CHAPMAN & HALL/CRC Research Notes in Mathematics 406

W Jäger, J Nečas, O John,
K Najzar, and J Stará (Editors)

Partial differential equations

Theory and numerical solution

CRC Press
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CHAPMAN & HALL/CRC
Research Notes in Mathematics Series

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CRC Press UK
Chapman & Hall/CRC Statistics and Mathematics
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CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
A CHAPMAN & HALL BOOK

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

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CONTENTS

Preface

The present volume constitutes the Proceedings of the conference Partial Differential Equations - Theory and Numerical Solutions. The conference took place in August 10 -16, 1998, in Praha. As part of the celebration of the 650th anniversary of the foundation of Charles University, it was organized under the auspices of the Charles University, Praha and the Ruprecht-Karls-Universitaet, Heidelberg. The conference was included among the satellite conferences of the International Mathematical Congress ICM'98 in Berlin. With its rich scientific programme, the conference provided an opportunity for almost 200 participants to gather in a convenient environment and to discuss the present status of several important directions in which partial differential equations have been developing recently.

The invited plenary speakers, as well as the invited speakers in sections, were carefully chosen by the members of the Programme Committee: H. Berestycki, M. Feistauer, B. Fiedler, A. Friedman, H. Gajewski, M. Golubitsky, J. Haslinger, G. C. Hsiao, T. J. R. Hughes, J. Kačur, D. Kroener, J. M. Morel, R. Rannacher, M. Renardy and I. Vrkoč. The care with which they performed their important task cannot be overestimated.

The scientific programme of the conference consisted of 15 plenary lectures and about 40 section lectures. Together with this invited part of the programme there were about 110 other presentations (short communications and posters).

This book consists mostly of the papers submitted by the invited speakers. They are completed by a limited number of the contributions based on the short communications. (In selecting these complementary papers, the articles containing new results were preferred to the survey papers.) The task of organizing the material for publication has been made easy by the thoroughness with which each author has converted the spoken to the written word. We hope that this volume will help to stimulate further research. We are grateful to CRC PRESS (U.K.) LLC publishing house for its prompt and efficient publication of the proceedings.

The organizers take pleasure in expressing their thanks to the participants - not only for the interesting talks but also for their contribution in creating the friendly and stimulating atmosphere of the conference. They also want to thank Charles University, Prague, for its moral support and Ruprecht-Karls-Universitaet, Heidelberg for its important contribution. We extend our thanks to our colleagues from the Institute of Mathematics of the Academy of Sciences of the Czech Republic for their substantial help with the organization. We cannot imagine how the conference could have been organized without the collaboration of the Prague School of Economy (conference building, perfect service and first class equipment) and the Institute of Chemical Technology (student residence).

Last but not least, we should like to express our gratitude to the brewery in the Moravian town of Litovel for sponsoring the conference party by providing excellent beer.

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On the Global in Time Solvability of the Cauchy-Dirichlet Problem to Nondiagonal Parabolic Systems with Quadratic Nonlinearities

A. A. Arkhipova

Abstract. In the case of two spatial variables the global solvability for nonlinear parabolic systems with quadratic nonlinearities in the gradient is proved. We assume that elliptic operator of the system has variational structure. The constructed solution is smooth almost everywhere and has at most a finite number of singular points.

Keywords: boundary value problems, nonlinear parabolic systems, solvability

Classification: 35J65

Let Ω be a bounded domain in $\mathbb{R}^n, n \geq 2, Q = \Omega \times (0,T), T > 0$ be fixed arbitrarily. Let $u : Q \mapsto \mathbb{R}^N, N > 1,$ be a solution of the problem

$$u_t - Lu = 0 \quad \text{in} \quad Q, \qquad u|_{\partial' Q} = \psi. \tag{1}$$

Here L is a nonlinear elliptic operator of the second order with *quadratic nonlinearity* in the gradient; $\partial' Q$ is the parabolic boundary of Q.

First, we suppose that L is a quasilinear operator, in such case *local* in time existence of classical solution of (1) was proved by H. Amann [1] and M. Giaquinta and G. Modica [7]. The existence of a solution on any time interval $[0,T]$ has not been studied yet. For the special case when L is the operator of *variational type* and $n = \dim\Omega = 2,$ the author has constructed global in time almost everywhere smooth in $\overline{Q}$ solution of (1).

Now we shall discuss this result and some of its generalizations. Further we assume that $\dim\Omega = 2$.

To describe L we consider the functional

$$\mathcal{E}_f[v] = \frac{1}{2}\int_\Omega \langle A(x,v)v_x \cdot v_x\rangle dx, \tag{2}$$

where the integrand

$$f(x,v,p) = \frac{1}{2}\langle A(x,v)p\cdot p\rangle = \frac{1}{2}\sum_{k,l}^{N}\sum_{\alpha,\beta}^{2} A_{kl}^{\alpha\beta}(x,v)p_\beta^l p_\alpha^k,$$

$$\forall x \in \overline{\Omega}, \quad v \in \mathbb{R}^N, \quad p \in \mathbb{R}^{2N}; \qquad v_x = \left\{ v^k_{x_\alpha} \right\}^{k \le N}_{\alpha \le 2}.$$

We suppose that the matrix $A = \left\{ A^{\alpha\beta}_{kl} \right\}$ satisfies the following conditions on the $\mathcal{M} = \overline{\Omega} \times \mathbb{R}^N$:

$$A^{\alpha\beta}_{kl}(x,v)\xi^l_\beta \xi^k_\alpha \ge \nu|\xi|^2, \quad \forall \xi \in \mathbb{R}^{2N}, \qquad \sup_{\mathcal{M}} \|A(x,v)\| \le \mu; \tag{3}$$

functions $A^{\alpha\beta}_{kl}$ are twice differentiable with respect to $x = (x_1, x_2)$ and $v = (v^1, \dots, v^N)$,

$$\begin{aligned} \sup_{\mathcal{M}} \sum_{k,l \le N} \sum_{\alpha,\beta,\gamma \le 2} \left| \left(A^{\alpha\beta}_{kl}(x,v) \right)'_{x_\gamma} \right| &= l_1 < +\infty, \\ \sup_{\mathcal{M}} \sum_{k,l,m \le N} \sum_{\alpha,\beta \le 2} \left| \left(A^{\alpha\beta}_{kl}(x,v) \right)'_{v^m} \right| &= a < +\infty, \\ \sup_{\mathcal{M}} \sum_{k,l,m,s \le N} \sum_{\alpha,\beta \le 2} \left| \left(A^{\alpha\beta}_{kl}(x,v) \right)''_{v^m v^s} \right| &= l_2 < +\infty. \end{aligned} \tag{4}$$

Let $L = \{L^{(k)}\}^{k \le N}$ be the Euler operator for $\mathcal{E}_f$, then system (1) has the form

$$u^k_t - \left(A^{\alpha\beta}_{kl}(x,u) u^l_{x_\beta} \right)_{x_\alpha} + \frac{1}{2} \left(A^{\alpha\beta}_{ml}(x,u) \right)'_{u^k} u^l_{x_\beta} u^m_{x_\alpha} = 0, \quad k = 1, \dots, N. \tag{5}$$

System (5) is the quasilinear parabolic system with quadratic nonlinearity in the gradient:

$$\left| f_u(x,u,p) \right| = \frac{1}{2} \left| \langle A'_u(x,u) p \cdot p \rangle \right| \le \frac{a}{2} |p|^2, \qquad \forall p \in \mathbb{R}^{2N},$$

(we *do not suppose* any smallness condition on the parameter a).

For some fixed $\alpha_0 \in (0,1)$ and $t_1, t_2 \in [0,T]$ we define the set

$$\begin{aligned} \mathcal{K}\{[t_1,t_2]\} = \Big\{ u : \overline{\Omega} \times [t_1,t_2] \mapsto \mathbb{R}^N \ \Big| \ & u, u_x, u_t, u_{xx} \in C^{\alpha_0}\left(\overline{\Omega} \times [t_1,t_2]; \delta \right), \\ & u_{xt} \in L^2\left(\Omega \times (t_1,t_2) \right) \Big\}, \end{aligned} \tag{6}$$

and write $v \in \mathcal{K}\{[t_1,t_2)\}$, if $v \in \mathcal{K}\{[t_1,\tau]\}$, $\forall \tau < t_2$.

We denote by δ usual parabolic metric;

$$u_{xx} = \left\{ u^k_{x_\alpha x_\beta} \right\}^{k \le N}_{\alpha,\beta \le 2}, \qquad u_{xt} = \left\{ u^k_{x_\alpha t} \right\}^{k \le N}_{\alpha \le 2}.$$

If u is a solution of (1), (5) then $\mathcal{E}_f[u]$ is called the energy of the problem and

$$\mathcal{E}_f[u(t), \Omega_R(x^0)] = \int_{\Omega_R(x^0)} f(x, u(x,t), u_x(x,t))dx$$

is called the local energy of the problem on the set $\Omega_R(x^0) = \Omega \cap B_R(x^0)$, $x^0 \in \overline{\Omega}$, at the moment t ($B_R(x^0)$ is a ball of radius R in $\mathbb{R}^2$ centered at x^0).

In [2] the following result was proved.

THEOREM 1 (Th. 0.3 [2]). *Let Ω be a bounded domain in $\mathbb{R}^2$ with $C^{2+\alpha_0}-$ smooth boundary $\partial\Omega$ and L be the operator defined by (5), $\psi = \psi(x)$. If conditions (3), (4) hold then for any $T > 0$ and $\psi \in \overset{0}{W}{}_2^1(\Omega)$ there exists a global solution of the problem (1) $u : Q \mapsto \mathbb{R}^N$, which is smooth on $\overline{\Omega} \times (0,T]\backslash\Sigma$ and the singular set Σ consists of at most finite many points (x^j, t^j), $j \leq M$. For some number $\varepsilon_0 > 0$ (depending on the parameters of the problem and $C^{1+1}-$ characteristic of $\partial\Omega$) and every point $(x^j, t^j) \in \Sigma$ the following inequality is valid:*

$$\overline{\lim}_{t \nearrow t^j} \mathcal{E}_f\left[u(t), \Omega_R(x^j)\right] \geq \varepsilon_0, \quad \forall R > 0.$$

Moreover,

(1) $u \in W(Q) = \left\{v \in L^\infty\left((0,T); \overset{0}{W}{}_2^1(\Omega)\right),\ \exists v_t \in L^2(Q)\right\}$, *and*

$$\sup_{[0,T]} \mathcal{E}_f[u(t)] \leq \mathcal{E}_f[\psi];$$

(2) u *is the unique solution of (1) with the pointed properties;*

(3) u *is a weak solution of (1) from the class $W(Q)$ in the following sense:*

$$\int_Q \left(u_t\eta + \langle A(x,u)u_x \cdot \eta_x\rangle + \frac{1}{2}\langle A'_u(x,u)u_x \cdot u_x\rangle\eta \right) dQ = 0,$$

$$\forall \eta \in L^2\left((0,T); \overset{0}{W}{}_2^1(\Omega)\right) \cap L^1\left((0,T), L^\infty(\Omega)\right);$$

(4) $u(\cdot, t) \to \psi$ *in $W_2^1(\Omega)$ as $t \to 0$.*

To prove this result we followed M. Struwe's idea of the construction of heat flows for harmonic maps in a two-dimensional case [11]. Our considerations were based on two important facts: 1) the result on the local classical solvability of (1) mentioned earlier and 2) some continuation theorem which is the main analytical result of [2]. Now we formulate the continuation theorem.

THEOREM 2 (Th. 0.2 [2]). *Let conditions (3), (4) hold, $\partial\Omega \in C^{2+\alpha_0}, \psi \in C^{2+\alpha_0}(\overline{\Omega}), \psi$ vanishes on $\partial\Omega$ with its first and second derivatives. Assume that $u \in \mathcal{K}\{[0,T_0)\}$ is a solution of (1), (5) for some $T_0 > 0$. There exist $\varepsilon_0 > 0$ and $R_0(\varepsilon_0) > 0$ such that if*

$$\sup_{[0,T_0]} \sup_{x^0 \in \overline{\Omega}} \mathcal{E}_f[u(t), \Omega_{R_0}(x^0)] < \varepsilon_0, \tag{7}$$

then $u \in \mathcal{K}\{[0,T_0]\}$. The number ε_0 depends on ν, μ, a, l_1, l_2 and C^{1+1}–characteristic of $\partial\Omega$.

Note that ε_0 is the same number in (7) and in Theorem 1.

In [2] we explained the proof of Theorem 2 in detail and here we only point the main steps of the proof. We shall write $Q_0 = \Omega \times \Lambda_0$, $\Lambda_0 = (0, T_0)$.

Step 1. As always, the variational structure of L guarantees that

$$\|u_t\|^2_{2,Q_0} \le \mathcal{E}_f[\psi], \qquad \sup_{\overline{\Lambda_0}} \mathcal{E}_f[u(t)] \le \mathcal{E}_f[\psi], \tag{8}$$

and

$$\mathcal{E}_f[u(t_2)] \le \mathcal{E}_f[u(t_1)] \quad \forall t_1, t_2 \in [0, T_0],\ t_1 < t_2. \tag{8_0}$$

Moreover, from the variational structure of L the local variant of the energy estimate follows:

$$\begin{gathered}\mathcal{E}_f[u(t_2), \Omega_R(x^0)] \le \mathcal{E}_f[u(t_1), \Omega_{2R}(x^0)] + \frac{16\mu^3(t_2 - t_1)}{\nu^2 R^2}, \\ \forall t_1, t_2 \in [0, T_0],\ t_1 < t_2,\ \forall x^0 \in \overline{\Omega},\ \forall R > 0.\end{gathered} \tag{9}$$

In the next steps we do not use the variational structure of L.

Step 2. With the help of the smallness energy condition (7) we derive that

$$\|u_{xx}\|_{2,Q_0} + \|u_x\|_{4,Q_0} \le k_1, \tag{10}$$

with the constant k_1 depending on ν, μ, l_1, l_2, a and C^{1+1}– characteristic of $\partial\Omega$.

To get (10) we straighten a part of $\partial\Omega$ and deduce the estimate like (10) in local coordinates. Here and later on we estimate the strong nonlinear term in (5) by using well-known multiplicative inequality ([8], Ch.1):

$$\|w\|^4_{4,D} \le 2\|w\|^2_{2,D}\|w_x\|^2_{2,D}, \quad \forall w \in \overset{0}{W}{}^1_2(D),\ \dim D = 2. \tag{11}$$

Step 3. Now finiteness of the integral $J = \int_{Q_0} |u_x|^4 dQ$ and inequality (11) allow to get the following result. There exists $t_1 \in (0, T_0)$ such that for any $\gamma < \gamma_1 = \frac{\nu}{2\mu}$ and $\tau_1 \ge t_1$

$$\begin{gathered}\sup_{[\tau_1, T_0)} \int_\Omega |u_t(x,t)|^{2+2\gamma} dx + \int_{\tau_1}^{T_0} \int_\Omega \left(|u_{xt}|^2 |u_t|^{2\gamma} + |u_t|^{3+2\gamma} \right) dx dt \le \\ \le c_0 \int_\Omega |u_t(x, \tau_1)|^{2+2\gamma} dx,\end{gathered} \tag{12}$$

with c_0, t_1 depending on ν, μ, a, l_2 , in addition, t_1 depends on the integral J.
Step 4. There exist $t_2 \in (t_1, T_0)$ and $\gamma_2 \in (0, \gamma_1)$ such that

$$\sup_{[\tau_2, T_0)} \int_\Omega |u_x(x,t)|^{2+2\gamma} dx \leq c\Big[\int_{\tau_2}^{T_0} \int_\Omega (|u_t|^{2+\gamma} + |u_x|^{2+2\gamma}) dx dt + \\ + \int_\Omega |u_x(x,\tau_2)|^{2+2\gamma} dx\Big], \qquad \forall \tau_2 \geq t_2,\ \gamma \leq \gamma_2, \tag{13}$$

where c and t_2 depend on $\nu, \mu, l_1, a, C^{1+1}-$ characteristic of $\partial\Omega$, and in addition, t_2 depends on J.

First, we derive this inequality in the local coordinates by attracting (11) once more. From (13), (12) and (10) it follows that

$$\sup_{[\tau_2, T_0)} \|u_x(\cdot,t)\|_{2+2\gamma,\Omega} \leq k_2\big(1 + \|u_x(\cdot,\tau_2)\|_{2+2\gamma,\Omega} + \|u_t(\cdot,\tau_2)\|_{2+2\gamma,\Omega}\big) \equiv m_0, \tag{14}$$

k_2 depends on the same parameters as k_1.

From now on we fix $\tau^* \in [t_2, T_0)$, $Q_* = \Omega \times \Lambda_*$, $\Lambda_* = (\tau^*, T_0)$. As $\|u_x(\cdot, T_0)\|_{p,\Omega} \leq \underline{\lim}_{t \nearrow T_0} \|u_x(\cdot,t)\|_{p,\Omega} \leq m_0$, $p = 2 + 2\gamma > 2$, and due to the imbedding theorem we have the estimate:

$$\sup_{\Lambda_*} \|u(\cdot,t)\|_{C^{\delta_0}(\overline{\Omega})} \leq m_1, \quad \delta_0 = \frac{\gamma}{1+\gamma}. \tag{15}$$

(All m_i, $i \geq 1$, depend on the same values as m_0.)
Step 5. From (15) and the first estimate in (8) it follows ([10], Lemma 4) that

$$\|u\|_{C^{\delta_0,\delta_1}(\overline{Q_*})} \leq m_2, \quad \delta_1 = \frac{\gamma}{2(1+2\gamma)}, \tag{16}$$

γ is fixed in (14).
Step 6. Let us look now at the system (5) as at the elliptic one for some fixed $t \in [\tau^*, T_0)$:

$$-\big(A_{kl}^{\alpha\beta}(x,u)u_{x_\beta}^l\big)_{x_\alpha} + b^k(x,u,u_x) = 0, \quad x \in \Omega, \quad u^k|_{\partial\Omega} = 0, \quad k \leq N, \\ |b(x,u,p)| \leq \frac{a}{2}|p|^2 + \Phi(x,t), \quad \Phi(x,t) = u_t(x,t), \tag{17}$$

from (12) it follows that $\|\Phi(\cdot,t)\|_{q,\Omega} \leq \text{const}$, $\forall t \in [\tau^*, T_0)$, for some $q > 2$.

To advance in estimating $\sup\limits_{\Lambda_*} \|u_x(\cdot,t)\|_{C^\theta(\overline{\Omega})}$ with some $\theta > 0$, we consider (17) in local coordinates and use the freezing coefficients method. First of all we prove that the derivatives $u_x(\cdot,t)$ are uniformly bounded on Λ_* in the norm of Morrey space $L^{2,2-\varepsilon}(\Omega)$, $\forall\varepsilon > 0$. It gives us estimate (15) with any $\delta_0 < 1$. Then we improve this result deriving the inequality

$$\sup_{\Lambda_*} \|u_x(\cdot,t)\|_{\mathcal{L}^{2,2+2\theta}(\Omega)} \le \text{const},$$

with some $\theta \in (0,1)$. At this step our considerations are close to papers [3,6]. According to the property of Campanato space $\mathcal{L}^{2,2+2\theta}(\Omega)$ the last inequality gives that

$$\sup_{\Lambda_*} \|u_x(\cdot,t)\|_{C^{\theta}(\overline{\Omega})} \le m_3. \tag{18}$$

Step 7. From (18) and (16) it follows ([9], Ch.2, Lemma 3.1) that for some $\beta_0 \in (0,1)$

$$\|u_x\|_{C^{\beta_0}(\overline{Q_*};\delta)} \le m_4. \tag{19}$$

Now we can attract the linear theory for parabolic systems to assert that $u \in \mathcal{K}\{[0,T_0]\}$. □

Remark 1. The monotonicity formula (8_0) guarantees that $u(\cdot,t_m) \to u_\infty(\cdot)$ weakly in $W_2^1(\Omega)$ as some sequence $t_m \to \infty$. There arise two opportunities: 1) all singularities appear in some finite interval $(0,T]$, and 2) the solution has singular points at the infinity. In the first situation $u(\cdot,t_m) \to u_\infty(\cdot)$ in $W_2^2(\Omega)$ and u_∞ appears to be smooth extremal of $\mathcal{E}_f$. Moreover, if $\mathcal{E}_f[\psi]$ is small enough then $u_\infty \equiv 0$ in Ω (u_∞ is the minimizer of $\mathcal{E}_f$). In the second situation the solution u has a finite number of singular points $x^1,\dots,x^M$ at the infinity. If to fix for any $r > 0$ the set $D_r = \overline{\Omega} \setminus \cup_{i=1}^M \Omega_r(x^i)$ then $u(\cdot,t_m) \to u_\infty(\cdot)$ in $W_2^2(D_r)$ for some sequence $t_m \to +\infty$, u_∞ being the almost everywhere smooth extremal of $\mathcal{E}_f$ (with finite number of singular points).

Remark 2. As it was pointed in [2] we can construct a global solution of (5) under nonzero boundary condition

$$\begin{gathered} u|_{t=0} = \varphi \in W_2^1(\Omega), \quad \exists\psi \in C^{2+\alpha_0}(\overline{\Omega}) \quad \text{such that} \quad u|_{\partial\Omega\times(0,T)} = \psi, \\ \varphi - \psi \in \overset{0}{W}{}_2^1(\Omega). \end{gathered} \tag{20}$$

Further we shall discuss the case of *nonlinear* operator L in (1). Let

$$\mathcal{E}_f[u] = \int_\Omega f(x,u,u_x)\,dx,$$

where $f(x,u,p)$ and its derivatives $f_p, f_u, f_{uu}, f_{pu}, f_{pp}, f_{px}$ are continuous functions on the set $\mathcal{M}_1 = \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{2N}$ and satisfy the following conditions

I.

$$\nu_0|p|^2 \le f(x,u,p) \le \mu_0(1+|p|^2), \tag{21}$$

$$|f_p| + |f_{px}| + |f_{pu}| \le \mu_1(1+|p|), \tag{22}$$

$$|f_u| + |f_{uu}| \le \mu_2(1+|p|^2), \tag{23}$$

$$f_{p^k_\alpha p^l_\beta}\xi^k_\alpha \xi^l_\beta \ge \nu|\xi|^2, \quad \forall \xi \in \mathbb{R}^{2N}, \qquad \|f_{pp}\| \le \mu, \tag{24}$$

where $\nu_0, \mu_0, \dots, \mu$ are some positive constants (no smallness condition on μ_2 is assumed).

II. For some $\alpha_0 \in (0,1)$ functions f_{pp}, f_{pu}, f_{px} and f_u satisfy Hölder condition with exponent α_0 on any compact set in $\mathcal{M}_1$.

III. Ω is a bounded domain in $\mathbb{R}^2$ with $C^{2+\alpha_0}-$ smooth boundary $\partial\Omega$.

Let $L = \{L^{(k)}\}^{k \le N}$ be the Euler operator of $\mathcal{E}_f$. We consider the following variant of the problem (1):

$$\begin{gathered} u^k_t - \frac{d}{dx_\alpha} f_{p^k_\alpha}(x,u,u_x) + f_{u^k}(x,u,u_x) = 0, \quad (x,t) \in Q, \ k \le N, \\ u|_{t=0} = \varphi(x), \qquad u|_{\partial\Omega \times (0,T)} = \psi(x). \end{gathered} \tag{25}$$

Now we formulate the global solvability result for (25) (generalization of Theorem 1).

THEOREM 3. *Let $T > 0$ be fixed arbitrarily and conditions I-III hold with some $\alpha_0 > 0$. Suppose that functions φ and ψ satisfy (20). Then there exists almost everywhere smooth in $\overline{\Omega} \times (0,T]$ solution u of Cauchy-Dirichlet problem (25). The description of the singular set Σ and other properties of $u(x,t)$ are just the same as in Theorem 1.*

To construct a solution with the properties mentioned in Theorem 3 (using M. Struwe's scheme) we need, as earlier, two main facts: 1) local in time classical solvability result and 2) the continuation theorem of smooth solutions. Both of the facts were proved by the author and we formulate them here as Theorems 4 and 5.

Suppose besides (20) that $\varphi \in C^{2+\alpha_0}(\overline{\Omega})$ and the compatibility condition is valid:

$$-\frac{d}{dx_\alpha} f_{p^k_\alpha}(x,\varphi(x),\varphi_x(x)) + f_{u^k}(x,\varphi(x),\varphi_x(x)) = 0, \quad x \in \partial\Omega. \tag{26}$$

THEOREM 4. *Let function f and its derivatives satisfy conditions I-II with some $\alpha_0 \in (0,1)$. Suppose that conditions III, (20), and (26) hold. Then for some $T_0 > 0$ there exists a unique solution u of (25) in the class $\mathcal{K}\{[0,T_0)\}$.*

We prove this result by the Leray-Shauder method as it was done in [9], Ch. 2, §6, for a single nonlinear parabolic equation. The derivation of *a priori* estimates of u and u_x in some Hölder spaces is the main analytical part of the method. We are able to deduce such estimates step by step as it was done in the proof of Theorem 2, if $T_0 > 0$ is small enough. It should be noted that there exists some difference in realizing these steps for the *nonlinear* case. In particular, at the sixth step we look at system (25) as a nonlinear elliptic system with fixed $t > 0$. To derive estimate (18) with some $\theta \in (0,1)$ we use S. Campanato's approach [5] for nonlinear systems in small dimensions and the author's result [4].

Remark 3. The assertion of Theorem 4 remains true if $f = f(x,t,u,p)$. In such case besides (21)-(24) on the set $\overline{\Omega} \times [0,T] \times \mathbb{R}^N \times \mathbb{R}^{2N}$ we suppose that

$$|f_{pt}| \leq \mu_1(1+|p|); \qquad |f_{ut}| + |f_t| \leq \mu_2(1+|p|^2). \tag{27}$$

THEOREM 5. *Suppose that conditions I-III hold, φ, ψ satisfy (20), (26), and for some $T_0 > 0$ $u \in \mathcal{K}\{[0,T_0)\}$ is a solution of (25). Then there exist $\varepsilon_0 > 0$ and $R_0(\varepsilon_0) > 0$ such that, if condition (7) is valid, then $u \in \mathcal{K}\{[0,T_0]\}$. The number ε_0 depends on the parameters in (21)-(24) and $C^{1+1}-$ characteristic of $\partial\Omega$.*

Remark 4. If f depends on t, conditions (27) are valid, and, moreover,

$$|f_t(x,t,u,p)| \leq \rho(t)(1+|p|^2), \quad \rho_0 = \int_0^\infty \rho(\tau)d\tau < +\infty, \quad \text{and} \quad \nu_0 > \rho_0, \tag{28}$$

then the assertion of Theorem 5 is true and ε_0 depends also on $(\nu_0 - \rho_0)^{-1}$.

If in addition to (27) and (28) we assume that $f_t(x,t,u,p) \leq 0$ for all arguments, then the conclusion of Theorem 3 is correct.

Remark 5. Under conditions of Theorem 3 we can study behavior of $u(x,t)$ when $t \to +\infty$. For some $t_m \to +\infty$ $u(\cdot,t_m) \to u_\infty(\cdot)$ in the same sense as it was described in Remark 1 for quasilinear case. Limit function u_∞ is smooth in $\overline{\Omega}$ extremal of $\mathcal{E}_f$ or it has a finite number of singular points.

This work was partially supported by Russian Fund for Fundamental Research (grant no. 96-01-00087) and Federal Russian Programme "Intergration" (grant no. 326.53).

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Boundary Element Solution of Scattering Problems Relative to a Generalized Impedance Boundary Condition

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Abstract. This paper addresses the solution of a scattering problem governed by the Helmholtz equation and relative to a boundary condition expressed through a second-order differential operator by a boundary element method. The usual approach, setting a boundary integral equation from a representation of the solution, cannot be directly applied then since derivatives of single- or double-layer potentials are not given explicitly by integrals converging in the usual sense. As a result, it is not suited for effective numerical computations. First, a suitable procedure bypasses the difficulty at the cost of tripling the number of unknowns relative to the case of a zero impedance, that is, with a Neumann boundary condition. In fact, a suitable lumping process in the computation of an integral permits to eliminate the two supplementary unknowns in the assembly process at the element level. Furthermore, the final linear system to be solved has a symmetric matrix exactly as the one involved in the double-layer solution of the problem relative to a Neumann boundary condition. Moreover, the former appears as a small perturbation of the latter for a nearly vanishing impedance.

Keywords: Impedance boundary condition, boundary element, integral equation, scattering

Classification: Primary 65R35, 65N35; Secondary 78A45

1. Motivation

Generalized impedance boundary conditions are generally used as effective boundary conditions in scattering problems. These boundary conditions are aimed to incorporate in an approximate way the effect of details or particular situations too complicated to be handled otherwise. Without being exhaustive, we can quote: imperfectly conducting scatterers, perfectly conducting obstacle coated by a thin dielectric layer or having a rough surface, approximate modeling of some types of cavities, non-reflecting boundary condition for infinite waveguides, interface condition in a domain decomposition method, etc. (cf. e.g., [27, 20, 2, 18, 28, 3, 16, 32, 23, 31, 7, 13, 15]).

To show how such a boundary condition can arise, we consider the following problem relative to the scattering of a TE electromagnetic wave by a perfectly conducting obstacle coated by a thin dielectric shell.

The scatterer is represented by a bounded domain Ω^+ of the plane with a smooth boundary Γ; n is the unit normal to Γ outwardly directed to Ω^+; the domain within which the wave propagates is the exterior domain $\Omega^- := \mathbb{R}^2 \setminus \overline{\Omega^+}$. The perfectly conducting obstacle is the interior domain bounded by the parallel curve Γ_δ to Γ:

$$\Gamma_\delta := \left\{x \in \mathbb{R}^2;\ x = m - \delta \mathbf{n}(m),\ m \in \Gamma\right\},$$

where δ is the thickness of the thin dielectric shell (see figure 1).

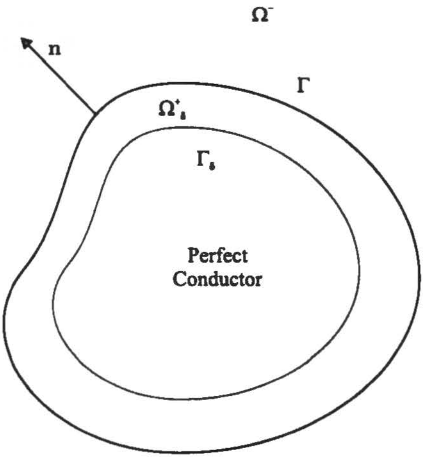

FIGURE 1. Perfect conductor coated by a thin shell

$$\Omega_\delta^+ := \left\{x \in \mathbb{R}^2;\ x = m - t\mathbf{n}(m),\ m \in \Gamma,\ 0 < t < \delta\right\}.$$

We will designate a generic point of Γ by m.

The magnetic field is everywhere normal to the plane of propagation in this case. The total wave is then represented by a function u giving its single nonzero component. It follows from Maxwell's equations and constitutive relations between fields and flux intensities that u is a solution to the following boundary-value problem of transmission-type:

$$\left.\begin{array}{l}
\Delta u + k^2 n^2 u = 0 \quad \text{in } \Omega_\delta^+, \\
\Delta u + k^2 u = 0 \quad \text{in } \Omega^-, \\
u^+ = u^-,\ \varepsilon^{-1}\partial_{\mathbf{n}} u^+ = \partial_{\mathbf{n}} u^-, \quad \text{on } \Gamma, \\
\partial_{\mathbf{n}} u = 0 \text{ on } \Gamma_\delta, \\
\lim_{|x|\to\infty} |x|^{1/2} \left(\partial_r (u - u^{\mathrm{inc}}) - ik(u - u^{\mathrm{inc}})\right) = 0,
\end{array}\right\} \tag{1.1}$$

where u^{inc} is a plane incident wave (or more generally is a wave produced in all the plane by sources having a compact support in Ω^-). In the above equations, k is the wave number in free space, n and ε, respectively, denote the index and the relative permittivity of the dielectric shell. Superscripts $+$ and $-$ indicate respective limits on Γ from within Ω^+ and Ω^-. However, if there is no risk of confusion, we drop this superscript. Finally, usual notation of PDE theory will be used without further comment.

The above problem can be restated as a boundary-value problem set only on Ω^- but involving a non-explicit boundary condition on Γ. Define the Neumann operator S_δ on Γ as follows. Given sufficiently smooth φ defined on Γ, let w be the solution to

$$\left.\begin{array}{l}\Delta w + k^2 n^2 w = 0 \quad \text{in } \Omega_\delta^+, \\ \partial_{\mathbf{n}} w = 0 \text{ on } \Gamma_\delta, \quad w = \varphi \text{ on } \Gamma,\end{array}\right\}$$

and let $S_\delta \varphi := \partial_{\mathbf{n}} w|_\Gamma$. Clearly, problem (1.1) now has the following setting:

$$\left.\begin{array}{l}\Delta u + k^2 u = 0 \quad \text{in } \Omega^-, \\ \partial_{\mathbf{n}} u = \varepsilon^{-1} S_\delta u \quad \text{on } \Gamma, \\ \lim_{|x|\to\infty} |x|^{1/2} \left(\partial_r (u - u^{\mathrm{inc}}) - ik(u - u^{\mathrm{inc}})\right) = 0.\end{array}\right\} \tag{1.2}$$

The main difficulty has hence been carried over the boundary condition which is expressed in terms of a non-explicit nonlocal operator. Fortunately enough, for a small δ, that is, in the case of a thin shell, S_δ can be approximated by a differential operator. This approach has been introduced in [14] and rigorously justified later in [8].

There exist two main ways to do such an approximation of S_δ: Taylor's expansion [14] and asymptotic analysis [8]. In this paper, another procedure, completely formal, will be used. It is transposed from an heuristic approach introduced in [21] to design higher order absorbing conditions starting from related radiation conditions on the circle at infinity. Indeed, we will use the improvement to this approach described in [1] which leads to more accurate absorbing boundary conditions. This construction has mainly two advantages. It leads more rapidly to the approximations of S_δ. However, its net benefit lies in the fact that it can give more accurate boundary conditions expressed in terms of differential as well as the inverses of elliptic differential operators but only of order ≤ 2. Such approximations of S_δ are beyond what can be reached by the above alternative methods, at least in a direct way. However, since it is not the main subject of this paper, we do not go further into this point and limit ourselves to the derivation of the 2-th order boundary condition obtained by Engquist and Nédélec [14]. We refer to a forthcoming work by Bartoli [5] for approximations of higher order.

So, for the moment consider the case

$$\Gamma := \left\{x \in \mathbb{R}^2;\ |x| = R\right\}, \quad \Gamma_\delta := \left\{x \in \mathbb{R}^2;\ |x| = R - \delta\right\}.$$

Thus the thin shell is

$$\Omega_\delta^+ = \left\{x \in \mathbb{R}^2;\ R - \delta < |x| < R\right\}.$$

Separation of cylindrical variables (r, θ) allows then to express S_δ in terms of a series expansion. More explicitly, express w by its Fourier series expansion

$$w(r\cos\theta, r\sin\theta) = \sum_{-\infty}^{+\infty} w_m(r)\, e^{im\theta},$$

$$w_m(r) = \tfrac{1}{2\pi} \int_0^{2\pi} w(r\cos\theta, r\sin\theta)\, e^{-im\theta}\, d\theta.$$

The Fourier coefficients are a solution to the following Sturm-Liouville two-point boundary-value problem:

$$\left.\begin{array}{l} w''_m(r) + r^{-1}w'_m(r) + (k^2n^2 - \frac{m^2}{r^2})w_m(r) = 0, \\ w'_m(R-\delta) = 0, \quad w_m(R) = \varphi_m, \end{array}\right\} \tag{1.3}$$

where the φ_m are the Fourier coefficients of φ defined by

$$\varphi_m := \frac{1}{2\pi}\int_0^{2\pi} \varphi(R\cos\theta, R\sin\theta)\, e^{-im\theta}\, d\theta.$$

Using the fact that Bessel J_m and Neumann Y_m functions constitute a fundamental system for the solutions to Bessel equation of order m (1.3), $w_m(r)$ has the following expression:

$$w_m(r) = \alpha_m J_m(kr) + \beta_m Y_m(kr).$$

Let $s := knR$ and $h := kn\delta$. Using the boundary conditions, we readily get

$$\alpha_m = -\frac{Y'_m(s-h)}{\Delta_m(h)}\varphi_m, \quad \beta_m = \frac{J'_m(s-h)}{\Delta_m(h)}\varphi_m,$$

with

$$\Delta_m(h) = J'_m(s-h)Y_m(s) - Y'_m(s-h)J_m(s).$$

Operator S_δ can then be made explicit through the Fourier-series expansion of $S_\delta\varphi$,

$$(S_\delta\varphi)(R\cos\theta, R\sin\theta) = \sum_{-\infty}^{+\infty} S_m(h)\varphi_m e^{im\theta},$$

$$S_m(h) = kn\frac{N_m(h)}{\Delta_m(h)}, \quad N_m(h) = J'_m(s-h)Y'_m(s) - Y'_m(s-h)J'_m(s).$$

Wave number k and index n being fixed, an asymptotic expansion in powers of h gives an expansion in powers of δ. Thus, Taylor series expansion yields

$$\begin{aligned} N_m(h) &= h(N_m^{(1)} + hN_m^{(2)} + \mathcal{O}(h^2)), \\ \Delta_m(h) &= \Delta_m^{(0)} + h\Delta_m^{(1)} + \mathcal{O}(h^2), \end{aligned}$$

with

$$\begin{aligned} N_m^{(1)} &= -J''_m(s)Y'_m(s) + Y''_m(s)J'_m(s), \\ N_m^{(2)} &= \tfrac{1}{2}\left(J'''_m(s)Y'_m(s) - Y'''_m(s)J'_m(s)\right), \\ \Delta_m^{(0)} &= J'_m(s)Y_m(s) - Y'_m(s)J_m(s), \\ \Delta_m^{(1)} &= -J''_m(s)Y_m(s) + Y''_m(s)J_m(s). \end{aligned}$$

Using the fact that J_m and Y_m are solutions to the Bessel equation of order m, $N_m^{(1)}$, $N_m^{(2)}$ and $\Delta_m^{(1)}$ can be expressed in terms of $\Delta_m^{(0)}$ only:

$$N_m^{(1)} = (\frac{m^2}{s^2} - 1)\Delta_m^{(0)}, \quad N_m^{(2)} = \frac{1}{2s}(\frac{3m^2}{s^2} - 1)\Delta_m^{(0)}, \quad \Delta_m^{(1)} = \frac{1}{s}\Delta_m^{(0)}.$$

A simple calculation then gives

$$S_m(h) = knh\frac{(\frac{m^2}{s^2}-1)+\frac{1}{2s}(\frac{3m^2}{s^2}-1)h}{1+\frac{h}{s}} + \mathcal{O}(h^3).$$

Expanding more completely the above expression and taking into account that the Fourier-series expansion interchanges derivative ∂_θ by multiplication by im, we arrive at the following approximate conditions.

- **Zero-order boundary condition** (the thin shell is completely neglected)

$$S_\delta^{(0)}\varphi = 0.$$

- **First-order boundary condition**

$$S_\delta^{(1)}\varphi = -\delta(\frac{1}{R^2}\partial_\theta^2 + k^2n^2)\varphi.$$

- **Second-order boundary condition**

$$S_\delta^{(2)}\varphi = -\left(\delta(\frac{1}{R^2}\partial_\theta^2 + k^2n^2) + \frac{\delta^2}{2}(\frac{1}{R^3}\partial_\theta^2 - \frac{k^2n^2}{R}\right)\varphi.$$

To obtain boundary conditions on arbitrary Γ, we use the formal approach introduced in [21] consisting in writing the above expressions in an intrinsic way by making the following substitutions:

$$\frac{1}{R}\partial_\theta \longleftrightarrow \partial_s, \quad \frac{1}{R}\longleftrightarrow \kappa,$$

where s is the curvilinear abscissa and κ is the related curvature defined by the following relations:

$$\tau := \partial_s m; \quad m \in \Gamma, \qquad \partial_s \mathbf{n} = \kappa\tau,$$

where τ is the unit tangent to Γ obtained by rotating $\mathbf{n}$ by $\pi/2$ in the direct sense. Note however that it is the improvement [1] mentioned above consisting in writing first the differential operator in a divergence form like, for example,

$$R^{-3}\partial_\theta^2 = R^{-1}\partial_\theta(R^{-1}R^{-1}\partial_\theta)$$

which leads to the right conditions which have been already rigorously obtained.

- **Neumann (or zero impedance) boundary condition**

$$\partial_{\mathbf{n}}u = \varepsilon^{-1}S_\delta^{(0)}u := 0. \tag{1.4}$$

- **First-order boundary condition of Engquist-Nédélec**

$$\partial_{\mathbf{n}}u = \varepsilon^{-1}S_\delta^{(1)}u := -\varepsilon^{-1}\delta(\partial_s^2 + k^2n^2)u. \tag{1.5}$$

- **Second-order boundary condition of Engquist-Nédélec**

$$\partial_{\mathbf{n}}u = \varepsilon^{-1}S_\delta^{(2)}u := -\varepsilon^{-1}\delta(\partial_s(1+\frac{\delta\kappa}{2})\partial_s + k^2n^2(1-\frac{\delta\kappa}{2}))u. \tag{1.6}$$

The related boundary-value problem, relative to one of the above boundary conditions (1.4), (1.5) or (1.6), is a particular form of the following general setting:

$$\left.\begin{array}{l} \Delta u + k^2 u = 0 \quad \text{in } \Omega^-, \\ \partial_{\mathbf{n}} u = -\partial_s(\alpha \partial_s u) + \beta u \quad \text{on } \Gamma, \\ \lim_{|x|\to\infty} |x|^{1/2} \left(\partial_r (u - u^{\mathrm{inc}}) - ik(u - u^{\mathrm{inc}})\right) = 0, \end{array}\right\} \tag{1.7}$$

α and β being two given functions defined on the boundary Γ.

This type of boundary condition relating the Neumann to the Dirichlet data is called a generalized boundary condition. We limit the exposure here to presenting how a boundary integral formulation with nice properties can be obtained for this problem and how it is solved by a Boundary Element Method (designated by BEM as usual in what follows). We refer to [9, 10, 29, 30, 6] for a complete mathematical as well as numerical analysis of the method and for further extension to the Leontovitch boundary condition for the full Maxwell's system.

2. The Boundary Integral Equation

A threefold objective is assigned to the reduction of the boundary-value problem (1.7) to a boundary integral system on Γ.

1. Its (main) unknowns must be the so-called physical unknowns

   $$\lambda := -u|_\Gamma, \quad \text{and} \quad p := -\partial_{\mathbf{n}} u|_\Gamma.$$

   Cauchy data in the terminology of PDE theory. In this way, existence of a solution to the integral equation will be assured *a priori*.
2. It must reduce to the formulation in terms of the physical unknowns of the problem relative to the Neumann boundary condition for $\alpha = \beta = 0$. (Since for the Neumann condition $p = 0$, the integral equation is obtained by representing u as a double-layer potential.)
3. All the integrals involved in the formulation must be weakly singular, that is, with a singularity integrable in the usual sense. Moreover, the final linear system to be solved must have a *symmetric matrix* and, for a given mesh of Γ, must have the same order than the related case of a Neumann boundary condition.

The starting point is as usual the following representation of the solution u to problem (1.7) given by the well-known formulae giving the outgoing solutions to Helmholtz equation (cf. e.g., [11, 12])

$$u(x) = u^{\mathrm{inc}}(x) + \mathcal{V}p(x) + \mathcal{N}\lambda(x), \quad x \notin \Gamma, \tag{2.1}$$

where $\mathcal{V}p$ is the single layer created by the density p

$$\mathcal{V}p(x) := \int_\Gamma G(x,y)p(y)\, d\Gamma(y), \quad x \notin \Gamma,$$

and $\mathcal{N}\lambda$ is the double layer created by the density λ

$$\mathcal{N}\lambda(x) := -\int_\Gamma \partial_{\mathbf{n}_y} G(x,y)\lambda(y)\, d\Gamma(y), \quad x \notin \Gamma.$$

The kernel $G(x,y)$ is that giving the outgoing solutions to the Helmholtz equation in $\mathbb{R}^2$,

$$G(x,y) := \tfrac{i}{4} H_0^{(1)}(k\,|x-y|),$$

where $H_0^{(1)} := J_0 + iY_0$ is the Hankel function of the first kind and of order 0.

It is of fundamental importance to note that the boundary condition results in the following relation linking densities p and λ:

$$p = (-\partial_s \alpha \partial_s + \beta)\lambda := Z\lambda. \tag{2.2}$$

A formal integral equation is formed by forcing the representation (2.1) to satisfy the boundary condition

$$\begin{aligned}(\partial_{\mathbf{n}} \mathcal{V} p)^- + (\partial_{\mathbf{n}} \mathcal{N}\lambda)^- + \partial_s \left(\alpha \partial_s (\mathcal{V}p + \mathcal{N}\lambda)^-\right) - \beta\,(\mathcal{V}p + \mathcal{N}\lambda)^- = \\ -\left(\partial_{\mathbf{n}} u^{\mathrm{inc}} + \partial_s \left(\alpha \partial_s u^{\mathrm{inc}}\right) - \beta u^{\mathrm{inc}}\right).\end{aligned}$$

Traces of single- and double-layers potentials are given by the following formulae (cf. e.g., [11, 12])

$$(\mathcal{V}p)^{\pm}(x) = Vp(x), \quad x \in \Gamma,$$

$$(\mathcal{N}\lambda)^{\pm}(x) = \pm\tfrac{1}{2}\lambda(x) + N\lambda(x), \quad x \in \Gamma,$$

$$(\partial_{\mathbf{n}} \mathcal{V}p)^{\pm}(x) = \pm\tfrac{1}{2}p(x) + Kp(x), \quad x \in \Gamma,$$

$$(\partial_{\mathbf{n}} \mathcal{N}\lambda)^{\pm}(x) = -\partial_s \left(V(\partial_s \lambda)\right)(x) - k^2 V(\lambda\tau)(x)\cdot\tau(x),$$

where V, N and K are the integral operators on Γ given by the weakly singular kernels

$$Vp(x) = \int_\Gamma G(x,y)p(y)\,d\Gamma(y),$$

$$N\lambda(x) = -\int_\Gamma \partial_{\mathbf{n}_y} G(x,y)\lambda(y)\,d\Gamma(y),$$

$$Kp(x) = \int_\Gamma \partial_{\mathbf{n}_x} G(x,y)p(y)\,d\Gamma(y).$$

Noting that K is the formal transpose $-N^{\mathsf{T}}$ of $-N$ for that

$$\int_\Gamma Kp\lambda'\,d\Gamma = -\int_\Gamma N\lambda' p\,d\Gamma,$$

and that the terms outside the integral sign cancel since p and λ are linked by relation (2.2), we are led to the "integral equation" which still remains formal:

$$-\partial_s V(\partial_s \lambda) - k^2 \tau\cdot V(\lambda\tau) - N^{\mathsf{T}} p + (\partial_s \alpha \partial_s - \beta)\,(Vp + N\lambda) = g \tag{2.3}$$

where the right-hand side g is related to the incident wave by

$$g = -\partial_{\mathbf{n}} u^{\mathrm{inc}} - \partial_s \left(\alpha \partial_s u^{\mathrm{inc}}\right) + \beta u^{\mathrm{inc}}.$$

Equation (2.3) has been called formal because the derivatives not under the integral sign cannot be moved inside and, hence, applied to the kernel. This prevents the equation from being directly suitable for designing an effective numerical method to solve problem (1.7). Following a different path, Nédélec [24], for the Laplace equation, and later Hamdi [17], for the Helmholtz one, have already remarked that it will be sufficient to use a variational formulation to get a formulation involving a weakly singular integral only as follows:

$$\int_\Gamma (\partial_n \mathcal{N}\lambda)^\pm \lambda' \, d\Gamma = \int_\Gamma \left(V(\partial_s\lambda)\,\partial_s\lambda' - k^2 V(\lambda\tau)\cdot\lambda'\tau\right) d\Gamma.$$

A similar approach is used to reformulate equation (2.3) as follows

$$\int_\Gamma (V(\partial_s\lambda)\partial_s\lambda' - k^2V(\lambda\tau)\cdot\lambda'\tau)\, d\Gamma - \int_\Gamma pN\lambda' \, d\Gamma$$
$$+ \int_\Gamma (V'p + N\lambda)\,(\partial_s\alpha\partial_s\lambda' - \beta\lambda')\, d\Gamma = \int_\Gamma g\lambda' \, d\Gamma.$$

Let p' be a test function, relative to unknown p, *linked to λ' exactly as p was linked to λ by (2.2)*:

$$p' = (-\partial_s\alpha\partial_s + \beta)\lambda' := Z\lambda'. \tag{2.4}$$

We have thus obtained the following variational equation,

$$a(\{p,\lambda\}, \{p',\lambda'\}) = \int_\Gamma g\lambda' \, d\Gamma, \tag{2.5}$$

where

$$\begin{aligned} a(\{p,\lambda\}, \{p',\lambda'\}) :=& \\ &\int_{\Gamma\times\Gamma} G(x,y)\,(\partial_s\lambda(y)\partial_s\lambda'(y) - k^2\lambda(y)\tau_y\cdot\lambda'(x)\tau_x)\, d\Gamma(y)d\Gamma(x) \\ &- \int_{\Gamma\times\Gamma} G(x,y)p(y)p'(x)\, d\Gamma(y)d\Gamma(x) \\ &- \int_{\Gamma\times\Gamma} \partial_{n_y}G(x,y)\,(\lambda'(y)p(x) + \lambda(y)p'(x))\, d\Gamma(y)d\Gamma(x). \end{aligned}$$

Remark 1. *Integral equation (2.5) only involves explicit integrals absolutely convergent in the usual meaning.*

If p and p' are eliminated from the formulation using relations (2.2) and (2.4), equation (2.5) lies in the class of Fredholm integral equations of the first kind and has a variational formulation expressed by a symmetric bilinear form in λ and λ'.

For $\alpha = \beta = 0$, equation (2.5) reduces to

$$\begin{aligned} \int_{\Gamma\times\Gamma} G(x,y)\,(\partial_s\lambda(y)\partial_s\lambda'(y) - k^2\lambda(y)\tau_y\cdot\lambda'(x)\tau_x)\, d\Gamma(y)d\Gamma(x)& \\ = -\int_\Gamma \partial_n u^{inc}\lambda' \, d\Gamma& \end{aligned} \tag{2.6}$$

which is the most efficient way to solve the boundary-value problem relative to a Neumann boundary condition.

The above remarks show that the threefold objective which has been initially assigned to the formulation is almost reached. However, it still remains an essential difference with the formulation (2.6) relative to a Neumann boundary condition. The direct elimination of p and p' would have as a consequence that the second-order derivatives of λ as well as those of λ' arise in the formulation. Apart from the difficulty of implementing a boundary element method with λ and λ' being polynomials of degree ≤ 3 over each element, this approach would require two degrees of freedom per node, namely twice of unknowns of what is involved in solving problem (2.6) on the same mesh. Thus, proceeding in this way would move us away from our initial program. The following treatment finishes the procedure.

We will consider relations (2.2) and (2.4) as constraints and deal with them through a system of Lagrange multipliers ℓ and ℓ' playing the role of a supplementary unknown and its related test function, respectively. More precisely, we consider the following problem whose solution $\{p, \lambda, \ell\}$ yields a solution $\{p, \lambda\}$ to (2.5). The problem is to find p, λ and ℓ so that

$$\left.\begin{array}{l} a(\{p,\lambda\},\{p',\lambda'\}) + b(\ell,\{p',\lambda'\}) = \int_\Gamma g\lambda'\, d\Gamma, \\ b(\ell',\{p,\lambda\}) = 0, \end{array}\right\} \tag{2.7}$$

for all p', λ' and ℓ' now without any explicit link between either p and λ or p' and λ'. We have denoted by

$$\begin{aligned} b(\ell,\{p',\lambda'\}) &= \int_\Gamma \ell\,(p' + \partial_s\alpha\partial_s\lambda' - \beta\lambda')\, d\Gamma \\ &= \int_\Gamma \ell p' d\Gamma - \int_\Gamma (\alpha\partial_s\ell\partial_s\lambda' + \beta\ell\lambda')\, d\Gamma. \end{aligned}$$

Remark 2. *Formulation (2.7) involves differential operators of order at most* 1 *so that its numerical solution can be obtained by a BEM with shape functions of degree* ≤ 1 *over each element only.*

The above formulation seems to triple the number of unknowns. We will see that, in fact, p *and* ℓ *can be eliminated at the element level during the assembly process only keeping* λ *as unknown in the final linear system to be solved.*

Clearly, any solution to problem (2.7) leads to a solution to problem (2.5). Thus, no spurious solution has been introduced by augmenting the problem. It is also interesting to examine the converse, to assure that problem (2.7) has a solution.

Let λ and p be a solution to problem (2.3). Set

$$\ell = Vp + N\lambda.$$

Since λ and p are linked by relation (2.2), equation (2.3) shows that $\{\lambda, p, \ell\}$ is solution to the following system:

$$\left.\begin{array}{l} -\partial_s(V(\partial_s\lambda)) - k^2V(\lambda\tau)\cdot\tau - N^{\mathsf{T}}p - Z\ell = g, \\ -N\lambda - Vp + \ell = 0, \\ -Z\ell + p = 0. \end{array}\right\} \tag{2.8}$$

Clearly, problem (2.7) is nothing else but a variational formulation of problem (2.8).

Remark 3. *It is easy to prove that the well-posedness of problem (2.3) is equivalent to the following two properties (cf. [29]):*

1. *Exterior problem (1.7) is well-posed.*
2. *The homogeneous associated cavity problem*

$$\left.\begin{aligned} &\Delta u + k^2 u = 0 \quad \text{in } \Omega^+, \\ &\partial_{\mathbf{n}} u^+ = -\partial_s(\alpha \partial_s u^+) + \beta u^+ \quad \text{on } \Gamma, \end{aligned}\right\} \tag{2.9}$$

has no spurious solution.

Finally, note that jump relations of single- and double-layer potentials give

$$\ell = \tfrac{1}{2} u^-.$$

Since $v := -u^{\mathrm{inc}}|_{\Omega^+}$ is a solution to the interior problem, this relation is a particular case of the following one:

$$\ell = \tfrac{1}{2}(v^+ + v^-),$$

where $v|_{\Omega^-}$ is the scattered wave relative to a general g as right-hand side in the boundary condition. In the same way, $v|_{\Omega^+}$ is the solution to problem (2.9) with a nonzero right-hand side equal to g (cf. [29]). This relation makes it possible to recover $v^\pm$ as well as $\partial_{\mathbf{n}} v^\pm$ by a simple post-processing of the solution. This is important in some decomposition domain methods where Cauchy data have to be transmitted through the interface.

3. The Boundary Element Method

As usual Γ is approximated by a polygonal curve, still denoted by Γ. The vertices $\mathbf{a}_1, \dots, \mathbf{a}_N$ lie on the initial curve and are numbered in the direction defined by the above curvilinear abscissa s. This yields a mesh on Γ that elements are given by segments

$$K_i := [\mathbf{a}_i, \mathbf{a}_{i+1}], \quad i = 1, \dots, N$$

with the usual convention $\mathbf{a}_{N+1} = \mathbf{a}_1$.

Next, all functions, unknown and test, involved in the formulation are approximated by a C^0-finite element method whose shape functions are linear over each element K_i. As a result, any of these functions can be characterized by a column-vector of length N whose components are its nodal values; as an example, λ is completely recovered from the vector $\{\lambda\} = \{\lambda_i\}_{i=1}^{i=N}$ in the following way:

$$\lambda_i := \lambda(\mathbf{a}_i), \quad \lambda|_{K_i}(m) = (1 - s/|K_i|)\lambda_i + (s/|K_i|)\lambda_{i+1},$$

where $s := |m - \mathbf{a}_i|$ and $|K_i|$ is the length of K_i.

The usual assembly process and evaluation of weakly singular integrals give the following matrices defined

$$\{\lambda'\}^{\mathsf{T}} D \{\lambda\} = \int_{\Gamma\times\Gamma} G(x,y)\left(\partial_s\lambda(y)\partial_s\lambda'(y) - k^2\lambda(y)\tau_y\cdot\lambda'(x)\tau_x\right) d\Gamma(y)d\Gamma(x),$$

$$\{p'\}^{\mathsf{T}} N \{\lambda\} = -\int_{\Gamma\times\Gamma} \partial_{\mathbf{n}_y} G(x,y)\lambda(y)p'(x)\, d\Gamma(y)d\Gamma(x),$$

$$\{p'\}^{\mathsf{T}} V \{p\} = \int_{\Gamma\times\Gamma} G(x,y)p(y)p'(x)\, d\Gamma(y)d\Gamma(x).$$

The key point of eliminating $\{p\}$ and $\{\ell\}$ lies in the fact that the integral

$$\int_{\Gamma} \ell p'\, d\Gamma$$

can be represented in an approximate way by a diagonal matrix M through a lumping process

$$\int_{\Gamma} \ell p'\, d\Gamma = \sum_{i=1}^{N}\int_{K_i} \ell p'\, dK_i \approx \sum_{i=1}^{N}\frac{|K_i|}{2}\left(\ell_i p'_i + \ell_{i+1}p'_{i+1}\right) = \{p'\}^{\mathsf{T}} M \{\ell\}.$$

Finally, computation of matrix Z involved is performed by usual finite element calculations through the relation

$$\{\lambda'\}^{\mathsf{T}} Z \{\ell\} = \int_{\Gamma}\left(\alpha\partial_s\ell\partial_s\lambda' + \beta\ell\lambda'\right) d\Gamma.$$

Indeed, matrices D, N and V are obtained from the computation of only the following elementary integrals:

$$V_{ij} = \int_{K_i\times K_j} G(x,y)\, dK_j(y)dK_i(x),$$

$$N_{ij} = -\int_{K_i\times K_j} \partial_{\mathbf{n}_y} G(x,y)\, dK_j(y)dK_i(x).$$

As an example, matrix D is obtained by assembling the following contributions:

$$\int_{K_i\times K_j} G(x,y)\left(\partial_s\lambda(y)\partial_s\lambda'(y) - k^2\lambda(y)\tau_y\cdot\lambda'(x)\tau_x\right) dK_j(y)dK_i(x) \approx V_{ij}\left(\frac{\lambda_{j+1}-\lambda_j}{|K_j|}\frac{\lambda'_{i+1}-\lambda'_i}{|K_i|} - k^2\frac{\lambda_{j+1}+\lambda_j}{2}\tau_j\cdot\frac{\lambda'_{i+1}+\lambda'_i}{2}\tau_i\right).$$

where τ_i is the unit vector directed from $\mathbf{a}_i$ to $\mathbf{a}_{i+1}$.

Denoting by $\{g\}$ the vector whose components are given by

$$g_i := g(\mathbf{a}_i),$$

the discrete system can be reduced to the following linear system:

$$\begin{bmatrix} D & -N^{\mathsf{T}} & -Z \\ -N & -V & M \\ -Z & M & 0 \end{bmatrix} \begin{bmatrix} \{\lambda\} \\ \{p\} \\ \{\ell\} \end{bmatrix} = \begin{bmatrix} M\{g\} \\ 0 \\ 0 \end{bmatrix}. \tag{3.1}$$

Matrix Z is symmetric. Since M is a diagonal invertible matrix, $\{p\}$ can be expressed in terms of $\{\lambda\}$ by only inverting a diagonal matrix

$$\{p\} = M^{-1}Z\{\lambda\} = H\{\lambda\} \tag{3.2}$$

reducing the solution to the following system:

$$\left(D - H^{\mathsf{T}}VH - (HN)^{\mathsf{T}} - HN\right)\{\lambda\} = M\{g\}. \tag{3.3}$$

Note that the matrix of this system is symmetric.

This is a direct generalization of the usual method when dealing with the Neumann boundary condition which corresponds to a vanishing value for α and β.

Note that no matrix product is needed for forming the coefficient matrix of system (3.3). The real overhead of solving the impedance boundary-value problem relative to the Neumann boundary-value one comes from the computation of elementary integrals N_{ij} relative to a double-layer potential.

To test the accuracy of the method, we solve the problem corresponding to the case where Γ is the unit circle with a constant value for α and β. Because in this case the RCS is exactly known from a Fourier-Hankel series expansion of the solution, we can compare the computed values with the exact ones.

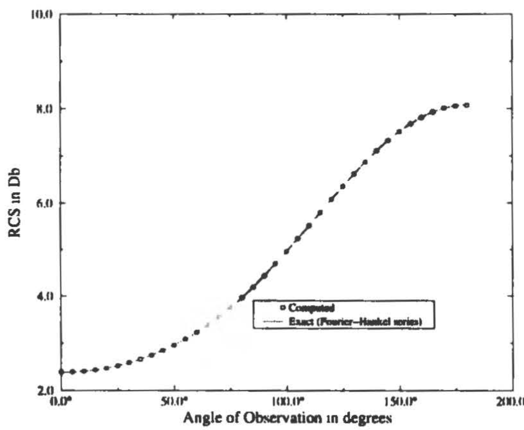

FIGURE 2. Bistatic RCS relative to $k = 1$, $\alpha = -\beta = 1$

For $\alpha = -\beta = 1$ and $k = 1$, figure 3 shows that the two curves cannot be distinguished. We have an uniform meshing of Γ with 90 nodes. As for the Neumann boundary condition, the error everywhere is $\leq 0.3\%$.

To test the stability of the method, we consider the case where now $\alpha = -\beta = 10^{-6}$. Figure 3 shows that, in this case, the solution either exact or approximated cannot be distinguished from that relative to the Neumann boundary condition.

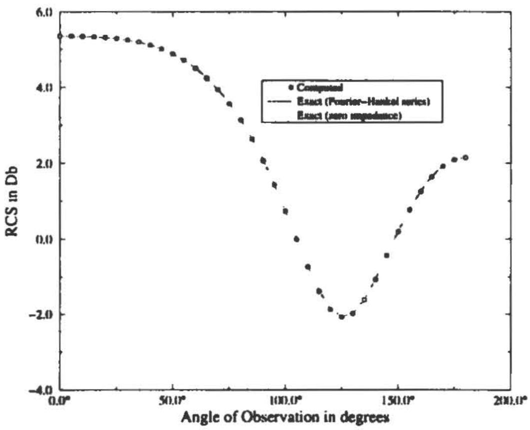

FIGURE 3. Bistatic RCS for nearly vanishing impedance

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Mathematical Analysis of Phase-Field Equations with Gradient Coupling Term

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Abstract. The article presents a basic mathematical analysis of equations in the phase-field model with special terms coupling the heat equation and the equation of phase which was induced by an extensive numerical work. The compactness technique is used to prove the existence and uniqueness of the weak solution of equations.

Keywords: Stefan problem with surface tension, reaction-diffusion equations, mean-curvature flow, compactness technique

Classification: 80A22, 82C26, 35A40

1. INTRODUCTION

The description of microscopic phenomena accompanying the solidification process of crystallic materials requires a simultaneous evaluation of the bulk enthalpy, the bulk and surface free energy of the system. Modern approach based on a non-sharp interpretation of the phase interface has the origin in the Cahn-Hilliard theory of solid - liquid phase transitions [5]. The equations have been analyzed from the point of view of the existence and uniqueness of solution, relation to the Stefan problem (see [4]), and convergence to the mean-curvature problem ([3]) in some special cases. The model based on phase-field equations exhibits a satisfactory qualitative agreement with a real situation (see [1]). The model is able to demonstrate important microscopic phenomena appearing in the metal solidification (dendritic or equiaxed growth, coarsening, ripening, reheating etc., see [1]). The question of quantitative agreement still remains open for the evolution of general, non-convex shapes like dendrites. Simulation of this phenomenon is at the limit of power of currently used computers and requires the use of more profound and sophisticated numerical algorithms, but also opens discussion about physical relevance of the phase-field model, or its particular parts. The behaviour of equations in question can be investigated from point of view of approximation of mean curvature, of influence of the diffusive interface on the release of latent heat, and approximate stability agreement with sharp-interface standards. These points have been discussed in [1] and [2] and lead in the first rank to a modification of coupling between the two equations proposed in view of better satisfaction of the sharp-interface relation — a Gibbs Thompson equation. As the numerical behaviour of the model was improved (as shown by a detailed analysis in [2]), it is useful to present a basic mathematical analysis of equations in question. Note that we discuss the original setting of the model containing physical constants, as this is convenient for the application of presented results.

2. Equations

The system of equations in [1] extensively used for the qualitative simulation of microstructure phenomena in solidification of crystallic materials reads as:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + L\frac{\partial p}{\partial t}, \\ \alpha\xi^2\frac{\partial p}{\partial t} &= \xi^2\Delta p + f_0(p) + F(u)\xi^2|\nabla p|, \end{aligned} \tag{1}$$

with initial conditions

$$u|_{t=0} = u_0\ , \quad p|_{t=0} = p_0,$$

and with boundary conditions of Dirichlet type

$$u|_{\partial\Omega} = 0\ , \quad p|_{\partial\Omega} = 0,$$

on Ω being a bounded domain in R^n with a C^2 boundary. Here, L, α, ξ are positive constants, and f_0 derivative of a double-well potential. The coupling function $F(u)$ is bounded and continuous, or even Lipschitz-continuous, $|.|$ denotes the Euclidean norm in R^n. We consider $f_0(p) = ap(1-p)(p-\frac{1}{2})$ with $a > 0$. For the sake of simplicity, $n = 2$ and boundary conditions are homogeneous. Obviously, the extension to higher dimensions and to other boundary conditions is possible.

In the physical context, the system (1) is treated as a regularization of the modified Stefan problem describing microstructure formation in solidification of a pure substance if $\xi \to 0$; see [6], [1]:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u \qquad \text{in } \Omega_s \text{ and } \Omega_l, && (2)\\ u|_{\partial\Omega} = 0, &\qquad u|_{t=0} = u_0, \\ \left.\frac{\partial u}{\partial n_\Gamma}\right|_s - \left.\frac{\partial u}{\partial n_\Gamma}\right|_l &= Lv_\Gamma, && (3)\\ F(u) &= -\kappa - \alpha v_\Gamma, && (4)\\ \Omega_s(t)|_{t=0} &= \Omega_{s0}, \end{aligned}$$

where Ω_s, Ω_l are solid and liquid phases, respectively, L is latent heat per unit volume, melting point is 0 and u temperature field. Discontinuity of heat flux on $\Gamma(t)$ is described by the Stefan condition (3), where v_Γ is the velocity in the direction of the outer normal $\mathbf{n}_\Gamma$ to Ω_s. The formula (4) is the Gibbs-Thompson relation on $\Gamma(t)$ whose mean curvature is denoted as κ. The parameter α is the coefficient of attachment kinetics. Following [1], the relation of (1) and (2-4) can be studied using asymptotical analysis.

The phase equation in (1) contains a modified coupling term $F(u)\xi^2|\nabla p|$, proposed by the author as a consequence of the level-set reformulation of the condition (4) using the definition of the boundary Γ as a manifold (see [2]). The paper presents a basic analysis for the phase-field equations (1). It deals with convergence of a semi-discrete scheme, and finally leads to the existence and uniqueness of the solution of the original system of equations.

3. Main result

First, we introduce the following notations:

$$(u, v) = \int_\Omega u(x)v(x)\,dx, \ \|u\| = \sqrt{\int_\Omega u(x)^2 dx} \text{ for } u, v \in L_2(\Omega),$$

$$(\nabla u, \nabla v) = \int_\Omega \nabla u(x) \cdot \nabla v(x)\,dx, \ \|\nabla u\| = \sqrt{\int_\Omega |\nabla u(x)|^2 dx} \text{ for } u, v \in H_0^1(\Omega).$$

We also notice that the assumptions on F imply that there are constants $C_F, L_F > 0$ such that $|F(s)| \le C_F, |F(s_1) - F(s_2)| \le L_F|s_1 - s_2|$ for all $s, s_1, s_2 \in \mathrm{R}$. We define the notion of the weak solution as usual in:

Definition 1. *Weak solution of the boundary-value problem with homogeneous Dirichlet boundary conditions for the phase-field equations is a couple of functions* $[u,p]$ *from* $(0,T)$ *to* $[H_0^1(\Omega)]^2$ *such that it satisfies*

$$\begin{aligned} \frac{d}{dt}(u - Lp, v) + (\nabla u, \nabla v) &= 0 \text{ a.e. in } (0,T), \\ u|_{t=0} &= u_0, \\ \alpha\xi^2 \frac{d}{dt}(p,q) + \xi^2(\nabla p, \nabla q) &= (f_0(p), q) + \xi^2(F(u)|\nabla p|, q) \text{ a.e. in } (0,T), \\ p|_{t=0} &= p_0 \end{aligned} \tag{5}$$

for each $v, q \in H_0^1(\Omega)$.

The continuous imbedding of $H^1(\Omega)$ into $L_s(\Omega)$ for each $s \in (1, +\infty)$ $(n = 2)$ ensures that $f_0(p) \in L_2(\Omega)$ for almost all $t \in (0,T)$. If $[u,p] \in [L_\infty(0,T; H_0^1(\Omega))]^2$ solves (5), then $[u,p]$ is a continuous mapping from $\langle 0, T\rangle$ to $[H^{-1}(\Omega)]^2$. Thus, the definition has proper sense. Our main results are contained in the following theorem. The proof by its virtue contains the investigation of convergence of a semi-discrete scheme based on the Faedo-Galerkin method.

Theorem 1. *Consider the problem (5) in a bounded domain* $\Omega \subset \mathrm{R}^2$ *with a* C^2 *boundary, and with* F *being a bounded continuous function. Assume that*

$$u_0, p_0 \in H^1(\Omega). \tag{6}$$

Then, there is a solution of the problem (5) satisfying

$$u, p \in L_\infty(0,T; H_0^1(\Omega)),\ p \in L_2(0,T; H^2(\Omega)),$$

$$\frac{\partial u}{\partial t}, \frac{\partial p}{\partial t} \in L_2(0,T; L_2(\Omega)).$$

Additionally, if F *is Lipschitz-continuous, the solution is unique.*

Proof. We derive a sequence of approximate solutions to the original problem. Assume that there is an orthonormal basis of the Hilbert space $L_2(\Omega)$ consisting of eigenvectors of the operator $-\Delta$ denoted as $\{v_i\}_{i\in\mathrm{N}}$, where $(\forall i \in \mathrm{N})(v_i \in \mathcal{C}^2(\Omega)\cap\mathcal{C}^1(\bar{\Omega}))$ with corresponding eigenvalues denoted as $\{\lambda_i\}_{i\in\mathrm{N}}$. Let $V_m = span\{v_i\}_{i\in\mathrm{N}_m}$ be a finite-dimensional subspace ($\mathrm{N}_m = \{1,\ldots,m\}$); $\mathcal{P}_m : L_2(\Omega) \to V_m$ be the projection operator. We seek for a solution of the auxiliary problem:

$$\begin{aligned}
\frac{d}{dt}(u^m - Lp^m, v) + (\nabla u^m, \nabla v) &= 0 \text{ a.e. in } (0,T),\ \forall v \in V_m,\\
u^m(0) &= \mathcal{P}_m u_0,\\
\alpha\xi^2\frac{d}{dt}(p^m, q) + \xi^2(\nabla p^m, \nabla q) &= (f_0(p^m), q) + \xi^2(F(u^m)|\nabla p^m|, q) \qquad (7)\\
&\qquad \text{a.e. in } (0,T),\ \forall q \in V_m,\\
p^m(0) &= \mathcal{P}_m p_0.
\end{aligned}$$

We use basic functions of V_m to express the solution of (7) as

$$u^m(t) = \sum_{i\in\mathrm{N}_m} \beta_i^m(t)v_i, \quad p^m(t) = \sum_{i\in\mathrm{N}_m} \gamma_i^m(t)v_i,$$

and to obtain a system of ODEs for the unknown functions of time: β_i^m, γ_i^m .

$$\begin{aligned}
\frac{d\beta_j^m}{dt} + \lambda_j\beta_j^m &= L\frac{d\gamma_j^m}{dt} \text{ in } (0,T), \qquad (8)\\
\beta_j^m(0) &= \beta_j^0, \text{ for each } j \in \mathrm{N}_m,\\
\alpha\xi^2\frac{d\gamma_j^m}{dt} + \xi^2\lambda_j\gamma_j^m &= (f_0(\sum_{i\in\mathrm{N}_m} \gamma_i^m v_i), v_j)\\
&\quad +\xi^2(F(\sum_{i\in\mathrm{N}_m} \beta_i^m v_i)|\sum_{i\in\mathrm{N}_m} \gamma_i^m \nabla v_i|, v_j) \text{ in } (0,T), \qquad (9)\\
\gamma_j^m(0) &= \gamma_j^0, \text{ for each } j \in \mathrm{N}_m,
\end{aligned}$$

where $\mathcal{P}_m u_0 = \sum_{i\in\mathrm{N}_m} \beta_i^0 v_i$, $\mathcal{P}_m p_0 = \sum_{i\in\mathrm{N}_m} \gamma_i^0 v_i$. Such a system has a maximal solution on the interval $\langle 0, T_m)$. We show that this interval does not depend on m, and show the appropriate convergence of the couple $[u^m, p^m]$. For this purpose, we prove an *a priori* estimate by multiplying (8) by $\frac{d\beta_j^m}{dt}$, and (9) by $\frac{d\gamma_j^m}{dt}$, and summing for $j \in \mathrm{N}_m$:

$$\begin{aligned}
\|\frac{\partial u^m}{\partial t}\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u^m\|^2 &= L(\frac{\partial p^m}{\partial t}, \frac{\partial u^m}{\partial t}),\\
\alpha\xi^2\|\frac{\partial p^m}{\partial t}\|^2 + \frac{\xi^2}{2}\frac{d}{dt}\|\nabla p^m\|^2 &= (f_0(p^m), \frac{\partial p^m}{\partial t}) + \xi^2(F(u^m)|\nabla p^m|, \frac{\partial p^m}{\partial t}). \qquad (10)
\end{aligned}$$

Using Schwarz and Young inequalities, we get

$$\begin{aligned}
\|\frac{\partial u^m}{\partial t}\|^2 + \frac{d}{dt}\|\nabla u^m\|^2 &\leq L^2\|\frac{\partial p^m}{\partial t}\|^2,\\
\frac{1}{2}\alpha\xi^2\|\frac{\partial p^m}{\partial t}\|^2 + \frac{\xi^2}{2}\frac{d}{dt}\|\nabla p^m\|^2 + \frac{d}{dt}(w_0(p^m), 1) &\leq \frac{C_F^2}{2\alpha}\xi^2\|\nabla p^m\|^2, \qquad (11)
\end{aligned}$$

where $w_0' = -f_0$. Combining these estimates, we have

$$\frac{1}{4}\alpha\xi^2\|\frac{\partial p^m}{\partial t}\|^2 + \frac{\alpha\xi^2}{4L^2}\|\frac{\partial u^m}{\partial t}\|^2 + \frac{\alpha\xi^2}{4L^2}\frac{d}{dt}\|\nabla u^m\|^2 + \frac{\xi^2}{2}\frac{d}{dt}\|\nabla p^m\|^2 + \frac{d}{dt}(w_0(p^m),1) \le \frac{C_F^2}{2\alpha}\xi^2\|\nabla p^m\|^2, \tag{12}$$

adding non-negative terms on the right-hand side,

$$\frac{1}{4}\alpha\xi^2\|\frac{\partial p^m}{\partial t}\|^2 + \frac{\alpha\xi^2}{4L^2}\|\frac{\partial u^m}{\partial t}\|^2 + \frac{\alpha\xi^2}{4L^2}\frac{d}{dt}\|\nabla u^m\|^2 + \frac{\xi^2}{2}\frac{d}{dt}\|\nabla p^m\|^2 + \frac{d}{dt}(w_0(p^m),1) \le \frac{C_F^2}{\alpha}\left(\frac{\xi^2}{2}\|\nabla p^m\|^2 + \frac{\alpha\xi^2}{4L^2}\|\nabla u^m\|^2 + (w_0(p^m),1)\right). \tag{13}$$

Integrating over $(0,t)$, we have

$$\left(\frac{\alpha\xi^2}{4L^2}\|\nabla u^m\|^2 + \frac{\xi^2}{2}\|\nabla p^m\|^2 + (w_0(p^m),1)\right)(t) \le \left(\frac{\alpha\xi^2}{4L^2}\|\nabla u^m\|^2 + \frac{\xi^2}{2}\|\nabla p^m\|^2 + (w_0(p^m),1)\right)(0)\exp\left(\frac{C_F^2}{\alpha}t\right). \tag{14}$$

The assumption of the theorem implies that $\nabla\mathcal{P}_m p_0, \nabla\mathcal{P}_m u_0 \in L_2(\Omega)$ (by (6)), and $\mathcal{P}_m p_0$ in $L_4(\Omega)$ are bounded regardless of m. Consequently, it implies that, regardless of m, $\nabla u^m, \nabla p^m$ are bounded in $L_\infty(0,T;L_2(\Omega))$, and p^m are bounded in $L_\infty(0,T;L_4(\Omega))$ for each finite time $T>0$. Integrating (14) over $(0,T)$, we get

$$\int_0^T \left(\frac{\alpha\xi^2}{4L^2}\|\nabla u^m\|^2 + \frac{\xi^2}{2}\|\nabla p^m\|^2 + (w_0(p^m),1)\right)(t)dt \le \left(\frac{\alpha\xi^2}{4L^2}\|\nabla u^m\|^2 + \frac{\xi^2}{2}\|\nabla p^m\|^2 + (w_0(p^m),1)\right)(0)\frac{\alpha}{C_F^2}\left(\exp(\frac{C_F^2}{\alpha}T) - 1\right), \tag{15}$$

We use this estimate for the integration of the relation (13), and we see that

$$\begin{aligned}
&\int_0^T \left(\frac{1}{4}\alpha\xi^2\|\frac{\partial p^m}{\partial t}\|^2 + \frac{\alpha\xi^2}{4L^2}\|\frac{\partial u^m}{\partial t}\|^2\right)(t)dt \\
&\quad + \left(\frac{\xi^2}{2}\|\nabla p^m\|^2 + \frac{\alpha\xi^2}{4L^2}\|\nabla u^m\|^2 + (w_0(p^m),1)\right)(T) \\
&\le \left(\frac{\xi^2}{2}\|\nabla p^m\|^2 + \frac{\alpha\xi^2}{4L^2}\|\nabla u^m\|^2 + (w_0(p^m),1)\right)(0) \\
&\quad + \frac{C_F^2}{\alpha}\int_0^T \left(\frac{\xi^2}{2}\|\nabla p^m\|^2 + \frac{\alpha\xi^2}{4L^2}\|\nabla u^m\|^2 + (w_0(p^m),1)\right)(t)dt \\
&\le \left(\frac{\alpha\xi^2}{4L^2}\|\nabla u^m\|^2 + \frac{\xi^2}{2}\|\nabla p^m\|^2 + (w_0(p^m),1)\right)(0)\exp(\frac{C_F^2}{\alpha}T).
\end{aligned} \tag{16}$$

Passing to a subsequence m', we have $u^{m'} \rightharpoonup u$ and $p^{m'} \rightharpoonup p$ in $L_2(0,T;H_0^1(\Omega))$. The non-linear terms in (1) require a stronger convergence result. Using the Aubin lemma on compact imbedding ([9]) with the setting

$$\{u^m\}_{m=1}^{\infty} \text{ bounded in } L_2(0,T;H_0^1(\Omega)), \{\frac{\partial u^m}{\partial t}\}_{m=1}^{\infty} \text{ bounded in } L_2(0,T;L_2(\Omega)),$$

$$\{p^m\}_{m=1}^{\infty} \text{ bounded in } L_6(0,T;H_0^1(\Omega)), \{\frac{\partial u^m}{\partial t}\}_{m=1}^{\infty} \text{ bounded in } L_2(0,T;L_2(\Omega)),$$

we see that $\{u^{m'}\}_{m'=1}^{\infty}$ converges strongly in $L_2(0,T;L_2(\Omega))$, and $\{p^{m'}\}_{m'=1}^{\infty}$ converges strongly in $L_6(0,T;L_6(\Omega))$. The polynomial form of f_0 implies the existence of the strong limit of $f_0(p^{m'})$ in $L_2(0,T;L_2(\Omega))$ being equal to $f_0(p)$. We also observe that the term $F(u^m)|\nabla p^m|$ is bounded in $L_2(0,T;L_2(\Omega))$, and, therefore, the subsequence converges weakly to a function $\tilde{F}$ in this space. In order to be able to pass toward the limit in (7), we prove that $\tilde{F} = F(u)|\nabla p|$. For this purpose, we show more about the regularity of p.

Lemma 1. *Under the assumptions of the theorem, the function p belongs to $L_2(0,T;H_0^1(\Omega)\cap H^2(\Omega))$.*

Proof. Multiply the equation of phase (9) by (q, v_j), for a $q \in \mathcal{D}(\Omega)$, and sum over N_m.

$$\alpha\xi^2(\frac{\partial p^{m'}}{\partial t}, \mathcal{P}_{m'}q) + \xi^2(\nabla p^{m'}, \nabla \mathcal{P}_{m'}q) = (f_0(p^{m'}), \mathcal{P}_{m'}q) + \xi^2(F(u^{m'})|\nabla p^{m'}|, \mathcal{P}_{m'}q). \quad (17)$$

We can pass to the limit in the sense of $\mathcal{D}'(0,T)$ by obtaining

$$\alpha\xi^2(\frac{\partial p}{\partial t}, q) + \xi^2(\nabla p, \nabla q) = (f_0(p), q) + \xi^2(\tilde{F}, q). \quad (18)$$

Consequently, the function p is continuous from $\langle 0,T\rangle$ into $L_2(\Omega)$. We rewrite the previous equality in the sense of $\mathcal{D}'(\Omega)$,

$$\alpha\xi^2\frac{\partial p}{\partial t} = \xi^2\Delta p + f_0(p) + \xi^2\tilde{F}. \quad (19)$$

As $\frac{\partial p}{\partial t}$, $f_0(p)$, $\tilde{F}$ belong to $L_2(0,T;L_2(\Omega))$, it follows that $\Delta p \in L_2(0,T;L_2(\Omega))$. The continuity of the operator Δ^{-1} mapping $L_2(\Omega)$ to $H^2(\Omega)\cap H_0^1(\Omega)$ – see [7] – implies that $p \in L_2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$. □

Next statement investigates the convergence of gradient.

Lemma 2. *The sequence $\nabla p^{m'}$ converges strongly to ∇p in $L_2(0,T;L_2(\Omega))$.*

Proof. The statement of the lemma is shown following the technique of [8]. Multiply the equation of phase (9) by $\gamma_i^m - \gamma_i$, where $p = \sum_{i\in\mathrm{N}} \gamma_i v_i$, sum over $i \in \mathrm{N}$ and integrate over $(0,T)$.

$$\alpha\xi^2\int_0^T(\frac{\partial p^{m'}}{\partial t}, p^{m'} - p)dt + \xi^2\int_0^T(\nabla p^{m'}, \nabla(p^{m'} - p))dt$$
$$= \int_0^T(f_0(p^{m'}), p^{m'} - p)dt + \xi^2\int_0^T(F(u^{m'})|\nabla p^{m'}|, p^{m'} - p)dt. \quad (20)$$

We add and subtract a term

$$\xi^2\int_0^T(\nabla p, \nabla(p^{m'} - p))dt$$

to the equality (20) knowing that it tends to 0 as

$$\nabla(p^{m'} - p) \to 0,$$

weakly in $[L_2(0,T;L_2(\Omega))]^2$, and

$$p^{m'} - p \to 0,$$

strongly in $L_2(0,T;L_2(\Omega))$, if $n \to \infty$. Then we have

$$\begin{aligned}
&\xi^2 \int_0^T (\nabla(p^{m'} - p), \nabla(p^{m'} - p))dt \\
&= -\alpha\xi^2 \int_0^T (\frac{\partial p^{m'}}{\partial t}, p^{m'} - p)dt + \int_0^T (f_0(p^{m'}), p^{m'} - p)dt \\
&+\xi^2 \int_0^T (F(u^{m'})|\nabla p^{m'}|, p^{m'} - p)dt + \xi^2 \int_0^T (\nabla p, \nabla(p^{m'} - p))dt.
\end{aligned}$$

As all terms in the right-hand side tend to 0 if $n \to \infty$, we see that $\nabla(p^{m'} - p) \to 0$ strongly in $[L_2(0,T;L_2(\Omega))]^2$, which gives the desired result. □

Lemma 3. *The sequence* $F(u^{m'})|\nabla p^{m'}|$ *converges weakly to* $F(u)|\nabla p|$ *in* $L_2(0,T;L_2(\Omega))$.

Proof. The sequence $|\nabla p^{m'}|$ converges strongly in $L_2(0,T;L_2(\Omega))$, as the same is valid for $\nabla p^{m'}$. The function F is bounded and continuous, and $u^{m'} \to u$ in $L_2(0,T;L_2(\Omega))$, then $F(u^{m'}) \to F(u)$ in $L_s(0,T;L_s(\Omega))$ for $1 < s < +\infty$. It implies that the sequence $F(u^{m'})|\nabla p^{m'}|$ converges strongly to $F(u)|\nabla p|$ in $L_{\frac{2s}{2+s}}(0,T;L_{\frac{2s}{2+s}}(\Omega))$. Then the boundedness of the sequence $F(u^{m'})|\nabla p^{m'}|$ in $L_2(0,T;L_2(\Omega))$ yields the statement of the lemma. □

Passage to the limit. Choose test functions $w, q \in \mathcal{D}(\Omega)$, multiply (8) by (w, v_j) and (9) by (q, v_j), sum over N_m. Then choose scalar functions $\varphi, \psi \in \mathcal{C}^1(\langle 0,T\rangle)$, for which $\varphi(T) = \psi(T) = 0$. Integrate both equations by parts over $(0,T)$. Knowing that

1. $\nabla p^{m'}$ converges strongly in $L_2(0,T;L_2(\Omega))$ to ∇p;
2. $\mathcal{P}_{m'}p_0$, $\mathcal{P}_{m'}u_0$ converge strongly to p_0, u_0 in $L_2(\Omega)$,
3. $F(u^{m'})|\nabla p^{m'}|$ converges weakly to $F(u)|\nabla p|$ in $L_2(0,T;L_2(\Omega))$,
4. $p^{m'}(0) = \mathcal{P}_{m'}p_0$, $u^{m'}(0) = \mathcal{P}_{m'}u_0$,

we are able to pass to the limit, and we obtain the following relations:

$$(u_0 - Lp_0, w)\varphi(0) - \int_0^T (u - Lp, w)\frac{d\varphi}{dt}dt + \int_0^T \varphi(\nabla u, \nabla w)dt = 0, \tag{21}$$

$$\begin{aligned}
&\alpha\xi^2(p_0, q)\psi(0) - \int_0^T \alpha\xi^2(p, q)\frac{d\psi}{dt}dt + \int_0^T \psi\{\xi^2(\nabla p, \nabla q) \\
&\quad -(f_0(p), q) - \xi^2(F(u)|\nabla p|, q)\}dt = 0.
\end{aligned}$$

If $\varphi, \psi \in \mathcal{D}(0,T)$, we have

$$\begin{aligned}
\frac{d}{dt}(u - Lp, w) + (\nabla u, \nabla w) &= 0, \\
\alpha\xi^2\frac{d}{dt}(p, q) + \xi^2(\nabla p, \nabla q) &= (f_0(p), q) + \xi^2(F(u)|\nabla p|, q).
\end{aligned} \tag{22}$$

The weak solution satisfies the initial condition. Indeed, in (21), by using scalar functions $\varphi, \psi \in C^1(\langle 0,T\rangle)$, for which $\varphi(T) = \psi(T) = 0$, we obtain

$$
\begin{aligned}
&(u(0) - Lp(0), w)\varphi(0) - \int_0^T (u - Lp, w)\frac{d\varphi}{dt}dt + \int_0^T \varphi(\nabla u, \nabla w)dt = 0,\\
&\alpha\xi^2(p(0), q)\psi(0) - \int_0^T \alpha\xi^2(p, q)\frac{d\psi}{dt}dt \\
&\quad + \int_0^T \psi\{\xi^2(\nabla p, \nabla q) - (f_0(p), q) - \xi^2(F(u)|\nabla p|, q)\}dt = 0.
\end{aligned}
\tag{23}
$$

Subtracting these equations from (21), we get

$$(u_0 - Lp_0 - u(0) + Lp(0), w)\varphi(0) = 0,\ (p_0 - p(0), q)\psi(0) = 0,\ \forall w, q \in \mathcal{D}(\Omega).$$

From this we see that $u(0) = u_0$, $p(0) = p_0$ in $L_2(\Omega)$.

In case when F is Lipschitz-continuous with the Lipschitz constant denoted by L_F, we prove **uniqueness** of the solution of (5). We consider two solutions of the problem (5), denoted by $[u_1, p_1]$ and $[u_2, p_2]$. Subtracting corresponding systems of equations and denoting $[u_{12}, p_{12}] = [u_1 - u_2, p_1 - p_2]$, multiplying the first equation by $u_{12} - Lp_{12}$ and the second equation by p_{12}, we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_{12} - Lp_{12}\|^2 + \|\nabla(u_{12} - Lp_{12})\|^2 + L(\nabla p_{12}, \nabla(u_{12} - Lp_{12})) &= 0 \text{ in } (0,T),\\
(u_{12} - Lp_{12})(0) &= 0,\\
\frac{1}{2}\alpha\xi^2\frac{d}{dt}\|p_{12}\|^2 + \xi^2\|\nabla p_{12}\|^2 &= (f_0(p_1) - f_0(p_2), p_{12})\\
&\quad +\xi^2(F(u)|\nabla p_1| - F(v)|\nabla p_2|, p_{12}) \text{ in } (0,T),\\
p_{12}(0) &= 0.
\end{aligned}
$$

Denote $\Psi(p_1, p_2) = -\frac{1}{2}a + \frac{3}{2}a(p_1 + p_2) - a(p_1^2 + p_1 p_2 + p_2^2)$. The *a priori* estimate guarantees that there is a constant $C_f > 0$ such that

$$\|\Psi(p_1, p_2)\| \le C_f \text{ in } (0,T)$$

(as implied by the continuous imbedding $H_0^1(\Omega) \hookrightarrow L_s(\Omega)$ for $s \in \langle 1, +\infty)$). Therefore (see [9]),

$$|(\Psi(p_1, p_2)p_{12}, p_{12})| \le \|\Psi(p_1, p_2)\|\|p_{12}\|^2_{L_4(\Omega)} \le C_f C_4\|p_{12}\|\|\nabla p_{12}\|,$$

where C_4 is the norm of the imbedding $H_0^1(\Omega)$ into $L_4(\Omega)$. Using the Poincaré, Young and Schwarz inequalities, we get

$$
\begin{aligned}
\frac{d}{dt}\|u_{12} - Lp_{12}\|^2 + \frac{1}{C_\Omega}\|u_{12} - Lp_{12}\|^2 &\le L^2\|\nabla p_{12}\|^2,\\
\frac{1}{2}\alpha\xi^2\frac{d}{dt}\|p_{12}\|^2 + \xi^2\|\nabla p_{12}\|^2 &\le C_f C_4\|p_{12}\|\|\nabla p_{12}\| + \xi^2 L_F\|u_{12}\|\|\nabla p_1\|_{L_4(\Omega)}\|p_{12}\|_{L_4(\Omega)}\\
&\quad +\xi^2 C_F\|\nabla p_{12}\|\|p_{12}\|,
\end{aligned}
$$

in $(0,T)$, where C_Ω appears in the Poincaré inequality. Considering the fact that there is a constant C_p for which

$$\int_0^T \|\nabla p_1\|^2_{L_4(\Omega)}dt \le C_4^2\int_0^T \|p_1\|^2_{H^2(\Omega)}dt \le C_p^2, \tag{24}$$

(because $\Delta p_1 \in L_2(0,T;L_2(\Omega))$), we obtain

$$\begin{aligned} \frac{d}{dt}\|u_{12} - Lp_{12}\|^2 &\leq L^2\|\nabla p_{12}\|^2 \text{ in } (0,T), \qquad (25) \\ \frac{1}{2}\alpha\xi^2\frac{d}{dt}\|p_{12}\|^2 + \frac{\xi^2}{2}\|\nabla p_{12}\|^2 &\leq (\xi^{-2}(C_fC_4 + C_F\xi^2)^2 + 2L^2\xi^2C_4^2\|\nabla p_1\|^2_{L_4(\Omega)}L_F^2)\|p_{12}\|^2 \\ &\quad +2C_4^2\|\nabla p_1\|^2_{L_4(\Omega)}L_F^2\xi^2\|u_{12} - Lp_{12}\|^2 \text{ in } (0,T). \end{aligned}$$

Combining these inequalities, we have in $(0,T)$:

$$\frac{d}{dt}\left(\frac{1}{2}\alpha\xi^2\|p_{12}\|^2 + \frac{\xi^2}{2L^2}\|u_{12} - Lp_{12}\|^2\right) \leq M(t)\left(\frac{1}{2}\alpha\xi^2\|p_{12}\|^2 + \frac{\xi^2}{2L^2}\|u_{12} - Lp_{12}\|^2\right)$$

with

$$M(t) = \frac{2(\xi^{-2}(C_fC_4 + C_F\xi^2)^2 + 2L^2\xi^2C_4^2\|\nabla p_1\|^2_{L_4(\Omega)}L_F^2)}{\min(\alpha,1)\xi^2}.$$

Such an inequality, together with (24) and with the initial conditions, implies that

$$p_{12}(t) = u_{12}(t) = 0 \text{ in } L_2(\Omega),\ \forall t \in (0,T),$$

as follows from the Gronwall lemma. □

Acknowledgment. A partial support of the Czech Grant Agency by a grant No. 205/96/0921 is acknowledged. The author also thanks the referee for many useful comments.

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Qualitative Properties of Positive Solutions of Elliptic Equations

H. BERESTYCKI

1. INTRODUCTION

This talk is intended as a survey of recent results about qualitative properties for positive solutions of semilinear elliptic equations of the type

$$\Delta u + f(u) = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \tag{1.2}$$

in various geometrical settings of unbounded domains Ω. The case of bounded domains will first be recalled in section 2 below.

Most of these results have been obtained by further developments of the *method of moving planes*. First introduced by Alexandrov [3] in the context of minimal surfaces, this method has been developed by Serrin [38] and by Gidas, Ni and Nirenberg [24] for semilinear elliptic equations. Serrin [38] used it to establish symmetry of the domain for a class of problems with overdetermined boundary conditions. It was developed into a systematic tool to prove symmetry properties of positive solutions of semilinear elliptic equations in the paper of Gidas, Ni and Nirenberg [24]. The method was further simplified and extended in the work of Nirenberg and myself [11].

This method of moving planes rests on various forms of the maximum principle. It is an extremely powerful approach for deriving qualitative properties of positive solutions to nonlinear elliptic equations.

In section 2 below, I will start by giving an outline of this method, following the version of [11], in the framework of bounded domains. This will make more apparent the difficulties which arise in dealing with unbounded domains. I will then go on to describe results as well as some (of the many) open problems for various types of unbounded domains: spherical symmetry in the space $\mathbb{R}^n$, monotonicity in a domain bounded by a Lipschitz graph, monotonicity and symmetry in a half space, one-dimensional symmetry in the space $\mathbb{R}^n$, and infinite cylinders. I will also discuss at the end of this lecture (see section 8) some related results for overdetermined boundary problems.

Some other approaches to the qualitative study of semilinear elliptic and parabolic equations have recently been developed which allow one to obtain somewhat different results and involve different types of arguments. One, which is due to F. Brock [16], rests on an ingenious construction of continuous Steiner symmetrization for functions (see for instance [28]). Another method developed by Hess and Poláčik [27] and by Matano and Ogiwara [31, 33, 34] is to consider the dynamical systems associated with these equations and deriving from them properties of the equilibrium states or even of the ω-limit sets. These results are in a different spirit. They concern *stable* solutions of the associated evolution equation.

Most of the results which are presented here have been obtained in a series of works in collaboration with L. Caffarelli and L. Nirenberg [5, 6, 7, 8] (see also the papers [11, 12, 13]).

In particular, concerning qualitative properties (such as uniqueness, monotonicity, asymptotic behaviour) of travelling fronts in unbounded domains, I refer the reader to the paper [12].

In all of what follows, it will always be assumed that f is locally Lipschitz continuous. A solution to (1.1)–(1.2) is always understood to be a function of class C^2 in the interior of Ω which is continuous up to the boundary.

2. The method of moving planes

Let us start by considering the case of bounded domains. The following result of Gidas, Ni and Nirenberg [24] is classical:

Theorem 2.1. *Suppose that $\Omega = B_R(0)$ is a sphere of radius R about the origin. Then any positive solution of (1.1)–(1.2) is spherically symmetric and decreases away from the origin, that is, $u = u(|x|)$ and $u_r < 0$ for $0 < r = |x| < R$.*

This, in fact, is a consequence of a more general statement:

Theorem 2.2. *Suppose that the domain Ω is convex in some direction, say x_1, and symmetric with respect to the plane $x_1 = 0$. Then any positive solution of (1.1)–(1.2) is symmetric in x_1 and decreases away from the plane $x_1 = 0$. That is, $u(x_1, \dots, x_n) = u(-x_1, \dots, x_n)$ in Ω and $u_{x_1}(x) < 0$ for $x_1 > 0$ in Ω.*

Actually, this was proved in [24] under the additional assumption that the boundary $\partial\Omega$ was smooth. This more general statement (which contains, e.g., the case of a rectangle or a cube) is in fact due to [11], the presentation of which follows.

To show this result, we consider for $\lambda < 0$ the domain $\Omega^\lambda := \{x \in \Omega,\ x_1 < \lambda\}$. For $x \in \Omega^\lambda$, we let $x^\lambda = (2\lambda - x_1, \dots, x_n)$ be the mirror image of x in the plane $x_1 = \lambda$. Consider the functions

$$u^\lambda(x) = u(x^\lambda) \quad \text{and} \quad w^\lambda = u^\lambda - u \quad \text{in} \quad \Omega^\lambda.$$

The principle of the moving planes method is to show that w^λ remains positive for all values of λ as the plane $\{x_1 = \lambda\}$ is moved all the way from when it first touches Ω on the left, say for some value $\lambda = \lambda^* < 0$, up to the maximal symmetry position, when $\lambda = 0$. Indeed, it is easily seen that this property implies that u is symmetric and decreasing away from the symmetry plane. A simple application of Hopf's lemma (using equation 2.1 below) then also shows that u satisfies $\frac{\partial u}{\partial x_1}(x) < 0$ for $x_1 > 0$ in Ω.

To achieve this goal, the key property which is used is that w^λ satisfies a linear elliptic equation. Indeed, u^λ satisfies the same equation as u:

$$\Delta u^\lambda + f(u^\lambda) = 0 \quad \text{in} \quad \Omega^\lambda.$$

Therefore, w^λ is a solution of a linear elliptic equation

$$\Delta w^\lambda + c^\lambda(x) w^\lambda = 0 \quad \text{in} \quad \Omega^\lambda, \tag{2.1}$$

where

$$c^\lambda(x) := \frac{f(u^\lambda(x)) - f(u(x))}{u^\lambda(x) - u(x)}. \tag{2.2}$$

Since f is Lipschitz continuous, c^λ is bounded in sup-norm independently of $\lambda \in (\lambda^*, 0)$. Furthermore, since $u > 0$ and by construction, it is easily seen that

$$w^\lambda \geq 0 \quad \text{on} \quad \partial\Omega_\lambda. \tag{2.3}$$

The claim is that (2.1)–(2.3) implies that w^λ remains positive for all values of $\lambda \in (\lambda^*, 0)$. This does not readily follow from the maximum principle applied to (2.1)–(2.3) because there is no assumption on the sign of c^λ. (The maximum principle would indeed apply directly if one were to assume that f is nonincreasing - a rather trivial case.) Instead, it involves a sophisticated continuation argument. I will now give the outline of this argument.

It hinges on a version of the maximum principle for small domains (see also [14]). In the present context, it can be stated as follows. Let L be a given constant; there exists a positive number δ, which depends on L and on the domain Ω for which the following holds. Let D be any subdomain of Ω such that its measure, $|D|$, satisfies $|D| \leq \delta$. Then if w satisfies

$$\Delta w + \gamma(x) w \leq 0 \quad \text{in} \quad D, \tag{2.4}$$

$$w \geq 0 \quad \text{on} \quad \partial D, \tag{2.5}$$

with $|\gamma(x)| \leq L$ for all $x \in D$ it must be the case that

$$w \geq 0 \quad \text{in} \quad D. \tag{2.6}$$

For more general statements and the proofs, we refer the reader to [11] and [14].

Applying this maximum principle to the above situation, we first see that $w^\lambda \geq 0$ when $0 < \lambda - \lambda^*$ is small enough. Next, suppose that $w^\lambda > 0$ for all values of λ such that $\lambda^* < \lambda < \mu$ and that μ is maximal for this property. The claim is that $\mu = 0$. We argue by contradiction and suppose that $\mu < 0$. Take a compact subset $K \subset \Omega_\mu$ with $|\Omega_\mu \setminus K| < \delta/2$. From the definition of μ it follows that $w^\mu \geq 0$ in Ω_μ and, in fact, by the strong maximum principle, $w^\mu > 0$ in Ω_μ. Therefore, for some $\eta > 0$, we have $w^\mu \geq \eta$ on K. By continuity, for $0 \leq \lambda - \mu$ small enough – say $\lambda \leq \mu + \varepsilon$ with $\varepsilon > 0$ – we see that $w^\lambda \geq \eta/2 > 0$ on K. This implies in particular that $w^\lambda > 0$ on ∂K. Since, by construction, $w^\lambda \geq 0$ on $\partial\Omega_\lambda$, it follows that $w^\lambda \geq 0$ on $\partial(\Omega_\lambda \setminus K)$. If ε is sufficiently small, the set $D := (\Omega_\lambda \setminus K)$ has measure smaller than δ. Hence, we may apply the previous maximum principle in the set D. We infer that $w^\lambda > 0$ first in D and then in all of Ω_λ since we already know it in K. This holds for all $\lambda < \mu+\varepsilon$, contradicting the maximality of μ. Hence, $\mu = 0$ and the proof is complete.

In [11], we also consider much more general equations of the type

$$F(x, u, Du, D^2u) = 0 \quad \text{in} \quad \Omega \tag{2.7}$$

that is completely nonlinear equations and I refer to that paper for precise statements.

For unbounded domains, the argument just outlined does not suffice, essentially because of the lack of maximum principle and for an underlying compactness argument in the continuation part. Three main approaches have been used to overcome this difficulty. One is to use some transformation such as an inversion or a Kelvin transform to get back to a bounded domain setting (an example is [18]). Another one is to apply the maximum principle in the unbounded part of the underlying domain, when for instance one makes assumptions on f which imply that c^λ (see 2.2 above) has a negative sign outside some large ball. Here an example is [29, 30]. Another one is to have special forms of the maximum principle exploiting the geometry of the domain. Some of the results presented here use the latter in their proofs.

3. Spherical symmetry in all of space

The case of positive solutions of 1.1 in all of space $\mathbb{R}^n$ was the first one to be studied and is indeed motivated by the study of ground states in some nonlinear field equations. One considers solutions of

$$\Delta u + f(u) = 0 \quad in \quad \mathbb{R}^n, \tag{3.1}$$

$$u > 0 \quad in \quad \mathbb{R}^n, \tag{3.2}$$

$$u(x) \to 0 \quad as \quad |x| \to \infty. \tag{3.3}$$

The limit in (3.3) is understood to be uniform.

The first result for this type of problem is due to Gidas, Ni and Nirenberg [25]. A more general version of this result was then obtained by Congming Li [29], who also extended it to the case of fully nonlinear equations in all of space. In these papers, it is essentially assumed that $f'(0) < 0$. This result has been further extended by Yi Li and W.M. Ni [30] (who also consider the case of fully nonlinear equations). Here is a result from this paper:

Theorem 3.1. *Suppose that f is of class C^1 and that on some interval $(0, s_0)$ (with $s_0 > 0$), the function f is nonincreasing with $f(0) = 0$. Then all positive solutions of (3.1) (3.3) must be radially symmetric about the origin (up to translation) and $u_r < 0$ for $r > 0$.*

Congming Li uses the moving plane method handling the unbounded part with the sign of c^λ (as was explained above). Indeed, in view of the assumption on f in theorem 3.1, one sees that $c^\lambda(x) \le 0$ outside some large enough sphere. Congming Li also considered in [29] fully nonlinear elliptic equations in all of space.

In a very recent paper, Serrin and Zou [39] have further extended these results to more general classes of quasilinear elliptic equations involving singular operators, like the p–Laplace operator, and are able to handle solutions which may have compact support.

4. Domains bounded by a Lipschitz graph

Here, one considers bounded positive solutions of (1.1)–(1.2) in an unbounded open set Ω defined by

$$\Omega = \{x \in \mathbb{R}^n;\ x_n > \varphi(x_1, \ldots, x_{n-1})\}, \tag{4.1}$$

where $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a globally Lipschitz continuous function. For $x \in \mathbb{R}^n$, we use coordinates $x = (x', x_n), x' = (x_1, \ldots, x_{n-1})$.

In this case, the specific issue we are concerned with is that of *monotonicity* properties for positive solutions of such problems.

We make the following assumptions :

$$0 < u \leq \sup u = M < \infty \quad \text{in} \quad \Omega, \tag{4.2}$$

$$f(s) > 0 \quad \text{on} \quad (0, \mu) \quad \text{and} \quad f(s) \leq 0 \quad \text{for} \quad s \geq \mu \tag{4.3}$$

for some $\mu > 0$ and, in addition, for some $0 < s_0 < s_1 < \mu$, f satisfies

$$f(s) \geq \delta_0 s \quad \text{on} \quad [0, s_0] \quad \text{for some} \quad \delta_0 > 0, \tag{4.4}$$

$$f(s) \quad \text{is non-increasing on} \quad (s_1, \mu). \tag{4.5}$$

Theorem 4.1. *Under conditions* (4.2)–(4.5) *above the solution u is monotonic with respect to x_n, that is, $\frac{\partial u}{\partial x_n} > 0$ in Ω.*

This result is proved in [7] by the *sliding method* (see [11] and the references therein) where various further properties are derived. In particular, we show that under the conditions of the theorem, the solution of (1.1)–(1.2) is unique in this case. Actually, theorem 4.1 yields monotonicity with respect to a whole cone of directions about the direction x_n.

This problem was motivated by an earlier work on regularity in some free boundary problems ([9]).

The first result of the nature of theorem 4.1 is due to M. Esteban and P. L. Lions [22]. They consider the case of a "coercive" Lipschitz graph, that is, they assume that

$$\lim_{|x'| \to \infty} \varphi(x') = +\infty. \tag{4.6}$$

Here is a more general version of their result (with no smoothness assumption on the graph):

Theorem 4.2. *Suppose φ: $\mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ is a locally Lipschitz graph satisfying (4.6) above. Let u be a positive solution of* (1.1)–(1.2) *(u is not assumed to be bounded) with f locally Lipschitz. Then $\frac{\partial u}{\partial x_n} > 0$ in Ω.*

This more general version includes for instance the case of a cone, which was not considered in [22]. It just suffices to observe that the method of moving planes as described above (like in [11]) applies with no changes since, in this case, the domain Ω^λ cut out by planes $\{x_n = \lambda\}$ is bounded.

If (4.6) does not hold, then, in general, the regions Ω^λ are not bounded, and the problem becomes more delicate. In fact, for unbounded solutions, one can construct examples (for which (4.6) is not satisfied) where the conclusion of theorem 4.2 fails.

In an earlier paper, S. Angenent [4] had proved the same result – under the same assumptions – as in theorem 4.1 for the particular case of a half space.

For a general geometry, it is an open problem to know under what more general assumptions on f or φ do the conclusions of Theorem 4.1 hold.

5. Half spaces

Consider now the case of a half space: $\Omega = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n, x_n > 0\}$. In this setting, one is interested in two types of properties: monotonicity with respect to x_n and symmetry – which in this case refers to *one-dimensional symmetry*, that is, that u is a function of x_n alone. Notice that these two properties are the exact analogues of the properties stated in theorem 2.1 if one thinks of the half space as a limiting case of a sphere tangent at the origin to the plane $x_n = 0$ and whose center goes to infinity.

The function f will always be assumed here to be (globally) Lipschitz continuous: $\mathbb{R}^+ \to \mathbb{R}$.

Let us start with the monotonicity property.

Theorem 5.1. *In the half space $\Omega = \mathbb{R}^n_+$, suppose that u is a solution of* (1.1)–(1.2), *$n = 2$ or when $n \geq 3$ that $f(0) \geq 0$. Then the function u satisfies*

$$\frac{\partial u}{\partial x_n} > 0 \quad in \quad \Omega.$$

It is still an open problem to know whether this property holds in general in dimension $n \geq 3$ in case $f(0) < 0$.

Under the additional hypotheses that u is bounded and that $f(0) \geq 0$ in all dimensions (also in dimension two), Theorem 5.1 was first proved by E.N. Dancer [20].

Let us now turn to symmetry. In [5] we prove the following:

Theorem 5.2. *Suppose that u is a bounded positive solution of* (1.1)–(1.2) *in a half space with*

$$M = \sup_\Omega u.$$

Then if $f(M) \leq 0$, the function u is symmetric, i.e., $u = u(x_n)$, and it is also monotonic, that is, $u_{x_n} > 0$ in Ω. Furthermore, $f(M) = 0$.

Actually, in [5], we consider more general equations of the type

$$\Delta u + g(x_n, u) = 0, \quad u > 0 \text{ in } \Omega. \tag{5.1}$$

Assuming that g is Lipschitz, that $t \to g(t,u)$ is nondecreasing in t, that $f(u) := \lim_{t\nearrow\infty} g(t,u)$ exists and is Lipschitz continuous in u and, lastly, that $g(t,M) \leq 0\ \forall t$, we prove symmetry – and monotonicity – of solution u of (5.1) and (1.2). Tehrani [41] has treated a more general form of (5.1):

$$\Delta u + g(x_n, u, |\nabla u|) = 0.$$

It should be noted that the condition that u be bounded is important. Indeed, one can find counter-examples to symmetry for unbounded solutions. For instance, in the half plane $\Omega = \{(x_1,x_2); x_2 > 0\}$, the function $u(x_1,x_2) = x_2 e^{x_1}$ is a positive solution of $\Delta u - u = 0$ with $u = 0$ on the boundary.

In [6], we have obtained still another symmetry result in low dimensions using a different approach. Here is one of the main results.

Theorem 5.3. *In the half space $\Omega = \mathbb{R}^n_+$ with $n = 2$ or 3, let u be a bounded positive solution of* (1.1)–(1.2). *In case, $n = 3$, assume further that $f(0) \geq 0$ and that $f \in C^1$. Then, u is symmetric, that is, $u = u(x_n)$.*

The proof of this result combines the monotonicity result theorem 5.1, the previous symmetry result, theorem 5.2 to which we reduce it together with a result about Schrödinger operators in the plane.

It is not known whether this result holds in general.

6. One-dimensional symmetry in all of space

The question of one–dimensional symmetry is also natural for monotone solutions in all of space. One such problem arises in a conjecture of De Giorgi [21]. It is about solutions of problems of the type

$$\Delta u + u - u^3 = 0 \quad \text{in} \quad \mathbb{R}^n. \tag{6.1}$$

Suppose that u satisfies $-1 < u < 1$ and that $\frac{\partial u}{\partial x_n} > 0$ in all of $\mathbb{R}^n$. Then the conjecture is to know whether u has one–dimensional symmetry. That is, are the level sets of u parallel hyperplanes? The answer is known in dimension two owing to works by Modica and Mortola [32] and a recent paper by Ghoussoub and Gui [23] which proves this result in full generality in the plane. They essentially derive the following.

Theorem 6.1. *In the plane $\mathbb{R}^2$, consider a bounded solution $u \in L^\infty(\mathbb{R}^2)$ of the equation*

$$\Delta u + f(u) = 0 \qquad \text{in all of } \mathbb{R}^2, \tag{6.2}$$

where f is an arbitrary continuously differentiable function. Suppose in addition that u is monotonic in some direction, say

$$\frac{\partial u}{\partial x_1} \geq 0 \qquad \text{in } \mathbb{R}^2. \tag{6.3}$$

Then u is a function of one variable only, that is, there exists a and b such that

$$u = u(ax_1 + bx_2). \tag{6.4}$$

The proof uses some spectral results for Schrödinger operators (see also a proof in [6]). Some results related to this problem appear in [17]. Related results about one–dimensional symmetry – and cases where it fails – are to be found in the papers [10, 15, 26].

7. Infinite cylinders

In our paper [8] we take up another class of unbounded domains: infinite cylinders or, more generally, product domains of the form

$$\Omega = \mathbb{R}^{n-j} \times \omega, \qquad \text{where } \omega \text{ is a bounded smooth domain in } \mathbb{R}^j.$$

We denote the variables in Ω by (x, y), $x \in \mathbb{R}^{n-j}$, $y \in \omega \subset \mathbb{R}^j$. It is not assumed that u is bounded. In fact, we establish that solutions u of (1.1)–(1.2) have at most exponential growth, that is,

$$u(x,y) \leq Ce^{\mu|x|} \qquad \forall x \in \mathbb{R}^{n-j}, \forall y \in \omega. \tag{7.1}$$

Assuming either that $j \geq 2$ or that $j = 1$ with $f(0) \geq 0$, we show that if ω is convex in some direction, say y_1, and symmetric with respect to the hyperplane $y_1 = 0$, then any solution u of (1.1)–(1.2) is symmetric about that hyperplane and decreases in y_1 for $y_1 > 0$. In [8] we also consider more general operators.

The case $f(0) < 0$ and $j = 1$, that is, when ω is an interval, say $\omega = (0, h)$, has been considered separately in [6]. It turns out to be somewhat particular. We prove the same symmetry result in this case in dimension two. The case of higher dimensional slab like domains of the type $\Omega = \mathbb{R}^{n-1} \times (0, h)$ with $n \geq 3$ is still open.

For other qualitative properties in infinite cylinders for travelling waves and fronts (such as monotonicity, uniqueness), I refer the reader to the papers [12, 13, 42]. A result about the symmetry of water waves has also been proved by Craig and Sternberg in [19].

8. Overdetermined problems

A well-known result of Serrin [38] concerns the determination of the shape of domains for which there exists a solution of the following overdetermined problem:

$$\Delta u + f(u) = 0 \quad \text{and} \quad u > 0 \quad \text{in} \quad \Omega, \tag{8.1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{8.2}$$

$$-\frac{\partial u}{\partial \nu} = k \quad \text{on} \quad \partial\Omega, \tag{8.3}$$

where k is a constant.

Using the method of moving planes, Serrin proved that if Ω is a bounded and smooth domain, then it is a sphere [38].

Very recently, this type of problem has been considered in the the setting of unbounded domains by W. Reichel [35, 36] and Aftalion and Busca [1]. The most recent work in this direction is by B. Sirakov [40], who has obtained a general result which I will now describe.

Consider the problem

$$\begin{cases} \nabla \cdot (g(|\nabla u|)\nabla u) + f(u, |\nabla u|) = 0\,, \quad u \geq 0\,, & \text{in } I\!R^n \setminus G \\ \text{Boundary Conditions} \quad \text{(BC)}, \end{cases} \tag{8.4}$$

where $G \subset I\!R^n$ is a bounded set such that $I\!R^n \setminus G$ is connected and

$$G = \bigcup_{i=1}^{k} G_i, \tag{8.5}$$

where k is an integer and G_i are bounded C^2–domains such that $\overline{G_i} \cap \overline{G_j} = \emptyset$ for $i \neq j$.

The elliptic operator and the nonlinearity are supposed to satisfy the following assumptions:

$$g \in C^2([0, \infty))\,, \, g(s) > 0 \quad \text{and} \quad (sg(s))' > 0 \text{ for all } s \geq 0\,, \tag{8.6}$$

the function $f(u, v) : [0, \infty) \times [0, \infty) \to I\!R$ is locally Lipschitz continuous and f is a non-increasing function of u for small positive values of u and v. Note that the Laplace operator as well as the mean curvature operator satisfy the assumption.

The overdetermined boundary conditions (BC) in this context are of the following type:

$$u = a_i > 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = \alpha_i \leq 0 \quad \text{on } \partial G_i\,,\ i = 1, \ldots, k, \tag{8.7}$$

where the a_i, α_i and β are constants. Furthermore, it is assumed that

$$\nabla u(x), u(x) \to 0 \quad \text{as } |x| \to \infty\,. \tag{8.8}$$

(No assumption needs to be imposed on ∇u if f does not depend on $|\nabla u|$.)

The main result of Sirakov is the following.

Theorem 8.1. *Under the above assumptions, if u is a solution of (8.4) satisfying (BC), then $k = 1$, and G is a ball centered at some point $x_0 \in I\!R^n$, u is spherically symmetric about the point x_0 and u is decreasing away from x_0.*

This result has applications in potential theory. Sirakov also derives this type of results in bounded domains of the type $\Omega \setminus G$ with overdetermined conditions on the boundary $\partial\Omega$. It extends earlier results of Alessandrini ([2]), Willms, Gladwell and Siegel ([43]) in bounded domains and of W. Reichel ([35], [36], [37]), A. Aftalion and J. Busca [1] in the case of exterior domains.

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Nonlinear Boundary Stabilization of the Wave Equation

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Abstract. We investigate the problem of the boundary stabilization of the wave equation in a bounded domain. Many authors have proved the exponential decrease of the energy function, with linear or nonlinear feedbacks, under restrictive geometrical assumptions. Here we obtain similar results by only assuming that the boundary of the domain is sufficiently smooth.

Keywords: mixed problems, singularities, controllability, stabilization

Classification: 35L05, 35Q72, 93B03, 93B07, 93C20, 93D15

Introduction

Let Ω be a bounded open set of $\mathbb{R}^n$ $(n \geq 3)$ such that its boundary $\partial\Omega$ satisfies in the sense of Nečas [11]:

$$\partial\Omega \text{ is of class } \mathcal{C}^2 . \tag{1}$$

Given $\mathbf{x}$ a point of $\partial\Omega$, we denote by $\nu(\mathbf{x})$ the normal unit vector pointing outward of Ω.
Let $\mathbf{x}_0$ be in $\mathbb{R}^n$. For $\mathbf{x}$ in $\mathbb{R}^n$, we define $m(\mathbf{x})$ by: $m(\mathbf{x}) = \mathbf{x} - \mathbf{x}_0$ as well as the two following subsets $\partial\Omega_N$ and $\partial\Omega_D$ of $\partial\Omega$ by:

$$\begin{aligned} &\partial\Omega_N = \{\mathbf{x} \in \partial\Omega \,/\, m(\mathbf{x}).\nu(\mathbf{x}) > 0\}\,; \\ &\partial\Omega_D = \partial\Omega \setminus \partial\Omega_N = \{\mathbf{x} \in \partial\Omega \,/\, m(\mathbf{x}).\nu(\mathbf{x}) \leq 0\}\,. \end{aligned} \tag{2}$$

We have $\partial\Omega_D \cup \partial\Omega_N = \partial\Omega$ and $\partial\Omega_D \cap \partial\Omega_N = \emptyset$. Furthermore, we assume that $\mathbf{x}_0$ is chosen such that

$$\begin{aligned} &\text{meas}(\partial\Omega_D) \neq 0\,, \ \text{meas}(\partial\Omega_N) \neq 0\,; \\ &\Gamma = \overline{\partial\Omega_D} \cap \overline{\partial\Omega_N} \text{ is a } \mathcal{C}^3\text{-manifold of dimension } n-2\,; \\ &\text{there exists a neighbourhood } \omega \text{ of } \Gamma \text{ such that} \\ &\quad \partial\Omega \cap \omega \text{ is a } \mathcal{C}^3\text{-manifold of dimension } n-1\,. \end{aligned} \tag{3}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that

$$g \text{ is non-decreasing;} \quad g \in \mathcal{C}^{0,1}(\mathbb{R})\,; \quad g(0) = 0\,. \tag{4}$$

We consider the following problem:

$$\begin{cases} u'' - \Delta u = 0\,, \text{ in } \Omega \times (0,+\infty)\,; \\ u = 0\,, \text{ on } \partial\Omega_D \times (0,+\infty)\,; \\ \dfrac{\partial u}{\partial \nu} = -(m.\nu)g(u')\,, \text{ on } \partial\Omega_N \times (0,+\infty)\,; \\ u(0) = u^0\,, \text{ in } \Omega\,; \\ u'(0) = u^1\,, \text{ in } \Omega\,, \end{cases} \tag{$\mathcal{W}$}$$

where $u' = \partial u/\partial t$ and $u'' = \partial^2 u/\partial t^2$.
Under reasonable assumptions on (u^0, u^1), this problem is well-posed.
We define the following Hilbert spaces:

$$V = \{v \in \mathrm{H}^1(\Omega)\,/\, v = 0\,, \text{ on } \partial\Omega_D\}\,; \quad \mathrm{H} = V \times \mathrm{L}^2(\Omega)\,.$$

We introduce a nonlinear operator $\mathcal{A}$ in H:

$$\mathcal{D}(\mathcal{A}) = \{(u,\hat{u}) \in V \times V\,/\, \Delta u \in \mathrm{L}^2(\Omega)\,;\ \frac{\partial u}{\partial \nu} = -(m.\nu)g(\hat{u}), \text{ on } \partial\Omega_N\}\,;$$
$$\mathcal{A}(u,\hat{u}) = (-\hat{u}, -\Delta u)\,, \quad \forall (u,\hat{u}) \in \mathcal{D}(\mathcal{A})\,.$$

One can show that $\mathcal{A}$ is a maximal monotone operator in H. So for some initial data (u^0, u^1) in $\mathcal{D}(\mathcal{A})$, problem $(\mathcal{W})$ has one and only one (strong) solution u satisfying

$$u \in \mathrm{W}^{1,\infty}(\mathbf{R}_+; V) \quad \text{and} \quad \Delta u \in \mathrm{L}^\infty(\mathbf{R}_+; \mathrm{L}^2(\Omega)).$$

The energy function of this solution is given by

$$E(t) = \frac{1}{2}\int_\Omega |\nabla u|^2(\mathbf{x},t)\,dx + \frac{1}{2}\int_\Omega |u'|^2(\mathbf{x},t)\,dx\,, \quad \forall t \geq 0.$$

Furthermore, one can prove that $\mathcal{D}(\mathcal{A})$ is dense in H. Thus, for every t in $\mathbf{R}_+$, the function $(u^0, u^1) \mapsto (u(t), \hat{u}(t))$ can be uniquely extended as a continuous semi-group S of contractions in H. For (u^0, u^1) in H, one may define the weak solution u of $(\mathcal{W})$ which satisfies $(u, u') \in C(\mathbf{R}_+; \mathrm{H})$.
Exponential decrease of the energy function has been obtained under restrictive assumptions. For the linear case, the reader can see [9], [4], [8], [7]. These results have been generalized in [10] and [1]. The nonlinear case has been addressed by many authors (see, for example, [3], [5]) under restrictive geometrical conditions (mainly, $\overline{\partial\Omega_D} \cap \overline{\partial\Omega_N} = \emptyset$).
We extend here the study of [1] by using a method close to [5]: we only assume (1)-(4) and a less restrictive geometrical hypothesis (5) which is given below. This hypothesis is satisfied when Ω is a convex set and $\mathbf{x}_0$ is an exterior point of Ω.
Under (3), Γ can be locally seen as a C^3-submanifold of $\partial\Omega_N$ of codimension 1. This allows us to define, in the tangent space, the normal unit vector pointing outward of $\partial\Omega_N$ ("from $\partial\Omega_N$ to $\partial\Omega_D$"). This vector will be denoted by $\tau(\mathbf{x})$. As in [1], we will assume that

$$\text{For almost every } \mathbf{x} \in \Gamma\,, \quad m(\mathbf{x}).\tau(\mathbf{x}) \leq 0\,. \tag{5}$$

Under assumptions (1)-(5), we obtain a regularity result concerning strong solutions of $(\mathcal{W})$, proposition 1. Then by using results given in [6], theorem 2 yields the decrease of the energy function E.

MAIN RESULTS

In [5], the stabilization result has been obtained by using a Rellich's identity. This is possible because of the regularity of the solution.
Here, with weak geometrical assumptions, the solution of $(\mathcal{W})$ is not regular enough and we are not able to use this identity. Nevertheless, the following proposition will be sufficient to get a similar stabilization result.

Proposition 1. — *Assume* (1)-(4). *For every* (u^0, u^1) *in* $D(\mathcal{A})$, *the solution* u *of* $(\mathcal{W})$ *satisfies*

$$(m.\nu)|\nabla u|^2 \in \mathrm{L}^1(\partial\Omega).$$

With further assumption (5), *we have*

$$2\int_\Omega \Delta u(m.\nabla u)\,dx \le (n-2)\int_\Omega |\nabla u|^2\,dx + 2\int_{\partial\Omega} \frac{\partial u}{\partial \nu}(m.\nabla u)\,d\sigma - \int_{\partial\Omega}(m.\nu)|\nabla u|^2\,d\sigma.$$

Proof.
We only have to notice that, under (4), for v in $\mathrm{H}^{1/2}(\partial\Omega)$, $g(v)$ belongs to $\mathrm{H}^{1/2}(\partial\Omega)$. And since $m.\nu$ belongs to $C^1(\partial\Omega)$, $(m.\nu)g(v) \in \mathrm{H}^{1/2}(\partial\Omega)$. So for every $(u, \hat{u})$ in $D(\mathcal{A})$, we have

$$\Delta u \in \mathrm{L}^2(\Omega) \quad \text{and} \quad \frac{\partial u}{\partial \nu} = -(m.\nu)g(\hat{u})\,, \text{ on } \partial\Omega_N\,.$$

Since $-(m.\nu)g(\hat{u})$ belongs to $\mathrm{H}^{1/2}(\partial\Omega)$, there exists some $\tilde{u} \in \mathrm{H}^2(\Omega)$ such that

$$\tilde{u} = 0\,, \text{ on } \partial\Omega \quad \text{and} \quad \frac{\partial \tilde{u}}{\partial \nu} = -(m.\nu)g(\hat{u})\,, \text{ on } \partial\Omega\,.$$

Then, $u - \tilde{u}$ belongs to V and satisfies

$$\Delta(u-\tilde{u}) \in \mathrm{L}^2(\Omega) \quad \text{and} \quad \frac{\partial(u-\tilde{u})}{\partial \nu} = 0\,, \text{ on } \partial\Omega_N\,.$$

Our proposition can be immediately deduced from results of [1].
We now present our main result for the stabilization of problem $(\mathcal{W})$.

Theorem 2. — *Assume* (1)-(5) *and suppose that there are* $p \in \mathbf{N}^*$ *and a positive constant* C *such that*

$$|g(x)| \ge C\mathrm{Min}(|x|, |x|^p)\,, \quad \forall x \in \mathbb{R}. \tag{6}$$

Then for every $(u^0, u^1) \in \mathrm{H}$, *there exists* $T > 0$ *such that the energy function of the solution of* $(\mathcal{W})$ *satisfies*

$$\text{if } p > 1: \quad E(t) \le Ct^{2/(1-p)}\,, \quad \forall t > T;$$
$$\text{if } p = 1: \quad E(t) \le E(0)\exp\left(1 - \frac{t}{C}\right)\,, \quad \forall t > T;$$

where, in the first case, C *depends on the initial energy* $E(0)$; *in the second case,* C *is independent of the initial data.*

Proof.
Proposition 1 is an essential tool of this proof where we follow [5].
We firstly prove the result for strong solutions of $(\mathcal{W})$ by applying some results of [6] (theorems 8.1 and 9.1).
The result is extended to weak solutions by using a density argument.
The reader will find details of proof in [2].

Remark. For the 2nd case, our result remains valid if Ω is convex or $\mathcal{C}^2$.
It is also valid for a polygonal domain Ω provided that, at each vertex of $\partial\Omega$ where the boundary condition changes, the angle of $\partial\Omega$ is less than π.

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Shape Derivative of Sharp Functionals Governed by Navier-Stokes Flow

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Abstract. The shape analysis of the Navier-Stokes equation has already been considered in the literature. Classical techniques, such as the Implicit Function Theorem, may be used to show that some functionals, the drag for example, are shape differentiable.

However, this property relies on results established for the basic regularity of the pressure and the velocity fields. Many other criteria of physical interest are out of this scope; we consider here the shape analysis of such functionals, for example, the (total) force exerted by the fluid on a body or the moment of these forces. The velocity and pressure fields u and p are assumed to be solutions of the stationary incompressible Navier-Stokes equation $-\nu\Delta u + [Du]u + \nabla p = f$ in Ω with the boundary condition $u|_\Gamma = 0$, $\Gamma = \partial\Omega$.

These new results are based on the so-called speed method which allows us to "bring back" vector fields from a perturbed domain to the initial one while preserving the divergence-free property. Regularity results are established for that correspondence and used to define and show some properties of the shape derivative u' and of the boundary shape derivative u'_Γ.

Keywords: shape optimization, shape gradient, Navier-Stokes equation

Classification: 35Q30, 49J20

1. Introduction

We study the shape differentiability of functionals of the solutions (u,p) of the Navier-Stokes equation in Ω_0 which is not defined for the minimal regularity of the solutions, for example because high-order derivatives or integrals on submanifolds appear in the expression of these functionals.

In the framework of the Speed Method (section 3.1), some perturbed sets $[s \mapsto \Omega_s]$ are associated with the initial set Ω_0; the velocities u_s solutions of the Navier-Stokes problem in the moving sets Ω_s are associated with fields u^s in the fixed set Ω_0 (section 3.2) which are solutions of the *Transported Navier-Stokes Equation* (section 3.3). The definition of this equation uses the functional spaces introduced in section 2. The Implicit Function Theorem is then used to prove some regularity of $[s \mapsto u^s]$ for the desired spatial regularity of u (section 3.4). This result allows us to show the existence of the shape derivatives of u and p for a given spatial regularity, and then through the development of a Tangential Calculus, their boundary shape derivatives (section 4). Theses objects are then intensively used to obtain the explicit form of the shape gradient of the force and the moment of the forces applied by a fluid on a part of its boundary (section 5).

2. THE NAVIER-STOKES EQUATIONS

The Navier-Stokes problem in an open and bounded set $\Omega \subset \mathbb{R}^3$ is classically written as

$$\left.\begin{array}{lll} -\nu\Delta u + [Du]u + \nabla p = f & \text{in } \Omega & (1.1) \\ \operatorname{div} u = 0 & \text{in } \Omega & (1.2) \\ u = 0 & \text{on } \partial\Omega, & (1.3) \end{array}\right\} \qquad (1)$$

where ν is the kinematic viscosity, u the velocity of the fluid, p the pressure and f the force. The study of this equation requires functional spaces built as subspaces (to deal with the fluid incompressibility (1.2) or the boundary value (1.3)) or quotients of Sobolev spaces (to handle forces defined up to a gradient): for any integer $m \geq 1$, we define

$$V^m(\Omega) = \{u \in H^m(\Omega; \mathbb{R}^3) \cap H_0^1(\Omega; \mathbb{R}^3), \operatorname{div} u = 0\} \qquad (2)$$

endowed with the $H^m(\Omega; \mathbb{R}^3)$ norm and for any integer $m \geq -1$, we set

$$W^m(\Omega) = H^m(\Omega; \mathbb{R}^3)/\{\nabla p, p \in H^{m+1}(\Omega; \mathbb{R})\} \qquad (3)$$

endowed with the quotient norm. The linear continuous mappings

$$i : V^m(\Omega) \to H^m(\Omega; \mathbb{R}^3) \quad \text{and} \quad \pi : H^m(\Omega; \mathbb{R}^3) \to W^m(\Omega) \qquad (4)$$

are respectively the canonical injection and the quotient mapping.

Remark 1. Strictly speaking, we did not uniquely define the mappings i and π, but only some sequences $(i_m)_{m\geq 1}$ and $(\pi_m)_{m\geq -1}$. It is obvious that when $u \in H^m(\Omega)$, for any integer m', $1 \leq m' \leq m$, $i_m(u) = i_{m'}(u)$, so i is uniquely determined. For π, we must first notice that for any $-1 \leq m' \leq m$, the mapping $\pi_m(f) \in W^m(\Omega) \mapsto \pi_{m'}(f) \in W^{m'}(\Omega)$ is well defined (that is, does not depends on the choice of f) and is a linear continuous injection, so we may identify $W^m(\Omega)$ with a subspace of $W^{m'}(\Omega)$. With this convention, $\pi_m(f) = \pi_{m'}(f)$ when the two expressions make sense, and π is also well defined. Moreover, $\pi(f) \in W^m(\Omega)$ if and only if f may be decomposed as $f = g + \nabla p$, $g \in H^m(\Omega; \mathbb{R}^3)$ and $p \in L^2(\Omega; \mathbb{R})$ (and not necessarily $p \in H^m(\Omega; \mathbb{R})$): the spaces $W^m(\Omega)$ characterize the best possible regularity of f "up to a gradient" (whatever its regularity is).

