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Estimates Near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions. I*

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Introduction

The Schauder estimates [33, 34] for second order linear elliptic differential equations play an important role in the existence theory for linear, and in particular, for non-linear elliptic equations. These estimates are for solutions of a second order linear elliptic equation in a domain $\mathfrak{D}$, with Hölder continuous coefficients[1] and are of two kinds. The first kind, "interior estimates", give bounds for the derivatives up to second order of the solution u, and their Hölder continuity, in any compact subset of the domain, the bounds depending on l.u.b. $|u|$ and on the moduli and Hölder continuity of the coefficients of the equation. The second kind, "estimates near the boundary", apply to solutions of the Dirichlet problem. Assuming the existence of Hölder continuous first and second derivatives of the solution on and near a smooth portion Γ of the boundary $\dot{\mathfrak{D}}$, one estimates these quantities in a suitable sub-domain of $\mathfrak{D}$ abutting Γ, the bounds depending as before on l.u.b. $|u|$ etc., and on estimates for the boundary values of u.

Simplified derivations of the estimates of Schauder have been given by Douglis and Nirenberg [10], Morrey [28], Miranda [25] and by Graves [12], where a comprehensive discussion of the estimates and existence theory is

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[1] A function g in a set in Euclidean space is said to satisfy a Hölder condition with exponent α, $0 < \alpha < 1$, if

$$[g]_\alpha = \text{l.u.b.} \frac{|g(P)-g(Q)|}{|P-Q|^\alpha} < \infty;$$

g is Hölder continuous (exponent α) in a domain if it satisfied a Hölder condition with exponent α in every compact subset of the domain.

presented. In [10] the "interior estimates" were extended to general elliptic systems of arbitrary orders. In this paper we shall derive "estimates near the boundary" for elliptic equations of arbitrary order under general boundary conditions, not merely the Dirichlet boundary conditions.[2] We have obtained these results for general elliptic systems, but for simplicity, we treat here in detail the theory of a single equation for one function. Systems will be treated in a forthcoming paper. We propose also to treat at some later time pointwise estimates near the boundary for overdetermined systems and boundary conditions.

In addition to the Schauder estimates we present analogous L_p estimates, for $p > 1$, up to the boundary. Interior L_p estimates can be obtained rather easily as has been noticed by several authors (Koševlev [19], Nirenberg [32], Browder [7]; also Greco [13], and Koševlev [18] for estimates up to the boundary for second order elliptic equations) with the aid of the Calderon-Zygmund theorem (see [8] and Theorem 2 of [9]). These boundary estimates have been obtained by us only recently; previously we proved such estimates only for $p = 2$. L_p estimates up to the boundary for elliptic boundary value problems have been announced earlier by Browder [6].[3]

Recently Koševlev established L_p estimates up to the boundary for the Dirichlet problem for elliptic equations of arbitrary order. We use our L_p estimates (see Section 11) in proving differentiability of solutions of nonlinear problems.

For the case $p = 2$ our result has now been derived by several people. Subsequent to our work Schechter [35] derived similar estimates for overdetermined systems which agree with ours in the case of one operator. The L_2 estimates overlap also with unpublished inequalities of Aronszajn and Smith extending [4]. Agmon [2], and independently Hörmander, have solved a general coerciveness problem (extending the fundamental paper [4] by Aronszajn) involving L_2 estimates which include ours, for $p = 2$, as a special case. Some of his results have recently been extended by Agmon to L_p, $p > 1$. Our method for deriving the L_p estimates, even for $p = 2$, differs considerably from the others and may be carried out fairly easily with the aid of a potential theoretic result, Theorem 3.3—once the explicit formulas for solutions of equations with constant coefficients, in a half-space, are known.

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[2]Estimates near the boundary for other than Dirichlet boundary conditions, including cases where the coefficients of the boundary operators are discontinuous ("mixed" boundary conditions) have been obtained by C. Miranda [26] for second order equations.

[3]Since this paper went to the printers there has appeared a more complete announcement of Browder's results which overlap considerable with ours. *Estimates and existence theorems for elliptic boundary value problems*, Proc. Nat. Acad. Sci. U.S.A., Vol. 45, 1959, pp. 365—372.

We shall consider a linear differential operator L with complex coefficients operating on functions $u(x)$ defined in a domain $\mathfrak{D}$ in $(n+1)$-space: $x = (x_1, \cdots, x_{n+1})$. L is assumed to be elliptic, i.e., if $L'(x; D) = L'(x; \partial/\partial x_1, \cdots, \partial/\partial x_{n+1})$ is the leading part of L (the part of highest order), then for every real vector $\Xi = (\xi_1, \cdots, \xi_{n+1}) \neq 0$ and for every point x in $\mathfrak{D}$, $L'(x; \Xi) \neq 0$. In case $n > 1$ it is easily seen that the order of L is even (Lopatinskii [21]). For this purpose set $\xi = (\xi_1, \cdots, \xi_n)$, $\xi_{n+1} = \tau$. For fixed real $\xi \neq 0$ consider the roots τ of the polynomial $L'(x; \xi, \tau)$. Clearly $L'(x; -\xi, -\tau) = 0$, and since the set of real non-vanishing ξ vectors is connected for $n > 1$ it follows that there are as many roots τ of $L'(x; \xi, \tau)$ with positive imaginary parts as with negative. Hence the degree of L' is even, equal to $2m$. In the special case of two variables, $n = 1$, this is not necessarily true, as we see for instance with the Cauchy Riemann operator $L = \xi + i\tau$.

In this paper we shall consider only operators L of even order $2m$. In the case of two variables we impose a condition on L:

For every point x in the closure $\bar{\mathfrak{D}}$ of $\mathfrak{D}$ we assume the

CONDITION ON L. *For every pair of linearly independent real vectors Ξ, Ξ' the polynomial in the variable τ, $L'(x; \Xi + \tau\Xi')$, has exactly m roots with positive imaginary part.*

We shall actually use this condition only at points x on the boundary $\dot{\mathfrak{D}}$ of $\mathfrak{D}$, and for Ξ, Ξ' the tangent and normal vectors, respectively, to $\dot{\mathfrak{D}}$. Because of ellipticity, however, the condition on L is invariant under deformation and change of sign of the vectors, and so holds generally provided it is satisfied at a particular point for some pair of independent vectors.

We shall also assume that L is uniformly elliptic, i.e., that for some positive constant A, the "ellipticity constant", the inequality

$$A^{-1}|\Xi|^{2m} \leqq |L'(x; \Xi)| \leqq A|\Xi|^{2m} \tag{1}$$

holds for every real vector Ξ, with $|\Xi| = (\sum \xi_i^2)^{1/2}$, and for every point x in the closure $\bar{\mathfrak{D}}$ of $\mathfrak{D}$.

We shall consider solutions $u(x)$ of

$$Lu = F \tag{2}$$

satisfying m boundary conditions on a portion Γ of the boundary $\dot{\mathfrak{D}}$. These conditions

$$B_j u = \Phi_j, \qquad j = 1, \cdots, m, \tag{3}$$

are expressed by differential operators $B_j(x; D)$, $j = 1, \cdots, m$, defined for x in Γ. Each B_j is of order m_j and has complex coefficients; denote its highest order part by B_j', $j = 1, \cdots, m$. The m_j are non-negative integers, and may be greater than $2m$, the order of m. We shall impose a certain algebraic

condition on the operators L', B'_j and shall then call the B_j "complementing boundary conditions". The algebraic condition is the following

COMPLEMENTING CONDITION. *At any point x of Γ let $\mathbf{n}$ denote the normal to $\mathfrak{D}$ and $\Xi \neq 0$ any real vector parallel to the boundary. We require that the polynomials, in τ, $B'_j(x; \Xi+\tau\mathbf{n})$, $j = 1, \cdots, m$, be linearly independent modulo the polynomial $\prod_{k=1}^{m} (\tau-\tau_k^+(\Xi))$, where $\tau_k^+(\Xi)$ are the roots of $L'(x; \Xi+\tau\mathbf{n})$ with positive imaginary parts.*

Without imposing any conditions, except an appropriate degree of smoothness, on the lower order coefficients of L, B_j we shall derive estimates up to the boundary for solutions of (2), (3) satisfying the conditions above, and also prove differentiability theorems for solutions near the boundary. In addition we show that these conditions are necessary for such estimates to hold.

Hörmander [15] has considered solutions of an equation with constant coefficients satisfying, on a planar portion of the boundary, conditions (3) with constant coefficients. He characterized completely, for hypoelliptic operators L, these operators B_j (with constant coefficients) for which solutions u of (2), (3) with F and Φ_j in C^∞ are necessarily themselves of class C^∞, and characterized for elliptic L those operators B_j for which the solutions are analytic at the boundary, in case F and the Φ_j are analytic. In case the operator L is elliptic, and L and the B_j are homogeneous, his condition on the B_j agrees exactly with the complementing boundary condition above. In the more general (non-homogeneous) case which he treats his condition also involves the lower order terms of the B_j. In order to obtain our estimates near the boundary for *arbitrary* lower order terms, our condition is however, as remarked above, necessary as well as sufficient.

Certain simple elliptic operators such as the Cauchy-Riemann operators mentioned before under, say, the condition that $\mathscr{R}e\, u = 0$ on the boundary, seem to be outside the scope of our theory for single operators. Although this is not a linear condition over the complex numbers it seems worthwhile to have a theory which includes this case. And in fact, as M. Schechter pointed out to us, if we break up the Cauchy-Riemann equation into real and imaginary parts (the way they are usually written) we find that they are covered by our theory for systems.

In deriving the Schauder estimates near the boundary we shall rely on the interior estimates given by Douglis, Nirenberg in [10]. These were derived for equations with real coefficients, and the proofs do not extend directly to equations with complex coefficients. However, any elliptic equation (or system) with complex coefficients may, on separation into real and imaginary parts, be written as a system with real coefficients, and this system is also elliptic. Thus we may safely refer to the results of [10].

We mention that the boundary operators for the Dirichlet problem

$$B_j = \left(\frac{\partial}{\partial \boldsymbol{n}}\right)^{j-1},$$

where $\boldsymbol{n}$ is the normal to the boundary, satisfy the "Complementing Condition" relative to *any* operator L for which the "Condition on L" holds. In [15] Hörmander characterizes all those operators B_j which satisfy the Complementing Condition relative to every operator satisfying the Condition on L.

This paper is primarily concerned with estimates. However, these suggest the possibility of new existence theorems. In particular, the fact that our estimates hold in the Dirichlet problem for any elliptic operator (for $n = 1$ we require the Condition on L) suggests that it should be possible to derive existence theorems for such operators L, not merely for operators that are strongly elliptic, to which, up to now, the existence theory for the Dirichlet problem has been confined. Suppose for example that L has highly differentiable coefficients so that, in particular, the formal adjoint L^* is well defined. Let $\|u\|_{j,L_2}$ denote the square root of the sum of integrals of the squares of all derivatives of u up to order j, and denote by H_{j,L_2} ($\mathring{H}_{j,L_2}$) the completion of C^∞ functions (with compact support) in $\mathfrak{D}$ with respect to this norm, $\| \ \|_{j,L_2}$. A direct (Hilbert space) approach to the Dirichlet problem would be: For given F in $H_{0,L_2} = L_2$, to find a function u in $S = H_{2m,L_2} \cap \mathring{H}_{m,L_2}$ satisfying

$$Lu = F,$$

if one assumes, say, that the only solution in S of $L^*u = 0$ is $u = 0$. From our estimates it follows easily that the range of $L : S \to L_2$ is closed. In trying to show that the range is dense we would be led, on supposing the contrary, to a function v in L_2 such that $(Lu, v) = 0$ for all u in S. Here $(\ , \ )$ is the L_2 scalar product in $\mathfrak{D}$. Thus v is a "weak solution" of $L^*v = 0$. If we had a differentiability theorem at the boundary asserting that such a weak solution belongs in fact to S, we would know by uniqueness for L^* that $v = 0$, hence that the range of L is dense. Thus the direct Hilbert space approach seems to require a strong differentiability theorem at the boundary for weak solutions.

Recently Agmon [3], using some results of this paper (in particular Section 2) has proved such a differentiability theorem at the boundary for weak solutions of a wide class of boundary value problems, and hence has obtained existence theorems for such problems—in particular for the Dirichlet problem. Also, M. Schechter [36], using a different and elegant Hilbert space argument which does not require such a strong differentiability theorem, has independently proved existence theorems for a wide class of boundary problems, including also the Dirichlet problem. These proofs all make use of the L_2 estimates up to the boundary.

Section 12 is concerned with straightforward applications of the estimates up to the boundary—chiefly to existence theorems. In particular, using the above results of Agmon, Schechter we show, under optimal conditions on the coefficients, that uniqueness for the Dirichlet problem for L implies existence. We also describe (Theorem 12.5) the approach to existence theory via the continuity method (in a slightly more general form than that used by Schauder [33]); in this connection some open questions are formulated.

Before giving a detailed description of the results of the paper we wish to point out some of the other main items. The basic work is concerned with systems (2), (3) having constant coefficients and only highest order terms, and for a domain $\mathfrak{D}$ which is a half-space. We obtain representations for solutions of such a system with the aid of explicitly constructed Poisson kernels. For two dimensions, $n = 1$, such kernels were constructed by Agmon [1]. In treating the general case we make use of an identity of F. John [17].

From these formulas we obtain, in the case of Dirichlet boundary data for a homogeneous operator with constant coefficients in a half-space, an extension of the maximum principle for second order operators. This asserts that if $u(p)$ is a solution of the homogeneous equation and is (for simplicity) of class C^∞ in the closed half-space, and if $u(P) = O(|P|^{m-1})$ for large $|P|$, then

$$\text{(4)} \qquad \underset{\mathfrak{D}}{\text{l.u.b.}}\ |D^{m-1}u| \leqq \text{constant} \cdot \underset{\dot{\mathfrak{D}}}{\text{l.u.b.}}\ |D^{m-1}u|,$$

provided the right side is finite. Here l.u.b. refers also to all derivatives D^{m-1} of the order $m-1$. We have not generalized this to equations with variable coefficients (however, in Section 9 we prove a general result involving the Hölder norms of the derivatives of order $m-1$). Recently Miranda [27] obtained a general result of the nature of (4) for strongly elliptic equations with variable coefficients in two dimensions; he uses the results of Agmon [1]. Previously he had proved such a result for the biharmonic equation in two dimensions [24].

In Section 3 we present some general potential theoretic results concerning certain convolution operators taking functions of n variables into functions of $n+1$ variables. Two of these are generalizations of results of Privaloff and M. Riesz. Except for Theorem 3.4 this section may be read without reference to the rest of the paper, and the results there, we feel, should prove useful in other problems.

We call attention also to Lemmas (9.1), (9.1)′, (9.1)″, and to Appendix 5 which is never referred to in the rest of the paper. Here we show that solutions having square integrable derivatives are in fact classical solutions

to which the interior estimates of [10] apply. Other appendices contain proofs of some of the results in the text.

One of the main features of the Schauder estimates is their use in nonlinear problems. In Theorem 12.5 we prove a local perturbation theorem for solutions of nonlinear elliptic equations involving a parameter. It is here that one sees the strength of the estimates.

We have included (Chapter III, and part of Chapter V) estimates for equations in integral, or variational, form. The remainder of the paper may be read independently.

Outline

Chapter I is concerned with the system with constant coefficients in the half-space, and potential theory.

Section 1. We formulate in a precise way the boundary value problem (2), (3) (with $F = 0$) in the half-space and do some preliminary algebra.

Section 2. We construct the Poisson kernels, and also prove the extended maximum principle, which is a special case of a more general result, Theorem 2.2.

Section 3. The potential theory.

Section 4. A general representation formula is obtained for solutions of (2), (3) (with $F \neq 0$) with the aid of the Poisson kernels and the fundamental solution for elliptic equations with constant coefficients.

Chapter II deals with the Schauder estimates. For $l \geqq l_0 = \max(2m, m_j)$ we estimate the Hölder continuity of derivatives up to order l of solutions of (2), (3) near the boundary in terms of l.u.b. $|u|$ and the given data F, Φ_j.

Section 5. We introduce the basic norms and state some simple calculus lemmas.

Section 6. With the aid of the potential theoretic results of Section 3 and the explicit formulas of Section 4 we prove the Schauder estimates for solutions of the equations with constant coefficients in a half-space considered in Chapter I. We also prove a general Liouville theorem for solutions of such equations.

Section 7. The Schauder estimates for equations with variable coefficients in general domains. Theorem 7.3.

Chapter III. We prove Schauder estimates for equations in integral, or variational, form such as arise from regular variational problems. These involve estimating derivatives of lower order than l_0.

Section 8. The constant coefficient case in a half-space.

Section 9. Equations with variable coefficients. Our main result, Theorem 9.3, contains as a special case for the Dirichlet problem the result that if we have an estimate of all derivatives up to order $k \geqq m-1$ on the boundary, and of their Hölder continuity (as functions on the boundary), then we can estimate these functions and their Hölder continuity in the entire domain.

Chapter IV is concerned with applications of the estimates and some comments.

Section 10. We prove that the Condition on L and the Complementing Condition are both necessary for our estimates (Schauder and L_p) to hold.

Section 11. Differentiability at the boundary for solutions of nonlinear elliptic equations satisfying nonlinear boundary conditions. The first variation of the equation and the boundary condition is required to satisfy (locally at the boundary) the Condition on L and the Complementing Condition.

Section 12. We first prove some obvious consequences of the Schauder estimates, such as the compactness of solutions and the finite dimensionality of the solutions of the homogeneous system. In Theorem 12.5 we describe the continuity method. We then prove Theorem 12.6, the perturbation theorem for nonlinear elliptic equations.

The rest of Section 12 is concerned mainly with the Dirichlet problem which we solve under mild hypotheses on the coefficients. For weakly positive elliptic equations we obtain a unique solution for the problem in case the Dirichlet data $(\partial/\partial \boldsymbol{n})^{j-1} u = \Phi_j$ belong to $C^{m-j+\alpha}$, $j = 1, \cdots, m$. The solution belongs to $C^{m-1+\alpha}$ in $\bar{\mathfrak{D}}$. See Theorems 12.10, 12.11.

Section 13. We sketch a general estimate of Schauder type for a certain class of semilinear elliptic equations (see Nagumo [30]).

Chapter V deals with the L_p estimates at the boundary.

Section 14. The constant coefficient case in the half-space.

Section 15. The general case.

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CHAPTER I

EQUATIONS WITH CONSTANT COEFFICIENTS IN A HALF-SPACE AND POTENTIAL THEORY

1. Preliminaries

We shall consider functions defined in domains of $(n+1)$-st dimensional Euclidean space, $n \geqq 1$, with coordinates $x_1, \cdots, x_{n+1}$ and use the notation

$$D_i = \frac{\partial}{\partial x_i}, \qquad D = (D_1, \cdots, D_{n+1}).$$

Greek letters β, γ, μ, ν, will be used to denote vectors $\beta = (\beta_1, \cdots, \beta_{n+1})$ with integral components $\beta_i \geqq 0$; however, Ξ will simply denote a vector $\Xi = (\xi_1, \cdots, \xi_{n+1})$ with scalar components, and we shall write

$$\Xi^\beta = \xi_1^{\beta_1} \cdots \xi_{n+1}^{\beta_{n+1}}, \qquad D^\beta = D_1^{\beta_1} \cdots D_{n+1}^{\beta_{n+1}}, \qquad |\beta| = \sum \beta_1 .$$

Thus a typical differentiation operation of order l will be written D^β with $|\beta| = l$. For convenience, when there is no danger of confusion, we will often denote such an operator by D^l.

The letter α will always denote a fixed positive number less than one.

Much of the work will be concerned with functions defined in a half-space given, say, by $x_{n+1} \geqq 0$. In that case, we shall write $t = x_{n+1}$ and $x = (x_1, \cdots, x_n)$, and shall denote the points by $P = (x, t)$. Vectors with $n+1$ components will be denoted by (ξ, τ), where $\xi = (\xi_1, \cdots, \xi_n)$. D_t will denote differentiation with respect to t, and $D_x = (D_1, \cdots, D_n)$ differentiation with respect to $x_1, \cdots, x_n$, D_x^l or D_x^β representing higher differentiations with respect to the x_i. If $P = (x, t)$, $Q = (y, \tau)$, we introduce the Euclidean distance

$$|P-Q| = (|x-y|^2 + (t-\tau)^2)^{1/2},$$
$$|x-y|^2 = \sum (x_i - y_i)^2.$$

The scalar product of real vectors x, y will be denoted by $x \cdot y = \sum x_i y_i$.

$\Sigma = \Sigma_r$ will denote the hemisphere in x, t-space given by $|x|^2 + t^2 < r^2$, $t \geqq 0$, and $\sigma = \sigma_r$ its planar boundary: $|x| < r$, $t = 0$.

A good part of the paper will deal with an elliptic operator $L(D) = L(D_x, D_t)$ with complex valued constant coefficients and having only terms of highest order. This will act on functions defined in a half-space $t \geqq 0$. The corresponding characteristic form $L(\xi, \tau)$ is different from zero for all real nonvanishing (ξ, τ). For real $\xi \neq 0$ consider the roots τ of the polynomial $L(\xi, \tau)$. If $n \geqq 2$, it follows easily that there are as many roots with positive imaginary parts as with negative imaginary parts. For, if τ is a root for given ξ, then $-\tau$ is a root for $-\xi$, and the assertion follows from

the fact that for $n > 1$ the set of real vectors $\xi \neq 0$ is connected. Thus in particular L is of even degree $2m$. In the special case of two variables, $n = 1$, this is not necessarily true, the Cauchy-Riemann equations $L(\xi, \tau) = \xi + i\tau$ being a counter-example. As in the introduction, we shall impose the following

CONDITION ON L. *$L(\xi, \tau)$ is of even order $2m$, and has, for each real $\xi \neq 0$, exactly m roots τ with positive imaginary parts.*

We shall also assume that L is uniformly elliptic, i.e., that for some constant A, the "ellipticity constant", the inequality

$$A^{-1}(|\xi|^2+\tau^2)^m \leqq |L(\xi, \tau)| \leqq A(|\xi|^2+\tau^2)^m \tag{1.1}$$

holds for all real (ξ, τ).

A number of quantities can be estimated in terms of A and m. Writing

$$L(\xi, \tau) = \sum_{i=0}^{2m} C_i(\xi)\tau^{2m-i}$$

we see from (1.1) that $A^{-1} \leqq C_0 \leqq A$, and we find successively for $i = 1, 2, \cdots$, that for $|\xi| = 1$, $|C_i(\xi)| \leqq$ constant depending only on A and m. From this it follows easily that there is a constant C, depending only on A and m, such that the roots $\tau(\xi)$ of $L(\xi, \tau)$ for real ξ, $|\xi| = 1$, satisfy

$$|\mathscr{I}m\, \tau(\xi)|^{-1}, \quad |\tau(\xi)| \leqq C. \tag{1.2}$$

Denote by $\tau_k^+(\xi)$ $(\tau_k^-(\xi))$, $k = 1, \cdots, m$, the roots of $L(\xi, \tau)$ with positive (negative) imaginary parts, and set

$$M^{\pm}(\xi, \tau) = \prod_1^m (\tau - \tau_k^{\pm}(\xi)) = \sum_{p=0}^m a_p^{\pm}(\xi)\tau^{m-p}. \tag{1.3}$$

We observe that

$$M^+(\xi, \tau) = (-1)^m M^-(-\xi, -\tau). \tag{1.4}$$

It is readily seen that the coefficients $a_p^{\pm}(\xi)$ are analytic functions of ξ for real nonvanishing ξ and are homogeneous of degree p.

With $M^+(\xi, \tau)$ we associate the polynomials (in τ) of degree j

$$M_j^+(\xi, \tau) = \sum_{p=0}^j \alpha_p^+(\xi)\tau^{j-p}, \qquad j = 0, \cdots, m-1. \tag{1.5}$$

We claim, and this is the reason for their definition, that they satisfy

$$\frac{1}{2\pi i}\oint_\gamma \frac{M_{m-1-j}^+(\xi, \tau)}{M^+(\xi, \tau)}\tau^k d\tau = \delta_{jk}, \qquad 0 \leqq j, k \leqq m-1, \tag{1.6}$$

where γ is a rectifiable Jordan contour in the complex plane enclosing all the roots $\tau_k^+(\xi)$ in its interior; δ_{jk} is the Kronecker delta.

Proof: By deforming the contour to a large circle about the origin whose

radius goes to infinity, we see immediately, since the highest order term of $M^+_{m-1-j}\tau^k$ is $a_0(\xi)\tau^{m-1-j+k}$, that (1.6) holds for $j \geqq k$. For $j < k$ the polynomial $M^+_{m-1-j}\tau^k$ differs from $\tau^{k-j-1}M^+$ by a polynomial Q of degree at most $k-1$, so that the left-hand side of (1.6) equals

$$\frac{1}{2\pi i}\int_\gamma \frac{Q}{M^+}\,d\tau.$$

Since the degree of $Q = k-1$ is less than $m-1$ we find, on deforming γ again to a large circle, that this integral is zero.

We consider now certain boundary operators given by m differential operators with constant (complex) coefficients $B_j(D)$ of order m_j with no lower order terms, $j = 1, \cdots, m$. Our aim in this and the next section is to study the following

BOUNDARY VALUE PROBLEM. *Let $\phi_j(x)$, $j = 1, \cdots, m$, be C^∞ functions of compact support in n-space. We seek a C^∞ function $u(x, t)$ in the half-space $t \geqq 0$ satisfying*

$$\begin{aligned} &L(D)u = 0, && t \geqq 0, \\ &B_j(D)u = \phi_j, && t = 0, \quad j = 1, \cdots, m. \end{aligned} \tag{1.7}$$

In order that this problem be well posed we shall, as in the introduction, impose the following algebraic condition on the operators L, B_j:

COMPLEMENTING CONDITION. *For every fixed real $\xi \neq 0$ the polynomials $B_j(\xi, \tau)$ (in τ) are linearly independent mod $M^+(\xi, \tau)$.*

In other words, for fixed $\xi \neq 0$, let

$$B'_j(\xi, \tau) = \sum_{k=1}^{m} b_{jk}(\xi)\tau^{k-1} = B_j(\xi, \tau) \pmod{M^+} \tag{1.8}$$

be the remainder when $B_j(\xi, \tau)$ is divided by $M^+(\xi, \tau)$ (each considered as a polynomial in τ). Then, the Complementing Condition means that

$$d(\xi) = \det \|b_{jk}(\xi)\| \neq 0 \qquad \text{for real } \xi \neq 0.$$

We note that $d(\xi)$ is analytic for real $\xi \neq 0$, and we introduce the determinant constant

$$\Delta = \min_{|\xi|=1} |d(\xi)|, \tag{1.9}$$

which is positive if the Complementing Condition is satisfied.

Remark. Because of (1.4) the Complementing Condition is equivalent to the condition that the B_j are linearly independent mod $M^-(\xi, \tau)$ for every real $\xi \neq 0$.

In solving the boundary value problem (in Section 2) we shall make use of the following: For fixed real $\xi \neq 0$ there exist polynomials (in τ) $N_j(\xi,\tau)$,

$j = 1, \cdots, m$, with coefficients which are analytic in ξ for ξ real $\neq 0$, such that

$$\frac{1}{2\pi i}\int_\gamma \frac{N_k(\xi, \tau)B_j(\xi, \tau)}{M^+(\xi, \tau)}\, d\tau = \delta_{kj}\,, \qquad j, k = 1, \cdots, m, \tag{1.10}$$

where γ is a contour in the upper half of the complex plane enclosing all the roots of $M^+(\xi, \tau)$ in its interior. Indeed, let $\|b^{jk}(\xi)\|$ be the inverse matrix of the $b_{jk}(\xi)$. Then the polynomials, in τ,

$$N_k(\xi, \tau) = \sum_{i=1}^{m} b^{ik}(\xi) M^+_{m-i}(\xi, \tau) \tag{1.11}$$

satisfy (1.10), since

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma \frac{N_k(\xi, \tau)B_j(\xi, \tau)}{M^+(\xi, \tau)} &= \frac{1}{2\pi i}\int_\gamma \frac{N_k B'_j}{M^+}\, d\tau \\
&= \sum_{q, p=1}^{m} b^{qk} b_{jp} \frac{1}{2\pi i}\int_\gamma \frac{M^+_{m-q}\tau^{p-1}}{M^+}\, d\tau \\
&= \sum_{q, p} b^{qk} b_{jp} \delta_{pq} \qquad \text{(in virtue of (1.6))} \\
&= \sum_{p} b^{pk} b_{jp} = \delta_{jk}\,.
\end{aligned}$$

We complete this section by examples showing that, if either the Condition on L or the Condition on the B_j is violated, then there are solutions of the homogeneous system (1.7), in which $\phi_j = 0$, with bounded derivatives up to any given order l, but such that the Hölder coefficient in the Hölder condition (with any exponent $\alpha < 1$) for derivatives of order l is as large as one pleases in any half-neighborhood of the boundary $t = 0$.

Suppose first that the Condition on L is satisfied but that the Complementing Condition on the B_j is violated. Then there is a real unit vector ξ such that the polynomials $B_j(\xi, \tau)$ (in τ) are linearly dependent mod $M^+(\xi, \tau)$. Hence, it follows easily that there exists a function $v(t) \not\equiv 0$ satisfying the following differential equation and initial conditions:

$$\begin{aligned}
&M^+(\xi, -iD_t)v = 0 && \text{for } t \geqq 0, \\
&[B_j(\xi, -iD_t)v]_{t=0} = 0 && \text{for } j = 1, \cdots, m.
\end{aligned} \tag{1.12}$$

Since each zero of $M^+(\xi, \tau)$ has a positive imaginary part, v and its derivatives tend to zero exponentially as $t \to +\infty$. Obviously, given an integer $l \geqq 0$ there exists a positive number a such that $D_t^l v(a) - D_t^l v(0) = c \neq 0$. For $\lambda \geqq 1$ define

$$v_\lambda(x, t) = \lambda^{-l} e^{i\lambda\xi\cdot x} v(\lambda t). \tag{1.13}$$

Since L and B_j are homogeneous we see that v_λ satisfies (1.7) with $\phi_j = 0$. Also, v_λ and its derivatives up to the order l are bounded uniformly for $t \geqq 0$, whereas

$$\left(\frac{\lambda}{a}\right)^{\alpha}\left|D_t^l v_\lambda\left(0, \frac{a}{\lambda}\right) - D_t^l v_\lambda(0, 0)\right| = \left(\frac{\lambda}{a}\right)^{\alpha}|c|$$

blows up of the order of λ^α as $\lambda \to \infty$.

Suppose now that the condition on L is violated (which is possible only for $n = 1$). Define also in this case

$$M^+(\xi, \tau) = \prod_1^{r(\xi)} (\tau - \tau_k^+(\xi)),$$

where $\tau_1^{(\xi)}, \cdots, \tau_{r(\xi)}^{(\xi)}$ are the $r(\xi)$ roots with positive imaginary part of $L(\xi, \tau)$. There exists in this case a unit vector $\xi = \pm 1$ such that $r(\xi) \geqq m+1$. Since, for such ξ, $M^+(\xi, \tau)$ is of degree $\geqq m+1$, whereas the number of boundary operators is m, it follows that there exists a function $v(t) \not\equiv 0$ which satisfies (1.12). The functions $v_\lambda(x, t)$ defined by (1.13) again are not subject to the estimates in question.

2. Solution of the Boundary Value Problem in a Half-Space; The Poisson Kernels

We consider the problem (1.7) and assume that the Condition on L and the Complementing Condition on the B_j are satisfied. In terms of the polynomials N_k of (1.11) we introduce in the half-space $t > 0$ the

POISSON KERNELS. For $m_j \geqq n$,

$$(2.1)\quad K_j(x, t) = \frac{\beta_j}{2\pi i}\int_{|\xi|=1} d\omega_\xi \left[\int_\gamma \frac{N_j(\xi, \tau)(x\cdot\xi + t\tau)^{m_j - n}}{M^+(\xi, \tau)} \log \frac{x\cdot\xi + t\tau}{i}\, d\tau\right];$$

for $0 \leqq m_j < n$,

$$(2.1)'\quad K_j(x, t) = \frac{\beta_j}{2\pi i}\int_{|\xi|=1} d\omega_\xi \left[\int_\gamma \frac{N_j(\xi, \tau)}{M^+(\xi, \tau)(x\cdot\xi + t\tau)^{n-m_j}} d\tau\right], \qquad j = 1, \cdots, m.$$

Here, and throughout, the principal branch of the logarithm in the complex plane slit along the negative real axis is taken, $d\omega_\xi$ is the area element on the unit sphere $|\xi| = 1$, and the β_j are absolute constants given by

$$(2.2)\qquad \begin{aligned} \beta_j &= -\frac{1}{(2\pi i)^n (m_j - n)!} && \text{if } m_j \geqq n, \\ \beta_j &= (-1)^{n-m_j}\frac{(n - m_j - 1)!}{(2\pi i)^n} && \text{if } 0 \leqq m_j < n. \end{aligned}$$

Finally γ is a Jordan contour in $\mathscr{I}m\,\tau > 0$ enclosing all the roots of $M^+(\xi, \tau)$ *for all* $|\xi| = 1$.

With the aid of the Poisson kernels we can write down the explicit solution of the boundary value problem (1.7).

THEOREM 2.1. *The function*

$$u(x, t) = \sum_j \int K_j(x-y, t)\phi_j(y)dy = \sum K_j * \phi_j \tag{2.3}$$

is a solution of problem (1.7).

Here integration is over the full n-space, $*$ denotes convolution. The ϕ_j, we recall, are C^∞ functions of compact support.

Our construction of the Poisson kernels, and their use in proving Theorem 2.1, are based on an identity due to F. John and used extensively in his book [17] (see p. 11). This asserts that a differentiable function ϕ with compact support in E_n may be represented in terms of plane waves by the formula[4]

$$\begin{aligned}\phi(x) = &-\frac{1}{(2\pi i)^n q!}\Delta_x^{(n+q)/2} \\ &\cdot \int \phi(y)\left(\int_{|\xi|=1}[(x-y)\cdot\xi]^q \log\frac{(x-y)\cdot\xi}{i}\,d\omega_\xi\right)dy,\end{aligned} \tag{2.4}$$

where q is any positive integer of the same parity as n, Δ_x represents the Laplacean $\sum D_i^2$, and $d\omega_\xi$ represents the area element on the unit sphere $|\xi| = 1$.

Before proving the theorem we apply the identities

$$\begin{aligned}&\frac{\mu!}{(\lambda+\mu)!}\left(\frac{d}{dz}\right)^\lambda\left[z^{\lambda+\mu}\left(\log\frac{z}{i}+c_{\lambda,\mu}\right)\right] = z^\mu\log\frac{z}{i}, && \mu \geqq 0, \quad \lambda \geqq 0, \\ &\frac{(-1)^{1+\mu}}{(\mu+\lambda)!(-1-\mu)!}\left(\frac{d}{dz}\right)^\lambda\left(z^{\lambda+\mu}\log\frac{z}{i}\right) = z^\mu, && \mu < 0, \quad \lambda+\mu \geqq 0,\end{aligned} \tag{2.5}$$

valid for integral λ, μ, the $c_{\lambda,\mu}$ being appropriate constants, to represent the Poisson kernels K_j in the form

$$K_j(x, t) = \Delta_x^{(n+q)/2}K_{j,q}(x, t), \qquad K_{j,q} = \Delta_x K_{j,q+2}\,. \tag{2.6}$$

Here q is a non-negative integer with the same parity as n and, for $m_j \geqq n$,

$$\begin{aligned}K_{j,q} = &\frac{\beta_j(m_j-n)!}{2\pi i(m_j+q)!}\int_{|\xi|=1} d\omega_\xi \\ &\cdot\left[\int_\gamma \frac{N_j(\xi,\tau)(x\cdot\xi+t\tau)^{m_j+q}\left(\log\dfrac{x\cdot\xi+t\tau}{i}+c_{n+q,m_j-n}\right)}{M^+(\xi,\tau)}\,d\tau\right],\end{aligned} \tag{2.6$'$}$$

and, for $m_j < n$,

[4]The formula in [17] is stated just for real valued $f(x)$, and the real part of the right-hand side is taken unnecessarily. The apparently more general formula (2.4), valid for complex $f(x)$, is, however, an immediate consequence of John's.

$$(2.6)'' \qquad K_{j,q} = \frac{\beta_j(-1)^{1+m_j-n}}{2\pi i(m_j+q)!(n-m_j-1)!} \cdot \int_{|\xi|=1} d\omega_\xi \left[\int_\gamma \frac{N_j(\xi,\tau)(x\cdot\xi+t\tau)^{m_j+q}}{M^+(\xi,\tau)} \log\frac{x\cdot\xi+t\tau}{i}\, d\tau\right].$$

It is easily seen that $K_{j,q}$ and all its derivatives up to order m_j+q-1 are continuous in the closed half-space $t \geqq 0$ (the same is true of derivatives of order m_j+q if $n \geqq 2$).

Proof of Theorem 2.1: Inspection shows that the Poisson kernels $K_j(x, t)$, and the kernels $K_{j,q}$, are analytic solutions of $Lu = 0$ for $t > 0$. Hence $u(x, t)$ is an analytic solution of the same equation in $t > 0$. Setting

$$(2.3)_j \qquad u_j = K_j * \phi_j\,,$$

it will suffice to show that u_j belongs to C^∞ in $t \geqq 0$ and that

$$(2.7) \qquad B_k(D)u_j = \delta_{kj}\phi_j(x) \qquad \text{for} \qquad t = 0.$$

Consider any partial derivative of order s of $u_j(x, t)$. Choosing an integer q of the same parity as n and such that $q \geqq s-m_j+1$ we have, for $t > 0$,

$$(2.8) \qquad \begin{aligned} D^s u_j &= D^s \int \Delta_x^{(n+q)/2} K_{j,q}(x-y, t)\phi_j(y)dy \\ &= D^s \int \phi_j(y)\Delta_y^{(n+q)/2} K_{j,q}(x-y, t)dy \\ &= \int D^s K_{j,q}(x-y, t)\cdot \Delta_y^{(n+q)/2}\phi_j(y)dy \end{aligned}$$

after partial integration, since ϕ_j is C^∞ with compact support. Since, as remarked above, $D^s K_{j,q}(x-y, t)$ is continuous in the closed half-space $t \geqq 0$, it follows that $D^s u_j$ can be extended as a continuous function in the entire closed half-space $t \geqq 0$. As s was arbitrary we have proved that $u_j \in C^\infty$ in $t \geqq 0$.

