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Boundary value problems for the Laplacian in convex and semiconvex domains

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Abstract

We study the fully inhomogeneous Dirichlet problem for the Laplacian in bounded convex domains in $\mathbb{R}^n$, when the size/smoothness of both the data and the solution are measured on scales of Besov and Triebel–Lizorkin spaces. As a preamble, we deal with the Dirichlet and Regularity problems for harmonic functions in convex domains, with optimal nontangential maximal function estimates. As a corollary, sharp estimates for the Green potential are obtained in a variety of contexts, including local Hardy spaces. A substantial part of this analysis applies to bounded semiconvex domains (i.e., Lipschitz domains satisfying a uniform exterior ball condition).

Keywords: Laplacian; Semiconvex domain; Convex domain; Lipschitz domain satisfying a uniform exterior ball condition; Besov and Triebel–Lizorkin spaces; Nontangential maximal function; Green operator; Poisson problem

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Contents

1. Introduction

The L^p-based Sobolev regularity of elliptic problems in a subdomain Ω of $\mathbb{R}^n$ is well understood when $1 < p < \infty$ and $\partial\Omega$ is sufficiently smooth. The classical reference is the paper [5] by S. Agmon, A. Douglis and L. Nirenberg; cf. also M.E. Taylor's monograph [79] for a more up-to-date account. More recent developments include extensions to Hardy spaces H^p, $0 < p \leqslant 1$, by E.M. Stein, S.G. Krantz and collaborators [12,14] and further, to Besov spaces, $B^{p,q}_\alpha$, and Triebel–Lizorkin spaces $F^{p,q}_\alpha$; cf. [6,72,80] and the references therein. The natural break-point of this theory is the case when Ω is a Lipschitz domain, i.e., satisfies a uniform cone condition. Informally speaking, Lipschitz domains make up the most general class of domains where a rich function theory can be developed, comparable in power and scope with that associated with the upper-half space $\mathbb{R}^n_+$. A paradigm example for this circle of ideas is as follows. Let $\mathbb{G}$ be the Green operator associated with the Dirichlet Laplacian in a domain $\Omega \subset \mathbb{R}^n$. That is, $\mathbb{G}f(x) = -\int_\Omega G(x,y)f(y)\,dy$, $x \in \Omega$, where $G(\cdot,\cdot)$ is the Green function for the Dirichlet Laplacian in Ω. For a reasonable domain Ω, the function $u := \mathbb{G}f$ solves

$$\Delta u = f \in W^{-1,2}(\Omega), \qquad u|_{\partial\Omega} = 0, \quad u \in W^{1,2}(\Omega), \tag{1.1}$$

where $W^{s,p}(\Omega)$ is the L^p-based Sobolev space of order s in Ω. It is then natural to ask the following question: *Under what assumptions on the domain Ω and the smoothness space, call it $\mathcal{X}$, do $\partial_j\partial_k u$ and f have the same amount of smoothness, measured in $\mathcal{X}$?* Clearly, this amounts to the boundedness of the operators

$$\partial_{x_j}\partial_{x_k}\mathbb{G} : \mathcal{X} \longrightarrow \mathcal{X}, \quad j,k = 1,2,\dots,n. \tag{1.2}$$

When $\partial\Omega \in C^\infty$ the operator $\partial_j\partial_k\mathbb{G}$ falls under the scope of the classical theory of singular integral operators of Calderón–Zygmund type. In particular, it maps $L^p(\Omega)$ boundedly into itself for any $1 < p < \infty$ – this is the point of view adopted in [5]. In fact, as proved in [12], (1.2) also holds when $\mathcal{X}$ is the local Hardy space $h^p(\Omega)$, $\frac{n}{n+1} < p \leqslant 1$. The situation is radically different in less smooth domains. A tantalizing hint of the complexity of the problem at hand transpires from the work of B. Dahlberg [19], where a bounded Lipschitz domain Ω is constructed, along with a function $f \in C^\infty(\overline{\Omega})$, such that $\nabla^2\mathbb{G}f \notin L^p(\Omega)$ for any $p > 1$ (this example has been further refined by D. Jerison and C.E. Kenig in [41] where the authors have constructed a bounded domain $\partial\Omega \in C^1$ and $f \in C^\infty(\overline{\Omega})$ with $\nabla^2\mathbb{G}f \notin L^1(\Omega)$).

It has long been understood that this regularity issue is intimately linked to the analytic and geometric properties of the underlying domain Ω. To illustrate this point, let us briefly consider the case when $\Omega \subset \mathbb{R}^2$ is a polygonal domain with at least one re-entrant corner. In this scenario, let $\omega_1, \dots, \omega_N$ be the internal angles of Ω satisfying $\pi < \omega_j < 2\pi$, $1 \leqslant j \leqslant N$, and denote by $P_1, \dots, P_N$ the corresponding vertices. Then the solution to the Poisson problem (1.1) with a datum $f \in L^2(\Omega)$ permits the representation

$$u = \sum_{j=1}^{N} \lambda_j v_j + w, \quad \lambda_j \in \mathbb{R}, \tag{1.3}$$

where $w \in W^{2,2}(\Omega)$ has zero boundary trace and, for each j, v_j is a function exhibiting a singular behavior at P_j of the following nature. Given $j \in \{1, \dots, N\}$, choose polar coordinates (r_j, θ_j) taking P_j as the origin and so that the internal angle is spanned by the half-lines $\theta_j = 0$ and $\theta_j = \omega_j$. Then

$$v_j(r_j, \theta_j) = \phi_j(r_j, \theta_j) r_j^{\pi/\omega_j} \sin(\pi\theta_j/\omega_j), \quad 1 \leqslant j \leqslant N, \tag{1.4}$$

where ϕ_j is a C^∞-smooth cut-off function of small support, which is identically one near P_j. In this scenario, $v_j \in W^{s,2}(\Omega)$ for every $s < 1 + (\pi/\omega_j)$, though $v_j \notin W^{1+(\pi/\omega_j),2}(\Omega)$. This implies that the best regularity statement regarding the solution of (1.1) is

$$u \in W^{s,2}(\Omega) \quad \text{for every } s < 1 + \frac{\pi}{\max\{\omega_1, \dots, \omega_N\}}, \tag{1.5}$$

and this fails for the critical value of s. In particular, this provides a geometrically quantifiable way of measuring the failure of the membership of u to $W^{2,2}(\Omega)$ for Lipschitz, piece-wise C^∞ domains exhibiting inwardly directed irregularities. For more details on the theory of elliptic regularity in domains with isolated singularities, the interested reader is referred to, e.g., [23,36, 48] and the references therein.

The issue of identifying those Sobolev–Besov spaces within which the natural correlation between the smoothness of the data and that of the solutions is preserved when the domain in question has a Lipschitz boundary was considered by D. Jerison and C.E. Kenig in the 1990s. In their ground breaking work [41], they were able to produce such an optimal 'well-posedness region' for the Poisson problem with Dirichlet boundary condition for the scalar, flat-space Laplacian in bounded, Euclidean Lipschitz domains in the context of Sobolev–Besov spaces. The main estimate in [41] is

$$\|\mathbb{G}f\|_{B^{p,p}_{\alpha+2}(\Omega)} \leqslant C\|f\|_{B^{p,p}_{\alpha}(\Omega)} \quad \text{for a suitable range } \mathcal{R} \text{ of indices } (\alpha, 1/p), \tag{1.6}$$

plus a similar inequality involving fractional Sobolev spaces. Here $\mathcal{R}$ depends exclusively on the Lipschitz character of the domain Ω (which, in the case of domains with corners, essentially amounts to the aperture of the smallest angle). See Section 4.1 for a discussion in this regard. After switching homogeneities, so that one seeks a harmonic function with a prescribed trace (in a Besov space on the boundary), the main step in [41] is establishing an atomic estimate in a certain end-point case. It is in this step that the authors rely on harmonic measure estimates. The full range of indices is then arrived at via interpolation with other, known results.

The counterexamples in [41] show that the range $\mathcal{R}$ appearing in (1.6) is optimal, but only if one insists that $p \geqslant 1$ (when all spaces involved are Banach). However, the Besov scale $B_\alpha^{p,p}$ naturally continues below $p = 1$, though the corresponding spaces are no longer locally convex. The consideration of the entire scales $B_\alpha^{p,q}$, $F_\alpha^{p,q}$, $0 < p, q < \infty$, is also natural both because Hardy spaces occur precisely when $p \leqslant 1$ on the Triebel–Lizorkin scale, and because Besov spaces with $p < 1$ offer a natural framework for certain types of numerical approximation schemes (a point eloquently made by R.A. DeVore and collaborators in a series of papers [22,24–26]).

The work in [41] has been extended in [28,66] to allow Neumann boundary condition and variable coefficient operators, and further, in [50,51], to allow data from $B_\alpha^{p,q}$, $F_\alpha^{p,q}$, $0 < p, q < \infty$, $\alpha \in \mathbb{R}$ for an optimal range of indices. In the case of Dirichlet boundary condition, these results are presented in Theorem 4.1. The reader is referred to (4.1)–(4.2) in Section 4.1 for a precise description of this sharp range of indices. Here we only wish to single out a corollary of this theorem to the effect that

$$\begin{aligned}&\text{if } \Omega \text{ is Lipschitz,}\\&\text{then the operators in (1.2) are bounded if } \mathcal{X} = h^p(\Omega) \text{ for } 1-\varepsilon < p < 1,\end{aligned} \tag{1.7}$$

where $\varepsilon > 0$ depends on the Lipschitz character of Ω. This provides a solution to a conjecture made by D.-C. Chang, S.G. Krantz and E.M. Stein in [13,14].

Roughly speaking, the goal of the present paper is to explore the extent to which the range $\mathcal{R}$ in (1.6) becomes larger if the underlying Lipschitz domain satisfies a uniform exterior ball condition (UEBC) or, somewhat more restrictively, is a convex domain. We wish to point out that it has been recently proved in [61] that the former class coincides with the class of semiconvex domains. The issue of regularity of Green potential associated with the Laplacian in these classes of domains has already received considerable attention. For example, according to the literature on this subject, the operators in (1.2) are bounded if Ω is a bounded Lipschitz domain and, in addition,

$$\Omega \text{ is convex and } \mathcal{X} = L^2(\Omega) \text{ [42,78]}, \tag{1.8}$$

$$\Omega \text{ satisfies a UEBC and } \mathcal{X} = L^2(\Omega) \text{ [2]}, \tag{1.9}$$

$$\Omega \text{ is convex and } \mathcal{X} = L^p(\Omega) \text{ with } 1 < p \leqslant 2 \text{ [3,31]}, \tag{1.10}$$

$$\Omega \text{ satisfies a UEBC and } \mathcal{X} = L^p(\Omega) \text{ with } 1 < p \leqslant 2 \text{ [39]}, \tag{1.11}$$

$$\Omega \text{ satisfies a UEBC and } \mathcal{X} = F_\alpha^{p,2}(\Omega) \text{ with } -1 \leqslant \alpha \leqslant 0 \text{ and } \tfrac{\alpha+1}{2} < \tfrac{1}{p} < 1 \text{ [31]}, \tag{1.12}$$

$$\Omega \subset \mathbb{R}^2 \text{ is convex and } \mathcal{X} = F_\alpha^{p,2}(\Omega) \text{ with } 0 < \alpha < 1 \text{ and } \tfrac{\alpha+1}{2} < \tfrac{1}{p} < 1 \text{ [32]}, \tag{1.13}$$

$$\Omega \text{ is convex and } \mathcal{X} = F_\alpha^{p,2}(\Omega) \text{ with } -1 \leqslant \alpha < 1 \text{ and } \tfrac{\alpha+1}{2} < \tfrac{1}{p} < 1 \text{ [33]}. \tag{1.14}$$

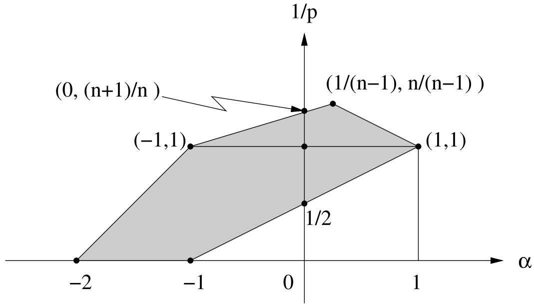

Fig. 1.

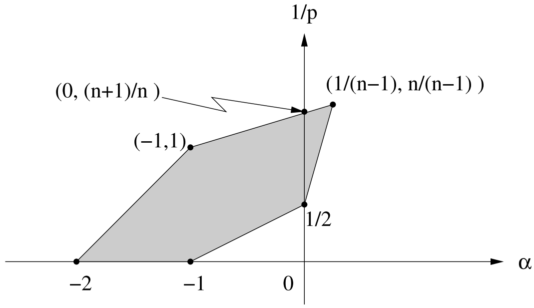

Fig. 2.

Related results have also been proved in [7–9,11,16,52–55,74,76]. The case of the Green operator associated with the Neumann Laplacian for $\mathcal{X} = L^p(\Omega)$ when Ω is a convex domains, or a Lipschitz domain satisfying a UEBC has been treated in [4,51,39]. As regards negative results, by means of counterexamples it has been shown in [4] that the range of p's in (1.10) is sharp on the Lebesgue scale $\{L^p(\Omega)\}_{1<p<\infty}$ with Ω arbitrary convex domain, while in [31] the author has proved that the range of indices in (1.14) is sharp on the scale $\{F_\alpha^{p,2}(\Omega)\}_{1<p<\infty,\,-1<\alpha<1}$, with Ω arbitrary convex domain.

Here we shall present a unified treatment as well as a significant extension of (1.8)–(1.14). To state one of our main results, consider the two-dimensional (open) regions in the $(\alpha, 1/p)$-plane (smoothness versus reciprocal integrability). See Figs. 1 and 2.

In Section 5 we shall prove the following (see Theorem 5.6 and Theorem 5.7):

Theorem 1.1. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and that $0 < p \leqslant \infty$, $\alpha \in \mathbb{R}$. In addition, suppose that one of the following two conditions is satisfied*:

(i) *the domain Ω is convex and either $(\alpha, 1/p)$ belongs to the open region in Fig.* 1*, or $p = \infty$ and $-2 < \alpha < -1$;*
(ii) *the domain Ω is semiconvex and either $(\alpha, 1/p)$ belongs to the open region in Fig.* 2*, or $p = \infty$ and $-2 < \alpha < -1$.*

Then the Green operators

$$\mathbb{G} : B^{p,q}_{\alpha}(\Omega) \longrightarrow B^{p,q}_{\alpha+2}(\Omega), \quad 0 < q \leqslant \infty, \tag{1.15}$$

$$\mathbb{G} : F^{p,q}_{\alpha}(\Omega) \longrightarrow F^{p,q}_{\alpha+2}(\Omega), \quad 0 < q < \infty, \tag{1.16}$$

are well defined, linear and bounded (assuming $p < \infty$ in the case of (1.16)*).*

Note that (1.14) becomes a particular case of (1.16). More precisely, on the scale of fractional Sobolev spaces, i.e., $F^{p,2}_{\alpha}(\Omega)$ with $1 < p < \infty$, $\alpha \in \mathbb{R}$, (1.16) holds, in addition to the range specified in (1.14), the region $-2 < \alpha < -1$, $0 < 1/p < \alpha + 2$.

The discrepancy in the ranges of indices in Theorem 1.1 corresponding to the cases when Ω is a semiconvex domain and when Ω is a convex domain, respectively, has to do with the nature of the point $(1, 1)$ (cf. Fig. 1 and Fig. 2). That the Green operator in the context of (1.16) for convex domains is bounded when $q = 2$ and the point with coordinates $(\alpha, 1/p)$, belonging to the region in Fig. 1, is near $(1, 1)$ is due to S.J. Fromm and D. Jerison [33] and it appears that their techniques make essential use of the convexity of Ω. Whether their result can be extended to the larger class of semiconvex domains remains an open question. We, nonetheless, wish to emphasize that once such an extension has been established, it is not difficult to expand, for the class of semiconvex domains, the region in Fig. 2 to the region in Fig. 1 based on the techniques developed here.

The version of Theorem 1.1 for arbitrary Lipschitz domains is reviewed in Section 4.1, following work in [50,51]. The mapping properties of the Green operator from Theorem 1.1 have several other remarkable consequences and, for the purpose of this introduction, we single out some of them. First, with $h^p(\Omega) := F^{p,2}_0(\Omega)$ (so that $h^p(\Omega)$ is a local Hardy space in Ω if $\frac{n}{n+1} < p \leqslant 1$, and the Lebesgue space $L^p(\Omega)$ if $1 < p < \infty$), we have the following corollary.

Corollary 1.2. *If $\Omega \subset \mathbb{R}^n$ is a bounded semiconvex domain, then for each $j, k \in \{1, \dots, n\}$ the operator*

$$\partial_j \partial_k \mathbb{G} : h^p(\Omega) \longrightarrow h^p(\Omega), \qquad \frac{n}{n+1} < p \leqslant 2, \tag{1.17}$$

is well defined, linear and bounded.

A few comments are in order. This is a satisfactory extension of (1.7) to the class of bounded semiconvex domains which encompasses (1.8)–(1.11). That one cannot allow $p \leqslant \frac{n}{n+1}$ in (1.17) even if the domain is smooth has already been observed in [12] (in [12] the authors also design appropriate Hardy spaces which permit such an extension in smooth domains; in this regard, see also S.G. Krantz's monograph [49]). In relation to the role played by the uniform exterior ball condition, let us also point out that mere Lipschitzianity for Ω does not even guarantee that (1.17) holds when $p = 1$. Indeed, by sharpening a counterexample due to B. Dahlberg [19], D. Jerison and C.E. Kenig have constructed in [41] an example of a bounded domain Ω with C^1 boundary and a function $f \in C^{\infty}_c(\Omega)$ with the property that $\partial_j \partial_k \mathbb{G} f \notin L^1(\Omega)$ for any $1 \leqslant j, k \leqslant n$. Next, by specializing (1.16) to the case when $p = 1$ and $q = 2$ yields the following result.

Corollary 1.3. *Let Ω be a bounded convex domain in $\mathbb{R}^n$ and assume that $-1 < \alpha < 1$. Then for each $j, k \in \{1, \dots, n\}$, the operator*

$$\partial_j \partial_k \mathbb{G} : F_\alpha^{1,2}(\Omega) \longrightarrow F_\alpha^{1,2}(\Omega) \tag{1.18}$$

is well defined, linear and bounded.

It should be noted that this corresponds to a borderline case (corresponding to $p = 1$) of the result (1.14), proved by S.J. Fromm and D. Jerison in [33]. Moreover, (1.18) can also be viewed as a higher-order regularity version of the Hardy space result (1.17) (with $p = 1$).

Moving on, if $0 < q \leqslant \infty$ we let $h^{p,q}(\Omega)$ stand for the Hardy–Lorentz space in Ω if $\frac{n}{n+1} < p \leqslant 1$, and for the standard Lorentz space $L^{p,q}(\Omega)$ if $1 < p < \infty$. In particular, corresponding to $q = \infty$, $h^{p,\infty}(\Omega)$ is the weak-Hardy space in Ω if $\frac{n}{n+1} < p \leqslant 1$, and the standard weak-Lebesgue space $L^{p,\infty}(\Omega)$ if $1 < p < \infty$. Other special cases of particular interest are listed below (proofs are given at the end of Section 5.1).

Corollary 1.4. *Consider a bounded semiconvex domain $\Omega \subset \mathbb{R}^n$. Then for each fixed $j, k \in \{1, \ldots, n\}$, the operators*

$$\partial_j \partial_k \mathbb{G} : h^1(\Omega) \longrightarrow L^1(\Omega), \tag{1.19}$$

$$\partial_j \partial_k \mathbb{G} : h^{p,q}(\Omega) \longrightarrow h^{p,q}(\Omega), \qquad \frac{n}{n+1} < p \leqslant 1,\ 0 < q \leqslant \infty, \tag{1.20}$$

$$\partial_j \partial_k \mathbb{G} : h^{p,\infty}(\Omega) \longrightarrow h^{p,\infty}(\Omega), \qquad \frac{n}{n+1} < p \leqslant 2, \tag{1.21}$$

$$\partial_j \partial_k \mathbb{G} : L^1(\Omega) \longrightarrow L^{1,\infty}(\Omega), \tag{1.22}$$

are bounded. Furthermore,

$$\Omega \subset \mathbb{R}^2 \quad \Longrightarrow \quad \mathbb{G} : h^1(\Omega) \longrightarrow C^0(\overline{\Omega}) \quad \textit{is bounded.} \tag{1.23}$$

The weak-type estimate implicit in (1.22) was apparently first discovered by B. Dahlberg, G. Verchota and T. Wolff in the 90s, who have established this based on the L^2-result from Theorem 5.5 and Calderón–Zygmund theory. See [3,4] and [31], for a discussion. Note that by interpolating this with the L^2-result from Theorem 5.5 via the real method yields (1.10). The weak-$(1,1)$ and L^p-boundedness properties of the second derivatives of the Green operator have been reproved by S.J. Fromm [31]. Another proof was given by V. Adolfsson in [3] where he established atomic estimates amounting to (1.19) and obtained the L^p result (1.10) interpolating between this and (5.15) via the complex method. The atomic estimate alluded to above was obtained by relying on the L^2 theory and the asymptotics at infinity for null-solutions of elliptic PDE's with L^∞ coefficients due to J. Serrin and H. Weinberger (an idea pioneered by B. Dahlberg and C.E. Kenig in [20]). The analogue of (1.10) in the case of Neumann boundary conditions has been resolved in the 90s by V. Adolfsson and D. Jerison in [4]. Here we give conceptually simple proofs to all of the above results, as well as present some new end-point estimates.

We would now like to elaborate on the sharpness of Theorem 1.1. In order to facilitate the subsequent discussion, call a point with coordinates $(\alpha, 1/p)$ "good" for the domain Ω if the Green operator $\mathbb{G}$ maps $F_\alpha^{p,2}(\Omega)$ boundedly into $F_{\alpha+2}^{p,2}(\Omega)$. We continue by recording here the following negative result from [31] (cf. Proposition 2, p. 232, [31]).

Proposition 1.5. *There exist a bounded convex domain $\Omega \subseteq \mathbb{R}^n$ and a function $f \in C^\infty(\overline{\Omega})$ with the property that $\mathbb{G}f \notin F^{p,2}_{\alpha+2}(\Omega)$ whenever $1 < p < \infty$, $-1 < \alpha < 1$, $1/p < (\alpha+1)/2$.*

In other words, there exist convex domains for which there are no good points in the triangle with vertices at $(-1,0)$, $(1,0)$ and $(1,1)$ (in the $(\alpha, 1/p)$ coordinate system). Hence, as a consequence of this and interpolation, the entire region below the line $1/p = (\alpha+1)/2$ contains no good points for such domains. Furthermore, from the comments following the statement of Corollary 1.2, we deduce that there exist smooth domains for which there are no good points above the line $1/p = (\alpha+n+1)/n$. Let us also note here the well-known fact that uniqueness for the inhomogeneous Dirichlet problem may fail in the region above the line $1/p = \alpha+2$ even in the case when the domain Ω is C^∞ (cf., e.g., the discussion at the top of p. 168 in [41]). For other pertinent counterexamples see [72].

Prior to presenting another consequence of Theorem 1.1 we discuss some background. Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, denote by γ_D the (Dirichlet) trace operator, i.e., the extension of $C^\infty(\overline{\Omega}) \ni u \mapsto u|_{\partial\Omega} \in \mathrm{Lip}(\partial\Omega)$ to a bounded, linear operator,

$$\gamma_D : F^{p,q}_{s+1/p}(\Omega) \longrightarrow B^{p,p}_s(\partial\Omega), \tag{1.24}$$

for, say, $1 < p, q < \infty$ and $0 < s < 1$. In the above context, γ_D is utterly ill-defined for $s \leqslant 0$ (e.g., the function $u \equiv 1$ belongs to the closure of $C^\infty_c(\Omega)$ in $F^{p,q}_{s+1/p}(\Omega)$ when $s \leqslant 0$), but this situation can be remedied if one restricts attention to suitable subspaces of $F^{p,q}_{s+1/p}(\Omega)$. A case in point is the question posed to one of the current authors by Gunther Uhlmann [83]. Specifically, motivated by problems in scattering theory by rough domains, Uhlmann has asked whether the implication

$$\partial\Omega \in C^\infty \quad \Longrightarrow \quad \gamma_D : \{u \in L^2(\Omega):\ \Delta u = 0 \text{ in } \Omega\} \longrightarrow B^{2,2}_{-1/2}(\partial\Omega) \quad \text{bounded}, \tag{1.25}$$

has any reasonable counterpart in the class of bounded Lipschitz domains. While the verbatim version of (1.25) is false in this case, it is nonetheless possible to consider a new scale of Besov spaces, $NB^{p,q}_{-s}(\partial\Omega)$, $1 < p, q < \infty$, $s \in (0,1)$ (see Definition 5.11 and Definition 5.14 for details), which is closely related to the standard Besov scale on $\partial\Omega$, and for which γ_D in (1.24) has a linear, bounded, onto extension

$$\widehat{\gamma}_D : \{u \in F^{p,q}_{1/p-s}(\Omega):\ \Delta u \in F^{p,q}_{1/p-s}(\Omega)\} \longrightarrow NB^{p,p}_{-s}(\partial\Omega), \tag{1.26}$$

for any bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ and any $1 < p, q < \infty$, $s \in (0,1)$. See Theorem 5.15. Making use of this noncanonical trace result we then prove at the end of Section 5.3 the following well-posedness result:

Theorem 1.6. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded semiconvex domain. Then for each $2 \leqslant p < \infty$, the problem*

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega, \quad u \in L^p(\Omega), \\ \widehat{\gamma}_D(u) = g \in NB^{p,p}_{-1/p}(\partial\Omega) \end{cases} \tag{1.27}$$

has a unique solution which, in addition, satisfies

$$\|u\|_{L^p(\Omega)} \leqslant C\|g\|_{NB^{p,p}_{-1/p}(\partial\Omega)} \tag{1.28}$$

where $C = C(\Omega, p) > 0$.

It is in the proof of this result that Theorem 1.1 plays a prominent role, since our approach relies on (1.16) and duality (see the proof of Theorem 5.19 for details).

In the last part of this section we wish to briefly elaborate on the method of proof for Theorem 1.1. A key step in our approach is establishing the well-posedness of the Poisson problem

$$\Delta u = f \in F^{p,q}_{s+1/p-2}(\Omega), \qquad \gamma_D(u) = g \in B^{p,p}_s(\partial\Omega), \quad u \in F^{p,q}_{s+1/p}(\Omega), \tag{1.29}$$

granted that $\Omega \subset \mathbb{R}^n$ is a bounded semiconvex domain and $0 < p, q < \infty$, $(n-1)\max\{0, \frac{1}{p} - 1\} < s < 1$. In [28,66,51] which deal with the case of Lipschitz domains, this issue was handled (for a more restrictive range of indices) by reducing matters to the case $f = 0$, and then taking

$$u = \mathcal{D}\left(\frac{1}{2}I + K\right)^{-1} g \quad \text{in } \Omega, \tag{1.30}$$

where $\mathcal{D}$ is the so-called harmonic double layer operator, K is its (principal-value) boundary version, and I is the identity (see Section 2.1 for more details on this matter). The limitations on the indices involved then stem from the demand that the inverse $(\frac{1}{2}I + K)^{-1}$ exists on $B^{p,q}_s(\partial\Omega)$. While in the class of semiconvex domains we once again reduce (1.29) to the case when $f = 0$, in stark contrast with the theory for Lipschitz domains from the aforementioned papers, the method of layer potentials no longer plays a central role in the subsequent considerations. Heuristically, this is due to the fact that the method of layer potentials does not distinguish, in principle, between a domain Ω and its complement $\mathbb{R}^n \setminus \overline{\Omega}$. As such, it is not expected that the boundary layer potentials will exhibit better mapping properties in Lipschitz domains satisfying a uniform exterior ball condition than they do in arbitrary Lipschitz domains.

Instead of (1.30) we are therefore led to considering alternative integral representation formulas, such as

$$u(x) = \int_{\partial\Omega} g\, d\omega^x = -\int_{\partial\Omega} \partial_{\nu(y)} G(x, y) g(y)\, d\sigma(y), \quad x \in \Omega. \tag{1.31}$$

Above, ω^x is the harmonic measure with pole at $x \in \Omega$, ∂_ν is the directional derivative along the outward unit normal to $\partial\Omega$, $d\sigma$ is the surface measure on $\partial\Omega$, and $G(\cdot,\cdot)$ is the Green function for the Dirichlet Laplacian in Ω. In this scenario, the estimates established for ω^x and $G(x, y)$ in [47] and [37] play a crucial role. We use them in order to first establish the solvability of $\Delta u = 0$, $u|_{\partial\Omega} = g$ with optimal estimates for the nontangential maximal function of u (this is done in Section 3 and Section 4.3). These are results of independent interest, which complement the work done by B. Dahlberg, D. Jerison, C.E Kenig, G. Verchota [18,40,20,84] in the case of the Laplacian in arbitrary Lipschitz domains. Our approach allows for a unified treatment of BVP's with nontangential maximal function estimates and inhomogeneous problems on Besov and Triebel–Lizorkin scales. See Section 4.2 and Section 4.4 for details about the transition between nontangential maximal function estimates for the Dirichlet problem and estimates for (1.29) involving Besov and Triebel–Lizorkin spaces. Theorem 1.1 is then proved in Section 5, by relying on the well-posedness of (1.29).

2. Function spaces on Lipschitz domains

This section is divided into five parts. In Section 2.1 we review the definition and geometrical properties of Lipschitz domains, introduce layer potentials, and recall the main results in [40,84, 20] pertaining to the well-posedness of the Dirichlet and Regularity problems in Lipschitz domains. In Section 2.2 we review the Besov and Triebel–Lizorkin spaces in $\mathbb{R}^n$ then, in Section 2.3 define Besov and Hardy spaces on Lipschitz surfaces. Finally, in Section 2.4 we discuss smoothness spaces in Lipschitz domains, whereas in Section 2.5 we record some useful identifications of the envelopes of certain nonlocally convex spaces.

2.1. Lipschitz domains and layer potentials

Recall that an open, bounded set Ω in $\mathbb{R}^n$ is called a bounded Lipschitz domain if for every $x_0 \in \partial\Omega$ there exist $b, c > 0$ with the following significance. There exist an $(n-1)$-plane $H \subset \mathbb{R}^n$ passing through x_0, a choice Z of the unit normal to H, and an open set

$$\mathcal{C} = \mathcal{C}(x_0, H, Z, b, c) := \{x' + tZ\colon\ x' \in H,\ |x' - x_0| < b,\ |t| < c\}, \tag{2.1}$$

called a local coordinate cylinder near x_0 (with axis along Z), such that

$$\mathcal{C} \cap \Omega = \mathcal{C} \cap \{x' + tZ\colon\ x' \in H,\ t > \varphi(x')\}, \tag{2.2}$$

$$\mathcal{C} \cap \partial\Omega = \mathcal{C} \cap \{x' + tZ\colon\ x' \in H,\ t = \varphi(x')\}, \tag{2.3}$$

$$\mathcal{C} \cap (\overline{\Omega})^c = \mathcal{C} \cap \{x' + tZ\colon\ x' \in H,\ t < \varphi(x')\}, \tag{2.4}$$

for some Lipschitz function $\varphi : H \to \mathbb{R}$ satisfying

$$\varphi(x_0) = 0 \quad \text{and} \quad |\varphi(x')| < c/2 \quad \text{if } |x' - x_0| \leqslant b. \tag{2.5}$$

In particular, if Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ then there exist finitely many local coordinate cylinders $\mathcal{C}_k = \mathcal{C}_k(x_k, H_k, Z_k, b_k, c_k)$ and Lipschitz functions $\varphi_k : H_k \to \mathbb{R}$, $1 \leqslant k \leqslant K$, such that

$$\partial\Omega \subseteq \bigcup_{1 \leqslant k \leqslant K} \mathcal{C}_k, \tag{2.6}$$

and the portion of the graph of each φ_k inside $\mathcal{C}_k$ coincides with $\partial\Omega$. The *Lipschitz character* of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ is determined by the number and size of the family $\{\mathcal{C}_k\}_{1 \leqslant k \leqslant K}$ as above, along with the quantity

$$\max\{\|\nabla\varphi_k\|_{L^\infty(\mathbb{R}^{n-1})}\colon\ 1 \leqslant k \leqslant K\}. \tag{2.7}$$

As is well known, for a Lipschitz domain Ω (bounded or unbounded), the surface measure $d\sigma$ is well defined on $\partial\Omega$ and there exists an outward pointing normal vector $\nu = (\nu_1, \ldots, \nu_n)$ at almost every point on $\partial\Omega$. In particular, this allows one to define the Lebesgue scale in the usual fashion, i.e., for $0 < p \leqslant \infty$,

$$L^p(\partial\Omega) := \left\{ f : \partial\Omega \to \mathbb{R}:\ f \text{ measurable, and } \|f\|_{L^p(\partial\Omega)} := \left(\int_{\partial\Omega} |f|^p \, d\sigma \right)^{1/p} < \infty \right\}. \tag{2.8}$$

When equipped with the surface measure and the Euclidean distance, $\partial\Omega$ becomes a space of homogeneous type (in the sense of Coifman and Weiss [15]). Hence, the associated Hardy–Littlewood maximal operator

$$Mf(x) := \sup_{r>0} \frac{1}{\sigma(\Delta_r(x))} \int_{\Delta_r(x)} |f(y)| \, d\sigma(y), \quad x \in \partial\Omega, \tag{2.9}$$

is bounded on $L^p(\partial\Omega)$ for each $p \in (1, \infty)$. Here and for the rest of the paper we denote by $B_r(x)$ the ball in $\mathbb{R}^n$ of radius r centered at x and set $\Delta_r(x) := B_r(x) \cap \partial\Omega$.