Let $\mathcal{D}(\Omega; \mathbb{R}^3)$ be the set of functions of $C^\infty(\mathbb{R}^3; \mathbb{R}^3)$ whose support is compactly included in Ω. We recall that if Ω is Lipschitz, $\mathcal{V}(\Omega) = \{v \in \mathcal{D}(\Omega; \mathbb{R}^3), \operatorname{div} v = 0\}$ is dense in $V^1(\Omega)$. Consequently,

Proposition 1. *The spaces $W^{-1}(\Omega)$ and $V^1(\Omega)'$ are isomorphic. Moreover,*

$$\begin{array}{lll} \forall f \in H^{-1}(\Omega; \mathbb{R}^3) \quad \pi(f) = 0 & \iff & \forall v \in V^1(\Omega), < f, v >= 0 \\ & \iff & \forall v \in \mathcal{V}(\Omega), < f, v >= 0. \end{array} \qquad (5)$$

Proof: The linear continuous operator $f \in H^{-1}(\Omega;\mathbb{R}^3) \mapsto f|_{V^1(\Omega)} \in V^1(\Omega)'$ is clearly onto and $f \in H^{-1}(\Omega;\mathbb{R}^3)$ belongs its kernel iff

$$\forall v \in V^1(\Omega),\ < f, v >= 0 \iff \exists p \in L^2(\Omega;\mathbb{R}),\ f = \nabla p \iff \pi(f) = 0$$

(see [10] for the first equivalence), which proves the first part of (5) and that $V^1(\Omega)' \simeq H^{-1}(\Omega;\mathbb{R}^3)/\{\nabla p,\ p \in L^2(\Omega;\mathbb{R})\} = W^{-1}(\Omega)$. The second equivalence follows by density. □

The operators π and i will be used to show some regularity results for some (one-variable or two-variable) mappings from $V^m(\Omega)$ to $W^{m'}(\Omega)$ which are built on mappings from $H^m(\Omega;\mathbb{R}^3)$ to $H^{m'}(\Omega;\mathbb{R}^3)$ (as it is shown in the following definition) whose regularity is known.

Definition 1. *For any integer n and mapping $T : (H^1(\Omega;\mathbb{R}^3))^n \to H^{-1}(\Omega;\mathbb{R}^3)$, we define the mapping $\bar{T} : (V^1(\Omega))^n \to W^{-1}(\Omega)$ by*

$$\bar{T} = \pi \circ T \circ (i \otimes i \otimes ... \otimes i). \tag{6}$$

In particular, this construction is applied for the operators A and B defined by

$$< Au, \varphi >_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = \int_\Omega \mathrm{D}u \cdot\cdot \mathrm{D}\varphi\, dx,\quad < B(u,v), \varphi >_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = \int_\Omega [\mathrm{D}u]v \cdot \varphi\, dx \tag{7}$$

for $u \in H^1_{\mathrm{loc}}(\Omega)$ and $v \in L^2_{\mathrm{loc}}(\Omega)$. These operators, used in the variational formulation of (1), have the following regularity:

Proposition 2. *For any integer $n \geq 1$, A is a continuous mapping $H^n(\Omega;\mathbb{R}^3) \to H^{n-2}(\Omega;\mathbb{R}^3)$. B is a continuous mapping $H^1(\Omega;\mathbb{R}^3) \times H^1(\Omega;\mathbb{R}^3) \to H^{-1}(\Omega;\mathbb{R}^3)$ and $H^n(\Omega;\mathbb{R}^3) \times H^n(\Omega;\mathbb{R}^3) \to H^{n-1}(\Omega;\mathbb{R}^3)$ for any integer $n \geq 2$.*

Sketch of the proof: The regularity of A is classical. The one of B is a consequence of the continuity of the trilinear form

$$(u, v, w) \in H^a(\Omega;\mathbb{R}) \times H^b(\Omega;\mathbb{R}) \times H^c(\Omega;\mathbb{R}) \mapsto \int_\Omega uvw\, dx$$

for nonnegative real numbers a, b, c such that $a + b + c > 3/2$ (see [4]). □

Proposition 1 shows that the usual variational formulation of the Navier-Stokes problem is equivalent to the equation

$$\nu\bar{A}u + \bar{B}(u,u) = \pi(f). \tag{8}$$

Moreover, the regularity of the Stokes equation combined with iterated evaluations of the regularity of $[\mathrm{D}u]u$ shows that if Ω is of class C^r, $r = \max(2, m+2)$, and $f \in H^m(\Omega;\mathbb{R}^3)$, $m \geq -1$, any solution u of the Navier-Stokes equation is in $H^{m+2}(\Omega;\mathbb{R}^3)$.

3. Transport

3.1. The Speed Method. In this section, we consider a *hold-all* D which contains the set Ω_0 filled by the fluid and a (time-dependent) vector field V defined on D which is used to define the perturbed sets Ω_s based on Ω_0: each point $x \in \Omega_0$ is continuously transported by the ODE defined by the field V. The parameter which controls the amplitude of the deformation is denoted by s.

Precisely, the hold-all D is assumed to be an open bounded set of class C^k, $k \geq 1$ and s is in a (possibly infinite) interval $I \subset \mathbb{R}$ such that $0 \in I$. The vector field V is assumed to be an element of the set $E^{n,k}(I, D)$ (or simply $E^{n,k}$) defined by

$$E^{n,k}(I, D) = \left\{V \in C^n(I; C^k(\overline{D}; \mathbb{R}^3)) \mid \forall s \in I,\ V(s) \cdot n = 0 \text{ on } \partial D\right\}, \tag{9}$$

where n denotes the unitary outer normal to D.

Then we may define the mapping $s \mapsto T_s(V)$ (or simply T_s when there is no possible confusion on the vector field) as the solution of

$$\forall s \in I,\ \frac{dT_s}{ds} = V(s) \circ T_s, \quad \text{and} \quad T_0 = \text{Id} \tag{10}$$

and also the perturbed sets $s \mapsto \Omega_s$ by

$$\Omega_s = T_s(\Omega_0). \tag{11}$$

We recall the following result which can be found in [9], [11]:

Proposition 3. *Let $V \in E^{n,k}(I, D)$, $n \geq 0$, $k \geq 1$, be a given vector field. Then*
i) $\forall s \in I$, $T_s(V) : D \to D$ and $\overline{D} \to \overline{D}$ are one-to-one mappings.
ii) $s \mapsto T_s(V) \in C^{n+1}(I; C^k(\overline{D}; \overline{D}))$ and $s \mapsto [T_s(V)]^{-1} \in C^n(I; C^k(\overline{D}; \overline{D}))$.
iii) $\forall s \in I, \forall x \in \overline{D}$, $\mathrm{D}T_s(x)$ is invertible, and the mappings $s \mapsto \mathrm{D}T_s$ and $s \mapsto [\mathrm{D}T_s]^{-1}$ are in $C^{n+1}(I; C^{k-1}(\overline{D}; \mathbb{R}^{3\times 3}))$.

As a first consequence of this proposition, the family of perturbed sets has its boundary regularity preserved for V smooth enough: if Ω_0 is of class C^r, $r \leq k$, then for any $s \in I$, Ω_s is also of class C^r.

Remark 2. In order to make the future calculations easier, we define

$$\gamma_s = \det(\mathrm{D}T_s) \quad \text{and} \quad C_s = \gamma_s^{-1}\mathrm{D}T_s. \tag{12}$$

Notice that, as a simple consequence of the Proposition 3, for $V \in E^{n,k}$, $n \geq 0$, $k \geq 1$, $\forall s \in I$, $\gamma_s^{-1} = \det([\mathrm{D}T_s]^{-1})$ exists and, moreover, γ_s and $\gamma_s^{-1} \in C^{n+1}(I; C^{k-1}(\overline{D}; \mathbb{R}))$. As $\gamma_0 = 1$, by continuity, $\forall s \in I$, $\gamma_s > 0$. Obviously, we also have the inversibility of C_s for any $s \in I$ and the regularity C_s and $C_s^{-1} \in C^{n+1}(I; C^{k-1}(\overline{D}; \mathbb{R}^{3\times 3}))$.

3.2. **Correspondence between vector fields.** The diffeomorphisms T_s defined in the previous section are used to build a correspondence between the mappings defined on the open and bounded set Ω_0 and those defined on Ω_s. As for any $s \in I$, T_s and T_s^{-1} are Lipschitz, the mapping $[v \mapsto v \circ T_s]$ is an isomorphism between $H_0^1(\Omega_s; \mathbb{R})$ and $H_0^1(\Omega_0; \mathbb{R})$ (see [8]) and the equation $\mathrm{D}[v \circ T_s] = [\mathrm{D}v \circ T_s]\mathrm{D}T_s$ holds.

Lemma 1. *Assume that Ω_0 is an open and bounded subset of $\mathbb{R}^3$ and that $V \in E^{0,2}$. Then the mapping $\mathbf{T}_s$, defined between functions $\Omega_0 \to \mathbb{R}^3$ and $\Omega_s \to \mathbb{R}^3$ by*

$$\mathbf{T}_s(u) \circ T_s = C_s u, \tag{13}$$

is an isomorphism between $V(\Omega_0)$ and $V(\Omega_s)$.

Proof: As for all $s \in I$, C_s and C_s^{-1} are $C^1(\overline{D};\mathbb{R}^{3\times3})$ (see remark 2), $\mathbf{T}_s$ is an isomorphism between $H_0^1(\Omega_0;\mathbb{R}^3)$ and $H_0^1(\Omega_s;\mathbb{R}^3)$. Moreover, for any $u \in H_0^1(\Omega_0;\mathbb{R}^3)$ and $\varphi \in H_0^1(\Omega_s;\mathbb{R}^3)$, we have

$$\int_{\Omega_s} \operatorname{div}(\mathbf{T}_s(u))\,\varphi\,dx = -\int_{\Omega_s} \mathbf{T}_s(u)\cdot\nabla\varphi\,dx = -\int_{\Omega_0} (\mathbf{T}_s(u)\circ T_s)\cdot(\nabla\varphi\circ T_s)\,\gamma_s\,dx$$

and as $\nabla(\varphi\circ T_s) = DT_s^*(\nabla\varphi\circ T_s)$,

$$\int_{\Omega_s} \operatorname{div}(\mathbf{T}_s(u))\,\varphi\,dx = -\int_{\Omega_0} (C_s^{-1}\mathbf{T}_s(u)\circ T_s)\cdot\nabla(\varphi\circ T_s)\,dx = \int_{\Omega_0} (\operatorname{div} u)\,\varphi\circ T_s\,dx = 0.$$

The mapping $\varphi \mapsto \varphi\circ T_s$ being an isomorphism between $H_0^1(\Omega_s;\mathbb{R})$ and $H_0^1(\Omega_0;\mathbb{R})$, $\operatorname{div}\mathbf{T}_s(u)$ if and only if $\operatorname{div} u = 0$ which achieves the proof. □

Remark 3. As $\mathbf{T}_s : H_0^1(\Omega_0;\mathbb{R}^3) \to H_0^1(\Omega_s;\mathbb{R}^3)$ is an isomorphism, so is its adjoint $\mathbf{T}_s^* : H^{-1}(\Omega_s;\mathbb{R}^3) \to H^{-1}(\Omega_0;\mathbb{R}^3)$.

3.3. **The Transported Navier-Stokes equations.** In the sequel, the objects (operators, vector fields, duality brackets) associated with the perturbed set Ω_s will be noted with a subscript s. Many of them are used to defined objects (by different means) associated with the initial set Ω_0; they are noted with the superscript s. Precisely, we define the following correspondences:

• Velocity fields $u_s \in V^1(\Omega_s)$ (solutions of the Navier-Stokes equation in Ω_s) and test functions $v_s \in V^1(\Omega_s)$ are associated with the fields $u^s \in V^1(\Omega_0)$ (resp. $v^s \in V^1(\Omega_0)$) by:

$$u^s = \mathbf{T}_s^{-1}(u_s) \text{ (resp. } v^s = \mathbf{T}_s^{-1}(v_s)).$$

• Force fields $f_s \in H^{-1}(\Omega_s;\mathbb{R}^3)$ (or in $H^{-1}(D;\mathbb{R}^3)$) are transported in $f^s \in H^{-1}(\Omega_0;\mathbb{R}^3)$ (or in $H^{-1}(D;\mathbb{R}^3)$) defined by:

$$f^s = \mathbf{T}_s^*(f_s).$$

• The (linear and bilinear) operators A_s and B_s from $H^1(\Omega_s;\mathbb{R}^3)$ to $H^{-1}(\Omega_s;\mathbb{R}^3)$ defined by (7) are associated with A^s and B^s linear and bilinear from $H^1(\Omega_0;\mathbb{R}^3)$ to $H^{-1}(\Omega_0;\mathbb{R}^3)$ by:

$$A^s = \mathbf{T}_s^*\circ A\circ\mathbf{T}_s \text{ and } B^s = \mathbf{T}_s^*\circ B\circ(\mathbf{T}_s\otimes\mathbf{T}_s). \tag{14}$$

With these notations, we have

Theorem 1. *Assume that $V \in E^{0,2}$ and that $f \in H^{-1}(D;\mathbb{R}^3)$. The field $u_s \in V^1(\Omega_s)$ is a solution of the Navier-Stokes equation in Ω_s if and only if $u^s \in V^1(\Omega_0)$ is a solution of*

$$\nu\bar{A}^s u^s + \bar{B}^s(u^s,u^s) = \pi(f^s). \tag{15}$$

The operators A^s and B^s satisfy for any u, v in $H^1(\Omega_0;\mathbb{R}^3)$

$$A^s u = C_s^*\operatorname{div}\left(\mathrm{D}[C_s u]\gamma_s^{-1}C_s^{-1}(C_s^*)^{-1}\right) \text{ and } B^s(u,v) = C_s^*\mathrm{D}(C_s u)\cdot v \tag{16}$$

and for $f \in L^2(D;\mathbb{R}^3)$, we have

$$f^s = [\mathrm{D}T_s]^*(f\circ T_s). \tag{17}$$

Remark 4. The pressure is not explicitly transported in theorem 1. Nevertheless, if $p_s \in L^2(\Omega_s;\mathbb{R})/\mathbb{R}$ is a pressure solution of the Navier-Stokes problem in Ω_s, that is if $\nu A_s u_s + B_s(u_s, u_s) + \nabla p_s = f$, then it can be verified that

$$\nu A^s u^s + B^s(u^s, u^s) + \nabla(p_s \circ T_s) = f^s.$$

It is therefore natural to define the transported pressure p^s by

$$p^s = p_s \circ T_s. \tag{18}$$

The operators involved in the equation (16) have the following regularity:

Proposition 4. *Assume that $V \in E^{0,m+1}$, $m \geq 1$. Then*

i) *$\forall s \in I$, A^s (resp. B^s) are linear (resp. bilinear) continuous from $H^m(\Omega_0;\mathbb{R}^3)$ to $H^{m-2}(\Omega_0;\mathbb{R}^3)$.*

ii) *$[s \mapsto A^s]$ and $[s \mapsto B^s]$ are continuously differentiable in these spaces.*

Proof: We know that for any integer n, $H^n(\Omega_0;\mathbb{R})$ is a $C^n(\overline{\Omega_0};\mathbb{R})$-topological module, so thanks to the regularity of $[s \mapsto \gamma_s]$, $[s \mapsto \gamma_s^{-1}]$, $[s \mapsto C_s]$ and $[s \mapsto C_s^{-1}]$ (see remark 2), we may consider the continuous mappings $\Lambda_s^1 : H^m(\Omega_0;\mathbb{R}^3) \to H^m(\Omega_0;\mathbb{R}^3)$, $\Lambda_s^2 : H^{m-1}(\Omega_0;\mathbb{R}^{3\times3}) \to H^{m-1}(\Omega_0;\mathbb{R}^{3\times3})$ and $\Lambda_s^3 : H^{m-2}(\Omega_0;\mathbb{R}^3) \to H^{m-2}(\Omega_0;\mathbb{R}^3)$, defined by $\Lambda_s^1(u) = C_s u$, $\Lambda_s^2(U) = U\gamma_s^{-1}C_s^{-1}(C_s^*)^{-1}$ and $\Lambda_s^3(u) = C_s^* u$.

As the operators A^s and B^s may be decomposed as sequences of linear (or bilinear) continuous operators

$$A^s = \Lambda_s^3 \circ \operatorname{div} \circ \Lambda_s^2 \circ \mathrm{D} \circ \Lambda_s^1 \text{ and } B^s = \Lambda_s^3 \circ B \circ (\Lambda_s^1 \otimes \mathrm{Id})$$

the property i) is proved. Moreover, as the mappings $[s \mapsto \Lambda_s^i]$ are of class C^1 on I, ii) follows. □

3.4. **Regularity of $[s \mapsto u^s]$.** Under the assumptions **A1** and **A2** described below, theorem 2 characterizes the regularity of $[s \mapsto u^s]$.

Assumption 1 (Regularity of the data). *Let $m \geq 2$ be a given integer.*

- *The initial set Ω_0 is an open connected subset included in the hold-all D.*

The sets Ω_0 and D are respectively of class C^{m+1} and C^{m+2}.

- *The fields V considered in the speed method are in $E^{1,m+2}(I,D)$.*
- *The force field f is in $H^{m-1}(D;\mathbb{R}^3)$.*

Assumption 2 (Nonsingularity of the Navier-Stokes Equation at u). *The velocity field u is a solution of the Navier-Stokes equation in Ω_0 such that the linearized Navier-Stokes equation*

$$-\nu\Delta v + [\mathrm{D}u]v + [\mathrm{D}v]u + \nabla q = g \tag{19}$$

has a unique solution $v \in V^1(\Omega_0)$ for any $g \in H^{-1}(\Omega_0;\mathbb{R}^3)$. Equivalently, the linear operator L from $V^1(\Omega_0)$ to $W^{-1}(\Omega_0)$ defined by $L(v) = \nu\bar{A} + \bar{B}(u,v) + \bar{B}(v,u)$ is an isomorphism.

In particular this assumption is satisfied for high viscosities or small force fields: for a given Ω_0, there is a $k(\Omega_0) > 0$ such that with $\nu^2/\|f\|_{H^{-1}(\Omega_0;\mathbb{R}^3)} > k(\Omega_0)$ this assumption holds (see [5, lemma 3.2, p. 300]). This threshold also ensures the unicity of the solution of the Navier-Stokes equation in Ω_0.

Theorem 2. *Assume that the assumptions* **A1** *and* **A2** *hold for a given solution u. Then there is on a neighbourhood $J \subset I$ of 0 a unique solution $u_s = u(\Omega_s)$ of the Navier-Stokes problem in Ω_s such that*
i) $u_0 = u$. ii) $[s \mapsto u^s] \in C^0(J; V^{m+1}(\Omega_0)) \cap C^1(J; V^m(\Omega_0))$.

Proof: Thanks to theorem 1, u_s is a solution of the Navier-Stokes equation in Ω_s iff u^s is a solution of $\bar{\phi}_s(u) = 0$ where

$$\begin{array}{rccl} \phi_s: & H^m(\Omega_0; \mathbb{R}^3) & \to & H^{m-2}(\Omega_0; \mathbb{R}^3) \\ & u & \mapsto & A^s u + B^s(u,u) - f^s. \end{array}$$

The result of the theorem is proved by two different versions of the Implicit Function Theorem for the same mapping but in different spaces: the mapping $(s,u) \mapsto \bar{\phi}_s(u)$ is considered as an application from $I \times V^{m+1}(\Omega_0)$ to $W^{m-1}(\Omega_0)$ and then stronger properties are exhibited for this mapping defined from $I \times V^m(\Omega_0)$ to $W^{m-2}(\Omega_0)$. These properties come from the results already shown for the operators A^s and B^s and from the regularity

$$[s \mapsto f \circ T_s] \in C^0(I; H^{m-1}(D; \mathbb{R}^3)) \cap C^1(I; H^{m-2}(D; \mathbb{R}^3))$$

which hold under the assumption **A1** (see [9]). Precisely, it holds

• The solution u of the Navier-Stokes problem in Ω_0 is in $V^{m+1}(\Omega_0)$ (see section 2) and *a fortiori* in $V^m(\Omega_0)$. It satisfies $\bar{\phi}_0(u) = 0$.

• The mapping $(s,u) \to \phi_s(u)$ is in $C^0(I \times H^{m+1}(\Omega_0); H^{m-1}(\Omega_0))$: $(s,u) \to A^s u + B^s(u,u)$ is in fact C^1 in these spaces (see proposition 4) and $[s \mapsto f^s]$ is in $C^0(I; H^{m-1}(\Omega_0; \mathbb{R}^3))$.

• The real $s \in I$ being fixed, for any $m \geq 1$, $u \mapsto \phi_s(u)$ is the sum of a linear continuous, a bilinear continuous and a constant mapping $H^{m+1}(\Omega_0; \mathbb{R}^3) \to H^{m-1}(\Omega_0; \mathbb{R}^3)$ and $H^m(\Omega_0; \mathbb{R}^3) \to H^{m-2}(\Omega_0; \mathbb{R}^3)$ (see proposition 4); $u \mapsto \phi_s(u)$ is therefore a $C^\infty(V^m(\Omega_0); W^{m-2}(\Omega_0))$ and $C^\infty(V^{m+1}(\Omega_0); W^{m-1}(\Omega_0))$ mapping. Consequently, $u \mapsto \bar{\phi}_s(u)$ as the same regularity (in both spaces).

• Proposition 4 also implies that for any $m \geq 1$ and a fixed $u \in V^m(\Omega_0)$, $[s \mapsto A^s u + B^s(u,u)]$ is in $C^1(I; H^{m-2}(\Omega_0))$. The same regularity also holds for $[s \mapsto f^s]$.

• As $\bar{\phi}_0 = \pi \circ \phi_0 \circ i$, we have $\partial_u \bar{\phi}_0(u)(v) = (\pi \circ \partial_u \phi_0(u) \circ i)(v)$ for any $u \in V^{m+1}(\Omega_0)$ (resp. $u \in V^m(\Omega_0)$). As $\partial_u \phi_0(u)(v) = Av + B(u,v) + B(v,u)$, $\partial_u \bar{\phi}_0(u) = L$, which is an isomorphism from $V^1(\Omega_0)$ to $W^{-1}(\Omega_0)$ (assumption 2). Repeated evaluations of the regularity of the bilinear terms show that in fact the solution v of $L(v) = g$ is in $V^{m+1}(\Omega_0)$ (resp. $V^m(\Omega_0)$) when $g \in H^{m-1}(\Omega_0; \mathbb{R}^3)$ (resp. $g \in H^{m-2}(\Omega_0; \mathbb{R}^3)$). □

4. The Shape Derivative u' and p'

From now on, we assume that the data are chosen such that **A1** and **A2** are satisfied for a given integer $m \geq 2$. Consequently, theorem 2 may be applied which allows to show that the solutions u and p of the Navier-Stokes problem are shape differentiable in H^m and H^{m-1} (see theorem 3) and that their shape derivatives u' and p' are solutions of a well-posed problem (see section 4.4).

4.1. Definition and basic properties. In this section, we recall the basic facts about the shape derivative of a mapping $\Omega \mapsto y(\Omega)$, defined for a given regularity of Ω. y is assumed to be a scalar field for a simplified exposition, but the adaptation to the vectorial case is obvious.

We say that y is *shape differentiable in H^n (resp. C^n) at Ω_0* if for any field V of a given regularity,

i) $[s \mapsto y(\Omega_s) \circ T_s]$ is differentiable in $H^n(\Omega_0; \mathbb{R})$ (resp. $C^n(\overline{\Omega_0}; \mathbb{R})$) at $s = 0$. Its derivative, noted $\dot{y}(\Omega_0)$ or simply $\dot{y}$, is the *material derivative*.

ii) $y(\Omega_0) \in H^{n+1}(\Omega_0; \mathbb{R})$ (resp. $y \in C^{n+1}(\overline{\Omega_0}; \mathbb{R})$).

and the *shape derivative* $y' \in H^n(\Omega_0; \mathbb{R})$ (resp. $C^n(\overline{\Omega_0}; \mathbb{R})$) is given by

$$y' = \dot{y} - \nabla y \cdot V(0). \tag{20}$$

In particular, i) and ii) hold when y satisfies the stronger assumption.

iii) $[s \mapsto y(\Omega_s) \circ T_s] \in C^0(I; H^{n+1}(\Omega_0; \mathbb{R})) \cap C^1(I; H^n(\Omega_0; \mathbb{R}))$

on a neighbourhood I of 0. If Ω_0 is at least C^{n+1}, we may recover y' as the derivative of an extension of $[s \mapsto y(\Omega_s)]$: we consider an linear extension $P : L^2(\Omega_0; \mathbb{R}) \to L^2(D; \mathbb{R})$ such that for any integer $r \leq n+1$, $H^r(\Omega_0; \mathbb{R})$ is mapped continuously into $H^r(D; \mathbb{R})$. We may then associate to the family $[s \mapsto y(\Omega_s)]$, the family $[s \mapsto Y_s]$ defined on the hold-all D by

$$Y_s \circ T_s = P(y(\Omega_s) \circ T_s). \tag{21}$$

As $[s \mapsto Y_s \circ T_s] \in C^0(I; H^{n+1}(D; \mathbb{R})) \cap C^1(I; H^n(D; \mathbb{R}))$, $[s \mapsto Y_s] \in C^1(I; H^n(D; \mathbb{R}))$ and

$$\frac{\partial Y_s}{\partial s}(0) = \frac{\partial Y_s \circ T_s}{\partial s}(0) - \nabla Y_0 \cdot V(0)$$

(direct adaptation of [9, prop 2.38, p.71]), y' is given as the restriction:

$$y' = \frac{\partial Y_s}{\partial s}(0) \Big|_{\Omega_0}.$$

Notice that the preceding construction *a priori* requires that $y(\Omega)$ is uniquely defined when Ω is given. This construction easily extends to the case of multiple solutions $y(\Omega)$ when the choice of a branch $[s \mapsto y(\Omega_s)]$ has been made.

Theorem 3. *Under the assumptions* **A1** *and* **A2**, *the branch of solutions u of the Navier-Stokes problem considered in theorem 2 is shape differentiable in H^m at Ω_0 and satisfies property iii) with $n = m$. Moreover, the associated pressure p is shape differentiable in H^{m-1} at Ω_0 and satisfies iii) with $n = m - 1$.*

Proof: The desired regularity of $[s \mapsto u(\Omega_s \circ T_s)]$ is a direct consequence of the regularity of $[s \mapsto u^s]$, given in theorem 2, and of the one of $[s \mapsto C_s]$, described in remark 2. The transported pressure also has analogous regularity, its gradient being given by $\nabla p^s = \nabla(p_s \circ T_s) = -\nu A^s u^s - B^s(u^s, u^s) + f^s$, and the mapping $[s \mapsto \nabla p^s] \in C^0(I; H^{m-1}(\Omega_0; \mathbb{R}^p)) \cap C^1(I; H^{m-2}(\Omega_0; \mathbb{R}^p))$. The fields p_s (and p^s) are defined up to a constant. We may set, for example, $\int_{\Omega_0} p_s \circ T_s \, dx = 0$, and get with that choice the regularity $[s \mapsto p_s \circ T_s] \in C^0(I; H^m(\Omega_0; \mathbb{R}^p)) \cap C^1(I; H^{m-1}(\Omega_0; \mathbb{R}^p))$. □

4.2. Tangential operators. We refer to [9] for the definition of the following usual tangential operators on the boundary Γ of a C^2 open and bounded set $\Omega \subset \mathbb{R}^3$:

- the tangential gradient $\nabla_\Gamma : H^1(\Gamma;\mathbb{R}) \to L^2(\Gamma;\mathbb{R}^3)$
- the tangential divergence $\mathrm{div}_\Gamma : H^1(\Gamma;\mathbb{R}^3) \to L^2(\Gamma;\mathbb{R})$
- the Laplace-Beltrami operator $\Delta_\Gamma : H^2(\Gamma;\mathbb{R}) \to L^2(\Gamma;\mathbb{R})$.

These tangential operators are connected to the corresponding operators in Ω: for any $y \in C^1(\overline{\Omega};\mathbb{R})$, $v \in C^1(\overline{\Omega};\mathbb{R}^3)$ and $z \in C^2(\overline{\Omega};\mathbb{R})$, we have

$$\nabla_\Gamma y = \nabla y|_\Gamma - \frac{\partial y}{\partial n} n \text{ and } \mathrm{div}_\Gamma v = \mathrm{div} v|_\Gamma - [\mathrm{D}v]n \cdot n \tag{22}$$

$$\Delta_\Gamma z = \Delta z|_\Gamma - \kappa \frac{\partial z}{\partial n} - \frac{\partial^2 z}{\partial n^2}, \tag{23}$$

where n is the unitary outer normal to Ω and κ the mean curvature of Γ. By density, equations (22) still hold for $y \in H^{\frac{3}{2}+\varepsilon}(\Omega;\mathbb{R})$, $v \in H^{\frac{3}{2}+\varepsilon}(\Omega;\mathbb{R}^3)$ and equation (23) for $z \in H^{\frac{5}{2}+\varepsilon}(\Omega;\mathbb{R})$, $\varepsilon > 0$.

Many tangential calculus formulas are shown easily with the use of the equations (22) to (23) for smooth functions and then extended by density. In particular, we will need the identity

$$\mathrm{div}_\Gamma(yv) = y\,\mathrm{div}_\Gamma v + \nabla_\Gamma y \cdot v \tag{24}$$

which is true for any $y \in C^1(\overline{\Omega};\mathbb{R})$ and $v \in H^{\frac{3}{2}+\varepsilon}(\Omega;\mathbb{R}^3)$.

Proposition 5 (Integration by part on Γ). *For all $(y,v) \in H^1(\Gamma;\mathbb{R}) \times H^1(\Gamma;\mathbb{R}^3)$ we have*

$$\int_\Gamma \nabla_\Gamma y \cdot v\, d\Gamma = -\int_\Gamma y\,\mathrm{div}_\Gamma v\, d\Gamma + \int_\Gamma \kappa y (v\cdot n) d\Gamma. \tag{25}$$

Lemma 2. *Assume that $u \in H_0^1(\Omega;\mathbb{R}^3) \cap H^2(\Omega;\mathbb{R}^3)$. Then we have*

$$\mathrm{D}u = [\mathrm{D}u]nn^* \text{ on } \Gamma. \tag{26}$$

Moreover, $\mathrm{div}\, u = 0$ *in Ω (that is, $u \in V^2(\Omega)$) implies that*

$$[\mathrm{D}u]n \cdot n = 0 \text{ and } [\mathrm{D}u]^* n = 0 \text{ on } \Gamma. \tag{27}$$

Proof: Let $y \in H_0^1(\Omega;\mathbb{R}) \cap H^2(\Omega;\mathbb{R})$. As $y|_\Gamma = 0$, $\nabla_\Gamma y = 0$ and the equation (22) leads to $\nabla y|_\Gamma = \frac{\partial y}{\partial n} n$. This result, used for $y = u^i$, $1 \le i \le 3$, proves the first part of the lemma. The second part is proved as follows: we use the decomposition (22) on the boundary. As $u|_\Gamma = 0$, $\mathrm{div}_\Gamma\, u = 0$ and we conclude that $[\mathrm{D}u]n \cdot n = 0$ on Γ. As $[\mathrm{D}u] = [\mathrm{D}u]nn^*$, $[\mathrm{D}u]^* = nn^*[\mathrm{D}u]^*$ and $[\mathrm{D}u]^* n = n(n^*[\mathrm{D}u]^* n) = n(n^*[\mathrm{D}u]n) = 0$. □

4.3. The boundary shape derivative. We may now define the boundary shape derivative of a mapping $\Omega \mapsto y(\Omega)$ which is shape differentiable in H^m, $m \ge 1$ (resp. in C^m, $m \ge 0$): it is the element of $H^{m-\frac{1}{2}}(\Gamma;\mathbb{R})$ (resp. of $C^m(\Gamma;\mathbb{R})$), defined by

$$y'_\Gamma = \dot{y}|_\Gamma - \nabla_\Gamma y \cdot V(0). \tag{28}$$

Obviously, using the equations (20) and (22), we also have

$$y'_\Gamma = y'|_\Gamma + \frac{\partial y}{\partial n}(V(0)\cdot n). \tag{29}$$

As the material derivative and tangential gradient which appear in formula (28) only depend on the values of y on the boundary Γ of Ω, we may more generally define the boundary shape derivative of a mapping $\Gamma \mapsto y(\Gamma)$ such that there is an extension of y in $H^m(\Omega;\mathbb{R})$ (resp. $C^m(\overline{\Omega};\mathbb{R})$) which is shape differentiable.

This is the way the boundary shape derivative of n may be defined (*via* a shape differentiable C^2 extension of n in Ω) and we have (see [2])

$$n'_\Gamma = -\nabla_\Gamma(V(0)\cdot n). \tag{30}$$

For two mappings y and z, respectively, shape differentiable in H^1 and in C^1, the product is shape differentiable in H^1 and its boundary shape derivative is given by

$$(yz)'_\Gamma = (y)'_\Gamma z + y(z)'_\Gamma. \tag{31}$$

Proposition 6. *Assume that y is shape differentiable in H^1. Then*

$$\frac{\partial}{\partial s}\left(\int_{\Gamma_s} y(\Gamma_s)\,d\Gamma\right)\Big|_{s=0} = \int_\Gamma y'_\Gamma\,d\Gamma + \int_\Gamma \kappa y(V(0)\cdot n)\,d\Gamma. \tag{32}$$

4.4. **PDE satisfied by u'. Adjoint Equation.** The assumption **A2** implies that the linearized Navier-Stokes problem with a right-hand side in $H^{-1}(\Omega_0;\mathbb{R}^3)$ and homogeneous Dirichlet boundary condition has a unique solution $u \in V^1(\Omega_0)$. The operator $L: V^1(\Omega_0) \to W^{-1}(\Omega_0)$ involved in abstract formulation $L(u) = \pi(g)$ of this equation being an isomorphism, the adjoint operator $L^*: V^1(\Omega_0) \to W^{-1}(\Omega_0)$ is also an isomorphism (we made the identification $W^{-1}(\Omega_0) \simeq V^1(\Omega_0)'$). The PDE form of the adjoint equation $L^*(\eta) = \pi(g)$ is $-\nu\Delta\eta + [\mathrm{D}u]^*\eta - [\mathrm{D}\eta]u + \nabla\zeta = g$ for $\eta \in V^1(\Omega_0)$ and $g \in H^{-1}(\Omega_0;\mathbb{R}^3)$. The well-posedness of the two problems

$$1)\begin{cases} -\nu\Delta v + [\mathrm{D}u]v + [\mathrm{D}v]u + \nabla q = 0 \\ \operatorname{div} v = 0 \\ v|_\Gamma = h \end{cases} \qquad 2)\begin{cases} -\nu\Delta\eta + [\mathrm{D}u]^*\eta - [\mathrm{D}\eta]u + \nabla\zeta = 0 \\ \operatorname{div}\eta = 0 \\ \eta|_\Gamma = h \end{cases} \tag{33}$$

for $h \in H^{1/2}(\Gamma;\mathbb{R}^3)$ follows directly from the existence of $g \in V^1(\Omega_0)$ such that $g|_\Gamma = h$ if h satisfies the compatibility condition

$$\int_\Gamma h\cdot n\,d\Gamma = 0 \tag{34}$$

(remember that Ω_0 is connected). Moreover, for both equations, and from the regularity of u implied by **A2**, the solutions v and η corresponding to a boundary condition $h \in H^{3/2}(\Gamma;\mathbb{R})$ are in $H^2(\Omega_0;\mathbb{R})$.

The problems (33.1) and (33.2) are used intensively in the calculations of the shape derivatives because of the following result:

Proposition 7. *The couple (u',p') is the only solution (v,q) of the linearized Navier-Stokes equation at u (33.1) with the boundary condition $v|_\Gamma = -[\mathrm{D}u]n(V(0)\cdot n)$.*

Proof: First we notice that $[\mathrm{D}u]n(V(0)\cdot n)\cdot n$ being zero on the boundary (see (27)), the compatibility condition (34) is satisfied. Moreover, for any $\varphi \in \mathcal{D}(\Omega_0;\mathbb{R}^3)$, for any $|s|$ being small enough, $\varphi \in \mathcal{D}(\Omega_s;\mathbb{R}^3)$ and any solution u_s of the Navier-Stokes equation in Ω_s satisfies

$$\begin{aligned}\int_{\Omega_s}(-\nu\Delta u_s + [\mathrm{D}u_s]u_s + \nabla p_s - f)\cdot\varphi\, dx &= \int_D(-\nu\Delta U_s + [\mathrm{D}U_s]U_s + \nabla P_s - f)\cdot\varphi\, dx\\ &= 0,\end{aligned}$$

where $[s\mapsto U_s]\in C^1(I;H^2(D;\mathbb{R}^3))$ and $[s\mapsto P_s]\in C^1(I;H^1(D;\mathbb{R}))$ are the extensions considered in the equation (21). The differentiation of this equation with respect to s gives

$$\int_D(-\nu\Delta U' + [\mathrm{D}U']U + [\mathrm{D}U]U' + \nabla P')\cdot\varphi\, dx = 0$$

which implies that u' and p' are solutions of (33.1) . The boundary condition is proved as follows: as $u|_\Gamma = 0$, for any $\varphi\in C^\infty(\overline{D};\mathbb{R}^3)$,

$$\int_{\Gamma_s} u\cdot\varphi\, d\Gamma = 0$$

using equation (32), we differentiate this equation for $s=0$. As $(u\cdot\varphi)'_\Gamma = u'_\Gamma\cdot\varphi$ and $u'_\Gamma = u'|_\Gamma + [\mathrm{D}u]n(V(0)\cdot n)$, we obtain

$$\int_\Gamma(u' + [\mathrm{D}u]n(V(0)\cdot n))\cdot\varphi\, d\Gamma = 0$$

which implies that $u'|_\Gamma = -[\mathrm{D}u]n(V(0)\cdot n)$. □

The calculation of the shape derivative of the functional considered in the next section also requires the following equations, given for divergence-free fields v and η in $H^2(\Omega;\mathbb{R}^3)$ and $u\in V^1(\Omega)$ by the repeated use of Green's formula.

$$\int_\Omega -\nu\Delta\eta\cdot v\, dx = \int_\Omega \eta\cdot(-\nu\Delta v)\, dx + \nu\int_\Gamma([\mathrm{D}v]n\cdot\eta - [\mathrm{D}\eta]n\cdot v)\, d\Gamma \tag{35}$$

$$\int_\Gamma [\mathrm{D}u]^*\eta\cdot v\, dx = \int_\Omega \eta\cdot[\mathrm{D}u]v\, dx \text{ and } \int_\Omega -[\mathrm{D}\eta]u\cdot v = \int_\Omega \eta\cdot[\mathrm{D}v]u\, dx \tag{36}$$

$$\int_\Omega \nabla\zeta\cdot v\, dx = \int_\Gamma \zeta(v\cdot n)\, d\Gamma \text{ and } \int_\Omega \eta\cdot\nabla q\, dx = \int_\Gamma(\eta\cdot n)q\, d\Gamma. \tag{37}$$

5. Shape derivative of sharp functionals

Geometrical Setting: In this section, the boundary Γ of Ω is the union of two disjoint two-dimensional submanifolds Γ_α and Γ_β. Let F and M be, respectively, the resultant of the forces and the moment of the forces exerted by the fluid on a piece Γ_α of the boundary. The vectors F and M are given by

$$F = \int_{\Gamma_\alpha}(-\sigma(u)n + pn)\, d\Gamma \text{ and } M = \int_{\Gamma_\alpha} x\times(-\sigma(u)n + pn)\, d\Gamma \tag{38}$$

with $\sigma(u) = \nu([Du]^* + [Du])$. Our aim in this section is to prove that $F(\Omega)$ and $M(\Omega)$ are *shape differentiable functionals* and to find the expression of their *Eulerian Derivative*. We recall (see [9]) that a functional $J(\Omega)$ is shape differentiable at Ω_0 if for any V regular enough, $s \mapsto J(\Omega_s)$ is differentiable at $s = 0$ and this derivative (the Eulerian derivative) $dJ(\Omega_0; V)$ is linear continuous in V.

Theorem 4. *Under the hypotheses* **A1** *with* $m = 2$ *and* **A2**, *both* F *and* M *are shape differentiable at* Ω_0. *For a given* $e \in \mathbb{R}^3$, *let* η *be the solution of the adjoint equation (33.2) with the boundary conditions* $\eta|_{\Gamma_\alpha} = e$ *and* $\eta|_{\Gamma_\beta} = 0$ *(resp.* $\eta|_{\Gamma_\alpha} = e \times x$ *and* $\eta|_{\Gamma_\beta} = 0$*).*

Then the Eulerian Derivative of $F_e = F \cdot e$ *(resp.* $M_e = M \cdot e$*) is given by*

$$dF_e(\Omega_0; V) = \int_{\Gamma_\alpha} (f \cdot e)(V(0) \cdot n)\, d\Gamma + \nu \int_\Gamma [D\eta]n \cdot [Du]n (V(0) \cdot n)\, d\Gamma \quad (39)$$

$$\text{resp.}\quad dM_e(\Omega_0; V) = \int_{\Gamma_\alpha} (x \times f + \nu[Du]n \times n) \cdot e\, (V(0) \cdot n)\, d\Gamma + \nu \int_\Gamma [D\eta]n \cdot [Du]n(V(0) \cdot n)\, d\Gamma. \quad (40)$$

Let $r \in C^\infty(D; \mathbb{R})$ be a function such that $r = 1$ in a neighbourhood of Γ_α and $r = 0$ in a neighbourhood of Γ_β. We set $g_F = re$ and $g_M = re \times x$. The values F_e and M_e may be studied as special cases of the functionals $J(\Omega) = J_1(\Omega) - J_2(\Omega)$, where

$$J_1(\Omega) = \int_\Gamma pn \cdot g\, d\Gamma, \quad J_2(u) = \int_\Gamma \sigma(u)n \cdot g\, d\Gamma$$

and g satisfies $\operatorname{div} g = 0$ in a neighbourhood of Γ and the compatibility condition (34). It is obvious that with $g = g_F$, $J(\Omega) = F_e$. Moreover, as $(x \times (-\sigma(u)n + pn)) \cdot re = (-\sigma(u)n + pn) \cdot (re \times x)$, $J(\Omega) = M_e$ with $g = g_M$. The two additional properties come easily from the equations $\operatorname{div} e = 0$ and $\operatorname{div}(e \times x) = \operatorname{rot}(e) \cdot x - \operatorname{rot}(x) \cdot e = 0$.

Shape Derivative of $J_1(\Omega)$: using the shape differentiability of p, g and n, and proposition 6, we obtain

$$dJ_1(\Omega_0; V) = \int_\Gamma p'_\Gamma (g \cdot n)\, d\Gamma + \int_\Gamma p(g \cdot n)'_\Gamma\, d\Gamma + \int_\Gamma \kappa(pn \cdot g)(V(0) \cdot n)\, d\Gamma.$$

As $p'_\Gamma = p'|_\Gamma + \frac{\partial p}{\partial n}(V(0) \cdot n)$ (see equation (29)), we have $dJ_1(\Omega; V) = A + \int_\Gamma p'(g \cdot n)\, d\Gamma$ with

$$A = \int_\Gamma \left(\frac{\partial p}{\partial n}(g \cdot n)(V(0) \cdot n) + p(g \cdot n)'_\Gamma + \kappa(pn \cdot g)(V(0) \cdot n) \right) d\Gamma.$$

• As $g'_\Gamma = g'|_\Gamma + [Dg]n(V(0) \cdot n)$ (straightforward adaptation of (29) to the vectorial case) and $n'_\Gamma = -\nabla_\Gamma (V(0) \cdot n)$ (equation (30)), we have $(g \cdot n)'_\Gamma = g'_\Gamma \cdot n + g \cdot n'_\Gamma = ([Dg]n \cdot n)(V(0) \cdot n) - g \cdot \nabla_\Gamma (V(0) \cdot n)$. Using the integration by parts formula on the boundary (25), we get

$$-\int_\Gamma pg \cdot \nabla_\Gamma (V(0) \cdot n)\, d\Gamma = \int_\Gamma \operatorname{div}_\Gamma (pg)(V(0) \cdot n)\, d\Gamma - \int_\Gamma \kappa(pn \cdot g)(V(0) \cdot n)\, d\Gamma$$

and finally, since $\operatorname{div} g = 0$ on Γ,

$$\begin{aligned} A &= \int_\Gamma \left(\frac{\partial p}{\partial n}(g \cdot n) + p[\mathrm{D}g]n \cdot n + \operatorname{div}_\Gamma(pg) \right) (V(0) \cdot n)\, d\Gamma \\ &= \int_\Gamma \operatorname{div}(pg)(V(0) \cdot n)\, d\Gamma = \int_\Gamma \nabla p \cdot g(V(0) \cdot n)\, d\Gamma. \end{aligned}$$

• We set $h = (g \cdot n)n$. The compatibility condition (34) being satisfied, we may introduce η_1, the unique solution of the adjoint equation (33.2). We have then

$$\begin{aligned} \int_\Gamma p'n \cdot g\, d\Gamma &= \int_\Gamma (p'\eta_1) \cdot n\, d\Gamma \\ &= \int_\Omega \operatorname{div}(p'\eta_1)\, dx \\ &= \int_\Omega \nabla p' \cdot \eta_1\, dx \quad (\text{as } \operatorname{div} \eta_1 = 0) \\ &= \int_\Omega (\nu\Delta u' - [\mathrm{D}u]u' - [\mathrm{D}u']u) \cdot \eta_1\, dx. \end{aligned}$$

Using equations (35) and (36) we come to

$$\int_\Gamma p'n \cdot g\, d\Gamma = \int_\Omega (\nu\Delta\eta_1 - [\mathrm{D}u]^*\eta_1 + [\mathrm{D}\eta_1]u) \cdot u'\, dx + \nu \int_\Gamma ([\mathrm{D}u']n \cdot \eta_1 - [\mathrm{D}\eta_1]n \cdot u')\, d\Gamma.$$

As $\nu\Delta\eta_1 - [\mathrm{D}u]^*\eta_1 + [\mathrm{D}\eta_1]u = \nabla\zeta_1$, equation (37) yields

$$\int_\Omega (\nu\Delta\eta_1 - [\mathrm{D}u]^*\eta_1 + [\mathrm{D}\eta_1]u) \cdot u'\, dx = \int_\Omega \nabla\zeta_1 \cdot u'\, dx = \int_\Gamma \zeta_1(u' \cdot n)\, d\Gamma = 0$$

due to $u'|_\Gamma = -[\mathrm{D}u]n(V(0) \cdot n)$ and $[\mathrm{D}u]n \cdot n = 0$ (see Lemma 2). On the other hand, $[\mathrm{D}u']n \cdot \eta_1 = (g \cdot n)[\mathrm{D}u']n \cdot n$ and on the boundary, $0 = \operatorname{div} u' = \operatorname{div}_\Gamma u' + [\mathrm{D}u']n \cdot n$. Accordingly, we have by integration by parts on Γ and because $u' = u'_\tau$

$$\nu \int_\Gamma [\mathrm{D}u']n \cdot \eta_1\, d\Gamma = -\nu \int_\Gamma (g \cdot n) \operatorname{div}_\Gamma u'\, d\Gamma = \nu \int_\Gamma \nabla_\Gamma(g \cdot n) \cdot u'\, d\Gamma. \tag{41}$$

• Finally, since $u'|_\Gamma = -[\mathrm{D}u]n(V(0) \cdot n)$, we end up with

$$dJ_1(\Omega_0; V) = \int_\Gamma (\nabla p \cdot g + \nu([\mathrm{D}\eta_1]n - \nabla_\Gamma(g \cdot n)) \cdot [\mathrm{D}u]n)\, (V(0) \cdot n)\, d\Gamma. \tag{42}$$

Shape Derivative of $J_2(\Omega)$: we notice that lemma 2 implies that $\sigma(u)n = \nu[\mathrm{D}u]n$. The shape differentiability of n, g and u (in H^2 for the latter) implies that

$$dJ_2(\Omega_0; V) = \int_\Gamma \nu([\mathrm{D}u]n \cdot g)'_\Gamma\, d\Gamma + \int_\Gamma \kappa\nu([\mathrm{D}u]n \cdot g)(V(0) \cdot n)\, d\Gamma.$$

As $([\mathrm{D}u]n \cdot g)'_\Gamma = [\mathrm{D}u]'_\Gamma n \cdot g + [\mathrm{D}u]n'_\Gamma \cdot g + [\mathrm{D}u]n \cdot g'_\Gamma$ and $[\mathrm{D}u]'_\Gamma = [\mathrm{D}u']|_\Gamma + [\mathrm{D}^2u]n(V(0) \cdot n)$, we have $dJ_2(\Omega; V) = B + \int_\Gamma \nu[\mathrm{D}u']n \cdot g_\tau\, d\Gamma$ with $g_\tau = g - (g \cdot n)n$ and

$$B = \int_\Gamma \nu \left((g \cdot n)[\mathrm{D}u']n \cdot n + (n^*[\mathrm{D}^2u]n \cdot g + [\mathrm{D}g]n \cdot [\mathrm{D}u]n + \kappa[\mathrm{D}u]n \cdot g)(V(0) \cdot n) \right) d\Gamma.$$

• The first term of B has already been calculated (see equation (41)) . The second is involved in the decomposition of the Laplace operator on the boundary (23). As $\Delta_\Gamma u = 0$ and $[\mathrm{D}u]u$=0 on Γ, $\nu n^*[\mathrm{D}^2 u]n = -\nu\kappa[\mathrm{D}u]n + \nabla p - f$ on Γ. Therefore,

$$B = \int_\Gamma ((\nabla p - f)\cdot g + \nu[\mathrm{D}g]n \cdot [\mathrm{D}u]n - \nu\nabla_\Gamma(g\cdot n))\,(V(0)\cdot n)\,d\Gamma.$$

• We set $h = g_\tau$. The compatibility condition (34) being satisfied, we may introduce η_2, the unique solution of the adjoint equation (33.2). The equation (35) yields

$$\nu\int_\Gamma [\mathrm{D}u']n\cdot g_\tau\,d\Gamma = \int_\Omega (-\nu\Delta\eta_2\cdot u' + \eta_2\cdot(\nu\Delta u'))\,dx + \nu\int_\Gamma [\mathrm{D}\eta_2]n\cdot u'\,d\Gamma.$$

Using $-\nu\Delta\eta_2 = -[\mathrm{D}u]^*\eta_2 + [\mathrm{D}\eta_2]u - \nabla\zeta$ and $\nu\Delta u' = [\mathrm{D}u]u' + [\mathrm{D}u']u + \nabla p'$, and then the equations (36) to (37), we come to

$$\nu\int_\Gamma [\mathrm{D}u']n\cdot g_\tau\,d\Gamma \;=\; \int_\Gamma \zeta(u'\cdot n)\,d\Gamma + \int_\Gamma (\eta_2\cdot n)p'\,d\Gamma + \nu\int_\Gamma [\mathrm{D}\eta_2]n\cdot u'\,d\Gamma$$

and the two first terms of the right-hand side vanish.

• We finally have

$$dJ_2(\Omega_0;V) = \int_\Gamma ((\nabla p - f)\cdot g + \nu[\mathrm{D}g]n\cdot[\mathrm{D}u]n - \nu\nabla_\Gamma(g\cdot n))\,(V(0)\cdot n)\,d\Gamma. \qquad (43)$$

Shape Derivative of $J(\Omega)$: with the expressions (42) and (43), we come to

$$dJ(\Omega_0;V) = \int_\Gamma (f\cdot g)(V(0)\cdot n) + \nu\int_\Gamma [\mathrm{D}(\eta - g)]n\cdot[\mathrm{D}u]n(V(0)\cdot n)\,d\Gamma, \qquad (44)$$

where $\eta = \eta_1 + \eta_2$ is the unique solution of the equation (33.2) with the boundary condition $\eta|_\Gamma = g$. This expression easily leads to formulas (39) and (40) of theorem 4 with the choices of $g = g_F$ and $g = g_M$.

6. Conclusion

The natural extension of this work is the study of the shape derivative of functionals that require even more regularity than the one considered in section 5. For example, for the functionals that characterize the uniformity of the pressure on a given body, the shape differentiability of p in H^1 (which corresponds to $m = 2$) is not sufficient. Notice also that with a slight adaptation, the case of non-homogeneous boundary conditions, which appear for example in the study of a body which moves in a fluid with a constant velocity, also fits in the preceding framework.

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Second-Order Shape Derivative for Hyperbolic PDEs

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Abstract. In this paper we study the second-order shape derivative for the solution to the wave equation with Dirichlet boundary conditions. We first prove the shape derivative exists for this problem. Specific complications arise when differentiating with respect to the domain hyperbolic PDEs; this requires an exclusive approach using the hidden regularity described in [7]. This study has been started in [1] and announced in [2]. We present here a new result where the data are no longer defined on the whole D and then restricted to the subsets of D but rather defined differently on each domain of D. Moreover, the differentiability is improved under stronger regularity of those functions. The Dirichlet boundary condition is now nonhomogeneous which was also necessary to consider the second-order shape derivative.

Keywords: hyperbolic partial differential equations, shape differentiability, domain optimization

Classification: 35L05, 65K10, 49-99

1. INTRODUCTION

1.1. Shape and Material Derivatives. We consider a bounded domain D in $\mathbf{R}^N$ and a family $\mathcal{O}_k$ of open domains Ω in D whose boundary $\Gamma = \partial\Omega$ is a C^k manifold oriented by the unitary normal field n outgoing to Ω. Throughout this paper we assume $k \geq 2$. Let T be a nonnegative real and $I = [0, T]$ be the *time interval.* We note $Q =]0; T[\times\Omega$ the *cylindrical evolution domain* and $\Sigma =]0, T[\times\Gamma$ the *lateral boundary* associated with any element Ω of the family $\mathcal{O}_k$. Let $\mathcal{E}_k$ be the set of $V \in C([0, S]; C^k(D, \mathbf{R}^N))$ with $\langle V, n_{\partial D}\rangle = 0$. For any $V \in \mathcal{E}_k$ we consider the flow mapping $T_s(V)$; we have $V(s)(x) = \left(\frac{\partial}{\partial s}T_s\right) \circ T_s^{-1}(x)$. For each $s \in [0, S[$, T_s is a one-to-one mapping from D onto D such that $T_0 = I$, $(s, x) \mapsto T_s(x)$ belongs to $C^1([0; S[, C^k(D; D))$ with $T_s(\partial D) = \partial D$ and $(s, x) \mapsto T_s^{-1}(x)$ belongs to $C([0; S[, C^k(D; D))$. Such transformations were studied in [3] and [4] where a full analysis of the situation was given.

The family $\mathcal{O}_k$ is stable under the perturbations $\Omega \mapsto \Omega_s(V) = T_s(V)(\Omega)$. We denote by Q_s the *perturbed cylinder* $]0; T[\times\Omega_s(V)$, $\Gamma_s = \partial\Omega_s$ and $\Sigma_s =]0, T[\times\Gamma_s$ the *perturbed lateral boundary.* We consider a map defined on the family $\mathcal{O}_k$

$$(f, g \circ \partial, \varphi, \psi) : \mathcal{O}_k \to \bigcup_{\Omega\in\mathcal{O}_k} (L^2(Q) \times H^1(\Sigma) \times H^1(\Omega) \times L^2(\Omega)),$$

verifying $f(\Omega) \in L^2(Q)$, $g(\Gamma) \in H^1(\Sigma)$, $\varphi(\Omega) \in H^1(\Omega)$, $\psi(\Omega) \in L^2(\Omega)$ with possible compatibilities conditions described later.

Let K be a coercive symmetric matrix with coefficients into $L^\infty(I, W^{2,\infty}(D)) \cap W^{1,\infty}(I, L^\infty(D))$. To each element $\Omega \in \mathcal{O}_k$ we associate the solution $y = y(\Omega)$ of the generalized wave problem

$$\left.\begin{array}{l} \partial_t^2 y - \operatorname{div}(K\nabla y) = f \text{ on } Q \\ y = g \text{ on } \Sigma, \quad y(0) = \varphi \text{ on } \Omega, \quad \partial_t y(0) = \psi \text{ on } \Omega. \end{array}\right\} \tag{1}$$

For any $V \in \mathcal{E}_k$ and $s \in [0; S]$ we set $y_s = y(\Omega_s) \in L^2(Q_s)$. Following [1], [2], [9], [6], [10] the mapping $\Omega \mapsto y(\Omega)$ is said to be *shape differentiable in* $L^2(I, H^m(D))$ if there exists Y in $C^1([0; S], L^2(I, H^m(D)))$ such that $Y(s, \cdot, \cdot)|_{Q_s} = y(\Omega_s)$ and $\frac{1}{s}(Y(s) - Y(0)) - \partial_s Y(0) \to 0$ in $L^2(I, H^m(D))$ as $s \to 0$. Then $\partial_s Y(0, \cdot, \cdot)|_Q$, which is the restriction to Q of the derivative with respect to the perturbation parameter s at $s = 0$, is independent of the choice of Y verifying those properties. We define, as well, the weak shape differentiability replacing the convergence in $L^2(I, H^m(D))$ by a weak convergence in that space.

Definition 1 (shape derivative). *The* shape derivative *is that unique element*

$$y'(\Omega; V) = \left(\frac{\partial}{\partial s} Y\right)\Big|_{s=0\ (t,x)\in Q} \in L^2(Q).$$

In this paper we shall study the shape differentiability for the Dirichlet-wave problem (1) under shape differentiability assumptions on the data $f(\Omega)$, $g(\Gamma)$, $\varphi(\Omega)$ and $\psi(\Omega)$. Those functions depend on the domain as opposed to simple restrictions to Ω of functions defined on D. This is required to do the second-order shape derivative.

In some weaker situations the solution $y(\Omega)$ will be only weakly shape differentiable in $L^1(I, H^{-1}(D))$. The definition is the same with $L^1(I, H^{-1}(D))$ instead of $L^2(I, H^m(D))$.

Following [6], [9], [10] the boundary data g is *shape boundary differentiable* in $L^1(I, H^p(\Gamma))$ if there exists G in $C^1([0; S], L^1(I, H^{p+\frac{1}{2}}(D)))$ such that $\frac{\partial}{\partial n_s} G(s) = 0$ on Σ_s and $G(s, \cdot, \cdot)|_{\Sigma_s} = g(\Gamma_s)$ on Σ_s. The restriction to Σ of the derivative with respect to the perturbation parameter s does not depend of the choice of the mapping G verifying those properties (*cf.* [6]). Similarly one defines the weak boundary shape differentiability.

Definition 2 (shape boundary derivative). *The element*

$$g'_\Gamma(\Gamma; V) = \left(\frac{\partial}{\partial s} G\right)\Big|_{s=0\ (t,x)\in\Sigma}$$

is the shape boundary derivative *of g.*

Definition 3 (material derivative). *The element $\dot{y}(\Omega; V)$ is the* material derivative *of y in $L^2(I, H^m(D))$ (*resp. *weakly in $L^2(I, H^m(D))$) if it is the limit in $L^2(I, H^m(D))$ (*resp. *weakly in $L^2(I, H^m(D))$) of $\frac{1}{s}(y(\Omega_s) \circ T_s - y(\Omega))$ as s tends to 0.*

Let V and W be two autonomous vector fields in $\mathcal{E}_k$.

Definition 4 (Second-Order Shape Derivative).

$$y''(\Omega; V, W) = (y'(\Omega; V))'(\Omega; W).$$

Definition 5 (Second-Order Boundary Shape Derivative).

$$g''_\Gamma(\Gamma; V, W) = (g'_\Gamma(\Gamma; V))'_\Gamma(\Gamma; W).$$

1.2. **The Results.** Let m and q be two integers with $q \geq 1$ (the subsequent results could be extended to positive reals m and q with $q \geq \frac{1}{2}$, the spaces being obtained through an interpolation; however, for the sake of shortness we shall restrict the study to integers indices only). Let $(\mathcal{H}_{m,q})$ be the hypothesis concerning the regularity of the data: f belongs to $L^1(I, H^m(\Omega))$, $\partial_t^{(m)} f$ belongs to $L^1(I, L^2(\Omega))$, g belongs to $H^{q+1}(\Sigma)$, φ belongs to $H^{m+1}(\Omega)$ and ψ belongs to $H^m(\Omega)$ with compatibility conditions and $\operatorname{supp}\varphi \Subset \Omega$, $\operatorname{supp}\Psi \Subset \Omega$. Assuming $(\mathcal{H}_{m,q})$ and $k = \min\{m, q\} + 1$, we refer to [7] and [8] for the existence and regularity of the solution to (1). For any integer $r \neq 0$ we note $Z^r(I, \Omega) = \bigcap_{i=0}^r C^i(I, H^{r-i}(\Omega))$ and $Z^0(I, \Omega) = C(I, L^2(\Omega)) \cap C^1(I, H^{-1}(\Omega))$. It was obtained that

$$y \in Z^{\min\{m,q\}+1}(I, \Omega) \quad \text{and} \quad \frac{\partial y}{\partial n} \in H^{\min\{m,q\}}(\Sigma).$$

One should note that the regularity on the boundary does not follow from the regularity in the domain using a trace theorem. For that reason it is called *hidden regularity*.

In order to derive the shape derivative properties of the mapping y, we introduce shape differentiability assumptions on the data. The hypothesis $(\mathcal{H}'_{m,q})$ stands for f is shape differentiable and the shape derivative $f' \in L^1(I, H^{m-1}(D))$. When $m \geq 2$, $\partial_t^{(m-1)} f' \in L^1(I, L^2(D))$ as well, g is boundary shape differentiable and the boundary shape derivative $g'_\Gamma \in H^q(\Gamma)$, φ is shape differentiable and the shape derivative φ' belongs to $H^m(D)$, and ψ is shape differentiable and the shape derivative ψ' belongs to $H^{m-1}(D)$. To derive the second-order shape derivative properties we introduce, as well, hypothesis of second-order shape differentiability on the data, and that hypothesis is denoted $(\mathcal{H}''_{m,q})$.

Theorem 1. *Let $m \geq 1$ and $q \geq 2$ be two integers, we assume $(\mathcal{H}_{m,q})$, $(\mathcal{H}'_{m,q})$, $(\mathcal{H}''_{m,q})$, and $k = \max\{2, \min\{m, q\} + 1\}$. The solution to (1) is*

- *weakly shape differentiable at the second order at $s = 0$ in $L^1(I, H^{-1}(D))$ if $m = 1$ or $q = 2$,*
- *weakly shape differentiable at the second order at $s = 0$ in $L^1(I, L^2(D))$ if ($m = 1$ or $q = 2$) and $f'' = 0$, $g''_\Gamma = 0$,*
- *weakly material differentiable at the second order at $s = 0$ in $L^2(I, H^1(D))$ if $m \geq 2$ and $q \geq 3$,*
- *strongly shape differentiable at the second order and strongly material differentiable at the second order at $s = 0$ in $L^2(I, H^{\min\{m-2,q-3\}}(D))$ if $m \geq 2$ and $q \geq 3$.*

The second-order shape derivative $y''(\Omega; V, W)$ belongs to $Z^{\min\{m,q\}-1}(I,\Omega)$ and is solution to

$$\left.\begin{array}{l} P(y'') = f'' \quad on\ Q \\ y' = g''_\Gamma - \frac{\partial}{\partial n}(y'(\Omega; W))\langle V, n\rangle + \langle \nabla_\Gamma g, \nabla_\Gamma \langle W, n\rangle\rangle \langle V, n\rangle \\ \qquad -\frac{\partial y}{\partial n}(\langle DV.n, n\rangle \langle W, n\rangle - \langle \nabla_\Gamma \langle W, n\rangle, V_\Gamma\rangle) \qquad on\ \Sigma \\ y''(0) = \varphi'' \quad on\ \Omega \\ \partial_t y''(0) = \psi'' \quad on\ \Omega. \end{array}\right\} \tag{2}$$

To prove this result we need to revisit the shape differentiability theory. We need the right-hand side, the boundary condition and the initial conditions to be defined differently for each domain Ω rather than being the restrictions to Ω of functions defined on D; moreover the Dirichlet boundary condition needs to be non-homogeneous.

2. Homogeneous Dirichlet B.C.

Let $(\mathcal{H}_m) = (g = 0) \wedge (\mathcal{H}_{m,q})$ and $(\mathcal{H}'_m) = (g = 0) \wedge (\mathcal{H}'_{m,q})$.

Theorem 2. *Let m be a positive integer, we assume $(\mathcal{H}_m)$, $(\mathcal{H}'_m)$, $k = \max\{m, 2\}$.*

- *If $m \geq 1$ then the solution to (1) is both shape and material differentiable at $s = 0$, strongly in $L^2(I, H^{m-1}(D))$.*
- *If $m = 1$ then the solution to (1) is material differentiable at $s = 0$, weakly in $L^2(I, H^1(D))$.*
- *If $m = 0$ and $f' = 0$ then the solution to (1) is shape differentiable at $s = 0$, weakly in $L^1(I, L^2(D))$.*
- *If $m = 0$ then the solution to (1) is shape differentiable at $s = 0$, weakly in $L^1(I, H^{-1}(D))$.*

The shape derivative $y' \in Z^m(I,\Omega)$ and is solution to

$$\left.\begin{array}{l} \partial_t^2 y' - \mathrm{div}\,(K\nabla y') = f' \quad on\ Q \\ y' = -\frac{\partial y}{\partial n}\langle V(0), n\rangle \quad on\ \Sigma, \quad y'(0) = \varphi' \quad on\ \Omega, \quad \partial_t y'(0) = \psi' \quad on\ \Omega, \end{array}\right\} \tag{3}$$

when applicable, the material derivative $\dot{y} \in Z^m(I,\Omega)$ and is solution to

$$\left\{\begin{array}{l} P(\dot{y}) = -\mathrm{div}\,(((\mathrm{div}\,V(0)).K - DV(0).K + {}^*(DV(0).K) + DK.V(0))\nabla y) \\ \qquad + \langle \nabla f, V(0)\rangle + f' + \mathrm{div}\,(V(0))\mathrm{div}\,(K\nabla y) \qquad on\ Q \\ \dot{y} = 0 \quad on\ \Sigma, \quad \dot{y}(0) = \langle \nabla\varphi, V(0)\rangle + \varphi' \quad on\ \Omega, \quad \partial_t\dot{y}(0) = \langle \nabla\psi, V(0)\rangle + \psi' \quad on\ \Omega. \end{array}\right.$$

The proof when $m \geq 1$ is based upon technical estimates of the energy. The implicit function theorem associated with a bootstrap technique gives the differentiability in higher spaces; however, it does not fully contain the result obtained with the estimates of the energy for the case $m = 1$. The situation is drastically different when $m = 0$ since the approach used when $m \geq 1$ cannot be extended. In that case we use the hidden regularity.

We will designate the regularity of the data as strong when $m \geq 1$ as opposed to the weak regularity of the data when $m = 0$. Moreover we will note $A = \mathrm{div}\,(K\nabla)$ and $P = \frac{\partial^2}{\partial t^2} - A$.

2.1. Material Derivative with Strong Regularity of the Data. We suppose $(\mathcal{H}_1)$, $(\mathcal{H}_1')$ and $k = 2$. Let λ be the coercivity constant and $\kappa = \max_{i,j} \|\partial_t K_{i,j}\|_{L^\infty(I\times D)}$. We suppose $\kappa < \frac{\lambda}{2T^2}$; one should remark that that requirement is always satisfied when K is time independent. Otherwise, the condition is satisfied on $]0; \sqrt{\lambda/(2\kappa)}[$. This condition may be dismissed since one can iterate the process.

Lemma 1. *Let $\theta \in Z_0^2(I, \Omega)$ and let*

$$a(\theta) = \frac{\lambda}{\lambda - 2\kappa T^2}\|P\theta\|_{L^1(I,L^2(\Omega))}$$

$$b(\theta) = \frac{\lambda}{\lambda - 2\kappa T^2}\int_\Omega (K(0,x)\nabla\theta(0,x).\nabla\theta(0,x) + (\partial_t\theta)^2(0,x))dx$$

then $\|\partial_t\theta\|_{L^\infty(I,L^2(\Omega))} \leq 2a(\theta) + \sqrt{b(\theta)}$ *and* $\|\theta\|_{L^\infty(I,H_0^1(\Omega))} \leq 2a(\theta) + \sqrt{b(\theta)}$.

Let us introduce the following notations:

$$a_s = \|\sqrt{\gamma_s}P_s w^s\|_{L^1(I,L^2(\Omega))}$$

$$b_s = \int_\Omega \langle K_s(0,x)\nabla w^s(0,x), \nabla w^s(0,x)\rangle + (\partial_t w^s(0,x))^2\, dx.$$

Proposition 1. $\forall\eta \in]0;1[, \exists\varepsilon^* > 0, \forall\varepsilon \in]0;\varepsilon^*[, \exists\alpha \in]1-\eta, 1+\eta[, \forall s \in [0;\varepsilon[,$

$$\alpha\|w^s\|_{L^\infty(I,H_0^1(\Omega))} \leq 2a_s + \sqrt{b_s} \quad \textit{and} \quad \alpha\|\partial_t w^s\|_{L^\infty(I,L^2(\Omega))} \leq 2a_s + \sqrt{b_s}.$$

Lemma 2. *When s is in the neighborhood of 0 we have $a_s \leq M_a$ and $b_s \leq M_b$ where M_a and M_b are independent of s.*

Proposition 2. *$\|w^s\|_{H^1(Q)}$ is bounded and $\exists w \in L^\infty(I, H_0^1(\Omega)) \cap W^{1,\infty}(I, L^2(\Omega))$ such that $w^{n_i} \rightharpoonup w$ weakly in $H^1(Q)$ (thus $w^{n_i} \to w$ in $L^2(Q)$).*

Proposition 3. *The function w belongs to $Z^1(I,\Omega)$ and is unique.*

For each sequence (s_n) of corollary 2 there is a function w. We prove the above proposition showing w is the solution of a well-posed problem.

Lemma 3. *For any sequence (s_n), $w(0) = \langle\nabla\varphi, V(0)\rangle + \varphi'$.*

Lemma 4. *There exists $\Xi \in L^2(Q)$ such that for any sequence (s_n), $Pw = \Xi$.*

Moreover, as a consequence of corollary 2 we have $w = 0$ on Σ. Hence

$$\begin{cases} P(w) = \Xi \ \text{ on } Q \\ w = 0 \ \text{ on } \Sigma, \quad w(0) = \langle\nabla\varphi, V(0)\rangle + \varphi' \ \text{ on } \Omega, \quad \partial_t w(0) = \langle\nabla\psi, V(0)\rangle + \psi' \ \text{ on } \Omega, \end{cases}$$

where $\varepsilon_K = \frac{1}{2}(DV(0).K + {}^*(DV(0).K))$ and Ξ is given by

$$\langle\nabla f, V(0)\rangle + f' + \operatorname{div}(V(0))\operatorname{div}(K\nabla y) - \operatorname{div}(((\operatorname{div}V(0)).K - 2\varepsilon_K + DK.V(0))\nabla y).$$

We refer to [7, theorem 2.1] and [7, section 4] to prove that w is unique and belongs to $Z^1(I,\Omega)$.

Proposition 4. *Under $(\mathcal{H}_1)$, $(\mathcal{H}_1')$ and $k=2$ there exists $\dot{y} \in Z^1(I,\Omega)$ such that*

$$\frac{y^s - y}{s} \rightharpoonup \dot{y} \text{ weakly in } L^2(I, H^1(D)) \text{ as } s \to 0$$

the material derivative $\dot{y}$ *is solution to the problem*

$$\begin{cases} P(\dot{y}) = \Xi \ \text{ on } Q \\ \dot{y} = 0 \ \text{ on } \Sigma, \quad \dot{y}(0) = \langle \nabla\varphi, V(0)\rangle + \varphi' \ \text{ on } \Omega, \quad \partial_t \dot{y}(0) = \langle \nabla\psi, V(0)\rangle + \psi' \ \text{ on } \Omega. \end{cases}$$

Corollary 1. *Under $(\mathcal{H}_1)$, $(\mathcal{H}_1')$ and $k=2$ there exists $\dot{y} \in Z^1(I,\Omega)$ such that for any* $\varepsilon \in]0;1]$

$$\frac{y^s - y}{s} \to \dot{y} \text{ strongly in } L^2(I, H^{1-\varepsilon}(D)) \text{ as } s \to 0.$$

Let us enhance the differentiability for greater values of m.

Proposition 5. *Let $m \geq 2$ be an integer and assume $(\mathcal{H}_m)$, $(\mathcal{H}_m')$, $k=m$. The shape differentiation takes place in $L^2(I, H^{m-1}(D))$.*

Proof. Let us consider

$$\begin{aligned} \mathcal{E}^m(I,\Omega) &= \{\theta \in Z^{m+1}(I,\Omega) \text{ s.t. } \theta|_\Sigma = 0,\ \theta(0) = \Phi_0,\ \partial_t\theta(0) = \Phi_1, \text{ and} \\ &\qquad P\theta \in L^1(I, H^m(D)),\ \partial_t^{(m)}(P\theta) \in L^1(I, L^2(D))\} \end{aligned}$$

and the norm $\|\cdot\|_{\mathcal{E}^m}$ defined by

$$\|\theta\|_{\mathcal{E}^m} = \|\theta\|_{Z^{m+1}(I,\Omega)} + \|P\theta\|_{L^1(I,H^m(\Omega))} + \left\|\partial_t^{(m)}(P\theta)\right\|_{L^1(I,L^2(\Omega))}.$$

Let Ψ be defined by

$$\begin{aligned} \Psi\ :\ [0;S[\times\mathcal{E}(I,\Omega) &\longrightarrow L^1(I, L^2(\Omega)) \\ (s,\theta) &\longmapsto P\theta - f\circ T_s + (P_s - P)y^s \end{aligned}$$

we have $D_\theta\Psi = P$, that function is continuous on $[0;S]\times\mathcal{E}$. Moreover, according to [7, theorem 2.2], $D_\theta\Psi(0,\theta)$ is an isomorphism between $\mathcal{E}^m(I,\Omega)$ and $\{f \in L^1(I, H^m(D)) \mid \partial_t^{(m)}(P\theta) \in L^1(I, L^2(D))\}$.

Let $\delta \in \mathcal{E}(I,\Omega)$ then $\Psi(s,\theta+\delta) - \Psi(0,\theta) = P(\theta+\delta) - f\circ T_s + (P_s - P)(\theta+\delta) - P\theta + f$ hence

$$\Psi(s,\theta+\delta) - \Psi(0,\theta) = P\delta - (f\circ T_s - f) + (P_s - P)(\theta+\delta)$$

since $f \in L^1(I, H^m(\Omega))$ we have $\frac{1}{s}(f\circ T_s - f) \to \nabla f.V(0)$ in $L^2(I, H^{m-1}(\Omega))$. Moreover, $(P_s - P)(\theta+\delta) = o(s,\delta)$; hence

$$\Psi(s,\theta+\delta) - \Psi(0,\theta) = s\nabla f.V(0) + o(s,\delta,P\delta)$$

and $\nabla f.V(0) \in L^2(I, H^{m-1}(\Omega))$.

Therefore Ψ is differentiable at $(0,\theta)$ in $L^2(I, H^{m-1}(D))$. The implicit function theorem applies: if $\theta \in \mathcal{E}$ satisfies $\Psi(s,\theta) = 0$ then θ is a function of s in the neighborhood of 0 and $\partial_s\theta$ exists.

Since $\Psi(s,y^s) = Py^s - f\circ T_s + P_s y^s - Py^s = P_s(y_s\circ T_s) - f\circ T_s = 0$ the differentiation takes place in $L^2(I, H^{m-1}(D))$. □

2.2. Shape Derivative with Strong Regularity of the Data. Let $\mathfrak{B}_m$ be an extension operator from $H^m(\Omega)$ onto $H^m(D)$ and $\mathfrak{B}^s_m = \theta \mapsto \mathfrak{B}_m(\theta \circ T_s) \circ T_s^{-1}$. For the sake of shortness we will also denote by $\mathfrak{B}_m$ the extension $id \otimes \mathfrak{B}_m$. The same remark applies to $\mathfrak{B}^s_m$.

Proposition 6. *$\frac{1}{s}(\mathfrak{B}^s_m y_s - \mathfrak{B}_m y) \to Y$ in $L^2(I \times D)$ as s tends to 0.*

Proof. This proposition comes from the decomposition

$$\frac{\mathfrak{B}^s_m y_s - \mathfrak{B}_m y}{s} = \frac{\mathfrak{B}^s_m y_s - \mathfrak{B}_m y^s}{s} + \mathfrak{B}_m\left(\frac{y^s - y}{s}\right).$$

The use of the previous sections for the second term of the right-hand side gives the result. □

Proposition 7. *$y' = Y|_Q$ is the shape derivative of y.*

Corollary 2. *$y' = \dot{y} - \langle \nabla y, V(0)\rangle$ on Q.*

We denote by y'_σ the shape derivative in σ. ($y' = y'_0$.)

Lemma 5. *Assume $\Theta \in C^1([0;S[, L^1(I \times D))$, we note $\theta_s = \Theta(s,\cdot,\cdot)$ and $\theta = \theta_0$, then*

$$\frac{\partial}{\partial s}\left(\int_{Q_s} \theta_s(x,t)\, dx\, dt\right)_{s=0} = \int_Q \partial_s\Theta(0,x,t)\, dx\, dt + \int_\Sigma \theta(x,t)\,\langle V(0), n\rangle\, d\Gamma\, dt.$$

Proposition 8. *$P(y') = f'$ on Q.*

Corollary 3. *The shape derivative y' is solution to*

$$\left.\begin{array}{l} P(y') = f' \quad on\ Q \\ y' = -\frac{\partial y}{\partial n}\langle V(0), n\rangle \quad on\ \Sigma, \quad y'(0) = \varphi' \quad on\ \Omega, \quad \partial_t y'(0) = \psi' \quad on\ \Omega. \end{array}\right\} \tag{4}$$

If y is shape differentiable at any $\Omega \in \mathcal{O}_k$ in any domain field V, then $y'(\Omega; V) = \frac{\partial}{\partial s} Y|_{\{0\}\times I \times \Omega}$, we have $\frac{\partial}{\partial s} Y|_{\{\sigma\}\times I\times\Omega_\sigma} = y'(\Omega_\sigma(V), V(\sigma)) = y'_\sigma$ and y'_σ is solution to

$$\left\{\begin{array}{l} P(y'_\sigma) = f'_\sigma \quad \text{on } Q_\sigma \\ y'_\sigma = -\frac{\partial y_\sigma}{\partial n_\sigma} V(\sigma).n_\sigma \quad \text{on } \Sigma_\sigma, \quad y'_\sigma(0) = \varphi'_\sigma \quad \text{on } \Omega_\sigma, \quad \partial_t y'_\sigma(0) = \psi'_\sigma \quad \text{on } \Omega_\sigma. \end{array}\right.$$

2.3. Shape Derivative with Weak Regularity of the Data. In this section, we suppose $(\mathcal{H}_0)$, $(\mathcal{H}'_0)$ and $k = 2$. According to [7, theorem 2.1] and [7, section 4] we have $\frac{\partial y}{\partial n} \in L^2(\Sigma)$; thus, using [7, theorem 2.3] and [7, section 4] on (4) we obtain that y' exists, is unique, and belongs to $Z^0(I, \Omega)$. Let us prove y' is the shape derivative of the solution to (1).

Since $\frac{\partial y}{\partial n}\langle V(0), n\rangle \in L^1(I, H^{m-1}(\Gamma))$, [7, theorem 2.1] and [7, section 4] say that (4) has a unique solution y' defined in $Z^0(I, \Omega)$. Hence the shape derivative may continue to exist even though the propositions to prove the existence of $\dot{y}$ do not apply anymore.

Remark 1. *It should be noticed that y' continues to exist while $\dot{y}$ does not exist anymore.*

From [7, theorem 2.1] and [7, section 4], the solution to (1) exists, is unique and belongs to $C(I, L^2(D))$. Let y be this solution. We denote by θ a function in $L^\infty(I, L^2(\Omega))$. Let us introduce $h(s) = \int_{Q_s} y_s\theta\, dx\, dt$ and $\bar{h}(s) = \int_{Q_s} y'_s\theta\, dx\, dt$.

Lemma 6. *The function h is absolutely continuous. More precisely $h(s) = h(0) + \int_0^s \bar{h}(\sigma)\, d\sigma$.*

This lemma may be proved using a mollifier sequence and the result of the previous section.

Proposition 9. $\frac{\mathfrak{B}_m^s y_s - \mathfrak{B}_m y}{s} \overset{*}{\rightharpoonup} y'$ *in $L^1(I, H^{-1}(D))$ as s tends to 0.*

We now suppose $\theta \in L^\infty(I, H^1(\Omega))$. To prove proposition 9, we are going to prove that h is differentiable. Since lemma 6 applies, we can do so by proving $\lim_{s \to 0} \bar{h}(s) = \bar{h}(0)$. That will be done using the next 4 lemmae.

Lemma 7. *Let $\Lambda_s \in Z^1(I, \Omega)$ be the solution to*

$$\left.\begin{array}{l} P(\Lambda_s) = \theta \quad \text{on } Q_s \\ \Lambda_s = 0 \quad \text{on } \Sigma_s, \quad \Lambda_s(T) = 0 \text{ on } \Omega_s, \quad \partial_t \Lambda_s(T) = 0 \text{ on } \Omega_s \end{array}\right\} \tag{5}$$

then $\bar{h}(s) = \int_{Q_s} f_s' \Lambda_s\, dx\, dt + \int_{\Sigma_s} \frac{\partial y_s}{\partial n_s} \frac{\partial \Lambda_s}{\partial n_s} \langle K\, n_s, n_s\rangle \langle V(s), n_s\rangle\, d\Gamma_s\, dt$.

Lemma 8. *Let $W(s) = \frac{1}{2} \langle K_s n, n\rangle \, \| {}^*DT_s^{-1} n\|^{-1} DT_s^{-1} V(s) \circ T_s$ then*

$$\bar{h}(s) = \int_Q f_s' \circ T_s \Lambda^s\, dx\, dt + \int_\Sigma \left(\left[\frac{\partial(y^s + \Lambda^s)}{\partial n}\right]^2 - \left[\frac{\partial y^s}{\partial n}\right]^2 - \left[\frac{\partial \Lambda^s}{\partial n}\right]^2 \right) \langle W(s), n\rangle\, d\Gamma\, dt$$

Lemma 9. *The function $s \mapsto \int_\Sigma (\frac{\partial}{\partial n}\theta^s)^2 \langle W(s), n\rangle\, d\Gamma\, dt$ is continuous in 0 for $\theta = y$, $\theta = \Lambda$ and $\theta = y + \Lambda$.*

Proof. Let $W(T) = 0$; this request is not restrictive since T can be taken as large as we want. Following [5] we use the *extractor* technique. By density, $\partial_t \theta^s$ and $\nabla \theta^s$ are continuous as $s \to 0$. Moreover, ${}^*DT_s^{-1} W(0)$ and $\nabla(DT_s^{-1}\, {}^*DT_s^{-1}) W(0)$ are bounded. □

Lemma 10. *The function $s \mapsto \int_{Q_s} f_s' \circ T_s \Lambda^s\, dx\, dt$ is continuous in 0.*

The differentiability $L^1(I, H^{-1}(\Omega))$ may be improved when $f' = 0$. Assume $\theta \in L^\infty(I, L^2(\Omega))$ then Λ_s, the solution to (5), exists and belongs to $Z^0(I, \Omega)$. Because $f' = 0$ we get

$$\bar{h}(s) = \int_{\Sigma_s} \frac{\partial y_s}{\partial n_s} \frac{\partial \Lambda_s}{\partial n_s} \langle K\, n_s, n_s\rangle \langle V(s), n_s\rangle\, d\Gamma_s\, dt.$$

The continuity is a consequence of lemmae 8 and 9 that does not require a better regularity of θ. This proves $\bar{h}$ is continuous in 0, hence

Proposition 10. *Assume $f' = 0$, $(\mathcal{H}_0)$, $(\mathcal{H}_0')$ and $k = 2$ then the solution to (1) is weakly shape differentiable in $L^1(I, L^2(D))$. Moreover, $y' \in Z^m(I, \Omega)$ and y' is solution to (3).*

3. Nonhomogeneous Dirichlet B.C

We consider the problem with a nonhomogeneous Dirichlet boundary condition (1). We consider a function G of definition 2.

Theorem 3. *Assume* $(\mathcal{H}_{m,q})$, $(\mathcal{H}'_{m,q})$ *and* $k = \max\{2, \min\{m,q\}\}$. *The solution to (1) is*

- *weakly shape differentiable at* $s=0$ *in* $L^1(I, H^{-1}(D))$ *if* $m=0$ *or* $q=1$.
- *weakly shape differentiable at* $s=0$ *in* $L^1(I, L^2(D))$ *if* ($m=0$ *or* $q=1$) *and* $f'=0$, $g'_\Gamma=0$.
- *weakly material differentiable at* $s=0$ *in* $L^2(I, H^1(D))$ *if* $m=1$ *and* $q=2$.
- *strongly shape differentiable and strongly material differentiable at* $s=0$ *in* $L^2(I, H^{\min\{m-1,q-2\}}(D))$ *if* $m \geq 1$ *and* $q \geq 2$.

Moreover,

$$y' \in Z^{\min\{m,q\}}(I, \Omega)$$

furthermore, y' *is solution to*

$$\left.\begin{array}{l} P(y') = f' \quad \text{on } Q \\ y' = g'_\Gamma - \frac{\partial y}{\partial n}\langle V(0), n\rangle \quad \text{on } \Sigma, \quad y'(0) = \varphi' \quad \text{on } \Omega, \quad \partial_t y'(0) = \psi' \quad \text{on } \Omega. \end{array}\right\} \tag{6}$$

Proof. The function $u = y - G(0)$ is solution to

$$\left\{\begin{array}{l} P(u) = f - PG(0) \quad \text{on } Q \\ u = 0 \quad \text{on } \Sigma, \quad y(0) = \varphi - G(0) \quad \text{on } \Omega, \quad \partial_t y(0) = \psi - \partial_t G(0) \quad \text{on } \Omega. \end{array}\right.$$

Since $G(0) \in H^{q+\frac{3}{2}}(Q)$ we get $PG(0) \in H^{q-\frac{1}{2}}(Q)$ thus $f - PG(0) \in L^1(I, H^{\min\{m,q-\frac{1}{2}\}})$. If $m \geq 1$ and $q \geq 2$ we also have

$$\partial_t^{(\min\{m,q-\frac{1}{2}\})}(f - PG(0)) \in L^1(I, L^2(\Omega)) \quad \text{and} \quad (f - PG(0))' \in L^1(I, H^{\min\{m-1,q-\frac{3}{2}\}}).$$

If $m \geq 2$ and $q \geq 3$, we have $\partial_t^{(\min\{m-1,q-2\})}(f - PG(0))' \in L^1(I, L^2(\Omega))$ as well.

Moreover, $\varphi - G(0) \in H^{q+1}(\Omega)$ and $\psi - \partial_t G(0) \in H^q(\Omega)$; therefore theorem 2 gives the shape differentiability of u. Furthermore, u' belongs to $Z^{\min\{m,q\}}(I, \Omega)$ and is solution to

$$\left.\begin{array}{l} P(u) = f' - (P(\partial_s G|_{s=0})) \quad \text{on } Q \\ u' = -\frac{\partial u}{\partial n}\langle V(0), n\rangle \quad \text{on } \Sigma \\ u'(0) = \varphi' - \partial_s G|_{s=0}(0) \quad \text{on } \Omega, \quad \partial_t u'(0) = \psi' - \partial_s \partial_t G|_{s=0}(0) \quad \text{on } \Omega \end{array}\right\} \tag{7}$$

using $\partial_s(PG) = P(\partial_s G|_{s=0})$ and $u' = y' - \partial_s G|_{s=0}$, (7) yields (6). Furthermore [7] and (6) give $y' \in Z^{\min\{m,q\}}(I, \Omega)$.

When $\min\{m,q\} \in [0;1[$, $f' = 0$ and $g'_\Gamma = 0$, proposition 10 applies and gives the weak shape differentiability in $L^1(I, L^2(D))$. □

Theorem 1 is a consequence of theorem 3. The only remaining difficulty lies in the characterization of y''. Indeed one has to compute the boundary shape derivative of $g'_\Gamma - \frac{\partial y}{\partial n}\langle V(0), n\rangle$. This problem is solved using the following lemma (*cf.* [6]).

Lemma 11. *We have*

$$(\nabla y)'_\Gamma(\Gamma; W) = \nabla(y'(\Omega; W))|_\Gamma + D^2 y.n \langle W, n\rangle$$
$$n'_\Gamma = -\nabla_\Gamma \langle W, n\rangle .$$

Remark 2. *Assuming enough regularity on the data, one can iterate the method presented here to prove the solution to (1) is shape analytic.*

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An Axiomatic Approach to Image Interpolation

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Abstract. We discuss possible algorithms for interpolating data given in a set of curves and/or points in the plane. We propose a set of basic assumptions to be satisfied by the interpolation algorithms which lead to a set of models in terms of possibly degenerate elliptic partial differential equations. The Absolute Minimal Lipschitz Extension model (AMLE) is singled out and studied in more detail. We show experiments suggesting a possible application, the restoration of images with poor dynamic range.

Keywords: image interpolation, degenerate elliptic problems, viscosity solutions

1. Introduction

Our purpose in this paper will be to discuss possible algorithms for interpolating scalar data given on a set of points and/or curves in the plane. Our main motivation comes from the field of image processing. A number of different approaches using interpolation techniques have been proposed in the literature for "perceptually motivated" coding applications [5, 18, 24]. The underlying image model is based on the concept of "raw primal sketch" [19]. The image is assumed to be made mainly of areas of constant or smoothly changing intensity separated by discontinuities represented by strong edges. The coded information, also known as *sketch data*, consists of the geometric structure of the discontinuities and the amplitudes at the edge pixels. In very low bit rate applications, the decoder has to reconstruct the smooth areas in-between by using the edge information. This can be posed as a scattered data interpolation problem from an arbitrary initial set (the sketch data) under certain smoothness constraints. For higher bit rates, the residual texture information has to be separately coded by means of a waveform coding technique, for instance, pyramidal or transform coding. In the following we assume that a set of curves and points is given and we want to construct a function interpolating these data. Several interpolation techniques using implicitely or explicitely the solution of a partial differential equation have been used in the engineering literature [4, 5, 6]. In the spirit of [1], our approach to the problem will be based on a set of formal requirements that any interpolation operator in the plane should satisfy. Then we show that any operator which interpolates continuous data given on a set of curves can be given as the viscosity solution of a degenerate elliptic partial differential equation of a certain type. The examples include the Laplacian operator and the minimal Lipschitz extension operator [2], [16] which is related to the work of J. Casas [5, 6].

Our plan is as follows. In Section 2 we introduce a formal set of axioms which should be satisfied by any interpolation operator in the plane and derive the associated partial differential equation. In Section 3 we discuss several examples of interpolation operators relating them to the set of axioms studied in the previous section. We shall concentrate our attention in a special one, the so-called AMLE model, giving existence and uniqueness results for the associated PDE. Its numerical analysis is given in Section 4. In Section 5, we give a generalization of the AMLE model when the data are singular. We conclude that the AMLE cannot be used in reconstructing occluded objects. Finally, in Section 6 we display some experimental results obtained by using the previous model. Although the whole theory will be developed in $\mathbb{R}^2$, there are very few alterations in order to extend it to $\mathbb{R}^n$. In this short note, we do not give any proof but the interested reader may find them in [8] and [9].

2. Axiomatic analysis of interpolation operators

We begin by recalling the definition of a continuous simple Jordan curve.

Definition 1. *A continuous function $\Gamma : [a,b] \to \mathbb{R}^2$ is a continuous simple Jordan curve if it is one-to-one in (a,b) and $\Gamma(a) = \Gamma(b)$. By Alexandroff Theorem, such a curve surrounds a bounded simply connected domain which we denote by $D(\Gamma)$.*

Let $\mathcal{C}$ be the set of continuous simple Jordan curves in $\mathbb{R}^2$. For each $\Gamma \in \mathcal{C}$, let $\mathcal{F}(\Gamma)$ be the set of continuous functions defined on Γ. We shall consider an interpolation operator as a transformation E which associates with each $\Gamma \in \mathcal{C}$ and each $\varphi \in \mathcal{F}(\Gamma)$ a unique function $E(\varphi, \Gamma)$ defined in the region $D(\Gamma)$ inside Γ satisfying the following axioms:

$(A1)$ Comparison principle:

$$E(\varphi, \Gamma) \leq E(\psi, \Gamma) \quad \text{for any } \Gamma \in \mathcal{C} \quad \text{and any } \varphi, \psi \in \mathcal{F}(\Gamma) \quad \text{with } \varphi \leq \psi.$$

$(A2)$ Stability principle:

$$E(E(\varphi, \Gamma)\mid_{\Gamma'}, \Gamma') = E(\varphi, \Gamma)\mid_{D(\Gamma')}$$

for any $\Gamma \in \mathcal{C}$, any $\varphi \in \mathcal{F}(\Gamma)$ and $\Gamma' \in \mathcal{C}$ such that $D(\Gamma') \subseteq D(\Gamma)$.

This principle means that no new application of the interpolation can improve a given interpolant. If this were not the case, we should iterate the interpolation operator indefinitely until a limit interpolant satisfying $(A2)$ is attained.

For the next principle, we denote by $SM(2)$ the set of symmetric two-dimensional matrices.

$(A3)$ Regularity principle: Let $A \in SM(2)$, $p \in \mathbb{R}^2 - \{0\}$, $c \in \mathbb{R}$ and

$$Q(y) = \frac{A(y-x, y-x)}{2} + \langle p, y-x \rangle + c$$

(where $\langle x, y \rangle = \sum_{i=1}^{2} x_i y_i$). Let $D(x,r) = \{y \in \mathbb{R}^2 : \|y - x\| \leq r\}$ and $\partial D(x,r)$ its boundary. Then

$$\frac{E(Q\mid_{\partial D(x,r)}, \partial D(x,r))(x) - Q(x)}{r^2/2} \to F(A,p,c,x) \quad \text{as } r \to 0+ \tag{1}$$

where $F : SM(2) \times \mathbb{R}^2 - \{0\} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function.
This assumption is much weaker than what it appears to be. Indeed, assume only that given A, p, c, x, we can find a C^2 function u such that $D^2u(x) = A$, $Du(x) = p$, $u(x) = c$, such that the differentiability assumption (1) holds (with u instead of Q). Then, arguing as in Theorem 1 in [9] it is easily proven that (1) holds for all C^2 functions and in particular for Q.

Together with these basic axioms, let us consider the following axioms which express obvious independence properties of the interpolation process with respect to the observer's standpoint and the grey level encoding scale.

$(A4)$ Translation invariance:

$$E(\tau_h\varphi, \Gamma - h) = \tau_h E(\varphi, \Gamma),$$

where $\tau_h\varphi(x) = \varphi(x+h)$, $h \in \mathbb{R}^2$, $\varphi \in \mathcal{F}(\Gamma)$, $\Gamma \in \mathcal{C}$. The interpolant of a translated image is the translated interpolant.

$(A5)$ Rotation invariance:

$$E(R\varphi, R\Gamma) = RE(\varphi, \Gamma),$$

where $R\varphi(x) = \varphi(R^t x)$, R being an orthogonal map in $\mathbb{R}^2$, $\varphi \in \mathcal{F}(\Gamma)$, $\Gamma \in \mathcal{C}$. The interpolant of a rotated image is the rotated interpolant.

$(A6)$ Grey scale shift invariance:

$$E(\varphi + c, \Gamma) = E(\varphi, \Gamma) + c$$

for any $\Gamma \in \mathcal{C}$, any $\varphi \in \mathcal{F}(\Gamma)$, $c \in \mathbb{R}$.

$(A7)$ Linear grey scale invariance:

$$E(\lambda\varphi, \Gamma) = \lambda E(\varphi, \Gamma) \quad \text{for any } \lambda \in \mathbb{R}.$$

$(A8)$ Zoom invariance:

$$E(\delta_\lambda\varphi, \lambda^{-1}\Gamma) = \delta_\lambda E(\varphi, \Gamma),$$

where $\delta_\lambda\varphi(x) = \varphi(\lambda x)$, $\lambda > 0$. The interpolant of a zoomed image is the zoomed interpolant.

Axioms $(A1)$, $(A3)$ and $(A4)$ to $(A8)$ are obvious adaptations from the axiomatic developed in [1]. As we shall always assume $(A4)$ and $(A6)$, F only depends on its first two arguments.
Let us write $G(A) = F(A, e_1)$, $A \in SM(2)$, $e_1 = (1,0)$. Then G is a continuous function of A. Given a matrix

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

let us write for simplicity $G(a,b,c)$ instead of $G(A)$. Then using an argument similar to the one developed in [1] we prove

Theorem 1. ([9]) *Assume that E is an interpolation operator satisfying $(A1)-(A8)$. Let $\varphi \in \mathcal{C}(\Gamma)$, $u = E(\varphi, \Gamma)$. Then u is a viscosity solution of*

$$G\left(D^2u\left(\frac{Du}{|Du|}, \frac{Du}{|Du|}\right), D^2u\left(\frac{Du}{|Du|}, \frac{Du^\perp}{|Du|}\right), D^2u\left(\frac{Du^\perp}{|Du|}, \frac{Du^\perp}{|Du|}\right)\right) = 0 \quad \text{in } D(\Gamma)$$
$$u_{|\Gamma} = \varphi. \tag{2}$$

In addition, $G(A)$ is a nondecreasing function of A satisfying $G(\lambda A) = \lambda G(A)$ for all $\lambda \in \mathbb{R}$.

We do not give the notion of the viscosity solution explicitly for the general model (2) and we refer to [9]. We shall specify this notion when needed for each concrete example below. From now on, we shall assume that the interpolation operator E satisfies $(A1)-(A8)$. Using the monotonicity of G, we can reduce the number of involved arguments inside G.

Proposition 1. *i) If G does not depend upon its first or its last argument, then it only depends on its last (resp. its first) argument. In other terms,*

$$\text{if} \quad G(\alpha, \beta, \gamma) = \hat{G}(\alpha, \beta), \quad \text{then} \quad G = \hat{G}(\alpha) = \alpha\hat{G}(1),$$

$$\text{if} \quad G(\alpha, \beta, \gamma) = \hat{G}(\beta, \gamma), \quad \text{then} \quad G = \hat{G}(\gamma) = \gamma\hat{G}(1).$$

$\alpha, \beta, \gamma \in \mathbb{R}$.
ii) If G is differentiable at 0 then G may be written as $G(A) = Tr(BA)$ where B is a nonnegative matrix.

Proposition 1 i) is due to the fact that $A \to A(\nu, \nu^\perp)$ ($\nu \in R^2$, $|\nu| = 1$, $\nu^\perp$ being the vector obtained by rotation of $\pi/2$ of ν) is not a monotone operator with respect to A.
Thus if we assume that G is differentiable at $(0,0,0)$ then we may write Equation (2) as

$$aD^2u\left(\frac{Du}{|Du|}, \frac{Du}{|Du|}\right) + 2bD^2u\left(\frac{Du}{|Du|}, \frac{Du^\perp}{|Du|}\right) + cD^2u\left(\frac{Du^\perp}{|Du|}, \frac{Du^\perp}{|Du|}\right) = 0, \tag{3}$$

where $a, c \geq 0$, $ac - b^2 \geq 0$ which is the same as to say that the symmetric matrix B with diagonal coefficients a and c and cross term b is nonnegative.

3. Examples

Example 1. Given $\Gamma \in \mathcal{C}$ and $\varphi \in \mathcal{F}(\Gamma)$ we consider $E_1(\varphi, \Gamma)$ to be the solution of

$$\Delta u = 0 \quad \text{in } D(\Gamma)$$
$$u_{|\Gamma} = \varphi. \tag{4}$$

The operator E_1 satisfies all axioms $(A1)-(A8)$ above. Just mention that the regularity axiom follows from the mean value theorem for the Laplace equation on a disk. It corresponds to the function $G(A) = -Tr(A)$, that is $G(a, b, c) = -(a + c)$.

We recall that this operator does not permit to interpolate points (see introduction). A more general situation is given by the so-called p-Laplacian

$$\begin{aligned} &div(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } D(\Gamma) \\ &u_{|\Gamma} = \varphi, \end{aligned} \tag{5}$$

where $p \geq 1$. Formally, after dividing by $|\nabla u|^{p-2}$, the above equation can be written as

$$(p-1)D^2u\left(\frac{Du}{|Du|}, \frac{Du}{|Du|}\right) + D^2u\left(\frac{Du^\perp}{|Du|}, \frac{Du^\perp}{|Du|}\right) = 0 \tag{6}$$

which is contained in the family of equations (3) with $a = p-1$, $b = 0$, $c = 1$. As it is known, the value of u can be fixed at point if and only if $p > 2$. This corresponds to the case $a > c$. By specifying radial solution in a ball, it can be seen that, unless $c = 0$, which corresponds to the case $p = \infty$, the gradient of u can be unbounded. The case of $p = \infty$ will be our next example.

Example 2. Our next example is more interesting and will be discussed in detail. Given a domain Ω with $\partial\Omega \in C$ and $\varphi \in \mathcal{F}(\partial\Omega)$ we consider $E_2(\varphi, \partial\Omega)$ to be the viscosity solution of

$$\begin{aligned} &D^2u\left(\frac{Du}{|Du|}, \frac{Du}{|Du|}\right) = 0 \quad \text{in } \Omega, \\ &u_{|\partial\Omega} = \varphi. \end{aligned} \tag{7}$$

We consider Equation (7) in the viscosity sense. Given $u \in C(\Omega)$ we say that u is a viscosity subsolution (supersolution) of (7) if for any $\psi \in C^2(\Omega)$ and any x_0 local maximum (minimum) of $u - \psi$ in Ω such that $D\psi(x_0) \neq 0$

$$D^2\psi(x_0)\left(\frac{D\psi(x_0)}{|D\psi(x_0)|}, \frac{D\psi(x_0)}{|D\psi(x_0)|}\right) \geq 0 \quad (\leq 0).$$

A viscosity solution is a function which is a viscosity sub- and supersolution.

Equation (7) was introduced by G. Aronsson in [2] and recently it has been studied by R. Jensen [16]. Indeed, in [2] the author considered the following problem: Given a domain Ω in $\mathbb{R}^n$, does a Lipschitz function u in Ω exist such that

$$\sup_{x\in\tilde{\Omega}} |Du(x)| \leq \sup_{x\in\tilde{\Omega}} |Dw(x)|,$$

for all $\tilde{\Omega} \subseteq \Omega$ and w such that $u - w$ is Lipschitz in $\tilde{\Omega}$ and $u = w$ on $\partial\tilde{\Omega}$. If it exists, such a function will be called an absolutely minimizing Lipschitz interpolant of $w|_{\partial\Omega}$ inside Ω or $AMLE$ for short. Notice that the above definition, if it defines uniquely u, immediately implies the stability of AMLE in the sense of $(A2)$. Then it was proved in [2] that if u is an $AMLE$ and is C^2 in Ω, then u is a classical solution of

$$D^2u(Du, Du) = 0 \quad \text{in } \Omega. \tag{8}$$

Later Jensen [16] proved that if u is an $AMLE$, then u solves (8) in the viscosity sense. Moreover, the viscosity solution is unique. We shall use the viscosity solution formulation of Equation (8). Given $u \in C(\Omega)$ we say that u is a viscosity subsolution (supersolution) of (8) if for any $\psi \in C^2(\Omega)$ and any x_0 local maximum (minimum) of $u - \psi$ in Ω

$$D^2\psi(x_0)(D\psi(x_0), D\psi(x_0)) \geq 0 \quad (\leq 0).$$

A viscosity solution is a function which is a viscosity sub- and supersolution. Then Jensen proved [16] a comparison principle between sub- and supersolutions of Equation (8) together with an existence result for boundary data in the space of functions $Lip_\partial(\Omega)$ which are Lipschitz continuous with respect to the distance $d_\Omega(x, y)$. We denote by $d_\Omega(x, y)$ the geodesic distance between x and y, i.e., the minimal length of all possible paths joining x and y and contained in Ω [16]. Observe that u is a viscosity subsolution (supersolution, solution) of (7) if and only if u is a viscosity subsolution (supersolution, solution) of (8). From this follows the corresponding comparison principle for solutions of (7).

Theorem 2. *Assume that v is a subsolution and w a supersolution of* (7) *(equivalently of* (8)*). If $v\,|_{\partial\Omega}$, $w\,|_{\partial\Omega} \in Lip_\partial(\Omega)$ then*

$$\sup_{x\in\Omega}(v - w) = \sup_{x\in\partial\Omega}(v - w). \tag{9}$$

Theorem 3. *Given $g \in Lip_\partial(\Omega)$, u is the AMLE of g into Ω if and only if u is the solution of* (8) *with $u\,|_{\partial\Omega} = g$.*

Let us state R. Jensen's existence result for (7) in a way that makes explicit the fact that we are able to interpolate a datum which is given on a set of curves and points. Let us consider a domain Ω whose boundary $\partial\Omega = \partial_1\Omega \cup \partial_2\Omega \cup \partial_3\Omega$ where $\partial_1\Omega$ is a finite union of rectifiable simple Jordan curves,

$$\partial_2\Omega = \cup_{i=1}^m C_i,$$

where C_i are rectifiable curves homeomorphic to a closed interval and

$$\partial_3\Omega = \{x_i : i = 1, ..., N\}$$

is a finite number of points. The boundary data to be interpolated is given by a Lipschitz function φ_1 on $\partial_1\Omega$, two Lipschitz functions $\varphi_{2+}^i, \varphi_{2-}^i$ on each curve C_i, which coincide on the extreme points of C_i, $i = 1, ..., m$ and a constant value u_i on each point x_i, $i = 1, ..., N$. We shall denote by C_i^+, C_i^- the same curve C_i where we take into account the direction of the normal $\nu_i^+(x)$, $\nu_i^-(x) = -\nu_i^+(x)$, $x \in C_i$ as in Figure 1. When we write $u_{|C_i^+} = \varphi_{2+}^i$ as in the next theorem we mean that $u(y) \to \varphi_{2+}^i(x)$ as $y \to x$ if $< y, \nu_i^+(x) >< 0$, $x \in C_i$ and similarly for $u_{|C_i^-} = \varphi_{2-}^i$.

Theorem 4. *Given Ω, φ_1, $\varphi_{2+}^i, \varphi_{2-}^i$, u_j, $i = 1, ..., m$, $j = 1, ..., N$, as above then there exists a unique Lipschitz viscosity solution u of*

$$
\begin{aligned}
&D^2u\left(\frac{Du}{|Du|},\frac{Du}{|Du|}\right) = 0 \quad \text{in } \Omega \\
&u_{|\partial_1\Omega} = \varphi_1 \\
&u_{|C_i^+} = \varphi_{2+}^i \\
&u_{|C_i^-} = \varphi_{2-}^i, \qquad i = 1, \dots, m \\
&u(x_i) = u_i, \qquad i = 1, \dots, N.
\end{aligned} \tag{10}
$$

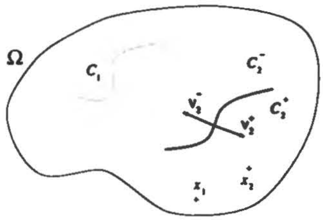

FIGURE 1. General domain of interpolation.

This result enables us to define the following interpolation operator. Given $\varphi \in Lip_\partial(\Omega)$, let $E_2(\varphi, \partial\Omega)$ be the viscosity solution of (7). Then

Theorem 5. *The operator E_2 defined above satisfies axioms $(A1) - (A8)$.*

For numerical reasons it is interesting to study the asymptotic behavior of the evolution problem corresponding to Equation (7). Then, under certain smoothness assumptions on the boundary data, we prove that the solution of the evolution problem converges to the solution of Equation (7). Let us consider the evolution equation

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = D^2u\left(\frac{Du}{|Du|},\frac{Du}{|Du|}\right) \quad \text{in } (0,+\infty)\times\Omega \\
&u(0,x) = u_0(x) \qquad x \in \Omega \\
&u(t,x) = \varphi(x) \qquad (t,x) \in (0,+\infty)\times\partial\Omega,
\end{aligned} \tag{11}
$$

where we suppose that $u_0(x) = \varphi(x)$ for all $x \in \partial\Omega$. We say that $u \in C([0,+\infty)\times\Omega)$ is a viscosity subsolution of (11) if $u(0,x) = u_0(x)$, $u(t,x) = \varphi(x)$ for all $(t,x) \in (0,+\infty)\times\partial\Omega$ and for any $\psi \in C^2((0,+\infty)\times\Omega)$ and any (t_0,x_0) local maximum of $u - \psi$ in $(0,+\infty)\times\Omega$

$$
\frac{\partial\psi}{\partial t}(t_0,x_0) \le D^2\psi(t_0,x_0)\left(\frac{D\psi(t_0,x_0)}{|D\psi(t_0,x_0)|},\frac{D\psi(t_0,x_0)}{|D\psi(t_0,x_0)|}\right), \tag{12}
$$

if $D\psi(t_0,x_0) \neq 0$ and

$$
\frac{\partial\psi}{\partial t}(t_0,x_0) \le sup_{|v|\le1} D^2\psi(t_0,x_0)(v,v),
$$

if $D\psi(t_0,x_0) = 0$. Similarly we define a viscosity supersolution. A viscosity solution is a viscosity sub- and supersolution.

Theorem 6. *Let Ω be a generalized domain in $\mathbb{R}^2$ as described above. Suppose that $\partial_1\Omega$, C_i^+, C_i^-, $i = 1, ..., m$, have a bounded curvature; the initial condition $u_0(x)$ and the boundary data $\varphi(x)$ have bounded second derivatives. Then there exists a unique continuous viscosity solution $u(t,x)$ of* (11) *such that $u(t)$ is Lipschitz for all $t > 0$ with uniformly bounded Lipschitz norm. Moreover, $u(t,.) \to u_\infty$, where u_∞ is the unique viscosity solution of* (10).

Geometric interpretation of Equation (8)

Proposition 2. *Let u be C^2 and satisfy $D^2u(Du, Du) = 0$. We define a gradient line as a curve $x(t)$, $t \in (a, b)$ such that*

$$Du(x(t)) \neq 0 \quad \text{and} \quad x'(t) = \frac{Du}{|Du|}(x(t)).$$

Then there is a constant C such that for every $t \in (a,b)$

$$|Du|(x(t)) = C.$$

Corollary 1. *Viscosity solutions of (7) are not necessarily C^2.*

For example, let u be defined on the boundary of the square $[0,1]^2$ by $u(0,1) = 1 = u(1,0)$, $u(0,0) = 0 = u(1,1)$ and u is affine on each side of the square. Let u_0 the AMLE of u. Assume by contradiction that u_0 is C^2. By symmetry of the datum and uniqueness, $u_0\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{2}$. Thus, there is some point y on the segment L joining $(0,1)$ and $(1,0)$ such that $Du_0(y) \neq 0$, and therefore a neighborhood of y in which $Du_0 \neq 0$. By symmetry again, $Du_0(y)$ is parallel to L. Thus, a segment of L containing y is a gradient line. By Proposition 2, on this segment we have $Du_0(z) = Du_0(y)$. So we conclude that the maximal segment is the whole line L. This is a contradiction with $u(0,1) = u(1,0)$. □

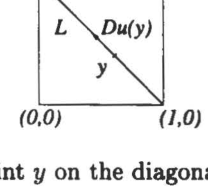

FIGURE 2. A point y on the diagonal with $Du_0(y) \neq 0$.

Example 3. Our next example is concerned with the curvature operator. Let Ω be a domain in $\mathbb{R}^2$ with Lipschitz boundary and φ be a Lipschitz continuous function on $\partial\Omega$. We consider the equation

$$\begin{aligned} &D^2u\left(\frac{Du^\perp}{|Du|}, \frac{Du^\perp}{|Du|}\right) = 0 \quad \text{in } \Omega, \\ &u_{|\partial\Omega} = \varphi. \end{aligned} \tag{13}$$

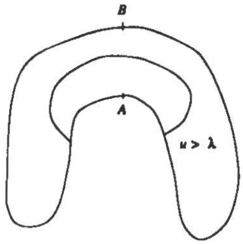

FIGURE 3. No viscosity solutions of (13) exist in this domain.

We consider solutions of (13) in the viscosity sense, which can be defined in the same way as solutions of (7) in Example 2. This model cannot be used as a model for interpolating data because of the following facts
a) There is no uniqueness of viscosity solutions of (13).
b) There are no viscosity solutions of (13) for general smooth curves $\partial\Omega$ and boundary data $\varphi \in \mathcal{F}(\partial\Omega)$. This is easily deduced from techniques developed in [7].

Indeed, let $\Omega = D(0,1)$, $\varphi(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2$, $\lambda_1 > \lambda_2$. Since on $\partial D(0,1)$, $x_1^2 + x_2^2 = 1$ and

$$\varphi(x_1, x_2) = \varphi(-x_1, x_2) = \varphi(x_1, -x_2),$$

the functions

$$u_1(x_1, x_2) = \varphi\left(\sqrt{1 - x_2^2}, x_2\right),$$

$$u_2(x_1, x_2) = \varphi\left(x_1, \sqrt{1 - x_1^2}\right)$$

are two viscosity solutions of (13) in $D(0,1)$ with the same boundary data.

Theorem 7. *There are no viscosity solutions of (13) for general smooth curves $\partial\Omega$ and boundary data $\varphi \in \mathcal{F}(\partial\Omega)$.*

The proof is based on simple heuristic arguments. A solution u of (13) is also a static solution of the evolution problem

$$\begin{aligned} &\frac{\partial v}{\partial t} = D^2 v\left(\frac{Dv^\perp}{|Dv|}, \frac{Dv^\perp}{|Dv|}\right) \quad \text{in } (0, +\infty) \times \Omega \\ &v(0, x) = u(x) \quad \text{in } \Omega \\ &v(t, x) = \varphi(x) \quad (t, x) \in (0, +\infty) \times \partial\Omega, \end{aligned} \tag{14}$$

which means that all level lines of u are moving by mean curvature. This is impossible unless the level lines of u are straight lines. In general, this is not possible as can be seen in Figure 3. Figure 3 depicts a nonconvex smooth domain with boundary data φ such that φ increases when we go from A to B along the boundary in the clockwise direction and then decreases symmetrically when going from B to A.

4. A Generalization of the AMLE in the plane

The results proved by Jensen assume that the boundary data are at least continuous. This assumption is obviously not satisfied when we deal with real images. Indeed objects in a scene are occluded by other ones and these occlusions create discontinuities (edges). What does the AMLE model become when trying to extend singularities? Can edges be preserved? Heuristically, the AMLE model is elliptic and thus regularizing and edges should be lost. Thanks to the generalization below, this pessimistic statement will be confirmed: The AMLE is a bad edges interpolation operator. For this task some interpolation operators are being studied [21]. On the other hand, the AMLE is very efficient for interpolation between level lines, as shown in the numerical experiments. We start with the following heuristic argument. Let Ω be the unbounded domain of $\mathbb{R}^2$ between two semi-lines crossing at the origin. We set φ constant on each semi-line. There exists a unique affine function of the argument interpolating φ in Ω. We call it interpolating spiral. It is also easy to check that the spiral is a twice differentiable *AMLE* in Ω. Hence, it is reasonable to look for a solution with the same behavior in the neighborhood of each singularity of the boundary value. More precisely, we can prove

Proposition 3. *The spiral is the unique bounded interpolant between two lines.*

Let us now give the generalization. Let Ω be a bounded domain in $\mathbb{R}^2$ and φ be defined on $\partial\Omega$ with possibly a finite set $Z \subset \partial\Omega$ of discontinuity points. We assume that out of Z, φ is uniformly Lipschitz continuous and that if $z \in Z$, then $\partial\Omega$ admits at z two semi-tangential directions. Note that φ admits a limit on both sides of any point of Z. For $z \in Z$, we shall also denote by a_z the interpolating spiral between the two tangent directions at z.

Definition 2. *We say that $u \in USC(\overline{\Omega}\backslash Z)$ (resp. $LSC(\overline{\Omega}\backslash Z)$) is viscosity subsolution (resp. supersolution) of $D^2u(Du, Du) = 0$ if and only if*

1. *for all $x \in \Omega$ and $\psi \in C^2(\Omega)$, if $u - \psi$ attains a local maximum (resp. minimum) at x then $D^2\psi(D\psi, D\psi) \geq 0$ (resp. ≤ 0) .*
2. *Let $z \in Z$ and $(x_n)_{n\in\mathbb{N}}$ such that $\lim x_n = z$ and $\lim \frac{x_n - z}{|x_n - z|} = \tau \in \mathbb{S}^1$. Then $\limsup u(x_n) \leq a_z(\tau)$ (resp. $\geq a_z(\tau)$).*

We say that u is solution if and only if it is both sub and supersolution.

We have the following generalized maximum principle.

Proposition 4 (Maximum Principle). *Let u and v bounded sub and supersolutions of (8). Assume that on $\partial\Omega\backslash Z$, $u \leq v$. Then in Ω, we also have $u \leq v$.*

By constructing approximated solutions, we can prove the following results.

Theorem 8. *Let φ be defined on $\partial\Omega$ as above. Then there exists a unique solution u of (8) such that $u = \varphi$ on $\partial\Omega\backslash Z$.*

Consequences of this result will be clear in numerical experiments.

5. Numerical analysis of the AMLE model

We shall use the AMLE model studied above as the basic equation to interpolate data given on a set of curves and/or a set of points which may be irregularly sampled. Thus, we discretize the equation

$$D^2u\left(\frac{Du}{|Du|},\frac{Du}{|Du|}\right)=0.$$

It is easy to see that there is a relation between iterative methods for the solution of elliptic problems and time stepping finite difference methods for the solution of the corresponding parabolic problems. Because of that and thanks to Theorem 6, we study the equation

$$\frac{\partial u}{\partial t}=D^2u\left(\frac{Du}{|Du|},\frac{Du}{|Du|}\right),$$

with corresponding initial and boundary data as in (11). Using an implicit Euler scheme we transform this evolution problem into a sequence of nonlinear elliptic problems. Thus, we may write the following implicit difference scheme in the image grid

$$u_{i,j}^{(n+1)}=u_{i,j}^{(n)}+\Delta t D^2u_{i,j}^{(n+1)}\left(\frac{Du_{i,j}^{(n+1)}}{|Du_{i,j}^{(n+1)}|},\frac{Du_{i,j}^{(n+1)}}{|Du_{i,j}^{(n+1)}|}\right) \tag{15}$$

$i,j=1,\ldots,N$. To solve the above nonlinear system we use a nonlinear over-relaxation method (NLOR). Writing the system as a set of $k=N^2$ algebraic equations, one for each unknown $u_{i,j}^{(n+1)}$ $(i,j=1,...,N)$,

$$f_p(x_1,x_2,\ldots,x_k)=0,\quad p=1,2,\ldots,k,$$

the basic idea of NLOR is to introduce a relaxation factor ω and iteratively compute

$$x_i^{(n+1)}=x_i^{(n)}-\omega\frac{f_i(x_1^{(n+1)},\ldots,x_{i-1}^{(n+1)},x_i^{(n)},\ldots,x_k^{(n)})}{f_{ii}(x_1^{(n+1)},\ldots,x_{i-1}^{(n+1)},x_i^{(n)},\ldots,x_k^{(n)})},\qquad i=1,2,\ldots,k$$

where $f_{ii}=\frac{\partial f_i}{\partial x_i}$. The convergence criterion can be shown to be the same as the over-relaxation method for linear systems, replacing the matrix by the Jacobian of the equations $f_p=0$, and stability is guaranteed for values of the relaxation parameter $0<\omega<2$.

6. Experimental Results

We display some experiments using the numerical scheme described in the previous section.

In Figure 4, the domain is a square and the boundary data we interpolate are equal to 255 on the vertical sides and 1 on horizontal sides. On the left, the numerical solution is presented. On the right we display a detailed view of the level lines near the upper-left corner. The spiral structure is clear. Figure 5 shows the behavior of the AMLE on edges. On the left Lena image, some domains are created. The middle image is the

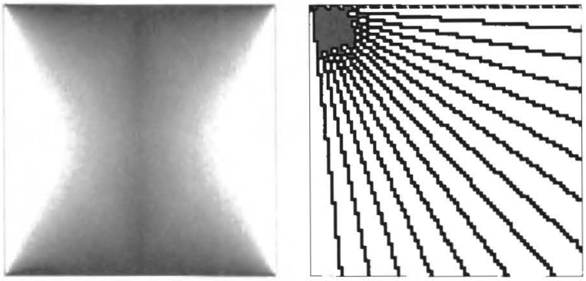

FIGURE 4. Discontinuous boundary data on a square.

FIGURE 5. Discontinuous boundary data.

numerical solution. We see that edges are poorly preserved. On the right image, we show level lines of the middle image. As foreseen, the spiral structure is visible.

Figure 6 shows how we can interpolate an image from the quantized level curves, obtaining a better result than the corresponding quantized image. Figure 6a displays the original image u which takes integer values between 0 and 255. Then we quantize it by giving the grey levels between $r\delta \leq u < (r+1)\delta$ the value $r\delta$, $r = 0, ..., M$, $M = [255/\delta]$. Figures 6c and 6e display the result of this operation on Figure 6a for values $\delta = 20$ and $\delta = 30$. Figure 6b displays the boundaries of the level sets $[u \geq r\delta]$ at the corresponding grey level $r\delta$ (here we have displayed the level sets for $\delta = 30$). We define the boundary values on the pixels belonging to the boundaries of the level sets B and the neighbouring pixels belonging to the boundary of the complement B'. For each pixel (i,j) we define $m(i,j) = inf\{r : u(i,j) \geq r\delta\}$, $M(i,j) = sup\{r : u(i,j) \geq r\delta\}$. Then we set $u(i,j) = m(i,j)\delta$ if $(i,j) \in B'$, $u(i,j) = M(i,j)\delta$ if $(i,j) \in B$, and we solve Equation (15) with these boundary data. The results are displayed in Figures 6d and 6f. In practice, the interpolation must keep smooth the regular regions of the image. So if we quantize the image at levels multiple of 30 (e.g.), the jump across the level line after quantization is either 0 or $30, 60$, etc. The behaviour of the algorithm is the following: if the jump $M(i,j) - m(i,j)$ is just 0, it is likely that the region around is smoothly perceptual, so our interpolation maintains it by giving a Lipschitz

Figure 6a

Figure 6b

Figure 6c

Figure 6d

Figure 6e

Figure 6f

FIGURE 6. Interpolation between level lines.

interpolation. If the jump across the level line is larger (e.g.) than 20, 30, etc., our decision is to maintain the jump because we consider that there must be an edge here. Since a jump larger than 20 is perceptible as edge, we maintain the existing edge by this choice, without significant attenuation or enhancement.

Acknowledgment: We acknowledge partial support by EC project MMIP, reference ERBCHRXCT 930095. The first and third authors were also partially supported by DGICYT project, reference PB94-1174. We would like to thank Pierre-Louis Lions for pointing out to us the work of R. Jensen on the AMLE model.

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A Class of Strong Resonant Problems via Lyapunov-Schmidt Reduction Method

J. Carmona

Abstract. We consider the problem of multiplicity of solutions for a class of nonlinear P.D.E. on a bounded domain $\Omega \subset \mathbb{R}^N$, with sufficiently smooth boundary. More precisely, I will be concerned with the following strong resonant problem, with Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = \lambda_k u + f(x,u) - g(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

The well-known Lyapunov-Schmidt reduction method and the ideas used by Amann-Ambrosetti-Mancini (Math. Z. 1978) to describe the range of a function defined in a finite dimensional space will be extended when the decay of f at infinity is "like" $|u|^\alpha u$. Finally inspired by a paper of Pellacci and Villegas [14] we try to extend our result to an equation on $\mathbb{R}^N$ and we obtain an existence result for the Schröedinger equation

$$-\Delta u + p(x)u = \mu_k u + f(x,u) - g(x), \; x \in \mathbb{R}^N, \; u \in H^1(\mathbb{R}^N).$$

Keywords: strong resonance, elliptic equations, degree theory

Classification: Primary 35J25, Secondary 35J60

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary, $\partial\Omega$. We study the existence of solution for the resonant boundary value problem

$$\begin{cases} -\Delta u = \lambda_k u + f(x,u) - g(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where f is a bounded Carathéodory function such that $f(x,\cdot) \in C^1$ a.e. $x \in \Omega$, $g \in L^2(\Omega)$ and $\{\lambda_n\}_{n\geq 1}$ is denoting the sequence of eigenvalues of the Laplacian operator with zero Dirichlet boundary condition. By solution of problem (1) we mean $u \in E = H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \nabla \varphi - \lambda_k \int_\Omega u\varphi - \int_\Omega f(x,u)\varphi + \int_\Omega g\varphi, \; \forall \varphi \in E.$$

From the work by Landesman and Lazer [12] for the case $\lambda_k = \lambda_1$, the first eigenvalue, the problem (1) has been extensively studied (see [1], [2], [3], [6], [10], [13] among others).

In particular, in [3] the authors proved that if f satisfies

(f_1) there exist constants $C_1, C_2 \in I\!R$, such that

$$\lambda_{k-1} < C_1 \leq \lambda_k + f_s'(x,s) \leq C_2 < \lambda_{k+p}$$

where p is the dimension of the eigenspace V_k associated to λ_k, and for $k=1$, $\lambda_{k-1} = \lambda_0 = -\infty$,

(f_2) there exists $f(\pm\infty) := \lim_{s\to\pm\infty} f(x,s)$, uniformly in x,

then (1) has at least one solution provided either that

$$\int_\Omega gz < \int_\Omega f(+\infty)z^+ + f(-\infty)z^-, \ \forall z \in V_k, \ \|z\|_{L^2} = 1,$$

or that

$$\int_\Omega gz > \int_\Omega f(+\infty)z^+ + f(-\infty)z^-, \ \forall z \in V_k, \ \|z\|_{L^2} = 1,$$

(where $z = z^+ + z^-, z^+ = \max\{z, 0\}$).

In this way, they extended the well-known Landesman-Lazer condition to the case of higher eigenvalues. Even more, the authors studied more subtle cases of "strong" resonance, namely $f(\pm\infty) = 0$. In fact they give sufficient conditions for g to obtain one or more solutions of (1) in terms of $\lim_{s\to\pm\infty} f(x,s)s$.

Other different "degrees of resonance" at infinity are distinguished by means of variational tools in [6]. Thus, the degrees of resonance are measured according to the behaviour of f and a primitive of it at infinity.

The scope of this note is to improve the previous results by handling more general cases of resonant terms $f(x,s)$. To be specific, if we denote by $L^1(V_k, p)$ to the set of measurable functions $b : \Omega \to I\!R$ such that

$$\int_{\Omega\setminus\Omega_z} |b(x)||z(x)|^p < +\infty, \ \forall z \in V_k,$$

where $\Omega_z = \{x \in \Omega, \ z(x) = 0\}$, then

Theorem A. *Suppose that hypotheses $(f_{1,2})$ are satisfied, and that there exists $-1 < \alpha < 1$ such that*

$$|f(x,s)||s|^{\alpha+1} \leq b(x), \tag{2}$$

$$\lim_{s\to\pm\infty} |s|^\alpha s f(x,s) = \mu(x), \ \textit{with} \int_{\Omega\setminus\Omega_z} \mu(x)|z(x)|^{-\alpha} > 0, \ \forall z \in V_k, \tag{3}$$

where $b(x), \mu(x) \in L^1(V_k, -\alpha)$, then

1. There exists ε such that equation (1) has solution if $\|P\hat{g}\| \leq \varepsilon$.

2. If $0 < \|P\hat{g}\| < \varepsilon$ then it has at least two solutions.

We point out that the degree of resonance is measured here by the behaviour at infinity of the nonlinearity $f(x,s)$. No assumptions are needed on the primitive of f. Condition (3) has also been used by [4] to study sufficient and necessary conditions for subcritical (to the left) or supercritical (to the right) bifurcations at infinity and to deduce complementary existence results.

We also study the extension of the previous results for bounded domains to the case $\Omega = \mathbb{R}^N$. In this case we improve the results in [14].

2. Proof of Theorem A

In our approach we use the well-known Lyapunov-Schmidt method to reduce the problem to a system of two equations: one of them is infinite dimensional and the other is finite dimensional. The first one is solved by means of a Global Inversion Theorem (see [3]). Hence, this allow us to reduce the problem (1) to a finite dimensional problem. More precisely, let $L, F : E \to E$ be the operators defined by

$$(Lu, \varphi) = -\int_\Omega \nabla u \nabla \varphi + \lambda_k \int_\Omega u\varphi, \ \forall \varphi \in E,$$
$$(Fu, \varphi) = \int_\Omega f(x,u)\varphi, \forall \varphi \in E,$$

(where $(\varphi, \phi) = \int_\Omega \nabla\varphi\nabla\phi$ is the inner product in E), and let $\hat{g} \in E$ be given by

$$(\hat{g}, \varphi) = \int_\Omega g\varphi, \ \forall \varphi \in E.$$

Then problem (1) can be written as

$$L(u) + F(u) = \hat{g}. \tag{4}$$

Considering $V_k^\perp$ the orthogonal subspace in $L^2(\Omega)$ to the eigenspace V_k and $W_k = V_k^\perp \cap E$, we can split $E = V_k \oplus W_k$ and for every $u \in E$, $u = v + w$ with $v \in V_k$, $w \in W_k$. Also let P and Q be the projectors defined in $L^2(\Omega)$ over V_k and W_k, respectively. Problem (4) is now equivalent to

$$L(w) + QF(v+w) = Q\hat{g}, \tag{5}$$
$$PF(v+w) = P\hat{g}. \tag{6}$$

The following lemma is contained in [3].

Lemma 2.1. *Let (f_1) be satisfied, then for every $g \in E$, (5) has one solution $w_g(v) \in W$, for every v in V_k. Moreover $w_g(v)$ is C^1 and is uniformly bounded on V_k.*

The previous result allows to reduce the system (5)-(6) to the study of $v \in V_k$ such that

$$PF(v + w_g(v)) = P\hat{g}.$$

With this end, we denote $\Gamma_g : V_k \to V_k$ the operator defined by

$$\Gamma_g(v) = PF(v + w_g(v)), \ v \in V_k.$$

As a consequence of Lemma 2.1, $\Gamma_g \in C^1$ and bounded on V_k.

Proof of Theorem A. We claim that there exists $r > 0$ such that

$$(v, \Gamma_g(v)) > 0, \ \forall v \in V_k, \ \|v\|_2 = r. \tag{7}$$

In another case there exists a sequence $v_n \in V_k$, tending strongly to infinity, such that

$$(v_n, \Gamma_g(v_n)) = \int_\Omega Pf(x, v_n + w_n)v_n = \int_\Omega f(x, v_n + w_n)v_n \leq 0. \tag{8}$$

Let t_n denote the norm of v_n and $z_n \in V_k$ be the normalized sequence of v_n; then since V_k is finite dimensional, there exists $z \in V_k \setminus \{0\}$ such that, up to a subsequence, $z_n \to z$ in every norm in V_k, in particular in $C^1(\overline{\Omega}) \cap V_k$.

By the unique continuation property (see [9]), $|\Omega_v| = \text{meas}\, \{x \in \Omega,\ v(x) = 0\} = 0$ for every $v \in V_k$, and we have

$$|v_n(x) + w_n(x)| = |t_n z_n(x) + w_n(x)| \to +\infty,\ \forall x \in \Omega \setminus \Omega_z,$$

hence, denoting $\Omega_1 = \{x \in \Omega,\ z(x)v_n(x) = 0\}$ we deduce from (8) that

$$\begin{aligned} 0 &\geq \int_\Omega f(x, v_n + w_n)t_n^\alpha v_n = \int_{\Omega\setminus\Omega_1} f(x, v_n + w_n)t_n^\alpha v_n \\ &= \int_{\Omega\setminus\Omega_1} f(x, v_n + w_n)|t_n z|^\alpha v_n |z|^{-\alpha}. \end{aligned} \tag{9}$$

In order to calculate the limit of the last integral above, we fix $x \in \Omega \setminus \Omega_1$ and note that for n large enough,

$$\frac{|t_n z(x)|^\alpha v_n(x)}{|v_n(x) + w_n(x)|^\alpha (v_n(x) + w_n(x))} = \frac{|z(x)|^\alpha z_n(x)}{|z_n(x) + \frac{w_n(x)}{t_n}|^\alpha (z_n(x) + \frac{w_n(x)}{t_n})} \to 1$$

so using (3), the sequence

$$\begin{gathered} f(x, v_n(x) + w_n(x))|t_n z(x)|^\alpha v_n(x)|z(x)|^{-\alpha} = \\ f(x, v_n(x) + w_n(x))|v_n(x) + w_n(x)|^\alpha (v_n(x) + w_n(x)) \\ \frac{|t_n z(x)|^\alpha v_n(x)|z(x)|^{-\alpha}}{|v_n(x) + w_n(x)|^\alpha (v_n(x) + w_n(x))} \to \mu(x)|z(x)|^{-\alpha}. \end{gathered}$$

On the other hand, using (2),

$$\begin{aligned} |f(x, v_n + w_n)t_n^\alpha v_n| &\leq b(x)\frac{|z_n|}{|z_n + \frac{w_n}{t_n}|^{\alpha+1}} = \\ b(x)|z_n|^{-\alpha}\frac{1}{|1 + \frac{w_n}{v_n}|^{\alpha+1}} &\leq c_\alpha b(x)|z(x)|^{-\alpha}, \end{aligned}$$

where c_α is a constant. The last estimate is obtained using the fact that $\frac{w_n}{v_n} \to 0$ uniformly in Ω and that for large n

$$\frac{1}{2}|z| \leq |z_n| \leq 2|z| \text{ in } \Omega \setminus \Omega_1 .$$

Hence, taking limit in (9) we have by Dominated Convergence Theorem,

$$0 \geq \int_{\Omega\setminus\Omega_1} f(x, v_n + w_n)|t_n z|^\alpha v_n |z|^{-\alpha} \to \int_{\Omega\setminus\Omega_1} \mu(x)|z|^{-\alpha} > 0,$$

which is a contradiction, proving (7).

Now we follow the ideas of [3]. Indeed (7) means that

$$d(\Gamma_g, B_r(0), 0) = 1,$$

so it is sufficient to choose $\varepsilon =$ dist $(0, \Gamma_g(\partial B_r(0))) > 0$ and to use that $d(\Gamma_g, \Omega, \cdot)$ is constant on $B_\varepsilon(0)$, to obtain the first assertion.