To verify (2.7) choose q so large that $q \geqq m_k-m_j+1$, $k = 1, \cdots, m$. Using (2.8) we have

$$(2.9) \qquad \begin{aligned} B_k(D)u_j(x, 0) &= \int \Delta_y^{(n+q)/2}\phi_j(y)\cdot B_k(D)K_{j,q}(x-y, 0)dy \\ &= \int \Delta_x^{(n+q)/2}\phi_j(x-y)B_k(D_y, D_t)K_{j,q}(y, 0)dy, \end{aligned}$$

after a change of variables.

Assume first that $k \neq j$. Using (2.6)', (2.6)'' we find that on $t = 0$

$$B_k(D)K_{j,q}(y, 0) = \text{constant}\cdot \int_{|\xi|=1} d\omega_\xi \cdot \left[\int_\gamma \frac{N_j(\xi,\tau)B_k(\xi,\tau)}{M^+(\xi,\tau)} (y\cdot\xi)^{m_j-m_k+q}\left(\log\frac{y\cdot\xi}{i} + \text{constant}\right) d\tau\right] = 0$$

since, according to (1.10),

$$\int_\gamma \frac{N_j B_k}{M^+}\, d\tau = 0.$$

Thus (2.7) is proved for $k \neq j$.

Next suppose $k = j$. If $m_j \geqq n$ we have, using (2.5), (2.6)′ and (1.10),

$$\begin{aligned} B_j(D)K_{j,q}(y, 0) &= \frac{\beta_j(m_j-n)!}{2\pi i q!}\int_{|\xi|=1} d\omega_\xi \\ &\quad \cdot \left[(y\cdot\xi)^q\left(\log\frac{y\cdot\xi}{i} + \text{constant}\right)\int_\gamma \frac{N_j B_j}{M^+}\, d\tau\right] \\ &= \frac{\beta_j(m_j-n)!}{q!}\int_{|\xi|=1}(y\cdot\xi)^q \log\frac{y\cdot\xi}{i}\, d\omega_\xi + \psi_q(y), \end{aligned} \tag{2.10$'$}$$

where $\psi_q(y)$ is a homogeneous polynomial of degree q.

Similarly if $m_j < n$ we find, using (2.6)″ and (1.10),

$$B_j(D)K_{j,q}(y, 0) = \frac{(-1)^{1+m_j-n}\beta_j}{(n-m_j-1)!\,q!}\int_{|\xi|=1}(y\cdot\xi)^q \log\frac{y\cdot\xi}{i}\, d\omega_\xi + \psi_q(y), \tag{2.10$''$}$$

where again ψ_q denotes a homogeneous polynomial of degree q.

From (2.9), (2.10)′, (2.10)″ we find, after inserting the value of β_j from (2.2) and rechanging variables, that

$$\begin{aligned} B_j(D)u_j(x, 0) &= -\frac{1}{(2\pi i)^n q!}\Delta_x^{(n+q)/2}\int \phi_j(y) \\ &\quad \cdot \int_{|\xi|=1}[(x-y)\cdot\xi]^q \log\frac{(x-y)\cdot\xi}{i}\, d\omega_\xi\, dy. \end{aligned} \tag{2.11}$$

Here we have used the fact that

$$\int \Delta_y^{(n+q)/2}\phi_j(y)\cdot\psi_q(x-y)dy = \int \phi_j(y)\cdot\Delta_x^{(n+q)/2}\psi_q(x-y)dy = 0$$

since ψ_q is a polynomial of degree q and is therefore annihilated by $\Delta_x^{(n+q)/2}$.

By John's identity (2.4) the right side of (2.11) equals $\phi_j(x)$, and the proof of the theorem is complete.

Remark 1. It is clear from the above proof that, if ϕ_j belongs to C^{n-m_j+2+s} for $s \geqq \max m_k$ and has compact support, $j = 1, \cdots, m$, then $u(x, t)$ is a solution of the problem (1.7) of class C^s in $t \geqq 0$.

Much stronger results will be proved in later sections.

We shall prove a lemma concerning the smoothness of the Poisson kernels. Using arguments similar to those employed by John in [17] Chapter 3 it can in fact be shown that each kernel $K_{j,q}(x, t)$ is analytic in $t \geqq 0$ except at the origin.

It is convenient to introduce the constant

$$E = A + b + \Delta^{-1} + n + m + \sum m_j, \tag{2.12}$$

where A is the ellipticity constant of (1.1); b is a bound for the coefficients of the operators L, B_j; Δ is the determinant constant (1.9).

LEMMA 2.1. *The kernels $K_{j,q}(x, t)$ are of class C^∞ in $t \geqq 0$, except at the origin, and satisfy*

$$|D^s K_{j,q}| \leqq C(s, E)(|x|^2+t^2)^{(m_j+q-s)/2}(1+|\log|P||), \qquad s \geqq 0. \tag{2.13}$$

Furthermore, if $s \geqq m_j+q+1$, then $D^s K_{j,q}$ is homogeneous of degree m_j+q-s, and the logarithmic term in the inequality may be omitted. Here $C(s, E)$ is a constant depending only on s, q and E.

The lemma is proved in Appendix 1. From (2.6) we obtain the following similar statement for the Poisson kernels K_j:

$$|D^s K_j| \leqq \bar{C}(s, E)(1+|\log|P||)(|x|^2+t^2)^{(m_j-n-s)/2}, \qquad s \geqq 0. \tag{2.13$'$}$$

Furthermore, if $s > m_j-n$, the kernel $D^s K_j$ is homogeneous of degree m_j-n-s, and the logarithmic term in (2.13)′ may be omitted. Here $\bar{C}(s, E)$ depends only on s and E.

Because of the reproducing properties (2.7) of K_j we can assert that (even for $k = j$)

$$B_k(D)K_j(x, 0) = 0 \qquad \text{for } x \neq 0, \qquad j, k = 1, \cdots, m, \tag{2.14}$$

and hence, with the aid of (2.13)′ and the theorem of the mean

$$|B_k(D)K_j(x, t)| \leqq \text{constant} \cdot t(1+|\log|P||)|P|^{m_j-m_k-n-1}, \tag{2.14$'$}$$

where the constant depends only on E.

Because $B_k K_{j,q}(x, 0) = 0$ for $x \neq 0$, $k \neq j$, as established in the proof of Theorem 2.1, we find in a similar way, with the aid of (2.13), that if $s = n+q+m_j-m_k \geqq 0$, then

$$|D_x^s B_k(D)K_{j,q}(x, t)| \leqq C(s, E, q)\frac{t}{(|x|^2+t^2)^{(n+1)/2}}, \qquad k \neq j. \tag{2.15}$$

Furthermore,

$$|B_j(D)K_j(x, t)| \leqq C(E)\frac{t}{(|x|^2+t^2)^{(n+1)/2}}. \tag{2.15$'$}$$

In Section 4 we shall make use of

Remark 2. Let $l_0 = \max(2m, m_j)$ and, for $j = 1, \cdots, m$, let $\phi_j(x)$ be a given function of class $C^{n+l_0-m_j+2}$ satisfying

$$D^k\phi_j(x) = O((1+\log|x|)|x|^{2m-n-1-m_j-k}), \qquad k = 0, \cdots, l_0-m_j,$$

for large $|x|$; let $D_x^{l_0-m_j}$ be a particular differential operator in the x variables of order l_0-m_j, and let $\tilde{u}_j(x, t)$ be defined for $t > 0$ by

$$\tilde{u}_j(x, t) = \sum_i \int D_x^{l_0-m_j} B_j(D_x, D_t)K_i(x-y, t)\phi_i(y)dy.$$

Then $\tilde{u}_j$ may be continued as a continuous function into the closed half-space $t \geqq 0$ and $\tilde{u}_j(x, 0) = D_x^{l_0-m_j}\phi_j(x)$.

Proof: Because of (2.13)′ and because of our conditions on the ϕ_j we see easily that partial integration is valid, with no contribution from infinity, so that $\tilde{u}_j$ may be written in the form

$$\tilde{u}_j = \sum_i \int B_j K_i(x-y, t) D_y^{l_0-m_j}\phi_i(y)dy.$$

Let $\zeta(y)$ be a C^∞ function with compact support which equals one in a sphere $|y| < R$ and satisfies $|\zeta| \leqq 1$. We may write

$$\tilde{u}_j = B_j w_1 + w_2,$$

where

$$w_1 = \sum_i \int K_i(x-y, t)\zeta(y) D_y^{l_0-m_j}\phi_i(y)dy,$$

$$w_2 = \sum_i \int B_j K_i(x-y, t)\,(1-\zeta(y)) D_y^{l_0-m_j}\phi_i(y)dy.$$

Using Remark 1 we see that $B_j w_1$ may be extended as a continuous function in $t \geqq 0$ and, for $|x| < \frac{1}{2}R$,

$$B_j w_1(x, 0) = D_x^{l_0-m_j}\phi_j(x).$$

For $|x| < \frac{1}{2}R$ consider the function w_2. Since the integrand is nonzero only for $|y| > R$, in which region $\frac{1}{2}|y| \leqq |x-y| \leqq \frac{3}{2}|y|$, we see from (2.14)′ and our assumptions on ϕ_j that

$$\begin{aligned} |w_2(x, t)| &\leqq \text{constant} \cdot t \int_{|y|>R} (1 + \log |y|)^2 |y|^{m_i-m_j-n-1} |y|^{2m-n-1-m_i-l_0+m_j} dy \\ &\leqq \text{constant} \cdot t \int_{|y|>R} (1 + \log |y|)^2 |y|^{2m-l_0-2n-2} dy \end{aligned}$$

which tends to zero as $t \to 0$. Thus $\tilde{u}_j(x, t)$ is continuous at every point $(x, 0)$ in $|x| < \frac{1}{2}R$ and assumes the asserted value there. Since R was arbitrary the proof is complete.

We conclude with an interesting result based on (2.15), (2.15)′. The result will make use of the following norm for any non-negative integer j:

$$[\phi(x)]_j = \sum_{|\beta|=j} \text{l.u.b. } |D^\beta \phi|,$$

and the trivial inequality

$$\int \frac{t}{(|x-y|^2+t^2)^{(n+1)/2}} \phi(y)dy \leqq \text{constant} \cdot \text{l.u.b. } |\phi|, \tag{2.16}$$

where the constant depends only on n, the dimension of the y-space.

THEOREM 2.2. *Consider the solution $u(x, t)$ given by* (2.3) *and let l be an integer satisfying $l \geqq m_j$, $j = 1, \cdots, m$. Then, for any $i = 1, \cdots, m$ and any differentiation $D_x^{l-m_i}$ in the x coordinates, the following inequality holds*:

$$\text{l.u.b. } |D_x^{l-m_i} B_i(D)u(x, t)| \leqq C(l, E) \sum_j [\phi_j]_{l-m_j},$$

where the l.u.b. *is taken over the full half-space* $t \geqq 0$.

As a special case, for the Dirichlet boundary conditions, with

$$B_j = D_t^{j-1}$$

we obtain

AN EXTENDED MAXIMUM PRINCIPLE (WEAK FORM). *The solution* (2.3) *of the Dirichlet problem*

$$Lu = 0, \qquad t > 0,$$
$$D_t^{j-1}u = \phi_j, \qquad t = 0, \quad j = 1, \cdots, m,$$

satisfies

$$\text{l.u.b. } |D^{m-1}u(x, t)| \leqq C(E) \sum_j [\phi_j]_{m-j},$$

or, equivalently,

$$\text{(2.17)} \qquad \text{l.u.b. } |D^{m-1}u(x, t)| \leqq C(E) \text{ l.u.b. } |D^{m-1}u(x, 0)|.$$

Here the least upper bounds are taken with respect to all derivatives of order $m-1$ *and on the left with respect to all* x, t *in the half-space, on the right with respect to all* x.

Proof of Theorem 2.2: It suffices for any fixed $j = 1, \cdots, m$ to prove that

$$\text{l.u.b. } |D_x^{l-m_i} B_i(D)K_j(x, t) * \phi_j(x)| \leqq C(l, E)[\phi_j]_{l-m_j}.$$

Suppose first that $j = i$, then we have

$$D_x^{l-m_i} B_i K_i(x, t) * \phi_i(x) = B_i K_i(x, t) * D^{l-m_i}\phi_i,$$

and the result follows from (2.15)′ with the aid of (2.16).

Consider now $j \neq i$. According to (2.6) we may write

$$I(x, t) = D_x^{l-m_i} B_i K_j(x, t) * \phi_j(x) = \int D_x^{l-m_i} B_i \Delta_x^{(n+q)/2} K_{j,q}(x-y, t)\phi_j(y)dy$$

and let q be fixed satisfying $q > l-m_j-n$. If $l-m_j$ is even we may write this, after partial integration, as

$$I = \int D_x^{l-m_i} B_i \Delta_x^{(n+q-l+m_j)/2} K_{j,q}(x-y, t)\Delta_y^{(l-m_j)/2}\phi_j(y)dy.$$

Applying (2.15) we find

$$|I| \leqq C_1(l, E, q) \int \frac{t}{(|x-y|^2+t^2)^{(n+1)/2}} \Delta_y^{(l-m_j)/2}\phi_j(y)dy,$$

from which, by (2.16), follows the inequality

$$\text{(2.18)} \qquad |I| \leqq \tilde{C}(l, E, q)|\phi_j|_{l-m_j}.$$

If finally $j \neq i$ and $l-m_j$ is odd, $l-m_j = 2k+1$, we write

$$I = \sum_r \int D_x^{l-m_i} D_{x_r} B_i \Delta_x^{(n+q)/2-k-1} K_{j,q}(x-y, t) D_{y_r} \Delta_y^k \phi_j(y)dy.$$

Appealing again to (2.15) and (2.16) we obtain the same estimate (2.18). This completes the proof of the theorem.

With the aid of the results of Section 6 we may derive (confining ourselves for simplicity to C^∞ functions)

AN EXTENDED MAXIMUM PRINCIPLE (STRONG FORM). *Consider C^∞ functions $\phi_j(x)$ with $[\phi_j]_{m-j}$ finite, $j = 1, \cdots, m$. There exists a unique C^∞ solution u of the Dirichlet problem*

$$\begin{aligned} Lu &= 0, & t &\geqq 0, \\ D_t^{j-1}u &= \phi_j, & t &= 0, \quad j = 1, \cdots, m, \end{aligned} \tag{2.19}$$

satisfying

$$u(P) = O(|P|^{m-1}) \qquad \text{for large } |P|. \tag{2.20}$$

This solution u satisfies

$$\text{l.u.b. } |D^{m-1}u(x, t)| \leqq \bar{c}(E) \sum_j [\phi_j]_{m-j}\,. \tag{2.21}$$

Thus any C^∞ solution $u(x, t)$ of $Lu = 0$ in $t \geqq 0$ which is $O(|P|^{m-1})$ for $|P|$ large, and whose derivatives $D^{m-1}u(x, 0)$ are bounded, satisfies

$$|D^{m-1}u(P)| \leqq \bar{c}(E) \text{ l.u.b. } |D^{m-1}u(x, 0)|. \tag{2.22}$$

Proof: From the Theorem of Liouville type at the end of Section 6 we see first that any two solutions of the problem differ by a polynomial v of degree $m-1$. However, the only such polynomial whose derivatives D_t^{j-1}, $j = 1, \cdots, m$, vanish on $t = 0$ is $v = 0$, so that the uniqueness is established.

To prove the existence we may, without loss of generality, assume that $D_x^k\phi_j(0) = 0$ for $0 \leqq k < m-j$, $j = 1, \cdots, m-1$; this is equivalent to adding a polynomial of degree $m-2$ to u. Let $\zeta(x)$ be a C^∞ function defined for all x, vanishing for $|x| \geqq 1$ and equal to unity for $|x| \leqq \frac{1}{2}$. For $\lambda > 0$ let u_λ be the solution given by (2.3) of

$$\begin{aligned} Lu_\lambda &= 0, & t &\geqq 0 \\ D_t^{j-1}u_\lambda &= \zeta(\lambda x)\phi_j(x), & t &= 0, \quad j = 1, \cdots, m. \end{aligned}$$

Setting $\sum_j [\phi_j]_{m-j} = M$ we see easily, because of our normalization of the ϕ_j at the origin, that

$$\sum_j [\zeta(\lambda x)\phi_j(x)]_{m-j} \leqq cM,$$

where c is an absolute constant independent of λ. Applying now the extended maximum principle in its weak form above we obtain the estimate

$$|D^{m-1}u_\lambda(P)| \leqq cC(E)M.$$

Using the Schauder estimates of Section 6 one can see rather easily that this implies that as $\lambda \to 0$ the u_λ converge to a solution of our problem which satisfies the desired inequalities (2.20), (2.21).

We remark that uniqueness does not hold if condition (2.20) is omitted since $u = t^m$ is a solution of the homogeneous system (2.19).

3. Potential Theoretic Considerations

3.1. In this section we shall study certain integral transforms of functions $f(x)$, $x = (x_1, \cdots, x_n)$, into functions $u(x, t)$, $t \geqq 0$. Let K be a kernel defined in the half-space $t \geqq 0$ and homogeneous of degree $-n$:

$$K(x, t) = \frac{\Omega\left(\frac{x}{|P|}, \frac{t}{|P|}\right)}{(|x|^2+t^2)^{n/2}}. \tag{3.1}$$

Here $|x| = (\sum x_i^2)^{1/2}$, $P = (x, t)$, $|P| = (|x|^2+t^2)^{1/2}$. Unless stated otherwise we shall assume throughout that $\Omega(x, t)$ is continuous on the half-sphere $t \geqq 0$, $|x|^2+t^2 = 1$, and satisfies a uniform Hölder condition at points P on the plane $t = 0$, i.e., if $P = (x, 0)$ and $Q = (y, t)$ are unit vectors, then for some positive constants κ and α', $\alpha' \leqq 1$,

$$|\Omega(P)-\Omega(Q)| \leqq \kappa \overline{PQ}^{\alpha'}, \qquad \max |\Omega| \leqq \kappa; \tag{3.2}$$

here $\overline{PQ}$ represents the geodetic distance on the unit sphere from P to Q. Additional smoothness assumptions under various circumstances will be stated when needed.

In addition we make the basic assumption

$$\int_{|x|=1} \Omega(x, 0) d\omega_x = 0, \tag{3.3}$$

where $d\omega_x$ represents element of surface area on the unit sphere $|x| = 1$. For $n = 1$ this assumption takes the form $\Omega(x, 0) = -\Omega(-x, 0)$.

We consider the transformation

$$u(x, t) = \int K(x-y, t) f(y) dy, \qquad t > 0, \tag{3.4}$$

integration being over the entire n-dimensional y-space.

Before describing our results we first write K as a sum:

$$\begin{aligned} K(x, t) &= \frac{\Omega\left(\frac{x}{|x|}, 0\right)}{(|x|^2+t^2)^{n/2}} + \frac{\Omega\left(\frac{x}{|P|}, \frac{t}{|P|}\right) - \Omega\left(\frac{x}{|x|}, 0\right)}{(|x|^2+t^2)^{n/2}} \\ &= K_1+K_2, \end{aligned} \tag{3.5}$$

and note some properties of these kernels.

In virtue of the condition (3.2) and the inequality $|\theta|^{\alpha'} \leqq C(\alpha')(\sin\theta)^{\alpha'}$ for $0 \leqq \theta \leqq \frac{1}{2}\pi$, where the constant $C(\alpha')$ depends only on α', we see that

$$\left|\Omega\left(\frac{x}{|P|}, \frac{t}{|P|}\right) - \Omega\left(\frac{x}{|x|}, 0\right)\right| \leqq \kappa C(\alpha')t^{\alpha'}(|x|^2+t^2)^{-\alpha'/2},$$

so that

$$|K_2(x, t)| \leqq C(\alpha')\kappa \frac{t^{\alpha'}}{(|x|^2+t^2)^{(n+\alpha')/2}}. \tag{3.6}$$

It follows, by setting $x = tv$, that

$$\begin{aligned} &\int |K_2(x, t)|dx \leqq C(\alpha')\kappa \int \frac{dv}{(|v|^2+1)^{(n+\alpha')/2}} \leqq C_1\kappa, \\ &\int |K_2(x, t)||x|^\alpha dx \leqq C(\alpha')\kappa t^\alpha \int \frac{|v|^\alpha}{(|v|^2+1)^{(n+\alpha')/2}} dv \\ &\qquad \leqq C_1\kappa t^\alpha, \qquad \alpha < \alpha', \end{aligned} \tag{3.7}$$

where C_1 depends only on α, α' and n.

In addition we easily verify

$$|K_1(x, t)-K_1(x', t)| \leqq C_1\kappa \frac{|x-x'|^{\alpha'}}{(|x|^2+t^2)^{(n+\alpha')/2}} \qquad \text{if } |x-x'| \leqq |x|, |x'|, \tag{3.8}$$

where C_1 depends only on α' and n.

By condition (3.3) it is clear that we can write u in the form

$$\begin{aligned} u(x, t) = &\int K(x-y, t)(f(y)-f(x))dy \\ &+ f(x)\int K_2(y, t)dy, \qquad t > 0, \end{aligned} \tag{3.9}$$

where by (3.7) we see that the last integral is convergent; in the first integral we mean the limit of the integral taken over large spheres about x with the radii going to infinity.

Suppose in addition to our assumptions above that $K(P)$ has derivatives up to order l with respect to t which are continuous in $t > 0$ and bounded on $|P| = 1$. Setting

$$g_j(t) = \int D_t^j K(x, t)dx, \qquad t > 0, \quad j \geqq 1,$$

we see readily that $g_j(t) = g_j(1)t^{-j}$ and $g_{j+1}(1) = -jg_j(1)$. Hence if $g_{j_0}(t) = 0$ for some $t > 0$ and a certain $j_0 > 0$, then $g_j(t) \equiv 0$ for all $t > 0$ and $j = 1, \cdots, l$. Moreover we have the following result related to the assumption (3.3):

LEMMA 3.1. *Condition* (3.3) *holds if and only if* $g_j(t) = 0$, $t > 0$, $1 \leqq j \leqq l$.

To prove the lemma it suffices to consider the case $j = 1$. Consider the

decomposition (3.5) of K. We know from (3.7) that for $t > 0$, $\int K_2(x, t)dx$ is absolutely convergent and, by the homogeneity, is constant. Hence its t derivative which equals $\int D_t K_2 dx$ vanishes. On the other hand

$$\int D_t K_1 dx = -nt \int \frac{\Omega\left(\frac{x}{|x|}, 0\right)}{(|x|^2+t^2)^{(n+2)/2}} dx$$
$$= -nt \int_{|y|=1} \Omega(y, 0)dy \int_0^\infty \frac{r^{n-1}}{(r^2+t^2)^{(n+2)/2}} dr.$$

Thus $\int D_t K dx$ and $\int_{|y|=1} \Omega(y, 0)\, dy$ vanish or differ from zero together, proving the lemma.

An immediate consequence is the following

COROLLARY. *Let $K(x, t)$ be a homogeneous kernel of degree $-n$ possessing continuous partial derivatives up to order l in the half-space $t \geqq 0$ (the origin excepted). Suppose further that K is a solution of a homogeneous partial differential equation with constant coefficients of order l, containing the term $D_t^l K$ with a nonvanishing coefficient. Then K satisfies* (3.3).

Indeed $D_t^l K$ is a linear combination of all the other l-th order derivatives each of which contains at least one differentiation with respect to an x variable, and hence has zero integral with respect to that x variable. Thus $\int D_t^l K\, dx = 0$, and the corollary follows from the lemma.

3.2. We first derive a theorem which we call of Privaloff type. It is a simple extension of the classical inequalities of Hölder, Korn, Lichtenstein, Giraud for integrals of the form (Cauchy principal value)

$$g(x) = \int K_1(x-y, 0)f(y)dy; \tag{3.4$'$}$$

see Mihlin [23] and Bers [5].

For $0 < \alpha < \alpha'$ we shall use the seminorms

$$[f]_\alpha = \underset{x \neq y}{\text{l.u.b.}} \frac{|f(x)-f(y)|}{|x-y)|^\alpha},$$
$$[u]_\alpha = \underset{P \neq Q}{\text{l.u.b.}} \frac{|u(P)-u(Q)|}{|P-Q|^\alpha}. \tag{3.10}$$

THEOREM 3.1. *Under the assumptions* (3.2), (3.3) *above, if $f(x)$ is in L_p for some finite $p \geqq 1$ and $[f]_\alpha < \infty$ for some positive $\alpha < \alpha'$, then $[u]_\alpha$ is finite and*

$$[u]_\alpha \leqq C_2 \kappa [f]_\alpha, \tag{3.11}$$

where C_2 depends only on α, α' and n.

Proof: Since $f \in L_p$, the integral (3.4) is absolutely convergent. We shall show that

$$\frac{|u(x,t)-u(x,t')|}{|t-t'|^\alpha}, \frac{|u(x,t)-u(x',t)|}{|x-x'|^\alpha} \leqq \text{constant} \cdot \kappa[f]_\alpha,$$

from which the general result (3.11) follows immediately.

We shall consider separately the contributions of K_1 and K_2 to K, and begin with the simpler case

(1) Suppose $K_1 = 0$. By (3.7) we obtain immediately

$$\begin{aligned} &|u(x,t)-u(x',t)| \\ &\quad \leqq \left|\int K_2(x-y,t)\,(f(y)-f(y+x'-x))dy\right| \\ &\quad \leqq \int [f]_\alpha \cdot |x-x'|^\alpha \int |K_2(x-y,t)|dy \\ &\quad \leqq C_1\kappa[f]_\alpha |x-x'|^\alpha, \end{aligned}$$

which is the desired estimate.

If $t' \geqq t$ and we set $t' = ts$ we find

$$\begin{aligned} I &\equiv u(x,t) - u(x,t') \\ &= \int (K_2(x-y,t)-K_2(x-y,t'))f(y)dy \\ &= \int (K_2(-y,t)-K_2(-y,t'))f(x+y)dy \end{aligned}$$

by an obvious change of variable. Making now another change of variable in the second term being integrated we find, from the homogeneity of K_2, that

$$I = \int K_2(-y,t)\,(f(x+y)-f(x+sy))dy,$$

so that

$$\begin{aligned} |I| &\leqq [f]_\alpha (s-1)^\alpha \int |K_2(-y,t)|\,|y|^\alpha dy \\ &\leqq C_1\kappa[f]_\alpha (s-1)^\alpha t^\alpha = C_1\kappa[f]_\alpha |t'-t|^\alpha, \end{aligned}$$

which again is the desired result—completing case (1).

(2) Suppose $K_2 = 0$. We first have, assuming $t' > t$ and using representation (3.9),

$$\begin{aligned} &|u(x,t) - u(x,t')| \\ &\quad \leqq \left|\iint_t^{t'} \frac{\partial K_1}{\partial t}(x-y,\tau)\,(f(y)-f(x))dy\,d\tau\right| \\ &\quad \leqq n\kappa[f]_\alpha \iint_t^{t'} \frac{\tau}{(|x-y|^2+\tau^2)^{n/2+1}}\,|x-y|^\alpha dy\,d\tau \\ &\quad \leqq C'\kappa[f]_\alpha \int_t^{t'} \tau^{\alpha-1}d\tau = \frac{C'}{\alpha}\kappa[f]_\alpha (t'^\alpha - t^\alpha) \\ &\quad \leqq \frac{C'}{\alpha}\kappa[f]_\alpha (t'-t)^\alpha, \end{aligned}$$

where C' depends only on α.

Thus to conclude the proof of the theorem we have only to show that, in case $K_2 = 0$,

$$\frac{|u(x,t)-u(x',t)|}{|x-x'|^\alpha} \leqq \text{constant} \cdot \kappa[f]_\alpha . \tag{3.12}$$

Set $|x-x'| = d$ and let S be the sphere with center x' and radius $2d$; let E be the exterior of S. Then, using (3.9), we may write

$$\begin{aligned} u(x,t) &- u(x',t) \\ &= \int_S K_1(x-y,t)\,(f(y)-f(x))dy \\ &- \int_S K_1(x'-y,t)\,((f(y)-f(x'))dy \\ &+ \int_E (K_1(x-y,t)-K_1(x'-y,t))\,(f(y)-f(x))dy \\ &+ (f(x')-f(x)) \int_E K_1(x'-y,t)dy. \end{aligned}$$

Because of condition (3.3) the last integral vanishes. Let us denote the other three terms on the right by I_1, I_2, I_3. Clearly

$$|I_1| \leqq \kappa[f]_\alpha \int_{|x-y|<3d} \frac{|x-y|^\alpha}{|x-y|^n}\, dy \leqq \text{constant} \cdot \kappa[f]_\alpha d^\alpha,$$

the constant depending only on α and n. The same estimate holds for $|I_2|$.

To estimate I_3 we observe that $d = |x-x'| \leqq |x-y|, |x'-y|$ for y in E, so that by (3.8),

$$\begin{aligned} |I_3| &\leqq C_1 \kappa[f]_\alpha |x-x'|^{\alpha'} \int_E \frac{|x-y|^\alpha}{|x-y|^{n+\alpha'}}\, dy \\ &\leqq C_1 \kappa[f]_\alpha d^{\alpha'} \int_{|y-x|>d} |x-y|^{\alpha-\alpha'-n}\, dy \\ &\leqq \text{constant} \cdot \kappa[f]_\alpha d^{\alpha'} \cdot d^{\alpha-\alpha'} = \text{constant} \cdot \kappa[f]_\alpha d^\alpha, \end{aligned}$$

where the constant depends only on α, α' and n. Note that the last integral converges because $\alpha < \alpha'$.

Combining these estimates for I_1, I_2, I_3, we obtain the desired estimate (3.12), completing the proof of the theorem.

Note that our last argument yields the classical estimate for (3.4)'

$$[g]_\alpha \leqq \text{constant} \cdot \kappa[f]_\alpha , \tag{3.11$'$}$$

the constant depending only on α, α' and n (this argument is, in fact, the usual argument.)

3.3. We turn now to an integral estimate, of Riesz type, for the function u defined by (3.4). We shall assume that f belongs to L_p, $1 < p < \infty$, and

denote its L_p norm by $|f|_{L_p}$. For fixed $t > 0$ we shall denote the L_p norm of $u(x, t)$, regarded as a function of x, by $|u|_{L_p, t}$. This result will not be used in any essential way in the paper, but is of interest.

THEOREM 3.2. *Under the conditions* (3.2), (3.3) *above,* $u(x, t)$ *is in* L_p *for each* t *and*

$$|u|_{L_p, t} \leqq C\kappa[f]_{L_p}, \qquad t > 0,$$

where C *depends only on* α', p *and* n.

We observe that for $K = K_2$, i.e., $K_1 = 0$, the theorem is trivial. According to (3.7), K_2 has uniformly bounded L_1 norm for $t =$ constant. Applying the well known result that convolution by a function in L_1 is a bounded mapping of L_p into L_p we obtain the desired estimate. Thus we need only consider $K = K_1$.

Theorem 3.2 is a straightforward extension of recent results of Calderon and Zygmund in the important papers [8, 9]. They proved such an estimate for the function g defined by (3.4)′ under our conditions on K_1 (in fact they showed that condition (3.2) can be considerably relaxed). (We shall refer to Theorem 2 of [9] as the Calderon-Zygmund theorem.) Our proof, given in Appendix 2, is modelled after that in [9]: We prove the result (for K_1) for $n = 1$ by a simple reduction to the classical case of the Hilbert transform for which the result is due to M. Riesz (see [39]). It is then extended to higher dimensions folllowing [9].

We remark that for $p = 2$ a simple proof using only Fourier transforms may be given; this is similar to the argument in [8] p. 89, and requires only measurability and boundedness of Ω.

3.4. The potential theoretic estimates used in deriving the L_p bounds make use of the following seminorms for $1 < p < \infty$ and j a non-negative integer. For functions $u(x, t)$ define

$$|u|_{j, L_p} = \left(\sum_{|\beta|=j} \int_{t>0} |D^\beta u|^p \, dx \, dt \right)^{1/p}. \tag{3.13}$$

For functions $f(x)$ we define a seminorm $|f|_{j-1/p, L_p}$ in the following way. Suppose that $f(x)$ is the boundary value of a function $v(x, t)$ in $t > 0$, $f(x) = v(x, 0)$, with finite norm $|v|_{j, L_p}$. Then define

$$|f|_{j-1/p, L_p} = \text{g.l.b. } |v|_{j, L_p}, \tag{3.13$'$}$$

where the greatest lower bound is taken over all such functions v.

We remark that for $p = 2$, $j > 0$, the norms (3.13)′ can be conveniently expressed in terms of the Fourier transform $\hat{f}(\xi)$ of $f(x)$. If we define

$$|f|^*_{j-1/2, L_2} = \left(\int (|\xi|^{j-1/2} |\hat{f}(\xi)|)^2 \, d\xi \right)^{1/2}, \tag{3.13$_2$}$$

then it may be seen that there is a constant C depending only on j and n

such that

$$(3.13)_2^* \qquad C^{-1}|f|_{j-1/2,L_2} \leqq |f|^*_{j-1/2,L_2} \leqq C|f|_{j-1/2,L_2}.$$

The norms $(3.13)_2$ which were brought to our attention by Hörmander (see for example Hörmander, Lions [16]) were used in the original draft of this paper. That draft, as far as integral estimates were concerned, dealt only with L_2 estimates $p = 2$.

Inequality $(3.13)_2^*$ suggests the following

CONJECTURE. *For real a define $f_a(x)$ as the inverse Fourier transform of $|\xi|^a \hat{f}(\xi)$. Now set*

$$(3.13)_p \qquad |f|^*_{j-1/p,L_p} = \text{the } L_p \text{ norm of } f_{j-1/p}.$$

Then there is a constant C depending only on $j > 0$, n and p such that

$$(3.13)_p^* \qquad C^{-1}|f|_{j-1/p,L_p} \leqq |f|^*_{j-1/p,L_p} \leqq C|f|_{j-1/p,L_p}.$$

We can now state our next main estimate (this surely holds under much weaker smoothness conditions on the kernel).

THEOREM 3.3. *Consider the transformation* (3.4). *Assume that the kernel K of* (3.1) *has continuous partial derivatives DK, D_t^2K in $t > 0$ which are bounded by κ on the hemisphere $|P| = 1$, and that K satisfies* (3.3). *If f is in L_p and has finite $|f|_{1-1/p,L_p}$ norm, then $u(x, t)$ has finite $|u|_{1,L_p}$ norm, and, in fact,*

$$|u|_{1,L_p} \leqq C\kappa|f|_{1-1/p,L_p},$$

where C is a constant depending only on p and n.

Theorem 3.3 is proved in Appendix 3. Our original proof of Theorem 3.3 for $p = 2$ which was phrased in terms of the kernels $(3.13)_2$ was based entirely on Fourier transforms, and required less smoothness of the kernel. The proof in the appendix makes constant use of the following simple consequence of the Calderon-Zygmund theorem.

LEMMA 3.2. *Let $G(x, t)$ be a measurable function defined in the half-space $t > 0$ and satisfying*

$$|G(P)| \leqq \frac{\Omega\left(\dfrac{P}{|P|}\right)}{|P|^{n+1}}.$$

Here Ω is a non-negative function defined on the hemisphere $S^+ : t > 0$, $|P| = 1$ with

$$\int_{S^+} \Omega^r \, d\omega_P < \kappa$$

for some $r > 1$ and some κ.

Consider the function

$$u(x, t) = \int_{s>0} G(x-y, t+s)v(y, s)dy\,ds,$$

where $v(x, t)$ is a given function belonging to L_p in $t > 0$, $p \geqq r/(r-1)$. Then $u \in L_p$ in $t > 0$, and

$$|u|_{0,L_p} \leqq \text{constant} \cdot |v|_{0,L_p}$$

with the constant depending only on r, κ, p and n.

Proof: Define a homogeneous kernel $K(P)$ of degree $-(n+1)$ in the full $P = x, t$-space E^{n+1} as follows: $K(P) = \Omega(P/|P|)|P|^{-n-1}$ for $t > 0$, and K is odd in t. Then, for $t > 0$, we have

$$\begin{aligned} |u(x, t)| &\leqq \int_{s>0} K(x-y, s+t)|v(y, s)|dy\,ds \\ &= \int_{E^{n+1}} K(x-y, s+t)|v(y, s)|dy\,ds \\ &= u^*(x, t), \end{aligned}$$

where we have extended $v(y, s)$ as zero for $s < 0$. Now the Calderon-Zygmund theorem (Theorem 2 of [9]) may be applied to the kernel K and it follows that

$$\int_{t>0} |u|^p\,dx\,dt \leqq \int_{E^{n+1}} |u^*|^p\,dx\,dt \leqq \text{constant} \cdot \int_{t>0} |v|^p\,dx\,dt,$$

proving the lemma.

Lemma 3.2 will actually be used only for continuous homogeneous kernels G of degree $-n-1$ in the half-space $t > 0$ for which

$$G(x, t) = O(\log 1/t) \qquad \text{as } t \to 0$$

uniformly for all points (x, t) with $|x| = 1$. Clearly such kernels satisfy the conditions of the lemma for all r, and hence the estimate holds for all finite $p > 1$.

3.5. We wish now to apply the preceding results to a kernel K of the form (see (2.6)′, (2.6)″)

$$K(x, t) = D^{m_j+q+n}K_{j,q}(x, t). \tag{3.14}$$

By Lemma 2.1 K is homogeneous of degree $-n$ and its first and second derivatives on $|P| = 1$ are bounded by a constant depending on E and q. Thus in order to apply Theorems 3.1, 3.2, 3.3 we need only verify that

$$\int_{|x|=1} K(x, 0)d\omega_x = 0. \tag{3.15}$$

But this follows from the corollary to Lemma 3.1; for, K satisfies $LK = 0$ in $t > 0$.

Applying now Theorems 3.1, 3.2, 3.3 we obtain the following result on which almost all our estimates will be based (the constants depend only on the arguments shown):

THEOREM 3.4. *Let* $u(x, t) = \int K(x-y, t)f(y)dy$, *with* K *a kernel of the form* (3.14)

$$K(x, t) = D^{m_j+q+n}K_{j,q}(x, t).$$

(a) *If* f *is in* L_p *for some finite* $p > 1$ *and* $[f]_\alpha < \infty$ *for some positive* $\alpha < 1$, *then* $[u]_\alpha$ *is finite and*

$$[u]_\alpha \leqq C(E, q, \alpha)[f]_\alpha .$$

(b) *If* f *belongs to* L_p, $p > 1$, *then so does* $u(x, t)$ *for every* t, *and*

$$|u|_{L_p, t} \leqq C(E, q, p)|f|_{L_p} .$$

(c) *If* f *belongs to* L_p *for some finite* $p > 1$ *and* $|f|_{1-1/p, L_p} < \infty$, *then* $|u|_{1, L_p}$ *is finite and*

$$|u|_{1, L_p} \leqq C(E, q, p)|f|_{1-1/p, L_p} .$$

3.6. In conclusion we mention a result of interest which is related to Theorem 3.1. We shall not make use of it, though it may be used, for instance, to give an alternative proof of a modification of Theorem 3.4(a); its proof is given in Appendix 4.