For a fixed parameter $\kappa > 0$ define the *nontangential approach regions* with vertex at $x \in \partial\Omega$ as

$$\Gamma_\kappa(x) := \{ y \in \Omega:\ |x - y| < (1 + \kappa)\,\mathrm{dist}(y, \partial\Omega) \}, \tag{2.10}$$

and, further, the *nontangential maximal operator* of a given function u in Ω by

$$(N_\kappa u)(x) := \sup\{ |u(y)|:\ y \in \Gamma_\kappa(x) \}, \quad x \in \partial\Omega. \tag{2.11}$$

As is well known,

$$\|N_\kappa u\|_{L^p(\partial\Omega)} \approx \|N_{\kappa'} u\|_{L^p(\partial\Omega)} \tag{2.12}$$

for every $\kappa, \kappa' > 0$ and $0 < p < \infty$. For further reference, let us also point out here that for each bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and any $p \in (0, \infty)$, $\kappa > 0$, there exists a finite constant $C = C(\Omega, p, \kappa) > 0$ such that

$$\|u\|_{L^{pn/(n-1)}(\Omega)} \leqslant C \|N_\kappa u\|_{L^p(\partial\Omega)}, \tag{2.13}$$

for every function u in Ω. See [58] for a proof. In the sequel, we shall often suppress the dependency of the nontangential maximal operator N_κ and of the nontangential approach region $\Gamma_\kappa(x)$ on the parameter κ, and simply write Nu in place of $N_\kappa u$ and $\Gamma(x)$ in place of $\Gamma_\kappa(x)$.

Next, define the nontangential point-wise trace by

$$u|_{\partial\Omega}(x) := \lim_{\substack{y \in \Gamma_\kappa(x) \\ y \to x}} u(y), \quad x \in \partial\Omega, \tag{2.14}$$

whenever the limit exists.

Given a Lipschitz domain $\Omega \subset \mathbb{R}^n$, consider next the first-order tangential derivative operators $\partial_{\tau_{jk}}$, acting on a compactly supported function ψ of class C^1 in a neighborhood of $\partial\Omega$ by

$$\partial_{\tau_{kj}}\psi := \nu_k(\partial_j\psi)|_{\partial\Omega} - \nu_j(\partial_k\psi)|_{\partial\Omega}, \quad j,k = 1,\dots,n. \tag{2.15}$$

For every $f \in L^1_{loc}(\partial\Omega)$ define the functional $\partial_{\tau_{kj}} f$ by setting

$$\partial_{\tau_{kj}} f : C^1_c(\mathbb{R}^n) \ni \psi \mapsto \int_{\partial\Omega} f(\partial_{\tau_{jk}}\psi)\, d\sigma. \tag{2.16}$$

When $f \in L^1_{loc}(\partial\Omega)$ has $\partial_{\tau_{kj}} f \in L^1_{loc}(\partial\Omega)$, the following integration by parts formula holds:

$$\int_{\partial\Omega} f(\partial_{\tau_{jk}}\psi)\, d\sigma = \int_{\partial\Omega} (\partial_{\tau_{kj}} f)\psi\, d\sigma, \quad \forall \psi \in C^1_c(\mathbb{R}^n). \tag{2.17}$$

For each $p \in (1,\infty)$ we can then define the Sobolev type space

$$L^p_1(\partial\Omega) := \{ f \in L^p(\partial\Omega) \colon\ \partial_{\tau_{jk}} f \in L^p(\partial\Omega),\ j,k = 1,\dots,n \}, \tag{2.18}$$

which becomes a Banach space when equipped with the natural norm

$$\|f\|_{L^p_1(\partial\Omega)} := \|f\|_{L^p(\partial\Omega)} + \sum_{j,k=1}^n \|\partial_{\tau_{jk}} f\|_{L^p(\partial\Omega)}. \tag{2.19}$$

If we introduce the tangential gradient of a real-valued function f defined on $\partial\Omega$ by

$$\nabla_{\tan} f := \left(\sum_{k=1}^n \nu_k \partial_{\tau_{kj}} f \right)_{1 \leqslant j \leqslant n}, \tag{2.20}$$

then for every $p \in (1,\infty)$, we have that $\|f\|_{L^p_1(\partial\Omega)} \approx \|f\|_{L^p(\partial\Omega)} + \|\nabla_{\tan} f\|_{L^p(\partial\Omega)}$, uniformly for $f \in L^p_1(\partial\Omega)$. For further use, let us also define here

$$L^p_{-1}(\partial\Omega) := \left\{ f + \sum_{j,k=1}^n \partial_{\tau_{jk}} g_{jk} \colon\ f, g_{jk} \in L^p(\partial\Omega) \right\}, \tag{2.21}$$

where $1 < p < \infty$, and note that

$$L^p_{-1}(\partial\Omega) = \left(L^{p'}_1(\partial\Omega)\right)^*, \qquad 1/p + 1/p' = 1. \tag{2.22}$$

Next, we discuss layer potential operators associated with a given Lipschitz domain $\Omega \subset \mathbb{R}^n$. To set the stage, we denote by E the canonical fundamental solution for the Laplacian $\Delta = \sum_{j=1}^n \partial_j^2$ in $\mathbb{R}^n$. That is,

$$E(x) := \begin{cases} \frac{1}{\omega_{n-1}(2-n)} \frac{1}{|x|^{n-2}} & \text{if } n \geqslant 3, \\ \frac{1}{2\pi} \log |x| & \text{if } n = 2, \end{cases} \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{2.23}$$

where ω_{n-1} is the surface measure of the unit sphere S^{n-1} in $\mathbb{R}^n$. Next, we recall the harmonic single layer and its boundary version given, respectively, by

$$\mathcal{S} f(x) := \int_{\partial\Omega} E(x-y) f(y)\, d\sigma(y), \quad x \in \Omega, \tag{2.24}$$

$$S f(x) := \int_{\partial\Omega} E(x-y) f(y)\, d\sigma(y), \quad x \in \partial\Omega. \tag{2.25}$$

We also recall here that

$$\mathcal{S} f|_{\partial\Omega} = S f \quad \text{on } \partial\Omega, \tag{2.26}$$

and that the following jump-formula for the normal derivative of the single layer potential operator holds

$$\partial_\nu \mathcal{S} f = \left(-\frac{1}{2} I + K^*\right) f \quad \text{a.e. on } \partial\Omega, \tag{2.27}$$

where I denotes the identity operator and, with p.v. denoting principal value, we have set

$$K^* f(x) := \frac{1}{\omega_{n-1}} \text{p.v.} \int_{\partial\Omega} \frac{\langle x-y, \nu(x)\rangle}{|x-y|^n} f(y)\, d\sigma(y), \quad x \in \partial\Omega. \tag{2.28}$$

Furthermore, if

$$\mathcal{D} f(x) := \frac{1}{\omega_{n-1}} \int_{\partial\Omega} \frac{\langle y-x, \nu(y)\rangle}{|x-y|^n} f(y)\, d\sigma(y), \quad x \in \Omega, \tag{2.29}$$

stands for the so-called harmonic double layer operator in Ω, then

$$\mathcal{D} f|_{\partial\Omega} = \left(\frac{1}{2} I + K\right) f \quad \text{a.e. on } \partial\Omega, \tag{2.30}$$

where

$$K f(x) := \frac{1}{\omega_{n-1}} \text{p.v.} \int_{\partial\Omega} \frac{\langle y-x, \nu(y)\rangle}{|x-y|^n} f(y)\, d\sigma(y), \quad x \in \partial\Omega, \tag{2.31}$$

is the formal adjoint of (2.28).

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ with the following significance.*

(i) *If* $1 < p < 2 + \varepsilon$ *and* $1/p + 1/p' = 1$*, the operators*

$$\frac{1}{2}I + K : L_1^p(\partial\Omega) \longrightarrow L_1^p(\partial\Omega), \tag{2.32}$$

$$\frac{1}{2}I + K : L^{p'}(\partial\Omega) \longrightarrow L^{p'}(\partial\Omega), \tag{2.33}$$

$$S : L^p(\partial\Omega) \longrightarrow L_1^p(\partial\Omega) \tag{2.34}$$

are invertible.

(ii) *If* $1 < p < 2 + \varepsilon$ *and* $1/p + 1/p' = 1$*, then the Dirichlet and Regularity problems*

$$(D)_{p'} \quad \begin{cases} \Delta u = 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} = f \in L^{p'}(\partial\Omega), \\ Nu \in L^{p'}(\partial\Omega), \end{cases} \qquad (R)_p \quad \begin{cases} \Delta v = 0 \quad \text{in } \Omega, \\ v|_{\partial\Omega} = g \in L_1^p(\partial\Omega), \\ N(\nabla v) \in L^p(\partial\Omega) \end{cases} \tag{2.35}$$

are uniquely solvable, and satisfy

$$\|Nu\|_{L^{p'}(\partial\Omega)} \leqslant C\|f\|_{L^{p'}(\partial\Omega)}, \qquad \big\|N(\nabla v)\big\|_{L^p(\partial\Omega)} \leqslant C\|g\|_{L_1^p(\partial\Omega)}, \tag{2.36}$$

for some finite constant $C = C(\partial\Omega, p) > 0$*. In addition, the solutions admit the integral representations in* Ω*:*

$$u = \mathcal{D}\bigg[\bigg(\frac{1}{2}I + K\bigg)^{-1} f\bigg], \qquad v = \mathcal{D}\bigg[\bigg(\frac{1}{2}I + K\bigg)^{-1} g\bigg] = \mathcal{S}\big[S^{-1}g\big]. \tag{2.37}$$

See [20,40,84] for a proof.

In closing, we briefly recall the Newtonian volume potential for the Laplacian. Specifically, given a function $f \in L^1(\Omega)$, we set

$$\Pi f(x) := \int_{\Omega} E(x-y) f(y)\, dy, \quad x \in \mathbb{R}^n, \tag{2.38}$$

and note that

$$\Delta \Pi f = f \quad \text{in } \Omega. \tag{2.39}$$

2.2. Smoothness spaces in the Euclidean setting

Here we briefly review the Besov and Triebel–Lizorkin scales in $\mathbb{R}^n$. One convenient point of view is offered by the classical Littlewood–Paley theory (cf., e.g., [72,80,81]). More specifically, let Ξ be the collection of all systems $\{\zeta_j\}_{j=0}^{\infty}$ of Schwartz functions with the following properties:

(i) there exist positive constants A, B, C such that

$$\begin{cases} \operatorname{supp}(\zeta_0) \subset \{x:\ |x| \leqslant A\}; \\ \operatorname{supp}(\zeta_j) \subset \{x:\ B2^{j-1} \leqslant |x| \leqslant C2^{j+1}\} \quad \text{if } j \in \mathbb{N}; \end{cases} \tag{2.40}$$

(ii) for every multi-index α there exists a positive, finite constant C_α such that

$$\sup_{x\in\mathbb{R}^n}\sup_{j\in\mathbb{N}} 2^{j|\alpha|}\left|\partial^\alpha \zeta_j(x)\right| \leqslant C_\alpha; \tag{2.41}$$

(iii)

$$\sum_{j=0}^{\infty}\zeta_j(x)=1 \quad \text{for every } x\in\mathbb{R}^n. \tag{2.42}$$

Fix some family $\{\zeta_j\}_{j=0}^\infty \in \Xi$. Also, let $\mathcal{F}$ and $S'(\mathbb{R}^n)$ denote, respectively, the Fourier transform and the class of tempered distributions in $\mathbb{R}^n$. Then the Triebel–Lizorkin space $F_s^{p,q}(\mathbb{R}^n)$ is defined for $s\in\mathbb{R}$, $0<p<\infty$ and $0<q\leqslant\infty$ as

$$F_s^{p,q}\big(\mathbb{R}^n\big) := \left\{ f\in S'\big(\mathbb{R}^n\big):\ \|f\|_{F_s^{p,q}(\mathbb{R}^n)} := \left\| \left(\sum_{j=0}^{\infty} \left|2^{sj}\mathcal{F}^{-1}(\zeta_j\mathcal{F}f)\right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty \right\} \tag{2.43}$$

(with a natural interpretation when $q=\infty$). The case $p=\infty$ is somewhat special, in that a suitable version of (2.43) needs to be used; see, e.g., [72, p. 9].

If $s\in\mathbb{R}$ and $0<p,q\leqslant\infty$ then the Besov space $B_s^{p,q}(\mathbb{R}^n)$ can be defined as

$$B_s^{p,q}\big(\mathbb{R}^n\big) := \left\{ f\in S'\big(\mathbb{R}^n\big):\ \|f\|_{B_s^{p,q}(\mathbb{R}^n)} := \left(\sum_{j=0}^{\infty} \left\|2^{sj}\mathcal{F}^{-1}(\zeta_j\mathcal{F}f)\right\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} < \infty \right\}. \tag{2.44}$$

A different choice of the system $\{\zeta_j\}_{j=0}^\infty \in \Xi$ yields the same spaces (2.43)–(2.44), albeit equipped with equivalent norms. Furthermore, the class of Schwartz functions in $\mathbb{R}^n$ is dense in both $B_s^{p,q}(\mathbb{R}^n)$ and $F_s^{p,q}(\mathbb{R}^n)$ provided $s\in\mathbb{R}$ and $0<p,q<\infty$.

It has long been known that many classical smoothness spaces are encompassed by the Besov and Triebel–Lizorkin scales. For example,

$$C^s\big(\mathbb{R}^n\big) = B_s^{\infty,\infty}\big(\mathbb{R}^n\big), \quad 0<s\notin\mathbb{Z}, \tag{2.45}$$

$$L^p\big(\mathbb{R}^n\big) = F_0^{p,2}\big(\mathbb{R}^n\big), \quad 1<p<\infty, \tag{2.46}$$

$$L_s^p\big(\mathbb{R}^n\big) = F_s^{p,2}\big(\mathbb{R}^n\big), \quad 1<p<\infty,\ s\in\mathbb{R}, \tag{2.47}$$

$$W^{k,p}\big(\mathbb{R}^n\big) = F_k^{p,2}\big(\mathbb{R}^n\big), \quad 1<p<\infty,\ k\in\mathbb{N}, \tag{2.48}$$

$$h^p\big(\mathbb{R}^n\big) = F_0^{p,2}\big(\mathbb{R}^n\big), \quad 0<p\leqslant 1. \tag{2.49}$$

Above, given $1<p<\infty$ and $s\in\mathbb{R}$, $L_s^p(\mathbb{R}^n)$ stands for the Bessel potential space defined by

$$\begin{aligned} L^p_s(\mathbb{R}^n) &:= \{(I-\Delta)^{-s/2}g:\ g\in L^p(\mathbb{R}^n)\} \\ &= \{\mathcal{F}^{-1}(1+|\xi|^2)^{-s/2}\mathcal{F}g:\ g\in L^p(\mathbb{R}^n)\}, \end{aligned} \tag{2.50}$$

equipped with the norm

$$\|f\|_{L^p_s(\mathbb{R}^n)} := \|\mathcal{F}^{-1}(1+|\xi|^2)^{-s/2}\mathcal{F}f\|_{L^p(\mathbb{R}^n)}. \tag{2.51}$$

As is well known, when the smoothness index is a natural number, say $s=k\in\mathbb{N}$, this can be identified with the classical Sobolev space

$$W^{k,p}(\mathbb{R}^n) := \left\{f\in L^p(\mathbb{R}^n):\ \|f\|_{W^{k,p}(\mathbb{R}^n)} := \sum_{|\gamma|\leqslant k}\|\partial^\gamma f\|_{L^p(\mathbb{R}^n)}<\infty\right\}, \tag{2.52}$$

i.e.,

$$L^p_k(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n), \quad k\in\mathbb{N}_0 := \mathbb{N}\cup\{0\},\ 1<p<\infty. \tag{2.53}$$

Also, $C^s(\mathbb{R}^n)$ and $h^p(\mathbb{R}^n)$ stand, respectively, for the Hölder and local Hardy spaces in $\mathbb{R}^n$ (cf. [35]). Recall that the latter class is the space of tempered distributions u in $\mathbb{R}^n$ with the property that the radial maximal function

$$u_{rad}(x) := \sup_{t\in(0,1)}\left|(\varphi_t * u)(x)\right|, \quad x\in\mathbb{R}^n, \tag{2.54}$$

belongs to $L^p(\mathbb{R}^n)$. Above, $0<p<\infty$ and $\varphi_t(x)=t^{-n}\varphi(x/t)$ where φ is a fixed Schwartz function with $\int_{\mathbb{R}^n}\varphi(x)\,dx\neq 0$. This space is equipped with the quasi-norm $\|u\|_{h^p(\mathbb{R}^n)} := \|u_{rad}\|_{L^p(\mathbb{R}^n)}$.

For a measure space (X,μ) and $0<p<\infty$, $0<q\leqslant\infty$, the Lorentz space $L^{p,q}(X)$ is defined as

$$L^{p,q}(X) := \{f: X\to\mathbb{R} \text{ measurable: } \|f\|_{L^{p,q}(X)}<\infty\}, \tag{2.55}$$

where, if $0<q<\infty$,

$$\|f\|_{L^{p,q}(X)} := \left(\frac{q}{p}\int_0^\infty \lambda^{q-1}\mu(\{x\in X:\ |f(x)|>\lambda\})^{q/p}\,d\lambda\right)^{1/q}, \tag{2.56}$$

and, corresponding to $q=\infty$ (i.e., for weak L^p-Lebesgue spaces),

$$\|f\|_{L^{p,\infty}(X)} := \sup_{\lambda>0}\left(\lambda\mu(\{x\in X:\ |f(x)|>\lambda\})^{1/p}\right). \tag{2.57}$$

Then the Hardy–Lorentz space $h^{p,q}(\mathbb{R}^n)$, $0<p<\infty$, $0<q\leqslant\infty$, is defined as the collection of tempered distributions u in $\mathbb{R}^n$ for which u_{rad} belongs to $L^{p,q}(\mathbb{R}^n)$, and $\|u\|_{h^{p,q}(\mathbb{R}^n)} := \|u_{rad}u\|_{L^{p,q}(\mathbb{R}^n)}$. In the Euclidean setting, these spaces have been studied in [29] (for $q=\infty$) and [1] (for $0<q\leqslant\infty$).

Following [29], $h^{p,\infty}(\mathbb{R}^n)$ will be referred to as a weak Hardy space. As is well known, while the space $h^{p,\infty}(\mathbb{R}^n)$ coincides with $L^{p,\infty}(\mathbb{R}^n)$ for $1 < p < \infty$, these two scales are not comparable when $0 < p \leqslant 1$. For nice functions f in $\mathbb{R}^n$ it is nonetheless true that

$$\|f\|_{L^{1,\infty}(\mathbb{R}^n)} \leqslant C_n \|f\|_{h^{1,\infty}(\mathbb{R}^n)}. \tag{2.58}$$

See the discussion in [29]. Furthermore, the following interpolation result holds

$$\big(h^{p_0}(\mathbb{R}^n), h^{p_1}(\mathbb{R}^n)\big)_{\theta,\infty} = h^{p,\infty}(\mathbb{R}^n), \tag{2.59}$$

granted that $0 < p_0, p_1 < \infty$, $\theta \in (0,1)$ and $1/p = (1-\theta)/p_0 + \theta/p_1$; see [30]. Here and everywhere else in the paper, $(\cdot,\cdot)_{\theta,q}$ stands for the real interpolation brackets. Returning to Hardy spaces, we also have

$$\big(h^{p,q_1}(\mathbb{R}^n), h^{p,q_2}(\mathbb{R}^n)\big)_{\theta,q} = h^{p,q}(\mathbb{R}^n), \tag{2.60}$$

provided $0 < p \leqslant 1$, $0 < q_0, q_1 \leqslant \infty$, $\theta \in (0,1)$ and $1/q = (1-\theta)/q_0 + \theta/q_1$; see [1]. Finally, it is also more or less folklore (see, e.g., [29]) that

$$h^1(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n) \hookrightarrow h^{1,\infty}(\mathbb{R}^n). \tag{2.61}$$

2.3. Besov and Hardy spaces on Lipschitz boundaries

Here we discuss the adaptation of certain smoothness classes to the situation when the Euclidean space is replaced by the boundary of a Lipschitz domain Ω. For $a \in \mathbb{R}$ set $(a)_+ := \max\{a, 0\}$. Consider three parameters p, q, s subject to

$$0 < p, q \leqslant \infty, \qquad (n-1)\left(\frac{1}{p} - 1\right)_+ < s < 1 \tag{2.62}$$

and assume that $\Omega \subset \mathbb{R}^n$ is the upper-graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. We then define $B_s^{p,q}(\partial\Omega)$ as the space of locally integrable functions f on $\partial\Omega$ for which the assignment $\mathbb{R}^{n-1} \ni x \mapsto f(x, \varphi(x))$ belongs to $B_s^{p,q}(\mathbb{R}^{n-1})$, the classical Besov space in $\mathbb{R}^{n-1}$. We equip this space with the (quasi-)norm

$$\|f\|_{B_s^{p,q}(\partial\Omega)} := \big\|f\big(\cdot, \varphi(\cdot)\big)\big\|_{B_s^{p,q}(\mathbb{R}^{n-1})}. \tag{2.63}$$

As far as Besov spaces with a negative amount of smoothness are concerned, in the same context as above we set

$$f \in B_{s-1}^{p,q}(\partial\Omega) \quad \Longleftrightarrow \quad f\big(\cdot, \varphi(\cdot)\big)\sqrt{1 + \big|\nabla\varphi(\cdot)\big|^2} \in B_{s-1}^{p,q}\big(\mathbb{R}^{n-1}\big), \tag{2.64}$$

$$\|f\|_{B_{s-1}^{p,q}(\partial\Omega)} := \Big\|f\big(\cdot, \varphi(\cdot)\big)\sqrt{1 + \big|\nabla\varphi(\cdot)\big|^2}\Big\|_{B_{s-1}^{p,q}(\mathbb{R}^{n-1})}. \tag{2.65}$$

As is well known, the case when $p = q = \infty$ corresponds to the usual (inhomogeneous) Hölder spaces $C^s(\partial\Omega)$, defined by the requirement that

$$\|f\|_{C^s(\partial\Omega)} := \|f\|_{L^\infty(\partial\Omega)} + \sup_{\substack{x\neq y\\ x,y\in\partial\Omega}} \frac{|f(x)-f(y)|}{|x-y|^s} < +\infty. \tag{2.66}$$

That is,

$$B_s^{\infty,\infty}(\partial\Omega) = C^s(\partial\Omega) \quad \text{for } s\in(0,1). \tag{2.67}$$

All the above definitions then readily extend to the case of (bounded) Lipschitz domains in $\mathbb{R}^n$ via a standard partition of unity argument. These Besov spaces have been defined in such a way that a number of basic properties from the Euclidean setting carry over to spaces defined on $\partial\Omega$ in a rather direct fashion. We continue by recording an interpolation result which is going to be very useful for us here. To state it, recall that $[\cdot,\cdot]_\theta$ stands for the complex interpolation brackets.

Proposition 2.2. *Suppose that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Also, assume that $0 < p, q, q_0, q_1 \leqslant \infty$ and that*

$$\begin{aligned} &\textit{either } (n-1)\left(\frac{1}{p}-1\right)_+ < s_0 \neq s_1 < 1,\\ &\textit{or } -1+(n-1)\left(\frac{1}{p}-1\right)_+ < s_0 \neq s_1 < 0. \end{aligned} \tag{2.68}$$

Then, with $0<\theta<1$, $s=(1-\theta)s_0+\theta s_1$,

$$\left(B_{s_0}^{p,q_0}(\partial\Omega), B_{s_1}^{p,q_1}(\partial\Omega)\right)_{\theta,q} = B_s^{p,q}(\partial\Omega). \tag{2.69}$$

Furthermore, if $s_0\neq s_1$ and $0<p_i,q_i\leqslant\infty$, $i=0,1$, satisfy $\min\{q_0,q_1\}<\infty$ as well as either of the following two conditions

$$\begin{aligned} &\textit{either } (n-1)\left(\frac{1}{p_i}-1\right)_+ < s_i < 1, \quad i=0,1,\\ &\textit{or } -1+(n-1)\left(\frac{1}{p_i}-1\right)_+ < s_i < 0, \quad i=0,1, \end{aligned} \tag{2.70}$$

then

$$\left[B_{s_0}^{p_0,q_0}(\partial\Omega), B_{s_1}^{p_1,q_1}(\partial\Omega)\right]_\theta = B_s^{p,q}(\partial\Omega), \tag{2.71}$$

where $0<\theta<1$, $s:=(1-\theta)s_0+\theta s_1$, $\frac{1}{p}:=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$ and $\frac{1}{q}:=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$.

Finally, if $1<p<\infty$, $0<q\leqslant\infty$ and $\theta\in(0,1)$ then

$$\left(L^p(\partial\Omega), L_1^p(\partial\Omega)\right)_{\theta,q} = B_\theta^{p,q}(\partial\Omega). \tag{2.72}$$

Fix a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and assume that $\frac{n-1}{n} < p \leqslant 1 < p_o \leqslant \infty$. Also, fix a threshold $\eta > 0$. Call a function $a \in L^1(\partial\Omega)$ an *inhomogeneous* (p, p_o)*-atom* if for some surface ball $\Delta_r \subseteq \partial\Omega$

$$\operatorname{supp} a \subseteq \Delta_r, \qquad \|a\|_{L^{p_o}(\partial\Omega)} \leqslant r^{(n-1)(\frac{1}{p_o}-\frac{1}{p})}, \quad \text{and}$$
$$\text{either } r = \eta, \quad \text{or } r < \eta \text{ and } \int_{\partial\Omega} a\, d\sigma = 0. \tag{2.73}$$

We then define $h^p_{at}(\partial\Omega)$ as the ℓ^p-span of inhomogeneous (p, p_o)-atoms, and equip it with the natural infimum-type quasi-norm. One can check that this is a "local" quasi-Banach space, in the sense that

$$h^p_{at}(\partial\Omega) \text{ is a module over } C^\alpha(\partial\Omega) \quad \text{for any } \alpha > (n-1)\left(\frac{1}{p} - 1\right). \tag{2.74}$$

Different choices of the parameters p_o, η lead to equivalent quasi-norms and

$$\big(h^p_{at}(\partial\Omega)\big)^* = C^{(n-1)(\frac{1}{p}-1)}(\partial\Omega). \tag{2.75}$$

We now proceed to discuss regular Hardy spaces defined on the boundary of a bounded Lipschitz domain. Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, and assume that $(n-1)/n < p \leqslant 1 < p_o \leqslant \infty$ are fixed. A function $a \in L^{p_o}_1(\partial\Omega)$ is called a *regular* (p, p_o)*-atom* if there exists a surface ball Δ_r for which

$$\operatorname{supp} a \subseteq \Delta_r, \qquad \|\nabla_{\tan} a\|_{L^{p_o}(\partial\Omega)} \leqslant r^{(n-1)(\frac{1}{p_o}-\frac{1}{p})}. \tag{2.76}$$

With $0 < \eta < \operatorname{diam}(\Omega)$ fixed, we next define

$$h^{1,p}_{at}(\partial\Omega) := \Big\{ f \in \operatorname{Lip}(\partial\Omega)' \colon\ f = \sum_j \lambda_j a_j,\ (\lambda_j)_j \in \ell^p \text{ and } a_j \text{ regular } (p, p_o)\text{-atom}$$
$$\text{supported in a surface ball of radius } \leqslant \eta \text{ for every } j \Big\}, \tag{2.77}$$

where the series converges in $\operatorname{Lip}(\partial\Omega)'$, and equip it with the natural infimum quasi-norm. We conclude this subsection by recording a useful Sobolev space-like characterization of the regular Hardy space (cf. [63] for a proof).

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and assume that $\frac{n-1}{n} < p \leqslant 1$ and that $p^* \in (1, \infty)$ is such that*

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n-1}. \tag{2.78}$$

Also, assume that $1 < q \leqslant p^*$. Then

$$h^{1,p}_{at}(\partial\Omega) = \big\{ f \in L^q(\partial\Omega):\ \partial_{\tau_{jk}} f \in h^p_{at}(\partial\Omega),\ 1 \leqslant j,k \leqslant n \big\} \tag{2.79}$$

and, in addition,

$$\|f\|_{h^{1,p}_{at}(\partial\Omega)} \approx \|f\|_{L^q(\partial\Omega)} + \sum_{j,k=1}^{n} \|\partial_{\tau_{jk}} f\|_{h^p_{at}(\partial\Omega)}. \tag{2.80}$$

2.4. Besov and Triebel–Lizorkin spaces in Lipschitz domains

In this subsection we review how the Besov and Triebel–Lizorkin spaces $B^{p,q}_s(\mathbb{R}^n)$, $F^{p,q}_s(\mathbb{R}^n)$, $0 < p,q \leqslant \infty$, $s \in \mathbb{R}$, originally considered in the entire Euclidean setting, can be defined on arbitrary open subsets of $\mathbb{R}^n$. Concretely, given an arbitrary open subset Ω of $\mathbb{R}^n$, we denote by $f|_\Omega$ the restriction of a distribution f in $\mathbb{R}^n$ to Ω. For $0 < p,q \leqslant \infty$ and $s \in \mathbb{R}$, both $B^{p,q}_s(\mathbb{R}^n)$ and $F^{p,q}_s(\mathbb{R}^n)$ are spaces of (tempered) distributions, hence it is meaningful to define

$$\begin{aligned} &A^{p,q}_s(\Omega) := \big\{ f \text{ distribution in } \Omega:\ \exists g \in A^{p,q}_s\big(\mathbb{R}^n\big) \text{ such that } g|_\Omega = f \big\}, \\ &\|f\|_{A^{p,q}_s(\Omega)} := \inf\big\{ \|g\|_{A^{p,q}_s(\mathbb{R}^n)}:\ g \in A^{p,q}_s\big(\mathbb{R}^n\big),\ g|_\Omega = f \big\}, \quad f \in A^{p,q}_s(\Omega), \end{aligned} \tag{2.81}$$

where $A = B$, or $A = F$. Throughout the paper, the subscript *loc* appended to one of the function spaces already introduced indicates the local version of that particular space.

The existence of a universal extension operator for Besov and Triebel–Lizorkin spaces in an arbitrary Lipschitz domain $\Omega \subset \mathbb{R}^n$ has been established by V.S. Rychkov in [73]. This allows transferring a number of properties of the Besov–Triebel–Lizorkin spaces in the Euclidean space $\mathbb{R}^n$ to the setting of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. Here, we only wish to mention a few of these properties. First, if $0 < p \leqslant \infty$, $0 < q < \infty$ and $s \in \mathbb{R}$, then

$$B^{p,\min(p,q)}_s(\Omega) \hookrightarrow F^{p,q}_s(\Omega) \hookrightarrow B^{p,\max(p,q)}_s(\Omega), \tag{2.82}$$

and, with $A \in \{B, F\}$,

$$A^{p,q_0}_{s_0}(\Omega) \hookrightarrow A^{p,q_1}_{s_1}(\Omega) \quad \text{if } s_1 < s_0,\ 0 < p, q_0,\ q_1 \leqslant \infty, \tag{2.83}$$

$$A^{p,q_0}_{s}(\Omega) \hookrightarrow A^{p,q_1}_{s}(\Omega) \quad \text{if } 0 < q_0 < q_1 \leqslant \infty,\ 0 < p \leqslant \infty. \tag{2.84}$$

Furthermore,

$$B^{p_0,p}_{s_0}(\Omega) \hookrightarrow F^{p,q}_s(\Omega) \hookrightarrow B^{p_1,p}_{s_1}(\Omega) \tag{2.85}$$

if $0 < p_0 < p < p_1 \leqslant \infty$ and $\frac{1}{p_0} - \frac{s_0}{n} = \frac{1}{p} - \frac{s}{n} = \frac{1}{p_1} - \frac{s_1}{n}$, whereas

$$F^{p_0,q_0}_{s_0}(\Omega) \hookrightarrow F^{p,q}_s(\Omega) \tag{2.86}$$

if $0 < p_0 \leqslant p \leqslant \infty$, $0 < q_0, q \leqslant \infty$, $\frac{1}{p} - \frac{s}{n} \geqslant \frac{1}{p_0} - \frac{s_0}{n}$, and

$$B^{p_0,q_0}_{s_0}(\Omega) \hookrightarrow B^{p,q}_s(\Omega) \tag{2.87}$$

if $0 < p_0 \leqslant p \leqslant \infty$, $0 < q_0, q \leqslant \infty$, $\frac{1}{p} - \frac{s}{n} > \frac{1}{p_0} - \frac{s_0}{n}$.

