-periodic solutions [traveling waves (spirals) and standing waves (ribbons) of Section III.2] and obtain new solutions, no longer spatially periodic. In a previous work, Collet and Eckmann [Co-Ec 86] studied time-dependent solutions for some similar evolution problems on the real line. Their analysis deals with propagating fronts, which have a time-periodic form in a moving frame (then losing the invariance under reflection symmetry). Another work dealing with a problem close to Section VII.2 is that of Renardy [Re 82], who treats reaction-diffusion equations on the real line. He focuses his analysis on two different types of nonspatially periodic solutions, the first one approaching a constant at infinity, the second one approaching periodic wave trains at infinity, the directions of propagation being opposite at both infinities. In Section VII.2, we begin by showing that it is possible to use the center manifold theorem to obtain bifurcating time-periodic solutions of Navier-Stokes equations in a cylindrical domain. This tool was not used by Renardy in [Re 82], but his problem was of a simpler structure. In fact, he did not construct all small bounded solutions and had no uniqueness result from his method. Using symmetry arguments and normal form theory, as in Section VII.1, we derive the reduced ordinary differential system for the amplitudes in the space variable that gives all bifurcating bounded solutions. It appears that one can completely solve at least the truncated system (at any order) and that this gives many unusual nonspatially periodic solutions. For the full system (untruncated), we are able to prove the persistence of most of these solutions. The physically most interesting solution is a "defect" solution occurring in the Couette-Taylor problem, which appears in the reduced system as a heteroclinic connection between the two symmetric regimes of spiral waves (similar to the solutions found by Renardy [Re 82]). This regime is indeed currently observed in experiments with counterrotating cylinders. We also prove the existence of another type of connection between standing waves and traveling waves and the existence of spatially quasi-periodic solutions. Unfortunately, it should be clear that we have no means to prove any stability result for these solutions, contrary to Renardy [Re 82] or Collet and Eckmann [Co-Ec 87]. However, it appears that our study gives, as in Section VII.1, a partial justification of the derivation of the Ginzburg-Landau equation from Navier-Stokes equations near threshold, when one is only interested in time-periodic solutions (see Section VII.3).

VII.1 Steady solutions in an infinite cylinder

VII.1.1 A center manifold for steady Navier-Stokes equations

The aim of this part is to write the steady Navier-Stokes equations in the infinite cylinder $Q = \Sigma \times \mathbb{R}$ in the form of a differential equation in the space variable z.

The Navier-Stokes equations are now written as (see (II.1))

$$\begin{cases} \dfrac{\partial V}{\partial t} + (V.\nabla)V + \nabla p = \nu \Delta V + f, \\ \qquad \nabla.V = 0 \quad \text{in } Q, \\ V = g(\mu,.) \quad \text{on } \partial Q = \partial\Sigma \times \mathbb{R}, \end{cases} \tag{VII.1}$$

where μ represents the set of parameters as defined in Subsection II.1.2 and f, g are functions of the cross-sectional variable $y \in \Sigma$ (or, $\partial\Sigma$) only, Σ being the annulus $R_1 < |y| < R_2$. The pressure is denoted by ρp, and the velocity vector field V is decomposed into a longitudinal component V_z and a transversal component $V_\perp$. In this section we only consider *steady solutions*, hence, variable t does not occur. We assume the existence of a family of z-independent solutions $V = V^{(0)}(\mu,.) \in C^1(\bar{\Sigma}, \mathbb{R}^3)$. In the Couette-Taylor problem, this is the Couette flow (see (II.1.3)).

We call (VII.1), without $\partial/\partial t$, a *reversible system*, because for every stationary solution V of (VII.1) another solution of the problem is the symmetric flow $\hat{V} = \mathbf{S}V$, defined by

$$\hat{V}(z,y) = [\hat{V}_z(z,y), \hat{V}_\perp(z,y)],$$
$$\text{with } \hat{V}_z(z,y) = -V_z(-z,y) \text{ and } \hat{V}_\perp(z,y) = V_\perp(-z,y),$$

where $(z,y) \in \mathbb{R} \times \Sigma$ [see (II.16)]. In the subsequent analysis we shall consider z as an "evolution" variable. Then the transformation $\mathbf{S}$ corresponds to apply the symmetry $V_z \to -V_z$ and reverse the "time." The Couette-Taylor problem (with $f \equiv 0$ and $g \equiv \Omega_i \times y|_{|y|=R_i}$) is reversible in this sense.

To carry out the bifurcation analysis we introduce the notation $U = V - V^{(0)}$, $W_z = -p/\nu$, $W_\perp = \partial U_\perp/\partial z$. We notice that $W = (W_z, W_\perp)$ has the same number of components as U. The incompressibility can be written in the form

$$\partial U_z/\partial z + \nabla_\perp.U_\perp = 0.$$

Moreover, setting $\mathfrak{V} = (U, W)$, equation (VII.1) *for steady solutions* takes the form

$$\frac{d\mathfrak{V}}{dz} = \mathcal{A}_\mu \mathfrak{V} + \mathcal{B}_\mu(\mathfrak{V}, \mathfrak{V}), \tag{VII.2}$$

where there are no longer differentiations in z on the right-hand side. Here the linear part $\mathcal{A}_\mu$:

$$D(\mathcal{A}_\mu) \to \mathcal{X} := \{(U,W) \in [H^1(\Sigma)]^3 \times [L^2(\Sigma)]^3;\ U = 0 \text{ on } \partial\Sigma\}$$

is a differential operator with domain

$$D(\mathcal{A}_\mu) := \{(U, W) \in [H^2(\Sigma)]^3 \times [H^1(\Sigma)]^3;\ U = \nabla_\perp.U_\perp = W_\perp = 0 \text{ on } \partial\Sigma\}.$$

Splitting $\mathcal{A}_\mu$ into a Stokes part $\mathcal{A}_{\mathrm{St}}$ and a convective part $\mathcal{L}_\mu$ we have $\mathcal{A}_\mu = \mathcal{A}_{\mathrm{St}} + \mathcal{L}_\mu$ and

$$\mathcal{A}_{\mathrm{St}}(\mathfrak{V}) = \mathcal{A}_{\mathrm{St}}(U, W) = \begin{pmatrix} -\nabla_\perp.U_\perp \\ W_\perp \\ -\Delta_\perp U_z + \nabla_\perp.W_\perp \\ -\Delta_\perp U_\perp - \nabla_\perp W_z \end{pmatrix}, \tag{VII.3}$$

$$\mathcal{L}_\mu(\mathfrak{V}) = \mathcal{L}_\mu(U, W) = \nu^{-1} \begin{pmatrix} 0 \\ 0 \\ (V_\perp^{(0)}.\nabla_\perp)U_z + (U_\perp.\nabla_\perp)V_z^{(0)} \\ -V_z^{(0)}\nabla_\perp.U_\perp \\ V_z^{(0)}W_\perp + (U_\perp.\nabla_\perp)V_\perp^{(0)} \\ +(V_\perp^{(0)}.\nabla_\perp)U_\perp \end{pmatrix}. \tag{VII.4}$$

The quadratic terms in (VII.2) take the form

$$\mathcal{B}_\mu(\mathfrak{V}, \mathfrak{V}) = \nu^{-1} \begin{pmatrix} 0 \\ 0 \\ (U_\perp.\nabla_\perp)U_z - U_z(\nabla_\perp.U_\perp) \\ U_z W_\perp + (U_\perp.\nabla_\perp)U_\perp \end{pmatrix}. \tag{VII.5}$$

Observe that $\mathcal{B}_\mu$ is a smooth mapping from $D(\mathcal{A}_\mu)$ into $\mathcal{X}$ since, in every product, one of the factors is in $H^2(\Omega)$ (which is included in $C^0(\bar{\Omega})$ by the Sobolev embedding theorem) and the other is in $H^1(\Omega)$.

The Couette flow satisfies $V_z^{(0)} = 0$, the reversibility of the problem now being expressed through the reflection $\hat{\mathbf{S}}$ defined by

$$\hat{\mathbf{S}}\mathfrak{V} = \hat{\mathbf{S}}(U, W) = (\mathbf{S}U, -\mathbf{S}W), \tag{VII.6}$$

and we then have

$$\hat{\mathbf{S}}\mathcal{A}_\mu = -\mathcal{A}_\mu\hat{\mathbf{S}}, \qquad \mathcal{B}_\mu \circ \hat{\mathbf{S}} = -\hat{\mathbf{S}}\mathcal{B}_\mu. \tag{VII.7}$$

We wish now to characterize all solutions V of (VII.1) (or, (VII.2)) existing in the unbounded region Q and which are close [with respect to the topology of the space $D(\mathcal{A}_\mu)$] to the trivial solution $V^{(0)}$, uniformly in z. Following the methods of [Ki 82] further developed in [Mi 86a, 88a], this can be achieved by constructing a center manifold for (VII.2). It should be noted that (VII.2) is not a usual evolution problem—indeed, it is derived from the elliptic problem (VII.1)—hence, the initial value problem would be ill posed. As we shall see below, the spectrum of $\mathcal{A}_\mu$ is infinite on both sides of the imaginary axis; however, there are only finitely many eigenvalues on the imaginary axis. To give rise to a locally invariant manifold for the flow of (VII.2) we need *the center manifold.*

To apply the result of [Mi 88a] we must show that the resolvent of $\mathcal{A}_\mu$ satisfies

$$||(\mathcal{A}_\mu - ik\mathrm{Id})^{-1}||_{\mathcal{L}(\mathcal{X})} = \mathrm{O}(|k|^{-1}) \tag{VII.8}$$

for $k \in \mathbb{R}$ and $|k| \to \infty$. This estimate is established in [Io-Mi-De] and it is shown that the spectrum of $\mathcal{A}_\mu$ is only composed of eigenvalues of finite multiplicities, not accumulating at a finite distance, and located in a sector of the complex plane centered on the real axis (see Figure VII.2). In fact, 0 is always an eigenvalue with the eigenvector $U = 0$, $W_\perp = 0$, $W_z =$ const. This corresponds to the *arbitrariness on the pressure* for any incompressible fluid. This eigenvalue is in fact multiple, since one can find a vector field of the form $U_\perp = W_\perp = W_z = 0$ and U_z of Poiseuille type, which is a generalized eigenvector belonging to the same 0 eigenvalue. To avoid the difficulty relative to the zero eigenvalue, the idea is to introduce in spaces $\mathcal{X}$ and $D(\mathcal{A}_\mu)$ a quotient norm on the component W_z, independent of any added constant. It is clear that the equation (VII.2) can be formulated on this quotient space and that the average value on a cross section of U_z, which is constant, appears as a parameter. In this section, we make the assumption $\int_\Sigma U_z\, dy = 0$. From now on, 0 is no longer an eigenvalue of the linear operator $\mathcal{A}_\mu$. Moreover, we shall see below that for $\mu = 0$ there are

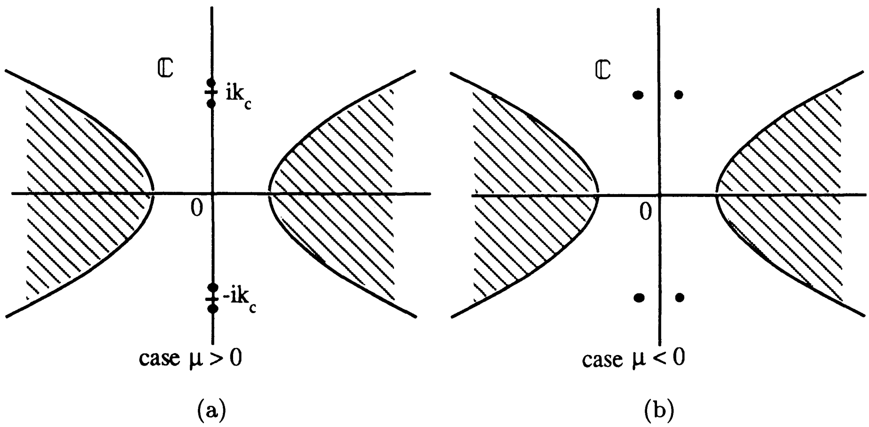


FIGURE VII.2. Location of the spectrum of $\mathcal{A}_\mu$ for $|\mu|$ close to 0: (a) $\mu > 0$, (b) $\mu < 0$.

only two eigenvalues on the imaginary axis. Hence, the remaining part of the spectrum is at a finite distance from this axis. All this ensures the possibility of using the result of [Mi 88a].

To characterize the center manifold we must now construct the spectral part $\mathfrak{X}_0$ corresponding to the spectrum lying on the imaginary axis. This amounts to solving the eigenvalue problem

$$\mathcal{A}_\mu \mathfrak{V} = \lambda \mathfrak{V} \tag{VII.9}$$

for $\lambda = \mathrm{i}k$, $k \in \mathbb{R}$. Note that, because of reversibility, eigenvalues come in pairs $\pm\lambda$, since (VII.9) is equivalent to $\mathcal{A}_\mu \hat{\mathbf{S}} \mathfrak{V} = -\lambda \hat{\mathbf{S}} \mathfrak{V}$.

At this point, it is important to remember that the linear part $d/dz - \mathcal{A}_\mu$ of (VII.2) is only a reformulation of the linear operator L_μ in (II.23). Hence, the eigenvalue problem (VII.9) with $\lambda = \mathrm{i}k$ is *equivalent* to the linear stability problem (see (II.28))

$$L_\mu(\hat{U}_k e^{\mathrm{i}kz}) = \sigma(\mu, \mathrm{i}k)\hat{U}_k e^{\mathrm{i}kz} \tag{VII.10}$$

with $\sigma(\mu, \mathrm{i}k) = 0$. Now, the neutral stability curve (see Figure VII.1(b)) given by $\mu = \mu_c(k)$ is precisely such that $\sigma_0(\mu_c(k), \mathrm{i}k) = 0$, where σ_0 is the eigenvalue of largest real part (see Figure VII.1(a)), supposed to be *real* here. So, we have

$$L_{\mu_c(k)}(\hat{U}_k e^{\mathrm{i}kz}) = 0. \tag{VII.11}$$

This is equivalent to the existence of an eigenvector $\mathfrak{V}(\mathrm{i}k)$ of $\mathcal{A}_{\mu_c(k)}$ such that

$$(\mathcal{A}_{\mu_c(k)} - \mathrm{i}k)\mathfrak{V}(\mathrm{i}k) = 0. \tag{VII.12}$$

Moreover, if we define a projection Π by $\Pi\mathfrak{V} = \Pi(U, W) = U$, we have $\Pi\mathfrak{V}(\mathrm{i}k) = \hat{U}_k$.

For $\mu = 0$, the only eigenvalues of $\mathcal{A}_\mu$ on the imaginary axis are $\pm \mathrm{i}k_c$; for $\mu > 0$, we see in Figure VII.1(a) that there are two pairs of eigenvalues on the imaginary axis, and for $\mu < 0$ they disappear from this axis. Because of reversibility, we know that the generic situation is that for $\mu = 0$ these eigenvalues are double and nonsemisimple (geometric multiplicity one). In fact, starting with the relation (VII.12) and using the property that $d\mu_c(k_c)/dk = 0$, we obtain

$$(\mathcal{A}_0 - \mathrm{i}k_c)\mathfrak{V}(\mathrm{i}k_c) = 0, \tag{VII.13}$$

and

$$(\mathcal{A}_0 - \mathrm{i}k_c) d\mathfrak{V}(\mathrm{i}k_c)/d\lambda = \mathfrak{V}(\mathrm{i}k_c). \tag{VII.14}$$

We set $\mathfrak{V}_0 = \mathfrak{V}(\mathrm{i}k_c)$ and $\mathfrak{V}_1 = d\mathfrak{V}(\mathrm{i}k_c)/d\lambda$. These vectors form a Jordan basis for the generalized eigenspace belonging to the eigenvalue $\mathrm{i}k_c$ of $\mathcal{A}_0$. We see that because $\hat{\mathbf{S}}\mathfrak{V}(\mathrm{i}k) = \mathfrak{V}(-\mathrm{i}k) = \bar{\mathfrak{V}}(\mathrm{i}k)$, we have the following representation of $\hat{\mathbf{S}}$ on the generalized eigenspace:

$$\hat{\mathbf{S}}\mathfrak{V}_0 = \bar{\mathfrak{V}}_0, \qquad \hat{\mathbf{S}}\mathfrak{V}_1 = -\bar{\mathfrak{V}}_1. \tag{VII.15}$$

In fact, this could always be assumed after an eventual change of basis (not necessary here).

Remark. Since $(d/dz - \mathcal{A}_0)\left[(z\mathfrak{V}_0 + \mathfrak{V}_1)e^{ik_c z}\right] = 0$, we see after differentiation of (VII.12) with respect to ik at $k = k_c$ that $\Pi\mathfrak{V}_0 = \hat{U}_{K_c}, \Pi\mathfrak{V}_1 = \frac{d}{d(ik)}\hat{U}|_{ik_c}$.

For $\mu = 0$ we know by construction the position of all pure imaginary eigenvalues of $\mathcal{A}_0$, and because of the properties of the spectrum described above, all other eigenvalues λ of $\mathcal{A}_\mu$ are bounded away from the imaginary axis as long as $|\mu|$ is small (shaded regions on Figure VII.2). The center manifold theorem we used until now is adapted for the search of small bounded solutions as t goes to ∞. This theorem works when there is no part of the spectrum of the linear operator lying on the right of the imaginary axis, at criticality. In the present case, the spectrum is symmetric with respect to the imaginary axis, and we are interested in solutions that are bounded as z goes at both $+$ and $-\infty$. However, there exists a version of the center manifold theorem that is adapted to elliptic problems. We can use this center manifold theorem proved in [Mi 88a] (see also [Va-Io]) to obtain the following result:

Theorem 1 *For small $|\mu|$, all steady solutions $V\colon Q \to \mathbb{R}^3$ of (VII.1) being sufficiently close to $V = V^{(0)}$ for all x in $\mathbb{R}$ satisfy a relation*

$$\mathfrak{V} = A\mathfrak{V}_0 + B\mathfrak{V}_1 + \bar{A}\bar{\mathfrak{V}}_0 + \bar{B}\bar{\mathfrak{V}}_1 + \mathfrak{Q}(\mu, A, \bar{A}, B, \bar{B}), \qquad \text{(VII.16)}$$

where $\mathfrak{Q}$ is a smooth function with $\mathfrak{Q} = O[|\mu|(|A| + |B|) + |A|^2 + |B|^2]$ and $V - V^{(0)} = \Pi\mathfrak{V}$. Moreover, (A, B) is a solution of a reduced equation for the complex amplitudes A and B

$$\begin{aligned}\frac{dA}{dz} &= ik_c A + B + f(\mu, A, \bar{A}, B, \bar{B}),\\ \frac{dB}{dz} &= ik_c B + g(\mu, A, \bar{A}, B, \bar{B})\end{aligned} \qquad \text{(VII.17)}$$

where $f, g = O(|\mu|(|A| + |B|) + |A|^2 + |B|^2)$. Moreover, due to reversibility, we have (see [Mi 86a])

$$\begin{aligned}f(\mu, \bar{A}, A, -\bar{B}, -B) &= -\overline{f(\mu, A, \bar{A}, B, \bar{B})},\\ g(\mu, \bar{A}, A, -\bar{B}, -B) &= \overline{g(\mu, A, \bar{A}, B, \bar{B})}\end{aligned} \qquad \text{(VII.18)}$$

and

$$\mathfrak{Q}(\mu, \bar{A}, A, -\bar{B}, -B) = \hat{\mathbf{S}}\mathfrak{Q}(\mu, A, \bar{A}, B, \bar{B}).$$

We have some freedom to choose coefficients of the expansions of f and g in (VII.17). These depend on the possible different choices in the expansion of $\mathfrak{Q}$. It is the aim of the next section to choose suitable coordinates (normal form as indicated in Section II.4) for the system (VII.17) in order to make possible its complete study, up to arbitrary high-order terms. Moreover, this system will be shown in Section VII.3 to be closely related to the steady (G-L) equation.

