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# The Porous Medium Equation

## Mathematical Theory

JUAN LUIS VÁZQUEZ

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# The Porous Medium Equation

## *Mathematical Theory*

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*To my wife Mariluz*

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# PREFACE

The heat equation is one of the three classical linear partial differential equations of second order that form the basis of any elementary introduction to the area of partial differential equations. Its success in describing the process of thermal propagation has known a permanent popularity since Fourier's essay *Théorie Analytique de la Chaleur* was published in 1822 [237] and has motivated the continuous growth of mathematics in the form of Fourier analysis, spectral theory, set theory, operator theory, and so on. Later on, it contributed to the development of measure theory and probability, among other topics.

The high regard of the heat equation has not been isolated. A number of related equations have been proposed both by applied scientists and pure mathematicians as objects of study. In a first extension of the field, the theory of linear parabolic equations was developed, with constant and then variable coefficients. The linear theory enjoyed much progress, but it was soon observed that most of the equations modelling physical phenomena without excessive simplification are nonlinear. However, the mathematical difficulties of building theories for nonlinear versions of the three classical partial differential equations (Laplace's equation, the heat equation and the wave equation) made it impossible to make significant progress until the twentieth century was well advanced. And this observation applies to other important nonlinear PDEs or systems of PDEs, like the Navier–Stokes equations.

The great development of functional analysis in the decades from the 1930s to the 1960s made it possible for the first time to start building theories for these nonlinear PDEs with full mathematical rigour. This happened in particular in the area of parabolic equations where the theory of linear and quasilinear parabolic equations in divergence form reached a degree of maturity reflected for instance in the classical books of Ladyzhenskaya et al. [357] and Friedman [239].

The aim of the present text is to provide a systematic presentation of the mathematical theory of the nonlinear heat equation

$$\partial_t u = \Delta(u^m), \quad m > 1\,, \tag{PME}$$

usually called the porous medium equation (PME), posed in $d$-dimensional Euclidean space, with interest in the cases $d = 1, 2, 3$ for the applied scientist, with no dimension restriction for the mathematician. $\Delta = \Delta_x$ represents the Laplace operator acting on the space variables. We will also study the complete form, $u_t = \Delta(|u|^{m-1}u) + f$, but in a less systematic way. Other variants appear in the literature but will be given less attention, since we keep to the idea of presenting a rather complete account of the main results and methods for the basic PME.

The reader may wonder why such a simple-looking variation of the famous and well-known heat equation (HE): $u_t = \Delta u$, needs a book of its own. There are several answers to this question: the theory and properties of the PME depart strongly from the heat equation; it contains interesting and sometimes sophisticated developments of nonlinear analysis; there are a number of interesting applications where this theory, with all its differences, is necessary and useful; and, finally, similar treatises have been written for individual equations with a strong personality. As for the latter argument, we have the example of the heat equation itself, described in the monographs by Cannon [148] and Widder [525], and also the Stefan problem that is closely related to the HE and the PME and was reported in the books of Cannon [148], Rubinstein [454] and Meirmanov [388].

Let us now comment on the first aspects listed some lines above. The theory that has been developed and we present in this text not only settles the main problems of existence, uniqueness, stability, smoothness, dynamical properties and asymptotic behaviour. In doing so, it contributes a wealth of new ideas with respect to the heat equation; great novelties occur also with respect to the standard nonlinear theories, represented by the theory of nonlinear parabolic equations in divergence form to which the porous medium equation belongs. This is due to the fact that the equation is not parabolic at all points, but *only degenerate parabolic*, a fact that has deep mathematical consequences, both qualitative and quantitative. On the other hand, and as a sort of compensation, the equation enjoys a number of nice properties due to its simple form, like scaling invariance. This aspect makes the PME an interesting benchmark in the development of nonlinear analytical tools for the quite general classes of nonlinear, formally parabolic equations that continue to make their way into the pure and applied sciences, and then into the mainstream of mathematics.

There are a number of physical applications where the simple PME model appears in a natural way, mainly to describe processes involving fluid flow, heat transfer or diffusion. Other applications have been proposed in mathematical biology, lubrication, boundary layer theory, and other fields. All of these reasons support the interest of its study both for the mathematician and the scientist.

### *Context*

In spite of the simplicity of the equation and of having some important applications, and due perhaps to its nonlinear and degenerate character, the mathematical theory of the PME has been only gradually developed in the last decades after the seminal paper of Oleinik et al. [408] in 1958; in the 1980s the theory was finally on firm ground and has been rounded up since then. The idea of the book arose out of the participation of the author in this progress in the last three decades. The immediate motivation for writing the text is the feeling that the time is ripe for a reasonably complete version of the mathematics of the PME, once the main mathematical issues have come to be fairly well understood, and every result receives a proof in the style of analysis. We are also aware of the need for researchers to apply to more complex models the wealth of techniques that work so well here, hence the need for clear and balanced expositions to learn the material. Therefore, we aim at providing a description of the questions of existence, uniqueness and the main properties of the solutions, whereby everything is derived from basic estimates using standard functional analysis and well-known PDE results. And we have tried to provide sound physical foundations throughout.

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# 1

# INTRODUCTION

## 1.1 The subject

### 1.1.1 *The porous medium equation*

The aim of the text is to provide a systematic presentation of the mathematical theory of the nonlinear heat equation

$$\partial_t u = \Delta(u^m), \quad m > 1, \tag{PME}$$

usually called the *porous medium equation*, with due attention paid to its closest relatives. The default settings are: $u = u(x, t)$ is a non-negative scalar function of space $x \in \mathbb{R}^d$ and time $t \in \mathbb{R}$, the space dimension is $d \geq 1$, and $m$ is a constant larger than 1. $\Delta = \Delta_x$ represents the Laplace operator acting on the space variables. We will refer to the equation by the label PME. The equation can be posed for all $x \in \mathbb{R}^d$ and $0 < t < \infty$, and then initial conditions are needed to determine the solutions; but it is quite often posed, especially in practical problems, in a bounded subdomain $\Omega \subset \mathbb{R}^d$ for $0 < t < T$, and then determination of a unique solution asks for boundary conditions as well as initial conditions.

This equation is one of the simplest examples of a nonlinear evolution equation of parabolic type. It appears in the description of different natural phenomena, and its theory and properties depart strongly from the heat equation, $u_t = \Delta u$, its most famous relative. Hence the interest of its study, both for the pure mathematician and the applied scientist. We will also discuss in less detail some important variants of the equation.

There are a number of physical applications where this simple model appears in a natural way, mainly to describe processes involving fluid flow, heat transfer or diffusion. Maybe the best known of them is the description of the flow of an isentropic gas through a porous medium, modelled independently by Leibenzon [367] and Muskat [394] around 1930. An earlier application is found in the study of groundwater infiltration by Boussisnesq in 1903 [123]. Another important application refers to heat radiation in plasmas, developed by Zel'dovich and coworkers around 1950 [533]. Indeed, this application was at the base of the rigorous mathematical development of the theory. Other applications have been proposed in mathematical biology, spread of viscous fluids, boundary layer theory, and other fields.

Most physical settings lead to the default restriction $u \geq 0$, which is mathematically convenient and currently followed. However, the restriction is not essential in developing a mathematical theory on the condition of properly defining the nonlinearity for negative values of $u$ so that the equation is still (formally) parabolic. The most used choice is the antisymmetric extension of the nonlinearity, leading to the so-called *signed PME*,

$$\partial_t u = \Delta(|u|^{m-1}u). \tag{sPME}$$

We will also devote much attention to this equation. For brevity, we will often write $u^m$ instead of $|u|^{m-1}u$ even if solutions have negative values in paragraphs where no confusion is to be feared. There is a second important extension, consisting of adding a *forcing term* in the right-hand side to get the complete form

$$\partial_t u = \Delta(|u|^{m-1}u) + f, \tag{cPME}$$

where $f = f(x, t)$. The full form is the natural framework of the abstract functional theory for the PME, and has also received much attention when $f = f(u)$ and represents effects of reaction or absorption. The dependence of $f$ on $\nabla u$ occurs when convection is taken into account. We will cover the complete form in the text, but the information on the qualitative and quantitative aspects is much less detailed in that generality, and we will not enter into the specific properties of reaction–diffusion models. Specially in the second part of the book, we want to concentrate on the plain equation (PME), hence the simple label for that case. The *complete porous medium equation* is also referred to as the PME with a source term, or the *forced PME*.

Equation (PME) for $m = 1$ is the famous *heat equation* (HE), that has a well documented theory, cf. Widder [525]. The equation can also be considered for the range of exponents $m < 1$. Some of the properties in this range are similar to the case $m > 1$ studied here, but others are quite different, and it is called the *fast diffusion equation* (FDE). Since it deserves a text of its own, the FDE will only be covered in passing in this book. Note that when $m < 0$ the FDE has to be written in the 'modified form'

$$\partial_t u = \Delta(u^m/m) = \mathrm{div}(u^{m-1}\nabla u)$$

to keep the parabolic character of the equation. This form of the equation allows us also to include the case $m = 0$ which reads $\partial_t u = \mathrm{div}(u^{-1}\nabla u) = \Delta \log(u)$, and is called logarithmic diffusion.

### 1.1.2 *The PME as a nonlinear parabolic equation*

The PME is an example of nonlinear evolution equation, formally of parabolic type. In a sense, it is the simplest possible nonlinear version of the classical heat equation, which can be considered as the limit $m \to 1$ of the PME. Written in its complete version and in divergence form,

$$\partial_t u = \mathrm{div}(D(u)\nabla u) + f, \tag{1.1}$$

we see that the *diffusion coefficient* $D(u)$ of the PME equals $mu^{m-1}$ assuming $u \geq 0$, and we have $D(u) = m|u|^{m-1}$ for signed solutions ($D(u) = |u|^{m-1}$ in the modified form). It is then clear that the equation is parabolic only at those points where $u \neq 0$, while the vanishing of $D(u)$ is recorded as saying that the PME *degenerates* wherever $u = 0$. In other words, the PME is a *degenerate parabolic equation.* The theory of nonlinear parabolic equations in divergence form deals with the class of nonlinear parabolic equations of the form

$$\partial_t u = \mathrm{div}\mathcal{A}(x, t, u, Du) + \mathcal{B}(x, t, u, Du), \tag{1.2}$$

where the vector function $\mathcal{A} = (A_1, \ldots, A_d)$ and the scalar function $\mathcal{B}$ satisfy suitable structural assumptions and $\mathcal{A}$ satisfies moreover ellipticity conditions. This topic became a main area of research in PDEs in the second half of the last century, when the tools of functional analysis were ready for it. The theory extends to systems of the same form, in which $u = (u_1, \ldots, u_k)$ is a vector variable, $\mathcal{A}$ is an $(m, d)$ matrix and $\mathcal{B}$ is an $m$-vector. Well-known areas, like reaction–diffusion, are included in this generality. There is a large literature on this topic, cf. e.g. the books [239, 357, 482] that we take as reference works.

The change of character of the PME at the level $u = 0$ is most clearly demonstrated when we perform the calculation of the Laplacian of the power function in the case $m = 2$; assuming $u \geq 0$ for simplicity, we obtain the form

$$\partial_t u = 2u\,\Delta u + 2|\nabla u|^2. \tag{1.3}$$

It is immediately clear that in the regions where $u \neq 0$ the leading term in the right-hand side is the Laplacian modified by the variable coefficient $2u$; on the contrary, for $u \to 0$, the equation simplifies into $\partial_t u \sim 2|\nabla u|^2$, the *eikonal equation* (a first-order equation of Hamilton–Jacobi type, that propagates along characteristics). A similar calculation can be done for general $m \neq 1$ after introducing the so-called *pressure variable*, $v = cu^{m-1}$ for some $c \geq 0$. We then get

$$\partial_t v = av\,\Delta v + b\,|\nabla v|^2, \tag{1.4}$$

with $a = m/c$, $b = m/(c(m-1))$. This is a fundamental transformation in the theory of the PME that allows us to get similar conclusions about the behaviour of the equation for $u, v \sim 0$ when $m \neq 2$. The standard choice for $c$ in the literature is $c = m/(m-1)$, because it simplifies the formulas ($a = m-1$, $b = 1$) and makes sense for dynamical considerations (to be discussed in Section 2.1), but $c = 1$ is also used. Mathematically, the choice of constant is not important.

Note that similar considerations apply to the FDE but then

$$D(u) = \frac{m}{|u|^{1-m}} \to \infty \quad \text{as} \quad u \to 0, \tag{1.5}$$

hence the name of fast diffusion which is well deserved when $u \sim 0$. The pressure can be introduced, but being an inverse power of $u$, its role is different from that in the PME. All this shows the kinship and differences from the start between the two equations.

In spite of the simplicity of the equation and of having some important applications, a mathematical theory for the PME has been developed at a slow pace over several decades, due most probably to the fact that it is a nonlinear equation, and also a degenerate one. Though the techniques depart strongly from the linear methods used in treating the heat equation, it is interesting to remark that some of the basic techniques are not very difficult nor need a heavy machinery. What is even more interesting, they can be applied in, or adapted to, the study of many other nonlinear PDEs of parabolic type. The study of the PME can provide the reader with an introduction to, and practice of some interesting concepts and methods of nonlinear science, like the existence of free boundaries, the occurrence of limited regularity, and interesting asymptotic behaviour.

## 1.2 Peculiar features of the PME

When considering the linear and quasilinear parabolic theories, the main questions are asked in comparison to what happens for the heat equation, which is the model from which these theories take their inspiration. Thus, the three main questions of existence, uniqueness, and continuous dependence are posed in the literature, as well as the questions of regularity, the validity of maximum principles, the existence of Harnack inequalities, and so on; in some sense, these comparative questions receive positive answers, though the analogy breaks at some points, thus originating novelty and interest.

### 1.2.1 *Finite propagation and free boundaries*

The same golden rule of comparison with the HE is applied to the theory developed in this book for the PME. The main questions can be posed, but then we see that such questions, though important, do not convey the special flavour of the equation. Indeed, the PME offers a number of very peculiar traits that separate it from the core of the parabolic theory. Mathematically, the difficulties stem from the degenerate character, i.e., the fact that $D(u)$ is not always positive. Explaining the consequences implies changing the way the heat equation theory is developed. We will be led to introducing dynamical concepts to account for the main qualitative difference, which is the property called *finite propagation* that will be precisely formulated and extensively explored in the text, especially in Chapters 14 and 15. This property is in strong contrast with one of the better known properties of the classical heat equation, the infinite speed of propagation, one of the most contested aspects of the HE on physical grounds. Let us express the contrast in simplest terms:

- HE: '*A non-negative solution of the heat equation is automatically positive everywhere in its domain of definition*'; to be compared with
- PME: '*Disturbances from the level* $u = 0$ *propagate in time with finite speed for solutions of the porous medium equation*'.

In a sense, the property of finite propagation supports the physical soundness of the PME to model diffusion or heat propagation.

A first consequence of the finite propagation property for the theory of the PME is that the strong maximum principle cannot hold. On the positive side, it means that, whenever the initial data are zero in some open domain of the space, the property of finite propagation implies the appearance of a *free boundary* that separates the regions where the solution is positive (i.e. where 'there is gas', according to the standard interpretation of $u$ as a gas density, see Chapter 2), from the 'empty region' where $u = 0$. Precisely, we define the free boundary as

$$\Gamma = \partial P_u \cap Q, \tag{1.6}$$

where $Q$ is the domain of definition of the solution in space-time,

$$\mathcal{P}_u = \{(x,t) \in Q : u(x,t) > 0\} \tag{1.7}$$

is the *positivity set*, and $\partial$ denotes boundary. Since $\Gamma$ moves as time passes, it is also called the *moving boundary.* In some cases, especially in one space dimension, the name *interface* is popular.

The theory of free boundaries, or propagation fronts, is an important and difficult subject of the mathematical investigation, covered for instance in the book by A. Friedman [240]. In principle, the free boundary of a nonlinear problem can be a quite complicated closed subset of $Q$. A main problem of the PME theory consists of proving that it is at least a Hölder continuous ($C^\alpha$) hypersurface in $\mathbb{R}^{d+1}$, and then to investigate how smooth it really is. Let us advance that it is often $C^\infty$ smooth, but not always.

Let us illustrate the two main situations that will be encountered. In the first of them, the space domain is $\mathbb{R}^d$, the initial data $u_0$ have compact support, i.e., there exists a bounded closed set $S_0 \subset \mathbb{R}^d$ such that $u_0(x) = 0$ for all $x \neq S_0$. In that case, we will prove that the solution $u(x,t)$ vanishes for all positive times $t > 0$ outside a compact set that changes with time. More precisely, if we define the *positivity set* at time $t$ as $\mathcal{P}_u(t) = \{x \in \mathbb{R}^d : u(x,t) > 0\}$, and the *support* at time $t$ as $\mathcal{S}_u(t)$ as the closure of $\mathcal{P}_u(t)$, then both families of bounded sets are shown to be expanding in time, or more precisely stated, non-contracting. Note that positivity sets and supports are not defined in the everywhere sense unless solutions are continuous; showing continuity of the solutions is a main issue in the PME theory, and it has been a hot topic in nonlinear elliptic and parabolic equations since the seminal papers of De Giorgi, Nash and Moser.

In the second scenario, the initial configuration 'has a hole in the support', i.e., there is a bounded subdomain $D_0 \neq \emptyset$ such that $u_0(x) = 0$ for every $x$ in the closure of $D_0$, and $u_0(x) > 0$ otherwise. Then, the solution has a possibly smaller hole for $t > 0$. The fact that this hole does disappear in finite time (it is filled up), motivates one of the most beautiful mathematical developments of the PME theory, the so-called focusing problem that we will study in Chapter 19.

### 1.2.2 The role of special solutions

Following a standard practice in applied nonlinear analysis and mechanics, before developing a fully fledged theory, the question is posed whether there exist special solutions in explicit or quasi-explicit form that serve as representative examples of the typical or peculiar behaviour. The answer to that question is positive in our case; a reduced number of representative examples have been found and they give both insight and detailed information about the most relevant questions, like existence, finite propagation, optimal continuity, higher smoothness, and so on.

A fundamental example of solution was obtained around 1950 in Moscow by Zel'dovich and Kompaneets [532] and Barenblatt [60], who found and analysed a solution representing heat release from a point source. This solution has the explicit formula

$$\mathcal{U}(x,t) = t^{-\alpha}\left(C - k\,|x|^2 t^{-2\beta}\right)_+^{\frac{1}{m-1}}, \tag{1.8}$$

where $(s)_+ = \max\{s,0\}$,

$$\alpha = \frac{d}{d(m-1)+2}, \qquad \beta = \frac{\alpha}{d}, \qquad k = \frac{\alpha(m-1)}{2\,m\,d} \tag{1.9}$$

and $C > 0$ is an arbitrary constant. The solution was subsequently found by Pattle [418] in 1959. The name *source-type solution* is due to the fact that it takes as initial data a Dirac mass: as $t \to 0$ we have $\mathcal{U}(x,t) \to M\,\delta(x)$, where $M$ is a function of the free constant $C$ (and $m$ and $d$). We will use the shorter term *source solution*, and very often the name *ZKB solution* that looks to us convenient. We recall that the names *Barenblatt solution* and *Barenblatt–Pattle solution* are found in the literature.

An analysis of this example shows many of the important features that we have been talking about. Thus, the source solution has compact support in space for every fixed time, since the free boundary is the surface given by the equation

$$t = c\,|x|^{d(m-1)+2}, \tag{1.10}$$

where $c = c(C,m,d)$. In physical terms, the disturbance propagates with a precise finite speed. This is to be compared with the properties of the Gaussian kernel,

$$E(x,t) = M\,(4\pi t)^{-d/2} \exp\left(-x^2/4t\right), \tag{1.11}$$

which is the source solution for the HE.

There are many other special solutions that have been studied and shed light on different aspects of the theory. Some of the most important will be carefully examined in Chapter 4 and then used in the theory developed in this text. They take the main forms of separate-variables solutions, travelling waves and self-similar solutions. Chapter 16 is entirely devoted to constructing solutions. They play a prominent role in Chapters 18 and 19, where the focusing solutions have a key part in settling the regularity issue.

## 1.3 Nonlinear diffusion. Related equations

The PME is but one example of partial differential equation in the realm of what is called nonlinear diffusion. Work in that wide area has frequent overlaps between the different models, both in phenomena to be described, results to be proved and techniques to be used. A quite general form of nonlinear diffusion equation, as it appears in the specialized literature, is

$$\partial_t H(x,t,u) = \sum_{i=1}^{d} \partial_{x_i}(A_i(x,t,u,Du)). \tag{1.12}$$

Suitable conditions should be imposed on the functions $H$ and $A_i$. In particular, $\partial_u H(x,t,u) \geq 0$ and the matrix $(a_{ij}) = (\partial_{u_j} A_i(x,t,u,Du))$ should be positive semidefinite. If we want to consider reaction and convection effects, the term $\mathcal{B}(x,t,u,Du)$ is added to the right-hand side. A theory for equations in such a generality has been in the making during the last few decades, but the richness of phenomena that are included in the different examples covered in the general formulation precludes a general theory with detailed enough information.

Progress has been quite remarkable on more specialized topics like ours. Let us mention next four natural extensions of the PME in that direction. Though they have some important traits in common with the PME, they are different territories and we think that the deep study deserves a separate text in each case.

(i) Fast diffusion. Much of the theory can be and has been extended to the simplest generalization of the PME consisting of the same formal equation, but now in the range of exponents $m < 1$. Since the diffusion coefficient $D(u) = |u|^{m-1}$ goes now to infinity as $u \to 0$, the equation is called in this new range the *fast diffusion equation*, FDE. In this terminology, the PME becomes a *slow diffusion equation.*

There are strong analogies and also marked differences between the PME and the FDE. For instance, the free boundary theory of the PME disappears for the FDE. We will only make small incursions into it. We refer to the monograph [515] and its references as a source of further information.

(ii) Filtration equations. A further extension is the *generalized porous medium equation,*

$$\partial_t u = \Delta\Phi(u) + f, \tag{GPME}$$

also called the *filtration equation*, specially in the Russian literature; $\Phi$ is an increasing function: $\mathbb{R}_+ \mapsto \mathbb{R}_+$, and usually $f = 0$. The diffusion coefficient is now $D(u) = \Phi'(u)$, and the condition $\Phi'(u) \geq 0$ is needed to make the equation formally parabolic. Whenever $\Phi'(u) = 0$ for some $u \in \mathbb{R}$, we say that the equation degenerates at that $u$-level, since it ceases to be strictly parabolic. This is the cause for more or less serious departures from the standard quasilinear theory, as we have already explained in the PME case.

An important role in the development of the topic of the filtration equation has been played by the *Stefan problem*, a simple but powerful model of phase transition, developed in the study of the evolution of a medium composed of water and ice. It can written as a filtration equation with

$$\Phi(u) = (u-1)_+ \quad \text{for } u \geq 0, \quad \Phi(u) = u \quad \text{for } u < 0. \tag{StE}$$

More generally, we can put $\Phi(u) = c_1(u-L)_+$ for $u \geq 0$, and $\Phi(u) = c_2\, u$ for $u < 0$, where $c_1, c_2$ and $L$ are positive constants. The Stefan problem and the PME have had a somewhat parallel history.

**Note** Due to the interest of other GPME models, we will develop a large part of the basic existence and uniqueness theory of this book for the GPME, and we will then specialize to the PME in the detailed analysis of the last part of the book.

(iii) $p$-LAPLACIAN EVOLUTIONS. There is another popular nonlinear degenerate parabolic equation:

$$\partial_t u = \mathrm{div}(|\nabla u|^{p-2}\nabla u), \tag{PLE}$$

called the *p-Laplacian evolution equation,* PLE, which has also attracted much attention from researchers. It is part of a general theory of diffusion with diffusivity depending on the gradient of the main unknown. It has a parallel, sometimes divergent, sometimes convergent theory. We can combine PME and PLE to get the so-called *doubly nonlinear diffusion equation*

$$\partial_t u = \mathrm{div}(|\nabla u^m|^{p-2}\nabla u^m). \tag{DNDE}$$

Though these equations have many similarities with the PME, we will not deal with them in this book.

(iv) PME WITH LOWER ORDER TERMS. These are equations of the form

$$\partial_t u = \Delta\Phi(x,u) + B(x,t,u,\nabla u). \tag{1.13}$$

We have written the general filtration diffusion, but $\Phi(s) = |s|^{m-1}s$ gives the PME. The lower order term takes several forms in the applications. The best known are:

(1) the form $B = f(u)$ is a homogeneous reaction term, and the full equation is then a PME-based reaction–diffusion model; when $f \leq 0$ we have the nonlinear diffusion-absorption model that has been studied extensively;
(2) when $B = a \cdot \nabla u^q$ we have a convection term; a famous example is the Burgers equation $u_t + uu_x = \mu\, u_{xx}$;
(3) when $B = |\nabla u|^2$ we have a diffusive Hamilton–Jacobi equation.

We can see these latter equations as particular cases of the complete PME, but this could be misleading: their theory is quite rich. Of particular interest are the equations of the form

$$\partial_t u = \Delta(|u|^{m-1}u) + \nabla \cdot (\mathbf{a}(x)u)), \quad \mathbf{a}(x) = \nabla V(x), \tag{1.14}$$

called *Fokker–Planck equations*. The extra term stands for a confining effect due to a potential $V$. In the case $V(x) = c\,|x|^2$ these equations are closely connected to the study of the asymptotic behaviour of the plain PME/HE/FDE after a convenient rescaling (see details in Chapter 18).

## 1.4 Contents

In a classical mathematical style, the foundation of the book is the study of existence, uniqueness, stability and practical construction of suitably defined solutions of the equation plus appropriate initial and boundary data. This theory uses the machinery of nonlinear functional analysis, as developed extensively in the last century. In the spirit of this theory, classical concepts of solution do not suffice, which leads to the introduction of suitable concepts of generalized solution, in the concrete form of *weak, limit, strong and mild solution*, among others.

### 1.4.1 *The main problems and the classes of solutions*

There are three main problems that are posed in parabolic theories:

- Problem A is the initial value problem in the whole space, $x \in \mathbb{R}^d$, $d \geq 1$, for a time $0 < t < T$ with $T$ finite or infinite. It is usually called the Cauchy problem, CP, and is considered the reference problem in the literature about the PME. It is usually posed for non-negative solutions without a forcing term ($u \geq 0$ and $f = 0$), but we will also study it for signed solutions, and with a forcing term.
- Problem B is posed in a subdomain $\Omega$ or $\mathbb{R}^d$, and the additional data include initial conditions and boundary conditions of Dirichlet type, $u(x,t) = g(x,t)$ for $x \in \partial\Omega$ and $0 < t < T$. The same observations on the sign of $u$ and on $f$ apply. By default $\Omega$ is bounded, $u \geq 0$, $f = 0$, and $g = 0$.
- Problem C is similar to Problem B, but the data on the lateral boundary are Neumann data, $\partial_n u^m(x,t) = h(x,t)$. By default, $\Omega$ is bounded and $f = 0$, $h = 0$.

There is a number of other problems posed on spatial domains $\Omega$ with more general conditions of mixed or nonlinear type. In one space dimension a typical problem is posed in a semi-infinite domain $\Omega = (0,\infty)$. Typical data in that case are $u(0,t) = C$ or $(u^m)_x(0,t) = 0$.

Once the problems are shown to be well-posed in suitable functional settings, the next question is the study of the main qualitative properties. Prominent among them is the phenomenon of finite propagation and its consequences in the form of free boundaries. The emphasis shifts now into dynamical considerations and differential geometry.

A third important subject related to both previous ones is optimal regularity. Let us illustrate it on the source-type solution. We have seen that it is continuous in its domain of definition $Q = \mathbb{R}^d \times \mathbb{R}_+$. However, it is not smooth at the free boundary, again a consequence of the loss of the parabolic character of the equation when $u$ vanishes. In fact, the function $u^{m-1}$ is Lipschitz continuous in $Q$ with jump discontinuities on $\Gamma$ (i.e., there exists a regularity threshold). On the contrary, the solution is $C^\infty$-smooth in $\mathcal{P}_u$. And we are interested in noting that though $u$ is not smooth on $\Gamma$, nevertheless the free boundary is a $C^\infty$ smooth surface given by the equation (1.10). However, not all free boundaries of solutions of the PME will be so smooth.

### 1.4.2 *Chapter overview*

The book is organized as follows. After this Introduction, we review the main applications in Chapter 2. This pays homage to the fundamental role played by these applications in motivating the mathematical research and supplying it with problems, intuitions, concepts and conjectures.

We continue with two preparatory chapters. In Chapter 3 we review the main facts and introduce the basic estimates we will need later in a classical framework. Chapter 4 examines the fundamental examples, and we use the opportunity to present in a simple and practical context some of the main topics of the theory, like the property of finite propagation, the appearance of free boundaries, the need for generalized solutions and the question of limited regularity. It even shows cases of blow-up and the evolution of signed solutions.

This gives way to the study of the classical problems of existence, uniqueness and regularity of a (generalized) solution for the tree main problems mentioned above. There have been two basic approaches to the existence theory for the PME in the literature: one of them is the so-called semigroup approach based on posing the problem in the abstract setting of ODEs in Banach spaces; the other one uses a priori estimates, approximation by related smooth problems (to which the estimates apply uniformly), and passage to the limit. Though both approaches have been fruitful, we have chosen to give priority to the latter, which uses as a cornerstone the preparatory work of Chapter 3. It is used in Chapters 5, 6 and 8 to study the Dirichlet boundary value problem, and in Chapter 9 to treat the Cauchy problem. An intermediate Chapter 7 establishes the continuity of the constructed solutions. Chapter 10 presents the semigroup approach which is very different in spirit and has had a fundamental importance in the historical development of the whole subject. The whole set of ideas is used Chapter 11 to treat the Neumann problem as well as the problems posed on Riemannian manifolds. This completes the first half of the book. Three remarks are in order:

(i) At this general level, there is an interest in considering not only the PME but rather a wider class of equations to which most of the methods apply. This is why a large part of the material is derived for the class of complete generalized porous medium equations,

$$\partial_t u = \Delta\Phi(u) + f.$$

(ii) For reasons of simplicity at this stage, most of the treatment is restricted to integrable data, a sound assumption on physical grounds, though not necessary from the point of view of mathematical analysis, as the sequel will show.

(iii) A main point of the study is the introduction of the different types of generalized solution that appear in the literature and are natural to the problem, and the careful analysis of their scope and mutual relationships.

With this foundation, the second part of the book enters into more peculiar aspects of the theory of the PME; existence with optimal data, free boundaries, self-similar solutions, higher regularity, symmetrization and asymptotics; though relying on the previous foundation, the new material is not necessarily more difficult, and the aspects it covers can probably be more attractive for many active researchers, both for theoretical or practical purposes.

Let us examine the contents of the different chapters in this part. The existence and uniqueness theory is complemented with two beautiful chapters on solutions for general classes of data, i.e., data that are not assumed to be either integrable or bounded. Chapter 12 covers the theory of solutions with so-called growing data. Optimal growth conditions are found that allows for a theory of existence and uniqueness. Chapter 13 extends the analysis to solutions whose initial value (so-called trace) is a Radon measure.

We are now ready for the main topics of the qualitative theory, which are covered in the next block of four chapters. The propagation properties, another fundamental topic in the PME theory, are discussed in detail in Chapter 14, including all questions related to finite propagation, free boundaries and evolution of the support.

The PME theory in several space dimensions presented many difficulties and was developed at a slow pace. Much of the earlier progress focused on understanding the basic questions in a one-dimensional setting. Actually, we have a much more detailed knowledge in that case, and we devote Chapter 15 to present the main features, like the 1D free boundary.

Chapter 16 contains the full analysis of self-similarity, which plays a big role in the theory of the PME.

Chapter 17 deals with the principles of symmetrization and concentration and their applications.

We devote the next three chapters to the questions of asymptotic behaviour as $t$ goes to infinity and higher regularity. Chapter 18 does the asymptotics for the Cauchy problem, and Chapter 20 for the homogeneous Dirichlet problem.

The former contains the famous result on stabilization of the integrable solutions of the PME towards the ZKB profile which is the analogue for $m > 1$ of the convergence towards the Gaussian profiles of the solutions of the heat equation. Since this convergence is a way of expressing the central limit theorem of probability theory, the convergence of the PME flow towards the ZKB is a nonlinear central limit theorem.

Chapter 19 examines the actual regularity of the solutions of the Cauchy problem; it concentrates on describing two of the main results for non-negative and compactly supported solutions: the Lipschitz continuity of the pressure and the free boundary for large times and the lesser regularity for small times of the so-called focusing solutions (or hole-filling solutions). Partial $C^\infty$ regularity is also shown according to Koch, and the concavity properties according to Daskalopoulos and Hamilton and Lee and Vázquez.

The last two chapters gather complements on the previous material. We devote Chapter 21 to collect further applications to the physical sciences.

We will use notations that are rather standard in PDE texts, like Evans [229], Gilbarg-Trudinger [261] or equivalent, which we assume known to the reader. A detailed summary of the main basic concepts and notations of real and functional analysis is contained in the Appendix. This chapter also contains a number of technical appendices on material that is used in the book and was considered not to have a place in the main flow of the text. One of these results is the proof of the lack of contractivity of the PME flow in $L^p$ spaces with $p$ large, which answers a question raised by some experts and posed some open problems.

### 1.4.3 *What is not covered*

This is a basic book on a very rich subject that keeps growing in many exciting directions. We list here some of the topics where much progress has been made and have been nevertheless left out of the presentation.

(1) The theory of the so-called limit cases of the PME. First, the limit $m \to 1$, where we can get either the heat equation or the eikonal equation, $u_t = |\nabla u|^2$, depending on the scaling of the data [50, 375]. We also have the limit as $m \to \infty$, leading to the famous Mesa problem [85, 141, 242, 463].

(2) The detailed treatment of the fast diffusion equation. The reader can find an expository account at a rather advanced level in the author's Lecture Notes [515]. A whole set of references is given.

(3) The more detailed study of the behaviour for large times, using recent work on gradient flows, optimal transportation and the entropy–entropy dissipation method [155, 413].

   Also, the question of asymptotic geometry, in particular the question of asymptotic concavity, cf. [196, 197, 365].

(4) The theory of viscosity solutions for the PME developed by Caffarelli and Vázquez, [144], see also [125, 332].

(5) More general boundary value problems: the general Dirichlet problem, and then Neumann and mixed problems.
(6) The Lagrangian approach and particle trajectories, as developed in [279, 389]. See also [515].
(7) Numerical computation of PME flows, see [232].
(8) Stochastic versions of the porous medium equation, as in the work of da Prato et al. [192].
(9) The porous medium equation posed on a Riemannian manifold [121, 413]. See Section 11.5 below.

Of course, we have left out the developments for parallel equations and models, though their mathematical development has been closely connected to that of the PME, like

(i) The combination of nonlinear diffusion and reaction or absorption. This is a classical area where a wide literature exists.
(ii) The combined models involving nonlinear diffusion and convection, like $u_t = \Delta\Phi(u) + \nabla \cdot \mathbf{F}(u)$. This has been a very active area of research in recent years.
(iii) Gradient flows and $p$-Laplacian equations, and their relation with the PME in 1D.
(iv) The detailed study of the so-called dual equation, $v_t = (\Delta u)^m$.

## 1.5 Reading the book

The whole book is aimed at providing a comprehensive coverage that hopes to be useful both to the beginning researcher as a text, and to the specialist as a reference. For that purpose, it is organized in blocks of different difficulty and scope.

While trying to present the most relevant basic results with whole proofs in each chapter, a parallel effort has been made to present an informative panorama of the relevant results known about the topics of the chapter. However, and especially in the second part of the book, many interesting results that can be easily traced and read in the sources were discussed more briefly by evident reasons of space. The more advanced sections have been marked with a star, $^*$. On the other hand, we have included the proof of many new results that the author felt were needed to complete the presentation and were not reported in the literature. Chapters contain detailed introductions where the topics to be covered are announced and commented upon, and are supplied with a final section of Notes (comments, historical notes or recommended reading) and a list of problems. Problems contain many bits of proofs and some are used in later chapters. Solving them is recommended to the reader, since we believe that the best way of reading mathematics is active reading. We also include some advanced problems; they are marked with a star, $^*$.

The first part of the book has been devised as an introductory course on nonlinear diffusion centred on the PME and the GPME. Selections of the text centred on the PME and versions of it have been taught as such to PhD students having previously followed courses in classical analysis, functional analysis and PDEs. Knowing some physics of continuous media or studying the subject in parallel is useful, but not required. Several selections are possible for one semester courses, the simplest one consisting of Chapters 2–11 plus 14, maybe jumping over most over 6 and 7. Relevant and elementary material is also contained in Chapters 15, 16, and 18. We will give extensive references when the material used is not standard.

This is a book in PDEs and analysis at a theoretical level but covering the interests of what is usually called applied analysis. We will pay a serious attention to some, say, classical applications, but the reader need not be an expert in any physical or natural science or engineering, since all relevant concepts will be clearly defined.

The reader will notice that the subject is rich in methods and results, but also in concepts and denominations, many taken from different branches of the applied sciences, others from different areas of mathematics. We will underline all new concepts by writing them in italics the first time they are precisely defined and referencing the relevant ones in the index.

We hope that the material will make it easier for the interested reader to delve into deeper or more specific literature. We have already mentioned that, although we concentrate most of our effort in examining the non-negative solutions of the PME, the natural functional framework leads the mathematician to work with the signed PME. A number of important issues are still open for signed solutions.

## Notes

*Some historical notes*

We have seen the important contribution of Zel'dovich and Kompaneets [532], 1950, who found the source solutions in a particular case, and Barenblatt [60], who performed a complete study of these solutions in 1952. After the work in the decade by Barenblatt et al. on self-similar solutions and finite propagation, cf. [71] and the book [63], the systematic theory of the PME can be said to have begun with the fundamental work of Oleĭnik and her collaborators Kalashnikov and Czhou around 1958 [408], who introduced a suitable concept of generalized solution and analysed both the Cauchy and the standard boundary value problems in one space dimension. The work was continued by Sabinina, [457], who extended the results to several space dimensions. The qualitative analysis was advanced by Kalashnikov and many authors followed. The survey of the last author contains a very complete reference list on the literature concerning different aspects of the PME and related equations at the time. For earlier history see the Notes of the next chapter.

Since the 1970s, the interest in the equation has touched many other scholars from different countries. Here are some important landmarks. Bénilan [79] and Crandall et al. [178, 180] constructed mild solutions, Brezis developed the theory of maximal monotone operators [128], Aronson studied the properties of the free boundary [35, 36, 37], Kamin began the analysis of the asymptotic behaviour [319, 320], and Peletier et al. studied self-similarity [54]. In the 1980s well-posedness in classes of general data was established in Aronson-Caffarelli [42] and Bénilan-Crandall-Pierre [91], and the study of solutions with measures as data was initiated in Brezis-Friedman [131] and advanced by Pierre [434] and Dahlberg-Kenig [187]. Basic continuity of solutions and free boundaries was proved by Caffarelli and Friedman [138, 139, 140] and refined by DiBenedetto [206, 207], Sacks [461] and a number of authors.

There exists today a relatively complete theory covering the subjects of existence and uniqueness of suitably defined generalized solutions, regularity, properties of the free boundary and asymptotic behaviour, for different initial and boundary-value problems. Their names will appear in the development.

*Previous reports on the PME and related equations*

The text has as a precedent the notes prepared on the basis of the course taught at the Université de Montréal in June–July of 1990, aimed at introducing the subject and its techniques to young researchers [508]. The material has been also used for graduate courses at the Universidad Autónoma de Madrid. It has several earlier precedents. A short survey was published by Peletier [425] in 1981 and has been much used. A much longer survey paper is due to Aronson [38], written in 1986. Another often cited contribution, more in the form of a summary but including a discussion of related nonlinear parabolic equations and a very extensive reference list is due to Kalashnikov [317] in 1987. These have been main references during these years. In his book on *Variational Principles and Free-Boundary Problems* [240], 1982, Friedman devotes a chapter to the PME because of its strong connection with free boundary problems. Recently, the book by four Chinese authors Wu, Yin, Li and Zhao [527], 2001, about nonlinear diffusion equations is worth mentioning.

Both PME and $p$-Laplacian equations are tied together as degenerate diffusions in DiBenedetto's book [209]. The book [469] by Samarski et al. is mainly devoted to reaction diffusion leading to blow-up but has wide information about PME, specially related to self-similarity. A similar observation applies to [255] by Galaktionov and the author which concentrates on asymptotic methods based on self-similarity and dynamical systems ideas. This book contains a chapter with the main facts about the PME that appear in the asymptotic studies.

A reference to the mathematics of diffusion is Crank [182] which contains a bulk of basic information on the classical applied topics and results. Conduction of heat in solids is treated by Carslaw and Jaeger [159]. A general text on reaction–diffusion equations is Smoller's [482]. The Stefan problem is covered in the already mentioned books by Rubinstein [454] and Meirmanov [388].

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# PART ONE

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# 2

# MAIN APPLICATIONS

The porous medium equation,

$$\partial_t u = \Delta_x u^m, \quad m > 1, \quad u = u(x,t), \tag{2.1}$$

is a prominent example of nonlinear partial differential equation. In the particular case $m = 2$ it is called Boussinesq's equation. We are going to describe a choice of the main applications found in the literature that have served as a motivation for the development of the mathematical theory. In Section 2.1 we describe the standard model of gas flow through a porous medium (Darcy–Leibenzon–Muskat), in Section 2.2 the model of nonlinear heat transfer (Zel'dovich–Raizer), in Section 2.3 Boussinesq's model of groundwater flow, and in Section 2.4 a model of population dynamics (Gurtin–McCamy). Further applications will be found in Chapter 21.

An understanding of this chapter is recommended since we will be using some of the images and names suggested by these applications.

## 2.1 Gas flow through a porous medium

The porous medium equation owes its name to its use in describing the flow of an ideal gas in a homogeneous porous medium. According to Leibenzon [367] and Muskat [394], this flow can be formulated from a macroscopic point of view in terms of the variables *density*, which we represent by $\rho$; *pressure*, represented by $p$; and *velocity*, represented by $\mathbf{V}$, which are functions of space $x$ and time $t$ (the former is a vector). These quantities are related by the following laws:

(i) *Mass balance*, also called continuity equation in fluid mechanics,

$$\varepsilon \rho_t + \nabla \cdot (\rho \mathbf{V}) = \mathbf{0}. \tag{2.2}$$

Here $\varepsilon \in (0,1)$ is the porosity of the medium, and $\nabla \cdot$ represents the divergence operator.

(ii) *Darcy's law*, an empirical law formulated in 1856 by the French engineer H. Darcy [193], which describes the dynamics of flows through porous media

$$\mu \mathbf{V} = -k \nabla p. \tag{2.3}$$

It replaces for that kind of media the usual Navier–Stokes law of standard fluid flows.

(iii) *State equation*, which for perfect gases asserts that

$$p = p_0\, \rho^\gamma, \tag{2.4}$$

where $\gamma$, is called the so-called *polytropic exponent.* Its values in the two main cases covered by this state law when applied to gases are: $\gamma = 1$ for isothermal processes, and $\gamma$ larger than 1 for adiabatic ones (for air at normal temperature, the value $\gamma = 1.405$ is derived from the experimental data). In any case $\gamma \geq 1$.

The parameters $\mu$ (the viscosity of the fluid), $\varepsilon$ (the porosity of the medium), $k$ (the permeability of the medium) and $p_0$ (the reference pressure) are assumed to be positive and constant, which constitutes an admissible simplification in many practical instances, but need not be the case in a more general situation. Accepting such hypothesis, an easy calculation allows us to reduce (2.2)–(2.4) to the form

$$\rho_t = c\, \Delta(\rho^m), \tag{2.5}$$

with exponent $m = 1 + \gamma$ and

$$c = \frac{\gamma k p_0}{(\gamma + 1)\varepsilon\mu}. \tag{2.6}$$

The constant $c$ can be easily scaled out (define for instance a new time, $t' = ct$), thus leaving us with the PME. Mathematically, we say that constants that can be scaled out play no role, though the engineer will need to take a look at them; this is an interesting philosophy that will be much used.

Observe that in the above applications the exponent $m$ is always equal or larger than 2. The mathematical theory to be developed below does not find many differences between the exponents $m$ as long as they are larger than 1, though the formulas look a bit simpler for $m = 2$. In all the formulas, the operators $\nabla\cdot = \mathrm{div}$, $\nabla = \mathrm{grad}$ and $\Delta$, the Laplacian, are supposed to act on the space variables $x = (x_1, \ldots, x_d)$.

In order to adapt the notation to the mathematical taste and also adapt to current usage in the PME, we will use the letter $u$ instead of $\rho$ for the density; and the letter $v$ is used for the pressure, which is exactly defined by the expression

$$v = \frac{m}{m-1} u^{m-1}, \tag{2.7}$$

so-called *mathematician's pressure.* This is an important definition that will be used frequently in the book. It allows to easily recover the above physical formulas with $m = k = \mu = 1$, that is, forgetting about physical constants. Thus, Darcy's law for the velocity is written in the form

$$\mathbf{V} = -\nabla v = -m\, u^{m-2} \nabla u, \tag{2.8}$$

and the mass balance can be written in the form $\partial_t u + \nabla \cdot \mathbf{j} = 0$, where the quantity $\mathbf{j} = u\,\mathbf{V}$ in this formula is called the *mass flux.*

### 2.1.1 *Extensions*

*Non-homogeneous media*

The consideration of flows where $\varepsilon$, $\mu$ and $k$ are not constant, but functions of space and maybe also time, provides us with a natural generalization of the PME. The equation is then written in the form suitable for inhomogeneous media, NHPME,

$$\varepsilon(x,t)\,\partial_t u = \nabla\cdot(c(x,t)\nabla u^m), \tag{2.9}$$

where $\varepsilon$ and $c$ are given positive functions (or even, non-negative).

*Filtration equation*

A quite different approach is assuming that the state law is not power-like, but has the form $p = p(\rho)$, as happens in general barotropic gases, and also that $k$ and $\mu$ may depend on $\rho$. In that case we get a final equation for the density of the form

$$\rho_t = \Delta\Phi(\rho) + f, \tag{2.10}$$

where $\Phi$ is a given monotone increasing function of $\rho$, $\rho \geq 0$. This is called the filtration equation or generalized porous medium equation. In our application, $\Phi'(\rho) = \rho k(\rho)p'(\rho)/\mu(\rho)\varepsilon$. The second term on the right-hand side, $f = f(x,t)$ represents mass sources or sinks distributed in the medium.

We can also combine both types of extensions. We leave the detail to the reader. See also Section 5.11 for more general variants of the PME and the filtration equation.

## 2.2 Nonlinear heat transfer

A quite important application, probably second in importance for the historic development of the field, happens in the theory of heat propagation with temperature-dependent thermal conductivity. The general equation describing such a process (in the absence of heat sources or sinks) takes the form

$$c\rho\,\frac{\partial T}{\partial t} = \mathrm{div}(\kappa\nabla T), \tag{2.11}$$

where $T$ is the temperature, $c$ the specific heat (at constant pressure), $\rho$ the density of the medium (which can be a solid, fluid or plasma) and $\kappa$ the thermal conductivity. In principle all these quantities are functions of $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$. In the case where the variations of $c$, $\rho$ and $\kappa$ are negligible, we obtain the classical heat equation. However, when the range of variation of the temperatures is large, say hundreds or thousands of degrees, such an assumption is not very reasonable.

(i) The simplest case of variable coefficients corresponds to constant $c$ and $\rho$ and variable $\kappa$, a function of temperature, $\kappa = \phi(T)$. We then write (2.11) in the form

$$T_t = \Delta\Phi(T). \tag{2.12}$$

The *constitutive function* $\Phi$ is given by

$$\Phi(T) = \frac{1}{c\rho}\int_0^T \kappa(s)\,ds. \tag{2.13}$$

This is sometimes called Kirchhoff's transform. We find again the filtration equation (2.10), but now in a completely different applied context. If the dependence is given by a power function

$$\kappa(T) = a\,T^n, \tag{2.14}$$

with $a$ and $n > 0$ constants, then we get

$$T_t = b\,\Delta(T^m) \quad \text{with} \quad m = n + 1, \tag{2.15}$$

and $b = a/(c\rho\, m)$, thus the PME but for the constant $b$ which is easily scaled out.

(ii) In case we also assume that $c\rho$ is variable, $c\rho = \psi(T)$, we still obtain a generalized PME, though we have to work a bit more. Thus, we introduce a new variable $T'$ by the formula

$$T' = \Psi(T) \equiv \int_0^T \psi(s)\,ds. \tag{2.16}$$

We then obtain the following equation for $T$:

$$\partial_t\psi(T) = \Delta\Phi(T) \tag{2.17}$$

which can also be written as a standard GPME in terms of the variable $T'$ by inverting (2.16), i.e. $\partial_t T' = \Delta F(T')$ with $F = \Phi \circ \Psi^{-1}$. Again, if the dependences are given by power functions we obtain the PME with an appropriate exponent.

Zel'dovich and Raizer [533] propose model (i) to describe heat propagation by radiation occurring in plasmas (ionized gases) at very high temperatures. In that case energy is transferred mainly by electromagnetic radiation (as well as by conduction and convection, but these are of lesser importance). According to the mentioned reference, the radiation thermal conductivity is defined as

$$\kappa = \frac{lc}{3}\,c_{\rm rad}, \quad c_{\rm rad} = aT^3, \tag{2.18}$$

where $c$ is speed of light, $l$ is Rosseland's mean free path and the form of the radiation specific heat $c_{\rm rad}$ comes from the law of black bodyradiation law. This is an approximation valid under circumstances called the 'optically thick' limit. If $l$ is supposed to be constant we obtain the PME with $m = 4$.[1]

However, $l$ is usually temperature dependent, $l \sim aT^n$, with different exponents depending on the type of high-energy approximations of the process. For multiply ionized gases the exponent $n$ ranges in the interval from 1.5 to 2.5, and then $m - 1 = n + 3 \sim 4.5$–$5.5$. This description is taken from Chapter X of [533] where further details can be found. Other references: Longmire [378], Ockendon et al. [404].

**Remark** The fast diffusion equation is found in plasma physics in a different context. Plasma diffusion with the Okuda–Dawson scaling implies a diffusion coefficient ($D \sim u^{-1/2}$) in equation (1.1) where $u$ is the particle density. This leads to the FDE with $m = 1/2$. See Berryman and Holland [106]. On the other hand, Berryman [105] reports that electron heat conduction in a plasma can be modelled with the PME with exponent $m = 3.5$.

## 2.3 Groundwater flow. Boussinesq's equation

We examine next another problem in fluid mechanics, this time related to liquids. It deals with the filtration of an incompressible fluid (typically, water) through a porous stratum, the main problem in groundwater infiltration. The model was developed first by Boussinesq in 1903 [123] and is related to the original motivation of Darcy [193]. See also Polubarinova-Kochina [439].

*Modelling*

We will impose the following simplifying assumptions:

(i) the stratum has height $H$ and lies on top of a horizontal impervious bed, which we label as $z = 0$;
(ii) we ignore the transversal variable $y$; and
(iii) the water mass which infiltrates the soil occupies a region described as

$$\Omega = \{(x, z) \in R : \ z \le h(x, t)\}. \tag{2.19}$$

In practical terms, we are assuming that there is no region of partial saturation.

This is an evolution model. Clearly, $0 \le h(x,t) \le H$ and the free boundary function $h$ is also an unknown of the problem. In this situation, we arrive at a system of three equations with unknowns the two velocity components $u, w$ and the pressure $p$ in a variable domain: one equation of mass conservation for an incompressible fluid and two equations for the conservation of momentum of the Navier–Stokes type. Add initial and boundary conditions to the recipe. The resulting system is too complicated and can be simplified for the practical computation after introducing a suitable assumption, the hypothesis of *almost horizontal flow*, i.e., we assume that the flow has an almost horizontal speed

[1] We will obtain the same exponent in the thin film example of Section 21.1.

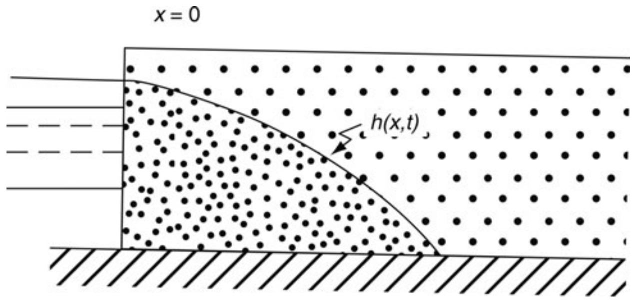


Figure 2.1: A schema of ground infiltration.

$\mathbf{u} \sim (u, 0)$, so that $h$ has small gradients. It follows that in the vertical component of the momentum equations

$$\rho \left( \frac{du_z}{dt} + \mathbf{u} \cdot \nabla u_z \right) = -\frac{\partial p}{\partial z} - \rho g,$$

we may neglect the inertial term (the left-hand side). Integration in $z$ gives for this first approximation $p + \rho g z =$ constant. We now calculate the constant on the free surface $z = h(x,t)$. If we impose continuity of the pressure across the interface, we have $p = 0$ (assuming constant atmospheric pressure in the air that fills the pores of the dry region $z > h(x,t)$). We then get

$$p = \rho g (h - z). \tag{2.20}$$

In other words, the pressure is determined by means of the *hydrostatic approximation.*

We go now to the mass conservation law which will give us the equation. We proceed as follows: we take a section $S = (x, x+a) \times (0, C)$. Then,

$$\varepsilon \frac{\partial}{\partial t} \int_x^{x+a} \int_0^h dy dx = -\int_{\partial S} \mathbf{u} \cdot \mathbf{n} \, dl, \tag{2.21}$$

where $\varepsilon$ is the porosity of the medium, i.e., the fraction of volume available for the flow circulation, and $\mathbf{u}$ is the velocity, which obeys Darcy's law in the form that includes gravity effects

$$\mathbf{u} = -\frac{k}{\mu} \nabla (p + \rho g z). \tag{2.22}$$

On the right-hand lateral surface we have $\mathbf{u} \cdot \mathbf{n} \approx (u, 0) \cdot (1, 0) = u$, i.e., $-(k/\mu) p_x$, while on the left-hand side we have $-u$. Using the formula for $p$ and differentiating in $x$, we get

$$\varepsilon \frac{\partial h}{\partial t} = \frac{\rho g k}{\mu} \frac{\partial}{\partial x} \int_0^h \frac{\partial}{\partial x} h \, dz. \tag{2.23}$$

We thus obtain *Boussinesq's equation*

$$h_t = \kappa\,(h^2)_{xx} \tag{2.24}$$

with constant $\kappa = \rho g k/2m\mu$. This is the PME with $m = 2$. It is a fundamental equation in groundwater infiltration. The system of nonlinear equations proposed in the initial model is reduced to solving a unique nonlinear heat equation that gives the height of the water mound. Once $h(x,t)$ is calculated, we may calculate the pressure via (2.20) and then the speed by means of Darcy's law.

We have made the final step of the derivation of Boussinesq's equation in one dimension for simplicity, but it generalizes immediately to several dimensions and gives

$$h_t = \kappa\,\Delta(h^2). \tag{2.25}$$

*Extension*

When there exists a water input into the porous stratum (by natural or artificial recharge), or an output (by sinks or pumping), the equation takes the complete form

$$h_t = \kappa\,\Delta(h^2) + f, \tag{2.26}$$

where function $f(x,z,t)$ reflects those effects. If we ideally assume that such effects take place at precise space locations, we are led to consider instead of a function $f$ a sum of Dirac masses, which gives rise to interesting mathematical problems.

**Remark** This is a fluid flow model and it involves a physical pressure that is given by the hydrostatic law (2.20), a function of $x$ and $z$. However, in average over $z$ it amounts to $c\,h(x)$, which is in accordance with our assumption that $v \sim u^{m-1}$ of Section 2.1.

## 2.4 Population dynamics

A very interesting example concerns the spread of biological populations. The simplest law regarding a population consisting of a single species is

$$\partial_t u = \mathrm{div}(\kappa\,\nabla u) + f(u), \tag{2.27}$$

where $u$ stands for the density or concentration of the species, and the reaction term $f(u)$ accounts for symbiotic interaction within the species; the medium is supposed to be homogeneous. According to Gurtin and McCamy [279], when populations behave so as to avoid crowding it is reasonable to assume that the *diffusivity* $\kappa$ is an increasing function of the population density, hence

$$\kappa = \phi(u), \quad \phi \quad \text{increasing}. \tag{2.28}$$

A realistic assumption in some particular cases is $\phi(u) = a\,u$. Disregarding the reaction term we obtain the PME with $m = 2$.

Of course, a complete study must take into account at least the reaction terms, and very often, the presence of several species. This leads to the consideration of nonlinear reaction–diffusion systems of equations of parabolic type containing lower order terms, whose diffusive terms are of PME type. Such equations and systems constitute therefore an interesting possibility of generalization of the theory of the PME. Similar equations appear in chemistry in the study of diffusive and reacting media.

## 2.5 Other applications and equations

The previous applications show how naturally the PME appears to replace the classical heat equation in processes of heat transfer or diffusion of a substance or population dispersal, whenever the assumption of constancy of the thermal conductivity (resp. diffusivity) cannot be sustained, and, instead, it is reasonable to assume that it depends in a power-like fashion (or almost power-like fashion) on the temperature (resp. density or concentration).

Once the theory for the PME began to be known, a number of applications have been proposed. Some of them concern the fast diffusion equation, the generalized PME and the inhomogeneous versions already commented. There are numerous examples with lower order terms, in the areas of reaction–diffusion, where the PME is only responsible for one of the various mechanisms of the equation or system.

We do not want to break the flow of the presentation of the theory with more applications at this point. Therefore, we devote Chapter 21 to describe a number of interesting applications for the reader's benefit. For applications of the fast diffusion equation we refer to the list of monograph [515].

## 2.6 Images, concepts and names taken from the applications

The presentation of the main applications of an equation or theory is a common practice in PDEs, and serves the purpose of justifying the attention paid to a particular topic, but also that of orienting the researcher in the difficult task of finding concepts and tools in the wild forest of applied nonlinear analysis.

It also serves another purpose that we want to stress here. It gives us the possibility of using a given application to put some flesh into the abstract thinking in the form of images, concepts and also a series of names that can be quite useful in coining a form of speech that allows for insight and communication.

Thus, starting with the name, it is quite common in the literature to talk about flows in porous media as the image behind the calculations. This brings us to talking about densities ($u$), pressures ($v$) and velocities ($-\nabla v$). Such speech will be quite useful, especially when studying the propagation aspects, like the existence of free boundaries. In this point of view, the integral

$$M(\Omega, t) := \int_\Omega u(x,t)\,dx$$

is called the *mass of gas* contained in volume $\Omega$ at time $t$. If $\Omega$ is the whole domain of definition, we call it the *total mass* at time $t$. An important issue of the theory is the conservation of the total mass in time (mass conservation law), which holds for some problems and does not for others (e.g., if mass is allowed to flow through the boundary).

We must remember, however, that for the pure mathematician all this is a manner of speech, since our theories are model-independent; for the applied mathematician, we must recall that the theory aspires to serve the needs of different applied areas, and will at times use the images and denominations of those other areas.

In our case, a quite important area is thermal propagation. Changing the letter for the unknown in equation (2.15), this application gives us the possibility of seeing equation $\partial_t u = \Delta u^m$ in terms of heat transfer, thus allowing us to assign the meaning of temperature to $u$, and temperature-dependent diffusivity to $D(u) = mu^{m-1}$. We remind the interested reader that Fourier's law is now written as $\Phi = -\kappa(u)\nabla u = -\nabla u^m$, where $\Phi$ is the heat flux as defined in standard heat theory. The total mass becomes now a *total thermal energy* (but for the constant factor $c\rho$ that we imagine put to 1).

The areas of population dynamics and chemistry add the possibility of viewing $u$ as a *concentration*, and now $D(u)$ is the concentration-dependent diffusivity. Concentration is the common concept in applications to nonlinear Diffusion processes.

## Notes

### *Some historical notes*

Let us review some of the early history, previous to the systematic theory, as far as we have discovered it. The French scientist J. Boussinesq seems to have been the first author to propose the porous medium equation as a mathematical model for a physical process [123] precisely to calculate the height of the water mound in groundwater infiltration. He used as basic flow law the one proposed by H. Darcy [193] in 1856, and under the so-called Dupuit assumption of small gradient [223]. Note that the exponent is $m = 2$.

It is historically remarkable that, even if the PME looks like an innocent nonlinear version of the heat equation, it took many years for it to be correctly posed (in classes of weak solutions) and solved.

In the 1930s the equation appeared again, this time for $m \geq 2$, in the study of gases in porous media, connected to oil extraction, in the works of two engineers, the Russian L. Leibenzon [367] and the American M. Muskat [394]. Polubarinova-Kochina [438] studied in 1948 the problem of groundwater infiltration into a porous stratum and proposed a self-similar solution that improved the knowledge of special solutions and their role in finite propagation.

Significant progress was made in Moscow in the 1950s, when Ya. Zel'dovich and collaborators studied heat propagation in plasmas and landed again on the porous medium equation, and its relative the filtration equation. Such simplified models are applicable for instance in the first stage after a nuclear explosion, when thermal waves are propagated in a gas that can still be considered stationary. Heat conduction happens mainly by radiation and the thermal conductivity is heavily dependent on temperature.

The mathematical study was seriously undertaken, attention to the presence of a front was duly paid, and the famous source-type solutions were found by Ya. Zel'dovich, A. Kompanyeets and B. Barenblatt, see the Introduction or Chapter 4. Finally, the theory of well-posedness started with O. Oleĭnik and her group around 1958. Basic results were obtained in Moscow in the early 1960s for the problem in several space dimensions (E. Sabinina, A. Kalashnikov, Yu. Dubinskii).

*Reading notes*

Earlier reference lists on applications of the PME can be found in: Berryman [105], Peletier [425], Lacey, Ockendon and Tayler [355], and Aronson [38], among other sources.

A general reference for the equations of fluid mechanics written with a mathematical audience in mind is Chorin and Marsden [171]. Interesting further reading on flows in porous media: Bear's books [76, 77], Barenblatt, Entov and Rhyzhik [67]. See also the author's lecture notes on flows in porous media [507]. A general reference for mathematical models in biology are Murray's two volumes [393].

Among the many works on nonlinear diffusion equations in population dynamics, let us mention the early papers of Aronson and Weinberger [53] and Aronson, Crandall and Peletier [46].

## Problems

**Problem 2.1** Scale out the constant $c$ in equation (2.5) by hiding it in the time variable, as indicated in the text. Do it also by hiding it in the space variable.

**Problem 2.2** Derive the equation of filtration of a gas with a barotropic law in an inhomogenous medium.

**Problem 2.3** Try to show formally that the mass conservation law should not hold for positive solutions of the PME defined in a domain $\Omega \subset \mathbb{R}^d$ and satisfying zero Dirichlet boundary conditions. How about zero Neumann conditions?

**Problem 2.4** Derive the equation satisfied by the pressure $v$, defined by (2.7), when the density obeys the PME with exponent $m$.

*Solution*: $v_t = (m-1)v\,\Delta v + |\nabla v|^2$.

**Problem 2.5** PRESSURE FOR THE FILTRATION EQUATION. When we consider equation (2.10), i.e., $\rho_t = \Delta\Phi(\rho)$, it can be written in the conservation form

$$\rho_t + \nabla \cdot (\rho\, \mathbf{V}) = \mathbf{0}.$$

(i) Show that this implies that $\mathbf{V} = -\nabla v$ if $v$ is defined as a function of $\rho$ by the formula

$$v = p(\rho) := \int_0^\rho \frac{\Phi'(s)}{s}\, ds,$$

whenever this integral is convergent.

(ii) Find the equation satisfied by $v$. [*Solution.* The equation is

$$v_t = a(v)\Delta v + |\nabla v|^2.$$

and $a(v) = \Phi'(\rho)$. See more in [125, 343].

(iii) Check that in the PME case this gives the usual formulas for the pressure and its equation.

# 3

# PRELIMINARIES AND BASIC ESTIMATES

This chapter covers preliminary material on parabolic equations needed to develop the main theories of the book. In this and the following chapters we work on subdomains of the Euclidean space $\mathbb{R}^d$ or the whole such space. However, we will see in Chapter 11 that the main facts of the theory extend in a natural way to equations posed on a Riemannian manifold.

We start with a review of useful properties of quasilinear parabolic equations. Next, Section 3.2 is devoted to non-degenerate versions of the generalized PME that will be used in approximating the degenerate cases. We derive for these better-behaved equations the basic estimates which will be used in developing the general theory for the class of possibly degenerate equations we have in mind.

We then specialize in Section 3.3 to properties that are formally satisfied by the PME; they will be justified in later chapters and used in the constructions of the different theories. Finally, Section 3.4 reviews the properties of the most popular alternative formulations of the PME.

In this chapter we consider solutions with changing sign. In most of the calculations $\Phi$ is not assumed to be a power. Sections 3.2 and 3.3 can be considered as basic material to be borrowed by later chapters.

## 3.1 Quasilinear equations and the PME

Let us review the properties of the solutions to quasilinear parabolic problems of the form

$$\partial_t u = \sum_{i=1}^{d} \frac{\partial}{\partial x_i} a_i(x, t, u, \nabla u) + b(x, t, u, \nabla u). \tag{3.1}$$

where $a_i(x, t, u, p_1, \dots, p_d)$ and $b(x, t, u, p_1, \dots, p_d)$ are called structural functions. They must satisfy certain conditions to ensure that a theory including existence, uniqueness and a certain regularity can be developed. The main condition is parabolicity to be explained presently. We will follow Ladyzhenskaya et al. [357], Friedman [239] or the more recent Lieberman [371] for reference to the classical theory of solutions of these equations.

### 3.1.1 *Existence of classical solutions*

In the classical theory, we assume that the structural functions $a_i(x, t, u, p_1, \dots, p_d)$ and $b(x, t, u, p_1, \dots, p_d)$ are bounded and $C^\infty$ in their arguments. The *uniform parabolicity* condition is formulated as follows: there exist constants $0 < c_1 < c_2 < \infty$ such that for every vector $\xi = (\xi_1, \dots, \xi_d)$, the following inequalities hold

$$c_1|\xi|^2 \le \sum_{i=1}^{d} \frac{\partial a_i}{\partial p_j}(x, t, u, u_{x_i})\, \xi_i \xi_j \le c_2 |\xi|^2. \tag{3.2}$$

Here are some of the most basic results under the classical assumptions:

(i) Given bounded and continuous initial data, the Cauchy problem can be solved and the solution $u(x,t)$ is unique, $C^\infty$ smooth in $Q = \mathbb{R}^d \times (0, \infty)$ and continuous down to $t = 0$, i.e., $u \in C(\mathbb{R}^d \times [0, \infty))$ and $u(x,0) = u_0(x)$. If the initial data are only bounded, then the initial data are taken only in the sense of a. e. convergence (and more precisely along time cones).

(ii) A main property in the theory of parabolic equations is the *maximum principle*, that is better termed the *comparison principle* in the nonlinear context. In the classical theory it takes a strong form that says:

*Strong maximum principle*

*Given two classical solutions $u(x,t)$ and $v(x,t)$ of the same equation of type* (3.1), *defined and continuous in $S = \mathbb{R}^d \times [0,T]$, if we assume that $u(x,0) \le v(x,0)$, then either $u = v$ everywhere in $S$, or $u < v$ everywhere in $S$.*

(iii) The existence, uniqueness and regularity theory of classical solutions extends to the mixed problems posed in cylindrical domains of the form $Q = \Omega \times (0,T)$ where $\Omega$ is a bounded domain of $\mathbb{R}^d$ with smooth boundary. Then, we have to give information not only of the initial data but also of data on the lateral boundary $\Sigma = \partial\Omega \times [0,T)$, which takes the form of Dirichlet data, Neumann data or some other versions that are found in the literature. This is why the problems are usually called 'initial and boundary value problems', IBVPs. If the initial and boundary data are compatible for $x \in \partial\Omega$ and $t = 0$, these mixed problems also have existence, uniqueness and regularity and the strong maximum principle holds: the same conclusion $u < v$ applies if $S = \Omega \times [0,T]$, $\Omega$ is a bounded open set with smooth boundary, and boundary data $u \le v$ are prescribed on $\Sigma = \partial\Omega \times [0,T]$.

### 3.1.2 *Weak theories and the PME*

In practice, the classical assumptions on $a_i$ and $b$ are not met in many problems of interest in the applied sciences. This is the origin of the weak theories, where relaxed conditions are accepted and then generalized solutions are obtained in Sobolev classes of weakly differentiable functions. The condition of uniform parabolicity is usually kept.

We will quote the results from the weak theory of non-degenerate quasilinear parabolic equations as the need arises. But let us mention that the strong maximum principle need not hold, and the typical comparison result states that if the data of a Cauchy problem are ordered by the relation $\leq$, so are the solutions a.e. This applies also to the Dirichlet and Neumann problems with suitable ordering of the boundary data.

We turn now to the PME example. The assumptions of smoothness fail in our case since the PME is a particular case of equation (3.1) where

$$a(x,t,u,p) = |u|^{m-1}p$$

with $u \in \mathbb{R}$, $p \in \mathbb{R}^d$, and $b = 0$. The main problem is non-uniform parabolicity; indeed, even for bounded non-negative solutions, condition (3.2) can only hold when $c_1 = 0$. This extension of the concept of parabolicity is called *degenerate parabolicity*. In physical speech, when thinking in terms of thermal propagation, it means that the thermal conductivity vanishes at zero temperature; in diffusion problems, we call it degenerate diffusivity. In any case and with any name, the study of the consequences of degenerate parabolicity is the reason of this book.

We still can save the classical theory as long as we consider 'non-degenerate' data $u_0$, i.e., data in the range $\varepsilon \leq u_0(x) \leq 1/\varepsilon$ with $\varepsilon > 0$. More generally, in the signed PME we may choose this option or $-1/\varepsilon \leq u_0(x) \leq -\varepsilon$. In order to solve the Cauchy problem for the PME with such data, we take $a(x,t,u,p) = m|u|^{m-1}p$ for $\varepsilon \leq |u| \leq 1/\varepsilon$, and extend the function as a linear function of $u$ and $p$ for $u$ near 0 or infinity, making a smooth connection around the values $u = \pm\varepsilon$ and $u = \pm 1/\varepsilon$. With these modifications, we pose the problem of finding a solution of the perturbed equation

$$\partial_t u = \operatorname{div}(\phi(u)\nabla u) = \Delta\Phi(u).$$

Since the degeneracy has been eliminated, there is a unique classical solution, and it satisfies the same bounds $\varepsilon \leq |u(x,t)| \leq 1/\varepsilon$. But this means that $u$ never takes values in the region of perturbed values, hence we get a classical solution of the PME. Let us state the result for the record.

**Theorem 3.1** (Classical solutions of the PME) *Assume that $u_0$ is a continuous function in $\mathbb{R}$ with*

$$\varepsilon \leq u_0(x) \leq 1/\varepsilon$$

*for some $\varepsilon > 0$ and all $x \in \mathbb{R}^d$. Then there exists a classical solution of the PME satisfying*

$$\varepsilon < u(x,t) < 1/\varepsilon$$

*for every $x \in \mathbb{R}^d$ and $0 < t < \infty$. If $u_0$ is $C^k$-smooth, so is $u$ at $t = 0$; if $u_0$ is only bounded, then the convergence to the initial data takes place a.e. along time cones.*

*The same applies to the signed PME with values in the range $-1/\varepsilon \leq u_0(x) \leq -\varepsilon$; a classical solution exists and $-1/\varepsilon < u(x,t) < -\varepsilon$.*

A similar argument applies to the Cauchy–Dirichlet problem when the boundary data satisfy the same condition $\varepsilon \leq u(x,t) \leq 1/\varepsilon$ for $x \in \partial\Omega$, $t \geq 0$, and are compatible with the initial data. It also applies to the problem with zero Neumann boundary conditions, and to other variants. The extensions to negative solutions also hold.

If, on the contrary, the data take zero values inside $\Omega$ the classical theories cannot apply, the strong maximum principle does not either, and there is no way to circumvent the weak theories. Moreover, a number of curious phenomena appear, like finite propagation and free boundaries, which are at the core of this book.

Let us finally recall that when data are unbounded we meet another problem, namely that $\partial a_i/\partial p_i = m|u|^{m-1}$ goes to infinity, so that the equation loses the upper bound on parabolicity. This is a different phenomenon, it will be less visible, but will also affect all calculations with large values of $u$, in the sense that the estimates will be different from the ones for the HE also in this case.

## 3.2 The GPME with good $\Phi$. Main estimates

Our aim is to establish an existence and uniqueness theory of generalized solutions for the PME, and also the generalized PME with quite general $\Phi$. This will be done in Chapter 5 and following in the class of weak solutions, and we will also obtain the most important properties of that class of solutions. The program offers a main difficulty the fact that the PME is a degenerate equation. Three other difficulties complicate the task: the generality of the nonlinearity $\Phi$, the generality of the data, and the sign of the solutions.

A standard approach to the construction of solutions for the PME and other degenerate cases will be approximation with non-degenerate problems. A quite useful choice, though not the only one possible, is approximation with a GPME having a nonlinearity $\Phi : \mathbb{R} \mapsto \mathbb{R}$ which is $C^2$ smooth and with $\Phi'(u) > 0$ for all $s \in \mathbb{R}$. Under such assumptions, the equation is parabolic non-degenerate, and we may apply standard quasilinear theory to obtain the existence and uniqueness of *classical solutions*, i.e., solutions such all the derivatives appearing in the equation exist and are continuous and the equation is satisfied everywhere in the space-time domain where we are working. This is what we will do in this section as a preliminary for the full treatment. We will assume the normalization $\Phi(0) = 0$, since this implies no loss of generality (the equation is invariant under addition of a constant to $\Phi$). We also ask the domain to have a smooth boundary, $\Gamma = \partial\Omega \in C^{2,\alpha}$. Actually, the consideration of inhomogeneous media recommends a bit more of generality and we will assume that $\Phi : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, it is smooth in both variables, it is strictly increasing in the second, and $\Phi(x, 0) = 0$ for all $x \in \Omega$.

There are two main problems: the homogeneous Dirichlet problem is

$$\partial_t u = \Delta\Phi(x, u) + f \quad \text{in } Q_T, \tag{3.3}$$

$$u(x, 0) = u_0(x) \quad \text{in } \Omega, \tag{3.4}$$

$$u(x, t) = 0 \quad \text{in } \Sigma_T, \tag{3.5}$$

where $Q_T = \Omega \times (0,T)$, and $\Sigma_T = \partial\Omega \times [0,T)$ is the lateral boundary. On the other hand, the homogeneous Neumann problem consists of equations (3.3), (3.4) and

$$\frac{\partial}{\partial\nu}\Phi(x,u) = 0 \quad \text{in } \Sigma_T, \tag{3.6}$$

where $\nu$ is the outer normal to the boundary $\partial\Omega$ As for the data, we will assume that $u_0$ and $f$ are bounded and $C^\alpha$ functions, and $u_0(x) = 0$ for $x \in \partial\Omega$. Under such assumptions, we apply the quasilinear theory to obtain the following existence result and, what is more important for later use, the main estimates on which the weak theory will be based.

**Theorem 3.2** *Under the above regularity assumptions, the Dirichlet problem* (3.3)–(3.5) *admits a classical solution $u$ in the space $C^{2,1}(\overline{Q})$. If $\Phi$, $u_0$ and $f$ are $C^\infty$, then so is $u$ in $Q$. Same results apply to the Neumann problem.*

The above Dirichlet problem usually serves to produce the approximate solutions that will be used to construct weak solutions of the PME and other cases of the filtration equation. Besides, it allows us to derive the main quantitative estimates on which the subsequent study is based. This is the content of the next subsections. The first two of them contain bounds for the solution. Next, we obtain the stability estimate in $L^1$ norm, one of most peculiar mathematical properties of these nonlinear diffusion processes. Three further estimates contain bounds for the derivatives that will be used to ensure compactness in the approximation processes.

### 3.2.1 *Maximum principle and comparison*

It applies to the solutions of both Dirichlet and Neumann problems. It has a simple form when $\Phi$ does not depend explicitly on $x$.

**Lemma 3.3** *If $\Phi = \Phi(u)$, then the solutions of the homogeneous Dirichlet or Neumann problem for equation* (3.3) *satisfy*

$$\|u\|_{L^\infty(Q_T)} \le \|u_0\|_{L^\infty(\Omega)} + T\|f\|_{L^\infty(Q_T)}. \tag{3.7}$$

**Proof** Let $M = \sup(u_0)$ and $N = \sup_Q f$. As an immediate consequence of the classical maximum principle, we have

$$u(x,t) \le M + Nt \quad \text{in } Q, \tag{3.8}$$

and a similar estimate applies as a lower bound. Hence, for bounded data $u_0$ and $f$ we have a bound on the solution.

The comparison principle holds for smooth solutions: if $u, \widehat{u}$ are solutions with initial data such that $u_0 \le \widehat{u}_0$ a.e. in $\Omega$ and $f \le \widehat{f}$ a.e. in $Q$, then $u \le \widehat{u}$ a.e. in $Q$. In particular, if $u_0, f \ge 0$ in $\Omega$, then $u \ge 0$ in $Q$. ∎

*An inhomogeneous extension*

In case $\Phi$ depends on $x$, things are not so simple. We have to make some assumption. Suppose to fix ideas that

$$\Delta_x \Phi(x, z) \leq K_1 + K_2 z \tag{3.9}$$

for all $x \in \mathbb{R}$ and all $z \geq M_0$ and constants $K_1, K_2 \geq 0$. We argue on a point where $u(x,t)$ touches from below the function

$$U(x, t) = M + Ct + \varepsilon. \tag{3.10}$$

If $u < U$ we get an estimate. If not, there is a first contact point $(x_0, t_0)$, and there the difference $f(x) = w(x, t_0) - \Phi(x, U(t_0))$, with $w = \Phi(x, u)$, attains a space maximum (equal 0). At that point $\Delta_x w \leq \Delta_x \Phi(x, U(t_0))$, and

$$\partial_t u(x_0, t_0) = f(x_0, t_0) + \Delta_x w(x_0, t_0) \leq N + \Delta_x \Phi(x, U(t_0)).$$

If moreover $U(x_0, t_0) > M_0$ we get

$$\partial_t u(x_0, t_0) \leq N + K_1 + K_2(M + \varepsilon) + K_2 C t_0.$$

on the other hand, at the first touching point $u_t(x_0, t_0) \geq U_t(x_0, t_0) = C$. Therefore, we avoid the touching point if

$$N + K_1 + K_2(M + \varepsilon) + K_2 C t_0 < C.$$

Suppose now that $t_0 \in [0, 1/(2K_2)]$. Then we may take $C = 2(N + K_1 + K_2(M + \varepsilon))$. Letting $\varepsilon \to 0$ we get the result

**Lemma 3.4** *Let us take the situation of Lemma* 3.3*, but now* $\Phi = \Phi(x, u)$*. If* (3.9) *holds, then*

$$u(x, t) \leq \min\{M + Ct, M_0\} \quad C = 2(N + K_1 + K_2 M) \tag{3.11}$$

*for all* $x \in \Omega$ *and* $0 < t < 1/(2K_2)$.

If we want to extend the time interval when $K_2 \neq 0$, we argue in time steps of $1/(2K_2)$. We get in this way a possible exponential increase in time. On the other, a similar argument applies to the negative part by using the change of variables $\widetilde{u} = -u$. The necessary bound for $\Delta_x \Phi$ now has the form

$$-\Delta_x \Phi(x, z) \leq K_1 - K_2 z \qquad \forall z \leq -M_0.$$

### 3.2.2 *Other boundedness estimates*

We now start the typical technique of the weak theories consisting in multiplying the equation by suitable *multipliers*, integrating in space or in space-time and then performing a number of integrations by parts and other calculus tricks. In our first example we take a function $p \in C^1(\mathbb{R})$ such that $p'(s) \geq 0$ for all $s \in \mathbb{R}$, and let $j$ be the primitive of $p$ with $j(0) = 0$. Then, if $\Phi$ does not depend explicitly on $x$ we have

$$\frac{d}{dt}\int j(u)\,dx = \int p(u)\partial_t u\,dx = -\int p'(u)\Phi'(u)|\nabla u|^2\,dx + \int f\,p(u)\,dx, \qquad (3.12)$$

with integrals in $\Omega$. The reader should check that this calculation applies to the solutions of both Dirichlet and Neumann problems. Since the term containing $|\nabla u|^2$ is negative, integrating in time from 0 to $t > 0$ we have

$$\int j(u(t))\,dx \le \int j(u_0)\,dx + \iint_{Q_t} f\,p(u)\,dxdt. \qquad (3.13)$$

If $f = 0$ this means that $J(u)(t) = \int j(u(x,t))\,dx$ is a monotone non-increasing function of time. Even if $f \neq 0$ we can get estimates. For instance, if $f$ is bounded and $p(u)/j(u)$ bounded as $u \to \infty$, we get boundedness of $\int j(u(t))\,dx$ for bounded times, see Problem 3.3. An interesting particular case happens when $j(s) = |s|^r$ for some $r > 1$. When $f = 0$ we get monotonicity of the $L^r$ norm

$$\frac{d}{dt}\int |u(t)|^r\,dx \le 0.$$

In case $\Phi$ depends on $x$, we above argument does not work because of the derivatives of $\Phi(x,z)$ with respect to $x$. We refrain from entering into the modifications which are not at immediate.

### 3.2.3 *The stability estimate. $L^1$ contraction*

This is a very important estimate which has played a key role in the PME and the GPME theory. It will allow us to develop existence, uniqueness and stability theory in the space $L^1(\Omega)$. Actually, the concept of $L^1$ contraction turns out to be a very powerful tool in the theory of nonlinear diffusion equations. There is no problem in admitting explicit dependence of $\Phi$ on $x$.

**Proposition 3.5** ($L^1$-contraction principle) *Let $u$ and $\widehat{u}$ be two smooth solutions, possibly of changing sign and with initial data $u_0$, $\widehat{u}_0$ and forcing terms $f$, $\widehat{f}$ respectively. We have for every $t > \tau \ge 0$*

$$\int_\Omega (u(x,t) - \widehat{u}(x,t))_+ dx \le \int_\Omega (u(x,\tau) - \widehat{u}(x,\tau))_+ dx + \int_\tau^t \int_\Omega (f - \widehat{f})_+\,dxdt. \qquad (3.14)$$

*As a consequence,*

$$\|u(t) - \widehat{u}(t)\|_1 \le \|u_0 - \widehat{u}_0\|_1 + \int_0^t \|f(s) - \widehat{f}(s)\|_1 ds. \qquad (3.15)$$

**Proof of the proposition** This result applies to the solutions of both Dirichlet and Neumann problems. It is so important that we give two quite different proofs.

**First proof** This is a standard proof in the literature. The technique goes as follows: Let $p \in C^1(\mathbb{R})$ be such that $0 \le p \le 1$, $p(s) = 0$ for

$s \le 0$, $p'(s) > 0$ for $s > 0$. Let $w = \Phi(x,u) - \Phi(x,\widehat{u})$ which vanishes on $\Sigma$ for the Dirichlet problem. Subtracting the equations satisfied by $u$ and $\widehat{u}$, multiplying by $p(w)$ and integrating in $\Omega$, and observing that $p(w) = 0$ on $\Sigma$, we have for $t > 0$

$$\begin{aligned}\int (u-\widehat{u})_t\, p(w)\, dx &= \int \Delta w\, p(w)\, dx + \int (f-\widehat{f})\, p(w)\, dx \\ &= -\int |\nabla w|^2 p'(w)\, dx + \int (f-\widehat{f})_+\, dx.\end{aligned}$$

Note the first term in the right-hand side is non-positive. Therefore, letting $p$ converge to the sign function $\text{sign}_0^+$, and observing that $\dfrac{\partial}{\partial t}(u-\widehat{u})_+ = (u-\widehat{u})_t\, \text{sign}_0^+(u-\widehat{u})$, cf. [261], and also observing that

$$\text{sign}_0^+(u-\widehat{u}) = \text{sign}_0^+(\Phi(x,u) - \Phi(x,\widehat{u})),$$

(a crucial fact based on the strict monotonicity of $\Phi$), we get

$$\frac{d}{dt}\int (u-\widehat{u})_+ dx \le \int (f-\widehat{f})_+\, dx,$$

which implies (3.14) for $u, \widehat{u}$. To obtain (3.15), combine (3.14) applied first to $u$ and $\widehat{u}$ and then to $\widehat{u}$ and $u$. The Neumann problem is completely analogous. ∎

**Second proof of the proposition** This contains two arguments, one for ordered solutions, another one for the maximum of two solutions.

**Lemma 3.6** *Assume that $u$ and $\widehat{u}$ are two smooth solutions such that $u_0 \le \widehat{u}_0$ and $f \le \widehat{f}$. Then, for every $t > 0$ we have $u(t) \le \widehat{u}(t)$ and*

$$\int (\widehat{u}(x,t) - u(x,t))\, dx \le \int (\widehat{u}_0(x) - u_0(x))\, dx + \iint_{Q_t} (\widehat{f} - f)\, dxdt. \tag{3.16}$$

This result is immediate. Note that in the case of the Neumann problem we have equality, for the Dirichlet problem only inequality. The second lemma is also elementary.

**Lemma 3.7** *Assume that $u$ and $\widehat{u}$ are two smooth solutions, and let $U$ be the solution with initial data $U(x,0) = \max\{u_0, \widehat{u}_0\}$ and forcing term $F = \max\{f, \widehat{f}\}$. Then, for every $t > 0$, $U(t) \ge \max\{u(t), \widehat{u}(t)\}$.*

In order to prove the contraction principle using these lemmas, we observe that for every $t > 0$ we have

$$U(t) - \widehat{u}(t) \ge \max\{u(t), \widehat{u}(t)\} - \widehat{u}(t) = (u(t) - \widehat{u}(t))_+$$

while equality holds at $t = 0$. Hence, by Lemma 3.6 we conclude that

$$\int_{\Omega} (u(t) - \widehat{u}(t))_+ dx \le \int_{\Omega} (U(t) - \widehat{u}(t))\, dx \le \int_{\Omega} (U(0) - \widehat{u}_0)\, dx + \iint_{Q_t} (F - \widehat{f})\, dxdt$$
$$= \int_{\Omega} (u_0 - \widehat{u}_0)_+ dx + \iint_{Q_t} (f - \widehat{f})_+\, dxdt.$$

This ends the proof. ■

Taking $\hat{u} = 0$ we get an interesting consequence.

**Corollary 3.8** *For every smooth solution and every $t > 0$*

$$\int_{\Omega} (u(x,t))_+ dx \le \int_{\Omega} (u_0(x))_+ dx + \iint_{Q_t} f_+(x,t)\, dxdt. \tag{3.17}$$

### 3.2.4 *The energy identity*

We want to control the derivatives of the solution or some function thereof in order to apply compactness arguments. With respect to spatial gradients, the natural function to control turns out to be $w = \Phi(x, u)$. In order to bound $\nabla w$ we need to introduce the function $\Psi$ which is the primitive of $\Phi$ with respect to $u$ with $\Psi(x, 0) = 0$, i.e.,

$$\Psi(x, s) = \int_0^s \Phi(x, \sigma)\, d\sigma. \tag{3.18}$$

Note that for the PME we have $\Psi(x, s) = |s|^{m+1}/(m + 1)$. Generally, $\Psi(x, u) \ge 0$ and moreover, $\Psi(x, u) \ge O(|u|)$ for all large $|u|$. On the other hand, $\Psi(x, u) \le |\Phi(x, u)u|$.

Since we are assuming that $\Phi$ is smooth and the solution is classical, we can multiply equation (3.3) by $\Phi(x, u)$ and integrate in $Q_T$ to obtain

$$\iint_{Q_T} |\nabla \Phi(x, u)|^2 dxdt + \int_{\Omega} \Psi(x, u(x, T))\, dx = \int_{\Omega} \Psi(x, u_0(x))\, dx$$
$$+ \iint_{Q_T} f\Phi(x, u)\, dxdt, \tag{3.19}$$

where we have integrated by parts in space in the term $\iint \Delta\Phi(x, u)\Phi(x, u)\, dxdt$ and integrated in time the term $\iint \Phi(x, u)u_t\, dxdt = \iint \Psi(x, u)_t\, dxdt$. This important formula will be used in Chapter 5 to provide key estimates in the existence theory for weak solutions. It is interesting therefore to supply some physical meaning to its terms. Thus, the estimate leads us to consider the expression

$$E_u(t) = \int_\Omega \Psi(x, u(t))\, dx \tag{3.20}$$

as a natural energy for the evolution, and then

$$DE(u) = \int_0^T \int_\Omega |\nabla\Phi(x, u)|^2\, dxdt \tag{3.21}$$

is the dissipated energy, while $\iint f\Phi(x, u)\, dxdt$, represents the work of the external forces. Formula (3.19) is known as the *energy identity.* If $f = 0$, it takes the simple form

$$\iint\limits_{Q_T} |\nabla\Phi(x, u)|^2 dxdt + \int\limits_\Omega \Psi(x, u(x, T))\, dx = \int\limits_\Omega \Psi(x, u_0(x))\, dx. \tag{3.22}$$

In case $f \neq 0$, we use Hölder's inequality to split the last term into

$$\frac{1}{4c} \iint f^2\, dxdt + c \iint \Phi(x, u)^2\, dxdt.$$

In the case of the homogeneous Dirichlet problem, the Poincaré inequality allows as to control the last term by the first term in the right-hand side. In this way we get

$$\frac{1}{2} \iint\limits_{Q_T} |\nabla\Phi(x, u)|^2 dxdt + \int\limits_\Omega \Psi(x, u(x, T))\, dx \le \int\limits_\Omega \Psi(x, u_0(x))\, dx + C \iint\limits_{Q_T} f^2\, dxdt, \tag{3.23}$$

where $C$ depends on $\Omega$ trough the constant in the Poincaré inequality. Since the right-hand side is bounded for every fixed $T > 0$, it follows that $\nabla\Phi(x, u)$ is bounded in $L^2(Q)$.

In the Neumann problem we still apply Hölder's inequality to the last term of (3.19); we can then bound $\iint \Phi(x, u)^2\, dxdt$ in terms of $\iint |\nabla\Phi(x, u)|^2\, dxdt$ and some $L^p$ norm of $\Phi$ (or even of $u$ is $\Phi$ behaves like a power). For that purpose we can use the boundedness estimates of Subsection 3.2.2. It then follows that it follows that $\nabla\Phi(x, u)$ is bounded in $L^2(Q)$.

*Local version*

An interesting version of this estimate proceeds by multiplying also by $\eta^2$, where $\eta$ is a smooth cut-off function, $0 \le \eta \le 1$. If the rest of the process is the same we get

$$\iint_{Q_T} |\nabla\Phi(x,u)|^2\eta^2\,dxdt + \int_\Omega \Psi(x,u(x,T))\eta^2\,dx$$
$$= \int_\Omega \Psi(x,u_0(x))\eta^2\,dx + \iint_{Q_T} f\,\Phi(x,u)\eta^2\,dxdt$$
$$- 2\iint_{Q_T} \Phi(x,u)(\nabla\eta\cdot\nabla\Phi(x,u))\eta\,dxdt.$$

Therefore,

$$\begin{aligned}&\iint_{Q_T} |\nabla\Phi(x,u)|^2\eta^2\,dxdt + 2\int_\Omega \Psi(x,u(x,T))\eta^2\,dx\\ &\quad\le 2\int_\Omega \Psi(x,u_0(x))\eta^2\,dx + \iint_{Q_T} f^2\eta^2\,dxdt + \iint_{Q_T} \Phi^2(u)(4|\nabla\eta|^2+\eta^2)\,dxdt.\end{aligned} \tag{3.24}$$

This allows to obtain local bounds in $L^2$ for $|\nabla\Phi(x,u)|$ when local bounds are available for $\Phi(x,u)$ and $f$, as well as for $\Psi(x,u_0)$ in $L^1_{\rm loc}(\mathbb{R}^d)$.

### 3.2.5 *Estimate of a time derivative*

The function whose time derivative we control is $z(x,t)=\mathcal{Z}(x,u(x,t))$, where the new function $\mathcal{Z}$ is defined in terms of $\Phi$ by

$$\mathcal{Z}(x,s)=\int_0^s (\Phi_u(x,s))^{1/2}ds. \tag{3.25}$$

Hence, $z_t=\Phi'(u)^{1/2}\partial_t u$. Note that since $2(\Phi_u(x,s))^{1/2}\le 1+\Phi_u(x,s)$, we have $|\mathcal{Z}|\le(1/2)|s+\Phi(x,s)|$, hence $|z|\le(1/2)|u+\Phi(x,u)|$. For the PME we have $z=c(m)\,|u|^{(m+1)/2}$.

In order to estimate $z_t$, we multiply the equation by $w_t$, with $w=\Phi(x,u)$, and integrate by parts in space to obtain (both for Dirichlet and Neumann problems)

$$\int w_t\,\partial_t u\,dx = \int w_t\,\Delta w\,dx + \int f w_t\,dx = -\int \nabla w\cdot\nabla w_t\,dx + \int f w_t\,dx$$
$$= -\frac{1}{2}\frac{d}{dt}\int |\nabla w|^2\,dx + \int f w_t\,dx,$$

where we have taken into account the fact that $w_t=0$ (or $\nabla_w=0$) on $\Sigma$. Moreover, this estimate has a simple form if $f=0$ upon integration in time

$$\iint_{Q_T^\tau} \Phi_u(x,u)|\partial_t u|^2\,dxdt + \frac{1}{2}\int_\Omega |\nabla w(x,T)|^2\,dx = \frac{1}{2}\int_\Omega |\nabla w(x,\tau)|^2\,dx. \tag{3.26}$$

If $f \neq 0$, there are several alternatives. Thus, multiplication by $\zeta(t)$, where $\zeta$ is a smooth function with $\zeta(0) = \zeta(T) = 1$, and integration in time can be used to obtain

$$\iint_{Q_T} \zeta\, \Phi_u(x,u)|\partial_t u|^2\, dxdt = \iint_{Q_T} \left\{ \frac{\zeta_t}{2} |\nabla \Phi(x,u)|^2 - (f\zeta)_t \Phi(x,u) \right\} dxdt. \tag{3.27}$$

The middle term is bounded in view of the energy estimate, and the last one is also if $f_t$ is bounded. Therefore, we get an estimate on the first, which is a non-negative expression, another kind of energy.

Without the extra condition on $f$, a typical approach consists of multiplication by $t$ and integration in time from 0 to $T$ to give

$$\begin{aligned} &\iint t\Phi_u(x,u)|\partial_t u|^2\, dxdt + \frac{T}{2}\int |\nabla w(x,T)|^2\, dx \\ &= \frac{1}{2}\iint |\nabla w|^2\, dxdt + \iint tf\Phi_u(x,u)\partial_t u\, dxdt. \end{aligned} \tag{3.28}$$

In order to obtain a uniform bound on the right-hand side, and since the term $\iint |\nabla w|^2\, dxdt$ is bounded, we only need to control the last term, that we may estimate as

$$\iint tf\Phi_u(x,u)\partial_t u\, dxdt \le \frac{1}{2}\iint t\Phi_u(x,u)f^2\, dxdt + \frac{1}{2}\iint t\Phi_u(x,u)|\partial_t u|^2\, dxdt.$$

The last term is absorbed by the first term of the previous expression. We get

$$\begin{aligned} \iint t\Phi_u(x,u)|\partial_t u|^2\, dxdt + T\int |\nabla w(x,T)|^2\, dx \le &\iint |\nabla w|^2\, dxdt \\ &+ \iint t\Phi_u(x,u)f^2\, dxdt \end{aligned} \tag{3.29}$$

and the last term is bounded for bounded $f$ and bounded $u$. This estimate means that for every $T > \tau > 0$ the integral $\int_\tau^T \int \Phi_u(x,u)|\partial_t u|^2\, dxdt = \iint z_t^2\, dxdt$ is bounded.

As a further alternative, we drop the multiplication by $t$ and integrate in time from $\tau$ to $T$ we get

$$\begin{aligned} &\iint_{Q^\tau} \Phi_u(x,u)|\partial_t u|^2\, dxdt + \frac{1}{2}\int_\Omega |\nabla w(x,T)|^2\, dx \\ &= \frac{1}{2}\int_\Omega |\nabla w(x,\tau)|^2\, dx + \iint_{Q_T^\tau} f\Phi_u(x,u)u_t\, dxdt. \end{aligned} \tag{3.30}$$

*Local version*

The same idea of multiplying also by $\eta^2$ allows as to derive local versions of the time derivative estimates under some assumptions. We have

$$\int w_t\, u_t\eta^2\, dx = -\int \nabla w \cdot \nabla w_t \eta^2\, dx + \int f w_t\eta^2\, dx - 2\int \nabla w w_t \nabla\eta\eta\, dx$$

so that

$$\int w_t\, u_t\eta^2\, dx + \frac{1}{2}\frac{d}{dt}\int |\nabla w|^2\eta^2\, dx = \int f w_t\eta^2\, dx - 2\int \nabla w w_t \nabla\eta\eta\, dx.$$

In order to proceed, assume that $u$ is bounded, $|u| \le M$, and that $\Phi_u(s) \le c|s|$ for $|s| \le M$ (note: this happens for the PME). Then, $|w_t| \ge c|u_t|$ and, integrating in space in $\Omega$ and in time from $\tau > 0$ to $T$ we get

$$\begin{aligned} &\frac{1}{c}\iint (w_t)^2\eta^2\, dxdt + \int |\nabla w(T)|^2\eta^2\, dx \\ &\quad\le \int |\nabla w(\tau)|^2\eta^2\, dx + C\iint f^2\eta^2\, dx + C\iint |\nabla w|^2|\nabla\eta|^2\, dxdt. \end{aligned} \tag{3.31}$$

The last term is supposed to be bounded by the first local estimate (3.24).

### 3.2.6 *The $BV$ estimates*

We consider the solutions of the Dirichlet problem. We differentiate the equation with respect to $t$ and put $v = \partial_t u$ to get the equation

$$\partial_t v = \Delta(\Phi_u(x,u)v). \tag{3.32}$$

We multiply by $p(\Phi_u(x,u)v)$ where $p$ is an approximation of the sign function with the properties already mentioned above: $p \in C^1(\mathbb{R})$ be such that $0 \le p \le 1$, $p(s) = 0$ for $s \le 0$, $p'(s) > 0$ for $s > 0$. Then, putting $w = \Phi_u(x,u)v$ and integrating in $\Omega$, we get

$$\int p(w)\partial_t v\, dx = \int \Delta(w)p(w)\, dx = -\int p'(w)|\nabla w|^2\, dx \le 0.$$

Therefore, $\int p(w)\partial_t v\, dx \le 0$. Now we let $p$ tend to the sign function and observe that $\operatorname{sign}(w) = \operatorname{sign}(v)$ and $\partial_t v \operatorname{sign}(v) = |v|_t$ a.e. to conclude that

$$\frac{d}{dt}\int |\partial_t u|\, dx \le 0. \tag{3.33}$$

Actually, we can let $p$ tend to the function $\operatorname{sign}^+$ (resp. $\operatorname{sign}^-$) to obtain the partial results

$$\frac{d}{dt}\int (\partial_t u)_+\, dx \le 0 \left(\frac{d}{dt}\int (\partial_t u)_-\, dx \le 0\right). \tag{3.34}$$

Together, they imply (3.33).

*Space estimates*

When $\Phi$ does not depend explicitly on $x$, this trick can be repeated with any space derivative, putting $v = \partial u/\partial x_i$. We also get

$$\frac{d}{dt}\int |\partial_i u|\,dx \le 0. \tag{3.35}$$

and the corresponding estimates for the positive and negative signs.

With these estimates we can control $u$ in the space $W^{1,1}(Q)$ if the initial data satisfy certain estimates. It is important to note that, when dealing with more general equations by approximation and passing to the limit in the approximations, the $L^1$ estimates obtained here may become estimates in the space of measures (since bounded sets in $L^1$ are not closed under weak convergence). Therefore, the estimates become estimates in the space of functions of bounded variation, $BV(Q)$.

## 3.3 Properties of the PME

The mathematical study of the PME and the GPME has a drawback common to all nonlinear theories: the absence of good representation formulas for the solutions in terms of the data; think of the role of the Gaussian kernel in the heat equation and the Green function in the Laplace equation. On the other hand, the very simplicity of the PME implies a number of interesting properties of other types, like scaling invariance, conservation laws and dissipation laws, that play a big role as technical tools. This properties hold for the GPME as long as $\Phi$ does not depend on $x$.

### 3.3.1 *Elementary invariance*

We assume the restriction $\Phi(u)$ in this subsection.

*Translations*

The HE, the PME, the FDE and more generally, the GPME, are invariant under displacement of the coordinate axes, since their behaviour is homogeneous in space and time. To be specific, if $u(x,t)$ is a solution of the PME defined in a space-time domain $Q$, then, for every $h \in \mathbb{R}^d$ and $\tau \in \mathbb{R}$ the function

$$\widehat{u}(y,s) = u(y-h, s-\tau) \tag{3.36}$$

is also a solution, now defined in the translated domain

$$Q' = Q + (h,\tau) = \{(x+h, t+\tau) : (x,t) \in Q\}. \tag{3.37}$$

*Space symmetries*

The PME and its relatives mentioned above are invariant under the symmetry with respect to a coordinate space hyperplane. Thus, if $u(x,t)$ is a solution in a domain $Q$, so is

$$\widehat{u}(y,t) = u(-y_1, y_2, \ldots, y_d, t), \tag{3.38}$$

defined in $Q'$ the domain of space-time that is symmetric of $Q$ respect to the symmetry w.r.t. $x_1 = 0$. The same happens for any other space variable. By iteration, we may consider symmetries in a number of coordinates.

*Space rotations*

Indeed, we can perform any rigid motion in space since the Laplacian commutes with all the transformations in the orthogonal group. If $A$ is the matrix of such a transformation and $u(x,t)$ is a solution in a domain $Q$, so is

$$\widehat{u}(y,t) = u(Ay, t), \tag{3.39}$$

defined in $Q' = \{(A^{-1}x, t) : (x,t) \in Q\}$. These arguments apply to any filtration equation $\partial_t u = \Delta\Phi(u)$.

*Sign change*

The filtration equation is invariant under the symmetry $u \mapsto -u$, if we change the nonlinearity $\Phi$ into $\widehat{\Phi}(s) = -\Phi(-s)$. To be precise, if a function $u$ is a solution of Problem HDP with initial data $u_0$ and nonlinearity $\Phi$, then $\widehat{u}(x,t) = -u(x,t)$ is a solution with data $\widehat{u}_0(x) = -u_0(x)$ and nonlinearity $\widehat{\Phi}(s) = -\Phi(-s)$.

### 3.3.2 *Scaling*

The HE, the PME and the FDE also share a powerful property inherited from the power-like form of the nonlinearity. This is the invariance under a transformation group of homotheties, usually known *the scaling group*. Indeed, whenever $u(x,t)$ is a classical solution of the equation $\partial_t u = \Delta(|u|^{m-1}u)$, the rescaled function

$$\tilde{u}(x,t) = K\ u(Lx, Tt) \tag{3.40}$$

is also a solution if the three real parameters $K, L, T > 0$ are tied by the relation

$$K^{m-1}L^2 = T. \tag{3.41}$$

We get in this way a two-parameter family of transformed solutions.

We can further restrict the family to a one-parameter family by imposing another condition. This happens for instance when the solutions are defined in the whole space and we impose the condition of preserving the $L^p$-norm of the data or the solution. In the first case, it reads

$$\int_{\mathbb{R}^d} |u|^p(x,0)\,dx = \int_{\mathbb{R}^d} K^p\,|u|^p(Lx,0)\,dx, \tag{3.42}$$

which implies the condition $K^p = L^n$. This allows as to determine two parameters in terms of the third, and we can choose at will the free parameter but for some exceptional cases. See Problem 3.2.

As a first practical application, in Section 4.4 we will impose the conservation of the $L^1$ norm in time and we will find the source solutions, probably the most relevant example of the whole theory.

Later on we will introduce classes of generalized solutions (weak, strong, mild, ...) and we will show that scaling applies to them.

### 3.3.3 *Conservation and dissipation*

These arguments apply to any equation $\partial_t u = \Delta\Phi(x,u)$ with $\Phi(x,0) = 0$ and $\Phi$ non-decreasing. The space domain is $\mathbb{R}^d$, $d \geq 1$.

*Mass conservation*

Given a classical solution $u(x,t)$ of the CP for the Filtration equation, we can multiply the equation by a cut-off function $\zeta(x)$ and integrate to obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d} u(x,t)\zeta(x)\,dx = \int \Delta\Phi(x,u)\zeta\,dx = \int \Phi(x,u)\Delta\zeta\,dx.$$

If $u$ is integrable in space and goes to zero at infinity then we may let $\zeta \to 1$ and get in the limit

$$\int u(x,t)\,dx = \int u(x,0)\,dx. \tag{3.43}$$

This is called mass conservation.

The same argument holds for the IBVP posed in a bounded domain with zero Neumann data $\partial_n\Phi(x,u) = 0$, since $\zeta = 1$ is an admissible multiplier (i.e., it does not produce extra boundary terms when integrating by parts). In the case of Dirichlet data $u = 0$ on the boundary (and $u \geq 0$), we do not get conservation but decrease

$$\frac{d}{dt}\int u(x,t)\,dx \leq 0. \tag{3.44}$$

These formal computations will be carefully justified for the classes of weak solutions that make up the bulk of our theory.

*Conservation of the first moment*

Assume that the solution of the Cauchy Problem for the filtration equation is such that the integral $\int |x|\,u(x,t)\,dx$ is finite. Then, using that $\zeta$ vanishes for large $|x|$ and that $\Delta x_i = 0$, we formally get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} x_i u(x,t)\zeta(x)\,dx &= \int \Delta\Phi(x,u)\,x_i\zeta\,dx \\ &= \int \Phi(x,u)x_i\Delta\zeta\,dx + 2\int \Phi(x,u)\partial_i\zeta\,dx.\end{aligned}$$

Passing to the limit $\zeta = 1$ we get the result

$$\frac{d}{dt}\int x_i u(x,t)\,dx = 0. \tag{3.45}$$

This result is still true in one dimension for the problem posed in $\Omega = (0,\infty)$ with zero boundary data at $x = 0$ and suitable decay at infinity, since the boundary terms obtained when integrating by parts both vanish. But the result is not true for the Dirichlet or Neumann problems in general.

*Conservation for the homogeneous Dirichlet problem*

When such a problem is posed in a bounded smooth domain $\Omega$ we may use as a multiplier the solution $\zeta$ of the problem

$$\Delta\zeta = -1 \quad \text{in } \ \Omega, \qquad \zeta = 0 \quad \text{in } \ \partial\Omega, \tag{3.46}$$

to get the estimate

$$\frac{d}{dt}\int \zeta(x)\,u(x,t)\,dx = -\int \Phi(u(x,t))\,dx. \tag{3.47}$$

*Dissipation and the $L^p$ norms*

The following formal computation works for the solutions of the CP that tend to zero at infinity, and also for the solutions of the homogeneous Dirichlet and Neumann problems:

$$\begin{aligned}\frac{d}{dt}\int |u|^p\,dx &= p\int |u|^{p-2}u\Delta\Phi(u)\,dx\\ &= -p(p-1)\int \Phi_u(u)|u|^{p-2}|\nabla u|^2\,dx \le 0,\end{aligned} \tag{3.48}$$

which shows that the $L^p$-norm decays with time. Moreover, integration in time gives

$$\int |u(x,t)|^p\,dx + p(p-1)\int_0^t \Phi_u(u)|u|^{p-2}|\nabla u|^2\,dx = \int |u_0(x)|^p\,dx. \tag{3.49}$$

The second integral is therefore finite when $t\to\infty$ and measures the amount of dissipation of the $L^p$-norm in time.

## 3.4 Alternative formulations of the PME and associated equations

There are some alternative formulations of the PME, where the lack of parabolicity is seen in a slightly different way. There are also some equation that can be derived from the PME $\partial_t u = \Delta(|u|^{m-1}u)$ through transformations.

### 3.4.1 *Formulations*

(i) One of alternative formulations consists in making the change of variables $|u|^{m-1}u = w$ (or simply, $w = u^m$ if $u \ge 0$). Formally, when $m > 1$ we arrive at the equation

$$\partial_t w = m\,|w|^n \Delta w, \tag{3.50}$$

with exponent $n = (m-1)/m \in (0,1)$. Now, we fall into the theory of nonlinear equations in non-divergence form, and it is immediately seen that the equation is parabolic for $|w| > 0$ and degenerates at $w = 0$.

(ii) The second change is the *pressure* formulation introduced in Section 1.1.2, that is used for non-negative solutions, mostly when $m > 1$, and uses the variable $v = cu^{m-1}$. If $c = m/(m-1)$ we get

$$\partial_t v = (m-1)\,v\Delta v + |\nabla v|^2, \tag{3.51}$$

which is again non-divergence. Cf. formulas (1.3), (1.4) and the physical interpretation of Section 2.1, see (2.7), hence the usual name of pressure variable that we will keep. It has an extra gradient term, but the nonlinearity of the right-hand side is homogeneous quadratic in $u$, and this is very useful for many calculations.

A technical detail: sometimes the equation is written as $\partial_t u = \Delta(u^m/m)$. Then we may define the pressure as $v = (1/(m-1))u^{m-1}$ and the equation for the pressure is still (3.51).

If the equation is the GPME the calculations are proposed as Problem 3.6.

### 3.4.2 *Dual equation*

Suppose that we have a smooth solution the GPME defined in the whole space $\mathbb{R}^d$ for $0 < t < T$ and let us assume that $u(t) \in L^1(\mathbb{R}^d)$ for all $t$. Assume also that $d \geq 3$. We can take the Newtonian potential of $u$ at every time $t > 0$ to get

$$v(t) = N(u(t)) = E_d \star u(t)$$

so that $v$ is a uniquely defined function in the Marcinkiewicz space $M^{d/(d-2)}(\mathbb{R}^d)$ and $\Delta v(t) = -u(t)$ (see more on potentials in Section A.6, and on Marcinkiewicz spaces in Section A.5). Then,

$$\partial_t v(x,t) = E_d \star (\Delta\Phi(x,u(t)) = -\Phi(x,u(t)).$$

In other words, $v$ solves the nonlinear evolution equation $\partial_t v = -\Phi(x,u)$, i.e.,

$$\partial_t v = \widetilde{\Phi}(\Delta v) \tag{3.52}$$

where $\widetilde{\Phi}(x,u) = -\Phi(x,-u)$. This is called the *dual filtration equation.* It is a formally parabolic, in principle degenerate, equation in non-divergence form. In the case of the PME the dual equation is just

$$\partial_t v = |\Delta v|^{m-1}\Delta v. \tag{3.53}$$

Some questions are better understood in terms of the dual equation satisfied by the potentials, see e.g. Chapter 13. The key point is that solutions of the dual equation have a better regularity since they are potentials. This is a fundamental calculation, obtained after differentiation and integration by parts

$$\frac{d}{dt}\int_{\mathbb{R}^d} |\nabla v|^2\,dx = -2\int_{\mathbb{R}^d} u\,\Phi(x,u)\,dx. \tag{3.54}$$

Another useful observation comes from integration of equation (3.52) in time. We get for $0 \le s < t < T$,

$$v(x,t) + \int_s^t \Phi(u(x,\tau))\,d\tau = v(s,t)$$

so that bounds on $v$ at time $s$ and a condition like $v \ge 0$ imply bounds on the integrated function $U(x,t) = \int \Phi(u)\,dt$. This function is the function to be controlled in some theories.

In dimensions $d = 1,2$ the potential approach in the whole space has some difficulties that we need not treat at this point. Let us give some details about the treatment in a bounded domain. Using the Green function with zero boundary conditions, $G = G_\Omega(x)$, as explained in Section A.6 to define $\mathcal{G}f \in W_0^{1,1}(\Omega)$ by $v(x,t) = \mathcal{G}u(t)(x) := \int_{\mathbb{R}^d} u(y,t)G_\Omega(x,y)\,dy$. Then, $-\Delta v(t) = u(t)$ again, but now we have

$$\partial_t v = \mathcal{G}(\partial_t u) = \mathcal{G}\Delta\Phi(u).$$

Now, for general smooth functions it is not true that $\mathcal{G}\Delta = -I$ (because of the boundary conditions) and we can only conclude that

$$\partial_t v = h - \Phi(u), \tag{3.55}$$

where $h$ is a harmonic function with boundary conditions $h|_{\partial\Omega} = \Phi(u)|_{\partial\Omega}$. Of course, if $\Phi(u)$ satisfies zero Dirichlet data we have $h = 0$ and the same type of dual equation holds.

We will return to potentials and dual equations in Chapter 13.

### 3.4.3 *The p-Laplacian equation in $d = 1$*

When $d = 1$ and $u$ is depends on two variables $(x,t) \in Q_T$, we can imagine that the filtration equation is just the condition that makes a differential form exact. A bit of reflection shows that such a form is $\omega = u\,dx + \Phi(x,u)_x dt$. Therefore, we can define a function of two variables

$$v(x,t) = v(x_0,t_0) + \int_\gamma (u\,dx + \Phi(x,u)_x dt) \tag{3.56}$$

along any piecewise continuous path that joins the fixed point $(x_0,t_0)$ to any $(x,t) \in Q_T$. The integral does not depend on the path $\gamma$. We easily see that $v_x = u$ and that $v$ satisfies the PDE

$$\partial_t v = (\Phi(x,v_x))_x. \tag{3.57}$$

If $\Phi(u) = |u|^{m-1}u$, then the equation for $v$ is the standard $p$-Laplacian equation

$$\partial_t v = (|v_x|^{p-2} v_x)_x, \quad p = m+1. \tag{3.58}$$

This calculation performed for classical solutions has to be justified in the theory when dealing with generalized solutions.

Unfortunately, this relationship between the equations (PME and PLE) does not extend to higher space dimensions.

## Notes

**Section 3.1.** References [357], [239] and [371] can be consulted as the need arises.

**Section 3.2.** The proof of the energy bound for $\nabla\Phi(u)$ and $z_t$ is adapted from Bénilan and Crandall [88].

**Section 3.3.** Scaling arguments are well known and very successful in the applied literature, see Barenblatt's book [63] and our Chapter 16. In Chapter 17 we will use scaling and special solutions that are scaling-invariant, in combination with symmetrization and mass comparison, as basic tools in obtaining basic estimates for the solutions.

**Section 3.4.** The dual equation was used in [202] in the study of extinction in fast diffusion, in [191] in the study of uniqueness of general weak solutions, and in [103] in the study of self-similarity and asymptotics.

The $p$-Laplacian equation has a very extensive literature, cf. the monograph [209].

## Problems

**Problem 3.1** Signed PME. Make the change of variables $w = |u|^{r-1}u$ for some $r > 0$. Obtain the equation for $w$

$$w_t = m\,|w|^n \Delta w + c|w|^{n-2} w |\nabla w|^2, \tag{3.59}$$

with $n = (m-1)/r$ and $c = m(m-r)/r$.

**Problem 3.2** Scaling transformation.

(i) Prove that the scaling transformation that preserves the PME and the $L^p$ norm of the data can be solved for $K$ and $L$ in terms of $T$ unless $n(1-m) = 2p$ (which implies $m < 1$ since $p \ge 1$). Find the explicit expressions

$$K = T^{n/(n(m-1)+2p)}, \quad L = T^{p/(n(m-1)+2p)}.$$

(ii) Find the admissible scaling if $n(1-m) = 2p$.

(iii) Explore the possibilities of taking $K$, $L$ and $T$ negative. Derive for $L = -1$ an invariance under symmetry. What happens when $K = -1$? Is $T = -1$ admissible?

**Problem 3.3** Consider the homogeneous GPME.

(i) Prove the boundedness of $\int j(u(t))\,dx$ in formula (3.13) for finite times when $f$ is bounded and $|p(u)| \le C_1 j(u) + C_2$.

(ii) Prove that for every $r \ge 1$

$$\int |u(t)|^r\,dx \le \int |u_0|^r\,dx + r\iint f|u|^{r-1}\,dxdt.$$

Derive from this that whenever $\int_T^\infty \|f(t)\|_r\,dt$ is bounded then $\|u(t)\|_r$ is uniformly bounded for $0 \le t < \infty$.

(iii) Put $f = 0$ and obtain the estimate

$$\int |u(t)|^r\,dx \le \int |u_0|^r\,dx.$$

**Problem 3.4** There is another useful conservation law for the GPME when the problem is posed in an exterior domain $\Omega = \mathbb{R}^d - K$, where $K$ is a compact set with smooth boundary. Prove that in dimensions $d \ge 3$ there exist a solution $\zeta > 0$ of

$$\Delta\zeta = 0 \quad \text{in } \Omega, \qquad \zeta = 0 \quad \text{in } \partial\Omega, \tag{3.60}$$

with the additional condition $\zeta \to 1$ as $|x| \to \infty$. Show that if $u$ is a classical solution of the exterior problem with $u = 0$ on $\partial\Omega$, and $u$ decays at infinity so that $u(\cdot,t)$ is integrable in space, then

$$\frac{d}{dt}\int \zeta(x)\,u(x,t)\,dx = 0.$$

This law is fundamental in the study of large time asymptotics done in [124].

**Problem 3.5** Prove the following local energy estimate as a variant of estimate (3.24). We take $f = 0$ for simplicity. For every $\eta \in C_c^2(Q_T)$, we have

$$\iint_{Q_T} |\nabla\Phi(u)|^2\eta\,dxdt = \frac{1}{2}\iint_{Q_T} (\Phi(u))^2\Delta\eta\,dxdt + \iint_{Q_T} \Psi(x,u)\eta_t.$$

This means that there is a bound for $\nabla\Phi(u)$ in $L^2_{\rm loc}(Q_T)$ in terms of the local norms of $\Phi(u)$ in $L^2_{\rm loc}(Q_T)$ and $\Psi(x,u)$ in $L^1_{\rm loc}(Q_T)$.

**Problem 3.6** Take the GPME $\partial_t u = \Delta\Phi(u)$ and take as new variable $w = \Phi(u)$. Putting $\beta(\cdot) = \Phi(\cdot)^{-1}$, the equation becomes

$$\partial_t\beta(w) = \Delta w. \tag{3.61}$$

This generalizes (3.50). If $\beta$ is differentiable the equation becomes

$$\beta'(w)\partial_t w = \Delta w. \tag{3.62}$$

which is convenient in the theory of fast diffusion. Write down the calculation for the so-called superslow diffusion equation where $\Phi(u) = e^{-1/u}$.

**Problem 3.7**

(i) In order to generalize the pressure change to the case $\partial_t u = \Delta\Phi(u)$, we write

$$v(x,t) = P(u(x,t)), \quad P(u) = \int_a^u \frac{\Phi'(u)}{u}\,du. \tag{3.63}$$

Writing $\nabla\Phi(u) = u\nabla v$, $\partial_t v = \Phi'(u)\partial_t u/u$, the equation for $v$ is then

$$\partial_t v = \sigma(v)\Delta v + |\Delta v|^2. \tag{3.64}$$

where $\sigma(v) = \Phi'(u)$. Work out the details and compare with (3.51).

(ii) Assume that $\Phi$ is $C^2$ for $u > 0$. Prove that $\sigma$ is $C^1$ at $v = 0$ if and only if there exists

$$\sigma'(0) = \lim_{u\to 0} \frac{u\Phi''(u)}{\Phi'(u)}.$$

(iii) Calculate the pressure in the superslow diffusion case $\Phi(u) = e^{-1/u}$, $u \ge 0$. See further details in [125].

**Problem 3.8** Derive carefully the associated equations (3.53), (3.55), and (3.58).

**Problem 3.9*** Try to derive the a priori estimates of Section 3.2 for an inhomogeneous equation of the form

$$\partial_t u = \sum_{i=1}^d \partial_{x_i}\left(a_{ij}(x,t)\partial_{x_j}\Phi(u)\right). \tag{3.65}$$

where $(a_{ij})$ is a symmetric positive-definite matrix defending smoothly on $x$ and $t$.

# 4

# BASIC EXAMPLES

In this chapter we present five interesting types of solutions that will play a role in the development of the theory: separate-variables solutions, travelling waves, source-type solutions, blow-up solutions and constant-height solutions. Other solutions, like dipoles and general fronts, serve to complete the picture.

We will use the presentation to introduce and use important concepts for the sequel, like scaling, limit solutions, finite propagation, free boundaries, existence under optimal conditions, blow-up, limited regularity, and initial traces. These questions will receive a full rigourous treatment later on.

Solutions with changing sign will also be considered; therefore the equation by default is the signed PME, $u_t = \Delta(|u|^{m-1}u)$. The main emphasis should be laid however on non-negative solutions of the standard PME. The last two sections can be skipped in a first reading.

## 4.1 Some very simple solutions

The PME admits a number of explicit solutions that play an role in developing the theory. Without any doubt, the simplest solutions are the ones that do not change in time, called *stationary solutions*. They satisfy the condition $u_t = 0$, hence $u$ depends only on the space variable, $u = u(x)$, and $w = u^m$ has to satisfy the equation

$$\Delta w = 0. \tag{4.1}$$

Therefore, any harmonic function $w(x)$ provides a stationary solution of the PME putting $u(x,t) = w(x)^{1/m}$ if $w \geq 0$, $u(x,t) = |w(x)|^{1/m}\text{sign}(w)$ for signed solutions. If in particular we ask for solutions defined and non-negative in the whole space, then such solutions must be constant. We call such solutions *trivial solutions*. They are the simplest solutions.

In one dimension the rest of the stationary solutions are linear functions, $u^m = Ax + B$, $A \neq 0$. If we insist on non-negativity, then we must restrict the definition to the hyperspace where $u > 0$. Thus, the solutions defined for $x > 0$ and vanishing at the lateral boundary $x = 0$ are given by the formula $u = Cx^{1/m}$, $C \in \mathbb{R}$. Note that they are not $C^1$ functions on the boundary! The restriction $x > 0$ is not necessary if signed solutions are admitted but then we have to worry about the concept of solution at the transition point $x = 0$. This will be the task of the next chapter. The same applies to stationary 'solutions' in two dimensions like

$$w(x,y,t) = x^2 - y^2 + c, \quad c \in \mathbb{R}$$

in $d = 2$ with $w = |u|^{m-1}u$. Here, the problematic locus is $x^2 = y^2 - c$.

## 4.2 Separation of variables

For our first model of non-trivial special solution, we follow the typical procedure of the Fourier approach for the linear heat equation (which is formally the case $m = 1$ of the PME), and we make the ansatz

$$u(x,t) = T(t)\, F(x). \tag{4.2}$$

This leads to separate equations for $T(t)$, the *time factor*, and $F(x)$, called the *space profile*:

$$T'(t) = -\lambda T(t)^m, \qquad \Delta F^m(x) + \lambda F(x) = 0. \tag{4.3}$$

The constant $\lambda$ is in principle arbitrary, but it serves to couple both equations. When it is zero, the solutions are stationary in time, a case already discussed. Assuming in the sequel that $\lambda \neq 0$, the first equation is easy to solve and gives

$$T(t) = (C + (m-1)\lambda t)^{-1/(m-1)}.$$

Therefore, we have reduced finding these special solutions of the PME to solving the nonlinear elliptic equation for $F$, the right-hand formula of (4.3). This is a nonlinear version of the eigenvalue problem to be solved in the Fourier analysis of the heat equation. The usual process is also the same: a domain $\Omega$ is chosen and boundary conditions are assigned on $\partial\Omega$; the boundary problem is then solved. But the results are remarkably different. As usual, the analysis depends on the sign of $\lambda$.

### 4.2.1 *Positive $\lambda$. Nonlinear eigenvalue problem*

The first curious feature of the nonlinear elliptic problem is that the general value $\lambda > 0$ can be reduced to $\lambda = 1$ by changing appropriately the value of $F$. In fact, if $F_1(x)$ is a solution of the equation with $\lambda = 1$,

$$\Delta(|F_1|^{m-1}F_1) + F_1 = 0. \tag{4.4}$$

the transformation

$$F(x) = \mu F_1(x), \quad \mu = \lambda^{1/(m-1)}, \tag{4.5}$$

is a solution of the original equation with $\lambda > 0$, $\lambda \neq 1$, $\Delta(|F|^{m-1}F) + \lambda F = 0$, and conversely. Equation (4.5) is the simplest case of what we call a scaling transformation.[1]

[1]We have seen in Subsection 3.3.2 transformations in which the space and time variables are also changed by homothety.

It is convenient to further change the variable $F$ into $G = |F|^{m-1}F$ and write the Nonlinear elliptic equation as

$$\Delta G(x) + |G(x)|^{p-1}G(x) = 0, \tag{4.6}$$

where $p = 1/m \in (0,1)$. When this equation is posed in a bounded domain with regular boundary and we take zero boundary conditions, there exists precisely one positive solution of the problem, and not many as in the linear case. In case $\Omega$ has a special shape, like a ball or a cube, the solution is easy to find by standard ODE methods, see Problem 4.1 below. In the case of a general bounded domain, the existence and uniqueness of a positive solution of (4.6) can be obtained by variational methods, as in [511]. We will find an indirect proof in the Chapter 5.

Summing up, the problem is quite different from the linear eigenvalue problem: there is existence and uniqueness of a positive solution for all $\lambda > 0$. Let us note that the solution is continuous (actually, Hölder continuous) up to the boundary of $\Omega$, and $C^\infty$ smooth inside $\Omega$.

Granted the existence of that solution, $F(x;\lambda,\Omega)$, the semi-explicit solution that we get for the PME has the form $u(x,t) = (C + (m-1)t)^{-1/(m-1)}F(x)$, that we can also write as

$$u(x,t) = ((m-1)(t-t_0))^{-1/(m-1)}F(x), \tag{4.7}$$

where $t_0$ is arbitrary. This form is called a separated variables solution. It is clear that the formula produces a classical solution of the PME in the space-time domain $\Omega \times (t_0, \infty)$ and takes on zero boundary data. There is no essential restriction in assuming that $t_0 = 0$ (since the difference is a time translation), but the family of solutions depends on $\Omega$ through the profile $F$ in a non-trivial way. Let us point out a quite strange feature: The initial data at $t = t_0$ is $u(x,t_0) \equiv +\infty$, something unheard of in the linear case.

**Note** The method does not produce any classical solution defined in the whole space $\mathbb{R}^d$ for the PME.

### 4.2.2 *Negative $\lambda = -l < 0$. Blow-up*

We get solutions with time factor

$$T(t) = (C - (m-1)l\,t)^{-1/(m-1)} = ((m-1)l\,(t_0 - t))^{-1/(m-1)},$$

which blows up in finite time. We can again reduce us to the case $l = 1$ and solve the elliptic equation for the profile

$$\Delta F^m(x) = F(x) \tag{4.8}$$

after a scaling. We cannot solve that problem in the same setting as before and obtain non-trivial solutions. But it is easy to find radially symmetric solutions defined in the whole space by solving the corresponding ODE. A particular solution is well-known, and will be studied below in Section 4.5, devoted to presenting blow-up solutions, see formula (4.44).

**Remark** We have concentrated on the values $m > 1$. Separable solutions can be constructed for $m < 1$, but they take the form $U(x,t) = (T-t)^{1/(1-m)}F(x)$. $F$ still solves an elliptic equation, see Problem 5.15.

## 4.3 Planar travelling waves

In the second model, we look for solutions of the form

$$u = f(\eta), \quad \eta = x_1 - ct \in \mathbb{R}. \tag{4.9}$$

This type of solution represents a wave that moves in time along an axis (here, $x_1$), without changing its shape. The form does not depend on the variables $x_2, \dots, x_d$, hence the name planar (that is usually omitted so that they are simply known as travelling waves, TWs for short). The parameter $c$ is the *wave speed*. We may assume that $c \neq 0$, since for $c = 0$ we find again the stationary solutions, and the case $c < 0$ can be reduced to $c > 0$ by a reflection (changing $u(x,t)$ into $u(-x,t)$ we find another solution of the equation moving in opposite direction). Note that a wave with $c > 0$ travels in the positive direction of the axis. Note finally that TWs are one-dimensional in their space dependence.

We have taken as wave direction a coordinate axis, but the invariance under rotations explained in Section 3.3 above allows us to find a wave that travels along any straight direction $\mathbf{n}$ of space $\mathbb{R}^d$. The formula would then be (4.9) with $\eta = \mathbf{x} \cdot \mathbf{n} - ct$.

Taking thus $c > 0$ fixed, and substituting (4.9) into la $u_t = \Delta u^m$ we arrive at the ODE

$$(f^m)'' + cf' = 0, \tag{4.10}$$

where prime indicates derivative respect to $\eta$. Integrating once we get

$$(f^m)' + cf = K, \tag{4.11}$$

with arbitrary integration constant $K \in \mathbb{R}$. In order to choose this constant we think of the situation where the wave advances against an 'empty region', i.e., we want $f(\eta) = f'(\eta) = 0$ for all $\eta \gg 0$. This condition leads to the conclusion that $K = 0$, so that (4.11) becomes

$$mf^{m-2}f' + c = 0, \tag{4.12}$$

which is easily integrated to give

$$\frac{m}{m-1}f^{m-1} = -c\eta + K_1 = c(\eta_0 - \eta). \tag{4.13}$$

This conclusion is very neat, since, according to our definition of the mathematical pressure, cf. formulas (1.4), (2.7), it means that the pressure is a linear function

$$v(x,t) = K_1 - c(x - ct) = c(x_0 + ct - x). \tag{4.14}$$

This is a perfectly valid classical solution of the PME in the expanding region $\{(x,t) : x < x_0 + ct\}$, where $u$ is positive.

*Analytical problems and ways of solution*

However, this conclusion is not satisfactory, since formula (4.14) fails to provide a solution of the PME in the whole space, which is the natural framework for a TW. The problem is serious; actually, $v$ becomes negative for $x > x_0 + ct$, a situation that goes squarely against the physics of many problems. One such problem was the heat transfer that the Moscow group was trying to solve around 1950. The way out of this dilemma is a crucial moment in the history of the PME, and a powerful argument in favour of the influence of the applications on the theory. It consists of two parallel moves: *drastically modifying formula* (4.14), and *abandoning the concept of classical solutions.* Both are quite natural today, but we are talking about 1950.

Indeed, the solution of the difficulty is quite natural and relies on the *strategy of the limit problem:* to solve an approximate problem for which the difficulty is not present, to pass to the limit and to examine the obtained result. Specifically, we take as boundary condition for equation (4.11)

$$f(\infty) = \varepsilon, \quad f'(\infty) = 0, \tag{4.15}$$

so that the problem is non-degenerate. We obtain for $K$ the value $K = \varepsilon\, c > 0$. We then write (4.11) as

$$f' = -c\frac{f-\varepsilon}{mf^{m-1}} \tag{4.16}$$

which is an ODE in separate-variable form, immediate to integrate, at least graphically and implicitly. We are interested in solutions $f \geq \varepsilon$.

**Proposition 4.1** *For every $\varepsilon \in (0,1)$ there exists a unique solution $f_\varepsilon(\eta)$ of equation* (4.16) *satisfying the initial condition $f_\varepsilon(0) = 1$. It has end condition $f_\varepsilon(\infty) = \varepsilon$. Moreover $f_\varepsilon : \mathbb{R} \to (\varepsilon, \infty)$ is a monotone decreasing and $C^\infty$ function such that $f_\varepsilon(-\infty) = \infty$. In the limit $\varepsilon \to 0$ we have*

$$\lim_{\varepsilon\to 0} \frac{m}{m-1} f_e^{m-1}(\eta) = c(\eta_0 - \eta)_+ \tag{4.17}$$

*with $\eta_0 = m/c(m-1)$. The limit is uniform in sets of the form $[a, \infty)$.*

We have taken the normalization value $f_\varepsilon(0) = 1$ without loss of generality. Since the equation is autonomous, we can get a one-parameter family of $C^\infty$ solutions $f > \varepsilon$ with $f \to \varepsilon$ as $\eta \to \infty$ by horizontal translation of the one obtained in the proposition. Since the proof of the proposition is based on a simple phase plane analysis, and we assume that the reader is familiar with the elements of that technique, we assign the task as Problem 4.2. We are also asking him/her to perform the graphical integration to get a visual evidence. Thanking

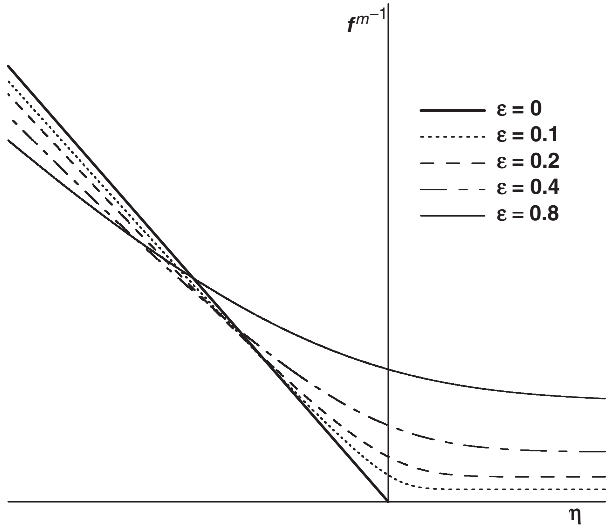


Figure 4.1: Travelling waves for $\varepsilon > 0$ and their limit.

the reader in advance, we will devote the spared space to discuss the meaning of the result.

### 4.3.1 *Limit solutions*

Inserting formula (4.17) into the form (4.9) and passing to the pressure, we get the formula

$$v(x,t) = c(x_0 + ct - x)_+. \tag{4.18}$$

Since it is obtained as a limit of perfectly safe classical solutions, a quite strong intuition developed in the applied sciences tells us that this qualifies as a valid physical solution in some sense to be made precise. In the meantime, we will call it a *limit solution.*

The introduction of limit solutions solves some problems and poses a number of other problems. Thus, we now have a concept of travelling wave to describe the movement of a mass of gas (or liquid, or heat) bordering on its right-hand side with empty space, a situation of enormous applied importance.

On the other hand, let us examine the problems. This limit solution is not a classical solution of the PME. Closer inspection shows that it is a broken version of the formula obtained by purely algebraic computations, (4.14), and it has a problem of differentiability at the line $x = x_0 + ct$, precisely, at the points where the equation passes from the classical state to the degenerate state. This set is called the *free boundary,* an important object of study as we have said.

Another problem of the concept of limit solution is the possibility of obtaining different solutions for the same problem, depending on the type of approximation used. We will introduce in subsequent chapters different concepts of *generalized solutions* that allow us to (i) solve the problem in a unique way, (ii) include the classical solutions if they exist, as well as their limits, and (iii) show existence in cases where there is no classical solution. Such an effort will prove that the limit solution we have just accepted is indeed a 'good solution'.

### 4.3.2 *Finite propagation and Darcy's law*

Travelling waves consist of space functions (called profiles) that propagate with constant speed without changing shape. They are also called *constant-shape fronts*. Such fronts exist also for the heat equation, but they have different form and properties. Let us examine more closely the differences between the PME and the HE. For the latter, the TWs have the form

$$u(x,t) = Ce^{c(ct-x)}.$$

On one hand, they are classical solutions. On the other hand, they are always positive, and reach the level $u = 0$ at $x = \infty$ after developing an infinitely long *exponential tail*. It is precisely this property of the heat equation, namely that non-negative solutions are actually positive everywhere, what is sometimes mentioned as an unphysical property of an otherwise quite effective model.

We have constructed an explicit (limit) solution of the PME that has a sharp and finite front separating the regions $\{u > 0\}$ and $\{u = 0\}$, and this front propagates in time with constant speed. This is the first appearance of the property of finite speed of propagation, usually called finite propagation, a fundamental property of the PME, that we will discuss at length later on.

#### *Darcy's law on the free boundary*

In trying to understand the lack of regularity of the TW at the free boundary, it is quite useful to go back to the modelling of a gas in a porous medium, Section 2.1. Putting $x_0 = 0$ without loss of generality, the pressure in the gas is given by

$$v(x,t) = c(ct - x)_+.$$

According to Darcy's law, the speed is $\mathbf{V} = -\nabla v = c\,\mathbf{e_1}$. In physical terms, every particle of whole mass of gas moves with the same speed and the pressure must grow linearly near the free boundary to account for Darcy's law. On the other hand, $v = 0$ on the empty region according to our mathematical definition (2.7).

Summing up, we have concluded that Darcy's law forces the gradient of $v$ to jump at the free boundary points $x = ct$.

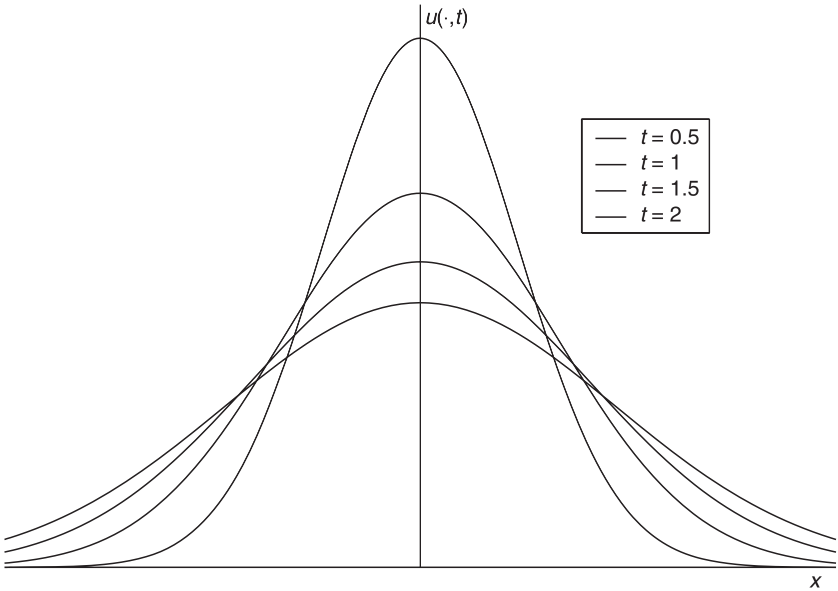


Figure 4.2: Fundamental solution of the heat equation.

## 4.4 Source-type solutions. Self-similarity

In the next example we look for the solution corresponding to PME flow starting from a finite mass concentrated at a single point of space, say, $x = 0$. A classical problem in the thermal propagation theory is to describe the evolution of a heat distribution after a point source release. In mathematical terms, we want to find a solution of the HE with initial data

$$u(x, 0) = M\,\delta(x), \tag{4.19}$$

where $M > 0$ and $\delta$ is Dirac's delta function. This is called in engineering a point source, hence the name *source solution* widely used in the Russian literature. Such type of solution is well-known in the case of the heat equation (i.e., for $m = 1$) and is called the *fundamental solution*, with formula

$$E(x, t) = M\,(4\pi t)^{-n/2} \exp\left(-x^2/4t\right). \tag{4.20}$$

The *Gaussian kernel*, as it is also known, plays a fundamental role in developing the PDE theory of the heat equation, and is also of paramount importance in the probabilistic approach to diffusion (central limit theorems).

This motivates the interest in the similar question about the existence of a source solution for our nonlinear diffusion equation, PME. Indeed, as we have indicated in Section 1.1, the source solution exists for $m > 1$ and, fortunately enough, it is explicitly given by a formula that we will now write as

$$\mathcal{U}(x,t;M)=t^{-\alpha}F(x\,t^{-\alpha/d}),\quad F(\xi)=(C-\kappa\,\xi^2)_+^{\frac{1}{m-1}}, \tag{4.21}$$

where

$$\alpha=\frac{d}{d(m-1)+2},\quad \kappa=\frac{(m-1)\alpha}{2md}. \tag{4.22}$$

We ask the reader to check that indeed it takes on a Dirac delta as initial trace, i.e., that

$$\lim_{t\to 0}\mathcal{U}(x,t)=M\,\delta_0(x), \tag{4.23}$$

in the sense of measures. The free parameter $C>0$ in formula (4.21) is in principle arbitrary; it can be uniquely determined by the condition of total mass, $\int U dx=M$, which gives the following relation between the 'mass' $M$ and $C$:

$$M=a(m,d)\,C^{\gamma},\quad \gamma=\frac{d}{2(m-1)\alpha}. \tag{4.24}$$

Note $a$ and $\gamma$ are functions of only $m$ and $d$; the exact calculation of $a$ is not needed at this point, cf. Section 17.5. We shall use quite often in the sequel the name ZKB solutions for the source solutions as explained in the Introduction; they are also quite widely known as Barenblatt solutions or Barenblatt–Pattle solutions. Notation: using the mass as parameter we denote it by $\mathcal{U}(x,t;M)$, or even $U_m(x,t;M)$ if the dependence on $m$ is important.

According to the previous calculations, putting

$$C=\kappa\xi_0^2\,M^{2(m-1)\alpha/d},$$

then formula (4.21) is transformed into

$$\mathcal{U}_m(x,t;M)=M\,\mathcal{U}_m(x,M^{m-1}t;1)=\frac{M^{2\alpha/d}}{t^{\alpha}}F_{m,1}\left(\frac{x}{(M^{m-1}t)^{\alpha/d}}\right), \tag{4.25}$$

where $F_{m,1}=(\kappa(\xi_0^2-\xi^2))_+^{1/(m-1)}$ is the profile with exponent $m$ and mass 1.

It is maybe a good idea to write the ZKB solution in terms of the pressure variable $v=u^{m-1}m/(m-1)$ and then we get the formula

$$\boxed{V_m(x,t;M)=\frac{(C\,t^{2\alpha/n}-b\,x^2)_+}{t}} \tag{4.26}$$

with $b=\alpha/2d$ and $C>0$ is a free parameter. We see that in terms of the pressure, the ZKB has a simpler expression, a parabolic shape for all $m>1$. This observation has strongly influenced a number of developments of the theory. About the shape see Problem 4.6.

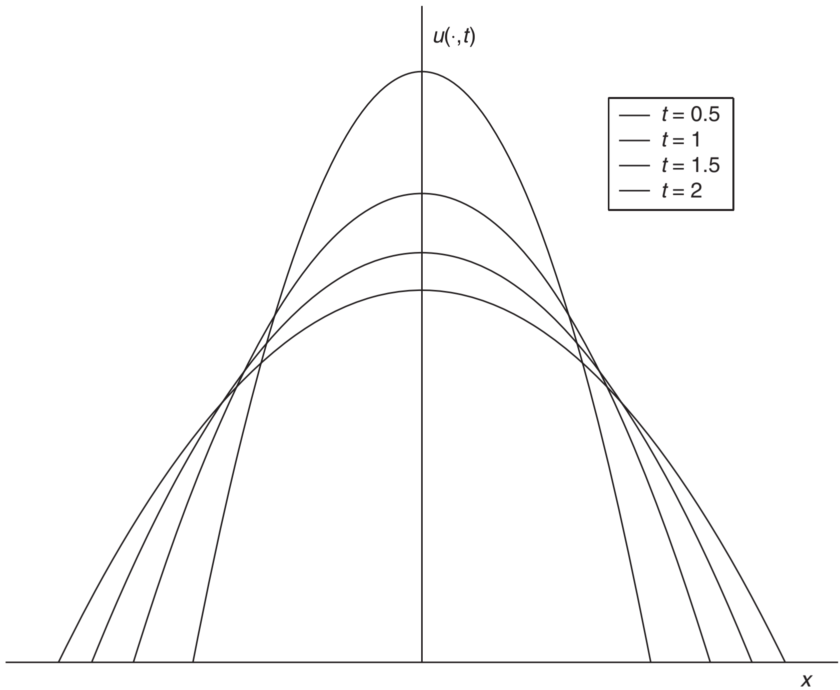


Figure 4.3: The ZKB solution of the PME.

We can also pass to the limit $m \to 1$ (with a fixed choice of the mass $M$) and obtain the fundamental solution of the heat equation,

$$\lim_{m\to 1} \mathcal{U}_m(x,t;M) = M\,E(x,t). \tag{4.27}$$

This is a relatively easy calculus result. We ask the reader to try his-her calculus ability.

### 4.4.1 *Comparison of ZKB profiles with Gaussian profiles. Anomalous diffusion*

We begin by pointing out that the ZKB solutions start from a point source and spread in space like $O(t^\beta)$. Since $\beta < 1/2$ for $m > 1$, this is a slower rate (i.e., with a smaller power) than the average spread rate $O(t^{1/2})$ of the Heat Equation; such a spread rate was indicated by Einstein in his famous 1905 paper [538] as the characteristic average spread rate of Brownian motion; observe furthermore than $\beta \to 0$ as $m \to \infty$. Such a deviation is not just a particular case, since we will show in Chapter 18 that the ZKB solutions represent the standard asymptotic behaviour of finite mass solutions both in size and spread rate; we conclude that the PME is an example of *anomalous diffusion* in the sense described for instance in [537].

Then, we notice the difference in propagation. While the HE solution travels immediately to the whole space, the PME solution is supported in the region $|x| \le r(t)$ behind the free boundary

$$r(t) = (C/\kappa)^{1/2} t^{\beta}.$$

Use of the maximum principle will allow us to conclude in later chapters that all weak solutions $u \ge 0$ of the PME with bounded and compactly supported initial data are located for positive and bounded times $t > 0$ in an expanding but still bounded region.

Secondly, the fundamental solution of the HE is a $C^\infty$ function, while the ZKB solutions are only Hölder continuous. Actually, the regularity depends on $m$ but the regularity of the pressure does not. It is Lipschitz continuous and not $C^1$, just as in the TWs.

Besides, it is not difficult to check that Darcy's law holds on the free boundary in the sense that

$$\lim_{|x| \to r(t)-} \nabla v = -r'(t), \tag{4.28}$$

where the limit is taken as $|x| \to r(t)$ but only for $|x| < r(t)$, i.e., only in the gas region. This formula equates the speed of the particles with the speed of the moving surface.

Let us finally say that the fundamental solution of the HE allows us to derive the whole theory of the equation using the representation formulas, a most powerful tool of linear analysis. This is not to be expected in the case of the PME where no valid equivalent of such formulas has been found. However, skillful use of the properties of the ZKB solution have allowed to obtain enormous progress in the theory of the PME. Therefore, carefully inspecting the properties of the ZKB is a most fruitful investment and we shall do it quite often. See in this direction Problem 4.5(vi). However, the development of the theory of the PME owes much to the other solutions mentioned in this chapter.

### 4.4.2 *Self-similarity. Derivation of the ZKB solution*

The most natural way of deriving formula (4.21) is using self-similarity, a most important concept in the theory that follows. It means that there is a scaling of the variables after which the ZKB become stationary solutions. Precisely, it holds that

$$u' = f(x'), \qquad \text{with} \qquad u' = ut^{\alpha}, \quad x' = xt^{-\beta}. \tag{4.29}$$

The self-similar form is then

$$\mathcal{U}(x,t) = t^{-\alpha} f(\eta), \quad \eta = xt^{-\beta}. \tag{4.30}$$

The exponents $\alpha$ and $\beta$ are called *similarity exponents*, and function $f$ is the *self-similar profile*. In particular, $\alpha$ is the density contraction rate, while $\beta$ is the space expansion rate. We have to determine exponents and profile so that the resulting function $U$ is a solution and has suitable additional data.

Self-similarity is a principal concept in mechanics and, generally speaking in the applied sciences. Note that the fundamental solution of the heat equation (4.20) is self-similar with exponents $\alpha = d/2$ , $\beta = 1/2$, and a Gaussian function as profile.

- In the case of the PME, we try the self-similar ansatz (4.30) in the PME. Since

$$\begin{aligned}\mathcal{U}_t &= -\alpha t^{-\alpha-1} f(\eta) + t^{-\alpha}\nabla f(\eta)\cdot x t^{-\beta-1}(-\beta)\\ &= -t^{-\alpha-1}(\alpha f(\eta) + \beta \nabla f(\eta)\cdot \eta),\end{aligned}$$

  and

$$\Delta(\mathcal{U}^m) = t^{-\alpha m}\Delta_x(f^m(xt^{-\beta})) = t^{-\alpha m-2\beta}\Delta_\eta(f^m)(\eta),$$

  the equation $\mathcal{U}_t = \Delta \mathcal{U}^m$ becomes

$$t^{-\alpha-1}(-\alpha f(\eta) - \beta\eta\cdot\nabla f(\eta)) = t^{-\alpha m-2\beta}\Delta f^m(\eta). \tag{4.31}$$

- We now eliminate the time dependence (this is a kind of separated variables argument). This implies a first relation between the exponents:

$$\alpha(m-1) + 2\beta = 1, \tag{4.32}$$

  and allows us to express one exponent in terms of the other (e.g., $\alpha$ in terms of $\beta$). We then get the *profile equation*,

$$\Delta f^m + \beta\eta\cdot\nabla f + \alpha f = 0, \tag{4.33}$$

  which is a nonlinear elliptic equation with a free parameter (say, $\beta$). We only need to specify the boundary or other conditions to get a well-specified *nonlinear eigenvalue problem.*
- We will see in Chapter 16 how to solve this problem for different values of $\beta$, and in another particular example in Section 4.6 below. In the present case, the ‘eigenvalue’ $\beta$ is fixed by means of a physical law, conservation of mass: $\int \mathcal{U}(x,t)\,dx =$ constant. When applied to the self-similar formula, it gives

$$\int \mathcal{U}(x,t)dx = \int t^{-\alpha} f(xt^{-\beta})\,dx = t^{-\alpha}t^{\beta n}\int f(\eta)d\eta = \text{ const}(t), \tag{4.34}$$

  which implies the relation $\alpha = d\beta$. Summing up, we have

$$\alpha(m-1) + 2\beta = 1, \qquad \alpha = \beta d, \tag{4.35}$$

  so that the exponents have the values:

$$\beta = \frac{1}{d(m-1)+2}, \quad \alpha = \frac{d}{d(m-1)+2}. \tag{4.36}$$

- We still have to solve equation (4.33) in $\mathbb{R}^d$ for these values of $\alpha$ and $\beta$. We want non-negative solutions. Since the problem is rotationally invariant we look for a radially symmetric solution, $f = f(r)$, $r = |x|$. We have

$$\frac{1}{r^{d-1}}(r^{d-1}(f^m)')' + \beta r f' + d\beta f = 0,$$

  which can be written as

$$(r^{d-1}(f^m)' + \beta r^n f)' = 0.$$

  This is a fortunate calculation, since we can integrate once to get

$$r^{d-1}(f^m)' + \beta(r^d f) = C. \tag{4.37}$$

  Boundary conditions enter: since we want $f \to 0$ as $r \to \infty$, we take $C = 0$, so that

$$(f^m)' + \beta r f = 0, \quad m f^{m-2} f' = -\beta r, \tag{4.38}$$

  hence

$$\frac{m}{m-1} f^{m-1} = -\frac{\beta}{2} r^2 + C, \quad f^{m-1} = A - \frac{\beta(m-1)}{2m} r^2. \tag{4.39}$$

This is the end of the integration. We have obtained the announced *quadratic profile for the pressure* of the source solution.

*Problems again*

(i) We find that the formula produces a smooth solution of the PME whenever $\mathcal{U} > 0$, but we again face the problem of negative values if we want this formal solution to serve as a solution in the whole space. Since this is precisely the situation we have encountered in the study of TWs, we know how to proceed. Approximate the delta function by a positive function solve the classical problems and pass to the limit. Unfortunately, we are not technically strong enough to perform that feat. But Zeldovich et al. found numerically that the result is similar: taking the maximum between 0 and the formal solution. In other words, cutting off the unwanted part of the profile. We thus arrive at formula (4.21).

This way of waving hands at the proofs is quite unsatisfying (but see Problems 4.4 to 4.6). We will devote the next chapters to develop the theory of weak solutions. We will prove that (4.21) is a weak solution. We will prove that when initial data are taken in a suitable class of integrable functions, and weak solutions are suitably defined, the weak solution of the problem exists and is unique; moreover, it is shown that classical solutions are weak solutions, and so are their limits.

(ii) Another problem appears. In Chapter 9 the class where weak solutions lie is $C([0,T) : L^1(\mathbb{R}^d))$. Now, $\mathcal{U}(\cdot,t) \in L^1(\mathbb{R}^d)$ for every $t > 0$, but not for $t = 0$, we have a problem with the initial data. We will have to enlarge the class of data

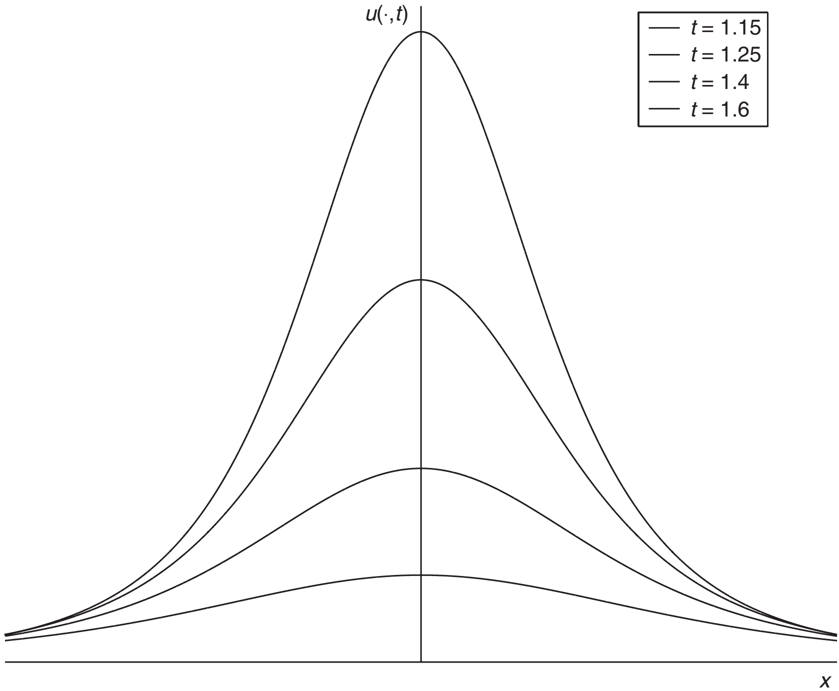


Figure 4.4: Source solution for FDE for $d = 3$, $m = 1/2$.

to measures in order to have a well-posed generalized theory that includes our favourite special solution.

This discussion leads to an important conclusion: the abstract theory has been strongly influenced by underlying physical considerations and special solutions.

### 4.4.3 *Extension to* $m < 1$

It was soon realized that the source solution also exists with many similar properties as long as $\alpha > 0$, i.e., it can be extended to the fast diffusion equation, $m < 1$, but only in the range $m_c < m < 1$, cf. [359], with

$$m_c = 0 \quad \text{for} \quad d = 1, 2, \qquad m_c = (d-2)/d \quad \text{for} \quad d \geq 3.$$

Formula (4.21) is basically the same, but now $m - 1$ and $k$ are negative numbers, so that $\mathcal{U}_m$ is everywhere positive with power-like tails at infinity. More precisely,

$$\mathcal{U}_m(x, t; M) = t^{-\alpha} F(x/t^{\alpha/d}), \quad F(\xi) = (C + \kappa_1 \xi^2)_+^{-\frac{1}{1-m}}. \tag{4.40}$$

with same value of $\alpha$ and $\kappa_1 = -\kappa = (1-m)\alpha/(2d)$.

## 4.5 Blow-up. Limits for the existence theory

Let us start by an elementary, but interesting observation. If $u(x,t)$ is a classical solution of the PME and it is given as a smooth expression of $x^2$ and $t$ in the form

$$u = F(x^2, t),$$

then

$$\widetilde{u} = F(-x^2, -t)$$

is again a classical solution of the equation. This trick is an extended form of the scaling transformations studied in Subsection 3.3.2. We may try the trick on the solutions of the last section, though they are not classical, and see what happens. Using it on equation (4.21) we get the formula

$$\widetilde{\mathcal{U}}(x,t) = (-t)^{-\alpha}\left(C + \kappa\,|x|^2(-t)^{-2\beta}\right)_+^{\frac{1}{m-1}} \tag{4.41}$$

with $\alpha$ and $\kappa$ given in (4.22). Let us examine this formula for the different values of the free constant $C$.

(i) When $C > 0$ formula (4.41) produces a function that is well defined and positive in the domain where $x \in \mathbb{R}^d$ and $t < 0$. It is moreover a classical solution of the PME in that domain and tends to infinity as $t \to 0$ at every point $x$. This is what we call *blow-up*.

It is customary to change the origin of time to some $T > 0$, write the solution as

$$\widetilde{\mathcal{U}}(x,t;C) = (T-t)^{-\alpha}\left(C + \kappa\,|x|^2(T-t)^{-2\beta}\right)^{\frac{1}{m-1}}, \tag{4.42}$$

and consider times $0 \le t \le T$ (or even $-\infty < t < T$). The formula is even easier in terms of the pressure

$$\widetilde{V}(x,t;C) = \frac{C(T-t)^{2\beta} + K\,|x|^2}{T-t} \tag{4.43}$$

where $K = \alpha/2d = m\kappa/(m-1)$, and $C > 0$ is arbitrary.

(ii) Case $C = 0$. We get an explicit solution whose pressure is a quadratic function that blows up in finite time

$$\widetilde{V}(x,t;0) = \frac{K\,|x|^2}{T-t}. \tag{4.44}$$

This is a classical blow-up solution for the pressure equation, it has a separate variables form, and is defined in $\{(x,t) : t < T\}$.

(iii) Case $C = -D^2 < 0$. In this case the solution that we obtain is not classical:

$$\widetilde{U}(x,t;-D) = (T-t)^{-\alpha}\left(K\,|x|^2(T-t)^{-2\beta} - D^2\right)_+^{\frac{1}{m-1}} \tag{4.45}$$

with a free boundary given by the hypersurface

$$|x| = DK^{-1/2}\,(T-t)^{\beta}. \tag{4.46}$$

Notice that in this case the empty region (where $u=0$) is a contracting hole located inside the support.

### 4.5.1 *Optimal existence versus blow-up*

The existence of solutions that blow up in finite time can be combined with the maximum principle to show that non-negative solutions of the PME whose initial data grow as $|x|\to\infty$ not less than $O(|x|^{2/(m-1)})$ must necessarily cease to exist at a time $T$ that can be estimated by using the above formulas. It is then shown that such a growth estimate is optimal, in the sense that solutions with data

$$u_0(x) = o((1+|x|^2)^{1/(m-1)}) \tag{4.47}$$

exist for all time (here, we use symbols $O$ and $o$ in the sense of Landau). This is no wonder, since similar transformations and blow-up solutions exist for the heat equation. But, whereas in the case of the HE the maximum admitted growth is square exponential, for the PME it is power-like with exponent $1/(m-1)$ (i.e., quadratic growth for the pressure). We conclude that as $m$ grows the class of existence decreases.

The question of existence for optimal classes of data will be investigated carefully in Chapter 12, where all the statements will be proved.

### 4.5.2 *Non-contractivity in uniform norm*

One of the most important properties of the class of filtration equations studied in Chapter 3 is the property of (non-strict) contraction with respect to the $L^1$ norm. This property is one of the cornerstones on which the general theory of the PME if founded. One may wonder if the PME evolution is also contractive with respect to other $L^p$ spaces. Actually, the heat equation is for all $p\in[1,\infty]$ and this is quite easy to prove and useful. We will show below that the PME is not contractive with respect to the $L^p$ norms for any $p>1$. The main idea is based on the following observation about the blow-up solutions (4.42), or in pressure terms (4.43). We note that for two different constants $0<C_1<C_2$ we have

$$\widetilde{V}(x,t;C_2) - \widetilde{V}(x,t;C_1) = \frac{C_2-C_1}{(T-t)^{\alpha(m-1)}}. \tag{4.48}$$

If $m=2$ this immediately implies that the $L^\infty$ norm of the difference of two solutions $\widetilde{U}_1$ and $\widetilde{U}_2$ increases with time like an inverse power of $T-t$, so that it actually goes to infinity.

The reader may object that our solutions are not bounded themselves. The adaptation will have to wait for the theory to be developed. Moreover, examples of non-contractivity will be constructed when $m \neq 2$ or when $p \in (1, \infty)$, but we will have to know a bit more about the theory. The proofs are contained in Section A.11.

## 4.6 Two solutions in groundwater infiltration

There are a number of self-similar solutions that play important roles in the theory. We present in this section two solutions for problems posed in a half line of one-dimensional space. They are motivated by the model introduced in Section 2.3 for groundwater infiltration into a horizontal porous stratum, which leads to the PME with $m = 2$ as we have shown. In an idealized situation, typical of the self-similar analysis, we assume that the stratum is horizontal with an impervious lower bed at $z = 0$ and point the $x$-axis in the direction perpendicular to the border, which is supposed to be $x = 0$. Forgetting the $y$ direction, the PME is posed for the variable $z = h(x, t)$ with $x > 0$ and boundary conditions

$$h(0, t) = H_0 > 0, \quad h(\infty, t) = 0. \tag{4.49}$$

The first condition represents border infiltration so that the groundwater level is kept constant at $x = 0$. There are then two main cases that have been studied.

### 4.6.1 *The Polubarinova-Kochina solution*

In case the height $H_0 > 0$ there exists a self-similar solution the form (4.30). The boundary condition is compatible with this form only if $\alpha = 0$. But then the compatibility with the equation (4.32) implies that $\beta = 1/2$. The solution is therefore written as

$$h(x, t = f(\eta), \quad \eta = x/t^{1/2}. \tag{4.50}$$

Corresponding initial data are

$$h(x, 0) = 0 \quad \text{for} \;\; x > 0. \tag{4.51}$$

This represents infiltration into an empty stratum from a lateral source with constant height, and was studied by Mrs. Polubarinova-Kochina in 1948.

The groundwater model problem is reduced to solving the ODE problem

$$(f(\eta)^m)'' + \frac{1}{2}\eta f'(\eta) = 0, \quad 0 < \eta < \infty \; f(0) = H_0, \quad f(\infty) = 0. \tag{4.52}$$

The constant $H_0$ is inessential and can be replaced by 1 without loss of generality by rescaling. The paper [438] makes a numerical study of this ODE and concludes that there is a correct solution that lands on the $x$-axis at a finite distance $\eta_*$, even if the slope does not go to zero at this point. We get in this way finite propagation in the way we have seen above. Figure 4.5 shows the solutions as well as nearby orbits obtained by shooting in the ODE problem with different

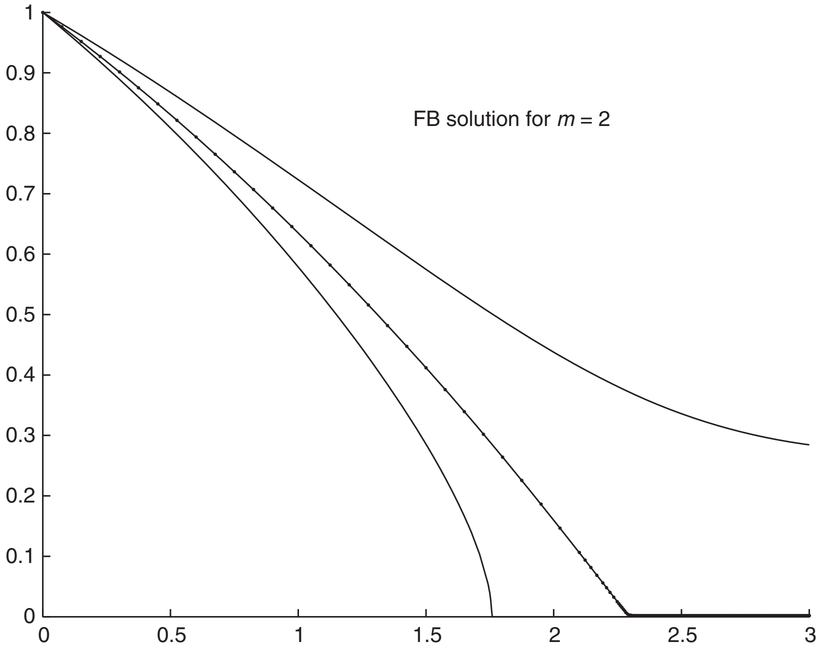


Figure 4.5: Groundwater solution and other orbits.

initial slopes. These calculations will be rigourously established in this book as the theory proceeds.