Second, if k is a nonnegative integer and $1 < p < \infty$, then

$$F_k^{p,2}(\Omega) = W^{k,p}(\Omega) := \{ f \in L^p(\Omega):\ \partial^\alpha f \in L^p(\Omega),\ |\alpha| \leqslant k \}, \tag{2.88}$$

the classical Sobolev spaces in Ω. Third, if $k \in \mathbb{N}_0$ and $0 < s < 1$, then

$$B_{k+s}^{\infty,\infty}(\Omega) = C^{k+s}(\Omega), \tag{2.89}$$

where

$$C^{k+s}(\Omega) := \Bigg\{ u \in C^k(\Omega):\ \text{with } \|u\|_{C^{k+s}(\Omega)} < \infty,\ \text{where} \\ \|u\|_{C^{k+s}(\Omega)} := \sum_{j=0}^{k} \big\| \nabla^j u \big\|_{L^\infty(\Omega)} + \sum_{|\alpha|=k} \sup_{x \neq y \in \Omega} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^s} \Bigg\}. \tag{2.90}$$

We conclude this subsection by recording a couple of useful lifting results on Besov and Triebel–Lizorkin spaces on bounded Lipschitz domains. The following has been proved in [60].

Proposition 2.4. *Let $1 < p, q < \infty$, $k \in \mathbb{N}$ and $s \in \mathbb{R}$. Then for any distribution u in the bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, the following implication holds*:

$$\partial^\alpha u \in A_s^{p,q}(\Omega), \quad \forall \alpha:\ |\alpha| = k \quad \Longrightarrow \quad u \in A_{s+k}^{p,q}(\Omega), \tag{2.91}$$

where, as usual, $A \in \{B, F\}$.

Going further, for $0 < p, q \leqslant \infty$, $s \in \mathbb{R}$, we set

$$A_{s,0}^{p,q}(\Omega) := \{ f \in A_s^{p,q}(\mathbb{R}^n):\ \operatorname{supp} f \subseteq \overline{\Omega} \}, \\ \|f\|_{A_{s,0}^{p,q}(\Omega)} := \|f\|_{A_s^{p,q}(\mathbb{R}^n)}, \quad f \in A_{s,0}^{p,q}(\Omega), \tag{2.92}$$

where we use the convention that either $A = F$ and $p < \infty$, or $A = B$. Thus, $B_{s,0}^{p,q}(\Omega)$, $F_{s,0}^{p,q}(\Omega)$ are closed subspaces of $B_s^{p,q}(\mathbb{R}^n)$ and $F_s^{p,q}(\mathbb{R}^n)$, respectively. Second, for $0 < p, q \leqslant \infty$ and $s \in \mathbb{R}$, we introduce

$$A_{s,z}^{p,q}(\Omega) := \{ f \text{ distribution in } \Omega:\ \exists g \in A_{s,0}^{p,q}(\Omega) \text{ with } g|_\Omega = f \}, \\ \|f\|_{A_{s,z}^{p,q}(\Omega)} := \inf\{ \|g\|_{A_s^{p,q}(\mathbb{R}^n)}:\ g \in A_{s,0}^{p,q}(\Omega),\ g|_\Omega = f \}, \quad f \in A_{s,z}^{p,q}(\Omega) \tag{2.93}$$

(where, as before, $A = F$ and $p < \infty$ or $A = B$) and, in keeping with earlier conventions,

$$L_{s,z}^p(\Omega) := F_{s,z}^{p,2}(\Omega) = \{ f \text{ distribution in } \Omega:\ \exists g \in F_{s,0}^{p,2}(\Omega) \text{ with } g|_\Omega = f \} \tag{2.94}$$

if $1 < p < \infty$, $s \in \mathbb{R}$. For further use, let us also make the simple yet important observation that the operator of restriction to Ω induces linear, bounded mappings in the following settings

$$\mathcal{R}_\Omega : A_s^{p,q}(\mathbb{R}^n) \longrightarrow A_s^{p,q}(\Omega) \quad \text{and} \quad \mathcal{R}_\Omega : A_{s,0}^{p,q}(\Omega) \longrightarrow A_{s,z}^{p,q}(\Omega) \tag{2.95}$$

for $0 < p, q \leqslant \infty$, $s \in \mathbb{R}$. For further use, let us now record a useful extension result, proved in [73], pertaining to the existence of a universal, linear extension operator. More specifically, we have

Proposition 2.5. *If Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, then there exists a linear operator* Ex *mapping $C_c^\infty(\Omega)$ into distributions on $\mathbb{R}^n$, and such that for any numbers $0 < p, q \leqslant \infty$, $s \in \mathbb{R}$, and $A \in \{B, F\}$,*

$$\mathrm{Ex} : A_s^{p,q}(\Omega) \longrightarrow A_s^{p,q}(\mathbb{R}^n) \tag{2.96}$$

boundedly, and

$$\mathcal{R}_\Omega \circ \mathrm{Ex} = I, \quad \textit{the identity operator on } A_s^{p,q}(\Omega). \tag{2.97}$$

Moving on, if $1 < p, q < \infty$ and $1/p + 1/p' = 1/q + 1/q' = 1$, then

$$\big(A_{s,z}^{p,q}(\Omega)\big)^* = A_{-s}^{p',q'}(\Omega) \quad \text{if } s > -1 + \frac{1}{p}, \tag{2.98}$$

$$\big(A_s^{p,q}(\Omega)\big)^* = A_{-s,z}^{p',q'}(\Omega) \quad \text{if } s < \frac{1}{p}. \tag{2.99}$$

Furthermore, for each $s \in \mathbb{R}$ and $1 < p, q < \infty$, the spaces $A_s^{p,q}(\Omega)$ and $A_{s,0}^{p,q}(\Omega)$ are reflexive.

There is yet another type of smoothness space which will play a significant role in this paper. Specifically, for $\Omega \subset \mathbb{R}^n$ Lipschitz domain, we set

$$\mathring{A}_s^{p,q}(\Omega) := \text{the closure of } C_c^\infty(\Omega) \quad \text{in } A_s^{p,q}(\Omega), \quad 0 < p, q \leqslant \infty,\ s \in \mathbb{R}, \tag{2.100}$$

where, as usual, $A = F$ or $A = B$. For every $0 < p, q < \infty$ and $s \in \mathbb{R}$, we then have

$$A_{s,z}^{p,q}(\Omega) \hookrightarrow \mathring{A}_s^{p,q}(\Omega) \hookrightarrow A_s^{p,q}(\Omega), \quad \text{continuously.} \tag{2.101}$$

Going further, Proposition 3.1 in [82] ensures that

$$\mathring{A}_s^{p,q}(\Omega) = A_s^{p,q}(\Omega) = A_{s,z}^{p,q}(\Omega), \quad A \in \{F, B\}, \tag{2.102}$$

whenever $0 < p, q < \infty$, $\max(1/p - 1, n(1/p - 1)) < s < 1/p$, and $\min\{p, 1\} \leqslant q < \infty$ in the case $A = F$. In particular, for $A \in \{B, F\}$, we have

$$\big(A_s^{p,q}(\Omega)\big)^* = A_{-s}^{p',q'}(\Omega) \quad \text{if } 1 < p, q < \infty \text{ and } -1 + \frac{1}{p} < s < \frac{1}{p}. \tag{2.103}$$

Other cases of interest have been considered in [51], from which we quote the following result.

Proposition 2.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then*

$$\mathring{F}_s^{p,q}(\Omega) = F_{s,z}^{p,q}(\Omega) \tag{2.104}$$

provided

$$\begin{gathered} 0 < p < \infty, \quad \min\{1,p\} \leqslant q < \infty, \quad \textit{and} \\ \exists k \in \mathbb{N}_0 \quad \textit{so that} \quad \max\left(\frac{1}{p} - 1, n\left(\frac{1}{p} - 1\right)\right) < s - k < \frac{1}{p}. \end{gathered} \tag{2.105}$$

Furthermore,

$$\mathring{B}_s^{p,q}(\Omega) = B_{s,z}^{p,q}(\Omega) \tag{2.106}$$

whenever

$$0 < p, q < \infty \quad \textit{and} \quad \exists k \in \mathbb{N}_0 \quad \textit{so that} \quad \max\left(\frac{1}{p} - 1, n\left(\frac{1}{p} - 1\right)\right) < s - k < \frac{1}{p}. \tag{2.107}$$

Next we record the following theorem from [51,63] which extends work done in [41].

Theorem 2.7. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that the indices p, s satisfy $\frac{n-1}{n} < p \leqslant \infty$ and $(n-1)(\frac{1}{p} - 1)_+ < s < 1$. Then the following hold*:

(i) *The restriction to the boundary extends to a linear, bounded operator*

$$\mathrm{Tr} : B_{s+\frac{1}{p}}^{p,q}(\Omega) \longrightarrow B_s^{p,q}(\partial\Omega) \quad \textit{for } 0 < q \leqslant \infty. \tag{2.108}$$

Moreover, for this range of indices, Tr *is onto, its null-space is given by*

$$\left\{u \in B_{s+\frac{1}{p}}^{p,q}(\Omega) \colon \mathrm{Tr}\, u = 0 \textit{ in } B_s^{p,q}(\partial\Omega)\right\} = B_{s+\frac{1}{p},z}^{p,q}(\Omega) \tag{2.109}$$

and has a linear, bounded, right inverse

$$\mathrm{Ex} : B_s^{p,q}(\partial\Omega) \longrightarrow B_{s+\frac{1}{p}}^{p,q}(\Omega). \tag{2.110}$$

(ii) *Similar considerations hold for*

$$\mathrm{Tr} : F_{s+\frac{1}{p}}^{p,q}(\Omega) \longrightarrow B_s^{p,p}(\partial\Omega) \tag{2.111}$$

with the convention that $q = \infty$ if $p = \infty$. More specifically, Tr *in* (2.111) *is a linear, bounded, operator which has a linear, bounded right-inverse*

$$\mathrm{Ex} : B_s^{p,p}(\partial\Omega) \longrightarrow F_{s+\frac{1}{p}}^{p,q}(\Omega). \tag{2.112}$$

Also, if $\min\{1, p\} \leqslant q < \infty$, *its null-space is given by*

$$\big\{u \in F^{p,q}_{s+\frac{1}{p}}(\Omega)\colon\ \mathrm{Tr}\, u = 0 \text{ in } B^{p,p}_s(\partial\Omega)\big\} = F^{p,q}_{s+\frac{1}{p},z}(\Omega). \tag{2.113}$$

Finally, corresponding to the limiting cases $s \in \{0, 1\}$ *of* (2.108)*, one has that*

$$\mathrm{Tr}: B^{p,1}_{\frac{1}{p}}(\Omega) \longrightarrow L^p(\partial\Omega), \qquad \mathrm{Tr}: B^{p,1}_{1+\frac{1}{p}}(\Omega) \longrightarrow L^p_1(\partial\Omega) \tag{2.114}$$

are well defined, bounded operators, whenever $1 < p < \infty$.

Recall that $(\cdot,\cdot)_{\theta,q}$ and $[\cdot,\cdot]_\theta$ denote, respectively, the real and complex method of interpolation. A proof of the following result can be found in [43].

Theorem 2.8. *Suppose* Ω *is a bounded Lipschitz domain in* $\mathbb{R}^n$. *Let* $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha_0 \neq \alpha_1$, $0 < q_0, q_1, q \leqslant \infty$, $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. *Then*

$$\big(F^{p,q_0}_{\alpha_0}(\Omega), F^{p,q_1}_{\alpha_1}(\Omega)\big)_{\theta,q} = B^{p,q}_\alpha(\Omega), \quad 0 < p < \infty, \tag{2.115}$$

$$\big(B^{p,q_0}_{\alpha_0}(\Omega), B^{p,q_1}_{\alpha_1}(\Omega)\big)_{\theta,q} = B^{p,q}_\alpha(\Omega), \quad 0 < p \leqslant \infty. \tag{2.116}$$

Furthermore, if $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, p_1 \leqslant \infty$ *and* $0 < q_0, q_1 \leqslant \infty$ *are such that*

$$\textit{either } \max\{p_0, q_0\} < \infty, \quad \textit{or } \max\{p_1, q_1\} < \infty, \tag{2.117}$$

then

$$\big[F^{p_0,q_0}_{\alpha_0}(\Omega), F^{p_1,q_1}_{\alpha_1}(\Omega)\big]_\theta = F^{p,q}_\alpha(\Omega), \tag{2.118}$$

where $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

On the other hand, if $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, p_1, q_0, q_1 \leqslant \infty$ *are such that*

$$\min\{q_0, q_1\} < \infty, \tag{2.119}$$

then also

$$\big[B^{p_0,q_0}_{\alpha_0}(\Omega), B^{p_1,q_1}_{\alpha_1}(\Omega)\big]_\theta = B^{p,q}_\alpha(\Omega), \tag{2.120}$$

where θ, α, p, q *are as above.*

Finally, the same interpolation results are valid if the spaces $B^{p,q}_\alpha(\Omega)$, $F^{p,q}_\alpha(\Omega)$ *are replaced by* $B^{p,q}_{\alpha,0}(\Omega)$ *and* $F^{p,q}_{\alpha,0}(\Omega)$, *respectively.*

Recall the discussion about Hardy–Lorentz spaces in $\mathbb{R}^n$ from the last part of Section 2.2. For an arbitrary bounded Lipschitz domain Ω in $\mathbb{R}^n$ and $0 < p < \infty$, $0 < q \leqslant \infty$, we set

$$h^{p,q}(\Omega) := \big\{u \in \mathcal{D}'(\Omega)\colon\ \exists v \in h^{p,q}\big(\mathbb{R}^n\big) \text{ such that } v|_\Omega = u\big\}, \tag{2.121}$$

equipped with the natural (quasi-)norm. A useful observation we wish to make here is as follows. Based on the existence of a universal extension operator from Proposition 2.5, it can be checked that the analogs of (2.58)–(2.61) continue to hold for the version of these spaces defined in Lipschitz domains. That is, there exists a finite constant $C = C(\Omega) > 0$ with the property that

$$\|f\|_{L^{1,\infty}(\Omega)} \leqslant C\|f\|_{h^{1,\infty}(\Omega)}, \tag{2.122}$$

for nice functions f in Ω. Also,

$$\big(h^{p_0}(\Omega), h^{p_1}(\Omega)\big)_{\theta,\infty} = h^{p,\infty}(\Omega), \tag{2.123}$$

provided $0 < p_0, p_1 < \infty$, $\theta \in (0,1)$ and $1/p = (1-\theta)/p_0 + \theta/p_1$, and

$$\big(h^{p,q_1}(\Omega), h^{p,q_2}(\Omega)\big)_{\theta,q} = h^{p,q}(\Omega), \tag{2.124}$$

granted that $0 < p \leqslant 1$, $0 < q_0, q_1 \leqslant \infty$, $\theta \in (0,1)$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. Finally,

$$h^1(\Omega) \hookrightarrow L^1(\Omega) \hookrightarrow h^{1,\infty}(\Omega), \tag{2.125}$$

in a bounded fashion. In closing, let us also point out that it is possible to provide an intrinsic description of $h^{p,q}(\Omega)$, for $\frac{n}{n+1} < p < \infty$, $0 < q \leqslant \infty$. More specifically, fix some $\varphi \in C_c^\infty(B(0,1))$ with $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$ and set $\varphi_t(x) := t^{-n}\varphi(x/t)$, $t > 0$. Define the radial maximal function of a distribution $u \in \mathcal{D}'(\Omega)$ as

$$u_{\Omega,rad}(x) := \sup\big\{\big|(\varphi_t * u)(x)\big|:\ 0 < t < \operatorname{dist}(x,\partial\Omega)\big\}, \quad x \in \Omega. \tag{2.126}$$

One can then use the results in [69,71,70] to show that if $\frac{n}{n+1} < p < \infty$ and $0 < q \leqslant \infty$, then $u \in \mathcal{D}'(\Omega)$ belongs to $h^{p,q}(\Omega)$ if and only if $u_{\Omega,rad} \in L^{p,q}(\Omega)$, and $\|u\|_{h^{p,q}(\Omega)} \approx \|u_{\Omega,rad}\|_{L^{p,q}(\Omega)}$.

2.5. *Envelopes of nonlocally convex spaces*

Let X be a quasi-normed space and, for each $0 < p \leqslant 1$, let $B_{X,p}$ be the absolutely p-convex hull of the unit ball in X, i.e.,

$$B_{X,p} := \left\{ \sum_{j=1}^n \lambda_j a_j:\ a_j \in X,\ \|a_j\|_X \leqslant 1,\ \sum_{j=1}^n |\lambda_j|^p \leqslant 1,\ n \in \mathbb{N} \right\}. \tag{2.127}$$

Set

$$|||x|||_p := \inf\{\lambda > 0:\ x/\lambda \in B_{X,p}\}. \tag{2.128}$$

Then, for each quasi-normed space X whose dual separates its points, we denote by $\mathcal{E}_p(X)$ the *p-envelope* of X, defined as the completion of X in the quasi-norm $|||\cdot|||_p$. The case $p = 1$ corresponds to taking the Banach envelope, i.e. the minimal enlargement of the space in question to a Banach space; cf. [45] for a discussion.

Several results are going to be of importance for us here. The first one essentially asserts that for a linear operator, being bounded, and being onto are stable properties under taking envelopes.

Proposition 2.9. *Let X, Y be two quasi-normed spaces and let $T : X \longrightarrow Y$ be a bounded, linear operator. Then, for each $0 < p \leqslant 1$, this extends to a bounded, linear operator $\widehat{T} : \mathcal{E}_p(X) \longrightarrow \mathcal{E}_p(Y)$. Furthermore, if T is onto, then so is $\widehat{T}$.*

See [57] for a proof in the case $p = 1$ which readily adapts (cf. [51]) to the above setting. Our next result explicitly identifies the envelopes of regular Hardy spaces on boundaries of Lipschitz domains.

Theorem 2.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $\frac{n-1}{n} < p \leqslant q \leqslant 1$. Then*

$$\mathcal{E}_q\big(h^{1,p}_{\mathrm{at}}(\partial\Omega)\big) = B^{q,q}_s(\partial\Omega), \quad \textit{where } s := 1 + (n-1)\left(\frac{1}{q} - \frac{1}{p}\right). \tag{2.129}$$

Again, see [57] for the case $q = 1$, and [51] for the general case.

Moving on, let $\Omega \subset \mathbb{R}^n$ be Lipschitz and assume that L is a constant coefficient, elliptic differential operator. For $0 < p < \infty$, $\alpha \in \mathbb{R}$, introduce the space

$$\mathbb{H}^p_\alpha(\Omega; L) := \big\{u \in \mathcal{D}'(\Omega)\colon\ Lu = 0 \text{ in } \Omega \text{ and } \|u\|_{\mathbb{H}^p_\alpha(\Omega;L)} < \infty\big\}, \tag{2.130}$$

where, with $\rho(x) := \operatorname{dist}(x, \partial\Omega)$, $x \in \mathbb{R}^n$, we set

$$\|u\|_{\mathbb{H}^p_\alpha(\Omega;L)} := \big\|\rho^{\langle\alpha\rangle-\alpha}\big|\nabla^{\langle\alpha\rangle}u\big|\big\|_{L^p(\Omega)} + \sum_{j=0}^{\langle\alpha-1\rangle}\big\|\nabla^j u\big\|_{L^p(\Omega)}. \tag{2.131}$$

Above, $\langle\alpha\rangle$ will denote the smallest nonnegative integer greater than or equal to α, and $\nabla^j u$ will stand for the vector of all mixed-order partial derivatives of order j of the components of u. The following theorem has been proved in [51].

Theorem 2.11. *If L is an elliptic, homogeneous, constant coefficient differential operator and Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, then*

$$\mathbb{H}^p_\alpha(\Omega; L) = \big\{u \in F^{p,2}_\alpha(\Omega)\colon\ Lu = 0 \textit{ in } \Omega\big\}, \tag{2.132}$$

$$\mathbb{H}^p_\alpha(\Omega; L) = \big\{u \in B^{p,p}_\alpha(\Omega)\colon\ Lu = 0 \textit{ in } \Omega\big\} \tag{2.133}$$

for each $\alpha \in \mathbb{R}$ and each $0 < p < \infty$.

Later on, we shall also make use of the fact that, given a null-solution of an elliptic PDE in a Lipschitz domain, its membership to a Triebel–Lizorkin space is unaffected by the selection of the second integrability index for this scale of spaces. More precisely, the following result has been proved in [43].

Theorem 2.12. *Let L be an elliptic, homogeneous, constant coefficient differential operator, and fix a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. Then for each $p \in (0, \infty)$ and $\alpha \in \mathbb{R}$,*

$$\textit{the space } F^{p,q}_\alpha(\Omega) \cap \operatorname{Ker} L \textit{ is independent of } q \in (0, \infty). \tag{2.134}$$

Finally, we conclude with a useful envelope identification result from [38].

Theorem 2.13. *Let L be a second-order, elliptic, homogeneous, differential operator with real, constant coefficients, and let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then, for any $\frac{n-1}{n} < p < q \leqslant 1$ one has*

$$\mathcal{E}_q\big(\big\{u \in C^\infty(\Omega):\ Lu = 0,\ N(\nabla u) \in L^p(\partial\Omega)\big\}\big) = \mathbb{H}^q_{s+\frac{1}{q}}(\Omega; L), \tag{2.135}$$

where $s := 1 + (n-1)(\frac{1}{q} - \frac{1}{p})$.

3. The Dirichlet and Regularity problems in semiconvex domains

This section contains two subsections. In Section 3.1 we discuss the Green function associated with the Dirichlet Laplacian in general Lipschitz domains. Next, in Section 3.2, we take up the task of establishing well-posedness results for the Dirichlet and Regularity problems, with non-tangential maximal function estimates, in semiconvex domains (equivalently, Lipschitz domains satisfying a uniform exterior ball condition).

3.1. The Green function in Lipschitz domains

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. The Green function for the Laplacian in Ω is the unique function $G : \Omega \times \Omega \to [0, +\infty]$ satisfying

$$G(\cdot, y) \in W^{1,2}\big(\Omega \setminus B_r(y)\big) \cap \mathring{W}^{1,1}(\Omega), \quad \forall y \in \Omega,\ \forall r > 0 \tag{3.1}$$

(with $\mathring{W}^{1,1}(\Omega)$ denoting the closure of $C^\infty_c(\Omega)$ in $W^{1,1}(\Omega)$), and

$$\int_\Omega \big\langle \nabla_x G(x, y), \nabla \varphi(x)\big\rangle\, dx = \varphi(y), \quad \forall \varphi \in C^\infty_c(\Omega). \tag{3.2}$$

Thus,

$$\begin{aligned} &G(x, y)|_{x\in\partial\Omega} = 0 \quad \text{for every } y \in \Omega, \text{ and}\\ &-\Delta G(\cdot, y) = \delta_y \quad \text{for each fixed } y \in \Omega, \end{aligned} \tag{3.3}$$

where the restriction to the boundary is taken in the sense of Sobolev trace theory, and δ_y is the Dirac distribution in Ω, with mass at y. See, e.g., [37] and [46]. As is well know, the Green function is symmetric, i.e.,

$$G(x, y) = G(y, x), \quad \forall x, y \in \Omega, \tag{3.4}$$

so that, by the second line in (3.3),

$$-\Delta G(x, \cdot) = \delta_x \quad \text{for each fixed } x \in \Omega. \tag{3.5}$$

Lemma 3.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then for each $y \in \Omega$ fixed,*

$$\big[\nabla_x G(x,y)\big]\big|_{x\in\partial\Omega} \quad \textit{exists for a.e. } x \in \partial\Omega, \tag{3.6}$$

where the boundary trace is taken in the sense of (2.14).

Proof. Upon recalling the fundamental solution for the Laplacian in $\mathbb{R}^n$ from (2.23), we observe that for each $y \in \Omega$ we may write

$$G(x,y) = -E(x-y) + v^y(x), \quad x \in \Omega, \tag{3.7}$$

where v^y solves (cf. Theorem 2.1)

$$\begin{cases} \Delta v^y = 0 \quad \text{in } \Omega, \\ v^y\big|_{\partial\Omega} = E(\cdot - y)|_{\partial\Omega} \in L^2_1(\partial\Omega), \\ N\big(\nabla v^y\big) \in L^2(\partial\Omega). \end{cases} \tag{3.8}$$

Indeed, since by (2.13) and (2.23) we have

$$v^y \in W^{1,\frac{2n}{n-1}}(\Omega) \cap C(\overline{\Omega}) \quad \text{and} \quad E \in W^{1,1}_{loc}\big(\mathbb{R}^n\big) \cap C^\infty\big(\mathbb{R}^n \setminus \{0\}\big), \tag{3.9}$$

and $v^y = E(\cdot - y)$ on $\partial\Omega$, (3.1) follows. Also, (3.2) is a direct consequence of (3.7) and the fact that v^y is harmonic in Ω. Finally, since for every $r > 0$ the function $-E(\cdot - y) + v^y$ is harmonic in $\Omega \setminus \overline{B_r(y)}$ and is $\geqslant 0$ on $\partial(\Omega \setminus \overline{B_r(y)})$ if $r > 0$ is small enough, the maximum principle ensures that $-E(\cdot - y) + v^y \geqslant 0$ on $\Omega \setminus \overline{B_r(y)}$, granted that $r > 0$ is small enough. Thus, $-E(\cdot - y) + v^y : \Omega \to [0,+\infty]$, so (3.7) is indeed the Green function for Ω.

Going further, (3.7) allows us to write

$$\nabla_x G(x,y) = -(\nabla E)(x-y) + \big(\nabla v^y\big)(x), \quad x, y \in \Omega, \tag{3.10}$$

and since $\nabla v^y|_{\partial\Omega}$ exists a.e. on $\partial\Omega$, we can conclude that (3.6) holds. □

As a consequence of Lemma 3.1,

$$\partial_{\nu(x)} G(x,y)|_{x\in\partial\Omega} := \nu(x) \cdot \Big(\lim_{\substack{z\in\Gamma_\kappa(x) \\ z\to x}} \nabla_z G(z,y)\Big) \tag{3.11}$$

exists for a.e. $x \in \partial\Omega$. Our next result provides a useful integral representation formula for the normal derivative of the Green function.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with outward unit normal ν and, as before, let $G(\cdot,\cdot)$ denote the associated Green function. Then for any fixed $y \in \Omega$,*

$$-\partial_{\nu(x)} G(x,y) = \left(\left(\frac{1}{2}I + K\right)^{-1}\right)^{*} \big(\partial_{\nu(\cdot)}\big[E(\cdot - y)\big]\big)(x) \tag{3.12}$$

for σ-a.e. $x \in \partial\Omega$, where both adjunction and inverse are taken in the sense of $L^2(\partial\Omega)$.

Proof. In general, for any two reasonably well-behaved functions u, v in Ω, the following Green formula holds

$$\int_{\Omega} \Delta u v - \int_{\Omega} u \Delta v = \int_{\partial\Omega} \partial_\nu u v \, d\sigma - \int_{\partial\Omega} u \partial_\nu v \, d\sigma. \tag{3.13}$$

Fix $x, y \in \Omega$, $x \neq y$ and use (3.13) for $u := E(x - \cdot)$ and $v := E(\cdot - y)$. Then, since $\Delta u = \delta_x$ and $\Delta v = \delta_y$, (3.13) becomes

$$\int_{\partial\Omega} \partial_{\nu(z)}\big[E(x-z)\big]E(z-y)\, d\sigma(z) = \int_{\partial\Omega} E(x-z)\partial_{\nu(z)}\big[E(z-y)\big]\, d\sigma(z), \tag{3.14}$$

that is,

$$\mathcal{D}\big(E(\cdot - y)|_{\partial\Omega}\big)(x) = \mathcal{S}\big(\partial_{\nu(\cdot)}\big[E(\cdot - y)\big]\big)(x). \tag{3.15}$$

Keeping now $y \in \Omega$ fixed, and letting x go nontangentially to the boundary, (3.15) implies

$$\left(\frac{1}{2}I + K\right)\big(E(\cdot - y)|_{\partial\Omega}\big) = S\big(\partial_{\nu(\cdot)}\big[E(\cdot - y)\big]\big), \quad \forall y \in \Omega. \tag{3.16}$$

If we now apply $-\frac{1}{2}I + K$ to both sides of (3.16), we obtain that

$$\left(-\frac{1}{2}I + K\right)S\big(\partial_{\nu(\cdot)}\big[E(\cdot - y)\big]\big) = \left(-\frac{1}{2}I + K\right)\left(\frac{1}{2}I + K\right)\big(E(\cdot - y)|_{\partial\Omega}\big), \quad \forall y \in \Omega. \tag{3.17}$$

To continue, we further specialize (3.13) to the case when $u := \mathcal{S}f$, $v := \mathcal{S}g$, where $f, g \in L^2(\partial\Omega)$ are arbitrary. On account of (2.26) and (2.27), this yields

$$\int_{\partial\Omega} \left(-\frac{1}{2}I + K^*\right) f Sg \, d\sigma = \int_{\partial\Omega} Sf \left(-\frac{1}{2}I + K^*\right) g \, d\sigma. \tag{3.18}$$

Given that $S : L^2(\partial\Omega) \to L^2(\partial\Omega)$ is self-adjoint, and f, g are arbitrary, this readily implies

$$KS = SK^* \quad \text{as operators on } L^2(\partial\Omega). \tag{3.19}$$

Consequently,

$$S^{-1}\left(\pm\frac{1}{2}I + K\right) = \left(\pm\frac{1}{2}I + K^*\right)S^{-1}, \tag{3.20}$$

where S^{-1} is regarded as a bounded mapping from $L^2_1(\partial\Omega)$ onto $L^2(\partial\Omega)$, while K and K^* are bounded epimorphisms of $L^2_1(\partial\Omega)$ and $L^2(\partial\Omega)$, respectively. With (3.19)–(3.20) in hand, (3.17) becomes

$$\left(-\frac{1}{2}I+K^*\right)\left(\partial_{\nu(\cdot)}\big[E(\cdot-y)\big]\right)=\left(\frac{1}{2}I+K^*\right)\left(-\frac{1}{2}I+K^*\right)S^{-1}\left(E(\cdot-y)|_{\partial\Omega}\right),\quad \forall y\in\Omega. \tag{3.21}$$

We now apply $(\frac{1}{2}I+K^*)^{-1}$ to (3.21) and obtain that, for each $y\in\Omega$,

$$\partial_{\nu(\cdot)}\big[E(\cdot-y)\big]-\left(\frac{1}{2}I+K^*\right)^{-1}\left(\partial_{\nu(\cdot)}\big[E(\cdot-y)\big]\right)=\left(-\frac{1}{2}I+K^*\right)S^{-1}\left(E(\cdot-y)|_{\partial\Omega}\right). \tag{3.22}$$

Recall (3.7) to first note that

$$-\partial_{\nu(\cdot)}G(\cdot,y)=\partial_{\nu(\cdot)}\big[E(\cdot-y)\big]-\partial_{\nu(\cdot)}v^y. \tag{3.23}$$

Moreover, since v^y solves (3.8), we know from (2.37) that

$$v^y=\mathcal{S}\left(S^{-1}\left(E(\cdot,y)|_{\partial\Omega}\right)\right), \tag{3.24}$$

so that the term in the right-hand side of (3.22) is equal to $\partial_\nu v^y$. Combining now (3.22), (3.23), and (3.24) we obtain that

$$-\partial_{\nu(\cdot)}G(\cdot,y)=\left(\frac{1}{2}I+K^*\right)^{-1}\left(\partial_{\nu(\cdot)}\big[E(\cdot-y)\big]\right), \tag{3.25}$$

from which (3.12) follows easily. □

It is useful to note that, as (3.12) and Theorem 2.1 show, for every $y\in\Omega$ fixed, we have

$$-\partial_{\nu(\cdot)}G(\cdot,y)\in L^p(\partial\Omega),\quad \forall p\in(1,2+\varepsilon), \tag{3.26}$$

whenever $\Omega\subset\mathbb{R}^n$ is a bounded Lipschitz domain and $\varepsilon=\varepsilon(\partial\Omega)>0$ is as in Theorem 2.1.