VII.1.2 Resolution of the four-dimensional amplitude equations

VII.1.2.1 The normal form

Now, to simplify the form of (VII.17), we put it into *normal form*. This, of course, can only arrange coefficients up to a given order, but it greatly simplifies the further analysis. It is shown in Elphick et al. [E-T-B-C-I] that a good choice of normal form associated with a critical linear operator such that

$$\mathcal{T}_0 = \begin{pmatrix} \mathrm{i}k_c & 1 & 0 & 0 \\ 0 & \mathrm{i}k_c & 0 & 0 \\ 0 & 0 & -\mathrm{i}k_c & 1 \\ 0 & 0 & 0 & -\mathrm{i}k_c \end{pmatrix} \tag{VII.19}$$

is as follows:

$$\begin{aligned} \frac{dA}{dz} &= \mathrm{i}k_c A + B + A\varphi_0[\mu; |A|^2; \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B)], \\ \frac{dB}{dz} &= \mathrm{i}k_c B + B\varphi_0[\mu; |A|^2; \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B)] \\ &\quad + A\varphi_1[\mu; |A|^2; \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B)]. \end{aligned} \tag{VII.20}$$

Moreover, this normalization process preserves the reversibility. This results easily from a proof similar to that for usual symmetries (see [E-T-B-C-I]) since $\hat{\mathbf{S}}$ is unitary. We notice that $|A|^2$ and $\frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B)$ are invariant under $\hat{\mathbf{S}}$, hence, property (VII.18) gives a pure imaginary function φ_0 and a real function φ_1. Finally, the system on the center manifold is now written as follows, up to order $\mathrm{O}(|A| + |B|)^N$, with arbitrary N:

$$\begin{aligned} \frac{dA}{dz} &= \mathrm{i}k_c A + B + \mathrm{i}AP[\mu; |A|^2; \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B)], \\ \frac{dB}{dz} &= \mathrm{i}k_c B + \mathrm{i}BP[\mu; |A|^2; \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B)] \\ &\quad + AQ[\mu; |A|^2; \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B)]. \end{aligned} \tag{VII.21}$$

Here P and Q are real polynomials in their last two arguments, with μ-dependent coefficients and such that $P(0,0,0) = Q(0,0,0) = 0$.

VII.1.2.2 Integrability of the reduced system

The system (VII.21) would be hamiltonian if there existed a scalar function $\varphi(\mu, u, v)$ such that

$$P = \frac{\partial \varphi}{\partial v}[\mu, |A|^2, \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B)], \qquad Q = -2\frac{\partial \varphi}{\partial u}[\mu, |A|^2, \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B)].$$

Then the hamiltonian would be

$$H = |B|^2 + \mathrm{i}k_c(A\bar{B} - \bar{A}B) + 2\varphi[\mu, |A|^2, \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B)]$$

and in this case the system (VII.21) would have the form

$$\frac{dA}{dz} = \frac{\partial H}{\partial \bar{B}} \qquad \frac{dB}{dz} = -\frac{\partial H}{\partial \bar{A}},$$

and it is known that such a system is integrable, the second integral being $(A\bar{B} - \bar{A}B)$.

In our case, we have *two* arbitrary functions P and Q; hence, (VII.21) is *not hamiltonian* in general. In fact, there is still a small arbitrariness in the choice of the normal form, and a conjecture is that one can find a hamiltonian normal form! However, our system (VII.21) is *integrable*, with again $(A\bar{B} - \bar{A}B)$ as an integral. The other integral can be found by looking for an expression similar to the above hamiltonian. In fact, if we define

$$G(\mu, u, v) = \int_0^u Q(\mu, s, v)\, ds, \tag{VII.22}$$

then, as could be checked easily, the following function is an integral:

$$H[\mu, |A|^2, |B|^2, v] \equiv |B|^2 - G[\mu, |A|^2, v]. \tag{VII.23}$$

where $v = {}^{\mathrm{i}}\!/\!_2(A\bar{B} - \bar{A}B)$ is the basic first integral.

We finally make the following change variables.

$$A = r_0 e^{\mathrm{i}(k_c z + \psi_0)}, \qquad B = r_1 e^{\mathrm{i}(k_c z + \psi_1)}. \tag{VII.24}$$

Then, the system (VII.21) becomes

$$\left\{\begin{aligned} \frac{dr_0}{dz} &= r_1 \cos(\psi_1 - \psi_0), \\ \frac{dr_1}{dz} &= r_0 \cos(\psi_1 - \psi_0) Q(\mu, r_0^2, K), \\ r_0 \frac{d\psi_0}{dz} &= r_1 \sin(\psi_1 - \psi_0) + r_0 P(\mu, r_0^2, K), \\ r_1 \frac{d\psi_1}{dz} &= r_1 P(\mu, r_0^2, K) - r_0 \sin(\psi_1 - \psi_0) Q(\mu, r_0^2, K), \end{aligned}\right. \tag{VII.25}$$

where the two integrals are now

$$\begin{aligned} r_0 r_1 \sin(\psi_1 - \psi_0) &= K, \\ r_1^2 - G(\mu, r_0^2, K) &= H. \end{aligned} \tag{VII.26}$$

If we set $u_0 = r_0^2$, $u_1 = r_1^2$, by taking account of (VII.26) the system (VII.25) reduces (for $u_0 u_1 \neq 0$) to

$$\left(\frac{du_0}{dz}\right)^2 = 4\{u_0[G(\mu, u_0, K) + H] - K^2\}, \tag{VII.27}$$

$$\left\{\begin{aligned} \frac{d\psi_0}{dz} &= P(\mu, u_0, K) + \frac{K}{u_0}, \\ \frac{d\psi_1}{dz} &= P(\mu, u_0, K) + \frac{K}{u_0} - K(u_0 u_1)^{-1}[u_0 Q(\mu, u_0, K) + u_1]. \end{aligned}\right. \tag{VII.28}$$

VII.1.2.3 Periodic solutions of the amplitudes equations

To study more precisely the behavior of the solutions, let us define the principal part of P and Q:

$$\begin{cases} P(\mu,u,v) &= p_1\mu + p_2 u + p_3 v + \mathrm{O}(|\mu|+|u|+|v|)^2, \\ Q(\mu,u,v) &= -q_1\mu + q_2 u + q_3 v + \mathrm{O}(|\mu|+|u|+|v|)^2. \end{cases} \tag{VII.29}$$

We can make precise the meaning of coefficients P_j and q_j by taking account of what we assumed on the eigenvalues of the linear operator $\mathcal{A}_\mu$ and also what we know of the steady spatially periodic solutions obtained in Section III.1 [see the calculated coefficients of the amplitude equation (III.6)].

For the linear operator occurring in (VII.21), the eigenvalues are

$$\mathrm{i}[k_\mathrm{c} + P(\mu,0,0)] \pm \sqrt{Q(\mu,0,0)}, \qquad \text{and the complex conjugate.} \tag{VII.30}$$

If $\mu > 0$, they correspond in Figure VII.1(a) to the four intersections of the curve $\mu > 0$ with the k-axis. This shows that $Q(\mu,0,0)$ is negative for $\mu > 0$. The generic situation is then when

$$q_1 > 0. \tag{VII.31}$$

From (VII.31) we can deduce the form of the neutral stability curve, given in Figure VII.1(b), by solving with respect to μ (implicit function theorem) the equation

$$[(k-k_\mathrm{c}) - P(\mu,0,0)]^2 + Q(\mu,0,0) = 0. \tag{VII.32}$$

This leads to the expansion

$$\mu_\mathrm{c}(k) = \frac{1}{q_1}(k-k_\mathrm{c})^2 - \frac{2p_1}{q_1^2}(k-k_\mathrm{c})^3 + \mathrm{O}(k-k_\mathrm{c})^4. \tag{VII.33}$$

The steady spatially periodic solutions obtained in Section III.1 correspond to stationary solutions of (VII.27) in u_0, while periodic solutions for u_0 lead to quasi-periodic solutions for A and B, because of the phases ψ_0, ψ_1. So, periodic solutions are given by such u_0's that cancel the right-hand side of (VII.27) and its derivative with respect to u_0. Therefore, we have

$$\begin{gathered} u_0[G(\mu,u_0,K) + H] - K^2 = 0, \\ G(\mu,u_0,K) + H + u_0 Q(\mu,u_0,K) = 0. \end{gathered} \tag{VII.34}$$

If we define α by

$$\alpha = P(\mu,u_0,K) + K/u_0, \tag{VII.35}$$

then, up to a phase shift, we have in (VII.24)

$$\psi_1 - \psi_0 = \pi/2 + l\pi, \qquad \psi_0 = \alpha z, \qquad \alpha = k - k_\mathrm{c}, \qquad l = 0 \text{ or } 1.$$

We take $l = 0$ or 1 in such a way that $r_1 > 0$ in (VII.26), and we can solve (by the implicit function theorem) with respect to K and μ the system (VII.35) with $Q + (\alpha - P)^2 = 0$. This gives

$$\mu - \mu_c(k) = \frac{q_2}{q_1} r_0^2 + \mathrm{O}[r_0^2(|\alpha| + r_0^2)]. \tag{VII.36}$$

When the bifurcation is *supercritical* and nondegenerate, i.e., when the cubic coefficient c in (III.6) is negative, we have

$$q_2 > 0, \tag{VII.37}$$

while the sign is reversed in the subcritical case.

Finally, as expected in the supercritical case, the "classical" steady spatially periodic solutions exist when the right-hand side of (VII.36) is positive, i.e., inside the parabolic region of Figure VII.1(b). We notice that in the present codimension-one problem, we have for a fixed value of the parameter μ a continuous set of periodic solutions, with wave numbers k belonging to the interval $\{k;\ \mu_c(k) < \mu\}$ (see Figure VII.1(b)). Since we recover solutions given by (III.15), we may also observe that these solutions of the normal form (VII.21) give in fact solutions of the full Navier-Stokes equations, even perturbed by the additional small terms. A proof of the existence of such solutions for the complete system, starting from the normal form (VII.21), is done in [Io-Pé].

VII.1.2.4 Other solutions of the amplitude equations

Let us set

$$K = |\mu|^{3/2}\kappa, \qquad H = \mu^2 h, \qquad h_m = q_1^2/2q_2. \tag{VII.38}$$

Then it is easy to show for $q_2 > 0$ and $\mu > 0$ that the only relevant cases are given by

$$0 < h < 4/3(h_m), \qquad (1+\rho)^2(1-2\rho) < (\kappa/\kappa_E)^2 < (1-\rho)^2(1+2\rho), \tag{VII.39}$$

where $\rho = \sqrt{1 - 3h/4h_m}$ and $\kappa_E = \left(4q_1^3/27q_2^2\right)^{1/2}$ (see shaded region in Figure VII.3). By "relevant case," we mean any case that leads to bounded solutions when z goes to both infinities. For $q_2 < 0$ and $\mu > 0$, we see that $h_m < 0$, and the only relevant cases are given by (see shaded region in Figure VII.4):

$$h > 0, \qquad (\kappa/\kappa_E)^2 < (1-\rho)^2(1+2\rho). \tag{VII.40}$$

For $q_2 < 0$ and $\mu < 0$, we see that $h_m < 0$, and the only relevant cases are given by (see shaded region in Figure VII.5)

$$h > h_m, \qquad (\kappa/\kappa_E)^2 < (1+\rho)^2(2\rho - 1). \tag{VII.41}$$

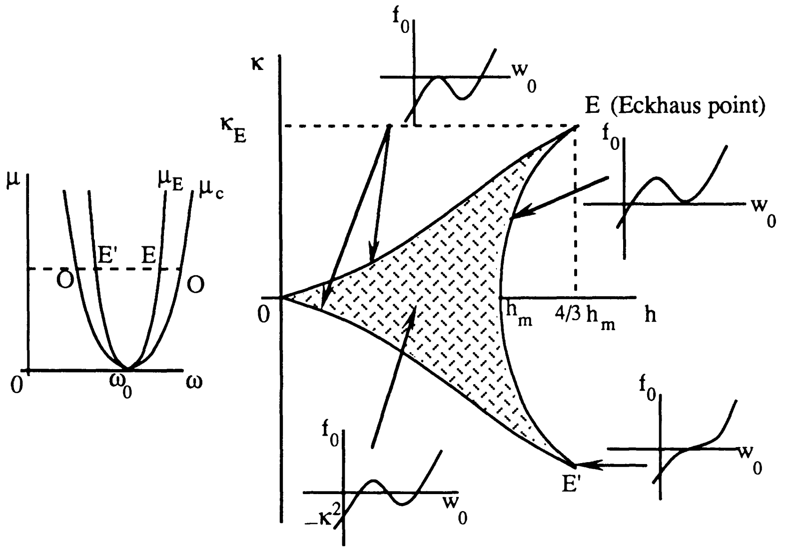


FIGURE VII.3. Different shapes of f_0 for $\mu > 0$, $q_2 > 0$ (supercritical bifurcation).

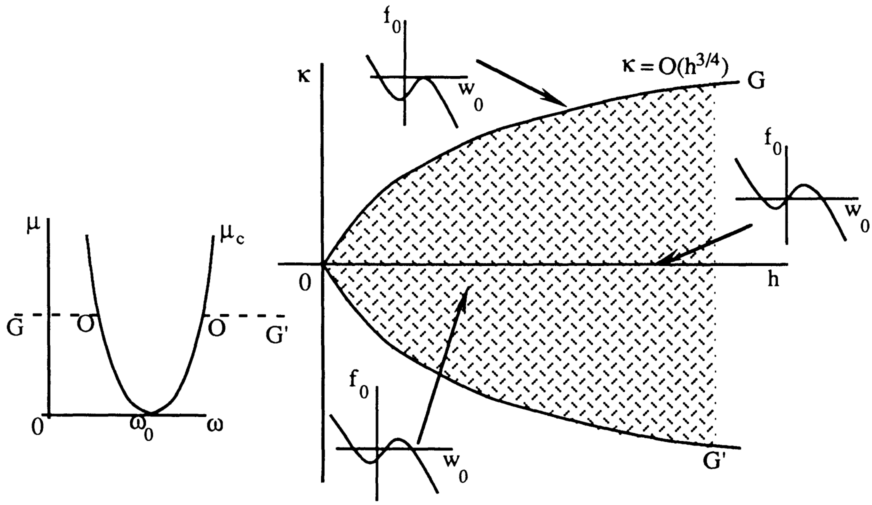


FIGURE VII.4. Different shapes of f_0 for $\mu > 0$, $q_2 < 0$ (subcritical bifurcation).

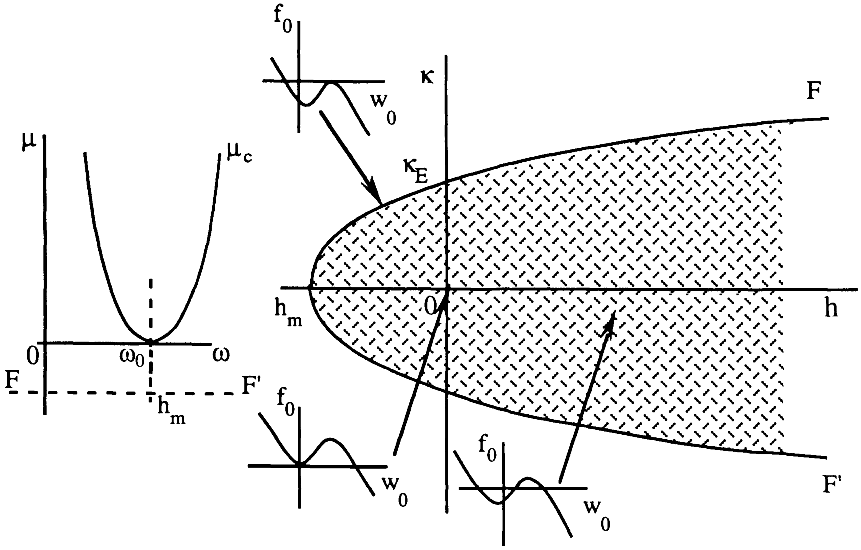


FIGURE VII.5. Different shapes of f_0 for $\mu < 0$, $q_2 < 0$ (subcritical bifurcation).

More precisely, if we define the function $f(u_0)$ by

$$f(u_0) = u_0[G(\mu, u_0, K) + H] - K^2, \tag{VII.42}$$

we then have $f(|\mu|w_0) = |\mu|^3 f_0(w_0) + O(|\mu|^{7/2})$, and $r_0^2/|\mu|$ lies between the two smallest (or largest) positive roots $w_0^{(1)}, w_0^{(2)}$ of f_0, when $q_2 > 0$ (or $q_2 < 0$), where f_0 is the following cubic polynomial:

$$f_0(w_0) \equiv \frac{q_2}{2} w_0^3 - \operatorname{sgn}(\mu) q_1 w_0^2 + h w_0 - \kappa^2. \tag{VII.43}$$

On Figures VII.3–5 we indicate the shape of f_0 varying in function of the parameters (h, κ).

On the boundary of the hatched areas of Figures VII.3–5, condition (VII.34) is satisfied (the cubic curve is tangent to the axis); hence, we obtain the steady solutions in $(u_0, u_1, \psi_1 \psi_0)$ given in Subsection VII.1.2.3. In addition to these solutions, we have, on the curve Eh_mE' of Figure VII.3, a *homoclinic loop*, solution in the three-dimensional space $(u_0, u_1, \psi_1 - \psi_0)$. Now, inside the open hatched regions of Figures VII.3–5 we have a family of *periodic* solutions $(u_0, u_1, \psi_1 - \psi_0)$ in the variable z. There is one exception given by the case when $q_2 < 0$ and $\mu < 0$ at the point $(h, \kappa) = 0$, where there is a *homoclinic solution going to 0 at both infinities* (see Figure VII.5).

VII.1.2.5 Quasi-periodic solutions

We already discussed the periodic solutions of the amplitude equation (VII.21). They correspond to the steady solutions in the space $(u_0, u_1, \psi_1 - \psi_0)$. Now, the periodic solutions $(u_0, u_1, \psi_1 - \psi_0)$ found in the interior of the hatched regions [see (VII.27)–(VII.28)], correspond to *quasi-periodic solutions* of (VII.21). In choosing suitably the origin in z, we obtain a family of solutions that possess the symmetry property

$$\hat{\mathbf{S}}(A_0, B_0)(z) = (\bar{A}_0, -\bar{B}_0)(z) = (A_0, B_0)(-z) \tag{VII.44}$$

and can be written in the form

$$\begin{aligned} A_0(z) &= r_0(\omega_1 z) e^{\mathrm{i}[\omega_0 z + \psi_0(\omega_1 z)]}, \\ B_0(z) &= r_1(\omega_1 z) e^{\mathrm{i}[\omega_0 z + \psi_1(\omega_1 z) + \pi/2]}, \end{aligned} \tag{VII.45}$$

where r_0, r_1, ψ_0, ψ_1 are 2π-periodic functions and r_0, r_1 are even while ψ_0, ψ_1 are odd. Functions $r_0, r_1, \psi_0, \psi_1, \omega_0$, and ω_1 depend on μ, h, κ, and in addition r_0 and r_1 are of order $\mathrm{O}(\mu^{1/2})$. Moreover, ω_0 and ω_1 are given by

$$\frac{2\pi}{\omega_1} = \int_{u_0^{(1)}}^{u_0^{(2)}} \frac{du}{\sqrt{f(u)}} \sim \frac{1}{\sqrt{|\mu|}} \int_{w_0^{(1)}}^{w_0^{(2)}} \frac{dw}{\sqrt{f_0(w)}}, \tag{VII.46}$$

$$\begin{aligned} \omega_0 = k_c + \hat{\alpha}, \qquad \hat{\alpha} &= \frac{\omega_1}{2\pi} \int_{u_0^{(1)}}^{u_0^{(2)}} \frac{[P(\mu, u, K) + K/u]\, du}{\sqrt{f(u)}} \\ &\sim \frac{\omega_1}{2\pi} \int_{w_0^{(1)}}^{w_0^{(2)}} \frac{\kappa\, dw}{w\sqrt{f_0(w)}}. \end{aligned} \tag{VII.47}$$

Let us define two functions $\alpha(h, \kappa)$ and $\beta(h, \kappa)$ as follows:

$$\begin{aligned} \omega_0 &= k_c + \sqrt{|\mu|}\alpha(h, \kappa) + \mathrm{O}(\mu), \\ \omega_1 &= \sqrt{|\mu|}\beta(h, \kappa) + \mathrm{O}(\mu). \end{aligned} \tag{VII.48}$$

Then, close to the boundary of the hatched region, it is shown in [Io-Lo] that

(i) when $(h, \kappa) \to$ curve Eh_mE', then $\beta(h, \kappa) \to 0$, $\alpha(h, \kappa) = \mathrm{O}(\kappa)$;

(ii) when $(h, \kappa) \to$ curve EOE', Fh_mF', or GOG', then $\beta(h, \kappa) = \mathrm{O}(1)$, $\alpha(h, \kappa) = \mathrm{O}(\kappa/h)$;

(iii) when $(h, \kappa) \to$ point E or E', then $\beta(h, \kappa) = \mathrm{O}(1 - 3h/4h_m)^{1/4}$, $\alpha(h, \kappa) \sim \sqrt{q_1/3}$.