Let now δ be $0 < \delta < \|P\hat{g}\| < \varepsilon$, taking into account that

$$\lim_{\|v\|\to\infty} \Gamma_g(v) = 0,$$

then there exists $r' > r$ such that $\Gamma_g(\partial B_{r'}) \subset B_\delta$. Moreover, since $\Gamma_g(V_k)$ is bounded, it is sufficient to choose $\bar{v} \in V_k \setminus \Gamma_g(V_k)$ to have $d(\Gamma_g, B_{r'}, \bar{v}) = 0$ and

$$d(\Gamma_g, B_{r'}, P\hat{g}) = d(\Gamma_g, B_{r'}, \bar{v}) = 0,$$

because $\delta < \|P\hat{g}\|$ and degree is constant in every connected component of $V_k \setminus \Gamma_g(\partial B_{r'}(0))$. Then $d(\Gamma_g, B_{r'} \setminus B_r, P\hat{g}) = d(\Gamma_g, B_{r'}, P\hat{g}) - d(\Gamma_g, B_r, 0) = -1 \neq 0$ and so the equation has at least other solution in $B_{r'} \setminus B_r$. □

Remark 2.2. *If λ_k is a simple eigenvalue then the conclusion of the previous theorem is that there exist $\underline{a}_g, \bar{a}_g \in \mathbb{R} \setminus \{0\}$, such that*

i. (1) has solution if and only if $\int_\Omega g\varphi_k \in \left[\underline{a}_g, \bar{a}_g\right]$.

ii. (1) has two nontrivial solutions if

$$\int_\Omega g\varphi_k \in \left[\underline{a}_g, \bar{a}_g\right] \setminus \left\{0, \underline{a}_g, \bar{a}_g\right\}.$$

In Theorem A the space $L^1(V_k, -\alpha)$ appears, and it will be useful to know if $L^r(\Omega) \subset L^1(V_k, -\alpha)$ for some r. In order to study this problem we recall (see [8]) that

Theorem 2.3. *(i) If $\partial\Omega \in C^2$ then $\varphi_1(x) \geq ad(x)$ for some positive constant a, where $d(x)$ is denoting the distance from x to the boundary of Ω.*
(ii) If $\partial\Omega$ satisfies an interior cone condition then there exists $\sigma \geq 1$ such that $\varphi_1(x) \geq ad(x)^\sigma$.

Remark 2.4.
- *If $\Omega \subset \mathbb{R}^2$ is a piecewise polygonal with vertex having an interior degree bigger than $\frac{\pi}{2}$, then $\sigma < 2$.*
- *If $\Omega = [0,1] \times [0,1]$, then $\sigma = 2$.*

We need to study some useful properties of the first positive eigenfunction for the Laplacian, $\varphi_1(\Omega)$.

Lemma 2.5. *Let Ω be a bounded domain satisfying the interior cone condition. Then*

1. *$\varphi_1^\theta \in H_0^1(\Omega)$, for every $\theta \in (\frac{1}{2}, 1]$,*
2. *$\frac{1}{\varphi_1^\vartheta} \in L^1(\Omega)$ for every $\vartheta < \frac{2}{\sigma} - 1$, where σ is defined in Theorem 2.3, (ii).*

Proof. We begin proving that $\frac{|\nabla\varphi_1|}{\varphi_1^\beta} \in L^2(\Omega)$ for every $\beta \in (0, \frac{1}{2})$.
We recall that φ_1 satisfies

$$\int_\Omega \nabla u \nabla \varphi_1 = \lambda_1 \int_\Omega u\varphi_1, \ \forall u \in H_0^1(\Omega),$$

now we take $f_n(\varphi_1)$ as test function, where

$$f_n(s) = \begin{cases} n^{2\beta}s & 0 < s < \frac{1}{n}, \\ s^{1-2\beta} & \frac{1}{n} \leq s < n, \\ n^{1-2\beta} & s \geq n, \end{cases}$$

that is C^1 piecewise with

$$f_n'(s) = \begin{cases} n^{2\beta} & 0 < s < \frac{1}{n}, \\ (1-2\beta)s^{-2\beta} & \frac{1}{n} < s < n, \\ 0 & s > n. \end{cases}$$

so we have that $f_n(\varphi_1) \in H_0^1(\Omega)$. Then we obtain

$$\int_\Omega |\nabla\varphi_1|^2 f_n'(\varphi_1) = \lambda_1 \int_\Omega \varphi_1 f_n(\varphi_1),$$

Hence, since $f_n(\varphi_1) \to f(\varphi_1)$ a.e. in Ω, and for $n > \|\varphi\|_\infty \varphi_1 f_n(\varphi_1) \in L^\infty(\Omega)$, using Dominated Convergence Theorem we have

$$\lambda_1 \int_\Omega \varphi_1 f_n(\varphi_1) \to \lambda_1 \int_\Omega \varphi_1^{2-2\beta},$$

and then for $n > \|\varphi\|_\infty$,

$$\begin{aligned} (1-2\beta)\int_\Omega |\nabla\varphi_1|^2\varphi_1^{-2\beta} &= \int_\Omega \liminf |\nabla\varphi_1|^2 f_n'(\varphi_1) \leq \\ \lim \int_\Omega |\nabla\varphi_1|^2 f_n'(\varphi_1) &= \lambda_1 \int_\Omega \varphi_1^{2-2\beta} < +\infty, \end{aligned}$$

so we have just proved that $|\nabla\varphi_1|\varphi_1^{-\beta} \in L^2(\Omega)$.
Since φ_1^θ is bounded, $\nabla\varphi_1^\theta = \theta\frac{\nabla\varphi_1}{\varphi_1^{1-\theta}}$ and $(1-\theta) \in [0, \frac{1}{2})$ we have that

$$\varphi_1^\theta \in H_0^1(\Omega).$$

Then recalling that (see [8])

$$H_0^1(\Omega) = H^1(\Omega) \cap \left\{ u : \int_\Omega \frac{|u|^2}{d^2} dx < \infty \right\}$$

and choosing $u = \varphi_1^\tau$ with $\tau \in (\frac{1}{2}, 1)$, we have using Theorem 2.3 that $\varphi_1^{2/\sigma} \geq a^{2/\sigma}d^2$, so

$$\frac{1}{\varphi_1^{2/\sigma}} \leq \frac{c}{d^2} \text{ and then } \frac{\varphi_1^{2\tau}}{\varphi_1^{2/\sigma}} \leq c\frac{\varphi_1^{2\tau}}{d^2},$$

where $c = a^{2/\sigma}$, and then

$$\int_\Omega \frac{\varphi_1^{2\tau}}{\varphi_1^{2/\sigma}} \le c \int_\Omega \frac{\varphi_1^{2\tau}}{d^2} < +\infty.$$

Hence, we have proved the integrability of $\frac{1}{\varphi_1^{\frac{2}{\sigma}-2\tau}}$, so if we let ϑ be defined by $\vartheta = (\frac{2}{\sigma} - 2\tau)$, we have that $\frac{2}{\sigma} - 2 \le \vartheta \le \frac{2}{\sigma} - 1$.

Note that the last inequality makes sense when $\sigma < 2$. Moreover, since $\sigma \ge 1$ we obtain that $\frac{2}{\sigma} - 2 \le 0$, and for negative values of ϑ the result is trivial. □

Remark 2.6. *When Ω is $[0,1] \times [0,1]$, then $\sigma = 2$, and the previous lemma gives no information; but in this case, we know explicitely φ_1 and $d(x)$, so by a direct computation we know that $\frac{1}{\varphi_1^\beta} \in L^1(\Omega)$ if and only if, $\beta < \frac{1}{2}$.*

Remark 2.7. *In Lemma 2.5 we have proved that whenever Ω is a bounded domain satisfying an interior cone condition, then*

$$L^r(\Omega) \subset L^1(V_1, p),\ \forall p > 1 - \frac{\sigma + 2r - 2}{r\sigma}.$$

When $\partial\Omega \in C^2$ then it is known that for every $z \in V_k$, $\Omega \setminus \Omega_z$ has a finite number of connected component, named nodal regions, and the restriction of z to each one can be considered as the first eigenfunction associated with λ_k, that is, the first eigenvalue for these nodal regions. If the boundary of each nodal region satisfies an interior cone condition (as it is the case for $N = 2$ (Courant Theorem, see [7])), and $\sigma_i(z)$ denote the number that assures the second assertion of Theorem 2.3 for each nodal region, then for $\sigma(z) = \max \sigma_i(z)$ we have, whenever $\sigma = \sup_{z \in V_k} \sigma(z)$ exists (that occur for example when V_k is one dimensional) and $\sigma < 2$, that

$$L^r(\Omega) \subset L^1(V_k, p),\ \forall p > 1 - \frac{\sigma + 2r - 2}{r\sigma}.$$

3. The case $\Omega = \mathbb{R}^N$

We deal now with the Schröedinger equation on $\mathbb{R}^N$:

$$- \Delta u + p(x)u - \mu_k u = f(x,u) - g(x),\ x \in \mathbb{R}^N\ u \in H^1(\mathbb{R}^N) \tag{10}$$

where $p(x) \in L^\infty(\mathbb{R}^N)$, such that there exists

$$\gamma := \liminf_{|x| \to \infty} p(x),$$

μ_k is an eigenvalue of finite multipicity of the Schröedinger operator defined by

$$S(u) = -\Delta u + p(x)u\ \forall u \in H^1(\mathbb{R}^N)$$

and $f(x,s) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function of class C^1 in s.

It is known that the discrete spectrum of S is finite or it acumulates on γ and lies in the interval $(-\infty, \gamma)$.

We will assume that the nonlinearity $f(x,s) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, such that almost every x in $\mathbb{R}^N$ is of class C^1 with respect to the variable s, and satisfying:

$$|f(x,s)| \leq h(x) \in L^2(\mathbb{R}^N) \tag{11}$$

$$-C \leq f_s'(x,s) \leq \gamma - \varepsilon_0 - \mu_k, \text{ where } C, \varepsilon_0 \in \mathbb{R}^+ . \tag{12}$$

$$\text{For almost every } x \in \mathbb{R}^N, \text{ there exists } \lim_{|s| \to \infty} f(x,s). \tag{13}$$

We adopt here the same decomposition that is made in [14], so consider μ_j such that $\mu_k < \mu_j < \gamma$ and $\mu_j > \gamma - \frac{\varepsilon_0}{2}$ or $\mu_j = \gamma$ if the discrete spectrum is finite.

We denote by $V = \bigoplus_{i<j} Ker(S - \mu_i)$, $V^\perp$ its orthogonal subspace on $L^2(\mathbb{R}^N)$ and $W = V^\perp \cap H^1(\mathbb{R}^N)$; then for every $u \in H^1(\mathbb{R}^N)$, $u = v + w$ where $v \in V$, $w \in W$. Also let P and Q be the projectors defined on $L^2(\mathbb{R}^N)$ to $V^\perp$, and K the continuous inverse of $S - \mu_k$ defined on $V^\perp$ and taking values on W. Applying P, Q and K to (10) we obtain the following equivalent system consisting again of an infinite dimensional equation and a finite dimensional one,

$$w = KQ(f(x, v+w) - g(x)), \tag{14}$$

$$S(v) - \mu_k v = P(f(x, v+w)) - P(g(x)). \tag{15}$$

Under hypotheses (12) and (13), it is known that (14) has only one solution $w = w_g(v)$ for every $v \in V$ (see Lemma 2.2 in [14]).

In order to study equation (15), we define again the function

$$\Gamma_g(v) = S(v) - \mu_k v - P(f(x, v + w_g(v))),$$

and we are looking for solutions on V of

$$\Gamma_g(v) = -P_g(x).$$

To this aim we again use Topological Brower Degree. Let $B_r = \{v \in V, \|v\| < r\}$ and $\pi : V \to Ker(S - \mu_k)$ be a projection. It is known (see Lemma 2.3 in [14]) that whenever $\deg(\Gamma_g + P(g), B_r, 0)$ is well defined, then

$$\deg(\Gamma_g + P(g), B_r, 0) = \deg((S - \mu_k)|_V + \pi, B_r, 0), \tag{16}$$

for some positive constant r. They use Landesman-Lazer Type conditions to prove that $\deg(\Gamma_g + P(g), B_r, 0)$ is well defined, obtaining their first existence theorem for (10). On the other hand, when strong resonance occurs, they also prove this when the decay of f is "like" $\frac{1}{s}$. We consider here more general nonlinearities. Consider as in the previous section $L^1(V, -\alpha)$ the set of measurable functions, $b : \mathbb{R}^N \to \mathbb{R}$, such that

$$\int_{\mathbb{R}^N} |b(x)||z|^{-\alpha} < +\infty,$$

for every $z \in Ker(S - \mu_k)$.

Theorem B. *Suppose that hypotheses (11), (12), (13) are satisfied, and that there exists* $-1 < \alpha < 1$ *such that*

$$|f(x,s)||s|^{\alpha+1} \leq b(x), \tag{17}$$

$$\lim_{s\to\pm\infty} |s|^\alpha s f(x,s) = \mu(x), \ \ and \int\limits_{\Omega\setminus\Omega_s} \mu(x)|z(x)|^{-\alpha} > 0, \ \forall z \in V_k, \tag{18}$$

where $b(x), \mu(x) \in L^1(V, -\alpha)$*; then there exists* ε *such that equation (10) has solution if* $\|P(g)\| \leq \varepsilon$.

Proof. Using 16 it suffices to prove that $d(\Gamma_g, B_r, 0)$ is well defined for sufficiently large values of $r > 0$. Let us argue by contradiction and suppose that there exists $v_n \in V$, with $\|v_n\| \to +\infty$, such that

$$\Gamma_g(v_n) = 0. \tag{19}$$

Since V is finite dimensional, the normalized sequence of v_n may be considered, up to a subsequence, strongly convergent in V and even in $Ker(S - \mu_k)$ (as we see taking limits in (19)). We consider then

$$v_n = \psi_n + z_n + w_n,$$

where $\psi_n \in Ker(S - \mu_k), w_n \in W$, so it is clear that $\|\psi_n\| \to +\infty$ and z_n, w_n are bounded.

Now, since S is a self-adjoint operator, we have

$$((S - \mu_k)z_n|\psi_n) = ((S - \mu_k)\psi_n|z_n) = 0,$$

hence,

$$\int_{\mathbb{R}^N} f(x, \psi_n + z_n + w_n)\psi_n = 0.$$

But a similar argument to that in Theorem A allows us to obtain a contradiction. □

Acknowledgment. *The author thanks the organizers of the conference "PDE Prague, 98" for their hospitality. I also thank Professor David Arcoya for his help and useful suggestions.*

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Some Theoretical and Numerical Aspects of Grade-Two Fluid Models

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Abstract. We present a brief review of some theoretical results on the solutions of grade-two fluid models. Then we discuss some finite-element schemes for solving numerically a grade-two fluid model in two dimensions.

0. INTRODUCTION

A fluid of grade n is a particular non-Newtonian Rivlin-Ericksen fluid (cf. [15]) whose extra stress tensor $\mathbf{T}_e$ has the form

$$\mathbf{T}_e = \mathbf{S}(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n),$$

where $\mathbf{A}_j$ are the Rivlin-Ericksen tensors defined recursively by

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T \quad , \quad \mathbf{L} = \nabla \mathbf{u},$$

$$\mathbf{A}_j = (\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla)\mathbf{A}_{j-1} + \mathbf{A}_{j-1}\mathbf{L} + \mathbf{L}^T\mathbf{A}_{j-1},$$

$\mathbf{S} = \sum_{i=1}^{n} \mathbf{S}_i$ and each $\mathbf{S}_i$ is a polynomial in the $\mathbf{A}_j$:

$$\mathbf{S}_1 = \eta \mathbf{A}_1 \ , \ \mathbf{S}_2 = \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2,$$

$$\mathbf{S}_3 = \beta_1 \mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(\operatorname{tr} \mathbf{A}_1^2)\mathbf{A}_1 \ , \ \text{etc}.$$

Thus a grade-two fluid has the constitutive equation,

$$\mathbf{T} = -\tilde{p}\mathbf{I} + \eta \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2, \tag{0.1}$$

where $\tilde{p}$ is the pressure and the parameters η, α_i and β_i are material constants. Substituting (0.1) into

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \operatorname{div} \mathbf{T} + \mathbf{f},$$

where $\mathbf{f}$ is an external force, expanding and dividing by the density, we obtain the equation of motion of a grade-two fluid:

$$\begin{aligned} \frac{\partial}{\partial t}(\mathbf{u} - \alpha_1 \Delta \mathbf{u}) - \nu \Delta \mathbf{u} + \mathbf{curl}(\mathbf{u} - (2\alpha_1 + \alpha_2)\Delta \mathbf{u}) \times \mathbf{u} \\ - (\alpha_1 + \alpha_2)\Delta(\mathbf{u} \cdot \nabla \mathbf{u}) + 2(\alpha_1 + \alpha_2)\mathbf{u} \cdot \nabla(\Delta \mathbf{u}) + \nabla p = \mathbf{f}, \end{aligned} \tag{0.2}$$

where the modified pressure p is related to $\tilde{p}$ by

$$p = \tilde{p} + \frac{1}{2}|\mathbf{u}|^2 - (2\alpha_1 + \alpha_2)(\mathbf{u} \cdot \Delta \mathbf{u} + \frac{1}{4}\operatorname{tr} \mathbf{A}_1^2).$$

It is considered an appropriate model for the motion of a water solution of polymers.

To be consistent with thermodynamics, Dunn and Fosdick show in [13] that the material constants must satisfy (cf. also [16], [14])

$$\eta \geq 0 \quad , \quad \alpha_1 \geq 0 \quad \text{and } \alpha_1 + \alpha_2 = 0 .$$

With these assumptions, setting $\alpha = \alpha_1$, (0.2) simplifies and leads to the equation of motion

$$\frac{\partial}{\partial t}(\mathbf{u} - \alpha\Delta\,\mathbf{u}) - \nu\Delta\,\mathbf{u} + \mathbf{curl}(\mathbf{u} - \alpha\Delta\,\mathbf{u}) \times \mathbf{u} + \nabla p = \mathbf{f} .$$

Then this equation is completed by the incompressibility condition and adequate boundary and initial conditions.

Even in this simplified form, this problem is difficult because its nonlinear term involves a third-order derivative, whereas its elliptic term is only a Laplace operator. The first successful proof of existence of solutions, for both the time-dependent and steady-state grade-two fluid models in two and three dimensions, with homogeneous Dirichlet boundary conditions, was written in the thesis of Ouazar [29] in 1981, and was later published by Cioranescu and Ouazar in [8] and [9]. These authors proved existence of solutions, with H^3 regularity in space, by looking for a velocity $\mathbf{u}$ such that $\mathbf{z} = \mathbf{curl}(\mathbf{u} - \alpha\,\Delta\,\mathbf{u})$ has L^2 regularity in space, introducing $\mathbf{z}$ as an auxiliary variable and discretizing the equations of motion (in variational form) by Galerkin's method in the basis of the eigenfunctions of the operator $\mathbf{curl}\,\mathbf{curl}(\mathbf{u} - \alpha\,\Delta\,\mathbf{u})$. As an easy exercise, with this approach, Ouazar proved in [29] that, in a simply connected plane domain with sufficiently smooth boundary, this problem always has a global solution in time for all positive parameters ν and α and all forces $\mathbf{f}$ with H^1 regularity. In 1986, the same method was applied by Amrouche in [1] to construct solutions of grade-three fluids.

Starting with [17] published in 1993, a renewal of interest for grade-two and three fluid models has given rise to a large number of publications. There is no space here to quote them all, but the reader will find in [34] a very extensive list of references. Compared with these recent publications, it turns out that in a bounded domain and with mildly smooth data $\mathbf{f}$ (i.e. in most practical situations), the construction introduced by Cioranescu and Ouazar is optimal, although the results they obtained in the 1980's were not optimal. They had not established global existence in time for small data, because they had not taken sufficient advantage, in their *a priori* estimates, of the damping effect of the viscous term $-\nu\Delta\,\mathbf{u}$. But once this damping effect is properly taken into account, global existence is derived from this construction with minimal restrictions on the size of the data, cf. [2] and [10].

Section 1 is devoted to a brief description and comparison of the main methods proposed by different authors for solving the equations of a grade-two fluid model. We shall see that the good methods (again, in most practical situations) are the ones that use the energy equality that stems from the equation of motion. And this property will prove vital for the numerical analysis of the finite element schemes proposed in Section 2.

We close this introduction with a short discussion on boundary conditions. Here we shall study only homogeneous Dirichlet or possibly non-homogeneous tangential

Dirichlet boundary conditions, i.e. $\mathbf{u} = \mathbf{g}$ with $\mathbf{g} \cdot \mathbf{n} = 0$ on the boundary. The reason is that this grade-two fluid model with a fully non-homogeneous Dirichlet boundary condition is not well-posed. There are examples in which the resulting problem has multiple solutions (cf. [24] and [26]), thus implying that additional boundary conditions should be imposed. However, it is not yet known what boundary conditions could be imposed in order to insure that the problem is well-posed. The reader can refer to [31] for a discussion on the boundary conditions for such fluids.

1. Constructions of solutions

Let Ω be a bounded domain in two or three dimensions, with a Lipschitz-continuous boundary $\partial\Omega$ (cf. [23]) and let $\mathbf{n}$ denote the unit normal to $\partial\Omega$, pointing ouside Ω. Our grade-two fluid model is: Find a vector-valued velocity $\mathbf{u}$ and a scalar pressure p, solution of

$$\frac{\partial}{\partial t}(\mathbf{u} - \alpha\Delta\,\mathbf{u}) - \nu\Delta\,\mathbf{u} + \mathbf{curl}(\mathbf{u} - \alpha\Delta\,\mathbf{u}) \times \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \times (0,T)\,, \tag{1.1}$$

with the incompressibility condition

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega \times (0,T)\,, \tag{1.2}$$

the homogeneous boundary condition

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \times (0,T)\,, \tag{1.3}$$

and initial condition

$$\mathbf{u}(0) = \mathbf{u}_0 \text{ in } \Omega\,, \text{ with } \operatorname{div} \mathbf{u}_0 = 0 \text{ in } \Omega \text{ and } \mathbf{u}_0 = \mathbf{0} \text{ on } \partial\Omega\,. \tag{1.4}$$

Let us describe the main points in the construction of Cioranescu and Ouazar. We refer to Nečas [28] and Lions and Magenes [27] for the definition and properties of the Sobolev spaces that we use below. To simplify the discussion, we assume that Ω is simply connected. As far as the regularity of the data is concerned, Cioranescu and Ouazar originally took $\mathbf{f}$ in $H^1(\Omega)^3$, $\mathbf{u}_0$ in $H^3(\Omega)^3$ and $\partial\Omega$ of class $\mathcal{C}^{3,1}$. However, it is sufficient that $\mathbf{curl}\,\mathbf{f} \in L^2(\Omega)^3$, $\partial\Omega$ be of class $\mathcal{C}^{2,1}$ and Ω can be multiply connected (cf. [10], [2], [3]). Following [29], we set

$$\mathbf{z} = \mathbf{curl}(\mathbf{u} - \alpha\,\Delta\,\mathbf{u})\,, \tag{1.5}$$

and we take formally the $\mathbf{curl}$ of (1.1); this leads to the transport equation

$$\alpha\frac{\partial}{\partial t}\mathbf{z} + \nu\mathbf{z} + \alpha\mathbf{u} \cdot \nabla\,\mathbf{z} - \alpha\mathbf{z} \cdot \nabla\,\mathbf{u} = \nu\mathbf{curl}\,\mathbf{u} + \alpha\mathbf{curl}\,\mathbf{f} \quad \text{in } \Omega \times (0,T)\,. \tag{1.6}$$

Their idea consists of discretizing (1.1) by Galerkin's method, using as basis the eigenfunctions of the operator $\mathbf{curl}\,\mathbf{curl}(\mathbf{u} - \alpha\,\Delta\,\mathbf{u})$. The effect of this special basis is that it yields a discretization of the transport equation (1.6). Thus, the discrete solution $\mathbf{u}_m$ and $\mathbf{z}_m = \mathbf{curl}(\mathbf{u}_m - \alpha\,\Delta\,\mathbf{u}_m)$ satisfies, respectively, the following energy equality and energy inequality (we drop the index m to simplify):

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}(t)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}(t)|^2_{H^1(\Omega)} + \nu|\mathbf{u}(t)|^2_{H^1(\Omega)} = (\mathbf{f}(t), \mathbf{u}(t))\,, \tag{1.7}$$

$$\frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{z}(t)\|^2_{L^2(\Omega)} + (\nu - \alpha\|\nabla\,\mathbf{u}(t)\|_{L^\infty(\Omega)})\|\mathbf{z}(t)\|^2_{L^2(\Omega)} \le (\nu\|\mathbf{curl}\,\mathbf{u}(t)\|_{L^2(\Omega)} + \alpha\|\mathbf{curl}\,\mathbf{f}(t)\|_{L^2(\Omega)})\|\mathbf{z}(t)\|_{L^2(\Omega)}, \tag{1.8}$$

where $|\cdot|_{H^1(\Omega)} = \|\nabla\cdot\|_{L^2(\Omega)}$. Clearly, (1.7) gives a uniform estimate for $|\mathbf{u}_m(t)|_{H^1(\Omega)}$ in terms of the *data exclusively* and (1.5) yields an upper bound for $\|\nabla\,\mathbf{u}_m(t)\|_{L^\infty(\Omega)}$ in terms of $\|\mathbf{z}_m(t)\|_{L^2(\Omega)}$. By substituting the former in the right-hand side of (1.8) and the latter in the left-hand side and assuming adequate assumptions on the data, we obtain a uniform estimate for $\|\mathbf{z}_m(t)\|_{L^2(\Omega)}$, from which (1.5) implies a uniform bound for $\|\mathbf{u}_m(t)\|_{H^3(\Omega)}$. Finally, the discretized Galerkin version of (1.1) and this bound show that $\|\mathbf{u}'_m(t)\|_{H^1(\Omega)}$ is also uniformly bounded. Those bounds allow one to pass to the limit in the Galerkin discretization of (1.1) and prove that the limiting function is a solution of (1.1)–(1.4) in $H^3(\Omega)^3$. It is a global solution in time for small data and a local solution, if no restriction is imposed on the data.

There are two reasons that explain the high performance of this method. First, the use of the special basis gives access to *all* the information contained in the three equations (1.1), (1.5) and (1.6). But since these equations are redundant, no fixed-point method can use these three equations in one shot. At first sight, one could think that the energy equality (1.7) can be by-passed and replaced by an estimate derived from (1.5). But this estimate, in terms of the unknown function $\mathbf{z}$, is very unfavorable compared to (1.7) and it imposes unnecessary restrictions on the size of the data.

The second reason is that, owing to the special basis, one does not need to solve the transport equation (1.6) (or the Euler equation that one obtains through other approaches) and this is a difficult equation considering that $\mathbf{z}$ belongs only to $L^2(\Omega)^3$. Little was known about (1.6) in 1981 and although more is known now (cf. [11], [12]), it is still an active subject and there are still open problems related to it.

Now let us discuss the two main methods which were proposed subsequently. The first was introduced by [17] in the case where $\alpha_1+\alpha_2 \neq 0$; but for the sake of simplicity, let us keep $\alpha_1+\alpha_2 = 0$. It consists of taking the Helmholtz decomposition of $\mathbf{u}-\alpha\,\Delta\,\mathbf{u}$ and substituting into (1.1). This yields a linearized "Euler" equation and a Stokes equation:

$$\alpha\frac{\partial}{\partial t}\mathbf{w} + \nu\mathbf{w} + \alpha\mathbf{u}\cdot\nabla\,\mathbf{w} + \alpha(\nabla\,\mathbf{u})^T\mathbf{w} - \nu\mathbf{u} + \nabla\,\hat{p} = \alpha\mathbf{f} \text{ in } \Omega\times(0,T), \tag{1.9}$$

where $\hat{p}$ is another function related to the pressure p,

$$\operatorname{div}\mathbf{w} = 0 \text{ in } \Omega\times(0,T) \quad , \quad \mathbf{w}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\times(0,T),$$

$$\mathbf{w}(0) = \mathbf{w}_0 \text{ in } \Omega\,, \text{ where } \mathbf{w}_0 \text{ is the Helmholtz decomposition of } \mathbf{u}_0\,,$$

$$\mathbf{u} - \alpha\,\Delta\,\mathbf{u} + \nabla\,q = \mathbf{w} \text{ in } \Omega\times(0,T),$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega\times(0,T) \quad , \quad \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega\times(0,T).$$

Then this system is solved by Schauder's fixed-point theorem and for sufficiently small data, it gives $\mathbf{w}$ in $H^3(\Omega)^3$. But by this process, the sharp estimate (1.7) arising from the original equation (1.1) is lost; furthermore enforcing $\mathbf{w}$ in $H^3(\Omega)^3$ (which is achieved by three differentiations of (1.9), thereby requiring an upper bound for fifteen terms) restricts even more the size of the data. It is worth noting that differentiating these equations should be avoided as much as possible because the number of terms in the equation grows exponentially with each differentiation.

The other method, proposed by [18], consists of solving the coupled system (1.1) and (1.6) by Schauder's fixed-point theorem, thereby constructing a solution $\mathbf{u}$ in $H^4(\Omega)^3$. Here, also, enforcing $\mathbf{u}$ in $H^4(\Omega)^3$ restricts unnecessarily the size of the data. But it is not easy to apply this method to less regular solutions because it loses the relation (1.5). This relation is recovered by proving that $\operatorname{div} \mathbf{z} = 0$. However $\operatorname{div} \mathbf{z}$ is a solution in $H^{-1}(\Omega)^3$ of a transport equation similar to (1.6) and this problem is not yet solved.

Other methods have been proposed (cf. for instance [4] and the work of Videman in [34]), but they are variants of the above Euler equation. They lose the estimate (1.7) and they are competitive only in less frequent situations, such as in the case of a flow in an exterior domain and in the case of a force $\mathbf{f}$ whose $\mathbf{curl}$ is not in $L^2(\Omega)^3$.

We finish this section with some theoretical results derived in [20] on the steady two-dimensional grade-two fluid model that we shall discretize in the next section. This problem can be solved for any value of α, on any domain and with tangential boundary conditions. For this, we define the spaces

$$H^1_T(\Omega) = \{\mathbf{v} \in H^1(\Omega)^2\,;\, \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}\,,$$

$$V = \{\mathbf{v} \in H^1_0(\Omega)^2\,;\, \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega\}\,,\ W = \{\mathbf{v} \in H^1_T(\Omega)\,;\, \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega\}\,,$$

$$H(\operatorname{curl},\Omega) = \{\mathbf{v} \in L^2(\Omega)^2\,;\, \operatorname{curl}\mathbf{v} \in L^2(\Omega)\} \text{ where } \operatorname{curl}\mathbf{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\,.$$

Then we assume that Ω is a bounded domain in $\mathbb{R}^2$, with a Lipschitz-continuous boundary $\partial\Omega$, $\mathbf{f}$ is a given function in $H(\operatorname{curl},\Omega)$, $\mathbf{g}$ is a given tangential vector field in $H^{1/2}(\partial\Omega)^2$ (i.e. $\mathbf{g}\cdot\mathbf{n} = 0$) and $\nu > 0$ and α are two given real constants. Our problem is: Find $\mathbf{u}$ and p such that

$$-\nu\Delta\,\mathbf{u} + \mathbf{curl}(\mathbf{u} - \alpha\Delta\,\mathbf{u}) \times \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega\,,$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega \quad , \quad \mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega\,.$$

By introducing the auxiliary variables

$$z = \operatorname{curl}(\mathbf{u} - \alpha\Delta\,\mathbf{u}) \quad , \quad \mathbf{z} = (0,0,z)\,,$$

and looking for a solution $\mathbf{u}$ such that z is in $L^2(\Omega)$, this problem can be split into a generalized Stokes equation and a transport equation

$$-\nu\,\Delta\,\mathbf{u} + \mathbf{z}\times\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega\,, \tag{1.10}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega\,, \tag{1.11}$$

$$\mathbf{u} = \mathbf{g} \quad \text{with } \mathbf{g}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega\,, \tag{1.12}$$

$$\nu\,z + \alpha\,\mathbf{u}\cdot\nabla\,z = \nu\operatorname{curl}\mathbf{u} + \alpha\operatorname{curl}\mathbf{f} \quad \text{in } \Omega\,, \tag{1.13}$$

with $(\mathbf{u}, p, z) \in H^1_T(\Omega) \times L^2_0(\Omega) \times L^2(\Omega)$. This formulation, which was introduced by Ouazar in [29] in view of the discretization, is equivalent to the original problem.

By using the Leray-Hopf's lifting (cf. [25]) of the tangential boundary value, and discretizing (1.13) by a standard Galerkin method, we easily prove the existence of one solution of (1.10)–(1.13) for all the above data. This solution satisfies the estimates

$$\|z\|_{L^2(\Omega)} \leq C_1(\|\mathbf{f}\|_{L^2(\Omega)} + |\alpha|\|\operatorname{curl}\mathbf{f}\|_{L^2(\Omega)}) + C_2\|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}, \tag{1.14}$$

$$|\mathbf{u}|_{H^1(\Omega)} \leq C_3\|\mathbf{f}\|_{L^2(\Omega)} + C_4\|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}\left(1 + C_5\|z\|_{L^2(\Omega)}\right), \tag{1.15}$$

$$\|p\|_{L^2(\Omega)} \leq C_6\|\mathbf{f}\|_{L^2(\Omega)} + C_7\|\mathbf{g}\|_{H^{1/2}(\partial\Omega)} + C_8|\mathbf{u}|_{H^1(\Omega)}\|z\|_{L^2(\Omega)}, \tag{1.16}$$

where C_i are constants that depend on ν but not on α.

However, it is much more difficult to prove that *all* solutions of (1.10)–(1.13) satisfy these bounds. This is equivalent to proving that the transport equation (1.13) has a unique solution for a given $\mathbf{u}$, and it then shows that all solutions z of (1.13) satisfy the Green's formula

$$\int_\Omega (\mathbf{u}\cdot\nabla z)z\,d\mathbf{x} = 0.$$

Clearly, as z is only in $L^2(\Omega)$, this formula requires a proof. Since the boundary of $\partial\Omega$ is not smooth, it is not a direct consequence of [11] and it deserves some discussion. Here the quantity $\frac{\mathbf{u}\cdot\nabla d}{d}$, where d is the distance to $\partial\Omega$, plays a crucial part. In [20], we prove that if $\partial\Omega$ is a curvilinear polygon of class $C^{1,1}$ (cf. [23]), then $\frac{\mathbf{u}\cdot\nabla d}{d} \in L^2$ in a neighborhood of $\partial\Omega$. This allows one to truncate any function φ in $W^{1,p}(\Omega)$ in such a way that the truncated function, say φ_ε, satisfies

$$\|\mathbf{u}\cdot\nabla\varphi_\varepsilon\|_{L^2(\Omega)} \leq C\|\mathbf{u}\|_{H^1(\Omega)}\|\varphi\|_{W^{1,p}(\Omega)}. \tag{1.17}$$

This has several interesting consequences; one of them is the above Green's formula that we can state here more precisely. Let

$$X_\mathbf{u} = \{z \in L^2(\Omega)\,;\, \mathbf{u}\cdot\nabla z \in L^2(\Omega)\},$$

where $\mathbf{u} \in W$. Then if $\partial\Omega$ is a curvilinear polygon of class $C^{1,1}$, we have

$$\forall z \in X_\mathbf{u}\,,\ \int_\Omega (\mathbf{u}\cdot\nabla z)z\,d\mathbf{x} = 0. \tag{1.18}$$

The proof of (1.18), found in [20], uses an argument of [12].

As an application, (1.18) yields the additional *a priori* estimate

$$\|\alpha\,\mathbf{u}\cdot\nabla z\|_{L^2(\Omega)} \leq \nu\sqrt{2}\,|\mathbf{u}|_{H^1(\Omega)} + |\alpha|\,\|\operatorname{curl}\mathbf{f}\|_{L^2(\Omega)}. \tag{1.19}$$

It also allows one to show that *each* solution of (1.10)–(1.13) converges strongly, as α tends to zero, to a solution $(\mathbf{u}^0, p^0)$ of the Navier-Stokes equations:

$$-\nu\,\Delta\,\mathbf{u}^0 + \operatorname{curl}\mathbf{u}^0 \times \mathbf{u}^0 + \nabla p^0 = \mathbf{f}\,,\ z^0 = \operatorname{curl}\mathbf{u}^0 \quad \text{in } \Omega,$$

$$\operatorname{div}\mathbf{u}^0 = 0 \quad \text{in } \Omega \quad,\quad \mathbf{u}^0 = \mathbf{g} \quad \text{on } \partial\Omega.$$

2. Some finite element schemes

The difficulties encountered in solving grade-two fluid models theoretically are amplified when solving them numerically, because of the high order of derivatives involved. However, the two-dimensional case does bring a major simplification, namely, by choosing appropriate schemes, all the numerical analysis can be performed without having to derive a uniform $W^{1,\infty}$ estimate for the discrete velocity. Of course, several schemes can be proposed to discretize this problem; the first two schemes we present here satisfy two criteria: existence of a solution in a polygonal domain without smallness restriction on the data, and error estimates. To derive error estimates in the transport equation, we find that we need either discrete velocities with exactly zero divergence, or we must discretize the transport equation by an adequate upwind scheme. In the first case, following [32] and [22], this leads us to work with triangular finite elements of degree at least four in each triangle. In the second case, triangular finite elements of degree at least three in each triangle are sufficient. Finally, the third scheme follows a least-squares method, for which we have yet no convergence analysis.

From now on, we assume that the domain Ω has a polygonal Lipschitz-continuous boundary, so it can be entirely triangulated. Let $h > 0$ be a discretization parameter and let $\mathcal{T}_h$ be a *regular* family of triangulations of $\overline{\Omega}$, consisting of triangles K with diameter h_K and with maximum mesh size h (cf. [7], [5]).

In the first scheme, we discretize the stream-function of $\mathbf{u}$. For the sake of briefness, we assume that Ω is simply connected; then each function $\mathbf{v} \in W$ (resp. $\mathbf{v} \in V$) has a unique stream-function $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ (resp. $\varphi \in H_0^2(\Omega)$): $\mathbf{v} = \mathbf{curl}\,\varphi = (\frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1})$. For $r \geq 5$, we consider the finite-element spaces S_h^r and $S_{h,c}^r$ consisting of C^1 piecewise-polynomials

$$S_h^r := \{f \in C^1(\overline{\Omega})\,;\, \forall K \in \mathcal{T}_h, f|_K \in \mathcal{P}_r\}\ ,\ S_{h,c}^r := S_h^r \cap H_0^2(\Omega)\,,$$

where $\mathcal{P}_r$ denotes the space of polynomials of degree less than or equal to r in two variables. Then we discretize W and V, respectively, by

$$W_h = \{\mathbf{curl}\, f\,;\, f \in S_h^r, f|_{\partial\Omega} = 0\}\ ,\ V_h = \{\mathbf{curl}\, f\,;\, f \in S_{h,c}^r\} \subset W_h\,,$$

and we denote by $G_{T,h}$ the trace space of W_h. The degrees of freedom of S_h^r are described in [21], the trace space $G_{T,h}$ is characterized and an adequate interpolation operator $\Pi \in \mathcal{L}(W^{l,p}(\Omega); S_h^r)$ is constructed, for $l \geq 2$ when $p = 1$ and $l > 1 + \frac{1}{p}$ when $p > 1$. It is shown that Π is a projection and it preserves individually homogeneous boundary conditions. Furthermore, Π has the following approximation property, for any numbers p and l such that $1 \leq p \leq \infty$ and $1 + 1/p < l \leq r + 1$ if $p > 1$, or $2 \leq l \leq r+1$ if $p = 1$, and any integer m such that $0 \leq m \leq l$:

$$\forall f \in W^{l,p}(\Omega)\,, \forall K \in \mathcal{T}_h\,, |f - \Pi f|_{W^{m,p}(K)} \leq C\, h_K^{l-m} \|f\|_{W^{l,p}(S_K)}\,, \tag{2.1}$$

where S_K is an adequate macro-element containing K. Then we define the approximation operator $P_h \in \mathcal{L}(W; W_h)$ by $P_h(\mathbf{v}) = \mathbf{curl}(\Pi\varphi)$, where φ is the stream-function of $\mathbf{v}$. The approximation properties of W_h are deduced from (2.1); in particular, there exists C independent of h_K, such that

$$\forall \mathbf{v} \in W, \forall K \in \mathcal{T}_h, \|\mathbf{v} - P_h(\mathbf{v})\|_{L^p(K)} \le C\, h_K^{2/p} \|\varphi\|_{W^{1+2/p,p}(S_K)}. \tag{2.2}$$

To simplify, we assume that the boundary data $\mathbf{g}$ is the trace of a known function $\mathbf{r} \in W$, but the method below is independent of the particular lifting $\mathbf{r}$. We discretize $\mathbf{u}$ in W_h and z in the standard finite element space

$$Z_h = \{\theta \in C^0(\overline{\Omega})\,;\, \forall K \in \mathcal{T}_h, \theta|_K \in \mathcal{P}_k\} \subset H^1(\Omega), \tag{2.3}$$

for an integer $k \ge 1$. Then our first scheme reads: Find $\mathbf{u}_h$ in W_h and $\mathbf{z}_h = (0, 0, z_h)$ with z_h in Z_h, such that

$$\forall \mathbf{v}_h \in V_h, \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) + (\mathbf{z}_h \times \mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \tag{2.4}$$

$$\mathbf{u}_h = \mathbf{g}_h = P_h(\mathbf{r})|_{\partial\Omega} \quad \text{on } \partial\Omega, \tag{2.5}$$

$$\forall \theta_h \in Z_h, \nu\,(z_h, \theta_h) + \alpha\,(\mathbf{u}_h \cdot \nabla z_h, \theta_h) = \nu\,(\operatorname{curl} \mathbf{u}_h, \theta_h) + \alpha\,(\operatorname{curl} \mathbf{f}, \theta_h). \tag{2.6}$$

The trace-preserving properties of P_h and (2.2) permit to construct a Leray-Hopf's lifting of $\mathbf{g}_h$, for h small enough, say $h \le h_0$; in turn this allows us to show the existence of a solution of (2.4)–(2.6) for all data and for all $h \le h_0$. All solutions of (2.4)–(2.6) satisfy the analogue of (1.14) and (1.15) with constants independent of h. With these uniform estimates and (1.18), we prove in [22] that $\mathbf{u}_h$ and z_h converge strongly to $\mathbf{u}$ and z, respectively. These results carry over to the pressure.

Establishing error estimates is more delicate because (2.6) does not give directly a bound for $\mathbf{u}_h \cdot \nabla z_h$. In [22], we turn this difficulty by associating with $\mathbf{u}_h$ and $\mathbf{u}$ the auxiliary transport equation: Find $\zeta = \zeta(h) \in L^2(\Omega)$ such that

$$\nu \zeta + \alpha\, \mathbf{u}_h \cdot \nabla \zeta = \nu \operatorname{curl} \mathbf{u} + \alpha \operatorname{curl} \mathbf{f}. \tag{2.7}$$

Since W_h is contained in W, (2.7) has a unique solution $\zeta \in L^2(\Omega)$ for any $\mathbf{u}_h$ in W_h and any $\mathbf{u}$ such that $\operatorname{curl} \mathbf{u}$ belongs to $L^2(\Omega)$. In addition, we have the following upper bounds:

$$\|\mathbf{u}_h \cdot \nabla(\zeta - z)\|_{L^2(\Omega)} \le \|(\mathbf{u}_h - \mathbf{u}) \cdot \nabla z\|_{L^2(\Omega)},$$

$$\|\zeta - z\|_{L^2(\Omega)} \le \frac{|\alpha|}{\nu} \|(\mathbf{u}_h - \mathbf{u}) \cdot \nabla z\|_{L^2(\Omega)}.$$

With this and a uniform bound for $\mathbf{u}_h$ in $L^\infty(\Omega)^2$, which can be established when $\mathcal{T}_h$ is uniformly regular (cf. [7]) and $\mathbf{g} \in H^{1/2+s}(\partial\Omega)^2$ for some $s > 0$, we can prove the following error estimate for small data and sufficiently smooth solutions:

$$\|z - z_h\|_{L^2(\Omega)} \le C\big(\inf_{\lambda_h \in Z_h} \|z - \lambda_h\|_{H^1(\Omega)} + |\mathbf{u} - \mathbf{u}_h|_{H^1(\Omega)}\big),$$

where C is independent of h. If we take $k = 1$ in the definition (2.3) of Z_h, this gives a scheme of only order one, which is low considering that $\mathbf{u}_h$ is a polynomial of degree four in each triangle.

The purpose of the second scheme is to reduce this degree by one. The high degree of $\mathbf{u}_h$ in the first scheme arises from the fact that $\operatorname{div} \mathbf{u}_h = 0$. This property is fundamental for solving (2.7). But it also plays an important part in passing to the limit in (2.6). Indeed, if $\mathbf{u}_h \in H_T(\Omega)$ with $\operatorname{div} \mathbf{u}_h \neq 0$, then

$$\int_\Omega (\mathbf{u}_h \cdot \nabla z_h)\theta_h \, d\mathbf{x} = -\int_\Omega (\operatorname{div} \mathbf{u}_h) z_h \, \theta_h \, d\mathbf{x} - \int_\Omega z_h(\mathbf{u}_h \cdot \nabla \theta_h) d\mathbf{x}$$

and, as we only have weak convergence of both $\operatorname{div} \mathbf{u}_h$ and z_h to begin with, we cannot conclude, without further information, that the limit of the first integral in the right-hand side is zero. In [22], we propose to eliminate this integral by choosing compatible spaces for z_h and the discrete pressure p_h. For this, we take a Hood-Taylor scheme with velocities that are polynomials of degree three (cf. [6]). More precisely, we introduce the finite-element spaces:

$$X_h = \{\mathbf{v} \in \mathcal{C}^0(\overline{\Omega})^2 \, ; \, \forall K \in \mathcal{T}_h \, , \, \mathbf{v}|_K \in \mathcal{P}_3^2, \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0\} \, , \; X_{h,0} = X_h \cap H_0^1(\Omega)^2 \, ,$$

$$M_h = \{q \in \mathcal{C}^0(\overline{\Omega}) \, ; \, \forall K \in \mathcal{T}_h \, , \, q|_K \in \mathcal{P}_2, \int_\Omega q \, d\mathbf{x} = 0\} \, ,$$

$$Z_h = \{\theta \in \mathcal{C}^0(\overline{\Omega}) \, ; \, \forall K \in \mathcal{T}_h \, , \, \theta|_K \in \mathcal{P}_1\} \, .$$

We know from [6] that the pair of spaces $(X_{h,0}, M_h)$ satisfies a uniform discrete inf-sup condition. It is easy to rewrite the first scheme in these spaces and derive existence of a discrete solution and convergence, without restriction on the data. But we do not have error estimates for such a scheme because we cannot use (2.7). Since the difficulty comes from the transport equation, we propose to discretize it by streamline diffusion; this will enable us to derive an estimate for $\mathbf{u}_h \cdot \nabla z_h$. Thus our second scheme reads: Find $(\mathbf{u}_h, p_h, z_h)$ in $X_h \times M_h \times Z_h$, such that

$$\forall \mathbf{v}_h \in X_{h,0} \, , \, \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) + (\mathbf{z}_h \times \mathbf{u}_h, \mathbf{v}_h) - (p_h, \operatorname{div} \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \, , \tag{2.8}$$

$$\forall q_h \in M_h \, , \, (q_h, \operatorname{div} \mathbf{u}_h) = 0 \, , \tag{2.9}$$

$$\mathbf{u}_h = \mathbf{g}_h = P_h(\mathbf{r})|_{\partial\Omega} \quad \text{on } \partial\Omega \, , \tag{2.10}$$

$$\begin{aligned} \forall \theta_h \in Z_h \, , \, \nu \, (z_h, \theta_h + \delta \mathbf{u}_h \cdot \nabla \theta_h) + \alpha \, (\mathbf{u}_h \cdot \nabla z_h, \theta_h + \delta \mathbf{u}_h \cdot \nabla \theta_h) \\ = \nu \, (\operatorname{curl} \mathbf{u}_h, \theta_h + \delta \mathbf{u}_h \cdot \nabla \theta_h) + \alpha \, (\operatorname{curl} \mathbf{f}, \theta_h + \delta \mathbf{u}_h \cdot \nabla \theta_h) \, , \end{aligned} \tag{2.11}$$

where $\mathbf{z}_h = (0, 0, z_h)$, δ is an arbitrary parameter such that the product $\alpha \, \delta$ is non-negative and, as above, $\mathbf{r}$ is any lifting of $\mathbf{g}$ in W.

A somewhat similar argument to the preceding one shows that, under the same assumptions, this scheme has at least one solution; all solutions satisfy the analogue of (1.14)–(1.16) with constants independent of h and furthermore $\alpha \, \delta \|\mathbf{u}_h \cdot \nabla z_h\|_{L^2(\Omega)}^2$ is bounded uniformly with respect to h. These uniform bounds yield weak convergence; but, in contrast to the first scheme, it does not seem obvious that, for all data, the limit functions are a solution of the original problem. So instead of showing convergence for a restricted set of data, we may as well restrict the data as in the first scheme and prove directly an error bound. This bound is similar to that for the first scheme; it shows that this second scheme is again of order one, cf. [22]. This is not yet satisfactory, considering that $\mathbf{u}_h$ is a polynomial of degree three in each triangle.

Our third scheme reduces by one the degree of $\mathbf{u}_h$; it uses a Hood-Taylor scheme of order two, but its numerical analysis is still an open problem. To simplify, we consider only homogeneous boundary conditions. Let

$$X_h = \{\mathbf{v} \in \mathcal{C}^0(\overline{\Omega})^2 \,;\, \forall K \in \mathcal{T}_h \,,\, \mathbf{v}|_K \in \mathcal{P}_2^2\} \cap H_0^1(\Omega)^2 \,,$$

$$M_h = \{q \in \mathcal{C}^0(\overline{\Omega}) \,;\, \forall K \in \mathcal{T}_h \,,\, q|_K \in \mathcal{P}_1, \int_\Omega q \, d\mathbf{x} = 0\} \,,$$

$$Z_h = \{\mathbf{z} \in \mathcal{C}^0(\overline{\Omega})^2 \,;\, \forall K \in \mathcal{T}_h \,,\, \mathbf{z}|_K \in \mathcal{P}_1^2\} \,.$$

We know from [19] that the pair of spaces (X_h, M_h) satisfies a uniform discrete inf-sup condition. With these spaces, we discretize a variant of (1.10) and (1.5) by the following least-squares method. For a given $\mathbf{z}_h \in Z_h$, find $\mathbf{u}_h^1 \in X_h$ solution of

$$\forall \mathbf{v}_h \in X_h \,,\, \alpha(\nabla \mathbf{u}_h^1, \nabla \mathbf{v}_h) + (\mathbf{u}_h^1, \mathbf{v}_h) = (\mathbf{z}_h, \mathbf{v}_h) \,,$$

and find $(\mathbf{u}_h^2, p_h^2) \in X_h \times M_h$ solution of

$$\forall \mathbf{v}_h \in X_h \,,\, \nu(\nabla \mathbf{u}_h^2, \nabla \mathbf{v}_h) + (\mathbf{curl}\, \mathbf{z}_h \times \mathbf{u}_h^2, \mathbf{v}_h) - (p_h^2, \operatorname{div} \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \,,$$

$$\forall q_h \in M_h \,,\, (q_h, \operatorname{div} \mathbf{u}_h^2) = 0 \,.$$

These two solutions depend on $\mathbf{z}_h$ and they are forced to coincide by computing $\mathbf{z}_h$ that minimizes the functional $|\mathbf{u}_h^1(\mathbf{z}_h) - \mathbf{u}_h^2(\mathbf{z}_h)|^2_{H^1(\Omega)}$. A gradient algorithm is used to approximate such $\mathbf{z}_h$. The numerical experiments described in [30] give good results, both for the steady and the time-dependent problem.

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Large Asymptotic Behaviour of Kolmogorov Equations in Hilbert Spaces

GIUSEPPE DA PRATO

1. INTRODUCTION

Let us consider a dynamical system in a separable Hilbert space H (norm $|\cdot|$, inner product $\langle\cdot,\cdot\rangle$) governed by a differential stochastic equation perturbed by a cylindrical Wiener process W in H,

$$\left.\begin{aligned} &dX(t) = (AX(t) + F(X(t)))dt + dW(t), \quad t \geq 0, \\ &X(0) = x \in H. \end{aligned}\right\} \tag{1.1}$$

where $A : D(A) \subset H \to H$ is a linear operator infinitesimal generator of a strongly continuous semigroup e^{tA} in H, and $F : H \to H$ is a nonlinear mapping.

Under suitable assumptions problem (1.1) has a unique solution $X(t,x)$ and one defines the *transition semigroup* P_t, $t \geq 0$ by setting ([1])

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))], \quad \varphi \in B_b(H). \tag{1.2}$$

$u(t,x) = P_t\varphi(x)$, is (formally) a solution to the following *Kolmogorov equation*,

$$\left.\begin{aligned} &D_t u = \frac{1}{2}\operatorname{Tr}[D^2u] + \langle Ax + F(x), Du\rangle, \\ &u(0) = \varphi, \end{aligned}\right\} \tag{1.3}$$

with $\varphi \in C_b(H)$. ([2]) The transition semigroup contains all information about the behaviour of system (1.1). We recall that a probability Borel measure ν on H is said to be *invariant* for P_t, $t \geq 0$ if for any $\varphi \in C_b(H)$ we have

$$\int_H P_t\varphi(x)\nu(dx) = \int_H \varphi(x)\nu(dx). \tag{1.4}$$

The invariant measure ν is said to be *ergodic* when

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T P_t\varphi(x)dt = \int_H \varphi(x)\nu(dx), \text{ a.e.} \tag{1.5}$$

---

[1] $C_b(H)$ is the Banach space of all uniformly continuous and bounded mappings from H into $\mathbb{R}$, endowed with the sup norm $\|\cdot\|_0$. $C_b^1(H)$ is defined in a similar way. $\mathbb{E}$ means expectation.

[2] We denote by $\operatorname{Tr} S$ the trace of the trace class operator S.

for all $\varphi \in C_b(H)$. We also recall that ν is said to be *weakly mixing* if there exists a subset $I \subset [0,+\infty[$ of relative measure 1 such that

$$\lim_{T\to\infty,\, T\in I} P_T\varphi = \int_H \varphi(x)\nu(dx), \text{ weakly in } L^2(H,\nu). \tag{1.6}$$

Finally ν is said to be *strongly mixing* if

$$\lim_{T\to\infty} P_T\varphi = \int_H \varphi(x)\nu(dx), \text{ weakly in } L^2(H,\nu). \tag{1.7}$$

Problem (1.3) has been been extensively studied when H is finite dimensional; see [10], [3],[11], [12].

In this paper we want to present some results for the asymptotic behaviour of Kolmogorov equation (1.3).

We recall in Section 2 some known results when $F = 0$, and in Section 3 the construction of the transition semigroup P_t, $t \geq 0$ when F is bounded. Moreover we show existence of an invariant measure ν.

In Section 4 we consider the extension of P_t, $t \geq 0$ to $L^2(H,\nu)$ and prove some regularity results that are exploited in Section 5 to study ergodicity and mixing of ν.

2. The Ornstein–Uhlenbeck semigroup

Here we assume

Hypothesis 2.1. (i) *A is the infinitesimal generator of a strongly continuous semigroup e^{tA} on H. There exists $\omega > 0$, such that*

$$\|e^{tA}\| \leq e^{-\omega t},\ t \geq 0, \tag{2.1}$$

(ii) *For any $t > 0$ the linear operator $e^{tA}e^{tA^*}$ is of trace class and*

$$\int_0^t Tr\,[e^{sA}e^{sA^*}]ds < +\infty.$$

We denote by $\mu_t = \mathcal{N}(0,Q_t)$ the Gaussian measure in H with mean 0 and covariance operator Q_t given by,

$$Q_t x = \int_0^t e^{sA}e^{sA^*}x\,ds,\ x \in H. \tag{2.2}$$

We set also $\mu = \mathcal{N}(0,Q)$ where

$$Qx = \int_0^{+\infty} e^{sA}e^{sA^*}x\,ds,\ x \in H. \tag{2.3}$$

We denote by $\{e_k\}$ a complete orthonormal system in H and by $\{\lambda_k\}$ a sequence of positive numbers such that

$$Qe_k = \lambda_k e_k,\ k \in \mathbf{N},$$

and by D_k the derivative in the direction e_k.

We recall that the Ornstein–Uhlenbeck semigroup R_t, $t \geq 0$ is defined by

$$R_t\varphi(x) = \int_H \varphi(e^{tA}x + y)\mu_t(dy),\ \varphi \in C_b(H). \tag{2.4}$$

It is convenient to introduce the set $\mathcal{E}_A(H)$ of all functions φ of the form

$$\varphi(x) = \text{Re} \sum_{k=1}^{n} e^{i\langle x, h_i\rangle}.$$

The following result is proved in [7].

Proposition 2.2. *Assume that Hypothesis 2.1 is fulfilled. Then semigroup R_t, $t \geq 0$ can be uniquely extended to a strongly continuous semigroup of contractions (denoted by the same symbol) on $L^2(H,\mu)$. Moreover $\mathcal{E}_A(H)$ is a core for the infinitesimal generator L of R_t, $t \geq 0$, and we have*

$$L\varphi = \frac{1}{2}\, Tr\,[D^2\varphi] + \langle Ax, D\varphi\rangle,\ \varphi \in \mathcal{E}_A(H). \tag{2.5}$$

Finally there exists the limit

$$\lim_{t\to\infty} R_t\varphi(x) = \int_H \varphi(y)\mu(dy),\ x \in H, \tag{2.6}$$

so that μ is ergodic and strongly mixing.

The following identities follow by a direct verification; see [1], [8],

$$\int_H L\varphi(x)\ \psi(x)d\mu = \int_H \langle QD\psi(x), A^*D\varphi(x)\rangle d\mu,\ \varphi,\psi \in \mathcal{E}_A(H), \tag{2.7}$$

$$\int_H L\varphi(x)\ \varphi(x)d\mu = -\frac{1}{2}\int_H |D\varphi(x)|^2 d\mu,\ \varphi \in \mathcal{E}_A(H). \tag{2.8}$$

From (2.8) it follows that

$$D(L) \subset W^{1,2}(H,\mu), \tag{2.9}$$

and that (2.8) holds for any $\varphi \in D(L)$.

We have the following regularity property of R_t, $t \geq 0$; see [7].

Proposition 2.3. *For any $t > 0$ and any $\varphi \in L^2(H,\mu)$ we have that $R_t\varphi \in W^{1,2}(H,\mu)$. Moreover the following estimate holds:*

$$\int_H |DR_t\varphi(x)|^2 d\mu \leq \frac{1}{t}\int_H |\varphi(x)|^2 d\mu. \tag{2.10}$$

Next corollary follows by Laplace transform.

Corollary 2.4. *For any $\lambda > 0$ and any $\varphi \in L^2(H,\mu)$ we have that $R(\lambda, L)\varphi \in W^{1,2}(H,\mu)$.* (3) *Moreover the following estimate holds:*

$$\int_H |DR(\lambda,L)\varphi(x)|^2 d\mu \leq \frac{\pi}{\lambda}\int_H |\varphi(x)|^2 d\mu. \tag{2.11}$$

We finally gives a simple description of the adjoint L^* of L under the following assumption:

[3] $R(\lambda, L) = (\lambda - L)^{-1}$ denotes the resolvent of L.

Hypothesis 2.5. *For any $t > 0$ we have*

$$e^{tA}Q(H) \subset Q(H). \tag{2.12}$$

Under this assumption we can consider the strongly continuous semigroup $e^{tA_1} = Qe^{tA^*}Q^{-1}$, where $A_1 = QA^*Q^{-1}$.

Proposition 2.6. *We have*

$$e^{tL^*}\varphi(x) = \int_H \varphi(e^{tA_1}x + y)\mu_{1,t}(dy), \ \varphi \in C_b(H),$$

where $\mu_{1,t} = \mathcal{N}(0, Q_{1,t})$, and

$$Q_{1,t}x = \int_0^t e^{sA_1}e^{sA_1^*}xds, \ x \in H. \tag{2.13}$$

Notice that $Q_{1,+\infty} = Q$. Arguing as before we find

$$D(L^*) \subset W^{1,2}(H, \mu). \tag{2.14}$$

3. Construction of the transition semigroup on $L^2(H, \mu)$

We assume here

Hypothesis 3.1. *$F : H \to H$ is Borel bounded,*

and we set

$$\|F\|_0 = \text{ esssup } \{|F(x)| : x \in H\}.$$

Then we consider the linear operator

$$K\varphi = L\varphi + \langle F(x), D\varphi\rangle, \ \varphi \in D(L).$$

Proposition 3.2. *Assume that Hypotheses 2.1 and 3.1 hold. Then the resolvent set $\rho(K)$ of K contains the half line $[\lambda_0, +\infty[$, where*

$$\lambda_0 = \sqrt{\pi}\ \|F\|_0. \tag{3.1}$$

Moreover K is the infinitesimal generator of a strongly continuous semigroup P_t, $t \geq 0$, in $L^2(H, \mu)$ and $\mathcal{E}_A(H)$ is a core for K.

Proof: Let $\lambda > \lambda_0$, and let $f \in L^2(H, \mu)$. Then, setting $\lambda\varphi - L\varphi = \psi$, the equation

$$\lambda\varphi - K\varphi = f$$

is equivalent to

$$\psi - T_\lambda\psi = f,$$

where

$$T_\lambda\psi = \langle F(x), DR(\lambda, L)\psi\rangle, \ \psi \in L^2(H, \mu).$$

Now by Corollary 2.4 it follows that $\|T_\lambda\| < 1$. Therefore $\lambda \in \rho(K)$ and we have

$$R(\lambda, K) = R(\lambda, L)(1 - T_\lambda)^{-1}.$$

Consequently

$$\|R(\lambda, K)\| \leq \frac{1}{\lambda - \lambda_1}, \ \lambda > \lambda_1,$$

so that K generates a strongly continuous semigroup. Finally $\mathcal{E}_A(H)$ is a core for K since it is a core for L. □

The following result is proved in [7].

Proposition 3.3. *Under the assumptions of Proposition 3.2, there exists a unique invariant measure ν for the semigroup P_t, $t \geq 0$. Moreover ν is absolutely continuous with respect to μ and the Radom–Nikodym derivative $\alpha := d\nu/d\mu$ is nonnegative and belongs to $W^{1,2}(H,\mu)$.*

We now can give a characterization of the adjoint K^* of K under the following additional assumption.

Hypothesis 3.4. (i) *F is continuously differentiable and bounded together with its derivative.*

(ii) *There exists a continuous and bounded function $F_1 : H \to H$, such that*

$$F_1(x) = \langle Q^{-1}x, F(x)\rangle,\ x \in Q(H).$$

The following result is proved in [5].

Proposition 3.5. *Assume that Hypotheses* 2.1, 2.5 *and* 3.4 *are fulfilled. Then the adjoint K^* of K in $L^2(H,\mu)$ is given by*

$$K^*\varphi = L^*\varphi - \langle F(x), D\varphi\rangle + (F_1(x) - \ \mathrm{div}\, F)\,\varphi,\ \varphi \in D(L^*). \tag{3.2}$$

We notice that the density α is the solution to

$$K^*\alpha = 0.$$

Let us notice that, setting $\psi = -\log\alpha$, then ψ is a solution to the following Hamilton–Jacobi equation,

$$L^*\psi - \langle F(x), D\psi\rangle - \frac{1}{2}\,|D\psi|^2 = F_1(x) - \ \mathrm{div}\, F. \tag{3.3}$$

4. Construction of the transition semigroup on $L^2(H,\nu)$

In this section we assume — besides Hypotheses 2.1 2.5, 3.4 — the following:

Hypothesis 4.1. *We have*

$$D\log\alpha \in L^2(H,\mu). \tag{4.1}$$

Under this assumption it follows that $L^2(H,\mu)$ is included, with continuous embedding, in $L^2(H,\nu)$.

We prove now an integration by parts formula.

Proposition 4.2. *For any $\varphi, \psi \in \mathcal{E}_A(H)$, we have*

$$\int_H D_h\varphi\ \psi d\nu = -\int_H \varphi\ D_h\psi d\nu + \frac{1}{\lambda_h}\int_H x_h\varphi\ \psi d\nu - \int_H \varphi\ \psi\ D\log\alpha d\nu. \tag{4.2}$$

Using this result it is easy to see that D_h is closable for any $h \in \mathbb{N}$. Thus space $W^{1,2}(H,\nu)$ can be defined as usual.

We consider now the linear operator in $L^2(H,\nu)$,

$$K_0\varphi = L\varphi + \langle F(x), D\varphi\rangle, \ \varphi \in \mathcal{E}_A(H).$$

The following result can be proved by using the integration by parts formula (4.2) and recalling that $-\log\alpha$ is a solution to the Hamilton–Jacobi equation (3.3); see [5].

Proposition 4.3. *Assume that Hypotheses* 2.1, 2.5, 3.4 *and* 4.1 *are fulfilled. Then for any $\varphi \in \mathcal{E}_A(H)$ the following identity holds:*

$$\int_H K_0\varphi\, \varphi d\nu = -\frac{1}{2}\int_H |D\varphi|^2 d\nu. \tag{4.3}$$

By (4.3) it follows that K_0 is dissipative in $L^2(H,\nu)$ and consequently is closable. We denote by K^ν its closure.

Proposition 4.4. *Assume that Hypotheses* 2.1, 2.5, 3.4 *and* 4.1 *are fulfilled. Then K^ν is m-dissipative. Moreover $D(K^\nu) \subset W^{1,2}(H,\nu)$ and for any $\varphi \in D(K^\nu)$ identity* (4.3) *holds. Finally the adjoint $(K^\nu)^*$ of K^ν in $L^2(H,\nu)$ is given by*

$$(K^\nu)^*\varphi = L^*\varphi + \langle A_1x - F(x) + D\log\alpha(x), D\varphi\rangle, \ \varphi \in D(L^*). \tag{4.4}$$

Proof: We first prove that $D(L) \subset D(K^\nu)$. In fact let $\varphi \in D(L)$ and let $\{\varphi_n\} \subset \mathcal{E}_A(H)$ be such that

$$\varphi_n \to \varphi, \ L\varphi_n \to L\varphi, \text{ in } L^2(H,\mu).$$

Then we have also

$$K_0\varphi_n \to L\varphi + \langle F, D\varphi\rangle, \text{ in } L^2(H,\mu).$$

Since $L^2(H,\mu) \subset L^2(H,\nu)$ it follows that $\varphi \in D(K^\nu)$ as required. Now by Proposition 3.2 it follows that $(\lambda - K^\nu)D(L) = L^2(H,\mu)$. Since $L^2(H,\mu)$ is dense in $L^2(H,\nu)$, K^ν is m-dissipative.

Now let $\varphi \in D(K^\nu)$ and let $\{\varphi_n\} \subset \mathcal{E}_A(H)$ be such that

$$\varphi_n \to \varphi, \ K_0\varphi_n \to K^\nu\varphi, \text{ in } L^2(H,\nu).$$

Then by (4.3) it follows

$$\int_H |D(\varphi_n - \varphi_m)|^2 d\nu = -2\int_H K^\nu(\varphi_n - \varphi_m)(\varphi_n - \varphi_m)d\nu.$$

Therefore the sequence $\{\varphi_n\}$ is Cauchy in $W^{1,2}(H,\nu)$ and $\varphi \in W^{1,2}(H,\nu)$.

Finally the last statement follows by using the integration by parts formula (4.2) and using the fact that $-\log\alpha$ is a solution to (3.3). □

Remark 4.5. By (4.4) it follows that K^ν is self-adjoint if and only if

$$D\log\alpha(x) = 2F(x) + Ax - A_1x, \ x \in D(A)\cap D(A_1). \tag{4.5}$$

In the following we shall denote by P_t^ν, $t \geq 0$, the semigroup generated by K^ν in $L^2(H,\nu)$.

5. Asymptotic properties of P_t^ν

In this section we assume that Hypotheses 2.1, 2.5, 3.4 and 4.1 are fulfilled. We denote by $\|\cdot\|$ the norm in $L^2(H,\nu)$.

We first study some regularity properties of the initial value problem

$$\begin{cases} u'(t) = K^\nu u(t), \\ u(0) = \varphi \in L^2(H,\nu), \end{cases} \tag{5.1}$$

whose solution is given by $u(t) = P_t^\nu \varphi$.

Proposition 5.1. *For any $\varphi \in L^2(H,\nu)$ and any $T > 0$ we have*

$$u \in C([0,T]; L^2(H,\nu)) \cap L^2(0,T; W^{1,2}(H,\nu)),$$

and the following identity holds

$$\|u(t)\|^2 + \int_0^t \|Du(s)\|^2 ds = \|\varphi\|^2. \tag{5.2}$$

Proof: Let first $\varphi \in D(K^\nu)$ so that

$$u \in C^1([0,T]; L^2(H,\nu)) \cap C([0,T]; D(K^\nu)).$$

In view of Proposition 4.4 we have

$$u \in C([0,T]; L^2(H,\nu)) \cap L^2(0,T; W^{1,2}(H,\nu)).$$

Multiplying (5.1) by $u(t)$, integrating on H with respect to ν, and taking into account (4.3), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 = \int_0^t K^\nu u(t)u(t)d\nu = -\frac{1}{2}\int_0^t \|Du(s)\|^2 ds.$$

Integrating between 0 and t we find (5.2). Now let $\varphi \in L^2(H,\nu)$ and let $\{\varphi_n\} \subset D(K^\nu)$ be such that

$$\varphi_n \to \varphi, \text{ in } L^2(H,\nu).$$

Then, setting $u_n(t) = P_t^\nu \varphi_n$, it follows by (5.2) that

$$\|u_n(t) - u_m(t)\|^2 + \int_0^t \|D(u_n(s) - u_m(s))\|^2 ds = \|\varphi_n - \varphi_m\|^2.$$

Therefore the sequence $\{u_n\}$ is Cauchy both in

$$C([0,T]; L^2(H,\nu)) \quad \text{and in } L^2(0,T; W^{1,2}(H,\nu)),$$

and the conclusion follows. □

We study now ergodicity and mixing of ν.

Theorem 5.2. *ν is ergodic.*

Proof: We have to show that if φ_0 is such that

$$P_t^\nu \varphi_0 = \varphi_0, \ \forall\, t \geq 0, \tag{5.3}$$

then φ_0 is constant ν a.s. In fact if (5.3) holds we have by (5.2), $D\varphi_0 = 0$ so that φ_0 is constant ν a.s. □

Theorem 5.3. *P_t^ν does not have angle variables so that ν is weakly mixing.*

Proof: Let $\lambda \in \mathbb{R}$ different from 0 and $\varphi \in L^2(H,\nu)$ be such that

$$P_t^\nu \varphi = e^{i\lambda t}\varphi, \; t \geq 0. \tag{5.4}$$

We have to show that $\varphi = 0$. Setting $\varphi = \varphi_1 + i\varphi_2$ with φ_1 and φ_2 real, (5.4) is equivalent to

$$K\varphi_1 = -\lambda\varphi_2, \quad K\varphi_2 = \lambda\varphi_1. \tag{5.5}$$

Taking into account (4.3) it follows

$$\int_H K\varphi_1\varphi_1 d\nu = -\frac{1}{2}\int_H |D\varphi_1|^2 d\nu = -\lambda\int_H \varphi_1\varphi_2 d\nu, \tag{5.6}$$

and

$$\int_H K\varphi_2\varphi_2 d\nu = -\frac{1}{2}\int_H |D\varphi_2|^2 d\nu = \lambda\int_H \varphi_1\varphi_2 d\nu. \tag{5.7}$$

Summing (5.6) and (5.7) gives

$$\int_H [|D\varphi_1|^2 + |D\varphi_2|^2] d\nu = 0.$$

Thus φ_1 and φ_2 are constant and by (5.5) it follows $\varphi_1 = \varphi_2 = 0$. □

To prove that ν is strongly mixing we need another regularity result for problem (5.1) and a further assumption,

Hypothesis 5.4. *For any $x \in H$ and $y \in D(A)$, we have*

$$\langle Ay, y\rangle + \langle DF(x)y, y\rangle \leq 0, \tag{5.8}$$

Proposition 5.5. *For any $\varphi \in W^{1,2}(H,\nu)$ and any $T > 0$ we have*

$$u \in C([0,T]; W^{1,2}(H,\nu)) \cap L^2(0,T; W^{2,2}(H,\nu)),$$

and

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|Du(t)\|^2 + \frac{1}{2}\int_H Tr\,[(D^2u(t))^2] d\nu \\ &= \int_H \langle ADu(t), Du(t)\rangle d\nu + \int_H \langle DF(x)Du(t), Du(t)\rangle d\nu. \end{aligned} \tag{5.9}$$

Proof: It is enough to prove (5.9) for $\varphi \in \mathcal{E}_A(H)$. We assume that $\{e_k\} \subset D(A)$ for simplicity and set

$$a_{i,j} = \langle Ae_j, e_i\rangle, \quad F_i(x) = \langle F(x), e_i\rangle, \; i,j \in \mathbb{N}.$$

Differentiating (5.1) with respect to x_h gives

$$\frac{d}{dt} D_h u(t) = K^\nu D_h u(t) + \sum_{i=1}^\infty a_{i,h} D_i u(t) + \sum_{i=1}^\infty D_h F_i(x) D_i u(t).$$

Multiplying by $D_h u(t)$, integrating with respect to ν, and taking into account (4.3), we find

$$\frac{1}{2}\frac{d}{dt}\|D_h u(t)\|^2+\frac{1}{2}\|DD_h u(t)\|^2$$

$$=\sum_{i=1}^{\infty}a_{i,h}\int_H D_i u(t)D_h u(t)d\nu+\sum_{i=1}^{\infty}\int_H D_h F_i(x)D_i u(t)D_h u(t)d\nu.$$

Summing up on h, the conclusion follows. □

Corollary 5.6. *Assume that Hypothesis 5.4 holds and let $\varphi\in L^2(H,\mu)$. Then we have*

$$\lim_{t\to+\infty}\|Du(t)\|=0. \tag{5.10}$$

Proof: Assume first that $\varphi\in W^{1,2}(H,\mu)$. Then by (5.9) it follows

$$\frac{d}{dt}\|Du(t)\|^2\le\|\varphi\|^2,$$

and so $\|Du(\cdot)\|$ is decreasing. On the other hand by (5.2) it follows

$$\int_0^{+\infty}\|Du(s)\|^2ds\le\|\varphi\|^2, \tag{5.11}$$

and therefore (5.10) must hold. Let now $\varphi\in L^2(H,\mu)$ and let $\{\varphi_n\}\subset W^{1,2}(H,\mu)$ be such that $\varphi_n\to\varphi$ in $L^2(H,\mu)$, and set $u_n(t)=P_t^\nu\varphi$. Then for any $T>0$ we have by Proposition 5.1

$$u_n\to u \text{ in } L^2(0,T;W^{1,2}(H,\mu)).$$

Since $\|Du_n(\cdot)\|$ is decreasing, then $\|Du(\cdot)\|$ is decreasing ν–a.s. and the conclusion follows arguing as before. □

We can prove now strongly mixing of ν. From now on we set

$$\overline{\varphi}=\langle\varphi,1\rangle_{L^2(H,\nu)}.$$

Theorem 5.7. *Let $\varphi\in L^2(H,\nu)$. Then*

$$\lim_{t\to+\infty}P_t\varphi=\overline{\varphi},\ \textit{weakly}. \tag{5.12}$$

Proof: Let $\{t_k\}\uparrow+\infty$ and $\psi\in L^2(H,\nu)$ be such that $u(t_k)$ converges weakly to ψ as $k\to+\infty$. By Corollary 5.6 it follows that $\lim_{k\to+\infty}Du(t_k)=0$. Since D is a closed operator in $L^2(H,\nu)$ it follows $D\psi=0$. Therefore $\psi=\overline{\varphi}$. The conclusion follows now from a standard argument. □

Corollary 5.8. *Assume, besides the assumptions of Theorem 5.7, that K^ν is symmetric. Then for any $\varphi\in L^2(H,\nu)$ we have*

$$\lim_{t\to+\infty}P_t\varphi=\overline{\varphi}\ in\ L^2(H,\nu). \tag{5.13}$$

Proof: We have

$$\|P_t\varphi\|^2 = \langle P_{2t}\varphi, \varphi\rangle_{L^2(H,\nu)}.$$

Therefore, letting t tend to infinity,

$$\lim_{t\to\infty} \|P_t\varphi\|^2 = (\overline{\varphi})^2.$$

Now the conclusion follows from Theorem 5.7. □

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On Existence Results for Fluids with Shear Dependent Viscosity – Unsteady Flows

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Abstract. New existence results for the system describing unsteady motion of incompressible fluids with shear dependent viscosity are presented. The nonlinear elliptic operator, which is related to the stress tensor, satisfies the p coercivity condition and is strictly monotone. We prove global existence of weak solutions if $p > \frac{2(d+1)}{d+2}$ for the space periodic problem; and for no-stick boundary conditions.

Keywords: Non-Newtonian fluid, shear dependent viscosity, unsteady flow, space-periodic boundary condition, weak solution, almost everywhere convergence

Classification: 35D05, 35Q35, 76A05, 76D99

1. Introduction

The unsteady motion of an incompressible fluid with a constant density[1] in a bounded domain $\Omega \subseteq I\!R^d$, $d \geq 2$, is described by the following system of partial differential equations:

$$\frac{\partial \mathbf{v}}{\partial t} + v_k \frac{\partial \mathbf{v}}{\partial x_k} - \operatorname{div} \mathbf{T}^E = -\nabla \pi + \mathbf{f}, \quad \operatorname{div} \mathbf{v} = 0, \tag{1.1}$$

where $\mathbf{v} = (v_1, \dots, v_d)$ is the velocity field, π represents the pressure and $\mathbf{f}$ denotes the vector of given body forces. All functions are evaluated at $(t,x) \in Q_T$, where $Q_T = (0,T) \times \Omega$, $T \in (0,\infty)$.

We denote the symmetric velocity gradient by $\mathbf{D}(\mathbf{v})$, i.e., $\mathbf{D}(\mathbf{v}) \equiv \frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathsf{T}})$. If the fluid is a fluid with shear dependent viscosity, then, by definition, the stress tensor $\mathbf{T}^E$ at $(t,x) \in Q_T$ is given by a tensorial function of $\mathbf{D}(\mathbf{v})$; it means

$$\mathbf{T}^E(t,x) = \mathbf{T}(t,x,\mathbf{D}(\mathbf{v}(t,x))). \tag{1.2}$$

If the fluid is in addition homogeneous, then (1.2) reduces to $\mathbf{T}^E(t,x) = \mathbf{T}(\mathbf{D}(\mathbf{v}(t,x)))$. Fluids with shear dependent viscosity represent one important class of non-Newtonian fluids, consisting of those fluids that have the ability to capture phenomena called shear thinning or shear thickening. All variants of power-law fluids that enjoy significant attention among engineers and physicists fall into this category. Typical examples are given by the formula

$$\mathbf{T}(\mathbf{D}(\mathbf{v})) = 2\nu_0\Big(\mu_0 + |\mathbf{D}(\mathbf{v})|^2\Big)^{\frac{p-2}{2}} \mathbf{D}(\mathbf{v}), \quad \nu_0 > 0, \mu_0 \geq 0, \tag{1.3}$$

[1] We suppose that the density is equal to 1, for simplicity.

where p is a real parameter. If $p = 2$ in (1.3), then the fluid is Newtonian; note that then (1.1) leads to the Navier-Stokes equations. If $p \in [1, 2)$, then the model (1.3) belongs to shear thinning fluids, while for $p > 2$ it captures shear thickening effects. We refer the interested reader to [9], where a detailed physical introduction and further references can be found.

In this paper we will present new results on global-in-time existence of a weak solution to the space periodic problem: to solve (1.1)–(1.2) such that

$$\begin{aligned} &\mathbf{v}, \pi \text{ are periodic in the variables } x_i \text{ with period } L_i\,, \\ &\mathbf{v}(0,x) = \mathbf{v}_0(x) \quad \text{in } \Omega \equiv (0, L_1) \times \cdots \times (0, L_d)\,. \end{aligned} \tag{1.4}$$

Here, $\mathbf{v}_0$ is a given periodic, divergence-free function; $L_i \in (0, \infty), i = 1, 2, \ldots, d$. We also assume that $\mathbf{f}$ is periodic and $\int_\Omega \mathbf{f}\, dx = \mathbf{0}$. The existence result presented below holds also for the following "no-stick" boundary conditions: to solve (1.1)–(1.2) in bounded domain Ω with Lipschitz boundary $\partial\Omega$ such that[2]

$$(\mathbf{v})_n = \mathbf{v} \cdot \mathbf{n} = 0 \text{ and } (\mathbf{Tn})_\tau = 0 \text{ at } (0,T) \times \partial\Omega, \quad \mathbf{v}(0,x) = \mathbf{v}_0(x) \text{ in } \Omega\,. \tag{1.5}$$

Here, $\mathbf{v}_0$ is a given divergence-free function satisfying $\mathbf{v}_0 \cdot \mathbf{n} = 0$ at the boundary.

We will assume that (for $p > 1$) the tensor $\mathbf{T} = (T_{ij})_{i,j=1}^d$, having continuous components, satisfies the conditions of p-coercivity of polynomial growth of order $p-1$ and of strict monotonicity. This means

$$\begin{aligned} &\exists c_1 > 0,\ \varphi_1 \in L^1(Q_T): \\ &\quad \mathbf{T}(t,x,\boldsymbol{\eta}) \cdot \boldsymbol{\eta} \geq c_1 |\boldsymbol{\eta}|^p - \varphi_1(t,x) \qquad \forall (t,x) \in Q_T\,, \boldsymbol{\eta} \in \mathbb{R}^{d\times d}_{sym} \end{aligned} \tag{1.6}$$

$$\begin{aligned} &\exists c_2 > 0,\ \varphi_2 \in L^{\frac{p}{p-1}}(Q_T): \\ &\quad |\mathbf{T}(t,x,\boldsymbol{\eta})| \leq c_2 |\boldsymbol{\eta}|^{p-1} + \varphi_2(x) \qquad \forall (t,x) \in Q_T\,, \boldsymbol{\eta} \in \mathbb{R}^{d\times d}_{sym} \end{aligned} \tag{1.7}$$

and

$$(\mathbf{T}(t,x,\boldsymbol{\eta}) - \mathbf{T}(t,x,\boldsymbol{\xi})) \cdot (\boldsymbol{\eta} - \boldsymbol{\xi}) > 0 \quad \forall (t,x) \in Q_T\,, \boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{R}^{d\times d}_{sym}\,, \boldsymbol{\eta} \neq \boldsymbol{\xi}\,. \tag{1.8}$$

Here, $\mathbb{R}^{d\times d}_{sym}$ denotes the space of symmetric real $d \times d$ matrices.

It is worth noticing that if $p \geq 2$ then (1.3) fulfils ($\forall \boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{R}^{d\times d}_{sym}\,, \boldsymbol{\eta} \neq \boldsymbol{\xi}$)

$$(\mathbf{T}(\boldsymbol{\eta}) - \mathbf{T}(\boldsymbol{\xi})) \cdot (\boldsymbol{\eta} - \boldsymbol{\xi}) \geq C |\boldsymbol{\eta} - \boldsymbol{\xi}|^p\,, \tag{1.9}$$

while for $p \in (1, 2)$

$$(\mathbf{T}(\boldsymbol{\eta}) - \mathbf{T}(\boldsymbol{\xi})) \cdot (\boldsymbol{\eta} - \boldsymbol{\xi}) \geq C(p-1) \frac{|\boldsymbol{\eta} - \boldsymbol{\xi}|^2}{(\mu_0 + |\boldsymbol{\eta}|^2 + |\boldsymbol{\xi}|^2)^{\frac{2-p}{2}}}\,. \tag{1.10}$$

By *Problem* $(ShV)_{per}$ we mean the problem of finding a solution to (1.1), (1.2), (1.6)–(1.8) and (1.4), while *Problem* $(ShV)_{stick}$ stands for the problem of finding a solution to (1.1), (1.2), (1.6)–(1.8) and (1.5).

There are essentially two different methods that were applied to prove the global existence of weak solutions without restricting the size of initial data and/or the

[2] The subscripts $(\cdot)_n$ and $(\cdot)_\tau$ denote the normal and tangential components of a vector $(\cdot)$; $\mathbf{n}$ represents the outer normal to $\partial\Omega$.

length of the time interval for *Problem* $(ShV)_{per}$. The first one, developed by O. A. Ladyzhenskaya and J. L. Lions (independently of each other) (cf. [4], [5]), is a combination of compactness and monotonicity arguments, which reveals to be applicable to *Problem* $(ShV)_{per}$ (or *Problem* $(ShV)_{stick}$ or the Dirichlet problem) if $p > \frac{3d+2}{d+2}$.

The second (regularity) method assumes the existence of a scalar potential to $\mathbf{T}^E$, which enables to use the second energy inequalities to estimate a certain fraction of the $W^{2,2}$-norm (or $W^{2,p}$-norm) versus the $W^{1,2}$-norm, integrated over time. This method is, for example, applicable to (1.1), (1.3) (however, μ_0 has to be positive if $p > 2$) and yields existence if $p > \frac{3d}{d+2}$ for *Problem* $(ShV)_{per}$ (see the monograph [8], or also [1], [9]). The same result holds also for the Cauchy problem as presented in [11]. If $p \geq 2$ and $d = 3$ the existence result for the Dirichlet problem is performed in [7]. In two dimensions, this method works for a large range of p's, namely for $p > 1$ for *Problem* $(ShV)_{per}$, and for $p > \frac{6}{5}$ for the Dirichlet problem; see [8]. We wish to remark that the main contribution of this second method does not lie in the existence theory, but in new regularity and uniqueness results (we refer to [8] or [7] for details).

In this paper, we incorporate the third, so-called truncation (or capacity) method and we show the existence of a weak solution if $p > \frac{2(d+1)}{d+2}$ for *Problem* $(ShV)_{per}$ and *Problem* $(ShV)_{stick}$. The method is based on a construction of a special bounded test function and strongly relies on the strict monotonicity of $\mathbf{T}$. Comparing the results of the second and third method, we see that the new results are worth, in higher dimensions, $d \geq 3$. Of course, the assumptions on $\mathbf{T}$ are slightly more restrictive compared to the first method; on the other hand, they are more general compared to the second method. Nevertheless, examples (1.3) satisfy all of them.

In the papers [3] and [12], the third method was applied[3] to the steady problem, and the existence of weak solutions was shown for $p \geq \frac{2d}{d+1}$. (In [12], the limiting case is not included.) Our aim was to find modifications that are needed to analyze the evolutionary case. The main difficulty is due to missing information on π which is overcome by constructing special divergence-free truncated function. The extension of the method to the Dirichlet problem up to $p > \frac{2(d+1)}{d+2}$ is open.

The introductory section is completed by a list of notation and some useful inequalities. After that, in Section 2, we define the notion of weak solutions to (1.1) and formulate our main theorem. Approximations are introduced in Section 3. The proof of the main theorem is presented in Section 4.

We use standard notation for function spaces: If $q > 1$, then $W^{1,q}(\Omega)$ denotes the Sobolev space of scalar-, vector- or tensor-valued functions in $L^q(\Omega)$ having first derivatives also in $L^q(\Omega)$.

[3] A different variant of this method has been used to solve different elliptic and parabolic problems before. Let us mention for example [2] and references cited there.

Next, we define the spaces S_q and $\mathcal{V}$: (i) for *Problem $(ShV)_{per}$* we set

$$S_q \equiv \{\Phi; \Phi_i \in W^{1,q}(\mathbb{R}^d),\ \Phi_i \text{ periodic },\ \int_\Omega \Phi_i\, dx = 0,\ i = 1,2,\dots,d\};$$
$$\mathcal{V} \equiv \{\Phi; \Phi_i \in C^\infty(\mathbb{R}^d),\ \Phi_i \text{ periodic }\}.$$

(ii) for *Problem $(ShV)_{stick}$* we put

$$S_q \equiv \{\Phi; \Phi_i \in W^{1,q}(\Omega),\ \Phi \cdot \mathbf{n} = 0 \ \text{ at } \partial\Omega\};$$
$$\mathcal{V} \equiv \{\Phi; \Phi_i \in C_0^\infty(\Omega)\}.$$

Finally, we define (in both cases) H as the closure of $\mathcal{V}$ with respect to the $L^2(\Omega)$-norm, and $V_q \equiv \{\mathbf{u} \in S_q; \operatorname{div} \mathbf{u} = 0\}$. By V_q^* we mean the dual space to V_q; the brackets $\langle \cdot, \cdot \rangle$ represent this duality. Finally, if $T \in (0,\infty)$, $I \equiv (0,T)$, and X is a Banach space, then $L^q(I;X)$ denote the Bochner spaces, $q \in [1,\infty]$.

Besides usual summation convention, we keep the convention that K represents a fixed constant that maximizes all *a priori* estimates used in the text, while c is a generic constant that does not depend on "important" parameters and can change from line to line. If $\mathbf{g}$, $\mathbf{h}$ are vector-valued functions then $(\mathbf{g},\mathbf{h}) \equiv \int_\Omega g_i h_i\, dx$ and $g_i h_i \in L^1(\Omega)$, while for tensor-valued functions $\boldsymbol{\eta}$, $\boldsymbol{\xi}$ the symbol $(\boldsymbol{\eta},\boldsymbol{\xi})$ denotes $\int_\Omega \eta_{ij}\xi_{ij}\, dx$ and $\eta_{ij}\xi_{ij} \in L^1(\Omega)$.

The Korn inequality is used during the proof. It says (cf. [10]): If $p > 1$ and Ω has Lipschitz boundary, then there exists a constant c_3 such that

$$\|\nabla \mathbf{v}\|_p \le c_3 \|\mathbf{D}(\mathbf{v})\|_p \quad \forall \mathbf{v} \in W_0^{1,p}(\Omega).$$

2. Definition of weak solutions and main theorem

Definition 1. *Let $\mathbf{v}_0 \in H$, $\mathbf{f} \in L^{p'}(I;V_p^*)$, $p' = \frac{p}{p-1}$. We say that $\mathbf{v} \in L^p(I;V_p) \cap L^\infty(I;H)$ is a weak solution to the Problem $(ShV)_{per}$ (or Problem $(ShV)_{stick}$) if*

$$\begin{aligned}\int_0^T \Big[-(\mathbf{v}(t), \frac{\partial \Phi(t)}{\partial t}) + (v_k \frac{\partial \mathbf{v}(t)}{\partial x_k}, \Phi(t)) + (\mathbf{T}(\mathbf{D}(\mathbf{v}(t))), \mathbf{D}(\Phi)(t)) \Big]\, dt \\ = \int_0^T \langle \mathbf{f}(t), \Phi(t) \rangle\, dt + (\mathbf{v}_0, \Phi(0)) \quad \forall \Phi \in C_0^\infty(-\infty, T; \mathcal{V}).\end{aligned} \tag{2.1}$$

With this definition we are ready to formulate our main theorem.

Theorem 1. *Let $p > \frac{2(d+1)}{d+2}$. Then there exist weak solutions $\mathbf{v}$ to Problem $(ShV)_{per}$ or Problem $(ShV)_{stick}$.*

In the next section we will study properties of weak solutions to an approximate problem. The proof of Theorem 1 is given in Section 4.

3. Approximations and their properties

We will use approximations to (1.1) which are based on η-mollification of the convective term. Let $\omega \in C_0^\infty(\mathbb{R}^d)$ be a usual mollification kernel with support in

$B_1(0) \equiv \{x \in \mathbb{R}^d : |x| < 1\}$ and $\int_{\mathbb{R}^d} \omega\, dx = 1$. For every $\eta > 0$ (small) we set $\omega_\eta(x) \equiv \frac{1}{\eta^d}\omega(\frac{x}{\eta})$ and put $\mathbf{v}_\eta \equiv \mathbf{v} * \omega_\eta$.

Then we look for solutions $\mathbf{v} = \mathbf{v}^\eta$ of the problem

$$\begin{aligned}
&\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}_\eta)_k \frac{\partial \mathbf{v}}{\partial x_k} - \operatorname{div}(\mathbf{T}(\mathbf{D}(\mathbf{v}))) = -\nabla\pi + \mathbf{f}, \quad \operatorname{div}\mathbf{v} = 0\,, \\
&\mathbf{v}(0,x) = \mathbf{v}_0(x) \quad \text{in } \Omega \text{ and } \quad \text{either (1.4) or (1.5) holds.}
\end{aligned} \tag{3.1}$$

Lemma 1. *For $p \geq \frac{2(d+1)}{d+2}$ and every $\eta > 0$ there exists a weak solution $\mathbf{v}^\eta$ to (3.1) such that*

$$\mathbf{v}^\eta \in C(I;H) \cap L^p(I;V_p), \qquad \frac{\partial \mathbf{v}^\eta}{\partial t} \in L^{p'}(I;V_p^*) \tag{3.2}$$

and the identity

$$\int_0^T \left(\langle \frac{\partial \mathbf{v}^\eta}{\partial t}, \mathbf{\Phi}\rangle + ((\mathbf{v}_\eta)_k \frac{\partial \mathbf{v}^\eta}{\partial x_k}, \mathbf{\Phi}) + (\mathbf{T}(\mathbf{D}(\mathbf{v}^\eta)), \mathbf{D}(\mathbf{\Phi}))\right) dt = \int_0^T \langle \mathbf{f}, \mathbf{\Phi}\rangle\, dt \tag{3.3}$$

is fulfilled for all $\mathbf{\Phi} \in L^p(I;V_p)$.

Moreover, $\{\mathbf{v}^\eta\}_{\eta>0}$ satisfy the following uniform estimates:

$$\operatorname*{ess\,sup}_{t\in I} \|\mathbf{v}^\eta(t)\|_2^2 + \|\nabla \mathbf{v}^\eta(t)\|_{p,Q_T}^p + \|\mathbf{D}(\mathbf{v}^\eta(t))\|_{p,Q_T}^p \leq K, \tag{3.4}$$

$$\int_0^T (\mathbf{T}(\mathbf{D}(\mathbf{v}^\eta)), \mathbf{D}(\mathbf{v}^\eta))\, dt \leq K, \qquad \|\mathbf{v}^\eta\|_{\frac{(d+2)p}{d},Q_T} \leq K, \tag{3.5}$$

$$\left\| (\mathbf{v}_\eta)_k \frac{\partial \mathbf{v}^\eta}{\partial x_k} \right\|_{s,Q_T} \leq K, \quad s \equiv \frac{p(d+2)}{2(d+1)} \quad (s \to 1 \ \textit{as}\ p \to \frac{2(d+1)}{d+2}), \tag{3.6}$$

$$\left\| \frac{\partial \mathbf{v}^\eta}{\partial t} \right\|_{Y^*} \leq K, \quad \textit{where}\ Y \equiv L^p(I;S_p) \cap L^{s'}(Q_T). \tag{3.7}$$

Remark. The constant K depends on $\|\mathbf{v}_0\|_2^2$, $\|\mathbf{f}\|_{L^{p'}(I;V_p^*)}$, d and constants in the Sobolev and Korn inequalities. It is the same in all estimates of Lemma 1 and also in estimates of $g^{\ell,m}$ below, i.e. it does not change from line to line. Without loss of generality we can suppose that $K > 1$. Note that definition of Y does not require that we restrict to the space of divergence-free functions.

Proof of Lemma 1: It can be carried out via the Galerkin method combined with monotone operator theory and compactness arguments (see the first method quoted in the introduction). The crucial point in applying this method consists of the observation that for fixed $\eta > 0$

$$(\mathbf{v}_\eta)_k \frac{\partial \mathbf{v}}{\partial x_k} \in L^{p'}(I;V_p^*) \quad \text{if } p \geq \frac{2(d+1)}{d+2}.$$

The reader can consult [5] or [4] for details. Having (3.2) and (3.3), one obtains (3.4), $(3.5)_1$ by taking $\mathbf{\Phi} = \mathbf{v}_\eta$ in (3.3). Next, $(3.5)_2$ follows from the imbedding of $L^\infty(I;H) \cap L^p(I;V_p)$ into $L^{\frac{(d+2)p}{d}}(Q_T)$. Further, (3.6) follows directly from $(3.4)_2$ and $(3.5)_2$.

Finally, consider $\Phi \in Y$. Then by the Helmholtz decomposition one writes $\Phi(t) = \Phi_0(t) + \nabla q(t)$, where $\Delta q = \operatorname{div} \Phi$ in Ω (for a.e. $t \in I$), q either periodic (for *Problem* $(ShV)_{per}$) or $\frac{\partial q}{\partial n} = 0$ at $\partial\Omega$ (for *Problem* $(ShV)_{stick}$), $\int_\Omega q\,dx = 0$ and $\nabla q \in Y$, $\|\nabla q\|_Y \le c\|\Phi\|_Y$. Then Φ_0 is a divergence-free function and keeps our boundary conditions. Therefore $\Phi_0 \in L^p(I;V_p) \cap L^{s'}(Q_T)$. (Let us remark that this is not true for the Dirichlet problem; see also [6], pp.83–86, for related comments.) In particular, we can test by Φ_0 in (3.3). Moreover, as ∇q is orthogonal to $\frac{\partial \mathbf{v}^\eta}{\partial t}$, we have

$$\sup_{\|\Phi\|_Y \le 1} \left| \int_0^T \langle \frac{\partial \mathbf{v}^\eta}{\partial t}, \Phi \rangle \, dt \right| = \sup_{\|\Phi\|_Y \le 1} \left| \int_0^T \langle \frac{\partial \mathbf{v}^\eta}{\partial t}, \Phi_0 \rangle \, dt \right| \tag{3.8}$$

and estimate (3.7) follows from (3.5)–(3.6) and the fact that $\|\Phi_0\|_Y \le c\|\Phi\|_Y$. Let us remark that to prove (3.8) we need slightly more information on integrability of $\frac{\partial \mathbf{v}^\eta}{\partial t}$. For example, knowing that there are potentials to $\mathbf{T}^\eta$, where $\mathbf{T}^\eta$ approximate $\mathbf{T}$ locally uniformly, then we can easily check (testing by $\frac{\partial \mathbf{v}^\eta}{\partial t}$ in (3.3)) that $\frac{\partial \mathbf{v}^\eta}{\partial t} \in L^2(Q_T)$, and (3.8) easily follows. We will not go into more details, but we wish to say that the typical example (1.3) posesses a potential; consequently, to verify L^2-integrability of $\frac{\partial \mathbf{v}^\eta}{\partial t}$ for example (1.3) is an easy exercise (see [7] for more details). □

A straightforward consequence of the uniform estimates (3.4) and (3.7) is the following lemma.

Lemma 2. *Under the assumptions of Lemma* 1 *there exists a sequence* $\eta_k \to 0$ *(as* $k \to \infty$*) and* $\mathbf{v} \in L^\infty(I;H) \cap L^p(I;V_p)$ *such that* $\mathbf{v}^k \equiv \mathbf{v}^{\eta_k}$ *satisfy*

$$\mathbf{D}(\mathbf{v}^k) \rightharpoonup \mathbf{D}(\mathbf{v}), \quad \nabla \mathbf{v}^k \rightharpoonup \nabla \mathbf{v} \quad \textit{weakly in } L^p(I;L^p(\Omega)), \tag{3.9}$$

$$\mathbf{v}^k \to \mathbf{v} \quad \textit{strongly in } L^r(I;L^q(\Omega)) \quad \forall r \in [1,\infty), \forall q \in [1, \frac{dp}{d-p}) \tag{3.10}$$

$$\mathbf{v}^k \to \mathbf{v} \quad \textit{almost everywhere in } Q_T. \tag{3.11}$$

Now, we take $\mathbf{v}^\ell, \mathbf{v}^m \in \{\mathbf{v}^k\}_{k\in\mathbb{N}}$ and define

$$g^{\ell,m} \equiv (|\nabla \mathbf{v}^\ell| + |\nabla \mathbf{v}^m|)^p + (1 + |\mathbf{T}(\mathbf{D}(\mathbf{v}^\ell))| + |\mathbf{T}(\mathbf{D}(\mathbf{v}^m))|)(|\mathbf{D}(\mathbf{v}^\ell)| + |\mathbf{D}(\mathbf{v}^m)|).$$

Obviously $g^{\ell,m}$ is uniformly bounded in $L^1(Q_T)$ by K and it holds

Lemma 3. *Under the assumptions of Lemma* 1 *there is a subsequence* $\{\mathbf{v}^n\} \subset \{\mathbf{v}^k\}$ *such that for every* $\varepsilon \in (0,1)$ *there exists* $L \le \frac{\varepsilon}{K}$ *independent of* ℓ, m *and* $E^{\ell,m} \equiv \{(t,x) \in Q_T \,:\, L^2 \le |\mathbf{v}^\ell(t,x) - \mathbf{v}^m(t,x)| < L\}$ *such that*

$$\|g^{\ell,m}\|_{1,E^{\ell,m}} \le \varepsilon\,. \tag{3.12}$$

PROOF: Assume that $\varepsilon \in (0,1)$ is given and set $L_0 \equiv \frac{\varepsilon}{K}$. Take $N \in \mathbb{N}$ large enough such that $N\varepsilon > K$. For $i = 1, 2, \ldots, N$, define iteratively $L_i = L_{i-1}^2$ and set $E_i^{\ell,m} \equiv \{(t,x) \in Q_T\,;\, L_i^2 \le |\mathbf{v}^\ell(t,x) - \mathbf{v}^m(t,x)| \le L_i\}$. For fixed ℓ, m the sets $E_i^{\ell,m}$ are disjoint. Therefore $\sum_{i=1}^N \|g^{\ell,m}\|_{1,E_i^{\ell,m}} \le K$ due to the uniform boundedness of $g^{\ell,m}$. Then certainly there exists $i_0(\ell,m)$ such that

$$\|g^{\ell,m}\|_{1,E_{i_0}^{\ell,m}} \leq \varepsilon.$$

For each ℓ, m, however, $i_0 \in \{1, 2, \dots, N\}$. Then necessarily there is a subsequence $\{\mathbf{v}^n\}$ of $\mathbf{v}^k$ such that $i_0(\ell, m)$ is the same, i.e. i_0 is independent of ℓ, m. Denote the corresponding L_{i_0} by L and $E_{i_0}^{\ell,m}$ by $E^{\ell,m}$. The proof is complete. □

Further, we introduce a double-sequence $\{\Psi^{\ell,m}\}_{\ell,m\in\mathbb{N}}$ by

$$\Psi^{\ell,m} \equiv (\mathbf{v}^\ell - \mathbf{v}^m)(1 - \min(1, \frac{1}{L}|\mathbf{v}^\ell - \mathbf{v}^m|)). \tag{3.13}$$

Then $\Psi^{\ell,m} = 0$ on $Q_L^{\ell,m} \equiv \{(t,x) \in Q_T \, : \, |\mathbf{v}^\ell - \mathbf{v}^m| \geq L\}$ and $|\Psi^{\ell,m}| \leq L$ otherwise. Thus, $\Psi^{\ell,m} \in L^\infty(Q_T) \cap L^p(I; S_p)$ and by Lebesgue's Dominated Convergence Theorem and (3.10)

$$\|\Psi^{\ell,m}\|_{r,Q_T} \longrightarrow 0 \quad \text{for all } r \in [1,\infty) \quad \text{as } \ell, m \to \infty.$$

Further we observe that for given $\varepsilon \in (0,1)$

$$\|\operatorname{div} \Psi^{\ell,m}\|_{p,Q_T} \leq 2\varepsilon. \tag{3.14}$$

Indeed, if χ denotes the characteristic function of the set $Q_T \setminus Q_L^{\ell,m}$, then

$$\operatorname{div} \Psi^{\ell,m} = \frac{(\mathbf{v}^\ell - \mathbf{v}^m)_i}{L} \frac{\partial(\mathbf{v}^\ell - \mathbf{v}^m)_r}{\partial x_i} \frac{(\mathbf{v}^\ell - \mathbf{v}^m)_r}{|\mathbf{v}^\ell - \mathbf{v}^m|} \chi.$$

Thus, by (3.12), the definition of $E^{\ell,m}$ and L we see

$$\begin{aligned}
&\|\operatorname{div} \Psi^{\ell,m}\|_{p,Q_T} \leq \|\nabla(\mathbf{v}^\ell - \mathbf{v}^m)\frac{1}{L}|\mathbf{v}^\ell - \mathbf{v}^m|\|_{p,Q_T\setminus Q_L^{\ell,m}} \\
=&\|\nabla(\mathbf{v}^\ell - \mathbf{v}^m)\frac{1}{L}|\mathbf{v}^\ell - \mathbf{v}^m|\|_{p,(Q_T\setminus Q_L^{\ell,m})\setminus E^{\ell,m}} + \|\nabla(\mathbf{v}^\ell - \mathbf{v}^m)\frac{1}{L}|\mathbf{v}^\ell - \mathbf{v}^m|\|_{p,E^{\ell,m}} \\
\leq& LK + \varepsilon \leq 2\varepsilon.
\end{aligned} \tag{3.15}$$

Next, we consider an auxiliary problem (for almost all $t \in I$, $s' = \frac{p(d+2)}{p(d+2)-2(d+1)}$)

$$\begin{aligned}
&-\Delta h^{\ell,m} = -\operatorname{div} \Psi^{\ell,m} \text{ in } \Omega, \quad \frac{\partial h^{\ell,m}}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega, \text{ or } h^{\ell,m} \text{ periodic;} \\
&\int_\Omega h^{\ell,m} dx = 0, \qquad \|h^{\ell,m}\|_{2,p} \leq c\|\operatorname{div} \Psi^{\ell,m}\|_p, \quad \|h^{\ell,m}\|_{1,s'} \leq c\|\Psi^{\ell,m}\|_{s'}.
\end{aligned} \tag{3.16}$$

The existence of $h^{\ell,m}$ satisfying (3.16) is standard. Set finally

$$\Phi^{\ell,m} = \Psi^{\ell,m} - \nabla h^{\ell,m}. \tag{3.17}$$

Note that $\Phi^{\ell,m} \in L^p(I; V_p) \cap L^{s'}(Q_T)$ ($\operatorname{div} \Phi^{\ell,m} = 0$, $\Phi^{\ell,m} \cdot \mathbf{n} = 0$ at $(0,T) \times \partial\Omega$ for *Problem $(ShV)_{stick}$*, and $\Phi^{\ell,m}$ is periodic for *Problem $(ShV)_{per}$*). In addition, from (3.16) we have in both cases for ℓ, m sufficiently large

$$\|\Phi^{\ell,m}\|_{s',Q_T} \leq \varepsilon. \tag{3.18}$$

4. Proof of Theorem 1

We are going to show that the following condition holds

$$\forall\theta \in(0,1)\,\forall\varepsilon_0 \in (0,1)\,\exists m_0 \in \mathbb{N}\,\forall \ell, m \geq m_0$$
$$0 < \mathcal{I} \equiv \int_0^T (\mathbf{T}(\mathbf{D}(\mathbf{v}^\ell)) - \mathbf{T}(\mathbf{D}(\mathbf{v}^m)), \mathbf{D}(\mathbf{v}^\ell - \mathbf{v}^m))^\theta\, dt < \varepsilon_0. \tag{4.1}$$

Disposing of (4.1) and assuming (1.8) (or one of the stronger conditions (1.9) or (1.10)) we see that

$$\mathbf{D}(\mathbf{v}^n) \to \mathbf{D}(\mathbf{v}) \quad \text{almost everywhere in } Q_T. \tag{4.2}$$

To show that $\mathbf{v}$ is a weak solution to (1.1) is standard (see [3]) making use of (4.2) and (3.10). In order to prove (4.1) we substract the weak formulation (3.3) for $\mathbf{v}^\ell$ from the weak formulation for $\mathbf{v}^m$ and take $\Phi^{\ell,m}$ defined in (3.17) on the place of the test function. We obtain the following identity (χ denotes the characteristic function of the set $Q_T \setminus Q_L^{\ell,m}$)

$$\begin{aligned}
\int_0^T \langle \frac{\partial \mathbf{v}^\ell}{\partial t} - \frac{\partial \mathbf{v}^m}{\partial t}, \Psi^{\ell,m}\rangle\, dt &+ \int_0^T (\mathbf{T}(\mathbf{D}(\mathbf{v}^\ell)) - \mathbf{T}(\mathbf{D}(\mathbf{v}^m)), \mathbf{D}(\mathbf{v}^\ell - \mathbf{v}^m)\chi)\, dt \\
&= \int_0^T \langle \frac{\partial \mathbf{v}^\ell}{\partial t} - \frac{\partial \mathbf{v}^m}{\partial t}, \nabla h^{\ell,m}\rangle\, dt \\
&+ \int_0^T (\mathbf{T}(\mathbf{D}(\mathbf{v}^\ell)) - \mathbf{T}(\mathbf{D}(\mathbf{v}^m)), \mathbf{D}(\nabla h^{\ell,m}))\, dt \\
&+ \int_0^T (\mathbf{T}(\mathbf{D}(\mathbf{v}^\ell)) - \mathbf{T}(\mathbf{D}(\mathbf{v}^m)), \mathbf{D}(\mathbf{v}^\ell - \mathbf{v}^m)\frac{|\mathbf{v}^\ell - \mathbf{v}^m|}{L}\chi)\, dt \\
&+ \int_0^T ((\mathbf{v}_{\eta_m})_k \frac{\partial \mathbf{v}^m}{\partial x_k} - (\mathbf{v}_{\eta_\ell})_k \frac{\partial \mathbf{v}^\ell}{\partial x_k}, \Phi^{\ell,m})\, dt \equiv \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4.
\end{aligned} \tag{4.3}$$

Before estimating $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$, $\mathcal{I}_4$ let us first observe that the first term equals $H(\mathbf{v}^\ell - \mathbf{v}^m)(T) - H(\mathbf{v}^\ell - \mathbf{v}^m)(0)$, where H is a nonnegative primitive function to $\Psi^{\ell,m}$. As $H(\mathbf{v}^\ell - \mathbf{v}^m)(0) = 0$, we see that the first term is nonnegative.

Next, integration by parts leads to $I_1 \equiv 0$ (see also (3.8) page 7). Further, by (3.16), (3.15) and the uniform L^1-bound for $g^{\ell,m}$

$$|\mathcal{I}_3| \leq c\,\|g^{\ell,m}\|_{1,Q_T}\|h^{\ell,m}\|_{2,p,Q_T} \leq cK\varepsilon.$$

$\mathcal{I}_2$ is estimated in the same way as $\operatorname{div}\Psi^{\ell,m}$ in (3.15) by using that the domain of integration is $Q_T \setminus Q_L^{\ell,m}$. Therefore $|\mathcal{I}_2| \leq c\epsilon$.

Finally, $\mathcal{I}_4$ is small due to (3.6) and (3.18). Then (4.1) follows from the proved estimates of the second term in (4.3) by the Hölder inequality and $(3.10)_2$, cf. [2]. □

Remark. Instead of estimating Cauchy sequences $\mathbf{v}^\ell - \mathbf{v}^m$, one can work more directly with $\mathbf{v}^m - \mathbf{v}$. This brings only minor changes; see [3] for details.

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J. Málek was partially supported by the grant GAČR no. 201/96/0228 and by grants of the SFB 256 at the University of Bonn.

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Stability of Propagating Fronts in Damped Hyperbolic Equations

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Abstract. We consider the damped hyperbolic equation

$$\varepsilon u_{tt} + u_t = u_{xx} + F(u) , \quad x \in \mathbf{R} , \quad t \geq 0 , \quad u \in \mathbf{R} ,$$

where ε is a positive, not necessarily small parameter. We assume that $F(0) = F(1) = 0$ and that F is concave on the interval $[0, 1]$. Under these assumptions, our equation has a continuous family of monotone propagating fronts (or travelling waves) indexed by the speed parameter $c \geq c_*$. Using energy estimates, we first show that the travelling waves are locally stable with respect to perturbations in a weighted Sobolev space. Then, under additional assumptions on the non-linearity, we obtain global stability results using a suitable version of the hyperbolic Maximum Principle. Finally, in the critical case $c = c_*$, we use self-similar variables to compute the exact asymptotic behavior of the perturbations as $t \to +\infty$. In particular, setting $\varepsilon = 0$, we recover several stability results for the travelling waves of the corresponding parabolic equation.

Keywords: damped hyperbolic equations, travelling waves, stability, asymptotic behavior, self-similar variables

Classification: 35B40, 35B35, 35B30, 35L05, 35C20

1. Introduction

Mathematical models for spreading and interacting particles or individuals are very common in chemistry and biology, especially in genetics and population dynamics. If the spatial spread of the particles is described by Brownian motion, these models usually take the form of reaction-diffusion equations or systems for the population densities [10,27]. Depending on the precise form of the interaction, such systems exhibit interesting solutions like propagating fronts or travelling waves, whose existence and stability properties have attracted a lot of attention in recent years [32].

It can be argued, however, that diffusion is not a realistic model of spatial spread for short times, because particles performing Brownian motion can move with arbitrarily high speed, and the directions of motion at successive times are uncorrelated. This drawback can be eliminated by replacing the diffusion process with a velocity jump process which is more satisfactory for short times and has equivalent long-time properties [17,22,31]. In one space dimension, this procedure leads to damped wave equations instead of reaction-diffusion systems [6,19,20,33]. Under the same assumptions as in the parabolic case, the damped hyperbolic equations also have travelling wave solutions [18] with analogous stability properties [14,16]. The aim of this paper is to review some of these stability results in the simplest case of a scalar equation with a non-linearity of "monostable" type.

We thus consider the damped hyperbolic equation

$$\varepsilon u_{tt} + u_t = u_{yy} + F(u) , \tag{1.1}$$

where $y \in \mathbf{R}$, $t \in \mathbf{R}_+$ and ε is a positive, *not necessarily small* parameter. We assume that the non-linearity $F \in C^2(\mathbf{R}, \mathbf{R})$ has the following properties:

(H1) $F(0) = F(1) = 0\ , \quad F'(0) > 0\ , \quad F'(1) < 0\ , \quad F''(u) < 0$ for $u \in (0,1]$.

In particular, $u \equiv 1$ is a stable equilibrium point of Eq.(1.1), and $u \equiv 0$ is unstable. For simplicity, we assume F being concave on $[0,1]$, but this condition is more restrictive than what we really need and could be relaxed in several ways. A typical non-linearity satisfying (H1) is $F(u) = u - u^m$, with $m \in \mathbf{N}$, $m \geq 2$.

Under the assumptions (H1), Eq.(1.1) has monotone travelling wave solutions (also called propagating fronts) connecting the equilibrium states $u = 1$ and $u = 0$. Indeed, setting

$$u(y,t) = h(\sqrt{1+\varepsilon c^2}y - ct) \equiv h(x) , \tag{1.2}$$

where $c > 0$, we obtain for the function h the ordinary differential equation

$$h''(x) + ch'(x) + F(h(x)) = 0\ , \quad x \in \mathbf{R}\ . \tag{1.3}$$

As is well-known [1,26], Eq.(1.3) has a solution satisfying $h'(x) < 0$ for all $x \in \mathbf{R}$, $h(-\infty) = 1$, $h(+\infty) = 0$ if and only if $c \geq c_* = 2\sqrt{F'(0)}$. This solution is unique up to a translation in the variable x. Therefore, for all $\varepsilon > 0$, Eq.(1.1) has a continuous family of monotone travelling waves indexed by the speed parameter $c \geq c_*$. Note that the actual speed of the wave is not c, but $c/\sqrt{1+\varepsilon c^2}$, a quantity which is bounded by $1/\sqrt{\varepsilon}$ for all $c \geq c_*$.

In the limit $\varepsilon \to 0$, Eq.(1.1) reduces to the semilinear parabolic equation $u_t = u_{yy} + F(u)$, which has been intensively studied since the pioneering work of Fisher [11] and Kolmogorov, Petrovskii and Piskunov [26]. Using the parabolic Maximum Principle and probabilistic techniques, the long-time behavior of a large class of solutions has been explicitly determined [1,2]. In a more general context, a local stability analysis of the travelling waves using functional-analytic methods has been initiated by Sattinger [30] and extended by many authors [7,23,25]. In particular, the decay rate in time of the perturbations in the critical case $c = c_*$ has been computed using renormalization techniques [3,13]. Since the equilibrium state $u = 0$ ahead of the front is linearly unstable, all these stability properties are restricted to perturbations which decay to zero at least as fast as h itself as $x \to +\infty$. Following [14,16], the aim of this paper is to show how these results can be extended to the hyperbolic case $\varepsilon > 0$.

To investigate the stability of the travelling wave (1.2) as a solution of (1.1), we go to a moving frame using the change of variables

$$u(y,t) = v(\sqrt{1+\varepsilon c^2}y - ct, t) \equiv v(x,t) ,$$

where $x = \sqrt{1+\varepsilon c^2}y - ct$. The equation for v is

$$\varepsilon v_{tt} + v_t - 2\varepsilon c v_{xt} = v_{xx} + c v_x + F(v) , \tag{1.4}$$

and, by construction, $h(x)$ is a stationary solution of (1.4). Setting $v(x,t) = h(x) + w(x,t)$, we obtain for the perturbation w the equation

$$\varepsilon w_{tt} + w_t - 2\varepsilon c w_{xt} = w_{xx} + c w_x + F'(h)w + \mathcal{N}(h,w)w^2 , \tag{1.5}$$

where

$$\mathcal{N}(h,w) = \int_0^1 (1-\sigma)F''(h+\sigma w)\,\mathrm{d}\sigma .$$

Rewriting in the usual way the second-order equation (1.5) as a first-order system for the pair (w, w_t), we shall study the stability of the origin $(w, w_t) = (0,0)$ in a weighted Sobolev space Z which we now describe.

For $k \in \mathbf{N}$, we denote by $\mathrm{H}^k = \mathrm{H}^k(\mathbf{R})$ the usual (real) Sobolev space of order k over $\mathbf{R}$, with $\mathrm{H}^0(\mathbf{R}) = \mathrm{L}^2(\mathbf{R})$. Following [30], we introduce for $s > 0$ the weight function $p_s(x) = 1 + \mathrm{e}^{sx}$ and, for $k \in \mathbf{N}$, we denote by H^k_s the Hilbert space $\mathrm{H}^k(\mathbf{R}, p_s^2\,\mathrm{d}x)$ defined by the norm

$$\|u\|^2_{\mathrm{H}^k_s} = \int_{\mathbf{R}} \left(\sum_{i=0}^{k} |\partial_x^i u(x)|^2\right) p_s(x)^2\,\mathrm{d}x . \tag{1.6}$$

Setting $\mathrm{L}^2_s = \mathrm{H}^0_s$, we define the product space $Z = \mathrm{H}^1_s \times \mathrm{L}^2_s$ equipped with the norm

$$\|(w_1, w_2)\|^2_Z = \|w_1\|^2_{\mathrm{H}^1_s} + \|w_2\|^2_{\mathrm{L}^2_s} . \tag{1.7}$$

Finally, it will be convenient to denote by Z_ε the space Z equipped with the ε-dependent norm

$$\|(w_1, w_2)\|^2_{Z_\varepsilon} = \|w_1\|^2_{\mathrm{H}^1_s} + \varepsilon\|w_2\|^2_{\mathrm{L}^2_s} . \tag{1.8}$$

The perturbation space Z clearly depends on the choice of $s > 0$ and becomes smaller when s increases. It is thus natural to look for the the smallest value of $s > 0$

for which the origin in (1.5) is linearly stable in Z. This is most conveniently done by setting $w(x,t) = \mathrm{e}^{-sx}\omega(x,t)$ and studying the equation for ω, namely:

$$\begin{aligned} &\varepsilon\omega_{tt} + (1+2\varepsilon cs)\omega_t - 2\varepsilon c\omega_{xt} = \\ &\quad \omega_{xx} + (c-2s)\omega_x + (F'(h) - cs + s^2)\omega + \mathcal{N}(h, \mathrm{e}^{-sx}\omega)\mathrm{e}^{-sx}\omega^2 \,. \end{aligned} \tag{1.9}$$

A straightforward computation in the Fourier variables shows that the origin in (1.9) is linearly stable in $\mathrm{H}^1 \times \mathrm{L}^2$ only if $F'(0) - cs + s^2 \le 0$. In fact, this condition can easily be inferred from the coefficient of ω in (1.9). Therefore, the largest perturbation space Z for which we can expect stability of the front h is obtained by choosing

$$s = \frac{1}{2}(c - \sqrt{c^2 - c_*^2}) \,. \tag{1.10}$$

Note that this value corresponds to the decay rate of h as $x \to +\infty$, since $h(x) \sim \mathrm{e}^{-sx}$ if $c > c_*$ and $h(x) \sim x\mathrm{e}^{-sx}$ if $c = c_*$ [1]. Thus, if (1.10) holds, the translations $h(x+x_0) - h(x)$ do not belong to the perturbation space H^1_s. In view of the translation invariance of Eq.(1.1), this is of course necessary to obtain an asymptotic stability result. In the sequel, we always assume that the condition (1.10) holds.

In this paper, we present three different results which show that the travelling wave h is stable with respect to perturbations in Z_ε or in a subspace of it. In Section 2, we give a local stability result valid for all $c \ge c_*$ and all $\varepsilon > 0$. Using appropriate energy functionals, we show that if $(w(0), w_t(0))$ is sufficiently small in Z_ε, then the solution $(w(t), w_t(t))$ of (1.5) stays in a neighborhood of the origin in Z_ε and converges to zero as $t \to +\infty$ in a slightly weaker norm. In the critical case $c = c_*$, we also give an estimate of the convergence rate. These results have been proved in [14] in the particular case where $F(u) = u - u^2$. Their proofs can be adapted, with minor changes, to cover the general case of a non-linearity F satisfying (H1).

Section 3 is devoted to global stability results. We first recall the Maximum Principle for hyperbolic equations [29] in a version adapted to our problem. Then, under the additional assumption that $F'(u)$ be strictly negative for $u \ge 1$, we show that the travelling wave h is stable with respect to "large" perturbations in Z_ε, provided some positivity conditions are satisfied. Furthermore, if $1 + 4\varepsilon F'(1) \ge 0$ and $F''(u) \le 0$ for $u \ge 0$, we obtain linear upper and lower bounds for the solutions of Eq.(1.5), as well as a decay rate in time for the quantity $\|p_s w(t)\|_{\mathrm{L}^\infty}$. The proofs of these results can also be found in [14] if $F(u) = u - u^2$.

In Section 4, we restrict our analysis to the critical case $c = c_*$, and we study the long-time behavior of the solutions (w, w_t) of (1.5) in a slightly smaller function space. In particular, we show that

$$w(x,t) = \frac{\alpha}{t^{3/2}} h'(x)\varphi^*\left(\frac{x\sqrt{1+\varepsilon c_*^2}}{\sqrt{t}}\right) + o(t^{-3/2}) \,, \quad t \to +\infty \,,$$

where $\alpha \in \mathbf{R}$ and $\varphi^* : \mathbf{R} \to \mathbf{R}$ is a *universal* profile. In the parabolic case $\varepsilon = 0$, this asymptotic expansion has been obtained by Gallay [13] using the renormalization group method [5] combined with resolvent estimates. We follow here a simpler and fairly different approach based on self-similar variables and energy estimates only. We refer to [16] for the detailed proof.

To conclude this introduction, we would like to point out the striking similarity between the stability results presented here and the corresponding statements in the parabolic case. This is an illustration of the more general fact that the long-time behavior of solutions to damped hyperbolic equations such as (1.1) is essentially parabolic [15]. A similar phenomenon can be observed in the context of hyperbolic conservation laws with damping [21,28].

2. Local Stability of the Travelling Waves

In this section, we show that the travelling wave h is stable with respect to sufficiently small perturbations in the space Z_ε. Our main result is:

Theorem 2.1. *Assume that (H1) holds, and let $\varepsilon_0 > 0$, $c \geq c_*$. Then there exist constants $\delta_0 > 0$ and $K_0 \geq 1$ such that, for all $0 < \varepsilon \leq \varepsilon_0$, the following result holds: for all $(\varphi_0, \varphi_1) \in Z_\varepsilon$ such that $\|(\varphi_0, \varphi_1)\|_{Z_\varepsilon} \leq \delta_0$, there exists a unique solution $(w, w_t) \in C^0([0,\infty), Z_\varepsilon)$ of (1.5) with initial data $(w(0), w_t(0)) = (\varphi_0, \varphi_1)$. Moreover, one has*

$$\|(w(t), w_t(t))\|_{Z_\varepsilon} \leq K_0 \|(\varphi_0, \varphi_1)\|_{Z_\varepsilon} , \quad t \geq 0 , \tag{2.1}$$

and

$$\lim_{t \to +\infty} \left(\|w(t)\|_{H^1} + \|(p_s w(t))_x\|_{L^2} + \|p_s w_t(t)\|_{L^2} \right) = 0 , \tag{2.2}$$

where $p_s(x) = 1 + e^{sx}$ and s is given by (1.10).

Sketch of the proof. We follow the lines of the proof of Theorem 1.1 in [14]. Let

$$\mathcal{N}_1(h) = \int_0^1 F''(\sigma h)\, d\sigma , \quad \mathcal{N}_2(h, w) = \frac{1}{2} \int_0^1 (1-\sigma)^2 F'''(h + \sigma w)\, d\sigma .$$

Given $0 < \varepsilon \leq \varepsilon_0$ and $c \geq c_*$, we assume that $(w, w_t) \in C^0([0,T], Z_\varepsilon)$ is a solution of (1.5) satisfying

(A1) $$\|w(t)\|_{H^1_s} \leq \delta , \quad t \in [0,T] ,$$

for some (sufficiently small) $\delta > 0$. As in (1.9), we set $w(x,t) = e^{-sx}\omega(x,t)$. To control the behavior of w on $[0,T]$, we introduce two families of energy functionals:

$$\begin{aligned}
E_0(t) &= \frac{1}{2} \int_{\mathbf{R}} \left(\varepsilon \omega_t^2 + \omega_x^2 - \omega^2 (h\mathcal{N}_1(h) + 2w\mathcal{N}_2(h,w)) \right) dx , \\
E_1(t) &= \int_{\mathbf{R}} \left((\tfrac{1}{2} + \varepsilon c s)\omega^2 + \varepsilon \omega \omega_t \right) dx , \\
E_2(t) &= \alpha_0 E_0(t) + E_1(t) ,
\end{aligned}$$

where $\alpha_0 = \max(2\varepsilon, 1/(2c^2))$, and

$$\begin{aligned}
\mathcal{E}_0(t) &= \frac{1}{2}\int_{\mathbf{R}} \left(\varepsilon w_t^2 + w_x^2 + s^2 w^2 - w^2(h\mathcal{N}_1(h) + 2w\mathcal{N}_2(h,w))\right) \mathrm{d}x \ , \\
\mathcal{E}_1(t) &= \int_{\mathbf{R}} \left((\tfrac{1}{2} - \varepsilon cs)w^2 + \varepsilon w w_t\right) \mathrm{d}x \ , \\
\mathcal{E}_2(t) &= \alpha_1 \mathcal{E}_0(t) + \mathcal{E}_1(t) + \alpha_2 (E_0(t)E_2(t))^{1/2} \ ,
\end{aligned}$$

where $\alpha_1 = \max(2\varepsilon, \theta/(2c^2))$ and θ, α_2 are positive constants. Note that the functionals E_i, $\mathcal{E}_i$ control the behavior of the perturbation w "ahead" and "behind" the front, respectively.

Arguing as in [14, Section 2], one can show that if δ, θ and α_2^{-1} are sufficiently small, then the functions E_0, E_2, $\mathcal{E}_2$ are non-negative and satisfy the differential inequalities

$$\begin{aligned}
&\frac{\mathrm{d}E_0}{\mathrm{d}t}(t) \le \frac{1}{2}(c^2 - c_*^2)\, E_0(t) \ , \quad \frac{\mathrm{d}E_2}{\mathrm{d}t}(t) + E_0(t) \le 0 \ , \\
&\frac{\mathrm{d}\mathcal{E}_2}{\mathrm{d}t}(t) + \alpha_3 \mathcal{E}_2(t) \le C_1 (E_0(t)E_2(t))^{1/2} \ , \quad t \in [0,T] \ ,
\end{aligned} \tag{2.3}$$

for some $\alpha_3, C_1 > 0$. In addition, there exists a constant $C_2 \ge 1$ such that

$$C_2^{-1} \|(w, w_t)\|_{Z_\varepsilon}^2 \le E_2(t) + \mathcal{E}_2(t) \le C_2 \|(w, w_t)\|_{Z_\varepsilon}^2 (1 + \Psi(\|w(t)\|_{\mathrm{L}^\infty})) \ , \tag{2.4}$$

where the function $\Psi : \mathbf{R}_+ \to \mathbf{R}_+$ is defined by

$$\Psi(K) = \sup_{0 \le u \le 1+K} |F'(u)| \ . \tag{2.5}$$

Combining the estimates Eqs.(2.3), (2.4), we obtain the bound

$$\|(w(t), w_t(t))\|_{Z_\varepsilon} \le C_0 \|(w(0), w_t(0))\|_{Z_\varepsilon} (1 + \Psi(\|w(0)\|_{\mathrm{L}^\infty}))^{1/2} \ , \quad t \in [0,T] \ , \tag{2.6}$$

where C_0 is a positive constant depending only on ε_0, c and F. Since the Cauchy problem for Eq.(1.5) in Z_ε is locally well-posed [14], this proves global existence of the solution (w, w_t) provided the right-hand side of (2.6) is smaller than the quantity δ appearing in (A1). Then, the differential inequalities (2.3) imply that $E_0(t)$ and $\mathcal{E}_2(t)$ converge to zero as $t \to +\infty$, thus proving (2.2). □

Remarks.

1) In the critical case $c = c_*$, it follows from (2.3) that E_0 is non-increasing in time, and that $tE_0(t) + t^{1/2}\mathcal{E}_2(t) \to 0$ as $t \to +\infty$. In particular, we have

$$\lim_{t \to +\infty} \left(t^{1/4}(\|w(t)\|_{\mathrm{H}^1} + \|w_t(t)\|_{\mathrm{L}^2}) + t^{1/2}(\|\omega_x(t)\|_{\mathrm{L}^2} + \|\omega_t(t)\|_{\mathrm{L}^2}) \right) = 0 \ . \tag{2.7}$$

This decay rate is not optimal; see Section 4 below.
2) Combining (2.1), (2.2), (2.7), we obtain

$$\lim_{t\to+\infty} \|p_s w(t)\|_{L^\infty} = 0 \ \text{ if } c > c_* \ , \quad \text{and} \quad \lim_{t\to+\infty} t^{1/4}\|p_s w(t)\|_{L^\infty} = 0 \ \text{ if } c = c_* \ . \tag{2.8}$$

3) All the estimates in the proof of Theorem 2.1 are uniform in ε for $\varepsilon \in (0, \varepsilon_0]$. In particular, taking the limit $\varepsilon \to 0$ everywhere, we obtain a proof of the local stability of the travelling wave h for the corresponding parabolic equation.
4) If $c = c_*$, the stability condition $F'(0) - cs + s^2 = (s - c_*/2)^2 \le 0$ implies $s = c_*/2$, hence (1.10) is the only possibility. On the other hand, if $c > c_*$ and s is chosen so that $F'(0) - cs + s^2 < 0$ (in contrast to (1.10)), one can show using the same spectral estimates as in the parabolic case that the origin in (1.5) is exponentially stable in Z_ε. The fastest decay rate in time for the perturbations is obtained if we set $s = \hat{s}(\varepsilon)$, where

$$\hat{s}(\varepsilon) = \frac{c}{2}\sqrt{\frac{1 + 4\varepsilon F'(0)}{1 + \varepsilon c^2}} \ .$$

3. Global Stability of the Travelling Waves

Throughout this section, we assume that the non-linearity F satisfies (H1) and

$$\text{(H2)} \qquad F'(u) \le -\mu < 0 \ , \quad u \ge 1 \ ,$$

for some $\mu > 0$. Under this additional assumption, it is known in the parabolic case that the front h is stable with respect to large perturbations in H^1_s satisfying a positivity condition. This property is a consequence of the classical Maximum Principle for parabolic equations. In this section, we show a similar global stability result for $\varepsilon > 0$ using a hyperbolic Maximum Principle.

Our starting point is the following crucial observation. Let $0 < \varepsilon \le \varepsilon_0$, $c \ge c_*$, $d \in (0, 1]$, and assume that $(w, w_t) \in C^0([0, T], Z_\varepsilon)$ is a solution of (1.5) satisfying, instead of (A1),

$$\text{(A2)} \qquad w(x, t) \ge -(1 - d)h(x) \ , \quad x \in \mathbf{R} \ , \quad t \in [0, T] \ .$$

In other words, we assume that $v(x, t) \equiv h(x) + w(x, t) \ge dh(x)$. Then it can be shown (see [14] in the case $F(u) = u - u^2$) that the *a priori* estimates (2.3), (2.4), (2.6) still hold for the solution (w, w_t), with constants C_0, C_1, C_2 depending only on ε_0, c, d and F. Thus, we can remove the smallness condition (A1) in the proof of Theorem 2.1 provided we are able to show that the positivity condition (A2) is satisfied for all times. This in turn can be obtained from the Maximum Principle under appropriate assumptions on the initial data.