Proposition 3.3. *Let $\Omega\subset\mathbb{R}^n$ be a bounded Lipschitz domain with outward unit normal ν and, as before, let $G(\cdot,\cdot)$ denote the associated Green function. Also, recall the parameter $\varepsilon=\varepsilon(\partial\Omega)>0$ from Theorem* 2.1. *Then for any $f\in L^p(\partial\Omega)$, $2-\varepsilon<p<\infty$, the unique solution of the Dirichlet problem*

$$(D)_p\quad \begin{cases}\Delta u=0 & \text{in } \Omega,\\ u|_{\partial\Omega}=f & \text{on } \partial\Omega,\\ Nu\in L^p(\partial\Omega),\end{cases} \tag{3.27}$$

can be expressed as

$$u(y)=-\int\limits_{\partial\Omega}\partial_{\nu(x)}G(x,y)f(x)\,d\sigma(x),\quad y\in\Omega. \tag{3.28}$$

Proof. By virtue of (2.37) and (3.12), for each $y \in \Omega$ fixed, we can write

$$\begin{aligned} u(y) &= \mathcal{D}\left(\left(\frac{1}{2}I + K\right)^{-1} f\right)(y) \\ &= \int_{\partial\Omega} \partial_{\nu(\cdot)}\left[E(\cdot - y)\right]\left[\left(\frac{1}{2}I + K\right)^{-1} f\right](x)\, d\sigma(x) \\ &= \int_{\partial\Omega} \left(\left(\frac{1}{2}I + K\right)^{-1}\right)^{*}\left(\partial_{\nu(\cdot)}\left[E(\cdot - y)\right]\right)(x) f(x)\, d\sigma(x) \\ &= -\int_{\partial\Omega} \partial_{\nu(x)} G(x, y) f(x)\, d\sigma(x), \end{aligned} \tag{3.29}$$

proving (3.28). □

Remark. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then, as is well known, there exists a family of probability measures $\{\omega^y\}_{y\in\Omega}$ on $\partial\Omega$ with the property that the unique solution of the classical Dirichlet problem

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u \in C^2(\Omega) \cap C^0(\overline{\Omega}), \qquad u|_{\partial\Omega} = f \in C^0(\partial\Omega), \tag{3.30}$$

can be represented as

$$u(y) = \int_{\partial\Omega} f(x)\, d\omega^y(x), \quad y \in \Omega. \tag{3.31}$$

Thus, from (3.30)–(3.31) and (3.28) it follows that, for each $y \in \Omega$, the harmonic measure with pole at y, i.e., $d\omega^y$, is absolutely continuous with respect to the surface measure, and the Radon–Nikodym derivative of the former with respect to the latter is given by

$$\frac{d\omega^y}{d\sigma} = -\partial_{\nu(\cdot)} G(\cdot, y), \quad y \in \Omega. \tag{3.32}$$

3.2. Dirichlet and Regularity problem with nontangential maximal function estimates

We debut by making the following definition.

Definition 3.4. An open set $\Omega \subset \mathbb{R}^n$ is said to satisfy a uniform exterior ball condition (henceforth abbreviated by UEBC), if there exists $r > 0$ with the following property: For each $x \in \partial\Omega$, there exists a point $y = y(x) \in \mathbb{R}^n$ such that

$$\overline{B_r(y)} \setminus \{x\} \subseteq \mathbb{R}^n \setminus \Omega \quad \text{and} \quad x \in \partial B_r(y). \tag{3.33}$$

The largest radius r satisfying the above property will be referred to as the UEBC constant of Ω.

Parenthetically, we note that a bounded open set Ω has a $C^{1,1}$ boundary if and only if Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ satisfy a UEBC. However, UEBC alone does allow the boundary to develop irregularities which are "outwardly directed." It is also important to note that any bounded, open convex subset of $\mathbb{R}^n$ satisfies a UEBC with constant r for any $r > 0$.

We continue to review a series of definitions and basic results.

Definition 3.5. Let $\mathcal{O}$ be an open set in $\mathbb{R}^n$. The collection of semiconvex functions on $\mathcal{O}$ consists of continuous functions $u : \mathcal{O} \to \mathbb{R}$ with the property that there exists $C > 0$ such that

$$2u(x) - u(x+h) - u(x-h) \leqslant C|h|^2, \quad \forall x, h \in \mathbb{R}^n \text{ with } [x-h, x+h] \subseteq \mathcal{O}. \tag{3.34}$$

The best constant C above is referred to as the semiconvexity constant of u.

Some of the most basic properties of the class of semiconvex functions are collected in the next two propositions below. Proofs can be found in, e.g., [10].

Proposition 3.6. *Assume that $\mathcal{O}$ is an open, convex subset of $\mathbb{R}^n$. Given a function $u : \mathcal{O} \to \mathbb{R}$ and a finite constant $C > 0$, the following conditions are equivalent*:

(i) *u is semiconvex with semiconvexity constant C*;
(ii) *u satisfies*

$$u\big(\lambda x + (1-\lambda)y\big) - \lambda u(x) - (1-\lambda)u(y) \leqslant C\frac{\lambda(1-\lambda)}{2}|x-y|^2, \tag{3.35}$$

for all $x, y \in \mathcal{O}$ and all $\lambda \in [0,1]$;
(iii) *the function $\mathcal{O} \ni x \mapsto u(x) + C|x|^2/2 \in \mathbb{R}$ is convex in $\mathcal{O}$*;
(iv) *there exist two functions, $u_1, u_2 : \mathcal{O} \to \mathbb{R}$ such that $u = u_1 + u_2$, u_1 is convex, $u_2 \in C^2(\mathcal{O})$ and $\|\nabla^2 u_2\|_{L^\infty(\mathcal{O})} \leqslant C$*;
(v) *for any $v \in S^{n-1}$, the* (*distributional*) *second-order directional derivative of u along v, i.e., $D_v^2 u$, satisfies $D_v^2 u \geqslant C$ in $\mathcal{O}$, in the sense that*

$$\int_{\mathcal{O}} u(x)\big(\mathrm{Hess}_\varphi(x)v\big)\cdot v\,dx \geqslant C\int_{\mathcal{O}} \varphi(x)\,dx, \quad \forall \varphi \in C_c^\infty(\mathcal{O}),\ \varphi \geqslant 0, \tag{3.36}$$

where $\mathrm{Hess}_\varphi := (\frac{\partial^2\varphi}{\partial x_j \partial x_k})_{1\leqslant j,k\leqslant n}$ *is the Hessian matrix of the function φ*;
(vi) *the function u can be represented as $u(x) = \sup_{i\in I} u_i(x)$, $x \in \mathcal{O}$, where $\{u_i\}_{i\in I}$ is a family of functions in $C^2(\mathcal{O})$ with the property that $\|\nabla^2 u_i\|_{L^\infty(\mathcal{O})} \leqslant C$ for every $i \in I$*;
(vii) *the same as* (vi) *above except that, this time, each function u_i is of the form $u_i(x) = a_i + w_i \cdot x + C|x|^2/2$, for some number $a_i \in \mathbb{R}$ and vector $w_i \in \mathbb{R}^n$.*

We also have

Proposition 3.7. *Suppose that $\mathcal{O}$ is an open subset of $\mathbb{R}^n$ and that $u : \mathcal{O} \to \mathbb{R}$ is a semiconvex function. Then the following assertions hold*:

(1) *The function u is locally Lipschitz in $\mathcal{O}$.*
(2) *The gradient of u (which, by Rademacher's theorem exists a.e. in $\mathcal{O}$) belongs to* $\mathrm{BV}_{loc}(\mathcal{O}, \mathbb{R}^n)$.
(3) *The function u is twice differentiable a.e. in $\mathcal{O}$ (Alexandroff's theorem). More concretely, for a.e. point x_0 in $\mathcal{O}$ there exists an $n \times n$ symmetric matrix $H_u(x_0)$ with the property that*

$$\lim_{x \to x_0} \frac{u(x) - u(x_0) - (x - x_0) \cdot \nabla u(x_0) + 2^{-1}(H_u(x_0)(x - x_0)) \cdot (x - x_0)}{|x - x_0|^2} = 0. \qquad (3.37)$$

Definition 3.8. A nonempty, proper, bounded open subset Ω of $\mathbb{R}^n$ is called semiconvex provided there exist $b, c > 0$ with the property that for every $x_0 \in \partial\Omega$ there exist an $(n-1)$-dimensional affine variety $H \subset \mathbb{R}^n$ passing through x_0, a choice N of the unit normal to H, and cylinder $\mathcal{C}$ as in (2.1) and some semiconvex function $\varphi : H \to \mathbb{R}$ satisfying (2.2)–(2.4) as well as (2.5).

It is then clear from Proposition 3.7, the definition of a Lipschitz domain at the beginning of Section 2.1, and Definition 3.8 that bounded semiconvex domains form a subclass of the class of bounded Lipschitz domains. The key feature which distinguishes the former from the latter is described in the theorem below, recently proved in [61].

Theorem 3.9. *Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty, bounded, open set. Then the following conditions are equivalent*:

(i) *Ω is a Lipschitz domain satisfying a UEBC*;
(ii) *Ω is a semiconvex domain.*

Our aim now is to show that in the case when Ω is a bounded semiconvex domain (hence, a Lipschitz domain satisfying a UEBC), then (3.28) solves $(D)_p$ for every $p \in (1, \infty)$. The well-posedness of $(D)_p$ for each $p \in (1, \infty)$ has been established in the case when $\partial\Omega \in C^1$ by E.B. Fabes, M. Jodeit and N.M. Rivière in [27] by relying on the method of layer potentials. By way of contrast, in the current setting, we shall use (3.28) and a key ingredient in this regard is an estimate proved by M. Grüter and K.-O. Widman (see Theorem 3.3(v) in [37]) to the effect that if $\Omega \subset \mathbb{R}^n$ is a domain which satisfies a UEBC then the Green function satisfies

$$\left|\nabla_x G(x, y)\right| \leqslant C \operatorname{dist}(y, \partial\Omega)|x - y|^{-n}, \quad \forall x, y \in \Omega, \qquad (3.38)$$

where C depends only on n and the UEBC constant of Ω. Based on this, we shall prove the following.

Theorem 3.10. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded semiconvex domain and suppose that $p \in (1, \infty)$ is arbitrary and fixed. Then for every datum $f \in L^p(\partial\Omega)$ the Dirichlet problem $(D)_p$ (cf.* (3.27)) *is uniquely solvable. Moreover, the solution u satisfies*

$$\|Nu\|_{L^p(\partial\Omega)} \leqslant C\|f\|_{L^p(\partial\Omega)}, \qquad (3.39)$$

and can be expressed as in (3.28).

Proof. In concert with Lemma 3.1, (3.38) yields (by taking the nontangential limit of x to an arbitrary boundary point) that for each fixed $y \in \Omega$,

$$\left|\nabla_x G(x,y)\right| \leqslant C \operatorname{dist}(y,\partial\Omega)|x-y|^{-n}, \quad \text{for a.e. } x \in \partial\Omega. \tag{3.40}$$

Note that this implies that

$$-\partial_{\nu(\cdot)} G(\cdot,y) \in L^\infty(\partial\Omega), \quad \forall y \in \Omega. \tag{3.41}$$

As a consequence, if $1 < p < \infty$ and $f \in L^p(\partial\Omega)$ is fixed, then u in (3.28) is a well-defined, harmonic function in Ω (for the latter claim see (3.5)). At the moment, our goal is to show that $Nu \in L^p(\partial\Omega)$. To this end, we remark that from (3.40) and (3.28) we have

$$\left|u(y)\right| \leqslant C \int_{\partial\Omega} \frac{\operatorname{dist}(y,\partial\Omega)}{|x-y|^n} \left|f(x)\right| d\sigma(x), \quad \forall y \in \Omega. \tag{3.42}$$

To continue, fix $\kappa > 0$ and consider an arbitrary boundary point, $y_0 \in \partial\Omega$. We claim that there exists a constant $C = C(\partial\Omega, \kappa) > 0$ such that

$$|x-y| \geqslant C\big(\operatorname{dist}(y,\partial\Omega) + |x-y_0|\big), \quad \forall y \in \Gamma_\kappa(y_0),\ x \in \partial\Omega. \tag{3.43}$$

Indeed, the fact that $|x-y| \geqslant \operatorname{dist}(y,\partial\Omega)$ is immediate. Next, pick $z \in \partial\Gamma_\kappa(y_0) \subset \overline{\Omega}$ such that $\operatorname{dist}(x,\Gamma_\kappa(y_0)) = |x-z|$. We therefore have $|y_0 - z| = (1+\kappa)\operatorname{dist}(z,\partial\Omega)$, hence we can write

$$|x-y_0| \leqslant |x-z| + |z-y_0| = |x-z| + (1+\kappa)\operatorname{dist}(z,\partial\Omega) \leqslant (2+\kappa)|x-z|. \tag{3.44}$$

Thus, $|x-y_0| \leqslant (2+\kappa)\operatorname{dist}(x,\Gamma_\kappa(y_0)) \leqslant (2+\kappa)|x-y|$, and (3.43) follows.

Going further and making use of (3.43) in (3.42) gives that

$$\left|u(y)\right| \leqslant C \int_{\partial\Omega} \frac{\operatorname{dist}(y,\partial\Omega)}{(\operatorname{dist}(y,\partial\Omega) + |x-y_0|)^n} \left|f(x)\right| d\sigma(x), \quad \forall y \in \Gamma_\kappa(y_0). \tag{3.45}$$

Let us fix $y \in \Gamma_\kappa(y_0)$ and set $r := \operatorname{dist}(y,\partial\Omega)$. We make use of a familiar argument based on decomposing $\partial\Omega$ into a (finite) family of dyadic annuli $\partial\Omega = \bigcup_{j=0}^N R_j(y_0)$, where $R_0(y_0) = \Delta_{2r}(y_0)$ and $R_j(y_0) := \Delta_{2^{j+1}r}(y_0) \setminus \Delta_{2^j r}(y_0)$ for $1 \leqslant j \leqslant N$. Estimating we can conclude that

$$\int_{R_j(y_0)} \frac{r}{(r+|x-y_0|)^n} \left|f(x)\right| d\sigma(x) \leqslant \frac{C}{r^{n-1}2^{jn}} \int_{\Delta_{2^{j+1}r}(y_0)} |f|\, d\sigma \leqslant C 2^{-j} Mf(y_0), \tag{3.46}$$

uniformly in $j \geqslant 1$, where M denotes the Hardy–Littlewood maximal function on $\partial\Omega$ (cf. (2.9)). Also,

$$\int_{R_0(y_0)} \frac{r}{(r+|x-y_0|)^n} \left|f(x)\right| d\sigma(x) \leqslant \frac{C}{r^{n-1}} \int_{\Delta_r(y_0)} |f|\, d\sigma \leqslant CMf(y_0), \tag{3.47}$$

thus, on account of (3.45)–(3.47), we obtain that $(Nu)(y_0) \leqslant C(Mf)(y_0)$. Since y_0 was arbitrarily selected in $\partial\Omega$, this proves that $Nu \leqslant CMf$ pointwise on $\partial\Omega$. Hence, for every $1 < p < \infty$,

$$\|Nu\|_{L^p(\partial\Omega)} \leqslant C(\Omega, p)\|f\|_{L^p(\partial\Omega)}, \tag{3.48}$$

by the boundedness of M on $L^p(\partial\Omega)$.

In summary, to show that u defined as in (3.28) is a solution of (3.27), we are left with proving that its nontangential boundary trace exists and equals the given datum f a.e. on $\partial\Omega$. Since we have proved that u defined as in (3.28) is harmonic and verifies (3.48), a Fatou-type theorem proved by B. Dahlberg in [18] gives that $u|_{\partial\Omega}$ exists a.e. on $\partial\Omega$. Consequently, if we consider the linear assignment $T : L^p(\partial\Omega) \to L^p(\partial\Omega)$, given by

$$L^p(\partial\Omega) \ni f \mapsto Tf := u|_{\partial\Omega} \in L^p(\partial\Omega), \tag{3.49}$$

then T is well defined and bounded, thanks to (3.48). In addition, Proposition 3.3 ensures that $Tf = f$ whenever $f \in L^2(\partial\Omega) \cap L^p(\partial\Omega)$. By density, we may therefore conclude that $Tf = f$ for every $f \in L^p(\partial\Omega)$, $1 < p < \infty$.

For the proof of the uniqueness part in the statement of Theorem 3.10, we shall need the fact that for each $p \in (1, \infty)$, the Regularity problem

$$(R)_p \quad \begin{cases} \Delta u = 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} = f \in L^p_1(\partial\Omega), \\ N(\nabla u) \in L^p(\partial\Omega) \end{cases} \tag{3.50}$$

has a solution u satisfying

$$\big\|N(\nabla u)\big\|_{L^p(\partial\Omega)} \leqslant C\|f\|_{L^p_1(\partial\Omega)}, \tag{3.51}$$

whenever $f \in L^p_1(\partial\Omega) \cap L^2_1(\partial\Omega)$. From Theorem 2.1, we know that this is the case when $1 < p \leqslant 2$, so it suffices to treat the case when $2 < p < \infty$. Assuming that $2 < p < \infty$, fix an arbitrary $f \in L^p_1(\partial\Omega) \hookrightarrow L^2_1(\partial\Omega)$ and, by relying on Theorem 2.1, we let u solve $(R)_2$ in Ω with datum f. In particular,

$$N(\nabla u), Nu \in L^2(\partial\Omega). \tag{3.52}$$

Next, fix a boundary point $z = (z', z_n) \in \partial\Omega$ and assume that $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function such that $\varphi(z') = z_n$ and, for some $T, R > 0$,

$$\begin{aligned} \Sigma_R &:= \big\{\big(x', \varphi\big(x'\big)\big)\colon x' \in \mathbb{R}^{n-1},\ \big|x' - z'\big| < R\big\} \subset \partial\Omega, \\ \mathcal{O}_{R,T} &:= \{x + te_n\colon x \in \Sigma_R,\ 0 < t < T\} \subset \Omega. \end{aligned} \tag{3.53}$$

For $\varepsilon \in (0, \varepsilon_o)$, with $\varepsilon_o > 0$ small, let us also set

$$\mathcal{O}_{R,T,\varepsilon} := \{x + te_n\colon x \in \Sigma_{R-\varepsilon},\ \varepsilon < t < T - \varepsilon\}. \tag{3.54}$$

Without loss of generality, we can assume that the origin in $\mathbb{R}^n$ belongs to $\mathcal{O}_{R,T,\varepsilon}$ and that $\mathcal{O}_{R,T}$ is star-like with respect to this point. With $p' \in (1,\infty)$ denoting the conjugate exponent for p, pick an arbitrary $h \in L^{p'}(\partial\Omega) \cap L^2(\partial\Omega)$ such that $\operatorname{supp} h \subseteq \Sigma_R$, and let $\widetilde{h}$ denote the extension of h by zero to a function in $L^{p'}(\partial\mathcal{O}_{R,T}) \cap L^2(\partial\mathcal{O}_{R,T})$ Also, let v solve $(D)_{p'}$ in $\mathcal{O}_{R,T}$ with datum $\widetilde{h}$. This solution is constructed according to the recipe presented in (3.28). Since the boundary datum belongs to $L^{p'}(\partial\mathcal{O}_{R,T}) \cap L^2(\partial\mathcal{O}_{R,T})$, the first part in the present proof ensures that

$$Nv \in L^{p'}(\partial\mathcal{O}_{R,T}) \cap L^2(\partial\mathcal{O}_{R,T}). \tag{3.55}$$

Moving on, set $\widetilde{v}(x) := v(x) - v(0)$, for $x \in \mathcal{O}_{R,T}$. Then $\widetilde{v}(0) = 0$, so that

$$w(x) := \int_0^1 \widetilde{v}(tx)\frac{dt}{t}, \quad x \in \mathcal{O}_{R,T}, \tag{3.56}$$

is well defined (note that the properties of v ensure that the integral is convergent) and, with $\nabla_\eta := \sum_{j=1}^n x_j\partial_j$ denoting the radial derivative, w is a harmonic function which is a normalized radial anti-derivative for $\widetilde{v}$. That is,

$$w(0) = 0 \quad \text{and} \quad \Delta w = 0, \qquad \nabla_\eta w = \widetilde{v} \quad \text{in } \mathcal{O}_{R,T}. \tag{3.57}$$

Going further, with $N_{R,T}$ denoting the nontangential maximal operator associated with the Lipschitz domain $\mathcal{O}_{R,T}$, we have

$$\begin{aligned} \|N_{R,T}(\nabla w)\|_{L^{p'}(\partial\mathcal{O}_{R,T})} &\leqslant C\|N_{R,T}(\nabla_\eta w)\|_{L^{p'}(\partial\mathcal{O}_{R,T})} = C\|N_{R,T}\widetilde{v}\|_{L^{p'}(\partial\mathcal{O}_{R,T})} \\ &\leqslant C\|N_{R,T}v\|_{L^{p'}(\partial\mathcal{O}_{R,T})} + C|v(0)| \leqslant C\|\widetilde{h}\|_{L^{p'}(\partial\mathcal{O}_{R,T})} + C|v(0)| \\ &\leqslant C\|h\|_{L^{p'}(\partial\Omega)}. \end{aligned} \tag{3.58}$$

Above, the first inequality is implied by Lemma 2.12 from [64] which, in turn, is inspired by earlier work in [77,84]. Also, the next-to-last inequality is a consequence of the fact that v solves $(D)_{p'}$ with datum $\widetilde{h}$, while the last inequality can be easily justified based on (2.13) and the mean-value theorem for the harmonic function v. In a similar fashion, we also have that

$$N_{R,T}(\nabla w) \in L^2(\partial\mathcal{O}_{R,T}). \tag{3.59}$$

Then, for any Lipschitz domain D, with outward unit normal ν, such that D is a relatively compact subset of $\mathcal{O}_{R,T}$, we have

$$\begin{aligned} \partial_\nu v = \partial_\nu\widetilde{v} &= \sum_{j,k=1}^n \nu_j\partial_j(x_k\partial_k w) = \sum_{j=1}^n \nu_j\partial_j w + \sum_{j,k=1}^n \nu_j x_k\partial_k\partial_j w \\ &= \partial_\nu w + \sum_{j,k=1}^n x_k(\nu_j\partial_k - \nu_k\partial_j)\partial_j w = \partial_\nu w + \sum_{j,k=1}^n x_k\partial_{\tau_{jk}}\partial_j w \quad \text{in } D, \end{aligned} \tag{3.60}$$

since $\sum_{j=1}^n \partial_j\partial_j w = 0$ in D. We now proceed by writing

$$\begin{aligned}
\int_{\partial\Omega} \partial_\nu u h\, d\sigma &= \int_{\partial\mathcal{O}_{R,T}} \partial_\nu u \widetilde{h}\, d\sigma = \int_{\partial\mathcal{O}_{R,T}} \partial_\nu u v\, d\sigma = \lim_{t\to 0^+} \int_{\partial\mathcal{O}_{R,T}} \partial_\nu\big[u(\cdot, t+e_n)\big] v\, d\sigma \\
&= \lim_{t\to 0^+}\Bigg(\lim_{\varepsilon\to 0^+} \int_{\partial\mathcal{O}_{R,T,\varepsilon}} \partial_\nu\big[u(\cdot + te_n)\big] v\, d\sigma\Bigg) \\
&= \lim_{t\to 0^+}\Bigg(\lim_{\varepsilon\to 0^+} \int_{\partial\mathcal{O}_{R,T,\varepsilon}} u(\cdot + te_n)\partial_\nu v\, d\sigma\Bigg) \\
&= \lim_{t\to 0^+}\Bigg(\lim_{\varepsilon\to 0^+} \int_{\partial\mathcal{O}_{R,T,\varepsilon}} u(\cdot + te_n)\Bigg(\partial_\nu w + \sum_{j,k=1}^{n} x_k \partial_{\tau_{jk}}\partial_j w\Bigg) d\sigma\Bigg) \\
&= \lim_{t\to 0^+}\Bigg(\lim_{\varepsilon\to 0^+} \int_{\partial\mathcal{O}_{R,T,\varepsilon}} \Bigg(u(\cdot + te_n)\partial_\nu w + \sum_{j,k=1}^{n} \partial_{\tau_{kj}}\big[x_k u(\cdot + te_n)\big]\partial_j w\Bigg) d\sigma\Bigg) \\
&= \int_{\partial\mathcal{O}_{R,T}} \Bigg(u\partial_\nu w + \sum_{j,k=1}^{n} \partial_{\tau_{kj}}(x_k u)\partial_j w\Bigg) d\sigma. \qquad (3.61)
\end{aligned}$$

The third equality above is implied by (3.52) and the fact that $h \in L^2(\partial\Omega)$, while the fourth uses (3.55) and the observation that for each small, fixed $t > 0$, the function $u(\cdot + te_n)$ is C^∞ in a neighborhood of $\overline{\mathcal{O}_{R,T}}$. Next, the fifth equality is a consequence of Green's formula for u, v which are harmonic in a neighborhood of $\overline{\mathcal{O}_{R,T,\varepsilon}}$, whereas the sixth equality is based on (3.60) (written for $D := \mathcal{O}_{R,T,\varepsilon}$). The seventh equality uses an integration by parts on the boundary and, finally, the last equality is implied by (3.52) and (3.59). Next, based on (3.61), (3.58) and Hölder's inequality we may estimate

$$\begin{aligned}
\Bigg|\int_{\partial\Omega} \partial_\nu u h\, d\sigma\Bigg| &\leqslant C\big[\|\nabla_{\tan} u\|_{L^p(\partial\mathcal{O}_{R,T})} + \|u\|_{L^p(\partial\mathcal{O}_{R,T})}\big]\|h\|_{L^{p'}(\partial\Omega)} \\
&\leqslant C\Bigg[\|u\|_{L^p_1(\partial\Omega)} + \Bigg(\int_{\partial\mathcal{O}_{R,T}\setminus\partial\Omega} |\nabla u|^p\, d\sigma\Bigg)^{1/p} \\
&\quad + \Bigg(\int_{\partial\mathcal{O}_{R,T}\setminus\partial\Omega} |u|^p\, d\sigma\Bigg)^{1/p}\Bigg]\|h\|_{L^{p'}(\partial\Omega)}. \qquad (3.62)
\end{aligned}$$

By raising both sides to the p-th power and suitably averaging the resulting estimate in R, we arrive at

$$\Bigg|\int_{\partial\Omega} \partial_\nu u h\, d\sigma\Bigg| \leqslant C\Bigg[\|u\|_{L^p_1(\partial\Omega)} + \Bigg(\int_{\Omega} |\nabla u|^p\, dx\Bigg)^{1/p} + \Bigg(\int_{\Omega} |u|^p\, dx\Bigg)^{1/p}\Bigg]\|h\|_{L^{p'}(\partial\Omega)}. \qquad (3.63)$$

To continue, fix some $\varepsilon > 0$ (to be specified later), and consider the compact set

$$D_\varepsilon := \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) \geqslant \varepsilon\}. \tag{3.64}$$

Then

$$\begin{aligned}\int_\Omega \big[|\nabla u|^p + |u|^p\big]\,dx &= \int_{D_\varepsilon} \big[|\nabla u|^p + |u|^p\big]\,dx + \int_{\Omega\setminus D_\varepsilon} \big[|\nabla u|^p + |u|^p\big]\,dx \\ &\leqslant \int_{D_\varepsilon} \big[|\nabla u|^p + |u|^p\big]\,dx + \varepsilon C \int_{\partial\Omega} \big[\big|N(\nabla u)\big|^p + |Nu|^p\big]\,d\sigma, \end{aligned} \tag{3.65}$$

where $C = C(\partial\Omega) > 0$ is a finite constant which only depends on the Lipschitz character of the domain Ω. To further bound the second-to-the-last integral above, employ interior estimates, the solvability of $(R)_2$ (and the fact that $p > 2$) in order to write

$$\left(\int_{D_\varepsilon} \big[|\nabla u|^p + |u|^p\big]\,dx\right)^{1/p} \leqslant C_\varepsilon \|u\|_{L_1^2(\partial\Omega)} \leqslant C_\varepsilon \|u\|_{L_1^p(\partial\Omega)}. \tag{3.66}$$

When used in concert with (3.63) and (3.66), estimate (3.65) yields

$$\left|\int_{\partial\Omega} \partial_\nu u h\,d\sigma\right| \leqslant \left[C_\varepsilon \|u\|_{L_1^p(\partial\Omega)} + \varepsilon^{1/p} C\left(\int_{\partial\Omega} \big[\big|N(\nabla u)\big|^p + |Nu|^p\big]\,d\sigma\right)^{1/p}\right] \|h\|_{L^{p'}(\partial\Omega)}, \tag{3.67}$$

with C independent of ε. In turn, the above estimate shows that there exists $C > 0$ such that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ with the property that

$$\|\partial_\nu u\|_{L^p(\Sigma_R)} \leqslant C_\varepsilon \|u\|_{L_1^p(\partial\Omega)} + \varepsilon C\big[\big\|N(\nabla u)\big\|_{L^p(\partial\Omega)} + \|Nu\|_{L^p(\partial\Omega)}\big]. \tag{3.68}$$

By covering $\partial\Omega$ with finitely many boundary patches Σ_R and adding up the corresponding estimates we therefore arrive at

$$\|\partial_\nu u\|_{L^p(\partial\Omega)} \leqslant C_\varepsilon \|u\|_{L_1^p(\partial\Omega)} + \varepsilon C\big[\big\|N(\nabla u)\big\|_{L^p(\partial\Omega)} + \|Nu\|_{L^p(\partial\Omega)}\big], \tag{3.69}$$

where C is independent of ε. Next, recall the integral representation formula

$$u = \mathcal{D}(u|_{\partial\Omega}) - \mathcal{S}(\partial_\nu u) \quad \text{in } \Omega, \tag{3.70}$$

where the boundary traces are taken using (3.52). Based on this and well-known algebraic manipulations (which involve integrating by parts on the boundary), for every $j \in \{1, \dots, n\}$ we then obtain

$$\partial_j u = \sum_{k=1}^n \partial_k \mathcal{S}(\partial_{\tau_{jk}} u) - \partial_j \mathcal{S}(\partial_\nu u) \quad \text{in } \Omega, \tag{3.71}$$

so that, by standard Calderón–Zygmund estimates,

$$\big\|N(\nabla u)\big\|_{L^p(\partial\Omega)} + \|Nu\|_{L^p(\partial\Omega)} \leqslant C\big[\|u\|_{L^p_1(\partial\Omega)} + \|\partial_\nu u\|_{L^p(\partial\Omega)}\big], \tag{3.72}$$

where $C = C(\Omega, p) > 0$. Combining this with (3.69), choosing $\varepsilon > 0$ sufficiently close to zero and absorbing the terms with small coefficients in the left-hand side then gives

$$\big\|N(\nabla u)\big\|_{L^p(\partial\Omega)} + \|Nu\|_{L^p(\partial\Omega)} \leqslant C\|u\|_{L^p_1(\partial\Omega)}, \tag{3.73}$$

where $C = C(\partial\Omega, p) > 0$. This proves (3.51).

Having established that the Regularity problem (3.50) has a solution u satisfying (3.51) whenever $p \in (1,\infty)$ and the boundary datum f belongs to $L^p_1(\partial\Omega) \cap L^2_1(\partial\Omega)$, we can now proceed with the proof of the uniqueness stated in Theorem 3.10. To get started, consider a family $\{\Omega_j\}_{j\in\mathbb{N}}$ of domains in $\mathbb{R}^n$ satisfying the following properties:

(i) Each Ω_j is a bounded Lipschitz domain satisfying a UEBC (hence, a semiconvex domain), whose Lipschitz character and UEBC constant are bounded uniformly in $j \in \mathbb{N}$;
(ii) For every $j \in \mathbb{N}$ one has $\overline{\Omega_j} \subset \Omega_{j+1} \subset \Omega$, and $\Omega = \bigcup_{j\in\mathbb{N}} \Omega_j$;
(iii) There exist bi-Lipschitz homeomorphisms $\Lambda_j : \partial\Omega \to \partial\Omega_j$, $j \in \mathbb{N}$, such that $\Lambda_j(x) \to x$ as $j \to \infty$, in a nontangential fashion;
(iv) There exist nonnegative, measurable functions ω_j on $\partial\Omega$ which are bounded away from zero and infinity uniformly in $j \in \mathbb{N}$, and which have the property that for each integrable function $g : \partial\Omega_j \to \mathbb{R}$ the following change of variable formula holds

$$\int_{\partial\Omega_j} g\, d\sigma_j = \int_{\partial\Omega} g \circ \Lambda_j \omega_j\, d\sigma, \tag{3.74}$$

where σ_j is the canonical surface measure on $\partial\Omega_j$.

Such a family of approximating domains has been constructed in [61].

We denote by $G_j(\cdot,\cdot)$ the Green function corresponding to each Ω_j, $j \in \mathbb{N}$. As seen in the proof of Lemma 3.1, if $y \in \Omega$ is arbitrary and fix, then for $j \in \mathbb{N}$ large enough we have

$$G_j(x,y) = -E(x-y) + v^{y,j}(x), \quad x \in \Omega_j, \tag{3.75}$$

where $v^{y,j}$ is the unique solution of the problem

$$\begin{cases} \Delta v^{y,j} = 0 \quad \text{in } \Omega_j, \\ v^{y,j}\big|_{\partial\Omega_j} = E(\cdot - y)|_{\partial\Omega_j} \in L^{p'}_1(\partial\Omega_j) \cap L^2_1(\partial\Omega_j), \\ N_j\big(\nabla v^{y,j}\big) \in L^{p'}(\partial\Omega_j), \end{cases} \tag{3.76}$$

where $1/p + 1/p' = 1$, and N_j is the nontangential maximal operator relative to Ω_j. That this Regularity problem has at least one solution is guaranteed by our earlier considerations pertaining to (3.50). In addition, the membership $N_j(\nabla v^{y,j}) \in L^{p'}(\partial\Omega_j)$ is uniform in j, which further yields that for each fixed $y \in \Omega$,

$$N_j\big(x \mapsto \nabla_x G_j(x,y)\big) \in L^{p'}(\partial\Omega_j) \quad \text{uniformly, for } j \in \mathbb{N} \text{ large enough.} \tag{3.77}$$

If now we take u to be a null-solution for the Dirichlet problem $(D)_p$, then $u \in C^\infty(\overline{\Omega}_j)$. Hence, for each $y \in \Omega$ fixed, by Proposition 3.3, (3.28) and (iii) in the above enumeration, we can write

$$u(y) = -\int_{\partial\Omega_j} \nu_j(x) \cdot (\nabla_x G_j)(x,y)(u|_{\partial\Omega_j})(x)\, d\sigma_j(x) \tag{3.78}$$

which, by (3.77), gives that

$$\big|u(y)\big| \leqslant C\bigg(\int_{\partial\Omega_j} |u|^p\, d\sigma_j\bigg)^{1/p}, \tag{3.79}$$

for some finite constant $C > 0$ which is independent of $j \in \mathbb{N}$. In concert with property (iv) listed above, this allows us to estimate

$$\big|u(y)\big|^p \leqslant C\int_{\partial\Omega} \big|u\big(\Lambda_j(x)\big)\big|^p\, d\sigma(x), \tag{3.80}$$

for some $C > 0$ independent of j. By Lebesgue Dominated Convergence Theorem (with the uniform domination provided by Nu; here, property (iii) is also used) we have that $u \circ \Lambda_j \to 0$ in $L^p(\partial\Omega)$ as $j \to \infty$. Thus, by letting $j \to \infty$ we obtain that $u(y) = 0$. Given that $y \in \Omega$ was arbitrary, we may therefore conclude that $u = 0$, as desired. □

We conclude this subsection with the following companion result for Theorem 3.10, dealing with the well-posedness of the Regularity problem $(R)_p$.