### 4.6.2 *The dipole solution*

There is a second solution that has the explicit form

$$U_{\mathrm{dip}}(x,t) = t^{-\frac{1}{m-1}}|x|^{1/m}\mathrm{sign}(x)\left(Ct^{\frac{m+1}{2m^2}} - \frac{m-1}{2m(m+1)}|x|^{\frac{m+1}{m}}\right)_+^{\frac{1}{m-1}}. \qquad (4.53)$$

This corresponds to self-similarity with exponents $\alpha = 1/m$ and $\beta = 1/(2m)$, and profile

$$f_{\mathrm{dip}}(\xi) = |\xi|^{1/m}\mathrm{sign}(\xi)\left(C - \kappa|\xi|^{\frac{m+1}{m}}\right)_+^{\frac{1}{m-1}}. \qquad (4.54)$$

We usually consider this solution as defined in the quarter of plane, $Q_1 = (0,\infty)\times(0,\infty)$, and then $U_{\mathrm{dip}} \geq 0$, and the mass $M(t) = \int_0^\infty u(x,t)\,dx$ decreases

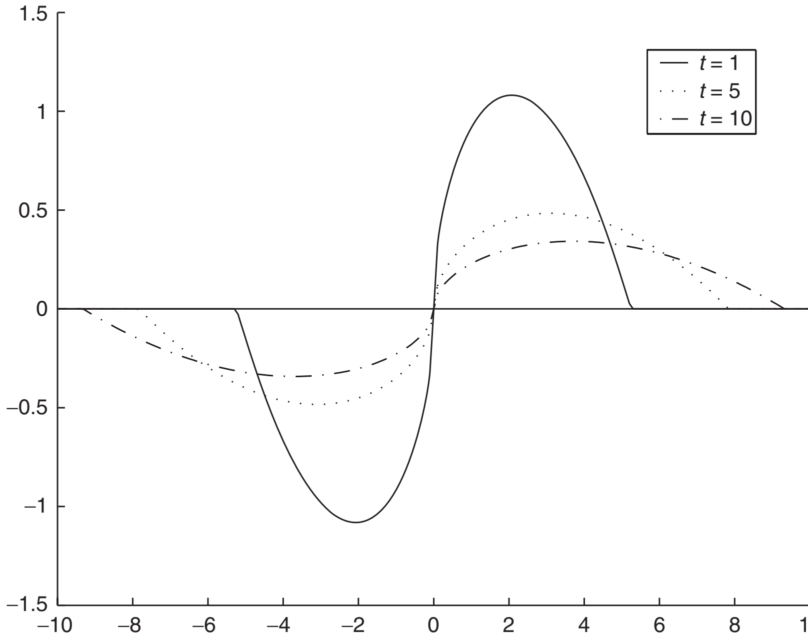


Figure 4.6: The dipole solution at different times, $m = 2$.

with time like $O(t^{-1/2m})$, while the momentum is conserved,

$$\int_0^\infty xu(x,t)\,dx = C_1, \tag{4.55}$$

cf. formula (3.45) of Section 3.3.3. A curious situation happens, namely that both the initial and boundary conditions vanish:

$$U_{\rm dip}(x,0) = 0 \quad \text{for all } x > 0, \quad U_{\rm dip}(0,t) = 0 \quad \text{for all } t > 0. \tag{4.56}$$

The question is then: should not the solution be trivial? An explanation of the negative answer comes from the realization that a singularity occurs (namely, $u$ is unbounded) as we approach the corner point $(x = 0, t = 0)$. We conclude that uniqueness of unbounded solutions is not necessarily true (even if the solution is non-negative and the divergence takes place only as $t \to 0$); this observation will affect the theory to be developed in this book. We will discuss in Section 13.5 a theory which allows for solutions with initial singularities like the dipole. Another illuminating observation consists of looking at the behaviour of the flux $(U_{\rm dip}^m)_x$ near zero. Indeed, we have $(U_{\rm dip}^m)_x \sim Ct^{-(2m+1)/m}$ which diverges as $t \to 0$.

The phenomenon is better understood if we consider the function as a signed solution of $u_t = (|u|^{m-1}u)_{xx}$ in $Q_2 = \mathbb{R} \times (0, \infty)$. Then, it is not difficult to see that for every test function $\zeta \in C_c^\infty(\mathbb{R})$

$$\lim_{t\to 0} \int_{\mathbb{R}} U_{\rm dip}(x,t)\zeta(x)\,dx = M\,\zeta'(0) \tag{4.57}$$

with $M = M(m, C) > 0$. In other words, $U_{\rm dip}(\cdot, t) \to M\,\delta'(x)$, where $\delta'(x)$ is the distributional derivative of the delta function. This is called in physics the elementary dipole, hence the name dipole solution of the PME for $U_{\rm dip}$. The data that are taken in the sense of the weak limit, (4.57), are called *initial traces*. They are a very natural and general form of data and will be considered in detail when the so-called theory with optimal data is done in Chapter 12.

The dipole solution can be extended to the FDE with $0 < m < 1$ but then it reads

$$U_{\rm dip}(x,t)^{1-m} = \frac{t\,|x|^{(1-m)/m}\mathrm{sign}(x)}{C\,t^{\frac{m+1}{2m^2}} + \frac{1-m}{2m(m+1)}\,|x|^{(m+1)/m}}. \tag{4.58}$$

Note the behaviour as $x \to \infty$, $u^{1-m} \sim ct/x^2$, which is the same as in the ZKB and is typical of the FDE, both in exponents and coefficients.

See Problem 4.12 for the derivation of the dipole solution from the source solution of the $p$-Laplacian equation and the limit $m \to 1$.

### 4.6.3 *Signed self-similar solutions*

The dipole solution opens the question of constructing other self-similar signed solutions of the PME in the whole space having compact support, maybe in a explicit or semi-explicit way. We will see in Chapter 16 how to construct self-similar solutions in a systematic way. The case of compactly supported solutions in $d = 1$ was analysed by Hulshof in [296]. The form is

$$u(x,t) = t^{-\alpha}U(\eta), \quad \eta = xt^{-\beta},$$

with the standard constraint $(m-1)\alpha + 2\beta = 1$; then, $U$ satisfies the differential equation

$$(|U|^{m-1}U)'' + \beta\eta U' + \alpha U = 0.$$

The theorem says that there exists a strictly decreasing sequence $\alpha_0 = 1/(m+1) < \alpha_1 = 1/m < \alpha_2 < \ldots \uparrow 1/(m-1)$ such that compactly supported similarity solutions of the type above exist if and only if $\alpha = \alpha_k$. The first exponent corresponds to the Barenblatt solution, the second to the dipole. The third was investigated in [103] where it was shown that the exponent is not derived from a conservation law (in other words, it is anomalous, and the simple extrapolation of the elementary algebraic conjecture breaks down).

It is also proved in [296] that $k$ equals the number of times $U(\eta)$ changes its sign and $U(\eta)$ is symmetric if $k$ is even (antisymmetric if $k$ is odd). The reader should note that for the wrong $\alpha$ (i.e., not in the list) self-similar solutions exist but they are compactly supported. Actually, they are not integrable.

A similar analysis holds for compactly supported, radially symmetric solutions of the PME in $\mathbb{R}^d$, $d > 1$.

## 4.7 General planar front solutions

As an extension of the theory of travelling waves, we now examine the class of solutions that propagate with a certain speed $c(t)$ and keep their space shape constant in time but for a scale factor $A(t)$. We call them *fronts*. The general form is then

$$u(x,t) = A(t)U(x_1 - s(t), x_2, \dots, x_d) \tag{4.59}$$

but for a possible rotation of the space direction. The PME implies then the differential equation

$$A'(t)U(\eta) - c(t)A(t)U'_{\eta_1} = A^m(t)\Delta U^m(\eta),$$

with $c(t) = s'(t)$. Since we want non-trivial solutions, we assume $A(t) \neq 0$. In that case, a simple separation of variables argument implies that the following two conditions must hold:

$$A'(t) = -\lambda A^m(t), \quad c(t)A(t) = \mu A^m(t). \tag{4.60}$$

The case $\lambda = 0$ allows us to recover the travelling waves of Section 4.3, and the case $\mu = 0$ gives $c(t) = 0$, hence, the solutions in separate-variables form.

New solutions are obtained when both parameters are non-zero. In that case, we get from the first equation

$$A(t) = \frac{1}{(C + \lambda(m-1)t)^{1/(m-1)}}, \quad c(t) = \frac{\mu}{(C + \lambda(m-1)t)}. \tag{4.61}$$

*Case* $\lambda > 0$

In this case the second equation integrates to give a speed of the form

$$s(t) = c_0 \log(C + \lambda(m-1)t),$$

which goes to infinity but slows down as $t \to \infty$, something that also happens for the ZKB solution (with a different rate though). The time factor also decreases in a power fashion, $A(t) = O(t^{-1/(m-1)})$, the same as the separate-variables solutions. The equation for the profile $U$ becomes

$$\Delta U^m(\eta) + \lambda U(\eta) + \mu U'_{\eta_1}(\eta) = 0, \tag{4.62}$$

which is a variation of the basic nonlinear elliptic equation (4.4). Using scaling, there is no loss of generality in reducing the case $\mu > 0$ to $\mu = 1$. The case of negative $\mu$ can be reduced to positive $\mu$ by reflection (which only changes the direction of the wave).

### 4.7.1 *Solutions with a blow-up interface*

The case $\lambda = -l < 0$ is more interesting, because it leads to interesting solutions whose interface blows up in a finite time. Indeed, in this case we get

$$s(t) = -c_0 \log(C - l(m-1)t) = c_0 \log(1/(T-t)) + c_1. \tag{4.63}$$

which blows up as $t \to T = C/l(m-1)$. This means that the location of the interface, if there is one, reaches infinity in finite time with a logarithmic rate. Also, the scale factor $A(t)$ blows up as $t \to T$. Let us calculate the profile $U$ in that case. The equation becomes

$$\Delta U^m(\eta) - lU(\eta) + \mu U'_{\eta_1}(\eta) = 0. \tag{4.64}$$

*Existence and analysis of a special solution*

Since this type of solution has a certain interest and seems not be described in the literature, we pay some attention to a particular solution in one space dimension. The exercise allows us to review some interesting ODE techniques.

Again, there is no loss of generality in fixing $\mu = 1$. In $d = 1$ we can write

$$(U^m)''(\eta) - lU(\eta) + U'(\eta) = 0. \tag{4.65}$$

Our next task is integrating this equation. We use a phase plane argument. Letting $V = -(U^m)'$, we have $V' = -lU + U'$. We get the system:

$$\begin{cases} \dfrac{dU}{d\eta} = -\dfrac{1}{m} V U^{1-m} \\ \dfrac{dV}{d\eta} = -lU - \dfrac{1}{m} V U^{1-m}, \end{cases} \tag{4.66}$$

so that

$$\frac{dV}{dU} = 1 + \frac{lmU^m}{V}. \tag{4.67}$$

We cannot integrate this ODE explicitly, but the usual qualitative techniques allow us to understand the existence and behaviour of the different solutions $(U(\eta), V(\eta))$. Let us recall that $dV/dU = 1$ is the equation for the TWs of Section 4.3, while $dV/dU = lmU^m/V$ describes the blow-up solution of Section 4.5 in one dimension. Therefore, we expect (4.67) to combine properties of both equations.

- Let us now examine the different orbits $V = V(U)$ in the first quadrant of the $(U, V)$ plane. We see that they are increasing with $dV/dU > 1$. They can get started at any point of the vertical axis, i.e., $U_0 = 0$ and $V_0 > 0$, and then the initial slope is $dV/dU = 1$. We can also shoot from the horizontal axis, $U_0 > 0$ and $V_0 = 0$ with initial slope $dV/dU = \infty$. Finally, there exists one separatrix solution between the two families, which starts from $(0, 0)$. Such a separatrix is unique by monotonicity arguments.

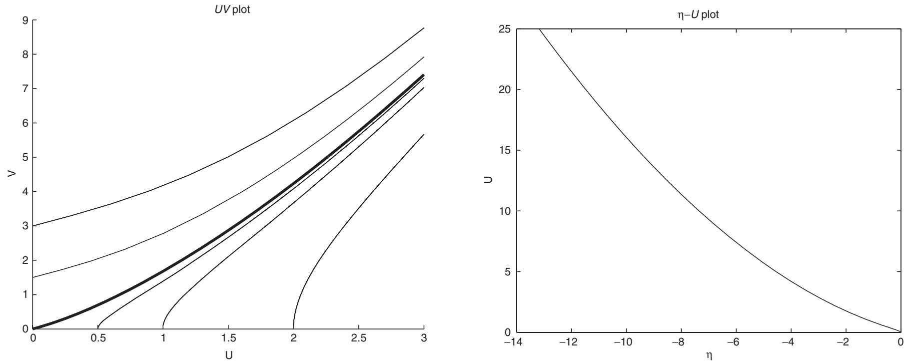


Figure 4.7: General front with blow-up. Left, the phase-plane with the solution in bold line. Right, the plot of $u$ versus $\eta$. Parameters $m = 2$, $l = 1$.

- Local analysis of that orbit near the origin is as follows: since $dV/dU > 1$, we have $V > U$, hence the equation simplifies for $U \sim 0$ to $dV/dU \sim 1$ (the last term is smaller, $O(U^{m-1})$), so that $V/U \to 1$. We can now use the definition of $V$ to conclude that there exists a constant $\eta_0$ such that

$$U(\eta)^{m-1} \sim \frac{m-1}{m}(\eta_0 - \eta)$$

as $\eta \to \eta_0$ with $\eta < \eta_0$; $\eta_0$ is an arbitrary constant, and we can take $\eta_0 = 0$ to normalize. We have obtained the correct behaviour near a free boundary.
- Behaviour for large values. As $U \to \infty$, we derive an estimate as follows: since $dV/dU > U^m/V$ we have $V^2 > C_1 U^{m+1}$; using the differential equation we conclude that $dV/dU < 1 + U^{(m-1)/2}$, hence $V < C_2 U^{(m+1)/2}$ for all large $U$. But that means that the equation can be simplified for all large $U$ to $dV/DU \sim mlU^m/V$, which gives the exact estimate to first order

$$V = c_3 U^{(m+1)/2} + \ldots, \quad c_3 = (2lm/(m+1))^{1/2}.$$

Recalling that $V = -mu^{m-1}U'$ and integrating, this gives the estimate as $\eta \to -\infty$:

$$U^{m-1} = c_4 \eta^2 + \ldots, \quad c_4 = \left(\frac{m-1}{2}\right)^2 \frac{2l}{m(m+1)}.$$

It follows that

$$\frac{m}{m-1} u^{m-1}(x,t) = \frac{1}{2(m+1)} \frac{x^2}{T-t} + \ldots,$$

as $x \to -\infty$. This is precisely the behaviour of the standard blow-up solution (4.44) in $d = 1$.

**Remark on the blow-up rate** As $t \to T$, and uniformly on bounded sets $|x| \le K$, the solution blows $U(x,t)$ up with the rate

$$U(x,t) \sim A(t)U(s(t)) \sim \frac{c}{(T-t)^{1/(m-1)}}(\log(1/(T-t)))^2. \tag{4.68}$$

This is faster than the blow-up rate of the standard blow-up solutions of Section 4.5. See the Open problem in the Problems.

## Notes

**Section 4.2.** The existence of the separate variable solution in a bounded domain was rigourously proved by Aronson and Peletier in [49]. If we eliminate the restriction of non-negativity, we may obtain an infinite family of solutions of the elliptic problem with increasingly complicated sign-change patterns, cf. [511].

The method can be applied to the fast diffusion equation, and then it produces solutions which vanish in finite time of the form

$$u(x,t) = ((1-m)(T-t))^{-1/(m-1)}F(x), \tag{4.69}$$

where $T$ is the extinction time, a free parameter. The existence of a positive profile vanishing on the boundary of a bounded domain is proved for $m > (d-2)/(d+2)$ while for $0 < m \le (d-2)/(d+2)$ the profile is positive in the whole space, cf. [99].

**Section 4.3.** Travelling waves are constant-shape fronts moving with constant speed $|\mathbf{V}| = c$ wherever $u > 0$, while in the empty region (where $u = 0$) the speed, defined as the gradient of the pressure $v$, is zero. This gives rise to a discontinuous function to represent $\mathbf{V} = -\nabla v$. This reminds us of the discontinuous solutions of standard gas dynamics and their shock waves, cf. [171, 175, 359]. Indeed, this analogy has been used to develop the theory, cf. [501].

**Section 4.4.** The origin of the source-type solutions has been explained in the Introduction and Chapter 2. The applied problem that motivated the studies was a problem in plasma physics, nothing to do with porous media! We will study their role as asymptotic patterns in Chapter 18.

The ZKB solutions can be continued algebraically for $m < (d-2)/d$, but they have different geometry, they do not solve the same initial value problem and they also lose their important role as asymptotic patterns for a large class of initial data. For a detailed analysis of this issue, cf. our monograph [515].

**Section 4.5.** Blow-up solutions will play a big role in setting limits to the existence theory, and as handy comparison functions.

The example of expansion of the $L^\infty$ norm of Subsection 4.5.2 seems to be due to the author.

**Section 4.6.** The first type of solutions of this section was studied by Polubarinova-Kochina in 1948 [438]. A study for the general equation $u_t = (D(u)u_x)_x$ was performed by [431] in 1955. This author finds cases with explicit solutions in [432]. These are interesting early references to mathematical work on the PME.

These solutions have appeared often in studies of groundwater infiltration. Solutions with changing sign in $d = 1$ were constructed by van Duijn et al. [221].

The dipole solution is due to Zel'dovich and Barenblatt [530] and has been used by Kamin and Vázquez [326] in describing the asymptotic behaviour of more general signed solutions. The extension to fast diffusion is new. For recent theory and experiments on dipole solutions see King and Woods [342].

A dipole solution for the signed PME in several space dimensions was constructed by Hulshof and Vázquez [299].

**Section 4.7.** General planar front solutions seem to be new in the literature. Our special solution shows that the free boundary may blow up in finite time.

### *Note on self-similarity*

This is a principal concept in mechanics and, generally speaking, in the applied sciences. Thus, it has been pointed out in many papers and corroborated by numerical experiments that similarity solutions furnish the asymptotic representation for solutions of a wide range of problems in mathematical physics. The reader is referred to the book of G. Barenblatt [63, 64] for a detailed discussion of this subject. As these books say, it must be borne in mind that, even if arisen to solve problems in the practical applications, self-similar solutions are ideal constructs that represent idealized situations and will only represent observed behaviour in a limit sense. However, it is discovered by the practical scientist that this sense has the deepest influence on the rest of the theory.

There are a number of other self-similar solutions that play a prominent role in the theory, like the hole-filling solutions of Graveleau and Aronson [48] that played a big role in the studies of optimal regularity. We will devote the whole Chapter 16 to the study of self-similarity. This set of ideas is better known in theoretical physics as the renormalization group.

### *Eternal solutions*

This is the name given to solutions that are defined for the whole time span, $-\infty < t < \infty$. The travelling wave solutions are eternal, and some of the planar fronts also are; the rest of the examples of this chapter exist either forward in time (separate-variables, source-type, dipole, constant-height solution) or backward in time (the blow-up solutions).

## Problems

**Problem 4.1** SEPARATED VARIABLES.

(i) Take $\Omega = B_R(0)$ and solve the nonlinear elliptic problem (4.5) by writing $F_1(x) = f(r)$, $r = |x|$, and solving the ODE for $g(r) = f^m(r)$

$$g''(r) + \frac{d-1}{r}\, g'(r) + g(r)^p = 0, \quad p = \frac{1}{m},$$

with $g(0) = h$, $g'(0) = 0$. Find $h > 0$ so that $g(R) = 0$.

(ii) Check that taking $\lambda \neq 1$ still produces the same family of solutions (4.7) in separated variables.

**Problem 4.2** TRAVELLING WAVES. Prove Proposition 4.1.
*Hint*: The function $f_\varepsilon$ is obtained by integration of equation (4.16). It is better to think that it defines $\eta$ in terms of $f$ in the range $f \geq \varepsilon$. Here is a work plan:

(i) Get the formula for $\eta = \eta_\varepsilon(f)$:

$$\eta = -\frac{m}{c} \int_1^f \frac{f^{m-1}}{f-\varepsilon} df.$$

(ii) Show that $\eta$ ranges from 0 to $\infty$ while $f$ goes from $\varepsilon$ to $\infty$, and the dependence is monotone decreasing. Note that $\eta_\varepsilon(1) = 0$ for every $\varepsilon \in (0,1)$.

(iii) Show that as $\varepsilon \to 0$ we get uniform converge on sets of the form $[1/a, a]$ to the solution of the limit equation

$$\frac{d\eta}{df} = -\frac{m}{c}\, f^{m-2}.$$

(iv) Conclude the announced result.

(v) Perform the explicit computation for $m = 2$ and pass to the limit in the obtained formula.

**Problem 4.3** SIGNED TRAVELLING WAVES. Construct a travelling wave with changing sign by considering the case $K < 0$ in formula (4.11). Show that $f \to K/c < 0$ as $\eta \to \infty$. Sketch the profile $f$ and determine the optimal regularity in terms of $m$.

*Solution:* $f \in C^{1/m}(\mathbb{R})$, with minimal regularity at the sign transition, $f = 0$.

Conclude from the analysis that the transition from plus to minus sign of these signed solutions implies a blow-up for the gradient of the pressure.

**Problem 4.4** TRAVELLING WAVES FOR THE GPME. Consider the existence of TWs for the equation $\partial_t u = \Delta\Phi(u)$ of the form (4.9), $u = f(x - ct)$.

(i) Show that the equation of the TW such that $u(x,t) \to 0$ as $x \to \infty$ is given by the ODE

$$\Phi(f)' + cf = 0, \tag{4.70}$$

which leads to the implicit expression

$$\int_1^f \frac{d\Phi(s)}{s} = c(\eta_1 - \eta). \tag{4.71}$$

(ii) Show that the TW has a finite interface if and only if

$$\int_0^1 \frac{d\Phi(s)}{s} < \infty. \tag{4.72}$$

(iii) Check that this happens for $\Phi(u) = u^m$ iff $m > 1$.

See continuation in Problem 15.11.

**Problem 4.5** The ZKB solution.

(i) Show that formula (4.21) is actually a classical solution of the PME in the region where it is positive.
(ii) Show that the initial data are taken in the sense of (4.23), i.e., that for every test function $\phi \in C_0(\mathbb{R}^d)$, $\phi \ge 0$ we have

$$\lim_{t \to 0} \int_{\mathbb{R}^d} \mathcal{U}(x,t)\phi(x)\,dx = M\,\phi(0).$$

(iii) Show that the pressure $V$ of the ZKB solution is Lipschitz continuous but not $C^1$ on the free boundary.
(iv) Check Darcy's law for the ZKB solution.
(v) Prove formula (4.27) for the convergence of ZKB profiles to Gaussian profiles as $m \to 1$. Determine in what sense and where the limit is taken.
(vi) Write the formula for the pressure and prove that

$$\Delta V = -\frac{C}{t} \tag{4.73}$$

in the set $\{(x,t) : U > 0\}$ (the solution is even concave on that set). This is a much used property of the ZKB pressure inside the solution support.

**Problem 4.6** Shape of the ZKB solution. Show that for $m = 2$ the shape of the density $u$ of the source solution in terms of $|x|$ for fixed time is a parabola. Show that for $m = 3$ it is an ellipse with vertical slope at the front. Show that for $m = 3/2$ it is a fourth-order polynomial with flat contact with the $x$-axis. Write the explicit expression for $d = 1$, $t = 1$ and $C = 1/15$.

**Problem 4.7** In later chapters we will be able to show that the ZKB solution is a limit solution by following this program: approximate the delta function by a sequence of positive functions $u_{0n}(x)$; solve the PME with these data and find classical solutions $u_n(x,t)$; pass to the limit as $n \to \infty$ and find the ZKB solution.

Study those chapters and perform this programme.

**Problem 4.8** Derive the formula for the velocity of the ZKB solutions and show that it is a discontinuous function. See more in Subsection 18.7.2.

**Problem 4.9** Write the formula for the pressure of solution (4.45) and show that it is not a $C^1$ function. Calculate the velocity. How does it evolve with time on the free boundary?

**Problem 4.10** BLOW-UP. Write the formulas of explicit blow-up solutions for the heat equation. The simplest has the form

$$U(x,t) = (T-t)^{-d/2} \exp\left(\frac{|x|^2}{4(T-t)}\right).$$

Find a whole family. Compare the rate of blow-up with the PME case.

**Problem 4.11** THE DIPOLE SOLUTION.

(i) Check that function $U_{\text{dip}}$ defined in (4.53) is singular near $(0,0)$ by checking that $\|U_{\text{dip}}(\cdot,t)\|_\infty = c(m,C)t^{-1/m}$, and the maximum for fixed $t$ is reached along a line of the form $x = c_1 t^{1/2m}$.
(ii) Calculate the relation between $M$ and $C$ in formula (4.57).
(iii) Calculate the constant $\kappa(m)$.
(iv) The free boundary of the dipole solution propagates like $|x| = O(t^{1/2m})$, while the ZKB propagates like $|x| = O(t^{1/(m+1)})$ in 1D. Find a justification for the smaller rate.

**Problem 4.12** THE BARENBLATT SOLUTION FOR THE $p$-LAPLACIAN EQUATION.

(i) Show that the function

$$W(x,t) = t^{-\frac{1}{m}} \left(C - k(m)|\xi|^{\frac{m+1}{m}}\right)_+^{\frac{m}{m-1}}$$

with $C > 0$ arbitrary, $\xi = x\,t^{-1/2m}$ and

$$k = \frac{m-1}{2m}(2m)^{-1/m},$$

is a generalized solution of the $p$-Laplacian equation $w_t = (|w_x|^{m-1}w_x)_x$ for every $m > 1$, in the sense that the equation is satisfied in the classical sense whenever $w_x \neq 0$, and it is $C^1$ function for all $(x,t)$. Cf. [68].
(ii) Prove that $W(\cdot,t) \to M\,\delta(x)$ for some $M = M(m,C) > 0$. This justifies the name of source-type solution.
(iii) Differentiate this solution with respect to $x$ to find the dipole solution of the PME, $U_{\text{dip}} = -W_x$.

(iv) Show that in the limit $m \to 1$ with constant moment we obtain the dipole of the heat equation in $d = 1$

$$U_{\text{dip}}(x, t; m = 1) = \frac{Mx}{t^{3/2} \exp\left(-x^2/4t\right)}. \tag{4.74}$$

**Problem 4.13**

(i) Use ODE techniques to obtain a description of the solutions of system (4.66) that enter the region $V < 0$. Draw the corresponding profiles $U(\eta)$.
(ii) Use ODE techniques to obtain a description of the solutions of Section 4.7 when $\lambda > 0$. Draw the corresponding profiles $U(\eta)$.

*Hint*: The analysis at the origin is delicate. ODE techniques are intensely studied in Chapter 16.

## Open problem

(i) The free boundary of the solution constructed in Section 4.7.1 blows up with a logarithmic rate, $s(t) = O(|\log(T - t)|)$. Are there any free boundary solutions of the PME with free boundaries which blow up in finite time with a power rate? The task is to construct one such solution in semi-explicit form.
(ii) The correction factor in the blow-up expression (4.68) is logarithmic. Is it the unique possible form? Are there any solutions with faster rates?

# 5

# THE DIRICHLET PROBLEM I. WEAK SOLUTIONS

In this long chapter we start the systematic study of the questions of existence, uniqueness and main properties of the solutions of the PME by concentrating on the first boundary-value problem posed in a spatial domain $\Omega$, which is a bounded subdomain of $\mathbb{R}^d$, $d \geq 1$. We focus on homogeneous Dirichlet boundary conditions, $u = 0$ on $\partial\Omega$, in order to obtain a simple problem for which a fairly complete theory can be easily developed as a first stage in understanding the theory of the PME. This is called the homogeneous Cauchy–Dirichlet problem, or more simply, the homogeneous Dirichlet problem.

Even if the main goal of the text is to develop a theory for non-negative solutions of the PME, the theory of this chapter can be safely done for the complete generalized porous medium equation, also called the filtration equation, $u_t = \Delta\Phi(u) + f$, under conditions that include the whole range of exponents $0 < m < \infty$ of the PME, HE and FDE. We pursue this course for four reasons: it does not imply undue extra effort, the generality can be illustrative of the functional analysis involved, it will be of help in the future, and finally the filtration equation is an important subject of study in itself. We recall that the full form is important for its application to the study of reaction-diffusion processes where the forcing term depends on $u$, $f = f(x, t, u)$, while convection processes include a term of the form $f = \sum_i \partial_i F_i(x, t, u)$.

The problem is shown to be well-posed globally in time in particular classes of generalized solutions, specifically, in a class of weak energy solutions. The main points for future reference are the definition of solution, Definition 5.4, the uniqueness result, Theorem 5.3, and the existence result, Theorem 5.7.

In this chapter we will use the symbols $Q = \Omega \times \mathbb{R}_+$, $Q_T = \Omega \times (0, T)$, $Q^\tau = \Omega \times (\tau, \infty)$, and $Q_T^\tau = \Omega \times (\tau, T)$. We also use the sloppier notation $Q^* = \Omega \times (\tau, T)$. $\sum_T = \partial\Omega \times [0, T)$ is the lateral boundary, $\sum = \partial\Omega \times [0, \infty]$. We recall the fact that when $\Omega$ is a bounded domain with Lipschitz boundary, then $H_0^1(\Omega)$ coincides with the restriction of the functions $u \in L^2(\Omega)$ that belong to $H^1(\mathbb{R}^d)$ when extended by zero in $\mathbb{R}^d \setminus \Omega$.

## 5.1 Introducing generalized solutions

A consequence of the degeneracy of the PME is that we do not expect to have classical solutions of the problem when the initial data take on the value $u = 0$, say, in an open subset of $\Omega$. Therefore, we need to introduce an appropriate concept of *generalized* solution of the equation. At the same time, we have to define in what sense the initial and boundary conditions are taken. In many cases, this latter information can be built into the definition of generalized solution.

There are different ways of defining generalized solutions, and we will explore in the book some natural choices, the most usual idea being that of multiplying the equation by suitable test functions, integrating by parts some or all of the terms, and asking for a regularity of the solution that allows this expression to make sense. Then, we say that the solution is a *weak solution*.

In any case, the concept of generalized solution changes the meaning of the term solution, so we have to be careful to ensure that the new definition makes good theoretical and practical sense. From the first point of view, we ask the theory to be well posed. Then, the new solutions must be defined so that they include all classical solutions whenever the latter exist (compatibility). Moreover, a concept of generalized solution will be useful if the problem becomes *well-posed* for a reasonably wide class of data, i.e., if a unique such solution exists for each set of data in a given class and it depends continuously on the data in the appropriate topologies.

As we will see, it can happen that several concepts of generalized solution arise naturally. It is then important to check that they agree in their common domain of definition (i.e., for data which are compatible with two of them). Selecting one them as the preferred definition depends of several factors, the most important being in principle that of having the largest domain. However, one could consider a more restrictive definition which still covers the applications in mind if it involves simpler statements or more natural concepts, or when it leads to simpler proofs of its basic properties.

Let us review the contents of the chapter in some detail. The study starts in Section 5.2 by considering a rather general setting, where a natural concept of *weak solution* is introduced to solve the complete filtration equation with zero boundary data and integrable initial data and forcing term. The idea is to lay the foundation of subsequent existence and uniqueness theories. The alternative of defining so-called *very weak solutions* is introduced and briefly commented, but will not be further developed for the moment, since the chapter is focused on weak solutions. The proof of uniqueness is quite immediate in that setting, Section 5.3. The existence of data for which there can be no classical solution immediately follows, thus justifying the need for a weak theory.

The class of weak solutions is rather general, and it is convenient in the development of the existence theory to restrict somewhat the generality in order to get simplicity and clarity, and to be able to make interesting calculations and approximations without undue effort in justifying them. In that sense, this chapter is based on the construction of the subclass of weak energy solutions, $\mathcal{WES}$, which are weak solutions that satisfy a version of the energy estimates which have been introduced in Section 3.2 in the classical setting. Such weak solutions form a class large enough for the purposes of the usual theory found in the literature. For instance, the class provides us with a unique solution when initial data are bounded, a safe assumption for many purposes. They also serve as a foundation for the more advanced topics of the next chapter, where we will strive for the largest generality of the data.

The construction of weak solutions in the case of general $\Phi$ and general data proceeds by approximation with smooth nonlinearities and data; the study of the question of existence under good assumptions on $\Phi$ and the data has been addressed in Section 3.2, and a number of important properties of the solutions have been obtained in that smooth setting. With this information at hand, the study of the problem with general $\Phi$ proceeds first for non-negative data, Section 5.4, and then for data with changing sign, Section 5.5. The technique is based on a priori estimates that force some restrictions on the data that are characteristic of weak energy solutions; thus, the initial data are restricted to the space $L_\Psi(\Omega)$, a subspace of $L^1(\Omega)$ (for the PME, it equals $L^{m+1}(\Omega)$), and the forcing term must belong to some dual space, cf. Theorem 5.7 and Corollary 5.6. These restrictions are compensated by the fact that such solutions enjoy energy inequalities, see formulas (5.20) or (5.39), that would be lost for more general data.

Let us note that, though the solution of the approximate problems by classical methods is performed in spatial domains with a smooth boundary, the main facts of the theory are formulated for Lipschitz domains. Such generality allows us to consider domains with corners, typical in many applications.

Once existence and uniqueness of these weak solutions are settled, we establish some of the main properties in Section 5.6.

We consider in Section 5.7 the problem with non-homogeneous boundary data (on a smooth boundary). This is the most general setting that will appear sometimes in the sequel. It departs a bit from the rest of this chapter, centred on the problem with zero boundary data, but provides insight and is used as an auxiliary tool, e.g., to understand different super- and subsolutions.

We recall that it is to be expected in a parabolic problem that the solutions enjoy some extra regularity properties. We will not address at this point the question of continuity for the weak solutions of the GPME we have constructed in the context of several space dimensions, because this involves heavy work that will be tackled in Chapter 7.

We continue the basic theory with the topics of universal bounds and maximal solutions for the PME and also for filtration equations with strongly superlinear $\Phi$. The existence of the universal bound is treated in Section 5.8.

In Section 5.9 we establish the existence of a special solution with infinite initial data. This solution is unique and acts as an absolute upper bound for all solutions of the Dirichlet problem. The existence of such a solution is a typical nonlinear effect, which is not possible in the linear theory. For the PME it takes the form $\tilde{U}(x,t) = f(x)\,t^{-\alpha}$ with decay rate $\alpha = 1/(m-1)$. Since it takes infinite initial data but it becomes bounded for positive times, this solution will be called *the Friendly Giant.*

We apply the same techniques in Section 5.10 to the fast diffusion equations with a different conclusion, extinction in finite time, and a comment on singular diffusion.

We end the chapter with a suggestion for advanced work: applying the techniques of this chapter to a number of more general equations of inhomogeneous media, Section 5.11.

The basic material is contained in Sections 5.2–5.7. The last section is an introduction to advanced reading.

## 5.2 Weak solutions for the complete GPME

We assume that $\Omega$ is a bounded domain in $\mathbb{R}^d$, $d \geq 1$, with Lipschitz continuous boundary $\Gamma = \partial\Omega$. We pose the homogeneous Dirichlet problem for the filtration equation in complete form, $u_t = \Delta\Phi(u) + f$. We make the following assumption on $\Phi$, which we call the constitutive function, or, in more familiar terms, the 'nonlinearity'.

($H_\Phi$) *The function $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous, strictly increasing and $\Phi(\pm\infty) = \pm\infty$. We also admit the normalization $\Phi(0) = 0$.*

These assumptions will be kept throughout the chapter unless we mention to the contrary. The PME and its signed counterpart are included as the special case $\Phi(s) = |s|^{m-1}s$ with $m > 1$. Note that the case $m = 1$ is also included. Relaxing the assumptions on $\Phi$ is possible with a small cost in complication that we have considered not necessary. The possible dependence of $\Phi$ on $x$ to account for the presence of so-called inhomogeneous media will be discussed in Section 5.11.

*Problem HDP*

*Given $u_0 \in L^1(\Omega)$, $f \in L^1(Q)$, find a locally integrable function $u = u(x,t)$ defined in $Q_T$, $T > 0$, that solves the set of equations*

$$u_t = \Delta\Phi(u) + f \qquad \text{in } Q_T, \tag{5.1}$$

$$u(x,0) = u_0(x) \qquad \text{in } \Omega, \tag{5.2}$$

$$u(x,t) = 0 \qquad \text{in } \Sigma_T. \tag{5.3}$$

*in a weak sense to be precisely defined. The time $T > 0$ can be finite or infinite. Moreover, we want to find $u$ in a suitable functional class that guarantees uniqueness and continuous dependence on the data.*

Though we will obtain solutions for all $T > 0$, i.e. with $T = \infty$, it is interesting for technical reasons to allow $T < \infty$.

We are going to introduce next precise definitions of what we understand by solution of Problem HDP. Since there are several options available for the concept of solution of the equation, and also for the sense in which the data are taken, it is important to carefully specify the choices we make at every instance.

### 5.2.1 *Concepts of weak and very weak solution*

First of all, we introduce a suitable concept of weak solution for the filtration equation in $Q_T$, avoiding at this moment any reference to initial or boundary data.

**Definition 5.1** *A weak solution of equation* (5.1) *in* $Q_T$ *is a locally integrable function,* $u \in L^1_{\rm loc}(Q_T)$*, such that*

*(i)* $w = \Phi(u) \in L^1_{\rm loc}(0, T : W^{1,1}_{\rm loc}(\Omega))$*;*
*(ii) the identity*

$$\iint_{Q_T} \{\nabla w \cdot \nabla \eta - u\eta_t\}\, dxdt = \iint_{Q_T} f\eta\, dxdt, \tag{5.4}$$

*holds for any test function* $\eta \in C^1_c(Q_T)$*.*

Equation (5.4) is obtained by extrapolating a property of classical solutions. Indeed, if $u$ is a smooth solution of the GPME in $Q_T$ and we multiply the equation by $\eta$ and integrate by parts we obtain (5.4). Observe that the equation is satisfied only in the sense that all these tests are true; this is called a weak sense. In particular, the definition does not require the derivatives appearing in equation (5.1) to be actual functions, they need merely exist in the sense of distributions.

Note that, in the PME the assumption $u \in L^1_{\rm loc}(Q_T)$ is implied by the condition $\Phi(u) \in L^1_{\rm loc}(Q_T)$, which is a part of (i). This implication is not necessarily true for more general $\Phi$, like the FDE with $m < 1$.

The previous definition of weak solution of the GPME is not the only possibility at hand. Actually, there is a very natural alternative where the regularity assumptions are relaxed by integrating once again in space, so that no space derivatives appear in the statement.

**Definition 5.2** *A very weak solution of equation* (5.1) *in* $Q_T$ *is a locally integrable function,* $u \in L^1_{\rm loc}(Q_T)$*, such that* $w = \Phi(u) \in L^1_{\rm loc}(Q_T)$*, and the identity*

$$\iint_{Q_T} \{w\, \Delta\eta + u\eta_t + f\eta\}\, dxdt = 0 \tag{5.5}$$

*holds for any test function* $\eta \in C^{2,1}_c(Q_T)$*.*

We can simply say that the equation is satisfied in the sense of distributions in $Q_T$ or that it is a *distributional solution.* But note that we are asking $u$ and $\Phi(u)$ to be integrable functions. We can also call these solutions *weak-0 solutions* to stress the fact that we do not use any derivatives of $u$ or $\Phi(u)$ in defining them, and then the weak solutions become *weak-1 solutions.* It is clear that all weak solutions are very weak solutions according to these definitions.

There are advantages and disadvantages to both definitions. We will work in this chapter with the concept of weak solution which seems to us suitable to develop the basic theory and present the main techniques.

**Remarks**

(1) In the definitions, we have chosen the space $L^1_{\rm loc}(Q_T)$ as base space for the sake of generality. However, the simplicity of the most common existence and uniqueness proof recommends replacing such a space by other smaller $L^p_{\rm loc}(Q_T)$ spaces, $1 < p \le \infty$. We recall that in the usual practice the weak solutions will be locally bounded, even continuous, either because we prove that a more general solution has this property, or because the author so assumes from the beginning.

(2) Since the weak theory is more restrictive than the very weak one, it allows for a simpler uniqueness proof. An existence theorem would be easier to prove in the more general context of very weak solutions, but we will obtain in this chapter a result on existence of weak solutions that is general enough for many purposes and allows us to develop interesting energy estimates.

On the other hand, very weak solutions allow for a more general theory that will be discussed in Section 6.2 as part of the more advanced topics. Quite strong uniqueness and comparison results will be proved.

(3) Note that we could go in the opposite direction of asking for more regularity than the theory provides; we will thus arrive at the quite useful concepts of *continuous weak solution* and *strong solution*. The former is discussed in Chapter 7 and used thereafter, while strong solutions are studied in Chapter 8; they are the preferred option in the study of the Cauchy problem in Chapter 9. These are not the only options: *mild solutions* will appear as a consequence of the semigroup approach of Chapter 10. See the related comment at the end of the chapter.

### 5.2.2 *Definition of weak solutions for the HDP*

The definition of weak solution we have proposed applies in the interior of the space-time domain (we usually say that it is a local weak solution), and does not take into account initial or boundary conditions, which are an essential part of Problem (5.1)–(5.3). Inserting the homogeneous boundary condition leads to the following standard definition.

**Definition 5.3** *A locally integrable function $u$ defined in $Q_T$ is said to be a weak solution of equation* (5.1) *with boundary condition* (5.3) *if*

(i) *$u \in L^1(\Omega \times (\tau, T-\tau))$ for all $\tau > 0$ and $w = \Phi(u) \in L^1_{\rm loc}(0, T : W^{1,1}_0(\Omega))$;*
(ii) *the identity*

$$\iint_{Q_T} \{\nabla\Phi(u)\cdot\nabla\eta - u\eta_t\}\,dxdt = \iint_{Q_T} f\eta\,dxdt \tag{5.6}$$

*holds for all test functions* $\eta \in C^1(\overline{Q}_T)$ *which vanish on* $\Sigma_T$*, and also for* $0 \le t \le \tau$*, and for* $T - \tau \le t \le T$ *for some* $\tau > 0$.

We may wonder where is the boundary condition (5.3) included in this formulation. The answer is that it is hidden in the functional space $W_0^{1,1}(\Omega)$, a typical trick of weak theories. Caution: note the change in the class of test functions.

The final step consists of including the initial data. There are several ways of doing it. Following [408] and [457], we propose a definition of solution of the whole problem.

**Definition 5.4** *A locally integrable function* $u$ *defined in* $Q_T$ *is said to be a weak solution of Problem* (5.1)–(5.3) *if*

(i) $u \in L^1(Q_T)$ *and* $w = \Phi(u) \in L^1(0, T : W_0^{1,1}(\Omega))$*;*
(ii) $u$ *satisfies the identity*

$$\iint\limits_{Q_T} \{\nabla\Phi(u) \cdot \nabla\eta - u\eta_t\}\, dxdt = \int\limits_{\Omega} u_0(x)\eta(x,0)dx + \iint\limits_{Q_T} f\eta\, dxdt \tag{5.7}$$

*for any function* $\eta \in C^1(\overline{Q}_T)$ *which vanishes on* $\Sigma$ *and for* $t = T$.

We call $\mathcal{WS}$ the class of functions thus obtained when $u_0 \in L^1(\Omega)$ and $f \in L^1(Q_T)$. The initial function $u_0$ of condition (5.2) is built into the integral formulation (5.7), and is actually satisfied in a very weak sense. The function should be selected so that the integral involving $u_0$ in (5.7) be well defined. The natural option in the present setting is asking that $u_0 \in L^1(\Omega)$, which contains all spaces $L^p(\Omega)$ for $p > 1$.

Note that the space of test functions can be modified in several ways without modifying the defined class of weak functions. This remark will be of much use. Thus, we may replace the condition $\eta \in C^1(\overline{Q}_T)$ by $\eta \in C^\infty(\overline{Q}_T)$ which reduces in principle the amount of test. But the full force of condition (ii) is recovered by approximation. In the other direction, we may enlarge to set of test functions to $\eta \in W^{1,\infty}(Q_T)$ as long as we may approximate it with functions $\eta_\varepsilon$ in the class stated in the definition. This technically means that the trace of $\eta$ on the parabolic boundary of $Q_T$ has to be zero.

As in the case of weak solutions, the definition of very weak solution can be made precise to include initial and boundary data. See Section 6.2.

### 5.2.3 *About the initial data*

The inclusion of the initial data into the definition of weak solution is not the only natural option. As an indication of the scope of the above definition and its alternatives, we indicate another natural way of defining a weak solution, and show that it is included in Definition 5.4.

**Proposition 5.1** *Let $u \in L^1(Q_T)$ be such that*

(i) $\Phi(u) \in L^1(0,T : W_0^{1,1}(\Omega))$;
(ii) *for any function $\eta \in C_c^\infty(Q_T)$, $u$ satisfies the identity*

$$\iint_{Q_T} \{\nabla\Phi(u)\cdot\nabla\eta - u\eta_t\}dxdt = \iint_{Q_T} f\eta\, dxdt; \tag{5.8}$$

(iii) *for every $t>0$ we have $u(t) \in L^1(\Omega)$ and $u(t) \to u_0$ as $t \to 0$ in $L^1(\Omega)$.*

*Then, $u$ is a weak solution to Problem* (5.1)–(5.3) *according to Definition* 5.4.

**Proof** Suppose that $u$ is as in the statement. We have to prove that (5.7) holds. Let $\eta$ be as in (5.7) but we also assume that it vanishes in a neighbourhood of $\Sigma$. We take a cut-off function $\zeta \in C^\infty(\mathbb{R})$, $0 \le \zeta \le 1$, such that $\zeta(t) = 0$ for $t<0$, $\zeta(t) = 1$ for $t \ge 1$ and $\zeta' \ge 0$, and let $\zeta_n(t) = \zeta(nt)$. Applying (5.8) with test function $\eta(x,t)\zeta_n(t)$ gives

$$\begin{aligned}\iint_{Q} \{\nabla\Phi(u)\cdot\nabla\eta - u\eta_t\}\zeta_n - \iint_{Q_T} f\eta\zeta_n &= \iint_{Q} u\eta\zeta_{n,t} = \iint_{Q_{1/n}} u\eta\zeta_{n,t} \\ &= \iint_{Q_{1/n}} (u-u_0)\eta\zeta_{n,t} + \iint_{Q_{1/n}} u_0(x)\eta(x,t)\zeta_{n,t}(t).\end{aligned} \tag{5.9}$$

Fix $\varepsilon > 0$ and let $n$ be so large that $\|u-u_0\|_1 \le \varepsilon$ for $0 \le t \le 1/n$. Then the first integral in the last line can be estimated as $\varepsilon\|\eta\|_\infty \int \zeta_{n,t}dt = \varepsilon\|\eta\|_\infty$ which vanishes as $n \to \infty$, $\varepsilon \to 0$. As for the last term, we get

$$\iint_{Q_{1/n}} u_0(x)\eta(x,t)\zeta_{n,t}(t)dxdt = \int_\Omega u_0(x)\eta\Big(x,\frac{1}{n}\Big)dx - \iint_{Q_{1/n}} u_0\eta_t\zeta_n dxdt,$$

and this tends to $\int_\Omega u_0(x)\eta(x,0)dx$ as $n \to \infty$, which proves (5.7) in this case.

It is very easy to see that (5.7) continues to hold when $\eta \in C^1(Q_T)$ with $\eta = 0$ on the boundary of $Q_T$ (*Hint*: approximate $\eta$ with $\eta_\varepsilon \in C_c^\infty$ and pass to the limit). ■

Actually, the definition of weak solution implies convergence to the initial data in a weaker sense. We ask the reader to prove the following statement (see Problem 5.2).

**Proposition 5.2** *If $u$ is a weak solution of HDP in the sense of Definition* 5.4 *then $u(t)$ converges to $u_0$ weakly in the sense that for every $\varphi \in C^1(\Omega)$ with $\varphi = 0$ on $\partial\Omega$ we have*

$$\lim_{t\to 0}\int_\Omega u(t)\varphi\, dx = \int_\Omega u_0\varphi\, dx. \tag{5.10}$$

Such poor convergence is immediate to obtain but not realistic. We will show below that the energy solutions constructed in this chapter, Sections 5.4 and 5.5, take the initial data in a much nicer way, indeed in the sense of strong convergence in $L^1(\Omega)$. Actually, this will apply to all limit solutions, as reflected in Theorem 6.2.

### 5.2.4 *Examples of weak solutions for the PME*

(i) *Compatibility. Classical implies weak.* Every classical solution of Problem (5.1)–(5.3) is automatically a weak solution of the problem. This is the required property of agreement between classical and weak concepts, a must for every reasonable generalized solution.

(ii) We continue with less trivial examples for the PME with $f = 0$. One of them is the separate-variables solution

$$u(x, t; C) = T(t)F(x), \tag{5.11}$$

where $T(t) = (C + (1-m)t)^{1/(m-1)}$ and $F > 0$ is the solution of a certain nonlinear elliptic equation that vanishes on the boundary, cf. Section 4.2. Since $F$ is $C^\alpha(\overline{\Omega})$ and $C^\infty$ in $\Omega$, and $F^m \in C^1(\overline{\Omega})$, it is clear that for every $C > 0$ this is a weak solution of the problem. Now, for $C = 0$ we obtain a limit solution that is perfect for $t > 0$, but takes on infinite values at $t = 0$ in the sense that

$$\lim_{t\to 0} u(x, t; 0) = \infty \quad \forall x \in \Omega. \tag{5.12}$$

We thus find a kind of giant solution that is infinite everywhere at $t = 0$. But since it becomes bounded and smooth for $t > 0$, it is rather a Friendly Giant. We refer to Section 5.9 for a detailed construction of this special solution. Separate-variables solutions with changing sign can also be constructed, cf. [511].

(iii) Another non-trivial solution of the PME with $f = 0$ is the explicit source-type solution $\mathcal{U}(x, t) = \mathcal{U}(x, t; C)$ of Section 4.4. This is not a weak solution according to our definition because of two reasons: its initial data are singular, and the boundary data are not necessarily 0. However, we can obtain from it weak solutions in our setting by the following method: take $x_0 \in \Omega$, let $\tau > 0$ and let the constant $C$ in $U$ be small enough. Then the function

$$w(x, t) = \mathcal{U}(x - x_0, t + \tau; C) \tag{5.13}$$

is a weak solution of the Dirichlet problem (5.1)–(5.3) in any time interval $(0, T)$ in which the free boundary lies inside of $\Omega$, i.e., if

$$T + \tau \le c\,\mathrm{dist}(x_0, \partial\Omega)^{d(m-1)+2},$$

cf. (1.10). Observe that $w$ is a weak solution but not a classical solution, which shows that the weak theory is a non-trivial extension of the classical theory. See Problem 5.3.

(iv) The dipole solution $U_{\rm dip}(x,t)$ given by formula (4.53) is a non-negative solution of the PME in any cylinder of the form $Q_1=(0,R)\times(0,T)$, hence $d=1$, as long as the free boundary does not reach the fixed boundary $x=R$. If we want integrable initial data we have to insert a time delay and replace it by $U_{\rm dip}(x,t+\tau)$.

When posed in the symmetric cylinder $Q_2=(-R,R)\times(0,T)$, it is a signed solution of the signed PME under the same conditions. The change of sign takes place at $x=0$, where we see that the solution is not $C^1$; to be precise, it is $C^{1/m}$ in $x$. This solution also shows that initial data more general than measures can occur in the theory with changing sign.

## 5.3 Uniqueness of weak solutions

The goal of the theory is to establish existence, uniqueness and other important properties of weak solutions of Problem (5.1)–(5.3). This will be done in the present chapter for the class of weak solutions under a small additional restriction. Moreover, the uniqueness of weak solutions is settled by means of an interesting and easy proof, based on using a quite specific test function.

**Theorem 5.3** *Under the additional assumption that* $\Phi(u)\in L^2(0,T:H_0^1(\Omega))$ *and* $u\in L^2(Q_T)$, *Problem* (5.1)–(5.3) *has at most one weak solution.*

**Proof** Suppose that we have two such solutions $u_1$ and $u_2$. We write $w_i=\nabla\Phi(u_i)$. By (5.7) we have

$$\iint_{Q_T} (\nabla(w_1-w_2)\cdot\nabla\eta-(u_1-u_2)\eta_t)\,dxdt=0 \tag{5.14}$$

for all test functions $\eta$. We want to use as a test function the one introduced by Oleĭnik,

$$\eta(x,t)=\begin{cases}\int_t^T(w_1(x,s)-w_2(x,s))\,ds & \text{if } 0<t<T\\ 0 & \text{if } t\ge T,\end{cases} \tag{5.15}$$

where $T>0$. Even if $\eta$ does not have the required smoothness, we may approximate it with smooth functions $\eta_\varepsilon$ for which (5.14) will hold with these test functions. Since

$$\begin{cases}\eta_t &=-(w_1-w_2)\in L^2(Q_T),\\ \nabla\eta &=\int_t^T(\nabla w_1-\nabla w_2)\,ds\in L^2(Q_T),\end{cases} \tag{5.16}$$

and moreover $\eta(t) \in H_0^1(\Omega)$ and $\eta(T) = 0$, we may pass to the limit $\varepsilon \to 0$ and (5.7) will still hold for $\eta$. Hence,

$$\iint_{Q_T} (w_1 - w_2)(u_1 - u_2)\, dxdt + \iint_{Q_T} (\nabla(w_1 - w_2)) \cdot \left( \int_t^T (\nabla w_1 - \nabla w_2)\, ds \right) dxdt = 0.$$

Integration of the last term gives

$$\iint_{Q_T} (w_1 - w_2)(u_1 - u_2)\, dx\, dt + \frac{1}{2} \int_\Omega \left\{ \int_0^T (\nabla w_1 - \nabla w_2)\, ds \right\}^2 dx = 0.$$

Since both terms are non-negative, we conclude that $u_1 = u_2$ a.e. in $Q$. ■

**Remark on the approximation** Given a function $\eta \in L^2(0, T : H_0^1(\Omega))$, we first cut it at height $n$ in the form

$$\eta_n = \max\{-n, \min\{n, \eta\}\}$$

to obtain a sequence of bounded functions $\eta_n \to \eta$ in $L^2(0, T : H_0^1(\Omega))$. In a second step, every $\eta_n$ is approximated by functions $\eta_{\varepsilon,n}$ as in Definition 5.4.

### 5.3.1 *Non-existence of classical solutions*

As a consequence of the uniqueness of weak solutions and the constructed examples, we have the following:

**Corollary 5.4** *There exist initial data for which Problem* HDP *for the PME does not admit a classical solution, even if the solution is non-negative and* $f = 0$.

**Proof** This is a rather standard argument. Firstly, we note that a classical solution of Problem (5.1)–(5.3) is necessarily a weak solution in our sense. Secondly, we remark that the particular example of weak solution $w(x, t)$ defined in (5.13) has the regularity of Theorem 5.3 and is not a classical solution. By the uniqueness result, there cannot be any other weak solution of (5.1)–(5.3) with the same data. Therefore, no classical solution exists for those data. ■

**Remark** Such an argument will apply to all the unique weak non-classical solutions that will be constructed in the sequel. Moreover, in later chapters we will have the opportunity of finding weak solutions of the PME corresponding to smooth initial data that cease to be classical after some time. One such example is presented in Problem 5.7. However, if the data are positive, then we will prove below that the solution stays classical.

### 5.3.2 *The subclass of energy solutions*

Uniqueness has been proved in a subclass of $\mathcal{WS}$ formed by the weak solutions such that $u \in L^2(Q_T)$ and $\Phi(u) \in L^2(0,T : H_0^1(\Omega))$. The extra regularity allows us to define the dissipated energy

$$DE(u) = \int_0^T \int_\Omega |\nabla\Phi(u)|^2 \, dxdt \tag{5.17}$$

and try to reproduce in this general context the energy calculation of Subsection 3.2.4. That estimate leads us to consider the expression

$$E(u) = E_u(t) := \int_\Omega \Psi(u(t)) \, dx \tag{5.18}$$

($\Psi$ is defined in (5.19)) as a natural energy for the evolution. In this chapter, solutions will be constructed in the class of square integrable functions such that $DE(u)$ and $E_u$ are finite. We will refer to this class as *weak energy solutions* $\mathcal{WES}$.

## 5.4 Existence of weak energy solutions for general $\Phi$. Case of non-negative data

We address in this section the existence of non-negative solutions $u$ with non-negative data, $u_0$ and $f$. This is the most typical problem that is solved for the PME. Though the results are superseded by the construction of Section 5.5 for data of any sign, the present construction uses less functional machinery and is the one currently found in the literature. The technique that we are going to use allows us to cover the following generality:

- $\Phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous and strictly increasing in $u$ with $\Phi(0+) = 0$;
- $\Phi(u)$ is smooth with $\Phi'(u) > 0$ for $u > 0$.

We want to construct solutions enjoying an energy estimate as discussed in Subsection 3.2.4. Such an estimate is essential in the existence proof that we give. We need to recall the function $\Psi$, the primitive of $\Phi$ defined as in (3.18):

$$\Psi(s) = \int_0^s \Phi(s) \, ds. \tag{5.19}$$

Concerning the initial data, such a setting leads us to assume that $u_0$ is a measurable function such that $\Psi(u_0(x)) \in L^1(\Omega)$. We call this space $X = L_\Psi(\Omega)$. It is a subspace of $L^1(\Omega)$. Note that for the PME, $X = L^{m+1}(\Omega)$. Concerning the forcing term $f$ we need the expression $\iint f\Phi(u)\, dxdt$ to make sense. This leads us to ask $f$ to belong to the dual space $Y$ of the space $L^2(0,T : H_0^1(\Omega))$ where $\Phi(u)$ lies. Since our interest in $f$ is minor (at least at this point), we will assume that $f$ is bounded for simplicity.

This is the existence and comparison result for weak solutions that we prove at this stage.

**Theorem 5.5** *Under the above assumptions on $\Phi$, there exists a weak solution of Problem* (5.1)–(5.3) *with initial data $u_0 \in L^1(\Omega)$, $\Psi(u_0) \in L^1(\Omega)$, $u_0 \geq 0$, and forcing term $f \geq 0$, $f$ bounded, where solution is understood in the weak sense of Definition* 5.4. *This solution is non-negative, and the time interval is unbounded ($T = \infty$).*

*We have $\Psi(u) \in L^\infty(0,T : L^1(\Omega))$ for all $T > 0$ and $\Phi(u) \in L^2(0,T : H_0^1(\Omega))$. An energy inequality is satisfied*

$$\iint_{Q_T} |\nabla\Phi(u)|^2 \, dxdt + \int_\Omega \Psi(u(x,T)) \, dx \leq \int_\Omega \Psi(u_0(x)) \, dx + \iint_{Q_T} f\Phi(u) \, dxdt. \tag{5.20}$$

*It is therefore a weak energy solution. The comparison principle holds for these solutions: if $u, \widehat{u}$ are weak solutions with initial data such that $u_0 \leq \widehat{u}_0$* a.e. *in $\Omega$ and $f \leq \widehat{f}$* a.e. *in $Q$, then $u \leq \widehat{u}$* a.e. *in $Q$. In particular, if $u_0, f \geq 0$ in $\Omega$, then $u \geq 0$ in $Q$.*

**Remark** The comparison principle is mentioned in the result because it is a basic property to be expected in a parabolic equation, linear or nonlinear, degenerate or not.

**Proof** It will be divided into several steps. Firstly, we will consider the case of smooth functions $u_0$ and $f$ and prove the existence result by approximation, compactness and monotone limit.

*First step: We assume that $\Gamma = \partial\Omega \in C^{2+\alpha}$, that $u_0$ is a non-negative and $C^2(\Omega)$ function with compact support in $\Omega$, and $f \geq 0$ is continuous and bounded in $\overline{Q}$.*

We begin by constructing a sequence of approximate initial data $u_{0n}$ which does not take the value $u = 0$, so as to avoid the degeneracy of the equation. That allows us to use the results of Section 3.2. We may simply put

$$u_{0n}(x) = u_0(x) + \frac{1}{n}\,. \tag{5.21}$$

Let $M = \sup(u_0)$ and $N = \sup_Q f$. We also approximate $f$ by a sequence of smooth functions $f_n$ in a monotone decreasing way, keeping the bound $0 \leq f_n \leq N_n = N + 1/n$. We now solve the problem

$$(u_n)_t = \Delta\Phi(u_n) + f_n \quad \text{in } Q, \tag{5.22}$$

$$u_n(x,0) = u_{0n}(x) \quad \text{in } \overline{\Omega}, \tag{5.23}$$

$$u_n(x,t) = 1/n \quad \text{on } \Sigma. \tag{5.24}$$

The maximum principle, which holds for classical solutions, implies that

$$\frac{1}{n} \leq u_n(x,t) \leq M + \frac{1}{n} + N_n t \quad \text{in} \quad \overline{Q}. \tag{5.25}$$

Therefore, we are dealing in practice with a uniformly parabolic problem. Actually, Problem (5.22)–(5.24) has a unique solution $u_n \in C^{2,1}(\overline{Q})$. The rigourous justification uses the already mentioned trick consisting of replacing equation (5.22) by

$$(u_n)_t = \operatorname{div}(a_n(u_n)\,\nabla u_n) + f_n, \tag{5.26}$$

where $a_n(u)$ is a positive and smooth function, $a_n(u) \geq c > 0$, and $a_n(u) = \Phi'(u)$ in the interval $[1/n,\, M + 1/n + NT]$. This equation is not degenerate and a unique solution $u_n$ of (5.26), (5.23), (5.24) exists in the space $C^{2,1}_{x,t}(\overline{Q})$ by the standard quasilinear theory of Chapter 3, and it satisfies (5.25). Moreover, by repeated differentiation and interior regularity results for parabolic equations, we are able to conclude that $u_n \in C^\infty(Q)$. Now, due to the definition of $a_n$, equations (5.22) and (5.26) coincide on the range of $u_n$. In this way, Problem (5.22)–(5.24) is solved in a classical sense and the degeneracy of the equation is avoided.

Moreover, again by the maximum principle

$$u_{n+1}(x,t) \leq u_n(x,t) \quad \text{ in } \overline{Q} \tag{5.27}$$

for all $n \geq 1$. Hence, we may define the function

$$u(x,t) = \lim_{n\to\infty} u_n(x,t), \qquad (x,t) \in \overline{Q}. \tag{5.28}$$

as a monotone limit of bounded non-negative functions. We see that $u_n$ converges to $u$ in $L^p(Q_T)$ for every $1 \leq p < \infty$. In order to show that this $u$ is the weak solution of Problem (5.1)–(5.3), we need to estimate the spatial gradient of $\Phi(u_n)$. First of all, from (5.25) we get

$$0 \leq u \leq M + Nt \quad \text{ in } \overline{Q}.$$

We control $\nabla\Phi(u_n)$ as in the energy identity of the Subsection 3.2.4. Since $u_n = 1/n$ on the lateral boundary, we have to multiply equation (5.22) by $\eta_n = \Phi(u_n) - \Phi(1/n)$. Integrating by parts in $Q_T$, we obtain

$$\begin{aligned}
\iint_{Q_T} |\nabla\Phi(u_n)|^2 dxdt &= \int_\Omega \{\Psi(u_{0n}(x)) - \Phi(1/n)u_{0n}(x)\}\,dx \\
&\quad - \int_\Omega \{\Psi(u_n(x,T)) - \Phi(1/n)u_n(x,T)\}\,dx \\
&\quad + \iint_{Q_T} f_n(\Phi(u_n) - \Phi(1/n))\,dxdt \\
&\leq \int_\Omega \Psi(u_0(x))\,dx + \int_\Omega \Phi(1/n)\,u_{0n}(x)\,dx + \iint_{Q_T} f_n\Phi(u_n)\,dxdt.
\end{aligned} \tag{5.29}$$

We may use the boundedness of $f$ in $L^2$ and the Poincarë inequality on the last term to estimate it in the form:

$$\iint_{Q_T} f\Phi(u_n)\,dxdt \le C\iint_{Q_T} f^2\,dxdt + C\iint_{\Sigma} \Phi(1/n)^2\,dS + \frac{1}{2}\iint_{Q_T} |\nabla\Phi(u_n)|^2\,dxdt.$$

We can absorb the influence of the last term into the first term of the left-hand side of (5.29), and the other two terms in this last formula are bounded. Then, since $T$ is arbitrary, it follows that $\{\nabla\Phi(u_n)\}$ is uniformly bounded in $L^2(Q)$, and therefore a subsequence of it converges to some limit $\psi$ weakly in $L^2(Q)$. Since also $\Phi(u_n) \to \Phi(u)$ everywhere, it follows that $\psi = \nabla\Phi(u)$ in the sense of distributions. The limit is uniquely defined so that the whole sequence must converge to it. Passing to the limit in (5.29), we get the energy identity transformed into the energy inequality (5.20).

On the other hand, since $u_n \in C(\overline{Q})$, $u_n(x,t) = 1/n$ on $\Sigma$ and $0 \le u \le u_n$, we have

$$\lim_{(x,t)\to\Sigma} u(x,t) = 0$$

with uniform convergence. Hence $\Phi(u(\cdot,t)) \in H_0^1(\Omega)$ for a.e. $t > 0$.

Finally, since $u_n$ is a classical solution of (5.1), it clearly satisfies (5.7) with $u_0$ replaced by $u_{0n}$. Letting $n \to \infty$ we obtain (5.7) for $u$. Therefore, $u$ is a weak solution of (5.1)–(5.3).

Let us remark, to end this step, that if we have data $(u_0, f)$ and $(\hat{u}_0, \widehat{f})$ such that $u_0 \le \hat{u}_0$ and $f \le \widehat{f}$, then the above approximation process produces ordered approximating sequences, $u_{0n} \le \hat{u}_{0n}$. By the classical maximum principle, we have $u_n \le \hat{u}_n$ for every $n \ge 1$. In the limit, $u \le \hat{u}$.

*Second step: We assume that $u_0$ is bounded and vanishes near the boundary, and $f$ is bounded and non-negative.*

The method of the previous step can still be applied, but now $f$ is approximated by a sequence of smooth functions $f_n$ that converge to $f$ a.e. According to the quasilinear theory, cf. [357], now the approximate solutions $u_n \in C^\infty(Q) \cap C^{2,1}(Q \cup \Sigma)$ are not continuous down to $t = 0$ unless the data are; instead, they take the initial data in $L^p(\Omega)$ for every $p < \infty$. Passage to the limit in $u_n$ is now based not on monotonicity, but on the $L^1$ dependence of the solutions on the data which is described in Subsection 3.2.3: it follows that $u_n \to u$ in $C([0,T) : L^1(\Omega))$. Since the functions are bounded, convergence also takes place in $C([0,T) : L^p(\Omega))$ for all $p < \infty$. The convergence of the gradients is unchanged and the proof ends as before. Comparison still applies.

*Third step: General case.*

For general $\Gamma$ and general data, $\Psi(u_0) \in L^1(\Omega)$, $u_0 \ge 0$, we first approximate the domain by an increasing family $\Omega_k$ of domains with $C^{2,\alpha}$ boundary $\Gamma_k$, we take an increasing sequence of cut-off functions $\zeta_k(x)$ which vanish near $\Gamma_k$, and consider the sequence of approximations of the initial data

$$u_{0k}(x) = \min\{u_0(x)\zeta_k(x), k\}. \tag{5.30}$$

Using Step 2 we solve Problem (5.1)–(5.3) with initial data $u_{0k}$ and forcing term $f_k = f\zeta_k$ to obtain a unique weak solution $u_k$ defined in $Q_k = \Omega_k \times (0, T)$. By the comparison remark, $u_{k+1} \geq u_k$ in $Q_k$ (note that $u_{k+1} \geq 0$ on $\Sigma_k$). On the other hand, by estimate (5.20) the family $\{\Psi(u_k)\}$ is uniformly bounded in $L^\infty(0, \infty : L^1(\Omega_k))$ and $\nabla\Phi(u_k)$ is likewise in $L^2(Q_k)$. Hence, extending $u_k$ by 0 in $\Omega \setminus \Omega_k$, we have that $\{u_k\}$ converges a.e. to a function $u \in L^\infty(0, \infty : L_\Psi(\Omega))$. On the other hand, $\nabla\Phi(u_k)$ is uniformly bounded, hence it converges weakly in $L^2(Q)$ to $\nabla\Phi(u)$, and (5.20) holds for $u$. It follows that $\Phi(u) \in L^2(0, \infty; H_0^1(\Omega))$. Finally, equation (5.7) is satisfied, as the reader may easily check passing to the limit in the similar expressions for $u_k$. ∎

*Motivation for the initial data*

We see from the proof that the choice of space for the initial data depends essentially on the energy estimate (5.20), which is a cornerstone of this chapter. A priori estimates are one of the most powerful and widely used tools in the study of PDE. This approach will be stressed in our treatment of the existence, uniqueness and qualitative properties of solutions to the different problems.

### 5.4.1 *Improvement of the assumption on $f$*

The approximation proof used above can be performed under weaker assumptions on the forcing term:

**Corollary 5.6** *The result of Theorem* 5.5 *on existence of weak energy solutions holds if $f \geq 0$, $f \in L^p(Q)$, with $p = 2d/(d+2)$ if $d \geq 3$; for some $p > 1$ if $d = 1, 2$.*

Actually, the technique of passage to the limit in the $L^1$ norm allows us to obtain a limit $u = \lim u_n(x, t)$ even if $f$ is not an $L^2$ function, as long as $f \in L^1(Q_T)$. However, if we want to obtain a weak solution in the sense of Definition 5.4 we need to keep a control for $\nabla\Phi(u)$ in $L^2(Q_T)$. In view of the energy estimate and Sobolev's embedding theorem, this is possible under the stated assumptions on $f$. These assumptions guarantee precisely that the energy estimate still holds with finite terms.

### 5.4.2 *Non-positive solutions*

The results of this section not only show the existence of weak solutions with non-negative data such that $\Psi(u_0) \in L^1(\Omega)$ and $f \geq 0$, but also the same problem with non-positive data $u_0 \leq 0$ in the same integrable class, and $f \leq 0$, under a similar assumption of regularity of $\Phi(s)$ for $s < 0$. Indeed, the filtration equation is invariant under the symmetry $u \mapsto \widehat{u} = -u$, if we change the nonlinearity

$\Phi$ into $\widehat{\Phi}(s) = -\Phi(-s)$. It is quite easy to check that, if a function $u$ is a weak solution of Problem HDP with initial data $u_0, f$, and nonlinearity $\Phi$, then $\widehat{u}(x,t) = -u(x,t)$ is a weak solution with data $\widehat{u}_0(x) = -u_0(x)$, $\widehat{f}(x,t) = -f(x,t)$ and nonlinearity $\widehat{\Phi}(s) = -\Phi(-s)$.

## 5.5 Existence of weak signed solutions

We address in this section the main problem of existence of signed solutions for the complete GPME. As in the previous section, our goal is to obtain weak energy solutions, which forces us to impose some conditions on the data. Such restrictions will eliminated in the following chapters by different techniques and with different concepts of solution. We assume that $\Phi$ satisfies the conditions $(H_\Phi)$ stated at the beginning of Section 5.2.

**Theorem 5.7** *Assume that $u_0 \in L_\Psi(\Omega)$ and $f \in L^p(Q)$, with $p = 2d/(d+2)$ if $d \ge 3$ ($f \in L^p(Q)$ for some $p > 1$ if $d = 1, 2$). Then, Problem* (5.1)–(5.3) *has a weak solution defined in an infinite time interval, $T = \infty$. We have $u \in L^\infty(0, T : L_\Psi(\Omega))$ and $\Phi(u) \in L^2(0, T : H_0^1(\Omega))$, and the energy inequality* (5.20) *holds. Comparison holds as in Theorem* 5.5.

**Proof** In our situation we cannot simply modify the initial data to obtain a problem with a classical solution, as we did in Theorem 5.5, since such approximations will necessarily be of changing sign, and the equation may degenerate (in the key case of the PME, it does so at the level $u = 0$, and that level cannot be avoided for solutions that change sign). Therefore, we also modify the equation into a non-degenerate parabolic equation by changing the nonlinearity $\Phi$ in the following form. We pick a sequence of functions $\Phi_n$ such that

(i) $\Phi_n \in C^\infty(\mathbb{R})$ and $\Phi_n'(u) > 0$ for every $n \ge 1$ and every $u \in \mathbb{R}$;
(ii) $\Phi_n \to \Phi$ uniformly on compact sets;
(iii) $\Phi_n(0) = 0$ for every $x \in \Omega$.

**Lemma 5.8** *The result holds when $u_0, f$ and $f_t$ are bounded, $\Gamma = \partial\Omega \in C^{2,\alpha}$, and $\Phi$ is locally Lipschitz continuous in $u$.*

**Proof of the lemma** (i) We fix $T > 0$ and consider the approximate equations

$$u_t = \Delta\Phi_n(u) + f_n \quad \text{in} \quad Q_T, \tag{5.31}$$

where $f_n$ is a smooth approximation converging to $f$ in $L^p(Q_T)$ for all $p < \infty$. We solve the problem formed by (5.31) with initial data

$$u_n(x,0) = u_{0n}(x) \qquad \text{in } \overline{\Omega}, \tag{5.32}$$

where $u_{0n} \in C_c^\infty(\overline{\Omega})$ approximates $u_0$ in $L_\Psi(\Omega)$; it will be convenient to ask that $|u_{0n}(x)| \le n$. We also impose boundary data

$$u_n(x,t) = 0 \qquad \text{on } \Sigma. \tag{5.33}$$

Since $\Gamma \in C^{2+\alpha}$, Problem (5.31)–(5.33) has a unique solution $u_n \in C^{2,1}(\overline{Q}) \cap C^\infty(Q)$, cf. [357, Theorem 6, p. 452]. Moreover, if $M_1 = \sup(-u_0)$, $M_2 = \sup u_0$, $N_1 = \sup(-f)$, and $N_2 = \sup f$, we get by the standard maximum principle

$$-M_1 - N_1 t \le u_n(x,t) \le M_2 + N_2 t \quad \text{in } \overline{Q}. \tag{5.34}$$

(ii) Let $w_n = \Phi_n(u_n)$. We want to control the spatial derivative $\nabla w_n$ uniformly in $n$ in terms of the initial data. As we know, the idea is to multiply the equation by $w_n$ and integrate by parts in $Q_\tau$ to obtain for every $T > 0$ an energy estimate of the form

$$\int_\Omega \Psi_n(u_n(x,T))\,dx + \iint\limits_{Q_T} |\nabla w_n|^2\,dxdt = \int_\Omega \Psi_n(u_{0n}(x))\,dx + \iint\limits_{Q_T} f_n w_n\,dxdt, \tag{5.35}$$

where $\Psi_n$ is the primitive of $\Phi_n$ with $\Psi_n(0) = 0$. Arguing as in (5.29), we conclude that the integral $\iint_Q |\nabla w_n|^2\,dxdt$ is bounded independently of $n$.

(iii) We produce next a compactness estimate in time as indicated in formula (3.27) for smooth solutions. The idea is to multiply the equation by $\zeta w_{n,t}$, with $w_n = \Phi_n(u_n)$ and $\zeta(t)$ a cut-off function, and integrate by parts in space to obtain the expression

$$\iint_{Q_T} \zeta\,\Phi_n'(u_n)|u_{n,t}|^2\,dxdt = \iint_{Q_T} \left\{ \frac{\zeta_t}{2}|\nabla\Phi_n(u_n)|^2 - (f\zeta)_t \Phi_n(u_n) \right\} dxdt.$$

We conclude from this, the previous energy estimate, and the assumption on $f$, that for every $\tau > 0$ the integral $\int_\tau^\infty \int \Phi_n'(u_n)|(u_n)_t|^2\,dxdt$ is uniformly bounded.

(iv) Under our assumptions, the $u_n$ are uniformly bounded by some $C_1$ in $Q_T$, and $\Phi$ is locally Lipschitz continuous, so that $\Phi_n'(s) \le C$ for all $n$ and for $|s| \le C_1$. Since $|(w_n)_t|^2 = (\Phi_n'(u_n)(u_n)_t)^2$, we conclude that $(w_n)_t \in L^2(\Omega \times (\tau,\infty))$ with bound independent of $n$.

The two previous estimates imply that the sequence $\{w_n\}$ is bounded in $H^1(Q^*)$, with $Q^* = \Omega \times (\tau, T)$. This allows us to pass to the limit $n \to \infty$ along a subsequence $\{n_j\}$ to obtain a function

$$w(x,t) = \lim_{j\to\infty} w_{n_j}(x,t), \tag{5.36}$$

and $w \in L^2(Q^*)$. Choosing a subsequence, the convergence $w_{n_j} \to w$ takes place almost everywhere. It is also clear that $w \in L^2(0,\infty : H_0^1(\Omega))$.

It is straightforward to check that, under these circumstances, the uniformly bounded sequence $\{u_{n_j}\}$ also converges to a bounded function $u$ a.e. and that $w = \Phi(u)$ a.e. Moreover, we have

$$\iint\limits_{Q_T} |\nabla w|^2\,dxdt \le \int_\Omega \Psi(u_0(x))\,dx + \iint f\,w\,dxdt. \tag{5.37}$$

By virtue of estimate (3.29) applied to the approximations, after passing to the limit we also have $t^{1/2}w \in L^\infty(0,\infty : H_0^1(\Omega))$ and

$$\frac{T}{2}\int_\Omega |\nabla w(x,T)|^2\,dx \le \iint_{Q_T} |\nabla w|^2\,dxdt + C(T). \tag{5.38}$$

No major difficulty arises in checking that $u$ satisfies the conditions of Definition 5.4, so that it is a weak solution of the problem. By uniqueness, we conclude that the whole sequence $u_n$ converges to $u$, the unique solution of the problem. Estimate (5.35) becomes in the limit

$$\int_\Omega \Psi(u(T))\,dx + \iint\limits_{Q_T} |\nabla\Phi(u)|^2\,dxdt \le \int_\Omega \Psi(u_0)\,dx + \iint\limits_{Q_T} f\Phi(u)\,dxdt, \tag{5.39}$$

while the time derivative estimate can be written as

$$\iint_{Q_T} \zeta\,(\mathcal{Z}(u)_t)^2\,dxdt = \iint_{Q_T} \left\{\frac{\zeta_t}{2}|\nabla\Phi(u)|^2 - (f\zeta)_t\Phi(u)\right\}dxdt, \tag{5.40}$$

with $\mathcal{Z}$ as in (3.25), or in the alternative form

$$\begin{aligned}\iint_{Q_T} t\Phi'(u)|u_t|^2\,dxdt + \frac{T}{2}\int_\Omega |\nabla w(T)|^2\,dx \\ \le \frac{1}{2}\iint_{Q_T} |\nabla w|^2\,dxdt + \iint_{Q_T} tfw_t\,dxdt.\end{aligned} \tag{5.41}$$

We also observe that in the limit of the approximations,

$$-M_1 - N_1 t \le u(x,t) \le \sup M_2 + N_2 t. \tag{5.42}$$

Let us recall at this stage that the maximum principle holds for the approximate problems. If we have two initial data $u_0, \hat u_0$ such that $u_0 \le \hat u_0$, $f \le \widehat{f}$, then the above approximation process can be performed so as to produce ordered approximating sequences, $u_n \le \hat u_n$. In the limit, $u \le \hat u$. Therefore, the proof is complete in this case. ■

**Lemma 5.9** *The result also holds when the previous condition on* $\Phi$ *is eliminated.*

**Proof** We have to tackle now the case where $\Phi$ is not Lipschitz continuous, for instance in the case of the FDE ($0 < m < 1$). In that case we cannot conclude that $w_t$ is bounded in some space and we need a slight modification in the passage to the limit of the previous step to arrive at the desired conclusion also in this case. Here is a way: we introduce the non-decreasing function $Z(s)$ defined by the differential rule, $dZ = \min\{ds, d\Phi(s)\}$, and its approximations

$$Z_n(s) = \int_0^s \min\{1, \Phi_n'(s)\}\,ds.$$

Clearly, the $Z_n$ are strictly increasing functions, uniformly Lipschitz continuous, we have $|Z_n(s)| \le |s|$, $|Z_n(s)| \le |\Phi_n(s)|$, and finally $Z_n(s) \to Z(s)$ locally uniformly in $\mathbb{R}$.

We then define $z_n(x,t) = Z_n(u_n(x,t))$, and immediately see that the sequence $z_n(x,t)$ is uniformly bounded in $Q_T$. Moreover, from steps (ii) and (iii) we conclude that $(z_n)_t$ and $\nabla z_n$ are uniformly bounded in $L^2(Q^*)$. Therefore, after passing to a subsequence, $z_n$ converges in $L^2(Q^*)$ and a.e. to a bounded function $z$. It is then easy to conclude that also $w_n \to w$, $u_n \to u$ weakly and a.e. and that $w = \Phi(u)$, since both are related to $z$ by continuous and increasing functions. See Problem 5.4. The rest is similar to Step (iv). ■

**Lemma 5.10** *The result also holds when the initial data $u_0$ is not bounded and/or $f$ and $f_t$ are not bounded.*

**Proof** We use approximations of these functions by functions $u_{0n}$, $f_n$ as in the lemma, and such that: $u_{0n}$ is uniformly bounded in $L_\Psi(\Omega)$ and $u_{0n} \to u_0$ in $L^1(\Omega)$; $f_n$ is uniformly bounded in $L^p(Q_T)$ and $f_n \to f$ in $L^1(Q_T)$. Using the $L^1$ stability result, Proposition 3.5, we conclude that $u_n$ converges in $L^\infty(0,T : L^1(\Omega))$ towards a function $u$. By the a priori estimates, $u \in L^\infty(0,T : L_\Psi(\Omega))$. Moreover, the energy argument used above implies that $w_n$ converges weakly to some $w \in L^2(Q_T)$ with $\nabla w_n$ converging in the same way to $\nabla w$. We also have $w = \Phi(u)$ a.e. In the limit of the weak formulation satisfied by $u_n$, we conclude that $u$ is a weak solution of the problem. ■

**End of proof of the theorem** We still need to consider the case where $\Gamma$ is not $C^{2+\alpha}$ smooth. As in the end of proof of Theorem 5.5, we approximate $\Omega$ by an increasing sequence $\Omega_k$ of domains strictly contained in $\Omega$ and having $C^{2+\alpha}$ boundary $\Gamma_k$, we take an increasing sequence of cut-off functions $\zeta_k(x)$ supported in $\Omega_k$, and define $u_{0k} = u_0\zeta_k$, $f_k = f\zeta_k$. Then, solving the problems with these data in $Q_k = \Omega_k \times (0,T)$, and extending $u_k$ by 0 in $\Omega \setminus \Omega_k$, the uniform estimates give boundedness of $\Psi(u_n)$ and compactness of the corresponding functions $z_k$, so that in the end $u_k \to u$, which is a solution of the desired problem in $Q_T$. ■

**Remarks**

(1) Every solution with changing sign obtained as a limit of this process is bounded above by the non-negative solution with data $u_0^+(x) = \sup\{u_0(x), 0\}$, $f_+(x,t) = \sup\{f(x,t), 0\}$, and below by the non-positive solution with initial data $u_0^-(x) = \inf\{u_0(x), 0\}$, and forcing term $f_-(x,t) = \min\{f(x,t), 0\}$.

(2) In the PME case, it is convenient to organize the approximation of $\Phi$ as follows: we first pick a function $\Phi_1 \in C^\infty(\mathbb{R})$ such that: (i) $\Phi_1(s) = \Phi(s)$ for $|s| \ge 1$; (ii) $\Phi_1(-s) = -\Phi(s)$; (iii) $\Phi_1$ is linear in the interval $(-1/2, 1/2)$, $\Phi = cs$; (iv) $\Phi_1$ is convex for $s \ge 0$.

We then define for every integer $n \ge 1$ the function

$$\Phi_n(s) = n^{-m}\Phi_1(ns). \tag{5.43}$$

Observe that $\Phi_n(s)$ is just $\Phi(s)$ if $|s| \ge 1/n$, while $\Phi = n^{1-m}s$ for $|s| \le 1/(2n)$.

(3) Let us recall that the convergence of $u_n$ to $u$ in the approximations takes place in $L^1(Q)$ without having to introduce any time delay.

### 5.5.1 *Constant boundary data*

We can also modify Theorem 5.7 to solve some problems with non-zero boundary data. We note that solving those problems implies introducing a suitable concept of solution, a task that will be performed in detail in Section 5.7. For the moment, we can use the recently proved theorem to solve the question with constant boundary data as follows. We observe that in the smooth case, the vertical displacement of a solution $u$ of the GPME $u_t = \Delta\Phi(u) + f$ produces another solution $\widetilde{u} = u + C$ of the GPME with a new nonlinearity, $\widetilde{\Phi}(s) = \Phi(s + C) - \Phi(C)$, that is also in the same class as $\Phi$. Namely, $\widetilde{u}_t = \Delta\widetilde{\Phi}(\widetilde{u}) + f$. Besides, if $u = 0$ on $\Sigma$, then $\widetilde{u} = C$ on $\Sigma$. We take this transformation as the definition of solution for the new boundary value problem. We immediately have:

**Corollary 5.11** *We can uniquely solve the Dirichlet problem for the GPME under the same assumptions on $u_0$ and $f$ but with constant non-zero boundary data $u = C$ on $\Sigma$. If $C > 0$ the solutions are larger than in the standard HDP; if $C < 0$ they are smaller.*

The comparison principle holds; it is easy in the smooth case, it is justified in the general case by approximation.

## 5.6 Some properties of weak solutions

Now that we know that weak solutions exist and are unique, we may proceed with the qualitative analysis. Though weak solutions with data which are not strictly positive need not be classical solutions, they enjoy some interesting regularity properties, some of them a consequence of the estimates satisfied by smooth solutions that we have presented in Section 3.2, and some others that will be derived as a consequence of new estimates. In the first type, let us mention:

- The energy inequality given by formulas (5.20) or (5.39) that asserts that $\Phi(u) \in L^2(0, T : H_0^1(\Omega))$ with a bound that depends only on the norm of $u_0$ in $L_\psi(\Omega)$ and the norm of $f$ in $L^p(Q_T)$. It also asserts that $u(t)$ is estimated in $L^\infty(0, T : L_\Psi(\Omega))$ in the same way.
- Time derivative control is given by (5.40), showing that $Z(u)_t$ is bounded in $L^2(Q_T^\tau)$ if $f, f_t \in L^p(Q_T)$. In the case of the PME, this means that $\partial_t(u^{(m+1)/2}) \in L^2(Q_T^\tau)$. When $\Phi'$ is bounded, also $\Phi(u)_t \in L^2(Q_T^\tau)$. The

same happens when $u$ is bounded for $t \ge \tau$ and $\Phi$ is locally Lipschitz. We will find in Section 5.8 a priori bounds for the sup norm of $|u|$ when $\Phi$ is superlinear at infinity and $f$ is bounded.

- When $\Phi'(u)$ is bounded, inequality (5.41) implies that $\nabla\Phi(u)$ is actually bounded in $L^\infty(\tau, T : H_0^1(\Omega))$. Same comment for bounded $u$ as before.

These estimates take on a much nicer form when applied to the incomplete equation, i.e., for $f = 0$, which is the case usually considered in the PME theory. Thus, the energy estimate becomes

$$\int_\Omega \Psi(u(x,T))\,dx + \iint_{Q_T} |\nabla w|^2\,dxdt \le \int_\Omega \Psi(u_0(x))\,dx, \tag{5.44}$$

which means that $\int_\Omega \Psi(u(x,t))\,dx$ is a non-increasing function of time, and that $\nabla\Phi(u)$ is square integrable in the whole $Q = \Omega \times (0,\infty)$. For future reference, we write this estimate in the case of the PME:

$$\frac{1}{m+1}\int_\Omega |u(x,T)|^{m+1}\,dx + \iint_{Q_T} |\nabla(|u|^{m-1}u)|^2\,dxdt \le \frac{1}{m+1}\int_\Omega |u_0(x)|^{m+1}\,dx. \tag{5.45}$$

On the other hand, the time derivative estimate reads

$$\iint_{Q_T} t\Phi'(u)|u_t|^2\,dxdt + \frac{T}{2}\int_\Omega |\nabla w(x,T)|^2\,dx \le \frac{1}{2}\iint_{Q_T} |\nabla w|^2\,dxdt, \tag{5.46}$$

or in the alternative forms, like

$$\iint_{Q_T^\tau} \Phi'(u)|u_t|^2\,dxdt + \frac{1}{2}\int_\Omega |\nabla w(x,T)|^2\,dx \le \frac{1}{2}\int_\Omega |\nabla w(x,\tau)|^2\,dx. \tag{5.47}$$

For the PME this estimate reads

$$2m\iint_{Q_T^\tau} |u|^{m-1}|u_t|^2\,dxdt + \int_\Omega |\nabla(|u|^{m-1}u)(x,T)|^2\,dx \le \int_\Omega |\nabla(|u|^{m-1}u)(x,\tau)|^2\,dx. \tag{5.48}$$

These estimates are satisfied with equality for classical solutions. The question will be discussed for weak solutions in Subsection 8.2.1.

- A very important property of the approximate equations is the contractivity with respect to the $L^1(\Omega)$ norm. This property passes to the limit and gives for two weak energy solutions $u$ and $\widehat{u}$ as in Theorem 5.7 the estimate

$$\|u(t) - \widehat{u}(t)\|_1 \le \|u_0 - \widehat{u}_0\|_1 + \int_0^t \|f(s) - \widehat{f}(s)\|_1 ds. \tag{5.49}$$

This implies the stability of such solutions. We will develop this issue in depth in the next chapter.

- We can also obtain a priori bounds in the norm $L^\infty(0,T:L^p(\Omega))$ when both $u_0$ and $f$ are $L^p$ functions by passing to the limit the estimates of Subsection 3.2.2. When $f=0$, we can also obtain monotonicity in all the $L^p$ norms, $1\le p<\infty$.

**Proposition 5.12** *In the situation of Theorem* 5.7, *if moreover the initial data belong to the space* $L^p(\Omega)$, $p\ge 1$, *and* $f=0$, *then* $u(\cdot,t)\in L^p(\Omega)$ *for any* $t>0$ *and*

$$\|u(\cdot,t)\|_p \le \|u_0\|_p. \tag{5.50}$$

**Proof** It is based on passing to the limit the estimate obtained for smooth solutions (3.13). In the case of the PME the complete calculation reads

$$\frac{4q(q+1)m}{(q+m)^2}\iint\limits_{Q_T}\left|\nabla\left(u^{\frac{q+m}{2}}\right)\right|^2 dxdt+\int_\Omega u^{q+1}(x,T)dx\le\int_\Omega u_0^{q+1}(x)dx, \tag{5.51}$$

valid for $q>0$. To get the case $p=1$ we pass to the limit as $q\to 0$. The proof is justified by approximation. ∎

## 5.7 Weak solutions with non-zero boundary data

We consider here the extension of the theory developed thus far in this chapter to the case where the boundary data are not homogeneous. Let us assume that $\Omega$ is a bounded domain in $\mathbb{R}^d$, $d\ge 1$, with regular boundary $\Gamma=\partial\Omega\in C^{2+\alpha}$; as in Section 5.2 we assume that $\Phi:\mathbb{R}\mapsto\mathbb{R}$ is a continuous increasing function with $\Phi(\pm\infty)=\pm\infty$. We pose the general Dirichlet problem for the filtration equation:

*Problem GDP*

*Given measurable functions* $u_0$ *in* $\Omega$, $g$ *in* $\Sigma_T$, *and* $f$ *in* $Q_T$, *find a locally integrable function* $u=u(x,t)$ *defined in* $Q_T$ *that solves the set of equations*

$$u_t=\Delta\Phi(u)+f \qquad \text{in } Q_T, \tag{5.52}$$

$$u(x,0)=u_0(x) \qquad \text{in } \Omega, \tag{5.53}$$

$$\Phi(u(x,t))=g(x,t) \qquad \text{in } \Sigma_T, \tag{5.54}$$

*in a weak sense to be precisely defined. The time* $T>0$ *can be finite or infinite.*

*Functional setting. Traces*

We want to find $u$ in a suitable functional class that guarantees uniqueness and continuous dependence on the data. Depending on that functional choice, suitable functional spaces are chosen for the data $u_0$, $f$ and $g$. Definition 5.1 is still good enough as a local weak solution. We need some changes to define a suitable concept of solution for the new problem that accounts for non-zero boundary data. We will ask $\Phi(u)\in L^2(0,T:H^1(\Omega))$, forgetting about the zero boundary conditions. Next, we need to recall some facts about the theory of *boundary traces*:

(i) Functions $f \in H^1(\Omega)$ have boundary values called *traces*, $T_{\partial\Omega} f$, on the boundary $\partial\Omega$; moreover, the linear *trace map* $T_{\partial\Omega}$ maps $H^1(\Omega)$ onto the space $H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$.[1]

(ii) In the time-dependent context, the trace operator can be naturally extended into a continuous linear map

$$T_\Sigma : L^2(0, T : H^1(\Omega)) \to L^2(0, T : H^{1/2}(\partial\Omega)) \subset L^2(\Sigma_T). \tag{5.55}$$

(iii) We will also need a further result. The trace operator admits a continuous lifting map, $j : H^{1/2}(\partial\Omega) \mapsto H^1(\Omega)$ such that $T_\Sigma(j(g)) = g$ for every $g \in H^{1/2}(\partial\Omega)$; we say that $j$ is a right inverse of $T_\Sigma$. This extends to a lifting map

$$J : L^2(0, T : H^{1/2}(\partial\Omega)) \mapsto L^2(0, T : H^1(\Omega)).$$

After these considerations, we propose the following definition.

**Definition 5.5** *Given $u_0 \in L^1(\Omega)$, $g \in L^2(0, T : H^{1/2}(\partial\Omega))$, and $f \in L^1(Q_T)$, a locally integrable function $u$ defined in $Q_T$ is said to be a weak solution of Problem* (5.52)–(5.54) *if*

(i) *$\Phi(u) \in L^2(0, T : H^1(\Omega))$, and $T_\Sigma(\Phi(u)) = g$;*

(ii) *$u \in L^2(\Omega \times (0, T))$;*

(iii) *$u$ satisfies the identity*

$$\iint\limits_{Q_T} \{\nabla\Phi(u) \cdot \nabla\eta - u\eta_t\}\, dxdt = \int\limits_\Omega u_0(x)\eta(x, 0)dx + \iint\limits_{Q_T} f\,\eta\, dxdt \tag{5.56}$$

*for any function $\eta \in C^1(\overline{Q}_T)$ which vanishes on $\Sigma$ and for $t = T$.*

Clearly, the weak solutions we have constructed for the homogeneous Dirichlet problem. HDP are the particular case of this definition which assumes zero boundary trace, $g = 0$. This theory covers also the existence for constant boundary data advanced in Corollary 5.11. Note, however, that the have restricted the generality of the discussion of the HDP to the case of weak energy solutions.

As in the homogeneous problem, the goal of the theory is to establish existence, uniqueness, continuous dependence and other important properties of weak solutions of the general Problem (5.52)–(5.54). The uniqueness of weak solutions as defined above is settled by exactly the same result as in Theorem 5.3, and even the proof is the same.

[1]References for this topic are e.g. Adams [4] or Dautray and Lions [198].

**Theorem 5.13** *Problem* (5.52)–(5.54) *has at most one weak solution.*

The reader should only notice that the test function $\eta$ defined in (5.15) is still acceptable because it continues to have zero boundary trace, and also that the data $u_0$ and $f$ disappear from the weak formulation when subtracting the expressions satisfied by the two solutions.

Concerning the existence theory, we repeat the assumptions on the initial data and forcing term made in Sections 5.4 and 5.5 and repeat the outline of the existence proofs with a suitable choice of boundary data. The choice is somewhat stricter:

**(HG)** We assume that there is a function $G \in L^2(0,T : H^1(\Omega))$ such that $g = T_\Sigma(G)$, and we assume further that $G, G_t, G_{tt} \in L^\infty(\Omega)$.

**Theorem 5.14** *Under the above assumptions on $G$, for every $u_0 \in L_\Psi(\Omega)$ and $f \in L^2(Q_T)$, there exists a weak solution of Problem GDP with $u \in L^\infty(0,\infty : L_\Psi(\Omega))$. The comparison principle applies to these solutions: if $u, \widehat{u}$ are weak solutions and $u_0 \le \widehat{u}_0$* a.e. *in $\Omega$, $f \le \widehat{f}$* a.e. *in $Q_T$, and $g \le \widehat{g}$* a.e. *in $\Sigma$, then $u \le \widehat{u}$* a.e. *in $Q_T$. In particular, If $u_0, f, g \ge 0$, then $u \ge 0$.*

**Proof** The proof we give follows the outline of Theorem 5.7. Therefore, we need only to stress the differences. We perform an approximation process where the data are bounded; we take as boundary value for the approximate solutions, $\Phi_n(u_n) = g_n(x,t)$, where $g_n$ is the trace on $\Sigma$ of a smooth and positive function $G_n$ that approximates $G$ in its space, $L^2(0,T : H^1(\Omega)) \cap L^\infty(Q_T)$. We call the solutions $u_n$ and put $w_n = \Phi_n(u_n)$. Multiplying the equation satisfied by the smooth solution $u_n$ by $\Phi_n(u_n) - G_n \in L^2(0,T : H_0^1(\Omega))$, we get in the usual way

$$\iint_{Q_T} \nabla\Phi_n(u_n)\cdot(\nabla\Phi_n(u_n) - \nabla G_n)\,dxdt + \iint_{Q_T} (\Phi_n(u_n) - G_n)u_{n,t}\,dxdt$$
$$= \iint_{Q_T} f_n\,(\Phi_n(u_n) - G_n)\,dxdt.$$

Hence,

$$\iint_{Q_T} |\nabla\Phi_n(u_n)|^2\,dxdt + \int_\Omega \Psi_n(x,u_n(T))\,dx + \iint_{Q_T} G_{n,t}u_n\,dxdt + \iint_\Omega u_n(0)G_n(0)\,dx$$
$$= \int_\Omega \Psi(x,u_{n,0})\,dx + \iint_{Q_T} f_n\,(\Phi_n(u_n) - G_n)\,dxdt + \iint_{Q_T} (\nabla\Phi_n(u_n)\cdot\nabla G_n)\,dxdt$$
$$+ \int_\Omega u_n(T)G_n(T)\,dx.$$

After some easy computations, and using the regularity of the data and Sobolev's embeddings, we may derive estimates of the form

$$\iint_{Q_T} |\nabla \Phi_n(u_n)|^2 \, dxdt \le C, \qquad \int_\Omega \Psi_n(u_n)\, dx \le C,$$

which are uniform in $n$ and in $t \in (0,T)$.

At least in the signed case, we also need an estimate on a time derivative just as in Theorem 5.7. We multiply the equation satisfied by $u_n$ by $\partial_t(w_n - G_n)$ and integrate by parts in space to obtain

$$\int_\Omega (w_n - G_n)_t \, (u_n)_t \, dx = -\int_\Omega \nabla w_n \cdot \nabla (w_n - G_n)_t \, dx + \int_\Omega f_n (w_n - G_n)_t \, dx.$$

Multiplying now by a smooth function $\zeta(t) \ge 0$ that vanishes for $t=0$ and $t=T$, and integrating in time and rearranging, we get (integrals in $Q_T$)

$$\begin{aligned}\iint \zeta \Phi_n'(x,u_n)|(u_n)_t|^2 \, dxdt = \frac{1}{2}\iint \zeta' |\nabla w_n|^2 \, dx - \iint (\zeta (G_n)_t)_t \, u_n \, dxdt \\ + \iint \nabla \zeta w_n \cdot \nabla (G_n)_t dxdt \\ - \iint (\zeta f_n)_t \, (w_n - G_n) \, dxdt.\end{aligned}$$

In this way, a uniform estimate is obtained for $\iint \Phi_n'(u_n)|(u_n)_t|^2 \, dxdt$.

The rest of the proof offers few novelties and is left to the reader as a long review exercise. Finally, the maximum principle applies to the approximate problems, and this property is conserved in the limit. ■

**Remarks**

(1) The regularity of $G_{tt}$ is not needed when treating non-negative solutions under the assumptions on $\Phi$ made in Theorem 5.5 using monotonicity. Also the assumption on $G_t$ may be relaxed.

(2) The condition on the forcing term for the result to be true can be weakened into $f \in L^p(Q)$, with $p = 2d/(d+2)$ if $d \ge 3$, and some $p > 1$ if $d = 1, 2$.

**Examples** All the examples of 'naive' solutions considered in Chapter 4 are all of them weak solutions of the GDP for the PME when restricted to a proper cylinder of the form $Q = \Omega \times (0,T)$. This applies for instance to the stationary solutions of the form $u = |w|^{1/m}\mathrm{sign}(w)$ with $\Delta w = 0$ in $\mathbb{R}^d$; when restricted to $x \in \Omega$ they are acceptable weak solutions with sign change.

The ZKB and the TW solutions show that weak non-negative solutions of the GDP need not be differentiable functions. But, since we also see that the lack of differentiability concerns only the free boundary, we may propose a compromise in the form of the concept of *classical free boundary solution*. We refer the reader to Problem 5.13 for this topic.

### 5.7.1 *Properties of radial solutions*

Weak solutions for the complete problem have properties that extend the ones derived in Section 5.6 for the homogeneous case. Instead of revising them, we will devote some space to consider the special properties of solutions in the so-called radially symmetric case in a homogeneous medium. We will use them in the study of initial continuity in Section 7.5.1.

We assume that the domain is a ball $\Omega = B_R(0)$, the data are radially symmetric, $u_0(x) = \phi(r)$, and also $f(x,t) = \psi(r,t)$ and the boundary data are constant in space, $g(x,t) = g(t)$.

**Proposition 5.15** (Property of radial symmetry) *Under those assumptions, the weak solution of Theorem* 5.14 *is also radially symmetric in the space variable, $u(x,t) = \widehat{u}(r,t)$.*

This follows from the invariance of the equation under orthogonal transformations plus the uniqueness for weak solutions that we have already proved. By abuse of language, we simply say that the solution is 'radial' and write $u = u(r,t)$, as well as $u_0(r)$, $f(r,t)$.

**Proposition 5.16** (Property of radial monotonicity) *Assume moreover that the radial profile is non-decreasing in $r$, i.e., $(u_0(r))' \ge 0$, that $\partial_r f(r,t) \ge 0$, and finally that $g'(t) \ge 0$ and $g(0) \ge u_0(R)$. Then, the solution satisfies $\partial_r u(r,t) \ge 0$.*

**Proof** The result is first proved for smooth solutions of filtration equations with smooth $\Phi$ such that $\Phi'(u) > 0$ and $\Phi''(u) \neq 0$, and smooth data $u_0$, $f$ and $g$ with $u_0'(r) \ge 0$ and $g(0) = u_0(R)$. In this case, the result is a consequence of the maximum principle applied to the equation for $v = u_r := \partial_r u(r,t)$:

$$v_t = \Delta_r(\Phi'(u)v) - \frac{d-1}{r^2}\Phi'(u)v + f_r(r,t),$$

where $\Delta_r$ is the radial version of the Laplacian. As boundary conditions we take $v = 0$ at $r = 0$ due to smoothness and symmetry. At $r = R$ we have $u = g$, which implies $u_t = g' \ge 0$ so that $\Delta\Phi(u) \ge 0$, i.e., $(r^{d-1}\Phi'(u)v)_r = 0$. In view of the values of $r = R$ and $\Phi'(0), \Phi''(0)$, we get an expression of the form

$$a(t)v_r + b(t)v^2 \ge 0,$$

with $a(t) > 0$. This implies that $v(R,t) \ge 0$ for all $t > 0$. Since $v(r,0) \ge 0$, the maximum principle implies that $v \ge 0$.

For general $\Phi$ and general radial data, the result follows by approximation. ■

Of course, the same result holds if we replace the condition of radially non-decreasing by radially non-increasing and change the signs of $g$; then we would get $\partial_r u(r,t) \le 0$.

## 5.8 Universal bound in sup norm

We investigate here a very well-known property of the PME, the boundedness for positive times of the solutions of Problem HDP. This bound holds also for the GPME with a strongly superlinear nonlinearity. It is a useful tool what will give us a convenient control on the solution used in many calculations.

**Proposition 5.17** *Every weak energy solution $u$ of Problem* HDP *for the complete PME ($m > 1$) with bounded $f$ (constructed by approximation with smooth functions) is bounded above in $Q_T^\tau = \Omega \times (\tau, T)$ for every $T > \tau > 0$. Moreover, we have a universal decay estimate of the form*

$$u(x,t) \le c(m,d) \left( R^{\frac{2}{m-1}} + (N/R^2)^{1/m} T^{\frac{1}{m-1}} \right) t^{-\frac{1}{m-1}}, \tag{5.57}$$

*where $c(m,d) > 0$, $R$ is the radius of a ball containing $\Omega$, and $N = \sup_{Q_T} f$.*

*The same result holds for the GPME under the following growth condition on $\Phi$: for all large $u \ge c_0$, $\Phi$ is $C^1$-smooth and*

$$\Phi'(u) \ge a \Phi^{(m-1)/m} \quad \textit{for some } a > 0, m > 1. \tag{5.58}$$

*Then, $R$ must be large and $c$ depends also on $c_0$.*

By *universal* we mean that the bound does not depend in any way on the size of the initial data we are considering, neither in the form of the expression nor in the constants that appear.

**Proof** We will use the fact that the solutions are constructed by approximation with smooth functions. We state and do the proof first for the PME, where the estimate is quite explicit and accurate.

(i) Let us first consider the case where $u_0$ is continuous and vanishes on $\partial\Omega$. We will construct an explicit supersolution $z(x,t)$ with which to compare the approximate solutions $u_n$ to (5.22)–(5.24).

In fact, we fix $T > 0$ and take a ball $B_R = B_R(0)$ of radius $R$ strictly containing $\Omega$, i.e., with $\Gamma = \partial\Omega \subset B_R$, and consider the function $z(x,t)$ defined in $B_R \times (0,T)$ by

$$z^m(x,t) = A(t+\tau)^{-\alpha}(R^2 - x^2) \tag{5.59}$$

for suitable constants $A, \tau$ and $\alpha > 0$ to be chosen presently. To begin with, we put $\alpha = m/(m-1)$. We want to prove that $u_n(x,t) \le z(x,t)$ in $Q_T$. This implies checking on the parabolic boundary: since function $z$ is positive in $B \times (0,\infty)$, for all large $n$ we have

$$u_n(x,t) = \frac{1}{n} < z(x,t) \quad \text{in } \Sigma,$$

if $A, \tau$ are kept fixed. Moreover, we choose $\tau$ small enough so that

$$u_{0n}(x) \le z(x,0).$$

Finally, we will obtain the inequality $z_t - \Delta(z^m) \ge f_n$ whenever

$$2dA \ge (t+\tau)^{m/(m-1)} f_n(x,t) + \frac{1}{m-1} A^{1/m} (R^2 - x^2)^{1/m} \tag{5.60}$$

for $|x| \le R$ and $0 \le t \le T$. This happens for instance if

$$A \ge c_1 R^{2/(m-1)}, \quad A \ge c_2 N T^{m/(m-1)}.$$

With these choices, and since $u_{n,t} - \Delta\Phi_n(u_n) - f_n = 0$, and $\Phi_n(z) = z^m$ due to the fact that $z(x,t) \ge 1/n$, the classical maximum principle implies that $u_n(x,t) \le z(x,t)$ in $Q_T$. Passing to the limit $n \to \infty$ and $\tau \to 0$, we get finally get

$$u(x,t) \le A^{1/m} t^{-\frac{1}{m-1}} \left(R^2 - x^2\right)^{1/m} \le A^{1/m} R^{2/m} t^{-\frac{1}{m-1}}. \tag{5.61}$$

By approximation, (5.61) holds for every weak solution obtained as a limit.

(ii) Let us now consider the GPME. We assume that $\Phi'(u) \ge c u^{m-1}$ for $u \ge n_0$, and we also assume that the approximate constitutive functions $\Phi_n$ satisfy the same condition $\Phi_n'(u) \ge c u^{m-1}$ for $u \ge C_0$. We repeat the outline of the previous proof, taking

$$w = \Phi_n(z_n(x,t)) := A(t+\tau)^{-\alpha}(R^2 - x^2) \tag{5.62}$$

with $R$ large enough so that $z_n$ will be larger than a certain constant $C_0$ on $\partial\Omega$ for $0 < t < T$. We have to pay attention to the supersolution condition for the equation that now reads $z_{n,t} - \Delta w - f_n \ge 0$ in $Q_T$, or, in another form,

$$w_t \ge \Phi_n'(z_n)(\Delta w - f_n).$$

Since $w_t, \Delta w < 0$ and $\Phi_n'(z) \ge c w^{(m-1)/m}$, this means that

$$|\Delta w| \ge f_n + \frac{1}{a} |w_t| w^{-(m-1)/m},$$

and we arrive at (5.60) but for a factor $1/c$ in the last term, that is not important. ■

## Remarks

(1) The existence of a universal upper bound is **not** true for the heat equation, $u_t = \Delta u$, simply because it is linear, so that given any solution $u(x,t) \ge 0$, we can also consider all multiples $cu(x,t)$, and this fact makes a universal bound impossible. The main requirement in order to obtain a universal upper bound is superlinearity of $\Phi$ at infinity.

(2) There are however estimates that imply boundedness for positive times when the nonlinearity has only linear growth, but then the $L^\infty$ norm of $u(t)$ must depend on the $L^1$ norm of the $u_0$ (or other convenient measure of the size of initial data). Symmetrization techniques are very useful in establishing such results. See Chapter 17.

(3) Since the bound is universal in its form, it will still be true when we extend the solutions to deal with $L^1(\Omega)$ initial data in the next chapter. Indeed, it is a universal bound.

(4) The estimate is accurate. Indeed, for $f = 0$ we will construct in the next section an actual exact solution that has the predicted decay for the PME, $O(t^{-1/(m-1)})$.

(5) A convenient condition of superlinearity that appears in the literature is

$$\frac{s\Phi'(s)}{\Phi(s)} \geq c > 1 \quad \text{for all} \quad s \geq c_0. \tag{5.63}$$

It is easy to prove that this implies $\Phi(s) \geq Cs^c$ for all large $s$, hence $\Phi'(s) \geq K\Phi^{1-1/c}$, the condition used in the proof with $c = m$.

(6) The growth assumption on $\Phi$ can also be weakened.

(7) On the other hand, the assumption on $f$ can be weakened; for instance the universal bound that we have obtained depends only on the $L^\infty$ norm of $F(x,t) = t^{m/(m-1)} f(x,t)$. However, improving $f$ is not a priority for us. We may also take $f$ in an $L^p$ space with large $p$.

We can also get a universal bound for the problem with boundary data.

**Proposition 5.18** *Let $u$ be a weak solution $u$ of* (5.52)–(5.54) *constructed by approximation with smooth functions, and assume that $f^+ \in L^\infty(Q_T)$, $g^+ \in L^\infty(\Sigma_T)$. If $\Phi$ is superlinear in the sense of Proposition* 5.17*, then, $u$ is bounded above in $Q^\tau$ for every $\tau > 0$, and we have a universal decay estimate of the form*

$$u(x,t) \leq F(t), \tag{5.64}$$

*where $F$ is a decreasing function of $t$ that depends on $\|f^+\|_\infty$, $\|g^+\|_\infty$, and the radius $R$ of a ball strictly containing $\Omega$. Moreover, for small $t > 0$ the estimate has the form*

$$u(x,t) \leq C(m,d)\, R^{\frac{2}{m-1}}\, t^{-\frac{1}{m-1}}\,. \tag{5.65}$$

**Proof** We only need to consider non-negative data and solutions. We still try a supersolution of the form (5.59),

$$z^m(x,t) = A(t+\tau)^{-m/(m-1)}(1 - bx^2).$$

We need to satisfy the conditions: $u_{0n}(x) \leq z(x,0)$, which offers no novelty; $u_n(x,t) < z(x,t)$ on $\Sigma$, which is satisfied if

$$A(1 - bR^2) \geq \|g^+\|_\infty (t+\tau)^{m/(m-1)};$$

and $z_t \geq \Delta(z^m) + f^+(x,t)$, which is implied by the two conditions

$$dbA \geq (t+\tau)^{m/(m-1)}\|f^+\|_\infty, \quad db(m-1)A^{(m-1)/m} \geq 1.$$

The result follows. ∎

## 5.9 Construction of the Friendly Giant

We want to explore now the question of how precise is the universal bound of the previous section. We investigate that issue by constructing a suitable solution that will later play a role in the theory. Considering for simplicity the case where $f = 0$, we show that there exists a special solution $\tilde{U}$ which is the largest element in the class of functions which are weak solutions of the Dirichlet problem in $Q$ in the sense of Definition 5.3 with $f = 0$. This solution is the *maximal solution* of the Cauchy–Dirichlet problem. It takes infinite initial data everywhere in $\Omega$. Following Dahlberg and Kenig, we call this solution the Friendly Giant. Moreover, when the equation is the PME, $\tilde{U}$ is a solution in separated-variables form; actually, it is the special solution discussed in Section 4.2, that is obtained here as a nice consequence of the general theory.

**Theorem 5.19** *Let us assume that $\Phi$ satisfies the growth condition* (5.58). *Then there exists a unique weak solution of the Dirichlet problem for the GPME with $f = g = 0$ that takes initial values $u_0(x,0) = +\infty$ and the divergence is uniform away from the boundary. This solution is an upper bound for all weak energy solutions of Problem* (5.1)–(5.3) *with $f = 0$. It is a decreasing function of time for all $x \in \Omega$.*

**Proof** (i) Weak solution is meant in the sense of Definition 5.3 and such that $u \in C((0,\infty) : L^1(\Omega))$ and for all $\tau > 0$ $v(x,t) = u(x,t+\tau)$ is a weak energy solution (this is assumed to simplify matters at this stage, cf. Section 6.5 below). For every integer $n \geq 1$ we solve the problem

$$(P_n) \quad \begin{cases} \partial_t u_n &= \Delta\Phi(u_n) \quad \text{in } Q, \\ u_n(x,0) &= n \quad\quad\quad\ \text{in } \Omega, \\ u_n(x,t) &= 0 \quad\quad\quad\ \text{on } \Sigma. \end{cases}$$

Let $u_n$ be the weak solution to this problem. Clearly, the sequence $\{u_n\}$ is monotone: $u_{n+1} \geq u_n$. We also know from Proposition 5.17 that for every $n$

$$u_n(x,t) \leq F(t) \quad \text{in} \quad Q, \tag{5.66}$$

where $F$ is a decreasing function of $t$ that does not depend on $n$. Therefore, we may pass to the limit and find a function

$$\widetilde{U}(x,t) = \lim_{n\to\infty} u_n(x,t),$$

also satisfying estimate (5.66). Let us examine the properties of $\widetilde{U}$:

As a monotone limit of bounded solutions $u_n$ in $Q^\tau$ such that the functions $\Phi(u_n)$ are bounded above by a function in $L^2(\tau,\infty : H_0^1(\Omega))$, it is straightforward to conclude that $\widetilde{U}$ is a weak solution of the Cauchy–Dirichlet problem for the GPME in any time interval $(\tau,\infty)$.

It is also clear that it takes on the value $\widetilde{U}(x,0) = +\infty$ everywhere in $\Omega$. The divergence is uniform thanks to a simple barrier argument: since the solutions $u_n(x,t)$ are continuous down to $t = 0$ at all interior points (see Proposition 7.13

for a proof), for every $\varepsilon > 0$ there exists $\tau > 0$ such that $u_n(x,t) \ge n - \varepsilon$ if $d(x, \partial\Omega) \ge \varepsilon$ and $0 < t < \tau$. Now, recall that $u_n \le \widetilde{U}$ to conclude.

(ii) Let us now prove that $\widetilde{U}$ is larger than any weak solution of the Cauchy–Dirichlet problem in $Q$ with $f = 0$. By Proposition 5.17 we know that every such solution satisfies

$$u(x,\tau) \le F(\tau) < \infty.$$

Taking $n \ge F(\tau)$, it follows from the maximum principle that

$$u(x,t+\tau) \le u_n(x,t) \le \widetilde{U}(x,t) \quad \text{in} \quad Q.$$

Using the fact that $u \in C([0,\infty) : L^1(\Omega))$ (see Section 6.1 for more details on this issue) and letting now $\tau \to 0$ we get $u(x,t) \le \widetilde{U}(x,t)$ in $Q$ as desired.

(iii) Next, we prove the uniqueness of the solution with $u(x,0) = +\infty$. Assume that $v$ is another such solution. Since we assume that $v(x,t+\tau)$ is a weak solution of problem (5.1)–(5.3), $v(x,\tau)$ must be an element in $H_0^1(\Omega)$, hence $v(x,2\tau)$ is bounded by Proposition 5.17. By comparison with the sequence $u_n$ we conclude that $v(x,t+2\tau) \le u_n(x,t)$ in $Q$ for some $n$ large enough. Letting $\tau \to 0$ we get

$$v(x,t) \le \widetilde{U}(x,t). \tag{5.67}$$

On the other hand, a function $v$ which has infinite initial values is larger than the solutions $u_n$, hence $v \ge \widetilde{U}$. The precise argument is as follows: the uniform divergence of $v$ at $t = 0$ and the contraction property imply that for any $n$ there is a small $\tau = \tau(n)$ such that

$$\int_\Omega (u_n(0) - v(\tau))_+ \, dx \le \varepsilon,$$

since $u_n(0) = n$. Therefore, $\int_\Omega (u_n(t) - v(t+\tau))_+ \, dx \le \varepsilon$ for every $t \ge 0$. In the limit, $\widetilde{U} \le v$. Putting both inequalities together, we get $v = \widetilde{U}$.

(iv) To prove the monotonicity in time, we fix $\tau > 0$ and observe that, by the a priori estimate, there exists $n_1 = n_1(\tau)$ such that for every $n \ge 1$

$$u_n(x,\tau) \le n_1 = u_{n_1}(x,0).$$

By the maximum principle, we conclude that $u_n(x,t+\tau) \le u_{n_1}(x,t)$ in $Q$. In the limit we have $\widetilde{U}(x,t+\tau) \le u_{n_1}(x,t)$ for every $t \ge 0$, hence

$$\widetilde{U}(x,t+\tau) \le \widetilde{U}(x,t) \quad \text{in} \quad Q. \tag{5.68}$$

This proves the monotonicity. ∎

**Theorem 5.20** *For the PME this special function has the separate-variables form*

$$\widetilde{U}(x,t) = t^{\frac{1}{m-1}}\, F(x)\,. \tag{5.69}$$

$\widetilde{U}$ *can be characterized as the maximal solution of the PME in* $Q$ *with zero Dirichlet conditions. Besides,* $g = F^m$ *is the unique positive solution of the nonlinear eigenvalue problem*

$$\Delta g + \frac{1}{m-1} g^{\frac{1}{m}} = 0\,, \quad g \in H_0^1(\Omega)\,. \tag{5.70}$$

**Proof** To show that $\widetilde{U}$ has the form (5.69), we introduce the scaling transformation

$$(\mathcal{T}u)(x,t) = \lambda u(x, \lambda^{m-1} t)\,, \quad \lambda > 0. \tag{5.71}$$

This transformation leaves the equation invariant (see Subsection 3.3.2 for more details on scaling). It is interesting to see what happens when it is applied to our latter sequence $\{u_n\}$: checking the initial and boundary values, we see that

$$(\mathcal{T}u_n)(x,t) = u_{\lambda n}(x,t) \quad \text{in} \quad Q. \tag{5.72}$$

Passing to the limit $n \to \infty$ in (5.72) we get

$$(\mathcal{T}\widetilde{U})(x,t) = \widetilde{U}(x,t), \tag{5.73}$$

which holds for every $(x,t) \in Q$ and every $\lambda > 0$. Fixing $(x,t)$ and setting $\lambda = t^{-1/(m-1)}$ we get (5.69) with $F(x) = \widetilde{U}(x,1)$.

The fact that $g = F^m$ satisfies (5.70) is also obvious. ■

**Remarks**

(1) The reader should compare this function with the similar situation for the linear case $m = 1$. Then, the solution of the equation equivalent to (5.70), i.e., $\Delta F + cF = 0$, is the sine,

$$F(x) = A\,\sin(\omega\, x), \quad \text{with} \quad \omega = \pi/|\Omega|, \tag{5.74}$$

$|\Omega|$ being the length of the interval $\Omega$, and $c = \omega^2$. Thus, we may say that for $m > 1$ the profile of the giant is a kind of *nonlinear sine function.* In the linear case we have a free parameter $A > 0$ which does *not* exist in the nonlinear case.

Moreover, $U = e^{-\lambda_1 t} F(x)$, $\lambda_1 = \omega^2$, is the asymptotic first approximation for non-negative solutions, but not an universal upper bound.

(2) The maximal solution shows that more general data are possible than those covered in this chapter. We will pursue this issue later in the book, starting with the next chapter. It is immediate to see that the present solution is also maximal with respect to the limit solutions defined there.

(3) There is no essential reason to consider maximal solutions only for forcing term $f = 0$. In fact, the proof of Theorem 5.19 goes through under the restriction $f \le C$ or even $f \le Ct^{-m/(m-1)}$. See Problem 5.11.

(4) The Friendly Giant will play a prominent role in the study of asymptotic behaviour of Section 20.1, where a new proof of existence will be given.

## 5.10 Properties of fast diffusion

We will mainly be interested in the PME equation where $\Phi(u) = |u|^{m-1}u$ with $m > 1$, and the associated properties like finite propagation and free boundaries. But the concepts and construction of solutions of this chapter apply equally well to the fast diffusion range $0 < m < 1$. There are, however, marked qualitative differences like *extinction* that we comment next.

### 5.10.1 *Extinction in finite time*

The techniques of Section 5.8 can be applied to the fast diffusion equation, $0 < m < 1$, but they lead to very different conclusions. In that case we have:

**Proposition 5.21** *Every weak energy solution $u$ of Problem* HDP *for the signed FDE with bounded initial data (and $f = 0$) vanishes identically after a finite time $T > 0$ with a bound that depends on $\|u_0\|_\infty$. Moreover, we have the upper estimate*

$$u(x,t) \le c(m,d)\, R^{-\frac{2}{1-m}}\, (T-t)^{\frac{1}{1-m}}\,, \tag{5.75}$$

*where $c(m,d) > 0$, $R$ is the radius of a ball containing $\Omega$ and $T \ge c_1(m,d) M^{1-m} R^2$ where $M = \sup_\Omega u_0$. Similar estimates apply to the negative part.*

**Proof** The construction is similar to the PME case of Proposition 5.17. We assume that the ball $B_R(0)$ of radius $R/2$ contains $\Omega$, and consider the function $z(x,t)$ defined in $B_{2R} \times (0,T)$ by

$$z^m(x,t) = A\,(T-t)^\alpha (4R^2 - x^2) \tag{5.76}$$

for suitable constants $A, T$, and $\alpha = m/(1-m)$; note the sign changes with respect to the PME case. We want to prove that there exist approximations as in the PME case such that $u_n(x,t) \le z(x,t)$ in $Q_T$. This implies checking on the parabolic boundary: since function $z$ is positive in $B_{2R} \times (0,\infty)$, for all large $n$ we have

$$u_n(x,t) = \frac{1}{n} < z(x,t) \quad \text{in } \Sigma,$$

if $A, T$ are kept fixed. Moreover, we choose $T$ large enough so that $u_{0n}(x) \le z(x,0)$. This happens if

$$M^m \le 4A\,T^\alpha R^2. \tag{5.77}$$

Next, we obtain the inequality $z_t - \Delta(z^m) \ge 0$ whenever

$$2dA \ge \frac{1}{1-m} A^{1/m}(4R^2 - x^2)^{1/m} \tag{5.78}$$

for $|x| \le 2R$ and $0 \le t \le T$. This happens if

$$A \le c_2 R^{-2/(1-m)}$$

with $c_2(m,d) > 0$. Our value for $c_2$ is $(2d(1-m))^{m/(1-m)}4^{-1/(1-m)}$. Note that when we choose the best value for $A$ according to this restriction, we get from (5.77) the condition $T \ge c_1(m,d)M^{1-m}R^2$. With these choices, and since $u_{n,t} - \Delta\Phi_n(u_n) = 0$, and $\Phi_n(z) = z^m$ due to the fact that $z(x,t) \ge 1/n$, the classical maximum principle implies that $u_n(x,t) \le z(x,t)$ in $Q_T$. Passing to the limit, we get $u(x,t) \le z(x,t)$. ■

**Corollary 5.22** *All weak energy solutions of the FDE with bounded initial data vanish in finite time.*

**Remark** Finding that the weak solution with non-trivial data becomes identically zero for a degenerate parabolic equation was in its day a big surprise. It is tied to the fact that the exponent is less than one. The simplest case of an evolution equation where the phenomenon of extinction happens is the ODE

$$\frac{du}{dt} = -u^p \quad \text{for} \quad 0 < p < 1,$$

with initial data $u(0) > 0$. Actually, there is a proof of the phenomenon of extinction based on energy inequalities that leads to an ODE like this one. Suppose that $d \ge 3$ and the solution is non-negative, and smooth so that the calculations are justified. Then, for every $q > 1$ we have by the usual methods of integration by parts applied to the FDE

$$\frac{d}{dt}\int_\Omega u^q\,dx = -q(q-1)m\int_\Omega u^{m+q-3}|\nabla u|^2\,dx \le -C(m,d,q,\Omega)\left(\int_\Omega u^p\right)^r$$

with $p = (m+q-1)d/(d-2)$ and $r = (d-2)/d$. We have used the Sobolev embedding in the last inequality. We now choose $q \ge d(1-m)/2$ so that $p \ge q$, and put $I_q = \int_\Omega u^q\,dx$ to get from the comparison of $L^p$ and $L^q$ norms

$$\frac{dI_q}{dt} \le -C\,I_q^\gamma, \qquad \gamma = 1 - \frac{1-m}{q} \in (0,1) \tag{5.79}$$

($\gamma = rp/q$). Integration of the ODE for $I_q$ leads to extinction in a finite time $T$ depending only on $m, q, \Omega$ and $I_q(0)$. This estimate is conserved when we make an approximation process. We leave it to the reader to prove the cases $d = 1, 2$ with similar conclusion that we state next.

**Proposition 5.23** *Extinction in finite time happens in the FDE for all* $0 < m < 1$ *if the initial data* $u_0$ *belong to the space* $L^q(\Omega)$ *with* $q > 1$, $q \ge d(1-m)/2$.

We will not pursue the study of extinction for fast diffusion since our aim is the study of the PME. But see Notes and Problem 5.15. The problem in the whole space is treated in full detail in the monograph [515].

### 5.10.2 *Singular fast diffusion*

The equation $u_t = \Delta u^m$ cannot be continued for $m \le 0$ because it is trivial for $m = 0$ and becomes inverse parabolic for $m < 0$. But the rescaled form

$$\partial_t u = \nabla \cdot (|u|^{m-1} \nabla u) \tag{5.80}$$

makes perfect sense as a *singular parabolic equation* (called singular because of the limit $D(u) = |u|^{m-1} \to \infty$ as $u \to 0$). It has appeared in several applications that have motivated the mathematical study in classes of non-negative solutions. The theory has some surprising features in the form of non-existence and non-uniqueness of solutions for bounded data. We refer the reader to the detailed study contained in the monograph [515]. We point out that in order to use the notation of this chapter we should consider for $m = -n < 0$ a nonlinearity of the form

$$\Phi(u) = c - \frac{1}{n} u^{-n}, \quad u > 0.$$

This falls out of our assumption $(H_\Phi)$ because of the limits $\Phi(0+) = -\infty$, $\Phi(+\infty) = c < \infty$. In the case $m = 0$ we have $\Phi(u) = \log(u)$ with the same conclusion.

## 5.11 Equations of inhomogeneous media. A short review

There are a number of extensions of the PME and its generalization the GPME that appear in the literature in the study of mass diffusion, and heat propagation of gas flow in non-homogeneous media. Here are some of the options.

(i) A natural generalization of the GPME in view of the existing theory of parabolic equations is the equation

$$\partial_t u = \sum_{i=1}^{d} \partial_{x_i} \left( a_{ij}(x,t)\, \partial_{x_j} \Phi(u) \right), \tag{5.81}$$

where $\Phi$ is as above and $(a_{ij})$ is a symmetric matrix of bounded measurable functions which is positive definite or at least non-negative. The equation is for instance suggested as a mathematical model for the flow of a gas in a non-homogeneous porous medium according to the model of Section 2.1 when the permeability or the viscosity depend on $x$ and/or $t$, see Subsection 2.1.1. We can think for instance of periodic media.

In order to re-do the theory of this chapter, the reader is advised to review the estimates of Section 3.2 and impose conditions on the derivatives of $a_{ij}$ on

$x$, $t$ and $u$. The main item, the parabolicity conditions may read

$$\Lambda^{-1}\xi^2 \le a_{ij}(x,t)\xi_i\xi_j \le \Lambda\xi^2 \qquad \text{for a.e. } (x,t)\in Q, \tag{5.82}$$

for all $\xi \in \mathbb{R}^d$, for some $\Lambda > 1$. We may also ask the coefficients $a_{ij}$ to be continuous or differentiable. As an example, Bertsch and Kamin study in [112] the one-dimensional version of this problem under the assumptions: (i) $\Phi(u) = u^m$, (ii) $a(x,t)$ is a $C^{2,2}$ function and satisfies (5.82); (iii) $u_0 \ge 0$ is bounded and continuous; and (iv) the space domain is $\mathbb{R}$.

(ii) More generally, we may consider an equation of the form

$$\partial_t u = \sum_1^d \partial_{x_i}(A_i(x,t,u,\nabla u)) \tag{5.83}$$

and derive an existence and uniqueness theory of weak solutions under convenient assumptions on the functions $A_i$. We may write $a_{ij}(x,t) = \partial_{p_j} A_i(x,t,u,\mathbf{p})\big|_{\mathbf{p}=\nabla u}$, and impose parabolicity conditions as before. The very influential paper of Alt and Luckhaus [11], 1983, treats the initial boundary value problems for quasilinear systems of the form

$$\partial_t b^j(u) - \nabla\cdot[a^j(b(u),\nabla_x u)] = f^j(b(u)), \tag{5.84}$$

$j = 1,\ldots,m$. General structure conditions (ellipticity of $a$ and subdifferentiability of $b$) allow for elliptic–parabolic equations, non-steady filtration problems and even Stefan problems. Existence, uniqueness and regularity results are established. Many subsequent papers have used and extended those results. This generality will be found below in extending the continuity results of Chapter 7, see e.g. DiBenedetto [207], and in extending the work on propagation, e.g. Antontsev [31] and Diaz-Véron [203].

The study of the so-called parabolic–elliptic boundary value problems has originated an extensive literature.

(iii) The previous models concern equations in divergence form, an important feature in developing the mathematical theory. The consideration of the gas flow model in porous media with variable porosity leads to the equation of the form (2.9): $\rho(x,t)\,\partial_t u = \nabla\cdot((c(x,t)\nabla u^m)$, or more generally

$$\rho(x,t)\,\partial_t u = \nabla\cdot(c(x,t)\nabla\Phi(u)), \tag{5.85}$$

which have non-divergent form (note that $\rho > 0$ is given). A particular instance of this mathematical model was proposed by Kamin and Rosenau [322], [452], in the study of thermal propagation in an unbounded medium. The equation has the form

$$\rho(x)\,\partial_t u = \Delta\Phi(u)\,, \tag{5.86}$$

where $u$ stands for the temperature and $\rho$ is the mass density. In the last case we fix the total mass

$$m = \int_\Omega \rho(x)\, dx,$$

which may be finite or infinite; $\Omega$ is a bounded domain or $\mathbb{R}^d$. The thermal energy is then

$$E(t) = \int_\Omega u(x,t)\rho(x)\, dx.$$

The authors pose the problem in $d = 1$, $\Omega = \mathbb{R}$, with finite total mass and finite initial energy. The assumptions on the equation structure are: $\rho(x)$ is smooth, and $\Phi$ satisfies $\Phi(0) = 0$, $\Phi'(0) \geq 0$, $\Phi'(u) > 0$ for $u > 0$; the initial data satisfy $0 \leq u_0(x) \leq M$. Existence and uniqueness of solutions for this problem can be obtained by methods that are variations of the ones of this chapter and have been developed by a number of authors for $d = 1$ and $d > 1$ both in the case of a bounded domain or in the case of the whole space (to be treated in Chapter 9). In the latter case, the behaviour of the density at infinity is a matter of concern. The typical assumption is power decay as $|x| \to \infty$:

$$\rho(x) \sim |x|^{-a}, \quad a > 0.$$

There is a great difference between the case $a \leq d$ (infinite mass) and $a > d$ (finite mass). The value $a = 2$ is critical. Let us mention that the problem in the whole space leads to interesting non-uniqueness results even for bounded initial data, cf. [225, 280, 448].

(iv) A simple inhomogeneous model that appears in the literature consists of the equation

$$\partial_t u = \Delta\Phi(x,u) + f. \tag{5.87}$$

This version already appears in the pioneering work of Oleĭnik et al. [408] (with $f = 0$). A convenient assumption on $\Phi$ is

**($H_\Phi$)** *The function $\Phi : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous in both variables and strictly increasing in the second. We also have $\Phi(x,0) = 0$ for all $x \in \Omega$.*

As indicated in Chapter 3, many of the basic estimates on which the theory relies can be easily adapted to this case, so that the whole theory of this chapter can be generalized. Additional assumptions on the dependence of $\Phi$ on $x$ will be needed to round up the existence theorems.

(v) We have mentioned one famous case in which the GPME involves a function $\Phi$ that is not strictly increasing, namely the Stefan problem, described in the Introduction, Section 1.3, to which many of the developments of this chapter apply. The combination of degenerate diffusion and the Stefan problem is treated by Bertsch et al. in [108].

At the other end, there is an interest in graphs $\Phi$ which have vertical parts, in other words, the inverse graph $c = \Phi^{-1}$ has a flat part. The corresponding equation

$$c(w)_t = \Delta w + f, \tag{5.88}$$

represents the so-called elliptic–parabolic problems, which also develop interesting free boundaries. Again, much of this chapter applies to such models. We refer for this topic to the work of J. Hulshof and coworkers [109, 294, 295].

(vi) A different question is the solution of forward–backward nonlinear heat equations of the form

$$\partial_t u = \Delta\, \Phi(u), \tag{5.89}$$

where $\Phi$ is a non-monotone function, typically with a cubic type structure: it is increasing for large and small values of $u$ but decreasing in an intermediate $u$-interval. The standard Dirichlet and Cauchy problems for this equation are ill-posed with the usual function spaces and topologies. Novick-Cohen and Pego [402] study the problem by means of a regularization of the form

$$\partial_t u = \Delta(\Phi(u) + \nu u_t), \qquad \nu > 0, \tag{5.90}$$

(Sobolev regularization), with Neumann boundary conditions $\mathbf{n}\cdot\nabla(\Phi(u) + \nu u_t) = 0$ on $\partial\Omega \times \mathbb{R}_+$ as a model for isothermal phase separation of a binary mixture. Padrön [416] finds this problem as a model of aggregating populations and uses the same regularization to find existence and uniqueness of global in time solutions of the HDP and certain regularity properties when $\Phi$ is coercive in some sense. The fine analysis of the weak limits and the hysteresis effects is done in Plotnikov [437] and Evans and Portilheiro [231].

These ill-posed problems can be regularized by a number of other methods with possibly different limits.

(vii) As a curiosity, Antontsev and Shmarev [33] have recently studied a model of porous medium equationwith variable exponent of nonlinearity:

$$u_t(x,t) - \operatorname{div}(|u|^{\gamma(x,t)}\nabla u(x,t)) = f(x), \tag{5.91}$$

for $(x,t) \in Q_T = \Omega \times (0,T]$ with initial data $u(0,x) = u_0(x)$, $x \in \Omega$, and Dirichlet boundary conditions $u(x,t) = 0$, $(x,t) \in \Gamma_T = \partial\Omega \times (0,T]$. They assume that $-1 < \gamma^- \le \gamma(x,t) \le \gamma^+ < +\infty$, for some given constants $\gamma^-, \gamma^+$. It is proved that the above-stated problem admits a unique weak solution if $\gamma(x,t) > 0$. Qualitative properties of the solution are derived in terms of the values of $\gamma$.

## Notes

**Section 5.2.** As we have explained above, solutions for the Cauchy, Dirichlet and Neumann problems were first announced by Oleĭnik [406], published in 1957, and explained in detail in [408], 1958. The case of one space dimension was considered, $f = 0$, and a class of so-called generalized solutions was introduced. Actually, a slightly more general equation was considered, $u_t = \Phi(x, u)_{xx}$ under convenient regularity assumptions on $\Phi$ and $u_0$. The uniqueness result, Theorem 5.3, follows the proof in [408]. Dubinskii [219] proves existence theorems for generalized solutions of the Dirchlet and the Cauchy problem for the PME and other much more general degenerating higher-order parabolic equations.

The semigroup approach to existence and uniqueness will be explained in Chapter 10. It has the advantage of allowing quite naturally for a greater generality for $\Phi$ which can then be a maximal monotone graph; this allows for instance to have graphs $\Phi$ with horizontal parts, like in the Stefan problem. Such problems are very important in theory and applications but they are not our concern.

A study of the properties of weak solutions to the Dirichlet problem was done by Aronson and Peletier in [49], who use a definition similar to our Definition 5.4. These works refer to non-negative solutions, but the semigroup approach applies to both signs.

The change to $L^2$ instead of $L^1$ as the basic space behind the functional setting is done for convenience in the uniqueness proof, and is then supported by the existence result, but our larger goal is to work in $L^1$, a space that has a prominent role in the complete theory. This aspect will be explored in the next chapters.

**Sections 5.4**, **5.5.** Usually, proofs of the existence of solutions with changing sign were done in the framework of semigroups, thus obtaining mild solutions. We have chosen to offer a comparative presentation for non-negative solutions and solutions of both signs, so that the reader can feel from the beginning the problems of extending the theory to the case of changing sign.

**Section 5.7.** A complete theory for the non-homogeneous problem can be developed on this foundation. We will not pursue such a line of work, since the present text must address other more urgent issues. The interested reader is offered a continuation of the investigation in Problem 5.9.

The boundary conditions are taken in the sense of traces. However, in many practical applications the weak solutions will also be continuous inside the domain and up to the boundary, so that the concept of trace is simple.

**Section 5.8.** The universal bound in sup norm is a very strong regularity result. It is used to propose a new definition of weak solution with finite energy in the next chapter.

**Section 5.9.** The existence of the special solution (5.69) is established in [49] by a different method, consisting in studying the elliptic equation (5.66). A more general result can be found in Dahlberg and Kenig [190] who introduced the term Friendly Giant in 1988.

The uniqueness of the Friendly Giant by elliptic methods is discussed in [511]. This survey paper reviews the Dirichlet problem for the PME with special attention to the asymptotic behaviour as $t \to \infty$. See Chapter 20.

**Section 5.10.** The theory of the family of fast diffusion equations with $1 > m > -\infty$ offers many theoretical surprises like instantaneous extinction, non-uniqueness, and lack of regularity. These aspects are studied in detail in the monograph [515].

The theory of solutions for the HDP for the GPME was studied by Evans in [227]. Under the assumption that $\Phi^{-1}$ is globally Lipschitz continuous, a unique solution is produced in the class of strong solutions (improved properties with respect to weak solutions, see Chapter 8).

The property of extinction in finite time was first proved by Sabinina [458, 459] for a class of one-dimensional parabolic equations of fast diffusion type in bounded intervals.

The extinction phenomenon for the GPME with general nonlinearities was studied by Diaz and Diaz who obtain in [202] the necessary and sufficient conditions on $\Phi$ for the existence of finite extinction time for solutions of the GPME in bounded domain. It reads

$$\int_0^u \frac{ds}{\Phi(s)} < \infty. \tag{5.92}$$

The study is generalized by a number of authors to the GPME with zero-order terms [315], [331], and with nonlinear boundary conditions in [369].

**Section 5.11.** Here are some additional observations:

(1) The study of reaction–diffusion equations with porous medium diffusion term has a very extensive literature that falls completely outside of the scope of our text. We refer to the book of Samarski et al. [469] which specializes in blow-up problems. For early references we can mention [46, 53]. The presence of convection terms has also been studied by a number of authors, cf. [446] and its references.

(2) The theory of equations in non-smooth domains is important in the applications but not often treated in the theory. We refer for recent work to [1, 2].

*Summary and perspective*

Let us recapitulate our progress thus far. We have posed the problem, introduced a concept of weak solution, and proved existence and uniqueness results in that framework for a suitable class of data that includes all bounded functions $u_0$ and $f$. The solutions belong to the energy class. Moreover, for $\Phi$ similar to the PME case the solutions for non-negative (or non-positive) data can be constructed as limits of classical solutions of the same equation after approximating the data, while for data of both signs the equation has to be approximated too. The solutions also satisfy the expected comparison theorem.

Though we have shown the order properties of the constructed solutions, the proof of continuous dependence will be left to the next chapter where it will be addressed by the $L^1$ technique, an important tool that deserves some attention.

We have started the qualitative analysis by showing that solutions are uniformly bounded for $t \geq \tau > 0$ for the kind of equations we want to study. A number of other properties have been established. This fits into the picture we had in mind.

The chapter covers the basic existence and uniqueness theory and some of the main properties. A large number of more advanced questions are left open and will be tackled in the following chapters. Note finally that for most of the results of this chapter, the restriction of superlinear growth on $\Phi$ is not needed and the PME with $m > 0$ is acceptable. Small changes are needed in the proofs, but we will leave to the reader such extensions into the realm of so-called fast diffusion, with the help of suitable literature.

Future chapters will introduce new definitions of generalized solution, like $L^1$ limit solutions, very weak solutions, continuous weak solutions, strong solutions and mild solutions, needed to account for more generality in the data, more general equations or different approach. And there are further options like entropy solutions, renormalized entropy solutions, viscosity solutions, kinetic solutions and dissipative solutions to be used in more general contexts, not needed at the basic level. We should not forget singular solutions which is a different direction. This variety is one of the aspects that makes the theory of nonlinear diffusion an active research field.

## Problems

**Problem 5.1** In the context of Problem PHD, check that a classical solution of Problem (5.1)–(5.3) is automatically a weak solution of the problem.

(ii) Prove that a weak solution in $Q_T$ is also a weak solution in $Q_{T_1}$ if $0 < T_1 < T$.

**Problem 5.2** The concept of initial data implicit in Definition 5.4 implies a weak form of convergence to the initial data as stated in Proposition 5.2. Prove it.

**Problem 5.3** Prove that function $w$ given by the ZKB formula (5.13) is a weak solution of the equation under the conditions stated below the formula.

*Hint*: In order to check the integral equalities (5.7) we may proceed as follows: First, we note that the function $w$ is $C^\infty$ away from the free boundary $|x| = r(t)$. We then divide $Q_T$ into two regions $Q_1 = \{(x,t) \in Q_T : \ |x| < r(t)\}$, where $u > 0$, and $Q_2 = Q_T \setminus Q_1$ where $u = 0$. The integrals are then reduced to $Q_1$. Now use the fact that $w$ is a classical solution of the equation inside $Q_1$ and also that $w^m$ is $C^1$ up to $|x| = r(t)$ to eliminate the boundary terms in the integrations by parts.

**Problem 5.4** (i) Complete the convergence parts of Lemma 5.9. In particular, show that $Z(s)$ and $Z_n(s)$ are strictly increasing continuous functions and that $Z_n(s) \to Z(s)$ uniformly on compacts; show that if $\Lambda = Z^{-1}$, $\Lambda_n = Z_n^{-1}$, they are also increasing continuous functions and that $\Lambda_n(s) \to \Lambda(s)$ uniformly on compacts; (ii) show that $u_n = \Lambda_n^{-1} z_n$ converges uniformly to $u$, the weak limit of $u_n$: show that $w_n = \Phi_n(\Lambda_n(z_n))$ also converges uniformly to $w$.

**Problem 5.5** Using (5.51), obtain a decay rate for the PME of the form

$$\iint_{Q^\tau} |(u^q)_t|^2 \, dxdt = O\left(\tau^{-\frac{2(q-1)}{m-1}}\right) \iint_{Q^\tau} |\nabla u^m|^2 dxdt. \tag{5.93}$$

*Hint*: We only need to observe that

$$(u^q)_t = (2q/(m+1))u^{q-(m+1)/2}(u^{(m+1)/2})_t$$

and recall that $u$ is bounded in $Q^\tau$ by Proposition 5.17. Combining inequalities (5.47) and (5.44) in $(\tau, T)$, with $T \to \infty$, with the $L^\infty$ estimate (5.57), we get (5.93).

**Problem 5.6** Prove that we have the following result for weak energy solutions of the PME:

$$\frac{8m}{(m+1)^2} \iint\limits_{Q_{12}} \left| \frac{d}{dt}(u^{(m+1)/2}) \right|^2 + \int\limits_{\Omega} |\nabla u^m(x,t_2)|^2 dx \le \int\limits_{\Omega} |\nabla u^m(x,t_1)|^2 dx, \tag{5.94}$$

where $0 < t_1 \le t_2$ and $Q_{12} = \Omega \times (t_1, t_2)$.

**Problem 5.7** Aronson's non-smoothness example. Take as domain a ball $\Omega = B_R(0)$, take smooth initial data $u_0(x)$, that are radially symmetric, and assume that $u_0(x) = c|x|^2$ for $0 \le x \le r_1 < R$, and is positive and integrable outside with $u_0(R) = 0$. Prove that in a finite time the solution cannot have a smooth pressure.

*Hint*: Write the equation for the pressure

$$p_t = (m-1)p\Delta p + |\nabla p|^2.$$

The solution is radially symmetric by uniqueness. As long as $p$ is smooth, it must be zero at $x = 0$. Now derive the equation for $\theta$, the Laplacian of the pressure:

$$\theta_t = (m-1)p\Delta\theta + 2m\nabla p \nabla\theta + (m-1)|\theta|^2 + 2\sum_{i,j}(\partial_{ij}^2 p)^2.$$

At $x = 0$ we get, as long as $p = 0$,

$$\theta_t = (m-1)|\theta|^2 + 2\sum_{ij}(\partial_{ij}^2 p)^2 \ge (m-1)|\theta|^2.$$

Since $\theta(0,0) = 2dc > 0$, integrating the inequality means that $\theta(0,t)$ blows up in finite time. At this time, $p$ cannot be $C^2$ in space nor $C^1$ in time.

**Problem 5.8** THE HEAT EQUATION. Adapt the theory of this chapter to the heat equation $u_t = \Delta u$. In particular;

(i) Use the methods of this section to prove existence, uniqueness and continuous dependence.
(ii) Show that the solutions are bounded for positive times but there cannot be a universal bound like (5.57).
(iii) Show that solutions are $C^\infty(Q)$ and not only continuous. A continuity higher than Hölder continuity is false for the PME due to the example of the ZKB solutions.

**Problem 5.9** THE NON-HOMOGENEOUS BOUNDARY PROBLEM.

(i) Prove the boundedness of solutions of Proposition 5.18 under the assumptions
$$t^{m/(m-1)}f,\ t^{m/(m-1)}g \quad \text{bounded.}$$
(ii) Prove an $L^1$ contraction result for fixed boundary data: If $u$ and $\widehat{u}$ are two weak solutions with data $(u_0, f, g)$ and $(\widehat{u}_0, \widehat{f}, g)$ resp., then
$$\|(u(t) - \widehat{u}(t))_+\|_1 \le \|(u_0 - \widehat{u}_0)_+\|_1 + \int_0^t \|(f(s) - \widehat{f}(s))_+\|_1 ds. \tag{5.95}$$
(iii)* Use this estimate to construct a theory of weak solutions with $L^1$ initial data and forcing term, and bounded and regular data on $\Sigma$.
(iv)* Consider the inhomogeneous boundary problem with less regular boundary.

**Problem 5.10*** Prove a universal $L^\infty$ bound as in Proposition 5.17 under weaker growth assumptions on $\Phi$.

**Problem 5.11** FRIENDLY GIANTS.

(i) Check that for every $\tau > 0$, $\widetilde{U}(x, t+\tau)$ is a weak solution of problem HDP.
(ii) Construct the special solution with initial data $U(x,0) = +\infty$ and forcing term $f = C > 0$ and show that it is the maximal solution for a certain class of weak solutions.
(iii) Do the same for the PME with $f = t^{-m/(m+1)}C$. Find the associated nonlinear elliptic problem and solve it.

**Problem 5.12** Continues the previous problem in $d = 1$.

(i) Compute numerically the *nonlinear sine function* $f_m(x)$ and discuss its shape as a function of $m$. Consider theoretically and numerically the limit situation $m \to \infty$. [*Hint*: use an appropriate variable in order not to lose the detail of the asymptotic information. See [511].]

(ii) Study the convergence as $m \to 1$ of the Friendly Giant to the linear approximant in the generality of bounded domains in several space dimensions. Note that a convenient scaling is needed.

**Problem 5.13** Classical free boundary solutions. We assume for simplicity that $\Phi(s)$ is smooth for $s > 0$. We propose the following definition:

**Definition 5.6** A function $u \geq 0$ defined in a closed cylinder $Q = \overline{\Omega} \times [0, T]$, $\Omega$ as before, is called a *classical free boundary solution* if there is a $C^1$ hypersurface $\Gamma \subset Q$ with normal not oriented along the $t$-axis, and such that

(i) $\Gamma = \partial\{u > 0\} \cap \{u = 0\}$;
(ii) $u \in C(Q)$, $u \in C^\infty(\{u > 0\})$; and
(iii) $\nabla_x \Phi(u)$ is continuous up to the free boundary $\Gamma$ and $\nabla_x \Phi(u) = 0$ on $\Gamma$.

(Other variants are possible but need not bother us now; the condition on the normal means that there is always a well-defined space normal.)

(i) Prove that the (delayed) ZKB and the TWs are classical free boundary solutions.
(ii) Prove that a classical free boundary solution is a weak solution and satisfies the energy estimates.

**Problem 5.14** Construct a separable solution of the FDE in the range $0 < m < 1$ of the form $U(x, t) = (T - t)^{1/(1-m)} F(x)$ by solving the elliptic equation for $F$. See Section A.9.1. Or solve the ODE for $F$ in case $\Omega$ is a ball and $F$ is radially symmetric.

**Problem 5.15*** Construct a Friendly Giant for the GPME in inhomogeneous media, $\Phi = \Phi(x, u)$. Derive a universal a priori bound.

**Problem 5.16*** Establish the main existence and uniqueness results of this chapter without the assumption that $\Phi$ is strictly increasing.

# 6

# THE DIRICHLET PROBLEM II. LIMIT SOLUTIONS, VERY WEAK SOLUTIONS AND SOME OTHER VARIANTS

We continue in this and the next chapter the analysis of the initial and boundary value problem. In Chapter 5 the GPME was considered, the Dirichlet problem was posed in a spatial bounded domain $\Omega$, and the problem was shown to be uniquely solvable in a class of weak solutions. It was also shown that these weak solutions are not always classical solutions. Some important questions were left open and are worth exploring, like: How general can the data be? Are there any natural and useful alternatives to the proposed definition of weak solution? Here, we address these questions and present extensions of the already developed theory. We recall that the central issue is to construct an existence theory as wide as possible and complement it with uniqueness and stability. Now, it is not automatic that the most natural class of data for existence purposes coincides with the class where uniqueness and stability can be proved. This is a standard source of complication in the theories, namely, combining well-posedness with having the widest possible (or at least wide enough) class of data.

We first discuss stability and limit solutions. A main property of the classical solutions examined in Chapter 3 is the continuous dependence with respect to the data, that is shown to take place in $L^1$ norm according to Proposition 3.5. This idea can be extended to prove continuous dependence of the weak solutions constructed in Chapter 5 with respect to the data. In this way, well-posedness is established. But once this is done, it is quite easy to perform an extension of the class of solutions to encompass merely integrable data. This is done however at the price of resorting to a new solution concept, *limit solution*. See Section 6.1. We solve in this way the homogeneous Dirichlet problem for the GPME with general $L^1$ data. Limit solutions will appear again in Chapter 10 in a slightly different guise associated to time discretizations, and that version will be called mild solutions. The equivalence of both approaches must be proved!

Limit solutions are a real extension of the concept of weak solution, but lack an intrinsic functional characterization other than the indirect statement that they are limits of weak solutions. Section 6.2 addresses this inconvenience by resorting to the concept of *very weak solution*. Uniqueness results are proved in that setting, cf. Theorem 6.5 and Corollary 6.7, which improve in a substantial way the uniqueness of weak solutions of Theorem 5.3. A key point of this section is the technique of duality used in the uniqueness proof, which is presented here in a simple setting. The section also includes the definition of trace of a solution at a given time.

In Section 6.3 we briefly explore the dependence of the solutions on the variation of the domain, a question of practical interest.

In Section 6.4 we specialize to the case $f = g = 0$ and present the main ideas of semigroups applied to the GPME in the context of limit solutions.

We then revisit the basic theory of weak solutions to address the issue of solutions with $L^1$ initial data. Such extension can be obtained at a low cost if $\Phi$ is superlinear and $f$ is assumed to be bounded. This is done in Section 6.5 and needs a modification of the old concept to accommodate the new data. The relation of both concepts of weak solution is carefully analysed.

Finally, we return to the question of possible generality of the data and present two further extensions of the theory already developed. In Section 6.6 we consider the existence of weak and limit solutions with more general initial data, taken in weighted spaces. Section 6.7 contains another extension: we now allow for data in the space $H^{-1}(\Omega)$; this is not a space of functions, but a space of distributions.

We can consider the material of this chapter as advanced reading, except for Sections 6.1 and 6.4 that are recommended at the basic level.

## 6.1 $L^1$ theory. Stability. Limit solutions

This section takes into account the $L^1$ contraction principle that we have proved in Section 3.2.3 for smooth solutions of the filtration equation, and that has appeared at some stages of the constructions of Chapter 5. We use this property to establish the stability of the constructed solutions and also to make an extension of the existence result.

### 6.1.1 *Stability of weak solutions*

It is easily seen that the $L^1$ contraction principle continues to hold in the limit for the weak solutions constructed from classical solutions by approximation. Let us explicitly state the property in our present setting.

**Proposition 6.1** *The statement of Proposition* 3.5 *holds for the weak solutions constructed in Theorem* 5.7. *In other words, for two weak energy solutions $u$ and $\widehat{u}$ with initial data $u_0$, $\widehat{u}_0$ and forcing terms $f$, $\widehat{f}$ respectively, we have for every $t > \tau \geq 0$*

$$\|(u(t) - \widehat{u}(t))_+\|_1 \leq \|(u(\tau) - \widehat{u}(\tau))_+\|_1 + \int_\tau^t \|(f(s) - \widehat{f}(s))_+\|_1 \, ds. \qquad (6.1)$$

**Remarks**

(1) This is a fundamental property that will allow us to develop existence, uniqueness and stability theory in the space $L^1(\Omega)$. For the moment

it serves the purpose of providing us with a stability result for the just constructed weak energy solutions. We recall that, as pointed out in Section 3.2.3, formula (6.1) implies the plain contraction:

$$\|u(t) - \widehat{u}(t)\|_1 \leq \|u_0 - \widehat{u}_0\|_1 + \int_0^t \|f(s) - \widehat{f}(s)\|_1 ds. \tag{6.2}$$

(2) This result implies the uniqueness of solutions of problem (5.1)–(5.3) by a new technique (the $L^1$ technique) which is completely different from that of Theorem 5.3. Indeed, estimate (6.1) not only implies $L^1$-dependence of solutions on data, but also the comparison principle, as stated at the end of Theorem 5.5: if $u_0 \leq \widehat{u}_0$ a.e. and $f \leq \widehat{f}$ a.e. in $Q$, then $(u_0 - \widehat{u}_0)_+ = 0$ a.e., then by estimate (6.1) it follows that $(u(t) - \widehat{u}(t))_+ = 0$ a.e., hence, $u(t) \leq \widehat{u}(t)$ a.e.

(3) The following is an important observation for the theory of the PME and related equations: the proof of the $L^1$ contraction principle does not depend on any particular properties of the nonlinearity $\Phi(u)$. It works in the same way whenever $\Phi$ is a monotone function. This has made the $L^1$ estimate a key item in the theory of the filtration equation $u_t = \Delta\Phi(u)$. On the contrary, similar estimates for $L^p$ norms with $p > 1$ do not exist if the filtration equation is not linear (i.e., unless we deal with the heat equation).

### 6.1.2 *Limit solutions in the $L^1$ setting*

The $L^1$ techniques are quite different in spirit from the energy estimates that form the core of the previous chapter. We pursue here the exploitation of such $L^1$ estimates to construct generalized solutions of a new type for more general data.

Indeed, the continuous dependence in $L^1$ norm stated in Proposition 6.1 allows us to introduce a concept of solution of Problem HDP for data $u_0 \in L^1(\Omega)$ and $f \in L^1(Q_T)$. This is done by approximation with a sequence of data $(u_{0n}, f_n) \in L_\psi(\Omega) \times L^\infty(Q_T)$ such that $u_{0n} \to u_0$ in $L^1(\Omega)$ and $f_n \to f$ in $L^1(Q_T)$. We may even take as data for the approximations bounded or continuous functions, since these subspaces are dense in $L^1$. The limit is well defined by virtue of estimate (6.1).

**Definition 6.1** *We call every such function a limit solution of Problem HDP for the GPME. The class is denoted as $\mathcal{LS}$.*

We obtain the following result for limit solutions.

**Theorem 6.2** *Let $\Phi$ be a monotone function as in Section* 5.2*. Then, for any $(u_0, f) \in L^1(\Omega) \times L^1(Q_T)$ there exists a unique $u \in C([0,\infty) : L^1(\Omega))$ that solves problem* HDP *in the sense of limit solutions. The weak solutions of Theorem* 5.7 *are limit solutions. The map: $(u_0, f) \mapsto u$ is an ordered contraction from $L^1(\Omega) \times L^1(Q_T)$ into $C([0,\infty) : L^1(\Omega))$ in the sense that* (6.1) *holds.*

**Proof** Note that the last statement implies continuous dependence in $L^1$ norms and means that the problem is well-posed in those spaces. We have to prove that the limit is independent of the approximating sequence, and also that it is continuous from $[0, \infty)$ into $L^1(\Omega)$.

(i) The independence of the approximating sequence is an easy consequence of the $L^1$ dependence estimate.

(ii) For the proof of continuity, assume first that $u_0$ is continuous in $\overline{\Omega}$ and $f$ bounded. Then, the method of initial barriers presented in detail in Section 7.5.1 proves that $u$ is continuous at $t = 0$. Hence, for every $\varepsilon > 0$ there is a $\tau > 0$ such that $\|u(h) - u(0)\|_1 \le \varepsilon$ if $0 < h < \tau$. By the $L^1$ stability estimate

$$\|u(t+h) - u(t)\|_1 \le \|u(h) - u(0)\|_1 \le \varepsilon$$

for every $t > 0$ and $0 < h < \tau$. It follows that $u \in C([0,T] : L^1(\Omega))$.

(iii) For any $u_0$, we approximate with functions $\widehat{u}_0$, $\widehat{f}$ as above and write, using Proposition 6.1,

$$\begin{aligned}\|u(\tau) - u_0\|_1 &\le \|u(\tau) - \widehat{u}(\tau)\|_1 + \|u_0 - \widehat{u}_0\|_1 + \|\widehat{u}(\tau) - \widehat{u}_0\|_1 \\ &\le 2\|u_0 - \widehat{u}_0\|_1 + \int_0^\tau \|f(s) - \widehat{f}(s)\|_1 \, ds + \|\widehat{u}(\tau) - \widehat{u}_0\|_1.\end{aligned}$$

Therefore, as $\widehat{u}_0 \to u_0$ and $\tau \downarrow 0$ we get $u(\tau) \to u_0$. This settles the continuity at $t = 0$. To settle it at any other time $t > 0$, we may displace the origin of time and argue as before at the times $t$ and $t + \tau$. ■

*Abstract dynamics*

We have arrived at an interesting concept, seeing solutions as continuous curves moving around in an infinite-dimensional metric space $X$ (here, the function space $L^1(\Omega)$). Viewing solutions as continuous curves in a general space is the starting point of the abstract theory of differential equations, a way that we will travel quite often. In the so-called abstract dynamics it is typical to forget the variable $x$ in the notation and look at the map $t \mapsto u(t) \in X$, where $u(t)$ is the abbreviated form for $u(\cdot, t)$.

**Remarks**

(1) Note that the theorem allows to define the value $u(t)$ of a limit solution (in particular, of a weak solution) $u$ at any time $t > 0$ as a well-defined element of $L^1(\Omega)$. Actually, in many cases, as when $\Phi$ is superlinear and $f$ is bounded, it is an element of $L^\infty(\Omega)$.

(2) If $u_0$ and $f$ are bounded the initial regularity is better. In that case the initial data are taken in the $L^p$ sense: $\tilde{u}(t) \to \tilde{u}(0)$ in $L^p(\Omega)$, for every $p < \infty$. We will see later that the solution $u(x,t)$ is Hölder continuous for all $t > 0$; if $u_0$ is continuous, then the convergence takes place uniformly in $x$ as $t \to 0$, see Section 7.5.1.

(3) Unfortunately, there are no equivalent $L^1$ estimates for the Dirichlet problem with non-homogeneous data $g \neq 0$.

We end this subsection with a simple but very useful consequence.

**Corollary 6.3** *Let $u$ be a limit solution with data $u_0 \in L^1(\Omega)$ and $f \in L^1(Q)$. If $t_1 > 0$, then $\tilde{u}(x,t) = u(x, t+t_1)$ is the limit solution with data $\tilde{u}_0(x) = u(x,t_1)$ and forcing term $\tilde{f}(x,t) = f(x,t+t_1)$.*

This important result is immediate for the approximations. We leave the details to the reader.

**Remark** Let us note that any concept of limit solution depends on the type of admissible approximations and on the functional setting in which limits are taken. The definition we propose applies in the $L^1$ setting. If needed, these solutions will be called $L^1$-limit solutions. For an extension see Section 6.6.

## 6.2 Theory of very weak solutions

The continuous dependence with respect to the $L^1$ norm is a powerful property. It has allowed us to extend the existence result for weak solutions of the preceding section and consider as data any non-negative function $u_0 \in L^1(\Omega)$ at the price of introducing the concept of limit solution, a function $u \in C([0,\infty) : L^1(\Omega))$ with $u(0) = u_0$ that is obtained as limit of weak energy solutions.

However, an important question remains: *Is the limit solution itself a weak solution according to Definition 5.4?* It turns out that in general we lose the control on $\nabla\Phi(u)$, which is important in giving a sense to identity (5.7). So, we are left with the problem of relating limit solutions to some weaker theory of solutions. Uniquely identifying the limit solutions as weak solutions in a certain sense is not an easy task. Though the text is not primarily intended to discuss the full theory of the GPME, we will explore in the sequel some aspects of the use of alternative theories of weak solutions to describe limit solutions.

We consider here the concept of very weak solution that was introduced in Definition 5.2 as a possible alternative to build a theory of generalized solutions. We recall that a very weak solution is a distribution solution with certain integrable derivatives. We apply the definition of very weak solution to the general Dirichlet problem with boundary data $\Phi(u) = g$ on $\Sigma_T$ as follows.[1] We assume that $u_0$, $f$ and $g$ are integrable functions in their respective domains.

**Definition 6.2** *An integrable function $u$ defined in $Q_T$ is said to be a very weak solution of Problem* (5.52)–(5.54) *if*

(i) $u, \Phi(u) \in L^1(Q_T)$;

[1] As before, $\Sigma_T = \partial\Omega \times [0,T)$ is the lateral boundary with measure $dSdt$; $\nu$ is the outer normal vector field.

(ii) *the identity*

$$\iint\limits_{Q_T} \{\Phi(u)\,\Delta\eta + u\eta_t + f\eta\}\,dxdt + \int\limits_{\Omega} u_0(x)\eta(x,0)dx = \int\limits_{\Sigma_T} g(x,t)\partial_\nu\eta(x,t)dSdt \tag{6.3}$$

*holds for any function* $\eta \in C^{2,1}(\overline{Q}_T)$ *which vanishes on* $\Sigma_T$ *and for* $t = T$.