3.1. The Hyperbolic Maximum Principle

Motivated by (1.4) or (1.5), we consider the hyperbolic operator L with constant coefficients

$$Lu = u_{xx} + 2\varepsilon c u_{xt} - \varepsilon u_{tt} + c u_x - u_t \ . \tag{3.1}$$

Assume that $\ell : \mathbf{R} \times [0,T] \to \mathbf{R}$ is a continuous function such that

$$\ell(x,t) \geq \underline{\ell} \ , \quad x \in \mathbf{R} \ , \quad t \in [0,T] \ , \tag{3.2}$$

where T is a positive number and $\underline{\ell} \in \mathbf{R}$ satisfies

$$1 + 4\varepsilon\underline{\ell} \geq 0 \ . \tag{3.3}$$

The following result is a consequence of the Maximum Principle given by Protter and Weinberger (see [29, Chapter 4, Theorem 1] or [14, Appendix A, Theorem A.1]).

Theorem 3.1. *Let $\varepsilon > 0$, $c > 0$, and assume that the conditions (3.2) and (3.3) are satisfied. If $(u(x,t), u_t(x,t))$ belongs to $C^0([0,T], H^1_{\mathrm{loc}}(\mathbf{R}) \times L^2_{\mathrm{loc}}(\mathbf{R}))$, with $u_{xx} + 2\varepsilon c u_{xt} - \varepsilon u_{tt}$ in $L^2_{\mathrm{loc}}(\mathbf{R} \times (0,T))$, and if*

$$\big(L + \ell(x,t)\big)u(x,t) \geq 0 \ , \quad \text{a.e.}(x,t) \in \mathbf{R} \times [0,T] \ , \tag{3.4}$$

$$u(x,0) \leq 0 \ , \quad \forall x \in \mathbf{R} \ , \tag{3.5}$$

$$\varepsilon u_t(x,0) - \varepsilon c u_x(x,0) + \frac{1}{2}u(x,0) \leq 0 \ , \quad \text{a.e. in } \mathbf{R} \ , \tag{3.6}$$

then $u(x,t) \leq 0$ for all $(x,t) \in \mathbf{R} \times [0,T]$.

As an application, we define for $d \in [0,1]$, $K \geq 0$ the function

$$\Lambda_d(K) = \inf \left\{ \frac{F(v) - F(u)}{v - u} \,\middle|\, 0 \leq u \leq d \, , \ u < v \leq 1{+}K \right\} \leq 0 \ . \tag{3.7}$$

We have the following result:

Proposition 3.2. *Assume that (H1), (H2) hold. Let $\varepsilon > 0$, $c \geq c_*$, $d \in [0,1]$ and let K be a non-negative constant such that*

$$1 + 4\varepsilon\Lambda_d(K) \geq 0 \ . \tag{3.8}$$

For some $T > 0$, assume that $(w, w_t) \in C^0([0,T], Z_\varepsilon)$ is a solution of (1.5) with initial data (φ_0, φ_1) satisfying

$$\varphi_0(x) \geq -(1-d)h(x) \ , \quad x \in \mathbf{R} \ , \tag{3.9}$$

$$\varepsilon\varphi_1(x) \geq \varepsilon c(\varphi_0'(x) + (1-d)h'(x)) - \frac{1}{2}(\varphi_0(x) + (1-d)h(x)) \,, \quad \text{a.e. } in\ \mathbf{R} \,. \quad (3.10)$$

Suppose moreover that

$$w(x,t) \leq K \,, \quad (x,t) \in \mathbf{R} \times [0,T] \,. \quad (3.11)$$

Then

$$w(x,t) \geq -(1-d)h(x) \,, \quad (x,t) \in \mathbf{R} \times [0,T] \,. \quad (3.12)$$

Proof. Without loss of generality, we may assume that $F'(u) \geq 0$ for all $u \leq 0$. Let $u(x,t) = dh(x) - v(x,t)$, where $v(x,t) = h(x) + w(x,t)$. Since $v(x,t)$ is a solution of (1.4), it is straightforward to verify that $(L + \ell)u(x,t) = F(dh(x)) - dF(h(x))$, where L is defined in (3.1) and

$$\ell(x,t) = \frac{F(v(x,t)) - F(dh(x))}{v(x,t) - dh(x)} \,.$$

In view of (H1), one has $F(dh(x)) - dF(h(x)) \geq 0$ for all $x \in \mathbf{R}$. Furthermore, since $v(x,t) \leq 1 + K$ by (3.11), we have $\ell(x,t) \geq \underline{\ell} = \Lambda_d(K)$. Indeed, this inequality follows immediately from (3.7) if $v(x,t) \geq dh(x)$; in the converse case, we observe that $\ell(x,t) \geq \min(F'(dh(x)), 0) \geq \Lambda_d(K)$. Thus (3.8) implies (3.3), and the conditions (3.9), (3.10) are nothing else as the hypotheses (3.5) and (3.6). Therefore Theorem 3.1 shows that $u(x,t) = dh(x) - v(x,t) \leq 0$ for all $(x,t) \in \mathbf{R} \times [0,T]$, which is (3.12). □

Remark. Theorem 3.1 suggests the definition of a partial order in $H^1_{\text{loc}}(\mathbf{R}) \times L^2_{\text{loc}}(\mathbf{R})$ as follows. We say that $(\varphi_0, \varphi_1) \leq (\psi_0, \psi_1)$ if

$$\varphi_0(x) \leq \psi_0(x) \,, \quad x \in \mathbf{R} \,,$$
$$\varepsilon\varphi_1(x) - \varepsilon c\varphi_0'(x) + \frac{1}{2}\varphi_0(x) \leq \varepsilon\psi_1(x) - \varepsilon c\psi_0'(x) + \frac{1}{2}\psi_0(x) \quad \text{a.e. in } \mathbf{R};$$

see (3.5), (3.6). Then, if $(\varphi_0, \varphi_1) \leq (\psi_0, \psi_1)$, the solution of the linear hyperbolic equation $(L + \ell)u(x,t) = 0$ satisfying $u(x,0) = \varphi_0(x)$, $u_t(x,0) = \varphi_1(x)$ stays for all $t \in \mathbf{R}_+$ below the solution of the same equation with initial data (ψ_0, ψ_1). This order has the property that we can write any $(\varphi_0, \varphi_1) \in H^1_{\text{loc}} \times L^2_{\text{loc}}$ as the sum of a "positive" part $(\varphi_0^+, \varphi_1^+) \geq 0$ and a "negative" part $(\varphi_0^-, \varphi_1^-) \leq 0$. This decomposition is unique if we impose that $(\varphi_0^+, \varphi_1^+) = 0$ whenever $(\varphi_0, \varphi_1) \leq 0$ and $(\varphi_0^-, \varphi_1^-) = 0$ whenever $(\varphi_0, \varphi_1) \geq 0$. The formulas for $(\varphi_0^\pm, \varphi_1^\pm)$ are given by

$$\begin{aligned} &\varphi_0^+(x) = \sup(0, \varphi_0(x)) \,, \\ &\varphi_1^+(x) = c(\varphi_0^+)'(x) - \frac{1}{2\varepsilon}\varphi_0^+(x) + \sup(0, (\varphi_1 - c\varphi_0' + \frac{1}{2\varepsilon}\varphi_0)(x)) \,, \end{aligned} \quad (3.13)$$

and

$$\begin{aligned}\varphi_0^-(x) &= \inf(0, \varphi_0(x))\,,\\ \varphi_1^-(x) &= c(\varphi_0^-)'(x) - \frac{1}{2\varepsilon}\varphi_0^-(x) + \inf(0, (\varphi_1 - c\varphi_0' + \frac{1}{2\varepsilon}\varphi_0)(x))\,.\end{aligned} \tag{3.14}$$

Remark that if $(\varphi_0, \varphi_1) \in Z_\varepsilon$, then $(\varphi_0^\pm, \varphi_1^\pm) \in Z_\varepsilon$. Moreover, it can be verified that $|\varphi_1^\pm(x)| \le |\varphi_1(x)| + c|\varphi_0'(x)|$ a.e. in $\mathbf{R}$.

3.2. A General Global Stability Result

Combining the *a priori* estimates of Section 2 and the Maximum Principle, we are now able to state and prove our main stability result:

Theorem 3.3. *Assume that (H1), (H2) hold, and let $\varepsilon_0 > 0$, $c \ge c_*$, $d \in (0,1]$. There exists a constant $C_0 \ge 1$ such that, for all $0 < \varepsilon \le \varepsilon_0$ and all $K > 0$ satisfying*

$$1 + 4\varepsilon\Lambda_d(K) \ge 0\,, \tag{3.15}$$

the following result holds: If $K_* > 0$ is such that

$$C_0 K_*(1 + \Psi(K_*))^{1/2} < K\,, \tag{3.16}$$

where Ψ is defined in (2.5), then for any $(\varphi_0, \varphi_1) \in Z_\varepsilon$ satisfying the inequalities (3.9), (3.10) and the bound $\|(\varphi_0, \varphi_1)\|_{Z_\varepsilon} \le K_*$, there exists a unique solution $(w, w_t) \in C^0([0,\infty), Z_\varepsilon)$ of (1.5) with initial data (φ_0, φ_1). Moreover, one has

$$\|(w(t), w_t(t))\|_{Z_\varepsilon} \le K\,, \quad w(x,t) \ge -(1-d)h(x)\,, \tag{3.17}$$

for all $x \in \mathbf{R}$, $t \in \mathbf{R}_+$, and (2.2), (2.7) hold.

Remark. The constant C_0 in (3.16) is the same as in (2.6).

Sketch of the proof. By Proposition 3.2, the solution (w, w_t) with initial data (φ_0, φ_1) satisfies $w(x,t) \ge -(1-d)h(x)$ as long as $w(x,t) \le K$. On the other hand, in view of (3.16), the *a priori* estimate (2.6) implies that $w(x,t) \le \|(w(t), w_t(t))\|_{Z_\varepsilon} < K$ as long as $w(x,t) \ge -(1-d)h(x)$. Combining these facts and using a contradiction argument, we show that the solution is globally defined and satisfies (3.17) for all times. The differential inequalities (2.3) then imply (2.2), (2.7). □

Remarks.

1) The relations (3.15), (3.16) imply that K (hence K_*) can be chosen very big if ε is sufficiently small. In this case, Theorem 3.3 shows that the travelling wave h is stable with respect to large perturbations, provided the positivity conditions (3.9), (3.10) are satisfied. Conversely, if ε is large, then K (hence K_*) has to be very small, and Theorem 3.3 reduces to a local stability result similar to Theorem 2.1.

2) As a simple example, consider the non-linearity $F(u) = u - u^2$ [14] which satisfies (H1), (H2). Then $\Lambda_d(K) = -(d+K)$, and the condition (3.15) reads $1 - 4\varepsilon(d+K) \geq 0$. Also $\Psi(K) = 1 + 2K$, hence the hypothesis $\|(\varphi_0, \varphi_1)\|_{Z_\varepsilon} \leq K_*$ can be replaced by

$$\|(\varphi_0, \varphi_1)\|_{Z_\varepsilon} \leq CK(1+K)^{-1/3} ,$$

where C is a sufficiently large positive constant.

3.3. Further Stability Results if $1 + 4\varepsilon F'(1) \geq 0$

The previous results are still incomplete for at least two reasons. First, if $c > c_*$, they fail to give a decay rate of the perturbations as $t \to +\infty$; see (2.8). Next, they do not provide any global existence result if $d = 0$, that is, if $v(x, 0) \geq 0$. Assuming that the non-linearity F satisfies the stronger assumption

$$\text{(H3)} \qquad F''(u) \leq 0 , \quad u \geq 0 ,$$

we can give a partial answer to both questions when $1 + 4\varepsilon F'(1) \geq 0$. Indeed, in this case, the Maximum Principle allows us to compare the solution $w(x,t)$ of (1.5) with solutions of the linear equations

$$\varepsilon\tilde{w}_{tt} + \tilde{w}_t - 2\varepsilon c\tilde{w}_{xt} = \tilde{w}_{xx} + c\tilde{w}_x + F'(h)\tilde{w} , \tag{3.18}$$

and

$$\varepsilon\tilde{w}_{tt} + \tilde{w}_t - 2\varepsilon c\tilde{w}_{xt} = \tilde{w}_{xx} + c\tilde{w}_x + F'(0)\tilde{w} . \tag{3.19}$$

We denote by $(\tilde{w}(t), \tilde{w}_t(t))$ the solution of (3.18) with initial data (φ_0, φ_1), and by $(\tilde{w}^\pm(t), \tilde{w}_t^\pm(t))$ the solution of (3.19) with initial data $(\varphi_0^\pm, \varphi_1^\pm)$, the "positive" and "negative" parts of (φ_0, φ_1) defined in (3.13), (3.14). Following the arguments in [14, Section 4], we obtain our last stability result:

Theorem 3.4. *Assume that (H1), (H3) hold and that $1+4\varepsilon F'(1) \geq 0$. Given $c \geq c_*$, $d \in [0,1]$, there exists a constant $C_3(c) \geq 1$ such that, for all $K \geq 0$ satisfying*

$$1 + 4\varepsilon\Lambda_1(K) \geq 0 ,$$

the following result holds: for any $(\varphi_0, \varphi_1) \in Z_\varepsilon$ satisfying (3.9), (3.10) and

$$\inf\left(\|(\varphi_0, \varphi_1)\|_{Z_\varepsilon} , \|(\varphi_0^+, \varphi_1^+)\|_{Z_\varepsilon}\right) \leq \frac{K}{C_3(c)},$$

there exists a unique solution $(w, w_t) \in C^0([0,\infty), Z_\varepsilon)$ of (1.5) with initial data (φ_0, φ_1). Moreover, we have

$$-(1-d)h(x) \leq w(x,t) \leq K , \tag{3.20}$$

and

$$\tilde{w}^-(x,t) \le w(x,t) \le \tilde{w}(x,t) \le \tilde{w}^+(x,t) ,$$

for all $x \in \mathbf{R}$, $t \in \mathbf{R}_+$. *Finally, if* $d > 0$ *and* $1 - 4\varepsilon F'(0) > 0$, *one has*

$$\lim_{t\to+\infty} t^{1/4} \left(\|p_s w(t)\|_{L^\infty} + \|w(t)\|_{H^1} + \|w_t(t)\|_{L^2}\right) = 0 .$$

Remark. If $(\varphi_0^+, \varphi_1^+) = 0$, *i.e.* if the initial data are non-positive, then (3.20) with $K = 0$ shows that the solution $w(x,t)$ remains non-positive for all times.

4. Asymptotic Expansions in the Critical Case $c = c_*$

In this section, we restrict ourselves to the critical case $c = c_*$, and we consider perturbations of the travelling wave h in a strict subspace of Z_ε. Using self-similar variables and energy estimates, we are able to compute explicitly the long-time asymptotics of the perturbations as $t \to +\infty$. In particular, we recover the results obtained by Gallay [13] in the parabolic case $\varepsilon = 0$.

Following Kirchgässner [25], we consider solutions of (1.4) of the following form:

$$v(x,t) = h(x) + h'(x) W\left(x, \frac{t}{1+\varepsilon c_*^2}\right) ,$$

i.e. we set $w(x,t) = h'(x)W(x, t/(1+\varepsilon c_*^2))$. Then W satisfies the equation

$$\eta W_{tt} + (1-\nu\gamma(x))W_t - 2\nu W_{xt} = W_{xx} + \gamma(x)W_x + h'(x)W^2\mathcal{N}(h(x), h'(x)W) , \quad (4.1)$$

where

$$\eta = \frac{\varepsilon}{(1+\varepsilon c_*^2)^2} , \quad \nu = \frac{\varepsilon c_*}{1+\varepsilon c_*^2} , \quad \gamma(x) = c_* + 2\frac{h''(x)}{h'(x)} , \quad x \in \mathbf{R} .$$

In (4.1) and in the sequel, the second argument of the function W is simply denoted by t, instead of $t/(1+\varepsilon c_*^2)$.

From [1] we know that the travelling wave h (with $c = c_*$) satisfies

$$h(x) = \begin{cases} 1 + \mathcal{O}(e^{\beta x}) & \text{as } x \to -\infty , \\ (a_1 x + a_2)e^{-c_* x/2} + \mathcal{O}(x^2 e^{-c_* x}) & \text{as } x \to +\infty , \end{cases}$$

where $a_1 > 0$, $a_2 \in \mathbf{R}$, and $\beta = \frac{1}{2}(-c_* + \sqrt{c_*^2 - 4F'(1)}) > 0$. Similar asymptotic expansions hold for the derivatives h', h'', hence

$$\gamma(x) = \begin{cases} \gamma_- + \mathcal{O}(e^{\beta x}) & \text{as } x \to -\infty , \\ 2/(x+x_0) + \mathcal{O}(x e^{-c_* x/2}) & \text{as } x \to +\infty , \end{cases} \quad (4.2)$$

where $\gamma_- = c_* + 2\beta = 2\sqrt{F'(0) - F'(1)}$ and $x_0 = (a_2/a_1 - 2/c_*)$. The hypothesis (H1) on F also implies that $\gamma'(x) < 0$ for all $x \in \mathbf{R}$.

To study the long-time behavior of the solution W of (4.1), we use the *scaling variables* or *self-similar variables* defined by

$$\xi = \frac{x}{\sqrt{t+t_0}} , \quad \tau = \log(t+t_0) ,$$

for some $t_0 > 0$. These variables have been widely used to study the long-time behavior of solutions to parabolic equations, in particular to prove convergence to self-similar solutions [4,8,9,12,24]. In [15], it has been shown by the authors that these variables are also a powerful tool in the framework of damped hyperbolic equations. Following [15], we define the rescaled functions U and V by

$$U(\xi,\tau) = e^{3\tau/2} W(\xi e^{\tau/2}, e^\tau - t_0) , \quad V(\xi,\tau) = e^{5\tau/2} W_t(\xi e^{\tau/2}, e^\tau - t_0) , \tag{4.3}$$

or equivalently

$$\begin{aligned} W(x,t) &= \frac{1}{(t+t_0)^{3/2}} U\left(\frac{x}{\sqrt{t+t_0}}, \log(t+t_0)\right) , \\ W_t(x,t) &= \frac{1}{(t+t_0)^{5/2}} V\left(\frac{x}{\sqrt{t+t_0}}, \log(t+t_0)\right) . \end{aligned} \tag{4.4}$$

Then $U(\xi,\tau), V(\xi,\tau)$ satisfy the system

$$\begin{aligned} & U_\tau - \frac{\xi}{2}U_\xi - \frac{3}{2}U = V , \\ & \eta e^{-\tau}\big(V_\tau - \frac{\xi}{2}V_\xi - \frac{5}{2}V\big) + (1 - \nu\gamma(\xi e^{\tau/2}))V - 2\nu e^{-\tau/2}V_\xi = \\ & \qquad U_{\xi\xi} + e^{\tau/2}\gamma(\xi e^{\tau/2})U_\xi + e^{-\tau/2}h'(\xi e^{\tau/2})U^2 N(\xi,\tau) , \end{aligned} \tag{4.5}$$

where $N(\xi,\tau) = \mathcal{N}(h(\xi e^{\tau/2}), e^{-3\tau/2}h'(\xi e^{\tau/2})U)$.

We now introduce function spaces for the rescaled perturbations U, V. For $\tau \geq 0$, we denote by $\mathcal{X}_\tau, \mathcal{Y}_\tau$ the Hilbert spaces of measurable functions on $\mathbf{R}$ defined by the norms

$$\begin{aligned} \|V\|_{\mathcal{Y}_\tau}^2 &= \int_{-\infty}^0 e^{2\beta\xi e^{\tau/2}} |V(\xi)|^2 \, d\xi + \int_0^{+\infty} (1+\xi^6)|V(\xi)|^2 \, d\xi , \\ \|U\|_{\mathcal{X}_\tau}^2 &= \|U\|_{\mathcal{Y}_\tau}^2 + \|U_\xi\|_{\mathcal{Y}_\tau}^2 . \end{aligned}$$

We denote by $\mathcal{Z}_\tau$ the product space $\mathcal{Z}_\tau = \mathcal{X}_\tau \times \mathcal{Y}_\tau$ equipped with the standard norm

$$\|(U,V)\|_{\mathcal{Z}_\tau}^2 = \|U\|_{\mathcal{X}_\tau}^2 + \|V\|_{\mathcal{Y}_\tau}^2 .$$

As is easily verified, if the functions (U, V) and (W, W_t) are related through (4.3) or (4.4), then $(U(\cdot,\tau), V(\cdot,\tau)) \in \mathcal{Z}_\tau$ if and only if the actual perturbation $w(x,t) = h'(x)W(x, t/(1+\varepsilon c_*^2))$ satisfies

$$\int_{-\infty}^{0} (w^2 + w_x^2 + w_t^2)(x,t)\,\mathrm{d}x + \int_0^\infty (1+x^4)\mathrm{e}^{c_* x}(w^2 + w_x^2 + w_t^2)(x,t)\,\mathrm{d}x \;<\; \infty\,. \quad (4.6)$$

Therefore, the perturbation space considered in this section is slightly smaller than the space Z defined in (1.7), due to the factor $(1 + x^4)$ in (4.6).

Before stating the main result of this section, we explain its content in a heuristic way. Taking formally the limit $\tau \to +\infty$ in (4.5) and using (4.2), we see that U satisfies the linear parabolic equation

$$U_\tau \;=\; \mathcal{L}_\infty U \;\stackrel{\text{def}}{=}\; U_{\xi\xi} + \Bigl(\frac{\xi}{2} + \frac{2}{\xi}\Bigr)U_\xi + \frac{3}{2}U \quad \text{if } \xi > 0\,, \quad U_\xi \;=\; 0 \quad \text{if } \xi \le 0\,. \quad (4.7)$$

Therefore, it is reasonable to expect that the long-time behavior of the solutions to (4.5) will be determined by the spectral properties of the operator $\mathcal{L}_\infty$ on $\mathbf{R}_+$, with Neumann boundary condition at $\xi = 0$. Now, as is easily verified, this limiting operator is just the image under the scaling (4.4) of the radially symmetric Laplacian operator in three dimensions. Indeed, if U and W are related through (4.4), the equation $U_\tau = \mathcal{L}_\infty U$ is equivalent to $W_t = W_{xx} + (2/x)W_x$, $x > 0$. This crucial observation allows to compute exactly the spectrum of $\mathcal{L}_\infty$ in various function spaces; see [15]. For instance, in the space $\mathrm{L}^2(\mathbf{R}_+, (1+\xi^6)\mathrm{d}\xi)$, the spectrum of $\mathcal{L}_\infty$ consists of a simple, isolated eigenvalue at $\lambda = 0$, and of "continuous" spectrum filling the half-plane $\{\lambda \in \mathbf{C} \,|\, \mathrm{Re}\,\lambda \le -1/4\}$. The eigenfunction corresponding to $\lambda = 0$ is the gaussian $\mathrm{e}^{-\xi^2/4}$. Therefore, we expect that the solution $U(\xi,\tau)$ of (4.5) converges as $\tau \to +\infty$ to $\alpha\varphi^*(\xi)$ for some $\alpha \in \mathbf{R}$, where

$$\varphi^*(\xi) \;=\; \frac{1}{\sqrt{4\pi}}\begin{cases} 1 & \text{if } \xi < 0\,, \\ \mathrm{e}^{-\xi^2/4} & \text{if } \xi \ge 0\,. \end{cases}$$

This function is normalized so that $\int_0^\infty \xi^2\varphi^*(\xi)\,\mathrm{d}\xi = 1$. Since $V = U_\tau - \frac{\xi}{2}U_\xi - \frac{3}{2}U$, we also expect that $V(\xi,\tau) \to \alpha\psi^*(\xi)$, where $\psi^* = -\frac{\xi}{2}\varphi^*_\xi - \frac{3}{2}\varphi^*$. It is crucial to note that Eq.(4.7) is independent of ε: this explains why the solutions of (4.1), hence of (1.1), behave for large times in a similar way to those of the corresponding parabolic equations.

Our last result shows that the heuristic arguments above are indeed correct:

Theorem 4.1. *Assume that (H1) holds, and let $\varepsilon > 0$, $c = c_*$. There exist $\tau_0 > 0$ and $\delta_0 > 0$ such that, for all $(U_0, V_0) \in \mathcal{Z}_{\tau_0}$ satisfying $\|(U_0, V_0)\|_{\mathcal{Z}_{\tau_0}} \le \delta_0$, the system (4.5) has a unique solution $(U, V) \in \mathcal{C}^0([\tau_0, +\infty), \mathcal{Z}_\tau)$ with $(U(\tau_0), V(\tau_0)) = (U_0, V_0)$. In addition, there exists $\alpha^* \in \mathbf{R}$ such that*

$$\|U(\tau) - \alpha^*\varphi^*\|^2_{\mathcal{X}_\tau} + \int_{\tau_0}^{\tau} \mathrm{e}^{-(\tau-\sigma)/2}\|V(\sigma) - \alpha^*\psi^*\|^2_{\mathcal{Y}_\sigma}\,\mathrm{d}\sigma \;=\; \mathcal{O}(\tau^2\mathrm{e}^{-\tau/2})\,,$$

as $\tau \to +\infty$.

Remark. We say that $(U, V) \in \mathcal{C}^0([\tau_0, +\infty), \mathcal{Z}_\tau)$ is a solution of the system (4.5) if there exists a solution $(W, W_t) \in \mathcal{C}^0([0, +\infty), \mathcal{Z}_0)$ of (4.1) such that (4.3), (4.4) hold, with $t_0 = e^{\tau_0}$.

In terms of the original variables, Theorem 4.1 implies that [16]

$$\sup_{x\in\mathbf{R}} \left(1 + \frac{e^{c_* x/2}}{1+|x|}\right) \left| w(x,t) - \frac{\alpha}{t^{3/2}} h'(x) \varphi^*\left(\frac{x\sqrt{1+\varepsilon c_*^2}}{\sqrt{t}}\right)\right| = \mathcal{O}(t^{-7/4} \log t) ,$$

as $t \to +\infty$, where $\alpha = \alpha^*(1+\varepsilon c_*^2)^{3/2}$, $w(x,t) = h'(x) W(x, t/(1+\varepsilon c_*^2))$ and W is given by (4.4). In the parabolic case $\varepsilon = 0$, this result has been obtained in [13] using slightly different function spaces. Remarkably enough, the asymptotic profile φ^* is *universal*: it does not depend on the initial data, nor on the parameter $\varepsilon \geq 0$, nor on the precise form of the non-linearity F.

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On the Equations of Melt-Spinning in Viscous Flow[1]

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Abstract. In this paper we provide an expository discussion of the equations of melt-spinning in the viscous regime. In particular, we shall address the issues of existence, uniqueness and regularity of solutions. To this end, we derive energy estimates for the solutions of certain boundary-initial value problems for hyperbolic equations of the first order. By virtue of weak* compactness arguments, these estimates enable the use of the Contraction Mapping Principle in suitable Banach spaces.

Keywords: nonisothermal extensional flow, first-order hyperbolic equation, energy estimates, weak* compactness

Classification: 35L60, 35Q35, 46N20

1. INTRODUCTION

In the polymer industry melt-spinning is used to manufacture long thin filaments from thermally stable polymeric melts. It is known as a particularly simple and effective technique (cf. [7], [9]). In this nonisothermal process fibers are formed by extruding the molten polymer from a pressurized reservoir through a circular orifice (spinneret), stretching and cooling the liquid jet, and winding the solidified filament on a take-up device (spool).

In this paper we shall discuss the equations governing melt-spinning for Newtonian fluids. We assume that the flow is axisymmetric and vertically downward. Cylindrical coordinates, centered at the spinneret exit, are reduced to a radial (r) and an axial (z) component. The flow is described by the fluid velocity $\mathbf{v}$, the fluid radius R and the fluid temperature T. Following [2, p. 2545], [6, pp. 427–429] and [7, pp. 58–63], we suppose a thin filament approximation: velocity $\mathbf{v}$, radius R and temperature T depend on the axial coordinate z and time t only. In addition, $\mathbf{v}$ has a nonvanishing axial component v only, i.e. $\mathbf{v} = (0, 0, v)$.

The fluid exits the spinneret at $z = 0$ with radius R_E, velocity v_E and temperature T_E. The velocity v_S of the take-up spool is larger than the exit velocity, so that the thread is actually stretched. In the flow region between extrusion and take-up the liquid jet is cooled by the ambient gas until the polymer reaches its solidification temperature T_S ($< T_E$) and freezes. We normalize the temperature of the environment

[1]This research was supported by the Department of Mathematics at VPI&SU and by the National Science Foundation under Grant DMS-9622735.

to 0, so that T_S is positive. The frozen polymer moves on to the spool with the velocity v_S.

Assuming dominating viscous forces and omitting air drag, surface tension and gravity, we obtain the following equations of change for the flow:

$$\text{Balance of Mass} \qquad \frac{\partial}{\partial t}\left(R^2(t,z)\right) + \frac{\partial}{\partial z}\left(v(t,z)\,R^2(t,z)\right) = 0 \tag{1.1}$$

$$\text{Balance of Momentum} \qquad \frac{\partial}{\partial z}\left(R^2(t,z)\,\frac{\partial}{\partial z}v(t,z)\right) = 0 \tag{1.2}$$

$$\text{Balance of Energy} \qquad \frac{\partial}{\partial t}T(t,z) + v(t,z)\,\frac{\partial}{\partial z}T(t,z) + \beta\,\frac{T(t,z)}{R(t,z)} = 0 \tag{1.3}$$

The β in Eq. (1.3) is a constant, positive heat transfer coefficient. The tensile viscosity is assumed as constant. Eqs. (1.1)–(1.3) are complemented by the boundary and initial conditions

$$\begin{aligned}
&\text{at } z = 0: && v = v_E, \quad R = R_E, \quad T = T_E, && (1.4)\\
&\text{at } T = T_S: && v = v_S, && (1.5)\\
&\text{at } t = 0: && R = R^0, \quad T = T^0. && (1.6)
\end{aligned}$$

To remove the implicit nature of Eqs. (1.1)–(1.6), we replace the free variable z by T. To this end, we formally assume $\frac{\partial}{\partial z}T \neq 0$. Thus for $T_S \leq \tilde{T} \leq T_E$, the equation $T(t,z) = \tilde{T}$ has a unique solution $z = z(t,\tilde{T})$. If we express the quantities in Eqs. (1.1)–(1.6) by using this formal transform, we recover the equations

$$A_t(t,T) = \frac{\alpha\,T}{\sqrt{A(t,T)}}\,A_T(t,T) - \frac{v_T(t,T)}{z_T(t,T)}\,A(t,T), \tag{1.7}$$

$$\frac{\partial}{\partial T}\left(\frac{v_T(t,T)}{z_T(t,T)}\,A(t,T)\right) = 0, \tag{1.8}$$

$$z_t(t,T) = \frac{\alpha\,T}{\sqrt{A(t,T)}}\,z_T(t,T) + v(t,T). \tag{1.9}$$

Here we have set $A \stackrel{\text{def}}{=} \pi\,R^2$ and $\alpha \stackrel{\text{def}}{=} \sqrt{\pi}\,\beta$, Eqs. (1.7)–(1.9) are accompanied by the boundary conditions

$$\begin{aligned}
A(t,T_E) &= A_E(t), && (1.10)\\
z(t,T_E) &= 0, && (1.11)\\
v(t,T_E) &= v_E(t), && (1.12)\\
v(t,T_S) &= v_S(t), && (1.13)
\end{aligned}$$

and the initial conditions

$$\begin{aligned}
A(0,T) &= A^0(T), && (1.14)\\
z(0,T) &= z^0(T). && (1.15)
\end{aligned}$$

In lieu of discussing Eqs. (1.1)–(1.6), we shall analyze the boundary-initial value problem posed in Eqs. (1.7)–(1.15), thereby justifying the exchange of z and T. To this end, we should obviously assume $z_T^0 < 0$.

2. THE FIRST-ORDER HYPERBOLIC EQUATION

Eqs. (1.7) and (1.9) suggest to study boundary-initial value problems of the form

$$u_t(t,x) = p(t,x)\,u_x(t,x) + f(t,x), \quad t \in [0,t_0], x \in [a,b], \tag{2.16}$$
$$u(0,x) = u^0(x), \quad x \in [a,b], \tag{2.17}$$
$$u(t,b) = u^b(t), \quad t \in [0,t_0]. \tag{2.18}$$

We shall interpret a function r of the form $r = r(t,x)$ as a function of t, that takes values in a function space over the spatial variable x. Therefore we may write $r(t)$ in lieu of $r(t,\cdot)$. All function spaces will be real. We shall also use several abbreviations: Let $r_1 < r_2$, $s_1 < s_2$, $t_1 > 0$ and m, n, $k \in \mathbb{N}_0$. Then we write
$||\cdot||_p$ for the norm on $L^p(r_1,r_2)$, $1 \le p \le \infty$,
$||\cdot||_{H^k}$ for the norm on $H^k(r_1,r_2)$,
$||\cdot||_{m,n}$ for the norm on $W^{m,\infty}([r_1,r_2];H^n(s_1,s_2))$,
$||\cdot||_{H^{m,n}}$ for the norm on $H^m([r_1,r_2];H^n(s_1,s_2))$.
The preceding norms will always be understood with respect to (w.r.t.) the entire domain of the particular functions, so that the numbers r_1, r_2, s_1 and s_2 are clear from the context. We shall also use the notation
$||\cdot||_{p,[t]}$ for the seminorms, defined for $0 \le t \le t_1$ by $||f||_{p,[t]} \stackrel{\text{def}}{=} ||f|_{[0,t]}||_p$,
$||\cdot||_{H^k[t]}$ for the seminorms, defined for $0 \le t \le t_1$ by $||f||_{H^k[t]} \stackrel{\text{def}}{=} ||f|_{[0,t]}||_{H^k}$,
$||\cdot||_{m,n,[t]}$ for the seminorms, defined for $0 \le t \le t_1$ by $||f||_{m,n,[t]} \stackrel{\text{def}}{=} ||f|_{[0,t]}||_{m,n}$,
$||\cdot||_{H^{m,n}[t]}$ for the seminorms, defined for $0 \le t \le t_1$ by $||f||_{H^{m,n}[t]} \stackrel{\text{def}}{=} ||f|_{[0,t]}||_{H^{m,n}}$.

2.1. Statement of the Main Result.

Definition 2.1. *We shall call a function u a solution of (2.16)–(2.18) if and only if*

$$u \in W^{1,\infty}([0,t_0];L^2(a,b)) \cap L^\infty([0,t_0];H^1(a,b)), \tag{2.19}$$
$$u \text{ satisfies Eq. (2.16)}, \tag{2.20}$$
$$u \text{ satisfies Eqs. (2.17) and (2.18) pointwise.} \tag{2.21}$$

Definition 2.2. *We shall call a function h on $[0,t_0] \times [a,b]$ boundary-regular if and only if*

$$h \in W^{1,\infty}([0,t_0];H^1(a,b)) \cap L^\infty([0,t_0];H^2(a,b)), \tag{2.22}$$
$$h_x(\cdot,a), h_x(\cdot,b) \in H^1(0,t_0). \tag{2.23}$$

Theorem 2.3. *Given functions f and p on $[0,t_0]\times[a,b]$, u^0 on $[a,b]$ and u^b on $[0,t_0]$ such that*

$$
\begin{aligned}
&p \text{ and } f \text{ are boundary-regular,} && (2.24)\\
&p>0 \text{ on } [0,t_0]\times[a,b], && (2.25)\\
&u^0\in H^2(a,b), && (2.26)\\
&u^b\in H^2(0,t_0), && (2.27)\\
&u^0(b)=u^b(0), && (2.28)\\
&u^b_t(0)=p(0,b)\,u^0_x(b)+f(0,b). && (2.29)
\end{aligned}
$$

Then the boundary-initial value problem (2.16)–(2.18) has a boundary-regular solution u such that

$$
\begin{aligned}
&u\in C^1([0,t_0];H^1(a,b))\cap C([0,t_0];H^2(a,b)), && (2.30)\\
&u \text{ is unique in } W^{1,\infty}([0,t_0];L^2(a,b))\cap L^\infty([0,t_0];H^1(a,b)). && (2.31)
\end{aligned}
$$

The proof of Theorem 2.3 will be outlined in the following sections. The uniqueness claim (2.31) is readily seen with the help of Gronwall's inequality.

2.2. The Compatibility Conditions.

Definition 2.4. *Given functions f and p on $[0,t_0]\times[a,b]$, u^0 on $[a,b]$ and u^b on $[0,t_0]$ such that*

$$
\begin{aligned}
&f,p\in C^\infty([0,t_0]\times[a,b]), && (2.32)\\
&p>0, && (2.33)\\
&u^0\in C^\infty([a,b]), && (2.34)\\
&u^b\in C^\infty([0,t_0]). && (2.35)
\end{aligned}
$$

Then we shall say that, for $N\in\mathbb{N}_0$, the functions u^0 and u^b are compatible of order N w.r.t. f and p if and only if the sequences (V^n) and (W^n), defined for $n\geq 0$ by

$$V^n(x) \stackrel{\text{def}}{=} [D^n_x u^0](x) \tag{2.36}$$

and

$$
\begin{aligned}
W^0(t) &\stackrel{\text{def}}{=} u^b(t), && (2.37)\\
W^{n+1}(t) &\stackrel{\text{def}}{=} (p(t,b))^{-1}\,(W^n_t(t)-[D^n_x f](t,b)) - && (2.38)\\
&\quad (p(t,b))^{-1}\left(\sum_{k=0}^{n-1}\binom{n}{k}W^{k+1}(t)\,[D^{n-k}_x p](t,b)\right),
\end{aligned}
$$

satisfy the compatibility conditions

$$V^n(b)=W^n(0)\quad \text{for } 0\leq n\leq N. \tag{2.39}$$

Theorem 2.5. *Suppose that the functions f, p, u^0 and u^b satisfy hypotheses (2.32)–(2.35). Then, for $N \in \mathbb{N}_0$, the boundary-initial value problem (2.16)–(2.18) has a unique solution u, satisfying*

$$u \in \bigcap_{k=0}^{N+1} C^k([0,t_0]; H^{N+1-k}(a,b)), \tag{2.40}$$

if and only if u^0 and u^b are compatible of order N w.r.t. f and p.

The proof of Theorem 2.5 is straightforward, but long and tedious, and shall therefore be omitted. See [1] for details.

2.3. Approximating Sequences.

Lemma 2.6. *Given a boundary-regular function h on $[0,t_0] \times [a,b]$. Let M be a constant such that*

$$M \geq ||h||_{1,1} + ||h||_{0,2}. \tag{2.41}$$

Then there exists a sequence $(h_n)_{n\in\mathbb{N}}$ of functions on $[0,t_0] \times [a,b]$ such that, for $n \in \mathbb{N}$,

$$h_n \in C^\infty([0,t_0] \times [a,b]), \tag{2.42}$$

$$h_n \longrightarrow h \text{ in } C([0,t_0]; H^1(a,b)) \text{ as } n \to \infty, \tag{2.43}$$

$$||h_n||_{H^{1,1}} \leq \sqrt{2t_0}\,(M+1), \tag{2.44}$$

$$||h_n||_{0,2} \leq M+1. \tag{2.45}$$

Moreover, for each $t \in (0,t_0]$, there exists $N \in \mathbb{N}$ such that

$$||h_n||_{H^{1,1}(t)} \leq \sqrt{2t}\,(M+1) \quad \text{for } n \geq N. \tag{2.46}$$

Proof. Define $\hat{h}$ by

$$\hat{h}(t) \stackrel{\text{def}}{=} \begin{cases} h(t_0) & \text{for } t > t_0, \\ h(t) & \text{for } 0 \leq t \leq t_0, \\ h(0) & \text{for } t < 0, \end{cases} \tag{2.47}$$

let H be given by

$$H(t,x) \stackrel{\text{def}}{=} \begin{cases} \hat{h}_x(t,b)\,(x-b) + \hat{h}(t,b) & \text{for } t \in \mathbb{R},\ x > b, \\ \hat{h}(t,x) & \text{for } t \in \mathbb{R},\ a \leq x \leq b, \\ \hat{h}_x(t,a)\,(x-a) + \hat{h}(t,a) & \text{for } t \in \mathbb{R},\ x < a, \end{cases} \tag{2.48}$$

and, by using a sequence of Friedrichs mollifiers (J_n), define the sequence $(H_n)_{n\in\mathbb{N}}$ by

$$H_n(t,x) \stackrel{\text{def}}{=} \int_{\mathbb{R}}\int_{\mathbb{R}} J_n(t-s)\,J_n(x-y)\,H(s,y)\,dy\,ds \quad \text{for } t,\ x \in \mathbb{R}. \tag{2.49}$$

A straightforward calculation yields that, for sufficiently large $K \in \mathbb{N}$, the sequence $(H_{n+K})_{n\in\mathbb{N}}$, restricted to $[0,t_0] \times [a,b]$, has the required properties (2.42)–(2.46).

□

2.4. Energy Estimates.

Lemma 2.7. *Given $\tilde{u}^b \in C^2([0,t^*])$ for some $t^* > 0$. Let $t_0 \in (0,t^*]$ and define $u^b \stackrel{\text{def}}{=} \tilde{u}^b|_{[0,t_0]}$. Suppose the boundary-initial value problem (2.16)–(2.18) has a solution u in $C^3([0,t_0]\times[a,b])$ with $f, p \in C^2([0,t_0]\times[a,b])$ and $u^0 \in C^2([a,b])$. Let $p^a \stackrel{\text{def}}{=} p(\cdot,a)$ and $p^b \stackrel{\text{def}}{=} p(\cdot,b)$. Then there exists a positive constant C, depending only on t^*, such that, for $0 \le t \le t_0$, the solution u obeys the estimates:*

$$\begin{aligned} ||u(t)||^2_{H^1} &\le \left(||u^0||^2_{H^1} + C\,||p||_{0,1}\left(||u^b||^2_{2,[t]} + ||u_x(\cdot,b)||^2_{2,[t]}\right) + t\,||f||^2_{0,1}\right)\cdot \\ &\quad \exp\{(C\,||p||_{0,2}+1)\,t\}, && (2.50) \\ ||u(t)||^2_{H^2} &\le \left(||u^0||^2_{H^2} + C\,||p||_{0,1}\left(||u^b||^2_{2,[t]} + ||u_x(\cdot,b)||^2_{2,[t]} + \right.\right. \\ &\quad \left.\left. ||u_{xx}(\cdot,b)||^2_{2,[t]}\right) + t\,||f||^2_{0,2}\right)\exp\{(C\,||p||_{0,2}+1)\,t\}, && (2.51) \\ ||u||^2_{1,1,[t]} &\le \left(||u^0||^2_{H^1} + ||p(0)\,u^0_x + f(0)||^2_{H^1} + C\,||u||_{0,2}\,||p||^2_{H^{1,1}[t]} + \right. \\ &\quad \left. ||f||^2_{H^{1,1}[t]} + C\,||p||_{0,1}\left(||u^b||^2_{H^1[t]} + ||u_x(\cdot,b)||^2_{H^1[t]}\right)\right)\cdot && (2.52) \\ &\quad \exp\{(C\,||p||_{0,2} + C\,||u||_{0,2}+1)\,t\}, \end{aligned}$$

where

$$\begin{aligned} ||u_x(\cdot,b)||_{2,[t]} &\le ||(p^b)^{-1}||_\infty\,(||u^b||_{H^1[t]} + C\sqrt{t}\,||f||_{0,1}), && (2.53) \\ ||u_x(\cdot,b)||_{H^1[t]} &\le C\,||(p^b)^{-1}||_\infty\,(||u^b||_{H^2[t]} + ||f||_{H^{1,1}[t]} + \\ &\quad ||(p^b)^{-1}||_\infty\,(||\tilde{u}^b||_{H^2} + ||f||_{0,1})\,||p||_{H^{1,1}[t]}), && (2.54) \\ ||u_{xx}(\cdot,b)||_{2,[t]} &\le ||(p^b)^{-1}||_\infty\,(||u_x(\cdot,b)||_{H^1[t]} + \\ &\quad C\,||p||_{0,2}\,||u_x(\cdot,b)||_{2,[t]} + C\sqrt{t}\,||f||_{0,2}). && (2.55) \end{aligned}$$

Moreover

$$\begin{aligned} ||u_x(\cdot,a)||^2_{H^1} &\le \left(||u^0||^2_{H^1} + ||p(0)\,u^0_x + f(0)||^2_{H^1} + C\,||u||_{0,2}\,||p||^2_{H^{1,1}} + \right. \\ &\quad \left. ||f||^2_{H^{1,1}} + C\,||p||_{0,1}\left(||u^b||^2_{H^1} + ||u_x(\cdot,b)||^2_{H^1}\right)\right)\cdot && (2.56) \\ &\quad ||(p^a)^{-1}||_\infty\,\exp\{(C\,||p||_{0,2} + C\,||u||_{0,2}+1)\,t_0\}. \end{aligned}$$

Proof. The constant C of the theorem is a generic constant that absorbs the imbedding constants of various Sobolev estimates of functions over $[a,b]$ and $[0,t^*]$. The stated estimates follow readily from elementary calculations involving Gronwall's inequality.

□

2.5. Existence and Regularity of Solutions.

Theorem 2.8. *Suppose that the functions f, p, u^0 and u^b satisfy hypotheses (2.24)–(2.29). Then the boundary-initial value problem (2.16)–(2.18) has a unique boundary-regular solution u such that*

$$u \in W^{1,\infty}([0,t_0]; H^1(a,b)) \cap L^\infty([0,t_0]; H^2(a,b)). \qquad (2.57)$$

Proof. Let $M \stackrel{\text{def}}{=} ||p||_{1,1} + ||p||_{0,2} + ||f||_{1,1} + ||f||_{0,2}$. By Lemma 2.6, for p and f there are C^∞-sequences $\left(p^k\right)$ and $\left(f^k\right)$, resp., with properties (2.42)–(2.45). There exist sequences $\left(u_k^b\right)$ and (u_k^0) such that, for $k \in \mathbb{N}$,

$$u_k^b \in C^\infty([0,t_0]), \quad u_k^0 \in C^\infty([a,b]), \tag{2.58}$$
$$u_k^0 \text{ and } u_k^b \text{ are compatible of order 3 w.r.t. } f^k,\, p^k, \tag{2.59}$$
$$u_k^b \longrightarrow u^b \text{ in } H^2(0,t_0) \text{ as } k \to \infty, \tag{2.60}$$
$$u_k^0 \longrightarrow u^0 \text{ in } H^2(a,b) \text{ as } k \to \infty. \tag{2.61}$$

The problems

$$u_t^{[k]}(t,x) = p^k(t,x)\, u_x^{[k]}(t,x) + f^k(t,x), \tag{2.62}$$
$$u^{[k]}(0,x) = u_k^0(x), \quad u^{[k]}(t,b) = u_k^b(t) \tag{2.63}$$

have solutions $u^{[k]}$ in $C^3([0,t_0] \times [a,b])$ by Theorem 2.5. For n, $m \in \mathbb{N}$ define

$$w^{n,m} \stackrel{\text{def}}{=} u^{[n]} - u^{[m]}. \tag{2.64}$$

By Lemma 2.7, $w^{n,m}$ obeys the estimate

$$\begin{aligned} ||w^{n,m}(t)||_{H^1}^2 \;\le\; & (||u_n^0 - u_m^0||_{H^1}^2 + C\,(1 + ||p^n||_{0,1} + \\ & ||p^n||_{0,1}\, ||(p^n(\cdot,b))^{-1}||_\infty^2)\, (||u_n^b - u_m^b||_{H^1} + \\ & \sqrt{t_0}\, (||p^n - p^m||_{0,1}\, ||u^{[m]}||_{0,2} + ||f^n - f^m||_{0,1}))^2) \cdot \\ & \exp\{(C\,||p^n||_{0,2} + 1)\, t_0\}. \end{aligned} \tag{2.65}$$

The quantities $||p^n||_{0,2}$, $||f^n||_{0,2}$, $||p^n||_{1,1}$ and $||f^n||_{1,1}$ are bounded according to (2.44)–(2.45). Hence $||u^{[m]}||_{0,2}$ is bounded according to (2.51). It follows that

$$w^{n,m} \longrightarrow 0 \text{ in } C([0,t_0]; H^1(a,b)) \text{ as } n,\, m \to \infty. \tag{2.66}$$

Hence $\left(u^{[n]}\right)$ is Cauchy in $C([0,t_0]; H^1(a,b)) \cap C^1([0,t_0]; L^2(a,b))$ with limit u. On the other hand, $\left(u^{[n]}\right)$ is bounded in $W^{1,\infty}([0,t_0]; H^1(a,b)) \cap L^\infty([0,t_0]; H^2(a,b))$ by estimates (2.51)–(2.52). By weak* compactness, the sequence $\left(u^{[n]}\right)$ contains a weak* convergent subsequence, say (v^n), in $W^{1,\infty}([0,t_0]; H^1(a,b)) \cap L^\infty([0,t_0]; H^2(a,b))$ with limit v. See [3] for weak and weak* topologies. By boundedness due to (2.54), (2.56), we may assume that the sequences $(v_x^n(\cdot,a))$, $(v_x^n(\cdot,b))$ converge weakly in $H^1(0,t_0)$. Note that the sequence (v^n) converges strongly in $L^2([0,t_0] \times [a,b])$ to its unique weak* limit v. Since the sequence $\left(u^{[n]}\right)$ also converges strongly in $L^2([0,t_0] \times [a,b])$, we have $u \equiv v$. Suppose v_a is the weak limit of $(v_x^n(\cdot,a))$ in $H^1(0,t_0)$. We remark that $(v_x^n(\cdot,a))$ converges strongly in $L^2(0,t_0)$ to its weak limit v_a. However, (v_x^n) is weakly convergent to u_x in $L^2([0,t_0]; H^1(a,b))$. Since the linear map $h \mapsto h(a)$ from $L^2([0,t_0]; H^1(a,b))$ to $L^2(0,t_0)$ is bounded, the sequence $(v_x^n(\cdot,a))$ converges weakly to $u_x(\cdot,a)$ in $L^2(0,t_0)$. Hence necessarily $u_x(\cdot,a) \equiv v_a$. Since the same argumentation works for $u_x(\cdot,b)$, the claim follows.

□

Theorem 2.9. *Suppose that the conditions of Theorem 2.8 hold. Then the solution u of the boundary-initial value problem (2.16)-(2.18) has the regularity property (2.30).*

Proof. The solution u has to satisfy the energy estimates of Lemma 2.7. Therefore we can conclude that

$$\limsup_{t\downarrow 0} \|u(t)\|_{H^2} \leq \|u^0\|_{H^2}. \tag{2.67}$$

Hence u is strongly right-continuous on $[0, t_0)$. We define $v(t,x) \stackrel{\text{def}}{=} u(t_0 - t, a + b - x)$. v also solves a problem of the form (2.16)–(2.18). Then the boundary-regularity of u yields the right-continuity of v according to (2.67).

□

3. SOLVABILITY OF THE EQUATIONS OF MELT-SPINNING

In this section we shall prove existence, uniqueness and regularity of solutions for the equations of melt-spinning, Eqs. (1.7)–(1.15). Our solution strategy is to prove the existence of a fixed-point for an appropriate solution operator. The main difficulty arises in the correct choice for the underlying metric space. Similar developments can be found elsewhere (see [4], [5], [8]).

3.1. Statement of the Main Result.

Definition 3.10. *We shall call a vector field (A, z, v), defined on $[0, t_0] \times [T_S, T_E]$, a solution of (1.7)-(1.15) if and only if*

$$A, z, v \in W^{1,\infty}([0, t_0]; H^1(T_S, T_E)) \cap L^\infty([0, t_0]; H^2(T_S, T_E)), \tag{3.68}$$

A, z, v satisfy Eqs. (1.7)-(1.9), (3.69)

A satisfies Eqs. (1.10), (1.14) pointwise, (3.70)

z satisfies Eqs. (1.11), (1.15) pointwise, (3.71)

v satisfies Eqs. (1.12), (1.13) pointwise. (3.72)

Theorem 3.11. *Given initial values A^0, z^0 on $[T_S, T_E]$ and boundary values A_E, v_E, v_S on $[0, t^*]$ for some $t^* > 0$ such that*

$$A^0, z^0 \in H^2(T_S, T_E), \tag{3.73}$$

$$A_E \in H^2(0, t^*), \tag{3.74}$$

$$v_E, v_S \in W^{1,\infty}(0, t^*), \tag{3.75}$$

$$A^0 > 0 \text{ on } [T_S, T_E], \tag{3.76}$$

$$z_T^0 < 0 \text{ on } [T_S, T_E]. \tag{3.77}$$

Also suppose that the compatibility conditions

$$A^0(T_E) = A_E(0), \quad z^0(T_E) = 0, \tag{3.78}$$

$$\dot{A}_E(0) = \frac{\alpha T_E}{\sqrt{A^0(T_E)}} A^0_T(T_E) + (v_S(0) - v_E(0)) \left(\int_{T_S}^{T_E} \frac{z^0_T(T)}{A^0(T)} dT \right)^{-1}, \tag{3.79}$$

$$\frac{\alpha T_E}{\sqrt{A^0(T_E)}} z^0_T(T_E) + v_E(0) = 0 \tag{3.80}$$

are satisfied. Then there exists a $t_0 \in (0, t^*]$ *such that the boundary-initial value problem (1.7)–(1.15) has a unique solution* (A, z, v) *on* $[0, t_0] \times [T_S, T_E]$. *This solution* (A, z, v) *has the properties:*

$$A, z \in C^1([0, t_0]; H^1(T_S, T_E)) \cap C([0, t_0]; H^2(T_S, T_E)), \tag{3.81}$$

$$A, z \in W^{2,\infty}([0, t_0]; L^2(T_S, T_E)), \tag{3.82}$$

$$v \in W^{1,\infty}([0, t_0]; H^1(T_S, T_E)) \cap C([0, t_0]; H^2(T_S, T_E)), \tag{3.83}$$

$$A,\ z,\ v \text{ are boundary-regular.} \tag{3.84}$$

Moreover, if

$$v_E, v_S \in C^1([0, t_0]), \tag{3.85}$$

then the solution (A, z, v) *has the additional properties:*

$$A, z \in \bigcap_{k=0}^{2} C^k([0, t_0]; H^{2-k}(T_S, T_E)), \tag{3.86}$$

$$v \in C^1([0, t_0]; H^1(T_S, T_E)) \cap C([0, t_0]; H^2(T_S, T_E)). \tag{3.87}$$

Remark 3.12 Solving the momentum balance (1.8) for v yields

$$v(t, T) = v_S(t) + (v_E(t) - v_S(t)) \int_{T_S}^{T} \frac{z_T(t, \tau)}{A(t, \tau)} d\tau \left(\int_{T_S}^{T_E} \frac{z_T(t, T)}{A(t, T)} dT \right)^{-1}. \tag{3.88}$$

and

$$\frac{v_T(t, T)}{z_T(t, T)} A(t, T) = (v_E(t) - v_S(t)) \left(\int_{T_S}^{T_E} \frac{z_T(t, T)}{A(t, T)} dT \right)^{-1}. \tag{3.89}$$

Hence we can eliminate Eq. (1.8) by (3.88)–(3.89) in Eqs. (1.7), (1.9).

From now on we shall assume that t^*, A^0, z^0, A_E, v_E and v_S are given once and for all such that hypotheses (3.73)–(3.80) hold.

3.2. The Solution Operator.

Definition 3.13. *Let* $h = h(t, T)$ *be a boundary-regular function with domain* $[0, t'] \times [T_S, T_E]$. *Then define the energy functional* $\mathcal{E}$ *by*

$$\mathcal{E}(h) \stackrel{\text{def}}{=} \left(||h||^2_{0,2} + ||h||^2_{1,1} + +||h_T(\cdot, T_S)||^2_{H^1} + ||h_T(\cdot, T_E)||^2_{H^1} \right)^{\frac{1}{2}}. \tag{3.90}$$

Definition 3.14. *For $L > 0$ and $t' \in (0, t^*]$, let $S(t', L)$ be the set of vector fields $(B, \xi)^T$ on $[0, t'] \times [T_S, T_E]$ such that*

$$B \text{ and } \xi \text{ are boundary-regular on } [0,t'] \times [T_S, T_E], \tag{3.91}$$

$$\mathcal{E}(B)^2 + \mathcal{E}(\xi)^2 \leq L^2, \tag{3.92}$$

$$B(0,T) = A^0(T) \text{ and } B(t, T_E) = A_E(t), \tag{3.93}$$

$$\xi(0,T) = z^0(T) \text{ and } \xi(t, T_E) = 0. \tag{3.94}$$

Definition 3.15. *We shall say that $t' \in (0, t^*]$ is admissible if and only if the set $S(t', L)$ is nonempty and all pairs $(B, \xi)^T \in S(t', L)$ have the properties*

$$B(t,T) > 0 \quad \text{and} \quad \int_{T_S}^{T_E} \frac{\xi_T(t,\tau)}{B(t,\tau)}\, d\tau \neq 0 \quad \text{for } t \in [0,t'],\ T \in [T_S, T_E]. \tag{3.95}$$

Lemma 3.16. *Suppose $S(t^*, L)$ is nonempty. Then there exists $t_0 \in (0, t^*]$ such that every $t \in (0, t_0]$ is admissible.*

This lemma follows from an easy calculation.

Definition 3.17. *Let $t' \in (0, t^*]$ be admissible. Define the solution operator $\Sigma_{t',L}$ on $S(t', L)$ by*

$$\Sigma_{t',L} : \begin{pmatrix} B \\ \xi \end{pmatrix} \longmapsto \begin{pmatrix} Y \\ \zeta \end{pmatrix}, \tag{3.96}$$

where $Y = Y(t,T)$ and $\zeta = \zeta(t,T)$ are the solutions (in the sense of Definition 2.1) of the boundary-initial value problems, stated on $[0,t'] \times [T_S, T_E]$,

$$Y_t = \frac{\alpha T}{\sqrt{B}} Y_T + (v_S - v_E) \left(\int_{T_S}^{T_E} \frac{\xi_T}{B}\, dT \right)^{-1}, \tag{3.97}$$

$$Y(0,T) = A^0(T), \quad Y(t, T_E) = A_E(t), \tag{3.98}$$

and

$$\zeta_t = \frac{\alpha T}{\sqrt{B}} \zeta_T + v_S + (v_E - v_S) \int_{T_S}^{T} \frac{\xi_T}{B}\, d\tau \left(\int_{T_S}^{T_E} \frac{\xi_T}{B}\, dT \right)^{-1}, \tag{3.99}$$

$$\zeta(0,T) = z^0(T), \quad \zeta(t, T_E) = 0. \tag{3.100}$$

3.3. **Application of the Contraction Mapping Principle.** In this final section we apply the Contraction Mapping Principle to the solution operator. To this end, it is necessary that $S(t, L)$ be a complete metric space. An appropriate metric is given in the following lemma.

Lemma 3.18. *Suppose $S(t, L)$ is nonempty. Then the metric $d(\cdot,\cdot)$, defined for $(B, \xi)^T$, $(\hat{B}, \hat{\xi})^T \in S(t, L)$ by*

$$d\left((B,\xi)^T, (\hat{B},\hat{\xi})^T\right) \stackrel{\text{def}}{=} \left(\left\| B - \hat{B} \right\|_{0,1}^2 + \left\| \xi - \hat{\xi} \right\|_{0,1}^2 \right)^{\frac{1}{2}}, \tag{3.101}$$

renders $S(t, L)$ a complete metric space.

Proof. d is clearly well-defined on $S(t, L)$. Suppose a sequence $(v^n) = (B^n, \xi^n)$ is Cauchy in this metric. Then v^n converges strongly, as $n \to \infty$, to a point v_0 in $(L^2([0,t] \times [T_S, T_E]))^2$. Since $\mathcal{E}(B^n)^2 + \mathcal{E}(\xi^n)^2 \leq L^2$ for all n, a subsequence of (v^n) is weak* convergent in $(W^{1,\infty}([0,t]; H^1(T_S, T_E)) \cap L^\infty([0,t]; H^2(T_S, T_E)))^2$ with weak* limit $w_0 \in S(t, L)$. The convergence is strong in $(L^2([0,t] \times [T_S, T_E]))^2$. Hence we deduce that $v_0 \equiv w_0$.

□

Theorem 3.19. *Suppose $S(t^*, L)$ is nonempty. Then there exists an admissible $t_0 \in (0, t^*]$ such that the solution operator $\Sigma_{t_0,L}$ is a contraction map on $S(t_0, L)$ w.r.t., the metric d given in (3.101).*

The proof of Theorem 3.19 is not complicated, but tedious (see [1]). The main conclusion we can draw is that the fixed-point of the mapping $\Sigma_{t_0,L}$ is the sought-for solution for the equations of melt-spinning.

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On Energy Estimates for Electro–Diffusion Equations Arising in Semiconductor Technology

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Abstract. The design of modern semiconductor devices requires the numerical simulation of basic fabrication steps. We investigate some electro–reaction–diffusion equations which describe the redistribution of charged dopants and point defects in semiconductor structures and which the simulations should be based on. Especially, we are interested in pair diffusion models. We present new results concerned with the existence of steady states and with the asymptotic behaviour of solutions which are obtained by estimates of the corresponding free energy and dissipation functionals.

Keywords: reaction–diffusion systems, drift–diffusion processes, pair diffusion models, steady states, asymptotic behaviour

Classification: 35B40, 35K45, 35K57, 78A35

1. Introduction

In the design of modern semiconductor devices and in the development of their technology, device and process simulation programmes turned out to be very important tools. The simulation of modern technologies requires the continuous improvement of the underlying physical models and their analytical and numerical investigation.

One of the main steps in the preparing of semiconductor devices is the redistribution of dopants connected with or followed after the doping. In order to explain this process, different models have been developed. Of special interest are pair diffusion models (see e. g. [2,4,10]). They consist of a set of reaction–diffusion equations for charged dopants, point defects and dopant–defect pairs coupled with a Poisson equation for the electrostatic potential of the inner electric field. Besides the mentioned species, electrons and holes have to be taken into account. But we assume that their kinetics is very fast, such that the Poisson equation is replaced by a nonlinear Poisson equation for the chemical potential of the electrons (see e. g. [7,10]).

Motivated by these considerations we investigate a rather general electro–reaction–diffusion system for m species X_i. We denote by ψ the chemical potential of the electrons, by $p_i(\psi)$ suitably chosen reference concentrations depending on ψ, and by u_i, $a_i = u_i/p_i(\psi)$, $\zeta_i = \ln a_i$ the concentration, the electrochemical activity and the electrochemical potential of the i-th species where all variables are suitably scaled. We assume that the set $\{1,\dots,m\}$ is split into two parts $\{1,\dots,m\} = J \cup J'$, and formulate the initial boundary value problem which we are interested in as follows:

$$\begin{aligned}
\frac{\partial u_i}{\partial t} + \nabla\cdot j_i + \sum_{(\alpha,\beta)\in\mathcal{R}} (\alpha_i-\beta_i)\,R_{\alpha\beta} &= 0 \quad \text{on } (0,\infty)\times\Omega, \\
\nu\cdot j_i &= 0 \quad \text{on } (0,\infty)\times\partial\Omega,\ i\in J; \\
\frac{\partial u_i}{\partial t} + \sum_{(\alpha,\beta)\in\mathcal{R}} (\alpha_i-\beta_i)\,R_{\alpha\beta} &= 0 \quad \text{on } (0,\infty)\times\Omega,\quad i\in J'; \\
-\nabla\cdot(\nabla\psi) + e(\psi) - \sum_{i=1}^{m} Q_i(\psi)u_i &= f \quad \text{on } (0,\infty)\times\Omega, \\
\nu\cdot\nabla\psi &= 0 \quad \text{on } (0,\infty)\times\partial\Omega; \\
u_i(0) &= U_i \quad \text{on } \Omega,\ i=1,\dots,m.
\end{aligned} \tag{1}$$

Here denote f a fixed charge density, $-e(\psi)$ the charge density of electrons and holes, and $Q_i(\psi) = -p_i'(\psi)/p_i(\psi)$ the charge of the i-th species depending on ψ, too. Only for species $i\in J$ there is a diffusive and convective transport given by the mass flux

$$j_i = -D_i(\psi)\Big[\nabla u_i + Q_i(\psi)\,u_i\nabla\psi\Big],\ i\in J.$$

But in all continuity equations source terms occur, generated by a lot of mass action type reactions of the form

$$\alpha_1X_1+\dots+\alpha_mX_m \rightleftharpoons \beta_1X_1+\dots+\beta_mX_m,\ (\alpha,\beta)\in\mathcal{R},$$

where α, $\beta\in\mathbb{Z}_+^m$ are the vectors of stoichiometric coefficients of the reaction and $\mathcal{R}$ describes the set of all reactions under consideration. The corresponding reaction rate $R_{\alpha\beta}$ is given by

$$R_{\alpha\beta}(u,\psi) = k_{\alpha\beta}(\psi)\Big[\prod_{i=1}^{m} a_i^{\alpha_i} - \prod_{i=1}^{m} a_i^{\beta_i}\Big],\ u\in\mathbb{R}^m,\ \psi\in\mathbb{R},\ a_i = \frac{u_i}{p_i(\psi)},\ (\alpha,\beta)\in\mathcal{R}.$$

If each species has a constant charge then

$$p_i(\psi) = \overline{u}_i\,e^{-q_i\psi},\ Q_i(\psi) = q_i,\ \overline{u}_i,\ q_i = \text{const},$$

and we arrive at a model which we have studied in great detail in [5,6,7,8] (for $J'=\emptyset$, but in a more general setting which is valid for heterostructures, too). There energy estimates, global existence and uniqueness of a solution and further qualitative properties of the solution have been established. For the pair diffusion model in [2,4] it holds

$$p_i(\psi) = \sum_{j\in J_i}\overline{u}_j e^{-q_j\psi},\ Q_i(\psi) = \frac{\sum_{j\in J_i} q_j\overline{u}_j e^{-q_j\psi}}{\sum_{j\in J_i}\overline{u}_j e^{-q_j\psi}}. \tag{2}$$

It is the aim of this paper to show that the energy estimates are valid in this new situation, too. More precisely, we prove that under some assumptions concerning the initial value and the structure of the underlying reaction system (see assumptions (II),

(III) later on) there exists a unique steady state to (1), and that the free energy along any solution to (1) remains bounded and decays monotonously and exponentially to its equilibrium value as t tends to infinity. From these assertions first global *a priori* estimates for solutions to (1) are obtained (see Corollary 1 – Corollary 3). We expect that based on these energy estimates further *a priori* estimates, and finally existence results could be derived. In this direction first results may be found in [12] (for $J' = \emptyset$ and for smooth data).

Notation. The notation of function spaces corresponds to that in [11]. By $\mathbb{Z}_+^m$, $\mathbb{R}_+^m$, L_+^p we denote the cones of nonnegative elements. For the scalar product in $\mathbb{R}^m$ we use a centered dot. If $u \in \mathbb{R}^m$ then $u \geq 0$ ($u > 0$) means $u_i \geq 0\, \forall i$ ($u_i > 0\, \forall i$); $\sqrt{u}$ denotes the vector $\{\sqrt{u_i}\}_{i=1,\dots,m}$, and analogously $\ln u$, e^u are to be understood. If u, $v \in \mathbb{R}^m$ then $uv = \{u_i v_i\}_{i=1,\dots,m}$ and analogously for u/v. Finally, if $u \in \mathbb{R}_+^m$ and $\alpha \in \mathbb{Z}_+^m$ then u^α means the product $\prod_{i=1}^m u_i^{\alpha_i}$. In our estimates positive constants, depending at most on the data of our problem, are denoted by c.

2. Formulation of the problem

First we summarize the basic assumptions (I) which we assume to be fulfilled up to the end of the paper:

$$
\left.\begin{aligned}
&\Omega \subset \mathbb{R}^2 \text{ bounded, Lipschitzian;}\\
&U \in L_+^\infty(\Omega, \mathbb{R}^m),\ f \in L^2(\Omega);\\
&e \in C^1(\mathbb{R}),\ |e(\psi)| \leq c\,\mathrm{e}^{c|\psi|},\ e'(\psi) \geq c > 0,\ \psi \in \mathbb{R};\\
&\mathcal{R} \subset \mathbb{Z}_+^m \times \mathbb{Z}_+^m;\\
&k_{\alpha\beta} \in C(\mathbb{R}),\ k_{\alpha\beta}(\psi) > 0,\ \psi \in \mathbb{R},\ (\alpha, \beta) \in \mathcal{R};\\
&\{1, \dots, m\} = J \cup J',\ J \cap J' = \emptyset;\\
&Q_i \in C^1(\mathbb{R}),\ |Q_i(\psi)| \leq c,\ Q_i'(\psi) \leq 0,\\
&p_i(\psi) = p_{i0}\,\mathrm{e}^{-P_i(\psi)},\ p_{i0} > 0,\ P_i(\psi) = \int_0^\psi Q_i(s)\,\mathrm{d}s,\ \psi \in \mathbb{R},\ i \in J \cup J';\\
&\text{for } i \in J\ :\ D_i \in C(\mathbb{R}),\ D_i(\psi) > 0,\ \psi \in \mathbb{R};\\
&\text{for } i \in J'\ :\ \text{there is a reaction of the form } R_{\alpha\beta} = k_{\alpha\beta}(\psi)\Big[\prod_{j\in J} a_j^{\alpha_j} - a_i^2\Big].
\end{aligned}\right\} \quad \text{(I)}
$$

The last two assumptions imply that there is a sufficiently high dissipation in the reaction–diffusion system, produced either by a diffusion–drift term ($i \in J$) or by a suitable quadratic reaction term ($i \in J'$). From (I) we easily find some further useful properties:

$$
\begin{aligned}
&p_i,\ P_i \in C^2(\mathbb{R}),\\
&p_i(\psi) > 0,\ p_i'(\psi) = -p_i(\psi)Q_i(\psi),\ p_i''(\psi) \ge 0,\ p_i(\psi), |p_i'(\psi)| \le c\,\mathrm{e}^{c|\psi|},\\
&|P_i'(\psi)| \le c,\ P_i''(\psi) \le 0,\ P_i(\psi) - Q_i(\psi)\psi \ge 0,\ \psi \in \mathbb{R},\ i = 1,\ldots,m;\\
&g \in C^1(\mathbb{R}) \text{ where } g(\psi) = e(\psi)\psi - \int_0^\psi e(s)\,\mathrm{d}s,\\
&g(\psi) \ge c\,\psi^2,\ g(\psi) \le c\,\mathrm{e}^{c|\psi|},\ \psi \in \mathbb{R};\\
&p_i(\psi),\ D_i(\psi),\ k_{\alpha\beta}(\psi) \ge c_R > 0,\ \psi \in \mathbb{R},\ |\psi| \le R.
\end{aligned}
\tag{3}
$$

Next we introduce the function spaces

$$Y = L^2(\Omega, \mathbb{R}^m), \quad X = \{u \in Y : u_i \in H^1(\Omega)\ \forall\, i \in J\}$$

and define the two operators

$$A : (X \cap L^\infty(\Omega, \mathbb{R}^m)) \times (H^1(\Omega) \cap L^\infty(\Omega)) \to X^*, \quad E : H^1(\Omega) \times Y \to (H^1(\Omega))^*,$$

$$
\begin{aligned}
\langle A(u,\psi), \overline{u}\rangle &= \int_\Omega \sum_{i\in J} D_i(\psi)\left[\nabla u_i + u_i Q_i(\psi)\nabla\psi\right]\cdot\nabla\overline{u}_i\,\mathrm{d}x\\
&+ \int_\Omega \sum_{(\alpha,\beta)\in\mathcal{R}} R_{\alpha\beta}(u,\psi)\sum_{i=1}^m(\alpha_i - \beta_i)\,\overline{u}_i\,\mathrm{d}x, \quad \overline{u} \in X,
\end{aligned}
$$

$$\langle E(\psi,u), \overline{\psi}\rangle = \int_\Omega \left\{\nabla\psi\cdot\nabla\overline{\psi} + \left[e(\psi) - \sum_{i=1}^m u_i Q_i(\psi) - f\right]\overline{\psi}\right\}\mathrm{d}x, \quad \overline{\psi} \in H^1(\Omega).$$

Because of (I) and of Trudinger's imbedding theorem [14] the operator $E(\cdot,u)$ turns out to be well defined on $H^1(\Omega)$ for every $u \in Y$. The precise formulation of the electro–diffusion system (1) now reads as follows:

$$
\begin{aligned}
&u'(t) + A(u(t),\psi(t)) = 0, \quad E(\psi(t),u(t)) = 0, \quad u(t) \ge 0 \quad \text{f.a.a. } t > 0,\\
&u(0) = U,\\
&u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, X) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^\infty(\Omega, \mathbb{R}^m)),\ u' \in L^2_{\mathrm{loc}}(\mathbb{R}_+, X^*),\\
&\psi \in L^2_{\mathrm{loc}}(\mathbb{R}_+, H^1(\Omega)) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^\infty(\Omega)).
\end{aligned}
\tag{P}
$$

3. The nonlinear Poisson equation

Here we summarize some results concerned with the nonlinear Poisson equation.

Lemma 1. *For any $u \in Y_+ = L^2_+(\Omega, \mathbb{R}^m)$ there exists a unique solution ψ to $E(\psi,u) = 0$. Moreover, there are a positive constant c and a monotonously increasing function $d \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|\psi - \overline{\psi}\|_{H^1} \le c\,\|u - \overline{u}\|_Y \quad \forall u, \overline{u} \in Y_+,\ E(\psi,u) = E(\overline{\psi},\overline{u}) = 0,$$

$$\|\psi\|_{L^\infty} \le c\Big\{1 + \sum_{i=1}^m \|u_i \ln u_i\|_{L^1} + d(\|\psi\|_{H^1})\Big\} \quad \forall u \in Y_+,\ E(\psi, u) = 0.$$

Proof. Since for $u \in Y_+$ the operator $E(\cdot, u)$ is strongly monotone uniformly with respect to u as well as hemicontinuous, and since for $\psi \in H^1(\Omega)$ the operator $E(\psi, \cdot)$ is Lipschitz continuous uniformly with respect to ψ, the first and second assertions are obvious. The third assertion is a consequence of Gröger's regularity result [9] and of Trudinger's imbedding theorem [14]. □

Later on we are interested in a modified version of the Poisson equation which is obtained by setting $u = ap(\psi)$, $a \in \mathbb{R}^m$. We define $\widetilde{E} : H^1(\Omega) \times \mathbb{R}^m \to (H^1(\Omega))^*$ by

$$\langle \widetilde{E}(\psi, a), \overline{\psi}\rangle = \int_\Omega \Big\{\nabla\psi \cdot \nabla\overline{\psi} + \Big[e(\psi) + \sum_{i=1}^m a_i p_i'(\psi) - f\Big]\overline{\psi}\Big\}\,\mathrm{d}x, \quad \overline{\psi} \in H^1(\Omega).$$

Lemma 2. *For any $a \in \mathbb{R}_+^m$ there exists a unique solution ψ to $\widetilde{E}(\psi, a) = 0$. Moreover, for every $R > 0$ there is a positive constant $c(R)$ such that*

$$\|\psi\|_{H^1},\ \|\psi\|_{L^\infty} \le c(R) \quad \forall a \in R_+^m,\ \|a\|_{\mathbb{R}^m} \le R,\ \widetilde{E}(\psi, a) = 0,$$

$$\|\psi - \overline{\psi}\|_{H^1} \le c(R)\,\|a - \overline{a}\|_{\mathbb{R}^m}\ \forall a, \overline{a} \in \mathbb{R}_+^m,\ \|a\|_{\mathbb{R}^m}, \|\overline{a}\|_{\mathbb{R}^m} \le R,\ \widetilde{E}(\psi, a) = \widetilde{E}(\overline{\psi}, \overline{a}) = 0.$$

Proof. Again for $a \in \mathbb{R}_+^m$ the operator $\widetilde{E}(\cdot, a)$ is strongly monotone uniformly with respect to a as well as hemicontinuous. From this the existence and uniqueness result follows. The estimate

$$c\|\psi\|_{H^1}^2 \le \langle \widetilde{E}(\psi, a) - \widetilde{E}(0, a), \psi\rangle \le \Big|\int_\Omega \Big[e(0) + \sum_{i=1}^m a_i p_i'(0) - f\Big]\psi\,\mathrm{d}x\Big| \le c(1 + \|a\|_{\mathbb{R}^m})\|\psi\|_{L^2}$$

yields the assertion on the H^1-norm of ψ. The assertion on the L^∞-norm of ψ is obtained from Lemma 1, for instance. Then the last assertion is obvious. □

4. The energy and dissipation functionals

In this section we investigate basic properties of the free energy functional and of other related functionals. First, we define $F_1, F_2 : Y_+ \to \mathbb{R}$ by

$$F_1(u) = \int_\Omega \Big\{\frac{1}{2}|\nabla\psi|^2 + g(\psi) + \sum_{i=1}^m u_i(P_i(\psi) - Q_i(\psi)\psi)\Big\}\,\mathrm{d}x,\ u \in Y_+ \tag{4}$$

where $\psi \in H^1(\Omega) \cap L^\infty(\Omega)$ is the unique solution to the Poisson equation $E(\psi, u) = 0$,

$$F_2(u) = \int_\Omega \sum_{i=1}^m \Big\{u_i\Big[\ln\frac{u_i}{p_i(0)} - 1\Big] + p_i(0)\Big\}\,\mathrm{d}x,\ u \in Y_+, \tag{5}$$

and set

$$F(u) = F_1(u) + F_2(u),\ u \in Y_+.$$

The value $F(u)$ can be interpreted as the free energy of the state u. Because of (3) it holds

$$F(u) \geq c\Big\{\|\psi\|^2_{H^1} + \sum_{i=1}^m \|u_i \ln u_i\|_{L^1} - 1\Big\}, \; u \in Y_+. \tag{6}$$

For $u, \overline{u} \in Y_+$ and corresponding $\psi, \overline{\psi} \in H^1(\Omega)$ with $E(\psi, u) = E(\overline{\psi}, \overline{u}) = 0$ we obtain

$$\begin{aligned} F_1(u) - F_1(\overline{u}) &= \int_\Omega \Big\{ \sum_{i=1}^m P_i(\overline{\psi})(u_i - \overline{u}_i) + \frac{1}{2}|\nabla(\psi - \overline{\psi})|^2 \\ &+ \int_{\overline{\psi}}^{\psi} (s - \overline{\psi})\, e'(s)\, \mathrm{d}s - \sum_{i=1}^m u_i \int_{\overline{\psi}}^{\psi} (s - \overline{\psi})\, Q_i'(s)\, \mathrm{d}s \Big\}\, \mathrm{d}x \\ &\geq (P(\overline{\psi}), u - \overline{u})_Y + c\,\|\psi - \overline{\psi}\|^2_{H^1} \geq (P(\overline{\psi}), u - \overline{u})_Y. \end{aligned} \tag{7}$$

From this relation it follows that F_1 is convex and continuous on the convex set Y_+. We extend F_1 to Y by setting $F_1(u) = +\infty$ for $u \in Y \setminus Y_+$. Then the extended functional $F_1 : Y \to \overline{\mathbb{R}}$ is proper, convex, lower semicontinuous, and subdifferentiable in each point $u \in Y_+$ where $P(\psi) \in \partial F_1(u)$. Because of properties of its integrand the functional F_2 is convex (see [3]) and continuous (see [7]) on Y_+. Again the extended functional $F_2 : Y \to \overline{\mathbb{R}}$, $F_2(u) = +\infty$ for $u \in Y \setminus Y_+$, is proper, convex and lower semicontinuous. For $u, \overline{u} \in Y_+$ with $\overline{u} \geq \delta > 0$ we obtain

$$\begin{aligned} F_2(u) - F_2(\overline{u}) &= \int_\Omega \Big\{ \sum_{i=1}^m \ln \frac{\overline{u}_i}{p_i(0)} (u_i - \overline{u}_i) + \sum_{i=1}^m \int_{\overline{u}_i}^{u_i} \ln \frac{s}{\overline{u}_i}\, \mathrm{d}s \Big\}\, \mathrm{d}x \\ &\geq (\ln \frac{\overline{u}}{p(0)}, u - \overline{u})_Y + \|\sqrt{u} - \sqrt{\overline{u}}\|^2_Y \geq (\ln \frac{\overline{u}}{p(0)}, u - \overline{u})_Y. \end{aligned} \tag{8}$$

Thus, F_2 is subdifferentiable in points $u \in Y_+$ with $u \geq \delta > 0$ and $\ln(u/p(0)) \in \partial F_2(u)$. Finally, we have to extend the introduced functionals to the space X^*. We define

$$\widetilde{F}_k = (F_k^*|_X)^* : X^* \to \overline{\mathbb{R}}, \; k = 1, 2,$$

where the star denotes the conjugation (see [3])

$$F_k^*(v) = \sup_{u \in Y} \Big\{(u, v)_Y - F_k(u)\Big\}, \; v \in Y, \quad (F_k^*|_X)^*(u) = \sup_{v \in X} \Big\{\langle u, v\rangle - F_k^*(v)\Big\}, \; u \in X^*.$$

Easily one verifies that $\widetilde{F}_k$ is proper, convex and lower semicontinuous, that $\widetilde{F}_k(u) = F_k(u)$ for $u \in Y_+$, and that it holds

$$P(\psi) \in \partial \widetilde{F}_1(u), \; u \in Y_+, \qquad \ln \frac{u}{p(0)} \in \partial \widetilde{F}_2(u), \; u \in X, \; u \geq \delta > 0. \tag{9}$$

Omitting the tilde we can summarize the properties of the free energy functional as follows.

Lemma 3. *The functional $F = F_1 + F_2 : X^* \to \overline{\mathbb{R}}$ is proper, convex and lower semicontinuous. For $u \in Y_+$ it can be evaluated according to* (4), (5). *The restriction*

$F|_{Y_+}$ is continuous. If $u \in X$ and $u \geq \delta > 0$ then

$$\zeta = P(\psi) + \ln \frac{u}{p(0)} = \ln \frac{u}{p(\psi)} \in X, \quad \zeta \in \partial F(u).$$

Next we study properties of the dual functional $F^* : X \to \overline{\mathbb{R}}$,

$$F^*(\zeta) = \sup_{u \in X^*} \Big\{ \langle u, \zeta \rangle - F(u) \Big\}, \quad \zeta \in X.$$

If F^* is subdifferentiable in ζ, $u \in \partial F^*(\zeta)$, or equivalently, $\zeta \in \partial F(u)$, then (see [3])

$$F^*(\zeta) = \langle u, \zeta \rangle - F(u), \quad \zeta \in \partial F(u). \tag{10}$$

Mainly we are interested in the special situation that $\zeta \in \mathbb{R}^m$. Therefore, let $\zeta \in \mathbb{R}^m$ be given, let $\psi \in H^1(\Omega) \cap L^\infty(\Omega)$ be the solution to the Poisson equation $\widetilde{E}(\psi, e^\zeta) = 0$, and define $u = p(\psi)\, e^\zeta$. Obviously, $u \in X$, $u \geq \delta > 0$ and thus $u \in \partial F^*(\zeta)$. By means of (10) we obtain

$$F^*(\zeta) = \int_\Omega \Big\{ \frac{1}{2} |\nabla \psi|^2 + \int_0^\psi e(s)\, ds - f\psi + \sum_{i=1}^m p_i(0) \Big[e^{\zeta_i - P_i(\psi)} - 1 \Big] \Big\}\, dx, \; \zeta \in \mathbb{R}^m.$$

We define the function

$$G = F^*|_{\mathbb{R}^m} : \mathbb{R}^m \to \mathbb{R}.$$

Because of (I), (3) we get the estimate

$$G(\zeta) \geq c \Big\{ \|\psi\|_{H^1}^2 + \|[\zeta - P(\psi)]^+\|_Y^2 - 1 \Big\}, \; \zeta \in \mathbb{R}^m, \tag{11}$$

and for $\zeta, \overline{\zeta} \in \mathbb{R}^m$ with corresponding $\psi, \overline{\psi}$ we find that

$$\begin{aligned}
G(\zeta) - G(\overline{\zeta}) &= \int_\Omega \sum_{i=1}^m p_i(\overline{\psi})\, e^{\overline{\zeta}_i} (\zeta_i - \overline{\zeta}_i)\, dx + \omega(\zeta, \overline{\zeta}), \\
\omega(\zeta, \overline{\zeta}) &= \int_\Omega \Big\{ \frac{1}{2} |\nabla(\psi - \overline{\psi})|^2 + \int_{\overline{\psi}}^\psi (\psi - s)\, e'(s)\, ds \\
&- \sum_{i=1}^m p_i(\overline{\psi})\, e^{\overline{\zeta}_i} \int_{\overline{\psi}}^\psi (\psi - s)\, Q_i'(s)\, ds \\
&+ \sum_{i=1}^m p_i(0) \int_{\overline{\zeta}_i - P_i(\overline{\psi})}^{\zeta_i - P_i(\psi)} (\zeta_i - P_i(\psi) - s)\, e^s\, ds \Big\} dx.
\end{aligned}$$

Let $R > 0$ be given. Because of Lemma 2 there exist constants $c_1(R), c_2(R) > 0$ such that for all $\zeta, \overline{\zeta}$ with $\|\zeta\|_{\mathbb{R}^m}, \|\overline{\zeta}\|_{\mathbb{R}^m} \leq R$ it holds

$$\begin{aligned}
\omega(\zeta, \overline{\zeta}) &\geq c_1(R) \Big\{ \|\psi - \overline{\psi}\|_{H^1}^2 + \|\zeta - \overline{\zeta} - (P(\psi) - P(\overline{\psi}))\|_{L^2}^2 \Big\}, \\
\omega(\zeta, \overline{\zeta}) &\leq c_2(R)\, \|\zeta - \overline{\zeta}\|_{\mathbb{R}^m}^2.
\end{aligned}$$

From these estimates we derive the following assertions.

Lemma 4. *The function $G = F^*|_{\mathbb{R}^m} : \mathbb{R}^m \to \mathbb{R}$ is strictly convex, continuous and Fréchet differentiable,*

$$\partial G(\zeta) = \int_\Omega e^\zeta p(\psi)\, dx, \quad \zeta \in \mathbb{R}^m, \quad \widetilde{E}(\psi, e^\zeta) = 0.$$

A further functional which we are interested in is the dissipation functional, or more precisely, a lower estimate of this functional. We define the set

$$M_D = \left\{u \in L^\infty_+(\Omega, \mathbb{R}^m) : \sqrt{a} \in X, \text{ where } a = u/p(\psi) \text{ and } E(\psi, u) = 0\right\}$$

and the functional $D : M_D \to \mathbb{R}$ by

$$\begin{aligned} D(u) &= \int_\Omega \Big\{ \sum_{i\in J} 4\, D_i(\psi) p_i(\psi) |\nabla \sqrt{a_i}|^2 \\ &+ \sum_{(\alpha,\beta)\in\mathcal{R}} 2\, k_{\alpha\beta}(\psi) |\sqrt{a}^\alpha - \sqrt{a}^\beta|^2 \Big\}\, dx, \ u \in M_D. \end{aligned} \tag{12}$$

Lemma 5. *For all $u \in M_D$ it holds $D(u) \geq 0$. If $u \in M_D$ and $D(u) = 0$ then $u = ap(\psi)$ where ψ is the solution to $E(\psi, u) = 0$ and*

$$a \in \mathbb{R}^m_+, \quad a^\alpha = a^\beta \quad \forall (\alpha, \beta) \in \mathcal{R}.$$

Proof. The first assertion is obvious. Now let $D(u) = 0$. Since all coefficients in (12) are strictly positive we obtain $a_i = \text{const} \geq 0$, $i \in J$, as well as $a^\alpha = a^\beta$, $(\alpha, \beta) \in \mathcal{R}$. Because of the last assumption in (I) we find that $a_i = \text{const} \geq 0$, $i \in J'$, too. □

5. Monotonicity and boundedness of the free energy

According to thermodynamic principles the free energy should monotonously decrease along solutions to the evolution problem (P). This property is indeed obtained from the following theorem.

Theorem 1. *Let (u, ψ) be a solution to (P) and set $a = u/p(\psi)$. Then $\sqrt{a} \in L^2_{\text{loc}}(\mathbb{R}_+, X)$, $u(t) \in M_D$ f.a.a. $t > 0$, and for every $\lambda \in \mathbb{R}_+$ it holds*

$$e^{\lambda t_2} F(u(t_2)) - e^{\lambda t_1} F(u(t_1)) - \int_{t_1}^{t_2} e^{\lambda t} \Big\{ \lambda F(u(t)) - D(u(t)) \Big\}\, dt \leq 0, \quad 0 \leq t_1 \leq t_2.$$

Proof. 1. Let (u, ψ) be a solution to (P), $\lambda \in \mathbb{R}_+$, $S = [t_1, t_2]$. Then $u \in H^1(S, X^*)$, $\psi \in L^2(S, H^1(\Omega))$, $P(\psi) \in L^2(S, X)$ and $\nabla P(\psi) = Q(\psi)\nabla\psi$. Because of (9) we find $P(\psi(t)) \in \partial F_1(u(t))$ f.a.a. $t \in S$. Therefore, the function $t \mapsto F_1(u(t))$ is absolutely continuous on S and there holds the chain rule (see [1])

$$\frac{\mathrm{d}}{\mathrm{d}t} F_1(u(t)) = \langle u'(t), P(\psi(t)) \rangle \text{ f.a.a. } t \in S.$$

2. We define $u^\delta = u + \delta$ for $\delta > 0$. Then $u^\delta \in H^1(S, X^*)$, $\ln(u^\delta/p(0)) \in L^2(S, X)$ and $\nabla \ln(u^\delta_i/p_i(0)) = \nabla u_i/(u_i + \delta)$, $i \in J$. Because of (9) we find $\ln(u^\delta(t)/p(0)) \in$

$\partial F_2(u^\delta(t))$ f.a.a. $t \in S$. Thus, the function $t \mapsto F_2(u^\delta(t))$ is absolutely continuous on S and

$$\frac{\mathrm{d}}{\mathrm{d}t} F_2(u^\delta(t)) = \Big\langle u'(t), \ln \frac{u^\delta(t)}{p(0)} \Big\rangle \text{ f.a.a. } t \in S.$$

3. Using these results and setting $\zeta^\delta = \ln(u^\delta/p(\psi)) \in L^2(S, X)$ we obtain

$$\mathrm{e}^{\lambda t}\Big[F_1(u(t)) + F_2(u^\delta(t))\Big]\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \mathrm{e}^{\lambda t}\Big\{\lambda\Big[F_1(u(t)) + F_2(u^\delta(t))\Big] + \langle u'(t), \zeta^\delta(t)\rangle\Big\}\,\mathrm{d}t.$$

Here we insert the evolution equation,

$$\langle u'(t), \zeta^\delta(t)\rangle = -\langle A(u(t), \psi(t)), \zeta^\delta(t)\rangle = -\langle A(u^\delta(t), \psi(t)), \zeta^\delta(t)\rangle + h^\delta(t) \text{ f.a.a. } t \in S$$

where $h^\delta = \langle A(u^\delta, \psi) - A(u, \psi), \zeta^\delta\rangle$. We set $a^\delta = u^\delta/p(\psi) = e^{\zeta^\delta}$. Then $\sqrt{a^\delta} \in L^2(S, X)$ and $\nabla\sqrt{a_i^\delta} = \frac{1}{2}\sqrt{a_i^\delta}\,\nabla\zeta_i^\delta$, $i \in J$. Some simple calculation yields

$$D_i(\psi)\Big[\nabla u_i^\delta + Q_i(\psi) u_i^\delta \nabla\psi\Big] \cdot \nabla\zeta_i^\delta = D_i(\psi) u_i^\delta |\nabla\zeta_i^\delta|^2 = 4\, D_i(\psi) p_i(\psi) |\nabla\sqrt{a_i^\delta}|^2, \; i \in J,$$

$$R_{\alpha\beta}(u^\delta, \psi)\,(\alpha - \beta)\cdot\zeta^\delta = k_{\alpha\beta}(\psi)\Big[\mathrm{e}^{\alpha\cdot\zeta^\delta} - \mathrm{e}^{\beta\cdot\zeta^\delta}\Big](\alpha - \beta)\cdot\zeta^\delta \geq 2\, k_{\alpha\beta}(\psi) |\sqrt{a^\delta}^{\,\alpha} - \sqrt{a^\delta}^{\,\beta}|^2,$$

and thus we obtain

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega \mathrm{e}^{\lambda t}\Big\{\sum_{i\in J} 4\, D_i(\psi) p_i(\psi) |\nabla\sqrt{a_i^\delta}|^2 + \sum_{(\alpha,\beta)\in\mathcal{R}} 2\, k_{\alpha\beta}(\psi) |\sqrt{a^\delta}^{\,\alpha} - \sqrt{a^\delta}^{\,\beta}|^2\Big\}\,\mathrm{d}x\mathrm{d}t \\
&\leq \int_{t_1}^{t_2} \mathrm{e}^{\lambda t}\,\langle A(u^\delta(t), \psi(t)), \zeta^\delta(t)\rangle\,\mathrm{d}t = H^\delta
\end{aligned} \tag{13}$$

where

$$H^\delta = \int_{t_1}^{t_2} \mathrm{e}^{\lambda t}\Big\{\lambda\Big[F_1(u(t)) + F_2(u^\delta(t))\Big] + h^\delta(t)\Big\}\,\mathrm{d}t - \mathrm{e}^{\lambda t}\Big[F_1(u(t)) + F_2(u^\delta(t))\Big]\Big|_{t_1}^{t_2}.$$

4. Now let $\delta \to 0$. First, we easily find that

$$H^\delta \to H = \int_{t_1}^{t_2} \mathrm{e}^{\lambda t} \lambda\, F(u(t))\,\mathrm{d}t - \mathrm{e}^{\lambda t}\, F(u(t))\,\Big|_{t_1}^{t_2}.$$

By Lebesgue's dominated convergence theorem we obtain that $\sqrt{a^\delta} \to \sqrt{a}$ in $L^2(S, Y)$, and at least for a subsequence, $\sqrt{a^\delta} \to \sqrt{a}$ a.e. on $S \times \Omega$. Fatou's lemma yields

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega \mathrm{e}^{\lambda t} \sum_{(\alpha,\beta)\in\mathcal{R}} 2\, k_{\alpha\beta}(\psi) |\sqrt{a}^{\,\alpha} - \sqrt{a}^{\,\beta}|^2\mathrm{d}x\mathrm{d}t \\
&\leq \liminf_{\delta\to 0} \int_{t_1}^{t_2}\int_\Omega \mathrm{e}^{\lambda t} \sum_{(\alpha,\beta)\in\mathcal{R}} 2\, k_{\alpha\beta}(\psi) |\sqrt{a^\delta}^{\,\alpha} - \sqrt{a^\delta}^{\,\beta}|^2\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{14}$$

Finally, let $i \in J$. Since $\psi \in L^\infty(S, L^\infty)$ and since the sequence H^δ is bounded because of (13) there exists a constant $c > 0$ such that $\|\nabla\sqrt{a_i^\delta}\|_{L^2(S,L^2)} \leq c$. From this

$$\nabla\sqrt{a_i} \in L^2(S, L^2), \quad \nabla\sqrt{a_i^\delta} \rightharpoonup \nabla\sqrt{a_i} \text{ in } L^2(S, L^2), \quad i \in J,$$

follows (see [13, Proposition 2.4]), and because of the weak lower semicontinuity of continuous quadratic functionals we arrive at

$$\int_{t_1}^{t_2}\int_\Omega e^{\lambda t}\sum_{i\in J} 4\,D_i(\psi)p_i(\psi)|\nabla\sqrt{a_i}|^2\,\mathrm{d}x\mathrm{d}t$$
$$\le \liminf_{\delta\to 0}\int_{t_1}^{t_2}\int_\Omega e^{\lambda t}\sum_{i\in J} 4\,D_i(\psi)p_i(\psi)|\nabla\sqrt{a_i^\delta}|^2\,\mathrm{d}x\mathrm{d}t. \tag{15}$$

Hence $\sqrt{a}\in L^2(S,X)$, and from (13) – (15) we obtain $\int_{t_1}^{t_2} e^{\lambda t}D(u(t))\,\mathrm{d}t \le H$. □

Corollary 1. *Along any solution (u,ψ) to* (P) *the free energy $F(u)$ remains bounded from above by its initial value $F(U)$ and decreases monotonously,*

$$F(u(t_2))\le F(u(t_1))\le F(U),\quad 0\le t_1\le t_2.$$

Moreover, there exists a constant c depending only on the data such that

$$\sum_{i=1}^m \|u_i\ln u_i\|_{L^\infty(\mathbb{R}_+,L^1(\Omega))}\le c,\quad \|u\|_{L^\infty(\mathbb{R}_+,L^1(\Omega,\mathbb{R}^m))}\le c,$$
$$\|\psi\|_{L^\infty(\mathbb{R}_+,H^1(\Omega))}\le c,\quad \|\psi\|_{L^\infty(\mathbb{R}_+,L^\infty(\Omega))}\le c$$

for any solution to (P).

Proof. The first assertion follows from Theorem 1 by setting $\lambda=0$. The remaining estimates are a consequence of (6) and Lemma 1. □

6. Invariants and steady states

We introduce the stoichiometric subspace $\mathcal{S}$ belonging to all reactions,

$$\mathcal{S}=\operatorname{span}\{\alpha-\beta:\ (\alpha,\beta)\in\mathcal{R}\}\subset\mathbb{R}^m.$$

By integrating the continuity equations over $(0,t)\times\Omega$ one easily verifies the following invariance property.

Lemma 6. *If (u,ψ) is a solution to (P) then $\int_\Omega\Big\{u(t)-U\Big\}\,dx\in\mathcal{S}$ for all $t\in\mathbb{R}_+$.*

Now we ask for steady states belonging to the evolution problem (P) which satisfy such an invariance property, too. Thus we have to solve the following problem.

$$A(u,\psi)=0,\quad E(\psi,u)=0,\quad u\ge 0,$$
$$\int_\Omega\Big\{u-U\Big\}\,\mathrm{d}x\in\mathcal{S}, \tag{S}$$
$$u\in X\cap L^\infty(\Omega,\mathbb{R}^m),\quad \psi\in H^1(\Omega)\cap L^\infty(\Omega).$$

Lemma 7. *If (u,ψ) is a solution to (S) then $u\in M_D$ and $D(u)=0$.*

Proof. The proof is similar to that of Theorem 1. Let (u,ψ) be a solution to (S), define $a = u/p(\psi)$ and set $u^\delta = u+\delta$, $a^\delta = u^\delta/p(\psi)$, $\zeta^\delta = \ln a^\delta$ where $\delta > 0$. Then

$$\zeta^\delta,\ \sqrt{a^\delta} \in X,\ \langle A(u^\delta,\psi),\zeta^\delta\rangle = h^\delta$$

where $h^\delta = \langle A(u^\delta,\psi) - A(u,\psi),\zeta^\delta\rangle \to 0$ if $\delta \to 0$. Furthermore, $\sqrt{a^\delta} \to \sqrt{a}$ in Y and because of the estimate

$$\int_\Omega \Big\{\sum_{i\in J} 4\,D_i(\psi)p_i(\psi)|\nabla\sqrt{a_i^\delta}|^2 + \sum_{(\alpha,\beta)\in\mathcal{R}} 2\,k_{\alpha\beta}(\psi)|\sqrt{a^\delta}^{\alpha} - \sqrt{a^\delta}^{\beta}|^2\Big\}\,dx \le \langle A(u^\delta,\psi),\zeta^\delta\rangle$$

we find that $\nabla\sqrt{a_i^\delta} \rightharpoonup \nabla\sqrt{a_i}$ in L^2, $i \in J$. Thus $\sqrt{a} \in X$, $u \in M_D$ and

$$0 \le D(u) \le \liminf_{\delta\to 0}\langle A(u^\delta,\psi),\zeta^\delta\rangle = 0. \quad \square$$

Next, we show that there is a correspondence between the set of steady states, i. e. the set of solutions to (S), and the set $\mathcal{A} \subset \mathbb{R}^m$ defined by

$$\mathcal{A} = \Big\{a \in \mathbb{R}^m_+ :\ a^\alpha = a^\beta\ \forall(\alpha,\beta)\in\mathcal{R},\ \int_\Omega \Big\{u - U\Big\}\,dx \in \mathcal{S},$$
$$\text{where } u = ap(\psi) \text{ and } \psi \text{ is the solution to } \widetilde{E}(\psi,a) = 0\Big\}.$$

Lemma 8. *If (u,ψ) is a solution to (S) then $a = u/p(\psi) \in \mathcal{A}$. Vice versa, if $a \in \mathcal{A}$ and u,ψ are chosen as in the definition of $\mathcal{A}$ then (u,ψ) is a solution to (S).*

Proof. The first assertion follows from Lemma 7 and Lemma 5. If $a \in \mathcal{A}$ then obviously $\psi \in H^1(\Omega)\cap L^\infty(\Omega)$, $u = ap(\psi) \in X \cap L^\infty(\Omega,\mathbb{R}^m)$, $u \ge 0$, $\nabla u_i = a_i p_i'(\psi)\nabla\psi = -u_i Q_i(\psi)\nabla\psi$, $i \in J$, $A(u,\psi) = 0$, and $E(\psi,u) = \widetilde{E}(\psi,a) = 0$. $\square$

Finally, we show that the set $\mathcal{A} \cap \operatorname{int}\mathbb{R}^m_+$ corresponds to the set of minimizers of the function $G_0 : \mathbb{R}^m \to \overline{\mathbb{R}}$ defined by

$$G_0(\zeta) = G(\zeta) + I_{\mathcal{S}^\perp}(\zeta) - \int_\Omega U\cdot\zeta\,dx, \quad \zeta \in \mathbb{R}^m.$$

This function is proper, convex and lower semicontinuous, $\operatorname{dom} G_0 = \mathcal{S}^\perp$. Because of the continuity of G (see Lemma 4) we obtain by the Moreau-Rockafellar theorem (see [3]) that

$$\partial G_0(\zeta) = \partial G(\zeta) + \partial I_{\mathcal{S}^\perp}(\zeta) - \int_\Omega U\,dx,\ \zeta \in \mathbb{R}^m. \tag{16}$$

Lemma 9. *If $a \in \mathcal{A} \cap \operatorname{int}\mathbb{R}^m_+$ then $\zeta = \ln a$ is a minimizer of G_0. Vice versa, if $\zeta \in \mathbb{R}^m$ is a minimizer of G_0 then $a = e^\zeta \in \mathcal{A} \cap \operatorname{int}\mathbb{R}^m_+$.*

Proof. Because of (16) and Lemma 4, ζ is a minimizer of G_0 if and only if

$$\zeta \in \mathcal{S}^\perp, \quad \partial G(\zeta) - \int_\Omega U\,dx = \int_\Omega \Big\{e^\zeta p(\psi) - U\Big\}\,dx \in \mathcal{S}$$

where ψ is the solution to $\widetilde{E}(\psi, e^\zeta) = 0$. The relation $\zeta \in \mathcal{S}^\perp$ is equivalent to $(\alpha - \beta) \cdot \zeta = 0 \ \forall (\alpha, \beta) \in \mathcal{R}$, or to $(e^\zeta)^\alpha = (e^\zeta)^\beta \ \forall (\alpha, \beta) \in \mathcal{R}$. Since the map $\zeta \mapsto e^\zeta$ is a bijection from $\mathbb{R}^m$ onto $\operatorname{int} \mathbb{R}^m_+$ all assertions of the lemma are obtained. □

Lemma 10. *The set $\mathcal{A} \cap \operatorname{int} \mathbb{R}^m_+$ contains at most one element. Furthermore, $\mathcal{A} \cap \operatorname{int} \mathbb{R}^m_+ \neq \emptyset$ if and only if the following condition is fulfilled:*

$$\int_\Omega U \cdot \overline{\zeta} \, dx > 0 \quad \forall \overline{\zeta} \in \mathcal{S}^\perp, \ \overline{\zeta} \geq 0, \ \overline{\zeta} \neq 0. \tag{II}$$

Proof. 1. The first assertion follows from Lemma 9 since the functions G and $G_0|_{\mathcal{S}^\perp}$ are strictly convex.
2. If $a \in \mathcal{A} \cap \operatorname{int} \mathbb{R}^m_+$ then for any $\overline{\zeta} \in \mathcal{S}^\perp$, $\overline{\zeta} \geq 0$, $\overline{\zeta} \neq 0$ we find

$$\int_\Omega U \cdot \overline{\zeta} \, \mathrm{d}x = \int_\Omega \sum_{i=1}^m a_i p_i(\psi) \overline{\zeta}_i \, \mathrm{d}x > 0.$$

3. Now let (II) be fulfilled. According to Lemma 9 we have to show that there is a minimizer of G_0. It is sufficient to verify the property $G_0(\zeta) \to +\infty$ if $\|\zeta\|_{\mathbb{R}^m} \to +\infty$. Suppose this to be false. Then there exist $R \in \mathbb{R}_+$ and a sequence $\zeta_n \in \mathcal{S}^\perp$ such that

$$\|\zeta_n\|_{\mathbb{R}^m} \to \infty, \quad G_0(\zeta_n) = G(\zeta_n) - \int_\Omega U \cdot \zeta_n \, \mathrm{d}x \leq R.$$

Using (11) this implies

$$c_1 \left\{ \|\psi_n\|^2_{H^1} + \|[\zeta_n - P(\psi_n)]^+\|^2_Y \right\} - (U, \zeta_n)_Y \leq R + c_2. \tag{17}$$

We set $\widetilde{\psi}_n = \psi_n / \|\zeta_n\|$, $\widetilde{\zeta}_n = \zeta_n / \|\zeta_n\|$, and assume that $\widetilde{\zeta}_n \to -\overline{\zeta}$ in $\mathbb{R}^m$ where $\overline{\zeta} \in \mathcal{S}^\perp$, $\overline{\zeta} \neq 0$. Because of (17) we find $\widetilde{\psi}_n \to 0$ in $H^1(\Omega)$, and $\overline{\zeta} \geq 0$ since P is Lipschitz continuous. Again using (17) we obtain $(U, \overline{\zeta})_Y \leq 0$ in contradiction to (II). □

There are examples of reaction–diffusion systems with steady states where the corresponding $a \in \mathcal{A}$ belongs to $\partial \mathbb{R}^m_+$ even if condition (II) is satisfied. But in many applications e. g. in semiconductor technology, this cannot happen. Therefore in our following considerations we shall assume that

$$\mathcal{A} \cap \partial \mathbb{R}^m_+ = \emptyset. \tag{III}$$

Then we may summarize our results concerned with the steady states to (P) as follows.

Theorem 2. *Let the additional assumption* (II) *be fulfilled. Then there exists a solution (u^*, ψ^*) to* (S) *with the following properties:*

$$a^* = u^* / p(\psi^*) \in \mathbb{R}^m, \ a^* > 0, \ \zeta^* = \ln a^* \in \mathcal{S}^\perp, \ u^* \geq c > 0 \ a.e. \ on \ \Omega.$$

If (III) *is fulfilled, too, then there is no other solution to* (S).

Corollary 2. *Assume* (II) *and let (u^*, ψ^*) be the solution to* (S) *as in Theorem* 2. *Then for any solution (u, ψ) to* (P) *it holds*

$$F(u^*) \leq F(u(t)) \quad \forall t \in \mathbb{R}_+, \quad \int_0^\infty D(u(t)) \, dt \leq F(U) - F(u^*).$$

Proof. This follows from (7), (8) with $\overline{u} = u^*$, $\overline{\psi} = \psi^*$ and from Theorem 1. □

7. Exponential decay of the free energy

In this section we shall prove that for any solution to the evolution problem (P) (with the initial value U) the free energy $F(u)$ decays exponentially to its equilibrium value $F(u^*)$ (where u^* belongs to that compatibility class which is generated by U) if the additional assumptions (II) and (III) are fulfilled. This will be a consequence of the following estimate of the free energy by the dissipation functional.

Theorem 3. *Let* (II) *and* (III) *be satisfied. Then for every* $R > 0$ *there exists a constant* $c_R > 0$ *such that*

$$F(u) - F(u^*) \le c_R D(u) \quad \forall u \in M_R$$

where

$$M_R = \Big\{u \in M_D\colon F(u) - F(u^*) \le R,\ \int_\Omega (u - U)\,dx \in \mathcal{S}\Big\}.$$

Proof. 1. For $u \in M_R$ let ψ, a be defined by $E(\psi, u) = 0$, $a = u/p(\psi)$. First let us note that there is a $c(R) > 0$ such that $\|\psi\|_{H^1}, \|\psi\|_{L^\infty} \le c(R)\, \forall u \in M_R$. Setting $\widetilde{F}(u) = F(u) - F(u^*)$ and using (7), (8) and Theorem 2, we obtain the estimates

$$c_1(R)\Big\{\|\sqrt{a/a^*} - 1\|_Y^2 + \|\psi - \psi^*\|_{H^1}^2\Big\} \le \widetilde{F}(u), \tag{18}$$

$$\widetilde{F}(u) \le c_2(R)\, \|u - u^*\|_Y^2, \tag{19}$$

$$D(u) \ge c_3(R)\widetilde{D}(u),\ \widetilde{D}(u) = \int_\Omega \Big\{\sum_{i\in J} \Big|\nabla\sqrt{a_i/a_i^*}\Big|^2 + \sum_{(\alpha,\beta)\in\mathcal{R}} \Big|\sqrt{a/a^*}^{\,\alpha} - \sqrt{a/a^*}^{\,\beta}\Big|^2\Big\}\,dx$$

for all $u \in M_R$ with positive constants $c_k(R)$. It remains to show that for every $R > 0$ there exists a $\widetilde{c}_R > 0$ such that

$$\widetilde{F}(u) < \widetilde{c}_R \widetilde{D}(u) \quad \forall u \in M_R \setminus \{u^*\}.$$

2. Suppose this assertion to be false. Then there exist $R > 0$ and sequences $c_n \in \mathbb{R}$, $u_n \in M_R$ such that $c_n \to +\infty$ and

$$0 < c_n \widetilde{D}(u_n) \le \widetilde{F}(u_n) \le R. \tag{20}$$

Let ψ_n, a_n be correspondingly defined. (18), (20) imply that $\|\psi_n\|_{H^1}, \|\sqrt{a_n}\|_Y \le c(R)$, $\widetilde{D}(u_n) \to 0$. Therefore $\psi_n \rightharpoonup \widehat{\psi}$ in H^1, $\psi_n \to \widehat{\psi}$ in L^2. For $i \in J$ we find $\widehat{a}_i \in \mathbb{R}_+$ with $\sqrt{a_{ni}} \to \sqrt{\widehat{a}_i}$ in H^1 and in each L^p. For $i \in J'$ we have at least $a_{ni} \to \widehat{a}_i \in \mathbb{R}_+$ in L^2 since for such i there is a special reaction for which

$$\int_\Omega \Big|\prod_{j\in J} \sqrt{a_{nj}/a_j^*}^{\,\alpha_j} - a_{ni}/a_i^*\Big|^2\,dx \to 0.$$

Fatou's lemma ensures that $\widehat{a}^\alpha = \widehat{a}^\beta\, \forall(\alpha,\beta) \in \mathcal{R}$. Setting $\widehat{u} = \widehat{a}p(\widehat{\psi})$ we get $u_n \to \widehat{u}$ in Y, and thus $\int_\Omega(\widehat{u} - U)\,dx \in \mathcal{S}$. The estimate $\|\psi_{n+p} - \psi_n\|_{H^1} \le c\,\|u_{n+p} - u_n\|_Y$ shows that $\psi_n \to \widehat{\psi}$ in H^1. Using properties of E we conclude that $E(\psi_n, \widehat{u}) \rightharpoonup E(\widehat{\psi}, \widehat{u})$ in $(H^1)^*$, $E(\psi_n, \widehat{u}) \to 0$ in $(H^1)^*$. Thus $E(\widehat{\psi}, \widehat{u}) = \widetilde{E}(\widehat{\psi}, \widehat{a}) = 0$ is obtained.

Summarizing, we have found that $\widehat{a} \in \mathcal{A}$. Now assumption (III) ensures that $\widehat{a} = a^*$, and correspondingly $\widehat{u} = u^*$, $\widehat{\psi} = \psi^*$. From (19) we conclude that $\widetilde{F}(u_n) \to 0$.
3. We set

$$w_n = \sqrt{a_n/a^*} - 1,\ \lambda_n = \sqrt{\widetilde{F}(u_n)},\ b_n = w_n/\lambda_n,\ y_n = (u_n - u^*)/\lambda_n,\ z_n = (\psi_n - \psi^*)/\lambda_n.$$

In the formula for the lower bound of the dissipation rate

$$\widetilde{D}(u_n) = \int_\Omega \Big\{ \sum_{i\in J} |\nabla w_{ni}|^2 + \sum_{(\alpha,\beta)\in\mathcal{R}} |(1+w_n)^\alpha - (1+w_n)^\beta|^2 \Big\} \,\mathrm{d}x$$

we use the binomial expansion

$$\prod_{i\in I}(1+w_{ni})^{\alpha_i} = 1 + \sum_{i\in I} \alpha_i w_{ni} + \omega(w_n), \quad \frac{1}{\lambda_n}|\omega(w_n)| \le c \sum_{i\in I} \Big\{ \lambda_n |b_{ni}|^2 + \lambda_n^{p_\alpha - 1} |b_{ni}|^{p_\alpha} \Big\}$$

where $p_\alpha = \max\{2, \sum_{i\in I} \alpha_i\}$. (18), (20) imply that $\|z_n\|_{H^1}, \|b_n\|_Y \le c(R)$, $\widetilde{D}(u_n)/\lambda_n^2 \to 0$. Therefore $z_n \rightharpoonup \widehat{z}$ in H^1, $z_n \to \widehat{z}$ in L^2. Now, for $i \in J$ we find $\widehat{b}_{ni} \in \mathbb{R}_+$ with $b_{ni} \to \widehat{b}_i$ in H^1 and in each L^p, moreover, $\lambda_n \|b_{ni}^2\|_{L^2} \to 0$. For $i \in J'$ from

$$\int_\Omega \Big| \frac{1}{\lambda_n} \Big\{ \prod_{j\in J} (1+w_{nj})^{\alpha_j} - 1 \Big\} - 2b_{ni} - \lambda_n b_{ni}^2 \Big|^2 \,\mathrm{d}x \to 0$$

it follows that $2b_{ni} + \lambda_n b_{ni}^2 \to 2\widehat{b}_i \in \mathbb{R}_+$ in L^2, and because of the estimate

$$\|b_{ni} - \widehat{b}_i\|_{L^2} \le c(R)\,(\|2b_{ni} + \lambda_n b_{ni}^2 - 2\widehat{b}_i\|_{L^2} + \lambda_n \|b_{ni}\|_{L^2})$$

we get $b_{ni} \to \widehat{b}_i$ in L^2 as well as $\lambda_n \|b_{ni}^2\|_{L^2} \to 0$. Fatou's lemma yields $(\alpha - \beta)\cdot \widehat{b} = 0$ $\forall (\alpha,\beta) \in \mathcal{R}$, or in other words, $\widehat{b} \in \mathcal{S}^\perp$. Setting $\widehat{y} = a^* p(\psi^*)(2\widehat{b} - Q(\psi^*)\widehat{z})$ we get

$$\|y_n - \widehat{y}\|_Y \le c(R) \Big\{ \|b_n - \widehat{b}\|_Y + \|z_n - \widehat{z}\|_{L^2} + \lambda_n \Big(\sum_{i=1}^m \|b_{ni}^2\|_{L^2} + 1 \Big) \Big\},$$

and thus $y_n \to \widehat{y}$ in Y. Consequently, $\int_\Omega \widehat{y}\,\mathrm{d}x \in \mathcal{S}$ and $(\widehat{b}, \widehat{y})_Y = 0$. Finally, we obtain

$$0 \ge \frac{1}{\lambda_n^2} \langle E(\psi^*, u_n) - E(\psi_n, u_n), \psi_n - \psi^* \rangle = -\int_\Omega \sum_{i=1}^m Q_i(\psi^*) z_n y_{ni} \,\mathrm{d}x,$$

and using $(\widehat{b}, \widehat{y})_Y = 0$, in the limit we arrive at

$$0 \ge -\int_\Omega \sum_{i=1}^m Q_i(\psi^*) \widehat{z} \widehat{y}_i \,\mathrm{d}x = \int_\Omega \sum_{i=1}^m (2\widehat{b}_i - Q_i(\psi^*)\widehat{z})\, \widehat{y}_i \,\mathrm{d}x = \int_\Omega \sum_{i=1}^m \frac{1}{a_i^* p_i(\psi^*)} |\widehat{y}_i|^2 \,\mathrm{d}x.$$

Therefore $\widehat{y} = 0$, and (19) gives the contradiction $1 \le c_2(R) \|y_n\|_Y^2 \to 0$. □

Corollary 3. *Let the assumptions* (II), (III) *be fulfilled. Then along any solution* (u, ψ) *to* (P) *the free energy* $F(u)$ *decays exponentially to its equilibrium value* $F(u^*)$,

$$F(u(t)) - F(u^*) \le e^{-\lambda t} (F(U) - F(u^*)) \quad \forall t \ge 0$$

where λ depends only on the data. Moreover, there is a constant c such that

$$\|u(t) - u^*\|_{L^1(\Omega,\mathbb{R}^m)},\ \|\psi(t) - \psi^*\|_{H^1(\Omega)} \le c\, e^{-\lambda t/2} \quad \forall t \ge 0$$

for any solution to (P).

Proof. Because of Corollary 1 and Lemma 6 for $R = \max\{1, F(U) - F(u^*)\} > 0$ it holds $u(t) \in M_R\, \forall t > 0$. We set $\lambda = 1/c_R$ and with Theorem 3 and Theorem 1 the first assertion is obtained. In Theorem 2 have we stated that $\zeta^* \in \mathcal{S}^\perp$. Then the estimates follow from (7), (8) by setting there $\overline{u} = u^*$, $\overline{\psi} = \psi^*$. □

8. Remarks

Remark 1. We consider a special version of the pair diffusion models in [2,4]. The meaning of the species and the used reactions are outlined in Figure 1. Here $m = 5$, $i = 1, \dots, 5$, or in the notation of Figure 1, $i = A, I, V, AI, AV$, $J = \{2, 3, 4, 5\}$, $J' = \{1\}$. The functions p_i, Q_i are given by (2), D_i, $k_{\alpha\beta}$ are similar averaged quantities and $e(\psi) = c \sinh \psi$. These functions have all properties which are required in (I).

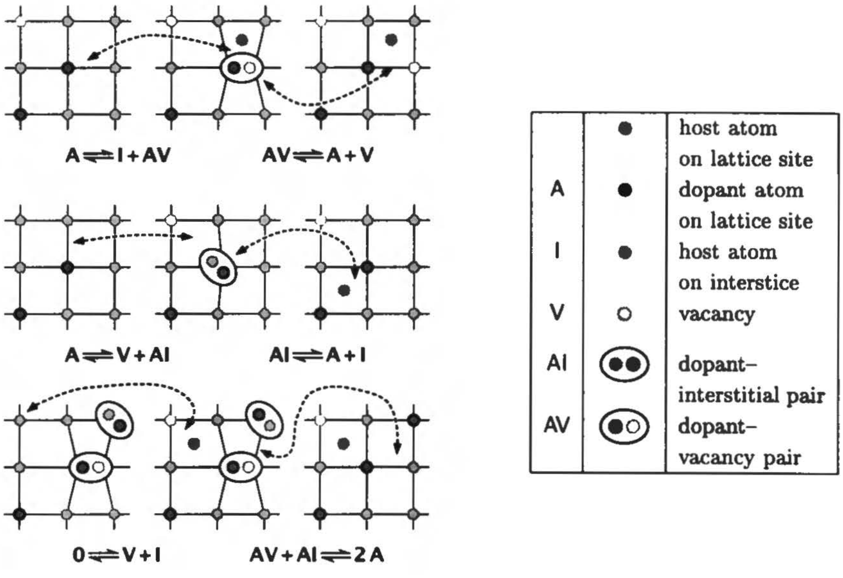

Figure 1: Species and reactions in the pair diffusion model (see [2,4]). The corresponding reaction rates are given by

$$R_{\alpha\beta} = k_{\alpha\beta}(\psi)\,(a_A - a_I\, a_{AV}), \qquad R_{\alpha\beta} = k_{\alpha\beta}(\psi)\,(a_{AV} - a_A\, a_V),$$
$$R_{\alpha\beta} = k_{\alpha\beta}(\psi)\,(a_A - a_V\, a_{AI}), \qquad R_{\alpha\beta} = k_{\alpha\beta}(\psi)\,(a_{AI} - a_A\, a_I),$$
$$R_{\alpha\beta} = k_{\alpha\beta}(\psi)\,(1 - a_V\, a_I), \qquad R_{\alpha\beta} = k_{\alpha\beta}(\psi)\,(a_{AV}\, a_{AI} - a_A^2)$$

where e. g. for the first reaction $\alpha = (1,0,0,0,0)$, $\beta = (0,1,0,0,1)$ and for the last one $\alpha = (0,0,0,1,1)$, $\beta = (2,0,0,0,0)$.

Next we find that $\dim \mathcal{S} = 3$, $\dim \mathcal{S}^{\perp} = 2$, $\mathcal{S}^{\perp} = \operatorname{span}\{(1,0,0,1,1),(0,1,-1,1,-1)\}$ such that there are two invariants the value of which is fixed by the initial state, namely

$$I_1(t) = \int_\Omega [u_A(t) + u_{AI}(t) + u_{AV}(t)]\,\mathrm{d}x = I_1(0),$$

$$I_2(t) = \int_\Omega [u_I(t) - u_V(t) + u_{AI}(t) - u_{AV}(t)]\,\mathrm{d}x = I_2(0) \quad \forall t \in \mathbb{R}_+.$$

Condition (II) means that $I_1(0) > 0$, and we easily verify that then assumption (III) is fulfilled, too. Thus our results can be applied to this special model.

Remark 2. In (1) the boundary conditions for the first set of continuity equations can be replaced by the following ones:

$$\nu \cdot j_i = \sum_{(\alpha,\beta)\in\mathcal{R}^\Gamma} (\alpha_i - \beta_i)\, R^\Gamma_{\alpha\beta} \quad \text{on } (0,\infty)\times\partial\Omega,\ i \in J,$$

$$R^\Gamma_{\alpha\beta}(x,u,\psi) = k^\Gamma_{\alpha\beta}(x,\psi)\Big[\prod_{i=1}^m a_i^{\alpha_i} - \prod_{i=1}^m a_i^{\beta_i}\Big],\ x\in\partial\Omega,\ u\in\mathbb{R}^m,\ \psi\in\mathbb{R},\ a_i = \frac{u_i}{p_i(\psi)},$$

where $(\alpha,\beta)\in\mathcal{R}^\Gamma$ and $\mathcal{R}^\Gamma \subset \{(\alpha,\beta)\in\mathbb{Z}^m_+\times\mathbb{Z}^m_+ \colon \alpha_i = \beta_i = 0\,\forall i\in J'\}$ describes a set of additional boundary reactions. We assume that for each $(\alpha,\beta)\in\mathcal{R}^\Gamma$ the function $k^\Gamma_{\alpha\beta} : \partial\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies the Carathéodory conditions and

$$c_{1,R}(x) \le k^\Gamma_{\alpha\beta}(x,\psi) \le c_{2,R}(x) \text{ f.a.a. } x\in\partial\Omega,\ \forall\psi\in\mathbb{R},\ |\psi|\le R,$$

$$c_{1,R},\ c_{2,R}\in L^\infty_+(\partial\Omega),\ \|c_{1,R}\|_{L^1(\partial\Omega)} > 0.$$

All boundary reactions must be included into the definition of the sets $\mathcal{S}$, $\mathcal{A}$, and in the definition of A, D boundary integrals have to be added. Then all assertions of the paper, especially the assertions of Corollary 1 – Corollary 3, remain valid.

Remark 3. Testing the Poisson equation $E(\psi,u) = 0$ by $\overline{\psi} = 1$ the global electroneutrality condition

$$\int_\Omega \Big[e(\psi(t)) - \sum_{i=1}^m Q_i(\psi(t))\,u_i(t) - f\Big]\,\mathrm{d}x = 0\ \forall t\in\mathbb{R}_+ \tag{21}$$

is obtained. If we want to use other boundary conditions for the Poisson equation in (1) then condition (21) must be taken onto account, too. Therefore, as in [7] we consider mixed boundary conditions of the form

$$\psi = \zeta_0 \quad \text{on } (0,\infty)\times\Gamma_D, \qquad \nu\cdot\nabla\psi + \tau\psi = \tau\zeta_0 \quad \text{on } (0,\infty)\times\Gamma_N$$

where the new unknown quantity $\zeta_0 : \mathbb{R}_+ \to \mathbb{R}$ (the electrochemical potential of electrons, i. e. the Fermi level) has to be determined by means of the nonlocal constraint (21). We assume that

$$\Gamma_D,\ \Gamma_N \text{ are disjoint open subsets of } \partial\Omega,\ \partial\Omega = \overline{\Gamma_D}\cup\overline{\Gamma_N},$$

$$\overline{\Gamma_D}\cap\overline{\Gamma_N} \text{ consists of finitely many points},$$

$$\tau\in L^\infty_+(\Gamma_N),\ \operatorname{mes}\Gamma_D + \|\tau\|_{L^1(\Gamma_N)} > 0.$$

The definition of E must be changed (see [7]) and F, F^* contain an additional boundary integral. Again, all assertions of Corollary 1 - Corollary 3 remain valid.

Remark 4. As in [6] analogous energy estimates and asymptotic properties can be derived for a discrete-time version of (1) using an implicit scheme of first order.

Acknowledgments. The authors are indebted to W. Röpke and N. Strecker for helpful discussion concerning the model equations. The second author is supported by the German Research Foundation (DFG).

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On the Boundary Conditions at the Contact Interface between Two Porous Media

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Abstract. We consider an incompressible creeping flow through a 2D porous medium containing two different types of pores. The two parts are separated by the interface $(0, L) \times \{0\}$. It is well-known that the effective flow in both parts is described by Darcy's law, with different permeabilities. For the periodic porous medium we prove that at the interface, continuity of the effective pressure and the effective normal velocity hold true. The proof uses the boundary layers for the Stokes operator in a heterogeneous periodic porous medium.

Keywords: homogenization, boundary layers, porous medium, interface, Stokes operator

Classification: 76D05, 35B27, 76Mxx, 35Qxx

1. Introduction

Flow through porous media, containing two or several subdomains with different micro-structures, arises in many applications. It is generally accepted that the effective conditions at interfaces, between subdomains, are the pressure continuity and the continuity of the normal components of Darcy's velocity (see e.g. Dagan[2]). However, one should not forget that Darcy's law is obtained after averaging the momentum equation and the orders of differential operators are changed. For an incompressible flow, we conclude immediately the continuity of the normal components of Darcy's velocity, at the interfaces. From the other side, the stress tensor completely changes in the weak limit and there are no *a priori* reasons to have the pressure continuity at the interfaces. Furthermore, it is known that, for contact between a porous medium and a channel flow, we don't have in general the pressure continuity and some additional terms appear. We refer to Jäger and Mikelić [3], Jäger and Mikelić [4] and Jäger, Mikelić and Neuss[5] for more details. In that situation engineers use the law of Beavers and Joseph (see Beavers and Joseph [1] and, for mathematical justification, Jäger and Mikelić [4]).

Here the difference between the geometrical structure of the subdomains is not so big and we don't expect that the averaging procedure leads to drastic changes. Finally, there are reasons to believe in good understanding of the problem by engineers.

The goal of this paper is to confirm rigorously the pressure continuity and in order to fix the ideas we introduce a model problem.

We consider a slow viscous two-dimensional incompressible flow in a domain Ω^ϵ consisting of the porous media $\Omega_1 = (0, L) \times \mathbf{R}_+$ and $\Omega_2 = (0, L) \times \mathbf{R}_-$, and the interface $\Sigma = (0, L) \times \{0\}$ between them. We assume that the structure of the porous media is periodic. Ω_1 is generated by translations of a cell $Z^\epsilon = \epsilon Z$, where Z is the standard cell, $Z = (0,1)^2$, consisting of an open set Z^*, $\partial Z^* \in C^\infty$, being strictly included in Z. Let $Z_F = Z \setminus \bar{Z}^*$ be connected and let χ_1 be the characteristic function of Z_F extended by periodicity to $\mathbf{R}^2$. We set $\chi_1^\epsilon(x) = \chi_1(\frac{x}{\epsilon})$, $x \in \mathbf{R}^2$, and define Ω_1^ϵ by $\Omega_1^\epsilon = \{x | x \in \Omega_1, \chi_1^\epsilon(x) = 1\}$. Ω_2 is also generated by translations of a cell Z^ϵ, but this time we suppose that Z strictly includes the open set Y^*, $\partial Y^* \in C^\infty$, and that $Y_F = Z \setminus \bar{Y}^*$. Let χ_2 be the characteristic function of Y_F. We set $\chi_2^\epsilon(x) = \chi_2(\frac{x}{\epsilon})$ on $\mathbf{R}^2$ and $\Omega_2^\epsilon = \{x | x \in \Omega_2, \chi_2^\epsilon(x) = 1\}$. Then $\Omega^\epsilon = \Omega_1^\epsilon \cup \Sigma \cup \Omega_2^\epsilon$. It is supposed that $\dfrac{L}{\epsilon} \in \mathbf{N}$ and $f \in C^\infty(\bar{\Omega})^2$, *suppf* is compact, f is L-periodic in x_1 .

The flow through Ω^ϵ is described by the Stokes system

$$-\Delta u^\epsilon + \nabla p^\epsilon = f \quad \text{in } \Omega^\epsilon \tag{1.1}$$

$$\text{div } u^\epsilon = 0 \quad \text{in } \Omega^\epsilon \tag{1.2}$$

$$u^\epsilon = 0 \quad \text{on } \partial\Omega^\epsilon \setminus \partial(\Omega_1 \cup \Omega_2) \tag{1.3}$$

$$\{u^\epsilon, p^\epsilon\} \quad \text{is } L\text{-periodic in } x_1 \tag{1.4}$$

$$\nabla u^\epsilon \in L^2(\Omega^\epsilon)^4 \quad \text{and } \nabla p^\epsilon \in L^2(\Omega^\epsilon)^2. \tag{1.5}$$

The choice of periodic boundary conditions in x_1 and unboundedness with respect to x_2 eliminates the effects of outer boundaries, which are of no importance for justifying the pressure continuity.

Before stating our results, we briefly discuss problem (1.1)-(1.5). We introduce the functional space W_ϵ by

$$\begin{aligned} W_\epsilon \;=\; & \Big\{ z \in H^1(\Omega^\epsilon)^2 : \text{ div } z = 0 \text{ a.e. in } \Omega^\epsilon, \\ & z = 0 \text{ on } \partial\Omega^\epsilon \setminus \partial(\Omega_1 \cup \Omega_2) \text{ and } z \text{ is } H^1\text{-periodic in } x_1 \Big\}. \end{aligned} \tag{1.6}$$

Then, using Poincaré's inequality, we easily get

Lemma 1.1. *Problem (1.1)-(1.5) has a unique solution $u^\epsilon \in W_\epsilon$. Furthermore, there exists $p^\epsilon \in L^2_{loc}(\Omega^\epsilon)$, $\nabla p^\epsilon \in L^2(\Omega^\epsilon)^2$ such that (1.1) holds in the sense of distributions. Finally, $\{u^\epsilon, p^\epsilon\} \in C^\infty(\Omega^\epsilon)^2 \times C^\infty(\Omega^\epsilon)$.*

Section 2 contains the detailed study of the homogenized problem. In addition we give all auxiliary problems used in this paper. Their solvability and properties of solutions were established in Jäger and Mikelić [3].

Our main results are stated in Section 3. Proofs are given in Sections 4 and 5.

2. Study of the homogenized problem and auxiliary results

We start with 2 classes of auxiliary problems defining the 2 permeabilities:
We are looking for $\{w^j, \pi^j\}$, $j = 1, 2$, satisfying

$$\left\{ \begin{array}{l} -\Delta_y w^j + \nabla_y \pi^j = e_j \text{ in } Z_F; \\ \text{div}_y w^j = 0 \text{ in } Z_F, \quad \displaystyle\int_{Z_F} \pi^j(y)dy = 0, \\ w^j = 0 \text{ on } \partial Z^*, \quad \{w^j, \pi^j\} \text{ is 1-periodic.} \end{array} \right. \tag{2.1}$$

Problem (2.1) admits a unique solution $\{w^j, \pi^j\} \in C^\infty(\bar{Z}_F)^3$ and the matrix

$$K^w_{ij} = \int_{Z_F} w^i_j(y)dy, \quad i,j = 1,2 \tag{2.2}$$

is symmetric and positive definite with $K^w_{11}, K^w_{22} > 0$.

Analogously, we consider the problem:

Find $\{v^j, \omega^j\} \in H^1(Y_F)^2 \times L^2(Y_F)$ such that

$$\begin{cases} -\Delta_y v^j + \nabla_y \omega^j = e_j \text{ in } Y_F; \\ \operatorname{div}_y v^j = 0 \text{ in } Y_F, \quad \int_{Y_F} \omega^j(y)dy = 0; \\ v^j = 0 \text{ on } \partial Y^*, \quad \{v^j, \omega^j\} \text{ is 1-periodic.} \end{cases} \tag{2.3}$$

The corresponding matrix is K^v, given by

$$K^v_{ij} = \int_{Y_F} v^i_j(y)dy, \quad i,j = 1,2. \tag{2.4}$$

Then the global permeability is defined by

$$K(x_2) = \begin{cases} K^w & \text{for} \quad x_2 > 0 \\ K^v & \text{for} \quad x_2 < 0 \end{cases} \tag{2.5}$$

and the homogenized problem reads:

Find $p^0 \in L^2_{loc}(\Omega), \nabla p^0 \in L^2(\Omega_1 \cup \Omega_2)^2$ such that

$$\begin{cases} -\operatorname{div}\{K \nabla p^0\} = -\operatorname{div}\{Kf\} \text{ in } \Omega \\ \\ \qquad p^0 \text{ is } L\text{-periodic in } x_1 \end{cases} \tag{2.6}$$

where $\Omega = \Omega_1 \cup \Omega_2 \cup \Sigma$.