Physically, the steady spatially quasi-periodic flows, obtained from the center manifold expression (VII.16), correspond to *large scale modulations in* *amplitude and phase on Taylor vortex types of solutions*. From one solution (VII.45) we obtain a two-parameter family of solutions by shifting the origin in z and by shifting independently the origin in the periodic function $r_0(\omega_1 z)$. The important question of what happens when we take account of all terms in the amplitude equations (VII.17), i.e., not only the polynomial normal form (VII.21), is solved in [Io-Lo] for quasi-periodic solutions. The difficulty comes from the classical small divisor problem, and here the reversibility property plays an essential role (analogous to techniques for Hamiltonian systems). To state the results clearly, we need to introduce some terminology. First, we recall that a *diophantine number* δ is a real number such that there exist $C > 0$ and $\tau \geq 0$ such that for any rational number p/q one has $|p + q\delta| \geq C|q|^{-(1+\tau)}$. It is known that the set of diophantine numbers is full with respect to the Lebesgue measure. Another useful notion is the *rotation number* of a flow on a $\mathbb{T}^2$ torus, introduced by H. Poincaré. In fact, we just need to know here that for a quasi-periodic flow the rotation number is just the ratio between the two independent frequencies. Now, let us denote by $\Delta(\mu)$ the hatched regions in Figures VII.3–5, and $X_0 = (h, \kappa)$ any point in $\Delta(\mu)$. One can show the following (see the theorem in [Io-Lo]):

Let δ be any diophantine number such that for a given $\mu \neq 0$ there exists $\bar{X}_0 \in \Delta(\mu)$ satisfying

$$\delta = \frac{\omega_1(\mu, \bar{X}_0)}{\omega_0(\mu, \bar{X}_0)} \quad \text{and} \quad \text{Det } (\mathrm{D}_{X_0}\Omega)_{X_0 = \bar{X}_0} \neq 0.$$

Then, for any X_0 on a curve γ_δ in Δ close to $\bar{X}_0$ and for μ close enough to 0, the system admits a quasi-periodic solution with rotation number δ close to that given by the normal form at $\bar{X}_0$.

Remark 1. Generically, almost all $\bar{X}_0 \in \Delta(\mu)$ define a diophantine number.

Remark 2. The result can be expressed in saying that "most of" the quasi-periodic solutions we found with normal form (VII.21) persist when adding the "flat terms," and *the persistence set in the hatched region $\Delta(\mu)$ can be locally represented by a product of a curve with a Cantor set.*

VII.1.2.6 Eckhaus points E and E'

A special limit case is obtained at points E and E' where the cubic curve f_0 has an order 3 contact with the axis. This situation corresponds to the limit when quasi-periodic solutions become periodic with a cancelling wave number β in (VII.48). We show in Section VII.3 that this special bifurcation from periodic solutions is associated with the "Eckhaus instability limit." This limit is, in fact, obtained by starting with the Ginzburg-Landau equation (see Section VII.3) and studying the linear stability of the classical spatially periodic solutions. It was shown by Eckhaus [Eck] that these solutions are stable (temporally) when (κ, μ) lies inside a parabolic region bounded by the graph of a function $\mu = \mu_E(k) \approx 3\mu_c(k)$.

Indeed, we have at the limit, $\omega_1 = 0$, and $\omega_0 = k_c + \sqrt{\mu q_1/3} + \cdots$ (see (VII.46)–(VII.47)), so the limiting periodic solution has a wave number k satisfying

$$\mu_E = \frac{3}{q_1}(k - k_c)^2 + O[(k - k_c)^3] \approx 3\mu_c(k) \quad \text{[see (7. 33)].} \tag{VII.49}$$

VII.1.2.7 Homoclinic solutions

We noticed in Subsection VII.1.2.4 that on the curve Eh_mE' there exist homoclinic solutions joining the same steady state at both infinities, in the space $(u_0, u_1, \psi_1 - \psi_0)$. In fact, once $u_0(z)$ is known, one deduces u_1 by $u_1 = G(\mu, u_0, K) + H$, and for $K \neq 0$

$$\psi_1 - \psi_0 = \pm \arctan\, [K^{-1}\sqrt{u_0(G(\mu, u_0, K) + H) - K^2}] + \pi/2 + l\pi, \tag{VII.50}$$

(+ sign when u_0 decreases) where $l = 0$ or 1, depending on the sign K, in such a way that r_0 and r_1 are positive. Typical graphs of $r_0, r_1, \psi_1 - \psi_0$ are indicated in Figure VII.6.

For $K = 0$, a special study is needed, since $\psi_1 - \psi_0 = l\pi$, and it is not hard to see that the homoclinic solution has the shape indicated in Figure VII.7, if we admit negative values for r_0 so that $\psi_1 - \psi_0$ is kept constant. For the amplitudes A and B, if $K \neq 0$, one can choose the origin such that these

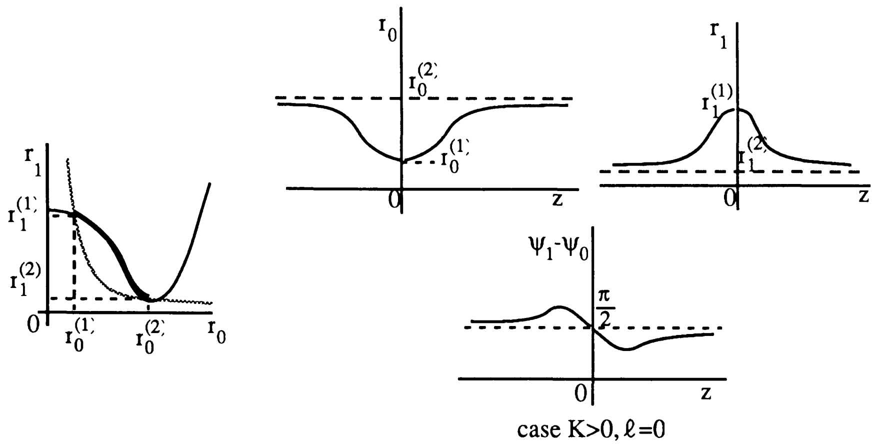


FIGURE VII.6. Homoclinic solutions on the curve Eh_mE' of Figure VII.3.

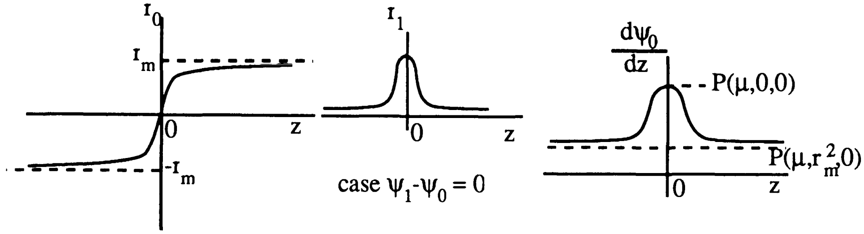


FIGURE VII.7. Homoclinic solution when $\kappa = 0$, $h = h_m$, for $\mu > 0$, $q_2 > 0$.

solutions have the following form (property (VII.44) holds):

$$A(z) = r_0(z)e^{i[k_c+\alpha]z+i\psi_0(z)},$$
$$B(z) = r_1(z)e^{i[k_c+\alpha]z+i[\psi_1(z)+\pi/2]}, \tag{VII.51}$$

where r_0, r_1 are even in z and tend toward constants at infinity while ψ_0, ψ_1 are odd and both tend to the same constants (mod π) when z tends toward $\pm\infty$. Notice that there are two solutions corresponding to $l = 0$ or 1 in (VII.50). These are the "*reversible*" solutions, homoclinic to a periodic one, with a phase shift between both infinities. When $K = 0$, one can also find two solutions that are invariant under the symmetry $\hat{\mathbf{S}}$ and $A|_{z=0} = 0$.

Remark 1. Physically, such homoclinic solutions would correspond to flows that are periodic in most of the domain, like the Taylor vortex solution obtained in Section III.1, except in a limited region where the amplitude and phase vary, the amplitude falling to a small value (maybe 0) in the middle of this region (where we are, locally in space, close to the basic Couette flow). Moreover, in general there is a phase shift between both infinities. The difference between the two "reversible-symmetric" solutions is a phase shift of half of the spatial period.

Remark 2. The delicate problem of what happens to the "pulse-like" and "front-like" solutions when we add the neglected terms of order $O(|A| + |B|)^N$ into (VII.21) is solved by Iooss and Pérouème [Io-Pé]. It is shown that the reversible-symmetric solutions persist under these perturbing flat terms. For the proof, reversibility of the system (VII.17) associated with the rotational invariance of its normal form (VII.21) is an essential tool.

Suppose now that the bifurcation to Taylor vortices is *subcritical*, as is the case for a small gap and slightly counterrotating cylinders [see the case $\eta = 0.95$ in Figure III.2]; then there is a homoclinic solution occurring for $\mu < 0$ (see Figure VII.5, for $h = \kappa = 0$) such that r_0 and r_1 go to 0 at infinity and $\psi_0 - \psi_1 = l\pi$. As above, there are two reversible-symmetric homoclinic solutions invariant under the symmetry $\hat{\mathbf{S}}(A, B)(z) = (\bar{A}, -\bar{B})(-z)$. Here

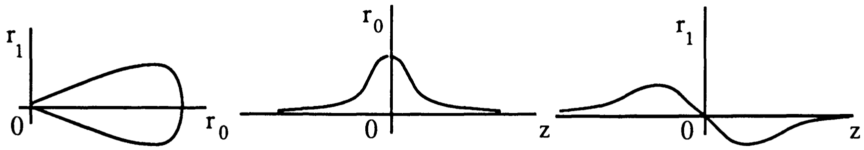


FIGURE VII.8. Homoclinic solution in the subcritical case, for $\mu < 0$.

we can take r_0 even and r_1 odd and $\psi_0 = \alpha z + \psi_0(z)$, with ψ_0 odd. We sketch in Figure VII.8 the shape of r_0 and r_1.

Remark 3. Physically, such homoclinic solutions would correspond to flows that are like the Couette flow in most of the domain, except in a limited region where the amplitude and phase vary and where, in the middle, the flow looks like the Taylor vortex flow (because of the form of the center manifold (VII.16).

Remark 4. The persistence, for the full system (VII.17), of these two reversible homoclinic solutions for $\mu < 0$, when the bifurcation is subcritical, is also proved in [Io-Pé].

VII.2 Time-periodic solutions in an infinite cylinder

VII.2.1 Center manifold for time-periodic Navier-Stokes equations

Starting again with Navier-Stokes system (VII.1) in the cylinder $Q = \Sigma \times \mathbb{R}$, we now search for *time-periodic solutions*. In this section we perform the center manifold reduction setting in a time-periodic functions space, the evolution variable being z, as in Section VII.1. The existence of a family of (z,t)-independent solutions $V = V^{(0)}(\mu, .) \in C^1(\bar{\Sigma}, \mathbb{R}^3)$ is given by the Couette flow solution. We are interested in the appearance of time-periodic solutions staying close to $V^{(0)}$ for all times t and all values of the axial variable $z \in \mathbb{R}$. We consider the situations corresponding to self-oscillating neutral modes, as studied in Section III.2, now with no restriction on any spatial periodicity, just imposing the boundedness of solutions. The method developed here will of course recover the well-known traveling wave solutions that are also periodic in z (see Section III.2). However, as the z-behavior is now not prescribed in advance, we also find new types of solutions that are quasi-periodic, homoclinic, or heteroclinic in the spatial direction, always modulated with some time frequency ω.

Looking for solutions with period $T = 2\pi/\omega$, it is convenient to introduce the scaled time $\tau = \omega t \in S^1$ so that the solutions will be 2π-periodic in τ and ω appears as a parameter in the equation. Using the notation of Section VII.1, equation (VII.1) takes the form

$$\frac{d\mathfrak{V}}{dz} = \mathcal{K}_{\mu,\omega}\mathfrak{V} + \mathcal{B}_\mu(\mathfrak{V}, \mathfrak{V}), \tag{VII.52}$$

where there are no longer z-derivatives on the right-hand side. With this notation, we have

$$\mathcal{K}_{\mu,\omega} = \mathcal{A}_\mu + \mathcal{L}_\mu + \omega\mathcal{E}, \tag{VII.53}$$

$$\mathcal{E}(\mathfrak{V}) = \begin{pmatrix} 0 \\ 0 \\ \partial U_z/\partial\tau \\ \partial U_\perp/\partial\tau \end{pmatrix}. \tag{VII.54}$$

From now on, we consider $\mathfrak{V} = (U, W)$ as a (vector-valued) function of $(\tau, y) \in S^1 \times \Sigma$, which will vary, according to (VII.52), along the axial variable $z \in \mathbb{R}$. In particular, we introduce the Hilbert spaces

$$\mathcal{H}^{\theta,s} = H^\theta[S^1, H^s(\Sigma)],$$

where $H^s(\Sigma)$ is the classical Sobolev space (see, for instance, Lions and Magenes [Li-Ma]) and $H^\theta[S^1, H^s(\Sigma)]$ denotes the space of $H^s(\Sigma)$-valued functions

$$u(\tau,.) = \sum_{n\in Z} u_n(.)e^{in\tau}, \quad \text{with } u_n \in H^s(\Sigma), \text{ such that}$$

$$||u||^2_{\theta,s} = \sum_{n\in Z}(1+n^2)^\theta ||u_n||^2_{H^s(\Sigma)} < \infty. \tag{VII.55}$$

We mainly need the cases $\theta = 0$ or 1 and $s = 0, 1$, or 2.

The phase space $\mathcal{X}$ is set equal to

$$\mathcal{X} = \{(U, W) \in (\mathcal{H}^{1,-1} \cap \mathcal{H}^{0,1}) \times (\mathcal{H}^{0,1})^2 \times (\mathcal{H}^{0,0})^3;\ U = 0 \text{ on } S^1 \times \partial\Sigma\}, \tag{VII.56}$$

and the linear operator $\mathcal{K}_{\mu,\omega}$ has domain

$$\begin{aligned} D(\mathcal{K}) = \{(U, W) \in \mathcal{X};\ & U \in (\mathcal{H}^{1,0} \cap \mathcal{H}^{0,2})^3,\ W \in (\mathcal{H}^{0,1})^3, \\ & \nabla_\perp . U_\perp = W_\perp = 0 \text{ on } S^1 \times \partial\Sigma\}. \end{aligned} \tag{VII.57}$$

For the nonlinear mapping $\mathcal{B}_\mu$ we then have the following theorem proved in [Io-Mi]:

Theorem 2. *For $\theta > (1 + \sqrt{17})/8 = 0.64\ldots$, the quadratic mapping $\mathcal{B}_\mu$, defined in (VII.5), is an analytic function from $D(\mathcal{K}^\theta) := [D(\mathcal{K}), \mathcal{X}]_\theta \subset$*

$(\mathcal{H}^{\theta,0} \cap \mathcal{H}^{0,1+\theta})^3 \times (\mathcal{H}^{0,\theta})^3$ *into the closed subspace* $\mathcal{Y} \subset \mathcal{X}$ *given by* $\mathcal{Y} := \{(U, W) \in \mathcal{X};\ U = 0\}$.

Remark. The interpolation functor $[.,.]_\theta$ is defined as in [Li-Ma]; in particular, we have

$$[H^s(\Sigma), H^t(\Sigma)]_\theta = H^{t+\theta(s-t)}(\Sigma), \qquad \text{whenever } 1/2 + t + \theta(s-t) \notin \mathbb{Z}.$$

In the analysis of the linear operator $\mathcal{K}_{\mu,\omega}$ some difficulties arise from the unsymmetric form of $\mathcal{A}_\mu$, involving the divergence equation in the first component and the pressure in the fourth. In fact, it is not possible to define $\mathcal{K}_{\mu,\omega}$ on a domain $D(\hat{\mathcal{K}})$ being compactly embedded in $\mathcal{X}$ such that the resolvent $(\hat{\mathcal{K}}_{\mu,\omega} - \lambda)^{-1} : \mathcal{X} \to D(\hat{\mathcal{K}})$ exists for some $\lambda \in \mathbb{C}$. One obvious reason is that $\mathcal{K}_{\mu,\omega}$ has an infinite-dimensional kernel spanned by $(U, W) = (0, 0, \alpha(\tau), 0)$ with arbitrary periodic functions α depending only on τ. Moreover, for each such α, the equation $\mathcal{K}_{\mu,\omega}(U, W) = (0, 0, \alpha, 0)$ has a solution $(U, W) = (\tilde{U}_z(\tau, y), 0, 0, 0)$. Note that this accounts for possibly prescribed pressure gradients in the cylinder that generate nonzero flux through each cross section.

Even cutting out this kernel does not resolve the problem. The problem arises because the pressure at time τ, which is $-W_z$ in our notation, is only a function of the velocity field at time τ. Hence the time dependence is not smoothed out, which would be necessary to obtain compactness.

To deal with the infinite-dimensional kernel, we define the projections

$$Pf = f - [f], \qquad [f](.) = \frac{1}{|\Omega|} \int_\Omega f(., y)\, dy, \tag{VII.58}$$

$$\mathfrak{Q}(U_z, U_\perp, W_z, W_\perp) = (PU_z, U_\perp, PW_z, W_\perp), \tag{VII.59}$$

and decompose $\mathfrak{V} \in \mathcal{X}$ into $\bar{\mathfrak{V}} + \tilde{\mathfrak{V}}$, where $\bar{\mathfrak{V}} = ([U_z], 0, [W_z], 0)$ and $\tilde{\mathfrak{V}} = \mathfrak{Q}\mathfrak{V}$. Applying projectors $I - \mathfrak{Q}$ and $\mathfrak{Q}$ to (VII.52), we obtain the system

$$\begin{cases} \dfrac{\partial}{\partial z}[U_z] = 0, \\ \dfrac{\partial}{\partial z}[W_z] = [-v\Delta_\perp(PU_z) + \nabla_\perp . W_\perp + (V_\perp^{(0)} . \nabla_\perp)PU_z \\ \quad +(U_\perp . \nabla_\perp)V_z^{(0)} - V_z^{(0)}\nabla_\perp . U_\perp + (U_\perp . \nabla_\perp)PU_z \\ \quad -PU_z(\nabla_\perp . U_\perp)] + \omega\dfrac{\partial}{\partial \tau}[U_z], \end{cases} \tag{VII.60}$$

$$\frac{d}{dz}\tilde{\mathfrak{V}} = \tilde{\mathcal{K}}_{\mu,\omega}\tilde{\mathfrak{V}} + \tilde{\mathcal{B}}_\mu([U_z], \tilde{\mathfrak{V}}), \tag{VII.61}$$

where
$$\tilde{\mathcal{K}}_{\mu,\omega} : \begin{cases} D(\tilde{\mathcal{K}}) = \mathfrak{Q}D(\mathcal{K}) \to \tilde{\mathcal{X}} = \mathfrak{Q}\mathcal{X}; \\ \qquad\qquad \tilde{\mathfrak{V}} \to \mathfrak{Q}\mathcal{K}_{\mu,\omega}\tilde{\mathfrak{V}}, \end{cases}$$

and
$$\tilde{\mathcal{B}}_\mu([U_z], \tilde{\mathfrak{V}}) = \mathfrak{Q}\mathcal{B}_\mu(\bar{\mathfrak{V}} + \tilde{\mathfrak{V}}, \bar{\mathfrak{V}} + \tilde{\mathfrak{V}}) \in \mathfrak{Q}\mathcal{Y} =: \tilde{\mathcal{Y}}.$$

Now, $[W_z]$ does not appear in $\tilde{\mathcal{B}}_\mu$ and $\beta(\tau) = [U_z(\tau, .)]$ is a z-independent volumic flux, according to (VII.60), which itself is a consequence of the incompressibility condition ($\nabla_\perp.U_\perp + \frac{\partial}{\partial z}U_z = 0$) and of the boundary condition $U_\perp = 0$ on $\partial\Omega$.

Hence, having fixed $\beta = \beta(\tau)$, we may first solve (VII.61), and then the mean value of the pressure $-[W_z]$ can be obtained by integrating (VII.60). In the following, we will restrict the analysis to the case when $[U_z] \equiv 0$, which is reasonable for the Couette-Taylor problem. We rely heavily on the reversibility property related to the invariance under reflection $z \to -z$. To keep this, we have to impose $[U_z] \equiv 0$, which also agrees with known experiments and numerical simulations—see the work by Raffaï and Laure [Ra-La 92] and [Raf] for details on comparisons between direct numerical computations, experimental, and those obtained by computing the amplitude equation. We shall recover the spatially periodic bifurcating solutions obtained by the classical method (see Section III.2), in the case when we use the functional frame defined in Subsection II.2.2.

A given time-varying $[U_z]$ could of course give rise to interesting phenomena. In particular, it would destroy the autonomy of the system and hence the symmetry under time shift, which is exploited below. A study with a nonzero constant $[U_z]$ would complete the study made in Section V.3.