As an extension of the definition, if $u$ satisfies a modified condition (ii) with inequality $\le$ (instead of equality) for every test function $\eta \ge 0$, then we call it a *very weak supersolution*; if the same happens with inequalities $\ge 0$, then $u$ is a *very weak subsolution* of the GPME.

We see that the present concept generalizes the work done so far.

**Example 6.1** Let $\Phi$ be a good nonlinearity in the sense of Section 3.2, let us assume that the data $f, g, u_0$ are smooth, and let us define a classical supersolution as a $C^{2,1}$ smooth function $u$ such that

$$\begin{cases} u_t \ge \Delta\Phi(u) + f & \text{in } Q_T, \\ u \ge g & \text{on } \Sigma_T. \end{cases} \tag{6.4}$$

Then $u$ is a supersolution in the present sense. The proof only needs a convenient integration by parts justified by the regularity we have. The same applies to classical subsolutions.

**Proposition 6.4** *The weak solution in the sense of Definitions* 5.4 *and* 5.5 *is a very weak solution in the present sense. All limit solutions of the homogeneous Dirichlet problem constructed in Subsection* 6.1.2 *are also very weak solutions.*

**Proof** The two first statements are clear by integration by parts. For the limit solutions, assume first the situation applied to a classical solution. Then, equation (6.3) holds. For limit solutions we perform a passage to the limit. The control of the $L^1(Q)$ norm of $u$ is guaranteed by the $L^1$ stability estimate. As for the control of the approximations $\Phi(u_n)$ in $L^1(Q)$, we need a further estimate that we will develop in Section 6.6. It is as follows: according to formula (6.27), for any pair of approximating solutions

$$\begin{aligned} &\int |u_n - u_m|\,\zeta(x)\,dx + \iint |\Phi(u_n) - \Phi(u_m)|\,dxdt \\ &\quad \le \int |u_{0n}(x) - u_{0m}(x)|\,\zeta(x)\,dx + \int_0^t \int |f_n(t) - f_m(t)|\,\zeta(x)\,dxdt. \end{aligned} \tag{6.5}$$

where $\zeta$ is the unique solution of the problem

$$\Delta\zeta = -1 \quad \text{in } \Omega, \qquad \zeta = 0 \quad \text{on } \partial\Omega.$$

This means that $\Phi(u_n)$ converges in $L^1(Q_T)$. By the monotonicity of $\Phi$, the limit is $\Phi(u)$ a.e. ∎

*Alternative definitions*

There are equivalent definitions of weak and very weak solution where integration in time is done in an interval $[t_1, t_2]$ with $0 < t_1 < t_2 < T$ and the values at the end-times $t_1$ and $t_2$ enter the definition. These versions appear often in the literature. We refer to Problems 6.3 and 6.4 for that interesting issue.

### 6.2.1 *Uniqueness of very weak solutions*

As commented above, the introduction of generalized solutions poses two related problems, first the problem of recognizing them as such when a candidate is given, then the problem of uniqueness of such objects. While the first problem leads naturally to the desire to relax the conditions in the definition of solution, the second is obviously easier if the definition of solution is stricter. Therefore, very weak solutions are likely to have a problem with uniqueness.

We present next a quite general uniqueness result for very weak solutions that imposes however some mild assumption on the integrability of the solutions. The main idea is solving a dual problem.

**Theorem 6.5** *Let $\Omega$ be a bounded domain with smooth boundary. Let $u_1$ be a very weak subsolution of the GPME defined in $Q_T$ for data $u_{01}, f_1, g_1$, and let $u_2$ be a very weak supersolution for data $u_{02}, f_2, g_2$. Assume moreover that both satisfy $u_i, \Phi(u_i) \in L^2(Q_T)$. If the data are ordered, $u_{01} \le u_{02}$ a.e., $f_1 \le f_2$ a.e., and $g_1 \le g_2$, then $u_1 \le u_2$ in $Q_T$.*

**Proof** (i) We write the weak inequalities satisfied by $u_1$ and $u_2$ with respect to a test function $\varphi \in C_0^{1,2}(\overline{Q_T})$. We subtract to get

$$0 \le \iint_{S_T} \{(u_1 - u_2)\varphi_t + (\Phi(u_1) - \Phi(u_2))\Delta\varphi\}\, dxdt.$$

We now write $u = u_1 - u_2$. Defining

$$a(x,t) = \frac{\Phi(u_1) - \Phi(u_2)}{u_1 - u_2}$$

where $u_1 \ne u_2$ and $a(x,t) = 0$ if $u_1 = u_2$, we may write $\Phi(u_1) - \Phi(u_2) = a(x,t)u(x,t)$ for a measurable function $a \ge 0$.

(ii) The next step is choosing a smooth test function $\theta(x,t) \ge 0$ compactly supported in $Q_T$ and solving the inverse-time problem

$$\begin{cases} \varphi_t + a_\varepsilon \Delta\varphi + \theta = 0 & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(x,T) = 0 & \text{for } x \in \Omega, \end{cases} \tag{6.6}$$

where $a_\varepsilon$ is a smooth approximation of $a$ such that $\varepsilon \le a_\varepsilon \le K$. Note that this is a correct parabolic problem if we define a new time as $t' = T - t$ (i.e., inverse time). Therefore, it has a smooth solution $\varphi \geq 0$. We then get for the difference $u = u_1 - u_2$ the estimate:

$$\iint_{Q_T} u\,\theta\,dxdt \leq \iint_{Q_T} |u||a - a_\varepsilon||\Delta\varphi|\,dxdt = J. \tag{6.7}$$

In view of the estimates that follow, we write the last term as

$$J \leq \left(\iint a_\varepsilon(\Delta\varphi)^2\,dxdt\right)^{1/2}\left(\iint \frac{|a - a_\varepsilon|^2}{a_\varepsilon}|u|^2\,dxdt\right)^{1/2}. \tag{6.8}$$

(iii) We need an a priori estimate for the term with $\Delta\varphi$. We multiply the equation satisfied by $\varphi$ by $\zeta\Delta\varphi$ where $1/2 \leq \zeta(t) \leq 1$ is a smooth and positive function for $0 \leq t \leq T$ with $\zeta_t \geq c > 0$. Integrating gives

$$\iint \varphi_t\zeta\Delta\varphi\,dxdt + \iint \zeta a_\varepsilon(\Delta\varphi)^2\,dxdt + \iint \zeta\theta\Delta\varphi\,dxdt = 0.$$

Integrating the first term by parts, using that $\varphi(x,T) = 0$, gives

$$\iint \zeta\varphi_t\Delta\varphi\,dxdt = -\iint \zeta\nabla\varphi\cdot\nabla\varphi_t\,dxdt \geq \frac{1}{2}\iint |\nabla\varphi|^2\,\zeta_t dxdt.$$

It follows that

$$\frac{1}{2}\iint |\nabla\varphi|^2\zeta_t\,dxdt + \iint \zeta a_\varepsilon(\Delta\varphi)^2\,dxdt \leq \iint \zeta(\nabla\theta\cdot\nabla\varphi)\,dxdt.$$

In view of the assumptions on $\zeta$, a very easy application of Hölder's inequality gives the desired estimate in the form

$$\iint_{Q_T} a_\varepsilon|\Delta\varphi|^2\,dxdt + \iint |\nabla\varphi|^2\,dxdt \leq C\iint_{Q_T} |\nabla\theta|^2\,dxdt.$$

This estimate allows to return to (6.7), (6.8) and get

$$\iint_{Q_T} u\,\theta\,dxdt \leq C\|\nabla\theta\|_2\left(\iint \frac{|a - a_\varepsilon|^2}{a_\varepsilon}|u|^2\,dxdt\right)^{1/2}. \tag{6.9}$$

(iv) At this stage we have to examine the way we construct the approximation so that the latter quantity goes to zero as $\varepsilon \to 0$, and the process is independent of $\theta$. We do it like this: given $\varepsilon > 0$ we select two height $K > \varepsilon > 0$ and define $a_{K,\varepsilon} = \min\{K, \max\{\varepsilon, a\}\}$ (we will be taking $K$ very large and $\varepsilon$ very small). We take smooth approximations $a_n \to a_{K,\varepsilon}$ in $L^p$ for all $p < \infty$. Then, we have

$$\begin{aligned}\iint |a - a_n|^2|u|^2\,dxdt \leq{}& 2\iint |a_{K,\varepsilon} - a_n|^2|u|^2\,dxdt\\ &+2\iint ((a-K)_+ + \varepsilon)^2|u|^2\,dxdt.\end{aligned}$$

Call the last integrals $I_1$ and $I_2$. The latter integrand is pointwise bounded by

$$2|u|^2(a^2+\varepsilon^2) = 2(\Phi(u_1)-\Phi(u_2))^2 + 2\varepsilon^2|u|^2,$$

where $\chi(a>k)$ is the characteristic function of the indicated set. Therefore, using the square integrability of $\Phi(u_i)$ and $u_i$, we may take $K$ large enough so that $I_2 \le (1/2)C\varepsilon^2$. Choosing now $n = n(\varepsilon, K)$ large enough we also get $I_1 \le C\varepsilon^2/2$. Then,

$$\iint |a-a_n|^2|u|^2\,dxdt \le C\varepsilon^2.$$

Since $a_n \ge \varepsilon$, we get in the end from (6.9) an estimate of the form

$$\iint_{Q_T} u\,\theta\,dxdt \le C\varepsilon^{1/2}\|\nabla\theta\|_2.$$

Finally, since $\varepsilon > 0$ was independent of $\theta$, we conclude that

$$\iint_{Q_T} u\,\theta\,dxdt \le 0\,.$$

By the arbitrary choice of the smooth test function $\theta \ge 0$, we get $u \le 0$ a.e. in $Q_T$. ■

The same line of proof can be used to treat the cases where the data $u_0$ and $f$ are not ordered. We get the $L^1$ dependence in another way.

**Theorem 6.6** *Let $\Omega$ be a bounded domain with smooth boundary. Let $u_1$ be a very weak subsolution of the GPME defined in $Q_T$ for data $u_{01}, f_1, g_1$, and let $u_2$ be a very weak supersolution for data $u_{02}, f_2, g_2$. Assume that both satisfy $u_i, \Phi(u_i) \in L^2(Q_T)$. Then, if $g_1 \le g_2$, we have for every $t_0 \in (0,T)$:*

$$\int_\Omega (u_1(x,t_0)-u_2(x,t_0))_+\,dx \le \int_\Omega (u_{01}(x)-u_{02}(x))_+\,dx + \int_0^{t_0}\int_\Omega (f_1-f_2)_+ dxdt. \tag{6.10}$$

**Proof** We repeat the proof, but taking now into account the differences $u_{01} - u_{02}$ and $f_1 - f_2$ that now do not disappear since they do not have a definite sign. In the end we get the inequality

$$\iint_{Q_T} (u_1-u_2)\,\theta\,dxdt \le M\int_\Omega (u_{01}(x)-u_{02}(x))_+\,\varphi(x,0)dx$$
$$+M\iint_{Q_T} (f_1-f_2)_+\varphi\,dxdt, \tag{6.11}$$

where $M$ is a uniform limit of the functions $\Phi(x,t)$ used in the preceeding proof.

Now, we choose $\theta = \psi(x)\rho_\varepsilon(t-t_0)$ with $0 \le \psi \le 1$ and $\rho_\varepsilon$ a standard smoothing kernel in one variable. By the maximum principle, we find that $\varphi$ is bounded by a function $C(t)$ with $C(T) = 0$, $C'(t) = -\rho_\varepsilon(t-t_0)$, which tends to the characteristic function $\chi_{[0,t_0]}(t)$. Hence, writing $u_{i\varepsilon}(x,t_0) = \int u_i(x,t)\rho_\varepsilon(t-t_0)\,dt$ we have

$$\int_\Omega (u_{1\varepsilon}(x,t_0) - u_{2\varepsilon}(x,t_0))\,\psi(x)\,dx \le \int_\Omega (u_{01}(x) - u_{02}(x))_+\,dx$$
$$+ \int_0^{t_0+\varepsilon} \int_\Omega (f_1 - f_2)_+ dxdt.$$

Passing to the limit $\varepsilon \to 0$ we get the inequality. ■

**Corollary 6.7** *Very weak solutions of Problem* HDP *for the GPME defined in $Q_T$ and such that $u, \Phi(u) \in L^2(Q_T)$ are uniquely determined by their data. They coincide therefore with the limit solutions. They are weak solutions if they also meet the conditions of Theorem* 5.7 *on the data.*

### 6.2.2 *Traces of very weak solutions*

The definition of very weak solution allows us to identify the value of the solution $u$ at almost every time $t \in (0,T)$ as a function $u(t) \in L^1(\Omega)$. But we would like to have a definite value at all times. This is possible with some extra work thanks to the theory of traces, that we start here. The result holds for a local very weak solution of the GPME $u_t = \Delta\Phi(u) + f$ in $Q_T = \Omega \times (0,T)$, in the sense that $u \in L^1(0,T : L^1_{\rm loc}(\Omega))$ and Definition 6.2 holds for every $\eta \in C^{2,1}(\overline{Q}_T)$ which vanishes for $t = T$ and near $\Sigma_T$; also, $f \in L^1(0,T : L^1_{\rm loc}(\Omega))$.

**Theorem 6.8** *Let $u$ be a local very weak solution of the GPME in the above sense. Then, for every $t \ge 0$ there exists a distribution $\mu(t)$ such that*

$$\lim_{s\to t} \int_{\mathbb{R}^d} u(x,s)\,\eta(x)\,dx = \langle \mu(t), \eta\rangle \tag{6.12}$$

*holds for all test functions $\eta \in C^2_0(\Omega)$. Moreover, for* a.e. *$t$ $\mu(t)$ is a measure with density $u$: $d\mu(t) = u(x,t)\,dx$. If $u \ge 0$, then $\mu(t)$ is a Radon measure.*

**Proof** (i) Take a test function $\varphi(x) \in C^\infty_c(\Omega)$ and define the function

$$L_\varphi(t) = \int_\Omega u(x,t)\varphi(x)\,dx, \tag{6.13}$$

which is a locally integrable function of $t \in (0,T)$, well-defined for a.e. $t$. We want to define $L_\varphi(t)$ for all $t$. In order to do that, we use a test function of the form $\eta(x,t) = \varphi(x)\theta(t)$ in the definition of very weak solution to get

$$-\int_0^T L_\varphi(t)\partial_t\theta\,dt = \iint_{Q_T} \{\Phi(u)\,\Delta\varphi(x) + f\varphi(x)\}\theta(t)\,dxdt. \tag{6.14}$$

Now take $0 \le t_1 < t < t_2 < T$, take a test function $\theta(t) \ge 0$ such that $\theta(t) = 0$ for $0 < t \le t_1$ and $t_2 \le t < T$, but $\theta(t) = 1$ for $t_1 + h < t < t_2 - k$. Then, pass to the limit as $h, k \to 0$ to obtain the function $\widetilde{\theta}(t) = 1$ for $t_1 \le t \le t_2$, and zero otherwise. Due to the local integrability of $f$ and $\Phi(u)$, the limit in the right-hand side exists for every $0 \le t_1 < t < t_2 < T$. Taking limits in the left-hand side we have

$$L_\varphi(t_2) - L_\varphi(t_1) = \int_{t_1}^{t_2} \int_\Omega \{\Phi(u)\,\Delta\varphi + f\varphi\}\,dxdt \tag{6.15}$$

for a.e. $t \in [0,T)$; let us call this set of times $\mathcal{T}_\varphi$, the Lebesgue points of $L_\varphi$. But the right-hand side makes sense for all $0 < t_1, t_2 < T$, and it is in fact a continuous function of $t_1, t_2$. Taking $t_1$ fixed in $\mathcal{T}_\varphi$ and $t_{2j} \to t$ with $t_{2j} \in \mathcal{T}_\varphi$, we may use the formula

$$X(t) = L_\varphi(t_1) + \int_{t_1}^{t} \int_\Omega \{\Phi(u)\,\Delta\varphi(x) + f\varphi(x)\}\,dxdt \tag{6.16}$$

as a definition of $L_\varphi(t)$ for all $0 < t < T$. It is easy to see that the limit is independent of $t_1$. Since it is finite for all $\varphi \in C_0^2(\Omega)$, we conclude that there is a linear functional on the set of functions $C_0^2(\Omega)$, a distribution $\mu(t)$, such that

$$\lim_{t_1 \to t, t \in \mathcal{T}} \int_\Omega u(x,t_1)\varphi(x)\,dx = \langle \mu(t), \varphi \rangle. \tag{6.17}$$

The limit has been taken as $t_1 \uparrow t$ but it is easy to see that the limit as $t_1 \downarrow t$ gives the same value. This formula is the definition of the trace of $u$ at time $t$. We recall that for a.e. time $t$ the trace is the value of $u(t)$; usually, we simply write $u(t)$ for the trace by abuse of notation. In that notation we can write the definition of very weak solution in the equivalent form

$$\int\limits_\Omega \{u(x,t_2)\eta(x,t_2) - u(x,t_1)\eta(x,t_1)\}\,dx = \iint\limits_{\Omega\times(t_1,t_2)} \{\Phi(u)\,\Delta\eta + u\eta_t + f\eta\}\,dxdt \tag{6.18}$$

for all $0 < t_1 < t_2 < T$ and all test functions $\eta \in C^{2,1}(Q_T)$ which are compactly supported in the space variable (uniformly in time).

(ii) Some properties of the family $\mu(t)$ are immediate. Thus, equation (6.16) implies that

$$\mu(t_2) - \mu(t_1) = \Delta \int_{t_1}^{t_2} \Phi(u)\,dt + \int_{t_1}^{t_2} f\,dt \tag{6.19}$$

in the sense of distributions, $\mathcal{D}'(\Omega)$. Usually, $\mu$ is a function, but it need not be in general. A sufficient condition is: if $u \in L^\infty_{\rm loc}(0,T : L^p_{\rm loc}(\Omega))$ with $1 < p < \infty$, then $\mu(t) \in L^p_{\rm loc}(\Omega)$ for all $t$. ■

We will make much use of traces in Chapter 13. We point out that the set of test functions that enter into formula (6.12) is $C_0^2(\Omega)$.

### 6.3 Problems in different domains

An interesting application of the preceding ideas happens when we consider the solutions of Problem HDP in two different domains $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^d$. On the one hand, we can compare the solution of the problem posed in $Q_2 = \Omega_2 \times (0, T)$ with data $u_{02}$, $f_2$, with the solution $u_1$ of the problem posed in $Q_1 = \Omega_1 \times (0, T)$ with initial data $u_{01}$, $f_1$, if we know that $u_2 \geq 0$.

**Proposition 6.9** *Let $u_1$ be the energy weak (or limit) solution of the HDP posed in $Q_1$ with data $u_{01}$, $f_1$ and let $u_2$ be the solution of the HDP posed in $Q_2$ with data $u_{02}$, $f_2$. If $u_2 \geq 0$ in $Q_2$, $u_{01}(x) \leq u_{02}(x)$ for $x \in \Omega_1$, and $f_1(x,t) \leq f_2(x,t)$ in $Q_1$, then*

$$u_1(x,t) \leq u_2(x,t) \quad \textit{for every} \quad (x,t) \in Q_1. \tag{6.20}$$

The proof relies on noting that we can easily take the approximations $u_{2,n}$ to solution $u_2$ in such a way that $u_{2,n}(x,t) \geq u_{1,n}$ on the parabolic boundary of $Q_1$, where $u_{1,n}$ are the approximations to solution $u_1$. Since the equation satisfied in $Q_1$ is the same but for the forcing term, the maximum principle implies that $u_{2,n}(x,t) \geq u_{1,n}(x,t)$ in $Q_1$. Note that a similar result holds if $u_2 \leq 0$ if we change all the inequalities.

On the other hand, we have a continuity result with respect to the domain.

**Proposition 6.10** *Let $\Omega_n$ a non-decreasing (resp. non-increasing) family of bounded domains with Lipschitz continuous boundary and let $\Omega$ be a domain with the same regularity. We assume that $\Omega = \bigcup_n \Omega_n$ (resp., $\overline{\Omega} = \bigcap_n \overline{\Omega}_n$). Let $u_n$ be the weak (or limit) solution of the HDP in $\Omega_n$ with data $(u_{0,n}, f_n)$, and let $u$ be the weak (or limit) solution of the HDP in $\Omega_n$ with data $(u_0, f)$. Under the assumption that $u_{0,n} \to u_0$ and $f_n \to f$ in $L^1$, we have $u_n \to u$ in the same norm.*

The convergence of the data is understood in the sense that we extend the data and solutions by 0 for $x \notin \Omega_n$, resp. $x \notin \Omega$, and then we assume that $u_{0,n} \to u_0$ in $L^1(\mathbb{R}^d)$ and $f_n \to f$ in $L^1(\mathbb{R}^d \times (0,T))$.

**Proof** (i) Assume first that the family $\Omega_n$ is increasing. We will need a metric fact: the boundary of $\Omega_n$ tends to $\partial\Omega$ in the sense that

$$d_n = \max\{d(x, \partial\Omega) : x \in \partial\Omega_n\}$$

tends to zero as $n \to \infty$. We leave to the reader to check that fact.

Assume to begin with that all the data are uniformly bounded, so that the solutions are too. Then, with the notation of Theorem 5.7, the estimates on $\nabla_x \Phi(u_n)$ and $\partial_t Z(u_n)$ are uniform locally in $\mathbb{R}^d \times (0,T)$, so that we can pass to the limit and obtain a bounded function $u(x,t)$ with convergence a.e. in $\Omega \times (0,T)$. The energy estimate passes to the limit and we obtain $\Phi(u_n) \to \Phi(u)$ weakly in $L^2(0,T : H^1(\mathbb{R}^d))$. We now observe that since the support of all functions $\Phi(u_n)$ is contained in $\overline{\Omega}$, the limit takes place in $L^2(0,T : H_0^1(\Omega))$.

In order to check that the equation is satisfied we try to pass to the limit in the weak formulation of the solution $u_n$ (given in formula (5.4)) for a test function $\eta$ as in the definition and with compact support in space. Since $d_n \to 0$, such a function is also an admissible test function for $u_n$ when $n$ is large enough. The weak convergences allow us to pass to the limit and show that $u$ is a solution in $\Omega \times (0,T)$ with the correct data.

(ii) We consider now the case where the family $\Omega_n$ is decreasing under the same boundedness assumptions on the data. The same argument shows that $u_n \to u$ and $\Phi(u_n) \to \Phi(u)$ weakly and a.e. in $\Omega \times (0,T)$. The weak formulation of the equation is now immediately satisfied. We have to justify that $\Phi(u(t)) \in H_0^1(\Omega)$ for a.e. $t$ and this follows from the fact that the support of $\Phi(u_n)$ is contained in $\overline{\Omega}_n \times [0,T]$ and the relation between $\Omega_n$ and $\Omega$ appearing in the statement.

(iii) Under any of the monotonicity assumptions on $\Omega_n$, if the data are general and converge in $L^1(\mathbb{R}^d)$, $L^1(\mathbb{R}^d \times (0,T))$ resp., we use the $L^1$ stability to conclude the result for limit solutions. ∎

We continue the study of the relation between the concept of solution in different nested domains in Problem 6.9.

## 6.4 Limit solutions build a semigroup

Let us now pay attention to the functional properties of the class of solutions generated by the GPME with $f = g = 0$. In that situation, and as we have pointed out, if $u(x,t)$ is a limit solution with data $u_0(x)$ and $\tau > 0$, then $v(x,t) = u(x,t+\tau)$ is the solution corresponding to data $v_0(x) = u(x,\tau)$. This allows us to show that the definition generates a very interesting functional object, a *semigroup of contractions.*

**Definition 6.3** (Semigroup) *Let $S_t$, $t \ge 0$, be a family of maps of a metric space $(E,d)$ into itself. It is called a semigroup if the following conditions hold*

(i) *$S_0$ is the identity map;*
(ii) *for every $t, s \ge 0$ we have*

$$S_{s+t} = S_t \circ S_s.$$

*In case we have*

(iii)

$$\lim_{t\to 0} S_t x = x$$

*for every $x \in E$, we say that $S_t$ is a strongly continuous semigroup, also known as a $C_0$ semigroup.*

Notice that in the usual notation $S_t$, the subscript $t$ does not indicate partial derivative. In order to avoid confusions we favour the notation $S(t)x$, but the standard notation is as it is.

There are different classes of semigroups considered in the literature. Thus, in the quite developed linear theory, the metric space is a normed space, or better a Banach space, and the maps: $u_0 \mapsto u(t) = S_t u_0$ are linear transformations in the linear space. It is called a *linear* semigroup. But the theory of nonlinear operators can dispense with that requirement, and $E$ is quite often a closed convex subset of a Banach space of functions. The denomination *nonlinear semigroup* refers to the general theory and includes in practice all semigroups, linear or not.

*Semigroups as a language. Types of semigroups*

This is not a book about semigroups. We rather think of semigroup theory as a convenient and motivating language in which our problems can be seen from a global point of view, which may also add some intuitions. Thus, a theory of existence and uniqueness is called existence of a semigroup, construction by approximation is seen as convergence of semigroups, a theory with comparison is termed an ordered semigroup, and the universal bound gives rise to a universally bounded semigroup.

Some questions are easier to understand in this new language. Thus, we will be interested in knowing whether our weak solutions are indeed bounded, or classical solutions, or at least continuous functions, or belong to a compact class. This translates in terms of classes of semigroups.

**Definition 6.4** *A semigroup acting on a metric space $E$ is called bounded if it maps bounded sets $K \subset E$ into bounded sets for every $t > 0$. If the bound of $S_t(K)$ is uniform for all $t > 0$, then we say that it is uniformly bounded.*

*It is called contractive if $S_t$ satisfies*

$$d(S_t x, S_t y) \le d(x, y).$$

*We also say that it is a semigroup of contractions. Actually, the more accurate term should be* non-expansive*, and contractive should be reserved for the case $d(S_t x, S_t y) < d(x, y)$, but the usual language in PDEs is as described.*

*A semigroup is called regularizing, or smoothing, if it maps the space into a subspace $F$ of smoother functions.*

*A semigroup is called compact if it maps bounded subsets of $E$ into compact subsets for every $t > 0$.*

The reference semigroups in diffusion theory are the ones generated by the heat equation. As is well known, all the above properties apply in the case of the HDP in a bounded domain. See Problem 6.4.

*The GPME semigroup*

Let us go back to the GPME with zero forcing term $f = 0$. We consider as linear space $X = L^1(\Omega)$, and as a special convex set

$$E = L^1(\Omega)_+ = \{g \in L^1(\Omega) : g \ge 0 \text{ a.e.}\}.$$

We define the maps $S_t : X \to X$ or $S_t : E \to E$ by

$$S_t(u_0) = u(t), \tag{6.21}$$

where $u_0 \in E$ and $u(t)$ is the limit solution of the HDP for the PME. We have proved the following result.

**Theorem 6.11** *The maps $S_t$ define a continuous semigroup of contractions in $X = L^1(\Omega)$, and $S_t$ preserves $E$. The semigroup is uniformly bounded. If $\Phi$ is superlinear, it is regularizing into $L^\infty(\Omega)$.*

Note that the semigroup property is equivalent to checking that, given a solution $u = u(t)$ with initial data $u_0$ and given a time $s > 0$, the solution with initial data $u(s)$ is

$$v(t) = u(s + t).$$

In other words, it is an existence and uniqueness theorem. At times we will refer to the contractions as $L^1$ contractions to make clear what is the norm used in the statement.

We will prove in the next chapter that bounded solutions are indeed $C^\alpha$ functions for some $\alpha \in (0, 1)$. In other words, we will prove that our semigroup regularizes from $L^1(\Omega)_+$ into $C^\alpha(\Omega)$. The same idea proves that the semigroup is compact. This property is important for many applications, for instance in the study of asymptotic behaviour.

## 6.5 Weak solutions with bounded forcing

We have extended the class of weak energy solutions into a larger class, the limit solutions, and we have mentioned that this new class enjoys the properties of well-posedness but lacks a good characterization as solutions of the equation. The characterization of limit solutions can be done by slightly modifying the concept of weak solution under some restriction on the forcing data. Recall that $f = 0$ is a current assumption anyway in the applications. Here, we admit bounded $f$. We also assume that $\Phi$ is superlinear as in Theorem 5.17.

It happens that, thanks to the universal bound, in passing to the limit $n \to \infty$ in the sequence $u_n$ considered in Section 6.1.2 and checking that $u$ is a weak solution, we encounter difficulties near $t = 0$. In general, $u$ does not satisfy the condition $\Phi(u) \in L^2(0, \infty : H_0^1(\Omega))$, which is important in giving a sense to identity (5.7), therefore we must change our definition of weak solution.

A convenient modification of the definition of weak solution to circumvent that difficulty and deal with solutions with $L^1$ data is as follows.

**Definition 6.5** *A non-negative function $u \in C([0, \infty) : L^1(\Omega))$ is said to be a weak solution of Problem* (5.1)–(5.3) *if*

(i) $\Phi(u) \in L^2_{\mathrm{loc}}(0, \infty : H_0^1(\Omega))$;

(ii) *$u$ satisfies the identity*

$$\iint_Q \{\nabla\Phi(u)\cdot\nabla\eta - u\eta_t - f\eta\}dxdt = 0 \tag{6.22}$$

*for any function $\eta \in C_0^1(\overline{Q})$ which vanishes everywhere for $0 < t < \tau$ for some $\tau > 0$.*

(iii) $u(0) = u_0$.

Note that $\Phi(u) \in L^2_{\rm loc}(0,\infty : H_0^1(\Omega))$ means that $\Phi(u) \in L^2(\tau, T : H_0^1(\Omega))$ for every $0 < \tau < T < \infty$, but not necessarily for $\tau = 0$. We immediately see that a weak solution in the sense of Definition 5.4 is also a weak solution in the present sense if we can ensure that it belongs to the class $C([0,\infty) : L^1(\Omega))$. We will come back to the relation between both definitions. Let us for the moment denote both concepts of solution, old and new, as weak-1 and weak-2.

**Theorem 6.12** *Let us assume that $\Phi$ is superlinear and $f$ is bounded. Then, there exists a unique weak-2 solution of Problem* (5.1)–(5.3) *with given initial data $u_0 \in L^1(\Omega)$. The comparison principle, the contraction principle, and the universal sup bound hold for this class of weak solutions.*

**Proof** (i) *Existence.* We construct approximations $u_n$ as indicated in Section 6.1.2 and pass to the limit using the $L^\infty$ estimate derived in Proposition 5.17, and the $L^1$ estimates of Propositions 3.5 and 6.1, plus the energy estimate (5.20), (5.39). The limit solution is a weak-2 solution, and the reader is asked to carefully verify the details.

(ii) *Uniqueness.* It relies on a rather tricky way of reducing the problem to the old uniqueness proof plus stability estimates. Let $u_1, u_2$ be weak-2 solutions of the problem with same initial data $u_0$. By the continuity assumption, given $\varepsilon > 0$, there exists $\tau > 0$ such that $\|u_1(t) - u_0\|_1$, $\|u_2(t) - u_0\|_1 < \varepsilon$ for $0 \le t \le \tau$.

Consider now the functions $\widetilde{u}_i(x,t) = u_i(x, t+\tau)$, $i = 1, 2$. Function $\widetilde{u}_i$ satisfies the assumptions of Proposition 5.1, hence it is a weak-1 solution of the same problem with initial data $u_i(x,\tau)$ (see also Problem 6.3). On the other hand, the assumption $\Phi(u) \in L^2_{\rm loc}(0,\infty : H_0^1(\Omega))$ implies that for a.e. $\tau > 0$, $\Phi(u_i(\tau)) \in L^2(\Omega)$; since $\Psi(u) \le |\Phi(u)u| \le C|\Phi(u)|^2$, for such a $\tau$ the weak-1 solution $\widetilde{u}_i(t)$ satisfies all the conclusions of Theorem 5.7, and also the $L^1$ dependence of Proposition 6.1. We thus get for $t > \tau$,

$$\begin{aligned}\|u_1(t) - u_2(t)\|_1 &= \|\widetilde{u}_1(t-\tau) - \widetilde{u}_2(t-\tau)\|_1\\ &\le \|\widetilde{u}_1(0) - \widetilde{u}_2(0)\|_1\\ &= \|u_1(\tau) - u_2(\tau)\|_1 < \varepsilon.\end{aligned}$$

We may now let $\varepsilon, \tau \to 0$ to get $u_1(t) = u_2(t)$ a.e. for every $t > 0$.

(iii) The validity of the sup bound (Proposition 5.17), the contraction principle (Proposition 3.5), and the comparison principle are just a consequence of the limit process. ■

*Eliminating the restriction of superlinearity*

The assumption of superlinearity of $\Phi$ is used to ensure the existence of a universal $L^\infty$ bound for the weak-1 solutions, which is used to prove that $u_\tau(x,t) = u(x, t+\tau)$ is a weak-1 solution for $\tau > 0$, hence $\nabla\Phi(u) \in L^2(\tau, T : L^2(\Omega))$.

It is to be noted that any $L^\infty$ bound depending on the initial $L^1$ or $L^p$ norm will do the job:

(i) We show in Section 7.7 that in one space dimension weak solutions are automatically bounded for $t \ge \tau > 0$. Hence, the assumption of superlinearity on $\Phi$ is not needed in that case.

(ii) Bounds can be obtained under much less stringent conditions on $\Phi$, like the one found by Bénilan and Berger [84] for $d \ge 3$:

$$\int_1^\infty \Phi(s)^{-d/(d-2)} ds < \infty,$$

and a similar growth condition as $s \to -\infty$. This condition is always implied by our standing assumption $|\Phi(u)| \ge c|u|$. Their bound for $|u(t)|$ depends on $\Phi$ and $\|u_0\|_1$. They put $f = 0$ and the proof is based on symmetrization techniques.

Therefore, Theorem 6.12 is true under such assumptions.

### 6.5.1 *Relating the concepts of solution*

We have been led to introduce two concepts of weak solution for the same initial and boundary value problem in Definitions 5.4 and 6.5. This is a bad situation, so we need to establish the relationship between both definitions and make a choice if possible. Fortunately for us, the relationship turns out to be clear and easy.

**Theorem 6.13** *Under the above assumptions on $\Phi$ and $f$, if $u_0 \in L_\Psi(\Omega)$, the concepts of weak-1 energy solution and weak-2 solution are equivalent. If $u_0 \in L^1(\Omega)$, the limit solution is a weak-2 solution.*

**Proof** (i) If $u$ is a weak-1 solution and $u_0 \in L_\Psi(\Omega)$, then it is also a weak-2 solution. Indeed, we have proved the continuity of the solution curve in Theorem 6.2. This part does not need any assumption on $\Phi$.

(ii) Suppose on the converse that $u$ is a weak-2 solution and $u_0 \in L_\Psi(\Omega)$. By uniqueness, it must be the weak-1 solution constructed in Theorem 5.5. ■

Both definitions have advantages, though Definition 6.5 seems to have the upper hand since it is an extension. It also has the advantage over Definition 5.4

that the comparison, boundedness and stability results proved in the last and this chapter are immediately seen to hold for all solutions with data in the larger class. We have started with the historical Definition 5.4 essentially because it has an easy uniqueness proof.

In comparison with the concept of limit solution, that has the advantage of a wider application, weak-2 solutions are easier to recognize by means of their characterization.

We can also prove that very weak solutions are weak solutions in some cases. Here is a first result in that direction. We use the notation $Q^* = \Omega \times (\tau, T)$.

**Proposition 6.14** *Let $u$ be a very weak solution of Problem* HDP *for the GPME and assume that $u \in C([0,T] : L^1(\Omega))$, $\Phi(u) \in L^2(Q^*)$ and $f \in L^p(Q^*)$ with $p$ large as before. Then, $u$ is the weak solution for positive times $t \geq \tau > 0$.*

This type of condition will be met quite often in the future.

## 6.6 More general initial data. The case $L^1_\delta$

In this section we extend the existence theory to data in the class of locally integrable functions that are allowed to diverge mildly at the boundary, since this more general setting fits nicely with the basic concept of $L^1$-stability. In order to develop such results, we have to introduce new estimates that are of interest in themselves. For simplicity we assume here that $\Omega$ has a $C^{2+\alpha}$ regular boundary.

We need some notation: we denote by $L^p_\delta(\Omega) = L^p(\Omega; \delta(x)dx)$ the class of functions $f \in L^p_{\text{loc}}(\Omega)$ such that

$$\int |f(x)|\, \delta(x) dx < \infty, \tag{6.23}$$

where $\delta(x) = d(x, \partial\Omega)$ is the distance from a point $x \in \Omega$ to the boundary $\partial\Omega$. Besides, let $\zeta$ be the unique solution of the problem

$$\Delta \zeta = -1 \quad \text{in } \Omega, \qquad \zeta = 0 \quad \text{on } \partial\Omega. \tag{6.24}$$

It is known that $\zeta \in C^\infty(\Omega)$, $\zeta > 0$ in $\Omega$, and whenever $\partial\Omega \in C^2$, then $\zeta(x)$ is $C^2$ up to the boundary and behaves like $\delta(x)$ in the sense that there exist constants $c_1, c_2 > 0$ such that

$$c_1 \delta(x) \leq \zeta(x) \leq c_2 \delta(x)$$

in a neighbourhood of the boundary.

**Theorem 6.15** *For every $u_0 \in L^1_\delta(\Omega)$ and $f \in L^1(0, T : L^1_\delta(\Omega))$ there exists a unique function $u \in C([0,\infty) : L^1_\delta(\Omega))$ which is a limit solution of the HDP for the GPME in the sense that it is obtained by approximation with weak solutions. We also have*

$$\int u(x,t)\zeta(x)\, dx + \iint \Phi(u)\, dxdt = \int u_0(x)\zeta(x)\, dx + \iint f\zeta\, dxdt, \tag{6.25}$$

*and*

$$\int |u(x,t)|\zeta(x)\,dx + \iint |\Phi(u)|\,dxdt \le \int |u_0(x)|\zeta(x)\,dx + \iint |f|\zeta\,dxdt. \quad (6.26)$$

*Moreover, the comparison principle holds for these solutions: if* $u, \widehat{u}$ *are two such solutions with initial data* $u_0, \widehat{u}_0 \in L_\Psi(\Omega)$*, and* $u_0 \le \widehat{u}_0$ a.e. *in* $\Omega$*,* $f \le \hat{f}$ *a.e. in* $Q$*, then* $u \le \widehat{u}$ a.e. *in* $Q$*. More precisely, for any two solutions we have*

$$\begin{aligned}&\int (u-\widehat{u})_+\zeta(x)\,dx + \iint (\Phi(u)-\Phi(\widehat{u}))_+\,dxdt\\ &\le \int (u_0(x)-\widehat{u}_0(x))_+\zeta(x)\,dx + \int_0^t \int (f(x,t)-\widehat{f}(x,t))_+\zeta(x)\,dxdt.\end{aligned} \quad (6.27)$$

**Proof** The proof should be easy after the developments of Chapter 5 and Section 6.1.2 once the new contraction inequality given by formula (6.27) is proved. We call such inequality the *weighted contraction principle.*

(i) Let us indicate the calculations to obtain formula (6.27) for smooth solutions. Let $p \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be such that $p(s) = 0$ for $s \le 0$, $p'(s) > 0$ for $s > 0$ and $0 \le p \le 1$, and let $j(r) = \int_0^r p(s)ds$ be a primitive of $p$. We subtract the equation for both solutions, multiply by $p(w)\zeta$ with $w = \Phi(u_1) - \Phi(u_2)$, and integrate by parts to get

$$\begin{aligned}&\int (u(x,t)-\widehat{u}(x,t))_t p(w)\zeta(x)\,dx\\ &= -\int p'(w)|\nabla w|^2\zeta\,dx - \int p(w)\nabla w\,\nabla\zeta\,dx + \int (f-\widehat{f})p(w)\zeta\,dx.\end{aligned}$$

Dropping the negative term and integrating the next one, we get

$$\int (u(x,t)-\widehat{u}(x,t))_t p(w)\zeta(x)\,dx \le \int j(w)\,\Delta\zeta\,dx + \int (f-\widehat{f})p(w)\zeta\,dx.$$

We now let $p$ tend to the function sign-plus and integrate in time. See a more detailed similar proof in Lemma 9.1.

(ii) We can now construct the limit solution by first approximating $u_0$ and $f$ with sequences of bounded functions $u_{0n}$ and $f_n$ that converge resp. to $u_0$ in $L^1_\delta(\Omega)$ and to $f$ in $L^1(0,T : L^1_\delta(\Omega))$. If $u_n$ is the sequence of solutions of the approximate problems, estimate (6.27) implies that

$$\begin{aligned}u_n &\to u \quad \text{in } L^\infty(0,T : L^1_\delta(\Omega)),\\ \Phi(u_n) &\to v \quad \text{in } L^1(0,T : L^1(\Omega)).\end{aligned}$$

By taking subsequences we may assume that the convergence takes place almost everywhere. It is then clear that $v(x,t) = \Phi(u(x,t))$ a.e.

(iii) The weighted contraction principle implies that all smooth approximations of this kind produce the same limit. It is also easy to prove that the weak energy solutions of the preceding chapter and the limit solutions of previous sections are particular cases of these solutions. Finally, we can use the data of such cases as approximations in the construction and still get the same limit. We leave these details to the reader as a training exercise.

(iv) The proof of the fact that $u \in C([0,\infty) : L^1_\delta(\Omega))$ copies the proof done in the previous sections for the $L^1$ case. We can be interested in the way the equation is satisfied since the energy inequality does not necessarily make sense as a relation between finite quantities. This topic will be further investigated in the next section. ■

Note that for a.e. $\tau > 0$ we have $\Phi(u(\tau)) \in L^1(\Omega)$; when $\Phi(s)$ has superlinear or linear growth as $(s) \to \infty$ we then have $u(\tau) \in L^1(\Omega)$ and the standard theory of $L^1$-limit solutions applies for $t \geq \tau$.

The solutions also enjoy the rest of estimates of weak solutions, like the energy estimate, once the origin of time is shifted a bit. If $\Phi$ is superlinear and $f$ is bounded, the solutions also enjoy the universal bound (5.57).

It is also clear that when $f = 0$ we have

**Theorem 6.16** *The HDP for the PME generates an ordered contraction semigroup in the space $L^1(\Omega; \zeta\, dx)$.*

**Remark** Very weak solutions can be considered with data in the weighted spaces $L^1_\delta$ as in the previous section. We leave the details as a problem.

## 6.7 More general initial data. The case $H^{-1}$

This section is devoted to a still different extension of the class of data, namely taking initial data in the space $H^{-1}(\Omega)$, dual of $H^1_0(\Omega)$. The difficulty does not lie now with the size but with the regularity: the data are not necessarily locally integrable functions. We take as $\Omega$ a bounded subset of $\mathbb{R}^d$ with $\Gamma = \partial\Omega \in C^{2+\alpha}$.

### 6.7.1 *Review of functional analysis*

The space $H = H^{-1}(\Omega)$ is defined as the dual of the Hilbert space $H^1_0(\Omega)$. It can be identified as the space of distributions that can be written in the form

$$f = f_0 + \sum_1^d \frac{\partial f_i}{\partial x_i}$$

for functions $f_0, f_1, \ldots, f_d \in L^2(\Omega)$. A key fact in the theory is the following: the map $A = -\Delta$ is an isomorphism from $H^1_0(\Omega)$ onto $H^{-1}(\Omega)$. Let us call its inverse $G$. For every $f \in H^{-1}(\Omega)$, $F = Gf$ is the weak solution of equation $\Delta F = -f$ with data $F = 0$ on $\Gamma$. We define a dot product in $H$ by means of the formula

$$\langle f_1, f_2 \rangle_H = \langle G(f_1), G(f_2) \rangle_{H^1_0} = \int \nabla F_1 \cdot \nabla F_2\, dx, \tag{6.28}$$

where we write $F_i = G(f_i)$. In this way, $H$ becomes a Hilbert space and $\|f\|_H = \|G(f)\|_{H_0^1}$. For more information, see [4, 373].

### 6.7.2 *Basic identities*

The basic calculations are better performed under the assumptions of Section 3.2: $\Phi : \mathbb{R} \mapsto \mathbb{R}$ is $C^2$ smooth, $\Phi(0) = 0$, and $\Phi'(u) > 0$ for all $s \in \mathbb{R}$; $u_0$ and $f$ are bounded and continuous functions, and $u_0(x) = 0$ for $x \in \partial\Omega$. Then $u$ is smooth and we have the following computations:

(i) We apply to all terms of the GPME the operator $G$ acting on the space functions for every fixed time $t$ to obtain the equation

$$U_t = -\Phi(u) + F, \tag{6.29}$$

where $U(\cdot, t) = G(u(\cdot, t))$, $F(\cdot, t) = G(f(\cdot, t))$.

(ii) The important new computation concerns the $H^{-1}$ norms:

$$\begin{aligned} \frac{d}{dt}\|u\|_H^2 = \frac{d}{dt}\|U\|_{H_0^1}^2 &= 2\int \nabla U \cdot \nabla U_t \, dx \\ &= -2\int \Delta U \, U_t \, dx = -2\int u\,\Phi(u)\,dx + 2\int uF\,dx. \end{aligned}$$

Since $\int uF\,dx = -\int (\Delta U)F\,dx = \int \nabla U \cdot \nabla F\,dx = \langle u, f\rangle_H$, we get

$$\frac{1}{2}\|u(t)\|_H^2 + \iint u\,\Phi(u)\,dxdt = \frac{1}{2}\|u_0\|_H^2 + \int_0^t \langle u(s), f(s)\rangle\,ds. \tag{6.30}$$

Therefore, the norm $\|u(t)\|_H$ stays bounded in any time interval with the following precise bound

$$\|u(t)\|_H \le \|u_0\|_H + \int_0^t \|f(s)\|_H\,ds.$$

(iii) This computation can be improved into a computation for the difference of two solutions $u_1, u_2$ with data $(u_{01}, f_1)$ and $(u_{02}, f_2)$ resp. We get

$$\frac{1}{2}\frac{d}{dt}\|u_1 - u_2\|_H^2 + 2\int (u_1 - u_2)(\Phi(u_1) - \Phi(u_2))\,dx = \langle u_1 - u_2, f_1 - f_2\rangle. \tag{6.31}$$

Note that the second term has a non-negative integrand precisely because of the assumption that $\Phi$ is monotone non-decreasing. This implies the estimate

$$\|u_1(t) - u_2(t)\|_H \le \|u_{01} - u_{02}\|_H + \int_0^t \|f_1(s) - f_2\|_H\,ds.$$

(iv) We now make use of the estimate on $\iint u\Phi(u)\,dxdt$. Indeed, since

$$d(s\Phi(s)) = sd\Phi(s) + \Phi(s)ds \ge \Phi(s)ds = d\Psi(s),$$

we have $\Psi(s) \le s\Phi(s)$ for every $s$. Therefore, we have

$$\iint \Psi(u)\,dxdt \le \iint u\Phi(u)\,dxdt \le C\|u_0\|_H^2 + C\left(\int_0^t \|f(s)\|_H\,ds\right)^2. \qquad (6.32)$$

This means that for a.e. $\tau > 0$ we have $\int u(\tau)\Phi(u(\tau))dx \in L^1(\Omega)$ and we enter into the energy calculations of weak energy solutions.

### 6.7.3 *General setting. Existence of $H^{-1}$ solutions*

Assume now that $\Phi$ is a monotone function as introduced in Section 5.2. All of the preceding estimates can be used together with a process of approximation and passage to the limit in order to obtain the following result.

**Theorem 6.17** *For any $u_0 \in H^{-1}(\Omega)$ and $f \in L^2(0,T : H^{-1}(\Omega))$ there exists a unique $u \in C([0,\infty) : H^{-1}(\Omega))$ obtained as limit of weak solutions of the HDP with data $(u_{0n}, f_n)$ that approximate $(u_0, f)$ in the indicated spaces. Moreover,*

(i) *for* a.e. $t > 0$, $\Phi(u(t)) \in L^1(\Omega)$, *and* (6.32) *holds;*
(ii) $u(t) \to u_0$ *as* $t \to 0$ *in the sense of* $H^{-1}(\Omega)$*;*
(iii) *the map:* $(u_0, f) \mapsto u$ *is a contraction from* $H^{-1}(\Omega) \times L^2(0,T : H^{-1}(\Omega))$ *into* $C([0,\infty) : H^{-1}(\Omega))$.

Note that the weak solutions of Theorem 5.7 are particular cases of $H^{-1}$ solutions.

*Case with no forcing. Time decay*

It is interesting to discuss the special properties of these solutions when $f = 0$. We immediately see that the norm $\|u(t)\|_H$ decreases in time. If we combine this with the already known fact that $J(t) = \int \Psi(u(t))\,dx$ is non-increasing in time, we then get the estimate

$$\int \Psi(u(t))\,dt \le \frac{C}{t}\|u_0\|_H^2. \qquad (6.33)$$

We also know from estimate (5.47) that $\int |\nabla\Phi(u(x,T))|^2\,dx$ is non-increasing in time so that

$$\int |\nabla\Phi(u(T))|^2\,dx \le (C/T)\iint |\nabla\Phi(u)|^2\,dxdt.$$

In this way, and using (5.39) a second decay estimate is obtained:

$$\int |\nabla\Phi(u(t))|^2\,dx \le \frac{C}{t}\int \Psi(u(t/2))\,dx \le \frac{C}{t^2}\|u_0\|_H^2. \qquad (6.34)$$

We have the following result

**Theorem 6.18** *When $f = 0$ the solutions of Theorem* 6.17 *are weak solutions in any interval* $t \in (\tau, T)$, *with* $\tau > 0$ *and the decay estimates* (6.33) *and* (6.34) *hold.*

*The GPME generates a semigroup of contractions in $H^{-1}(\Omega)$. This semigroup is compact.*

It is interesting to compare these decay estimates with the actual decay of the Friendly Giant that we have constructed in the previous chapter. In the case of the PME, the explicit formula (5.69) implies that

$$\int_\Omega U(t)^{m+1}\,dx = O(t^{-m/(m-1)}), \quad \int_\Omega |\nabla U^m|^2\,dx = O(t^{-2m/(m-1)}),$$

which improve the exponents of the above a priori decay estimates for large $t$, but are worse for small $t$. We ask the reader to think about this fact.

**Remark** We will prove in the next section that in the case of the PME, $u_t = \Delta(|u|^{m-1}u)$, the solutions have better regularity; they are actually strong solutions.

## Notes

**Section 6.1.** In dealing with limit solutions we must bear in mind that since the weak energy solutions are also constructed by an approximation method, they can be justly called limit solutions. The point to be stressed in the new class is that we lack at this moment a functional characterization of the set $\mathcal{LS}$ as solutions in some weak or similar sense.

Stability is proved, though with respect to a different norm, $L^1$. This reflects the different type of estimate involved. The mixture of norms is typical of nonlinear problems. Actually, the new technique produces a new solution concept, the limit solution.

**Section 6.2.** The duality proof of Theorem 6.5 is inspired in the proof by Kamin for the Stefan Problem [318] and by Kalashnikov [313] who studied the case $d = 1$ of the GPME; the idea was used for the PME and $d > 1$ by Bénilan, Crandall and Pierre [91]. See more uses in Chapters 12 and 13. The method of proof of uniqueness theorem for evolutionary differential equations based on duality is originally due to Holmgren, see [482, Chapter 5], and was adapted by Oleinik for nonlinear equations. Traces are treated in [191].

**Section 6.4.** The generation of semigroups by abstract nonlinear differential equations was a main subject of research in the late 1960s and early 1970s. A main reference is Crandall and Liggett's [180]. We will study that aspect in greater detail in Chapter 10.

**Section 6.6.** We have basically followed [88].

**Section 6.7.** The problem in $H^{-1}$ was investigated in the framework of the theory of contractive semigroups in Hilbert spaces by Brezis [127, 128]. We will study that theory in Chapter 10.

## Problems

**Problem 6.1** Show that the construction of limit solutions can be performed for boundary data $g$ as in Theorem 5.14. Show that we can prove $L^1$ continuous dependence on $u_0$ and $f$, but not on $g$.

**Problem 6.2** Prove the statements of Theorem 6.8 in detail. In particular:

(i) Show that the definition of very weak solution can be written in the equivalent form

$$\int_\Omega \{u(x,t_2)\eta(x,t_2) - u(x,t_1)\eta(x,t_1)\}\,dx = \iint_{\Omega\times(t_1,t_2)} \{\Phi(u)\,\Delta\eta + u\eta_t + f\eta\}\,dxdt \tag{6.35}$$

for all $0 < t_1 < t_2 < T$ and all test functions $\eta \in C^{2,1}(Q_T)$ which are compactly supported in the space variable (uniformly in time).

(ii) Show that we have

$$u(t_2) - u(t_1) = \Delta \int_{t_1}^{t_2} \Phi(u)\,dt + \int_{t_1}^{t_2} f\,dt \tag{6.36}$$

in the sense of distributions, $\mathcal{D}'(\Omega)$; the values of $u(t)$ are the traces.

**Problem 6.3** Prove that the following definition of weak solution is equivalent to Definition 6.5 for the PME. The difference lies in the explicit occurrence of the initial and end-values of the solution.

**Definition 6.6** *A non-negative function $u \in C([0,\infty) : L^1(\Omega))$ is said to be a weak solution of Problem* (5.1)–(5.3) *if*

(i) $\Phi(u) \in L^2_{\mathrm{loc}}(0,\infty : H^1_0(\Omega))$;

(ii) *for every $0 < t_1 < t_2$, $u$ satisfies the identity*

$$\iint_{Q_{12}} \{\nabla(\Phi(u))\cdot\nabla\eta - u\eta_t\}dxdt = \int_\Omega u(x,t_1)\eta(x,t_1)\,dx - \int_\Omega u(x,t_2)\eta(x,t_2)\,dx \tag{6.37}$$

*for any function $\eta \in C^1(Q)$ that vanishes on the lateral boundary $\Sigma = \partial\Omega \times (0,\infty)$; here, $Q_{12} = \Omega \times (t_1,t_2)$;*

(iii) $u(0) = u_0$.

**Problem 6.4** Show that for the HE a semigroup of contractions is generated in all spaces $L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, and not only in $L^1(\mathbb{R}^d)$.

**Problem 6.5** Continuous dependence on $\Phi$. Prove that when $\Phi_\varepsilon$ is smooth and approximates $\Phi$ then the solutions of the Dirichlet problem converge as $\varepsilon \to 0$.

**Problem 6.6**

(i) Develop the theory for general initial data of Section 6.6 for the GPME.
(ii) Construct very weak solutions with data in weighted spaces (i.e., $L^1_\delta$).

**Problem 6.7** Repeat the theory of weak-2 solutions of Section 6.5 for the GPME $u_t = \Delta\Phi(u) + f$ when $\Phi$ satisfies the assumptions of [84] and $f$ is bounded.

**Problem 6.8** Repeat the theory of weak-2 solutions for $H^{-1}$ initial data when $f \neq 0$ but is still regular.

**Problem 6.9** Restriction of non-negative supersolutions.

(i) Let $u$ be a non-negative very weak supersolution to the GPME posed in a domain $\Omega_1$ with zero boundary data (in the sense of Definition 6.2 and subsequent comment). Let $\Omega$ be a domain strictly contained in $\Omega_1$. Show that $u$ is still a supersolution of the GPME posed in $\Omega$ with zero boundary data.
(ii) Show that the result is not true if we replace supersolution by solution or subsolution.

*Hint*: Part (ii) is easy by construction of examples. In part (i) we take the definition of supersolution

$$\iint_{Q_T} \{\Phi(u)\,\Delta\eta + u\eta_t + f\eta\}\,dxdt + \int_\Omega u_0(x)\eta(x,0)dx \le 0 \tag{6.38}$$

for any non-negative function $\eta \in C^{2,1}(\overline{Q}_T)$ which vanishes on $\Sigma$ and for $t = T$. In order to prove this result we proceed as follows. First, we extend $\eta$ to $\Omega_1 \times (0,T)$ by putting $u(x,t) = 0$ when $x \neq \Omega$. We make convolution with a smooth kernel $\rho_\varepsilon \ge 0$ to obtain a smooth function $\eta_\varepsilon$ that is acceptable as a test function for $u$ as a supersolution in $Q_1 = \Omega_1 \times (0,T)$. Therefore, we have

$$I := \iint_{Q_1} \{\Phi(u)\,\Delta\eta_\varepsilon + u\eta_{\varepsilon,t} + f\eta_\varepsilon\}\,dxdt + \int_{\Omega_1} u_0(x)\eta_\varepsilon(x,0)dx \le 0.$$

We now observe that $\eta_\varepsilon, |\eta_{\varepsilon,t}|$ are uniformly bounded for all $\varepsilon > 0$ small, and $\Delta\eta_v$ is uniformly bounded below (though not above near the boundary of $\Omega$ where $\Delta\eta$ has a Dirac delta). We separate the integral in tree regions: the interior region $Q_i$ where $x \in \Omega$ and $d(x,\partial\Omega) \ge 1/n$, the exterior $Q_e$ where $x \in \Omega_1 \setminus \Omega$ and $d(x,\partial\Omega) \ge 1/n$ and the neighbourhood of the boundary $Q_b$ where $d(x,\partial\Omega) < 1/n$. Write integral $I$ as $I_i + I_e + I_b$. Prove that $I_e = 0$ if $\varepsilon$ is small.

Prove also that $I_b \geq -\delta$ if $\varepsilon$ and $1/n$ are small (use the integrability of $u$ and $\Phi(u)$). Conclude that $I_i \leq \delta$. Take the limit to get the desired result.

**Project*** Extend as much as possible of the theory of this chapter to equations of the form $u_t = \Delta\Phi(x, u)$.

**Open problem** Prove that any weak-1 solution in the sense of Definition 5.4 is a very weak solution in the sense of Definition 6.2.

Are weak-1 solutions always limit solutions? Same question for weak-2 solutions.

# 7

# CONTINUITY OF LOCAL SOLUTIONS

In this chapter we address a main issue of the theory, namely, the continuity of the solutions for times $t > 0$. This is a necessary complement to the existence results of the previous chapters. In view of its application to different problem settings to appear later, the solutions are only assumed to be *local solutions*, i.e., weak solutions defined in a subdomain of space-time, and initial and boundary conditions will not matter. Such solutions have arisen as solutions of the Dirichlet problem, and they will appear in the sequel as solutions of the Cauchy problem, the Neumann problem, or from other possibilities. The equation we treat in this chapter is a generalized version of the GPME.

The question of continuity is introduced in Section 7.1. The precise problem and conditions are stated in Section 7.2; the main result, Theorem 7.1, asserts the uniform equicontinuity of bounded solutions with a definite modulus of continuity that depends only on the bounds on the data and the structural conditions of the equation. The proofs are organized in Sections 7.3 and 7.4. The continuity result is a major fact of the theory, and the proof is rather long and difficult.

The application to the weak solutions constructed in Chapter 5 and the questions of initial and boundary regularity are discussed in Section 7.5.

Once continuity is proved, the natural question is to know how regular the solutions of the PME and related equations are. A first step in that direction is Hölder continuity which is proved for the PME in Section 7.6.

A much simpler proof of continuity in the case of one space dimension is presented in Section 7.7. It holds under weaker assumptions on the data and equation. Hölder continuity with explicit exponents is obtained.

The existence of classical positive solutions is briefly discussed in Section 7.8.

Continuity is a typical property of parabolic equations, linear or nonlinear, and even degenerate equations like the PME enjoy this property. But there are limits in the direction of so-called singular coefficients. Examples of those limits will be given in the short Section 7.9.

This chapter covers the continuity questions that are relevant at this point of the theory. The question of higher regularity will be taken up in earnest in Chapter 19.

## 7.1 Continuity in several space dimensions

A typical result of the quasilinear elliptic and parabolic theories says that bounded functions, or even functions in some Lebesgue space, say $L^2$, that satisfy in a weak sense an equation of such types (i.e., elliptic or parabolic) with certain structural assumptions, are in fact Hölder continuous with Hölder exponents and constants depending only on the $L^2$ norm of the solution and the bounds in the structure assumptions. This is the content of the much celebrated regularity results of De Giorgi, Nash and Moser in the late 1950s, cf. [201, 390, 396], and they have been extended in the following decades to wide classes of equations, first of linear type and then quasilinear.

The question is then posed to prove the continuity of weak solutions (or other types of solutions) to nonlinear elliptic and parabolic equations of degenerate type, under convenient assumptions on the data, coefficients and nonlinearities of the problem.

The equation under consideration in this chapter is basically the GPME that we will write according to the convention of papers that deal with the continuity issue in the form

$$\partial_t \beta(v) = \Delta v + f, \tag{7.1}$$

after the change of variables $v = \Phi(u)$, so that $u = \beta(v)$ where $\beta$ is the inverse function of $\Phi$. Assuming that $\Phi$ is strictly increasing with total range, $\mathrm{Im}(\Phi) = \mathbb{R}$, then $\beta = \Phi^{-1}$ is a monotone and continuous function defined in $\mathbb{R}$.

We will not need to deal with solutions of specific initial and boundary value problems; our requirement is that the solutions will be defined in a space-time domain; we will refer to them as *local solutions*. It is convenient to assume that the domain is a parabolic cylinder of the form $Q = B_R(x_0) \times (t_1, t_2)$, or even $Q = B_R(0) \times (0, T)$, which implies no loss of generality in view of the local form of our results and the invariance of the equations.

The type of solution on which the continuity estimates are proved in the literature can be one of the weak types we have introduced. Now, we have seen that in the GPME case the solutions of the HDP have been constructed as limits of classical solutions. Taking that fact into account, and accepting for the moment that this will be a rule in the future, a convenient approach to deriving regularity results for general weak solutions is to show that a class of classical solutions enjoy continuity estimates in the form of a modulus of continuity that depends on constants or functions that are calculated only in terms of integrals of the solution and the structure of the equation. We then use such solutions to construct classical approximating sequences $u_n$ to the weak solution $u$ under consideration; the equicontinuity of $u_n$ with a certain modulus will imply the continuity of $u$ with the same modulus if we can show that the estimates are uniform for the approximating sequence. This is a usual approach in the field of PDEs; it was followed by Caffarelli and Evans in their fundamental work on the Stefan problem [137], and used also by P. Sacks in [461] to study the general filtration equation that covers the Stefan problem and the PME.

*Notations*

- We follow the notations of preceding chapters. For a measurable set $E \subset \mathbb{R}^d$ or $E \subset \mathbb{R}^{d+1}$, $|E|$ or meas$(E)$ denotes the Lebesgue measure. We will use the notation $\{u \ge k\}$ for the set of points in the domain of $u$ where $u \ge k$.
- We need to recall the concept of *modulus of continuity*. It is a continuous and non-decreasing real function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ with $\omega(0) = 0$. A function $f$ is continuous with modulus $\omega$ in a domain $\Omega$ if

$$|f(x) - f(x')| \le \omega(|x - x'|) \tag{7.2}$$

for every $x, x' \in \Omega$. A similar definition applies for functions $u(x,t)$ defined in $Q$. Particular cases are $\omega(s) = Cs$, called Lipschitz continuity, and $\omega(s) = Cs^\alpha$ with $0 < \alpha < 1$, called Hölder continuity. When

$$\int_0^1 \frac{ds}{\omega(s)} < \infty$$

we talk about Dini continuity.
- The equation and main computations will be performed on parabolic or space-time cylinders. It will be convenient to use the so-called parabolic scale in those cylinders. Thus, we will use the parabolic cylinders with base at a point $P(x_0, t_0) \in \mathbb{R}^{d+1}$ defined as

$$Q_R(P) = \{(x,t) : \ |x - x_0| < R, \ t_0 - R^2 < t < t_0\}. \tag{7.3}$$

These are the correct half-neighbourhoods where the estimates of the parabolic theory are naturally performed. In stating the final continuity conclusions we will use the full neighbourhoods

$$Q_R^*(P) = \{(x,t) : \ |x - x_0| < R, \ |t - t_0| < R^2\}. \tag{7.4}$$

- The parabolic boundary of a cylinder $Q = \Omega \times (0,T)$ is the subset of $\partial Q$ formed by the initial section and the lateral boundary,

$$\partial_p Q = \overline{\Omega} \times \{0\} \cup \Sigma, \quad \Sigma = \partial\Omega \times [0,T].$$

- We will use the functional space

$$V_2(Q_T) = L^2(0,T : H_0^1(\Omega)) \cap L^\infty(0,T : L^2(\Omega)).$$

with norm given by

$$\|u\|^2_{V_2(Q_T)} = \sup_{0 \le t \le T} \int u(\cdot,t)^2\,dx + \iint_{Q_T} |\nabla u|^2\,dxdt.$$

We have a continuous embedding from this space into $L^{2(d+2)/d}(Q_T)$ with embedding constant depending only on $d$, cf. the textbook [357].

- We denote by $\fint_B f(x)\,dx$ the average of an integral over a ball $B = B_R(x_0)$, i.e.,

$$\fint_B f(x)\,dx = \frac{1}{|B|}\int_B f(x)\,dx, \quad |B_R(x_0)| = \omega_d R^d.$$

The same notation applies to a cylinder

$$\fint_Q f(x,t)\,dxdt = \frac{1}{|Q|}\iint_Q f(x,t)\,dxdt.$$

## 7.2 Problem, assumptions and result

Following Sacks [461] we consider a problem of the form

$$\partial_t \beta(v) = \Delta v + F(x,t,v). \tag{7.5}$$

The following structural assumptions are made on the functions $\beta$ and $F$:

- For every $s \neq 0$, $0 < \beta'(s) < \infty$ and $\beta'$ satisfies the following Property **B**:

*there exist functions* $0 < \mu_1(\delta) < \mu_2(\delta)$ *defined in* $\mathbb{R}^+$ *such that*

$$\mu_1(\delta) \le \beta'(s) \le \mu_2(\delta) \qquad \textit{for every } |s| \ge \delta > 0. \tag{7.6}$$

Without loss of generality we may assume that $\mu_1$ is monotone increasing and $\mu_2$ is monotone decreasing. We assume the normalization condition $\beta(0) = 0$, which entails no loss of generality. Moreover, we assume the regularity $\beta \in C^2(\mathbb{R})$, but this assumption will not affect the modulus of continuity.

Property **B** implies that the equation is uniformly parabolic in regions where $v$ is not near zero. However, we point out that the nonlinearity $\beta(v)$ is not supposed to have any good behaviour at $v = 0$. It may even have a jump, as in the Stefan problem case, where

$$\beta(v) = cv + H(v),$$

$H$ being the Heaviside function, $H(v) = \mathrm{sign}_0^+(v)$. In the PME case,

$$\beta(v) = |v|^{1/m}\mathrm{sign}(v)$$

which is not Lipschitz continuous at $v = 0$ if $m > 1$; at least it does not degenerate near $v = 0$, $\beta'(v)$ is bounded below away from zero for $v \approx 0$. We have in that case $\mu_2(r) = (1/m)|r|^{-(m-1)/m}$, while $\mu_1$ depends on the maximum of $|v|$, say, $M$: $\mu_1(r) = (1/m)M^{-(m-1)/m}$.

- About $F$, following our line of thought, we assume that $F \in C^1(Q_T \times \mathbb{R})$ but that regularity will not affect the modulus of continuity. Practically, we may assume that it is a measurable function of its three arguments and moreover, that it is bounded whenever $v$ varies in a bounded interval

$$|F(x,t,v)| \le C_0(K) \quad \text{if} \quad |v| \le K.$$

for some continuous function $C_0$.

Here is the main continuity result.

**Theorem 7.1** *Let $v$ be a classical solution of equation* (7.5) *defined in the cylinder $Q = B_R(0) \times (0,T)$. Then, $v$ is Hölder continuous in every subdomain $Q'$ strictly contained in $Q$ with a space-time modulus of continuity $\omega$ that depends only on*

$$d,\ \mu_1(\cdot),\ \mu_2(\cdot),\ \mathrm{dist}(Q', \partial_p Q),\ \text{and}\ \ C_1, \tag{7.7}$$

*where* $C_1 = \max\{\|v\|_{L^\infty(Q)}, \|\beta(v)\|_{L^\infty(Q)}, \|F(\cdot,\cdot,v)\|_{L^\infty(Q)\|}\}$.

The modulus of continuity is usually written in the form

$$|v(x_1,t_1) - v(x_2,t_2)| \le \omega_{\mathrm{data},Q'}(|x_1 - x_2| + |t_1 - t_2|^{1/2}),$$

for every pair of points $(x_i, t_i) \in Q'$, $i = 1, 2$. Note that the result asserts the continuity of $v = \Phi(u)$. In the PME this implies also the continuity of $u = \beta(v)$ since $\beta$ is continuous. But for cases like the Stefan Problem, the continuity of $u$ is not true.

Note also that the values taken by $\beta$ and $F$ for $v > \|v\|_{L^\infty(Q)}$ or $v < -\|v\|_{L^\infty(Q)}$ are not important for the result.

The next two sections will be devoted to the proof of this difficult result. The main difficulty lies at the value $v = 0$, where the equation is not uniformly parabolic. The difficult technical work will be concentrated in proving that near a point $P = (x_0, t_0)$ where $v = 0$, we can find a shrinking family of parabolic cylinders $Q_{R_k}(P)$ where $|v| \le M_k$, and $M_k, R_k \to 0$. Moreover, we will show that the sequences $M_k, R_k$ depend only on the data (7.7). This represents a uniform modulus of continuity at every point where $v$ vanishes.

The extension to the rest of the points is then comparatively easy, since the equation is uniformly parabolic and for such equations the equicontinuity result is known under quite general assumptions, both for divergence and non-divergence equations, cf. [353, 357] respectively.

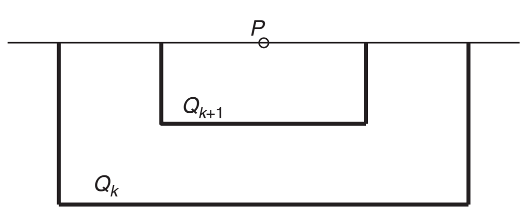


Figure 7.1: A schema of the cylinders.

## 7.3 Lemmas controlling the size of $v$

In our first technical result, we assume that $v \leq M$ in a cylinder $Q_R(P) \subset Q_T$ based at a point $P = (x_0, t_0)$, and prove that $v \geq M/2$ in the smaller cylinder $Q_{R/2}(P)$ under a smallness condition on the integral average of $M - v$. This introduces the first technical function related to the future modulus of continuity, $H_1$.

**Lemma 7.2** *Assume that the hypotheses of Theorem* 7.1 *are satisfied. Then, there exists a non-decreasing function* $H_1(r)$ *defined for* $r > 0$, *with* $0 < H_1(r) < r/2$, *and such that the conditions*

(i) $v \leq M$ *in* $Q_R(P) \subset Q_T$, *with* $0 < M \leq C_1$; *and*
(ii)

$$\fint_{Q_R(P)} (M - v)\, dxdt \leq H_1(M),$$

*imply that*

$$v \geq M/2 \quad in \quad Q_{R/2}(P).$$

**Proof** We denote by $C$ different constants depending on $d$ and $C_1$, and write $Q(R) = Q_R(P)$. We set $w = M - v \geq 0$ in $Q(R)$. It satisfies the equation

$$\beta'(M - w)w_t = \Delta w - F(x, t; M - w).$$

Local energy estimates come from a calculation performed on $w$ with respect to different heights $k \in [0, M/2]$. We proceed as follows: let $\zeta$ be a smooth cut-off function that vanishes near $\partial_p Q(R)$, multiply the equation by $(w - k)_+\zeta^2$, and integrate in $Q'(R, \tau) = B_R(x_0) \times (t_0 - R^2, \tau)$ for some $\tau \in (t_0 - R^2, t_0)$. We get

$$\begin{aligned}\iint \beta'(M - w)w_t(w - k)_+\zeta^2 dxdt = &- \iint \nabla w \cdot \nabla((w - k)_+\zeta^2)\, dxdt \\ &- \iint F(x, t, M - w)(w - k)_+\zeta^2\, dxdt.\end{aligned}$$

The first term on the right is computed as

$$\begin{aligned}&- \iint |\nabla(w - k)_+|^2\zeta^2\, dxdt - 2\iint (w - k)_+\zeta\nabla(w - k)_+ \cdot \nabla\zeta\, dxdt \\ &\quad \leq \frac{1}{2}\iint |\nabla(w - k)_+|^2\zeta^2\, dxdt + C\iint (w - k)_+^2|\nabla\zeta|^2\, dxdt.\end{aligned}$$

Thus,

$$\begin{aligned}\iint \beta'(M - w)w_t(w - k)_+\zeta^2\, dxdt + \frac{1}{2}\iint |\nabla(w - k)_+|^2\zeta^2\, dxdt \\ \leq C\iint (w - k)_+^2(\zeta^2 + |\nabla\zeta|^2)\, dxdt.\end{aligned} \tag{7.8}$$

Moreover, the second term on the left may be replaced by

$$\frac{1}{2}\iint |\nabla(w-k)_+\zeta|^2\,dxdt$$

since the difference may be absorbed into the right-hand side. We now transform the first term in (7.8). We introduce the function $B(r) = B_k(r)$ by the formula

$$B(r) = \int_0^r \beta'(M-k-s)s\,ds. \tag{7.9}$$

This is convenient since

$$\partial_t(B((w-k)_+)) = \beta'(M-w)(w-k)_+w_t,$$

which appears in the integrand of the first term of (7.8). The growth of $B$ will play a role in the estimates. In order to go further, we need to examine that issue. We need an auxiliary result:

**Lemma 7.3** *Let $M > 0$, assume that $\mu_1(s) = \mu_1(-s)$ and define for $r \ge 0$*

$$\widetilde{B}(r,M) = \frac{1}{r^2}\int_0^r \mu_1(M-s)s\,ds, \qquad \mu_3(r) = \frac{1}{16r^2}\int_0^r\int_0^s \mu_1(t)\,dtds.$$

*Then, $\mu_3$ is non-decreasing and $\widetilde{B}(r,m) \ge \mu_3(r)$ for $0 \le r \le 4M$.*

**Proof** By direct calculation,

$$d\mu_3/dr = \frac{1}{16r^2}\int_0^r \mu_1(s)\,ds - \frac{1}{8r^3}\int_0^r\int_0^s \mu_1(t)\,dtds,$$

which is non-negative because the function $\int_0^r \mu_1(s)\,ds$ is convex. We also have

$$d\widetilde{B}(r,M)/dr \le 0$$

for $0 \le r \le M$. Therefore,

$$\widetilde{B}(r,M) \ge \widetilde{B}(M,M) = \frac{1}{M^2}\int_0^M \mu_1(M-s)s\,ds \ge \frac{1}{M^2}\int_0^M\int_0^s \mu_1(t)\,dtds \ge \mu_3(M).$$

On the other hand, for $M \le r \le 4M$ we have

$$\widetilde{B}(r,M) \ge \frac{1}{r^2}\int_0^M \mu_1(M-s)s\,ds \ge \frac{1}{16M^2}\int_0^M \mu_1(M-s)s\,ds \ge \mu_3(M).$$

This completes the proof. Note that in the PME case $\mu_1(r) = a\,M^{-(m-1)/m}$, hence $\widetilde{B}(r,M) = bM^{-(m-1)/m}$ and $\mu_3(r) = cM^{-(m-1)/m}$ with $b = a/2$, $c = a/32$. ∎

Resuming the proof of the main lemma, we consider again the function $B$ and observe that

$$B((w-k)_+) \geq \int_0^{(w-k)_+} \mu_1(M-k-s)sds = (w-k)_+^2 \widetilde{B}((w-k)_+, M-k).$$

Since $(w-k)_+ \in [0, 4(M-k)]$, we have the lower estimate for $B$:

$$B((w-k)_+) \geq (w-k)_+^2 \mu_3(M-k) \geq (w-k)_+^2 \mu_3(M/2).$$

On the other hand,

$$\begin{aligned} B((w-k)_+) \leq (w-k)_+ \int_0^{(w-k)_+} \beta'(M-k-s)\, ds = (w-k)_+[\beta(M-k) \\ -\beta(M-w)], \end{aligned}$$

and this can be bounded above by $C(w-k)_+$, with $C = \max\{\beta(M), -\beta(-M)\}$. After this, we can go back to the first term of (7.8), that is estimated as follows:

$$\begin{aligned} \iint \beta'(M-w)w_t(w-k)_+\zeta^2 &= \int B((w-k)_+)\zeta^2\big|_{t=\tau}\, dx - 2\iint B((w-k)_+)\zeta\zeta_t \\ &\geq \mu_3(M/2)\int_{B_R(x_0)} ((w-k)_+\zeta)^2\big|_{t=\tau}\, dx \\ &\quad -C\iint (w-k)_+|\zeta_t|. \end{aligned}$$

Putting this estimate into (7.8), we get

$$\begin{aligned} \mu_3(M/2)\int_{B_R(x_0)} ((w-k)_+\zeta)^2\big|_{t=\tau}\, dx + \frac{1}{4}\iint |\nabla(w-k)_+\zeta|^2\, dxdt \\ \leq C\iint (w-k)_+^2(\zeta^2 + |\nabla\zeta|^2 + |\zeta_t|)\, dxdt. \end{aligned}$$

By truncating $\mu_1$ from above, we may assume that $\mu_3(M/2) \leq 1/4$. Taking the supremum of the above expression over $\tau \in [t_0 - R^2, t_0]$ we get

$$\|(w-k)_+\zeta\|^2_{V_2(Q(R))} \leq \frac{C}{\mu_3(M/2)}\iint_{D_k} (\zeta^2 + |\nabla\zeta|^2 + |\zeta_t|)\, dxdt \tag{7.10}$$

where $D_k = \{w \geq k\} \cap \operatorname{supp}(\zeta)$. This completes the basic local energy estimate.

We can still use the embedding from $V_2(Q(R)$ from this space into $L^{2(d+2)/d}(Q(R))$ to get the same estimate with left-hand side in that space

$$\|(w-k)_+\zeta\|^2_{L^{2(d+2)/d}(Q(R))} \leq \frac{C}{\mu_3(M/2)}\iint_{D_k} (\zeta^2 + |\nabla\zeta|^2 + |\zeta_t|)\, dxdt. \tag{7.11}$$

Iteration step: We perform the iteration process in a nested sequence of cylinders $Q_j = Q(R_j)$, $j \geq 0$, with a decreasing sequence of radii

$$R_j = \frac{1}{2}R(1 + 2^{-j}).$$

Put also $k_j = (M/2)(1 - 2^{-j}) \to M/2$, and let

$$J_j = \frac{1}{|Q_0|} \iint_{Q_j} (w - k_j)_+^2 \, dxdt.$$

Let $\zeta_j$ be a smooth test function with $0 \leq \zeta_j \leq 1$, $\zeta_j = 1$ in $Q_{j+1}$, $\zeta_j = 0$ near $\partial_p Q_j$, and such that

$$|\nabla \zeta_j|^2, |\zeta_{j,t}| \leq \frac{C4^j}{R^2}.$$

By Hölder's inequality we have

$$|Q_0| J_{j+1} \leq \left( \iint_{Q_{j+1}} (w - k_{j+1})_+^{2(d+2)/d} dxdt \right)^{d/(d+2)} |Q_j \cap \{w \geq k_{j+1}\}|^{2/(d+2)}. \quad (*)$$

We apply now the energy estimate (7.11) with $k = k_{j+1}$, $\zeta = \zeta_j$. Putting $\gamma = d/(d+2)$ and $D_j = \{w \geq k_{j+1}\} \cap Q_j$. we get

$$\begin{aligned} \left( \iint_{Q_{j+1}} (w - k_{j+1})_+^{2(d+2)/d} dxdt \right)^{\gamma} &\leq \left( \iint_{Q_j} ((w - k_{j+1})_+ \zeta_j)^{2(d+2)/d} dxdt \right)^{\gamma} \\ &\leq \frac{C}{\mu_3(M/2)} \iint_{D_j} (\zeta^2 + |\nabla \zeta|^2 + |\zeta_t|^2) \, dxdt \\ &\leq \frac{C4^j}{R^2 \mu_3(M/2)} |Q_j \cap \{w \geq k_{j+1}\}|. \qquad (**) \end{aligned}$$

Now, since

$$(k_{j+1} - k_j)^2 |Q_j \cap \{w \geq k_{j+1}\}| \leq \iint_{Q_j} (w - k_j)_+^2 \, dxdt,$$

we have

$$|Q_j \cap \{w \geq k_{j+1}\}| \leq \frac{C4^j R^{d-2}}{M^2} J_j.$$

Combining this with (*) and (**) we arrive at

$$J_{j+1} \leq \frac{C4^j}{R^2 \mu_3(M/2)} \left( \frac{C4^j R^{d-2}}{M^2} J_j \right)^{1+2/(d+2)},$$

hence

$$J_{j+1} \leq \frac{C_2}{M^{2(d+4)/(d+2)} \mu_3(M/2)} (4^j)^{2(d+3)/(d+2)} J_j^{1+2/(d+2)}. \qquad (7.12)$$

This kind of superlinear iterative relation is studied in the classical book [357], where it is proved that $J_j \to 0$ as $j \to \infty$ under a condition on the size of the initial data of the form

$$J_0 \le C_* \, M^{d+4} (\mu_3(M/2))^{(d+2)/2}, \tag{7.13}$$

for some $C_* = C_*(d, C_2)$. We now take

$$H_1(M) = \min\left\{ \frac{C_*}{2C_1} (M^{d+4} (\mu_3(M/2))^{(d+2)/2}, \frac{M}{2} \right\}.$$

If

$$\frac{1}{|Q_R|} \iint_{Q_R} (M - u)\, dxdt \le H_1(M),$$

then $J_0$ fulfils the necessary condition, so that

$$0 = \lim_{j\to\infty} J_j = \frac{1}{|Q_R|} \iint_{Q_{R/2}} \left( w - \frac{M}{2} \right)_+^2 dxdt.$$

We conclude that $w \le M/2$ in $Q(R/2)$, which completes the proof. ■

This result has the following corollary, that controls the fraction of the measure of a cylinder where $v$ is substantially smaller than $M$. The definition of function $H(r)$ is important for the final result.

**Corollary 7.4** *Under the hypotheses of Theorem* 7.1 *there exists an increasing function $H(r)$ with $0 < H(r) < r/2$ for $r > 0$, such that the conditions*

(i) *$v \le M$ in $Q_R(P) \subset Q_T$, with $0 < M \le C_1$; and*
(ii) *there exists a point $P_1 \in \overline{Q_{R/2}(P)}$ such that $v(P_1) < M/2$,*

*imply that*

$$|Q_R(P) \cap \{v \le M - H(M)\}| \ge H(M)\, |Q_R(P)|. \tag{7.14}$$

**Proof** We only need to put $H(r) = H_1(r)/(2C_1 + 1))$, where $H_1$ is the function defined in Lemma 7.2. If the conclusion is false then $\int_{Q(R)} (M - v)\, dxdt$ can be computed as

$$\int_{Q(R)\cap\{v\le M-H(M)\}} (M - v)\, dxdt + \int_{Q(R)\cap\{v>M-H(M)\}} (M - v)\, dxdt.$$

The last integral is bounded above by $H(M)|Q(R)|$ and the first by $2C_1|Q_R(P) \cap \{v \le M - H(M)\}|$. Using estimate (7.14) with reversed inequality we get

$$\int_{Q(R)} (M - v)\, dxdt \le H_1(M)|Q(R)|.$$

The conclusion of Lemma 7.2 contradicts the existence of the point $P_1$. ■

We proceed now with the last preliminary estimate. We fix a subdomain $Q'$ compactly embedded in $Q_T$ and let $D = \mathrm{dist}(Q', \partial_p Q_T)$. We assume that $v$ is a solution of equation (7.5) as in Theorem 7.1, that $P \in Q'$ and $R$ is such that $Q(R) = Q_R(P) \subset Q_T$. Besides, we modify $v$: given $\varepsilon, n > 0$ it is easy to find a function $g = g_{\varepsilon,n} \in C_c^\infty(\mathbb{R})$ such that $g \geq 0$, $0 \leq g' \leq 1$, $g'' \geq 0$, and

$$(s - \varepsilon - (1/n))_+ \leq g(s) \leq (s - \varepsilon)_+$$

We then define the function

$$z(x,t) = g_{\varepsilon,n}(v(x,t)). \tag{7.15}$$

We point out that the choice of the constant $\varepsilon$ will play an important role in the iterative construction of Proposition 7.10. On the contrary, $n$ is only chosen for smoothness reasons and can be any large number.

We derive the following lemma that shows how an upper bound for $z$ in a cylinder can be improved (i.e., lowered) when we shrink the cylinder, if a certain technical condition on level sets is fulfilled.

**Lemma 7.5** *Let $M, \varepsilon, \theta$ and $n$ be positive constants, and let $z$ be defined by* (7.15)*. Under the hypotheses of Theorem* 7.1*, there exist constants $R_*$ and $\sigma_*$ such that $0 < R_* \leq D/2$, $0 < \sigma_* \leq M/2$, and the following holds: if $0 < R \leq R_*$ and*

(i) $z \leq M$ *in* $Q(R)$*;*
(ii) $|Q(R) \cap \{z = 0\}| \geq \theta\, |Q(R)|$,

*then*

$$z \leq M - \sigma_* \qquad \text{in } Q(\kappa_d \theta R).$$

*Moreover, $R_*$ and $\sigma_*$ depend only on $M, \varepsilon, \theta$ and the data* (7.7)*.*

Before attacking the proof of this result, we need some preparatory work. We first see that $z$ satisfies a parabolic inequality.

**Lemma 7.6** *Under the above hypotheses, $z$ satisfies*

$$z_t - a(x,t)\Delta z \leq b(x,t).$$

*with functions $a(x,t), b(x,t)$ such that*

$$\delta \leq a(x,t) \leq \delta^{-1}, \quad |b(x,t)| \leq \delta^{-1},$$

*for some $\delta > 0$ depending only on $\varepsilon, \mu_2(\varepsilon), \mu_3(\varepsilon)$ and $C_1$.*

**Proof** By direct computation,

$$\beta'(v) z_t - \Delta z \leq |F|$$

pointwise in $Q_T$. Take now $\overline{v}(x,t) = \max(v(x,t), \varepsilon)$ and set

$$a(x,t) = \frac{1}{\beta'(\overline{v}(x,t))}, \quad b(x,t) = \frac{|F(x,t,v(x,t))|}{\beta'(\overline{v}(x,t))},$$

to have the conclusion of the lemma, recalling that $z_t = \nabla z = \Delta z = 0$ on the set $\{v \leq \varepsilon\}$. ■

We now recall the work by Krylov and Safonov [353] on Harnack inequalities for linear parabolic equations in non-divergence form, which proves the following positivity result.

**Proposition 7.7** *Assume $w$ is a smooth function satisfying*

(i) $w \geq 0$ *in* $Q(R)$*;*
(ii) $w_t - a_{ij}\, w_{x_i x_j} \geq 0$ *in* $Q(R)$ *under the conditions*

$$\delta\, |\xi|^2 \leq a_{ij}\xi_i\xi_j, \quad \xi \in \mathbb{R}^d, \quad \|a_{ij}\|_\infty \leq \delta^{-1}, \quad \delta > 0;$$

(iii) $|Q(R) \cap \{w \geq 1\}| \geq \theta\, |Q(R)|$.

*Then, there is a constant $a_0 = a_0(d, \delta, \theta) > 0$ such that*

$$w(x, t_0) \geq a_0 \quad \textit{for} \ \ |x - x_0| \leq R/2.$$

This result can be easily extended as follows.

**Corollary 7.8** *Under the assumptions of Proposition* 7.7 *there exists a constant $a_0(d, \delta, \theta) > 0$ such that*

$$w(x,t) \geq a_0 \quad \textit{for} \ \ (x,t) \in Q(\kappa\theta R) = Q_{\kappa\theta R}(P), \tag{7.16}$$

*where $\kappa(d, \theta) > 0$ is given by*

$$\kappa(d)^2 = \min\left\{1/5, \inf_{\theta\in[0,1]} (1 - (1-\theta/2)^{2/(d+2)})/\theta^2\right\}. \tag{7.17}$$

**Proof** For $t \in [t_0 - (\alpha R)^2, t_0]$ with $\alpha = \kappa\,\theta$, we have $Q_{\sqrt{1-\alpha^2}R}(x_0, t) \subset Q_R(P)$. Also,

$$|Q_R(P) - Q_{\sqrt{1-\alpha^2}R}(x_0,t)| = (1 - (1-\alpha^2)^{(d+2)/2})\, |Q_R(P)|,$$

and this is less than $(\theta/2)|Q_R(P)|$ by the definition of $\kappa$. Therefore,

$$|Q_{\sqrt{1-\alpha^2}R}(x_0,t) \cap \{w \geq 1\}| \geq \frac{\theta}{2}\, |Q_R(P)| \geq \frac{\theta}{2}\, |Q_{\sqrt{1-\alpha^2}R}(x_0,t)|.$$

We may now apply Proposition 7.7 to $w$ in each of the cylinders $Q_{\sqrt{1-\alpha^2}R}(x_0,t)$ to conclude that there exists a constant $a_0 > 0$ such that $w(x,t) \geq a_0$ if

$$|x - x_0| \leq \frac{1}{2}\sqrt{1-\alpha^2}R, \qquad t_0 - (\alpha R)^2 \leq t \leq t_0.$$

But since $\alpha^2 \leq 1/5$, we have $\alpha \leq (\sqrt{1-\alpha^2})/2$ and the positivity conclusion follows in $Q(\alpha R) = Q(\kappa\theta R)$. ■

Applying this corollary to the function

$$w(x,t) = \frac{2}{M}\left(M + \left(\frac{t - t_0 + R^2}{\delta}\right) - z(x,t)\right),$$

we finally get

**Corollary 7.9** *Let $z$ be a smooth function satisfying*

(i) $z \leq M$ *in* $Q(R)$*;*
(ii) $z_t - a_{ij} z_{x_i x_j} \leq b$ *in* $Q(R)$ *under the conditions*

$$\delta\,|\xi|^2 \leq a_{ij}\xi_i\xi_j, \quad \xi \in \mathbb{R}^d, \quad \|a_{ij}\|_\infty, \|b\|_\infty \leq \delta^{-1}, \quad \delta > 0;$$

(iii) $|Q(R) \cap \{z \leq (M/2)\}| \geq \theta\,|Q(R)|$,

*Then, there is a constant $a_0 = a_0(d,\delta,\theta) > 0$ such that*

$$z(x,t) \leq M - \frac{Ma_0}{2} - \frac{R^2}{\delta} \quad in \ \ Q(\kappa\theta R).$$

We may now proceed with the *proof of Lemma* 7.5. By Lemma 7.6, $z$ defined by (7.15) satisfies conditions (i), (ii) and (iii) of Corollary 7.9 with $\delta$ depending only on $\varepsilon$ and the data (7.7). Corollary 7.9 applies to give a positivity constant $a_0$ depending on this $\delta$ and the given $\theta$. We now set

$$R_* = \min\left(\frac{d}{2}, \left(\frac{Ma_0\delta}{4}\right)^{1/2}\right), \qquad \sigma_* = \min\left(\frac{M}{2}, \frac{Ma_0}{2}\right).$$

Then, Corollary 7.9 implies that

$$z \leq M - \frac{Ma_0}{2} - \frac{R^2}{\delta} \leq M - \sigma_*$$

in $Q(\kappa\theta R)$ provided that $R \leq R_*$. ■

## 7.4 Proof of the continuity theorem

We proceed in three steps: behaviour near a vanishing point, behaviour near a non-vanishing point, and final step.

### 7.4.1 *Behaviour near a vanishing point*

We have the following result that sums up the main difficulty of the continuity argument. It relies on Corollary 7.4 and Lemma 7.5.

**Proposition 7.10** *Under the hypotheses of Theorem* 7.1*, let $v(P) = 0$ at a point $P = (x_0, t_0) \in Q'$. Then there exist sequences $M_k, R_k \downarrow 0$ depending only on the* *data list* (7.7), *such that*

$$|v(x,t)| \le M_k \quad \textit{in} \;\; Q_{R_k}(P).$$

**Proof** We fix the values $\varepsilon = M - H(M)$, $\theta = H(M)$ and change $M$ into $H(M)$ in the functions $\sigma_*(M,\varepsilon,\theta)$ and $R_*(M,\varepsilon,\theta)$ introduced in Lemma 7.5, and define the functions

$$\sigma(M) = \sigma_*(H(M), M - H(M), H(M)),$$

$$R(M) = R_*(H(M), M - H(M), H(M)).$$

Function $H$ is defined in Corollary 7.4. For $0 < M \le C_1$ we have

$$0 < \sigma(M) \le H(M)/2, \quad 0 < R_*(M) \le D/2.$$

We now define iteratively the sequences $M_k$ and $R_k$:

$$M_1 = C_1, \quad M_{k+1} = M_k - \sigma(M_k)$$
$$R_1 = R(M_1), \quad R_{k+1} = \min\left(R(M_{k+1}), \kappa\, H(M_k) R_k\right).$$

These definitions depend only on the stated data. Both sequences tend to zero as $k \to \infty$. Clearly, the stated conclusion holds for $k = 1$.

Suppose now that $k \ge 1$ and $|v| \le M_k$ in $Q(R_k)$. We consider the function $z = g_{\varepsilon,n}(v(x,t))$ for the choice $\varepsilon = \varepsilon(M_k) = M_k - H(M_k)$, $n > 0$. Then,

$$z \le (v - \varepsilon(M_k)))_+ \le H(M_k) \quad \text{in} \;\; Q(R_k).$$

Since $v(P) = 0$, by Corollary 7.4, and using the exact value of $\varepsilon(M_k) = M_k - H(M_k)$, we have

$$|Q(R_k) \cap \{z = 0\}| \ge |Q(R_k) \cap \{v \le M_k - H(M_k)\}| \ge H(M_k)\,|Q(R_k)|.$$

This motivates the choice of $\theta$ in the definition of $\sigma(M)$ and $R(M)$. Since $R_k \le R(M_k)$, using the definition of $R(M)$ and Lemma 7.5 we obtain the improvement

$$z \le H(M_k) - \sigma(M_k) \quad \text{in} \;\; Q(\kappa H(M_k) R_k).$$

Writing $v \le (v - \varepsilon(M_k) - (1/n))_+ + \varepsilon(M_k) + (1/n)$, we translate it into the following estimate for $v$ in $Q(R_{k+1})$

$$v \le H(M_k) - \sigma(M_k) + \varepsilon(M_k) + (1/n) \le M_k - \sigma(M_k) + (1/n) = M_{k+1} + (1/n).$$

Since this is true for all $n > 0$, we must have $v \le M_{k+1}$ in $Q(R_{k+1})$.

In the same way we can prove that $-v \le M_{k+1}$ in $Q(R_{k+1})$. This completes the proof. ■

### 7.4.2 *Behaviour near a non-vanishing point*

We now consider the solution $v$ near a point $P$ where $v(P) \ne 0$ and show that it remains bounded away from zero in some full neighbourhood $Q^*_R(P)$. It relies on Corollary 7.4 and the preceding proposition.

**Proposition 7.11** *Let the hypotheses of Theorem* 7.1 *be satisfied, and let* $M_k, R_k$ *be the sequences constructed in Proposition* 7.10. *Let* $P \in Q$ *with* $\operatorname{dist}(P, \partial_p Q) \geq 2D$ *and suppose that*

$$M_{k_0+1} \leq v(P) \leq M_{k_0}.$$

*for some* $k_0$. *Then, we have*

$$|v(x,t)| \geq \frac{1}{2} M_{k_0} \quad in \;\; Q^*_{\widetilde{R}_{k_0}}(P),$$

*where* $\widetilde{R}_{k_0} = \min(R_{k_0+1}, R_{k_0}/2)$.

**Proof** Let $u(x_1, t_1) < M_k/2$ at some point $P_1 = (x_1, t_1)$ in $Q^*_{\widetilde{R}_{k_0}}(P)$. We examine two possibilities: if $t_1 \leq t_0$ then by Corollary 7.4 the induction argument of Proposition 7.10 may be carried out until the $k_0$-th step. It follows that $v(x,t) \leq M_{k+1}$ in $Q_{R_{k_0+1}}(P)$, a contradiction.

If $t_1 \geq t_0$ then $P \in Q_{R_{k_0+1}}(P_1)$ and again the induction argument of Proposition 7.10 may be carried out until the $k_0$-th step. It follows that $v(x,t) \leq M_{k+1}$ in $Q_{R_{k_0+1}}(P_1)$, a contradiction. ■

### 7.4.3 *End of proof*

Let $\varepsilon > 0$ and let $P = (x_0, t_0) \in Q'$. We must find $\eta > 0$ depending only on $\varepsilon$ such that

$$|v(x_1, t_1) - v(x_0, t_0)| \leq \varepsilon$$

for all $P_1 = (x_1, t_1) \in Q'$ such that $\operatorname{dist}(P, P_1) \leq \eta$. We may assume that $L = v(P) \geq 0$, otherwise we apply the argument to $-v$.

If $L \leq \varepsilon/3$. Then by Proposition 7.11 there exists $\eta_1 > 0$ depending only on $\varepsilon$ and the data such that

$$v(x,t) < 2\varepsilon/3 \quad \text{ for } \; \operatorname{dist}(P, P_1) \leq \eta_1.$$

Thus, $|v(x_1, t_1) - v(x_0, t_0)| \leq \varepsilon$ in that case.

On the other hand, if $L > \varepsilon/3$, by the same Proposition 7.11

$$v(x,t) \geq \varepsilon/6 \quad \text{ for } \; \operatorname{dist}(P, P_1) \leq \eta_2,$$

i.e., in a cylinder $Q^*_{\widetilde{R}}(P)$, where $\widetilde{R}$ depends only on $\varepsilon$ and the data. In this cylinder $v$ satisfies a linear equation of the form

$$u_t = a(x,t)\Delta v + b(x,t)$$

which is uniformly parabolic, since

$$a(x,t) = \frac{1}{\beta'(v(x,t))}, \qquad b(x,t) = \frac{F(x,t,v(x,t)}{\beta'(v(x,t))},$$

and $\varepsilon/6 \le v \le C_1$ in that cylinder. The bounds on the coefficients depend only on $\varepsilon$ and the data. The theory of linear parabolic equations of [353], cf. Theorem 4.2, implies that $v$ is continuous in $Q_{\widetilde{R}/2}(P)$ with a modulus of continuity that depends on $\varepsilon$ and the data. In conclusion, we may find $\eta_3 > 0$ such that

$$|v(P_1) - v(P)| \le \varepsilon \qquad \text{if } \operatorname{dist}(P_1, P) \le \eta_3.$$

Take now $\eta = \min(\eta_1, \eta_3)$ to finish the proof. ■

## 7.5 Continuity of weak solutions of the Dirichlet problem

We now apply the continuity result to the weak solutions constructed in the previous chapter. In this way we can settle the question of interior regularity.

**Corollary 7.12** *The weak energy solutions we have constructed in Chapter* 5 *are continuous functions in $Q_T = \Omega \times (0,T)$ when $u_0, f$ and $g$ are bounded.*

The proof consists only of noticing that in any inner subdomain $Q' \subset Q_T$, the sequences of approximating classical solutions are uniformly equicontinuous with a fixed modulus of continuity. Actually, we may reduce the assumptions to: $u$ and $f$ are locally bounded, but this needs some work in justifying the approximations.

### 7.5.1 *Initial regularity*

We have shown in the previous chapter that weak energy solutions, and even all limit solutions, take on the initial data in the sense of strong convergence in $L^1(\Omega)$. However, when the initial data have some continuity property, that continuity is reflected in the way the solution behaves for $t \approx 0$. This is the standard result about initial pointwise continuity.

**Proposition 7.13** *If $u$ is a weak energy solution as in Theorems* 5.7 *or* 5.14 *with bounded data, $u_0$, $f$ (and $g$), and $u_0$ is continuous at a point $x_0 \in \Omega$, then $u(x,t)$ is continuous at $(x_0, 0)$.*

Our proof of the proposition uses a technical lemma based on the previous study of radial solutions.

**Lemma 7.14** *If moreover the data are radially symmetric, non-decreasing in $r$ and $u_0$ is continuous at $r = 0$, while $f$ is bounded, then $u(r,t)$ is continuous at $r = 0$, $t = 0$. In the non-homogeneous case, we also assume that $g$ is constant, $g \ge u_0(R)$.*

**Proof of the lemma** The lower bound is easy: we use the facts that $g \ge u_0(r) \ge u_0(0) = a$ and $f \ge -N$ to conclude from the maximum principle that

$$u(r,t) \ge a - Nt,$$

see formula (5.34) in the proof of Theorem 5.7. We conclude that $\liminf_{t\to 0} u(r,t) \ge a$ for every $r \ge 0$.

As for the upper bound, we argue as follows: fix $\varepsilon > 0$ and let $\rho$ be a small radius such that $u_0(r) \le a+\varepsilon$ for $r \le 2\rho$; and let $\zeta(r)$ be a radial cut-off function supported in the annulus $\rho/2 \le r \le 3\rho$ with value 1 in the annulus $\rho < r < 2\rho$. Then, from the definition of weak solution we get

$$\int_\Omega (u(x,t)-u_0(x))\zeta\,dx = \int_0^t \int_\Omega \{\Phi(u)\Delta\zeta + f\zeta\}\,dx,$$

which goes to zero as $t \to 0$, hence it is less than $\varepsilon$ for $0 < t < \tau$. Now,

$$\int_\Omega (u(x,t)-u_0(x))\zeta\,dx \ge \int_\rho^{2\rho} (u(\rho,t)-u_0(2\rho))\,r^{d-1}dr = C(u(\rho,t)-u_0(2\rho))\rho^d.$$

Since $\rho$ is fixed, we conclude that $\limsup_{t\to 0} u(\rho,t) \le a+\varepsilon$. Putting both things together, the continuity of $u(r,t)$ at $(0,0)$ follows. ■

**Proof of the proposition** Let $u_0(x_0) = a$. We may assume without loss of generality that $x_0 = 0$, and let $B_R(0) \subset \Omega$. Let $f \le N$ and let $K$ be an upper bound of $|u|$ in $Q_T$.

In order to get an upper bound for $u$ near $(0,0)$, we introduce the radial weak solution $u_1$ defined in $Q_1 = B_R(0) \times (0,T)$ with radial initial data $u_{01}(r) \ge u(x)$ $(r = |x|)$, $f_1(x,t) = N_2$, and boundary data $u_1(R,t) = K$. By comparison, we have $u(x,t) \le u_1(r,t)$. If we assume that $u_1(r,0)$ is increasing, then $u_1(r,t)$ is continuous at $(0,0)$. But we may take $u_1(0,0)$ as close as $u_0(0) = a$ as we want. It follows that

$$\limsup_{(x,t)\to(0,0)} u(x,t) \le a.$$

The lower bound is similar, using a bound from below. ■

**Remark.** These types of functions, usually solutions of auxiliary problems, that are constructed on purpose to serve as upper (or lower) bounds are called *barriers* and will be of much use in many situations. They can be upper or lower barriers. Usually, they are solutions of the same equation with different data and are also called supersolutions and subsolutions resp. See more on this topic in Section 6.2 and Subsection 8.2.2.

The above proposition is the main step behind a much more appealing result.

**Corollary 7.15** *A modulus of continuity in $x$ for a bounded weak solution of the GPME at a time $t_0 \ge 0$ implies a modulus of continuity in $t$ at $t = t_0+$, and the modulus of continuity in time depends only on $\Phi$, $d$, the modulus of continuity of $u_0$ and the $L^\infty$ norm of the data. If the space modulus is uniform in a certain strip $S = \Omega \times [t_1, t_2]$, then $u$ is continuous in $S$.*

**Proof** The first part is immediate after the proof of the proposition. The argument applies to $t_0 > 0$ by translation of the origin of time. The notation $t = t_0+$ means that $u$ is continuous in time for $t \geq t_0$.

As for the second part, we have a uniform space modulus and a uniform time modulus for positive time increments. It is then immediate that the time modulus works also for negative time increments. We ask the reader to prove this calculus fact. ∎

### 7.5.2 *Boundary regularity*

We now address the remaining question of boundary regularity.

**Proposition 7.16** *Bounded weak solutions of the Dirichlet problem are continuous up to the lateral boundary with a modulus of continuity that depends only on the conditions of Theorem* 7.1 *and the modulus of continuity of the boundary data.*

**Proof** (i) Let $u$ be a solution of Problem (5.1)–(5.3). We have to prove that $u$ is continuous at the lateral boundary $\partial\Omega \times (0, \infty)$ and to control the modulus of continuity near that boundary. We have two options, either to re-do the previous theory from the start in a neighbourhood of a point $P_0 = (x_0, t_0)$ of the lateral boundary, or to develop some ad hoc theory using the method of barriers. The first approach could be considered natural and allows us to revise the contents of the main result. We refer the reader to Ziemer's [536] for details on how to proceed.

(ii) In the case of the homogeneous Dirichlet problem the second approach is quite easy using barriers: a possible argument uses the fact that $u$ is bounded above by the separated-variable solution (Friendly Giant). That solution serves as a continuous upper barrier and minus this function will be the lower barrier.

Another more general argument is as follows: we approximate the zero boundary data by a positive constant $g = \varepsilon$, we also raise the initial data by $\varepsilon$ and obtain in this way a smooth supersolution $u_\varepsilon \leq u$ that is continuous up to the lateral boundary and has a modulus of continuity that depends on the arguments explained in Theorem 7.1 and on $\varepsilon$. Since $u_\varepsilon \downarrow u$, the conclusion follows in this case.

(iii) Assume now that $g$ is bounded and continuous at a boundary point $P_0 = (x_0, t_0) \in \Sigma$ and let $g(x_0, t_0) = c_0 > 0$. The barrier from above is immediate, arguing by raising the data as before. The construction of a lower barrier is not so immediate. We may proceed locally: we select a small full parabolic neighbourhood $Q^*$ centred at $P_0$, define $Q' = Q^* \cap Q_T$ and construct a continuous subsolution $u_1$ with boundary data $c_0 - \varepsilon$ near $P_0$ and very negative otherwise, $u = -M$. The initial data for $t_1 = t_0 - r^2$ are $u(x, t_1) = -M$. If we prove that $u_1$ is continuous, it will be enough for our purposes. The construction of such a subsolution is not difficult with the results of later chapters or by direct trial in the case of the PME. ■

## 7.6 Hölder continuity for porous media equations

Under the general assumptions in the above class of equations, only a modulus of continuity is achieved as an answer to the question of how regular the solutions are. But the PME has a simple power structure in its nonlinearity and this helps in getting Hölder continuity, which is the standard regularity in the De Giorgi–Nash–Moser tradition. This is the corresponding result. We take $\Omega$ a domain (or even an open rectangle) in $\mathbb{R}^d$, $Q_T = \Omega \times (0,T)$ and $Q_T^\varepsilon = \Omega^\varepsilon \times (\varepsilon^2 \times T)$ where $\Omega^\varepsilon = \{x \in \Omega : d(x, \partial\Omega) > \varepsilon\}$,

**Theorem 7.17** *Let $u$ be a weak energy solution of the porous medium equation defined in the cylinder $Q_T$ and assume that $u$ is bounded, $\|u\|_{L^\infty(Q_T)} \le M$. Then there are positive constants $C > 0$ and $\alpha \in (0,1)$ such that for every pair of points $(x_1,t_1), (x_2,t_2) \in Q_T^\varepsilon$ we have*

$$|u(x_1,t_1) - u(x_2,t_2)| \le C(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\alpha/2}). \tag{7.18}$$

*The constants $C$ and $\alpha$ depend only on $M$ and $\varepsilon$.*

**Proof** This result was proved by DiBenedetto and Friedman [211] in 1985. The result is achieved by working on cylinders suitably scaled to reflect in a precise quantitative way the power-like degeneracy of the equation. We will follow their proof, with references to the original paper for some technical parts. In addition to the standard parabolic cylinders $Q_R(P)$ defined in (7.3), we will use some types with dimensions adapted to the scale structure of the equation. They are

$$\mathcal{Q}^\varepsilon_{R_0}(P_0) = B_{R_0}(x_0) \times (t_0 - R_0^{2-\varepsilon}, t_0).$$

where $P_0 = (x_0, t_0)$ is a point in $Q_T$, and

$$\mathcal{Q}_R(P_0, \omega) = B_R(x_0) \times (t_0 - R^2\omega^{-\alpha}, t_0),$$

$$\mathcal{Q}^\rho_R(P_0, \omega) = B_R(x_0) \times (t_0 - \frac{1}{2}\rho R^2\omega^{-\alpha}, t_0)$$

where $\alpha = \frac{m-1}{m}$. Note the special time scale in all cases.

(i) We present the details of the proof for non-negative solutions. Let $v = u^m$ which satisfies the equation

$$\partial_t(v^q) = \Delta v, \quad q = 1/m.$$

We take a fixed $P_0 = (x_0, t_0) \in Q_T$ and let $R_0 > 0$ such that the cylinder $\mathcal{Q}_0 = \mathcal{Q}^\varepsilon_{2R_0}(P_0)$ is contained in $Q_T$. There is no lack of generality in the arguments that follow in assuming that $x_0 = 0$ and $t_0 = 0$ (by translation of the axes). We then drop the reference to $P_0$ in the cylinders. Let

$$\mu^+ = \sup\{v(x,t) : (x,t) \in \mathcal{Q}_0\}, \quad \mu^- = \inf\{v(x,t) : (x,t) \in \mathcal{Q}_0\}.$$

and pick a number

$$\omega > \mu^+ - \mu^- := \operatorname{osc}(v; \mathcal{Q}_0).$$

Take now $R \in (0, R_0]$ and suppose first that

$$\omega^\alpha > R^\varepsilon, \tag{7.19}$$

where $\alpha = (m-1)/m$. Then, the cylinder

$$\mathcal{Q}_R(\omega) = B_R(0) \times (-R^2\omega^{-\alpha}, 0) \tag{7.20}$$

is contained in $\mathcal{Q}_0$, hence

$$\operatorname{osc}(v; \mathcal{Q}_R(\omega)) < \omega. \tag{7.21}$$

(ii) Let us assume further that the infimum is small in relative size, in the sense that

$$\mu^- < \omega/4.$$

Under this assumption we derive some inequalities for $v$. We multiply the equation satisfied by $v$ by $\pm(v-k)^\pm\zeta^2$ where $k > 0$ and $\zeta$ is a cut-off function in $\mathcal{Q}_R(\omega)$ which equals 1 in the subcylinder

$$\mathcal{Q}^* = \mathcal{Q}_R(\omega; \sigma_1, \sigma_2) = B_{(1-\sigma_1)R}(0) \times (-(1-\sigma_2)R^2\omega^{-\alpha}, 0).$$

Put $t_1 = -(1-\sigma_2)R^2\omega^{-\alpha}$ and $R_1 = R - \sigma_1 R$. We get

$$\operatorname{ess\,sup}_{t_1<t<0} \int_{B_{R_1}} \left( \int_0^{(v-k)^\pm} (k \pm \xi)^{\frac{1}{m}-1} \xi \, d\xi \right) dx + \|\nabla (v-k)^\pm\|^2_{2,\mathcal{Q}^*}$$

$$\le \frac{C}{(\sigma_2 R)^2} \|(v-k)^\pm\|^2_{2,\mathcal{Q}^*} + C \iint_{\mathcal{Q}_R(\omega)} \left( \int_0^{(v-k)^\pm} (k \pm \xi)^{\frac{1}{m}-1} \xi \, d\xi \right) \zeta_t \, dx dt.$$

We shall now use this expression with the choice $(v-k)^-$ and $k = \mu^- + \omega\, 2^{-s}$, $s \ge 1$. By the current assumption $\mu^- < \omega/4$, we have the lower bound

$$\int_0^{(v-k)^-} (k - \xi)^{\frac{1}{m}-1} \xi \, d\xi \ge \frac{1}{2} k^{-\alpha} ((v-k)^-)^2 \ge C\omega^{-\alpha}((v-k)^-)^2,$$

as well as the upper bound

$$\int_0^{(v-k)^-} (k - \xi)^{\frac{1}{m}-1} \xi \, d\xi \le C\omega^{1/m}(v-k)^-.$$

Note that $(v-k)^- \leq \omega$. Taking $\zeta$ such that $0 \leq \zeta_t \leq 2\omega^\alpha/(\sigma_2 R^2)$, we obtain

$$\omega^{-\alpha} \text{ess sup}_{t_1<t<0} \|v(t) - (\mu^- + 2^{-s}\omega))^-\|^2_{2,B(R_1)} + \|\nabla(v - (\mu^- + 2^{-s}\omega))^-\|^2_{2,\mathcal{Q}^*}$$

$$\leq C\frac{\omega^2}{\sigma_1 R^2}\int_{t_1}^0 |A_{s,R}(t)| + C\frac{\omega^{\alpha+1+1/m}}{\sigma_2 R^2}\int_{t_1}^0 |A_{s,R}(t)|,$$

where $A_{s,R}(t)$ is the set $\{x \in B_R(x_0) : \; v(x,t) < \mu^- + 2^{-s}\omega\}$. It follows that

$$\text{ess sup}_{t_1<t<0} \|v(t) - (\mu^- + 2^{-s}\omega))^-\|^2_{2,B(R_1)} + \omega^\alpha \|\nabla(v - (\mu^- + 2^{-s}\omega))^-\|^2_{2,\mathcal{Q}^*}$$

$$\leq C\omega^{2+\alpha}\left[(\sigma_1 R)^2 + (\sigma_2 R^2)^{-1}\right]\int_{t_1}^0 |A_{s,R}(t)|\, dt. \tag{*}$$

(iii) We now need two technical lemmas which describe the process of bound improvement by suitable reduction of the domain in its two aspects.

*Claim 1*

*There exists a number $\rho \in (0,1)$, independent of $\omega$, such that if*

$$\textit{meas}\left\{(x,t) \in \mathcal{Q}_R(\omega) : \; v(x,t) < \mu^- + \frac{1}{2}\omega\right\} < \rho\,|\mathcal{Q}_R(\omega)| \tag{7.22}$$

*then*

$$v(x,t) > \mu^- + \omega/4 \qquad \forall (x,t) \in \mathcal{Q}_{R/2}(\omega). \tag{7.23}$$

The outline of the proof is as follows: We first renormalize the problem by making the change of time variable

$$\tau = \omega^\alpha t.$$

Then, the cylinders $\mathcal{Q}_R(\omega)$ and $\mathcal{Q}_R(\omega;\sigma_1,\sigma_2)$ become

$$\mathcal{Q}_R = B_R(0) \times (-R^2, 0), \quad \text{and}$$
$$\mathcal{Q}_R^* = \mathcal{Q}_R(\sigma_1,\sigma_2) = B_{R-\sigma_1 R}(0) \times (-(1-\sigma_2)R^2, 0).$$

Write also

$$\overline{v}(x,\tau) = v(x,\omega^{-\alpha}\tau), \quad \overline{A} = \{y \in B_R(0) : \; \overline{v}(y,\tau) < \mu^- + 2^{-s}\omega\}$$

We can then write expression (*) in the form

$$\|\overline{v} - (\mu^- + 2^{-s}\omega))^-\|^2_{V^{1,0}(\mathcal{Q}_R^*)} \leq C\omega^2\left[(\sigma_1 R)^2 + (\sigma_2 R^2)^{-1}\right]\textstyle\int_{-R^2}^0 |\overline{A}_{s,R}(t)|\, dt. \tag{7.24}$$

Under these circumstances we can use the standard techniques of bound improvement used in previous sections. The details in this case are contained in

[210, Lemma 6.5]. Since the constant $C$ in (7.24) is independent of $\omega$, also $\rho$ is.

A second technical lemma is needed to cover the possibility of bound improvement from above.

*Claim 2*

*Suppose now that* (7.22) *is not satisfied. Then, there exists a number* $s_0$*, independent of* $\omega$*, such that*

$$v(x,t) \le \mu^+ - \frac{\omega}{2^{s_0}} \quad \forall (x,t) \in \mathcal{Q}^\rho_{R/2}(\omega). \tag{7.25}$$

We refrain from giving the details of these lengthy technical arguments for which we refer to the paper and references.

(iv) We may now prove the theorem. We have assumed that $\mu^- < \omega/4$. If $\operatorname{osc}(v; \mathcal{Q}_R(\omega)) < \omega$ and moreover (7.22) holds, then by Claim 1, the oscillation of $v$ in $\mathcal{Q}_{R/2}(\omega)$ is less than $(1-1/4)\omega = (3/4)\omega$. On the other hand, if it does not hold, then Claim 2 implies that the oscillation is less then $(1-2^{-s_0})\omega$ in $\mathcal{Q}^\rho_{R/2}(\omega)$. Setting $\eta = 1 - 2^{-s_0}$ we have in both cases

$$\operatorname{osc}_{\mathcal{Q}^\rho_{R/2}(\omega)} v \le \eta\, \omega, \tag{7.26}$$

which is the desired improvement of (7.21).

We now perform an iterative process. By the existence of the cylinder $\mathcal{Q}_0$ and condition $\omega_0^\alpha > R_0^\varepsilon$ we arrived at the starting conclusion, formula (7.21), that the oscillation in $\mathcal{Q}^\rho_R(\omega_0)$ is less than $\omega_0$. Using the preceding improvement argument under the assumption

$$\mu^- < \frac{\omega}{4},$$

and proceeding inductively from the initial values $\omega_0, R_0$, we conclude that there exist sequences $\{\omega_n\}$ and $\{R_n\}$ such that $\omega_0 = \omega$, and

$$\omega_{n+1} = \eta\, \omega_n, \quad R_{n+1} = c_0 R_n, \tag{7.27}$$

such that $\operatorname{osc}_{\mathcal{Q}_{R_n}} v \le \eta^n\, \omega_0$. The constant $c_0$ is determined by the condition $\mathcal{Q}_{R_{n+1}}(\omega_{n+1}) \subset \mathcal{Q}^\rho_{R_n}(\omega_n)$, hence

$$c_0 \le \frac{1}{2}\eta^{\alpha/2}\rho^{1/2}.$$

Note that the numbers $0 < \eta, \rho < 1$ are the product of the technical lemmas where the structure of the equation has a direct effect. Relating $\eta$ to $R_n$ we get

$$\operatorname{osc}_{\mathcal{Q}_{R_n}} v \le C(R_n/R_0)^\sigma \tag{7.28}$$

for some $\sigma > 0$ which is determined by the condition

$$2\eta^{1/\sigma} \le \eta^{\alpha/2}\rho^{1/2},$$

hence, it can be taken independent of $R_0$ and $\mu$. Putting $\rho = \eta^\gamma$, this yields Hölder continuity with exponent $\sigma < 2/(\alpha+\gamma)$.

(v) Let us consider now the case where the infimum is not comparatively small, i.e., $\mu^- \ge \omega/4$. Since $\omega_0^\alpha > R_0^\varepsilon$, we then have

$$\inf_{\mathcal{Q}_0} v \ge \frac{1}{4} R_0^{\varepsilon/\alpha}.$$

We may then rescale the equation in the $x$ or $t$ direction so as to obtain a uniformly parabolic operator, to which we may apply standard local estimates. Going back to the original coordinates we easily get the dimensional form

$$|v(x_1,t_1) - v(x_0,t_0)| \le C\, R_0^{-\sigma} \left(|x_1 - x_0|^\beta + |t_1 - t_0|^{\beta/2}\right)$$

for suitable $\sigma > 0$ and $\beta > 0$.

(vi) We have completed the proof in case $\omega^\alpha > R_0^\varepsilon$ holds. If it does not hold the situation is in some sense better since the oscillation is small. In order to apply the preceding scheme we try to see if it holds for $R_0/2$ instead of $R_0$, or if eventually there is a $k$ such that

$$\operatorname{osc}(v; \mathcal{Q}^\varepsilon_{2^{-k}R_0}) \le \left(\frac{R_0}{2^{k-1}}\right)^{\varepsilon/\alpha}$$

for some $k \ge 2$ an integer. Then there is no problem with step (iv), and step (v) will work if $\sigma$ is small enough. In that case we have to take care of the situation around $P_0$ and around $P_1$. Finally, the case $k = \infty$ can be treated is a similar way. The proof for non-negative solutions is complete.

(vii) The proof extends to signed solutions of the PME, $u_t = \Delta(|u|^{m-1}u)$.

Outline of proof of this extension: we notice that when $\mu^- > 0$ no change is needed. If on the contrary, $\mu^- < 0$, we argue as follows: when $\mu^- < -\omega/4$ then for the levels $k = \mu^- + 2^{-s}\omega$, $s \ge 3$, there is non-degeneracy on $\{(v-k)^- > 0\}$ as before. If $-\omega/4 < \mu^- < 0$, we can work with the levels

$$k = \mu^- + \frac{\omega}{3} + \frac{\omega}{2^s}$$

for which there is non-degeneracy on $\{(v-k)^- > 0\}$. We conclude that any local weak energy solution of the PME is Hölder continuous. ■

**Remark** The authors of [211] state the result for non-negative solutions of the generalized porous medium equation

$$u_t - \sum_{i,j} (a^{k,l}(x,t,\nabla u)(u^m)_{x_l})_{x_k} = f(x,t,u,\nabla u) \tag{7.29}$$

which is an anisotropic version of the PME. The solutions under consideration are local in the sense that they are defined in a cylinder $Q_T = \Omega \times (0,T)$ without reference to boundary conditions. The following structure assumptions are made: $m > 1$ and

$$\textstyle\sum_{i,j} a^{k,l}\xi_k\xi_l \geq c_0|\xi|^2, \quad |a^{k,l}| \leq c_1,$$
$$|f(x,t,u,\nabla u)| \leq c_2|\nabla u^m| + c_3,$$

for some constants $c_0, c_1, c_2, c_3 > 0$.

**Theorem 7.18** *Let $u$ be a weak energy solution of porous medium equation (7.29) defined in a cylinder $Q_T$ and assume that $u$ is bounded, $\|u\|_{L^\infty(Q_T)} \leq M$, and the structure assumptions hold. Then there are positive constants $C > 0$ and $\alpha \in (0,1)$ such that for every pair of points $(x_1,t_1),(x_2,t_2) \in Q_T^\varepsilon$ we have*

$$|u(x_1,t_1) - u(x_2,t_2)| \leq C(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\alpha/2}). \tag{7.30}$$

*The constants $C$ and $\alpha$ depend only on $M$, $\varepsilon$ and the structure constants $c_i$.*

## 7.7 Continuity of weak solutions in 1D

As we have seen, the question of continuity of weak solutions of the GPME in the context of several space dimensions involves heavy work. But this important question can be easily settled when the problem is posed in one space dimension, $d = 1$, since it involves quite simple calculations. We address the problem in the general class of weak solutions defined in a cylinder $Q = I \times (0,T)$ with $I = (a,b)$ a bounded interval in $\mathbb{R}$, without reference to its boundary or initial conditions.

The idea is very simple: in view of the regularity of weak solutions proved in Theorem 5.7 we consider as main function $w = \Phi(u)$ and use the following calculus lemma.

**Lemma 7.19** *Let $w$ be a function in $L^2(Q)$ for some cylinder $Q = I \times (0,T)$ and let $w_x \in L^\infty(0,T : L^2(I))$ and $w_t \in L^2(Q)$. Then, $w$ admits a continuous representative in $C^{1/2,1/4}(\overline{Q})$.*

**Proof** (i) Let us assume that $w$ is a smooth function in $Q$. We will obtain uniform estimates for the Hölder norm of $w$ in terms of the norms of $w_x$ and $w_t$ in the stated spaces. The uniform continuity in $x$ is easy from the standard inequality

$$|w(x,t) - w(y,t)| \leq \|w_x(\cdot,t)\|_2\,|x-y|^{1/2}. \tag{7.31}$$

This means that $w$ is uniformly Hölder continuous as a function of $x$ in the rectangle $[a,b] \times (0,T)$.

The continuity in time takes some more effort, and comes from a calculation in the spirit of interpolation theory. We fix times $0 \leq t < t' = t + h \leq T$, a point $x_0 \in I$ and take a space interval of the form $J = (x_0 - \delta, x_0]$ or $J = [x_0, x_0 + \delta)$ contained in $I$. One of the two possibilities holds if $\delta \leq (b-a)/2$. Then, we calculate

$$\delta\,|w(x_0,t')-w(x_0,t)| \le \int_J |w(y,t')-w(y,t)|\,dy + \int_J (|w(x_0,t')-w(y,t')|+|w(x_0,t)-w(y,t)|)\,dy.$$

The last two terms can be evaluated using (7.31). The first integral can be evaluated as equal or less than

$$\int_t^{t'}\int_J |w_t|\,dyds \le \|w_t\|_{L^2(Q)}(h\,\delta)^{1/2}.$$

Putting it together we get

$$|w(x_0,t)-w(x_0,\tau)| \le C\|w_x\|\,\delta^{1/2}+C\|w_t\|\,\delta^{-1/2}h^{1/2}.$$

When $h=t-\tau$ is small, optimization of this formula with respect to $\delta$ happens for $\delta = C'\|w_t\|\,h^{1/2}/\|w_x\|$, and we get the desired estimate

$$|w(x_0,t)-w(x_0,\tau)| \le C''\|w_x\|^{1/2}\|w_t\|^{1/2}\,h^{1/4}. \tag{7.32}$$

(ii) If $w$ is not smooth we approximate it by smooth functions, obtain the uniform bounds as above and pass to the limit. We obtain a continuous solution satisfying estimates (7.31) and (7.32). ∎

**Corollary 7.20** *Bounded weak solutions of Problem* (5.1)–(5.3) *are continuous functions in $Q_T\cup\Sigma$. More precisely, if $\Phi$ is locally Lipschitz continuous, then $w=\Phi(u)$ belongs to the class $C^{1/2,1/4}(\overline{Q^\tau})$ for every $\tau>0$, and the Hölder norm depends only on the norm of $u_0$ and $f$ in $L_\Psi(\Omega)$ and $L^2(Q_T)$ resp. For general $\Phi$, the function $Z(u)$ defined in Lemma* 5.9 *has this regularity property.*

**Proof** Function $Z(u)$ is defined by the rule, $dZ(u)=\min\{du,d\Phi(u)\}$. The proof relies on recalling the estimates obtained in the proof of Theorem 5.7, that have been summarized at the beginning of Section 5.6. The continuity at the boundary has already been discussed in the lemma. ∎

**Remarks** (1) We have proved in Section 5.8 that solutions of the GPME are bounded for all positive times if $\Phi$ is strongly superlinear and $f$ bounded. Actually, in one dimension the condition of bounded $f$ is sufficient. The argument is as follows: by virtue of estimate (7.31) since $w_x\in L^2(I)$ for a.e. $t$, $u(t)$ is bounded. But once $u(t)$ is bounded, say by $M$, then $u(t')$ is bounded by $M+(t'-t)N$ for all $t'>t$, where $N=\|f\|_\infty$. Therefore, $u$ is uniformly bounded on sets of the form $0<\tau\le t\le T$.

(2) The regularity stated in Corollary 7.20 is not the best possible regularity result that can be obtained, but it serves our purpose at this time: it contains a quite clear statement of uniform continuity, it holds under assumptions that include the solutions we have constructed and others that will be encountered,

and it has a quite simple proof. The question of optimal regularity in one space dimension is addressed in Chapter 15.

*Continuity of radial solutions*

There is no difficulty in repeating the outline of the preceding proof for radial solutions defined in any annular domain of the form $\{r_1 < |x| < r_2\}$, since the equation is quite similar. However, near the origin the calculations are different because of the weight $r^{d-1}$ in the Laplace operator, and we have to resort to the general theory developed in previous sections.

## 7.8 Existence of classical solutions

Once a solution of the equation is constructed in some generalized sense, it is an important point to decide if it is indeed a classical solution. Though we know that in general this will not be the case, it can happen under additional requirements on the data. We prove next that when the initial data are smooth and positive inside $\Omega$, the equation is parabolic non-degenerate and we obtain a classical solution by essentially using the standard quasilinear theory.

**Proposition 7.21** *Let $u_0 \in C(\overline{\Omega})$ be positive in $\Omega$ and vanish on its boundary $\Gamma$, let $f \geq 0$ be $C^\infty$ smooth and let $u$ be the corresponding weak solution of the PME. Then $u \in C^\infty(Q) \cap C(\overline{Q})$, $u$ is positive in $Q$ and vanishes on $\Sigma$. If $f \in C^{2k+\alpha,k+\alpha/2}(Q)$, then $u \in C^{2k+2+\alpha,k+1+\alpha/2}(Q)$.*

**Proof** The first step is proving that for every point $x_0 \in \Omega$ where $u_0(x_0) > 0$ we will have $u(x_0, t) > 0$ for every $t > 0$. In the case of the PME this is done by the classical method of *barriers*, comparing $u$ with a suitable source-type solution that solves the equation with $f = 0$. Actually, if $B = B_r(x_0)$ is a ball of radius $r$ where $u_0$ is positive, say $u_0(x) \geq c > 0$ for $x \in B$, we consider the Barenblatt function

$$\bar{u} = U(x - x_0, t + 1; C).$$

We may choose $C$ small enough so that $u_0(x) \geq \bar{u}(x, 0)$ in $B$, and also that the support of $\bar{u}$ is contained in $Q_T$ for a given $T > 0$. This support is of the form $\mathcal{S} = \{(x,t) : c|x - x_0|^\gamma < (t+1)\}$ with $\gamma = d(m-1) + 2$ (cf. (0.5)), $\bar{u} \in C^\infty(\mathcal{S})$ and $\bar{u}$ vanishes on the lateral boundary of $\mathcal{S}$.

Hence, by the classical maximum principle applied in $\mathcal{S} \cap Q_T$ to $\bar{u}$ and a smooth approximation to $u$ we conclude that $u \geq \bar{u}$ in $\mathcal{S}$, hence $u(x,t)$ is bounded uniformly away from 0 in a neighbourhood of the form $N = B_1 \times (0, T)$, $B_1 = B_r(x_0)$.

Therefore, when taking the limit $u_n \to u$ in the approximation process of Theorem 5.5, we can apply in $N$ the regularity theory of quasilinear non-degenerate parabolic equations, and conclude that $u \in C^\infty(N)$ and the initial data are taken continuously in $B_1$.

The fact that $u$ vanishes continuously on $\Sigma$ is a simple consequence of the approximation process (5.21)–(5.24). In fact, $u \le u_n, u_n \in C^\infty(\overline{Q})$ and $u_n(x,t) = \frac{1}{n}$ on $\Sigma$. ■

Of course, if moreover $u_0$ is smooth, e.g. if $u_0 \in C^k(\Omega)$ for some $k > 0$, this regularity is reflected in the regularity of $u$ near $t = 0$, according to the same quasilinear theory. We leave it to the reader to prove similar results for the GPME when $\Phi$ is smooth for $u \neq 0$. The regularity holds then at points where $u_0(x) \neq 0$.

## 7.9 Extensions

Here are some complements to the information of this chapter.

### 7.9.1 *Fast diffusions*

Chen and DiBenedetto study in [167] the question of Hölder estimates of solutions of singular parabolic equations with measurable coefficients of the $p$-Laplacian type, and the methods apply to the PME equations in the fast diffusion range. The Harnack inequality for non-negative solutions of singular parabolic equations is proved in [168]. The equation may have bounded measurable coefficients and have the form $u_t - (a_{ij}(x,t)|u|^{m-1}u_{x_i})_{x_j} = 0$ with $0 < m < 1$. The authors use an iteration method in the context of quasilinear singular parabolic equations that is different from the classical iteration techniques of E. De Giorgi [201].

### 7.9.2 *When continuity fails*

There are two scenarios that come in progression when the nonlinearity of the GPME $u_t = \Delta\Phi(u)$ is allowed to be singular at $u = 0$, i.e., when we allow $\Phi'(0+) = \infty$.

(i) Weak non-negative solutions of the fast diffusion $u_t = \Delta(u^m)$, $m < 1$, need not be bounded, a fortiori they are not continuous, when $m \le m_c = (d-2)/d$. See the study of this question in the Lecture Notes [515]. The exponent $m_c$ is sharp, since weak solutions with locally integrable data are locally bounded and continuous for $m > m_c$, see [190, 286, 435].

(ii) On the other hand, when $m \le 0$ even bounded solutions need not be continuous. Examples of bounded discontinuities, called *needles,* are given in [513]. The dimension can even be one and the equation is written as $u_t = (u^{m-1}u_x)_x$.

### 7.9.3 *Equations with measurable coefficients*

Here $(a_{ij})$ is a symmetric matrix of bounded measurable functions which satisfies the ellipticity condition

$$\Lambda^{-1}\xi^2 \le a_{ij}(x)\,\xi_i\xi_j \le \Lambda\xi^2 \quad \text{a.e. in } \mathbb{R}$$

for all $\xi \in \mathbb{R}^d$, for some $\Lambda > 1$. The equation is suggested as a mathematical model for the flow of a gas in a non-homogeneous porous medium.

### 7.9.4 *Other*

Hölder estimates for solutions of doubly nonlinear degenerate parabolic equations are studied by a number of authors.

## Notes

**Section 7.1.** Continuity of the solutions of the PME in the several-dimensional context is due to several authors in slightly different contexts: the first several dimensional results seems to be due to Caffarelli and Friedman [139], 1979, who study non-negative solutions of the plain PME in the whole space and obtain a logarithmic modulus of continuity. In [140], 1980, the authors prove that weak solutions are locally Hölder continuous with free boundaries which are locally Hölder continuous surfaces. Then, Gilding and Peletier [269], 1981, treated the homogeneous Dirichlet problem. Solutions of both signs are treated a bit later for more general classes of equations by different authors: thus, Caffarelli and Evans [137] study equations that include the two-phase Stefan problem and has $f = 0$; DiBenedetto [206, 207], Sacks [461], and Ziemer [536] treat rather general classes of degenerate parabolic equations of the form

$$\beta(u)_t = \mathrm{div} A(x, t, u, Du) + B(x, t, u, Du)$$

under structural conditions on $A$ and $B$. Technical conditions are imposed on $\beta'$ near $u = 0$. The standard assumption is that $\beta'$ is locally bounded from above and below. The local lower bound on $\Phi'$ is eliminated in Sacks [461], 1983, see also DiBenedetto [215], 1985.

We repeat that the assumption of smoothness of the solution made in Subsection 7.2 is made for convenience in justifying the calculations, and the conclusions will then apply to solutions obtained as limits of smooth solutions. The same is true about the $C^2$ assumption on $\beta$. However, the approach of working directly with less smooth weak solutions is followed by some authors.

The question of time regularity can be reduced to obtaining first space regularity thanks to the work of Kruzhkov [351], 1967. He proved that for bounded solutions of a wide class of parabolic equations Hölder continuity of $u$ with respect to the spatial variable $x$, with exponent $\alpha \in (0, 1]$ implies Hölder continuity in time with exponent $\alpha^* = \alpha/(2 + \alpha)$. This exponent was improved by Gilding [262] to $\alpha^{**} = \alpha/2$. We will return to the precise exponents in Chapter 15 for the PME in $d = 1$ and in Chapter 19 for $d > 1$.

**Section 7.2.** The statement of the continuity Theorem and the proof performed in Sections are taken from the paper [461] by P. Sacks. Actually, that paper treats a somewhat more general equation with convection term

$$\partial_t \beta(v) = \Delta v + \mathbf{q} \cdot (\nabla \gamma(v)) + F(x,t,v)$$

with $\mathbf{q} \in C^1(Q_T)$, $\gamma \in C^2(\mathbb{R})$. On the other hand, DiBenedetto [207] treats equations of the more general form

$$\partial_t \beta(u) = \nabla \cdot \overrightarrow{A}(x,t,u,\nabla_x u) + B(x,t,u,\nabla_x u) \ni 0, \tag{7.33}$$

where the increasing function $\beta$ may have a jump at $u = 0$. Structural assumptions are imposed on $\beta$, $\overrightarrow{A}$ and $B$.

**Section 7.6.** Caffarelli and Friedman proved in [140] that for a non-negative solution of the porous medium equation $u_t = \Delta u^m$, $m > 1$, the boundary of the set $[u > 0]$ is a locally Hölder continuous surface and as a consequence that the solution itself is locally Hölder continuous. Free boundaries will be studied in depth in Chapter 14.

The main part of DiBenedetto and Friedman's paper [211] is devoted to proving Hölder continuity for the gradient of local weak solutions of degenerate parabolic systems

$$D_t u^j - \operatorname{div}(|\nabla u|^{p-2} \nabla u^j) = F_j(x,t,\nabla u),$$

in $m$ unknowns, $1 \le j \le m$. Better regularity than a certain Hölder exponent cannot be achieved in view of the explicit examples of solutions with free boundaries, like the ZKB family and the travelling waves. This phenomenon of limited regularity is a general property of solutions with moving free boundaries, as we will see in Chapter 14 devoted to study propagation and free boundaries.

**Section 7.7.** The study of one-dimensional continuity will be continued in Chapter 15 and is intimately related to the properties of the free boundary or interface. Optimal regularity will be found.

The topic of determination of optimal regularity in the several dimensional setting still offers many open problems. We will return to it in Chapter 19.

## Problems

**Problem 7.1** Prove that the iterative relation (7.12) has a solution that tends to zero as $j \to \infty$ if $J_0$ is small enough, as indicated in (7.13).

**Problem 7.2**

(i) Complete the details of the proof of Theorem 7.17 as indicated in the text.
(ii) Prove Theorem 7.18.

# 8

# THE DIRICHLET PROBLEM III. STRONG SOLUTIONS

We devote the present chapter to addressing the question of how regular actually are the solutions constructed in previous chapters. We recall the results of Chapter 7 where the continuity of bounded solutions was established, but the results of this chapter take another direction.

Here, we begin to concentrate our interest towards the PME, so that $\Phi(s) = |s|^{m-1}s$ for some $m > 1$ and take forcing term $f = 0$, a case most often found both in the theory and the applications. We consider the solutions of the homogeneous Dirichlet problem for the signed PME, posed in a bounded spatial domain with initial data $u_0 \in L^1(\Omega)$. In this setting, we describe solutions with a better regularity than the one provided by the weak solutions of Chapter 5.

In Section 8.1 we address the question of further regularity of the time derivative $u_t$. Both in the case $u \geq 0$ and in the signed case, we prove that $u_t$ is a locally integrable function.

This allows us to introduce in Section 8.2 the more stringent concept of solution called *strong solutions*, i.e., weak solutions such that both $u_t$ and $\Delta\Phi(u)$ are locally integrable functions.

Strong solutions have nice calculus properties. Some of those properties are examined in detail. We also discuss the concepts of super- and subsolutions, important technical tools in developing the theory.

We denote by $\mathcal{M}(\Omega)$ is the space of bounded and signed Radon measures in a subdomain $\Omega$ of the Euclidean space of any dimension.

## 8.1 Regularity for the PME. Bounds for $u_t$

To begin with, we recall the results we have derived in the preceding chapters for the GPME, rephrasing them in terms of the PME with $f = 0$. We know that for any $u_0 \in L^1(\Omega)$ there exists a unique weak solution (as in Definition 6.5) of the signed PME that is bounded for positive times. The universal estimate in terms of the Friendly Giant gives the following upper bound for all solutions

$$|u(x,t)| \leq C\, t^{-\frac{1}{m-1}} \tag{8.1}$$

for a constant $C = C(\Omega, m)$. Moreover, the solutions are continuous for $t > 0$ and we have uniform estimates on $\nabla(|u|^{m-1}u)$ and $(|u|^{(m+1)/2})_t$ in $L^2(\tau, T : L^2(\Omega))$ with $\tau > 0$.

### 8.1.1 *Bounds for $u_t$ if $u \geq 0$*

Unfortunately, none of the previous estimates allows for a direct control of the derivative $u_t$ appearing in the equation. We obtain next a universal estimate for $u_t$. Such an estimate is a quite useful tool. Though such estimates exist for signed solutions, the strongest one happens when $u \geq 0$. We complement the result with an improvement into the form of $L^p$ integrability for some $p > 1$.

Let us start with the universal bound.

**Lemma 8.1** *All non-negative weak or limit solutions of problem* HDP *for the PME satisfy the estimate*

$$u_t \geq -\frac{u}{(m-1)t} \tag{8.2}$$

*in the sense of distributions in $Q_T$*

**First proof** Let $u = u_n$ be one of the approximate solutions to Problem (5.1)–(5.3). Consider the function

$$z := (m-1)tu_t + u. \tag{8.3}$$

A simple computation shows that $z$ is a solution in $Q$ of the equation

$$z_t = \Delta(mu^{m-1}z). \tag{8.4}$$

Also, that $z(x,t) = u(x,t) \geq 0$ on $\Sigma$ and $z(x,0) \geq 0$ for all $x \in \Omega$. Hence, by the standard maximum principle $z(x,t) \geq 0$, which is equivalent to (8.2). In this case we obtain a pointwise inequality.

We now pass to the limit in (8.2) to obtain the estimate for any limit solution of the HDP, as formulated in equations (5.1)–(5.3). This can only be done on the weak or distributional form of the inequality, which is obtained by multiplying by a test function $\varphi \in C_c^\infty(Q)$, $\varphi \geq 0$, and integrating by parts, i.e.

$$\int\!\!\int \left(\frac{1}{(m-1)t}u\varphi - u\varphi_t\right) dxdt \geq 0.$$

**Second proof** The reader may wonder how we found the precise combination

$$z = u + (m-1)tu_t$$

to which the maximum principle can be applied. There is a beautiful and simple argument based on scaling which produces such a magic function. It is as follows: given a smooth solution $u$ and a constant $\lambda > 1$, we consider the function

$$\tilde{u}(x,t) = \lambda u(x, \lambda^{m-1}\, t). \tag{8.5}$$

This is again a solution of the PME. Moreover, for $\lambda > 1$ we have $\tilde{u}(x,0) = \lambda u(x,0) \geq u(x,0)$, hence by the maximum principle $\tilde{u} \geq u$ in $Q$. Now differentiate

(8.5) with respect to $\lambda$ and put $\lambda = 1$. We get

$$0 \le \frac{d}{d\lambda}\tilde{u}(x,t)|_{\lambda=1} = u(x,t) + (m-1)tu_t(x,t),$$

namely (8.2). ∎

The fact that both estimates hold in the sense of distributions does not mean that $u_t$ is a function. At least, since $u$ is the limit of a sequence $\{u_n\}$ for which $(u_n)_t$ is locally bounded below uniformly in $n$, $u_t$ is in principle a Radon measure.

We continue the study in the context of non-negative solutions of the PME to prove that $u_t$ is actually an integrable function. For that purpose, we have to use Lemma 8.1 and combine it with the estimate for $(u^{(m+1)/2})_t$ into an $L^p$-estimate for $u_t$. This is rather technical. We use the following result.

**Lemma 8.2** *Let $K$ be a subset of $\mathbb{R}^d$ with finite measure, let $I = [t_0, t_1]$ and assume that $v$ is a function defined in $K \times I$ that satisfies*

(i) $v \in L^\infty(I : L^1(K)), v \ge 0, \partial_t v \ge 0$;
(ii) $v^\lambda$ *and* $\dfrac{d}{dt}(v^\lambda) \in L^r(K \times I)$ *for some* $\lambda, r > 1$.

*Then,* $\dfrac{d}{dt}v \in L^p(K \times I)$ *for every* $p \in [1, p_1)$, *where*

$$p_1 = \frac{r\lambda}{r(\lambda-1)+1} \in (1, r).$$

**Proof** Without loss of generality we may assume that $v \ge \varepsilon > 0$ in $K \times I$ by replacing $v$ by $v + \varepsilon$ since our estimates will not depend on $\varepsilon$. Now, for any $p \in (1, r)$ and $\nu \in (0, p)$ we have

$$|\frac{dv}{dt}|^p = \left|\frac{1}{\lambda}\frac{dv^\lambda}{dt}\right|^\nu \left|v^{\sigma-1}\frac{dv}{dt}\right|^{p-\nu}$$

where $1 - \sigma = \nu(\lambda - 1)/(p - \nu)$. We choose $\nu$ such that $p - \nu + (\nu/r) = 1$, that is

$$\nu = \frac{(p-1)r}{r-1}.$$

Clearly, $0 < \nu < p$. Moreover, we obtain for $\sigma$ the value

$$\sigma = 1 - \frac{r(p-1)(\lambda-1)}{r-p}$$

so that $\sigma > 0$ if $p < p_1$. With the assumption we have in $K \times I$

$$\iint \left|\frac{dv}{dt}\right|^p \le \frac{1}{\lambda^\nu}\left(\iint |\frac{dv^\lambda}{dt}|^r\right)^{\nu/r}\left(\iint v^{\sigma-1}\left|\frac{dv}{dt}\right|\right)^{p-\nu}.$$

Finally, the last integral is estimated at every fixed time as

$$\frac{1}{\sigma}\int v^\sigma dx \le \frac{1}{\sigma}(\text{meas}K)^{1-\sigma}\left(\int v dx\right)^\sigma .$$

■

These calculations must be justified for general functions by approximation.

**Corollary 8.3** *Any non-negative weak solution of Problem* (5.1)–(5.3) *satisfies* $u_t \in L^p_{\text{loc}}(Q)$ *for any* $p \in [1,(m+1)/m)$.

**Proof** Again, we may restrict ourselves to classical solutions by approximation. If $u$ is the solution, then

$$v(x,t) = tu(x,t^{m-1})$$

satisfies the conditions of Lemma 8.2. Observe in particular that $v_t \ge 0$ is a consequence of (8.2). By estimate (5.41) of Theorem 5.7, we may take $\lambda = \frac{m+1}{2}$, $r = 2$, hence $p_1 = (m+1)/m$. ■

As a consequence of Proposition 5.12, Lemma 8.1, and Corollary 8.3 we have

**Corollary 8.4** *For any non-negative weak solution we have* $tu_t \in L^\infty(0,\infty : L^1(\Omega))$ *and*

$$\int u_t\, dx \le 0, \qquad t\|u_t(t)\|_1 \le \frac{2}{m-1}\|u_0\|_1. \tag{8.6}$$

**Proof** In case $u$ is smooth the first inequality follows from (5.50) for $p = 1$. Since $u_t = (u_t)^+ - (u_t)^-$, and $|u_t| = (u_t)^+ + (u_t)^-$, we have

$$\int (u_t)^+ dx \le \int (u_t)^- dx \text{ and } \int |u_t|\, dx = \int (|u_t^+| + |u_t^-|)\, dx \le 2\int |u_t^-|\, dx.$$

We now use (8.2) to obtain (8.6)-right. ■

**Remark** If $\Delta u_0^m$ is bounded below as in (8.17) the bound (8.6)-right for $u_t$ can be improved and $u_t \in L^\infty(0,\infty : L^1(\Omega))$. See Problem 8.1.

### 8.1.2 *Bound for $u_t$ for signed solutions*

There is also a weaker universal estimate for signed solutions

**Lemma 8.5** *All weak solutions of problem* HDP *for the PME have a distributional time derivative $u_t$ in the space $L^\infty((\tau,\infty) :\ \mathcal{M}(\mathbb{R}^d)$ for all $\tau > 0$ with a bound of the form*

$$\|u_t(t)\|_{\mathcal{M}(\mathbb{R}^d)} \le \frac{2\|u_0\|_1}{(m-1)t}. \tag{8.7}$$

**Proof** Arguing as in the second proof of the Lemma 8.1, if $u$ is a solution with data $u_0$ and $\lambda$ is a positive constant, then

$$\tilde{u}(x,t) = \lambda u(x,\lambda^{m-1}\, t)$$

is the solution with data $\tilde{u}_0(x) = \lambda u_0(x)$. Now fix $t$ and $h > 0$ and put $\lambda^{m-1}t = t + h$ so that $\lambda > 1$. Then,

$$u(x, t+h) - u(x,t) = \lambda^{-1}\tilde{u}(x,t) - u(x,t),$$

which can split into $(\lambda^{-1} - 1)\tilde{u}(x,t) + (\tilde{u}(x,t) - u(x,t))$. Using $L^1$ contractivity to estimate the last term, we get

$$\|u(x,t+h) - u(x,t)\|_1 \leq (\lambda^{-1} - 1)\|\tilde{u}_0\|_1 + (\lambda - 1)\|u_0\|_1. \tag{8.8}$$

In the limit $h \to 0$ (with $t$ fixed), we have $\lambda \to 1$ and $(\lambda - 1)/h \to 1/((m-1)t)$. Therefore, we get for $h \approx 0$

$$\|u(x,t+h) - u(x,t)\|_1 \leq \frac{2\|u_0\|_1}{(m-1)t}(h + o(h)). \tag{8.9}$$

This implies that $u_t$, the limit of the time-increment quotients, is a Radon measure, and satisfies estimate (8.7), since the norm $\|\cdot\|_1$ goes over in the limit to the norm in the space $\mathcal{M}(\Omega)$. ■

Actually, we know that for $t \geq \tau > 0$ our solutions are bounded and also $C^\infty$ on the set $\{u \neq 0\}$. There is a general result of measure theory that says that under such circumstances, if $u_t$ is a bounded Radon measure, then it must be a plain integrable function. Such a result is proved in a slightly more general form in Lemma A.2 for the reader's convenience. In this way we conclude

**Corollary 8.6** *The solutions of problem* HDP *for the PME have a distributional time derivative $u_t$ in the space $L^\infty((\tau,\infty) : L^1(\mathbb{R}^d))$ with the bound*

$$\|u_t(t)\|_1 \leq \frac{2\|u_0\|_1}{(m-1)t}. \tag{8.10}$$

**Remark** Estimates (8.2) and (8.7) lose their information as $t \to 0$. This is quite natural since the estimates are universal and the initial data need not be good.

## 8.2 Strong solutions

We take into account the regularity just proved to propose a new concept of solution that appears naturally in the literature. Let us begin by a general statement: a locally integrable function $u$ for which all the derivatives which appear in an equation are functions rather than distributions and such that the Fequation is satisfied a.e. in its domain is called a *strong* solution of that equation.

For equation (5.1) these requirements amount to the following:

(i) $u$, $\Phi(u)$, $u_t$, $\Delta\Phi(u) \in L^1_{\rm loc}(Q)$;
(ii) $u_t = \Delta\Phi(u) + f$ as locally integrable functions in $Q$ (i.e., almost everywhere).

This is the definition of strong solution for the PME posed in $Q$, where no reference to initial or boundary data is made.

A precise definition of strong solution for a problem, like (5.1)–(5.3), asks for functional spaces which allow to define in what sense the initial and boundary data are taken. Again, a convenient choice of spaces should allow both for existence for a suitable class of data and, on the other side, for uniqueness.

In our case, the estimates obtained in the previous subsection imply the following result when $\Phi(s) = |s|^{m-1}s$ and $f = 0$.

**Theorem 8.7** *For every $u_0 \in L^1(\Omega)$, the weak solution of the HDP for the PME is a strong solution in the following sense:*

(i) $u^m \in L^2(\tau,\infty : H_0^1(\Omega))$ *for every* $\tau > 0$*;*

(ii) $u_t$ *and* $\Delta u^m \in L^1_{\rm loc}(0,\infty : L^1(\Omega))$ *and* $u_t = \Delta u^m$ a.e. *in* $Q$*;*

(iii) $u \in C([0,T) : L^1(\Omega))$ *and* $u(0) = u_0$.

For brevity, we often write $u^m$ instead of $|u|^{m-1}u$. Conditions (i) and (iii) have been already established. As for (ii), we have even proved that $u_t \in L^\infty_{\rm loc}(0,\infty : L^1(\Omega)) \cap L^p_{\rm loc}(Q)$ for $0 < p < p_1$ if $u \ge 0$. Using equation (6.22) with $\eta \in C_c^\infty(Q)$, we conclude that $\nabla(u^m)$ has $u_t$ as its weak divergence, hence $\Delta(u^m) \in L^p_{\rm loc}(Q)$ with $p > 1$ as in Corollary 8.3. By standard theory, all the second spatial derivatives of $u^m$ belong to $L^p_{\rm loc}(Q)$. Moreover, the PME is satisfied in $Q$. ■

We give next a summary of the additional properties of the solution

**Theorem 8.8** *The strong solution of the above problem also satisfies:*

(i) $u \in L^\infty(Q^\tau)$ *and the* $L^\infty$ *bound* (5.57) *holds.*

(ii) $\nabla(u^\gamma) \in L^2(Q^\tau)$ *for every* $\gamma > m/2$ *and the bounds* (5.20), (5.50) *and* (5.51) *hold.*

(iii) $tu_t \in L^\infty(0,\infty : L^1(\Omega))$ *and the bounds* (5.47), (5.93) and (8.10) hold.

(iv) *If* $u_0 \ge 0$, *then* $u_t \in L^p_{\rm loc}(\Omega)$ *for* $1 \le p < p_1$ *and the bounds* (8.2) *and* (8.6) *hold.*

(v) *For every two solutions* $u, \widehat{u}$ *we have the contraction estimates* (6.1), (6.2). *In particular,* $u_0 \le \widehat{u}_0$ *implies* $u \le \widehat{u}$ *in* $Q$.

(vi) *For every* $t \ge \tau \ge 0$ *and every* $1 \le p \le \infty$ *we have* $\|u(t)\|_p \le \|u(\tau)\|_p$.

(vii) *If* $u_0 \in C(\Omega), u_0(x) > 0$ *for* $x \in Q$ *and* $u_0(x) = 0$ *for* $x \in \partial\Omega$, *then* $u$ *is a classical solution, positive in* $Q$.

**Remark** The condition $u \in C([0,\infty) : L^1(\Omega))$ does not look essential in the definition. Nevertheless, it is natural since we want to view our solution as a continuous curve in some functional space, in this case $t \in [0,\infty) \to u(t) \in L^1(\Omega)$. Anyway, in our case it does not mean any extra condition, since $\Phi(u) \in$

$L^2_{\rm loc}(0,\infty : H^1_0(\Omega))$ clearly implies $u \in L^2_{\rm loc}(0,\infty : L^1(\Omega))$ which together with $u_t \in L^1_{\rm loc}(0,\infty : L^1(\Omega))$ gives $u \in C((0,\infty) : L^1(\Omega))$. We make the assumption of continuity at $t = 0$ in order to satisfy the initial condition $u(0) = u_0$. ■

### 8.2.1 *The energy identity. Dissipation*

One of the benefits of the improved regularity of strong solutions is found in making (easier) proofs of results that depend on integration and/or taking limits. Let us prove here that strong solutions of the PME satisfy the energy identity that had been stated in formulas (5.20), (5.39) as inequality.

**Proposition 8.9** (Energy identity) *For the strong solution of Theorem* 8.7 *we have the energy identity*

$$(m+1)\iint_{Q_{12}} |\nabla u^m|^2\,dx\,dt + \int |u|^{m+1}(x,t_2)\,dx = \int |u|^{m+1}(x,t_1)\,dx \tag{8.11}$$

*where* $Q_{12} = \Omega \times (t_1, t_2)$ *and* $0 \le t_1 < t_2$. *This translates into the dissipation law*

$$\frac{d}{dt}\int_\Omega |u|^{m+1}\,dx = -(m+1)\int |\nabla u^m|^2\,ds, \tag{8.12}$$

*valid for* a.e. $t > 0$.

**Proof** For a classical solution the calculation is easy: we have

$$\partial_t(|u|^{m+1}) = (m+1)u^m u_t,$$

where we write $u^m$ instead of $|u|^m \mathrm{sign}(u)$. Then,

$$\frac{d}{dt}\int_\Omega |u|^{m+1}\,dx = (m+1)\int_\Omega u^m u_t\,dx = (m+1)\int_\Omega u^m \Delta u^m\,dx$$

and this equals $-(m+1)\int |\nabla u^m|^2\,ds$.

For a strong solution, since $u_t$ is integrable and $u$ is also bounded, the first displayed line is true for a.e. time. Then, we approximate $u^m$ by smooth functions $\varphi_n$ so that also the spatial gradient converges weakly in $L^2$ to $\nabla u^m$. We get for a.e. $t$

$$\int_\Omega \Delta u^m \varphi_n\,dx + \int_\Omega (\nabla u^m \cdot \nabla\varphi_n)\,dx = 0,$$

so that in the limit the equation

$$\iint u^m \Delta u^m h(t)\,dxdt + \iint (\nabla u^m \cdot \nabla u^m) h(t)\,dxdt = 0$$

holds for every test function $h(t)$ with compact support in $(0,T)$. We conclude that

$$\iint (u^{m+1})_t\,h(t)\,dxdt = -\iint |\nabla u^m|^2 h(t)\,dxdt.$$

Letting $h$ converge to the characteristic function of the interval $[t_1, t_2]$ we obtain (8.11). This easily gives (8.12) for a.e. time $t$. ■

We have taken as 'energy' the expression $\int |u|^{m+1}\,dx$ for convenience because of its simpler form. We could perform a similar calculation and obtain a dissipation formula for powers $\int |u|^p\,dx$ with $p > 1$, using as a starting point Proposition 5.12.

### 8.2.2 *Super- and subsolutions. Barriers*

The property of comparison enjoyed by the semigroup of weak solutions has an interesting technical consequence that is much employed in the theory, namely the possibility of getting estimates by using comparisons with functions that are not exact solutions but satisfy a suitable inequality. This has been done in a certain sense in Section 5.7 by considering problems with a forcing term and a boundary condition. Super- and subsolutions have been considered in Section 6.2 for the very weak theory, see specially Theorem 6.6.

**Definition 8.1**

(A) *A non-negative function $u \in C((0,\infty) : L^1(\Omega))$ is said to be a strong supersolution of the GPME in $Q = \Omega \times (0,T)$ if*

(i) $\Phi(u) \in L^2_{\rm loc}(0,\infty : H^1(\Omega))$, *and* $u_t \in L^1_{\rm loc}(0,\infty : L^1(\Omega))$;
(ii) *$u$ satisfies the inequalities*

$$\iint_Q \{\nabla\Phi(u)\cdot\nabla\eta + u_t\eta - f\eta\}\,dxdt \ge 0 \tag{8.13}$$

*for any function $\eta \in C_0^1(Q)$, $\eta \ge 0$.*

(B) *A strong supersolution of the GPME in $Q = \Omega \times (0,T)$ is said to be a strong supersolution of the homogeneous Dirichlet problem if $u \in C([0,\infty) : L^1(\Omega))$ and*

(iii) $u(0) \ge u_0$.

A similar definition applies to subsolutions.

**Definition 8.2**

(A) *A non-negative function $u \in C((0,\infty) : L^1(\Omega))$ is said to be a strong subsolution of the GPME in $Q = \Omega \times (0,T)$ if*

(i) $\Phi(u) \in L^2_{\rm loc}(0,\infty : H^1(\Omega))$, *and* $u_t \in L^1_{\rm loc}(0,\infty : L^1(\Omega))$;
(ii) *$u$ satisfies the inequalities*

$$\iint_Q \{\nabla\Phi(u)\cdot\nabla\eta + u_t\eta - f\eta\}dxdt \le 0 \tag{8.14}$$

*for any function $\eta \in C_0^1(Q)$, $\eta \ge 0$.*

(B) *A strong subsolution of the GPME in* $Q = \Omega \times (0, T)$ *is said to be a strong subsolution of Problem* (5.1)–(5.3) *if* $u \in C([0, \infty) : L^1(\Omega))$ *and*

(iii) $u(0) \le u_0$.

It is clear that any strong solution is at the time a strong supersolution and a strong subsolution. We recall that functions with the regularity of the above definitions have traces on the lateral boundary, $\Sigma = \partial\Omega \times (0, T)$ if the boundary is a smooth surface, and the trace $T_\Sigma(\Phi(u)) \in L^2(0, T : W^{1/2}(\partial\Omega)) \subset L^2(\Sigma)$. The weak solutions we have constructed for the homogeneous Dirichlet problem have trace zero, $Tr_\Sigma(\Phi(u)) = 0$. We have the following comparison result.

**Theorem 8.10** *It* $u$ *is a strong supersolution of problem HDP with data* $u_0$, $f$, *and* $v$ *is a strong subsolution of the same problem with data* $v_0$, $g$, *and* $u_0 \ge v_0$, $f \ge g$ *and we assume that the trace of* $u$ *on the lateral boundary* $\Sigma$ *is* a.e. *larger than the trace of* $v$. *Then, for every* $t > 0$ *we have* $u(t) \ge v(t)$ a.e. *in* $\Omega$.

Theorem 8.10 is a consequence of the following lemma.

**Lemma 8.11** *Given* $u$ *a strong supersolution of problem HDP with data* $u_0$, $f$, *and* $v$ *a strong subsolution of the same problem with data* $v_0$, $g$, *and assume that* $Tr_\Sigma(u) \ge Tr_\Sigma(v)$. *Then, we have*

$$\|[v(t) - u(t)]_+\|_{L^1(\Omega)} \le \|[v_0 - u_0]_+\|_{L^1(\Omega)}. \tag{8.15}$$

**Proof** We copy from the proof of Proposition 3.5, but now a prior approximation is used. Let $p \in C^1(\mathbb{R})$ be such that $0 \le p \le 1$, $p(s) = 0$ for $s < 0$, $p'(s) > 0$ for $s > 0$, and consider a sequence $w_n$ of $C^1$ approximations of $\Phi(v) - \Phi(u)$ in $L^2(\tau, T : H^1(\Omega))$. We may also ask that $w_n \le 0$ on $\Sigma$ and converges a.e. to $\Phi(v) - \Phi(u)$. Since $\nabla p(w_n) = p'(w_n)\nabla w_n$ and $p(w_n) = 0$ on $\Sigma$, we can take as test function $\eta = p(w_n)h(t)$ with $0 \le h \le 1$ smooth. We get

$$\iint (\nabla(\Phi(v) - \Phi(u)) \cdot \nabla p(w_n))\, h(t) dx dt - \iint (v - u)_t p(w_n(x, t)) h(t)\, dx dt \le 0,$$

Now, the first integral converges to

$$\iint |\nabla(\Phi(v) - \Phi(u))|^2 p'(\Phi(v) - \Phi(u))\, h(t) dx dt,$$

which is non-negative, while the second integral converges to

$$\iint (v - u)_t p(\Phi(u) - \Phi(v))\, h(t) dx dt.$$

Therefore, letting $p$ converge to the sign function $\operatorname{sign}_0^+$, we get

$$\iint (v - u)_t \operatorname{sign}_0^+ (\Phi(u) - \Phi(v))\, h(t) dx dt \le 0.$$

We now observe that $\mathrm{sign}_0^+(\Phi(u)-\Phi(v)) = \mathrm{sign}_0^+(u-v)$, and

$$\frac{d}{dt}[v-u]^+ = (v-u)_t \,\mathrm{sign}_0^+(v-u),$$

cf. [261]. Hence,

$$\iint \partial_t([v-u]_+)\,h(t)dxdt \le 0.$$

After some calculation this means that

$$\frac{d}{dt}\int [v-u]^+ dx \le 0,$$

which implies that $\|[v(t)-u(t)]_+\|_1$ is non-increasing in time. This proves the result. ∎

**Remark** See the definition of classical super- and subsolutions in Problem 8.5.

*Use as barriers*

The standard way of employing supersolutions is as follows. We want to estimate the behaviour of a strong solution $u$ of the PME in terms of its initial data $u_0$ ($f=0$). We construct a more or less explicit supersolution $u_1$ with equal or larger initial data. By Theorem 8.10, the explicit supersolution is on top of the solution, $u_1 \ge u$, and estimates from above for $u$ can be performed on $u_1$. We say that $u_1$ is an *upper barrier* for $u$. The reader will realize at this moment that the proof of the universal bound of Proposition 5.17 uses such a barrier argument.

Another easy application of the preceding theory happens when we consider the solutions of the PME in two different domains $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^d$, as in Section 6.3. It is immediate that any strong supersolution (subsolution) of the PME in a domain $\Omega_2$ is automatically a strong supersolution (subsolution) of the PME in a smaller domain $\Omega_1$. In the case of Problem HDP, the solution of the problem posed in $Q_2 = \Omega_2 \times (0,T)$ with initial data $u_{02}$ is automatically a supersolution of the problem posed in $Q_1 = \Omega_1 \times (0,T)$ with initial data $u_{01}$ such that $u_{01}(x) \le u_{02}(x)$ for $x \in \Omega_1$.

**Corollary 8.12** *In the above situation $\Omega_1 \subset \Omega_2$, let $u_1$ be the solution of the HDP posed in $Q_1$ with data $u_{01}$ and let $u_2$ be the solution of the HDP posed in $Q_2$ with data $u_{02}$. Then, we have*

$$u_1(x,t) \le u_2(x,t) \quad \textit{for every} \quad (x,t) \in Q_1. \tag{8.16}$$

The same type of comment applies to subsolutions used as *lower barriers*. The use of barriers and the corresponding construction of 'artificial approximate solutions' is a whole line of work for some very skillful specialists.