Obviously problem (2.6) admits a solution $p^0 \in H^1_{loc}(\Omega_1 \cup \Omega_2)$, unique up to a constant. It is important to establish the regularity properties of p^0. Due to the periodicity and the geometry, we have immediately

Lemma 2.1. *$p^0 \in C^\infty_{per}([0,L]; H^1_{loc}(\mathbf{R}_+ \cup \mathbf{R}_-))$, i.e. p^0 is C^∞ with respect to x_1.*

Unfortunately, there is a jump of derivatives with respect to x_2 on Σ. However we have the following regularity result.

Proposition 2.2. *Let $f = 0$ on $[0,L] \times (-\delta, \delta)$ for some $\delta > 0$. Then. $p^0 \in C^\infty([0,L] \times [0,+\infty)) \cap C^\infty([0,L] \times (-\infty, 0])$ and $p^0(x_1, -0) = p^0(x_1, +0)$ on Σ.*

Proof: Let us fix $x_1 \in (0, L)$. It will play the role of a dummy variable. We write (2.6) in the form

$$-\frac{d}{dx_2}\left(K_{22}\frac{dp^0}{dx_2} + 2K_{21}\frac{\partial p^0}{\partial x_1} + (Kf)_2 + K_{12}\frac{\partial f_2}{\partial x_1}\right) =$$

$$K_{11}\frac{\partial^2 p^0}{\partial x_1^2} + K_{11}\frac{\partial f_1}{\partial x_1} \in C(\bar{\Omega}) \qquad \text{in } \Omega_1 \cup \Omega_2 \tag{2.7}$$

$$K_{22}^v \frac{dp^0}{dx_2}(x_1, -0) - K_{22}^w \frac{dp^0}{dx_2}(x_1, +0) =$$

$$((K^v - K^w)f)_2(x_1, 0) + (K_{21}^w - K_{21}^v)\frac{\partial p^0}{\partial x_1}(x_1, 0) \quad \text{on } \Sigma \tag{2.8}$$

$$p^0(x_1, -0) = p^0(x_1, +0) \quad \text{on } \Sigma. \tag{2.9}$$

After simple change of coordinates $x_2 \to \dfrac{x_2}{K_{22}}$ we write (2.7)-(2.9) as

$$\begin{cases} -\dfrac{d^2 p^0}{dx_2^2} = g_i \in L^q_{loc}(\Omega_i) \text{ in } \Omega_i, \quad i = 1, 2, \quad \forall q < +\infty \\[2ex] \left[\dfrac{dp^0}{dx_2}\right]_\Sigma = h \in C(\Sigma), \quad [p^0]_\Sigma = 0. \end{cases} \tag{2.10}$$

At this stage, we redefine p^0 by setting

$$\pi^0 = \begin{cases} p^0 + x_2 e^{-x_2} h(x_1) & \text{in } \Omega_1 \\[1ex] p^0 & \text{in } \Omega_2. \end{cases} \tag{2.11}$$

Then $\dfrac{d^2\pi^0}{dx_2^2} \in L^q_{loc}(\Omega)$, $\forall q < +\infty$ and $\pi^0 \in W^{2,q}_{loc}(\Omega)$. Consequently, $p^0 \in C^\infty_{per}([0, L];$ $W^{2,q}_{loc}(\mathbf{R}_+ \cup \mathbf{R}_-))$ and $\left.\dfrac{\partial p^0}{\partial x_2}\right|_\Sigma \in W^{1-1/q,q}(\Sigma)$.

Now we differentiate the equation (2.7) with respect to x_2 and arguing as above obtain $\dfrac{\partial p^0}{\partial x_2} \in C^\infty_{per}([0, L]; W^{2,q}_{loc}(\mathbf{R}_+ \cup \mathbf{R}_-))$.

Repeating the argument, we obtain the regularity from the statement of the proposition.

□

Corollary 2.3. *For every $\alpha \in \mathbf{N}^2$ there exists $\delta_0(\alpha) > 0$ such that $|D^\alpha p^0(x)| \leq Ce^{-\delta_0(\alpha)|x_2|}$ for $x_2 > x^*(\alpha)$.*

Next we need auxiliary problems correcting the compressibility effects caused by corrections. They are exactly the same as in Jäger and Mikelić [3]:

We are looking for $\gamma^{i,j}$, $i, j = 1, 2$, satisfying

$$\begin{cases} \operatorname{div}_y \gamma^{i,j} = w_i^j - \dfrac{1}{|Z_F|} K_{ij}^w & \text{in } Z_F \\ \\ \gamma^{i,j} = 0 \text{ on } \partial Z^*; \quad \gamma^{i,j} \text{ is 1-periodic.} \end{cases} \tag{2.12}$$

For existence of a function $\gamma^{i,j} \in C^\infty(\bar{Z}_F)^2$ satisfying (2.12) we refer to Jäger and Mikelić [3].

The corresponding problem in Y_F reads

$$\begin{cases} \operatorname{div}_y \delta^{i,j} = v_i^j - \dfrac{1}{|Y_F|} K_{ij}^v & \text{in } Y_F \\ \\ \delta^{i,j} = 0 \text{ on } \partial Y^*; \quad \delta^{i,j} \text{ is 1-periodic.} \end{cases} \tag{2.13}$$

The final class of auxiliary problems are ones corresponding to the boundary layers due to the difference between w^j and v^j on $S = (0,1) \times \{0\}$.

Let $Z^- = \bigcup_{k=1}^{+\infty} \{Y_F - (0,k)\}$, $Z^+ = \bigcup_{k=0}^{+\infty} \{Z_F + (0,k)\}$, $S = (0,1) \times \{0\}$ and $Z_{BL} = Z^- \cup S \cup Z^+$. Let $[a]_S(\cdot, 0) = a(\cdot, +0) - a(\cdot, -0)$. We consider the following problem for $j = 1, 2$

$$-\Delta_y w^{j,bl} + \nabla_y \pi^{j,bl} = 0 \text{ in } Z^+ \cup Z^- \tag{2.14}$$

$$\operatorname{div}_y w^{j,bl} = 0 \text{ in } Z^+ \cup Z^- \tag{2.15}$$

$$\left[w^{j,bl}\right]_S = v^j(\cdot, -0) - w^j(\cdot, +0) + e_2 \int_0^1 \left(w_2^j(y_1, +0) - v_2^j(y_1, -0)\right) dy_1 \quad \text{on } S \tag{2.16}$$

$$\left[(\nabla_y w^{j,bl} - \pi^{j,bl}\mathbf{I})e_2\right]_S (\cdot, 0) = (\nabla v^j - \omega^j \mathbf{I})(y_1, -0)e_2 - (\nabla w^j - \pi^j \mathbf{I})(y_1, +0)e_2 \text{ on } S \tag{2.17}$$

$$w^{j,bl} = 0 \text{ on } \partial Z_{BL} \setminus ((\{0\} \cup \{1\}) \times \mathbf{R}) \tag{2.18}$$

$$\{w^{j,bl}, \pi^{j,bl}\} \text{ is 1-periodic.} \tag{2.19}$$

Remark: Since $K_{2j}^w = \int_0^1 w_2^j(y_1, y_2) dy_1, \forall y_2 \in (0,1)$ and $K_{2j}^v = \int_0^1 v_2^j(y_1, y_2) dy_2, \forall y_2 \in (0,1)$, the condition (2.16) can be written as

$$[w^{j,bl}]_S = v^j(\cdot, -0) - w^j(\cdot, +0) + (K_{2j}^w - K_{2j}^v)e_2 \quad \text{on } S. \tag{2.20}$$

□

Now arguing as in the proof of Theorem 3.15 from Jäger and Mikelić [3] we obtain the following result.

Proposition 2.4. *The problem (2.14)-(2.19) has a unique solution* $w^{j,bl} \in W = \{z \in L^2(Z_{BL}) : z \in H^1(Z^+),\ z \in H^1(Z^-),\ z = 0$ *on* $\partial Z_{BL} \setminus (\{0\} \cup \{1\}) \times \mathbf{R}, \operatorname{div}_y z = 0$ *on* $Z^+ \cup Z^-$ *and* z *is* H^1*-periodic in* $y_1\}$.

Moreover, $w^{j,bl} \in C^\infty_{loc}(Z^+ \cup Z^-)^2$ *and there exists* $\gamma_0 > 0$ *such that*

$$|\nabla_y w^{j,bl}(y)| \le Ce^{-\gamma_0 |y_2|} \quad \text{and} \quad |w^{j,bl}(y)| \le Ce^{-\gamma_0 |y_2|}. \tag{2.21}$$

Finally, there exists $\pi^{j,bl} \in C^\infty_{loc}(Z^+ \cup Z^-)$ *such that* $\nabla_y \pi^{j,bl} \in L^2(Z^+ \cup Z^-)$, *(2.14) holds and there are constants* $\gamma_0 > 0, C^j_\infty, C^j_\pi$ *such that*

$$|\nabla_y \pi^{j,bl}(y)| + |\pi^{j,bl}(y) - H(y_2)C^j_\pi - H(-y_2)C^j_\infty| \le Ce^{-\gamma_0|y_2|} \tag{2.22}$$

At $S = (0,1) \times \{0\}$, $w^{j,bl} \in W^{2,q}(S)^2$ *and* $\pi^{j,bl} \in W^{1,q}(S)$, $\forall q \in (1, +\infty)$.

Next we need the boundary layers corresponding to the functions $\gamma^{j,i}$ and $\delta^{j,i}$:

$$-\Delta_y \gamma^{j,i,bl} + \nabla_y \pi^{j,i,bl} = 0 \text{ in } Z^+ \cup Z^- \tag{2.23}$$

$$\operatorname{div}_y \gamma^{j,i,bl} = \mathcal{H}^{j,i} \left\{ \frac{e^{y_2} H(-y_2)}{\int_{Z^-} e^{y_2} dy_2 dy_1} + \frac{e^{-y_2} H(y_2)}{\int_{Z^+} e^{-y_2} dy} \right\} \text{ in } Z^+ \cup Z^- \tag{2.24}$$

$$[\gamma^{j,i,bl}]_s = -\gamma^{j,i}(y_1, +0) + \delta^{j,i}(y_1, -0) \text{ on } S \tag{2.25}$$

$$\left[(\nabla_y \gamma^{j,i,bl} - \pi^{j,i,bl}\mathbf{I})e_2\right]_S = 0 \text{ on } S \tag{2.26}$$

$$\gamma^{j,i,bl} = 0 \text{ on } \partial Z_{BL} \setminus (\{0\} \cup \{1\}) \times \mathbf{R} \tag{2.27}$$

$$\{\gamma^{j,i,bl}, \pi^{j,i,bl}\} \text{ is 1-periodic in } y_1 \tag{2.28}$$

where $\mathcal{H}^{j,i} = \int_0^1 (\delta_2^{j,i}(y_1, -0) - \gamma_2^{j,i}(y_1, +0))dy_1$.

Analogously to Proposition 2.3 we obtain existence of the unique solution $\gamma^{j,i,bl}$ which exponentially tends to zero for large $|y_2|$. For $\pi^{j,i,bl}$ we obtain an analogue to (2.22).

3. Statement of the main results

In this section we give two results. The first result is obtained under the hypothesis:

$$\begin{cases} f \in C_0^\infty(\Omega)^2, \ f \text{ is } L\text{-periodic in } x_1 \\ \text{and } f = 0 \text{ on } [0, L] \times (-\delta, \delta) \text{ for some } \delta > 0 \end{cases} \tag{3.1}$$

and the second under the much weaker assumption:

$$f \in C_0^\infty(\Omega)^2, \ f \text{ is } L\text{-periodic in } x_1 \tag{3.2}$$

Theorem 3.1 *Let us suppose (3.1). Then we have*

$$\left\| \frac{u^\epsilon}{\epsilon^2} - \sum_{k=1}^2 \left(H(x_2) w^k(\frac{x}{\epsilon}) + H(-x_2) v^k(\frac{x}{\epsilon}) \right) \left(f_k(x) - \frac{\partial p^0}{\partial x_k}(x) \right) \right\|_{L^2(\Omega)^2} \le C\sqrt{\epsilon} \tag{3.3}$$

and

$$\frac{u^\epsilon}{\epsilon^2} \rightharpoonup K^w(f - \nabla p^0)H(x_2) + K^v(f - \nabla p^0)H(-x_2) \tag{3.4}$$

weakly in $L^2(\Omega)^2$.

Furthermore, there exists an extension $\tilde{p}^\epsilon$ *of the pressure such that*

$$\left\| \frac{1}{1 + |x_2|} (\tilde{p}^\epsilon - p^0) \right\|_{L^2(\Omega)} \le C\sqrt{\epsilon}. \tag{3.5}$$

The next result corresponds to the assumption (3.2):

Theorem 3.2 *Let us suppose (3.2). Then we have*

$$\left\| \frac{u^\epsilon}{\epsilon^2} - \sum_{k=1}^{2} \left(H(x_2) w^k(\frac{x}{\epsilon}) + H(-x_2) v^k(\frac{x}{\epsilon}) \right) \left(f_k(x) - \frac{\partial p^0}{\partial x_k}(x) \right) \right\|_{L^2(\Omega)^2} \leq C\epsilon^{1/8} \tag{3.6}$$

and there exists an extension $\tilde{p}^\epsilon$ of the pressure such that

$$\left\| \frac{1}{1+|x_2|} (\tilde{p}^\epsilon - p^0) \right\|_{L^2(\Omega)} \leq C\epsilon^{1/8}. \tag{3.7}$$

Finally, the convergence (3.4) holds true.

Corollary 3.3

$$e_2 \frac{u^\epsilon}{\epsilon^2}\bigg|_{\Sigma} \rightharpoonup K^w(f - \nabla p^0)e_2 = K^v(f - \nabla p^0)e_2 \tag{3.8}$$

weakly in $L^2(\Sigma)$.

4. Proof of Theorem 3.1

In the proof which follows we will frequently use the space

$$V_{per}(\Omega^\epsilon) = \{ z \in H^1(\Omega^\epsilon)^2 : z = 0 \text{ on } \partial\Omega^\epsilon \setminus \partial\Omega \text{ and } z \text{ is } L\text{-periodic in } x_1 \}. \tag{4.1}$$

Then we start with the approximation corresponding to Darcy's law. We define $u^{0,\epsilon}$ and $p^{0,\epsilon}$ by

$$u^{0,\epsilon}(x) = \begin{cases} \sum_{k=1}^{2} w^{k,\epsilon}(x) \left(f_k(x) - \frac{\partial p^0}{\partial x_k}(x) \right), & x \in \Omega_1 \\ \sum_{k=1}^{2} v^{k,\epsilon}(x) \left(f_k(x) - \frac{\partial p^0}{\partial x_k}(x) \right), & x \in \Omega_2 \end{cases} \tag{4.2}$$

$$p^{0,\epsilon}(x) = \begin{cases} \sum_{k=1}^{2} \pi^{k,\epsilon}(x) \left(f_k(x) - \frac{\partial p^0}{\partial x_k}(x) \right), & x \in \Omega_1 \\ \sum_{k=1}^{2} \omega^{k,\epsilon}(x) \left(f_k(x) - \frac{\partial p^0}{\partial x_k}(x) \right), & x \in \Omega_2 \end{cases} \tag{4.3}$$

where

$$\begin{cases} w^{k,\epsilon}(x) = w^k(\frac{x}{\epsilon}); \; v^{k,\epsilon}(x) = v^k(\frac{x}{\epsilon}) \\ \pi^{k,\epsilon}(x) = \pi^k(\frac{x}{\epsilon}) \text{ and } \omega^{k,\epsilon}(x) = \omega^k(\frac{x}{\epsilon}). \end{cases} \tag{4.4}$$

Then we set

$$\begin{cases} U_0^\epsilon = \frac{u^\epsilon}{\epsilon^2} - u^{0,\epsilon}, & x \in \Omega^\epsilon \\ P_0^\epsilon = \frac{p^\epsilon}{\epsilon^2} - \frac{p^0}{\epsilon^2} - \frac{1}{\epsilon} p^{0,\epsilon}, & x \in \Omega^\epsilon. \end{cases} \tag{4.5}$$

Our first estimate is given by the following proposition

Proposition 4.1. *Let $\phi \in V_{per}(\Omega^\epsilon)$. Then we have*

$$\left|\int_{\Omega_1^\epsilon\cup\Omega_2^\epsilon} (\nabla U_0^\epsilon \nabla\phi - P_0^\epsilon \operatorname{div}\phi)\right| \le$$
$$C\left\{\|\nabla(f-\nabla p^0)\|_{L^2(\Omega)^4} + \epsilon\|\Delta(f-\nabla p^0)\|_{L^2(\Omega)^2}\right\}\|\nabla\phi\|_{L^2(\Omega^\epsilon)^4}$$
$$+\left|\sum_{j=1}^{2}\int_\Sigma \left\{\nabla(w^{j,\epsilon}-v^{j,\epsilon}) - \epsilon^{-1}(\pi^{j,\epsilon}-\omega^{j,\epsilon})\mathbf{I}\right\} e_2\phi\left(f_j - \frac{\partial p^0}{\partial x_j}\right)\right|. \tag{4.6}$$

Proof: We start with the weak formulation corresponding to (1.1)-(1.5):

$$\int_{\Omega_1^\epsilon\cup\Omega_2^\epsilon} \nabla\frac{u^\epsilon}{\epsilon^2}\nabla\phi - \int_{\Omega_1^\epsilon\cup\Omega_2^\epsilon} \epsilon^{-2}p^\epsilon \operatorname{div}\phi = \int_{\Omega^\epsilon}\epsilon^{-2}f\phi, \quad \forall\phi\in V_{per}(\Omega^\epsilon). \tag{4.7}$$

The introduction of U_0^ϵ and P_0^ϵ corresponds to the elimination of the forcing term. We have:

$$\int_{\Omega_1^\epsilon\cup\Omega_2^\epsilon} \nabla U_0^\epsilon\nabla\phi - \int_{\Omega_1^\epsilon\cup\Omega_2^\epsilon} P_0^\epsilon \operatorname{div}\phi =$$
$$\int_\Sigma \sum_{j=1}^{2}\left\{B_j^\epsilon + \left(\nabla w^{j,\epsilon} - \epsilon^{-1}\pi^{j,\epsilon} - \nabla v^{j,\epsilon} + \epsilon^{-1}\omega^{j,\epsilon}\mathbf{I}\right)\left(f_j - \frac{\partial p^0}{\partial x_j}\right)\right\}e_2\phi + \sum_{i=1}^{2}\int_{\Omega_i^\epsilon}\phi\sum_{j=1}^{2}A_\epsilon^j \tag{4.8}$$

where

$$A_\epsilon^j = w^{j,\epsilon}\Delta\left(f_j - \frac{\partial p^0}{\partial x_j}\right) + \left(2\nabla w^{j,\epsilon} - \frac{1}{\epsilon}\pi^{j,\epsilon}\mathbf{I}\right)\cdot\nabla\left(f_j - \frac{\partial p^0}{\partial x_j}\right) \quad \text{in } \Omega_1^\epsilon \tag{4.9}$$

and

$$B_j^\epsilon = (w^{j,\epsilon} - v^{j,\epsilon})\otimes\nabla\left(f_j - \frac{\partial p^0}{\partial x_j}\right) \quad \text{on } \Sigma. \tag{4.10}$$

Following Jäger and Mikelić [3] we obtain

$$\left|\sum_{i=1}^{2}\int_{\Omega_i^\epsilon}\phi\sum_{j=1}^{2}A_\epsilon^j\right| \le C\|\nabla\phi\|_{L^2(\Omega^\epsilon)^4}\cdot\left\{\|\nabla(f-\nabla p^0)\|_{L^2(\Omega)^4} + \epsilon\|\Delta(f-\nabla p^0)\|_{L^2(\Omega)^2}\right\} \tag{4.11}$$

and

$$\left|\int_\Sigma\left(\sum_{j=1}^{2}B_j^\epsilon\right)e_2\phi\right| \le C\sqrt{\epsilon}\,\|\nabla\phi\|_{L^2(\Omega^\epsilon)^4}\cdot\|\nabla(f-\nabla p^0)\|_{L^2(\Sigma)^4}. \tag{4.12}$$

This proves the estimate (4.6).

□

Now we would like to test (4.6) with $\phi = U_0^\epsilon$ but $U_0^\epsilon \notin H^1(\Omega^\epsilon)^2$ since it has a jump on Σ.

We proceed as in Jäger and Mikelić [3] and introduce the corresponding boundary layers.

Let $\{w^{j,bl}, \pi^{j,bl}\}$ be given by (2.14)-(2.19). Then we set

$$w^{j,bl,\epsilon}(x) = w^{j,bl}\left(\frac{x}{\epsilon}\right), \quad x \in \Omega^\epsilon \tag{4.13}$$

$$\pi^{j,bl,\epsilon}(x) = \pi^{j,bl}\left(\frac{x}{\epsilon}\right), \quad x \in \Omega^\epsilon \tag{4.14}$$

and obtain the second correction:

Proposition 4.2. *Let $\phi \in V_{per}(\Omega^\epsilon)$ and let*

$$U_1^\epsilon = \frac{u^\epsilon}{\epsilon^2} - u^{0,\epsilon} - \sum_{j=1}^{2} w^{j,bl,\epsilon}(x)\left(f_j - \frac{\partial p^0}{\partial x_j}\right) \tag{4.15}$$

$$P_1^\epsilon = \epsilon^{-2}p^\epsilon - \epsilon^{-2}p^0 - \epsilon^{-1}p^{0,\epsilon} - \sum_{j=1}^{2} \pi^{j,bl,\epsilon}(x)\left(f_j - \frac{\partial p^0}{\partial x_j}\right). \tag{4.16}$$

Then we have

$$\left|\int_{\Omega_1^\epsilon \cup \Omega_2^\epsilon} (\nabla U_1^\epsilon \nabla \phi - P_1^\epsilon \operatorname{div} \phi)\right| \le$$
$$C\left\{\|\nabla(f - \nabla p^0)\|_{L^2(\Omega)^4} + \sqrt{\epsilon}\,\|\Delta(f - \nabla p^0)\|_{L^2(\Omega)^2}\right\} \cdot \|\nabla\phi\|_{L^2(\Omega)^4}. \tag{4.17}$$

Proof: The variational equation for $\{U_1^\epsilon, P_1^\epsilon\}$ reads

$$\int_{\Omega_1^\epsilon \cup \Omega_2^\epsilon} (\nabla U_1^\epsilon \nabla \phi - P_1^\epsilon \operatorname{div} \phi) =$$
$$\int_\Sigma \sum_{j=1}^{2} (K_{2j}^w - K_{2j}^v) e_2 \otimes \nabla\left(f_j - \frac{\partial p^0}{\partial x_j}\right) e_2 \phi + \sum_{i=1}^{2} \int_{\Omega_i^\epsilon} \phi \sum_{j=1}^{2} (A_\epsilon^j - A_\epsilon^{j,bl}) \tag{4.18}$$

where

$$A_\epsilon^{j,bl} = \left(2\nabla w^{j,bl,\epsilon} - \frac{1}{\epsilon}\pi^{j,bl,\epsilon}\mathbf{I}\right) \nabla\left(f_j - \frac{\partial p^0}{\partial x_j}\right) + w^{j,bl,\epsilon}\Delta\left(f_j - \frac{\partial p^0}{\partial x_j}\right).$$

Now the estimate (4.17) follows from the properties of the functions $w^{j,bl,\epsilon}$ and $\pi^{j,bl,\epsilon}$. □

We see easily that $[U_1^\epsilon]_\Sigma = 0$, but

$$\operatorname{div} U_1^\epsilon = -\sum_{k=1}^{2} (w^{k,\epsilon}H(x_2) + v^{k,\epsilon}H(-x_2) + w^{k,bl,\epsilon})\nabla\left(f_k - \frac{\partial p^0}{\partial x_k}\right). \tag{4.19}$$

Now we use the results from section 1.2.18 in Jäger and Mikelić [3] on the *a priori* estimate for the pressure through the velocity. We have

Lemma 4.3. (Jäger and Mikelić [3]) *There is an extension $\tilde{P}_1^\epsilon$ of the pressure correction P_1^ϵ such that*

$$\left\|\frac{\tilde{P}_1^\epsilon}{1 + |x_2|}\right\|_{L^2(\Omega)} \le \frac{C}{\epsilon}\left\{\|\nabla U_1^\epsilon\|_{L^2(\Omega^\epsilon)^4} + \|\nabla(f - \nabla p^0)\|_{L^2(\Omega)^4} + \sqrt{\epsilon}\,\|\Delta(f - \nabla p^0)\|_{L^2(\Omega)^2}\right\}. \tag{4.20}$$

It implies

$$\left|\int_{\Omega_1^\epsilon\cup\Omega_2^\epsilon} P_1^\epsilon \text{ div } \phi\right| \le \frac{C}{\epsilon}\|(1+|x_2|) \text{ div } \phi\|_{L^2(\Omega^\epsilon)}\cdot$$
$$\left\{\|\nabla U_1^\epsilon\|_{L^2(\Omega^\epsilon)^4} + \|\nabla(f-\nabla p^0)\|_{L^2(\Omega)^4} + \sqrt{\epsilon}\,\|\Delta(f-\nabla p^0)\|_{L^2(\Omega)^2}\right\}. \quad (4.21)$$

If we test (4.21) wih $\phi = U_1^\epsilon$, then we see that $\|(1+|x_2|) \text{ div } U_1^\epsilon\|_{L^2(\Omega)} \sim O(1)$. Therefore, we have no choice but to correct div U_1^ϵ as well.

Let $\gamma^{j,i}, \delta^{j,i}, \gamma^{j,i,bl}$ and $\pi^{j,i,bl}$ be given by (2.12),(2.13) and (2.23)-(2.28), respectively. Then we set

$$\gamma^{j,i,\epsilon}(x) = \epsilon\gamma^{j,i}\left(\frac{x}{\epsilon}\right), \quad x\in\Omega_1^\epsilon\cup\Sigma \quad (4.22)$$

$$\delta^{j,i,\epsilon}(x) = \epsilon\delta^{j,i}\left(\frac{x}{\epsilon}\right), \quad x\in\Omega_2^\epsilon\cup\Sigma \quad (4.23)$$

$$\gamma^{j,i,bl,\epsilon}(x) = \epsilon\gamma^{j,i,bl}\left(\frac{x}{\epsilon}\right), \quad x\in\Omega^\epsilon \quad (4.24)$$

$$\pi^{j,i,bl,\epsilon}(x) = \pi^{j,i,bl}\left(\frac{x}{\epsilon}\right), \quad x\in\Omega^\epsilon. \quad (4.25)$$

Now we are in situation to establish the third correction:

Proposition 4.4. *Let $\phi\in V_{per}(\Omega^\epsilon)$ and let*

$$U_2^\epsilon = U_1^\epsilon + \sum_{i,j=1}^{2} H(x_2)\gamma^{j,i,\epsilon}\frac{\partial}{\partial x_i}\left(f_j - \frac{\partial p^0}{\partial x_j}\right)$$
$$+\sum_{i,j=1}^{2}(H(-x_2)\delta^{j,i,\epsilon} + \gamma^{j,i,bl,\epsilon})\frac{\partial}{\partial x_i}\left(f_j - \frac{\partial p^0}{\partial x_j}\right) \quad (4.26)$$

$$P_2^\epsilon = P_1^\epsilon + \sum_{i,j=1}^{2}\pi^{j,i,bl,\epsilon}\frac{\partial}{\partial x_i}\left(f_j - \frac{\partial p^0}{\partial x_j}\right). \quad (4.27)$$

Then $U_2^\epsilon\in V_{per}(\Omega)^\epsilon)$ and

$$\|(1+|x_2|) \text{ div } U_2^\epsilon\|_{L^2(\Omega)} \le$$
$$C\sqrt{\epsilon}\left\{\|\nabla(f-\nabla p^0)\|_{L^2(\Omega)^4} + \sqrt{\epsilon}\,\|\nabla\nabla(f-\nabla p^0)\|_{L^2(\Omega)^8}\right\} \quad (4.28)$$

$$\left|\int_{\Omega_1^\epsilon\cup\Omega_2^\epsilon}(\nabla U_2^\epsilon\nabla\phi - P_2^\epsilon \text{ div } \phi)\right| \le$$
$$C\Big\{\|\nabla(f-\nabla p^0)\|_{L^2(\Omega)^4} + \|\nabla\nabla(f-\nabla p^0)\|_{L^2(\Omega)^8}$$
$$+\epsilon^{5/2}\|\Delta\nabla(f-\nabla p^0)\|_{L^2(\Omega)^4}\Big\}\cdot\|\nabla\phi\|_{L^2(\Omega)^4}. \quad (4.29)$$

Proof: It is along the same lines as the proof of Proposition 4.2 and of the analogous results from Jäger and Mikelić [3] and we omit it.

□

Now we are able to complete the proof of Theorem 3.1.

Proof of Theorem 3.1

We test (4.29) with $\phi = U_1^\epsilon$. Then

$$\int_{\Omega_1^\epsilon \cup \Omega_2^\epsilon} |\nabla U_2^\epsilon|^2 dx \le C \|\nabla U_2^\epsilon\|_{L^2(\Omega)^4} \cdot \|\nabla (f - \nabla p^0)\|_{H^2(\Omega)^4}$$
$$+ \|(1+|x_2|)^{-1} \tilde{P}_2^\epsilon\|_{L^2(\Omega)} \cdot \|(1+|x_2|) \text{ div } U_2^\epsilon\|_{L^2(\Omega)}.$$

Now we use the estimates (4.20) and (4.28) and obtain

$$\|\nabla U_2^\epsilon\|^2_{L^2(\Omega)^4} \le \frac{C}{\sqrt{\epsilon}} \|\nabla U_2^\epsilon\|_{L^2(\Omega)^4} \|\nabla (f - \nabla p^0)\|_{H^2(\Omega)^4} + C\sqrt{\epsilon}\, \|\nabla (f - \nabla p^0)\|_{H^2(\Omega)^4}. \tag{4.30}$$

(4.30) implies (3.3), (3.4) and (3.5).

□

5. Proof of Theorem 3.2

We note that it is possible to approximate f by f^δ, satisfying (3.1) such that

$$\frac{1}{\sqrt{\delta}} \|f - f^\delta\|_{L^2(\Omega)^2} + \sqrt{\delta}\, \|\nabla (f - f^\delta)\|_{L^2(\Omega)^4} +$$
$$\delta\sqrt{\delta}\, \|\nabla\nabla (f - f^\delta)\|_{L^2(\Omega)^{16}} + \delta^{5/2} \|\Delta \nabla (f - f^\delta)\|_{L^2(\Omega)^4} \le C. \tag{5.1}$$

Then $u^{\delta\epsilon}$ and $p^{0\epsilon}$ are corresponding solutions to (1.1)-(1.5) and (2.6), respectively, with f replaced by f^δ.

It is easy to see that

$$\begin{cases} \left\| \dfrac{u^{\epsilon\delta}}{\epsilon^2} - \dfrac{u^\epsilon}{\epsilon^2} \right\|_{L^2(\Omega^\epsilon)^2} \le C \|f - f^\delta\|\|_{L^2(\Omega^\epsilon)^2} \\ \|\nabla (p^{0\delta} - p^0)\|_{L^2(\Omega)^4} \le C \|f - f^\delta\|\|_{L^2(\Omega)^2}. \end{cases} \tag{5.2}$$

Furthermore, the estimates (4.29) and (4.30) imply

$$\left\| \frac{u^{\epsilon\delta}}{\epsilon^2} - \sum_{j=1}^{2} \{H(x_2) w^{j,\epsilon} + H(-x_2) v^{j,\epsilon}\} \left(f_j^\delta - \frac{\partial p^{0\delta}}{\partial x_j} \right) \right\|_{L^2(\Omega)^2}$$
$$\le C\sqrt{\epsilon} \left\{ \|\nabla (f^\delta - \nabla p^{0\delta})\|_{H^1(\Omega)^4} + \epsilon^{5/2} \|\Delta (f^\delta - \nabla p^{0\delta})\|_{L^2(\Omega)^4} \right\}$$
$$\le C\sqrt{\epsilon} \left\{ \frac{1}{\sqrt{\delta}} + \frac{1}{\delta\sqrt{\delta}} + \frac{\epsilon^{5/2}}{\delta^{5/2}} \right\}. \tag{5.3}$$

Now (5.1), (5.2) and (5.3) imply that it is optimal to choose $\delta = \epsilon^{1/4}$. The estimates (3.6)-(3.7) immediately follow.

□

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On the Semistatic Limit for Maxwell's Equations

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Abstract. In this paper a L^p-theory for the Maxwell system in the semistatic limit case in an exterior domain is described. The problem under consideration is of mixed type, since the possibly nonlinear electric conductivity vanishes on a certain subset of the domain. The solution is obtained from a singular perturbation of the full Maxwell system including the displacement current.

Keywords: Maxwell's equations, singular perturbations, degenerate and mixed type PDE, asymptotic behavior

Classification: 35B25, 35B40, 35D05, 35Q60

1. Introduction

This paper concerns the initial-boundary value problem for Maxwell's equations [7] in an exterior domain; that is the system

$$\operatorname{curl} \mathbf{h}_\varepsilon = \varepsilon \partial_t \mathbf{E}_\varepsilon + \mathbf{S}(t, x, \mathbf{E}_\varepsilon) + \mathbf{j}_0 \tag{1.1}$$

$$\operatorname{curl} \mathbf{E}_\varepsilon = -\mu \partial_t \mathbf{h}_\varepsilon \tag{1.2}$$

supplemented by the initial-boundary conditions

$$\vec{n} \wedge \mathbf{E}_\varepsilon = 0 \text{ on } [0, \infty) \times \partial\Omega, \tag{1.3}$$

$$\mathbf{E}_\varepsilon(0) = \mathbf{E}_0 \text{ and } \mathbf{h}_\varepsilon(0) = \mathbf{h}_0. \tag{1.4}$$

Here $\Omega \subset \mathbb{R}^3$ is a domain with bounded complement, $\mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon$ denote the electric and magnetic field, respectively, which depend on the time $t \geq 0$ and the space-variable $x \in \Omega$. On the perfectly conducting boundary $\partial\Omega$ the tangential component of $\mathbf{E}$ must vanish, which is expressed by boundary condition 1.3. $\mathbf{j}_0$ is a prescribed external current, which is assumed to be divergence-free. The charge current $\mathbf{S}(t, x, \mathbf{E}_\varepsilon)$ may depend nonlinearly on the electric field, with the property that

$$\mathbf{S}(t, x, \mathbf{E}) = 0 \text{ if } x \in \Omega_0 \stackrel{\text{def}}{=} \Omega \setminus \overline{G}, \tag{1.5}$$

with some bounded set $G \subset \Omega$. The precise assumptions on $\mathbf{S}$ are given in the second section. In 1.2 $\mu \in L^\infty(\mathbb{R}^3)$ is the magnetic susceptibility, which is assumed to be uniformly positive. The dielectric susceptibility $\varepsilon > 0$ is considered as a parameter in the above system.

In many situations the displacement current $\varepsilon \partial_t \mathbf{E}_\varepsilon$ is small in comparison with the charge currents and is often neglected. Therefore, it is the aim of this paper to investigate the singular limit $\varepsilon \to 0$. By letting $\varepsilon \to 0$ one obtains formally from 1.1-1.4 the quasi-stationary Maxwell system

$$\text{curl } \mathbf{h} = \mathbf{S}(t, x, \mathbf{E}) + \mathbf{j}_0 \tag{1.6}$$

$$\text{curl } \mathbf{E} = -\mu \partial_t \mathbf{h} \tag{1.7}$$

$$\text{div } \mathbf{E} = 0 \text{ on } [0, \infty) \times \Omega_0 \tag{1.8}$$

supplemented by the initial-boundary conditions

$$\vec{n} \wedge \mathbf{E} = 0 \text{ on } [0, \infty) \times \partial\Omega \tag{1.9}$$

$$\text{and } \mathbf{h}(0) = \mathbf{h}_0 \text{ on } \Omega. \tag{1.10}$$

This problem has been investigated in [8] in the case that the domain Ω is bounded and $\mathbf{S}$ is linear with respect to $\mathbf{E}$ and uniformly positive. See also [3] and [4], where a temperature-dependent electrical conductivity is considered. However, in this paper Ω is an exterior domain and the electrical conductivity vanishes on a subset $\Omega_0 = \Omega \setminus \overline{G}$. Note that $(\mathbf{E}, \mathbf{h})$ is not uniquely determined by 1.6-1.10, since $(\mathbf{E} + \mathbf{F}, \mathbf{h})$ also solves 1.6-1.10 provided that curl $\mathbf{F} = 0$ on Ω, $\vec{n} \wedge \mathbf{F} = 0$ on $\partial\Omega$, $\mathbf{F} = 0$ on G and div $\mathbf{F} = 0$ on Ω_0. In order to also determine the electric field $\mathbf{E}$ uniquely, the following boundary condition is imposed.

$$\int_{\partial C_k} \vec{n}\mathbf{E}(t, x) dS = Q_k \text{ for all } t \in [0, \infty), k \in \{1, .., N\}. \tag{1.11}$$

Here C_k, $k = 1, .., N$ denote the connected components of $C \stackrel{\text{def}}{=} I\!R^3 \setminus \overline{\Omega_0}$, the complement of the "vacuum-region" Ω_0. The physical meaning of 1.11 is that the total electric charge Q_k on each C_k is prescribed. (In the case that C is not connected it is not sufficient to prescribe only $\int_{\partial C} \vec{n}\mathbf{E} dS$ in order to achieve uniqueness.)
The initial data must obey the compatibility conditions

$$\text{div } \mathbf{E}_0 = 0, \quad \text{curl } \mathbf{h}_0 = \mathbf{j}_0(0) \text{ on } \Omega_0 = \Omega \setminus G \tag{1.12}$$

$$\text{and } \int_{\partial C_k} \vec{n}\mathbf{E}_0(x) dS = Q_k \text{ for all } k \in \{1, .., N\}. \tag{1.13}$$

Note that 1.6-1.11 can be considered as a problem of mixed type. It is elliptic on Ω_0 (with respect to the space variable x for fixed time) and parabolic on $(0, \infty) \times G$. It is shown in this paper that 1.6-1.11 admits for given $\mathbf{h}_0$, $\mathbf{j}_0$ and Q_k a unique global weak solution $(\mathbf{E}, \mathbf{h})$. Moreover, it is shown that $(\mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon) \xrightarrow{\varepsilon \to 0} (\mathbf{E}, \mathbf{h})$ in a suitable weak topology.
Finally, the asymptotic behavior of the solution to 1.6-1.11 for $t \to \infty$ is investigated in the case $\mathbf{j}_0 = 0$.
It is shown that $||\mathbf{h}(t)||_{L^2}$ decays exponentially provided that

$$\int_\Omega \mu \mathbf{h}_0 \mathbf{g} dx = 0 \text{ for all } \mathbf{g} \in L^2(\Omega) \text{ with curl } \mathbf{g} = 0.$$

This condition includes div $(\mu \mathbf{h}_0) = 0$ on Ω and $\vec{n}\mathbf{h} = 0$ on $\partial\Omega$ weakly. The aim of this article is to summarize the results in [6] without presenting detailed proofs.

2. Notation, definitions and assumptions

For an arbitrary open set $K \subset \mathbb{R}^3$ the space of all infinitely differentiable functions with compact support contained in K is denoted by $C_0^\infty(K)$. Moreover, $\mathcal{D}'(K)$ is the space of distributions on K.
For $p \in [1,\infty)$ we define $H^p_{curl}(K)$ as the space of all $\mathbf{E} \in L^p(K)$ with curl $\mathbf{E} \in L^p(K)$. $\overset{0}{H^p}_{curl}(K)$ denotes the set of all $\mathbf{E} \in H^p_{curl}(K)$, such that

$$\int_K \mathbf{E} \text{ curl } \mathbf{h} - \mathbf{h} \text{ curl } \mathbf{E} dx = 0 \text{ for all } \mathbf{h} \in H^{p^*}_{curl}(K),$$

which includes a weak formulation of the boundary condition $\vec{n} \wedge \mathbf{E} = 0$ on ∂K. Here $\frac{1}{p} + \frac{1}{p^*} = 1$. $W^{1,p}(K)$ is the usual first-order Sobolev space consisting of all functions in $L^p(K)$ whose distributional gradient belongs to $L^p(K)$. $\overset{0}{W}{}^{1,p}(K)$ denotes the closure of $C_0^\infty(K)$ in $W^{1,p}(K)$.
In the sequel $\Omega \subset \mathbb{R}^3$ is an open set with bounded complement, $G \subset \Omega$ and $\Omega_0 \overset{\text{def}}{=} \Omega \backslash \overline{G}$. It is assumed that $\mathcal{C} \overset{\text{def}}{=} \mathbb{R}^3 \setminus \overline{\Omega_0} = \mathcal{C}_1 \cup .. \cup \mathcal{C}_N$ is a bounded Lipschitz-domain, where $\mathcal{C}_k$ are its connected components with $\overline{\mathcal{C}}_j \cap \overline{\mathcal{C}}_k = \emptyset$ if $j \neq k$.
$\mu \in L^\infty(\mathbb{R}^3)$ is an uniformly positive function.
Let $R_0 > 0$, such that $\overline{\mathcal{C}} \subset B_{R_0} \overset{\text{def}}{=} \{|x| < R_0\}$.
Next, let $p_0 \in (2,6)$, such that for all $p \in [2,p_0]$ the following holds:

$$\varphi \in \overset{0}{W}{}^{1,2}(\Omega_0 \cap B_{2R_0}) \text{ and}$$

$$\Delta\varphi \in W^{-1,p}(\Omega_0 \cap B_{2R_0}) \overset{\text{def}}{=} \left(\overset{0}{W}{}^{1,p^*}(\Omega_0 \cap B_{2R_0})\right)^* \tag{2.14}$$

$$\Longrightarrow \nabla\varphi \in L^p(\Omega_0 \cap B_{2R_0}).$$

Since $\Omega_0 \cap B_{2R_0} = B_{2R_0} \setminus \overline{\mathcal{C}}$ is a Lipschitz-domain, it follows from the result in [2] that there exist such $p_0 > 2$.
The following assumptions are imposed on $\mathbf{S} : [0,\infty) \times \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$.

$$\mathbf{S}(t,x,\mathbf{y}) = 0 \text{ if } x \in \Omega_0 = \Omega \setminus \overline{G}, \tag{2.15}$$

$$\mathbf{S}(\cdot,\cdot,\mathbf{y}) \in L^\infty_{loc}([0,\infty), L^\infty(\Omega)) \text{ for all fixed } \mathbf{y} \in \mathbb{R}^3. \tag{2.16}$$

Next, $\mathbf{S}$ is assumed to be Lipschitz-continuous, i.e. there exists $L \in (0,\infty)$, such that

$$|\mathbf{S}(t,x,\mathbf{y}) - \mathbf{S}(t,x,\tilde{\mathbf{y}})| \leq L|\mathbf{y} - \tilde{\mathbf{y}}| \text{ for all } t \geq 0, \mathbf{y}, \tilde{\mathbf{y}} \in \mathbb{R}^3 \text{ and } x \in \Omega. \tag{2.17}$$

Moreover,

$$|\mathbf{S}(t,x,\mathbf{y})| \leq C_0\gamma(x)|\mathbf{y}| \text{ and } \mathbf{y}\mathbf{S}(t,x,\mathbf{y}) \geq \gamma(x)|\mathbf{y}|^2 \tag{2.18}$$

for all $\mathbf{y} \in \mathbb{R}^3, t \geq 0$ and $x \in G$ with some $\gamma \in L^\infty(G)$, $\gamma > 0$ and $C_0 \in (0,\infty)$.
Finally, $\mathbf{S}$ is assumed to be monotone, i.e.

$$(\mathbf{y} - \tilde{\mathbf{y}})(\mathbf{S}(t,x,\mathbf{y}) - \mathbf{S}(t,x,\tilde{\mathbf{y}})) > 0 \text{ for all } \mathbf{y} \neq \tilde{\mathbf{y}} \in \mathbb{R}^3, t \geq 0 \text{ and } x \in G. \tag{2.19}$$

The following compatibility conditions will be imposed on p_0 and γ.

$$\gamma^{-1} \in L^{r_0/(2-r_0)}(G) \text{ with } r_0 \stackrel{\text{def}}{=} p_0^*. \tag{2.20}$$

All these assumptions are fulfilled in the frequently occurring linear case

$$\mathbf{S}(t,x,\mathbf{y}) = \sigma(t,x)\mathbf{y} \text{ for all } \mathbf{y} \in \mathbb{R}^3, t \geq 0 \text{ and } x \in G \tag{2.21}$$

with some positive $\sigma \in L^{\infty}_{loc}([0,\infty), L^{\infty}(G))$ satisfying $\sigma(t,x) \geq \gamma(x)$ with γ as above. Let $\mathcal{G}$ and $\mathcal{G}^*$ be the weighted L^2-space consisting of all functions $\mathbf{f} : G \to \mathbb{C}^6$ and $\mathbf{F} : G \to \mathbb{C}^6$ with

$$||\mathbf{f}||^2_{\mathcal{G}} \stackrel{\text{def}}{=} \int_G |\mathbf{f}|^2 \gamma dx < \infty \text{ and } ||\mathbf{F}||^2_{\mathcal{G}^*} \stackrel{\text{def}}{=} \int_G |\mathbf{F}|^2 \gamma^{-1} dx < \infty, \text{ respectively.}$$

It follows from 2.18 and Hölder's inequality that

$$\mathcal{G} \subset L^{r_0}(G) \tag{2.22}$$

and for $\mathbf{f} \in \mathcal{G}$ and $t \geq 0$ one has

$$\mathbf{S}(t,x,\mathbf{f}) \in \mathcal{G}^* \subset L^2(G) \tag{2.23}$$

with $||\mathbf{S}(t,x,\mathbf{f})||_{\mathcal{G}^*} = ||\mathbf{S}(t,x,\mathbf{f})\gamma^{-1/2}||_{L^2(G)} \leq K||\mathbf{f}||_{\mathcal{G}}$ with some $K \in (0,\infty)$ independent of $t, \mathbf{f}$.

Next, some function spaces are introduced.
Let $Z \stackrel{\text{def}}{=} \{\varphi \in L^6 | \nabla\varphi \in L^2(\mathbb{R}^3)\}$. It is a Hilbert-space endowed with the scalar product $< \varphi, \psi >_Z \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \varepsilon \nabla\varphi \nabla\overline{\psi} dx$.
Let Z_0 be the closed subspace consisting of all $\varphi \in Z$ with $\nabla\varphi = 0$ on $\mathcal{C}$, i.e. which are constant on each component $\mathcal{C}_k$.
For $\mathbf{E} \in L^1_{loc}(\overline{\Omega})$ with $\operatorname{div} \mathbf{E} = 0$ on Ω_0 define

$$q_j(\mathbf{E}) \stackrel{\text{def}}{=} \int_{\Omega} \mathbf{E}\nabla\chi_j dx = \int_{\Omega_0} \mathbf{E}\nabla\chi_j dx, \tag{2.24}$$

where $\chi_j \in C_0^{\infty}(\mathbb{R}^3)$ obey $\chi_j|_{\mathcal{C}_k} = \delta_{j,k}$. Due to the condition $\operatorname{div} \mathbf{E} = 0$ this defintion is independent of the choice of the functions χ_j.
Now, the following function spaces are defined.
Let for $q \in [1,2]$ Y_0^q be the space of all measurable functions $\mathbf{E} : \Omega \to \mathbb{C}^6$, such that $\mathbf{E} \in L^q(\Omega \cap B_{R_0})$ and $\mathbf{E} \in L^6(\mathbb{R}^3 \setminus B_{R_0})$.
Recall that $\overline{G} \subset B_{R_0}$ and $\mathbb{R}^3 \setminus \Omega \subset B_{R_0}$.
Y_1^q is defined as the space of all $\mathbf{E} \in Y_0^q$ with the property that
$\operatorname{div} \mathbf{E} = 0$ on Ω_0, $\operatorname{curl} \mathbf{E} \in L^2(\Omega)$ and $\varphi\mathbf{E} \in \overset{0}{H}{}^q_{curl}(\Omega)$ for all $\varphi \in C_0^{\infty}(\mathbb{R}^3)$.
The latter condition means $\vec{n} \wedge \mathbf{E} = 0$ on $\partial\Omega$.
The following norms are introduced:

$$||\mathbf{E}||_{Y_0^q} \stackrel{\text{def}}{=} ||\mathbf{E}||_{L^6(\mathbb{R}^3 \setminus B_{R_0})} + ||\mathbf{E}||_{L^q(\Omega \cap B_{R_0})},$$

$$|\mathbf{E}|_{Y_1^q} \stackrel{\text{def}}{=} ||\operatorname{curl} \mathbf{E}||_{L^2(\Omega)} + ||\mathbf{E}||_{L^q(G)} + \sum_{k=1}^{N} |q_k(\mathbf{E})|$$

$$\text{and } ||\mathbf{E}||_{Y_1^q} \stackrel{\text{def}}{=} |\mathbf{E}|_{Y_1^q} + ||\mathbf{E}||_{Y_0^q}.$$

Next, let $\tilde{\mathcal{X}}$ be the space of all $\mathbf{h} \in H^2_{curl}(\Omega)$ with curl $\mathbf{h} \in \mathcal{G}^*$ and curl $\mathbf{h} = 0$ on Ω_0. Moreover, let $\mathcal{X}$ be the space of all $\mathbf{E} \in Y_1^{r_0} \cap \mathcal{G}$, such that

$$\int_\Omega \mathbf{h} \text{ curl } \mathbf{E} dx = \int_G \mathbf{E} \text{ curl } \mathbf{h} \text{ for all } \mathbf{h} \in \tilde{\mathcal{X}}. \tag{2.25}$$

Let $\mathcal{W}$ be the space of all $T \in \mathcal{D}'((0,\infty) \times \Omega) \cap L^2_{loc}([0,\infty), \mathcal{G})$, such that $T = \partial_t \mathbf{A}$ with some $\mathbf{A} \in L^\infty_{loc}([0,\infty), \mathcal{X}) \cap W^{1,2}_{loc}([0,\infty), \mathcal{G})$. For $\mathbf{A} \in L^\infty_{loc}([0,\infty), \mathcal{X}) \cap W^{1,2}_{loc}([0,\infty), \mathcal{G})$ and $T \stackrel{\text{def}}{=} \partial_t \mathbf{A} \in \mathcal{W}$ we define $\tilde{q}_k(T) \in \mathcal{D}'((0,\infty))$ by $\tilde{q}_k(T) \stackrel{\text{def}}{=} \frac{d}{dt} q_k(\mathbf{A}(t))$.
Next, the notion of weak solutions to 1.6 - 1.11 is given precisely.

Definition 1. *Let $\mathbf{j}_0 \in L^2_{loc}([0,\infty), L^2(\Omega))$, $\mathbf{h}_0 \in L^2(\Omega)$ and $Q_k \in \mathbb{R}$. Then $(\mathbf{E}, \mathbf{h}) \in \mathcal{W} \times C([0,\infty), L^2(\Omega))$ is called a solution to 1.6 - 1.11, if*

$$\textit{curl}\, \mathbf{h} = \mathbf{S}(t,x,\mathbf{E}) + \mathbf{j}_0, \qquad \textit{curl}\, \mathbf{E} = -\mu \partial_t \mathbf{h}, \tag{2.26}$$

$$\tilde{q}_k(\mathbf{E}) = Q_k \textit{ for all } k \in \{1,..,N\}, \tag{2.27}$$

$$\textit{and} \quad \mathbf{h}(0) = \mathbf{h}_0. \tag{2.28}$$

Note that by the definition of $\mathcal{W}$ it follows $\mathbf{E}|_{(0,\infty)\times G)} \in L^2_{loc}([0,\infty), \mathcal{G})$. Therefore $\mathbf{S}(\mathbf{E}) \in L^2_{loc}([0,\infty), L^2(\Omega) \cap \mathcal{G}^*)$.

3. A basic estimate

The following estimate will be used frequently.

Theorem 1. *For $r \in [r_0, 2]$, i.e. $r^* \le p_0$, there exists a constant $K_r \in \mathbb{R}^+$, such that for all $\mathbf{E} \in Y_1^r$ the estimate*

$$||\mathbf{E}||_{Y_0^r} = ||\mathbf{E}||_{L^6(\mathbb{R}^3 \setminus B_{R_0})} + ||\mathbf{E}||_{L^r(\Omega \cap B_{R_0})}$$

$$\le K_r |\mathbf{E}|_{Y_1^r} = K_r \left(|| \textit{ curl } \mathbf{E}||_{L^2(\Omega)} + ||\mathbf{E}||_{L^r(G)} + \sum_{k=1}^N |q_k(\mathbf{E})| \right)$$

holds. In particular $|\cdot|_{Y_1^r}$ and $||\cdot||_{Y_1^r}$ are equivalent norms on Y_1^r.

By 2.22 and the previous theorem the space $\mathcal{X}$ is complete with respect to the norm

$$||\mathbf{E}||_{\mathcal{X}} \stackrel{\text{def}}{=} || \text{ curl } \mathbf{E}||_{L^2(\Omega)} + ||\mathbf{E}|_G||_{\mathcal{G}} + \sum_{k=1}^N |q_k(\mathbf{E})| \text{ for } \mathbf{E} \in \mathcal{X}.$$

The estimate

$$||\mathbf{E}||_{L^6(\{|x|>R_0\})} + ||\mathbf{E}||_{L^{r_0}(\Omega \cap B_{R_0})} \le K||\mathbf{E}||_{\mathcal{X}} \text{ for all } \mathbf{E} \in \mathcal{X} \tag{3.29}$$

holds.

The proof of this estimate involves the following extension theorem for the spaces $H^p_{curl}(\Omega)$:

Lemma 1. *Let $\Omega_1 \subset \mathbb{R}^3$ be a bounded Lipschitz-domain. Then there exists a bounded operator $T : L^1(\Omega_1, \mathbb{C}^6) \to L^1(\mathbb{R}^3, \mathbb{C}^6)$ with the following properties.*
i) $(T\mathbf{w})(x) = \mathbf{w}(x)$ *for all* $\mathbf{w} \in L^1(\Omega_1), x \in \Omega_1$.
ii) Let $p \in [1, \infty)$. *Then* $T\mathbf{w} \in L^p(\mathbb{R}^3, \mathbb{C}^6)$ *and* $||T\mathbf{w}||_{L^p(\mathbb{R}^3)} \leq C_p||\mathbf{w}||_{L^p(\Omega_1)}$ *for all* $\mathbf{w} \in L^p(\Omega_1)$.
iii) Let $p \in [1, \infty)$. *Then* $T\mathbf{w} \in H^p_{curl}(\mathbb{R}^3)$ *and* $||T\mathbf{w}||_{H^p_{curl}(\mathbb{R}^3)} \leq C_p||\mathbf{w}||_{H^p_{curl}(\Omega_1)}$ *for all* $\mathbf{w} \in H^p_{curl}(\Omega_1)$.

Also the following $W^{1,p}$-regularity theorem for solutions of the elliptic boundary-value problem

$$\Delta\psi_0 = \text{ div } \mathbf{F} \text{ on } \Omega_0, \quad \psi_0 \text{ constant on } \partial C_k$$
$$\text{and } \int_{\partial C_k} \vec{n}\nabla\psi_0 dS = 0 \text{ for all } k \in \{1, .., N\}$$

is used.

Lemma 2. *Let* $p \in [2, p_0]$, $\mathbf{F} \in L^p(\Omega_0) \cap L^2(\Omega_0)$ *and* $\psi_0 \in Z_0$ *with*

$$\int_{\Omega_0} \nabla\psi_0\nabla\psi dx = \int_{\Omega_0} \mathbf{F}\nabla\psi dx \textit{ for all } \psi \in Z_0.$$

Then $\psi_0 \in W^{1,p}(B_{2R_0})$ *and*

$$||\nabla\psi_0||_{L^p(B_{2R_0})} + \sum_{k=1}^{N} |\psi_0|_{C_k}| \leq K_3(||\mathbf{F}||_{L^p(\Omega_0)} + ||\mathbf{F}||_{L^2(\Omega_0)})$$

with some $K_3 \in (0, \infty)$ *independent of* $\mathbf{F}$.

This follows essentially from 2.14. Finally, the proof of theorem 1 uses the following lemma, which is related to Poincare's lemma.

Lemma 3. *i) Let* $\mathbf{E} \in L^2(\mathbb{R}^3)$ *with div* $\mathbf{E} = 0$ *on* $\mathbb{R}^3$.
Then there exists a unique $\mathbf{h} \in L^6(\mathbb{R}^3)$ *with curl* $\mathbf{h} = \mathbf{E}$ *and div* $\mathbf{h} = 0$ *on* $\mathbb{R}^3$ *in the sense of distributions.*
ii) There exists a bounded operator $\mathcal{T} : \mathcal{U} \to L^6(\mathbb{R}^3, \mathbb{C}^6)$ *such that curl* $(\mathcal{T}\mathbf{E}) = \mathbf{E}$ *on* Ω_0 *for all* $\mathbf{E} \in \mathcal{U}$.

Here $\mathcal{U} \subset L^2(\Omega_0)$ is the space of all $\mathbf{E} \in L^2(\Omega_0)$ with $\int_{\Omega_0} \mathbf{E}\nabla\psi dx = 0$ for all $\psi \in Z_0$. This is the weak formulation of div $\mathbf{E} = 0$ on Ω_0 and $\int_{\partial C_k} \vec{n}\mathbf{E} dS = 0$ for all $k = 1, .., N$.

4. The semistatic limit for ME

In the sequel let $\mathbf{j}_0 \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$, such that there exists some $\mathbf{g}_0 \in L^2_{loc}(\mathbb{R}, H^2_{curl}(\Omega)) \cap W^{1,1}_{loc}(\mathbb{R}, L^2(\Omega))$, with

$$\mathbf{j}_0(t) = \text{ curl } \mathbf{g}_0(t), \tag{4.30}$$

in particular $\mathbf{j}_0(t)$ is assumed to be divergence-free.
Moreover let $(\mathbf{E}_0, \mathbf{h}_0) \in X_0 = L^2(\Omega)$ with

$$\mathbf{g}_0(0) - \mathbf{h}_0 \in \overline{\mathcal{X}}, \tag{4.31}$$

where the closure is taken in the $L^2(\Omega)$-topology. Recall that $\tilde{\mathcal{X}}$ denotes the space of all $\mathbf{h} \in H^2_{curl}(\Omega)$ with curl $\mathbf{h} = 0$ on Ω_0 and curl $\mathbf{h} \in \mathcal{G}^*$. By 4.30 it follows that the latter assumption implies that $\mathbf{h}_0$ obeys condition 1.12.
It is assumed that $\mathbf{E}_0$ satisfies

$$\text{div } \mathbf{E}_0 = 0 \text{ on } \Omega_0, \quad Q_k \stackrel{\text{def}}{=} q_k(\mathbf{E}_0) \tag{4.32}$$

First uniqueness for problem 2.26 - 2.28 is shown in [6].
For this purpose suppose $(\mathbf{E}^{(1)}, \mathbf{h}^{(1)})$ and $(\mathbf{E}^{(2)}, \mathbf{h}^{(2)})$ are solutions to 2.26 - 2.28 in the sense of definition 1 with $\mathbf{E}^{(k)} \in \mathcal{W}$ and $\mathbf{h}^{(k)} \in C([0,\infty), L^2(\Omega))$ and $\mathbf{E}^{(k)} = \partial_t \mathbf{A}^{(k)}$, where $\mathbf{A}^{(k)} \in L^\infty_{loc}([0,\infty), \mathcal{X}) \cap W^{1,2}_{loc}([0,\infty), \mathcal{G})$.
Let $\mathbf{A} \stackrel{\text{def}}{=} \mathbf{A}^{(1)} - \mathbf{A}^{(2)}$, $\mathbf{E} \stackrel{\text{def}}{=} \mathbf{E}^{(1)} - \mathbf{E}^{(2)}$, $\mathbf{h} \stackrel{\text{def}}{=} \mathbf{h}^{(1)} - \mathbf{h}^{(2)}$,
and $\mathbf{J} \stackrel{\text{def}}{=} [\mathbf{S}(\mathbf{E}^{(1)}) - \mathbf{S}(\mathbf{E}^{(2)})] \in L^2_{loc}([0,\infty), \mathcal{G}^*)$.
The energy balance

$$\frac{1}{2}||\mu^{1/2}\mathbf{h}(T)||^2_{L^2} = -\int_0^T \int_G \partial_t \mathbf{A}(t) \text{ curl } \mathbf{h}(t) dx dt \tag{4.33}$$

$$= -\int_0^T \int_G \mathbf{E}(t)\mathbf{J}(t) dx dt = \int_0^T \int_G (\mathbf{E}^{(2)}) - \mathbf{E}^{(1)})[\mathbf{S}(\mathbf{E}^{(1)}) - \mathbf{S}(\mathbf{E}^{(2)})] dx dt$$

for all $T > 0$ is obtained after a regularization with respect to the time variable. By the monotonicity assumption 2.19 on $\mathbf{S}$ this implies

$$\mathbf{h} = 0 \text{ and } \partial_t \mathbf{A}|_{[0,\infty)\times G} = \mathbf{E}|_{[0,\infty)\times G} = 0 \text{ on } (0,\infty) \times G. \tag{4.34}$$

Since $\mathbf{E} = \partial_t \mathbf{A}$ it follows

$$\partial_t \text{ curl } \mathbf{A} = \text{ curl } \mathbf{E} = \partial_t(\mu \mathbf{h}) = 0. \tag{4.35}$$

Moreover,

$$\frac{d}{dt} q_k(\mathbf{A}(t)) = \tilde{q}(\mathbf{E}(t)) = \tilde{q}(\mathbf{E}^{(1)}(t)) - \tilde{q}(\mathbf{E}^{(2)}(t)) = 0. \tag{4.36}$$

Since $\mathbf{A} \in L^\infty_{loc}(\mathbb{R}, Y_1^{ro})$, it follows from 4.34 - 4.36 and lemma 1 that $\mathbf{A}$ is constant with respect to t and hence also $\mathbf{E} = \partial_t \mathbf{A} = 0$. □

Now, Maxwell's equations including the displacement current

$$\varepsilon \partial_t \mathbf{E}_\varepsilon = \text{ curl } \mathbf{h}_\varepsilon - \mathbf{j}_0 - \mathbf{S}(t, x, \mathbf{E}_\varepsilon), \tag{4.37}$$

$$\mu \partial_t \mathbf{h}_\varepsilon = - \text{ curl } \mathbf{E}_\varepsilon, \tag{4.38}$$

are considered, where $\varepsilon > 0$ is a parameter. 4.37, 4.38 is supplemented by the initial-boundary conditions

$$\vec{n} \wedge \mathbf{E}_\varepsilon = 0 \text{ on } (0,\infty) \times \partial\Omega, \tag{4.39}$$

$$\mathbf{E}_\varepsilon(0,x) = \mathbf{E}_0(x), \mathbf{h}_\varepsilon(0,x) = \mathbf{h}_0(x). \tag{4.40}$$

For the weak formulation of 4.37 - 4.40 the operator B defined by

$$B(\mathbf{E}, \mathbf{h}) \stackrel{\text{def}}{=} (\text{ curl } \mathbf{h}, -\mu^{-1} \text{ curl } \mathbf{E})$$

for $(\mathbf{E},\mathbf{h}) \in D(B) \stackrel{\text{def}}{=} \overset{0}{H^2}_{curl}(\Omega) \times H^2_{curl}(\Omega)$ is introduced. It turns out that B is a densely defined skew self-adjoint operator in the Hilbert-space $X_0 \stackrel{\text{def}}{=} L^2(\Omega, \mathbb{C}^6)$ endowed with the scalar product $< (\mathbf{E},\mathbf{h}), (\mathbf{F},\mathbf{g}) >_{X_0} \stackrel{\text{def}}{=} \int_\Omega (\mathbf{E}\overline{\mathbf{F}} + \mu \mathbf{h}\overline{\mathbf{g}})dx$.
Setting $\mathbf{w}_\varepsilon \stackrel{\text{def}}{=} (\varepsilon^{1/2}\mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon - \mathbf{g}_0)$ 4.37-4.40 reads as

$$\partial_t \mathbf{w}_\varepsilon = \varepsilon^{-1/2} B\mathbf{w}_\varepsilon + \varepsilon^{-1/2}\mathcal{R}(\mathbf{w}_\varepsilon)(t) + \mathbf{f}_0 \tag{4.41}$$

$$\text{and } \mathbf{w}_\varepsilon(0) = \mathbf{w}_{\varepsilon,0} \stackrel{\text{def}}{=} (\varepsilon^{1/2}\mathbf{E}_0, \mathbf{h}_0 - \mathbf{g}_0(0)),$$

where $\mathbf{f}_0 \stackrel{\text{def}}{=} -(0, \partial_t \mathbf{g}_0)$.
Here $\mathcal{R} : L^2_{loc}([0,\infty), X_0) \to L^1_{loc}([0,\infty), X_0)$ is defined by

$$(\mathcal{R}(\mathbf{E},\mathbf{h}))(t,x) \stackrel{\text{def}}{=} -(\mathbf{S}(t,x,\mathbf{E}(t,x)), 0) \text{ for } (\mathbf{E},\mathbf{h}) \in L^2_{loc}([0,\infty), X_0).$$

The unique weak solution $\mathbf{w}_\varepsilon \in C([0,\infty), X_0)$ to 4.41 is given by the variation of constant formula

$$\mathbf{w}_\varepsilon(t) = \exp(\varepsilon^{-1/2} tB)\mathbf{w}_{\varepsilon,0} \tag{4.42}$$

$$+ \int_0^t \exp(\varepsilon^{-1/2}(t-s)B)[\varepsilon^{-1/2}\mathcal{R}(\mathbf{w}_\varepsilon)(s) + \mathbf{f}_0(s)]ds,$$

where $\exp(tB), t \in \mathbb{R}$ is the unitary group generated by B. Since $\mathcal{R}$ is Lipschitz-continuous as an operator from $L^2((0,T), L^2(\Omega))$ to $L^1((0,T), L^2(\Omega))$, it follows from a standard result that this integral equation has a unique solution
$\mathbf{w}_\varepsilon = (\varepsilon^{1/2}\mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon) \in C([0,\infty), X_0)$; see [9], chapter 6.
In particular the energy balance

$$\frac{1}{2}\frac{d}{dt}||\mathbf{w}_\varepsilon(t)||^2_{X_0} = < \varepsilon^{-1/2}\mathcal{R}(\mathbf{w}_\varepsilon)(t) + \mathbf{f}_0(t), \mathbf{w}_\varepsilon(t) >_{X_0} \tag{4.43}$$

$$= -\int_G \mathbf{E}_\varepsilon(t)\mathbf{S}(t,x,\mathbf{E}_\varepsilon(t))dx + \int_\Omega \mu(\mathbf{g}_0(t) - \mathbf{h}_\varepsilon(t))\partial_t \mathbf{g}_0(t)dx$$

holds, since $\exp(tB)$ is unitary in X_0.
The aim of the following considerations is the investigation of the limit $\varepsilon \to 0$.
Let $\mathbf{A}_\varepsilon \in C(\mathbb{R}, L^2(\Omega))$ be defined by

$$\mathbf{A}_\varepsilon(t) \stackrel{\text{def}}{=} \int_0^t \mathbf{E}_\varepsilon(s)ds \text{ for } t \geq 0. \tag{4.44}$$

By using 4.43 and theorem 1 the following lemma is obtained.

Lemma 4. *The family*

$$(\mathbf{w}_\varepsilon = (\varepsilon^{1/2}\mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon - \mathbf{g}_0))_{\varepsilon>0} \textit{ is bounded in } L^\infty((0,T), X_0), \tag{4.45}$$

$$(\mathbf{E}_\varepsilon)_{\varepsilon>0} = (\partial_t \mathbf{A}_\varepsilon)_{\varepsilon>0} \textit{ is bounded in } L^2((0,T), \mathcal{G}) \tag{4.46}$$

$$\textit{and } (\mathbf{A}_\varepsilon)_{\varepsilon>0} \textit{ is bounded in } L^\infty((0,T), \mathcal{X}) \subset L^\infty((0,T), Y_1^{ro} \cap \mathcal{G}), \tag{4.47}$$

for every $T > 0$. Moreover,

$$div\, \mathbf{E}_\varepsilon(t) = 0 \textit{ on } \Omega_0, \quad q_k(\mathbf{E}_\varepsilon(t)) = Q_k. \tag{4.48}$$

$$and\quad curl\ \mathbf{A}_\varepsilon(t) = \mu[\mathbf{h}_0 - \mathbf{h}_\varepsilon(t)] \in L^2(\Omega). \tag{4.49}$$

By the previous lemma 4 there exist $\mathbf{A} \in L^\infty_{loc}(I\!R, \mathcal{X}) \cap W^{1,2}_{loc}([0,\infty), \mathcal{G})$ and a sequence $\varepsilon_n, n \in I\!N$ with $\lim_{n\to\infty} \varepsilon_n = 0$, such that

$$\mathbf{A}_{\varepsilon_n} \xrightarrow{n\to\infty} \mathbf{A} \text{ in } L^\infty((0,T), L^6(\{|x| > R_0\})) \text{ weak } -*, \tag{4.50}$$

$$L^\infty((0,T), L^{r_0}(\Omega_0 \cap B_{R_0})) \text{ weak } -* \text{ and in } W^{1,2}((0,T), \mathcal{G}) \text{ weakly}$$

$$\text{with curl } \mathbf{A}_{\varepsilon_n} \xrightarrow{n\to\infty} \text{curl } \mathbf{A} \text{ in } L^\infty((0,T), L^2(\Omega)) \text{ weak } -*,$$

Since $\mathbf{S}(\cdot, \mathbf{E}_{\varepsilon_n})_{n\in I\!N}$ is bounded in $L^2((0,T), \mathcal{G}^*)$ by 2.18 and 4.46, it can also be assumed that

$$\mathbf{S}(\cdot, \mathbf{E}_{\varepsilon_n}) \xrightarrow{n\to\infty} \mathbf{J} \text{ in } L^2((0,T), \mathcal{G}^*) \text{ weakly.} \tag{4.51}$$

From 4.44, 4.48 and 4.50 it follows that

$$q_k(\mathbf{A}(t)) = tQ_k \text{ for all } k \in \{1, .., N\} \text{ and} \tag{4.52}$$

$$\mathbf{E}_{\varepsilon_n}|_{(0,\infty)\times G} = \partial_t \mathbf{A}_{\varepsilon_n}|_{(0,\infty)\times G} \xrightarrow{n\to\infty} \mathbf{e} \overset{\text{def}}{=} \partial_t \mathbf{A}|_{(0,\infty)\times G} \tag{4.53}$$

in $L^2((0,T), \mathcal{G})$ weakly, in particular $\mathbf{A}(t)|_G = \int_0^t \mathbf{e}(s)ds \in \mathcal{G}$.
Moreover, it follows from 4.49 and 4.50 that

$$\mathbf{h}_{\varepsilon_n} = \mathbf{h}_0 - \mu^{-1} \text{ curl } \mathbf{A}_{\varepsilon_n} \xrightarrow{n\to\infty} \mathbf{h} \overset{\text{def}}{=} \mathbf{h}_0 - \mu^{-1} \text{ curl } \mathbf{A} \tag{4.54}$$

in $L^\infty((0,T), L^2(\Omega))$ weak $-*$.
Since $||\varepsilon_n \mathbf{E}_{\varepsilon_n}||_{L^\infty((0,T),L^2(\Omega))} \xrightarrow{n\to\infty} 0$ by 4.45, it follows easily from 4.37 that

$$\text{curl } \mathbf{h} = \mathbf{J} + \mathbf{j}_0 \in L^2_{loc}([0,\infty), L^2(\Omega)), \tag{4.55}$$

in particular $\mathbf{h}(t) - \mathbf{g}_0(t) \in \tilde{\mathcal{X}}$ with

$$\text{curl } (\mathbf{h} - \mathbf{g}_0) = \mathbf{J} \in L^2_{loc}([0,\infty), \mathcal{G}^*).$$

Next, let $\mathbf{E} \overset{\text{def}}{=} \partial_t \mathbf{A} \in \mathcal{W} \subset \mathcal{D}'((0,\infty) \times \Omega)$.
It follows from 4.52 and 4.54 that $\partial_t(\mu \mathbf{h}) = -$ curl $\mathbf{E}$ and $\tilde{q}_t(\mathbf{E}) = Q_k$.

Theorem 2. $(\mathbf{E}, \mathbf{h}) \in \mathcal{W} \times C([0,\infty), L^2(\Omega))$ *is the unique solution of problem 2.26 - 2.28 in the sense of definition 1. Moreover,*

$$\mathbf{A}_\varepsilon \xrightarrow{\varepsilon\to 0} \mathbf{A} \text{ in } L^\infty((0,T), Y_1^{r_0}) \text{ weak } -* \tag{4.56}$$

and in $W^{1,2}((0,T), \mathcal{G})$ *weakly, in particular*

$$\mathbf{E}_\varepsilon \xrightarrow{\varepsilon\to 0} \mathbf{E} = \partial_t \mathbf{A} \text{ in } \mathcal{D}'((0,\infty) \times \Omega) \text{ and in } L^2((0,T), \mathcal{G}) \text{ weakly} \tag{4.57}$$

$$\text{and } \mathbf{h}_\varepsilon \xrightarrow{\varepsilon\to 0} \mathbf{h} = \mathbf{h}_0 - \mu^{-1} \text{ curl } \mathbf{A} \text{ in } L^\infty((0,T), L^2(\Omega)) \text{ weak } -*. \tag{4.58}$$

Finally,

$$\mathbf{S}(\cdot, \mathbf{E}_\varepsilon) \xrightarrow{\varepsilon\to 0} \mathbf{S}(\cdot, \mathbf{E}) \text{ in } L^{1,2}((0,T), \mathcal{G}^*) \text{ weakly} \tag{4.59}$$

For the detailed proof see [6].

5. Asymptotic behavior: Decay of the magnetic field

In this section the asymptotic behavior for $t \to \infty$ of solutions to 2.26-2.28 in the case

$$\mathbf{j}_0 = 0 \tag{5.60}$$

is investigated. By the assumptions on $\mathbf{S}$ there exists a constant $K_3 \in (0,\infty)$, such that

$$||\mathbf{S}(t,\cdot,\mathbf{E})||^2_{L^2(G)} \le K_3 \int_G \mathbf{S}(t,x,\mathbf{E}(t,x)) \cdot \mathbf{E}(t,x)dx \tag{5.61}$$

for all $\mathbf{E} \in L^2_{loc}([0,\infty),\mathcal{G})$.
In the sequel let $H_{curl,0}(\Omega)$ be the space of all $\mathbf{h} \in H^2_{curl}(\Omega)$ with curl $\mathbf{h} = 0$ on Ω. It is a closed subspace of $L^2(\Omega)$. Its orthogonal complement with respect to the scalar product $< \mathbf{h}, \mathrm{g} >_\mu \stackrel{\mathrm{def}}{=} \int_\Omega \mu \mathbf{h}\overline{\mathbf{g}}dx$ is denoted by X_h.
Let P be the orthogonal projection on $X_h \subset L^2(\Omega)$ with respect to $< \cdot,\cdot >_\mu$.
Since $\nabla\psi \in H_{curl,0}(\Omega)$ for all $\psi \in H^1(\Omega)$, it follows that each $\mathbf{h} \in X_h$ obeys

$$\text{div } (\mu\mathbf{h}) = 0 \text{ on } \Omega \text{ and } \vec{n}\mathbf{h} = 0 \text{ on } \partial\Omega \text{ weakly} \tag{5.62}$$

in the sense that $-\int_\Omega \mu\mathbf{h}\nabla\psi dx = 0$ for all $\varphi \in H^1(\Omega)$.
Next, let $(\mathbf{E},\mathbf{h}) \in \mathcal{W} \times C([0,\infty), L^2(\Omega))$ be the solution to 2.26-2.28. In order to apply a compactness theorem, it is assumed that $\mathbb{R}^3 \setminus \overline{\Omega}$ is also a Lipschitz-domain. The solution $(\mathbf{E},\mathbf{h})$ satisfies the energy balance

$$\frac{1}{2}\frac{d}{dt}||\sqrt{\mu}\mathbf{h}(t)||^2_{L^2} = -\int_G \mathbf{S}(t,x,\mathbf{E}(t,x))\mathbf{E}(t,x)dx. \tag{5.63}$$

(This is obtained from a regularization with respect to the time variable as in the proof of uniqueness.) Now it follows from 5.61 and 5.63 that there exists some $\delta > 0$ (depending only on $\mathbf{S}$), such that

$$\frac{d}{dt}\frac{1}{2}||\sqrt{\mu}\mathbf{h}(t)||^2_{L^2} \le -\delta|| \text{ curl } \mathbf{h}(t)||^2_{L^2(\Omega)}. \tag{5.64}$$

The aim of the following considerations is to estimate $||\mathbf{h}||_{L^2}$ by $||$ curl $\mathbf{h}||_{L^q}$ for $\mathbf{h} \in X_h$.

Lemma 5. *Let* $\mathbf{h} \in L^2(\mathbb{R}^3)$ *with curl* $\mathbf{h} \in L^{6/5}(\mathbb{R}^3)$ *and div* $(\mu\mathbf{h}) \in L^{6/5}(\mathbb{R}^3)$. *Then*

$$||\mathbf{h}||_{L^2(\mathbb{R}^3)} \le K(|| \textit{ curl } \mathbf{h}||_{L^{6/5}(\mathbb{R}^3)} + || \textit{ div } (\mu\mathbf{h})||_{L^{6/5}(\mathbb{R}^3)})$$

with some $K \in (0,\infty)$ *independent of* $\mathbf{h}$.

A consequence of the result in [5] is

Lemma 6. *Let* $q \in (6/5, 2]$ *and* $(\mathbf{h}_n)_{n\in\mathbb{N}}$ *be a sequence in* X_h,
such that (curl $\mathbf{h}_n)_{n\in\mathbb{N}}$ *is bounded in* $L^q(\Omega)$,
supp (curl $\mathbf{h}_n) \subset B_R$ *for some* $R \in (R_0,\infty)$ *independent of* n.
Then $(\mathbf{h}_n)_{n\in\mathbb{N}}$ *is precompact in* $L^2(\Omega)$.

With the aid of the previous lemmata the following estimate can be proved.

Theorem 3. *Let $R \in (0, \infty)$. Then there exists a constant $K_{q,R} \in (0, \infty)$, such that for all $\mathbf{h} \in X_h \cap H^2_{curl}(\Omega)$ with curl $\mathbf{h}(x) = 0$ if $|x| > R$ the estimate*

$$||\mathbf{h}||_{L^2(\Omega)} \leq K_{q,R}|| \; curl \; \mathbf{h}||_{L^q(\Omega \cap B_R)}$$

holds.

Now the asymptotic behavior of the solution $(\mathbf{E}, \mathbf{h})$ is investigated. Let $\mathbf{h}^{(1)}(t) \stackrel{\text{def}}{=} P\mathbf{h}(t) \in X_h$.

Theorem 4. *There exists some $C, d > 0$, such that*

$$||\mathbf{h}^{(1)}(t)||_{L^2(\Omega)} \leq C \exp(-dt).$$

Proof: Suppose that $\mathbf{E} = \partial_t \mathbf{A}$ with some $\mathbf{A} \in L^\infty_{loc}([0, \infty), \mathcal{X})$. Then $\mu\mathbf{h} + \text{ curl } \mathbf{A} \in L^\infty_{loc}([0, \infty), L^2(\Omega))$ satisfies

$$\partial_t \left[\mu\mathbf{h} + \text{ curl } \mathbf{A}\right] = 0 \text{ and hence}$$

$$\mathbf{h}(t) + \mu^{-1}\mathbf{A}(t) = \mathbf{h}_2 \text{ for all } t \in (0, \infty) \tag{5.65}$$

with some $\mathbf{h}_2 \in L^2(\Omega)$.
Since $H_{curl,0}(\Omega) \subset \tilde{\mathcal{X}}$, it follows from 2.25 that μ^{-1} curl $\mathbf{A}(t) \in X_h$. Hence 5.65 yields

$$\mathbf{h}(t) = \mathbf{h}^{(1)}(t) + (1 - P)\left(\mathbf{h}_2 - \mu^{-1}\mathbf{A}(t)\right) = \mathbf{h}^{(1)}(t) + (1 - P)\mathbf{h}_2$$

and therefore

$$||\sqrt{\mu}\mathbf{h}(t)||^2_{L^2} = ||\sqrt{\mu}\mathbf{h}^{(1)}(t)||^2_{L^2} + ||\sqrt{\mu}(1 - P)\mathbf{h}_2||^2_{L^2} \tag{5.66}$$

Since curl $\mathbf{h}^{(1)}(t) =$ curl $\mathbf{h}(t)$, it follows from 5.64, 5.66 and lemma 3 that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||\sqrt{\mu}\mathbf{h}^{(1)}(t)||^2_{L^2} &= \frac{1}{2}\frac{d}{dt}||\sqrt{\mu}\mathbf{h}(t)||^2_{L^2} \\ \leq -\delta|| \text{ curl } \mathbf{h}(t)||^2_{L^2(G)} &= -\delta|| \text{ curl } \mathbf{h}^{(1)}(t)||^2_{L^2(G)} \\ &\leq -K^{-2}_{2,R_0}\delta||\mathbf{h}^{(1)}(t)||^2_{L^2(G)}. \end{aligned}$$

This completes the proof. □

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Application of Relaxation Schemes and Method of Characteristics to Degenerate Convection-Diffusion Problems[1]

J. Kačur

Abstract. A new approximation scheme is proposed to solve a nonlinear degenerate convection-diffusion problem. The method is based on a relaxation scheme developed for degenerate parabolic problems in [JK$_2$], [K$_2$] and on the method of characteristics [P], [DR].

Keywords: degenerate diffusion, convection-diffusion, relaxation methods, method of characteristics

Introduction

In this paper we propose a new approximation scheme to solve degenerate nonlinear convection-diffusion problems of the form

$$\partial_t b(u) + div(\bar{F}(t,x,u) - k(t,x,u)\nabla u) = f(t,x,u) \tag{1}$$

in $(0,T) \times \Omega, \Omega \subset R^N$ is a bounded domain with a Lipschitz continuous boundary $\partial\Omega$, $T < \infty$. We consider the mixed boundary condition

$$u = 0 \quad \text{on} \quad I \times \Gamma_1, \qquad -k(t,x,u)\nabla u, \nu = g(t,x,u) \quad \text{on} \quad I \times \Gamma_2 \tag{2}$$

where $I = (0,T)$, $\Gamma_1, \Gamma_2 \subset \partial\Omega, \Gamma_1 \cap \Gamma_2 = \emptyset$ and $mes_{N-1}\Gamma_1 + mes_{N-1}\Gamma_2 = mes_{N-1}\partial\Omega$ (Γ_1, Γ_2 are open in $\partial\Omega$). Together with (1),(2) we consider the initial condition

$$b(u(x,0)) = b(u_0(x)) \qquad \text{in} \quad \Omega. \tag{3}$$

We assume that $b(s)$ is strictly increasing in s, $\bar{F}(t,x,s)$ is Lipschitz continuous in x, s and f, g are sublinear in u. This model includes a large variety of engineering problems.

[1]This work was partially supported by Slovak Grant VEGA and by Grant Agency of Czech Republic.

Our method of approximation (1)-(3) is based on the relaxation method developed in [JK$_1$], [JK$_2$], [K$_1$], [K$_2$] and on the method of characteristics; see [P], [DR], etc. The degeneracies $b'(s) = 0$ resp. $b'(s) = \infty$ in some points $s \in R$ are included. The degeneracy $b'(0) = \infty$ corresponds to porous media-type equation with convection. Thus the convective term generated by $\bar{F}(t,x,s)$ can be dominated and the corresponding velocity filed is dependent on the solution u. We shall assume that $\bar{F}$ is Lipschitz continuous in s. Our method can also be used to the systems of type (1) which are coupled via the terms $\bar{F}, k, f$. A slight modification of our method can also be used for Navier-Stokes system and the convergence is proved only under the assumption that $u \in L_\infty(\Omega \times I)$ without any additional regularity assumptions.

In [JK$_2$], [K$_2$] we have introduced relaxation schemes for (1) without convective phenomenon (i.e. $\bar{F} \equiv 0$). Using standard notation $\tau = \frac{T}{n}(n \in N)$, $u_i \approx u(t_i, x), t_i = i\tau (i = 1, \ldots, n)$ the relaxation scheme reads

$$\begin{gathered}\lambda_i(u_i - u_{i-1}) - \tau div(k_i \nabla u_i) = \tau f_i \qquad \text{on} \quad \Omega \\ u_i = 0 \qquad \text{on} \quad \Gamma_1, \qquad -k_i \nabla u_i . \nu = g_i \qquad \text{on} \quad \Gamma_2\end{gathered}$$

where $k_i = k(t_i, x, u_{i-1}), f_i = f(t_i, x, u_{i-1}), g_i = g(t_i, x, u_{i-1})$ and $\lambda_i \in L_\infty(\Omega)$ is a relaxation function which has to satisfy the "convergence condition"

$$|\lambda_i - \frac{b_n(u_i) - b_n(u_{i-1})}{u_i - u_{i-1}}| < \tau^\beta \quad (\beta \in (0,1)),$$

$b_n(s)$ is a suitable regularization of b ($b_n(s) \to b(s)$ locally uniformly). This scheme is implicit and we determine λ_i, u_i in an iterative way

$$\begin{gathered}\lambda_{i,k-1}(u_{i,k} - u_{i-1}) - \tau div(k_i \nabla u_{i,k}) = \tau f_i \qquad \text{in} \quad \Omega \\ u_{i,k} = 0 \quad \text{on} \quad \Gamma_1; \quad -k_i \nabla u_{i,k} . \nu = g_i \qquad \text{on} \quad \Gamma_2\end{gathered}$$

putting

$$\lambda_{i,k} := \frac{b_n(u_{i,k}) - b_n(u_{i-1})}{u_{i,k} - u_{i-1}}$$

for $k = 1, \ldots,$ where $\lambda_{i,0} = b'_n(u_{i-1})$. The numerical efficiency of this scheme (or its slight modifications) has been demonstrated in [B, JK$_1$, KHK, M$_1$, M$_2$] etc.

The method of characteristics applied to the model problem $\partial_t u + \bar{v}.\nabla u - \Delta u = 0$ with e.g. homogeneous Dirichlet boundary conditions reads

$$\frac{u_i - u_{i-1} \circ \varphi^i}{\tau} - \Delta u_i = 0$$

where $\varphi^i(x) = x - \tau \bar{v}(t_i, x)$ is an approximation (Euler explicit) of characteristics

$$\frac{dX(t;s,x)}{dt} = \bar{v}(t, X(t;s,x)), \qquad X(s;s,x) = x$$
$$\varphi^i(x) \approx X(t_{i-1}; t_i, x).$$

Then u_i can be interpreted as an approximation of composition of two phenomena in the time interval (t_{i-1}, t_i) where $u_{i-1} \circ \varphi^i$ represents the transport (without diffusion) of u_{i-1} along the approximated characteristics and u_i approximates (Euler implicit) the diffusion of the initial profile $u_{i-1} \circ \varphi^i$ time length $\tau = t_i - t_{i-1}$. Here it is substantial that the characteristics $X(t;s,x)$ (and also $\varphi^i(x)$) do not intersect each other (otherwise the transport cannot be realized). This can be guaranteed if $|\nabla \bar{v}|_\infty \le c_v$ ($L_\infty(\Omega \times I)$ norm), since $det(J\varphi^i(x)) = 1 - \tau det(J\bar{v}) \ge q > 0$ if $\tau \le \tau_{c_v}$ ($J\bar{v}$ is Jacobian of $\bar{v}$). In our model (1) the transport part of the equation is $\partial_t b(u) + div \bar{F}(t,x,u) = 0$ and its (corresponding) velocity field is $\bar{v} = \frac{\bar{F}'_u(t,x,u)}{b'(u)}$ since formally we have

$$\partial_t u + \frac{\bar{F}'_u(t,x,u)}{b'(u)}.\nabla u = -\frac{div_x \bar{F}(t,x,u)}{b'(u)}.$$

Even if the solution of (1) would be smooth we cannot guarantee $|\nabla \frac{\bar{F}'_u(t,x,u)}{b'(u)}|_\infty < c < \infty$.

In Section 1 we present our approximation scheme and discuss the convergence of the method. In Section 2 we discuss the approximation scheme and the convergence for Navier-Stokes system. In Section 3 we present some numerical experiments.

1. Approximation scheme

Let us denote $\omega_h * z$ the convolution of the mollifier ω_h with z where

$$\omega_h(x) = \frac{1}{h^N} \cdot \omega_1(\frac{x}{h}) \quad \text{and} \quad \omega_1(x) = \varkappa exp\left(\frac{|x|^2}{|x|^2 - 1}\right), \quad \text{for} |x| \le 1, \quad \omega_1(x) = 0$$

for $|x| > 1$ and $\int \omega_1(x) dx = 1$. Denote by

$$\varphi^i_\nu(x) := x - \tau \omega_h * \left(\frac{\bar{F}'_u(t_i, x, u_{i-1})}{\nu(x)}\right), 0 < \nu \in L_\infty(\Omega), \quad h = \tau^\omega, 0 < \omega < 1$$

and let $\tilde{u}$ denote an extension of u on $\Omega^* \supset \bar{\Omega}$ such that $\tilde{u}(x) = u(x)$ in Ω and $\|u\|_{W^1_2(\Omega^*)} \le c\|u\|_{W^1_2(\Omega)}$ (Sobolev space). Under $u(y)$ for $y \in \Omega^*$ we shall understand $\tilde{u}(y)$.

Our concept of approximation of (1) is a combination of a relaxation scheme and the method of characteristics in the following form:

$$\begin{aligned} \lambda_i(u_i - u_{i-1} \circ \varphi^i_{\mu_i}) - \tau div(k_i \nabla u_i) = -\tau(f_i - div_x \bar{F}(t_i, x, u_{i-1})) \equiv \tau H_i \quad \text{in } \Omega \\ u_i = 0 \qquad \text{on } \Gamma_1, -k_i \nabla u_i.\bar{\nu} = g_i \qquad \text{on } \Gamma_2 \end{aligned} \tag{4}$$

where the following "convergence conditions (5), (6)" are assumed:

$$\|\mu_i - G_i(\mu_i)\|_0 < \tau^\alpha \tag{5}$$

where

$$G_i(\nu) = \int_0^1 b_n'(u_{i-1} + s(u_{i-1}\circ\varphi_\nu^i - u_{i-1}))ds \equiv \frac{b_n(u_{i-1}) - b_n(u_{i-1}\circ\varphi_\nu^i)}{u_{i-1} - u_{i-1}\circ\varphi_\nu^i},$$

$\alpha \in (0,1)$ ($\|.\|_0$ is L_2 norm) and $\lambda_i \in L_\infty(\Omega)$ with

$$\left\|\lambda_i - \frac{b_n(u_i) - b_n(u_{i-1}\circ\varphi_{\mu_i}^i)}{u_i - u_{i-1}\circ\varphi_{\mu_i}^i}\right\|_0 < \tau^\beta, \qquad \beta \in (0,1). \tag{6}$$

This scheme is implicit and to guarantee (5),(6) we propose the iterations

$$\begin{aligned} \lambda_{i,k-1}(u_{i,k} - u_{i-1}\circ\varphi_{\mu_i}^i) - \tau div(k_i, \nabla u_{i,k}) &= \tau H_i \\ u_{i,k} = 0 \quad \text{on} \quad \Gamma_1, \quad -k_i\nabla u_{i,k}.\nu &= g_i, \quad \text{on} \quad \Gamma_2 \\ \lambda_{i,k} := \frac{b_n(u_{i,k}) - b_n(u_{i-1}\circ\varphi_{\mu_i}^i)}{u_{i,k} - u_{i-1}\circ\varphi_{\mu_i}^i}&, \lambda_{i,0} = b_n'(u_{i-1}). \end{aligned} \tag{7$_k$}$$

These iterations are not coupled with (5). If $\|\lambda_{i,k_0} - \lambda_{i,k_0-1}\|_0 < \tau^\beta$ are satisfied then we put $u_i := u_{i,k}$, $\lambda_i \equiv \lambda_{i,k-1}$.

To obtain $\varphi_{\mu_{i+1}}^{i+1}$ we propose fixed-point-type iterations

$$\mu_{i,l} = G_{i+1}(\mu_{i,l-1}), \quad l = 1, \dots \tag{8$_l$}$$

and when $\|\mu_{i,l_0} - \mu_{i,l_0-1}\|_0 < \tau^\alpha$ then we put $\mu_{i+1} := \mu_{i,l_0-1}$ and obtain (5) (with $i+1$ instead of i). Then we continue with (4) on the next time level t_{i+1}.

To formulate the convergence result, we specify our assumptions. By c (with or without indices) we denote generic positive constant.

H$_1$) b is an increasing, absolutely continuous function satisfying $b(0) = 0$. We assume that there exist $b_n \in C^1(R), b_n(0) = 0$ ($\tau \equiv \tau_n = \frac{T}{n}$) with $b_n'(s)$ locally Lipschitz continuous satisfying:

i) $b_n(s) \to b(s)$ locally uniformly;
ii) $c\tau^d \le b_n'(s) \le c\tau^{-\gamma} \quad \forall s \in R, d, \gamma \in (0,1)$;
iii) $\sup_{|z|<K} |b_n(z)| \le C(K) < \infty \quad \forall 0 < K < \infty$;
iv) $\min\{b'(s), \varepsilon\} \le Cb_n'(s)$ for some $\varepsilon > 0$;
v) $b_n''(s) \le c\tau^{-\rho}, \rho \in (0,1)$;

where d, γ, ρ are some parameters of the method.

H$_2$) $k(t,x,s) : I \times \Omega \times R \to R^{N\times N}$ is continuous and

$$C_1|\xi|^2 \le (k(t,x,s)\xi,\xi) \le C_2|\xi|^2;$$

H$_3$) $\bar{F}(t,x,s), \bar{F}'_s \equiv \partial_s \bar{F}(t,x,s) : I \times \Omega \times R^1 \to R^N$ are continuous and

$$|\partial_s \bar{F}(t,x,s)| \le C, |\partial_s^2 \bar{F}(t,x,s)| \le C, |div \bar{F}(t,x,s)| + |\partial_x \bar{F}'_s(t,x,s)| \le$$
$$\le C(L(t,x) + |s|) \text{ for a.e. } (t,x) \in Q_T = I \times \Omega, s \in R.$$

We assume that $\bar{F}(t,x,s)$ can be extended on $\Omega^* \supset \bar{\Omega}$ so that the estimates hold true for $x \in \Omega^*$ and $L \in L_2(\Omega^* \times I)$;

H$_4$) $f(t,x,s), g(t,x,s)$ are continuous in their variables and

$$|f(t,x,s)| \le C(1+|s|),$$
$$|g(t,x,s)| \le C(1+|s|).$$

H$_5$) $u_0 \in W_2^1(\Omega) \cap L_\infty(\Omega)$.

We denote the standard functional spaces by $L_2 = L_2(\Omega), L_\infty(\Omega)$, $V = \{v \in W_2^1(\Omega); v = 0 \text{ on } \Gamma_1\}, L_2(I,V)$; see [KJF]. By $\|.\|_0, \|.\|_\infty, \|.\|, \|.\|_{\Gamma_2}$ we denote the norms in $L_2, L_\infty(\Omega), W_2^1(\Omega), L_2(\Gamma_2)$, respectively. We denote by $(u,v) = \int_\Omega uvdx$, $(u,v)_{\Gamma_2} = \int_{\Gamma_2} uvdx$ and V^* the dual space to V. In the sequel we drop the variable x in the terms $k, f, g, \bar{F}, H, \lambda, \mu$.

We use the concept of variational solution. Let $V \equiv \{v \in W_2^1(\Omega) : v|\Gamma_1 = 0\}$ (W_2^1 being the Sobolev space) and let $< u,v >$ represents the duality between V^* and V.

Definition 9. *$u \in L_2(I,V)$ is a variational solution of (1)-(3) iff*

i) $b(u) \in L_1(\Omega), \partial_t b(u) \in L_2(I,V^*)$;

ii) $\int_I < \partial_t b(u), v > + \int_I \int_\Omega div \bar{F}(t,u) v + \int_I (k(t,u)\nabla u, \nabla v) + \int_I (g(t,u),v)_{\Gamma_2} =$
$= \int_I (f(t,u),v), \quad \forall v \in L_2(I,V)$

iii) $\int_I < \partial_t b(u), v > = \int_I \int_\Omega (b(u) - b(u_0))\partial_t v \qquad \forall v \in L_2(I,V) \cap L_\infty(Q_T), \partial_t v \in L_\infty(Q_T)$, $v(T) = 0$.

The existence of a variational solution u is proved in [AL]; see also [K$_1$]. The uniqueness is proved by F.Otto in [O] under some additional restrictions on $\bar{F}, f, g$.
Also $u_i \in V$ in (4) is considered as a variational solution of (4).

$$(\lambda_i(u_i - u_{i-1} \circ \varphi^i_{\mu_i}), v) + \tau(k_i \nabla u_i, \nabla v) + \tau(g_i, v)_{\Gamma_2} = \tau(H_i, v) \qquad \forall v \in V. \quad (10)$$

If $u_{i-1} \circ \varphi^i_{\mu_i} \in L_2(\Omega)$ then the existence of u_i in (10) is given by the Lax-Milgram theorem. The proposed method is based on the following lemma.

Lemma 11. *If $\mu_i = G_i(\nu)$ for any $0 < \nu \in L_\infty(\Omega)$ and $\omega + d < 1$ then $0 < c_1 < det(J\varphi^i_{\mu_i}) < c_2$ for $\tau \le \tau_0 > 0$ uniformly for $i = 1, \dots, n$.*

For the proof see [K$_3$]. As a consequence $\varphi^i_{\mu_i}(x)$ possess inverse and both $\varphi^i_{\mu_i}$ and $\left(\varphi^i_{\mu_i}\right)^{-1}$ are Lipschitz continuous and

$$\frac{1}{2}|x-y| \le |\varphi^i_{\mu_i}(x) - \varphi^i_{\mu_i}(y)| \le 2|x-y|. \tag{12}$$

Then if $u_i \in L_2(\Omega)$, then also $u_{i-1} \circ \varphi^i \in L_2(\Omega)$. Under some more special structure of data, e.g.,

$$i)\ f, g, div_x \bar{F}(t,x,s) \equiv 0;$$
$$ii)\ g \equiv 0, \lambda_i \ge q > 0 \text{ (for some } q \in R), L \in L_\infty(Q_T) \text{ in H3) ;}$$
$$iii)\ f(t,x,s)s \ge 0, g(t,x,s)s \ge 0, H(t,x,s).s \ge 0$$

it can be proven (see [K$_3$])

$$\|u_i\|_\infty \le c \qquad i = 1,\ldots,n. \tag{13}$$

In this section we shall assume (13) without additional restriction on data.

The convergence of iterations $(8)_l$ can be guaranteed in the following theorem (see [K$_3$]).

Theorem 14. *Suppose (13) and $\|\nabla u_i\|_0 \le c \quad \forall n, i = 1,\ldots,n$. If $\rho + d + \frac{N}{2}\omega < 1$ then the iterations $(8)_l$ are convergent, provided $\tau \le \tau_0$ ($\tau_0 > 0$ being sufficiently small).*

Remark 15. The regularization $b_n(s)$ of $b(s)$ in H$_1$ (see i–v) is not so much restrictive with respect to asymptotic behavior of b'_n, b''_n. For example, let us consider $b(s) = |s|^p \operatorname{sgn} s, p \in (0,1)$. We can take for $b_n(s)$

$$b_n(s) = \begin{cases} (s + \frac{1}{n^\delta})^p - \frac{1}{n^{p\delta}} & \text{for } s_1 \ge s \ge 0 \\ -|s - \frac{1}{n^\delta}|^p + \frac{1}{n^{p\delta}} & \text{for } -s_1 < s < 0 \end{cases}, s_1 = s_1(n) = n^\delta - \frac{1}{n^\delta}$$

$b_n(s) = b_n(s_1) + b'_n(s_1)(s - s_1)$ for $s \ge s_1$ and similarly for $s \le -s_1$. Then we have $c_1\tau^{(1-p)\delta} \le b'_n(s) \le c_2\tau^{-(1-p)\delta}$ (i.e. $d = \gamma = (1-p)\delta$ for an $\delta > 0$) and $|b''_n(s)| \le c\tau^{-\rho}$ with $\rho = (2-p)\delta$. We can verify easily i–v in H$_1$.

By means of $u_i, i = 1,\ldots,n$ from (10) we construct Rothe's function $\bar{u}^n(t) = u_i$ for $t \in (t_{i-1}, t_i\rangle$, $i = 1,\ldots,n$, $\bar{u}^n(0) = u_0$.

Our convergence result can be formulated in the form

Theorem 16. *Suppose H_1-H_5. Suppose (13) and (4)-(6) with $\alpha > d$. Let μ_i in (8) be of the form $\mu_i = G_i(\mu_{i-1,l_0})$ (see $(8)_l$). If $\omega + d < 1, \gamma + 2d + \rho < \omega$ then $\bar{u}^{\bar{n}} \to u$ in $L_s(Q_T)$ $\forall s > 1$ and $\bar{u}^{\bar{n}} \rightharpoonup u$ in $L_2(I,V)$, where $\bar{u}^n$ is from (4)-(6), (22) and u is a variational solution of (1). If the variational solution u is unique then the original sequence $\{\bar{u}^n\}$ is convergent.*

For the proof see [K$_3$]. A stronger convergence result can be obtained under the assumption

$$
\begin{gathered}
0 < \varepsilon \le b'(s) \le M < \infty \quad \text{a.e. in } \dot{R} \quad (d = \gamma = 0 \text{ in } \mathrm{H}_1) \\
\text{and assumptions (5), (6) are satisfied with } \|.\|_\infty \\
\text{instead of } \|.\|_0.
\end{gathered} \tag{17}
$$

Theorem 18. *Let the assumptions of Theorem 16 and the assumptions (17) be satisfied. Then $\bar{u}^{\bar{n}} \to u$ in $L_2(I, V)$ where u is a variational solution of (1)-(3) and $\{\bar{u}^n\}$ are from (10).*

Similar convergence results can be obtained when elliptic problem (10) is projected to a finite dimensional space V_λ (e.g., by FEM) provided $V_\lambda \to V$ for $\lambda \to 0$ in canonical sense. Then in the place of $\bar{u}^n$ we obtain $\bar{u}_\alpha$ with $\alpha = (\tau, \lambda), \alpha \to 0$ (see [K3]).

2. Convergence of the method for Navier-Stokes problem

Let us consider the Navier-Stokes problem

$$
\begin{gathered}
\partial_t \mathbf{u} + (\mathbf{u}\nabla).\mathbf{u} - \gamma\Delta\mathbf{u} = -\nabla p + \mathbf{f}(t, x, \mathbf{u}) \qquad \text{in } \Omega \times I \\
\mathbf{u} = 0 \text{ on } \partial\Omega \times I, \quad \mathbf{u}(x, 0) = \mathbf{u}_0 \quad \text{and} \\
div\ \mathbf{u} = 0
\end{gathered} \tag{19}
$$

where $\mathbf{u}\nabla = (u_1\partial_{x_1}, u_2\partial_{x_2}, \dots, u_n\partial_{x_n})$. Denote by $\mathcal{V} = \{\mathbf{v} \in C_0^\infty(\Omega) : div\ \mathbf{v} = 0\}$ and $\mathbf{V} = \bar{\mathcal{V}}$ where the closure is in the norm of the space $\left[\overset{\circ}{W}{}_2^1(\Omega)\right]^N$ (Cartesian product of $\overset{\circ}{W}{}_2^1(\Omega)$), $N = 2, 3$. We denote by $\mathbf{L}_2(\Omega) = [L_2(\Omega)]^N$, $(\varphi, \psi) = \int_\Omega \varphi\psi dx$, $(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^N \int_\Omega u_i v_i dx$, $((\mathbf{u}, \mathbf{v})) = \sum_{i=1}^N (\nabla u_i, \nabla v_i)$ and $b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{i,j=1}^N \int_\Omega u_j \partial_{x_j} v_i w_i dx$ which is well defined for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}$ since $W_2^1(\Omega) \hookrightarrow L_4(\Omega)$ (continuous imbedding, $N = 2, 3$). The variational solution to (19) is defined as $\mathbf{u} \in L_\infty(I, \mathbf{V}) \cap L_2(I, \mathbf{L}_2)$ satisfying

$$
\begin{gathered}
\frac{d}{dt}(\mathbf{u}, \mathbf{v}) + ((\mathbf{u}, \mathbf{v})) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = (\mathbf{f}(t, \mathbf{u}), \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V} \\
\text{with} \qquad \mathbf{u}(0) = \mathbf{u}_0.
\end{gathered} \tag{20}
$$

We assume that $\mathbf{f}(t, x, s)$ is continuous in its variables and

$$
|\mathbf{f}(t, x, s)| \le c(1 + |s|). \tag{21}
$$

The existence of the variational solution can be found, e.g., in [F]. The uniqueness is proved for $N = 2$ in [LP] and for $N = 3$ it is an open problem. If $\mathbf{u} \in L_8(I, \mathbf{L}_4)$ then the variational solution is also unique for $N = 3$. The method of characteristics has been applied to (19) in [P], [S] etc. and its convergence has been proved under rather strong regularity assumptions.

We apply our concept of the method of characteristics from Section 1 and prove its convergence without the assumption $\|\mathbf{u}\|_\infty \le c$ (norm in $\mathbf{L}_\infty(\Omega \times I)$). In that case we can truncate the convective term and apply our concept of approximation to (see (19))

$$\begin{gathered}\partial_t \mathbf{u} + (\mathbf{u}^K \nabla).\mathbf{u} - \gamma \Delta \mathbf{u} = -\nabla p + \mathbf{f}(t,x,\mathbf{u}) \quad \text{in } \Omega \times I \\ \mathbf{u} = 0 \text{ on } \partial\Omega \times I, \quad \mathbf{u}(x,0) = \mathbf{u}_0 \\ div\ \mathbf{u} = 0\end{gathered} \tag{18_K}$$

where $\mathbf{u}^K = (u_1^K, \dots, u_N^K)$ with $\varphi^K := \begin{cases} \varphi(x) & \text{for } |\varphi(x)| < K, \\ K sgn\ \varphi(x) & \text{for } |\varphi(x)| \ge K. \end{cases}$

Remark 22. Suppose that $\mathbf{f}(t,x,s)$ is Lipschitz continuous in s. It can be proved that $(18)_K$ possess the unique solution in the case $N = 2$. In the case $N = 3$ it holds true under the additional assumption that a variational solution of (19) or $(18)_K$ is an element of $\mathbf{L}_\infty(I, \mathbf{V})$.

The unique variational solution of $(18)_K$ coincides with the bounded variational solution of (19).

Now our modified method of characteristics applied to $(18)_K$ reads

$$\left(\frac{\mathbf{u}_i - \mathbf{u}_{i-1} \circ \varphi^i}{\tau}, \mathbf{v} \right) + \gamma((\mathbf{u}_i, \mathbf{v})) = (\mathbf{f}_i, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V} \tag{23}$$

for $i = 1, \dots, n$ ($\mathbf{u}_i \approx \mathbf{u}(x, t_i), t_i = i\tau$) where $\varphi^i(x) := x - \omega_h * \mathbf{u}_{i-1}^K, h = \tau^\omega$ for any $\omega \in (0,1)$.

Problem (23) is the stationary Stokes problem for determination of $\mathbf{u}_i \in \mathbf{V}$. The existence is guaranteed by Lax-Millgram argument provided $\mathbf{u}_{i-1} \circ \varphi^i \in \mathbf{L}_2(\Omega)$. In this case (since $|\mathbf{u}_i^K|_\infty \le K$) we verify that $\varphi^i(x)$ with its inverse are Lipschitz continuous for $h = \tau^\omega$ for any $\omega \in (0,1)$ and hence $\mathbf{u}_{i-1} \circ \varphi^i \in \mathbf{L}_2(\Omega)$ for $\mathbf{u}_{i-1} \in \mathbf{L}_2(\Omega)$. In our case we cannot prove the boundedness $\|\mathbf{u}_i\|_\infty \le c < \infty$ as in Section 1 since (23) represents a system coupled through $\mathbf{V}$. Thus we shall not assume (13) in this section.

We can prove the *a priori* estimates

Lemma 24. *The estimates*

$$\max_{1 \le i \le n} |\mathbf{u}_i|^2 \le c, \quad \sum_{i=1}^n \|\mathbf{u}_i\|^2 \tau \le c \tag{25}$$

hold true uniformly for $i = 1, \dots, n$.

Proof. We put $\mathbf{v} = \mathbf{u}_i \tau$ into (23) and sum it up for $i = 1, \dots, j$. Then we obtain

$$\begin{gathered}\sum_{i=1}^j (\mathbf{u}_i - \mathbf{u}_{i-1}, \mathbf{u}_i) + \gamma \sum_{i=1}^j \|\mathbf{u}_i\|^2 \tau \le \sum_{i=1}^j |(\mathbf{u}_{i-1} - \mathbf{u}_{i-1} \circ \varphi^i, \mathbf{u}_i)| + \\ + \sum_{i=1}^j |\mathbf{u}_i|^2 \tau + c.\end{gathered} \tag{26}$$

The first term on R.H.S. we estimate in the following way

$$\begin{aligned}\mathbf{u}_{i-1} - \mathbf{u}_{i-1} \circ \varphi^i &= -\tau(\omega_h * \mathbf{u}^K_{i-1}\nabla) \int_0^1 \tilde{\mathbf{u}}_{i-1}(x + s(\varphi^i(x) - x))ds = \\ &= -\tau \left(\mathbf{u}^K_{i-1}\nabla\right) \int_0^1 \tilde{\mathbf{u}}_{i-1}(x + s(\varphi^i(x) - x))ds + \\ +\tau \left(\left[\mathbf{u}^K_{i-1} - \omega_h * \mathbf{u}^K_{i-1}\right] \nabla\right) &\int_0^1 \tilde{\mathbf{u}}_{i-1}(x + s\varphi^i(x) - x))ds \equiv -\tau J^1_i + \tau J^2_i\end{aligned} \tag{27}$$

where $\tilde{\mathbf{u}}_{i-1}$ is a prolongation of $\mathbf{u}_{i-1}$ on $\Omega^* \supset \bar{\Omega}$ ($\|\tilde{\mathbf{u}}\|_{\mathbf{W}^1_2(\Omega^*)} \le \|\mathbf{u}\|_{\mathbf{W}^1_2(\Omega)}$). Then we can estimate the first term on R.H.S. in (26) by

$$\begin{aligned}\varepsilon \sum_{i=1}^j \tau \int_\Omega |J^1_i|^2 dx + c(\varepsilon) \sum_{i=1}^j \|\mathbf{u}_i\|_0^2 \tau + \sum_{i=1}^j \tau \int_\Omega |J^2_i|^2 dx \le \\ \le \varepsilon \sum_{i=1}^j \|\nabla \mathbf{u}_i\|_0^2 \tau + c(\varepsilon, K) \sum_{i=1}^k \|\mathbf{u}_i\|_0^2 \tau.\end{aligned}$$

Since $\sum_{i=1}^j (\mathbf{u}_i - \mathbf{u}_{i-1}, \mathbf{u}_i) \ge \frac{1}{2}\|\mathbf{u}_j\|_0^2 - \frac{1}{2}\|\mathbf{u}_0\|_0^2$. Then the Gronwall argument in (26) yields the *a priori* estimate (25).

Lemma 28. *The sequence $\{\mathbf{u}_i\}$ is compact in $\mathbf{L}_2(Q_T)$ i.e. $\exists \mathbf{u} \in L_2(I, \mathbf{V})$ such that $\bar{\mathbf{u}}^{\bar{n}} \to \mathbf{u}$ in $\mathbf{L}_2(Q_T)$.*

Proof. We sum up (23) for $i = j+1, \ldots, j+k$ and then we put $\mathbf{v} = \mathbf{u}_{j+k} - \mathbf{u}_j$ and again we sum it up for $j = 1, \ldots, n-k$. The first term in (23) we split in the form

$$\mathbf{u}_i - \mathbf{u}_{i-1} \circ \varphi^i = (\mathbf{u}_i - \mathbf{u}_{i-1}) + (\mathbf{u}_{i-1} - \mathbf{u}_{i-1} \circ \varphi^i). \tag{29}$$

Then using the *a priori* estimates of Lemma 24 we successively obtain

$$\sum_{j=1}^{n-k} (\mathbf{u}_{j+k} - \mathbf{u}_j, \mathbf{u}_{j+k} - \mathbf{u}_j)\tau \le ck\tau$$

from which we conclude

$$\int_I \|\bar{\mathbf{u}}^n(t+z) - \bar{\mathbf{u}}^n(t)\|_0^2 dt \le cz \qquad \text{for any } 0 < z \le z_0 \tag{30}$$

and for all n ($\bar{\mathbf{u}}^n(t)$ is the corresponding Rothe's function). From the estimate (see $(24)_2$)

$$\|\bar{\mathbf{u}}^n\|_{L_2(I,\mathbf{V})} \le c$$

we deduce

$$\int_\Omega |\bar{\mathbf{u}}^n(t, x+y) - \bar{\mathbf{u}}^n(t,x)|^2 dy \le c|y|$$

which together with (29) gives

$$\int_{Q_T} |\bar{\mathbf{u}}^n(t+z, x+y) - \bar{\mathbf{u}}^n(t,x))^2 dx dt \le c(z + |y|).$$

Then Kolmogorov's argument implies the compactness of $\{\bar{\mathbf{u}}^n\}$. The rest is a consequence of Lemma 24.

Theorem 31. *Let $\{\mathbf{u}_i\}_{i=1}^n$ be the solution of the Stokes problem (23). Then $\bar{\mathbf{u}}^n \to \mathbf{u}$ in $L_2(I, \mathbf{V})$ where $\mathbf{u}$ is a variational solution of $(18)_K$.*

Proof. Let $\mathbf{v} \in C^1(I, \mathcal{V})$ be a smooth test function, $\mathbf{v}(x,t) = 0$ for $t = T$. We put it into (23) and integrate it over $(0,T)$. In the first term $J_1 = J_1^1 + J_1^2$ in (23) we use (29) and obtain ($\mathbf{v}_i = \mathbf{v}(x, t_i)$)

$$J_1^1 = \sum_{i=1}^n (\mathbf{u}_i - \mathbf{u}_{i-1}, \mathbf{v}_i) + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (\mathbf{u}_i - \mathbf{u}_{i-1}, \mathbf{v}(s) - \mathbf{v}_i) ds.$$

Easily we obtain that the second term converges to 0 with $n \to \infty$. Using Abel's summation in the first term we obtain

$$\sum_{i=1}^n (\mathbf{u}_i - \mathbf{u}_{i-1}, \mathbf{v}_i) \to -\int_I (\mathbf{u}, \partial_t \mathbf{v}) - (\mathbf{u}_0, \mathbf{v}(x, 0)).$$

We can prove

$$J_1^2 \equiv \sum_{i=1}^n (\mathbf{u}_{i-1} - \mathbf{u}_{i-1} \circ \varphi^i, \mathbf{v}) \to ((\mathbf{u}^K \nabla)\mathbf{u}, \mathbf{v})$$

since $\bar{\mathbf{u}}^n \to \mathbf{u}$ a.e. in Q_T, $\nabla \bar{\mathbf{u}}^n \rightharpoonup \mathbf{u}$ in $L_2(I, \mathbf{V})$ and $\mathbf{v} \in C^1(I, \mathcal{V})$. Similarly we obtain

$$J_2 \equiv \int_I ((\bar{\mathbf{u}}^n, \mathbf{v}))dt \to \int_I ((\mathbf{u}, \mathbf{v}))dt$$

$$J_3 \equiv \int_I (\mathbf{f}(t, \bar{\mathbf{u}}^n(t - \tau)), \mathbf{v})dt \to \int_I (\mathbf{f}(t, \mathbf{u}), \mathbf{v})dt$$

and hence we deduce that $\mathbf{u}$ is a variational solution of $(18)_K$. To prove the strong convergence of $\{\bar{\mathbf{u}}^n\}$ we shall need $\partial_t \bar{\mathbf{u}}^n \rightharpoonup \partial_t \mathbf{u}$ in $L_2(I, \mathbf{V}^*)$. We obtain from (23) (using (29)) by duality argument

$$\left\| \frac{\mathbf{u}_i - \mathbf{u}_{i-1}}{\tau} \right\|_* \le c_1 + c_2 \|\mathbf{u}_i\|$$

where (26) has been used. Hence we deduce

$$\|\partial_t^\tau \bar{\mathbf{u}}^n\|_{L_2(I,\mathbf{V}^*)} \le c \qquad \left(\partial_t^\tau \bar{\mathbf{u}}^n \equiv \frac{\mathbf{u}_i - \mathbf{u}_{i-1}}{\tau} \text{ for } t \in (t_{i-1}, t_i)\right)$$

and $\partial_t^\tau \bar{\mathbf{u}}^n \rightharpoonup \partial_t \mathbf{u}$ in $L_2(I, \mathbf{V}^*)$. Moreover, we use integration by parts formula

$$\int_0^t (\partial_t \mathbf{u}, \mathbf{u}) ds = \frac{1}{2}\|\mathbf{u}(t)\|_0^2 - \frac{1}{2}\|\mathbf{u}_0\|_0^2 \tag{32}$$

and

$$\liminf \int_0^t (\partial_t^\tau \bar{\mathbf{u}}^n, \bar{\mathbf{u}}^n) \ge \frac{1}{2}\|\mathbf{u}(t)\|_0^2 - \frac{1}{2}\|\mathbf{u}_0\|_0^2. \tag{33}$$

We put $\mathbf{v} = \mathbf{u} - \bar{\mathbf{u}}^n$ into (23) and integrate it over $(0, t)$. The first term we split in the form (29). Using (32), (33) in parabolic terms we obtain

$$\liminf_{n\to\infty} \int_0^t (\partial_t^\tau \bar{\mathbf{u}}^n, \bar{\mathbf{u}}^n - \mathbf{u}) ds \ge 0. \tag{34}$$

We can estimate

$$\left|\int_0^t (\bar{\mathbf{u}}^n - \bar{\mathbf{u}}^n \circ \varphi, \bar{\mathbf{u}}^n - \mathbf{u}) ds\right| \to 0 \quad \text{for } n \to \infty. \tag{35}$$

Since $\bar{\mathbf{u}}^n \to \mathbf{u}$ in $L_2(I, \mathbf{L}_2)$. Similarly we have

$$\left|\int_0^t (\mathbf{f}(t, \bar{\mathbf{u}}^n(t-\tau)), \bar{\mathbf{u}}^n - \mathbf{u}) ds\right| \to 0 \quad \text{for } n \to \infty. \tag{36}$$

The elliptic term we estimate as follows:

$$\begin{aligned}\int_0^t ((\bar{\mathbf{u}}^n, \bar{\mathbf{u}}^n - \mathbf{u})) ds &= \int_0^t ((\bar{\mathbf{u}}^n - \mathbf{u}, \bar{\mathbf{u}}^n - \mathbf{u})) dt + \int_0^t ((\mathbf{u}, \bar{\mathbf{u}}^n - \mathbf{u})) dt \ge \\ &\ge c \int_0^t \|\bar{\mathbf{u}}^n - \mathbf{u}\|^2 dt - \mathcal{O}(1)\end{aligned}$$

($\mathcal{O}(1)$ denotes any term c_n with $c_n \to 0$ for $n \to \infty$). Finally we obtain

$$c \int_0^t \|\bar{\mathbf{u}}^n - \mathbf{u}\|^2 dt \le \int_0^t ((\bar{\mathbf{u}}^n - \mathbf{u}, \bar{\mathbf{u}}^n - \mathbf{u})) + \mathcal{O}(1) \le \mathcal{O}(1), \quad \forall t \in I$$

from which we obtain the required result.

3. Some numerical experiments

We present a numerical solution of a mathematical model

$$\partial_t(\theta u + \rho S) + div\ (\bar{v}u - D\nabla u) = 0$$
$$\rho\partial_t S = k(\Psi(u) - S), \Psi(C) = \kappa C^p, p \in (0,1)$$

describing contaminant transport with sorption governed by Freundlich isotherm. In the equilibrium mode, i.e. $k \to \infty$, we obtain $S = \Psi(u) = \kappa u^p$. Then $b(u) = \theta u + \rho\kappa u^p$ (here $b'(0) = \infty$) where θ is the porosity of the porous media, $\bar{v}$ is the velocity field of the water and u is the concetration of the contaminant. This problem has been studied in [Kn], [DDG], [DKn], etc...

We use the following data:

$$\partial_t\left(\frac{1}{2}u + 1.5u^p\right) + \partial_x(3u - 0.05\partial_x u) = 0$$

with $p = 1.2; 1; 0.8; 0.6; 0.4$. We consider $\Omega = (0,100), T = 30$ and the Dirichlet boundary conditions $u(t,100) = 0$ and

i) $u(t,0) = 1$,
ii) $u(t,0) = 0$

with the corresponding initial conditions

i) $u(0,x) = u_0^1(x)$,
ii) $u(0,x) = u_0^2(x)$

where $u_0^i(x)$ $(i = 1,2)$ are piecewise linear functions of the following form: $u_0^1(x) = 1$ for $x \in (0,0.1)$, $u_0^1(0.2) = 0, u_0^1(x) = 0$ for $x > 0.2$; $u_0^2(x) = 0$ for $x \in (0,100)\backslash(0.1,0.5)$, $u_0^2(0.2) = u_0^2(0.4) = 1$. The solutions are drawn in three time moments for various p ($p = 1.2$ - dash-dash-dotted line, $p = 1.0$ - full line, $p = 0.8$ - dash-dotted line, $p = 0.6$ - dashed line, $p = 0.4$ - dotted line) in Figure 1 for $t = 2$, in Figure 2 for $t = 4$ and in Figure 3 for $t = 6$ in the case *i*. The case *ii* is drawn in Figure 4 for $t = 1$, in Figure 5 for $t = 2$ and in Figure 6 for $t = 6$.

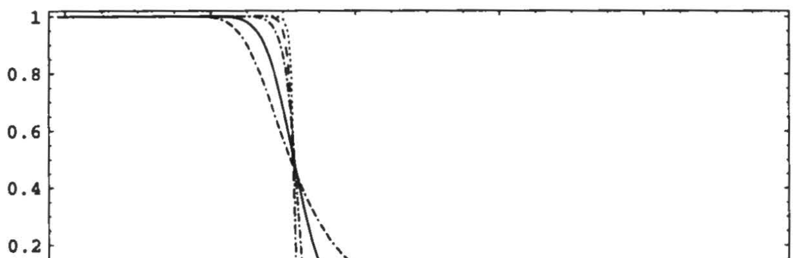

Figure 1

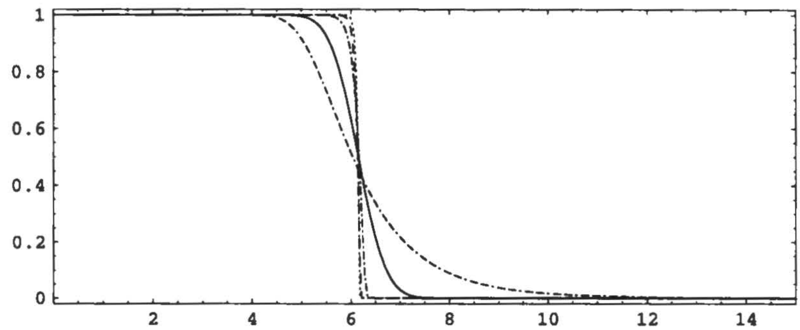

FIGURE 2

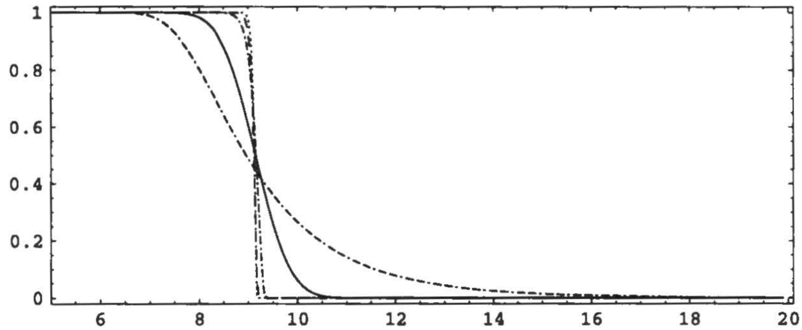

FIGURE 3

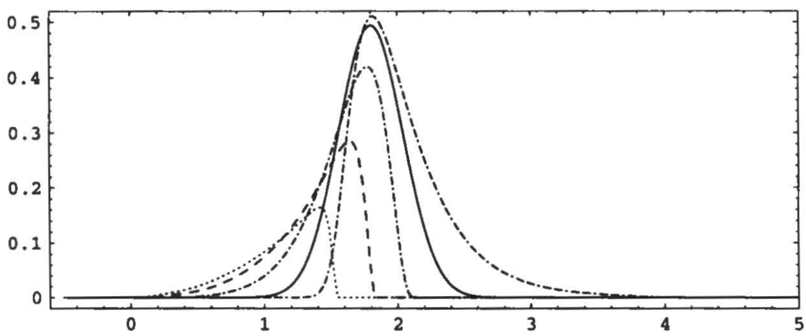

FIGURE 4

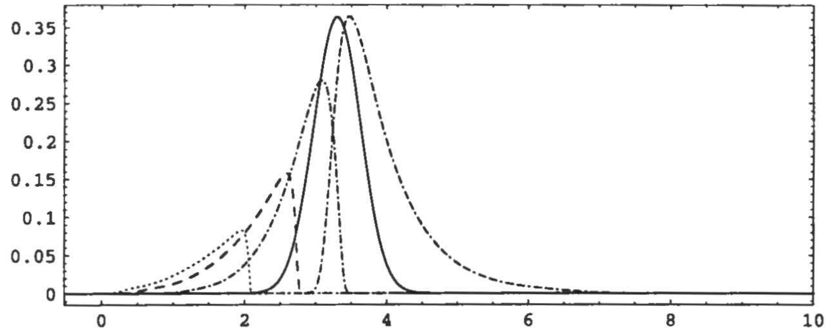

FIGURE 5

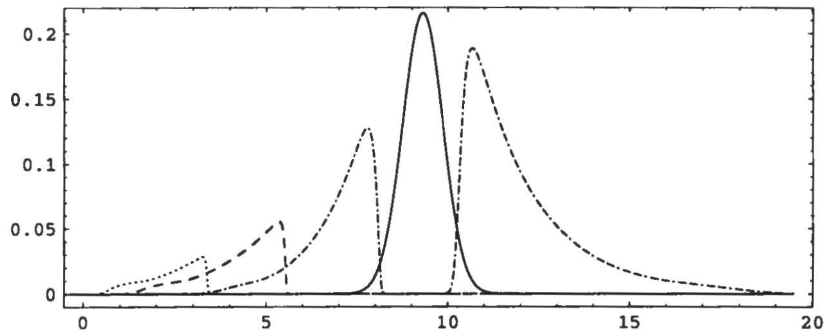

FIGURE 6

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Symmetrization – or How to Prove Symmetry of Solutions to a PDE

B. Kawohl

Abstract. Many partial differential equations are Euler equations of variational problems, and this circumstance can be used to derive symmetry or monotonicity properties of their solutions. I survey some recent results on symmetrization methods, i.e. on procedures in which a supposedly nonsymmetric minimizer is deformed into a symmetric one with lower energy. I also compare this variational approach with arguments involving the maximum principle. There are specific advantages and disadvantages of both methods. Another symmetry approach uses the unique continuation principle. Finally I address the recently emerging subject of isoparametric surfaces.

Keywords: symmetrization, symmetry, rearrangement, unique continuation, isoparametric surfaces, Ginzburg-Landau

Classification: 35B05, 49K10, 26D10, 35J20

"Nihil est sine ratione" (Nothing is without a reason) was - according to G.Polya [46] - a fundamental idea of W.G.Leibniz. He phrased it in Latin, the lingua franca of the time when Charles University was founded 650 years ago. With my lecture in today's Latin I follow a suggestion of the organizing committee to talk about symmetry. Symmetry of solutions is often taken for granted in symmetric problems, but by no means automatically true [36, 37]. There are many discussions of this topic in the mathematical literature, e.g. [10, 45, 46, 54]. In this lecture I focus on symmetry of elliptic equations, and here in particular on symmetrization methods. For parabolic equations I refer to [31] and the lecture of P.Poláčik. Other methods to prove symmetry, for instance via stability arguments, are listed in [36].

I. What is symmetrization?

Let me present the geometric idea of symmetrization in one dimension by means of a simple example. Symmetrization maps the nonsymmetric function $u_a(x)$ into a symmetric function $u^*(x) = u_0(x)$.

$$u_a(x) = \begin{cases} \frac{1+x}{1+a} \text{ in } [-1,a] \\ \frac{1-x}{1-a} \text{ in } [a,1] \end{cases} \quad \mapsto \quad u^*(x) = 1 - |x| \text{ in } [-1,1]$$

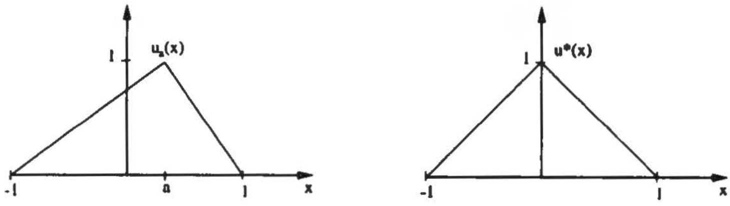

Figure 1: The graphs of u_a and u^*

Remark 1:

One immediately observes the following two properties, which seem to be predestined for use in variational problems.

- $\int_{-1}^{1} F(u_a)\,dx = \int_{-1}^{1} F(u^*)\,dx$ (Cavalieri's principle) for any Borel measurable function F.
- $g(a) := \int_{-1}^{1} |du_a/dx|^p\,dx = (1+a)^{1-p} + (1-a)^{1-p} \geq \int_{-1}^{1} |du^*/dx|^p\,dx = g(0)$, because g is even and (even strictly) convex.

Definitions:

For $u(x) \in W_0^{1,p}(\Omega, I\!R^+)$ consider the level sets $\Omega_c(u) := \{\, x \in I\!R \mid u(x) \geq c \,\}$. Then their rearrangements are defined as

$$\Omega_c^*(u) := \begin{cases} [-|\Omega_c(u)|/2, |\Omega_c(u)|/2] & \text{if } \Omega_c(u) \neq \emptyset, \\ \emptyset & \text{otherwise,} \end{cases}$$

and $u^*(x) := \sup \{\, c \in I\!R \mid x \in \Omega_c^*(u) \,\}$, which is defined on $\Omega^* = \Omega_0^*(u)$, is called the **symmetrically decreasing rearrangement** of u. Note that the level sets $\Omega_c(u^*)$ of u^* coincide with $\Omega_c^*(u)$ by construction.

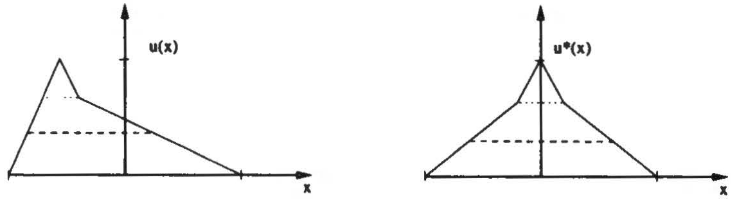

Figure 2: The graphs of u and its symmetrically decreasing rearrangement u^*

To symmetrize a function of several variables, we have to symmetrize its level sets in a suitable way. One of them is **Steiner symmetrization** with respect to $y = 0$.

For $u(x) \in W_0^{1,p}(\Omega, I\!R^+)$ with $\Omega \subset I\!R^n$ set $x = (x', y)$ with $x' \in I\!R^{n-1}$. Now let us consider the one-dimensional level sets $\Omega_c(u(x', \cdot)) := \{\, y \in I\!R \mid u(x', y) \geq c \,\}$, and define

$$\Omega_c^*(u(x',\cdot)) := \begin{cases} [-|\Omega_c(u(x',\cdot))|/2, |\Omega_c(u(x',\cdot))|/2] & \text{if } \Omega_c(u(x',\cdot)) \neq \emptyset, \\ \emptyset & \text{otherwise,} \end{cases}$$

$$\Omega_c^*(u) := \bigcup_{x' \in \mathbb{R}^{n-1}} \{x'\} \times \Omega_c^*(u(x',\cdot)), \quad \text{the } \textit{Steiner symm.} \text{ of } \Omega_c(u),$$

$$u^*(x) := \sup \{ c \in \mathbb{R} \mid x \in \Omega_c^* \}, \qquad \text{the } \textit{Steiner symmetrization}$$

or symmetrically decreasing rearrangement of u in direction y.