In [Io-Mi] the following results on the linear operator $\tilde{\mathcal{K}}_{\mu,\omega}$ are proved:

Theorem 3. *There is a positive δ such that the resolvent $(\tilde{\mathcal{K}}_{\mu,\omega} - \lambda)^{-1} : \tilde{\mathcal{X}} \to D(\tilde{\mathcal{K}})$ is a meromorphic function of $\lambda \in \mathbb{C}_\delta := \{\lambda \in \mathbb{C};\ |Re\,\lambda| < \delta(1 + |\mathrm{Im}\,\lambda|)\}$ (see Figure VII.9).*

(i) *In $\mathbb{C}_\delta$ there are only finitely many eigenvalues of $\tilde{\mathcal{K}}_{\mu,\omega}$ (i.e., poles of $(\tilde{\mathcal{K}}_{\mu,\omega} - \lambda)^{-1}$) each having a finite-dimensional generalized eigenspace. Functions in these eigenspaces are finite sums of terms like $e^{in\tau}\mathfrak{V}_n(y)$.*

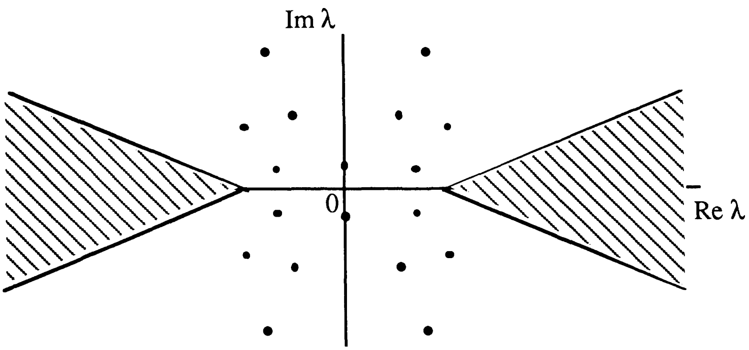


FIGURE VII.9. Spectrum of $\tilde{\mathcal{K}}_{\mu,\omega}$. The complement of region $\mathbb{C}_\delta$ is hatched.

(ii) *Moreover, for any fixed $\theta \in [0,1]$ and for $\lambda \in \mathbb{C}_\delta$ with $|\lambda| \to \infty$ the following estimates hold:*

$$||(\tilde{\mathcal{K}}_{\mu,\omega} - \lambda)^{-1}||_{\mathcal{L}(\tilde{\mathcal{X}},\tilde{\mathcal{X}})} = \mathcal{O}(1), \tag{VII.62}$$

$$||(\tilde{\mathcal{K}}_{\mu,\omega} - \lambda)^{-1}||_{\mathcal{L}(\tilde{\mathcal{Y}},\mathfrak{Q}D\mathcal{K}^{\theta}))} = \mathcal{O}(1/|\lambda|^{1-\theta}). \tag{VII.63}$$

Remarks. (i) The decay estimate (VII.63) only holds for the restricted resolvent $(\tilde{\mathcal{K}}_{\mu,\omega} - \lambda)^{-1}|_{\tilde{\mathcal{Y}}}$, but this is sufficient, as the nonlinear terms $\tilde{\mathcal{B}}_\mu([U_x], \mathfrak{V})$ have only values in $\tilde{\mathcal{Y}}$.

(ii) We cannot expect $(\mathcal{K}_{\mu,\omega} - \lambda)^{-1}$ to be meromorphic in the whole complex plane. In fact, it is shown in an example treated in [Io-Mi] that the eigenvalues may have accumulation points.

Theorem 3 implies that $\tilde{\mathcal{K}}_{\mu,\omega}$ has for any (μ,ω) only a finite-dimensional invariant space belonging to isolated spectral part on the imaginary axis; let us call it $\mathcal{X}^0$ for $(\mu,\omega) = (0,\omega_0)$ (to be specified in Subsection VII.2.2), and define the corresponding $\tilde{\mathcal{K}}_{0,\omega_0}$- invariant projection $\mathcal{P}^0$ by the Dunford integral

$$\mathcal{P}^0 = \frac{1}{2i\pi} \int_{\Gamma_0} (\lambda - \tilde{\mathcal{K}}_{0,\omega_0})^{-1}\, d\lambda,$$

where $\Gamma_0 \subset \mathbb{C}_\delta$ is a curve surrounding exactly the eigenvalues on the imaginary axis. Hence, $\mathcal{K}^0 = \tilde{\mathcal{K}}_{0,\omega_0}|_{\mathcal{X}_0}$ has only eigenvalues on the imaginary axis. It can be shown now that the center manifold theory, as developed in [Mi 86a] or [Va-Io 89], is applicable, when, according to Theorem 2, θ is chosen in the interval $\big((1+\sqrt{17})/8, 1\big)$. We have

Theorem 4. *For each integer k there are neighborhoods $\mathcal{O}_k$ of 0 in $\mathcal{X}^0$ and o_k of $(0,\omega_0)$ in $\mathbb{R}^2$ and a C^k function $\Phi : o_k \times \mathcal{O}_k \to (\mathrm{Id} - \mathcal{P}^0)\tilde{\mathcal{X}}$ such that the manifold*

$$M_c = \{\tilde{\mathfrak{V}} = \tilde{\mathfrak{V}}_0 + \Phi(\mu,\omega,\tilde{\mathfrak{V}}_0);\ (\mu,\omega,\tilde{\mathfrak{V}}_0) \in o_k \times \mathcal{O}_k\}$$

contains every small bounded solution of (VII.61) and (VII.52). Moreover, every solution of the reduced equation

$$\begin{aligned}\frac{d}{dz}\tilde{\mathfrak{V}}_0 = \mathcal{K}^0\tilde{\mathfrak{V}}_0 + \mathcal{P}^0\tilde{\mathcal{B}}_\mu([U_z], \tilde{\mathfrak{V}}_0 + \Phi(\mu,\omega,\tilde{\mathfrak{V}}_0))\\ +\mathcal{P}^0(\tilde{\mathcal{K}}_{\mu,\omega} - \mathcal{K}^0)[\tilde{\mathfrak{V}}_0 + \Phi(\mu,\omega,\tilde{\mathfrak{V}}_0)]\end{aligned} \tag{VII.64}$$

yields through $\tilde{\mathfrak{V}} = \tilde{\mathfrak{V}}_0 + \Phi(\mu,\omega,\tilde{\mathfrak{V}}_0)$ a solution of (VII.61).

If the system (VII.61) is equivariant with respect to some symmetry or if it is reversible, then so is (VII.64).

One special symmetry of our problem arises from the autonomy of (VII.1), which propagates to (VII.61) provided that $[U_z]$ is independent of τ. From Theorem 3(i) we know that $\tilde{\mathfrak{V}}_0 \in \mathcal{X}^0$ is of the form

$$\tilde{\mathfrak{V}}(\tau, y) = \sum_{r=1}^{N} A_r e^{im_r\tau}\mathfrak{V}_r(y) \quad \text{(real sum)},$$

so the reduced equation (VII.64) can now be written in terms of the complex vector $\mathcal{A} = (A_1, \dots, A_N)$.

VII.2.2 Spectrum of $\tilde{\mathcal{K}}_{\mu,\omega}$ near criticality

Let us come back to the Navier-Stokes equations (VII.1), linearized around $V^{(0)}$. This system is translationally invariant, so let us denote by $(z, y) \to \hat{U}_k(y)e^{ikz}$ an eigenvector belonging to the eigenvalue $\sigma(\mu, ik)$. The classical hydrodynamical instability threshold (criticality) occurs when the eigenvalue of largest real part $\sigma_0(\mu, ik)$ satisfies the following properties (where criticality is defined by $\mu = 0$):

$$\operatorname{Re}\sigma_0(0, ik_c) = 0, \qquad \frac{\partial}{\partial k}\operatorname{Re}\sigma_0(0, ik_c) = 0, \qquad \frac{\partial}{\partial \mu}\operatorname{Re}\sigma_0(0, ik_c) > 0, \tag{VII.65}$$

and $\operatorname{Re}\sigma_0(\mu, ik)$ is maximum at $k = k_c$, $\mu = 0$ (see Figure VII.1(a) with $\operatorname{Re}\sigma_0$ instead of σ_0). In what follows, we assume that $k_c \neq 0$ and $\operatorname{Im}\sigma_0(0, ik_c) = \omega_0 \neq 0$ as it is the case in Section III.2. It results from these properties that one can write the Taylor expansion of $\sigma_0(\mu, \lambda)$ for (μ, λ) in $\mathbb{R} \times \mathbb{C}$ near $(0, ik_c)$ under the form (indices r and i mean real and imaginary parts):

$$\sigma_0(\mu, \lambda) = i\omega_0 + a\mu + e_1(\lambda - ik_c) + e_2(\lambda - ik_c)^2 + e_3\mu^2 + e_4\mu(\lambda - ik_c) + \text{ h.o.t.}, \tag{VII.66}$$

where

$$a_r > 0, \qquad e_1 \in \mathbb{R}, \qquad e_{2r} > 0.$$

We check that the neutral stability curve of Figure VII.1(b), given by $\operatorname{Re}\sigma_0(\mu, ik) = 0$, takes the form

$$\mu = \mu_c(k) = \frac{e_{2\mathrm{r}}}{a_{\mathrm{r}}}(k - k_c)^2 + \text{ h.o.t.}, \tag{VII.67}$$

and, on this curve, we have

$$\operatorname{Im}\sigma_0(\mu_c(k), ik) = \omega_c(k) = \omega_0 + e_1(k - k_c) + \left(\frac{a_{\mathrm{i}} e_{2r}}{a_r} - e_{2\mathrm{i}}\right)(k - k_c)^2 + \text{ h.o.t.}. \tag{VII.68}$$

Notice that $\bar{\sigma}_0(\mu, ik)$ is an eigenvalue belonging to $\bar{\hat{U}}_k(y)e^{-ikz}$, however, one cannot use expression (VII.66) for λ near $-ik_c$.

We wish to link the above knowledge of the spectrum of the traditional linearized operator for Navier-Stokes equations to the spectrum of our new linear operator $\tilde{\mathcal{K}}_{\mu,\omega}$ for (μ, ω) near $(0, \omega_0)$ (analogously to Section VII.1). When $\operatorname{Re}\sigma_0(\mu, ik) = 0$, i.e., when $\mu = \mu_c(k)$, we obtain a time-periodic solution of the linearized Navier-Stokes equations of the form $\hat{U}_k(y)e^{ikz}e^{i\omega t}$, with $\omega = \omega_c(k)$. With our formulation in $\mathfrak{V} = (U, W)$, this corresponds to the existence of an eigenvalue ik for $\mathcal{K}_{\mu,\omega}$; hence, since $k \neq 0$, to the same eigenvalue for $\tilde{\mathcal{K}}_{\mu,\omega}$, with an eigenvector of the form $(\tau, y) \to \mathfrak{V}(y)e^{i\tau}$. In addition, we also observe that $\bar{\mathfrak{V}}(y)e^{-i\tau}$ is an eigenvector belonging to the eigenvalue $-ik$ of $\tilde{\mathcal{K}}_{\mu,\omega}$. So, for $\mu = \mu_c(k)$, $\omega = \omega_c(k)$, we know eigenvalues $\pm ik$ of $\tilde{\mathcal{K}}_{\mu,\omega}$ on the imaginary axis, and, by construction, other eigenvalues of this operator do not belong to this axis. Our original system is invariant under the reflection symmetry $z \to -z$, and $V^{(0)}$ respects this invariance, since $V_z^{(0)} = 0$. This invariance propagates to the perturbed Navier-Stokes system satisfied by U and also on the linearized system. We know that if $\hat{U}_k(y)e^{ikz}$ is an eigenvector, so is $\mathbf{S}\hat{U}_k(y)e^{-ikz}$ belonging to the same eigenvalue and we have $\sigma_0(\mu, ik) = \sigma_0(\mu, -ik)$.

Our system (VII.52) is now *reversible*. This means, as in Section VII.1, that we have the anticommutation properties

$$\hat{\mathbf{S}}\mathcal{K}_{\mu,\omega} = -\mathcal{K}_{\mu,\omega}\hat{\mathbf{S}}, \qquad \hat{\mathbf{S}}\mathcal{B}_{\mu} = -\mathcal{B}_{\mu^\circ}\hat{\mathbf{S}}. \tag{VII.69}$$

We then have for the system (VII.61)

$$\hat{\mathbf{S}}\tilde{\mathcal{K}}_{\mu,\omega} = -\tilde{\mathcal{K}}_{\mu,\omega}\hat{\mathbf{S}}, \qquad \hat{\mathbf{S}}\tilde{\mathcal{B}}_{\mu}([U_z], \tilde{\mathfrak{V}}) = -\tilde{\mathcal{B}}_{\mu}(-[U_z], \hat{\mathbf{S}}\tilde{\mathfrak{V}}), \tag{VII.70}$$

so, this system is also reversible whenever $[U_z] = 0$.

For all reversible systems, if λ is an eigenvalue, then $-\lambda$ is an eigenvalue too, corresponding to the symmetric eigenvector. Finally, for $(\mu, \omega) = (0, \omega_0)$ we have two double semisimple eigenvalues $\pm ik_c$ with the following eigenvectors:

$$\begin{cases} \mathfrak{V}_0(y)e^{i\tau}, \hat{\mathbf{S}}\bar{\mathfrak{V}}_0(y)e^{-i\tau} \in ik_c, \\ \bar{\mathfrak{V}}_0(y)e^{-i\tau}, \hat{\mathbf{S}}\mathfrak{V}_0(y)e^{i\tau} \in -ik_c. \end{cases} \tag{VII.71}$$

Now, what happens for the spectrum of $\tilde{\mathcal{K}}_{\mu,\omega}$ if (μ, ω) lies in a neighborhood of $(0, \omega_0)$, not necessarily on the neutral curve $\mu = \mu_c(k)$, $\omega = \omega_c(k)$?

We may observe, more generally in (VII.66), that if $\operatorname{Re}\sigma_0(\mu, \lambda) = 0$, we have an eigenvalue λ for $\tilde{\mathcal{K}}_{\mu,\omega}$ where $\omega = \operatorname{Im}\sigma_0(\mu, \lambda)$. Here λ is not necessarily purely imaginary, so we obtain a new information on the spectrum of $\tilde{\mathcal{K}}_{\mu,\omega}$. Let us solve with respect to λ the system

$$\operatorname{Re}\sigma_0(\mu, \lambda) = 0, \qquad \omega = \operatorname{Im}\sigma_0(\mu, \lambda).$$

Provided that the coefficient e_1 is not zero, we obtain

$$\begin{aligned}\lambda(\mu, \omega) = ik_c &- \frac{1}{e_1}[a\mu - i(\omega - \omega_0)] - \frac{e_2}{e_1^3}[a\mu - i(\omega - \omega_0)]^2 \\ &- \frac{e_3}{e_1}\mu^2 + \frac{e_4}{e_1^2}\mu[a\mu - i(\omega - \omega_0)] + \text{ h.o.t.}\end{aligned} \tag{VII.72}$$

and $\operatorname{Re}\lambda = 0$ gives again the neutral curve, now under the form (see Figure VII.10)

$$\mu = \mu^c(\omega) = \frac{e_{2r}}{a_r e_1^2}(\omega - \omega_0)^2 + \text{ h.o.t.}. \tag{VII.73}$$

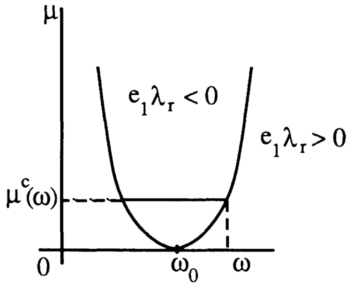


FIGURE VII.10. Curve $\lambda_r(\mu, \omega) = 0$.

It results that, when (μ, ω) is near $(0, \omega_0)$, the spectrum of $\tilde{\mathfrak{K}}_{\mu,\omega}$ contains four simple eigenvalues close to the imaginary axis, $\pm\lambda$, and $\pm\bar{\lambda}$, symmetric with respect to both axes in the complex plane, where λ is given by formula (VII.72).

Remark. The semisimplicity of eigenvalues is ensured here by the condition $e_1 \neq 0$ in (VII.66). If $e_1 = 0$, we then obtain Jordan blocks as in Section VII.1, by differentiating the following identity with respect to k at $(\mu, \omega, k) = (0, \omega_0, k_c)$:

$$\tilde{\mathfrak{K}}_{\mu_c(k),\omega_c(k)} \mathfrak{V}_k e^{i\tau} = ik\mathfrak{V}_k e^{i\tau}.$$

VII.2.3 Resolution of the four-dimensional amplitude equations. New solutions

Following the results of Subsection VII.2.2 and Theorem 4, the bifurcating time-periodic solutions, bounded for $z \in \mathbb{R}$, are solutions of the following system of complex amplitudes equations:

$$\begin{cases} \dfrac{dA}{dz} &= ik_c A + N_0(\mu, \omega, A, B, \bar{A}, \bar{B}), \\ \dfrac{dB}{dz} &= -ik_c B + N_1(\mu, \omega, A, B, \bar{A}, \bar{B}), \end{cases} \tag{VII.74}$$

where A and B are defined by the decomposition of the four-dimensional vector $\tilde{\mathfrak{V}}_0$ in $\mathfrak{X}^0$ and

$$\tilde{\mathfrak{V}} = Ae^{i\tau}\mathfrak{V}_0(y) + \bar{A}e^{-i\tau}\bar{\mathfrak{V}}_0(y) + Be^{i\tau}\mathbf{S}\mathfrak{V}_0(y) + \bar{B}e^{-i\tau}\mathbf{S}\bar{\mathfrak{V}}_0(y) \\ +\Phi(\mu, \omega, A, \bar{A}, B, \bar{B}), \tag{VII.75}$$

the vector field Φ taking values in $\tilde{\mathfrak{X}}$ and giving explicitly the center manifold.

We know from Theorem 4 that (VII.74) is equivariant under the group σ_α (time shift $\tau \to \tau + \alpha$) and is reversible. This first means that, for any real α, we have for $j = 0, 1$

$$N_j(\mu, \omega, e^{i\alpha}A, e^{i\alpha}B, e^{-i\alpha}\bar{A}, e^{-i\alpha}\bar{B}) = e^{i\alpha}N_j(\mu, \omega, A, B, \bar{A}, \bar{B}). \quad \text{(VII.76)}$$

Secondly, reversibility of (VII.74) has to be understood with the representation of the action of $\mathbf{S}$ on (A, B), which is by construction $(A, B) \to (B, A)$. For (VII.74) this leads to

$$N_0(\mu, \omega, B, A, \bar{B}, \bar{A}) = -N_1(\mu, \omega, A, B, \bar{A}, \bar{B}). \quad \text{(VII.77)}$$

Now, we can also put the system (VII.74) into "normal form." This means that we can choose suitable coordinates such that our system, truncated at any fixed arbitrary order, looks as simple as possible. This normalization results directly from the structure of the linear operator in (VII.74), which is diagonal here. The result (see [E-T-B-C-I], for instance) is that one can choose coordinates such that the truncated vector field (P_0, P_1) commutes with the action of the fundamental group of the linear part: $(A, B) \to (e^{ik_c z}A, e^{-ik_c z}B)$, $z \in \mathbb{R}$. We note that this group action differs from the action of σ_α.

It is then easy to see that the normal form of the amplitude equation (VII.74) is as follows:

$$\begin{cases} \dfrac{dA}{dZ} = ik_c A + AP[\mu, \omega, |A|^2, |B|^2], \\ \dfrac{dB}{dZ} = -ik_c B - BP[\mu, \omega, |B|^2, |A|^2], \end{cases} \quad \text{(VII.78)}$$

where P is a polynomial in its last two arguments taking complex values, such that $P[0, \omega_0, 0, 0] = 0$, and where, by construction,

$$ik_c + P(\mu, \omega, 0, 0) = \lambda(\mu, \omega) \quad \text{(VII.79)}$$

is given by formula (VII.72).

Remark. It should be possible to show that the coefficients of the principal part in $|A|^2$ and $|B|^2$ in polynomial P are the same as in usual Hopf bifurcation computations in the presence of O(2) symmetry, as in Section III.2, multiplied by the factor $(e_1)^{-1}$.

The full system (VII.74) may be written as a vector field of the form (VII.78) completed by adding a vector field $(R_0, R_1) = O[(|A| + |B|)^N]$ that satisfies invariance properties (VII.76)–(VII.77). This remark is useful for studying the precise structure of bounded solutions of (VII.74).