THEOREM 3.1A. *Let* $K(x, t)$ *be an infinitely differentiable kernel for* $t > 0$ *satisfying the following two conditions*:

(i) *For every* C^∞ *function* $f(x)$ *with compact support the function*

$$u(x, t) = \int K(x-y, t)f(y)dy$$

is of class C^∞ *in* $t \geqq 0$.

(ii) *There is an integer* $h \geqq n$ *such that for* $t > 0$

$$|D^s K(x, t)| \leqq C_s(|x|^2+t^2)^{-(s+h)/2}, \qquad s = 1, 2, \ldots, \tag{3.16}$$

where the C_s *form an increasing sequence of constants.*

Assume that the support of $f(x)$ *is contained in some fixed sphere* $|x| \leqq R$. *Then, for* $0 < \alpha < 1$,

$$\sum [D^l u]_\alpha \leqq C(l) \sum [D^{l+h-n} f]_\alpha , \qquad l \geqq 0,$$

where the constant $C(l)$ *is independent of* f. *Here summation is over all derivatives of the orders shown.*

4. A Representation Formula in the Inhomogeneous Boundary Value Problem

In Section 2, with the aid of the Poisson kernels, we constructed solutions of (1.7)

$$\begin{aligned} Lu &= f(x, t) & t &> 0 \\ B_j u &= \phi_j(x), & t &= 0, \quad j = 1, \cdots, m, \end{aligned} \tag{4.1}$$

in case $f \equiv 0$. We propose now to construct solutions with $f \not\equiv 0$. Our aim here will not be to form solutions, although with our formulas such solutions can be easily written down, but rather to obtain a representation formula for a C^∞ function u with compact support in $t \geqq 0$ in terms of $Lu(x, t)$ and $B_j u(x, 0)$.

An essential tool is the fundamental solution $\Gamma(P-\bar{P})$, $P = (x, t)$, $\bar{P} = (\bar{x}, \bar{t})$ of the elliptic equation $Lu = 0$ with singularity at $P = \bar{P}$. We refer here to the elegant discussion given by F. John in his book [17], pp. 69—70. There he constructs a fundamental solution having the form

$$\Gamma(P) = |P|^{2m-n-1}\psi\left(\frac{P}{|P|}\right) + q(P)\log|P|, \tag{4.2}$$

where $q(P)$ is a polynomial of degree $2m-n-1$ for $n+1$ even, $2m \geqq n+1$, and $q(P)$ is zero otherwise; $\psi(Q)$ is an analytic function on $|Q| = 1$. From (4.2) it follows that

$$|D^s\Gamma(P)| \leqq \text{constant} \cdot |P|^{2m-n-1-s} \tag{4.3}$$

holds for (a) $s \geqq 0$, in case $n+1$ is odd or $n+1$ is even and greater than $2m$, (b) $s > 2m-n-1$ if $n+1$ is even and not greater than $2m$. If $n+1$ is even and $0 \leqq s \leqq 2m-n-1$, then

$$|D^s\Gamma(P)| \leqq \text{constant} \cdot |P|^{2m-n-s-1}(1+|\log|P||). \tag{4.3$'$}$$

Inspection of the explicit formulas (in [17]) for the fundamental solution shows that the constants in (4.3), (4.3)′ depend only on s, m, n and the ellipticity constant A.

We consider now a C^∞ solution of (4.1) with compact support in $t \geqq 0$ and seek a suitable representation. It will be convenient to extend f to the whole $(n+1)$-space to be of class C^N for N sufficiently large. This may be achieved, for instance, with a appropriate choice of constants λ_p by setting

$$f = f_N(x, t) = \begin{cases} \sum_{p=0}^{N} \lambda_p f(x, -pt) & \text{for } t < 0 \\ f(x, t) & \text{for } t \geqq 0; \end{cases} \tag{4.4}$$

the λ_p depend only on N. Having chosen some large N we now set

$$v(P) = v_N(P) = \int \Gamma(P-\bar{P}) f_N(\bar{P})\, d\bar{P}, \tag{4.5}$$

integration being carried out over the full $\bar{x}$, $\bar{t}$-space. The function v is of class C^{N+2m-1} in the full space (in fact it is in C^∞ for $t \neq 0$) and satisfies $Lv = f_N$. We set further

$$B_j v(x, 0) = \psi_j(x). \tag{4.6}$$

At this point it would be very convenient if we could assert that the following representation for u holds:

$$u(x, t) = v(x, t) + \sum_{j=1}^{m} \int K_j(x-y, t)(\phi_j(y)-\psi_j(y))dy.$$

However. the integrals on the right may not converge. As a substitute we shall obtain a corresponding representation for the derivatives of order l_0 of u, where $l_0 = \max(2m, m_j)$. (Such a formula can also be obtained for some derivatives of lower order.)

THEOREM 4.1. *If u is a C^∞ solution with compact support in $t \geqq 0$ satisfying* (4.1), *then the representation formulas*

$$D^{l_0}u(x, t) = D^{l_0}v(x, t) + \sum_{j=1}^{m} \int D^{l_0}K_j(x-y, t)(\phi_j(y)-\psi_j(y))dy \tag{4.7}$$

hold for $t > 0$. Here v and ψ are defined by (4.5), (4.6), *and f_N is a sufficiently smooth extension of f to the whole space.*

Since u has compact support it is easy to derive from (4.7) a representation of u itself.

In later sections (see corollaries to Theorem 6.1, Theorem 15.1) it is shown that the C^∞ hypothesis in Theorem 4.1 may be replaced by appropriate finite differentiability of u.

If we set $u-v = w$, $\phi_j-\psi_j = B_j w|_{t=0} = \omega_j$, then it is to be proved that

$$D^{l_0}w(x, t) = \sum \int D^{l_0}K_j(x-y, t)\omega_j(y)dy. \tag{4.8}$$

We have from (4.3), (4.3)′: for large $|P|$, $|x|$ and $s \geqq 0$

$$\begin{aligned} D^s w(P) &= O(|P|^{2m-n-1-s}(1+\log|P|)), \\ D_x^{s-m_j}\omega_j(x) &= O(|x|^{2m-n-1-s}(1+\log|x|)), \end{aligned} \tag{4.9}$$

where the logarithmic terms can be dropped if $s > 2m-n-1$. With the aid of (2.13)′ it is seen that the integrals on the right of (4.8) are convergent.

The proof of Theorem 4.1 makes use of a simple uniqueness lemma.[5]

LEMMA 4.1. *Let u be a function of class C^{l_0} in $t \geqq 0$ satisfying*

$$Lu = 0, \qquad t \geqq 0, \tag{4.10}$$

$$B_j u = 0, \qquad t = 0, \quad j = 1, \cdots, m. \tag{4.10$'$}$$

Assume that u and its derivatives up to the order l_0 are absolutely integrable on each plane $t =$ constant > 0, the integrals converging uniformly in t in every finite interval $0 < \varepsilon \leqq t \leqq R$. Assume also that u and its derivatives up to the

[5]The assumptions here can be relaxed considerably. See end of Section 6.

order l_0 possess uniformly bounded L_2 norms on the planes $t =$ constant $\geqq 0$. Finally, suppose that

$$\iint_{t>0} |u(x, t)|^2 dx\, dt < \infty.$$

Then $u \equiv 0$.

Proof: Introducing the Fourier transform of u with respect to the x variables,

$$\hat{u}(x, t) = (2\pi)^{-n/2} \int e^{-ix\cdot\xi} u(x, t) dx, \qquad \xi = (\xi_1, \cdots, \xi_n),$$

we see that, for $t > 0$, $\hat{u}$ is a continuous function in (ξ, t) possessing continuous derivatives with respect to t up to the order l_0. Furthermore, we see that

$$L\left(\xi, \frac{1}{i} D_t\right) \hat{u} = 0, \qquad t > 0. \tag{4.11}$$

Hence, it follows that, for a fixed ξ, $\hat{u}(\xi, t)$ is a linear combination of exponentials $e^{\mu t}$ (possibly multiplied by polynomials), where μ is a root of $L(\xi, -i\mu) = 0$. By our last assumption on u we find that $\int |\hat{u}(\xi, t)|^2 dt < \infty$ for almost all ξ so that for these ξ the exponentials μ have a negative real part. For these ξ, $\hat{u}(\xi, t)$ is a solution of the ordinary differential equation of order $m : M^+(\xi, -iD_t)u = 0$, see page 632. By continuity the same equation holds for all ξ. If we show in addition that

$$B_j\left(\xi, \frac{1}{i} D_t\right) \hat{u} = 0, \qquad t = 0, \quad j = 1, \cdots, m, \tag{4.11$'$}$$

the result will follow. Since according to our Complementing Condition of page 633 the B_j are linearly independent mod M^+, we can infer from (4.11)′ that $D_t^k \hat{u}(\xi, 0) = 0$, $0 \leqq k \leqq m - 1$, and hence that $\hat{u}(\xi, t) \equiv 0$. But then $u(x, t) \equiv 0$.

To establish (4.11)′ it will suffice to show that for every infinitely differentiable function $\varphi(\xi)$ of compact support we have

$$\lim_{t=+0} \int \varphi(\xi) B_j\left(\xi, \frac{1}{i} D_t\right) \hat{u}(\xi, t) d\xi = 0.$$

But this follows readily from (4.10)′ and Parseval's formula, since

$$\begin{aligned} i^{m_j} \lim_{t=+0} \int \varphi(\xi) B_j\left(\xi, \frac{1}{i} D_t\right) \hat{u} d\xi &= \lim_{t=+0} \int \hat{\varphi}(x) B_j u(x, t) dx \\ &= \int \hat{\varphi}(x) B_j u(x, 0) dx = 0. \end{aligned}$$

Here the fact that L_2 norms are uniformly bounded is used when passing to the limit under the integral sign.

Proof of Theorem 4.1: Consider the right-hand side of (4.8) for various derivatives D^{l_0}. As is clear from the obvious compatibility relations which they satisfy for N sufficiently large, they are the derivatives of order l_0 of a C^{2l_0+2} function $g(x, t)$. We easily see that

$$D^{l_0-2m} Lg = 0, \qquad t > 0,$$

and with the aid of Remark 2 (after Theorem 2.1) that

$$D_x^{l_0-m_j} B_j g = D_x^{l_0-m_j} \omega_j, \qquad t = 0, \quad j = 1, \cdots, m.$$

We intend to prove first that the square integrals on planes $t =$ constant $\geqq 0$ of all derivatives of the form $D^k g$, $l_0 \leqq k \leqq 2l_0 + 1$, are uniformly bounded, next that all derivatives of the form $D_x D^k g$, $l_0 \leqq k < 2l_0 + 1$, are square integrable in the upper half-space $t > 0$, and that the same derivatives are absolutely integrable on planes $t =$ constant > 0, the convergence being uniform in any interval $0 < \varepsilon \leqq t \leqq R$. The theorem then follows easily; for, from (4.9) the corresponding derivatives of w have this property and if $h = w - g$, we have

$$\begin{aligned} &D^{l_0-2m} Lh = 0, && t > 0, \\ &D_x^{l_0-m_j} B_j h = 0, && t = 0, \end{aligned} \tag{4.12}$$

with h satisfying the same integrability conditions as g. Lemma 4.1 may then be applied to the functions $D_x^{l_0+1} h$ and we conclude that $D_x^{l_0+1} h \equiv 0$. Thus, $D_x^{l_0} h$ depends only on t. Since, however, $D_x^{l_0} h$ is square integrable on $t =$ constant we also have $D_x^{l_0} h \equiv 0$. Hence all derivatives of h of the form $D^{l_0} D_x^{l_0} h$ vanish. Operating on

$$D_t^{l_0-2m} Lh = 0, \qquad t > 0, \tag{4.12$'$}$$

with $D_x^{l_0-1} D_t$ we infer that $D_t^{l_0+1} D_x^{l_0-1} h = 0$. Operating next on (4.12)$'$ with $D_x^{l_0-2} D_t^2$ we see that $D_t^{l_0+2} D_x^{l_0-2} h = 0$. Operating in turn with $D_t^{l_0+k} D_x^{l_0-k}$, $k = 1, 2, \cdots, l_0$, we conclude in this way that $D^{2l_0} h \equiv 0$ or that h is a polynomial. Since the derivatives $D^{l_0} h$ are square integrable on $t =$ constant, they vanish, i.e., (4.8) is proved.

Thus to conclude the proof of the theorem we show that for $l_0 \leqq l \leqq 2l_0+1$ each function

$$\int D^l K_j(x - y, t)\omega_j(y)dy$$

has uniformly bounded L_2 norms on the planes $t =$ constant. Also that for $l_0 \leqq l \leqq 2l_0$ the first derivatives of these functions are square integrable in the entire half-space $t > 0$, and finally that these last derivatives are absolutely integrable on $t =$ constant > 0, the convergence being uniform in any interval $0 < \varepsilon \leqq t \leqq R$. Consider then a typical term

$$I_j(x, t) = \int D^l K_j(x-y, t)\omega_j(y)dy$$

for $l \geqq l_0$. By (2.6)

$$I_j(x, t) = \int D^l \Delta_x^{(n+q)/2} K_{j,q}(x-y, t)\omega_j(y)dy.$$

If $l - m_j$ is even we may write this, after partial integration, in the form

$$I_j = \int D^l \Delta_x^{(n+q-l+m_j)/2} K_{j,q}(x-y, t)\Delta_y^{(l-m_j)/2}\omega_j(y)dy, \tag{4.13}$$

q being chosen so that $n + q - l + m_j$ is even and positive. By (2.13) and (4.9) there is no contribution from infinity; we see further that for $l_0 \leqq l \leqq 2l_0 + 1$ any term $D_y^{l-m_j}\omega_j$, and hence $\Delta_y^{(l-m_j)/2}\omega_j$, is square integrable. It follows, furthermore, that the functions $D_y^{l-m_j}\omega_j(y)$ for $l_0 \leqq l \leqq 2l_0$ have finite $|\ |_{1-\frac{1}{2}, L_2}$ norms (see (3.13)'); for, $D_y^{l-m_j}\omega_j(y)$ is the boundary value of the function $v_0(y, t) = D_y^{l-m_j}\omega_j(y)\ \zeta(t)$, where $\zeta(t)$ is a C^∞ function of t in $t \geqq 0$ which equals one for $t = 0$ and vanishes for $t > 1$. The function v_0 then has square integrable first derivatives in the half-space $t > 0$.

Applying Theorem 3.4 (b) and (c) for $p = 2$ we find for $l_0 \leqq l \leqq 2l_0 + 1$ that I_j has uniformly bounded L_2 norm on each plane $t =$ constant, and that, for $l_0 \leqq l \leqq 2l_0$, the first derivatives of I_j are square integrable in the half-space $t > 0$.

Furthermore, writing

$$DI_j = \int D^{l+1} \Delta_x^{(n+q-l+m_j)/2} K_{j,q}(x - y, t)\Delta_y^{(l-m_j)/2}\omega_j(y)dy,$$

and noting that $\Delta_y^{(l-m_j)/2}\omega_j(y)$ is absolutely integrable and that, using (2.13),

$$|D^{l+1} \Delta_x^{(n+q-l+m_j)/2} K_{j,q}(x)| \leqq \text{constant} \cdot (|x|^2 + t^2)^{-n-1},$$

we conclude that DI_j is absolutely integrable on every plane $t =$ constant > 0, the convergence being uniform in any interval $0 < \varepsilon \leqq t \leqq R$.

In case $l - m_j$ is odd, $l - m_j = 2k + 1$, we write

$$I_j = \sum_i \int D^l D_{x_i} \Delta_x^{(n+q)/2-k-1} K_{j,q}(x-y, t) D_{y_i} \Delta_y^k \omega_j(y)dy, \tag{4.13$'$}$$

and use the same argument as above deriving the stated integrability properties for each member of the last sum. This completes the proof.

CHAPTER II

THE SCHAUDER ESTIMATES

5. Notation

Let $\mathfrak{D}$ be a domain in E_k (not necessarily bounded) and denote by $\dot{\mathfrak{D}}$ and $\bar{\mathfrak{D}}$ its boundary and closure, respectively. $C^l(\mathfrak{D})$ $(C^l(\bar{\mathfrak{D}}))$ will denote the space of functions possessing continuous derivatives up to order $l \geqq 0$ in $\mathfrak{D}$ $(\bar{\mathfrak{D}})$. For f in $C^l(\mathfrak{D})$ we define the pseudonorms,

$$[f]_l = [f]_l^{\mathfrak{D}} = \text{l.u.b. } |D^l f|,$$

where the least upper bound is taken over all derivatives of order l and over the domain $\mathfrak{D}$, and the norm

$$|f|_l = |f|_l^{\mathfrak{D}} = \sum_{j=0}^{l} [f]_j \,.$$

The subclass of those functions in $C^l(\bar{\mathfrak{D}})$ whose derivatives of order l satisfy a uniform Hölder condition of order α, $0 < a < 1$, is denoted by $C^{l+\alpha}(\mathfrak{D})$. For $f \in C^{l+\alpha}(\mathfrak{D})$ we define the pseudonorm

$$[f]_{l+\alpha} = \text{l.u.b. } \frac{|D^l f(P) - D^l f(Q)|}{|P-Q|^\alpha},$$

where the l.u.b. is over $P \neq Q$ in $\mathfrak{D}$ and all derivatives of order l, and the norm

$$|f|_{l+\alpha} = |f|_l + [f]_{l+\alpha} \,.$$

Thus $[f]_a$ and $|f|_a$ are defined for every number $a \geqq 0$.

We shall make use of the following lemma which is just a consequence of the theorem of the mean.

LEMMA 5.1.[6] *Suppose $\mathfrak{D}$ in E_k has the property that there is a $\delta > 0$ such that every point $P \in \mathfrak{D}$ is the extremity of a segment of length δ lying entirely in $\mathfrak{D}$. Then, given $a \geqq 0$, there are constants $c = c(k, \delta, a)$, $d = d(k, \delta, a)$ depending only on the variables indicated, such that*

$$[f]_b \leqq c[f]_a^{b/a}[f]_0^{1-b/a} + d|f|_0, \qquad 0 \leqq b \leqq a.$$

If $\mathfrak{D}$ is the entire half-space $t > 0$, then we may take $d = 0$.

We shall make use of some other definitions. Let $\Sigma = \Sigma_R : |x|^2 + t^2 < R^2$, $t \geqq 0$, be a half-sphere in x, t-space with $x = (x_1, \cdots, x_n)$. Let $\sigma = \sigma_R$ denote its planar boundary $|x|^2 < R^2$, $t = 0$. For the linear spaces $C^l(\Sigma)$, $C^l(\sigma)$, and $C^{l+\alpha}(\Sigma)$, $C^{l+\alpha}(\sigma)$, $0 < \alpha < 1$, we introduce the seminorms

[6] See for instance the proofs of analogous results in Miranda [26] Section 33 and Douglis, Nirenberg [10] Section 2.

$$[f]_{p,l} = [f]^{\Sigma}_{p,l} = \text{l.u.b. } d_P^{p+l}|D^l f(P)| \leqq R^{p+l}|f|^{\Sigma}_l$$

for p any integer with $p+l \geqq 0$. If $p+l < 0$ we define $[f]_{p,l} = 0$. Here the least upper bound is taken over all derivatives of order l and all points P in Σ, d_P denotes the distance from P to the spherical part of the boundary of Σ, $|x|^2+t^2 = R^2$. Correspondingly we also define

$$\begin{aligned} |f|_{p,l} &= \sum_{j=0}^{l} [f]_{p,j}\,, \\ [f]_{p,l+\alpha} &= \underset{4|P-Q|<d_P,\, d_Q}{\text{l.u.b.}} \; d_P^{p+l+\alpha} \frac{|D^l(P)-D^l(Q)|}{|P-Q|^{\alpha}} \\ &\leqq R^{p+l+\alpha}|f|^{\Sigma}_{l+\alpha}\,, \\ |f|_{p,l+\alpha} &= |f|_{p,l}+[f]_{p,l+\alpha}\,. \end{aligned}$$

The norms $|\phi|_{p,a}$ for functions in $C^a(\sigma)$ are defined in the same way. Finally we set

$$\begin{aligned} [u]_{0,a} &= \widetilde{[u]}_a\,, \\ |u|_{0,a} &= \widetilde{|u|}_a\,. \end{aligned} \tag{5.1}$$

The new seminorms are subject to inequalities analogous to those of Lemma 5.1, proved in a similar way with the aid of the theorem of the mean.

LEMMA 5.2.[6] *Given $p \geqq 0$, $a \geqq b$, there is a constant $\tilde{C} = \tilde{C}(n, p, a)$ (depending only on the variables indicated) such that*

$$[f]_{p,b} \leqq \tilde{C}[[f]^{b/a}_{p,a}[f]^{1-b/a}_{p,0}+[f]_{p,0}].$$

6. The Schauder Estimates for Equations with Constant Coefficients

We consider, as in the previous sections, equations (4.1) with constant coefficients in the half-plane $t \geqq 0$, i.e.,

$$\begin{aligned} Lu &= f, && t > 0, \\ B_j u &= \phi_j\,, && t = 0. \end{aligned} \tag{6.1}$$

Set $l_0 = \max\,(2m, m_j)$ and let l be an integer $\geqq l_0$. Using the notation of Section 5 we shall assume throughout this section that $u(x, t) \in C^{l_0+\alpha}$ and that $f \in C^{l-2m+\alpha}$ in $t \geqq 0$, $\phi_j \in C^{l-m_j+\alpha}$ on $t = 0$, for some positive $\alpha < 1$.

We derive first a basic estimate, for functions with compact support, from which the general inequalities will be deduced.

THEOREM 6.1. *In addition to the preceding conditions, assume that u has compact support. Then $u \in C^{l+\alpha}$ and*

$$[u]_{l+\alpha} \leqq \text{constant } ([f]_{l-2m+\alpha} + \sum_{j=1}^{m} [\phi_j]_{l-m_j+\alpha}), \tag{6.2}$$

where the constant depends only on l, α and the characteristic constant E.

The various norms in (6.2) refer clearly to the half-space $t \geqq 0$ for u and f, and to the plane $t = 0$ for the $\phi_j(x)$.

Proof: The proof of (6.2) is based on the representation of $u(x, t)$, obtained in Section 4, and on Theorem 3.4(a).

Assume first that u is of class C^∞. According to Theorem 4.1 we have, by $(l-l_0)$-fold differentiation of (4.7),

$$\begin{aligned} D^l u &= D^l v + \sum_{j=1}^{m} I_j\,, \\ I_j &= \int D^l K_j(x-y, t)(\phi_j(y)-\psi_j(y))dy \\ &= D^l K_j * (\phi_j-\psi_j), \end{aligned} \tag{6.3}$$

where $v = v_N$ and ψ_j are defined by (4.5), (4.6), with N sufficiently large. We see easily that for any fixed l for N sufficiently large the extended function $f = f_N$ (this enters into the definition of v) of (4.4) satisfies

$$[f_N]_{k-2m+\alpha} \leqq \text{constant} \cdot [f]_{k-2m+\alpha}\,, \qquad l_0 \leqq k \leqq l,$$

where the norm on the left is taken over the whole x, t-space and that on the right over the half-space $t > 0$. The constant depends only on n and N (chosen sufficiently large).

The function $v = v_N$ given by (4.5) satisfies the inequality

$$[v]_{l+\alpha} \leqq \text{constant} \cdot [f_N]_{l-2m+\alpha}\,,$$

where the constant depends only on l, α and E. This inequality is by now well known (see for instance Bers [5]), and may be derived in fact from Theorem 3.1 for the case $t = 0$ (the Hölder-Giraud case, see page 647). Thus, applying the previous inequality, we have

$$[v]_{l+\alpha} \leqq \text{constant} \cdot [f]_{l-2m+\alpha}\,,$$

both norms here being taken in the half-space $t > 0$. From (4.6) we infer that

$$[\psi_j]_{l-m_j+\alpha} \leqq \text{constant} \cdot [v]_{l+\alpha} \leqq \text{constant} \cdot [f]_{l-2m+\alpha}\,, \qquad l_0 \leqq k \leqq l. \tag{6.4}$$

In order to apply Theorem 3.4(a) we use the representations (4.13), (4.13)′ of the I_j:

$$I_j = D^l \Delta^{(n+q-l+m_j)/2} K_{j,q} * \Delta^{(l-m_j)/2}(\phi_j-\psi_j) \tag{6.5}$$

for $l-m_j$ even,

$$I_j = \sum_i D^l D_i \Delta^{(n+q)/2-k-1} K_{j,q} * D_i \Delta^k \omega_j\,, \tag{6.5$'$}$$

for $l-m_j = 2k+1$. Here $q > l-m_j-n$. We may now apply Theorem 3.4(a) to each term in I_j. Using (6.4) we find

$$(6.6)\qquad \begin{aligned}[I_j]_\alpha &\leqq c(E,\alpha)[\phi_j-\psi_j]_{l-m_j+\alpha}\\ &\leqq \text{constant}\cdot([\phi_j]_{l-m_j+\alpha}+[f]_{l-2m+\alpha}).\end{aligned}$$

From (6.3) we find $[D^l u]_\alpha \leqq [D^l v]_\alpha + \sum [I_j]_\alpha$, and, by (6.4), (6.6),

$$[D^l u]_\alpha \leqq \text{constant}\cdot([f]_{l-2m+\alpha}+\sum_{j=1}^{m}[\phi_j]_{l-m_j+\alpha}),$$

which is equivalent to (6.2).

Suppose now that u is merely of class $C^{l_0+\alpha}$, not C^∞, in $t \geqq 0$. In order to carry through the proof it suffices to have the representation (6.3) for $l = l_0$, i.e., (4.7). By differentiation we then obtain (6.3) for the value of l employed, and may proceed as above. But the representation (4.7) for functions $u(x, t)$ of compact support and of class $C^{l_0+\alpha}$ in $t \geqq 0$ is easily obtained by suitably approximating u by C^∞ functions u_n and applying inequality (6.2) for $l = l_0$ to the u_n. (In Section 8 such an approximation procedure is described for a more elaborate situation.)

This completes the proof of the theorem; incidentally we have proved

COROLLARY 1. *The representation* (4.7) *holds for functions u of class $C^{l_0+\alpha}$, $0 < \alpha < 1$, and compact support in $t \geqq 0$.*

Theorem 6.1 contains the basic estimates used to establish the Schauder estimates for equations with variable coefficients. Before treating the general equation we shall need a preliminary consequence of Theorem 6.1 for the special equations (6.1). The notation is that of Section 5.

THEOREM 6.2. *Let $u(x, t)$ be a bounded solution of* (6.1) *of class $C^{l_0+\alpha}$, $0 < \alpha < 1$, in the half-sphere $\Sigma = \Sigma_R : |x|^2+t^2 < R^2$, $t \geqq 0$. Assume that for fixed $l \geqq l_0$,*

$$K = [f]_{2m,\, l-2m+\alpha} + \sum_j [\phi_j]_{m_j,\, l-m_j+\alpha}$$

is finite. Then u is of class $C^{l+\alpha}$ and

$$(6.7)\qquad \widetilde{[u]}_{l+\alpha} \leqq \text{constant}\cdot(K+\widetilde{[u]}_0),$$

where the constant depends only on l, α and E.

Proof: Because of the homogeneity of the norms in (6.7) we may assume that $R = 1$.

(a) We treat first the case that $u \in C^{l+\alpha}$ in Σ. We may then assume that $\widetilde{[u]}_{l+\alpha}$ is finite. Otherwise we first apply (6.7) to u in $\Sigma_{R-\varepsilon}$ and then let $\varepsilon \to 0$ obtaining (6.7) for Σ.

Let P, Q be two points in Σ with $4|P-Q| \leqq d_P, d_Q$ and such that

$$(6.8)\qquad \tilde{A} = d_P^{l+\alpha}\frac{|D^l u(P)-D^l u(Q)|}{|P-Q|^\alpha} \geqq \tfrac{1}{2}\widetilde{[u]}_{l+\alpha}.$$

Let $\zeta(x, t)$ be a non-negative C^∞ function which is identically one in

$|x|^2+t^2 \leqq (|P|+\frac{1}{2}d_P)^2$, and vanishes for $|x|^2+t^2 \geqq (|P|+\frac{3}{4}d_P)^2$, so that $\zeta(P) = \zeta(Q) = 1$. The function ζ can be so chosen that

$$|D^k\zeta| \leqq C(k, n)d_P^{-k}, \qquad k \geqq 0, \tag{6.9}$$

where $C(k, n)$ depends only on k and n. We may apply Theorem 6.1 to the function $v = \zeta u$ to obtain the inequality

$$[v]_{l+\alpha} \leqq \text{constant} \cdot ([Lv]_{l-2m+\alpha} + \sum_j [B_j v(x, 0)]_{l-m_j+\alpha}). \tag{6.10}$$

Now

$$D^{l-2m}Lv = \zeta D^{l-2m}Lu + \Sigma_{k>0} \text{ coefficients} \cdot D^k\zeta \cdot D^{l-k}u.$$

Since $D^k\zeta$ vanishes except for the points in Σ whose distance from its spherical boundary is between $\frac{1}{4}d_P$ and $\frac{1}{2}d_P$ we find fairly easily with the aid of (6.9) and the theorem of the mean, that for $k > 0$

$$[D^k\zeta \cdot D^{l-k}u]_\alpha \leqq \text{constant} \cdot d_P^{-(l+\alpha)}\widetilde{|u|}_{l-k+1},$$

where the constant depends only on α, k and n. Furthermore we find

$$[\zeta D^{l-2m}Lu]_\alpha \leqq \text{constant} \cdot d_P^{-(l+\alpha)}([Lu]_{2m,\, l-2m+\alpha} + \widetilde{|u|}_l).$$

Thus we have

$$[Lv]_{l-2m+\alpha} \leqq \text{constant} \cdot d_P^{-(l+\alpha)}([Lu]_{2m,\, l-2m+\alpha} + \widetilde{|u|}_l).$$

Similarly we find

$$[B_j v(x, 0)]_{l-m_j+\alpha} \leqq \text{constant} \cdot d_P^{-(l+\alpha)}([\phi_j]_{m_j,\, l-m_j+\alpha} + \widetilde{|u|}_l).$$

Here the constants depend only on l, α and n.

Since $\tilde{A} \leqq d_P^{l+\alpha}[v]_{l+\alpha}$ we find, from the preceding paragraph and inequalities (6.10), (6.8), the inequality

$$\tfrac{1}{2}\widetilde{[u]}_{l+\alpha} \leqq \text{constant} \cdot ([Lu]_{2m,\, l-2m+\alpha} + \sum [\phi_j]_{m_j,\, l-m_j+\alpha} + \widetilde{|u|}_l).$$

By Lemma 5.2, the term constant $\cdot\, \widetilde{|u|}_l$ on the right may be estimated by $\frac{1}{4}\widetilde{[u]}_{l+\alpha}$ + constant $\cdot\, \widetilde{[u]}_0$, which on insertion in the preceding yields the desired inequality (6.7).

(b) Suppose now merely that $u \in C^{l_0+\alpha}$ and that $l > l_0$. By case (a) we know that (6.7) holds for $l = l_0$. It follows that for $\Sigma' = \Sigma_{R-\varepsilon}$, $\Sigma'' = \Sigma_{R-2\varepsilon}$ we have

$$[u]^{\Sigma''}_{l_0+\alpha} \leqq \text{constant} \cdot ([f]^{\Sigma'}_{l_0-2m+\alpha} + \sum [\phi_j]^{\sigma}_{l_0-m_j+\alpha} + [u]^{\Sigma'}_0), \tag{6.11}$$

where the constant depends on ε. We propose to show with the aid of (6.11) that $u \in C^{l+\alpha}$. We shall carry out only the first step, the proof that $u \in C^{l_0+1+\alpha}$; the succeeding steps are similar (induction may be used). To prove that $u \in C^{l_0+1+\alpha}$ apply (6.11) to a tangential difference quotient u^h of u, $h = (h_1, \cdots, h_n)$, $|h| < \varepsilon$,

$$u^h = \frac{u(x+h, t)-u(x, t)}{|h|}.$$

We find that $[u^h]^{\Sigma''}_{l_0+\alpha}$ is bounded by a constant independent of h. It follows that u has derivatives of class C^α of the form $D_x D^{l_0} u$. Equation (6.1) being valid, $D_t^{l_0+1} u$ also belongs to class C^α, so that in fact $u \in C^{l_0+1+\alpha}$; and so on for higher derivatives.

This completes the proof of Theorem 6.2.

We conclude this section with an extension of Theorem 6.1.

THEOREM 6.3. *Let $u(x, t)$ be a solution of* (6.1) *of class $C^{l_0+\alpha}$, $0 < \alpha < 1$, in the half-space $t \geqq 0$. Assume that $[f]_{l-2m+\alpha}$ and $\sum_j [\phi_j]_{l-m_j+\alpha}$, for fixed $l \geqq l_0$, are finite and that*

$$M_0 = \varliminf_{R\to\infty} R^{-(l+\alpha)} \max_{\Sigma_R} |u|$$

is finite. Then u is of class $C^{l+\alpha}$ and

$$[u]_{l+\alpha} \leqq \text{constant} \cdot ([f]_{l-2m+\alpha} + \sum [\phi_j]_{l-m_j+\alpha} + M_0),$$

where the constant depends only on l, α and E.

Proof: Let P, Q be arbitrary points in $t \geqq 0$. If we apply Theorem 6.2 to u in $\Sigma = \Sigma_R$, $R \geqq 10(|P|+|Q|)$, we find

$$\begin{aligned} d_P^{l+\alpha} \frac{|D^l u(P)-D^l u(Q)|}{|P-Q|^\alpha} &\leqq \text{constant} \cdot ([f]^{\Sigma}_{2m, l-2m+\alpha} \\ &\quad + \sum_j [\phi_j]^{\sigma}_{m_j, l-m_j+\alpha} + [\widetilde{u}]^{\Sigma}_0) \\ &\leqq \text{constant} \cdot R^{l+\alpha}([f]_{l-2m+\alpha} \\ &\quad + \sum_j [\phi_j]_{l-m_j+\alpha} + R^{-(l+\alpha)} \max_{\Sigma_R} |u|). \end{aligned}$$

Letting $R \to \infty$ through a suitable sequence, and noting that $d_P/R \to 1$, we obtain the desired result.

We note here as an immediate corollary a

THEOREM OF LIOUVILLE TYPE. *Let $u(x, t)$ be a solution of* (6.1) *of class $C^{l_0+\alpha}$, $0 < \alpha < 1$, in the upper half-space $t \geqq 0$. Assume that the functions f, ϕ_j are polynomials of degrees k, $k+2m-m_j$, respectively, and that $\varliminf_{R\to\infty} R^{-(k+2m+\alpha)} \max_{\Sigma_R} |u| = 0$. Then u itself is a polynomial of degree $k+2m$.*

7. Equations with Variable Coefficients

The transition from equations with constant coefficients to equations with variable coefficients is routine (see [10]) but tedious and we shall not carry out every detail. We first consider such equations, as before, in the

$(n+1)$-dimensional half-space $x, t,\ t \geqq 0$. The point (x, t) will be denoted by P. Our equations with variable coefficients have the form

$$(7.1)\qquad \begin{aligned} L(P; D)u(P) &= F(P), && t \geqq 0 \\ B_j(x; D)u(P) &= \Phi_j(x), && t = 0,\quad j = 1, \cdots, m. \end{aligned}$$

Here L and B_j are differential operators of orders $2m$ and m_j respectively; the coefficients of B_j are independent of t. More explicitly, using the notation of page 631,

$$L = \sum_{|\beta| \leqq 2m} a_\beta(P) D^\beta,$$

$$B_j = \sum_{|\gamma| \leqq m_j} b_{j,\gamma}(x) D^\gamma.$$

We write $L = L' + L''$, $B_j = B_j' + B_j''$, where L' and B_j' are the principal parts of L and B_j, i.e., the parts of highest order. We shall make the following assumptions:

(i) *Condition on L.* We assume first that L is uniformly elliptic, that is, that there exists a constant $A > 0$ such that the characteristic form associated with L' satisfies (1)

$$A^{-1}(|\xi|^2 + \tau^2)^m \leqq |L'(P; \xi, \tau)| \leqq A(|\xi|^2 + \tau^2)^m$$

for all real $\xi = (\xi_1, \cdots, \xi_n)$ and real τ, and all points P in the half-space $t \geqq 0$. In addition, in the case of two variables ($n = 1$) we assume that for each P on $t = 0$ and real $\xi \neq 0$ exactly half of the roots τ of the polynomial $L'(P; \xi, \tau)$ lie in the upper half of the complex τ-plane. Furthermore all coefficients of L are bounded in absolute value by a constant b.

(ii) *Complementing Condition of the boundary operators relative to L.* The coefficients of the B_j are bounded in absolute value by the constant b. Furthermore, for every fixed $P^* = (x^*, 0)$ the system with constant coefficients consisting of the elliptic operator $L'(P^*, D)$ and the boundary operators $B_j'(P^*, D)$, $j = 1, \cdots, m$, satisfies the Complementing Condition of page 633 uniformly for all P^*. That is, if Δ_{P^*} denotes the "determinant constant" (1.9) of the system, then

$$\Delta_{P^*} \geqq \Delta > 0$$

for some fixed constant Δ.

(iii) *The coefficients are smooth.* For fixed $l \geqq l_0 = \max(2m, m_j)$, $0 < \alpha < 1$, we shall assume that F and the coefficients of L belong to $C^{l-2m+\alpha}$ and have $|\ |_{l-2m+\alpha}$ norms bounded by k, while Φ_j and the coefficients of B_j belong to $C^{l-m_j+\alpha}$ and have their $|\ |_{l-m_j+\alpha}$ norms bounded by k. (In Theorem 7.2 we consider solutions in a half-sphere $\Sigma = \Sigma_R$ and there, of course, the coefficients of the operators L, B_j are only required to be defined in Σ and to have finite norms there.)

In treating equations in integral (or variational) form we shall assume a slightly altered form of (iii).

In extending the theorems of Section 6 we use the notation of Section 5.

In the following, constants depending only on A, Δ, b, k, l, n, a will be denoted by $C_1, C_2, \cdots$.