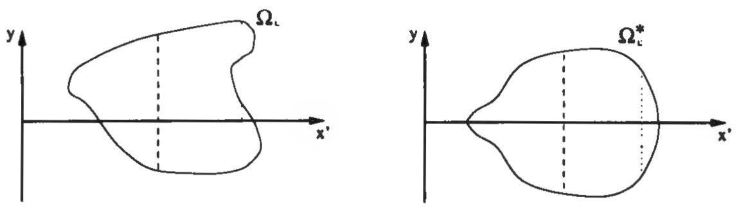

Figure 3: A level set Ω_c and its Steiner symmetrization with respect to y

Another suitable way to symmetrize level sets and functions is known as **Schwarz symmetrization** (not Schwartz symmetrization please!), named after Hermann Amandus Schwarz, in which for $u(x) \in W_0^{1,p}(\Omega, \mathbb{R}^+)$ with $\Omega \subset \mathbb{R}^n$ the level sets $\Omega_c(u) := \{ x \in \mathbb{R}^n \mid u(x) \geq c \}$ are replaced by concentric balls centered at zero. Therefore Schwarz symmetrization amounts to defining

$$\Omega_c^*(u) := \begin{cases} \text{ball of same volume as } \Omega_c \text{ centered at } 0 & \text{if } \Omega_c \neq \emptyset, \\ \emptyset & \text{otherwise,} \end{cases}$$

$$u^*(x) := \sup \{ c \in \mathbb{R} \mid x \in \Omega_c^*(u) \}.$$

Let me remark in passing that Schwarz symmetrization can be obtained as a limit of consecutive Steiner symmetrizations (similar to making dumplings or snowballs).

Remark 2:
The following inequalities are common to all these rearrangements; see for instance [33].

- $\int_\Omega F(u)\, dx = \int_{\Omega^*} F(u^*)\, dx$ (Cavalieri's principle)
- $\int_\Omega |\nabla u|^p\, dx \geq \int_{\Omega^*} |\nabla u^*|^p\, dx$ (Polya-Szegö inequality) for $p \in (0, \infty)$
- $\int_{\Omega^*} u^*\ v^*\ dx \geq \int_\Omega u\ v\ dx$ (Hardy-Littlewood inequality)

Application 1:
The Krahn Faber inequality $\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$ is a typical example for the usefulness of Remark 2. Recall that the first Dirichlet eigenvalue for the Laplace

operator on any domain Ω is characterized as $\lambda_1(\Omega) = \min \{R(v;\Omega) \; ; \; 0 \not\equiv v \in H_0^{1,2}(\Omega)\}$ and $R(v;\Omega) = ||\nabla v||_2/||v||_2$, where $||\cdot||_2$ denotes the norm in $L^2(\Omega)$. Therefore

$$\lambda_1(\Omega) = \frac{\int_\Omega |\nabla u_1|^2 \, dx}{\int_\Omega |u_1|^2 \, dx} \geq \frac{\int_{\Omega^*} |\nabla u_1^*|^2 \, dx}{\int_{\Omega^*} |u_1^*|^2 \, dx} = R(u_1^*, \Omega^*) \geq \min_v R(v, \Omega^*) = \lambda_1(\Omega^*),$$

and **equality** holds **only if** Ω is a ball. This last observation about equality was (as I learned in 1993) first made by F.Schulz in his diploma thesis [49]. It rests on the observation that sometimes one can discuss the equality sign in the second inequality of Remark 2. In fact Schulz did this 1977 for the first eigenfunction and for the Dirichlet integral $\int_\Omega |\nabla u|^2 \, dx$, while I discussed it in 1984 [32] for integrals of the type $\int_\Omega F(x', u)G(|\nabla u|) \, dx$ with positive F, strictly convex and strictly increasing G and e.g. analytic functions u. Moreover, I showed that for $F \equiv 1$ and $G(t) = t^p$ and for a dense subset of $W^{1,p}(\Omega)$ one could discuss equality and conclude that it implies symmetry of u. But be careful. There are also functions, even C^∞ functions, with large sets of critical points for which this conclusion is wrong. As a counterexample, imagine a radially symmetric cutoff function η_1, centered at zero, with a flat top $B_1(0)$, and add to it another radially symmetric cutoff function, centered at $x_0 \neq 0$, such that supp $\eta_2 \subset B_1(0)$. Then the Dirichlet integrals of $\eta = \eta_1 + \eta_2$ and η^* coinincide, but $\eta \not\equiv \eta^*$. As for Schwarz symmetrization, these results were later refined by J.E.Brothers and W.T.Ziemer [12, 13] and by R.Tahraoui [52]. Incidentally, the result of Brothers and Ziemer was first presented at a winter school 1987 in Srni, organized by Charles University. Brothers and Ziemer considered integrals of type $\int_\Omega A(|\nabla u|) \, dx$ under Schwarz symmetrization, with $A(0) = 0$, A increasing and $A^{1/p}$ convex for some $p \in [1, \infty)$. They discussed the equality sign in $\int_\Omega A(|\nabla u|) \, dx \geq \int_\Omega A(|\nabla u^*|) \, dx$ under the assumptions $p > 1$ and that $\{x \in \Omega^* \; ; \; |\nabla u^*(x)| = 0\} \cap u^{*-1}(0, M)$ is a nullset. Here M =ess sup $u^* \leq \infty$. Fortunately, solutions of elliptic equations are often almost nowhere flat, so that these results are applicable for our purposes. Tahraoui considered more general integrals of type $\int_\Omega F(u, |\nabla u|) \, dx$ under Schwarz symmetrization, where F satisfies natural growth assumptions and is convex in its second argument [52]. Finally, Brock considered integrals of type $\int_\Omega F(x, u, \nabla u) \, dx$ under Steiner symmetrization with structural assumptions on F that recover all previous (weak) inequalities and that are too detailed to be reproduced here; see [7] and also [33, p.83].

Application 2:

Nonlinear ground states on $\mathbb{R}^n$ are in a simple setting characterized as minimal energy solutions of

$$\begin{aligned} -\Delta u = \lambda f(u) \quad &\text{and} \quad u > 0 \quad \text{in } \mathbb{R}^n, \\ u(x) \to 0 \quad &\qquad\qquad\qquad \text{as } |x| \to \infty. \end{aligned}$$

Modulo suitable assumptions on f, there is a metatheorem which states that any ground state u is radial and radially decreasing. This approach was used for instance in [4] or [25]. Of course there are other ways to get such results via the moving plane method; see [29] and [39]. To be a little more precise, in the recent paper [25] of M.Flucher and S.Müller the ground states are characterized as maximizing $\int_{\mathbb{R}^n} F(v)\,dx$ over all functions v with Dirichlet integral not exceeding 1. Here $F' = f$, $0 \le F(t) \le c|t|^{2^*}$ with $2^* = 2n/(n-2)$ and F is upper semicontinuous and $F \not\equiv 0$ in L^1-sense.

This includes the case that $f(0) = f'(0) = 0$, which was not covered in [29]. In fact, in [29] the assumption $f(0) = 0$ and $f'(0) < 0$ was used, and only later removed in the paper of Y.Li and W.M.Ni. I refer to the lecture of H.Berestycki during this meeting for further details.

Application 3:

Oddness and monotonicity, even one-dimensionality in cylinders can be proved via Steiner symmetrization as well. To this end let $\Omega = \omega \times \mathbb{R}$ with $\omega \subset \mathbb{R}^{n-1}$ and consider minimizers u of the Ginzburg-Landau energy $J(v) = \int_\Omega 2|\nabla v|^2 + (v^2-1)^2\,dx$ among those functions v which go to ± 1 as $x_n \to \pm\infty$.

Claim 1:

The nodal surface of u is a plane; without loss of generality it is the plane $\{x_n = 0\}$, so that $u(x',0) = 0$, and u is odd in x_n, i.e. $u(x',x_n) = -u(x',-x_n)$ in Ω.

Proof: Substitute $w := (1-u^2)$, then $w \ge 0$ and $w \to 0$ as $|x_n| \to \infty$. For every $x' \in \omega$ the function $w(x',\cdot)$ has a maximum of height 1 somewhere. Under this tranformation the energy is converted into a **convex functional**

$$\tilde{J}(w) := \int_\Omega \frac{|\nabla w|^2}{2(1-w)} + w^2\,dx.$$

Incidentally, the convexity of this transformed functional $\tilde{J}$ might explain the somewhat unexpected uniqueness-, symmetry- and Liouville-theorems for Ginzburg-Landau type problems. To prove the oddness of u, Steiner-symmetrize w and set $\tilde{u}(x',x_n) := \operatorname{sign}(x_n)\sqrt{1-w^*}$. Then it is easy to see that $J(\tilde{u}) \le J(u)$ and equality holds only if u is odd and monotone. More can be shown, however.

Claim 2:

Not only the nodal surface, but every level surface of u is a plane, and therefore u depends only on x_n.

Proof: For each $c \in (-1,1)$ the level surface $\{(x',x_n); u(x) = c\}$ can be parametrized by x' as $x_n = f_c(x')$. Replace it by its average $x_n = \fint_\omega f_c(x')\,dx'$, i.e. replace the level sets of $w = 1-u^2$ by cylinders with cross section ω and with plane ends. It can be shown that this decreases energy.

This approach can be found in a nice paper of G.Carbou [17]. It was recently generalized by F.Brock [8] to apply to travelling waves in cylinders, generalizing previous results of H.Berestycki & L.Nirenberg [3].

What about boundary conditions other than Dirichlet? We just saw an example with zero Neumann data on the lateral boundary $\partial\omega \times I\!R$. Here are two more.

Application 4:

Steiner symmetry for nonlinear boundary conditions can sometimes be achieved without symmetrization arguments. The following theorem is new, and its assumptions are far from optimal. Rather than specialize in generalizations, I just want to expose the idea of proof.

Theorem 1:

Let $\Omega \subset I\!R^n$ be Steiner symmetric in x_n and consider minimizers u of the functional $J(v) = \int_\Omega \{|\nabla v|^2 - f(x)v\}\, dx + \int_{\partial\Omega} j(v)\, d\sigma$ with $f \in C^1(\Omega)$ nonnegative and Steiner symmetric and $j \in C^1(I\!R)$ convex and even. Then u is Steiner symmetric, i.e. symmetrically decreasing in x_n.

Note that u satisfies the nonlinear boundary condition $-\partial u/\partial\nu = \partial j(u)$ on $\partial\Omega$.

Proof of Theorem 1: By strict convexity a minimizer u is unique, thus symmetric. Moreover it is nonnegative, because otherwise we can replace it by $|u|$. It remains to show that u is symmetrically decreasing in x_n or equivalently, that u decreases in x_n on $\Omega_> := \Omega \cap \{x_n > 0\}$. Clearly $u_{x_n} = 0$ on the hyperplane $\Omega \cap \{x_n = 0\}$ because of the symmetry, and $u_{x_n} \le 0$ on $\partial\Omega \cap \{x_n > 0\}$ due to the boundary condition, thus $w = u_{x_n} \le 0$ on $\partial\Omega_>$. But now we can apply the maximum principle to w. In fact u satisfies the Euler equation $-\Delta u = f$ in Ω. Therefore differentiation with respect to x_n yields $-\Delta w = f_{x_n} \le 0$ in $\Omega_>$. Thus $w < 0$ in $\Omega_>$ and u is Steiner symmetric.

Remark 3:

It seems to be difficult to prove Theorem 1 via Steiner symmetrization. It can be shown that the boundary integral in J decreases under symmetrization. The L^2-product of u and f increases under symmetrization due to the Hardy-Littlewood inequality in Remark 2. Thus the second term in J decreases. What about the Dirichlet integral? It can in fact increase under Steiner symmetrization, unless we know already that for a.e. x' the minimum of $u(x', x_n)$ is attained on the boundary points. Otherwise the obstruction from [33, p.56] causes problems. An example (due to my student Miroslaw Michaelowski) for which the Dirichlet integral increases under Steiner symmetrization is provided by the following function. Let $\Omega := (0,1] \times (-2,2) \cup \{(x,y)|\ x \in [1,2),\ |y| < 3 - x\}$ and $u(x,y) := 0$ if $|y| < 1$ and $u(x,y) := |y| - 1$ otherwise. Then a simple calculation shows that $\int_\Omega |\nabla u|^2 dx = 3$, while $\int_\Omega |\nabla u^*|^2 dx = 4$.

I wish to point out, however, that Steiner symmetrization can be sucessfully applied to functions which are periodic in x_n. This observation is used for instance in circular symmetrization, where a disk in $I\!R^2$ is interpreted as a rectangle in polar coordinates, by identifying $x' = r$ and $x_n = \phi$, see [33, Ch.II.9] or [21].

Second proof of Theorem 1: If we are willing to assume C^1-smoothness of f and $\partial\Omega$, we can argue that the function $w = u_{x_n}$ satisfies the equation $-\Delta w - f_{x_n}(x)w = 0$ in Ω and $x_n w \leq 0$ on $\partial\Omega$. Hence $w \leq 0$ in $\Omega \cap \{x_n > 0\}$ by the maximum principle.

Application 5:
Optimal insulation of bodies against heat loss was recently studied in [21]. Let Ω be the unit disk, and let u be a stationary temperature in Ω, subject to uniform heating, i.e. $-\Delta u = 1$ in Ω. On the boundary $\partial\Omega$ we have Newton's law of cooling $-(\partial u/\partial\nu) = hu$, where h represents conductivity. We want to maximize insulation in the sense that the average temperature should be maximized among admissible variations of h. Depending on the material which bounds $\partial\Omega$, the conductivity constant h is allowed to attain two values h_1 and h_2 with $0 < h_1 < h_2 < \infty$, but the proportion of $\partial\Omega$ on which $h = h_1$ is prescribed. Its location is free. Mathematically this leads to the problem of minimizing

$$J(v,h) := \int_\Omega \{|\nabla v|^2 - 2v\}\, dx + \int_{\partial\Omega} hv^2\, ds$$

over $v \in W^{1,2}(\Omega)$; $h : \partial\Omega \to \{h_1, h_2\}$ and $\int_{\partial\Omega} h\, ds = c \in (2\pi h_1, 2\pi h_2)$.

One can think of the subset $\partial\Omega_1$ of $\partial\Omega$ on which $h = h_1$ as a wall, and of $\partial\Omega_2 = \partial\Omega \setminus \partial\Omega_1$ as windows in a room. Then one of the the results in [21] states that having a single large window is preferable to having many small ones. This result was obtained by means of circular symmetrization. A similar problem was studied in [24], where the first eigenvalue was minimized under mixed Dirichlet-Neumann boundary conditions. These occur as limits $h_2 \to \infty$ and $h_1 \to 0$ in our formulation.

II. Continuous symmetrization

Until now my **message** has been: Global minimizers of variational problems are symmetric or monotone. An apparent advantage of symmetrization lies in the fact that often it requires less regularity than for the moving plane method. Disadvantages of symmetrization until now seemed to be a) that the method did not apply to nonvariational problems and b) that it did not yield results for local minimizers or critical points.

At least the second one of these difficulties has recently been overcome. **F.Brock** has been able to apply **continuous symmetrization** methods to local minimizers of variational problems and to critical points. For functions of a single variable this concept had been developed by H.Matano and myself over ten years ago [42, 34], and for functions of several variables with convex level sets it goes back about 50 years to G.Polya and G.Szegö [47, pp.201–204]. At the beginning of my lecture I presented an example of continuous symmetrization. Remember the function u_a from Figure 1? Consider a as a parameter for a family $\{u_a\}$ of equimeasurable functions and send a to zero. Then u_a tends to $u_0 = u^*$. It is easy to extend this approach to functions with convex level sets, and this was done in [47].

In continuous symmetrization a more general function u is continuously deformed into a family of equimeasurable functions u^t and, as $t \to \infty$, into u^*. To this end the level sets of u are shifted toward their final location with a speed that depends on their shape and their distance to the final location. The details of the definition are too long to be reproduced here; see [5, 6]. However, for u_a from Figure 1 above, Brock's u^t corresponds to our u_b with $b = a\, e^{-t}$. To give you a flavor of Brock's results I list one of them.

Theorem 2: (Brock [6, Theorem 9.1])
Let Ω be Steiner-symmetric in x_n, $p \in (1,\infty)$, $u \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ a nonnegative solution of

$$-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(x', x_n, u) \quad \text{in } \Omega,$$

and let f be symmetrically strictly decreasing in x_n and continuous. Then $u = u^*$.

The idea of the proof proceeds as follows. Test the differential equation with $u^t - u$. Then by the convexity of the function $s \mapsto s^p$ and since u_t is close to u for small t

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(u^t - u)dx &= \int_\Omega f(x,u)(u^t - u)dx \qquad (= o(t)) \\ &\le \frac{1}{p}\int_\Omega (|\nabla u^t|^p - |\nabla u|^p)\, dx \ \le\ 0\end{aligned}$$

Thus

$$\int_\Omega (|\nabla u^t|^p - |\nabla u|^p)\, dx = o(t)$$

and this implies via a delicate quantitative analysis that u is symmetric. To get a feeling for this qualitative analysis consider the function $g(a)$ defined in Remark 1 and note that $g(a+t) - g(a) = o(t)$ if and only if $a = 0$.

F.Brock has many other results, for instance on mean curvature type equations, on equations in $\mathbb{R}^n$ et cetera. Further applications of continuous symmetrization have recently emerged, namely,

• Applications to overdetermined boundary value problems. More precisely, if u solves $-\Delta_p u = f(u)$ in D, $u = 0$ and $\partial u/\partial\nu = c$ on ∂D, where $D \subset \mathbb{R}^n$ is simply connected and convex, and where f is continuous and nonincreasing, then D is a ball. This result in [9] is a considerable generalization of [26], where $f \equiv 1$.

Another variant is given in [19, Sect. 4], this time for solutions of $-\Delta u = \lambda_1 u$ under the same boundary conditions, and again assuming convexity of Ω. Here continuous Steiner symmetrization leads to the well-known conclusion that Ω is a ball.

• Results on the attainability of eigenvalues by D.Bucur, G.Buttazzo, I.Figueiredo [13]. The Krahn Faber inequality states that given the volume of Ω the range of $\lambda_1(\Omega)$ is a half-infinite interval. In [13] the authors look for the range of the vector $(\lambda_1(\Omega), \lambda_2(\Omega))$, given the volume of Ω, and prove via continuous rearrangement that it is a closed set E in $\mathbb{R}^2$ which is convex in both cartesian directions. This shows, in particular, that given $\lambda_1(\Omega)$ one can maximize or minimize $\lambda_2(\Omega)$ and vice versa.

• More general shape optimization problems require the convergence $W_0^{1,p}(\Omega^t) \to W_0^{1,p}(\Omega^s)$ as $t \to s^-$ in a suitable sense (of Mosco) in order to prove existence results. Such a convergence has been derived by D.Bucur and A.Henrot via continuous symmetrization in [14].

III. Symmetry without symmetrization

Let me now come to a somewhat different subject. There are domains with reflection symmetries which are not Steiner symmetric, for instance spherical annuli in $\mathbb{R}^n$, i.e. $\Omega = B_{R_2}(0) \setminus B_{R_1}(0)$ with $0 < R_1 < R_2 < \infty$. How about symmetry of solutions on those domains?

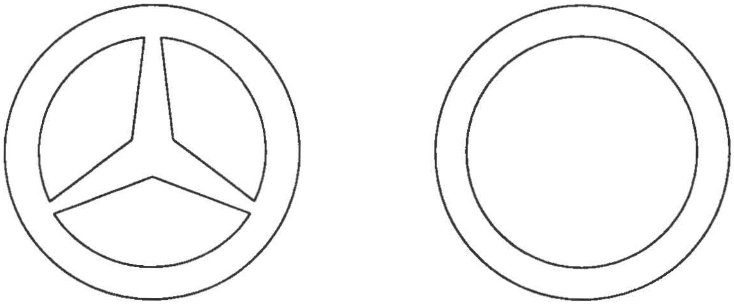

Figure 4: Domains without Steiner symmetry

One successful approach to proving symmetry is based on the **unique continuation principle.**

Theorem 3: (Lopez [41])
Let u minimize $J(v) = \int_\Omega \{|\nabla v|^2 + F(v)\}\,dx$ and let Ω be symmetric in x_1. Then u is symmetric in x_1.

For the proof cut Ω in half, into $\Omega_r = \Omega \cap \{x_1 > 0\}$ and its reflection Ω_l. Then the energy $J(u)$ is equidistributed,

$$\int_{\Omega_l} \{|\nabla u|^2 + F(u)\}\ dx = \int_{\Omega_r} \{|\nabla u|^2 + F(u)\}\ dx.$$

Otherwise, if say the left-hand side is smaller than the right-hand side, we could define a function

$$\tilde{u}(x_1, \dots x_n) = \begin{cases} u(x_1, \dots x_n) & \text{if } x \in \Omega_l, \\ u(-x_1, \dots x_n) & \text{if } x \in \Omega_r, \end{cases}$$

such that $J(\tilde{u}) < J(u)$; contradicting the minimality of u.

Therefore we see that if u minimizes J, so does $\tilde{u}$. But $u = \tilde{u}$ in Ω_l. Consequently $u \equiv \tilde{u}$ on all of Ω, provided the solutions of the associated Euler equation have the unique continuation property, i.e. if f is for instance Lipschitz continuous.

This trick was used by O.Lopez, and it works even for some elliptic systems such as the Lamé system with variable Lamé constants (see [1]), but it is not entirely new. M.E.Gurtin and H.Matano used it for instance in 1988 to prove monotonicity results in [30, p.308].

On **annuli** there can be symmetry breaking, but then the solutions still show some symmetry under certain group actions.

This effect was shown in [18, Sec.3], [22], [25, p.148 and 163], [33, pp.95–99], [38] and [40] for positive solutions of

$$\begin{aligned} -\Delta u &= f(u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and for suitable f with subcritical growth. In these papers prototypes for f are $f(u) = u^p - u$ or $f(u) = u^p$.

IV. Isoparametric surfaces

are level sets of solutions to the overdetermined system

$$-\Delta u = f(u) \quad \text{in } \Omega \quad \text{and} \quad |\nabla u| = g(u) > 0 \quad \text{in } \Omega. \tag{$*$}$$

The following result is well known among differential geometers, but maybe not so well known in the PDE community. It was found around 1920 in attempts to describe the propagation of wave fronts in geometrical optics.

Theorem 4:
If u solves system $(*)$ above in a connected domain $\Omega \subset I\!R^n$, if f is C^0 and g is C^1, then the level surfaces of u are either planes, spheres or cylinders, i.e. cartesian products of spheres and planes.

For a recent proof in the three-dimensional case, see [48]; for the general history of Theorem 4 and its generalizations to noneuclidean geometry I refer to [53]. Before giving a generalization of this result, let me demonstrate its usefulness.

Application 6:

In 1971 H.Weinberger published an elegant approach to a special overdetermined problem

$$-\Delta u = 1 \qquad \text{in } D \subset I\!R^n$$

$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = c \neq 0 \qquad \text{on } \partial D$$

for a bounded domain D, and showed that D is a ball. This way he recovered a special case of J.Serrin's seminal results [50]. His strategy of proof went through first showing

$$|\nabla u|^2 + \frac{2}{n} u = c^2 \quad \text{in } D$$

and to derive radial symmetry from this [55]. If we set $g(u) = \sqrt{c^2 - \frac{2}{n}u^2}$, and identify Ω with $D \setminus \{\text{set of critical points of } u\}$, we can apply Theorem 4 to see that D must be a ball.

It seems only natural that Weinberger's trick might extend to more general semilinear equations $-\Delta u = f(u)$, and that then $P(x) = |\nabla u|^2 + \frac{2}{n} F(u)$ should be constant. Let me dismiss any hope about this by pointing out that P is not even constant for the linear function $f(u) = \lambda_1(D)u$ and if D is a ball.

Application 7:

An open problem of DeGiorgi, that was already presented in H.Berestycki's lecture, reads precisely as follows: "Consider a solution $u \in C^2(I\!R^n)$ of $\Delta u = u^3 - u$ in $I\!R^n$ such that $|u| \leq 1$, $\frac{\partial u}{\partial x_n} > 0$ in the whole $I\!R^n$. Is it true that for every $\lambda \in I\!R$ the sets $\{u = \lambda\}$ are hyperplanes, at least if $n \leq 8$?"

In [36] I gave an explanation why the dimension $n \leq 8$ makes things easier. L.Caffarelli, N.Garofalo and F.Segala [16] and later X.Chen [20] showed that the auxiliary function $P(u, x) := |\nabla u(x)|^2 - 2F(u(x))$ is nonpositive in $I\!R^n$. Here $F(u) = (u^2 - 1)^2/4$. This tells us that the "kinetic energy" or Dirichlet integral in the corresponding functional is dominated by the "potential energy" term, except that both terms are infinite. Moreover, under the further assumption, that there is an $x_0 \in I\!R^n$ such that $P(u, x_0) = 0$ they were able to derive a Liouville-type result, namely $P \equiv 0$. Once this result is known one can see that P_ν, the partial derivative of u in direction $-\nabla u$ must vanish. This leads to the observation, see [36], that the $(n-1)$ dimensional level surfaces of u are minimal surfaces, and then a famous theorem of Bernstein applies, provided $n - 1 \leq 7$ or $n \leq 8$. Both in [16] and [20] the one dimensionality of u is derived for any n, provided P is constant. But in this case we can cut their proof short by observing that one can identify $g(u)$ with $\sqrt{2F(u)}$ and apply Theorem 4 to reach the conclusion. Incidentally, N.Ghoussoub and C.Gui have recently found another proof of DeGiorgi's conjecture without assuming anything about P, but unfortunately only under the restriction $n = 2$ or 3; see [27].

Application 8:

O.Mendez and W.Reichel looked recently at the overdetermined exterior boundary value problem

$$\begin{aligned} -\Delta u &= 0 && \text{in } \mathbb{R}^n \setminus D \\ u = c > 0 \ \text{ and } \ \frac{\partial u}{\partial \nu} &= -1 && \text{on } \partial D \end{aligned}$$

with D a bounded convex domain $D \subset \mathbb{R}^n$ and $n \geq 3$, and showed that D is a ball [43].

They showed first that the auxiliary function

$$P(x) = |\nabla u(x)|^2 \; u(x)^{(2-2n)/(n-2)}$$

equals a constant C in $\mathbb{R}^n \setminus D$. If we set $g(u) = \sqrt{C} u(x)^{(n-1)/(n-2)}$ and apply Theorem 4, we can recover their result.

After these applications you might bear with me, if I present another result in this spirit. It is not essential that $(*)$ contains the Laplace operator. To fix ideas let $\Omega \subset \mathbb{R}^n$ be a connected domain and suppose that $u : \Omega \to \mathbb{R}$ solves the following system of differential equations:

$$-\text{div} \; \big(\, a(u, |Du|^2) \; Du \, \big) = f(u, |Du|^2) \quad \text{in } \Omega, \tag{1}$$

$$|Du| = g(u) > 0 \qquad \text{in } \Omega. \tag{2}$$

Here a and g are assumed to be of class C^1 and $f \in C^0$. For $a \equiv 1$, i.e. if (1) has the special structure $-\Delta u = f(u)$, we recover Theorem 4. However, not even the ellipticity of the operator in (1) is needed to derive its conclusion.

Theorem 5:

Suppose that $a > 0$, then the level surfaces $\Omega_d(u) := \{ \, x \in \Omega \mid u(x) = d \, \}$ of solutions u to (1), (2) *are subsets of planes, cylinders or spheres.*

Proof: Without loss of generality we may assume that $g \equiv 1$, otherwise we substitute

$$\tilde{u}(x) := \int_0^{u(x)} \frac{1}{g(t)} \, dt$$

for u. If we denote differentiation with respect to x_j by subscript $_j$, and let p stand for $|\nabla u|^2$, we can rewrite (1) using Einstein's summation convention as

$$-a(u,p)\Delta u - 2a_p(u,p)u_i u_j u_{ij} - a_u(u,p)p = f(u,p). \tag{3}$$

Consequently, observing that $p = g \equiv 1$

$$-\Delta u = \frac{2a_p(u,1)u_i u_j u_{ij} + a_u(u,1) + f(u,1)}{a(u,1)}. \tag{4}$$

But the right-hand side of (4) looks almost as if it depends only on u. Let $\nu(x) = -\nabla u(x)/|\nabla u(x)|$ be the direction of steepest descent of u in x. Then the term $u_i u_j u_{ij}$ can be expressed as $u_\nu^2 u_{\nu\nu}$, but $g_\nu = (|\nabla u|^2)_\nu = 2u_\nu u_{\nu\nu} \equiv 0$. Therefore (4) degenerates to

$$-\Delta u = \tilde{f}(u) = [a_u(u,1) + f(u,1)]/a(u,1), \tag{5}$$

a variant of (1). But now the proof of Theorem 5 is reduced to Theorem 4.

Remark 3:
Let $H(x)$ denote the mean curvature of the level surface through x. One can then rewrite Δu as $u_{\nu\nu} + (n-1)H(x)u_\nu$ and (3) as (see for example [35])

$$-(n-1)\ a(u,p)\ H(x)\ u_\nu - [a(u,p) + 2pa_p(u,p)]\, u_{\nu\nu} = a_u(u,p)p + f(u,p),$$

and, using $u_{\nu\nu} \equiv 0$, arrive at

$$H(x) = \frac{a_u(u,1) + f(u,1)}{(n-1)a(u,1)} =: h(u) \qquad \text{in } \Omega,$$

i.e. at the fact that the mean curvature of each level surface is constant and the constant depends only on the level u. Other equations that u satisfies, once (1) (respectively, (5)) and $u_\nu = -1$ hold, are

$$\Delta_q u = \operatorname{div}(|\nabla u|^{q-2}\nabla u) = |\nabla u|^{q-2}[\Delta u + (q-2)u_{\nu\nu}] = \Delta u = \tilde{f}(u)$$

for any $q \in (1,\infty)$, or

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \frac{\tilde{f}(u)}{\sqrt{2}} \qquad \text{in } \Omega,$$

i.e., the mean curvature of the graph of u depends only on its level u. (Incidentally, if we know in addition that the level surfaces of u are compact, a famous result of Alexandrov implies that they are spheres.) This shows how overdetermined the system (1), (2) is.

Remark 4:
Let me conclude with an open problem. In [2] the authors look for a variant of Serrin's result, i.e. for special symmetries of solutions to overdetermined boundary value problems on unbounded domains, and state: "It is tempting to conjecture that a much more general result is true. Assuming that Ω is a smooth domain with Ω^c connected and that there exists a positive bounded solution of"

$$\begin{aligned} \Delta u + f(u) = 0 \qquad &\text{and} \qquad u > 0 \qquad &&\text{in } \Omega, \\ u = 0 \qquad &\text{and} \qquad \frac{\partial u}{\partial n} = c \neq 0 \quad &&\text{on } \partial\Omega, \end{aligned}$$

"for some Lipschitz function f, then Ω is either a half-space, a ball, the complement of a ball or a circular-cylinder type of domain: $\mathbb{R}^j \times B$ with B a ball."

But such a problem seems to be amenable via a P-function, and maybe one can solve it by a combination of results in [51] with Theorem 4 above.

This concludes my little survey. I hope to have demonstrated that there is more than one way to prove symmetry. I learned Liouville theorems in Prague during my time in 1979 as a postdoctoral student of J.Nečas. Therefore I feel entitled to say: Universitas Carolina vivat crescat et floreat!

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A Bubble–Type Stabilization of the Q_1/Q_1–Element for Incompressible Flows

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Abstract. Starting with the unstable finite Q_1/Q_1–element a general class of bubble-type spaces V_h^2 is constructed and used for enlarging the space V_h^1 approximating the velocity. It is shown that the new pair of spaces satisfies the Babuška–Brezzi condition. Using the techniques of reduced discretizations and static condensation some new and old stabilized schemes are obtained.

Keywords: Babuška–Brezzi condition, stabilization, Stokes problem

Classification: 65N30

1. Introduction

In this paper we introduce a general class of stable finite element spaces suitable for a numerical solution of the Stokes equations

$$-\nu\,\Delta\,\boldsymbol{u} + \nabla p = \boldsymbol{f}\,, \qquad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega\,, \qquad \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega\,. \tag{1}$$

Here, $\boldsymbol{u}$ is the velocity and p is the pressure in a linear viscous fluid contained in a bounded domain $\Omega \subset \mathbf{R}^2$ with a polygonal boundary $\partial\Omega$. The parameter $\nu > 0$ is the kinematic viscosity and $\boldsymbol{f}$ is an external body force, e.g. the gravity. Denoting

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x\,, \qquad b(\boldsymbol{v},p) = -\int_\Omega p\operatorname{div}\boldsymbol{v}\,\mathrm{d}x\,,$$

the usual weak formulation of (1) reads: Given $\nu > 0$ and $\boldsymbol{f} \in H^{-1}(\Omega)^2$, find $\boldsymbol{u} \in H_0^1(\Omega)^2$ and $p \in L_0^2(\Omega)$ such that

$$\nu\,a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) - b(\boldsymbol{u},q) = \langle \boldsymbol{f},\boldsymbol{v}\rangle \qquad \forall\,\boldsymbol{v} \in H_0^1(\Omega)^2,\ q \in L_0^2(\Omega)\,, \tag{2}$$

where $L_0^2(\Omega)$ consists of $L^2(\Omega)$ functions having zero mean value on Ω. It can be shown that this problem has a unique solution (cf. [9], p. 80, Theorem 5.1).

A conforming finite element discretization of (2) reads: Find $\boldsymbol{u}_h \in \mathrm{V}_h$ and $p_h \in \mathrm{Q}_h$ satisfying

$$\nu\,a(\boldsymbol{u}_h,\boldsymbol{v}_h) + b(\boldsymbol{v}_h,p_h) - b(\boldsymbol{u}_h,q_h) = \langle \boldsymbol{f},\boldsymbol{v}_h\rangle \qquad \forall\,\boldsymbol{v}_h \in \mathrm{V}_h,\ q_h \in \mathrm{Q}_h\,, \tag{3}$$

where $\mathrm{V}_h \subset H_0^1(\Omega)^2$ and $\mathrm{Q}_h \subset L_0^2(\Omega)$ are finite element spaces based on a triangulation $\mathcal{T}_h$ of Ω. In this paper, we shall consider only triangulations consisting of rectangles T

(cf. Section 2), thus, we implicitly assume that Ω can be decomposed into rectangles. We are interested in equal order spaces

$$\begin{aligned}\mathrm{V}_h^1 &= \{v \in H_0^1(\Omega)^2;\ v|_T \in Q_1(T)^2 \ \ \forall\, T \in \mathcal{T}_h\}\,,\\ \mathrm{Q}_h &= \{q \in H^1(\Omega) \cap L_0^2(\Omega);\ q|_T \in Q_1(T) \ \ \forall\, T \in \mathcal{T}_h\}\end{aligned}$$

for approximating the velocity and the pressure, respectively. However, this pair of spaces does not satisfy the Babuška–Brezzi condition

$$\exists\, \beta > 0 : \qquad \sup_{v_h \in \mathrm{V}_h^1 \setminus \{0\}} \frac{b(v_h, q_h)}{|v_h|_{1,\Omega}} \geq \beta\, \|q_h\|_{0,\Omega} \quad \forall\, q_h \in \mathrm{Q}_h,\ h > 0\,,$$

which often causes that the problem (3) with $\mathrm{V}_h = \mathrm{V}_h^1$ is not solvable or that its solution contains spurious oscillations. One way to suppress these oscillations and to assure the solvability is to add some extra terms to the discretization (3) (cf. e.g. [6], [10], [5]). Another way is to enlarge the space V_h^1 by a space V_h^2 so that the Babuška–Brezzi condition is satisfied. Here we start with the second possibility and construct a general class of spaces V_h^2 assuring the fulfillment of the Babuška–Brezzi condition. Then, by using the concept of reduced discretizations [11] and static condensation, we shall show that, for suitable spaces V_h^2, the V_h^1–component of u_h and the function p_h are solutions of the stabilized methods of [6] and [10].

In case of the mini element [1], which is defined by enriching continuous piecewise linear functions by cubic bubble functions, the close relation between stabilized methods and Galerkin methods with bubble functions was already established in [13] and [3]. In an abstract framework, this equivalence was investigated for linear problems in [2]. It was also realized that bubble functions can help to design new stabilized methods (cf. e.g. [7], [8]).

There are a lot of further papers devoted to investigations of discretizations stabilized using bubble functions, but most of them are restricted to *triangular* elements and to *linear* problems. In this paper, we deal with quadrilateral elements and, in addition, we consider more general functions than the standard bubble functions used before. Apart from investigating the relations to some well-known stabilized methods, we shall also derive, eliminating a suitable space V_h^2 from the discretization, a new type of stabilization which can also be applied to the Navier–Stokes equations [12].

The space V_h^2 added to V_h^1 to satisfy the Babuška–Brezzi condition will be defined in a general way as

$$\mathrm{V}_h^2 = \mathrm{span}\{\varphi^i\, t^i\}_{i=1}^{N_h}\,, \tag{4}$$

where $\varphi^i \in H_0^1(\Omega)$ and $t^i \in \mathbf{R}^2$ are some suitable functions and vectors, respectively. The proof of the Babuška–Brezzi condition for the spaces $\mathrm{V}_h \equiv \mathrm{V}_h^1 \oplus \mathrm{V}_h^2$ and Q_h, which uses some ideas of [4] and a modification of the Verfürth trick [14], requires that the functions φ^i have localized supports and that, for any φ^i, there exists a point $A^i \in \overline{\Omega}$

such that

$$\int_\Omega \frac{\partial q_h}{\partial t^i}\,\varphi^i\,\mathrm{d}x = \frac{\partial q_h}{\partial t^i}(A^i)\int_\Omega \varphi^i\,\mathrm{d}x \qquad \forall\, q_h \in \mathrm{Q}_h\,,\ i = 1,\dots,N_h\,, \tag{5}$$

$$|q_h|^2_{1,T} \le C\,h_T^2 \sum_{\substack{i=1,\\ A^i\in T}}^{N_h} \left|\frac{\partial q_h}{\partial t^i}(A^i)\right|^2 \qquad \forall\, q_h \in \mathrm{Q}_h,\ T\in\mathcal{T}_h\,. \tag{6}$$

The plan of the paper is as follows. In Section 2, we introduce some notations and summarize the assumptions on the triangulations and the functions φ^i needed for proving the Babuška–Brezzi condition in Section 3. In Section 4, we give examples for the functions φ^i defining the supplement spaces V_h^2. In Section 5, we investigate discretizations obtained from (3) by eliminating the V_h^2–component of $\boldsymbol{u}_h$. In particular, we show the equivalence to the stabilized methods of [6] and [10] and establish a new type of stabilization.

2. Assumptions and notations

We assume that we are given a family $\{\mathcal{T}_h\}_{h>0}$ of triangulations of Ω consisting of closed rectangular elements T having the usual compatibility properties. Let $h_T \equiv \operatorname{diam}(T) \le h$ for any $T \in \mathcal{T}_h$ and let the ratio of the longest and the shortest edge of each T satisfy

$$\frac{h_T^{max}}{h_T^{min}} \le C_1$$

with some constant C_1 independent of h.

Before introducing the functions φ^i used for defining V_h^2 in (4), we define the points A^i mentioned in the introduction. Thus, for any triangulation $\mathcal{T}_h$, we define a set $\{A^i\}_{i=1}^{N_h}$ of points $A^i \in \overline{\Omega}$ different from the vertices of $\mathcal{T}_h$ and we assume that

$$\operatorname{card}\{A^i;\ A^i \in T\} \le C_2 \qquad \forall\, T \in \mathcal{T}_h\,,$$

where the constant C_2 is independent of h. In order to be able to use the same function φ in (4) twice (each time with another $\boldsymbol{t}^i$), we admit $A^i = A^j$ for $i \ne j$. For any A^i, we denote by P^i the union of the elements containing A^i, i.e., P^i consists of one or two elements. Further, for any A^i, we introduce a unit vector $\boldsymbol{t}^i \in \mathbf{R}^2$ and if A^i belongs to an edge of some $T \in \mathcal{T}_h$, we require that $\boldsymbol{t}^i$ coincides with the direction of this edge. We choose a fixed Cartesian coordinate system with axes parallel to the edges of the triangulation and, for any $\boldsymbol{t}^i = (t_1^i, t_2^i)$, we define $\boldsymbol{n}^i = (t_2^i, t_1^i)$. For getting the property (6), we assume that, for any $T \in \mathcal{T}_h$, there exist indices $i, j, k \in \{1,\dots,N_h\}$ such that A^i, A^j, $A^k \in T$ and

$$S_h^{ijk} = (\boldsymbol{n}^i \times \boldsymbol{n}^j)\,\boldsymbol{n}^k \cdot A^k + (\boldsymbol{n}^j \times \boldsymbol{n}^k)\,\boldsymbol{n}^i \cdot A^i + (\boldsymbol{n}^k \times \boldsymbol{n}^i)\,\boldsymbol{n}^j \cdot A^j$$

satisfies

$$|S_h^{ijk}| \ge C_3\,h_T\,, \tag{7}$$

where the constant C_3 is independent of T and h, the vector product $\boldsymbol{a}\times\boldsymbol{b}$ is defined as $a_1\,b_2 - b_1\,a_2$ and $\boldsymbol{n}^i \cdot A^i$ is defined as $\boldsymbol{n}^i \cdot (A^i - \boldsymbol{0})$.

Remark 1. If, particularly, $\boldsymbol{n}^i = \boldsymbol{n}^j \perp \boldsymbol{n}^k$, then

$$|S_h^{ijk}| = |\boldsymbol{n}^i \cdot (A^i - A^j)|\,,$$

which illustrates the meaning of (7).

Finally, for each A^i, we introduce a function $\varphi^i \in H_0^1(\Omega)$ such that

$$\operatorname{supp}\varphi^i \subset P^i\,, \tag{8}$$

$$\boldsymbol{n}^i \cdot \int_T (x - A^i)\,\varphi^i(x)\,\mathrm{d}x = 0 \qquad \forall\, T \subset P^i\,, \tag{9}$$

$$\left|\int_\Omega \varphi^i\,\mathrm{d}x\right| \geq C_4\,|P^i|\,, \tag{10}$$

$$|\varphi^i|_{1,\Omega} \leq C_5\,, \tag{11}$$

where the constants C_4, $C_5 > 0$ are independent of h. We shall see (cf. Lemma 1 in the next section) that from (9) the property (5) can be concluded. The functions $\{\varphi^i\,\boldsymbol{t}^i\}_{i=1}^{N_h}$ are assumed to be linearly independent.

3. Babuška–Brezzi condition

In this section, we prove that, under the assumptions made in Section 2, the spaces $\mathrm{V}_h \equiv \mathrm{V}_h^1 \oplus \mathrm{V}_h^2$ and Q_h satisfy the Babuška–Brezzi condition with a constant independent of h. First, in Lemmas 1 and 2, we prove the validity of (5) and (6). Then, in Lemma 3, we establish a Babuška–Brezzi condition with a "wrong" norm of q_h and, finally, in Theorem 1, we prove the desired Babuška–Brezzi condition applying the modified Verfürth trick.

Lemma 1. *We have*

$$\int_\Omega \frac{\partial q_h}{\partial \boldsymbol{t}^i}\,\varphi^i\,\mathrm{d}x = \frac{\partial q_h}{\partial \boldsymbol{t}^i}(A^i)\int_\Omega \varphi^i\,\mathrm{d}x \qquad \forall\, q_h \in \mathrm{Q}_h\,,\ i = 1,\ldots,N_h\,.$$

Proof. Consider any $T \subset P^i$ and let $q_h(x) = \xi_0 + \xi_1\,x_1 + \xi_2\,x_2 + \xi_3\,x_1\,x_2$ for any $x \in T$, where $\xi_0,\ldots,\xi_3$ are some real numbers. Denoting $A^i = (a_1, a_2)$, we have for $x \in T$

$$\frac{\partial q_h}{\partial x_1}(x) = \frac{\partial q_h}{\partial x_1}(A^i) + \xi_3\,(x_2 - a_2)\,, \qquad \frac{\partial q_h}{\partial x_2}(x) = \frac{\partial q_h}{\partial x_2}(A^i) + \xi_3\,(x_1 - a_1)\,.$$

Hence

$$\frac{\partial q_h}{\partial \boldsymbol{t}^i}(x) = \frac{\partial q_h}{\partial \boldsymbol{t}^i}(A^i) + \xi_3\,\boldsymbol{n}^i \cdot (x - A^i)$$

and the lemma follows using (9). □

Lemma 2. *For any $T \in \mathcal{T}_h$, we have*

$$|q_h|_{1,T}^2 \leq \frac{54}{C_3^2}\,|T| \sum_{\substack{i=1,\\ A^i \in T}}^{N_h} \left|\frac{\partial q_h}{\partial \boldsymbol{t}^i}(A^i)\right|^2 \qquad \forall\, q_h \in \mathrm{Q}_h\,. \tag{12}$$

Proof. Consider any $T \in \mathcal{T}_h$ and let A^1, A^2, $A^3 \in T$ be points satisfying (7). Let us set for $x \in T$ and $i = 1, 2, 3$

$$q^i(x) = \{n^{i+1}(n^{i-1} \cdot A^{i-1}) - n^{i-1}(n^{i+1} \cdot A^{i+1})\} \cdot (-x_1, x_2) + (n^{i+1} \times n^{i-1})\, x_1 x_2\,,$$

with the convention that $i-1 \equiv 3$ for $i = 1$ and $i+1 \equiv 1$ for $i = 3$. An easy calculation shows that any $q_h \in \mathrm{Q}_h$ satisfies

$$q_h(x) = \xi_0 + \frac{1}{S_h^{123}} \sum_{i=1}^{3} \frac{\partial q_h}{\partial t^i}(A^i)\, q^i(x) \qquad \forall\, x \in T\,,$$

where ξ_0 is a real number. It can be shown that $\left|\dfrac{\partial q^i}{\partial x_j}(x)\right| \leq 3\, h_T$, $j = 1, 2$, and hence we obtain (12). □

Lemma 3. *We have*

$$\sup_{v_h \in \mathrm{V}_h^2 \setminus \{0\}} \frac{b(v_h, q_h)}{|v_h|_{1,\Omega}} \geq \frac{C_3 C_4}{C_5 \sqrt{108\, C_1 C_2}} \sqrt{\sum_{T \in \mathcal{T}_h} h_T^2\, |q_h|_{1,T}^2} \qquad \forall\, q_h \in \mathrm{Q}_h\,. \tag{13}$$

Proof. Consider any $v_h \in \mathrm{V}_h^2$. Then $v_h = \sum_{i=1}^{N_h} \alpha^i \varphi^i t^i$ and we have

$$|v_h|_{1,\Omega}^2 = \sum_{T \in \mathcal{T}_h} |v_h|_{1,T}^2 \leq C_2 \sum_{T \in \mathcal{T}_h} \sum_{\substack{i=1,\\ A^i \in T}}^{N_h} |\alpha^i|^2\, |\varphi^i|_{1,T}^2 \leq C_2 C_5^2 \sum_{i=1}^{N_h} |\alpha^i|^2\,. \tag{14}$$

Applying Lemma 1, we obtain

$$b(v_h, q_h) = \int_\Omega v_h \cdot \nabla q_h \, \mathrm{d}x = \sum_{i=1}^{N_h} \alpha^i \int_\Omega \frac{\partial q_h}{\partial t^i} \varphi^i \, \mathrm{d}x = \sum_{i=1}^{N_h} \alpha^i \frac{\partial q_h}{\partial t^i}(A^i) \int_\Omega \varphi^i \, \mathrm{d}x\,.$$

Setting

$$\alpha^i = \frac{\partial q_h}{\partial t^i}(A^i) \int_\Omega \varphi^i \, \mathrm{d}x\,,$$

it follows that

$$b(v_h, q_h) = \sum_{i=1}^{N_h} |\alpha^i|^2 = \left[\sum_{i=1}^{N_h} |\alpha^i|^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^{N_h} \left|\frac{\partial q_h}{\partial t^i}(A^i)\right|^2 \left|\int_\Omega \varphi^i \, \mathrm{d}x\right|^2\right]^{\frac{1}{2}},$$

which implies (13) owing to (10), (12) and (14). □

Theorem 1. *There exists a constant $C_6 > 0$ independent of h such that the spaces $\mathrm{V}_h = \mathrm{V}_h^1 \oplus \mathrm{V}_h^2$ and Q_h satisfy the Babuška–Brezzi condition*

$$\sup_{v_h \in \mathrm{V}_h \setminus \{0\}} \frac{b(v_h, q_h)}{|v_h|_{1,\Omega}} \geq C_6\, \|q_h\|_{0,\Omega} \qquad \forall\, q_h \in \mathrm{Q}_h\,. \tag{15}$$

Proof. Applying the modified Verfürth trick presented in [5], pp. 255–256, we obtain

$$\sup_{v_h \in \mathrm{V}_h \setminus \{0\}} \frac{b(v_h, q_h)}{|v_h|_{1,\Omega}} \geq C\, \|q_h\|_{0,\Omega} - \tilde{C} \sqrt{\sum_{T \in \mathcal{T}_h} h_T^2\, |q_h|_{1,T}^2} \qquad \forall\, q_h \in \mathrm{Q}_h$$

with some constant C, $\widetilde{C} > 0$ independent of h and (15) follows from Lemma 3. $\square$

4. Examples of the finite element space V_h^2

In this section we give examples of functions φ^i satisfying the assumptions (8)–(11). Denoting by $\widehat{T} = [0,1]^2$ the reference element, we first introduce functions $\widehat{\varphi}_1 \in Q_2(\widehat{T}) \cap H_0^1(\widehat{T})$, $\widehat{\varphi}_2 \in Q_2(\widehat{T})$ and $\widehat{\varphi}_3 \in Q_3(\widehat{T}) \cap H_0^1(\widehat{T})$ defined for $\widehat{x} \in \widehat{T}$ by

$$\widehat{\varphi}_1(\widehat{x}) = \widehat{x}_1\,(1-\widehat{x}_1)\,\widehat{x}_2\,(1-\widehat{x}_2)\,, \qquad \widehat{\varphi}_2(\widehat{x}) = \widehat{x}_1\,(1-\widehat{x}_1)\,(1-\widehat{x}_2)\,(\tfrac{1}{2}-\widehat{x}_2)\,,$$
$$\widehat{\varphi}_3(\widehat{x}) = (3-5\,\widehat{x}_2)\,\widehat{\varphi}_1(\widehat{x})\,.$$

It is easy to verify that

$$\int_{\widehat{T}} (\widehat{x} - C_{\widehat{T}})\,\widehat{\varphi}_1(\widehat{x})\,\mathrm{d}\widehat{x} = 0\,, \qquad \int_{\widehat{T}} \widehat{x}_2\,\widehat{\varphi}_2(\widehat{x})\,\mathrm{d}\widehat{x} = 0\,, \qquad \int_{\widehat{T}} \widehat{x}_2\,\widehat{\varphi}_3(\widehat{x})\,\mathrm{d}\widehat{x} = 0\,, \tag{16}$$

where $C_{\widehat{T}} = (\frac{1}{2}, \frac{1}{2})$ is the barycentre of $\widehat{T}$.

We denote by $\mathcal{E}_h$ the set of all edges E of the triangulation $\mathcal{T}_h$, by S_E the midpoint of each edge E, by P_E the union of the elements possessing the edge E, by $\boldsymbol{t}_E$ a unit vector in the direction of E, by $\boldsymbol{n}_E$ the vector (t_{E2}, t_{E1}) and by C_T the barycentre of each element T. For any edge E of an element T, we define a rectangle R_T^E as depicted in Figure 1 and we denote by R_E the union of the rectangles R_T^E possessing the edge E. Further, for any element T, we define a rectangle R_T (cf. Figure 1) and rectangles R_T^1, R_T^2, R_T^3, R_T^4 with barycentres A_T^1, A_T^2, A_T^3, A_T^4, respectively (cf. Figure 2). These points are defined as the images of the points

$$(\tfrac{3-\sqrt{3}}{6}, \tfrac{1}{2})\,, \qquad (\tfrac{3+\sqrt{3}}{6}, \tfrac{1}{2})\,, \qquad (\tfrac{1}{2}, \tfrac{3-\sqrt{3}}{6})\,, \qquad (\tfrac{1}{2}, \tfrac{3+\sqrt{3}}{6})$$

in the reference element $\widehat{T}$. That assures that

$$\int_T g\,\mathrm{d}x = \frac{|T|}{2}\,[g(A_T^1) + g(A_T^2)] \quad \text{for} \quad g(x) = (\xi_0 + \xi_1\,x_1)\,q(x)\,, \tag{17}$$
$$\int_T h\,\mathrm{d}x = \frac{|T|}{2}\,[h(A_T^3) + h(A_T^4)] \quad \text{for} \quad h(x) = (\eta_0 + \eta_1\,x_2)\,q(x)\,, \tag{18}$$

where $q \in Q_1(T)$ and ξ_0, ξ_1, η_0, η_1 are arbitrary real numbers. The dimensions of the rectangles R_T^i are $\frac{3-\sqrt{3}}{6}\,a \times \frac{3-\sqrt{3}}{6}\,b$.

For any $T \in \mathcal{T}_h$, we define functions φ_T^1, φ_T^2, ψ_T^1, ψ_T^2, ψ_T^3, $\psi_T^4 \in H_0^1(\Omega)$ by

$$\varphi_T^1|_T = \widehat{\varphi}_1 \circ F^{-1} \quad \text{with} \quad F(\widehat{T}) = T\,, \qquad \varphi_T^1|_{\Omega\setminus T} = 0\,,$$
$$\varphi_T^2|_{R_T} = \widehat{\varphi}_1 \circ F^{-1} \quad \text{with} \quad F(\widehat{T}) = R_T\,, \qquad \varphi_T^2|_{\Omega\setminus R_T} = 0\,,$$
$$\psi_T^i|_{R_T^i} = \widehat{\varphi}_1 \circ F^{-1} \quad \text{with} \quad F(\widehat{T}) = R_T^i\,, \qquad \psi_T^i|_{\Omega\setminus R_T^i} = 0\,, \qquad i = 1,2,3,4\,,$$

where F always is a regular affine mapping. Note that

$$\int_\Omega \psi_T^i\,\mathrm{d}x = \int_\Omega \psi_T^1\,\mathrm{d}x\,, \qquad |\psi_T^i|_{1,\Omega} = |\psi_T^1|_{1,\Omega} \qquad \forall\, T \in \mathcal{T}_h,\ i = 1,2,3,4\,. \tag{19}$$

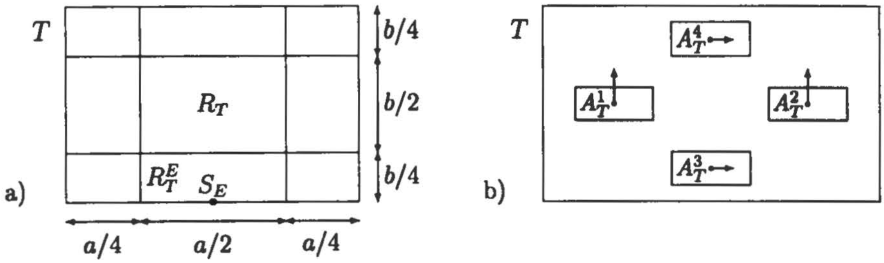

FIGURE 1. Subdivisions of an element T

For edges $E \in \mathcal{E}_h$, we define functions φ_E^1, $\varphi_E^2 \in H_0^1(\Omega)$ satisfying $\varphi_E^1|_{\Omega\setminus P_E} = 0$, $\varphi_E^2|_{\Omega\setminus R_E} = 0$ and

$$\begin{array}{llll} \varphi_E^1|_T = \widehat{\varphi}_2 \circ F^{-1} & \text{with } F(\widehat{T}) = T & \forall\, T \subset P_E & \text{if } E \not\subset \partial\Omega, \\ \varphi_E^2|_{R_T^E} = \widehat{\varphi}_2 \circ F^{-1} & \text{with } F(\widehat{T}) = R_T^E & \forall\, T \subset P_E & \text{if } E \not\subset \partial\Omega, \\ \varphi_E^2|_{R_T^E} = \widehat{\varphi}_3 \circ F^{-1} & \text{with } F(\widehat{T}) = R_T^E & \forall\, T \subset P_E & \text{if } E \subset \partial\Omega, \end{array}$$

where F always is a reagular affine mapping satisfying $F(\frac{1}{2},0) = S_E$.

It follows from (16) that, for any $T \in \mathcal{T}_h$ and any $E \in \mathcal{E}_h$ (in case of φ_E^1 only for $E \not\subset \partial\Omega$), we have

$$\int_T (x - C_T)\, \varphi_T^i(x)\, \mathrm{d}x = 0\,, \quad n_E \cdot \int_T (x - S_E)\, \varphi_E^i(x)\, \mathrm{d}x = 0\,, \qquad i = 1,2\,,$$
$$\int_T (x - A_T^i)\, \psi_T^i(x)\, \mathrm{d}x = 0\,, \qquad i = 1,2,3,4\,.$$

Clearly, all the functions also satisfy (10) and (11). Thus, setting

$$\begin{array}{lll} \mathrm{W}_h^1 = \mathrm{span}\{\varphi_E^1\, t_E\}_{E\not\subset\partial\Omega}\,, & \mathrm{W}_h^2 = \mathrm{span}\{\varphi_T^1\, e^1\}_{T\in\mathcal{T}_h}\,, & \mathrm{W}_h^3 = \mathrm{span}\{\varphi_T^1\, e^2\}_{T\in\mathcal{T}_h}\,, \\ \mathrm{W}_h^4 = \mathrm{span}\{\varphi_E^2\, t_E\}_{E\in\mathcal{E}_h}\,, & \mathrm{W}_h^5 = \mathrm{span}\{\varphi_T^2\, e^1\}_{T\in\mathcal{T}_h}\,, & \mathrm{W}_h^6 = \mathrm{span}\{\varphi_T^2\, e^2\}_{T\in\mathcal{T}_h}\,, \\ \mathrm{W}_h^7 = \mathrm{span}\{\psi_T^1\, e^2,\, \psi_T^2\, e^2,\, \psi_T^3\, e^1,\, \psi_T^4\, e^1\}_{T\in\mathcal{T}_h}\,, & & \end{array}$$

where $e^1 = (1,0)$ and $e^2 = (0,1)$, we can define V_h^2 e.g. as one of the following spaces

$$\mathrm{V}_h^{2,1} = \mathrm{W}_h^1 \oplus \mathrm{W}_h^2 \oplus \mathrm{W}_h^3\,, \quad \mathrm{V}_h^{2,2} = \mathrm{W}_h^4\,, \quad \mathrm{V}_h^{2,3} = \mathrm{W}_h^4 \oplus \mathrm{W}_h^5 \oplus \mathrm{W}_h^6\,, \quad \mathrm{V}_h^{2,4} = \mathrm{W}_h^7\,.$$

For all these spaces, the assumption (7) is satisfied (cf. Remark 1) so that the Babuška–Brezzi condition holds. Since $\mathrm{V}_h^{2,1}$ consists only of piecewise biquadratic functions, we particularly infer that the Babuška–Brezzi condition is satisfied for the pair V_h, Q_h with V_h consisting of piecewise biquadratic functions from $H_0^1(\Omega)^2$. Note that the functions from $\mathrm{W}_h^2 \oplus \mathrm{W}_h^3$ are needed for the validity of the Babuška–Brezzi condition for $\mathrm{V}_h^1 \oplus \mathrm{V}_h^{2,1}$, Q_h only on those elements which have two or three edges on $\partial\Omega$.

5. Stabilized methods for the Stokes problem

In this section, we investigate the relation between modified Galerkin methods with bubble functions and stabilized methods for the Stokes problem (1). Following [11], we consider a reduced discretization of the Stokes equations given by: Find $\boldsymbol{u}_h^1 \in \mathrm{V}_h^1$, $\boldsymbol{u}_h^2 \in \mathrm{V}_h^2$ and $p_h \in \mathrm{Q}_h$ satisfying

$$\begin{aligned}
\nu\, a(\boldsymbol{u}_h^1, \boldsymbol{v}_h^1) \qquad\qquad\quad + b(\boldsymbol{v}_h^1, p_h) &= \langle \boldsymbol{f}, \boldsymbol{v}_h^1 \rangle \quad && \forall\, \boldsymbol{v}_h^1 \in \mathrm{V}_h^1\,, && (20)\\
\nu\, a(\boldsymbol{u}_h^2, \boldsymbol{v}_h^2) + b(\boldsymbol{v}_h^2, p_h) &= 0 && \forall\, \boldsymbol{v}_h^2 \in \mathrm{V}_h^2\,, && (21)\\
b(\boldsymbol{u}_h^1, q_h) + b(\boldsymbol{u}_h^2, q_h) \qquad\qquad\quad &= 0 && \forall\, q_h \in \mathrm{Q}_h\,, && (22)
\end{aligned}$$

the solution of which has, for $\boldsymbol{f} \in L^2(\Omega)^2$, the same convergence properties as the solution of the original problem (3).

We assume that the basis functions of the space V_h^2 satisfy $a(\varphi^i\, \boldsymbol{t}^i, \varphi^j\, \boldsymbol{t}^j) = 0$ for $i \neq j$, which holds e.g. for the spaces $\mathrm{V}_h^{2,2}$, $\mathrm{V}_h^{2,3}$ and $\mathrm{V}_h^{2,4}$. Using the basis representation $\boldsymbol{u}_h^2 = \sum_{j=1}^{N_h} \alpha^j\, \varphi^j\, \boldsymbol{t}^j$ and setting $\boldsymbol{v}_h^2 = \varphi^i\, \boldsymbol{t}^i$ in (21), we get

$$\alpha^i\, \nu\, a(\varphi^i\, \boldsymbol{t}^i, \varphi^i\, \boldsymbol{t}^i) + b(\varphi^i \boldsymbol{t}^i, p_h) = 0\,.$$

Eliminating $\boldsymbol{u}_h^2$ from (20)–(22) and applying Lemma 1, we obtain a stabilized Q_1/Q_1 discretization in the form: Find $\boldsymbol{u}_h^1 \in \mathrm{V}_h^1$ and $p_h \in \mathrm{Q}_h$ satisfying

$$\begin{aligned}
\nu\, a(\boldsymbol{u}_h^1, \boldsymbol{v}_h^1) + b(\boldsymbol{v}_h^1, p_h) &= \langle \boldsymbol{f}, \boldsymbol{v}_h^1 \rangle \quad && \forall\, \boldsymbol{v}_h^1 \in \mathrm{V}_h^1\,, && (23)\\
b(\boldsymbol{u}_h^1, q_h) - c_h(p_h, q_h) &= 0 && \forall\, q_h \in \mathrm{Q}_h\,, && (24)
\end{aligned}$$

where the stabilizing term is given by

$$c_h(p_h, q_h) = \sum_{i=1}^{N_h} \frac{\partial p_h}{\partial \boldsymbol{t}^i}(A^i)\, \frac{\partial q_h}{\partial \boldsymbol{t}^i}(A^i)\, \frac{\left|\int_\Omega \varphi^i\, \mathrm{d}x\right|^2}{\nu\, |\varphi^i|_{1,\Omega}^2}\,.$$

For the spaces $\mathrm{V}_h^{2,2}$, $\mathrm{V}_h^{2,3}$ and $\mathrm{V}_h^{2,4}$ defined in the preceding section, the stabilizing term can be respectively written as

$$c_h^2(p_h, q_h) = \sum_{E \in \mathcal{E}_h} \frac{\delta_E\, h_E^4}{\nu}\, \frac{\partial p_h}{\partial \boldsymbol{t}_E}(S_E)\, \frac{\partial q_h}{\partial \boldsymbol{t}_E}(S_E)\,, \tag{25}$$

$$c_h^3(p_h, q_h) = \sum_{E \in \mathcal{E}_h} \frac{\delta_E\, h_E^4}{\nu}\, \frac{\partial p_h}{\partial \boldsymbol{t}_E}(S_E)\, \frac{\partial q_h}{\partial \boldsymbol{t}_E}(S_E) + \sum_{T \in \mathcal{T}_h} \frac{\delta_T\, h_T^4}{\nu}\, \nabla p_h(S_T) \cdot \nabla q_h(S_T)\,, \tag{26}$$

$$c_h^4(p_h, q_h) = \sum_{T \in \mathcal{T}_h} \frac{\gamma_T\, h_T^2}{\nu} \int_T \nabla p_h \cdot \nabla q_h\, \mathrm{d}x\,, \tag{27}$$

where h_E denotes the length of the edge E and the parameters

$$\delta_E = \frac{\left|\int_\Omega \varphi_E^2\, \mathrm{d}x\right|^2}{h_E^4\, |\varphi_E^2|_{1,\Omega}^2}\,, \qquad \delta_T = \frac{\left|\int_\Omega \varphi_T^2\, \mathrm{d}x\right|^2}{h_T^4\, |\varphi_T^2|_{1,\Omega}^2}\,, \qquad \gamma_T = \frac{2\left|\int_\Omega \psi_T^1\, \mathrm{d}x\right|^2}{h_T^2\, |T|\, |\psi_T^1|_{1,\Omega}^2}$$

are bounded from below and from above by positive constants independent of h. The relation (27) follows from (17)–(19) and from the fact that $\partial p_h/\partial x_k|_T$ is a linear function which does not depend on x_k, $k = 1, 2$. In the sums of (26), it is sufficient to consider only those terms which assure that the assumption (7) is satisfied. For instance, if $\nabla p_h(S_T) \cdot \nabla q_h(S_T)$ is present for a given T, we need only one edge $E \subset T$ in the first sum of (26). Note also that the matrix corresponding to (25) has only five entries in each row. The matrix corresponding to (26) has nine entries per row in general. The stabilizations (25) and (26) are new whereas in (27) we recovered a stabilization originally introduced and studied by Brezzi and Pitkäranta [6].

Finally, using (20)–(22) with $\langle \boldsymbol{f}, \boldsymbol{v}_h^2 \rangle$ instead of the right-hand side of (21) (a consistent reduced discretization), we obtain a stabilized problem which differs from (23)–(24) by the right-hand side of (24) which is, for $\boldsymbol{f} \in L^2(\Omega)^2$, given by

$$-\sum_{i=1}^{N_h} \frac{\partial q_h}{\partial t^i}(A^i) \int_\Omega \boldsymbol{f} \cdot \boldsymbol{t}^i \varphi^i \, \mathrm{d}x \, \frac{\int_\Omega \varphi^i \, \mathrm{d}x}{\nu \, |\varphi^i|_{1,\Omega}^2} .$$

For $\mathrm{V}_h^2 = \mathrm{V}_h^{2,4}$ and a piecewise bilinear function $\boldsymbol{f}$, we infer using (17)–(19) that the modified continuity equation reads

$$b(\boldsymbol{u}_h^1, q_h) + \sum_{T \in \mathcal{T}_h} \frac{\gamma_T h_T^2}{\nu} \int_T (\boldsymbol{f} - \nabla p_h) \cdot \nabla q_h \, \mathrm{d}x = 0 . \tag{28}$$

This stabilization is identitical with the Petrov–Galerkin formulation of the Stokes equations introduced in [10]. Increasing the number of the bubble functions, we can obtain (28) also for $\boldsymbol{f}$ being a piecewise higher degree polynomial.

Acknowledgment

The authors would like to thank the DAAD for supporting this research (grant KZ A/98/18415).

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Instability for Incompressible and Inviscid Fluids

H. KOCH

Abstract. We discuss well-posedness questions in $C^{1,\alpha}$ in all dimensions. For two space dimensions it is shown that stability of a stationary solution in $C^{1,\alpha}$ implies that all flow lines are periodic with a single (not necessarely minimal) period T. In particular all shear flows are unstable. Moreover the Fredholm spectrum of the linearized group is an annulus with radii determined by the stagnation points, provided stagnation points exist.

Keywords: Euler equation, inviscid fluid, instability, transport

Classification: 376C05, 35Q3576F10, 35L60

1. INTRODUCTION

Incompressible inviscid fluids are described by the Euler equation. Physically the only inner volume force is due to the pressure. This implies that the (divergence free) velocity field gets transported (as one form) up to the addition of a gradient vector field.

There is an immediate consequence: the vorticity satisfies a (tensorial) transport equation. This means that the vorticity (the outer derivative of the velocity vector field considered as one form) at time t is the push-forward under the flow of the initial vorticity.

In this work we study consequences of this transport property for well-posedness and stability questions. Our interest has two sources: the desire to obtain a deeper understanding of the Euler equation and of quasilinear hyperbolic equations (by considering the Euler equation as a model case for quasilinear hyperbolic equations).

There are many structural peculiarities of the Euler equation:

1. The vorticity ξ is the exterior derivative of the velocity considered as 1–form. It is a skew symmetric matrix. Its determinant is constant along flow lines. This leads to infinitely many invariants in even dimensional spaces.
2. If the maximal existence interval of the solution is finite then the supremum norm of the vorticity blows up. This immediately leads to global existence in 2-D.
3. There is a multi-particle interpretation of the Euler equation.
4. The evolution of the linearized problem is described by a Fourier integral operator. The canonical relation however is the simplest possible one: it is induced from the flow map of the space variables.

The first local existence result is due to Wolibner [8] and, more or less independently, to Hölder [2] (there is a discussion about the relation of the two papers in the introduction of the second paper and a note of the editors at the beginning of it). They show that boundedness of the vorticity implies Hölder continuity of the generated flow, which then implies that Hölder continuity of the vorticity is preserved albeit with a time-dependent Hölder exponent. This suffices for bounding the gradient of the velocity and boot-strap arguments imply the result.

A different proof, which also gives local existence in higher dimensions, is due to Kato and Ponce [4], Beale, Kato and Majda [3]. There are two important facts: First

$$\frac{d}{dt} \ln \|u(s)\|_{H^{1+s}} \leq c\|u(t)\|_{C^1}$$

and secondly

$$\|u(t)\|_{C^1} \leq c\|\xi(t)\|_\infty \left(1 + \ln_+ \frac{\|u(t)\|_{H^{s+1}}}{\|\xi(t)\|_\infty}\right)$$

for $s > n/2$ where $\ln_+$ denotes the positive part of the logarithm. Combining both estimates and using Gronwall's inequality one obtains existence until the vorticity blows up - a result proven by Beale, Kato and Majda [3] in $\mathbb{R}^n$ and by Ferrari [1] in domains with boundary.

We shall reconsider well-posedness questions in all space dimensions in $C^{1,\alpha}$ using arguments which generalize to many other function spaces. The case $n = 2$ is different from the case $n > 2$. There the proof simplifies and results are stronger: one obtains global solutions. The velocity vector fields with Dini continuous vorticity provide an attractive setting for well-posedness questions. Our approach seems to interpolate between the approach of [8] and [4].

We want to obtain qualitative information about solutions. A by-product of the existence proof in this work is the estimate for $n = 2$

$$\|u(t)\|_{C^1} \leq ce^{c\|u(0)\|_{C^1} t}\|u(0)\|_{C^{1,\alpha}}. \tag{1}$$

It would be desirable to know whether this estimate is sharp, i.e. are there solutions for which $\sup_x |\nabla u(x,t)|$ grows exponentially? On the other hand the L^p-norm of the vorticity is conserved and thus $\|\nabla u(t)\|_{L^p}$ is uniformly bounded in t for all $p < \infty$.

Finally we turn to stability issues: we almost completely and explicitly describe the essential spectrum for the linearized equation. We give a strong necessary condition for (linearized and nonlinear) stability which shows that, among other things, all two-dimensional shear flows are nonlinearly (Ljapunov) unstable.

Linear instability criteria have been found by Vishik [6] for spatial periodic problems and applied by Friedlander and Vishik [7]. Moreover Friedlander, Strauss and Vishik [5] have shown that some particular shear flows are unstable. In this paper we restrict ourselves to a study of stability in two dimensions. We use completely different methods which yield precise results in domains with boundaries.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary Γ. Let $C^{k,\alpha}(\Omega)$ with a nonnegative integer k and $0 \leq \alpha \leq 1$ be the standard space of functions with Hölder continuous derivatives. We denote by $L^p(\Omega)$ and $W^{k,p}(\Omega)$ with nonnegative integer k and $1 \leq p \leq \infty$ the standard Lebesgue and Sobolev spaces.

2.1. The Euler equation. We study the Euler equation for incompressible inviscid fluids:

$$
\begin{array}{rcll}
u_t + (u \cdot \nabla)u + \nabla p &=& 0 & \text{in } \Omega \times (0,T) \\
\nabla \cdot u &=& 0 & \text{on } \Omega \times (0,T) \\
u \cdot \nu &=& 0 & \text{on } \Gamma \times (0,T) \\
u(.,0) &=& u_0 & \text{on } \Omega.
\end{array} \tag{2}
$$

Here u is the velocity vector field, p is the pressure which is determined up to a function $p_0(t)$, T is a positive number and u_0 is the initial datum which satisfies $u_0 \cdot \nu = 0$ at Γ. One easily sees that the kinetic energy $\frac{1}{2}\int_\Omega |u|^2\,dx$ is formally conserved.

The velocity vector field u defines an evolution operator Φ by

$$
\begin{array}{rcl}
\partial_t \Phi(x,t,s) &=& u(\Phi(x,t,s),t) \\
\Phi(x,t,t) &=& x.
\end{array} \tag{3}
$$

Clearly

$$
|\nabla \Phi(x,t,s)| \le exp\Big(\int_s^t \sup_x |\nabla u(x,\sigma)|d\sigma\Big) \tag{4}
$$

where $|v|$ denotes the length of the vector and $|\nabla u| = \sup_{|v|=1,|w|=1} |(v\cdot\nabla)(u\cdot w)|$. We will use a similar notation for higher tensors.

2.2. A representation of the solution. Let $\psi = \Phi(t,0)^{-1}$ and define

$$
\Phi_* u(x) := \sum_j (\partial_i \psi^j u^j(\psi(x)).
$$

Let Q be the standard projection to solenoidal vector fields.

Lemma 2.1. *Suppose that u is sufficiently smooth and tangential at the boundary. Suppose that Φ is defined by* (3). *Suppose that $u(t) = Q\Phi(t,0)_* u_0$. Then u satisfies the Euler equation in the sense that there exists p such that (u,p) satisfies* (2).

2.3. Dini continuity. We define the space C_D of Dini-continuous functions as the subset of functions of L^∞ for which the norm

$$
\|f\|_{C_D} := \max\left\{\int_0^1 \sup_{|x-y|\le\sigma} |f(x)-f(y)|\frac{d\sigma}{\sigma}, \|f\|_{L^\infty}\right\}
$$

is finite. A simple check reveals that C^∞ is dense in C_D. There are three important properties of the space C_D:

1. The imbedding $C_D \subset L^\infty$ is compact.
2. Suppose that the operator $T : L^2(\Omega) \to L^2(\Omega)$ has a kernel k which satisfies

$$
|k(x,y)| \le c|x-y|^{-n} \text{ and } T(1) \in L^\infty. \tag{5}
$$

Then T defines a unique operator from C_D to $C(\bar\Omega)$ and

$$
\|Tf\|_\infty \le c_0 \|f\|_{C_D}.
$$

3. There are logarithmic interpolation estimates:

$$\|f\|_{C_D} \leq \frac{\|f\|_\infty}{\alpha}\left(\frac{1}{\alpha} + 2\ln_+\left(\frac{\alpha\|f\|_\alpha}{\|f\|_\infty}\right)\right)$$

holds whenever the right-hand side is finite.

2.4. **Relation between velocity and vorticity: $n \geq 2$.** The spatial derivative of u is a square matrix depending on x and t. Its symmetric part is the tensor $D = \frac{1}{2}(\nabla u + \nabla u^t)$ and its antisymmetric part is the vorticity $\xi = \frac{1}{2}(\nabla u - \nabla u^t)$. There are two constraints for the velocity vector field u: it is divergence free and its normal component is zero at Γ, i.e. there is no flux through the boundary.

The velocity and the vorticity ξ are related by the problem

$$\begin{array}{rcll} \Delta u^j & = & 2\partial_i \xi^{ij} & \text{in } \Omega \\ u \cdot \nu & = & 0 & \text{on } \Gamma \\ t_j \nu_i(\partial_i u^j - 2\xi^{ij}) & = & s^{ij} u^j t_i & \text{on } \Gamma \end{array} \tag{6}$$

for all tangential vectors t and s^{ij} is the second fundamental form of Γ.

Equations (6) are the Euler-Lagrange equations of critical points of

$$\frac{1}{2}\int_\Omega |\nabla u|^2 + 2\xi^{ij}\partial_i u^j dx + \frac{1}{2}\int_\Gamma s^{ij} u^i u^j.$$

As a consequence the null space is finite dimensional, u is smooth if ξ is smooth, and $T : \xi \to P\xi \to \nabla u$ where ∇u is orthogonal to the null space and P is the projection to the range, is a singular integral operator with kernel satisfying (5).

3. Well-posedness

We shall construct a local solution as the fixed point of the map $u \to v$ where v is the projection of the transported initial velocity vector field. For simplicity we only formulate the result for Hölder spaces. The case of Sobolev spaces is more tedious, but it does not require new arguments.

Theorem 3.1. *Suppose that the bounded domain $\Omega \subset \mathbb{R}^n$ has a smooth boundary $\partial\Omega$ with the exterior unit normal ν. Suppose that $u_0 \in C^{1,\alpha}(\Omega)$ satisfies $u_0 \cdot \nu = 0$ and $\nabla \cdot u_0 = 0$. Then there exist $t_0 > 0$ and a unique solution $u \in C^{1,\alpha}(\Omega \times [0, t_0])$ to (2).*

Sketch of the proof: Let Φ be the evolution operator defined by the vector field u. The solution is characterized as fixed point of the map

$$u \to v = Q\Phi(t)_* u_0.$$

At first sight this map seems to lead to a loss of derivatives. But it implies that the vorticity is again transported as a tensor by u. There the loss of derivatives disappears. The result follows by careful but elementary estimates and standard fixed point arguments. □

Remark 3.2. The proof implies that the first derivatives of u grow at most exponentially if $n = 2$. It would be interesting to understand whether this is optimal, i.e. whether there are solutions for which the derivatives of the velocity grow exponentially or not. To the knowledge of the author the estimate by $e^{ce^{ckt}}$ for $\|u(.,t)\|_{C^{1,\alpha}}$ is the best which is known up to now.

4. Instability for the nonlinear problem

Let $n = 2$ in the sequel. Nonlinear instability is a consequence of the following result and the next section.

Theorem 4.1. *Let $u \in C^{k+2,\alpha}$ be a solution with initial velocity u_0. Let $\mu(T) = \|\nabla\phi(.,0,T)\|_\infty$ and let ε and R be nonnegative. There exists a solution v in $C^{k+1,\alpha}$ with*

$$\|v(0) - u(0)\|_{C^{k+1,\alpha}} \leq R \text{ and } \|v(T) - u(T)\|_{C^{k+1,\alpha}} \geq R(\mu(T) - \varepsilon)^{k+\alpha}.$$

If $k \geq 1$, $1 < p < \infty$ then there exist solutions such that

$$\|v(0) - u^0\|_{W^{k+1,p}} \leq R \text{ and } \|v(T) - u(T)\|_{W^{k+1,p}} \geq R(\mu(T) - \varepsilon)^{k}.$$

Proof. The proof consists of several steps.

Step 1: There exists a point x_0 and a unit vector ω such that, with $x_1 = \Phi(x_0, T, 0)$, $(\omega \cdot \nabla)\Phi(x_1, T, 0) = \mu(T)$. We suppose that x_0 lies in the interior. Otherwise we approximate x_0 from the interior. We will essentially work in a neighborhood of this point.

Step 2: Let $\eta_0 \in C_0^\infty(B_0(1))$ have mean zero. There exists a unique Ψ with $\Delta\Psi = \eta_{\delta,\rho}$ and $\Psi|_\Gamma = 0$. We set $v_{\delta,\rho}$ the solution with initial data $u_0 + (\partial_2\Psi, -\partial_1\Psi)$, $w = v - u$ and η the vorticity of w. We omit the indices δ and ρ in the notation of w and η. The solution $v_{\delta,\rho}$ has the desired properties for properly chosen ρ and δ. Up to a term of minor order η_0 is transported by the vector field v, which is close to the vector field u. The support of η_0 gets distorted under the flow. This causes the growth of the C^α norm of the vorticity. □

5. Stationary solutions

There are many stationary solutions to the 2-d Euler equation. Suppose that

$$\begin{aligned} \Delta\psi &= f(\psi) && \text{on } \Omega \\ \psi &= 0 && \text{on } \Gamma \end{aligned} \tag{7}$$

for some C^1 function f. Then $u = (\partial_2\psi, -\partial_1\psi)$ is a stationary solution to the Euler equation. In particular any velocity profile for a shear flow gives a stationary solution.

Every stationary solution is the Hamiltonian vector field to the Hamiltonian function ψ with respect to the symplectic structure on $\mathbb{R}^2$ which comes from the identification with $\mathbb{C}$.

Let Φ be the flow defined by this vector field. The next lemma is surprising and crucial for the sequel.

Lemma 5.1. *Let Λ be the maximal real part of the eigenvalues of the linearization at stagnation points (which we define to be 0 if there is no stagnation point). Then*

$$\frac{\ln \sup_x |\nabla\Phi(x,t)|}{|t|} \to \Lambda \text{ as } t \to \pm\infty.$$

The proof makes heavy use of two dimensionality and the fact that u_0 is a Hamiltonian vector field.

Lemma 5.2. *Let u be a sufficiently Hamiltonian vector field on Ω, tangential at the boundary. Let Φ be the associated flow. If there exists c such that*

$$|\nabla\Phi(x,t)| \le c$$

independently of t and x then there exists $T > 0$ such that

$$\Phi(x,T) = x.$$

Proof. If there is no regular point of the Hamiltonian ψ then $u(x) = 0$ for all x. Then the assertion is trivial. Let x be a regular point of ψ. The orbit through x is periodic. There is a neighborhood consisting of periodic orbits. We choose a maximal connected neighborhood U with this property. A check of $\nabla\Phi(x,T)$ at multiples of the period shows that the period is locally constant since otherwise $\nabla\Phi$ grows linearly. The boundary of U consists of elliptic fixed points. Hence U is dense if Ω is connected. This proves the assertion. □

Remark 5.3. This is all we need for Ljapunov instability for most stationary solutions of the Euler equation. For example all stationary solutions are unstable in domains with more than one hole. All shear flows are unstable.

6. The linear equation

Let (u,p) be a stationary sufficiently smooth solution to the Euler equation. In this section we study the linearized problem:

$$\begin{aligned} \dot v_t + (u\cdot\nabla)v + (v\cdot\nabla)u + \nabla q &= 0 \quad \text{in } \Omega\times(0,T)\\ \nabla\cdot v &= 0 \quad \text{on } \Omega\times(0,T)\\ v\cdot\nu &= 0 \quad \text{on } \Gamma\times(0,T)\\ v &= v_0 \quad \text{on } \Omega. \end{aligned}$$

Let $S_X(t)$ be the semi-group defined by this equation on the function space X which we shall specify later. We denote the set where $S_X(t) - \mu$ is not Fredholm by the Fredholm spectrum. Here we complexify space and operator in the natural way. In this section we determine the Fredholm spectrum of $S(t)$: if $\Lambda > 0$ then it is an annulus with radii $e^{\pm c_X \Lambda}$. Outside the Fredholm spectrum of $S_X(t)$ there are only isolated eigenvalues of finite multiplicity. There are two steps: a reduction to a composition operator, and a study of the spectrum of the composition operator. We begin with the reduction. Let Φ be the flow generated by u and let $\tilde S_X(t)$ be the semigroup $X \ni f \to f\circ\Phi(-t) \in X$ for suitable function spaces.

Lemma 6.1. 1. *Let $X = W^{k,p}(\Omega)$, $W_0^{k,p}(\Omega)$ with $k \geq 0$ and $1 < p < \infty$, their dual, or $C^{k,\alpha}(\Omega)$ with $0 < k + \alpha$. Then $\tilde{S}_X$ is a (weak*) continuous semigroup.*

2. *Let $X = W^{k,p}(\Omega)$ with $k \geq 0$ or $C^{k+1,\alpha}$ with $k + \alpha > 0$. Then S_X is a (weak*) continuous semigroup.*
3. *Let A be the map $\xi_0 \to u_0$ defined by $u_0 = (\partial_2\Psi, -\partial_1\Psi)$ with $\Delta\Psi = \xi_0$ and $\Psi = 0$ at the boundary. Then, if $k \geq 0$*

$$\|S(t)u_0 - A\tilde{S}(t)\xi_0\|_{W^{k+1,p}} \leq c\|u_0\|_{W^{k,p}}$$

and, if in addition $0 < \alpha < 1$

$$\|S(t)u_0 - A\tilde{S}(t)\xi_0\|_{C^{k+1,\alpha}} \leq c\|u_0\|_{C^{k,\alpha}}.$$

Remark 6.2. In particular the difference is compact. Hence the Fredholm spectrum of both operators coincides.

Theorem 6.3. *The spectrum of $\tilde{S}_X(t)$ and the Fredholm spectrum of $\tilde{S}_X$ coincide. Moreover*

1. *If $X = C^{k,\alpha}$, $k \geq 0$, $k + \alpha > 0$ and $\Lambda > 0$ then the spectrum of $\tilde{S}_X(t)$ is the annulus*

$$\{z \in \mathbb{C} : e^{-(k+\alpha)\Lambda|t|} \leq |z| \leq e^{(k+\alpha)\Lambda|t|}\}.$$

2. *If $X = W^{k,p}$, k an integer and $1 \leq p \leq \infty$ and $\Lambda > 0$ then the spectrum is the annulus*

$$\{z \in \mathbb{C} : e^{-|k|\Lambda|t|} \leq |z| \leq e^{|k|\Lambda|t|}\}.$$

Let $\mu(t) = \inf_x |\nabla\Phi(x,t)|$. It is not hard to obtain bounds for the operator $\tilde{S}_X(t)$. We state the simplest of them.

Lemma 6.4. *The following estimates hold:*

$$\|\xi \circ \Phi\|_{L^p} = \|\xi\|_{L^p},$$

$$\|\xi \circ \Phi\|_{W^{1,p}(\Omega)} \leq \mu(-t)\|\xi\|_{W^{1,p}(\Omega)}$$

$$\|\xi \circ \Phi\|_{C^\alpha} \leq \mu(-t)^{-\alpha}\|\xi\|_{C^\alpha}.$$

Proof. This is obvious. □

There is an immediate consequence of this: the spectrum is contained in the annulus given in Theorem 6.3. It remains to construct elements in the spectrum. We shall prove that all points inside this annulus are Fredholm spectrum. The following lemma is a link between properties of spectrum and analytic estimates. There are two different cases: $\Lambda = 0$ and $\Lambda > 0$. We consider here only the second one and we only state the result for the simplest function space.

Lemma 6.5. *Suppose that $0 < \alpha \leq 1$. Let $T > 0$, $e^{-\alpha\Lambda T} \leq |\mu| \leq e^{\alpha\Lambda T}$, $\varepsilon > 0$, and $X = C^{1,\alpha}$. Then there exists an infinite dimensional closed subspace L of the dual space $(C^\alpha)^*$ such that*

$$|l(\tilde{S}(T)v - \mu v)| \leq \varepsilon\|l\|_{(C^{1,\alpha})^*}\|v\|_{C^\alpha}$$

for all $v \in C^\alpha$ *and* $l \in L$.

Theorem 6.3 is an immediate consequence.

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The Kuramoto–Sakaguchi Nonlinear Parabolic Integrodifferential Equation[1]

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Abstract. Global in time existence and uniqueness of classical solutions to a certain nonlinear parabolic partial differential equation, containing an integral term, are proved. Smoothness regularity and time–independent estimates for all partial derivatives are also obtained. Such an equation is of a non–standard type, and governs the time evolution of certain populations of infinitely many nonlinearly coupled random oscillators, described by a model first proposed by Kuramoto and Sakaguchi.

Classification: 35K10, 35K99

INTRODUCTION

In this paper we analyze a model, earlier derived formally by Kuramoto [5] and Sakaguchi [8], to describe the dynamical behavior of a population of infinitely many nonlinearly coupled random oscillators.

It seems that numerous phenomena in Biology, Medicine, and Physics can be modeled by a system of Langevin equations, sometimes in the so–called mean–field coupling model, where each oscillator feels the global effects of the others essentially in the same way (see [8, 9], e.g.).

In [1, 9], stability and bifurcations of the so–called Kuramoto–Sakaguchi model for nonlinearly coupled oscillators with randomly distributed frequencies, and subject to independent external white noises, have been analyzed in the "thermodynamic limit" of infinitely many oscillators. Following [1, 9], in such a limit, a nonlinear Fokker–Planck–type equation for the one–oscillator probability density, $\rho(\theta, t, \omega)$, was obtained. This model equation is studied in this paper.

The authors of [1] pay attention to stationary and time–periodic states, and investigate questions of linear and nonlinear stability of these states. However, we cannot refer to the literature ([4, 6], e.g.) to show solvability and boundedness of solutions, uniformly in time. These questions are discussed in the present paper.

The rest of the paper is organized as follows. Section 1 is devoted to some of the basic properties of solutions and auxiliary results are established there. In Section 2, we give a local existence theorem of classical solutions. Section 3 is devoted to obtaining time–independent estimates for the solutions; global in time existence then follows. This section represents the high points of the paper.

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[1]This work was supported, in part, by the Russian Foundation for Basic Research under grant no. 96-01-01618 and by the GNFM of the Italian C.N.R.

1. The basic problem and some properties of solutions

Consider the following problem (see [9], e.g.):

$$\frac{\partial \rho}{\partial t} = D\,\frac{\partial^2 \rho}{\partial \theta^2} - \omega\,\frac{\partial \rho}{\partial \theta} - K\,\frac{\partial}{\partial \theta}\left[\rho(\theta,t,\omega)\int_0^{2\pi}\int_{-\infty}^{+\infty} \sin(\varphi-\theta)\,\rho(\varphi,t,\omega)\,g(\omega)\,d\omega\,d\varphi\right], \tag{1.1}$$

$$\rho(\theta,0,\omega) = \rho_0(\theta,\omega), \tag{1.2}$$

$$\rho\big|_{\theta=0} = \rho\big|_{\theta=2\pi}, \qquad \left.\frac{\partial \rho}{\partial \theta}\right|_{\theta=0} = \left.\frac{\partial \rho}{\partial \theta}\right|_{\theta=2\pi}. \tag{1.3}$$

Throughout the paper we assume that: the initial value of the probability density, $\rho_0(\theta,\omega)$, is $(\mathbf{a_1})$ 2π–periodic in θ; $(\mathbf{a_2})$ smooth, say $\rho_0 \in C^\infty$ in both variables, for simplicity (this implies, in particular, that ρ will be uniformly bounded with respect to the frequency parameter, ω); $(\mathbf{a_3})$ positive, and $(\mathbf{a_4})$ normalized, $\|\,\rho_0\,\|_{L_1(\theta)} = 1$; the frequency distribution density $g(\omega)$ is $(\mathbf{b_1})$ nonnegative; $(\mathbf{b_2})$ integrable, $g \in L^1(\mathbf{R})$; and $(\mathbf{b_3})$ compactly supported, $\operatorname{supp} g \subset [-N,N]$; the "free parameter" ω in (1.1) must be picked up from $\operatorname{supp} g$.

We do *not* discuss, in this paper, the singular case when the coefficient ω in equation (1.1) is allowed to grow unboundedly. Moreover, the assumption $\omega \in \operatorname{supp} g$ suffices to describe a number of phenomena in Physics and Biology [2, 11].

In [3], an apparently similar equation was studied. There are, however, several important differences, since in (1.1)–(1.3) an additional integral with respect to ω, explicit dependence on ω, and periodicity with respect to θ appears.

In this section, we give a simple but important property of solutions to problem (1.1)–(1.3) that we will use in the sequel.

Lemma 1.1. *If there is a smooth solution $\rho(\theta,t,\omega)$ to problem (1.1)–(1.3), then it remains positive for all times $t \geq 0$.*

Lemma 1.2. *The solution $\rho(\theta,t,\omega)$ to the problem (1.1)–(1.3) remains normalized for all times $t \geq 0$ and all ω's,*

$$\|\rho\|_{L^1} := \int_0^{2\pi} \rho(\theta,t,\omega)\,d\theta = 1, \tag{1.4}$$

provided that the initial distribution $\rho_0(\theta,\omega)$ is normalized $\|\,\rho_0\,\|_{L_1}$.

For detailes of the proofs, here and in the sequel, see [7].

2. Existence and uniqueness of solutions

The basic idea for proving existence of solutions to problem (1.1)–(1.3) is standard. Theorem 2.1 below establishes local solvability. Unfortunately, we cannot refer to well–known results on solvability because of the points given in Section 1, by which the present problem differs from those already studied in the literature. Therefore, we shall give a sketch of the proof, paying attention to the peculiarities of our problem.

Theorem 2.1. *In the assumptions above, there exists $T > 0$, such that there is a classical solution $\rho(\theta, t, \omega)$ to the problem (1.1)–(1.3) for $t \in [0, T]$. Such a solution is bounded uniformly in ω, along with all its space derivatives $\frac{\partial^m \rho}{\partial \theta^m}$.*

Sketch of the proof. We construct a sequence of successive approximations and prove its convergence, as well as appropriate estimates, by a compactness argument. Consider the sequence $\{\rho_n(\theta, t, \omega)\}$ of solutions to the linear problems

$$\frac{\partial \rho_n}{\partial t} = D \frac{\partial^2 \rho_n}{\partial \theta^2} - [\omega + B_n(\theta, t)] \frac{\partial \rho_n}{\partial \theta} - \frac{\partial B_n}{\partial \theta}(\theta, t) \rho_n, \quad n = 1, 2, \dots, \tag{2.1}$$

with initial and boundary conditions as in (1.2)–(1.3), the coefficients B_n being given by

$$B_n(\theta, t) := K \int_0^{2\pi} \int_{-\infty}^{+\infty} \sin(\varphi - \theta) \rho_{n-1}(\varphi, t, \omega) g(\omega) \, d\omega \, d\varphi, \quad n = 1, 2, \dots. \tag{2.2}$$

As a first approximation, we choose the initial distribution $\rho_0(\theta, t, \omega) \equiv \rho_0(\theta, \omega)$.

As Lemmas 1.1 and 1.2 can be applied, the successive approximations, $\rho_n(\theta, t, \omega)$ above, are all *positive* and *normalized*: $\|\rho_n\|_{L^1} = 1$.

As is known, the fundamental solution to equation $u_t = u_{\theta\theta} - \omega u_\theta$ is given by

$$G(\theta, t, \omega) := \frac{1}{2\sqrt{\pi t}} e^{-\frac{(\theta - \omega t)^2}{4t}}, \tag{2.3}$$

see [2], e.g. Extending all functions $\rho_n(\theta, t, \omega)$ periodically on R as functions of θ, we obtain from (2.1), (2.2) the representation

$$\begin{aligned} \rho_n(\theta, t, \omega) &= \int_{-\infty}^{+\infty} G(\theta - \xi, t, \omega) \rho_0(\xi, \omega) \, d\xi - K \int_0^t \int_{-\infty}^{+\infty} G(\theta - \xi, t - \tau, \omega) \\ &\times \frac{\partial}{\partial \xi} \left[\rho_n(\xi, \tau, \omega) \int_0^{2\pi} \int_{-\infty}^{+\infty} \sin(\varphi - \xi) \rho_{n-1}(\varphi, \tau, \omega) \, g(\omega) \, d\varphi d\omega \right] d\xi \, d\tau. \end{aligned} \tag{2.4}$$

Therefore, we have for the differences $\rho_n(\theta, t, \omega) - \rho_{n-1}(\theta, t, \omega)$, integrating by parts,

$$\begin{aligned} \rho_n - \rho_{n-1} &= -K \int_0^t \int_{-\infty}^{+\infty} \frac{\partial G}{\partial \xi} \left[(\rho_n - \rho_{n-1}) \int_0^{2\pi} \int_{-\infty}^{+\infty} \sin(\varphi - \xi) \rho_{n-1} \, g \, d\varphi \, d\omega \right. \\ &+ \left. \rho_{n-1} \int_0^{2\pi} \int_{-\infty}^{+\infty} \sin(\varphi - \xi)(\rho_{n-1} - \rho_{n-2}) \, g \, d\varphi \, d\omega \right] d\xi \, d\tau. \end{aligned} \tag{2.5}$$

After standard calculations, from (2.5) we can prove that

$$\max_{\theta, \omega, \tau} |\rho_n - \rho_{n-1}| \le M^{n-1} \left(\sqrt{t}\right)^{n-1} \max_{\theta, \omega, \tau} |\rho_1 - \rho_0|. \tag{2.6}$$

Therefore, the sequence of successive approximations $\{\rho_n(\theta,t,\omega)\}$ converges for $t \in [0,T]$ with T sufficiently small, such that $M\sqrt{T} < 1$. As before, we can show that all derivatives of ρ_n are bounded uniformly in n, and hence an existence theorem follows by the usual compactness arguments, which completes the proof. □

The next basic result can now be established.

Theorem 2.2. *Under conditions of Theorem 2.1 the solution to problem (1.1)–(1.3) is unique.*

Again, the proof is standard, but because of the nonclassical form of equation (1.1), we cannot refer to the existing literature. For details, again, we refer the reader to [7].

3. Basic estimates of solutions

In this section we prove that the solution ρ to problem (1.1)–(1.3) is bounded, *uniformly in t*. The key technique we shall use is based on obtaining "energy-like estimates." Multiply both sides of equation (1.1) by ρ and integrate with respect to θ. Note that, because of smoothness and periodicity of $\rho(\theta,t,\omega)$, in all calculations below, all terms out of the integrals (arising from integration by parts) vanish. Thus, we obtain after simple calculations

$$\frac{\partial}{\partial t}\int_0^{2\pi} \rho^2\,d\theta + 2D\int_0^{2\pi} \rho_\theta^2\,d\theta = K\int_0^{2\pi} \rho^2\left(\int_0^{2\pi}\int_{-\infty}^{+\infty} \cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\right)d\theta. \tag{3.1}$$

The next integral identity can be obtained multiplying both sides of equation (1.1) by $\rho_{\theta\theta}$ and integrating with respect to θ:

$$\frac{\partial}{\partial t}\int_0^{2\pi} \rho_\theta^2\,d\theta + 2D\int_0^{2\pi} \rho_{\theta\theta}^2\,d\theta = K\int_0^{2\pi} (3\rho_\theta^2+\rho^2)\left(\int_0^{2\pi}\int_{-\infty}^{+\infty} \cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\right)d\theta. \tag{3.2}$$

Multiplying both sides of equation (3.1) by the constant $3KA/(2D)+1$, and summing side by side to equation (3.2), we conclude that the "energy-like" functional

$$R(t,\omega) := \int_0^{2\pi} \left[\left(\frac{3KA}{2D}+1\right)\rho^2 + \rho_\theta^2\right]d\theta \tag{3.3}$$

satisfies

$$\frac{\partial R(t,\omega)}{\partial t} + (2D\ +\ 3KA)\int_0^{2\pi} \rho_\theta^2\,d\theta + 2D\int_0^{2\pi} \rho_{\theta\theta}^2\,d\theta$$
$$= K\int_0^{2\pi}\left[3\rho_\theta^2 + \left(\frac{3KA}{2D}+2\right)\ \rho^2\right]\int_0^{2\pi}\int_{-\infty}^{+\infty} \cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\,d\theta. \tag{3.4}$$

From (3.4), after technical calculations, we get

$$\frac{\partial R(t,\omega)}{\partial t} + D\int_0^{2\pi} \rho_\theta^2\,d\theta \le C_1 \tag{3.5}$$

with a proper C_1. Consider the two possibilities, that either the function $R(t,\omega)$ satisfies the inequality $\frac{\partial R(t,\omega)}{\partial t} \geq 0$, or $\frac{\partial R(t,\omega)}{\partial t} < 0$.

In the first case we have

$$\int_0^{2\pi} \rho_\theta^2 \, d\theta \leq \frac{C_1}{D}, \tag{3.6}$$

and, consequently, we conclude that for the time t then $\frac{\partial R(t,\omega)}{\partial t} \geq 0$, the inequality $\rho(\theta,t,\omega) \leq C_2$ holds. From latter relation and (3.6), it is easy to see that quantity $R(t,\omega)$ defined in (3.3) is bounded,

$$R(t,\omega) \leq C_3, \tag{3.7}$$

for every t where $\frac{\partial R(t,\omega)}{\partial t} \geq 0$. If $\frac{\partial R(t,\omega)}{\partial t} < 0$ holds, instead, for some open interval $t \in (t_0,t_1)$, obviously $R(t) \leq R(t_0)$ and for $t = t_0$ estimate (3.7) holds. Therefore we have the following lemma.

Lemma 3.1. *The quantity $R(t,\omega)$ in (3.3) remains uniformly bounded as in (3.7) as long as the solution $\rho(\theta,t,\omega)$ to problem (1.1)–(1.2) remains smooth.*

We stress that (3.5) would give a time–dependent estimate for R, after integrating in time. Lemma 3.1 yields, instead, a time–independent estimate.

As the function $R(t,\omega)$ remains bounded for all times t in the interval of existence of the solution $\rho(\theta,t,\omega)$, the estimate

$$\int_0^{2\pi} \rho_\theta^2 \, d\theta \leq C_4 \tag{3.8}$$

follows from (3.3). Finally, the main result of this section can be obtained as a theorem, by (3.8):

Theorem 3.1. *The classical solution $\rho(\theta,t,\omega)$ to problem (1.1)–(1.2) remains bounded, uniformly in t, ω, and N for $\omega \in [-N,N]$, as long as it exists.*

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Scattering of Acoustical and Electromagnetic Waves by Some Canonical Obstacles

E. MEISTER, A. PASSOW

Abstract. *A. Sommerfeld* published in 1896 his habilitation thesis studying the diffraction of plane time–harmonic acoustical waves by a half–plane or a wedge. He used series expansions and reformulation by integrals. These *Sommerfeld integrals* were systematically used by Russian authors. Their approach is called the *Maliushinets method.* Western authors preferably use the *Wiener–Hopf method,* based on the Fourier transforms and *symbol factorization* of the related boundary integral equations.
Many new results, including electromagnetic and elastodynamic waves with different kind of boundary conditions, different media, plane surfaces, and aperiodic initial boundary conditions, using generalized eigenfunction expansions, have been achieved during the last decade. Still a lot of open problems exist. The paper reports on more recent results obtained together with co-workers.

Keywords: diffraction theory, canonical domains, Wiener–Hopf method, factorization

1. INTRODUCTION

A. SOMMERFELD was one of the first who formulated and treated a *canonical boundary value problem* for the Helmholtz equation which governs time–harmonic scalar waves. In 1896 he presented his famous Goettingen Habilitation thesis [21], dealing with half–planes and wedges. He used series expansions w.r.t. the polar angle and Riemann surface concepts to arrive at the solutions for the Dirichlet boundary value problem at the two faces $\Gamma_1 := \{(x, y) \in \mathbf{R}^2 : x \geq 0,\, y = 0\}$ and $\Gamma_2 := \{(x, y) \in \mathbf{R}^2 : x = r\cos\alpha,\, y = r\sin\alpha,\, r \geq 0\}$ with $\alpha = 0$ or $0 < \alpha \leq \pi/2$. The *Sommerfeld integrals* could be achieved also by the *Wiener–Hopf method* [16, 22]. Many papers have been written on more general situations, like semi–infinite waveguides of different cross–sections or systems of a finite or infinite number of parallel screens even with very general linear boundary conditions or generalizing to elastodynamical wave scattering by screen–like semi–infinite cracks [1, 4, 13, 16]. Here we cannot evaluate all the enumerable papers on the subject but want to concentrate on some more recent results concerning electromagnetic boundary–transmissions problems.

2. THE CANONICAL BOUNDARY–TRANSMISSIONS PROBLEMS

The simplest canonical boundary value problems apply to n–dimensional half–spaces $\mathbf{R}^n_{\pm}$, $n \geq 2$, for linear, elliptic partial differential equations or systems of such equations with constant coefficients. The theory of these is now standard and can be found in many textbooks [6] applying an $(n-1)$–dimensional Fourier–transformation w.r.t. $\underline{x}' = (x_1, \dots, x_{n-1})$ for $\varphi \in \mathcal{S}(\mathbf{R}^n)$ or $f \in \mathcal{S}'(\mathbf{R}^n)$, the Schwartz spaces:

$$\mathcal{F}_{\underline{x}'\to\underline{\xi}'}\varphi = \hat{\varphi}(\underline{\xi}', x_n) := \int_{\mathbf{R}^{n-1}_{\underline{x}'}} e^{i<\underline{x}',\underline{\xi}'>} \varphi(\underline{x}', x_n)\, d\underline{x}' \tag{1}$$

and $\hat{f} \in \mathcal{S}'(\mathbf{R}^{n-1}_{\underline{\xi}'})$ defined by

$$(\hat{f}, \varphi) = (f, \hat{\varphi}), \quad \forall \varphi \in \mathcal{S} \tag{2}$$

acting as a continuous linear functional taking into account that $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a linear topological isomorphism. Then the $\mathcal{F}$–transformation algebraizes the operations of differentiation and partial convolution. By means of the $(n-1)$–dimensional $\mathcal{F}$–transform the partial differential equations are reduced to ordinary differential equations containing the $\mathcal{F}$–variable $\underline{\xi}'$ in polynomial form.
Let us take as a simple example the Helmholtz equation in $\mathbf{R}^n_\pm$ which leads to

Definition 1 (Boundary value problem (B.v.p.) for $\mathbf{R}^n_+$). *Find $u \in C^2(\mathbf{R}^n_+) \cap C^1_0(\overline{\mathbf{R}^n_+})$ (classically) or $\in H^1_{loc}(\mathbf{R}^n_+, \Delta) \cap \mathcal{S}'(\mathbf{R}^n_{\underline{\xi}'})$ (in the weak sense), such that for given $F \in C^1_0(\overline{\mathbf{R}^n_+})$ or $\in L^2(\overline{\mathbf{R}^n_+})$, $f \in C^1_0(\mathbf{R}^{n-1}_{\underline{x}'})$ or $\in H^{1/2-m}(\mathbf{R}^{n-1}_{\underline{x}'})$, $m = 0, 1$, and constant $k = k_1 + ik_2$ with $k_1 > 0$ and $k_2 \geq 0$*

$$(\Delta_n + k^2)u = F \qquad in \quad \mathbf{R}^n_+ \tag{3}$$

in the strong (pointwise) or weak (distributional) sense. Additionally u has to fulfill the Dirichlet boundary condition

$$B_0[u] := u|_{\mathbf{R}^{n-1}_{\underline{x}'}} = f_0 \in C^1_0(\mathbf{R}^{n-1}_{\underline{x}'}), \tag{4}$$

or the Neumann condition

$$B_1[u] := u|_{\mathbf{R}^{n-1}_{\underline{x}'}} = -\left.\frac{\partial}{\partial x_n}u\right|_{\mathbf{R}^{n-1}_{\underline{x}'}} = f_1 \in C^1_0(\mathbf{R}^{n-1}_{\underline{x}'}), \tag{5}$$

or the impedance boundary condition

$$B_2[u] := \left.\left(-\frac{\partial}{\partial x_n} + ip\right)u\right|_{\mathbf{R}^{n-1}_{\underline{x}'}} = f_{imp} \in C_0(\mathbf{R}^{n-1}_{\underline{x}'}). \tag{6}$$

In the Sobolev space setting the boundary functions are assumed as traces from u such that $f_0 \in H^{1/2}_{loc}(\mathbf{R}^{n-1}_{\underline{x}'}) \cap \mathcal{S}'(\mathbf{R}^{n-1}_{\underline{x}'})$ and $f_1, f_{imp} \in H^{-1/2}_{loc}(\mathbf{R}^{n-1}_{\underline{x}'}) \cap \mathcal{S}'(\mathbf{R}^{n-1}_{\underline{x}'})$.

Remark 1. *It is well known that one has to add further conditions on the behavior of u as $r := |\underline{x}| \to \infty$ in $x_n \geq 0$. If $k_2 > 0$ one can look, e.g., for decaying functions or such from the smaller space $H^1(\mathbf{R}^n_+; \Delta)$. For $k = k_1 > 0$ one puts the* Sommerfeld radiation condition

$$\frac{\partial}{\partial r}u - iku = o(r^{\frac{1-n}{2}}) \qquad as \quad r \to \infty \tag{7}$$

to guarantee uniqueness of these b.v.p.'s.

Now the second fundamental situation in diffraction theory is given by

Definition 2 (Transmission problem (T.p.) for $\mathbf{R}^n_\pm$:). *For given $F = (F_+, F_-)^T \in C_0(\overline{\mathbf{R}^n_+}) \times C_0(\overline{\mathbf{R}^n_-})$ or $\in L^2(\mathbf{R}^n_+) \times L^2(\mathbf{R}^n_-)$, and $f \in C^1_0(\overline{\mathbf{R}^n_{\underline{x}}})$, $g \in C_0(\overline{\mathbf{R}^n_{\underline{x}}})$ (classically) or $f \in H^{1/2}_{loc}(\mathbf{R}^{n-1}_{\underline{x}'}) \cap \mathcal{S}'(\mathbf{R}^{n-1}_{\underline{x}'})$, $g \in H^{-1/2}_{loc}(\mathbf{R}^{n-1}_{\underline{x}'}) \cap \mathcal{S}'(\mathbf{R}^{n-1}_{\underline{x}'})$ (weakly) and constants $k^\pm = k^\pm_1 + ik^\pm_2$ with $k^\pm_1 > 0$ and $k^\pm_2 \geq 0$, $\varrho^\pm$ similar, find a function $u = (u_+, u_-)^T \in \left(C^2(\mathbf{R}^n_+) \times C^2(\mathbf{R}^n_-)\right) \cap \left(C^1_0(\overline{\mathbf{R}^n_+}) \times C^1_0(\overline{\mathbf{R}^n_-})\right)$ (classically) or $\in \left(H^1_{loc}(\mathbf{R}^n_+;\Delta) \times H^1_{loc}(\mathbf{R}^n_-;\Delta)\right) \cap \left(\mathcal{S}'(\overline{\mathbf{R}^n_+}) \times \mathcal{S}'(\overline{\mathbf{R}^n_-})\right)$ (weakly) such that*

$$(\Delta_n + (k^\pm)^2)u^\pm = F^\pm \qquad in \quad \mathbf{R}^n_\pm \tag{8}$$

in the strong or weak sense, respectively. Additionally the two transmission conditions *have to hold*

$$\mathrm{Tr}_0\, u := u^+|_{\mathbf{R}^{n-1}_{\underline{x}'}} - u^-|_{\mathbf{R}^{n-1}_{\underline{x}'}} = f, \tag{9}$$

$$\mathrm{Tr}_1\, u := \frac{1}{\varrho^+}\frac{\partial u^+}{\partial x_n}\bigg|_{\mathbf{R}^{n-1}_{\underline{x}'}} - \frac{1}{\varrho^-}\frac{\partial u^-}{\partial x_n}\bigg|_{\mathbf{R}^{n-1}_{\underline{x}'}} = g. \tag{10}$$

Remark 2. *In order to get a well-posed problem one has to add conditions concerning the behavior of the solution u for $r = |\underline{x}| \to \infty$ in $\mathbf{R}^n_\pm$. They could be similar like in Remark 1: global L^2-integrability if $k^\pm \in \mathbf{C}^{++}$ or fulfilling the appropriate radiation conditions like (7) with different k's. For incident waves*

$$u^\pm_{inc}(\underline{x}) \sim e^{ik^\pm \langle \underline{e}, \underline{x} \rangle}, \tag{11}$$

with $|\underline{e}| = 1$ one arrives at the well-known Snellius law *at the interface $\partial\mathbf{R}^n_+ = \partial\mathbf{R}^n_- = \mathbf{R}^{n-1}_{\underline{x}'}$.*

Remark 3. *The above-mentioned boundary and transmission conditions may be generalized to other differential equations and systems of them of elliptic type for smoothly bounded finite domains $\Omega \subset \mathbf{R}^n$ and $\Omega' := \mathbf{R}^n \backslash \overline{\Omega}$. There exists a fully developed theory making use of integral representations with fundamental solutions for the Lamé equations of linear elastodynamics, thermo-elastodynamics, and also for Maxwell's system — which for the stationary case is not even elliptic* [2, 7, 9, 10, 14].

There has been a growing interest by physicists and engineers in the behavior of structures having singularities in their geometry, like edges and vertices, under different constant or time–changing loads, like microwaves in radar technique or seismic waves in geophysics, but also the scattering and absorption of sound waves and pulses by traffic noise shielding walls. Nondestructive testing and tomography — as inverse scattering problems — rely very strongly on the well-understood behavior of diffracted fields near the corners, at far distances, shortly after a wave hit the obstacle or a long time afterward.
As the *main canonical obstacles* there are considered those with *semi–infinite* boundaries, like half- and quarter–planes, cones, octants, and pyramids. They model locally the most important geometrical singularities, not including cusps and needles.

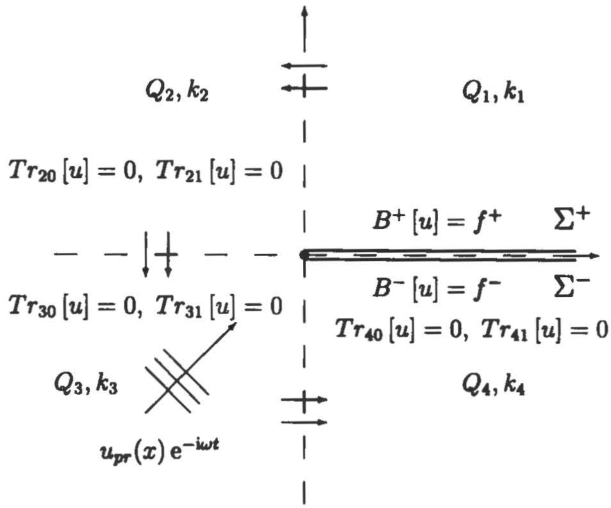

FIGURE 1. Sommerfeld problem with four media

The actual physical behavior of the scattered fields also strongly depends on the types of boundary/transmission conditions like in Definitions (1) and (2), but there is now a great demand to improve the models of boundary material response to impinging waves. The combination of different materials to *composites* in elasticity or microwave electrical circuits lead to the mixed problems for thin layers of dielectrics and metal coatings on such substrates [20]. Of great interest also are the *chiral media*. One may find some information about the state of the art in [14].
In the following two figures, you will see the basic canonical boundary–transmission problems for the scalar wave fields which arise in acoustics and for specially polarized electromagnetic waves having only one component of the electric or magnetic field vectors parallel to the infinite line–like edges. The goal in the scalar diffraction theory is to solve the initial boundary–transmission problems for the *d'Alembert wave equation*

$$\left(\frac{1}{c_j^2}\frac{\partial^2}{\partial t^2} - \Delta\right) U(\underline{x}, t) = F(\underline{x}, t) \tag{12}$$

in the *time half-cylinder* $\Omega_T := \Omega \times (0, T)$, $0 < T \leq \infty$ classically, i.e. $U \in C^2(\Omega_T) \cap C^1(\overline{\Omega_T})$ or in Sobolev–space valued functions with special weights w.r.t. time t to allow for a Laplace transformation

$$\tilde{U}(\underline{x}, s) := \int_0^\infty e^{-ts} U(\underline{x}, t)\, dt \tag{13}$$

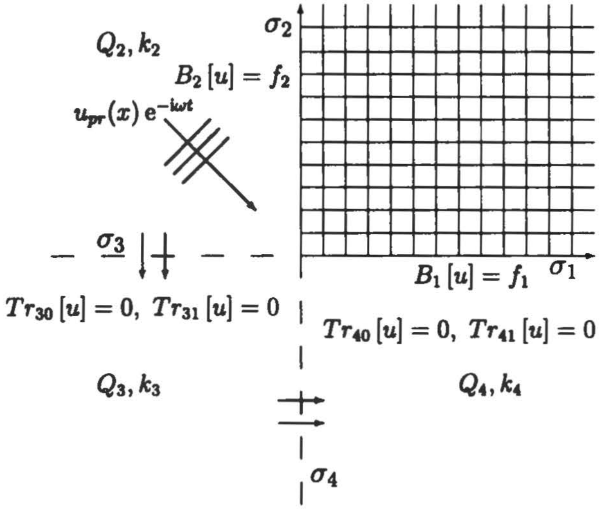

FIGURE 2. Wedge diffraction problem with three media

existing as parameter-dependent Sobolev space functions from $H^1_{loc}(\Omega;\Delta) \cap \mathcal{S}'(\Omega;\Delta)$ when there are given (compatible) initial data

$$\left.\begin{aligned} U(\underline{x},+0) &= U_0(\underline{x}) \\ \tfrac{\partial}{\partial t}U(\underline{x},+0) &= U_1(\underline{x}) \end{aligned}\right\} \in \mathcal{S}'(\Omega). \tag{14}$$

The main questions in this context are the short–time and the long–time behavior, particularly the validity of the *limiting amplitude principle* and the existence of the *general wave operators* of scattering theory.

Remark 4. *In the case of finite scatterers Ω such that $\Omega' = \mathbf{R}^n \backslash \overline{\Omega}$ is an* exterior domain *a lot is known about those questions* [10], *but in the case of semi–infinite obstacles, still a lot has to be done starting with the representation of the resolvents of the stationary boundary–transmission problems, particularly for the vectorial cases in electro- and elastodynamics.*

3. Electrodynamical Boundary–Transmission Problems

Let there be given a canonical domain $\Omega \subset \mathbf{R}^3$ with a piecewise smooth boundary $\partial\Omega$. Then we have

Definition 3. *If there are given the electric current density $\underline{J} \in \mathbf{R}^3$ and the electrical charge density $\varrho \in \mathbf{R}$, then we look for a quadruple of vectorfields $\in \mathbf{R}^3$, the electric fieldstrength $\underline{E}$, the magnetic fieldstrength $\underline{H}$, the electric displacement $\underline{D}$ and the* *magnetic induction* $\underline{B}$ *fulfilling the* Maxwell equations

$$\operatorname{curl} \underline{E} + \partial_t \underline{B} = \underline{0}, \qquad \operatorname{div} \underline{B} = 0, \tag{15}$$

$$\operatorname{curl} \underline{H} - \partial_t \underline{D} = \underline{J}, \qquad \operatorname{div} \underline{D} = \varrho. \tag{16}$$

Remark 5. *This system of eight partial differential equations has to be solved — in the classical or weak form — for* $(\underline{x}, t) \in \Omega_T = \Omega \times (0, T)$ *with additional boundary–transmission conditions on* $\partial\Omega_T = \partial\Omega \times [0, T]$, *where on* Ω *initial conditions are prescribed. Here the different types of boundary–transmission conditions will be formulated after specifying the* constitutive equations

Definition 4. *The material equations are*

$$\underline{D}(\underline{x}, t) = \underline{\mathcal{D}}\left(\underline{E}(\underline{x}, t), \underline{H}(\underline{x}, t)\right) \tag{17}$$

$$\underline{B}(\underline{x}, t) = \underline{\mathcal{B}}\left(\underline{E}(\underline{x}, t), \underline{H}(\underline{x}, t)\right) \tag{18}$$

which in general are nonlinear functions and in general depend on time history which causes hysteresis effects. Neglecting these influences, we arrive at the simplest situation of linear dependency

$$\underline{D}(\underline{x}, t) = \underline{\underline{\varepsilon}}\, \underline{E}(\underline{x}, t) \tag{19}$$

$$\underline{B}(\underline{x}, t) = \underline{\underline{\mu}}\, \underline{H}(\underline{x}, t) \tag{20}$$

$$\underline{J}(\underline{x}, t) = \underline{\underline{\sigma}}\, \underline{E}(\underline{x}, t) \quad \textit{Ohm's law} \tag{21}$$

$\underline{\underline{\varepsilon}}$, $\underline{\underline{\mu}}$ *and* $\underline{\underline{\sigma}}$ *are called, respectively, the* permittivity, *the* permeability, *and the* conductivity *tensors. If they are multiples of the identity tensor* I_3 *the material is called* isotropic, *otherwise* anisotropic.

By energy considerations with the *Poynting vector* it can be seen that for lossless media the relations

$$\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}^{\dagger} := (\underline{\underline{\varepsilon}}^{*})^{T} \qquad \text{and} \qquad \underline{\underline{\mu}} = \underline{\underline{\mu}}^{\dagger} := (\underline{\underline{\mu}}^{*})^{T} \tag{22}$$

hold, which specialize to $\Im\{\varepsilon\} = \Im\{\mu\}$ for $\underline{\underline{\varepsilon}} = \varepsilon I_3$ and $\underline{\underline{\mu}} = \mu I_3$ in the isotropic case.

Definition 5. *If* $\underline{n}_s$ *denotes the unit normal vector to a surface* $\mathcal{S} \subset \mathbf{R}^3$ *being sufficiently smooth and* $\underline{U} \in \mathbf{R}^3$ *will be a vector field continuous on* $\mathcal{S}$ *— or having integrable traces* $\underline{U}|_S$ *of fields* $\in H^m_{loc}(\Omega) \cap \mathcal{S}'(\Omega)$, $m \in \mathbf{N}_0$, $\mathcal{S} \subset \overline{\Omega}$, *then we have*

$$\underline{n}_s \wedge \underline{E}_{sc}|_S = \underline{f}_t \qquad \textit{and} \qquad \underline{n}_s \wedge \underline{H}_{sc}|_S = \underline{g}_t \tag{23}$$

as electric *and* magnetic boundary conditions *for the scattered fields given by*

$$\underline{E}_{sc} := \underline{E}_{tot} - \underline{E}_{inc}, \qquad \underline{H}_{sc} := \underline{H}_{tot} - \underline{H}_{inc} \tag{24}$$

which may be augmented by conditions for the normal components w.r.t. S

$$< \underline{n}_s, \underline{\underline{\varepsilon}}\, \underline{E} > |_S = f_n \qquad \textit{and} \qquad < \underline{n}_s, \underline{\underline{\mu}}\, \underline{H} > |_S = g_n. \tag{25}$$

These boundary conditions hold reasonably for electrically or magnetically ideal conducting surfaces at high frequency, but have to be replaced by such who model better the absorption and reflection properties of the scatterer's surface S. In the following sections we shall concentrate on the application of the so-called *Leontovich boundary conditions* of different orders, the simplest being given by

$$\underline{n}_s \wedge \underline{E}|_S = \underline{\underline{Z}}\underline{n}_s \wedge (\underline{n}_s \wedge \underline{H})|_S \tag{26}$$

with the *impedance matrix* $\underline{\underline{Z}} \in \mathbf{C}^{3\times 3}$. Many authors [8, 19, 20] studied this electromagnetic boundary value problem for smooth surfaces in the isotropic case. Like for acoustics or elastodynamics there are scattering problems for different, not ideally conducting media in $\mathbf{R}^3$-space leading to

Definition 6. *Let there be given at least two domains Ω and $\Omega' \subset \mathbf{R}^3$ such that $\overline{\Omega} \cup \overline{\Omega'} = \mathbf{R}^3$ and $\Omega \cap \Omega' = \emptyset$ but $S = \overline{\Omega} \cap \overline{\Omega'} \neq \emptyset$ being an oriented piecewise smooth surface. Let $\underline{\underline{\varepsilon}}$, $\underline{\underline{\varepsilon}}'$ and $\underline{\underline{\mu}}$, $\underline{\underline{\mu}}'$ be permittivity and permeability tensors, respectively, continuously depending on $\underline{x} \in \overline{\Omega}$ and $\underline{x} \in \overline{\Omega'}$, and being strictly Hermitean — here for $\underline{\underline{\varepsilon}}(\underline{x})$ —*

$$\underline{\xi}^T \underline{\underline{\varepsilon}}(\underline{x})\underline{\xi} \geq c|\underline{\xi}|^2 \qquad \forall \underline{\xi} \in \mathbf{C}^3, \quad \underline{x} \in \Omega, \tag{27}$$

then $\underline{E}$ and $\underline{H}$ fulfill the transmission conditions at the interface S, *if*

$$\begin{aligned} \underline{n}_s \wedge \underline{E}_{tot} &= \underline{n} \wedge \underline{E}'_{tot}, & (28)\\ \underline{n}_s \wedge \underline{H}_{tot} &= \underline{n} \wedge \underline{H}'_{tot}, & (29)\\ < \underline{n}_s, \underline{\underline{\varepsilon}}\,\underline{E}_{tot} > &= < \underline{n}_s, \underline{\underline{\varepsilon}}'\underline{E}' >, & (30)\\ < \underline{n}_s, \underline{\underline{\mu}}\,\underline{H}_{tot} > &= < \underline{n}_s, \underline{\underline{\mu}}'\underline{H}' >, & (31) \end{aligned}$$

This problem has been discussed and solved by boundary integral equations [3] *where a* Silver–Müller radiation condition *has to be involved for $\Omega' := \mathbf{R}^3 \backslash \overline{\Omega}$ an* exterior domain:

$$\left.\begin{aligned} \omega\underline{\underline{\mu}}'\left(\tfrac{\underline{x}}{|\underline{x}|} \wedge \underline{H}\right) + k'\underline{E} &= \mathcal{O}\left(\tfrac{1}{r^2}\right)\\ \omega\underline{\underline{\varepsilon}}'\left(\tfrac{\underline{x}}{|\underline{x}|} \wedge \underline{E}\right) - k'\underline{E} &= \mathcal{O}\left(\tfrac{1}{r^2}\right) \end{aligned}\right\} \quad \text{for} \quad r = |\underline{x}| \to \infty; \tag{32}$$

see also [7].

4. The Sommerfeld–Half–plane Problem for Anisotropic Dielectric Screens

Let us consider the following boundary–transmission problem (b.t.p.) represented by Figure 3.

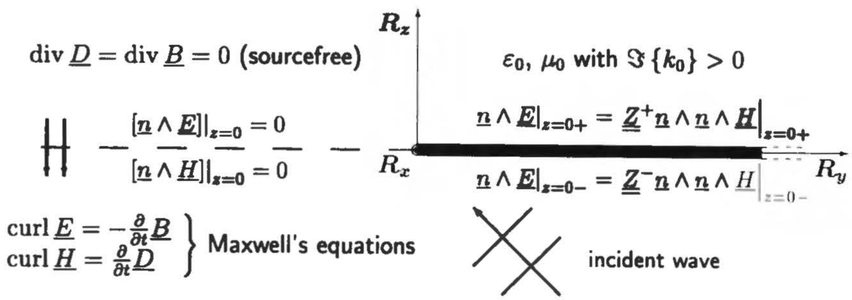

Figure 3: Sommerfeld Problem with anisotropic half-plane

If we split off the incident electromagnetic wave field assuming a time–harmonic dependence $\underline{\mathbf{E}} = \underline{E}\mathrm{e}^{-i\omega t}$ and $\underline{\mathbf{H}} = \underline{H}\mathrm{e}^{-i\omega t}$ with complex amplitude $\underline{E}$, $\underline{H} \in \mathbf{C}^3$ and frequency ω for the scattered field, we are lead with the Sommerfeld screen Σ and its complementary screen Σ^c given by

$$\Sigma,\ \Sigma^c := \left\{(x, y, z) \in \mathbf{R}^3; x \gtrless 0, y \in \mathbf{R}, z = 0\right\} \tag{33}$$

and

Definition 7. *The* anisotropic Bessel potential spaces *are defined by*

$$H_{r,s}(\mathbf{R}^3_{\pm}) := \left\{u \in L_2(\mathbf{R}^3_{\pm}) : \mathcal{F}^{-1}(1+\xi^2)^{r/2}(1+\eta^2)^{s/2}\mathcal{F}_{\substack{x\to\xi \\ y\to\eta}}\, u \in L_2(\mathbf{R}^3_{\pm})\right\} \tag{34}$$

for all pairs $r, s \in \mathbf{R}$. For $r = s$ we write for short $H_s(\mathbf{R}^3_{\pm})$ to the following

Problem $\mathcal{S}$: Find a function $\underline{u}' := (\underline{E}, \underline{H})^T \in \mathbf{C}^6$ with $\underline{E}$, $\underline{H} \in [L_2(\mathbf{R}^3)]^3$ and restrictions

$$\underline{u}'_{\pm} := \underline{u}'|_{\mathbf{R}^3_{\pm}} \in [H_{0,1}(\mathbf{R}^3_{\pm}) \times H_{1,0}(\mathbf{R}^3_{\pm}) \times L_2(\mathbf{R}^3_{\pm})]^2 \tag{35}$$

being weak solutions of the Maxwell equations in $\mathbf{R}^3_{\pm}$.

$$\operatorname{curl} \underline{E} - i\omega\mu_0\underline{H} = \underline{0}, \qquad \operatorname{curl} \underline{H} + i\omega\varepsilon_0\underline{E} = \underline{0} \tag{36}$$

and their traces on $z = 0$ fulfill the conditions

$$\underline{E},\ \underline{H} \in H_{-\frac{1}{2},\frac{1}{2}}(\mathbf{R}^2) \times H_{\frac{1}{2},-\frac{1}{2}}(\mathbf{R}^2) \times H_{-\frac{1}{2}}(\mathbf{R}^2) \tag{37}$$

and the *Leontovich–boundary conditions* on $\Sigma^{\pm}$

$$\pm\underline{e}_z \wedge \underline{E}^{\pm} = \underline{\underline{Z}}'\left(\underline{e}_z \wedge \left(\underline{e}_z \wedge \underline{H}^{\pm}\right)\right) + \underline{f}'_{\pm} \tag{38}$$

and the transmission conditions on $\Sigma^c_{\pm}$

$$\underline{e}_z \wedge \left(\underline{E}^{+} - \underline{E}^{-}\right)\Big|_{z=0} = 0, \qquad \underline{e}_z \wedge \left(\underline{H}^{+} - \underline{H}^{-}\right)\Big|_{z=0} = 0. \tag{39}$$

In the case of traces — which exist on $z = 0$ in the sense of distributions —we use the same notations with $\mathbf{R}^2$ replacing $\mathbf{R}^3_\pm$.
In order that boundary and transmission conditions fit together we have to claim

$$\underline{f}'_+ - \underline{f}'_- \in [\tilde{H}_{-1/2}(\Sigma)]^3 \tag{40}$$

having supports on $\Sigma \subset \mathbf{R}^2$.
Due to the interface $z = 0$, being $\mathbf{R}^2$, the third components $(\underline{f}'_\pm)_3$ vanish and we may put

$$\underline{f}'_\pm = (\underline{f}^\pm, 0)^T \qquad \text{and} \qquad \underline{\underline{Z}}'_\pm = \begin{pmatrix} \underline{\underline{Z}}^\pm & 0 \\ 0 & 0 \end{pmatrix} \tag{41}$$

with vectors $\underline{f}^\pm \in \mathbf{C}^2$ and tensors $\underline{\underline{Z}}' \in \mathbf{C}^{2\times 2}$.
Due to the special geometry we may reformulate the problem $\mathcal{S}$ by the following

Problem $\mathcal{P}$: Find a function $\underline{u} = (\underline{u}_1, \underline{u}_2)^T \in \mathbf{C}^4$ such that $\underline{u}_l \in [L_2(\mathbf{R}^3)]^2$; $l = 1, 2$; with

$$\underline{u}^\pm = \underline{u}|_{\mathbf{R}^3_\pm} \in [H_{0,1}(\mathbf{R}^3_\pm) \times H_{1,0}(\mathbf{R}^3_\pm)]^2 \tag{42}$$

which is a weak solution of

$$\frac{\partial}{\partial z}\underline{u} = \begin{pmatrix} 0 & -(\mathrm{i}\omega\varepsilon_0)^{-1}(k_0^2\underline{\underline{I}}_2 + \underline{\underline{D}})\underline{\underline{N}} \\ (\mathrm{i}\omega\mu_0)^{-1}(k_0^2\underline{\underline{I}}_2 + \underline{\underline{D}})\underline{\underline{N}} & 0 \end{pmatrix} \underline{u} \tag{43}$$

in $\mathbf{R}^3_\pm$ with the differential matrix operator

$$\underline{\underline{D}} := \begin{pmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} \end{pmatrix} \tag{44}$$

and the $\pi/2$–rotation operator

$$\underline{\underline{N}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{45}$$

and fulfills the *Leontovich-boundary conditions*

$$\pm\underline{\underline{N}}\,\underline{u}^\pm_{10} + \underline{\underline{Z}}^\pm \underline{u}^\pm_{20} = \underline{f}^\pm \qquad \text{on} \quad \Sigma^\pm \tag{46}$$

and the *transmission condition*

$$\underline{u}^+_0 - \underline{u}^-_0 = 0 \qquad \text{on} \quad \Sigma^c \tag{47}$$

for $\underline{u}^\pm_{l,0} \in [H_{-\frac{1}{2},\frac{1}{2}}(\mathbf{R}^2) \times H_{\frac{1}{2},-\frac{1}{2}}(\mathbf{R}^2)]^2$; $l = 1, 2$; the traces of the electrical and magnetic tangential fields on $z = 0$. These data $\underline{f}^\pm$ have to fulfill the *compatibility condition*

$$\underline{f}^+ - \underline{f}^- \in [\tilde{H}_{-\frac{1}{2}}(\Sigma)]^2. \tag{48}$$

It can be shown then (see [17]) that the problems $\mathcal{S}$ and $\mathcal{P}$ are equivalent, after eliminating the normal components of the field vectors giving an equivalent 4×4 system for the four tangential components. Using then a 2D–Fourier–transformation w.r.t. (x, y), we get

Theorem 1 (Representation of tangential components). *A vector function* $\underline{u} \in [L_2(\mathbf{R}^3)]^4$ *with*

$$\underline{u}^{\pm} = \underline{u}|_{\mathbf{R}^3_\pm} \in [H_{0,1}(\mathbf{R}^3_\pm) \times H_{1,0}(\mathbf{R}^3_\pm)]^2 \tag{49}$$

is a (weak) solution of problem $\mathcal{P}$ iff it can be represented as

$$\underline{u}(x,y,z) = \mathcal{F}^{-1}_{\substack{\xi \to x \\ \eta \to y}} \left\{ \hat{\underline{u}}_0^+(\xi,\eta) \mathrm{e}^{-t(\xi,\eta)z} \chi_+(z) + \hat{\underline{u}}_0^-(\xi,\eta) \mathrm{e}^{t(\xi,\eta)z} \chi_-(z) \right\} \tag{50}$$

where $\hat{\underline{u}}_0 = \mathcal{F}\underline{u}_0$, $\chi_\pm$ are the characteristic functions of $\mathbf{R}_\pm$, the square root $t(\xi,\eta) := \sqrt{\xi^2+\eta^2-k_0^2}$ for $(\xi,\eta) \in \mathbf{R}^2$ and $\sqrt{\xi^2-k_0^2}$ continuable into the cut ζ-plane with branch cuts $\Gamma_\pm := \{\zeta \in \mathbf{C} : \zeta = \pm(k+\mathrm{i}\tau), \tau \geq 0\}$, $k \in \mathbf{C}^{++}$. The $\mathcal{F}$-transformed traces are given by

$$\hat{\underline{u}}_0^\pm(\xi,\eta) = \hat{\alpha}_\pm(\xi,\eta) \begin{pmatrix} 1 \\ \frac{\xi\eta}{\xi^2-k_0^2} \\ 0 \\ \pm\frac{\mathrm{i}\omega\varepsilon\, t(\xi,\eta)}{\xi^2-k_0^2} \end{pmatrix} + \hat{\beta}_\pm(\xi,\eta) \begin{pmatrix} 0 \\ \mp\frac{\mathrm{i}\omega\mu\, t(\xi,\eta)}{\xi^2-k_0^2} \\ 1 \\ \frac{\xi\eta}{\xi^2-k_0^2} \end{pmatrix} \tag{51}$$

and the eigenvector factors $\hat{\alpha}_\pm$ and $\hat{\beta}_\pm$ of the eigenvalue problem

$$(\underline{\underline{M}} + \mathrm{i}t\underline{\underline{I}}_4)\hat{\underline{u}}_0 = 0 \tag{52}$$

with

$$\underline{\underline{M}} = \begin{pmatrix} 0 & 0 & \frac{\xi\eta}{\omega\varepsilon_0} & \frac{k_0^2-\xi^2}{\omega\varepsilon_0} \\ 0 & 0 & -\frac{k_0^2-\eta^2}{\omega\varepsilon_0} & -\frac{\xi\eta}{\omega\varepsilon_0} \\ -\frac{\xi\eta}{\omega\mu_0} & -\frac{k_0^2-\xi^2}{\omega\mu_0} & 0 & 0 \\ \frac{k_0^2-\eta^2}{\omega\mu_0} & \frac{\xi\eta}{\omega\mu_0} & 0 & 0 \end{pmatrix}. \tag{53}$$

If the tangential components on $z=0$ are known, the Maxwell equations give

Theorem 2 (Representation formula for the normal components). *The two normal components of the electrical and magnetic field vectors from $[L_2(\mathbf{R}^3_\pm)]^2$ are given by*

$$\begin{aligned} v_1(x,y,z) &= \mathcal{F}^{-1}\left\{\frac{\xi t(\xi,\eta)}{\xi^2-k_0^2}\left(\mathrm{e}^{-t(\xi,\eta)z}\mathcal{F}\alpha_+\chi_+(z) - \mathrm{e}^{t(\xi,\eta)z}\mathcal{F}\alpha_-\chi_-(z)\right)\right\} \\ &- \mathcal{F}^{-1}\left\{\frac{\mathrm{i}\omega\mu_0\eta}{\xi^2-k_0^2}\left(\mathrm{e}^{-t(\xi,\eta)z}\mathcal{F}\beta_+\chi_+(z) - \mathrm{e}^{t(\xi,\eta)z}\mathcal{F}\beta_-\chi_-(z)\right)\right\} \end{aligned} \tag{54}$$

and

$$\begin{aligned} v_2(x,y,z) &= \mathcal{F}^{-1}\left\{\frac{\mathrm{i}\omega\varepsilon_0\eta}{\xi^2-k_0^2}\left(\mathrm{e}^{-t(\xi,\eta)z}\mathcal{F}\alpha_+\chi_+(z) + \mathrm{e}^{t(\xi,\eta)z}\mathcal{F}\alpha_-\chi_-(z)\right)\right\} \\ &+ \mathcal{F}^{-1}\left\{\frac{t(\xi,\eta)\xi}{\xi^2-k_0^2}\left(\mathrm{e}^{-t(\xi,\eta)z}\mathcal{F}\beta_+\chi_+(z) - \mathrm{e}^{t(\xi,\eta)z}\mathcal{F}\beta_-\chi_-(z)\right)\right\}. \end{aligned} \tag{55}$$

Remark 6. *If $\alpha_\pm$, $\beta_\pm \in H_{-\frac{1}{2},\frac{1}{2}}$ then the traces $\underline{u}_0^\pm \in H_{-\frac{1}{2},\frac{1}{2}}(\mathbf{R}^2) \times H_{\frac{1}{2},-\frac{1}{2}}(\mathbf{R}^2)$ which results from the ξ,η–orders of the anisotropic Fourier–multipliers in*

$$\underline{u}_0^\pm(x,y) = \mathcal{F}^{-1}\begin{pmatrix} 1 \\ \frac{\xi\eta}{\xi^2-k_0^2} \\ 0 \\ \pm\frac{i\omega\varepsilon t}{\xi^2-k_0^2} \end{pmatrix}\mathcal{F}\alpha_\pm + \mathcal{F}^{-1}\begin{pmatrix} 0 \\ \mp\frac{i\omega\mu_0 t}{\xi^2-k_0^2} \\ 1 \\ \frac{\xi\eta}{\xi^2-k_0^2} \end{pmatrix}\mathcal{F}\beta_\pm. \tag{56}$$

Remark 7. *This Sommerfeld half-plane problem may be generalized to higher order anisotropic Leontovich boundary conditions. On the other hand one gets the well-known scalar impedance boundary conditions of first and higher order as they were studied in* [5, 11, 17].