## Notes

**Section 8.2.1.** Some of the estimates are more or less classical in nonlinear parabolic equations.

The first proof of the control of $u_t$ from below in Lemmas 8.1 follows the proof of Caffarelli and Friedman in [138], while an argument close to the second proof was used in [145] in the study of the regularity of the Cauchy problem.

Lemma 8.5 is due to Bénilan and Crandall [89] using the clever homogeneity arguments that have wider applicability. Lemma 8.2 is due to Bénilan [82]. These estimates are crucial in establishing that the weak solution is strong.

**Section 8.2.** Strong solutions are the preferred choice in the works of many authors, like Bénilan, but note that they need a nicer equation than the usual rule in nonlinear filtration. A convenient reference is the paper by Bénilan and Gariepy [93], where the authors prove that any $L^\infty(Q)$ distributional solution $u$ of the initial value problem $u_t = \Delta\Phi(u) + \operatorname{div} F(u) + f$ on $Q \equiv (0,T) \times \Omega$ with $u(0,\cdot) = u_0$ is a strong $L^1_{\rm loc}(Q)$ solution. It is assumed that $u_0 \in L^\infty(\Omega)$, $f \in L^2_{\rm loc}(Q)$, $\Phi \in C^1(\mathbb{R})$, $F \in C^1(\mathbb{R}^d)$, $\Phi' > 0$ a.e., and there exists $\sigma \in C(\mathbb{R})$ such that $|F'|^2 \le \sigma\Phi'$.

## Problems

### Problem 8.1

(i) Extend estimate (8.2) to an $L^\infty$ estimate down to $t = 0$ if $u_0$ if $\Delta u_0^m$ is conveniently controlled from below. Prove that when

$$(m-1)\Delta u_0^m \ge -a u_0 \tag{8.17}$$

for some constant $a > 0$, then

$$u_t \ge -\frac{au}{(m-1)(at+1)}. \tag{8.18}$$

*Hint*: Compare the functions $z_1 = (m-1)(at+1)u_t + au$ and $z_2 = 0$: both are solutions of (8.4) in $Q$ and $z_1 \ge z_2$ on the parabolic boundary of $Q$. Hence, by the maximum principle which is again justified by approximation, we obtain $z_1 \ge z_2$, i.e., the desired estimate.

(ii) Show that condition (8.17) is implied for instance by the pressure bound

$$m\Delta u_0^{m-1} \ge -a.$$

(iii) Show that under the above conditions, the bound (8.6)-right for $u_t$ can be improved and $u_t \in L^\infty(0,\infty : L^1(\Omega))$.

**Problem 8.2**

(i) Show that the lower bound (8.2) can be obtained for the solutions of the Dirichlet problem for the PME with $f, g \neq 0$ if

$$mf + (m-1)tf_t \geq 0, \quad g + (m-1)tg_t \geq 0$$

in the sense of distributions in their respective domains.

(ii) Show that $u_0 \geq 0$ and $\Delta u_0^m \leq 0$ imply that $u_t \leq 0$ in $Q_T$ when the forcing and boundary data vanish, $f = g = 0$.

*Hint*: Approximate by smooth solutions and write the equation for $u_t$.

**Problem 8.3** Prove that formulas (5.51) are satisfied as identities for the solutions of the PME. Derive the dissipation formula

$$\frac{d}{dt}\int_\Omega u^{q+1}\,dx = -\frac{4q(q+1)m}{(q+m)^2}\int \left|\nabla\left(u^{\frac{q+m}{2}}\right)\right|^2 dx. \tag{8.19}$$

for a.e. time $t > 0$.

**Problem 8.4** Combine estimates (5.94) and (5.20) we get estimates of the left-hand side of (5.94) in terms of $\int u_0^{m+1}dx$. Thus,

$$\iint_Q t\left|\frac{d}{dt}(u^{(m+1)/2})\right|^2 dxdt \leq \frac{m+1}{8m}\int_\Omega u_0^{m+1}(x)\,dx.$$

Conclude that

$$\iint_{Q^\tau} \left|\frac{d}{dt}(u^{(m+1)/2})\right|^2 dxdt \leq \frac{m+1}{8m\tau}\int_\Omega u(x,\tau)^{m+1}(x)\,dx.$$

**Problem 8.5** Assume that $u \geq 0$ is a continuous function in $Q$ such that $u_t$ and $\Delta u^m$ are continuous, and satisfies

$$u_t \geq \Delta u^m$$

in $Q$. Show that it is a strong supersolution of the PME. If $u$ is continuous at $t = 0$ with initial trace $u(x,0) \geq u_0(x)$, show that $u$ is a supersolution of the Problem (5.1)–(5.3). This is called a classical supersolution.

State a similar result for classical subsolutions and prove it.

**Problem 8.6** The estimates for $u_t$ of Section 8.1 apply to the FDE since they can be proved by scaling methods that rely on the power nonlinearities with exponent $m \neq 1$; precisely the linear case is excluded!

(i) Prove Lemma 8.1 in the form

$$u_t \leq \frac{u}{(1-m)t}. \tag{8.20}$$

(ii) Prove the analogous to Lemma 8.5.

**Open problem** Does the dissipation law (8.12) hold for all times?

# 9

# THE CAUCHY PROBLEM. $L^1$-THEORY

This chapter is devoted to study the Cauchy problem, or pure initial value problem, for the PME and the GPME in $d$-dimensional space, $d \geq 1$. In order to focus on our main objective, we concentrate the main effort on the PME with zero forcing term. We will work with integrable initial data, $u_0 \in L^1(\mathbb{R}^d)$. Consequently, we consider solutions which are integrable with respect to the space variables, so-called *solutions with finite mass*, and develop the corresponding $L^1$ theory. We will establish well-posedness for the Cauchy problem in this setting, which is the one most often found in the literature and the applications.

We pose the problem in the class of strong solutions in Section 9.1 and prove first uniqueness and $L^1$-stability.

We address in Section 9.2 the existence of non-negative solutions for non-negative initial data, which is a standard restriction in the applications. The problem enjoys then some nice properties absent in the signed case. We investigate these properties, among them the fundamental estimate in Section 9.3, and the boundedness of the solutions for $t \geq \tau > 0$ in Section 9.4. The contents up to this moment is absolutely basic material, whose careful study is required.

We solve the problem of existence of signed solutions in Section 9.5. We prove the conservation of mass in Subsection 9.5.1, and special properties in Section 9.6. A main physical property of the PME is the finite propagation property. The topic is presented in Subsection 9.6.3. It allows us to introduce a main geometrical object, the free boundary, that will be the main object of study of Chapter 14. This is also basic material, required in the sequel.

Many ideas and estimates are common to the study of the Dirichlet problem in previous chapters. The main new feature arising in the study of the Cauchy problem lies in the fact that we have to take into account the behaviour of the data and solutions as $|x| \to \infty$. In some sense, working with data and solutions with finite mass is a way of expressing that the solutions and data are *small at infinity*. However, the mathematical theory is concerned with more general data which may grow at infinity. Such a study is left for the advanced Chapters 12 and 13.

We devote Section 9.7 to extend the results to the homogeneous Dirichlet problem posed in a possibly unbounded subdomain of $\mathbb{R}^d$, thus completing the theory developed in Chapters 5, 6, and 8.

Keeping with the spirit of previous chapters, we turn our attention at the end of the chapter to the Cauchy problem for the GPME in Section 9.8. We still work in the $L^1$ framework. The idea is to obtain solutions of the Cauchy problem as limits of solutions of the Dirichlet problem posed in a bounded domain, e.g., a ball, and then let the bounded domain tend to the whole of $\mathbb{R}^d$. Many of the ideas of previous chapters are used in this setting.

In this chapter we will use the symbols $Q = \mathbb{R}^d \times \mathbb{R}_+$ and $Q_T = \mathbb{R}^d \times (0, T)$. An important constant of the theory appears in the estimates, $\alpha = d/(d(m-1)+2)$. Even if solutions have negative values we often write $u^m$ instead of $|u|^{m-1}u$ for the sake of brevity. Sections 9.7 and 9.8 can be skipped in a first reading.

## 9.1 Definition of strong solution. Uniqueness

Let us consider the initial value problem, CP:

$$\begin{cases} u_t &= \Delta(|u|^{m-1}u) \quad \text{in } Q \\ u(x,0) &= u_0(x) \qquad\qquad \text{for } x \in \mathbb{R}^d, \end{cases} \tag{9.1}$$

where $m > 1$ and $u_0 \in L^1(\Omega)$. We will pay special attention to the case $u_0 \geq 0$ that produces solutions $u \geq 0$. No difficulties arise in restricting time to the interval $0 \leq t \leq T$ and replacing $Q$ by $Q_T$. Following the motivation of Chapter 8, we will first give a suitable definition of strong solution for our initial value problem and then prove existence, uniqueness and a series of basic properties of such solutions.

**Definition 9.1** *We say that a function $u \in C([0,\infty) : L^1(\mathbb{R}^d))$ is a strong $L^1$ solution of problem* (9.1) *if*

(i) $/\ |u|^{m-1}u \in L^1_{\text{loc}}(0,\infty : L^1(\mathbb{R}^d))$ *and* $u_t, \Delta(|u|^{m-1}u) \in L^1_{\text{loc}}(Q)$*;*
(ii) $u_t = \Delta(|u|^{m-1}u)$ a.e. *in* $Q$*;*
(iii) $u(t) \to u_0$ *as* $t \to 0$ *in* $L^1(\mathbb{R}^d)$*.*

Equivalently, we could have said in (ii) that $u_t = \Delta(u^m)$ in the sense of distributions in $\mathcal{D}(Q)$. In the rest of the chapter strong solution will always mean strong $L^1$-solution. Our first step in the study of strong solutions will be to establish the crucial $L^1$-order-contraction property, similar to Proposition 3.5.

**Proposition 9.1** *Let $u_1, u_2$ be two strong solutions of Problem* (9.1) *in $Q_T$. For every $0 < t_1 < t_2$ we have*

$$\int [u_1(x,t_2) - u_2(x,t_2)]_+ dx \leq \int [u_1(x,t_1) - u_2(x,t_1)]_+ dx. \tag{9.2}$$

**Proof** Let $p \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be such that $p(s) = 0$ for $s \leq 0$, $p'(s) > 0$ for $s > 0$ and $0 \leq p \leq 1$, and let $j(r) = \int\limits_0^r p(s)ds$ be a primitive of $p$. We will choose $p$ as an approximation to the sign function

$$\text{sign}_0^+(r) = 1 \text{ if } r > 0, \quad \text{sign}_0^+(r) = 0 \text{ if } r \leq 0, \tag{9.3}$$

hence $j$ will approximate the function $s \mapsto [s]_+$. Moreover, consider a cut-off function $\zeta_1 \in C_c^\infty(\mathbb{R}^d)$ such that $0 \le \zeta_0 \le 1$, $\zeta_1(x) = 1$ if $|x| \le 1$, $\zeta_1(x) = 0$ if $|x| \ge 2$ and let $\zeta = \zeta_n(x) = \zeta_1(x/n)$. As $n \to \infty, \zeta_n \uparrow 1$.

We subtract the equations satisfied by $u_1$ and $u_2$, multiply by $\eta = p(u_1^m - u_2^m)\,\zeta$ and integrate on $S = \mathbb{R}^d \times [t_1, t_2]$ to obtain, with $w = u_1^m - u_2^m$,

$$\iint (u_1 - u_2)_t\, p(w)\zeta = \iint \Delta w\, p(w)\zeta. \tag{9.4}$$

Now, approximate $w$ by means of a smooth kernel sequence $\rho_n$. If $w_n = w * \rho_n$ (here, $*$ denotes convolution) we have $w_n \to w$, $\nabla w_n \to \nabla w$ and $\Delta w_n \to \Delta w$ in $L^1_{\rm loc}(Q)$ and almost everywhere for a subsequence, so that $p(w_n) \to p(w)$ a.e. Moreover,

$$\iint p(w_n)\Delta w_n \zeta + \iint p'(w_n)|\nabla w_n|^2\zeta + \iint p(w_n)\nabla w_n \cdot \nabla\zeta = 0.$$

We observe that the second integral is uniformly bounded above, since the first and the third are uniformly bounded. Letting $n \to \infty$ we get by Fatou's lemma

$$\iint p'(w)|\nabla w|^2\zeta \le -\iint p(w)\Delta w\zeta - \iint p(w)\nabla w \cdot \nabla\zeta. \tag{9.5}$$

Hence, returning to (9.4) we get

$$\begin{aligned}\iint (u_1 - u_2)_t\, p(w)\zeta &\le -\iint p'(w)|\nabla w|^2\zeta - \iint p(w)\nabla w \cdot \nabla\zeta \\ &\le -\iint p(w)\nabla w \cdot \nabla\zeta = -\iint \nabla j(w)\cdot\nabla\zeta \qquad (9.6)\\ &= \iint j(w)\Delta\zeta \le \iint |w|\ |\Delta\zeta|,\end{aligned}$$

where integration is understood on $Q_T$. Letting now $p$ tend to $\text{sign}_0^+$ and observing that our regularity justifies the formula $\dfrac{d}{dt}[u_1 - u_2]_+ = \text{sign}_0^+(u_1 - u_2)\frac{d}{dt}$ $(u_1 - u_2)$, we get after performing the time integration,

$$\begin{aligned}\int [u_1(x,t_2) - u_2(x_1,t_2)]_+\zeta\, dx &\le \int [u_1(x,t_1) - u_2(x,t_1)]_+\zeta\, dx \\ &\quad + \|\Delta\zeta\|_\infty \iint\limits_{S\cap\{|x|>n\}} |w(x,t)|\, dx\, dt.\end{aligned} \tag{9.7}$$

We let now $n \to \infty$ to obtain (9.2), since $w \in L^1(t_1, t_2 : L^1(\mathbb{R}^d))$ and $\|\Delta\zeta_n\|_\infty = \|\Delta\zeta_1\|_\infty/n^2$. ■

**Remark** The proof of Proposition 9.1 actually uses the following requirements on $u$ and $u^m$: $u \in C([0,\infty) : L^1_{\rm loc}(\mathbb{R}^d))$, $u^m \in L^1_{\rm loc}(Q)$ and

$$\iint_{S_n} (u_1^m - u_2^m)_+(x,t)\,dxdt = o(n^2) \text{ as } n \to \infty, \tag{9.8}$$

where $S_n = \{(x,t) : n \le |x| \le 2n, \quad t_1 \le t \le t_2\}$ with $0 < t_1 < t_2$, which are weaker than our Definition 9.1. Therefore, Proposition 9.1 also holds under the above hypotheses, if (9.8) holds uniformly for $0 \le t \le t_2$ and if the initial data are taken continuously in $L^1_{\rm loc}(\mathbb{R}^d)$. We shall use this remark later on.

Again, as in Chapter 5, we obtain *uniqueness* and *comparison* as simple consequences of this result.

**Theorem 9.2** *Problem* (9.1) *has at most one strong solution. If $u_1, u_2$ are strong solutions with initial data $u_{01}, u_{02}$ resp. and $u_{01} \le u_{02}$ are in $\mathbb{R}^d$, then $u_1 \le u_2$* a.e. *in $Q$. In particular, if $u_{01} = u_{02}$* a.e. *then $u_1 = u_2$* a.e. *The map $u_0 \mapsto u(t)$ is an ordered contraction in $L^1(\mathbb{R}^d)$ (wherever defined, see below).*

**Examples** Many of the examples mentioned so far in Chapters 4 and 5 are examples in the new context, but not all.

(1) Though the source-type solution $U(x,t)$ fails to be a strong solution of the Cauchy problem because of the singularity of its initial data, any time-delayed version $u(x,t) = U(x, t+\tau)$ with $\tau > 0$ is indeed a strong solution. Moreover, $U \ge 0$.

(2) The dipole solution $U_d(x,t)$ of formula (4.53) is an example of one-dimensional signed solution with finite mass, once a proper delay is inserted to account for the singularity of the initial data. The signed, compactly supported solutions constructed in Subsection 4.6.3 are further examples of signed solutions of the PME.

(3) The constant functions $u(x,t) = c$ are strong solutions at the local level, but when $c \ne 0$ they fail to satisfy the finite mass criterion. They will be included in the extended theory of Chapter 12. Also the travelling waves of Section 4.3 will be included at that moment.

## 9.2 Existence of non-negative solutions

We proceed next with the construction of non-negative solutions. We start by the case of bounded initial data by using an approximation process and the results of the previous chapters. The existence result for general initial data in $L^1(\mathbb{R}^d)$ will follow once we show that every solution is bounded for $t \ge \tau > 0$, which will be done in Section 9.4.

**Theorem 9.3** *For every non-negative function $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ there exists a strong solution $u \ge 0$ of Problem* (9.1). *Moreover, $u_t \in L^p_{\rm loc}(Q)$ for*

*$1 \le p < (m+1)/m$ and*

$$u_t \ge -\frac{u}{(m-1)t} \quad in \quad \mathcal{D}'(Q), \tag{9.9}$$

$$\|u_t(\cdot,t)\|_1 \le \frac{2\|u_0\|_1}{(m-1)t}. \tag{9.10}$$

*If $u_0 \in L^p(\mathbb{R}^d)$ for $1 \le p \le \infty$, then $u(t) \in L^p(\mathbb{R}^d)$ and*

$$\|u(t)\|_p \le \|u_0\|_p. \tag{9.11}$$

*Moreover, the map $u_0 \mapsto u(t)$ is an ordered contraction in $L^1(\mathbb{R}^d)$.*

**Proof** (i) We begin by assuming that $u_0$ is not only bounded and integrable over $\mathbb{R}^d$, but also that it is strictly positive, $C^\infty$ smooth and all its derivatives are bounded in $\mathbb{R}^d$. Finally, (8.17) holds. Under these conditions we construct a strong and classical solution. For that we consider the Cauchy-Dirichlet problems

$$(P_n) \quad \begin{cases} u_t \quad = \Delta(u^m) & \text{in} \quad Q_n = B_n(0) \times (0,\infty), \\ u(x,0) = u_{0n}(x) & \text{for} \quad |x| \le n, \\ u(x,t) \;= 0 & \text{for} \quad |x| = n,\ t \ge 0, \end{cases}$$

where $u_{0n} = u_0\zeta_n$, $\{\zeta_n\}$ being a cut-off sequence with the following properties: $\zeta_n \in C^\infty(\mathbb{R}^d)$, $\zeta_n(x) = 1$ for $|x| \le n-1$, $\zeta_n(x) = 0$ for $|x| \ge n$, $0 < \zeta_n(x) < 1$ for $n-1 < |x| < n$, the derivatives of the $\zeta_n$ up to second order are bounded uniformly in $x \in \mathbb{R}^d$, and $n \ge 2$. Finally, $\Delta\zeta_n^{m-1}$ is uniformly bounded below.

By the results of Chapter 5 (Theorem 5.5 and Proposition 7.21), $(P_n)$ admits a unique classical solution $u_n \in C^\infty(Q_n) \cap C(\overline{Q}_n)$ and $u_n > 0$ in $Q_n$. In particular, $u_{n+1}$ will be a classical solution of the PME in $Q_n$ with positive boundary data and initial data larger than $u_{0n}$. We conclude from the classical maximum principle that $u_{n+1} \ge u_n$ in $Q_n$, i.e., the sequence $\{u_n\}$ is monotone. Moreover, we get from the two previous chapters uniform estimates for

(a) $\{u_n\}$ in $L^\infty(0,\infty : L^p(B_n(0)))$, $1 \le p \le \infty$;
(b) $\{(u_n)_t\}$ in $L^\infty(0,\infty : L^1(B_n(0))) \cap L^p_{\text{loc}}(Q_n)$ for $1 \le p < p_1$;
(c) $\{u_n^m\}$ in $L^2(0,\infty : H_0^1(B_n(0)))$.

Since all of these estimates involve bounds which are independent of $n$, we may pass to the limit $n \to \infty$ and obtain a positive function $u \in L^\infty(0,\infty : L^p(\mathbb{R}^d))$ for every $p \in [1,\infty)$, such that $u_t, u^m, \Delta u^m$ belong to the same spaces to which $(u_n)_t, u_n^m, \Delta(u_n^m)$ belonged, and equation (9.1) holds in $Q$.

To check the smoothness of $u$, we first observe that in a neighbourhood $N \subset \overline{Q}$ of any point $(x_0,t) \in \overline{Q}$, $u_n(x,t)$ is defined and positive, say $u_n(x,t) \ge c > 0$ for every $(x,t) \in N$ if $n > n_0$. Since the sequence $\{u_n\}$ is monotone non-decreasing and bounded, the interior regularity theory for uniformly parabolic quasilinear equations gives uniform bounds for all the derivatives of $u_n, n \ge n_0$, in a smaller neighbourhood of $(x_0, t)$. In the limit we conclude that $u \in C^\infty(\overline{Q})$. Moreover for $t = 0$ we get $u(x,t) = u_0(x),\ x \in \mathbb{R}^d$.

We have proved that, under the present assumptions, $u$ is classical solution of Problem (9.1). To comply with our definition of strong solution, we still have to check the continuity of $u = u(t)$ as a map from $[0, \infty)$ into $L^1(\mathbb{R}^d)$. It is a consequence of the fact that $u \in L^\infty(0, \infty : L^1(\mathbb{R}^d))$ (for instance, by (5.50)) and $u_t \in L^\infty(0, \infty : L^1(\mathbb{R}^d))$ (cf. Remark to Corollary 8.4) so that $u$ is absolutely continuous from $[0, \infty)$ into $L^1(\mathbb{R}^d)$.

Estimates (9.9), (9.10), (9.11) are an easy consequence of similar estimates for the Cauchy-Dirichlet problem after passing to the limit. In particular, we have $0 \le u(x,t) \le \|u_0\|_\infty$.

(ii) If $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ does not fulfill the above requirements, we approximate it by a sequence $\{u_{0n}\}$ of such functions. We may always do in such a way that $\|u_{0n}\|_1 \le \|u_0\|_1$, $\|u_{0n}\|_\infty \le \|u_0\|_\infty$, $u_{0n} \to u_0$ in $L^1(\mathbb{R}^d)$. Let $u_n$ be the solution with data $u_{0n}$. It follows from Proposition 9.1 that $u_n$ converges in $C([0,\infty) : L^1(\mathbb{R}^d))$ to a function $u$ and $u(0) = u_0$.

Again estimates (a), (b), (c) of the previous step will hold uniformly in $n$ so that passing to the limit $n \to \infty$ produces a strong solution of (9.1), which satisfies the estimates (9.9), (9.10), (9.11). ■

## 9.3 The fundamental estimate for the CP

Perhaps the most significant novelty of the Cauchy problem (with data $u_0 \ge 0$) is the existence of a lower bound for the Laplacian of the pressure. Indeed, we have

**Proposition 9.4** *Let $v = mu^{m-1}/(m-1)$. Then,*

$$\Delta v \ge -\frac{\alpha}{t} \qquad \textit{with} \qquad \alpha = \frac{d}{d(m-1)+2}. \tag{9.12}$$

The inequality is understood in the sense of distributions in $Q$. This bound has been used so often in the theory of non-negative solutions in the whole space (which is the most treated theory) that we consider it the fundamental estimate for the Cauchy problem. It is usually known as the Aronson–Bénilan estimate after its authors. Let us also remark that (9.12) is optimal in the sense that equality is actually attained by the source-type or ZKB solutions, which are a kind of worst case with respect to this bound, a fact which has interesting consequences.

**Proof** (i) The formal derivation of the estimate is very simple. We first write the PDE satisfied by the pressure $v$, i.e.,

$$v_t = (m-1)v\Delta v + |\nabla v|^2. \tag{9.13}$$

Then we write the equation satisfied by $p = \Delta v$ by differentiating (9.13) twice. We have

$$p_t = (m-1)v\Delta p + 2m\nabla v \cdot \nabla p + (m-1)p^2 + 2\sum_{i,j}\left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)^2 .$$

Since

$$\sum_{i,j}(a_{ij})^2 \geq \sum_i (a_{ii})^2 \geq \frac{1}{d}\left(\sum_i a_{ii}\right)^2 ,$$

we get

$$\mathcal{L}(p) \equiv p_t - (m-1)v\Delta p - 2m\nabla v \cdot \nabla p - \left(m - 1 + \frac{2}{d}\right)p^2 \geq 0.$$

Here $\mathcal{L}$ is a quasilinear parabolic operator with smooth variable coefficients, since we consider $v$ as a given function of $x$ and $t$. We now apply $\mathcal{L}$ to the trial function

$$P(x,t) = -\frac{C}{t+\tau} \tag{9.14}$$

and observe that $\mathcal{L}(P) \leq 0$ if and only if $C \geq \alpha = 1/\left(m - 1 + (2/d)\right)$. We fix $C = \alpha$. By choosing $\tau$ small enough we may also obtain

$$p(x,0) \equiv \Delta v(x,0) \geq P(x,0) \equiv -\frac{C}{\tau}, \tag{9.15}$$

from which the classical maximum principle should allow us to conclude that $p \geq P$ in $Q$. Letting $\tau \to 0$ we would then obtain a pointwise inequality $\Delta v \geq -\alpha/t$.

(ii) The application of the maximum principle is justified when considering classical solutions of (9.13) such that $v, \nabla v$ and $p = \Delta v$ are bounded and $v$ is bounded below away from 0 so that the equation is uniformly parabolic. Therefore, we need to construct *new approximate solutions*. This we do as follows. We may always restrict ourselves to initial data $u_0$ which are bounded, smooth and positive, thanks to Proposition 9.1. Consider now initial data

$$u_{0\varepsilon}(x) = u_0(x) + \varepsilon, \quad \varepsilon > 0. \tag{9.16}$$

According to [357], there exists exactly one function $u_\varepsilon \in C^\infty(\overline{Q})$ that solves (9.1) with initial data $u_{0\varepsilon}$, and $\varepsilon \leq u_\varepsilon \leq M + \varepsilon$, where $M = \|u_0\|_\infty$. Moreover, by interior regularity results all the derivatives of $u_\varepsilon$ are bounded in $Q$. In particular, equation (9.1) is uniformly parabolic on $u_\varepsilon$. It follows that the fundamental estimate (9.12) holds for $v_\varepsilon$, the pressure of $u_\varepsilon$.

Now, if we prove that $v_\varepsilon \to v$ as $\varepsilon \to 0$ in $L^1_{\text{loc}}(Q)$, then (9.12) will still hold in the limit for $v$, though only in distribution sense, i.e.

$$\iint \left(v\Delta\varphi - \frac{\alpha}{t}\varphi\right) dxdt \geq 0 \tag{9.17}$$

for every $\varphi \in C_c^\infty(Q)$, $\varphi \geq 0$. Therefore, the proof is complete with the following convergence result. ■

**Lemma 9.5** *As* $\varepsilon \to 0$ $u_\varepsilon \to u$ *locally uniformly in* $Q$.

**Proof** The result is a consequence of the general theory to be developed later in Chapter 12. However, we will give an ad hoc proof at this point for the reader's convenience. We first observe that, by the maximum principle, the family $\{u_\varepsilon\}$ is non-increasing as $\varepsilon \downarrow 0$. It is also easy to establish that every $u_\varepsilon$ is above the solution $u$ with initial data $u_0$ (*Hint*: compare $u_\varepsilon$ with the approximations $u_n$ to $u$ constructed in step 1 of Theorem 5.5 in the domain $Q_n$ and let $n \to \infty$). Since $u$ is strictly positive in $\overline{Q}$ and $u_\varepsilon \geq u$, and thanks again to the interior regularity results, not only $\{u_\varepsilon\}$ converges to a function $\widehat{u}$, but also the derivatives converge, so that $\widehat{u}$ is a $C^\infty$ solution of Problem (9.1) in $Q$, $\widehat{u}(\cdot, 0) = u_0$ and $\widehat{u} \geq u$. ■

To conclude that $\widehat{u} = u$ we still need some control of $u^m$ as $|x| \to \infty$, as in (9.8), to be able to apply Theorem 9.2. We use the following result

**Lemma 9.6** *For every* $\varepsilon$ *and* $t > 0$ *we have*

$$\int (u_\varepsilon(x,t) - \varepsilon)\, dx \leq \int u_0(x)\, dx. \tag{9.18}$$

**Proof** Formally, we have $\int u_{\varepsilon,t}\, dx = \int \Delta u_\varepsilon^m\, dx = 0$, hence

$$\int (u_\varepsilon(x,t) - \varepsilon)\, dx = \int (u_{0\varepsilon}(x) - \varepsilon)\, dx = \int u_0(x)\, dx.$$

More rigourously, we approximate $u_\varepsilon$ with the solution $u_{\varepsilon n}$ of the following Cauchy-Dirichlet problem

$$\begin{cases} u_t = \Delta(u^m) & \text{in } Q_n \\ u(x,0) = u_{0n}(x) + \varepsilon & \text{for } |x| \leq n \\ u(x,t) = \varepsilon & \text{for } |x| = n \text{ and } t \geq 0, \end{cases}$$

for which we argue as in Chapter 5 and get a contraction formula as (6.1), which we apply to $u_{\varepsilon n}$ and $\widehat{u}_n = \varepsilon$ to get (9.18) for $u_{\varepsilon n}$. Letting $n \to \infty$ we obtain that $u_{\varepsilon n}$ converges (the sequence is compact by the interior regularity theory) to a solution of (9.1) which is $u_\varepsilon$ by uniqueness. In the limit (9.18) holds.

Going back now to the main argument, we let $\varepsilon \to 0$ to obtain

$$\int \widehat{u}(x,t)dx \leq \int u_0(x)dx.$$

It follows that $\widehat{u}(t) \in L^\infty(0,\infty : L^1(\mathbb{R}^d)) \cap L^\infty(Q)$, hence by the Remark to Proposition 9.1 we conclude that $\widehat{u} = u$ in $Q$. This ends the proof of the fundamental estimate. ■

Estimate (9.12) is exact for the ZKB solutions. It implies the following improvement of (9.9), (9.10).

**Corollary 9.7** $u_t \in L^\infty_{\rm loc}(0,\infty : L^1(\mathbb{R}^d))$ *and*

$$u_t \quad \ge -\frac{\alpha u}{t} \qquad in \quad \mathcal{D}'(Q), \tag{9.19}$$

$$t\|u_t\|_1 \qquad \le 2\alpha\|u_0\|_1. \tag{9.20}$$

**Proof** The first inequality is a consequence of

$$v_t = (m-1)v\Delta v + |\nabla v|^2 \ge (m-1)v\Delta v,$$

together with $v_t/v = (m-1)u_t/u$ and (9.12). For the second one argue as in Corollary 8.4. Again the calculations are justified for smooth solutions and hold in the limit for every solution. ■

## 9.4 Boundedness of the solutions

We are now in a position to prove that all solutions are bounded for $t \ge \tau > 0$, the so-called $L^1$–$L^\infty$ *smoothing effect*. The proof is not so easy as in the Dirichlet problem of Chapter 5; compare with Proposition 5.17.

**Proposition 9.8** *For every $t > 0$ we have*

$$u(x,t) \le C\,\|u_0\|_1^\sigma\, t^{-\alpha}, \tag{9.21}$$

*where $\sigma = 2/(d(m-1)+2)$, $\alpha = d/(d(m-1)+2)$ and $C > 0$ depends only on $m$ and $d$. The exponents are sharp.*

The result will be derived as a consequence of the fundamental estimate (9.12), thanks to the following result.

**Lemma 9.9** *Let $g$ be any non-negative, smooth, bounded and integrable function in $\mathbb{R}^d$ such that*

$$\Delta(g^{m-1}) \ge -K \tag{9.22}$$

*for some $m > 1$ and $K > 0$. Then $g \in L^\infty(\mathbb{R}^d)$ and $\|g\|_\infty$ depends only on $m$,$K$, $d$ and $\|g\|_1$ in the form*

$$\|g\|_\infty \le C(m,d)\,\|g\|_1^\rho\, K^\sigma, \tag{9.23}$$

*with $\rho = 2/(2+d(m-1))$ and $\sigma = d/(2+d(m-1))$.*

For a proof of this calculus lemma see Section A.8. Given the result, it suffices to fix $t > 0$, and put $g(x) = u(x,t)$ and $K = \alpha(m-1)/mt$ (see formula (9.12)) to obtain Proposition 9.8 in the case where the solution $u$ is positive everywhere, hence smooth. The general case is done by approximation.

Formula (9.21) not only asserts that solutions with $L^1$ data are bounded for positive times, but also gives a very precise quantitative estimate of the bound. In fact, the exponents appearing in the formula can be derived from the general boundedness statement thanks to a scaling argument. Since this kind of argument has wider applicability, we give here a proof of this implication.

**Lemma 9.10** *Suppose that for all solutions of the PME with $\|u_0\|_1 \le 1$ we have at $t = 1$ the uniform bound $\|u(\cdot, 1)\|_\infty \le C$ with $C = C(m, d) > 0$. Then* (9.21) *necessarily holds.*

**Proof** Let $u$ be any solution of (9.1) with $\|u_0\|_1 = M > 0$. Now, if we consider the rescaled function

$$\widehat{u}(x, t) = Ku(Lx, Tt),$$

with constants $K, L, T > 0, \widehat{u}$ is again a solution of (9.1) if

$$K^{m-1}L^2 = T.$$

On the other hand, $\|\widehat{u}_0\|_1 = 1$ if

$$KM = L^d.$$

Both equalities are satisfied for $T$ arbitrary, $K = M^{-\sigma}T^\alpha$, $L = (M^{m-1}T)^\beta$ with $\beta = \alpha/d$. Under these conditions our assumptions say that $\widehat{u}(x, 1) \le C$. Then,

$$u(x, T) = K^{-1}\widehat{u}(Lx, 1) \le C/K = CM^\sigma T^{-\alpha}. \qquad \blacksquare$$

It is interesting to remark that if we calculate the decay rate of the Barenblatt solution in the sup norm, we find that formula (9.21) holds with a certain precise constant. We will show in Chapter 17 that the constant corresponding to the Barenblatt solution is the *optimal* constant in inequality (9.21). This means that the Barenblatt solutions solve an extremal problem, that of maximizing $\sup_x u(x, t)$ for given $t > 0$ and given $\|u_0\|_1 = M$.

We point out that these arguments are quite different from the boundedness proof in Chapter 5, compare with Proposition 5.7. The same techniques (or interpolation) can be used to prove a more general version of the smoothing effect:

**Proposition 9.11** *For every $t > 0$ and $1 \le p < q \le \infty$ we have*

$$\|u(t)\|_q \le C\|u_0\|_p^\gamma t^{-\sigma} \tag{9.24}$$

*whenever $u_0 \in L^p(\mathbb{R}^d)$. The constants $C, \gamma$ and $\sigma$ depend only $m, p, q$ and $d$.*

We leave it to the reader to fill in the details and also to calculate the explicit values of $\gamma$ and $d$, which are given again by a scaling argument. See also [515], Chapters 2, 3.

At this stage we can complete the proof of existence of a non-negative solution for every $u_0 \in L^1(\mathbb{R}^d)$, $u_0 \ge 0$, using the fact that all the approximations are uniformly bounded functions for $t \ge \tau > 0$. But actually, we may as well address the problem of existence without any sign restriction.

## 9.5 Existence with general $L^1$ data

We study now the existence of solutions without the sign restriction on the data or solutions.

**Theorem 9.12** *For every $u_0 \in L^1(\mathbb{R}^d)$ there exists a unique strong solution $u$ of Problem* (9.1) *such that $u \in C([0,\infty) : L^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d \times (\tau,\infty))$ for every $\tau > 0$. The solution satisfies estimates* (9.10) and (9.11)*; $|u|$ satisfies the $L^\infty$ estimate*

$$|u(x,t)| \le C\,\|u_0\|_1^\sigma\, t^{-\alpha}, \tag{9.25}$$

*with $C, \sigma, \alpha$ as in Proposition* 9.8.

**Proof** (i) We approximate $u_0$ with a sequence of functions $u_{0n} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ converging to $u_0$, say

$$u_{0n}(x) = \max(-n, \min(u_0(x), n))\,\chi_{B_n(0)}(x). \tag{9.26}$$

We may apply to those data the results of existence of solutions of the homogeneous Cauchy-Dirichlet problem in bounded domains $\Omega = B_{2n}(0)$ derived in previous sections, to obtain solutions $u_n$.

In order to pass to the limit, we examine the available estimates:

(a) Since $\|u_{0n}\|_1 \le \|u_0\|_1$, there is an estimate in $L^\infty(0,\infty : L^1(B_{2n}(0))$ independent of $n$:

$$\|u_n(t)\|_1 \le \|u_0\|_1 \quad \forall t, n.$$

(b) Since the solutions $u_n$ are bounded above by the solutions $v_n$ of the problem with non-negative data $v_{n0}(x) = \max(u_{0n}(x), 0)$, by the boundedness result of previous section, the sequence $\{u_n(\cdot,t)\}$ is also bounded above in $L^\infty(\mathbb{R}^d)$ uniformly in $n$ and $t$ for $t \ge \tau > 0$. A similar argument shows that it is uniformly bounded below.

(c) Using Theorem 5.7, uniform estimates hold for $\nabla(u_n^m)$ in the space $L^2(\tau,\infty : L^2(B_n(0)))$:

$$\int_\tau^T \int_{B_{2n}(0)} |\nabla u_n^m|^2\, dxdt \le \frac{1}{m+1} \int_{B_{2n}(0)} |u_n(x,\tau)|^{m+1}\, dx \le C < \infty.$$

(d) Lemma 8.5 and Corollary 8.6 imply that we have a uniform bound on $u_{n,t}$ of the form

$$\|u_{n,t}(t)\|_1 \le \frac{2\|u_0\|_1}{(m-1)t}. \tag{9.27}$$

(e) The continuous dependence $C((0,\infty) : L^1(\mathbb{R}^d))$ follows from the last estimate. For $t = 0$ we need another argument based on approximation, barriers, and the $L^1$-stability property (6.1), cf. Theorem 6.2, (ii). We leave these details to the reader.

Since all of these estimates involve bounds which are independent of $n$, we may pass to the limit $n \to \infty$ and obtain a function $u \in L^\infty(0,\infty : L^p(\mathbb{R}^d))$ for every $p \in [1,\infty)$, such that $u_t, u^m, \Delta u^m$ belong to the same spaces to which $(u_n)_t, u_n^m, \Delta(u_n^m)$ belonged, and equation (9.1) holds in $Q$.

(ii) Uniqueness of strong solutions was settled in Proposition 9.1. ■

Let us list some of the properties that these solutions satisfy that come as direct consequence of the proof.

**Proposition 9.13**

(i) *The solutions are continuous functions of* $(x,t)$ *in* $Q$ *with a uniform modulus of continuity for* $t \ge \tau > 0$.

(ii) *For* $t \ge \tau > 0$ *the solutions are energy weak solutions with the regularity* $|\nabla u^m| \in L^2(\mathbb{R}^d \times (\tau,\infty))$ *and*

$$\int_\tau^T \int_{\mathbb{R}^d} |\nabla u^m|^2\, dxdt + \frac{1}{m+1}\int_{\mathbb{R}^d} |u(x,T)|^{m+1}\, dx = \frac{1}{m+1}\int_{\mathbb{R}^d} |u(x,\tau)|^{m+1}\, dx.$$

*This estimate holds down to* $\tau = 0$ *if* $u_0 \in L^{m+1}(\mathbb{R}^d)$.

(iii) *The maximum principle holds, and even formula* (9.2).

(iv) *If* $u_0$ *is non-negative, then* $u \ge 0$ *and estimates* (9.12) *and* (9.19) *hold.*

(v) *If* $u_0$ *is strictly positive and continuous, then* $u \in C^\infty(Q) \cap C(\overline{Q})$ *and is a classical solution of* (9.1).

**Proof** The question of continuity has been settled for bounded solutions in a local setting in Chapter 7 and the results apply here. For the equality sign in (ii) see Section 8.2.1. The rest is also easy. ■

**Remarks**

(1) We point out that the pointwise derivative estimates (9.12), (9.19) are typical of non-negative solutions and need not be true for solutions of changing sign. Of course they hold for negative solutions (with reversed inequality).

(2) Note also that estimate (9.19) improves the constant of (9.9) (a fact that is not so important for $m > 1$ but has a strong influence on the theory for $m < 1$, see [515]).

(3) Moreover, if $u_0$ is smooth this is reflected in the smoothness of $u$ down to $t = 0$ that holds at all points where $u_0(x) \neq 0$.

**Corollary 9.14** *The strong solutions of the Cauchy problem* (9.1) *for the PME form an ordered contraction semigroup in the space* $L^1(\mathbb{R}^d)$.

**Definition 9.2** *The class of solutions constructed in this section is the most frequently encountered in the literature, specially when* $u_0, u \geq 0$. *We will refer to it as the class* $\mathcal{S}_1$. *We shall sometimes write* $u(t) = S_t(u_0)$ *for the function* $u(\cdot, t)$ *where* $u$ *is the strong solution on this class with data* $u_0$.

These notations are useful in Chapter 12.

### 9.5.1 *Mass conservation*

The solutions of the Cauchy problem (9.1) have an important conservation property, not enjoyed by the solutions of the Cauchy-Dirichlet problem.

**Proposition 9.15** *For every* $t > 0$ *we have*

$$\int u(x,t)dx = \int u_0(x)dx. \tag{9.28}$$

**Proof** We take a cut-off function $\zeta_n$ as in Theorem 9.3 and integrate by parts as follows:

$$\begin{aligned}\int u(x,t)\zeta_n(x)dx - \int u_0(x)\zeta_n(x)dx &= \iint u_t\zeta_n\, dxdt \\ &= \iint \Delta u^m \zeta_n dxdt \\ &= \iint u^m \Delta\zeta_n dxdt \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

The calculation is justified if $u$ is smooth and bounded. For general $u$ it follows by approximation, using Proposition 9.1. ■

This law is usually called *conservation of total mass*, or *mass conservation law*. The motivation is as follows: when $u \geq 0$, we will interpret a strong solution $u = u(t)$ of the Cauchy problem as the density distribution at time $t$ of a certain substance that evolves in time according to the PME while keeping the whole mass constant. In the case of the Cauchy-Dirichlet problem posed in a domain $\Omega$ such a mass is not conserved because a part of it flows out through the boundary $\partial\Omega$. Conservation is also true for the solutions of the homogeneous Neumann problem, see Chapter 11.

Note that the law is true for signed solutions, where the interpretation is not just the same.

### 9.5.2 *More properties of $L^1$ solutions*

We investigate further the regularity of the constructed solutions. In particular, we show that the initial data are taken in the most standard sense of weak solutions. For $t \geq \tau > 0$ they are local weak energy solutions.

**Proposition 9.16** *If* $u$ *is a solution of the class* $\mathcal{S}_1$ *with initial data* $u_0 \in L^1(\mathbb{R}^d)$, *then* $|u|^m \in L^1(S)$ *for all sets* $S = B_R(0) \times [0,T]$, $R > 0$,

$0 < T < T(u_0)$. *Moreover, for all* $\eta \in C_c^\infty(\mathbb{R}^d \times [0, T(u_0))$ *we have*

$$\iint \{|u|^{m-1}u\Delta\eta + u\eta_t\}\, dxdt = \int u_0(x)\eta(x,0)\, dx. \tag{9.29}$$

*Moreover,* $|u|^{m-1}u \in H^1_{\text{loc}}((\mathbb{R}^d) \times (0, T(u_0))$.

**Proof** Let $\theta \in C_c^\infty(\mathbb{R})$ with $0 \le \theta \le 1$ and $\theta(x) = 0$ for $|x| \ge R \ge r$. In view of the local boundedness estimate (9.21) for the solutions of the class $\mathcal{S}_1$, for $T > 0$ we have

$$\iint_{Q_T} |u|^m\theta\, dxdt \le c\, R^2 \|u_0\|_1^{2\beta(m-1)} \iint_S |u(x,s)| s^{-\beta d(m-1)}\, dsdx.$$

Using the uniform bound in $L^1$ we get

$$\iint_{Q_T} |u|^m\theta\, dxdt \le c(R)\|u_0\|_1^\delta T^{-2\beta}, \quad \delta = 2\beta(m-1) + 1.$$

This proves the first claim.

As for the integration formula, if we admit that the estimates in $H^1_{\text{loc}}((\mathbb{R}^d) \times (0, T(u_0))$ apply uniformly to the approximations $u_n$, then we can pass to the limit in the formulas for the approximations with time origin $t = \tau > 0$. We then pass to the limit $\tau \to 0$. ■

### 9.5.3 *Sub- and supersolutions. More on comparison*

Theorem 9.2 allows us to compare solutions of the Cauchy problem. However, in many cases we will be interested in functions which either are defined in a subdomain of $Q$ or are not exact solutions of (9.1). A first observation is the following: if $u$ is a strong solution of the Cauchy problem and $\Omega \subset \mathbb{R}^d$ is any bounded space domain, then $u$ is both a supersolution and a subsolution of the PME in $Q = \Omega \times (0, T)$, and will have a non-negative trace on the lateral boundary $\Sigma = \partial\Omega \times (0, T)$. We are now in a position to apply the results of Sections 6.2.1 and 8.2.2 and obtain comparison results.

In our context, the natural definitions of super- and subsolution are as follows: a function $u$ defined in a subdomain $S$ of $Q$ is called a (strong) *supersolution* of (9.1) in $S$ if $u, u^m, u_t$ and $\Delta u^m \in L^1_{\text{loc}}(Q)$ and $u_t \ge \Delta u^m$ a.e. in $S$. A *subsolution* is defined in a similar way, only $u_t \le \Delta u^m$.

We ask the reader to check that a strong supersolution (resp. subsolution) of the CP becomes a weak supersolution (resp. subsolution) of the DP when restricted to $Q = \Omega \times (0, T)$ if $Q \subset S$. In particular, this applies to the solutions of the CP defined in $Q = \mathbb{R}^d \times (0, T)$ are supersolutions of the HDP when restricted to a subdomain of the form $\Omega \times (0, T)$ with $\Omega$ a subdomain of $\mathbb{R}^d$.

We present in Problem 9.1 a useful variant of Proposition 9.1.

We can also modify the above results to provide comparison for a subsolution and a supersolution defined in unbounded domains, see Section 11.4.

## 9.6 Solutions with special properties

The semigroup generated by the PME in the whole space has interesting properties when the class of data is restricted. We will discuss in the sequel two different scenarios: symmetric data and compactly supported data.

### 9.6.1 *Invariance and symmetry*

The PME enjoys a number of invariance and symmetry properties that we have already mentioned at the formal level in Chapter 3. There is no difficulty in proving that they hold for the CP studied in this chapter:

(i) Invariance under space and time translations. If $u$ is a strong solution of the PME defined in $Q = \mathbb{R}^d \times (0,T)$, and $a \in \mathbb{R}^d$, $\tau \geq 0$, then $\widetilde{u}(x,t) = u(x+a, t+\tau)$ is a strong solution of the PME defined in $Q' = \mathbb{R}^d \times (-\tau, T-\tau)$. Moreover, if $\tau > 0$, it is solution of the PME posed in $Q'' = \mathbb{R}^d \times (0, T-\tau)$ with initial data $\widetilde{u}(x,0) = u(x,\tau)$.

(ii) Scaling. If $u$ is a strong solution of the CP for the PME defined in $Q = \mathbb{R}^d \times (0,T)$, and $k, l$ are positive constants, then

$$\widetilde{u}(x,t) = ku(lx, k^{m-1}l^2 t)$$

is again a solution of the CP for the PME defined in $Q' = \mathbb{R}^d \times (0,T')$ with $T' = Tk^{1-m}l^{-2}$.

(iii) Symmetry. If $u$ is a strong solution of the CP for the PME defined in $Q = \mathbb{R}^d \times (0,T)$, then

$$\widetilde{u}(x,t) = u(-x,t)$$

is again a strong solution of the CP for the PME defined in $Q$ with initial data $\widetilde{u}(x,0) = u_0(-x)$.

(iv) Rotation. If $u$ is a strong solution of the CP for the PME defined in $Q = \mathbb{R}^d \times (0,T)$, and $R$ is a rotation of the space around the origin, then

$$\widetilde{u}(x,t) = u(Rx,t)$$

is again a strong solution of the CP for the PME defined in $Q$ with initial data $\widetilde{u}(x,0) = u_0(Rx)$. We can generalize that into all orthogonal transformations.

(v) Radial symmetry. As a conclusion of the previous result, we derive the following property: if $u$ is a strong solution of the CP for the PME defined in $Q = \mathbb{R}^d \times (0,T)$, $u_0(x) = f_0(|x|)$, then the solution is also radially symmetric with respect to the space variable,

$$u(x,t) = f(|x|,t).$$

(vi) Planar symmetry. If $u$ is a strong solution of the CP for the PME defined in $Q = \mathbb{R}^d \times (0, T)$, then

$$\widetilde{u}(x,t) = u(-x_1, x_2, \ldots, x_d, t)$$

is again a strong solution of the CP for the PME defined in $Q$ with initial data $\widetilde{u}(x, 0) = u_0(-x_1, x_2, \ldots, x_d)$. Moreover, this holds when we consider symmetries with respect to a hyperplane, $H$ of $\mathbb{R}^d$.

### 9.6.2 *Aleksandrov's reflection principle*

All of the above properties hold for signed solutions. In the case of non-negative solutions, a more general result can be obtained, called Aleksandrov's reflection principle. We need some notation. Any $H$, hyperplane of $\mathbb{R}^d$, divides $\mathbb{R}^d$ into two half spaces $\Omega_1(H)$ and $\Omega_2(H)$. We denote by $\pi = \pi_H$ the specular symmetry that maps a point $x \in \Omega_1$ into its symmetric image with respect to $H$, $\pi(x) \in \Omega_2$.

**Lemma 9.17** *Let $u \geq 0$ be solution of the Cauchy problem for the PME with initial data $u_0 \in L^1(\mathbb{R}^d)$ and assume that for a given hyperplane $H$ we have*

$$u_0(\pi_H(x)) \leq u_0(x) \tag{9.30}$$

*for all $x \in \Omega_1(H)$. Then, for all times*

$$u(\pi_H(x), t) \leq u(x, t), \quad x \in \Omega_1(H). \tag{9.31}$$

**Proof** By rotation and translation we may assume that $H = \{x_1 = 0\}$, so that

$$\pi(x_1, \ldots, x_n) = (-x_1, \ldots, x_n).$$

and $\Omega_1 = \{x_1 > 0\}$. By approximation we may assume that the solutions are continuous and even smooth, even at $t = 0$. We consider in $\widehat{Q} = \Omega_1 \times (0, \infty)$ the solution $u_1 = u$ and a second solution

$$u_2(x, t) = u(\pi(x), t).$$

By the symmetry invariance, $u_2$ is also a strong solution of the PME; it has initial values $u_2(x, 0) \leq u_1(x, 0)$ by assumption. The boundary values on $\Sigma = H \times (0, T)$ are the same. If we are able to justify the maximum principle for these solutions, then

$$u_2(x, t) \leq u_1(x, t)$$

in $\widehat{Q}$, which proves the result. When $\Omega_1$ is a bounded set, this justification is contained in Theorem 8.10. In our unbounded situation, we have to extend the result as indicated in the previous section: we take approximations of $u$ by solutions of the Cauchy-Dirichlet problem posed in $B_R(0)$ with zero boundary data, show the result in that case, and pass to the limit as $R \to \infty$. This is precisely the way solutions of the CP are constructed in Section 9.2. ∎

As the reader may have observed, there is nothing very particular of the PME in the proof. Indeed, the reflection principle holds for the typical parabolic equations with space independent coefficients, like the heat equation, the fast diffusion equation, the $p$-Laplacian equation, Stefan problem, reaction–diffusion, and so on. The consequences of Aleksandrov's principle on the behaviour of solutions and free boundaries are discussed in Section 14.6.2.

### 9.6.3 *Solutions with compactly supported data*

We consider now the behaviour of solutions with compactly supported initial data. Then the spatial support is bounded for all $t > 0$, which is the simplest version of the property of finite propagation. We sometimes use in that case the expression 'compactly supported solutions' for simplicity by abuse or language. We get the following property for the evolution of the support of the solution.

**Proposition 9.18** *Let $u$ be the strong solution to Problem* (9.1) *with initial data $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, and assume that $u_0$ is supported in a bounded set of $\mathbb{R}^d$. Then for every $t > 0$ the support of $u(\cdot, t)$ is a bounded set.*

**Proof** (i) For $u_0 \geq 0$, it consists merely of noting that we can find a delayed Barenblatt solution centred for instance at 0 that lies on top of $u_0$ a.e.:

$$u_0(x) \leq U(x, \tau; M).$$

By Theorem 9.2 we get $u(x,t) \leq U(x, t+\tau; M)$, hence the result.

(ii) If $u_0$ has changing sign, we bound it above by the solution $u_1 = S_t(u_0^+)$ and below by the solution $u_2 = S_t(-u_0^-)$. ■

This result is complemented by estimates from above and below for the expansion of the support when $u_0 \geq 0$.

**Proposition 9.19** *Assume moreover $u_0 \geq 0$ and $u_0$ is not zero. Then, given any point $x_0$ where $u_0(x_0) > 0$ in a neighbourhood, there exist constants $c_1, c_2$ such that*

$$B_{R_1(t)}(x_0) \subset \{x : u(x,t) > 0\} \subset B_{R_2(t+1)}(x_0) \tag{9.32}$$

*holds for $t \geq 0$, where $R_i(t) = c_i t^{\alpha/d}$ and $c_i = c_i(u_0)$. The estimate from above is true for signed solutions.*

**Proof** (i) If there is a point $x_0$ such that $u_0(x) \geq c > 0$ in a neighbourhood of $x_0$, then, we can find a Barenblatt solution centred at $x_0$ above and below $u_0$: there exist $M$ and $M'$, $\tau$ and $\tau'$ such that

$$\mathcal{U}(x - x_0, \tau'; M') \leq u_0(x) \leq \mathcal{U}(x - x_0, \tau; M).$$

By Theorem 9.2 we get $\mathcal{U}(x - x_0, t + \tau'; M') \leq u(x,t) \leq \mathcal{U}(x - x_0, t + \tau; M)$, hence the result. Note that there is no problem with the upper bound even without the assumption $u_0(x) \geq c > 0$.

(ii) We may eliminate the assumption of strict positivity on the initial data and still get a lower bound for times $t \ge \tau > 0$. Since $u$ is continuous there exist $x_1 \in \mathbb{R}^d$ and $M'$ and $\tau' > 0$ such that

$$u(x,\tau) \ge U(x - x_1, \tau'; M').$$

By the comparison theorem it follows that for every $t \ge \tau$ we have

$$u(x,t) \ge U(x - x_1, t + \tau' - \tau; M'),$$

hence

$$\{x : u(x,t) > 0\} \supset B_{r(t+\tau'-\tau)}(x_1), \tag{9.33}$$

which gives the desired lower bound. ∎

**Remarks**

(1) The upper bound can still be obtained for all compactly supported data $u_0 \in L^1(\mathbb{R}^d)$. This needs however a further tool that will be developed in Chapter 14, see Proposition 14.24.

(2) Since the initial data of Proposition 9.18 are dense in $L^1(\mathbb{R}^d)$, and the semigroup is contractive, we conclude that solutions with compact support of the form (9.32) form a dense set of strong solutions with respect to the norm of $C([0,\infty) : L^1(\mathbb{R}^d))$.

(3) The solutions of the Cauchy problem (9.1) having compact support as described above are automatically solutions of the HDP (5.1)–(5.3) in any domain $Q = \Omega \times (0,T)$ such that the support of $u(t)$ is contained in $\overline{\Omega}$ for $0 \le t \le T$. By choosing as $\Omega$ a ball with a very large radius we may conserve this property for a time as large as desired. When this requirement is no longer satisfied, they become super-solutions because of the boundary condition.

Note that classical free boundary solutions can be defined as in Section 5.13 of Chapter 5. We will pursue further the study of the evolution of solutions with compact support in Chapter 14, Section 14.6, where the emphasis is laid on monotonicity and the location of the free boundary, in Chapter 19 where the main question is regularity, and in Chapter 18 which is devoted to asymptotic behaviour.

### 9.6.4 *Solutions with finite moments*

Let us concentrate again on non-negative solutions. For any function $f \in L^1(\mathbb{R}^d)$, $f \ge 0$ the *moment of order* $p \ge 0$, or $p$-moment, is the integral

$$M_p(f) = \int_{\mathbb{R}^d} |x|^p f(x)\, dx,$$

finite or infinite. We introduce the spaces $\mathcal{X}_p$ as the set of integrable and non-negative functions with finite $p$-moment,

$$\mathcal{X}_p = \{f \in L^1(\mathbb{R}^d) : f \geq 0, M_p(f) < \infty\}.$$

We want to show that the PME semigroup preserves the class $\mathcal{X}_p$ for every $p \geq 0$. Note that for $p = 0$, $\mathcal{X}_0$ is the usual class of integrable and non-negative functions, and conservation of mass answers the question, since it implies that the 0-moment is constant.

Moments are used in the theories of probability and diffusion to evaluate the way a stochastic process or a mass distribution, represented by $u(x,t)$, spread in time. In probability the total mass $M_0 = 1$; when this is not the case in the PME, we may use rescaling to reduce all the calculations to the case $M_0(u_0) = 1$. There are several choices, but we favour the rescaling

$$u_0(x,t) = k\widetilde{u}(x, k^{m-1}t), \tag{9.34}$$

and put $k = M_0(u_0)$ to obtain $M_0(\widetilde{u}_0) = 1$.

Two further elementary observations: by interpolation (Hölder) we get for every $0 < p < q$

$$M_p(f) \leq M_0(f)^{1-(p/q)} M_q(f)^{p/q}.$$

Therefore (or using the fact that $|x|^p \leq |x|^q + C(p,q)$) we have

$$M_p(f) \leq M_q(f) + C(p,q)M_0(f).$$

After these preliminaries, we proceed with the main result on the evolution of moments.

**Proposition 9.20** *Let $u$ be the solution of the CP with data $u_0 \in L^1(\mathbb{R}^d)$, $u_0 \geq 0$. If for some $p \geq 2$ we have $M_p(u_0) < \infty$, then the moment $M_p(u(t))$ is finite for every $t \geq 0$ and we have the growth estimate for large times: $M_p(t) = O(t^{p\beta})$, where $\beta = (d(m-1)+2)^{-1} < 1/2$. The same is true for $p \in (0,2)$ if we assume moreover that $u_0 \in \mathcal{X}_2$.*

**Proof** We recall that $O(t^\beta)$ is the expansion rate of the ZKB solution. For brevity we call $M_p(t) = M_p(u(t))$. (i) Iterative calculation. For every $p \geq 0$ we get the formal computation

$$\frac{d}{dt} M_{p+2}(t) = \int |x|^{p+2} u_t \, dx = \int |x|^{p+2} \Delta u^m \, dx.$$

If $u$ is assumed to be small enough at infinity, we get after integration by parts

$$\frac{d}{dt} M_{p+2}(t) = (p+2)(d+p) \int |x|^p u^m \, dx \leq (p+2)(d+p) \|u(t)\|_\infty^{m-1} M_p(t).$$

Use now use the estimate $\|u(t)\|_\infty \le CM_0^{2\beta}t^{-d\beta}$, see formula (9.21), to get the basic iteration formula

$$\frac{d}{dt}M_{p+2}(t) \le c_1 t^{2\beta-1}M_0^{2(m-1)\beta}M_p(t), \tag{9.35}$$

with $0 < 2\beta < 1$ and $c_1 = c_1(p,d,m) > 0$. For initial data with the generality of the statement we argue by approximation with compactly supported or fast decaying solutions.

(ii) We now use induction on $p$. Starting from $M_0(t) = \|u_0\|_1$ constant, the induction step allows us to obtain when $p \ge 2$ is an even integer the estimate $M_p(t) = O(t^p\beta)$.

(iii) For the rest of the cases, the assumption $u_0 \in \mathcal{X}_p$ with $p > 2$ implies $u_0 \in \mathcal{X}_2$; by the above iterative formula we reduce the calculation to the lower moments $M_{p-2}(t), \ldots$ until we reach $M_q(t)$ for $0 < q < 2$. Let us tackle that case: assuming that $u_0 \in \mathcal{X}_2$, we use interpolation to get

$$M_q(t) \le M_0(t)^{(2-q)/2}M_2(t)^{q/2} \le O(t^{q\beta}). \tag{9.36}$$

This ends the proof for $p \le 2$. The rate for $p > 2$ is obtained by induction. ■

**Remarks**

(1) Computing the moments on the ZKB solution (an easy calculation since it is self-similar) we see that the asymptotic formula $M_p(t) = O(t^{p\beta})$ has the best possible rate. Indeed, if $u(x,t) = U(x,t+\tau;M_0)$ is a ZKB solution then

$$M_p(u(t)) = c(m,c)M_0^{1+(m-1)\beta p}(t+\tau)^{p\beta}. \tag{9.37}$$

This is one of the many instances where ZKB solutions will prove to be the model for the rest of the $L^1$ solutions of the CP problem.

(2) The above result leaves a small gap, namely, proving that $M_p(t)$ is finite and has the correct growth under the sole assumption that $u_0 \in \mathcal{X}_p$ when $0 < p < 2$. In fact, the result is true, but it needs some extra work that we discuss next, and we also show precise large-time estimates.[1]

**Proposition 9.21** *Let $u$ be the solution of the CP with data $u_0 \in \mathcal{X}_p$ for some $p > 0$. Then, we also have*

$$M_p(t) \le M_p(0) + C_p M_0^{1+p(m-1)\beta}t^{p\beta} \tag{9.38}$$

*and $C_p$ depends actually on $m, d, p$ (and $M_{p-2}(0)$ if $p > 2$). Also, the double integral $I_p = \int_0^t \int_{\mathbb{R}^d}(1+|x|^2)^{(p-2)/2}u^m\,dx$ is finite, and*

$$M_p(t) - M_p(0) = p(d+p-2)\int_0^t \int |x|^{p-2}u^m\,dx. \tag{9.39}$$

[1]We recommend skipping this result in a first reading.

**Proof** (i) Starting from $M_0(t) = \|u_0\|_1$ constant, the iteration step allows us to obtain formula (9.38) when $p \geq 2$ is an even integer, since integration of (9.35) gives

$$M_{p+2}(t) \leq M_{p+2}(0) + c_1 \int_0^t (M_p(0) + C_p M_0^{1+p(m-1)\beta} t^{p\beta}) M_0^{2(m-1)\beta} t^{2\beta-1}\, dt$$
$$\leq M_{p+2}(0) + cM_p(0) M_0^{2(m-1)\beta} t^{2\beta} + C_p' M_0^{1+(p+2)(m-1)\beta} t^{(p+2)\beta}.$$

We get (9.38) with a leading constant $C_p = c(m,d,p) M_0^{p(m-1)\beta}$. Formula (9.39) holds so that $I_p$ is finite and grows in $t$ like $M_p(t)$.

(ii) Recall that by scaling we may always assume $M_0 = 1$ and then the obtained dependence on $t$ has to be replaced by dependence on $M_0^{m-1}t$.

(iii) We tackle next the case $0 < p < 2$. It will be convenient to replace the moments by the following modified moments,

$$\widetilde{M}_p(t) = \int u\,(1 + |x|^2)^{p/2}\, dx.$$

Then, we have

$$\frac{d}{dt}\widetilde{M}_p(t) = p(p+d-2)\int \frac{u^m}{(1+x^2)^{1-(p/2)}}\,dx + pd\int \frac{u^m}{(1+x^2)^{2-(p/2)}}\,dx.$$

In any case,

$$\frac{d}{dt}\widetilde{M}_p(t) \leq C\int \frac{u^m}{(1+x^2)^{(2-p)/2}}\,dx \leq \int \frac{u^m}{|x|^{2-p}}\,dx.$$

Here we use a more advanced technique. According to the theory of symmetrization to be developed in a later chapter, the last integral for $u(\cdot,t)$ is bounded above by the same type of integral when the initial data is a Dirac mass, $u_0(x) = M_0\,\delta(x)$. Such a solution is the ZKB $U(x,t;1)$, and then the integral is explicit, and can be approximated by

$$I \leq c(m,d) t^{p\beta-1}.$$

By integration, the result holds. This ends the proof if $d \geq 2$ or $d = 1$ and $p > 1$, so that the coefficient $p(p+d-2) > 0$.

(iv) Consider now the case $d = 1$, $0 < p \leq 1$ and let us control $I_p$. If $p < 1$ the two integrals have opposite sign but the first is dominant for $|x| \geq 1$, while the integral for $|x| \leq 1$ is of the order to $C\int t^{-m\beta}$ for $t$ large, which is integrable in time. It only remains to consider the case $p = 1$, $d = 1$, where the computation allows us to control only the integral $\iint u(1+x^2)^{-3/2}\, dx$. Note also the computation

$$\frac{d}{dt}\int u(x,t)|x|\, dx = 2u^m(0,t),$$

which predicts also the correct size $M_1(t) = M_1(0) + c(M_0^{m-1}t)^\beta$; symmetrization implies also the best constant is obtained when $u$ is a ZKB. ■

We propose another method in Problem 9.10, where an almost optimal rate is obtained, of the form $M_p(t) = O(t^{p'\beta})$ for all $p' > p$.

The study of moments can be extended to signed solutions. With the appropriate definition of moment, the above results can be extended, and the proofs can be obtained from the maximum principle by comparison with positive solutions. We refrain from entering into the details and differences.

### 9.6.5 *Centre of mass and mean deviation*

The most important moments in the applications are those with $p = 1$ and $p = 2$ that we discuss in more detail next. For $p = 1$ we may introduce the *linear moments along each axis*,

$$M_{1,i}(t) = \int u(x,t)x_i\,dx, \quad i = 1, 2, \ldots, d. \tag{9.40}$$

This is a very important quantity when we think of $u(x,t)$ as a mass distribution, since then the vector $\widehat{x} = (\widehat{x}_1, \ldots, \widehat{x}_d)$ defined by

$$\widehat{x}_i(t) = \frac{M_{1,i}(t)}{M_0(t)}$$

is called the *centre of mass* of the distribution $u(\cdot, t)$. In probability the notation $\langle x_i \rangle$ is used and the integral $\int u(x,t)dx = 1$.

**Proposition 9.22** *For every $u_0 \in \mathcal{X}_1$ the centre of mass of a solution of the PME is finite and an invariant of the motion.*

**Proof** The formal calculation is as follows

$$\frac{d}{dt}M_{1,i}(t) = \int u_t(x,t)x_i\,dx = -\int u^m \Delta x_i\,dx = 0.$$

This is true for smooth solutions of approximate problems and holds in the limit by approximation. ■

Once we know that the centre of mass does not move, we may translate the origin to that point and normalize the solution to have $\widehat{x}_i = 0$ for all $i$. In that case, we pass to the second moment which becomes, after renormalization, the square of the *mean deviation* of the mass distribution,

$$\sigma^2(u(t)) = \frac{M_2(t)}{M_0(t)}.$$

Our previous analysis if that case gives the precise estimate

$$M_2(t) = M_2(0) + 2d\int_0^t \int u^m(x,t)\,dx \le M_2(0) + c(m,d)M_0^{1+2(m-1)\beta}t^{2\beta}.$$

Therefore, we have

**Proposition 9.23** *The following estimate holds for the solutions of the PME in class $\mathcal{X}_2$:*

$$\sigma(t) = O(M_0^{(m-1)\beta} t^\beta). \tag{9.41}$$

*Again, this estimate has an exact rate.*

Note that as $m \to 1$ we get the well-known rate of the heat equation and Brownian motion, see Problem 9.7.

## 9.7 The Cauchy-Dirichlet problem in unbounded domains

The strategy developed in constructing solutions of the Cauchy problem from the solutions of the Cauchy-Dirichlet problem in expanding bounded domains can be used to solve the Cauchy-Dirichlet problem posed in an unbounded domain $\Omega \subset \mathbb{R}^d$. We assume that the boundary is locally a Lipschitz-continuous hypersurface of $\mathbb{R}^d$. We refer briefly to the Cauchy-Dirichlet problem as the Dirichlet problem.

More precisely, the homogeneous Dirichlet problem is well posed and Theorem 9.12 is true, conveniently restated so that $\mathbb{R}^d$ becomes $\Omega$ throughout. The proof is done by solving Dirichlet problems in domains

$$\Omega_n = \Omega \cap B_n(0),$$

and approximating the data $u_0(x)$, $x \in \Omega$, into a sequence of functions $u_{0n}$ defined in $\Omega_n$ as in formula (9.26).

In dealing with non-negative solutions we have to recall that the pointwise derivative estimates (9.12), (9.19) are typical of the Cauchy problem and are not necessarily valid for non-negative solutions of the present Dirichlet problems. But we can use an important bound:

**Lemma 9.24** *If $u_n$ is the solution of the approximate problem with non-negative initial data, and $\widetilde{u}_n = S_t(u_{0n})$ are the solutions of the Cauchy problem with the same data, then*

$$0 \le u_n(x,t) \le \widetilde{u}_n(x,t) \le S_t(u_0) \quad \forall x \in \Omega_n,\ t > 0.$$

In view of the $L^\infty$ bound (9.25) for the Cauchy problem, this gives uniform bounds for the sequence $u_n$ when $t \ge \tau > 0$, and helps in passing to the monotone limit. The proof for changing sign solutions now offers no novelties. The uniqueness proof of Proposition 9.1 need not be changed and Proposition 9.13 holds but for the last part of (iv).

On the other hand, mass conservation does not hold in general and the symmetry properties apply only if the domain has the same symmetry property. The application of the Aleksandrov principle is not easy.

Finally, the property of compact support is a simple consequence of the comparison of the solutions of Dirichlet problem with the corresponding solutions of the Cauchy problem, so that Proposition 9.18 and its upper estimate are true. The lower estimates of Proposition 9.19 depend on the possible collision of the support of the solution with the lateral boundary $\Sigma = \partial\Omega \times (0,T)$. This issue will be investigated in Chapter 14, see Subsection 14.2.2.

*Non-homogeneous Dirichlet problems*

There is also interest in solving such problems as in Section 5.7, now in unbounded domains. The most famous case concerns the so-called exterior problems, where the domain is the complement of the closure of a bounded domain of $\mathbb{R}^d$, typically the exterior of a ball.

Another very typical problem of this kind is posed in $d = 1$ on a semi-infinite domain $\Omega = (0,\infty)$, the so-called half line. Typical data in that case are of Dirichlet type $u(0,t) = C$ or Neumann type, $(u^m)_x(0,t) = 0$. The existence and uniqueness theory offers no difficulties. See Problem 9.18.

## 9.8 The Cauchy problem for the GPME

We pose the Cauchy problem for the GPME in complete form, $u_t = \Delta\Phi(u) + f$. We follow closely the approach of Chapter 5. We assume that the constitutive function $\Phi$ is a continuous and increasing function : $\mathbb{R} \to \mathbb{R}$, $\Phi(0) = 0$, and has at least linear growth at infinity in the sense that $|\Phi(s)| \ge c|s| > 0$ for some $c > 0$ and all large $|s|$. Here is the problem statement.

**Problem CP** *Given $u_0 \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $f \in L^1_{\text{loc}}(Q_T)$, find a locally integrable function $u = u(x,t)$ defined in $Q_T$, $T > 0$, that solves the set of equations*

$$\begin{aligned} u_t &= \Delta\Phi(u) + f \qquad &&\textit{in } Q_T, \\ u(x,0) &= u_0(x) &&\textit{in } \Omega, \end{aligned} \tag{9.42}$$

*in a sense to be precisely defined.*

The time $T > 0$ can be finite or infinite. Moreover, we want to find $u$ in a suitable functional class that guarantees existence, uniqueness and continuous dependence on the data. To that effect, the data $(u_0, f)$ will have to be chosen in suitable functional spaces.

### 9.8.1 *Weak theory*

In a first step, we introduce a suitable concept of weak solution. This does not differ at all from the concepts introduced in Definitions 5.1 and 5.2. The most general definition concerns the class of very weak solutions of equation GPME in $Q_T$, which are functions $u \in L^1_{\text{loc}}(Q_T)$ with $\Phi(u) \in L^1_{\text{loc}}(Q_T)$ and such that

$$\iint_{Q_T} \{\Phi(u)\,\Delta\eta + u\eta_t + f\eta\}\,dxdt = 0 \tag{9.43}$$

holds for any test function $\eta \in C_c^\infty(Q_T)$. We assume that $f \in L^1_{\text{loc}}(Q_T)$. In the more restrictive concept of weak solution of the GPME, we also ask that $\nabla\Phi(u) \in$

$L^1_{\rm loc}(Q_T)$ and the equation takes the form

$$\iint\limits_{Q_T} \{\nabla\Phi(u)\cdot\nabla\eta - u\eta_t - f\eta\}\,dxdt = 0. \tag{9.44}$$

After inserting the initial conditions, as in Chapter 5, a suitable definition of weak solution for Problem CP is

**Definition 9.3** *A locally integrable function $u$ defined in $Q_T$ is said to be a weak solution of Problem* CP *if*

(i) $\Phi(u) \in L^2(0,T:H^1(\mathbb{R}^d))$;
(ii) *$u$ satisfies the identity*

$$\iint\limits_{Q_T} \{\nabla\Phi(u)\cdot\nabla\eta - u\eta_t\}\,dxdt = \int\limits_{\mathbb{R}^d} u_0(x)\eta(x,0)dx + \iint\limits_{Q_T} f\eta\,dxdt \tag{9.45}$$

*for any function $\eta \in C^1(\overline{Q}_T)$ which vanishes for $t = T$ and has uniformly bounded support in the space variable.*

We get existence and uniqueness results that are very similar to what was derived for the Dirichlet problem in bounded domains. In the spirit of Theorem 5.7, we get a basic existence result as follows: we define $L_\Psi(\mathbb{R}^d)$ as the space of measurable functions $u_0$ defined in $\mathbb{R}^d$ and such that $\Psi(u_0) \in L^1(\mathbb{R}^d)$. Recall that $\Psi$, the primitive of $\Phi$ defined in (3.18):

$$\Psi(s) = \int_0^s \Phi(r)\,dr.$$

Let $X = L_\Psi(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, $Y = L^\infty(Q_T) \cap L^1(Q_T)$, for some $0 < T \le \infty$. In contrast with the situation of the Dirichlet problem studied in Chapter 5, it is important to note that we do not have an ordering of the $L^p$ spaces in the present situation; in particular, we cannot assert that $L_\psi(\mathbb{R}^d)$ is a subspace of any $L^p(\mathbb{R}^d)$, only that $L_\psi(\mathbb{R}^d) \subset L^2_{\rm loc}(\mathbb{R}^d)$.

The following result parallels Theorem 5.7.

**Theorem 9.25** *Let $u_0 \in X$ and $f \in Y$. Then, Problem* CP *has a unique weak solution defined in the full time interval* $(0,T)$, *and* $\nabla\Phi(u) \in L^2(Q_T)$. *Moreover, we also have $u \in L^\infty((0,T):X)$. The solution is obtained as limit of weak solutions of HDP problems.*

**Proof** (i) Uniqueness parallels Theorem 5.3. We formulate the result as an independent lemma for ease of reference. The proof offers no real changes.

**Lemma 9.26** *Problem* CP *has at most one weak solution if also $u \in L^2(Q_T)$.*

(ii) The standard way in which the solution of the Cauchy problem is obtained as limit of the weak solutions of the HDP problems is as follows: we take domains $\Omega_n = B_n(0)$; in each of them the initial data $u_{0n}(x)$ is the restriction of $u_0$ to

$\Omega_n$; we may also use a cut-off function to make it take zero boundary value at $|x| = n$ in a continuous way: we take zero boundary data on $\Sigma_n$; we then solve the HDP Problem to get a weak solution $u_n$; finally, we need to pass to the limit as $n \to \infty$.

We also have uniform estimates for the masses of $u_n$ due to the $L^1$ stability estimate

$$\int_{B_n} |u_n(t)|\,dx \le \int_{B_n} |u_{0n}|\,dx + \int_0^t \int_{B_n} |f_n(x,\tau)|\,dxd\tau \le C_1$$

for $0 \le t \le T$; this means a uniform control of the quantities $|\{X : |u_n(x,t)| > k\}|$ which are bounded above by a constant $C_k$ that does not depend on $t \in [0,T]$ or $n$. Moreover, $C_k \to 0$ as $K \to \infty$.

The energy inequality (5.20) applies to all approximate solutions in the form

$$\int_{B_n} \Psi(u_n(T))\,dx + \iint_{Q_n} |\nabla\Phi(u_n)|^2\,dxdt \le \int_{B_n} \Psi(u_{0n})\,dx + \iint_{Q_n} f\Phi(u_n)\,dxdt. \tag{9.46}$$

where $Q_n = B_n(0) \times (0,T)$. In view of the assumptions on $u_0$ and $f$, when $u_0$ is also bounded the right-hand side is uniformly bounded (for the last term observe that $\Phi(u_n)$ is bounded in $L^\infty(Q)$ while $f$ is integrable). We thus get uniform boundedness in $n$ of three sequences:

$$\int_{B_n} \Psi(u_n(T))\,dx \le C_3, \quad \iint_{Q_n} \Phi(u_n)^2\,dxdt \le C_4, \quad \iint_{Q_n} |\nabla\Phi(u_n)|^2\,dxdt \le C_5.$$

The constants $C_1, \ldots, C_5$ do not depend on $n$ or $t$. When $u_0$ is not assumed to be bounded we need to show that $\Phi(u(t)) \in L^2(0,T : H^1(\mathbb{R}^d))$. We may go back to formula (9.46) and estimate in a finer way the contribution of the last term. If $d \ge 3$ we use the Sobolev imbedding to get

$$\begin{aligned} |\iint_{Q_n} \Phi(u_n) f_n dxdt| \le C_6 &\left(\iint_{Q_n} |\nabla\Phi(u_n)|^2 dxdt\right)^{1/2} \\ &\times \left(\int_0^T ||f_n||^2_{L^{2n/n+2}(B_n)} dt\right)^{1/2}. \end{aligned}$$

The last factor is bounded since $f \in Y$ and we can absorb the other one into the left-hand side and conclude as before. Note that the constants $C_1, \ldots, C_6$ do not depend on $n$ or $t$.

We ask the reader to complete the details also for $d = 1, 2$. Use the extra assumption that $f$ is compactly supported to simplify the calculation.

(iii) When $u_0 \ge 0$ the sequence $u_n$ is monotone non-decreasing and we have uniform bounds on the norms of the spaces stated in the result. We can pass to the limit and obtain a weak solution of the Cauchy problem.

(iv) For data of any sign we may perform a double passage to the limit using monotonicity or we can use compactness.

The first approach is outlined in Problem 9.9.

The second approach uses the local energy estimates (3.24) and (3.31) if $\Phi$ is locally Lipschitz continuous. Or it can use the continuity with a uniform modulus if the weaker assumption of Theorem 7.1 is met.

We leave both options as exercises for the reader. ■

**Corollary 9.27**

(i) *The solution belongs to the usual space* $C([0,T) : L^1(\mathbb{R}^d))$.

(ii) *Comparison holds as in Theorem* 5.5, *and consequently* $u_0 \geq 0$ *and* $f \geq 0$ *imply* $u \geq 0$ a.e.

(iii) *We have the usual* $L^1$ *stability: for any two solutions the following inequality holds*

$$\|(u(t) - \widehat{u}(t))_+\|_1 \leq \|(u(\tau) - \widehat{u}(\tau))_+\|_1 + \int_\tau^t \|(f(s) - \widehat{f}(s))_+\|_1 \, ds. \tag{9.47}$$

(iv) *If* $u_0 \in L^p(\mathbb{R}^d)$ *for some* $p \in [1, \infty]$, *then* $u(t) \in L^p(\mathbb{R}^d)$ *for all* $t > 0$ *and the* $L^p$ *norm is non-increasing in time.*

(v) *Under the Lipschitz continuity hypothesis* B *of Theorem* 7.1 *on* $\Phi$, *the solution is continuous with a modulus of continuity with the usual dependence.*

**Remark** It is possible to relax the assumptions of Theorem 9.25 and still get weak solutions. Different ideas of Chapters 5, 6 and 8 can be easily adapted. We consider that this is not a priority at this point. We prefer to consider the theory with $L^1$ data.

### 9.8.2 *Limit $L^1$ theory*

We first remark that, when we work in the whole space $\mathbb{R}^d$ there is no order relation between the functional spaces $L^p$ for different $p \in [1, \infty]$, hence the theory for $L^1$ data loses its character of extension of other $L^p$ theories. It is however an extension of the theory posed in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

On the other hand, the extension to $L^1$ data is quite natural on mathematical grounds in view of the powerful $L^1$ stability that the equation enjoys and we have proved for the case of data $L^1 \cap L^\infty$ just treated. Moreover, the physical interpretation of non-negative $L^1$ solutions as solutions with finite mass is very appealing; or in the theory of heat propagation, it means a solution with finite thermal energy.

In any case, the theory for $L^1$ provokes the introduction of limit solutions starting from the results of the previous subsection, that is completely parallel to Section 6.1.2 for the HDP. We have

**Theorem 9.28** *For any* $(u_0, f) \in L^1(\mathbb{R}^d) \times L^1(Q_T)$ *there exists a unique function* $u \in C([0,\infty) : L^1(\mathbb{R}^d))$ *that solves Problem* CP *in the sense of limits of the weak solutions of Theorem* 9.25. *The map:* $(u_0, f) \mapsto u$ *is an order contraction from* $L^1(\Omega) \times L^1(Q_T)$ *into* $C([0,\infty) : L^1(\Omega))$. *Properties* (ii), (iii), (iv) *of Corollary* 9.27 *also hold. Property* (v) *holds if the solution is locally bounded.*

Specializing the result to the case $f = 0$, there is no problem in proving that the strong solutions of the Cauchy problem form a semigroup of contractions in $L^1(\mathbb{R}^d)$.

We refrain at this point from discussing the theory of very weak solutions for the Cauchy problem of the GPME by lack of space and leave it as an advanced topic. We refer the reader to Section 6.2 for the same topic in the Dirichlet problem.

### 9.8.3 *Relating the Cauchy-Dirichlet and Cauchy problems*

We can compare the non-negative solutions of the homogeneous Cauchy-Dirichlet and the solutions of a corresponding Cauchy problem.

**Proposition 9.29** *Let* $u_0 \in L^1(\Omega)$, $u_0 \ge 0$, *and let* $u_D$ *be the solution of the HDP for the GMPE and let* $u_C$ *the solution of the Cauchy problem with initial data* $\widetilde{u}_0(x)$ *such that* $\widetilde{u}_0(x) \ge u_0(x) \ge 0$ *for* $x \in \Omega$. *Let* $f = \widetilde{f} = 0$. *Then,*

$$0 \le u_D(x,t) \le u_C(x,t) \quad \text{in} \;\; Q.$$

*The result also holds when* $0 \le f \le \widetilde{f}$ *in* $\Omega$.

**Proof** The proof is immediate in the classical case by the maximum principle (note that $u_D = 0 \le u_N$ on $\Sigma$, and $u_D = u_N$ for $t = 0$). Passing to the limit we get the result for all limit solutions. ∎

We can also prove that solutions of the Cauchy-Dirichlet problems in expanding domains converge to the solution of the Cauchy problem if the data converge conveniently. We leave this result as an exercise for the reader, see Problem 9.19.

## Notes

**Section 9.1.** As explained in preceding chapters, pioneering work is due to Oleĭnik and collaborators in one space dimension. Sabinina [457] made the extension to several dimensions in 1961.

**Section 9.3.** The fundamental estimate is due to Aronson and Bénilan [40], 1979. The authors point out its optimality by checking it on the Barenblatt solutions and use the estimate in establishing existence of a strong solution of the Cauchy problem with $L^1$ data.

**Section 9.4.** The boundedness of the solutions was first obtained by and Bénilan [81], 1976, and Véron [520], 1979. The proof given here, based on the fundamental estimate, is new (and considerably shorter). The Barenblatt solutions as extremal solutions for the $L^1$–$L^\infty$ effect will be discussed in Chapter 17. In this way, very sharp versions of the effect will be obtained.

Estimates of the form $\Delta v \geq -C$ play a role in the theory of Hamilton–Jacobi equations, cf. e.g. [374]. Such functions are called semi-subharmonic functions.

Most of the results of Chapter 5 have immediate adaptation to the Cauchy problem, but not all. In particular, the universal bound of Section 5.8 is not true in $\mathbb{R}^d$. This is easily understood when we assert that the constants will be acceptable solutions in our theory (once extended).

**Section 9.6.2.** The reflection principle is a quite important tool in the analysis of propagation properties, but also in the general theory of elliptic and parabolic equations. It was introduced by A.D. Aleksandrov [6, 7]. It is also known and used as the *moving plane method.* Pioneering applications of Aleksandrov's reflection principle are due to Serrin [476]. A famous application of symmetrization phenomena for nonlinear elliptic and parabolic problems is described by Gidas, Ni and Nirenberg in [260]. Another symmetrization argument, based on Aleksandrov's reflection principle, is given in Section 5 of [325]. The application to the PME is presented in the book [255].

**Section 9.6.3.** The control of the growth of the support as $t \to \infty$ (formula (9.32)) was first obtained in $d = 1$ by Knerr [343], 1977. Sharp results will described in Chapter 18.

**Section 9.6.4.** The study of moments seems to be new.

**Section 9.8.** We can also consider the problem in the $H^{-1}$ context along the lines of Section 6.7. We leave the work to Problem 9.11.

The questions of further regularity and strong solutions have been studied in the chapter in the case of the PME without forcing. When applied to the GPME, or the PME with forcing, these questions are left to the interested reader as further topics of study.

*Fast diffusion equations*

The theory of the Cauchy problem can be repeated to a large extent for the FDE in the range $0 < m < 1$. Thus, the existence of a semigroup of solutions $u \in C([0,\infty) : L^1(\mathbb{R}^d))$, the maximum principle and $L^1$ contraction hold, cf. [79]. There are however remarkable differences like the absence of free boundaries. Also for $m < (d-1)/d$ the conservation of mass is lost and actually many solutions extinguish in finite time. This very interesting topic falls out of the scope of this volume. We refer the reader to the monograph [515] where extensive references are given. See also Problem 9.12.

## Problems

**Problem 9.1** Using a modification of the arguments of Proposition 9.1, prove the following result.

**Lemma 9.30** *Let $\Omega$ be a bounded subset of $\mathbb{R}^d$ with $C^1$ boundary, let $S = \Omega \times I \subset Q$, with $I = (t_1, t_2)$, and let $u_1$ be a subsolution, $u_2$ a supersolution of* (9.1) *in $S$. Assume moreover that $u_1$ and $u_2$ are continuous in $\overline{S}$ and $u_1 \le u_2$ on $\partial\Omega \times I$. Then, for every $t \in [t_1, t_2]$*

$$\int [u_1(x_t) - u_2(x,t)]_+ dx \le \int [u_1(x,t_1) - u_2(x,t_2)]_+ dx. \tag{9.48}$$

*In particular, if $u_1(\cdot, t_1) \le u_2(\cdot, t_1)$ in $\Omega$ we have $u_1 \le u_2$ in $S$.*

**Problem 9.2** THE HEAT EQUATION. Adapt the theory of this chapter to the heat equation $u_t = \Delta u$. In particular,

(i) Prove that it generates a semigroup of contractions in all spaces $L^p_+(\mathbb{R}^d)$, $1 \le p \le \infty$, and not only in $L^1_+(\mathbb{R}^d)$.
(ii) Prove the fundamental estimate and the bundedness estimates and show that they coincide with the limit $m \to 1$ of the ones calculated in this chapter.
(iii) Prove the conservation of mass and centre of mass.
(iv) Repeat the calculation of the estimates for the moments in the case of the heat equation. Then, put formally $m = 1$ in the results of this chapter and compare the results.
(iv) Prove that the time rate of the estimate for the mean deviation $\sigma(t)$ in Proposition 9.23 is exact by calculating a lower bound with a ZKB solution.

**Problem 9.3** SPACE DECAY

(i) Show that when the initial data satisfy a bound of the form $0 \le u_0(x) \le C/(1+|x|^2)^\alpha$ with $\alpha > 0$ then for every $t > 0$ we have $u(x,t) = O(|x|^{-2\alpha})$ as $|x| \to \infty$ and the estimate holds uniformly in $t \in (0,T)$, $T$ finite.
(ii) Prove a similar estimate for $0 \le u_0(x) \le Ce^{\alpha|x|}$.
(iii) Extend to the heat equation and the fast diffusion equation if possible.

*Hint*: For (i) Use a supersolution of the form $U(x,t) = C(x_1 - ct)^{-2\alpha}$. Generalize the estimate by rotation invariance. For (iii) the result about the power decay is true for the FDE depending on the power, the exponential decay is never true (see [515]).

**Problem 9.4** ELLIPSOIDAL BLOW-UP SUPERSOLUTIONS. Consider the following formulas for the pressure

$$v = \frac{1}{T-t} \sum_i k_i x_i^2. \tag{9.49}$$

This is a variation of the blow-up solution (4.41) with ellipsoids as level sets. Let $k_1 = \max k_i$ and $\lambda_i = k_i/k_1$. Show that it is a supersolution if

$$k_1(2 + (m-1)\sum_i \lambda_i) \le \frac{1}{2}.$$

The ellipsoids are elongated along the axes $x_2, \ldots, x_d$.

**Problem 9.5** Prove the energy estimates equivalent to the ones obtained in the two previous chapters. Prove the decay estimates of Proposition 9.11.

**Problem 9.6** Take the GPME with a function $\Phi \in C^3$ and write the equation for the pressure $v = P(u)$ as in Problem 3.7. Now write the equation for $p = \Delta v$ in the form

$$p_t = \sigma(v)\Delta p + 2(\sigma'(v)+1)\nabla v \cdot \nabla p + \sigma'(v)p^2 + 2\sum_{i,j}\left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)^2 + \sigma''(v)|\nabla v|^2 p. \tag{9.50}$$

Assuming that $\sigma''(v) \le 0$ and $\sigma'(v) \ge a \ge 0$ obtain the a priori estimate

$$\Delta v \ge -\frac{C}{t}, \quad C = \frac{d}{da+2}. \tag{9.51}$$

Compare with the case $\Phi(u) = c\,u^m$.

**Problem 9.7**

(i) Use the mass conservation property to prove that when $u$ is a non-negative solution in $\mathcal{S}_1$, then $u(t)$ cannot be trivial for $t > 0$ unless $u_0 \equiv 0$.
(ii)* Such an assertion for signed solutions is an open problem.

**Problem 9.8** Removable singularities. There are cases in which a solution of the PME is obtained by some process and we know that it is a weak solution of the equation unless at one or several points, where such property is under question. We usually say that the solution may have one or several singularities. Under suitable assumptions such singularities can be removed. Here is an example:

(i) Prove that a non-negative and weak solution of the PME defined in $Q^* = (\mathbb{R}^d \setminus \{0\}) \times (0,T)$, $d \ge 2$, that is bounded is also a solution in $Q = \mathbb{R}^d \times (0,T)$. In other words, the singularity can be removed.
(ii) Show that this is not true for $d = 1$.

*Hint*: Here is a standard proof: take a smooth cut-off function $0 \le \psi \le 1$ that vanishes near $x = 0$ and is 1 for $|x| \ge 1$ and put $\psi_r(x) = \psi(x/r)$. We now write the weak formulation of the PME with respect to a test function $\varphi(x,t) = \zeta(x,t)\psi_r(x)$ where $\zeta| \in C_c^\infty(\mathbb{R}^d \times (0,\infty))$. Since $\varphi = 0$ near $x = 0$, this test function is admissible for the solution with a bounded singularity. Since $u_\infty$ is bounded, the limit $r \to 0$ shows that it is a solution of the PME for all $t > 0$, $x \in \mathbb{R}^d$.

**Problem 9.9** Double monotonicity. Make an alternative existence proof for the signed part of Theorem 9.25, using the trick of double monotonicity.
*Idea:*

(i) Use the approximations

$$u_{0,m,n}(x) = u_0(x)^+ \chi_{B_n(0)} - u_0(x)^+ \chi_{B_m(0)} \tag{9.52}$$

in balls $B_r = B_r(0)$ with $r \geq \max(m,n)$. Solve the HDP in $\Omega = B_r$ with data $u_{0,m,n}$ to get a solution $u_{r,m,n}$.

(ii) Use the uniform bounds to pass to the limit as $r \to \infty$ for fixed $m, n$ and prove that the limit $u_{m,n}$ is a weak solution of the Cauchy problem with data $u_{0,m,n}(x)$, $x \in \mathbb{R}^d$. Show that $u_{m,n}$ is monotone non-decreasing with $n$, non-increasing with $m$.

Pass the monotone limit in $n \to \infty$ for $m$ fixed to obtain a weak solution $u_m$. Pass now to the monotone limit $m \to \infty$ to get the final solution $u$. Show that it is a weak solution.

**Problem 9.10** Show that the construction of solutions of Theorem 9.25 can be done under the assumptions

$$u_0 \in L_\Psi(\mathbb{R}^d), \quad f \in L^p(Q_T) \text{ with } p = 2d/(d+2) \text{ if } d \geq 3.$$

Check which properties of Corollary 9.27 still hold.

**Problem 9.11** Construct a theory for the Cauchy problem in the setting of $H^{-1}(\Omega)$, following the lines of Section 6.7.

**Problem 9.12** FAST DIFFUSION EQUATIONS. Much of the theory of this chapter has an equivalent for fast diffusion equations, at least when $m > (d-2)/d$.

(i) Prove the fundamental estimate, Proposition 9.4, for $(d-2)/d < m < 1$.

(ii) Prove that in that range we have bounds from above and below for $u_t$:

$$\frac{Cu}{t} \leq u_t \leq \frac{u}{(1-m)t}.$$

**Problem 9.13** Investigate the behaviour of the moments for the solutions of the homogeneous Cauchy-Dirichlet problem of Chapters 5 and 6. Show that the result is trivial. What does it say?

**Problem 9.14*** Prove that the $p$-moments are finite for $0 < p < 2$ under the assumption $u_0 \in \mathcal{X}_p$ without using symmetrization.

*Hint*: Using the modified moments, $\widetilde{M}_p(t) = \int u\,(1+|x|^2)^\varepsilon\,dx$ where $\varepsilon = p/2 \in (0,1)$, we have

$$\frac{d}{dt}\widetilde{M}_p(t) = p(p+d-2)\int \frac{u^m}{(1+x^2)^{1-\varepsilon}}dx + pd\int \frac{u^m}{(1+x^2)^{2-\varepsilon}}dx.$$

In any case we have

$$\frac{d}{dt}\widetilde{M}_p(t) \leq C\int \frac{u^m}{(1+x^2)^{1-\varepsilon}}dx.$$

We use Hölder's inequality on the last integral

$$\int \frac{u^m}{(1+x^2)^{1-\varepsilon}}dx \le \|u(t)\|_\infty^{m-1+\delta}\left(\int u\,(1+|x|^2)^\varepsilon\,dx\right)^{1-\delta}\left(\int \frac{dx}{(1+|x|^2)^\gamma}\right)^\delta.$$

with the relation $\gamma\delta = 1-\varepsilon\delta$ between the two new parameters. We need to impose the conditions $\gamma > d/2$ (for the last integral to be bounded), and $0 < \delta < 1$. This is perfectly compatible with the relation since $\delta(\gamma+\varepsilon) = 1$. Making such a choice, we arrive at the inequality

$$\frac{d}{dt}\widetilde{M}_p(t) \le C\widetilde{M}_p(t)^{1-\delta}t^{-d\beta(m-1+\delta)} = C\widetilde{M}_p(t)^{1-\delta}t^{\beta(2-d\delta)-1},$$

which can be integrated to give

$$\widetilde{M}_p(t)^\delta \le \widetilde{M}_p(0)^\delta + Ct^{\beta(2-d\delta)},$$

hence $\widetilde{M}_p(t) \le \widetilde{M}_p(0) + Ct^{\kappa\beta}$ with $\kappa = (2-d\delta)/\delta$. We want this to be $2\varepsilon$. It implies $\delta = 1/(d+2\varepsilon)$, which in turn implies $\gamma = d/2$, the limit case that is excluded. We therefore obtain the correct exponent plus a bit, $\widetilde{M}_p(t) \le C + Ct^{\beta p'}$ for all $p' > p$.

**Problem 9.15** Extend the study of the evolution of moments to signed solutions. Use as definition

$$M_p(f) = \int_{\mathbb{R}^d} |x|^p\, f(x)\,dx,$$

or the absolute version

$$M_p(f) = \int_{\mathbb{R}^d} |x|^p\, |f(x)|\,dx.$$

Compare the results.

**Problem 9.16*** Write the law of conservation of mass and the evolution of the centre of mass for signed solutions of

$$\partial_t u = \Delta(|u|^{m-1}u) + f. \tag{9.53}$$

Find the formulas

$$\frac{dM}{dt} = \int_{\mathbb{R}^d} f(x,t)\,dx, \qquad \frac{d}{dt}\int_{\mathbb{R}^d} x_i\, u(x,t)\,dx = \int x_i\, f(x,t)\,dx. \tag{9.54}$$

State the conditions under which they hold and the sense in which they do.

**Problem 9.17*** Continuous dependence on $\Phi$. Prove that when $\Phi_\varepsilon$ is smooth and approximates $\Phi$, then the solutions of the Cauchy problem with $\Phi_\varepsilon$ converge as $\varepsilon \to 0$ to the solution with $\Phi$. See in this respect [88].

**Problem 9.18** State and prove the existence and uniqueness theorem for the non-homogeneous Dirichlet problem for the GPME in the half line, $x \in (0,\infty)$. Same in an exterior domain, $\Omega \setminus \overline{G}$, where $G$ is a bounded set of $\mathbb{R}^d$, $d \ge 2$.

**Problem 9.19** Prove the following result: *Let $\Omega_n$ an expanding sequence of smooth domains such that $\bigcup_n \Omega_n = \mathbb{R}^d$. Let $u_{0n} \in L^1(\Omega_n)$, $u_0 \in L^1(\mathbb{R}^d)$ and assume that $u_{0n} \to u_0$ in the obvious $L^1(\mathbb{R}^d)$ sense (i.e., extending $u_{0n}$ by zero outside $\Omega_n$). Let $u_n$ be the limit solution of the HDP in $Q_n = \Omega_n \in \mathbb{R}_+$ with data $u_{on}$, and let $u$ be the limit solution of the CP in $Q = \mathbb{R}^d \times \mathbb{R}_+$. Prove that $u_n \to u$ in $C([0,\infty) : L^1(\mathbb{R}^d))$.*

**Problem 9.20*** Prove the following comparison result for very weak solutions. *Two bounded and non-negative, ordered very weak solutions $u_1, u_2$ of the filtration equation GPME with the same data are the same if $\Phi$ is Lipschitz continuous on the range of the solutions.*

**Idea of the proof** Let $u(x,t) = u_1(x,t) - u_2(x,t) \geq 0$. Then, $u(x,0) = 0$. Let us define

$$M(t) = \int_{\mathbb{R}^d} u(x,t)\varphi(x)\, dx$$

for some smooth $\varphi \in L^1(\mathbb{R}^d)$ to be chosen below. Integrating by parts we get

$$M'(t) = \frac{d}{dt}\int_{\mathbb{R}^d} (u_1 - u_2)\varphi\, dx = \int_{\mathbb{R}^d} \Delta\varphi\,(\Phi(u_1) - \Phi(u_2))\, dx.$$

We now choose $\varphi$ as the solution of the equation $-\Delta\varphi + \varphi = \psi$, for some smooth $\psi \in L^1(\mathbb{R}^d)$, $\psi \geq 0$. It is well known that a unique $\varphi$ exists, $\varphi > 0$ and $\int \varphi\, dx = \int \psi\, dx$. Using also the fact that $\Phi$ is monotone and Lipschitz continuous we get

$$\begin{aligned} M'(t) &= \int_{\mathbb{R}^d} \varphi(\Phi(u_1) - \Phi(u_2))\, dx - \int_{\mathbb{R}^d} \psi(\Phi(u_1) - \Phi(u_2))\, dx \leq K \int_{\mathbb{R}^d} u\varphi\, dx \\ &= KM(t), \end{aligned}$$

where $K$ is an upper bound of $\Phi'$ in the range of $u_1$ and $u_2$. Note that the last term before the inequality is negative and can be dropped. Since $M(0) = 0$, we conclude from the differential inequality $M'(t) \leq KM(t)$ that $M \equiv 0$, hence uniqueness. We ask the reader to justify this calculation. ■

**Problem 9.21*** Minimal non-negative solutions. Prove that for every given $u_0 \in L^1(\Omega)$ and $f \in L^1(Q_T)$, with $u_0, f \geq 0$, the limit solution of the CP is the minimal element among all non-negative very weak solutions such that $u, \Phi(u) \in L^2_{\text{loc}}(Q_T)$.

*Hint*: Solve approximate problems with bounded data $u_{0n}, f_n \geq 0$ increasing to $u_0, f$ and posed $Q_n = B_n \times (0,T)$, where $B_n$ is the ball with centre 0 and radius $R_n = n$. If the (weak energy) solution is $u_n$, we have $0 \leq u_n \leq u_{n+1}$. By Problem 6.9, any non-negative very weak solution $U$ of the CP with data $u_0, f$ is a supersolution for the restricted problems. According to Theorem 6.5, we have $u_n \leq U$ in $Q_n$. We may now pass to the limit and define the candidate to minimal solution

$$u_{\min}(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{9.55}$$

Showing that this definition is minimal, it is independent of the construction and coincides with the limit solution is immediate.

**Problem 9.22*** Read Chapter 17 and complete the details of the end of proof of Proposition 9.21.

**Problem 9.23*** Prove the mass conservation law for very weak solutions.

# 10

# THE PME AS AN ABSTRACT EVOLUTION EQUATION. SEMIGROUP APPROACH

In this chapter we address the question of construction of solutions of the GPME by viewing it as an abstract evolution equation, more precisely as an ordinary differential equation with values in a Hilbert or Banach space. In this approach to evolution problems we use the shortened notation $u(t)$ instead of $u(x,t)$ since emphasis is laid on the $t$-dependence and $u(t)$ is for every $t$ an element of a functional space thanks to its remaining $x$-dependence. We are interested in solving abstract Cauchy problems of the form

$$\frac{du}{dt} + A(u) = f, \quad u(0) = u_0, \tag{10.1}$$

where $A$ is a (possibly nonlinear) operator acting in a Banach space $X$, and the solution $u$ is supposed to be a function from a time interval $[0,T]$ into $X$. In the classical setting $A$ is a continuous linear operator and then we are able to find a differentiable solution $u \in C^1([0,T) : X)$ if $u_0 \in X$ and $f \in C([0,T) : X)$ that can be written as

$$u(t) = e^{At}u_0 + \int_0^t e^{(t-s)A} f(s)\, ds, \tag{10.2}$$

a so-called variation of constants formula.

Our aim here is to treat possibly nonlinear and discontinuous operators like the ones corresponding to the GPME and other parabolic and even hyperbolic equations. In the case of a linear operator, the typical example of such problems is the heat equation; the operator is then $A = -\Delta$, minus the Laplacian acting on a space of integrable functions, say $L^2(\Omega)$, or more generally $L^p(\Omega)$ with $1 \le p \le \infty$, where $\Omega$ is for instance a bounded domain of $\mathbb{R}^d$ with smooth boundary.

In the linear case, the answer to the question of existence at this abstract level is given by the theory developed by E. Hille, Y. Yosida and R. Phillips in the 1930s which applies to linear operators $A$ which are maximal monotone in a Hilbert space $H$ or $m$-accretive in a Banach space $X$. This theory is covered in the classical books.

The extension to nonlinear operators takes two directions of interest for us. One of them is the theory of *maximal monotone operators* in Hilbert spaces and the second one the theory of $m$-accretive operators in Banach spaces. Both have played a role in the development of the PME theory and we devote this chapter to present the relevant results.

A practical side of this approach is the construction of approximate solutions by the implicit version of the famous Euler method, which is called in this context the implicit time discretization (ITD) scheme. This means that even if the context may be very abstract, the numerical implementation is quite natural.

When the forcing term is zero, a semigroup is constructed. Actually, the approach of this section is usually called the semigroup approach and the ensuing solutions obtained by approximation are sometimes called semigroup solutions.

The outline of the chapter is as follows. Section 10.1 deals with the theory of maximal monotone operators in Hilbert spaces; its application to the PME allows us to recover the construction of solutions with data in $H^{-1}$ of Section 6.7.

Section 10.2 introduces the time discretizations, and the concepts of mild solutions and the accretive operators in Banach spaces. Since our main interest is not the theory of monotone or accretive operators which are covered in the specialized literature, we will give a number of useful results without proofs in both sections.

Section 10.3 applies the theory of accretive operators to the filtration equation. In that context, let us point out another quite important aspect of the theory: each of the steps of the ITD scheme consists in solving a *nonlinear elliptic problem*, and the study of such problems is quite important in itself, while the connection between the ensuing parabolic and elliptic theories has been a source of progress on both sides of the dividing line. We also establish the relation of the new concept of solution, i.e., *mild solution*, with the solution concepts of previous chapters. Peculiar nonlinearities give rise to a semigroup with curious properties, as shown in Subsection 10.3.3.

We end the chapter with the new ideas of mass transportation and gradient flows, Section 10.4, and a review of different extensions to more general equations where new concepts of solution are needed, Section 10.5. This is advanced reading.

The Notes contain reading suggestions and references to early work in the semigroup approach to these evolution equations.

We point out that the abstract theory can be applied in various settings; this is actually its strongest point. In the next chapter we will see it applied to Neumann problems and problems on manifolds.

## 10.1 Maximal monotone operators and semigroups

### 10.1.1 *Generalities on maximal monotone operators*

In the whole section we follow notations and results of [129] and [128] to which we refer for further details. Let $H$ be a Hilbert space over the reals with scalar product denoted either as $u \cdot v$ or as $\langle u, v\rangle$ and norm denoted by $|u|$. The notation $u_n \to u$ denotes the strong convergence in $H$ and $u_n \rightharpoonup u$ the weak convergence. We identify the dual $H'$ of $H$ with $H$ in the standard way. If $C$ is a closed convex subset in $H$, then $\mathrm{Proj}_C x$ denotes the projection of an element $x$ on $C$.

**Definition 10.1** *A single-valued nonlinear monotone operator $A$ in a Hilbert space is a map $A$ from a subset of $D(A) \subset H$, called the domain of $A$, into $H$, and such that for every $u_1, u_2 \in D(A)$ we have*

$$\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq 0. \tag{10.1}$$

*We say that $A$ is dissipative if $-A$ is monotone.*

However, the theory of nonlinear monotone operators in Hilbert spaces deals naturally with multivalued operators, so care has to be taken from the beginning to get used to the correct concepts and notation. The following modifications apply to the definition:

(i) The map $A$ goes from $D(A) \subset H$ into the set of parts of $H$ (denoted by $\mathcal{P}(H) = 2^H$), so that for every $u \in D(A)$, $A(u)$ is a subset of $H$, not an element of $H$.
(ii) The monotonicity assumption then reads: for every $u_1, u_2 \in D(A)$ and every $v_1 \in A(u_1)$, $v_2 \in A(u_2)$ we have

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq 0. \tag{10.2}$$

Note that the single-valued case is recovered when $A(u)$ is a singleton for every $u \in H$ (i.e., the set $A(u)$ consists of a unique element that we also call $A(u)$ in that case). In the sequel, our monotone operators are allowed to be nonlinear and multivalued. We will not assume that $D(A) = H$ which is usually false, the closure of $D(A)$ is generally a convex closed subset of $H$. We will denote by $R(A)$ the range of $A$, a subset of $H$. Note that there is no problem in defining the inverse $A^{-1}$ of a multivalued operator (actually, this property is one the reasons for using multivalued maps). For more details on operators see Section A.2. Finally, if $u \in D(A)$ we denote by $A^o u$ the element with minimal norm in $Au$. We will follow in this chapter the multivalued notation. However, for most purposes the reader may assume that the operators are single-valued, that we may write $A(u)$ instead of $v \in A(u)$, and that the signs $\in$ and $\ni$ may be replaced by $=$.

The following property is fundamental in the study of monotone operators.

**Proposition 10.1** *Let $A$ be a nonlinear operator in $H$. Then $A$ is monotone if and only if the following property holds: for every $u_1, u_2 \in D(A)$, every $v_1 \in A(u_1)$, $v_2 \in A(u_2)$, and every real number $\lambda > 0$ we have*

$$|u_1 - u_2| \leq |u_1 - u_2 + \lambda(v_1 - v_2)|. \tag{10.3}$$

We can rephrase the result by defining the (multivalued) operator $B_\lambda = I + \lambda A$, and then $J_\lambda(A) = B_\lambda^{-1} = (I + \lambda A)^{-1}$. In that notation, $A$ is monotone iff $J_\lambda(A)$ is a (non-strict) contraction. Note that $J_\lambda(A)$ is necessarily single-valued since it is a contraction. The operators $J_\lambda(A)$ are called resolvent operators.

In order to compare operators, we view them as graphs in $H \times H$.

**Definition 10.2** *An operator is maximal monotone if its graph is a maximal element among all monotone operators in H.*

Maximal monotone operators (m.m.o. for short) are a basic tool in the existence theory of this section. Proposition 2.2 of [128] characterizes maximal monotone operators.

**Proposition 10.2** *Let $A$ be an monotone operator in a Hilbert space. Then it is maximal monotone if and only if one the following equivalent conditions is met:*

(i) $R(I + A) = H$;
(ii) *for every $\lambda > 0$, $J_\lambda(A)$ is a contraction defined on the whole of $H$.*

Property (i) is called the *range condition.* In our practice, $A$ will be a differential operator, think of $A = -\Delta$ acting on a subset $D(A)$ of $H = L^2(\Omega)$ with Neumann or Dirichlet boundary conditions. Then, the range condition can be interpreted as saying the formula

$$u + Au = f$$

covers all possible $f$ in $H$. In other words, it is *an existence theorem for a stationary elliptic equation.* This is the crux of the game.

The next observation is that any monotone operator $A$ can be extended into a larger operator $B$ which is maximal monotone. As we said, extension means that the graph is larger. The extension need not be unique, but it is unique if $R(I + A)$ is dense in $H$. In that case we call it $\overline{A}$.

One of the main classes of maximal monotone operators in the applied fields is given by the following result (see [128], Example 2.3.4).

**Proposition 10.3** *Let $\Psi$ be a proper convex function in $H$. If $\Psi$ is lower semicontinuous, then its subdifferential $\partial\Psi$ is a maximal monotone operator.*

Let us revise the definitions. A proper convex function $\Psi$ is defined for all $x \in H$ and takes values in $\mathbb{R} \cup \{+\infty\}$. We denote by $D(\Psi) = \{x \in H : \Psi(x) < +\infty\}$, this set is non-empty if $\Psi$ is proper. Convexity means that for every $x, y \in H$ and a positive number $\lambda \in (0, 1)$ we have

$$\Psi(\lambda x + (1 - \lambda)y) \le \lambda\Psi(x) + (1 - \lambda)\Psi(y).$$

Next, $\Psi$ is lower semicontinuous (l.s.c.) if for every $x \in H$ we have

$$\liminf_{y \to x} \Psi(y) \ge \Psi(x).$$

The *subdifferential* of a convex function is defined as the multivalued map $A = \partial\Psi$ such that $v \in A(u)$ if for every $x \in H$ we have

$$\Psi(x) \ge \Psi(u) + \langle v, x - u\rangle$$

Clearly, $D(\partial\Psi) \subset D(\Psi)$.

We need yet another technical tool: in the next subsection we will be using the *Yosida approximations* of a maximal monotone operator $A$:

$$A_\lambda = \frac{1}{\lambda}(I - J_\lambda(A)) \subset J_\lambda \circ A.$$

These approximations are single-valued Lipschitz continuous operators (with Lipschitz constant $1/\lambda$). They serve to approximate $A$: as $\lambda \to 0$ we have $A_\lambda x \to A^o x$ for every $x \in D(A)$, while $|A_\lambda x| \to +\infty$ for $x \notin D(A)$.

### 10.1.2 *Evolution problem associated to an m.m.o. Semigroup*

Let us now consider the evolution equation (10.1) that we now write in the compatible notation

$$\frac{du}{dt} + A(u) \ni 0, \quad u(0) = u_0, \tag{10.4}$$

where $A$ is a monotone operator acting in the Hilbert space $H$ and we have eliminated the forcing term. Let us also take initial data $u(0) = u_0 \in H$. Can we solve this abstract Cauchy problem? The answer is given by the following result (Theorem 3.1 of [128]).

**Theorem 10.4** *If $A$ is maximal monotone and $u_0 \in D(A)$, the abstract CP has a unique solution $u \in C([0,\infty) : H)$ in the sense that*

(i) *for every $t > 0$, $u(t) \in D(A)$;*

(ii) *$du/dt \in L^\infty(0,\infty; H)$ (derivative in the sense of distributions) and*

$$\|du/dt\|_{L^\infty} \le |A^o(u_0)|;$$

(iii) *the inclusion $0 \in A(u(t)) + \dfrac{du}{dt}(t)$ holds for* a.e. $t > 0$*;*

(iv) *$u(t) \to u_0$ in $H$ as $t \to 0$.*

There are other properties of this unique solution:

(v) $u$ admits at every $t \ge 0$ a right derivative $d^+u/dt$ and

$$0 \in \frac{d^+u}{dt} + A^o(u(t));$$

(vi) The function $A^o(u(t))$ is right continuous and moreover $|A^o(u(t))|$ is non-increasing in time;

(vii) for every two solutions $u_1, u_2$ and every $0 \le s \le t$ we have

$$|u_1(t) - u_2(t)| \le |u_1(s) - u_2(s)|. \tag{10.5}$$

When the operator is linear this is just the Hille–Yosida theorem. The standard proof is based in replacing operator $A$ by the Yosida approximations $A_\lambda$, $\lambda > 0$. Since these approximations are Lipschitz continuous operators, the existence of solutions $u_\lambda$ of the CP in that case follows the classical ODE proof. Then the limit $\lambda \to 0$ is taken.

Recalling what was said in Section 6.1 about $L^1$-limits of contractive evolutions and in Section 6.4 about semigroups we conclude that

**Corollary 10.5** *If $A$ is a maximal monotone operator, equation* (10.4) *defines a continuous semigroup of (non-strict) contractions defined in the convex set $K = \overline{D(A)} \subset H$.*

In order to solve the Cauchy problem in a limit sense for all $u_0 \in \overline{D(A)}$ we take $u_{0n} \in D(A)$ converging to $u_0$ in $H$, find the solutions $u_n$ and pass to the limit using the contraction property (iv). As in Section 6.4, we will write $S_t u_0 = u(t)$ if $u$ is the solution of problem (10.4) with $u_0 \in \overline{D(A)}$. In most applications $D(A)$ is dense so that $K = H$. In view of the form of writing our equation, in order to conform to standard notation we must say that *the semigroup is generated by* $-A$.

One of the problems of the abstract theory is allowing for limit solutions that fall out of the original domain where the concept of operator (in our case, differential equation) is clear. This is avoided in the maximal monotone case by Theorem 3.3 of [128] which asserts that under a mild condition on $D(A)$ (namely that it has non-empty interior) then $S_t u_0 \in D(A)$ for all $t > 0$ if $u_0 \in \overline{D(A)}$, even if $u_0$ is not in $D(A)$.

### 10.1.3 *Complete evolution equation*

The theory of evolutions with m.m.o.s covers also the Cauchy for the complete equation

$$\frac{du}{dt} + A(u) \ni f, \quad u(0) = u_0 \tag{10.6}$$

where $f \in L^1(0, T : H)$ for some $T > 0$. The corresponding concepts of solutions are weak solution and strong solution, see Definition 3.1 of [128].

**Definition 10.3**

(i) *A function $u \in C([0, \infty) : H)$ is a called strong solution of Problem* (10.6) *if it is absolutely continuous as an $H$-valued function of time, if moreover for* a.e. $t > 0$ *we have $u(t) \in D(A)$ and the inclusion $f \in \frac{du}{dt} + A(u(t))$ holds.*

(ii) *A function $u \in C([0, \infty) : H)$ is called a weak solution of Problem* (10.6) *if it is a limit of strong solutions $u_n$ with data $f_n$ such that $f_n \to f$ in $L^1(0, T : H)$ and $u_n \to u$ in $L^\infty(0, T : H)$.*

This parallels the theory that we have seen for the PME in Chapters 5, 6 and 8, but Brezis calls here weak what was called there limit solution. See the concept of mild solution in the next section. Here is the main result, see Theorem 3.4 of [128].

**Theorem 10.6** *Under the above conditions, a MMO, $u_0 \in \overline{D(A)}$ and $f \in L^1(0, T : H)$, there exists a unique weak solution of Problem* (10.6).

We will be interested in cases where $A$ is a subdifferential, $A = \partial\Psi$ of a convex function. In that case we have extra regularity. Let $\min \Psi = 0$ and $K = \{u \in H : \Psi(u) = 0\}$.

**Theorem 10.7** *If $A = \partial\Psi$ and $f \in L^2(0,T:H)$, then every weak solution of* (10.6) *is in fact strong. Moreover we have the energy estimate*

$$\left(\int_0^T t\,|\frac{du}{dt}|^2\,dt\right)^{1/2} \le \left(\int_0^T |f(t)|^2\,dt\right)^{1/2} + \frac{1}{\sqrt{2}}\int_0^T |f(t)|\,dt + \frac{1}{\sqrt{2}}\mathrm{dist}\,(u_0,K). \tag{10.7}$$

*Therefore, $\sqrt{t}\dfrac{du}{dt} \in L^2(0,T:H)$. Moreover, when $u(0) \in D(\Psi)$ we have $\dfrac{du}{dt} \in L^2(0,T:H)$ and*

$$\left(\int_0^T |\frac{du}{dt}|^2\,dt\right)^{1/2} \le \left(\int_0^T |f(t)|^2\,dt\right)^{1/2} + \sqrt{\Psi(u_0)}. \tag{10.8}$$

*In this case, the function $\Psi(u(t)$ is absolutely continuous in* $[0,T]$.

See more details in [128], Theorem 3.6.

### 10.1.4 *Application to the GPME*

Let us consider a bounded domain $\Omega$ in $\mathbb{R}^d$ and let $\Lambda = -\Delta$ be the canonical isomorphism from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$ and let $G = \Lambda^{-1}$ the Green operator. We take $H = H^{-1}(\Omega)$ with the usual inner product

$$\langle f,g\rangle_{H^{-1}} = \langle \Lambda^{-1}f,g\rangle_{H_0^1\times H^{-1}}$$

Let now $j$ be a l.s.c. function $:\mathbb{R}\mapsto\mathbb{R}\cap\{+\infty\}$ with $j$ not identically infinity, and let $\varphi = \partial j$. Assume also that $j(r)/|r| \to \infty$ as $|r| \to \infty$ so that $R(\varphi) = \mathbb{R}$. For $u \in H^{-1}(\Omega)$ define

$$\Psi(u) = \int_\Omega j(u)\,dx \tag{10.9}$$

whenever $u \in L^1(\Omega)$ and $j(u) \in L^1(\Omega)$, and define $\Psi(u) = +\infty$ otherwise. Then,

**Proposition 10.8** *The function $\Psi$ is convex and lower semicontinuous in $H = H^{-1}(\Omega)$, so that its subdifferential is a maximal monotone operator in $H$. This subdifferential $\partial\Psi$ is characterized as follows: $f \in \partial\Psi(u)$ if and only if*

$$Gf(x) \in \varphi(u(x)) \quad \text{a.e. } in\ \Omega.$$

Recall that $G = \Lambda^{-1}$. This result is proved by Brezis in [127], Theorem 17. In case $\varphi$ is single-valued we can write $f \in \partial\Psi(u)$ in the clearer form

$$f(x) = -\Delta\varphi(u(x)) \qquad \text{a.e.},$$

which is the differential operator associated to the GPME. We are thus able to solve equation $u_t - \Delta\varphi(u) = f$ in the new abstract formulation.

**Theorem 10.9** *For every $u_0 \in H^{-1}(\Omega)$ and $f$ absolutely continuous from $[0,T]$ into $H^{-1}(\Omega)$ there exists a unique strong solution $u \in C([0,T] : H^{-1}(\Omega))$ of the problem*

$$\begin{cases} \partial_t u = \Delta\varphi(u) + f & \text{in } \ \Omega \times (0,T) \\ \varphi(u(x,t)) = 0 & \text{on } \ \partial\Omega \times (0,T) \\ u(x,0) = u_0 & \text{for } \ t = 0. \end{cases} \tag{10.10}$$

*We also have*

$$t\,\varphi(u) \in L^\infty(0,T : H_0^1(\Omega)), \quad t\,\partial_t u \in L^\infty(0,T : H^{-1}(\Omega)),$$
$$t\,u \in L^\infty(0,T : L^1(\Omega)).$$

*Moreover, for every $t > 0$ we have $u(\cdot,t) \in L^1(\Omega)$, and $\varphi(u(\cdot,t)) \in H_0^1(\Omega)$.*

See [127], corollary 31. Note that when $f = 0$ we find a semigroup in $H$.

*Comparison and important observation*

These results allow us to recover the existence and uniqueness results obtained in Section 6.7, in particular Theorem 6.17. Actually, Theorem 10.7 seems to say that we have a strong solution, which looks like better regularity, but the context is deceiving: strong means here $H^{-1}$-strong, i.e., that $u_t \in H^{-1}(\Omega)$ for a.e. $t > 0$, and this is what weak solutions of the GMPE are because of the equation $u_t = \nabla F$ with $F = \nabla\varphi(u) \in L^2(\Omega)$.

## 10.2 Discretizations, mild solutions and accretive operators

We now discuss a new proposal for solving abstract differential equations that is based on time discretization and has proved to be quite powerful in the applications to nonlinear diffusion and other related contexts. We consider an abstract Cauchy problem written in the form

$$\frac{du}{dt} + A(u) \ni f, \quad u(0) = u_0, \tag{10.11}$$

where $A$ is a (possibly nonlinear and multivalued) operator acting now in a Banach space $X$. We recall that in many cases the operator is single-valued and we can write the equation as $du/dt + A(u) = f$, which is more familiar and comfortable. We take $f \in L^1(0,T : X)$ for some or all $T > 0$.

**Definition 10.4**

(i) *When $f \in C(0,T : X)$ we say that $u : (0,T) \mapsto X$ is a classical abstract solution of the abstract equation if $u \in C^1((0,T) : X)$ and e for all $0 < t < T$ and such that $u'(t) + A(u(t)) \ni f$.*

(ii) *Moreover, it is a classical solution of the initialvalue problem if $u$ is also continuous at $t = 0$ and $u(t) \to u_0$ as $t \to 0$.*

**Definition 10.5**

(i) *A function* $u \in C((0,T):X)$ *is called a strong solution of the abstract equation if it is absolutely continuous and differentiable* a.e. *as an* $X$-*valued function of time and for* a.e. $0<t<T$ *we have* $u(t) \in D(A)$ *and the inclusion* $f \in \frac{du}{dt} + A(u(t))$ *holds* a.e.

(ii) *Moreover, it is a strong solution of the whole problem if* $u$ *is also continuous at* $t=0$ *and* $u(t) \to u_0$ *as* $t \to 0$.

### 10.2.1 *The ITD method*

The method of implicit time discretization proposes to solve the abstract ODE problem by approximating the solution in the following way:

We take a partition $\mathcal{P} = \{0 \le t_0 < t_1 < \ldots < t_{N-1} \le t_N \le T\}$, and we pose the problem of solving as the system of difference relations

$$\frac{U_i - U_{i-1}}{h_i} + AU_i \ni F_i, \quad i = 1, 2, \ldots, N \tag{10.12}$$

where $h_i = t_i - t_{i-1}$ is the time step, and $\{F_1, \ldots, F_N\}$ is a discretization of $f$ adapted to the partition $\mathcal{P}$ (we use capitals for the discretized values in order to better identify the 'discrete part' from the 'continuous parts'). This process is called a **discretization** of equation $u'(t) + Au(t) \ni f$ relative to the given time partition and $f$-values; we can describe it as

$$D(A; \mathcal{P}, F_i) = D(A; t_0, t_1, \ldots, t_N; F_1, \ldots, F_N).$$

In the terminology of numerical analysis, it is an implicit time discretization (note: in order to perform the practical computation it has to be completed with some kind of space discretization, a second step that we will *not* take in our present theory).

The resulting equations can be written as

$$h_i\, A(U_i) + U_i \ni U_{i-1} + h_i\, F_i. \tag{10.13}$$

If $A$ is an elliptic operator, for instance the Laplacian or a quasilinear variant thereof, then (10.13) is a (quasilinear) elliptic equation. Solving such equations is a main problem of concern for the theory developed here. The initial step is solved by assigning to $U_0$ the valued prescribed by the theory as $u_0$ or an approximation of it. The iterative system of equations can be written as

$$U_i = (I + h_i A)^{-1}(U_{i-1} + h_i\, F_i), \tag{10.14}$$

the so-called *implicit time discretization scheme*. The problem is treatable in this way if the *resolvent operators*

$$J_\lambda(A) = (I + \lambda A)^{-1}, \quad \lambda > 0, \tag{10.15}$$

have good properties as operators defined in $X$, or a suitable subspace thereof. When this is the case, solving the evolution problem reduces to solving the cascade of relations (10.14) to obtain a *discrete approximate solution* $\{U_i\}$. This discrete set can be pieced together into a function defined for all $t \in [0,T]$ in various natural forms. A standard form is by forming a piecewise constant function $u_D^{(1)}$

$$u_D^{(1)}(t) = U_i \quad \text{for} \ \ t_{i-1} < t \le t_i, \ i = 1, 2, \ldots, N.$$

Another form that produces a continuous interpolation, $u_D^{(2)}(t) \in C([0,T] : X)$, is

$$u_D^{(2)}(t) = \frac{t - t_{i-1}}{h_i} U_i + \frac{t_1 - t}{h_i} U_{i-1} \quad \text{if } t_{i-1} \le t \le t_i,$$

The extension to the end intervals $[0, t_0)$ and $(t_N, T)$ when they are not empty can be done in a constant way. In any of these ways, we form an *approximate solution* subordinate to the discretization.

We have assumed that the time interval is $(0,T)$. We can translate these concepts to any time interval $I = (a,b) \subset \mathbb{R}$ without loss of generality.

### 10.2.2 *Problem of convergence. Mild solutions*

Once the approximations are constructed, the main mathematical question is the convergence of these approximations to some identifiable solution of the problem. This will hopefully happen when the partition is refined. As usual in such a general setting, we have to accept solutions which are neither classical nor strong, at least in principle. We need to define the concept of *$\varepsilon$-discretization.* It is a discretization $D$ as before such that

$$0 \le t_0 < \varepsilon, \quad T - \varepsilon < t_N \le T, t_i - t_{i-1} < \varepsilon \quad \text{for} \ \ i = 1, 2, \ldots, N, \tag{10.16}$$

and also

$$\sum_{i=1}^{N} \int_{t_{i-1}}^{t_1} \|f(s) - F_i\| \, ds < \varepsilon. \tag{10.17}$$

This is a way of expressing that the discrete problems have data $\varepsilon$-close to the continuous problem (10.11).

We can now state the main concept of solution, the mild solution. We assume that the time interval is $I = [a,b]$. See [90].

**Definition 10.6** *Let $f \in L^1_{\mathrm{loc}}(I : X)$.* (i) *A mild solution of problem* (10.11) *in $I$ is a function $u \in C(I : X)$ that is obtained as uniform limit of $\varepsilon$-discretizations of the problem. Namely, for every $\varepsilon > 0$ there exists an $\varepsilon$-discretization $D_\varepsilon$ with solution $u_\varepsilon$ in $I$ and*

$$\|u(t) - u_\varepsilon(t)\|_X < \varepsilon \quad \textit{for} \ \ t_0 \le t \le t_N.$$

(ii) *A mild solution on an arbitrary time interval $J$ is a function $u \in C(J : X)$ whose restriction to any compact subinterval $I \subset J$ is a mild solution of the problem in $I$.*

Compatibility of the above definition implies that part (ii) applies also when $J$ is compact to a mild solution according to part (i). We leave this as an exercise, [90].

Therefore, the concept of generalized solution that is proposed is a kind of limit solution, this time a limit of semidiscretized problems. The introduction of a new concept of solution needs to be accompanied by the usual compatibility result

**Proposition 10.10** *Let $f \in L^1_{\mathrm{loc}}(I : X)$ and let $u$ be a strong solution of the abstract problem* (10.11) *in $I = (0, T)$. Then $u$ is a mild solution.*

For the proof see Problem 10.2. The new type of solution is needed mainly in nonlinear and discontinuous contexts. The following result states conditions under which it is not needed.

**Proposition 10.11** *Let $f \in L^1_{\mathrm{loc}}(I : X)$ and let $A$ be a single-valued operator defined in a closed domain $D(A)$ of $X$ which is continuous on $D(A)$. Let $u$ be a mild solution of the abstract problem* (10.11) *in $I = (0, T)$. Then, it is a strong solution and satisfies for every $0 < t < T$ the equation*

$$u(t) = u(0) - \int_0^t Au(s)\, ds + \int_0^t f(s)\, ds.$$

*If moreover $f \in C(I : X)$, then $u$ is a classical solution.*

This result is not true in the general setting we work with here, but it can be true under special conditions. Criteria under which a mild solution is strong in the nonlinear setting will be given in Proposition 10.18 after we have introduced accretive operators. For subdifferential operators in Hilbert spaces see Theorem 10.7. The definition of mild solution being indirect, it is not easy to give criteria on the solution itself to identify it in a unique way. Here is some help.

**Proposition 10.12** *Let $f \in L^1_{\mathrm{loc}}(I : X)$ and let $u$ be a mild solution of problem* (10.11) *in $I = (0, T)$. Then,*

(i) *For every $t \in I$ we have $u(t) \in \overline{D(A)}$.*
(ii) *The continuation in time of a mild solution defined in a time interval $I_1$ with a mild solution defined in an overlapping time interval $I_2$ is a mild solution in $I_1 \cup I_2$ if they agree on $I_1 \cap I_2$.*
(iii) *The limit $u$ of mild solutions $u_n$ is a mild solution if $\|u(t) - u_n(t)\|_X \to 0$ uniformly in $t \in I$ and the forcing terms $f_n$ converge to $f$ in $L^1(I; X)$.*

Let us point out the connection with semigroups.

**Definition 10.7** *Given an operator $A$ we call $D_0(A)$ the set of $x \in X$ such that there exists exactly one mild solution of the abstract ODE: $u'(t) + Au(t) = 0$ existing in $I = (0, \infty)$ and such that $\lim_{t\to 0} u(t) = x$. In that case we write $S_A(t)x = u(t)$.*

**Proposition 10.13** *The family of maps $S_A(t) : D_0(A) \mapsto D_0(A)$, $t \geq 0$, forms a strongly continuous semigroup on the metric space $E = D_0(A) \subset X$ in the sense of Definition* 6.3.

We say that $S_A$ is the semigroup generated by $A$. The problem of this result is that $D_0(A)$ may be quite small.

### 10.2.3 *Accretive operators*

The problem of convergence of the ITD and existence of mild solutions for a wide class of data had a positive answer with the work of Crandall and Liggett [180], who posed this problem for the class of accretive operators.

**Definition 10.8**

(i) *A nonlinear operator in a Banach space is called accretive if the resolvent operator $J_\lambda(A) = (I + \lambda A)^{-1}$ is a (non-strict) contraction defined in $X$ for all $\lambda > 0$.*
(ii) *We say that an accretive operator $A$ in a Banach space $X$ is $m$-accretive if it satisfied the range condition*

$$R(I + \lambda A) = X, \quad \text{for all } \lambda > 0. \tag{10.18}$$

(iii) *In the Banach setting we say that $A$ is a dissipative operator if and only if $-A$ is accretive.*

The definition of accretive operator implies that the equation $x + \lambda Ax = y$ has at most one solution for $y \in X$; $m$-accretivity means that it does have one. Accretivity can be characterized in several ways: one of them uses a technical tool, the *bracket* (sometimes called the Sato bracket), which is defined as

$$[x, y]_+ = \lim_{\lambda\to 0+} \frac{\|x + \lambda y\| - \|x\|}{\lambda}. \tag{10.19}$$

It is the right-hand derivative of the norm of $x \in X$ in the direction of $y \in X$. A standard result say that an operator $A$ is accretive if and only if $[x_1 - x_2, y_1 - y_2]_+ > 0$ for all $x_i \in D(A)$, $y_i \in Ax_i$, $i = 1, 2$. It is also proved that the bracket $[\cdot, \cdot]_+$ is an upper-semicontinuous map: $X \times X \to \mathbb{R}$.

When $A$ is $m$-accretive we can solve the 'discretized problems' (10.14) for every partition discretization $D$. An $m$-accretive operator is a maximal element in the set of accretive operators but the converse is not true and the concepts are not equivalent for general Banach spaces, hence the need for the (rather awkward) new name. Let us also note the in a Hilbert space the concepts of accretive and monotone operator coincide and $m$-accretive becomes maximal monotone. In that case, maximal monotone is the same as maximal element among monotone operators.

Earlier results on the theory of accretive semigroups were concerned with linear operators and the generation of a semigroup. Here is a classical result of Lumer and Phillips.

**Proposition 10.14** *The linear operator $A$ generates a $C_0$ semigroup $S_t$ of linear contractions in $X$ if and only if $A$ is $m$-accretive and $D(A)$ is dense in $X$. In that case we have the exponential formula*

$$S_t u_0 = \lim_{n\to\infty} (J_{t/n}(A))^n u_0 = \lim_{n\to\infty} \left(I + \frac{t}{n}A\right)^{-n} u_0. \tag{10.20}$$

Another standard result says that accretivity implies uniqueness of the strong solutions.

**Proposition 10.15** *Let $u_1, u_2$ strong solutions of the ODE $u' + Au = f$ with resp. data $u_{01}, u_{02}$ and forcing terms $f_1, f_2$ existing in the time interval $I = [0, T)$. Then for every $0 < t < T$*

$$\|u_1(t) - u_2(t)\| \le \|u_{01} - u_{02}\| + \int_0^t \|f_1(s) - f_2(s)\|\, ds. \tag{10.21}$$

There is a stronger version of the concept of accretive operator, much used by Bénilan and coauthors, called *$T$-accretive operator*, that combines the contraction property with the maximum principle typical of elliptic and parabolic equations. It is applicable in spaces $X$ where we can define the positive part of an element $f \in X$, as in the typical Lebesgue function spaces in $\mathbb{R}^d$ or $\Omega$ that we have been using. To be precise, an operator $A$ is $T$-contractive if for every $\lambda > 0$ $J_\lambda(A)$ is a $T$-contraction, which means that

$$\|(J_\lambda(A)f_1 - J_\lambda(A)f_2)_+\|_X \le \|(f_1 - f_2)_+\|_X$$

where $(\cdot)_+$ denotes the positive part. It is then clear how this is related with comparison arguments: if $f_1 \le f_2$, we have $(f_1 - f_2)_+ = 0$, and the $T$-contraction implies $J_\lambda(A)f_1 \le J_\lambda(A)f_2$.

There is also a relaxed form of accretivity. An operator $A$ is called *$\omega$-accretive* (for some $\omega > 0$) if $A_\omega = A + \omega I$ is accretive. The use of this concept is tied to the following observation at the formal level for linear A: if $u$ is a solution of the ODE: $u'(u) + Au(t) = f(t)$, then $v(t) = u(t)e^{-\omega t}$ solves the equation

$$v'(t) + Av(t) + \omega v(t) = g(t) := f(t)e^{-\omega t}. \tag{10.22}$$

Thus, the ability to solve equations with $A_\omega$ suffices to solve equations with $A$ after a change of variables that allows for an extra exponential increase in time.

### 10.2.4 The Crandall–Liggett theorem

We now address the main topic of the theory of accretive operators, namely the existence of mild solutions for Problem (10.11). Let us now consider the non-forcing case $f=0$. The convergence is given by the famous Crandall–Liggett theorem. A preliminary observation. The solution of the implicit steps of the ITD scheme is possible is $R(I+\lambda A)=X$ for every $\lambda>0$, but is still possible under the condition $D(A)\subset R(I+\lambda A)$. In order to be able to take limits with good properties it turns out that we need to impose the so-called *range condition*

$$\overline{D(A)}\subset R(I+\lambda A) \quad \forall\lambda>0. \tag{10.23}$$

An $m$-accretive operator satisfies of course this range condition. But the generality encompasses interesting cases like the following: $\overline{D(A)}$ is a closed convex set $K\subset X$ and $R(I+\lambda A)=K$ for all $\lambda>0$.

**Theorem 10.16** *Let $A$ be an accretive operator in $X$ that satisfies the range condition. Then, for any $u_0\in\overline{D(A)}$ the limit*

$$S_t(A)u_0=\lim_{n\to\infty}(J_{t/n})^n u_0 \tag{10.24}$$

*exists uniformly on compact subsets of $[0,\infty[$. Moreover, the family of operators $S_t(A)$, $t>0$, is a strongly continuous semigroup of contractive mappings of $\overline{D(A)}\subset X$.*

By analogy with the linear theory we may write $S_t u_0:=e^{-tA}u_0$. Formula (10.24) is called the *Crandall–Liggett exponential formula* for the nonlinear semigroup generated by $-A$. A further statement in Crandall and Liggett's work estimates the speed of convergence as follows:

$$\|J^n_{t/n}u_0-e^{-tA}u_0\|\le\frac{t}{\sqrt{n}}\|y\|+2\|u_0-x\| \tag{10.25}$$

for every $(x,y)\in A$. The kind of generalized solution of the Cauchy problem obtained in this way has been termed *mild solution* in [89].

*Complete equation*

We now consider the complete equation where $f\neq 0$. This is the theorem on existence and uniqueness of a type of generalized solution that we call mild solution. We take again $A$ an $m$-accretive operator in $X$. For given $u_0\in\overline{D(A)}$ and $f\in L^1(0,T:X)$ we solve the discretized problems (10.13) for a sequence of partitions $\mathcal{P}_n$ with diameter tending to 0 and construct the discretized solutions $u_n(t)$.

**Theorem 10.17** *Let $A$ be an $m$-accretive operator in a Banach space $X$ and let $u_0\in\overline{D(A)}$ and $f\in L^1(0,\infty:X)$. Then, the abstract problem* (10.11) *has a unique mild solution that is obtained as limit of the solutions $u_n$ of*

*$\varepsilon_n$-discretizations of the problem by the ITD scheme, $\varepsilon_n \to 0$:*

$$u(t) := \lim_{n\to\infty} u_n(t) \tag{10.26}$$

*and the limit is uniform on compact subsets of* $[0,\infty[$ *and does not depend on the particular choice of discretization. Moreover, $u \in C([0,\infty) : X)$ and for every two solutions $u_1$, $u_2$ we have*

$$\|u_1(t) - u_2(t)\|_X \le \|u_1(s) - u_2(s)\|_X + \int_s^t \|f_1(\tau) - f_2(\tau)\|_X \, d\tau \tag{10.27}$$

*for every $0 \le s < t$.*

A similar result is true when $A_\omega$ is $m$-accretive for some $\omega \ge 0$ but growth factors appear in (10.27). The theorem as formulated by Bénilan in [79], 1972, says more, namely that the unique mild solution exists on the condition that we can produce the sequence of solutions $u_n$ of $\varepsilon_n$-discretized problems with $\varepsilon_n \to 0$.

A mild solution need not be strong, it may not be differentiable. But if it is, we have the following result.

**Proposition 10.18** *Let $u$ be a mild solution of equation* (10.11) *where $A$ is an $m$-accretive operator in $X$ and $f \in L^1(0,T : X)$. If $t_1 \in (0,T)$ is Lebesgue point of $f$ and $u$ is differentiable at $t = t_1$, then*

$$u'(t_1) + Au(t_1) \ni f(t_1).$$

*As a consequence, if $u$ is a mild solution and $u \in W^{1,1}(0,T : X)$ then $u$ is a strong solution.*

The question is then to decide when is $u \in W^{1,1}$. We recall that an RK space is a Banach space having the Radon–Nikodym property, see Section A.1.

**Proposition 10.19** *Let $u$ be as above with $A$ accretive in an RK space $X$, let $f \in BV(0,T : X)$ and $u_0 \in D(A)$. Then, $u \in W^{1,1}(0,T : X)$ and $u$ is a strong solution.*

The quest for a direct characterization of mild solutions leads to the concept of *integral solution* [80].

**Definition 10.9** *Let $A$ be an $\omega$-accretive and $f \in L^1(0,T : X)$. A function $u \in C([0,T) : X)$ is an integral solution of equation* (10.11) *if it satisfies*

$$\|u(t) - x\| \le \|u(s) - x\| + \omega \int_s^t \|u(\tau) - x\| \, d\tau + \int_s^t [u(\tau) - x, f(\tau) - y]_+ \, d\tau \tag{10.28}$$

*for all $(x,y) \in A$ and all $0 < s < t < T$.*

A strong solution is an integral solution. Uniform limits of integral solutions are also integral solutions. This is a relevant result connecting mild and integral solutions.

**Proposition 10.20** *Under the above conditions on $A$ and $f$, we have*

(i) *every mild solution of problem* (10.11) *is also an integral solution;*
(ii) *the initial value problem* (10.11) *has at most one mild solution;*
(iii) *if it has a mild solution, then it is the unique integral solution of the problem.*

## 10.3 Mild solutions of the filtration equation

We take as our main example the equation

$$u_t = \Delta\Phi(u) + f, \tag{10.29}$$

where $\Phi$ is a maximal monotone graph (m.m.g.) with $0 \in \Phi(0)$ and $f = f(x,t)$ is defined in $Q_T = \mathbb{R}^d \times (0,T)$. Actually, the correct notation is $u_t = \Delta v + f$, $v \in \Phi(u)$, as the reader might have guessed.

When $\Phi$ is a smooth function and $\Phi'$ is bounded above and below away from zero, i.e., $0 < c < \Phi'(u) < 1/c < \infty$, the equation is a quasilinear parabolic equation and we can apply the standard quasilinear theory, as mentioned before. But here we may consider a general setting in which $\Phi$ is only non-decreasing, so that the equation can be singular or degenerate parabolic. Such a generality has interesting applications that we have already pointed out, but the reader not used to or not willing to deal with the corresponding complications will do well in assuming that $\Phi$ is a continuous and strictly monotone function as in Chapter 5 and he will not lose the main action since our main interest lies in treating the PME, the HE, the FDE and close relatives. Note that even if $\Phi$ is smooth the equation can be degenerate at points where $\Phi'(u) = 0$.

However, the natural generality of the theory of this chapter leads to considering $\Phi$ as a m.m.g., which is an extension of the filtration equations considered in Chapter 5 and following. In order to discuss equation in the framework of the accretivity theory and obtain mild solutions we need associate to the formal expression $A_\Phi = -\Delta\Phi(u)$ an $m$-accretive operator $A_\Phi$ acting in a Banach space $X$ of functions on $\Omega$ with. In view of the formal estimates of Chapter 3 and the weak theories of Chapter 6 the space where we can find a contraction property is $X = L^1(\Omega)$. Attempts to find accretivity properties in $L^p$ spaces, $p > 1$, have failed.

Proving the $m$-accretivity of a suitable definition of the formal operator $A_\Phi$ implies discussing the existence, uniqueness and contractivity properties of equation $u + A_\Phi u \ni f$, i.e., $-\Delta\Phi(u) + u \ni f$. Putting $\beta = \Phi^{-1}$ we arrive at the semilinear elliptic equation written in standard form as

$$-\Delta v + \beta(v) \ni f. \tag{10.30}$$

Indication of the domain and precise boundary conditions complete the specification of the problem. We will discuss next several of the most common options.

### 10.3.1 *Problems in bounded domains*

In the case of a bounded domain with a regular boundary we can pose the homogeneous Dirichlet problem for equation (10.29), i.e., the one treated in Chapters 5 and 6, in the present abstract framework. To do that we define an abstract operator $\mathcal{A} = \mathcal{A}_\Phi : L^1(\Omega) \to L^1(\Omega)$ in the domain

$$D(\mathcal{A}) = \Big\{ u \in L^1(\Omega) : \exists v \in W_0^{1,1}(\Omega) : \Delta v \in L^1(\Omega), \quad v(x) \in \Phi(u(x)) \text{ a.e.} \Big\}$$

by the formula $\mathcal{A}(u) = -\Delta v$. This operator is $m$-$T$-accretive in $L^1(\Omega)$ and has a dense domain, as proved by Brezis and Strauss [133], 1973, see also [256]. This means that we have the following result.

**Corollary 10.21** *Let $\Phi$ be a maximal monotone graph (m.m.g.) with $0 \in \Phi(0)$. Then, $\mathcal{A}$ is an $m$-accretive operator in $X = L^1(\Omega)$. Moreover, it is $T$-accretive. The domain is dense in $X$.*

*The homogeneous Dirichlet problem for equation* (10.29) *with initial data $u_0 \in L^1(\Omega)$ and $f \in L^1(Q)$ admits a unique mild solution $u \in C([0,\infty) : L^1(\Omega))$. The maximum principle applies to those solutions.*

*In particular, $\mathcal{A}$ generates an order preserving semigroup of contractions in $L^1(\Omega)$ given by $S_t : u_0 \to u(t)$ which solves the Cauchy problem for the GPME with $f = 0$ in the sense of mild solutions.*

The result can be extended by replacing $-\Delta$ by the second-order elliptic operator $Lu = -D_j(a_{ij} D_i u) + D_i(a_i u) + au (D_i = \partial/\partial x_i)$, where $a_{ij}\xi_i\xi_i \geq \alpha|\xi|^2$, $\alpha > 0$, $\xi \in \mathbb{R}^d$, $x \in \Omega$ and $a \geq 0$, $a + \sum D_i a_i \geq 0$, $x \in \Omega$, see [133].

*Comparison of results*

Once we have on the table different concepts of solution, an urgent task of the theoretical analysis is to determine the relationship among them.

First of all, there are particular situations where the relationship is clear. Thus, in the case of the PME with $f = 0$, we have established in Chapter 8 that weak solutions are indeed strong. Now, strong solutions are mild solutions by a general result, Proposition 10.10.

Under the assumptions of Chapter 5 we have constructed weak energy solutions for good data. Now, it is almost immediate to see that these solutions satisfy the condition to be integral solutions. By the result (iii) of Proposition 10.20 we conclude that the mild solution coincides with this solution. Passing to the limit for general data using the results of Section 6.1 for limit solutions and Proposition 10.12 (iii) for mild solutions, we arrive at the following general result.

**Proposition 10.22** *Let us consider the homogeneous Dirichlet problem for the GPME posed in a bounded domain $\Omega$ with data $u_0 \in L^1(\Omega)$ and $f \in L^1(Q)$. Then, the concepts of limit solution defined in Section* 6.1 *and mild solution defined here are equivalent.*

### 10.3.2 *Problem in the whole space*

Let us first consider the case $\Omega = \mathbb{R}^d$. We take right-hand side $f \in L^1(\mathbb{R}^d)$. This is the setting proposed by Bénilan, Brezis and Crandall in their well-known 1975 paper [87]. This is their main result.

**Theorem 10.23** *Let $f \in L^1(\mathbb{R}^d)$ and let $\beta$ a maximal monotone graph in $\mathbb{R}^2$ with $\beta(0) = 0$. Then*

(i) *If $d \geq 3$, there exists a unique $v \in M^{d/(d-2)}(\mathbb{R}^d)$ which solves* (10.30) *in the weak sense. Moreover, $|\nabla v| \in M^{d/(d-1)}(\mathbb{R}^d)$.*

(ii) *If $d = 1, 2$, and under the additional assumption that 0 belongs to the interior of the $R(\beta)$, there exists a unique solution (but for a constant in some cases) which belongs to $W^{1,\infty}(\mathbb{R}^2)$ if $d = 1$, and to $W^{1,1}_{\rm loc}(\mathbb{R}^2)$ with $|\nabla v| \in M^2(\mathbb{R}^2)$ if $d = 2$.*

(iii) *The possible non-uniqueness happens only for very special cases where $\beta^{-1}(0)$ is an interval $I = [a, b]$, and then $\beta(v) = 0$, $R(v)$ is strictly included in $I$ and $\Delta v = f$.*

(iv) *In all cases the function $u = \Delta v + f$ is well-defined, and the map*

$$T : f \mapsto u$$

*is a contraction in $L^1(\mathbb{R}^d)$. More precisely, for functions $f_i \in L^1(\mathbb{R}^d)$, $i = 1, 2$, and corresponding $u_i = \Delta v_i + f_i$ we have*

$$\int [u_1 - u_2]_+ dx \leq \int [f_1 - f_2]_+ dx. \tag{10.31}$$

*This is technically called a $T$-contraction.*

Here, $M^p(\mathbb{R}^d)$ denotes the Marcinkiewicz or weak $L^p$ space, which is described in Section A.5. $R(v)$ denotes the range of $v$.

Let us translate the results of the theorem for our purposes: recall that $\beta = \Phi^{-1}$; we define the nonlinear operator $\mathcal{A} = \mathcal{A}_\Phi : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ in dimensions $d \geq 3$ in the domain

$$\begin{aligned} D(\mathcal{A}) = \Big\{ & u \in L^1(\mathbb{R}^d) : \exists v \in W^{1,1}_{\rm loc}(\mathbb{R}^d) : \Delta v \in L^1(\mathbb{R}^d), \\ & |\nabla v| \in M^{n/(n-1)}(\mathbb{R}^d), \quad \text{and} \quad u(x) \in \beta(v(x)) \text{ a.e.} \Big\} \end{aligned} \tag{10.32}$$

by the formula $\mathcal{A}(u) = -\Delta v$, with modifications on the space to which $u$ belongs if $d = 1, 2$ as indicated in the theorem. Then, we see that theorem 10.23 allows us to solve in a unique way the equation

$$\mathcal{A}(u) + u = f,$$

and the map $f \mapsto u = J_1(\mathcal{A})f$ is a contraction in $L^1(\mathbb{R}^d)$. Since we may clearly replace $\mathcal{A}$ by $\lambda\mathcal{A}$ in the theorem for any $\lambda > 0$, we can conclude that

**Corollary 10.24** *Let $\Phi$ be a maximal monotone graph (m.m.g.) with $0 \in \Phi(0)$. Then, $\mathcal{A}$ is an m-T-accretive operator in $X = L^1(\mathbb{R}^d)$ with dense domain.*

*Therefore, the Cauchy problem for equation (10.29) with initial data $u_0 \in L^1(\mathbb{R}^d)$ and $f \in L^1(Q)$ admits a unique mild solution $u \in C([0,\infty) : L^1(\mathbb{R}^d))$. The maximum principle applies to those solutions.*

*In particular, $\mathcal{A}$ generates an order preserving semigroup of contractions in $L^1(\mathbb{R}^d)$ given by $S_t : u_0 \to u(t)$ which solves the Cauchy problem for the GPME with $f = 0$ in the sense of mild solutions.*

The operator $\mathcal{A}$ is the appropriate realization of the formal operator $-\Delta\Phi$, $\Phi = \beta^{-1}$, in the $L^1(\mathbb{R}^d)$ context. The last part of the statement comes from the Crandall–Liggett Theorem 10.16.

There are a number of results that have been proved using the implicit time discretization and mild solution. An important instance is the following result on continuous dependence of Ph. Bénilan and M. G. Crandall [88].

**Proposition 10.25** *Let $\Phi$ be a m.m.g in $\mathbb{R}^2$ such that $0 \in \Phi(0)$. The mild solutions of the Cauchy problem for equation $u_t = \Delta\Phi(u)$ with initial data in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ depend continuously on $\Phi$ in the sense of convergence in $C([0,\infty) : L^1(\mathbb{R}^d))$ under the following conditions: either $d = 1, 2$, or $d \geq 3$ and*

$$\int_a^\infty r^{d-1}\beta(-r^{2-d})\,dr = -\int_a^\infty r^{d-1}\beta(r^{2-d})\,dr = \infty$$

*for some $a > 0$,with $\beta = \Phi^{-1}$ (more precisely, $\beta = (\Phi^{-1})^o$).*

The condition on $\beta$ is not technical, there are counterexamples when it is not satisfied. For further results in this direction, cf. [172]. They obtain an error estimate of the form

$$\|u_1(t) - u_2(t)\|_{L^1(\mathbb{R}^d)} \leq 4td\,\|u_0\|_{TV(\mathbb{R}^d)} \sup_{s\in\mathbb{R}} |\sqrt{\Phi_1'} - \sqrt{\Phi_2'}| \tag{10.33}$$

where $TV$ indicates norm in the space of functions of bounded variation.

**Remark** There is no difficulty in generalizing these results to the more general equation

$$-\sum_{i,j} \partial_i(a_{ij}\partial_j u) + \beta(x,u) = f \tag{10.34}$$

with coefficients $a_{ij}$ which are bounded measurable functions in $\Omega$ satisfying the ellipticity hypothesis

$$\sum_{i,j} a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \tag{10.35}$$

for some constant $\lambda > 0$ and all vectors $\xi \neq 0$, when $\beta(x,u)$ is a Carathéodory function, i.e., measurable in $x$ for all $u$ and continuous in $u$ for almost all $x$, with the conditions that it is monotone increasing in $u$ (for a.e. $x$) and uniformly bounded in $x$ for bounded $u$. More general conditions are sometimes useful and have been studied in the literature but need not concern us here.

*Comparison of results*

Arguments like the ones used in the previous section allow us to prove a result similar to Proposition 10.22 equating limit solutions and mild solutions.

### 10.3.3 *Cauchy problem with a peculiar nonlinearity*

This is just a complement to the previous theory. Of interest for some of the applications in diffusion is the special possibility that arises in dimensions $d = 1, 2$ of considering the elliptic equation (10.30) with maximal monotone graphs that do not satisfy the normalization condition $0 \in \beta(0)$. We take graphs such that $\beta > 0$ everywhere, and insist in keeping the condition that $v \in \beta(u)$ still be an integrable function, $v \in L^1(\mathbb{R}^n)$. This forces zero to be the infimum of $R(\beta)$. It follows that $\beta(s) \to 0$ as $s \to -\infty$, and also $u \to -\infty$ as $|x| \to \infty$.

*Dimension $d = 1$*

This was studied by Crandall and Evans [178]. The main result is

**Theorem 10.26** *Let $\beta(\mathbb{R}) \subset (0,\infty)$. Then the problem*

$$-u'' + \beta(u) \ni f, \qquad u'(\pm\infty) = 0, \tag{10.36}$$

*is solvable for every $f \in L^1_+(\mathbb{R}) = \{f \in L^1(\mathbb{R}), \int f(x)dx > 0\}$, iff $\beta$ is integrable at $-\infty$:*

$$\int_{-\infty}^{u} \beta(s)ds < \infty \quad \forall u \in D(\beta). \tag{10.37}$$

*The map $f \mapsto u'' + f \in \beta(u)$ is an $L^1$-contraction with domain $L^1_+(\mathbb{R})$.*

Typical examples of such m.m.g.'s are

$$\beta(s) = Ce^{as} \qquad \text{for some } \; C, a > 0,$$

$$\beta(s) = C|s|^{-p} \qquad \text{for some } \; C > 0 \text{ and } \; p > 1.$$

In the first case the domain is $\mathbb{R}$, in the second $D(\beta) = (-\infty, 0)$. This theorem allows us to solve the evolution equation $u_t = (\Phi(u))_{xx}$ where $\Phi = \beta^{-1}$, under the above conditions on $\beta$. Very interesting cases are very singular diffusion equations $u_t = \Phi(u)_{xx}$ like

$$u_t = (\log u)_{xx}, \quad \text{and} \quad u_t = (-u^{-m})_{xx} = m\left(\frac{u_x}{u^{m+1}}\right)_x, \tag{10.38}$$

where $0 < m < 1$. The behaviour of the solutions at infinity is such that

$$u(x,t) \to 0, \ \Phi(u(x,t)) \to -\infty \quad \text{as} \ |x| \to \infty.$$

*The option with non-zero flux at infinity*

We are expecting a problem like $-u'' + \beta(u) \ni f$ to have only one solution when posed in $\mathbb{R}$. This is not the case for the peculiar nonlinearities. The solution thus constructed is not the only possibility of a well-posed problem with $v \in L^1(\mathbb{R})$. Solutions with non-zero 'Neumann conditions at infinity' can be constructed; they were investigated in [450].

**Theorem 10.27** *Let $\beta(\mathbb{R}) \subset (0,\infty)$ and $\int_{-\infty} \beta(s)ds < \infty$. Then, the problem*

$$\begin{cases} -u'' + \beta(u) \ni f \\ u'(-\infty) = a, \quad u'(\infty) = -b, \end{cases} \tag{10.39}$$

*is solvable for every pair of constants $a, b \geq 0$ if $f \in L^1_+(\mathbb{R})$ and $\int f(x)dx > a + b$. The solution constructed in* [178] *is maximal in this set and corresponds to $a = b = 0$.*

The result implies the possibility of solving the nonlinear diffusion equations (10.38) with data

$$\Phi(u)_x \to a \quad \text{as} \ |x| \to -\infty, \quad \Phi(u)_x \to -b \quad \text{as} \ |x| \to \infty,$$

for any pair of flux data $a, b \geq 0$. See more details in [450, 515].

*Dimension $d = 2$*

This was studied in [499]. The typical example is in this case the exponential.

**Theorem 10.28** *Let $\beta(\mathbb{R}) \subset (0,\infty)$. For every $f \in L^1(\mathbb{R}^2)$ with $\int f(x)dx > 0$ there is a solution of the problem*

$$-\Delta u + \beta(u) \ni f \tag{10.40}$$

*in the class: $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$, $u \geq 0$, $|\nabla u| \in M^2(\mathbb{R}^2)$, $\Delta u \in L^1(\mathbb{R}^2)$ and the mass condition*

$$\int \Delta u \, dx = 0 \tag{10.41}$$

*if for every $b > 0$ we have*

$$\int_{-\infty} \beta(t) \exp(-bt)dt < \infty. \tag{10.42}$$

*Moreover, if we replace the hypothesis on $\beta$ by the weaker condition that* (10.42) *holds for all $b > b_0$ then equation* (10.40) *admits solutions with the mass condition $\int \Delta u dx = c$ if $\int f(x)dx > c$.*

The main application is the solution of the logarithmic diffusion equation $u_t = \Delta \log u$ in $\mathbb{R}^2$ with non-trivial flux data at infinity. For a complete study see [515], Chapter 8.

By contrast, the Dirichlet problem in a bounded domain with positive $\beta$ with zero Dirichlet data admits no solutions.

## 10.4 Time discretization and mass transfer problems

The theory of mass transfer problems and optimal transportation in the spirit of Monge and Kantorovich is the object of much current research focused on the possibility of using methods of nonlinear partial differential equations and the calculus of variations. It has a close connection with Monge–Ampère equations and convex analysis. It also adds a new way of looking at some problems of different disciplines.

An example of interesting application is the approximation to the solution of nonlinear diffusion equations via time discretization, but this time associated to energy minimization at every time step. This should be looked at as an alternative to the ITD of Section 10.2.1. The idea proposed by F. Otto [413] sees the HE, the PME and other nonlinear diffusion equations as *gradient flows* in a space of measures endowed with a convenient metric given by the Wasserstein distance, a tool developed in probability theory. This is a sketch of how it works.

- We initiate the time-step procedure, by taking a small step size $h > 0$ and an initial profile $u_0 \in \mathcal{P}$, where

$$\mathcal{P} = \left\{ f \in L^1(\mathbb{R}^d) : \ f \geq 0, \int f\, dx = 1 \right\}.$$

The iteration step is defined through the following rule: given $u_k \in \mathcal{P}$ we define $u_{k+1} \in \mathcal{P}$ as the minimizer of the functional

$$E_k(v) = \int_{\mathbb{R}^d} \beta(v)\, dx + \frac{1}{h} d(v, u_k)^2, \tag{10.43}$$

among all $v \in \mathcal{P}$. Here, $d(v,u)$ means the Wasserstein distance defined in $\mathcal{P}$ by the formula

$$d(u,v)^2 = \frac{1}{2} \inf \left\{ \iint |x-y|^2\, d\mu(x,y) \right\} \tag{10.44}$$

the infimum taken over all non-negative Radon measures $\mu$ whose projections (marginals) are $u(x)\, dx$ and $v(y)\, dy$.

The proposed minimization depends on the choice of the function $\beta$ entering the first energy term, which is a kind of potential energy, while the term $d(v, u_k)/2$ penalizes the difference between $v$ and $u_k$ in the form of the standard cost of the mast transfer. We assume that $\beta$ is a convex real function with superlinear growth. Obviously, we assume that $\int \beta(u_0)\, dx$ is finite.

Convexity and weak convergence arguments show that there exists a minimizer of the problem, which we call $u_{k+1}$. In this way, given a step $h$ and an end time $T = Nh$, we can construct a discrete solution $u_0, u_1, \ldots, u_N$ that we can join by linear interpolation into a continuous curve $u^{(h)} \in C([0,T]; \mathcal{P})$.

- The main question is now: what happens when the step size $h$ goes to 0?

We have to prove that $u^{(h)}$ converges to some $u$ in $L^1_{\text{loc}}(\mathbb{R}^d \times (0,T))$.

Then, the following holds

**Theorem 10.29** *The limit $u$ is a weak solution of the GMPE $u_t = \Delta\Phi(u)$ in $Q = \mathbb{R}^d \times (0,T)$ with initial data*

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^d,$$

*where $\Phi$ is related to $\beta$ by $\Phi'(u) = \beta''(u)u$.*

In order to get the HE, $\Phi(u) = u$, the choice of $\beta$ is $\beta(u) = u\,\log(u)$ and the integral $\int \beta(u)\, dx$ is the usual entropy of the mass distribution $u$. For the PME, $\Phi(u) = u^m$, so that $\beta(u) = u^m/(m-1)$, which gives rise to a non-standard entropy that we shall find again in Section 18.6.

- A main property of this approach is the contractivity property: the PME semigroup is contractive with respect to the Wasserstein distance: for any two solutions, $u_i(t)$, $i = 1, 2$, we have

$$d(u_1(t), u_2(t)) \le d(u_1(0), u_2(0)), \tag{10.45}$$

cf. Carrillo, McCann and Villani [152]. This parallels the contractivity in $L^1(\mathbb{R}^d)$ that is used in the Crandall–Liggett approach. Moreover, in $d = 1$ we have better properties: the semigroup in contractive with respect to the whole family Wasserstein distances $d_p$ defined by replacing definition (10.44) by

$$d_p(u,v)^2 = \frac{1}{2} \inf \left\{ \iint |x-y|^2 \, d\mu/x, y) \right\} \tag{10.46}$$

for $1 \le p < \infty$, with the expected extension to $p = \infty$. Though this property is true for the HE in several space dimensions, $d_p$-contractivity is false for the PME at least for large $p$, as is proved by Vázquez [514]. This again parallels the absence of $L^p$ contractivity for the PME for large $p$ that we establish in Section A.11.

## 10.5 Other concepts of solution

We have seen the appearance of the concept mild solution, a kind of limit solution naturally produced in the abstract setting when dealing with accretive operators. We have only seen in passing the integral solution. The reader may wonder if this is the end of the presentation of different solution versions. The answer if yes for the purposes of this book, but no for the general theory of nonlinear diffusion.

The consideration of more general diffusion equations has led to difficult problems in the study of existence and uniqueness. One type of such problems arises from the combination of diffusion with convection. A general class of elliptic–parabolic–hyperbolic degenerate equations of the form

$$\partial_t b(u) - \Delta\Phi(u) + \sum_{i=1}^{d} \partial_{x_1} F_i(u) = f, \tag{10.47}$$

under sufficiently general assumptions on the functions $b(u)$, $\Phi(u)$, $F_i(u)$: $b$ and $\Phi\colon \mathbb{R} \to \mathbb{R}$ are continuous and non-decreasing, $b(0) = \Phi(0) = 0$, $F \in C(\mathbb{R};\mathbb{R}^d)$, $F_j(0) = 0$ for $1 \le j \le d$. Since $b$ and $\Phi$ are not strictly increasing, the above formulations include Stefan problems, filtration problems, etc. The equation includes as the special case when $\Phi(u) \equiv 0$ the multi-dimensional hyperbolic conservation laws, so that it has attracted considerable interest throughout the last decades. In this case, the correct class of weak solutions, the so-called *entropy solutions*, was identified rather early by Kruzhkov [352], 1970. After the work Volpert and Hudjaev [522], a general uniqueness proof was given by Carrillo in [149] for equations including isotropic diffusions. There are many extensions; for instance, Bénilan and Touré [95] consider entropy solutions for $u_t = a(\cdot, u, \phi(\cdot, u)_x)_x + v$.