THEOREM 7.1. *Let $u(P)$ be of class $C^{l+\alpha}$ in $t \geqq 0$ and be a solution of the boundary value problem* (7.1) *in the half-space $t \geqq 0$. Assume* (i)—(iii) *and assume that $|u|_{l+\alpha}$ is finite. Then*

$$|u|_{l+\alpha} \leqq C_1([F]_{l-2m+\alpha} + \sum [\Phi_j]_{l-m_j+\alpha} + |u|_0). \tag{7.2}$$

THEOREM 7.2. *Let $u(P)$ be a solution of* (7.1) *of class $C^{l_0+\alpha}$* in the hemisphere $\Sigma = \Sigma_R$, $R \leqq 1$, *and assume* (i)—(iii). *Then u belongs to $C^{l+\alpha}$ in Σ and*

$$\widetilde{|u|}_{l+\alpha} \leqq C_2(|F|_{2m,\, l-2m+\alpha} + \sum |\Phi_j|_{m_j,\, l-m_j+\alpha} + \widetilde{|u|}_0). \tag{7.3}$$

We shall first prove Theorem 7.2 using Theorem 7.1. In the proof we shall assume that we already know that u belongs to $C^{l+\alpha}$ in Σ, for this fact may be derived from the estimate (7.3) for $l = l_0$, as in step (b) of the proof of Theorem 6.2, by taking difference quotients. In the proofs of both Theorems 7.1 and 7.2 we make use of Lemmas 5.1, 5.2 in the following form: For every $\varepsilon > 0$ there are constants C_1, $\tilde{C}_1$ depending only on ε, l, n, α such that: 1) for u in the half-space $t \geqq 0$ we have

$$|u|_l \leqq \varepsilon[u]_{l+\alpha} + C_1|u|_0, \tag{7.4}$$

2) in $\Sigma = \Sigma_R$ we have

$$\widetilde{|u|}_l \leqq \varepsilon\widetilde{[u]}_{l+\alpha} + \tilde{C}_1\widetilde{|u|}_0. \tag{7.5}$$

Proof of Theorem 7.2.: The reduction of the theorem to Theorem 7.1 is just the same as the reduction of Theorem 6.2 to Theorem 6.1, and we merely sketch it. Again we may assume that $\widetilde{|u|}_{l+\alpha}$ is finite; otherwise we may first derive (7.3) for u in $\Sigma_{R-\varepsilon}$ and then let $\varepsilon \to 0$, obtaining (7.3) in Σ_R. We may also suppose that $\widetilde{[u]}_{l+\alpha} > \frac{1}{2}\widetilde{|u|}_{l+\alpha}$; for otherwise, (7.3) follows easily from (7.5) with $\varepsilon = \frac{1}{2}$. Thus there are two points P, Q in Σ with $4|P-Q| \leqq d_P, d_Q$, and a particular derivative $D^l u$ such that

$$d_P^{l+\alpha} \frac{|D^l u(P) - D^l u(Q)|}{|P-Q|^\alpha} > \tfrac{1}{2}\widetilde{|u|}_{l+\alpha}.$$

Let ζ be the non-negative function defined in the proof of Theorem 6.2. Applying Theorem 7.1 to the function $v = \zeta u$ we obtain the inequality

$$[v]_{l+\alpha} \leqq C_1([Lv]_{l-2m+\alpha} + \sum |B_j v(x, 0)|_{l-m_j+\alpha} + |v|_0).$$

Following the argument of part (a) of the proof of Theorem 6.2 and using hypothesis (iii) we can establish the inequalities

$$d_P^{l+\alpha}[Lv]_{l-2m+\alpha} \leqq C_3(|Lu|_{2m,\,l-2m+\alpha}+\widetilde{|u|}_l),$$
$$d_P^{l+\alpha}[B_j v(x,0)]_{l-m_j+\alpha} \leqq C_3(|\Phi_j|_{m_j,\,l-m_j+\alpha}+\widetilde{|u|}_l).$$

The remainder of the proof of the theorem follows that of part (a) of the proof of Theorem 6.2 with the aid of (7.5).

We turn now to Theorem 7.1. Its proof involves replacing the coefficients of L' and B_j' (the leading parts of L and B_j) by their fixed values at some point, and taking all remaining terms in the equations (7.1) to the right-hand sides. Then we apply Theorem 6.2 and use (7.4) to estimate these extra terms on the right.

Proof of Theorem 7.1: As in the proof of Theorem 7.2 we may suppose that $[u]_{l+\alpha} > \frac{1}{2}|u|_{l+\alpha}$ so that there are two points P_0, Q, where we may take $P_0 = (0, \cdots, 0, t_0)$, $Q = (y, \tau)$, $\tau \geqq t_0 > 0$, and a particular derivative $D^l u$ such that

$$\tilde{A} = \frac{|D^l u(P_0) - D^l u(Q)|}{|P_0 - Q|^\alpha} > \tfrac{1}{2}|u|_{l+\alpha}\,. \tag{7.6}$$

Now let λ be a positive constant which will be determined later as a function of $A, \Delta, b, k, l, n, \alpha$. We distinguish a number of cases according to the relative positions of P_0 and Q.

(a) $|P_0 - Q| \geqq \lambda$. Then we have

$$\tilde{A} \leqq 2\lambda^{-\alpha}[u]_l$$

which, combined with (7.6) and (7.4), yields (7.2).

(b) $|P_0 - Q| < \lambda$ and $t_0 \geqq 2\lambda$. Then u is a solution of (7.1) in the sphere with center at P_0 and radius 2λ. Since Q is in the concentric sphere with radius λ the desired estimate for $\tilde{A}$, and hence for $|u|_{l+\alpha}$, follows from the interior estimates of Schauder type that were established by Douglis and Nirenberg [10].

The final case is

(c) $|P_0 - Q| < \lambda$, $t_0 < 2\lambda$. Consider the system of equations (7.1) in $\Sigma = \Sigma_{8\lambda}$ and write it in the form

$$\begin{aligned} L'(P_0\,;D)u(x,t) &= (L'(P_0\,;D) - L)u(x,t) + F(x,t) \\ &\equiv f(x,t), && t \geqq 0 \\ B_j'(O;D)u &= (B_j'(O;D) - B_j)u + \Phi_j(x) \equiv \phi_j(x), && t = 0, \end{aligned}$$

where O is the origin, the functions f, ϕ_j being defined by these identities. Since P_0, Q lie well in the interior of $\Sigma = \Sigma_{8\lambda}$ we may apply Theorem 6.2 to obtain

$$\lambda^{l+\alpha}\tilde{A} \leqq C_4([f]^{\Sigma}_{2m,\,l-2m+\alpha} + \sum_j [\phi_j]^{\sigma}_{m_j,\,l-m_j+\alpha} + |u|_0).$$

We shall estimate the terms on the right. Clearly

$$[f]^{\Sigma}_{2m,\,l-2m+\alpha} \leqq [F]^{\Sigma}_{2m,\,l-2m+\alpha} + [(L'(P_0;D)-L'(P;D))u(P)]^{\Sigma}_{2m,\,l-2m+\alpha}+[L''u]^{\Sigma}_{2m,\,l-2m+\alpha}\,.$$

The first term on the right is not greater than $C_5\lambda^{l+\alpha}[F]_{l-2m+\alpha}$. Since the coefficients of L belong to $C^{l-2m+\alpha}$ we see that the coefficients of $L'(P_0;D)-L'(P;D)$ are bounded by constant $\cdot\, k\cdot\lambda^{\alpha}$ (see condition (iii)), and a straightforward analysis shows that

$$[(L'(P_0;D)-L'(P;D))u(P)]^{\Sigma}_{2m,\,l-2m+\alpha} \leqq C_5\lambda^{l+\alpha}(\lambda^{\alpha}[u]_{l+\alpha}+|u|_l).$$

Similarly we find, with the aid of the theorem of the mean, that

$$[L''u]^{\Sigma}_{2m,\,l-2m+\alpha} \leqq C_6\lambda^{l+\alpha}|u|_l\,.$$

Combining these inequalities we obtain

$$\lambda^{-l-\alpha}[f]^{\Sigma}_{2m,\,l-2m+\alpha} \leqq C_5[F]_{l-2m+\alpha} + C_5\lambda^{\alpha}[u]_{l+\alpha}+(C_5+C_6)|u|_l\,.$$

A similar argument yields

$$\lambda^{-l-\alpha}[\phi_j]^{\Sigma}_{m_j,\,l-m_j+\alpha} \leqq C_7[\Phi_j]_{l-m_j+\alpha} + C_7\lambda^{\alpha}[u]_{l+\alpha}+C_7|u|_l\,;$$

inserting into the above inequality for $\tilde{A}$ we obtain

$$\tilde{A} \leqq C_8\{[F]_{l-2m+\alpha}+\sum_j[\Phi_j]_{l-m_j+\alpha}\}+C_8\lambda^{\alpha}|u|_{l+\alpha} +C_8|u|_l+C_4\lambda^{-l-\alpha}|u|_0\,.$$

We now fix λ by setting

$$C_8\lambda^{\alpha}=\tfrac{1}{4}.$$

From (7.6) and the last inequality it follows that

$$|u|_{l+\alpha} \leqq 4C_8\{[F]_{l-2m+\alpha}+\sum[\Phi_j]_{l-m_j+\alpha}\}+C_9|u|_l\,.$$

By (7.4) we have $C_9|u|_l \leqq \frac{1}{2}|u|_{l+\alpha}+C_{10}|u|_0$, and the desired result (7.2) follows.

This completes the proof of Theorem 7.1, which, however, is not fully satisfactory in that we assume the boundedness of u, in contrast to the assumptions in Theorem 6.3. It should be clear, however, that our method of proof would yield an estimate should we allow the solution to grow with a certain speed at infinity, provided we made certain assumptions on the behaviour of the lower order coefficients near infinity. This would require a more careful use of Lemmas 5.1, 5.2.

We shall establish finally the Schauder estimates for a general domain $\mathfrak{D}$, which may be unbounded; the result, even for a half-space as considered in Theorem 7.1, will be a sharper form of the theorem.

In the domain $\mathfrak{D}$ (in $(n+1)$-space) with boundary $\dot{\mathfrak{D}}$ and closure $\bar{\mathfrak{D}}$

we consider a bounded solution of the elliptic equation

$$Lu = F,$$

which satisfies certain boundary conditions

$$B_j u = \Phi_j, \qquad j = 1, \cdots, m,$$

on a portion Γ of the boundary $\dot{\mathfrak{D}}$ of $\mathfrak{D}$; Γ may be the entire boundary. As before, L and B_j are differential operators of orders $2m$ and m_j; we set $l_0 = \max(2m, m_j)$ and let $l \geqq l_0$, $0 < \alpha < 1$, be fixed. Consider a (possibly unbounded) subdomain $\mathfrak{A}$ of $\mathfrak{D}$ with the property that $\dot{\mathfrak{A}} \cap \dot{\mathfrak{D}}$ lies in the interior of Γ (regarding these as sets in the n-dimensional boundary of $\mathfrak{D}$). Under certain smoothness hypotheses on the boundary of $\mathfrak{D}$ and under conditions analogous to (i)—(iii) we shall, roughly, establish the estimate

$$|u|^{\mathfrak{A}}_{l+\alpha} \leqq \text{constant } ([F]^{\mathfrak{D}}_{l-2m+\alpha} + \sum [\Phi_j]^{\Gamma}_{l-m_j+\alpha} + |u|^{\mathfrak{D}}_0),$$

where the norms for the Φ_j will be defined in a fairly obvious way, and where the constant depends on the domains $\mathfrak{A}$, $\mathfrak{D}$ and on certain parameters (as in Theorem 7.1), but is independent of u.

We now describe our assumptions. We shall assume first of all that the portion Γ of the boundary is of class $C^{l+\alpha}$. More precisely we shall assume the following concerning $\mathfrak{A}$ and $\mathfrak{D}$. There is a positive number d such that each point P in $\mathfrak{A}$ within a distance d of $\dot{\mathfrak{D}}$ has a neighborhood U_P with the properties: (a) $\overline{U}_P \cap \dot{\mathfrak{D}} \subset \Gamma$, (b) U_P contains the sphere about P with radius $\frac{1}{2}d$, (c) the set $\overline{U}_P \cap \overline{\mathfrak{D}}$ can be mapped in a one-to-one way onto the closure of a hemisphere $\Sigma_{R(P)}$, $R(P) \leqq 1$, in $(n+1)$-space, with $\overline{U}_P \cap \dot{\mathfrak{D}}$ mapping onto the flat part of the hemisphere, by a mapping T_P which, together with its inverse, is of class $C^{l+\alpha}$; *in fact each component of the mapping, and its inverse, is assumed to have finite* $|\ |_{l+\alpha}$ *norm (where it is defined) bounded by a constant* κ *independent of* P.

Our assumptions preclude the boundary of the domain following back on itself or pinching into a narrow bottleneck.

Concerning the equation and the boundary conditions we shall assume that under each such mapping T_P these go into a system (7.1) in $\Sigma_{R(P)}$ satisfying the hypotheses (i)—(iii)—with the relevant constants A, Δ, etc. independent of P. In addition we assume the coefficients of L and B_j to have finite norms $|\ |_{l-2m+\alpha}$ and $|\ |_{l-m_j+\alpha}$ in $\mathfrak{D}$, respectively, and L to be uniformly elliptic in $\mathfrak{D}$, i.e., for some constant A (as in condition (i)) we have

$$A^{-1}|\Xi|^{2m} \leqq |L'(P, \Xi)| \leqq A|\Xi|^{2m}$$

for real $\Xi = (\xi_1, \cdots, \xi_{n+1})$. After the transformation T_P the functions Φ_j are defined on the flat part σ_P of the boundary of $\Sigma_{R(P)}$. We set

$$\text{(7.7)} \qquad \underset{P}{\text{l.u.b.}}\ |\Phi_j|^{\sigma_P}_{l-m_j+\alpha} = |\Phi_j|^{\Gamma}_{l-m_j+\alpha}.$$

We assume finally that u is bounded and is of class $C^{l_0+\alpha}$ in $\mathfrak{D}+\Gamma$. We now state our main estimate, which includes the results of Theorem 7.1 and essentially of Theorem 7.2.

THEOREM 7.3. *The solution u is of class $C^{l+\alpha}$ in $\mathfrak{A}$ and*

$$|u|_{l+\alpha}^{\mathfrak{A}} \leqq \bar{C}(|F|_{l-2m+\alpha}^{\mathfrak{D}} + \sum |\Phi_j|_{l-m_j+\alpha}^{\Gamma} + |u|_0^{\mathfrak{D}}), \tag{7.8}$$

where the constant $\bar{C}$ depends only on d, κ, and the constants E, α, k and l. (Recall (2.12), $E = A+b+\Delta^{-1}+n+m+\Sigma m_j$.)

Proof: It follows from the interior estimates of Douglis, Nirenberg [10] that $u \in C^{l+\alpha}$ in $\mathfrak{D}$. Furthermore near the boundary we may perform one of the mappings T_P described above, and so imagine that we are dealing with a solution of (7.1) in $\Sigma_{R(P)}$. From Theorem 7.2, we may conclude that $u \in C^{l+\alpha}$ at boundary points of $\mathfrak{A}$ belonging to $\dot{\mathfrak{D}}$, and hence in all of $\mathfrak{A}$.

Consider now any fixed point P in $\mathfrak{A}$. If its distance from $\dot{\mathfrak{D}}$ is greater than d, we may apply the interior Schauder estimates of [10] and infer that the derivatives of u up to order l at P and the difference quotients

$$\frac{|D^l u(P) - D^l u(Q)|}{|P-Q|^\alpha} \qquad \text{for } |P-Q| < \tfrac{1}{4}d$$

are bounded by the right-hand side of (7.8), with the constant depending only on d and E, α, k, l. If the distance of P to $\dot{\mathfrak{D}}$ is not greater than d, we may perform the mapping T_P and consider the transformed equation, with boundary conditions, in $\Sigma_{R(P)}$. It follows easily from our assumptions that the distance of the image of the sphere about P with radius $\frac{1}{4}d$ from the curved boundary of $\Sigma_{R(P)}$ is greater than a constant depending only on d, κ, R. We may then apply Theorem 7.2 in $\Sigma_{R(P)}$ and we find that the derivatives of u up to order l at P and the difference quotients

$$\frac{|D^l u(P) - D^l u(Q)|}{|P-Q|^\alpha} \qquad \text{for } |P-Q| < \tfrac{1}{4}d$$

are bounded by the right-hand side of (7.8), with the constant depending only on d, κ, and $A, \cdots, \alpha$.

Combining these results we see that $|u|_l^{\mathfrak{A}}$ and the difference quotients

$$\frac{|D^l u(P) - D^l u(Q)|}{|P-Q|^\alpha}, \qquad P, Q \in \mathfrak{A},$$

are bounded by the right-hand side of (7.8) except in the case that $|P-Q| > \frac{1}{4}d$. But in that case the difference quotient is bounded by $|u|_l^{\mathfrak{A}} \cdot (4/d)^\alpha$, and the proof of the theorem is complete.

REMARK 1. *In case $\mathfrak{D}$ is bounded and $\Gamma = \dot{\mathfrak{D}}$, we may take $\mathfrak{A} = \mathfrak{D}$ and replace the term $|u|_0^{\mathfrak{D}}$ in* (7.8) *by the L_1 norm of u*

$$\int_{\mathfrak{D}} |u| dV.$$

This follows easily from a general inequality which may be stated as follows: For every $\varepsilon > 0$ there is a constant C depending only on ε and $\mathfrak{D}$ such that

$$|u|_0^{\mathfrak{D}} \leqq \varepsilon |u|_1^{\mathfrak{D}} + C \int_{\mathfrak{D}} |u| dV.$$

REMARK 2. *In case $\mathfrak{D}$ is bounded, $\Gamma = \dot{\mathfrak{D}}$, and the solution u of class $C^{l_0+\alpha}$ is unique, we may take $\mathfrak{A} = \mathfrak{D}$ and omit the term $|u|_0^{\mathfrak{D}}$ on the right of* (7.8). *The constant in* (7.8) *then depends on the equation but is still independent of u.*

Proof: The remark is a consequence of the following assertion: If the solution is unique, then

$$|u|_0^{\mathfrak{D}} \leqq \text{constant } (|Lu|_{l-2m+\alpha}^{\mathfrak{D}} + \sum |B_j u|_{l-m_j+\alpha}^{\Gamma}), \tag{7.9}$$

where the constant depends only on the equation, i.e., is independent of u. Assume that (7.9) does not hold. Then there is a sequence of functions u_n in $C^{l+\alpha}$ with $|u_n|_0^{\mathfrak{D}} = 1$ such that the terms in the brackets of the right-hand side of (7.9) go to zero for the corresponding u_n as $n \to \infty$. From (7.8) we see that the norms $|u_n|_{l+\alpha}$ are uniformly bounded. It follows that a subsequence of the u_n converge in the norm $|\ |_l^{\mathfrak{D}}$ to a solution u in $C^{l+\alpha}(\overline{\mathfrak{D}})$ of the limit equation, which is homogeneous. By uniqueness $u \equiv 0$ contradicting the fact that $|u|_0^{\mathfrak{D}} = 1$.

CHAPTER III

SCHAUDER ESTIMATES FOR EQUATIONS IN INTEGRAL OR VARIATIONAL FORM

8. The Constant Coefficient Case

We consider again a solution of the differential equation $Lu = F$ satisfying $B_j u = \Phi_j$ on the boundary. But now the equations are to be written in a special form and we assume less differentiability of the solution. The arguments will be much the same as in the preceding sections, and we will mainly describe the crucial modifications and additions that are necessary.

In this section we consider the constant coefficient case (4.1), or (6.1), in the half-space $t \geqq 0$, i.e.,

$$\begin{aligned} Lu &= f, & t &> 0, \\ B_j u &= \phi_j, & t &= 0. \end{aligned} \tag{8.1}$$

Let now $l \leqq l_0$ be a fixed integer which is not less than the maximum order of differentiation with respect to t that occurs in the B_j. We shall suppose that (in the notation of page 631)

$$f = \Sigma D^\beta f_\beta, \qquad \phi_j = \sum D_x^\gamma \phi_{j,\gamma}, \tag{8.2}$$

and write

$$B_j = \sum b_{j,\nu} D_x^\nu D_t^{m_j - |\nu|}. \tag{8.3}$$

Here the summations are for $|\beta| \leqq \max(0, 2m-l)$, $|\gamma| \leqq \max(0, m_j - l)$, and $m_j - l \leqq |\nu| \leqq m_j$, $j = 1, \cdots, m$.

The functions f_β, $\phi_{j,\gamma}$ will be assumed to belong to C^α in $t \geqq 0$ for fixed positive $\alpha < 1$ if $l < 2m$, $l < m_j$. If $l \geqq 2m$ we assume $f = f_0$ to be of class $C^{l-2m+\alpha}$; if $l \geqq m_j$ we assume $\phi_j = \phi_{j,0}$ to be of class $C^{l-m_j+\alpha}$. Moreover, we shall assume that the solution u is of class $C^{l+\alpha}$ in $t \geqq 0$. But then we have to explain in what sense (8.1) is satisfied. It is the following integral (weak) sense: For every C^∞ function $\zeta(x, t)$ with compact support in $t > 0$ and for every C^∞ $\zeta(x)$ with compact support, we require that

$$\begin{aligned} &\iint u(x,t) L(D_x, D_t)\zeta(x,t)\,dx\,dt \\ &\qquad = \sum_\beta (-1)^{|\beta|} \iint f_\beta(x,t) D^\beta \zeta(x,t)\,dx\,dt, \\ &\sum_\nu (-1)^{|\nu|} \int D_t^{m_j - |\nu|} u(x,0) b_{j,\nu} D_x^\nu \zeta(x)\,dx \\ &\qquad = \sum_\gamma (-1)^{|\gamma|} \int \phi_{j,\gamma}(x) D_x^\gamma \zeta(x)\,dx, \qquad j = 1, \cdots, m. \end{aligned} \tag{8.1$'$}$$

Our aim is to prove analogues of Theorems 6.1 and 6.2.

For functions u, f_β, $\phi_{j,\gamma}$ defined in $\Sigma = \Sigma_R$ we shall make use of the notation

$$\kappa_0 = \sum_{|\beta| \leqq 2m-l} R^{2m-|\beta|}([f_\beta]_0^\Sigma + [f_\beta]_\alpha^\Sigma R^\alpha) \qquad \text{if } l < 2m,$$

$$\kappa_0 = \sum_{k=0}^{l-2m} [f]_k^\Sigma R^{2m+k} + [f]_{l-2m+\alpha}^\Sigma R^{l+\alpha} \qquad \text{if } l \geqq 2m.$$

For $j = 1, \cdots, m$,

$$\kappa_j = \sum_{|\gamma| \leqq m_j - l} R^{m_j - |\gamma|}([\phi_{j,\gamma}]_0^\sigma + [\phi_{j,\gamma}]_\alpha^\sigma R^\alpha) \qquad \text{if } l < m_j,$$

$$\kappa_j = \sum_{k=0}^{l-m_j} [\phi_j]_k^\sigma R^{m_j+k} + [\phi_j]_{l-m_j+\alpha}^\sigma R^{l+\alpha} \qquad \text{if } l \geqq m_j.$$

THEOREM 8.1. *Let u be a solution of (8.1) of class $C^{l+\alpha}$ in $t \geqq 0$. Assume that u, f_β vanish outside the unit hemisphere Σ_1, while the $\phi_{j,\gamma}$ vanish outside the unit sphere σ_1. Then, with $R = 1$ above,*

$$[u]_{l+\alpha} \leqq \text{constant} \cdot \sum_{j=0}^m \kappa_j,$$

the constant depending only on α and E.

THEOREM 8.2. *Let u be a bounded solution of* (8.1)′ *of class $C^{l+\alpha}$ in the hemisphere $\Sigma = \Sigma_R$, and assume that $\sum_0^m \kappa_j$ is finite. Let P, Q be two points in $\Sigma_{6/7R}$. Then for any derivative $D^l u$ we have*

$$d_P^{l+\alpha} \frac{|D^l u(P) - D^l u(Q)|}{|P-Q|^\alpha} \leqq \text{constant} \cdot \Big(\sum_0^m \kappa_j + \sum_{k=0}^{l} [u]_k R^k\Big), \tag{8.4}$$

where the constant depends only on α and E.

Theorem 6.2 is an analogue of Theorem 2 of [10]; Theorem 8.2 is an even closer analogue of Theorem 2′ of [10].

Let us now prove Theorem 8.2 assuming first that Theorem 8.1 holds. The inequality is invariant under uniform stretching of the independent variables so that we may assume $R = 1$. Our proof follows that of Theorem 6.2. We consider again the function $\zeta(x, t)$ of page 661 and shall apply Theorem 8.1 to the function $v = \zeta u$. However, we must be careful to write Lv and $B_j v(x, 0)$ in a special form. Thus we write

$$\begin{aligned} Lv &= \zeta Lu + \sum_{k>0} \text{coefficients} \cdot D^k \zeta \cdot D^{2m-k} u \\ &= \sum_{\beta,\beta'} \text{coefficient} \cdot D^{\beta'}(f_\beta D^{\beta-\beta'} \zeta) \\ &\qquad + \sum \text{coefficient} \cdot D^\beta (D^\varepsilon u \cdot D^\delta \zeta), \end{aligned}$$

where the last summation is over $|\beta| + |\delta| + |\varepsilon| = 2m$, $|\delta| > 0$, $|\varepsilon| < l$, $|\beta| \leqq \max(0, 2m-l)$.

Similarly

$$\begin{aligned} B_j v &= \sum_{\gamma,\gamma'} \text{coefficient} \cdot D_x^{\gamma'}(\phi_{j,\gamma} D_x^{\gamma-\gamma'} \zeta) \\ &\qquad + \sum \text{coefficient} \cdot D_x^\gamma (D^\varepsilon u \cdot D^\delta \zeta), \end{aligned}$$

where the last summation is over $|\gamma| + |\varepsilon| + |\delta| = m_j$, $|\delta| > 0$, $|\varepsilon| < l$, $|\gamma| \leqq m_j - l$.

Recalling that the left-hand side of (8.4) is not greater than $d_P^{l+\alpha}[v]_{l+\alpha}$, we apply Theorem 8.1 and follow the proof of Theorem 6.2 to obtain (8.4). This involves estimating a large number of terms. We shall indicate the estimate for a typical term occurring in the expression for Lv. For $R = 1$, so that $\frac{1}{7} \leqq d_P \leqq 1$, we have

$$d_P^{l+\alpha}[f_\beta D^{\beta-\beta'} \zeta]_\alpha \leqq \text{constant} \cdot \big([f_\beta]_\alpha d_P^{l+\alpha-|\beta|+|\beta'|} + |f|_0 d_P^{l-|\beta|+|\beta'|}\big).$$

Note that the power of d_P on the right may be negative. It is in order to render this harmless that the assumption $d_P \geqq \frac{1}{7}R$ is made.

The other terms are estimated in a similar way, and we shall regard the proof of Theorem 8.2 as complete.

We turn now to Theorem 8.1 which is proved, as was Theorem 6.1, with the aid of the representation formula given by Theorem 4.1. We must however modify this formula and extend it to solutions of (8.1)′.

We remark first that we may assume $f_\beta \equiv 0$ for $|\beta| < 2m-l$. For, if T represents the operator: integration with respect to t from (x, ∞) to (x, t), then we may represent $D^\beta f_\beta$ with $|\beta| < 2m-l$ in the form

$$D^\beta f_\beta = D^\beta D_t^{2m-l-|\beta|}(T^{2m-l-|\beta|} f_\beta).$$

Furthermore, we clearly have

$$[T^{2m-l-|\beta|} f_\beta]_0 + [T^{2m-l-|\beta|} f_\beta]_\alpha \leqq \text{constant} \cdot ([f_\beta]_0 + [f_\beta]_\alpha).$$

Thus we shall assume $f_\beta \equiv 0$ for $|\beta| < 2m-l$. Consider the representation formula (4.7), where the function v is defined in (4.5) in terms of a "smooth" extension f_N of the function f to the entire space. The particular way in which the f_N was constructed is not too important. Let us now modify f_N by first constructing extensions $f_{\beta,N}$ of the functions f_β by, say, formulas (4.4), and then defining $f_N = \sum D^\beta f_{\beta,N}$; here N is chosen sufficiently large. With this new definition of f_N the representation (4.7) still holds (for C^∞ data) with $\psi_j(x) = B_j v(x, 0)$ and, for $P = (x, t)$,

$$v(P) = \sum D^\beta \int \Gamma(P-\bar{P}) f_{\beta,N}(\bar{P}) d\bar{P}, \tag{8.5}$$

where summation is for $|\beta| = \max(0, 2m-l)$.

In case $l < m_j$ it will be convenient to express B_j in the form

$$B_j = \sum b_{j,\gamma,\mu} D_x^\gamma D^\mu,$$

where summation is for $|\gamma| = m_j - l$, $|\gamma| + |\mu| = m_j$.

The new representation formula to be established asserts the existence of a polynomial V of degree at most $l_0 - l - 1$ such that

$$\begin{aligned} D^l u(x, t) + V &= D^l v(x, t) \\ &\quad + \sum_{j,\gamma} \int D^l D_x^\gamma K_j(x-y, t) \phi_{j,\gamma}(y) dy - \sum_{j=1}^m I_j(x, t), \end{aligned} \tag{8.6}$$

where

$$I_j = \sum \int D^l b_{j,\gamma,\mu} D_x^\gamma K_j(x-y, t) D^\mu v(y, 0) dy \qquad \text{if } l < m_j,$$

$$I_j = \int D^l K_j(x-y, t) B_j v(y, 0) dy \qquad \text{if } l \geqq m_j.$$

Before proving (8.6) we observe that the integrals on the right converge provided that, in $t \geqq 0$, $f_\beta \in C^\alpha$ if $l < 2m$, $f \in C^{l-2m+\alpha}$ if $l \geqq 2m$, $\phi_{j,\gamma} \in C^\alpha$ if $l < m_j$, $\phi_j \in C^{l-m_j+\alpha}$ if $l \geqq m_j$. This may be seen with the aid of (8.5), (4.3), (4.3)′ and (2.13)′ just as in the proof of Theorem 4.1.

Theorem 8.1 follows from (8.6) in essentially the same way as Theorem 6.1 did from the representation (4.7), and the details of the argument may be left to the reader. We confine ourselves here only to a few remarks concerning the proof. Theorem 3.4(a) is to be applied to all boundary integrals in (8.6) with the exception of the integrals

$$\int D^l D_x^\gamma K_j(x-y, t)\phi_{j,\gamma}(y)dy, \qquad |\gamma| < m_j - l.$$

The estimation of the Hölder continuity in, say, $|P| < 2$ of these integrals follows in a more elementary way, since here the kernels are not very singular on $t = 0$. We note further that in estimating the right-hand side of (8.6) one has to distinguish the cases $l < 2m$, $l \geqq 2m$ in addition to the two cases for I_j. In this way one estimates the Hölder continuity of the right-hand side of (8.6) in $|P| < 2$. Since $u(P) \equiv 0$ for $|P| > 1$, this estimate may then be used to estimate the polynomial V, in a fairly obvious way, leading to the desired estimate for $D^l u$ itself.

Thus in order to complete the discussion of Theorems 8.1, 8.2 we shall verify (8.6). Suppose first that all functions concerned are of class C^∞. Then not only are the integrals on the right of (8.6) convergent, but in fact their derivatives may be obtained by differentiation under the integral signs. and (8.6) is an immediate consequence of our representation formula (4.7),

We have now to eliminate the C^∞ hypothesis, by approximation of u by means of C^∞ functions as in the proof of Theorem 6.1. However, this approximation is not so obvious here, and we shall carry it out. To this end we smooth out the function u by convolution with a C^∞ kernel (the Friedrichs mollifier). Let $j(r) \geqq 0$ be a C^∞ function of one variable with support in $|r| \leqq 1$, such that

$$\int_{-\infty}^{\infty} j(r)dr = 1.$$

Set for $\varepsilon, \varepsilon' > 0$

$$J_\varepsilon u(x, t) = \varepsilon^{-n} \int \prod_{i=1}^{n} j\left(\frac{x_i - y_i}{\varepsilon}\right) u(y_1, \cdots, y_n, t)dy$$

in $t \geqq 0$, integration being over entire n-space, and

$$J_{\varepsilon,\varepsilon'} u(x, t) = \frac{1}{\varepsilon'} \int_0^\infty j\left(\frac{t+\varepsilon'-s}{\varepsilon'}\right) J_\varepsilon u(x, s)ds;$$

$J_{\varepsilon,\varepsilon'} u$ is C^∞ in $t \geqq 0$, while $J_\varepsilon u$ is C^∞ in the x variables.

It is easily seen that $J_{\varepsilon,\varepsilon'} u$ is a solution of

$$LJ_{\varepsilon,\varepsilon'} u = \sum D^\beta J_{\varepsilon,\varepsilon'} f_\beta,$$

while $J_\varepsilon u$ satisfies (8.1)′ with f_β, $\phi_{j,\gamma}$ replaced by $J_\varepsilon f_\beta$, $J_\varepsilon \phi_{j,\gamma}$.

Let D^l be any differentiation operator of order l. According to (8.6) for C^∞ functions we have, with v defined by (8.5), and $V(\varepsilon, \varepsilon')$ a suitable polynomial of degree $l_0 - l - 1$ with coefficients depending on $\varepsilon, \varepsilon'$,

$$\begin{aligned} D^l J_{\varepsilon,\varepsilon'} u + V(\varepsilon, \varepsilon') &= D^l J_{\varepsilon,\varepsilon'} v \\ &\quad + \sum_j \int D^l K_j(x-y, t) B_j[J_{\varepsilon,\varepsilon'} u(y, \tau) - J_{\varepsilon,\varepsilon'} v(y, \tau)]_{\tau=0} dy. \end{aligned}$$

Now let $\varepsilon' \to 0$ keeping $\varepsilon > 0$ fixed. Then we find, from the fact that

$u \in C^{l+\alpha}$ and the related assumptions of f_β and $\phi_{j,\gamma}$, that for a suitable polynomial $V(\varepsilon)$,

$$D^l J_\varepsilon u + V(\varepsilon) = D^l J_\varepsilon v + \sum_j \int D^l K_j(x-y, t) B_j[J_\varepsilon u(y, \tau) - J_\varepsilon v(y, \tau)]_{\tau=0} dy.$$

It is important to remark that the expressions $D^l J_\varepsilon v$, $B_j J_\varepsilon v$ are well defined.

Now we may replace $B_j J_\varepsilon u$ at $t = 0$ by $\sum D_y^\gamma J_\varepsilon \phi_{j,\gamma}$, and may also write

$$\int D^l K_j(x-y, t) B_j J_\varepsilon v\, dy = \sum \int D^l b_{j,\gamma,\mu} D_x^\gamma K_j(x-y, t) D^\mu J_\varepsilon v\, dy$$

if $l < m_j$, so that

$$D^l J_\varepsilon u + V(\varepsilon) = D^l J_\varepsilon v + \sum_{j,\gamma} \int D^l D_x^\gamma K_j(x-y, t) J_\varepsilon \phi_{j,\gamma} dy - \sum_{j=1}^m \tilde{I}_j,$$

where

$$\tilde{I}_j = \sum \int D^l b_{j,\gamma,\mu} D_x^\gamma K_j(x-y, t) D^\mu J_\varepsilon v\, dy \qquad \text{if } l < m_j,$$
$$\tilde{I}_j = \int D^l K_j(x-y, t) B_j J_\varepsilon v\, dy \qquad \text{if } l \geqq m_j.$$

The identity (8.6) itself then follows by letting $\varepsilon \to 0$.

9. Variable Coefficients

In this section we shall treat equation (7.1) with variable coefficients in general domains, but we start by considering equations in the hemisphere $\Sigma = \Sigma_R$, $R \leqq 1$, in $t \geqq 0$ and solutions u of class $C^{l+\alpha}$, $0 < \alpha < 1$, where $l_0 \geqq l \geqq$ maximum order of t differentiation occurring in the B_j.

For the operators, for f and for ϕ, we assume the forms

$$\begin{aligned} L &= \sum D^\beta a_{\beta,\mu}(P) D^\mu, \\ B_j &= \sum D_x^\gamma b_{j,\gamma,\mu}(x) D^\mu, \\ F &= \sum D^\beta F_\beta, \quad \Phi_j = \sum D_x^\gamma \Phi_{j,\gamma}, \end{aligned} \tag{9.1}$$

with summations over $|\beta| \leqq \max(0, 2m-l)$, $|\mu| \leqq l$, $|\gamma| \leqq \max(0, m_j - l)$, $j = 1, \cdots, m$.

Now to specify our assumptions. We shall assume (i) and (ii) of Section 7, where by the coefficients of the operators L, B_j we mean now the functions $a_{\beta,\mu} b_{j,\gamma,\mu}$. In addition we assume

$$\text{(iii)}' \qquad \begin{aligned} &F_\beta \in C^\alpha \text{ if } l < 2m, \quad F \in C^{l-2m+\alpha} \text{ if } l \geqq 2m, \\ &\Phi_{j,\gamma} \in C^\alpha \text{ if } l < m_j, \quad \Phi \in C^{l-m_j+\alpha} \text{ if } l \geqq m_j. \end{aligned}$$

Furthermore we shall assume, respecting the coefficients, that the following norms are bounded by k: $|a_{\beta,\mu}|_\alpha$ if $l < 2m$, $|a_{0,\mu}|_{l-2m+\alpha}$ if $l \geqq 2m$, $|b_{j,\gamma,\mu}|_\alpha$ if $l < m_j$, $|b_{j,0,\mu}|_{l-m_j+\alpha}$ if $l \geqq m_j$.

We shall assume the solution to be of class $C^{l+\alpha}$ in Σ_R. By solution we mean the following analogue of (8.1)′: For every C^∞ function $\zeta(x, t)$ with

compact support in the interior of Σ_R and for every C^∞ function $\zeta(x)$ with compact support in $|x| < R$ we require that

$$
\begin{aligned}
&\sum (-1)^{|\beta|} \iint a_{\beta,\mu} D^\mu u \cdot D^\beta \zeta(x, t) dx\, dt \\
&\qquad = \sum (-1)^{|\beta|} \iint F_\beta D^\beta \zeta(x, t) dx\, dt, \\
&\sum (-1)^{|\gamma|} \int b_{j,\gamma,\mu} D^\mu u(x, 0) \cdot D_x^\gamma \zeta(x) dx \\
&\qquad = \sum (-1)^{|\gamma|} \int \Phi_{j,\gamma} D_x^\gamma \zeta(x) dx, \qquad j = 1, \cdots, m.
\end{aligned} \tag{9.2}
$$

We give now an analogue of Theorem 7.2. (The constant C_2' in the statement depends only on the same parameters as the constant C_2 of Theorem 7.2.)