5. The Wiener–Hopf system for the anisotropic Leontovich–boundary value problem of a screen

Denoting by $H^+_{r,s}(\mathbf{R}^2)$ for $r,s \in \mathbf{R}$ the closed subspace of $H_{r,s}(\mathbf{R}^2)$ having support on

$$\overline{\mathbf{R}^2_+} := \left\{(x,y) \in \mathbf{R}^2 : y > 0\right\} \tag{57}$$

and by

$$\underline{\phi}^+ := \underline{u}_0^+ - \underline{u}_0^- \in [H^+_{-\frac{1}{2},\frac{1}{2}}(\mathbf{R}^2) \times [H^+_{\frac{1}{2},-\frac{1}{2}}(\mathbf{R}^2)]^2 \tag{58}$$

$$= \mathcal{F}^{-1}\begin{pmatrix} 1 & 0 & -1 & 0 \\ \frac{\xi\eta}{\xi^2-k_0^2} & -\frac{i\omega\mu_0 t(\xi,\eta)}{\xi^2-k_0^2} & -\frac{\xi\eta}{\xi^2-k_0^2} & -\frac{i\omega\mu_0 t(\xi,\eta)}{\xi^2-k_0^2} \\ 0 & 1 & 0 & -1 \\ \frac{i\omega\varepsilon_0 t(\xi,\eta)}{\xi^2-k_0^2} & \frac{\xi\eta}{\xi^2-k_0^2} & \frac{i\omega\varepsilon_0 t(\xi,\eta)}{\xi^2-k_0^2} & -\frac{\xi\eta}{\xi^2-k_0^2} \end{pmatrix}\mathcal{F}\begin{pmatrix} \alpha_+ \\ \beta_+ \\ \alpha_- \\ \beta_- \end{pmatrix} \tag{59}$$

which is in $[H^+_{-\frac{1}{2},\frac{1}{2}}(\mathbf{R}^2) \times H^+_{\frac{1}{2},-\frac{1}{2}}(\mathbf{R}^2)]^2$ if $\underline{\gamma} := (\alpha_+,\beta_+,\alpha_-,\beta_-)^T \in [H_{-\frac{1}{2},\frac{1}{2}}(\mathbf{R}^2)]^4$. It's easy to see then that the four–vectors $\underline{\gamma}$ and $\underline{\phi}^+$ are connected via

$$\underline{\gamma} = B\underline{\phi}^+ \tag{60}$$

with the invertible matrix pseudo–differential operator

$$B : \mathcal{F}^{-1}\underline{\underline{\sigma}}_B\mathcal{F} : [H^+_{-\frac{1}{2},\frac{1}{2}}(\mathbf{R}^2) \times H^+_{\frac{1}{2},-\frac{1}{2}}(\mathbf{R}^2)]^2 \to [H_{-\frac{1}{2},\frac{1}{2}}(\mathbf{R}^2)]^4 \tag{61}$$

with the symbol matrix

$$\underline{\underline{\sigma}}_B = \frac{1}{2}\begin{pmatrix} 1 & 0 & -\frac{\xi\eta}{t_1} & \frac{\xi^2-k_0^2}{t_1} \\ \frac{\xi\eta}{t_2} & -\frac{\xi^2-k_0^2}{t_2} & 1 & 0 \\ -1 & 0 & -\frac{\xi\eta}{t_1} & \frac{\xi^2-k_0^2}{t_1} \\ \frac{\xi\eta}{t_2} & -\frac{\xi^2-k_0^2}{t_2} & -1 & 0 \end{pmatrix} \tag{62}$$

where $t_1 := i\omega\varepsilon_0 t(\xi,\eta)$ and $t_2 := i\omega\mu_0 t(\xi,\eta)$. If we write the impedance tensors in the form

$$\underline{\underline{Z}}^\pm := \begin{pmatrix} a^\pm & b^\pm \\ c^\pm & d^\pm \end{pmatrix} \tag{63}$$

we can transform the Leontovich conditions into the equation

$$r_+ C\underline{\gamma} = \underline{f} = (\underline{f}^+, \underline{f}^-)^T \tag{64}$$

with the pseudo-differential operator

$$C = \mathcal{F}^{-1}\underline{\underline{\sigma}}_C\mathcal{F} : [H_{-\frac{1}{2},\frac{1}{2}}(\mathbf{R}^2)]^4 \to [H_{-\frac{1}{2}}(\mathbf{R}^2)]^4 \tag{65}$$

having the symbol $\underline{\underline{\sigma}}_C =$

$$\begin{pmatrix} -\frac{\xi\eta}{\xi^2-k_0^2} + \frac{t_1 b^+}{\xi^2-k_0^2} & \frac{t_2}{\xi^2-k_0^2} + a^+ + \frac{b^+\xi\eta}{\xi^2-k_0^2} & 0 & 0 \\ 1 + \frac{t_1 d^+}{\xi^2-k_0^2} & c^+ + \frac{d^+\xi\eta}{\xi^2-k_0^2} & 0 & 0 \\ 0 & 0 & \frac{\xi\eta}{\xi^2-k_0^2} - \frac{t_1 b^-}{\xi^2-k_0^2} & \frac{t_2}{\xi^2-k_0^2} + a^- + \frac{b^-\xi\eta}{\xi^2-k_0^2} \\ 0 & 0 & -1 - \frac{t_1 d^-}{\xi^2-k_0^2} & c^- + \frac{d^-\xi\eta}{\xi^2-k_0^2} \end{pmatrix}. \tag{66}$$

The relation $\det \underline{\underline{\sigma}}_c \neq 0$ is the condition of invertibility which leads to the conditions for $\underline{\underline{Z}}(\xi, \eta)$:

$$\underline{\xi}' \underline{\underline{Z}}^{\pm} \underline{\xi}'^T + \mathrm{i}k_0 Z_0 \left(\frac{\det \underline{\underline{Z}}^{\pm}}{Z_0^2} + 1 \right) t + (\mathrm{i}k_0)^2 \operatorname{tr} \underline{\underline{Z}}^{\pm} \neq 0 \tag{67}$$

for $\underline{\xi}' \in \mathbf{R}^2$. In the 2D-case, $\xi = 0$, we have

$$t(\eta) \neq -\frac{\mathrm{i}k_0}{2d^{\pm}} \left(Z_0 \det \underline{\underline{Z}}^{\pm} + \frac{1}{Z_0} \mp \sqrt{\left(Z_0 \det \underline{\underline{Z}}^{\pm} + \frac{1}{Z_0} \right)^2 - 4a^{\pm}d^{\pm}} \right). \tag{68}$$

In the classical isotropic case of impedance conditions $\underline{n} \wedge \underline{E} = \underline{\underline{Z}}\,\underline{n} \wedge (\underline{n} \wedge \underline{H})$ corresponding to

$$\frac{\partial E_n}{\partial n} - ik_0 \frac{Z}{Z_0} E_n = 0 \qquad \text{and} \qquad \frac{\partial H_n}{\partial n} - ik_0 \frac{Z_0}{Z} H_n = 0 \tag{69}$$

the necessary conditions read

$$t \neq \begin{cases} -i\omega\varepsilon_0 Z, \\ -i\omega\frac{\mu_0}{Z}. \end{cases} \tag{70}$$

Introducing the formula for $\underline{\gamma}$ (60) into (64), we arrive at a first relation of the Wiener–Hopf type

$$r_+ C B \underline{\phi}^+ = \underline{f} \tag{71}$$

After adding and substracting pairwise the corresponding four equations, we obtain the Wiener–Hopf equation

$$\tilde{W} \underline{\phi}^+ = \underline{\tilde{f}} \tag{72}$$

acting on the Dirichlet data jump across Σ with given data vector $\underline{\tilde{f}} := (\underline{f}^+ + \underline{f}^-, \underline{f}^+ - \underline{f}^-)^T$ and $\tilde{W} = r_+ W$ the restriction of the pseudo-differential operator

$$W := \mathcal{F}^{-1}\underline{\underline{\sigma}}\mathcal{F} : [H_{-\frac{1}{2},\frac{1}{2}}(\mathbf{R}^2) \times H_{\frac{1}{2},-\frac{1}{2}}(\mathbf{R}^2)]^2 \to [H_{-\frac{1}{2}}(\mathbf{R}^2)]^4 \tag{73}$$

with the Fourier–symbol $\underline{\underline{\sigma}} =$

$$\begin{pmatrix} \frac{a_+\xi\eta+b_+(\eta^2-k_0^2)}{ik_0Z_0t} & -1-\frac{a_+(\xi^2-k_0^2)+b_+\xi\eta}{ik_0Z_0t} & a_- & b_- \\ 1+\frac{c_+\xi\eta+d_+(\eta^2-k_0^2)}{ik_0Z_0t} & -\frac{c_+(\xi^2-k_0^2)+d_+\xi\eta}{ik_0Z_0t} & c_- & d_- \\ \frac{a_-\xi\eta+b_-(\eta^2-k_0^2)}{ik_0Z_0t} & -\frac{a_-(\xi^2-k_0^2)+b_-\xi\eta}{ik_0Z_0t} & Z_0\frac{\eta^2-k_0^2}{ik_0t}+a_+ & -Z_0\frac{\xi\eta}{ik_0t}+b_+ \\ \frac{c_-\xi\eta+d_-(\eta^2-k_0^2)}{ik_0Z_0t} & -\frac{c_-(\xi^2-k_0^2)+d_-\xi\eta}{ik_0Z_0t} & -Z_0\frac{\xi\eta}{ik_0t}+c_+ & Z_0\frac{\xi^2-k_0^2}{ik_0t}+d_+ \end{pmatrix} \tag{74}$$

where $a_\pm = a^+ \pm a^-$. All of this gives

Theorem 3 (Equivalence). *Problem $\mathcal{P}$ is uniquely solvable iff the Wiener–Hopf operator $\tilde{W}$ is invertible. In that case it holds with the trace vector $\underline{\gamma}$ of $\underline{u}$*

$$\underline{\phi}^+ = B^{-1}\underline{\gamma} = \mathcal{F}^{-1}\begin{pmatrix} 1 & 0 & -1 & 0 \\ \frac{\xi\eta}{\xi^2-k^2} & -\frac{t_2}{\xi^2-k^2} & -\frac{\xi\eta}{\xi^2-k^2} & -\frac{t_2}{\xi^2-k^2} \\ 0 & 1 & 0 & -1 \\ \frac{t_1}{\xi^2-k^2} & \frac{\xi\eta}{\xi^2-k^2} & \frac{t_1}{\xi^2-k^2} & -\frac{\xi\eta}{\xi^2-k^2} \end{pmatrix}\mathcal{F}\underline{\gamma} \tag{75}$$

for the solution of the Wiener–Hopf equation. If, on the other hand , $\underline{\phi}^+$ is a solution of equation (74) the function $\underline{u}$ given by Theorem 1 from which follows the data vector $\underline{\gamma}$ via $\underline{u}_0^\pm$ gives by

$$\underline{\gamma} = B\underline{\phi}^+ \tag{76}$$

a solution of problem $\mathcal{P}$.

Now it can be shown [17] that the system of the Wiener–Hopf equations (74) is not uniquely solvable, since $\tilde{W}$ has a nontrivial kernel. (It is shown there for the special case of $\underline{\underline{Z}}^+ = \underline{\underline{Z}}^- = \underline{\underline{Z}}$ where the 4×4–symbol matrix $\underline{\underline{\sigma}}$ decouples into two 2×2 blocks.) From the theory of impedance boundary value problems for the Sommerfeld half–plane in the case of the Helmholtz equation [5, 12] it is known that there exists a unique solution in smoother spaces $H_{1+\epsilon}(\mathbf{R}^2_+) \times H_{1+\epsilon}(\mathbf{R}^2_-)$. So we shall modify our Problem $\mathcal{P}$ to

Problem $\mathcal{P}_\epsilon$: Find a function $\underline{u} := (\underline{u}_1, \underline{u}_2)^T \in [L_2(\mathbf{R}^3)]^4$ with

$$\underline{u}^\pm = \underline{u}|_{\mathbf{R}^3_\pm} \in \left[H_{\epsilon,1+\epsilon}\left(\mathbf{R}^3_\pm\right) \times H_{1+\epsilon,\epsilon}\left(\mathbf{R}^3_\pm\right)\right]^2, \tag{77}$$

being a weak solution to the modified Maxwell equations

$$\frac{\partial}{\partial z}\underline{u} = \begin{pmatrix} 0 & -(i\omega\epsilon_0)^{-1}\left(k_0^2\underline{\underline{I}}_2 + \underline{\underline{D}}\right)\underline{\underline{N}} \\ (i\omega\mu_0)^{-1}\left(k_0^2\underline{\underline{I}}_2 + \underline{\underline{D}}\right)\underline{\underline{N}} & 0 \end{pmatrix}\underline{u} \tag{78}$$

with the matrix differential operator (44) and the rotational matrix $\underline{\underline{N}}$ in (45). The traces $\underline{u}_{l0}^\pm \in H_{-\frac{1}{2}+\epsilon,\frac{1}{2}+\epsilon}(\mathbf{R}^2) \times H_{\frac{1}{2}+\epsilon,-\frac{1}{2}+\epsilon}(\mathbf{R}^2)$ of the electrical, $l = 1$, and the magnetic, $l = 2$, field tangential components on $z = 0$ fulfill the boundary (46) and transmission conditions (47) on Σ and Σ^c, respectively. The given screen data have to satisfy the compatibility condition

$$\underline{f}^+ - \underline{f}^- \in \left[\tilde{H}_{-\frac{1}{2}+\varepsilon}(\Sigma)\right]^2. \tag{79}$$

The representation Theorems 1 and 2 remain valid. For $\varepsilon \neq n \in \mathbf{N}_0$ the compatibility condition holds trivially, since then the spaces $[\tilde{H}_{-\frac{1}{2}+\varepsilon}(\Sigma)]^2$ coincide with $[H_{-\frac{1}{2}+\varepsilon}(\Sigma)]^2$. The problem $\mathcal{P}_\varepsilon$ then may be equivalently reduced to the Wiener–Hopf system $\tilde{W}_\varepsilon \underline{\gamma} = \underline{f}$ with a pseudo-differential operator W_ε with the same symbol matrix $\underline{\underline{\sigma}}$ (74) such that

$$W_\varepsilon := \mathcal{F}^{-1}\underline{\underline{\sigma}}\mathcal{F} : [H^+_{-\frac{1}{2}+\varepsilon,\frac{1}{2}+\varepsilon}(\mathbf{R}^2) \times H^+_{\frac{1}{2}+\varepsilon,-\frac{1}{2}+\varepsilon}(\mathbf{R}^2)]^2 \to [H_{-\frac{1}{2}+\varepsilon}(\mathbf{R}^2)]^4 \tag{80}$$

is invertible iff the problem $\mathcal{P}_\varepsilon$ is uniquely solvable.
In the case of $\varepsilon \neq \mathbf{N}_0$ the system $\tilde{W}_\varepsilon \underline{\phi}^+ = \underline{\gamma}$ on Σ may be transformed into an equivalent 4×4–system of Wiener–Hopf type acting on $[L_2(\cdot, \mathbf{R})]^4$ with supports on $\overline{\mathbf{R}^+}$ and ξ acting as a parameter. Using Bessel potential operators

$$\Lambda_{+,\varepsilon} = \mathcal{F}^{-1}\underline{\underline{\sigma}}_{+,\varepsilon}\mathcal{F} : \left[H_{\frac{1}{2}+\varepsilon}(\cdot, \mathbf{R}) \times H_{-\frac{1}{2}+\varepsilon}(\cdot, \mathbf{R})\right]^2 \to [L_2(\cdot, \mathbf{R})]^4, \tag{81}$$

$$\Lambda_{-,\varepsilon} = \mathcal{F}^{-1}\underline{\underline{\sigma}}_{-,\varepsilon}\mathcal{F} : [L_2(\cdot, \mathbf{R})]^4 \to \left[H_{-\frac{1}{2}+\varepsilon}(\cdot, \mathbf{R})\right]^4, \tag{82}$$

with symbol matrices

$$\underline{\underline{\sigma}}_{+,\varepsilon} = \operatorname{diag}\left(t_+^{1+2\varepsilon}, t_+^{-1+2\varepsilon}, t_+^{1+2\varepsilon}, t_+^{-1+2\varepsilon}\right) \tag{83}$$

$$\underline{\underline{\sigma}}_{-,\varepsilon} = t_-^{1-2\varepsilon}\underline{\underline{I}}_4, \tag{84}$$

containing the factors from

$$t = \sqrt{\xi^2 + \eta^2 - k_0^2} = t_-(\cdot, \eta)t_+(\cdot, \eta) \qquad \text{w.r.t.} \quad \eta \tag{85}$$

and after fixing $\xi = 0$ we can replace the Wiener–Hopf operator $\tilde{W}_\varepsilon$ by

$$\tilde{W}_0(\varepsilon) = \mathcal{P}^+ W_0(\varepsilon)\Big|_{[L_2^+(\cdot,\mathbf{R})]^4} : \left[L_2^+(\cdot, \mathbf{R})\right]^4 \to \left[L_2^+(\cdot, \mathbf{R})\right]^4, \tag{86}$$

with

$$W_0(\varepsilon) := \mathcal{F}^{-1}\underline{\underline{\sigma}}_\varepsilon\mathcal{F} : [L_2(\cdot, \mathbf{R})]^4 \to [L_2(\cdot, \mathbf{R})]^4 \tag{87}$$

and the symbol matrix $\underline{\underline{\sigma}}_\varepsilon$ similar to (74) but now containing the factor $(t_+/t_-)^{1-2\varepsilon}$ in front and some places t_-/t_+. It follows

$$\underline{\underline{\sigma}}_\varepsilon = \left(\frac{t_+}{t_-}\right)^{1-2\varepsilon} \underline{\underline{\sigma}}\operatorname{diag}\left(t_+^{-2}, 1, t_+^{-2}, 1\right) \tag{88}$$

after all. The special case of 3D–lifting onto L_2 when $\underline{\underline{Z}}_+ = \underline{\underline{Z}}_-$, which reduces to the decoupled 2×2–symbol matrices of the electrical, $\underline{\underline{\sigma}}_\varepsilon^e$, and the magnetic, $\underline{\underline{\sigma}}_\varepsilon^m$, part was treated in [17]. This simplifies further in the case of 2D–lifting onto $[L_2(\mathbf{R})]^2$. In the 3D–case we obtain

$$\tilde{W}_0(\varepsilon)\underline{\Phi}^+ = \tilde{\Lambda}_{-,\varepsilon}^{-1}\underline{f} \tag{89}$$

with

$$\underline{\Phi}^+ := \tilde{\Lambda}_{+,\varepsilon}\underline{\phi}^+, \tag{90}$$

where $\tilde{W}_0(\varepsilon)$ has a symbol with piecewise continuous elements on $\dot{\mathbf{R}}$ having jumps at infinity due to $t_+/t_- \to \pm 1$ for $\eta \to \pm\infty$, ξ fixed.
So we may use the fact of the equivalence of Fredholmness of the operator $\tilde{W}_0(\varepsilon)$ and the general factorizability of its symbol [15] to get

Proposition 1. *The Wiener–Hopf operator $\tilde{W}_0(\varepsilon)$ is Fredholm iff*

$$\varepsilon \neq \frac{n-1}{2}, \qquad n \in \mathbf{N}, \tag{91}$$

and has then

$$\operatorname{Ind}\tilde{W}_0(\varepsilon) = 0 \qquad \textit{for} \quad \frac{1}{4} \leq \varepsilon < \frac{1}{2}. \tag{92}$$

Remark 8. *One has to show that the* augmented symbol matrix

$$\underline{\underline{\sigma}}'_\varepsilon = \begin{cases} \underline{\underline{\sigma}}_\varepsilon(\cdot,\eta+0)\mu + (1-\mu)\underline{\underline{\sigma}}_\varepsilon(\cdot,\eta) & : \ \eta \in \mathbf{R},\ \mu \in [0,1], \\ \underline{\underline{\sigma}}_\varepsilon(\cdot,-\infty)\mu + (1-\mu)\underline{\underline{\sigma}}_\varepsilon(\cdot,\infty) & : \ \mu \in [0,1], \end{cases} \tag{93}$$

has a nowhere vanishing determinant.

Remark 9. *In the two-dimensional case of electrically or magnetically transverse polarized waves one ends up with the two scalar impedance boundary conditions. The corresponding 2×2-symbol matrices of the 2D-lifted Wiener–Hopf system runs as*

$$\left(\frac{t_+}{t_-}\right)^{1-2\varepsilon} \begin{pmatrix} -1 - \frac{\lambda_0^+ + \lambda_0^-}{2t} & \frac{\lambda_0^+ - \lambda_0^-}{2t}\frac{t_-}{t_+} \\ -\frac{\lambda_0^+ - \lambda_0^-}{2t} & \frac{t_-}{t_+} + \frac{\lambda_0^+ + \lambda_0^-}{2t}\frac{t_-}{t_+} \end{pmatrix} \tag{94}$$

which is well known [5].

6. Factorization and explicit representation of the scattered field

In this section we consider the 2D–problem with identical layers on both screens of the half-plane. The impedance matrix now reads

$$\underline{\underline{Z}} = Z_0 \begin{pmatrix} \frac{a}{ik_0} & ik_0 b \\ \frac{c}{ik_0} & ik_0 d \end{pmatrix}. \tag{95}$$

Moreover, we restrict ε to the interval $\left[\frac{1}{4}, \frac{1}{2}\right)$. The Wiener–Hopf operator $\tilde{W}_0(\varepsilon)$ is then a Fredholm operator with index zero. This yields our symbols for the electric Dirichlet–jumps

$$\tilde{W}_0^{\mathrm{e}}(\varepsilon) = \mathcal{P}^+\mathcal{F}^{-1}\underline{\underline{\sigma}}_\varepsilon^{\mathrm{e}}\mathcal{F} : \left[L_2^+(\mathbf{R})\right]^2 \to \left[L_2^+(\mathbf{R})\right]^2, \tag{96}$$

with

$$\underline{\underline{\sigma}}_{\varepsilon}^{\mathrm{e}} = \left(\frac{t_+}{t_-}\right)^{1-2\varepsilon} \begin{pmatrix} b\frac{t_-}{t_+} & -1-\frac{a}{t} \\ \left(\frac{1}{t}+d\right)\frac{t_-}{t_+} & -\frac{c}{t} \end{pmatrix} \tag{97}$$

and for the magnetic Dirichlet–jumps

$$\tilde{W}_0^{\mathrm{m}}(\varepsilon) = \mathcal{P}^+\mathcal{F}^{-1}\underline{\underline{\sigma}}_{\varepsilon}^{\mathrm{m}}\mathcal{F} : \left[L_2^+(\mathbf{R})\right]^2 \to \left[L_2^+(\mathbf{R})\right]^2, \tag{98}$$

$$\underline{\underline{\sigma}}_{\varepsilon}^{\mathrm{m}} = \left(\frac{t_+}{t_-}\right)^{1-2\varepsilon} \begin{pmatrix} \frac{Z_0}{ik_0}\left(1+\frac{a}{t}\right)\frac{t_-}{t_+} & ik_0 Z_0 b \\ \frac{Z_0}{ik_0}\frac{c}{t}\frac{t_-}{t_+} & ik_0 Z_0\left(\frac{1}{t}+d\right) \end{pmatrix}. \tag{99}$$

Both symbols thus differ only by permutation of lines and columns and by a scalar factor:

$$\underline{\underline{\sigma}}_{\varepsilon}^{\mathrm{e}} = \underline{\underline{\sigma}}_{\varepsilon}^{\mathrm{m}} \begin{pmatrix} 0 & -\frac{ik_0}{Z_0} \\ \frac{1}{ik_0 Z_0} & 0 \end{pmatrix} \quad \text{and} \quad \underline{\underline{\sigma}}_{\varepsilon}^{\mathrm{m}} = \underline{\underline{\sigma}}_{\varepsilon}^{\mathrm{e}} \begin{pmatrix} 0 & ik_0 Z_0 \\ -\frac{Z_0}{ik_0} & 0 \end{pmatrix}. \tag{100}$$

This yields

Theorem 4 (Factorization). *The Wiener–Hopf operator $W_0^{\mathrm{e/m}}(\varepsilon)$ is invertible for $\varepsilon \in \left[\frac{1}{4}, \frac{1}{2}\right)$. The inverses, under the assumption* $\det \underline{\underline{Z}} = 0$, *are given by*

$$\tilde{W}_{0,\mathrm{e}}^{-1}(\varepsilon) = \mathcal{F}^{-1}\left(\underline{\underline{\sigma}}_{\varepsilon,+}^{\mathrm{e/m}}\right)^{-1} P^+ \left(\underline{\underline{\sigma}}_{\varepsilon,-}^{\mathrm{e/m}}\right)^{-1} : \left[L_2^+(\mathbf{R})\right]^2 \to \left[L_2^+(\mathbf{R})\right]^2 \tag{101}$$

with the Cauchy-projector $P^+ = \mathcal{F}^{-1}\mathcal{P}\mathcal{F}$ and the inverted positive and negative factors of the symbol $\left(\underline{\underline{\sigma}}_{\varepsilon,\pm}^{\mathrm{e/m}}\right)^{-1}$, i.e.,

$$\left(\underline{\underline{\sigma}}_{\varepsilon,+}^{\mathrm{e}}\right)^{-1} = \begin{pmatrix} t_+^{2\varepsilon} g_+^{-1} & t_+^{2\varepsilon} h_+ g_+^{-1} \\ -\frac{a}{c} t_+^{2\varepsilon-2} g_+^{-1} & \left(1-\frac{a}{c}t_+^{-1}h_+g_+^{-1}\right)t_+^{2\varepsilon-1} \end{pmatrix} \tag{102}$$

and for the second factor

$$\left(\underline{\underline{\sigma}}_{\varepsilon,-}^{\mathrm{e}}\right)^{-1} = \begin{pmatrix} \left(t_-^{-1}g_-^{-1}+h_-\right)t_-^{1-2\varepsilon} & \frac{a}{c}h_- t_-^{1-2\varepsilon} \\ t_-^{1-2\varepsilon} & \frac{a}{c}t_-^{1-2\varepsilon} \end{pmatrix} \tag{103}$$

with the Wiener factors $g_\pm \in \mathcal{W}^\pm$ of the function

$$g = b + \frac{a}{c}\left(\frac{1}{t}+\frac{a}{t^2}\right) = g_- g_+. \tag{104}$$

The split $h = P^+h + P^-h = h_+ + h_-$ for

$$h = \frac{1+\frac{a}{t}}{g_- t_-} \tag{105}$$

is calculated via Cauchy–projectors. The inverses of the operators $\underline{\underline{\sigma}}_\varepsilon^{\mathrm{m}}$ may be calculated with the relation

$$\left(\underline{\underline{\sigma}}_\varepsilon^{\mathrm{m}}\right)^{-1} = \begin{pmatrix} 0 & -\frac{ik_0}{Z_0} \\ \frac{1}{ik_0 Z_0} & 0 \end{pmatrix} \left(\underline{\underline{\sigma}}_\varepsilon^{\mathrm{e}}\right)^{-1}. \tag{106}$$

We assume now, for the more general physical case $\det \underline{\underline{Z}}^{\pm} \neq 0$, that on both banks $\Sigma^{\pm}$ of the Sommerfeld screen Σ the impedance matrix is the same, i.e.,

$$\underline{\underline{Z}}^{+} = \underline{\underline{Z}}^{-} = \underline{\underline{Z}} = Z_0 \begin{pmatrix} \frac{1}{ik_0}a & ik_0 b \\ \frac{1}{ik_0}c & ik_0 d \end{pmatrix} \tag{107}$$

and assume additionally $\frac{\partial}{\partial x} = 0$ corresponding to $\xi = 0$, then we are led to a block structure

$$\underline{\underline{\sigma}}_\varepsilon = \begin{pmatrix} \underline{\underline{\sigma}}_{\varepsilon,1} & 0 \\ 0 & \underline{\underline{\sigma}}_{\varepsilon,2} \end{pmatrix} \quad \in \mathbf{C}^{4\times 4} \tag{108}$$

which decouples the 4×4–Wiener–Hopf system into one 2×2 for the electric and one for the magnetic tangential components. It can be proved that the corresponding Wiener–Hopf operator $\tilde{W}_0(\varepsilon)$ may be inverted by a Neumann series expansion for the inverse symbol

$$\underline{\underline{\sigma}}_\varepsilon^{-1} = \lambda^{-1} \sum_{n=0}^{\infty} \left(\underline{\underline{I}}_2 - \lambda^{-1} \underline{\underline{\sigma}}_\varepsilon\right)^n \tag{109}$$

if a certain inequality containing the data of $\underline{\underline{Z}}$ is fulfilled, which is too long to be written here [17].

7. Final remarks

In this paper a certain type of generalized Leontovich boundary conditions on the faces of a Sommerfeld half–plane type screen $\Sigma \subset \mathbf{R}^3$ was studied using the Fourier–transform technique w.r.t. the coordinates $(x, y) \in \mathbf{R}^2$. In general anisotropic situation on $\Sigma^{\pm}$ the electromagnetic field cannot be seperated into two fields, like for the simple boundary conditions $\underline{E}_{tan} = 0$ or $\underline{H}_{tan} = 0$ on $\Sigma^{\pm}$. The existing plane waves are calculated by solving an eigenvalue problem for a 4×4 matrix which reduces to a block matrix of 2×2 submatrices only in special cases.
Still open for the general anisotropic case is the explicit factorization of the lifted symbol matrix. The corresponding initial boundary–transmission problems are still challenging as well. One has to find the general resolvent operator with the representation of its kernel function by generalized eigenfunctions [18, 23] for the scalar case in electrodynamics or acoustics for the Dirichlet and Rawlin's mixed boundary conditions on $\Sigma^{\pm}$. The corresponding problems for arbitrary wedges K_α or octants $O_{++} \subset \mathbf{R}^3$ are unsolved, too.

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PDEs, Motion Analysis and 3D Reconstruction from Movies

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Abstract. We study a non-linear second-order PDE related to the image processing problem called "structure from motion." Its close relation to the monodimensional heat equation permits to define weak solutions and establish existence and uniqueness properties. We also point out a variational interpretation and present numerical simulations.

Keywords: non-linear partial differential equations, variational methods, weak solutions, image processing

Classification: 35K55, 35D05. 35A15, 68U10

1. INTRODUCTION

Throughout this paper, we study the partial differential equation

$$u_t = u_{\theta\theta} - 2\frac{u_\theta}{u_x}u_{\theta x} + \left(\frac{u_\theta}{u_x}\right)^2 u_{xx}, \qquad (DCMA)$$

where $u(x,\theta,t)$ is a real-valued function depending on three scalar variables x (space), θ (time) and t (scale), and submitted to some boundary conditions. This is a degenerate parabolic equation which describes an anisotropic diffusion process: the right term is the second derivative of u evaluated in the direction $(-u_\theta/u_x, 1)$. Equation (DCMA) was introduced in [7] as the only multiscale analysis of movies compatible with the depth recovery (the so-called structure from motion problem). Evans also studied it as a way to extend the heat equation to multi-valued functions (see [6]).

The goal of structure from motion is to compute the tridimensional structure of objects that are seen by a camera from different points of view. In theory, two images are sufficient to recover the 3D structure of observed parts (this is the principle of stereovision), but in practice long sequences of images are used to guarantee robustness. Formally, this leads to consider *movies*, represented as functions u of two spatial coordinates x, y and the time coordinate θ. In the following, we shall only deal with *gray-level* movies, for which the scalar value $u(x,y,\theta)$ measures the intensity received by the camera at point (x,y) of the image plane and at time θ.

If the camera is moving along the x axis, pointing in a perpendicular direction Z toward a lambertian fixed surface, then it produces a movie u which ideally satisfies

$$-\frac{u_\theta}{u_x}(x,y,\theta) = -\frac{V(\theta)}{Z(M)}, \qquad (1)$$

where $V(\theta)$ is the camera speed at time θ and $Z(M)$ the depth of the physical point M projected in (x, y) on the focal plane at time θ. In practice, the computation of the left term $-u_\theta/u_x$ (which corresponds to the apparent velocity v induced on the image plane by the camera motion) requires some smoothing process, first because the low sampling rates in the digitization of real movies make the use of finite difference estimations hazardous, second in order to take advantage of the redundancy of the depth information contained in the movie.

The most relevant smoothing process found so far consists precisely of the multiscale representation $u(x, y, \theta, t)$ of a raw movie u_0, obtained as the solution u of the (DCMA) evolution satisfying the initial condition $u(x, y, \theta, 0) = u_0(x, y, \theta)$. Notice that the y variable is not involved in this equation and can be removed in its mathematical analysis. The following results justify in some way the use of the DCMA (see [8] or [9] for more precise formulations).

Theorem 1. *The DCMA is the only regular semigroup $T_t : u_0(\cdot) \mapsto u(\cdot, t)$ which is*

- *monotone: if $u_1 \leqslant u_2$, then $T_t u_1 \leqslant T_t u_2$ at any scale $t \geqslant 0$,*
- *contrast invariant: if $g : \mathbb{R} \to \mathbb{R}$ is continuous, then $T_t\, g(u) = g(T_t u)$,*
- *Galilean invariant: T_t commutes with the change of Galilean referential $(x, \theta) \mapsto (x - \alpha\theta + x_0, \theta + \theta_0)$,*
- *zoom invariant: T_t commutes with spatial homotheties $(x, \theta) \mapsto (\lambda x, \theta)$.*

Theorem 2. *The Depth Compatible Multiscale Analysis (DCMA) is the only regular monotone semigroup preserving the depth map of ideal movies: if u_0 has interpretation $Z(X, Y)$ (depth map) and $V_0(\theta)$ (camera speed), then $T_t u_0$ has interpretation $Z(X, Y)$ (same depth map) and $V(\theta, t)$, where $V(\theta, 0) = V_0(\theta)$ and $V_t = V_{\theta\theta}$ (that is, the camera speed interpretation is smoothed by the monodimensional heat equation).*

2. Classical and weak solutions of the DCMA

Equation (DCMA) can be written under the form $u_t = F(D^2u, Du)$, where Du and D^2u are the first and second derivatives of u and F is an elliptic operator (that is, nondecreasing with respect to D^2u for the natural order on symmetric 2x2 matrices). However, since F is not continuous, the theory of viscosity solutions of Crandall, Ishii and Lions (see [4]) does not apply. Even known generalizations to discontinuous functions F (e.g. Evans and Spruck [5] and Chen, Giga and Goto [3] for the Mean Curvature Motion) do not work for the DCMA, because the singularity at points where $u_x = 0$ is too strong. However, it is possible to give a definition of weak solutions of (DCMA) ensuring uniqueness and existence for some class of initial conditions by noticing (like Evans in [6]) that the DCMA is the level set formulation of the linear heat equation. More precisely, if u solves (DCMA) and if we parameterize a level curve $u(x, \theta, t) = cst$ under the form $x = \varphi(\theta, t)$, then this level curve should evolve according to the one-dimensional heat equation $\varphi_t = \varphi_{\theta\theta}$.

2.1. Classical solutions of the DCMA. For the reason we explained before, we forget the y variable in the following, and a movie is defined on $\mathbb{R} \times \overline{I}$, with either $I =]\theta_1, \theta_2[$ or $I = S^1$. In the space variable, a periodization has no meaning in terms of scene interpretation, so that we shall rather suppose that u tends toward some constant when x grows to infinity (we shall say that u is "constant at infinity").

Definition 1. *For $c = (c^-, c^+) \in \mathbb{R}^2$ and $n \geqslant 0$, C_c^n is the space of movies $u \in C^n(\mathbb{R} \times I)$ such that*

$$\sup_{\theta \in \overline{I}} |u(-x,\theta) - c^-| + |u(x,\theta) - c^+| \to 0 \quad \text{as} \quad x \to +\infty. \tag{2}$$

From now on, we write $\Omega = \mathbb{R} \times I \times]0, +\infty[$, i.e. $\overline{\Omega}$ is the domain of movie analyses.

Definition 2. *For $c \in \mathbb{R}^2$ and $n, p \geqslant 0$, $C_c^{n,p}$ is the space of movie analyses $u \in C^0(\overline{\Omega})$ such that*

1. $\forall T > 0, \quad \sup_{\theta \in \overline{I}, t \leqslant T} |u(-x,\theta,t) - c^-| + |u(x,\theta,t) - c^+| \to 0 \quad \text{as} \quad x \to +\infty,$
2. *on Ω, $(x,\theta,t) \mapsto u(x,\theta,t)$ is of class C^n with respect to (x,θ) and C^p with respect to t.*

Definition 3. *Given $u_0 \in C_c^0$, we say that u is a classical solution of the DCMA associated with the initial datum u_0 if*

(i) $u \in C_c^{2,1}$,

(ii) *on* $\Omega = \mathbb{R} \times I \times]0, +\infty[$, $\begin{cases} u_t = u_{\theta\theta} - 2\dfrac{u_\theta}{u_x} u_{\theta x} + (\dfrac{u_\theta}{u_x})^2 u_{xx} & \text{when} \quad u_x \neq 0, \\ u_t = 0 \quad \text{when} \quad u_x = 0. \end{cases}$

(iii) $\forall (x,\theta,t) \in \partial\Omega, \quad u(x,\theta,t) = u_0(x,\theta)$.

This definition ensures the uniqueness result thanks to the following.

Proposition 1 (comparison principle). *Let u and $\tilde{u}$ be classical solutions of the DCMA associated with initial data u_0 and $\tilde{u}_0$, respectively. If $u_0 \leqslant \tilde{u}_0$, then $u \leqslant \tilde{u}$ on Ω.*

The proof is rather classical: it consists of showing that for any $\alpha, T, R > 0$, the map $(x,\theta,t) \mapsto u(x,\theta,t) - \tilde{u}(x,\theta,t) - \alpha t$ attains its max value on the boundary of $[-R, R] \times \overline{I} \times [0, T]$, and then of sending α to zero and R to infinity.

Corollary 1 (uniqueness). *A classical solution of the DCMA associated with a given initial datum $u_0 \in C_c^2$ is unique.*

In order to ensure the existence of classical solutions of the DCMA, we now restrain the space of initial data.

Definition 4. *For $n \geqslant 1$, we write $\mathcal{V}_c^n$ the space of movies $u \in C_c^n$ for which there exists a movie $v \in C_0^{n-1}$ such that*

$$u_\theta + v u_x = 0 \quad \text{on} \quad \mathbb{R} \times \overline{I}. \tag{3}$$

v is called a velocity map of u. In addition, the space $\mathcal{V}_c^{n,p}$ is defined as elements of $C_c^{n,p}$ admitting a velocity map $v \in C_0^{n-1,p}$.

Remark: Consider a movie $u \in \mathcal{V}_c^n$. When $u_x(x,\theta) = 0$, (3) implies $u_\theta(x,\theta) = 0$, and if $n \geqslant 2$, differentiating (3) with respect to θ and x shows that $u_{\theta\theta} + 2vu_{\theta x} + v^2 u_{xx} = 0$ as soon as $u_x = 0$. A consequence is that if $u \in \mathcal{V}_c^{2,1}$ is a classical solution of the DCMA, then any velocity map v of u satisfies on Ω

$$\begin{cases} u_\theta + vu_x = 0 \\ u_t = u_{\theta\theta} + vu_{\theta x} + v^2 u_{xx}. \end{cases} \tag{4}$$

We now build explicit solutions of the DCMA. As we said before, the main idea is to notice that the trajectories (i.e. the curves $x(\theta)$ along which u is constant) are smoothed by the monodimensional heat equation. For that purpose, we need to introduce the natural domain I^* for such trajectories. If $I =]\theta_1, \theta_2[$ then $I^* = I$, and if $I = S^1$, then $I^* = \mathbb{R}$ (the natural injection $S^1 \hookrightarrow [0, 2\pi[\subset \mathbb{R}$ being implicit). To simplify the notations, we suppose in the following that $0 \in \overline{I}$.

Definition 5. *A map $\varphi \in C^n(\mathbb{R} \times I^*)$ $(n \geqslant 0)$ is a θ-**graph** of $u \in C_c^n$ if*

1. *for any $\theta \in \overline{I^*}$, the map $x \mapsto \varphi(x,\theta)$ is increasing and bijective (and φ_x does not vanish if $n \geqslant 1$).*
2. *for any $(x,\theta) \in \mathbb{R} \times \overline{I^*}$,*
$$u(\varphi(x,\theta),\theta) = u(x,0), \tag{5}$$
3. *for any $x \in \mathbb{R}$, $\varphi(x,0) = x$, and if $I = S^1$, then for any $(x,\theta) \in \mathbb{R} \times \overline{I^*}$,*
$$\varphi(x,\theta + 2\pi) = \varphi(\varphi(x,2\pi),\theta), \tag{6}$$
4. $\sup_{|x| \geqslant R, \theta \in \overline{I}} |\varphi_\theta(x,\theta)| \to 0$ as $R \to +\infty$ *(in a generalized sense if $n = 0$).*

Remark: Notice that in Condition 4, the sup is taken for $\theta \in \overline{I}$ and not for $\theta \in \overline{I^*}$. If $n = 0$, the term $|\varphi_\theta(x,\theta)|$ must be replaced by

$$\limsup_{h \to 0} \left| \frac{\varphi(x,\theta+h) - \varphi(x,\theta)}{h} \right|.$$

Proposition 2. *A movie $u \in C_c^n$ $(n \geqslant 2)$ belongs to $\mathcal{V}_c^n$ if and only if it admits a θ-graph of class C^n.*

Proposition 3. *Let $u_0 \in \mathcal{V}_c^n$ $(n \geqslant 2)$, and φ_0 be a θ-graph of u_0 of class C^n. Define $(x,\theta,t) \mapsto \varphi(x,\theta,t)$ as the unique solution of the monodimensional heat equation*

$$\frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial \theta^2} \tag{7}$$

on $\Omega^* = \mathbb{R} \times I^* \times]0, +\infty[$ submitted to the boundary condition

$$\forall (x,\theta,t) \in \partial\Omega^*, \quad \varphi(x,\theta,t) = \varphi_0(x,\theta). \tag{8}$$

Then the unique map $u : \overline{\Omega} \to \mathbb{R}$ defined by

$$\forall (x,\theta,t) \in \overline{\Omega}, \quad u(\varphi(x,\theta,t),\theta,t) = u_0(x,0) \tag{9}$$

belongs to $\mathcal{V}_c^{n,n}$ and is a classical solution of the DCMA associated with the initial datum u_0.

We think it is worth explaining here the link between the (DCMA) and the monodimensional heat equation stated in Proposition 3. Let us note $\mathbf{z}_1 = (\varphi(\mathbf{z}), \theta, t)$ for a given $\mathbf{z} \in \Omega$. If $u_x(\mathbf{z}_1) = 0$, differentiating (9) with respect to t yields

$$\varphi_t(\mathbf{z})u_x(\mathbf{z}_1) + u_t(\mathbf{z}_1) = u_t(\mathbf{z}_1) = 0$$

as expected. If $u_x(\mathbf{z}_1) \neq 0$, we obtain $u_t(\mathbf{z}_1) = -\varphi_t(\mathbf{z})u_x(\mathbf{z}_1)$,

$$\frac{d}{d\theta}\Big(u_0(x,0)\Big) = 0 = \varphi_\theta(\mathbf{z})u_x(\varphi(\mathbf{z}),\theta,t) + u_\theta(\varphi(\mathbf{z}),\theta,t), \qquad \text{and}$$

$$\begin{aligned}
\frac{d^2}{d\theta^2}(u_0(x,0)) = 0 &= \frac{d}{d\theta}\Big(\varphi_\theta(\mathbf{z})u_x(\varphi(\mathbf{z}),\theta,t) + u_\theta(\varphi(\mathbf{z}),\theta,t)\Big) \\
&= \varphi_{\theta\theta}(\mathbf{z})u_x(\mathbf{z}_1) + \varphi_\theta^2(\mathbf{z})u_{xx}(\mathbf{z}_1) + 2\varphi_\theta(\mathbf{z})u_{x\theta}(\mathbf{z}_1) + u_{\theta\theta}(\mathbf{z}_1) \\
&= \left(-u_t + u_{\theta\theta} - 2\frac{u_\theta}{u_x}u_{\theta x} + (\frac{u_\theta}{u_x})^2 u_{xx}\right)(\mathbf{z}_1).
\end{aligned}$$

Hence, u is a classical solution of the DCMA associated with the initial datum u_0. We now have the following.

Proposition 4 (existence). *Given an initial datum $u_0 \in \mathcal{V}_c^n$ ($n \geqslant 2$), there exists a unique classical solution of the DCMA, and it belongs to $\mathcal{V}_c^{n,n}$.*

Proposition 3 proves that the DCMA equation is a scalar formulation of the monodimensional heat equation (7), like two other important equations of image processing: the Mean Curvature Motion and the Affine Morphological Scale Space, which can be obtained by axiomatic formulations as well (see [1]). The difference between them only comes from the intrinsic parameter of the level lines: the Euclidean abscissa for the Mean Curvature Motion, the affine abscissa for the Affine Scale space. For the DCMA, the natural parameter is the time θ, which means that level lines are not considered as curves but as *graphs*. This remark allowed us to prove the existence of weak solutions for the DCMA, but in certain cases only: precisely, when the level lines of the initial datum can be described by graphs.

2.2. **Weak solutions of the DCMA.** We define weak (only continuous) solutions of the DCMA as uniform limits of classical solutions.

Definition 6. *Given a movie $u_0 \in C_c^0$, we say that a map $u \in C_c^{0,0}$ is a weak solution of the DCMA associated with the initial datum u_0 if*

$$\forall (x,\theta,t) \in \partial\Omega, \;\; u(x,\theta,t) = u_0(x,\theta)$$

and if there exists a sequence $(u^\varepsilon)_{\varepsilon>0}$ of classical solutions of the DCMA associated with the initial datum u_0 such that $u^\varepsilon \to u$ uniformly on $\overline{\Omega}$ when $\varepsilon \to 0$.

Proposition 5 (uniqueness). *A weak solution of the DCMA associated with a given initial datum is unique.*

Proposition 6 (existence). *Call $\overline{\mathcal{V}_c}$ the topological closure of $\mathcal{V}_c^2$ with respect to the $\|\cdot\|_\infty$ norm. Then, given $u_0 \in \overline{\mathcal{V}_c}$, there exists a unique weak solution u of the DCMA associated with the initial datum u_0.*

Once again, the uniqueness property results from a comparison principle. The existence can be shown using the approximation of the initial datum by elements of $\mathcal{V}_c^2$ and the existence property for regular solutions. One can also build an explicit weak solution, using the construction (the monodimensional heat equation) of Proposition 3.

Definition 7. *We write $\mathcal{V}_c^0$ the space of movies $u \in C_c^0$ which admit a continuous θ-graph.*

Proposition 7. *Let $u_0 \in \mathcal{V}_c^0$, and φ_0 be a θ-graph of u_0. Define $(x,\theta,t) \mapsto \varphi(x,\theta,t)$ as the unique solution of the monodimensional heat equation (7) submitted to the boundary condition (8). Then the unique map u defined from φ by (9) is a weak solution of the DCMA.*

A consequence of this characterization of weak solutions is that a weak solution of the DCMA associated with an initial datum $u_0 \in \mathcal{V}_c^n$ admits a kind of velocity movie as soon as $t > 0$, as stated by

Corollary 2. *Let u be the weak solution of the DCMA associated with an initial datum $u_0 \in \mathcal{V}_c^0$. If u is locally Lipschitz in the x variable, then there exists a continuous map v defined on $\Omega = \mathbb{R} \times I \times]0,+\infty[$ such that on Ω,*

$$\begin{aligned} u(x+\tau v(x,\theta,t),\theta+\tau,t) &= u(x,\theta,t)+o(\tau) \\ \text{and} \quad u(x+\tau v(x,\theta,t),\theta+\tau,t-\frac{\tau^2}{2}) &= u(x,\theta,t)+o(\tau^2). \end{aligned}$$

Notice that this property is a generalization of (4).

2.3. **Further existence properties.** In the previous sections, we did not prove the existence of (weak or classical) solutions of the DCMA in the general case, that is to say when the initial datum admits no θ-graph. In fact, we do not believe that the DCMA admits a solution in general, at least a solution in the sense we defined. When the initial datum u_0 admits a θ-graph, the DCMA is obtained by applying the linear monodimensional heat equation to the level lines of u_0. For an ordinary continuous map u_0, the level lines have no reason to be graphs in the θ variable, since to a given value of θ, several values of x will correspond in general. Hence, defining general solutions of the DCMA is somewhat equivalent to defining solutions of the heat equation for multi-valued data. It is in that spirit that in [6] Evans studied (DCMA) as the limit when $\varepsilon \to 0$ of the more regular equation

$$u_t = \frac{u_x^2 u_{\theta\theta} - 2u_x u_\theta u_{x\theta} + u_\theta^2 u_{xx}}{u_x^2 + \varepsilon^2 u_\theta^2}. \tag{10}$$

Equation (10) admits viscosity solutions because it is more or less the Mean Curvature Motion (actually, the case $\varepsilon = 1$ is exactly the Mean Curvature Motion). He noticed that in the general case (that is, when the level lines of the initial datum are not graphs), the regularizing effects of the heat equation are so strong that the limit of approximate solutions is not continuous at scale $t = 0$, because the level lines are constrained to become graphs instantaneously.

3. Variational interpretation of the DCMA

Proposition 8. *The DCMA induces on v a flow associated with the minimization of*

$$\mathcal{E}(v) = \frac{1}{2}\iint (v_\theta + vv_x)^2\, dx d\theta. \tag{11}$$

Let us consider the functional $\mathcal{E}(v)$ defined by (11) on compactly supported movies of class C^2. Differentiating $\mathcal{E}$ yields, after integrations by parts,

$$D_v\mathcal{E}(h) = -\iint \frac{\mathrm{D}^2 v}{\mathrm{D}\theta^2} h\, dx d\theta,$$

where $\frac{\mathrm{D}}{\mathrm{D}\theta} = \frac{\partial}{\partial\theta} + v\frac{\partial}{\partial x}$ represents the total derivative operator. Then, for a classical solution of the DCMA $u \in \mathcal{V}_0^{4,1}$ associated with a compactly supported initial datum and admitting a velocity map v, one has

$$\frac{d}{dt}\Big(E(v)\Big) = -\iint \left(\frac{\mathrm{D}^2 v}{\mathrm{D}\theta^2}\right)^2 dx d\theta,$$

which means that the flow induced on v by the DCMA is associated with the minimization of $\mathcal{E}$. This minimization property proves that the DCMA "idealizes" movies and tends to give them a coherent depth interpretation as scale increases, since the apparent acceleration $Dv/D\theta = v_\theta + vv_x$ is globally decreasing.

4. Numerical scheme

In order to apply the DCMA evolution to real movies, we need to devise a numerical scheme. A "naive" discretization of the partial derivatives of u cannot be used, because in practice it is well known that the time discretization is not thin enough. Moreover, such a discretization is not likely to satisfy the axioms that we imposed to the DCMA. This is the reason why we focus our attention on an inf-sup scheme. To this end, given a movie $u : \mathbb{R}^2 \times \overline{I} \to \mathbb{R}$, we define

$$\begin{aligned} IS_h u(x_0, y_0, \theta_0) &= \inf_{v\in\mathbb{R}} \sup_{-h\leqslant\theta\leqslant h} u(x_0 + v\theta, y_0, \theta_0 + \theta), \\ SI_h u(x_0, y_0, \theta_0) &= \sup_{v\in\mathbb{R}} \inf_{-h\leqslant\theta\leqslant h} u(x_0 + v\theta, y_0, \theta_0 + \theta), \\ \text{and } \; T_h u &= \frac{1}{2}\left(IS_h u + SI_h u\right). \end{aligned}$$

We have a consistency result (see [9] for a proof) at points where u_x does not vanish.

Theorem 3. *If u is a bounded movie locally C^3 near $\mathbf{z}_0$, with $u_x(\mathbf{z}_0) \neq 0$, then*

$$T_h u(\mathbf{z}_0) = u(\mathbf{z}_0) + \frac{1}{2} h^2 u_{\xi\xi}(\mathbf{z}_0) + O(h^3),$$

and the $O(h^3)$ is uniform in a neighborhood of $\mathbf{z}_0$.

Theorem 3 proves the consistency of the numerical scheme given by the iteration of T_h with respect to the DCMA evolution. Due to the h^2 coefficient in the expansion of T_h, it is natural to consider the numerical scheme which associates, to a given movie u_0 and a scale $t \geqslant 0$, the sequence of movies $(u_{n,t})_{n\geqslant 1}$ given by

$$u_n = (T_{h_n})^n u_0, \quad \text{with} \quad h_n = \sqrt{2t/n},$$

and satisfying the boundary constraint $u_n(x, y, \theta) = u_0(x, y, \theta)$ on $\partial(\mathbb{R}^2 \times I)$. Thanks to Theorem 3, we know that such a scheme is consistent, and one could prove that u_n converges toward the DCMA of u_0 when the partial derivative of u_0 with respect to x never vanishes. In the general case, the existence of a solution is not guaranteed, even if numerically the monotonicity of the scheme ensures the convergence of the algorithm. In fact, at singular points where no velocity can be defined, the scheme should produce an instantaneous evolution, as stated by the following.

Proposition 9. *Let $P(x, \theta)$ be a polynomial with degree at most two and such that $P_x(x_0, \theta_0) = 0$. Then in (x_0, θ_0) we have, as $h \to 0$,*

$$T_h P = P + \frac{h}{2} |P_\theta| \operatorname{sgn}(P_{xx}) + O(h^2).$$

Proposition 9 suggests that the numerical scheme we proposed may induce a projection of the initial datum from $\mathcal{C}_c^0$ to $\mathcal{V}_c^0$, defined by the asymptotic state of

$$u_t = \begin{cases} |u_\theta| \operatorname{sgn}(u_{xx}) & \text{if} \quad u_x = 0, \\ 0 & \text{else.} \end{cases}$$

Notice that if we follow Evans (see [6]) and consider the DCMA as the limit of (10), we obtain a different projection operator in general.

One may notice the extreme simplicity of the algorithm we presented: in particular, it can be implemented very easily on a massive parallel machine. Our optimized code in C language for one iteration consists of only 23 instructions. Numerical simulations realized with this algorithm are presented in Figure 1.

5. Conclusion

We presented a study of the DCMA equation, based on its interpretation as a level set formulation of the monodimensional heat equation. When the initial condition admits trajectories, we prove the existence and uniqueness of weak solutions. For general initial conditions, difficulties appear because the time variable imposes one to consider level curves as graphs. Defining solutions in that case would probably require a weaker formulation of the DCMA allowing occlusion fronts to arise and propagate. The variational interpretation we pointed out might then be helpful to build the proper

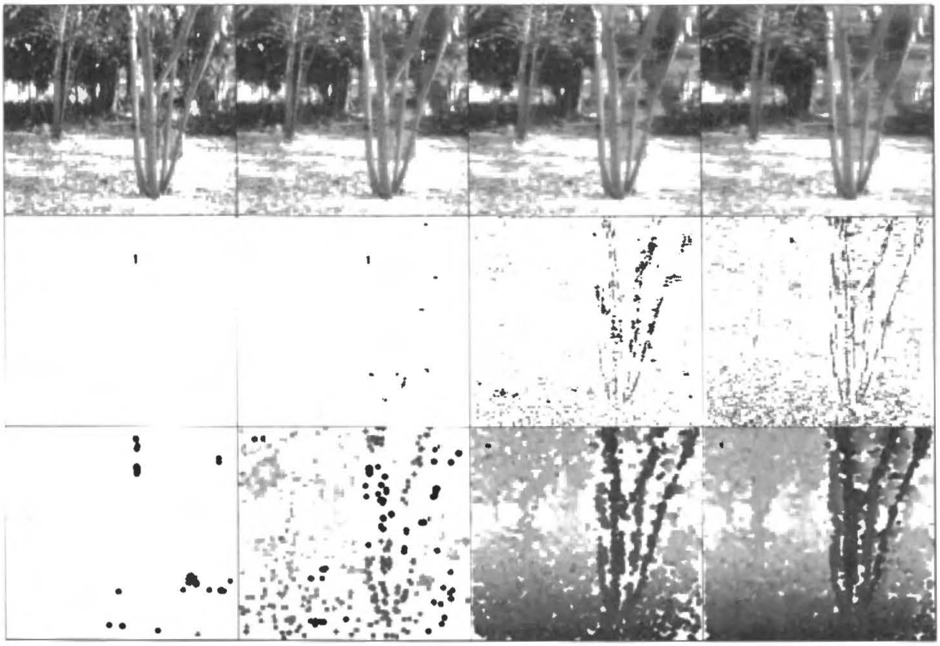

FIGURE 1. Computation of the velocity field (minimum of 15 matchings).

The following results were produced from a real movie produced by the SRI center (see [2]*) and available with anonymous ftp at* periscope.cs.umass.edu*. The four images on the first row are taken from four different movies: each image is the 20th image (over 64) of the movie it belongs to: column 1: original "TREES" movie; column 2: movie processed with DCMA (5 iterations); column 3: processed movie (15 iterations); column 4: processed movie (31 iterations). Then the velocity field of each movie was computed on the 20th image simply by looking for trajectories with a matching constraint of 15 images. These velocities are represented on row 2: the white color means points where no matching was found with respect to the constraint, and the grey scale (from light grey to black) measures the velocity from 0.0 to 2.0 pixels per image. On the third row, the velocity images of row 2 were simply "dilated" to produce more readable results. Notice how the velocity information, which is almost nonexistent on the original movie (for the matching constraint we imposed), progressively appears on the DCMA as the scale increases. Since the distance of objects to the image plane is inversely proportional to their velocity, closest points appear in black and farthest ones in light grey. On the last image of row 3, we distinguish the foreground tree in black, the ground from black to middle grey, the background tree in middle grey, and the far background in light grey.*

definition of solution in the case of occlusions. It is not sure, however, that an inf-sup-like scheme would still exist then and allow to estimate *indirectly* the velocity field.

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Steady Flow of Viscoelastic Fluid Past an Obstacle — Asymptotic Behaviour of Solutions

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Abstract. We study steady flows of a certain class of viscoelastic fluids past an obstacle in two and three space dimensions with non-zero velocity prescribed at infinity. Decomposing the original problem into the (modified) Oseen problem and transport equations we prove for sufficiently small data the existence of solutions with almost the same asymptotic properties at infinity as the fundamental solution to the Oseen problem.

Keywords: viscoelastic fluid, Oseen problem, steady transport equation, wake region, asymptotic properties of solutions

Classification: 35B40, 76A10

The steady flow of an incompressible fluid is described by the following system of equations:

$$\nabla \cdot \mathbf{v} = 0 \tag{1}$$

$$\varrho(\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = \nabla \cdot \mathbf{T} + \varrho \mathbf{f}\,, \tag{2}$$

where $\mathbf{v} = (v_1, \dots, v_N)$ is the velocity field, p the pressure, $\varrho = \text{const}$ the density, $\mathbf{f}$ the external force and $\mathbf{T}$ the extra stress tensor, which characterizes the model of the fluid. Equation (1) expresses the balance of mass, equation (2) the balance of momentum.

We shall assume that $\mathbf{T}$ obeys the following differential equation:

$$\mathbf{T} + \lambda \frac{\mathcal{D}_a \mathbf{T}}{\mathcal{D}t} + \mathbf{B}(\mathbf{D}, \mathbf{T}) = 2\eta \mathbf{D}\,, \tag{3}$$

where

$$\frac{\mathcal{D}_a \mathbf{T}}{\mathcal{D}t} = (\mathbf{v} \cdot \nabla)\mathbf{T} + \mathbf{T}\mathbf{W} - \mathbf{W}\mathbf{T} - a(\mathbf{D}\mathbf{T} + \mathbf{T}\mathbf{D}) \tag{4}$$

represents objective derivative of a symmetric tensor in the stationary case, $\mathbf{D}$ is the symmetric part and $\mathbf{W}$ the skew part of the gradient of velocity, $\mathbf{B}(\mathbf{D}, \mathbf{T})$ will be in our case a bilinear tensor-valued function, λ characterizes the relaxation time, $\eta > 0$ is a viscosity and $a \in [-1; 1]$ is a given real parameter (see e.g. [1] for other possible choices of $\mathbf{B}$ and for characteristics of other models of viscoelastic fluids).

Let us mention that our model (1)–(3) still involves several physically reasonable models like e.g. the lower convective, corotational and upper convective Maxwell fluid.

[1]Supported by the Grant Agency of the Czech Republic, grant No. 201/96/0228

Following [11] we can reformulate the system (1)–(3). Namely, denoting

$$\mathbf{F}(\nabla\mathbf{v},\mathbf{T}) = -\lambda\mathbf{T}(\nabla\mathbf{v})^T - \lambda(\mathbf{TW}-\mathbf{WT}) + \lambda a(\mathbf{DT}+\mathbf{TD}) - \mathbf{B}(\mathbf{D},\mathbf{T}) \tag{5}$$

$$\mathbf{G}(\nabla\mathbf{v},\mathbf{T}) = \lambda(\mathbf{TW}-\mathbf{WT}) - \lambda a(\mathbf{DT}+\mathbf{TD}) + \mathbf{B}(\mathbf{D},\mathbf{T}) \tag{6}$$

we end up with the following system:

$$\begin{gathered} -\eta\Delta\mathbf{v} + \varrho(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\pi = \varrho\mathbf{f} + \nabla\cdot[\mathbf{F}(\nabla\mathbf{v},\mathbf{T}) - \lambda\varrho((\mathbf{v}\cdot\nabla)\mathbf{v})\otimes\mathbf{v}+ \\ +\lambda\varrho\mathbf{f}\otimes\mathbf{v} + \lambda p\nabla\mathbf{v}^T] \\ \nabla\cdot\mathbf{v} = 0 \\ p + \lambda(\mathbf{v}\cdot\nabla)p = \pi \\ \mathbf{T} + \lambda(\mathbf{v}\cdot\nabla)\mathbf{T} + \mathbf{G}(\nabla\mathbf{v},\mathbf{T}) = 2\eta\mathbf{D}(\mathbf{v})\,. \end{gathered} \tag{7}$$

We now formulate the boundary value problem for the system (5)–(7). Let $\Omega = \mathbb{R}^N \setminus \mathcal{B}$ be an exterior domain in $\mathbb{R}^N$; without loss of generality we can assume $\mathcal{B} \subset B_1(\mathbf{0})$ and $\mathbf{0} \in \mathcal{B}$.

We shall assume $\mathbf{v} = \mathbf{0}$ at $\partial\Omega$ and $\mathbf{v} \to \mathbf{v}_\infty = \beta\mathbf{e}_1$ as $|\mathbf{x}| \to \infty$. Throughout this paper we shall assume the constant velocity at infinity $\mathbf{v}_\infty$ small but non-zero. Denoting $\mathbf{u} = \mathbf{v} - \mathbf{v}_\infty$ we get

$$\begin{gathered} -\eta\Delta\mathbf{u} + \varrho\beta\frac{\partial\mathbf{u}}{\partial x_1} + \nabla\pi = \\ = \varrho\mathbf{f} + \nabla\cdot\Big[\mathbf{F}(\nabla\mathbf{u},\mathbf{T}) - \lambda\varrho((\mathbf{u}\cdot\nabla)\mathbf{u})\otimes\mathbf{u} - \varrho\mathbf{u}\otimes\mathbf{u} - \lambda\varrho\beta^2\frac{\partial\mathbf{u}}{\partial x_1}\otimes\mathbf{e}_1 - \\ -\lambda\varrho\beta\Big(\frac{\partial\mathbf{u}}{\partial x_1}\otimes\mathbf{u} + ((\mathbf{u}\cdot\nabla)\mathbf{u})\otimes\mathbf{e}_1\Big) + \lambda\varrho\mathbf{f}\otimes(\mathbf{u}+\beta\mathbf{e}_1) + \lambda p(\nabla\mathbf{u})^T\Big] \end{gathered} \tag{8}$$

$$\begin{gathered} \mathbf{u} = -\beta\mathbf{e}_1 \text{ at } \partial\Omega \\ \mathbf{u} \to \mathbf{0} \text{ as } |\mathbf{x}| \to \infty \end{gathered} \tag{9}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{10}$$

$$\pi = p + \lambda((\mathbf{u}+\mathbf{v}_\infty)\cdot\nabla)p \tag{11}$$

$$\mathbf{T} + \lambda((\mathbf{u}+\mathbf{v}_\infty)\cdot\nabla)\mathbf{T} + \mathbf{G}(\nabla\mathbf{u},\mathbf{T}) = 2\eta\mathbf{D}(\mathbf{u})\,, \tag{12}$$

where $\mathbf{F}(\nabla\mathbf{v},\mathbf{T}) = \mathbf{F}(\nabla\mathbf{u},\mathbf{T})$, $\mathbf{G}(\nabla\mathbf{v},\mathbf{T}) = \mathbf{G}(\nabla\mathbf{u},\mathbf{T})$ satisfy (5) and (6), respectively. We linearize the system (8)–(9) around $\mathbf{u} = \mathbf{0}$. Denoting $A(\mathbf{u}) = -\eta\Delta\mathbf{u} + \beta^2\lambda\varrho\frac{\partial^2\mathbf{u}}{\partial x_1^2}$ we introduce the operator

$$\mathcal{M}: \quad (\mathbf{w},s) \mapsto (\mathbf{u},\pi)\,,$$

where

$$\begin{gathered} A(\mathbf{u}) + \varrho\beta\frac{\partial\mathbf{u}}{\partial x_1} + \nabla\pi = \\ = \varrho\mathbf{f} + \nabla\cdot\Big[\mathbf{F}(\nabla\mathbf{w},\mathbf{T}) - \lambda\varrho((\mathbf{w}\cdot\nabla)\mathbf{w})\otimes\mathbf{w} - \varrho\mathbf{w}\otimes\mathbf{w} - \\ -\lambda\varrho\beta\Big(\frac{\partial\mathbf{w}}{\partial x_1}\otimes\mathbf{w} + ((\mathbf{w}\cdot\nabla)\mathbf{w})\otimes\mathbf{e}_1\Big) + \lambda\varrho\mathbf{f}\otimes(\mathbf{w}+\beta\mathbf{e}_1) + \lambda p(\nabla\mathbf{w})^T\Big] \end{gathered} \tag{13}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{14}$$

$$\begin{aligned} \mathbf{u} \to 0 \text{ as } |\mathbf{x}| \to \infty \\ \mathbf{u} = -\beta \mathbf{e}_1 \text{ at } \partial\Omega \end{aligned} \tag{15}$$

$$p + \lambda((\mathbf{w} + \mathbf{v}_\infty) \cdot \nabla)p = s \tag{16}$$

$$\mathbf{T} + \lambda((\mathbf{w} + \mathbf{v}_\infty) \cdot \nabla)\mathbf{T} + \mathbf{G}(\nabla\mathbf{w}, \mathbf{T}) = 2\eta\mathbf{D}(\mathbf{w})\,. \tag{17}$$

We have decomposed the original problem into a Oseen-like problem (13)–(15) and transport equations (16)–(17). Due to the presence of the Oseen-like problem we expect (compare with [3], [12] for the incompressible or [9], [2] for the compressible Navier–Stokes equations) that the structure of the solutions corresponds to the structure of the fundamental Oseen tensor (i.e. the existence of a wake region). Denoting $s(\mathbf{x}) = |\mathbf{x}| - x_1$ and $\eta_b^a(\mathbf{x}) = (1 + |\mathbf{x}|)^a(1 + s(\mathbf{x}))^b$ we would like to prove that for $N = 2$

$$u_1\eta^{\frac{1}{2}}_{\frac{1}{2}},\, u_2\eta^1_0,\, \nabla\mathbf{u}\eta^1_1 \in L^\infty(\Omega)$$

and for $N = 3$

$$\mathbf{u}\eta^1_1,\, \nabla\mathbf{u}\eta^{\frac{3}{2}}_{\frac{3}{2}} \in L^\infty(\Omega)\,,$$

or something very close to this.

For this purpose it would be easier to involve the additional term on the left-hand side, $\beta^2\lambda\varrho\frac{\partial^2\mathbf{u}}{\partial x_1^2}$, into the right-hand side and instead of (13) we could consider

$$-\eta\Delta\mathbf{u} + \varrho\beta\frac{\partial\mathbf{u}}{\partial x_1} + \nabla\pi = N_1(\mathbf{w}, \mathbf{T}, p, \mathbf{f}) \tag{18}$$

with $N_1(\mathbf{w}, \mathbf{T}, p, \mathbf{f}) = N(\mathbf{w}, \mathbf{T}, p, \mathbf{f}) - \varrho\beta^2\lambda\frac{\partial^2\mathbf{w}}{\partial x_1^2}$, $N(\mathbf{w}, \mathbf{T}, p, \mathbf{f})$ denoting the right-hand side of (13). Unfortunately, in such a case we would loose the possibility to study the asymptotic behaviour of solutions to the original non-linear problem. This is connected with the fact that the weighted estimates for solutions to the Oseen problem are obtained from the integral representation of solutions (see e.g. [6]).

The right-hand side of (18) contains one linear term. But the weighted estimates for kernels representing second gradient of the fundamental Oseen tensor are not optimal. For example, for $R = D^2\mathcal{O}_{ij}$, where $\mathcal{O}_{ij}$ is the fundamental Oseen tensor (see e.g. [4]) we can only show that

$$Tf = \text{v.p.}(R * f)$$

maps $L^p(\mathbf{R}^N; \eta_b^a)$ into $L^p(\mathbf{R}^N; \eta_b^{a-\varepsilon})$ for any $\varepsilon > 0$[1] and for some a, b depending on N. This implies that we cannot leave the linear term on the right-hand side and we must involve it into the left-hand side. The weighted estimates for the transport equations are proved in [7].

Now, on the left-hand side of (13) we have an operator which is similar to the (classical) Oseen problem. We shall call such a problem modified Oseen problem. It can be verified (see [10]) that the properties of the modified Oseen problem are similar to those of the Oseen problem and sometimes we can directly apply the results for the

[1] More precisely, there appear some logarithmic factors, see [6].

(classical) Oseen problem (as e.g. the weighted L^p–estimates); sometimes only very small changes are needed.

Let us study the linearized system (13)–(17). Our proof is based on the following version of the Banach fixed point theorem.

Theorem 1. *Let X, Y be Banach spaces such that X is reflexive and $X \hookrightarrow Y$. Let H be non-empty, closed, convex and bounded subset of X and let $T : H \mapsto H$ be a mapping such that*

$$\|T(u) - T(v)\|_Y \leq \kappa \|u - v\|_Y \quad \forall u, v \in H,$$

$0 \leq \kappa < 1$. Then T has a unique fixed point in H.

We combine the results for the (modified) Oseen problem (see [10] or for the (classical) Oseen problem see [4], [5]) with the results for the transport equation (see [7]) and using Theorem 1 we show the following (see e.g. [10]).

Theorem 2. *Let $\Omega \in C^{k+1}$ be an exterior domain in $\mathbb{R}^3$ and let one of the following assumptions be satisfied:*

(i) $\mathbf{f} \in D_0^{-1,2}(\Omega) \cap W^{k,2}(\Omega)$, $k \geq 2$
(ii) $\mathbf{f} \in D_0^{-1,2}(\Omega) \cap L^2(\Omega) \cap W^{k,p}(\Omega)$, $k \geq 1$, $3 < p \leq 4$
(iii) $\mathbf{f} \in W^{1,q}(\Omega) \cap W^{k,p}(\Omega)$, $k \geq 1$, $p \in (3; \infty)$, $q = \frac{4}{3}$ $(q = \frac{6}{5}$ *if* $k = 1)$
(iv) $\mathbf{f} \in W^{1,\frac{4}{3}}(\Omega) \cap W^{k,2}(\Omega)$, $k \geq 2$

and let β_0 and the corresponding norms of $\mathbf{f}$ be sufficiently small. Then for any $\beta \in (0; \beta_0]$ there exists a solution to the problem (8)–(9). Moreover[2]

(i) $\mathbf{u} \in L^4(\Omega)$; $\nabla \mathbf{u}, \pi, p \in W^{k,2}(\Omega)$
(ii) $\mathbf{u} \in L^4(\Omega) \cap D^{1,2}(\Omega)$; $p, \pi \in L^2(\Omega)$; $\nabla^2 \mathbf{u}, \nabla p, \nabla \pi \in W^{k-1,p}(\Omega)$
(iii) $\mathbf{u} \in L^{\frac{2q}{2-q}}(\Omega) \cap D^{1,\frac{4q}{4-q}}(\Omega)$; $\nabla^2 \mathbf{u}, \nabla p, \nabla \pi \in L^q(\Omega) \cap W^{k-1,p}(\Omega)$; $p, \pi \in L^{\frac{3q}{3-q}}(\Omega)$
(iv) $\mathbf{u} \in L^{\frac{2q}{2-q}}(\Omega) \cap D^{1,\frac{4q}{4-q}}(\Omega)$; $\nabla^2 \mathbf{u}, \nabla p, \nabla \pi \in L^q(\Omega) \cap W^{k-1,2}(\Omega)$; $p, \pi \in L^{\frac{3q}{3-q}}(\Omega)$.

In two space dimensions we have for

$$S^k_{p,q} = \Big\{(\mathbf{u}, \pi); u_2 \in L^{\frac{2q}{2-q}}(\Omega), \nabla u_2 \in L^q(\Omega), \mathbf{u} \in L^{\frac{3q}{3-2q}}(\Omega),$$
$$\nabla \mathbf{u} \in L^{\frac{3q}{3-q}}(\Omega) \cap L^p(\Omega), \nabla^2 \mathbf{u}, \nabla \pi \in W^{1,q}(\Omega) \cap W^{k,p}(\Omega)\Big\}.$$

Theorem 3. *Let $\mathbf{f} \in W^{2,q}(\Omega) \cap W^{k,p}(\Omega)$, $1 < q < \frac{6}{5}$, $k \geq 2$ and let $\|\mathbf{f}\|_{2,q} + \|\mathbf{f}\|_{k,p}$ and β_0 be sufficiently small. Let $\Omega \subset \mathbb{R}^2$ be an exterior domain of the class C^{k+1}. Then for any $\beta \in (0; \beta_0]$ there exists a solution to the problem (8)–(9) such that $(\mathbf{u}, p) \in S^k_{p,q}$.*

Let us also remark that we can look at the construction of the solution as a limit of the sequence defined

$$(\mathbf{w}^i, s^i) = \mathcal{M}(\mathbf{w}^{i-1}, s^{i-1}), \quad i \geq 1.$$

In order to prove the weighted estimates it is therefore sufficient to show that the operator $\mathcal{M}$ defined above maps sufficiently small balls in some weighted spaces into themselves.

[2] Let us recall that π plays the role of the effective pressure; the real pressure is p.

As mentioned above, the main tool will be weighted estimates from [6] applied on the integral representation to the modified Oseen problem together with the weighted estimates of the solution to the transport equation. We shall assume that the right-hand side can be written in the divergence form, $\mathbf{f} = \nabla \cdot \mathbf{h}$, use the integral representation for solutions to the modified Oseen problem (see [10] or [4], [8] in analogical situation for the (classical) Oseen problem) and apply the Green theorem (for the velocity, its first gradient and the pressure) on the volume terms. Next, using the estimates from [6] on the volume terms, calculating directly the estimates of the surface terms and combing this with the weighted estimates of the solutions to the transport equation we show that $\mathcal{M}$ maps sufficiently small balls in some weighted spaces into themselves and therefore we have (see [10] for further details)

Theorem 4. *Let* $\mathbf{f} = \nabla \cdot \mathbf{h}$ *and one of the following conditions be satisfied:*

(i) $\mathbf{h} \in L^2(\Omega)$, $\mathbf{f} \in W^{k,2}(\Omega)$, $k \geq 3$
(ii) $\mathbf{h} \in L^2(\Omega)$, $\mathbf{f} \in L^2(\Omega) \cap W^{k,p}(\Omega)$, $k \geq 2$, $p \in (3;4]$
(iii) $\mathbf{h} \in L^1_{loc}(\overline{\Omega})$, $\mathbf{f} \in W^{1,q}(\Omega) \cap W^{k,r}(\Omega)$, $q = \frac{6}{5}$ *if* $k = 1$, $q = \frac{4}{3}$ *if* $k \geq 2$

with the corresponding norms sufficiently small. Let $\Omega \in C^{k+1}$ *be an exterior domain in* $\mathbb{R}^3$. *Moreover let*

$$\mathbf{h}, \mathbf{f}, \nabla \mathbf{f} \in L^\infty(\Omega; \eta_1^2(\cdot)) \tag{19}$$

and let $\beta = |\mathbf{v}_\infty|$ *and* $\|\mathbf{h}, \mathbf{f}, \nabla \mathbf{f}\|_{\infty,(\eta_1^2(\cdot))}$ *be sufficiently small.*

Then $(\mathbf{v}, p)$ *solution to the problem (1)–(3) constructed in Theorem 2 has the following asymptotic properties*

$$\begin{aligned} \mathbf{u} = \mathbf{v} - \mathbf{v}_\infty &\in L^\infty(\Omega; \eta_1^1(\cdot)) \\ \nabla \mathbf{v}, \nabla^2 \mathbf{v} &\in L^r(\Omega; \eta_{\frac{3}{2}-\frac{1}{r}}^{\frac{3}{2}-\frac{3}{r}}(\cdot)) \\ p, \nabla p &\in L^r(\Omega; \eta_0^{2-\frac{4}{r}}(\cdot)), \end{aligned} \tag{20}$$

where $r \in [4; \infty)$ *is in the cases (i) and (ii) arbitrary, while in the case (iii) corresponds to the integrability of the right-hand side.*

Theorem 5. *Let* $\mathbf{f} = \nabla \cdot \mathbf{h}$ *and let* $\mathbf{h} \in L^1_{loc}(\overline{\Omega})$, $\mathbf{f} \in W^{2,q}(\Omega) \cap W^{k,p}(\Omega)$, $q \in (1; \frac{6}{5})$, $k \geq 2$ *with the norms sufficiently small. Let* $\Omega \in C^{k+1}$ *be an exterior domain in* $\mathbb{R}^2$. *Moreover, let*

$$\mathbf{h}, \mathbf{f}, \nabla \mathbf{f} \in L^\infty(\Omega; \eta_{\frac{1}{2}}^{\frac{3}{2}}(\cdot)) \tag{21}$$

and let $\beta = |\mathbf{v}_\infty|$ *and* $\|\mathbf{h}, \mathbf{f}, \nabla \mathbf{f}\|_{\infty,(\eta_{\frac{1}{2}}^{\frac{3}{2}}(\cdot))}$ *be sufficiently small.*

Then $(\mathbf{v}, p)$ *solution to the problem (1)–(3) constructed in Theorem 3 has the following asymptotic properties*

$$
\begin{aligned}
u_1 = v_1 - \beta \mathbf{e}_1 &\in L^\infty(\Omega; \eta^{\frac{1}{2}}_{\frac{1}{2}}(\cdot)) \\
v_2 &\in L^\infty(\Omega; \eta^1_0(\cdot)|\ln(2+\cdot)|^{-1}) \\
\nabla \mathbf{v}, \nabla^2 \mathbf{v} &\in L^r(\Omega; \eta^{1-\frac{2}{r}}_{1-\frac{1}{r}}(\cdot)) \\
p, \nabla p &\in L^r(\Omega; \eta_0^{1-\frac{3}{r}}(\cdot)),
\end{aligned}
\tag{22}
$$

where $r \in (5; \infty)$.

Remark 1. We can weaken the assumptions on the right-hand side to get still precise asymptotic structure of $\mathbf{u}$. Then we loose some information on the pressure or on the gradients of $\mathbf{u}$. Namely, in three space dimensions, for $\mathbf{h}, \mathbf{f}, \nabla \mathbf{f} \in L^\infty(\Omega; \eta^2_{\frac{1}{2}}(\cdot))$, we get $\nabla \mathbf{v}$, $\nabla^2 \mathbf{v} \in L^r(\Omega; \eta^{\frac{3}{2}-\frac{3}{r}}_{1-\frac{1}{r}}(\cdot))$, while for $\mathbf{h}, \mathbf{f}, \nabla \mathbf{f} \in L^\infty(\Omega; \eta_1^{\frac{3}{2}}(\cdot))$ we have $\nabla \mathbf{v}$, $\nabla^2 \mathbf{v} \in L^r(\Omega; \eta^{\frac{3}{2}-\frac{3}{r}}_{1-\frac{1}{r}}(\cdot))$ and π, $\nabla \pi \in L^r(\Omega; \eta^{\frac{3}{2}-\frac{4}{r}}_{\frac{1}{2}}(\cdot))$. In both cases, $\mathbf{v} - \mathbf{v}_\infty \in L^\infty(\Omega; \eta^1_1(\cdot))$. In two space dimensions then for $\mathbf{h}, \mathbf{f}, \nabla \mathbf{f} \in L^\infty(\Omega; \eta^1_{\frac{1}{2}}(\cdot))$ (indeed, with a sufficiently small norm) we would get $(22)_{1,2,4}$ and instead of $(22)_3$ only $\nabla \mathbf{v}, \nabla^2 \mathbf{v} \in L^r(\Omega; \eta^{1-\frac{2}{r}}_{\frac{1}{2}-\frac{1}{r}}(\cdot))$.

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Properties of Optimal Control Problems for Elliptic Equations

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Abstract. The existence of various extensions for optimal control problems governed by a single elliptic equation or by an elliptic system is considered. The descriptions of weak and strong closures of the set of feasible states of a single equation are given. These properties are compared with properties of the case of elliptic systems.

Keywords: optimal control, elliptic equations, extension

Classification: 49J20

1. INTRODUCTION

Consider the optimal control problem

$$(P) \qquad \begin{cases} I(\overline{u}) \to \min, \\ div\, A(x)\,(\nabla \overline{u} + g) = div\, f \text{ in } \Omega, \\ \overline{u} = (u_1, \cdots, u_m) \in H_0^1(\Omega; \mathbf{R}^m), \ A \in \mathbf{A}, \end{cases}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded Lipschitz domain and $\mathbf{A}$ is the set of admissible matrices. In practical problems the matrix $A(x)$ describes properties of the media at the point $x \in \Omega$. We shall mainly discuss the sets $\mathbf{A}$ of the type

$$\mathbf{A} = \{A \in L_2(\Omega; \mathbf{R}^{nm\times nm}) \mid A(x) \in M \ a.e.\ x \in \Omega\}$$

where M is a given set of symmetric $nm \times nm$ matrices.

It is well known that the problem (P) does not have, as a rule, an optimal solution. This property caused the search for various extensions of (P). There are two different cases: the first one is where I is weakly continuous; the second one is where I is continuous only in the strong topology.

It is clear that the first case can be fully solved in the framework of G-convergence and G-closed sets. The main problem here is that there are only few cases of M where an effective description of the G-closure of $\mathbf{A}$ is known. In reality, the exact knowledge of the G-closure of $\mathbf{A}$ is not necessary for a successful extension of (P). There exist extensions, different from the G-closure, which solve the problem. An example of such an extension is given in Section 3.

To describe the situation in the second case where the cost functional is not weakly continuous introduce the following notation.

$$Z(\mathbf{A}, g, f) = \{\overline{u} \in H_0^1(\Omega; \mathbf{R}^m) \mid div\, A(x)(\nabla \overline{u} + g) = div\, f \text{ in } \Omega, \ A \in \mathbf{A}\}.$$

It is clear that the first step in the investigation of possible extensions of (P) is the description of the strong closure of $Z(\mathbf{A}, g, f)$. If $m \leq n-1$ then the strong closure of $Z(\mathbf{A}, g, f)$ is equal to $Z(\overline{co}\mathbf{A}, g, f)$. Unfortunately, this property does not hold for $m \geq n$. Moreover, as will be shown in Section 4, there does not exist, in general, a larger set $\mathbf{A}_*$ such that $Z(\mathbf{A}_*, g, f)$ is equal to the strong closure of $Z(\mathbf{A}, g, f)$ for every g, f.

2. Preliminaries

Let $n \geq 2$, $m \geq 1$ be integers and let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain. Let R_{sym} be the space of all constant symmetric $nm \times nm$ matrices.

A set $Q \subset \mathbf{R}^n$ is said to be regular if

(i) Q is a bounded Lipschitz domain;
(ii) Q is homeomorphic to the unite ball of $\mathbf{R}^n$.

For Ω and for an arbitrary regular domain Q introduce the following spaces:

$$L = L_2(\Omega; \mathbf{R}^{nm}),$$
$$H = \{v \in L \mid v = \nabla \overline{u},\ \overline{u} \in H_0^1(\Omega; \mathbf{R}^m)\}$$

$$L(Q) = L_2(Q; \mathbf{R}^{nm}),$$
$$V(Q) = cl\,\{\eta \in L(Q) \mid \eta = (\sum_{i=1}^{n} \tfrac{\partial}{\partial x_i}\vartheta_{i1}^1, \cdots, \sum_{i=1}^{n} \tfrac{\partial}{\partial x_i}\vartheta_{in}^1, \cdots$$
$$\cdots, \sum_{i=1}^{n} \tfrac{\partial}{\partial x_i}\vartheta_{i1}^m, \cdots, \sum_{i=1}^{n} \tfrac{\partial}{\partial x_i}\vartheta_{in}^m),\ \vartheta_{ij}^s = -\vartheta_{ji}^s,\ \vartheta_{ij}^s \in C_0^\infty(Q),$$
$$i, j = 1, \cdots, n,\ s = 1, \cdots, m\},$$

$$\mathcal{H}(Q) = \{\, h \in L(Q) \mid h = \nabla \overline{u},\ \overline{u} \in H^1(Q; \mathbf{R}^m),\ \Delta \overline{u} = 0\},$$
$$\mathcal{N} = \{\, w \in L_2(\Omega; \mathbf{R}^{nm} \times \mathbf{R}^m \mid w = (\overline{w}, w'),\ div\,\overline{w} = w'\}.$$

Here and what follows for a given subset X_o of a Banach space X by $cl\, X_o$ and $wcl\, X_o$ we denote the closure of X_o in the strong topology and in the weak topology of X, respectively.

From the representation of divergence-free vectors and properties of Helmholtz's projections (see, for instance, Temam [8] and Zhikov et al. [10]), it follows that for every regular $Q \subset \mathbf{R}^n$

$$L(Q) = H(Q) \oplus V(Q) \oplus \mathcal{H}(Q).$$

In what follows we always assume that the set $\mathbf{A}$ of admissible matrices belongs to $L_2(\Omega; R_{sym})$ and that there exist positive constants $0 < \nu < \mu$ such that

$$\langle A(x)z, z\rangle \geq \nu |z|^2,\ |A(x)z| \leq \mu |z|\ a.e.\ x \in \Omega, ; \forall z \in \mathbf{R}^{nm},\ \forall A \in \mathbf{A}.$$

For a given set $E \subset \mathbf{R}^n$ by $|E|$ we shall denote Lebesgue measure of E.

3. Representation of $wcl\, Z(\mathbf{A}, g, f)$

The aim of this section is to give a description of a larger set $W\mathbf{A}$ such that

$$Z(W\mathbf{A}, g, f) = wcl\, Z(\mathbf{A}, g, f) \text{ for every } g, f \in L.$$

A set $Q \subset \mathbf{R}^n$ is said to be standard cube iff

$Q = \{x \in \mathbf{R}^n \mid x_{0i} - \tau \leq x_i \leq x_{0i} + \tau,\ i = 1, \cdots, n\}$ for some $x_0 \in \mathbf{R}^n$ and $\tau > 0$.

Definition 1. A finite family of sets $\{\Omega_0; Q_1; \cdots; Q_N\}$ is the $(\varepsilon\prime, \varepsilon\prime\prime, \delta)$ - partition of Ω iff

(i) $\Omega_0, Q_1, \cdots, Q_N$ are pairwise disjoint subsets of Ω and $\Omega = \Omega_0 \cup Q_1 \cup \cdots \cup Q_N$;

(ii) Q_j, $j = 1, \cdots, N$ are standard cubes;

(iii) diam $Q_j \leq \delta, dist(\partial\Omega, Q_j) \geq \varepsilon''$, $j = 1, \cdots, N$;

(iv) $|\Omega_0| < \varepsilon'$;

(v) $\delta < \varepsilon''$ and $\varepsilon'(8\varepsilon'')^{-1} \geq \int\limits_{\partial\Omega} dS$.

We always assume that $0 < \delta < \varepsilon'' < \varepsilon'$ and that ε' is small enough.

Definition 2. For a fixed set $\mathbf{A}$ of matrices the set $W\mathbf{A}$ consists of all matrices A_0 such that

(i) $A_0 \in L_2(\Omega; R_{sym})$;

(ii) for every $\varepsilon_0' > 0$, $\varepsilon_0'' > 0$, $\delta_0 > 0$ there exists a $(\varepsilon', \varepsilon'', \delta)$ - partition $\{\Omega_0; Q_1, ..., Q_N\}$ of Ω with $\varepsilon' < \varepsilon_0$, $\varepsilon'' < \varepsilon_0''$, $\delta < \delta_0$ such that for every N-tuple $\{(z'^1, z''^1); \cdots; (z'^N, z''^N)\} \in (\mathbf{R}^{nm} \times \mathbf{R}^{nm})^N$ there exists a sequence $\{A_k\} \subset \mathbf{A}$ such that

$$\begin{aligned}
&\int\limits_{Q_j} [\langle A_0(x) z'^j, z'^j\rangle + \langle A_0^{-1}(x) z''^j, z''^j\rangle] dx \\
&\geq \lim_{k\to\infty} \inf_{v \in H(Q_j)\, \eta \in V(Q_j)} \int\limits_{Q_j} [\langle A_k(x)(v + z'^j), v + z'^j\rangle + \\
&\langle A_k^{-1}(x)(\eta + z''^j), \eta + z''^j\rangle] dx, \\
&j = 1, \cdots, N.
\end{aligned} \tag{1}$$

Theorem 1. For every fixed $\mathbf{A}$ the set $W\mathbf{A}$ has the following properties:

(i) if $\langle A(x)z, z\rangle \geq \nu|z|^2$, $|A(x)z| \leq \mu|z|$ a.e. $x \in \Omega$, $\forall z \in \mathbf{R}^{nm}$, $\forall A \in \mathbf{A}$ then the same estimates are valid for all matrices $A_0 \in W\mathbf{A}$;

(ii) the set $W\mathbf{A}$ is "self closed" in the sense: if the set $W(W\mathbf{A})$ is constructed from $W\mathbf{A}$ in the same way as the set $W\mathbf{A}$ is constructed from $\mathbf{A}$ then $W(W\mathbf{A}) = W\mathbf{A}$;

(iii) the G-closure $G\mathbf{A}$ of the set $\mathbf{A}$ belongs to $W\mathbf{A}$;

(iv) for every $g, f \in L$ $Z(W\mathbf{A}, g, f) = wcl\, Z(\mathbf{A}, g, f)$;

(v) the set $W\mathbf{A}$ is G-closed.

Proof. We shall give here only a very short sketch of the proof. The full proofs will be published elsewhere.

The main idea of the proof is to consider optimal control problems:

$$
\begin{aligned}
&I(\overline{u}) = \alpha \int_\Omega (\overline{u}-\overline{a})^2 dx \to min\\
&div\, A(x)(\nabla\overline{u} + g) - \beta\overline{u} = div\, f \text{ in } \Omega,\\
&\overline{u} \in H_0^1(\Omega; \mathbf{R}^m),\ A \in \mathbf{A},
\end{aligned}
\tag{2}
$$

with $0 < \alpha < \beta$ and arbitrary $\overline{a} \in L_2(\Omega; \mathbf{R}^n)$ (for fixed $g, f \in L$).

These problems are equivalent to (see, for instance, Raitums[2])

$$
\inf_{A\in\mathbf{A}} \inf_{\overline{u}\in H_0^1(\Omega;\mathbf{R}^m)} \inf_{w\in\mathcal{N}} J(A, g_0, f_0, \alpha, \beta, \overline{a}, \overline{u}, w) \tag{3}
$$

where $g_0 = 1/2g,\ f_0 = 1/2f$ and

$$
\begin{aligned}
&J(A, g_0, f_0, \alpha, \beta, \overline{a}, \overline{u}, w)\\
&= \int_\Omega [\langle A(x)(\nabla\overline{u} + g_0), \nabla\overline{u} + \overline{g}_0\rangle + \langle A(x)^{-1}(\overline{w} + f_0), \overline{w} + f_0\rangle - 2\langle f_0, \nabla\overline{u}\rangle - 2\langle g_0, \overline{w}\rangle +\\
&+\beta|\overline{u}|^2 + \alpha|\overline{u}-\overline{a}|^2 + (\beta-\alpha)^{-1}|w' + \alpha(\overline{u}-\overline{a})|^2 - 2\,\langle f_0, g_0\rangle]dx.
\end{aligned}
$$

The equivalence of (2) and (3) are understood in the sense that for every admissible $\mathbf{A}$ (even for a set consisting of one matrix) the prices of problems (2) and (3) coincide.

A careful examination of the functional J gives that the passage from $\mathbf{A}$ to $W\mathbf{A}$ preserves the price of the problem (3) (for arbitrary fixed g_0, f_0, α, β, $\overline{a}$). Standard techniques from the homogenization theory ensure that $G\mathbf{A} \subset W\mathbf{A}$. Both these properties together with the arbitrarity of $\overline{a} \in L_2(\Omega; \mathbf{R}^m)$ and $0 < \alpha < \beta$ give the statement (iv) of Theorem 1. Statement (i) is a simple consequence from relationships (1). Finally, by means of appropriate approximations and applying estimates of the type (1) to subdomains of Q_j one gets statement (ii). □

Corollary 1. Let the set $\mathbf{A} = \mathbf{A}_0$ be defined as

$$
\mathbf{A}_0 = \{A \in L_2(\Omega; R_{sym}) \mid A(x) \in M_0 \text{ a.e. } x \in \Omega\},
$$

where M_0 is a fixed subset of R_{sym}. Then the set $W\mathbf{A}_0$ has the representation

$$
W\mathbf{A}_0 = \{A \in L_2(\Omega; R_{sym}) \mid A(x) \in WM_0 \text{ a.e. } x \in \Omega\}
$$

where

$$
WM_0 = \{B_0 \in R_{sym} \mid \langle B_0 z', z'\rangle + \langle B_0^{-1} z'', z''\rangle \geq F(z', z'')\ \forall z', z'' \in \mathbf{R}^{nm}\} \tag{4}
$$

and the function F is defined as

$$
\begin{aligned}
F(z', z'') = &\inf_{B(x)\in M_0} \inf_{v\in H(Q)} \inf_{\eta\in V(Q)} \frac{1}{|Q|} \int_Q [\langle B(x)(v + z'), v + z'\rangle\\
&+\langle B^{-1}(x)(\eta + z''), \eta + z''\rangle]dx
\end{aligned}
\tag{5}
$$

with Q being a fixed standard cube (or an arbitrary fixed convex regular domain).

The set WM_0, defined by relationships (3) and (4), is larger, in general, than the set GM_0 of matrices which generates the set $G\mathbf{A}$ in the same way as the set M_0 generates the set $\mathbf{A}$. It appears that for the so-called multiloaded case with $m = n$ the set WM_0 coincides with GM_0. More precisely, let K be a fixed closed set of positive definite symmetric $n \times n$ matrices D and let M_1 consist of all diagonal block $nn \times nn$ matrices of the type

$$\begin{pmatrix} D & & 0 \\ & \ddots & \\ 0 & & D \end{pmatrix}, \; D \in K.$$

Then $WM_1 = GM_1$.

This property follows from a more general result.

Theorem 2. Let $m \geq 1$, let $M_0 \subset R_{sym}$ and let there exist positive constants $0 < \nu < \mu$ such that

$$\langle Az, z \rangle \geq \nu |z|^2, \; |Az| \leq \mu |z| \; \forall z \in \mathbf{R}^{nm}, \; \forall A \in M_0.$$

Then the set GM_0 consists of all matrices $B_0 \in R_{sym}$ such that

$$\sum_{s=1}^{nm} [\langle B_0 z'^s, z'^s \rangle + \langle B_0^{-1} z''^s, z''^s, z''^s \rangle] \geq \mathcal{S}(\mathcal{Z}) \; \forall \mathcal{Z} \in [\mathbf{R}^{nm} \times \mathbf{R}^{nm}]^{nm} \tag{6}$$

where

$$\begin{aligned} &\mathcal{Z} = ((z'^1, z''^1), \cdots, (z'^{nm}, z''^{nm})), \\ &\mathcal{S}(\mathcal{Z}) = \inf_{A(x) \in M_0} \{ \sum_{s=1}^{nm} \inf_{v_s \in H(Q)\, \eta_s \in V(Q)} \frac{1}{|Q|} \int_Q \Big[\langle A(x)(v_s + z'^s), v_s + z'^s \rangle \\ &+ \langle A^{-1}(x)(\eta_s + z''^s), \eta_s + z''^s \rangle \Big] dx \}. \end{aligned} \tag{7}$$

Here Q is a fixed standard cube (or a fixed convex regular domain).

Proof. The function $\mathcal{S}$ plays for the multisystem

$$div\, A(x)(\nabla \overline{u}^s + g^s) = div\, f^s \text{ in } \Omega, \; \overline{u}^s \in H_0^1(\Omega; \mathbf{R}^m), s = 1, \cdots, nm, \tag{8}$$

the same role as the function F has for the initial system. Hence, the set M_* of all matrices B_0 which satisfy (6) defines the weak closure of sets of feasible states for (8). By appropriate choice of g^s, f^s in (8) one can obtain that for every $B_0 \in M_*$ there is a matrix $A_0 \in G\mathbf{A}_0$ (the set $\mathbf{A}_0$ is defined by the set M_0 as in Corollary 1) such that

$$A_0 = B_0 + A',$$

where every column of A' is equal to some element $\eta \in V(Q)$. After that, by means of expanding A' by periodicity and standard reasonings from the theory of homogenization we have that there exists a constant matrix $A_* \in GM_0$ such that $B_0 = A_*$, which gives the inclusion $M_* \subset GM_0$. In turn, the inclusion $GM_0 \subset M_*$ is obvious.

Corollary 2. Let the assumptions of Theorem 2 be fulfilled. Then the set GM_0 is defined as the intersection of a convex set $M \subset R_{sym} \times R_{sym}$ by the manifold $\{(C, D) \in R_{sym} \times R_{sym} \mid D = C^{-1}\}$. The same property has the set WM_0.

It is easy to see that the function F (or the function $\mathcal{S}$) remains unchanged if we replace the set M_0 in (5) (or in (7)) by the set GM_0. That means that the set GM_0 fully defines the set $W\mathbf{A}_0$. On the other hand, the relationships (5) and (7) can be used for inner and outer approximations of the sets WM_0 and GM_0, respectively.

The exact evaluation of the function F is a serious problem. We only point out that F coincides with a natural analogue of a quasiconvex envelope (see Dacorogna [1]) for the function F_0,

$$F_0(z', z'') = \inf_{B \in M_0} [\langle Bz', z'\rangle + \langle B^{-1}z'', z''\rangle],$$

in the space $H(Q) \times V(Q)$. The same property has the function S.

As far as the set M from Corollary 2 is concerned, we can only point out that the statement of Corollary 2 gives a better understanding of the structure of the sets WM_0 and GM_0.

As was mentioned in Introduction, the case $m = 1$ has many special properties. We shall illustrate them when the set $\mathbf{A} = \mathbf{A}_1$,

$$\begin{aligned} &\mathbf{A}_1 = \{A \in L_2(\Omega; \mathbf{R}^{n\times n}) \mid A(x) = \sum_{j=1}^{N} \theta_j(x) R_j(x) B_j R_j^*(x) \text{ a.e.}, x \in \Omega, \\ &\theta_j(x) = 0 \text{ or } 1, \int_\Omega \theta_j(x)dx \le d_j, \ R_j(x) \in SO(n) \text{ a.e. } x \in \Omega, \\ &j = 1, \cdots, N, \ \theta_1(x) + \cdots + \theta_N(x) = 1\}, \end{aligned} \tag{9}$$

where $d_1 + \cdots + d_N \ge |\Omega|$.

This case corresponds to the optimal material layout problem governed by a single equation. Here B_j, $j = 1, \cdots, N$, are given positive definite symmetric matrices which prescribe properties of the involved materials. Integral constraints on the functions θ_j mean that there are restrictions on the volumes occupied by the corresponding materials. The presence of matrices R_j means that all rotations of materials are allowed.

The set $\mathbf{A}_1$ does not satisfy assumptions of Corollary 1; nevertheless there exists an effective description of a set $\mathbf{A}_{1*}$ which plays the same role as the set $W\mathbf{A}$.

Proposition 1. Let the set $\mathbf{A}_1$ be defined by (9) and let

$$\begin{aligned} &\mathbf{A}_{1*} = \{A \in L_2(\Omega; \mathbf{R}^{n\times n}) \mid A(x) = A^*(x), \ \lambda_1(A)(x) \le \lambda_n(A)(x), \\ &\big(\lambda_1(A)(x)^{-1}, \lambda_n(A)(x)\big) \le \Big(\sum_{j=1}^{N} p_j(x)\lambda_1(B_j)^{-1}, \sum_{j=1}^{N} p_j(x)\lambda_n(B_j)\Big), \\ &p_j \in L_\infty(\Omega), \ \int_\Omega p_j dx \le d_j, \ p_j(x) \ge 0, \ j = 1, \cdots, N, \\ &p_1(x) + \cdots + p_N(x) = 1\}, \end{aligned}$$

where $\lambda_1(A)(x)$ and $\lambda_n(A)(x)$ stand for the smallest and the largest eigenvalues of the matrix $A(x)$, respectively, and the inequality $(a, b) \le (c, d)$ means that $a \le c$ and $b \le d$. Then for every $g, f \in L$ the set $wcl\, Z(\mathbf{A}_1, g, f)$ is equal to $Z(\mathbf{A}_{1*}, g, f)$.

The proof of this result was obtained independently by Tartar [7] and Raitums [3].

4. The strong closure of sets of feasible states

For the case of a single equation, i.e. $m = 1$, there exist simple effective descriptions for the strong closure $cl\, Z(\mathbf{A}, g, f)$ of the set $Z(\mathbf{A}, g, f)$ of feasible states generated by the equation

$$div\, A(x)(\nabla u + g) = div\, f \text{ in } \Omega, \ u \in H_0^1(\Omega),$$

with $A \in \mathbf{A}$. We mention results by Tartar [6] and Raitums [4] which show that if the set $\mathbf{A} = \mathbf{A}_0$ is defined as in Corollary 1 (with $m \leq n-1$), then for every $g, f \in L$

$$cl\, Z(\mathbf{A}_0, g, f) = Z(\overline{co}\mathbf{A}_0, g, f).$$

If the set $\mathbf{A}$ coincides with the set $\mathbf{A}_1$ from Proposition 1 (with $m = 1$), then for every $g, f \in L$

$$cl\, Z(\mathbf{A}_1, g, f) = Z(\overline{co}\mathbf{A}_1, g, f).$$

The situation changes drastically if one considers the case of systems with $m \geq n$.

To illustrate that we refer to one result due Zaitsev [9]. Let $m = n \geq 2$, $N \geq 2$ and let the set $\mathbf{A}_2$ be defined as

$$\begin{aligned} &\mathbf{A}_2 = \{A \in L_2(\Omega; R_{sym}) \mid A(x) \in M_2 \text{ a.e. } x \in \Omega\}, \\ &M_2 = \{\alpha_1 I; \cdots, \alpha_N I\}, \end{aligned}$$

where α_i, $i = 1, \cdots, N$, are different positive constants and I is the identity matrix. Then there does not exist any family $\mathcal{L}$ of linear bounded invertible operators

$$\mathcal{A} : H_0^1(\Omega; \mathbf{R}^m) \to \left(H_0^1(\Omega; \mathbf{R}^m)\right)^*$$

such that for every $f \in L$

$$cl\, Z(\mathbf{A}_2, 0, f) = \{\overline{u} \in H_0^1(\Omega; \mathbf{R}^m) \mid \overline{u} = \mathcal{A}^{-1} f,\ \mathcal{A} \in \mathcal{L}\}.$$

This result shows that one must seek other types of extensions for optimal control problems with nonweakly continuous cost functionals and $m = n \geq 2$.

We would like to point out here that the above-mentioned result corresponds to the so-called multiloaded case, in some sense the simplest case of elliptic systems.

We conclude with a description of the strong closure of $Z(\mathbf{A}, g, f)$ for the case $m = n \geq 2$ with $\mathbf{A} = \mathbf{A}_3$,

$$\mathbf{A}_3 = \{A \in L_2(\Omega; R_{sym}) \mid A(x) = B_1 \text{ or } B_2 \text{ a.e. } x \in \Omega\},$$

where B_1 and B_2 are given positive definite symmetric $nn \times nn$ matrices.

Denote by $\mathcal{K}(g, f)$ the set of all pairs $(\overline{u}, \theta) \in H_0^1(\Omega; \mathbf{R}^n) \times L_\infty(\Omega)$ such that

$$\begin{cases} div\,(B_1 + \theta(x)(B_2 - B_1))(\nabla\overline{u} + g) = div\, f \text{ in } \Omega, \\ \theta(x) = 0 \text{ or } 1 \text{ if } rank(B_2 - B_1)(\nabla\overline{u}(x) + g(x)) = n, \\ 0 \leq \theta(x) \leq 1 \text{ otherwise.} \end{cases}$$

Then for every $g, f \in L$

$$cl\, Z(\mathbf{A}, g, f) = \{\overline{u} \mid (\overline{u}, \theta) \in \mathcal{K}(g, f)\}.$$

This representation for $g = 0$ is given in Raitums [5] and it is easy to see that it is true for arbitrary $g, f \in L$.

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On the Galerkin Method for Semilinear Parabolic-Ordinary Systems

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Abstract. We consider a general system of n_1 semilinear parabolic partial differential equations and n_2 ordinary differential equations, with locally Lipschitz continuous nonlinearities. We analyse the well-posedness of this problem, exploiting the tools of the semigroups theory, and derive other further regularity results and conditions for the boundedness of the solution.

We define the Galerkin semidiscrete approximation to the system and derive optimal order error estimates in L^2 norm, under various assumptions on the nonlinear terms, on the finite dimensional subspaces in which the approximation is sought and on the regularity of the exact solution. As a by-product, we can also show that the approximate solution is globally defined and bounded.

Keywords: semilinear parabolic-ordinary systems, Galerkin approximation

Classification: 35K57, 65N30

1. Semilinear parabolic-ordinary systems

In this work we study a general class of reaction-diffusion problems, described by systems of weakly coupled semilinear parabolic and ordinary differential equations, with Neumann boundary conditions and Lipschitz continuous nonlinearities. Examples of such systems can be found in electrocardiology [1], [2] and have already been studied analytically by Pao [4].

Let Ω be a convex bounded domain of $\mathbb{R}^n$, $n = 1, 2, 3$ and let the boundary $\partial\Omega$ be $C^{2+\alpha}$, $\alpha \in (0, 1)$. Let's indicate by $\nu = (\nu_1, \dots, \nu_n)^T$ the outward normal to $\partial\Omega$. Let's consider the following semilinear system of n_1 parabolic equations and n_2 ordinary differential equations, with nonhomogeneous Neumann boundary conditions:

$$\begin{array}{lll} \partial_t u_i + \mathcal{L}_i u_i = g_i(x, t, \mathbf{u}) & in\ \Omega\times]0, T] & (i = 1, \dots, n_1) \\ \partial_t u_i = g_i(x, t, \mathbf{u}) & in\ \Omega\times]0, T] & (i = n_1 + 1, \dots, n_1 + n_2) \\ \mathcal{B}_i u_i = q_i(x, t) & in\ \partial\Omega\times]0, T] & (i = 1, \dots, n_1) \\ u_i(x, 0) = u_{i,0}(x) & in\ \Omega & (i = 1, \dots, n_1 + n_2) \end{array} \tag{1}$$

where $\mathcal{L}_i$ and $\mathcal{B}_i$, $i = 1, \dots, n_1$, are operators of the form

$$\mathcal{L}_i w := -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(a_{jk}^{(i)}(x)\frac{\partial w}{\partial x_k}\right) + \sum_{j=1}^{n} b_j^{(i)}(x)\frac{\partial w}{\partial x_j} + a_0^{(i)}(x) w, \tag{2}$$

$$B_i w := \frac{\partial w}{\partial \nu_{\mathcal{L}}} = \sum_{j,k=1}^{n} a_{jk}^{(i)}(x) \frac{\partial w}{\partial x_k} \nu_j; \tag{3}$$

the matrices $\left(a_{jk}^{(i)}\right)$ are symmetric and the operators $\mathcal{L}_i$ are uniformly elliptic in $\overline{\Omega}$; $b_j^{(i)}, a_0^{(i)} \in C(\overline{\Omega})$, $a_{jk}^{(i)} \in C^1(\overline{\Omega})$ and the functions $q_i(x,t) \in C^{1+\alpha}([0,T]; C^{1+\alpha}(\partial\Omega))$. The ellipticity of $\mathcal{L}_i$ implies that the boundary conditions are of *non tangential* type, i.e.

$$\sum_{k=1}^{n} \left(\sum_{j=1}^{n} a_{jk}^{(i)} n_j \right) n_k \neq 0, \quad \forall x \in \partial\Omega.$$

Let's consider first the homogeneous case $q_i \equiv 0$, $i = 1, 2, \dots, n_1$.

The well-posedness of problem (1) can be reduced to the study of an abstract evolution equation in the space $X = C(\overline{\Omega})^{n_1+n_2}$, by defining the *realization of* $-\mathcal{L}_i$ *in* $X_i = C(\overline{\Omega})$ *with homogeneous first-order oblique boundary conditions*

$$D(A_i) := \{u \in \bigcap_{p \geq 1} W^{2,p}(\Omega) : \mathcal{L}_i u \in C(\overline{\Omega}),\ B_i u = 0 \text{ on } \partial\Omega\}, \qquad A_i u = -\mathcal{L}_i u.$$

We recall that, for $n = 1$, $D(A_i) = \{u \in C^2(\overline{\Omega}) : B_i u_{|\partial\Omega} = 0\}$ and that (see [3] for the general theory), for every $i = 1, \dots, n_1$, the *resolvent set* $\rho(A_i)$ and the *resolvent operator* $R(\lambda, A_i)$ are defined as $\rho(A_i) := \{\lambda \in \mathbb{C} : \exists (\lambda I - A_i)^{-1} \in L(X_i)\}$ and $R(\lambda, A_i) := (\lambda I - A_i)^{-1}$, $\quad \forall \lambda \in \rho(A_i)$, where $L(X_i)$ is the space of the continuous linear functionals of the Banach space X_i.

Definition 1.1. *Let X be a Banach space, with norm $\|\cdot\|_X$. A linear operator $A : D(A) \subset X \to X$ is said to be sectorial if there are constants $\omega \in \mathbb{R}$, $\theta \in]\pi/2, \pi[$, $M > 0$ such that*

$$\begin{cases} (i) & \rho(A) \supset S_{\theta,\omega} := \{\lambda \in \mathbb{C} : \lambda \neq \omega,\ |\arg(\lambda - \omega)| < \theta\}, \\ (ii) & \|R(\lambda, A)\|_{L(X)} \leq \frac{M}{|\lambda - \omega|}\ \forall \lambda \in S_{\theta,\omega}. \end{cases}$$

Given a sectorial operator A in X, we define the intermediate spaces between X and $D(A)$ $(0 < \alpha < 1)$ as $D_A(\alpha, \infty) := \{x \in X : t \mapsto v(t) = \|t^{1-\alpha} A e^{tA} x\|_X \in L^\infty(0,1)\}$, endowed with the norm $\|x\|_{D_A(\alpha,\infty)} = \|x\|_X + \|v\|_{L^\infty(0,1)}$. It can be proved [3] that $D(A) \subset D_A(\alpha, \infty) \subset \overline{D(A)}$, $0 < \alpha < 1$.

In the case of (2) and (3), it can be shown that A_i is sectorial in X_i and that

$$\overline{D(A_i)} = C(\overline{\Omega}), \qquad D_{A_i}(\alpha, \infty) = \begin{cases} C^{2\alpha}(\overline{\Omega}), & \text{if } \alpha < 1/2, \\ C^{2\alpha}_{B_i}(\overline{\Omega}), & \text{if } \alpha > 1/2, \end{cases}$$

where $C^{2\alpha}_{B_i}(\overline{\Omega})$ is the subspace of $C^{2\alpha}(\overline{\Omega})$ whose functions φ satisfy $B_i \varphi = 0$ on $\partial\Omega$.

To simplify the exposition, we consider the case of a system of the form (1) with $n_1 = 1$ and $n_2 = 2$. The following results can be easily extended to the general case.

Let's define the operator A in the space X

$$D(A) = D(A_1) \times C(\overline{\Omega}) \times C(\overline{\Omega}), \quad A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where 0 represents the *null operator*. The following theorem can be easily verified.

Theorem 1.2. *The operator A is sectorial in X and $\overline{D(A)} = X$. Moreover $D_A(\alpha,\infty) = D_{A_1}(\alpha,\infty) \times C(\overline{\Omega})^2$, $0<\alpha<1$.*

Set in (1)

$$u(t)(x) := \mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))^T,$$
$$f(t,u)(x) := (g_1(x,t,\mathbf{u}(x)), g_2(x,t,\mathbf{u}(x)), g_3(x,t,\mathbf{u}(x)))^T,$$
$$u_0 := (u_{1,0}(x), u_{2,0}(x), u_{3,0}(x))^T.$$

We get the (semilinear parabolic) abstract evolution equation

$$\begin{cases} u'(t) = Au(t) + f(t,u(t)) & t \in]0,T] \\ u(0) = u_0. \end{cases} \tag{4}$$

Local existence and uniqueness results for the abstract problem (4) are stated by the following [3].

Theorem 1.3. *Let $A : D(A) \subset X \to X$ be a sectorial operator and $f : [0,T]\times X \to X$ be a continuous function locally Lipschitz continuous with respect to u, i.e. $\forall R>0$ $\exists L = L(R) > 0$ such that*

$$\|f(t,u) - f(t,v)\|_X \le L\|u-v\|_X, \qquad \forall t \in [0,T] \quad \forall u,v \in B(0,R), \tag{5}$$

where $B(0,R)$ is the ball centred in 0 with ray R in X. Moreover suppose that there exists $\alpha \in (0,1)$ such that for every $R>0$

$$\|f(t,v) - f(s,v)\|_X \le C(R)(t-s)^\alpha, \qquad 0 \le s \le t \le T, \ \|v\|_X \le R. \tag{6}$$

The following statements hold:

(i) if $u_0 \in \overline{D(A)}$, then (4) has a unique maximal classical solution $u \in C([0,T'[;X) \cap C(]0,T'[;D(A)) \cap C^1(]0,T'[;X)$, $0 < T' \le T$;

(ii) if $u_0 \in D(A)$ and $Au_0 + f(0,u_0) \in \overline{D(A)}$, then u is a maximal strict solution of (4), i.e. $u \in C([0,T'[;D(A)) \cap C^1([0,T'[;X)$;

(iii) if $u_0 \in D(A)$ and $Au_0 + f(0,u_0) \in D_A(\alpha,\infty)$ for some $\alpha \in (0,1)$, then for every $b < T'$ we have $u', Au \in C^\alpha([0,b];X)$, $u' \in B([0,b];D_A(\alpha,\infty))$, where $B([a,b];Y)$ is the space of the bounded functions: $[a,b] \to Y$.

Among all the possible global existence results, we only recall the following [3].

Theorem 1.4. *Suppose there exists $C>0$ such that*

$$\|f(t,u)\|_X \le C(1 + \|u\|_X), \qquad \forall u \in X, \ \forall t \in [0,T]. \tag{7}$$

Let $u : [0,T'[\to X$ be a local solution of (4). Then u is bounded in $[0,T'[$ with values in X and therefore it is a global solution of (4).

Now, let us consider again the concrete problem (1). Suppose the functions g_i, $i = 1, 2, 3$, are continuous and for $0 < \alpha < 1$ satisfy

for every $R > 0$ *there exist some positive constants* $K_i = K_i(R)$ *such that*

$$\begin{aligned} &|g_i(x,t,\mathbf{u}) - g_i(x,s,\mathbf{w})| \le K_i(|t-s|^\alpha + |\mathbf{u}-\mathbf{w}|_1) \\ &\quad \text{for } x \in \overline{\Omega},\ t,s \in [0,T],\ |\mathbf{u}|_1, |\mathbf{w}|_1 \le R, \end{aligned} \tag{8}$$

where $|\mathbf{u}|_1 = \sum_{i=1}^3 |u_i|$.

A direct application of the abstract theorems 1.3 and 1.4 to the reaction-diffusion problem (1), via its abstract formulation (4), yields the following theorems. See [6] for the details of the proof.

Theorem 1.5. *Under condition (8), the following statements hold:*

(i) if $u_{i,0} \in C(\overline{\Omega})$, *for* $i = 1, 2, 3$, *then (1) with* $q_i = 0$ *has a unique maximal classical solution* $\mathbf{u}$ *continuous in* $\overline{\Omega} \times [0, T'[$, *with* $\mathbf{u}$ *differentiable with respect to* t *in* $\overline{\Omega} \times]0, T'[$ *and* $u_1(\cdot, t) \in W^{2,p}(\Omega)$ *for every* $p \ge 1$ *and* $0 < t < T' < T$;

(ii) if $u_{1,0} \in \bigcap_{p\ge 1} W^{2,p}(\Omega)$, $\mathcal{B}_1 u_{1,0} = 0$, $u_{2,0}, u_{3,0} \in C(\overline{\Omega})$ *and* $\mathcal{L}_1 u_{1,0} \in C(\overline{\Omega})$, *then* $\mathbf{u}$ *is a maximal strict solution of (1), i.e.* $(u_i)_t$, $\mathcal{L}_1 u_1 \in C(\overline{\Omega} \times [0, T'[)$, *for* $i = 1, 2, 3$;

(iii) if $u_{1,0} \in \bigcap_{p\ge 1} W^{2,p}(\Omega)$, $\mathcal{B}_1 u_{1,0} = 0$, $u_{2,0}, u_{3,0} \in C(\overline{\Omega})$ *and* $-\mathcal{L}_1 u_{1,0} + g_1(\cdot, 0, \mathbf{u}_0) \in C^{2\alpha}(\overline{\Omega})$, *for some* $\alpha \in (0, 1/2)$, *or* $-\mathcal{L}_1 u_{1,0} + g_1(\cdot, 0, \mathbf{u}_0) \in C^{2\alpha}_{\mathcal{B}_1}(\overline{\Omega})$, *for some* $\alpha \in (1/2, 1)$, *then* $\partial_t u_1$ *belongs to* $C^{2\alpha,\alpha}(\overline{\Omega} \times [0, b])$, $A_1 u_1$ *belongs to* $C^{0,\alpha}(\overline{\Omega} \times [0, b])$, *for every* $b < T'$. *In particular, if* $\alpha > 1/2$ *then* $(u_1)_{x_i t}$ *is continuous for* $i = 1, \ldots, n$.

Theorem 1.6. *Suppose all the hypotheses of Theorem 1.5 hold. If the nonlinear terms satisfy*

$$\exists C > 0 \ s.\ t. \quad \sum_{i=1}^{3} |g_i(x,t,\mathbf{u})| \le C(1 + |\mathbf{u}|_1), \quad \forall t \in [0,T],\ \forall x \in \overline{\Omega},\ \forall \mathbf{u} \in \mathbb{R}^3, \tag{9}$$

$i = 1, 2, 3$, *then the solution* $\mathbf{u}$ *is global.*

In the case of nonhomogeneous boundary conditions, the abstract theory cannot be applied directly, since $D(A_1)$ is endowed with homogeneous boundary conditions. Usually, in these cases, it is useful to consider the boundary terms q_i as traces at $\partial\Omega \times [0, T]$ of $\mathcal{B}_i Q_i$, where Q_i are sufficiently smooth functions defined in $\overline{\Omega} \times [0, T]$, so that the nonhomogeneous problem can be reduced to a homogeneous one in the unknowns $\tilde{u}_1 = u_1 - Q_1$ and $\tilde{u}_i = u_i$, $i = 2, 3$, and then the previous Theorem 1.5 can be applied. More precisely, we can set

$$Q_i(\cdot, t) = \mathcal{N}_i q_i(\cdot, t), \qquad 0 \le t \le T,$$

where $\mathcal{N}_i$ is an operator which belongs to $L(C^\theta(\partial\Omega), C^{\theta+1}(\overline{\Omega}))$ for every $\theta \in [0, 1]$ (for $\theta \in [0, 1+\alpha]$ if $\partial\Omega$ is uniformly $C^{2+\alpha}$, $0 \le \alpha < 1$) such that

$$\mathcal{B}_i(\mathcal{N}_i f)_{|\partial\Omega} = f, \quad \forall f \in C(\partial\Omega).$$

Then the following theorem holds (see [6] for further details of the proof).

Theorem 1.7. *Suppose that $\partial\Omega$ is uniformly $C^{2+\alpha}$ for some $\alpha \in (0,1)$, that (8) holds and that $q_1 \in C^{1+\alpha}([0,T];C^{1+\alpha}(\partial\Omega))$. The following statements hold:*

(i) if $u_{i,0} \in C(\overline{\Omega})$, for $i = 1,2,3$, then (1) has a unique maximal classical solution $\mathbf{u}$ continuous in $\overline{\Omega} \times [0,T'[$, with $\mathbf{u}$ differentiable with respect to t in $\overline{\Omega}\times]0,T'[$ and $u_1(\cdot,t) \in W^{2,p}(\Omega)$ for every $p \geq 1$ and $0 < t < T' < T$;

(ii) if $u_{1,0} \in \bigcap_{p\geq 1} W^{2,p}(\Omega)$, $\mathcal{B}_1 u_{1,0}(x) = q_1(x,0)$ for $x \in \partial\Omega$, $u_{2,0}, u_{3,0} \in C(\overline{\Omega})$ and $\mathcal{L}_1 u_{1,0} \in C(\overline{\Omega})$, then $\mathbf{u}$ is a maximal strict solution of (1), i.e. $(u_i)_t$, $\mathcal{L}_1 u_1 \in C(\overline{\Omega} \times [0,T'[)$, for $i = 1,2,3$;

(iii) if $u_{1,0} \in \bigcap_{p\geq 1} W^{2,p}(\Omega)$, $\mathcal{B}_1 u_{1,0}(x) = q_1(x,0)$ for $x \in \partial\Omega$, $u_{2,0}, u_{3,0} \in C(\overline{\Omega})$ and $-\mathcal{L}_1 u_{1,0} + g_1(\cdot,0,\mathbf{u}_0) \in C^{2\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1/2)$, or $-\mathcal{L}_1 u_{1,0} + g_1(\cdot,0,\mathbf{u}_0) \in C^{2\alpha}(\overline{\Omega})$ and $\mathcal{B}_1(-\mathcal{L}_1 u_{1,0} + g_1(\cdot,0,\mathbf{u}_0)) = \partial_t q_1(\cdot,0)$ on $\partial\Omega$, for some $\alpha \in (1/2,1)$, then $\partial_t u_1$ belongs to $C^{2\alpha,\alpha}(\overline{\Omega} \times [0,b])$, $\mathcal{A}_1 u_1$ belongs to $C^{0,\alpha}(\overline{\Omega} \times [0,b])$, for every $b < T'$.

2. Galerkin approximation

Let us now introduce the Galerkin semidiscretization of system (1). The convergence of such method is studied according to the smoothness of the solution of (1) and to the approximation properties of the finite dimensional spaces V_h used.

Let's indicate by $\mathcal{A}^{(i)}(.,.)$ the bilinear form associated to the operator $\mathcal{L}_i$

$$\mathcal{A}^{(i)}(w,v) := \int_\Omega \left[\sum_{j,k=1}^{n} a_{jk}^{(i)} \frac{\partial w}{\partial x_k}\frac{\partial v}{\partial x_j} + \sum_{j=1}^{n} b_j^{(i)} v \frac{\partial w}{\partial x_j} + a_0^{(i)} wv \right] dx.$$

We can suppose without restriction that the bilinear forms $\mathcal{A}^{(i)}(.,.)$ are continuous and coercive in $H^1(\Omega)$, i.e. there exist some constants $\alpha^{(i)} > 0$ such that

$$\mathcal{A}^{(i)}(v,v) \geq \alpha^{(i)} \|v\|_1^2 \qquad \forall v \in H^1(\Omega). \tag{10}$$

Let us consider again the case $n_1 = 1$ and $n_2 = 2$, leaving to the reader the extension to the general case. Suppose that the data are such that (1) has a unique bounded soluton $\mathbf{u} = (u_1,u_2,u_3)^T$, which is sufficiently smooth for all our purposes. Therefore, under the hypothesis specified in Section 1, let us consider the following weak formulation of problem (1):

$$\text{find } u_1 \in L^2(0,T;H^1(\Omega)) \cap C^0([0,T];L^2(\Omega)),\ u_2,u_3 \in C^0([0,T];L^2(\Omega))$$
$$\text{with } \operatorname{supess}_{\Omega\times]0,T[}|u_i| < \infty,\ i = 1,2,3 \text{ such that}$$

$$\frac{d}{dt}(u_1(t),\varphi)+\mathcal{A}^{(1)}(u_1(t),\varphi) = (g_1(x,t,u_1(t),u_2(t),u_3(t)),\varphi)+(q_1(t),\varphi)_{\partial\Omega}, \quad \forall\varphi \in H^1(\Omega)$$
$$\frac{d}{dt}(u_i(t),\psi) = (g_i(x,t,u_1(t),u_2(t),u_3(t)),\psi), \quad \forall\psi \in L^2(\Omega) \quad i = 2,3$$
$$u_i(0) = u_{i,0}, \quad i = 1,2,3. \tag{11}$$

Let us define the Galerkin semidiscretization of (1). Fix the families of subspaces $\{V_h\}_{h>0}$ of $H^1(\Omega)$ and $\{W_h\}_{h>0}$ of $L^2(\Omega)$, with finite dimension N_h, and define the Galerkin semidiscrete problem:

$$\text{find } u_1 :]0,T[\to V_h,\ u_2,u_3 :]0,T[\to W_h, \text{ s. t. } \forall\varphi_h \in V_h\ \forall\psi_h \in W_h$$

$$\frac{d}{dt}(u_{1,h}(t), \varphi_h) + \mathcal{A}^{(1)}(u_{1,h}(t), \varphi_h) = (g_1(x, t, u_{1,h}(t), u_{2,h}(t), u_{3,h}(t)), \varphi_h) + (q_1(t), \varphi_h)_{\partial\Omega},$$

$$\frac{d}{dt}(u_{i,h}(t), \psi_h) = (g_i(x, t, u_{1,h}(t), u_{2,h}(t), u_{3,h}(t)), \psi_h), \quad i = 2, 3$$

$$u_{i,h}(0) = u_{i,0,h}, \quad i = 1, 2, 3, \tag{12}$$

where $u_{1,0,h} \in V_h$ and $u_{i,0,h} \in W_k$, for $i = 2, 3$, are suitable approximations of the initial data.

In this section we study the properties of the semidiscrete system (12). The proof of the convergence of the semidiscrete solution to $\mathbf{u}$ in the presence of locally Lipschitz continuous nonlinear terms needs some L^∞ estimates on the semidiscrete solution to be provided, in order to assure it doesn't blow up. Therefore, we first analyse the case of globally Lipschitz continuous reaction terms. It must be noted that, in both cases, the obtained results are independent of the particular choice of the spaces V_h and W_h, since they are based only on the smoothness of the solution $\mathbf{u}$ and on the approximation properties of such spaces.

Let us now consider the case of functions g_i, $i = 1, 2, 3$, globally Lipschitz continuous with respect to the variable $\mathbf{u}$, i.e. we suppose there exist some constants $K_i > 0$ such that

$$|g_i(x, t, \mathbf{u}) - g_i(x, t, \mathbf{w})| \leq K_i |\mathbf{u} - \mathbf{w}|_1 \qquad \forall \mathbf{u}, \mathbf{w} \in \mathbb{R}, \ x \in \overline{\Omega}, \ t \in [0, T]. \tag{13}$$

In this case, the global Lipschitz continuity of g_i yields the global Lipschitz continuity of the right-hand side of system (12) and hence, by standard existence theorems for these systems, the semidiscrete solution is globally defined. Let us indicate by $\mathbf{u}_h = (u_{1,h}, u_{2,h}, u_{3,h})^T$ the unique solution of (12); note that the solution $\mathbf{u} = (u_1, u_2, u_3)^T$ of (1) is also solution to (11).