Let us just study here the bounded solutions of the normal form (VII.78). For this, let us introduce polar coordinates:

$$A = r_0 e^{i(k_c z + \psi_0)}, \qquad B = r_1 e^{-i(k_c z + \psi_1)}. \quad \text{(VII.80)}$$

We then obtain a system in r_0 and r_1 uncoupled from the phases

$$\begin{cases} \dfrac{dr_0}{dz} = r_0 P_{\mathrm{r}}(\mu,\omega,r_0^2,r_1^2) = r_0[\lambda_r(\mu,\omega) + b_r r_0^2 + c_r r_1^2 + \text{ h.o.t.}], \\ \dfrac{dr_1}{dz} = -r_1 P_{\mathrm{r}}(\mu,\omega,r_1^2,r_0^2) = -r_1[\lambda_r(\mu,\omega) + c_r r_0^2 + b_r r_1^2 + \text{ h.o.t.}]; \end{cases} \tag{VII.81}$$

$$\begin{cases} \dfrac{d\psi_0}{dz} = P_{\mathrm{i}}(\mu,\omega,r_0^2,r_1^2) = \lambda_{\mathrm{i}}(\mu,\omega) - k_{\mathrm{c}} + b_{\mathrm{i}} r_0^2 + c_{\mathrm{i}} r_1^2 + \text{ h.o.t.}, \\ \dfrac{d\psi_1}{dz} = P_{\mathrm{i}}(\mu,\omega,r_1^2,r_0^2) = \lambda_{\mathrm{i}}(\mu,\omega) - k_{\mathrm{c}} + c_{\mathrm{i}} r_0^2 + b_{\mathrm{i}} r_1^2 + \text{ h.o.t.}. \end{cases} \tag{VII.82}$$

We are reduced to studying the two-dimensional vector field in (r_0, r_1), which appears to be *integrable*. We can of course restrict the analysis to the quadrant $r_0 \geq 0$, $r_1 \geq 0$. In fact, if $b_{\mathrm{r}} \neq 0$, and $c_{\mathrm{r}} - b_{\mathrm{r}} \neq 0$, there is an explicit integral for the system (VII.81) truncated at the cubic order:

$$H(r_0, r_1) = (r_0 r_1)^{2b_{\mathrm{r}}/(c_{\mathrm{r}}-b_{\mathrm{r}})}[\lambda_{\mathrm{r}}/b_{\mathrm{r}} + r_0^2 + r_1^2]. \tag{VII.83}$$

We have no explicit integral for the higher-order system, but it is not hard to show that the trajectories are very similar to those obtained for the cubic vector field. Equilibria other than 0 are $(r_0, 0), (0, r_0)$, which exist for $\lambda_{\mathrm{r}} b_{\mathrm{r}} < 0$, and (r_1, r_1), which exists for $\lambda_{\mathrm{r}}(b_{\mathrm{r}} + c_{\mathrm{r}}) < 0$, and such that

$$P_{\mathrm{r}}(\mu,\omega,r_0^2,0) = 0, \qquad P_{\mathrm{r}}(\mu,\omega,r_1^2,r_1^2) = 0. \tag{VII.84}$$

The axes $r_0 = 0$ and $r_1 = 0$ are invariant manifolds, on which the dynamics is easy to determine. Equilibrium points $(r_0, 0)$ and $(0, r_0)$ are saddles for $\lambda_r(b_{\mathrm{r}} - c_{\mathrm{r}}) > 0$, while they are nodes for $\lambda_{\mathrm{r}}(b_{\mathrm{r}} - c_{\mathrm{r}}) < 0$, and the equilibrium (r_1, r_1) is a saddle for $c_{\mathrm{r}}^2 - b_{\mathrm{r}}^2 < 0$, while it is a center if $c_{\mathrm{r}}^2 - b_{\mathrm{r}}^2 > 0$. Trajectories cut the diagonal orthogonally, except at the equilibrium points 0 and (r_1, r_1). If $(r_0(z), r_1(z))$ is a solution of (VII.81), then $(r_1(-z), r_0(-z))$ is also a solution (symmetric with respect to the diagonal). Since the divergence of the vector field has the sign of $(r_0^2 - r_1^2)b_{\mathrm{r}}$, it does not cancel except on the diagonal; hence, any closed orbit is symmetric with respect to this diagonal. The trajectories connecting equilibrium points are explicit on (VII.83), and the proof of their existence for the higher-order system (VII.81) follows from perturbation arguments and is left to the reader. We give in Figure VII.11 the phase portraits in the plane (r_0, r_1) in the case $\lambda_{\mathrm{r}} > 0$ (which is not a real restriction), depending on the values of the main nonlinear coefficients (b_{r} and c_{r}) in P_{r}.

Equilibrium solution 0 corresponds to the basic solution $V^{(0)}$ of (VII.1) (Couette flow). Now, the solution of the form $(r_0, 0)$ gives

$$\psi_0 = \beta_0 z + \varphi_0, \quad \text{with } k_0 = k_{\mathrm{c}} + \beta_0 = \lambda_{\mathrm{i}}(\mu,\omega) + b_{\mathrm{i}} r_0^2 + \text{ h.o.t.}, \tag{VII.85}$$

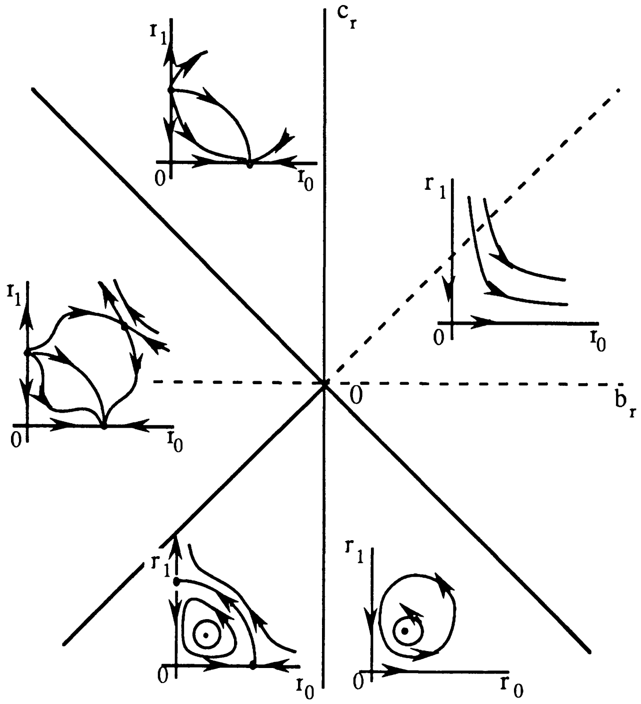


FIGURE VII.11. Phase portraits ($\lambda_r > 0$) depending on coefficients of P_r.

and the corresponding flow $\tilde{\mathfrak{V}}_0$ on the center manifold has the form

$$\tilde{\mathfrak{V}}_0(\tau, z, y) = r_0 e^{i[k_0 z+\tau+\varphi_0]}\mathfrak{V}_0(y) + \text{ c.c.}, \tag{VII.86}$$

which leads to the invariance properties $\sigma_{k_0\alpha}\mathbf{T}_{-\alpha}\tilde{\mathfrak{V}} = \tilde{\mathfrak{V}}$, $\mathbf{T}_{2\pi/k_0}\tilde{\mathfrak{V}} = \tilde{\mathfrak{V}}$ (as on the center manifold), where we now denote by $\mathbf{T}_\alpha$ the representation of the axial shift $z \to z+\alpha$. This solution then corresponds to traveling waves as well as the symmetric solution $(0, r_0)$ of (VII.81), which travels in the opposite direction. We recover, in fact, one of the spirals obtained when assuming spatial periodicity. The *spiral structure of the traveling waves* results from the structure of $\mathfrak{V}_0(y)$, which is a vector field function of the radial coordinate r multiplied by $\exp(im\theta)$, leading to a velocity vector field only depending on $(r, kx + \omega t + m\theta)$ (see Subsection III.2.3). Until now, we only showed that this result is true on the truncated system (at any order!); we need to show the persistence of such solutions when taking account of the flat terms (R_0, R_1), which are not in normal form. This is completely done in [Io-Mi], where it is shown that we have indeed *traveling waves for the full problem.*

The solution of the form (r_1, r_1) gives

$$\psi_j = \beta_1 z + \varphi_j, \quad j = 0, 1, \qquad \text{with } k_1 = k_c + \beta_1 = \lambda_i(\mu, \omega) + (b_i + c_i) r_1^2 + \text{ h.o.t.}, \tag{VII.87}$$

and the corresponding flow $\tilde{\mathfrak{V}}_0$ on the center manifold has the form

$$\tilde{\mathfrak{V}}_0(\tau, z, y) = r_1 e^{i[\tau + k_1 z + \varphi_0]} \mathfrak{V}_0(y) + r_1 e^{i[\tau - k_1 z - \varphi_1]} \hat{\mathbf{S}} \mathfrak{V}_0(y) + \text{ c.c.}. \tag{VII.88}$$

This principal part of the flow satisfies the invariance properties

$$\mathbf{T}_{2\pi/k_1} \tilde{\mathfrak{V}}_0 = \tilde{\mathfrak{V}}_0, \qquad \sigma_\pi \mathbf{T}_{\pi/k_1} \tilde{\mathfrak{V}}_0 = \tilde{\mathfrak{V}}_0, \tag{VII.89}$$

and

$$\hat{\mathbf{S}} \tilde{\mathfrak{V}}_0(\tau, z, y) = \tilde{\mathfrak{V}}_0(\tau, -z, y), \quad \text{provided that } \varphi_0 = -\varphi_1. \tag{VII.90}$$

Moreover, any solution in the family is obtained by letting the linear operators $\sigma_{(\varphi_1 - \varphi_0)/2}$ and $\mathbf{T}_{-(\varphi_1 + \varphi_0)/2k_1}$ act successively on the solution given for $\varphi_0 = \varphi_1 = 0$. These properties propagate on the center manifold, due to the commutativity of $\mathbf{T}$ and σ with $\mathbf{\Phi}$ in (VII.75); hence, they also apply to the full solution $\tilde{\mathfrak{V}}$ of (VII.61). It is then clear that we recover "the standing waves" (ribbons) obtained classically when assuming spatial periodicity (see Subsection III.2.3). It is shown in [Io-Mi] that *this family of standing waves indeed exists for the full system* (VII.74), even in considering "flat" high-order terms (R_0, R_1).

Remark. If $e_1 < 0$, we see in Figure VII.10 that the case of Figure VII.11 $(\lambda_r > 0)$ corresponds to $\mu > \mu^c(\omega)$. Then, we observe that the situation $b_r < 0$, $(b_r - c_r) > 0$, which gives *saddle points* for the traveling waves, corresponds in fact to the situation when, in the classical spatially periodic analysis, they are *attracting nodes* for the usual stability analysis in the class of spatially periodic solutions (see Subsection III.2.1). The same remark holds for the standing waves when $(b_r + c_r) < 0$ and $(b_r - c_r) < 0$. Now, if $e_1 > 0$, the same case corresponds to the phase portraits for $\lambda_r < 0$.

Closed orbits in the (r_0, r_1) plane correspond to periodic solutions $[r_0(z), r_1(z)]$ with some period denoted by H. Let us show that this leads to spatially quasi-periodic solutions with two basic frequencies. In fact, we can first choose the origin in z such that $r_0(0) = r_1(0)$ (there are two such points); then $r_0(z) = r_1(-z)$ by the uniqueness of the solution of (VII.81) and we have, after integrating the phases:

$$\begin{aligned} \psi_0 &= \beta_z + h_0(z) + \varphi_0, \\ \psi_1 &= \beta_z - h_0(-z) + \varphi_1, \end{aligned} \tag{VII.91}$$

where h_0 is H-periodic in z with 0 mean value and

$$\beta = \frac{1}{H} \int_0^H P_i(\mu, \omega, r_0^2(z), r_0^2(-z))\, dz. \tag{VII.92}$$

Now the principal part $\tilde{\mathfrak{V}}_0$ of the flow has two fundamental spatial periods, namely, H and $2\pi/k$, where $k = k_c+\beta$, and this property clearly propagates on the full solution $\tilde{\mathfrak{V}}$. For any "good" fixed value of μ and ω, there is a *four-parameter family* of such solutions: one parameter corresponds to the level curves in the (r_0, r_1) plane, and the three other parameters correspond to independent shifts on the phases (φ_0, φ_1) and on the origin in z. The free shift on the time origin is then included in the phase shifts. This shows that we have a large family of solutions both spatially quasi-periodic and time-periodic, for the normal form. It is shown in [Io-Mi] that *most of these solutions persist* when we study the complete vector field (including flat terms), in a way analogous to the result given in the steady case in Subsection VII.1.2.5.

About the heteroclinic connections between saddles in Figure VII.11 (joining the two symmetric traveling waves), it is shown in [Io-Mi] that the reversible solutions persist for the complete system. In addition, it is shown that the separatrices joining the standing waves (ribbons) with the traveling waves (spirals) and trajectories connecting symmetric spirals persist for the complete vector field. Other heteroclinic connections do not persist in general. All these solutions physically look like juxtapositions of the two limiting regimes connected by a region of the space of size $O(\lambda_r^{-1})$. In the case of a heteroclinic connection of the two symmetric spiral wave regimes, the solution exhibits a *"defect"* (see Coullet et al. [Co-El-Gi-Le] for details and references on defects in waves). This solution is clearly observed in experiments. It has the form of two spiral waves, symmetric with respect to a horizontal plane, traveling in opposite axial directions and rotating in the same azimuthal direction (see Figure VII.12).

For more general problems, the results about homoclinic and heteroclinic connections are set up in [Io-Mi] as follows:

Theorem 5. *For a system (VII.1) invariant under reflection symmetry, and undergoing an oscillatory instability, let us consider the situation for $[U_z] = 0$ and the generic case where $\lambda_r b_r < 0$. Then there are two symmetric traveling waves, space- and time-periodic solutions of the Navier-Stokes equations. Moreover, if $\lambda_r(b_r - c_r) > 0$, then there exists at least 2 different time-periodic solutions (defects) connecting these two symmetric regimes, all with respectively the same (shifted by half of the period) flows at infinity.*

Theorem 6. *For a system (VII.1) invariant under reflection symmetry, and undergoing an oscillatory instability, let us consider the situation for $[U_z] = 0$, and the generic case where $\lambda_r(b_r + c_r) < 0$. Then there exists a standing waves regime solution of the Navier-Stokes equations, both time and space periodic. Moreover, if $\lambda_r b_r < 0$ and $b_r^2 - c_r^2 > 0$, then there is a family of solutions connecting these standing waves at $-\infty$ to the traveling waves TW_0 at $+\infty$ and there exists the symmetric solution connecting the traveling waves TW_1 at $-\infty$ to the standing waves at $+\infty$.*

In the Couette-Taylor problem, the traveling waves are the spirals and the standing waves are the ribbons. If it is clear that the *"defect solution"*

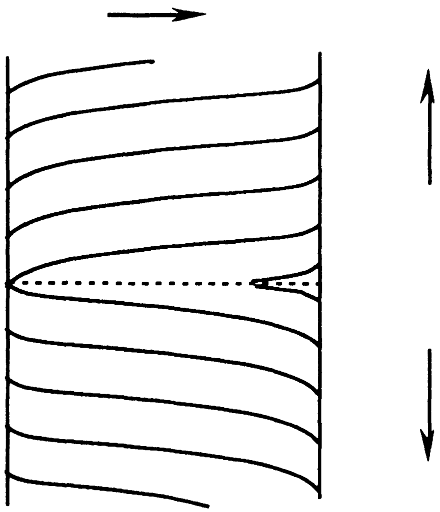


FIGURE VII.12. Defect solution ("source" on this picture), connecting two symmetric regimes of spirals. See Figure I.6.

with symmetric spirals is currently observed, this is not yet the case for other solutions. Of course, one must not forget that no stability result is given for all these new solutions. The stability results that are in this book, until now, are essentially restricted to spatially periodic solutions (hence including the initial perturbation).

To go further in the analysis, we need a compromise that now loses some mathematical rigor. It is the aim of the next section to introduce Ginzburg-Landau envelope equations, which keep the full time dependency, and allow nonspatially periodic solutions. The problem of the mathematical justification of the derivation of the full equations from Navier-Stokes equations is still an open mathematically challenging problem (see Mielke [Mi92] for recent progress on this topic).

VII.3 Ginzburg-Landau equation

We start with the full equation (II.23) for perturbations U and study the stability of the maximally symmetric solution $U = 0$. Usual classical linear theory of hydrodynamical stability deals with perturbations of the form $\hat{U}_k e^{ikz}$, where $\hat{U}_k$ is a function of variables lying in Σ. The corresponding eigenvalues of L_μ are denoted by $\sigma(\mu, ik)$:

$$L_\mu(\hat{U}_k e^{ikz}) = \sigma(\mu, ik)\hat{U}_k e^{ikz}. \qquad \text{(VII.93)}$$

For each k, there is an infinite set of eigenvalues $\{\sigma_m; m \in \mathbb{N}\}$, and if the only restriction on the behavior in z is the boundedness of vector fields, it is clear that the set of all eigenvalues $\{\sigma_m(\mu, ik)\}$ is not discrete (since k can vary continuously). Once $2\pi/h$ periodicity is assumed, one only allows k to take multiple values of h. It is a classical result that the full spectrum of L_μ is then discrete, as remarked in Section II.3.

In addition, the effect of the symmetry $z \to -z$ leads to

$$\sigma(\mu, -\mathrm{i}k) = \sigma(\mu, \mathrm{i}k), \qquad \hat{U}_{-k} = S\hat{U}_k. \tag{VII.94}$$

Assuming, as in Section VII.1, that *the eigenvalue with largest real part* σ_0 *is real*, the classical theory leads to a neutral stability curve $\mu = \mu_c(k)$ (even function) defined by $\sigma_0(\mu, \mathrm{i}k) = 0$, as indicated at Figure VII.1. We have a family of curves σ_0 as a function of k, parametrized in μ, and we see that for $\mu > 0$ there are two symmetric intervals (see Figure VII.1(a)) where the wave number k gives an exponential growth of the perturbation. In the (k, μ) plane, if we look at a fixed value of $\mu > 0$, then the values of $|k|$ giving points inside the parabolic region $\mu > \mu_c(k)$ lead to instability (see Figure VII.1(b)), while outside of this region perturbations $\hat{U}_k e^{\mathrm{i}kz}$ are damped.

According to the formulation described in Chapter II, the axial periodicity of solutions is chosen a priori. If we choose a period $2\pi/h$, then criticality is given by the multiple of h giving *the lowest point* on the neutral curve $\mu = \mu_c(k)$. This point is denoted by k (close to k_c if h is small). For positive μ and large periods the number of excited wave modes is of order $\sqrt{\mu}/h$, but the nonlinear interactions are strong (order 1). Hence, the classical bifurcation theory is only valid for $|\mu| = \mathrm{O}(h^2)$. It is precisely the aim of the Ginzburg-Landau equation to take account of the interactions of all the excited wave numbers k near k_c.

Let us consider the analysis assuming $2\pi/h$-periodicity. We can mimic the analysis made in Chapter III: denote the critical eigenvectors, belonging to the 0 eigenvalue, by $\hat{U}_k e^{\mathrm{i}kz}; \bar{\hat{U}}_k e^{-\mathrm{i}kz}$, where $S\hat{U}_k = \bar{\hat{U}}_k (= \hat{U}_{-k})$. The central part is then parametrized as follows:

$$X = \mathcal{A}\hat{U}_k e^{\mathrm{i}kz} + \bar{\mathcal{A}}\bar{\hat{U}}_k e^{-\mathrm{i}kz}, \tag{VII.95}$$

and the amplitude equation reads

$$\frac{d\mathcal{A}}{dt} = a_k[\mu - \mu_c(k)]\mathcal{A} + b_k\mathcal{A}|\mathcal{A}|^2 + \cdots, \tag{VII.96}$$

where a_k and b_k are *real and even functions of* k and we have

$$\sigma_0(\mu, \mathrm{i}k) = [\mu - \mu_c(k)](a_k + \text{ h.o.t.}).$$

In fact, we reconsider the problem solved in Chapter III in *leaving free the wave number* k *in the neighborhood of* k_c. Note that for our purpose we assume that $a_{k_c} > 0$, $b_{k_c} < 0$, to have a supercritical bifurcation (to Taylor vortices here).