The theory concerns bounded solutions. When data are taken in $L^1$ the concept of entropy solution is no more sufficient and *renormalized entropy solutions* have been introduced to fill the gap. After the work [86] on conservation laws, the uniqueness of this type of solutions is proved in [150] for the class of degenerate elliptic–parabolic problems associated with the equation

$$\partial_t b(v) = \operatorname{div} a(v, Dv) + f.$$

Main results of this paper are also the proof that bounded weak solutions are renormalized solutions and that renormalized solutions satisfy the contraction property as stated in their Theorem 2.3.

**Theorem 10.30** *Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary. Under some assumptions on the nonlinearities of the equation $b$ and $a$, if for $i = 1, 2$ we let $v_{0i} : \Omega \to \overline{\mathbb{R}}$ be measurable with $b(v_{0i}) \in L^1(\Omega)$, $f_i \in L^1(Q)$ and $v_i$ are renormalized solutions of the HDP with data $(v_{0i}, f_i)$, $i = 1, 2$, then there* *exists* $\kappa \in \operatorname{sign}^+(v_1 - v_2)$ *such that for* a.e. $0 < t < T$,

$$\int_\Omega (b(v_1)(t) - b(v_2)(t))_+ dx \le \int_\Omega (b(v_{01}) - b(v_{02}))_+ dx + \int_0^t \int_\Omega \kappa\, (f_1 - f_2)\, dxdt.$$

*In particular, for any* $v_0 : \Omega \to \overline{\mathbb{R}}$ *measurable with* $b(v_0) \in L^1(\Omega)$, $f \in L^1(Q)$, *there is uniqueness of* $u = b(v)$ *for* $v$ *renormalized solution of the problem.*

Hence, these solutions are integral solutions in the sense defined above. Existence of renormalized solutions for general $L^1$ data is established in [16]. Let us also mention that much of the work on elliptic parabolic equations originates from Alt-Luckhaus' paper [11] and an important contractivity argument was introduced by Otto [412].

The case of anisotropic diffusion leads to new difficulties which are quite interesting from the mathematical point of view. Let us consider the general class of non-isotropic degenerate parabolic–hyperbolic equations of the form

$$\partial_t u + \sum_{i=1}^{d} (A_i(u))_{x_1} + \sum_{i,j=1}^{d} (A_{ij}(u))_{x_i x_j} = 0. \tag{10.48}$$

Recently, Chen and Perthame came up with what appears to be a good definition of solution in that setting. In [166] they introduced both *entropy* and *kinetic solutions* for (10.48) and proved their equivalence. See also [428] for the kinetic approach. A typical assumption on the matrix is that $(A_{ij}(u))$ is a symmetric $d \times d$ matrix of the form $A(u) = \sigma(u)\sigma(u)^T \ge 0$, $\sigma \in (L^\infty_{\text{loc}})^{d\times K}$, with $K \in [1, d]$.

Evans and Portilheiro proposed another notion of weak solutions for conservation laws included in (10.48), called *dissipative solutions*, based on the properties of accretive operators. This was shown in [440] to be equivalent, in the case of the conservation laws, to the entropy solution of Kruzhkov. The equivalence of entropy and dissipative solutions for the diffusive equation (10.48) was settled by Perthame and Souganidis [429] using a modification of the definition of dissipative solution. As an application they prove the strong convergence of a general relaxation-type approximation for such equations.

Bendahmane and Karlsen use the concept of renormalized entropy solutions in [78] to study this type of quasilinear anisotropic degenerate parabolic equations with integrable data $u_0 \in L^1(\mathbb{R}^d)$, $F \in L^1(Q_T)$ extending [166].

In a completely different direction, Caffarelli and Vázquez [144] propose to study the existence and uniqueness of solutions of the Cauchy problem for the PME in the context of *viscosity solutions*. The equation is better formulated in terms of the pressure variable and reads

$$\partial_t p = (m-1)p\, \Delta p + |\nabla p|^2, \tag{10.49}$$

and it is assumed that $p$ is continuous, non-negative and bounded. The concept of viscosity solution is a modification of the concept introduced by Crandall, Evans and Lions [179]. The study has been extended to the GPME in [125], the equation being written

$$\partial_t p = a(p)\,\Delta p + |\nabla p|^2. \tag{10.50}$$

Well-posedness is shown for continuous viscosity solutions with $p \ge 0$. But the proper definition and well-posedness of viscosity solutions is still an open problem in the case of signed solutions.

## Notes

The question of generation of continuous linear semigroups in Hilbert or Banach spaces is treated by a wide number of authors, like Davies [199], Goldstein [270], Pazy [420], Tanabe [487] or Yosida [529]. In many books the equation is written in the form

$$\frac{du}{dt} = A(u) + f,$$

so the operator has the reversed sign, and the positivity properties that we call maximal monotonicity and $m$-accretivity have to be replaced by negativity properties: dissipativity and $m$-dissipativity.

**Section 10.1.** Ideas and results in this section on maximal monotone operators are taken from Brezis' monograph [128]. See also Section A.1. The characterization of the operator corresponding to the GPME as a maximal monotone in $H^{-1}(\Omega)$ is due to Brezis, cf. [127], 1971.

Early results on semigroup generation by maximal monotone operators go back to Komura [349], 1967, who proved that any strongly continuous semigroup on a closed convex subset $K$ of a Hilbert space $H$ was generated by a maximal monotone operator $A$ such that $\overline{D(A)} = K$.

There are various interesting references to work on nonlinear evolution equations of the form $dA(u)/dt + B(u) \ni f$ involving monotone operators. Let us mention [213] which has applications to nonlinear parabolic equations, pseudoparabolic equations, and elliptic-parabolic systems.

**Section 10.2.** This section is inspired by two major contributions to the nonlinear diffusion theory, Crandall and Liggett's work [180], 1971, and Bénilan's thesis [79], 1972. Accretive operators were studied by many authors at the time, see for instance Kato [327, 328] and Brezis and Pazy [132], 1970. We follow the definitions of Bénilan, Crandall and Pazy's unpublished monograph [90]. Well-known expositions of the subject are Crandall's [176, 177] and Evans' [228]. We have found useful the survey by Bénilan and Wittbold [97] with a number of interesting open problems and the appendix of Andreu-Vaillo et al. [17].

The concept of mild solution as limit of the solutions of discretized problems is to be compared with the introduction of limit solutions from approximation with smooth problems in Section 6.1. Actually, the contractivity estimates at the foundations of both approaches are just the same. There are a number of studies on the error estimates of such implicit schemes in the general context, like [400, 401].

The concept of bracket and the realization in different $L^p$ spaces was studied by Sato [470].

**Section 10.3.** Interest in treating the GPME $u_t = \Delta\Phi(u)$ by semigroup methods is prominent in the already mentioned works of Bénilan, Brezis and Crandall in the early 1970s, where further references can be found. At that time it was studied by Vol'pert and Hudjaev [522]. We have also mentioned the work on the special cases in $d = 1, 2$, that opens the path to much recent work on fast and superfast diffusion equations, cf. the Lecture Notes [515]. At the time the dual equation $u_t = \Phi(\Delta u)$ was also treated and its accretivity in $L^\infty$ shown, like in Konishi's [350].

The convergence of the numerical schemes to solve the PME has been much studied. Early references are Raviart [445] and Graveleau and Jamet [274]. Close to this chapter is the paper by Berger, Brezis and Rogers [100]. The error estimate (10.33) is due to Cockburn and Gripenberg. Similar estimates are known for conservation laws after the work of Kruzhkov. Numerical methods were tried for self-similar solutions of the form $u = F(x/t^{1/2})$ at an early date, cf. Polubarinova-Kochina in 1948, [438], Philìp in 1955 [431]. Numerical investigations for density-dependent diffusion equations are reported in Crank's book [182] in 1956.

It was soon realized that in some respect fast diffusion equations can be better behaved than porous medium equations. The following result is proved in Evans [227]:

**Proposition 10.31** *Let $\Phi$ strictly increasing with $\Phi(0) = 0$ and $\Phi^{-1}$ Lipschitz continuous. Let us pose the GPME in a bounded domain with zero Dirichlet conditions, let $u_0 \in L^1(\Omega)$ and $f = 0$. Then, $u(t) = S_t u_0$ is a strong solution, differentiable* a.e. *into $L^1(\Omega) \cap L^2(\Omega)$. Moreover, $u_t \in L^2_{\mathrm{loc}}(Q)$.*

**Section 10.4.** Mass transfer problems and optimal transportation are treated by Villani in [521], see also the lectures of [14] and Caffarelli's [136].

The topic of gradient flows for nonlinear diffusion reported here originated in the work of Otto [413], see also Jordan, Kinderlehrer, and Otto [311]. A nice exposition with short proofs of the above assertions can be found in Evans' survey paper [230]. A full account of gradient flows in spaces of probability measures is found in the recent monograph [15] by Ambrosio, Gigli and Savarè. This is a fruitful approach which also applies to a wide variety of problems.

For the application to nonlinear diffusion equations and contractivity properties see also [5, 156] and the references of Section 18.6.

## Problems

**Problem 10.1** Prove that a linear operator $A$ acting on a finite dimensional Euclidean space is monotone if and only if its matrix referred to an orthonormal base is positive semidefinite. Prove that it is automatically maximal monotone. Do the same analysis for accretivity and $m$-accretivity.

**Problem 10.2** Find a simple example of an ODE in $X = \mathbb{R}$ where the strong solution of $u' + Au = f$ is not classical by choosing a discontinuous $f$.

**Problem 10.3** Compare the definition of strong solution for the PME proposed in Section 8.2 with the definition that follows from the abstract theory, Definition 10.6, applied to operator $A = -\Delta\Phi$ of Subsection 10.3.1. Do the same comparison for the Cauchy problem of Subsection 10.3.2. Conclude that they coincide.

**Problem 10.4** Show the compatibility of parts (i) and (ii) of Definition 10.6: if $J = I$ is compact and $u$ is a mild solution according to part (i), then it is also according to part (ii).

*Hint*: Restrict the discretizations to the subintervals $I' \subset J = I$.

**Problem 10.5** Prove Proposition 10.10.

*Hint*: This is the idea in [90], Proposition 1.4: take a strong solution $u$ of problem (10.11), take points $t_k$ where $u$ is differentiable, the equation is satisfied and they are Lebesgue point for $f$. Then put

$$f_k = f(t_k) + \frac{u(t_k) - u(t_{k-1}}{t_k - t_{k-1}} - u'(t_k).$$

Show then that this is a $\varepsilon$ discretization and that the discretized solution $U(t_k) = u(t_k)$ is a good $\varepsilon$-approximation.

**Problem 10.6** Justify the change of variables leading to formula (10.22) for classical and strong solutions. Formulate the Crandall-Liggett Theorem 10.16 for $A_\omega$ accretive operators. Formulate also Theorem 10.17 in the same way.

**Problem 10.7** Prove the assertion at the end of Subsection 10.3.2 equating limit solutions and mild solutions.

**Project*** Analyse the existence of mild solutions and a semigroup for the Homogeneous Neumann problem for the GPME in a bounded domain.

**Open problem** It is not known what are the conditions under which mild solutions of the GPME are indeed very weak solutions or even distributional solutions.

# 11

# THE NEUMANN PROBLEM AND PROBLEMS ON MANIFOLDS

In this chapter we complete the investigation of previous chapters on the Dirichlet and Cauchy problems by applying the techniques to other important problems. We select two directions, the Neumann boundary conditions and the problems posed on manifolds. The text goes at a quick pace in the parts that are applications of already presented techniques and can be considered as training. Many variations are possible and some are proposed as suggestions to the reader.

The first sections deal with the questions of existence, uniqueness and properties of the Neumann problem posed in a bounded domain of $\mathbb{R}^d$. The problem posed in the classical setting with smooth and non-degenerate nonlinearity function $\Phi$ has been addressed in Chapter 3 and a number of basic estimates were also derived. It was shown there that the theory for the Dirichlet and the Neumann problems has many similarities at the basic level, as well as some important differences, like mass conservation.

Here, we address the problem with non-smooth or degenerate $\Phi$ and integrable initial data in classes of weak solutions. The theory relies heavily of what has been developed to solve the Dirichlet problem. The material serves as a reminder of the techniques learned in Chapters 5, 6 and 8. In Section 11.1 we introduce the problem and concepts of weak solution, prove a uniqueness result and present examples. In most of the chapter we take $f = 0$ to simplify the presentation. Section 11.2 reviews the theory of existence and uniqueness of weak solutions and limit solutions. Section 11.3 proves better estimates and boundedness of solutions in the case of the PME.

We devote Section 11.4 to examine the mixed problems and problems posed in exterior space domains. This is intended as a reference for researchers.

The second main topic of this chapter is the theory of PME and GPME on Riemannian manifolds, Section 11.5. This is wide open field. We choose the compact case for simplicity; the already developed techniques fit nicely to this case.

## 11.1 Problem and weak solutions

We assume that $\Omega$ is a bounded domain in $\mathbb{R}^d$, $d \geq 1$, with a $C^2$ boundary $\Gamma = \partial\Omega$ for simplicity. We pose the homogeneous Neumann problem for the filtration equation in complete form, $u_t = \Delta\Phi(u) + f$. In this study we will use the symbols $Q = \Omega \times \mathbb{R}_+$, $Q_T = \Omega \times (0,T)$, $Q^\tau = \Omega \times (\tau, \infty)$, and $Q_T^\tau = \Omega \times (\tau, T)$;

$\Sigma_T = \partial\Omega \times [0, T)$ is the lateral boundary. We denote by $\nu = \nu(x,t)$ the outer normal to the boundary $\partial\Omega$ which is well defined everywhere if $\Omega$ is $C^1$. We assume that $\Phi$ is a continuous and increasing function : $\mathbb{R} \to \mathbb{R}$. The PME, the HE and the FDE are included and signed solutions are admitted.

*Problem HNP*

*Given* $u_0 \in L^1(\Omega)$, $f \in L^1(Q)$, *find a locally integrable function* $u = u(x,t)$ *defined in* $Q_T$, $T > 0$, *that solves the set of equations*

$$u_t = \Delta\Phi(u) + f \quad \text{in } Q_T, \tag{11.1}$$

$$u(x,0) = u_0(x) \qquad \text{in } \Omega, \tag{11.2}$$

$$\frac{\partial}{\partial\nu}\Phi(u(x,t)) = 0 \qquad \text{in } \Sigma_T. \tag{11.3}$$

*in a weak sense to be precisely defined. The time* $T > 0$ *can be finite or infinite.*

Moreover, we want to find $u$ in a suitable functional class that guarantees uniqueness and continuous dependence on the data. Though we will obtain solutions for all $T > 0$, i.e. with $T = \infty$, it is interesting for technical reasons to allow $T < \infty$.

### 11.1.1 *Concept of weak solution*

First of all, the definition of weak solution and very weak solution (distribution solution) for the filtration equation was introduced in Definitions 5.1 and 5.2. These definitions are local (i.e., tested only in the interior of the space-time domain), hence valid as starting point for the Neumann problem. Note also that we could take $\mathcal{D}(Q_T) = C_c^\infty(Q_T)$ as space for the test functions and recover the validity of the larger test spaces by density. The remarks made in Subsection 5.2.1 apply.

We now take a step further and insert the homogeneous boundary condition to get the following definition.

**Definition 11.1** *A locally integrable function $u$ defined in $Q_T$ is said to be a weak solution of equation* (11.1) *with Neumann boundary condition* (11.3) *if*

(i) $\Phi(u) \in L^1_{\rm loc}(0, T : W^{1,1}(\Omega))$, *and* $u \in L^1(\Omega \times (\tau, T-\tau))$ *for all* $\tau > 0$*;*
(ii) $u$ *satisfies the identity*

$$\iint_{Q_T} \{\nabla\Phi(u)\cdot\nabla\eta - u\eta_t\}\,dxdt = \iint_{Q_T} f\eta\,dxdt \tag{11.4}$$

*for all test functions* $\eta \in C^1(\overline{Q}_T)$ *which vanish on for* $0 \le t \le \tau$, *and also for* $T - \tau \le t \le T$ *for some* $\tau > 0$.

We may wonder where is the boundary condition (11.3) included in this formulation. The answer is that it is hidden in the absence of the boundary term $\iint_\Sigma \partial_\nu \Phi(u)\eta\, dSdt$ that should appear in the integration by parts that is needed to obtain formula (11.4) from the original GPME; this is again a trick of weak theories. Caution: note that the class of test functions is larger than in the Dirichlet problem; it can be replaced by the smaller class where we also impose the condition that $\partial_\nu \eta = 0$ on $\Sigma$ and the definition is equivalent. See the explanation proposed as Problem 11.1.

The final step consists of including the initial data. There are several ways of doing it. We propose

**Definition 11.2** *A locally integrable function $u$ defined in $Q_T$ is said to be a weak solution of the HEN Problem* (11.1)–(11.3)

(i) $u \in L^1(Q_T)$, $\Phi(u) \in L^1(0, T : W^{1,1}(\Omega))$;
(ii) *$u$ satisfies the identity*

$$\iint_{Q_T} \{\nabla\Phi(u) \cdot \nabla\eta - u\eta_t\}\, dxdt = \int_\Omega u_0(x)\eta(x,0)dx + \iint_{Q_T} f\eta\, dxdt \tag{11.5}$$

*for any function $\eta \in C^1(\overline{Q}_T)$ which vanishes for $t = T$.*

Call $\mathcal{C}$ the class of test functions; it can be replaced by the smaller class $\mathcal{C}_{n0}$ (subscript stands for Neumann zero) where we also impose the condition that $\partial_\nu \eta = 0$ on $\Sigma$ and the definition is equivalent. We may even assume that $\eta$ is $C^\infty$ smooth. In fact, the restricted class must be chosen if we consider the concept of very weak solution.

**Definition 11.3** *An integrable function $u$ defined in $Q_T$ is said to be a very weak solution of Problem* (11.1)–(11.3) *if $u, \Phi(u) \in L^1(Q_T)$, and $u$ satisfies the identity*

$$\iint_{Q_T} \{\Phi(u)\, \Delta\eta + u\eta_t + f\eta\}\, dxdt + \int_\Omega u_0(x)\eta(x,0)dx = 0 \tag{11.6}$$

*for any function $\eta \in C^{2,1}_{x,t}(\overline{Q}_T)$ which vanishes for $t = T$ and such that $\partial_\nu \eta = 0$ on $\Sigma$.*

This parallels Definition 6.2 for the Dirichlet problem. Note that the conditions on the test functions are the natural ones when we have a classical solution of the HNP and want to write the weak or very weak formulations using integration by parts.

We will work preferentially with weak solutions. We call $\mathcal{WS}$ the class of weak solutions obtained when $u_0 \in L^1(\Omega)$ and $f \in L^1(Q_T)$. The initial function $u_0$ of condition (11.2) is built into the integral formulation (11.5), and is actually satisfied in a very weak sense. The considerations made in Chapter 5 about the way in which the initial data are taken in the Dirichlet problem (Propositions 5.1 and 5.2) apply also in this case

### 11.1.2 *Examples of solutions of the HNP*

(i) Let simplest solution of the HNP is given by the constant functions $u = c$ that solve the problem with constant initial data and $f = 0$. The constant $c$ need not be positive.

(ii) Take now $d = 1$. Let $u$ be a (classical, weak) solution of the Cauchy problem in $Q_T$ with periodic and symmetric data

$$u_0(x + 2a) = u_0(x), \qquad u_0(x) = u_0(-x)\ \forall x \in \mathbb{R}, \tag{11.7}$$

for some $a > 0$. Then, $u$ restricted to $Q_a = (0, a) \times (0, T)$ is a (classical, weak) solution of the Neumann problem in that domain.

(iii) A similar result holds when $d > 1$ and $u_0$ is symmetric and periodic in all coordinate directions. Symmetry means here that for all $i = 1, 2, \ldots, d$ we have

$$u_0(x) = u_0(S_i x) \quad \forall x \in \mathbb{R}$$

where $S_1(x_1, \ldots, x_d) = (-x_1, x_2, \ldots, x_d)$ and so on for $S_2, \cdot, S_d$. Periodicity means that there exist $a_i > 0$, $i = 1, 2, \ldots, d$, such that

$$u_0(x + a_i \mathbf{e}_i) = u_0(x) \quad \forall x \in \mathbb{R}, \tag{11.8}$$

where $\mathbf{e}_i$ is the unit vector along the $i$-th coordinate. Then we take $\mathcal{B}_a = (0, a_1) \times \ldots \times (0, a_n)$ and $Q_a = \mathcal{B}_a \times (0, T)$. Under these assumptions, $u$ restricted to $Q_a$ solves the Neumann Problem.

(iv) In the PME case, there is a family of solutions that can be obtained by the method of separation of variables introduced in Section 4.2. We write $u(x,t) = G(t)\,F(x)$, insert into the equation and find $G(t) = ((m-1)\lambda t + C)^{-1/(m-1)}$ for the time dependent factor, while the profile function $F$ must solve

$$\begin{cases} \Delta(|F|^{m-1}F) + \lambda F = 0 & \text{on } \Omega \\ \partial_\nu(|F|^{m-1}F) = 0 & \text{on } \partial\Omega. \end{cases} \tag{11.9}$$

The parameter $\lambda > 0$ can be scaled out. There is a first solution $\lambda = 0$, $F = 1$ (or constant), and a sequence of other solutions which change sign $F_n$. This is a consequence of the theory of nonlinear elliptic equations, and was proved in [8] to which we refer the reader for a proof.

## 11.2 Existence and uniqueness for the HNP

The essentials of the theory of energy weak solutions and limit solutions offers few novelties with respect to the Dirichlet problem developed in Chapters 5ff.

### 11.2.1 *Uniqueness and energy solutions*

There is no novelty in the following result that we have established for the Dirichlet problem.

**Theorem 11.1** *Under the additional assumption that* $\Phi(u) \in L^2(0,T : H^1(\Omega))$ *and* $u \in L^2(Q_T)$, *Problem* (5.1)–(5.3) *has at most one weak solution.*

The proof is left as an exercise for the reader, following Theorem 5.3. This result and the a priori estimates of Chapter 3 show the importance of restricting ourselves to the subclass of weak energy solutions, $\mathcal{WES}$, where $DE(u)$ and $E(u)$ (as defined in (5.17), (5.18)) are finite.

### 11.2.2 *Existence and properties for good data*

The results we have obtained for the Dirichlet problem still hold. In particular, when $u_0$ belongs to the space $L_\Psi(\Omega)$ we have a good theory of energy weak solutions.

**Theorem 11.2** (i) *Under the stated conditions on* $\Phi$, *assume that* $u_0 \in L_\Psi(\Omega)$ *and* $f = 0$ *for simplicity. Then, Problem* (11.1)–(11.3) *has a weak energy solution defined in an infinite time interval,* $T = \infty$. *We have* $u \in L^\infty((0,T) : L_\Psi(\Omega))$ *and* $\Phi(u) \in L^2(0,T : H_0^1(\Omega))$, *and the energy inequality*

$$\iint_{Q_T} |\nabla\Phi(u)|^2 \, dxdt + \int_\Omega \Psi(u(x,T))\, dx \le \int_\Omega \Psi(u_0(x))\, dx. \tag{11.10}$$

*holds. Moreover,* $u \in C([0,\infty) : L^1(\Omega))$.

(ii) *This solution is obtained as limit of classical solutions of approximate problems.*

(iii) *Comparison holds: if* $u, \widehat{u}$ *are weak solutions with initial data such that* $u_0 \le \widehat{u}_0$ a.e. *in* $\Omega$, *then* $u \le \widehat{u}$ a.e. *in* $Q$. *In particular, if* $u_0$ *in* $\Omega$, *then* $u \ge 0$ *in* $Q$.

(iv) *Mass conservation: for every* $t > 0$ *we have*

$$\int_\Omega u(t)\, dx = \int_\Omega u_0\, dx. \tag{11.11}$$

(v) *For every two solutions* $u$ *and* $\widehat{u}$ *with initial data* $u_0$, $\widehat{u}_0$ *resp., we have for every* $t > \tau \ge 0$

$$\|(u(t) - \widehat{u}(t))_+\|_1 \le \|(u(\tau) - \widehat{u}(\tau))_+\|_1. \tag{11.12}$$

**Proof** Existence is obtained in the limit of the solutions $u_n$ of approximate problems following the proof of Theorem 5.7 for the Dirichlet case. The difference is that now $\partial_\nu u_n(x,t) = 0$ on $\Sigma$, and this provokes small changes in the proofs that we leave to the reader. Comparison holds in the same way. The continuity is proved as part of Theorem 6.2. Mass conservation is immediate since the property was proved in Subsection 3.3.3 in the classical case. Contraction holds as in Proposition 6.1. ■

A number of other properties derived for the Dirichlet problem still hold. like the ones derived in Section 5.6 which are based on the estimates of Chapter 3. We have an estimate for $u_t$

$$\iint_{Q_T} t\Phi'(u)|u_t|^2\,dxdt + \frac{T}{2}\int_\Omega |\nabla w(T)|^2\,dx \le \frac{1}{2}\iint_{Q_T} |\nabla w|^2\,dxdt. \tag{11.13}$$

However, the universal bound in sup norm of Section 5.8 cannot be true since all constants are solutions. This and mass conservation are the first marked difference between the properties of both problems.

On the other hand, the local results of Chapter 7 apply to show that bounded solutions are continuous in $Q_T$. The problem of showing that weak solutions are bounded is a main new problem of this chapter.

### 11.2.3 *Existence for $L^1$ data*

Using the property of contraction we can extend the class of initial data by taking limits, as was done in Subsection 6.1.2 for the Dirichlet case.

**Theorem 11.3** *Let $\Phi$ be a monotone function as above. Then, for any $u_0 \in L^1(\Omega)$ there exists a unique $u \in C([0,\infty) : L^1(\Omega))$ that solves problem PHD in the sense of limit solutions. The weak energy solutions of Theorem* 11.2 *are limit solutions. The map: $u_0 \mapsto u$ is an ordered contraction from $L^1(\Omega)$ into $C([0,\infty) : L^1(\Omega))$ in the sense that* (11.12) *holds for limit solutions.*

Let us write the limit solution $u(t)$ obtained for data $u_0 \in L^1(\Omega)$ as $S_t u_0$. Then, we have as in Theorem 6.11:

**Corollary 11.4** *The maps $S_t$ define a continuous semigroup of contractions in $X = L^1(\Omega)$, and $S_t$ preserves $E = L^1(\Omega)_+$. The semigroup is uniformly bounded.*

Finally, let us compare the Dirichlet and Neumann problems.

**Proposition 11.5** *Let $u_0 \in L^1(\Omega)$, $u_0 \ge 0$, and let $u_D$ be the solution of the HDP and $u_N$ the solution of the HNP, both with $f = 0$. Then $0 \le u_D(x,t) \le u_N(x,t)$ in $Q$.*

**Proof** The proof is immediate in the classical case by the maximum principle (note that $u_D = 0 \le u_N$ on $\Sigma$, and $u_D = u_N$ for $t = 0$). Passing to the limit we get the result for all limit solutions. ■

**Remark** We point out that when $u_0$ is bounded and positive everywhere, then the problem is no longer degenerate and the standard quasilinear parabolic theory proves that the solution $u(x,t)$ is a positive and smooth classical solution.

### 11.2.4 *Neumann problem and abstract ODE theory*

The existence of solutions of the HNP can also be studied in the framework of the abstract ODE theory developed in Chapter 10 by either the maximal monotone approach or the Crandall–Liggett approach. The second idea is used by Alikakos and Rostamian in their paper [8]. We leave this topic as a study project for the reader.

### 11.2.5 *Convergence to the Cauchy problem*

The relation between the Neumann problem and the Cauchy problem is clarified by the following result.

**Theorem 11.6** *Let $\Omega_n$ an expanding sequence of smooth domains such that $\bigcup_n \Omega_n = \mathbb{R}^d$. Let $u_{0n} \in L^1(\Omega_n)$, $u_0 \in L^1(\mathbb{R}^d)$, and assume that $u_{0n} \to u_0$ in the obvious $L^1(\mathbb{R}^d)$ sense (i.e., extending $u_{0n}$ by zero outside $\Omega_n$). Let $u_n$ be the limit solution of the HNP in $Q_n = \Omega_n \in \mathbb{R}_+$ with data $u_{on}$, and let $u$ be the limit solution of the CP in $Q = \mathbb{R}^d \times \mathbb{R}_+$. Prove that $u_n \to u$ in $C([0,\infty) : L^1(\mathbb{R}^d))$.*

See Problem 11.9.

## 11.3 Results for the HNP with a power equation

As in the Dirichlet case, a number of 'advanced properties' are only true, or at least they are more easily derived, when $\Phi(u) = |u|^{m-1}u$, i.e., for the pure PME. We can derive new estimates for all solutions of the HNP based on the homogeneity of the equation. We have

**Lemma 11.7** *All non-negative weak or limit solutions of problem* HDP *for the PME satisfy the estimate*

$$u_t \geq -\frac{u}{(m-1)t} \tag{11.14}$$

*in the sense of distributions in $Q_T$. Any non-negative weak solution of problem* (5.1)–(5.3) *satisfies $u_t \in L^p_{\rm loc}(Q)$ for any $p \in [1,(m+1)/m)$.*

The proof of the first fact is similar to the second proof we gave of Lemma 8.1 because the scaling properties are the same. For the last statement see Corollary 8.3. We can use this inequality to prove boundedness of the solutions with data in $L^1(\Omega)$, a main result of the theory.

**Theorem 11.8** *Every limit solution $u(x,t)$ of the HNP with initial data $u_0 \in L^1(\mathbb{R}^d)$ is bounded above for all $t > 0$. More precisely, if $\int u_0^+(x)\,dx = M > 0$ then*

$$u(x,t) \leq M\,F(M^{m-1}t) \quad \forall t > 0, \tag{11.15}$$

*where $F : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-increasing function that depends only on $m$ and $\Omega$.*

**Proof** We recall that $u_0^+(x) = \max\{u_0(x), 0\}$.

(i) First of all some reductions. By the maximum principle we need only consider the case $u_0 \geq 0$. Next, we can use the scaling $\widehat{u}(x,t) = Mu(x, M^{m-1}t)$ to reduce the positive mass to $M = 1$ without changing the space domain. By approximation, there is also no loss of generality in assuming that $u_0$ is bounded, positive and smooth so that the solution is classical.

(ii) We take $t > 0$ and use estimate (11.14) to write the equation as

$$-\Delta u^m(t) \le c(t)u(t)$$

with $c(t) = ((m-1)t)^{-1}$. We now fix $t > 0$ and put $g = u^m(t)$. Then, $g$ is a smooth positive subsolution of the nonlinear elliptic problem

$$\begin{cases} \Delta g + cg^q = 0, & \text{in } \Omega, \\ \partial_\nu g = 0 & \text{on } \partial\Omega, \end{cases} \tag{11.16}$$

with $q = 1/m \in (0,1)$. We also have the mass constraint

$$\int_\Omega g^q(x)\,dx = 1. \tag{11.17}$$

(iii) Let us prove that the solutions of the last problem are bounded. The conclusion is easy in dimension $d = 1$. In that case we prove that $|g'| \le c$, and then $|g| \le C(c,\ell)$ where $\ell = \text{length}(\Omega)$. That implies the conclusion with

$$F(t) = [C(((m-1)t)^{-1}, \ell)]^{1/m}.$$

(iv) In case $d \ge 2$, we proceed in two steps: improvement of integrability and boundedness of highly integrable $f$. The last part consists in using the equation to show that as a subsolution of the Laplace–Poisson equation, the function $g = u(t)^m$ is bounded if we prove that the right-hand side of equation (11.16), $h = g^q = u(t)$, belongs to the space $L^r(\Omega)$ with $r > d/2$.

We have to establish that regularity and we do this by iteration. We use the regularity theory of the Laplace equation $\Delta g = h$, with homogeneous boundary conditions, to solve $-\Delta G = h$ with the given boundary and mass conditions. Since $h \in L^1(\Omega)$, the theory implies that $G \in L^p(\Omega)$ for all $p \le d/(d-2)$. By comparison, so does $g$, and this means that we can assume that $h = g^q \in L^p(\Omega)$ for all $p \le p_1 = dm/(d-2)$; we thus get an improvement of the regularity of $h$. Successive rounds of the same iteration procedure produce numbers $p_n$ such that

$$p_{n+1} = \frac{mdp_n}{d - 2p_n}$$

as long as $p_n < d/2$. In this way we get $p_{n+1} > d/2$ in a finite number of steps and the proof is complete.

(vi) Using the maximum principle we can prove that $F = F(t; m, \Omega)$ is non-increasing in $t$ and monotone in $\Omega$. ■

**Remark** Contrary to the proofs of boundedness for the Dirichlet and Cauchy problems, this proof is not based on identifying the worst problem, in other words the extremal situation. As a consequence, the form of function $F$ in the estimate is rather vague.

The result is better when the total mass is zero. This is yet another novelty with respect to previous problems. We have

**Theorem 11.9** *For solutions with mass $M = 0$ the $L^\infty$ bound takes the form*

$$|u(x,t)| \le C(m,\Omega)\, t^{-1/(m-1)} \tag{11.18}$$

*The result is sharp.*

**Proof** The proof we present uses a technique of energy inequalities that we have already used in the study of extinction for fast diffusion in Subsection 5.10.1. Let $d \ge 3$. By the usual methods of integration by parts we have for every $q > 1$

$$\frac{d}{dt}\int_\Omega |u|^q dx = -q(q-1)m\int_\Omega |u|^{m+q-3}|\nabla u|^2 dx.$$

We now recall that the standard Sobolev embedding into $L^{p*}(\Omega)$ holds for $W^{1,p}(\Omega)$ functions with zero average. We want to apply that result to the right-hand side with $f(x) = u(x,t)^{(m+q-1)/2}$; now, in this case $f(x)^\varepsilon = u(x,t)$ has average zero, with $\varepsilon = 2/(m+q-1)$. It can be checked that the embedding is still true, hence

$$\frac{d}{dt}\int_\Omega |u|^q dx \le -C(m,d,q,\Omega)\left(\int_\Omega |u|^s\right)^r$$

with $s = (m+q-1)d/(d-2)$ and $r = (d-2)/d$. We now choose $q \ge d(1-m)/2$ so that $s \ge q$, and put $X_q = (\int_\Omega |u|^q dx)^p$ to get from the comparison of $L^p$ norms

$$-X_q(t)^{q-1}\frac{dX_q}{dt} \ge CX_s^{m+q-1}, \quad \text{hence} \quad -\frac{dX_q}{dt} \ge CX_q^m \tag{11.18$a$}$$

where $C$ denotes different constants depending on $m, d, q, \Omega$. Integrating the ODE we get

$$X_q(t) \le (X_q(0)^{1-m} + C(m-1)t)^{-1/(1-m)}$$

The calculation is justified by approximation. Taking now $q > d/2$ and starting the calculation again we get the same conclusion for $L^\infty$-norm. We leave these last details to the reader, as well as the cases $d = 1, 2$.

Sharpness is derived from the existence of solutions in the separated variables form. ∎

**Remarks**

(1) Solutions with non-zero mass do not decay to zero at all as $t \to \infty$. We show this fact in Chapter 20.

(2) Another limit of interest is the behaviour near $t = 0$. We propose to address this question in Problem 11.8.

## 11.4 Other boundary value problems

There are some interesting initial and boundary value problems whose theory can be obtained by means of the techniques and concepts introduced in the above chapters. We want to present some relevant cases next.

### 11.4.1 *Exterior problems*

The problems are posed in an exterior domain $\Omega$, i.e., the complement in $\mathbb{R}^d$ of the closure of a bounded domain. Let $\Gamma = \partial\Omega$. We pose the mixed problem consisting of solving the equation together with initial conditions and homogeneous Dirichlet conditions at the lateral boundary.

*Problem EHDP*

*Given $u_0 \in L^1_{\text{loc}}(\Omega)$, $f \in L^1_{\text{loc}}(Q)$, find a locally integrable function $u = u(x,t)$ defined in $Q_T$, $T > 0$, that solves the set of equations*

$$u_t = \Delta\Phi(u) + f \qquad \text{in } Q_T, \tag{11.19}$$

$$u(x,0) = u_0(x) \qquad \text{in } \Omega, \tag{11.20}$$

$$u(x,t) = 0 \qquad \text{in } \Sigma_T. \tag{11.21}$$

*in a weak sense to be precisely defined.*

At this level there is no difference with the HDP treated in Chapter 5 for bounded domains. Such a difference appears in the form of the integrability conditions under which we can solve the problem and which affect the behaviour as $|x|$ goes to infinity. Existence of solutions can be obtained as limit of problems posed in bounded domains, typically $\Omega_R = \Omega \cap B_R(0)$. This is the approach followed in Chapter 9 to obtain the solutions of the Cauchy problem. We refrain from further details which can be developed with some care.

*One-dimensional problem in the half line*

A slightly different but simple problem that falls into this class is the problem posed in a semi-infinite domain $\Omega = (0,\infty) \in \mathbb{R}$. Typical data in that case are $u(0,t) = C$ (Dirichlet) or $(u^m)_x(0,t) = 0$ (Neumann). Both can be reduced to the Cauchy problem by standard tricks. In the Dirichlet case we extend the initial data to $x < 0$ by means of the antisymmetric definition

$$\widetilde{u}_0(x) = -u_0(-x). \tag{11.22}$$

The solution $\widetilde{u}$ of the Cauchy problem exists and is unique under the condition $\int |u_0(x)|\,dx < \infty$. The symmetry condition is conserved for all times, $\widetilde{u}(x,t) = -\widetilde{u}(-x,t)$. From the continuity of the equation we conclude that $\widetilde{u}(0,t) = 0$. This shows that the restriction of $\widetilde{u}$ to $x > 0$ is therefore a solution of the original problem.

We leave as a problem the construction of the solution of the Neumann problem by means of symmetric extension $\widetilde{u}_0(x) = u_0(-x)$.

### 11.4.2 *Mixed problems*

The problem with mixed conditions consists in selecting an open subset of the space boundary $\Gamma_0 \subset \partial\Omega$ where we impose homogeneous boundary conditions while we impose Neumann conditions on $\Gamma_1 = \partial\Omega \setminus \Gamma_1$. Bénilan has shown in [81] that this problem can be formulated together with the Dirichlet and the Neumann cases as a unique abstract problem in a suitable setting. Such an approach uses the ideas of Chapter 10 and is a very interesting topic that we recommend as an advanced project.

The problem can also be put for non-zero data and more general equations. As an example of a practical case, this is the version of [330], proposed as a model of filtration in partially saturated porous media:

$$
\begin{aligned}
\partial_t b(u) - \nabla\cdot a[\nabla u + k(b(u))] &= f && \text{in } \Omega \times (0,\infty) \\
u &= h(t,x) && \text{on } \Gamma_0 \times (0,\infty) \\
\nu \cdot a[\nabla u + k(b(u))] &= g(t,x) && \text{on } \Gamma_1 \times (0,\infty)
\end{aligned}
$$

### 11.4.3 *Nonlinear boundary conditions*

Another way of generalizing the boundary conditions that often appears in the literature consists of using as boundary condition the relation

$$
w_\nu + \gamma(w) \ni 0 \tag{11.23}
$$

where $\gamma$ is a monotone increasing function (or even a maximal monotone graph, see Section A.3). This is already contained in Bénilan's thesis, [79]. A basic reference for the study of $L^1$ estimates is Bénilan, Crandall and Sacks's paper [92]. See also Mazon and Toledo [386] and Igbida [304]. Here is a complete problem

$$
\begin{cases}
\partial_t u - \Delta w = 0, & w \in \varphi(u) \quad \text{in} \quad (0,+\infty) \times \Omega, \\
-\partial_\nu w \quad \in \gamma(w) & \text{on} \quad (0,+\infty) \times \partial\Omega, \\
u(0) \quad = u_0 & \text{in} \quad \Omega,
\end{cases}
$$

where $\Omega$ is a bounded domain of $\mathbb{R}^d$ with smooth boundary $\partial\Omega$ and $u_0 \in L^1(\mathbb{R}^d)$; $\varphi$ and $\gamma$ are two maximal monotone graphs in $\mathbb{R}$ with domains $D(\varphi) = \mathbb{R}$ and $D(\gamma) = \mathbb{R}$ or $D(\gamma) = \{0\}$, respectively, and satisf $0 \in \varphi(0) \cap \gamma(0)$. The above problem includes the homogeneous Dirichlet and Neumann boundary conditions as particular cases.

We may also modify the above results to provide comparison for a subsolution and a supersolution defined in unbounded domains, for instance when $\Omega = (-\infty, 0)$ in one space dimension, or $\Omega = \mathbb{R}^d - B$, where $B$ is a ball. We need to impose conditions on the initial and lateral boundary plus integrability on the supersolution as $|x| \to \infty$, $t > 0$, like (9.8).

### 11.4.4 *Dynamic boundary conditions*

There is still a type of boundary conditions called dynamical, which include a time derivative in the boundary expression. Here is a typical example: Su [486] poses the problem of solving the nonlinear degenerate diffusion equation

$$\partial_t\theta(u) - \nabla\cdot a(\theta(u),\nabla u) = f(\theta(u))$$

in $(0,T)\times\Omega$, where $a(z,r)$ satisfies a certain ellipticity condition, $\theta(z)$ is non-decreasing, $f(z)$ is continuous, and $\Omega\subset\mathbb{R}^d$. The boundary condition is

$$\partial_t\beta(u) + a(\theta(u),\nabla u)\cdot\nu = g(\beta(u)$$

on $(0,T)\times\Gamma$, where $\beta(z)$ is also non-decreasing, $g(z)$ is continuous, $\nu$ is the outward normal to $\partial\Omega$, and $\Gamma\subset\partial\Omega$. The author proves existence and uniqueness as well as error estimates. A more advanced mathematical study is performed b Igbida and Kirane [307] on the simplified equations

$$\partial_t u - \Delta w = 0, \quad u\in\beta(w),$$

where $\beta$ is a maximal monotone graph in $\mathbb{R}$, with boundary data

$$\partial_t z + \partial_\eta w = 0$$

where $z=\rho(w)$, $\rho\colon\mathbb{R}\to\mathbb{R}$ is continuous and non-decreasing, and $\partial_\eta$ is the derivative in the outward normal direction.

### 11.4.5 *Boundary conditions of combustion type*

The so-called boundary conditions of combustion type consist in prescribing on the free boundary both the value $u=0$ and the normal derivative of $w=\Phi(u)$,

$$|\partial_\nu w| = C\neq 0, \tag{11.24}$$

for a GPME of the form $u_t=\Delta w$, $w=\Phi(u)$. This is really a free boundary problem. It has been proposed as way of modelling the propagation of flames in the limit of high activation energy [143] using as equation the heat equation. The model with the PME as diffusion equation was studied by Barenblatt and Vázquez in [68].

## 11.5 The porous medium flow on a Riemannian manifold

Let $(M,g)$ be a $d$-dimensional smooth Riemannian manifold, which we assume compact, connected and with boundary $\partial M$. Let $\overline{M}=M\cup\partial M$. We want to solve the initial value problem for the GPME posed on $M$, with appropriate boundary conditions if needed. For concepts and notations of Riemannian geometry we refer to the textbooks [164, 218, 285]. We will denote by $\langle\cdot,\cdot\rangle$ the scalar product induced by $g$ on $T_xM$, by $\nabla f$ the gradient of a differentiable function and by $\Delta=\Delta_g$ the Laplace–Beltrami operator on the manifold, using the analysts' convention about the sign; in coordinates $x = (x_1, \ldots, x_d)$, we have

$$\Delta_g f = \frac{1}{\sqrt{|g|}}\, \partial_{x_i} \left( g^{ij} \sqrt{|g|}\, \partial_{x_j} f) \right) = g^{ij} \frac{\partial^2 f}{\partial x_i x_j} - \Gamma^k_{ij} g^{ij} \frac{\partial f}{\partial x_k},$$

where we have used the summation convention[1]; here $(g^{ij})$ is the inverse matrix to $g = (g_{ij})$, and $|g|$ denotes the determinant of the matrix $g$ that gives the metric in those coordinates. The volume element on $M$ is a positive measure $dv_g$ expressed in coordinates as $dv_g(x) = \sqrt{|g|}\, dx_1 \cdots dx_d$; it is usually abbreviated as $dx$ when no confusion is to be feared with the Lebesgue measure on the charts. In that notation we have

$$\int_M u \Delta v\, dx = - \int_M \langle \nabla u, \nabla v \rangle\, dx = \int_M v \Delta u\, dx$$

whenever $u, v \in C^2(M)$ and they vanish on $\partial M$. We have $\int_M \Delta f\, dv_g = 0$ for every $f \in C^2(M)$ when $\partial M = \emptyset$. This is also true when $\partial M \neq \emptyset$ but $\partial_\nu f = 0$ on $\partial M$. The Sobolev embeddings and Poincaré inequality are important tools that are available for these manifolds, see [285, Chapter 2].

### 11.5.1 *Initial value problem*

No boundary condition is needed if the manifold is boundaryless, $\partial M = \emptyset$. This happens in the most typical cases, namely, the $d$-dimensional sphere $\mathbb{S}_d$ (which has constant positive curvature) and the flat torus, $\mathbb{T}_d$ (which is the quotient of $\mathbb{R}^d$ by the discrete group $\mathbb{Z}_d$ and has zero curvature). The problem is to find a solution of

$$\text{(P(M,I))} \qquad \begin{cases} \partial_t u = \Delta\Phi(u) + f & \text{in } \ Q = M \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{for } x \in M. \end{cases}$$

given suitable data $u_0$ and $f$. We will build on the theory of previous chapters.

(i) A convenient starting point is the theory of maximal monotone operators developed in Section 10.1 and applied in Subsection 10.1.4 to the Dirichlet and Cauchy problems for the GPME. The arguments of that subsection apply without any significant change and we obtain a weak solution of Problem (P(M,I)) whenever $u_0 \in H^{-1}(M)$, $f \in L^2(Q)$. Moreover, the maximum principle applies so that for bounded initial data and forcing term the solution is bounded in the usual sense:

$$\|u(t)\|_\infty \le \|u_0\|_\infty + t \|f\|_{L^\infty(Q_T)}$$

whenever $0 \le t \le T$. In case $f = 0$, we obtain a strong continuous semigroup in $H^{-1}(M)$. Besides, when $\beta = \Phi^{-1}$ satisfies the conditions of Chapter 7, the regularity theory applies without changes and we have a continuous solution with a uniform modulus of continuity.

[1]This definition reverses the sign of the Laplacian with respect to geometry books; $\Delta$ is a dissipative operator in analysis, but positive in geometry.

(ii) Suppose now that $\Phi$ is smooth and non-degenerate, i.e., $0 < c_1 < \Phi'(u) < c_2 < +\infty$. Then, we can apply the higher regularity of solutions of uniformly parabolic equations, which is a local theory, to prove that the continuous weak solutions are indeed smooth classical solutions of the equation. We thus prove that Theorem 3.2 holds for the solutions of Problem (3.3)–(3.5) when $\Omega$ stands for a compact manifold $M$ without boundary.

After that we can recover the whole set of a priori estimates of Chapter 3 which are based on integrations by parts, notably the $L^1$ contraction and the bounds on $\int j(u(t))\,dx$, $\iint |\nabla(u^m)|^2\,dxdt$ and $\iint \Phi'(u)u_t^2\,dxdt$ (where $dx = dv_g(x)$). An interesting point to be noted is that conservation of mass holds for these solutions

$$\int_M u(x,t)\,dv_g(x) = \int_M u_0(x)\,dv_g(x).$$

(iii) With this classical foundation, we may recover the whole theory of weak energy solutions developed in Chapter 5, as well as the theory of limit and very weak solutions developed in Chapter 6. There is then no difficulty in developing the application of the theory of $m$-accretive operators in $L^1(\Omega)$ as in Section 10.3 of the previous chapter. We have

**Theorem 11.10** (i) *Under the assumptions on $M$ and $\Phi$, for every $u_0 \in L^1(M)$ and $f \in L^1(Q_T)$, $T > 0$, there is a unique mild solution $u \in C([0,\infty) : L^1(\Omega))$ of Problem* (P(M,I)) *defined in $Q_T$*

(ii) *Comparison holds: if $u, \widehat{u}$ are weak solutions with initial data such that $u_0 \le \widehat{u}_0$* a.e. *in $M$, and $f \le \hat{f}$ in $Q_T$, then $u \le \widehat{u}$* a.e. *in $Q$. In particular, if $u_0 \ge 0$, $f \ge 0$, then $u \ge 0$ in $Q$.*

(iii) *For every two solutions $u$ and $\widehat{u}$ with initial data $u_0$, $\widehat{u}_0$ resp., we have for every $t > \tau \ge 0$*

$$\|(u(t) - \widehat{u}(t))_+\|_1 \le \|(u(\tau) - \widehat{u}(\tau))_+\|_1 + \int_\tau^t ||(f(s) - \hat{f}(s))_+||_1 ds. \tag{11.25}$$

(iv) *If $u_0$ and $f$ are bounded, the solution is continuous. These conditions can be relaxed.*

(v) *When $f = 0$, the GPME generates an ordered semigroup of $L^1$ contractions on the manifold $M$, and mass is conserved in time: for every $t > 0$ we have*

$$\int_\Omega u(t)\,dx = \int_\Omega u_0\,dx. \tag{11.26}$$

*Moreover, the semigroup is bounded in all $L^p$ spaces, $1 \le p \le \infty$, and for all $0 \le s < t$ we have*

$$\|u(t)\|_p \le \|u(s)\|_p \le \|u_0\|_p. \tag{11.27}$$

We ask the reader to perform these proofs as an exercise.

### 11.5.2 *Initial value problem for the PME*

The special theory developed for equation $u_t = \Delta(|u|^{m-1}u)$ in the case of the Dirichlet problem in Chapter 8, and in Section 11.3 for the Neumann problem, is easily adapted to the flow on a compact manifold $M$. To start with, we ask the reader to prove the $u_t$ bound of Bénilan and Crandall which was valid for both mentioned problems.

**Proposition 11.11** *Any non-negative solution of $u_t = \Delta u^m$ on $M \times (0, T)$ satisfies*

$$u_t \geq -\frac{u}{(m-1)t}. \tag{11.28}$$

We prove next the equivalent of the Aronson–Bénilan estimate, which we have established for the Cauchy problem in Proposition 9.4.

**Proposition 11.12** *Let $M$ have non-negative Ricci curvature, let $u$ be a non-negative solution of $u_t = \Delta u^m$ on $M \times (0, T)$, and let the pressure be defined as usually, $v = mu^{m-1}/(m-1)$. Then,*

$$\Delta v \geq -\frac{\alpha}{t}. \tag{11.29}$$

*where $\alpha = d/(d(m-1)+2)$.*

**Proof** The formal derivation of the estimate is quite similar to Proposition 9.4. We write the PDE satisfied by the pressure $v$, which continues to be

$$v_t = (m-1)v\Delta v + |\nabla v|^2, \tag{11.30}$$

where now $\Delta$ and $\nabla$ are differential operators on $M$. In order to justify this and following calculations, we consider classical solutions of the PME so that $v$ and its derivatives are bounded and $v$ is bounded below away from 0 so that the equation is uniformly parabolic. Then we write the equation satisfied by $p = \Delta v$ by differentiating (11.30) twice. We have

$$p_t = (m-1)v\Delta p + 2(m-1)\nabla v \cdot \nabla p + (m-1)p^2 + \Delta(\nabla v \cdot \nabla v).$$

Now we have to evaluate the last term on a manifold and this differs from the Euclidean case. The calculation is usually done by using the formula

$$\Delta(\nabla f, \nabla g) = 2\Gamma_2(f, g) + \nabla f \cdot \nabla \Delta g + \nabla g \cdot \nabla \Delta f,$$

which defines the differential bilinear form $\Gamma_2$. The Bochner–Lichnerowicz formula says that (cf. [101])

$$\Gamma_2(f, f) = \|\mathrm{Hess}(f)\|_2^2 + \mathrm{Ric}(\nabla f, \nabla f)$$

where Ric is the Ricci curvature tensor and Hess is the Hessian, a symmetric bilinear form on the tangent space involving second derivatives; $\|\mathrm{Hess}(f)\|_2^2$ is the corresponding Hilbert–Schmidt norm. If the Ricci curvature is minorized by $\rho$ and we use the Cauchy–Schwartz inequality on $\|\mathrm{Hess}(f)\|_2^2$ we get

$$\Gamma_2(f,f) \geq \frac{1}{d}(\Delta f)^2 + \rho|\nabla f|^2.$$

Therefore,

$$p_t \geq (m-1)v\Delta p + 2m\nabla v \cdot \nabla p + (m-1)p^2 + \frac{2}{d}p^2 + 2\rho|\nabla v|^2.$$

Finally, since we assume that Ric is non-negative, i.e., $\rho \geq 0$, we get

$$p_t \geq (m-1)v\Delta p + 2m\nabla v \cdot \nabla p + \left(m - 1 + \frac{2}{d}\right)p^2,$$

See Proposition 9.4 and its proof. We can write the last result as

$$\mathcal{L}(p) \equiv p_t - (m-1)v\Delta p - 2m\nabla v \cdot \nabla p - \left(m - 1 + \frac{2}{d}\right)p^2 \geq 0,$$

where $\mathcal{L}$ is a quasilinear parabolic operator with smooth, variable coefficients, since we consider $v$ as a given function of $x$ and $t$. We now apply $\mathcal{L}$ to the trial function

$$P(x,t) = -\frac{C}{t+\tau} \tag{11.31}$$

and observe that $\mathcal{L}(P) \leq 0$ if and only if $C \geq \alpha = 1/\left(m - 1 + (2/d)\right)$. We fix $C = \alpha$. By choosing $\tau$ small enough we may also obtain

$$p(x,0) \equiv \Delta v(x,0) \geq P(x,0) \equiv -\frac{C}{\tau}, \tag{11.32}$$

from which the classical maximum principle should allow us to conclude that $p \geq P$ in $Q$. Letting $\tau \to 0$ we would then obtain a pointwise inequality $\Delta v \geq -\alpha/t$. The application of the maximum principle is justified. ■

Then, we can address the problem of boundedness and recover the conclusions obtained for the Neumann problem. There are no essential changes in the proofs.

**Proposition 11.13** *The $L^\infty$ estimates of Theorem* 11.8 *for data $u_0 \in L^1(M)$ and the sharper estimates of Theorem* 11.9 *for the case $\int u_0(x)\,dx = 0$ hold for the PME on a compact manifold without boundary.*

### 11.5.3 *Homogeneous Dirichlet, Neumann and other problem*

The homogeneous Dirichlet problem is

$$(\mathrm{P}_M) \quad \begin{cases} \partial_t u = \Delta\Phi(u) + f & \text{in } \ Q = \Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{for } x \in \Omega \\ u(x,t) = 0 & \text{on } \ \partial M. \end{cases}$$

given suitable data $u_0$ and $f$. The homogeneous Neumann problem is

$$(\mathrm{P}_M) \quad \begin{cases} \partial_t u = \Delta\Phi(u) + f & \text{in } \ Q = \Omega \times (0,\infty), \\ \partial_\nu \Phi(u(x,t)) = 0 & \text{on } \ \partial M. \end{cases}$$

given suitable data $u_0$ and $f$. We leave the many details of the 'standard' theory as exercises. This applies also to the mixed or nonlinear boundary problems.

## Notes

*Notes on the Neumann problem*

The assumptions that $\Omega$ has a $C^2$ boundary and that $f = 0$ are made for simplicity and lack of space. The extension is however interesting.

**Section 11.1.** The Neumann problem has been relatively less studied in the literature. Alikakos and Rostamian wrote a fundamental paper [8] in 1981. Their approach is based on semigroups and they concentrate on asymptotic behaviour.

**Section 11.2.** This section borrows heavily from previous chapters. It can be used as a tool for comparison and review.

**Section 11.3.** The proofs of boundedness seem to be new.

**Section 11.4.** The section is intended as reference on active directions.

*Notes on flows on manifolds*

**Section 11.5.** The flow of the heat equation on a Riemannian manifold has been much investigated [101, 199, 200, 453]. nonlinear elliptic equations have also been studied, see e.g. [117]. On the contrary, there is scarce evidence for the PME on manifolds. The presentation of the elementary theory made here seems to be new. Mention of the topic is made by Otto [413] in his approach to the PME as a gradient flow. Asymptotics are studied by Bonforte and Grillo [121]. Fast diffusion equations on manifolds have been treated in connection with the Yamabe and Ricci problems, see in that respect Aubin [55] and Hamilton [283].

There is an extensive literature for interrelation of PDEs and Riemannian manifolds, see [55, 285].

## Problems

### Problem 11.1

(i) Show that a function $f \in C^1(\overline{\Omega})$ can be approximated by functions $f_n \in C^1(\overline{\Omega})$ that satisfy $\partial_\nu f_n = 0$ on $\partial\Omega$ and such that $f_n \to f$ uniformly and $|\nabla f_n|$ is uniformly bounded.

(ii) Prove that the class of test functions in Definition 11.2 can be taken indistinctly to be $\mathcal{C}$ as in the stated definition or $\mathcal{C}_{n0}$ where we also impose the condition that $\partial_\nu \eta = 0$ on $\Sigma$. The same happens in Definitions 11.1.
(iii) Show that we can even take $C^\infty$ functions in $\mathcal{C}_{n0}$ is $\Omega$ is smooth.

*Hint*: (i) By standard tricks we may first localize the problem in a neighbourhood of a point of the boundary, map this point to $x = 0$, straighten the boundary to look like $x_1 = 0$, and then assume that $\Omega$ is the half space $x_1 < 0$ and $\eta$ has compact support in $H = \{x_1 \le 0\}$. A possible construction of the approximation consists of taking a smooth cut-off function $\theta(s)$ such that $0 \le \theta \le 1$, $\theta = 0$ for $s \ge 2$ and $\theta(s) = 1$ for $s \le 1$, and defining

$$f_\varepsilon(x_1, \dots, x_d) = \int_{-\infty}^{x_1} \partial_{x_1} f(y, x_2, \dots, x_d)\, \theta\left(-\frac{y}{\varepsilon}\right)\, dy.$$

Then, $f_\varepsilon \in C^1(H)$, it has compact support, $\partial_{x_1} f_\varepsilon = 0$ for all $x_1 \ge -\varepsilon$ and $f_\varepsilon$ approximates $f$ as needed.

**Problem 11.2** Prove that the examples of solutions of the Neumann problem derived in Subsection 11.1.1 from the Cauchy problem are actually weak solutions.

**Problem 11.3** Extend the existence theory of Section 11.2 to the case $f \ne 0$ following the ideas of Chapters 5 and 6. Note that the uniqueness theorem does not need a new proof.

**Problem 11.4***

(i) Use the conservation law (11.11) to prove that there cannot be a non-negative solution with infinite initial data (a Friendly Giant) for the Neumann problem. Show that this happens also on compact Riemannian manifolds.
(ii) Investigate the existence of a Friendly Giant for the mixed problem with Dirichlet data on a subset of the boundary with non-zero measure, and Neumann data otherwise.

**Problem 11.5** Investigate the problem with non-zero Neumann boundary data, as done in Section 5.7 for the Dirichlet problem.

**Problem 11.6** Investigate the extension of the theory to a Lipschitz domain.

**Problem 11.7** On the $L^\infty$ bound with non-zero mass. Let $\Omega$ be a bounded domain and let $\Omega_R = R\,\Omega$ an homothetical domain. Let $F_R$ be the function in the $L^\infty$ bound of Proposition 11.8 with domain $\Omega_R$. Prove that

$$F_R(t; m, M) = F_1(tR^{-(d(m-1)+2)}; m, M).$$

*Hint*: Use rescaling to pass from a solution in $\Omega_R$ with mass $M$ to a solution in $\Omega$ with the same mass.

**Problem 11.8** BEHAVIOUR NEAR $t = 0$. Prove that the boundedness function of Proposition 11.8 satisfies

$$\liminf_{t\to 0} F(t)t^{1/(d(m-1)+2)} > 0.$$

Is it optimal?

*Hint*: Use the ZKB as candidates in the result.

**Problem 11.9** CONVERGENCE TO THE CP Prove the following result: *Let $\Omega_n$ an expanding sequence of smooth domains such that $\bigcup_n \Omega_n = \mathbb{R}^d$. Let $u_{0n} \in L^1(\Omega_n)$, $u_0 \in L^1(\mathbb{R}^d)$ and assume that $u_{0n} \to u_0$ in the obvious $L^1(\mathbb{R}^d)$ sense (i.e., extending $u_{0n}$ by zero outside $\Omega_n$). Let $u_n$ be the limit solution of the HNP in $Q_n = \Omega_n \in \mathbb{R}_+$ with data $u_{on}$, and let $u$ be the limit solution of the CP in $Q = \mathbb{R}^d \times \mathbb{R}_+$. Prove that $u_n \to u$ in $C([0,\infty) : L^1(\mathbb{R}^d))$.*

**Problem 11.10** Study the relation of Dirichlet and Neumann problems on the line; study the mixed problem.

*Hint*: Use separation of variables; use the examples of sines and cosines as motivation for the space part of linear diffusion; use extensions to relate them.

**Problem 11.11** Compare the Neumann and Cauchy problems in some cases. Example: $u_0 \geq 0$ and $f \geq 0$ have compact support in a ball inside $\Omega$.

**Problem 11.12** PROJECT. Analyse the existence of mild solutions and a semigroup for the homogeneous Neumann problem for the GPME in a bounded domain.

**Problem 11.13** Show that for a $d$-dimensional surface in $\mathbb{R}^{d+1}$ defined explicitly by the equation $x_{d+1} = f(x_1, \dots, x_d)$ with the induced metric we have for $i, j = 1, \dots, d$:

$$g_{ii} = 1 + (\partial_{x_i} f)^2, \quad g_{ij} = (\partial_{x_i} f)(\partial_{x_j} f) \quad \text{if } i \neq j,$$

so that $|g| = 1 + |\nabla f|^2$.

**Problem 11.14** Prove that statements made in Section 11.5 about the solution of the GPME on a manifold.

(i) State the existence of a weak solution in the space $H^{-1}(M)$.
(ii) State the existence and uniqueness of classical solutions.
(iii) State the result about existence and uniqueness of weak energy solutions as in Chapter 5. Construct a Friendly Giant is possible.
(iv) Discuss the existence of limit and very weak solutions.
(v) Apply the Crandall–Liggett theorem.

**Projects** Study the Neumann problem on unbounded domains, or in manifolds with infinite volume and with boundary. Asymptotics in that case are unknown.

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# PART TWO

# 12

# THE CAUCHY PROBLEM WITH GROWING INITIAL DATA

In this chapter we examine the general conditions on the initial data under which the PME can be solved. The emphasis is on non-negative solutions and the data are locally integrable functions, but we will also treat signed solutions.

Section 12.1 introduces the problem and the main growth assumptions. In Section 12.2 we derive the well-known Aronson–Caffarelli estimate for non-negative solutions, which can be seen as a lower bound for the value of the solutions at time $t > 0$ in terms of the initial data or, alternatively, as an upper bound for the possible behaviour of the initial data of the solutions. The former aspect is the typical Harnack approach to parabolic regularity, while the latter aspect is what interests us most in this chapter.

Section 12.3 studies the existence of solutions for a class of initial data $u_0 \in L^1_{\rm loc}(\mathbb{R}^d)$ with controlled growth as $|x| \to \infty$. The main result is Theorem 12.8 on the existence of solutions local in time, complemented by Theorem 12.9 on the existence of solutions global in time. The proof is based in obtaining estimates of the solutions in suitable weighted norms. Comparison with the a priori estimate of the allowed growth of the previous section shows that the initial conditions we assume are optimal when the data are non-negative.

The uniqueness of the class of solutions with optimal growth rate we have obtained is established in Section 12.4 by using a version of Holmgren's duality method. See the best result in Theorem 12.11.

Section 12.5 deals with further properties of the solutions and Section 12.6 studies the properties of some particular classes of solutions obtained under special assumptions on the data. As a complement to the results of this section, we consider in Section 12.7 the questions of boundedness of solutions defined locally and their approximation by smooth solutions.

We also devote a short Section 12.8 to construct solutions in conical domains, which allows us to show that growth rates as high as desired can be admissible if the aperture of the cone is small enough.

In this chapter we will use the fixed values

$$\lambda = d(m-1) + 2, \quad \gamma = \frac{\lambda}{(m-1)} = d + \frac{2}{m-1}, \quad \alpha_0 = \frac{\gamma}{2}.$$

The letters $c_i$ will denote different positive constants depending only on $m$ and $d$.

## 12.1 The Cauchy problem with large initial data

We are here concerned with the existence of solutions of the Cauchy problem

$$\begin{cases} u_t = \Delta(|u|^{m-1}u) & \text{in } Q_T, \\ u(x, 0 = u_0(x) & x \in \mathbb{R}^d. \end{cases} \tag{12.1}$$

We take $m > 1$ and want to solve this problem for a class of initial data as large as possible, and we accept that the existence time may depend on the data $u_0$. We take as inspiration the well-known results for the linear case $m = 1$, where such a problem can be solved for the class of locally integrable initial data such that

$$\int u_0(x) e^{-c|x|^2}\, dx < \infty \tag{12.2}$$

for some $c \geq 0$. This is called the square exponential growth condition. It is known that under such assumption a solution can be constructed in the time interval $0 < t < T = 4/c$, cf. [524]. Moreover, when the data are non-negative, condition (12.2) is also necessary for Problem (12.1) with $m = 1$ to have a non-negative solution in some time interval $(0, T)$.

In the case of the PME, we have seen in Section 4.5 that there exist explicit solutions with increasing initial data that blow up in a finite time. The growth rate of such solutions is quadratic in terms of the pressure, i. e., $v_0(x) = O(|x|^2)$, which in terms of $u$ reads

$$u_0(x) = O(|x|^{2/(m-1)}). \tag{12.3}$$

Here, we will prove that the appropriate growth condition for the existence of solutions of the Cauchy problem (12.1) with $m > 1$ is an averaged form of that growth condition that can be written as

$$\sup_{R \geq 1} R^{-\gamma} \int_{|x| \leq R} u_0(x)\, dx < \infty. \tag{12.4}$$

More precisely, the existence time of the solutions will be shown to depend on the limit

$$\ell(u_0) := \lim_{r \to \infty} \sup_{R \geq r} R^{-\gamma} \int_{|x| \leq R} u_0(x)\, dx < \infty. \tag{12.5}$$

Indeed, when we make the calculation of the blow-up time $T$ for the solutions (4.42), (4.44) we find $V(x, 0) \sim A|x|^2$ and $T = K/A$, with $K = 1/2\lambda$. In terms of (12.5) we get the expression

$$T = \frac{c(m, d)}{(\ell(u_0))^{m-1}}.$$

This is the type of expression that will be found as a lower bound in the general theory below.

## 12.2 The Aronson–Caffarelli estimate

Before we proceed with the construction of solutions in classes of growing initial data, we will derive the Aronson–Caffarelli estimate, a relation between the values of the solution at $t = 0$ and $t > 0$ that shows a very precise restriction on the admissible growth of the initial data of any non-negative solution. Such relations are known collectively as Harnack inequalities; we will review the classical meaning in the Notes of this chapter. The existence theory to developed later confirms that the restriction is sharp.

As a motivation, we recall that in the range $m > 1$ the equation has the property of finite propagation, exemplified by the ZKB solutions that exhibit an empty zone or zero-set $Z = \{(x,t) : u(x,t) = 0\}$ besides the occupied zone $\mathcal{P} = \{(x,t) : u(x,t) > 0\}$ (called positivity set), separated by a sharp interface or free boundary $\Gamma = \partial\mathcal{P} \cap Q$. Therefore, the onset of positivity must take some time at points which lay at $t = 0$ inside the zero-set. This means that a Harnack inequality of the classical type cannot be true. There are versions that adapt very well to the properties of the nonlinear equations. To begin with, we remark that the ZKB solutions indicate that all points become eventually positivity points, and this property is proved for general non–negative solutions.

The problem is then to find a quantitative statement of the eventual positivity of solutions. A natural inequality in this direction was obtained by Aronson and Caffarelli in 1983 [42] and it has played a major role in developing the theory of the equation under general assumptions on the data.

**Theorem 12.1** *There exists a constant $C = C(m,d) > 0$ such that the following estimate holds for all non-negative solutions of the PME:*

$$\int_{B_r(x_0)} u(x,0)\,dx \leqslant C\left(r^{\lambda/(m-1)}T^{-1/(m-1)} + T^{d/2}u^{\lambda/2}(x_0,T)\right) \qquad (12.6)$$

*with $\lambda$ as before; $x_0 \in \mathbb{R}^d$, and $r, T > 0$ are arbitrary.*

We present a version of the proof taken from [162] that uses in a direct way the smoothing effects we have derived, and is quite simple. We begin with a lemma in which sizes are taken as unity.

**Lemma 12.2** *Let $u \in \mathcal{C}(\mathbb{R}^d \times [0,T])$ be a non-negative solution of the PME in $Q_1 = B_1(0) \times (0,1)$. Let*

$$M = \int_{B_1} u(x,0)\,dx. \qquad (12.7)$$

*There exist positive constants $M_0 = M_0(d,m)$ and $k = k(d,m)$ such that for $M \geqslant M_0$*

$$u(0,1) \geqslant k\,M^{2/\lambda}. \qquad (12.8)$$

**Proof** This is the combination of several steps. Again, the letter $C$ will denote different positive constants that depend only on $d$ and $m$.

By comparison we may assume that $u_0$ is supported in the unit ball $B_1$. Indeed, for general $u_0$, then $u_0$ is greater than $u_0\eta$, $\eta$ being a suitable cut-off function compactly supported in $B_1$ and less than one. Thus, if $v$ is the solution with initial data $u_0\eta$ (existence and uniqueness are well-known in this case), we obtain

$$\int_{B_1} u(x,0)dx \geqslant \int_{B_1} u_0\eta\, dx = M,$$

and if the lemma holds true for $v$, then

$$u(0,1) \geqslant v(0,1) \geqslant kM^{2/\lambda}.$$

We may then take the domain of definition as $Q = \mathbb{R}^d \times (0,\infty)$.

By using the standard smoothing effect, cf. Theorem 9.8 or the more detailed results of Chapter 17, we know the a priori estimate for the solution

$$0 \leqslant u(x,t) \leqslant C\, M^{2/\lambda}\, t^{-d/\lambda} \tag{12.9}$$

and the a priori estimate for the support at time $t$,

$$\operatorname{supp} u(\cdot,t) \subset B_R(t), \qquad R(t) = C\, M^{(m-1)/\lambda}\, t^{1/\lambda}. \tag{12.10}$$

Note that if $M$ is large this radius is much larger than 1 at $t = 1$.

The reflection argument of Aleksandrov, see Subsection 9.6.2 means that for $|x| \geq 2$ we have

$$u(0,t) \geqslant u(x,t). \tag{12.11}$$

By conservation of mass we know that

$$\int u(x,t)\, dx = \int u_0(x)\, dx. \tag{12.12}$$

The first term can be split into the integrals

$$\int_{|x|\geqslant 2} u(x,t)\, dx + \int_{|x|\leqslant 2} u(x,t)\, dx,$$

and the last term can be estimated by

$$C\, M^{2/\lambda}\, t^{-d/\lambda} 2^d.$$

We conclude that

$$Cu(0,t)\,(R(t)^d - 2^d) \geqslant \int_{|x|\geq 2} u(x,t)\, dx \geqslant M - C\, M^{2/\lambda}\, t^{-d/\lambda} 2^d,$$

hence for $t = 1$,

$$C\, u(0,1)\,(M^{d(m-1)/\lambda} - 2^d) \geqslant M - C\, M^{2/\lambda}.$$

If $M > 1$ is large enough there are constants $c_1, c_2(m,d)$ such that

$$u(0,1) \geqslant c_1 M^{2/\lambda} - c_2 M^{\gamma/\lambda}.$$

Where $\gamma < 2$. Hence, there exists some constants $M_0$ and $k$ such that

$$c_1 M^{2/\lambda} - c_2 M^{\gamma/\lambda} \geqslant k M^{2/\lambda}$$

holds for every $M \geqslant M_0$, and this proves the lemma. ■

We can now prove the full Harnack-type inequality:

**Proof of Theorem 12.1** We can use the previous lemma on $(t, T)$ since $u \in \mathcal{C}^0(Q_T)$, perform the transformation

$$u^*(x,t) = r^{-2/(m-1)} T^{1/(m-1)} u(rx, Tt) \tag{12.13}$$

as in [42, p. 361], and look at the equation satisfied by $u^*$. It is the same. Noting that

$$\int_{|x|\leq 1} u^*(y,t)dy = r^{-(d+\frac{2}{m-1})} T^{\frac{1}{m-1}} \int_{|x|\leq r} u(x, Tt)\, dx,$$

we may apply the already derived formula (12.8) to $u^*$ to conclude that

$$\int_{B_r(0)} u(x,0)\, dx \leqslant C\, T^{d/2} u^{\lambda/2}(0,T)$$

on the condition that

$$\int_{B_r(0)} u(x,0)\, dx \geq r^{\lambda/(m-1)} T^{-1/(m-1)} M_0.$$

Formula (12.6) is another form of writing that conclusion for $x_0 = 0$. But the choice of origin is indifferent. ■

### 12.2.1 *Precise a priori control on the initial data*

For our purposes, it is convenient to write inequality (12.6) as

$$R^{-\lambda/(m-1)} \int_{B_R(x_0)} u(x,0)\, dx \leqslant C \left( T^{-1/(m-1)} + T^{d/2} R^{-\lambda/(m-1)} u^{\lambda/2}(x_0, T) \right) \tag{12.14}$$

which proves that left-hand side must be bounded independently of $R \geq 1$, and also that

$$\ell(u_0) \leq C T^{-1/(m-1)},$$

for some $C = C(m,d)$. This means that $\ell(u_0) = 0$ is a necessary condition for global-in-time solvability in the class of non-negative solutions.

We recall that no equivalent necessary condition is known for solutions of changing sign.

## 12.3 Existence under optimal growth conditions

We proceed now with the construction of solutions in classes of growing data under the growth restriction that has been found to be necessary. No sign restriction is enforced.

### 12.3.1 *Functional preliminaries*

We need some definitions. For $f \in L^1_{\rm loc}(\mathbb{R}^d)$ and $r > 0$ we let

$$\|f\|_r = \sup_{R \geq r} R^{-\gamma} \int_{|x| \leq R} |f(x)|\, dx < \infty. \tag{12.15}$$

Note that if $\|f\|_r$ is finite for some $r > 0$ then it is finite for all $r > 0$. Define the space $X = X(\mathbb{R}^d)$ as

$$X(\mathbb{R}^d) := \{f \in L^1_{\rm loc}(\mathbb{R}^d) : \|f\|_r < \infty\}. \tag{12.16}$$

We equip this space with the norm $\|\cdot\|_1$. It is a Banach space, and any norm $\|\cdot\|_r$, $r > 0$, is an equivalent norm. For $f \in X$ we define

$$\ell(f) = \lim_{r \to \infty} \|f\|_r. \tag{12.17}$$

We define the space $X_0 = X_0(\mathbb{R}^d)$ as

$$X_0 = \{f \in X : \ell(f) = 0\}.$$

Note that $L^1(\mathbb{R}^d) \subset X_0 \subset X \subset L^1_{\rm loc}(\mathbb{R}^d)$ with continuous inclusions. We have $\ell(f) = 0$ for all $f \in L^1(\mathbb{R}^d)$. Actually, $\ell(f) = 0$ if $f \in L^1_{\rm loc}(\mathbb{R}^d)$ and $f(x)/|x|^s \to 0$ with $s \leq 2/(m-1)$ as $|x| \to \infty$. It is easy to see that whenever $f_n \to f$ in $L^1_{\rm loc}(\mathbb{R}^d)$, and $r > 0$, then

$$\|f\|_r \leq \liminf_{n \to \infty} \|f_n\|_r.$$

It will be convenient to introduce another equivalent norm on $X$. We take a function $\phi \in C_c^\infty(\mathbb{R}_+)$ with $\phi(s) = 1$ for $|s| \leq 1/2$, $\phi(s) = 0$ for $s \geq 1$ and define $\varphi(x) = \phi(|x|)$. We then define

$$|f|_r = \sup_{R \geq r} R^{-\gamma} \int_{\mathbb{R}^d} f(x) \varphi(x/R)\, dx. \tag{12.18}$$

One easily sees that $|f|_r$ and $\|f\|_r$ are equivalent norms on $X$ related by constants independent of $r \geq 1$.

The spaces $L^1(\rho_\alpha)$ will be introduced below.

### 12.3.2 *Growth estimates for good solutions*

The construction of the solutions will proceed in the usual way by (i) approximation with good data, (ii) solution of the simple problems, and (iii) passage to the limit.

About steps (i) and (ii), we recall that for data $u_0 \in X_* = L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ there exists a unique solution $u(t) \in C([0,\infty) : L^1(\mathbb{R}^d))$, $u \in L^\infty(Q)$, and the solution map $S : (u_0, t) \mapsto u(t)$ satisfies a number of nice properties:

- $S(u_0, \cdot) \in C([0,\infty) : L^1(\mathbb{R}^d))$;
- $L^1$ contraction: $\|S(u_0,t) - S(v_0,t)\|_{L^1} \le \|u_0 - v_0\|_{L^1}$ for all $t \ge 0$;
- the comparison principle holds;
- boundedness: $-\|u_0^-\|_{L^\infty(\mathbb{R}^d)} \le S(u_0,t) \le \|u_0^+\|_{L^\infty(\mathbb{R}^d)}$;
- invariance under sign changes: $S(-u_0,t) = -S(u_0,t)$.

We call this class of solutions $\mathcal{S}$ see page 204, Definition 9.2. Moreover, for data $u_0 \ge 0$ we have $u \ge 0$ and

- second-order estimate: $\Delta u^{m-1} \ge -C/t$.

Finally, if the data are positive everywhere, then the solution is smooth. We will also assume that the data decrease fast as $|x| \to \infty$. The restricted class of resulting smooth solutions will be called $\mathcal{S}_1$.

Our main task at this moment is to derive suitable a priori estimates that allow us to control the approximate solutions in a uniform way in the passage to the limit.

In that direction. we will obtain some estimates that are true for the solutions of the PME in the class $S$ with constants that depend on the norms of the space $X$. We try the estimates first for $\mathcal{S}_1$ where the justification of the computations is immediate.

We first need a technical result, that extends the $L^\infty$-estimate of Lemma 9.9.

**Lemma 12.3** *Let $f \ge 0$ be a bounded measurable function in $\mathbb{R}^d$ satisfying : $\Delta u^{m-1} \ge -C$. Then, there exists a constant $K(d,m) > 0$ for $m > 1$ such that when $1 \le r \le R$*

$$\frac{1}{R^2}\|f\|^{m-1}_{L^\infty(B_R)} \le K(C^{d(m-1)/\lambda}|f|_r^{2(m-1)/\lambda} + |f|_r^{m-1}). \tag{12.19}$$

We give a proof of an estimate that implies this fact in Lemma 22.4. Here are the first estimates that control the solutions in terms of growing norms.

**Lemma 12.4** *Let $u$ be a solution of the PME in the class $\mathcal{S}$. Then, for $1 \le r \le R$ and $0 < t < T_r(u_0) := c_1/|u_0|_r^{m-1}$, we have*

$$|u(t)|_r \le c_2 |u_0|_r \tag{12.20}$$

*and*

$$R^{-2/(m-1)}\|u(t)\|_{L^\infty(B_R(0))} \le c_3\,|u_0|_r^{2/\lambda} t^{-d/\lambda}. \tag{12.21}$$

*The positive constants $c_1, c_2$ and $c_3$ depend only on $m > 1$ and $d$.*

**Proof** (i) We first assume that $u$ is smooth non-negative in the class $\mathcal{S}_1$. We take a function $\varphi(x) = \phi(|x|)$ as in the definition (12.18). We have

$$\begin{aligned}\frac{d}{dt}\int u(x,t)\varphi(x/R)\,dx &= \int \Delta u^m(x,t)\,\varphi(x/R)\,dx\\ &= \int u^m(x,t)\,\Delta(\varphi(x/R))\,dx\\ &= R^{-2}\int u^m\,(x,t)(\Delta\varphi)(x/R)\,dx.\end{aligned}$$

Then, we integrate in time

$$\begin{aligned}\int u(x,t)\varphi(x/R)\,dx &= \int u_0(x)\varphi(x/R)\,dx + R^{-2}\iint u^m\,(\Delta\varphi)(x/R)\,dxdt\\ &\le \int u_0(x)\varphi(x/R)\,dx + cR^{-2}\int_0^t d\tau\|u(\tau)\|^{m-1}_{L^\infty(B_R)}\int_{B_R} u(\tau)\,dx\\ &\le \int u_0(x)\varphi(x/R)\,dx + c\int_0^t d\tau R^{-2}\|u(\tau)\|^{m-1}_{L^\infty(B_R)}\\ &\quad\times\int_{B_R} u(\tau)\varphi(x/2R)\,dx.\end{aligned}$$

The constant $c = c(m,d)$ may change from line to line. We can multiply this inequality by $R^{-\lambda/(m-1)}$ and take the supremum in $R \ge r$ on both sides to get an inequality in terms of $N(t) = |u(t)|_r$:

$$N(t) \le N(0) + c\int_0^t (\sup_{R\ge r} R^{-2}\|u(\tau)\|^{m-1}_{L^\infty(B_R)})N(\tau)\,d\tau. \tag{12.22}$$

In view of the second-order estimate for non-negative solutions and Lemma 12.3 we get

$$R^{-2}\|u(t)\|^{m-1}_{L^\infty(B_R)} \le c(t^{-d/\lambda(m-1)}C|u(t)|_r^{2(m-1)/\lambda} + |u(t)|_r^{m-1})$$

Therefore, $N(t)$ satisfies the integral inequality:

$$N(t) \le N(0) + c\int_0^t (\tau^{-d(m-1)/\lambda}N(\tau)^{1+(2(m-1))/\lambda} + N(\tau)^m)\,d\tau. \tag{12.23}$$

In order to obtain a direct bound as a consequence of this inequality, we consider the case of equality which leads to the differential equation

$$\widetilde{N}'(t) = c(t^{-d(m-1)/\lambda}\widetilde{N}(t)^{1+(2(m-1))/\lambda} + \widetilde{N}(t)^m) \tag{12.24}$$

with $\widetilde{N}(0) = N(0)$. Note that for small $t$ the first term on the right-hand side is dominant over the second, and also that $\alpha(m-1) < 1$ so that the factor $t^{-\alpha(m-1)}$ is integrable in time. An estimate in a first time interval is therefore obtained by integrating the ODE

$$H'(t) = 2c\,t^{-d(m-1)/\lambda}H(t)^{1+(2(m-1))/\lambda}, \tag{12.25}$$

with initial value $H(0) = N(0)$. This solution is explicitly given by

$$H(t)^{2(m-1)/\lambda} = 1/(N(0)^{-2(m-1)/\lambda} - 2c(m-1)t^{2/\lambda}).$$

Comparing equations (12.24) and (12.25) we see that as long as

$$H^m(t) \le t^{-d(m-1)/\lambda} H(t)^{1+2\lambda^{-1}(m-1)},$$

this solution $H(t)$ will be a supersolution for equation (12.24), hence for (12.23), and we will have

$$N(t) \le \widetilde{N}(t) \le H(t).$$

Such a condition happens if $tH^{m-1} \le 1$, which in view of the formula for $H(t)$ holds if

$$0 \le t \le T_1 = c_1/N(0)^{m-1}.$$

Note that this time interval is not essentially improvable since the formula for $H(t)$ blows up in a finite time with a similar expression (as also does the actual formula for $N(t)$).

Since $H$ is increasing, we conclude that in the time interval $[0, T_1]$ function $H(t)$ is bounded above by $H(T_1) = c_5 N(0)$. Since $N(t) \le H(t)$ in that interval, this completes the proof of estimate (12.20).

(ii) As for estimate (12.21) in the class $\mathcal{S}_1$, it easily follows from this estimate, estimate $\Delta u^{m-1} \ge -C/t$, and Lemma 12.3 that

$$R^{-2}\|u(t)\|^{m-1}_{L^\infty(B_R(0))} \le c\,(t^{-d(m-1)/\lambda}|u_0|_r^{2(m-1)/\lambda} + |u_0|_r^{m-1})$$

as long as $1 \le r \le R$ and $0 \le t \le c_4/|u_0|_r^{m-1}$. But then $t|u_0|_r^{m-1} \le c_4$ and we can transform the last summand in the right-hand side into the desired form to obtain (12.20).

(iii) In order to extend the result to the class $\mathcal{S}$, we first consider non-negative initial data and obtain the above inequalities for smooth approximations $u_n \in \mathcal{S}_1$. Since the estimates are uniform for all the $u_n$ and $u_n \to u$ in $C([0,\infty) : L^1(\mathbb{R}^d))$, the results hold for $u$.

The results are also true for non-positive solutions, $u \le 0$, by the invariance of the equation under sign changes. Finally, if $u_0$ has changing sign, we consider the solutions $u_1, u_2 \in \mathcal{S}$ with respective initial data

$$u_1(x,0) = \max\{u_0(x), 0\}, \quad u_2(x,0) = \min\{u_0(x), 0\}$$

Then the estimates apply to $u_1 \ge 0$ and $u_2 \le 0$; since

$$u_2 \le u \le u_1,$$

the result follows for $u$. ∎

**Corollary 12.5** *Estimate* (12.21) *implies that for* $0 < t < T_r(u_0) = c_1/|u_0|_r^{m-1}$ *we have*

$$t^{d/\lambda}|u(x,t)| \le c_3 \|u_0\|_r^{2/\lambda} \max\{r^2, |x|^2\}^{1/(m-1)}. \tag{12.26}$$

*Hence,*

$$\frac{|u(x,t)|}{(1+|x|^2)^{1/(m-1)}} \le c_3 \, \|u_0\|_r^{2/\lambda} t^{-d/\lambda}. \tag{12.27}$$

A similar proof allows us to derive some continuous dependence of the solution on the data.

**Lemma 12.6** *Let $u$ and $v$ be solutions in the class $\mathcal{S}$ with initial data $u_0, v_0$ resp. Then, for every $r \ge 1$ and $0 < t < \min\{T_r(u_0), T_r(v_0)\}$ we have*

$$\|u(t) - v(t)\|_r \le \exp\left(B_1 t^{2/\lambda}\right) \|u_0 - v_0\|_r, \tag{12.28}$$

*where $B_1$ depends on* $\max\{\|u_0\|_r, \|v_0\|_r\}$.

**Proof** (i) We only need to perform the proof when $u_0 \le v_0$ so that $u \le v$, since otherwise we may derive the result for the solutions $w_1$ and $w_2$ with initial data

$$w_1(x,0) = \min\{u_0(x), v_0(x)\}, \quad w_2(x,0) = \max\{u_0(x), v_0(x)\}$$

Note that $w_1, w_2 \in \mathcal{S}$, $w_1 \le u, v \le w_2$ so that $|u - v| \le w_2 - w_1$. At $t = 0$ we have $w_2(0) - w_1(0) = |u(0) - v(0)|$.

(ii) Assume then that $u_0 \ge v_0$, so that $u - v \ge 0$. We subtract the equation and multiply by a test function $\eta(x,t) = \zeta(t)\varphi(x/R)$, with $\varphi$ as before and $\zeta \in C_c^\infty(\mathbb{R}_+)$. From the definition of weak solution, we get

$$\begin{aligned} -\iint_Q (u-v)(x,t)\zeta_t(t)\varphi(x/R)\, dxdt &= \iint \zeta(t)\Delta(u^m - v^m)(x,t)\, \varphi(x/R)\, dx \\ &= R^{-2} \iint \zeta(t)(u^m - v^m)\, (\Delta\varphi)(x/R)\, dxdt \end{aligned}$$

where we have written $u^m$ instead of $|u|^{m-1}u$ and $v^m$ instead of $|v|^{m-1}v$ for brevity. Putting now

$$M_R(t) = \int (u-v)(x,t)\varphi(x/R)\, dx,$$

and arguing as in the proof of Lemma 12.20, we have

$$-\int M_R(t)\zeta_t(t)\, dt \le \frac{c}{R^2} \int_0^\infty \int_{\mathbb{R}^d} \max\{|u|,|v|\}^{m-1}(u-v)\varphi(x/2R)\zeta(t)\, dxdt.$$

Using estimate (12.21) to estimate $|u|^{m-1}$ and $|v|^{m-1}$ in $B_R$ with $r \le R$, putting $B_0 = \max\{|u_0|_r, |v_0|_r\}$, and letting $\zeta$ converge to the characteristic function of the interval $[0, t_1]$ we get:

$$M_R(t_1) - M_R(0) \le \int_0^{t_1} \frac{c}{t^{\lambda^{-1}d(m-1)}}\, B_0^{2/\lambda} M_{2R}(t)\, dt.$$

We now multiply this inequality by $R^{-\lambda/(m-1)}$ and take the supremum in $R \ge r$ on both sides to get an inequality in terms of $N(t) = |u(t) - v(t)|_r$

$$N(t) \le N(0) + c\, B_0^{2/\lambda} \int_0^t \frac{N(\tau)}{\tau^{d(m-1)/\lambda}}\, d\tau.$$

This is to be compared with the solution of the exact relation

$$H'(t) = kH(t)\, t^{d(m-1)/\lambda}, \quad H(0) = |u_0 - v_0|_r.$$

We take $k = c\, B_0^{2/\lambda} = c \max\{|u_0|_r^{2/\lambda}, |v_0|_r^{2/\lambda}\}$. Note that $\alpha(m-1) < 1$, so that the singularity at $t = 0$ is integrable. The solution is

$$H(t) = H(0) \exp{(k\, t^{2/\lambda})}.$$

Since $N(t) \le H(t)$, the proof is complete. ■

### 12.3.3 *Estimates in the spaces $L^1(\rho_\alpha)$*

The preceding result establishes continuous dependence with respect to a natural norm, $\|\cdot\|_r$. Space $X$ however has bad properties of approximation by functions in $X_*$ with respect to that norm. For instance, all $L^1$ functions have $\ell(f) = 0$, while a general function of $X$ does not.

In order to avoid that difficulty, the authors of [91] introduce the weighted spaces

$$L^1(\rho_\alpha) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^d) : \int f\, \rho_\alpha\, dx < \infty\},$$

where $\rho_\alpha(x) = (1 + |x|^2)^{-\alpha}$. It is easy to see that

(i) $L^1(\mathbb{R}^d) \subset L^1(\rho_\alpha)$ if $\alpha \ge 0$; and
(ii) $X \subset L^1(\rho_\alpha)$ for $\alpha > \lambda/(2(m-1))$, and $L^1(\rho_\alpha) \subset X$ if $\alpha \le \lambda/(2(m-1))$,

and the inclusions are dense. Note finally that

$$\Delta\rho_\alpha = -\frac{2\alpha}{1 + |x|^2}\, (d + (d - 2\alpha - 2)|x|^2),$$

so that $|\Delta\rho_\alpha| \le C_\alpha\, \rho_\alpha/(1 + |x|^2)$.

We can translate Corollary 12.26 into this framework. Indeed, estimate (12.26) implies that

$$\sup_{x\in\mathbb{R}^d} \frac{|u(x,t)|}{(1 + |x|^2)^{1/(m-1)}} \le \frac{c_r}{t^{d/\lambda}} \|u_0\|_r^{2/\lambda}$$

which means that the function $v(x,t) := u(x,t)\rho_{1/(m-1)}(x)$ is uniformly bounded in sets of the form $\mathbb{R}^d \times [\tau, T_r(u_0)]$ with $0 < \tau < T_r(u_0)$, $r \geq 1$.

The dependence result we need is

**Lemma 12.7** *Let $u$ and $v$ be solutions in the class $\mathcal{S}$ with initial data $u_0, v_0$ resp. Then, for every $\alpha \in \mathbb{R}$, $r \geq 1$ and $0 < t < \min\{T_r(u_0), T_r(v_0)\}$ we have*

$$\|u(t) - v(t)\|_{L^1(\rho_\alpha)} \leq \exp\left(B_2 t^{2/\lambda}\right) \|u_0 - v_0\|_{L^1(\rho_\alpha)}, \tag{12.29}$$

*where $B_2$ depends on $\|u_0\|_r, \|v_0\|_r, \alpha$ and $r$.*

**Proof** We begin with

$$\frac{d}{dt}\int |u(x,t) - v(x,t)|\,\rho_\alpha(x)\,dx \leq \int |u^m(x,t) - v^m(x,t)|(\Delta\rho_\alpha)\,dx.$$

In view of the estimate for $\Delta\rho_\alpha$ we get

$$\frac{d}{dt}\int |u(x,t) - v(x,t)|\,\rho_\alpha(x)\,dx \leq mC_\alpha \int \frac{\max\{|u|^{m-1}, |v|^{m-1}\}}{1+|x|^2}\,|u - v|\rho_\alpha\,dx.$$

Let $r \geq 1$. For $r \leq R \leq |x| \leq 2R$ we have

$$\frac{|u(x,t)|^{m-1}}{1+|x|^2} \leq \frac{4R^2}{1+R^2}\,\frac{\|u(t)\|^{m-1}_{L^\infty(B_{2R})}}{4R^2},$$

while for $|x| \leq r$

$$\frac{|u(x,t)|^{m-1}}{1+|x|^2} \leq r^2\,\frac{\|u(t)\|^{m-1}_{L^\infty(B_r)}}{r^2}.$$

We conclude that

$$\sup_{x\in\mathbb{R}^d} \frac{|u(x,t)|^{m-1}}{1+|x|^2} \leq c_r \sup_{R\geq r} \frac{\|u(t)\|^{m-1}_{L^\infty(B_R)}}{R^2}$$

We may now continue as in Lemma 12.6. ■

### 12.3.4 *Existence results*

The estimates of the previous section are enough to allow for the proof of the following existence result. Given $u_0 \in X$ we define

$$T(u_0) = \frac{c_1}{(\ell(u_0))^{m-1}}, \tag{12.30}$$

and we put $T = \infty$ if $\ell(u_0) = 0$. We also put $T_r(u_0) = c_1/\|u_0\|_r$. The positive constants $c_1, c_2, c_3$ are defined in Lemma 12.4.

**Theorem 12.8** *For every $u_0 \in X$ we can construct a solution $u(t) = U(u_0, t)$ of the PME Cauchy problem* (12.1) *defined in the time interval $0 \leq t < T(u_0)$. The solution has the following properties:*

(i) *(Boundedness in X) For* $r \ge 1$ *and* $0 < t < T_r(u_0)$

$$\|u(t)\|_r \le c_3 \|u_0\|_r. \tag{12.31}$$

(ii) *(Local pointwise boundedness) For* $1 \le r \le R$ *and* $0 < t < T_r(u_0)$ *we have*

$$\|u(t)\|_{L^\infty(B_R(0))} \le c_2 \|u_0\|_r^{2/\lambda} R^{-2/(m-1)} t^{-d/\lambda}. \tag{12.32}$$

*The function* $v(x,t) := u(x,t)\rho_{1/(m-1)}(x)$ *is uniformly bounded in sets of the form* $\mathbb{R}^d \times [\tau, T_r(u_0)]$ *with* $0 < \tau < T_r(u_0)$, $r \ge 1$, *and estimates* (12.26), (12.27) *hold.*

(iii) *u is a weak solution of the PME in* $\mathbb{R}^d \times (0, T(u_0))$ *in the sense of Definition* 5.1.

(iv) *(Initial data) For* $\alpha > \alpha_0 = \lambda/(2(m-1))$ *we have* $u \in C([0, T_r(u_0) : L^1(\rho_\alpha))$ *and* $u(0) = u_0$.

(v) *(Continuity) u is Hölder continuous in* $Q_T$.

(vi) *(Continuous dependence) If* $u$, $v$ *are the solutions with initial data* $u_0, v_0 \in X$, *then for every* $r \ge 1$ *and* $0 < t < \min\{T_r(u_0), T_r(v_0)\}$ *we have estimate* (12.28) *in the norm* $\|\cdot\|_r$ *and estimate* (12.29) *in* $\|\cdot\|_{L^1(\rho_\alpha)}$ *for all* $\alpha$.

(vii) *If* $u_0 \in L^1(\mathbb{R}^d)$, *then* $u(t)$ *is the solution constructed in Chapter 9.*

(viii) *The comparison principle holds: if* $u_0, v_0 \in X$ *and* $u_0 \le v_0$ *then* $u(t) \le v(t)$ *for every* $t \in (0, T(v_0))$.

**Proof** Clearly, $T_r(u_0) \le T(u_0)$ for all $r$ and $T_r(u_0)$ converges monotonically to $T(u_0)$ as $r \to \infty$.

(a) We consider first the case $u_0 \in X_+$, i. e., $u_0 \in X$ and $u_0 \ge 0$. We perform a standard regularization of the data into

$$u_{0n}(x) = \begin{cases} \min\{u_0(x), n\} & \text{if } |x| \le n, \\ u_{0n}(x) = 0 & \text{if } |x| > n. \end{cases}$$

Chapter 9 asserts the existence and uniqueness of a strong solution $u_n \ge 0$ to this problem. The $u_n$ are continuous functions in $Q$. The sequence $u_n$ is monotonically increasing in $n$. The a priori estimates hold uniformly in $n$, hence the limit

$$u(x,t) = \lim_{n\to\infty} u_n(x,t)$$

is finite a.e for $0 < t < T(u_0)$. In this way the map $u_0 \mapsto u$ is well defined on $X_+$, and the comparison principle holds in the limit. Moreover, the estimates of Lemma 12.4 hold in the limit, so that $u(x,t)$ is locally bounded, and hence continuous. It is immediate that $u$ is a very weak local solution of the PME in $Q_T$. In this case we have the second-order estimate:

$$\Delta u^{m-1} \ge -C/t \quad \mathcal{D}'(Q).$$

We still have to check that the initial data are taken. Actually, we may use the uniform continuous dependence estimates of Lemma 12.7, which depend only on the $|\cdot|_r$ norm, $r \ge 1$, and on $\|u_0 - v_0\|_{L^1(\rho_\alpha)}$. We now recall that for $\alpha > \alpha_0 = \lambda/(2(m-1))$ we have $X \subset L^1(\rho_\alpha)$ and also

$$\|u_{0n}\|_r \text{ is increasing to } \|u_0\|_r, \quad \|u_0 - u_{0n}\|_{L^1(\rho_\alpha)} \to 0.$$

Taking then $\alpha > \alpha_0$, we conclude that $u_n$ is a Cauchy sequence in $C([0, T_r(u_0) : L^1(\rho_\alpha))$, and thus converges to $u$ in that norm. In other words, we can prove that $u(t)$ converges to $u_0$ in $L^1(\rho_\alpha)$ the estimate of the lemma, plus the uniform continuity of the approximations $u_n$ in time in the norm $L^1$, the embedding $L^1(\mathbb{R}^d) \to L^1(\rho_\alpha)$, and the triangle inequality.

(b) In case $u_0$ is non-positive, we use the same procedure on $-u_0$ and then apply the rule $U(-u_0, t) = -U(u_0, t)$. In the general case where $u_0$ has both signs, we use the approximations

$$u_{0n}(x) = \begin{cases} \min\{n, \max\{u_0(x), -n\}\} & \text{if } |x| \le n, \\ u_{0n}(x) = 0 & \text{if } |x| > n. \end{cases} \tag{12.33}$$

Since $u_{0n} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, we get for every $n$ a solution $u_n$ in $\mathcal{S}$ and $|u_n| \le n$. Now the convergence is not monotone and we have to rely on Lemma 12.7 to guarantee the existence of a limit. ∎

Our construction allows us to define the local solution mapping $U : D \to X$ in the set

$$D = \{(u_0, t) : \ u_0 \in X, \ 0 < t < T(u_0)\} \subset X \times \mathbb{R}_+$$

by the law: $(u_0, t) \mapsto U(u_0, t)$. We call the class of obtained solutions $\mathcal{U}$.

We also derive global solutions with special properties in the case $\ell(u_0) = 0$.

**Theorem 12.9** *When $u_0 \in X_0$ the solution $u = U(u_0, t)$ is global in time and continuous into $X$, $u \in C([0, \infty) : X)$. Moreover, $u(t) \in X_0$ for every $t \ge 0$ and*

$$\lim_{|x| \to \infty} \frac{u(x,t)}{(1 + |x|^2)^{1/(m-1)}} = 0. \tag{12.34}$$

**Proof** (i) If $u_0 \in X_0$ and $u_{0n}$ is defined as in formula (12.33), we have

$$\lim_{n \to \infty} \|u_0 - u_{0n}\|_r = 0 \tag{12.35}$$

Actually, if we take $1 \le r \le r_0$ and then $R \ge r$ we have

$$R^{-\gamma} \int_{|x| \le R} |u_0 - u_{0n}|\, dx \le R^{-\gamma} \int_{|x| \le r_0} |u_0 - u_{0n}|\, dx + 2R^{-\gamma} \int_{r_0 \le |x| \le R} |u_0|\, dx.$$

Take now supremum over $R \ge r$ for $r, r_0$ fixed to get

$$\|u_0 - u_{0n}\|_r \le r^{-\gamma} \int_{|x| \le r_0} |u_0 - u_{0n}|\, dx + 2\|u_0\|_{r_0}$$

Now, for fixed $r_0$ we have $\int_{|x|\le r_0} |u_0 - u_{0n}|\,dx \to 0$ as $n\to\infty$. Hence,

$$\limsup_{n\to\infty} \|u_0 - u_{0n}\|_r \le 2\|u_0\|_{r_0}.$$

Then, we recall that $u_0 \in X_0$ implies that $\|u_0\|_{r_0} \to 0$ as $r_0 \to \infty$. The result follows.

(ii) We know that the approximations $u_n \in C([0,\infty) : L^1(\mathbb{R}^d)) \hookrightarrow C([0,\infty) : X)$. The initial convergence (12.35) and the continuous dependence result of Lemma 12.6 imply that $u_n$ is a Cauchy sequence in $C([0, T_r(u_0)) : X)$ for all $r > 0$. Letting $r \to \infty$ we have $T_r(u_0) \to \infty$. We conclude that the limit $u \in C([0,\infty) : X)$. We now use the fact that $X_0$ is closed in $X$ to conclude that $u \in C([0,\infty) : X_0)$.

Estimate (12.34) is an easy consequence of estimate (12.27) as $r \to \infty$. ■

## 12.4 Uniqueness of growing solutions

This is a first uniqueness result for solutions which may grow as $|x| \to \infty$.

**Theorem 12.10** *Let $u_1, u_2$ be distributional solutions of the PME in $Q_T$, $T > 0$, such that*

*(i) $u_1, u_2 \in C([0,T] : L^1_{\rm loc}(\mathbb{R}^d))$;*
*(ii) $u_i\,\rho_{1/(m-1)} \in L^\infty(Q_T)$;*
*(iii) $u_1(t) - u_2(t) \to 0$ as $t \to 0$.*

*Then, $u_1 = u_2$.*

**Proof** We use the duality method introduced in Section 6.2 in the study of very weak solutions. It has to be adapted to the setting in the whole space. It suffices to prove that $u_1(T) = u_2(T)$ since $T > 0$ can be changed. For brevity, we write $u_i^m$ instead of $|u_i|^{m-1}u_i$, $i = 1, 2$.

(i) We write the weak inequalities satisfied by $u_1$ and $u_2$ with respect to a test function $\psi \in C_c^\infty(\overline{Q_T})$. We subtract to get

$$\int\!\!\int_{Q_T} \{(u_1 - u_2)\psi_t + (u_1^m - u_2^m)\Delta\psi\}\,dxdt = \int (u(T) - v(T))\psi\,dx. \qquad (12.36)$$

We now write $u = u_1 - u_2$. Defining

$$a(x,t) = \frac{u_1^m - u_2^m}{u_1 - u_2}$$

and $a(x,t) = mu_1^{m-1}$ if $u_1 = u_2$, we may write $u_1^m - u_2^m = a(x,t)u(x,t)$ for a measurable function $0 \le a$.

(ii) The next step offers a variation with respect to Theorem 6.5. We choose radii $R > R_0 + 1 > 0$ and a smooth test function $\theta(x) \ge 0$, compactly supported in

$B_{R_0}(0)$, $0 \le \theta \le 1$, and solve the inverse-time problem in $Q_{RT} = B_R(0) \times (0,T)$

$$\begin{cases} \varphi_t + a_n \Delta\varphi = 0 & \text{in } B_R(0) \times (s,t) \\ \varphi = 0 & \text{on} \Sigma_R = \partial B_R(0) \times (s,t) \\ \varphi(x,t) = \theta & \text{for } x \in B_R(0), \end{cases} \tag{12.37}$$

where $a_n$ is a smooth approximation of $a$ such that $1/n \le a_n \le K$. This is a correct parabolic problem in inverse time that has a smooth solution $\varphi_{R,n} \ge 0$. We also need a smooth cut-off function $\eta_\varepsilon$, $0 < \varepsilon < 1/2$, such that

$$0 \le \eta_\varepsilon \le 1, \quad \eta_\varepsilon(x) = 1 \text{ for } |x| < R - 2\varepsilon, \quad \eta_\varepsilon(x) = 0 \text{ for } |x| \ge R - \varepsilon,$$
$$\|\nabla\eta_\varepsilon\|_\infty \le c/\varepsilon, \qquad \|\Delta\eta_\varepsilon\|_\infty \le c/\varepsilon^2.$$

We put now the test function $\psi$ in (12.36) equal to $\varphi_{R,n}\eta_\varepsilon$ to get for the difference estimate (dropping the subindexes $R, n$ and $\varepsilon$ in $\varphi$ and $\eta$ for brevity):

$$\begin{aligned} &\int (u_1(T) - u_2(T))\theta \, dx \\ &= \iint_{Q_T} (u_1 - u_2)\eta(a - a_\varepsilon)\Delta\varphi + \iint_{Q_T} (u_1^m - u_2^m)(2\nabla\eta\nabla\varphi + \varphi\Delta\eta) \, dxdt. \end{aligned} \tag{12.38}$$

We denote the last two integrals by $I$ and $J$. They are functions of $\varepsilon$, $n$ and $R$. Our aim is to perform the limit when $\varepsilon \to 0$ and $n, R \to \infty$ and find that $I, J \to 0$. In that case the conclusion

$$\int_{\mathbb{R}^d} (u_1(T) - u_2(T))\theta \, dx = 0 \tag{12.39}$$

would follow, which proves that $u_1(T) = u_2(T)$.

(iii) We estimate $J = J_{R,n,\varepsilon}$ as follows:

$$|J| \le c \int_0^T \int_{R-2\varepsilon<|x|<R} |u_1^m - u_2^m|(|\nabla\varphi|/\varepsilon + |\varphi|/\varepsilon^2) \, dx.$$

But $\varphi = 0$ for $|x| = R$, which means that the gradient can be approximated in a small neighbourhood of the lateral boundary by the normal derivative $\partial\varphi/\partial\nu$. Let this neighbourhood take the form $N = \{R - 2\varepsilon < |x| < R, 0 < t < T\}$. Also, $\varphi$ can be estimated by distance to the boundary times a bound for the gradient. Summing up, if we keep $R$ and $n$ fixed and take the limit as $\varepsilon \to 0$, we get

$$J_{R,n} = \limsup_{\varepsilon\to 0} J_{R,n,\varepsilon} \le c\, R^{d-1} \left( \sup_{\Sigma_R} \frac{\partial\varphi}{\partial\nu} \right) \lim_{\varepsilon\to 0} \sup_N |u_1^m - u_2^m|.$$

Using the growth estimate (ii) of the solutions we get

$$J_{R,n} \le c\, R^{d-1-(2m/(m-1))} \left( \sup_{\Sigma_R} \left| \frac{\partial\varphi}{\partial\nu} \right| \right).$$

After choosing a convenient sequence $a_n$, we will prove below an estimate of the form

$$\sup_{\Sigma_R}\left|\frac{\partial\varphi}{\partial\nu}\right| \le c\,R^{-2\beta}, \tag{12.40}$$

for some

$$\beta > \frac{d-1}{2} + \frac{m}{m-1}.$$

In that case

$$J_{R,n} \le c\,R^{\sigma}, \quad \sigma = d-1-(2m/(m-1))-2\beta < 0. \tag{12.41}$$

The constant $c$ depends on $m, d, \theta, R_0$, but not on $n$ or $R$. Therefore, we will have $J_{R,n} \to 0$ as $R, n \to \infty$, as desired.

(iv) We estimate $I = I_{R,n,\varepsilon}$ as follows:

$$|I|^2 \le \iint_{Q_{RT}} |u_1-u_2|^2 \frac{(a-a_n)^2}{a_n}\,dxdt \iint a_n|\Delta\varphi_n|^2\,dxdt,$$

which does not depend on $\varepsilon$. We need an a priori estimate of the term with $\Delta\varphi_n$. We multiply equation (12.37) satisfied by $\varphi_n$ by $\Delta\varphi_n$. Integrating by parts gives

$$\frac{1}{2}\int |\nabla\varphi_n(0))|^2\,dxdt + \iint a_n(\Delta\varphi_n)^2\,dxdt = \frac{1}{2}\int |\nabla\theta)|^2\,dxdt.$$

This implies that

$$|I|^2 \le c(R)\iint_{Q_{RT}} \frac{(a-a_n)^2}{a_n}\,dxdt.$$

At this stage we have to construct the approximations $a_n$ to $a$ so that the latter quantity goes to zero as $n \to \infty$, and the process is independent of $\theta$. A similar process has been done in Theorem 6.5. We extend $a$ to the whole strip $Q_T$ by 0, and then perform a smoothing with kernel $\rho_n$ and raise the result by $1/n$,

$$a_n = a * \rho_n + (1/n).$$

Then $a_n \in C^\infty(Q_T)$, $a_n \ge 1/n$, and moreover

$$a_n(x,t) \le K(1+|x|^2)$$

in $Q_T$, where

$$K = 1 + m\max\{\|u_1^{m-1}\|_{L^\infty(Q_T)}, \|u_2^{m-1}\|_{L^\infty(Q_T)}\}.$$

which is finite by assumption. With this choice we get after some easy calculations

$$\iint_{Q_{RT}} \frac{(a-a_n)^2}{a_n}\,dxdt \le n\left(\frac{1}{n^2} + \frac{TR^d}{n^2}\right).$$

(see details in [91, page 80]). Since $n>1$ is independent of $R$, $T$ or $\theta$, we may pass to the limit $n\to\infty$ to conclude that

$$\lim_{n\to\infty}|I_{R,n,\varepsilon}|=0.$$

We may now pass to the limit in formula (12.38) first in $\varepsilon\to 0$ and then in $n\to\infty$ to get

$$\left|\int (u_1(T)-u_2(T))\,\theta\,dx\right|\le c\,R^{-\sigma},$$

where $\sigma(m,d)>0$ and $c$ does not depend on $R$. Letting $R\to\infty$ we get formula (12.39) and uniqueness follows.

(v) We are left with the proof of the technical estimate (12.40). We refer to [91, pages 78, 79]. ■

We present next the proof of uniqueness for the class of solutions $\mathcal{U}$ constructed in Section 12.3.

**Theorem 12.11** *Let $u$ be a distributional solution of the PME in $Q_T$, $T>0$, such that*

(i) $u\in C([0,T]:L^1_{\rm loc}(\mathbb{R}^d))\cap L^\infty(0,T:X)$;
(ii) *For all* $\tau>0$, $u\,\rho_{1/(m-1)}\in L^\infty(\mathbb{R}^d\times(\tau,T)$.

*Then,* $u=U(u(0),t)$ *for* $0\le t\le \min(T,T(u(0)))$ .

**Proof** (i) By Theorems 12.8 and 12.10 we know that for every small $\tau>0$ there exists a solution in the class $\mathcal{U}$ with initial data $u(\tau)$ defined in a certain time interval and moreover

$$u(t+\tau)=U(u(\tau),t) \tag{12.42}$$

for $0<t<\min(T-\tau,T(u(\tau))$. If we may assume that $u(\tau)$ converges to $u(0)$ in some $L^1(\rho_\alpha)$ with $\alpha>0$, then the continuous dependence result (12.29) asserted in part (vi) of Theorem 12.8 implies that the right-hand side of (12.42) converges as $\tau\to 0$ (with fixed $t>0$) to $U(u(0),t)$ in $L^1(\rho_\alpha)$. Since the left-hand side converges to $u(t)$ in $L^1_{\rm loc}(\mathbb{R}^d)$, the conclusion $u(t)=U(u(0),t)$ follows once we note that

$$T_r(u(0))\ge\limsup_{\tau\to 0}T_r(u(\tau)).$$

(ii) We will now prove that the assumption (i) of the Theorem implies that $u(\tau)$ converges to $u(0)$ in some $L^1(\rho_\alpha)$ with $\alpha>\alpha_0$. In fact, if we split the norm of $|u(0)-u(\tau)|$ in $L^1(\rho_\alpha)$ in two parts,

$$\int_{|x|<R}|u(0)-u(\tau)|\rho_\alpha dx+\int_{|x|\ge R}|u(0)-u(\tau)|\rho_\alpha dx,$$

the first one goes to zero as $\tau \to 0$ because of the $L^1_{\rm loc}$-continuity. As for the second, it is small if $R$ is large enough because both $u(0)$ and $u(\tau)$ are bounded in $X$ (uniformly in $\tau$). We propose the last calculation to the reader as Problem 12.4. ■

## 12.5 Further properties of the solutions

The case of global solutions can be nicely expressed in terms of semigroups.

**Corollary 12.12** *The PME generates a bounded continuous semigroup in the space $X_0$. It is just the restriction of the solution map $U$ to the set $D_1 = X_0 \times (0, \infty)$. It extends the semigroup $S$ defined in Section 9 for data in $L^1(\mathbb{R}^d)$.*

The solutions in the class $\mathcal{U}$ need not be smooth. In any case we have

**Proposition 12.13** *Solutions in the class $\mathcal{U}$ are uniformly Hölder continuous in $Q_T$, $T = T(u_0)$. Moreover, when the initial function is continuous and positive, the solution is $C^\infty$ smooth. Non-negative solutions can be approximated by smooth solutions in the norm $L^\infty(0, T : L^1_{\rm loc}(\mathbb{R}^d))$ for $T < T(u_0)$.*

The result should be easy to prove for the reader in view of the previous regularity theory and the construction of solutions.

We show next that the initial data are taken in the most standard sense of weak solutions and that for $t \ge \tau > 0$ they are local weak energy solutions.

**Proposition 12.14** *If $u$ is a solution of the class $\mathcal{U}$ with initial data $u_0 \in X$, then $|u|^m \in L^1(S)$ for all sets $S = B_R(0) \times [0, T]$, $R > 0$, $0 < T < T(u_0)$. Besides, for all $\eta \in C_c^\infty(\mathbb{R}^d \times [0, T(u_0)))$*

$$\iint \{|u|^{m-1} u \Delta \eta + u \eta_t\}\, dxdt = \int u_0(x) \eta(x, 0)\, dx. \tag{12.43}$$

*Moreover, $u^m \in H^1_{\rm loc}(\mathbb{R}^d \times (0, T(u_0)))$.*

**Proof** Let $\theta \in C_c^\infty(\mathbb{R})$ with $0 \le \theta \le 1$ and $\theta(x) = 0$ for $|x| \ge R \ge r$. In view of the local boundedness estimates for the solutions $\mathcal{U}$, if $T < T_r(u_0)$ we have

$$\iint_{Q_T} |u|^m \theta\, dxdt \le c\, R^2 \|u_0\|_r^{2(m-1)/\lambda} \iint_S |u(x, s)| s^{-d(m-1)/\lambda}\, dsdx.$$

Using now (12.31) we get

$$\iint_{Q_T} |u|^m \theta\, dxdt \le c(R) \|u_0\|_r^{\delta} T^{-2/\lambda}, \quad \delta = 2\lambda^{-1}(m-1) + 1.$$

This proves the first claim.

(i) Let us now prove the $H^1$ regularity of $u^m$. We only have to repeat the idea of local estimates of Subsections 3.2.4 and 3.2.5. The first refers to controlling space derivatives. We take a smooth test function $\eta(x) \in C_c^\infty(\mathbb{R}^d)$ and use $u^m \eta$

as a test function on the equation satisfied by $u$ to get, after an integration by parts, the formula

$$\iint_{Q_T} |\nabla u^m|^2 \eta \, dxdt = \frac{1}{m+1} \iint_{Q_T} |u|^{m+1} \eta_t \, dx + \frac{1}{2} \iint_{Q_T} |u|^{2m} m \Delta \eta \, dxdt \qquad (12.44)$$

(see Problem 3.5). In view of the estimate for the local boundedness of $u$, we conclude that the local energy $\iint_K |\nabla u^m|^2 \, dxdt$ is uniformly bounded on compact subsets $K$ of $Q_T$ in terms of $K$ and the initial norm $\|u_0\|_r$. This calculation is justified on the approximations used in the construction of solutions. Since the estimate is uniform, it holds also in the limit.

(ii) In order to get a bound for $(u^m)_t$, we use multiplication by $(u^m)_t \eta$, where $\eta = \zeta(x/R)^2$ for some smooth test function $\zeta(x)$ with $\zeta(x) = 0$ for $|x| \geq 2$, $\zeta = 1$ for $|x| \leq 1$. We get after integration

$$\int_{\mathbb{R}^d} (u^m)_t u_t \eta \, dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\nabla u^m|^2 \eta \, dx = - \int_{\mathbb{R}^d} (u^m)_t (\nabla u(x,t)^m \cdot \nabla \eta) \, dx.$$

Using Hölder in the last term, we get a bound for the last term of the form

$$C \|\nabla \zeta\|_\infty \left( \int_{|x| \leq 2R} |\nabla u^m|^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^d} |(u^m)_t|^2 \eta \, dx \right)^{1/2}.$$

Put now

$$a^2(t) := \int [(u^m)_t]^2 \eta \, dx \leq m \|u^{m-1}(t)\|_{L^\infty(B_{2R})} \int_{\mathbb{R}^d} (u^m)_t u_t \eta \, dx. \qquad (12.45)$$

Then we have

$$a(t)^2 + \sigma(t) \frac{d}{dt} \int_{\mathbb{R}^d} |\nabla u^m|^2 \eta \, dx \leq c(R) \sigma(t) a(t) \left( \int_{|x| \leq 2R} |\nabla u^m|^2 \, dx \right)^{1/2} \qquad (12.46)$$

where $\sigma(t) = (m/2) \|u^{m-1}(t)\|_{L^\infty(B_{2R})}$. In order to get a clean conclusion from this inequality we observe that the algebraic relation $X^2 + Y \leq AX$, $A > 0$, implies that $Y \leq A^2/4$. We deduce that

$$\frac{d}{dt} \int_{\mathbb{R}^d} |\nabla u^m|^2 \eta \, dx \leq \frac{c(R)}{4} \sigma(t) \int_{|x| \leq 2R} |\nabla u^m|^2 \, dx$$

Using this and the uniform estimate (12.44) we conclude that $\|\nabla u^m(t)\|_{L^2(B_R)}$ is bounded locally uniformly in time $0 < t < T(u_0)$ for all $R > 0$. Going back to (12.46) and integrating in time we get

$$\int_{t_1}^{t_2} a^2(t) \, dt \leq c(R, t_1, t_2) < \infty$$

for all $0 < t_1 < t_2 < T(u_0)$. We have proved that $u^m$, $\nabla u^m$ and $(u^m)_t$ are bounded in $L^2_{\rm loc}(\mathbb{R}^d \times T(u_0))$ with local norms depending on the domain and the norms of $u_0$.

(iii) As for the integration formula (12.43), if we admit that the estimates in $H^1_{\rm loc}((\mathbb{R}^d) \times (0, T(u_0))$ apply uniformly to the approximations $u_n$, then we can pass to the limit in the formulas for the approximations with time origin $t = \tau > 0$. We then pass to the limit $\tau \to 0$. ■

## 12.6 Special solutions

We consider some consequences of the optimal theory when we impose conditions on the initial data. Thus, the existence of non-negative solutions for non-negative data has already been seen, and it is clear that radially symmetric initial data produce solutions that are radially symmetric for all times. We have also seen that $L^1$ data produce the bounded solutions constructed in Chapter 9.

Imposing other restrictions on the initial data leads to some types of interesting particular solutions. One of these types consists of the solutions with interfaces that will discussed in great generality in Chapter 14.

Another class are the *self-similar solutions*, of the form

$$U(x,t) = t^{-\alpha} f(x\, t^{-\beta}),$$

like the ZKB solution (1.8). Many of these solutions arise when the initial data have the form of a pure power $u_0(x) = A\,|x|^{-\gamma}$. Since such solutions are quite important for the general theory, we will devote the whole Chapter 16 to them. The study will show that there actually exist solutions with a growth $u(x,t) = O(|x|^p)$ for all rates $0 < p \le 2/(m-1)$. When $p < 2/(m-1)$ the solutions are global in time.

### 12.6.1 *Bounded solutions*

We have seen that for $u_0 \in L^1(\mathbb{R}^d)$ we recover the bounded solutions of Chapter 9. Now, it is interesting that bounded solutions for $t \ge \tau > 0$ can be obtained under a somewhat milder condition.

**Proposition 12.15** *Let $u_0 \in L^1_{\rm loc}(\mathbb{R}^d)$ be such that*

$$\sup_{x\in\mathbb{R}^d} \int_{B_1(x)} |u_0(y)|\, dy = \|u_0\|_{1,u} < \infty. \tag{12.47}$$

*Then, the solution $u = U(u_0, t)$ is global in time, bounded for all $t > 0$ and*

$$|u(x,t)| \le \frac{c_1}{t^{d/\lambda}} \|u_0\|_{1,u}^{2/\lambda} + c_2 \|u_0\|_{1,u} \tag{12.48}$$

*with constants $c_i$ depending only on $m$ and $d$.*

**Proof** Boundedness for a non-negative solution follows from Lemma A.4. The final result is obtained by rescaling. ■

Functions satisfying (12.47) form a space, $L^1_u(\mathbb{R}^d)$, larger than $L^1(\mathbb{R}^d)$ and smaller than $X_0$. The subscript in the new norm $\|\cdot\|_{1,u}$ indicates uniform integrability.

For a proof of the result see [91, Proposition 1.3]. Results about other intermediate spaces are given in that reference.

### 12.6.2 *Periodic solutions*

We say that a function $f(x)$ defined in $\mathbb{R}^d$ is periodic with a period $\mathbf{e} \in \mathbb{R}^d$ if

$$f(x+\mathbf{e}) = f(x) \quad \forall x \in \mathbb{R}^d. \tag{12.49}$$

In the case of $L^1_{\rm loc}$ functions, we consider the identity a.e. We say for short that $f$ is $\mathbf{e}$-periodic.

Let us now consider the class $X_{\mathbf{e}}$ of functions $f \in X$, as defined in (12.16), such that $f$ is also periodic with period $\mathbf{e}$. As an immediate consequence of the existence and uniqueness theory of this chapter and the invariance of the equation under translations, we have the following corollary.

**Corollary 12.16** *If $u_0 \in X_{\mathbf{e}}$, then the solution of the PME constructed in the class $\mathcal{U}$ is $\mathbf{e}$-periodic for all times $t > 0$.*

We consider next the interesting case where $f$ is periodic with respect to a base of vectors $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. In that case the solution is periodic in all directions of space, and we consider it as defined in the basic parallelepid

$$P(\mathcal{B}) = \{x = \sum_i \lambda_i \mathbf{e}_i, \quad 0 \le \lambda_i \le 1, \ \sum_i \lambda_i = 1\}$$

with periodic boundary conditions.

**Proposition 12.17** *Let $u_0 \in L^1_{\rm loc}(\mathbb{R}^d)$ be periodic with respect to a base $\mathcal{B}$. Then, the solution of the PME is also $\mathcal{B}$-periodic and is bounded for all positive times.*

Let last assertion comes from the easy observation that under the stated conditions, $u_0 \in L^1_u(\mathbb{R}^d)$. Moreover, we have

$$\|u_0\|_{1,u} \le c(d) \int_P |u_0|\, dx.$$

### 12.6.3 *Problems in a half space*

The theory can also be specialized to problems in a half space with Dirichlet or Neumann lateral conditions. Let us consider for instance the Dirichlet problem. Assume that $\Omega$ is the half line $I = (0, \infty)$. In that case we can solve the Dirichlet problem for equation $u_t = (|u|^{m-1}u)_{xx})$ with boundary data

$$u(0,t) = 0, \quad \forall t > 0, \tag{12.50}$$

and initial data

$$u_0(x) = \phi(x) \in L^1_{\rm loc}([0,\infty) \tag{12.51}$$

as follows: we extend the data to $x<0$ in an antisymmetric way:

$$u_0(x,t) = -\phi(-x).$$

If this extended $u_0$ belongs to the space $X(\mathbb{R})$, then there exists a solution $u$ in the class $\mathcal{U}$. The solution will be locally bounded and continuous for $t>0$. The initial symmetry will be conserved in time, so that for all $x\in\mathbb{R}, t>0$ we have

$$u(-x,t) = -u(x,t).$$

This means in particular that $u(0,t)=0$ for all $t>0$, so the restriction of $u$ to the domain $\overline{I}\times(0,T)$ solves the Dirichlet problem in $Q_+$. Let us define

$$\ell_+(\phi) = \lim_{r\to\infty} \|\phi\|_{r,+} \tag{12.52}$$

where the norm $\|\phi\|_{r,+}$ is defined just as $\|\phi\|_r$ of (12.15) with integrals in $(0,R)$, $R>r$.

This is the end result

**Theorem 12.18** *Let us pose the* HDP *for the PME in $Q_T = I\times(0,T)$, where $I$ is the half line $I=(0,\infty)$ with initial data* (12.51)*. If $\phi\in L^1_{\rm loc}([0,\infty)$ and satisfies the growth condition $\ell_+(\phi)<\infty$, then there is a unique solution in the class $U_{1d}$. If $\ell_+(\phi)=0$, the solution is global in time.*

A similar result can be obtained for the homogeneous Neumann problem if we extend the data given on $(0,\infty)$ in a symmetric way to $(-\infty,0)$.

### 12.6.4 *Problems in intervals*

Similar tricks allow us to solve Dirichlet or Neumann problems in an interval. Let us take this time the homogeneous Neumann problem posed in a space interval $(0,a)$. We proceed as follows: we first extend the initial data to $(-a,0)$ by symmetry; then these extended data are extended by periodicity to the whole of $\mathbb{R}$, the period being $2a$. In this way we obtain a locally integrable and periodic function $\widetilde{u}_0(x)$ that gives rise to a solution $\widetilde{u}(x,t)$ that is bounded for all $t>0$. The restriction of $\widetilde{u}(x,t)$ to the space interval $(0,a)$ is the solution of the HNP we are looking for. The same thing can be done for the Dirichlet problem with antisymmetric extension. See problem 12.9.

## 12.7 Boundedness of local solutions

We address here the problem of boundedness of local solutions of the PME, i. e., defined on a bounded subdomain $S$ of $\mathbb{R}^{d+1}$. The following lemma is proved for smooth solutions using Moser's iterations procedure [390, 391]. We will use the notations $S(\rho)=\{(x,t): |x|<\rho, -\rho^2<t\le 0\}$ and $|E|$ to denote the Lebesgue measure of the set $E$.

**Lemma 12.19** *Let $u$ be a smooth positive solution of the PME, $m > 1$, defined in $S(2\rho)$. There is a constant $\sigma(\mu, d) > 0$ such that for every $p > m$ there is $C_p = C(p, m, d) > 0$ and*

$$\sup_{S(\rho)} u(x,t) \le 1 + C_p \left( \frac{1}{|S(2\rho)|} \int_{S(2\rho)} u^p \, dxdt \right)^{\sigma/p} . \tag{12.53}$$

See [187] for the proof. A related subject is the approximation of solutions by smooth solutions.

**Lemma 12.20** *Let $u$ be a continuous weak solution defined in a rectangle $R = \overline{S(\rho)}$. Then it is uniformly approximable by $C^\infty$ smooth solutions defined in $R$.*

## 12.8 The PME in cones and tubes. Higher growth rates

In view of the preceding sections, one could conjecture that the growth condition $u_0 \le C(|x|^2 + 1)^{1/(m-1)}$ is optimal for the PME posed in an infinite domain. One way of showing that this is not so, even for non-negative solutions, is to consider conical or tubular domains.

### 12.8.1 *Solutions in conical domains*

Given an open subset $A \in \mathbb{S}^{d-1}$, we call cone of vertex 0 and base $A$ the set

$$\mathcal{C}(A) = \{x = r\sigma : \; r > 0, \; \sigma \in A\}. \tag{12.54}$$

By translation we can construct cones with vertex at any point $x_0 \in \mathbb{R}$. Usually, $A$ is a spherical ball whose angular radius is called aperture. Let us now solve the eigenvalue problem for the Laplace-Beltrami operator in the spherical domain $A$ with zero Dirichlet boundary conditions. We find a first eigenfunction $F(\sigma)$, which is smooth and positive and has eigenvalue $\lambda_1 = \lambda_1(A)$. Then, the function

$$W(x) = r^p F(\sigma)$$

is a harmonic function in $\mathcal{C}(A)$ with zero boundary data on $\partial\mathcal{C}(A)$ if $p^2 + (d-2)p = \lambda_1$. This equation has always one positive root $p_1 > 0$ and on negative root $p_2$. It is also clear that $p_1$ and $-p_2 \to \infty$ as $\lambda_1 \to \infty$. As the base $A$ of the cone goes to zero then $\lambda_1 \to \infty$ and then $p_1 \sim \lambda_1^{1/2} \to \infty$. In the particular case $d = 2$ and $A = (-\alpha/2, \alpha/2)$, we have $W(x) = r^p \cos(p\sigma)$, with $p = \pi/\alpha$ and $\lambda_1 = \pi^2/\alpha^2$.

Function $W$ allows to construct a stationary continuous solution of the PME equation

$$\widehat{U}(x) = W^{1/m}(x).$$

We can use this function as an upper barrier to solve the PME in a cylinder with conical space domain, $Q = \mathcal{C}(A) \times (0, \infty)$. Using monotone approximation with Dirichlet problems in bounded domains as done in Chapter 9 for the

Cauchy problem, we can construct a weak solution for all measurable initial data $u_0(x) \geq 0$ such that

$$u_0(x) \leq \widehat{U}(x) = r^{p/m} F^{1/m}(\sigma). \tag{12.55}$$

Then, $u(x,t) \leq \widehat{U}(x)$ for all times. Note that when $\lambda_1$ is large enough, the allowed growth rate $q = p/m$ can be as large as we like.

*Application to signed solutions*

The existence of solutions in cones has some consequences for the existence theory of signed solutions. Let us explain the idea in simple terms. Take $d = 2$ and let $\mathcal{A}$ be the sector with total angle $\alpha = \pi/n$ for some integer $n \geq 1$. We can think of $\mathbb{R}^2$ as the union of $2n$ sectors obtained by rotation of the original one by angles of $\pi k/n$. Let us call these sectors $\mathcal{A}_1, \ldots, \mathcal{A}_{2n}$.

We now solve the problem in the first sector $\mathcal{A}_1$ with some initial data $u_{01} \geq 0$, and then in $\mathcal{A}_2$ with data $u_{02}(x) = -u_{01}(Rx)$, where $R$ denotes rotation of angle $\alpha$, and we proceed with the rest of the sectors by rotation and change of sign. In this way we construct a signed solution of the PME in the whole domain $\mathbb{R}^2$ which has growth rate as $|x| \to \infty$ as large as you like. The conclusion is

**Theorem 12.21** *The theory of signed solutions of the PME does not have a natural growth restriction on the data.*

This is of course a well-known fact in the theory of the heat equation.

### 12.8.2 *Solutions in tubes*

We can go even further when the aperture of the cone goes to zero. We will discuss next the existence theory on tubular domains where no growth restriction is needed. The precise setting is as follows: we consider non-negative solutions $u = u(x,t)$ posed in a space-time domain $Q = \Omega \times (0, \infty)$, where $\Omega$ is a tubular space domain of the form

$$\Omega = \mathbb{R} \times D,$$

and $D$ is an open subset of $\mathbb{R}^d$. We make no special smoothness assumptions on $\Omega$. Note that the boundary of the tube is $\partial\Omega = \mathbb{R} \times \partial D$. We denote the space variable in $\Omega$ by $x = (y, z)$ with $y \in \mathbb{R}$, $z \in D \subset \mathbb{R}^n$. We use the notation $\Delta$ for the Laplace operator with respect to the $N = d + 1$ variables $x$, while $\Delta' = \Delta_z$ is the Laplace operator in the $d$ last variables $z = (x_2, \ldots, x_N)$.

We want to solve the HDP for the PME in this domain taking as initial data any non-trivial function

$$u(x, 0) = u_0(x) \in L^1(\Omega), \quad u_0 \geq 0. \tag{12.56}$$

We also take zero Dirichlet boundary data,

$$u = 0 \qquad \text{on } \Sigma = \partial\Omega \times (0 < t < \infty). \tag{12.57}$$

We refer to our initial and boundary value problem as THDP.

The definition, existence and uniqueness of weak solutions is similar to what we have seen before, Chapters 5 and 9. A *non-negative weak solution* of problem THDP is a non-negative function $u \in C([0,\infty) : L^1(\Omega))$ such that $u^m \in L^2((t_1,t_2) : H_0^1(\Omega))$ for every $0 < t_1 < t_2 < \infty$, the PME is satisfied in the sense of distributions and $u(\cdot,t) \to u_0$ in $L^1(\Omega)$ as $t \to 0$. The solution depends continuously on the data in the $L^1(\Omega)$-norm, cf. Chapters 5 and 6.

Moreover, the solution enjoys several properties that relate it to Cauchy problem in the whole space and to Dirichlet problems in bounded domains and make the construction easier.

(i) COMPARISON WITH THE CAUCHY PROBLEM, CP. We can compare the solutions of our problem with the solutions of the Cauchy problem posed in the whole space $\mathbb{R}^N = \mathbb{R}^{d+1} \supset \Omega$. In particular, we have the $L^\infty$-estimate for solutions $\widehat{u}$ of CP

$$\widehat{u}(x,t) \le C\|\widehat{u}_0\|_{L^1(\mathbb{R}^N)}t^{-\gamma}, \qquad \gamma = \frac{N}{N(m-1)+2}. \tag{12.58}$$

This estimate applies to our equation after putting $N = d+1$ and defining $\widehat{u}_0$ as $u_0$ extended by zero outside of $\Omega$. It then follows that

$$u(x,t) \le \widehat{u}(x,t)$$

everywhere in $Q$. Consequently, the solutions are bounded for $t > 0$. However, the rate given by (12.58) is not very accurate, so we will turn to the HDP.

(ii) COMPARISON WITH THE DIRICHLET PROBLEM, HDP. In a similar way, one proves that the solution of our problem THDP in $\Omega$ is bounded above by the solution of the HDP posed in $D$ with constant data $\widehat{u}_0(x) = \|u(\cdot,\tau)\|_\infty$ after displacement of the time origin (so that $u(\cdot,\tau)$ is bounded). If $\widehat{u}$ is the solution of this HDP, we have the a priori estimate based on the existence of the Friendly Giant for the HDP in the bounded domain $D$. Moreover, the solution $\widehat{u}(x,t)$ can be considered as a solution in $\Omega$ that happens to be independent of $y$. An easy comparison gives:

$$u(y,z,t) \le \widehat{u}(z,t+\tau) \le F(z)\,(t+\tau)^{-1/(m-1)} \le F(z)\,t^{-1/(m-1)}.$$

This is a very useful estimate with correct size. The end form does not depend on the initial data for the original problem THDP. It allows to prove the following result

**Theorem 12.22** *For every $u_0 \in L^1_{\rm loc}(\overline{\Omega})$, $u_0 \ge 0$, the initial and boundary value problem* THDP *admits a unique strong solution $u \ge 0$ and the universal a priori estimate holds*

$$u(x,t) \le F(z)\,t^{-1/(m-1)}, \quad x = (y,z). \tag{12.59}$$

Note that the allowed class of initial data does not have any growth restriction.

*Application to signed solutions*

Here is a simple setting: $d = 2,\ D = (0, 1)$. By filling the whole plane with translated copies of $\Omega_1 = \mathbb{R} \times (0, 1)$, solving problem THDP in $\Omega_1$ with highly increasing initial data $u_{01}$, and then pasting in $\Omega_n$ copies with alternative signs plus and minus we can construct signed solutions of the PME in $\mathbb{R}^2$ with growth rate as $|x| \to \infty$ as big as we like. To extend the result to more than two dimensions, just put blank variables $x_3, \ldots, x_d$ into an oscillating solution $u(x_1, x_2, t)$ as above.

## Notes

**Section 12.1.** The fundamental work of Kalashnikov [313], see also [317], though treating only the one-dimensional problem, already considers the extension to the GPME. The suitable growth condition is then

$$\Phi'(u_0(x)) = O(|x|^2)$$

which coincides with the growth (12.3) for the PME.

**Section 12.2** The Aronson–Caffarelli estimate applies only to non-negative solutions. As Section 12.8 shows, there is no restriction on the maximal growth for signed solutions. The question of the maximal growth allowed for signed solutions under convenient restrictions on the class is interesting and remains essentially open to the author's knowledge. Remember the Tychonov example for the heat equation.

It is to be noted that the theory of the FDE is quite different: existence and uniqueness of non-negative solutions can be obtained in the case of the FDE with $0 < m < 1$ with *no growth restriction* on the initial data, cf. [190, 286]. Chasseigne and Vázquez [161] have constructed solutions for initial data that are not even locally finite measures (they are Borel measures), though in that case permanent singularities arise.

*Harnack inequalities*

Let us expand a bit on the point of view of Harnack inequalities and lower bounds in connection with the Aronson–Caffarelli estimate. We recall that the *classical Harnack inequality* states that any positive harmonic function defined in a ball $B_{2r}(0)$ of $\mathbb{R}^d$ satisfies the following inequality

$$\sup_{B_r(0)} u(x) \le C \inf_{B_r(0)} u(x),$$

where $C > 0$ is a constant that depends only on the dimension $d$. The result extends to non-negative solutions of the heat equation defined in a cylinder

$Q = B_{2r}(0) \times (0, 4r^2)$ in the form

$$\sup_{Q_1} u(x,t) \le C \inf_{Q_2} u(x),$$

where again $C = C(d)$ and $Q_1 = B_r(0) \times (r^2, 2r^2)$, $Q_2 = B_r(0) \times (3r^2, 4r^2)$. Note that the cylinder $Q_2$ where the inf is taken comes later in time than the cylinder $Q_1$ where the sup is taken.

These results have to be compared with the results of Section 12.2 for the PME.

**Section 12.3.** The contents of this section is an edited version of the existence part of the fundamental paper by Bénilan, Crandall and Pierre [91]. Our value of $\lambda$ is written $\lambda/N$ in that reference.

**Section 12.4.** Most of the contents follows closely [91]. The last part of proof of Theorem 12.11 introduces a very simplified argument.

The existence and uniqueness results will be extended to measures as initial data in the next chapter.

**Section 12.8.** This material on cones and signed solutions is new. The determination of the optimal growth rates on a conical domain is an **open problem**. When the cone is a quadrant of the $d$-dimensional space the problem of large time behaviour has been studied in [120].

Tubular domains are studied in [511]. Theorem 12.22 is new. Problems posed in tubes can be important for some applications (it is a favourite setting for combustion problems).

## Problems

### Problem 12.1

(i) Show that the norms $\|\cdot\|_r$, $r > 0$, on $X$ are equivalent with constants depending on $r$. Show that they decrease as $r$ increases. Show that the norms $\|\cdot\|_r$ and $|\cdot|_r$ are equivalent with constants independent of $r$ if $r \ge 1$.

(ii) Show that $\ell(f)$ is a continuous functional on $X$, so that $X_0$ is a closed subspace. Show that $\ell(f) = 0$ for all $f \in L^1(\mathbb{R}^d)$, and that $\ell(f) = 0$ if $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $f(x)/|x|^\gamma \to 0$ with $\gamma \le 2/(m-1)$.

(iii) Show that the embeddings $L^1(\mathbb{R}^d) \subset X_0 \subset X \subset L^1_{\text{loc}}(\mathbb{R}^d)$ are continuous.

(iv) Show that $L^1(\mathbb{R}^d)$ is dense in $X_0$.

(v) Show that whenever $f_n, f \in L^1_{\text{loc}}(\mathbb{R}^d)$, $f_n \to f$ in $L^1_{\text{loc}}(\mathbb{R}^d)$, and $r > 0$, then

$$|f|_r \le \liminf_{n\to\infty} |f|_r.$$

*Hint*: For (iv) note that if $f \in X$ and $f_n = f\,\chi(B_n)$, then

$$R^{-\gamma}\int_{|x|\le R} |f-f_n|\,dx = R^{-\gamma}\int_{n\le|x|\le R} |f|\,dx \le \|f\|_n$$

which must go to zero as $n\to\infty$ if $f\in X_0$.

**Problem 12.2**

(i) Show that for every $\alpha\in\mathbb{R}$, $L^1(\rho_\alpha)$ is a Banach space with norm

$$\|f\|_{L^1(\rho_\alpha} = \int |f|\,\rho_\alpha\,dx.$$

(ii) Show that $L^1(\mathbb{R}^d)\subset L^1(\rho_\alpha)$ if $\alpha\ge 0$ with dense inclusion.

(iii) Show that $X\subset L^1(\rho_\alpha)$ for $\alpha>\alpha_0=\lambda/(2(m-1))$, and that $L^1(\rho_\alpha)\subset X$ if $\alpha\le\alpha_0$.

**Problem 12.3** Prove Corollary 12.5. Prove that the estimates apply to the solutions of Theorem 12.8.

**Problem 12.4** Prove that the inclusion $X\subset L^1(\rho_\alpha)$ with $\alpha>\alpha_0$ is small at infinity in the following sense. If $f\in X$ and $f_R$ is defined as $f_R(x)=f(x)$ for $|x|\ge R$, $f(x)=0$ otherwise, then

$$\|f_R\|_{L^1(\rho_\alpha)}\le cR^{-2(\alpha-\alpha_0)}\|f\|_X$$

*Hint*: Note that for every $R>r>1$ we have $\int_{|x|\le R}|f|dx\le R^\gamma\|f\|_r$. This means that

$$\int_{\{2^nR\le|x|\le 2^{n+1}R\}} |f|\,\rho_\alpha dx \le \|f\|_r(2^{(n+1)}R)^\gamma(2^nR)^{-2\alpha}.$$

Use now the fact that $2\alpha>\gamma=2\alpha_0$ and sum in $n$.

**Problem 12.5** Prove Proposition 12.13.

**Problem 12.6** Work out the details of the existence and uniqueness theory of the PME in conical domains sketched in Section 12.8. Same for tubular domains.

**Problem 12.7** CONSERVATION OF THE POSITIVE SIGN OF $\Delta v$. Prove that for every non-negative solution of the PME in $\mathbb{R}^d$ with initial pressure such that $\Delta v_0\ge 0$ we have for all times $\Delta v(t)\ge 0$ for all times. Conclude that $v_t\ge 0$ in $Q$ so that $v$ is monotone non-decreasing in time.

*Hint*: Prove first that whenever $v_0$ is smooth bounded and positive and $\Delta v_0\ge -c$ then

$$\Delta v(t)\ge -\frac{c}{1+(c/\alpha)t},\quad \alpha=\frac{d}{d(m-1)+2}$$

by an adaptation of the argument of Proposition 9.4.

**Problem 12.8** Prove that whenever $u$ is a non-negative solution of the PME in $\mathbb{R}^d$ with initial data such that $\Delta u_0^m \geq 0$ we have $u_t \geq 0$ in $Q$.

**Problem 12.9**

(i) Complete the details of the construction of the solution of the Neumann problem in an interval in $d = 1$ proposed in Subsection 12.6.4.
(ii) Do the same for the Dirichlet problem by antisymmetric extension.
(iii) Do a similar process for the problem with Dirichlet condition on one side and Neumann on the other. In that case we need an antisymmetric extension on one end, symmetric on the other, and the periodic extension with period $4a$.

**Problem 12.10** Construct solutions in a conical domain $\mathcal{C}$ with a singularity at $x = 0$. Estimate the allowed growth rate at the origin.

# 13

# OPTIMAL EXISTENCE THEORY FOR NON-NEGATIVE SOLUTIONS

Most of this chapter is devoted to study the existence and uniqueness of solutions of the Cauchy problem for the PME posed in the whole space which take a Radon measure as initial data. We recall (see Section A.4) that a Radon measure $\mu$ is in principle defined as a (real-valued) linear map on $C_c(\mathbb{R}^d)$. The main restriction to fully develop this theory is non-negativity, $u \geq 0$. The Riesz theorem allows to associate to a Radon measure a regular locally finite Borel measure, also called $\mu$. Note the alternative notations for integrals with respect to a measure, $\int f(x)\,\mu(dx)$ and $\int f(x)d\mu(x)$.

In Section 13.1 we construct limit solutions for data measures with the growth condition found as optimal in the previous chapter (in the non-negative case). The assumption of non-negativity is not needed in this section. This is the beginning of the optimal theory of the Cauchy problem for the PME.

The theory is continued in Section 13.2 where we prove that any non-negative solution defined in a domain $Q_T$ has a unique initial trace. In Sections 13.3 and 13.4 we prove that the initial trace determines the solution in a unique way. This is a landmark in the theory of the PME and completes the basic theory of the Cauchy problem developed in previous chapters.

Section 13.5 complements these results with an outline of the study for the homogeneous Dirichlet problem posed in bounded domain, thus continuing the theory developed in Chapters 5–8. There are similarities and some marked differences, to quote (i) there exists a special solution, the Friendly Giant, with infinite initial trace, and (ii) the initial trace is not a simple object, but a pair of measures, one of them supported in $\Omega$, and the other one on the boundary $\partial\Omega$. The explicit example of the dipole solution shows that this latter trace is important.

Section 13.6 shows that general very weak and non-negative solutions of the PME defined in an open space-time domain are actually continuous functions, i.e., continuous weak solutions. This means that there is no loss of generality in assuming continuity in this setting.

Section 13.7 contains a number of related topics where progress is under way.

As in the previous chapter we fix the values of $\lambda = d(m-1)+2$ and $\gamma = \lambda/(m-1)$.

## 13.1 Measures as initial data. Initial trace

We are going to extend the existence theorem obtained in the previous chapter to cover the case where the initial data are allowed to be Radon measures. Non-trivial limit solutions are obtained as limits of approximations with solutions having locally integrable data as in the previous chapter. This procedure works if the initial measure satisfies the growth condition similar to the one satisfied by the locally integrable data serving as initial data in Chapter 12. It turns out that the main difference with the existence result of Theorem 12.8 will be the weaker form in which the initial data are taken. This motivates the introduction of the concept of trace.

**Definition 13.1** *Let $u$ be a solution of the PME defined in $Q_T$. A Radon measure $\mu$ defined in $\mathbb{R}^d$ is called the initial trace of $u$ if*

$$\lim_{t\downarrow 0} \int_{\mathbb{R}^d} u(x,t)\,\zeta(x)\,dx = \int_{\mathbb{R}^d} \zeta(x)\,\mu(dx) \tag{13.1}$$

*holds for all test functions $\zeta \in C_c(\mathbb{R}^d)$.*

In other words, the family of measures $\mu(t) = u(\cdot,t)dx$ converges vaguely to $\mu$ as $t \to 0$, see Section A.4. Note that whenever an initial trace exists satisfying (13.1), then it is unique as a Radon measure. We also recall that locally bounded sequences of measures have subsequences that converge vaguely.

**Example 13.1** We see that all solutions with initial data given by locally integrable functions having the prescribed growth, as constructed in the previous chapter, satisfy $u \in C([0,T) : L^1_{\rm loc}(\mathbb{R}^d))$. This clearly implies that the initial trace is equivalent to initial data in that case.

On the other hand, the ZKB solutions have as initial data a Dirac delta, which falls out of that class of data but satisfies (13.1), i.e., the Dirac mass is an initial trace. Including such type of solutions is the motivation for the study of initial traces and their role in the existence and uniqueness theory.

*Existence*

The first problem we will address is existence of solutions with a Radon measure $\mu$ as initial trace. The assumption of non-negativity is not needed in this section. The necessary growth condition reads:

(GC) *There exists a constant $C > 0$ such that for every $R > 0$,*

$$|\mu|(B_R(0)) \le C\,(R^\gamma + 1). \tag{13.2}$$

We call $\mathcal{M}_\gamma$ the space of Radon measures satisfying (13.2). Note that the space depends on both the exponent $m$ and the space dimension $d$. For measures

$\mu \in \mathcal{M}_\gamma$ we define the functional $\ell(\mu)$ analogously to (12.5):

$$\ell(\mu) := \lim_{r\to\infty} \sup_{R\ge r} R^{-\gamma}|\mu|(\overline{B}_r) < \infty, \tag{13.3}$$

where $\overline{B}_r = \{|x| \le R\}$. Similarly, we can modify formula (12.15) to define $\|\mu\|_r$ for $r > 0$. The growth condition (13.2) can be written as $\|\mu\|_r < \infty$ for all $r \ge 1$. Note also that $\ell(\mu) = \lim_{r\to\infty} \|\mu\|_r$.

We obtain next existence of limit solutions for data $\mu \in \mathcal{M}_\gamma$. The positive constants $c_1, c_2, c_3$ are the same as in Theorem 12.8.

**Theorem 13.1** *Let $\mu$ be a Radon measure on $\mathbb{R}^d$ satisfying the growth condition (GC). Then there is a function $u(x,t)$ defined in the time interval $0 < t < T(\mu)$ with*

$$T(\mu) = \frac{c_1}{(\ell(\mu))^{m-1}} \tag{13.4}$$

*such that:* (i) *The map $t \mapsto u(t)$ belongs to $C((0,T(\mu)) : L^1_{\rm loc}(\mathbb{R}^d))$.*

(ii) *For every $R > r > 0$ and $0 < t < T_r(\mu) = c_1\|\mu\|_r^{m-1}$ we have*

$$\|u(t)\|_{L^\infty(B_R(0))} \le c_2\|\mu\|_r^{2/\lambda} R^{2/(m-1)} t^{-d/\lambda}; \tag{13.5}$$

*and $\|u(t)\|_r \le c_3\|\mu\|_r$ for $0 < t < T_r(\mu)$.*

(iii) *For every $\varepsilon > 0$, $u(x,t+\varepsilon)$ is a solution of the PME in the class $\mathcal{U}$ for $0 < t < T_\varepsilon$; this class of solutions with limited growth is defined in Chapter 12.*

(iv) *We have $|u|^m \in L^1(S)$ for all sets $S = B_R(0) \times [0,T]$, $R > 0$, $0 < T < T(u_0)$. Moreover, for every $\zeta \in C_c^\infty(\mathbb{R}^d \times [0,T(\mu)))$ we have*

$$\int\!\!\int (u\zeta_t + |u|^{m-1}u\,\Delta\zeta)\,dxdt = \int_{\mathbb{R}^d} \zeta(x,0)\mu(dx). \tag{13.6}$$

(v) *We have $|u|^{m-1}u \in H^1_{\rm loc}((\mathbb{R}^d) \times (0,T(\mu))$.*

Note that in (i) no strong continuity down to $t = 0$ is proved. Information about how the initial data is taken is contained in (iv), which implies in particular that $\mu$ is the initial trace of $u$ (taking $\zeta = \zeta(x)$).

**Proof** Take a mollifier sequence $\rho_n$ and define $\mu_n = \mu * \rho_n$. Then $\mu_n$ is an admissible function for the PME in the sense of the last chapter, $\mu_n \in X$, and moreover $\|\mu_n\|_r$ converges to $\|\mu\|_r$ as $n \to \infty$. Using Theorem 12.8, we obtain existence of solutions $u_n$ with initial data $\mu_n$ and satisfying (i)–(v) with $\mu_n$ instead of $\mu$. These estimates are uniform in $n$ for $0 < t < T(\mu) - \varepsilon$. In particular, the functions $u_n$ are locally bounded uniformly in $n$.

We want to pass to the limit a.e. in the sequence $u_n$. This convergence is a consequence of the compactness of $u_n^m$ in $L^2_{\rm loc}(\mathbb{R}^d \times (0,T(\mu)-\varepsilon))$. Indeed, we can use the argument of Proposition 12.14 to see that $u_n^m \in H^1_{\rm loc}((\mathbb{R}^d) \times (0,T(\mu_n))$

with bounds that are uniform in $n$. It is then easy to see that

$$u(x,t) = \lim_{n\to\infty} u_n(x,t)$$

is a solution and satisfies conditions (i)–(iii) and (v). For condition (iv) we use again the argument of Proposition 12.14 to show that if (i), (ii), (iii) hold, then

$$\iint_{Q_T} |u|^m \theta \, dxdt \le c(R)\|\mu\|_r^\delta T^{-2/\lambda}, \quad \delta = 2\lambda^{-1}(m-1)+1.$$

hence $|u|^m \in L^1(S)$ for all sets $S = B_R(0) \times [0,T]$, $R>0$, $0<T<T(u_0)$. Formula (13.6) follows easily. ■

## 13.2 Existence of initial traces in the CP

We continue with the question of existence of initial traces for non-negative weak solutions of the PME posed in the space-time domain $Q_T = \mathbb{R}^d \times (0,T)$ for some $T>0$. By solution we shall always understand non-negative weak solution.

**Theorem 13.2** *Let $u$ be a continuous and non-negative weak solution of the PME defined in the domain $Q_T$ for some $T>0$. Then $u$ has a unique initial trace $\mu$ that is generally speaking a Radon measure. Moreover, there exists a constant $C = C(m,d) > 0$ such that for every $x_0 \in \mathbb{R}^d$ and $r>0$, $0<t<T$,*

$$\oint_{B_r(x_0)} \mu(dx) \le C\left(r^{2/(m-1)}t^{-1/(m-1)} + \frac{t^{d/2}}{r^d}\, u^{\lambda/2}(x_0,t)\right). \tag{13.7}$$

**Proof** Existence along subsequences. The main previous tool is the Aronson–Caffarelli estimate, cf. Theorem 12.1, or [42]. This estimate can be applied to the solution with initial time $\varepsilon \in (0,T)$ and end-time $T$ to get the bound

$$\int_{B_r(0)} u(x,\varepsilon)\, dx \le C\left(r^{\lambda/(m-1)}(T-\varepsilon)^{-1/(m-1)} + (T-\varepsilon)^{d/2}u^{\lambda/2}(0,T)\right), \tag{13.8}$$

where $\lambda = d(m-1)+2>0$. This means that the sequence of regular measures $\mu_\varepsilon = u(x,\varepsilon)\,dx$ with parameter $\varepsilon>0$ is uniformly bounded on every ball. Thus, there exists a subsequence $\varepsilon_k \downarrow 0$ such that $u(x,\varepsilon_k)\,dx$ converges vaguely to a Radon measure that we call $\mu$. This means that

$$\lim_{k\to\infty}\int_{\mathbb{R}^d} u(x,\varepsilon_k)\zeta(x)\,dx = \int_{\mathbb{R}^d}\zeta(x)\,\mu(dx).$$

In other words, $\mu$ is an initial trace along that subsequence of times approaching zero. It is moreover clear that the $\mu$ obtained in the limit $\varepsilon_k \to 0$ satisfies the growth estimate (13.7).

Uniqueness of the initial trace. We have to prove that the initial measure $\mu$ does not depend on the sequence of times $\varepsilon_k \downarrow 0$. The tool to prove this uniqueness is the control of the speed at which material escapes from a given ball. First, we consider a solution initially supported in a ball and control the size outside the ball at later times.

**Lemma 13.3** *Let $u$ be a weak solution that is continuous for $0 < t \le T$. Assume that* $\mathrm{supp}(u(\cdot,0)) \subset B_1(0)$ *and also that*

$$\int_{\mathbb{R}^d} u(x,0)\,dx \le M_1. \tag{13.9}$$

*Then there exists a constant $C = C(d) > 0$ such that for every $\eta > 0$ we have*

$$u(x,t) \le \frac{M_1}{C\eta^{(3d-1)/2}} \tag{13.10}$$

*for every $|x| \ge 1+\eta$ and $0 < t < T$.*

**Proof** We use the Aleksandrov reflection principle (cf. Section 9.6.2) to show that for any $x_0$ such that $|x_0| \ge 1+\eta$ there is a cone $\mathcal{K}(x_0)$ of points $x_1$ such that the hyperplane $\Phi(x_0,x_1)$ perpendicular to the segment $\overline{x_1x_0}$ at its middle point leaves $B_1(0)$ on the same side as $x_1$ so that we conclude that

$$u(x_1,t) \ge u(x_0,t)$$

for all $0 < t < T$. Then we have

$$M_1 \ge \int_{\mathbb{R}^d} u(x,0)\,dx = \int_{\mathbb{R}^d} u(x,t)\,dx \ge \int_{\mathcal{K}(x_0)} u(x,t)\,dx \ge |\mathcal{K}(x_0)|\,u(x_0,t).$$

The cone $\mathcal{K}$ is defined as the set of points $x_1$ such that $|x_1 - x_0| \le \eta/2$ and

$$\cos\theta \ge \frac{1+\eta/4}{1+\eta}$$

where $\theta$ denotes the angle between the vectors $x_0$ and $x_0 - x_1$. Then we have (see the corresponding graph)

$$d(\Pi(x_0,x_1),0) = |x_0|\cos\theta - \frac{1}{2}|x_0 - x_1| \ge (1+\eta)\frac{1+\eta/4}{1+\eta} - \frac{\eta}{4} = 1,$$

which is the condition we needed to apply Aleksandrov. The volume of $\mathcal{K}$ is then of the order of $O(\ell^d\theta^{d-1})$. In this case we have $\ell = \eta/2$ and $\theta^2 \approx \eta$ for small $\eta$ so that

$$|\mathcal{K}| \ge C\,\eta^{(3d-1)/2}$$

for small $\eta$. The conclusion follows. ■

**Lemma 13.4** *Let $u$ be a solution as before. Then, for every $\varepsilon > 0$ there exists $\tau > 0$ such that*

$$\mathrm{supp}(u(\cdot,t)) \subset B_{1+\varepsilon}(0) \tag{13.11}$$

*for every $0 < t < \tau$. This $\tau$ depends on $d, m, \varepsilon, M_1, T$.*

**Proof** Take the ZKB solution $U_M$ with a mass $M > M_1$ to be adjusted soon. Given $\varepsilon > 0$ one can check from the explicit formula that there exists a time $t_0$ and a mass $M$ such that

$$\operatorname{supp}(U_M(\cdot, t_0)) = B_{1+\varepsilon}(0),$$

and also

$$\min_{0<t<T} U_M(x, t+t_0) \geq \frac{M_1}{C(\varepsilon/4)^{(3d-1)/2}}$$

holds for $|x| = 1 + \varepsilon/4$. Both $t_0$ and $M$ depend on $d, m, \varepsilon, M_1, T$. We try parabolic comparison between $u(x,t)$ and $U_M(x, t+t_0)$ in the space domain $\Omega = \mathbb{R}^d \setminus B_{1+\varepsilon/4}(0)$ and $0 < t < T$. On the initial time we have

$$u(x, 0) = 0 \leq U_M(x, t_0) \quad \text{for } x \in \Omega,$$

while on the lateral boundary

$$u(x,t) \leq \frac{M_1}{C(\varepsilon/4)^{(3d-1)/2}} \leq U_M(x, t+t_0).$$

Therefore, $u(x,t) \leq U_M(x, t+t_0)$ in $\Omega \times (0, T)$. Now let $\tau$ be defined by

$$\operatorname{supp}(U_M(\cdot, \tau + t_0)) = \overline{B}_{1+\varepsilon}(0).$$

The result follows. ■

We will return to the propagation question treated here in Theorem 14.6. We can now prove that, though mass may get out of a given ball, it does slowly and the amount that is left can be controlled from below.

**Lemma 13.5** *Let $u$ be a continuous solution as before. If*

$$0 < M_0 \leq \int_{B_1(0)} u(x, 0)\, dx \leq M_1$$

*then for each $\varepsilon > 0$ there exists $\tau = \tau(d, m, \varepsilon, M_1, T)$ such that*

$$\int_{B_{1+\varepsilon}(0)} u(x,t)\, dx \geq M_0$$

*for all $t \in (0, \tau)$.*

**Proof** Let $w$ be the solution with initial data $w(x,0) = u_0(x)\chi_{B_1(0)}(x)$. We have

$$\int_{\mathbb{R}^d} w(x,0)\, dx = M \in [M_0, M_1].$$

The comparison results say that $w(x,t) \leq u(x,t)$ in $Q_T$. On the other hand, since the support of $w(x,0)$ is contained in $B_1(0)$, Lemma 13.4 implies that there exists a $\tau > 0$ such that

$$\int_{B_{1+\varepsilon}} w(x,t)\,dx = M$$

if $0 < t < \tau$. It follows that $\int_{B_{1+\varepsilon}} u(x,t)\,dx \geq M$ in that time interval. ■

**Proof of uniqueness in Theorem 13.2** Suppose that $u(t)$ converges weakly to a measure $\mu_1$ along a sequence of times $\varepsilon_k \downarrow 0$ and to $\mu_2$ along another sequence $\tau_j \downarrow 0$. We recall that these $\mu_i$ are positive linear maps on $C_c(\mathbb{R}^d)$ and also set operators, and as such they are regular finite Borel measures.

Estimate (13.7) must hold for both measures. The Aronson–Caffarelli estimate in the form (13.8) allows us to control from above the mass of the solution $u(t)$ uniformly for all times $t \in (0, T/4)$. We can then use Lemma 13.5 to show that there exists $\tau(d, m, \varepsilon, M_1, T) > 0$ such that

$$\int_{B_{1+\varepsilon}(0)} u(x, t+s)\,dx \geq \int_{B_1(0)} u(x,s)\,ds$$

whenever $s \in (0, T/4)$ and $0 < t < \min(T/4, \tau)$. Now we fix $t > 0$ and set $s = \varepsilon_k$; taking the limit as $k \to \infty$ and using the continuity of $u$, we get

$$\int_{B_{1+\varepsilon}(0)} u(x,t)\,dx \geq \lim_{\varepsilon_k \to 0} \int_{\mathbb{R}^d} u(x,s)\zeta_1(x)\,ds \geq \mu_1(B_{1-\varepsilon}(0)).$$

where $\zeta_1$ is a cut-off function which is one in $B_{1-\varepsilon}(0)$ and 0 outside of $B_1(0)$. Letting now $t = \tau_j$ and $j \to \infty$ we get

$$\mu_2(B_{1+2\varepsilon}(0)) \geq \lim_{\varepsilon_k \to 0} \int_{\mathbb{R}^d} u(x,s)\zeta_2(x)\,ds \geq \mu_1(B_{1-\varepsilon}(0)).$$

Using a scaling of the radius we can write this as

$$\mu_2(B_1(0)) \geq \mu_1(B_{1-\varepsilon}(0)).$$

We may now let $\varepsilon \to 0$ (and use the properties of measures) to get

$$\mu_2(B_1(0)) \geq \mu_1(B_1(0)).$$

Reversing the roles we get

$$\mu_1(B_1(0)) \geq \mu_2(B_1(0)).$$

By scaling and translation, these inequalities are true for every ball $B = B_r(x_0)$, hence $\mu_1(B) = \mu_2(B)$ for all $B$. We conclude that $\mu_1 = \mu_2$. ■

## 13.3 Pierre's uniqueness theorem

We turn our attention to the question of uniqueness of solution given the initial trace. This is the first result in that direction. Let $Q_T = \mathbb{R}^d \times (0,T)$ and $Q_T^\tau = \mathbb{R}^d \times (\tau, T)$. We write $\Phi(u) = u^m$.

**Theorem 13.6** *Let $u_1, u_2 \in L^1(Q_T) \cap L^\infty(Q_T^\tau)$ be two non-negative solutions of the PME in the sense of distributions in $Q_T$. If $d = 1, 2$ assume also that $\Phi(u_i) \in L^1(Q_T)$. If the initial traces of both solutions coincide, then $u_1 = u_2$* a.e.