In the following we suppose that the family $\{V_h\}$ satisfies the approximation property

$$\lim_{h\to 0} \inf_{\varphi_h \in L^\infty(0,T;V_h)} \sup_{[0,T]} h^j \|\varphi(t) - \varphi_h(t)\|_j = 0 \quad \forall \varphi \in L^\infty(0, T; H^1(\Omega)), \tag{14}$$

where $j = 0, 1$; while for the family $\{W_h\}$ the following property holds:

$$\lim_{h\to 0} \inf_{\psi_h \in L^\infty(0,T;W_h)} \sup_{[0,T]} \|\psi(t) - \psi_h(t)\|_0 = 0 \quad \forall \psi \in S, \tag{15}$$

where $S := L^\infty(0, T; L^2(\Omega)) \cap \{f : \ f(t) \in C_0^\infty(\Omega) \text{ a.e. } t \in [0, T]\}$. We state, without proof [6], the following.

Lemma 2.1. *Let $\{W_h\}$ be a family of subspaces of $L^2(\Omega)$ satisfying (15). For every $w \in C^0([0, T]; L^2(\Omega))$ we have*

$$\lim_{h\to 0} \inf_{w_h \in L^\infty(0,T;W_h)} \sup_{[0,T]} \|w(t) - w_h(t)\|_0 = 0.$$

We have the following convergence result in the L^2-norm.

Theorem 2.2. *Suppose that g_i, $i = 1,2,3$, satisfies the condition (13) and that the solution of (11) is such that $u_1 \in C^0([0,T];L^2(\Omega)) \cap W^{1,\infty}(0,T;H^1(\Omega))$ and $u_2, u_3 \in C^1([0,T];L^2(\Omega))$. If the families $\{V_h\}$ and $\{W_h\}$ satisfy (14) and (15), respectively, then the solution of (12) exists for $t \leq T$ and converges in the L^2-norm to the solution* $\mathbf{u}$ *of (11).*

The analysis of the error between $\mathbf{u}_h$ and $\mathbf{u}$ can be accomplished comparing the Galerkin solution $u_{1,h}(t)$ to the *elliptic projection* $W_1(t)$ of the solution $u_1(t)$ of (11) onto V_h, defined for a.e. $t \in]0,T[$ by

$$\mathcal{A}^{(1)}(W_1(t), \varphi_h) = \mathcal{A}^{(1)}(u_1(t), \varphi_h) \qquad \forall \varphi_h \in V_h. \tag{16}$$

As it is well known, the error $\rho_1(t) = u_1(t) - W_1(t)$ satisfies for $t \leq T$ and for some $\bar{C} > 0$

$$\|\rho_1(t)\|_0 \leq \bar{C}h \inf_{\varphi_h \in V_h} \|u_1(t) - \varphi_h\|_1. \tag{17}$$

The rest of the error in v_h is then $\theta_1(t) = u_{1,h}(t) - W_1(t) \in V_h$.

In the following C will denote constants, not necessarily the same at different occurrences, which are independent of h and the functions involved. Similarly, c will denote constants which are independent of h but which may depend on the solution $\mathbf{u}(t)$ of (11). We shall assume that the initial values are chosen in such a way that

$$\|u_{1,0,h} - W_1(0)\|_0 \leq ch^{\mu}, \ \|u_{2,0,h} - u_{2,0}\|_0 \leq ch^{\mu'}, \ \|u_{3,0,h} - u_{3,0}\|_0 \leq ch^{\mu'}, \tag{18}$$

for some $\mu, \mu' > 0$, where $W_1(t)$ is defined by (16).

Proof. Let us subtract (11) from (12), for fixed t. Taking into account (16), setting $\varphi = \varphi_h = \theta_1(t)$ and using Cauchy-Schwarz's inequality, the coerciveness of $\mathcal{A}^{(1)}$, Lipschitz condition for g_1 and Young's inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$, we obtain for a.e. $t < T$

$$\frac{1}{2}\frac{d}{dt}\|\theta_1\|_0^2 \leq \frac{1}{2}\|(\rho_1)_t\|_0^2 + \frac{1}{2}(5 + K_1^2)\|\theta_1\|_0^2 + \frac{K_1^2}{2}\|\rho_1\|_0^2 + \frac{K_1^2}{2}\|U_2\|_0^2 + \frac{K_1^2}{2}\|U_3\|_0^2,$$

where $U_2(t) = u_{2,h}(t) - u_2(t)$ and $U_3(t) = u_{3,h}(t) - u_3(t)$ for $t \in [0,T]$. Since time differentiation commutes with elliptic projection, by (17) we have $\|(\rho_1)_t\|_0 \leq \bar{C}h \inf_{\varphi_h \in V_h} \|(u_1)_t(t) - \varphi_h\|_1$; hence

$$\frac{1}{2}\frac{d}{dt}\|\theta_1\|_0^2 \leq C_1\|\theta_1\|_0^2 + C_2\|U_2\|_0^2 + C_2\|U_3\|_0^2 + C_3h^2\|u_1 - \varphi_h\|_1^2 + C_4h^2\|(u_1)_t - \bar{\varphi}_h\|_1^2, \tag{19}$$

where $C_1 = (5 + K_1^2)/2 > 0$, $C_2 = K_1^2/2 > 0$, $C_3 = C_2\bar{C}^2$, $C_4 = \bar{C}^2/2$ and φ_h, $\bar{\varphi}_h$ are arbitrary elements of V_h.

Now, let us consider the first ordinary differential equation, in its weak form; we have for $t < T$

$$((U_2)_t(t), \psi_h) = (g_2(x,t,u_{1,h},u_{2,h},u_{3,h}) - g_2(x,t,u_1,u_2,u_3), \psi_h) \qquad \forall \psi_h \in W_h. \tag{20}$$

Set $\psi_h = u_{2,h}(t) - w_h \in W_h$, where w_h is an arbitrary element of W_h. As before, this yields

$$\frac{1}{2}\frac{d}{dt}\|U_2\|_0^2 \le C_5\|U_2\|_0^2 + \frac{1}{2}\|(U_2)_t\|_0^2 + \frac{5}{2}\|u_2 - w_h\|_0^2 + K_2^2\|\theta_1\|_0^2 + K_2^2\|\rho_1\|_0^2 + K_2^2\|U_3\|_0^2, \tag{21}$$

with $C_5 = 2 + K_2^2$. Moreover, choosing $(u_{2,h})_t - \phi_h \in W_h$ in (20), where ϕ_h is an arbitrary element of W_h, we have for every $t \in]0, T[$

$$\|(U_2)_t\|_0^2 \le \|(u_2)_t - \phi_h\|_0^2 + C_6\|\theta_1\|_0^2 + C_6\|\rho_1\|_0^2 + C_6\|U_2\|_0^2 + C_6\|U_3\|_0^2,$$

where $C_6 = 10K_2^2$. The last inequality can be combined with (21) yielding for $t < T$

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\|U_2\|_0^2 \;&\le \tfrac{1}{2}\|(u_2)_t - \phi_h\|_0^2 + \tfrac{5}{2}\|u_2 - w_h\|_0^2 + C_5'\|U_2\|_0^2 + C_6'\|\theta_1\|_0^2 + \\ &+ C_6'\|\rho_1\|_0^2 + C_6'\|U_3\|_0^2,\end{aligned} \tag{22}$$

for suitable constants C_5' e C_6'.

Similarly, the second ordinary differential equation yields the inequality

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\|U_3\|_0^2 \;&\le \tfrac{1}{2}\|(u_3)_t - \bar{\phi}_h\|_0^2 + \tfrac{5}{2}\|u_2 - \bar{w}_h\|_0^2 + \\ &+ C_7'\|U_3\|_0^2 + C_8'\|\theta_1\|_0^2 + C_8'\|\rho_1\|_0^2 + C_8'\|U_2\|_0^2,\end{aligned} \tag{23}$$

for $\bar{\phi}_h, \bar{w}$ arbitrary elements of W_h and for suitable constants $C_7', C_8' > 0$.

Now, we sum up (19), (22) and (23) and integrate over $[0, t]$, $t < T$; using (17), for a suitable constant C we get

$$\|\theta_1(t)\|_0^2 + \|U_2(t)\|_0^2 + \|U_3(t)\|_0^2 \le \|\theta_1(0)\|_0^2 + \|U_2(0)\|_0^2 + \|U_3(0)\|_0^2 +$$

$$+ \int_0^t [\|(u_2)_t(s) - \phi_h\|_0^2 + 5\|u_2(s) - w_h\|_0^2 + \|(u_3)_t(s) - \bar{\phi}_h\|_0^2 + 5\|u_3(s) - \bar{w}_h\|_0^2 +$$

$$+ Ch^2\|u_1(s) - \varphi_h\|_1^2 + Ch^2\|(u_1)_t(s) - \bar{\varphi}_h\|_1^2] ds + C\int_0^t (\|\theta_1(s)\|_0^2 + \|U_2(s)\|_0^2 + \|U_3(s)\|_0^2)\, ds,$$

where $\varphi_h, \bar{\varphi}_h \in L^\infty(0, T; V_h)$ and $\phi_h, \bar{\phi}_h, w_h, \bar{w}_h \in L^\infty(0, T; W_h)$ are arbitrary elements.

Gronwall's lemma yields $\forall t < T$

$$\|\theta_1(t)\|_0^2 + \|U_2(t)\|_0^2 + \|U_3(t)\|_0^2 \le [\|\theta_1(0)\|_0^2 + \|U_2(0)\|_0^2 + \|U_3(0)\|_0^2 +$$

$$+ \int_0^T (\|(u_2)_t(s) - \phi_h(s)\|_0^2 + 5\|u_2(s) - w_h(s)\|_0^2 + \|(u_3)_t(s) - \bar{\phi}_h(s)\|_0^2 + 5\|u_3(s) - \bar{w}_h(s)\|_0^2 +$$

$$+ Ch^2\|u_1(s) - \varphi_h(s)\|_1^2 + Ch^2\|(u_1)_t(s) - \bar{\varphi}_h(s)\|_1^2)\, ds] e^{CT},$$

and hence

$$\|\mathbf{u}_h(t) - \mathbf{u}(t)\|_0^2 \le 2[\|\theta_1(0)\|_0^2 + \|U_2(0)\|_0^2 + \|U_3(0)\|_0^2 +$$

$$+ \int_0^T (\|(u_2)_t(s) - \phi_h(s)\|_0^2 + 5\|u_2(s) - w_h(s)\|_0^2 + \|(u_3)_t(s) - \bar{\phi}_h(s)\|_0^2 + 5\|u_3(s) - \bar{w}_h(s)\|_0^2 +$$

$$+ Ch^2\|u_1(s) - \varphi_h(s)\|_1^2 + Ch^2\|(u_1)_t(s) - \bar{\varphi}_h(s)\|_1^2)\, ds] e^{CT} + 2\bar{C}h^2 \inf_{\varphi_h \in V_h} \|u_1(t) - \varphi_h\|_1^2. \tag{24}$$

The arbitrariness of the test functions implies

$$\|\mathbf{u}_h(t) - \mathbf{u}(t)\|_0^2 \le 2e^{CT}[\|\theta_1(0)\|_0^2 + \|U_2(0)\|_0^2 + \|U_3(0)\|_0^2 +$$
$$+T \inf_{\phi_h \in L^\infty(0,T;W_h)} \sup_{[0,T]} \|(u_2)_t - \phi_h\|_0^2 + 5T \inf_{w_h \in L^\infty(0,T;W_h)} \sup_{[0,T]} \|u_2 - w_h\|_0^2 +$$
$$+T \inf_{\bar\phi_h \in L^\infty(0,T;W_h)} \sup_{[0,T]} \|(u_3)_t - \bar\phi_h\|_0^2 + 5T \inf_{\bar w_h \in L^\infty(0,T;W_h)} \sup_{[0,T]} \|u_3 - \bar w_h\|_0^2 +$$
$$+CTh^2 \inf_{\varphi_h \in L^\infty(0,T;V_h)} \sup_{[0,T]} \|u_1 - \varphi_h\|_1^2 + CTh^2 \inf_{\bar\varphi_h \in L^\infty(0,T;V_h)} \sup_{[0,T]} \|(u_1)_t - \bar\varphi_h\|_1^2] +$$
$$+2\bar C h^2 \inf_{\varphi_h \in V_h} \|u_1(t) - \varphi_h\|_1^2,$$

and from (14),(15), (18) and from Lemma 2.1, we have the convergence as $h \to 0$. □

In order to have an optimal error estimate in L^2-norm, we have to introduce further assumptions on the spaces V_h and W_h, and hypothesis on the spatial smoothness of the solution.

Definition 2.3. *A family of spaces $\{X_h\}$ is said to be of class $S_{k,\mu}$ (with $k \le \mu$) if $X_h \subset H^k(\Omega)$ and there exists a constant $C > 0$ such that*

$$\forall w \in H^\mu(\Omega), \qquad \inf_{w_h \in X_h} \sum_{j=0}^{k} h^j \|w - w_h\|_j \le C h^\mu \|w\|_\mu.$$

Since we are dealing with a second-order partial differential system coupled with ordinary differential equations, in the sequel we shall assume that $\{V_h\}$ is of class $S_{1,\mu}$ ($\mu \ge 2$) and $\{W_h\}$ is of class $S_{0,\mu'}$ ($\mu' \ge 1$).

We have the following convergence result.

Theorem 2.4. *Suppose that g_i, $i = 1,2,3$, satisfies the Lipschitz condition (13) and that the family $\{V_h\}$ is of class $S_{1,\mu}$ ($\mu \ge 2$) and $\{W_h\}$ of class $S_{0,\mu'}$ ($\mu' \ge 1$). Assume that the solution $\mathbf{u}$ is such that $u_1 \in H^1(0,T;H^\mu(\Omega))$ and $u_2, u_3 \in H^1(0,T;H^{\mu'}(\Omega))$. Then we have the following error estimate*

$$\|\mathbf{u}_h - \mathbf{u}\|_0 \le c h^{\min(\mu,\mu')}, \qquad \forall t \in [0,T],$$

where c depends on the constants K_i and on the solution $\mathbf{u}$.

Proof. The thesis follows easily from the error estimate (24) and from the assumptions on the spaces V_h and W_h. □

Let us move to the case of locally Lipschitz continuous functions g_i. In this case the proof of the convergence needs further assumptions on the families $\{V_h\}$ and $\{W_h\}$, since in general local Lipschitz continuity of the g_i's does not guarantee global existence of the solution to (12) either and therefore we have to provide estimates which assure that the semidiscrete solution does not blow up. The idea of the proof was used by Thomée [7] for general semilinear equations of parabolic type and extends to a system of the form (1) what proved in [5].

In the sequel we will assume that the families $\{V_h\}$ and $\{W_h\}$ are of class $\mathcal{S}_{1,\mu}$ and $\mathcal{S}_{1,\mu'}$ respectively, and satisfy the following *inverse inequality*:

$$\|\varphi_h\|_\infty \le C_1 h^{-\nu} \|\varphi_h\|_0 \quad \forall \varphi_h \in V_h,\ h \le h_1, \quad \text{for some } \nu \text{ and } h_1, \tag{25}$$

$$\|\psi_h\|_\infty \le C_2 h^{-\nu'} \|\psi_h\|_0 \quad \forall \psi_h \in W_h,\ h \le h_2, \quad \text{for some } \nu' \text{ and } h_2. \tag{26}$$

We start by observing that, for fixed h, system (12) has a unique local solution in $[0, T_h]$, for some $T_h \le T$. In the analysis developed below, we shall show that, under suitable assumptions on $\{V_h\}$ and $\{W_h\}$ and for h sufficiently small, T_h can be taken to be equal to T. Hence, global solvability of system (12) follows. This can be achieved by providing an estimate of the error $\mathbf{u}_h - \mathbf{u}$ in the L^∞-norm, which allows to derive boundedness of the semidiscrete solution $\mathbf{u}_h$. Therefore, assume that the following approximation properties hold:

$$\lim_{h\to 0} \sup_{[0,T]} \inf_{w_h \in V_h} \{\|w(t) - w_h\|_\infty + h^{-\nu}\|w(t) - w_h\|_0\} = 0 \quad \forall w \in L^\infty(0,T; H^\mu(\Omega)), \tag{27}$$

$$\lim_{h\to 0} \sup_{[0,T]} \inf_{w_h \in W_h} \{\|w(t) - w_h\|_\infty + h^{-\nu'}\|w(t) - w_h\|_0\} = 0 \quad \forall w \in L^\infty(0,T; H^{\mu'}(\Omega)), \tag{28}$$

for some $\nu, \nu' > 0$. Let us indicate by $\mathbf{u}_h = (u_{1,h}, u_{2,h}, u_{3,h})$ the unique maximal local solution of (12). Moreover, let Σ be the range of the solution $\mathbf{u}$. Let us fix $\delta > 0$ sufficiently large as to include the initial datum $\mathbf{u}_{0,h} = (u_{1,0,h}, u_{2,0,h}, u_{3,0,h})$ in a closed neighborhood Σ_δ of Σ and let $K_i > 0$ be constants such that the Lipschitz condition (8) holds in Σ_δ.

Heuristically we might argue that, since the approximate solution $\mathbf{u}_h$ is always going to be close to $\mathbf{u}$, it belongs to Σ_δ. In order to show that this is the case, we have to provide maximum norm estimates for the approximation error, since closeness in the sense of L^2 or H^1 does not automatically imply that $\mathbf{u}_h$ belongs to Σ_δ for small h.

Now we state the main theorem of this section.

Theorem 2.5. *Assume that g_i, $i = 1, 2, 3$, satisfies the Lipschitz condition (8). Let $\{V_h\}$ be of class $\mathcal{S}_{1,\mu}$ ($\mu \ge 2$) and $\{W_h\}$ of class $\mathcal{S}_{1,\mu'}$ ($\mu' \ge 2$) and suppose that (25), (26), (27) and (28) hold for some ν, ν', with $\nu < \mu$ and $\nu' < \mu'$. Then, if the solution $\mathbf{u}$ is such that $u_1 \in H^1(0,T; H^\mu(\Omega))$ and $u_2, u_3 \in H^1(0,T; H^{\mu'}(\Omega))$, there exists an h_0 such that, for $h \le h_0$, the solution $\mathbf{u}_h$ of (12) exists for $t \le T$, and for these t we have*

$$\|\mathbf{u}_h - \mathbf{u}\|_0 \le c\, h^{\min(\mu,\mu')}.$$

Proof. Let t^h be the largest number less than or equal to T such that $\mathbf{u}_h$ exists and belongs to Σ_δ for $t \le t^h$, i.e. $t^h = \sup\{s \le T \ :\ \mathbf{u}_h(t) \in \Sigma_\delta\ \forall t \le s\}$.

Similarly to Theorem 2.2, we get the estimate (24) for $t < t^h$ and, taken into account the assumptions on V_h and W_h and the smoothness of $\mathbf{u}$, we have

$$\|(u_{1,h}(t), u_{2,h}(t), u_{3,h}(t)) - (u_1(t), u_2(t), u_3(t))\|_0 \le c h^{\bar{\mu}}, \quad \bar{\mu} = \min(\mu, \mu'). \tag{29}$$

Moreover, for $t < t^h$, we have

$$\|\mathbf{u}_h - \mathbf{u}\|_\infty \le \|u_{1,h} - u_1\|_\infty + \|u_{2,h} - u_2\|_\infty + \|u_{3,h} - u_3\|_\infty \tag{30}$$

Let us consider the first term in the right-hand side of (30). Let w_h be an arbitrary element of V_h; from (25), it follows

$$\begin{aligned}\|u_{1,h} - u_1\|_\infty &\le \|u_{1,h} - w_h\|_\infty + \|w_h - u_1\|_\infty \le Ch^{-\nu}\|u_{1,h} - w_h\|_0 + \|w_h - u_1\|_\infty \le \\ &\le Ch^{-\nu}\|u_{1,h} - u_1\|_0 + Ch^{-\nu}\|w_h - u_1\|_0 + \|w_h - u_1\|_\infty \\ &\le ch^{\mu-\nu} + C \inf_{w_h \in V_h} \{h^{-\nu}\|u_1(t) - w_h\|_0 + \|u_1(t) - w_h\|_\infty\} < \delta/6 \qquad h \le h_1,\end{aligned}$$

since $\nu < \mu$ and (27) holds. Similar results can be found for the other two terms in (30). Therefore

$$\forall t < t^h. \qquad \|\mathbf{u}_h(t) - \mathbf{u}(t)\|_\infty < \delta/2, \qquad h \le h_0 \tag{31}$$

where h_0 is sufficiently small and independent of t^h. Hence we may conclude by continuity that t^h cannot be smaller than T, that is $t^h = T$ for $h \le h_0$ and, by (29),

$$\|\mathbf{u}_h(t) - \mathbf{u}(t)\|_0 \le ch^{\bar{\mu}} \qquad \forall t \le T. \square$$

Remark 2.6. *In particular, estimate (31) yields the boundedness of the semidiscrete solution* $\mathbf{u}_h$.

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Modelling the Dynamic Contact Angle

B. Schweizer

Abstract. We consider a fluid system with a free boundary and with a point of contact of the free boundary with a solid wall. We use the instationary Navier–Stokes equations and include surface tension. The correct choice of equations for the angle of contact is an open problem. We present an argument that favors the condition of a constant angle. Since solutions are necessarily of low order of regularity, we have to show the well-posedness of the equations.

Keywords: free boundary, Stokes equations, contact angle

Classification: 76D45

Equations for the contact angle

We consider the 2-dimensional situation of a domain $\Omega_t = \{(x,y) | 0 < x < L, -1 < y < h_t(x)\}$. The upper boundary is the graph of a function $h_t(x)$; we denote the normal vector by n. Let the velocity field $v = (v_1, v_2)$ and the pressure p satisfy the Navier–Stokes equations in Ω_t. With the stress-tensor $T(v,p) := -pI_2 + \nu(\nabla v + (\nabla v)^T)$, the mean curvature $\mathcal{H}(h)$, and a force f the equations on the free boundary are

$$n \cdot T(v,p) = f + \mathcal{H}(h) n, \qquad \mathcal{H}(h) := \partial_x \left(\frac{\partial_x h}{\sqrt{1 + |\partial_x h|^2}} \right), \tag{1}$$

$$\partial_t h = v_2 + (\partial_x h)\, v_1. \tag{2}$$

It is well-known that the no-slip boundary condition is incompatible with a movement of the point of contact — no solution v of class H^1 can be expected. Following e.g. [1] we impose the Navier conditions on the lateral boundary,

$$v_1 = 0, \qquad \partial_x v_2 = \pm \gamma v_2,$$

with positive sign on the left boundary, negative sign on the right boundary. If a wetted solid surface Σ carries the energy $\sigma|\Sigma|$ and a free surface Γ the energy $\beta|\Gamma|$, then we can calculate the energy decay without specifying any equation in the contact points P_l and P_r:

$$\frac{d}{dt}\left\{ \frac{1}{2}\int_\Omega |v|^2 + \beta|\Gamma| + \sigma|\Sigma| \right\} = -2\nu \int_\Omega |S_v|^2 - \nu \int_\Sigma \gamma |v_2|^2 \tag{3}$$

$$+ \int_\Gamma v \cdot f + v_2(P_l)\left[\sigma - \beta \sin(\Theta_0 - \pi/2)\right] + v_2(P_r)\left[\sigma - \beta\sin(\Theta_1 - \pi/2)\right].$$

The first line describes the loss of energy by friction. Suppose that the system has no loss of energy in the points P_i. Then, for $f = 0$, the square brackets must vanish and we impose that the contact angle is a constant and coincides with the static angle.

The argument can be improved. In [3] we consider two equations for a discrete time-step. The solution to the equations with prescribed contact angle is denoted by (v^0, h^0). We compare it with the following construction: The second line of (3) vanishes if intermolecular forces in P_i result in a point force of appropriate strength. We regularize this force with a parameter ε and denote the smooth solution by $(v^\varepsilon, h^\varepsilon)$. The result in [3] is

$$\lim_{\varepsilon\to 0}(v^\varepsilon, h^\varepsilon) = (v^0, h^0) \qquad \text{in } L^2(\Omega) \times H^1(0, L). \tag{4}$$

This result suggests to close the system by prescribing the static contact angle. Observe that for regular solutions we do not have to pose a contact angle equation: If a scheme satisfies $\frac{1}{\Delta t}(h^{n+1} - h^n) = v_2^{n+1}$ then in the case of sufficient regularity there holds

$$\frac{1}{\Delta t}(\partial_x h^{n+1}(0) - \partial_x h^n(0)) = \partial_x v_2^{n+1}(P_l) = \gamma v_2^{n+1}(P_l) = \gamma \frac{1}{\Delta t}(h^{n+1}(0) - h^n(0)). \tag{5}$$

This gives a boundary condition for h^{n+1} that closes the time-step equations.

Well posedness of the equations with prescribed contact angle

Equation (5) can be used to make a statement on the time-dependent equations with a prescribed contact angle. The relation $\partial_t h'(0) = \gamma\partial_t h(0)$ will not be satisfied by the physical system — we cannot expect $v(t)$ to be of class $H^2(\Omega)$. The work of Renardy [2] opens the possibility to derive existence results for free boundary systems in function spaces of spatial regularity $v(t) \in H^s(\Omega)$, $s \in (1, 2)$. In this approach the equations are differentiated with respect to time. The estimates for time derivatives are combined stationary estimates.

Regularity results for the stationary equations in domains with corners are available for weighted Sobolev spaces [4]. In fractional Sobolev spaces and for a 90° contact angle they can be derived with elementary methods:

Lemma: Consider a rectangle Ω. Let (V, P) satisfy the linear equations

$$\begin{aligned} -\nu\Delta V + \nabla P &= F, & \nabla\cdot V &= 0 & &\textit{in } \Omega,\\ V_2 &= v_\Gamma & & & &\textit{on } \Gamma,\\ \nu\, e_2\cdot T(V,P) - \beta\Delta_x h e_2 &= \phi & & & &\textit{on } \Gamma,\\ \partial_1 V_2 &= \gamma_i V_2, & V_1 &= 0, & &\textit{on } \Sigma_i,\\ h'(0) = 0 &= h'(L). \end{aligned} \tag{6}$$

Then V satisfies maximal regularity estimates in $H^s(\Omega)$, $s \in (1, 2)$.

Proof: There holds an energy estimate for $V \in H^1(\Omega)$. Consider the two solutions (u, p) and (v, q) of the homogeneous Stokes equations with boundary conditions

$$\begin{aligned} &h'(0) = 0 = h'(L); \quad u_1 = 0, \quad \partial_1 u_2 = 0 & &\textit{on } \Sigma_i,\\ &u_2 = v_\Gamma, \quad T_{12}(u,p) = 0, \quad T_{22}(u,p) - \Delta_x h = g & &\textit{on } \Gamma, \end{aligned}$$

$$\begin{aligned} &v_2 = 0, \quad \partial_2 v_1 = \gamma_i V_2 & &\textit{on } \Sigma_i,\\ &T_{12}(v,q) = 0, \quad v_2 = 0 & &\textit{on } \Gamma. \end{aligned}$$

We recover $V = u + v$ if we set $g = -T_{22}(v, q)$. For fixed $s \in (1, 2)$ we conclude successively: v solves a Stokes equation with data $\psi = \gamma_i V_2|_\Sigma$. The boundary conditions along Γ allow to reflect v_1 even and v_2 odd across Γ; the continuation still satisfies a Stokes equation. Consider the odd continuation of ψ. V satisfies energy

estimates in H^1 and therefore ψ is bounded in $H^{1/2}(\Sigma_i)$. The odd continuation is bounded in $H^{s-3/2}$. Standard results yield estimates for $v \in H^s(\Omega)$.
The Stokes equation for u allows a reflection across Σ_i. The data $g \in H^{s-3/2}(\Gamma)$ together with its even continuation are bounded. The remaining system is of standard type and without corners. Using regularity of this mixed type Stokes system we end up with an *a priori* estimate for u and V in the fractional Sobolev space $H^s(\Omega)$.

Using this lemma and following the ideas of Renardy [2] we can conclude the well-posedness of the nonlinear instationary equations for the prescribed contact angle $\Theta = \frac{\pi}{2}$.

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L^1-decay and the Stability of Shock Profiles

D. SERRE

Abstract. We consider both the relaxation and the viscous approximations of a scalar conservation law. It turns out that the shock profiles are the same. However, the stability analysis differs a lot. We show how the L^1-stability of shock profiles is related to the L^1-stability of constants under zero-mass perturbations. We give a new proof of the latter in the viscous case. For the semi-linear relaxation, we prove this stability in the linear case; the semi-linear one remains an open problem.

Nous considérons les approximations d'une lois de conservation scalaire, par relaxation semi-linéaire ou par viscosité. Dans les deux cas, les profils de choc sont les mêmes, bien que l'analyse de stabilité diffère de manière significative. Nous montrons que la stabilité dans L^1 de ces profils est liée à la stabilité dans L^1 des constantes, pour des perturbations de masse nulle. Nous donnons une nouvelle preuve de celle-ci dans le cas visqueux. Concernant la relaxation, nous prouvons la stabilité dans le cas linéaire ; le cas semi-linéaire reste ouvert.

Keywords: conservation laws, shock profiles, dynamical systems

Classification: 35L65, 35L45, 65M06

THE L^1-DECAY PROBLEM

Let us consider a scalar conservation law

$$u_t + f(u)_x = 0, \quad x \in R, t > 0, \tag{1}$$

supplemented by an initial datum $u(\cdot, 0) = u_0$. The flux f is a given C^2 function. The datum u_0 belongs to $L^1 + L^\infty$; the solution is well-defined since, thanks to Kruzkhov's analysis [5], a semi-group $t \mapsto T(t)$ may be defined on this space with the following properties:

Contraction:: If $a - b \in L^1$, then $T(t)a - T(t)b \in L^1$ and $\|T(t)a - T(t)b\|_1 \leq \|a - b\|_1$.

Mass conservation:: Similarly,

$$\int_R (T(t)a - T(t)b)dx = \int_R (a - b)dx.$$

Comparison:: If $a(x) \leq b(x)$ almost everywhere, then $T(t)a \leq T(t)b$. In particular, $t \mapsto \|T(t)a\|_\infty$ is non-increasing.

This solution, called the "entropy" solution, is known to be the strong limit of the one obtained by viscous approximation:

$$u_t + f(u)_x = \tau u_{xx}, \quad u(\cdot, 0) = u_0, \tag{2}$$

as $\tau \to 0^+$. Recently, it has also been shown (see [2, 9]) to be the limit of the relaxation approximation

$$u_t + v_x = 0, \quad v_t + a^2 u_x = \frac{1}{\tau}(f(u) - v), \tag{3}$$

with $(u, v)_{t=0} = (u_0, v_0)$, as $\tau \to 0^+$. Thus, v_0 is any given bounded datum and u_0 is assumed to be bounded. The constant $a > 0$ is assumed to be large enough so that it dominates *a priori* the velocities $f'(u)$. Such a semi-linear relaxation has been introduced by Shi Jin and Zhouping Xin in [4] for numerical purposes. It belongs to the more general class of relaxation models considered by T.-P. Liu [7] and subsequent authors.

Most simple solutions of (1) are the so-called shock waves

$$u(x,t) = \begin{cases} u_l, & x < st, \\ u_r, & x > st, \end{cases}$$

where $u_{r,l}$ are constants and (u_r, u_l, s) satisfy the Rankine-Hugoniot relation

$$f(u_l) - f(u_r) = s(u_l - u_r), \tag{4}$$

plus the Oleinik condition. The latter claims the existence of a heteroclinic solution U of the differential equation

$$U' = f(U) - f(u_l) - s(U - u_l),$$

with $U(-\infty) = u_l$ and $U(+\infty) = u_r$.

It turns out that U may be used to build travelling waves of either (2) or (3), which approach the shock wave as $\tau \to 0^+$. In the viscous case, let us define

$$u^\tau(x,t) = U\left(\frac{x - st}{\tau}\right),$$

whereas in the relaxation case, we define

$$u^\tau(x,t) = U\left(\frac{x - st}{\tau(a^2 - s^2)}\right), \quad v^\tau(x,t) = s(u^\tau(x,t) - u_l) + f(u_l).$$

Here we encounter the so-called "sub-characteristic" condition:

$$a > \sup\{|f'(w)|; w \in I(u_l, u_r)\},$$

where $I(b, c)$ denotes the interval with bounds b and c.

As is well-known (an analysis of the linear stability is convincing), the uniform stability of an approximation as $\tau \to 0^+$ is strongly related to the asymptotic stability as $t \to +\infty$ with a fixed value of τ, say $\tau = 1$. On the other hand, the semi-groups defined by either (2) or (3) share the properties of T. This is clear for the viscous approximation, whereas it has to be slightly adapted for the relaxation in the following way: we define the characteristic variables $w, z := au \pm v$, which satisfy

$$w_t + aw_x = f\left(\frac{w+z}{2a}\right) + \frac{z-w}{2}, \quad z_t - az_x = -f\left(\frac{w+z}{2a}\right) + \frac{w-z}{2}.$$

The right-hand sides are monotonous increasing/decreasing with respect to w or z, within the box $[A, C] \times [B, D]$, provided $a \geq \sup\{|f'(u)|; A + B \leq 2au \leq C + D\}$. This box is then a positively invariant domain, so that the semi-group has the following properties (see [9]):

: It is contracting for the L^1-distance

$$d_1((u,v),(u',v')) := \|w-w'\|_1 + \|z-z'\|_1.$$

: It is mass-preserving, provided $v_0 - v_0'$ "vanishes at infinity":

$$\int_R (u-u')(x,t)dx = \int_R (u_0 - u_0')dx.$$

: Let (w_0, z_0) and $(\hat{w}_0, \hat{z}_0)$ be two initial data such that $w_0 \le \hat{w}_0$ and $z_0 \le \hat{z}_0$ almost everywhere, then the corresponding solutions satisfy $w \le \hat{w}$ and $z \le \hat{z}$.

For all these reasons, we are interested in the L^1-stability of shock profiles, which reads as follow. Given an initial datum u_0 or (u_0, v_0), so that $u_0 - U \in L^1$ (in the viscous case), or $u_0 - U(\cdot/(a^2-s^2)) \in L^1$, does the solution behave asymptotically the same than the travelling wave in the L^1-norm? In other words, do we have

$$\lim_{t\to+\infty} \|u(t) - \bar{u}(t)\|_1 = 0,$$

where $\bar{u}$ stands for the travelling wave? The general answer is NO, because of the mass conservation of the semi-groups: a necessary condition is that the mass of the initial disturbance $u_0 - \bar{u}(0)$ be zero. However, this condition may be fulfilled if $u_r \neq u_l$, to the price of a shift of the profile U. For instance, in the viscous case,

$$\int_R (u_0(x) - U(x+h))dx = \int_R (u_0 - U)dx + h(u_l - u_r)$$

vanishes when choosing

$$h := \frac{1}{u_r - u_l}\int_R (u_0 - U)dx.$$

Thus we may assume, without loss of generality, that the initial disturbance $u_0 - U$ has zero mass.

A definitive result in the viscous case has been obtained in [3].

Theorem 1 (H. Freistühler, D. Serre 1998). *Let U be a shock profile ($u_r \neq u_l$) of the viscous conservation law (2), with velocity s. Let $u_0 \in U + L^1$ have a zero disturbance mass:*

$$\int_R (u_0 - U)dx = 0.$$

Then

$$\lim_{t\to+\infty} \|u(t) - U(\cdot - st)\|_1 = 0.$$

The proof of this result splits in two parts, of equal importances. The first one considers the case where u_0 takes values in $I(u_l, u_r)$ (see [10]). It uses arguments of dynamical systems: compactness, ω-limit set, Lyapunov functions and Lasalle's invariance principle. Let us note that thanks to the contraction property of the semi-group, one always may restrict to a dense subset of data; here it consists of those data which satisfy inequalities of the form $U(\cdot + \alpha) \le u_0 \le U(\cdot + \beta)$ for convenient constants α, β, which depend on u_0. Similar arguments have been used in the relaxation approximation (see [8]), with the following result.

Theorem 2 (C. Mascia, R. Natalini 1996). *Let $\bar{u}$ be a shock profile (with $u_r \neq u_l$), with velocity s, for the relaxation system (3) and let $\bar{v} := s(\bar{u} - u_l) + f(u_l)$ be its associated flux. Let $u_0 \in \bar{u}(0) + L^1$ and $v_0 \in \bar{v}(0) + L^1$ be initial data. We assume that*

- *the disturbance has zero mass:*

$$\int_R (u_0 - \bar{u})dx = 0,$$

- *(w_0, z_0) takes values in the invariant box $[au_- + f(u_-), au_+ + f(u_+)] \times [au_- - f(u_-), au_+ - f(u_+)]$, where $u_- = \min\{u_l, u_r\}$, $u_+ = \max\{u_l, u_r\}$,*
- *$a > \sup\{|f'(m)|; m \in [u_-, u_+]\}$.*

Then

$$\lim_{t\to+\infty} \|u(t) - \bar{u}(t)\|_1 + \|v(t) - \bar{v}(t)\|_1 = 0.$$

We conjecture actually the more general statement, which mimics Theorem 1.

Conjecture 1. *Let $\bar{u}$ be a shock profile (with $u_r \neq u_l$), with velocity s, for the relaxation system (3) and let $\bar{v} := s(\bar{u}-u_l)+f(u_l)$ be its associated flux. Let $u_0 \in \bar{u}(0)+L^1$ and $v_0 \in \bar{v}(0) + L^1$ be an initial data. Let define $A := (\inf_x w_0(x) + \inf_x z_0(x))/2a$ and $B := (\sup_x w_0(x) + \sup_x z_0(x))/2a$. We assume that*

- *the disturbance has zero mass:*

$$\int_R (u_0 - \bar{u})dx = 0,$$

- *$a > \sup\{|f'(m)|; m \in [A, B]\}$.*

Then

$$\lim_{t\to+\infty} \|u(t) - \bar{u}(t)\|_1 + \|v(t) - \bar{v}(t)\|_1 = 0.$$

Let us remark that the sub-characteristic condition was written here in such a way to ensure the global existence of the solution (u, v), as well as the contraction property and the maximum principle. This conjecture is a bit more general than Theorem 2 in that we do not limit the size of the initial data by a box *a priori* defined by the shock values $u_{l,r}$.

The second part of the proof of Theorem 1 consists of proving that $(u(t) - u_+)^+$ and $(u(t) - u_-)^-$ both tend to zero in L^1-norm, so that the distance of $u(t)$, to the set of data u_0 considered in the first part, tends to zero as $t \to +\infty$. Then the contraction property implies that $\|u(t) - \bar{u}(t)\|_1$ tends to zero.

The way to prove that $\|(u(t) - u_+)^+\|_1$ tends to zero is the following. Because $u_0 - U$ is integrable and $U(x)$ tends to u_- $(< u_+)$ in some direction, there is a $\hat{u}_0$ so that

$$u_0 \leq \hat{u}_0, \quad \hat{u}_0 \in u_+ + L^1, \quad \int_R (\hat{u}_0 - u_+)dx = 0.$$

Then, because of the comparison principle, $u(t) \leq \hat{u}(t)$, where $\hat{u}$ is the solution of (2) with data $\hat{u}_0$. Therefore the conclusion follows from the following theorem.

Theorem 3 (H. Freistühler, D. Serre 1998). *Let u_+ be a constant and let $u_0 \in u_+ + L^1$ be such that*

$$\int_R (u_0 - u_+)dx = 0. \tag{5}$$

Then

$$\lim_{t\to+\infty} \|u(t) - u_+\|_1 = 0.$$

The same argument may be used in the case of relaxation model (3): keeping v_0 unchanged and choosing $\hat{u}_0$ as above, one defines $\hat{w}_0 := a\hat{u}_0 + v_0$ and $\hat{z}_0 := a\hat{u}_0 - v_0$, so that $w_0 \le \hat{w}_0$ and $z_0 \le \hat{z}_0$. Therefore, $w \le \hat{w}$ and $z \le \hat{z}$. It follows that

$$u = \frac{w+z}{2} \le \frac{\hat{w}+\hat{z}}{2} = \hat{u},$$

although we cannot compare v with $\hat{v}$. Thus $\|(u(t) - u_+)^+\|_1$ will tend to zero as $t \to +\infty$ as soon as $\|\hat{u}(t) - u_+\|_1$ does. We conclude that Conjecture 1 is a consequence of the following.

Conjecture 2. *Let full u_+ be a constant and let $u_0 \in u_+ + L^1$, $v_0 \in f(u_+) + L^1$ be an initial data. Let's define $A := (\inf_x w_0(x) + \inf_x z_0(x))/2a$ and $B := (\sup_x w_0(x) + \sup_x z_0(x))/2a$. We assume that*

- *the disturbance has zero mass: $\int_R (u_0 - u_+)dx = 0$,*
- $a > \sup\{|f'(m)|; m \in [A, B]\}$.

Then

$$\lim_{t\to+\infty} \|u(t) - u_+\|_1 + \|v(t) - f(u_+)\|_1 = 0.$$

Let us make two remarks. First, we do not need to assume any integral condition on v_0, since the system (3) preserves only the mass of u (or $u - u_+$ here). Second, the condition of zero mass is a meaningful restriction in Theorem 3 and Conjecture 2, since we cannot come back to this case by a shift of the constant state. When the integral does not vanish, we expect that a diffusion wave develops as t increases. In both cases, this wave spreads on a domain of width $\sqrt{t}$ and decays in L^∞ as $t^{-1/2}$; however, it does not decay in L^1.

We shall prove Conjecture 2 in the linear case.

Theorem 4. *Let f be affine: $f(u) = bu + c$ with $|b| < a$. Let u_+ be a constant and let $u_0 \in u_+ + L^1$, $v_0 \in f(u_+) + L^1$ be such that the disturbance has zero mass:*

$$\int_R (u_0 - u_+)dx = 0.$$

Then

$$\lim_{t\to+\infty} \|u(t) - u_+\|_1 + \|v(t) - f(u_+)\|_1 = 0.$$

1. A new proof of Theorem 3

Making the changes $u \mapsto u - u_+$, $f \mapsto f - f(u_+)$ and $x \mapsto x - f'(u_+)t$, one may assume the following: $u_+ = 0$, $f(0) = f'(0) = 0$. Therefore $f(u) = \mathcal{O}(u^2)$ as $u \to 0$. The assumption is $u_0 \in L^1$, with

$$\int_R u_0(x)dx = 0, \tag{6}$$

and we must prove

$$\lim_{t\to+\infty} \|u(t)\|_1 = 0. \tag{7}$$

1.1. A first reduction. Thanks to the contraction property, we may restrict to a dense subset of data, say those which belong to $L^1 \cap L^\infty$. The energy identity

$$(u^2)_t + g(u)_x + 2u_x^2 = (u^2)_{xx},$$

where $g'(u) = 2uf'(u)$, shows that $t \mapsto \|u(t)\|_2$ is bounded, while $t \mapsto \|u_x(t)\|_2$ is square-integrable. By Ladyzenskaya inequality, we conclude that $t \mapsto \|u(t)\|_\infty^4$ is integrable. However, it is monotonous non-increasing (maximum principle). Thus it converges to zero and there is a time T so that $\|u(T)\|_\infty \le 1$.

Therefore we may assume from now on that $\|u_0\|_\infty \le 1$, so that $\|u(t)\|_\infty \le 1$. Since $f(v) = \mathcal{O}(v^2)$ at the origin, there exists a constant M such that

$$|f(u(x,t))| \le Mu(x,t)^2, \quad x \in R, t > 0.$$

1.2. A nonlinear inequality. For such a data, we define

$$A(t) := \sup_{0 \le s \le t} \|u(s)\|_2 s^{1/4},$$

which is a non-decreasing function. By the Duhamel's principle,

$$u(t) = K^t * u_0 - \int_0^t (\partial_x K^{t-s}) * f(u(s))ds,$$

where K is the heat kernel. Using a Young inequality, we obtain

$$\|u(t)\|_2 \le \|K^t\|_2 \|u_0\|_1 + \int_0^t \|\partial_x K^{t-s}\|_2 \|f(u(s))\|_1 ds.$$

This yields

$$\begin{aligned} \|u(t)\|_2 t^{1/4} &\le c_1\|u_0\|_1 + c_2 M t^{1/4} \int_0^t (t-s)^{-3/4} \|u(s)\|_2^2 ds \\ &\le c_1\|u_0\|_1 + c_2 M\, A(t)^2 t^{1/4} \int_0^t (t-s)^{-3/4} s^{-1/2} ds \\ &= c_1\|u_0\|_1 + c_3 M A(t)^2. \end{aligned}$$

Therefore

$$A(t) \le c_1\|u_0\|_1 + c_3 M A(t)^2. \tag{8}$$

Actually, $t \mapsto u(t)$ is continuous with values in L^2, so that $t \mapsto A(t)$ is continuous and vanishes at $t = 0$. We now make the restriction

$$\|u_0\|_1 < \delta, \tag{9}$$

where $\delta := (4c_1c_3M)^{-1}$. Then the roots of the equation $X = c_1\|u_0\|_1 + c_3MX^2$ are real distinct positive numbers. By continuity, $A(t)$ remains smaller than the least one. This root clearly is bounded by $c_4\|u_0\|_1$. Therefore:

Lemma 1. *Let $u_0 \in L^1 \cap L^\infty$ be such that $\|u_0\|_\infty \le 1$ and $\|u_0\|_1 \le \delta$. Then*

$$\|u(t)\|_2 \le c_4\|u_0\|_1 t^{-1/4}, \quad t > 0.$$

Let us point out that up to now, we did not use the assumption (6). Let us also point out that this lemma is not the best possible. Abourjaily and Bénilan [1] proved that $\|u(t)\|_2 \leq (2t)^{-1/4}\|u_0\|_1$, without smallness assumption on the data. Their idea consists of writing the energy equality

$$\|u\|_2 \frac{d}{dt}\|u\|_2 + \|u_x\|_2^2 = 0,$$

then to write a Sobolev-Gagliardo-Nirenberg inequality $\|u\|_2^3 \leq 2\|u\|_1^2\|u_x\|_2$. Last, one uses the decay of $\|u\|_1$. They obtain the following inequality:

$$2\|u_0\|_1^4 \frac{d}{dt}\|u\|_2 + \|u\|_2^5 \leq 0,$$

from which the conclusion follows easily.

1.3. The decay for small data. We still assume that $\|u_0\|_\infty \leq 1$ and $\|u_0\|_1 \leq \delta$. We use again a Young inequality:

$$\begin{aligned}
\|u(t)\|_1 &\leq \|K^t * u_0\|_1 + \int_0^t \|\partial_x K^{t-s}\|_1 \|f(u(s))\|_1 ds \\
&\leq \|K^t * u_0\|_1 + c_5 \int_0^t \|u(s)\|_2^2 \frac{ds}{\sqrt{t-s}} \\
&\leq \|K^t * u_0\|_1 + c_6 \|u_0\|_1^2 \int_0^t \frac{ds}{\sqrt{t(t-s)}} \\
&= \|K^t * u_0\|_1 + c_7\|u_0\|_1^2.
\end{aligned}$$

Because of (6), we know that $\|K^t * u_0\|_1$ tends to zero as $t \to +\infty$. Therefore

$$l := \lim_{t\to+\infty} \|u(t)\|_1 \leq c_7\|u_0\|_1^2.$$

However, since $\|u(s)\|_1$ and $\|u(s)\|_\infty$ decay, $u(s)$ satisfies all the assumptions used above, so that, replacing $t = 0$ by $t = s$, we also have $l \leq c_7\|u(s)\|_1^2$. Passing to the limit as $s \to +\infty$, it comes $l \leq c_7 l^2$, where we already know that $l \leq \|u_0\|_1$. Finally, we have

Lemma 2. *Let $u_0 \in L^1 \cap L^\infty$ satisfy (6) and*

$$\|u_0\|_\infty \leq 1, \quad \|u_0\|_1 \leq \min(\delta, 1/c_7).$$

Then

$$\lim_{t\to+\infty} \|u(t)\|_1 = 0.$$

1.4. The conclusion. It works as in the proof of [3]. Let define X to be the metric space of measurable functions $v \in L^1 \cap L^\infty$ such that $\|v\|_\infty \leq 1$ and $\int_R v dx = 0$, endowed with the L^1-distance. It is convex, thus a connected space. It is positively invariant by the flow defined by equation (2), because of maximum principle and mass conservation. Let B be the attraction basin, in X, of the origin. On one hand, the semi-group is L^1-contracting, so that B is closed. On the other hand, we proved (Lemma 2) that B is a neighbourhood of $u = 0$; since the semi-group is continuous, this implies that B is an open set. Finally, B is non-void. We conclude that $B = X$.

Now we recall that the solution enters X provided $u_0 \in L^1 \cap L^\infty$. Thus Theorem 3 is valid with the condition that u_0 be bounded. Last, by density, this condition is removed.

QED

2. The case of linear relaxation

Here we consider the linear system with damping, which is the linear case of the relaxation model. Up to a change of variable, we choose $c = 0$ and $u_+ = 0$:

$$u_t + v_x = 0, \quad v_t + a^2 u_x = bu - v, \tag{10}$$

where $|b| < a$. Here we prove Theorem 4, that is

$$\left(u_0, v_0 \in L^1 \text{ and } \int_R u_0 dx = 0\right) \Longrightarrow \lim_{t\to+\infty} \|u(t)\|_1 + \|v(t)\|_1 = 0.$$

As before, the contraction property allows us to restrict to a dense subset of initial data, namely to bounded data of compact support. Let (u_0, v_0) be such a data, supported in $[-L, L]$. Then $(u(t), v(t))$ is supported in $[-L - at, L + at]$. Moreover, $\int_R u(x,t)dx$ remains null. Therefore we may introduce a potential function $p(t)$, compactly supported, so that $u = p_x$. Then $v = p_t$.

Let us now define

$$E(u,v,p) := a^2u^2 + v^2 + \frac{1}{2}p^2 - pv, \quad Q(u,v) := a^2u^2 + v^2 - 2buv,$$

which are positive definite quadratic forms. The following identity holds:

$$\partial_t E + \partial_x \left(2a^2uv + \frac{b}{2}p^2 - a^2pu\right) + Q = 0.$$

Integration on $R \times (0,T)$ yields

$$\int_R E(u,v,p)_{t=T}dx + \int_0^T \int_R Q(u,v)dxdt = \int_R E(u_0,v_0,p_0)dx < +\infty.$$

This implies that u and v belong to $L^2_{x,t}$.

On the other hand, (10) may be rewritten in characteristic form as

$$w_t + aw_x = \gamma z - \delta w, \quad z_t - az_x = \delta w - \gamma z,$$

where $\gamma, \delta = 1/2 \pm b/2a$ are positive numbers. The functional

$$J_1(t) := \|w(t)\|_1 + \|z(t)\|_1$$

is a Lyapunov function (this is a particular case of Natalini's results). Let l be its limit as $t \to +\infty$. The Cauchy-Schwarz inequality gives

$$l \le \|w(t)\|_1 + \|z(t)\|_1 \le \sqrt{2(L+at)}\,(\|w(t)\|_2 + \|z(t)\|_2),$$

so that

$$\frac{l^2}{L+at} \le 4(\|w(t)\|_2^2 + \|z(t)\|_2^2).$$

Since the right-hand side is integrable on $(0,+\infty)$, l must be zero.

QED

2.1. The case $b = \pm a$. Let us consider for instance the case $b = a$. Then $\delta = 0$ and the equations

$$w_t + aw_x = z, \quad z_t - az_x = -z$$

can be integrated by hands. One obtains

$$u(t) = u_0(\cdot - at) + L^t z_0,$$

where

$$L^t z_0(x) := e^{-t} z_0(x + at) - z_0(x - at) + \int_0^t e^{-s} z_0(x - at + 2as)ds.$$

Using the shift $\tau_{at} : y \mapsto y + at$, one has

$$(\tau_{at} L^t z_0)(y) = e^{-t} z_0(y + 2at) - z_0(y) + \int_0^t e^{-s} z_0(y + 2as)ds.$$

The first term of the right-hand side converges to zero in L^1. The second one converges since it does not depend on time. The last one is the convolution by the kernel

$$m^t(s) := \frac{1}{2a} e^{s/2a} \chi_{[-2at,0]}.$$

By Young's inequality, it converges in L^1 to $m^\infty * z_0$. Finally,

$$\lim_{t \to +\infty} \|u(t) - (u_0 + Mz_0)(\cdot - at)\|_1 = 0,$$

where we define

$$MZ(y) := \int_0^{+\infty} e^{-t} Z(y + 2at)dt - Z(y).$$

This calculus shows that the conclusion of Theorem 4 does not hold in this borderline case: $u(t)$ does not decay to zero in general. We may give a more precise statement as follows. The condition for $\|u(t)\|_1$ converging to zero is $Mz_0 + u_0 = 0$. This is equivalent to the differential equation $z_0' = u_0' - u_0/2a$. Since u_0 is given with the condition (6), we know that there exists a unique absolutely continuous function p_0 such that $p_0' = u_0$ and $p_0(\pm\infty) = 0$. Then $z_0 = u_0 - p_0/2a$. The hypothesis $z_0 \in L^1$ is ensured if we assume moreover that $u_0 \in L^1((1 + |x|)dx)$.

Acknowledgment: The author is very much indebted to Ph. Bénilan and Kevin Zumbrun, for fruitful discussions on this topic.

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On Positive Solutions of the Equation $\Delta U + f(|x|)U^p = 0$ in $\mathbb{R}^n$, $n > 2$

TADIE

Abstract. Positive solutions of the problem (E) below when f is nonnegative and decreasing have been investigated thoroughly by many authors (see references herein). Here we provide some sufficient conditions for their existence even in some cases where f (or f') changes sign. Some estimates of the solutions are provided.

Keywords: semilinear elliptic equations, super-sub solutions methods

Classification: 35B05, 35J65

1. Introduction

Consider the problem

$$\begin{cases} E(U) := \Delta U + f(|x|)U^p = 0 \quad \text{in} \quad \mathbb{R}^n;\ n \geq 3; \\ U(0) > 0; \quad u \geq 0 \quad \forall x \in \mathbb{R}^n \end{cases} \tag{E}$$

where $f \in C^1(\ (0,\ \infty)\ )$ is positive at least in some $(0,\ \rho)$ and satisfies everywhere for some $C_2 \geq 0$

$$|f(|x|)| \leq \frac{C_2 + |x|^\alpha}{(1+|x|)^\theta}; \quad 2+\alpha > 0; \quad \theta \geq 0. \tag{f}$$

This work investigates some conditions for the existence of radial solutions (i.e. $u(x)$ depends only on $r := |x|$) of (E). Such solutions will be in $C^2([0,\ \infty))$ if f is continuous at $r = 0$ and in $C([0,\ \infty)) \bigcap C^2((0,\ \infty))$ otherwise. The case where $f(r) \geq 0$ and decreasing has been thoroughly investigated in the literature (e.g. [2] - [6]). The novelty here is that we allow f not to be decreasing in the whole $\mathbf{R}_+$. When f is positive in some $(0,\ \rho)$, (E) has a solution which is positive in some $[0,\ R)$ and can be extended by a solution in the whole $\mathbf{R}_+$ (see e.g. [5]).
The super- and subsolutions will be the main tools for our investigation. So we recall some results for nonnegative and continuous f:

R1: ([6], [7])
The existence of a decreasing weak supersolution v (i.e $v \in C^1(\mathbf{R}_+)$; $v \geq 0$; $v' \leq 0$ and $E(v) \leq 0$ in $\mathbf{R}_+$ a.e.) is a sufficient condition for the existence of a classical (nonradial) solution and of a weak (in $C^1(\mathbf{R}_+)$) radial one, each of which is bounded above by v;

R2: ([1] , [2])
Let f be decreasing; if a classical solution u of (E) satisfies for large $|x|$ and some a_0, $k>0$;
$u(x)=a_0|x|^{-k}+g(x)$ where $|g(x)|\searrow 0$ and $|\nabla g(x)|=o(|x|^{-k-2})$
then u is radially symmetric around some $x_0\in\mathbb{R}^n$;
R3: ([7])
For more general $f(r,w)$, if a radial solution u is positive in $[0,R)$ with $u(R)=0$ the Pokhozhaev's identity reads with $F(r,T):=\int_0^T f(r,s)ds$

$$\begin{aligned}&R^n u'(R)^2=\int_0^R r^{n-1}Hf(u)dr \quad \text{where}\\ &Hf(\phi)(r):=2nF(r,\phi)+2rF_r(r,\phi)-(n-2)\phi f(r,\phi).\end{aligned} \tag{Po}$$

(3i) If $Hf(T)\le 0\quad \forall T>0$, then (E) has a classical and radial positive solution; if in addition $\int_0^\infty rf(r,T)dr=+\infty\quad \forall T>0$, then that solution decays to 0 at ∞;
(3ii) Let v be a radial and decreasing weak supersolution of (E).
If for some $t_0>0$
(a) $Hf(v)<0$ for $r>t_0$ and $Hf(\phi)\le 0$ for $r>t_0$ and any $0\le\phi\le v$;
(b) $\exists R_0>t_0;\quad \forall R>R_0\quad \int_0^{t_0} r^{n-1}Hf(v)dr+\int_{t_0}^R r^{n-1}Hf(v/2)dr<0$,
then (E) has a radial and classical solution u such that $0\le u\le v$;
R4: ([9])
Let u be a positive solution of (E) with
$u(x)=O(\,|x|^{-b}\,)$ at ∞; $b>0$ and $f(x)\simeq|x|^{-\tau}$ at ∞; $\tau\ge 0$.
Then if $\tau+pb>n$, $b=n-2$.
Main results:

Theorem 1.1. *(Uniqueness)*
Consider in $I_\rho:=(0,\ \rho)$, $(\rho\in\,]0,\infty])$ *the "radial" problem in* $\mathbb{R}^n$

$$\begin{cases}\Delta U+g(r,U)=0; & r:=|x|>0\\ U'(0)=0; \quad U>0 \quad in \quad I_\rho; & U(\rho)=0\end{cases} \tag{P}$$

where $g\in C([0,\ \infty))\bigcap C^1((0,\ \infty))$. *If* $\forall r\in I_\rho$ *and* $T>0$
1) $\partial g(r,T)/\partial T<0$ *or*
2) $\Psi(r,T):=\partial\{\,g(r,T)/T\,\}/\partial T$ *does not change sign,*
then (P) has at most one solution in $C^2([0,\ \rho))$.

Theorem 1.2.
Assume that f *is nonnegative and* $|f(|x|)|\simeq|x|^\alpha/(1+|x|)^\theta$.
1) For any $p>(n+\alpha-\theta)/(n-2)$, *(E) has a (nonradial) solution* U *and a weak radial one in* $C^1(\mathbb{R}_+)$ *such that* $|x|^{\frac{2+\alpha-\theta}{p-1}}\,U(x)$ *is bounded.*
2) i) Let $p\ge(n+2+2\alpha)/(n-2)$ *and* $\theta<2+\alpha$. *Then (E) has a slowly decaying (i.e.* $r^{n-2}U$ *is unbounded) radial solution.*

ii) Let $\theta < 2+\alpha$ *and* $p > p_\alpha := (n+2+2\alpha-2\theta)/(n-2)$; *then*
a) if $n < 2(\alpha+2-\theta)$ *or otherwise*
b) $1 < m < 1+ 2(\alpha-\theta)/(n+2\theta-2\alpha-2)$ *with*
$p = P(m,\alpha,\theta) := 1+m(2+\alpha-\theta)/(n-2)$,
(E) has a (not necessarily radial) solution U *such that* $U \simeq r^{2-n}$ *at* ∞. *Moreover,* U *is radial if* f *is decreasing,* $\theta \le 1+\alpha$ *and*
$p_\alpha < p < (n+2+2\alpha)/(n-2)$.

Theorem 1.3.
a) If either $Hf(1)(r) := \{n+2-p(n-2)\}f(r)+2rf'(r) \le 0 \quad \forall r>0$
b) or $\exists R_1 > 0$ *such that* $Hf(1)(r) > 0$ *in* $(0,\ R_1)$, $Hf(1)(r) \le 0$ *for* $r \ge R_1$ *and* $\forall \tau > 0 \quad \lim_{\rho\nearrow\infty} \int_{R_1}^{\rho} r^{n-1} Hf(\tau)(r)dr = -\infty$;
then (E) has a radial solution.

Theorem 1.4.
Assume that f *is nonnegative,* f' *changes sign and for some* $R_0 > 0$, $\forall t > 0 \quad Hf(t)(r) > 0$ *in* $(0,\ R_0)$ *and* $Hf(t)(r) < 0$ *for* $r > R_0$. *Then*

1) $Hu := \Delta u + f(r)u^p = 0$ *has a radial solution* u_0 *such that* $0 < u_0 \le v$ *in* $[0,\ R_0]$; u_0 *has an extension* U_0, *which is a radial solution of* $Hu = 0$ *in* $\mathbb{R}_+$; *it could be a solution of (E) or a crossing solution (i.e.* $\exists R > R_0$ *such that* $U_0(R) = 0$ *and* $U_0'(R) < 0$*).*

2) If in addition $\forall t > 0 \quad \lim_{\rho\nearrow\infty} \int_{R_0}^{\rho} r^{n-1} Hf(t)(r)dr = -\infty$ *then (E) has a solution* U *such that* $0 \le U \le v$.

Remarks

1) Theorem 1.3 applies also when $f(r,u) := f_0(r)u^p + g(u)$ with f_0 satisfying (f) and specially when $g(u) < 0$ for $u > 0$.
2) If for f in Theorem 1.2, $\alpha \in (-2,\ 0)$, then (E) has a radial and continuous solution which is positive in some $[0,\ \rho_1)$. In fact it is enough to notice that for $U(r) = u(0) - \int_0^r dt\{\int_0^t (s/t)^a f(s)u(s)^p ds\}$ and $f_1(s) := s^{-\alpha}f(s)$, $|U(r)-u(0)| \le \sup_{[0,\ r]}\{t^{\alpha+2}f_1(t)u(t)^p\}$ for small $r > 0$.

2. Preliminaries

A supersolution for (E) ([6]): define for some $b, A > 0$

$$v(r) := (1+(Ar)^{2+\alpha})^{-b};\quad 2+\alpha > 0. \tag{2.1}$$

Simple calculations show that for $C(m) := (\frac{n-2}{m})^2\,(m-1)$,

$$\begin{cases} \text{for} \quad b = \dfrac{n-2}{m(2+\alpha)} \quad \text{and} \quad m > 1, \\ \Delta v + \dfrac{C(m)A^{2+\alpha}r^{\alpha}\,v^p}{(1+Ar)^{\theta}} \le 0 \quad \text{in} \quad [0,\infty) \\ \forall p \ge P(m,\alpha,\theta) := 1 + \dfrac{m(2+\alpha-\theta)}{n-2}. \end{cases} \tag{2.2}$$

For $f(r, v) = r^{\alpha}(1 + Ar)^{-\theta}\, v^p$,

$$Hf(v)(r) = \frac{v^{p+1}\, r^{\alpha}}{(p+1)(1+Ar)^{\theta+1}} \{Ar(n+2+2\alpha-2\theta-p(n-2)\,)+ \\ +\; n+2+2\alpha-p(n-2)\} \tag{2.3}$$

whence

$$(i) \quad p \geq \frac{n+2+2\alpha}{n-2} \Longrightarrow Hf(v)(r) \leq 0 \quad \forall r > 0 \tag{2.4}$$

$$(ii) \quad P_\alpha := \frac{n+2+2\alpha-2\theta}{n-2} < p < \frac{n+2+2\alpha}{n-2} \Longrightarrow Hf(v)(r) < 0$$
$$\text{if} \quad r > R_A := \frac{n+2+2\alpha-p(n-2)}{\{p(n-2)+2\theta-(n+2+2\alpha)\}A}. \tag{2.5}$$

If a decreasing and bounded function f satisfies for some $R_1 > 0$,

$$\{n+2-p(n-2)\}f(r) + 2rf'(r) < 0 \quad \forall r > R_1, \tag{2.6}$$
$$\text{then for} \quad \Delta u + f(r)u^p = 0, \quad Hf(u)(r) < 0 \quad \text{if} \quad r > R_1. \tag{2.7}$$

Lemma 2.1.
If V is a decreasing, radial and continuous piecewise C^2 supersolution of (E) in $[0, R]$ with $V(R) > t > 0$, then there is a radial solution U of (E) such that $t/2 < U \leq V$ in $[0, R]$.

Proof. It is an application of Theorem 2.3 of [6]. □

As a supplement to Theorem 1.3, let $g \in C^1([0,\infty)^2)$ be such that for some $I_0 := [0, R_0]$, $\forall u > 0$ and $g(r, u) := f(r)u^p + g_0(u)$ with $g_0 \leq 0$

$$g(r, u) \geq 0 \quad \text{in} \quad I_0 \quad \text{and} \quad g(r, u) < 0 \quad \text{otherwise}. \tag{2.8}$$

Theorem 2.2.
Let g be as in (2.8) and $R > R_0$. If there is a positive radial and decreasing weak supersolution v of (E), then $Gu := \Delta u + g(r, u) = 0$ has a classical and radial solution u such that $0 \leq u \leq v$ in I_R and $u(R) = v_*(R)$ where $v_*(R)$ denotes any element of $(0,\ v(R)\,)$.
Moreover u can be extended to a decreasing solution of $Gu = 0$ in $\mathbb{R}_+$ such that $u \leq v$.

Proof.
Let $w \in C(I_0)$ with support in I_0 be piecewise C^1, nonnegative and nonincreasing such that $\Delta w + g(r, w) \geq 0$ and $w < v$ in I_0 (see [7]).
Let $B > 0$ be such that $\forall u \in [w(r), v(0)]$, $r \in I_R$

$g(r,u)+Bu$ and $\partial g(r,u)/\partial u \; + B$ are strictly positive.
Define on $C(I_R) := C([0,R])$ the operator T by
$T\phi = \Phi \Leftrightarrow \Delta\Phi - B\Phi = -g_1(r,\phi) := -\{g(r,\phi)+B\phi\}$ in I_R;
$\Phi'(0)=0$; $\Phi(R) = v_*(R)$.
For $\phi = v$, $\Delta(v-\Phi) \leq B(v-\Phi)$ in I_R or for $a := n-1$
$(r^a(v'-\Phi'))' \leq Br^a(v-\Phi)$ in I_R; $r^a(v'-\Phi')|_{r=0} = 0$. If we suppose that $\Phi > v$ in $[0,\tau)$, $\tau \leq R$ then $v' < \Phi'$ and v "decreases" faster than Φ in I_R; That conflicts with the fact that $v_*(R) = \Phi(R)$. Thus $\Phi \leq v$ in I_R. Similarily we obtain that $w \leq \Phi$ in I_R.
So T maps $C(I_R)$ to $E := \{u \in C(I_R) \; ; \; w \leq u \leq v \text{ in } I_R\}$.
The continuity of g and the fact that $v \in C^2$ imply that $TC(I_R) \subset E \bigcap C^2(I_R)$ by elliptic theory. Because E is closed and bounded, T has a fixed point u, say, in $E\bigcap C^2(I_R)$ (i.e. $\Delta u - Bu = -g(r,u) - Bu$ in I_R) which is such a required solution.
For the desired extension, consider the sequence $(u_i)_{i\geq 2}$ where for
$R_i := iR_0$, u_i is the solution of
$Gu = 0$ in $[0, R_i]$; $u_i'(0)=0$; $u_i(r) = v_*(r)$ for $r \geq R_i$.
For any $k > 2$, $Gu_k = 0$ in $[0, R_k]$; $u_k|_{[0, R_l]} \equiv u_l$ $\forall l \leq k$. An inductive limit of the sequence (see [6]) is such a required solution. □

Lemma 2.3.
Let $u \in C^2(\mathbf{R}_+)$ be a positive solution of

$$\Delta u + \frac{r^\alpha u^\gamma}{(1+r)^\theta} = 0 \tag{2.9}$$

where $\gamma > 0$, $\theta \geq 0$, $2+\alpha > 0$ and for some $k > 0$ $u(r) = O(r^{-k})$ as $r \nearrow \infty$. Then there are $R, C > 0$ such that for $C_k = k(k-n+2)C$,
$\psi(r) := u(r) - Cr^{-k} > 0$ for $r > R$ and as $r \nearrow \infty$

$$\begin{aligned} &\psi(r) = O(\,r^{2+\alpha-\theta-k\gamma} \; + \; |C_k| r^{-k}\,); \\ &|\nabla\psi(x)| = O(\,r^{1+\alpha-\theta-k\gamma} \; + \; |C_k| r^{-k-1}\,). \end{aligned} \tag{2.10}$$

Proof. It is an application of Lemma 6 of [6] as the introduction of the factor r^α brings no major difficulty. □

Lemma 2.4.
Assume that $\alpha > \theta$ and let V_m be the supersolution in (2.2) corresponding to $p = P(m,\alpha,\theta)$; $m > 1$. If (1) $n < 2+2\alpha-2\theta$ or otherwise
(2) $1 < m < 1 + 2(\alpha-\theta)/\{n+2\theta-2-2\alpha\}$
then any positive solution of (2.9) whose supersolution is V_m satisfies $U(x) = O(\,|x|^{2-n}\,)$ at ∞.

Proof.
At ∞ $U \simeq |x|^{-b}$; $b \geq (n-2)/m \;=\; (2+\alpha-\theta)/(p-1)$ and $f(x) \simeq |x|^{-(\theta-\alpha)}$. If the conditions displayed in (1) or (2) hold then $\theta - \alpha + bp > n$ and R4 applies. □

3. Proof of the Theorems

Proof of Theorem 1.1. Let u and v be two solutions of (P) with $u(0) > v(0)$ and $a := n-1$ (note that if $u(0) = v(0)$ then $u \equiv v$ in I_ρ is a well-known result).
1) As $\{r^a(u'-v')\}' = r^a\{g(r,v)-g(r,u)\}$ in I_ρ with $(u'-v')|_{r=a} = 0$, there is $R > 0$ such that $(u'-v') > 0$ in $(0,\ R)$. Thus $u(r) - v(r) \geq u(0) - v(0) > 0$ in $[0,\ R]$ and $u > v$ as long as $v \geq 0$.
2) Let $W(r) := u'(r)v(r) - u(r)v'(r)$. From the equations of u and v,
$vu'' + (a/r)vu' = -vg(r,u)$ and $uv'' + (a/r)uv' = -ug(r,v)$ whence
$(r^a W)' = r^a uv\{g(r,v)/v \ - \ g(r,u)/u\}$ with $W(0) = W(\rho) = 0$.
This implies that $\int_0^\rho r^a uv\{g(r,v)/v \ - \ g(r,u)/u\}dr = 0$ and that cannot hold if $\forall r > 0, \quad t \mapsto g(r,t)/t$ is monotone in $t > 0$.■

Proof of Theorem 1.2. 1) For $p > (n+\alpha-\theta)/(n-2)$,
$\exists m > 1; \quad p = P(m,\alpha,\theta)$ whence v in (2.1)-(2.2) is a supersolution for (E) and R1 applies.
2)As $\theta < 2+\alpha, \quad \exists m > 1$ such that $p := p_\alpha = P(m,\alpha,\theta)$.
i)If $p \geq (n+2+2\alpha)/(n-2)$ then $Hf \leq 0\ \forall r > 0$ and if
$\theta < 2+\alpha, \quad \int_0^\infty rf(r,T)dr = +\infty$ and 3i) applies. The slow decaying of the solution follows from Lemma 2.5 of [4] .
ii) From (2.4), $p > p_\alpha \Longrightarrow Hf \leq 0$ in some (R,∞).
Lemma 2.4 implies that $U(x) \simeq O(|x|^{2-n})$ at ∞. Finally in (2.10) we have $|\nabla\psi| = o(|x|^{-n})$ if $\theta \leq 1+\alpha$. R2 completes the proof. ■

Proof of Theorem 1.3. a) It follows from 3i) of R3.
b) Let v be a positive and decreasing supersolution of (E). Suppose that $f > 0$ in $[0, R_1)$. By Lemma 2.1 and Theorem 2.2, $\forall R > R_1$, (E) has a radial solution u such that $0 \leq u \leq v$ in $[0, R]$ with $u(R) = v_*(R)$.
Define $Hf(V)(r) := \min\{Hf(v)(r) \mid r \in [R_1,\ R]\}$. R can be chosen such that $\int_0^{R_1} r^{n-1}Hf(v)(r)dr + \int_{R_1}^R r^{n-1}Hf(V)(r)dr = 0$. For the nonincreasing extension of these u and that R (see Theorem 2.2),

$$\int_0^R r^{n-1}Hf(u)(r)dr < \int_0^{R_1} r^{n-1}Hf(v)(r)dr + \int_{R_1}^R r^{n-1}Hf(V)(r)dr = 0$$

and from (Po), $u \geq 0 \quad \forall r > 0$; in fact for $r > R$ we cannot have $u(r) = 0$ and $u'(r) < 0$. ■

Proof of Theorem 1.4. Part 1) follows from Theorem 2.2 and the principle of the extension of local solutions for elliptic equations. Part 2) is an application of Theorem 1.3.■

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Singularity Formation for the Stefan Problem

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1. INTRODUCTION

The purpose of this note is to review several results concerning singularity formation for the one-phase Stefan problem in both the ordinary (or classical) situation as well as in the undercooled setting.

The Stefan problem is a well-known mathematical model that has been extensively used to describe the evolution of many phase transition problems. In order to keep a particular situation in mind, we can consider the Stefan problem as a model that describes the evolution of an interface separating liquid water and ice. Neglecting surface tension as well as kinetic effects and assuming by simplicity that the temperature of the ice is constant, we obtain in dimensionless units the following model for the temperature of the liquid phase:

$$\frac{\partial T}{\partial t} = \Delta T \quad \text{in} \quad \Omega(t) , \tag{1}$$

$$T = 0 \quad \text{in} \quad \partial\Omega(t) , \tag{2}$$

$$V_N = \frac{\partial T}{\partial N} \quad \text{in} \quad \partial\Omega(t) , \tag{3}$$

$$T(x,0) = T_0(x) \quad \text{in} \quad \Omega(0) . \tag{4}$$

In (1)-(4), T stands for the temperature of the liquid phase. The domain $\Omega(t) \subset \mathbb{R}^N$, $N = 2,3$, is the region occupied for the water. Equation (3) known as Stefan law states that in order to melt an amount of ice we have to provide some quantity of heat that is called latent heat. In (3) V_N stands for the normal velocity. We will assume that N is the outer normal to the domain $\partial\Omega(t)$. Finally, $T_0(x)$ in (4) is the initial distribution of temperature of the liquid phase.

Problem (1)-(4) is known as the one-phase Stefan problem. This model can be analyzed in two particular situations that are rather different from the physical and mathematical viewpoint. In the classical (or ordinary) Stefan problem we assume that $T_0(x)$ in (4) is a nonnegative function. A simple argument using the maximum principle shows that in this case, assuming that a solution of (1)-(4) exists, the domain $\Omega(t)$ expands. On the contrary, if $T_0(x) \leq 0$ we are in the so-called undercooled case. In such a case the domain $\Omega(t)$ would shrink. The first situation (classical Stefan problem) corresponds to the case in which the temperature of water is above the melting point of the ice. Nevertheless, in the undercooled Stefan problem the growth of ice is due to the fact that the temperature of water is below that corresponding to the melting point. From the physical point of view, for reasons that will be apparent later, there is a major difference between both the classical and the undercooled cases. In the first one the evolution of the boundary is expected to be smooth. On the contrary, the interface of the undercooled Stefan problem should develop complicated dendritic structures.

In this paper we restrict ourselves to the analysis of several singularity formation mechanisms in both the classical and undercooled cases. The main reason to analyze singularity formation mechanisms for (1)-(4) is that some physical effects, like the surface tension and the so-called kinetic undercooling, that are usually small and for this reason have not been taken into account in (1)-(4), could become relevant near singularities.

The literature available concerning the Stefan problem is huge in mathematical as well as in physical journals. Most of the rigorous mathematical results are concerned with the classical Stefan problem. Among the most relevant mathematical results on that problem, we can mention that a local solvability theory in Hölder spaces for the Stefan problem is available (cf. [16]). We also observe that a local partial regularity theory for the classical Stefan problem was developed in the works by Caffarelli (cf. [3], [4] and also [9]).

2. SINGULARITY FORMATION FOR THE UNDERCOOLED STEFAN PROBLEM

In this section we describe several singularity formation mechanisms for the undercooled Stefan problem. First we consider the one-dimensional case. In this situation, problem (1)-(4) might be written as:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \quad x < s(t), \tag{5}$$

$$T = 0, \quad x = s(t), \tag{6}$$

$$\frac{ds}{dt}(t) = -\frac{\partial T}{\partial x}, \quad x = s(t), \tag{7}$$

$$T(x,0) = T_0(x), \quad x < s(0). \tag{8}$$

The solvability of (5)-(8) can be found in [10] under very general assumptions on the initial data. If $T_0(x) \geq 0$ it can be shown, using a suitable barrier function, that the solutions of (5)-(8) are global in the sense that $\left|\frac{ds}{dt}(t)\right|$ remains bounded in bounded sets of t. In fact if $\inf(T_0(\cdot)) > -1$, a similar result can be obtained using suitable travelling wave solutions as barriers. If on the contrary $\inf(T_0(\cdot)) < -1$ the solutions of (5)-(8) might develop singularities in a finite time, a fact that was firstly derived in [18]. By singularity formation for (5)-(8) we understand that the velocity of the interface $\left|\frac{ds}{dt}(t)\right|$ becomes unbounded in finite time. Taking into account (6), (7), this implies that the temperature function becomes very steep near the interface as one approaches the time of formation of the singularity that we will denote from now on by t^*.

In the paper [8] several questions were analyzed about the possible continuation of the solutions of (5)-(8) after the formation of the singularity.

A complete classification of all the possible asymptotics of $(T(x,t), s(t))$ as $t \to (t^*)^-$ was recently derived in [12]. At the time of formation of the singularity the interface $s(t)$ approaches to a point that we will assume to be the origin without loss of generality. It then turns out that the interface exhibits one of the following behaviours:

$$s(t) = 2(t^* - t)^{\frac{1}{2}} \left|\log\left(\left|\log(t^* - t)\right|\right)\right|^{\frac{1}{2}} (1 + o(1)) \quad \text{as } t \to (t^*)^- , \tag{9}$$

or:

$$s(t) = C(t^* - t)^{\frac{1}{m}} (1 + o(1)) \quad \text{as } t \to (t^*)^- , \tag{10}$$

where $C > 0$ and $m = 3, 4, ...$

In [12] the asymptotics of the temperature $T(x,t)$ as $t \to (t^*)^-$, as well as the final profile at the singularity time were also computed. More precisely, in the cases (9), (10) above we have:

$$T(x,t^*) \sim -1 - \frac{1}{2\left|\log\left(\left|\log(|x|)\right|\right)\right|} \quad \text{as } x \to 0^- , \tag{11}$$

$$T(x,t^*) \sim -1 - K|x|^{m-2} \quad \text{as } x \to 0^- ,\ m = 3, 4, ... \tag{12}$$

where $K \in \mathrm{R}$.

In higher dimensions the situation is rather different. Notice that (6) as well as (11), (12) imply that $T(x,t)$ develops a jump near the interface of value -1 as t approaches t^* in the one-dimensional case. On the contrary, it was pointed out in [15] that in higher dimensional situations another mechanisms of singularity formation should exist, for which the jump of the temperature (or undercooling) at the origin would be arbitrarily small. In [19] a mechanism of cusp formation in dimensions two and three for (1)-(4) was described using matched asymptotic expansions. Detailed asymptotics for the temperature and the inteface profile were derived there. For instance, it was obtained in [19], that in the two-dimensional case the interface behaves asymptotically at the time of formation of the singularity as:

$$|x_2| \sim K \frac{|x_1|}{\left|\log\left(\left|\log|x_1|\right|\right)\right|^{\frac{1}{2}}} \quad \text{as } |x_1| \to 0 ,\ K > 0 . \tag{13}$$

where we have assumed, without loss of generality, that a singularity appears at $x = 0$. For this mechanism of singularity formation the undercooling at $x = 0$ can take any value in the interval $(-1, 0)$. The constant K in (13) is a function of the local undercooling at the singularity, and it turns out that it approaches zero if the undercooling approaches zero, and on the contrary K approaches $+\infty$ if the undercooling approaches -1. In other words, as the undercooling approaches -1 the cusp formation mechanism above resembles more the one-dimensional singularity previously described.

Formula (13) describes the asymptotics of the interface just at the time of formation of the singularity. It is interesting to remark that for times $t < t^*$, the interface is smooth. At distances $(t^* - t)^{\frac{1}{2}} |\log(|\log(t^* - t)|)|^{\frac{1}{2}}$ from the origin, the interface resembles a parabola of size $\frac{(t^*-t)^{\frac{1}{2}}}{|\log(|\log(t^*-t)|)|^{\frac{1}{2}}}$. The onset of such a parabola is due to the fact that the tip of the cusp can be approximated, after a suitable rescaling, by an explicit travelling wave of (1)-(3) that was discovered by Ivantsov (cf. [14]).

In [11] the problem of the stability of the singularity mechanism so far described was addressed. More precisely, in that paper, the evolution of small perturbations of one-dimensional solutions with the asymptotics (9) was considered.

Detailed asymptotics were obtained showing that under small perturbations, a cusp formation as in (13) takes place. These results strongly suggested that cusp formation as described in [19] is the generic singularity formation mechanism. However, it was recently obtained in [20] that besides the singularity formation mechanisms previously described, other possibilities exist. More precisely, the way in which a corner formation can occur in two and three dimensions was examined. It is apparent from the arguments therein that many other singularity formation mechanisms can be obtained for (1)-(4) in the undercooled case.

The reason behind such a high degree of arbitrariness in the possible singularities of the undercooled Stefan problem is related to a well-known physical instability, that is called the Mullins-Sekerka instability (cf. [17]) and that we will describe shortly. In the analysis of Mullins and Sekerka, small perturbations of a planar front solving (1)-(3), moving at a constant velocity c in the undercooled case, were considered. Let us write the position of the interface as

$$x_1 = ct + f(x_2, t) \tag{14}$$

where $f(x_2, t)$ is a small perturbation. Solving the free boundary problem in the linearized approximation we derive the following catastrophical growth for high frequencies:

$$\hat{f}(k, t) = \alpha(k)\, e^{|k|t} \tag{15}$$

where $\hat{f}(k, t)$ denotes the Fourier transform of $f(x_2, t)$ with respect to the x_2-variable. The catastrophic instability (15) is usually cut for high frequencies adding to (1)-(4) the effect of surface tension. Actually (15) could raise doubts about the solvability of (1)-(4). In fact solvability results for (1)-(4) obtained in [19] were derived under the assumption of analyticity on the initial data and the interfaces. Nevertheless, although (1)-(4) can be solved (locally in time) under such stringent assumptions, the instability (15) finally shows up in the possibility of having an extremely rich set of possible singularity formation mechanisms. A problem that requires further research is the analysis of the effect on the singularities of the usual smoothing mechanisms that must be introduced in (1)-(4), namely, surface tension and kinetic effects.

3. SINGULARITY FORMATION FOR THE CLASSICAL STEFAN PROBLEM

The understanding of the structure of singularities that might appear for (1)-(4) in the classical case (i.e. for $T_0(x) \geq 0$) is somehow more complete than for the undercooled situation. In the paper [1] a rather thought-out description of the structure of singularities has been given by using asymptotic methods. The main results of [1] will be reviewed here.

There are different ways in which the interfaces of the classical Stefan problem can develop singularities, i.e. points of nonsmoothness. The simplest one is just the collision between two regions where $T > 0$ that move independently and collide at a particular point. Such a collision generically occurs at a point where the curvature of both interfaces at the contact point is nonzero. We have analyzed the way in which both interfaces move after the contact. The two regions that were evolving independently become connected after the collision by means of a hole of size $a(t^* + t)^{\frac{1}{2}}$, where $a > 0$ is a parameter that depends on the distribution of temperature as well as the curvature at the point of collision. The smoothing after nongeneric collisions has also been considered in [1].

Another singularity mechanism that can take place for the classical Stefan problem is the cusp formation. In [1] we have discussed in detail different mechanisms for such a phenomenon, both in two and three space dimensions. We should point out that the mechanisms of the cusp formation found in [1] for the classical Stefan problem are rather unstable, since they correspond to a transition between a non-graph-like interface yielding in its evolution one or several holes of ice and a smooth interface after a while, and a graph-like interface that, when evolving, remains smooth all the time.

We have described in detail the manner of the singularity formation and the asymptotics of the interface and the temperature at different scales in [1]. We point out that the behaviour of these functions varies drastically at different length scales, although it can be described by a sequence of nested boundary layers. As $t \to (t^*)^-$, the tip of the interface can be approximated by a Ivantsov's parabola, as it happened in the undercooled case. In [2] we provide a rigorous construction of one of the simplest cusp-forming solutions for (1)-(4) in the classical case.

Another type of singular behaviour that can take place in the classical Stefan problem is the closing of a domain occupied by ice to a point. In [1] we have discussed in detail the existence of shrinking ellipses and ellipsoids (or circles and spheres in particular cases). In the two-dimensional case the characteristic length of the shrinking ellipses is $(t^* - t)^{\frac{1}{2}} \exp\left(-\frac{\sqrt{2}}{2}|\log(t^* - t)|^{\frac{1}{2}}\right)$ as $t \to (t^*)^-$. In the three-dimensional case we find shrinking ellipsoids of size $\frac{(t^*-t)^{\frac{1}{2}}}{|\log(t^*-t)|}$ as $t \to (t^*)^-$. Rigorous constructions of this type of solutions in the radially symmetric case can be found in [13].

There exist other singularity formation mechanisms that have been described in the same work, as for instance segment-like and disk-like domains shrinking to a point.

However, since those structures are rather unstable under small perturbations, we will not pursue their discussion here.

4. STEFAN PROBLEM AND HOMEOIDS

In this section we will describe some connections between the Stefan problem and the geometric figures known as homeoids. Let us consider a bounded domain D containing the origin. This last restriction is not completely needed, but we will assume it here for simplicity. We will say that D is a homeoid if the gravitational field generated by the density $\chi_{\lambda D} - \chi_D$ in the component of $\mathrm{R}^N \setminus (\lambda D \Delta D)$ that contains the origin is identically zero. By χ_G we mean the characteristic function of G and Δ stands for a symmetric difference of two sets.

The first example of a homeoid is the ellipse (or ellipsoid in three dimensions) as was proved by Newton (cf. [5]). On the other hand, it was proved in [7] and [6] that the only examples of homeoids are just ellipses and ellipsoids.

In the course of our analysis of shrinking ellipsoids (cf. [1]) we found a rather surprising rigidity result:

Theorem 1. *Assume that $u \geq 0$ solves the problem:*

$$\Delta u = H(u) \quad in \ \mathbb{R}^N \ , \ N \geq 2 \, . \tag{16}$$

Suppose that the set $\{u = 0\}$ is bounded. Then the domain $\{u = 0\}$ is an ellipsoid (an ellipse if $N = 2$).

In equation (16), $H(u)$ is the Heaviside function ($H(s) = 1$ for $s > 0$, $H(s) = 0$ otherwise), and equation (16) has to be understood in the sense of variational inequalities.

The fact that the only possible domains compatible with equation (16) are ellipsoids suggests some kind of relation with the homeoid problem. Actually, it turns out that equation (16) that arises in a natural way in the study of the Stefan problem is some kind of "integrated form" of the homeoid problem. Conversely, the homeoid problem is, in some sense, the first variation of (16). To make this statement more precise, notice that given any solution $u(x)$ of (16), we can derive another solution of (16) by means of the rescaling

$$u\lambda(x) = \lambda^2 u\left(\frac{x}{\lambda}\right) .$$

Let us define the function:

$$\varphi\lambda(x) = u\lambda(x) - u(x) .$$

This function solves the problem:

$$\Delta\varphi\lambda = \chi_{\lambda D} - \chi_D \ ,$$

where $D = \{u = 0\}$. Notice also that $\varphi\lambda$ vanishes on the set $\lambda D \Delta D$. A local analysis of the asymptotics of $u(x)$ as $|x| \to \infty$ shows that $\varphi\lambda$ is, up to an additive constant, the Newtonian potential associated to the density $\chi_{\lambda D} - \chi_D$. It then follows that D

is a homeoid. Conversely, given any homeoid D, we can assume (after the addition of a constant) that the corresponding Newtonian potential vanishes at D. Rescaling the homeoid and adding the corresponding potentials, we can obtain a nonnegative solution of (16) that vanishes on the set D. Along the lines of this argument we have constructed a new proof of the "converse Newton's theorem" in [1] that states that all the homeoids are ellipsoids, taking as starting point the theorem stated above.

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On the Construction of Interior Spike Layer Solutions to a Singularly Perturbed Semilinear Neumann Problem

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Abstract. A singularly perturbed semilinear Neumann problem which models chemotaxis and pattern formation is considered. We establish the existence of interior spike layer solutions for any bounded domain and the location of the interior spike is given. Our construction uses the very properties of the distance function which arises in the estimate of the exponentially small terms.

Keywords: Spike layer, Singular perturbation, Viscosity solution

Classification: Primary 35B40, 35B45; Secondary 35J40

1. INTRODUCTION

The aim of this paper is to construct a family of single-peaked interior spike layer solutions to the following singularly perturbed elliptic problem:

$$\begin{cases} \epsilon^2 \Delta u - u + u^p = 0 \ \text{ in } \Omega, \\ u > 0 \ \text{ in } \ \Omega \text{ and } \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, Ω is a bounded smooth domain in R^N, $\epsilon > 0$ is a constant, the exponent p satisfies $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$ and $1 < p < \infty$ for $N = 2$ and $\nu(x)$ denotes the normal derivative at $x \in \partial\Omega$.

Equation (1.1) is known as the stationary equation of the Keller-Segal system in chemotaxis. It can also be seen as the limiting stationary equation of the so-called Gierer-Meinhardt system in biological pattern formation; see [11] for more details.

In the pioneering papers of [7], [9] and [10], Lin, Ni and Takagi established the existence of least-energy solutions and showed that for ϵ sufficiently small the least-energy solution has only one local maximum point P_ϵ and $P_\epsilon \in \partial\Omega$. Moreover, $H(P_\epsilon) \to \max_{P \in \partial\Omega} H(P)$ as $\epsilon \to 0$, where $H(P)$ is the mean curvature of P at $\partial\Omega$. In [11], Ni and Takagi constructed boundary spike solutions for axially symmetric domains. The author in [21] studied the general domain case and showed that for single boundary spike solutions, the boundary spike must approach a critical point of the mean curvature; on the other hand, for any nondegenerate critical point of $H(P)$, one can construct boundary spike solutions with spike approaching that point. When $p = \frac{N+2}{N-2}$, similar results for the boundary spike layer solutions have been obtained by [1], [2], [3], [8], [18], [23], etc.

In all the above papers, only *boundary* spike layer solutions are obtained and studied. It remains a question whether or not *interior* spike layer solutions exist for problem (1.1). It was proved in [22] that under some geometric conditions, one can construct single interior spike solutions. However the geometric conditions are very restricted. It is the purpose of this paper to construct solutions by using the very properties of the distance function. We note that the distance function has already appeared in the study of the corresponding Dirichlet problem in [12], [19].

In this paper, we shall construct a family of single-peaked interior spike solutions to problem (1.1) for ϵ sufficiently small and any bounded smooth domain. We first state a more general theorem.

Theorem 1.1. *Let $\Lambda \subset \Omega$ be an open set such that*

$$\max_{P\in\partial\Lambda} d(P,\partial\Omega) < \max_{P\in\Lambda} d(P,\partial\Omega) \tag{1.2}$$

Then there exists $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$, problem (1.1) has a solution u_ϵ with the property that u_ϵ has exactly one local maximum point P_ϵ in Ω, $P_\epsilon \in \Lambda, u_\epsilon(P_\epsilon) \to w(0)$ and $u_\epsilon(\cdot + P_\epsilon) \to 0$ in $C^1_{loc}(\overline{\Omega} - P_\epsilon\backslash\{0\})$, where w is the unique solution of the following problem:

$$\begin{cases} \Delta w - w + w^p = 0 \text{ in } R^N \\ w > 0, w(0) = \max_{y\in R^N} w(y) \\ w(y) \to 0 \text{ as } |y| \to 0 \end{cases} \tag{1.3}$$

Moreover, $d(P_\epsilon, \partial\Omega) \to \max_{P\in\Lambda} d(P,\partial\Omega)$ as $\epsilon \to 0$.

We have the following important corollary.

Corollary 1.2. *Let Ω be a bounded smooth domain. Then for ϵ sufficiently small, (1.1) has a single-peaked solution with interior peak near the most centered part of the domain (i.e., the points which achieve the maximum of the distance function).*

A particular example is a domain with k–handles (see Figure 1). In this case, Theorem 1.1 asserts that there are at least k solutions to problem (1.1) and each handle contributes a single-peaked interior spike solution. Note that in Theorem 1.1 we do allow some degeneracy of the distance function.

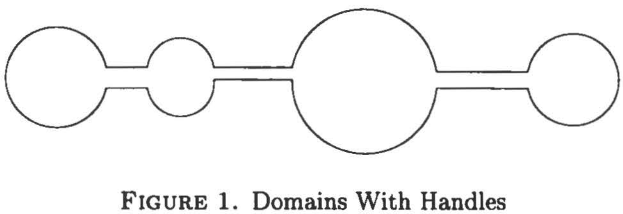

FIGURE 1. Domains With Handles

Remark: 1. By Theorem 1.1, if the function $d(P,\partial\Omega)$ has k strictly local maximum points, then for ϵ sufficiently small, problem (1.1) has at least k solutions. An interesting question is to construct multi-peaked solutions at multiple local maximum points of the distance function. Partial progress has been made in this direction.

2. By Corollary 1.2, interior single-peaked solutions always exist. We believe that this is the first result of this kind for problem (1.1).

To introduce the main idea of the proof of Theorem 1.1, we first need to give some necessary notations and definitions.

Let w be the unique solution of (1.2). It is known (see [5], [6]) that w is radially symmetric, decreasing and

$$\lim_{|y|\to\infty} w(y)e^{|y|}|y|^{\frac{N-1}{2}} = c_0 > 0.$$

For any smooth bounded domain U, we set $P_U w$ to be the unique solution of

$$\begin{cases} \Delta u - u + w^p = 0 \text{ in } U, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } U. \end{cases} \tag{1.4}$$

Let

$$\langle u, v\rangle_\epsilon = \epsilon^{-N}(\epsilon^2 \int_\Omega \nabla u \nabla v + \int_\Omega uv)$$

$$\|u\|_\epsilon^2 = \langle u, u\rangle_\epsilon.$$

For $P \in \Omega$, we set

$$\Omega_{\epsilon,P} = \{y : \epsilon y + P \in \Omega\}$$

$$E_{\epsilon,P} = \{v \in H^1(\Omega) : \langle v, P_{\Omega_{\epsilon,P}} w\rangle_\epsilon = \langle v, \frac{\partial P_{\Omega_{\epsilon,P}} w}{\partial P_i}\rangle_\epsilon = 0, i = 1, ..., N\}.$$

Our approach is based on establishing the existence of critical points of the functional

$$J_\epsilon(u) = \frac{\epsilon^2 \int_\Omega |\nabla u|^2 + \int_\Omega u^2}{(\int_\Omega u^{p+1})^{\frac{2}{p+1}}}, u \in H^1(\Omega) \tag{1.5}$$

via seeking critical points of

$$K_\epsilon(P, v) = J_\epsilon(P_{\Omega_{\epsilon,P}} w + v) \tag{1.6}$$

over

$$M = \{(P, v) : P \in \overline{\Lambda}, v \in E_{\epsilon,P}, \|v\|_\epsilon \le \eta\}.$$

To this end, we first solve v by minimizing K_ϵ for fixed P and obtain a solution $v_{\epsilon,P}$ which is C^1 in P. Then we maximize K_ϵ among $P \in \overline{\Lambda}$.

This paper is organized as follows. In Section 2, we study the properties of $P_{\Omega_{\epsilon,P}} w$. We set up the technical framework in Section 3 and we solve v in Section 4. We study a maximization problem in Section 5. Finally we prove Theorem 1.1 in Section 6. The proofs of some technical lemmas are given in the Appendix.

Throughout this paper, unless otherwise stated, the letter C will always denote various generic constants which are independent of ϵ, for ϵ sufficiently small.

2. Projection of w

In this section, we study properties of the function $P_{\Omega_{\epsilon,P}}$ as introduced in Section 1. Recall that $P_{\Omega_{\epsilon,P}}w$ is the unique solution of

$$\begin{cases} \Delta v - v + w^p = 0 \text{ in } \Omega_{\epsilon,P}, \\ \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega_{\epsilon,P}, \end{cases} \tag{2.1}$$

where $\Omega_{\epsilon,P} := \{y | \epsilon y + P \in \Omega\}$.

Set

$$\varphi_{\epsilon,P}(x) = w(\frac{|x-P|}{\epsilon}) - P_{\Omega_{\epsilon,P}}w(y), \ \epsilon y + P = x.$$

Then $\varphi_{\epsilon,P}(x)$ satisfies

$$\begin{cases} \epsilon^2 \Delta v - v = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = \frac{\partial}{\partial \nu} w(\frac{|x-P|}{\epsilon}) & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

It is immediately seen that on $\partial\Omega$

$$\begin{aligned} \frac{\partial}{\partial \nu} w(\frac{|x-P|}{\epsilon}) &= \frac{1}{\epsilon} w'(\frac{|x-P|}{\epsilon}) \frac{< x-P, \nu >}{|x-P|} \\ = -\frac{1}{\epsilon}\Big(|x-P|^{-(N-1)/2} \cdot \epsilon^{+\frac{N-1}{2}} e^{-\frac{|x-P|}{\epsilon}} & \big(1+O(\epsilon)\big)\Big) \frac{< x-P, \nu >}{|x-P|} \\ &= -\epsilon^{\frac{N-3}{2}} e^{-\frac{|x-P|}{\epsilon}} \big(1+O(\epsilon)\big) \frac{< x-P, \nu >}{|x-P|^{\frac{N+1}{2}}}. \end{aligned} \tag{2.3}$$

To analyze $P_{\Omega_{\epsilon,P}}w$, we introduce another linear problem. Let $P^D_{\Omega_{\epsilon,P}}w$ be the unique solution of

$$\begin{cases} \epsilon^2 \Delta v - v + w^p = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Set

$$\varphi^D_{\epsilon,P} = w - P_{\Omega_{\epsilon},P}w, \psi^D_{\epsilon,P}(x) = -\epsilon \log \varphi^D_{\epsilon,P}(x).$$

Then $\psi^D_{\epsilon,P}$ satisfies

$$\begin{cases} \epsilon \Delta v - |\nabla v|^2 + 1 = 0 & \text{in } \Omega, \\ v = -\epsilon \log(w(\frac{|x-P|}{\epsilon})) & \text{on } \partial\Omega. \end{cases}$$

Note that for $x \in \partial\Omega$

$$\begin{aligned} \psi^D_{\epsilon,P}(x) &= -\epsilon \log\left((\frac{|x-P|}{\epsilon})^{-\frac{N-1}{2}} e^{-\frac{|x-P|}{\epsilon}} (1+O(\epsilon)) \right) \\ &= |x-P| + \frac{N-1}{2} \epsilon \log(\frac{|x-P|}{\epsilon}) + O(\epsilon^2) \\ &= |x-P| + \frac{N-1}{2} \epsilon \log(\frac{|x-P|}{\epsilon}) + O(\epsilon). \end{aligned}$$

By the results of Section 4 of [12] and Section 3 in [20], we have

Lemma 2.1. (1) $\frac{\partial \psi^D_{\epsilon,P}}{\partial \nu} = -(1+O(\epsilon))\frac{< x-P, \nu >}{|x-P|}$,

(2) $\psi^D_{\epsilon,P}(x) \longrightarrow \psi_0(x) = \inf_{z\in\partial\Omega}(|z-x|+|z-P|)$ *as* $\epsilon \to 0$ *uniformly in* $\bar{\Omega}$. *In particular* $\psi_0(P) = 2d(P,\partial\Omega)$.

Let us now compare $\varphi_{\epsilon,P}(x)$ and $\varphi^D_{\epsilon,P}(x)$. To this end, we introduce another function. Let U_ϵ be the solution for the following problem:

$$\begin{cases} \epsilon^2\Delta U_\epsilon - U_\epsilon = 0 \text{ in } \Omega, \\ U_\epsilon = 1 \text{ on } \partial\Omega. \end{cases}$$

Set

$$\Psi_\epsilon = -\epsilon \log(U_\epsilon).$$

Then by Theorem 1 of [4], we have

$$\Psi_\epsilon(x) = d(x,\partial\Omega) + O(\epsilon), \frac{\partial \Psi_\epsilon}{\partial \nu} = 1 + O(\epsilon)$$

and

$$|U_\epsilon(x)| \leq Ce^{-\frac{d(x,\partial\Omega)}{\epsilon}}.$$

Moreover

$$\frac{U_\epsilon(\epsilon y + P)}{U_\epsilon(P)} \leq Ce^{(1+\sigma_0)|y|}. \tag{2.4}$$

Then we have

Lemma 2.2. *Then there exist* $\eta_0, \alpha_0 > 0, \epsilon_0 > 0$ *such that for* $\epsilon < \epsilon_0$, *we have*

$$-(1+\eta_0\epsilon)\varphi^D_{\epsilon,P} - Ce^{-\beta(1+\alpha_0)d(P,\partial\Omega)}U_\epsilon < \varphi_{\epsilon,P} < -(1-\eta_0\epsilon)\varphi^D_{\epsilon,P} + Ce^{-\beta(1+\alpha_0)d(P,\partial\Omega)}U_\epsilon.$$

Proof: We first assume that Ω is convex with respect to P. Namely, there is a constant $c_0 > 0$ such that

$$\langle x-P, \nu_x\rangle \geq c_0 > 0$$

for all $x \in \partial\Omega$, where ν_x is the unit normal at $x \in \partial\Omega$.Then on $\partial\Omega$ we have

$$\begin{aligned} \frac{\partial \varphi^D_{\epsilon,P}}{\partial \nu} &= e^{-\frac{\psi^D_{\epsilon,P}(x)}{\epsilon}}(-\frac{1}{\epsilon})\frac{\partial \psi^D_{\epsilon,P}(x)}{\partial \nu} \\ &= -\frac{1}{\epsilon}(w)\frac{\partial \psi^D_{\epsilon,P}(x)}{\partial \nu} \\ &= \frac{1}{\epsilon}w(1+O(\epsilon))(1+O(\epsilon))\frac{< x-P,\nu >}{|x-P|} \\ &= -(1+O(\epsilon))\frac{\partial \varphi^D_{\epsilon,P}}{\partial \nu}. \end{aligned}$$

Note that since Ω is convex with respect to P, we have $\frac{\partial \varphi^D_{\epsilon,P}}{\partial \nu} < 0$, hence by comparison principles

$$-(1+\eta_0\epsilon)\varphi^D_{\epsilon,P} \leq \varphi_{\epsilon,P} \leq -(1-\eta_0\epsilon)\varphi^D_{\epsilon,P}.$$

Now for general Ω, we can choose a subdomain Ω_1 such that

(1) $\Omega_1 \subset \Omega$,

(2) There is $\delta > 0$ such that $\bar{B}_{d(z,\partial\Omega)}(z) \cap \partial\Omega = \bar{B}_{d(z,\partial\Omega)}(z) \cap \partial\Omega_1$ for $|z-P| < \delta$,

(3) Ω_1 is convex. Moreover

$$< z - P, \nu_x > \geq \nu_0 > 0$$

for $|z-P| < \delta, \nu_x$ being the outer normal of x at $\partial\Omega_1$.

Then on $\partial\Omega_1 \cap \partial\Omega = \Gamma_1$, we have

$$\frac{\partial \varphi_{\epsilon,P}}{\partial \nu} \leq -(1+O(\epsilon))\frac{\partial \varphi^D_{\epsilon,P}}{\partial \nu}.$$

On $\partial\Omega \backslash \Gamma_1$, we have

$$|\frac{\partial \varphi^D_{\epsilon,P}}{\partial \nu}| \leq Ce^{-(1+2\alpha_0)\beta d(P,\partial\Omega)}$$

$$\frac{\partial \varphi_{\epsilon,P}}{\partial \nu} \leq Ce^{-(1+2\alpha_0)\beta d(P,\partial\Omega)} \leq Ce^{-(1+\alpha_0)\beta d(P,\partial\Omega)}\frac{\partial U_\epsilon}{\partial \nu}.$$

By comparison principle, we get the inequality.

Lemma 2.2 is thus proved. □

By Lemma 2.2, we have that

$$\psi_\epsilon(P) := -\epsilon \log\left(-\varphi_{\epsilon,P}(P)\right) \to 2d(P,\partial\Omega)$$

since

$$\varphi_{\epsilon,P}(P) = (-1+O(\epsilon))\varphi^D_{\epsilon,P}(P) + O(e^{-(2+\alpha_0)\beta d(P,\partial\Omega)}).$$

Let

$$V_{\epsilon,P}(y) = \frac{1}{\varphi_{\epsilon,P}(P)} \cdot \varphi_{\epsilon,P}(x).$$

Then $V_{\epsilon,P}(0) = 1$ (hence $V_{\epsilon,P}(y) > 0$ by Harnack inequality) and we have

Lemma 2.3. *For every sequence $\epsilon_k \to 0$, there is a subsequence $\epsilon_{k\ell} \to 0$ such that $V_{\epsilon_{k\ell},P} \to \bar{V}$ uniformly on every compact set of R^N, where $\bar{V}$ is a positive solution of*

$$\begin{cases} \Delta u - u = 0 \text{ in } R^N, \\ u > 0 \text{ in } R^N \text{ and } u(0) = 1. \end{cases} \tag{2.5}$$

Moreover for any $c_1 > 0$, $\sup_{z \in \Omega_{\epsilon_{k\ell},P}} e^{-(1+c_1)|z|}\left|V_{\epsilon_{k\ell},P}(z) - \bar{V}\right| \to 0$ *as* $\epsilon_{k\ell} \to 0$.

Proof: By Lemma 2.2, we have

$$\begin{aligned} |V_{\epsilon,P}(y)| &= |(w - P_{\Omega_{\epsilon,P}} w)\frac{1}{\varphi_{\epsilon,P}(P)}| \\ &\leq C\frac{\varphi^D_{\epsilon,P}(P)}{\varphi_{\epsilon,P}(P)} + C\frac{1}{\varphi_{\epsilon,P}(P)}e^{-\beta(1+\alpha_0)d(P,\partial\Omega)}U_\epsilon \\ &\leq Ce^{(1+\sigma_0)|y|} + Ce^{-\beta(1-\alpha_0)d(P,\partial\Omega)}U_\epsilon \text{ (by Lemma 4.6 in [12])} \\ &\leq Ce^{(1+\sigma_0)|y|} + CU_\epsilon(x)/U_\epsilon(P) \text{ (since } U_\epsilon(P) = Ce^{betad(P,\partial\Omega)}) \\ &\leq Ce^{(1+\sigma_0)|y|}. \end{aligned}$$

By a local compactness argument, we have that $\lim_{\epsilon \to 0} V_{\epsilon,P} = \bar{V}$ and $\bar{V}$ satisfies (2.4). □

We have the following key computations.

Lemma 2.4. *For any $P \in \overline{\Lambda}$ and ϵ sufficiently small*

$$J_\epsilon(P_{\Omega_{\epsilon,P}} w) = \epsilon^{\frac{(p-1)N}{p+1}} (A)^{(p-1)/(p+1)} [1 - \gamma A^{-1} (e^{-\beta\psi_\epsilon(P)}) + o(e^{-\beta\psi_\epsilon(P)})] \tag{2.6}$$

where $\beta = \frac{1}{\epsilon}$, $A = \int_{R^N} w^{p+1}$, $\gamma := \int_{R^N} w^p u_0^* > 0$ and u_0^* is the unique radial solution of (2.4).

Proof:

Note that by Lemma 2.3 and similar arguments as in the proof of Lemma 5.1 of [12] we have

$$\epsilon^2 \int_\Omega |\nabla P_{\Omega_{\epsilon,P}} w|^2 + \int_\Omega |P_{\Omega_{\epsilon,P}} w|^2 = \epsilon^N \int_{\Omega_{\epsilon,P}} w^p P_{\Omega_{\epsilon,P}} w$$

$$= \epsilon^N \left(\int_{\Omega_{\epsilon,P}} w^{p+1} + \int_{\Omega_{\epsilon,P}} w^p (P_{\Omega_{\epsilon,P}} w - w) \right)$$

$$= \epsilon^N \left(\int_{R^N} w^{p+1} - \varphi_{\epsilon,P}(P) \int_{\Omega_{\epsilon,P}} w^p V_{\epsilon,P} + o(\varphi_{\epsilon,P}(P)) \right)$$

$$= \epsilon^N (A - \varphi_{\epsilon,P}(P)\gamma + o(\varphi_{\epsilon,P}(P)))$$

since

$$\gamma = \int_{R^N} w^{p+1} \bar{V} = \int_{R^N} w^{p+1} u_0^*$$

for any solution $\bar{V}$ of (2.4).

Similarly we have

$$\int_\Omega (P_{\Omega_{\epsilon,P}} w)^{p+1} = \epsilon^N (A - (p+1)\varphi_{\epsilon,P}(P)\gamma + o(\varphi_{\epsilon,P}(P)).$$

Hence

$$J_\epsilon(P_{\Omega_{\epsilon,P}} w) = \epsilon^{N(p-1)/(p+1)} \frac{A - \gamma\varphi_{\epsilon,P}(P) + o(\varphi_{\epsilon,P}(P))}{(A - (p+1)\gamma\varphi_{\epsilon,P}(P) + o(\varphi_{\epsilon,P}(P)))^{2/(p+1)}}$$

$$= \epsilon^{\frac{(p-1)N}{p+1}} (A)^{(p-1)/(p+1)} [1 - \gamma A^{-1} (e^{-\beta\psi_\epsilon(P)}) + o(e^{-\beta\psi_\epsilon(P)})].$$

3. Technical Framework

Recall that

$$J_\epsilon(u) = \frac{\epsilon^2 \int_\Omega |\nabla u|^2 + \int_\Omega u^2}{(\int_\Omega u^{p+1})^{\frac{2}{p+1}}}, u \in H^1(\Omega).$$

If u is a critical point of J_ϵ, u satisfies on Ω the equation

$$\epsilon^2 \Delta u - u + l(u)|u|^{p-1} u = 0$$

where

$$l(u) = \frac{\epsilon^2 \int_\Omega |\nabla u|^2 + \int_\Omega u^2}{(\int_\Omega u^{p+1})} \tag{3.1}$$

(note that $l(u) > 0$). This will allow us to look for u_ϵ as critical points of J_ϵ.

Let $a_0 > 0$ be a fixed positive number (to be defined later). As in [20], for each $a > 0$, we define

$$F_{a_0} = \{P_{\Omega_\epsilon,P} w(\frac{x-P}{\epsilon}) \mid d(P, \partial\Omega) > a_0\} \tag{3.2}$$

$$V_a = \{(\alpha, P) \in R \times \Omega \mid |\alpha - 1| < a, d(P, \partial\Omega) > a_0\}. \tag{3.3}$$

Recall that for each $u, v \in H^1(\Omega)$, we define

$$\langle u, v\rangle_\epsilon = \epsilon^{-N} \int_\Omega (\epsilon^2 \nabla u \cdot \nabla v + uv).$$

We denote $\langle u, u\rangle_\epsilon$ as $\|u\|_\epsilon^2$.

Set $d(u, F_{a_0}) = \inf_{v \in F_{a_0}} \|u - v\|_\epsilon$.

The following decomposition lemma will be proved in Appendix A.

Lemma 3.1. *If $u \in H^1(\Omega)$ such that $d(u, F_{a_0})$ is small enough and a is small, then the problem*

$$\textit{minimize } \|u - \alpha P_{\Omega_\epsilon,P} w\|_\epsilon^2$$

with respect to (α, P) has a unique solution in the open set V_a.

By Lemma 2.2 of [22], we have

Lemma 3.2. *There exists $\delta_0 > 0$ such that if u_ϵ is a single-peaked solution with interior local maximum P_ϵ, then $d(P_\epsilon, \partial\Omega) \geq \delta_0 > 0$.*

We now set $a_0 = \delta_0$. By Lemmas 3.1 and 3.2, there exists a diffeomorphism between a neighborhood of the possible *single interior peak* solutions of (1.1) we are interested in and the open set

$$M_\eta^1 = \left\{ (\alpha, P, v) \,\middle|\, \begin{array}{l} (\alpha, P, v) \in R_+ \times \Omega \times H^1(\Omega), |\alpha - 1| < \eta, \\ d(P, \partial\Omega) > a_0, v \in E_{\epsilon,P}, \|v\|_\epsilon < 2\eta \end{array} \right\} \tag{3.4}$$

with $\eta > 0$ some suitable constant and

$$E_{\epsilon,P} = \left\{ v \in H^1(\Omega) \,\middle|\, \langle v, P_{\Omega_\epsilon,P} w\rangle_\epsilon = \langle v, \frac{\partial}{\partial P_i} P_{\Omega_\epsilon,P} w\rangle_\epsilon = 0, i = 1, ..., N \right\}. \tag{3.5}$$

Let us now define the functional

$$\begin{aligned} K_\epsilon^1 : \quad & M_\eta \to R \\ & m = (\alpha, P, v) \to \epsilon^{-\frac{(p-1)N}{p+1}} J_\epsilon(\alpha P_{\Omega_\epsilon,P} w + v) \end{aligned}$$

and

$$K_\epsilon(P, v) = K_\epsilon^1(1, P, v)$$

where

$$(P, v) \in M := \left\{ (P, v) \,\middle|\, \begin{array}{l} (P, v) \in \Omega \times H^1(\Omega), \\ d(P, \partial\Omega) > a_0, v \in E_{\epsilon,P}, \|v\|_\epsilon < \eta \end{array} \right\}.$$

Since K_ϵ^1 is homogeneous of degree 0, it follows from Lemma 3.1 and Lemma 3.2 that

Proposition 3.3. *$(P, v) \in M$ is a critical point of K_ϵ if and only if $u = P_{\Omega_{\epsilon,P}} w + v$ is a critical point of J_ϵ, i.e. if and only if there exists $(\alpha, \beta) \in R \times R^N$ such that*

$$(E_P) \qquad \frac{\partial K_\epsilon}{\partial P_i} = \sum_{j=1}^{n} \beta_j \langle \frac{\partial^2 P_{\Omega_{\epsilon,P}} w}{\partial P_i \partial P_j}, v \rangle_\epsilon \tag{3.6}$$

$$(E_v) \qquad \frac{\partial K_\epsilon}{\partial v} = \alpha P_{\Omega_{\epsilon,P}} w + \sum_{j=1}^{N} \beta_j \frac{\partial P_{\Omega_{\epsilon,P}} w}{\partial P_j}. \tag{3.7}$$

Therefore to find a critical point of K_ϵ, it is enough to find (P, v) satisfying (E_v) and (E_P). In Section 4, we first solve v by minimizing and then solve P by maximizing.

4. Minimizing v

In this section, we solve (E_v) for fixed $P \in \overline{\Lambda}$ and ϵ sufficiently small.

Proposition 4.1. *There exist $\eta_0 > 0, \epsilon_0 > 0$ such that for $\epsilon \leq \epsilon_0, \eta \leq \eta_0$ there exists a smooth map which to any (ϵ, P) such that $(P, 0) \in M_{\eta_0}$ associates $v_{\epsilon,P} \in E_{\epsilon,P}$, $\|v_{\epsilon,P}\|_\epsilon < \eta$ such that (E_v) is satisfied for some $(\alpha, \beta) \in R \times R^N$. Such a $v_{\epsilon,P}$ is unique, minimizes $K_\epsilon(P, v)$ with respect to v in $\{v \in E_{\epsilon,P} | \ \|v\|_\epsilon < \eta\}$ and we have the estimate*

$$\frac{\partial v_{\epsilon,P}}{\partial \nu} = 0 \quad on \ \partial\Omega, \ \|v_{\epsilon,P}\|_\epsilon \leq O(|\varphi_{\epsilon,P}(P)|^{\frac{1+\sigma}{2}}), \tag{4.1}$$

where $\sigma = \min\,(p-1, 1)$.

Proof: We first expand $K_\epsilon(P, v)$ at $(P, 0)$; we have

$$K_\epsilon(P, v) = K_\epsilon(P, 0) + f_{\epsilon,P}(v) + Q_{\epsilon,P}(v) + R_{\epsilon,P}(v), \tag{4.2}$$

where

$$K_\epsilon(P, 0) = \epsilon^{(p-1)N/(p+1)} J_\epsilon(P_{\Omega_{\epsilon,P}} w)$$

$$f_{\epsilon,P}(v) = -2\left(\int_{\Omega_{\epsilon,P}} w^p P_{\Omega_{\epsilon,P}} w\right)\left(\int_{\Omega_{\epsilon,P}} (P_{\Omega_{\epsilon,P}} w)^{p+1}\right)^{-\frac{p+3}{p+1}} \left[\int_{\Omega_{\epsilon,P}} (Pw)^p v\right]$$

$$Q_{\epsilon,\alpha,P}(v) = \left(\int_{\Omega_{\epsilon,P}} (P_{\Omega_{\epsilon,P}} w)^{p+1}\right)^{-\frac{2}{p+1}}$$

$$\times \Big\{\int_{\Omega_{\epsilon,P}} (|\nabla v|^2 + v^2) - p\left(\int_{\Omega_{\epsilon,P}} w^p P_{\Omega_{\epsilon,P}} w\right)\left(\int_{\Omega_{\epsilon,P}} (P_{\Omega_{\epsilon,P}} w)^{p+1}\right)^{-1} \int_{\Omega_{\epsilon,P}} (P_{\Omega_{\epsilon,P}} w)^{p-1} v^2$$

$$+(p+3)\left(\int_{\Omega_{\epsilon,P}} w^p P_{\Omega_{\epsilon,P}} w\right)\left(\int_{\Omega_{\epsilon,P}} (P_{\Omega_{\epsilon,P}} w)^p v\right)^2\Big\}$$

and $R_{\epsilon,P}$ satisfies

$$R_{\epsilon,P}(v) = O(\|v\|_\epsilon^{min(3,p+1)}), \tag{4.3}$$

$$R'_{\epsilon,P}(v) = O(\|v\|_\epsilon^{min(2,p)}), \tag{4.4}$$

$$R''_{\epsilon,P}(v) = O(\|v\|_\epsilon^{min(1,p-1)}). \tag{4.5}$$

Since $f_{\epsilon,P}(v)$ is continuous from $E_{\epsilon,P}$ equipped with $\langle \cdot, \cdot \rangle_\epsilon$ scalar product we may write

$$f_{\epsilon,P}(v) = \langle F_{\epsilon,P}, v \rangle_\epsilon \quad \text{for some } F_{\epsilon,P} \in E_{\epsilon,P}. \tag{4.6}$$

Since $Q_{\epsilon,P}$ is a continuous quadratic form on $E_{\epsilon,P}$, there exists a continuous and symmetric operator $L_{\epsilon,P} \in L(E_{\epsilon,P})$ — linear functional on $E_{\epsilon,P}$ — such that

$$Q_{\epsilon,P}(v) = \langle L_{\epsilon,P}v, v\rangle_\epsilon. \tag{4.7}$$

Moreover, by the same argument in [Lemma 4.2, [19]], there exists $\rho > 0$ such that for ϵ small enough, and η small enough, we have

$$Q_{\epsilon,P}(v) \geq \rho\|v\|_\epsilon^2, \quad \text{for all } v \in E_{\epsilon,P}. \tag{4.8}$$

Therefore $L_{\epsilon,P}$ is a coercive operator whose modulus of coercivity is bounded from below independently on ϵ, P.

The derivative of K_ϵ with respect to v on $E_{\epsilon,P}$ may be written

$$F_{\epsilon,P} + 2L_{\epsilon,P}v + O(\|v\|_\epsilon^2).$$

Using the implicit function theorem, we derive the existence of a C^2−map T which to each (ϵ, P) associated $v_{\epsilon,P} \in E_{\epsilon,P}$ such that

$$\frac{\partial K_\epsilon}{\partial v}(P, v)|_{E_{\epsilon,P}} = 0$$

and

$$\|v_{\epsilon,P}\|_\epsilon = O(\|F_{\epsilon,P}\|_\epsilon). \tag{4.9}$$

Moreover, since $v_{\epsilon,P}$ minimizes K_ϵ over $E_{\epsilon,P}$, $\frac{\partial v_{\epsilon,P}}{\partial \nu} = 0$ on $\partial\Omega$.

We now claim that

$$|f_{\epsilon,P}(v)| \leq O(|\varphi_{\epsilon,P}(P)|^{\frac{1+\sigma}{2}})\|v\|_\epsilon, \tag{4.10}$$

which by (4.6), (4.9) and (4.10) proves Proposition 4.1.

By definition

$$|f_{\epsilon,P}(v)| \leq C|\int_{\Omega_{\epsilon,P}} (P_{\Omega_{\epsilon,P}}w)^p v|.$$

Since

$$\int_{\Omega_{\epsilon,P}} (P_{\Omega_{\epsilon,P}}w)^p v = \int_{\Omega_{\epsilon,P}} ((P_{\Omega_{\epsilon,P}}w)^p - w^p)v \text{ (since } v \in E_{\epsilon,P}).$$

We now calculate, by Lemma 2.3,

$$\int_{\Omega_{\epsilon,P}} |(P_{\Omega_{\epsilon,P}}w)^p - w^p|^2 \leq O(|\varphi_{\epsilon,P}(P)|^{1+\sigma}).$$

Thus (4.10) is established. □

5. A Maximizing Problem

In this section, we study a maximizing problem. We shall prove

Proposition 5.1. *Let $v = v_{\epsilon,P}$ be the solution given by Proposition 4.1. Then for ϵ small, the following maximizing problem*

$$\max\{K_\epsilon(P, v_{\epsilon,P}) : P \in \overline{\Lambda}\}$$

has a solution $P_\epsilon \in \Lambda$.

Proof: Since $J_\epsilon(P_{\Omega_\epsilon,P}w + v_{\epsilon,P})$ is continuous in P, the maximizing problem has a solution. Let $J_\epsilon(P_{\Omega_\epsilon,P_\epsilon}w + v_{\epsilon,P_\epsilon})$ be the maximum. Then by Proposition 4.1, we have

$$K_\epsilon(P_\epsilon, v_{\epsilon,P_\epsilon}) = K_\epsilon(P_\epsilon, 0) + f_{\epsilon,P_\epsilon}(v_{\epsilon,P_\epsilon}) + O(\|v_{\epsilon,P_\epsilon}\|_\epsilon^2)$$
$$= K_\epsilon(P_\epsilon, 0) + O(e^{-(1+\sigma)\beta\psi_\epsilon(P_\epsilon)}).$$

Similarly for $P \in \overline{\Lambda}$

$$K_\epsilon(P, v_{\epsilon,P}) = K_\epsilon(P, 0) + O(e^{-(1+\sigma)\beta\psi_\epsilon(P)}).$$

By Lemma 2.4, we have

$$K_\epsilon(P, v_{\epsilon,P}) = (A)^{(p-1)/(p+1)}[1 - \gamma A^{-1}(e^{-\beta\psi_\epsilon(P)}) + o(e^{-\beta\psi_\epsilon(P)})].$$

Hence we have

$$e^{-\beta\psi_\epsilon(P_\epsilon)} \leq e^{-\beta\psi_\epsilon(P)} + o(e^{-\beta\psi_\epsilon(P_\epsilon)})$$

for any $P \in \overline{\Lambda}$.

So $d(P_\epsilon, \partial\Omega) \to \max_{P\in\Lambda} d(P, \partial\Omega)$ as $\epsilon \to 0$ and by the hypotheses of Theorem 1.1, $P_\epsilon \in \Lambda$. This completes the proof of Proposition 4.1. □

6. Proof of Theorem 1.1

In this section, we apply results in Section 4 and Section 5 to prove Theorem 1.1 and Corollary 1.2.

Proofs of Theorem 1.1 and Corollary 1.2

By Propositions 3.3 and 4.1, there exists ϵ_0 such that for $\epsilon < \epsilon_0$ we have a C^1 map which, to any $P \in \overline{\Lambda}$, associates $v_{\epsilon,P} \in E_{\epsilon,P}$ and $(P, v_{\epsilon,P})$ satisfies E_v of Section 3, for some $(\alpha, \beta) \in R \times R^N$. By Proposition 5.1, we have $P_\epsilon \in \Lambda$, achieving the maximum of the maximization problem in Proposition 5.1. Let $v_\epsilon = v_{\epsilon,P_\epsilon}$ then

$$D_{P_i}K_\epsilon(P_\epsilon, v_\epsilon) = 0, i = 1, ..., N,$$

where

$$D_{P_i}K_\epsilon(P_\epsilon, v_\epsilon) = \frac{\partial K_\epsilon}{\partial P_i} + \langle\frac{\partial K_\epsilon}{\partial v}, \frac{\partial v_\epsilon}{\partial P_i}\rangle_\epsilon$$

Note that

$$\langle\frac{\partial K_\epsilon}{\partial v}, \frac{\partial v_\epsilon}{\partial P_i}\rangle_\epsilon = \alpha\langle P_{\Omega_\epsilon,P_\epsilon}w, \frac{\partial v_\epsilon}{\partial P_i}\rangle_\epsilon + \sum_{i=1}^N \beta_j\langle\frac{\partial P_{\Omega_\epsilon,P_\epsilon}w}{\partial P_j}, \frac{\partial v_\epsilon}{\partial P_i}\rangle_\epsilon$$
$$= -\sum_{i=1}^N \beta_j\langle\frac{\partial^2 P_{\Omega_\epsilon,P_\epsilon}w}{\partial P_j\partial P_i}, v_\epsilon\rangle_\epsilon$$

since

$$\langle\frac{\partial P_{\Omega_\epsilon,P_\epsilon}w}{\partial P_i}, v_\epsilon\rangle_\epsilon = 0$$

which implies

$$\langle\frac{\partial^2 P_{\Omega_\epsilon,P_\epsilon}w}{\partial P_i\partial P_j}, v_\epsilon\rangle_\epsilon = -\langle\frac{\partial P_{\Omega_\epsilon,P_\epsilon}w}{\partial P_j}, \frac{\partial v_\epsilon}{\partial P_i}\rangle_\epsilon.$$

So

$$\frac{\partial K_\epsilon}{\partial P_i}(P_\epsilon, v_\epsilon) = \sum_{j=1}^{N} \beta_j \langle \frac{\partial^2 P_{\Omega_\epsilon, P_\epsilon} w}{\partial P_i \partial P_j}, v_\epsilon \rangle_\epsilon, i = 1, ..., N.$$

Hence (E_P) is satisfied and by Proposition 3.3, $w_\epsilon = P_{\Omega_\epsilon, P_\epsilon} w + v_\epsilon$ is a critical point of K_ϵ. Hence $u_\epsilon = l(w_\epsilon) w_\epsilon$ is a solution of (1.1), where $l(w_\epsilon)$ is defined in (3.1).

By our construction, it is easy to see that $l(w_\epsilon) \to 1, u_\epsilon > 0$ in Ω. Let P_ϵ^1 be the local maximum point of u_ϵ; then we have $d(P_\epsilon^1, \partial\Omega) \to \max_{P \in \Lambda} d(P, \partial\Omega)$. It is easy to see that u_ϵ satisfies all the properties of Theorem 1.1.

Finally, Corollary 1.2 can be easily proved by taking $\Lambda = \{x : d(x, \partial\Omega) > \frac{1}{2} \max_{x \in \Omega} d(x, \partial\Omega)\}$.

7. Appendix A. Decomposition lemma

In this appendix, we shall prove the decomposition lemma 3.1 in Section 3. We start with some lemmas.

Lemma 7.1. *Let (ϵ_k) be a sequence with $\epsilon_k > 0$, $\lim_{k \to \infty} \epsilon_k = 0$. Let $P_k, \tilde{P}_k \in \Omega_{a_0}, \alpha_k, \tilde{\alpha}_k \in (\frac{1}{2}, 2)$ be such that*

$$\lim_{k \to \infty} \|\alpha_k P_{\Omega_\epsilon, P_k} w - \tilde{\alpha}_k P_{\Omega_{\epsilon_k}, \tilde{P}_k} w\|_{\epsilon_k} = 0. \tag{7.1}$$

Then we have

$$\lim_{k \to \infty} |\alpha_k - \tilde{\alpha}_k| = 0, \tag{7.2}$$

$$\lim_{k \to \infty} |\frac{P_k - \tilde{P}_k}{\epsilon_k}| = 0. \tag{7.3}$$

Proof: We have

$$\begin{aligned}
&\|\alpha_k P_{\Omega_{\epsilon_k}, P_k} w - \tilde{\alpha}_k P_{\Omega_{\epsilon_k}, \tilde{P}_k} w\|_{\epsilon_k} \\
&= \|\alpha_k P_{\Omega_\epsilon, P_k} w - \tilde{\alpha}_k P_{\Omega_{\epsilon_k}, \tilde{P}_k} w(\cdot - \frac{\tilde{P}_k - P_k}{\epsilon})\|_{\epsilon_k} \\
&= \|(\alpha_k - \tilde{\alpha}_k) P_{\Omega_{\epsilon_k}, P_k} w + \tilde{\alpha}_k (P_{\Omega_\epsilon, P_k} w - P_{\Omega_{\epsilon_k}, \tilde{P}_k} w(\cdot - \frac{\tilde{P}_k - P_k}{\epsilon}))\|_{\epsilon_k} \\
&\geq |\alpha_k \| P_{\epsilon_k, P_k} w\|_{\epsilon_k, \Omega} - l\tilde{pha}_k \|P_{\epsilon_k, P_k} w\|_{\epsilon_k, \Omega}|.
\end{aligned}$$

Since both $P_k, \tilde{P}_k \in \Omega_{a_0}$, $\|P_{\epsilon_k, P_k} w\|_{\epsilon_k} \to \|w\|_{H^1(R^N)}$ and $\|P_{\epsilon_k, \tilde{P}_k} w\|_{\epsilon_k, \Omega} \to \|w\|_{H^1(R^N)}$. Hence $|\alpha_k - \tilde{\alpha}_k| = o(1)$.

On the other hand, suppose (7.3) is not satisfied; we have that by passing to a subsequense, $|\frac{P_k - \tilde{P}_k}{\epsilon_k}| \to a$ with $0 < a \leq \infty$.

If $0 < a < \infty$, then $\|P_{\Omega_\epsilon, P_k} w - P_{\Omega_{\epsilon_k}, \tilde{P}_k} w(\cdot - \frac{\tilde{P}_k - P_k}{\epsilon})\|_{\epsilon_k, \Omega} \to \|w - w(\cdot - a)\|_{H^1(R^N)} \neq 0$.

If $a = \infty$, then $\|P_{\Omega_\epsilon, P_k} w - P_{\Omega_{\epsilon_k}, \tilde{P}_k} w(\cdot - \frac{\tilde{P}_k - P_k}{\epsilon})\|_{\epsilon_k, \Omega} \to 2\|w\|_{H^1(R^N)} \neq 0$. In any case, we reach a contradiction with (7.1). □

We now prove Lemma 3.1. We will follow closely to Appendix B of [20]. We argue by contradiction. Suppose there exists $\epsilon_k \to 0, \eta_k \to 0$ such that

$$\inf_{v \in \Lambda_{a_k}} \|u - v\|_{\epsilon_k} < \eta_k,$$

and $(\alpha_k, P_k), (\tilde{\alpha}_k, \tilde{P}_k) \in \Lambda_{\eta_k}$, such that if $v^k = u_k - \alpha_k P_{\Omega_{\epsilon_k, P_k}} w, \tilde{v}^k = u_k - \tilde{\alpha}_k P_{\Omega_{\epsilon_k, \tilde{P}_k}} w$,

$$\langle v^k, P_{\Omega_{\epsilon_k, P_k}} w \rangle_{\epsilon_k} = 0, \tag{7.4}$$

$$\langle v^k, \frac{\partial}{\partial P_i} P_{\Omega_{\epsilon_k, P_k}} w \rangle_{\epsilon_k} = 0, \tag{7.5}$$

$$\langle \tilde{v}^k, P_{\Omega_{\epsilon_k, \tilde{P}_k}} w \rangle_{\epsilon_k} = 0, \tag{7.6}$$

$$\langle \tilde{v}^k, \frac{\partial}{\partial \tilde{P}_i} P_{\Omega_{\epsilon_k, \tilde{P}_k}} w \rangle_{\epsilon_k} = 0. \tag{7.7}$$

Let $a_k = \frac{P_k - \tilde{P}_k}{\epsilon_k}, \mu_k = \alpha_k - \tilde{\alpha}_k$. Then by Lemma 7.1, $|a_k| = o(1), |\mu_k| = o(1)$.
We denote C as various constants which do not depend on k. We first observe that

$$|w^p(y) - w^p(y - a_k)| \le C|a_k| w^p(y). \tag{7.8}$$

By the Maximum Principle

$$|P_{\Omega_{\epsilon_k, P_k}} w - P_{\Omega_{\epsilon_k, \tilde{P}_k}} w| \le C|a_k| w(y). \tag{7.9}$$

The rest of the proof is exactly the same as those of Appendix B in [20]; we omit the details. □

Acknowledgment. This research is supported by an Earmarked Grant from RGC of Hong Kong.

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