THEOREM 9.1. *Let u be a solution of* (9.2) *of class $C^{l+\alpha}$ in $\Sigma = \Sigma_R$, $R \leqq 1$, and assume* (i) *and* (ii) *of Section* 7 *and* (iii)′. *Then*

$$
\widetilde{|u|}_{l+\alpha} \leqq C_2' \left(\sum_{j=0}^{m} K_j + \widetilde{|u|}_0 \right), \tag{9.3}
$$

provided the right side is finite.

Here

$$
\begin{aligned}
K_0 &= \sum_\beta |F_\beta|_{2m-|\beta|,\alpha} && \text{if } l < 2m, \\
K_0 &= |F|_{2m,\, l-2m+\alpha} && \text{if } l \geqq 2m,
\end{aligned}
$$

and for $j = 1, \cdots, m$

$$
\begin{aligned}
K_j &= \sum_\gamma |\Phi_{j,\gamma}|_{m_j-|\gamma|,\alpha} && \text{if } l < m_j, \\
K_j &= |\Phi|_{m_j,\, l-m_j+\alpha} && \text{if } l \geqq m_j.
\end{aligned}
$$

Our proof of this theorem is modeled not as much on that of Theorem 7.2 as on the proofs of Theorems 1 and 4′ of [10]. We also make use of the interior estimate of Theorem 4′ (with $s = 2m-l$, $\sigma = 0$, in the notation of that theorem, in case $l < 2m$).

Proof of Theorem 9.1: As in previous arguments we may assume that $\widetilde{|u|}_{l+\alpha}$ is finite; otherwise we may first derive (9.3) for u in $\Sigma_{R-\varepsilon}$ and then let $\varepsilon \to 0$. We may also suppose that $\widetilde{[u]}_{l+\alpha} > \frac{1}{2}\widetilde{|u|}_{l+\alpha}$; for, otherwise (9.3) follows easily from (7.5) with $\varepsilon = \frac{1}{2}$. Thus there are two points P_0, Q in Σ with $4|P_0-Q| \leqq d_{P_0}, d_Q$ and a particular derivative $D^l u$ such that

$$
\tilde{A} = d_{P_0}^{l+\alpha} \frac{|D^l u(P_0) - D^l u(Q)|}{|P_0 - Q|^\alpha} > \tfrac{1}{2}\widetilde{|u|}_{l+\alpha}. \tag{9.4}
$$

We may take $P_0 = (x_0, t_0)$, $Q = (y, \tau)$, $\tau \geqq t_0 > 0$.

Let $\lambda \leqq \frac{1}{6}$ be a positive constant which will be determined later as a function of $R, A, \Delta, \cdots, \alpha$. We distinguish several cases according to the relative positions of P_0, Q.

(a) $|P_0-Q| \geqq \lambda R$. Then

$$\tilde{A} \leqq \text{constant} \cdot \lambda^{-\alpha}[\widetilde{u}]_l .$$

Combined with (9.4) and (7.5) this yields easily the result (9.3).

(b) $t_0 \geqq \frac{1}{2}d_{P_0}$. Then u is a solution of (9.2) in the sphere with center P_0 and radius $\frac{1}{2}d_{P_0}$. Since Q lies in the concentric sphere with half the radius the result follows again with the aid of the interior estimates of [10].

(c) $|P_0-Q| < \lambda R$, $2\lambda R \leqq t_0 \leqq \frac{1}{2}d_{P_0}$. Then as is easily seen the sphere with center P_0 and radius $2\lambda R$ lies in Σ_R. Since Q is in the concentric sphere with half the radius, the desired estimate for $\tilde{A}$, and hence for $[\widetilde{u}]_{l+\alpha}$, follows from the interior estimates (Theorem 4′ in case $l < 2m$, otherwise Theorem 4) of [10].

The last case is

(d) $|P_0-Q| < \lambda R$, $t_0 < \frac{1}{2}d_{P_0}$ and $t_0 < 2\lambda R$. In this case the hemisphere Σ' with center $(x_0, 0)$ and radius $r = \min(\frac{7}{8}d_{P_0}, 4\lambda R)$ lies in Σ_R; in fact its distance from the curved boundary of Σ_R is at least $\frac{1}{8}d_{P_0}$. Furthermore the points P_0, Q lie in the concentric hemisphere with radius $\frac{6}{7}r$. We are thus in a position to apply Theorem 8.2 to u in Σ' once we rewrite the equations as

$$L'(P_0\,;D)u \equiv \sum_{|\beta|+|\mu|=2m} D^\beta a_{\beta,\mu}(P_0)D^\mu u(P) = \sum D^\beta f_\beta(P),$$

where

$$f_\beta(P) \equiv \sum_{|\beta|+|\mu|=2m} (a_{\beta,\mu}(P_0)-a_{\beta,\mu}(P))D^\mu u(P) - \sum_{|\beta|+|\mu|<2m} a_{\beta,\mu}(P)D^\mu u(P)+F_\beta,$$

and at $t = 0$

$$B_j'(x_0\,;D)u \equiv \sum_{|\gamma|+|\mu|=m_j} D_x^\gamma b_{j,\gamma,\mu}(x_0)D^\mu u = \sum D_x^\gamma \phi_{j,\gamma},$$

where

$$\phi_{j,\gamma} \equiv \sum_{|\gamma|+|\mu|=m_j} (b_{j,\gamma,\mu}(x_0)-b_{j,\gamma,\mu}(x))D^\mu u(x) - \sum_{|\gamma|+|\mu|<m_j} b_{j,\gamma,\mu}(x)D^\mu u(x)+\Phi_{j,\gamma}.$$

These equations should of course be understood in the integral sense analogous to (9.2).

We claim now that by applying Theorem 8.2 we may obtain the inequality

$$\tilde{A} \leqq \tilde{C}\left(\sum_{j=0}^{m} K_j + r^\alpha|\widetilde{u}|_{l+\alpha}+|\widetilde{u}|_l\right), \tag{9.4$'$}$$

where the constant $\tilde{C}$ depends only on E, α, l.

To see this we shall only consider some typical terms which occur in applying Theorem 8.2. We observe first that in Σ' we have

$$|a_{\lambda,\mu}(P_0)-a_{\lambda,\mu}(P)| \leqq kr^\alpha \text{ (see condition (iii)}').$$

Suppose $l < 2m$. Then

$$r^l|(a_{\lambda,\mu}(P_0)-a_{\lambda,\mu}(P))D^\mu u|_0^{\Sigma'} \leqq \text{constant} \cdot kr^\alpha|\widetilde{u}|_l$$

since $r \leqq \frac{7}{8} d_P$ and the distance of Σ' from $|x|^2+t^2 = R^2$ is at least $\frac{1}{8}d_P$; also

$$r^{l+\alpha}[(a_{\lambda,\mu}(P_0)-a_{\lambda,\mu}(P))D^\mu u]_\alpha^{\Sigma'} \leqq \text{constant} \cdot kr^\alpha|\widetilde{u}|_{l+\alpha}.$$

Furthermore, if $|\beta| < 2m-l$,

$$r^{2m-|\beta|}(|a_{\beta,\mu}D^\mu u|_0^{\Sigma'}+r^\alpha|a_{\beta,\mu}D^\mu u|_\alpha^{\Sigma'}) \leqq \text{constant} \cdot kr|\widetilde{u}|_{l+\alpha}$$

with similar estimates for the terms occurring in $\phi_{j,\gamma}$ and for $l \geqq 2m$. Since $r \leqq r^\alpha$ and $\sum_{k=0}^{l} [u]_k^{\Sigma'} R^k \leqq \text{constant} \cdot |\widetilde{u}|_l$, we are led to (9.4)′.

We now choose λ as the largest number satisfying

$$\lambda \leqq \tfrac{1}{6}, \qquad \tilde{C}(4\lambda R)^\alpha \leqq \tfrac{1}{4}.$$

(Note that with this choice of λ the factor $\lambda^{-\alpha}$ occurring in case (a) is bounded independent of R for $R \leqq 1$.) Combining (9.4), (9.4)′ and using (7.5) in the usual way we obtain (9.3), completing the proof of Theorem 9.1.

Before passing on to general domains we prove an extension of Theorem 9.1. Consider again the situation of Theorem 9.1 and let us make the following additional assumptions. Let p be an integer $\geqq l$ of Theorem 9.1. Replace (iii)′ by the stronger assumption

$$\text{(iii)}'' \qquad \begin{aligned} &F_\beta \in C^{p-l+\alpha} \text{ if } l < 2m, \ F \in C^{p-2m+\alpha} \text{ if } l \geqq 2m, \\ &\Phi_{j,\gamma} \in C^{p-l+\alpha} \text{ if } l < m_j, \ \Phi \in C^{p-m_j+\alpha} \text{ if } l \geqq m_j. \end{aligned}$$

Furthermore we shall assume that the coefficients have the following norms bounded by k: $|a_{\beta,\mu}|_{p-l+\alpha}$ if $l < 2m$, $|a_{0,\mu}|_{p-2m+\alpha}$ if $l \geqq 2m$, $|b_{j,\gamma,\mu}|_{p-l+\alpha}$ if $l < m_j$, $|b_{j,0,\mu}|_{l-m_j+\alpha}$ if $l \geqq m_j$.

THEOREM 9.2. *Under the conditions of Theorem* 9.1 *and under the stronger hypothesis* (iii)″ *the solution u is of class $C^{p+\alpha}$ in Σ and*

$$|\widetilde{u}|_{p+\alpha} \leqq C_2'' \left(\sum_{j=0}^{m} K_j' + |\widetilde{u}|_0\right),$$

where C_2'' depends only on p and the same parameters which enter in the dependence of C_2.

The K_j' are defined as follows: each constant K_j of Theorem 9.1 is a sum of norms of the form $|\ |_{a,b}$. The constant K_j' is formed by replacing b by $p-l+b$ in the terms of K_j.

In proving Theorem 9.2 we shall make use of the following lemma which is proved in Appendix 6.

LEMMA 9.1. *Let v be of class $C^{1+\alpha}$ in the interior of Σ_R and be a solution of*

$$D_t^k v = \sum_{|\nu|<k} D^\nu v_\nu \tag{9.5}$$

in Σ_R for some fixed $k > 0$, where the functions v, $D_x v$, v_ν belong to C^α in Σ_R. Then $D_t v \in C^\alpha$ in Σ_R.

We understand (9.5), as usual, in the distribution sense, i.e. that, for every C^∞ function $\zeta(x, t)$ with compact support in the interior of Σ_R, we have

$$(-1)^k \iint v D_t^k \zeta \, dx \, dt = \sum (-1)^{|\beta|} \int v_\beta D^\beta \zeta \, dx \, dt.$$

Lemma 9.1 has an L_p, $p > 1$, analogue which can be proved in a similar way. See the end of Appendix 6.

LEMMA 9.1′. *Let v be a solution of (9.5) in Σ_R having first derivatives in L_p in every subdomain whose closure lies in the interior of Σ_R. Assume that for every $\delta > 0$ the functions v, $D_x v$, v_ν belong to L_p in $\Sigma_{R-\delta}$. Then the same is true for $D_t v$.*

Before proving Theorem 9.2 we mention that the last part of its proof yields the following useful result

LEMMA 9.1″. *Let v be a solution of an equation*

$$Lv = f$$

of order k in the interior of a hemisphere Σ_R and assume, for simplicity, that the coefficients of the equation belong to C^∞ in the closure of Σ_R. Assume that the flat boundary is nowhere characteristic for the equation (which need not however be elliptic). Assume that the derivatives up to order $j < k$ of the solution belong to $C^\alpha(L_p, p > 1)$, in every domain whose closure lies in the interior of Σ_R. Assume also that for every $\delta > 0$ the functions $D^i u$, $D_x D^i u$, $0 \leqq i < j$, belong to $C^\alpha(L_p)$ in $\Sigma_{R-\delta}$. Then the same is true for $D_t^j u$.

Proof of Theorem 9.2: We shall carry out the proof only for the case $p = l+1$. To treat higher p, for instance $p = l+2$, we may, because of our assumptions on the coefficients, and because we have the result for $p = l+1$, write the system (9.1) (or (9.2)) in the same form, but with l replaced by $l+1$, after carrying out some first order differentiations.

To prove the theorem for the case $p = l+1$ it suffices to show that $u \in C^{l+1+\alpha}$ in Σ_R. The desired estimate then is to be obtained from Theorem 9.1 applied to (9.1), rewritten as indicated with l replaced by $l+1$. By the interior estimates, Theorem 4′ of [10], we know that $u \in C^{l+1+\alpha}$ in the interior of Σ_R.

We start the proof by showing that derivatives of the form $D_x D^l u$ belong to C^α in Σ_R. This we do by taking difference quotients of (9.1) in x-directions. If h is a vector perpendicular to the t-axis, then for $|h| < \delta/2$ the difference quotient

$$g^h(x) = \frac{g(x+h, t) - g(x, t)}{|h|}$$

of a function $g(x)$ defined in Σ_R is well defined for $P = (x, t)$ in $\Sigma_{R-\delta/2}$. We can see easily that $u^h(x)$ is a solution of the differenced equations

$$Lu^h = \sum D^\beta a_{\beta,\mu}(P)D^\mu u^h(P) = \sum D^\beta(F^h_\beta - a^h_{\beta,\mu}(P)D^\mu u(P+h)),$$

$$B_j u^h = \sum D^\gamma_x b_{j,\gamma,\mu}(x)D^\mu u^h(x, 0) = \sum D^\gamma_x(\Phi^h_{j,\gamma} - b^h_{j,\gamma,\mu}(x)D^\mu u(x+h, 0)),$$

in the integral sense of course. Applying Theorem 9.1 we find that since $\Sigma_{R-\delta}$ is a distance $\delta/2$ from the curved boundary of $\Sigma_{R-\delta/2}$,

$$|u^h|^{\Sigma_{R-\delta}}_{l+\alpha} \leqq \text{constant independent of } h.$$

Letting $h \to 0$ we find, since h may have any x-direction, that $|D_x u|^{\Sigma_{R-\delta}}_{l+\alpha}$ is finite for any fixed $\delta > 0$, i.e., $D_x D^l u \in C^\alpha$ in Σ_R.

To conclude the proof of the theorem we need only show that $D_t^{l+1} u \in C^\alpha$. In case $l \geqq 2m-1$ this follows from the fact that we may solve for $D_t^{l+1} u$ in terms of $D_t^{l-2m+1} F$ and the other derivatives $D_x D^l u$ from the equation $D_t^{l-2m+1}(Lu - F) = 0$. Thus we need only consider the case $l < 2m-1$.

Let us write equation (9.1) in the form

$$D_t^{2m-l}(aD_t^l u) + \text{other terms} = \sum D^\beta F^\beta,$$

where $a = a_{\beta,\mu}$ for $\beta = (0, \cdots, 0, 2m-l)$, $\mu = (0, \cdots, 0, l)$. Because of ellipticity, $a \neq 0$. Because of our conditions on the coefficients, and because we have shown that $D_x D^l u \in C^\alpha$, we may write the equation in the form

$$D_t^{2m-l}(aD_t^l u) = \sum_{|\nu| < 2m-l} D^\nu v_\nu,$$

where the v_ν belong to C^α in Σ_R. We may therefore apply Lemma 9.1 to the function $v = aD_t^l u$ to conclude that $D_t(aD_t^{\tilde{l}} u) \in C^\alpha$. It now follows easily, since $a \neq 0$, that $D_t^{l+1} u$ exists and is in C^α in Σ_R, completing the proof of the theorem.

Finally we describe briefly the analogue of Theorem 7.3, i.e., Schauder estimates for a uniformly elliptic equation (9.1) in integral form in a general domain $\mathfrak{D}$ which as in Section 7 may be unbounded. Here the boundary conditions are satisfied on a portion Γ of the boundary. We assume that in terms of local coordinates x, t near a point on the boundary Γ, with $t = 0$ on Γ, the boundary conditions have the form of (9.1) (or really (9.2)). It is easily seen that the maximum order of differentiation with respect to t which occurs in the B_j is independent of the particular choice of local coordinates. We take $l \leqq l_0$ to be fixed and not less than this number at every boundary point. Let $p \geqq l$ be a fixed integer.

The statement of the result is very similar to Theorem 7.3. In fact, we shall assume that we have the same geometric configuration except that the boundary is assumed to be of class $C^{\tilde{l}+\alpha}$ (as defined there) where $\tilde{l} = \max(l_0, p)$. Under the local mappings T_P, described in the condition

for Theorem 7.3, we shall assume that the equation and boundary conditions go over into the form (9.1) (really (9.2)) and satisfy the hypotheses (i), (ii), (iii)″ of Theorem 9.2.

We shall assume that the norms $|\Phi_{j,\gamma}|^{\sigma_P}_{p-l+\alpha}$ if $l < m_j$, or $|\Phi|^{\sigma_P}_{p-m_j+\alpha}$ if $l \geqq m_j$, and $|F_\beta|^{\Sigma_{R(P)}}_{p-l+\alpha}$ if $l < 2m$, or $|F|_{p-2m+\alpha}$ if $l \geqq 2m$, are bounded by a constant K. We assume finally that u is bounded and is of class $C^{l+\alpha}$ in $\mathfrak{D}+\Gamma$. Then we have

THEOREM 9.3. *The solution u is of class $C^{p+\alpha}$ in $\bar{\mathfrak{A}}$ and*

$$|u|^{\mathfrak{A}}_{p+\alpha} \leqq \text{constant} \cdot (K+|u|^{\mathfrak{D}}_0), \tag{9.6}$$

where the constant depends only on d, κ, p and the constants occurring in conditions (i), (ii), (iii)″.

The proof of the theorem is similar to that of Theorem 7.3 using Theorem 9.2 in place of Theorem 7.2, and will not be repeated.

Again we have, in case $\mathfrak{D}$ is bounded, $\Gamma = \dot{\mathfrak{D}}$, $\mathfrak{A} = \mathfrak{D}$:

REMARK 1. *We may replace the term $|u|^{\mathfrak{D}}_0$ by the L_1 norm of u.*

REMARK 2. *If the solution of class $C^{p+\alpha}$ is unique, then we may omit altogether the term $|u|^{\mathfrak{D}}_0$ on the right of* (9.6). *The constant in the resulting inequality then depends on the system but is independent of u.*

The proofs of these remarks are the same as the proofs of the corresponding remarks after Theorem 7.3.

CHAPTER IV

APPLICATIONS OF THE SCHAUDER ESTIMATES AND COMMENTS

10. Necessity of the Complementing Condition

Up to now we have always assumed the Condition on L and the Complementing Condition specified in (i), (ii) of page 626 to hold. We shall now show that these conditions are necessary. Consider solutions of (7.1) in $\Sigma = \Sigma_R$. The Condition on L required that at every boundary point x on $t = 0$ the principal part L' of L should have the property that for real $\xi = (\xi_1, \cdots, \xi_n) \neq 0$ the polynomial $L'(x; \xi, \tau)$ have exactly m roots on either side of the real τ-axis. We propose to show here that if either this condition or the Complementing Condition on the B_j fails to hold at a single boundary point on $t = 0$, say at the origin, then it is impossible to estimate $|u|^{\sigma_{R-\delta}}_{l_0+\alpha}$, $\delta > 0$, in terms of $|Lu|^{\Sigma}_{l_0-2m+\alpha}$, $|B_j u|^{\Sigma}_{l_0-m_j+\alpha}$ and $|u|_0$—even if the operators have C^∞ coefficients. In fact we will show that there is no constant

C such that for every C^∞ function $u(x, t)$ with compact support in Σ_R the following holds:

$$(10.1) \qquad |u|_{l_0+\alpha} \leqq C(|Lu|_{l_0-2m+\alpha} + \sum |B_j u(x, 0)|_{l_0-m_j+\alpha} + |u|_0).$$

Assume first that (10.1) holds for some C independent of u. If $L'(P; D)$, $B_j'(x; D)$ are the leading parts of L and B_j, set

$$L(P; D)u = L'(0; D)u + \bar{L}(P; D)u,$$
$$B_j(x; D)u = B_j'(0; D)u + \bar{B}_j(x; D)u.$$

Then from (10.1) we have, for u with compact support in Σ,

$$|u|_{l_0+\alpha} < C(|L'(0; D)u|_{l_0-2m+\alpha} + \sum |B_j'(0; D)u(x, 0)|_{l_0-m_j+\alpha}$$
$$+ |\bar{L}u|_{l_0-2m+\alpha} + \sum |\bar{B}_j u(x, 0)|_{l_0-m_j+\alpha} + |u|_0).$$

If the support of u is contained in Σ_r and if r is small enough, $r \leqq r_0$, we have

$$C(|\bar{L}u|_{l_0-2m+\alpha} + |\bar{B}_j u(x, 0)|_{l_0-m_j+\alpha}) \leqq \tfrac{1}{2}|u|_{l_0+\alpha} + \tilde{c}|u|_0$$

for some constant $\tilde{c}$. This follows, with the aid of (7.4) and the fact that the leading coefficients in Σ_r differ little from their values at the origin, if r is sufficiently small. Thus we find that if u has support in Σ_r and $r \leqq r_0$, r_0 independent of u, then

$$(10.1)' \qquad |u|_{l_0+\alpha} \leqq 2C(|L'(0; D)u|_{l_0-2m+\alpha}$$
$$+ \sum |B_j'(0; D)u(x, 0)|_{l_0-m_j+\alpha} + 2(C+\tilde{c})|u|_0 .$$

Let us now suppose that the Condition on L is satisfied but that our Condition on the B_j is violated at the origin. Then consider the function v_λ defined by (1.13), with $l = l_0$, relative to operators $L'(0; D)$, $B_j'(0; D)$. The function v_λ satisfies the homogeneous system

$$L'(0; D)v_\lambda = 0, \qquad t \geqq 0,$$
$$B_j'(0; D)v_\lambda = 0, \qquad t = 0, \quad j = 1, \cdots, m.$$

Now set

$$(10.2) \qquad u_\lambda(x, t) = v_\lambda(x, t)\zeta(x, t),$$

where ζ is a C^∞ function with support in Σ_{r_0}, and equal to one at the origin. Since

$$L'(0; D)u_\lambda \qquad \text{and} \qquad B_j'(0; D)u_\lambda(x, 0)$$

involve derivatives of v_λ only up to orders $2m-1$ and m_j-1, respectively, we see from the construction of v_λ that as $\lambda \to \infty$ the norms

$$|L'(0; D)u_\lambda|_{l_0-2m+\alpha}, \quad |B_j'(0; D)u_\lambda(x, 0)|_{l_0-m_j+\alpha}, \quad |u_\lambda|_0,$$

remain bounded (in fact tend to zero). On the other hand $|u_\lambda|_{l_0+\alpha}$ blows up like λ^α—contradicting (10.1)′.

Suppose now that our Condition on L is violated at the origin, which may happen for $n = 1$. Then let v_λ be the function defined again at the end of Section 1 relative to $L'(0; D)$, $B_j'(0; D)$ with $l = l_0$. If we define as before $u_\lambda(x, t)$ by (10.2) we find again that for λ large u_λ does not satisfy (10.1)'.

We may see similarly that our conditions are necessary for the Schauder estimates for equations in integral form and also, as is shown again by the functions given in (10.2), for the L_p estimates of Chapter V.

11. Differentiability for Nonlinear Equations

Consider a nonlinear differential equation of order $2m$ in $(n+1)$-space,

$$F(x, u, Du, \cdots, D^{2m}u) = 0, \tag{11.1}$$

which is elliptic with respect to a solution u, i.e., the first variation of the operator F with u and its derivatives inserted in the coefficients is an elliptic operator.

The latter operator is a linear operator on a function w, and is of the form

$$Lw = \sum \frac{\partial F}{\partial D^\beta u}(x, u(x), \cdots, D^{2m}u(x))D^\beta w(x).$$

We shall prove a differentiability theorem near the boundary for a solution satisfying m boundary conditions (which may be nonlinear)

$$F_j(x, u, \cdots, D^{m_j}u) = 0, \qquad j = 1, \cdots, m, \tag{11.2}$$

for x on a portion Γ of the boundary. Since differentiability is a local property we shall assume that Γ is an open subset of an n-dimensional hyperplane, in fact, we shall assume that the solution is defined in a hemisphere $\Sigma = \Sigma_R$ and satisfies the boundary conditions on the flat part of the boundary. The first variation of the operators F_j with respect to a particular function u are the operators

$$B_j w = \sum_{|\beta| \leq m_j} \frac{\partial F_j}{\partial D^\beta u}(x, u(x), \cdots, D^{m_j}u(x))D^\beta w(x).$$

We first prove differentiability theorems using only the Schauder estimates, and show later how these may be improved with the aid of the L_p estimates of Chapter V.

If the linear system made up of L and the boundary operators B_j satisfies the conditions (i) and (ii) of Section 7, then Theorem 7.2 yields the following differentiability theorem.

THEOREM 11.1. *Let $l_0 = \max(2m, m_j)$ and assume that $u \in C^{l_0+\alpha}$ in Σ_R. Let $l \geqq l_0$ be a fixed integer and assume that F and F_j have Hölder continuous*

(exponent α) derivatives up to orders $l-2m$ and $l-m_j$, respectively, with respect to all of their arguments. Then u belongs to $C^{l+\alpha}$ in Σ_R.

The proof follows familiar lines. With the aid of the interior Schauder estimates it has been proved (Theorem 5 of [10]) that, in the interior of the domain, $u \in C^{l+\alpha}$. This was done by differencing the equation for two points, x, $x+\delta x$ and obtaining a linear elliptic differential equation for the difference quotient. The interior estimates then yielded uniform estimates for the difference quotient which were independent of δx, and letting $\delta x \to 0$ the desired differentiability was obtained. This standard method works here, as in the proofs of Theorems 7.2 and 9.2. On taking differences parallel to the flat boundary face and applying Theorem 7.2 we obtain for fixed positive δ a uniform estimate for the norm $|\ |_{l_0+\alpha}^{\Sigma_{R-\delta}}$ of the tangential difference quotients of u from which it follows that derivatives of u of the form $D_x D^{l_0} u$ belong to C^α in Σ_R. Since we may solve for the remaining derivative $D_t^{l_0+1} u$ of order l_0+1 from the equation $F(x, \cdots, D^m u) = 0$ — differentiated with respect to t — it follows that $u \in C^{l_0+1+\alpha}$ in Σ_R. Continuing in this way we obtain the result.

Consider now a nonlinear equation in integral form such as would arise from a problem in the calculus of variations, and a solution satisfying nonlinear boundary conditions. Again we shall imagine the solution to be given in Σ_R and to satisfy the boundary conditions on the flat face:

$$
\begin{aligned}
F(x, t, u, \cdots, D^{2m} u) &\equiv \sum_\beta D^\beta a_\beta(x, t, u, \cdots, D^l u) \\
&= 0 \text{ in } \Sigma_R, \\
\sum D_x^\gamma b_{j,\gamma}(x, u, \cdots, D^l u) &= 0 \text{ on } t = 0, \qquad j = 1, \cdots, m.
\end{aligned} \tag{11.3}
$$

Here summation is for $|\beta| \leqq \max(0, 2m-l)$, $|\gamma| \leqq \max(0, m_j - l)$, and $l < l_0$ is a fixed number which is not less than the maximum order of differentiation with respect to t occurring in the boundary operators. By the integral form of the problem we mean that for every C^∞ function $\zeta(x, t)$, with support in the interior of Σ_R, and for every C^∞ function $\zeta(x)$, with support in $|x| < R$, $u(x, t)$ should satisfy (analogous to (9.2))

$$
\sum (-1)^{|\beta|} \iint a_\beta(x, t, u, \cdots, D^l u) D^\beta \zeta(x, t) dx\, dt = 0,
$$

$$
\sum (-1)^{|\gamma|} \int b_{j,\gamma}(x, u, \cdots, D^l u) D_x^\gamma \zeta(x) dx = 0.
$$

We assume our solution to be of class $C^{l+\alpha}$ in Σ_R and the first variation of the system with respect to u to be elliptic and to satisfy conditions (i), (ii) of Section 7. The first variation is described by the linear system (meant in the integral form of course)

$$Lw = \sum_{\beta,\mu} D^\beta \frac{\partial a_\beta}{\partial D^\mu u}(x, t, u(x, t), \cdots, D^l u(x, t))D^\mu w,$$

$$B_j w = \sum_{\gamma,\mu} D_x^\gamma \frac{\partial b_{j,\gamma}}{\partial D^\mu u}(x, u(x, 0), \cdots, D^l u(x, 0))D^\mu w(x, 0), \qquad j = 1, \cdots, m,$$

for a function $w(x, t)$.

THEOREM 11.2. *Assume that $u \in C^{l+\alpha}$ in Σ_R. Let $p \geqq l$ be a fixed integer and assume that the a_β have Hölder continuous (exponent α) derivatives up to order $p-l$ if $l < 2m$, or $p-2m$ if $l \geqq 2m$, while the $b_{j,\gamma}$ have Hölder continuous derivatives (exponent α) up to order $p-l$ if $l < m_j$, or $p-m_j$ if $l \geqq m_j$. Then u belongs to $C^{p+\alpha}$ in Σ_R.*

Proof: From the interior differentiability theorems (see Section 8 of [10]) we know that $u \in C^{p+\alpha}$ in the interior of Σ_R. Our proof that $u \in C^{p+\alpha}$ in Σ_R is similar to that above and to the proof of Theorem 9.2. We shall consider here only the case $p = l+1$. By taking tangential (to the flat boundary) differences of the system and applying Theorem 9.1 we show in the usual way that derivatives of the form $D_x D^l u$ belong to C^α in Σ_R. We have only to show that $D_t^{l+1} u$ belongs to C^α in Σ_R.

In case $l \geqq 2m-1$ this follows from the fact that we may solve for D_t^{l+1} in terms of the derivatives $D_x D^l u$ and lower order derivatives from the equation $D_t^{l-2m+1} F = 0$; in this the ellipticity of the equation is used. In case $l < 2m-1$ we observe that, since the system is elliptic with respect to u, there is a term in (11.3) of the form

$$D_t^{2m-l} a(x, t, u, \cdots, D^l u)$$

with $\partial a/\partial D_t^l u \neq 0$. Because of our conditions on the a_β, and because we have shown that $D_x D^l u \in C^\alpha$ in Σ_R, we may write the equation (11.3) in the form (to be understood in the integral sense)

$$D_t^{2m-l} a(x, t, \cdots, D^l u) = \sum_{|\nu|<2m-l} D^\nu u_\nu,$$

where the u_ν belong to C^α in Σ_R. We may therefore apply Lemma 9.1 to a to conclude that

$$D_t a(x, t, \cdots, D^l u) \in C^\alpha \text{ in } \Sigma_R.$$

Since $\partial a/\partial D_t^l u \neq 0$ it follows easily that $D_t^{l+1} u \in C^\alpha$ in Σ_R. This completes the proof of the theorem.

We now indicate how the differentiability Theorems 11.1, 11.2 may be sharpened with the aid of the L_p estimates of Chapter V. Using Theorems 15.3, 15.3′, we may, in most cases, replace the assumptions $u \in C^{l_0+\alpha}$ and $u \in C^{l+\alpha}$ in these respective theorems by the weaker assumptions $u \in C^{l_0}$ and $u \in C^l$.

We start first with standard interior differentiability results [32].

THEOREM 11.3. *Let u be a solution of* (11.1) *of class C^{2m} in a domain. Assume that F is once continuously differentiable with respect to its arguments and that the associated first variation, the operator L, is elliptic. Then $u \in C^{2m+\alpha'}$ for every positive $\alpha' < 1$. If F has Hölder continuous (exponent α) derivatives up to order $l - 2m > 0$, then $u \in C^{l+\alpha}$.*

For equations in integral form a similar result maintains:

THEOREM 11.4. *Let u be a solution of the differential equation given in* (11.3) *and be of class C^l in a domain. Assume that the a_β are once continuously differentiable with respect to their arguments, and the associated first variation is elliptic. Then $u \in C^{l+\alpha'}$ for every positive $\alpha' < 1$. If the a_β have Hölder continuous (exponent α) derivatives up to order j, then $u \in C^{l+j+\alpha}$.*

The proof of the first statements in the first (respectively second) theorem is carried out in the usual way by taking difference quotients, and applying the interior L_p estimates to estimate the L_p norms of the derivatives of order $2m$ (l) of the difference quotients of u, choosing $p = (n+1)/(1-\alpha')$. It follows from the well known results of Sobolev [37], see Lemma A5.1 part (c) in Appendix 5, that the derivatives of order $2m-1$ ($l-1$) of the difference quotients of u satisfy a uniform Hölder condition (exponent α') in any compact subdomain, and hence that the same is true for the derivatives of u of order $2m$ (l).

The remainder of the theorems follows from the differentiability theorems in [10].

Turning now to differentiability at the boundary, consider again a solution of (11.1), (11.2) in Σ_R and assume that the linear system of first variation satisfies, as before, the conditions (i) and (ii) of Section 7. Then we have the following

THEOREM 11.1′. *Let $l_1 = \max(2m, m_j + 1)$, and assume that $u \in C^{l_1}$ in Σ_R. Assume that F and F_j have continuous derivatives up to orders $l_1 - 2m + 1$ and $l_1 - m_j + 1$, respectively, with respect to all their arguments. Then u belongs to $C^{l_1+\alpha'}$ in Σ_R for every positive $\alpha' < 1$.*

Consider finally a system in integral form (11.3), with the first variation, as before, satisfying conditions (i) and (ii) of Section 7,

$$F(x, t, u, \cdots, D^{2m} u) = \sum D^\beta a_\beta(x, \cdots, D^l u) = 0 \text{ in } \Sigma_R,$$
$$\sum D_x^\gamma b_{j,\gamma}(x, \cdots, D^{l-1} u) = 0 \quad \text{on} \quad t = 0, \qquad j = 1, \cdots, m;$$

here summation is for $|\beta| \leqq \max(0, 2m - l)$, $|\gamma| \leqq \max(0, m_j - l + 1)$, and l is to be greater than the maximum order of differentiation with respect to t in the B_j.

THEOREM 11.2′. *Assume that $u \in C^l$ in Σ_R, and assume that with respect to all their arguments, the a_β have continuous derivatives of first order if*

$l \leqq 2m$, or up to order $l - 2m + 1$ if $l \geqq 2m$, while the $b_{j,\gamma}$ have continuous derivatives of first order if $l \leqq m_j$, or up to order $l - m_j + 1$ if $l \geqq m_j$. Then u belongs to $C^{l+\alpha'}$ in Σ_R for every positive $\alpha' < 1$.

The proofs of the theorems follow the now familiar procedure. In each case one differences the equations in a direction parallel to the boundary, obtaining a linear system for the corresponding difference quotient of u, and applies Theorems 15.3, 15.3′ for $p = (n + 1)/(1 - \alpha')$ (see the Corollary of Theorem 15.2). In this way, in either theorem, one obtains a uniform Hölder condition (exponent α') in a compact subset of Σ_R for the derivatives of order $l - 1$ of the tangential difference quotients of u, from which it follows that derivatives of the form $D_x D^{l-1} u$ belong to $C^{\alpha'}$ in Σ_R. In the case of Theorem 11.1′, the Hölder continuity of the remaining derivative $D_t^l u$ follows from the fact that by applying the implicit function theorem to the equation this derivative may be expressed as a function of the others and lower order derivatives.

In the case of Theorem 11.2′, in order to infer the Hölder continuity of the derivative $D_t^l u$, we set for $\beta = (0, \ldots, 0, l)$

$$a_\beta(x, t, u(x, t), \cdots, D^l u(x, t)) = v(x, t).$$

It is easily seen that it suffices to prove the Hölder continuity of $v(x, t)$ in Σ_R. From the L_p estimate of the previous paragraph we see that in every $\Sigma_{R-\delta}$ the derivatives of u of the form $D_x D^l u$ belong to L_p for $p = (n + 1)/(1 - \alpha')$. Hence the first derivatives $D_x v$ belong to L_p in $\Sigma_{R-\delta}$. Thus it suffices to prove only that $D_t v$ belongs also to L_p in $\Sigma_{R-\delta}$. But one sees easily, via the differential equation, that v satisfies an equation of the form (9.5), to which Lemma 9.1′ may be applied to give the desired result.

Theorems 11.1, 11.1′ together yield a general differentiability theorem at the boundary for nonlinear problems, as do Theorems 11.2, 11.2′ for many problems in integral form.

12. Existence and Properties of Solutions

12.1. We shall discuss mainly the use of the Schauder estimates of Section 7. Similar remarks apply to the estimates for equations in integral form of Section 9 and to the L_p estimates of Chapter V. We consider the equation

$$Lu = F \tag{12.1}$$

in a (for simplicity) bounded domain $\mathfrak{D}$ with the boundary conditions

$$B_j u = \Phi_j, \qquad j = 1, \cdots, m, \tag{12.2}$$

over the entire boundary $\dot{\mathfrak{D}}$, and assume that these satisfy all the conditions of Theorem 7.3 with $\mathfrak{A} = \mathfrak{D}$ and l equal, say, to l_0. Our solution u, as well as

F and Φ_j, are assumed to be of classes $C^{l_0+\alpha}$, $C^{l_0-2m+\alpha}$, $C^{l_0-m_j+\alpha}$, respectively. We say that the system (12.1), (12.2) is solvable if it admits a solution u in $C^{l_0+\alpha}$ for every such F, Φ_j, and that uniqueness holds if the only solution in $C^{l_0+\alpha}$ of the homogeneous system is $u = 0$.

Some immediate consequences of Theorem 7.3 are

THEOREM 12.1. *There are at most finitely many linearly independent solutions of the homogeneous system (i.e., with $F = \Phi_j \equiv 0$).*

THEOREM 12.2 (COMPACTNESS). *Let u_i be a sequence of uniformly bounded solutions of the systems*

$$L_i u_i = F_i \qquad \text{in } \mathfrak{D}$$
$$B_{j,i} u_i = \Phi_{j,i} \qquad \text{on } \dot{\mathfrak{D}}, \qquad j = 1, \cdots, m,$$

which satisfy, uniformly for all i, the conditions of Theorem 7.3. Assume that the coefficients of the systems tend uniformly to the corresponding coefficients of L and B_j, and that F_i and $\Phi_{j,i}$ tend uniformly to F and Φ_j, as $i \to \infty$. Then a subsequence of the u_i converges in the norm $|u|_{l_0}$ to a solution in $C^{l_0+\alpha}$ of the limit system.