**Proof** (i) We first remark that if $u$ is any weak solution of the PME as in the theorem, then, for every $0 < s < t < T$ we have

$$u(t) - u(s) = \Delta \int_s^t \Phi(u(\sigma))\, d\sigma \tag{13.12}$$

in the sense of distributions in $\mathbb{R}^d$, see Definition 5.2. Note that we have skipped writing the $x$-dependence in the formula, as usual. The proof is done by using test functions of the form $\zeta(x,t) = \theta_n(t)\varphi(x)$ in the definition of weak solution and passing to the limit in $n$. The exact statement is given in Theorem 6.8, see formula (6.19) and also Problem 6.2. On the other hand, we know that $u(s) \to \mu$ in the vague sense as $s \to 0$, where $\mu$ is the initial trace of $u$ whose existence has been proved in the previous section.

Then we remark that the function

$$F(x; s, t) = \int_s^t \Phi(u(x,\sigma))\, d\sigma$$

is non-negative and monotone in both $s$ and $t$. Under the conditions of the theorem $\Delta_x F(x; s, t)$ is bounded in $L^1(\mathbb{R}^d)$ for all $0 < s < t < T$, uniformly as $s$ varies in the interval $(0, t)$. It follows from potential theory (cf. [87]) that $F(x; s, t)$ is uniformly bounded in the Marcinkiewicz space $L^{p_*,\infty}(\mathbb{R}^d)$ with $p_* = d/(d-2)$, which means that it is uniformly bounded in $L^p_{\rm loc}(\mathbb{R}^d)$ for every $1 \le p < p_*$, $0 < s < t$. It follows that

$$F(x, s, t) \to F(x, 0, t) = \int_0^t \Phi(u(x,\sigma))\, d\sigma$$

as $s \to 0$, with convergence in $L^1_{\rm loc}(\mathbb{R}^d)$.

We can take the Newtonian potential $v(t) = N(u(t)) := E_d \star u(t)$ (see more on potentials in Section A.6). This is a continuous function in $L^q(\mathbb{R}^d)$ for every $p < q \le \infty$ and then we have for every $0 < s < t < T$

$$v(t) - v(s) = E_d \star (\Delta \int_s^t \Phi(u(\sigma))\, d\sigma) = - \int_s^t \Phi(u(\sigma))\, d\sigma$$

with equality for a.e. $x \in \mathbb{R}^d$. Note that we have used the uniqueness of solution of the equation $-\Delta u = f$ in this functional setting. This means that $v(x,t)$ is monotone non-increasing and continuous function of $t$. Consequently, it must have a limit $v(0)$ as $t \to 0$. Moreover, this limit takes place in $L^1_{\rm loc}(\mathbb{R}^d)$. It easily follows that $v(0)$ is a lower semicontinuous function and moreover

$$v(0) = N(\mu).$$

(ii) Let us now take a fixed and small $0 < h < T$ and consider the difference $u_1(t) - u_2(t+h)$. We get

$$u_1(t) - u_2(t+h) = u_1(s) - u_2(s+h) + \Delta \int_s^t (\Phi(u_1(\sigma)) - \Phi(u_2(\sigma+h)))\, d\sigma$$

as distributions in $\mathbb{R}^d$. We may now let $s \to 0$ to get

$$u_1(t) - u_2(t+h) = \mu - u_2(h) + \int_0^t (\Phi(u_1(\sigma)) - \Phi(u_2(\sigma+h)))\, d\sigma. \qquad (13.13)$$

(iii) Integrated equation. We can write this expression in a convenient way after introducing the function

$$g(t) := \int_0^t (\Phi(u_1(\sigma)) - \Phi(u_2(\sigma+h)))\, d\sigma + v_2(h) - v_1(0) \qquad (13.14)$$

where $v_i(t)$ is the Newtonian potential of $u_i(t)$. Then, (13.13) becomes

$$u_1(t) - u_2(t+h) = \Delta g(t).$$

Taking potentials we get equivalently $g(t) = v_2(t+h) - v_1(t)$. We may also differentiate $g$ with respect to $t$ to get $g_t = \Phi(u_1(x,t)) - \Phi(u_2(x,t+h))$, which means that $g$ is a solution of the 'integrated equation'

$$g_t = a(x,t)\Delta g \qquad (13.15)$$

where $a(x,t)$ is defined as

$$a(x,t) = \frac{\Phi(u_1(x,t)) - \Phi(u_2(x,t+h))}{u_1(x,t) - u_2(x,t+h)}$$

whenever $u_1(x,t) \neq u_2(x,t+h)$ and we put $a(x,t) = \Phi'(u_1(x,t))$ if $u_1(x,t) = u_2(x,t+h)$. Note that (13.15) is a linear equation, and the coefficient is a non-negative function, and moreover, it belongs to $L^\infty(\mathbb{R}^d \times (\tau, T))$ for every $\tau > 0$. If $a$ were positive and regular enough, we could use the theory of linear parabolic equations in non-divergence form, cf. [357]. In that respect we point out that $v_1(0) = N(\mu) = v_2(0)$ since the Theorem assumes that the initial traces coincide. Therefore,

$$\lim_{t\to 0} g(t) = v_2(h) - v_1(0) = v_2(h) - v_2(0) \leq 0.$$

If the maximum principle can be applied to equation (13.15), then $g(t) \leq 0$ for all $t > 0$. Therefore, $v_2(t+h) \leq v_1(t)$ for every $t, h > 0$ with $t + h < T$. Letting $h \to 0$ we will get

$$v_2(t) \leq v_1(t).$$

The converse will also be true, hence $v_1 = v_2$, and this implies $u_1 = u_2$ a.e. The rest of the proof is devoted to justify this argument.

(iv) An involved maximum principle. The method is a variant of the duality method used in the proof of uniqueness and comparison of very weak solutions of the GPME in Subsection 6.2.1. Formally, it consists in multiplying equation (13.15) by the solution $\psi$ of the problem

$$\psi_t + \Delta(a\psi) = 0, \quad \psi(x, T_1) = \theta(x)$$

where $\theta \in C_c^\infty(\mathbb{R}^d)$ and $T_1 + h < T$. This should give (after integration by parts)

$$\int_{\mathbb{R}^d} g(T_1)\,\theta\,dx = \int_{\mathbb{R}^d} g(s)\,\psi(s)\,dx$$

for all $0 < s < T_1$. Then, we have to show that the right-hand side has a non-positive limit when $s \to 0$.

We have to justify both claims. For the first one we regularize $a$ into $a_n$ in the usual way. We replace function $a$ by a sequence of smooth approximations $a_n \ge 0$ defined in $Q_1 = \mathbb{R}^d \times (0, T_1)$ such that

(a) the $a_n$ are bounded as well as $|\nabla a_n|$ and $\Delta a_n$ in $Q_1$;
(b) for every fixed $\tau \in (0, T_1)$, the $a_n$ are bounded in $\mathbb{R}^d \times (\tau, T_1)$ uniformly in $n$;
(c) $a_n$ converges to $a$ a.e. in $Q_1$ as $n \to \infty$.

We then take $a_{n\varepsilon} = a_n + \varepsilon$ for some $\varepsilon > 0$ and solve the problem in inverse time

$$\begin{cases} \partial_t\psi + \Delta((a_n + \varepsilon)\,\psi) = \theta & \text{in } Q_1 \\ \psi(x, T_1) = 0 & \text{for } x \in \mathbb{R}^d. \end{cases} \tag{13.16}$$

This problem has a unique non-negative $C^\infty$ smooth solution $\psi_n$ and $\psi_n(t) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ for all $t$ [357]. Moreover, $\int \psi_n(t)\,dx = \int \theta\,dx$ for all $0 < t < T_1$.

Performing the announced multiplication and integration with $\psi_n$ instead of $\psi$ we get

$$\begin{aligned} \int_{\mathbb{R}^d} g(T_1)\,\theta\,dx - \int_{\mathbb{R}^d} g(s)\,\psi_n(s)\,dx &= \int_s^{T_1}\int_{\mathbb{R}^d} (g_t\psi_n + g\partial_t\psi_n)\,dxdt \\ &= \int_s^{T_1}\int_{\mathbb{R}^d} (a(\sigma) - a_n(\sigma) - \varepsilon)\,\psi_n(\sigma)\Delta g(\sigma))\,dxd\sigma. \end{aligned} \tag{13.17}$$

In order to pass to the limit in $n$, $\varepsilon$ we need some estimates. We introduce the potential $H_n = E_d \star \psi_n$ (so that $\Delta H_n = -\psi_n$). We multiply equation (13.16) by $H_n$ and integrate to obtain:

$$\int_s^{T_1}\int_{\mathbb{R}^d} (a_n + \varepsilon)\psi_n^2\,dxdt = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla H_n(T_1)|^2 - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla H_n(s)|^2. \tag{13.18}$$

This implies that $\psi_n$ is uniformly bounded in $L^2(Q_1)$. Hence, it has a weakly convergent subsequence, $\psi_n \rightharpoonup \psi_\varepsilon$. Then, $(a_n + \varepsilon)\psi_n$ also converges weakly in

$L^2(\tau, T_1 : L^2(\mathbb{R}^d))$ to $w = (a+\varepsilon)\psi_\varepsilon$ (for all $\tau > 0$). We conclude that the limit $\psi_\varepsilon$ satisfies an integrated form of (13.16):

$$\psi_\varepsilon(t) - \psi_\varepsilon(s) = -\Delta \int_x^t (a+\varepsilon)\psi_\varepsilon \, dx \tag{13.19}$$

for all $0 \le s < t \le T_1$. In the limit $n \to \infty$ we still have $\int \psi_\varepsilon \, dx \le \int \theta \, dx$ and in fact equality holds

$$\int_{\mathbb{R}^d} \psi_\varepsilon(t) \, dx = \int_{\mathbb{R}^d} \theta \, dx$$

because of the last identity. Moreover, we can assert that $\psi_\varepsilon(t)$ converges to $\psi_\varepsilon(t)$ in the weak topology $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ for all $t > 0$.

We may now pass to the limit in $n$ in equation (13.17). We use the fact that $a_n \to a$ a.e. and is uniformly bounded for $t \ge \tau > 0$; the fact that $\Delta g(t) = u_1(t) - u_2(t+h)$ is bounded in $L^p(\mathbb{R}^d \times (\tau, T_1))$ for all $p$ and finally that $\psi_n \rightharpoonup \psi_\varepsilon$ to conclude that

$$\int_{\mathbb{R}^d} g(T_1)\, \theta \, dx - \int_{\mathbb{R}^d} g(s)\, \psi_n(s) \, dx = -\varepsilon \int_s^{T_1} \int_{\mathbb{R}^d} \psi_\varepsilon(\sigma) \Delta g(\sigma) \, dx d\sigma. \tag{13.20}$$

Next, we have to pass to the limit $\varepsilon \to 0$. For any $0 < s < T_1$ the right-hand side is bounded by

$$\varepsilon \|\Delta g\|_{L^\infty(\mathbb{R}^d \times (s, T_1))} \int_s^{T_1} \|\psi_\varepsilon(s)\|_{L^1(\mathbb{R}^d)}.$$

This converges to zero since we have proved that $\|\psi_\varepsilon\|_1$ is bounded above by $\int \theta \, dx$. In order to pass to the limit in the second term of (13.20) we use again potentials. If $H_\varepsilon = E_d \star \psi_\varepsilon$, then (13.19) gives

$$H_\varepsilon(t) - H_\varepsilon(s) = \int_x^t (a+\varepsilon)\psi_\varepsilon \, dx. \tag{13.21}$$

Since $\psi_\varepsilon \ge 0$, we have

$$0 \le H_\varepsilon(s) \le H_\varepsilon(t) \le H_\varepsilon(T_1) = E_d \star \theta. \tag{13.22}$$

The functions $H_\varepsilon$ are therefore uniformly bounded in $L^p(\mathbb{R}^d \times (0, T_1))$ for $p > d/(d-2)$. One can find a convex combination of those $H_\varepsilon$ converging strongly and a.e. to a limit $H$ in $L^p(\mathbb{R}^d \times (0, T_1))$ for some $p \in (d/(d-2), \infty)$. Since $\psi_\varepsilon(t)$ is bounded in $L^p(\mathbb{R}^d)$ uniformly in $t$ and $\varepsilon$, we can assume that the same combinations of $\psi_\varepsilon(t)$ converge to a measure $\nu(t) \in \mathcal{M}^+(\mathbb{R}^d)$ in the weak-* topology $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$. The passage to the limit in (13.20) needs the convergence in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$. This is due to the fact that for $0 < s < T_1$ we have

$$\lim_\varepsilon \int_{\mathbb{R}^d} \psi_\varepsilon(s) \, dx = \int_{\mathbb{R}^d} \theta \, dx = \int_{\mathbb{R}^d} d\nu(s). \tag{13.23}$$

To prove this fact, we observe that for every $0 < s < T_1$ the integral $\int_x^{T_1} (a+\varepsilon)\,\psi_\varepsilon$ is bounded in $L^1(\mathbb{R}^d)$. Hence, by (13.19) there exists $\rho(s) \in \mathcal{M}^+(\mathbb{R}^d)$ such that

$$\theta - \nu(s) = -\Delta\rho(s) \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

This and the mass equality $\int_{\mathbb{R}^d} \psi_\varepsilon(t)\,dx = \int_{\mathbb{R}^d} \theta\,dx$ imply (13.23). We may now pass to the limit $\varepsilon \to 0$ in (13.20) and conclude that there exist a family of measures $\nu(s) \in \mathcal{M}^+(\mathbb{R}^d)$ such that for every $0 < s < T_1$

$$\int_{\mathbb{R}^d} g(T_1)\,\theta\,dx = \int_{\mathbb{R}^d} g(s)\,d\nu(s). \tag{13.24}$$

Moreover, the potentials $H(s) = E_d \star \nu(s)$ satisfy (13.22).

The last step is passing to the limit $s \to 0$ in (13.24). Note first that $g(s) = v_2(s+h) - v_1(s)$. By the monotonicity results for potentials we have for $0 < s < s_0 < T_1$:

$$\int_{\mathbb{R}^d} g(T_1)\,\theta\,dx \le \int_{\mathbb{R}^d} (v_2(h) - v_1(s_0))\,d\nu(s) = \int_{\mathbb{R}^d} (u_1(h) - u_1(s_0))\,H(s) \tag{13.25}$$

(see remark below for the integration by parts). We know that $\int d\nu(s)$ is uniformly bounded in $s$, and also that $H(s)$ decreases pointwise as $s \to 0$. Hence, $\nu = -\Delta H(0+) \in \mathcal{M}^+(\mathbb{R}^d)$, and the last inequality gives

$$\int_{\mathbb{R}^d} g(T_1)\,\theta\,dx \le \int_{\mathbb{R}^d} (u_1(h) - u_1(s_0))\,H(0+) = \int_{\mathbb{R}^d} (v_2(h) - v_1(s_0))\,d\nu. \tag{13.26}$$

We may now let $s_0 \downarrow 0$ to get by the monotonicity of $v_1$

$$\int_{\mathbb{R}^d} g(T_1)\,\theta\,dx \le \int_{\mathbb{R}^d} (v_2(h) - v_1(0))\,d\nu.$$

Recall now that $v_2(x,s) \le v_2(x,0) = v_1(x,0)$ for all $x \in \mathbb{R}^d$. It follows that

$$\int_{\mathbb{R}^d} g(T_1)\,\theta\,dx \le 0$$

for all test functions $\theta \in \mathcal{D}^+(\mathbb{R}^d)$. This is the relation we were looking for and ends the proof in dimensions $d \ge 3$.

**Proofs in dimensions $d = 1, 2$** The above method has to be strongly modified due to the bad properties of the potentials in those dimensions. We refrain from the lengthy proofs by lack of space and refer the reader to [434]. ■

### Remarks

(1) The proof holds for the GPME $u_t = \Delta\Phi$ when $\Phi : [0,\infty) \to [0,\infty)$ is a locally Lipschitz function, non-decreasing and $\Phi(0) = 0$. This is the generality stated in the original paper [434].

(2) We have used the fact that given two measures $\mu, \nu \in \mathcal{M}^+(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} (E_d \star \mu)\, d\nu = \int_{\mathbb{R}^d} (E_d \star \nu)\, d\mu$$

whether the integral is finite or not. Thus, in (13.25) or (13.26), $u_2(h) - u_1(s_0)$ is a good function, integration by parts works in that case. With $h = 0$ this would not be justified. Note that $H(0+)$ is the decreasing limit of the potentials $H(s)$. It is generally not an l.s.c. potential itself but it is equal a.e. to $E_d \star \nu$.

## 13.4 Uniqueness without growth restrictions

We now discuss the results where the growth assumptions of Theorems 12.10 and 13.6 are eliminated from the statement. Before proving the Dahlberg–Kenig uniqueness result, we establish a preliminary result about existence of minimal solutions under quite general assumptions.

**Theorem 13.7** *There exists a minimal element in the set of non-negative distributional solutions of the PME defined in $Q_T$, $T > 0$, such that $u^m$ is locally in $L^2$, and having a given measure $\mu$ as initial trace.*

**Proof** Locally in $L^2$ means that $u^m \in L^2(B_R \times (s,t))$ for every $R > 0$ and every $0 < s < t < T$. We recall that given $\mu$ we can construct at least one $u$ non-negative distributional solution. We can take it continuous but that is not important in the argument.

(i) In the first step we construct a solution with smaller initial data that lies below a given solution. This is done as follows: we take a cut-off function $\alpha \in C_c^\infty(\mathbb{R}^d)$, $0 \le \alpha \le 1$, and let $v_n(x,t)$ be the weak solutions of the PME with initial data

$$v_n(x,0) = \alpha(x) u(x, 1/n)$$

for $n = 1, 2, \ldots$ This falls into the theory of Chapter 9 and such solutions are limits of solutions of Dirichlet problems in balls. By the comparison result proved in Problem 6.9 we have

$$v_n(x,t) \le u\left(x, t + \frac{1}{n}\right) \quad \text{in } \; Q_{T-1/n}.$$

We want to take the limit $n \to \infty$ if possible. Note that the solutions $v_n$ are uniformly bounded for $t \ge \tau > 0$ and uniformly Hölder continuous, hence a subsequence converges uniformly on compact sets of $Q_T$ to a continuous non-negative solution $v$ of the PME and $v \le u$ in $Q_T$. This reminds us of the proof that there exists a minimal solution done in Problem 9.21. We still have to check the initial trace, i.e., that for a smooth test function $\zeta$ we have

$$\lim_{t \to 0} \int_{\mathbb{R}^d} v(x,t)\, \zeta(x)\, dx = \int_{\mathbb{R}^d} \alpha(x)\, \zeta(x) d\mu(x).$$

Take a smooth function $\theta(t)$ such that $\theta(t) = 1$ for $t$ near $0$ and $\theta(t) = 0$ for $t > T/2$. Let $0 < \tau < T/2 < s < T$ such that $\theta(\tau) = 1$. Then, taking test function

$\eta(x,t)=\theta(t)\zeta(x)$ in the definition of very weak solution we get

$$-\int_{\mathbb{R}^d} v(x,\tau)\zeta(x)\,dx = \int_{\mathbb{R}^d}\int_\tau^s (v^m\Delta\eta + v\partial_t\eta)\,dxdt. \tag{13.27}$$

Cf. Problem 6.2. Call $A_n(t)$ the integral in the right-hand side with time integration from 0 to $t$ and $v$ replaced by $v_n$. Then,

$$-\int v(x,\tau)\zeta\,dx = \lim_{n\to\infty}(A_n(s)-A_n(\tau)).$$

As in (13.27) we know that

$$A_n(s) = -\int_{\mathbb{R}^d} v_n(x,0)\zeta(x)\,dx = -\int_{\mathbb{R}^d}\alpha(x)u(x,1/n)\zeta(x)\,dx,$$

and this converges to $-\int\alpha(x)\zeta(x)\,d\mu(x)$ as $n\to\infty$ by the definition of initial trace of $u$. We still need to prove that $A_n(\tau)\to 0$ uniformly in $n$ as $\tau\to 0$. To this end, we point out the constructed solution $v$ satisfies the bounds of the previous chapter, hence (with $\lambda = d(m-1)+2$ ),

$$\begin{aligned}
&\int_0^\tau\int_{|x|\le L} v^m(x,t)\,dxdt\\
&\le \left(\sup_{0<t<\tau}\int_{|x|\le L} v(x,t)\,dx\right)\int_0^\tau \|v(t)\|_{L^\infty(B_L(0))}\,dt\\
&\le C(\sigma,u(0,\sigma))(1+L^2)\left(\sup_{0<t<\tau}\int_{|x|\le L} v(x,t)\,dx\right)\int_0^\tau t^{-d(m-1)/\lambda}\\
&\le C(\sigma,L,u(0,\sigma))\tau^{2/\lambda}.
\end{aligned}$$

This means that $A_n(\tau)\to 0$ as $\tau\to 0$ uniformly in $n$. We conclude that $v$ has initial trace $\alpha(x)d\mu(x)$.

(ii) We next identify $v$ in terms of its trace. For that we examine some properties of $v_n$ and $v$. First of all, conservation of mass holds for the sequence $v_n$:

$$\int_{\mathbb{R}^d} v_n(x,t)\,dx = \int_{\mathbb{R}^d}\alpha(x)u(x,1/n)\,dx.$$

We conclude that the there exists $C(\alpha)<\infty$ such that

$$\int v_n(x,t)\,dx \le C(\alpha) \tag{13.28}$$

uniformly in $n\ge 1$ and $t>0$. Besides, the solutions $v_n$ have initial data with uniformly bounded compact support. It follows that for $t>0$ the function $v_n(\cdot,t)$ is compactly supported. Moreover, there exist $\tau_0, R_0$ independent of $n$ such that

$$\operatorname{supp} v_n(\cdot,t)\subset B_{R_0}(0)\quad \forall\ 0<t<\tau_0. \tag{13.29}$$

In the limit $n \to \infty$, properties (13.28) and (13.28) hold for $v$. Moreover, $\iint v^m \, dxdt < \infty$. It follows from Theorem 13.6 that $v$ is uniquely determined by its initial trace, i.e., by $\mu$ and $\alpha$. We can write $v = V(\mu, \alpha)$. Even if we have not yet proved uniqueness for $u$ in terms of $\mu$, we have proved uniqueness for $V(\mu, \alpha)$.

(iii) We now take a sequence $\alpha_j \in C_c^\infty(\mathbb{R}^d)$, $j = 1, 2, \ldots$, such that $\alpha_j \le \alpha_{j+1}$ and $\alpha_j(x) = 1$ for $\|x\| \le j$. The comparison principle applies to the previous construction, hence $V(\mu, \alpha_j)$ is a monotone non-decreasing sequence and we can define the limit

$$v^* = \lim_{j\to\infty} V(\mu, \alpha_j),$$

and the inequality $v^*(x,t) \le u(x,t)$ holds everywhere in $Q_T$. Let us check the initial trace of $v^*$. By monotonicity it has to be equal to or larger than that of $V(\mu, \alpha_j)$ for every $j$ and equal to or less than that of $u$. The conclusion it that $v^*$ has initial trace $\mu$, just like $u$. We have proved that there exists a minimal solution for the problem. ■

We now state the Dahlberg–Kenig uniqueness result, where a continuity assumption is needed. We will return to the question of continuity in Section 13.6.

**Theorem 13.8** *Let $u_1, u_2 \ge 0$ be continuous distributional solutions of the PME in $Q_T$, $T > 0$, such that*

$$\lim_{t\to 0} (u_1(t) - u_2(t)) = 0 \tag{13.30}$$

*in the sense of distributions in $\mathbb{R}^d$. Then, $u_1 = u_2$ in $Q_T$.*

**Proof** Since the assumptions of the preceding theorem are satisfied, we know that there is a minimal element having the same initial data, let us call it $v^* = v^*(\mu)$. Taking $u_2 = v^*(\mu)$, we have $u_1 \ge u_2$. We formulate as an independent lemma the proof that then $u_1 \le u_2 = v^*(\mu)$. Since $v^*$ only depends on the initial data, the uniqueness result follows. ■

**Lemma 13.9** *Let $u_1, u_2 \ge 0$ be continuous distributional solutions of the PME in $Q_T$, $T > 0$, such that*

$$\lim_{t\to 0} \int_{|x|\le R} (u_2(t) - u_1(t))^+ \, dx = 0 \tag{13.31}$$

*for every $R > 0$. Then $u_2 \le u_1$ in $Q_T$.*

**Proof** We want to prove that for every test function $\theta \in C_c^\infty(\mathbb{R}^d)$, $\theta \ge 0$, and every $0 < s < T$ we have

$$\int_{\mathbb{R}^d} (u_2(t) - u_1(t)) \, dx \le 0. \tag{13.32}$$

The proof is a variation of Theorem 12.10. We fix $\theta$ and take radii $R > R_0 > 0$ such that the support of $\theta$ is contained in $B_{R_0}(0)$. We take also $0 < \delta < s$. We define

$$a(x,t) = \frac{u_2^m - u_1^m}{u_2 - u_1}$$

and $a(x,t) = mu_1^{m-1}$ if $u_1 = u_2$, we may write $u_2^m - u_1^m = a(x,t)u(x,t)$, $u = u_2 - u_1$ for a measurable function $0 \le a$. We then solve the inverse-time problem in $Q_{RT} = B_R(0) \times (0,T)$

$$\begin{cases} \varphi_t + a_n \Delta\varphi = 0 & \text{in } Q_{RT} \\ \varphi = 0 & \text{on } \Sigma_R = \partial B_R(0) \times (0,T) \\ \varphi(x,T) = \theta & \text{for } x \in B_R(0), \end{cases} \tag{13.33}$$

where $a_n$ is a smooth approximation of $a$ such that $1/n \le a_n \le K$. This parabolic problem in inverse time has a smooth solution $\varphi_{R,n} \ge 0$. We also need a smooth cut-off function $\eta_\varepsilon$, $0 < \varepsilon < 1/2$, such that

$$0 \le \eta_\varepsilon \le 1, \quad \eta_\varepsilon(x) = 1 \text{ for } |x| < R - 2\varepsilon, \quad \eta_\varepsilon(x) = 0 \text{ for } |x| \ge R - \varepsilon,$$
$$\|\nabla\eta_\varepsilon\|_\infty \le c/\varepsilon, \qquad \|\Delta\eta_\varepsilon\|_\infty \le c/\varepsilon^2.$$

We put now the test function $\psi$ in (12.36) equal to $\varphi_{R,n}\eta_\varepsilon$ to get the estimate (dropping the subindexes $R, n$ and $\varepsilon$ in $\varphi$ and $\eta$ for brevity):

$$\begin{aligned} &\int (u_2(s) - u_1(s))\theta \, dx \\ &\quad = \int (u_2(\delta) - u_1(\delta))\varphi_{R,n} \, dx \iint_{Q_T} (u_2 - u_1)\eta(a - a_\varepsilon)\Delta\varphi \\ &\qquad + \iint_{Q_T} (u_2^m - u_1^m)(2\nabla\eta\nabla\varphi + \varphi\Delta\eta) \, dxdt. \end{aligned} \tag{13.34}$$

By arguing like in Theorem 12.10 and using the estimates we conclude for the difference $u = u_1 - u_2$

$$\int u(x,s)\theta(x) \, dx \le A_\beta \int_{|x|\le R} u^+(x,\delta)(1+|x|^2)^{-\beta} \, dx + A_\delta R^{-1}. \tag{13.35}$$

for convenient $\beta$. Next, we let $R \to \infty$ to find that

$$\int u(x,s) \, dx \le A_\beta \int_{|x|\le R} u^+(x,\delta)(1+|x|^2)^{-\beta} \, dx. \tag{13.36}$$

Finally, we let $\delta \to 0$.

$$J = \limsup_{\delta\to 0} \int_{\mathbb{R}^d} u^+(x,\delta)(1+|x|^2)^{-\beta} \, dx \tag{13.37}$$

$$\le \sum_{k=N}^{\infty} \sup_{0<t<T/2} \int_{D_k} u^+(x,\delta) \, 2^{-2k\beta} \, dx \tag{13.38}$$

by the hypotheses of the Lemma, with $D_k = \{s^k < |x| < 2^{k+1}\}$. The Aronson–Caffarelli estimate gives then

$$\sup_{0<t<T/2} \int_{|x|<2^{k+1}} u^+(x,\delta)\, dx \le C_T\, 2^{(k+1)(d+2/(m-1))}$$

so that

$$J \le \sum_{k=N}^{\infty} 2^{(k+1)(d+2/(m-1))-2k\beta}$$

and this is convergent for $2\beta > d + 2/(m-1)$, hence tends to zero as $N \to \infty$. We conclude that

$$\int u(x,s)\theta(x)\, dx \le 0,$$

and since $\theta \ge 0$ was an arbitrary test function, we conclude that $u \le 0$. ■

In the proof we need the following lemma where the role of the continuity assumption appears.

**Lemma 13.10** *Let $u \ge 0$ be a continuous distributional solution of the PME in $Q_T$, $T > 0$, Then for every $0 < t < T$ we have*

$$u(x,t) \le C_t(u)(1 + |x|^2)^{1/(m-1)}. \tag{13.39}$$

*The constant can be written as $C_t(u) = C_T\, C(u(0,T))\, t^{-d/(d(m-1)+2)}$ as $t \to 0$.*

For the proof we refer to [187].

## 13.5 Dirichlet problem with optimal data

We will now address the question of existence of initial traces for solutions of the homogeneous Dirichlet problem posed in a bounded domain, and the questions of existence and uniqueness of solutions given an initial trace. These topics exhibit the main features of the study just performed for the Cauchy problem, but they offer at the same time a number of quite interesting novelties, exemplified by the existence of dipole solutions with a singularity at the boundary of the initial domain.

We do not have space to describe in full detail the results obtained by Dahlberg and Kenig in their paper [189], but we will give a complete summary since it is very important for comparison reasons to get an idea of the type of results that are to be expected in nonlinear diffusion problems, and more generally, in reaction–diffusion problems.

The study of [189] applies to the class of non-negative and continuous very weak solutions of the HDP, that we will abbreviate as c.w. solutions, and the class as $\mathcal{CWS}$. The nonlinearity $\Phi$ is assumed to be continuous, increasing, with $\Phi(0) = 0$ as in Chapter 5. Moreover, the superlinearity condition of Section 5.9

takes the form

$$0 < a \le \frac{u\Phi'(u)}{\Phi(u)} \le \frac{1}{a} \qquad \text{for } u > 0 \tag{13.40}$$

plus

$$1 + a \le \frac{u\Phi'(u)}{\Phi(u)} \qquad \text{for } u \ge u_0 \tag{13.41}$$

for some constants $a, u_0 > 0$. We may normalize $u_0 = 1$ and $\Phi(1) = 1$.

### 13.5.1 *The special solution*

As we have said, a marked difference of the HDP is the existence of the special solution constructed in Section 5.9 and called the Friendly Giant, which takes the separate-variables form (5.69)

$$\widetilde{U}(x,t) = t^{\frac{1}{m-1}}\, F(x). \tag{13.42}$$

in the case of the PME. Let us state a characterization of this special object.

**Lemma 13.11** *Let $u$ be a solution of the HDP in $\mathcal{CWS}$.*

(i) *$u$ is the Friendly Giant if and only if*

$$\sup_{t>0} \int_\Omega u(x,t)\delta(x)\, dx = \infty. \tag{13.43}$$

(ii) *Let $w(t) = Gu(t)$ be its potential. If*

$$\lim_{t\to 0} w(x,t) = +\infty \tag{13.44}$$

*everywhere, then $u$ is the Friendly Giant.*

### 13.5.2 *The double trace results*

The first novel result is the existence of a modified type of trace. We will call the distance function $d(x, \partial\Omega) = \rho(x)$ to avoid confusion with Dirac's delta function.

**Theorem 13.12** *Let $u$ be a solution of the* HDP *in the class $\mathcal{CWS}$ and assume that $u$ is not the Friendly Giant. Then, there exist a non-negative Radon measure $\mu$ in $\Omega$ and a non-negative Radon measure $\lambda$ on $\partial\Omega$ such that*

$$\int_\Omega \rho(x)\, d\mu(x) < \infty, \qquad \int_{\partial\Omega} d\lambda(x) < \infty, \tag{13.45}$$

*and for every test function $\eta \in C^\infty(\mathbb{R}^d)$ such that $\eta = 0$ on $\partial\Omega$ we have*

$$\lim_{t\to 0} \int_\Omega u(x,t)\eta(x)\, dx = \int_\Omega \eta(x)\, d\mu(x) + \int_{\partial\Omega} \partial_\nu \eta(x)\, d\lambda(x), \tag{13.46}$$

*where $\partial_\nu \eta$ denotes the normal derivative in the direction of the outward normal.*

Note that both measures are uniquely determined by the solution $u$ as is easily checked. The result suggests a modification of the definition of trace proposed in Definition 13.1 for the CP in order to include the so-called *boundary trace.*

**Definition 13.2** *We call the pair $(\mu, \lambda)$ as in the theorem the initial trace for the c.w. solution $u$ of the HDP; $\mu$ is the interior initial trace and $\lambda$ is the boundary initial trace.*

**Example 13.2** The most typical example of the new situation is maybe the *dipole solution* constructed in Subsection 4.6.2 for the PME in the one-dimensional setting. In this context, we have to consider the equation as posed in a half line, say $\Omega = (0, \infty)$, but we can restrict consideration to any large interval due to the property of compact support. A careful inspection shows that the initial trace is composed of a trivial interior component $\mu = 0$ and a non-trivial boundary trace, $\lambda = M\,\delta(x)$, a Dirac mass, see formula (4.57).

A dipole solution in several dimensions $n \geq 2$ has been constructed by Hulshof and Vázquez [299] for the problem posed in a half space, $\Omega = \{x : x_1 > 0\}$. It is given by a self-similar formula.

**Example 13.3** An example where there is only a trace of the type $\mu$ but not integrable is simple if we allow for a small change of setting. Consider the solution in $d = 1$ with initial data $u_0(x) = c/x$ for $x \neq 0$. We pose the equation in the half line $x > 0$ with boundary conditions $u = 0$ at $x = 0$ for $t > 0$. According to the theory this solution exists even if $u_0$ is not integrable in a neighbourhood of $x = 0$, because $x\, u_0 \in L^1_{\rm loc}(\mathbb{R})$. By the uniqueness result below, the solution must be self-similar of the form

$$u(x,t) = t^{-1/(m+1)} f(x/t^{1/(m+1)}) \tag{13.47}$$

(see Section 16.3 for this argument). The profile is bounded with one maximum for $x > 0$ and $f(x) \sim c/x$ as $x \to \infty$. Thus, the behaviour of the solution for every $t > 0$ is

$$u(x,t)\|_\infty = C\, t^{-1/(m+1)}.$$

Note that at infinity

$$u(x,t) \sim \frac{c}{x} \quad \text{as} \quad x \to \infty,$$

for every $t > 0$, so that for any later time the solution is locally integrable but not globally integrable in space due to the open end $x = \infty$. We can solve this problem in the whole line we may perform an antisymmetric extension, see Subsection 12.6.3, and then we obtain an interesting example of signed solution.

Let us now examine the questions of existence and uniqueness of solutions.

**Theorem 13.13** *Let $u_1$ and $u_2$ two solutions in $\mathcal{CWS}$ with the same initial trace in the above sense. Then $u_1 = u_2$ everywhere in $Q_T = \Omega \times (0, T)$.*

The existence of solutions offers no novelties

**Theorem 13.14** *Given a pair of non-negative Radon measures as in Theorem* 13.12, *there exists a c.w. solution* $u$ *of the HDP defined in* $Q = \Omega \times (0, \infty)$ *and having* $(\mu, \lambda)$ *as initial trace.*

**Corollary 13.15** *Every c.w. solution of the HDP defined in a domain* $Q_T = \Omega \times (0, T)$ *can be extended uniquely to* $Q = \Omega \times (0, \infty)$.

## 13.6 Weak implies continuous

An important result also due to Dahlberg and Kenig [191] shows that a non-negative very weak solution of the PME is in fact continuous, so that the restriction made in previous sections of this chapter is not a loss of generality.

**Theorem 13.16** *If* $u \in L^m_{\text{loc}}(Q)$, $Q \subset \mathbb{R}^{d+1}$, *is a non-negative distributional solution of* $\partial u/\partial t = \Delta u^m$, $m > 1$, *then* $u$ *is locally Hölder continuous*

There are two steps in the proof.

(i) To prove that a very weak solution in the space required in the definition has a minimum extra regularity, actually that $u \in L^{m+1}_{\text{loc}}(Q)$. This is based on regularity theory.

(ii) To prove that $u \in L^{m+1}_{\text{loc}}(Q)$ is enough to show continuity. This is based on the study of the dual equation satisfied by the potentials.

## 13.7 Complements

We list here some related topics under construction.

### 13.7.1 *Signed solutions*

One of the possible extensions of the above theories is to solutions with changing sign. There, progress is partial. We will point out some characteristic features that strongly deviate from the theory of non-negative solutions.

(i) A case that has been studied is the problem posed in one dimension with integrable data of zero total mass. In that case, we can integrate in space and then the new variable

$$v(x,t) = \int_{-\infty}^{x} u(s,t)\, ds$$

obeys the $p$-Laplacian equation. We recall that the equation is written in (3.58) as

$$\partial_t v = (|v_x|^{p-2} v_x)_x, \quad p = m + 1.$$

The existence of solutions for this equation is well-known, and differentiation gives information on the original PME with zero mass at all times. This procedure has been studied by Kamin and Vázquez in [326] and we refer there the reader for details. It allows us to include the dipole solution (4.53) as a solution for all $x \in \mathbb{R}$. It is remarkable that this solution has an *initial trace at* $t = 0$ *that is not a measure but a distribution* (the derivative of the delta function). The consequence that we draw is that the theory of initial traces for solutions with changing sign cannot be done completely in terms of measures, even if they are signed measures.

(ii) But the initial trace can fail to exist because of the oscillations of the limit (13.1) in Definition 13.1 as $t \to 0$. This is shown by means of the self-similar solutions constructed in [505] and reported in Theorem 16.8. These solutions are periodic in space in a certain representation.

(iii) The mentioned theorem contains still another frustrating possibility: there are solutions with zero initial trace in the sense of that definition, that are nevertheless not trivial.

*Other domains*

There is no difficulty in extending these theories to exterior domains, since the presence of the boundary measure offers no new difficulty, being supported on a compact set, $\partial\Omega$. The extension to other unbounded domains does not seem to have been explored. Uniqueness question should not be easy.

*Neumann problem*

An interesting subject to be explored.

*Other equations*

As we have mentioned in the previous chapter, the theory of the FDE is quite different: in the good interval $m_c = (d-2)_+/d < m < 1$, initial traces of locally bounded and non-negative solutions defined in $\mathbb{R}^d \times (0,T)$ exist and determine the solution, but they are not subject to any growth condition. In the range $0 < m < m_c$ the situation gets more complicated, see [435].

The theory of initial traces has an interest in all kinds of evolution problems, related to the question: which initial information determines the solution. As examples of related equations, we have examined with E. Chasseigne [162] the existence of initial traces for the diffusion–absorption equations of the form

$$u_t = \Delta u^m - u^p,$$

for different ranges of $m$ and $p$. On the other hand, the standard initial trace of the pressure equation

$$v_t = v\,\Delta v + \kappa|\nabla v|^2$$

determines the non-negative solutions when $\kappa < 0$, but it has to be redefined in a very special way in order to determine the solution when $\kappa \le 0$, see [163]. This means that the theory of initial traces needs new understanding in the area of non-divergent equations.

## Notes

The results of this chapter are motivated by the similar theory for the heat equation as developed by Widder [524]. It is proved that every non-negative solution of the heat equation existing in a strip $Q_T$ has an initial trace $\mu$ that has quadratic exponential growth in average

$$\int_{\mathbb{R}^d} e^{-ax^2}\mu(dx) \le \infty$$

for $a = c/T$ and some $c > 0$. On the other hand, this restriction allows us to construct solutions in a strip even for signed initial measures and to prove uniqueness in the class of solutions with that growth.

**Section 13.1.** The proof follows Bénilan, Crandall and Pierre in [91], Proposition 1.6. It can be seen as the natural continuation of the results of the previous chapter.

**Section 13.2.** The proof of existence and uniqueness of initial traces is taken from Aronson-Caffarelli' s [42]. Our previous existence and uniqueness theory simplifies the proofs.

Let us review some extensions:

(1) the initial trace of a solution of a porous medium equation with bounded measurable coefficients is studied by Cho and Choe in [170]. This is the result: let $u$ be a non-negative weak solution of the degenerate parabolic equations $u_t = (a_{ij}(x)u^m u_{x_i})_{x_j}$ in $\mathbb{R}^d \times (0,T), m > 0$ where $(a_{ij}(x))$ is a bounded measurable positive-definite symmetric matrix. Then, there is a unique $\sigma$-finite Borel measure on $\mathbb{R}^d$, which is the initial trace of $u$, and has the standard growth condition of the Aronson–Caffarelli lemma with a constant that depends on $m, d$, the maximum eigenvalues of matrix $(a_{ij}(x))$ and $u(0,T_0)$.

(2) Existence and uniqueness of an initial trace (in the form of a local measure) for non-negative solutions of the GPME $\Delta\varphi(u) - u_t = 0$ is studied by Ughi [495] under the condition that $\Phi(u)$ is close to $u^m$ in a precise sense. Results in that direction are obtained by Fabricant et al. [234].

**Section 13.3.** The proof follows Pierre's [434]. An important precedent is due to Brezis and Crandall [130], 1979, who proved existence for the initial value problem for $u_t - \Delta\varphi(u) = 0$ for general graphs $\varphi$ for bounded solutions such that the difference $u_1 - u_2$ belongs to $L^1(Q)$.

**Section 13.4.** We follow to a large extent Dahlberg and Kenig's [187]. The proof of Theorem 13.7 extends to the GPME; we have refrained from the interesting generality for lack of space but ask the reader to work on it.

**Section 13.5.** The material on the initial traces for the Dirichlet problem is taken from Dahlberg and Kenig's [189]. The dipole is due to [530] in $d = 1$ and the several dimensional one to [299]. These dipole solutions do not have conservation of mass, but they have conservation of the first moment (see Subsections 3.3.3 and 9.6.4). The properties of solutions like the dipole in several dimensions have been less studied.

**Section 13.7.** Some key references are given in the text.

## Problems

**Problem 13.1** Generalize Pierre's uniqueness theorem to the filtration equation $u_t = \Delta\Phi(u)$ posed in a bounded domain $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary conditions $\phi(u) = 0$ on the regular boundary $\partial\Omega$.

*Hint*: Use as definition of potential the unique solution of $\Delta v(t) = -u(t)$ in $\Omega$ with $v(t) = 0$ on the boundary. This method works for all $d \geq 1$.

**Problem 13.2** Prove Corollary 13.15.

**Problem 13.3*** Project. Study the existence and properties of initial traces for the HDP posed in an exterior domain $\Omega = \mathbb{R}^d \setminus K$, with $K$ compact.

**Problem 13.4*** Project. Study the existence and properties of initial traces for the HNP posed in a bounded domain.

**Problem 13.5** Show that the stationary solutions of the form $u = |w|^{1/m}\mathrm{sign}(w)$ with $\Delta w = 0$ in $\mathbb{R}^d$ are acceptable weak solutions with sign change that do not satisfy (for $d > 1$) the condition of limited growth of the non-negative solutions. Check the examples of Section 4.1. Find signed solutions with all growths.

**Open problem** Find conditions (maybe growth conditions) under which the theory of signed solutions parallels the theory of the non-negative case.

# 14

# PROPAGATION PROPERTIES

This chapter is an introduction to the study of the properties of the support and the free boundary of the solution in the several dimensional setting. We have already remarked that the diffusivity $D(u) = m|u|^{m-1}$ vanishes in the PME at the level $u = 0$. This degeneracy causes an important phenomenon to occur, i.e., finite speed of propagation of disturbances from 0; this is known in the literature as *finite propagation* (FP). We have observed this phenomenon on the source-type solutions in the form of compact support of the solution at any time $t > 0$. Finite propagation appears in the travelling wave solutions as sidewise propagation. We have also established the result for the solutions of the Cauchy problem for the PME in Section 9.6.3.

We want to discuss here in detail the property of finite propagation and its consequences for the PME. Attention will be focused on non-negative solutions. The whole chapter is written having in mind the class of non-negative and continuous weak solutions of the PME defined in a cylinder $Q = \Omega \times (0, T)$ with $\Omega$ a bounded set and $T$ finite or infinity. We have proved that comparison holds, in particular with respect to the continuous strong super- and subsolutions that we will use. We make in principle no reference to boundary conditions nor initial data. This was already done in Chapter 7. We refer to the class of solutions in this chapter as *local solutions* whenever the qualifier is needed or convenient.[1] Finally, let us note that we do not assume as a principle that initial data are taken. When this is assumed, the data are taken as in previous chapters: either continuously, in the sense of $L^1$-convergence, or in the weaker sense of trace.

In Section 14.1 we introduce the basic definitions. Section 14.2 discusses the basic propagation properties, like persistence, penetration and the hole-filling lemma. Section 14.3 covers the initial behaviour, in particular the characterization of waiting times. Our aim is to cover the full generality of the theory developed in previous chapters, which is usually not found in the literature.

Section 14.5 treats the topic of Hölder regularity of the free boundary, following the fundamental work of Caffarelli and Friedman for the Cauchy problem in the whole space. It also describes the free boundary as a geometrical object.

[1] We could consider solutions defined in more general space-time domains, but that generality complicates the results and does not add to the basic theory.

Section 14.6 gathers some properties of the Cauchy problem in the whole space. A particularly important result is obtained for solutions with compactly supported data: the free boundary is a Lipschitz surface after a finite time.

In the whole chapter it is assumed that solutions are non-negative. Only a final section contains some remarks about propagation for signed solutions.

Free boundaries can be quite complicated geometrical objects and their study has been one of the difficult issues in porous medium theory. Very detailed information about the motion and properties of free boundaries will be obtained in the next chapter for flows in one space dimension, where the geometry is simpler. More advanced several dimensional theory is contained in Chapter 19.

## 14.1 Basic definitions. The free boundary

Before we prove the detailed propagation results, we need some definitions and notations. Let $u$ be a continuous solution defined in $Q = \Omega \times (0, T)$.

The **positivity set** of $u$ is the set

$$\mathcal{P}_u = \{(x,t) \in Q : u(x,t) > 0\}. \tag{14.1}$$

Its time sections are defined for fixed $t \in (0,T)$ as

$$\mathcal{P}_u(t) = \{x \in \Omega : u(x,t) > 0\} \tag{14.2}$$

in $\mathbb{R}^d$. All solutions of the PME under consideration are continuous; therefore, $\mathcal{P}_u$ is an open set in $\mathbb{R}^{d+1}$, and the sections $\mathcal{P}_u(t)$ are open subsets of $\mathbb{R}^d$. The reader will remember that for the classical heat equation and a solution $u \geq 0$, we have either $\mathcal{P}_u(t) = \Omega$, or the empty set and then $\mathcal{P}_u = Q$ or a subset or the form $Q' = \Omega \times (t_1, T)$.

The **support** of $u$, $\mathcal{S}_u$, is defined as the closure of $\mathcal{P}_u$ in $Q$. The complement of $\mathcal{P}_u$ is called the *vanishing set* or *zero set*, $\mathcal{Z}_u$ (note that this is true because $u \geq 0$, otherwise the negativity set has to be considered).

*Free boundary*

The boundary of the positivity set in $Q$,

$$\Gamma_u = \partial \mathcal{P}_u \cap Q, \tag{14.3}$$

is called the free boundary (also called **interface**, specially in one-dimensional problems). It is clear that $\mathcal{S}_u$ and $\Gamma_u \subset \mathcal{S}_u$ are closed sets, and so is $\mathcal{Z}_u$; moreover, $\Gamma_u = \mathcal{S}_u \cap \mathcal{Z}_u$. The interior of $\mathcal{Z}_u$ is the largest open set where $u$ vanishes.

The free boundary is a very important object since it represents the region separating the 'occupied region', $\{u > 0\}$, from the 'empty region', $\{u = 0\}$, in the terminology of fluids in porous media. Studying the movement and regularity of free boundaries is one of main goals of the mathematical theory of the PME. A simple example of such a free boundary happens for the ZKB solution (5.13) centred at a point $x_0 \in \Omega$,

$$w(x,t) = U(x - x_0, t; M)$$

with mass $M > 0$. Its free boundary is the hypersurface with equation

$$|x - x_0| = r(t; M) = c(M^{m-1}t)^{\beta}, \tag{14.4}$$

where $\beta = (d(m-1)+2)^{-1} < 1/2$ and $c = c(m,d) > 0$. When considered with space domain $\mathbb{R}^d$, the support $\mathcal{S}_U(t)$ is then the closed ball with centre $x_0$ and expanding radius $r(t; M)$. When considered as restricted to a space domain $\Omega$, it is the intersection of that ball with $\Omega$. In the theory of the homogeneous Dirichlet problem of Chapter 5, it is an acceptable solution as long as the free boundary does not reach the exterior boundary $\partial\Omega$ (this happens though in a finite time). Afterwards, it is a solution of the non-homogeneous boundary-value problem.

For every time $t$ we may define $\mathcal{S}_u(t)$ as the section of $\mathcal{S}_u$ at time $t$, in other words, $\mathcal{S}_u(t) = \{x : (x,t) \in \mathcal{S}\}$. But it is also natural to consider the closure of $\mathcal{P}_u(t)$ in $\Omega$,

$$\mathcal{S}_u^*(t) = \overline{\mathcal{P}_u(t)}.$$

It has to be proved that both definitions coincide. The inclusion $\mathcal{S}_u^*(t) \subset \mathcal{S}_u(t)$ is immediate from the topological definitions. The other inclusion is a consequence of the theory to be developed below. See Problem 14.1.

We will drop the subscript $u$ and write $\mathcal{P}$, $\mathcal{S}$, $\Gamma$, and so on, whenever mention of the solution is deemed unnecessary.

**Note** In more general equations and contexts, where solutions need not be continuous, but are locally integrable functions or measures, measure theoretical definitions are used: the zero set is defined as the largest measurable set where the measure $d\mu = u\,dxdt$ vanishes, $\mu(\mathcal{Z}) = 0$; for continuous functions both concepts coincide. $\mathcal{P}_u$ is defined a.e. and is not necessarily open, and so are its sections for a.e. time. A particularly useful and non-trivial application is the section at $t = 0$ of a solution that is continuous for $t > 0$ but not down to $t = 0$. We will study such an issue in Section 14.3 below.

## 14.2 Evolution properties of the positivity set

We will derive some of the main properties of the positivity set of a solution as it evolves with time. Let us make two preliminary observations. On the one hand, note that since the solutions are continuous, $\mathcal{P}_u(t)$ is an open set of $\Omega$ for every $t > 0$. On the other hand, $\mathcal{P}_u(t)$ cannot be empty for $t \approx 0$ if the data are non-trivial (since, as seen in previous chapters, the solution is continuous as a function $u(t)$ with values in $L^1_{\rm loc}(\Omega)$). But it can be zero for some time and then become non-empty as time passes. The following representative situation illustrates this point: the domain is $Q = B_1(0) \times (0,\infty)$ and $u$ is a ZKB solution supported at $t = 0$ in a ball $B_r(x_0)$ with $|x_0| > r + 1$. Then, $\mathcal{P}_u(t)$ is empty for some $0 < t \le t_1$ and non-empty later.

### 14.2.1 *Persistence*

Let us now prove that the positivity set of a solution expands with time.

**Proposition 14.1** *The family* $\{\mathcal{P}(t)\}_{t>0}$ *is non-contracting, i.e.* $\mathcal{P}(t_1) \subset \mathcal{P}(t_2)$ *for every* $0 < t_1 < t_2$.

**Proof** For the solutions of the HDP, CP or HNP of previous sections, it follows from estimate (8.2), which just means that the function $z(t) = u(x,t)t^{1/(m-1)}$ is non-decreasing for every fixed $x \in \Omega$. Hence if $z(t_1) > 0$ and $t_2 > t_1$ we have $z(t_2) > 0$, i.e. $x \in \Omega(t_2)$.

A second proof for general solutions proceeds by comparison with the separated variable solution $\widetilde{U}(x,t)$ constructed in Section 5.9, formula (5.69). In fact, given $x_1 \in \mathcal{P}(t_1)$, there exists a radius $R > 0$ and a time delay $\tau > 0$ such that $B_R(x_1) \subset \mathcal{P}(t_1)$ and

$$u(x,t_1) \geq \widetilde{U}(x,t_1+\tau) = (t_1+\tau)^{\frac{1}{m-1}} f(x).$$

Since $\widetilde{U}(x,t_1+\tau) = 0$ on the lateral boundary of the cylinder $Q_1 = B_R(x_1) \times (t_1,\infty)$, it follows from the comparison result (see Corollary 8.12) that $u(x,t) \geq \widetilde{U}(x,t+\tau)$ in $Q_1$, hence $u > 0$ there. It means that $B_R(x_1) \subset \mathcal{P}(t)$ for every $t > t_1$. ■

This result does not establish that actual expansion takes place, it is rather a non-contraction result. We call this property *persistence*. It is also called *retention* property, since $u(t)$ retains its positivity at any given point when time increases.

### 14.2.2 *Expansion and penetration of the support*

Our next task is to prove that expansion does take place. The basic tool for showing that is comparison with a small ZKB solution. This is done as follows.

**Lemma 14.2** *Let* $u$ *be a non-trivial local solution of the PME in* $Q$ *and let* $t_1 > 0$ *and* $x_1 \in \mathcal{P}(t_1)$. *Then there exist* $M_1$ *and* $\tau_1$ *such that, for every* $t > t_1$ *such that* $r(t+\tau_1) \leq d(x_1, \partial\Omega)$, *we have*

$$\mathcal{P}(t) \supset B_{r(t+\tau_1;M_1)}(x_1), \tag{14.5}$$

*where* $r(t;M)$ *is the function given in* (14.4).

**Proof** Recall that for $x \in \Omega$ we define distance to the boundary as

$$d(x,\partial\Omega) = \sup\{d(x,y) : y \in \partial\Omega\}.$$

Arguing as before, there exists a radius $R > 0$ and a time delay $\tau_1 > 0$ such that $B_R(x_1) \subset \mathcal{P}(t_1)$ and

$$u(x,t_1) \geq w(x,t_1) = U(x-x_1, t_1+\tau_1; M_1).$$

By the comparison theorem, it follows that

$$u(x,t) \geq w(x,t) = U(x-x_1, t+\tau_1; M_1),$$

as long as $w$ is a solution of the HDP, i.e., as long as its free boundary does not reach $\partial\Omega$. In other words, we have (14.5). ■

This result gives a quantitative estimate of the property of *penetration* of the substance into the empty region. This estimate will be shown to be exact when considering penetration into the whole space later in this chapter. In a bounded domain, a problem arises when the expanding ball meets the boundary, and the subsolution ceases to be valid. But then we may start the argument from any ball contained in the expanded positivity set and wait until the new subsolutions meet the boundary in their turn. We may wonder if this process allows the solution to reach the whole domain after a finite time, or if it takes infinite time. This is a general question that may be asked in diffusion processes. In our model the answer is positive.

**Theorem 14.3** *Let $u$ be a non-trivial local solution of the PME defined in $\Omega \times (0,\infty)$, $\Omega$ connected. Then, every point of $\Omega$ is absorbed in finite time by the positivity set of $u$, i.e.,*

$$\bigcup_{t>0} \mathcal{P}(t) = \Omega. \tag{14.6}$$

*Moreover, any compact subset $K$ of $\Omega$ is covered by $\mathcal{P}$ in a finite time $T$ that depends on $m, d$, the initial data, the geometry of $\Omega$ and $d(K;\partial\Omega)$.*

**Proof** (i) If the domain is a ball the last result is rather trivial: in a finite sequence of steps we prove that the centre of the ball is reached in finite time, then reaching the whole boundary is immediate by using the above ZKB subsolution.

(ii) For general domains, the question needs a more careful argument. Given $\delta > 0$ small, we introduce the sets

$$K_\delta(\Omega) = \{x \in \Omega : d(x, \partial\Omega) \geq \delta\}.$$

These are closed subsets of $\Omega$, non-empty and even connected if $\delta$ is small. Taking one of these compact subsets, we recover it by a finite number of balls of radius $r \leq \delta/4$. If $r$ is small enough, we can pick one of them, say the one centred at $x_1$, so that we are in the situation of Lemma 14.2: at time $t_1$ we have

$$u(x, t_1) \geq w(x, t_1) := U(x - x_1, t_1 + \tau_1; M_1)$$

in the ball $B_r(x_1)$. We wait now until the support of $w$ expands to become a ball of radius $3r$. This happens at a time $t_2$ that depends only on the given parameters. In this time it does not reach the boundary of $\Omega$, because $d(x_1) \geq 4r$. It follows from the lemma that at $t_2$ the positivity set of $u$ covers all neighbouring balls, i.e., those with $d(x_i, x_1) < 2r$. It is immediate to see that in this situation

$$u(x, t_2) \geq U(x - x_1, t_2 + \tau_1; M_1) \geq U(x - x_i, t_2 + \tau_1; M_i) \quad \text{in} \quad B_r(x_i),$$

with $M_i$ a fraction of $M_1$.

We can now iterate the application of Lemma 14.2 to this set of balls and reach in a finite time the second set of neighbouring balls. In a finite number of steps we reach the whole of $K_\delta(\Omega)$. This proves the last result in the case $K \subset K_\delta(\Omega)$ with a constant that depends on $m, d, \delta$, the geometry of $\Omega$ and the initial data through $x_1, \tau_1, M_1$. But any compact $K$ is contained in a $K_\delta(\Omega)$. ■

In case the boundary $\partial\Omega$ is regular, then it is completely reached in a finite time.

**Theorem 14.4** *Let $u$ be a continuous non-trivial solution of the PME defined in $\Omega \times (0, \infty)$, $\Omega$ connected. If $\partial\Omega$ has the property of the uniform interior ball, then there exists a time $T^* = T(\Omega, u)$ such that $\mathcal{P}(t) = \Omega$ for all $t \ge T^*$. The time depends on the parameters of the last result plus the minimal internal radius on $\partial\Omega$.*

**Proof** By assumption, every point $x_0$ on the boundary $\partial\Omega$ admits an inner ball of radius $r_0 > 0$, say $B_{r_0}(x_0')$. But the points at distance $r_0$ are reached in a finite time, even all the balls $B_{r_0/2}(x_0')$ are covered after a time $T_1$. Using those balls as basis for the application of Lemma 14.2, we conclude that the whole boundary is reached in a time $T^* = T_1 + t_*$. The final time $T^*$ depends only on $m$, $d$, the geometry of $\Omega$ and the parameters $x_1, \tau_1, M_1$ of the initial data. ■

**Remarks**

(1) The property of the interior ball is implied by the regularity $\partial\Omega \in C^2$, which is standard in the theory. In that case, $\partial\Omega$ has even a tubular neighbourhood. Our result does not need however, any conditions of curvature on the outer side of $\partial\Omega$. For instance, a domain with a corner or a spike pointing inside is perfectly admissible.

(2) Very accurate penetration results will be obtained in the last section for solutions defined in the whole space, $\Omega = \mathbb{R}^d$.

(3) A certain condition on $\partial\Omega$ is needed for the result to hold. In Subsection 14.3.3 below we construct examples of that situation when $\partial\Omega$ has corners pointing outside.

### 14.2.3 *Finite propagation*

We address now the remaining and main question: propagation proceeds in a finite way. This property can be conveniently formulated as follows:

**Definition 14.1** (Finite propagation)

(i) *We say that finite propagation holds for a certain class of solutions of an evolution process if given two times $0 \le t_1, t_2$, the support of the solution at* *time $t_2$ is included in a neighbourhood of radius $D(|t_2 - t_1|)$ of the support of $u(t_1)$, where $D$ is a continuous function $\mathbb{R}_+ \mapsto \mathbb{R}_+$ with $D(0+) = 0$.*

(ii) *If $D$ is independent of the solution under consideration, we call it uniform finite propagation.*

(iii) *If $D(s) \le Cs$ for some $C > 0$ we say that propagation has finite speed.*

The study of finite propagation is simplified when extra assumptions are made like radial symmetry or bell-shaped form of the initial data, so that tricks like the ones presented in Subsection 9.6.2 can be used. We will address here the general situation. It is then convenient to start by observing that the main difficulty in the study of propagation is caused by the presence of 'holes' in the support, that are filled in some finite time, the so-called *hole-filling problem.* We present next a basic lemma in the study of this topic.

**Lemma 14.5** (Hole-filling lemma) *Let $u \ge 0$ be a bounded local solution of the PME posed in $Q = \Omega \times (0, T_1)$ and let us assume that it takes (continuously or in the sense of $L^1$ convergence) initial data given by a bounded function, $u_0 \in L^\infty(\Omega)$, and let us also assume that $u_0$ vanishes* a.e. *in a ball $B_R(x_0) \subset \Omega$. Then, there is a time $T = T(\|u\|_\infty)$ such that for every $0 < t < T$ the solution $u(t)$ vanishes at least in a smaller ball $B_{R(t)}(x_0)$ with $0 < R(t) \le R$. The function $R(t)$ is monotone non-increasing. Moreover, we have the bounds*

$$T \ge c_1 R^2 H^{1-m}, \tag{14.7}$$

*and*

$$R(t) \ge R - c_2 (H^{m-1} t)^{1/2}. \tag{14.8}$$

*where $c_1$ and $c_2$ depend on $m$, $d$ and $H = \|u\|_\infty$.*

**Proof** Let $H$ be the $L^\infty$ norm of $u$ in $B_R(x_0) \times (0, \infty)$. We use a comparison argument with respect to the family of supersolutions formed by the quadratic blow-up solutions $\widetilde{U}(x, t)$ with pressure given by formula (4.44), i.e.

$$\widetilde{V}(x, t) = k_1 \frac{|x - x_1|^2}{T - t}, \tag{14.9}$$

where $k_1 = 2\alpha/d$, and we have centred the solution at a point $x_1 \in B_R(x_0)$. We adjust the parameter $T$ in dependence of $H$ and put

$$d(x_1) = d(x_1, \partial B_R(x_0)) = R - |x_1 - x_0|. \tag{14.10}$$

We compare $u$ and $\widetilde{U}(x, t)$ in the cylinder $Q_1 = B_R(x_0) \times (0, T)$. Since both are solutions of the Dirichlet problem in $Q_1$, we only need to compare them on the parabolic boundary. It is clear that $u(x, 0) \le \widetilde{U}(x, 0)$ for all $x \in B_R(x_0)$ since $u_0$ vanishes there. On the other hand, the boundary condition $u(x, t) \le \widetilde{U}(x, t)$ for $|x - x_0| = R$ holds if

$$\frac{m}{m-1} H^{m-1} \le k_1 d(x_1)^2 / T. \tag{14.11}$$

We can write the condition as

$$c(m,d)^2 H^{m-1} T \geq d(x_1)^2. \tag{14.12}$$

Under this condition with equality, we conclude that $u(x,t) \leq \widetilde{U}(x,t)$ in $Q$. In particular,

$$u(x_1,t) \leq U(x_1,t) = 0 \quad \text{for } 0 < t \leq T.$$

Putting $t = T$, this proves that $R(t) \geq R - c_m (H^{m-1} t)^{1/2}$, which implies the result. ■

**Remarks**

(1) As a consequence of the penetration results, we know the hole must disappear in a finite time. The behaviour near the time of disappearance, so-called *focusing time*, is not like (14.7). That behaviour is quite interesting and non-trivial and will be studied in Chapter 19, Section 19.2.

(2) Our estimates do not calculate the exact vanishing set, but only a lower bound $R(t)$ for the radius of the ball where $u$ vanishes.

(3) The whole estimate does not involve any control of the $L^1$ norm. This suggests that it is true for the much larger class of all bounded solutions, that has been studied as a part of the contents of Chapter 12. In that class the proof can be optimized by considering only the worst case with respect to the estimate under consideration, which is realized by the solution $\overline{u}$ with initial data

$$\overline{u}_0(x) = 0 \quad \text{for } \ x \in B_R(x_0), \quad \overline{u}_0(x) = H \quad \text{otherwise}. \tag{14.13}$$

This solution is bounded for all times and radially symmetric, supported in the complement of the ball $B_{R(t)}(0)$. Now, $R(t)$ is the exact size of the vanishing set and also serves as a lower bound for the vanishing set of the rest of the solutions considered in Lemma 14.5, since the maximum principle is established in the larger class of solutions. These considerations show the convenience of having a theory that applies to wider classes of solutions. The moral could be: many proofs are simpler and more natural in the context of a more general theory.

(4) Using that approach and the study of one-dimensional or radial problems of Chapter 15, we may show that estimate (14.7) has the exact power of $H$ and $t$ for $t \sim 0$, though the constant can be improved. We can also show that the asymptotic estimate for $t \sim 0$ is exact for the actual radius $R(t)$ of the special solution $\overline{u}$. See Problems 14.2 and 14.3.

Using the lemma, we obtain the following propagation result.

**Theorem 14.6** *Uniformly bounded local solutions of the PME defined in $Q_T$, where $\Omega$ is a subdomain of $\mathbb{R}^d$, have the property of uniform finite propagation. Moreover, the support $\mathcal{S}(t)$ of any such solution expands continuously in time. Indeed, if $\Omega$ is bounded there exist uniform constants $\delta$ and $C > 0$ such that for* *every $0 < h < \delta$ the support at time $t+h$ is included in the neighbourhood of radius $d$ the support at time $t$ with $d = ch^{1/2}$:*

$$\mathcal{S}(t+h) \subset \mathcal{S}(t) + B_{ch^{1/2}}(0). \tag{14.14}$$

We can now answer the question raised in Section 14.1 about the definition of support.

**Corollary 14.7** *For every uniformly bounded solution of the PME as above and every $t > 0$, we have $\lim_{h\to 0} \mathcal{S}(t+h) = \mathcal{S}(t)$ and $\mathcal{S}(t) = \mathcal{S}^*(t)$.*

**Proof** For the last statement, notice that a point $x_0 \notin \mathcal{S}^*(t)$ has a small ball $B_r(x_0)$ that is outside the positivity set of $u$ for $t_0 < t < t_0 + h$ if $h$ is small, hence $(x_0, t_0)$ is not in $\mathcal{S}$. ■

As a consequence of the results of this section, we may assert the existence of a free boundary in quite general circumstances.

**Theorem 14.8** *Let $u$ be a continuous and bounded strong solution of the PME defined in a space-time cylinder $Q = \Omega \times [0, T)$ and assume that $u(x, 0)$ vanishes in a ball $B \subset \Omega$. Then, the free boundary is a non-empty set.*

Hole-filling near and at the boundary is proposed in Problem 14.4.

## 14.3 Initial behaviour. Waiting times

Given a solution $u$ defined in $Q = \Omega \times (0, \infty)$, we define its initial sets as

$$\mathcal{S}_0(u) = \bigcap \{\mathcal{S}_u(t) : t > 0\}, \qquad \mathcal{P}_0(u) = \bigcap \{\mathcal{P}_u(t) : t > 0\}. \tag{14.15}$$

As limits of monotone families these sets exist, $\mathcal{S}_0(u)$ is closed and $\mathcal{P}_0(u) \subset \mathcal{S}_0(u)$ †.[2] They might be empty or the whole of $\Omega$. The points of $\mathcal{P}_0(u)$ are called *points of immediate positivity*. If $x_0 \notin \mathcal{P}_0(u)$ then $u(x_0, t)$ will become positive only after some time, and then we say that *$u$ has a waiting time at $x_0$* defined by the formula

$$\tau_u(x_0) = \inf\{t > 0 : u(x_0, t) > 0\} = \inf\{t > 0 : x_0 \in \mathcal{P}_u(t)\}. \tag{14.16}$$

The prototype of solution of the PME with a waiting time is the blow-up solution with pressure given by the formula

$$v(x, t) = \frac{K\,|x|^2}{T - t}, \tag{14.17}$$

with $K = (2(d(m-1)+2))^{-1}$ (cf. Section 4.5). We see that $v_0(x) = C|x|^2$ with $C = K/T$. Therefore, the waiting time can be explicitly calculated as

$$T = \frac{K\,x^2}{v_0(x)}.$$

[2] As the intersection of a countable family of open sets $\mathcal{P}_0(u)$ is a $G_\delta$ set.

In case $u$ solves an initial problem, like $u \in C([0,\infty) : L^1_{\rm loc}(\Omega))$ with $u(t) \to u_0$ as $t \to 0$ we may wonder what is the relation of the sets $\mathcal{P}_0(u)$ and $\mathcal{S}_0(u)$ with the positivity set and support of the initial trace $u_0$.

If $u$ is continuous down to $t = 0$ with initial data $u_0$, then we define $\mathcal{P}(u_0) = \{x \in \Omega : u_0(x) > 0\}$, $\mathcal{S}(u_0)$ its closure in $\Omega$; continuity implies that every positivity point of $u_0$ will stay positive for later times, i.e.,

$$\mathcal{P}(u_0) \subset \mathcal{P}_0(u).$$

However, both sets need not be the same: there may be points while $u_0$ vanishes while $u(x,t)$ may be positive for all times $t > 0$. We have the following preliminary result characterizing those points of immediate positivity.

**Proposition 14.9** *Let $u$ be a bounded solution of the PME defined in $Q_T$ with continuous initial data $u(x,0)$, and let $x_0 \in \Omega$. If there is a constant $A > 0$ such that the pressure satisfies*

$$v_0(x) \le A\,|x - x_0|^2, \tag{14.18}$$

*then $u(x_0,t)$ vanishes for some time $0 < t < t_* = k_1(m,d)/A$. On the contrary, if for some $r_0 > 0$ and all $0 < |x| < r_0$*

$$v_0(x) \ge A\,|x - x_0|^2, \tag{14.19}$$

*then $u(x_0,t)$ is positive after some time $t > t_* = k_2(m,d)/A$.*

**Proof** The proof of the first assertion is based on comparison in a cylinder $Q' = B_r(x_0) \times (0,T)$ with a blow-up solution of the form (14.17) with large constant $K$ (also the boundary data have to be compared). The key point is observing that the blow-up solution has a positive waiting time that can be estimated explicitly.

The second assertion is based on comparison with a ZKB solution. Let us give some details of this process. We may assume that $x_0 = 0$ and that $r_0$ is small enough. Take $x_1$ at a distance $r_1$ from the origin and put a delayed ZKB solution (4.21), (5.13) centred at $x_1$ below $v_0$. This happens if the constants $C$ and $\tau$ are such that

$$C\tau^{2\beta} \le k(r - r_1)^2 + A\tau r^2$$

for every $r$ (note that $k = k(m,d)$ but $C$ and $\tau$ are in principle free). The inequality holds if we put $\tau = 1/A$ and $C\tau^{2\beta} \le c(m,d)r_1$. But then the free boundary of the ZKB subsolution reaches the origin (after travelling a distance $r_1$) in a time $t_1$ given by

$$C(t_1 + \tau)^{2\beta} = kr_1^2$$

This gives for $t_1$ an estimate of the form $t_1 = O(\tau) = O(1/A)$. ∎

**Corollary 14.10** *Under the condition that there exists the limit*

$$\lim_{x\to x_0} u_0(x)\,|x-x_0|^{-2/(m-1)} = c \ge 0, \tag{14.20}$$

*the waiting time at $x_0$ is positive if and only if the limit is finite, $c<\infty$.*

### 14.3.1 *Waiting times for general solutions of the Cauchy problem*

An exact characterization of the existence of a positive waiting time at $x_0$ holds when the limit of $u_0$ is replaced by a limit of averages in balls $B_r(x_0)$. This needs some better estimates that have been derived in the previous chapters for solutions of the Cauchy problem.

(i) We first look for lower estimates. There is a way of writing the Aronson–Caffarelli result of Section 12.2 that is more convenient when we are interested in the positivity property. The estimate can also be seen as an a priori lower bound on the positivity of solutions for positive times. Let $u$ be a solution with initial data $u_0 \ge 0$ and let

$$M_R(x_0) = \int_{B_R(x_0)} u_0(x)\,dx.$$

Then we have the following reformulation of Theorem 12.1.

**Proposition 14.11** *For every solution with locally integrable data and for every $R, t > 0$ and $x_0 \in \mathbb{R}^d$ there are constants $c_1, c_2 > 0$ depending only on $m, n$ such that*

$$u(x_0,t) \ge c_1\, M_R(x_0)^{2/\lambda} t^{-d/\lambda} \tag{14.21}$$

*if the time is not too small, precisely when*

$$t \ge t_* = c_2 R^\lambda M_R(x_0)^{1-m}, \quad \lambda = d(m-1)+2. \tag{14.22}$$

**Remarks**

(1) Since $\alpha = d/\lambda$ is the exponent in the $L^1$–$L^\infty$ smoothing effect, we have found that for large times an estimate similar to the upper estimate of the $L^1$-$L^\infty$ is proved. However, the time where it begins to apply at a point $x_0$ depends on the initial mass around $x_0$ in and inverse power way according to formula (14.22).

(2) The exponents in the expressions are exact, since they have been obtained by scaling considerations. However, we do not give a method to calculate the best constants. $R$ plays the role of a free parameter.

(ii) We now look for upper estimates. Let

$$N(u_0,x_0) = \sup_{R>0} \{(M_R(x_0))^{m-1}/R^\lambda\}. \tag{14.23}$$

The following result is a consequence of Lemma 12.4.

**Proposition 14.12** *Let $u \ge 0$ be a general solution of the PME. There exists a constant $c_1'(m,d) > 0$ such that*

$$u(x_0, t) = 0 \quad \textit{for} \;\; 0 < t < c_1'/N(u_0, x_0). \tag{14.24}$$

**Proof** By a general solution we understand a solution in the class $\mathcal{U}$ of Chapter 12. Assume without loss of generality that $x_0 = 0$. We point out that $|u_0|_r^{m-1} \le N(u_0, 0)$. Assume that $N(u_0, 0)$ is finite, otherwise there is nothing to prove. As a consequence of the mentioned lemma, we have

$$\|u(t)\|^{m-1}_{L^\infty(B_R(0))} \le c_3 R^2\, |u_0|_r^{2(m-1)/\lambda} t^{-d(m-1)/\lambda} \le c_3 R^2\, N(u_0, 0)^{2/\lambda} t^{-d(m-1)/\lambda} \tag{14.25}$$

for $1 \le r \le R$ and $0 < t < T_r(u_0) := c_1'/|u_0|_r^{m-1}$. By scaling we can eliminate the restriction $r \ge 1$ and get in the limit $R \to 0$ the estimate $u(x_0, t) = 0$ for all $0 < t < c_1'/N(u_0, 0)$. This proves the result. ∎

These two results may be combined into a full characterization of the existence of a waiting time for the non-negative solutions of the Cauchy problem.

**Theorem 14.13** *Let $u$ be a non-negative solution of the Cauchy problem for the PME with locally integrable initial data. The waiting time at $x_0$ is positive if and only if $N(u_0, x_0) < \infty$. Moreover, there exist constants $c_1, c_1' > 0$ depending only on $m, d$ such that*

$$c_1'(m,d)/N(u_0, x_0) \le \tau_u(x_0) \le c_1(m,d)\,/N(u_0, x_0). \tag{14.26}$$

### 14.3.2 *Addendum for comparison. Positivity for the heat equation*

An optimal lower estimate of the type we are discussing is very easily obtained in the case of the heat equation. It says

**Proposition 14.14** *If $u$ is a non-negative solution of the HE in $Q = \mathbb{R}^d \times (0, T)$, then for all $t, R > 0$ and $x_0 \in \mathbb{R}^d$ we have*

$$u(x_0, t) \ge (4\pi)^{-d/2} M_R(x_0)\, t^{-d/2}\, e^{-R^2/4t}. \tag{14.27}$$

*and the exponents and constant are optimal.*

**Proof** The representation formula says that

$$u(x_0, t) \ge (4\pi t)^{-d/2} \int_{\mathbb{R}^d} u_0(y)\, e^{-(y-x_0)^2/4t}\, dy$$

so that the best estimate in terms of $M_R(x_0)$ consists in forgetting the part of $u_0$ not supported in $B_R(x_0)$ and displacing all the mass in this ball to the boundary. We get the solution corresponding to a $M$ times the Dirac delta located at a point of the boundary. ∎

In comparison with the results of the previous subsection, we see that here the characteristic time is $t_* = cR^2$ and that formula (14.22) is respected. We notice that this positivity estimate consists of two time periods: an increasing lower bound for an initial time interval $0 < t < t_c$ plus an exponentially decreasing bound for all later times. In the case $m > 1$ we cannot ensure positivity in the initial period where waiting times may occur.

### 14.3.3 *Examples of infinite waiting time near a corner*

We have proved that an interior point of a local solution cannot have an infinite waiting time because of the property of penetration, Theorem 14.3, and we have given a quantitative estimate for the case of solutions of the Cauchy problem in Proposition 14.11. We construct here examples of solutions defined in a sector of $\mathbb{R}^d$ which exhibit an infinite waiting time at the corner point, i.e., the vertex. What we show is that the support of our solution does not reach the vertex in finite time if the solid angle is small enough.

We recall that given an open subset $A \in \mathbb{S}^{d-1}$, the cone of vertex 0 and base $A$ the set $\mathcal{C}(A) = \{x = r\sigma : \ r > 0, \sigma \in A\}$. We have also proved in Section 12.8 that there exist in such domains stationary solutions of the form

$$\widehat{U} = |x|^q f(\sigma)$$

where $f$ is positive and smooth in $A$ and vanishes on $\partial A$. Then we have

**Proposition 14.15** *Let $\mathcal{C}(A)$ be a cone in $\mathbb{R}^d$ with small angle so that there exists a supersolution of the PME of the form described above and $q \geq 2/(m-1)$. Then, there exist solutions of the homogeneous Dirichlet problem for the PME in the sector with a support that lies away from the vertex for all finite times.*

**Proof** (i) We note that $\widehat{U}$ is invariant under the similarity transformation

$$u_\lambda(x,t) = (T_\lambda u)(x,t) := \lambda^q u(x/\lambda, \mu t)$$

for every $\lambda, \mu > 0$, and the PME is also invariant under this transformation if $\mu = \lambda^{q(m-1)-2}$ holds.[3] Note the relation of supports: $\mathcal{S}_{u_\lambda}(t) = \lambda \mathcal{S}_u(\mu t)$. Hence, if we define

$$r_u(t) = \inf\{|x| : x \in \mathcal{P}_u(t)\},$$

we have $r_{u_\lambda}(t) = \lambda r_u(\mu t)$.

(ii) Let us now consider the solution $\overline{u}(x,t)$ with initial data

$$\overline{u}(x,0) = \widehat{U}(x) \quad \text{for} \quad |x| \geq 1, \quad u_0(x) = 0 \quad \text{for} \quad 0 < |x| < 1.$$

Due to the finite propagation property we know that for a time $\tau > 0$ the solution keeps away from the ball centred at 0 with radius $a < 1$. Due to the supersolution condition we have

$$u(x,1) \leq \widehat{U}(x,1) := \widehat{U}(x).$$

[3]See Chapter 16 for more on similarity transformations.

But the similarity transformation with parameter $\lambda = a$ produces a solution $\overline{u}_1 = T_a\overline{u}$ such that $\overline{u}_1(x,t) \le \widehat{U}(x)$, and $\overline{u}_1(x,0) = 0$ iff $|x| < a$. We conclude that $\overline{u}(x,\tau) \le (T_a\overline{u})(x,0)$, and by parabolic comparison

$$\overline{u}(x, t+\tau) \le (T_a\overline{u})(x,t) \quad \text{for all} \quad t > 0.$$

Comparing the supports and using the conclusion of step (i), we get

$$r_{\overline{u}}(t+\tau) \ge a r_{\overline{u}}(\mu t).$$

This is the basic relation that has to be iterated.

(iii) We use the last formula to get $r_{\overline{u}}(\tau + \frac{\tau}{\mu}) \ge a r_{\overline{u}}(\tau) \ge a^2$, and then

$$r_{\overline{u}}(t_k) \ge a^{k+1} \quad \text{for} \quad t_k = \tau \sum_{j=0}^{k} \mu^{-j}.$$

Our assumption is that $\mu \ge 0$. Hence, the sequence $t_k \to \infty$. The result follows. ∎

**Remark** The conclusion of infinite waiting time at $x = 0$ applies to any solution with data bounded above by $\overline{u}(x,0)$. It also applies to subdomains of $\mathcal{C}(A)$ with a corner point at 0.

On the other hand, there is no need to work in an infinite domain. The conclusion applies by comparison to a bounded domain obtained by cutting the sector and putting Dirichlet boundary conditions on the new boundary (or Neumann conditions in the appropriate geometry).

## 14.4 Hölder continuity and vertical lines

Our next analysis describes the fundamental work by Caffarelli and Friedmann [140] that improves on the continuity results of Chapter 7. This work is restricted to the Cauchy problem, since it uses the fundamental estimate $\Delta u^{m-1} \ge -C$. The main result of [140] is regularity of the free boundary.

**Theorem 14.16** *If $u$ is a non-negative solution of the Cauchy problem posed in the whole space, then $\Gamma_u$ is a Hölder continuous hypersurface in $\mathbb{R}^{d+1}$. Moreover, the pressure is locally Lipschitz, hence the solution itself is locally Hölder continuous.*

The proof relies on two propagation results: one that quantifies the average of the pressure near a point that advances, while the other quantifies the advance of a solution that has a large pressure behind. The methods of proof consist essentially of classical potential theory, coupled with comparison principles and the regularizing effect $u_t \ge -Cu$.

The first lemma shows that if a solution has a hole at a certain time and has a small pressure around a bit later, then it has a controlled propagation rate.

**Lemma 14.17** *Let $v \ge 0$ be the pressure of a bounded solution of the PME such that $\Delta v \ge -C$ in $Q = \mathbb{R}^d \times (0,T)$. There exist positive constants $\eta, c > 0$*

*depending only on $m, d$ such that the following is true: if $v \ge 0$ is the pressure of a bounded solution such that $v$ vanishes at $t = t_0$ in a ball:*

$$v(x, t_0) = 0 \quad \text{for all} \;\; x \in B_R(x_0), \tag{14.28}$$

*for some $x_0 \in \mathbb{R}^d$, $R > 0$, $0 < t_0 < T$, and if $t_0 + \sigma < T$ and*

$$\fint_{B_R(x_0)} v(x, t_0 + \sigma)\, dx \le \frac{cR^2}{\sigma}. \tag{14.29}$$

*for $0 < \sigma < \eta$, then*

$$v(x, t_0 + \sigma) = 0 \quad \text{for all} \;\; x \in B_{R/6}(x_0). \tag{14.30}$$

**Corollary 14.18** *Under the above assumptions, if (14.28) holds and $(x_0, t_0 + \sigma) \in \Gamma_u$, then*

$$\fint_{B_R(x_0)} v(x, t_0 + \sigma)\, dx > \frac{cR^2}{\sigma}. \tag{14.31}$$

The second technical result goes in the converse direction:

**Lemma 14.19** *Under the above assumptions, if*

$$\fint_{B_R(x_0)} u^m(x, t_0)\, dx \ge \nu \left(\frac{R^2}{\sigma}\right)^{m/(m-1)} \tag{14.32}$$

*for some $0 < \sigma \le \eta$, then*

$$u^m(x_0, t_0 + \lambda\sigma) \ge c \left(\frac{R^2}{\sigma}\right)^{m/(m-1)} \tag{14.33}$$

*where $c = \nu^{m-1}/\lambda C_0$, and $C_0 = C_0(m, d) > 0$.*

We refer to [140] for the proofs of this technical material. Note that the assumption $\Delta v > -C$ is not necessarily satisfied by the solutions of the Cauchy problem from $t = 0$, but it holds after any time delay. This restriction is the only place where the Cauchy problem is used, otherwise we can prove it for local solutions.

With these lemmas we can obtain a result that clarifies the existence of vertical segments in the free boundary, and shows that once the free boundary begins to move, it never stops.

**Lemma 14.20** *If the free boundary $\Gamma_u$ contains two points $(x_0, t_1)$ and $(x_0, t_2)$ with $0 < t_1 < t_2$, then the whole line $\{(x_0, t) :\; 0 < t < t_2\}$ is contained in $\Gamma_u$.*

**Proof** If the assertion is not true, we can choose a first time $t_1$ such that all points $(x_0, t)$ with $t \in (0, t_1)$ are not $\Gamma_u$. Let $t_0$ be one such time, so that there exists $r > 0$ such that

$$u(x, t_0) = 0 \quad \text{for} \;\; x \in B_r(x_0).$$

Without loss of generality we may assume that $t_1 - t_0$ is sufficiently small. We now use the technical lemmas: applying Corollary 14.18 with $\sigma = t_1 - t_0$, we get

$$\fint_{B_r(x_0)} v(x, t_1)\, dx \geq \frac{cr^2}{t_1 - t_0}.$$

We then apply Lemma 14.19 with $\lambda\sigma = t_*$, $\nu = 1$ provided that $t_* - t_1 \geq C(t_1 - t_0)$ for some constant $C$, and we obtain the inequality $v(x_0, t_*) > 0$. Since $t_*$ can be arbitrarily close to $t_1$, this contradicts the assumption that $(x_0, t_*) \in \Gamma_u$. ■

As a consequence of this result, we conclude that all vertical segments of the free boundary are indeed *waiting time segments.*

The lemmas are also used in the proof of Theorem 14.16. This is the more technical version of that result that is proved in [140].

**Proposition 14.21** *Let $(x_0, t_0)$ be a point of $\Gamma_u$ with $\tau_0 \geq \eta_0 > 0$, and assume that it is not contained in a waiting line segment. Then there are constants $C, \gamma, h_0 > 0$ such that*

$$u(x,t) = 0 \qquad \text{if} \quad |x - x_0| \leq C(t_0 - t)^\gamma, \quad t_0 - h_0 < t < t_0 \quad (14.34)$$

$$u(x,t) > 0 \qquad \text{if} \quad |x - x_0| \leq C(t_0 - t)^\gamma, \quad t_0 < t < t_0 + h_0. \quad (14.35)$$

We refrain from the proofs of these results for lack of space since the reference is classical and easily available.

## 14.5 Describing the free boundary by the time function

We have defined the free boundary $\Gamma_u$ of a non-negative and continuous local solution $u$ of the PME as the boundary in $Q = \Omega \times (0, T)$ of the positivity set of $u$. We may describe it by means of the (waiting) time function of formula (14.16):

$$\tau_u(x) = \inf\{t > 0 : u(x,t) > 0\} = \inf\{t > 0 : x \in \mathcal{P}_u(t)\}.$$

The free boundary behaviour is reflected in the properties of $\tau$. Let us review some of the known properties of $\tau$: by Theorem 14.3, this function is well-defined and finite for every $x \in \Omega$. It is clear from the definitions that the graph of $\tau$ is contained in the free boundary $\Gamma_u$; actually, we will see that most of times it coincides with $\Gamma_u$. Theorem 14.4 implies that $\tau$ is a bounded function if $\Omega$ is bounded with smooth boundary (note also that $\tau(x)$ is bounded if $u \in C(\overline{\Omega})$ and $u$ is strictly positive on the boundary). On the other hand, Corollary 14.10 gives conditions under which $\tau(x) = 0$. In particular, $\tau(x)$ vanishes identically in $\Omega$ if the initial data are continuous and positive, or when the pressure vanishes with a linear rate. We recall that $\{x \in \Omega : \tau(x) = 0\}$ is the set of immediately positive points. On the other hand, the set of points with a positive waiting time, $\{x \in \Omega : \tau(x) > 0\}$, may be considered as the set of essential zeros of the initial data as regards the PME flow.

**Proposition 14.22** *The function $\tau_u$ is upper semicontinuous, but need not be continuous.*

**Proof** The upper semicontinuity is a simple consequence of the definition and the fact that $u$ is continuous. A situation where $\tau_u$ is not continuous is given by a solution with continuous initial data $u_0(x)$ that are positive for $x \neq x_0$ but $u_0$ vanishes in the form (14.18) as $x \to x_0$. Then, $\tau(x) = 0$ for all $x \neq x_0$, while $\tau(x_0) > 0$. ■

When we deal with the Cauchy problem, the results of the preceding section give some better information on function $\tau_u$.

**Theorem 14.23** *The function $\tau_u$ is Hölder continuous at all points $x \notin \mathcal{S}_0$. Moreover, if the free boundary contains a vertical segment at $x = x_0$, then*

$$\{x_0\} \times (0, \tau(x)) \subset \Gamma_u.$$

*Indeed, $\Gamma_u$ is composed of the graph of $\tau_u$ completed by those vertical segments.*

The investigation of the local behaviour of the free boundary is continued in Chapter 15 in one space dimension and in Chapter 19 for $d > 1$.

## 14.6 Properties of solutions in the whole space

We will now examine some results that can be obtained when dealing with solutions of the Cauchy problem defined in the whole space, $\Omega = \mathbb{R}^d$ for all times $t > 0$.

### 14.6.1 *Finite propagation for $L^1$ data*

Our task is extending the hole-filling estimates of Lemma 14.5 to solutions in the $L^1$ class, i.e., with data that are not necessarily bounded.

**Proposition 14.24** *Let $u$ be the strong solution to the Cauchy problem* (9.9), (9.10) *with initial data $u_0 \in L^1(\mathbb{R}^d)$, $u_0 \geq 0$, and assume that $u_0$ vanishes* a.e. *in a ball $B_R(x_0)$. Then, for every $t > 0$ small enough $u$ vanishes in a smaller ball $B_{R(t)}(x_0)$ with $0 < R(t) < R$. The function $R(t)$ is monotone non-increasing and we have an estimate for $t \approx 0$ of the form*

$$R(t) \approx R - c(M^{m-1}t)^{\alpha/d}. \tag{14.36}$$

*where $M = \|u_0\|_1$ and $c = c(m,d)$.*

**Proof** (i) The scaling argument allows us to reduce the case to $M = 1$, $R = 1$ by using transformation (14.46) with $K = MR^{-d}$ which implies

$$R(t) = R\, s(M^{m-1}R^{-2-(m-1)d}t),$$

where $s(t)$ is the estimate for $M = R = 1$.

(ii) Therefore, we set out to prove the result for $M = R = 1$. We select a sequence of times $t_k = 2^{-k}t \to 0$ and estimate the decrease of the hole in those times, i.e., we will consider the evolution in the different time intervals $I_k = [t_k, t_k]$. We will use the $L^\infty$ smoothing effect of Theorem 9.8 to assert that at each initial time $t = t_k$ of the evolution we have

$$K_k = \|u(t_k)\|_\infty \le c(m,d)\, t_k^{-\alpha}.$$

Therefore, using Lemma 14.5 we get

$$R(t_{k-1}) \ge R(t_k) - cK_k^{(m-1)/2}(t_{k-1} - t_k)^{1/2}.$$

Note that this kind of estimate needs $K_k^{m-1} t_k \ll R_k^2 \sim R^2$, which is true for small $t$ and/or large $k$. Now, $t_{k-1} - t_k = t_k$, hence

$$R(t_k) - R(t_{k-1}) \le c_1 t_k^{-\alpha(m-1)/2+1/2} = c_1 t_k^{\alpha/d}.$$

This series is convergent and gives the desired result.

(iii) The scrupulous reader may have a problem in starting the iteration. In that case, he should prove the result for an approximate solution with bounded initial data, where the number of steps is finite, and then pass to the limit. ■

### 14.6.2 *Monotonicity properties for solutions with compact support*

We have introduced Aleksandrov's reflection principle in Subsection 9.6.2 during the study of the Cauchy problem. This principle is quite useful when applied to the study of the monotonicity properties of the solutions with compactly supported data, and more specifically, the properties of their free boundaries. We begin by presenting the following *monotonicity lemma* along outgoing directions.

**Lemma 14.25** *Let $u \ge 0$ be a solution of the Cauchy problem for the heat equation, or the PME with initial data supported in the ball $B_R(0)$, $R > 0$. Then for every $x_0 \in \mathbb{R}^d$ such that $|x_0| > R$ and every $t > 0$, $u(x,t)$ is monotone non-increasing along the ray $l(x_0) = \{x = sx_0 : s \ge 1\}$ in the sense that*

$$u(s_2 x_0, t) \le u(s_1 x_0, t) \quad \textit{if } s_2 \ge s_1 \ge 1. \tag{14.37}$$

**Proof** The application of Alexandrov's reflection principle proceeds as follows: we draw the hyperplane $H$ which is mediatrix between the points $x = s_2 x_0$ and $y = s_1 x_0$ in the above situation. It is easy to see that $H$ divides the space $\mathbb{R}^N$ into two half spaces, one $\Omega_1$ which contains $y$ and the support of $u_0$ and another one, $\Omega_2$, which contains $x$ and where $u_0 = 0$. We consider now the initial and boundary-value problem in $\widehat{Q} = \Omega_1 \times (0,\infty)$. Two particular solutions of this problem are compared: one of them is $u_1$, the restriction of $u$ to $\widehat{Q}$, another one is

$$u_2(z,t) = u(\pi(z), t), \quad z \in \Omega_1,$$

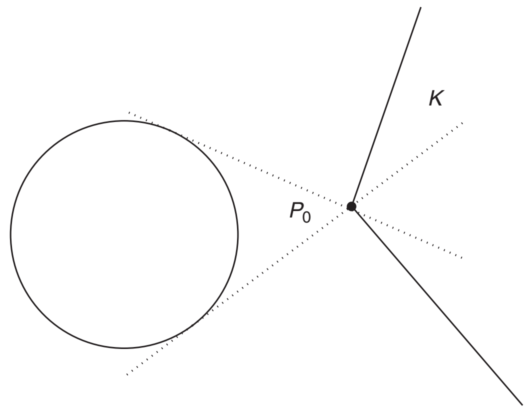


Figure 14.1: The monotone cone of directions.

where $\pi$ is the specular symmetry with respect to the hyperplane $H$. By the reflection principle,

$$u_1(z,t) \geq u_2(z,t) \quad \text{for } z \in \Omega_1,\ t > 0.$$

Putting $z = y$ we have $\pi(z) = x$ so that $u(y,t) \geq u(x,t)$ as desired. ■

Actually, the result can be sharpened into monotonicity along the cone of directions with vertex $x_0$, axis along the same direction and a certain amplitude (angle) that allows for the previous argument with hyperplanes to be applied. We denote by $\mathbf{e}_\theta$ the unit vector in the direction $\theta \in \mathbb{S}^{d-1}$. We denote by $\alpha(x,y)$ the angle between two vectors $x, y \in \mathbb{R}^d$. Let finally $\phi(x_0)$ be the angle with which the ball $B_R(0)$ is seen from $x_0$, $\sin\phi(x_0) = R/|x_0|$. This is the full result.

**Proposition 14.26** *Let $u \geq 0$ be a solution of the Cauchy problem for the heat equation, or the PME with initial data supported in the ball $B_R(0)$, $R > 0$. Then for every $x_0 \in \mathbb{R}^N$ such that $|x_0| > R$ and every $t > 0$, $u(x,t)$ is monotone non-increasing along the rays $l(x_0,\theta) = \{x = x_0 + s\mathbf{e}_\theta : s \geq 1\}$, with angle $\alpha(\theta, x_0) \leq \alpha(x_0)$, $\alpha(x_0) = \pi/2 - \phi(x_0)$.*

The rays in the result form a cone $K(x_0, \alpha_0)$ with vertex at $x_0$, axis $x_0$ and aperture $\alpha_0$. The property of monotonicity along cones of directions has been an essential tool in the development of the regularity theory of free boundaries of the PME.

This result can be improved by the application of Aleksandrov's reflection principle as follows. We study the large time distribution of the level sets of solutions of the PME in the form of a *monotonicity lemma* as is used in [145].

**Proposition 14.27** *Let $u \geq 0$ be a solution of the Cauchy problem for the PME with initial data supported in the ball $B_R(0)$, $R > 0$. Then for every $x$ such that*

*$|x| > 2R$ and every $r < |x| - 2R$, $r > 0$, we have*

$$u(x,t) \le \inf_{|y|=r} u(y,t). \tag{14.38}$$

**Proof** To use Aleksandrov's reflection principle, we draw the hyperplane $H$ which is mediatrix between the points $x$ and $y$ in the above situation. It is easy to see that $H$ divides the space $\mathbb{R}^N$ into two half spaces, we define $\Omega_1$ and $\Omega_2$ as before, perform the specular transformation $\Pi$, apply comparison, and finally conclude that $u(y,t) \ge u(x,t)$ as desired. ■

The arguments apply without changes to the heat equation, the $p$-Laplacian equation and other parabolic equations as long as they respect specular symmetry and the maximum principle. It follows from here that the lower level lines tend to be almost spherical.

### 14.6.3 *Free boundary behaviour*

We can use these results to obtain a conclusion on the behaviour of the solution and its free boundary for large times. We assume again that $u \ge 0$ is a solution of the Cauchy problem for the heat equation or the PME, with initial data supported in the ball $B_R(0)$, $R > 0$. We introduce the concept of maximum and minimum support radius:

$$r_{\max}(t) = \max\{|x| \in \Gamma(t)\}, \quad r_{\min}(t) = \min\{|x| \in \Gamma(t)\}. \tag{14.39}$$

**Corollary 14.28** *There exists a time $T = T(u_0)$ after which the initial ball $B_R(0)$ is contained in the positivity set and the free boundary $\Gamma(t)$ is located outside of $B_R(0)$. Then, for fixed $t > T(u_0)$, $\Gamma(t)$ is a Lipschitz hypersurface in $\mathbb{R}^d$ written in polar coordinates*

$$|x| = f(\theta, t), \quad \textit{for } \theta \in \mathbb{S}^{d-1}. \tag{14.40}$$

*Moreover, estimate* (14.37) *holds and we also have*

$$r_{\max}(t) - r_{\min}(t) \le 2R. \tag{14.41}$$

*Therefore, $r_{\max}(t)/r_{\min}(t) \to 1$ as $t \to \infty$.*

**Proof** $\Gamma(t)$ covers eventually all $B_R(0)$ by the expansion argument of Proposition 9.19. Lipschitz continuity comes from Propositions 14.26 and 14.27 that allow us to put an inner space cone and an outer space cone at every point of the free boundary. Note that the aperture $\alpha(x_0)$ of those cones increases as $t \to \infty$ so that the free boundary tends to be locally flat, a fact that needs a proof which will be discussed in Chapter 19. Estimate (14.41) is a consequence of Proposition 14.27. The last limit follows from this and the estimate $r(t) = O(t^{\beta})$, $\beta = 1/(d(m-1)+2)$, obtained in Proposition 9.19. Actually, we have $r_{\max}(t)/r_{\min}(t) = 1 + O(t^{-\beta})$. ■

The main conclusion of this analysis is that as time grows the free boundary becomes rounded like a ball, a form of the property called *asymptotic symmetry*. We will prove in Chapter 18 that this radially symmetric pattern to which a general solution evolves with time is precisely the ZKB solution of the same mass. In that context, our result says that, concerning the free boundary, the error in the *asymptotic symmetrization* is at most a constant.

## 14.7 Propagation of signed solutions

In case $u$ is a continuous solution with two signs, the definitions of Section 14.1 have to be modified and completed. We have the positivity set, $\mathcal{P}_u = \{(x,t) \in Q : u(x,t) > 0\}$, an open set of $Q = \Omega \in (0,T)$, with its time sections $\mathcal{P}_u(t) = \{x \in \Omega : u(x,t) > 0\} \in \Omega$, but also a negativity set

$$\mathcal{N}_u = \{(x,t) \in Q : u(x,t) < 0\} \tag{14.42}$$

which is also an open subset of $Q$ with time sections $\mathcal{N}_u(t)$. The zero set is now

$$\mathcal{Z}_u = \{(x,t) \in Q : u(x,t) = 0\} = Q \setminus (\mathcal{P}_u \cup \mathcal{N}_u), \tag{14.43}$$

a closed subset of $Q$. The support is now $\mathcal{S}_u = (\overline{\mathcal{P}}_u \cup \overline{\mathcal{N}}_u) \cap Q$ and the free boundary

$$\Gamma_u = (\partial\mathcal{P}_u \cap Q) \cup (\partial\mathcal{N}_u \cap Q). \tag{14.44}$$

It is clear that $\mathcal{S}_u$ and $\Gamma_u \subset \mathcal{S}_u$ are closed sets, and $\Gamma_u = \mathcal{S}_u \cap \mathcal{Z}_u$. The interior of $\mathcal{Z}_u$ is the largest open set where $u$ vanishes.

In the case of two signs, we lose the properties of persistence and expansion, since the positive part can push the negative part or be pushed by it, so that the free boundary moves back and forth. Some basic results can be obtained by solving the problem with data $u^+ = \max\{u, 0\}$ on the parabolic boundary of $Q$, let us call it $u_1 \ge 0$, and then with data $-u^- = \min\{u, 0\}$ to obtain $u_2 \le 0$. By the maximum principle, $u_2 \le u \le u_1$, and this allows us to bound the possible expansion of the positive part and the negative part of the solution $u$:

$$u^+ \le u_1, \quad u^- \ge -u_2.$$

We next use the following result.

**Lemma 14.29** *Any contraction of the support of $u^+$ is due to the expansion of the support of $u^-$ and conversely. More precisely, if $x_0 \in \Omega$, $0 < t_1 < t_2 < T$, and*

$$u(x_0, t_1) > 0, \quad u(x_0, t_2) \le 0$$

*then $x_0 \in \overline{\mathcal{N}_u(t)}$ for some $t \in (t_1, t_2]$.*

We leave the easy proof to the reader. With the help of this result, we can prove the following continuity theorem that extends Theorem 14.6.

**Proposition 14.30** *There exists a continuous function* $\delta_u : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\delta(0) = 0$ *such that*

$$d(\Gamma_u(t), \Gamma_u(t+h)) \le \delta_u(h), \tag{14.45}$$

*where* $d(\cdot,\cdot)$ *denotes the distance between sets.*

The waiting time of a signed solution defined in the whole space can be infinite, as the dipole solutions (4.53) show. Examples of solutions with advancing or receding free boundaries are the signed travelling waves of Problem 4.3.

*Further results*

Bertsch and Kamin [111] assume certain monotonicity properties on the initial function, and prove that there exists a time $T \ge 0$ after which the regions where $u < 0$ and $u > 0$ are separated by an interface $x = \zeta(t)$ such that $\zeta$ is continuously differentiable on $(T, \infty)$. See also [467].

## Notes

**Section 14.2.** The first estimate of the section to control the growth of the support is already in early papers. Penetration and retention are mentioned by Knerr in [344].

Theorems 14.3 and 14.4 show that every point of the space is eventually reached by the diffusing substance, a property that was not obvious a priori. The proofs are based on [49], in the second one the regularity of the boundary is weakened. In this version they are new (though the general result was known).

**Section 14.2.3.** The problem of hole filling has attracted much attention from researchers in recent years because of the connected problem of limited regularity at the focusing point that we will discuss in Chapter 19. The hole-filling Lemma 14.5 and its extension Proposition 14.24 seem to be new.

**Section 14.3.** The necessary and sufficient condition of non-zero waiting time, Theorem 14.13, is taken from [145]. The waiting time was first characterized by [498] in one space dimension.

More details about lower estimates are given in the Lecture Notes [515], where comparison with the heat equation and fast diffusion is done. Note that both HE and FDE have infinite speed of propagation for non-negative solutions so that no interfaces arise.

The example of infinite waiting time at a corner point for Dirichlet data in Subsection 14.3.3 is new.

**Section 14.5.** The Hölder regularity of the free boundary, a main result in the regularity theory of the PME, is due to Caffarelli and Friedman [140], 1980. The vertical line theorem, Theorem 14.20, was first proved by Knerr [343] in $d = 1$, but his proof does not extend to $d > 1$.

Cho and Choe [169] consider the Cauchy problem for the symmetric degenerate parabolic equation

$$u_t - \sum_{i,j=1}^{n} D_j[a_{ij}(x)D_i(u^m)] = 0$$

for $(x,t) \in \mathbb{R}^d \times (0,\infty)$ for $x \in \mathbb{R}^d$, with $m > 1$ and assuming that $u(x,0) = u_0(x) \geq 0$ is continuous and compactly supported. Here $(a_{ij})$ is a symmetric matrix of bounded measurable functions which satisfies the ellipticity condition

$$\Lambda^{-1}\xi^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda\,\xi^2 \quad \text{a.e. in } \mathbb{R}$$

for all $\xi \in \mathbb{R}^d$, for some $\Lambda > 1$. They establish the Hölder regularity of the expanding free boundary. The asymptotic behaviour as $t \to \infty$ is also studied.

Propagation properties for the GPME and other variants, like doubly nonlinear equations, are studied by different authors, like [383].

**Section 14.6.** For the precise asymptotic behaviour of solutions and free boundaries when the data are compactly supported see Chapter 18 and references.

The monotonicity estimate (14.38) is of great use in the study of regularity of PME solutions and their free boundaries [42, 145] among many other works. It follows from here that the lower level lines tend to be almost spherical. In particular, this applies to the free boundary, cf. [325].

**Section 14.7.** Not much is known about free boundaries in the general situation of signed solutions cf. Section 15.9 of the next chapter for $d = 1$. The problem of Hölder continuity of free boundaries of signed solutions seems to be open.

*Various*

There are a number of works extending the property of finite speed of propagation of non-negative solutions of the Cauchy–Dirichlet problem to porous media equations with lower order terms (absorption or convection terms). We refer for instance to the book by Samarski et al. [469].

The method of *energy estimates* has been used quite effectively by Antontsev and collaborators [31], and Diaz and Véron [203]. Equations of quite general form like $\partial\psi(u)/\partial t - \operatorname{div}\mathcal{A}(t,x,u,Du) + \mathcal{B}(t,x,u,Du) + \mathcal{C}(t,x,u) = 0$ can be treated by this method.

## Problems

**Problem 14.1** Let $u \geq 0$ be a local weak solution of the PME. Prove that $\mathcal{S}_u(t_0) = \mathcal{S}_u^*(t_0)$.

*Hint*: Since $\mathcal{S}_u(t_0) = \overline{\mathcal{P}_u} \cap \{t = t_0\}$ and $\mathcal{S}_u^*(t_0) = \overline{\mathcal{P}_u \cap \{t = t_0\}}$, the inclusion $\mathcal{S}_u^*(t_0) \subset \mathcal{S}_u(t_0)$ is immediate from point topology. For the converse inclusion, use the expansion of Proposition 14.1 and the continuity of Theorem 14.6.

**Problem 14.2** Apply rescaling to obtain the dependence on $H$ and $R$ of the function $R(t)$ of the hole-filling Lemma 14.5.

*Hint*: When $H \neq 1$ or $R \neq 1$ we use a scaling transformation

$$u(x,t) = K\tilde{u}(R^{-1}x, K^{m-1}R^{-2}t). \tag{14.46}$$

If $K = H$, then $\tilde{u}$ has sup norm 1. In this way the previous result extends by replacing $t$ by $H^{m-1}R^{-2}t$, and the space is expanded by the factor $R$. In other words, we have the formula

$$R(t) = R\,s(H^{m-1}R^{-2}t), \tag{14.47}$$

where $s(t)$ is the radius for the vanishing ball of the rescaled solution with data unity. With this we arrive at formula (14.8) with a better constant $c = 2((m-1)/m)^{m-1}$ that depends only on $m$ and not on dimension. The rate holds when $K^{m-1}t \ll R^2$.

**Problem 14.3** Prove formula (14.8) with a constant for small times $c = 2((m-1)/m)^{m-1}$ that depends only on $m$ and not on dimension:

$$R(t) \approx R - c(K^{m-1}t)^{1/2} \tag{14.48}$$

as $t \to 0$, where $K = \|u\|_\infty$ by using as supersolutions the family of spherical TWs

$$u(x,t) \le W = \frac{m}{m-1}(A(|x| + c(t-T) - B)_+^{m-1}$$

in the domain $Q_1 = B_R(x_0) \times (0,T)$ for suitable $T, A, B$ and $c$. Without loss of generality we may put $x_0 = 0$.

**Problem 14.4***

(i) Prove the following result about hole-filling near the boundary: *Let $u \ge 0$ be a bounded local solution to the PME posed in $Q = \Omega \times (0,T_1)$ with initial data $u_0 \in L^\infty(\Omega)$, and let us also assume that $u_0$ vanishes in a open set $G$ such that $\overline{G} \cap \partial\Omega$ contains a ball $B_r(x_0)$, $x_0 \in \partial\Omega$. Then, there is a time $T = T(u)$ such that for every $0 < t < T$ the solution $u(t)$ vanishes at least in a smaller set $B_{R(t)}(x_0) \in \Omega$ with $0 < R(t) \le R$.*

(ii) See if the way the support advances can be controlled as in Lemma 14.5.

**Problem 14.5** Use the blow-up solution (14.17) to show that $\Gamma_u \neq \partial\mathcal{S}_u \cap Q$ when there are isolated waiting time points. Find general conditions for equality to occur.

**Problem 14.6** Complete the details of the proof of Proposition 14.9.

**Problem 14.7** Open problem on waiting times in corners.

(i) Take a bounded domain with a corner point pointing outside with a small angle. Construct a solution that takes infinite time to arrive at the corner.

(ii) Show that when the angle is large, the arrival time is finite.
(iii) Define in two dimensions the critical angle and investigate the rates.
(iv) Think of the use for the theory for solutions with changing sign.

*Hint*: read Section 12.8.

**Problem 14.8** Read Chapter 12 and prove the assertion of Remark (2) after Lemma 14.5. Calculate the second-order correction in formula (14.48) and show that it depends on the space dimension.

**Problem 14.9** Prove Lemma 14.29 and Proposition 14.30.

**Problem 14.10*** Construct (at least numerically) an example of a signed solution in $d = 1$ with an interface consisting of a curve that advances and then goes back. Construct interfaces that change direction several times.

# 15

# ONE-DIMENSIONAL THEORY. REGULARITY AND INTERFACES

When trying to understand the behaviour and regularity of solutions and interfaces, the special case of the Cauchy problem in one spatial dimension offers the advantages of a rather complete theory and comparatively easy treatment, together with a great richness in detail. In fact, much (but not all) of what is to be found in more general situations is already present at this level. This recommends the material of this chapter for an elementary introduction to some of the main topics of the PME.

We will deal almost exclusively with non-negative solutions. The theory of the Cauchy problem is treated first. We begin in Section 15.1 by a detailed analysis of the regularity of the pressure, for which Lipschitz continuity is proved both in space and time; this regularity is optimal for solutions with interfaces, as the later analysis shows.

Section 15.2 introduces new comparison results. Shifting comparison, intersection comparison and lap number count are quite useful in the study of interfaces.

The study of interfaces is begun in Section 15.3. The growth of the interface is estimated and the waiting time analysed.

Subsection 15.4 deals with some of the main issues of the theory, the regularity of the interface and the Darcy law, which is proved in the form (15.61). This is completed in Section 15.5 by showing $C^1$ regularity at moving interfaces. This result is shown to be false at the so-called corner point, so that Lipschitz is the optimal regularity of 1D interfaces if no restrictions are placed on the data.

Though we have introduced the special study of 1D for the Cauchy problem, the main flavour of the regularity results for solutions and interfaces is local in nature and can be safely developed for *local solutions*. We devote Section 15.6 to deriving the basic estimates for local solutions, and Section 15.7 to extending the previous theory to cover the properties of local interfaces. Section 15.8 studies the question of higher regularity, including the derivation of $C^\infty$ and analytic regularity.

A short Section 15.9 deals with the behaviour of solutions and interfaces for changing-sign solutions.

Many of the results of this section are easily adapted to radially symmetric solutions in several dimensions. However, some important results, like the Lipschitz continuity of non-negative solutions for positive times are not true in that case! We refer the reader to Chapter 19 for further analysis of the equation in several space dimensions.

## 15.1 Cauchy problem. Regularity of the pressure

Let $u$ be the solution of the Cauchy problem

$$(CP) \quad \begin{cases} u_t = (u^m)_{xx} & \text{for } (x,t) \in Q, \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \tag{15.1}$$

with $u_0 \in L^1(\mathbb{R}), u_0 \geq 0$, $u_0 \not\equiv 0$. We denote by $v$ its pressure which satisfies (at least on the positivity set)

$$v_t = (m-1)vv_{xx} + v_x^2. \tag{15.2}$$

From Chapter 9 we know that for $t \geq \tau > 0$ the solution is bounded and in fact

$$u(x,t) \leq c(m)\, M^{\frac{2}{m+1}} t^{-\frac{1}{m+1}}, \tag{15.3}$$

where $M = \int u_0\, dx$ is the initial mass. We also know that $v_{xx}$ is bounded below independently of the initial data:

$$v_{xx} \geq -\frac{1}{(m+1)t}. \tag{15.4}$$

This is called the fundamental estimate in Proposition 9.4; in one dimension it becomes a semiconvexity estimate and this strong geometrical fact helps in developing the one-dimensional theory. Note that comparison with Barenblatt solutions shows that both estimates have sharp exponents, and in the second one also a sharp constant.[1]

We can obtain Lipschitz continuity of the pressure starting from these two estimates. Let us tackle first the spatial derivative

**Theorem 15.1** *For every $t > 0$ $v(\cdot,t)$ is Lipschitz continuous in $\mathbb{R}$ and more precisely we have* a.e. *in $Q$*

$$|v_x(x,t)|^2 \leq \frac{2\|v(\cdot,t)\|_\infty}{(m+1)t}. \tag{15.5}$$

**Proof** This is based on a following simple calculus lemma:

**Lemma 15.2** *Let $f$ be a bounded $C^2$ real function such that $0 \leq f \leq N$ and $f'' \geq -C$. Then*

$$|f'(x)|^2 \leq 2NC. \tag{15.6}$$

Our estimate follows if $v$ is smooth by putting $f = v(\cdot,t)$, $N = \|v(\cdot,t)\|_\infty$ and $c = 1/(m+1)t$. If $u$ is not a smooth solution, we can use approximation.

[1] The sharp constant for formula (15.3) will be obtained in Chapter 19.

To prove (15.6) we assume that there is point $x \in \mathbb{R}$ where $|f'(x_0)| = a > (2NC)^{1/2}$. Take for instance $f'(x_0) > a$. Since $f(x_0) \geq 0$ and $f''(x) \geq -c$ for every $x$ we get with $y = x - x_0$

$$f(x) \geq ay - Cy^2/2 \quad \text{ for } y > 0.$$

Now, the maximum of the right-hand side, attained at $y = a/C$, is $a^2/2C \leq f(x_0 + C) \leq N$, a contradiction. ■

**Corollary 15.3** *For every $t > 0$ we have*

$$|v_x(x,t)| \leq cM^\lambda t^{-\mu}, \tag{15.7}$$

*where $M = \|u_0\|_1$ and*

$$\lambda = \frac{m-1}{m+1}, \quad \mu = \frac{m}{m+1}, \quad \text{ and } \ c = c(m).$$

*Equality holds for the fundamental solutions at the interface.*

In fact, the right-hand side of (15.7) equals $r'(t)$ the derivative of the free boundary of the ZKB solution. Since $-v_x$ is the velocity of propagation of point particles, see the example of gases in porous media in Chapter 2, the estimate says that quickest propagation corresponds for a given mass to the Barenblatt solutions on their interfaces.

In case the initial velocity is bounded we have the following estimate.

**Proposition 15.4** *For every $t > 0$*

$$|v_x(x,t)| \leq \|v_{0,x}\|_\infty. \tag{15.8}$$

**Proof** Assume that $v$ is a smooth solution with $0 < M_1 \leq v \leq M_2$ and $|v_0'| \leq C$. Then $|u_0'|$ is also bounded. Applying the interior regularity theory to $u_t = (u^m)_{xx}$ we find that $u_x$, and consequently $v_x$, are uniformly bounded in domains of the form $R = [x_0 - \ell, x_0 + \ell] \times [0, T]$ with $\ell, T > 0$ with constant depending only on $\ell, T$, hence uniformly in $x \in \mathbb{R}$, $0 \leq t \leq T$. Now we apply the maximum principle to the equation satisfied by $z = v_x$

$$z_t = (m-1)vz_{xx} + (m+1)zz_x, \tag{15.9}$$

which can be viewed as linear uniformly parabolic in $z$ with bounded coefficients $(m-1)v$, $(m+1)z$. This proves (15.8). For general solutions we can use approximation. ■

We can also prove Lipschitz continuity in time. A lower bound for $v_t$ is an immediate consequence of equation (15.2) and the fundamental estimate. In fact

$$v_t \geq (m-1)vv_{xx} \geq -\frac{(m-1)v}{(m+1)t}. \tag{15.10}$$

An upper bound is more difficult and not as precise. We have

**Theorem 15.5** *The pressure $v$ is also Lipschitz continuous in time for $t \geq \tau > 0$. More precisely, if $v$ is bounded we have*

$$v_t + (m-1)v_x^2 \leq \frac{8m\|v\|_\infty}{(m+1)t} \qquad \text{a.e. } in\ \ Q. \tag{15.11}$$

**Proof** By approximation, we may assume that $v$ is positive and smooth. We consider the function

$$P := v_t + \alpha v_x^2 = (m-1)vv_{xx} + (\alpha+1)v_x^2 \tag{15.12}$$

for some $\alpha \geq 0$ to be chosen later. We have

$$\begin{aligned} P_x &= (m-1)vv_{xxx} + (m+2\alpha+1)v_x v_{xx} \\ P_{xx} &= v_{txx} + 2\alpha(v_{xx}^2 + v_x v_{xxx}) \\ P_t &= (m-1)vv_{xxt} + (m-1)v_t v_{xx} + 2(\alpha+1)(m+1)v_x^2 v_{xx} \\ &\quad +2(\alpha+1)(m-1)vv_x v_{xxx}. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \mathcal{L}(P) &\equiv P_t - (m-1)vP_{xx} - 2v_x P_x \\ &= (m-1)\, v_{xx}[v_t + 2\alpha v_x^2 - 2\alpha v v_{xx}] \\ &= \frac{1}{v}[P - (\alpha+1)v_x^2]\left[\alpha\left(1+\frac{2(\alpha+1)}{m-1}\right)v_x^2 - \left(\frac{2\alpha}{m-1}-1\right)P\right] \equiv f(x,t,P) \end{aligned}$$

We want to find a supersolution of the form $P_1 = C/t$ for the equation $\mathcal{L}(P) = f(x,t,P)$. This means that

$$-\frac{C}{t^2} \geq \frac{1}{v}\left[\frac{C}{t} - (\alpha+1)v_x^2\right]\left[\alpha\left(1+\frac{2(\alpha+1)}{m-1}\right)v_x^2 - \left(\frac{2\alpha}{m-1}-1\right)\frac{C}{t}\right] \tag{15.13}$$

Let $N = \|v\|_\infty$. By (15.5) we have $v_x^2 = 2N\theta/(m+1)t$ with $0 \leq \theta \leq 1$. Let us also write

$$C = \frac{2kN}{(m+1)}.$$

Then (15.13) holds if for every $\theta \in [0,1]$

$$\frac{1}{2}k(m+1) + (k - (\alpha+1)\theta)\left(\alpha\left(1+\frac{2(\alpha+1)}{m-1}\right)\theta - \left(\frac{2\alpha}{m-1}-1\right)k\right) \leq 0. \tag{15.14}$$

This is satisfied if, for instance, $\alpha = m-1$ and $k = 4m$. By the parabolic maximum principle, we conclude then that $P \leq C/t$, i.e. (15.11). ∎

**Remarks**

(1) Of course, if we combine (15.11) with (15.3) we obtain an estimate of the form

$$|v_t|,\ |v_x|^2 \le c\|u_0\|_1^{\frac{2(m-1)}{m+1}}\, t^{-\frac{2m}{m+1}}. \tag{15.15}$$

(2) We shall see below that Lipschitz continuity is the best possible regularity for the pressure of any solution with interfaces, since both $v_x$ and $v_t$ are discontinuous across any moving interface piece.

As for the regularity of the density $u$, the result is indirect, obtained as a consequence of the above result for the pressure, which seems to be a more natural variable in this respect. We have

**Theorem 15.6** *Let $u$ be a non-negative solution of the PME with $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then $u$ and $(u^m)_x$ are continuous in $Q$. Moreover,*

(i) *if $1 < m < 2$, then $u_t = (u^m)_{xx}$ and $u_t$ is continuous in $Q$;*

(ii) *if $1 < m \le 2$, then $tu_t = t(u^m)_{xx} \in L^\infty(Q)$;*

(iii) *if $m > 2$, then $u_t = (u^m)_{xx} \in L^\infty(\delta, T; L^p_{loc}(\mathbb{R}))$ for any $1 \le p < 1 + \frac{1}{m-2}$ and $\delta > 0$.*

**Proof** We know that $u \in C^\infty([u > 0])$ and that $v$ is Lipschitz continuous in $Q$. Moreover,

$$|(u^m)_x| = |uv_x| \le u\left(\frac{2N}{(m+1)t}\right)^{1/2}. \tag{15.16}$$

Therefore, $(u^m)_x$ is continuous in $Q$ with $(u^m)_x \equiv 0$ on $[u = 0]$. Also, from (15.10) we have

$$u_t = (u^m)_{xx} \ge uv_{xx} \ge -\frac{u}{(m+1)t}, \tag{15.17}$$

while for $1 < m \le 2$ we obtain from (15.11)

$$(u^m)_{xx} = u_t = \frac{u^{2-m}}{m}v_t \le \frac{8N}{(m+1)t}u^{2-m}. \tag{15.18}$$

It follows that $tu_t \in L^\infty(Q)$. Also, if $1 < m < 2$ then $u_t$ is continuous with $u_t \equiv 0$ on $[u = 0]$.

In the case $m > 2$, we obtain (iii) by applying the following lemma with $w(x) = u^m(t, x)$, $\beta = (m-1)/m$ and $\gamma = (m-2)/m$. ∎

**Lemma 15.7** *Let $w$ be a non-negative continuous and bounded real function satisfying*

$$(w^\beta)_{xx} \ \ge -c \quad \textit{in } \mathcal{D}'(\mathbb{R}) \textit{ for some } 0 < \beta < 1, \tag{15.19}$$

$$w_{xx} \le cw^{-\gamma} \quad \textit{in } \mathcal{D}'([w > 0]) \textit{ for some } \gamma > 0. \tag{15.20}$$

*Then,* $w_{xx} \in L^p_{\mathrm{loc}}(\mathbb{R})$ *for any* $p : 1 \le p < 1 + \frac{1-\beta}{\gamma}$.

**Proof** As in Theorem 15.1, assumption (15.19) implies

$$|(w^\beta)_x|^2 \le 2c\|w\|_\infty^\beta.$$

Hence, $w \in C^1(\mathbb{R})$ with $|w_x| \le \frac{1}{\beta} w^{1-\beta}(2c\|w\|_\infty^\beta)^{1/2}$. Also

$$w_{xx} \ge \frac{w^{1-\beta}}{\beta}(w^\beta)_{xx} \ge -\frac{c}{\beta} w^{1-\beta} \text{ in } \mathcal{D}'(\mathbb{R}).$$

We conclude that $w_x$ is locally of bounded variation in $\mathbb{R}$ since, by (15.20), $w_x$ is locally Lipschitz on $[w > 0] \subset [w_x \ne 0]$ and $w_{xx} \in L^1(\mathbb{R})$ ($w_{xx}$ does not change the set $[w_x = 0] \subset [w = 0]$).

We have for $1 \le p < 1 + (1-\beta)/\gamma$

$$\begin{aligned} |w_{xx} + \frac{c}{\beta} w^{1-\beta}|^p &= \left(w^\gamma w_{xx} + \frac{c}{\beta} w^{1-\beta+\gamma}\right)^{p-1} \left(\frac{w_{xx}}{w^{\gamma(p-1)}} + \frac{c}{\beta} w^{1-\beta-\gamma(p-1)}\right) \\ &\le C\left(w^{\sigma+\beta-1} w_{xx} + \frac{c}{\beta} w^\sigma\right) \qquad \text{a.e. on } [w > 0], \end{aligned}$$

with

$$C = \left(c + \frac{c}{\beta}\|w\|_\infty^{1-\beta+\gamma}\right)^{p-1}, \ \sigma = 1 - \beta - \gamma(p-1) > 0.$$

Now

$$\begin{aligned} w^{\sigma+\beta-1} w_{xx} &= \tfrac{1-\beta}{\beta\sigma}(w^\sigma (w^\beta)_x)_x - \tfrac{1-\beta-\sigma}{\beta\sigma} w^\sigma (w^\beta)_{xx} \\ &\le \tfrac{1-\beta}{\beta\sigma}(w^\sigma (w^\beta)_x)_x + \tfrac{c\gamma(p-1)}{\beta\sigma} w^\sigma. \end{aligned}$$

Then

$$\left|w_{xx} + \frac{c}{\beta} w^{1-\beta}\right|^p \le \frac{C(1-\beta)}{\beta\sigma}[(w^\sigma (w^\beta)_x)_x + cw^\sigma]$$

and

$$\int_{-R}^{R} \left|w_{xx} + \frac{c}{\beta} w^{1-\beta}\right|^p < \infty$$

for any finite $R > 0$. ■

We remark that the regularity stated in (ii), (iii) of Theorem 15.6 is the best possible in that direction as can be seen on the ZKB solutions $U(x,t;M)$. In any case, comparison with the results of Theorems 15.1 and 15.5 for $v$ shows that the pressure is a more natural variable to state regularity for the PME.

## 15.2 New comparison theorems

There are special forms of comparison valid for one-dimensional flows that are widely used in the study of interfaces.

### 15.2.1 *Shifting comparison*

In establishing the propagation rates of solutions with interfaces it will be very convenient to compare our solutions with other solutions which lie to the right or the left of them in the $(x, t)$-plane in the sense of mass distributions (we can also think of probability densities if the total mass is one). This is possible thanks to the following *comparison by shifting*.

**Theorem 15.8** *Let $u_1, u_2$ be two solutions of the Cauchy problem whose initial data $u_{01}, u_{02}$ satisfy for every $x \in \mathbb{R}$ the inequality*

$$\int_x^\infty u_{01}(y)dy \le \int_x^\infty u_{02}(y)dy. \tag{15.21}$$

*Then, for every $t > 0$ we have*

$$\int_x^\infty u_1(y,t)dy \le \int_x^\infty u_2(y,t)dy. \tag{15.22}$$

**Remarks**

(1) The result is true not only for integrable initial data but also for finite non-negative measures, $u_{0i} \in \mathcal{M}$. The proof follows by approximation. Actually, the total mass need not be finite, only the convergence at $x = +\infty$ matters for the validity of Theorem 15.8.

(2) The same result is true if we replace $\int_x^\infty$ by $\int_{-\infty}^x$. In fact, integrals from $x = -\infty$ are the standard choice to define the distribution function in probability theory:

$$F_{u(t)}(x) := \int_{-\infty}^x u(y,t)dy = M - \int_x^\infty u(y,t)dy. \tag{15.23}$$

**Proof of the lemma** (i) Our result is just the usual comparison result applied to the integrated or *mass-distribution* variables

$$w_i(x,t) = -\int_x^\infty u_i(y,t)dx, \quad i = 1, 2, \tag{15.24}$$

which formally satisfies the integrated equation

$$w_t = ((w_x)^m)_x = m w_x^{m-1} w_{xx} \tag{15.25}$$

with initial data $w_i(x,0) = -\int_x^\infty u_{0i}(y)dy$ and limits at infinity

$$\begin{cases} \text{(a)} \quad \lim_{x\to+\infty} w_i(x,t) = 0 \\ \text{(b)} \quad \lim_{x\to-\infty} w_i(x,t) = -M_i \equiv -\int_{-\infty}^{\infty} u_{0i}(y)dy. \end{cases} \tag{15.26}$$

Since by (15.21) $w_1(x,0) \ge w_2(x,0)$ and $M_1 \le M_2$, the maximum principle would imply that $w_1 \ge w_2$ in $Q$.

(ii) In order to apply the classical maximum principle to the solutions (15.25), we assume that the $u_i$ are bounded, smooth and positive and note that $w_{i,x} = u_i > 0$, and that the limits (15.26) and the initial data are taken uniformly, i.e., $w_i$ is continuous in $[-\infty,+\infty]\times[0,\infty)$. Then, if $w_1 < w_2$ somewhere we argue at the points where $(w_2-w_1)e^{-\lambda t}, \lambda > 0$, attains a positive maximum to arrive at a contradiction with (15.25) since putting $w = w_2 - w_1$ we have $w_t \ge \lambda w > 0, w_x = 0$ and $w_{xx} \le 0$.

The above regularity for $w_i$ can be obtained by assuming that the $u_{0i}$ is strictly positive, bounded and smooth and we also assume the bound $v_{0i}'' \ge -C$. We drop the subscript in the sequel. Under these conditions we have obtained in Chapter 9 a classical, bounded and positive solution $u$ to the PME taking the initial data continuously. Clearly, we have $\partial w/\partial x = u$. On the other hand, integration of the PME over the rectangle $S = (-\infty,x)\times(t,t+h)$ for $(x,t)\in Q$, $h > 0$ gives

$$\int_x^\infty u(y,t+h)dy - \int_x^\infty u(y,t)dy = -\int_t^{t+h} (u^m)_x(x,\tau)d\tau. \tag{15.27}$$

In the limit $h\to 0$ (15.27) gives $\partial w/\partial t = (u^m)_x$. Therefore (15.25) holds, the equation is strictly parabolic and $u$ is a classical solution, continuous down to $t = 0$.

We are left with the limits (15.26). It is here that we use the bound $v_0'' \ge -C$. This implies that $(u_0^m)'' \ge u_0 v_0'' \ge -Cu_0$. Under these conditions we have, cf. Problem 8.1,

$$u_t \ge -\frac{u}{(m-1)t + 1/a}. \tag{15.28}$$

Now fix $T > 0$. For $0 < t < T$ we have $u_t + bu \ge 0$ with $b = 1/((m-1)t + 1/a)$, so that $u(x,t) \le u(x,T)\exp(+bT)$ and thus

$$|w(x,t)| = \int_x^\infty u(y,t)dt \le e^{bT}|W(x,T)|$$

Since $w(x,T)\to 0$ as $x\to+\infty$ we conclude that (15.26.a) holds uniformly in $0 \le t < T$. Similarly (15.26.b)).

(iii) Finally, for general $u_0$ we use approximation. ■

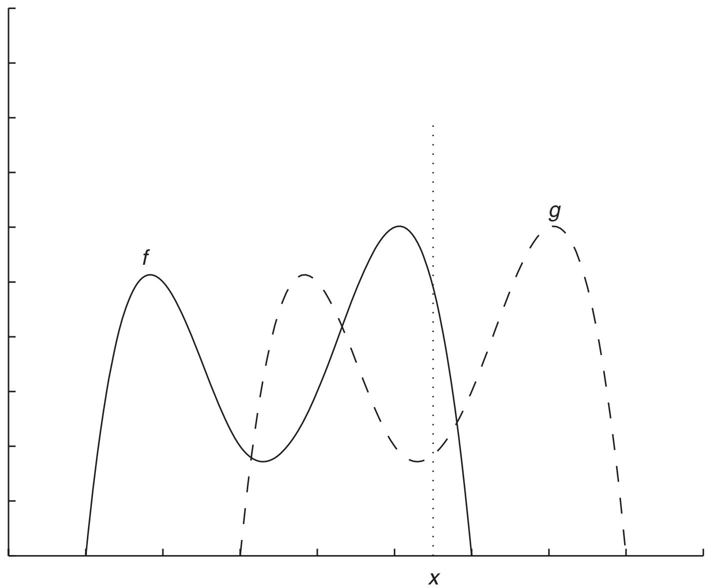


Figure 15.1: A situation in which $f \succ g$.

*Notation and interpretation*

For two functions $f, g \in L^1(\mathbb{R})$, $f, g \geq 0$, we will write $f \succ g$ when

$$\int_x^\infty f(y)dy \leq \int_x^\infty g(y)dy. \tag{15.29}$$

Thinking intuitively in terms of mass distributions in the typical case when $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g = M < \infty$, we may say that the mass of $f$ is 'located to the left' of the mass of $g$. This way of talking is justified in the Monge–Kantorovich theory of optimal transportation, cf. [521, 514]. In this view the theorem asserts that there is an initial ordering then this ordering continues to hold for all $t > 0$, i.e., $u_{01} \succ u_{02}$ implies $u_1(t) \succ u_2(t)$ for all $t > 0$. Note that the condition of same mass is not needed in our theorem.

### 15.2.2 *Counting intersections and lap number*

A readily observed property of the solutions of parabolic equations is that the complexity of the graph of a solution at each fixed time does not get more complicated as times passes. There is a way of quantifying this property that works for one-dimensional flows and is based on counting the number of space oscillations of a solution and proving that this number, an integer, does not go up in time. This property holds for a wide class of linear of nonlinear parabolic equations with suitable boundary conditions (so that new oscillations coming from the boundary are avoided).

*Intersection number*

We begin with the classical Sturm result for smooth solutions of one-dimensional linear parabolic equations (dating back to [485] from 1836).

**Definition 15.2** *For any continuous function $f(x)$, $x \in I = [a, b]$, the* intersection number *or* zero-crossing number $Z(f) = Z_I(f)$ *counts the sign changes of $f$ over $I$, and is precisely defined as the number of connected components of $\{x \in I :\ u(x,t) \neq 0\}$ minus one. Alternatively, $Z_I(f)$ is the supremum over all natural $k$ such that there exist $k+1$ points from $I$, $x_0 < x_1 < x_2 < \ldots < x_k$, satisfying*

$$f(x_j) \cdot f(x_{j+1}) < 0 \quad \text{for all } j = 0, 1, 2, \ldots, k-1. \tag{15.30}$$

*Consider now a linear parabolic equation*

$$u_t = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u \tag{15.31}$$

*posed in the rectangle $S = (a, b) \times (0, T)$. Given a constant $\tau \in (0, T)$, we denote by $\partial S_\tau$ the parabolic boundary of the domain $S_\tau = S \cap \{t < \tau\}$, i.e., the lateral sides and the bottom of the boundary of $S_\tau$. Given a solution $u$ defined on $S_\tau$, the positive and negative sets of $u$ are defined as follows:*

$$U_\tau^+ = \{(x,t) \in S_\tau,\ u(x,t) > 0\}, \quad U_\tau^- = \{(x,t) \in S_\tau,\ u(x,t) < 0\}. \tag{15.32}$$

*We state next the first sturm Theorem on sign changes.*

**Theorem 15.9** *Let $a$, $b$, $c$ be continuous, bounded and $a \geq \mu > 0$ in $S$. Let $u(x,t)$ be a solution of* (15.31) *in $S$ which is continuous on $\overline{S}$.*

(i) *Suppose that on $\partial S_\tau$ there are precisely $n$ (respectively $m$) disjoint intervals where $u$ is positive (resp. negative). Then $U_\tau^+$ (resp. $U_\tau^-$) has at most $n$ (resp. $m$) connected components in $S_\tau$ and the closure of each component must intersect $\partial S_\tau$ in at least one interval.*

(ii) *The intersection number of $u(\cdot, \tau)$ on $I$ is not greater than the number of sign changes of $u$ on $\partial S_\tau$. This number is $Z(\tau; u)$.*

A connected component is a maximal open connected subset. This version of the result follows Sattinger [471]. It admits natural extensions to the Cauchy problem or other problems in unbounded domains, under necessary assumptions on the initial and boundary data and functional setting, if we can control intersections of the solutions at infinity. What is more important, the result extends to semilinear and quasilinear parabolic equations since in the classical case the counting argument can be repeated, and in the case of continuous weak solutions the conclusion holds in the limit of the approximations with solutions of classical problems. But the argument is strictly one-dimensional.

In the particular case of the Cauchy problem for the PME posed in $\mathbb{R}$ or the Dirichlet or Neumann problems, no new components of the positivity of negativity set may arise at the lateral boundaries. We conclude

**Corollary 15.10** *For a solution of the PME in the above situations the number $Z(\tau, u)$ is non-increasing in time.*

Note the the profiles $u(t)$ of solutions of the PME typically have intervals where the profile vanishes. These intervals do not count as crossings; thus, the profile of the ZKB solution has $Z(t,u) = 0$ for all times and the dipole solution has $Z(t,u) = 1$ for all times. See more examples in Problems 15.8 and 15.9.

*Intersection comparison*

The intersection number count is very often applied to the difference of two solutions, which satisfies some kind of linearized equation. In this way, we control the intertwining of the graphs of the two solutions. The corollary applies also to these differences and defines a functional $Z(t; u_1, u_2) = Z(t; u_1 - u_2)$ that is non-increasing in time. Using this tool is usually referred to as intersection comparison.

**Caution** Care must be taken for signed solutions with growing oscillating behaviour as $x \to \infty$ since new oscillation can come from infinity!

*Intersections for radial solutions*

Note that this tool cannot be applied to general solutions in dimension $d > 1$, but the particular case of radially symmetric solutions is allowed since the Laplacian operator is then written as

$$\frac{1}{r^{N-1}}(r^{N-1}(u^m)_r)_r \tag{15.33}$$

which is compatible with the theorem, even if there is a singularity at $t = 0$. The result about monotonicity of the intersection number functional applies to radially symmetric solutions of the PME in $\mathbb{R}^d$.

*Lap number*

A closely related notion is the *lap number* defined by Matano [385] as the number of changes of monotonicity of a continuous and piecewise monotone $f$ as a function of $x \in D$. Let $k$ be a positive integer number such that we can choose $k + 1$ distinct points $a = x_0 < x_1 < x_2 < \ldots < x_k = b$ such that

$$(f(x_{i+1}) - f(x_i))\,(f(x_i) - f(x_{i-1})) < 0$$

for $i = 1, \ldots, k-1$. The lap number $l(f)$ is the supremum of the possible numbers $k$. It is understood that $l(f) = 0$ when $f$ is a constant function, and $l(f) = \infty$ when $f$ is not a piecewise monotone function. We can split $l(f)$ into $l^+(f)$ that counts those intervals $(x_i, x_{i+1})$ of increasing monotonicity and $l^-(f)$ that counts decreasing monotonicities. We have $l(f) = l^+(f) + l^-(f)$.

The definitions imply that $Z(f) \leq l(f)$. There is a close connection between both concepts for $C^1$ functions since the increasing or decreasing monotonicity of a function is equivalent to the positivity or negativity of the derivative. The monotonicity of the lap number of a solution of a quasilinear parabolic equation can be proved in this way. It applies in particular to the solutions of the one-dimensional PME and GPME.

**Examples** The ZKB solution has intersection number 0 and lap number 2, while the dipole solution has intersection number 1 and lap number 3. We ask the reader to imagine solutions with $Z = 0$ and $l$ large.

## 15.3 The interface

We now turn our attention to the existence and properties of free boundaries of non-negative solutions of the Cauchy problem. They are usually called *interfaces* in the one-dimensional setting. As in Chapter 14, $\mathcal{P} = \mathcal{P}_u$ denotes the positivity set of a solution $u \geq 0$, which is an open subset of $\mathbb{R}^{1+1}$.

### 15.3.1 *Generalities*

In order for a solution $u$ to have an interface, the initial data have to vanish somewhere. This can happen in three ways. Indeed, let

$$a = \operatorname{ess\,inf}\{x : \ u_0(x) > 0\}, \quad b = \operatorname{ess\,sup}\{x : \ u_0(x) > 0\}. \tag{15.34}$$

Then it can happen that $b < \infty$ and then $\mathcal{P}(t)$ will be bounded above for all $t > 0$ and an interface will appear to the right of $\mathcal{P}_u$ in the $(x,t)$-plane. We will describe such an interface in detail. Likewise, if $a > -\infty$, there will be an interface to the left of $\mathcal{P}$. Finally, if $u_0$ vanishes in some interval inside $[a, b]$, it will take a time to fill up this gap, during which inner interfaces occur.

The properties of these three types of interfaces are similar. Therefore, we will concentrate on the study of the *right-hand interface* $x = s(t)$, where

$$s(t) = \sup\{x : u(x,t) > 0\}, \quad t > 0. \tag{15.35}$$

We will write $s_r(t)$ when it is necessary to distinguish it from other interfaces. By the persistence property (Section 14.2), in order for $s(t)$ to be finite we need to assume that $b$ is finite. We recall that $u_0 \in L^1(\mathbb{R})$, $u_0 \geq 0$ and $\int u_0(x)dx = M > 0$. We first use comparison by shifting to estimate $s(t)$.

**Proposition 15.11** *Assume that $b < \infty$. Then the function $s : [0,\infty) \to [0,\infty)$ is continuous and non-decreasing. Moreover, $s(0) = b$ and*

$$b \leq s(t) \leq b + r(t) \tag{15.36}$$

*where $r(t) = c_m(M^{m-1}t)^{\frac{1}{m+1}}$ represents the interface of the ZKB solution with same mass centreed at $x = 0$. Moreover, if $a > -\infty$ we have*

$$a + r(t) \leq s(t). \tag{15.37}$$

**Proof** (i) Let us deal first with the upper estimate. We need the following consequence of the shifting comparison result.

**Lemma 15.12** *Let $u_1$ and $u_2$ be two solutions with initial data $u_{01}, u_{02}$ and right-hand interfaces $s_1(t), s_2(t)$, respectively. If $u_{01} \succ u_{02}$, i.e., if*

$$\int_x^\infty u_{01}(y)dy \le \int_x^\infty u_{02}(y)dy, \tag{15.38}$$

*then $s_1(t) \le s_2(t)$ for every $t > 0$. This holds in particular if $\int u_{01} = \int u_{02}$ and for every $x \in \mathbb{R}$*

$$\int_{-\infty}^x u_{01}(y)dy \ge \int_{-\infty}^x u_{02}(y)dy. \tag{15.39}$$

**Proof** If (15.38) holds, then for every $t > 0$

$$\int_x^\infty u_1(y,t)dy \le \int_x^\infty u_2(y,t)dy,$$

so that for $x > s_2(t)$ both integrals vanish, hence $u_2(s,t) = 0$ if $x > s_2(t)$, i.e., $s_1(t) \le s_2(t)$. The last assertion is immediate. ∎

(ii) Let us proceed with the proof of the upper estimate in the proposition. The idea is to displace (shift) the initial mass distribution $u_0$ to the right and concentrate the whole mass at $x = b$ to obtain a Dirac mass $\bar{u}_0(x) = M\,\delta(x-b)$. Of course, $u_0 \succ \bar{u}_0$ and both have the same mass. The desired conclusion would then follow from Lemma 15.12 if it were not for the fact that $\bar{u}_0 \notin L^1(\mathbb{R})$. This is not an essential difficulty and we can for instance extend our existence theory to cover measures as initial data (as we have done in Chapter 13) and Lemma 15.12 still holds.

Though this extension is interesting and immediate, we may avoid the effort at this stage. We may for instance use as comparison function a displaced and delayed ZKB of the form $\widehat{u}(x,t) = \mathcal{U}(x - b - r(\tau), t + \tau; M)$ for some $\tau > 0$, and then

$$\widehat{u}_0(x) = \mathcal{U}(x - b - r(\tau), \tau; M). \tag{15.40}$$

We have $u_0 \succ \widehat{u}_0$. By Lemma 15.12

$$s(t) \le \widehat{s}(t) = b + r(t+\tau) + r(\tau). \tag{15.41}$$

Now let $\tau \to 0$ to obtain $s(t) \le b + r(t)$.

(iii) The lower estimate is obtained in the same fashion by displacement of the whole mass to the left and comparison if the additional assumption $a > -\infty$ holds.

(iv) The fact that $s(t)$ is non-decreasing is a consequence of the property of expanding supports, Proposition 14.1. The fact that $s$ is continuous follows from

Theorem 14.6. Let us give a simple direct proof: we first show that $s$ is continuous at 0, i.e. $s(0+) = b$. Clearly, the limit exists since $s$ is monotone. Moreover $s(0+) \le b + r(0) = b$ by the previous comparison. Now if $s(0+) = c < b$ we arrive at a contradiction as follows: we know that

$$M_1 = \int_c^b u_0(x)dx > 0.$$

Take $u_{01} = u_0$ and $u_{02}(x) = M_1\delta(x - c)$. Then, (15.38) holds (reserved) so that

$$s(t) = s_1(t) \ge s_2(t) = r(t; M_1) + c,$$

hence $s(0+) \ge c$ and since $c$ is arbitrary less than $b$, we have $s(0+) \ge b$. Finally, for the continuity at a time $t_0 > 0$, we know by the monotonicity of $s$ that $s(t_0-) \le s(t_0) \le s(t_0+)$. Moreover, displacing the origin of time from $t = 0$ to $t = t_0$ allows us to apply the arguments above to show that $s(t_0+) = s(t_0)$. Finally, $s(t_0-) \ge s(t_0)$ follows from the continuity of $u$. ■

The solution does indeed propagate outside its initial support and, in fact, it penetrates into the whole positive axis: $x > 0$. Moreover, for large times the penetration proceeds at the rate of the fundamental solution with same mass:

**Proposition 15.13** *If the support of $u_0$ is bounded above, then as $t \to \infty$ we have*

$$\lim_{t\to\infty} \frac{s(t)}{r(t)} = 1, \tag{15.42}$$

*where $r(t)$ is the Barenblatt radius as in* (15.36). *If $u_0$ is compactly supported, then we have the more precise estimate: $s(t) = r(t) + O(1)$.*

**Proof** The proof is immediate from estimates (15.36), (15.37) when the support of $u_0$ is contained in the compact interval $[a, b]$. Actually, we get

$$|s(t) - r(t)| \in [a, b] = O(1).$$

When $a = -\infty$ we argue by approximation. We only have to show that $\liminf(s(t)/r(t)) \ge 1$. Let $\varepsilon > 0$. There exists $x_1 < 0$ such that

$$\int_{-\infty}^{x_1} u_0(y)dy < \varepsilon.$$

Consider now the fundamental solution $\widehat{u}(x, t) = \mathcal{U}(x - x_1, t; M - \varepsilon)$. It is clear that $\widehat{u}_0 \succ u_0$, therefore

$$s(t) \ge \widehat{s}(t) = x_1 + c_m((M - \varepsilon)^{m-1}t)^{\frac{1}{m+1}},$$

from which our result follows letting $t \to \infty$ and then $\varepsilon \to 0$. ■

The convergence of $s(t)$ towards $r(t)$ will be examined in Chapter 18 in the context of several space variables. We continue with the behaviour of the solution while it expands.

**Proposition 15.14** *In the domain $\{b < x < s(t), t > 0\}$ the solution is positive and monotone non-increasing in $x$, i.e. $u_x \le 0$.*

**Proof** We use an interesting technique, the *reflection principle*, already described in Section 9.6.2. In fact, we consider for $a \ge 0$ the domain $R = (-\infty, a) \times (0, \infty)$ and the solutions $u_1(x,t) = u(x,t)$ and $u_2(x,t) = u(2a-x,t)$. (Here we use the symmetry of the equation.) Since $u_1(x,0) \ge u_2(x,0) \equiv 0$ and $u_1 = u_2$ on the lateral boundary $x = a, 0 \le t < \infty$, we conclude from parabolic comparison that $u_1(x,t) \ge u_2(x,t)$ in $R$, i.e. $(y = a - x)$

$$u(a-y,t) \ge u(a+y,t)$$

for every $y > 0$. This proves that $u$ is monotone. By our definition (1.2) $u(x,t)$ is then positive for $0 < x < s(t)$. ∎

### 15.3.2 *Left-hand interface and inner interfaces*

When $a > -\infty$ we may define the *left-hand interface*

$$s_l(t) = \inf\{x : u(x,t) > 0\}, \quad t > 0. \tag{15.43}$$

The properties are similar but symmetric under the map $x \mapsto -x$.

There can also appear *inner interfaces*. Let us explain the situation with a simple case. We assume that the initial function $u_0$ is continuous and has the following structure: there are two intervals $I_1 = (a_1, b_1)$ and $I_2 = (a_2, b_2)$ where $u_0 > 0$, while $u_0(x) = 0$ otherwise. Let $-\infty < a_1 < b_1 < a_2 < b_2 < \infty$. Let us analyse the form of the solution and interfaces for this problem.

We can then solve the initial-value problem for initial data $u_0^{(1)}$ which takes only the part contained in $I_1$ different from zero, and obtain a solution $u^{(1)}(x,t)$ with interfaces $s_l^{(1)}(t) < s_r^{(1)}(t)$. In the same way, we can solve the problem with initial data $u_{02}$ supported in $I_2$ to get a solution $u^{(2)}(x,t)$ with interfaces $s_l^{(2)}(t) < s_r^{(2)}(t)$. Now the following important observation applies: until $s_r^{(1)}(t)$ (which travels to the right) meets $s_l^{(2)}(t)$ (which travels to the left) both solutions have disjoint supports, so that the solution with data $u_0 = u_{01} + u_{02}$ is just

$$u(x,t) = u^{(1)}(x,t) + u^{(2)}(x,t).$$

This time $t_i$ must be finite (question to the reader: why?). We easily conclude that in the time interval $0 < t < t_i$ the free boundary of $u$ is composed of these four connected components, $s_l^{(1)}(t) < s_r^{(1)}(t) < s_l^{(2)}(t) < s_r^{(2)}(t)$. The two intermediate ones are called the inner interfaces. They become connected to each other at the time $t_i$ at which they meet (they must, by the property of full

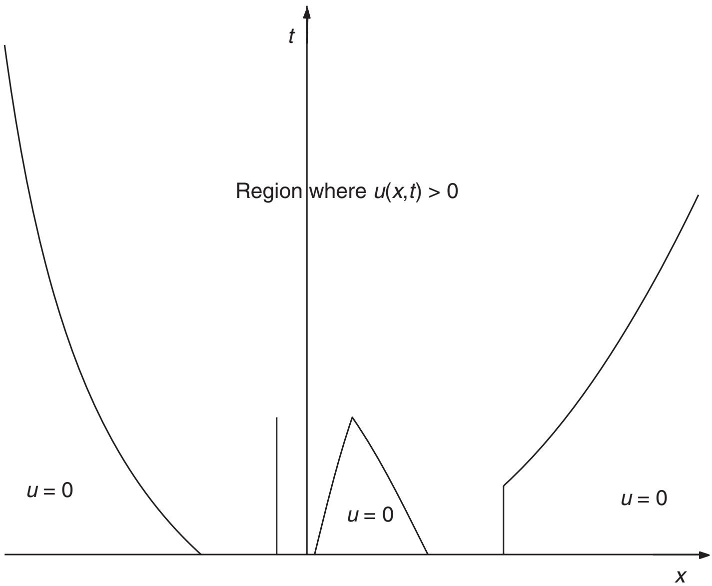


Figure 15.2: Different interfaces for a PME solution.

expansion, prove it). For $t > t_i$ we have

$$u(x,t) \geq \max\{u^{(1)}(x,t), u^{(2)}(x,t)\}$$

(by the maximum principle) without equality sign; only the two extreme interfaces survive, and the two inner pieces disappear.

This situation is easy to generalize. Many different geometries may occur: we can have for instance more than two intervals of positivity, even infinitely many intervals. More general solutions can have for a time at most a countable number of connected components of the free boundary, which we call elementary interfaces. We can also have a situation with inner interfaces but no external interfaces. However, there is no essential novelty with respect to the previous arguments. Indeed, it is clear that all inner interfaces can be considered either as left-hand interfaces or right-hand interfaces depending on their position with respect to the adjoining positivity set. The case where they are both things at the same time corresponds to a non-zero waiting time at an isolated point.

### 15.3.3 *Waiting time*

We have introduced the concept of waiting time in the general several-dimensional setting in Section 14.3. But we will develop the subject from scratch in this more elementary context.

We consider solutions of the initial value problem for the PME which vanish in a certain interval. The degeneracy of the equation not only causes finite propagation and the corresponding interfaces. It may even occur that the interface is stationary for a while if the mass distribution near the border of the support is very small. This is a metastable state; after a certain finite time, called the *waiting time*, the interface will begin to move.

Without loss of generality we may assume that the point under question is $x = 0$ and that the right-hand interface starts there. Similar argument applies when $x = 0$ is the starting point of the left interface or an inner interface. We want to characterize the occurrence of a positive waiting time

$$t^* = \sup\{t \geq 0 : s(t) = 0\}. \tag{15.44}$$

This is the result that gives also a quantitative estimate.

**Theorem 15.15** *The waiting time is always finite. In the above situation, it is positive if and only if*

$$\int_x^0 u_0(y)dy = O(|x|^{\frac{m+1}{m-1}}) \tag{15.45}$$

*as $x \uparrow 0$. More precisely, there exist constants $T_2 > T_1 > 0$ depending only on $m$, such that*

$$\frac{T_1}{A^{m-1}} \leq t^* \leq \frac{T_2}{A^{m-1}}, \tag{15.46}$$

*where $A = A(u_0)$ is given by*

$$A(u_0) = \sup_{x<0} \left( |x|^{-\frac{m+1}{m-1}} \int_x^0 u_0(y)dy \right). \tag{15.47}$$

**Proof** (i) The lower bound in (15.46) relies on comparison with the explicit quadratic solution whose pressure is given by

$$w(x,t) = \begin{cases} \dfrac{x^2}{C - 2(m+1)t}, & \text{for } x < 0 \\ 0 & \text{for } x \geq 0, \end{cases} \tag{15.48}$$

which exist for $0 < t < T(C) = C/2(m+1)$ and blow up as $t \uparrow T(C)$. Observe that the interface for $w$ is stationary during its entire lifetime. To begin with, standard comparison with $w$ produces the following result.

**Lemma 15.16** *Let $u$ be a solution whose initial pressure $u_0$ vanishes for $x \geq 0$ and satisfies*

$$v_0(x) \leq a|x|^2 \quad \textit{for } x < 0. \tag{15.49}$$

*Then, $v(x,t) \leq w(x,t)$ for $0 < t < T(C)$ with $C = 1/a$. In particular,*

$$t^* \geq \frac{1}{2(m+1)a}. \tag{15.50}$$

If we calculate $A(u_0)$ for $v_1(x) = ax^2$ we find

$$A^{m-1} = a\frac{m-1}{m}\left(\frac{m-1}{m+1}\right)^{1/(m-1)}, \tag{15.51}$$

so that the lower bound (15.46) holds in this case with

$$T_1 = \frac{1}{2m}\left(\frac{m-1}{m+1}\right)^{m/(m-1)}. \tag{15.52}$$

For a general $u$ such that $A(u_0) < \infty$ we use shifting comparison. We consider the solution $\widehat{u}$ with initial pressure $\widehat{v}_0$ such that

$$\widehat{v}_0(x) = 0 \quad \text{for} \quad x > 0, \qquad \widehat{v}_0(x) = a|x|^2 \quad \text{for} \quad x_1 < x < 0,$$

with $a$ given by (15.51) and $x_1 \ll 0$, and $\widehat{v}_0(x) = 0$ for $x < x_1$. Then, assumption (15.45) implies that $u_0 \prec \widehat{u}_0$, so that $t^*(u) \geq t^*(\widehat{u})$, the latter being estimated below by (15.50).

(ii) For the upper bound in (15.46) we take any $x_1 < 0$ and consider the solution $\widehat{u}$ with initial data $\widehat{u}_0(x) = M(x_1)\delta(x - x_1)$, where

$$M(x_1) = \int_{x_1}^{0} u_0(y)dy.$$

Then $\widehat{u}_0 \succ u_0$ so that $s(t) \geq \widehat{s}(t) = x_1 + c_m(M(x_1)^{m-1}t)^{\frac{1}{m+1}}$. This gives an estimate for $t^*$ of the form

$$t^* \leq \frac{|x_1|^{m+1}}{c_m^{m+1}M(x_1)^{m-1}}. \tag{15.53}$$

Taking the infimum with respect to $x_1$ in the second member we obtain the upper bound of (15.46), with

$$T_2 = \frac{1}{c_m^{m+1}} = \frac{m-1}{2m(m+1)}B\left(\frac{m}{m-1}, \frac{1}{2}\right)^{m-1}.$$

The accuracy of estimate (15.46) depends on the ratio $\mu(m) = T_2/T_1$. It follows from (15.52), (15.3.3) that $\mu \to 1$ as $m \to 1$, indeed $\mu(1+\varepsilon) = 1 + 0(\varepsilon \log \varepsilon)$ as $\varepsilon \to 0$. On the contrary $\mu$ grows like $2^m$ as $m \to \infty$. Observe that both $T_1$ and $T_2$ correspond to particular shapes, the former square quadratic, with concentrated mass away from the interface the latter. Hence, both constants are sharp. ■

Nevertheless, an exact computation of the waiting time is possible for initial data with a particular shape.

**Proposition 15.17** *Let $v_0(x) = 0$ for $x > 0$ and*

$$ax^2 \geq v_0(x) \geq ax^2 + 0(x^2), \quad a > 0, \tag{15.54}$$

*for $x < 0$. Then* (15.50) *is exact, i.e.*

$$t^* = \frac{1}{2(m+1)a}. \tag{15.55}$$

*Moreover, as $x \uparrow 0$*

$$v(x,t) = \frac{x^2}{2(m+1)(t^* - t)} + 0(x^2), \tag{15.56}$$

*locally uniformly in $t \in (0, t^*)$.*

**Proof** (i) Let $T = 1/(2(m+1)a)$. Only the inequality $t^* \le T$ needs to be shown in (15.55) since $t^* \ge T$ follows from (15.50). We may also consider that $v_0$ is continuous and that the function $h(x) = a - v_0(x)x^{-1}$ is decreasing as $x \uparrow 0$, the general case being then proved by approximation and use the maximum principle.

(ii) Under the stated assumptions, we consider the family of rescaled solutions of the PME $\{u_\lambda\}$ with pressures defined by

$$v_\lambda(x,t) = \lambda^2 v\left(\frac{x}{\lambda}, t\right), \quad \lambda > 0. \tag{15.57}$$

Then, $v_\lambda(x,0)$ converges as $\lambda \to \infty$ to the function $a|x|^2$ for $x < 0$ and zero for $x > 0$. Local estimates allow us to pass to the limit in the solutions and get

$$\lim_{\lambda\to\infty} v_\lambda(x,t) = V(x,t) = \frac{a|x|^2}{1 - 2(m+1)at}$$

for $x < 0$ and $0 < t < T$, and the convergence is uniform on compact subsets. Hence,

$$|x|^{-2} v(x,t) \to \frac{a}{1 - 2(m+1)at} = \frac{1}{2(m+1)(T-t)}$$

uniformly in intervals of the form $\varepsilon < t < T - \varepsilon$, $\varepsilon > 0$.

(iii) To end the proof, we show that $t^* \le T$. For that we take $\varepsilon > 0$ and apply Theorem 15.15 to the solution $u_\varepsilon(x,t) = u(x, t + T - \varepsilon)$. Calculating $A(u_\varepsilon(0))$ from (15.47) and substituting into (15.46) gives

$$t^*_\varepsilon \le T_2/A(u_\varepsilon)^{m-1} = 0(\varepsilon).$$

Therefore, $t^* = t^*_\varepsilon + T - \varepsilon \le T + 0(\varepsilon)$. Letting $\varepsilon \to 0$, we conclude. ∎

A simple application of the shifting comparison allows to obtain an upper bound for general solutions. Let

$$L(u_0) = \liminf_{x\uparrow 0} \left( |x|^{-\frac{m+1}{m-1}} \int_x^0 u_0(y)dy \right). \tag{15.58}$$

Clearly, $0 \le L(u_0) \le A(u_0)$. Then,

**Corollary 15.18** *If $L(u_0) > 0$ we have*

$$t^* \le \frac{T_1}{L^{m-1}}. \tag{15.59}$$

Estimate (15.59) relies only on the values of $v_0(x)$ for $x$ close to 0, it is a *local* estimate for $t^*$. Notice that (15.46) and (15.59) give an exact value for the waiting time whenever $A(u_0) = L(u_0)$. If, on the contrary, $L$ is an actual limit as $x \to 0$ and estimate (15.59) fails, this can only be due to the influence of $u_0(x)$ for $x$ far from 0, which is reflected in $t^*$ through $A(u_0)$ and formula (2.3). We may then speak of a waiting time determined by the *global* shape of $u_0$. The most clear example of this is a big lump of mass concentrated near a point $x_1 \ll 0$ together with a thin density $u_0(x) > 0$ in $x_1 < x < 0$ such that $L(u_0) = 0$. This solution will have a waiting time roughly given by (15.46)-right. Such a situation is studied in [45].

## 15.4 Equation of the interface and Lipschitz continuity

We now address some of the main issues of the theory, the Lipschitz continuity of the interface and the Darcy law. We continue to work with the right-hand interface $x = s(t)$ for definiteness. The next results are formulated in terms of one-sided partial derivatives in space and time, $D_x^{\pm} v$, $D_t^{\pm} v$, as well as $D_t^{\pm} s(t) = D^{\pm} s(t)$. We will also use the simpler notations $D_x^- v(s(t),t) = v_x(s(t)-,t)$ and so on.

**Theorem 15.19** *For every $t > 0$ there exist the limits*

$$D_x^- v(s(t),t) = \lim_{x \uparrow s(t)} v_x(x,t), \quad D_t^+ s(t) = \lim_{h \downarrow 0} \frac{1}{h}[s(t+h) - s(t)]. \tag{15.60}$$

*Moreover, the Darcy law holds in the form*

$$D_t^+ s(t) = -D_x^- v(s(t),t) \tag{15.61}$$

*The equation is also valid at $t = 0$ if $D_x^- v_0(s(0))$ exists.*

**Proof** The existence of the first limit is a simple application of the fundamental estimate $v_{xx} \ge -C/t$. In fact, this estimate implies that

$$\varphi(x,t) = v(x,t) + \frac{Cx^2}{2t}$$

is a convex function of $x$ for a.e. $t > 0$. Since $\varphi$ is continuous, convexity holds for all $t > 0$. Therefore, for every $x_0 \in \mathbb{R}$, $t_0 > 0$ the function $\varphi(\cdot, t_0)$ admits one-sided derivatives at $x_0$, $\varphi_x(x_0+, t_0)$ and $\varphi_x(x_0-, t_0)$, the functions $x \mapsto \varphi_x(x_0\pm, t_0)$ are non-decreasing and $\varphi_x(x_0-, t_0) \le \varphi_x(x_0+, t_0)$. Consequently, there exist $v_x(x_0+, t_0)$, $v_0(x_0-, t_0)$ and $v_x(x_0-, t_0) \le v_0(x_0+, t_0)$. Observe that $v_x(s(t_0)+, t_0) = 0$ and $v_x(s(t_0)-, t_0) \le 0$.

To prove existence of $D^+s(t)$ and the equation (15.61) we use comparison in a neighbourhood of a point of the interface $P_0 = (x_0, t_0), t_0 > 0$, $x_0 = s(t_0)$, with a linear pressure wave of the form

$$L_c(x,t) = c(c(t-t_0) - (x-x_0))_+$$

for suitable $c > 0$ (see Section 4.3). We proceed as follows: let $\gamma = v_x(x_0-, t_0)$. Given $\varepsilon > 0$, we will compare $v(x,t)$ and $z_0(x,t)$ with $c = \gamma + \varepsilon$ in a domain of the form $R = R_{\delta,\tau} = \{x_0 - \delta < x < x_0 + \delta, t_0 < t < t_0 + \tau\}$ with $\delta, \tau > 0$. Now, if $\delta$ is small enough it happens that $v \le L_c$ on the base segment $\{x_0 - \delta \le x \le x_0 + \delta, t = t_0\}$. Even more, $v(x_0 - \delta, t_0) \le L_c(x_0 - \delta, t_0)$. Fixing $\delta$ and choosing now $\tau$ small we will have $v \le z_c$ on the left-hand boundary (by continuity), while on the right-hand boundary (since $s(t)$ is continuous, $v(x_0 + \delta, t) = 0$ for $t_0 \le t \le t_0 + \delta$).