Let us now *suppress the $2\pi/h$-periodicity assumption.* As we saw in Figure VII.1, this gives for the wave number k, two intervals where perturbations increase exponentially for the linear problem. The fact that these intervals are small and centered at $\pm k_c$ leads to the idea of making the same decomposition of U as in Chapter III, with a complex amplitude $\mathcal{A}$, except that we now allow $\mathcal{A}$ to *depend slowly on z*. In this way, modulations due to values of k near k_c are taken into account. This idea, applied to the bidimensional Bénard convection problem, was initiated in [New-Wh], [Seg], and [DP-Ec-Seg]. The *envelope equation* for $\mathcal{A}$, now a partial differential equation, is most usually called the Ginzburg-Landau equation. In this section we derive this equation *in a formal way*, since it has not been mathematically justified so far, unlike the Landau equation (VII.96) as we have shown in Chapter III.

Let us introduce useful notations:

$P = (p_0, p_1, p_2, \ldots, p_n, 0, 0\ldots) \in \mathbb{N}^{\mathbb{N}}$ will also be denoted $(p_0, p_1, p_2, \ldots, p_n)$, and we define

$$\mathcal{A}^{(P)} \stackrel{\text{def}}{=} \mathcal{A}^{p_0}(\partial_z\mathcal{A})^{p_1}\ldots(\partial_z^n\mathcal{A})^{p_n}, \qquad |P| = p_0 + p_1 + p_2 + \cdots + p_n,$$

where $\partial_x^n\mathcal{A}$ is the nth derivative of $\mathcal{A}$ with respect to x. We decompose U as follows:

$$U = \mathcal{A}(z,t)\hat{U}_{k_c}e^{ik_cz} + \bar{\mathcal{A}}(z,t)\bar{\hat{U}}_{k_c}e^{-ik_cz} + \Phi(\mu, \mathcal{A}, \bar{\mathcal{A}}, \partial_z), \tag{VII.97}$$

where we formally write

$$\Phi(\mu, \mathcal{A}, \bar{\mathcal{A}}, \partial_z) = \sum_{r\in\mathbb{N},\ P\&Q\in\mathbb{N}^{\mathbb{N}}} \mu^r\Phi_{rPQ}\mathcal{A}^{(P)}\bar{\mathcal{A}}^{(Q)}e^{ik_c(|P|-|Q|)z} \tag{VII.98}$$

where the Φ_{rPQ} are vector functions depending on the transverse variable in Σ only. In all expansions we formally consider that, due to the slow variable, for $|P| = |Q|$ and $\Sigma jp_j < \Sigma jq_j$ then $|\mathcal{A}^{(P)}| \ll |\mathcal{A}^{(Q)}|$. This gives a partial order for decreasing the magnitudes of the terms in the expansions. Since we have to replace U in (II.23), we must now define how to apply operators L_μ and $N(\mu, .)$ to products of a scalar function of the slow variable with a $2\pi/k_c$-periodic vector function (considered as quickly varying). We can define the following expansions for α slowly varying and for any sufficiently smooth vector function Y of the transverse variables in Σ:

$$\begin{aligned} L_\mu(\alpha e^{nik_cz}Y) &= \alpha L_\mu(e^{nik_cz}Y) + \partial_z\alpha L_\mu^{(1)}(e^{nik_cz}Y) \\ &\quad + \partial_z^2\alpha L_\mu^{(2)}(e^{nik_cz}Y) + \cdots, \end{aligned} \tag{VII.99}$$

$$\begin{aligned} N(\mu, \alpha e^{nik_cz}Y) &= \alpha^2 N(\mu, e^{nik_cz}Y) + \alpha\partial_z\alpha N_\mu^{(1)}(e^{nik_cz}Y) + \cdots \\ &= \sum_{|P|=2} \alpha^{(P)}N_\mu^P(e^{nik_cz}Y), \end{aligned} \tag{VII.100}$$

where we denote $N_\mu^P = N(\mu, .)$ if $P = (2, 0, \ldots)$ and $N_\mu^P = N_\mu^{(1)}$ if $P = (1, 1)$. In these expansions, ∂_z might be considered as a small parameter. The construction of operators $L_\mu^{(j)}$ and N_μ^P, on the subspace of $2\pi/k_c$-periodic vector functions, is given later. One has to realize that these expansions are infinite because the pressure gradient in the Navier-Stokes equations leads to a nonlocal operator (the pressure is indeed a nonlocal function of the velocity vector field). However, they are well defined since the fast variable occurs at some power of $e^{ik_c z}$ as a factor in all terms. Moreover, if α is a polynomial in z, expansions (VII.99)–(VII.100) are finite and there is no restriction on the nature of the variable in α (formulas are exact). The slowness of the z dependence of α is, in fact, used to give a meaning to the expansions (VII.99)–(VII.100).

We wish to obtain an envelope equation of the form

$$\frac{\partial \mathcal{A}}{\partial t} = f(\mu, \mathcal{A}, \bar{\mathcal{A}}, \partial_z), \tag{VII.101}$$

where f has an expansion of the form

$$f(\mu, \mathcal{A}, \bar{\mathcal{A}}, \partial_z) = \sum_{r \in \mathbb{N},\ P \& Q \in \mathbb{N}^{\mathbb{N}}} \mu^r f_{rPQ} \mathcal{A}^{(P)} \bar{\mathcal{A}}^{(Q)}. \tag{VII.102}$$

In fact, the same symmetry arguments as for the Landau equation lead to the properties

$$\begin{aligned} f(\mu, \mathcal{A}e^{i\phi}, \bar{\mathcal{A}}e^{-i\phi}, \partial_z) &= e^{i\phi} f(\mu, \mathcal{A}, \bar{\mathcal{A}}, \partial_z), \\ f(\mu, \bar{\mathcal{A}}, \mathcal{A}, -\partial_z) &= \overline{f(\mu, \mathcal{A}, \bar{\mathcal{A}}, \partial_z)}. \end{aligned} \tag{VII.103}$$

Hence the principal part of (VII.101) can be written as (Ginzburg-Landau equation)

$$\begin{aligned} \frac{\partial \mathcal{A}}{\partial t} = {} & c_0 \mu \mathcal{A} + ie_1 \mu \partial_z \mathcal{A} + e_2 \partial_z^2 \mathcal{A} + ie_3 \partial_z^3 \mathcal{A} + d_0 \mathcal{A} |\mathcal{A}|^2 \\ & + id_1 |\mathcal{A}|^2 \partial_z \mathcal{A} + id_2 \mathcal{A}^2 \partial_z \bar{\mathcal{A}} + \cdots, \end{aligned} \tag{VII.104}$$

where the coefficients $c_0, e_1, e_2, d_0, d_1, \ldots$ are real. This is a partial differential equation once one decides to truncate at some finite order. These differential operators that are defined by their expansions are in fact pseudo-differential operators. We notice that the linear part has to be such that for $\mathcal{A} = \mathcal{A}_0(t) e^{i\alpha z}$ we recover the eigenvalue

$$\sigma_0(\mu, ik) = c_0 \mu - e_1 \mu \alpha - e_2 \alpha^2 + e_3 \alpha^3 \cdots, \tag{VII.105}$$

where $k = k_c + \alpha$. The neutral curve of Figure VII.1(b) is then given by

$$\mu = \mu_c(k) = \frac{e_2}{c_0} \alpha^2 + \frac{e_1 e_2 - e_3 c_0}{c_0^2} \alpha^3 + \cdots, \qquad \alpha = k - k_c. \tag{VII.106}$$

It results from (VII.105)–(VII.106) that $c_0 > 0$ for the right change of stability when μ increases and $e_2 > 0$ for a minimum on the curve of Figure VII.1(b) to occur at the point $\mu = 0$. Moreover, equation (VII.104) also contains the case when solutions are $2\pi/h$-periodic; hence, by setting

$$\mathcal{A} = \mathcal{A}_0(t)e^{\mathrm{i}\alpha z}, \qquad \alpha = k - k_{\mathrm{c}}$$

in (VII.104), we should recover the Landau equation (VII.96). This observation (see [Ku]) leads to the relations

$$\begin{aligned} a_k &= c_0 - e_1\alpha + \mathrm{O}(\alpha^2), \\ b_k &= d_0 + (d_2 - d_1)\alpha + \mathrm{O}(\alpha^2). \end{aligned} \tag{VII.107}$$

To compute the coefficients of (VII.104), we proceed exactly like for the Landau equation [we thank P. Coullet for showing us this direct derivation of (G-L)], where here we must identify monomials $\mu^r\mathcal{A}^{(p)}\bar{\mathcal{A}}^{(Q)}$.

Now, several natural questions arise:

(i) If the coefficient e_2 is not small, physicists just keep linear terms up to second-order derivatives and at most first derivatives in nonlinear terms. How could we justify that higher-order derivatives do not play an important physical role?

(ii) Once truncated at this order, one finds, for instance, solutions such that at some point $\mathcal{A} = 0$ while $\partial_z\mathcal{A} \neq 0$. This is in contradiction with our assumption for the formal derivation of the Ginzburg-Landau equation (VII.104): ($|\mathcal{A}^{(p)}| << |\mathcal{A}^{(Q)}|$ with $P = (1, 0, \ldots)$ and $Q = (0, 1, 0, \ldots)$).

The purpose of what follows is to answer precisely these two questions—but only when considering *steady solutions*, as in Section VII.2.

We now want to rewrite the fourth-order system (VII.21) into the form of a second-order complex equation to allow comparison with the steady (G-L) equation [(VII.104) without the term $\partial\mathcal{A}/\partial t$]. Let us set

$$\begin{aligned} A' &= Ae^{-\mathrm{i}k_{\mathrm{c}}z}, \\ B' &= \left\{B + \mathrm{i}AP[\mu, |A|^2, \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B)]\right\}e^{-\mathrm{i}k_{\mathrm{c}}z}, \end{aligned} \tag{VII.108}$$

then

$$u' = |A'|^2 = u, \qquad v' = \frac{\mathrm{i}}{2}(A'\bar{B}' - \bar{A}'B') = v + uP(\mu, u, v).$$

Hence, by the implicit function theorem, we get

$$v = v' - u'[p_1\mu + p_2u' + p_3v' + \cdots]. \tag{VII.109}$$

Now the system (VII.21) becomes

$$\frac{d^2A'}{dz^2} + A'Q'(\mu, u', v') + \mathrm{i}\frac{dA'}{dz}P'(\mu, u', v') = 0, \tag{VII.110}$$

where $u' = |A'|^2$, $v' = \frac{i}{2}(A'\frac{d\bar{A}'}{dz} - \bar{A}'\frac{dA'}{dz})$, and

$$\begin{aligned} Q'(\mu, u', v') &= -\{Q[\mu, u', v(u', v')] + P^2[\mu, u', v(u', v')] \\ &\quad +2v'\frac{\partial P}{\partial u}[\mu, u', v(u', v')]\} \\ P'(\mu, u', v') &= -2P[\mu, u', v(u', v')] - 2u'\frac{\partial P}{\partial u}[\mu, u', v(u', v')]. \end{aligned}$$

Let us write more explicitly the principal part of (VII.110). We first have

$$\begin{aligned} Q'(\mu, u', v') &= q_1\mu - q_2u' - v'(2p_2 + q_3) + \cdots, \\ P'(\mu, u', v') &= -2p_1\mu - 4p_2u' - 2p_3v' + \cdots; \end{aligned}$$

hence the principal part reads

$$\begin{aligned} &\frac{d^2A'}{dz^2} + q_1\mu A' - 2ip_1\mu\frac{dA'}{dz} - q_2A'|A'|^2 + \frac{i}{2}(q_3 - 6p_2)|A'|^2\frac{dA'}{dz} \\ &-\frac{i}{2}(2P_2 + q_3)A'^2\frac{d\bar{A}'}{dz} + p_3\left\{A'|\frac{dA'}{dz}|^2 - \bar{A}'\left(\frac{dA'}{dz}\right)^2\right\} = 0. \quad \text{(VII.111)} \end{aligned}$$

Now, if we come back to the definition of $\mathbf{A}$ in (VII.21) and consider the projection $\Pi\mathfrak{V} = U$ (see Subsection VII.1.1), we have in fact a decomposition of the velocity field of the form

$$U(z) = A'(z)\hat{U}_{k_c}e^{ik_cz} + \frac{dA'}{dz}(z)\hat{U}_1e^{ik_cz} + \text{ c.c. } + \psi\left(\mu, A', \bar{A}', \frac{dA'}{dz}, \frac{d\bar{A}'}{dz}\right), \quad \text{(VII.112)}$$

where "c.c." means "complex conjugate," $\hat{U}_1 = \frac{d}{d(ik)}\hat{U}|_{k_c}$, and where we have taken into account the fact that $B - dA'/dze^{ik_cz}$ is of higher order and can be incorporated into ψ. This decomposition looks very much like decomposition (VII.97) used for obtaining the (G-L) equation. In fact, in deriving (VII.97) we consider the terms in $\partial_z\mathcal{A}$ of order higher than $\mathcal{A}$, and hence they belong to Φ. Note that in the study of solutions of (VII.110), performed in Section VII.1, we obtained, in particular, solutions where $|dA'/dz| >> |A'|$ in the neighborhood of some front or pulse and even for a family of quasi-periodic solutions, which then violate the imposed condition on $\partial_z\mathcal{A}$ and $\mathcal{A}$ for the (G-L) equation.

Finally, equation (VII.111) associated with decomposition (VII.112) and the steady (G-L) equation associated with decomposition (VII.97) must give the same solutions in U. The steady (G-L) equation reads

$$c_0\mu\mathcal{A}+ie_1\mu\partial_z\mathcal{A}+e_2\partial_z^2\mathcal{A}+ie_3\partial_z^3\mathcal{A}+d_0\mathcal{A}|\mathcal{A}|^2+id_1|\mathcal{A}|^2\partial_z\mathcal{A}+id_2\mathcal{A}^2\partial_z\bar{\mathcal{A}}+\ldots=0. \quad \text{(VII.113)}$$

We see that this equation is (unfortunately) of *infinite* order (in fact, the differential operators are pseudo-differential operators); hence, there is a problem in the identification of the two equations (VII.113) and (VII.111).

However, we already obtained (VII.33) and (VII.36), which lead to the correspondences

$$q_1 = \frac{c_0}{e_2}, \qquad -q_2 = \frac{d_0}{e_2}, \qquad -2p_1 = \frac{e_1}{e_2} - \frac{e_3 c_0}{e_2^2} \tag{VII.114}$$

Hence we check the terms in $\mu A'$, $\frac{d^2A'}{dz^2}$, and $A'|A'|^2$ and now have a rule for finding the coefficient of $\mu dA'/dz$ that takes into account coefficients of $\mu\partial_z \mathcal{A}$ and of $\partial_z^3\mathcal{A}$ in (VII.113). In fact, it may be obtained by replacing $\partial_z^2\mathcal{A}$ by $-c_0/e_2\mu\mathcal{A}$ in (VII.113). This suggests an idea for obtaining (VII.110) from the steady (G-L) equation: (i) write the (G-L) equation (VII.113) in the form $\partial_z^2\mathcal{A} =$ right-hand side; (ii) replace on the right-hand side all $\partial_z^p\mathcal{A}$, $p \geq 2$, by the derivative of order p-2 of the full right-hand side; and then (iii) iterate the process indefinitely. The resulting equation, truncated at some arbitrary order, is a second-order complex ODE of the form

$$\partial_2^2\mathcal{A} = h(\mu, \mathcal{A}, \bar{\mathcal{A}}, \partial_z\mathcal{A}, \partial_z\bar{\mathcal{A}}),$$

which satisfies the invariances (see (VII.21))

$$\begin{aligned} h(\mu, e^{i\phi}\mathcal{A}, e^{-i\phi}\bar{\mathcal{A}}, e^{i\phi}\partial_z\mathcal{A}, e^{-i\phi}\partial_z\bar{\mathcal{A}}) &= e^{i\phi}h(\mu, \mathcal{A}, \bar{\mathcal{A}}, \partial_z\mathcal{A}, \partial_z\bar{\mathcal{A}}), \\ h(\mu, \bar{\mathcal{A}}, \mathcal{A}, -\partial_z\bar{\mathcal{A}} - \partial_z\mathcal{A}) &= \overline{h(\mu, \mathcal{A}, \bar{\mathcal{A}}, \partial_z\mathcal{A}, \partial_z\bar{\mathcal{A}})}. \end{aligned}$$

Hence, it is clear that this equation contains more terms in the power expansion than (VII.110). For instance, there are two additional types of monomials in $(\mathcal{A}, \bar{\mathcal{A}}, \partial_z\mathcal{A}, \partial_z\bar{\mathcal{A}})$ of degree 3, and six additional for degree 5. Provided we could give a meaning to the steady (G-L) equation (VII.113), a relevant conjecture would be the following.

Conjecture. *Assume $e_2 \neq 0$; then the ODE (VII.113) plays the role of a normal form for the steady (G-L) equation on a four-dimensional center manifold.*

However, we may observe that for terms such that the total order of derivation is lower than or equal to 1, we have not suppressed any monomial in this normalization; hence, we might identify corresponding coefficients. Finally, if the conjecture is correct, we obtain a new form for the principal part of equation (VII.113)

$$\begin{aligned} \partial_z^2\mathcal{A} + \frac{c_0}{e_2}\mu\mathcal{A} + \frac{d_0}{e_2}\mathcal{A}|\mathcal{A}|^2 + \mathrm{i}\left[\frac{e_1}{e_2} - \frac{c_0 e_3}{e_2^2}\right]\mu\partial_z\mathcal{A} + \mathrm{i}\left[\frac{d_1}{e_2} - \frac{2d_0 e_3}{e_2^2}\right]|\mathcal{A}|^2\partial_z\mathcal{A} \\ +\mathrm{i}\left[\frac{d_2}{e_2} - \frac{d_0 e_3}{e_2^2}\right]\mathcal{A}^2\partial_z\bar{\mathcal{A}} + \cdots = 0, \end{aligned} \tag{VII.115}$$

which leads, in addition to (VII.89), by identification with (VII.86) to the relations:

$$q_3 = \frac{d_1 - 3d_2}{2e_2} + \frac{d_0 e_3}{2e_2^2}, \qquad p_2 = -\frac{d_1 + d_2}{4e_2} + \frac{3d_0 e_3}{4e_2^2}. \tag{VII.116}$$

Remark. By this method, we cannot identify the coefficient p_3 in (VII.111) because terms in $\mathcal{A}\,|d\mathcal{A}/dz|^2$ and $\bar{\mathcal{A}}\,(d\mathcal{A}/dz)^2$ do not occur with their difference in (VII.115). The "normalization" process, which corresponds to a nonlinear change of variable on $\mathcal{A}$, is necessary to obtain this combination. This would consist, starting with (VII.115), in computing the normal form associated with a four-dimensional linear operator having 0 as a quadruple eigenvalue, with two conjugate two-dimensional Jordan blocks (see [Io-Mi 92]), respecting the invariances properties of (VII.115) and its complex conjugate.

Let us now consider the situation studied in Section VII.2 and come back to the velocity vector field U (perturbation of the Couette flow). We have, in fact, the decomposition

$$U = Ae^{\mathrm{i}\tau}\hat{U}_{k_c}(y) + \bar{A}e^{-\mathrm{i}\tau}\bar{\hat{U}}_{k_c}(y) + Be^{\mathrm{i}\tau}\mathbf{S}\hat{U}_{k_c}(y) + \bar{B}e^{-\mathrm{i}\tau}\mathbf{S}\bar{\hat{U}}_{k_c}(y) \\ +\psi(\mu,\omega,A,\bar{A},B,\bar{B}), \qquad \text{(VII.117)}$$

where at first order A and B satisfy

$$\frac{dA}{dz} = \mathrm{i}k_cA - (e_1)^{-1}[a\mu - \mathrm{i}(\omega-\omega_0)]A + A(b|A|^2 + c|B|^2),$$
$$\frac{dB}{dz} = -\mathrm{i}k_cB + (e_1)^{-1}[a\mu - \mathrm{i}(\omega-\omega_0)]B - B(b|B|^2 + c|A|^2), \qquad \text{(VII.118)}$$

and $\tau = \omega t$. Let us denote $A'(z,t) = A(z)e^{\mathrm{i}(\omega-\omega_0)t-\mathrm{i}k_cz}$ and $B' = B(z)_e^{\mathrm{i}(\omega-\omega_0)t+\mathrm{i}k_cz}$. Then (VII.118) becomes

$$\frac{\partial A'}{\partial t} = a\mu A' + e_1\frac{\partial A'}{\partial z} + A'(e_1b|A'|^2 + e_1c|B'|^2),$$
$$\frac{\partial B'}{\partial t} = a\mu B' - e_1\frac{\partial B'}{\partial z} + B'(e_1b|B'|^2 + e_1c|A'|^2). \qquad \text{(VII.119)}$$

Since we factor out the quick variations in z and t, we may assume that A' is a slowly varying function of z and t, then taking account of the little segment of excited wave numbers near k_c for $\mu > 0$ small (see Figure VII.1(b)). This system is a partial differential equation, which is the principal part of a system of Ginzburg-Landau type, which might be obtained in the same way as for (VII.104). Obviously, if we expand $P(\mu,\omega,0,0)$ at higher orders we get higher-order derivatives in t for A', B'. These should be eliminated as we did for derivatives of order higher than 2 in z in (VII.113). In looking at the expression for the eigenvalue $\sigma0(\mu,\mathrm{i}k)$ we also see that its expansion corresponds to the linear part of (VII.119) and that terms of higher orders in $\mathrm{i}(k-k_c)$ correspond to higher-order derivatives in z. These can be eliminated, in the study of time-periodic solutions, using the fact that e_1 is supposed to be strictly of order 1, by the same trick as indicated in going from (VII.113) to (VII.115) in the steady case.