Theorem 3.11. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded semiconvex domain and suppose that $p \in (1,\infty)$ is arbitrary and fixed. Then for every datum $f \in L^p_1(\partial\Omega)$ the Regularity problem $(R)_p$ (cf.* (3.50)) *is uniquely solvable. Moreover, the solution u satisfies*

$$\big\|N(\nabla u)\big\|_{L^p(\partial\Omega)} \leqslant C\|f\|_{L^p_1(\partial\Omega)}, \tag{3.81}$$

and can be expressed as in (3.28)*.*

Proof. Uniqueness follows from the corresponding theory in Lipschitz domains (in which case (3.50) is well posed for $1 < p < 2+\varepsilon$, some $\varepsilon = \varepsilon(\Omega) > 0$) and the fact that $L^p(\partial\Omega) \subseteq L^2(\partial\Omega)$ if $p \geqslant 2$ (as $\partial\Omega$ has finite measure).

Fix now $1 < p < \infty$ and $f \in L^p_1(\partial\Omega)$, and let u be as in (3.28). Then, applying Theorem 3.10, we have that $\Delta u = 0$, $Nu \in L^p(\partial\Omega)$ and $u|_{\partial\Omega}$ exists and equals f σ-a.e. on $\partial\Omega$. As such, to show that u is a solution of (3.50), all we have to prove is that $N(\nabla u) \in L^p(\partial\Omega)$. To this end, we pick $f_j \in L_1^p(\partial\Omega) \cap L_1^2(\partial\Omega)$ such that $f_j \to f$ in $L_1^p(\partial\Omega)$ as $j \to \infty$ and, for each $j \in \mathbb{N}$, we let u_j be as in (3.50)–(3.51) when f is replaced by f_j. Then (2.13) gives

$$\|u - u_j\|_{L^{\frac{pn}{n-1}}(\Omega)} \leqslant C \big\| N(u - u_j)\big\|_{L^p(\partial\Omega)} \leqslant C\|f - f_j\|_{L^p(\partial\Omega)}, \tag{3.82}$$

where for the last inequality in (3.82) we have used (3.48). Since the sequence $\{f_j\}_j$ converges to f in $L^p(\partial\Omega)$, from (3.82) we conclude that $u_j \to u$ in $L^{\frac{np}{n-1}}(\Omega)$ as $j \to \infty$. Now take $\mathcal{O}$ an arbitrary compact subset of Ω and let $d := \operatorname{dist}(\mathcal{O}, \partial\Omega)$. Then, using interior estimates for harmonic functions, for each n-tuple $\alpha \in \mathbb{N}_0^n$ (recall that $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$), and for each $x \in \mathcal{O}$ we have

$$\begin{aligned}\big|\partial^\alpha(u - u_j)(x)\big| &\leqslant \frac{C}{d^{|\alpha|}} \fint_{B_{d/2}(x)} |u - u_j| \\ &\leqslant C(\mathcal{O})\bigg(\int_\Omega |u - u_j|^{\frac{np}{n-1}}\bigg)^{\frac{n-1}{np}} \to 0 \quad \text{as } j \to \infty.\end{aligned} \tag{3.83}$$

Consequently, from (3.83) we can conclude that

$$\partial^\alpha u_j \to \partial^\alpha u \quad \text{uniformly on compact sets of } \Omega,\ \forall\alpha \in \mathbb{N}_0^n. \tag{3.84}$$

Fix $\kappa > 0$ and, for $\varepsilon > 0$, consider the nontangential approach region truncated away from the boundary $\Gamma_\kappa^\varepsilon(y_0) := \{y \in \Gamma_\kappa(y_0)\colon |y - y_0| > \varepsilon\}$, where $y_0 \in \partial\Omega$. Then clearly

$$\|\nabla u\|_{L^\infty(\Gamma_\kappa^\varepsilon(y_0))} \leqslant \|\nabla u_j\|_{L^\infty(\Gamma_\kappa^\varepsilon(y_0))} + \|\nabla u - \nabla u_j\|_{L^\infty(\Gamma_\kappa^\varepsilon(y_0))}, \tag{3.85}$$

so by taking $\liminf_{j\to\infty}$ in (3.85) and using (3.84) we obtain

$$\|\nabla u\|_{L^\infty(\Gamma_\kappa^\varepsilon(y_0))} \leqslant \liminf_{j\to\infty} \|\nabla u_j\|_{L^\infty(\Gamma_\kappa^\varepsilon(y_0))} \leqslant \liminf_{j\to\infty} N(\nabla u_j)(y_0). \tag{3.86}$$

Since $y_0 \in \partial\Omega$ is arbitrary, (3.86) implies that for each $p \in (1, \infty)$,

$$\big[N(\nabla u)(y)\big]^p \leqslant \liminf_{j\to\infty}\big[N(\nabla u_j)(y)\big]^p, \quad \forall y \in \partial\Omega. \tag{3.87}$$

Employing now Fatou's lemma in concert with (3.87) we can write that

$$\big\|N(\nabla u)\big\|_{L^p(\partial\Omega)}^p \leqslant \liminf_{j\to\infty}\big\|N(\nabla u_j)\big\|_{L^p(\partial\Omega)}^p \leqslant C\liminf_{j\to\infty}\|f_j\|_{L_1^p(\partial\Omega)}^p = C\|f\|_{L_1^p(\partial\Omega)}^p, \tag{3.88}$$

where for the second inequality in (3.88) we have used the fact that $f_j \in L_1^p(\partial\Omega) \cap L_1^2(\partial\Omega)$ and that the u_j's are as in (3.50)–(3.51) with f replaced by f_j. This completes the proof of the well-posedness of (3.50). $\square$

Remark. In the proof of Theorems 3.10–3.11, we have established that, in the case of the Laplace operator, the solvability of $(D)_p$ implies the solvability of $(R)_{p'}$, $1/p+1/p'=1$, in a bounded Lipschitz domain (assumed to satisfy a UEBC, although this condition did not play a crucial role for this particular task). This was done in a direct, fairly self-contained fashion. It should be noted, however, that, recently, a closely related result, valid for more general operators, has been proved in [75].

4. The Poisson problem on Besov and Triebel–Lizorkin spaces

This section is divided into four parts. In Section 4.1 we review the well-posedness results for the Poisson problem for the Laplacian with a Dirichlet boundary condition on the scales of Besov and Triebel–Lizorkin spaces in arbitrary Lipschitz domains. In Section 4.2 we then proceed to study similar issues in the class of semiconvex domains. Here we deal with the case when the boundary Besov spaces are Banach. As a preamble to completing this study, in Section 4.3 we establish the well-posedness of the Regularity problem with atomic data, and nontangential maximal function estimates, in semiconvex domains. Finally, in Section 4.4, we deal with the most general version of the Poisson problem formulated on the scales of Besov and Triebel–Lizorkin spaces.

4.1. Known results in the class of Lipschitz domains

This section contains a description of known results in the class of Lipschitz domains. The following result appears in [50,51].

Theorem 4.1. *For each bounded, connected Lipschitz domain Ω in $\mathbb{R}^n$ there exists $\varepsilon=\varepsilon(\Omega)\in(0,1]$ with the following significance. Assume that $\frac{n-1}{n}<p\leqslant\infty$, $(n-1)(\frac{1}{p}-1)_+<s<1$, are such that either one of the four conditions*

$$\begin{aligned}
&(\mathrm{I}): && \frac{n-1}{n-1+\varepsilon}<p\leqslant 1 \quad \textit{and} \quad (n-1)\left(\frac{1}{p}-1\right)+1-\varepsilon<s<1;\\
&(\mathrm{II}): && 1\leqslant p\leqslant\frac{2}{1+\varepsilon} \quad \textit{and} \quad \frac{2}{p}-1-\varepsilon<s<1;\\
&(\mathrm{III}): && \frac{2}{1+\varepsilon}\leqslant p\leqslant\frac{2}{1-\varepsilon} \quad \textit{and} \quad 0<s<1;\\
&(\mathrm{IV}): && \frac{2}{1-\varepsilon}\leqslant p\leqslant\infty \quad \textit{and} \quad 0<s<\frac{2}{p}+\varepsilon,
\end{aligned} \tag{4.1}$$

is satisfied, if $n\geqslant 3$, and either one of the following three conditions

$$\begin{aligned}
&(\mathrm{I}'): && \frac{2}{1+\varepsilon}\leqslant p\leqslant\frac{2}{1-\varepsilon} \quad \textit{and} \quad 0<s<1;\\
&(\mathrm{II}'): && \frac{2}{3+\varepsilon}<p<\frac{2}{1+\varepsilon} \quad \textit{and} \quad \frac{1}{p}-\frac{1+\varepsilon}{2}<s<1;\\
&(\mathrm{III}'): && \frac{2}{1-\varepsilon}<p\leqslant\infty \quad \textit{and} \quad 0<s<\frac{1}{p}+\frac{1+\varepsilon}{2},
\end{aligned} \tag{4.2}$$

is satisfied, if $n = 2$. Then for $0 < q \leqslant \infty$, the problem

$$\Delta u = f \in B^{p,q}_{s+1/p-2}(\Omega), \qquad \operatorname{Tr} u = g \in B^{p,q}_{s}(\partial\Omega), \quad u \in B^{p,q}_{s+1/p}(\Omega), \tag{4.3}$$

has a unique solution (which, in addition, satisfies natural estimates). As a consequence, if $u = \mathbb{G} f$ is the unique solution of (4.3) with $g = 0$, then the operator

$$\mathbb{G} : B^{p,q}_{\alpha}(\Omega) \longrightarrow B^{p,q}_{\alpha+2,z}(\Omega) \tag{4.4}$$

is an isomorphism whenever p, q, s are as above and $\alpha := s + 1/p - 2$.

Similar results hold for Triebel–Lizorkin spaces. More specifically, retain the same assumptions on the indices p, q, s as before and, in addition, assume that $p, q < \infty$. Then the problem

$$\Delta u = f \in F^{p,q}_{s+1/p-2}(\Omega), \qquad \operatorname{Tr} u = g \in B^{p,p}_{s}(\partial\Omega), \quad u \in F^{p,q}_{s+1/p}(\Omega), \tag{4.5}$$

has a unique solution (once again satisfying natural estimates), so that $\mathbb{G}$ also extends isomorphically to

$$\mathbb{G} : F^{p,q}_{\alpha}(\Omega) \longrightarrow F^{p,q}_{\alpha+2,z}(\Omega) \tag{4.6}$$

whenever p, q, s are as above, $\alpha := s + 1/p - 2$, and, in addition $\min\{1, p\} \leqslant q$. *Without the latter condition,*

$$\mathbb{G} : F^{p,q}_{\alpha}(\Omega) \longrightarrow F^{p,q}_{\alpha+2}(\Omega) \tag{4.7}$$

is a well-defined and bounded operator. These results are sharp in the class of Lipschitz domains. For $\partial\Omega \in C^1$ one can take $\varepsilon = 1$.

It should be pointed out that the portion of this theorem corresponding to $p = q \in (1, \infty)$ for the Besov scale, and $1 < p < \infty$, $q = 2$ for the Triebel–Lizorkin scale has been earlier worked out in [41]. Subsequently, a new approach which also permitted the treatment of the Neumann boundary condition was devised in [28]. This has been further extended to the case of variable coefficient operators in [66].

Moving on, we remark that, for each $n \geqslant 2$, the collection of all points with coordinates $(\alpha, 1/p)$ such that α, p are as in the statement of Theorem 4.1 can be identified with a two-dimensional hexagonal region. More precisely, if $n \geqslant 3$, the region is depicted in Fig. 3, while Fig. 4 depicts the region corresponding to $n = 2$.

If $\Omega \subset \mathbb{R}^n$ is an arbitrary bounded Lipschitz domain, we set

$$h^p(\Omega) := \begin{cases} h^p_{at}(\Omega) & \text{for } \frac{n}{n+1} < p \leqslant 1, \\ L^p(\Omega) & \text{for } 1 < p < \infty. \end{cases} \tag{4.8}$$

When specialized to the Triebel–Lizorkin scale with $q = 2$ and $\alpha = 0$, Theorem 4.1 shows that there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that the operator

$$\partial_{x_j}\partial_{x_k}\mathbb{G} : h^p(\Omega) \longrightarrow h^p(\Omega), \quad 1 \leqslant j, k \leqslant n, \tag{4.9}$$

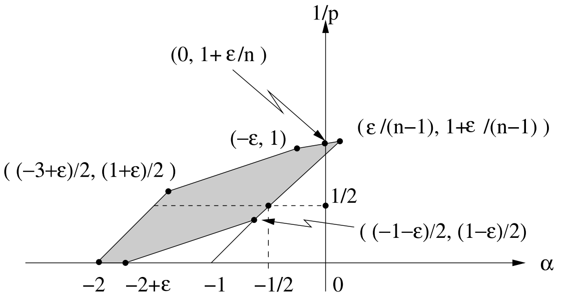

Fig. 3.

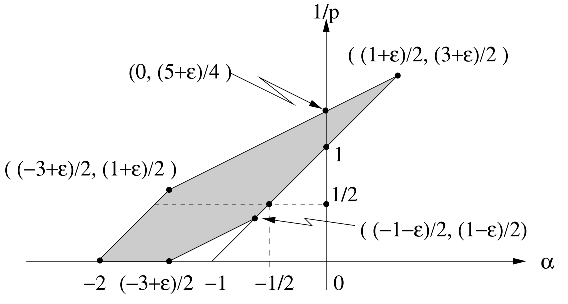

Fig. 4.

is bounded, provided that $1 - \varepsilon < p < 1$. This result, which provides a solution to a conjecture made by D.-C. Chang, S.G. Krantz and E.M. Stein (cf. [13,14]), is sharp in the class of Lipschitz domains. As pointed out in Section 1, B. Dahlberg [19] has constructed a Lipschitz domain for which (4.9) fails for the entire L^p scale, $1 < p < \infty$ (cf. also [41] for a refinement of this counterexample).

Nonetheless, when Ω is a bounded convex domain, we shall show that the operators in (4.9) are actually bounded for $\frac{n}{n+1} < p \leqslant 2$. This will eventually be obtained in Section 5.1, after the build-up in the remainder of Section 4.

4.2. The Dirichlet problem on Besov and Triebel–Lizorkin spaces

For the purpose of studying the Poisson problem in Besov and Triebel–Lizorkin spaces (a task which we take up later, in Section 4.4), it is important to be able to relate nontangential maximal function estimates to membership to these smoothness spaces. A first result in this regard is recalled below (cf. [68] for a proof). To state it, given a function u and $k \in \mathbb{N}$, set $|\nabla^k u| := \sum_{|\gamma| \leqslant k} |\partial^\gamma u|$.

Theorem 4.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that L is a homogeneous, constant* (*real*) *coefficient, symmetric, strongly elliptic system of differential operators of order $2m$. Then if $u \in F^{p,q}_{m-1+1/p}(\Omega)$ for some $\frac{n-1}{n} < p \leqslant 2$, $0 < q < \infty$ and $Lu = 0$ in Ω, it follows that $N(\nabla^{m-1} u) \in L^p(\partial\Omega)$ and a natural estimate holds.*

Our next theorem addresses the converse direction, i.e., passing from nontangential maximal function estimates to membership to Besov spaces.

Theorem 4.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that $1 < p \leqslant 2$. Also, let L be a homogeneous, constant (real) coefficient, symmetric, strongly elliptic system of differential operators of order $2m$. If the function u is such that $Lu = 0$ in Ω and $N(\nabla^{m-1}u) \in L^p(\partial\Omega)$, then $u \in B^{p,2}_{m-1+1/p}(\Omega)$, plus a natural estimate.*

Proof. Let u be as in the hypotheses of the theorem and, for an arbitrary multi-index γ with $|\gamma| = m-1$, set $v := \partial^\gamma u$. Our aim is to show that $v \in B^{p,2}_{1/p}(\Omega)$ since, granted this, Proposition 2.4 yields the desired conclusion. The strategy is to combine the techniques in [41] (where the case $L = \Delta$ has been treated), with the results from [21]. To get started, we briefly recall the so-called trace method of interpolation. Given a compatible couple of Banach spaces A_0, A_1 and $1 \leqslant p < \infty$, $\theta \in (0,1)$, set $(A_0, A_1)_{\theta,p}$ for the intermediate space obtained via the standard real interpolation method (cf., e.g., [6, Chapter 3]). Then, if $1 \leqslant p_0, p_1 < \infty$ are such that $1/p = (1-\theta)/p_0 + \theta/p_1$, we have

$$\|w\|_{(A_0,A_1)_{\theta,p}} \approx \inf\left\{\left(\int_0^\infty \|t^\theta f(t)\|_{A_0}^{p_0}\frac{dt}{t}\right)^{1/p_0} + \left(\int_0^\infty \|t^\theta f'(t)\|_{A_1}^{p_1}\frac{dt}{t}\right)^{1/p_1}\right\}, \tag{4.10}$$

uniformly for $w \in (A_0, A_1)_{\theta,p}$, where the infimum is taken over all functions $f : (0,\infty) \to A_0 + A_1$ with the property that f is locally A_0-integrable, f' (taken in the sense of distributions) is locally A_1-integrable, and such that $\lim_{t\to 0^+} f(t) = w$ in $A_0 + A_1$. See Theorem 3.12.2 on p. 73 in [6]. In our case, since for every $1 < p < \infty$ we have

$$B^{p,2}_{1/p}(\Omega) = \big(W^{1,p}(\Omega), L^p(\Omega)\big)_{1-1/p,2} \tag{4.11}$$

(cf. (2.88) and (2.115)), it suffices to show that there exists some $C = C(\Omega) > 0$ such that the infimum of

$$\int_0^\infty \|t^{1-1/p} f(t)\|^2_{W^{1,p}(\Omega)}\frac{dt}{t} + \int_0^\infty \|t^{1-1/p} f'(t)\|^2_{L^p(\Omega)}\frac{dt}{t} \tag{4.12}$$

taken over all functions $f : (0,\infty) \to L^p(\Omega) + W^{1,p}(\Omega)$ with the property that f is locally $W^{1,p}(\Omega)$-integrable, f' (taken in the sense of distributions) is locally $L^p(\Omega)$-integrable, and satisfying $\lim_{t\to 0^+} f(t) = v$ in $L^p(\Omega) + W^{1,p}(\Omega)$, is bounded by $C\|N(\nabla^{m-1}u)\|^2_{L^p(\partial\Omega)}$. Note that since v belongs to $W^{1,p}(\mathcal{O})$ for any $\mathcal{O} \subset\subset \Omega$, we only need to prove the corresponding estimate for a small, fixed neighborhood of the boundary. Since the domain Ω is Lipschitz, it suffices to prove the latter in the case when the boundary point is the origin and for some constant $r > 0$, depending only on Ω, $B_r(0) \cap \partial\Omega$ is part of the graph of a Lipschitz function φ with $\varphi(0) = 0$ (otherwise we just use a suitable rigid motion). In addition, we can assume that

$$\operatorname{dist}\big((x', y), \partial\Omega\big) \approx y - \varphi(x'), \quad \forall x = (x', y) \in B_r(0). \tag{4.13}$$

Next, we choose

$$\begin{array}{lcl} \eta \in C_c^\infty\big(B_r(0)\big), \quad |\eta| \leqslant 1, & & \theta \in C_c^\infty((-r,r)), \quad |\theta| \leqslant 1, \\ & \text{and} & \\ \eta(x) = 1 \quad \text{for } |x| \leqslant r/2, & & \theta(t) = 1 \quad \text{for } |t| < r/2. \end{array} \tag{4.14}$$

If we now consider the function f such that $(f(t))(x', y) := \eta(x)v(x', y + t)\theta(t)$ if $x = (x', y)$, then clearly f is locally $W^{1,p}(\Omega)$-integrable, f' is locally $L^p(\Omega)$-integrable, and $\lim_{t\to 0^+} f(t) = \eta v$. Thus, it is enough to prove that

$$\int_0^\infty \big\|t^{1-1/p} f(t)\big\|^2_{W^{1,p}(\Omega)} \frac{dt}{t} + \int_0^\infty \big\|t^{1-1/p} f'(t)\big\|^2_{L^p(\Omega)} \frac{dt}{t} \leqslant C \big\|N\big(\nabla^{m-1}u\big)\big\|^2_{L^p(\partial\Omega)}. \tag{4.15}$$

To this end, first we note that since $2 - 2/p - 1 > -1$ (and Ω has finite measure),

$$\begin{aligned} I := \int_0^\infty \big\|t^{1-1/p} f(t)\big\|^2_{L^p(\Omega)} \frac{dt}{t} &\leqslant \int_0^r \bigg(\int_{B_{2r}(0)\cap\Omega} \big|v\big(x', y\big)\big|^p \, dx'\, dy \bigg)^{2/p} t^{2-2/p-1}\, dt \\ &\leqslant C\|v\|^2_{L^p(\Omega)} \leqslant C\|v\|^2_{L^{pn/(n-1)}(\Omega)} \\ &\leqslant C\big\|N(v)\big\|^2_{L^p(\partial\Omega)}, \end{aligned} \tag{4.16}$$

where the last step uses (2.13). Second, we have

$$\begin{aligned} II &:= \int_0^\infty \bigg(\int_\Omega \big(|\eta(x)|\big|\nabla v\big(x', y + t\big)\big|\big|\theta(t)\big|t^{1-1/p}\big)^p \, dx \bigg)^{2/p} \frac{dt}{t} \\ &\leqslant C \int_0^r t^{2-2/p} \bigg(\int_{|x'|<r} \int_t^r \big|\nabla v\big(x', \varphi\big(x'\big) + s\big)\big|^p \, dx'\, ds \bigg)^{2/p} \frac{dt}{t} \\ &= C \int_0^r \bigg(\int_t^r h(s)\, ds \bigg)^{2/p} t^{1-2/p}\, dt, \end{aligned} \tag{4.17}$$

where

$$h(s) := \int_{|x'|<r} \big|\nabla v\big(x', \varphi\big(x'\big) + s\big)\big|^p \, dx'. \tag{4.18}$$

To continue, we recall Hardy's inequality:

$$\bigg(\int_0^\infty \bigg(\int_x^\infty g(y)\, dy \bigg)^p x^{\alpha-1}\, dx \bigg)^{1/p} \leqslant \frac{p}{r} \bigg(\int_0^\infty \big(yg(y)\big)^p y^{\alpha-1}\, dy \bigg)^{1/p}, \tag{4.19}$$

which holds for $g \geqslant 0$ measurable, $p \geqslant 1$ and $\alpha > 0$. If we apply this inequality with $g := h\chi_{[0,r]}$, $\alpha = 2 - 2/p$ and p replaced by $2/p$, we obtain that

$$\begin{aligned}\int_0^r \left(\int_t^r h(s)\,ds\right)^{2/p} t^{1-2/p}\,dt &\leqslant C\int_0^\infty \big(sh(s)\chi_{[0,r]}(s)\big)^{2/p} s^{1-2/p}\,ds\\ &= C\int_0^r h(s)^{2/p} s\,ds\\ &\leqslant C\Bigg[\int_{|x'|<r}\left(\int_0^r \big|\nabla v\big(x',\varphi\big(x'\big)+s\big)\big|^2 s\,ds\right)^{p/2} dx'\Bigg]^{2/p}.\end{aligned}\tag{4.20}$$

For the last inequality in (4.20) we have used Minkowski's inequality. Next we claim that

$$\int_0^r \big|\nabla v\big(x',\varphi\big(x'\big)+s\big)\big|^2 s\,ds \leqslant C\int_{\Gamma(x',\varphi(x'))} \big|\nabla v(z)\big|^2 \operatorname{dist}(z,\partial\Omega)^{2-n}\,dz,\tag{4.21}$$

uniformly, for $(x',\varphi(x'))\in B_r(0)\cap\partial\Omega$. To justify (4.21), fix $(x',\varphi(x'))\in B_r(0)\cap\partial\Omega$ and note that there exists $\lambda>0$ such that $B_{\lambda s}(x',\varphi(x')+s)\subset\Gamma(x',\varphi(x'))$ for all $s\in(0,r)$. Using the fact that v is a null-solution for the elliptic operator L in Ω, interior estimates give that

$$\big|\nabla v\big(x',\varphi\big(x'\big)+s\big)\big|^2 \leqslant C \fint_{B_{\lambda s}(x',\varphi(x')+s)} \big|\nabla v(z)\big|^2\,dz,\quad \forall s\in(0,r).\tag{4.22}$$

Hence, by choosing λ small enough (relative to the Lipschitz character of Ω), we may write (for some $0<c_0<c_1<\infty$)

$$\begin{aligned}\int_0^r \big|\nabla v\big(x',\varphi\big(x'\big)+s\big)\big|^2 s\,ds &\leqslant C\int_0^r s^{1-n}\int_{\Gamma(x',\varphi(x'))} \big|\nabla v(z)\big|^2 \chi_{|z-(x',\varphi(x')+s)|<\lambda s}\,dz\,ds\\ &\leqslant C\int_{\Gamma(x',\varphi(x'))} \big|\nabla v(z)\big|^2\left(\int_0^\infty s^{1-n}\chi_{|z-(x',\varphi(x')+s)|<\lambda s}\,ds\right)dz\\ &\leqslant C\int_{\Gamma(x',\varphi(x'))} \big|\nabla v(z)\big|^2\left(\int_{c_0\operatorname{dist}(z,\partial\Omega)}^{c_1\operatorname{dist}(z,\partial\Omega)} s^{1-n}\,ds\right)dz\\ &= C\int_{\Gamma(x',\varphi(x'))} \big|\nabla v(z)\big|^2 \operatorname{dist}(z,\partial\Omega)^{2-n}\,dz.\end{aligned}\tag{4.23}$$

This concludes the justification of (4.21).

Moving on, we make use of (4.21) in order to write

$$\begin{aligned}
&\int_{|x'|<r}\left(\int_0^r\left|\nabla v\big(x',\varphi(x')+s\big)\right|^2 s\,ds\right)^{p/2}dx\\
&\quad\leqslant C\int_{|x'|<r}\left(\int_{\Gamma(x',\varphi(x'))}\left|\nabla v(y)\right|^2\operatorname{dist}(y,\partial\Omega)^{2-n}\,dy\right)^{p/2}dx'\\
&\quad\leqslant C\left\|\mathcal{A}\big(\nabla^{m-1}u\big)\right\|^p_{L^p(B\cap\partial\Omega)}\leqslant C\left\|N\big(\nabla^{m-1}u\big)\right\|^2_{L^p(\partial\Omega)}, \qquad (4.24)
\end{aligned}$$

where, generally speaking,

$$(\mathcal{A}w)(x):=\left(\int_{\Gamma(x)}\left|\nabla w(y)\right|^2\operatorname{dist}(y,\partial\Omega)^{2-n}\,dy\right)^{\frac12},\quad x\in\partial\Omega, \qquad (4.25)$$

is the area-function, and the last inequality in (4.24) is due to [21]. Combining (4.17), (4.20), and (4.24) we obtain that

$$\int_0^\infty\left(\int_\Omega\left(|\eta(x)|\left|\nabla v\big(x',y+t\big)\right||\theta(t)|t^{1-1/p}\right)^p dx\right)^{2/p}\frac{dt}{t}\leqslant C\left\|N\big(\nabla^{m-1}u\big)\right\|^2_{L^p(\partial\Omega)}. \qquad (4.26)$$

Since the left-hand side of (4.15) can be bounded by a linear combination of the terms I and II, the estimates on I and II readily yield (4.15). □

For the applications we have in mind, the following regularity result is particularly useful.

Corollary 4.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and $1<p<\infty$. If v is such that $\Delta v=0$ in Ω and $N(v)\in L^p(\partial\Omega)$, then $v\in B^{p,\max\{p,2\}}_{1/p}(\Omega)$.*

Proof. The case $1<p\leqslant 2$ is already contained in Theorem 4.3. In the case when $p\geqslant 2$, we may rely on the (first) representation in (2.37) and the fact that

$$\mathcal{D}:L^p(\partial\Omega)\longrightarrow B^{p,p}_{1/p}(\Omega),\quad 2\leqslant p<\infty, \qquad (4.27)$$

in a bounded fashion; cf. [66] for a proof. □

Having established Corollary 4.4, we are now ready to deal with the Dirichlet problem with data from Besov spaces $B^{p,q}_s$ with $1<p<\infty$.

Theorem 4.5. *Let Ω be a bounded semiconvex domain in $\mathbb{R}^n$. Then for every $1<p<\infty$, $0<q\leqslant\infty$ and $s\in(0,1)$, the Dirichlet problem*

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega, \\ \operatorname{Tr} u = f \in B_s^{p,q}(\partial\Omega), \\ u \in B^{p,q}_{s+\frac{1}{p}}(\Omega) \end{cases} \tag{4.28}$$

has a unique solution. In addition, there exists $C = C(\Omega, p, q, s) > 0$ *such that the solution* u *of* (4.28) *satisfies*

$$\|u\|_{B^{p,q}_{s+\frac{1}{p}}(\Omega)} \leqslant C\|f\|_{B_s^{p,q}(\partial\Omega)}. \tag{4.29}$$

Similar results are also valid on the Triebel–Lizorkin scale. More precisely, if $1 < p < \infty$, $0 < q < \infty$ *and* $s \in (0,1)$ *then the Dirichlet problem*

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega, \\ \operatorname{Tr} u = f \in B_s^{p,p}(\partial\Omega), \\ u \in F^{p,q}_{s+\frac{1}{p}}(\Omega), \end{cases} \tag{4.30}$$

has a unique solution, which also satisfies

$$\|u\|_{F^{p,q}_{s+\frac{1}{p}}(\Omega)} \leqslant C(\Omega, p, q, s)\|f\|_{B_s^{p,p}(\partial\Omega)}. \tag{4.31}$$

Proof. At a formal level, let PI (acronym for Poisson integral) denote the solution operator for the Dirichlet problem in Ω. That is, $u := \mathrm{PI}\, f$ satisfies

$$\Delta u = 0 \quad \text{in } \Omega \quad \text{and} \quad u|_{\partial\Omega} = f \quad \text{on } \partial\Omega. \tag{4.32}$$

From Theorem 3.10, Theorem 3.11 and Corollary 4.4 it follows that

$$\mathrm{PI} : L^p(\partial\Omega) \longrightarrow B^{p,\max\{p,2\}}_{1/p}(\Omega), \tag{4.33}$$

$$\mathrm{PI} : L^p_1(\partial\Omega) \longrightarrow B^{p,\max\{p,2\}}_{1+1/p}(\Omega) \tag{4.34}$$

are bounded, linear operators, which act in a compatible fashion for each $p \in (1, \infty)$. Using this, (2.72) and (2.116), we may then conclude that PI extends as a bounded, linear operator

$$\mathrm{PI} : B_s^{p,q}(\partial\Omega) \longrightarrow B^{p,q}_{s+1/p}(\Omega) \tag{4.35}$$

whenever $1 < p < \infty$, $0 < q \leqslant \infty$, $s \in (0, 1)$. This shows that (4.28) has a solution which satisfies (4.29). To show that such a solution is unique, assume that u is a null-solution for (4.28). Then $u \in B^{p,p}_{1/p}(\Omega) = F^{p,p}_{1/p}(\Omega)$ according to (2.83) and (2.82). Since also $\Delta u = 0$, $Nu \in L^p(\partial\Omega)$ if $1 < p \leqslant 2$, by Theorem 4.2, so that $u = 0$ in this case, by the uniqueness part in Theorem 3.10. There remains to treat the case when $2 < p < \infty$, in which scenario we shall use the fact that there exist s_0, p_0 for which $B^{p,q}_{s+1/p}(\Omega) \hookrightarrow B^{p_0,q}_{s_0+1/p_0}(\Omega)$ and such that the problem (4.3) formulated with s, p replaced by s_0, p_0 is well posed (as discussed in Theorem 4.1). This forces $u = 0$, as desired.