The proofs of Theorems 12.1 and 12.2 are immediate.

Consider the systems L, B_j of (12.1), (12.2). For ε, $k' > 0$ we define an (ε, k')-neighborhood of the system as follows: A system $\tilde{L}$, $\tilde{B}_j$, $j = 1, \cdots, m$, (these operators are of the same orders as L, B_j) belongs to the (ε, k')-neighborhood, if the coefficients in $\tilde{L}(\tilde{B}_j)$ differ from the corresponding coefficients in $L(B_j)$ by less than ε, and have norms $|\ |_{l_0-2m+\alpha}$ ($|\ |_{l_0-m_j+\alpha}$) bounded by k'. Here $k' \geqq$ the constant k occurring in (iii) of Theorem 7.3.

THEOREM 12.3. *Assume that uniqueness holds for the given system* (12.1), (12.2). *Then it is true that*: (a) *For any $k' > k$ there is an $\varepsilon > 0$ such that for any system $\tilde{L}$, $\tilde{B}_j$ in the (ε, k')-neighborhood of the given one we have, for $u \in C^{l_0+\alpha}$,*

$$|u|_{l_0+\alpha} \leqq \text{constant} \cdot (|\tilde{L}u|_{l_0-2m+\alpha} + \sum_j |\tilde{B}_j u|_{l_0-m_j+\alpha}),$$

with the constant depending only on the given system and on k'. Thus uniqueness holds in the (ε, k')-neighborhood. (b) *Assume that for some k' there is a system in every (ε, k')-neighborhood of the given one which is solvable. Then also the given one is solvable.*

Proof: With the aid of Theorem 12.2, part (b) follows easily from part (a), and we shall only prove part (a). Its proof is similar to that of Remark 2 after Theorem 7.3. Suppose that part (a) is not true. Then there is a sequence of systems L_i, $B_{j,i}$, $i = 1, 2, \cdots$, satisfying the conditions of Theorem 12.2, and a sequence of functions u_i in $C^{l_0+\alpha}$ such that $|u_i|_{l_0+\alpha} = 1$, and

$$\sigma_i = (|L_i u_i|_{l_0-2m+\alpha} + \sum_j |B_{j,i} u_i|_{l_0-m_j+\alpha}) \to 0 \text{ as } i \to \infty.$$

For i sufficiently large, the systems satisfy uniformly the conditions of Theorem 7.3 (with different constants) so that, by (7.8) (for $\mathfrak{A} = \mathfrak{D}$),

$$1 = |u_i|_{l_0+\alpha} \leqq \text{constant} \cdot (\sigma_i + |u_i|_0).$$

By Theorem 12.2 a subsequence of the u_i converges in the norm $|\ |_{l_0}$ to a solution u in $C^{l_0+\alpha}$ of the homogeneous original system. But by the preceding inequality this limit solution u cannot be zero, contradicting the uniqueness assumption.

THEOREM 12.4. *Let L be a differential operator of order $2m$, and M a differential operator of lower order, such that L and $L+M$ satisfy all the conditions of Theorem 7.3 relative to given boundary operators B_j. Assume that for every set of data $F \in C^{l_0-2m+\alpha}$, $\Phi_j \in C^{l_0-m_j+\alpha}$ there exists a unique solution in $C^{l_0+\alpha}$ of*

$$Lu = F \text{ in } \mathfrak{D}, \qquad B_j u = \Phi_j \text{ on } \dot{\mathfrak{D}}.$$

Consider the problem

$$(L+\lambda M)u = F \text{ in } \mathfrak{D}, \qquad B_j u = \Phi_j \text{ on } \dot{\mathfrak{D}}. \tag{12.3}$$

Except for a discrete set of values of λ the system (12.3) *has one and only one solution in $C^{l_0+\alpha}$ for arbitrary $F \in C^{l_0-2m+\alpha}$, $\Phi_j \in C^{l_0-m_j+\alpha}$. Furthermore if for some λ there is at most one solution in $C^{l_0+\alpha}$ of* (12.3) (*i.e., uniqueness holds*), *then in fact a solution exists for that λ.*

Proof: Let Φ_j be given values of the boundary expressions $B_j u$, and let v be the solution of $Lv = 0$, $B_j v = \Phi_j$ on $\dot{\mathfrak{D}}$. We may subtract v from u and thus reduce our problem to one with homogeneous boundary conditions

$$(L+\lambda M)u = F \text{ in } \mathfrak{D}, \qquad B_j u = 0 \text{ on } \dot{\mathfrak{D}}. \tag{12.4}$$

Denote by $L^{-1}F$ the solution w of $Lw = F$ with homogeneous boundary data. Then our equation may be written in the form

$$u + \lambda L^{-1} M u = L^{-1} F. \tag{12.5}$$

Since the order of M is at most $2m-1$ we see, for $u \in C^{l_0}$, that

$$|L^{-1} M u|^{\mathfrak{D}}_{l_0+\alpha} \leqq \text{constant } |u|^{\mathfrak{D}}_{l_0},$$

with the constant independent of u.

Consider now equation (12.5) for functions u in C^{l_0}. Since the set $|u|_{l_0+\alpha} \leqq$ constant is compact in the Banach space C^{l_0} it follows that $L^{-1}M$ is a completely continuous transformation of C^{l_0} into C^{l_0}. We see furthermore that a solution u in C^{l_0} of (12.5) belongs also to $C^{l_0+\alpha}$ so that the equations (12.4) and (12.5) are completely equivalent. The desired result follows from the Riesz theory for completely continuous operators applied to equations (12.5).

12.2. We now describe the continuity method in a slightly more general form than the usual.

THEOREM 12.5. *Let L_t, $B_{j,t}$ be a family of systems depending on a parameter t, $0 \leqq t \leqq 1$, satisfying uniformly in t the conditions of Theorem 7.3. Assume that the coefficients of L_t vary continuously with respect to t in the $|\ |_{l_0-2m+\alpha}$ norm and those of $B_{j,t}$ continuously with respect to t in the $|\ |_{l_0-m_j+\alpha}$ norm; here we assume that the m_j are independent of t. Consider the system*

$$(12.6) \qquad (L_t+\lambda)u = F \text{ in } \mathfrak{D}, \qquad B_{j,t}u = \Phi_j \text{ on } \dot{\mathfrak{D}}, \qquad j = 1, \cdots, m,$$

for arbitrary F in $C^{l_0-2m+\alpha}$, Φ_j in $C^{l_0-m_j+\alpha}$. Assume (a) *for $t=0$ and $\lambda=0$, the system* (12.6) *is uniquely solvable,* (b) *for every t there is a complex number λ_t such that uniqueness holds for* (12.6) *with $\lambda = \lambda_t$. Then for each t the system* (12.6) *has one and only one solution in $C^{l_0+\alpha}$ for arbitrary F, Φ_j —except possibly for a discrete set of values of λ. Furthermore uniqueness for* (12.6) *(for any t, λ) implies existence (for the same t, λ).*

QUESTION 1. *Does the theorem hold if assumption* (b) *is dropped?*

QUESTION 2. *For $n > 1$ can any two elliptic operators L_0, L_1 of order $2m$ be connected by a one parameter family of elliptic operators (depending continuously on t) L_t, $0 \leqq t \leqq 1$, reducing for $t = 0$ to the given L_0 and for $t = 1$ to the given L_1?*

Concerning Question 2 we mention that for $n = 1$ the set of elliptic operators satisfying the Condition on L is connected. This is not difficult to show.

In general the continuity method described here raises the problem of characterizing the class of boundary value problems L, B_j that can be connected to any given one by a family of such problems satisfying our conditions.

Proof of Theorem 12.5: For simplicity we treat the case that the $B_{j,t} = B_j$ are independent of t. As in the preceding proof it suffices to consider (12.6) with homogeneous boundary data $B_j u = 0$ on $\dot{\mathfrak{D}}$. Denote by $\dot{C}^{l_0+\alpha}$ the subspace of the Banach space $C^{l_0+\alpha}$ consisting of those functions satisfying the homogeneous boundary conditions $B_j u = 0$.

We may regard L_t as a bounded transformation of $\dot{C}^{l_0+\alpha}$ into $C^{l_0-2m+\alpha}$. Let T be the set of points on the interval $0 \leqq t \leqq 1$ for which our assertion holds; by Theorem 12.4, T contains $t = 0$. We shall show that T is both open and closed—and hence is the entire interval. The openness is simple. Suppose $t \in T$ and let $\tilde{\lambda}$ be such that $L_t+\tilde{\lambda}$ maps $\dot{C}^{l_0+\alpha}$ one-to-one onto $C^{l_0-2m+\alpha}$. According to Remark 2 after Theorem 7.3, $L_t+\tilde{\lambda}$ has a bounded inverse. The operator $L_{t+\delta t}-L_t$ has small norm for small $|\delta t|$, and it follows by a standard argument that $L_{t+\delta t}+\tilde{\lambda}$ has a bounded inverse for $|\delta t|$ small. That $t+\delta t$ lies in T follows now from Theorem 12.4 (with $L = L_{t+\delta t}+\tilde{\lambda}$, $M =$ identity).

To prove that T is closed suppose that $\{t_i\}$ is a sequence of points in T converging to t. We will show that $L_t+\lambda_t$ is invertible; this will imply as above, via Theorem 12.4, that $t \in T$.

Consider the operators $L_i = L_{t_i}$. Since for each i the operator $L_i+\lambda$ has an inverse (except for a discrete set of λ), there is a sequence $\lambda_i \to \lambda_t$ such that the operators $L_i+\lambda_i$ are invertible, and the result follows from Theorem 12.3(b).

12.3. We now state a general perturbation theorem for nonlinear elliptic equations. It is in fact in proving such a result that the strength of the Schauder estimates becomes apparent; we know of no other way to establish this result.

THEOREM 12.6. *Consider a nonlinear differential equation and nonlinear boundary conditions* (11.1), (11.2) *depending on a parameter* τ,

$$\begin{aligned} &F_\tau(x, u, \cdots, D^{2m}u) = 0 \text{ in } \mathfrak{D}, \\ &F_{j,\tau}(x, u, \cdots, D^{m_j}u) = 0 \text{ in } \dot{\mathfrak{D}}, \qquad j = 1, \cdots, m, \end{aligned} \tag{12.7}$$

and let u_0 *be a solution in* $C^{l_0+\alpha}$ *for* $\tau = 0$, *say* $u_0 \equiv 0$. *Assume that* F_τ, $F_{j,\tau}$ *have continuous first and second derivatives with respect to* τ, *and that* F_τ, $F_{j,\tau}$ *and these derivatives have Hölder continuous (exponent* α*) derivatives up to orders* l_0-2m *and* l_0-m_j, *respectively, with respect to the other arguments. Assume further that the linear system for a function* $w(x)$,

$$\begin{aligned} Lw &= \sum_{|\beta|\leq 2m} \frac{\partial F_\tau}{\partial D^\beta u}(x, 0, \cdots, 0)D^\beta w = f \text{ in } \mathfrak{D}, \\ B_j w &= \sum_{|\gamma|\leq m_j} \frac{\partial F_{j,\tau}}{\partial D^\gamma u}(x, 0, \cdots, 0)D^\gamma w = \phi_j \text{ on } \dot{\mathfrak{D}} \quad j = 1, \cdots, m, \end{aligned} \tag{12.8}$$

satisfy all the conditions of Theorem 7.3 *and possess a unique solution* w *in* $C^{l_0+\alpha}$ *for every* f *in* $C^{l_0-2m+\alpha}$ *and* ϕ_j *in* $C^{l_0-m_j+\alpha}$. *Then for* $|\tau|$ *sufficiently small the system* (12.7) *possesses a unique solution in* $C^{l_0+\alpha}$.

With the aid of the Schauder estimates of Theorem 7.3 the proof is routine. Let $W_1[f]+W_2[\phi_1, \cdots, \phi_m]$ be the solution of (12.8), where $W_1[f]$ is the solution of $Lw = f$ with homogeneous boundary data, and $W_2[\phi_1, \cdots, \phi_m]$ is the solution of $Lw = 0$ with the boundary data of (12.8).

Let us now write (12.7) in the form

$$\begin{aligned} -\tau Lu &= F_\tau(x, u, \cdots, D^{2m}u) - \tau Lu = R[u] \text{ in } \mathfrak{D}, \\ -\tau B_j u &= F_{j,\tau}(x, u, \cdots, D^{2m}u) - \tau B_j u = R_j[u] \text{ on } \dot{\mathfrak{D}}, \quad j = 1, \cdots, m, \end{aligned}$$

or

$$-u = \frac{1}{\tau} W_1[R[u]] + \frac{1}{\tau} W_2[R_1[u], \cdots, R_m[u]].$$

For τ small the right-hand side is small, of first order in τ. Using

iterations one may prove the existence of a solution in $C^{l_0+\alpha}$ for τ sufficiently small.

12.4. From now on we shall confine our remarks to a special linear problem, namely, the Dirichlet problem: $B_j = \partial^{j-1}/\partial \boldsymbol{n}^{j-1}$, where $\boldsymbol{n}$ denotes the normal to the boundary. Here $m_j = j-1$, $l_0 = 2m$, and the Complementing Condition is automatically satisfied. For convenience we shall assume that the boundary $\dot{\mathfrak{D}}$ is of class C^∞. A boundary of class $C^{2m+\alpha}$ can then be handled by a suitable approximation procedure.

Until recently the Dirichlet problem was treated only for strongly elliptic operators L, in the sense of Vishik [38] (see [31]), an operator L of order $2m$ being strongly elliptic if after multiplication by a suitable function its real part is elliptic. Thus after multiplication by the factor, we may write for a strongly elliptic operator

$$(-1)^m \mathscr{R}e\, L(x; \Xi) \geqq a(x)|\Xi|^{2m}, \tag{12.9}$$

where $a(x)$ is a suitable positive function.

As mentioned in the introduction, recently both Schechter [36] and Agmon [3] using L_2 estimates and Hilbert space theory obtained general existence theorems for a wide variety of boundary value problems. In particular, for an elliptic operator L (satisfying the Condition on L of the introduction), having sufficiently differentiable coefficients so that the formal adjoint operator L^* is well defined, they proved the following: There exists a solution of the Dirichlet problem for L provided that uniqueness holds for the Dirichlet problem for L^*—uniqueness being in the class of solutions having, say, square integrable derivatives up to order $2m$ in the domain $\mathfrak{D}$. Using this result it is easy to prove the following theorem (in the same way, using the results of Chapter V, we may prove a similar theorem for solutions having square integrable (or L_p) derivatives up to order $2m$ in $\mathfrak{D}$, assuming boundedness of the coefficients and continuity of the leading ones).

We always assume that the Condition on L is satisfied.

THEOREM 12.7. *If the elliptic operator L has coefficients in C^α, and if the Dirichlet problem*

$$Lu = F \text{ in } \mathfrak{D}, \qquad \frac{\partial^{j-1}}{\partial \boldsymbol{n}^{j-1}} u = \Phi_j \text{ on } \dot{\mathfrak{D}}, \qquad j = 1, \cdots, m,$$

has at most one solution in $C^{2m+\alpha}$ (i.e. uniqueness holds), then in fact it has a solution for every F in C^α and Φ_j in $C^{2m-j+1+\alpha}$.

Proof: It suffices to consider only homogeneous boundary conditions, $\Phi_j = 0$.

Assume first that all coefficients in L and B_j are, say, C^∞ functions. We shall make use of the following norms for functions in $\mathfrak{D}$, where integra-

tion extends over $\mathfrak{D}$, for a non-negative integer j:

$$\|\zeta\|_{j,L_2} = \Big[\sum_{|\beta|\leq j} \int |D^\beta \zeta|^2 dx\Big]^{1/2}. \tag{12.10}$$

We denote by H_{j,L_2} ($\mathring{H}_{j,L_2}$) the closure in the norm $\|\ \|_{j,L_2}$ of C^∞ functions (with compact support) in $\mathfrak{D}$. Denote by S the intersection of H_{2m,L_2} and $\mathring{H}_{m,L_2}$. By the results of Schechter and Agmon the Dirichlet problem for the formal adjoint operator L^*, $L^*v = F$, with homogeneous boundary conditions, has for F in $H_{0,L_2} = L_2$ a solution v belonging to S. The solution v can be made unique by the additional requirement that it should lie in the orthogonal (with respect to L_2 scalar product in $\mathfrak{D}$) complement of the null space of L^*, i.e., in the set of solutions of the homogeneous equation (the set is finite dimensional). Denote by V the intersection of the complement with S, and denote by $\bar{V}$ the complex conjugates of the functions in V. The map $L^* : V \to L_2$ has an inverse L^{*-1}.

We propose first to find a solution u in $\bar{V}$ of $Lu = F$, for given F in H_0; if F is in C^∞ in $\bar{\mathfrak{D}}$, the solution u belongs also to C^∞ in $\bar{\mathfrak{D}}$. Our proof of the existence of such a u is based on the observation that

$$L - \bar{L}^* = M$$

is an operator of order $< 2m$. Here $\bar{L}^*$ is the operator whose coefficients are the complex conjugates of the coefficients of L^*. Clearly the map $\bar{L}^* : \bar{V} \to L_2$ has a well-defined inverse $\bar{L}^{*-1}$. For u in $\bar{V}$ the equation $(\bar{L}^* + M)u = F$ is equivalent to the equation

$$u = -\bar{L}^{*-1} M u + \bar{L}^{*-1} F.$$

Since the operator M is of order lower than $2m$ the operator $\bar{L}^{*-1}M$ is a compact operator. Hence by the Riesz theory for compact operators the last equation has a solution if and only if uniqueness holds. But if $u \in \bar{V}$ is a solution of

$$u = -\bar{L}^{*-1} M u,$$

then it is also a solution of $Lu = 0$, and hence of class C^∞ in $\bar{\mathfrak{D}}$ and so, by our uniqueness assumption, is identically zero. Thus the equation above, and hence $Lu = F$, admits a solution in $\bar{V}$.

It follows then, in a standard way, that the null space of L^* is in fact zero, so that $V = S$. Thus if uniqueness holds for L it holds also for L^*.

The solution u obtained above is in C^∞ in $\bar{\mathfrak{D}}$ provided that F is. If now we are given a function F in C^α we may approximate it by C^∞ functions F_i, solve the corresponding equations $Lu_i = F_i$ and use Theorems 12.2—12.6 to conclude that there is a solution $u \in C^{2m+\alpha}$ of $Lu = F$. Thus we have shown, for a system with C^∞ coefficients, that uniqueness for the Dirichlet problem implies solvability.

To complete the proof, i.e., to dispose of the C^∞ hypothesis we approximate the system by systems with C^∞ coefficients, and the desired result follows from Theorem 12.3.

This completes the proof of the Theorem.

12.5. In view of Theorem 12.4 and Theorem 12.7 it is natural to ask the following question (which is a special case of Question 1 after Theorem 12.5). *For a given elliptic operator L of* (12.1), *does there exist a constant λ such that uniqueness holds for the Dirichlet problem for the operator $L+\lambda$?*

For strongly elliptic operators with sufficiently smooth coefficients the existence of such a λ follows from Gårding's inequality (see [11]). Recently Morrey [29] announced that such a uniqueness result also holds for strongly elliptic L with coefficients in C^α. We include here an elementary proof of a slightly more general result, Theorem 12.8. At the same time, Browder has also found a proof of Morrey's uniqueness result.

DEFINITION. We shall say that L is weakly positive semidefinite if

$$(12.9)' \qquad (-1)^m \mathscr{R}e\, L'(x; \Xi) \geqq 0 \text{ for real } \Xi,$$

where L' is the leading part of L.

An immediate consequence of (12.9)', via Fourier transforms, is that for any x_0 in $\mathfrak{D}$ the operator $L(x_0\,; D)$ with constant coefficients satisfies

$$(12.11) \qquad \mathscr{R}e\, (L'(x_0\,; D)\zeta, \zeta) \geqq 0$$

for every C^∞ function ζ with compact support. Here (,) represents the L_2 scalar product in $\mathfrak{D}$.

In particular, any strongly elliptic operator, i.e., one satisfying (12.9), is weakly positive semidefinite.

THEOREM 12.8. *Let L satisfy condition* (1) *and be weakly positive semidefinite. Assume that the coefficients of L are bounded and that the leading coefficients are continuous. For λ sufficiently large positive we have*

$$(12.12) \qquad ||u||^2_{2m, L_2} \leqq \text{constant} \cdot ||(L+\lambda)u||^2_{0, L_2} \text{ for } u \in S,$$

where the constant is independent of u; hence we have uniqueness of the Dirichlet problem for $L+\lambda$.

Here we have used the notation occurring in the proof of Theorem 12.7; $S = H_{2m, L_2} \cap \mathring{H}_{m, L_2}$.

As an immediate consequence of Theorems 12.8, 12.4, 12.7 we have the following statement which contains the results of [29].

COROLLARY. *Let the operator L in* (12.1) *have coefficients in $C^\alpha(\bar{\mathfrak{D}})$ and assume that it is weakly positive semidefinite. Then for all except possibly a discrete set of values of λ, the Dirichlet problem for the operator $L+\lambda$ is uniquely solvable. Furthermore uniqueness implies solvability.*

The corollary may also be proved without the aid of Theorem 12.7, using instead the existence theory for strongly elliptic operators, and Theorem 12.3(b), for a weakly positive semidefinite operator may be approximated by solvable, strongly elliptic operators with C^∞ coefficients.

In proving the theorem we shall make use of the integral estimate proved in Chapter V (here we take the case $p = 2$)

$$(12.13) \qquad ||u||^2_{2m, L_2} \leqq c(||Lu||^2_{0, L_2} + ||u||^2_{0, L_2}), \qquad u \in S,$$

where the constant c is independent of u. We may assume that $c \geqq 1$.

Proof of Theorem 12.8: We have

$$(12.14) \qquad ||(L+\lambda)u||^2_{0, L_2} = ||Lu||^2_{0, L_2} + 2\lambda\, \mathscr{R}e(Lu, u) + \lambda^2 ||u||^2_{0, L_2}.$$

If we can show that

$$(12.15) \qquad 2\lambda\, \mathscr{R}e(Lu, u) \geqq -\frac{1}{2c}\, ||u||^2_{2m, L_2} - \tfrac{1}{2}\lambda^2 ||u||^2_{0, L_2} - C' ||u||^2_{0, L_2},$$

where C' depends only on L, then the inequality (12.12) follows; for, combining (12.14), (12.13) and (12.15) we have

$$\begin{aligned} ||(L+\lambda)u||^2_{0, L_2} &\geqq \frac{1}{c}\, ||u||^2_{2m, L_2} - ||u||^2_{0, L_2} - \frac{1}{2c}\, ||u||^2_{2m, L_2} \\ &\quad - \tfrac{1}{2}\lambda^2 ||u||^2_{0, L_2} - C' ||u||^2_{0, L_2} + \lambda^2 ||u||^2_{0, L_2} \\ &\geqq \frac{1}{2c}\, ||u||^2_{2m, L_2} + (\tfrac{1}{2}\lambda^2 - C' - 1) ||u||^2_{0, L_2}, \end{aligned}$$

from which (12.12) follows for λ sufficiently large.

The proof of (12.15) has some similarity to the proof of Gårding's inequality [11] for the non-constant coefficient case. From (12.11) we obtain immediately: If L satisfies the conditions of Theorem 12.8, and if its leading coefficients differ from their values at a point P_0 by less than ε, then for u in S

$$(12.16) \qquad \mathscr{R}e(Lu, u) \geqq -\varepsilon ||u||_{2m, L_2} ||u||_{0, L_2} - C ||u||_{2m-1, L_2} ||u||_{0, L_2},$$

where C depends only on the bound of the coefficients of L. This comes from writing $L(P; D) = L(P_0; D) +$error term, and applying (12.11) to $L(P_0; D)$; the contribution of the error term is clearly bounded from below by the right-hand side of (12.16).

In the following we shall use the notation $C_{j,k}$ to mean an expression which may change from case to case, but which is bounded in absolute value by constant $\cdot\, ||u||_{j, L_2} \cdot ||u||_{k, L_2}$, the constant differing from case to case depending only on L. $C_1, C_2, \cdots$ will denote constants depending only on L.

Let

$$1 = \sum_1^N \omega_j^2$$

be a partition of unity in $\bar{\mathfrak{D}}$, the ω_j being real C^∞ functions of compact support such that, in $\bar{\mathfrak{D}} \cap$ support of any ω_j, the leading coefficients differ from their values at a point by less than $1/4c$. (That this is possible follows from the continuity of the leading coefficients, but it may clearly be possible under weaker hypotheses.) Then

$$2\lambda\, \mathscr{R}e(Lu, u) = 2\lambda \sum_j \mathscr{R}e(\omega_j Lu, \omega_j u) = 2\lambda \sum \mathscr{R}e(L(\omega_j u), \omega_j u) + \lambda C_{2m-1,0}\,.$$

Applying (12.14) to the term $\mathscr{R}e\ (L(\omega_j u), \omega_j u)$ we find

$$\begin{aligned} 2\lambda\, \mathscr{R}e\ (Lu, u) &\geqq -2 \sum_j \frac{\lambda}{4c}\, ||\omega_j u||_{2m, L_2} ||\omega_j u||_{0, L_2} - \lambda C_{2m-1,0} \\ &\geqq -\frac{1}{4c} \sum_j ||\omega_j u||^2_{2m, L_2} - \frac{\lambda^2}{4c} \sum ||\omega_j u||^2_{0, L_2} - \lambda C_{2m-1,0} \\ &= -\frac{1}{4c}\, ||u||^2_{2m, L_2} - \frac{\lambda^2}{4c}\, ||u||^2_{0, L_2} - \lambda C_{2m-1,0} \\ &\qquad - C_{2m, 2m-1}\,. \end{aligned} \tag{12.17}$$

For the last terms we have

$$\begin{aligned} |\lambda C_{2m-1,0}| + |C_{2m,2m-1}| &\leqq C_1 \lambda ||u||_{2m-1, L_2} ||u||_{0, L_2} \\ &\quad + C_2 ||u||_{2m, L_2} ||u||_{2m-1, L_2} \\ &\leqq \frac{\lambda^2}{4c}\, ||u||^2_{0, L_2} + \frac{1}{8c}\, ||u||^2_{2m, L_2} \\ &\qquad + C_3 ||u||^2_{2m-1, L_2}\,. \end{aligned} \tag{12.18}$$

By a known estimate (see Lemma 14.1) we have

$$C_3 ||u||^2_{2m-1, L_2} \leqq \frac{1}{8c}\, ||u||^2_{2m, L_2} + C_4 ||u||^2_0$$

so that, inserting in (12.18), we find

$$|\lambda C_{2m-1,0}| + |C_{2m,2m-1}| \leqq \frac{\lambda^2}{4c}\, ||u||^2_{0, L_2} + \frac{1}{4c}\, ||u||^2_{2m, L_2} + C_4 ||u||^2_0\,.$$

Inserting this, in turn, into (12.17) and using the fact that $c \geqq 1$ we obtain (12.15) with $C' = C_4$.

We conclude with some remarks about equations in integral form. We mention first that the preceding results have analogues for equations in integral form which we will not bother listing. However, we shall state the

analogue of Theorem 12.7. Again assume the boundary to be C^∞ and $B_j = \partial^{j-1}/\partial \boldsymbol{n}^{j-1}$.

THEOREM 12.9. *Let L be written in integral form as in Theorem 9.3 with $l \geqq m-1$. In terms of local coordinates (x, t) with the boundary represented by $t = 0$, the equation thus has the form of (9.2). Assume that there is at most one solution of the equation of class $C^{l+\alpha}$. Then there exists a solution for F_β in C^α and Φ_j in $C^{l-j+1+\alpha}$.*

The proof is similar to the proof of Theorem 12.7.

We shall mention some consequences of the theorem. Consider L written in integral form with $l = m$:

$$Lu = \sum_{|\beta|, |\mu| \leqq m} D^\beta a_{\beta,\mu} D^\mu u, \tag{12.19}$$

where all coefficients belong to $C^{1+\alpha}$. Assume that the operator is weakly positive in the sense that for every C^∞ function ζ with compact support in $\mathfrak{D}$ the inequality

$$\mathscr{R}e \sum_{\beta,\mu} (-1)^{|\beta|} \int a_{\beta,\mu} D^\mu \zeta \cdot D^\beta \bar{\zeta}\, dx \geqq c \int |\zeta|^2 dx \tag{12.20}$$

holds with c a fixed positive constant.

Let $\Phi_j \in C^{m-j+\alpha}$, $j = 1, \cdots, m$, be given functions on the boundary. We seek a solution in $C^{m-1+\alpha}$ of

$$\begin{aligned} &Lu = 0 \quad \text{in } \mathfrak{D}, \\ &\frac{\partial^{j-1}}{\partial \boldsymbol{n}^{j-1}} u = \Phi_j \text{ on } \dot{\mathfrak{D}}, \qquad j = 1, \cdots, m. \end{aligned} \tag{12.21}$$

THEOREM 12.10. 1) *The system (12.21) admits a unique solution u in $C^{m-1+\alpha}$.* 2) *If all the $a_{\beta,\mu}$ belong to $C^{k+\alpha}$ and $\Phi_j \in C^{k+m-j+\alpha}$, then u belongs to $C^{k+m-1+\alpha}$.*

Proof: The second part of the theorem follows from Theorem 9.3. To prove the first part we see from the conditions on the coefficients that we may also write the equation (12.19) in integral form with $l = m-1$. By Theorem 12.9 we need only prove that the solution u_0 of class $C^{m-1+\alpha}$ of the homogeneous Dirichlet problem is zero. But by Theorem 9.3 this solution is of class $C^{m+\alpha}$, and applying (12.20) which extends to such functions we find that $u_0 \equiv 0$.

Suppose now we are given L in the form

$$\begin{aligned} &Lu = \sum a_\beta D^\beta u = 0 \text{ in } \mathfrak{D}, \\ &\frac{\partial^{j-1}}{\partial \boldsymbol{n}^{j-1}} u = \Phi_j \text{ on } \dot{\mathfrak{D}}, \qquad j = 1, \cdots, m, \end{aligned} \tag{12.22}$$

with $\Phi_j \in C^{m-j+\alpha}$. Assume that all the coefficients belong to C^α and that for

$|\beta| \geqq m$, $a_\beta \in C^{|\beta|-m+1+\alpha}$. Assume also that the operator L is weakly positive, i.e., for every C^∞ function ζ with compact support

$$\int L\zeta \cdot \bar{\zeta}\, dx \geqq c \int |\zeta|^2 dx$$

for some positive constant c. Then as an easy consequence of Theorem 12.10 we have

THEOREM 12.11. *The system* (12.22) *has a unique solution u which belongs to $C^{2m+\alpha}$ in $\mathfrak{D}$ and to $C^{m-1+\alpha}$ in $\bar{\mathfrak{D}}$.*

13. Schauder Estimates for Semilinear Equations

Consider an elliptic operator L and boundary conditions B_j satisfying all the conditions of Theorem 7.3. Here we shall take $\mathfrak{D}$ to be a bounded domain, $\Gamma = \dot{\mathfrak{D}}$, $\mathfrak{A} = \mathfrak{D}$, and, for simplicity, $l = l_0$. Consider the system

$$\begin{aligned} Lu &= F && \text{in } \mathfrak{D}, \\ B_j u &= \Phi_j && \text{on } \dot{\mathfrak{D}}, \qquad j = 1, \cdots, m, \end{aligned}$$

where now we permit F and Φ_j to depend on u and some of its derivatives

$$\begin{aligned} F &= F(x, u, \cdots, D^{2m-1}u), \\ \Phi_j &= \Phi_j(x, u, \cdots, D^{m_j-1}u). \end{aligned}$$

Nagumo [30] has shown for a second order equation that it is still possible to obtain Schauder estimates analogous to those of Theorem 7.3 provided the nonlinear terms do not grow too fast. This is done by using Lemma 5.1 in its original form rather than its watered down form (7.4). This can be carried over directly to the general situation. We wish to estimate $|u|_{l_0+\alpha}$ in terms of $|u|_0$ and known data.

Suppose that for any function $u(x) \in C^{l_0+\alpha}$ inserted into F the resulting function

$$\tilde{F}_u(x) = F(x, u(x), \cdots, D^{2m-1}u(x))$$

satisfies

$$|\tilde{F}_u|_{l_0-2m+\alpha} \leqq KI + \rho(J),$$

where (a) I is a finite sum of terms each of which is a finite product $\prod [u]_{a_i}^{p_i}$ with $a_i < l_0+\alpha$ and $\sum p_i a_i < l_0+\alpha$, (b) J is a similar finite sum of finite products with each $a_i < l_0+\alpha$ and $\sum p_i a_i = l_0+\alpha$; K is a constant, and $\rho(S)$ is a fixed function defined for positive S which is $o(S)$ as $S \to \infty$ (12.6). This property would for instance be satisfied by a function F which consists of a finite sum of terms each of which is a finite product $\prod_j (D^{i_j}u)^{q_j}$ times a smooth function of x, with $i_j < 2m$ and q_j positive integers, and $\sum i_j q_j < 2m$.

Similarly about Φ_j we assume that for any function $u \in C^{l_0+\alpha}$ inserted into Φ_j the resulting function

$$\tilde{\Phi}_{j,u}(x) = \Phi_j(x, u(x), \cdots, D^{m_j-1}u(x))$$

satisfies

$$|\tilde{\Phi}_{j,u}|_{l_0-m_j+\alpha} \leqq KI+\rho(J).$$

Then we have

THEOREM 13.1. *If a bound for $|u|_0$ is known it is possible to estimate $|u|_{l_0+\alpha}$.*

The proof of the theorem follows rather directly from Theorem 7.3 with the aid of Lemma 5.1, and will be left to the reader.

It is also possible to replace $\rho(J)$ in the assumption by constant times J but the result then holds—or at any rate this proof evidently works—only if the constant is sufficiently small, the degree of smallness depending on the size of $|u|_0$. (See Nagumo [30] for a careful analysis for a second order equation.)

CHAPTER V

L_p THEORY

14. Constant Coefficients

The letter p will denote a fixed number > 1. We first define some integral pseudonorms and norms analogous to those in Section 5. We consider functions in a domain $\mathfrak{D}$ in $(n+1)$-space which, for simplicity, is assumed to be either a half-space $x_{n+1} > 0$ or a bounded domain of class C^2 as defined in the discussion preceding Theorem 7.3, and we define

$$\begin{aligned} |[f]|_j &= |[f]|_{j,L_p} = \left(\sum_{|\beta|=j} \int |D^\beta f|^p \, dx\right)^{1/p}, \\ \|f\|_j &= \|f\|_{j,L_p} = \left[\sum_{i \leqq j} (|[f]|_i)^p\right]^{1/p}. \end{aligned} \tag{14.1}$$

We define by H_{j,L_p} (see Section 12) the completion of C^∞ functions in $\mathfrak{D}$ with respect to the norm $\| \ \|_{j,L_p}$. Clearly, H_{j,L_p} forms a Banach space.

We shall make use of the following analogue of Lemma 5.1 (see, for instance, [31] for a proof).

LEMMA 14.1. *Suppose $i < j$, then for any $\varepsilon > 0$ there is a constant c depending only on ε, i, j, p and $\mathfrak{D}$ such that*

$$|[f]|_{i,L_p} \leqq \varepsilon|[f]|_{j,L_p}+c||f||_{0,L_p}\,.$$

For functions ϕ defined on the boundary $\dot{\mathfrak{D}}$ of $\mathfrak{D}$, and for j a positive integer, we introduce related classes of functions, seminorms and norms similar to those in (3.13)′.

$H_{j-1/p,L_p}$ is to be the class of functions ϕ which are the boundary values of functions v belonging to H_{j,L_p} in $\mathfrak{D}$. In this class we introduce the norms

$$\begin{aligned} |[\phi]|_{j-1/p,L_p} &= \text{g.l.b. } |[v]|_{j,L_p}\,, \\ ||\phi||_{j-1/p,L_p} &= \text{g.l.b. } ||v||_{j,L_p}\,, \end{aligned} \tag{14.2}$$

where in both cases g.l.b. is taken over all functions v in H_{j,L_p} which equal ϕ on the boundary.

From now on we shall usually write the abbreviated forms $|[\phi]|_{j-1/p}$, $||u||_j$ etc., omitting L_p.

The following special result describing the effect on the norm $||\phi||_{j-1/p,L_p}$ by a change of independent variables will be needed.

Let ψ be defined on the boundary of a domain $\mathfrak{D}$ and assume that ψ vanishes outside a subset γ of the boundary $\dot{\mathfrak{D}}$. Assume that there is a ($(n+1)$-dimensional) neighborhood U of γ such that $\bar{U} \cap \mathfrak{D}$ can be mapped in a one-to-one way onto the closure of a hemisphere Σ_R in $(n+1)$-space, with $\bar{U} \cap \dot{\mathfrak{D}}$ mapping onto the flat part of the hemisphere. Assume that the mapping T, together with its inverse, has continuous derivatives up to order j bounded by a constant κ. We consider the hemisphere Σ_R as lying in our half-space $t \geqq 0$. The function ψ goes over into the function

$$\phi(x) = \psi(T^{-1}(x, 0)).$$

LEMMA 14.2. *There is a constant C depending only on κ, p, j, n, and the distance from γ to the boundary of U such that*

$$C^{-1}||\psi||_{j-1/p,L_p} \leqq ||\phi||_{j-1/p,L_p} \leqq C||\psi||_{j-1/p,L_p}\,.$$

Proof: We shall prove only the first inequality, the other following in the same way. Let $v(x, t)$ be a function belonging to H_{j,L_p} in the upper half-space $t \geqq 0$ which equals $\phi(x)$ on the boundary and satisfies $||v||_j \leqq 2||\phi||_{j-1/p}$. Let $\sigma(x, t)$ be a C^∞ function defined in $t \geqq 0$ with support on Σ_R, and such that $\sigma = 1$ on $T\gamma$. Under the mapping T the function $v(x, t)\sigma(x, t)$ goes over into a function u with support in $U \cap \mathfrak{D}$, which equals ψ on γ. One easily verifies that

$$||u||_j \leqq \text{constant} \cdot ||v||_j\,,$$

the constant depending only on κ and the distance from γ to the boundary of U, and the lemma follows.

In deriving L_p estimates up to the boundary we shall restrict ourselves

to solutions of elliptic equations in a bounded domain $\mathfrak{D}$, satisfying the boundary conditions on the entire boundary. In general, the problem being local, one can

(a) obtain estimates near the boundary, analogous to those in Sections 7 and 9, for solutions satisfying boundary conditions on merely a *portion* of the boundary,

(b) consider unbounded domains.