VIII
Small Gap Approximation

This chapter basically reproduces the content of the paper by Demay, Iooss, and Laure [De-Io-La]. The classical choice of scales indicated in Subsection II.1.5.1 will not be discussed below, and we refer the reader to the bibliography mentioned in the introduction of this chapter. However, it should be clear that a systematic use of the symmetries indicated in Subsection II.1.5.1 would give a new insight on the existing results.

VIII.1 Introduction

We consider the flow between rotating concentric cylinders, in the limiting case when the radius ratio $\eta = R_1/R_2$ is very close to 1. There are several possible relevant choice of scales *leading to different limiting systems.* When both cylinders rotate in the same direction, and the average value of the rotation rates is large (of the order of the square of the inverse of the gap) a specific choice of scales and the limit $\eta \to 1$ lead to a limiting flow between moving parallel plates with an additional Coriolis term (see II.19). In fact, this model appears to be very analogous to the Bénard problem for the bidimensional perturbations (see Nagata [Na 86,88]), so there is no hope of obtaining oscillatory instabilities.

In what follows we are precisely interested in *oscillatory instabilities* for the small gap case. The basic assumption is now that the Reynolds numbers built with rotation rates of the inner and outer cylinders are very large, of the order $(\eta - 1)^{-1/2}$. The two dimensionless parameters are now the two Taylor numbers $\mathcal{T}_j$, $j = 1$ and 2. In addition, we want to study the large scale effects, which means, for instance, that the height of the cylinders is large with respect to the radius of the inner cylinder. It is known, from Section VII.3, that the classical ordinary differential equations for amplitudes, which drive the dynamics on a low-dimensional center manifold in the case of a spatial periodicity assumption, are now replaced by partial differential Ginzburg-Landau equations, which now take into account the interactions between critical modes with spatial wave numbers close to the critical one. For the small gap case studied here, contrary to the Bénard convection problem that is isotropic in the two extended directions, we have here an example of an anisotropic system.

The case $\mathcal{T}_2 = 0$ (outer cylinder fixed) was studied by Tabeling [Tab]. He computed the coefficients of the Ginzburg-Landau equation and studied the phase dynamics of the Taylor vortex system. An open question was whether or not the present limiting equations lead to an oscillatory instability when $\mathcal{T}_2 < 0$ (while $\mathcal{T}_1 > 0$), $|\mathcal{T}_2|$ being sufficiently large. Our work first modifies the limiting equations used in [Tab], adding a limiting "pressure term" that was mistakenly forgotten in previous works (luckily this term was not active). The present chapter shows that the most critical mode is becoming oscillatory as $\mathcal{T}_2$ decreases. However, these excited modes can only lead to standing waves for some values of $\mathcal{T}_2$, and they are always unstable with respect to homogeneous perturbations. The aim of the present analysis is to give precisely the coefficients of the Ginzburg-Landau equations in this physical situation. Let us notice that the phenomenology of defects associated with traveling waves was more specially studied by Coullet et al. [Co-El-Gi-Le], [Co-Le] and by Lega [Le]. For the case of static patterns see also Kramer et al. [Kr-Bo-Pe-Zi]. Nevertheless, the present Ginzburg-Landau equations have never been studied numerically, and it would be interesting to make such computations in order to have an idea of observable solutions. In addition, two interesting codimension-two bifurcations are proved to occur in this example—the first corresponds to the point where Taylor vortices are becoming subcritical, and the second occurs when the horizontal critical modes start becoming oscillatory.

VIII.2 Choice of scales and limiting system

We now consider the case when the inner and possibly the outer Reynolds numbers $R_j\Omega_j d/\nu$, $j = 1, 2$ (ν being the kinematic viscosity) are very large when η is close to 1. The choice of scale we present follows mainly Tabeling [Tab] but differs in result. The parameters are here $\mathcal{T}_j$, $j = 1, 2$, defined by

$$\mathcal{T}_j = R_j\Omega_j d/\nu\sqrt{2(1-\eta)}, \quad j = 1, 2 \quad [\mathcal{T}_1 \text{ is } > 0, \text{ while } \mathcal{T}_2 \text{ may be } \leq 0], \tag{VIII.1}$$

where $d = R_2 - R_1$.

Remark. When $\Omega_2 = 0$, $\mathcal{T}_1^2 = 2\bar{R}\Omega_j^2 d^3/\nu^2$ is the classical *Taylor number.*

VIII.2.1 Choice of scales

Let us consider the Navier-Stokes equations in cylindrical coordinates written in (II.7)–(II.8) and make another choice of scales. We choose a new set of dimensionless variables defined by the following relations, as a function of the nondimensional cylindrical coordinates introduced in Subsection II.1.2:

$$x = r - \frac{1+\eta}{2(1-\eta)}, \qquad y = \theta\sqrt{\frac{2\eta}{1-\eta}}, \qquad z = z. \tag{VIII.2}$$

In addition, we define the scales of components of the velocity (in cylindrical coordinates) in such a way that the incompressibility condition is recovered in the new cartesian coordinates

$$v_r = (\mathcal{R})^{-1}v_x^*, \quad v_\theta = (\mathcal{R})^{-1}\sqrt{\frac{\eta}{2(1-\eta)}}v_y^*, \quad v_z = (\mathcal{R})^{-1}v_z^*; \tag{VIII.3}$$

$$\rho^{-1}\frac{\partial p}{\partial r} = (\mathcal{R})^{-1}H_x^*, \qquad \rho^{-1}\frac{1}{r}\frac{\partial p}{\partial \theta} = (\mathcal{R})^{-1}\sqrt{\frac{\eta}{2(1-\eta)}}H_y^*,$$
$$\rho^{-1}\frac{\partial p}{\partial z} = (\mathcal{R})^{-1}H_z^*. \tag{VIII.4}$$

VIII.2.2 Limiting system

Substituting new variables in equations (II.7)–(II.8) and neglecting terms vanishing for $\eta = 1$, we obtain the dimensionless limit equations (where "$*$" are omitted):

$$\begin{aligned}
\frac{\partial v_x}{\partial t} + ((V.\nabla)V)_x + \frac{1}{2}v_x^2 &= \tilde{\Delta}v_x - H_x,\\
\frac{\partial v_y}{\partial t} + ((V.\nabla)V)_y &= \tilde{\Delta}v_y - H_y,\\
\frac{\partial v_z}{\partial t} + ((V.\nabla)V)_z &= \tilde{\Delta}v_z - H_z,\\
\nabla.V &= 0,
\end{aligned} \tag{VIII.5}$$

where $\tilde{\Delta}$ is the usual Laplace operator in the *only two* coordinates x and z, and due to (VIII.4) and (VIII.2),

$$\frac{\partial H_x}{\partial z} = \frac{\partial H_z}{\partial x}, \qquad \frac{\partial H_y}{\partial z} = \frac{\partial H_y}{\partial x} = 0.$$

It appears that H_x and H_z are the derivatives with respect to x and z of a function p (of coordinates x, y, z) and that H_y is a function of y only. Hence equations (VIII.5) can be written in the form:

$$\begin{aligned}
\frac{\partial V}{\partial t} &= \tilde{\Delta}V + \begin{pmatrix} v_y^2/2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ f(y) \\ 0 \end{pmatrix} - (V.\nabla)V - \tilde{\nabla}p,\\
\nabla.V &= 0,
\end{aligned} \tag{VIII.6}$$

where functions H_x, H_z, and H_y are replaced by the gradient $\tilde{\nabla}p$ on the two coordinates x and z and the arbitrary scalar function $f(y)$. The boundary conditions are

$$(v_x, v_y, v_z)_{|x=-1/2} = (0, \mathfrak{T}_1, 0), \qquad (v_x, v_y, v_z)_{|x=1/2} = (0, \mathfrak{T}_2, 0).$$

The stationary solution that corresponds to the Couette solution is given by

$$V^{(0)} = (0, v^{(0)}(x), 0), \quad \text{with} \quad v^{(0)}(x) = (\mathfrak{T}_2 - \mathfrak{T}_1)x + \frac{(\mathfrak{T}_1 + \mathfrak{T}_2)}{2}, \tag{VIII.7}$$

and the perturbation U now satisfying ((II.22) is then proved)

$$\frac{\partial U}{\partial t} = \tilde{\Delta} U - v^{(0)}(x)\frac{\partial U}{\partial y} + \begin{pmatrix} v^{(0)}(x)u_y + \dfrac{u_y^2}{2} \\ -v'^{(0)}(x)u_x + f(y) \\ 0 \end{pmatrix} - (U.\nabla)U - \tilde{\nabla} q,$$
$$\nabla.U = 0, \tag{VIII.8}$$

with the boundary conditions $U_{|x=\pm 1/2} = 0$, and where $f(y)$ is arbitrary. We notice here the difference between system (VIII.8) given here and the one used in [Tab]. This additional arbitrary term $f(y)$ is not active in [Tab], while here it will be important for solutions depending on y. We shall notice that this arbitrary choice is of great importance in Fredholm alternatives for computation of bifurcation coefficients.

Remark. The system (VIII.8) is odd in the fact that it loses its elliptic character (unlike the Stokes equation) in the y-coordinate.

Let us finally observe that this system is translation invariant in y-and z-directions and is invariant under the $z \to -z$ symmetry.

VIII.3 Linear stability analysis

Let us linearize the system (VIII.8) and look for eigenmodes of the form

$$\begin{aligned} U &= \hat{U}(x)e^{\mathrm{i}(\alpha z + \beta y)}, \\ q &= \hat{p}(x)e^{\mathrm{i}(\alpha z + \beta y)}, \\ f &= \hat{C}e^{\mathrm{i}(\alpha z + \beta y)} \quad \text{with } \hat{C} = 0 \text{ if } \alpha \neq 0 \end{aligned} \tag{VIII.9}$$

belonging to some eigenvalue σ (not necessarily real). We then obtain (with $D \equiv d/dx$)

$$\begin{aligned} (D^2 - \alpha^2 - \sigma - \mathrm{i}\beta v^{(0)}(x))\hat{u}_x + v^{(0)}(x)\hat{u}_y - D\hat{p} &= 0, \\ (D^2 - \alpha^2 - \sigma - \mathrm{i}\beta v^{(0)}(x))\hat{u}_y + (\mathfrak{T}_1 - \mathfrak{T}_2)\hat{u}_x + \hat{C} &= 0, \\ (D^2 - \alpha^2 - \sigma - \mathrm{i}\beta v^{(0)}(x))\hat{u}_z - \mathrm{i}\alpha\hat{p} &= 0, \\ D\hat{u}_x + \mathrm{i}\beta\hat{u}_y + \mathrm{i}\alpha\hat{u}_z &= 0, \end{aligned} \tag{VIII.10}$$

with the boundary conditions

$$\hat{u}_x = \hat{u}_y = \hat{u}_z = 0 \quad \text{in } x = \pm 1/2. \tag{VIII.11}$$

Let us notice that the scalar $\hat{C}$ does not play any role in the linear analysis as criticality is reached for nonzero α. However, this term is important because system (VIII.10) with boundary conditions (VIII.11) becomes singular if α and $\hat{C}$ are equal to zero. This term is in fact used in the computation of nonlinear coefficients of the amplitude equation corresponding to oscillatory bifurcation. In such a case, we have to solve the linear system (VIII.10) with a nonzero right-hand side for $\alpha = 0$ and $\beta \neq 0$, and it appears that there are too many boundary conditions if the arbitrary constant $\hat{C}$ is forgotten.

The most unstable mode is given by the eigenvalue σ_0 with the largest real part. We have

$$\sigma_0 = \sigma_0(\alpha, \beta, \mathcal{T}_1, \mathcal{T}_2), \tag{VIII.12}$$

and the neutral stability curve in the $(\mathcal{T}_1, \mathcal{T}_2)$ plane corresponds to

$$\mathcal{T}_{1c} = \min_{\alpha,\beta} \mathcal{T}_1(\alpha, \beta, \mathcal{T}_2) \tag{VIII.13}$$

with $\mathcal{T}_1$ solution of $\mathrm{Re}(\sigma_0(\alpha, \beta, \mathcal{T}_1, \mathcal{T}_2)) = 0$. This gives in particular the critical wave numbers α_c, β_c and the neutral curve (computed numerically), which is given in Figure VIII.1.

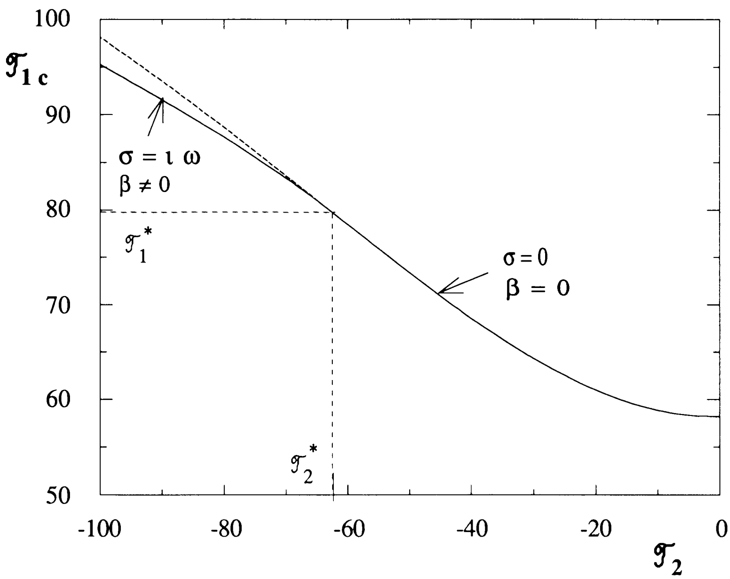


FIGURE VIII.1. Neutral stability curve.

A way of finding the neutral curve is to solve with respect to $\mathcal{T}_1$ the equation $\mathrm{Re}(\sigma_0(\alpha,\beta,\mathcal{T}_1,\mathcal{T}_2)) = 0$ and to study the following functions for fixed $\mathcal{T}_2$:

$$\mathcal{T}_1'(\beta,\mathcal{T}_2) = \min_{\alpha} \mathcal{T}_1(\alpha,\beta,\mathcal{T}_2). \tag{VIII.14}$$

These functions are shown in Figure VIII.2. We observe that the minima are obtained for $\beta = 0$ until a critical value $\mathcal{T}_2^*$ (~ -62.5) of $\mathcal{T}_2$. For $\mathcal{T}_2 < \mathcal{T}_2^*$ the minima correspond to positive values of β, starting at 0 for $\mathcal{T}_2^*$. This fact was a priori not obvious, since in previous careful analysis, as in Langford et al. [La-Ta-Ko-Sw-Go 90], it can be observed that when η is close to 1 the most critical oscillatory nonaxisymmetric modes (integer azimuthal wave number $m \neq 0$) come together while the inner and outer Reynolds numbers go to $\pm\infty$. However, taking the values (*plotted in Figure 2*) of Langford et al. for the interaction $\{m = 0, m = 1\}$, completed by the values given by Demay and Iooss [De-Io] for $\eta = 0.95$, we obtain Table VIII.1.
Table VIII.1 clearly shows the coherence of our result with these extremal numerical values (not very precise because when η is close to 1, some numerical problem occur in the computations).

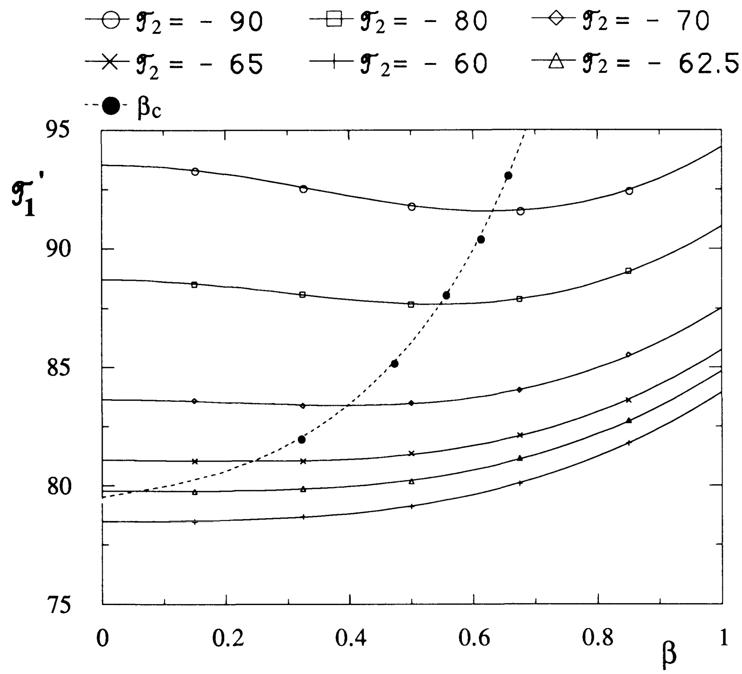


FIGURE VIII.2. Graphs of $\mathcal{T}_1'(\beta,\mathcal{T}_2)$ for fixed $\mathcal{T}_2$.

TABLE VIII.1.

η	$\mathcal{T}_1$	$\mathcal{T}_2$
0.883	91.5	−62.4
0.95	83.6	−61.0
0.975	79.9	−61.4

It will be useful in what follows to write explicitly the eigenvalues of largest real part. We then have to distinguish three cases.

(i) $\mathcal{T}_2 > \mathcal{T}_2^*$ ($\mathcal{T}_2^* = -62.5$). In this case, $\beta_c = 0$. The eigenvalue is real when $\beta = 0$, and

$$\sigma_0(\alpha, -\beta, \mathcal{T}_1, \mathcal{T}_2) = \overline{\sigma_0(\alpha, \beta, \mathcal{T}_1, \mathcal{T}_2)}, \tag{VIII.15}$$

so the Taylor expansion may be written, near the neutral curve, as

$$\begin{aligned}\sigma_0(\alpha, \beta, \mathcal{T}_1, \mathcal{T}_2) = \mathrm{i}a_1\beta + a_2(\mathcal{T}_1 - \mathcal{T}_{1c}(\mathcal{T}_2)) - a_3(\alpha^2 - \alpha_c^2)^2 - a_4\beta^2 \\ -\mathrm{i}a_5\beta(\alpha^2 - \alpha_c^2) + \cdots,\end{aligned} \tag{VIII.16}$$

where coefficients a_1, a_2, a_3, a_4, a_5 are real, where we take account of the symmetry $z \longrightarrow -z$ (σ_0 is even in α), and where we denote by $\mathcal{T}_1 = \mathcal{T}_{1c}(\mathcal{T}_2)$ the equation of the neutral curve.

The coefficients a_1, a_2, a_3, a_4, a_5 were computed in particular by Tabeling [Tab] in the case when $\mathcal{T}_2 = 0$. In Table VIII.2 we give a comparison between our computations and these previous ones and give some indications of the behavior of these coefficients as functions of $\mathcal{T}_2$.

We observe that the agreement with Tabeling is very good for a_1, a_2 but not for a_3, a_4, a_5. In fact, the value of a_3 we find is more satisfactory, since it corresponds to the value −3.9 deduced from the work of Brand and Cross [Br-Cr]. For a_4 and a_5 we propose new values, thinking that they are more reasonable.

TABLE VIII.2. The first row (marked*) shows Tabeling values. For the comparison with [Tab] we set $a_1 = v_0\mathcal{T}_1$, and $a_2 = c_0/\mathcal{T}_1$.

$\mathcal{T}_2$	$\mathcal{T}_{1c}$	α_c	a_1	a_2	$8a_3\alpha_c^2$	a_4	$2a_5\alpha_c$	c
0*			−30.63	0.449	1.97	2.61	0.395	40.2
0	58.22	3.127	−30.63	0.450	3.83	1.916	0.289	0.047
−10	58.87	3.136	−26.95	0.459	3.81	2.576	0.467	0.032
−20	60.99	3.155	−24.57	0.481	3.78	3.299	0.723	0.020
−30	64.32	3.187	−23.58	0.516	3.73	3.873	1.047	0.012
−40	68.56	3.239	−23.97	0.562	3.66	3.919	1.371	0.005
−50	73.38	3.319	−25.66	0.615	3.59	2.966	1.556	−0.001
−60	78.48	3.429	−28.43	0.671	3.51	0.735	1.427	−0.005

TABLE VIII.3. Critical values occurring in (VIII.17).