Next, specializing (4.35) to the case when $q = p$ and making use of the fact that for every $0 < r < \infty$

$$B^{p,p}_{s+1/p}(\Omega) \cap \ker \Delta = F^{p,p}_{s+1/p}(\Omega) \cap \ker \Delta = F^{p,r}_{s+1/p}(\Omega) \cap \ker \Delta \tag{4.36}$$

(cf. (2.134)), we obtain that

$$\mathrm{PI} : B^{p,p}_{s}(\partial\Omega) \longrightarrow F^{p,q}_{s+1/p}(\Omega) \tag{4.37}$$

is well defined, linear and bounded whenever $1 < p < \infty$, $0 < q < \infty$, $s \in (0, 1)$. This proves that (4.30) has a solution which satisfies (4.31). Uniqueness can be proved as in the case of the Besov spaces, and this finishes the proof of the theorem. □

The case $p = \infty$ on the Besov scale in Theorem 4.5 is discussed separately below.

Theorem 4.6. *Let Ω be a bounded semiconvex domain in $\mathbb{R}^n$. Then for each $\alpha \in (0, 1)$ and $0 < q \leqslant \infty$, the Dirichlet problem*

$$\begin{cases} u \in B^{\infty,q}_{\alpha}(\Omega), \\ \Delta u = 0 \quad \text{in } \Omega, \\ \mathrm{Tr}\, u = f \in B^{\infty,q}_{\alpha}(\partial\Omega) \end{cases} \tag{4.38}$$

has a unique solution, which satisfies

$$\|u\|_{B^{\infty,q}_{\alpha}(\Omega)} \leqslant C(\Omega, \alpha)\|f\|_{B^{\infty,q}_{\alpha}(\partial\Omega)} \tag{4.39}$$

for some finite constant $C = C(\Omega, q) > 0$.

Proof. To begin with, we consider the case $q = \infty$ corresponding to the Dirichlet problem with data and solutions on the Hölder scale (indeed, $B^{\infty,\infty}_{\alpha}(\Omega) = C^{\alpha}(\overline{\Omega})$ and $B^{\infty,\infty}_{\alpha}(\partial\Omega) = C^{\alpha}(\partial\Omega)$). In this scenario, the claims in the statement of the theorem follow from the work in [59]. More specifically, since a bounded Lipschitz domain Ω satisfying a UEBC also satisfies a uniform exterior cone condition with any angle $\theta \in (0, \pi)$, and since the critical Hölder index associated to the angle π is $\alpha_\pi = 1$ (see [59] for definitions), Theorem 2.5 in [59] gives that the problem (4.38) with $q = \infty$ is well posed for any $\alpha \in (0, \alpha_\pi) = (0, 1)$, and the solution satisfies

$$\|u\|_{C^{\alpha}(\overline{\Omega})} + \sup_{x \in \Omega} \bigl[\operatorname{dist}(x, \partial\Omega)^{1-\alpha} \bigl|\nabla u(x)\bigr|\bigr] \leqslant C(\Omega, \alpha)\|f\|_{C^{\alpha}(\partial\Omega)}. \tag{4.40}$$

Hence,

$$\mathrm{PI} : B^{\infty,\infty}_{\alpha}(\partial\Omega) \longrightarrow B^{\infty,\infty}_{\alpha}(\Omega), \quad \alpha \in (0, 1), \tag{4.41}$$

is well defined, linear and bounded. With this in hand, we can then allow $q = \infty$ to be replaced by any $0 < q < \infty$ via real interpolation (cf. (2.69) and (2.116)). This proves existence and estimates for (4.38) when $0 < q \leqslant \infty$. Uniqueness then follows from the uniqueness part in the first part of the proof and elementary embeddings. □

4.3. The Regularity problem with data from Hardy spaces

The first priority is to establish a trace result in the regular Hardy space $h^{1,p}_{at}(\partial\Omega)$, where $\frac{n-1}{n} < p \leqslant 1$, for a function u harmonic in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, for which $N(\nabla u) \in L^p(\partial\Omega)$. As a preamble, we record a couple of useful lemmas.

Lemma 4.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, with outward unit normal ν, and assume that $\frac{n-1}{n} < p \leqslant 1$. Then there exists a finite constant $C = C(\partial\Omega, p) > 0$ such that for any divergence-free vector field $\vec{F} : \Omega \to \mathbb{R}^n$ with harmonic components for which $N(\vec{F}) \in L^p(\partial\Omega)$ there holds*

$$\nu \cdot \vec{F} \in h^p_{at}(\partial\Omega) \quad \textit{and} \quad \|\nu \cdot \vec{F}\|_{h^p_{at}(\partial\Omega)} \leqslant C \big\|N(\vec{F})\big\|_{L^p(\partial\Omega)}, \tag{4.42}$$

with $\nu \cdot \vec{F}$ on $\partial\Omega$ defined in the following sense. Let Z be a coordinate cylinder for $\partial\Omega$, with axis in the direction of a unit vector (pointing into Ω) denoted by e_n, and pick a function $\zeta \in C^\infty_c(\mathbb{R}^n)$ with $\operatorname{supp}\zeta \subset Z$. *Then*

$$\lim_{\varepsilon \to 0^+} \int\limits_{Z\cap\partial\Omega} \nu(x) \cdot \vec{F}(x + \varepsilon e_n)\zeta(x)\, d\sigma(x) = \int\limits_{\partial\Omega} \nu \cdot \vec{F}\zeta\, d\sigma, \tag{4.43}$$

where the last integral above stands for the paring between $h^p_{at}(\partial\Omega)$ and $\operatorname{Lip}(\partial\Omega)$.

This has been proved in [38], via an approach akin to the work of J.M. Wilson [85]. We continue by recording a result which can, in essence, be attributed to Hardy (a proof, based on ideas due to R. Brown, appears in [68]).

Lemma 4.8. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Assume u is a null-solution of a homogeneous, constant coefficient, elliptic differential operator L in Ω, and that $N(\nabla u) \in L^p(\partial\Omega)$ for some $0 < p < n-1$. Then there exists a constant $C = C(\partial\Omega) > 0$ such that*

$$\|Nu\|_{L^{p^*}(\partial\Omega)} \leqslant C \big\|N(\nabla u)\big\|_{L^p(\partial\Omega)} + C\|Nu\|_{L^p(\partial\Omega)}, \tag{4.44}$$

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n-1}$.

We are now ready to state and prove the trace result alluded to at the beginning of this subsection. In order to facilitate the subsequent exposition, given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, introduce

$$h^{1,p}(\partial\Omega) := \begin{cases} h^{1,p}_{at}(\partial\Omega) & \text{for } \frac{n-1}{n} < p \leqslant 1, \\ L^p_1(\partial\Omega) & \text{for } 1 < p < \infty. \end{cases} \tag{4.45}$$

Proposition 4.9. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume $\frac{n-1}{n} < p < \infty$. Then there exists $C = C(\Omega, p) > 0$ with the property that for each function u which is harmonic in Ω and has $N(\nabla u) \in L^p(\partial\Omega)$, there holds*

$$u|_{\partial\Omega} \in h^{1,p}(\partial\Omega) \quad \textit{and} \quad \|u|_{\partial\Omega}\|_{h^{1,p}(\partial\Omega)} \leqslant C \big\|N(\nabla u)\big\|_{L^p(\partial\Omega)} + C\|Nu\|_{L^p(\partial\Omega)}. \tag{4.46}$$

Proof. The conclusions in (4.46) when $1 < p < \infty$ are essentially well known (cf., e.g., the discussion in [84]), so we will concentrate on the interval $\frac{n-1}{n} < p \leqslant 1$. Assume that this is the case and let Ω, u be as in the statement of the proposition. Also, let p^* be as in (2.78). By Lemma 4.8, there exists a constant $C = C(\Omega, p) > 0$ such that (4.44) holds. Thus, in particular,

$$u|_{\partial\Omega} \in L^{p^*}(\partial\Omega) \quad \text{and} \quad \|u\|_{L^{p^*}(\partial\Omega)} \leqslant C\big\|N(\nabla u)\big\|_{L^p(\partial\Omega)} + C\|Nu\|_{L^p(\partial\Omega)}. \tag{4.47}$$

Next, for a fixed pair of indices, $j, k \in \{1, \dots, n\}$, we introduce the vector field

$$\vec{F}_{jk} := (\partial_k u)e_j - (\partial_j u)e_k \quad \text{in } \Omega, \tag{4.48}$$

where $\{e_\ell\}_{1\leqslant \ell\leqslant n}$ is the standard orthonormal basis in $\mathbb{R}^n$. Note that

$$\begin{gathered} N(\vec{F}_{jk}) \in L^p(\partial\Omega), \quad \vec{F}_{jk} \text{ has harmonic components}, \\ \operatorname{div} \vec{F}_{jk} = \partial_j\partial_k u - \partial_k\partial_j u = 0 \quad \text{in } \Omega, \\ \nu \cdot \vec{F}_{jk} = \nu_j\partial_k u - \nu_k\partial_j u = \partial_{\tau_{jk}} u \quad \text{on } \partial\Omega, \end{gathered} \tag{4.49}$$

where $\nu = (\nu_1, \dots, \nu_n)$ is the outward unit normal to $\partial\Omega$. Then (4.47), Proposition 2.3 and Lemma 4.7 give that $u|_{\partial\Omega} \in h^{1,p}_{at}(\partial\Omega)$ and

$$\begin{aligned} \|u|_{\partial\Omega}\|_{h^{1,p}_{at}(\partial\Omega)} &\approx \|u\|_{L^{p^*}(\partial\Omega)} + \sum_{j,k=1}^{n} \|\partial_{\tau_{jk}} u\|_{h^p_{at}(\partial\Omega)} \\ &\leqslant C\big\|N(\nabla u)\big\|_{L^p(\partial\Omega)} + C\|Nu\|_{L^p(\partial\Omega)}. \end{aligned} \tag{4.50}$$

This finishes the proof of (4.46). □

The main result of this subsection is the following well-posedness theorem.

Theorem 4.10. *Let Ω be a bounded semiconvex domain in $\mathbb{R}^n$ and assume that $\frac{n-1}{n} < p \leqslant 1$. Then the problem*

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} = f \in h^{1,p}_{at}(\partial\Omega), \\ N(\nabla u) \in L^p(\partial\Omega) \end{cases} \tag{4.51}$$

has a unique solution and there exists a constant $C > 0$ independent of f such that

$$\big\|N(\nabla u)\big\|_{L^p(\partial\Omega)} \leqslant C\|f\|_{h^{1,p}_{at}(\partial\Omega)}. \tag{4.52}$$

Proof. To begin with, note that Proposition 4.9 ensures that the trace in (4.51) exists in $h^{1,p}_{at}(\partial\Omega)$, so the problem is meaningfully formulated. To proceed, fix $\frac{n-1}{n} < p \leqslant 1$ and let f be an $h^{1,p}_{at}(\partial\Omega)$ atom. For the existence part, it suffices to show that there exists $C = C(\Omega, p) > 0$ such that if u is the solution of $(R)_2$ with datum f then

$$\big\|N(\nabla u)\big\|_{L^p(\partial\Omega)} \leqslant C. \tag{4.53}$$

This estimate also proves (4.52). To justify it, pick $x_0 \in \partial\Omega$ and $r > 0$ such that $\operatorname{supp} f \subseteq \Delta_r(x_0)$ and

$$\|\nabla_{\tan} f\|_{L^\infty(\partial\Omega)} \leqslant r^{-(n-1)\frac{1}{p}}. \tag{4.54}$$

In particular, we also have that

$$\|f\|_{L^\infty(\partial\Omega)} \leqslant C r^{1-(n-1)\frac{1}{p}}. \tag{4.55}$$

Following a standard technique, we shall prove (4.53) by estimating separately the L^p-norm of $N(\nabla u)$ near x_0, and away from x_0. Near x_0, we make use of Hölder's inequality and the well-posedness of the L^2-Regularity problem combined with (4.54) in order to write

$$\begin{aligned}
\int_{\Delta_{100r}(x_0)} \big|N(\nabla u)\big|^p\,d\sigma &\leqslant C\Bigg(\int_{\Delta_{100r}(x_0)} \big|N(\nabla u)\big|^2\,d\sigma\Bigg)^{p/2} \cdot r^{(n-1)(1-p/2)} \\
&\leqslant C r^{(n-1)(1-p/2)} \big\|N(\nabla u)\big\|^p_{L^2(\partial\Omega)} \\
&\leqslant C r^{(n-1)(1-p/2)} \|\nabla_{\tan} f\|^p_{L^2(\partial\Omega)} \leqslant C.
\end{aligned} \tag{4.56}$$

To estimate the contribution from integrating $N(\nabla u)^p$ away from x_0, fix $x \in \Omega$ and let $d\omega^x$ denote the harmonic measure of Ω with pole at x (cf. (3.30)–(3.31)). Making use of (4.55), we have

$$\big|u(x)\big| = \Bigg|\int_{\partial\Omega} f\,d\omega^x\Bigg| \leqslant C r^{1-(n-1)\frac{1}{p}} \omega^x\big(\Delta_r(x_0)\big). \tag{4.57}$$

Let $T(x_0,r) := B_r(x_0) \cap \Omega$ be the "tent" region above $\Delta_r(x_0)$ and let $A_r(x_0)$ be the corresponding "corkscrew" point, i.e., a point in $T(x_0,r)$ satisfying $\operatorname{dist}(A_r(x_0), \partial\Omega) = r \approx |A_r(x_0) - x_0|$. Then, if $G(\cdot,\cdot)$ is the Green function for the Laplacian with homogeneous Dirichlet boundary condition, we have

$$\frac{\omega^x(\Delta_r(x_0))}{G(x, A_r(x_0))} \approx r^{n-2}, \quad \forall x \in \Omega \setminus T(x_0, 2r). \tag{4.58}$$

This has been proved in [18], [47, p. 476] using the so-called comparison principle (for non-negative harmonic functions vanishing on a portion of the boundary) and Harnack's inequality. Combining (4.57) and (4.58) we obtain that

$$\big|u(x)\big| \leqslant C r^{-(n-1)(\frac{1}{p}-1)} G\big(x, A_r(x_0)\big), \quad \forall x \in \Omega \setminus T(x_0, 2r). \tag{4.59}$$

Bring in the following estimate, proved by M. Grüter and K.-O. Widman (cf. Theorem 3.3(ii) in [37]),

$$G(x,y) \leqslant C \operatorname{dist}(y, \partial\Omega)|x-y|^{1-n}, \quad \forall x, y \in \Omega, \tag{4.60}$$

where C depends only on n and the UEBC constant of Ω. Based on this, for $\alpha \in (0,1)$ fixed and $x \in \Omega \setminus T(x_0, 2r)$ arbitrary we have

$$
\begin{aligned}
G\big(x, A_r(x_0)\big) &\leqslant C \operatorname{dist}\big(A_r(x_0), \partial\Omega\big)\big|x - A_r(x_0)\big|^{1-n} \leqslant Cr|x - x_0|^{1-n} \\
&\leqslant\ C_\alpha r^\alpha |x - x_0|^{2-n-\alpha}. \qquad (4.61)
\end{aligned}
$$

Hence, (4.61) can be further used in (4.59) to conclude that

$$
\big|u(x)\big| \leqslant C_\alpha r^{\alpha-(n-1)(\frac{1}{p}-1)}|x - x_0|^{2-n-\alpha}, \quad \forall x \in \Omega \setminus T(x_0, 2r),\ \forall \alpha \in (0,1). \qquad (4.62)
$$

Next, consider the annuli $R_j(x_0) := \Delta_{2^{j+1}r}(x_0) \setminus \Delta_{2^j r}(x_0)$, $j \in \mathbb{N}$ and, for each j, introduce the following truncated nontangential maximal functions:

$$
\big(N^{j,r} w\big)(x) := \sup\big\{\big|w(y)\big|:\ y \in \Gamma(x) \text{ with } |y - x| \geqslant 2^{j+1}r\big\}, \quad x \in \partial\Omega, \qquad (4.63)
$$

$$
(N_{j,r} w)(x) := \sup\big\{\big|w(y)\big|:\ y \in \Gamma(x) \text{ with } |y - x| < 2^{j+1}r\big\}, \quad x \in \partial\Omega. \qquad (4.64)
$$

Combining interior estimates for u with (4.62) and the fact that for each $\alpha \in (0,1)$ one has $1 - n - \alpha < 0$, we have that for $x \in R_j(x_0)$

$$
\begin{aligned}
\big|\nabla u(y)\big| &\leqslant \frac{C}{(2^j r)^{n+1}} \int\limits_{B_{2^j r}(y)} \big|u(z)\big|\, dz \\
&\leqslant \frac{C_\alpha}{(2^j r)^{n+1}} r^{\alpha-(n-1)(\frac{1}{p}-1)} \int\limits_{B_{2^j r}(y)} |z - x_0|^{2-n-\alpha}\, dz \\
&\leqslant C_\alpha r^{\alpha-(n-1)(\frac{1}{p}-1)}|x - x_0|^{1-n-\alpha}, \quad \forall y \in \Gamma(x) \text{ with } |y - x| \geqslant 2^{j+1}r. \qquad (4.65)
\end{aligned}
$$

Thus, on account of (4.65) we obtain that for each fixed $\alpha \in (0,1)$,

$$
N^{j,r}(\nabla u)(x) \leqslant C_\alpha r^{\alpha-(n-1)(\frac{1}{p}-1)}|x - x_0|^{1-n-\alpha}, \quad \forall\, x \in R_j(x_0),\ \forall j \in \mathbb{N}. \qquad (4.66)
$$

As a consequence, if $\alpha \in (0,1)$ is fixed, then for every $j \in \mathbb{N}$ we have

$$
\begin{aligned}
\int\limits_{R_j(x_0)} \big|N^{j,r}(\nabla u)(x)\big|^p\, d\sigma(x) &\leqslant C_\alpha r^{\alpha p-(n-1)(1-p)}\big(2^j r\big)^{(1-n-\alpha)p}\big(2^j r\big)^{n-1} \\
&= C_\alpha\big(2^j\big)^{(n-1)(1-p)-\alpha p}. \qquad (4.67)
\end{aligned}
$$

Since $\frac{n-1}{n} < p \leqslant 1$, it follows that $0 \leqslant (n-1)(\frac{1}{p} - 1) < 1$ and if we choose

$$
(n-1)\bigg(\frac{1}{p} - 1\bigg) < \alpha < 1, \qquad (4.68)
$$

then $(n-1)(1-p) - \alpha p < 0$.

Going further, for each $j \in \mathbb{N}$, consider a Lipschitz domain $D_j(x_0, r)$ whose Lipschitz character is bounded by a constant independent of j, r and x_0, and such that

$$D_j(x_0, r) \subseteq \Omega, \qquad \operatorname{diam}\bigl(D_j(x_0, r)\bigr) \approx 2^j r, \qquad R_j(x_0) \subseteq \partial D_j(x_0, r) \cap \partial\Omega. \tag{4.69}$$

Then we have

$$\begin{aligned}
\int_{R_j(x_0)} \bigl|N_{j,r}(\nabla u)\bigr|^p \, d\sigma &\leqslant C\bigl(2^j r\bigr)^{(n-1)(1-\frac{p}{2})} \Biggl(\int_{R_j(x_0)} \bigl|N_{j,r}(\nabla u)\bigr|^2 \, d\sigma \Biggr)^{p/2} \\
&\leqslant C\bigl(2^j r\bigr)^{(n-1)(1-\frac{p}{2})} \Biggl(\int_{\partial D_j(x_0,2r)} \bigl|N_{j,r}(\nabla u)\bigr|^2 \, d\sigma \Biggr)^{p/2} \\
&\leqslant C\bigl(2^j r\bigr)^{(n-1)(1-\frac{p}{2})} \Biggl(\int_{\partial D_j(x_0,2r)\setminus\partial\Omega} |\nabla_{\tan} u|^2 \, d\sigma \Biggr)^{p/2} \\
&\leqslant C\bigl(2^j r\bigr)^{(n-1)(1-\frac{p}{2})} \Biggl(\int_{\partial D_j(x_0,2r)\setminus\partial\Omega} |\nabla u|^2 \, d\sigma \Biggr)^{p/2},
\end{aligned} \tag{4.70}$$

where the first inequality in (4.70) is Hölder's, the second one is trivial while the third one uses the well-posedness of the L^2-Regularity problem in $D_j(x_0, 2r)$ and the fact that $\nabla_{\tan} u = 0$ on $\partial\Omega \setminus \Delta_{2r}(x_0)$.

Let us now momentarily digress and point out that, in general, if D is a Lipschitz domain in $\mathbb{R}^n$ with diameter $\operatorname{diam}(D)$, and u is a solution of the L^2-Regularity problem for Δ in D, then

$$\int_{\partial D} \bigl|N(\nabla u)\bigr|^2 \, d\sigma \leqslant C \int_{\partial D} |\nabla_{\tan} u|^2 \, d\sigma + \frac{C}{(\operatorname{diam}(D))^2} \int_{\partial D} |u|^2 \, d\sigma, \tag{4.71}$$

with the constant in (4.71) depending only on the Lipschitz character of D. This can be seen by first assuming that D has $\operatorname{diam}(D) = 1$ and then rescaling the estimate to obtain the general case. Now (4.71) continues to hold if u is replaced bu $u - c$, for $c \in \mathbb{R}$. In particular, by choosing $c := ⨍_{\partial D} u \, d\sigma$, and recalling Poincaré's inequality:

$$\Bigl\| u - ⨍_{\partial D} u \, d\sigma \Bigr\|_{L^2(\partial D)} \leqslant C \operatorname{diam}(D) \|\nabla_{\tan} u\|_{L^2(\partial D)}, \tag{4.72}$$

where C again depends only on the Lipschitz character of D, we obtain that

$$\int_{\partial D} \bigl|N(\nabla u)\bigr|^2 \, d\sigma \leqslant C \int_{\partial D} |\nabla_{\tan} u|^2 \, d\sigma, \tag{4.73}$$

with C depending only on the Lipschitz character of D. It is precisely (4.73) (with $D := D_j(x_0, 2r)$) which is used to prove the third inequality in (4.70).

Raising the resulting inequality from (4.70) to the power $\frac{2}{p}$, re-denoting r by ρ, and then averaging for $r/2 \leqslant \rho \leqslant 2r$, we arrive at

$$\Bigl(\int_{R_j(x_0)} \bigl|N_{j,r}(\nabla u)(x)\bigr|^p \, d\sigma(x)\Bigr)^{2/p} \leqslant C\bigl(2^j r\bigr)^{(n-1)(\frac{2}{p}-1)-1} \int_{\{x\in\Omega:\ |x-x_0|\approx 2^j r,\ \operatorname{dist}(x,\partial\Omega)\leqslant C2^j r\}} \bigl|\nabla u(x)\bigr|^2 \, dx. \tag{4.74}$$

Next, since $u = 0$ on $\partial\Omega \setminus \Delta_r(x_0)$, we may invoke boundary Cacciopolli's inequality to further bound the last term in (4.74) and obtain

$$\begin{aligned}
\Bigl(\int_{R_j(x_0)} \bigl|N_{j,r}(\nabla u)\bigr|^p \, d\sigma\Bigr)^{2/p} &\leqslant C\bigl(2^j r\bigr)^{(n-1)(\frac{2}{p}-1)-3} \int_{\{x\in\Omega:\ |x-x_0|\approx 2^j r,\ \operatorname{dist}(x,\partial\Omega)\leqslant C2^j r\}} \bigl|u(x)\bigr|^2 \, dx \\
&\leqslant C_\alpha \bigl(2^j r\bigr)^{(n-1)(\frac{2}{p}-1)-3} r^{2[\alpha-(n-1)(\frac{1}{p}-1)]} \bigl(2^j r\bigr)^{2(2-n-\alpha)} \bigl(2^j r\bigr)^n \\
&= C_\alpha \bigl(2^j\bigr)^{2[(n-1)(\frac{1}{p}-1)-\alpha]},
\end{aligned} \tag{4.75}$$

where for the second inequality we have used (4.62). At this point, we select α as in (4.68) and combine (4.67) and (4.75) to conclude that

$$\begin{aligned}
\int_{\partial\Omega\setminus\Delta_{2r}(x_0)} \bigl|N(\nabla u)\bigr|^p \, d\sigma &\leqslant C\sum_{j=1}^{\infty}\Bigl(\int_{R_j(x_0)} \bigl|N^{j,r}(\nabla u)\bigr|^p \, d\sigma + \int_{R_j(x_0)} \bigl|N_{j,r}(\nabla u)\bigr|^p \, d\sigma\Bigr) \\
&\leqslant C\sum_{j=1}^{\infty}\bigl(2^j\bigr)^{(n-1)(\frac{1}{p}-1)-\alpha} + C\sum_{j=1}^{\infty}\bigl(2^j\bigr)^{2[(n-1)(\frac{1}{p}-1)-\alpha]} \\
&= C < +\infty.
\end{aligned} \tag{4.76}$$

Having established this, (4.53) now follows from (4.56) and (4.76). As mentioned earlier, this shows that (4.51) has a solution, which also satisfies (4.52).

There remains to prove that this solution is unique. To this end, assume that the function u solves the homogeneous version of (4.51). Then (2.13) implies that $u \in W^{1,q}(\Omega)$ where $q := pn/(n-1) > 1$. With this in hand, the desired conclusion then follows from the uniqueness part in Theorem 4.5, after embedding $W^{1,q}(\Omega)$ into $F^{p,2}_{s+1/p}(\Omega)$ for some $p \in (1,\infty)$ and $s \in (0,1)$ sufficiently small. $\square$

4.4. *The fully inhomogeneous problem*

The first-order of business is to convert the well-posedness result from Theorem 4.10 in which the size of the solution is measured using the nontangential maximal operator, into a well-posedness result on Besov and Triebel–Lizorkin scales.

Theorem 4.11. *Let Ω be a bounded semiconvex domain in $\mathbb{R}^n$ and assume that $\frac{n-1}{n} < p \leqslant \infty$, $0 < q \leqslant \infty$ and $(n-1)(\frac{1}{p}-1)_+ < s < 1$. Then the Dirichlet problem* (4.28) *has a unique solution which, in addition, satisfies* (4.29).

Moreover, if $\frac{n-1}{n} < p < \infty$, $0 < q < \infty$ and $(n-1)(\frac{1}{p}-1)_+ < s < 1$ then the Dirichlet problem (4.30) *also has a unique solution, which satisfies* (4.31).

Proof. Of course, the novel case here is when $\frac{n-1}{n} < p \leqslant 1$ since otherwise the claims are covered by Theorem 4.5 and Theorem 4.6. Assume that $\frac{n-1}{n} < p < 1$ and recall the Poisson integral operator PI introduced in the course of the proof of Theorem 4.5. Theorem 4.10 then ensures that

$$\mathrm{PI}: h^{1,p}_{at}(\partial\Omega) \longrightarrow \bigl\{u \in C^\infty(\Omega):\ \Delta u = 0 \text{ in } \Omega,\ N(\nabla u) \in L^p(\partial\Omega)\bigr\} \tag{4.77}$$

is well defined, linear and bounded. Consequently, if $\frac{n-1}{n} < p < q \leqslant 1$ then

$$\mathrm{PI}: \mathcal{E}_q\bigl(h^{1,p}_{\mathrm{at}}(\partial\Omega)\bigr) \longrightarrow \mathcal{E}_q\bigl(\bigl\{u \in C^\infty(\Omega):\ \Delta u = 0 \text{ in } \Omega,\ N(\nabla u) \in L^p(\partial\Omega)\bigr\}\bigr) \tag{4.78}$$

is, by Proposition 2.9, linear and bounded. Based on this, (2.129), (2.135) and (2.132), we may therefore conclude that the operator

$$\mathrm{PI}: B^{q,q}_s(\partial\Omega) \longrightarrow \mathbb{H}^q_{s+\frac{1}{q}}(\Omega;\Delta) = F^{q,2}_{s+\frac{1}{q}}(\Omega) \cap \ker\Delta \tag{4.79}$$

is well defined, linear and bounded, provided $s := 1 + (n-1)(\frac{1}{q} - \frac{1}{p})$. Finally, by bringing in (2.134) and observing that $(n-1)(\frac{1}{q}-1) < s < 1$ as $\frac{n-1}{n} < p < q \leqslant 1$, we obtain that

$$\mathrm{PI}: B^{q,q}_s(\partial\Omega) \longrightarrow F^{q,r}_{s+\frac{1}{q}}(\Omega) \tag{4.80}$$

is well defined, linear and bounded, whenever $\frac{n-1}{n} < q \leqslant 1$, $(n-1)(\frac{1}{q}-1) < s < 1$ and $0 < r < \infty$. This ensures that the Dirichlet problem (4.30) has a solution which satisfies (4.31). The uniqueness of such a solution is then a consequence of the uniqueness part in Theorem 4.5 and the fact that $B^{p,q}_{s+1/p}(\Omega)$ embeds into some space $B^{p_0,q}_{s_0+1/p_0}(\Omega)$ with $1 < p_0 < \infty$ and $s_0 \in (0,1)$.

Moving on, the fact that (4.80) is a linear and bounded operator for $\frac{n-1}{n} < q \leqslant 1$, $(n-1)(\frac{1}{q}-1) < s < 1$ and $0 < r < \infty$, implies, via the real interpolation formulas (2.69) and (2.115) that

$$\mathrm{PI}: B^{p,q}_s(\partial\Omega) \longrightarrow B^{p,q}_{s+\frac{1}{q}}(\Omega) \tag{4.81}$$

is well defined, linear and bounded, whenever $\frac{n-1}{n} < p \leqslant 1$, $(n-1)(\frac{1}{p}-1) < s < 1$ and $0 < q \leqslant \infty$. In particular, the Dirichlet problem (4.28) has a solution which satisfies (4.29). Finally, the uniqueness of such a solution is then proved much as before. $\square$

The main result in this subsection is the theorem below, dealing with the fully inhomogeneous problem for the Laplacian.

Theorem 4.12. *Let Ω be a bounded semiconvex domain in $\mathbb{R}^n$. Then for every $\frac{n-1}{n} < p \leqslant \infty$, $0 < q \leqslant \infty$ and $(n-1)(\frac{1}{p}-1)_+ < s < 1$, the Dirichlet problem*

$$\begin{cases} \Delta u = f \in B^{p,q}_{s+\frac{1}{p}-2}(\Omega), \\ \operatorname{Tr} u = g \in B^{p,q}_{s}(\partial\Omega), \\ u \in B^{p,q}_{s+\frac{1}{p}}(\Omega) \end{cases} \tag{4.82}$$

has a unique solution. In addition, there exists $C = C(\Omega, p, q, s) > 0$ such that the solution u of (4.82) *satisfies*

$$\|u\|_{B^{p,q}_{s+\frac{1}{p}}(\Omega)} \leqslant C\|f\|_{B^{p,q}_{s+\frac{1}{p}-2}(\Omega)} + C\|g\|_{B^{p,q}_{s}(\partial\Omega)}. \tag{4.83}$$

Similar results are also valid on the Triebel–Lizorkin scale. More precisely, if $\frac{n-1}{n} < p < \infty$, $0 < q < \infty$ and $(n-1)(\frac{1}{p}-1)_+ < s < 1$, then the Dirichlet problem

$$\begin{cases} \Delta u = f \in F^{p,q}_{s+\frac{1}{p}-2}(\Omega), \\ \operatorname{Tr} u = g \in B^{p,p}_{s}(\partial\Omega), \\ u \in F^{p,q}_{s+\frac{1}{p}}(\Omega) \end{cases} \tag{4.84}$$

has a unique solution, which also satisfies

$$\|u\|_{F^{p,q}_{s+\frac{1}{p}}(\Omega)} \leqslant C\|f\|_{F^{p,q}_{s+\frac{1}{p}-2}(\Omega)} + C\|g\|_{B^{p,p}_{s}(\partial\Omega)}. \tag{4.85}$$

Proof. We look for a solution to (4.82) in the form $u := [\Pi f_o]|_\Omega + v$, where f_o is an extension of f to a compactly supported distribution in $B^{p,q}_{s+\frac{1}{p}-2}(\mathbb{R}^n)$, and v solves

$$\Delta v = 0 \quad \text{in } \Omega, \qquad \operatorname{Tr} v = g - \operatorname{Tr}[\Pi f_o] \in B^{p,q}_{s}(\partial\Omega), \quad v \in B^{p,q}_{s+\frac{1}{p}}(\Omega). \tag{4.86}$$

That this problem is well formulated and such a function v exists, then follows from the fact that $[\Pi f_o]|_\Omega \in B^{p,q}_{s+\frac{1}{p}}(\Omega)$, Theorem 2.7 and Theorem 4.5. This proves that (4.82) always has a solution which satisfies (4.83). The uniqueness of such a solution follows from the corresponding uniqueness part in Theorem 4.5. Altogether, we have that (4.82) is well posed, and the argument for (4.84) is analogous. □

Consider the (open) pentagonal region in Fig. 5.

Then Theorem 4.12 states that the inhomogeneous problems (4.82), (4.84) are well posed if $0 < q \leqslant \infty$ whenever the point with coordinates $(s, 1/p)$ belongs to the open shaded region in Fig. 2 (with the convention that $q \neq \infty$ for the Triebel–Lizorkin scale, while for the Besov scale the bottom segment is also included).

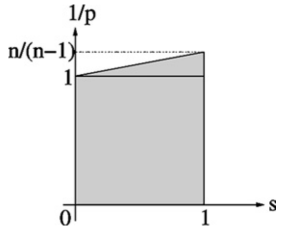

Fig. 5.