Therefore, by the local comparison result, which is part of Theorem 5.14, we have $v(x,t) \le L_c(x,t)$ in $R$. In particular, the interface of $v$, $x = s(T)$, lies to the left of that of $L_c$, i.e.

$$x(t) \le x_0 + (\gamma + \varepsilon)(t - t_0).$$

Since $s(t_0) = x_0$, we get $s(t) - s(t_0) \le (\gamma + \varepsilon)(t - t_0)$, hence dividing by $h = t - t_0$ and letting first $h \downarrow 0$ and then $\varepsilon \downarrow 0$ gives

$$\limsup_{h \downarrow 0} \frac{1}{h} \left[ s(t_0 + h) - s(t_0) \right] \le \gamma. \tag{15.62}$$

If $\gamma = 0$, (15.62) implies that $D^+s(t_0) = 0$ and we are done. If $\gamma > 0$ we may repeat the above comparison argument to prove that $L_c \le v$ in a domain $R_{s,\tau}$ if $c = \gamma - \varepsilon > 0$ and $s, \tau$ are small. In this way we get

$$\liminf_{h \downarrow 0} \frac{1}{h} \left[ s(t_0 + h) - s(t_0) \right] \ge \gamma,$$

which completes the proof of (15.61). It is clear that the argument works at $t = t_0$ if $v_x(0-, 0) = v_0'(0-)$ exists. ■

Combining equation (15.61) with estimates (15.5), (15.7), we obtain

**Corollary 15.20** *The interface function $s(t)$ is Lipschitz continuous for $t \ge \tau > 0$. Precisely,*

$$|D_t^+ s(t)| \le \left( \frac{2\|v(\cdot,t)\|_\infty}{(m+1)t} \right)^{1/2}, \quad |D^+s(t)| \le C(m)\|u_0\|_1^\lambda \, t^{-\mu} \tag{15.63}$$

*with $\lambda = (m-1)/(m+1)$, $\mu = m/(m+1)$.*

### 15.4.1 *Semiconvexity*

Before we proceed with the proof of regularity, we introduce the following geometric property that improves on the Lipschitz continuity. It is most easily formulated if we reparametrize time in a power way.

**Lemma 15.21** *The function* $S(\tau) = s(\tau^{m+1})$ *is convex in* $(0, \infty)$.

**Proof** We begin by choosing a point on the interface $P_0 = (x_0, t_0)$, $x_0 = s(t_0)$ where $D^+s(t_0) = \gamma > 0$. We then insert a Barenblatt solution under the profile $u(\cdot, t_0)$ having first-order contact at $P_0$ (on the left). This is done as follows: we consider the solution $\widehat{u}(x,t) = U(x - x_1, t; M)$, where $U$ is given by formula (1.8), and determine the centre $x_1$ and mass $M$ by means of the contact conditions

$$\text{(i)}\ \ \widehat{u}(x_0, t_0) = u(x_0, t_0) = 0;$$

$$\text{(ii)}\ \ \widehat{v}_x(x_0-, t_0) = v_x(x_0-, t_0) = -\gamma.$$

Since $r(t) = c(M^{m-1}t)^{1/(m+1)}$ and $\widehat{v}_x(x_0-, t_0) = r'(t_0)$, the second condition determines $M$, while (i) implies that $x_1 = x_0 - r(t_0)$. We now supplement (i), (ii) with the observation that whenever $\widehat{v}(\cdot, t_0) > 0$ we have

$$\text{(iii)}\ v_{xx}(\cdot, t_0) \geq -\tfrac{1}{(m+1)t_0} = \widehat{v}_{xx}(\cdot, t_0).$$

These three conditions imply at once that $v(x, t_0) \geq \widehat{v}(x, t_0)$ for every $x \in \mathbb{R}$. By the comparison theorem, we have

$$v(x,t) \geq \widehat{v}(x,t) \qquad \text{for } x \in \mathbb{R}, t \geq t_0.$$

Consequently, the interfaces satisfy

$$s(t) \geq \widehat{s}(t) = x_1 + r(t) = s(t_0) + r(t) - r(t_0)$$

for $t \geq t_0$, i.e. lies to the right of $r$. We recall that by (i), (ii) and (15.61), $s(t)$ and $r(t)$ have first-order contact at $P_0$. Replace now $t$ by $\tau = t^{1/(m+1)}$ and observe that

$$D^+S(\tau_0) = (m+1)\tau_0^m D^+s(\tau_0^{m+1}) = (m+1)t_0^{\frac{m}{m+1}} r'(t_0) = cM^{\frac{m-1}{m+1}}.$$

We finally get for $\tau \geq \tau_0$

$$S(\tau) - S(\tau_0) \geq D^+S(\tau_0)(\tau - \tau_0). \tag{15.64}$$

Hitherto we have assumed that $\gamma > 0$. However, for $\gamma = 0$ (in particular, if $0 < t < t^*$ when $t^* > 0$) $D^+S(\tau_0) = 0$ and (15.64) reduces to the monotonicity statement $S(\tau) \geq S(\tau_0)$ for $\tau \geq \tau_0$. Consequently, (15.64) is valid for all $\tau_0 > 0$.

Condition (15.64) means in geometrical terms that $S$ lies to the right of its one-sided tangent $D^+S(\tau_0)(\tau - \tau_0) + S(\tau_0), \tau \geq \tau_0$. This property characterizes $S$ as a convex function. ∎

A convex function has one-sided derivatives $D^{\pm}S(\tau)$ for all $\tau > 0$ and $D^-S(\tau_1) \leq D^+S(\tau_1) \leq D^+S(\tau_2)$ for $0 < \tau_1 < \tau_2$. It follows that $D^{\pm}s(t)$ exists for all $t > 0$ and

$$D^- s(t_1) \le D^+ s(t_1) \le (t_2/t_1)^{\frac{m}{m+1}} D^- s(t_2). \tag{15.65}$$

We observe that proving $C^1$ continuity is reduced to showing that $D^- s(t) = D^+ s(t)$, which may fail at most for a countable set. On the other hand, since

$$S''(\tau) = (m+1)^2 \tau^{2m} s''(\tau^{m+1}) + m(m+1)\tau^{m-1} s'(\tau^{m+1}),$$

the convexity of $S$ can be equivalently formulated as:

**Proposition 15.22** (Semiconvexity of the interface) *On any right-hand interface we have*

$$s''(t) + \frac{m}{(m+1)t} s'(t) \ge 0. \tag{15.66}$$

*This holds in the sense of distributions (measures). The inequality is reversed for left-hand interfaces.*

We remark that (15.66) holds with exact equality for the Barenblatt solutions, since their interface is $r(t) = c(m, M)\, t^{\frac{1}{m+1}}$. The corresponding function $S(\tau)$ is a straight line. In fact Lemma 15.21 is proved by comparing with fundamental solutions and using this remark.

We end this section with a consequence of semiconvexity:

**Corollary 15.23** *For every $t > t^*$ we have $D^\pm s(t) > 0$.*

**Proof** If $t > t^*$ there are necessarily times $t_1 \in (t^*, t)$ such that $D^+ s(t_1) > 0$. Use (15.65) to conclude. ■

In terms of motion, this says that *once the interface starts to move it never stops.* This means that no later metastable situations will occur after the initial waiting time has elapsed.

## 15.5 $C^1$ regularity

Our next step is to establish the $C^1$ regularity of $s(t)$ and $v(x,t)$ on the moving interface for times $t > t^*$ after the waiting time. The main idea is to show that the solution behaves in an approximately linear way near any moving interface point.

### 15.5.1 *Local linear behaviour and $C^1$ regularity near moving points*

We call the piece of interface $\{(s(t), t) : t > t^*\}$ the moving (righ-hand) interface. We are going to show that it looks locally like a linear pressure wave (4.18) of the form $v(x,t) = c(a + ct - x)_+$ with velocity $c = s'(t) > 0$.

In order to perform the local analysis we consider a typical point $P_0 = (x_0, t_0)$ such that $t_0 > t^*$, $x_0 = s(t_0)$. By an inner neighbourhood of $P_0$ we understand a set of the form $N_\varepsilon = N_\varepsilon(P_0) = Q_\varepsilon(P_0) \cap \{u > 0\}$, where $Q_\varepsilon(P_0) = (x_0 - \varepsilon, x_0 + \varepsilon) \times (t_0 - \varepsilon, t_0 + \varepsilon)$ and we assume that $t_0 - \varepsilon < t^*$. An inner neighbourhood is informally referred as a one-sided neighbourhood ('on the side of the fluid'). Let us put $\gamma = D_t^+ s(t_0) > 0$.

**Theorem 15.24** *In an inner neighbourhood of any point $P_0 = (x_0, t_0)$, $x_0 = s(t_0)$, such that $t_0 > t^*$, we have the linear approximation of the pressure*

$$v(x,t) = L_\gamma(x - x_0, t - t_0) + o(|x - x_0| + |t - t_0|), \tag{15.67}$$

*where $L_\gamma$ is the linear front*

$$L_\gamma(x,t) = \gamma(\gamma t - x)_+. \tag{15.68}$$

*We have*

$$D^+ s(t_0) = D^- s(t_0), \tag{15.69}$$

*hence, $\gamma = s'(t_0)$, which implies the $C^1$ continuity of $s$. Moreover, $v_x$ and $v_t$ admit limits as $(x,t) \to (x_0, t_0)$ with $x < s(t)$, so-called inner limits. We have the fundamental interface relations:*

$$s'(t_0) = -v_x(P_0) \quad \text{and} \quad v_t(P_0) = v_x^2(P_0), \tag{15.70}$$

*valid for all $t_0 > t^*$, where the values of $v_x(P_0)$ and $v_t(P_0)$ are understood as the inner limits.*

**Proof** The proof is split into several steps, and the main idea is the blow-up technique that we explain next.

(i) We perform the scaling

$$v_\delta(x,t) = \frac{1}{\delta} v(x_0 + \delta x, t_0 + \delta t) \tag{15.71}$$

with parameter $0 < \delta < 1$. The idea is to perform an invariant scaling transformation with the parameter $\delta$ around an interface point $(x_0, t_0)$ and then let $\delta \to 0$ (blow-up) in order to study the situation there. It is easy to see that $v_\delta$ satisfies the same PME equation but it is defined in the rescaled domain $D_\delta = \mathbb{R} \times (-(t_0 - t^*)/\delta, \infty)$. A key point of this transformation is that it has the same uniform estimate for the first derivatives since $v_{\delta,x} = v_x$ and $v_{\delta,t} = v_t$. As for the second derivative, we have

$$v_{\delta,xx}(x,t) \geq -\frac{\delta}{(m+1)(\delta t + t_0)}, \tag{15.72}$$

where the right-hand side goes to zero as $\delta \to 0$, an important fact in what follows. Moreover, by the interface equation (15.61) the right-hand interface $x = s_\delta(t)$ for $v_\delta$ passes through the origin with (the right-hand) velocity

$$D^+ s_\delta(0) = \gamma.$$

By the semiconvexity result, we also know that there exists the left velocity $D^- s_\delta(0) = D^- s(t_0) = \gamma_1$, which can be in principle different from $\gamma$ (this is precisely the problem to overcome). Moreover,

$$\lim_{t\uparrow 0} D^+ s_\delta(t) = D^- s_\delta(0) = \gamma_1, \quad \lim_{t\downarrow 0} D^+ s_\delta(t) = \gamma. \tag{15.73}$$

The interface equation relates these velocities with the slopes at the interfaces.

(ii) Next, we pass to the limit as $\delta \to 0$. We use the Lipschitz continuity of solutions and interfaces together with known interior regularity to conclude that the family $v_\delta$ is compact in a Hölder $(x,t)$-space on compact subsets, so that along a sequence it converges to a certain limit

$$V(x,t) = \lim_{\delta_n \to 0} v_{\delta_n}(x,t). \tag{15.74}$$

It is clear that $V$ is a continuous and non-negative weak solution of the pressure equation in the whole plane $\{-\infty < x, t < \infty\}$, and that it has at $t = 0$ the linear profile

$$V(x,0) = [-\gamma x]_+. \tag{15.75}$$

On the other hand, the second derivative estimate passes to the limit as

$$V_{xx} \ge 0. \tag{15.76}$$

We want to conclude that $V(x,t)$ coincides with $L_\gamma(x,t)$ which ends the proof of the linear behaviour in first approximation. The conclusion is immediate for $t > 0$ by uniqueness of weak solutions, so we only have to check what happens for $t < 0$.

(iii) Let us discuss the limit $V$ for $t < 0$. This is a kind of backward uniqueness result. We apply a simple geometric approach based on the semiconvexity estimate. Using (15.73) and the convexity it is easy to see that for $t < 0$ the solution $V$ is positive in the region $x - \gamma_1 t < 0$ with interface $x = \gamma_1 t$ and it has there a slope

$$D_x^- V(\gamma_1 t, t) = -\gamma_1,$$

and in view of the convexity a lower estimate holds

$$V(x,t) \ge L_{\gamma_1}(x,t), \quad t < 0. \tag{15.77}$$

This immediately implies in the limit $t \to -0$ that $\gamma_1 \le \gamma$. We will prove later that equality holds. For the moment we continue as follows. It is known (and proved by approximation) that the derivative $v_x$ of the solutions of the PME in $d = 1$ satisfies the maximum principle. Since $V_x$ attains the value $-\gamma$ at $t = 0$, it follows that $\|V_x(\cdot,t)\|_\infty$ is equal or larger than this value for all $t < 0$. Now, the function $V(x,t)$ is convex in $x$ for every $t$ so that the maximum value of $-V_x$ is taken at $x = -\infty$. It must be precisely $\gamma$, not larger, otherwise $V$ would resemble a travelling wave with a larger speed and the interface would eventually move faster than $x = \gamma t$, a contradiction. It follows that

$$\lim_{x\to-\infty} V_x = -\gamma. \tag{15.78}$$

Suppose now that equality holds, $\gamma_1 = \gamma$. Since $V_x$ starts with the value $-\gamma_1$ at the interface and it is monotone and ends with value $-\gamma$, we conclude that $V_x$ is constant on the positivity set of $V$ when $\gamma = \gamma_1$, so that $V = L_\gamma$ and the proof is complete when we undo the scaling.

(iv) Next, let us examine the possibility $\gamma_1 < \gamma$ and we arrive at a contradiction. In that case we still have (15.77) and (15.78), and

$$\gamma_1 \le -v_x \le \gamma$$

for $x < \gamma_1 t$, $t < 0$. Let us prove next that

$$V(x,t) \ge L_\gamma(x,t) \tag{15.79}$$

for $t < 0$. We need another estimate, this time on $v_{tt}(x,t)$. It takes the form

$$|v_{tt}| \le \frac{C}{|x - x_0|},$$

valid on small triangular regions of the form

$$R = \{(x,t) :\ x_0 - \delta < x < x_0,\ 0 < 2\gamma_1(t_0 - t) < (x_0 - x)\}$$

with $\delta > 0$. This comes from the scaling arguing by contradiction on a sequence of points $(x_n, t_n) \to (x_0, t_0)$ in $R$, putting $\delta_n = x_0 - x_n$, using the uniform convergence of the $v_\delta$ and their derivatives and the equality

$$v_{\delta,tt} = \delta v_{tt}.$$

Once this is established, we can derive the estimate

$$|v_{\delta,tt}| \le \frac{C}{|x|}$$

on growing triangles, and in the limit the same inequality is true for $V$ in an infinite triangle backwards in time. Using Taylor's formula at a point $(x,t)$ with $x < 2\gamma_1\, t < 0$, we get

$$V(x,t) = V(x,0) + tV_t(x,0) + \frac{t^2}{2}\, V_{tt}(x,s).$$

The last term is bounded by $C\,t^2/x$. The other two can be computed explicitly at $t = 0$ and amount to the travelling wave. We conclude that

$$V(x,t) - L_\gamma(x,t) \ge -\frac{C}{|x|}.$$

Taking the limit as $x \to -\infty$ we see that it goes to zero. Since the function is convex, it must be non-negative everywhere.

The final step of the linear front analysis consists in proving that equality holds in (15.79). This is an easy consequence of the strong maximum principle since both members are solutions of the pressure equation of the PME, which is uniformly parabolic with smooth coefficients in the positivity domain $\{v \ge \varepsilon > 0\}$. Since both solutions coincide for $t \ge 0$ and they are ordered, they must also coincide before. As a summary, this means that the 'initial' data (15.75) prescribe the unique TW solution

$$V(x,t) \equiv [-\gamma(x-\gamma t)]_+$$

for any $t \approx 0$.

(v) The continuity of $s'(t)$ follows from the equality $D^+s(t_0) = D^-s(t_0)$ plus the two partial statements derived from the semiconvexity:

$$D^-s(t_0) = \lim_{t \uparrow t_0} D^+s(t), \qquad D^+s(t_0) = \lim_{t \downarrow t_0} D^+s(t).$$

(vi) The $C^1$ smoothness of the pressure away from the interfaces is standard since the equation is parabolic non-degenerate there. Near the interfaces it follows from the blow-up argument of the previous theorem, as does the fact that $v\, v_{xx} \to 0$ as $(x,t)$ approaches the interface, hence the limit equations (15.70). They represent a more classical form of the interface dynamics derived in the previous section. ■

**Remark**

(1) We recall that the pressure has discontinuous gradient across the interfaces, the values involved in formula (15.70) are the inner limits, taken from the positivity set $\{v > 0\}$.

(2) The linear front approximation and the $C^1$ regularity extend by symmetry to moving points on a left-hand interface, and also to moving points in inner interfaces (note that at the merging point of two interfaces, we can only think of interfaces backwards in time).

### 15.5.2 *Limited regularity. Interfaces with a corner point*$^\star$

We prove here that Lipschitz continuity is the best possible global regularity for general interfaces of the Cauchy problem by exhibiting solutions for which $t^* > 0$ and $D^+s(t^*) > 0$ while of course $D^-s(t^*) = 0$, so that $s$ is not differentiable at $t^*$. Since we have just proved $C^1$ regularity of the interfaces of non-negative solutions of the one-dimensional Cauchy problem for times after the waiting time (and we will improve it below into $C^\infty$ smoothness), and the interface is given by a constant function from $t = 0$ to the waiting time $t^*$, the only possibility of lack of regularity must take place precisely at $t = t^*$.

The possible behaviour at the waiting time was investigated by Lacey, Ockendon and Tayler [355] by constructing self-similar solutions both for $t > t^*$ (what we call in Chapter 16 forward self-similarity) and for $t < t_*$ (backward self-similarity) and joining them in a correct way at $t = t^*$ to form a solution that has a stationary interface for $0 < t < t^*$ and starts to move for $t > t^*$ like $s(t) - s(t_*) = c\,(t - t^*)^\delta$ for some $\delta > 1$. This means that the onset of movement takes place slowly, in a $C^1$ way. The study of general classes of solutions for which $C^1$-start occurs was done by Aronson, Caffarelli and Kamin in [44].

However, the existence of solutions with an abrupt onset of movement was proved soon afterwards by Aronson, Caffarelli and Vázquez [45]. The idea is simply explained as follows. Let us think of the porous medium as describing the spread of a viscous fluid, cf. the modelling in Section 21.1. Suppose that we have an initial datum for the pressure of the form

$$v_0(x) = (a^2 - x^2)_+$$

so that the right-hand interface is strictly advancing and given by the formula $s(t) = (ct + d)^{1/(m+1)}$ with $s(0) = a$ (cf. the ZKB solutions). We now take a distance $b > a$ and add to the initial data a very thin film of fluid so that the new initial pressure $\widetilde{v}_0(x) \ge v_0(x)$ has support in $x \in [-a, b]$ and the conditions to have a non-trivial waiting time are met at $x = b$. For instance, we put

$$\widetilde{v}_0(x) \le x(b - x)^{2+\varepsilon} \quad \text{for } x - c < x < b.$$

Then the bulk of the solution $\widetilde{v}(x, t)$ will still be similar to $v(x, t)$. What is more, the particles of the fluid will propagate behind the thin film very much like in the Barenblatt solution, i.e., all of them with positive speed. The technical part comes in proving that the waiting time of $\widetilde{v}$, let us call it, $\widetilde{t}^*$, will be slightly less than that of $v$, and the interface speed at $t = t^*+$ will be similar to that of $v$ at its waiting time, i.e., positive. This means that there will be a corner point of the interface at $t = \widetilde{t}^*$.

We will not state the general result of [45], Theorem 1, which is rather technical, but its corollary.

**Theorem 15.25** *Let $u$ be a non-negative solution of the one-dimensional PME in the whole line and let the initial pressure be a continuous and function $v_0$ supported in $[a, b]$ which satisfies for $x < b$, $x \to b$ the estimate*

$$v_0(x) = o((b - x)^2). \tag{15.80}$$

*Then the waiting time $t^*$ at $x = b$ is finite and positive and*

$$D^+ s(t_*) > 0. \tag{15.81}$$

*The same is true under the mass restriction*

$$\int_x^b u_0(x)\,dx = o((b - x)^{(m+1)/(m-1)}). \tag{15.82}$$

There is a corollary of this result:

**Corollary 15.26** *Lipschitz continuity is the best possible regularity for interfaces of non-negative solutions of the one-dimensional PME in the whole line.*

This result is also true for local solutions that we study in the next section.

### 15.5.3 *Initial behaviour*

The Lipschitz result applies to $t > 0$. At $t = 0+$ we have self-similar examples of solutions with different initial growth of the type $s(t) - s(0) \sim C\,t^{\gamma}$ with $\gamma = 1/(m+1)$. According to the propagation result of Lemma 14.5, the worst (i.e. minimal $\gamma$ ) for bounded solutions is $\gamma = 1/2$. See Problem 15.7.

A way of obtaining a solution with a $C^1$ solution for all times $t \geq 0$ is to impose the condition that $D_x^* v_0$ exists as $x \to s(0)$.

## 15.6 Local solutions. Basic estimates

We have discussed the continuity of weak solutions in the many-dimensional case in Chapter 7 using some heavy machinery. On the contrary, the continuity of solutions in the $d = 1$ setting has been settled by quite simple means, even with optimal bounds.

We now assume that $u$ is a one-dimensional local weak solution, i.e., a solution defined in a domain of space-time; it is also non-negative. We assume for convenience that the domain is a rectangle $S = (a, b) \times (0, T)$ and that $u$ is continuous in the closure $\overline{S}$; $\partial_p S$ denotes the parabolic boundary of $S$. We put $d(x) = \min\{a - x, x + a\}$, the distance from a point $x \in (-a, a)$ to the boundary.

We will still denote by $v$ the pressure variable. We denote $N = \|v\|_{L^\infty(S)}$.

We know that the maximum principle applies to these local weak solutions. We also know that they can be constructed as limits of smooth solutions. Moreover, the local regularity theory, cf. Section 7.8, shows that $u$ is $C^\infty$ in the positivity set $P_u$. But the regularity near an interface is under question.

### 15.6.1 *The local estimate for $v_x$*

The local theory relies on Aronson's a priori estimate for the velocity of the flow. Here is the version with precise dependence obtained in [287].

**Lemma 15.27** *Under the above assumptions there is a constant $c_1(m) > 0$ such that*

$$v_x^2(x,t) \leq c(m)\left(\frac{N}{t} + \frac{N^2}{d^2(x)}\right) \tag{15.83}$$

*holds for every local weak solution defined in $S$.*

**Proof** This is done by using the Bernstein technique. See Problem 15.10. ■

15.6.2 *The local lower estimate for $v_{xx}$*

The semiconvexity inequality (15.4) has played a big role in developing the theory for solutions of the Cauchy problem. In the case of local solutions the following estimate gives local semiconvexity for the pressure.

**Lemma 15.28** *Under the above assumptions there is a constant $c_2(m) > 0$ such that*

$$v_{xx}(x,t) \geq -\frac{1}{(m+1)t} - c_2(m)\frac{N}{d^2(x)} \tag{15.84}$$

*holds in the sense of distributions for every local weak solution defined in $S$.*

**Proof** As in Proposition 9.4, we write the equation for $p = v_{xx}$, which in this one-dimensional setting reads:

$$\mathcal{L}(p) \equiv p_t - (m-1)vp_{xx} - 2m\, v_x p_x - (m+1)\, p^2 = 0. \tag{15.85}$$

Let us now pick a point $(x_0, t_0) \in S$ and set $r = d(x_0)$- We shall find a lower bound for $p = v_{xx}$ in the subrectangle $S_1 = (x_0 - r, x_0 + r) \times (0, t_0)$. We consider in $S_1$ the function

$$P(x,t) = -\frac{1}{(m+1)t} - \frac{A}{(r^2 - (x-x_0)^2)^2}.$$

We want to find $A > 0$ such that $P$ is a subsolution for the boundary value problem satisfied by $p$ in $S_1$. The boundary comparison is immediate since $P = -\infty$ on $\partial_p S_1$. On the other hand, we consider (15.85) as a semilinear equation in $p$ with variable coefficients that depend on the value of $v$ and $v_x$ for our solution under consideration. Putting $\omega = r^2 - (x-x_0)^2$, we get

$$\begin{aligned}\mathcal{L}(P) = {}& \frac{1}{(m+1)t^2} + (m-1)v\left\{\frac{4A}{\omega^3} + \frac{24A(x-x_0)^2}{\omega^4}\right\} + \frac{8mv_x A\,(x-x_0)}{\omega^3} \\ & -(m+1)\left(\frac{1}{(m+1)t} + \frac{A}{\omega^2}\right)^2.\end{aligned}$$

We now use the bound $v \leq N$ and the bound (15.83) for $v_x$ in the form

$$|v_x| \leq c^{1/2}\left(\frac{N}{t} + \frac{N^2}{d^2(x)}\right)^{1/2} \leq (cN/t)^{1/2} + c^{1/2}N/d(x).$$

Using also the fact that $|x - x_0| \leq r$, that $0 \leq \omega \leq r^2$, and putting $c_1 = c^{1/2}$, we get

$$-\omega^4\mathcal{L}(P) \geq A\left(A(m+1) + \frac{2}{t}\omega^2 - 28(m-1)r^2N - 8mc_1\frac{r\omega N}{d(x)} - 8mc_1\frac{\omega r N^{1/2}}{t^{1/2}}\right).$$

Use now the inequalities:

$$8mc_1\frac{\omega r N^{1/2}}{t^{1/2}} \le \frac{2\omega^2}{t} + 8m^2c_1^2r^2N, \qquad \frac{\omega}{d(x)} = \frac{r^2-(x-x_0)^2}{r-|x-x_0|}$$
$$= r+|x-x_0| \le 2r,$$

to get

$$-\omega^4\mathcal{L}(P) \ge A(A(m+1)-cr^2N),$$

and this quantity is non-negative if $A \ge c_2(m)r^2N$. This proves that $P$ is a subsolution for the equation satisfied by $p$ in $S_1$.

We can now apply the maximum principle to $p$ and $P$ to conclude that $p(x,t) \ge P(x,t)$ in $S_1$. In the particular case $x = x_0$, $t = t_0$ we get

$$p(x_0,t_0) \ge -\frac{1}{(m+1)t} - \frac{A}{r^4} = -\frac{1}{(m+1)t} - \frac{c_3 N}{r^2}$$

This ends the proof. ∎

With this result we can improve the bound for $v_x$ and get a bound for $v_t$.

**Corollary 15.29** *Under the above assumptions, we have*

$$|v_x|^2 \le \frac{2N}{(m+1)t} + \frac{2c_2N^2}{d(x)^2}, \qquad v_t \ge -\frac{(m-1)}{(m+1)t}\,v - \frac{c_3N}{d(x)^2}\,v. \tag{15.86}$$

Use Lemma 15.2 to prove the first one. A local bound from above for $v_t$ is also available, but the proof will be delayed to Chapter 19, see Theorem 19.4 where a result is proved in several space dimensions.

### 15.6.3 *Boundary behaviour*

The previous local analysis does not take into account the form of the boundary conditions, Dirichlet, Neumann or otherwise. Concrete results about such behaviour can be obtained in all particular cases of interest. Problem 15.12 deals with a typical result for the non-negative solutions of the homogeneous Dirichlet problem.

## 15.7 Interfaces of local solutions

We still assume that $u$ is a local weak solution, i.e., a solution defined in a domain of space-time; it is also non-negative and continuous. We assume for convenience that the domain is a rectangle $S = (-a,a)\times(0,T)$ and that $u$ is continuous in the closure $\overline{S}$; $\partial_p S$ denotes the parabolic boundary of $S$. It is clear that the positivity sets $\mathcal{P}_u(t) = \{x \in (-a,a) : \ u(x,t)a > 0\}$ are formed by an at most countable union of disjoint open intervals.

Due to the study of propagation done in Chapter 14 and what was said in Subsection 15.3.1, we know that the interface

$$\Gamma_u = \partial P_u \cap S$$

consists of several connected components, typically a right-hand interface, a left-hand interface, and maybe some inner interfaces. This last part may even consist of countably many connected components and take the form of waiting time lines. As for the dynamics, the main difference is between a moving part and a stationary part, as before. This is the main result:

**Theorem 15.30** *The interface of a solution of the PME consists of stationary interface lines and smooth moving interface curves. Moreover, for every point* $(x_0, t_0)$ *lying on a moving interface curve, there is a small neighbourhood* $\mathcal{N} \subset S$ *such that* $\Gamma \cap \mathcal{N}$ *can be described as a curve* $x = s(t)$, $t \in J = (t_0 - \varepsilon, t_0 + \varepsilon)$, *and*

$$v \in C^1(\mathcal{N} \cap \overline{P_u}), \qquad s \in C^1(J), \tag{15.87}$$

$$s'(t) = -v_x(s(t), t), \qquad s''(t) = m\, s'(t)\, v_{xx}(s(t), t). \tag{15.88}$$

We announce here the result on $s''$ for easy reference though it is part of the higher regularity to be proved in Section 15.8 below. Indeed, $v \in C^\infty(\mathcal{N} \cap \overline{P_u})$ and $s \in C^\infty(J)$. An immediate consequence of the $v_t$ estimate in Corollary 15.29, we can derive the persistence property of Proposition 14.1. Therefore, the disjoint intervals that form the positivity sets $\mathcal{P}_u(t)$ expand in time, with the possibility of merging after some time. These sets may also lie touching each other at an isolated waiting time line. In the rest of the cases, they are strictly expanding as we will show.

### 15.7.1 *Review of the regularity in the local case*

We may now recover many of the results proved for the solutions of the Cauchy problem in slightly modified form. Thus, given a compact subset $K$ of $S$, we know that $v$ and $v_x$ are uniformly bounded with bounds depending on the distance from $K$ to the parabolic boundary of $S$, and also $v_{xx} \geq -C(K)$. It immediately follows that

- Theorem 15.19 holds at any free boundary points of $K$. In particular, Darcy's law holds at all free boundary points interior to $S$.
- The semiconvexity of the right-hand interfaces should be modified to read

  $$s''(t) + C(K)s'(t) \geq 0, \tag{15.89}$$

  for some universal $C(K) > 0$, and the inequality is reversed for left-hand interfaces. Corollary 15.23 holds so when an interface starts to move it never stops.

- Theorem 15.24 about the linear behaviour near a moving interface holds so that $C^1$ regularity of such an interface portion holds. The proofs contain only minor modifications left to the reader as training.

## 15.8 Higher regularity

The proof of higher regularity for moving interfaces of non-negative solutions has two main steps. The first one is obtaining an estimate for the second derivative of the pressure; another one is to bootstrap from this situation to the full regularity.

### 15.8.1 *Second derivative estimate*

We will assume the following situation: there is a rectangle $R = [a, b] \times (t_1, t_2)$ with $0 < t_1 < t_2$ where a PME solution $u$ is defined, it has a $C^1$ interface $x = s(t)$ moving to the left with non-zero speed $s'(t) < 0$; moreover $a < s(t) < b$ for all $t \in [t_1, t_2]$ and $u > 0$ on the set

$$D = \{(x, t) \in R : \ s(t) \le x \le b\}.$$

Let us call $\Gamma_D$ the part of the interface in $D$. Note that we are working with a left-hand interface. The same argument would work for a right-hand interface, and also for an inner interface before the meeting point. Here is the intermediate result:

**Proposition 15.31** *Under those assumptions, $v_{xx}$ is locally bounded in $D$.*

**Proof** (i) We already know that $u$ is $C^\infty$ smooth at all points where it is positive, and also that $v_{xx}$ is bounded below according to formula (15.84):

$$v_{xx} \ge -\frac{1}{(m+1)t} - c_2(m)\frac{N}{d^2(x)}$$

in the sense of distributions. Hence, it suffices to find an upper bound for $v_{xx}$ near points of the interface. At those points we construct an explicit upper barrier for $v_{xx}$ by taking advantage of the fact that the possible divergence of $v_{xx}$ at the interface is controlled by

$$v v_{xx} = \frac{1}{m-1}(v_t - (v_x)^2) \to 0 \tag{15.90}$$

as the point $(x, t) \in D$ moves to $P_0 = (s(t_0), t_0) \in \Gamma_D$. Let $a = -s'(t_0) > 0$

(ii) Local setting: we recall that we have already proved that the functions $v, v_t, v_x$ and $v v_{xx}$ are continuous in a relative neighbourhood $N(P_0)$ of $P_0$ in $D$. Moreover, if we are given $\varepsilon > 0$ small (there will be a limitation that will be explained below) we can find such a neighbourhood of the form

$$N_{\eta,\delta}(P_0) = \{(x, t) : \ t_0 - \eta \le t \le t_0 + \eta, \ s(t) \le x \le s(t) + \delta\} \tag{15.91}$$

so that for all $(x, t) \le N_{\eta,\delta}(P_0)$, $x > s(t)$, we have

$$0 < a - \varepsilon \le -s'(t) \le a + \varepsilon, \qquad a - \varepsilon \le v_x(x, t) \le a + \varepsilon, \tag{15.92}$$

as well as $|vv_{xx}| \le F(\delta,\eta) \le \varepsilon$ (of course, $\eta$ and $\delta$ depend on $\varepsilon$). It follows by integration that

$$(a-\varepsilon)(x-s(t)) < v(x,t) < (a+\varepsilon)(x-s(t)) \tag{15.93}$$

in $N_{\eta,\delta}$. Now let us call $t_1^* = t_0 - \eta$ and introduce a new line

$$s^*(t) = s(t_1^*) - b(t-t_1^*), \quad b = a+2\varepsilon. \tag{15.94}$$

Clearly, $s^*(t) < s(t)$ for $t \in (t_0-\eta, t_0+\eta)$.

(iii) We now prove that there is a small relative neighbourhood $N_{\eta,\delta}(P_0)$ of $P_0$ in $D$ such that $v_{xx}$ is bounded by a constant $C$. We will use the method of barriers. We need to take $\varepsilon$ small enough so that $\varepsilon < a/4m$. We write the equation that is satisfied by $p = v_{xx}$ in $\mathcal{P}_u$:

$$\mathcal{L}(p) := p_t(m-1)vp_{xx} - 2mv_xp_x - (m+1)p^2 = 0. \tag{15.95}$$

We shall construct a barrier for $p$ in $N_{\eta,\delta}$ of the form

$$\phi(x,t) = \frac{\alpha}{x-s(t)} + \frac{\beta}{x-s^*(t)} \tag{15.96}$$

where $\alpha$ and $\beta$ are positive constants. We must choose these constants so that $\mathcal{L}(\phi) \ge 0$ in $N_{\eta,\delta}$. Actually, we will show that this can be done for arbitrarily small $\alpha > 0$. It is easy to verify that

$$\mathcal{L}(\phi) \ge \frac{\alpha}{(x-s(t))^2}\{s' - 2(m-1)\frac{v}{x-s} + 2mv_x - 2(m+1)\alpha\}$$
$$+ \frac{\beta}{(x-s^*(t))^2}\{(s^*)' - 2(m-1)\frac{v}{x-s^*} + 2mv_x - 2(m+1)\beta\}$$

Using the estimates on $s'$, $v$, $v_x$ and the definition of $s^*$ we arrive at

$$\mathcal{L}(\phi) \ge \frac{\alpha}{(x-s(t))^2}\{a-(4m-1)\varepsilon - 2(m+1)\alpha\}$$
$$+ \frac{\beta}{(x-s^*(t))^2}\{a - 4m\varepsilon - 2(m+1)\beta\}.$$

Since $\varepsilon < a/4m$, if we fix

$$\beta = \frac{a-4m\varepsilon}{2(m+1)}, \tag{15.97}$$

then, $\beta > 0$ and $\mathcal{L}(\phi) \ge 0$ in $N_{\eta,\delta}$ if $\alpha$ is small enough.

(iv) We now perform the comparison between $p$ and $\phi$. Due to the conditions $\mathcal{L}(\phi) \ge 0 = \mathcal{L}p$, we need only compare them on the parabolic boundary. In view of the estimate $|vv_{xx}| \le \varepsilon$ and the bounds (15.93) for $v$ we have

$$v_{xx} < \frac{\varepsilon}{(a-\varepsilon)(x-s)} \quad \text{in } N_{\eta,\delta}, \text{ so that} \quad v_{xx}(s(t)+\delta) < \frac{\varepsilon}{(a-\varepsilon)\,\delta}.$$

for $t_1^* < t < t_0 + \eta$. Next, we use the mean value theorem and (15.94) to get

$$s(t) + \delta - s^*(t) = \delta + (a + 2\varepsilon + s'(\tau))(t - t_1^*)$$
$$\leq \delta + 3\varepsilon(t - t_1) \leq \delta + 6\eta\varepsilon$$

for some $\tau \in (t_1^*, t_0 + \eta)$. The final quantity can be made less than $2\delta$ if $\eta$ is small enough, $\eta \leq \delta/6\varepsilon$. We then get

$$\phi(s(t) + \delta, t) \geq \frac{\beta}{2\delta} \geq \frac{\varepsilon}{(a - \varepsilon)\delta} \geq v_{xx}(s(t) + \delta),$$

which settles the comparison on the right-hand lateral boundary of $N_{\eta,\delta}$. Moreover, at the initial time $t = t_1^*$ we have

$$\phi(x, t_1^*) \geq \frac{\beta}{x - s(t_1^*)} \geq \frac{\varepsilon}{(a - \varepsilon)(x - s(t_1^*))} \geq v_{xx}(x, t_1^*).$$

Finally, we have to make a comparison on the free boundary, which seems impossible since it is precisely the place where we still do not know whether $v_{xx}$ diverges. But the comparison can be done in a line $\Gamma'$ parallel to the free boundary $\Gamma_D$ obtained by displacement at a distance $\delta' \ll \delta$, since we know that in any case $v_{xx}$ diverges at a lesser rate than $\phi$. Indeed at this distance $\delta'$ we have on $\Gamma'$:

$$\delta' v_{xx} \leq \frac{1}{a - \varepsilon} v v_{xx} \leq \frac{F(\eta, \delta')}{a - \varepsilon}, \qquad \delta' \phi(x, t) \leq \alpha,$$

and we only need $F(\eta, \delta') < \alpha(a - \varepsilon)$ to conclude. This is possible even if $\alpha$ is small since $F(\eta, s) \to 0$ as $s \to 0+$ because of (15.90).

By the comparison principle for parabolic equations [357] we conclude that

$$v(x, t) \leq \phi(x, t) = \frac{\alpha}{x - s(t)} + \frac{\beta}{x - s^*(t)}$$

in $N_{\eta,\delta}$ if $\beta$ is given by (15.97), while $\alpha$ can be as small as we like. Letting $\alpha \to 0$ and restraining the time interval to $I = (t_0 - \eta/2, t_0 + \eta/2)$ we get a uniform bound

$$v(x, t) \leq \frac{\beta}{x - s^*(t)} \leq \frac{2\beta}{\varepsilon\eta},$$

in $R_{\eta/2,\delta}(P_0)$. ■

**Remark** The above argument also shows that $C$, $\delta$ and $\eta$ may be chosen to depend continuously on $t_0$.

### 15.8.2 $C^\infty$ *regularity of* $v$ *and* $s(t)^*$

We now turn to the problem of estimating the rest of the derivatives of $v$ with respect to $x$ and $t$ near the moving boundary. We contend that only the pure space derivatives $v^{(j)} = (\partial_x)^j v$ have to be estimated, since the time derivatives and the mixed derivatives can be derived from the equation upon repeated differentiation in time. Thus, the equation itself shows that a bound for $v, v_x$ and $v_{xx}$ implies a bound for $v_t$ (which was already known anyway).

Estimating the higher derivatives $v^{(j)}$ is done in an iterative way by using a barrier argument similar to the one used for $v^{(2)}$ with some differences: (i) in the estimate of $v^{(2)}$ we were able to use the a priori information that $v^{(2)}(x,t) = 0(1/d)$ where $d = d(x,t) = x - s(t)$. For $j \geq 3$ the information that can be derived is

$$v^{(j)}(x,t) = O(1/d).$$

However, to compensate for this, for $j \geq 3$ we can write the equation satisfied by $v^{(j)}$ as a linear equation with coefficients depending on the previous derivatives,

$$\mathcal{L}_j(v^{(j)}) = v_t^{(j)} - (m-1)vv_{xx}^{(j)} - (2 + j(m-1))v_x v_x^{(j)} - c_{mj} v_{xx} v^{(j)} \\ - \sum_{l=1}^{j^*} d_{mjl} v^{(l)} v^{(j+2-l)}$$

where $j^* = [j/2] + 1$ and $c_{mj}$ and $d^{mjl}$ are constants that depend on their indices. This can be exploited in a step where the barrier is improved.

Here is the result that we can get.

**Proposition 15.32** *Let $P_0 = (x_0, t_0)$ be a point on the moving interface. For each integer $j \geq 2$ there exist constants $C_j, \delta$ and $\eta$ depending only on $P_0$, $m, j$ and $u$ such that*

$$|(\partial_x)^j v(x,t)| \leq C_j \quad \text{in} \quad N_{\eta,\delta}(P_0). \tag{15.98}$$

We refer for the details of the proof to [51], pages 337–341. In fact, the calculation of the growth of $C_j$ with $j$ shows that $v$ belongs to a Gevrey class, a fact that is improved in the next subsection, and has been generalized to several space dimensions by Koch [347], see Section 19.4.

### 15.8.3 *Higher interface equations and convexity properties**

Once we have proved that $v$ is $C^\infty$ in $\mathcal{P}_u$ and up to the moving parts of the free boundary, it is easy to use the Darcy formula $s'(t) = -v_x(s(t),t)$ to obtain formulas for all the derivatives of $s$. For instance, we have

$$s''(t) = -v_{xx}(s(t),t)s'(t) - v_{xt}(s(t),t) = -mv_x(s(t),t)v_{xx}(s(t),t). \tag{15.99}$$

We ask the reader to perform the computation of $v_{xt}$ and obtain the final formula. Here is a consequence:

**Corollary 15.33** *Assume that the initial pressure $v_0$ is continuous, zero for $x > 0$ and convex and positive for $x < 0$. Then, by Problem* 12.7 *we know that*

$v_{xx} \geq 0$ *for all* $(x,t) \in Q$. *We conclude from* (15.99) *that* $s(t)$ *is a convex function of time.*

### 15.8.4 *Concavity results*

In one space dimension the conservation of pressure concavity for the PME flow was proved by Bénilan and Vázquez [96]. We consider the PME that we write in terms of the pressure as $v_t = (m-1)v\, v_{xx} + (v_x)^2$ in $Q = \mathbb{R} \times (0,\infty)$, with $m > 1$ and initial data $v(x,0) = v_0(x)$ for $x \in \mathbb{R}$, where $v_0$ is continuous, non-negative and vanishes outside an interval $(a,b)$ with $v_0(x) > 0$ for $x \in (a,b)$. We assume moreover that

$$v_{0,xx} \leq -C \leq 0 \quad \text{for } a \leq x \leq b.$$

Then, recalling that there exist two continuous monotone interface curves $x = \zeta_1(t)$, $x = \zeta_2(t)$ such that $\zeta_1(0) = a$, $\zeta_2(0) = b$ and $\mathcal{P}_u(t) = \{x : \zeta_1(t) < x < \zeta_2(t)\}$, and using a Trotter–Kato formula, the following 'concavity inequalities' are shown to hold:

$$v_{xx} \leq -\frac{C}{1+(m+1)Ct} \qquad \text{in } \mathcal{P},$$

and for $i = 1,2$

$$(-1)^i(\zeta_i'' + \frac{mC}{1+(m+1)Ct}\,\zeta_i') \leq 0 \qquad \text{in } \ \mathcal{D}'(\mathbb{R}^+).$$

As a consequence of equation (15.99), the right-hand interface function $x = s(t)$ is a concave function. By symmetry, the left-hand interface curve must be convex.

### 15.8.5 *Analyticity*$^\star$

Consider a non-negative solution of the PME $u_t = (u^m)_{xx}$ with compactly supported initial data. Angenent proved in [20] that the (outer) free boundary is real analytic after the waiting time. The proof uses the a priori estimates derived in this chapter up to $v_{xx}$ plus the maximal regularity theory of Da Prato and Grisvard.

The author continues the investigation in paper [23] with two main results. In the first place, it is proved that the moving parts of the free boundary completely determine the solution of the PME in that time interval: let $u_1$ and $u_2$ be two solutions of the Dirichlet–Cauchy problem in a rectangle $Q = (-a,a) \times (-\tau,\tau)$ with $u_i(x,t) > 0, x \in (-a, s_i(t))$ and $u_i(x,t) = 0, x \in (s_i(t), a)$ where $s_i$ are two strictly increasing functions. If $s_1(0) = s_2(0)$ then either $u_1$ and $u_2$ coincide for $t \leq 0$ or else there exists a $k > 0$ such that the derivatives of order $k$ of the $s_i$ at $t = 0$ are different. The argument extends to radially symmetric solutions in several dimensions.

In the case of an analytic free boundary, the author proves that the pressure function $u^{m-1}$ is real analytic in the set where it is positive. This result, combined with the previous result about the analyticity of the free boundary for the solutions of the Cauchy problem, implies that the pressure is the positive part of a real analytic function when the support of the solution is expanding.

## 15.9 Solutions and interfaces for changing-sign solutions*

The study of the signed solutions of the one-dimensional PME is less advanced since most of the derivative estimates derived in this chapter do not apply. In any case, the existence theory holds as described in the general chapters on the Dirichlet, Cauchy and Neumann problems, initial data in $L^p$, $p \in [1, \infty)$ produce bounded solutions, and bounded solutions are Hölder continuous. Shifting comparison, intersection comparison and lap number count apply.

The study is closely related to the study of solutions of the $p$-Laplacian equation that is obtained from the PME by integration, as explained in Subsection 3.4.3. Then the positivity or negativity of $u$ is reflected in the oscillations of the integral $v$ since $v_x = u$.

A typical situation is the configuration where $u_0$ is negative in say $-\infty < x < 0$ and positive for $0 < x < \infty$. By the lap number theory, the lap number will be 1 or 0, so that we can define an interface $x = s(t)$ separating the regions where $u < 0$ and $u > 0$ for times $t > 0$. Bertsch and Hilhorst [110] study a situation of this type and prove under certain monotonicity hypotheses on the initial function that there exists a time $T \geq 0$ after which the $s(t)$ is continuously differentiable on $(T, \infty)$. See also [465, 467].

Changing-sign solutions may have infinite waiting times, as the explicit dipole solution (4.53) shows. Self-similar signed solutions are constructed in Chapter 16 with all possible lap numbers. The asymptotic behaviour of those solutions is discussed in Chapter 18.

## Notes

**Section 15.1.** The Lipschitz continuity of the pressure with respect to the space variable was shown by Aronson [35] in 1969, who also observed that the estimate was optimal on the source-type solutions. By the way, this paper led to the interest in studying the properties of the pressure. Our simple proof of Theorem 15.1, based on the fundamental estimate, can be found in [287]. Theorems 15.5 and 15.6 and Lemma 15.7 are taken from [83].

Let us add to the regularity results of this chapter a further result of general interest: in his Kentucky Notes [83], Bénilan proves the boundedness of the flux $\Phi(u)_x$ of the GPME $u_t = \Phi(u)_{xx}$ of the following form: if $0 \leq u_0 \leq C$ and $\Phi$ is a function like in Chapter 5 with $\Phi(0) = 0$, then

$$|\partial_x \Phi(u)|^2 \leq \left(\frac{\Phi(C)u}{t}\right). \tag{15.100}$$

There is also a general result for parabolic equations in $d = 1$ that allows us to translate $C^\alpha$ regularity in $x$ into $C^{\alpha/2}$ regularity in $t$, cf. Gilding [262]. Note that for the PME the result is better, we have $C_x^\alpha$ and $C_t^\alpha$ with $\alpha = 1/(m-1)$ if $m > 2$. See more in Section 19.3.

Regularity results were also obtained by a number of authors, cf. [74].

**Section 15.2.** The method of comparison by shifting was introduced in [498] and used to obtain comparison of interfaces just as will be done in the next section. The proof presented here is direct, while [498] used the (semigroup approach) discretization in time and reduces Theorem 15.8 to proving a similar result for the corresponding elliptic problem $-(u^m)_{xx} + u = f$.

For the topic of intersection comparison theory we refer to the accounts of the books [255] and [245]. The principle was used extensively by Galaktionov and coworkers, see [469]. The lap number became a regular tool in the recent research after the influential paper by Matano [385] in 1982. Other interesting contributions are due to Brunovski and Fiedler [134] and Angenent [22], who proves a very important additional result: under certain assumptions on the structure of a parabolic equation, the number of zeros of a solution $u(t,x)$ is a discrete subset of $\mathbb{R}$, even if it was not at $t = 0$. This is not true in general for the PME, a degenerate equation, as we will show in Example (ii) of Theorem 16.8.

The details on how to justify the monotonicity in time of the intersection or lap number in the case of the PME can be seen in Chapter 2 of our book [255]. Functionals like these that are non-increasing in time are called *weak Lyapunov functionals* for the evolution. We will find more of such functionals in the chapters on asymptotic behaviour.

The intersection and lap number results have interesting consequences in the study of long time behaviour when the evolution solution is expected to stabilize towards an equilibrium solution and it is found that there exist stable equilibrium solutions with a highly inhomogeneous spatial pattern. Such patterns cannot be formed if one starts from a simpler-shaped initial function with low lap number. Indeed, these numbers can be viewed as order structures preserved by the semigroup associated to the PME. See in this respect Angenent [22] or in other settings [24] and [243].

These properties are also true for the many other nonlinear degenerate equations, like the $p$-Laplacian equation.

**Section 15.3.** In obtaining the first interface bounds the use of the shifting principle can be replaced by using the intersection number as is done in Section 2.5 of book [255].

The waiting time analysis follows work of Aronson, Caffarelli, Kamin and Vázquez [44, 45]. The characterization of the positive waiting time appeared in [500]. The $C^1$ regularity is due to Caffarelli and Friedman [138]; we follow here the proof of [44]. The sharp semiconvexity inequality follows [498].

The numerical calculation of the waiting time is a delicate question; it has been studied for the one-dimensional porous medium equation by Tomoeda et al. in a series of papers [395, 493]. Properties of waiting-time solutions of the porous medium equation applied to viscous gravity currents are studied numerically by

Gratton and Vigo [272], who show a structure of corner layers. Early numerical work on interfaces in one-dimensional PME is due to Graveleau and Jamet [274], from year 1971.

**Section 15.6.** The local results are mainly taken from our paper [504] where the local semiconvexity inequality, Lemma 15.28, was established. The local bound for the speed, Lemma 15.27, had been proved by Aronson in 1970 using the Bernstein technique and represented a breakthrough in the study of the regularity of the PME. The sharp version we present is due to [287].

A local second-order estimate of the form $\Delta v \geq -C$ generalizing the fundamental estimate (9.12) is not known in the many-dimensional case. This is a long-standing **open problem** of the theory of the PME.

**Section 15.7.** The basic material on local regularity is taken from [504].

The upper second-order estimate is taken from [51]. The $C^\infty$ regularity was proved in that paper, and simultaneously by Höllig and Kreiss [292] using a technique of iterated weighted norms. They work with compactly supported solutions.

The analyticity of the interfaces is due to Angenent [20] and [23].

*Further comments*

(1) A detailed description of the asymptotic behaviour for solutions with compact support in one and several variables will be done in Chapter 18, while the study of the Dirichlet problem is done in Chapter 20. As a consequence of those results, it is proved that the waiting times of local non-negative solutions are always finite.

(2) Non-negative solutions of the FDE for $0 < m < 1$ become immediately positive, hence no interfaces arise. The solutions are $C^\infty$ smooth.

## Problems

### Problem 15.1

(i) Prove the shifting comparison result for the solutions of the elliptic equation

$$u - \beta(u)_{xx} = f. \tag{15.101}$$

(ii) Use implicit time discretization to obtain the parabolic version of the shifting comparison result, Theorem 15.8.

**Problem 15.2** Derive the properties of the left-hand interface (15.43), and the inner interfaces if they exist.

### Problem 15.3

(i) Construct an example of solution with a non-zero waiting time at an isolated point.
(ii) Construct a periodic example with infinitely many positive waiting times.

(iii) Construct an example compact support and infinitely many positive waiting times.

**Problem 15.4*** Suppose that $u \geq 0$ is a solution of the Cauchy problem with initial data supported in $x < 0$ and a waiting time at $x = 0$ for $0 < t < t^*$. Prove that the Darcy relations

$$s'(t) = -v_x(P) = 0, \quad \text{and} \quad v_t(P) = v_x^2(P) = 0, \tag{15.102}$$

hold near the stationary interface point $P = (0, t)$ with $0 < t < t^*$. The values of $v_x$ and $v_t$ are still inner limits.

*Hint*: The problem is not easy. The a priori estimates are derived in [45].

**Problem 15.5** Prove by simple means that the support of a non-negative solution defined in $S = (a, b) \times (O, \infty)$ becomes positive everywhere in $(a, b)$ if $t \geq t(u_0)$. Estimate $t(u_0)$.

**Problem 15.6** (i) Prove Theorem 15.25 about the existence of interfaces with one corner point.
(ii) Find sufficient conditions to have a global $C^1$ interface.

**Problem 15.7**

(i) Use shifting comparison to prove that the maximal initial growth rate of the right-hand interface of solutions of the PME on the line with initial data supported on $x < 0$ is given by the Barenblatt solution with a given mass. Therefore,

$$s(t) \leq C\, t^{1/(m+1)}$$

for all small $t > 0$.

(ii) Show that the worst exponent is $1/2$ in case we also know that the initial data is (locally) bounded.

(iii) Use the self-similar results of Chapter 16 to construct solutions with interfaces that grow with exact exponent $\gamma$ for all $\gamma$ in the interval $(1/(m+1), \infty)$.

**Problem 15.8** We want to know more about lap numbers. Examine the possibilities of decrease of the lap number. The counter $l(t, u)$ can go down one unit by losing a peak (relative maximum) or a valley (relative minimum) to the boundary, it can lose two units by interior merging a peak with a valley, it can also lose three units by merging two peaks with the intermediate valley; there are possibilities for higher orders of decrease but they are less probable in terms of the perturbations of the data.

Show examples of the typical situations, either theoretically or numerically

**Problem 15.9** Examples of infinite lap number are easy in the whole line, by considering a periodic solution with changing sign. Give an explicit example.

A less trivial example consists of integrable solutions. Construct an example by taking initial data consisting of damped copies of a hump (e.g., a Barenblatt profile) placed at increasingly distant places and assign to them alternating signs.

An even less trivial example is constructed in Section 16.7. There a signed solution is constructed that has infinitely many oscillations near $x = 0$ at every time $t > 0$.

**Problem 15.10** Prove the Bernstein estimate in the sharp form (15.83).

SKETCH

(i) We may assume that the space interval was $(0, d)$. Reduce the calculation with norm $N$ at time $t_0$ to the case $N = 1$, $t_0 = 1$ by using the transformation $v(x,t) = N\widetilde{v}(Lx, t/t_0)$ with $L^2 = 1/(t_0 N)$. The new function is defined in a rectangle $S' = \{(x,t) : 0 < x < a',\ \ 0 < t \le 1\}$ with $b' = Lb$. In those circumstances we only have to prove that $v_x^2(x,1) \le c(1 + (1/d^2))$.

(ii) Put $v = \varphi(w)$ for a certain $\varphi$ that maps $[0,1]$ onto $[0,1]$ with $\varphi' \ge c_1 > 0$ and $\varphi'' < 0$ and $(\varphi''/\varphi')' \le 0$. Prove that $w$ satisfies the equation

$$w_t = (m-1)\varphi(w)\, w_{xx} + \left\{ \varphi' + \frac{\varphi(w)\varphi''(w)}{\varphi'(w)} \right\} (w_x)^2.$$

Differentiate to get

$$w_{xt} = (m-1)\varphi(w)\, w_{xxx} + A(x,t)\, w_x w_{xx} + B(x,t)\,(w_x)^3, \tag{15.103}$$

where

$$A(x,t) = (m+1)\varphi' + \frac{2\varphi(w)\varphi''(w)}{\varphi'(w)}, \qquad B(x,t) = 2\varphi'' + \left.\left(\frac{\varphi''}{\varphi'}\right)' \varphi \right|_{w(x,t)}.$$

(iii) Choose $\varphi(w) = aw - bw^2$ in such a way that $\varphi(1) = 1$, $\varphi'(w) > c_1 > 0$, and $B \le -c_2 < 0$ for $0 \le w \le 1$.

(iv) Write $z = t\zeta^2(x)(w_x)^2$ for a certain test function $\zeta(x)$ compactly supported in $(0, d)$ and $\zeta = 1$ in $(d/4, 3d/4)$. Examine the conditions at a non-trivial maximum of $z$ in $S'$. At that point, $P_1$ we will have $z_t \ge 0, \quad z_x = 0, \quad z_{xx} \le 0$. This means that

$$\zeta^2\,(w_x)^2 + 2t\zeta^2\, w_x w_{xt} \ge 0, \quad 2\zeta\zeta_x (w_x)^2 + 2\zeta^2\, w_x w_{xx} = 0,$$

which for $z \ne 0$ simplify into $(w_x)^2 + 2t w_x w_{xt} \ge 0$, $\zeta\, w_{xx} = -\zeta_x w_x$; we also have

$$(2\zeta\zeta_{xx} + 2(\zeta_x)^2)\,(w_x)^2 + 8\zeta\zeta_x\, w_x w_{xx} + 2\zeta^2\,(w_{xx})^2 + 2\zeta^2\, w_x w_{xxx} \le 0.$$

Combining these inequalities with the equation for $w_{xt}$ we get the bound for $w_x$ and putting the term $(w_x)^4$ coming from $(w_x)^3$ in (15.103) in the left-hand side we get at $P_1$:

$$-2Bt\zeta^2 (w_x)^4 \le D(x,t)\,(w_x)^2 + E(x,t)\,(w_x)^3,$$

where $D, E$ are bounded with a factor $O(1 + 1/d^2)$ (the dependence on $\zeta$ makes for the factor $1/d^2$). See details in [287], pages 156, 157. We can cancel $(w_x)^2$. This means that $z$ is bounded at $P_1$, hence in $S'$. We may now put $t = 1$ and $b/4 \le x \le 3b/4$ to get

$$(w_x)^2(x, 1) = z(x, 1) \le \max z \le C(1 + d^{-2}).$$

Note now that $|v_x| \le k|w_x|$ where $k = \max\{\phi'(s) : 0 \le s \le 1\}$.

**Problem 15.11** Interfaces for the generalized porous medium equation. Continuing with Problem 4.4, we assume that the GMPE $\partial_t u = \partial^2_{xx}\Phi(u)$ has a nonlinearity that satisfies the condition to have finite travelling waves:

$$\int_0^U \frac{d\Phi(s)}{s} = \int_0^z \frac{dz}{\beta(z)} < \infty. \tag{15.104}$$

where $\beta = \Phi^{-1}$ is the inverse graph of $\Phi$.

(i) Generalize for this equation the shifting comparison theorem.
(ii) Prove that under condition (15.104), the finite propagation property holds and there are interfaces of the there kinds explained in the text. Peletier [424] shows that this is a necessary and sufficient condition for the existence of an interface, and this is true for several initial and boundary value problems.
(iii) Caffarelli and Friedman state in [138] that under convenient assumptions on $\Phi$ that looks like a power near $u = 0$ the moving interfaces are semi-convex and $C^1$ smooth in the moving parts.

**Problem 15.12** Regularity up to the fixed boundary. Let us consider a non-negative solution of the PME defined in a rectangle $S = I \times (0, T)$ with zero Dirichlet conditions and assume that

$$0 \le u \le H, \quad \partial_t u \ge -Au.$$

Assume also that $(u^m)_t \in L^2(S)$. Then, $u^m$ is Lipschitz continuous with respect to the $x$ variable and Hölder continuous with exponent $1/3$ with respect to $t$, with coefficients that depend only on $m$, $d$, $H$ and $A$.

**Outline of the proof** The regularity in the inside is already settled, so we worry about the regularity up to boundary. Note that the assumptions we mention are satisfied by general solutions for $t \ge \tau > 0$.

(i) *Lipschitz continuity in space.* Let $w(x, t) = u^m(x, t)$. We have

$$w_{xx} = u_t \ge -Au \ge -AH.$$

Using this and $0 \le u \le H$, an elementary interpolation lemma gives the result. Fixing $x_0 \in I$ and $t \in (0, T)$ and calling $c_0 \ge w(x_0, t)$, $c_1 = w_x(x_0, t)$ we have for every $x \in I$

$$H^m \geq w(x,t) \geq c_0 + c_1(x-x_0) - \frac{1}{2}AH(x-x_0)^2.$$

We have to conclude from this that $c_1$ is bounded.

(ii) *Hölder continuity in time.* This is a consequence of another interpolation lemma using the fact that $w_t \in L^2(Q)$ and $w_x \in L^\infty(Q')$. Indeed, let $K$ the Lipschitz constant for $w_x$ just obtained. For $0 < t_1 < t_2 = t_1 + \tau < T$ and $x_0, x \in I = (a,b)$, we write

$$\begin{aligned} |w(x_0,t_2) - w(x_0,t_1)| &\leq |w(x,t_2) - w(x,t_1)| + |w(x_0,t_2) - w(x,t_2)| \\ &\quad + |w(x_0,t_1) - w(x,t_1)| \\ &\leq |w(x,t_2) - w(x,t_1)| + 2K|x-x_0|. \end{aligned}$$

Writing $w(x,t_2) - w(x,t_1) = \int_{t_1}^{t_2} w_t\,dt$ and integrating in $x \in (x_0-h,x_0)$ or in $(x_0,x_0+h)$ (this is done to ensure that the interval is contained in $I$) we get

$$h|w(x_0,t_2) - w(x_0,t_1)| \leq Kh^2 + \int_{t_1}^{t_2}\int_{I'} |w_t|dxdt$$

Applying now Hölder's inequality we get

$$|w(x_0,t_2) - w(x_0,t_1)| \leq Kh + \|w_t\|_2^{1/2} h^{-1/2}\tau^{1/2}.$$

Minimization in $h$ with the constraint $h \leq (b-a)/2$ gives for small $\tau \leq \tau_0$

$$|w(x_0,t_1+\tau) - w(x_0,t_1)| \leq K_1\tau^{1/3}$$

for some $K_1 > 0$ that depends on $K$, $\|w_t\|_2$ (while $\tau_0$ depends also on $b-a$). This completes the result. ■

**Project*** Radial solutions. Extend the results of this chapter to radially symmetric solutions in several dimensions whenever possible.

# 16

# FULL ANALYSIS OF SELF-SIMILARITY

This chapter is devoted to studying the class of solutions of the PME that are invariant under the scaling group in the variables $(x, t, u)$, and take therefore the so-called self-similar form. Since the similarity analysis is based on the power form of the nonlinearity in the equation, it applies equally to the HE and FDE, though the details vary more or less. As a consequence of this chapter the small set of already available explicit solutions will be enriched with a large series of (families of) solutions exhibiting interesting behaviour of different types. These solutions are explicit up to solving a planar ODE system. This is an almost trivial task for our computers, if properly prepared; on the other hand, the modern qualitative tools allow us to derive all relevant information without the computation.

This is a summary of the chapter:

After a detailed analysis of the application of the scaling group to the PME, Section 16.1, we show that the self-similar solutions can be classified into tree different types: forward, backward and exponential self-similarity, Section 16.2.

In Section 16.4 we introduce the technique of phase-plane analysis that allows to obtain a rather complete description of these solutions for all parameters (under the restriction of radial symmetry in several dimensions). An alternative phase plane is introduced in Section 16.5 to clarify the behaviour at infinity of the previous plane. The tools are completed in Section 16.6 with the study of sign-change trajectories through inversion.

Oscillating signed solutions are studied in Section 16.7 and two special solutions are constructed that are important in the existence and uniqueness theory of signed solutions.

The special features of self-similar solutions of Type II are examined in Section 16.8. Two short sections contain supplementary material.

The generality of the arguments and the deep consequences for the theory recommend the material of this chapter for an elementary introduction to some of the main topics of the PME. Besides, it supplies the possibility of doing interesting numerical experimentation resulting in beautiful graphs that reveal an elegant dynamical structure.

## 16.1 Scale invariance and self-similarity

Scale invariance  is a very basic idea coming from mechanics, originated in the analysis of the consequences of the change of units of measurement on the mathematical form of the laws of physics. It can also be viewed as a particular aspect of the study of the invariance of differential equations under general groups of transformations, a discipline that has known great development after the work of S. Lie and had moments of glory in the contributions of E. Noether.

Let us examine the application of *scaling transformations* to the PME in some detail. Let $u = u(x,t)$ be a solution of the PME or the FDE,

$$\partial_t u = \Delta(|u|^{m-1}u). \tag{16.1}$$

We apply the group of dilations in all the variables

$$u' = Ku, \qquad x' = Lx, \qquad t' = Tt, \tag{16.2}$$

and impose the condition that $u'$ so expressed as a function of $x'$ and $t'$, i.e.,

$$u'(x',t') = Ku\left(\frac{x'}{L},\frac{t'}{T}\right), \tag{16.3}$$

has to be again a solution of (16.1).[1] Then:

$$\frac{\partial u'}{\partial t'} = \frac{K}{T}\frac{\partial u}{\partial t}\left(\frac{x'}{L},\frac{t'}{T}\right), \qquad \Delta_{x'}(u')^m = K^m L^{-2}\Delta_x(u^m)\left(\frac{x'}{L},\frac{t'}{T}\right).$$

Hence, (16.3) will be a solution if and only if $KT^{-1} = K^m L^{-2}$, i.e.,

$$K^{m-1} = L^2 T^{-1}. \tag{16.4}$$

We thus obtain a two-parametric transformation group acting on the set of solutions of (16.1). Assuming that $m \neq 1$, we may choose as free parameters $L$ and $T$, so that it can be written as

$$u'(x',t') = L^{\frac{2}{m-1}} T^{\frac{-1}{m-1}} u(x,t) = \left(\frac{L^2}{T}\right)^{\frac{1}{m-1}} u\left(\frac{x'}{L},\frac{t'}{T}\right).$$

Using standard letters for the independent variables and putting $u' = \mathcal{T}u$, we get:

$$(\mathcal{T}u)(x,t) = L^{\frac{2}{m-1}} T^{-\frac{1}{m-1}}\, u\left(\frac{x}{L},\frac{t}{T}\right). \tag{16.5}$$

The conclusion is:

**Lemma 16.1** *If $u$ is a solution of the PME in a certain class of solutions $\mathcal{S}$ that is closed under dilations in $x,t$ and $u$, then $\mathcal{T}u$ given by* (16.5) *is again a solution of the PME in the same class.*

The dilations are usually called *rescalings*. Family $\mathcal{T}$ with parameters $L,T$ is usually referred to as the *similarity transformations* for the PME, as well as the *scaling transformations*, or simply *rescalings*. We also speak of the scaling group

[1] Note: here, the primes denote new variables and not derivatives. The letters used being indifferent, we usually find the scaling formula written as $u'(x,t) = Ku(x/L, t/T)$.

or renormalization group. The reader is asked to check that weak solutions, very weak solutions, strong solutions, mild solutions and limit solutions are classes admitting the property of rescaling; consequently, $\mathcal{T}$ is well defined on them.

### 16.1.1 *Subfamilies*

In practice, we often use one of the free parameters to force $\mathcal{T}$ to preserve some important behaviour of the orbit. This allows us to classify the family of all scale-invariant solutions into subfamilies by the corresponding new relation. Analytically, the basic idea is to impose a new relation between the two independent parameters, say $K$ and $L$; this allows to reduce the transformation to a one-parameter family of scaled functions.

(i) It is typical to assume that the new relation takes the form $K = L^{-\gamma}$ for some $\gamma \in \mathbb{R}$. Such a relation comes from considerations from physics or from analysis, as we will see in the sequel. Then, we can express $K$ and $L$ in terms of $T$ in the form

$$K = T^{-\alpha}, \qquad L = T^{\beta}, \tag{16.6}$$

with the correct scaling exponents given by

$$\alpha(m,\gamma) = \frac{\gamma}{\gamma(m-1)+2}, \quad \beta(m,\gamma) = \frac{1}{\gamma(m-1)+2} \tag{16.7}$$

unless $\gamma = -2/(m-1)$. Note that $\gamma = \alpha/\beta$. The transformation becomes finally

$$(\mathcal{T}u)(x,t) = T^{-\alpha} u\left(x/T^{\beta}, t/T\right).$$

Let us write the scaling factor as $\lambda = 1/T$. Then, the solution is

$$\widetilde{u}_\lambda(x,t) = (\mathcal{T}_\lambda u)(x,t) = \lambda^{\alpha} u(\lambda^{\beta} x, \lambda\, t). \tag{16.8}$$

Let us put a representative example: the ZKB solution $U_M(x,t)$, given by formula (1.8), has a constant mass; actually, this characterizes the solution uniquely. Imposing thus the condition of mass conservation at $t = 0$ we get

$$\int_{\mathbb{R}^d} (\mathcal{T}u_0)(x)\, dx = \int_{\mathbb{R}^d} u_0(x)\, dx, \tag{16.9}$$

namely,

$$\int_{\mathbb{R}^d} K\, u_0\left(\frac{x}{L}\right) dx = \int_{\mathbb{R}^d} u_0(x)\, dx.$$

It easily follows that $K\, L^d = 1$. We get the one-parameter family $\mathcal{T}_\lambda u$ with the well-known exponents

$$\alpha = \frac{d}{d(m-1)+2}, \quad \beta = \frac{1}{d(m-1)+2}. \tag{16.10}$$

Note the formula for the change of the initial data: $\widetilde{u}_{0,\lambda}(x) = (\mathcal{T}_\lambda u_0)(x) = \lambda^\alpha u_0(\lambda^\beta x)$.

(ii) We still have to analyse the case $\gamma = -2/(m-1)$. In that situation, the new relation $KL^\gamma = 1$ is incompatible with the old one, $K^{m-1}T = L^2$ unless $T = 1$. Therefore, the transformation group is still one-parametric, but the form is

$$\widetilde{u}_\mu(x,t) = (\mathcal{T}_\mu)(x,t) = \mu^{-2/(m-1)} u(\mu x, t), \tag{16.11}$$

where we have put $\mu = 1/L$. The reader will realize that the blow-up solution (4.44) is invariant under this scaling transformation.[2]

### 16.1.2 *Invariance implies self-similarity*

We now look for the special solutions that are themselves invariant under the scaling group. We call these solutions self-similar. This is an individual property, as opposed to the previous concept that was a property of a whole class of solutions. Self-similarity means that $(\mathcal{T}u)(x,t) = u(x,t)$ for all $(x,t)$ in the domain of definition, which we also assume to be scale invariant. Though several choices are possible, the standard option is to choose $(x,t) \in Q = \mathbb{R}^d \times (0,\infty)$. Then,

$$u(x,t) = Ku\left(\frac{x}{L}, \frac{t}{T}\right) \tag{16.12}$$

holds for every $(x,t) \in Q$ and every admissible $K, L, T > 0$. We have at least the relation $K^{m-1}T = L^2$, imposed by conserving the equation.

- Let us examine the concept of self-similarity more closely. Self-similarity with respect to the full two-parameter group implies that $K, L$ and $T$ are only restricted by the last condition. In that case, we may fix $(x,t)$ with $|x| \neq 0$, $t > 0$, choose $T = t$ and $L = |x|$, and in this way get

$$u(x,t) = |x|^{\frac{2}{m-1}} t^{-\frac{1}{m-1}}\, u(\omega, 1) = F(\omega)(x^2/t)^{1/(m-1)}, \tag{16.13}$$

  where $\omega = x/|x| \in \mathbb{S}^{d-1}$. This is a very special form, the one taken by some of the blow-up solutions of Section 4.5.
- However, the most common option is to consider solutions that are invariant under the transformations in a one-parameter subfamily. When $\gamma \neq -2/(m-1)$, the scaling group is given by formula (16.8), so that

$$u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)$$

[2] This paragraph makes no sense for $m = 1$. We will avoid discussing the linear heat equation in the sequel since it does not fit with our parameter choices. But the reader should bear in mind that many results agree.

for every $(x,t) \in Q$ and every $\lambda > 0$. Fixing $(x,t)$ and taking $\lambda = 1/t$, we get

$$\boxed{u(x,t) = t^{-\alpha} u(x\,t^{-\beta}, 1) = t^{-\alpha} F(x\,t^{-\beta}),} \tag{16.14}$$

and $\alpha$ y $\beta$ are related by $\alpha(m-1) + 2\beta = 1$. This is the standard form of the self-similar solutions, that we will call *self-similarity of Type I*. In that case, $F(\eta) = u(\eta, 1)$ is called the *profile* of the solution.

- However, we can also think of the solutions defined for negative time and then we fix $(x,t)$ with $t < 0$ and put $\lambda = -1/t$ (so that $\lambda > 0$) to get

$$u(x,t) = (-t)^{-\alpha} u(x\,(-t)^{-\beta}, 1) = (-t)^{-\alpha} F(x\,(-t)^{-\beta}).$$

In that case, it is more common to set the final time at a certain $T > 0$ and think of the solutions defined for $-\infty < t < T$ or for $0 < t < T$. The standard form of *self-similarity of Type II* is then $u(x,t) = (T-t)^{-\alpha}$ $F(x\,(T-t)^{-\beta})$. with the same relation between and $\alpha$ y $\beta$ as before. The names *forward self-similarity* for Type I and *backward self-similarity* for Type II are used to refer to the direction in which the time stretches to infinity respectively. For reasons of convenience in stating the most usual results, we will change the sign of $\alpha$ and $\beta$ and use in this case the form

$$\boxed{u(x,t) = (T-t)^{\alpha} F(x\,(T-t)^{\beta}).} \tag{16.15}$$

with the relation $(m-1)\alpha + 2\beta + 1 = 0$. See more in Section 16.8.

- Finally, we have the case $\gamma = -2/(m-1)$, i.e., $\alpha(1-m) = 2\beta$. The scaling group is given by (16.11), hence

$$u(x,t) = \mu^{-2/(m-1)} u(\mu x, t),$$

so that, putting $\mu = 1/|x|$ we get

$$u(x,t) = |x|^{2/(m-1)} F(\omega, t). \tag{16.16}$$

This is a similarity of a quite different type, related to (16.13).

## 16.2 Three types of time self-similarity

We now view the subject of self-similarity from a different angle. We want to examine the class of solutions of the PME that take on the general self-similar form

$$u(x,t) = A(t) F(B(t)x) \tag{16.17}$$

for functions $A(t)$ and $B(t)$ to be selected. The reader will see that these are the solutions that possess a profile $F(\eta)$ that is constant in time when both $x$ and $u$ are properly rescaled, or in other words *zoomed*, in a way given by

$$u = A(t)\,F, \quad x = \eta/B(t).$$

Substituting into the equation gives

$$A'(t)F(\eta) + A(t)(\nabla F(\eta)\cdot x)B'(t) = A^m(t)B(t)^2\Delta F^m(\eta).$$

Since we want non-trivial solutions, we assume $A(t) \neq 0$, $B(t) \neq 0$. In that case, a simple separation of variables argument implies that the following two conditions must hold:

$$A'(t) = -\lambda A^m(t)B(t)^2, \quad B'(t) = -\mu A^{m-1}(t)B^3(t) \tag{16.18}$$

with parameters $\lambda, \mu \in \mathbb{R}$, and the profile must satisfy the equation in $\eta \in \mathbb{R}^d$

$$\Delta F^m(\eta) + \lambda F(\eta) + \mu(\nabla F \cdot \eta) = 0. \tag{16.19}$$

We have to solve system (16.18) to find the possible zooming factors, $A(t)$ and $B(t)$. It is clear that there are solutions in the form of power functions

$$A(t) = (a+bt)^{-\alpha}, \quad B(t) = (a+bt)^{-\beta},$$

and this holds with $\lambda = \alpha b$, $\mu = \beta b$ if

$$(m-1)\alpha + 2\beta = 1. \tag{16.20}$$

We recover in this way the self-similar forms of Types I and II described above. The sign of the parameter $b$ is essential, denoting forward or backward self-similarity, while for $b = 0$ the solutions are just the harmonic constant profiles of Section 4.1. For different values of the exponents we recover other solutions of Chapter 4. Note also that for $\lambda = 0$ we get the solution of power form with $\alpha = 0$, $\beta = 1/2$.

Let us examine the general solution of system (16.18) for $A$ and $B$ in search of the possible novelties. Putting $X = A^{1-m}$, $Y = 1/B^2$, we have

$$X' = \lambda(m-1)Y^{-1}, \quad Y' = 2\mu X^{-1}.$$

This implies finally

$$\lambda(m-1)XX'' = -2\mu(X')^2.$$

Immediate integration of this equation gives $X' = cX^a$ for $a = -2\mu/\lambda(m-1)$ if $\lambda \neq 0$ (the case $\lambda = 0$ is already studied), and $c$ an arbitrary constant. Therefore, unless $a = 1$ we get a power expression for $X$ that leads to the already found expressions for $A$ and $B$.

*Exponential self-similarity*

We still have the possibility $a = 1$ in the preceding analysis, and then $X(t) = Ce^{ct}$, so that $A(t) = A_0\, e^{-ct/(m-1)}, B(t) = B_0\, e^{ct/2}$. It is more convenient to write this as

$$A(t) = A_0\, e^{-2\sigma t}, \quad B(t) = B_0\, e^{(m-1)\sigma t}$$

with parameter $\sigma \in \mathbb{R}$. The form of the *self-similarity of Type III* is therefore

$$\boxed{u(x,t) = e^{-\alpha t} F(e^{-\beta t} x),} \tag{16.21}$$

with $\alpha = 2\sigma$ and $\beta = -(m-1)\sigma$. This type is also called *exponential self-similarity.* In comparison with formulas (16.14) and (16.15), the relation of the similarity exponents is now

$$\alpha(m-1) + 2\beta = 0. \tag{16.22}$$

We have absorbed the constants $A_0$ and $B_0$ into $F$. The equation of the profile of Type III is

$$\Delta F^m(\eta) - \frac{2\beta}{m-1} F(\eta) + \beta \nabla F \cdot \eta = 0. \tag{16.23}$$

While self-similar solutions of Type I are certain to exist for all $t > 0$ but not necessarily for $t < 0$, and solutions of Type II exist for $t < T$ but not necessarily later, self-similar solutions of Type III are necessarily eternal solutions, which live for $-\infty < t < \infty$.

## 16.3 Self-similarity and existence theory

The preceding analysis of the scaling transformation combined with the existence theory of Chapters 12 and 13 allow us to obtain self-similar solutions almost for free for the PME. The idea is to consider the solution of the Cauchy problem for scale invariant data and show that this solution must be self-similar.

Assume that the initial data have the form

$$u_0(x) = A(\sigma)\,|x|^{-\gamma}, \quad \sigma = \frac{x}{|x|}, \tag{16.24}$$

where $A$ is a bounded function in $\mathbb{S}^{d-1}$, and $\gamma \in \mathbb{R}$. We know from the theory of Chapter 12 that the Cauchy problem for the PME ($m > 1$) has a unique non-negative weak solution corresponding to such data if and only if $-2/(m-1) < \gamma < d$. Let $u$ be the solution for such data. The scaling transformation (16.3) is now applied to $u$ to produce another solution $\tilde{u}$. Now, if $K = L^{-\gamma}$ we have as initial data

$$\tilde{u}(x,0) = KA\,|x/L|^{-\gamma} = u(x,0).$$

By uniqueness of solutions, we conclude that $u(x,t) = \tilde{u}(x,t)$, i.e.,

$$u(x,t) = \tilde{u}(x,t) = K\; u(x/L, t/T).$$

Since the scaling group operates under the conditions $K^{m-1} = L^2 T^{-1}$, we have $K = T^{-\gamma\beta}$, $L = T^{\beta}$ with $\beta = 1/(\gamma(m-1)+2)$. Repeating the argument of Subsection 16.1.2 we get the following result.

**Theorem 16.2** *For every initial data of the form* (16.24) *with $A$ a bounded function in $\mathbb{S}^{d-1}$, and $-2/(m-1) < \gamma < d$, the PME has a unique solution*

*$u(x,t)$; this solution is self-similar of Type I:*

$$u(x,t) = t^{-\alpha} F(x\, t^{-\beta}), \tag{16.25}$$

*where $\beta = \gamma/\alpha$ and*

$$\frac{1}{\alpha} = m - 1 + \frac{2}{\gamma} \tag{16.26}$$

*($\alpha = 0$ and $\beta = 1/2$ if $\gamma = 0$). If $A \geq 0$ then $F \geq 0$; if $A$ is constant, then $F$ is a radial function.*

The range of $\gamma$ is chosen so that $u_0 \in L^1_{loc}(\mathbb{R}^d)$ and the optimal growth condition is satisfied. The reader is asked to draw the graph of function (16.26), where he will notice that $\alpha$ is a monotone function of $\gamma$ , that $\alpha = 0$ for $\gamma = 0$, and that the interval $-2/(m-1) < \gamma < d$ is mapped onto $-\infty < \alpha < \alpha(d)$, where $\alpha(d)$ is the ZKB exponent. Note that this upper bound, coming from the theory of Chapter 12, is not essential for the theory of self-similar solutions that allow for sign changes; it can be improved with dipole solutions and other solutions with changing sign.

The argument also shows that the solution in self-similar form must live forever. Suppose for instance that $u$ were only defined in a finite time interval $0 < t < T_1$. Fix now $T > 1$ and define $K$, $L$ and $\tilde{u}$ as before. The latter is then defined for $0 < t < T_1 T$. But the argument shows then that $u = \tilde{u}$ in the common time interval $0 < t < T_1$. This means that $u$ is extended by $\tilde{u}$ to the interval $0 < t < TT_1$. By iteration, we may take $T_1 = \infty$.

*Heat equation and fast diffusion equation*

The same reasoning applies to the HE, and Theorem 16.2 applies in the range $-\infty < \gamma < d$ as corresponds to taking the limit $m \to 1$.

When applying the argument to the FDE, we have to find the range where weak solutions exist and are unique. When $0 < m < 1$ this range includes all locally integrable functions with no growth restriction, so Theorem 16.2 applies with $-\infty < \gamma < d$. As we have already pointed out, a main feature of fast diffusion is that solutions may vanish identically in finite time. This can happen even for non-negative solutions of the Cauchy problem if $m < (d-2)/d$, $d > 2$, with $\gamma = 2/(1-m)$, and is exemplified in self-similar solutions of Type II. See more details on this topic in [515].

## 16.4 Phase-plane analysis

We now specialize to radially symmetric solutions. In that case we can find self-similar solutions of the PME by direct ODE methods.[3] This is a quite popular technique that produces the whole family of solutions under various assumptions, and it also provides very detailed information.

[3] As we will explain below this restriction is not needed in $d = 1$.

### 16.4.1 *The autonomous ODE system*

We begin the study looking for solutions in the Type I self-similar form

$$U(x,t) = t^{-\alpha} F(\xi), \quad \xi = |x|\, t^{-\beta}, \tag{16.27}$$

for the PME/FDE equation written in the form[4]

$$\partial_t u = \Delta(|u|^{m-1} u/m) = \nabla \cdot (|u|^{m-1} \nabla u). \tag{16.28}$$

There is no major problem at this stage in letting $m$ be also less than 1; $m = 1$ is excluded though, since we will be dividing by $m-1$. We recall that parameters $\alpha$ and $\beta$ must be related by (16.20), i.e.,

$$(m-1)\alpha + 2\beta = 1. \tag{16.29}$$

We get the ODE (in terms of the variable $\xi$)

$$\xi^{1-d}(\xi^{d-1}|F(\xi)|^{m-1}F'(\xi))' + \beta\xi\, F'(\xi) + \alpha\, F(\xi) = 0. \tag{16.30}$$

Now comes the interesting point of this method.

**Proposition 16.3** *Let $U$ be a self-similar solution of the PME of the form* (16.27)*, and let us introduce the following variables:*

$$\xi = e^r, \qquad X(r) = \frac{\xi F'}{F}, \qquad \text{and} \qquad Y(r) = \xi^2 |F|^{1-m}. \tag{16.31}$$

*Functions $X(r)$ and $Y(r)$ satisfy the following autonomous system:*

$$\boxed{\begin{array}{c} \dot{X} = (2-d)X - mX^2 - (\alpha + \beta X)Y, \\ \dot{Y} = (2 + (1-m)X)Y, \end{array}} \tag{16.32}$$

*where $\dot{X} = dX/dr$.*

The system is derived in Problem 16.3. Note that, properly speaking, it is not a unique ODE system but a family of systems with two parameters that can be studied for its own sake. The analysis of this system is not difficult using the typical phase plane techniques developed in the theory of ODEs. We will make this analysis below in the most relevant cases.

In the case of application to the PME we need condition (16.29) to relate $\alpha$ and $\beta$, and thus it becomes a system with one free parameter, either $\alpha$ or $\beta$. It is common to use the parameter $\gamma = \alpha/\beta$ so that

$$\alpha = \frac{\gamma}{\gamma(m-1)+2}, \quad \beta = \frac{1}{\gamma(m-1)+2}. \tag{16.33}$$

[4]Dividing the right-hand side of the equation by $m$ is a harmless normalization, equivalent to using time $t' = t/m$, but it is very important in fast diffusion for $m$ near or below 0, as we do in [515]. Therefore, we keep such a normalization here.

We will call it in the sequel System (S1). It gives a very detailed knowledge of the class of self-similar solutions of the PME/FDE equation. Note that positive and negative branches of signed solutions are represented on the same points.

Note finally that we are restricting ourselves to radially symmetric solutions in dimensions $d \geq 2$. In dimension $d = 1$ this restriction can be eliminated by studying separately the domains $x > 0$ and $x < 0$ (apply reflection to the latter to see that the system is the same). A global solution is then obtained by gluing together both pieces with the correct matching condition: continuity of $u$ and $(u^m)_x$ at $x = 0$.

### 16.4.2 *Analysis of system (S1)*

The phase-plane analysis revolves around the consideration of invariant regions, the existence and properties of the critical points together with 'the behaviour at infinity'.

*Invariance and trajectories*

It is very easy to see that the line $Y = 0$ is invariant and has simple dynamics given by the equation

$$\dot{X} = (2-d)X - mX^2,$$

which makes $X = 0$ an attractor for $d > 3$, a repeller for $d = 1$ and a degenerate critical point for $d = 2$. The other critical point is $X = (2-d)/m$ which is a repeller for all $d \geq 3$ and an attractor for $d = 1$. We ask the reader to draw the simple flow diagrams.

We also see that the upper half plane $H^+ = \{Y > 0\}$ is an invariant region. This is where self-similar solutions of Type I will live. Every trajectory $(X(r), Y(r))$ of the system in that region will give rise to two self-similar solutions of the PME, one positive and one negative, which will be defined in a maximal $\xi$ interval, either finite or infinite. Actually, the invariance of the system under translations of $r$ (it is autonomous) implies that we get not two branches but two one-parameter families. It is easy to see that changing $r$ into $r + c$ implies a similarity transformation on the self-similar solution that is obtained. Actually, if $c = \log \lambda$, we have

$$|\widetilde{F}(\xi)|^{m-1} = \frac{\xi^2}{\widetilde{Y}(r)} = \frac{\xi^2}{Y(r+c)} = \lambda^{-2}\frac{(\lambda\xi)^2}{Y(\log(\lambda\xi))} = \lambda^{-2}|F(\lambda\xi)|.$$

On the other hand, the flow at $X = 0$ is given by $\dot{X} = -\alpha Y$, $\dot{Y} = 2Y$, We conclude that it enters the first quadrant $Q_1 = \{X > 0,\ Y > 0\}$ with positive slope it $\alpha < 0$, and the second quadrant if $\alpha > 0$. The latter situation happens for the PME if $\gamma > 0$ (for the FDE, the conditions are $m > (d-2)/d$ and $\gamma > 0$ or $m < (d-2)/d$ and $\gamma < 0$). For $\gamma = 0$ the vertical axis is a trajectory, representing constant solutions. The signs of $\alpha$ and $\beta$ will play a big role in the analysis that follows.

*Critical points*

There are at most three critical points for the system in the whole plane, depending on the parameters. Indeed, the second line of system (16.32) selects the values $Y = 0$ and $X = 2/(m-1)$. For $Y = 0$ there exist two critical points:

$$P_0 = (0,0), \quad P_1 = ((2-d)/m, 0).$$

The latter is defined if $m \neq 0$, and is different from $P_0$ if $d \neq 2$. The third option is the point $P_2 = (X_B, Y_B)$ with the choice $X_B = 2/(m-1)$ given by the second line. Then, the first line vanishes at the so-called isocline of vertical slopes,

$$Y = -\frac{X(mX + d - 2)}{\beta(\gamma + X)},$$

which for $X = X_B$ gives

$$Y_B = -X_B(2 + d(m-1)) = -\frac{2\,(2 + d(m-1))}{m-1}.$$

The point $P_2$ is defined for $m \neq 1$ and coincides with $P_1$ for $m = (d-2)/d < 1$ (unless $d = 2$, when $P_1$ is not defined).

When we look for Type I self-similar solutions, we have to consider only the $Y \geq 0$ part of the $XY$-plane. For $m > 1$ there are two critical points in that region, $P_0$ and $P_1$ (both of them coincide when $d = 2$, which is a special and easier case). The local analysis of these points is straightforward. We always assume that $m > 0$.

**Proposition 16.4** *The linearization of system* (S1) *around* $P_0 = (0,0)$ *has matrix*

$$\begin{pmatrix} 2-d & -\alpha \\ 0 & 2 \end{pmatrix},$$

*with eigenvalues* $\lambda_1 = 2 - d$ *and* $\lambda_2 = 2$ *and corresponding eigenvectors* $\mathbf{e_1} = (1,0)$ *and* $\mathbf{e_2} = (-\alpha, d)$. *Thus,* $P_0$ *is a saddle when* $d > 2$, *a repeller when* $d < 2$, *and a saddle-node for the bifurcation value* $d = 2$.

In any case there is always a solution branch getting out of $(0,0)$ along the direction $\mathbf{e_2}$, i.e., $Y/X \sim -d/\alpha$. It corresponds to solution profiles which start from $\xi = 0$ with any height $F(0) = a > 0$ and $F'(0) = 0$. For $d > 2$ this is the only solution that exits $(0,0)$.

**Proposition 16.5** *The linearization of system* (S1) *around* $P_1 = (\frac{2-d}{m}, 0)$ *has matrix*

$$\begin{pmatrix} d-2 & \dfrac{\beta d - 1 - \alpha}{m} \\ 0 & \dfrac{d(m-1)+2}{m} \end{pmatrix},$$

*with eigenvalues* $\lambda_1 = d - 2$ *and* $\lambda_2 = (d(m-1)+2)/m$, *and corresponding eigenvectors* $e_1 = (1, 0)$ *and* $e_2 = (\beta d - 1 - \alpha,\ 2m - d + 2)$. *Thus, if* $d(m-1) + 2 > 0$ *(i.e., for PME or for FDE with* $m$ *near 1),* $P_1$ *is a repeller when* $d > 2$, *a saddle when* $d < 2$, *and a saddle-node for the bifurcation value* $d = 2$. *The eigenvalue is double for the so-called critical exponent* $m_c = (d-2)/2$.

For $m > (d-2)/d$ there is always a solution branch getting out of $P_1$ along the direction $\mathbf{e_2}$. If $d > 2$ this corresponds to a singularity at the origin of the form $|F(\xi)| \sim \xi^{-(d-2)/m}$. These solutions do not appear in the standard PME theory. For $d = 1$ the singularity is bounded and takes the form $|F(\xi)| \sim \xi^{1/m}$ which represents change of sign at $\xi = 0$, giving rise to antisymmetric solutions in the whole line $x \in \mathbb{R}$. We will not consider the cases $m \leq (d-2) = d$ that can be studied by similar methods but lead away from our present interest. The interested reader can consult [515], Chapters 3 and 5.

The study of the linearization around the point $(X_B, Y_B)$ is more complicated but we will need it less in the standard applications. See Problem 16.5.

*Periodic orbits*

For $m > 1$ there are no periodic orbits of system (S1) in the half plane because all orbits come out of the critical points $P_0$ and $P_1$ or from infinity and must go to infinity eventually. If $0 < m < 1$ the critical point $P_2$ comes into the half plane $H^+$ when $(d-2)/d < m < 1$, which gives rise to the singular solution $F = c\xi^{-2/(1-m)}$. Linearization around this point shows that it is a saddle so that no periodic solutions can appear around it.

*Analysis of infinity*

In order to complete the behaviour of the trajectories as $X$, $Y$ or both diverge, the usual idea in the dynamical systems literature is to compactify the plane in order to see clearly the dynamics at those diverging points, Poincaré compactification cf. [426]. There are different versions. One of them is to introduce rescaled polar coordinates

$$X = \frac{\rho}{1-\rho}\cos\phi, \qquad Y = \frac{\rho}{1-\rho}\sin\phi. \tag{16.34}$$

In this way the orbits are mapped into the region $\{0 \leq \rho \leq 1, 0 \leq \phi \leq \pi\}$. The points of infinity of the $(X, Y)$ plane correspond to the line $\rho = 1$ of the new system. This approach is taken for instance by Hulshof in [296] who writes the new system in $d = 1$ and finds the critical points at infinity, with $\rho = 1$ and

$$\phi = 0,\ \arctan(-1/\beta),\ \pi/2,\ \pi.$$

The plot uses the idea of the polar variables to renormalize the $XY$-plane by defining $\widetilde{X} = \rho X/R$ and $\widetilde{Y} = \rho Y/R$, $R = (X^2 + Y^2)^{1/2}$, $\rho = R/(1+R)$. We will use another way of addressing the problem of infinity in the next sections, consisting in inversions of one or both variables.

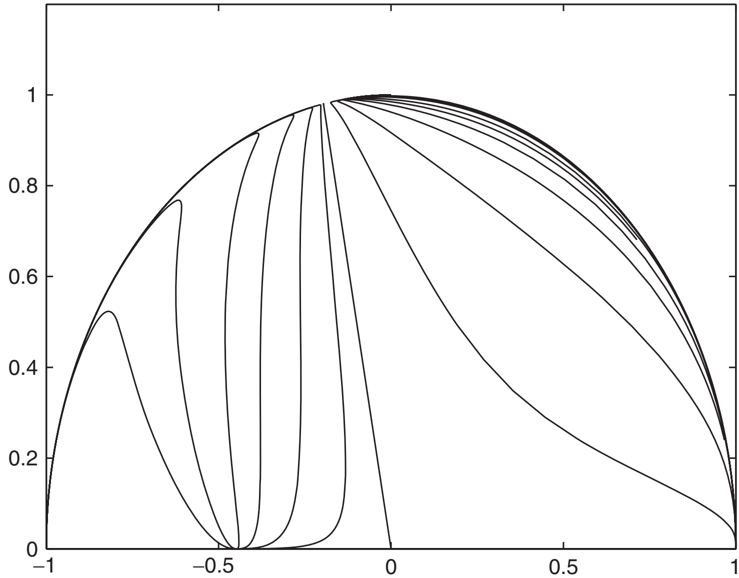


Figure 16.1: The compactified $XY$-plane for $d = 3$, $m = 2$, $\gamma = d$.

### 16.4.3 *Some special solutions. Straight lines in phase plane*

When looking for special solutions of the form (16.27), it is interesting to consider the solutions that correspond to straight line orbits in the $(X, Y)$ plane. We have the trivial solutions $F = 0$, which correspond to orbits in the axis $Y = 0$. Apart from them we have

(i) The constant solutions $F = C$ occur for $\gamma = 0$ with $X = 0$, $Y > 0$.

(ii) Solution lying on a vertical line, $\dot{X} = 0$, occur for $\gamma = (2 - d)/m$, $X = -\gamma$ and variable $Y > 0$, so that

$$F^m = C\,|\xi|^{2-d}.$$

They have a singularity at $\xi = 0$ that implies that the PME is not satisfied at $x = 0$ for $u$.

(iii) In the class of slanted lines the only admissible one occurs for $\gamma = d$ and takes the form

$$dX + \alpha Y = 0,$$

which for $\alpha > 0$ (which is equivalent to $m > m_c = (d - 2)/d$) gives the ZKB solution. For completeness, let us say that there is also the possibility $\alpha < 0$ (equivalent to $m < m_c$), which gives a solution in the quadrant

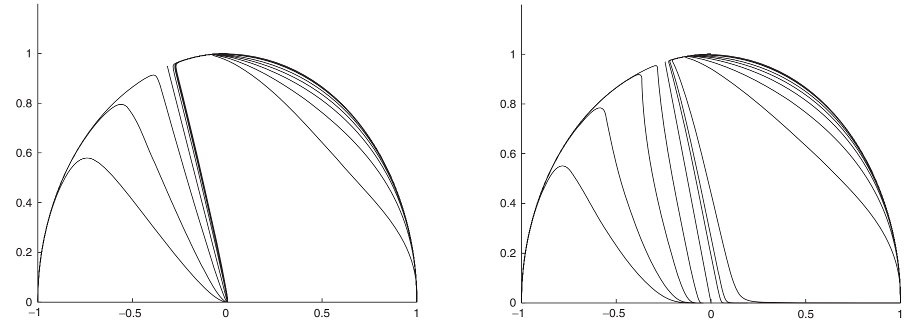


Figure 16.2: Standard $XY$-planes for $d = 1$ (left) and $d = 2$ (right), with $m = 2$, $\gamma = d$.

$Q = \{X > 0,\ Y > 0\}$ which is not globally defined in space. Its precise profile is

$$F^{1-m} = \frac{1}{(C - c|\xi|^2)_+},$$

where $C > 0$ is arbitrary and $c = |\alpha|(1-m)/2d$.

### 16.4.4 *The special dimensions*

The analysis of the phase plane $(X, Y)$ in this dimensions has special properties because of the different location of the critical points and the different dynamics on the horizontal axis. Thus, point $P_1$ moves to the positive $X$ axis in $d = 1$, and merges with $P_0$ in $d = 2$. But in any orbits enter $H^+$ from them and the dynamics near infinity is not changed.

There is another noticeable change in $d = 1$. The point $P_1$ represents a behaviour of the form $X(\xi) \sim 1/m$, which means that $(F^m)'(0) \sim c \neq 0$. This is the behaviour for sign change at $\xi = 0$.

## 16.5 An alternative phase plane

There is another way of attacking the study of non-negative solutions of the PME/FD of the forward self-similar form (16.27), i.e., $U(x,t) = t^{-\alpha}F(\xi)$ with $\xi = |x|\,t^{-\beta}$ and $\alpha$ and $\beta$ satisfying $\alpha(m-1) + 2\beta = 1$. It leads to a new phase plane that allows us to resolve some of the difficulties of the study of system (S1) when the orbits go to infinity. We concentrate in the case $m > 1$ to focus attention and minimize the discussion.

Let us make first a direct derivation and then relate both approaches. We will work with the pressure variable $v = u^{m-1}/(m-1)$. We know that this variable satisfies the equation

$$v_t = (m-1)v\Delta v + |\nabla v|^2, \qquad x \in \mathbb{R}^n \setminus \{0\}.$$

We look for radially symmetric solutions. This time we will write the self-similar solutions in the form

$$v = \frac{|x|^2}{t}\,\Phi(|\xi|), \qquad \xi = x\,t^{-\beta}. \tag{16.35}$$

Comparing with the self-similar form we see that the profiles are related by

$$|\xi|^2\Phi(|\xi|) = \frac{1}{m-1}F^{m-1}(|\xi|). \tag{16.36}$$

We want to write a system of ODEs for $\Phi$ that hopefully leads to a phase plane that is as easy to examine as (16.32) but hopefully offers new insights. We observe that the factor $|x|^2/t$ has the same dimensions as the pressure (length squared over time). Thus, $\Phi$ is adimensional. The ordinary differential equation satisfied by $\Phi$ is

$$\begin{aligned}(m-1)\Phi|\xi|^2\Phi'' + \big((m-1)(d+3)+4\big)\Phi|\xi|\Phi' \\ +\big(2d(m-1)+4\big)\Phi^2 + |\xi|^2(\Phi')^2 + \beta|\xi|\Phi' + \Phi = 0.\end{aligned} \tag{16.37}$$

This equation can be made autonomous by introducing the new independent variable $\tau = \log|\xi|$. Written as a first-order system the resulting equations are

$$\left.\begin{aligned}&\dot\Phi = \Psi, \\ &(m-1)\dot\Psi = -[(m-1)(d+2)+4]\Psi - [2d(m-1)+4]\Phi - \frac{\Psi}{\Phi}(\Psi+\beta) - 1,\end{aligned}\right\} \tag{16.38}$$

where the dot denotes differentiation with respect to $\xi$. This is our first representation of the self-similar solutions. System (16.37) is singular at $\Phi = 0$; we remove the singularity by making the nonlinear change of variable given implicitly by $d\tau/ds = \Phi(\xi)$. Then, $\Phi(s)$ and $\Psi(s)$ satisfy

$$\left.\begin{aligned}&\frac{d\Phi}{ds} = \Phi\Psi, \\ &(m-1)\frac{d\Psi}{ds} = -[(m-1)(d+2)+4]\Phi\Psi - [2d(m-1)+4]\Phi^2 - \Psi(\Psi+\beta) - \Phi\end{aligned}\right\}. \tag{16.39}$$

Observe that this change of variable reverses the flow in the $\{\Phi < 0\}$ region. We are mainly looking for positive solutions; thus, we want to draw conclusions about the $\Phi \geq 0$ part of the $\Phi\Psi$–plane. We call this system (S2).

*Relating both phase planes*

It is quite easy to see that both representations are related by the formulas

$$(m-1)\,X(\xi) = \frac{\Psi(\xi)}{\Phi(\xi)} + 2, \qquad (m-1)Y(\xi) = \frac{1}{\Phi(\xi)}. \tag{16.40}$$

In the converse direction we have $\Phi = 1/((m-1)Y)$ and

$$\frac{X}{Y} = \Psi + 2\Phi. \tag{16.41}$$

Since we are assuming that $m > 1$, $Y > 0$ implies $\Phi > 0$ and $X \le 0$ implies $\Psi/\Phi < -2$. On the other hand, the transformation from one phase plane to the other is singular, which makes it interesting because what one does not see in one of them can probably be seen in the other one. Thus,

- the critical point $P_0 = (0,0)$ in the $XY$ plane is mapped into $P_0^* = (\infty, \infty)$ in terms of $(\Phi, \psi)$ with $\Psi/\Phi = -2$;
- the critical point $P_1 = ((2-d)/m, 0)$ is mapped into $P_1^* = (\infty, \infty)$ in the direction $\Psi/\Phi = -(d(m-1)+2)/m$;
- finally, $P_2 = (X_B, Y_B)$ is mapped into $P_2^* = (\Phi_B, \Psi_B)$

$$\Phi_B = -\frac{1}{2\,(2 + d(m-1))}, \qquad \Psi_B = 0. \tag{16.42}$$

What is more interesting, we can map the whole set of asymptotic directions $Y/X = \lambda$ with $Y \to \infty$ into the lines

$$\Psi + 2\Phi = \frac{1}{\lambda}$$

and take the limit as $\Phi \to 0+$. This means that the dynamics of system (S2) near the vertical axis $\Phi = 0$ is equivalent to the behaviour at infinity of system (S1).

*Analysis of the new plane*

Again, the analysis of this phase plane revolves around invariant lines and regions, the existence and properties of the critical points, together with 'the behaviour at infinity'.

We immediately see that the region $\Phi > 0$ (which corresponds to $Y > 0$) is invariant and the solutions we are studying now live in it. Also the line $\Phi = 0$ is invariant and has two critical points on it, $\Psi = 0$ and $\Psi = -\beta$. The flow goes down for all large $\Psi$. If $\beta > 0$ then $\Psi = 0$ is an attractor on this line, $\Psi = -\beta$ a repeller, if $\beta < 0$ it is the other way around.

Let us examine the critical points: there are two critical points in this representation on the vertical axis, $P_3 = (0,0)$ and $P_4 = (0, -\beta)$, plus the point $P_2^* = (\Phi_B, \Psi_B)$. The last point lies in the half plane $\Phi < 0$ and is not to be considered if $m > 1$.

**Proposition 16.6** *The critical point* $P_3 = (0,0)$ *is a saddle-node of system* (16.39). *The linearization of* (16.39) *around* $P_3$ *has matrix*

$$\begin{pmatrix} 0 & 0 \\ -\frac{1}{m-1} & -\frac{\beta}{m-1} \end{pmatrix},$$

*with eigenvalues* $\lambda_1 = 0$ *with eigenvector* $\mathbf{e_1} = (\beta, -1)$ *and* $\lambda_2 = -\beta/(m-1)$ *along the vertical axis,* $\mathbf{e_2} = (0, 1)$.

**Proof** Centre manifolds are tangent to $\mathbf{e_1}$. Since the flow enters the origin along this direction from the half plane $\Phi > 0$, it is a node on this part, the one that interests us. All the other trajectories in this half plane enter the origin parallel to this direction. ∎

*Interpretation*

Approaching this point corresponds in the $XY$-plane to approaching the asymptote $X = -\gamma = -\alpha/\beta$, $Y = \infty$. This is a way of recovering possible asymptotes in the $XY$-plane in the form of finite points in the new plane. This option means that the solution profile behaves like $|x|^{-\gamma}$ either as $\xi \to \infty$ (if $\beta > 0$). The remaining entering direction corresponds to the axis and does not count.

**Proposition 16.7** *The critical point* $P_4 = (0, -\beta)$ *is a saddle. The linearization of* (16.39) *around* $P_4$ *has matrix*

$$\begin{pmatrix} -\beta & 0 \\ \frac{\beta((m-1)(d+2)+4)-1}{m-1} & \frac{\beta}{m-1} \end{pmatrix},$$

*with eigenvalues* $\lambda_1 = -\beta$ *with eigenvector* $\mathbf{e_1} = (1, a)$ *and* $\lambda_2 = \beta/(m-1)$ *along the vertical axis,* $e_2 = (0, 1)$. *The value of* $a$ *is*

$$a = \frac{1 - \beta((m-1)(d+2)+4)}{m\beta}.$$

*Interpretation*

The stable manifold in the direction $\mathbf{e_1}$ corresponds to solutions that enter the point $P_4$ for a finite value of the parameter, $\xi \to \xi_0 < \infty$. Approaching this point corresponds in the $XY$-plane to approaching $X \to -\infty$ and $Y = -\infty$. Since the value $\xi_0$ is finite, the self-similar solution has compact support. Moreover, we have

$$\frac{1}{m-1}(F^{m-1})' = \frac{\xi X}{Y} = \xi(\Psi + 2\Psi) \to -\xi_0\,\beta,$$

which is the correct version of Darcy's law on the free boundary $\xi = \xi_0$. This point plays a role in the analysis of the solutions with compact support. Let us review the argument about the Darcy law: the velocity of the free boundary in the normal direction, $\mathbf{v}_n$, satisfies

$$\mathbf{v}_n = -\nabla v \cdot \vec{n} \tag{16.43}$$

at points of the free boundary where it is smooth, as in this case. Indeed, for solutions of the form (16.35) the free boundary is given by

$$|x| = e^{\tau_0} t^{\beta} = \xi_0\, t^{\beta}. \tag{16.44}$$

Hence, (16.43) implies that $\Psi(\xi_0) = -\beta$.

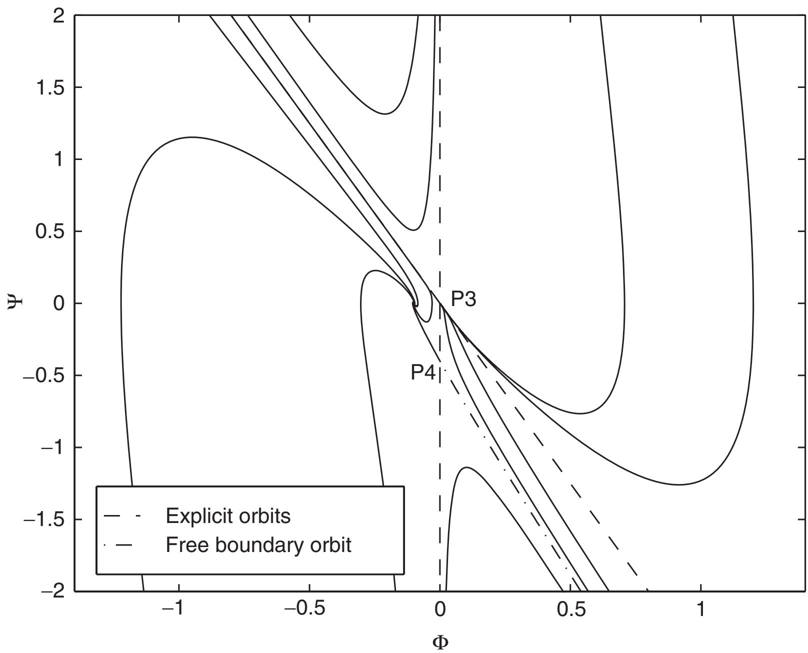


Figure 16.3: ΨΦ–plane for $d = 3$, $m = 2$ and some $\beta > 0$.

## 16.6 Sign-change trajectories. Complete inversion

In view of the previous analysis, we still have to analyse the orbits that go to infinity in both variables of the $XY$-plane. It is clear from the analysis of system (S2) that the such trajectories correspond in the $\Phi\Psi$-plane to trajectories tangent to the vertical axis, and they are outgoing when the limit is $\Psi = +\infty$ and incoming if $\Psi \to -\infty$. In view of (16.41) the trajectories in the $XY$ plane satisfy $Y/X \to 0$. It is then an easy task to solve approximately system (S1) and find the asymptotic behaviour as $X \to \infty$, $Y \to \infty$ and $Y/X \to 0$. We have

$$\frac{dY}{dX} \sim \frac{(1-m)XY}{-mX^2} \tag{16.45}$$

which leads first to the expression $Y \sim c\,X^{\frac{m-1}{m}}$. Using this in the definitions for $X$ and $Y$ we conclude that the profiles they represent touch the axis at a finite distance, $F(\xi_0) = 0$, with a slope $(F^m)'(\xi_0) \neq 0$, i.e.,

$$\lim_{\xi \to \xi_0} (F^m)'(\xi = c_1 \neq 0.$$

They are therefore the profiles of changing sign solutions. In order to better visualize this 'point at infinity' of the $XY$-plane, that we shall call $P_5$, we may

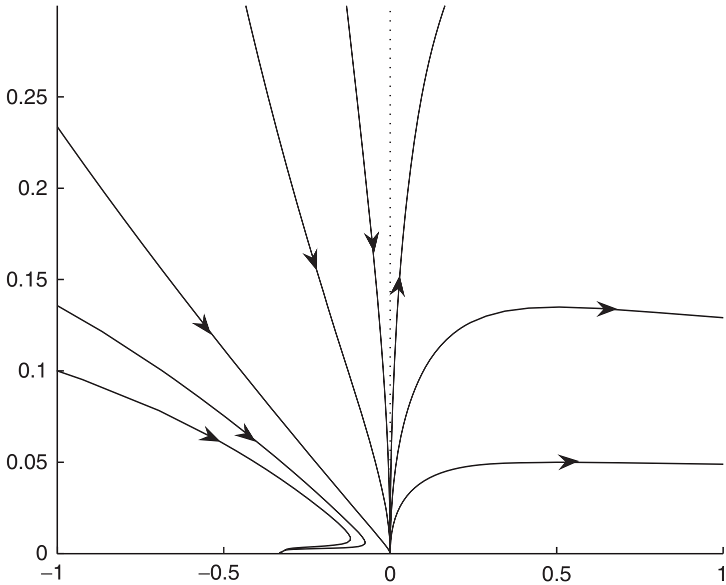


Figure 16.4: The inverted plane $(x, y)$ for $d = 3$, $m = 2$, $\gamma = d$.

perform the complete inversion

$$x = \frac{1}{X}, \qquad y = \frac{1}{Y}, \tag{16.46}$$

thus arriving at system (S3):

$$\left.\begin{aligned} \dot{x} &= \frac{1}{y}(my + \beta x + (d-2)xy + \alpha x^2), \\ \dot{y} &= \frac{1}{x}\,((m-1) - 2x)y. \end{aligned}\right\} \tag{16.47}$$

We view that this system does not define any trajectories on the vertical axis $x = 0$, nearby trajectories slip downwards along that axis for $x < 0$, while they slip upwards for $x > 0$. The orbits on the first quadrant starting from the node $(0, 0)$ are the orbits coming from infinity with behaviour (—16.45), i.e., $y \sim c\,x^{\frac{m-1}{m}}$. On the second quadrant the situation is more complicated because we have incoming orbits into zero (i.e., outgoing in the $XY$-plane) of two types:

- the 'linear entrance' $y \sim -\beta x$, and
- the 'slow trajectories' of type $y \sim -c\,|x|^{\frac{m-1}{m}}$.

The first is the entrance into the free boundary, point $P_4$ while the last is $P_5$. This completes the analysis. Note the vertical asymptotes that appeared as point $P_3$ in system (S2) appear here as entrance into $P_4^* = (-1/\gamma, 0)$.

**Remark** Notice that $y$ in the third system equals $\Phi$ in the second. In fact, system (S2) can be considered as a partial inversion. It has over (S3) the advantage of separating $P_4$ and $P_5$ while in system (S3) both are merged at the origin. Strategies to split into simpler, separate points the analysis of complicated critical points into simpler, separate points are standard in the dynamical systems literature.

*The mechanism of change of sign*

A change of sign in a profile $F$ at a value $\xi_0 > 0$ implies outgoing the half plane $H^+$ through 'point' $P_5$ on the left (as $X \to -\infty$) and appearing in the next branch on the right (as $X \to +\infty$). Remember that branches of negative and positive $u$ are undistinguishable in the $(X, Y)$ representation. In order to connect two branches at the point of sign change and still get a weak solution, we need equality of the flux

$$\lim_{\xi \to \xi_0} (|F|^{m-1}F)'(\xi)$$

on both sides. In view of the above analysis, this means that we have to make sure the constant in the asymptotic expression $Y \sim c\,X^{\frac{m-1}{m}}$ on the outgoing part (for $X \to -\infty$) matches the corresponding constant at the beginning of the next branch (for $X \to +\infty$).

### 16.6.1 *Global analysis. Applications*

Summing up, we have found a maximum of six extended critical points in the compactified analysis. In the PME case and for forward self-similarity, only five are used. Combining the information supplied by system (S2) we can complete the analysis of the behaviour of the solutions of system (S1) contained in the upper half space $H^+$. We now know that orbits going into infinity must do so in the directions $Y/X = -1/\beta$, $\infty$, or $Y/X = 0$, and we have classified these points: the first is free boundary point, the second is the outgoing direction to the asymptote $X = -\gamma$, the third is the solution that touches the axis with $(F^{m-1})' \neq 0$. The orbits may emanate from $P_0$ (they are bounded solutions) or from $P_1$ (they have a singularity at $\xi = 0$), or come from infinity (changing sign solutions).

*Initial value problems*

We use the information of this analysis in the study of the properties of solutions with data of the form $u_0(x) = A\,|x|^{-\gamma}$ with $A > 0$. We assume that $m > 1$. As we have said, we only need to consider orbits where $\{Y > 0\}$.

- Assume first that $\gamma > 0$ (decreasing profiles) so that $\alpha > 0$. We work in the quadrant $Q_2 = \{X < 0,\ Y > 0\}$. If $\gamma < d$ the existence theory says that there exists a bounded solution and that $F$ is monotone decreasing. We start from $(0, 0)$ in the $(X, Y)$ plane and move into $Q_2$ along the outgoing direction. As for the end values, since $\beta > 0$ we have to look at the limit as $\xi \to \infty$ to recover the initial situation, i.e., as $r \to \infty$. We have

$F(\xi) \sim A\xi^{-\gamma}$ as $\xi \to \infty$, hence

$$X(\infty) = -\gamma, \quad Y(r) \sim \xi^{-\gamma(1-m)+2}$$

as $r \to \infty$, and the latter goes to $\infty$ since $\beta > 0$.

When $\gamma = d$ the trajectory starting from $P_0$ is the straight line that corresponds to the ZKB solution and it ends in the free boundary point $P_4$.

If $\gamma > d$ then the existence theory does not guarantee a solution in the whole plane so that starting from $(0,0)$ leads into another critical point of the compactified plane. We ask the reader to check that it must be the change of sign 'point', $P_5$. So the solutions undergo one or several sign changes. This is carefully studied in [296].

- Let now $\gamma < 0$ (increasing profiles). If $0 < -\gamma < 2/(m-1)$, then, $\alpha < 0$ so that so that starting from $P_0$ we enter the quadrant $Q_1 = \{X > 0,\ Y > 0\}$. The existence theory comes again into play to ensure that we get a trajectory corresponding to an increasing profile and that $X \to -\gamma > 0$ with a vertical asymptote in the $(X,Y)$ plane.
- For $\gamma < 2/(m-1)$ the phase plane situation changes dramatically. But this deserves a separate treatment that will be done just below.

*One-dimensional signed solutions*

In one dimension there is no problem in considering solutions for data of the form

$$u_0(x) = c_1 x^{-\gamma} \quad \text{for} \quad x > 0, \qquad u_0(x) = -c_2|x|^{-\gamma} \quad \text{for} \quad x < 0, \tag{16.48}$$

with $c_1, c_2 > 0$, which leads to signed solutions. In the symmetric case $c_1 = c_2$, the self-similar solution must have a zero for $\xi = 0$. This was explained in Subsection 16.4.4 as the trajectory starting from $P_1$. When $c_1 \neq c_2$ the sign change takes place at a place different from $\xi = 0$.

## 16.7 Beyond blow-up growth. The oscillating signed solution

Let us continue the discussion of the existence of radial self-similar solutions of Type I started in Subsection 16.6.1. We assume that $m > 1$. Taking data of the form $u_0(x) = A\,|x|^{\gamma}$, we know that for

$$-\frac{2}{m-1} < \gamma < d \tag{16.49}$$

the Cauchy problem for the PME must have a unique non-negative weak solution, since these bounds imply that $u_0 \in L^1_{loc}(\mathbb{R}^d)$ and the growth conditions of Chapter 12 hold. On the other hand, Section 16.3 implies that the unique solution must be self-similar of the form (16.27). In view of equation (16.33), we see that the growth factor $t^{-\alpha}$ has an exponent $\alpha$ that goes to infinity as $\gamma \downarrow -2(/m-1)$. This limitation is in agreement with the fact that growth of the order of $u_0(x) \sim |x|^{2/(m-1)}$ leads to blow-up in finite time, as exemplified by the explicit solutions (4.43), (4.44).

Our concern here is what is the situation when we try to solve the ODE system in the case where $\gamma < -2/(m-1)$ so that the growth condition of the existence theory will be violated, and no solution can come from initial data of the form $u_0(x) = A\,|x|^\gamma$. The questions are: *What do the trajectories of the ODE system represent? Where do they come from in terms of initial data of the PME?*

If we go to the phase plane $(X, Y)$ we see definite changes with respect to the situation for $-\gamma < 2/(m-1)$. Indeed, for $-\gamma > 2/(m-1)$, the following happens:

$$\alpha > \frac{2}{m-1} \quad \text{and} \quad \beta < 0,$$

the trajectory starting from $P_0$ enters the second quadrant, the free boundary point $P_4$ moves to the first quadrant and its direction $\mathbf{e_1}$ becomes outgoing, and finally the line of horizontal slopes $X = 2/(m-1)$ trades places with the vertical asymptote $X = -\gamma$. A qualitative analysis of the phase plane shows that shooting from $P_0$ can only go into the sign change point $P_5$ in the limit $X \to -\infty$, the only attractor in the second quarter of the $XY$-plane. We remark that trajectories coming from the first quadrant can have originated at three possible directions: the vertical asymptote $X = -\gamma > 0$, the free boundary 'point' $P_4$ or the sign-change 'point' $P_5$ in its limit for $X \to \infty$.

Some striking consequences of this novel situation were studied in Vázquez [505] in dimension $d = 1$. It begins with the study of the profile functions and shows in Theorem 1 that all solutions of the ODE for $F$ have infinitely many oscillations as $\xi \to \infty$, and in Corollary 5 that the peaks of the profile grow like $F_{\max} - F_{\min} \sim c\,\xi^{2/(m-1)}$. Because of the oscillations and the fact that $F \to 0$ as $\xi \to 0$, all solutions decay in time like $u(0,t) = O(t^{1/(m-1)})$ even if the factor $t^{-\alpha}$ is highly increasing.

Once this preliminary analysis is done, two particular solutions are constructed:

**Theorem 16.8** (i) *There exists a particular self-similar solution $u_1(x,t)$ with a profile that vanishes for $|\xi| \le \xi_0$ for some $\xi_0 > 0$.*

(ii) *There exists a particular self-similar solution $u_2(x,t)$ with a profile that oscillates infinitely many times in $0 < \xi < \infty$ in a rescaled periodic fashion, in the sense that*

$$F(\lambda\xi) = \lambda^{2/(m-1)} F(\xi). \tag{16.50}$$

The proofs in [505] rely on direct analysis of the profile $F$ and the flux $(|F|^{m-1}F)'$ (called $U$ and $V$ in the paper), but the use of the $XY$-phase plane is quite convenient to get an alternative form of the proofs.

*Consequences*

In view of the form of the solutions

$$u(x,t) = t^{-\alpha} F(x\, t^{-\beta})$$

and the fact that $\beta < 0$, we see that solution $u_1(x,t)$ vanishes in a set of the form

$$|x| \le \xi_0\, t^{\beta}$$

i.e., a ball with a radius that goes to infinity as $t \to 0$ and to zero as $t \to \infty$ (a shrinking hole lasting for an infinite time). The conclusion for the optimal theory of Chapter 13 is that

**Corollary 16.9** *Solution $u_1$ has a trivial initial trace $\mu \equiv 0$ in the sense of Definition* 13.1*, but this solution is not global in time and non-trivial for all times $t > 0$.*

We have explained this graphically as 'a mass coming from infinity'. As for solution $u_2(x,t)$, the consequence is

**Corollary 16.10** *Solution $u_2(x,t)$ has no initial trace because of the oscillations.*

These results mark a strong difference between the theory of non-negative solutions of the PME and the theory of general solutions of changing sign.

## 16.8 Phase plane for Type II

This is the type of similarity that corresponds to formula (16.15):

$$U(x,t) = (T-t)^{\alpha} F(x\,(T-t)^{\beta}), \qquad \text{for } x \in \mathbb{R}^d,\ t < T. \tag{16.51}$$

We point out again the change the signs of the exponents, because this makes the signs positive in typical applications and the ODE formally similar. We recall that these solutions extend backwards in time as far as $t \to -\infty$. The present similarity exponents are related by

$$\alpha(m-1) + 2\beta = -1 \tag{16.52}$$

(whereas the relation was $\alpha(m-1) + 2\beta = 1$ for Type I). If the equation is written as $u_t = \Delta(|u|^{m-1}u/m)$, the equation for the profile $F = F(\eta)$ is

$$\Delta f^m + m\alpha f + m\beta\,(\eta \cdot \nabla f) = 0 \tag{16.53}$$

for $\eta \in \mathbb{R}^d$; this is the same as in Type I.

Let us now concentrate on radially symmetric profiles $F = F(\xi)$, $\xi = |\eta| > 0$. The ODE for the profile $F$ is the same, (16.30), and Proposition 16.3 is still valid. The difference is noticed in the fact that when we write $\alpha = \gamma\beta$, then

$$\beta = \frac{1}{\gamma(1-m) - 2}, \quad \alpha = \frac{\gamma}{\gamma(1-m) - 2}, \tag{16.54}$$

which is just the opposite of the formulas (16.33) for Type I similarity. Proceeding just as there, we introduce new variables

$$X = \frac{\xi\, f'(\xi)}{f(\xi)}, \quad Z = \xi^2 f^{1-m}, \tag{16.55}$$

and $r = \log \xi$. Then, we get the system

$$\left.\begin{aligned} \dot{X} &= (2-d)X - mX^2 - \beta(\gamma + X)Z, \\ \dot{Z} &= (2 + (1-m)X)Z, \end{aligned}\right\} \tag{16.56}$$

where $\dot{X} = dX/dr$, $\dot{Z} = dZ/dr$. With respect to system (S1) with the same value of $\gamma$, there is only a difference, the sign of $\beta$ as a function of $\gamma$ and the variable $Z$ instead of $Y$.

*Using the same plane for both similarities*

Let us explain this last detail. There is a way of reducing the study of both types of similarity to the same phase-plane. Indeed, if $(X(r), Z(r))$ is an orbit of system (16.56), and we put $X_1(r) = X(r)$ and $Y_1(r) = -Z(r)$, then $(X_1(r), Y_1(r))$ is an orbit of system (16.32) with the value of $\beta$ in terms of $\gamma$ given by (16.29), as corresponds to Type I self-similarity (i.e., $\beta_1(\gamma) = -\beta(\gamma)$, and also $\alpha_1(\gamma) = -\alpha(\gamma)$). Moreover, $X$ and $X_1$ travel in the same direction, though $Z$ and $Y_1$ are reflections of each other (with respect to the $X$-axis). Summing up, a symmetry around the $X$-axis allows us to pass from the study of the first system to the study of the second. We use the convention that the upper half plane of system (16.32) represents Type I Self-similarity, while the lower half plane of the same system portrays the orbits of Type II self-similarity.[5]

**Examples** We have seen this type of self-similar solutions in the blow-up study of Section 4.5. One of the most impressive uses of Type II self-similarity for the PME are the focusing solutions of Aronson and Graveleau [48] with a hole in the support that will be described in Section 19.2. In that case both exponents $\alpha$ and $\beta$ in formula (16.51) are positive.

Type II solutions are popular in fast diffusion associated to extinction phenomena. In that case exponent $\alpha$ in formula (16.51) is positive but $\beta$ may change sign. See the detailed study in the monograph [515].

## 16.9 Other types of exact solutions

The explicit or semi-explicit solutions of the PME obtained in this chapter are based on the exploitation of the property of invariance under the scaling group. This and radial symmetry allows for a dramatic dimension reduction so that in the end only an ODE system has to be solved.

[5] We warn the reader that the use of the lower phase plane can easily lead to confusion in assigning the signs of the different parameters and variables. We introduce this variant here because it quite often comes in the literature on self-similarity.

The PME has a larger group of transformations that includes scaling and translations in space and time. Different combinations of these aspects lead to other forms of solutions invariant under different subgroups of the whole invariance group. An example in the direction of other invariant solutions that are not generally speaking *scaling self-similar* is the case of the travelling waves considered in Section 4.3 which are invariant under space-time translation along the lines $x = x_0 + c\,\mathbf{e}\,t$, with variable parameter $x_0 \in \mathbb{R}$ and fixed direction $\mathbf{e}$. A more general case are the general planar fronts of Section 4.7 which combine travelling wave behaviour with dilations in $u$.

We present next an example of a truly non-self-similar solution of the PME that has been communicated to me by Luc Tartar [491].

### 16.9.1 *Ellipsoidal solutions of ZKB type*

We consider the following formulas for the pressure of the PME in the whole space

$$v(x,t) = \langle A(t)x, x\rangle + B(t), \tag{16.57}$$

where for every $t > 0$ $A(t)$ is a symmetric matrix and $B(t)$ is a scalar. This approach is suggested by the theory of 'invariant subspaces.' The formula produces a solution of the PME if $A(t)$ and $B(t)$ satisfy

$$A' = -2(m-1)TA + 4A^2, \quad B' = -2(m-1)TB, \tag{16.58}$$

where $T(t)$ is minus the trace of $A(t)$. The first of these equations is a matrix Riccati equation that can be solved separately. Then $B(t)$ follows in one time integration. In order to solve the equation for $A$ we assume that it is symmetric and refer to the eigenvector basis so that it takes diagonal form. Since we are interested in negative values of $A$ we write

$$A(t) = \operatorname{diag}(-\lambda_1(t), \ldots, -\lambda_d(t)). \tag{16.59}$$

Then, we have $T(t) = \sum_i \lambda_i(t)$ and

$$\lambda_i'(t) = -4\lambda_i(t)^2 - 2(m-1)T(t)\,\lambda_i(t). \tag{16.60}$$

This is a coupled system of $d$ ODEs that can be solved for all initial data. Tartar suggests the following way to integrate it: assume that $T(t)$ has been calculated and define a function $h(t)$ by

$$h''(t) = -2(m-1)T(t)h'(t), \quad h(0) = 0, \quad h'(0) = 1. \tag{16.61}$$

Then, we can express each $\lambda_i$ as

$$\lambda_i(t) = \frac{a_i h'(t)}{1 + 4a_i h(t)}. \tag{16.62}$$

Next, we calculate $h$ by solving

$$h''(t) = -2(m-1)h'(t)\sum \frac{a_i h'(t)}{1+4a_i h(t)}.$$

which gives

$$h'(t) = \left(\,\Pi_i(1+4a_i h(t))\,\right)^{-(m-1)/2}. \tag{16.63}$$

A further integration in time gives $h$ as a function of $t$, or more explicitly, $t$ as a function of $h$. It is then immediate to calculate $A$ and also

$$B(t) = b_0 h'(t). \tag{16.64}$$

The support of the solutions at time $t \geq 0$ is the ellipsoidal region where $-\langle A(t)x, x\rangle \leq B(t)$, i.e.,

$$\lambda_1(t)x_1^2 + \ldots + \lambda_d(t)x_d^2 \leq b_0 h'(t). \tag{16.65}$$

If the $a_i$ are different, the solution cannot be self-similar because it changes shape with time. See further details on this family of solutions in Problem 16.12.

In Problem 16.13 we propose new blow-up solutions by a modification of the data in the above construction.

**Remark** There are many works on classes of semi-explicit solutions that are not self-similar and play a big role in other nonlinear parabolic equations. A systematic study of those questions depends on a Lie group analysis of the symmetries of the equation and techniques like the Bäcklund transformation. We will not enter into this topic which is very important for the investigation of wide classes of solutions but will not be essential for our presentation. We will mention the basic works of Ovsiannikov [414] and Bluman et al. [118, 119]. All of these methods are aimed at dimension reduction which allows in the end to arrive at a problem that has explicit or so-considered semi-explicit solutions (usually, solutions of ODE systems). Exact solutions are obtained in a number of cases. See also [288, 336, 337, 456, 469] and the many references. Recently, Gandarias [257, 258] worked on diffusion-convection equations and found symmetry reductions and exact solutions by using Lie group methods, the non-classical method developed by Bluman and Cole and found potential symmetries. She gives references to previous work. Invariant subspaces are also very much used, see e.g. Galaktionov [246].

## 16.10 Self-similarity for GPME

In the case of the GPME with a nonlinearity $\Phi$ which is not a power there is no possibility of obtaining self-similar solutions that perform any kind of scaling in the $u$ variable. This reduces enormously the class of similarity solutions.

- We we want to find self-similar solutions of the Types I and II (formulas (16.27) and (16.51) respectively); the exponent of the self-similarity formulas must be $\alpha = 0$, which means that $\beta$ must be $\pm 1/2$. In this way, we may find self-similar solutions of the form

$$u(x,t) = F(\eta), \quad \eta = x\,(1 \pm t)^{-1/2} \tag{16.66}$$

(we have put the 1 in the time variable for aesthetic reasons, displacement of the time origin in inconsequential). The equation for the profile is

$$\Delta_\eta \Phi(F(\eta)) + \frac{1}{2}\,\eta \cdot \nabla F(\eta) = 0. \tag{16.67}$$

The Polubarinova solutions of Subsection 4.6.1 belong to this class.

- The travelling wave solutions are also admissible for the GMPE. These equations are not available for inhomogeneous media. The analysis of these solutions by Peletier [423, 424] lead to establishing the necessary and sufficient condition for finite propagation of the GPME in $d = 1$. See Problem 16.11.

## Notes

**Section 16.1.** Typical books on similarity analysis are Bluman and Cole [118] or Sedov [475]. Sachdev's book [460] investigates exact solutions to nonlinear partial differential equations and systematic methods for finding them. In the case of the PME, fundamental work is due to Barenblatt with his famous books [62, 63, 64].

**Section 16.2.** The existence of three types of self-similarity is described for instance in [263, 267, 268], who study the one-dimensional case. We have discussed the list of three families for instance in [236]. The exponential type is seldom mentioned or used, but the idea that it leads to eternal solutions is interesting. We warn the reader that the signs of the similarity exponents are taken with any variations of the signs in the literature; this can make the direct application of the results dangerous.

**Section 16.3.** The FDE is treated in some detail in [515], where references are given.

The reader will have noticed that only general information is used on the equation: existence and uniqueness for certain data (and only locally in time), and existence of a scaling transformation. Therefore, the argument applies not only to the PME and the FDE, but also to the $p$-Laplacian equation and many other equations, not necessarily parabolic. Note also that the scaling group algebra is different for different equations, hence the difference in the obtained similarity exponents.

**Section 16.4.** This kind of transformation goes back to Barenblatt [60] and Jones [310] and is used in many papers.

The alternative formulation of the phase plane is found in a close version in Aronson and Graveleau's [48]; the relation of the formulations is discussed in [442].

Other versions of the ODE system are used by Lacey at al. [355], and by Bernis et al. [103]. They are all equivalent with pros and cons.

**Section 16.8.** The idea of using the same plane for both self-similarities is taken from [355]. The study of the phase plane system for the different values of $m$, $d$ and $\gamma = \alpha/\beta$ is quite complicated and gives rise to a wealth of *semi-explicit solutions* with quite different qualitative behaviour that is supposed to reflect most if not all the properties of the PME or the FDE. See a detailed analysis about the later equation in [515].

**Section 16.10.** Early reference to the solutions with the $x/(t+1)^{1/2}$ scaling is the work on ground infiltration of [438]. The case of general $\Phi$ is studied in [222, 423, 424], among many later references. Solutions with this self-similarity play a big role in the theory of the Stefan problem.

## Problems

### Problem 16.1

(i) Show that the self-similar solutions (16.13) that enjoy the property of full self-similarity have a profile $F$ that satisfies

$$\Delta_{\mathrm{LB}} F(\omega) + \frac{1}{m-1} F = cF^m$$

for a certain constant $c(m,n) > 0$. Determine this constant ($\Delta_{\mathrm{LB}}$ denotes the Laplace-Beltrami operator on the unit sphere).

(ii) Take $F$ constant and find the blow-up solution (4.44).

(iii) Write the equation of the profile for the solutions (16.16).

### Problem 16.2

(i) Find self-similar solutions of Type I that cannot be continued backwards in time.

(ii) Find self-similar solutions of Type II that cannot be continued forwards in time.

**Problem 16.3** Perform the derivation of system (16.32) as follows: The second line is immediate from the definition of $Y$. The derivation of the first line is as follows: the first term of (16.30) gives

$$\xi^{1-d}(\xi^{d-1}|F|^{m-1}F'(\xi))' = \xi^{1-d}(\xi^{d-2}|F|^{m-1}FX)' = \xi^{-d}(\xi^d F\, X/Y)'.$$

Working it out we get

$$\frac{\dot{X}}{Y}\, F - \frac{X}{Y^2}\, \dot{Y} F + \frac{X^2}{Y} F + d\, \frac{X}{Y}\, F.$$

The whole equation then reads

$$\frac{\dot{X}}{Y}\, F - \frac{X}{Y^2}\, \dot{Y} F + \frac{X^2}{Y} F + d\, \frac{X}{Y}\, F + \beta X\, F + \alpha\, F = 0.$$

Use now $\dot{Y} = (2 + (1 - m)X)Y$ to get

$$\dot{X} - (2 + (1 - m)X)X + X^2 + d\,X + (\beta X + \alpha)Y = 0.$$

This completes the derivation. Note that all the computations are valid for either $m > 1$ or $m < 1$ but not for $m = 1$.

**Problem 16.4** Perform the phase-plane analysis of Section 16.4 for $m = 1$.

**Problem 16.5** Perform the local analysis near the point $P_B = (X_B, Y_B)$.

**Problem 16.6**

(i) Perform the phase-plane analysis of Section 16.4 for $d = 2$ and $m > 1$ and obtain a precise classification of trajectories in the different cases. Get the collection of plots in the standard and compactified $XY$-plane.
(ii) Same problem for $0 < m < 1$.

**Problem 16.7**

(i) Perform the phase-plane analysis of Section 16.4 for $d = 1$ and obtain a precise classification of trajectories in the different cases.
(ii) Show the role of point $P_1$ and draw a plot of the trajectories.
(iii) Show that for $\gamma = 1$ we obtain a family of signed solutions of the PME that have infinite mass on both sides but still decay like $\|u(t)\| = O(t^{-1/(m+1)})$ like the non-negative solutions with finite mass.

**Problem 16.8** (Previous problem continued) The profile equation (16.30) is easy to integrate when $d = 1$ and $\gamma = 1$, since the $\alpha = \beta = 1/(m + 1)$ and the profile equation gives

$$|F(\eta)|^{m-1}F'(\eta) = C - \beta\eta\, F(\eta) = 0, \qquad \eta = x\, t^{-\beta}. \tag{16.68}$$

(i) Take constant $C = 0$ and $F(0) = 1$ to find the explicit ZKB solution.
(ii) Take $C = 1$ and $F(0) = 0$ to find the solution of the PME with initial data $u(x, 0) = c/x$. Draw the antisymmetric profile $F$.

**Problem 16.9*** Write the ODE analysis for self-similar solutions of the exponential type in the spirit of Section 16.8. Point out similarities and differences. Find typical examples of eternal solutions of this form.

**Problem 16.10*** Prove the statements of Theorem 16.8 using the phase plane as a tool to investigate the properties of the profiles $F$.

**Problem 16.11** Analyse the existence of travelling wave solutions for the GPME $\partial_t u = \Delta\Phi(u)$. Show that the necessary and sufficient condition to have non-negative travelling waves with a free boundary is the convergence of the integral

$$\int_0^u \frac{d\Phi(s)}{s} \tag{16.69}$$

for $u > 0$. Find in that case the implicit equation of the profile.

**Problem 16.12** Let us consider the ellipsoidal solutions (16.57) with $A(t)$, $B(t)$ given as in Subsection 16.9.1.

(i) Show that when we put all $\lambda_i(0) = a_i$ equal we get a solution with radial symmetry such that

$$(1 + 4ah)^{(2+d(m-1)/2} = 2(m-1)at + 1.$$

Check that this solution is the ZKB.

(ii) Show that in any case, when $a_i > 0$ for all $i = 1, \ldots, d$, then $h(t)$ tends as $t \to \infty$ to the same value as in the ZKB solution, $\lambda_i(t)/\lambda_j(t) \to 1$ and the get in the limit the ZKB in the sense of the scaling that we discuss in whole detail in Chapter 18. Indeed, this must be so because of Theorem 18.1. In particular, show that $h(t) \sim c\,t^{2\beta}$ for all large $t$ ($\beta = 1/(d(m-1)+2)$ as in the theorem).

(iii) In the same case $a_i > 0$ show that support length along axis $X_i$ is given by

$$l_i^2(t) = b_0\,(4h(t) + (1/a_i)). \tag{16.70}$$

This means that $h(t)$ measures also in this case the size of the support up to a distortion factor that goes to 1 as $t \to \infty$. Make precise the distortion as

$$\frac{l_i(t)}{l_j(t)} = 1 + O(t^{-2\beta}), \tag{16.71}$$

with a coefficient that depends on $a_i - a_j$.

(iv) Consider now that case where some of the $a_i = 0$ and make a similar analysis.

**Problem 16.13** We want to find heavily asymmetric blow-up solutions. We propose to take $d = 2$ and use as initial values $a_1 = a > 0$ and $a_2 = -b < 0$. The same analysis produces solutions with a hyperbolic support that blow-up in finite time.

(i) Find the solution with initial data

$$v_0(x, y) = (y^2 - x^2)_+.$$

Describe the free boundary. Find a free boundary that is not flat for $B > 0$.

(ii) Work in $d = 3$ with $a_1 > 0$, $a_2 = 0$ and $a_3 < 0$.

# 17

# TECHNIQUES OF SYMMETRIZATION AND CONCENTRATION

A modern approach to the existence and regularity theory of partial differential equations relies on obtaining suitable a priori estimates in terms of the information available on the data, typically in the form of norms in appropriate functional spaces. Following such ideas, this chapter is devoted to obtain basic estimates for the PME and related equations using as main tools the techniques of symmetrization and mass concentration comparison, combined with scaling properties.

In Section 17.1 we review the main concepts from symmetrization theory and introduce the comparison of mass concentrations.

In Sections 17.2 and 17.3 we derive the basic comparison results for the elliptic equations to which the PME is reduced when it is solved by ITD method.

Comparison theorems for the evolution are then derived in Section 17.4. The application to the PME is contained in Section 17.5.

## 17.1 Functional preliminaries

This section collects the main ideas on rearrangement and symmetrization that we shall be using. Let $\Omega$ be a domain in $\mathbb{R}^d$, not necessarily bounded, possibly $\mathbb{R}^d$. We denote by $\operatorname{meas}\Omega = |\Omega|$ the Lebesgue measure of $\Omega$ and by $\mathcal{L}(\Omega)$ the set of (classes of) Lebesgue measurable real functions defined in $\Omega$ up to a.e. equivalence.

For every function $f$ defined and measurable in $\Omega$ we define the distribution function $\mu_f$ of $f$ by the formula

$$\mu_f(k) = \operatorname{meas}\,\{x : |f(x)| > k\}. \tag{17.1}$$

We denote by $\mathcal{L}_0(\Omega)$ the space of measurable functions in $\Omega$ such that $\mu_f(k)$ is finite for every $k > 0$. If $\Omega$ has finite measure, then $\mathcal{L}_0(\Omega) = \mathcal{L}(\Omega)$, otherwise $\mathcal{L}_0(\Omega)$ contains the measurable functions that tend to zero at infinity in a weak sense. All $L^p(\Omega)$ spaces with $1 \le p < \infty$ are contained in $\mathcal{L}_0(\Omega)$.

### 17.1.1 *Rearrangement*

A measurable function $f$ defined in $\mathbb{R}^d$ is called *radially symmetric* (or radial for short) if $f(x) = \tilde{f}(r)$, $r = |x|$. It is called *rearranged* if it is non-negative, radially symmetric, and $\tilde{f}$ is a non-increasing function of $r > 0$. For definiteness, we also impose that $\tilde{f}$ be left-continuous at every jump point. We will often write $f(x) = f(r)$ by abuse of notation. A similar definition applies to functions defined on a ball $B = B_R(0) = \{x \in \mathbb{R}^d : \ |x| < R\}$.

### 17.1.2 *Schwarz symmetrization*

For every bounded domain $\Omega$, the *symmetrized domain* is the ball $\Omega^* = B_R(0)$ having the same volume as $\Omega$, i.e.,

$$|\Omega| := \text{meas}(\Omega) = \omega_d R^d. \tag{17.2}$$

The precise value $\omega_d$ of the volume of the unit ball in $\mathbb{R}^d$ is $\omega_d = 2\pi^{d/2}/(d\Gamma(d/2))$, where $\Gamma$ is Euler's Gamma function. We put $(\mathbb{R}^d)^* = \mathbb{R}^d$. For a function $f \in \mathcal{L}_0(\Omega)$ we define the *spherical rearrangement* of $f$ (also called the symmetrized function of $f$) as the unique rearranged function $f^*$ defined in $\Omega^*$ which has the same distribution function as $f$, i.e., for every $k > 0$

$$\mu_f(k) := \text{meas}\{x \in \Omega : |f(x)| > k\} = \text{meas}\{x \in \Omega^* : |f^*(x)| > k\}. \tag{17.3}$$

The quantity is finite for every $k > 0$ by the assumption $f \in \mathcal{L}_0(\Omega)$. Then,

$$f^*(x) = \inf\{k > 0 : \text{meas}\{y : |f(y)| > k\} < \omega_n |x|^n\}. \tag{17.4}$$

A rearranged function coincides with its spherical rearrangement. Sometimes the name symmetric decreasing rearrangement is used. The following Hardy–Littlewood formula is well-known and illustrates the relation between $f$ and $f^*$:

$$\int_{B_R(0)} f^*\, dx = \sup\left\{\int_E |f|\, dx : \ E \subset \Omega, \text{meas}(E) \le \text{meas}(B_R)\right\}. \tag{17.5}$$

There is also an immediate relation between distribution functions and $L^p$ integrals given by the formulas

$$\int_\Omega |f|^p\, dx = -\int_0^\infty k^p\, d\mu(k) = p\int_0^\infty k^{p-1}\mu(k)\, dk, \tag{17.6}$$

and

$$\int_{\Omega\cap\{|f|\ge a\}} |f|^p\, dx = -\int_a^\infty k^p\, d\mu(k) = p\int_a^\infty k^{p-1}\mu(k)\, dk + a\mu(a). \tag{17.7}$$

Since the distribution functions of $f$ and $f^*$ are identical, conservation of integrals $\int_\Omega |f|^p\, dx = \int_\Omega (f^*)^p\, dx$ holds for every $p \in wehave[1,\infty)$. Moreover, for every convex, non-negative and symmetrical real function $\Phi$ we have

$$\int_\Omega \Phi(f)\, dx = \int_\Omega \Phi(f^*)\, dx. \tag{17.8}$$

Note finally that $f^*$ is continuous if $f$ is. There is another related function often used in the proofs, namely the one-dimensional symmetric representation, defined by means of the formula

$$f_*(s) = f^*(r), \quad s = \omega_n r^n. \tag{17.9}$$

Then, $f_*$ is defined in the interval $[0, |\Omega|]$, with $|\Omega| = \text{meas}(\Omega)$. Notice that

$$f_*(s) = \inf\{t \geq 0 : \mu(t) < s\}, \tag{17.10}$$

which makes $f_*$ a *generalized inverse* of $\mu_f$.

### 17.1.3 *Mass concentration*

The comparison of mass concentrations is a basic notion in our approach to getting estimates for elliptic and parabolic equations. The precise definition that was introduced in [496] is as follows:

**Definition 17.1** *Given two radially symmetric functions* $f, g \in L^1_{\text{loc}}(\mathbb{R}^n)$ *we say that* $f$ *is more concentrated than* $g$, *if for every* $R > 0$,

$$\int_{B_R(0)} f(x)\, dx \geq \int_{B_R(0)} g(x)\, dx, \tag{17.11}$$

*i.e.*,

$$\int_0^R f(r) r^{n-1}\, dt \geq \int_0^R g(r) r^{n-1}\, dt. \tag{17.12}$$

In this section we will write $f \succ g$ for the situation of (17.11). The partial order relationship $\succ$ will be called *comparison of mass concentrations*. We can also write $f \succ g$ in the form $g \prec f$. A similar definition applies to radially symmetric and locally integrable functions defined on a ball $B = B_R(0)$. In the case of rearranged functions this notion coincides with the comparison introduced by Hardy and Littlewood [284], which is also used by Bandle in her book; but the present definition does not ask for the condition of rearrangement, only radial symmetry, and the difference is used below.

In fact, the natural way of looking at the concept is to view it as a comparison between two radially symmetric measures, $d\mu_f = f(x)dx$ and $d\mu_g = g(x)dx$. Then the comparison reads,

$$\mu_f(B_R(0)) \geq \mu_g(B_R(0)) \quad \text{for every} \quad R > 0. \tag{17.13}$$

In this formulation, comparison can be considered for general radially symmetric Radon measures. Measures are natural data for elliptic and parabolic equations, see Chapter 13.

The comparison of concentrations can be formulated in an equivalent way when the functions are rearranged, thanks to a powerful equivalence result, which seems to be essentially due to Hardy and Littlewood. This is the precise formulation.

**Lemma 17.1** *Let $f, g \in L^1(\Omega)$ be rearranged functions defined in $\Omega = B_R(0)$ and let $g \to 0$ as $|x| \to R$. Then $f \succ g$ if and only if for every convex non-decreasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ we have*

$$\int_\Omega \Phi(f(x))dx \geq \int_\Omega \Phi(g(x))dx. \tag{17.14}$$

*The result is also valid when $R = \infty$ and $f, g \in L^1_{\rm loc}(\mathbb{R}^d)$, $g \to 0$ as $|x| \to \infty$.*

For a proof we refer to [512], Appendix. Let us mention an elementary property that we will use later: if $f$ and $g \in L^1(\Omega)$ and are non-negative, then

$$(f+g)^* \prec f^* + g^*.$$

The proof follows immediately from the characterization (17.5). Actually, the relation is true for all combinations of the form $af + bg$ with $a, b > 0$ constant.

## 17.2 Concentration theory for elliptic equations

We have seen in Chapter 10 the way of reducing the construction of solutions of the PME and related equations by means of implicit time discretization ITD, which leads to the solution of a certain type of semilinear elliptic equations. We develop here the idea of concentration comparison for nonlinear elliptic equations with symmetric data and solutions. Our model equation is

$$-\Delta u + \beta(u) = f, \tag{17.1}$$

posed in a domain $\Omega \subset \mathbb{R}^d$. Here, we may assume to simplify that $\beta$ is a continuous and monotone increasing function of its argument $u \in \mathbb{R}$. In the generality of maximal monotone graphs we have to write the equation in the form of set inclusion,

$$-\Delta u + \beta(u) \ni f. \tag{17.2}$$

For normalization we usually take $\beta(0) \ni 0$ (but see exceptions in chapter 10). Note that we have $v(x) := \Delta u(x) + f(x) \in \beta(u(x))$ a.e. We may use the inverse graph $\varphi = \beta^{-1}$ to write $u(x) \in \varphi(v(x))$ a.e., so that the equation looks formally

$$-\Delta\varphi(v) + v \ni f. \tag{17.3}$$

We will call $f$ the *forcing term* of the equation. We assume that $f \in L^1_{\rm loc}(\Omega)$ (exceptionally, it can be a measure). A *solution* of (17.2) is then a pair $(u, v) \in W^{1,1}_{\rm loc}(\Omega) \times L^1_{\rm loc}(\Omega)$ such $\Delta u = v - f$ in $\mathcal{D}'(\Omega)$ and $v(x) \in \beta(u(x))$ a.e. Well-posedness of the problem needs a more definite setting, including boundary conditions. Typical options will be discussed below. It must be observed that the results and applications we have in mind (i.e., the ITD) lead us to focus more on $v$ than on $u$, contrary to standard elliptic theory and practice.

### 17.2.1 *Solutions are less concentrated than their data*

Our first result in the area of concentration shows that the solutions $v$ of equations like (17.1) are less concentrated than the forcing term $f$ from which they originate, under certain conditions that are often fulfilled, though not always. In physical terms, this reflects the spreading effect due to diffusion represented by the Laplace operator.

We recall that, in our definition, concentration is a concept that applies to radially symmetric functions. We work in a space $\Omega$ that is either a finite ball $B_R(0)$ or the whole space $\mathbb{R}^d$ (i.e., $R = \infty$). There exist some differences in the results, but the argument is the same.

**Proposition 17.2** *Let $(u(r), v(r))$ be a solution pair of equation (17.1) with $f(r)$ a radially symmetric and decreasing function in $L^1_{\rm loc}(\Omega)$. There is an alternative:*

(i) *Standard case: $f \succ v$, and then $v$ and $u$ are monotone non-increasing.*
(ii) *Increasing case: $f \succ v$ does not hold, and then $u$ is increasing in $r$ in an interval of the form $I = (a, R)$, $v$ is non-decreasing with $v > f$ in $I$, and*

$$\int_\Omega (v(x) - f(x))\, dx > 0. \tag{17.4}$$

*Moreover, in this case the function $X(r) := \int_0^r s^{n-1}\,(v(s) - f(s))\, ds$ is positive and grows in $I$.*

**Proof** For $0 < r < R$ we define the continuous functions

$$V(r) := \frac{1}{d\omega_d} \int_{B_r(0)} v(x) dx = \int_0^r v(s) s^{d-1}\, ds,$$
$$F(r) := \frac{1}{d\omega_d} \int_{B_r(0)} f(x) dx = \int_0^r f(s) s^{d-1}\, ds.$$

Integration of the equation $v = f - \Delta u$ in $B_r(0)$ gives the basic relation:

$$V(r) - F(r) = r^{d-1} u'(r). \tag{17.5}$$

(i) The first case of the alternative happens when $V(r) \le F(r)$ for all $r \ge 0$, which is an equivalent way of expressing that $f \succ v$. Then, we can show that $v(r)$ and $u(r)$ are monotone non-increasing. Argument for $u$: it follows from (17.5) that $u'(r) \le 0$. If $\beta$ is single-valued or $u$ is strictly decreasing then we immediately conclude that $v(r)$ is also monotone. We have to establish monotonicity for $v$ in the case where $\beta$ is multivalued. Now, we can see that the only possibility for $v$ to be non-monotone is when $u$ fails to be strictly decreasing, say $u(r) =$ constant in an interval $J = a_1 < r < a_2$. But then the equation implies that $v = f$ in $J$, hence it must be non-increasing also there.

(ii) Assume now that $f \succ v$ does not hold. Then the set

$$G_\varepsilon := \{r \ge 0 \;:\; V(r) > F(r) + \varepsilon\}$$

is a non-empty open set of $(0,R)$ for some small $\varepsilon>0$. In principle, it is a union of disjoint intervals of the form $G_\varepsilon=\bigcup_i(a_i,b_i)$. Formula (17.5) implies that $u(r)$ is a strictly increasing function inside $G_\varepsilon$, $u'(r)\ge\varepsilon\, r^{1-d}>0$. By the monotonicity of $\beta$ and the relation $v(r)\in\beta(u(r))$ a.e., we conclude that $v(r)$ is also non-decreasing. Since $f$ is non-increasing, it follows that

$$r^{1-d}(V-F)'=v-f$$

is non-decreasing in $G_\varepsilon$. Now, $V-F$ is continuous and $V(0)-F(0)=0$, We conclude that whenever $G_\varepsilon$ is non-empty it can only be an open interval of the form $I=(a_\varepsilon,R)$ for some $a_\varepsilon\in(0,R)$ and then $V-F$ is positive and non-decreasing in $I$. It follows that $(V-F)'$, hence $v-f$ must be positive at a certain point, say $c\in I$. Since $v-f$ is non-decreasing, this means that it does not tend to zero as $r\to R$. We conclude that both $u(r)$ and $V-F$ are increasing in $G_\varepsilon=(a_\varepsilon,R)$. ■

Notice that in the increasing case, if $R=\infty$ we can estimate the growth of $u(r)$ and $V(r)-F(r)$. Suppose that $v-f$ tends to a finite limit $C>0$ as $r\to\infty$; then we have $V-F\sim Cr^d/d$ so that, using (17.5), $u'(r)\sim C_1\,r$. Therefore, $u$ grows at least quadratically $r\to\infty$.

However, we want to work in the standard case in the application to the PME that we will consider below. As a consequence of the preceding analysis, this case holds under several additional assumptions that are found in practice. Recall that $f$ was assumed to be radially symmetric and non-increasing, but we do not necessarily assume that $f\ge 0$ (in which case it is rearranged).

**Corollary 17.3** *Under the conditions of Proposition* 17.2*, the standard case happens if one of the following situations occur:*

(i)
$$\int_\Omega (v-f)\,dx\le 0; \tag{17.6}$$

(ii) *$R$ is finite and $u$ takes Neumann data $u'(R)\le 0$;*
(iii) *$R$ is finite, $f\ge 0$, and $v$ takes a value $v(R)\le 0$;*
(iv) *$R=\infty$, and $u$ is bounded; or*
(v) *$R=\infty$, $f\ge 0$, $v\in\mathcal{L}_0(\mathbb{R}^d)$ (in particular, if $v\in L^p(\mathbb{R}^d)$ for any $p<\infty$). In this last case, $v$ is rearranged.*

**Proof** The first assertions are easy. As for the last, we have seen in the proof of the proposition that in part (ii) $v-f$ is positive in $I$. If $f\ge 0$ then $v$ is positive and non-decreasing, hence it goes to a positive or infinite limit, which contradicts $v\in\mathcal{L}_0(\mathbb{R}^d)$. We are thus in the standard case, so that $v$ is non-increasing and goes to zero as $r\to\infty$. See Remark (1) below in this respect. ■

**Remarks**

(1) It often happens that the integral of formula (17.6) vanishes, since it is the form of the form of conservation of mass when we discretize the evolution problem. Such an equality is a way of showing that the pointwise inequality $v \le f$ does not hold in general, for then it would imply that $v = f$ and $\Delta u = 0$. On the other hand, if we have equality of integrals and also $f(r) \to 0$ as $r \to \infty$, then $v(r) \to 0$, i.e., $v \in \mathcal{L}_0(\mathbb{R}^d)$. Then, both $f$ and $v$ are rearranged.

(2) The results are false if $f$ is not monotone. Imagine for instance that $d = 1$ and $f$ is radially symmetric, non-negative and compactly supported in the interval $[1, 3]$ and assume moreover that the part contained in $\{r > 0\}$ is rearranged around the point $r = 2$. Assume that $\beta(u) = u$, so $u$ solves $u - u'' = f$ with $u \ge 0$. It follows from the strong maximum principle that $u > 0$ everywhere, hence $v \prec f$ is not true.

(3) The increasing case of Proposition 17.2 cannot be discarded without some assumptions. Consider for instance the case $n = 1$, $\sigma = 1$, $f = 0$ and $\beta(u) = u$. We have an increasing solution of the form $u(r) = \mathrm{Cosh}(r)$, which is symmetric and increasing in the half line. Examples in several dimensions, or having non-trivial $f$, are left as exercises.

(4) Even if $f$ is non-negative, it is not proved here that in the standard case $u(r)$ and/or $v(r)$ are always non-negative. If this happens, usually as a consequence of the maximum principle and the fact that $0 \in \beta(0)$, then the conclusion of the standard case is that they are rearranged. Generally speaking, we could have $v(r)$ converging to a $C \in \mathbb{R}$ or to minus infinity. If $f \to 0$, then the condition $f \succ v$ implies that $C \le 0$.

(5) We are assuming that $v(r)$ and $u(r)$ are radially symmetric functions. In the general elliptic theory, this property follows from the radial symmetry of the data, the fact that the problem is invariant under rotations, plus the uniqueness of solutions in a certain class.

(6) The above considerations are easier when $\beta$ is smooth and strictly monotone $0 < \beta'(u) < \infty$. An argument of approximation justified by the theory allows us to pass to the general case. But we have shown that the direct argument works easily in the present case.

### 17.2.2 *Integral super- and subsolutions*

The proof of the preceding result does not need to deal with an exact solution. It is sufficient that $u$ is a *subsolution* in the sense that $-\Delta u + \beta(u) \le f$. We can go one step further. We define a **radial integral subsolution** of equation (17.1) as a pair of radial functions $(u, v) \in (L^1_{\mathrm{loc}}(\Omega))^2$ such that

$$f \succ -\Delta u + v, \quad v(r) \in \beta(u(r)) \text{ a.e.} \tag{17.7}$$

Sometimes we refer to the subsolution only as $u$, since $v$ is then defined in terms of $u$ and $f$. In the same way we define a **radial integral supersolution**, using $f \prec -\Delta u + v$ instead of $f \succ -\Delta u + v$.

**Proposition 17.4** *Let $f(r)$ be a radial function in $L^1_{\rm loc}(\mathbb{R}^d)$ and let $(u(r), v(r))$ be a radial integral subsolution of equation* (17.1). *Then, either*

(i) *$f \succ v$; or*
(ii) *the second option of Proposition* 17.2 *holds.*

The proof is the same with very minor changes, once we remark that the new assumption reads as

$$r^{n-1}u'(r) \geq V(r) - F(r), \tag{17.8}$$

which implies that $u$ is strictly increasing in $G_\varepsilon$ (the reader should at this point re-do that part of the proof of Proposition 17.2). Note that weak subsolutions in the sense of formula (17.7) need not be rearranged or even monotone if we do not impose the condition such a priori. The second part of the alternative is in any case true.

### 17.2.3 *Comparison of solutions*

Our next result is the comparison of radial solutions for different data.

**Theorem 17.5** *Let $f_i(r)$, $i = 1, 2$ be two radially symmetric functions in $L^1_{\rm loc}(\Omega)$, let $u_i(r)$ be the respective solutions of* (17.1), *and let $v_i = f_i + \Delta u_i$. Assume that $f_1 \succ f_2$. Then either*

(i) *(standard case) $v_1 \succ v_2$, and then*
$$\int_\Omega (v_2(x) - v_1(x))\, dx \leq 0;$$
*or*
(ii) *$v_1 \succ v_2$ does not hold, and then the functions $u(r) = u_2(r) - u_1(r)$ and*
$$\int_{B_r(0)} (v_2(x) - v_1(x))\, dx$$
*are both positive and increasing for all large $r$. The same result holds when $u_1$ is an integral supersolution and $u_2$ is an integral subsolution.*

**Proof** We argue as before. We introduce the functions $u = u_2 - u_1$, $v = v_2 - v_1$, $f = f_2 - f_1$,

$$V(r) := \int_0^r v(r)\, r^{d-1} dr, \quad F(r) := \int_0^r f(r)\, r^{d-1} dr,$$

and derive from the equation the relation

$$V(r) - F(r) \leq r^{d-1}(u_2'(r) - u_1'(r)). \tag{17.9}$$

If we assume that $u_1$ and $u_2$ are exact solutions then equality holds in this formula, for super- and subsolutions we have inequality. By assumption, $F \le 0$. In the standard case we have $V \le 0$. To explore what happens otherwise, we consider the sets

$$G = \{r : V(r) > 0\}, \quad G_\varepsilon = \{r : V(r) > \varepsilon\}.$$

In case (ii) $G_\varepsilon$ is non-empty for some small $\varepsilon$, we conclude that in $G_\varepsilon$ $u_2' > u_1'$, hence $u_2' > u_1'$ and $u = u_1 - u_2$ is strictly increasing in $G$. Since $V(0) = 0$ and $V$ becomes positive, it must be increasing somewhere in $G$, say at $c \in G$, hence $v(c) = v_2(c) - v_1(c) > 0$ and by the monotonicity of $\beta$, $u_2(c) \ge u_1(c)$, so that $u_2 > u_1$ for $r > c$, $r \in G$. It follows that $v = v_2 - v_1 \ge 0$ for $r \ge c$, $r \in G$, hence $V$ is non-decreasing in $G$, which must be an interval going to infinity or to $R$, and so are the sets $G_\varepsilon$ for small $\varepsilon > 0$.

Summing up, in the non-standard case we have as $r \to R \le \infty$ that (a) $V(r) \ge \varepsilon > 0$ and is non-decreasing, and (b) $u = u_2 - u_1$ is positive and increasing. ■

**Remarks** As remarked above, we are interested in the standard case, hence we discuss several variants of the theorem where the assumption $u_i \in \mathcal{L}_0(\Omega)$ is replaced by other conditions. Thus,

(1) The standard case holds if (a) $\int_\Omega (v_2(x) - v_1(x))\,dx \le 0$, or (b) $\Omega = \mathbb{R}^d$ and $u_i \in \mathcal{L}_0(\mathbb{R}^d)$, or (c) $\beta^{-1}(0) = \{0\}$ and $v_i \in \mathcal{L}_0(\mathbb{R}^d)$.

(2) In many instances we know that $f_i$ and $v_i$ are integrable and $\int v_i = \int f_i$. Then we have $V(R) - F(R) \le 0$, and the standard case follows.

(3) In dimensions $d = 1, 2$ and with $\sigma = 1$ (Laplacian case) the assumption that $G$ is non-empty leads to the conclusion that $u_2 - u_1 \to \infty$ (*Hint*: integrate (17.9)). Thus, a condition that $u_i$ be bounded is enough to get the standard case. This is in particular the case if $v_i \in \mathcal{L}_0(\mathbb{R}^d)$ and $\beta^{-1}(0)$ is a bounded interval, $[a, b]$. For $d \ge 3$ such condition does not work, see Section 10.3.

## 17.3 Symmetrization and comparison. Elliptic case

We tackle next the second main technique of this chapter, Schwarz symmetrization for elliptic equations, a subject which has an extensive literature. Here, we review the basic theory since it leads at the end of the section to the presentation of the interaction between both techniques. Such interaction needs a different way of looking at the standard symmetrization inequality in terms of concentration comparison.