$\mathcal{T}_2$	$\mathcal{T}_{1c}$	ω	α_c	β_c
−65	81.04	−5.97	3.49	0.24
−70	83.39	−12.22	3.54	0.40
−80	87.67	−17.49	3.60	0.55
−90	91.58	−20.97	3.63	0.63
−100	95.27	−23.66	3.67	0.69

TABLE VIII.4. Coefficients occurring in (VIII.17).

$\mathcal{T}_2$	a_1	b_1	a_2	a_3	a_4	a_5
−60	−29.1	−.05	.678 − i.091	.036 − i.003	1.43 − i3.65	.211 − i.004
−70	−28.6	−.07	.667 − i.151	.036 − i.005	3.79 − i5.49	.257 − i.006
−80	−27.7	−.11	.653 − i.210	.035 − i.008	7.45 − i6.82	.357 + i.011
−90	−26.7	−.14	.645 − i.242	.034 − i.010	1.04 − i7.45	.454 + i.048
−100	−25.7	−.16	.641 − i.263	.034 − i.012	1.32 − i7.89	.055 + i.097

(ii) $\mathcal{T}_2 < \mathcal{T}_2^*$. In this case $\beta_c \neq 0$, and we can again write the Taylor expansion of σ_0 as

$$\begin{aligned}\sigma_0(\alpha,\beta,\mathcal{T}_1,\mathcal{T}_2) = \mathrm{i}\omega + \mathrm{i}a_1(\beta-\beta_c) + a_2(\mathcal{T}_1 - \mathcal{T}_{1c}(\mathcal{T}_2)) + \mathrm{i}b_1(\alpha^2-\alpha_c^2)\\ -a_3(\alpha^2-\alpha_c^2)^2 - a_4(\beta-\beta_c)^2\\ -\mathrm{i}a_5(\beta-\beta_c)(\alpha^2-\alpha_c^2)+\cdots \end{aligned} \tag{VIII.17}$$

but the coefficients a_2, a_3, a_4, a_5 are no longer real (ω, a_1, and b_1 are real). The values of basic coefficients in function of $\mathcal{T}_2$ are given in Table VIII.3.

(iii) *Codimension-two situations.* A first codimension-two situation comes from the fact that the main nonlinear coefficient (c in (VIII.22)) in the amplitude equation vanishes for $\mathcal{T}_2 = -48.5$, $\mathcal{T}_{1c} = 72.62$.

A second interesting codimension-two situation is when ($\mathcal{T}_1$, $\mathcal{T}_2$) is close to $(\mathcal{T}_1^*, \mathcal{T}_2^*)$ and β_c is near 0. In fact, a relevant Taylor expansion of the eigenvalue σ_0 is

$$\sigma_0(\alpha,\beta,\mathcal{T}_1,\mathcal{T}_2) = \mathrm{i}a_1\beta+\mu-a_3(\alpha^2-\alpha_c^2)^2+\nu\beta^2-\mathrm{i}a_5\beta(\alpha^2-\alpha_c^2)-a_6\beta^4+\cdots, \tag{VIII.18}$$

where we have

$$\begin{aligned}\mu &= a_2(\mathcal{T}_1-\mathcal{T}_1^*) + a_2'(\mathcal{T}_2-\mathcal{T}_2^*),\\ \nu &= b_4(\mathcal{T}_1-\mathcal{T}_1^*) + b_4'(\mathcal{T}_2-\mathcal{T}_2^*).\end{aligned} \tag{VIII.19}$$

All coefficients are real and given in Tables VIII.5 and VIII.6.

TABLE VIII.5. Critical values and coefficients in (VIII.16) and (VIII.18) for the codimension-two situations.

$\mathcal{T}_2$	$\mathcal{T}_{1c}$	α_c	a_1	a_2	$-8a_3\alpha_c^2$	a_4	$2a_5\alpha_c$
-48.5	72.62	3.305	-25.33	0.606	-3.591	3.19	1.54
-62.5	79.77	3.462	-29.25	0.685	-3.497	10^{-4}	1.33

TABLE VIII.6. Coefficients occurring in relations (VIII.18) and (VIII.19).

$\mathcal{T}_2^*$	a_6	a_2	a_2'	b_4	b_4'
-62.5	5.95	0.685	0.351	-0.228	-0.311

VIII.4 Ginzburg-Landau equations

VIII.4.1 Case (i)

Let us first consider the case where the critical eigenvalue is 0 and the critical wave number in the azimuthal direction is 0. In what follows we denote the critical eigenmodes by

$$\zeta_0 = \hat{U}(x)e^{\mathrm{i}\alpha_c z} \quad \text{and} \quad \bar{\zeta}_0, \tag{VIII.20}$$

where $\hat{U}(x)$ is changed into $\bar{\hat{U}}(x)$ under the symmetry through the $(z=0)$ plane.
We then decompose the perturbation U of the Couette flow as follows:

$$U = A(y,z,t)\zeta_0 + \bar{A}(y,z,t)\bar{\zeta}_0 + \Phi(\mu, A, \bar{A}, \partial_y, \partial_z), \tag{VIII.21}$$

where we consider that y and z are "*slow variables*" in the amplitude function A, hence, ∂_y, ∂_z can be considered as small parameters ($\partial_y A, \partial_z A$, $\partial^n_{y\cdot\cdot yz\cdot\cdot z}A, \partial_y\bar{A}, \partial_z\bar{A}, \ldots$ are considered to be independent in the expansions). In (VIII.24) Φ replaces the usual expression for the center manifold, taking account of the dependency into y and z of the amplitude A.

The partial differential system that holds for A and $\bar{A}$ has to satisfy the symmetries of the problem, as is well known. Here we have equivariances with respect to the translations in y and z, and the symmetry $z \longrightarrow -z$ which changes $(A, \partial_y, \partial_z)$ into $(\bar{A}, \partial_y, -\partial_z)$. For the principal part, we have the following Ginzburg-Landau equation:

$$\frac{\partial A}{\partial t} = a_1\frac{\partial A}{\partial y} + a_2(\mathcal{T}_1 - \mathcal{T}_{1c}(\mathcal{T}_2))A + 4a_3\alpha_c^2\frac{\partial^2 A}{\partial z^2} + a_4\frac{\partial^2 A}{\partial y^2} + 2\mathrm{i}a_5\alpha_c\frac{\partial^2 A}{\partial y\partial z} - cA|A|^2, \tag{VIII.22}$$

where we take into account the form (VIII.16) of the eigenvalue σ_0 and we only keep the first relevant nonlinear term. The computation of the nonlinear coefficient c is classical, using the Fredholm alternative. The numerical results have been obtained with a normalization of the eigenvector ζ_0 in such a way that the azimuthal velocity fulfills $\hat{u}_y(x = 0) = 1$ (they are reported in Table VIII.2). One can show that the nonlinear coefficient c vanishes at $\mathcal{T}_2 \sim -42.5$ and there exists a discrepancy at $\mathcal{T}_2 = 0$ between our result and that of Tabeling. Our value seems more reasonable because it has been checked by a limit process as η tends to 1. In fact, the value of c can be deduced from previous nonlinear coefficients c_η where $\eta \neq 1$ of [De-Io] by using the formula

$$c = \lim_{\eta \to 1} (c_\eta \eta / \mathcal{T}_1^*). \tag{VIII.23}$$

The qualitative behavior of solutions for $\mathcal{T}_2$ larger than -42.5 is the same as for Tabeling's system and can be described by the equation

$$\frac{\partial A}{\partial t} = \mu A + \frac{\partial^2 A}{\partial z^2} + \frac{\partial^2 A}{\partial y^2} + \mathrm{i}a \frac{\partial^2 A}{\partial y \partial z} - A|A|^2 \tag{VIII.24}$$

after the following changes of variables and scales

$$y' = (y + a_1 t)/\sqrt{a_4}, \quad A' = A\sqrt{c}, \quad z' = z/2\alpha_c\sqrt{a_3}, \quad a = a_5/\sqrt{a_3 a_6}. \tag{VIII.25}$$

By the way, we eliminate the advection term "$a_1 \partial A/\partial y$", which corresponds to a constant propagation velocity in the azimuthal direction. Solutions of (VIII.24) of the form

$$A = Q e^{\mathrm{i}(qy + pz + \Omega t)}, \qquad Q = \mu - p^2 - q^2 \text{ and } \Omega = -apq, \tag{VIII.26}$$

give the classical periodic patterns of Taylor vortices if $q = 0$ with wave number $(\alpha_c + p)$ and, if $q \neq 0$, a small modulation in the azimuthal direction (helicoidal wave). Let us notice that if $q = 0$, the solution (VIII.26) is stationary, and if $q \neq 0$, the azimuthal modulation is due to the anisotropy of the problem (coefficients a_1 and a_5). The study of the stability of steady flows ($q = 0$) made by Tabeling [Tab] shows that the most dangerous mode is again given by the Eckhaus instability phenomenon. One can generalize this result and prove that the solution (VIII.26) is stable with respect to perturbations of an infinitely small wave number if

$$\mu \geq 3(q^2 + p^2). \tag{VIII.27}$$

In the neighborhood of the situation where the bifurcation becomes subcritical, we must develop the previous envelope equation up to fifth order. With notations (VIII.23) and (VIII.24), we again obtain an equation as follows:

$$\frac{\partial A}{\partial t} = a_1\frac{\partial A}{\partial y} + \mu A + 4a_3\alpha_c^2\frac{\partial^2 A}{\partial z^2} + a_4\frac{\partial^2 A}{\partial y^2} + 2\mathrm{i}a_5\alpha_c\frac{\partial^2 A}{\partial y\partial z} - cA^2\bar{A}$$
$$+\mathrm{i}b_1A^2\frac{\partial \bar{A}}{\partial z} + \mathrm{i}b_2A\bar{A}\frac{\partial A}{\partial z} + c_1A^2\frac{\partial \bar{A}}{\partial y} + c_2A\bar{A}\frac{\partial A}{\partial y} + dA^3\bar{A}^2, \quad \text{(VIII.28)}$$

where μ and c are two small parameters and where all nonlinear coefficients are given in Table VIII.7.

After the change of variables and scales (VIII.32), one can write (suppressing the primes)

$$\frac{\partial A}{\partial t} = \mu A + \frac{\partial^2 A}{\partial z^2} + \frac{\partial^2 A}{\partial y^2} + \mathrm{i}a\frac{\partial^2 A}{\partial y\partial z} - \frac{\partial^4 A}{\partial y^4} + \varepsilon A|A|^2$$
$$+\mathrm{i}f_1A^2\frac{\partial \bar{A}}{\partial z} + \mathrm{i}f_2A\bar{A}\frac{\partial A}{\partial z} + g_1A^2\frac{\partial \bar{A}}{\partial y} + g_2A\bar{A}\frac{\partial A}{\partial y} + A^3\bar{A}^2 \quad \text{(VIII.29)}$$

with $f_\mathrm{i} = \dfrac{b_\mathrm{i}}{2\alpha_c\sqrt{da_3}}$, $g_\mathrm{i} = \dfrac{c_\mathrm{i}}{\sqrt{da_4}}$, $\varepsilon = -\dfrac{c}{\sqrt{d}}$.

We have the following basic "homogeneous" solutions

$$A = Qe^{\mathrm{i}(pz+qy+\Omega t)}, \qquad Q^4 - Q^2(\varepsilon + pf) = \mu - p^2 - q^2,\ \Omega = -(apq + gQ^2q), \quad \text{(VIII.30)}$$

where $f = f_1 - f_2$ and $g = g_1 - g_2$. There exists a unique solution (VIII.30) inside the domain defined by $\mu \geq p^2 + q^2$, and the stability of this solution is given by the conditions $\alpha_{20} > 0$ and $4\alpha_{20}\alpha_{02} - \alpha_{11}^2 > 0$, where

$$\begin{aligned}
\alpha_{20} &= \left(\frac{1}{Q^2} - \frac{gg_1}{\sqrt{\Delta}} + \frac{g^2q^2}{Q^2\Delta}\right),\\
\alpha_{11} &= \frac{q}{\Delta}\left(-ag + 2f_2 + 4\frac{P}{Q^2} + g_1g\frac{-2p + fQ^2}{\sqrt{\Delta}} - 2g^2pq^2\frac{-2p + fQ^2}{Q^2\Delta}\right),\\
\alpha_{02} &= 1 - \frac{-2p + fQ^2}{2\sqrt{\Delta}}\left(-\frac{2p + (f_1 + f_2)Q^2}{Q^2} + g^2q^2\frac{-2p + fQ^2}{Q^2\Delta}\right),\\
\Delta &= (\varepsilon + pf)^2 + 4(\mu - p^2 - q^2).
\end{aligned}$$

It can be shown numerically that there is a region in the (p, q, μ) space where the solution (VIII.30) is indeed stable. However, a general precise discussion analogous to that made in [Ec-Io] and [Do] is still needed here.

TABLE VIII.7. Nonlinear coefficients for the codimension two situations.

$\mathcal{T}_2$	c	b_1	b_2	c_1	c_2	d
−48.5	10^{-4}	−0.008	−0.011	0.358	0.029	$1.39\ 10^{-4}$
−62.5	−0.006	−0.005	−0.009	0.373	0.019	$1.54\ 10^{-4}$

VIII.4.2 Case (ii)

We now consider the neighborhood of the second codimension-two situation. The critical wave number in the azimuthal direction is 0, and the critical eigenvalue is zero. We obtain the following Ginzburg-Landau type of equation

$$\begin{aligned}\frac{\partial A}{\partial t} =& a_1\frac{\partial A}{\partial y} + \mu A + 4a_3\alpha_c^2\frac{\partial^2 A}{\partial z^2} - \nu\frac{\partial^2 A}{\partial y^2} + 2\mathrm{i}a_5\alpha_c\frac{\partial^2 A}{\partial y\partial z} - a_6\frac{\partial^4 A}{\partial y^4} - cA|A|^2\\ &+\mathrm{i}b_1A^2\frac{\partial \bar{A}}{\partial z} + \mathrm{i}b_2A\bar{A}\frac{\partial A}{\partial z} + c_1A^2\frac{\partial \bar{A}}{\partial y} + c_2A\bar{A}\frac{\partial A}{\partial y} + dA^3\bar{A}^2, \qquad \text{(VIII.31)}\end{aligned}$$

where μ and ν are *two small parameters* defined by (VIII.19). This equation differs from (VIII.28) by the annulation of coefficients a_4 and now coefficient c is no longer small (c is negative). After a change of variables and scales, defined by

$$\begin{aligned}y' = (y + a_1 t)/(a_6)^{1/4}, \qquad A' = Ad^{1/4}, \qquad z' = z/2\alpha_c\sqrt{a}_3,\\ a = a_5/(a_3)^{1/2}(a_6)^{1/4}, \qquad \varepsilon = -c/\sqrt{d}, \qquad \text{(VIII.32)}\end{aligned}$$

one can write (suppressing the primes)

$$\begin{aligned}\frac{\partial A}{\partial t} =& \mu A + \frac{\partial^2 A}{\partial z^2} - \nu\frac{\partial^2 A}{\partial y^2} + \mathrm{i}a\frac{\partial^2 A}{\partial y\partial z} - \frac{\partial^4 A}{\partial y^4} + \varepsilon A|A|^2\\ &+\mathrm{i}f_1A^2\frac{\partial \bar{A}}{\partial z} + \mathrm{i}f_2A\bar{A}\frac{\partial A}{\partial z} + g_1A^2\frac{\partial \bar{A}}{\partial y} + g_2A\bar{A}\frac{\partial A}{\partial y} + A^3\bar{A}^2, \qquad \text{(VIII.33)}\end{aligned}$$

where coefficients, except f_j and g_j, are given in Tables VIII.4–VIII.6 and where $\varepsilon \cong 0.5$. Since ε is positive and of order 1, it is clear that homogeneous solutions such as (VIII.30) are always unstable, since one has only to consider "small" solutions in norm.

VIII.4.3 Case (iii)

We now consider the case when the critical wave number in the azimuthal direction is not 0, the critical eigenvalue being pure imaginary. In what follows we denote the critical eigenmodes by

$$\zeta_0 = \hat{U}(x)e^{\mathrm{i}(\alpha_c z+\beta_c y)}, \qquad \zeta_1 = \hat{V}(x)e^{\mathrm{i}(-\alpha_c z+\beta_c y)}, \text{ and } \quad \bar{\zeta}_0, \bar{\zeta}_1, \qquad \text{(VIII.34)}$$

where $\hat{V}(x)$ is the image of $\hat{U}(x)$ under the symmetry $z \to -z$.
We then decompose the perturbation U of the Couette flow as follows:

$$\begin{aligned}U =& A(y,z,t)\zeta_0 + \bar{A}(y,z,t)\bar{\zeta}_0 + B(y,z,t)\zeta_1 + \bar{B}(y,z,t)\bar{\zeta}_1\\ &+\Phi(\mu, A, \bar{A}, B, \bar{B}, \partial_y, \partial_z), \qquad \text{(VIII.35)}\end{aligned}$$

and, as in Subsection VIII.4.1, the partial differential system that holds for $A, B, \bar{A}, \bar{B}$ has to respect the symmetries of the problem. Here again

TABLE VIII.8. Nonlinear coefficients occurring in (VIII.36)–(VIII.37).

$\mathcal{T}_2$	b	c
-65	$0.0048 + \mathrm{i}0.0079$	$-0.0042 - \mathrm{i}0.0103$
-70	$0.0034 + \mathrm{i}0.0115$	$-0.0046 - \mathrm{i}0.0166$
-80	$0.0025 + \mathrm{i}0.0137$	$-0.0040 - \mathrm{i}0.0233$
-90	$0.0028 + \mathrm{i}0.0147$	$-0.0028 - \mathrm{i}0.0279$
-100	$0.0034 + \mathrm{i}0.0153$	$-0.0016 - \mathrm{i}0.0320$

we have equivariances with respect to the translations in y and z and the symmetry $z \longrightarrow -z$ which changes $(A, B, \partial_y, \partial_z)$ into $(B, A, \partial_y, -\partial_z)$. It results that, for the principal part, we have the following Ginzburg-Landau system of equations:

$$\begin{aligned}\frac{\partial A}{\partial t} &= [\mathrm{i}\omega + a_2(\mathcal{T}_1 - \mathcal{T}_{1\mathrm{c}}(\mathcal{T}_2))]A + a_1\frac{\partial A}{\partial y} + 2b_1\alpha_\mathrm{c}\frac{\partial A}{\partial z} + 4a_3\alpha_\mathrm{c}^2\frac{\partial^2 A}{\partial z^2} \\ &\quad + a_4\frac{\partial^2 A}{\partial y^2} + 2\mathrm{i}a_5\alpha_\mathrm{c}\frac{\partial^2 A}{\partial y \partial z} + bA|A|^2 + cA|B|^2, \qquad \text{(VIII.36)}\end{aligned}$$

$$\begin{aligned}\frac{\partial B}{\partial t} &= [\mathrm{i}\omega + a_2(\mathcal{T}_1 - \mathcal{T}_{1\mathrm{c}}(\mathcal{T}_2))]B + a_1\frac{\partial B}{\partial y} - 2b_1\alpha_\mathrm{c}\frac{\partial B}{\partial z} + 4a_3\alpha_\mathrm{c}^2\frac{\partial^2 B}{\partial z^2} \\ &\quad + a_4\frac{\partial^2 B}{\partial y^2} - 2\mathrm{i}a_5\alpha_\mathrm{c}\frac{\partial^2 B}{\partial y \partial z} + cB|A|^2 + bB|B|^2. \qquad \text{(VIII.37)}\end{aligned}$$

A system similar to (VIII.36)–(VIII.37) with no cross derivative was studied by Coullet and Lega [Co-Le]. Neglecting spatial dependencies, equations (VIII.36) –(VIII.37) possess two types of nontrivial solutions, namely, the traveling waves ($A \neq 0$ and $B = 0$) and the standing waves ($|A| = |B|$). The values of nonlinear coefficients b and c give a criterion for existence and stability of such solutions (see [C-I]). The numerical computations (see Table VIII.8) show unfortunately that only the standing waves may exist (when $\mathrm{Re}\, b + \mathrm{Re}\, c < 0$) and they are always unstable since $\mathrm{Re}\, b - \mathrm{Re}\, c > 0$. As a conclusion, let us just say that the different systems we established here, with specific values of the coefficients, should be studied numerically, to be able to say more about their solutions.

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