We conclude this subsection by formally stating a result about the mapping properties of the Poisson integral operator

$$\mathrm{PI}\, f(x) := -\int_{\partial\Omega} \partial_{\nu(x)} G(x,y) f(x)\, d\sigma(x), \quad y \in \Omega, \tag{4.87}$$

whose proof is implicit in what we established so far.

Theorem 4.13. *Let Ω be a bounded semiconvex domain in $\mathbb{R}^n$ and assume that $\frac{n-1}{n} < p \leqslant \infty$ and $(n-1)(\frac{1}{p}-1)_+ < s < 1$. Then the operators*

$$\mathrm{PI} : B_s^{p,q}(\partial\Omega) \longrightarrow B_{s+\frac{1}{p}}^{p,q}(\Omega), \quad 0 < q \leqslant \infty, \tag{4.88}$$

$$\mathrm{PI} : B_s^{p,p}(\partial\Omega) \longrightarrow F_{s+\frac{1}{p}}^{p,q}(\Omega), \quad 0 < q < \infty, \tag{4.89}$$

are well defined and bounded (*assuming $p < \infty$ in the case of* (4.89)).

5. Further results for the Poisson problem

This section is organized into three subsections. In Section 5.1 we study the mapping properties of the Green potential (i.e., the solution operator for the inhomogeneous Dirichlet Laplacian). Then, in Section 5.2, we revisit the issue of traces in Besov spaces, for the purpose of establishing some results outside the standard range of indices. Finally, in Section 5.3, we use all these results to state and prove certain Poisson problems for the Laplacian with data exhibiting a nonstandard amount of smoothness.

5.1. Mapping properties of the Dirichlet Green operator

With $G(\cdot,\cdot)$ the Green function associated with a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, define the Green operator $\mathbb{G}$ by setting

$$(\mathbb{G} f)(x) := -\int_{\Omega} G(x,y) f(y)\, dy, \quad x \in \Omega, \tag{5.1}$$

for any $f \in C^\infty(\overline{\Omega})$. It follows that $\Delta(\mathbb{G} f) = f$ in Ω and $[\mathbb{G} f]|_{\partial\Omega} = 0$, i.e., for each $f \in C^\infty(\overline{\Omega})$ the function $u := \mathbb{G} f$ solves the inhomogeneous Dirichlet problem

$$\Delta u = f \quad \text{in } \Omega, \qquad \operatorname{Tr} u = 0 \quad \text{on } \partial\Omega. \tag{5.2}$$

The issue of extending (5.1) as a smoothing operator of order two on the Besov and Triebel–Lizorkin scales in a given bounded semiconvex domain is discussed below.

Corollary 5.1. *Let Ω be a bounded semiconvex domain in $\mathbb{R}^n$. Assume that $\frac{n-1}{n} < p \leqslant \infty$, $0 < q \leqslant \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$, and set $\alpha := s + \frac{1}{p} - 2$. Then the Green operator* (5.1) *extends as a linear and bounded operator – indeed, an isomorphism – in each of the cases*

$$\mathbb{G} : B^{p,q}_{\alpha}(\Omega) \longrightarrow B^{p,q}_{\alpha+2,z}(\Omega), \tag{5.3}$$

$$\mathbb{G} : F^{p,q}_{\alpha}(\Omega) \longrightarrow F^{p,q}_{\alpha+2,z}(\Omega), \tag{5.4}$$

assuming that $p, q < \infty$ and $\min\{1, p\} \leqslant q$ *in the case of* (5.4). *Without the latter condition,*

$$\mathbb{G} : F^{p,q}_{\alpha}(\Omega) \longrightarrow F^{p,q}_{\alpha+2}(\Omega) \tag{5.5}$$

is a well-defined and bounded operator.

Proof. This is a direct consequence of Theorem 4.12 (with $g = 0$) and (2.109), (2.113). □

The case of Hardy spaces deserves special mention. Recall (4.8).

Corollary 5.2. *Let Ω be a bounded semiconvex domain in $\mathbb{R}^n$. Then for each $\frac{n}{n+1} < p \leqslant 2$, the Green operator*

$$\mathbb{G} : h^p(\Omega) \longrightarrow F^{p,2}_{2,z}(\Omega) \tag{5.6}$$

is an isomorphism. In particular,

$$\partial_j \partial_k \mathbb{G} : h^p(\Omega) \longrightarrow h^p(\Omega) \tag{5.7}$$

is well defined and bounded for each $j, k \in \{1, \dots, n\}$, provided $\frac{n}{n+1} < p \leqslant 2$.

Proof. This is a direct consequence of Corollary 5.1 and the identification $h^p(\Omega) = F^{p,2}_0(\Omega)$ for $\frac{n}{n+1} < p < \infty$. □

Moving on, for a vector field $w = (w_1, w_2, \dots, w_n)$ whose components are distributions in an open set $\Omega \subset \mathbb{R}^n$ we define its curl, $\operatorname{curl} w$, to be the vector field with n^2 components given by

$$(\operatorname{curl} w)_{jk} = \partial_j w_k - \partial_k w_j, \quad j, k = 1, \dots, n. \tag{5.8}$$

Also, if $\Psi = (\Psi_{jk})_{1 \leqslant j,k \leqslant n}$ is a vector field whose components are distributions in Ω, set

$$(\operatorname{Div} \Psi)_k = \sum_{j=1}^{n} \partial_j (\Psi_{jk} - \Psi_{kj}), \quad k = 1, \dots, n. \tag{5.9}$$

We then make the following definition.

Definition 5.3. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain with outward unit normal ν. Let $w = (w_1, \dots, w_n)$ be a vector field with components in $L^2(\Omega)$ such that $\operatorname{curl} w$ also has components in $L^2(\Omega)$. Then $\nu \times w$ is the unique vector field with n^2 components in $B^{2,2}_{-1/2}(\partial\Omega) = (B^{2,2}_{1/2}(\partial\Omega))^*$ which satisfies the following property. If Ψ is any vector field with n^2 components in $W^{1,2}(\Omega)$ and $\psi = \operatorname{Tr}\Psi$, with the trace taken component-wise, then

$$\langle \nu \times w, \psi\rangle = \int_\Omega \langle \operatorname{curl} w, \Psi\rangle\, dx + \int_\Omega \langle w, \operatorname{Div}\Psi\rangle\, dx, \tag{5.10}$$

where the pairing in the left-hand side of (5.10) is that between $B^{2,2}_{-1/2}(\partial\Omega)$ and $B^{2,2}_{1/2}(\partial\Omega)$.

It is not difficult to check (using the fact that $\operatorname{Tr}: W^{1,2}(\Omega) \to B^{2,2}_{1/2}(\partial\Omega)$ is onto, and that $C^\infty_c(\Omega)$ is dense in the space of functions from $W^{1,2}(\Omega)$ with vanishing trace) that (5.10) unambiguously defines $\nu \times w$ as a functional in $(B^{2,2}_{1/2}(\partial\Omega))^*$.

We now record a useful regularity result proved in [62] and [67] (these references also contain more general results, formulated in the language of differential forms).

Proposition 5.4. *Assume that $\Omega \subset \mathbb{R}^n$ is a semiconvex domain with outward unit normal ν. Then,*

$$\begin{aligned}&\bigl\{w \in L^2(\Omega)\colon \operatorname{div} w \in L^2(\Omega),\ \operatorname{curl} w \in L^2(\Omega),\ \nu \times w = 0\bigr\}\\ &\quad = \bigl\{w \in W^{1,2}(\Omega)\colon \nu \times w = 0\bigr\}\end{aligned} \tag{5.11}$$

and, in addition, there exists a finite constant $C = C(\Omega) > 0$ such that

$$\|w\|_{W^{1,2}(\Omega)} \leqslant C\bigl(\|w\|_{L^2(\Omega)} + \|\operatorname{div} w\|_{L^2(\Omega)} + \|\operatorname{curl} w\|_{L^2(\Omega)}\bigr), \tag{5.12}$$

whenever $\nu \times w = 0$.

We now proceed to discuss a special well-posedness result which, in the case of domains satisfying more restrictive conditions than semiconvexity, is due to J. Kadlec and G. Talenti in the 60s; see, e.g., [2,3,31,36,39,42,65,78], for other related versions.

Theorem 5.5. *Let Ω be a bounded semiconvex domain in $\mathbb{R}^n$. Then the boundary value problem*

$$\begin{cases}\Delta u = f \in L^2(\Omega),\\ u \in W^{2,2}(\Omega),\\ \operatorname{Tr} u = 0 \quad \text{on } \partial\Omega,\end{cases} \tag{5.13}$$

is well posed. In particular, there exists a finite constant $C = C(\Omega) > 0$ such that

$$\|u\|_{W^{2,2}(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}. \tag{5.14}$$

As a corollary,

$$\mathbb{G} : L^2(\Omega) \longrightarrow W^{2,2}(\Omega) \tag{5.15}$$

is well defined, linear and bounded.

Proof. Given $f \in L^2(\Omega) \hookrightarrow W^{-1,2}(\Omega)$, Lax–Milgram's lemma ensures that there exists a unique function $u \in W_0^{1,2}(\Omega)$ such that $\Delta u = f$. Thus, it suffices to show that the vector field $w := \nabla u \in L^2(\Omega)^n$ actually belongs to $W^{1,2}(\Omega)^n$ (plus a natural estimate). This, in turn, is a direct consequence of (5.11)–(5.12) upon noticing that $\operatorname{div} w = \Delta u = f \in L^2(\Omega)$, $\operatorname{curl} w = 0$, and that $\nu \times w = 0$. The latter equality is (in light of (5.9) and (5.10)) a consequence of the fact that if $u \in W_0^{1,2}(\Omega)$ then

$$\sum_{j,k=1}^{n} \int_{\Omega} (\partial_k u) \partial_j (\Psi_{jk} - \Psi_{kj})\, dx = 0, \tag{5.16}$$

for any family of functions Ψ_{jk} from $W^{1,2}(\Omega)$. Indeed, (5.16) is readily verified when $u \in C_c^\infty(\Omega)$ (integrating by parts and using simple symmetry considerations), so a simple density argument concludes the proof. □

The proof of the above theorem makes essential use of Proposition 5.4 which, in turn, involves successive integrations by parts (cf. [67]). Thus, it makes essential use of the Hilbert character of L^2 and, as such, it does not readily extend to L^p-based Sobolev space with $p \neq 2$. Our goal is to establish the following extension.

Theorem 5.6. *Let Ω be a bounded semiconvex domain in $\mathbb{R}^n$. Assume that $\alpha \in \mathbb{R}$ and $0 < p \leqslant \infty$ are such that either $p = \infty$ and $-2 < \alpha < -1$, or*

$$\max\left\{0, \frac{\alpha+1}{2}, \frac{(n+1)\alpha}{2} + \frac{1}{2}\right\} < \frac{1}{p} < \min\left\{\alpha + 2, \frac{n+1+\alpha}{n}\right\}. \tag{5.17}$$

In geometrical terms, condition (5.17) *is equivalent to the membership of the point with coordinate $(\alpha, 1/p)$ to the interior of the open pentagonal region in Fig.* 2 (*in Section* 1). *Then, with $A \in \{B, F\}$, the Green operator*

$$\mathbb{G} : A_\alpha^{p,q}(\Omega) \longrightarrow A_{\alpha+2}^{p,q}(\Omega) \tag{5.18}$$

is well defined, linear and bounded for any $0 < q \leqslant \infty$ when $A = B$, and any $0 < q < \infty$ when $A = F$.

Proof. This follows from Corollary 5.1, Theorem 5.5 and complex interpolation. □

In the special case when Ω is actually a convex domain, Theorem 5.6 can be further refined as follows.

Theorem 5.7. *Let Ω be a bounded convex domain in $\mathbb{R}^n$. Assume that $\alpha \in \mathbb{R}$ and $0 < p \leqslant \infty$ are such that either $p = \infty$ and $-2 < \alpha < -1$, or*

$$\max\left\{0, \frac{\alpha+1}{2}\right\} < \frac{1}{p} < \min\left\{\alpha+2, \frac{n+1+\alpha}{n}, \frac{n-1-\alpha}{n-2}\right\}. \tag{5.19}$$

Geometrically, (5.19) *is equivalent to the membership of the point with coordinate* $(\alpha, 1/p)$ *to the interior of the open pentagonal region in Fig.* 1 (*in Section* 1). *Then the Green operators*

$$\mathbb{G} : B^{p,q}_{\alpha}(\Omega) \longrightarrow B^{p,q}_{\alpha+2}(\Omega), \quad 0 < q \leqslant \infty, \tag{5.20}$$

$$\mathbb{G} : F^{p,q}_{\alpha}(\Omega) \longrightarrow F^{p,q}_{\alpha+2}(\Omega), \quad 0 < q < \infty, \tag{5.21}$$

are well defined, linear and bounded (*assuming* $p < \infty$ *in* (5.21)).

Proof. In the special case in which

$$0 < \frac{\alpha+1}{2} < \frac{1}{p} < 1 \quad \text{and} \quad q = 2 \tag{5.22}$$

the fact that $\mathbb{G}$ in (5.21) is bounded was proved in [33]. By repeatedly interpolating via the complex method (cf. (2.118)) between this region and

$$0 < \frac{1}{p} < \alpha + 2 < 1 \quad \text{and} \quad 0 < q < \infty, \tag{5.23}$$

we can gradually extend the range of q's until we eventually obtain that

$$\begin{aligned} &\text{if } 0 < \frac{\alpha+1}{2} < \frac{1}{p} < 1 \text{ and } 0 < q < \infty \\ &\text{then the operator } \mathbb{G} : F^{p,q}_{\alpha}(\Omega) \longrightarrow F^{p,q}_{\alpha+2}(\Omega) \text{ is well defined and bounded.} \end{aligned} \tag{5.24}$$

Hence, the claim about (5.21) follows by further interpolating this result with (5.4). Finally, the claim about (5.20) is a consequence of (5.21) and the method of real interpolation (cf. (2.115)). □

Remark. When more geometric information is available about the domain Ω, the results in Theorems 5.6–5.7 can be refined accordingly. To illustrate this point, consider the case when $\Omega \subset \mathbb{R}^3$ is a Lipschitz polyhedron, and denote by ω the largest angle between its adjacent faces. In this context, it has been proved in [23, Corollary 18.18] that if

$$-\frac{3}{2} < \alpha < \alpha_o := \min\left\{\frac{3}{2}, \frac{\pi}{\omega} - 1\right\} \quad \text{and} \quad \alpha \neq -\frac{1}{2}, \tag{5.25}$$

then the Laplace operator

$$\Delta : F^{2,2}_{2+\alpha}(\Omega) \cap F^{2,2}_{1,z}(\Omega) \longrightarrow F^{2,2}_{\alpha}(\Omega) \tag{5.26}$$

is an isomorphism. Thus, in this setting, $\mathbb{G} : F^{2,2}_{\alpha}(\Omega) \longrightarrow F^{2,2}_{2+\alpha}(\Omega)$ is a bounded operator. Consequently, if in addition we also have $\omega < \pi$, then new results can be obtained by interpolating this with the results from Theorems 5.6–5.7. As a concrete example, the above analysis shows that for a given bounded convex polyhedron $\Omega \subset \mathbb{R}^3$ the operator $\mathbb{G} : L^p(\Omega) \longrightarrow W^{2,p}(\Omega)$ is bounded for all $1 < p < 2\alpha_0 + 2$, where α_0 is as in (5.25). As already mentioned in the introduction, such a result fails for $p > 2$ in the class of arbitrary bounded convex domains.

We conclude this subsection by providing the proof of Corollary 1.4 stated in Section 1.

Proof of Corollary 1.4. The claim about (1.19) is immediate from (5.7) (with $p = 1$) and the fact that $h^1(\Omega) \hookrightarrow L^1(\Omega)$ (cf. (2.125)). That (1.20)–(1.21) are also bounded operators is a consequence of (5.7) and (2.123)–(2.124). Next, (2.122) and (2.125) give that for every $j, k \in \{1, \dots, n\}$,

$$\|\partial_j \partial_k \mathbb{G} f\|_{L^{1,\infty}(\Omega)} \leqslant C \|\partial_j \partial_k \mathbb{G} f\|_{h^{1,\infty}(\Omega)} \leqslant C \|f\|_{h^{1,\infty}(\Omega)} \leqslant C \|f\|_{L^1(\Omega)}, \tag{5.27}$$

uniformly, for reasonable functions f. A density argument then proves the claim about (1.22). Finally, the claim in (1.23) then follows from (5.6) (with $p = 1$) and the fact that $F_2^{1,2}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ if $\Omega \subset \mathbb{R}^2$ (see Theorem 1, Section 2.2.4 in [72] for the latter embedding when $\Omega = \mathbb{R}^2$). $\square$

5.2. Trace theory outside of the canonical range

In this subsection we study the action of the trace operator Tr in border line cases (cf. Theorem 5.8), and settings when one either has more smoothness (cf. Theorem 5.10), or less smoothness (cf. Theorem 5.15), than in (2.108), (2.111).

First, we discuss some limiting cases ($s = 1$ and $s = 0$) of Theorem 2.7. To state this result (which extends work in [31] where the case $p = q = 2$ and $s = 1$ has been treated), recall (4.45).

Theorem 5.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain such that either Ω or $\mathbb{R}^n \setminus \overline{\Omega}$ is semiconvex. Assume that $\frac{n-1}{n} < p \leqslant 2$ and $0 < q < \infty$. Then the boundary trace operator in Theorem 2.7 further extends to the limiting cases*

$$\mathrm{Tr} : F^{p,q}_{1+1/p}(\Omega) \longrightarrow h^{1,p}(\partial\Omega) \tag{5.28}$$

and

$$\mathrm{Tr} : F^{p,q}_{1/p}(\Omega) \longrightarrow L^p(\partial\Omega), \tag{5.29}$$

as a well-defined, linear and bounded mapping.

Proof. Assume first that Ω is a bounded semiconvex domain (hence a bounded Lipschitz domain satisfying a UEBC). The extension of the trace operator we seek is given by

$$\widetilde{\mathrm{Tr}} u := \big[u - \mathbb{G}(\Delta u)\big]\big|_{\partial\Omega}. \tag{5.30}$$

Above, the restriction to the boundary is understood in the sense of Proposition 4.9, when $u \in F^{p,q}_{1+1/p}(\Omega)$. Indeed, in this scenario, $\Delta u \in F^{p,q}_{1/p-1}(\Omega)$ and if we set $v := u - \mathbb{G}(\Delta u)$ then v is a well-defined function which belongs to $F^{p,q}_{1+1/p}(\Omega)$, by Theorem 5.6 (used with

$\alpha = 1/p - 1$). Moreover, v is harmonic in Ω and, by applying Theorem 4.2 (with $L = \Delta^2$, so that $m = 2$) we obtain that $N(\nabla v) \in L^p(\partial\Omega)$ and that $\|N(\nabla v)\|_{L^p(\partial\Omega)} \leqslant C\|v\|_{F^{p,q}_{1+1/p}(\Omega)}$. Hence, we can use Proposition 4.9 to obtain that $v|_{\partial\Omega} \in h^{1,p}(\partial\Omega)$ with appropriate control. This shows that $\widetilde{\mathrm{Tr}} : F^{p,q}_{1+1/p}(\Omega) \to h^{1,p}(\partial\Omega)$ is a well-defined, linear and bounded operator. Given that $\mathrm{Tr}(\mathbb{G}(\Delta u)) = 0$ whenever u is sufficiently smooth, we may also conclude that $\widetilde{\mathrm{Tr}}$ in (5.30) is an extension of Tr in Theorem 2.7.

This completes the proof of the claims made about (5.28) when the bounded domain Ω is semiconvex. In the case when $\mathbb{R}^n \setminus \overline{\Omega}$ is semiconvex, fix an open ball $B \subset \mathbb{R}^n$ which contains $\overline{\Omega}$, and consider $D := B \setminus \overline{\Omega}$. Then D is a bounded Lipschitz domain which satisfies a UEBC (hence, semiconvex), and we set $\mathbb{G}_D$ for the Green operator associated with it. Also, recall the universal extension operator Ex from Proposition 2.5, and set Ex_D for the composition of Ex with the restriction to D. In particular, $\mathrm{Ex}_D : F^{p,q}_\alpha(\Omega) \to F^{p,q}_\alpha(D)$, $\alpha \in \mathbb{R}$, is linear, and bounded. After this preamble, the proof proceeds as before if we consider

$$\widetilde{\mathrm{Tr}}u := \big[\mathrm{Ex}\,u - \mathbb{G}_D\big(\Delta(\mathrm{Ex}\,u)\big)\big]\big|_{\partial\Omega}, \tag{5.31}$$

in place of (5.30). Finally, the treatment of (5.29) is similar, the only differences being using Theorem 4.2 with $L = \Delta$ (hence $m = 1$), and invoking Dahlberg's Fatou-type theorem for harmonic functions in Lipschitz domains (cf. [18]) in place of Proposition 4.9. □

Remarks. (i) Note that, as particular cases of (5.28)–(5.29), we have that

$$\mathrm{Tr} : B^{p,p}_{\frac{1}{p}}(\Omega) \longrightarrow L^p(\partial\Omega), \qquad \mathrm{Tr} : B^{p,p}_{1+\frac{1}{p}}(\Omega) \longrightarrow L^p_1(\partial\Omega) \tag{5.32}$$

are bounded operators whenever Ω is as in the statement of Theorem 5.8 and $1 < p \leqslant 2$. Since, in general, $B^{p,1}_{s+\frac{1}{p}}(\Omega) \hookrightarrow B^{p,p}_{s+\frac{1}{p}}(\Omega)$ for $s \in \{0,1\}$ and $1 < p < \infty$, the fact that the operators in (5.32) are bounded is then an improvement over the claim made about (2.114) in Theorem 2.7, albeit for the current, more restrictive, class of domains and range of indices.

(ii) It is interesting to note that, via a standard localization argument (involving a smooth partition of unity), the above result can be extended to all domains which locally are as in Theorem 5.8 (informally speaking, bounded Lipschitz domains $\Omega \subset \mathbb{R}^n$) such that Ω satisfies a uniform exterior, or interior, ball condition, near each boundary point. For example, this trace result holds in the class of all polygonal domains in $\mathbb{R}^2$.

We now turn our attention to the case when traces are considered for functions exhibiting either a strictly larger, or strictly lower amount of smoothness than in (2.108), (2.111). This requires some preliminaries and, for the sake of uniformity, we choose to define the Dirichlet and Neumann traces (formally) by

$$\gamma_D(u) := \mathrm{Tr}\,u, \qquad \gamma_N(u) := \nu \cdot \mathrm{Tr}(\nabla u), \tag{5.33}$$

on the boundary of a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$, with outward unit normal ν. The following higher-order trace result has been proved in [56].

Theorem 5.9. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and denote by ν the outward unit normal to $\partial\Omega$. Also, suppose that $0 < p, q \leqslant \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$. Fix a number p^* such that $1 < p^* < (\frac{1}{p} - \frac{s}{n-1})^{-1}$ if $p \leqslant 1$, and $p^* := p$ if $p > 1$. Then the operator*

$$\gamma : B^{p,q}_{1+s+\frac{1}{p}}(\Omega) \longrightarrow \left\{(g_0, g_1) \in L^{p^*}_1(\partial\Omega) \oplus L^{p^*}(\partial\Omega):\ \nabla_{\tan} g_0 + g_1\nu \in \left(B^{p,q}_s(\partial\Omega)\right)^n\right\},$$
$$\gamma(u) := \left(\gamma_D(u), \gamma_N(u)\right), \quad u \in B^{p,q}_{1+s+\frac{1}{p}}(\Omega), \tag{5.34}$$

is well defined, linear, bounded, onto, and has a linear, bounded right inverse. Above, the space $\{(g_0, g_1) \in L^{p^*}_1(\partial\Omega) \oplus L^{p^*}(\partial\Omega):\ \nabla_{\tan} g_0 + g_1\nu \in (B^{p,q}_s(\partial\Omega))^n\}$ is considered equipped with the natural norm

$$(g_0, g_1) \mapsto \|g_0\|_{L^{p^*}_1(\partial\Omega)} + \|g_1\|_{L^{p^*}(\partial\Omega)} + \|\nabla_{\tan} g_0 + g_1\nu\|_{(B^{p,q}_s(\partial\Omega))^n}. \tag{5.35}$$

Furthermore, the null-space of the operator (5.34) *is given by*

$$\operatorname{Ker}\gamma := \left\{u \in B^{p,q}_{1+s+\frac{1}{p}}(\Omega):\ \gamma_D(u) = \gamma_N(u) = 0\right\} = B^{p,q}_{1+s+\frac{1}{p},z}(\Omega). \tag{5.36}$$

Finally, similar results are valid for the Triebel–Lizorkin scale, i.e. for

$$\gamma : F^{p,q}_{1+s+\frac{1}{p}}(\Omega) \longrightarrow \left\{(g_0, g_1) \in L^{p^*}_1(\partial\Omega) \oplus L^{p^*}(\partial\Omega):\ \nabla_{\tan} g_0 + g_1\nu \in \left(B^{p,p}_s(\partial\Omega)\right)^n\right\},$$
$$\gamma(u) := \left(\gamma_D(u), \gamma_N(u)\right), \quad u \in F^{p,q}_{1+s+\frac{1}{p}}(\Omega), \tag{5.37}$$

provided $p, q < \infty$. In this case, if $\min\{1, p\} \leqslant q < \infty$ then the null-space of (5.37) *is given by*

$$\operatorname{Ker}\gamma := \left\{u \in F^{p,q}_{1+s+\frac{1}{p}}(\Omega):\ \gamma_D(u) = \gamma_N(u) = 0\right\} = F^{p,q}_{1+s+\frac{1}{p},z}(\Omega). \tag{5.38}$$

We are now ready for the first main result of this subsection.

Theorem 5.10. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with outward unit normal ν. Also, suppose that $0 < p, q \leqslant \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$. Fix $1 < p^* < (\frac{1}{p} - \frac{s}{n-1})^{-1}$ if $p \leqslant 1$, and $p^* := p$ if $p > 1$. Then the boundary trace operator*

$$\operatorname{Tr} : B^{p,q}_{1+s+1/p}(\Omega) \to \left\{g \in L^{p^*}_1(\partial\Omega):\ \nabla_{\tan} g \in \left[B^{p,q}_s(\partial\Omega)^n\right]_{\tan}\right\} \tag{5.39}$$

is well defined, linear, bounded and onto. Above, $[B^{p,q}_s(\partial\Omega)^n]_{\tan}$ denotes the tangential components of vector fields on $\partial\Omega$ with components in $B^{p,q}_s(\partial\Omega)$, and this is considered equipped with the natural (*quasi-*)*norm*

$$g \mapsto \|g\|_{L^{p^*}_1(\partial\Omega)} + \inf\left\{\|\vec{h}\|_{B^{p,q}_s(\partial\Omega)^n}:\ \vec{h} \in B^{p,q}_s(\partial\Omega)^n \text{ and } \nabla_{\tan} g = \vec{h}_{\tan}\right\}. \tag{5.40}$$

Finally, similar claims are valid for

$$\operatorname{Tr}: F^{p,q}_{1+s+1/p}(\Omega) \to \left\{g \in L_1^{p^*}(\partial\Omega): \nabla_{\tan} g \in \left[B_s^{p,p}(\partial\Omega)^n\right]_{\tan}\right\}, \tag{5.41}$$

granted that, in addition, $p, q < \infty$.

Proof. Fix $u \in B^{p,q}_{1+s+1/p}(\Omega)$ and note that

$$\nabla_{\tan}[\operatorname{Tr} u] = \left[\operatorname{Tr}(\nabla u)\right]_{\tan} \in \left[B_s^{p,q}(\partial\Omega)^n\right]_{\tan}, \tag{5.42}$$

by Theorem 2.7. Since $B_s^{p,q}(\partial\Omega) \hookrightarrow L^{p^*}(\partial\Omega)$, we also obtain that $\operatorname{Tr} u \in L_1^{p^*}(\partial\Omega)$, plus a natural estimate. Altogether, this shows that the operator in (5.39) is well defined, linear, and bounded. To prove that Tr in (5.39) is onto, fix an arbitrary function $g \in L_1^{p^*}(\partial\Omega)$ such that $\nabla_{\tan} g \in [B_s^{p,q}(\partial\Omega)^n]_{\tan}$. Then there exists $\vec{h} \in B_s^{p,q}(\partial\Omega)^n \hookrightarrow L^{p^*}(\partial\Omega)^n$ such that $\nabla_{\tan} g = \vec{h}_{\tan} = \vec{h} - (\vec{h}\cdot\nu)\nu$. In particular, if we take $g_1 := \vec{h}\cdot\nu \in L^{p^*}(\partial\Omega)$, then $\nabla_{\tan} g + g_1\nu = \vec{h} \in B_s^{p,q}(\partial\Omega)^n$. In light of Theorem 5.9, this shows that there exists $u \in B^{p,q}_{1+s+1/p}(\Omega)$ such that $\operatorname{Tr} u = g$. The argument in the case of (5.41) is similar, and this completes the proof of the theorem. □

Remarks. (i) Let p, q, s, p^* be as in the statement of Theorem 5.10. Then, if Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ with outward unit normal $\nu = (\nu_1, \dots, \nu_n)$ for which the multiplication operator with ν_j is bounded from $B_s^{p,q}(\partial\Omega)$ into itself, for $1 \leqslant j \leqslant n$, then

$$\left\{g \in L_1^{p^*}(\partial\Omega): \nabla_{\tan} g \in \left[B_s^{p,q}(\partial\Omega)^n\right]_{\tan}\right\} = \left\{g \in L_1^{p^*}(\partial\Omega): \nabla_{\tan} g \in B_s^{p,q}(\partial\Omega)^n\right\}. \tag{5.43}$$

Thus, if $\partial\Omega \in C^{1,\varepsilon}$ with $\varepsilon > s$, then the spaces appearing in (5.43) further take the more familiar form $B^{p,q}_{1+s}(\partial\Omega)$.

(ii) Although, as proved above, the trace operator in (5.41) is onto, this does not, generally speaking, have a universal, linear, bounded, right-inverse, even in the class of convex domains. Indeed, if this were the case, then $X_s^p(\partial\Omega) := \{g \in L_1^{p^*}(\partial\Omega): \nabla_{\tan} g \in [B_s^{p,p}(\partial\Omega)^n]_{\tan}\}$ would be a retract of $F^{p,q}_{1+s+1/p}(\Omega)$. In turn, this would imply that $\{X_s^p(\partial\Omega)\}_{p,s}$ is a complex interpolation scale (in the sense that $[X_{s_0}^{p_0}(\partial\Omega), X_{s_1}^{p_1}(\partial\Omega)]_\theta = X_s^p(\partial\Omega)$ if $\theta \in (0,1)$ and $1/p = (1-\theta)/p_0 + \theta/p_1$). Now, $(\Delta, \operatorname{Tr}): F^{p,q}_{1+s+1/p}(\Omega) \to F^{p,q}_{s+1/p-1}(\Omega) \oplus X_s^p(\partial\Omega)$ is a bounded, linear operator for all indices and, granted the current working assumption, would be an isomorphism when $p = q = 2$ and $s = 1/2$, whenever Ω is a bounded convex domain (here Theorem 5.5 is also used). As a consequence of the fact that being an isomorphism is a stable property on complex interpolation scales (cf., e.g., [44] for general results of this type), we would then be able to conclude (specializing the above discussion to the case when $s = 1 - 1/p$ and $q = 2$) that the problem $\Delta u = f \in L^p(\Omega)$, $u \in W^{2,p}(\Omega)$, $\operatorname{Tr} u = 0$ on $\partial\Omega$, continues to be uniquely solvable for some $p > 2$. This, however, contradicts the counterexamples in [31,4].

To proceed, it is natural to make the following definition (extending earlier considerations in [34]).

Definition 5.11. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and denote by ν the outward unit normal to $\partial\Omega$. Also, suppose that $1 < p, q < \infty$ and $0 < s < 1$. Then introduce

$$NB_s^{p,q}(\partial\Omega) := \big\{g \in L^p(\partial\Omega):\ g\nu_j \in B_s^{p,q}(\partial\Omega),\ 1 \leqslant j \leqslant n\big\}, \tag{5.44}$$

where the ν_j's are the components of ν. This space is equipped with the natural norm

$$\|g\|_{NB_s^{p,q}(\partial\Omega)} := \sum_{j=1}^{n} \|g\nu_j\|_{B_s^{p,q}(\partial\Omega)}. \tag{5.45}$$

As we shall see momentarily, the above spaces are closely related to the standard Besov scale on $\partial\Omega$ (the acronym *NB* stands for "new Besov"), to which they reduce if the domain is sufficiently smooth.

Lemma 5.12. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and fix $1 < p, q < \infty$, $0 < s < 1$. Then $NB_s^{p,q}(\partial\Omega)$ is a reflexive Banach space which embeds continuously into $L^p(\partial\Omega)$.*

Furthermore, if the multiplication operators by the components of the unit normal are bounded on $B_s^{p,q}(\partial\Omega)$, then $NB_s^{p,q}(\partial\Omega) = B_s^{p,q}(\partial\Omega)$. Thus, in particular, this is the case when Ω is a bounded domain whose boundary is of class $C^{1,\varepsilon}$, with $\varepsilon > 1/p$.