(a) and (b) give rise to technical points similar to those, treated in detail, for the Schauder theory. It is only in order to avoid these, and to shorten the discussion, that we make the restrictions. Furthermore, except for some special results, we shall not consider equations in integral form. These may be treated by similar arguments with the aid of Lemma 9.1′.

In this section we consider only the constant coefficient elliptic operator L, of (6.1), containing only terms of order $2m$. We recall first the basic L_p estimate used in deriving the interior L_p estimates (see [32, 19, 7]).

THEOREM 14.1′. *If u has compact support and if its derivatives of order $2m$ belong to L_p, $p > 1$, then*

$$\int |D^{2m} u|^p \, dx \leqq \text{constant} \cdot \int |Lu|^p \, dx.$$

Furthermore, if l is an integer $> 2m$, and if $|[Lu]|_{l-2m, L_p}$ is finite, so is $|[u]|_{l, L_p}$, and

$$|[u]|_{l, L_p} \leqq \text{constant} \, |[Lu]|_{l-2m, L_p} .$$

Here the constants depend only on p, n and the constant A of (1) *in the introduction.*

The first statement follows from the representation

$$D^{2m} u = D^{2m} \Gamma * Lu + \text{constant} \cdot Lu,$$

where $*$ denotes convolution (in all variables), and Γ is the fundamental solution (4.2); the kernel $D^{2m}\Gamma$ satisfies the conditions of the Calderon-Zygmund theorem, which yields the result. The second follows by the usual difference quotient procedure.

We mention, in passing, that for the equations in integral form of Section 8 (see (8.1)′), namely

$$Lu = \sum D^\beta f_\beta ,$$

$|\beta| \leqq 2m-l$, for some fixed $l < 2m$, if u and f_β all have support in the unit sphere, and if u has derivatives of order l in L_p, the estimate

$$\int |D^l u|^p \, dx \leqq \text{constant} \cdot \sum_\beta \int |f_\beta|^p \, dx$$

is valid. This is proved again from

$$D^l u = \sum_\beta D^l D^\beta \Gamma * f_\beta + \sum_\beta c_\beta f_\beta , \qquad c_\beta \text{ constants,}$$

by applying the Calderon-Zygmund theorem to the terms with $|\beta| = 2m-l$, the other terms involving less singular kernels being handled by more elementary means. Equations in integral form will not be mentioned again in this section.

We consider now, in the half-space, the system (6.1), or

$$\begin{aligned} Lu &= f, & t &> 0, \\ B_j u &= \phi_j , & t &= 0, \quad j = 1, \cdots, m, \end{aligned} \tag{14.3}$$

with constant coefficients. Let $l_1 = \max(2m, m_j+1)$, and let l be an integer $\geqq l_1$.

THEOREM 14.1. *Assume that $u(P)$ vanishes for $|P| \geqq 1$ and belongs to H_{l,L_p} in the half-space $t \geqq 0$. Then*

$$||u||_l \leqq \text{constant} \cdot (||f||_{l-2m} + \sum_j ||\phi_j||_{l-m_j-1/p}), \tag{14.4}$$

where the constant depends only on l, p and the characteristic constant E.

Proof: We see easily that it suffices to prove this inequality for C^∞ functions, and to prove that $|[u]|_{l,L_p}$ is bounded by the right side of (14.4). Our proof makes use of the representation formula (6.3)

$$D^l u = D^l v + \sum I_j ,$$

and (6.5), (6.5)′. Here v is given by (4.5). Since the kernels $D^{2m}\Gamma$ satisfy the conditions of the Calderon-Zygmund theorem we have

$$|[v]|_l \leqq \text{constant} \cdot |[f_N]|_{l-2m} \leqq \text{constant} \cdot |[f]|_{l-2m} .$$

It follows that

$$|[D_x^{l-m_j-1} \psi_j]|_{1-1/p} \leqq |[v]|_l \leqq \text{constant} \cdot |[f]|_{l-2m} .$$

To complete the derivation of the inequality we show that for each I_j

$$|[I_j]|_0 \leqq \text{constant} \cdot \sum_{|\beta|=l-m_j-1} |[D_x^\beta(\phi_j - \psi_j)]|_{1-1/p} . \tag{14.5}$$

This together with the preceding inequalities yields the desired estimate for $|[u]|_l = |[u]|_{l,L_p}$. Consider I_j given, say, by (6.5). It may be written in the form

$$I_j = \sum_k D_{x_k} I_{j,k} ,$$

where

$$I_{j,k} = D^l \Delta^{(n+q-l+m_j)/2} K_{j,q} * D_k \Delta^{(l-m_j-1)/2} (\phi_j - \psi_j).$$

The estimate (14.5) then follows by applying Theorem 3.4(c) to this representation of the function $I_{j,k}$. In a similar way we treat I_j in the form (6.5)′. This completes the proof of Theorem 14.1.

The following is a simple

COROLLARY. *Under the conditions of Theorem* 14.1 *the representation formula* (4.7) *holds with* l_0 *replaced by* l_1.

15. L_p Estimates for Equations with Variable Coefficients

We consider first the system (7.1) in a hemisphere $\Sigma = \Sigma_R$, and let u be a solution with support in Σ_r, $r < R$. Let $l_1 = \max(2m, m_j+1)$ and let l be a fixed integer $\geqq l_1$.

We shall assume conditions (i) and (ii) of Section 7 and, in place of (iii),

(iii)$_{L_p}$: The coefficients of L belong to C^{l-2m} and have $|\ |_{l-2m}$ norms bounded by k, while the coefficients of B_j belong to C^{l-m_j} and have $|\ |_{l-m_j}$ norms, on $t = 0$, bounded by k. F and the functions Φ_j are assumed to have finite $||\ ||_{l-2m}$ and $||\ ||_{l-m_j-1/p}$ norms, respectively.

In the following, constants $r_1, C_1, C_2, \cdots$ depend only on E, k, l, p *and the modulus of continuity of the leading coefficients of* L.

THEOREM 15.1. *If* $||u||_{l_1}$ *is finite and if* $r \leqq r_1$, *then* $||u||_l$ *is also finite and*

$$||u||_l \leqq C_1(||F||_{l-2m} + \sum ||\Phi_j||_{l-m_j-1/p} + ||u||_0). \tag{15.1}$$

Proof: We shall assume that we already know that $||u||_l$ is finite; for, this may be derived from (15.1) for $l = l_0$ by the usual trick of taking difference quotients, as in Theorems 6.2, 7.2.

Using the notation of Section 7 we write the system (7.1) in the form

$$\begin{aligned} L'(0; D)u(P) &= F(P) + (L'(0; D) - L'(P; D))u(P) - L''(P; D)u(P) \\ &= f(P) \\ B_j'(0; D)u(x, 0) &= \Phi_j(x) + (B_j'(0; D) - B_j'(x; D))u(x, 0) - B_j''(x; D)u(x, 0) \\ &= \phi_j(x), \qquad j = 1, \cdots, m, \end{aligned} \tag{15.2}$$

and proceed to estimate $||f||_{l-2m}$, $||\phi_j||_{l-m_j-1/p}$.

If

$$L'(P; D) = \sum_{|\beta|=2m} a_\beta(P) D^\beta,$$

then we easily see that

$$\begin{aligned} &||(L'(0; D) - L'(P; D))u(P)||_{l-2m} \\ &\qquad \leqq \text{constant} \cdot (\max_\beta |a_\beta(P) - a_\beta(0)|_0^{\Sigma_r} ||u||_l + k||u||_{l-1}). \end{aligned}$$

Furthermore,

$$||L''(P; D)u(P)||_{l-2m} \leqq \text{constant} \cdot k||u||_{l-1}.$$

Combining these we find

$$(15.3)\qquad \|f\|_{l-2m} \leqq \|F\|_{l-2m}+C_2(\|u\|_{l-1} + \max_\beta |a_\beta(P)-a_\beta(0)|_0^{\Sigma_r}\|u\|_l).$$

To estimate some of the terms in ϕ_j we have, from the definition of the norm,

$$\|(B_j'(0;D)-B_j'(x;D)-B_j''(x;D))u(x,0)\|_{l-m_j-1/p} \leqq \text{constant}\cdot\|(B_j'(0;D)-B_j'(x;D)-B_j''(x;D))u(x,t)\|_{l-m_j}.$$

Since the first derivatives of the coefficients of B_j' are bounded by k we find easily that this expression is bounded by

$$\text{constant}\cdot k(r\|u\|_l+\|u\|_{l-1}).$$

Thus

$$(15.3)'\qquad \|\phi_j\|_{l-m_j-1/p} \leqq \|\Phi_j\|_{l-m_j-1/p}+C_2\|u\|_{l-1}+C_2 r\|u\|_l .$$

Applying Theorem 14.1 to (15.2) and using (15.3), (15.3)′, we obtain the inequality

$$\|u\|_l \leqq C_3[\|F\|_{l-2m} + \sum_j \|\Phi_j\|_{l-m_j-1/p}+\|u\|_{l-1} + (r + \max_\beta |a_\beta(P)-a_\beta(0)|_0^{\Sigma_r})\|u\|_l].$$

Since the a_β are continuous there is a constant r_1 such that

$$C_3(r + \max_\beta |a_\beta(P)-a_\beta(0)|_0^{\Sigma_r}) \leqq \tfrac{1}{2} \qquad \text{for} \qquad r = r_1 ,$$

so that for $r \leqq r_1$ we have

$$\|u\|_l \leqq 2C_3(\|F\|_{l-2m} + \sum_j \|\Phi_j\|_{l-m_j-1/p}+\|u\|_{l-1}).$$

By Lemma 14.1

$$2C_3\|u\|_{l-1} \leqq \tfrac{1}{2}\|u\|_l+C_4\|u\|_0 .$$

Inserting this into the preceding we obtain the desired inequality (15.1).

The following well known analogue for interior estimates is proved in the same way as Theorem 15.1 using Theorem 14.1′ in place of Theorem 14.1.

THEOREM 15.1′. *Let L be the operator occurring in Theorem* 15.1 *and let u be a function, in $(n+1)$-space, belonging to H_{2m,L_p} and vanishing outside a sphere of radius r. Then if $r < r_1$ (possibly a different constant than the one occurring in Theorem* 15.1), *u satisfies* (15.1) (*with a different constant C_1 and with no boundary terms*).

From now on we use r_1 to denote the smaller of the two constants called r, in Theorems 15.1, 15.1′.

We turn now to a general domain $\mathfrak{D}$ which for simplicity we take to be bounded. As mentioned in Section 14, it will be clear from the discussion that our results can be extended to a wide class of unbounded domains. Let u be a solution of

$$(15.4)\qquad \begin{aligned} Lu &= F && \text{in } \mathfrak{D}, \\ B_j u &= \Phi_j && \text{on } \dot{\mathfrak{D}}, \qquad j = 1, \cdots, m, \end{aligned}$$

L and B_j being differential operators of orders $2m$, m_j; set $l_1 = \max(2m, m_j+1)$ and let $l \geqq l_1$ be fixed. We shall assume the boundary $\dot{\mathfrak{D}}$ of $\mathfrak{D}$ to be of class C^l in the following sense: The boundary $\dot{\mathfrak{D}}$ can be covered by a finite number of ($(n+1)$-dimensional) neighborhoods U_i such that each $\bar{U}_i \cap \bar{\mathfrak{D}}$ can be mapped in a one-to-one way onto the closure of a hemisphere Σ_{R_i}, $R_i \leqq 1$ in $(n+1)$-space, with $\bar{U}_i \cap \dot{\mathfrak{D}}$ mapping onto the flat face of the hemisphere, by a mapping T_i which together with its inverse has continuous derivatives up to order l bounded by a constant κ.

Concerning the equation and the boundary condition we shall assume first of all that L is uniformly elliptic, and that under each mapping T_i the system (15.4) goes into a system (7.1) in Σ_{R_i} satisfying the hypotheses (i) and (ii) of Section 7. In addition we assume that the coefficients of L belong to C^{l-2m} and have $|\ |_{l-2m}$ norms bounded by k, while the coefficients of B_j belong to C^{l-m_j} and have $|\ |_{l-m_j}$ norms (defined say as the maximum$_i$ of these norms over the flat faces of Σ_{R_i}) bounded by k. F and the functions Φ_j are assumed to have finite $||\ ||_{l-2m}$ and $||\ ||_{l-m_j-1/p}$ norms, respectively, (see (14.1), (14.2)).

THEOREM 15.2. *If $||u||_{l_1}$ is finite, then $||u||_l$ is also finite and*

$$(15.5)\qquad ||u||_l \leqq \bar{C}(||F||_{l-2m} + \sum_j ||\Phi_j||_{l-m_j-1/p} + ||u||_0),$$

where $\bar{C}$ depends only on κ, E, k, l, p, the domain $\mathfrak{D}$, and the modulus of continuity of the leading coefficients of L.

Proof: Let $\sum_1^N \omega_\sigma \equiv 1$ be a partition of unity in $\bar{\mathfrak{D}}$ by C^∞ functions ω_σ having the following property: If the support of ω_σ is not in the interior of $\mathfrak{D}$, then it is in the interior of one of the U_i, $U_{i(\sigma)}$. Furthermore we require that the support of ω_σ, and the image of $\bar{\mathfrak{D}} \cap$ support of ω_σ under the mapping $T_{i(\sigma)}$, is contained in a hemisphere of radius r_1, where r_1 is the constant entering in Theorems 15.1, 15.1′. We intend to show that for each σ, $||\omega_\sigma u||_l$ is bounded by the right side of (15.5) plus $\bar{C} \cdot ||u||_{l-1}$. It follows that $||u||_l \leqq \sum_\sigma ||\omega_\sigma u||_l \leqq$ the right side of (15.5) (with a suitable constant $\bar{C}$) plus $\bar{C} \cdot ||u||_{l-1}$. With the aid of Lemma 14.1 the inequality (15.5) is then obtained in the usual way.

With the aid of Theorem 15.1 we shall treat the case that the support of ω_σ is not contained in the interior of $\mathfrak{D}$. The other (interior) case, which is similar, is handled in the same way using Theorem 15.1′.

Let us consider a fixed ω_σ. Under the mapping $T_{i(\sigma)}$, ω_σ goes over into a function ω with support in Σ_{r_1}, u goes over into a function v which we shall consider only in the support of ω. Consider first the case $l = l_1$.

Because of our assumption on the mappings T_i

$$(15.6) \qquad ||\omega_\sigma u||_l \leqq \text{constant} \cdot ||\omega v||_l .$$

Let us denote the operators L, B_j, after the mapping $T_{i(\sigma)}$, by $\tilde{L}$ and $\tilde{B}_j$. According to Theorem 15.1

$$(15.7) \qquad ||\omega v||_l \leqq C_1(||\tilde{L}(\omega v)||_{l-2m} + \sum ||\tilde{B}_j(\omega v)||_{l-m_j-1/p} + ||\omega v||_0).$$

We easily see that

$$(15.8) \qquad \begin{aligned} &||\tilde{L}(\omega v)||_{l-2m} \leqq \text{constant} \cdot (||F||_{l-2m} + ||u||_{l-1}), \\ &||\omega v||_0 \leqq \text{constant} \cdot ||u||_0 . \end{aligned}$$

From Lemma 14.2 it follows also that

$$(15.9) \qquad ||\tilde{B}_j(\omega v)||_{l-m_j-1/p} \leqq \text{constant} \cdot ||B_j(\omega_\sigma u)||_{l-m_j-1/p} .$$

Combining these inequalities we find

$$(15.10) \qquad ||\omega_\sigma u||_l \leqq \text{constant} \cdot (||F||_{l-2m} + ||u||_{l-1} + \sum_j ||B_j(\omega_\sigma u)||_{l-m_j-1/p}).$$

Thus the desired estimate for $||\omega_\sigma u||_l$ will follow from the estimate

$$(15.11) \qquad ||B_j(\omega_\sigma u)||_{l-m_j-1/p} \leqq \text{constant} \cdot (||\Phi_j||_{l-m_j-1/p} + ||u||_{l-1}).$$

Now on $\dot{\mathfrak{D}}$

$$B_j(\omega_\sigma u) = \omega_\sigma \Phi_j + \sum_{\substack{|\beta|+|\gamma| \leqq m_j \\ |\gamma| < m_j}} c_{j,\beta,\gamma} D^\beta \omega_\sigma D^\gamma u,$$

where the functions $c_{j,\beta,\gamma}$ are of class C^{l-m_j}. These coefficients may be extended into the support of ω_σ as functions $d_{j,\beta,\gamma}$ in C^{l-m_j} because of our conditions on the boundary $\dot{\mathfrak{D}}$. Therefore, by definition,

$$\begin{aligned} ||c_{j,\beta,\gamma} D^\beta \omega_\sigma D^\gamma u||_{l-m_j-1/p} &\leqq ||d_{j,\beta,\gamma} D^\beta \omega_\sigma D^\gamma u||_{l-m_j} \\ &\leqq \text{constant} \cdot ||u||_{l-1} . \end{aligned}$$

To estimate $||\omega_\sigma \Phi_j||_{l-m_j-1/p}$ we note that since $||\Phi_j||_{l-m_j-1/p}$ is finite there is a function v_j in $\mathfrak{D}$ which equals Φ_j on the boundary and which satisfies $||v_j||_{l-m_j} \leqq 2||\Phi_j||_{l-m_j-1/p}$. Therefore

$$\begin{aligned} ||\omega_\sigma \Phi_j||_{l-m_j-1/p} \leqq ||\omega_\sigma v_j||_{l-m_j} &\leqq \text{constant} \cdot ||v_j||_{l-m_j} \\ &\leqq \text{constant} \cdot ||\Phi_j||_{l-m_j-1/p} . \end{aligned}$$

Combining these inequalities and using the triangle inequality for these norms, we obtain the inequality (15.11). This completes the proof of the theorem for $l = l_1$.

Suppose now that $l = l_1 + 1$. Having a bound for $||u||_{l-1}$ we can obtain bounds for the right-hand sides of (15.8) and (15.11), and hence also for (15.9). Applying Theorem 15.1 we again see that (15.7), and hence (15.10), holds for $l = l_1 + 1$. Using (15.11) we obtain the desired estimate for $||\omega_\sigma u||_l$.

Knowing that now $||u||_{l_1+1}$ is finite, and having an estimate for it, we prove in the same way that $||u||_{l_1+2}$ is finite, and obtain the desired bound for it. And so on.

In analogy with Remarks 1, 2 after Theorem 7.3 we have here:

The term $||u||_0$ *on the right of* (15.5) *may be replaced by the* L_1 *norm of* u, *and it may be omitted altogether if the solution having derivatives up to order* l_1 *in* L_p *is unique* (*in either case the constant* $\bar{C}$ *has to be changed*).

We mention again that with the aid of the examples in Section 10 we see also that the Condition on L and the Complementing Condition are necessary conditions for the L_p estimate to hold.

We conclude with some corollaries of Theorems 15.1, 15.2. The first follows from Theorem 15.2 with the aid of a special case of well known results of Sobolev ([37], see Lemma A5.1 part c, in Appendix 5). It will be clear that a more general form of the following corollary can be derived from the results of Sobolev.

COROLLARY. *Under the conditions of Theorem* 15.2, *if* p *exceeds the dimension* $n+1$, *so that* $\alpha = 1-(n+1)/p > 0$, *then* $|u|_{l-1+\alpha}$ *is majorized by the right-hand side of* (15.5).

The following results are used in the proofs of Theorems 11.1′, 11.2′.

THEOREM 15.3. *Consider again a solution of* (7.1) *in* Σ_R, *and assume that all the conditions of Theorem* 15.1 *are satisfied with the exception that* u *is no longer assumed to vanish near the curved boundary of* Σ_R. *Then, for every* $r < R$, (i) $||u||_l^{\Sigma_r}$ *is finite,* (ii) *if* $\zeta(x, t)$ *is a* C^∞ *function defined in* Σ_R *which is identically one in* Σ_r *and zero outside* $\Sigma_{(R+r)/2}$ *we have*

$$||u||_l^{\Sigma_r} \leqq \text{constant } (||\zeta F||_{l-2m} + \sum ||\zeta(x, 0)\Phi_j||_{l-m_j-1/p} + ||u||_{l_1-1}^{\Sigma_r})$$

where the constant is independent of u.

A similar inequality holds, in fact, with $||u||_{l_1-1}^{\Sigma_R}$ on the right replaced by $||u||_0^{\Sigma_R}$, but we shall not bother proving this.

For $l = l_1$ the theorem is proved (we omit details) by applying Theorem 15.1 to the function ζu; the proof for the general case is similar.

Because of its application in the proof of Theorem 11.2′ we state the following analogue of Theorem 15.3 for equations in integral form, omitting the proof. Consider the system (9.1) in Σ_R, written in a somewhat modified form.

$$\begin{aligned} L &= \sum D^\beta a_{\beta,\mu}(P) D^\mu \\ B_j &= \sum D_x^\gamma b_{j,\gamma,\delta}(x) D^\delta \\ F &= \sum D^\beta F_\beta, \qquad \Phi_j = \sum D_x^\gamma \Phi_{j,\gamma} \end{aligned}$$

with summations over $|\beta| \leqq \max(0, 2m-l)$, $|\mu| \leqq l$, $|\gamma| \leqq \max(0, m_j - l + 1)$,

$|\delta| \leqq l-1$, and $j = 1, \cdots, m$. Here l is to be greater than the maximum order of t differentiation occurring in the B_j.

In place of $(\text{iii})_{L_p}$ we assume

$(\text{iii})'_{L_p}$: The F_β belong to L_p if $l < 2m$, $F \in H_{l-2m, L_p}$ if $l \geqq 2m$; for every function $\zeta(x, t)$ as above, $\zeta(x, 0)\Phi_{j,\gamma}$ has finite $\| \ \|_{1-1/p}$ norm if $l \leqq m_j$, while $\zeta(x, 0)\Phi_j \in H_{l-m_j+1-1/p}$ if $l > m_j$. Concerning the coefficients, we assume that the $a_{\beta,\mu}$ are continuous for $|\beta| + |\mu| = 2m$, and that the following are bounded by k: $|a_{\beta,\mu}|_0$ if $l < 2m$, $|a_{\beta,\mu}|_{l-2m}$ if $l \geqq 2m$, $|b_{j,\gamma,\delta}|_1$ if $l \leqq m_j$, $|b_{j,0,\delta}|_{l-m_j}$ if $l > m_j$.

THEOREM 15.3′. *If $||u||_l$ is finite, then for any $r < R$ and $\zeta(x, t)$ as above we have*

$$||u||_l^{\Sigma_r} \leqq \text{constant} \cdot \Big(\sum_{j=0}^{m} k_j + ||u||_{l-1}^{\Sigma_R}\Big),$$

with the constant independent of u. Here

$$k_0 = \sum ||F_\beta||_0 \quad \text{if } l < 2m,$$
$$k_0 = ||F||_{l-2m} \quad \text{if } l \geqq 2m,$$

and for $j = 1, \cdots, m$

$$k_j = \sum_\gamma ||\zeta(x, 0)\Phi_{j,\gamma}||_{1-1/p} \qquad \text{if } l \leqq m_j,$$
$$k_j = ||\zeta(x, 0)\Phi_{j,\gamma}||_{l-m_j+1-1/p} \quad \text{if } l > m_j.$$

For completeness, though no use is made of it, we add the following interior analogue of Theorem 15.1′ for an equation in integral form

$$Lu = \sum D^\beta a_{\beta,\mu} D^\mu u = \sum D^\beta f_\beta,$$

$|\beta| \leqq 2m - l$, $|\mu| \leqq l$, for some fixed $l < 2m$. We assume, for simplicity that the f_β and $D^\mu u$, $|\mu| \leqq l$, belong to L_p in $\mathfrak{D}$, that L satisfies (1) and has bounded cccfficients, and that the leading coefficients $a_{\beta,\mu}$ i.e. those with $|\beta| + |\mu| = 2m$, are continuous.

THEOREM 15.1″. *For any compact subdomain $\mathfrak{A}$ of $\mathfrak{D}$ we have*

$$\sum_{|\mu| \leqq l} \int_{\mathfrak{A}} |D^\mu u|^p \, dx \leqq \text{constant} \cdot \int_{\mathfrak{D}} (\sum |f_\beta|^p + |u|^p) dx.$$

Appendix 1

Proof of Lemma 2.1

Consider the kernels $K_{j,q}$ given by (2.6)′, (2.6)″ and choose for γ the contour composed of the circular arc $|\tau| = 2C$, $\mathscr{I}m\, \tau \geqq 1/2C$ and the chord joining its extremities (see (1.2)). For $s \geqq m_j + q + 1$ the homogeneity of

$D^sK_{j,q}$ asserted in the lemma then follows obviously from (2.6)′, (2.6)″. We see also from these formulas that it suffices to establish (2.13) for points $P = (x, t)$ on the unit hemisphere $|x|^2+t^2 = 1$, $t > 0$. Since for $|\xi| = 1$, τ on γ we have $|N_k(\xi, \tau)/M^+(\xi, \tau)| \leqq C_1(E)$ (see (1.11)) the inequality (2.13) for $s < m_j+q+1$ follows immediately.

The letters $C_1(E)$, $C_2(E)$, $\cdots$ will denote constants depending only on E of (2.12), while $C_1(s, E)$, $C_2(s, E)$, $\cdots$ depend also on s.

If $s \geqq m_j+q+1$ we find, setting $s-m_j-q = \sigma$,

$$D^sK_{j,q} = \int_{|\xi|=1} d\omega_\xi \int_\gamma \frac{F(\xi, \tau)d\tau}{(x \cdot \xi+t\tau)^\sigma}, \tag{A1.1}$$

where $F(\xi, \tau)$ is an analytic function of ξ, τ on $|\xi| = 1$, $\tau \in \gamma$ such that F and all the derivatives of order $\leqq l$ are bounded by a constant depending only on E and l. In order to prove (2.13),

$$|D^sK_{j,q}| \leqq C(s, E), \tag{A1.2}$$

for the case $s \geqq m_j+q+1$ it will suffice to establish for (x, t) on the hemisphere and for all $s \geqq m_j+q+1$ the estimate

$$|D^sK_{j,q}| \leqq \frac{C_1(s, E)}{t}, \tag{A1.3}$$

the boundedness of the derivatives on the unit hemisphere following by integrating the derivatives twice from some fixed point on the sphere.

Because of (A1.1) the estimate (A1.2) certainly holds on the part of the hemisphere with $t \geqq \frac{1}{2}$, so it suffices to consider the ring R: $|x|^2+t^2 = 1$, $0 < t < \frac{1}{2}$. Let $\zeta(r)$ be a C^∞ function on the interval $[-1, 1]$ such that $|\zeta| \leqq 1$, $\zeta \equiv 1$ for $|r| < \frac{1}{2}$, $\zeta \equiv 0$ for $\frac{3}{4} \leqq |r| \leqq 1$. We may then write (A1.1) in the form

$$\begin{aligned} D^sK_{j,q} &= \int_{|\xi|=1}\int_\gamma \frac{F(\xi, \tau)}{(x \cdot \xi+t\tau)^\sigma}\,\zeta(\xi \cdot x)d\omega_\xi d\tau \\ &\quad + \int_{|\xi|=1}\int_\gamma \frac{F(\xi, \tau)}{(x \cdot \xi+t\tau)^\sigma}\,(1-\zeta(\xi \cdot x))d\tau\, d\omega_\xi \\ &= I_1+I_2, \end{aligned}$$

where the two integrals define I_1 and I_2.

We first estimate I_2. Since the integrand vanishes except for $|x \cdot \xi| > \frac{1}{2}$ we see easily that there we have, for $\tau \in \gamma$, $|x \cdot \xi+t\tau|^{-1} \leqq C_2(E)$. Hence

$$|I_2| \leqq C_2^\sigma \int_{|\xi|=1}\int_\gamma |F(\xi, \tau)||d\tau|d\omega_\xi = C_3(s, E).$$

Consider finally I_1. Let $\eta = T_x\xi$ be a rotation in E_n which takes $x = (x_1, \cdots, x_n)$ to $(|x|, 0, \cdots, 0)$. Making the change of variable $\eta = T_x\xi$ we have

$$I_1(x, t) = \int_{\substack{|\eta|=1 \\ |\eta_1| \leq \sqrt{3/4}}} \int_\gamma \frac{F(\ {}_x^{-1}\eta, \tau)\zeta(\eta_1|x|)}{(|x|\eta_1+t\tau)^\sigma}\, d\tau\, d\omega_\eta .$$

Denote by η' the generic point $(\eta_2, \cdots, \eta_n)$ in E_{n-1}.[7] Then

$$I_1 = \int_{|\eta'|=1} d\omega_{\eta'} \int_{-\sqrt{3/4}}^{\sqrt{3/4}} d\eta_1 \int_\gamma \frac{F(\ {}_x^{-1}\eta, \tau)\zeta(\eta_1|x|)}{(|x|\eta_1+t\tau)^\sigma}\, (1-\eta_1^2)^{(n-3)/2}\, d\tau.$$

Integrating by parts $\sigma-1$ times with respect to η_1 we find

$$|I_1| = \frac{1}{(\sigma-1)!\,|x|^{\sigma-1}} \left| \int_{|\eta'|=1} d\omega_{\eta'} \int_{-\sqrt{3/4}}^{\sqrt{3/4}} d\eta_1 \right.$$
$$\left. \cdot \int_\gamma \frac{\left(\dfrac{d}{d\eta_1}\right)^{\sigma-1} [F(T_x^{-1}\eta, \tau)\zeta(\eta_1|x|)(1-\eta_1^2)^{(n-3)/2}]}{|x|\eta_1+t\tau}\, d\tau \right| .$$

Since now $|x| > \frac{1}{2}$ on the ring R, and since the numerator of the last integrand is bounded by $C_4(s, E)$ in absolute value, we find

$$|I_1| \leq \frac{C_5(s, E)}{t} .$$

Combining this estimate with the previous ones we obtain (A1.3) — proving the lemma.

Appendix 2

Proof of Theorem 3.2

We restate the theorem for $K = K_1$, i.e., $K_2 = 0$.

Let

(A2.1) $$u(x, t) = \int K(x-y, t)f(y)dy = K * f$$

with

$$K(x, t) = \frac{\Omega\left(\dfrac{x}{|x|}\right)}{(|x|^2+t^2)^{n/2}}$$

satisfying (3.2), (3.3).

Then if f is in L_p, $p > 1$, so is $u(x, t)$, for each t, and

(A2.2) $$|u|_{L_p, t} \leq c\kappa|f|_{L_p}, \qquad t > 0,$$

where c depends only on n and p, i.e., is independent of t.

As one easily sees it suffices to establish the estimate (A2.2) for C^∞ functions $f(x)$ with compact support.

[7] The following has to be modified somewhat when $n = 1$ or 2.

Consider first the case $n = 1$. For our given kernel K we have $\Omega(x) = \kappa'$ (sign x), where $\kappa' \leqq \kappa$ is a constant. To treat it we shall rely upon the classical result due to M. Riesz: for $K_0(x, t) = x/(x^2+t^2)$, the inequality

$$\begin{aligned} |K_0 * f|_{L_p, t} &= \left| \int K_0(x-y, t) f(y) dy \right|_{L_p, t} \\ &\leqq c(p) |f|_{L_p}, \qquad 1 < p < \infty, \end{aligned}$$

holds, with $c(p)$ depending only on p. Now consider the function

$$\gamma(x, t) = K(x, t) - \kappa' K_0(x, t).$$

It has finite L_1 norm in the variable x, and this norm is independent of t. But then we have

$$\begin{aligned} |K * f|_{L_p, t} &\leqq |\kappa' K_0 * f|_{L_p, t} + |\gamma * f|_{L_p, t} \\ &\leqq \kappa c(p) |f|_{L_p} + |\gamma|_{L_1, t} |f|_{L_p} \\ &\leqq \text{constant} \cdot \kappa |f|_{L_p}. \end{aligned}$$

Thus Theorem 3.2 is proved for $n = 1$. This fact will be used in treating the higher dimensional case.

Turning now to arbitrary $n > 1$ we treat first the case of an odd kernel $\Omega(x) = -\Omega(-x)$ (for this case we require only that Ω be measurable and bounded by κ). The formula (A2.1) may be written in the form

$$\begin{aligned} u(x, t) &= \int K(-y, t) f(x+y) dy \\ &= \int_{|\eta|=1} \Omega(-\eta) d\omega_\eta \int_0^\infty \frac{r^{n-1}}{(r^2+t^2)^{n/2}} f(x+r\eta) dr, \end{aligned}$$

in polar coordinates. Because Ω is odd we may write this in the form

$$u(x, t) = \tfrac{1}{2} \int \Omega(-\eta) d\omega_\eta \int_{-\infty}^\infty \frac{|r|^{n-1} (\text{sign } r)}{(r^2+t^2)^{n/2}} f(x+r\eta) dr.$$

Let $I(x; \eta)$ denote the inner integral. The kernel

$$\frac{|r|^{n-1} (\text{sign } r)}{(r^2+t^2)^{n/2}}$$

satisfies the conditions of Theorem 3.2 for $n = 1$. Applying the theorem we find that the integral of $|I(x; \eta)|^p$ along the line parallel to η through some point x_0 is bounded by a constant times the integral of $|f(x)|^p$ along the same line, the constant depending only on n and p.

Now

$$|u|_{L_p, t} \leqq \text{constant} \cdot \kappa \left[\int d\omega_\eta \int |I(x; \eta)|^p dx \right]^{1/p}.$$

Integrating first with respect to lines, in the x-space, parallel to η and then with respect to axes perpendicular to these we find, by the above, that

$$|u|_{L_p,t} \leqq \text{constant} \cdot \kappa |f|_{L_p}.$$

This completes the case of an odd kernel. We may again assert as before that Theorem 3.2 is proved in general for kernels that are odd in x.

Since any kernel K can be written as the sum of an odd and an even kernel it suffices now to consider an even kernel $\Omega(x) = \Omega(-x)$. We may assume that f is of class C^∞ and has compact support. Consider the kernels (not involving t)

$$K_m(x) = -i\pi^{-(k+1)/2}\Gamma[\tfrac{1}{2}(k+1)]\,\frac{x_m}{|x|^{n+1}}, \qquad m = 1, \cdots, n.$$

In [9] Calderon and Zygmund proved that, if

$$f_m(x) = K_m * f = \int K_m(x-y)f(y)dy,$$

the integral being the limit, as $\varepsilon \to 0$, of the integral over $|x-y| > \varepsilon$, then

$$f = \sum K_m * f_m.$$

Furthermore,

$$|f_m|_{L_p} \leqq A_p |f|_{L_p}, \qquad 1 < p < \infty,$$

where A_p depends only on p and n.

We may therefore write

$$u = K * f = \sum K * K_m * f_m = \iint K(x-y, t)K_m(y-z)f_m(z)dz\,dy.$$

The functions $\Gamma_m(x, t) = K * K_m$ are clearly homogeneous of degree $-n$ and odd in x. It may be shown, as in [9], that the kernels Γ_m satisfy the conditions of Theorem 3.2 and that

$$u = \sum \Gamma_m * f_m.$$

Having established Theorem 3.2 for odd kernels, we find

$$\begin{aligned} |u|_{L_p,t} &\leqq \text{constant} \cdot \kappa \sum |f_m|_{L_p} \\ &\leqq \text{constant} \cdot \kappa \cdot A_p |f|_{L_p}, \end{aligned}$$

completing the proof.

Appendix 3

Proof of Theorem 3.3

We first prove a related, but more elementary, result which, however, would be sufficient with the aid of Lemma 9.1′ for obtaining all our L_p estimates for solutions of elliptic boundary value problems.

THEOREM A3.1. *Suppose all the conditions in Theorem* 3.3 *to be satisfied with the exception of condition* (3.3). *Then each x derivative $D_x u$ belongs to L_p in $t > 0$, and*

$$|D_x u|_{0,L_p} \leqq C\kappa |f|_{1-1/p,L_p},$$

where C depends only on p and n.

Proof: From the definition of the seminorms for f it follows that there exists a function $v(x, t)$ in $t \geqq 0$ with

$$\text{(A3.1)} \qquad v(x, 0) = f(x) \quad \text{and} \quad |v|_{1,L_p} \leqq 2|f|_{1-1/p,L_p}.$$

Since $f \in L_p$ it also follows easily that $v(x, t) \in L_p$ on every hyperplane $t =$ constant, and that

$$\text{(A3.2)} \qquad \left(\int |v(x, t)|^p dx\right)^{1/p} \leqq \text{constant} \cdot t^{1-1/p} \quad \text{as} \quad t \to \infty.$$

From the identity

$$\begin{aligned} D_s(v(y, s)K(x-y, t+s)) \\ = D_s v \cdot K(x-y, t+s) + vD_s K \end{aligned}$$

we obtain, after integration with respect to s, y over the strip $0 \leqq s \leqq T$,

$$\text{(A3.3)} \qquad \begin{aligned} -u(x, t) &= -\int v(y, 0)K(x-y, t)dy \\ &= \int_{0<s<T} D_s v(y, s) \cdot K(x-y, t+s)dy\,ds \\ &\quad + \int_{0<s<T} v(y, s)D_s K(x-y, t+s)dy\,ds \\ &\quad - \int v(y, T)K(x-y, t+T)dy. \end{aligned}$$

Operating with $D_i = \partial/\partial x_i$ we find, with the aid of a partial integration which is easily justified,

$$\begin{aligned} -D_i u = &\iint_{0<s<T} D_s v(y, s) \cdot D_i K(x-y, t+s)dy\,ds \\ &+ \iint D_i v(y, s) \cdot D_s K(x-y, t+s)dy\,ds \\ &- \int v(y, T)D_i K(x-y, T+t)dy. \end{aligned}$$

From (A3.2) and the inequality

$$|D_i K(y, t)| \leqq \text{constant} \cdot (|y|^2+t^2)^{-(n+1)/2}$$

we find readily that the last, n-dimensional, integral is $O(T^{-(n+1)/p})$ as $T \to \infty$. Hence letting $T \to \infty$ we obtain the formula

$$\begin{aligned} -D_i u = &\iint_{s>0} D_s v(y, s) \cdot D_i K(x-y, t+s)dy\,ds \\ &+ \iint_{s>0} D_i v(y, s) \cdot D_s K(x-y, t+s)dy\,ds. \end{aligned}$$

Since the kernels $D_i K$, $D_t K$ satisfy the conditions of Lemma 3.2 with, in fact, $\Omega =$ constant, we obtain, on applying the lemma,

$$|D_i u|_{0,L_p} \leqq \text{constant} \cdot |v|_{1,L_p}.$$

In virtue of (A3.1) the theorem follows.