Proof. Obviously we have

$$g = \sum_{j=1}^{n} \nu_j(g\nu_j) \quad \text{for any function } g \in L^p(\partial\Omega), \tag{5.46}$$

so that, in particular, $\|g\|_{L^p(\partial\Omega)} \leqslant n\|g\|_{NB_s^{p,q}(\partial\Omega)}$. Consequently, we obtain that the natural inclusion $NB_s^{p,q}(\partial\Omega) \hookrightarrow L^p(\partial\Omega)$ is bounded. If $\{g_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $NB_s^{p,q}(\partial\Omega)$ then, for each $j \in \{1,\dots,n\}$, $\{g_k\nu_j\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $B_s^{p,q}(\partial\Omega)$ and, from what we have proved so far, $\{g_k\}_{k\in\mathbb{N}}$ converges in $L^p(\partial\Omega)$ to some $g \in L^p(\partial\Omega)$. It follows that $\{g_k\nu_j\}_{k\in\mathbb{N}}$ converges in $L^p(\partial\Omega)$ to $g\nu_j$ for each $j \in \{1,\dots,n\}$. With this in hand, it is then easy to conclude that g is the limit of $\{g_k\}_{k\in\mathbb{N}}$ in $NB_s^{p,q}(\partial\Omega)$. This proves that $NB_s^{p,q}(\partial\Omega)$ is Banach. Next, if we consider

$$\Phi : NB_s^{p,q}(\partial\Omega) \longrightarrow \big[B_s^{p,q}(\partial\Omega)\big]^n, \qquad \Phi(g) := (g\nu_j)_{1\leqslant j\leqslant n}, \tag{5.47}$$

it follows that Φ is an isometric embedding, which allows identifying $NB_s^{p,q}(\partial\Omega)$ with a closed subspace of the reflexive space $[B_s^{p,q}(\partial\Omega)]^n$. As is well known, this implies that $NB_s^{p,q}(\partial\Omega)$ is also reflexive. Finally, the claims in the second part in the statement of the lemma are direct consequences of (5.46). $\square$

Our interest in the space $NB_s^{p,q}(\partial\Omega)$, $1 < p, q < \infty$, $0 < s < 1$, stems from the fact that this arises naturally when considering the Neumann trace operator acting from

$$\big\{u \in A_{1+s+1/p}^{p,q}(\Omega):\ \gamma_N(u) = 0\big\} = A_{1+s+1/p}^{p,q}(\Omega) \cap A_{s+1/p,z}^{p,q}(\Omega), \quad A \in \{B, F\}, \tag{5.48}$$

considered as a closed subspace of $A^{p,q}_{1+s+1/p}(\Omega)$ (hence, a Banach space when equipped with the inherited norm). More specifically, we have

Lemma 5.13. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and that $1 < p, q < \infty$ and $0 < s < 1$. Then the Neumann trace operator γ_N considered in the contexts of*

$$\gamma_N : B^{p,q}_{1+s+1/p}(\Omega) \cap B^{p,q}_{s+1/p,z}(\Omega) \longrightarrow NB^{p,q}_s(\partial\Omega), \tag{5.49}$$

$$\gamma_N : F^{p,q}_{1+s+1/p}(\Omega) \cap F^{p,q}_{s+1/p,z}(\Omega) \longrightarrow NB^{p,p}_s(\partial\Omega), \tag{5.50}$$

is, in each case, well defined, linear, bounded, onto and with a linear, bounded right inverse. In addition, the null-spaces of γ_N in (5.49) *and* (5.50) *are, respectively, $B^{p,q}_{1+s+1/p,z}(\Omega)$ and $F^{p,q}_{1+s+1/p,z}(\Omega)$, so that, in particular,*

$$NB^{p,q}_s(\partial\Omega) \text{ is isomorphic to } \frac{B^{p,q}_{1+s+1/p}(\Omega) \cap B^{p,q}_{s+1/p,z}(\Omega)}{B^{p,q}_{1+s+1/p,z}(\Omega)}, \tag{5.51}$$

$$NB^{p,p}_s(\partial\Omega) \text{ is isomorphic to } \frac{F^{p,q}_{1+s+1/p}(\Omega) \cap F^{p,q}_{s+1/p,z}(\Omega)}{F^{p,q}_{1+s+1/p,z}(\Omega)}. \tag{5.52}$$

Proof. To prove the well-definiteness of (5.49), note that if $u \in B^{p,q}_{1+s+1/p}(\Omega) \cap B^{p,q}_{s+1/p,z}(\Omega)$ then (2.109) and Theorem 5.9 give

$$\big(0, \gamma_N(u)\big) = \gamma(u) \in \big\{(g_0, g_1) \in L^p_1(\partial\Omega) \oplus L^p(\partial\Omega) \colon\ \nabla_{\tan} g_0 + g_1\nu \in \big(B^{p,q}_s(\partial\Omega)\big)^n\big\} \tag{5.53}$$

from which we deduce that $\gamma_N(u) \in NB^{p,q}_s(\partial\Omega)$ and $\|\gamma_N(u)\|_{NB^{p,q}_s(\partial\Omega)} \leqslant C\|u\|_{B^{p,q}_{1+s+1/p}(\Omega)}$ for some $C = C(\Omega, p, q, s) > 0$ independent of u. This shows that (5.49) is well defined, linear and bounded. Moving on, denote by $\mathcal{E}$ a linear, bounded, right inverse for γ in (5.34). Then, if

$$\iota : NB^{p,q}_s(\partial\Omega) \to \big\{(g_0, g_1) \in L^p_1(\partial\Omega) \oplus L^p(\partial\Omega) \colon\ \nabla_{\tan} g_0 + g_1\nu \in \big(B^{p,q}_s(\partial\Omega)\big)^n\big\} \tag{5.54}$$

is the injection given by $\iota(g) := (0, g)$, for every $g \in NB^{p,q}_s(\partial\Omega)$, it follows that the composition $\mathcal{E} \circ \iota : NB^{p,q}_s(\partial\Omega) \to B^{p,q}_{1+s+1/p}(\Omega) \cap B^{p,q}_{s+1/p,z}(\Omega)$ is a linear, bounded, right inverse for the operator γ_N in (5.49). As a consequence, this operator is onto. Finally, the fact that the null-space of γ_N in (5.49) is precisely $B^{p,q}_{1+s+1/p,z}(\Omega)$ follows from its definition and the last part in the statement of Theorem 5.9. The proof on the Triebel–Lizorkin scale is analogous and this finishes the proof of the lemma. □

Our goal is to use the above Neumann trace result in order to extend the action of the trace operator Tr from Theorem 2.7 to the space $\{u \in A^{p,q}_{1/p-s}(\Omega) \colon\ \Delta u \in A^{p,q}_{1/p-s}(\Omega)\}$, which we consider equipped with the graph norm $u \mapsto \|u\|_{A^{p,q}_{1/p-s}(\Omega)} + \|\Delta u\|_{A^{p,q}_{1/p-s}(\Omega)}$, $A \in \{B, F\}$. To state our next result, we agree that, given a Banach space $\mathcal{X}$, the pairing ${}_{\mathcal{X}^*}\langle \Lambda, X\rangle_{\mathcal{X}}$ is the duality matching between a functional $\Lambda \in \mathcal{X}^*$ and a vector $X \in \mathcal{X}$. In order to streamline notation, let us also make the following definition.

Definition 5.14. Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and $1 < p, q < \infty$, $0 < s < 1$, set

$$NB_{-s}^{p,q}(\partial\Omega) := \big(NB_s^{p',q'}(\partial\Omega)\big)^*, \tag{5.55}$$

where $1/p + 1/p' = 1/q + 1/q' = 1$.

We have

Theorem 5.15. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and that $1 < p, q < \infty$, $1/p + 1/p' = 1/q + 1/q' = 1$, and $0 < s < 1$. Then there exists a unique linear, bounded operator*

$$\widehat{\gamma}_D : \big\{u \in B_{1/p-s}^{p,q}(\Omega):\ \Delta u \in B_{1/p-s}^{p,q}(\Omega)\big\} \longrightarrow NB_{-s}^{p,q}(\partial\Omega) \tag{5.56}$$

which is compatible with the trace Tr *from Theorem* 2.7, *in the sense that, for each smoothness index $\alpha \in (1/p, 1 + 1/p)$, one has*

$$\widehat{\gamma}_D(u) = \operatorname{Tr} u \quad \text{for every } u \in B_\alpha^{p,q}(\Omega) \text{ with } \Delta u \in B_{1/p-s}^{p,q}(\Omega). \tag{5.57}$$

Furthermore, this extension of the trace operator has dense range and it allows for the following generalized integration by parts formula

$$\begin{aligned} &{}_{NB_s^{p',q'}(\partial\Omega)}\big\langle \gamma_N(w), \widehat{\gamma}_D(u)\big\rangle_{NB_{-s}^{p,q}(\partial\Omega)} \\ &\quad = {}_{(B_{1/p-s}^{p,q}(\Omega))^*}\langle \Delta w, u\rangle_{B_{1/p-s}^{p,q}(\Omega)} - {}_{(B_{1/p-s}^{p,q}(\Omega))^*}\langle w, \Delta u\rangle_{B_{1/p-s}^{p,q}(\Omega)}, \end{aligned} \tag{5.58}$$

valid for every

$$u \in B_{1/p-s}^{p,q}(\Omega) \quad \text{with } \Delta u \in B_{1/p-s}^{p,q}(\Omega) \quad \text{and} \quad w \in B_{1+s+1/p'}^{p',q'}(\Omega) \cap B_{s+1/p',z}^{p',q'}(\Omega). \tag{5.59}$$

Finally, similar results are valid for the Triebel–Lizorkin scale, in which case

$$\widehat{\gamma}_D : \big\{u \in F_{1/p-s}^{p,q}(\Omega):\ \Delta u \in F_{1/p-s}^{p,q}(\Omega)\big\} \longrightarrow NB_{-s}^{p,p}(\partial\Omega), \tag{5.60}$$

in a linear and bounded fashion.

Proof. Let $u \in B_{1/p-s}^{p,q}(\Omega)$ be such that $\Delta u \in B_{1/p-s}^{p,q}(\Omega)$. We attempt to define a functional $\widehat{\gamma}_D(u) \in (NB_s^{p'q'}(\partial\Omega))^*$ as follows. Assume an arbitrary function $g \in NB_s^{p'q'}(\partial\Omega)$ has been given. By Lemma 5.13, there exists some $w \in B_{1+s+1/p'}^{p',q'}(\Omega) \cap B_{s+1/p',z}^{p'q'}(\Omega)$ such that $\gamma_N(w) = g$ and $\|w\|_{B_{1+s+1/p'}^{p',q'}(\Omega)} \leqslant C\|g\|_{NB_s^{p'q'}(\partial\Omega)}$ for some constant $C = C(\Omega, p, q, s) > 0$, independent of g. We then set

$$\begin{aligned} {}_{NB_s^{p',q'}(\partial\Omega)}\big\langle g, \widehat{\gamma}_D(u)\big\rangle_{(N_s^{p',q'}(\partial\Omega))^*} &:= {}_{(B_{1/p-s}^{p,q}(\Omega))^*}\langle \Delta w, u\rangle_{B_{1/p-s}^{p,q}(\Omega)} \\ &\quad - {}_{(B_{1/p-s}^{p,q}(\Omega))^*}\langle w, \Delta u\rangle_{B_{1/p-s}^{p,q}(\Omega)}. \end{aligned} \tag{5.61}$$

The first-order of business is to show that the above definition does not depend on the particular choice of w, with the properties listed above. By linearity, this comes down to proving the following claim: If u is as before and $w \in B^{p',q'}_{1+s+1/p'}(\Omega) \cap B^{p',q'}_{s+1/p',z}(\Omega)$ is such that $\gamma_N(w) = 0$ then

$$ {}_{(B^{p,q}_{1/p-s}(\Omega))^*}\langle \Delta w, u\rangle_{B^{p,q}_{1/p-s}(\Omega)} = {}_{(B^{p,q}_{1/p-s}(\Omega))^*}\langle w, \Delta u\rangle_{B^{p,q}_{1/p-s}(\Omega)}. \tag{5.62} $$

However, since Lemma 5.13 gives that $w \in B^{p'q'}_{1+s+1/p',z}(\Omega)$, it follows that w can be approximated with functions form $C^\infty_c(\Omega)$ in the norm of $B^{p',q'}_{1+s+1/p'}(\Omega)$, by Proposition 2.6. With this in hand, (5.62) follows via a standard limiting argument. Thus, formula (5.61) yields a well-defined functional $\widehat{\gamma}_D(u) \in (NB^{p',q'}_s(\partial\Omega))^*$ which satisfies

$$ \|\widehat{\gamma}_D(u)\|_{NB^{p,q}_{-s}(\partial\Omega)} \leqslant C\big[\|u\|_{B^{p,q}_{1/p-s}(\Omega)} + \|\Delta u\|_{B^{p,q}_{1/p-s}(\Omega)}\big]. \tag{5.63} $$

Thus, the operator (5.56) is well defined, linear and bounded. Also, by definition, this operator will satisfy (5.58).

Next, we will show that (5.57) is valid whenever $\alpha \in (1/p, 1 + 1/p)$. Fix such a number α along with some function $u \in B^{p,q}_\alpha(\Omega)$ with $\Delta u \in B^{p,q}_{1/p-s}(\Omega)$. In particular, $\mathrm{Tr}\, u \in B^{p,q}_{\alpha-1/p}(\partial\Omega)$. We shall make use of a density result to the effect that if

$$ 1 < p, q < \infty, \qquad -1 + 1/p < \beta < 1/p, \qquad \alpha < 2 - \beta, \tag{5.64} $$

then

$$ C^\infty(\overline{\Omega}) \hookrightarrow \big\{u \in B^{p,q}_\alpha(\Omega)\colon\ \Delta u \in B^{p,q}_\beta(\Omega)\big\} \quad \text{densely}, \tag{5.65} $$

where the latter space is equipped with the natural graph norm $u \mapsto \|u\|_{B^{p,q}_\alpha(\Omega)} + \|\Delta u\|_{B^{p,q}_\beta(\Omega)}$. On the scale of L^2-based Sobolev spaces, this appears as Lemma 1.5.3.9 on p. 60 of [36] when $\alpha = 1$, $\beta = 0$, and in [17] when $\alpha < 2$, $\beta = 0$. The case of Besov spaces considered here is dealt with similarly. Then (5.57) follows as soon as we show that

$$ \begin{aligned} {}_{L^{p'}(\partial\Omega)}\big\langle\gamma_N(w), \mathrm{Tr}\, u\big\rangle_{L^p(\partial\Omega)} &= {}_{(B^{p,q}_{1/p-s}(\Omega))^*}\langle \Delta w, u\rangle_{B^{p,q}_{1/p-s}(\Omega)} \\ &\quad - {}_{(B^{p,q}_{1/p-s}(\Omega))^*}\langle w, \Delta u\rangle_{B^{p,q}_{1/p-s}(\Omega)} \end{aligned} \tag{5.66} $$

whenever $u \in C^\infty(\overline{\Omega})$ and $w \in B^{p',q'}_{1+s+1/p'}(\Omega)$ with $\mathrm{Tr}\, w = 0$. To this end, consider $w_j \in C^\infty(\overline{\Omega})$, $j \in \mathbb{N}$, such that $w_j \to w$ in $B^{p',q'}_{1+s+1/p'}(\Omega)$. Then passing to the limit $j \to \infty$ in Green's formula

$$ \int_{\partial\Omega} \gamma_N(w_j)\,\mathrm{Tr}\, u\, d\sigma - \int_{\partial\Omega} \mathrm{Tr}\, w_j\, \gamma_N(u)\, d\sigma = \int_\Omega \Delta w_j u\, dx - \int_\Omega \Delta u w_j\, dx, \tag{5.67} $$

readily yields (5.66). This concludes the proof of (5.57).

Next, we aim to establish the uniqueness of an operator satisfying (5.56)–(5.57). This, however, is again a consequence of (5.64)–(5.65). There remains to show that the trace operator (5.56) has dense range. With this goal in mind, granted (5.64)–(5.65), it suffices to show that

$$\big\{u|_{\partial\Omega}\colon\ u\in C^\infty(\overline{\Omega})\big\}\ \text{is a dense subspace of}\ NB^{p,q}_{-s}(\partial\Omega). \tag{5.68}$$

Going further, (5.68) will follow as soon as we prove that

$$\text{if}\ \Lambda\in\big(NB^{p,q}_{-s}(\partial\Omega)\big)^*\ \text{vanishes on}\ \mathrm{Tr}\big[C^\infty(\overline{\Omega})\big]\ \text{then necessarily}\ \Lambda=0. \tag{5.69}$$

To this end, fix a functional Λ as in the first part of (5.69), recall (5.55) and note that since $NB^{p',q'}_s(\partial\Omega)$ is a reflexive Banach space, continuously embedded into $L^{p'}(\partial\Omega)$ (cf. Lemma 5.12), we may conclude that $\Lambda\in NB^{p',q'}_s(\partial\Omega)\hookrightarrow L^{p'}(\partial\Omega)$. Together with Lemma 5.13, this further shows that there exists a function $w\in B^{p',q'}_{1+s+1/p'}(\Omega)\cap B^{p',q'}_{s+1/p',z}(\Omega)$ with the property that $\gamma_N(w)=\Lambda$. Consequently,

$${}_{NB^{p',q'}_s(\partial\Omega)}\big\langle\gamma_N(w),\mathrm{Tr}\,u\big\rangle_{NB^{p,q}_{-s}(\partial\Omega)}=0\quad\text{for all}\ u\in C^\infty(\overline{\Omega}). \tag{5.70}$$

With (5.70) in hand, the integration by parts formula (5.58) in Theorem 5.15 then yields

$${}_{(B^{p,q}_{1/p-s}(\Omega))^*}\langle\Delta w,u\rangle_{B^{p,q}_{1/p-s}(\Omega)}={}_{(B^{p,q}_{1/p-s}(\Omega))^*}\langle w,\Delta u\rangle_{B^{p,q}_{1/p-s}(\Omega)}\quad\text{for all}\ u\in C^\infty(\overline{\Omega}). \tag{5.71}$$

On the other hand, since $w\in B^{p',q'}_{1+s+1/p'}(\Omega)$, for every $u\in C^\infty(\overline{\Omega})$ we may write

$$\begin{aligned}{}_{(B^{p,q}_{1/p-s}(\Omega))^*}\langle\Delta w,u\rangle_{B^{p,q}_{1/p-s}(\Omega)}&=\int\limits_{\partial\Omega}\gamma_N(w)\,\mathrm{Tr}\,u\,d\sigma-\int\limits_{\partial\Omega}\mathrm{Tr}\,w\,\gamma_N(u)\,d\sigma\\&\quad+{}_{(B^{p,q}_{1/p-s}(\Omega))^*}\langle w,\Delta u\rangle_{B^{p,q}_{1/p-s}(\Omega)}.\end{aligned} \tag{5.72}$$

Upon recalling that we also have $w\in B^{p',q'}_{s+1/p',z}(\Omega)$ and $\Lambda=\gamma_N(w)$, based on (5.71)–(5.72) we deduce that

$$\int\limits_{\partial\Omega}\Lambda\,\mathrm{Tr}\,u\,d\sigma=0\quad\text{for all}\ u\in C^\infty(\overline{\Omega}). \tag{5.73}$$

Since, as is well known,

$$\mathrm{Tr}\big[C^\infty(\overline{\Omega})\big]\ \text{is dense in}\ L^p(\partial\Omega), \tag{5.74}$$

it follows from (5.73) that $\Lambda=0$ in $L^{p'}(\partial\Omega)$, completing the justification of (5.69). The case of the operator (5.60) is treated analogously and this finishes the proof of the theorem. □

We conclude this section with a discussion aimed at showing that, in the class of bounded Lipschitz domains $\Omega\subset\mathbb{R}^n$, the space $NB^{p,q}_s(\partial\Omega)$ is nontrivial. One line of reasoning is to note that, thanks to (5.51)–(5.52), this comes down to checking whether

$$B^{p,q}_{1+s+1/p,z}(\Omega) \neq B^{p,q}_{1+s+1/p}(\Omega) \cap B^{p,q}_{s+1/p,z}(\Omega),$$
$$F^{p,q}_{1+s+1/p,z}(\Omega) \neq F^{p,q}_{1+s+1/p}(\Omega) \cap F^{p,q}_{s+1/p,z}(\Omega). \tag{5.75}$$

Take the second relation in (5.75) in the case when $n = 2$, $p = q = 2$ and $s = 1/2$. In this setting, consider the sector $\Omega_\alpha := \{re^{i\theta}\colon\ 0 < \theta < \alpha,\ 0 < r < 1\}$ where $\alpha \in (0, \pi)$. Also, pick $\psi \in C^\infty(\mathbb{R}^2)$ with $\psi(z) = 1$ for $|z| \leqslant 1/4$ and $\psi(z) = 0$ for $|z| > 1/2$. Then the function $u(z) := \psi(z)\,\mathrm{Im}(z^{\pi/\alpha}) = \psi(re^{i\theta})r^{\pi/\alpha}\sin(\pi\theta/\alpha)$, if $z = re^{i\theta}$, satisfies

$$\int_{\Omega_\alpha} |\nabla^2 u|\,dx\,dy \leqslant C \int_0^{1/2} r^{2(\pi/\alpha-2)+1}\,dr < +\infty, \tag{5.76}$$

since $\alpha < \pi$. Consequently, for $\alpha \in (0, \pi)$, we have that $u \in W^{2,2}(\Omega_\alpha)$. Since also $\mathrm{Tr}\,u = 0$ on $\partial\Omega_\alpha$, we may conclude that

$$u \in F^{2,2}_2(\Omega_\alpha) \cap F^{2,2}_{1,z}(\Omega_\alpha). \tag{5.77}$$

As such, the desired conclusion follows as soon as we show that $u \notin F^{2,2}_{2,z}(\Omega_\alpha)$. To justify this, note that near the origin $u(z) = \mathrm{Re}(z^{\pi/\alpha})$, a harmonic function in Ω_α, which has $v(z) := \mathrm{Im}(z^{\pi/\alpha})$ as a harmonic conjugate in Ω_α. Thus, by the Cauchy–Riemann equations, we have $|\gamma_N(u)| = |\nabla_{\tan}(v|_{\partial\Omega_\alpha})|$ a.e. near the point $0 \in \partial\Omega_\alpha$. Now, since $v|_{\partial\Omega_\alpha}$ is not constant near $0 \in \partial\Omega_\alpha$, its tangential gradient does not vanish identically, so $\gamma_N(u) \neq 0$ on $\partial\Omega_\alpha$. Thus, $u \notin F^{2,2}_{2,z}(\Omega_\alpha)$, as wanted. In general, the aforementioned nontriviality statement can be seen from the following result.

Corollary 5.16. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and that $1 < p, q < \infty$, $0 < s < 1$. Then*

$$\big\{u|_{\partial\Omega}\colon\ u \in C^\infty(\overline{\Omega})\big\} \text{ is a dense subspace of } NB^{p,q}_{-s}(\partial\Omega). \tag{5.78}$$

Proof. Given that the map (5.56) has dense range, this is a consequence of (5.64)–(5.65). □

5.3. Noncanonical Poisson problems

Having extended the action of the trace operator beyond the canonical range of Theorem 2.7, here the goal is to establish the well-posedness of the Poisson problem, for the Laplacian with Dirichlet boundary condition, in settings when one either has more smoothness (cf. Theorem 5.17), or less smoothness (cf. Theorem 5.19), than in Theorem 4.12.

Theorem 5.17. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and that $0 < p, q \leqslant \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$. Set $\alpha := 1/p + s - 1$. In addition, suppose that either*

(i) *the domain Ω is convex and* (5.19) *holds, or*
(ii) *the domain Ω is semiconvex and* (5.17) *holds.*

Then the boundary value problem

$$\begin{cases} \Delta u = f \in B^{p,q}_{s+1/p-1}(\Omega), \\ u \in B^{p,q}_{1+s+1/p}(\Omega), \\ \operatorname{Tr} u = g \quad \text{on } \partial\Omega, \end{cases} \tag{5.79}$$

has a solution if and only if

$$g \in L^{p^*}_1(\partial\Omega) \quad \text{and} \quad \nabla_{\tan} g \in \big[B^{p,q}_s(\partial\Omega)^n\big]_{\tan}, \tag{5.80}$$

where $p^* := (\frac{1}{p} - \frac{s}{n-1})^{-1} \in (1, \frac{n-1}{n-2})$ *if* $p \leqslant 1$, *and* $p^* := p$ *if* $p > 1$. *Furthermore, in the case when* g *is as in* (5.80), *the solution* u *of* (5.79) *is unique.*

Finally, a similar statement is valid on the Triebel–Lizorkin scale, i.e. for

$$\begin{cases} \Delta u = f \in F^{p,q}_{s+1/p-1}(\Omega), \\ u \in F^{p,q}_{1+s+1/p}(\Omega), \\ \operatorname{Tr} u = g \quad \text{on } \partial\Omega \end{cases} \tag{5.81}$$

(*assuming that, in addition,* $p, q < \infty$). *In this case,* (5.81) *is solvable, if and only if*

$$g \in L^{p^*}_1(\partial\Omega) \quad \text{and} \quad \nabla_{\tan} g \in \big[B^{p,p}_s(\partial\Omega)^n\big]_{\tan}, \tag{5.82}$$

and the solution is unique, whenever the boundary datum is as in (5.82).

Proof. In one direction, if (5.79) has a solution u, then Theorem 5.10 gives that $g = \operatorname{Tr} u$ satisfies (5.80). Conversely, take g as in (5.80), and let $v \in B^{p,q}_{1+s+1/p}(\Omega)$ be such that $\operatorname{Tr} v = g$ (this is possible by Theorem 5.10). If we now set $w := \mathbb{G}(\Delta v - f)$ then by Theorem 5.6 when Ω is a semiconvex domain, and by Theorem 5.7 when Ω is convex, it follows that

$$\begin{cases} \Delta w = \Delta v - f \in B^{p,q}_{s+1/p-1}(\Omega), \\ w \in B^{p,q}_{1+s+1/p}(\Omega), \quad \operatorname{Tr} w = 0. \end{cases} \tag{5.83}$$

Hence, $u := v - w$ solves (5.79). Uniqueness for (5.79) follows from the uniqueness part in Theorem 4.12 and standard embedding results on the Besov scale in Ω. Finally, the case of (5.81) is treated analogously. □

Specializing the above theorem to the case when $1 < p \leqslant 2$ and $s = 1 - 1/p$ then yields the following.

Corollary 5.18. *Assume that* $\Omega \subset \mathbb{R}^n$ *is a bounded semiconvex domain. Then for each* $1 < p \leqslant 2$, *the problem*

$$\Delta u = f \in L^p(\Omega), \quad u \in W^{2,p}(\Omega), \qquad \operatorname{Tr} u = g \quad \text{on } \partial\Omega, \tag{5.84}$$

has a unique solution if and only if

$$g \in L_1^p(\partial\Omega) \quad \textit{and} \quad \nabla_{\tan} g \in \left[B_s^{p,p}(\partial\Omega)^n\right]_{\tan}. \tag{5.85}$$

In the case when $g = 0$, this corollary is well known; cf. [42,3,31,50].

The case when the Poisson problem is formulated with the Dirichlet boundary condition interpreted in the sense of Theorem 5.15 is discussed next. Before doing so, we note that this theorem can be regarded as the dual statement corresponding to Theorem 5.17 (this will become more apparent from an examination of its proof).

Theorem 5.19. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and that $1 < p, q < \infty$, $0 < s < 1$. Denote by p', q' the conjugate exponents of p, q, and set $\alpha := 1/p + s - 1$. In addition, suppose that either*

(i) *the domain Ω is convex and* (5.19) *holds, or*
(ii) *the domain Ω is semiconvex and* (5.17) *holds.*

Then each of the following boundary value problems

$$\begin{cases} \Delta u = f \in F^{p',q'}_{1/p'-s}(\Omega), \\ u \in F^{p',q'}_{1/p'-s}(\Omega), \\ \widehat{\gamma}_D(u) = g \in NB^{p',p'}_{-s}(\partial\Omega), \end{cases} \qquad \begin{cases} \Delta u = f \in B^{p',q'}_{1/p'-s}(\Omega), \\ u \in B^{p',q'}_{1/p'-s}(\Omega), \\ \widehat{\gamma}_D(u) = g \in NB^{p',q'}_{-s}(\partial\Omega) \end{cases} \tag{5.86}$$

has a unique solution, which in addition satisfies a natural estimate.

Proof. Consider the first problem in (5.86) (the second one is treated similarly), and observe that, by subtracting a suitable Newtonian potential, there is no loss of generality in assuming that $f = 0$. To continue, assume that $g \in \{\psi|_{\partial\Omega}\colon \psi \in C^\infty(\overline{\Omega})\}$ and set

$$u(y) := -\int_{\partial\Omega} \partial_{\nu(x)} G(x,y) g(x)\, d\sigma(x), \quad y \in \Omega. \tag{5.87}$$

Since $u|_{\partial\Omega} = g$ and $\Delta u = 0$ in Ω, via a density argument based on Corollary 5.16 and Theorem 5.15, it suffices to show that there exists a finite constant $C = C(\Omega, p, q, s) > 0$ such that

$$\|u\|_{F^{p',q'}_{1/p'-s}(\Omega)} \leqslant C\|g\|_{NB^{p',q'}_{-s}(\partial\Omega)}. \tag{5.88}$$

With this goal in mind, we proceed by duality and note that for every $v \in C^\infty(\overline{\Omega})$, we have

$$\begin{aligned} \int_\Omega u(y)v(y)\,dy &= -\int_{\partial\Omega} g(x)\partial_{\nu(x)}\left(\int_\Omega G(x,y)v(y)\,dy\right) d\sigma(x) \\ &= \int_{\partial\Omega} g\,\partial_\nu(\mathbb{G}v)\,d\sigma = \int_\Omega (\gamma_N \circ \mathbb{G})^* g(y)v(y)\,dy, \end{aligned} \tag{5.89}$$

so that, further,

$$u = (\gamma_N \circ \mathbb{G})^* g. \tag{5.90}$$

Above, $(\gamma_N \circ \mathbb{G})^*$ is the adjoint of the composition

$$\gamma_N \circ \mathbb{G} : F^{p,q}_{s+1/p-1}(\Omega) \xrightarrow{\mathbb{G}} F^{p,q}_{1+s+1/p}(\Omega) \cap F^{p,q}_{s+1/p,z}(\Omega) \xrightarrow{\gamma_N} NB^{p,q}_s(\partial\Omega). \tag{5.91}$$

Since by Theorem 5.6, Theorem 5.7 and Lemma 5.13 each of the above arrows is a bounded assignment, it follows from this, Definition 5.14 and (2.103) that

$$(\gamma_N \circ \mathbb{G})^* : NB^{p',q'}_{-s}(\partial\Omega) \longrightarrow F^{p',q'}_{1/p'-s}(\Omega) \tag{5.92}$$

is a well-defined, linear and bounded operator. Wit this in hand, (5.88) follows from (5.90).

At this stage, there remains to establish uniqueness. Thus, assume that u satisfies

$$\Delta u = 0 \quad \text{in } \Omega, \quad u \in F^{p',q'}_{1/p'-s}(\Omega), \qquad \widehat{\gamma}_D(u) = 0 \quad \text{in } NB^{p',q'}_{-s}(\partial\Omega). \tag{5.93}$$

Then, for an arbitrary $v \in (F^{p',q'}_{1/p'-s}(\Omega))^* = F^{p,q}_{1/p+s-1}(\Omega)$, consider $w := \mathbb{G}v$ in Ω and note that $w \in F^{p,q}_{1+s+1/p}(\Omega) \cap F^{p,q}_{s+1/p,z}(\Omega)$ by the aforementioned mapping properties of the Green operator. Then the version of the integration by parts formula (5.58)–(5.59) written for the Triebel–Lizorkin scale (and the conjugate integrability indices) gives that

$$\begin{aligned}
&{}_{(F^{p',q'}_{1/p'-s}(\Omega))^*}\langle v, u\rangle_{F^{p',q'}_{1/p'-s}(\Omega)} \\
&\quad = {}_{(F^{p',q'}_{1/p'-s}(\Omega))^*}\langle \Delta w, u\rangle_{F^{p',q'}_{1/p'-s}(\Omega)} \\
&\quad = {}_{NB^{p,p}_s(\partial\Omega)}\big\langle \gamma_N(w), \widehat{\gamma}_D(u)\big\rangle_{NB^{p',p'}_{-s}(\partial\Omega)} + {}_{(F^{p',q'}_{1/p'-s}(\Omega))^*}\langle w, \Delta u\rangle_{F^{p',q'}_{1/p'-s}(\Omega)} \\
&\quad = 0 + 0 = 0.
\end{aligned} \tag{5.94}$$

In turn, since $v \in (F^{p',q'}_{1/p'-s}(\Omega))^*$ was arbitrary, the Hahn–Banach theorem gives that $u = 0$, as desired. □

In closing, we note that Theorem 1.6 is obtained by specializing the above theorem to the case when $1 < p \leqslant 2$ and $s = 1/p$ (and readjusting notation).

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