Proof of Theorem 3.3: In virtue of Theorem A3.1 it is only necessary to establish the estimate

$$|D_t u|_{0,L_p} \leqq \text{constant} \cdot |f|_{1-1/p,L_p}\,. \tag{A3.4}$$

We proceed as above. Let v again be the function satisfying (A3.1). We obtain from (A3.3) the representation

$$\begin{aligned} -D_t u(x,t) &= \iint_{s>0} D_s v(y,s)\cdot D_t K(x-y,t+s)dy\,ds \\ &\quad + \iint_{s>0} v(y,s)D_t^2 K(x-y,t+s)dy\,ds = I_1+I_2\,. \end{aligned} \tag{A3.5}$$

Since $D_t K$ satisfies the conditions of Lemma 3.2, we have as before

$$|I_1|_{0,L_p} \leqq \text{constant}\cdot |D_t v|_{0,L_p} \leqq \text{constant}\cdot |f|_{1-1/p,L_p}\,.$$

Hence to prove (A3.4), and thus the theorem, it suffices to prove that in $t>0$

$$|I_2|_{0,L_p} \leqq \text{constant}\cdot \sum_{i=1}^{n} |D_i v|_{0,L_p}\,, \tag{A3.6}$$

where

$$I_2(x,t) = \iint_{s>0} v(y,s)D_t^2 K(x-y,t+s)dy\,ds. \tag{A3.7}$$

We observe that, in virtue of Lemma 3.1, condition (3.3) implies the condition

$$\int D_t^2 K(y,t)dy = 0, \qquad t>0. \tag{A3.8}$$

The inequality (A3.6) then follows from the following lemma, applied to the kernel $J(x,t) = D_t^2 K(x,t)$. (Also Theorem A3.1 follows directly from this lemma.)

LEMMA A3.1. *Let $J(P)$ be a continuous kernel, homogeneous of degree $-(n+2)$, in $t>0$, which is bounded by κ on $|P|=1$ and satisfies the condition: for every $t>0$*

$$\int J(x,t)dx = 0. \tag{A3.9}$$

Let $v(x,t)$ have finite $|\ |_{1,L_p}$ norm. Then the function

$$I(x,t) = \iint_{s>0} v(y,s)J(x-y,t+s)dy\,ds$$

satisfies the inequality

$$|I|_{0,L_p} \leqq C\kappa \sum_{i=1}^{n} |D_i v|_{0,L_p}\,, \tag{A3.10}$$

where C depends only on p and n.

Proof: We may assume that $\kappa = 1$, so that

$$|J(P)| \leqq |P|^{-n-2}. \tag{A3.11}$$

The word "constant" will be used to denote various constants depending only on n.

We distinguish two cases.

Case A; $n = 1$. With x now a single variable introduce the function

$$L(x, t) = \int_{-\infty}^{x} J(y, t)dy, \qquad t > 0.$$

Clearly L is continuous on $t > 0$, homogeneous of degree -2 and satisfies $D_x L = J$. We claim, furthermore, that

$$|L(x, t)| \leqq \frac{\text{constant}}{x^2+t^2}. \tag{A3.12}$$

By the homogeneity it suffices to verify this for $t = 1$, $|x| \leqq 1$ and for $|x| = 1$, $0 < t < 1$; the former case following trivially from (A3.11), we consider only the second case. By (A3.11) we have

$$|L(-1, t)| \leqq \int_{-\infty}^{-1} \frac{dy}{(y^2+t^2)^{3/2}} \leqq \text{constant}.$$

Next it follows from (A3.9) that

$$L(1, t) = \int_{-\infty}^{1} J(y, t)dy = -\int_{1}^{\infty} J(y, t)dy.$$

As in the case for $x = -1$ we find that this is bounded in absolute value by a constant. Thus (A3.12) is established.

Returning to I we have, on integrating by parts,

$$I = \iint D_y v(y, s) \cdot L(x-y, t+s)dy\,ds.$$

By (A3.12) the kernel L satisfies the conditions of Lemma 3.2 with $\Omega =$ constant, which then yields the desired estimate (A3.10).

Case B; $n \geqq 2$. We claim first that J may be assumed to satisfy the additional conditions

$$\int x_i J(x, t)dx = 0, \qquad t > 0, \quad i = 1, \cdots, n. \tag{A3.13}$$

Indeed if (A3.13) does not hold we may replace J by

$$J_0(x, t) = J(x, t) - \sum_{i=1}^{n} c_i M_i(x, t),$$

where

$$M_i = \frac{x_i}{(|x|^2+t^2)^{(n+3)/2}} = D_i M,$$

$$M = -\frac{1}{n+1} (|x|^2+t^2)^{-(n+1)/2},$$

and the constants c_i are chosen so that the kernel J_0 satisfies conditions (A3.13), i.e.,

$$c_i = nt \int x_i J dx \cdot \left(\int \frac{|x|^2}{(|x|^2+1)^{(n+3)/2}} dx \right)^{-1}.$$

By the homogeneity of J the value of c_i so obtained is indeed a constant. We see also from (A3.11) that $|c_i| \leqq$ constant.

Since the kernels M_i trivially satisfy (A3.9) we see that the kernel J_0 satisfies all the conditions of Lemma A3.1 and, in addition, (A3.13). Furthermore, inequality (A3.10) holds in case the kernel M_i is substituted for J. To see this last fact we observe that, setting M_i in place of J and integrating by parts, we have

$$\begin{aligned} I &= \iint v(y, s) M_i(x-y, t+s) dy\, ds \\ &= \iint D_i v(y, s) \cdot M(x-y, t+s) dy\, ds, \end{aligned}$$

and the desired inequality (A3.10) follows from Lemma 3.2 applied to the kernel M.

Thus to prove the lemma it suffices to prove (A3.10) for J_0, i.e., for a kernel J satisfying the additional conditions (A3.13).

To this end, let $F(x)$ be the fundamental solution with singularity at the origin, of the Laplace equation $\Delta_x \phi = 0$:

$$\begin{aligned} F(x) &= \text{constant} \cdot |x|^{2-n} \qquad \text{for } n \geqq 3, \\ F(x) &= \text{constant} \cdot \log |x| \qquad \text{for } n = 2. \end{aligned}$$

The following representation formula holds almost everywhere for a function $g(x)$ which is continuous and such that $g(x)(1+|x|)^{1-n}$ is absolutely integrable in the whole space:

$$g(x) = \sum_{i=1}^{n} D_i \int g(y) D_i F(x-y) dy.$$

Applying this to the function $J(x, t)$ for a fixed $t > 0$ we have

$$J(x, t) = \sum D_i L_i(x, t),$$

where we have set

$$L_i(x, t) = \int J(y, t) \cdot D_i F(x-y) dy. \tag{A3.14}$$

Hence I can be written in the form

$$\begin{aligned} I &= \sum \iint_{s>0} v(y, s) D_i L_i(x-y, t+s) dy\, ds \\ &= \sum \iint_{s>0} D_i v(y, s) \cdot L_i(x-y, t+s) dy\, ds, \end{aligned}$$

and Lemma A3.1 will follow from Lemma 3.2 provided the kernels $L_i(x, t)$ satisfy the conditions of that lemma.

The continuity and homogeneity of the L_i are easily established, as is a bound for $|L_i(x, t)|$ for $|x| \leqq 1$, $t > \frac{1}{2}$, and we shall merely establish here the following estimate for L_i, enabling us (see the remarks after Lemma 3.2) to apply the lemma:

$$|L_i(x, t)| \leqq \text{constant} \cdot \left|\log \frac{1}{t}\right| \quad \text{for} \quad |x| = 1, \qquad t < \tfrac{1}{2}. \tag{A3.15}$$

For $|x| = 1$ define

$$F_i(y; x) = D_i F(x-y) - D_i F(x) + \sum_{j=1}^{n} y_j D_j D_i F(x). \tag{A3.16}$$

Then by Taylor's formula we see that $F_i(y; x)$ satisfies for $|x| = 1$

$$\begin{aligned} &|F_i(y; x)| \leqq \text{constant} \cdot |y|^2 \quad \text{for} \quad |y| \leqq \tfrac{1}{2}, \\ &|F_i(y; x)| \leqq \text{constant} \cdot (|x-y|^{1-n} + |y|) \quad \text{for} \quad |y| \geqq \tfrac{1}{2}. \end{aligned} \tag{A3.17}$$

Using (A3.9), (A3.13), (A3.14) and (A3.16) we see that, for i fixed,

$$\begin{aligned} L_i(x, t) &= \int J(y, t) D_i F(x-y) dy = \int J(y, t) F_i(y; x) dy \\ &= \int_{|y| \leqq 1/2} J F_i dy + \int_{|y| > 1/2} J F_i dy \\ &= J_1(x, t) + J_2(x, t). \end{aligned}$$

By (A3.17) and (A3.11) we have

$$\begin{aligned} |J_1(x, t)| &\leqq \int_{|y| \leqq 1/2} |J(y, t) F_i(y; x)| dy \\ &\leqq \text{constant} \cdot \int_{|y| \leqq 1/2} \frac{|y|^2}{(|y|^2 + t^2)^{(n+2)/2}} dy \\ &\leqq \text{constant} \cdot \int_{|y| \leqq 1/2} \frac{dy}{(|y|^2 + t^2)^{n/2}} \\ &= \text{constant} \cdot \int_0^{1/2} \frac{r^{n-1}}{(r^2 + t^2)^{n/2}} dr \\ &\leqq \text{constant} \cdot \int_0^{1/2} \frac{dr}{(r^2 + t^2)^{1/2}} \\ &= \text{constant} \cdot \int_0^{1/2t} \frac{dr}{(r^2 + 1)^{1/2}} \\ &\leqq \text{constant} \cdot \log \frac{1}{t} \quad \text{for} \quad t < \tfrac{1}{2}. \end{aligned}$$

For $J_2(x, t)$ we find in a similar manner:

$$\begin{aligned} |J_2(x, t)| &\leqq \text{constant} \cdot \int_{|y| > 1/2} |y|^{-n-2} (|x-y|^{1-n} + |y|) dy \\ &\leqq \text{constant}. \end{aligned}$$

Combining these inequalities for J_1 and J_2 we obtain the inequality (A3.15) for L_i. This completes the proof of Lemma A3.1, and hence of Theorem 3.3.

Appendix 4

Proof of Theorem 3.1A

Our proof makes use of the following

LEMMA. *Suppose $u(x, t)$ is once continuously differentiable in the half-space $t > 0$, and satisfies for some positive $\alpha < 1$ and every first derivative Du,*

$$|Du| \leqq At^{\alpha-1}.$$

Then

$$[u]_\alpha \leqq A\left(1+\frac{2}{\alpha}\right).$$

Proof: Consider two points (x, t), (y, τ) with $t \leqq \tau$; denote their distance apart by d. Then we have

$$\begin{aligned} d^{-\alpha}|u(x,t)-u(y,\tau)| &\leqq d^{-\alpha}\int_t^{t+d}|D_t u(x,s)|ds \\ &\quad + d^{-\alpha}\left|\int_\tau^{t+d} D_t u(y,s)ds\right| \\ &\quad + d^{-\alpha}|u(x,t+d)-u(y,t+d)| \\ &\leqq d^{-\alpha}2A\int_t^{t+d} s^{\alpha-1}ds + d^{-\alpha}A(d+t)^{\alpha-1}d \end{aligned}$$

(here we have used our assumption on Du, and the theorem of the mean)

$$\begin{aligned} &\leqq \frac{2A}{\alpha}d^{-\alpha}d^{\alpha}+A \\ &= A\left(1+\frac{2}{\alpha}\right). \end{aligned}$$

Proof of Theorem 3.1A: Set $\sum [D^{l+h-n}f]_\alpha = F$. In virtue of the lemma it suffices to show that

$$|D^{l+1}u(x,t)| \leqq \text{constant}\cdot F\cdot t^{\alpha-1}. \tag{A4.1}$$

Since f vanishes for $|x| \geqq R$, f and its derivatives up to order $l+h-n$ are bounded in absolute value by constant $\cdot$ F. Thus we see immediately from (3.16) and the expression for u that

$$|D^{l+1}u(x,t)| \leqq \text{constant}\cdot F\cdot(|x|^2+t^2)^{-(l+1+h)/2} \quad \text{for} \quad |x|^2+t^2 \geqq 4R^2,$$

from which the following estimate follows easily:

$$|D^{l+1}u(x,t)| \leqq \text{constant}\cdot F\cdot t^{\alpha-1} \quad \text{for} \quad |x|^2+t^2 \geqq 4R^2. \tag{A4.2}$$

Thus we need only consider the region $|x|^2+t^2 < 4R^2$. For x fixed, $|x| \leqq 2R$, denote by $P(y)$ the osculating polynomial of degree $l+h-n$ of $f(y)$ at the point x so that $f(y)-P(y) = o(|x-y|^{l+h-n})$ as $y \to x$. With the aid of Taylor's formula it is easily seen that, in fact,

$$|f(y)-P(y)| \leqq \text{constant} \cdot F \cdot |y-x|^{l+h-n+\alpha}. \tag{A4.3}$$

The coefficients of the polynomial, being constants times derivatives of f at x, are also bounded by constant $\cdot F$.

Let $\zeta(y)$ be a non-negative C^∞ function which is identically one for $|y| \leqq 3R$ and vanishes for $|y| \geqq 3R+1$. Write $D^{l+1}u(x, t)$ as a sum, for $|x| \leqq 2R$,

$$\begin{aligned} D^{l+1}u(x, t) &= \int D^{l+1}K(x-y, t)(f(y)-P(y))\zeta(x-y)dy \\ &\quad + \int D^{l+1}K(x-y, t)P(y)\zeta(x-y)dy \\ &= I_1+I_2 . \end{aligned}$$

From (A4.3) and (3.16) we have

$$\begin{aligned} |I_1| &\leqq \text{constant} \cdot \int \frac{|x-y|^{l+h-n+\alpha}}{(|x-y|^2+t^2)^{(l+1+h)/2}}\, dy \\ &= \text{constant} \cdot t^{\alpha-1} \int \frac{|v|^{l+h-n+\alpha}}{(|v|^2+1)^{(l+1+h)/2}}\, dv \\ &= \text{constant} \cdot t^{\alpha-1}, \end{aligned}$$

where we have made the change of variable $y = x+vt$.

In order to estimate I_2 for $|x| \leqq 2R$, $t \leqq 2R$, we note that I_2 is a sum of terms of the form

$$\begin{aligned} &\text{coefficient} \cdot \int \prod (y_i-x_i)^{k_i} D^{l+1}K(x-y, t)\zeta(x-y)dy \\ &= \text{coefficient} \cdot \int \prod y_i^{k_i} D^{l+1}K(-y, t)\zeta(-y)dy \end{aligned}$$

with $\sum k_i \leqq l+h-n$. We now invoke hypothesis (i) of the theorem to ensure that these integrals are bounded for $t \leqq 2r$ (this is the only place that (i) is used), so that

$$|I_2| \leqq \text{constant} \cdot F, \qquad |x|, t \leqq 2R.$$

This estimate, combined with the estimate above for I_1, and (A4.2) yields the desired estimate (A4.1).

Appendix 5

An Improvement of the Interior Schauder Estimates

Consider a uniformly elliptic equation

$$Lu = F \tag{A5.1}$$

of order $2m$ in a domain $\mathfrak{D}$ and suppose that F, and the coefficients of L, belong to C^α in $\mathfrak{D}$ for some fixed positive $\alpha < 1$. In [10] we proved the following interior Schauder estimate (see Theorems 1 and 4 there): if $u \in C^{2m+\alpha}$ in $\mathfrak{D}$, then for every subdomain $\mathfrak{A} \subset \overline{\mathfrak{A}} \subset \mathfrak{D}$

$$|u|^{\mathfrak{A}}_{2m+\alpha} \leqq \text{constant} \cdot (|F|^{\mathfrak{D}}_{\alpha} + |u|^{\mathfrak{D}}_{0}), \tag{A5.2}$$

the constant being independent of u.

We shall extend this to solutions which are not assumed to be of class $C^{2m+\alpha}$. We shall, in fact, consider generalized solutions having derivatives in, say, L_2.

THEOREM A5.1. *If u has square integrable derivatives up to order $2m$ (in the sense of Friedrichs, Sobolev) and satisfies* (A5.1) *almost everywhere, then in fact u belongs to $C^{2m+\alpha}$ in $\mathfrak{D}$, and hence satisfies* (A5.2).

For second order elliptic equation E. Hopf [14] proved that solutions which are in C^2 belong to $C^{2+\alpha}$; we shall follow his method of proof. In the course of the proof we shall make use of the following well known

LEMMA A5.1 (*Sobolev* [37]). *Let $f(x)$ be a function with support in the unit sphere and belonging to L_p, $1 < p < \infty$, in $(n+1)$-space, and consider the function*

$$g(x) = \int \frac{f(y)}{|x-y|^\lambda}\, dy, \qquad \lambda < n+1.$$

Then, denoting the L_p norm in the unit sphere S_1 of a function ϕ by $\|\phi\|^{S_1}_{L_p}$ we have, setting $a = n+1-\lambda-(n+1)/p$,

(a) for $a < 0$, $\|g\|^{S_1}_{L_q} \leqq \text{constant} \cdot \|f\|^{S_1}_{L_p}$ where $\dfrac{1}{q} = -\dfrac{a}{n+1}$,

(b) for $a = 0$, $\|g\|^{S_1}_{L_q} \leqq \text{constant} \cdot \|f\|^{S_1}_{L_p}$ for every finite $q > 0$,

(c) for $a > 0$, $a \neq$ *integer*, $|g|^{S_1}_{a} \leqq \text{constant} \cdot \|f\|^{S_1}_{L_p}$,

(d) for $a > 0$, $a =$ *integer*, $|g|^{S_1}_{b} \leqq \text{constant} \cdot \|f\|^{S_1}_{L_p}$, $\qquad 0 \leqq b < a$,

where the constants depend only on n, λ, p, q and b.

We shall use the lemma in showing by successive steps that u has derivatives in L_p locally, for some $p > 2$, and then using this information to show that its derivatives are in L_p for some still higher p. Repeating this

process we find eventually that u belongs to $C^{2m+\varepsilon}$ for some $\varepsilon > 0$. Then we use a result of E. Hopf [14] to conclude that $u \in C^{2m+\alpha}$. It will become clear from the proof that we could have assumed u and its derivatives up to order $2m$ to be in L_p for any $p > 1$.

Proof of Theorem A5.1: Since the argument is local we may suppose that $\mathfrak{D}$ lies in the unit sphere. Let k denote a bound on the coefficients of L and on their norms $|\ |_\alpha^{\mathfrak{D}}$. Let $L'(P, D)$ be the terms of highest order in $L(P, D)$. For any fixed point P_0 let $\Gamma(P_0\,;\,P)$ be the fundamental solution (4.2) of F. John [17] of the operator $L'(P_0\,,\,D)$ with coefficients equal to their values at P_0. It is important to observe, from the explicit formula [17] for Γ, that for $|P| = 1$, $D_P^{2m}\Gamma(P_0\,;\,P)$ is of class C^α in P_0 and, in fact, its $|\ |_\alpha^{\mathfrak{D}}$ norm with respect to P_0 is bounded by a constant independent of P for $|P| = 1$.

For any subdomain $\mathfrak{A} \subset \bar{\mathfrak{A}} \subset \mathfrak{D}$ let $\zeta(P)$ be a non-negative function with compact support in $\mathfrak{D}$ which is identically one on $\mathfrak{A}$. For any C^∞ function $u(P)$ in $\mathfrak{D}$ we have

$$\zeta(P)u(P) = \int \Gamma(P_0\,;\,P-Q)L'(P_0\,,\,D_Q)\zeta(Q)u(Q)dQ,$$

so that for $P \in \mathfrak{A}$, $j \leqq 2m$,

$$\begin{aligned} D_P^j u(P) = \int \Big\{ & D_P^j \Gamma(P_0\,;\,P-Q)L(\zeta u) \\ & + D_P^j \Gamma(P_0\,;\,P-Q)(L'(P_0, D_Q) - L(Q, D))\zeta u \Big\}\, dQ + c_j L(\zeta u), \end{aligned}$$

where $c_j = c_j(P_0)$ is a constant depending on D^j, $c_j = 0$ for $j < 2m$, and where for $j = 2m$ the integrals are taken in the sense of the Cauchy principal value. If now we set $P_0 = P \in \mathfrak{A}$ we find for $0 \leqq j \leqq 2m$

$$\begin{aligned} D_P^j u(P) = & \int D^j \Gamma(P, P-Q)L(\zeta u)dQ + c_j L(\zeta u), \\ & + \int D^j \Gamma(P, P-Q)(L'(P, D) \\ & - L(Q, D))\zeta(Q)u(Q)dQ, \end{aligned} \tag{A5.3}$$

where $D^j\Gamma(P, P-Q)$ involves only differentiation with respect to the second variable. Here $c_j = c_j(P)$ is Hölder continuous in P.

We claim now that (A5.3) holds also for functions u having only square integrable derivatives up to order $2m$ in the sense that the difference of the two sides of (A5.3) has vanishing L_2 norm in $\mathfrak{A}$. To see this one approximates such a function by C^∞ functions and checks the convergence of both sides of (A5.3) on going to the limit in the approximation. For $j < 2m$ the desired convergence follows easily with the aid of Lemma A5.1 and the bounds (4.3), (4.3)′ for the derivatives of $\Gamma(P_0\,;\,P)$. For $k = 2m$ it follows from certain known properties of the operator

$$f \to Tf = \int D^{2m}\Gamma(P; P-Q)f(Q)dQ. \tag{A5.4}$$

(a) The operator T takes L_p boundedly into L_p, $1 < p < \infty$ (this result is contained in Theorem 2 of Calderon, Zygmund [9]); in particular this holds for $p = 2$.

(b) T takes C^α boundedly into C^α (this result is well known, see for instance Mihlin [23]).

Before proceeding we write (A5.3) in the form

$$D^j u(P) = I_{1,j}+I_{2,j}+I_{3,j}+I_{4,j}, \qquad 0 \leqq j \leqq 2m, \tag{A5.3$'$}$$

where

$$\begin{aligned} I_{1,j}(P) &= \int D^j\Gamma(P; P-Q)\zeta(Q)F(Q)dQ + c_j F(P), \\ I_{2,j}(P) &= \int D^j\Gamma(P; P-Q) \sum_{\substack{|\beta|+|\gamma|\leqq 2m \\ |\gamma|<2m}} C_{\beta,\gamma}(Q)\cdot D^\beta\zeta(Q)\cdot D^\gamma u(Q)dQ, \\ I_{3,j}(P) &= -\int D^j\Gamma(P; P-Q)L''(Q, D)(\zeta(Q)u(Q))dQ, \\ I_{4,j}(P) &= \int D^j\Gamma(P; P-Q) \sum_{|\beta|=2m} (a_\beta(P)-a_\beta(Q))D^\beta(\zeta u)dQ; \end{aligned} \tag{A5.5}$$

here the functions $C_{\beta,\gamma}$, a_β are in C^α, the $a_\beta(P)$ being the leading coefficients of L, and $F = Lu$.

We note that, since $f \in C^\alpha$, also $I_{1,j} \in C^\alpha$ in $\mathfrak{A}$, by the above remark. We note furthermore that

$$|I_{4,2m}(P)| \leqq \text{constant}\cdot\int \frac{1}{|P-Q|^{n+1-\alpha}} \sum_{|\beta|=2m} |D^\beta(\zeta u)|dQ. \tag{A5.6}$$

If we now apply Lemma A5.1, using (4.3) and (4.3)$'$, we find that for $j \leqq 2m-1$, $I_{i,j} \in L_{q_1}$, $i = 2, 3, 4$, where $1/q_1 = \frac{1}{2}-1/(n+1)$ if $n > 1$, or q_1 is an arbitrary positive number if $n = 1$. Since $\mathfrak{A}$ was an arbitrary compact subdomain of $\mathfrak{D}$ it follows that $D^j u$, $j \leqq 2m-1$, belongs to L_{q_1} in any compact subset of $\mathfrak{D}$. We shall now assume that this is known and consider again the terms $I_{2,j}, \cdots, I_{4,j}$ for some fixed $\mathfrak{A}$ and ζ. (We shall not look at $I_{1,j}$ since it is always in C^α.)

If we apply the L_p preservation property (a) of T (see (A5.4)) to the terms $I_{2,2m}$, $I_{3,2m}$, we find them to be in L_{q_1}. Applying the lemma to (A5.6) we find that $I_{4,2m}$ is in L_{p_1} for

$$\frac{1}{p_1} = \frac{1}{2} - \frac{\alpha}{n+1}.$$

Since in any case we may take $q_1 > p_1$ it follows that

$$D^j u \in L_{p_1}, \qquad j \leqq 2m,$$

in any compact subdomain of $\mathfrak{D}$. Again we may use this fact to improve our knowledge of u.

Operating in this way, applying again Lemma A5.1 and the L_p preservation property of T, we find in general that if $D^j u \in L_p$, $p > 1$, $j \leqq 2m$, in any compact subdomain of $\mathfrak{D}$, then in fact, for $j \leqq 2m$,

(a) $D^j u \in L_{(1/p-\alpha/(n+1))^{-1}}$ in any compact subdomain if $\dfrac{1}{p} > \dfrac{\alpha}{n+1}$,

(b) $D^j u \in L_q$ for any q if $\dfrac{1}{p} = \dfrac{\alpha}{n+1}$,

(c) $D^j u \in C^\varepsilon$ for some positive ε if $\dfrac{1}{p} < \dfrac{\alpha}{n+1}$.

Continuing thus for a *finite* number of steps we find that for $j \leqq 2m$, $D^j u \in C^\varepsilon$ for some positive ε. But then for $j < 2m$, $D^j u \in C^1$.

Consider finally the terms $I_{2,2m}$, $I_{3,2m}$, $I_{4,2m}$. Since $D^j u \in C^1$ for $j < 2m$ it follows from the C^α preservation property (b) of T that $I_{2,2m}$ and $I_{3,2m}$ belong to C^α. Thus we need only consider the term $I_{4,2m}$. The fact that this term belongs to C^α is contained in the corollary to Lemma 3 of Section 2 of E. Hopf's paper [14].

This completes the proof of Theorem A5.1.

The method of proof also applies to uniformly elliptic equations in integral form, (9.1) (or really (9.2)) (see also Section 8 of [10])

$$Lu = \sum D^\beta a_{\beta,\mu} D^\mu u = \sum D^\beta F_\beta, \tag{A5.7}$$

where the summation is over $|\beta| \leqq 2m-l$, $|\mu| \leqq l$, for fixed $l < 2m$. Assuming that F_β, $a_{\beta,\mu} \in C^\alpha(C^0)$ for $|\beta| = 2m-l$ ($|\beta| < 2m-l$) we have

THEOREM A5.2. *If u has square integrable derivatives up to order l and satisfies* (A5.7) *in the integral form, see* (9.2), *then u belongs to $C^{l+\alpha}$.*[8]

The proof of Theorem A5.2 is similar to that of Theorem A5.1, the fundamental identity (A5.3) being replaced by: for $j \leqq l$

$$\begin{aligned} D^j u(P) = &\sum_\beta (-1)^{|\beta|} \int D^j D_Q^\beta [\Gamma(P; P-Q)\zeta(Q)] F_\beta(Q) dQ \\ &+ \sum_{\substack{|\beta| \leqq 2m-l \\ |\mu|+|\gamma| \leqq l \\ |\mu| < l}} (-1)^{|\beta|} \int D^j D_Q^\beta \Gamma(P; P-Q) a_{\beta,\mu,\gamma} D^\gamma \zeta \cdot D^\mu u dQ \\ &- \sum_{|\beta|+|\mu| < 2m} (-1)^{|\beta|} \int D^j D_Q^\beta \Gamma(P; P-Q) a_{\beta,\mu}(Q) D_Q^\mu(\zeta u) dQ \\ &+ \sum_{|\beta|+|\mu| = 2m} (-1)^{|\beta|} \int D^j D_Q^\beta \Gamma(P; P-Q)(a_{\beta,\mu}(P) - a_{\beta,\mu}(Q)) D_Q^\mu(\zeta u) dQ. \end{aligned}$$

[8]Thus in Theorems 1 and 4 of [10] the conditions that u_j belong to $C^{t_j+\alpha}$ can be relaxed to: u_j has square integrable derivatives up to order t_j. Similarly in Theorem 4′ of [10] the conditions that u_j belong to $C^{\sigma+t_j+\alpha}$ may be relaxed to: u_j has square integrable derivatives up to order $\sigma+t_j$.

It is interesting that *by a smoothing process one may prove directly, with the aid of* (A5.2) *that a solution of* (A5.1) *belonging to* C^{2m} *belongs also to* $C^{2m+\alpha}$. We feel that it is worthwhile to include this proof which follows the procedure of Friedrichs in his first application of his "mollifier".

The proof involves constructing functions which approximate u uniformly in any given compact subdomain $\mathfrak{A} \subset \overline{\mathfrak{A}} \subset \mathfrak{D}$ and which have uniformly bounded norm $|\ |_{2m+\alpha}^{\mathfrak{A}}$. To this end we convolve u with a C^∞ function of compact support (the Friedrichs mollifier) which we first describe. Let $j(r)$ be a non-negative C^∞ function of one variable with support in $|r| \leqq 1$, and such that

$$\int_{-\infty}^{\infty} j(r)dr = 1.$$

Let $\mathfrak{B}$ be a domain such that $\overline{\mathfrak{A}} \subset \mathfrak{B} \subset \overline{\mathfrak{B}} \subset \mathfrak{D}$. Let $\varepsilon > 0$ be less than $\frac{1}{2}(n+1)^{-1/2}$ times the distance from $\overline{\mathfrak{A}}(\overline{\mathfrak{B}})$ to the boundary of $\mathfrak{B}(\mathfrak{D})$. If $x = (x_1, \cdots, x_{n+1})$ denotes the independent variables in $\mathfrak{D}$, consider in $\mathfrak{B}$ the C^∞ function

$$u_\varepsilon(x) = \varepsilon^{-(n+1)} \int \prod_{i=1}^{n+1} j\left(\frac{x_i - y_i}{\varepsilon}\right) u(y)dy,$$

integration being over the entire $(n+1)$-dimensional space. Set $\prod_i j(x_i) = j(x)$.

Our aim is to establish a bound for $|u_\varepsilon|_{2m+\alpha}^{\mathfrak{A}}$ independent of ε; since u_ε and its derivatives up to order $2m$ converge uniformly to u and its corresponding derivatives in $\mathfrak{A}$ the theorem will follow. To obtain the desired estimate it suffices by the interior estimate (A5.1) to establish a bound for $|Lu_\varepsilon|_\alpha^{\mathfrak{D}}$ independent of ε. (Clearly $|u_\varepsilon|_0^{\mathfrak{B}} \leqq |u|_0^{\mathfrak{D}}$). To this end it suffices to establish such a bound for $|Lu_\varepsilon - (Lu)_\varepsilon|_\alpha^{\mathfrak{B}}$ since, as is easily seen, $|(Lu)_\varepsilon|_\alpha^{\mathfrak{B}} \leqq |Lu|_\alpha^{\mathfrak{D}}$. Thus it suffices, in particular, to establish such a bound for a typical term in L, $a(x)D^k u(x)$, $k \leqq 2m$, i.e., to prove

$$|g(x, \varepsilon)|_\alpha^{\mathfrak{B}} \leqq \text{constant independent of } \varepsilon, \tag{A5.8}$$

where

$$\begin{aligned} g(x; \varepsilon) &= aD^k u_\varepsilon - (aD^k u)_\varepsilon \\ &= \varepsilon^{-(n+1)} \int j\left(\frac{x-y}{\varepsilon}\right) (a(x) - a(y)) D^k u(y)dy. \end{aligned}$$

The function $j(x)$ and the coefficient $a(x)$ satisfy Hölder conditions

$$\frac{|j(x) - j(x')|}{|x - x'|^\alpha} \leqq K, \qquad \frac{|a(x) - a(x')|}{|x - x'|^\alpha} \leqq k. \tag{A5.9}$$

Let M denote a bound for $|D^k u|$ in the set of points that are closer to $\dot{\mathfrak{B}}$ than to $\dot{\mathfrak{D}}$.

Consider arbitrary points x, x' in $\mathfrak{B}$. We have

$$
\begin{aligned}
|g(x,\varepsilon)-g(x',\varepsilon)| &= \varepsilon^{-(n+1)}\int\left[j\left(\frac{x-y}{\varepsilon}\right)-j\left(\frac{x'-y}{\varepsilon}\right)\right][a(x)-a(y)]D^k u(y)dy \\
&\quad + \varepsilon^{-(n+1)}\int j\left(\frac{x'-y}{\varepsilon}\right)(a(x)-a(x'))D^k u(y)dy.
\end{aligned}
\tag{A5.10}
$$

We note that the first integrand on the right vanishes unless $|y_i-x_i| < \varepsilon$, $i = 1, \cdots, n$, or $|y_i-x_i'| < \varepsilon$, $i = 1, \cdots, n$, so that the first term on the right is bounded by I_1+I_2, where

$$
I_1 = M\varepsilon^{-(n+1)}\int_{|y_i-x_i|<\varepsilon}\left|j\left(\frac{x-y}{\varepsilon}\right)-j\left(\frac{x'-y}{\varepsilon}\right)\right||a(x)-a(y)|dy,
$$

$$
I_2 = M\varepsilon^{-(n+1)}\int_{|y_i-x_i'|<\varepsilon}\left|j\left(\frac{x-y}{\varepsilon}\right)-j\left(\frac{x'-y}{\varepsilon}\right)\right|(|a(x)-a(x')|+|a(x')-a(y)|)dy.
$$

With the aid of (A5.9) we see that

$$
I_1 \leqq \text{constant}\cdot MK\left|\frac{x-x'}{\varepsilon}\right|^\alpha k\varepsilon^\alpha = \text{constant}\cdot MK|x-x'|^\alpha
$$

and

$$
\begin{aligned}
I_2 &\leqq 2Mk|x-x'|^\alpha + \text{constant}\cdot MK\left|\frac{x-x'}{\varepsilon}\right|^\alpha k\varepsilon^\alpha \\
&\leqq \text{constant}\cdot Mk(1+K)|x-x'|^\alpha.
\end{aligned}
$$

Similarly the second term on the right of (A5.10) is bounded by

$$
Mk|x-x'|^\alpha.
$$

Here the constants depend only on n. Combining these estimates we obtain the desired Hölder inequality (A5.8).

Appendix 6

Proof of Lemma 9.1

In proving the lemma we shall show that for any $\delta > 0$ we have

$$
\begin{aligned}
|D_t v|_\alpha^{\Sigma_{R-\delta}} &\leqq \text{constant}\cdot(|v|_0^{\Sigma_{R-\delta/2}} \\
&\quad + \sum_{i=1}^n |D_i v|_\alpha^{\Sigma_{R-\delta/2}} + \sum |v_\nu|_\alpha^{\Sigma_{R-\delta/2}},
\end{aligned}
\tag{A6.1}
$$

where the constant depends only on n, k, R, α and δ.

It suffices to prove this for C^∞ functions v_ν, v in $\Sigma_{R-\delta/2}$, since we may approximate v in $\Sigma_{R-\delta/2}$ by the C^∞ functions $J_{\varepsilon,\varepsilon}v$ (defined on page 673) for

ε sufficiently small, and we see that $J_{s,\varepsilon}v$ satisfies

$$D_t^k J_{s,\varepsilon} v = \sum D^\nu J_{s,\varepsilon} v_\nu .$$

If then (A6.1) holds for $J_{s,\varepsilon}v$ and $J_{s,\varepsilon}v_\nu$, it follows for v and v_ν by letting $\varepsilon \to 0$.

We prove (A6.1), for C^∞ functions in $\Sigma_{R-\delta/2}$ by extending v across the planar boundary $t = 0$ with the aid of the formula (4.4), slightly modified. First we write (9.5) in a form distinguishing the x- and t-derivatives on the right

$$D_t^k v = \sum_{|\mu|+i<k} D_x^\mu D_t^i v_{\mu,i} , \tag{A6.2}$$

where $\nu = (\mu, i)$, $v_{\mu,i} = v_\nu$. We now extend v in a manner similar to that of (4.4) by defining for $t < 0$ and $|t|$ small

$$v(x, t) = v_N(x, t) = \sum_{p=1}^{N} \lambda_p f\left(x, -\frac{t}{p}\right),$$

N large. Here the λ_p are to be such that the extended function v is of class C^{N-1} in the full sphere $S_{R-\delta/2} : |x|^2+t^2 < (R-\delta)^2$, i.e., so as to satisfy

$$\sum_{p=1}^{N} \lambda_p \left(-\frac{1}{p}\right)^j = 1, \qquad j = 0, \cdots, N-1. \tag{A6.3}$$

At the same time we also extend the $v_{\mu,i}(x, t)$ for $t < 0$ by the formulas

$$v_{\mu,i}(x, t) = \sum_p \lambda_p \left(-\frac{1}{p}\right)^{k-i} v_{\mu,i}\left(x, -\frac{t}{p}\right)$$

and verify that (A6.2) holds for $t < 0$. By (A6.3) the extended functions $v_{\mu,i}$ are of class $C^{N-k+i-1}$ in $S_{R-\delta/2}$. Furthermore each norm occurring on the right of (A6.1), taken over $S_{R-\delta/2}$ instead of $\Sigma_{R-\delta/2}$, is less than a constant times the corresponding norm over $\Sigma_{R-\delta/2}$, the constant depending only on k and N.

If now, say, $N = 4k$ we see that the function v satisfies in $S_{R-\delta/2}$ the elliptic equation with constant coefficients

$$(D_t^{2k} + \sum_{i=1}^{n} D_i^{2k})v = \sum D_x^\mu D_t^{i+k} v_{\mu,i} + \sum_{i=1}^{n} D_i^{2k-1}(D_i v), \tag{A6.4}$$

in which the right member is regarded as known. Since $\Sigma_{R-\delta}$ is a compact subset of $S_{R-\delta/2}$, we find from the simple interior estimates for equations with constant coefficients (see for instance Theorem 2′ of [10]) that (A6.1) holds with each norm on the right replaced by the same norm over $S_{R-\delta/2}$ instead of $\Sigma_{R-\delta/2}$. By the remark above the inequality therefore holds with the norm over $\Sigma_{R-\delta/2}$ as in (A6.1).

Lemma 9.1′ is proved by applying the Calderon-Zygmund theorem to (A6.4) in a fairly standard way; see Section 14.

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