Let us consider the problem

$$-\sum_{i,j} \partial_i(a_{ij}\partial_j u) + b(x, u) = f \tag{17.1}$$

in a bounded open set $\Omega \in \mathbb{R}^d$, or in $\mathbb{R}^d$. The coefficients $a_{ij}$ are bounded measurable functions in $\Omega$ satisfying the ellipticity hypothesis

$$\sum_{i,j} a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \tag{17.2}$$

for some constant $\lambda > 0$ all vectors $\xi \neq 0$. Without loss of generality, we may take $\lambda = 1$ by replacing $f$ by $f/\lambda$. We assume that the function $b(x, u)$ is measurable, continuous and non-decreasing in $u$ for fixed $x$, and bounded in $x$ uniformly for bounded $u$. We will also assume that

$$b(x, u)\, u \geq 0 \quad \text{for a.e. } x \text{ and all } u \tag{17.3}$$

The second member $f$ is a measurable function in some Lebesgue $L^p$ space, though other spaces like Marcinkiewicz spaces also appear in the literature. We take boundary conditions of Dirichlet type

$$u(x) = 0 \quad x \in \partial\Omega. \tag{17.4}$$

The *symmetrized problem* is posed in the ball $\Omega^* = B_R(0)$ and consists of the symmetrized equation

$$-\Delta\overline{u} = f^* \quad \text{in } \Omega^*, \tag{17.5}$$

where $f^*$ is the spherical rearrangement of $f$, with boundary conditions

$$\overline{u}(x) = 0 \qquad \text{on } \partial\Omega^*. \tag{17.6}$$

### 17.3.1 *Standard symmetrization result revisited*

The classical result of symmetrization theory says that, in the absence of the zero-order term, the symmetric rearrangement of $u$, that we write $u^*$, can be compared pointwise with the solution of the symmetrized problem. Let us review that result, since forms the outline of the proof of our main comparison theorem for symmetrized functions in equations with lower-order terms.

**Theorem 17.6** *Let us assume that* $f \in L^2(\Omega)$, $f \geq 0$ *and that* $u \in H_0^1(\Omega)$ *is a weak solution of equation* (17.1) *under the above hypotheses. Then,*

$$u^*(r) \leq \overline{u}(r) \qquad \text{for all} \;\; r \in (0, R), \tag{17.7}$$

**Proof** The steps of the proof, as described, e.g. in Talenti [488], are as follows:

(i) Write equation (17.1) in variational form as

$$\sum_{i,j} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\, dx = \int_\Omega g(x, u) v\, dx \tag{17.8}$$

for test functions $v \in H_0^1(\Omega)$, where $g(x, u) = f(x) - b(x, u)$. Taking $f \geq 0$ we have $u \geq 0$ by the maximum principle. Let us now write $a(\nabla u, \nabla v) \equiv \sum a_{ij}\partial_i u \partial_j v$.

(ii) We calculate for a.e. $t>0$ the derivative $\frac{d}{dt}\int_{\Omega(t)} a(\nabla u,\nabla u)dx$, where $\Omega(t)=\{u>t\}$. Taking as test function $v=(u-t)_+$ in (17.8) we get

$$\int_{\{u>t\}} a(\nabla u,\nabla u)dx = \int_{\{u>t\}} g(x,u)v(x)dx.$$

It is then proved that for a.e. $t\in(0,\sup\text{ess}(u))$ we have

$$\frac{d}{dt}\int_{\{u>t\}} gvdx = -\int_{\{u>t\}} g\,dx.$$

It follows that

$$-\frac{d}{dt}\int_{\{u>t\}} a(\nabla u,\nabla u)dx = \int_{\{u>t\}} g(x,u)dx. \tag{17.9}$$

(iii) A very elementary step using the ellipticity assumption allows us to conclude that

$$-\frac{d}{dt}\int_{\{u>t\}} a(\nabla u,\nabla u)dx \geq -\frac{d}{dt}\int_{\{u>t\}} |\nabla u|^2 dx \geq 0.$$

We transform in this way equality (17.9) into

$$-\frac{d}{dt}\int_{\{u>t\}} |\nabla u|^2 dx \leq \int_{\{u>t\}} g(x,u)dx. \tag{17.10}$$

(iv) We need to transform the left-hand side. Using the Cauchy–Schwartz inequality, we get:

$$\frac{1}{h}\int_{\{t<u<t+h\}} |\nabla u|dx \leq \left(\frac{1}{h}\int_{\{t<u<t+h\}} |\nabla u|^2 dx\right)^{1/2}\left(\frac{1}{h}\int_{\{t<u<t+h\}} dx\right)^{1/2}.$$

from which, after recalling the definition of distribution function $\phi=\phi_u(t)$ of the function $u$, we get in the limit $h\to 0$

$$\left(-\frac{d}{dt}\int_{\{t<u\}} |\nabla u|\,dx\right)^2 \leq \left(\frac{d}{dt}\int_{\{t<u\}} |\nabla u|^2 dx\right)(-\phi'(t))$$

which by (17.10) is equal or less than $(-\phi'(t))\int_{\{u>t\}} g\,dx$, hence

$$\left(-\frac{d}{dt}\int_{\{t<u\}} |\nabla u|\,dx\right)^2 \leq (-\phi'(t))\int_{\{u>t\}} g\,dx. \tag{17.11}$$

(v) We now use heavier artillery: the Fleming–Rishel formula says that for a.e. $t$

$$P_\Omega(\{u>t\}) = -\frac{d}{dt}\int_{\{u>t\}} |\nabla u|\,dx, \tag{17.12}$$

while De Giorgi's isoperimetric inequality can be written as

$$P_\Omega(\{u > t\}) \geq d\omega_d^{1/d}\phi(t)^{\frac{d-1}{d}}. \tag{17.13}$$

Using both formulas, (17.11) becomes

$$d^2\omega_d^{2/d}\phi(t)^{2-\frac{2}{d}} \leq (-\phi'(t))\int_{\{u>t\}} g(x,u)\,dx. \tag{17.14}$$

(vi) Moreover, $g(x) = f(x) - b(x,u)$, and by the properties we have assumed on $b$, we have $\int_{\{u>t\}} b(x,u)dx \geq 0$, therefore

$$d^2\omega_d^{2/d}\phi(t)^{2-\frac{2}{d}} \leq (-\phi'(t))\int_{\{u>t\}} f(x)\,dx. \tag{17.15}$$

(vii) We use the Hardy–Littlewood theorem to estimate

$$\int_{\{u>t\}} f dx \leq \int_{\{u^*>t\}} f^* dx = \int_{B_r} f^*(x)dx,$$

where $|B_r| = |\{u > t\}|$, i.e. $\omega_d r^d = \phi(t)$. Let us also introduce the notation $F(s) = \int_{B_r} f^*(x)dx$, with $s = \omega_d r^d$. Substituting into (17.15), we get the inequality

$$d^2\omega_d^{\frac{2}{d}}\phi(t)^{2-\frac{2}{d}} \leq (-\phi'(t))\int_{\{u^*>t\}} f^*(x)dx = F(\phi(t)). \tag{17.16}$$

At this stage we recall that (17.16) is satisfied with equality by the symmetrized problem. Indeed, we have

$$-d\omega_d r^{d-1}\overline{u}'(r) = \int_{B_r} f^*(x)dx, \tag{17.17}$$

The comparison we are looking for follows easily since, using the fact that for a.e. $r$ we have $\phi(\overline{u}(r)) = d\omega_d r^d$, it follows that (17.16) can be written as

$$-d\omega_d r^{d-1}(u^*)'(r) \leq \int_{B_r} f^*(x)dx. \tag{17.18}$$

Using the boundary condition $u^*(R) = \overline{u}(R) = 0$ we conclude the inequality. For the sequel, we notice that (17.18) is a formulation as an integral subsolution. ■

### 17.3.2 *General symmetrization-concentration comparison*

We deal now with the presence of the lower-order term. In the previous result its effect has been eliminated, but this leads to a poorer understanding and poorer estimates. We are forced to keep track of the term because of our intention of applying the results to the study of parabolic problems by means of implicit time discretization. It happens that, in the spirit of our end of the previous proof, there is a simple modification that naturally leads to concentration comparison. In this way we can compare the result obtained by first solving and then rearranging with the result of the reverse procedure, i.e., first rearranging and then solving the symmetrized problem. In fact, using the results of Section 17.2 we can do better: to compare the symmetrized problem at once with a radial problem for different forcing term $f$ (and even different nonlinearity $\beta$). This is our main result.

**Theorem 17.7** *Next to the assumptions of the preceding theorem on $f$ and $u$, we suppose that $b(x,s)s \ge \beta(s)s$ for all $s$, where $\beta$ is a maximal non-decreasing function with $\beta(0)=0$ and let $v=\beta(u)$. Let $\overline{u}(r)$, $0<r<R$ be an integral supersolution of the radial problem*

$$-\lambda\Delta\overline{u} + \beta_1(\overline{u}) \ni \overline{f}(r), \tag{17.19}$$

*with boundary condition $\overline{u}(R) \ge 0$, where $\overline{f}$ is a radial function in $L^1(\Omega^*)$ such that $\overline{f} \succ f^*$, and $\beta_1$ is a maximal non-decreasing function such that $\beta_1^{-1} \prec \beta^{-1}$. Under these assumptions, we conclude that the two radial functions $v^*(r)$ and $\overline{v}(r)$ are ordered:*

$$v^* \prec \overline{v}. \tag{17.20}$$

**Proof** (i) We repeat the previous proof with the following modification at the end. In formula (17.15) we observe that

$$\int_{\{u>t\}} g(x,u)dx \ge \int_{\{u>t\}} f(x)\,dx - \int_{\{u>t\}} \beta(u)\,dx. \tag{17.21}$$

This means the needed result: $u^*$ is a radial integral subsolution of equation

$$-\lambda\Delta u + \beta(u) = f^*(r). \tag{17.22}$$

(ii) The comparison is now a consequence of Theorem 17.5 of Section 17.2. The fact that we are in the standard case is ensured by the Dirichlet conditions $u=0$ on $\partial\Omega$ and the non-negative condition for $\overline{u}$. ∎

*Generalization*

We have refrained from stating the theorem with multivalued graphs for simplicity. However, in view of what we have seen up to now, the same result holds if $\beta$ and $\beta_1$ are maximal monotone graphs in $\mathbb{R}^2$ with the corresponding modifications: $0 \in \beta_i(0)$; $\overline{v}(r) \in \beta_1(\overline{u}(r))$ is specified as part of the definition of supersolution, and $v^*(r) \in \beta(u^*(r))$ a.e.

The concentration statement can be reformulated in terms of standard norms by means of Lemma 17.1. After some elementary calculations we get

**Corollary 17.8** *Under the assumptions of Theorem* 17.7*, for every convex non-decreasing function* $\Phi : [0,\infty) \to [0,\infty)$ *with* $\Phi(0) = 0$ *we have*

$$\int_{B_R} \Phi(v^*(x))\, dx \le \int_{B_R} \Phi(\overline{v}(x))\, dx. \tag{17.23}$$

*In particular, for every* $1 \le p \le \infty$ *we have*

$$\|v^*\|_p \le \|\overline{v}\|_p. \tag{17.24}$$

The same conclusion holds for other norms like the natural norm in the Marcinkiewicz spaces $M^p$, $1 < p < \infty$. These spaces will play an important role for the smoothing effects obtained in our work [515], and are discussed in detail there.

### 17.3.3 *Problem in the whole space*

The results of the symmetrization-concentration theorem apply equally when we pose the problem in $\mathbb{R}^d$. In fact, we can use two approaches: either (i) to derive the estimates directly, or (ii) use approximation of solutions in the whole space by solutions in a sequence of bounded balls with radii $R \to \infty$.

When working in the whole space it is important to recall that the classical conclusion $u^* \le \overline{u}$, or the inequality $v^* \le \overline{v}$ cannot be true in general, since we have in most cases equality for the masses,

$$\int_{\Omega^*} v^*(x)dx = \int_\Omega v(x)dx = \int_\Omega f(x)dx, \quad \int_{\Omega^*} \overline{v}(x) = \int_{\Omega^*} f^*(x)dx.$$

It would follow in that case from $v^* \le \overline{v}$ that $v^* \equiv \overline{v}$. But this is not true in general.

## 17.4 Comparison theorems for the evolution

We consider now mild solutions $u$ of the GPME with nonlinearity $\varphi$ a maximal monotone graph in $\mathbb{R}^2$ with $0 \in \varphi(0)$; initial data $u_0 \ge 0$ are taken in $L^1(\mathbb{R}^n)$ and $f$ is integrable in $Q = \mathbb{R}^d \times (0,T)$. Via the Crandall–Liggett Theorems 10.16 and 10.17, we derive the following evolution version of the comparison results for elliptic equations.

**Theorem 17.9** *Let* $u$ *be the mild solution of the Cauchy problem for the GPME with data* $u_0$*, nonlinearity* $\varphi$ *and second member* $f$ *under the above assumptions. Let* $v$ *be the solution of a similar problem with radially symmetric data* $v_0(r) \ge 0$*, nonlinearity* $\psi$ *and second member* $g(r,t) \ge 0$*. Assume moreover that*

(i) $u_0^* \prec v_0$;

(ii) $\psi \prec \varphi$ and $\varphi(0) = \psi(0) = 0$;

(iii) $f^*(\cdot,t) \prec g(\cdot,t)$ for every $t \ge 0$.

*Then, for every $t \geq 0$*

$$u^*(\cdot, t) \prec v(\cdot, t). \tag{17.25}$$

**Remark** Note the order reversal in condition (ii). It is quite natural that for larger diffusivities to produce solutions that are more spread out and have lesser concentration.

**Proof** Using the Crandall–Liggett result we are reduced to comparing the discretization steps, which consist of elliptic problems as those treated in Sections 17.2 and 17.3. It is important to realize that comparison of concentrations between the discretized versions of the solutions is inherited in every step of the iteration.

We proceed as follows. In the first step, between $t_0 = 0$ and $t_1 = t/N$, we start from a datum $u_0$, forcing term $f_0$, and obtain a solution of the elliptic problem

$$-h\Delta\phi(u) + u = u_0 + h\, f_0,$$

which is a form of (17.3). Let us call the solution $u_1$. We symmetrize it into $u_1^*$ and it becomes a $\prec$-subsolution of the radially symmetric version of the problem with right-hand side $u_0^* + hf_0^*$. Note that this expression is more concentrated than $(u_0 + hf_0)^*$. We compare this solution with the radially symmetric solution of the elliptic equation appearing in the first iteration step with data $v_0 + hg_0$ and with nonlinearity $\psi$. By Theorem 17.7, we get

$$u_1^* \prec v_1.$$

In the second step we have to solve an elliptic problem three times: the first elliptic equation with data $u_1 + hf_1$ to get the second step of the discretized solution, $u_2$; the symmetrized version with data $u_1^* + hf_1^*$ to get some radial solution $w_2$; and the radial version with nonlinearity $\psi$ and data $v_1 + hg_1$ to get a radial solution $v_2$. The same type of comparison gives

$$u_2^* \prec w_2 \prec v_2.$$

The process is then continued for all the steps. Therefore, the comparison of concentrations works at all levels. To end the proof, the limit is taken as the time-step length goes to 0. This is where the Crandall–Liggett theorem enters. ∎

**Corollary 17.10** *In particular, under the assumptions of Theorem 17.9, or every $t \geq 0$ and every $p \in [1, \infty]$ we have comparison of $L^p$ norms,*

$$\|u(\cdot, t)\|_p \leq \|v(\cdot, t)\|_p. \tag{17.26}$$

Note that the terms of (17.26) can also be infinite for some or all values of $p$.

**Remarks** There is no problem in adapting the main result, Theorem 17.9, to the initial and boundary value problem posed in a bounded domain $\Omega \subset \mathbb{R}^d$

with zero Dirichlet conditions, since the elliptic theory is ready. We leave the easy details to the reader since there is no real difference.

The next result uses Proposition 17.2 and is a form of a general principle, that we describe as the law of *decreasing concentration of solutions of nonlinear diffusion.*

**Theorem 17.11** *Let $\varphi$ a m.m.g. in $\mathbb{R}^2$ with $0 \in \varphi(0)$. Let $0 \le u_0$ be a radially symmetric function in $L^1(\mathbb{R}^n)$. If $u(x,t)$ is the mild solution of the Cauchy problem for the GPME with no forcing term, $f = 0$, then*

$$u(.,t) \prec u(.,s) \prec u_0 \quad \text{for} \quad t \ge s \ge 0. \tag{17.27}$$

## 17.5 Smoothing effect and decay for the PME with $L^1$ functions or measures as data

We come to the main point of our analysis, comparison with the corresponding worst case, which allows us to derive the quantitative expression of the smoothing effect with exact exponents (i.e., rates), and also to obtain the best constants in the inequalities. In other words, we solve *optimal problems.*

The comparison relies on the observation that, when dealing with the porous medium equation, there is a worst case with respect to the symmetrization and concentration comparison theorem of Section 17.4 in the class of solutions with the same initial mass $\|u_0\|_1 = M$. It is just the solution $U$ with initial data a Dirac mass or ZKB solution. This special solution has been mentioned in Subsection 1.2.2 and Section 4.4, it exists for $m > 1$ and is explicitly given by formula (1.8). In order to keep the agreement with reference [515] where the following precise computations are carried out and the discussion is performed in greater detail, we will use the modified version or the PME written in the form that is convenient for even negative values of $m$, that is,

$$\frac{\partial u}{\partial t} = \nabla \cdot (u^{m-1} \nabla u), \tag{17.28}$$

the only difference a factor $m$ in the diffusivity that can be absorbed by a time rescaling. The solution is then

$$U(x,t;M) = t^{-\alpha} F(x/t^{\alpha/d}), \quad F(\xi) = (C - k\,\xi^2)_+^{\frac{1}{m-1}} \tag{17.29}$$

where

$$\alpha = \frac{d}{d(m-1)+2}, \quad k = \frac{(m-1)\alpha}{2d}. \tag{17.30}$$

The remaining parameter $C > 0$ in formula (17.29) is in principle arbitrary; it can be uniquely determined by the mass condition $\int U dx = M$, which gives the following relation between the 'mass' $M$ and $C$:

$$M = D\,C^{\gamma}, \quad D = d\,\omega_d \int_0^{\infty} (1 - k\,y^2)_+^{1/(m-1)}\, y^{d-1}\,dy, \quad \gamma = \frac{d}{2(m-1)\alpha} \tag{17.31}$$

($d$ and $\gamma$ are functions of only $m$ and $d$; the exact calculation of $D$ will be performed later). Using the mass as parameter we denote it by $U(x,t;M)$ or $U_m(x,t;M)$.

After these remarks, we get the following result that sharpens Proposition 9.8.

**Theorem 17.12** *Let $u$ be the solution of equation $\partial_t u = \Delta u^m$ in the range $m > m_c$ with initial datum $u_0 \in L^1(\mathbb{R}^d)$. Then, for every $t > 0$ we have $u(t) \in L^\infty(\mathbb{R}^d)$ and moreover there is a constant $c(m,n) > 0$ such that*

$$|u(x,t)| \le c(m,d)\, \|u_0\|_1^{\sigma}\, t^{-\alpha}, \tag{17.32}$$

*with $\alpha$ given in* (17.30) *and $\sigma = 2\alpha/d$. The best constant is attained by the ZKB solution and is given by formulas* (17.34), (17.38), (17.40) *below.*

*The same result holds when $u_0$ belongs to the space $\mathcal{M}(\mathbb{R}^d)$ of bounded and non-negative Radon measures if $\|u_0\|_1$ is replaced by $\|u_0\|_{\mathcal{M}(\mathbb{R}^d)}$.*

**Proof** (i) It is clear that the worst case with respect to the symmetrization-and-concentration comparison in the class of solutions with the same initial mass $M$ is just the ZKB solution $U$ with initial data a Dirac mass, $u_0(x) = M\delta(x)$.

(ii) Actually, there is a difficulty in taking $U$ as a worst case in the comparison, namely that $U(x,0;M)$ is not a function but a Dirac mass. We can solve this technical problem by extending the theory to bounded measures as initial data, which seems the most natural way, and indeed such a theory has been developed and offers no problem in the present context, cf. [434]. However, we prefer to stay at a more elementary level and overcome the difficulty by approximation.

We take first a solution with bounded initial data, $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. We then replace $U(x,t;M)$ by a slightly delayed function $U(x,t+\tau;M)$, which is a solution with initial data $U(x,\tau;M)$, bounded but converging to $M\delta(x)$ as $\tau \to 0$. It is then clear that for a small $\tau > 0$ such solution is more concentrated than $u_0$. From the comparison theorem we get

$$|u(x,t)| \le \|U(\cdot,t+\tau;M)\|_\infty = c(m,n) M^{\sigma} (t+\tau)^{-\alpha}, \tag{17.33}$$

which of course implies (17.32). The result for general $L^1$ data or measures follows by approximation and density once it is proved for bounded $L^1$ functions. ■

### 17.5.1 *The calculation of the best constant*

We are reduced to performing the computation of the best constant for the ZKB solution. We have

$$\|U(t)\|_\infty = C^{1/(m-1)} t^{-\alpha} = D^{-2\alpha/d} M^{2\alpha/d} t^{-\alpha}.$$

Computing $D$ is an exercise involving Euler's beta and gamma functions. For $m > 1$ we obtain

$$D = k^{-d/2} d\omega_d \int_0^1 (1-s^2)^{1/(m-1)} s^{d-1} ds = \frac{1}{2} k^{-d/2} d\omega_d B\left(\frac{d}{2}, \frac{m}{m-1}\right),$$

and for $m < 1$

$$D = k^{-d/2} d\omega_d \int_0^\infty (1+s^2)^{-1/(1-m)} s^{d-1} ds = \frac{1}{2} k^{-d/2} d\omega_d B\left(\frac{d}{2}, \frac{1}{m-1} - \frac{d}{2}\right).$$

We thus conclude that inequality (17.32)–(17.30) holds with the precise constant

$$c(m,d) = \left( \frac{\alpha(m-1)}{2d\pi} \left\{ \frac{\Gamma(m/(m-1) + d/2)}{\Gamma(m/(m-1))} \right\}^{2/d} \right)^{\alpha}. \tag{17.34}$$

for $m > 1$,

### 17.5.2 *Cases $m \le 1$*

We can pass to the limit $m \to 1$ (with a fixed choice of the mass $M$) in the ZKB solutions and obtain the fundamental solution of the heat equation,

$$E(x,t) = M\,(4\pi t)^{-d/2} \exp\left(-x^2/4t\right). \tag{17.35}$$

Therefore, $E(x,t;M) = U_1(x,t;M)$. Note the difference: $U_m$ has compact support in the space variable for all $m > 1$, while $E$ is positive everywhere with exponential tails at infinity.

It was later realized that the source solution also exists with many similar properties as long as $\alpha > 0$, i.e., it can be extended to the fast diffusion equation, $m < 1$, but only in the range $m_c < m < 1$, with

$$m_c = 0 \quad \text{for} \quad d = 1, 2, \qquad m_c = (d-2)/d \quad \text{for} \quad d \ge 3.$$

Formula (17.29) is basically the same, but now $m - 1$ and $k$ are negative numbers, so that $U_m$ is everywhere positive with power-like tails at infinite. More precisely,

$$U_m(x,t;M) = t^{-\alpha} F(x/t^{\alpha/d}), \quad F(\xi) = (C + k_1\,\xi^2)_+^{-\frac{1}{1-m}}. \tag{17.36}$$

with same value of $\alpha$ and $k_1 = -k = (1-m)\alpha/2n$. It is maybe useful to write the complete expression as

$$U_m(x,t;M)^{1-m} = \frac{t}{C\,t^{2\alpha/d} + k_1\,x^2}, \quad C = a(m,n) M^{-2(1-m)\alpha/d}. \tag{17.37}$$

The calculation of the best constant in the smoothing effect proceeds much as before and we get

$$c(m,d) = \left( \frac{\alpha(1-m)}{2d\pi} \left\{ \frac{\Gamma(1/(1-m))}{\Gamma(1/(1-m) - d/2))} \right\}^{2/d} \right)^{\alpha} \tag{17.38}$$

for $m_c < m < 1$. There are some interesting cases worth commenting. First, the expression is quite simple for $n = 2$, since $\alpha = 1/m$ and an immediate calculation gives

$$c(m, 2) = (4\pi)^{-1/m}. \tag{17.39}$$

On the other hand, taking the limit in both expressions (for $m > 1$ and $m < 1$) as $m \to 1$ we get the best constant for the $L^1$–$L^\infty$ effect for the heat equation

$$c(1, d) = (4\pi)^{-d/2}. \tag{17.40}$$

Finally, $\lim_{m\to\infty} c(m, d) = 1$ for all $d$, while there is an alternative in the limits in the lower end: $\lim_{m\to m_c} c(m, d) = \infty$ for $d \geq 3$, while the same limit is 0 for $d = 2$.

*Correction for other versions of the PME*

If the PME is written in the usual form $u_t = \Delta u^m$, or even if a constant coefficient $a > 0$ is inserted, $u_t = a\Delta u^m$, $a > 0$, the only difference with the previous calculation is the constant $ma$ that multiplies the right-hand side. There is a simple way to take into account this constant, which consists in incorporating it into the time variable and writing the usual equation in the form

$$\frac{\partial u}{\partial t'} = \nabla \cdot (u^{m-1}\nabla u), \quad t' = amt. \tag{17.41}$$

The above formulas apply exactly to this formulation with $t$ replaced by $t'$. In this way, the ZKB is corrected in the value of $k$ that becomes

$$k = \frac{(m-1)\alpha}{2adm},$$

and Theorem 17.12 holds with $c(m, n)$ of formula (17.32) replaced by

$$c'(m, d) = c(m, d)/(am)^\alpha. \tag{17.42}$$

## 17.6 Smoothing exponents and scaling properties

The reader will have observed that the estimates, and the intermediate calculations leading to them, contain a number of exponents that may seem abstruse at first glance. Since, on the other hand, they may have a certain importance in the applications, we would like to be able to predict them in an easy way. This is indeed possible once we realize that they are closely related to the scaling properties of the equation introduced in Section 3.3, cf. formulas (3.40) and (3.41) and studied further in Section 16.1.

Let us show how this property is applied to reduce the proof of existence of the $L^1$–$L^\infty$ smoothing effect of Theorem 17.12. Indeed, we will prove the following

**Proposition 17.13** *The smoothing effect* (17.32) *follows from the case* $M = 1$, $t = 1$, *i.e., if we are able to prove that for all functions* $u_0 \in L^1(\mathbb{R}^d)$ *with* $u_0 \geq 0$

*and* $\int u_0\,dx \le 1$, *the following estimate holds:*

$$u(x,1) \le c. \tag{17.43}$$

**Proof** We have already seen that PME is invariant under a two-parameter scaling group, so that whenever $u(x,t)$ is a solution of the equation, also $\tilde u(x,t) = K\ u(Lx,Tt)$ is a solution, if the $K, L$ and $T$ satisfy

$$K^{m-1}L^2 = T. \tag{17.44}$$

Next, given any solution $u(x,t)$ with data $u_0 \in L^1(\mathbb{R}^d)$, $u_0 \ge 0$ and $\int u_0\,dx = M$, we can choose $K, L, T$ so that $\tilde u$ fulfils on top of the above properties the requirement of having $L^1$-mass 1:

$$\int \tilde u_0(x)\,dx = K\int u_0(Lx,0)\,dx = KL^{-d}M.$$

We have two conditions, $K^{m-1}L^2 = T$ and $KM = L^d$, hence

$$L = M^{(m-1)\delta}T^{\delta}, \quad K = M^{-2\delta}T^{d\delta}$$

with $\delta = 1/(d(m-1)+2)$ and free parameter $T$. But now, taking $t = 1$ and fixing the free parameter $T > 0$ at will we have

$$\|u(T)\|_\infty = \frac{1}{K}\|\tilde u(1)\|_\infty = \frac{c}{K} \le c\,M^{2\delta}T^{-d\delta}.$$

We only have to change the letter $T$ into $t$ to obtain the desired result with constant $C$, cf. formula (17.32). ∎

A similar calculation can be done if we know the bound at any other time $t_0 > 0$.

Summing up, the difficulty in the derivation of the decay formula (17.32) does not lie in the exponents attached to mass $M = \|u_0\|_1$ and time $t$, because these are determined by the scaling properties of the equation, and can be calculated by some simple algebra once we have established that the map $u_0 \mapsto u(t)$ admits a bound for a certain time $t_0$ and a certain mass $M = \|u_0\|_1$. The real novelty of our result lies therefore in the existence of a finite constant and the calculation of its precise value, which we see as an optimization problem. Really minimum effort was needed in doing this part once it was discovered that it is attained by a special solution.

## 17.7 Smoothing effect and time decay from $L^p$

Next, we consider the question of boundedness for the PME when initial data are chosen in the Lebesgue space $L^p$, $p \in (1,\infty)$.

**Theorem 17.14** *For every* $u_0 \in L^p(\mathbb{R}^d)$, $1 < p \le \infty$, *and every* $t > 0$ *we have* $u(t) \in L^\infty(\mathbb{R}^d)$ *and*

$$|u(x,t)| \le c\,\|u_0\|_p^{\sigma_p}\,t^{-\alpha_p} \tag{17.45}$$

*with suitable exponents* $\alpha_p$ *and* $\sigma_p$ *and a best constant* $c = c(m,n,p,\infty) > 0$.

**Proof** Using the scaling techniques as described in Section 17.6, it is not difficult to prove that the only possible exponents are a priori given by

$$\alpha_p = \frac{d}{d(m-1)+2p}, \quad \sigma_p = \frac{2p}{d(m-1)+2p}, \tag{17.46}$$

so the only remaining task is proving that there exists a finite constant and to determine whether it is attained or not. Though this question could look like a mere extension of the preceding results, it offers several new and interesting perspectives. In particular, Marcinkiewicz spaces appear as the natural setting for the estimates.

We prove the present result as a consequence of the $L^1$–$L^\infty$ case and the comparison of equations with different diffusivities. Take a solution $u$ with data in $u_0 \in L^p(\mathbb{R}^d)$, take $\varepsilon > 0$ and consider the function $v_\varepsilon = u - \varepsilon$. This function is the solution of a filtration equation (FE$_\varepsilon$): $v_t = \Delta\Phi_\varepsilon(v)$ with

$$\Phi_\varepsilon(s) = \frac{1}{m}\left((s+\varepsilon)^m - \varepsilon^m\right).$$

Clearly, for every $s \geq 0$ we have $\Phi_\varepsilon'(s) = (s+\varepsilon)^{m-1} \geq \Phi_0'(s)$. This means that for non-negative values, the new diffusivity is larger than the original one, hence by the comparison result, Theorem 17.5, the $L^\infty$-norm of non-negative solutions goes down if the initial data are kept and $\varepsilon$ is increased.

Since $v_\varepsilon$ has changing sign, we need to consider the solution $\tilde{v}_\varepsilon(x,t)$ of $v_t = \Delta\Phi_\varepsilon(v)$ with initial data

$$\tilde{v}_\varepsilon(x,0) = (u_0(x) - \varepsilon)_+ \leq u_0(x).$$

We have $\tilde{v}_{0,\varepsilon} \in L^1(\mathbb{R}^d)$. Moreover, since $\|u_0\|_p^p \geq \varepsilon^p\,|\{u_0 \geq \varepsilon\}|$, we get

$$\|\tilde{v}_{0,\varepsilon}\|_1 \leq \|u_0\|_1 \leq \|u_0\|_p |\{u_0 \geq \varepsilon\}|^{(p-1)/p} \leq \frac{\|u_0\|_p^p}{\varepsilon^{p-1}}. \tag{17.47}$$

The comparison of diffusivities implies that $\|\tilde{v}_\varepsilon(t)\|_\infty \leq \|U(t)\|_\infty$, where $U(x,t)$ is the solution of the PME having as initial data the symmetrization of $\tilde{v}_\varepsilon(x,0)$. Since the smoothing effect has been proved in that case, see Theorem 17.12, we get

$$\|\tilde{v}_\varepsilon(t)\|_1 \leq c(m,d)\|\tilde{v}_{0,\varepsilon}\|_1^{\sigma'} t^{-\alpha'} \leq c(m,d)\|u_0\|_p^{p\,\sigma'} t^{-\alpha'}\varepsilon^{-(p-1)\,\sigma'},$$

where we have used formula (17.47) and the notation $\alpha' = \alpha(1,\infty)$ and $\sigma‘ = \sigma(1,\infty)$.

We next observe that $v(x,t) = u(x,t) - \varepsilon \leq \tilde{v}_\varepsilon(x,t)$ for every $x \in \mathbb{R}^d$ and $t > 0$, by the standard comparison theorem for solutions of equation (FE$_\varepsilon$). Therefore,

$$\|u(t)\|_\infty \leq \varepsilon + c(m,d)\|u_0\|_p^{p\,\sigma'} t^{-\alpha'}\varepsilon^{-(p-1)\,\sigma'}.$$

We can minimize this expression in $\varepsilon$ since $\varepsilon > 0$ was taken arbitrary. This gives the desired result with correct exponents,

$$\alpha = \frac{p\sigma'}{1+(p-1)\sigma'}, \qquad \sigma = \frac{\alpha'}{1+(p-1)\sigma'},$$

that agree with (17.46), and some constant that need not be the optimal constant $c(m,n,p,\infty)$. ■

**Remark** Note that when we consider the embeddings into $L^P$, $p<\infty$, we have

$$\lim_{m\to\infty} \alpha(m,d;1,p) = 0, \quad \lim_{m\to\infty} \sigma(m,d;1,p) = \frac{1}{p}.$$

**Notes**

This chapter follows closely the paper [512].

**Section 17.1.** The topics of rearrangement and symmetrization are covered in many classical texts, for more details, we refer e.g. to the books [56, 98, 329], or the articles [488] and [490]. The present notes are taken from [512].

An alternative technique for the study of symmetrization that is well suited for nonlinear parabolic problems was developed by Bénilan and coworkers [3], [84].

The standard symmetrization adapts well for the inhomogeneous equations in divergence form like

$$\partial_t u - \sum_{i,j} \partial_i(a_{ij}\partial_j \Phi(u)) + b(x,u) = f.$$

On the contrary, the application to equations like

$$\rho(x)\,\partial_t u = \Delta\Phi(u)) + f$$

needs serious modifications. A theory is developed by Reyes and Vázquez in [447] using the concept symmetrization with respect to a measure.

**Section 17.2.** Actually, much more general equations can be treated. In reference [512] we replace the Laplace operator in the first term of the equation by a whole class of operators of interest both in the theory of nonlinear analysis and in the applications to diffusive phenomena. We will consider operators $\mathcal{A}$ of the form

$$-\mathcal{A}(u) = -\mathrm{div}\,(A(\nabla u)) \tag{17.1}$$

(and some variants thereof). We need the vector function $A:\mathbb{R}^d \to \mathbb{R}^d$ to be strictly monotone. This means that for two different vectors $w_1, w_2 \in \mathbb{R}^d$, $w_1 \neq w_2$, the following property holds

$$\langle A(w_1) - A(w_2), w_1 - w_2\rangle > 0. \tag{17.2}$$

The comparison results are extended in [512] to cover the case of different nonlinearities $\beta$.

**Section 17.7.** We may also cover the FDE with $m > m_c$. The results are extended in [515] to initial data in the natural Marcinkiewicz space $M^p(\mathbb{R}^d)$.

### Problems

**Problem 17.1** Prove the following comparison result when the relation of concentrations is not exactly satisfied.

**Corollary 17.15** *Let $f_i(r)$, $i = 1, 2$ be two radially symmetric functions in $L^1_{\mathrm{loc}}(\Omega)$, let $u_i(r)$ be the respective solutions of* (17.1)*, and let $v_i = f_i - -\Delta u_i$. Assume that there is a constant $C > 0$ such that*

$$\int_\Omega f_2(x)\, dx \le \int_\Omega f_1(x)\, dx + C.$$

*Then either we have* (i) (standard case)

$$\int_\Omega v_2(x)\, dx \le \int_\Omega v_1(x)\, dx + C;$$

*or otherwise* (ii): *the function $u(r) = u_2(r) - u_1(r)$ is positive and increasing for all large $r$ and*

$$\int_{B_r(0)} (v_2(x) - v_1(x))\, dx$$

*is larger than $C$ and increasing for all large $r$. The same result holds when $u_1$ is an integral supersolution and $u_2$ is an integral subsolution.*

# 18

## ASYMPTOTIC BEHAVIOUR I. THE CAUCHY PROBLEM

We initiate in this chapter the study of the behaviour of solutions of the PME for large times. This is a subject on which the mathematical investigation of the PME has been most active, and we will devote much attention to it. The cornerstone of our presentation is the interplay between asymptotic behaviour and self-similarity. We will also see the large time behaviour as giving rise to the formation of patterns.

On a general level, it has been pointed out in many papers and corroborated by numerical experiments that similarity solutions furnish the asymptotic representation for solutions of a wide range of problems in mathematical physics. The reader is referred to the books of G.I. Barenblatt [63, 64] for a detailed discussion of this subject. Self-similar solutions and scaling techniques will play a prominent role in our asymptotic study of this and the next chapters.

The closest motivation of our study comes from the theory of the linear heat equation, $m = 1$, which is the standing reference in diffusion theory. The asymptotic behaviour of the typical initial and boundary value problems in usual classes of solutions is a well researched subject for the HE. Both the asymptotic patterns and the rates of convergence are known under various assumptions. Thus, the most classical result for the Cauchy problem says that under the assumptions of non-negative and integrable initial data $u_0 \in L^1(\mathbb{R}^d)$, $u_0 \geq 0$, there is convergence of the solution of the Cauchy problem towards a multiple of the Gaussian kernel: $u(x,t) \sim M\,G(x,t)$, where

$$G(x,t) := \frac{M}{(4\pi t)^{N/2}} \exp\{-|x|^2/4t\}, \tag{18.1}$$

and $M = \int u_0(x)\,dx$ is the mass of the solution (space integration is performed by default in $\mathbb{R}^d$). When the heat equation is viewed as the PDE expression of the basic linear diffusion process in probability theory, the functions $u(\cdot,t)$ with mass 1 are viewed as the probability distributions of a stochastic process and formula (18.1) is a way of formulating the central limit theorem.

In the case of the Cauchy–Dirichlet problem posed in a bounded domain $\Omega \subset \mathbb{R}^N$, it is well known that the asymptotic shape of any solution with non-negative initial data in $L^2(\Omega)$ approaches one of the special *separated-variables* *solutions*

$$u_1(x,t) = c\,T_1(t)\,F_1(x)\,. \tag{18.2}$$

Here $T_1(t) = e^{-\lambda_1 t}$, where $\lambda_1 = \lambda_1(\Omega) > 0$ is the first eigenvalue of the Laplace operator in $\Omega$ with zero Dirichlet data on $\partial\Omega$, and the space pattern $F_1(x)$ is the corresponding positive and normalized eigenfunction. The constant $c > 0$ is determined as the coefficient of the $L^2(\Omega)$-projection of $u_0$ on $F_1$.

In the case of the Neumann problem posed in a bounded domain with zero boundary data the solutions of the heat equation are known to stabilize to a constant profile and the rate of convergence is exponential in time, with coefficients corresponding to the higher eigenvalues of the Laplacian.

We will explore next the analogues of these results for the PME. This first chapter is devoted to the Cauchy problem posed in $\mathbb{R}^d$. We start by the investigation of the long-time behaviour of solutions of the PME with data $u_0 \in L^1(\mathbb{R}^d)$ in order to prove that for large $t$ all such solutions can be described in first approximation by the one-parameter family of ZKB solutions $\mathcal{U}(x,t;C)$ given by formulas (1.8)–(1.9). This result, contained in Theorem 18.1, is one of the highlights of the whole text. When applied to non-negative solutions with mass 1 *it is the precise statement of the nonlinear central limit theorem for the evolution generated by the PME.* But note that the mathematical result does not have a mass restriction, not even a sign restriction.

Section 18.2 contains the proof of the asymptotic theorem for non-negative solutions using the so-called four step method, based on rescaling and compactness. The convergence of supports and interfaces for compactly supported data occupies Section 18.3.

Section 18.4 examines the so-called continuous scaling and the associated Fokker–Planck equations. This alternative approach is very fruitful in a variety of settings.

We devote two sections to deriving alternative proofs of the main convergence result for the PME based on standard implementations of the idea of *Lyapunov function.* In Section 18.5 we use as functional the $L^1$-norm of the difference with respect to a Barenblatt solution.

Section 18.6 introduces another functional, the entropy. Convergence is also proved by this method, but there is an extra benefit: convergence rates can be calculated. Section 18.7 delves on the peculiarities of the asymptotic behaviour in one space dimension; they allow us to establish optimal convergence rates.

Section 18.8 contains the proof of the asymptotic convergence for signed solutions, and the extension to cover integrable forcing terms. Section 18.9 is an introduction to the special properties of the large time behaviour of the FDE.

The study of asymptotic behaviour is quite popular in the mathematical literature. We gather a collection of interesting directions on large time behaviour of Cauchy problems for nonlinear diffusions of PME type in the final Section 18.10.

## 18.1 ZKB asymptotics for the PME

Consider the class of solutions of the PME posed in $Q = \mathbb{R}^d \times (0,\infty)$ with initial data $u_0 \in L^1(\mathbb{R}^d)$. Let us recall the formulas for the ZKB solutions,

$$\mathcal{U}(x,t;C) = t^{-\alpha}\left(C - k\,|x|^2 t^{-2\beta}\right)_+^{\frac{1}{m-1}}\,, \tag{18.3}$$

where $(s)_+ = \max\{s,0\}$,

$$\alpha = \frac{d}{d(m-1)+2}, \quad \beta = \frac{\alpha}{d}, \quad k = \frac{\beta(m-1)}{2\,m}. \tag{18.4}$$

The constant $C > 0$ is free and can be used to adjust the mass of the solution: $\int_{\mathbb{R}^d} \mathcal{U}(x,t;C)\,dx = M > 0$. It follows that

$$C = c(m,d)\, M^{2(m-1)/(d(m-1)+2)}. \tag{18.5}$$

Let us write $\mathcal{U}_M$ for the solution with mass $M$ and $F_M$ for its profile. This is the precise statement of the asymptotic convergence result.

**Theorem 18.1** *Let $u(x,t)$ be the unique weak solution of the Cauchy problem with initial data $u_0 \in L^1(\mathbb{R}^N)$, and $\int_{\mathbb{R}^d} u_0\,dx = M > 0$. Let $\mathcal{U}_M$ be the ZKB solution with the same mass as $u_0$. As $t \to \infty$ the solutions $u(t)$ and $\mathcal{U}_M$ are increasingly close and we have*

$$\lim_{t\to\infty} \|u(t) - \mathcal{U}_M(t)\|_1 = 0. \tag{18.6}$$

*Convergence holds also in $L^\infty$ norm in the proper scale:*

$$\lim_{t\to\infty} t^{\alpha}\|u(t) - \mathcal{U}_M(t)\|_\infty = 0 \tag{18.7}$$

*with $\alpha = d/[d(m-1)+2]$. Moreover, for every $p \in (1,\infty)$ we have*

$$\lim_{t\to\infty} t^{\alpha(p-1)/p}\|u(t) - \mathcal{U}_M(t)\|_{L^p(\mathbb{R}^d)} = 0. \tag{18.8}$$

*Let now $\int_{\mathbb{R}^d} u_0\,dx = -M \le 0$. The same result is true with $\mathcal{U}_M$ replaced by $-\mathcal{U}_M(x.t)$ when $M < 0$, and by $\mathcal{U}_0(x,t) = 0$ if $M = 0$.*

### Remarks

(1) According to this result, all the information the solution remembers from the initial configuration after a large time is reduced in first approximation only to the mass $M$, since the pattern and the rate are supplied by the equation. This is a very apparent manifestation of the equalizing effect of the diffusion.

(2) The PME replaces the Gaussian profile by the ZKB profile as the asymptotic pattern. This pattern has sharp fronts at a finite distance and no space tails, exponential or otherwise.

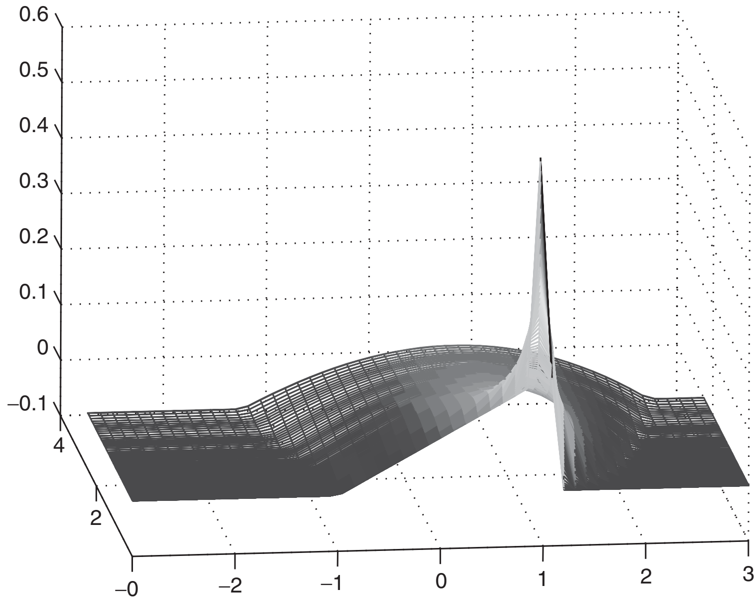


Figure 18.1: ZKB as intermediate asymptotics from heavily asymmetric data.

(3) Note that the end result of the evolution is the zero level, $u(x,t) \to 0$ as $t \to \infty$. The non-trivial ZKB pattern is visible in finite time (actually, quite soon, see Figure 18.1). For this reason it is called in the literature *intermediate asymptotics.*

(4) For a given solution $u$, there is only one correct choice of the constant $C = C(u_0)$ in these asymptotic estimates, since trying two ZKB solutions with different constants in the above formulas produces non-zero limits. In that sense, the estimates are sharp.

(5) The last result follows from (18.6) and (18.7) by simple interpolation, but (18.7) and (18.6) are (to an extent) independent.

(6) The result is optimal in the sense that we cannot get better convergence rates in the general class of solutions $u_0 \in L^1(\mathbb{R}^d)$, even if we assume $u_0 \geq 0$. The following construction settles this question.

*Counterexample*

*Given any decreasing function $\rho(t) \to 0$, there exists a solution of the Cauchy problem with integrable and non-negative initial data of mass $M > 0$ such that*

$$lim\ sup_{t\to\infty} \frac{(u(0,t) - \mathcal{U}(0,t;M))\, t^{\alpha}}{\rho(t)} = \infty. \tag{18.9}$$

*Moreover, we can also get*

$$\lim\sup_{t\to\infty}\frac{\|u(t)-\mathcal{U}(t;M)\|_1}{\rho(t)}=\infty. \tag{18.10}$$

*We can also ask the solution to be radially symmetric with respect to the space variable.*

We will give below several proofs of Theorem 18.1. The construction of the counterexample is given in Section A.10. We point out that convergence rates can be obtained if the initial data are ‘small at infinity’, technically under assumptions on the moments or on the decay as $|x|\to\infty$, see Subsection 18.6.1. Actually, this is a quite active topic of the theory of the PME and also the heat equation.

(7) If the initial data has a finite second moment (which for the probability distribution means finite standard deviation), then the convergence can be extended to the moment

**Corollary 18.1A** *Let $u_0$ be such that $\int|u_0(x)|(1+|x|^2)dx<\infty$. Then, the second moment of $u(x,t)$ converges to that of the corresponding ZKB solutions in the sense that*

$$\int u(x,t)|x|^2dx=c(m,d)M^{2(m-1)\beta}t^{2\beta}+o(t^{2\beta})$$

*as $t\to\infty$. The first term in the right-hand side is just $\int\mathcal{U}(x,t)|x|^2dx$.*

The proof is quite simple and we leave it as an exercise, Problem 18.12.

(8) The convergence takes also place in the Wasserstein distance $d_2$ for some scaling of the solution to be discussed in Section 18.4, cf. [413]. But this is a different story that we will skip here.

## 18.2 Proof of convergence for non-negative solutions

We proceed next with the proof of the case $u_0\ge 0$. We will follow the ‘four-step method’, a general plan to prove asymptotic convergence devised by S. Kamin and Vázquez in 1988 [325], who settled in this way the asymptotic behaviour both for the $p$-Laplacian equation, $u_t=\nabla\cdot(|\nabla u|^{p-2}\nabla u)$, and for the PME. For convenience, we divide the proof into several steps.

*Step 1. Rescaling*

In order to observe the asymptotic behaviour of the orbit $u(t)$, $t>0$, of the Cauchy problem we rescale it according to the Barenblatt exponents, i.e., the exponents of the ZKB solution. We have discussed the family of scaling transformations in Section 16.1. Let us recall the story in the present context: given a solution $u=u(x,t)\ge 0$ of the PME in the class of strong solutions with finite mass constructed in Chapter 9, we obtain a family of solutions

$$\widetilde{u}_\lambda(x,t)=(\mathcal{T}_\lambda u)(x,t)=\lambda^\alpha u(\lambda^\beta x,\lambda t) \tag{18.11}$$

with initial data $\widetilde{u}_{0,\lambda}(x) = (\mathcal{T}_\lambda u_0)(x) = \lambda^\alpha u_0(\lambda^\beta x)$. The exponents $\alpha$ and $\beta$ are related by

$$\alpha(m-1) + 2\beta = 1, \tag{18.12}$$

so that all the $\widetilde{u}_\lambda$ are again solutions of the PME. This is the scaling formula that we call the $\lambda$-scaling or **fixed scaling**. It is in fact a family of scalings with free parameter $\lambda > 0$, that performs a kind of **zoom** on the solution.

In the present application we have another constraint that allows us to fix both $\alpha$ and $\beta$ to the desired values (the Barenblatt values). It is the condition of mass conservation. We only need to impose it at $t = 0$,

$$\int_{\mathbb{R}^d} (\mathcal{T}_\lambda u_0)(x)\, dx = \int_{\mathbb{R}^d} u_0(x)\, dx, \tag{18.13}$$

and we get $\alpha = d\,\beta$. Together with (18.12), this implies that $\alpha$ and $\beta$ have the values (18.4) of the theorem. To end this part, note the following important property: the source-type solutions are invariant under this mass conserving $\lambda$-rescaling, i.e.

$$\mathcal{U}_M(t) = \mathcal{T}_\lambda(\mathcal{U}_M(t)).$$

*Step 2. Uniform estimates and compactness*

We want to show that the family $\{\widetilde{u}_\lambda(t)\ \lambda > 0\}$, is uniformly bounded and even relatively compact in suitable functional spaces. This is an important step where we put to work the estimates derived in Chapter 9. It is crucial that the rescaling performed in the previous step and the estimates match, otherwise this step could not work.

To begin with our case, the family is uniformly bounded in $L^1(\mathbb{R}^d)$ for all $t$ positive:

$$\int_{\mathbb{R}^d} \widetilde{u}_\lambda(x,t) dx = \int_{\mathbb{R}^d} \lambda^\alpha u(\lambda^\beta x, \lambda t)\, dx = \int_{\mathbb{R}^d} u(y, \lambda t)\, dy = M < \infty. \tag{18.14}$$

Using now the $L^\infty$ bound (9.21), we get

$$\|\widetilde{u}_\lambda(\cdot, 1)\|_\infty = \lambda^\alpha \|u(\cdot, \lambda)\|_\infty \le \lambda^\alpha \frac{M^{2\alpha/d}}{\lambda^\alpha} C = C M^{2\alpha/d} \tag{18.15}$$

independently of $\lambda$, and in the same way

$$\|\widetilde{u}_\lambda(\cdot, t_0)\|_\infty = \lambda^\alpha \|u(\cdot, t_0\lambda)\|_\infty \le \lambda^\alpha \frac{M^{2\alpha/d}}{\lambda^\alpha t_0^\alpha} C = C\, M^{2\alpha/d}\, t_0^{-\alpha}. \tag{18.16}$$

Control of the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ means control of all norms $\|\cdot\|_p$ for all $p \in [1, \infty]$. Thus, $\|\widetilde{u}_\lambda(\cdot, t)\|_p$ are equibounded for all $p \in [1, \infty]$.

Next, we take $t_0 > 0$ so that, by the regularizing effect, $u(t_0) \in L^{m+1}(\mathbb{R}^d)$. The energy estimates (9.46) give

$$\int_{\mathbb{R}^d} |\nabla \widetilde{u}_\lambda^m(x,t)|^2 dx \le C(t_0, \|\widetilde{u}_\lambda(x,t_0)\|_{L^{m+1}}) \tag{18.17}$$

for $t \ge t_0 > 0$. Now, the $\|\widetilde{u}_\lambda(t)\|_{L^{m+1}}$ are equibounded, hence $\|\nabla \widetilde{u}_\lambda^m(x,t)\|_{L^2}$ are equibounded (for $t \ge t_0 > 0$). Moreover, we have a $u_t$ estimate of the form

$$\|\frac{\partial \widetilde{u}_\lambda}{\partial t}(t)\|_{L^1} \le C\frac{\|\widetilde{u}_\lambda(t)\|_{L^1}}{t} \tag{18.18}$$

Using the same argument, we conclude that the norms $\|(\widetilde{u}_\lambda)_t(t)\|_{L^1}$ are equibounded if $t \ge t_0 > 0$.

**Compactness** We now use the Rellich–Kondrachov theorem, reviewed in Section A.1. Let us now recall our situation for the family $\{\widetilde{u}_\lambda\}_{\lambda \ge 1}$ for $t \ge t_0 > 0$:

$$\widetilde{u}_\lambda(x,t) \in L^\infty_{x,t} \subset L^1_{\rm loc}, \quad \frac{\partial \widetilde{u}_\lambda}{\partial t}(x,t) \in L^\infty_t(L^1_x) \subset L^1_{x,t}\,(t \in (t_0,t_1)),$$

and

$$\nabla_x \widetilde{u}_\lambda^m \in L^2_{x,t} \subset L^1_{x,t}.$$

All spaces are local in time in the sense that they exclude $t = 0$.

There is another way to compactness which needs the more advanced estimates of the local regularity theory of Chapter 7. It implies that the family $\widetilde{u}_\lambda$ is also Hölder continuous, more precisely the family is bounded in local Hölder spaces, hence equicontinuous on compact subsets of $Q$. By the Ascoli–Arzelà theorem, this implies compactness in the uniform norm, but on compact domains. Here is the conclusion of this step:

**Lemma 18.2** *The family $\{\widetilde{u}_\lambda\}_{\lambda>1}$ is relatively compact locally in $L^1_{x,t}$ and also in $C_{loc}(Q)$. Also is the family $\{\widetilde{u}_\lambda^m\}_{\lambda>1}$.*

*Step 3. Passage to the limit*

We can now take a sequence $\{\lambda_k\} \to \infty$ and assert that $\widetilde{u}_{\lambda_k}$ converges in $L^1_{\rm loc}(Q)$ (and in $C_{loc}(Q)$) to some function $U$:

$$\lim_{\lambda\to\infty} \widetilde{u}_\lambda(x,t) = U(x,t). \tag{18.19}$$

We need to study the properties of such limit functions $U(x,t)$.

**Lemma 18.3** *Any limit $U$ is a non-negative weak solution of the PME satisfying uniform bounds in $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$ for all $t \ge \tau > 0$.*

**Proof** It is clear that, as a consequence of the passage to the limit, $U$ is non-negative. Also, $U(t)$ is uniformly bounded in $L^1$ and $L^\infty$ for $t \ge t_0 > 0$, according to formulas (18.14), (18.16). Let us check that $U$ is a weak solution. We have already remarked that our uniform estimates are not good near $t = 0$. In view of this, we restrict the test functions to the class

$$\varphi \in C_c^\infty(\mathbb{R}^d \times (0, \infty)),$$

so that $\varphi$ vanishes in a neighborhood of $t = 0$, and write the weak form of the equation for $\widetilde{u}_\lambda$ as

$$\int\int \{\widetilde{u}_\lambda \varphi_t - \nabla \widetilde{u}_\lambda^m \nabla \varphi\} dxdt = 0. \tag{18.20}$$

With our estimates

$$\begin{cases} \widetilde{u}_\lambda \to U & \text{locally in } L^1_{x,t} \\ \nabla \widetilde{u}_\lambda^m \to \nabla U^m & \text{in } L^2_{x;t,\mathrm{loc}} \text{weak}, \end{cases}$$

we may pass to the limit in this expression (along a subsequence $\{\lambda_n\} \to \infty$) to get

$$\iint \{U\varphi_t - \nabla_x U^m \cdot \nabla_x \varphi\} \, dxdt = 0. \tag{18.21}$$

This means that $U$ is a weak solution of the PME. But the question of the initial trace is not settled. ■

*Step 4. Identification of the limit*

Thus far, we have posed the dynamics in the form of an initial value problem and we have introduced a method of rescaling which has allowed to obtain, after passage to the limit, one or several new solutions of the original problem. These solutions, which, we call the asymptotic dynamics, form a special subset of the set of all orbits of our dynamical system and represent the (scaled) asymptotic behaviour of the original orbits. Their complete description becomes our main problem, a problem that may turn out to be difficult to solve in some contexts.

In the present case the asymptotic dynamics turns out to be quite simple. We want to prove that the limit $U$ along any sequence $\{\lambda_n\} \to \infty$ is necessarily $\mathcal{U}_M$. Both $U$ and $\mathcal{U}_M$ are solutions of the PME for $t > 0$, enjoying a number of similar bounds. In order to identify them we only need to check their initial data and use a suitable uniqueness theorem for the Cauchy problem. The necessary uniqueness theorem is available thanks to M. Pierre's work [434] and has been explained in Theorem 13.6: we only need to check the coincidence of initial traces. Let us then worry about them. At first sight it looks easy:

**Lemma 18.4** *As $\lambda \to \infty$ we have $\lim \widetilde{u}_{0,\lambda}(x) \to M\delta(x)$ in the sense of bounded measures.*

**Proof** As $\lambda \to \infty$, since $\alpha = d\,\beta > 0$,

$$\int_{\mathbb{R}^d} \widetilde{u}_{0\lambda}(x)\,\varphi(x)\,dx = \int_{\mathbb{R}^d} \lambda^\alpha u_0(\lambda^\beta x)\varphi(x)\,dx = \int_{\mathbb{R}^d} u_0(y)\varphi(y/\lambda^\beta)\,dy\,,$$

which converges to $\int_{\mathbb{R}^d} u_0(y)\,\varphi(0)\,dy$ for all $\varphi \in C_c^\infty(\mathbb{R}^d)$, $\varphi \geq 0$. We have used the mass value: $\int_{\mathbb{R}^d} u_0(y)\,dy = M$. ∎

**The problem of the double limit** Unfortunately, the fact that the initial data for $\widetilde{u}_\lambda$ converge to $M\,\delta(x)$ does not justify by itself that $U(t)$ takes initial data $M\,\delta(x)$, because we do not control the evolution of the $\widetilde{u}_\lambda$ near $t = 0$ in a uniform way and a discontinuity might be taking place near $t = 0$ in the limit $\lambda \to \infty$. This is a typical case of double limits,

$$\lim_{t\to 0}\lim_{\lambda\to\infty} \widetilde{u}_\lambda(x,t) = \lim_{\lambda\to\infty}\lim_{t\to 0} \widetilde{u}_\lambda(x,t)\,?$$

Preparing for a correct analysis, the first thing to do is to check that $U$ and $\mathcal{U}_M$ have the same mass, i.e., that $U$ has mass $M$. Since

$$\int_{\mathbb{R}^d} \widetilde{u}_\lambda(x,t)\,dx = M,$$

and $\widetilde{u}_{\lambda_k}$ converges to $U$ in $L^1_{x,t}$-strong locally, we have $\widetilde{u}_{\lambda_k}(t) \to U(t)$ for a.e. $t$ in $L^1_x(\mathbb{R}^d)$ locally and a.e. in $(x,t) \in Q$. By Fatou's lemma

$$\int_{\mathbb{R}^d} U(t)\,dx \leq \lim_{k\to\infty} \int_{\mathbb{R}^d} \widetilde{u}_{\lambda_k}(x,t)\,dx = M,$$

hence the mass is equal or less. We have again met a difficulty. This difficulty is in principle essential. There are examples for rather simple equations in the nonlinear parabolic area where the initial data are not trivial but the whole solution disappears in the limit![1]

**Compactly supported solutions** Here the only way the discontinuity can happen is by mass escaping to infinity, since there is only a mechanism at play, diffusion. In view of this difficulty we change tactics and try to establish the result under an extra assumption: *we take $u_0$ a bounded, $0 \leq u_0 \leq C$, and compactly supported function,* $\mathrm{supp}(u_0) \subset B_R(0)$.

Then, $\mathrm{supp}(\widetilde{u}_{0\lambda}) \subset B_{R/\lambda^\beta}(0)$. Moreover, there exists a source-type solution of the form $V(x,t) = \mathcal{U}_{M'}(x, t+1)$ with $M' \gg M$ such that $V(x,0) = \mathcal{U}_{M'}(x,1) \geq u_0(x)$. Then,

$$\widetilde{u}_\lambda(x,0) = \lambda^\alpha u_0(x\lambda^\beta, 0) \leq \lambda^\alpha \mathcal{U}_{M'}(x\lambda^\beta, 1) = \mathcal{U}_{M'}\left(x, \frac{1}{\lambda}\right),$$

where in the last equality we have used the invariance of $\mathcal{U}$ under $\mathcal{T}_\lambda$. We conclude from the maximum principle that

$$\widetilde{u}_\lambda(x,t) \leq \mathcal{U}_{M'}\left(x, t + \frac{1}{\lambda}\right), \tag{18.22}$$

[1] Should such a 'disaster' happen, we refer to it as an *initial layer of discontinuity*, an interesting object of study. See [515], Chapter 9.

and in the limit $U(x,t) \le \mathcal{U}_{M'}(x,t)$. The bound solves all our problems since it implies that the support of the family $\{\widetilde{u}_\lambda(t)\}$ is uniformly small for all $\lambda$ large and $t$ close to zero. Indeed, we observe the relation between the radii of the supports of a solution and its rescaling:

$$R_\lambda(t) = \frac{1}{\lambda^\beta} R(\lambda t). \tag{18.23}$$

It follows that the support of $\widetilde{u}_\lambda(t)$ is contained in a ball of radius

$$R = C\,(M')^{(m-1)\beta}\left(t+\frac{1}{\lambda}\right)^\beta \tag{18.24}$$

with $C = C(m,n)$. Now we can proceed.

**Lemma 18.5** *The limit $U$ has mass $M$ for all $t > 0$.*

This is a consequence of the dominated convergence theorem since $U$ is bounded above by a big source-type (ZKB) solution.

**Lemma 18.6** *Under the present assumptions on $u_0$, we have $U(x,t) \to M\delta(x)$ as $t \to 0$, i.e.,*

$$\lim_{t\to 0}\int_{\mathbb{R}^d} U(x,t)\varphi(x)dx = M\varphi(0) \tag{18.25}$$

*for all test functions $\varphi \in C_c^\infty(\mathbb{R}^d)$.*

**Proof** Since $M = \int U(x,t)\,dx$, we have for $t > 0$

$$\begin{aligned}\left|\int U(x,t)\varphi(x) - M\varphi(0)\,dx\right| &\le \int |U(x,t)||\varphi(x)-\varphi(0)|dx\\ &\le \int_{|x|\le\delta} |U(x,t)||\varphi(x)-\varphi(0)|dx\\ &\quad + \int_{|x|>\delta} |U(x,t)||\varphi(x)-\varphi(0)|dx = (*).\end{aligned}$$

By continuity there exists $\delta > 0$ such that $|\varphi(x)-\varphi(0)| \le \varepsilon/2M$ if $|x| \le \delta$. Besides, $\varphi$ is bounded so that

$$|\varphi(x)-\varphi(0)| \le 2C \quad (\varphi \in C_c^\infty)\,.$$

Since $U$ vanishes for $|x| \ge \delta$ if $t$ is small enough we get

$$(*) \le M\frac{\varepsilon}{2M} + 2C\int_{|x|>\delta} |U(x,t)| \le K\varepsilon\,.$$

**Conclusion** Using the uniqueness result, Theorem 13.6, we can identify $U$. Hence, for $t = 1$ we have $\widetilde{u}_{\lambda_n}(x,1) \to \mathcal{U}_M(x,1)$ in $L^1_{\rm loc}(\mathbb{R}^d)$. Now, the $\widetilde{u}_\lambda$ have compact support which is uniformly bounded in $\lambda$. It follows that

$$\widetilde{u}_{\lambda_n}(x,1) \to \mathcal{U}_M(x,1) \qquad \text{in } L^1\text{-strong.}$$

(We pass from local to global convergence.) The limit is thus independent of the sequence $\{\lambda_n\}$. It follows that the whole family $\{\widetilde{u}_\lambda\}$ converges to $\mathcal{U}_{\mathcal{M}}$ as $\lambda \to \infty$.

**General data** We still have to deal with data which do not have compact support. The proof in this case implies some non-trivial extra effort that we delay for the moment, see Subsection 18.2.1.

*Step 5. Rephrasing the result*

The argument has concluded (for compactly supported data at this moment), but we still have to write the conclusion in the original variables and scales. So actually the 'four-step method' is rather a 'five-step method', having a simple end step. Let $F_M(x) = \mathcal{U}_M(x,1)$. We have just proved that

$$\lim_{\lambda\to\infty} \|\lambda^\alpha u(\lambda^\beta x, \lambda) - F_M(x)\|_{L^1} = 0,$$

which means with $y = \lambda^\beta x$ that

$$\lim_{\lambda\to\infty} \int \lambda^\alpha |u(y,\lambda) - \lambda^{-\alpha} F_M(y/\lambda^\beta)| \lambda^{-\beta d}\, dy = 0.$$

Noting that $\mathcal{U}_M(y,\lambda) = \lambda^{-\alpha} F_M(y/\lambda^\beta)$ and that $\alpha = \beta d$, we arrive at

$$\lim_{\lambda\to\infty} \int |u(y,\lambda) - \mathcal{U}_M(y,\lambda)|\, dy = 0,$$

i.e., replacing $\lambda$ by $t$,

$$\lim_{t\to\infty} \|u(y,t) - \mathcal{U}_M(y,t)\|_{L^1_y} = 0.$$

This is the asymptotic formula (18.6). On the other and, using the uniform Hölder continuity of the $\widetilde{u}_\lambda$ we conclude that

$$\widetilde{u}_\lambda(x,1) \to U_M(x,1) \tag{18.26}$$

uniformly on compact subsets, hence uniformly in $\mathbb{R}^d$ in the present situation. After rephrasing the result in terms of $u$, we have the uniform estimate (18.7). As we said, estimate (18.8) follows by interpolation. Theorem 18.1 is proved for the class of bounded and compactly supported initial data. In view of the smoothing effect, boundedness is not a restriction for large times if we start with $u_0 \in L^1(\mathbb{R}^d)$. ■

### 18.2.1 *Completing the general case*

We now extend the result from compactly supported initial data to the whole class of data $u_0 \in L^1(\mathbb{R}^d)$, $u_0 \geq 0$ by a density argument. Given $\varepsilon > 0$ we construct an approximation $\widetilde{u}_0$ which is bounded and compactly supported and such that

$$\|\widetilde{u}_0 - u_0\|_1 \le \varepsilon, \qquad \int_{\mathbb{R}^d} \widetilde{u}_0(x)\,dx = \widetilde{M}.$$

We first tackle the $L^1$ estimate (18.6). In order to prove this formula for $u(x,t)$ we only have to use the triangle formula plus the $L^1$ contraction property:

$$\|u(t) - U_M(t)\|_1 = \|u(t) - \widetilde{u}(t)\|_1 + \|\widetilde{u}(t) - U_{\widetilde{M}}(t)\|_1 + \|U_{\widetilde{M}}(t) - U_M(t)\|_1.$$

Now, $|M - \widetilde{M}| \le \varepsilon$, hence $\|U_{\widetilde{M}}(t) - U_M(t)\|_1 \le \varepsilon$. By the contraction principle,

$$\|u(t) - \widetilde{u}(t)\|_1 \le \|u(0) - \widetilde{u}(0)\|_1 \le \varepsilon.$$

Thus, we get $\|u(t) - U_M(t)\|_1 \le \varepsilon + \delta(t) + \varepsilon = 2\varepsilon + \delta(t)$, where $\delta(t) \to 0$ as $t \to \infty$ according to the result proved for the special compactly supported solutions. As $t \to \infty$ we get

$$\lim_{t\to\infty} \|u(t) - U_M(t)\|_1 \le 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof of the $L^1$ estimate in Theorem 18.1.

*Comment on the method*

The method we have used so far in the proof of Theorem 18.1 can be applied to different equations and systems as long as they possess good scaling properties that are relevant for the asymptotics, and as long as the identification step has some nice characteristic which enables us to determine the solution obtained as limit. We note that no essential use is made of the maximum principle, which is replaced as a main argument by compactness. This makes the method in principle well suited for systems and higher-order equations.

*Uniform convergence. Local regularizing effect*

Once the basic $L^1$ convergence result is proved, we turn next to the question of uniform convergence in $\mathbb{R}^d$, formula (18.7). This improvement of convergence is a consequence of the compactness of the family of scalings $\{\widetilde{u}_\lambda\}$ in the class of bounded continuous functions already used with new estimates of local type to control the behaviour of the 'tails at infinity'. Of course, the tail analysis is not needed for solutions with compactly supported data.

When there are tails we argue as follows: take a very large radius $R_1 \gg 1$, in particular larger than the radius of the support of $\mathcal{U}_M(x,1)$. In the time interval $1/2 < t < 1$ we have uniform convergence of $\widetilde{u}_\lambda$ towards $\mathcal{U}_M$ in the ball of radius $R_1$ (uniform also in time). Now we have to examine the outer region, $\mathcal{O}_1 = \{|x| > R_1\}$. We know that $\widetilde{u}_\lambda \ge \mathcal{U}_M$ because $\mathcal{U}_M$ vanishes identically there.

Moreover, the mass

$$\int_{\mathcal{O}_1} \widetilde{u}_\lambda(x,t)\,dx \le \varepsilon(\lambda).$$

Clearly, $\varepsilon \to 0$ as $\lambda \to \infty$; the reason is that $\widetilde{u}_\lambda$ and $\mathcal{U}_M$ have the same total mass and we have shown that they are almost identical for $|x| = R_1$. Under these circumstances, we want to prove that there is a function $C(\varepsilon)$ with $C(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$\widetilde{u}_\lambda(x,1) \le C(\varepsilon) \quad \text{for} \;\; |x| \ge R_1, \tag{18.27}$$

and the proof of Theorem 18.1 for $u_0 \ge 0$ will be complete. In other words, we want to translate small $L^1$ norms into small $L^\infty$ norms at later times. The technical tool to do that is the following result.

**Lemma 18.7** *Let $g$ be any non-negative, smooth, bounded function in the ball $B_2 = B_{2R}(a) \subset \mathbb{R}^d$, and assume that $g \in L^1(B_2)$ and*

$$\Delta(g^s) \ge -K \tag{18.28}$$

*for some $s$ and $K > 0$. Then $g \in L^\infty(B_1)$ with $B_1 = B_R(a)$) and $\|g\|_\infty$ depends only on $s$, $K$, $d$, $R$ and $\|g\|_{L^1(B_2)}$. If $\|g\|_1$ is very small compared with $R$ the estimate takes the form*

$$\|g\|_{L^\infty(B_1)} \le C(s,d)\,\|g\|^{\rho}_{L^1(B_2)}\,K^{\sigma}\,, \tag{18.29}$$

*with $\rho = 2/(2s+d)$ and $\sigma = d/(2s+d)$.*

The exact condition for (18.29) to hold is $\|g\|_1^s \le RK^{sd+2}$. This result has an interest in itself and is proved in Section A.8.

In the present situation we apply the lemma with $g = \widetilde{u}_\lambda(t)$, $a$ a point with $|a| > 2R_1$, $2R < R_1$, $s = m-1$, $K = \alpha(m-1)/mt$. We conclude that $\widetilde{u}_\lambda$ is uniformly small for $|x| \ge 3R_1/2$ for all $\lambda$ large enough and $1/2 < t \le 1$. This is the missing tail estimate to complete estimate (18.7). ■

## 18.3 Convergence of supports and interfaces

We assume in this section that $u_0$ is compactly supported and describe the asymptotic shape and size of the support as $t \to \infty$. We may assume without loss of generality that $u_0$ is continuous and non-trivial and that 0 belongs to the positivity set of $u_0$. We introduce the *minimal* and *maximal radius*,

$$\begin{cases} r(t) = \sup\{r > 0 : \; u(x,t) > 0 \text{ in } B_r(0)\}, \\ R(t) = \inf\{r > 0 : \; \operatorname{supp}(u(x,t)) \subset B_r(0)\}. \end{cases} \tag{18.30}$$

Since the source-type solution $\mathcal{U}_M(x,t)$ is given by formula (18.3), its support is the ball of radius

$$\mathcal{R}(t) = \xi_0(m,n)(M^{m-1}\,t)^{\beta} = C_0\,t^{\beta}\,. \tag{18.31}$$

**Theorem 18.8** *As $t \to \infty$ we have*

$$\lim_{t\to\infty} \frac{r(t)}{\mathcal{R}(t)} = \lim_{t\to\infty} \frac{R(t)}{\mathcal{R}(t)} = 1. \tag{18.32}$$

**Proof** The fact that the limits in (18.32) are equal or larger than 1 is a direct consequence of the uniform convergence of Theorem 18.1. On the contrary, the fact that for large $t$

$$R(t) \le (1+\varepsilon)\mathcal{R}(t)$$

needs a proof. Of course, we know that a large Barenblatt solution with some delay is a supersolution, hence there is a constant $C > 1$ such that for all large $t$

$$R(t) \le C\mathcal{R}(t).$$

On the other hand, we know that the mass contained in exterior sets of the form

$$\Omega_\varepsilon = \{|x| > (1+\varepsilon)\mathcal{R}(t)\}$$

is less than $\varepsilon$ for all large $t$. By Lemma 18.7 there is uniform convergence to 0 in this region as $t \to \infty$, hence if the support is larger than the support of $\mathcal{U}$ the excess region takes the form of a *thin tail.*

We show next that the possible tail must disappear as time grows by means of a comparison with slow travelling waves. This is done as follows: if we define the ratio $s(t) = R(t)/t^\beta$, we must prove that

$$\limsup_{t\to\infty} s(t) = C_0.$$

Assume by contradiction that this limit is $C > C_0$ and take a very large time $t_1$ for which the ratio $s(t_1) \ge C - \varepsilon$, with $\varepsilon$ very small. By scaling we can reduce that time to $t_1 = 1$. Since the ratio has limsup $C$ we have $R(1/2) \le (C+\varepsilon)/2^\beta = d < C$. On the other hand, by the uniform convergence $u \to \mathcal{U}$, we may also assume that $u \le \varepsilon$ for $|x| \ge \mathcal{R}(1) + \varepsilon = C_0 + \varepsilon$, and $t \in [1/2, 1]$. Let $d_1 = \max\{d, C_0\}$, which we may take such that $d_1 < C - 4\varepsilon$ for $\varepsilon$ small. Now, we compare $u$ with the explicit travelling wave solution $\widehat{u}$ with small speed $\varepsilon$ defined as

$$\widehat{u}^{m-1} = \frac{m-1}{m}\,(\varepsilon(t-1/2) + \varepsilon + d_1 - x_1)_+$$

where $x_1$ is the first coordinate of $x$. Comparison takes place in the region: $\{t \in [1/2, 1],\ x_1 \ge d_1\}$. By inspecting the parabolic boundary, we easily show that $u \le \widehat{u}$ there. Since $\widehat{u}$ vanishes for $x_1 \ge d_1 + \varepsilon + \varepsilon(t-1/2)$ we conclude that $u$ vanishes at $t = 1$ for $x_1 \ge d + 2\varepsilon$. We may rotate the axes in the previous argument, hence we conclude that $u(x,1) = 0$ for $|x| \ge d_1 + 2\varepsilon$ and this is a contradiction with $R(1) \ge C - \varepsilon$. The tail is eliminated. ∎

Theorem 18.8 is a manifestation of the property of *asymptotic symmetrization*, a very interesting property of diffusive processes. This result should be combined with the monotonicity properties along cones proved in

Subsection 14.6.2 that imply also that for large times the free boundary is almost spherical in a precise way,

$$0 \le R(t) - r(t) \le 2R(0). \tag{18.33}$$

## 18.4 Continuous scaling version. Fokker–Planck equation

A different way of implementing the scaling of the orbits of the Cauchy problem and proving the previous facts consists of using the **continuous rescaling**, which in this case is written in the form

$$\theta(\eta,\tau) = t^{\alpha} u(x,t), \quad \eta = x\,t^{-\beta}, \quad \tau = \ln t, \tag{18.34}$$

with $\alpha$ and $\beta$ the standard similarity exponents given by (18.4). Then $t^\alpha$ and $t^\beta$ are called the *scaling factors* (or *zoom factors*), while $\tau$ is called the *new time*. With respect to the $\lambda$-scaling, we see that the zoom factors change continuously with time, hence the name. We may also call it time-adapted rescaling.

This version of the scaling technique has a very appealing dynamical flavour and it will appear often in the sequel. The reader should note that *every problem has its corresponding zoom factors that have to be determined as a part of the analysis.*

In our problem, the new orbit $\theta(\tau)$ satisfies the equation

$$\theta_\tau = \Delta(\theta^m) + \beta\eta\cdot\nabla\theta + \alpha\theta.$$

This is the continuously rescaled equation in the general case. In the present scaling we have $\alpha = d\,\beta$, so that we can write the result as

$$\theta_\tau = \Delta(\theta^m) + \beta\nabla\cdot(\eta\,\theta). \tag{18.35}$$

This is a particular case of the so-called Fokker–Planck equations which have the general form

$$\partial_t u = \Delta(|u|^{m-1}u) + \nabla\cdot(\mathbf{a}(x)u), \quad \mathbf{a}(x) = \nabla V(x). \tag{18.36}$$

The extra term stands for a confining effect due to a potential $V$. In our case $V(x) = \beta\,|x|^2/2$. The study of Fokker–Planck equations is interesting in itself but will be conducted here only as the rescaled version of the PME.

Going back to our problem, the orbit $\theta(\tau)$ is bounded uniformly in $L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. The source-type (ZKB) solutions transform into the stationary profiles $F_M$ in this transformation, i.e., $F(\eta)$ solves the nonlinear elliptic problem

$$\Delta f^m + \beta\eta\cdot\nabla f + \alpha f = 0. \tag{18.37}$$

The boundedness and compactness arguments developed before apply here, and we may pass to the limit and form the $\omega$-limit, which is the set

$$\omega(\theta) = \{f \in L^1(\Omega) : \exists\ \{\tau_j\} \to \infty \text{ such that } \ \theta(\tau_j) \to f\}. \tag{18.38}$$

The convergence takes place in the topology of the functional space in question, here any $L^p(\Omega)$, $1 \le p < \infty$ (strong).

The rest of the proof consists in showing that the $\omega$-limit is just the Barenblatt profile $F_M$. The argument can be translated in the following way. Corresponding to the sequence of scaling factors $\lambda_n$ of the previous subsection, we take a sequence of **delays** $\{s_n\}$ and define

$$\theta_n(\eta, \tau) = \theta(\eta, \tau + s_n). \tag{18.39}$$

The family $\{\theta_n\}$ is precompact in $L^\infty_{\text{loc}}(\mathbb{R}_+ : L^1(\mathbb{R}^d))$ hence, passing to a subsequence if necessary, we have

$$\theta_n(\eta, \tau) \to \widetilde{\theta}(\eta, \tau). \tag{18.40}$$

Again, it is easy to see that $\widetilde{\theta}$ is a weak solution of (18.35) satisfying the same estimates. The end of the proof is identifying it as a stationary solution, $\widetilde{\theta}(\eta, \tau) = F(\eta)$. This has been done in the previous proof by the other scaling method, the *fixed rescaling*. In this view, the PME evolution is seen as a trend to the equilibrium configuration (which is given by the Barenblatt profile of the same mass).

Theorem 18.1 can now be used to characterize the stationary solutions.

**Theorem 18.9** *The profiles $F_M$ can be characterized as the unique solution of equation* (18.37) *such that $f \in L^1(\mathbb{R}^d)$, $f^m \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $f \geq 0$. The conditions $f^m \in H^1(\mathbb{R}^d)$, $f \in C(\mathbb{R}^d)$ are true, but not needed in the proof.*

**Proof** Any other solution $f$ of the stationary equation can be taken as initial data for the evolution equation (18.35) and then Theorem 18.1 proves that the corresponding evolutionary solution converges to the source-type solution with the same mass, $F_M$. Now, the solution $u(x,t) = t^{-\alpha} f(x\, t^{-\beta})$ is an admissible solution of the PME which converges in the rescaling to $f$. Therefore, $f = F_M$. ∎

## 18.5 A Lyapunov method

The work of A.M. Lyapunov has had a lasting influence on the studies of stability not only for ordinary differential equations but also for general dynamical systems, and in particular for PDEs (which are infinite-dimensional dynamical systems). However, it must be noted that it is not always easy to find the way of applying Lyapunov's second method to nonlinear heat equations. We present here an implementation of that idea.

*Functional*

Given an orbit $\{u(t)\}$ of the PME having mass $M > 0$, we introduce the functional

$$J_u(t) = \int_{\mathbb{R}^d} |u(x,t) - \mathcal{U}_M(x,t)|\, dx. \tag{18.41}$$

It is clear from the contraction property that $J_u(t)$ is non-increasing in $t$. We get the following result.

**Lemma 18.10** *There exists the limit* $J_\infty = \lim_{t\to\infty} J_u(t) \ge 0$.

Note that $J_u(t)$ becomes zero only if $u(t)$ coincides with the ZKB solution for some $t_1 > 0$ and then the equality holds for all $t \ge t_1$ and the asymptotic result is trivial. Otherwise $J_u(t) > 0$ for all $t > 0$. We have to examine this case.

*Limit solutions*

We perform Steps 1, 2 and 3 of the proof in Section 18.2 to obtain a sequence $\{\lambda_k\} \to \infty$ such that

$$\widetilde{u}_{\lambda_k}(x,t) \to U(x,t) \tag{18.42}$$

in $L^1(\mathbb{R}^d \times (t_1,t_2))$. The limit $U$ is again a solution of the PME. It is non-trivial and has mass $M$ (this is easy for compactly supported solutions and then true for the rest by approximation, as we saw).

*Invariance principle*

One of the key features of the use of Lyapunov functions is the following asymptotic invariance property.

**Lemma 18.11** *The Lyapunov function is constant on limit orbits, i.e.,* $J_U$ *does not depend on* $t$.

**Proof** The Lyapunov function is translated to the rescaled family $\widetilde{u}_\lambda$ by the formula

$$J_{\widetilde{u}_\lambda}(t) = \int_{\mathbb{R}^d} |\widetilde{u}_\lambda(x,t) - \mathcal{U}_M(x,t)|\, dx = J_u(\lambda t). \tag{18.43}$$

It follows that for fixed $t > 0$, we have

$$\lim_{\lambda\to\infty} J_{\widetilde{u}_\lambda}(t) = \lim_{\lambda\to\infty} J_u(\lambda t) = J_\infty.$$

On the other hand, we see that $J_u$ depends in a lower-semicontinuous way on $u$. Moreover, it is continuous under the passage to the limit that we have performed. That means that for every $t > 0$, $J_U(t) = J_\infty$. ■

*A limit solution is a source-type solution*

In order to identify $U$, the next result we need is the following.

**Lemma 18.12** *Consider the orbit of* $u(t)$ *with mass* $M > 0$ *and with connected support for* $t \ge t_0$. *Then the function* $J_u(t)$ *is strictly decreasing in any time interval* $(t_1,t_2)$, $t_0 < t_1 < t_2$, *unless* $u = \mathcal{U}_M$ *or both solutions have disjoint supports in that interval.*

**Proof** We consider for $t \geq t_1 > 0$ the solution $w$ of the PME with initial data at $t = t_1$

$$w(x, t_1) = \max\{u(t_1), v(t_1)\}, \tag{18.44}$$

where we put $v = \mathcal{U}_M$ for easier notation. Clearly, $w \geq u$ and $w \geq v$, hence

$$w(t) \geq \max\{u(t), v(t)\}, \quad t > t_1.$$

Moreover, we have $w(x, t_1) - u(x, t_1) = (v(x, t_1) - u(x, t_1))_+$ and $w(x, t_1) - v(x, t_1) = (u(x, t_1) - v(x, t_1))_+$ so that

$$J_u(t_1) = \int_{\mathbb{R}^d} (w(t_1) - u(t_1))\, dx + \int_{\mathbb{R}^d} (w(t_1) - v(t_1))\, dx,$$

while for general $t > t_1$,

$$J_u(t) + 2\int_{\mathbb{R}^d} (w(t) - \max\{u(t), v(t)\})\, dx = \int_{\mathbb{R}^d} (w(t) - u(t))\, dx + \int_{\mathbb{R}^d} (w(t) - v(t))\, dx.$$

Both integrals on the right-hand side are non-increasing in time by the contraction principle, hence constancy of $J_u$ in an interval $[t_1, t_2]$ implies that

$$w(t_2) = \max\{u(t_2), v(t_2)\}. \tag{18.45}$$

In order to examine the consequences of this equality we use the strong maximum principle.

**Lemma 18.13** *Two ordered solutions of the PME cannot touch for $t > 0$ wherever they are positive.*

This is a standard result for classical solutions of quasilinear parabolic equations, cf. [357]. It follows that (18.45) is then possible on any connected open set $\Omega$ where $w(\cdot, t_2) > 0$ under three circumstances:

(i) $w(t_2) = u(t_2) > v(t_2)$; or
(ii) $w(t_2) = v(t_2) > u(t_2)$; or
(iii) $w(t_2) = u(t_2) = v(t_2)$.

Since the support of the source-type solution is a ball and the support of $u$ is also connected, we conclude the result of Lemma 18.12. ■

**Note** If $M$ is not the mass of $u$, there is still another possibility for constant $J_u$, namely that the solutions are different but ordered: either $u(t) \geq \mathcal{U}_M(t)$ or $u(t) \leq \mathcal{U}_M(t)$.

We may now conclude the proof of Theorem 3.1 by this method in the case where $u_0$ has compact support, so that by standard properties of the propagation of support, it is connected after a certain time $t_0$. Since the source-type solution penetrates into the whole space eventually in time and $U$ has a non-contracting support, it follows that for large $t$ the supports of $U$ and $\mathcal{U}_M$ do intersect. Since both solutions cannot be ordered because they have the same mass, $J_U(t)$ must be zero since it is not strictly decreasing by Lemma 18.11. We have thus proved that $J_\infty = 0$ and

$$U = \mathcal{U}_M, \tag{18.46}$$

which identifies all possible limits of rescalings as the unique source-type solution with the same mass. This ends the proof. The extension to general data is done by density, we omit the details. ■

*Continuous rescaling*

One way of proving the previous facts is by using the continuous rescaling, formula (18.34). As explained in the previous section, taking a sequence of delays $\{s_n\}$ we define

$$\theta_n(\eta,\tau) = \theta(\eta,\tau+s_n),$$

and passing to the limit yields

$$\theta_n(\eta,\tau) \to \widetilde{\theta}(\eta,\tau). \tag{18.47}$$

Again it is easy to see that $\widetilde{\theta}$ is a weak solution of (18.35) satisfying the same estimates. For $\theta$ the Lyapunov function is translated into

$$J_\theta(t) = \int_{\mathbb{R}^d} |\theta(\eta,\tau) - F_M(\eta)|\, d\eta, \tag{18.48}$$

and we see that it is continuous under the passage to the limit we have performed. Let us examine now the situation when $J_\infty > 0$. Then $\widetilde{\theta} \neq F_M$ and the orbit of $\widetilde{\theta}$ has a strictly decreasing functional, so that for $\tau_2 = \tau_1 + h$ we have

$$J_{\widetilde{\theta}}(\tau_1) - J_{\widetilde{\theta}}(\tau_2) = c > 0.$$

Since $\widetilde{\theta}$ is the limit of the $\theta_n$ we get for all large enough $n$,

$$J_{\theta_n}(\tau_1) - J_{\theta_n}(\tau_1+h) \geq c/2.$$

But this means that for all $n$ large enough,

$$J_\theta(\tau_1+s_n) - J_\theta(\tau_1+s_n+h) \geq c/2.$$

This contradicts the fact that $J_\theta$ has a limit. The proof is complete. ■

**Comment** As we had announced, the proof of this section uses several steps of the former with a completely different end. It contains some fine regularity results that can make it difficult to apply in more general settings. However, some of these difficulties can be overcome by other means. LaSalle's invariance principle is a powerful tool in dynamical Systems [361], worth knowing also in this context.

## 18.6 The entropy approach. Convergence rates

A different Lyapunov approach is based on the existence of the so-called *entropy functional* or *Newman functional* [397]. It is defined as follows: given a solution $u$ we perform the scaling transformation (18.34) to obtain a rescaled orbit $\theta(\tau)$ that obeys equation (18.35). We then define

$$H_\theta(\tau) = \int_{\mathbb{R}^d} \left\{ \frac{1}{m-1}\theta^m + \frac{\beta}{2}\eta^2\,\theta \right\} d\eta\,, \tag{18.49}$$

where $\theta = \theta(\eta,\tau)$ and $\beta = 1/(d(m-1)+2)$ is the similarity exponent. We will also write $H_u(t)$ taking into account the equivalence $u \mapsto \theta$. $H_\theta$ represents a measure of the entropy of the mass distribution $\theta(\tau)$ at any time $\tau \ge 0$ which is well adapted to the renormalized PME evolution. Note that the entropy need not be finite for all solutions; a sufficient condition is

$$u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \qquad \int x^2 u_0(x)\,dx < \infty. \tag{18.50}$$

These properties will then hold for all times. Note that the restriction of boundedness is automatically satisfied for positive times, and the $L^\infty$ norm for positive times depends only on the initial $L^1$ norm and $t$ (see Section 17.5). The requirement that the second moment be initially finite is essential (the evolution of the second moment has been studied in Subsections 9.6.4 and 9.6.5).

Actually, we can calculate the variation of the entropy in time along an orbit of the Fokker–Planck equation.

**Lemma 18.14** *Let $H_\theta$ be the functional* (18.49) *for a bounded smooth solution of the Cauchy problem satisfying*

$$\int_{\mathbb{R}^d} (1+x^2)\,u_0(x)\,dx < \infty. \tag{18.51}$$

*Then, we have*

$$\frac{dH_\theta}{d\tau} = -I_\theta, \quad \textit{where } I_\theta(\tau) = \int \theta \left| \nabla \left( \frac{m}{m-1}\theta^{m-1} + \frac{\beta}{2}\eta^2 \right) \right|^2 d\eta \ge 0. \tag{18.52}$$

**Proof** In order to analyse the evolution of $H$ let us put for a moment

$$H(\tau) = \int_{\mathbb{R}^d} \{ \frac{1}{m-1}\theta(\eta,\tau)^m + \lambda\,|\eta|^2\theta(\eta,\tau)\}\,d\eta,$$

with $\lambda > 0$. We perform the following formal computation:

$$\begin{aligned}\frac{dH}{d\tau} &= \int \left(\frac{m}{m-1}\theta^{m-1} + \lambda\,|\eta|^2\right)\,\theta_\tau\,d\eta \\ &= \int \left(\frac{m}{m-1}\theta^{m-1} + \lambda\,|\eta|^2\right)(\Delta\theta^m + \beta\nabla\cdot(\eta\,\theta))\,d\eta \\ &= -\int \nabla\left(\frac{m}{m-1}\theta^{m-1} + \lambda\,|\eta|^2\right)\cdot(\nabla\theta^m + \beta\eta\theta)\,d\eta \\ &= -\int \theta\,\nabla\left(\frac{m}{m-1}\theta^{m-1} + \lambda\,|\eta|^2\right)\cdot\nabla\left(\frac{m}{m-1}\theta^{m-1} + \frac{\beta}{2}|\eta|^2\right)\,d\eta.\end{aligned}$$

For the precise value $\lambda = \beta/2$ we can write this quantity as (18.52), which proves that $H_\theta$ is a Lyapunov function, i.e., it is monotone along orbits. ■

These computations are easily justified for classical solutions which decay quickly at infinity. A density argument shows that for all solutions the entropy is a non-increasing functional. This is enough for the argument that follows. But it could be easier to work with regular solutions for which the lemma holds and pass to the limit after the proof is complete.

*Limit orbits and invariance*

As in the previous section, we pass to the limit along sequences $\theta_n(\tau) = \theta(\tau + s_n)$ to obtain limit orbits $\widetilde{\theta}(\tau)$, on which the Lyapunov function is constant, hence $dH_{\widetilde{\theta}}/d\tau = 0$.

*Identification step*

The proof of asymptotic convergence concludes in the present instance in a new way, by analysing when $dH_\theta/d\tau$ is zero. Here is the crucial observation that ends the proof: *the second member of* (18.52) *vanishes if and only if* $\theta$ *is a Barenblatt profile.* ■

**Corollary 18.15** *The entropy of a class of rescaled solutions with finite second moments and fixed mass* $M > 0$ *attains a minimum for the corresponding Barenblatt profile* $\theta_\infty = F_M$. *We have*

$$\lim_{\tau\to\infty} H_\theta(\tau) = H(F_M). \tag{18.53}$$

**Definitions** The difference $H(\theta|\theta_\infty) = H(\theta) - H(\theta_\infty)$ is called the *relative entropy*. Function $I_\theta$ is called the *entropy production*

**Remark on the linear case** The use of entropy functionals comes from the theory of linear diffusion, $m = 1$, where the rescaling is defined by

$$\theta(\eta,\tau) = t^{d/2}u(x,t), \quad \eta = x\,t^{-1/2}, \quad \tau = \ln t, \tag{18.54}$$

(note that usually $t$ in the scaling factors in replaced by $(1+t)$ but this is irrelevant for the asymptotics), the Fokker–Planck equation is

$$\theta_\tau = \Delta(\theta) + \frac{1}{2}\nabla\cdot(\eta\,\theta). \tag{18.55}$$

and the entropy reads

$$H_\theta(\tau) = \int_{\mathbb{R}^d}\left\{\theta\,\log\theta + \frac{\beta}{2}\eta^2\,\theta\right\}\,d\eta\,, \tag{18.56}$$

The form of Lemma 18.14 is different but the conclusion is similar.

### 18.6.1 *Rates of convergence*

The entropy functional can also be used to improve the convergence result by obtaining rates of convergence. This is done by computing $d^2H_\theta/d\tau^2$, the so-called Bakry–Emery analysis in the linear case which has been adapted to the PME by Carrillo and Toscani [155] and improved in [151]. The final result of the second derivative computation is

$$\frac{dI}{d\tau} = -2\beta\,I(\tau) - R(\tau), \tag{18.57}$$

for a certain term $R\ge 0$. Since we know by the previous analysis that $H(\theta|\theta_\infty)$ and $I_\theta$ go to zero as $\tau\to\infty$, we conclude the following result

**Theorem 18.16** *Under the regularity assumptions we have*

$$I(\tau)\le A\,e^{-2\beta\tau}. \tag{18.58}$$

*This implies the inequalities for all* $\tau>0$

$$0\le H(\theta|\theta_\infty)(\tau)\le\frac{1}{2\beta}I(\tau),\quad H(\theta|\theta_\infty)\le B\,e^{-2\beta\tau}. \tag{18.59}$$

This is the result about exponential decay of the entropy and entropy production (in logarithmic time for the PME, in real time for the Fokker–Planck equation). Let us remark that estimate (18.59)-left is a relation between integrals of a function and integrals of the gradient that can be paralleled to the Sobolev inequalities. Actually, it can be derived independently as a functional inequality (related to the so-called Gross logarithmic Sobolev inequalities). Together with (18.52) it gives an alternative way to obtain the exponential decay of $H$ without differentiating $I$, cf. [216].

In order to get a convergence result in familiar terms we have to relate relative entropies with the usual norms. This is done by means of functional inequalities like the following Csiszar–Kullback inequality.

**Lemma 18.17** *Let* $m>1$*, let* $\theta_\infty$ *be a Barenblatt profile and let* $\theta\ge 0$ *be an integrable function with the same mass and same compact support. Then, there* *exists a constant* $D(m, M) > 0$ *such that*

$$\|\theta - \theta_\infty\|_1^2 \le D\, H(\theta|\theta_\infty). \tag{18.60}$$

As a consequence decay rates are obtained in [151] for the solutions of the PME (cf. Theorem 32 and Remark 34)

**Corollary 18.18** *Let* $u$ *be a non-negative solution of the PME in the whole space and assume that the entropy of the initial data is finite. Then the convergence towards the ZKB stated in Theorem* 18.1 *happens with an estimate of the right-hand sides in formulas* (18.6)–(18.8) *of the form* $O(t^{-\gamma})$, *where*

$$\gamma = \beta \quad \text{if} \;\; 1 < m \le 2, \qquad \gamma = \frac{2\beta}{m} \quad \text{if} \;\; m \ge 2, \tag{18.61}$$

*and* $\beta = 1/(d(m-1)+2)$. *This means a renormalized error of* $t^{-\gamma}$. *The results are of course also valid for the HE [34, 154], and the relative rate is* $t^{-1/2}$.

The rates of convergence obtained here apply to solutions with bounded entropy, where the second moment is bounded at all times. These rates have to be confronted with the explicit examples at our disposal. There are three perturbations of the ZKB solutions that serve as test.

(i) The solutions obtained by displacement in space of the ZKB, $u_1(x,t) = \mathcal{U}(x - x_0, t)$ for some $x_0 \ne 0$. It is clear that this solution tends to the ZKB once the proper rescaling is done, like in (18.34). An easy calculation shows that the relative error in $L^1(\mathbb{R}^d)$ norm or in the location of the interface behaves like $O(t^{-\beta})$. This shows that the estimates of Corollary 18.18 are optimal for $1 < m \le 2$, even if we restrict ourselves to compactly supported solutions.

(ii) The (compactly supported) ellipsoidal solutions of Subsection 16.9.1. It is not difficult to check (see Problem 16.12) that the error of asymmetry is of the order or $O(t^{-2\beta})$ when compared with $\|v(t)\|_\infty$. This relative error translates into the $L^1(\mathbb{R}^d)$ norm and location of the interface. The new error shows that solutions that have a centre of mass at zero are expected to have better convergence error rates than the general case (i).

(iii) The ZKB solutions delayed in time, $u_3(x,t) = \mathcal{U}(x, t+\tau)$ for some $\tau > 0$. In this case the relative error is $O(1/t)$. This better behaviour happens for a solution with centre of mass at $x = 0$ and radial symmetry.

We will get back to the question of rates of convergence in Subsection 18.7.3 where optimal rates are found in dimension $d = 1$ for solutions with compactly supported data. We will show that errors behave like $O(1/t)$ in relative size, once the origin of space is displaced to the centre of mass so as to make the first moment zero. In several space dimensions, relative rates of order $O(1/t)$ are also obtained if the data are compactly supported and radially symmetric (see Problem 18.11).

## 18.7 Asymptotic behaviour in one space dimension

The study of one-dimensional flows allows us to obtain more refined estimates due to the simpler geometry of the solutions and supports. We consider here the Cauchy problem for the PME

$$u_t = (u^m)_{xx} \tag{18.62}$$

with initial data

$$u(x,0) = u_0(x) \geq 0 \tag{18.63}$$

and we assume that $u_0$ is integrable with $\int u_0(x)\,dx = M$, a conserved quantity of the evolution. We assume that $u_0$ is compactly supported with finite outer ends

$$a_1 = \inf\{x \in \mathbb{R} : \; u_0(x) \geq 0\}, \quad a_2 = \sup\{x \in \mathbb{R} : \; u_0(x) \geq 0$$

Let $l_0 = a_2 - a_1$. By the known regularity theory we may assume that $u_0$ is continuous without loss of generality (after a possible delay of the origin of time). We call $s_1(t)$ and $s_2(t)$ the outer interfaces. We already know that after a finite time all inner interfaces disappear:

**Proposition 18.19** *After a finite time, $t \geq T^* = (l_0/c_m)^{m+1}M^{1-m}$, the positivity set $\mathcal{P}_u(t)$ equals the interval $(s_1(t), s_2(t))$ with expanding borders.*

Hence assuming that $u > 0$ in $(s_1(t), s_2(t))$ means no loss of generality for the asymptotic study. Our previous study in this chapter shows that $u(x,t)$ converges to the ZKB profile in rescaled variables, and the interfaces converge to the ZKB interfaces in the sense that the quotient goes to 1.

### 18.7.1 *Adjusting the centre of mass. Improved convergence*

We got in [498] the following improvement of the interface approximation by using as the asymptotic model the ZKB solution with same mass and located at the centre of mass of our original solution

$$x_0 = \frac{1}{M}\int_{\mathbb{R}} x\, u_0(x)\,dx. \tag{18.64}$$

Notice that we have proved in Chapter 9 that the centre of mass in an invariant of the PME evolution.

**Theorem 18.20** *Let $u \geq 0$ be a solution of the PME with compact support in space, let $\widetilde{u}(x,t) = \mathcal{U}(x - x_0, t; M)$ be the ZKB solution with same mass and same centre of mass as $u$, and let $s_i(t)$, $i = 1, 2$, be its left-hand and right-hand interfaces, resp.*

$$R_i(t) = x_0 + (-1)^i c_m (M^{m-1}t)^{1/(m+1)}.$$

*As $t \to \infty$ we have the following results:*

$$(-1)^i(s_i(t) - R_i(t)) \downarrow 0, \qquad t\,|s_i'(t) - R_i'(t)| \to 0. \tag{18.65}$$

In other words, adjusting not only the mass but also the parameter of space translation in the ZKB solution allows us to obtain a relative factor of $o(t^{-1/(m+1)})$ in the determination of the interface and its speed for large times. Writing $R(t) = c_m(M^{m-1}t)^{1/(m+1)}$, we have

$$\begin{aligned} s_i(t) - x_0 &= (-1)^i R(t)(1 + o(t^{-1/(m+1)})), \\ s_i'(t) &= (-1)^i R'(t)(1 + o(t^{-1/(m+1)})). \end{aligned} \tag{18.66}$$

In absolute terms, the error of the interface goes to zero without a further estimate of a rate.

We divide the proof of this result into a series of lemmas

**Lemma 18.21** *For $t > 0$ we have for the right-hand interface $s(t) = s_2(t)$;*

$$s'(t) \le R'(t) \quad \textit{and} \quad s'(t)/R'(t) \to 1 \quad \textit{as} \;\; t \to \infty. \tag{18.67}$$

*Moreover, there exists $c \in [a, b]$ such that*

$$s(t) - R(t) \to c \quad \textit{as} \;\; t \to \infty \tag{18.68}$$

*and the limit is monotone non-increasing.*

**Proof** We have proved in Proposition 15.22 that

$$s''(t) + \frac{m}{(m+1)t} s'(t) = \mu(t) \ge 0.$$

where $\mu$ is a non-negative measure. This means that $s'(t)t^{m/(m+1)}$ is a non-decreasing function, so that it has a limit as $t \to \infty$, finite of infinite. But we also know the asymptotic behaviour of $s(t)$ in a first approximation

$$s(t)/R(t) \to 1,$$

cf. Proposition 15.13. It follows that the limit is finite, and in fact

$$s'(t)/R'(t) \to 1 \quad \text{as} \;\; t \to \infty.$$

Moreover, the limit is taken in a non-decreasing way so that for all $t > 0$ we have $s'(t) \le R'(t)$. This means that we have a new monotone limit

$$c = \lim_{t\to\infty} (s(t) - R(t))$$

The estimates of Proposition 15.11 imply that $a \le c \le b$. ∎

**Lemma 18.22** *We have*

$$\lim_{t\to\infty} t(s'(t) - R'(t)) = 0. \tag{18.69}$$

**Proof** We have $(ts'(t))' = s'(t)/(m+1) + t\mu(t)$, hence

$$ts'(t) = \frac{s(t)-s(1)}{m+1} + f(t), \quad f(t) = \int_1^t t\, d\mu(t).$$

As the integral of a non-negative measure, $f$ is a non-negative monotone non-decreasing function. It has a limit as $t \to \infty$. Let us assume that it is finite. Using (18.68), for all large $t$ we have

$$s'(t) = R'(t) + \left(K - \frac{s(1)}{m+1}\right)\frac{1}{t} + o(1/t).$$

If $K \neq s(1)/(m+1)$, integration of this expression implies that $s(t) - R(t)$ diverges as $t \to \infty$, which contradicts the assertion (18.68). Therefore, $K = s(1)/(m+1)$ and the conclusion follows. If the limit of $f$ is infinite, the contradiction is easier. ■

The same analysis applies of course to the left-hand interface and we obtain a monotone limit $c'$ for the difference $s_1(t) - x_0 + R(t)$.

**Lemma 18.23** *We have $c = c' = x_0$.*

**Proof** We examine the dispersion of the solution which is defined as $l(t) = s_2(t) - s_1(t)$ and the relative dispersion $d(t) = l(t) - 2R(t)$. Our results imply that $d(t) \to d_\infty = c - c'$. We want to prove that the case $c \neq c'$, i.e., non-zero asymptotic relative dispersion, leads to contradiction.

We use the fundamental inequality $v_{xx} \geq -1/((m+1)t)$ and the boundary conditions $v(s_2(t),t) = 0$, $v_x(s_2(t),t) = -s_2'(t)$ to estimate

$$v(x,t) \geq -\frac{y^2}{2(m+1)t} + s_2'(t)y$$

where $y = s_2(t) - x$. Using the fact that $s_2'(t) = R'(t) + \varepsilon(t/)/t$ we conclude that $v > 0$ for $y \leq 2(m+1)ts'(t) = 2R(t) + 2(m+1)\varepsilon$. This means that $l \geq 2R$ so that $d_\infty \geq 0$.

In case $d_\infty > 0$ we shoot from the other interface to find that

$$v(x,t) \geq -\frac{y^2}{2(m+1)t} + |s_1'(t)|y$$

where now $y = x - s_1(t)$. The analysis of the superposition of the two estimates shows that the mass of the solution is more than $M$, a contradiction (see the details of this easy but lengthy computation in [498], page 518). ■

**Lemma 18.24** *We have $c = x_0$.*

**Proof** (i) The first step is to prove that the sharp limits on the interface behaviour plus the former analysis on the bounds for the profile can be combined to give estimates of the form

$$|v_x(x,t) + \frac{x-c}{(m+1)t}| = o(1/t),$$
$$v(x,t) - V(x-c,t;M) = o(t^{-m/(m+1)}).$$

The last inequality is uniform in $x \in \mathbb{R}$, but the former applies for $x \in (s_1(t), s_2(t))$. The proof consists in arguing by contradiction using the inequality $v_x x \geq -1/((m+1)t)$; in other words, a calculus lemma about errors with respect to parabolas.

(ii) The second step is to use this information and the law of conservation of the centre of mass to identify $c = x_0$. This is easy for $1 < m \leq 2$, since we can estimate increments as

$$|\delta u(t)| \leq \|v(t)\|_\infty^{(2-m)/(m-1)} |\delta v|.$$

Then,

$$M\,|x_0 - c| \leq \int_{\mathbb{R}} x|u(x,t) - \mathcal{U}(x-c,t;M)|\,dx$$
$$= O(R^2(t))\,O(t^{(2-m)/(m+1)})\,o(t^{-m/(m+1)}) = o(1).$$

The case $m > 2$ is also a calculus lemma but the details are longer, cf. [498], page 520.

As we have explained in the proof of the last lemma, the study of the sharp interface behaviour leads to a better estimates on the pressure and its derivative $v_x$. We recall that the velocity $V = -v_x$ in the particle approach described in Section 2.1. ■

**Theorem 18.25** *We also have as $t \to \infty$*

$$t^{m/(m+1)}|v(x,t) - \overline{v}(x,t)| \to 0 \quad \textit{uniformly in } \mathbb{R}, \tag{18.70}$$

$$t\,|v_x(x,t) + \frac{x-x_0}{(m+1)t}| \to 0 \quad \textit{uniformly in } \mathcal{P}_u(t). \tag{18.71}$$

Since $\overline{v}(\cdot,t) = O(t^{-(m-1)(m+1)})$, we see that these estimates imply extra convergence rates in the sense of Section 18.6 with relative factor $t^{-1/(m+1)}$ in the error estimate for the pressure. The translation for the density is immediate when $1 < m \leq 2$.

**Remark** The result of Theorem 18.25 is true for the right-hand interface for solutions with infinite support as $x \to -\infty$ under the assumption of finite centre of mass (hence, finite moment) but no assumption of compact support. The proof is done by approximation with compactly supported solutions, see Problem 18.9.

### 18.7.2 *Closer analysis of the velocity. N-waves*

We have seen that the convergence of solutions towards a self-similar profile happens not only at the level of densities $u$ or pressures $v$, but also for the respective velocities, $-v_x$. Let us take a closer look at the model profile corresponding to the self-similar solution $U(x,t,M)$. It is given by the function

$$V_x = \begin{cases} -\dfrac{x}{(m+1)t} & \text{for } |x| < R(t) \\ 0 & \text{for } |x| > R(t). \end{cases} \tag{18.72}$$

We see that $V_x$ has a jump discontinuity at the fronts $x = \pm R(t)$. This profile is popular in the literature on conservation laws where it receives the appropriate name of $N$-wave. It is well known that $N$-waves represent the asymptotic profiles of solutions of equations of the type $u_t + f(u)_x = 0$ under different assumptions on $u(x,0)$ and $f$, cf. e.g. [362, 377]. Thus, the equation

$$u_t + (|u|^n)_x = 0, \quad n > 1,$$

admits $N$-wave solutions of the type

$$N(x,t:p,q) = \begin{cases} \dfrac{x}{nt} & \text{if } -pt^{1/n} < x < qt^{1/n} \\ 0 & \text{if } x > qt^{1/n} \text{ or } x < pt^{1/n}, \end{cases} \tag{18.73}$$

for $p, q > 0$ arbitrary. We observe that our $N$-waves form a one-parameter family (parametrized by $M$) of symmetric $N$-waves with $n = m+1$. In conservation laws, the discontinuities are called shocks and are governed by Rankine–Hugoniot conditions. In our case, we have the same type of discontinuities, though not for $v$ but for $v_x$ (i.e., not for the pressure but for the velocity) and they are governed by Darcy's law. The mathematical formula is the same, the speed of the interface is proportional to the shock jump.

When we try to prove the convergence of $v_x$ to $V_x$ there is no problem inside their common support as formula (18.71) shows. However, there is a problem due to the error in the determination of the fronts since the derivatives $v_x$ and $V_x$ have jump discontinuities on their respective fronts, and this creates a large and localized $L^\infty$ error for $v_x - V_x$. The appropriate form of formulating the convergence result is convergence in the *graph norm* of $\mathbb{R}^2$. The following result is proved in [501] by a careful revision of the proof of Theorem 18.25.

**Theorem 18.26** *For every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that whenever $t \ge t_\varepsilon$ we have*

$$\begin{aligned} &t^{\frac{m}{m+1}}|v_x(x,t) + \frac{x}{(m+1)t}| \le \varepsilon && \textit{if } |x| \le R(t)(1-\varepsilon), \\ &t^{\frac{m}{m+1}}|v_x(x,t)| \le \varepsilon && \textit{if } |x| \ge R(t)(1+\varepsilon), \\ &-\varepsilon \le V_x(x,t)t^{\frac{m}{m+1}}\operatorname{sign}(x) && \textit{if } |\,|x| - r(t)| < \varepsilon R(t). \end{aligned} \tag{18.74}$$

In terms of rescaled solutions the estimates just mean that the graph of the rescaled function $\tilde{v}(x,1)$ lies in an $\varepsilon$-neighbourhood of the standard $N$-wave at $t=1$. In particular this means that for large times the solution looks like a parabolic lump propagating in both opposite directions, with maximal speed $|v_x| \approx R(t)/(m+1)t = 0(t^{-m/(m+1)})$ along the approximate fronts $x \approx R(t)$. Velocities are small $v_x = o(t^{-\frac{m}{m+1}})$ for $x \approx 0$, or for $|x| \gg c\,R(t)$ with $c>1$. Finally, $v_x$ is negative in the interval

$$\varepsilon(m+1)t^{\frac{1}{m+1}} < x < (1-\varepsilon)c_m(M^{m-1}t)^{\frac{1}{m+1}}, \tag{18.75}$$

and $v_x > 0$ in the interval symmetric to this one.

Paper [501] also proves convergence of the velocity $-v_x$ to the ZKB $N$-wave for solutions with data $u_0 \ge 0$, $u_0 \in L^1(\mathbb{R})$, without the assumption of compact support, but in this case the rates are worse; in rescaled variables we have convergence in the graph norm without a rate, and this cannot be improved without further assumptions on the data.

### 18.7.3 *The quest for optimal rates*

Better convergence results can be obtained if such conditions are assumed. We mention two situations involving radially symmetric solutions.

*Symmetric solutions*

The case of compactly supported and radially symmetric solutions. Then we have

**Theorem 18.27** *The results of Theorems* 18.20 *and* 18.25 *hold with improved relative error rates of size* $O(t^{-1})$ *instead of* $O(t^{-1/(m+1)})$. *The rate is optimal.*

The proof is given in [498], Theorem B, page 521. It is based on concentration comparison in the sense of Chapter 17; the argument is very easy by comparison with the ZKB solution $\mathcal{U}(x,t;M)$ and a delayed version of it $\mathcal{U}(x,t+\tau;M)$, that is less concentrated than $u_0$ (more exactly, the comparison of concentrations holds for some $t_1>0$). Then we have

$$R(t) \le s(t) \le R(t+\tau) = R(t)(1+O(1/t)). \tag{18.76}$$

This is the key difference. Since the comparison functions satisfy the estimate, it cannot be improved. One of the good points of this argument is that it works in several space dimensions (always under conditions of radial symmetry).

*Eventual concavity*

The case of compactly supported data without the assumption of symmetry was addressed by Aronson and Vázquez [51] who showed that all non-negative and compactly supported solutions of the PME in $d=1$ are eventually pressure concave and satisfy the estimates of paper [96] from a time on. This is the main result.

**Theorem 18.28** *For every solution of the PME in $d=1$ with compactly supported, integrable and non-negative initial data there exists a time $T>0$ such that the pressure $v(x,t)$ is a concave function of $x$ in $\mathcal{P}_u$, the interface $s_2(t)$ is a concave function, $s_1(t)$ is a convex function for $t \ge T$. Moreover, for $t$ large we have*

$$v_{xx}(x,t) = -\frac{1}{(m+1)t} + O\left(\frac{1}{t^2}\right) \tag{18.77}$$

*valid for $s_1(t) < x < s_2(t)$, and*

$$s_i''(t) = (-1)^i R''(t)\left(1 + O\left(\frac{1}{t}\right)\right). \tag{18.78}$$

*Estimates with relative error $O(1/t)$ follow from this for $v$, $v_x$, $s_i(t) - x_0$ and $s_i'(t)$.*

We will not give the proof of the important result (18.77), which uses again the delayed solutions $\mathcal{U}(x,t+\tau;M)$ but the comparison argument is different. The 1D regularity theory of Chapter 15 is also used. We refer to [51], pages 341–347, and [96], pages 91–92.

The estimates for the interface and its derivatives are immediate from (18.77) using formula (15.99) which can be written as

$$s''(t) = ms'(t)v_{xx}(s(t),t)$$

integrating and using what is already known about $s(t)$ from Theorem 18.20.

## 18.8 Asymptotic behaviour for signed solutions

We are going to complete the proof of Theorem 18.1 by examining the behaviour of signed solutions with integrable data. The first steps of the proof of Section 18.2 are the same. Only the identification of the initial data and the uniqueness theorem are under scrutiny. The following result solves the main difficulty.

**Theorem 18.29** *If the negative part of $u_0$ is compactly supported and the total mass is positive $\int u_0(x)\,dx = M > 0$, then $u(t) \ge 0$ after a finite time.*

**Proof** (i) By the maximum principle, it is enough to consider the case where $u_0$ is compactly supported, so we will only discuss this case. By conservation of mass, $\int u(x,t)\,dx = M$ is larger than zero for all time. We consider the solutions $u_1$ and $u_2$ with initial data $u_0^+$ and $-u_0^-$, respectively, i.e., $u_1(t) = S_t u_0^+$ and $u_2 = S_t(-u_0^-)$, where $S_t$ if the PME semigroup. For all $t>0$ we have $u_2(t) \le u(t) \le u_1(t)$ and $u_2(t) \le 0 \le u_1(t)$. We conclude from Section 18.3 that the support of $u_1$ is approximately a ball of radius $c(\int u_0^+ dx)^{(m-1)\beta} t^\beta$, which contains the support of $u_2(t)$. This means that some cancellation must be taking place. We

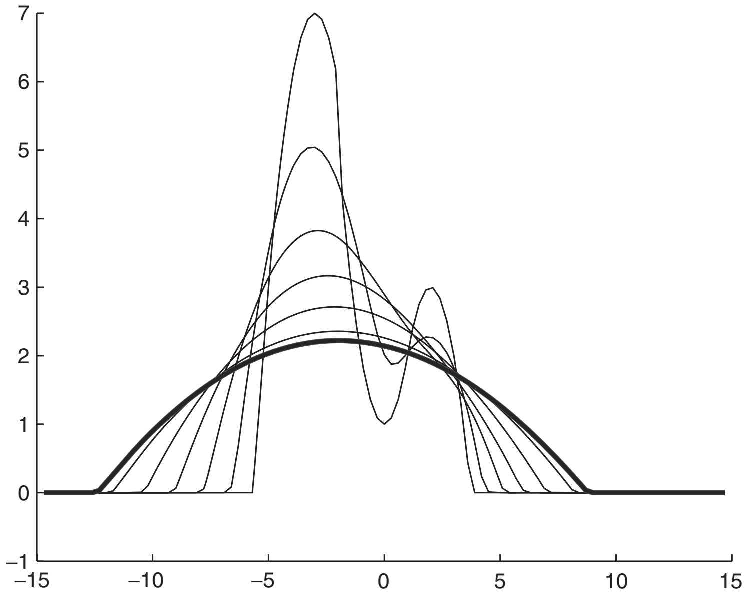


Figure 18.2: Eventual concavity of a solution.

examine that cancellation by looking at the mass function

$$M_+(t) := \int u^+(x,t)\,dx.$$

As a consequence of the $L^1$ contraction principle, cf. Propositions 3.5 and 9.1, this non-negative function is non-increasing in time along solutions, i.e., it is another Lyapunov functional. We have for all times

$$M_+(t) \le \int u_1(x,t)\,dx = M_+(0).$$

We also have the functional $M_+(t) := \int u^-(x,t)\,dx$ with similar properties and

$$M = \int u(x,t)\,dx = M_+(t) - M_-(t).$$

which means that $M_+(t) > M_-(t)$ and also that $M_+(t) \ge M$ for all times.

(ii) Let us define

$$\lim_{t\to\infty} M_+(t) = M_\infty \ge M.$$

We want to prove that $M_\infty = M$, hence that $\lim_{t\to\infty} M_-(t) = 0$. This comes from the following result

**Lemma 18.30** *As $\lambda \to \infty$*

$$\mathcal{T}_\lambda u^+(x,t) \to \mathcal{U}(x,t;M_\infty), \quad \mathcal{T}_\lambda u^-(x,t) \to \mathcal{U}(x,t;M_\infty - M).$$

**Proof** Consider the solution $v_\tau$ starting at time $t=\tau$ with initial data $u^+(x,\tau)$. By Theorem 18.1 already proved for non-negative solutions, we have

$$\mathcal{T}_\lambda v_\tau(x,1) \to \mathcal{U}(x,t;M_+(\tau))$$

as $\lambda \to \infty$. Therefore, for $\lambda$ and $\tau$ large enough we obtain

$$|\mathcal{T}_\lambda v_\tau(x,1) - \mathcal{U}(x,t;M_\infty)| \le \varepsilon,$$

since $M_+(\tau) \to M_\infty$. We now recall that $u \le v_\tau$ for all $t \ge \tau$. Hence,

$$\mathcal{T}_\lambda u^+(x,1) \le \mathcal{U}(x,t;M_\infty) + \varepsilon$$

if $\lambda$ is large enough. On the other hand, by the results of Step 2 of Section 18.2 we know that the family $(\mathcal{T}_\lambda u^+)(x,1)$ is uniformly bounded and its supports are contained in a fixed ball. Moreover, the family $\mathcal{T}_\lambda u^+$ is equicontinuous on compact subsets. All this implies that along a sequence $\lambda_k \to \infty$, $(\mathcal{T}_\lambda u^+)(x,1)$ converges to a function $f$ such that $0 \le f \le \mathcal{U}(x,1;M_\infty)$ and

$$\int f(x)\,dx = \lim \int (\mathcal{T}_\lambda u^+)(x,1)\,dx = \lim M_+(\tau) = M_\infty.$$

These two conditions on $f$ imply that necessarily $f = \mathcal{U}(x,1;M_\infty)$. The uniqueness of the limit implies that the whole family converges. The proof of the negative parts is similar. ■

The above convergences imply that as $t \to \infty$

$$t^\alpha u^+(0,t) \to c(m,d)M_\infty^{2\beta}, \quad t^\alpha u^+(0,t) \to c(m,d)(M_\infty - M)^{2\beta}.$$

The only possibility of making both estimates compatible is $M = M_\infty$.

(iii) We may now complete the proof of positivity in finite time. Take $\varepsilon$ very small and let $t \ge t(\varepsilon)$ so that

$$\int u^-(x,t)\,dx < \varepsilon.$$

According to Section 18.3 the support of the solution $v$ with initial data $u^-(x,t(\varepsilon)$ is of the order of $O(\varepsilon^{(m-1)\beta}t^\beta)$ for all large $\beta$. By the maximum principle, so is the support of $u^-(x,t)$. This means that for all $\lambda \le \lambda(\varepsilon)$

$$\mathcal{T}_\lambda u(x,1) \ge 0 \quad \text{if } |x| \ge c\varepsilon^{(m-1)\beta}.$$

Now, the converge of the positive parts implies that

$$\mathcal{T}_\lambda u(x,1) \ge \mathcal{U}(x,1;M_\infty) - \varepsilon,$$

which is positive in a ball of radius $(c-\varepsilon)\,M_\infty^{(m-1)\beta}$ if $\lambda$ is large. We conclude that for $\lambda \ge \lambda_1$ we have $\mathcal{T}_\lambda u(x,1) \ge 0$, or equivalently that $u(x,t) \ge 0$ for all large $t \ge T$ and some $T = T(u_0)$. ∎

Once this result is established, Theorem 18.1 holds for signed solutions with compact support and $M > 0$. By density, the result also holds without the restriction of compact support. Note that the first steps of the proof of Section 18.2 are unchanged. We leave the details to the reader. In particular, we prove that for all $\int u_0\,dx > 0$ we have

$$\lim_{t\to\infty} t^\alpha u(x,t) \ge 0. \tag{18.79}$$

The case $M < 0$ is solved by symmetry. Finally, the case $M = 0$ is obtained by sandwiching it between solutions with non-zero mass. It must be noticed that the limit profile in this case is $\mathcal{U} = 0$, so the result we obtain is

$$\lim_{t\to\infty} t^\alpha u(x,t) = 0 \tag{18.80}$$

with uniform convergence; this *does not* give (any non-trivial) first-order approximation.

### 18.8.1 *Actual rates for $M = 0$*

We will only work in one space dimension where we can integrate to find solutions of the $p$-Laplacian equation with $p = m+1$. We have obtained in Chapter 4 a dipole solution with decay rate $O(t^{1/m})$. This is the more general result we get:

**Theorem 18.31** *For every $\alpha \in [1/(m+1), 1/m)$ there exist initial data $u_0 \in L^1(\mathbb{R})$ with $\int u_0(x)\,dx = 0$ such that the solution $u(x,t)$ of the PME behaves as $t \to \infty$ like*

$$u(x,t) \approx t^{-\alpha} F(x\,t^{-\beta}) \tag{18.81}$$

*with $\alpha(m-1) + 2\beta = 1$ so that $\beta \in [1/2m, 1/(m+1))$ and $\alpha \in (1/(m+1), 1/m]$. These solutions have one sign change.*

We propose this result as Problem 18.5. The end cases are on one end the dipole solutions and on the other one the source-type solutions. A complete family of compactly supported solutions with an increasing number of sign changes and correspondingly higher decay rates have been constructed by Hulshof [297] by phase plane methods. For the case of the fast diffusion this sequence is constructed in [103] under the restriction $m > (d-1)/d$. These solutions show different explicit rates for solutions with an increasing number of *laps* (ups and downs). The intermediate asymptotics of the porous medium equation with sign changes is further studied by Hulshof et al. in [298]. In particular, this paper describes the way in which sign changes disappear in finite time.

Convergence of the signed solutions to the dipole is shown by Kamin and Vázquez in [326] in the following situation.

**Theorem 18.32** *Let $u_0 \in L^1(\mathbb{R})$, $\int u_0(x)\,dx = 0$, and assume that the first moment is non-zero,*

$$\int |x|u_0(x)\,dx < \infty, \qquad \int xu_0(x)\,dx \neq 0.$$

*Then there is a dipole solution $z(x,t) = U_{dip}(x-a,t)$ such that*

$$\lim_{t\to\infty} t^{1/m}|u(x,t) - z(x,t)| = 0.$$

*Moreover, the outer free boundaries grow like*

$$s_{\pm}(t) = \pm c\, t^{1/2m} + a + o(1).$$

### 18.8.2 *Asymptotics for the PME with forcing*

The natural extension of the result about asymptotic behaviour of signed solutions of the PME concerns the influence of a forcing term in the form of an integrable right-hand side in the equation.

**Theorem 18.33** *Let $u$ be the mild solution of the Cauchy problem for the PME with initial data $u_0(x) \in L^1(\mathbb{R}^d)$ and $f \in L^1(Q)$ (no sign restriction is imposed). Then,*

$$\lim_{t\to\infty} \|u(t) - \mathcal{U}_{M'}(t)\|_1 = 0\,, \tag{18.82}$$

*where*

$$M' = \int_{\mathbb{R}^d} u_0(x)\,dx + \int\int_Q f\,dxdt. \tag{18.83}$$

**Proof** We consider the solution $u_n(x,t)$ of the problem with same initial data and forcing term $f_n$ such that

$$f_n(x,t) = f(x,t) \quad \text{if } t < n, \qquad f_n(x,t) = 0 \quad \text{if } t \geq n\,.$$

The mild solution becomes a standard weak continuous solution for $t \geq n$. We can think of this solution as having initial data at $t = n$ of the form $u_n(x,n) = u(x,n)$. Mass conservation now reads for $t \geq n$:

$$M_n = \int u_n(x,t)\,dx = \int u(x,n)\,dx = \int u_0(x)\,dx + \int_0^n \int f(x,s)\,dxds.$$

By our previous results, for $t \gg n$ large enough, $u_n$ approaches the source-type profile with mass $M_n$. This means that

$$\lim_{t\to\infty} \|u_n(t) - \mathcal{U}_{M_n}(t)\|_1 = 0.$$

On the other hand, $\lim_{t\to\infty} M_n = M'$. Finally, the contraction property for mild solutions implies that

$$\|u(t) - u_n(t)\|_1 \le \int_0^t \int |f - f_n|\, dxdt = \int_n^\infty \int_{\mathbb{R}^d} |f|\, dxdt \to 0$$

as $n \to \infty$. The proof is complete. ∎

The question of uniform convergence depends on the regularity of the forcing term. See Problem 18.7.

### 18.8.3 *Asymptotic expansions*

The finer properties of large time behaviour will be studied in the next chapter. Let us only remark here that the asymptotic behaviour is improved into an asymptotic expansion in the works of Barenblatt and Zel'dovich [72] and Angenent [19] that hold in $d = 1$. These questions are open for $d > 1$.

## 18.9 Introduction to the fast diffusion case

The extension of the asymptotic properties proved above to exponent $m = 1$ gives as a consequence results that are well-known for the classical heat equation. It is interesting to remark that the proof given here applies (with inessential minor changes) and is very different from the usual proof, based on the representation formula.

We can even go below $m = 1$ and prove similar asymptotic results for the FDE but only for some exponents $m < 1$. To start with, we need two basic ingredients.

(i) A theory of well-posedness for the Cauchy problem. As we have said, the main results of Chapter 9 apply also in this case with minor easy changes if $(d-2)/d < m < 1$. The main novelty is that non-negative solutions are positive everywhere and $C^\infty$-smooth, which is rather good news in this context.

(ii) The second ingredient is the model of asymptotic behaviour. The integrable source-type solution exists just for $m > m_c = (d-2)/d$ and it can be conveniently written in the form

$$\mathcal{U}_M(x,t) = \left(\frac{Ct}{|x|^2 + At^{2\beta}}\right)^{1/(1-m)} = \frac{K\,t^{-\alpha}}{[A + (x\,t^{-\beta})^2]^{1/(1-m)}} \tag{18.84}$$

where $\beta = 1/[2 - d(1-m)]$ is positive precisely in that range, $\alpha = d\beta$, $C = 2m/\beta(1-m)$ is a fixed constant, $K = C^{1/(1-m)}$, and $A > 0$ is an arbitrary constant that can be determined as a decreasing function of the mass $M = \int \mathcal{U}(x,t)\, dx$, $A = k(m,N)\, M^{-\gamma}$ with $\gamma = 2(1-m)\beta$.

In dimensions $d = 1, 2$ the whole range $0 < m < 1$ is covered. However, the *critical exponent*, $m_c = 1 - (2/d)$, is larger than zero for $d \ge 3$. It is then proved that for $0 < m < m_c$ no solution of the ZKB type exists (i.e., self-similar with constant positive mass). The value $m_c = (d-2)/d$ is related to some Sobolev embedding exponents as the reader will easily realize.

### 18.9.1 *Stabilization with convergence in relative pointwise error*

It can be checked that the convergence results of Theorem 18.1 hold true for $m > m_c$, and the proofs given above are true but for minor details. However, the fact that the solutions of the fast diffusion equation do not have the property of conserving compact supports, but rather develop tails at infinity of a certain power-like form gives rise to a very interesting estimate formulated in terms of *relative pointwise error*, or in other words, as *weighted convergence*, that we present next. It requires suitable behaviour of the initial data as $|x| \to \infty$ (similar in decay to the Barenblatt solution).

**Theorem 18.34** *Under the assumption that $u_0 \geq 0$ is bounded and that*

$$u_0(x) = O(|x|^{-2/(1-m)}) \qquad as \quad |x| \to \infty,$$

*we have the asymptotic estimate*

$$\lim_{t\to\infty} \frac{|u(x,t) - \mathcal{U}(x,t;M)|}{\mathcal{U}(x,t;M)} \to 0 \tag{18.85}$$

*uniformly in $x \in \mathbb{R}^d$. The condition on the initial data can be weakened into the integral estimate*

$$\int_{|y-x|\leq|x|/2} |u_0(y)|\, dy = O(|x|^{N-\frac{2}{1-m}}) \quad as \quad |x| \to \infty\,. \tag{18.86}$$

In particular, we have $\|u(t) - \mathcal{U}(t;M)\|_1 \to 0$ as $t \to \infty$ (like case $p = 1$ of Theorem 18.1), and $t^\alpha|u(x,t) - \mathcal{U}(x,t;M)| \to 0$ as $t \to \infty$ uniformly in $x$ (case $p = \infty$), but estimate (18.85) is much more precise because the convergence is uniform with weight

$$\rho = (|y|^2 + c)^{1/(1-m)}, \qquad y = x\,t^{-\beta}.$$

For the detailed proof we refer to [509]. Convergence with rates by the entropy method is established in [157]. The cases $m \leq m_c$ have different asymptotics. The subject is studied in great detail in our text [515].

### 18.9.2 *Solutions of the FDE that remain two-signed*

We have proved in Theorem 18.29 that signed solutions of the PME with compactly supported initial data $u_0$ with positive integral $\int u_0\, dx = M > 0$ become positive in finite time. We propose to the reader to prove a similar result for the HE in Problem 18.3. However, the result is surprisingly false for the FDE. Here is a counterexample.

**Proposition 18.35** *Let $u$ be the solution of the FDE in $d = 1$ with exponent $0 < m < 1$ and assume that the initial data $u_0$ are as follows: $u_0$ is continuous in $\mathbb{R}$, positive for $-1 < x < 0$, negative for $0 < x < 1$ and zero otherwise. We also assume that $\int u_0\,dx \neq 0$ and the lap number is 3. Then, $u(\cdot, t)$ takes positive and negative values for all times and the intersection number is always 1.*

**Proof** We will only give the main details of the proof for reasons of space and we ask the reader to fill in the details as a working project, see Problem 18.3.

(i) We may assume that $M = \int u_0\,dx > 0$. The integral will be the same for all times $t > 0$. The lap number assumption implies that $u_0$ has one maximum for $-1 < x < 0$ and one minimum for $0 < x < 1$. According to Subsection 15.2.2, we have $Z(u_0) = 1$ and then $Z(u(t)) = 1$ for a time; later on the intersection number may become zero at a certain time $T > 0$, or it may stay 1 for all time. Let us see the meaning of the two options: In the latter case there will be a change-of-sign interface $x = s(t)$ defined for $0 < t < \infty$ separating the region $\{x < s(t)\}$ where $u(x,t) < 0$ from $\{x > s(t)\}$ where $u(x,t) > 0$ (we have to use the fact that non-negative solutions of the FDE in a parabolic region are actually positive by the property of infinite propagation). On the other hand, if $Z$ becomes 0 the sign change is lost at $T$ and the solution must become positive ever after.

(ii) We want to rule out the latter possibility. Since the integral $\int u(x,t)\,dx = M > 0$ for all time, the only way of losing the interface is through an asymptote

$$\lim_{t \uparrow T} s(t) = +\infty. \tag{18.87}$$

The analysis is based on the behaviour of the positive and negative parts as $x \to +\infty$. We consider the solution $u_1(x,t) > 0$ with data $u_1(x,0) = u_0^+(x)$ and $u_2(x,t) < 0$ with data $u_2(x,0) = -u_0^-(x)$. We have $u_1(x,t) \le u(x,t) \le u_1(x,t)$. A key point in what follows is the expression of the behaviour of the solution as $x \to +\infty$: by using the ZKB solution and the dipole solutions as barriers, we obtain in first approximation

$$\begin{cases} u_1(x,t) = c_m\,(t/x^2)^{1/(1-m)}\,(1 + O(x^{-(m+1)/m})), \\ u_2(x,t) = -c_m\,(t/(x-1))^2)^{1/(1-m)}\,(1 + O(x^{-(m+1)/m})). \end{cases}$$

Hence independent of the respective masses in first approximation. Moreover, the behaviour of $u$ is the same as $u_2$ as $x \to \infty$ as long as there exists an interface.

(iii) Let us now take some $T > 0$ and prove that the interface is finite for $0 < t < T$. By continuity of the solutions with respect to the data, there is a small time $\tau > 0$ such that $s(\tau)$ is finite. Moreover, we may take $a > s(\tau)$ (depending on $T$) such that

- $u(x,\tau) \le -U_{dip}(x - a_1; t)$ for all $x > a$;
  This implies choosing a convenient $a_1 > a$, and also a mass for the dipole;

- for all $\tau < t < T$ we have

$$u(a,t) \le c_m\,(t/x^2)^{1/(1-m)}(1 + Cx^{-(m+1)/m})) \le -U_{\rm dip}(x - a_1, t).$$

If this holds then we have comparison with the solution $\widetilde{u}(x,t) = -U_{\rm dip}(x - a_1, t)$ in the rectangle $R = (a, \infty) \times (\tau, T)$ which means that $u(x,t) < 0$ for $\tau < t < T$ and $x > a_1$. ■

## 18.10 Various topics

### 18.10.1 *Asymptotics of non-integrable solutions*

We have considered in this chapter the class of integrable data and correspondingly the solutions are integrable in the space variable for all times. This is a reasonable assumption from the point of view of many applications and we have given preference to it, but it is not the only option, and previous chapters have dealt with solutions with non-integrable, even growing data.

The asymptotic behaviour for non-integrable solutions was studied by Alikakos and Rostamian in [9] in classes of data with power behaviour as $|x| \to \infty$. The behaviour follows in each case a corresponding self-similar solutions from the classes obtained in Chapter 16. A more complicated analysis for solutions with data that decay like $|x|^{-d}$ is done by Kamin and Ughi [324]. The interesting behaviour is

$$\lim_{t\to\infty} \tau^l |u(x,t)/B\log\tau - \mathcal{U}(x,\tau)| = 0$$

where $t \cong B^{1-m}(\tau/(\log(\tau))^{m-1})$, $B$ is some constant depending on $n$, $m$ and the data, and $\mathcal{U}$ is the ZKB solution with $M = 1$. It follows from this that $u(x,t) \cong B\log(\tau)\mathcal{U}(x,\tau)$ for large $t$. The latter expression is called a 'reconstructed similarity solution'.

There is a literature on the whole issue that has many open problems. We have no more space here to devote to this topic.

The analysis is specially clear in space dimension $d = 1$ when we consider the long term effect of the growth of the initial data as $x \to -\infty$ on solutions that have a free boundary on the right-hand side. More details on the behaviour of solutions and interfaces can be found in [500].

It the data grow fast enough, say at minus infinity, $u_0(x) = O(x^{2/(m-1)})$, then the solution may blow up in finite time as the blow-up solutions of Section 4.5 show. A solution whose interface blows up in a finite time is constructed in 4.7.1.

### 18.10.2 *Asymptotics of filtration equations*

There are many studies of the long time behaviour of the GPME under different assumptions on $\Phi$, specially for non-negative solutions.

There are on the one hand studies where the nonlinearity behaves like a power for small $u > 0$. If the data are integrable and non-negative then we expect the long-time behaviour to be similar to the behaviour of the PME. We refer the reader to the recent work of Carrillo, Di Francesco and Toscani [153], where explicit conditions are given on how similar $\Phi$ must be to $u^m$ for the asymptotic simplification into the ZKB profiles to take place.

### 18.10.3 *Asymptotics of superslow diffusion*

Cases where $\Phi$ is very far from being a power appear in the literature. This happens in the *superslow diffusion equations*, defined by the property that

$$\lim_{u\to 0} \frac{\Phi'(u)}{u^p} = 0 \quad \text{for all } p > 0. \tag{18.88}$$

A typical example is the following initial value problem

$$u_t = \Delta(e^{-u}) \quad \text{in } Q = \mathbb{R}^d \times (0,\infty) \tag{18.89}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}^d, \tag{18.90}$$

where the initial function $u_0 \not\equiv 0$ is bounded, integrable and non-negative. Existence and uniqueness of a non-negative weak solution and comparison theorems for this problem follow from the results of Chapter 9. The solution is smooth at any point of positivity.

The study of the asymptotic behaviour for this problem has been performed in [252], see also Chapter 3 of the book [255], in the case of one space dimension, $d = 1$. It is convenient to state the main result in terms of the function

$$v = e^{-1/u}. \tag{18.91}$$

Then, $0 \le v(x,t) < 1$ in $Q$, and $v(x,t)$ solves the quasilinear equation

$$v_t = v\,(\ln v)^2\, v_{xx} \quad \text{in } Q. \tag{18.92}$$

The asymptotic behaviour of $v(x,t)$ is exactly described by the following result.

**Theorem 18.36** *Under the above assumptions on $u_0$, we have*

$$\lim_{t\to\infty} t\,v(\eta(\ln t),t) = F_a(\eta) \equiv \frac{1}{2}(a^2 - \eta^2)_+ \tag{18.93}$$

*uniformly for $\eta \in \mathbb{R}$, where $a$ is one half of the initial mass:*

$$a = \frac{1}{2}\int u_0(x)\,dx > 0. \tag{18.94}$$

If we translate this result (18.93) to the function $u(x,t)$ by means of the inverse transformation

$$u(x,t) = -1/\ln v(x,t), \tag{18.95}$$

we get the asymptotic formula

$$\lim_{t\to\infty} (\ln t)\, u(\eta(\ln t),t) = 1 \tag{18.96}$$

uniformly in any set $\{|\eta| \le c\}$, where $c \in (0, a)$ is a constant, while for $|\eta| \ge a$ we have

$$\lim_{t\to\infty} (\ln t)\, u(\eta(\ln t), t) = 0. \qquad (18.97)$$

Thus, in terms of the initial variable $u(x,t)$ we observe a new asymptotic pattern, a *mesa-like profile.* Notice that the only parameter which appears in the formulas is the normalized length of the support of $u$, namely $2a$. This parameter is easily calculated from the law of conservation of mass $\int u(x,t)dx = \text{constant}$, since for large $t$ it follows from (18.96), (18.97) that $\int u(x,t)dx \approx 2a$ and $\|u_0\|_1 = 2a$. Any further information about the asymptotic spatial structure of the solution as $t \to \infty$ is lost in the $u$ variable (in first approximation).

We also obtain a precise result on the asymptotic behaviour of interfaces of every compactly supported solution.

**Theorem 18.37** *Assume moreover that $u_0$ has a compact support. Then as $t \to \infty$,*

$$s_+(t) \equiv \sup\{x \in \mathbb{R} : u(x,t) > 0\} = a\, \ln t + O(1), \qquad (18.98)$$

$$s_-(t) \equiv \inf\{x \in \mathbb{R} : u(x,t) > 0\} = -a\, \ln t + O(1). \qquad (18.99)$$

The reader should compare these results with the behaviour of the PME which has a power-like nonlinearity.

### 18.10.4 *Asymptotics of the PME in inhomogeneous media*

We consider the model of PME in inhomogeneous media (5.86) proposed by Kamin and Rosenau [322, 452],

$$\rho(x,t)\, \partial_t u = \Delta\Phi(u), \qquad (18.100)$$

see Section 5.11. Take for simplicity $\Phi(u) = u^m$ and $u \ge 0$. In the case of finite mass and finite energy, the main asymptotic theorem says: as $t \to +\infty$, the unique solution of the Cauchy problem tends to $u = E/m$, i.e., there is an isothermalization to a positive average temperature, as arises in the case of a bounded domain with Neumann boundary conditions; this is very different from the case of diffusion in homogeneous media, where the average temperature is zero.

In order to get a closer idea of the process of asymptotic stabilization, paper [247] studies the asymptotic behaviour of these problems in $d = 1$ when the weight function $\rho(x)$ behaves as $|x| \to \infty$ like $|x|^{-\alpha}$ with $0 < \alpha < 2$, including cases of a medium with infinite mass, $0 < \alpha < 1$, but focusing on the cases with finite mass, $\alpha \in [1, 2]$. It is proved that for the medium of infinite mass, there exist self-similar Barenblatt solutions that give the asymptotic behaviour. However, for a medium of infinite mass, we have isothermalization for the solutions of the Cauchy problem; in the range $1 < \alpha < 2$ the precise asymptotic behaviour (i.e., the way the solutions spread for large $x$ and $t$) is given by a completely new type of self-similar solution of the form

$$U(x,t) = f(x/t^{\beta}), \qquad \beta = \frac{1}{2-\alpha}. \tag{18.101}$$

Observe that this function does not decay in time! The cases $\alpha > 2$ have a still different behaviour, faster rate of thermalization.

The asymptotic behaviour of solutions of a porous medium equation with bounded measurable coefficients is studied by Cho and Choe in [169].

### 18.10.5 *Asymptotics for systems*

The large time behaviour of a system of degenerate parabolic equations is studied by Bertsch and Kamin in [113]. The system

$$\begin{cases} \rho_t = (\phi_1(T)\rho^{a_1}\rho_x)_x \\ (\rho T)_t = (\phi_2(T)\rho^{a_2+1}T_x)_x + (T\phi_1(T)\rho^{a_1}\rho_x)_x \end{cases}$$

is posed in $Q = \mathbb{R} \times \mathbb{R}^+$. It is a simplified model describing heat and mass transfer in a one-dimensional plasma, introduced by Rosenau and Hyman [451].

### 18.10.6 *Other*

The influence of capillary effects in groundwater flow in a porous medium leads to the model

$$u_t + \gamma|u_t| = \Delta(u^m) \tag{18.102}$$

that has been studied by Hulshof and Vázquez in [300]. This is a non-divergent equation for which the suitable concept of solution is the viscosity solution. A maximal viscosity solution is constructed and the asymptotic behaviour towards a self-similar solution, a modified Barenblatt solution with an anomalous exponent, is proved. The question of uniqueness of viscosity solutions remains open even for non-negative data. A similar kind of solutions is taken up by Feireisl et al. [235] for more general equations of the form $u_t = F(x,t,u,Du,D^2u)$.

The fundamental solution of the anisotropic porous medium equation is constructed by Song and Jian in [484]. This should allow for the proof of asymptotic stability.

## Notes

The general topic of asymptotic behaviour is very large and important in itself, but is addressed here in a particular setting where precise results can be obtained and a rather complete view can be given in a restricted space. Even for the porous medium equation, there are many other settings in which large-time behaviour can be studied. Important references for asymptotic behaviour for dissipative systems are Hale [282], Ladyzhenskaya [356], Temam [492], but our results are very different in style and methods. A wide survey of results with the present and related methods is contained in the book by Galaktionov and Vázquez [255].

**Section 18.1.** The first proof of the main convergence result of Theorem 18.1 for the PME in several dimensions appeared in a celebrated paper by A. Friedman and Kamin in 1980 [241] using a method of optimal lower barriers; however, the uniform convergence result was established only for non-negative and compactly supported data. If the data are not compactly supported, they prove local uniform convergence on expanding balls. As a precedent, the result was proved in one space dimension by Kamin in 1973 [319]. The complete proof of Theorem 18.1 with uniform convergence for non-negative data in $L^1(\mathbb{R}^d)$ was done by Vázquez in [509]. In that paper seven different proofs are given. The cases of signed solutions and forcing term were also included.

**Section 18.7.** The first results of asymptotic behaviour of the PME and the GPME in $d = 1$ were due to Kamin [319, 320], 1973. The control of the growth of the support as $t \to \infty$ with a power formula was first obtained in $d = 1$ by Knerr [343], 1977. The sharp results comparing with the centreed Barenblatt solutions are due to Vázquez [498], 1983. The $N$-wave behaviour of the velocity is studied in [501].

**Section 18.8.** The convergence theorem for signed solutions was proved by Kamin and Vázquez in [326]. The result for solutions with forcing term was published in [509].

The intermediate asymptotics of the porous medium equation with sign changes is studied by Hulshof, King and Bowen [298] where a formal classification of self-similar and non-self-similar scenarios is found for the disappearance of sign changes.

**Section 18.9.** The properties of fast diffusion equations with subcritical exponents $-\infty < m < m_c$ offer many surprises, like extinction in finite time, non-existence of solutions even for bounded data, non-uniqueness under similar conditions. The case of logarithmic diffusion is important in many respects. We refer to [515] for details on these issues.

The convergence in relative error of Theorem 18.34 is taken from [509]. The result about conservation of two signs for infinite time, Proposition 18.35, is new.

**Section 18.10.** The large time behaviour of equation (18.100) in several space dimensions is still being investigated.

## Problems

**Problem 18.1** Complete the details of the proof of Theorem 18.1 for signed solutions after Theorem 18.29. Show formulas (18.79) and (18.80).

**Problem 18.2** Show that whenever $0 \le u_0(x) \le C\,|x|^{-a}$ with $a > d$ we have a limit for the rate of convergence in Theorem 18.1.

*Hint*: Use the construction of the counterexample in a quantitative way.

**Problem 18.3** EVENTUAL POSITIVITY

(i) Use the representation formula to prove that a solution of the heat equation with integrable initial data $u_0 \in L^1(\mathbb{R}^d)$ and positive mass $\int_{\mathbb{R}^d} u_0(x)\,dx > 0$ becomes positive in finite time if the negative part, $u_0^-(x)$, is compactly supported.

(ii) Take $d = 1$ and take as initial data

$$u_0(x) = \delta(x) - \varepsilon\delta(x - a).$$

Calculate the approximate location of the point of minimum of $u(\cdot, t)$ for different times $t > 0$.

(iii) Investigate the failure of the result of eventual positivity for the FDE with $0 < m < 1$ and $d = 1$. Prove in detail Proposition 18.35.

(iv) Find an example with $Z = 2$ where the intersections disappear and the solution becomes positive in finite time.

**Problem 18.4**

(i) Assume that $u_0 \geq 0$ is locally integrable and $\int u(x, 0)\,dx = \infty$. Prove that

$$\lim_{t\to\infty} t^\alpha u(x, t) = +\infty \tag{18.103}$$

uniformly on sets of the form $|x| \leq C\,t^\beta$, with $\alpha$ and $\beta$ as in Theorem 18.1.

(ii) Show self-similar examples where the convergence is uniform in $\mathbb{R}^d$ and examples where it is not.

(iii) Show that the restriction $u_0 \geq 0$ cannot be eliminated. Is there a simple result for signed solutions with infinite mass?

**Problem 18.5** This problem is based on Problem 4.12. We ask the reader to construct solutions of the PME in $d = 1$ with one sign change by the following method. Solve the $p$-Laplacian equation

$$w_t = (|w_x|^{m-1} w_x)_x$$

with initial data

$$w(x, 0) = A\,|x|^{-p}$$

with $0 < p < 1$. Scaling methods produce a self-similar solution much as we did in Chapter 16 for the PME. Differentiate to obtain $u = -w_x$ a signed solution of the PME with zero mass and one sign change. Prove Theorem 18.31.

**Problem 18.6** Construct a solution of the signed PME that has finite positive mass but does become positive in finite time.

*Hint*: It must not have compact support. See an example in [326], page 43.

**Problem 18.7** Find conditions on the forcing term $f$ so that the solutions of the complete PME satisfy a uniform convergence result like (18.7), of course with $M$ replaced by $M'$.

**Problem 18.8** Complete the proofs of Lemmas 18.23 and 18.24.

**Problem 18.9** Prove the result of Theorem 18.20 with the assumption of finite centre of mass but no assumption of compact support. It is not clear whether Theorem 18.25 holds under those assumptions. The result about the centre of mass is not known in several space dimensions, though we conjecture that it is true.

**Problem 18.10** Prove Theorem 18.28.

*Hint*: Use the references of the text.

**Problem 18.11**

(i) Prove eventual concavity for radially symmetric solutions of the PME in $d > 1$ with compactly supported data (the result is anyway guaranteed by Theorem 19.18).
(ii) Obtain sharp asymptotic rates of convergence.

**Problem 18.12** (i) Prove Corollary 18.1A
(ii) Prove also that

$$\int_{\mathbb{R}^d} |u(t) - \mathcal{U}(t)| x^2 dx = o(t^{2\beta}) \quad \text{as} \quad t \to \infty$$

*Hint*: (i) For smooth solutions we calculate

$$\frac{d}{dx} \int (u(x, t - \mathcal{U}(x,t)))|x|^2 dx = 2d \int (u^m - \mathcal{U}^m) dx$$

which can be bounded above by $C(||u(t)||_\infty^{m-1} + ||\mathcal{U}(t)||_\infty^{m-1})||u(t) - \mathcal{U}(t)||_1)$. Integrate in time to obtain the result. We leave the easy details to the reader.
(ii) Use an approximation of the function sign $(u - \mathcal{U})$ as a multiplier before calculating the previous integral.

# 19

# REGULARITY AND FINER ASYMPTOTICS IN SEVERAL DIMENSIONS

In this chapter we investigate more closely the precise regularity that is to be expected for non-negative solutions of the PME that have free boundaries. In view of the results of Chapters 14 and 15, this is a problem that needs investigation for flows in more than one dimension, $d > 1$. In order to be concise, we focus on the Cauchy problem for non-negative solutions of the PME, a context that offers the most interesting results to date and the simplest setting. We recall the work of Caffarelli and Friedman [140], who proved that weak non-negative solutions of the Cauchy problem are locally Hölder continuous with free boundaries which are locally Hölder continuous surfaces; Di Benedetto and Friedman [211] established Hölder continuity of bounded local solutions $u \geq 0$ of the PME. But the question of the precise exponent or bounds for it was left open. On the other hand, we have shown that positive solutions are automatically smooth and classical solutions, so the problem of regularity is posed for solutions with a free boundary. The simplest setting is asking the initial data to be compactly supported, which implies that $u(\cdot, t)$ will also be compactly supported for all $t$; we abbreviate this situation by saying that the solution is compactly supported (with an innocent *abus de langage*).

In Section 19.1 we describe the results of Caffarelli, Vázquez and Wolanski [145], that show that non-negative solutions with compactly supported initial data have pressures that are Lipschitz continuous functions after a certain time that depends on the data. This phenomenon is called *eventual regularity*; similar phenomena happen also for other nonlinear evolution equations in different forms. In view of the behaviour of the ZKB solutions, Lipschitz continuity of the pressure is the best that can be achieved as for global regularity. The result is sharpened by Caffarelli and Wolanski [146] who showed that under certain conditions on the initial data, the free boundary has $C^{1,\alpha}$ regularity.

In the opposite direction, in Section 19.2 we present focusing solutions constructed by Aronson et al. which develop a singularity for the speed in finite time, precisely where a hole in the initial support is filled; in terms of regularity, this means that the pressure is no longer Lipschitz continuous somewhere. The singularity takes place at an isolated point and a precise time, thus signalling the fact that the singular set is usually quite small.

In Section 19.3 we prove the transfer of regularity that happens in the PME: Lipschitz continuity with respect to the space variable implies the same type of continuity in time. Of course, the assumption is not always true.

In Section 19.4 we report on more refined regularity results for large times. Thanks to Koch's thesis, [347] we know that for large times the free boundaries of the solutions are $C^\infty$ smooth hypersurfaces and the pressure is $C^\infty$ in the positivity set *and up to the free boundary.*

Section 19.5 we discuss two subjects: one of them is the conservation of the initial regularity when the data are smooth; conservation can happen either locally in time or globally in time. Then, we present the property of asymptotic concavity: it is proved the pressure of a compactly supported solution becomes a concave function in its support for all large times, hence the free boundary (and all the level sets) are convex hypersurfaces.

In Section 19.6 we report on other results on precise Hölder regularity.

## 19.1 Lipschitz and $C^1$ regularity for large times

We consider the solution $u \ge 0$ of the Cauchy problem for the PME with initial data $u_0$ supported in a ball. By translation we may assume that the ball is centreed at $x = 0$ for simplicity. Let $B_{R_0}(0)$ be the smallest ball containing the support of $u_0$. By the results of Chapter 14 we know that the support of $u(\cdot, t)$ overflows $B_{R_0}(0)$ after a finite time that we call $T_0$. We recall that the free boundary is described by a Lipschitz continuous surface after that time, Corollary 14.28, and moreover this surface is increasingly flatter in the sense of larger aperture angles for the inner and outer cones.

We pursue here the study of such solutions to obtain bounds for the pressure $v$ and its spatial gradient $\nabla v$ for $t > T_0$, and we then derive from it the Lipschitz continuity of the free boundary $\Gamma \cap \{t > T_0\}$.

### 19.1.1 *Lipschitz continuity for the pressure*

We start with a preliminary result.

**Lemma 19.1** *For any $\varepsilon > 0$ there exists $T_1 = T_1(\varepsilon, R_0, u_0)$ such that $|\nabla v| \le \varepsilon v$ if $|x| \le R_0$ and $t \ge T_1$.*

**Proof** Since $|\nabla v|/v = (m-1)|\nabla u|/u$, it is enough to prove the result for $u$ instead of $v$. Let us define the family of rescaled functions

$$u_k(x,t) = k^d\, u(kx, k^{d(m-1)+2}t)$$

in $Q$ with parameter $k > 0$. We know that they are again solutions of the PME. We have proved in Theorem 18.1 (in a completely independent way) that this family converges as $t \to \infty$ to the ZKB solution (1.8) having the same mass as $u$

$$\int_{\mathbb{R}^d} u(x,t)\, dx = M = \int_{\mathbb{R}^d} \mathcal{U}(x,t;C)\, dx.$$

Moreover, the convergence $u_k \to u$ takes places in the uniform norm of $C(\mathbb{R}^d \times (\tau, \infty))$ for every $\tau > 0$. Therefore, there exists $k_0$ such that for $k \ge k_0$

and $1/2 < t < 2$ we have

$$0 < c_1 \le \frac{1}{2}\mathcal{U}(x,t) \le u_k(x,t) \le 2\mathcal{U}(x,t) \le C_2$$

if $|x| \ge R_1 = R_0(c,d,m)$. In particular, we have

$$u(y, k^{d(m-1)+2}t) \ge c_1 k^{-d} \qquad \text{if } \; |y| \le kR_1. \tag{19.1}$$

Taking $k_0$ large enough we can say that (19.1) holds for every $y \in B_{R_0}(0)$.

Now we realize that $u_k$ is a uniformly bounded sequence of solutions of the PME in $B_{R_0}(0) \times (1/2, 2)$. We have mentioned in the Introduction that the equation can be written in the divergence form $u_t = \nabla \cdot (a(x,t)\,\nabla u)$, and we have also shown in Chapters 3 and 7 that when the coefficient $a(x,t) = m\,u^{m-1}$ is bounded above and below away from zero the equation becomes uniformly parabolic and the solutions are smooth with derivatives locally bounded in terms of the bounds for $u$. We conclude that there exist a constant such that

$$|\nabla u_k(x,1)| \le C_2 \quad \text{if } \; |x| \le R_1/2,$$

and $C_1$ depends only on $u$ and $R_1$. This means that

$$|\nabla u(y, k^{d(m-1)+2})| \le C_2 k^{-(d+1)}. \tag{19.2}$$

Putting $t = k^{d(m-1)+2}$, inequalities (19.1) and (19.2) give

$$|\nabla u(x,t)| \le C_3 t^{-1/(d(m-1)+2)} u(x,t). \tag{19.3}$$

with $C_3 = C_2/C_1$ and this holds whenever $|x| \le (1/2)kR_1$, for $t = k^{d(m-1)+2}$, $k \ge k_0$. In particular, if $|x| \le R_0$ and $t \ge T_1$ with $T_1$ large enough. ∎

We are ready to obtain bounds for $v_t$ and $\nabla v$ for large $t$.

**Theorem 19.2** *There exists a time $T_1 = T_1(u_0) > 0$ and a constant $c = c(m,d) > 0$ such that for every $t > T_1$ and every $x \in \mathbb{R}^d$*

$$|\nabla v(x,t)| \le c\left(\left(\frac{v}{t}\right)^{1/2} + \frac{|x|}{t}\right) \tag{19.4}$$

*and*

$$-\frac{d(m-1)\,v}{(d(m-1)+2)\,t} \le v_t(x,t) \le \left(\frac{v}{t-T} + \frac{|x||\nabla v|}{t-T}\right). \tag{19.5}$$

**Proof** For every $\varepsilon > 0$ we define the family of solutions of the PME in $Q$ that in pressure variable read

$$v_\varepsilon(x,t) = (1+\varepsilon)^{-1} v((1+\varepsilon)x, (1+\varepsilon)t + t_0), \tag{19.6}$$

where $t_0$ is a large time to be chosen below. We want to show first that if $t_0$ is large enough the inequality

$$v_\varepsilon(x,0) \le v_{\varepsilon=0}(x,0) \equiv v(x,t_0) \tag{19.7}$$

holds for every $\varepsilon \in (0,1)$ and $x \in \mathbb{R}^d$. Suppose that this is true. Then the comparison results that we have proved for the solutions of the PME imply that

$$v_\varepsilon(x,t) \le v_{\varepsilon=0}(x,t) \equiv v(x,t+t_0)$$

holds for $\varepsilon \in (0,1)$, $x \in \mathbb{R}^d$ and $t > 0$. Differentiating $v_\varepsilon$ with respect to the parameter $\varepsilon$ at $\varepsilon = 0$ gives

$$v(x,t+t_0) - t\, v_t(x,t+t_0) - x \cdot \nabla v(x,t+t_0) \ge 0$$

for all $t > 0$. Hence,

$$(t-t_0)\, v_t(x,t) \le v(x,t) - x \cdot \nabla v(x,t) \tag{19.8}$$

for all $x \in \mathbb{R}^d$ and $t > t_0$. This is the upper bound for $v_t$. The lower bound is standard from the equation and the fundamental estimate (9.4) and we get

$$v_t = (m-1)v\,\Delta v + |\nabla v|^2 \ge |\nabla v|^2 - \frac{\Lambda}{t}\,v, \quad \Lambda = \frac{(m-1)d}{(d(m-1)+2)}. \tag{19.9}$$

Combining both inequalities gives

$$|\nabla v|^2 \le \left(\frac{\Lambda}{t} + \frac{1}{t-t_0}\right) v + \frac{|x|}{t-t_0}\,|\nabla v|. \tag{19.10}$$

Now let $T = 2t_0$ and the estimate (19.4) easily follows. The estimate for $v_t$ is then immediate. This would end the proof.

We still have to check inequality (19.7). We write

$$v_\varepsilon(x,0) - v(x,t_0) = -\frac{\varepsilon}{(1+\varepsilon)}\, v_\varepsilon((1+\varepsilon)x,t_0) + \{v((1+\varepsilon)x,t_0) - v(x,t_0)\}.$$

If $|x| \ge R_0$, it follows from the monotonicity Lemma 14.25 that $v((1+\varepsilon)x,t_0) \le v(x,t_0)$ so that the conclusion $v_\varepsilon(x,0) - v(x,t_0) \le 0$ follows. On the other hand, when $|x| \le R_0$, we estimate the last term as

$$|v((1+\varepsilon)x,t_0) - v(x,t_0)| \le \varepsilon |x|\, |\nabla v(\xi,t_0)|$$

for some point $\xi$ in the segment joining $x$ to $\varepsilon\, x$. We know that for $t_0$ large enough $v$ is positive and smooth, $0 < c < v(x,t_0) < C$ in the ball of radius $2R_0$, and by Lemma 19.1 we have $|\nabla v(x,t_0)| \le \delta\, v \le \delta\, C$. Therefore,

$$v_\varepsilon(x,0) - v(x,t_0) \le -\frac{\varepsilon}{2}\, c + 2R_0\delta\, C.$$

This quantity is negative as soon as $\delta < \varepsilon c/4R_0C$, which happens if $t_0$ is large enough since the convergence to the ZKB profile implies in particular that $C/c \to 1$ as $t_0 \to \infty$. ■

As a consequence, we have the following large-time estimates:

**Corollary 19.3** *For compactly supported solutions, both $v_t$ and $|\nabla v|$ are bounded for $t \ge T_1 + h > T_1$. Actually, we have the estimates, uniformly* *in* $x \in \mathbb{R}^d$,

$$|v_t|,\ |\nabla v|^2 \le C\,t^{-\gamma}, \quad \gamma = \frac{2d(m-1)+2}{d(m-1)+2}. \tag{19.11}$$

These are the precise rates of the ZKB solutions. The proof is based on the estimate for $u$, $u(x,t) \le C\,t^{-d/(d(m-1)+2)}$, obtained in Proposition 9.8, plus the support estimates of Proposition 9.19 and the above estimates for the derivatives.

An interesting question in view of Theorem 19.2 is to estimate the first time after which $v_t$ and $|\nabla v|$ become bounded. Here is an answer to that question. Let

$$T(u_0) = \inf\{t > 0 : \mathcal{P}_u(t) \supset \overline{B_{R_0}(0)}\}. \tag{19.12}$$

Then, we have

**Theorem 19.4** *Both* $v_t$ *and* $|\nabla v|$ *are bounded functions in* $\mathbb{R}^d \times (\tau, \infty)$ *for every* $\tau > T(u_0)$.

We refrain from giving the proof, which is a modification of Theorem 19.2; for reasons of space, the reader is referred to [145], Theorem 2. A basic step is the equivalent of estimate (19.10) that now reads

$$|\nabla v|^2 \le \left(\frac{\Lambda}{t} + \frac{2}{h}\right) v + \frac{2|x|}{h}\,|\nabla v| \tag{19.13}$$

for $0 < h < \tau - T(u_0)$, $0 < h < h_0$ and $\tau < t < T_1$.

### 19.1.2 *Lipschitz continuity of the free boundary*

As a consequence of the estimates for the first derivatives of $v$, we have the following result, which improves Corollary 14.28. We want to describe the part of the free boundary $\Gamma(u)$ of a compactly supported solution $u \ge 0$ of the PME (as used in this chapter) for times $t > T(u_0)$. We will call this part $\Gamma_1(u)$. It is clear that the space coordinates of the points of $\Gamma_1(u)$ satisfy $|x| > R_0$.

**Theorem 19.5** *The late free boundary* $\Gamma_1(u)$ *is a Lipschitz hypersurface in* $\mathbb{R}^{d+1}$ *written in polar coordinates*

$$|x| = f(\theta, t), \tag{19.14}$$

*where* $f : \mathbb{S}^{d-1} \times (T(u_0), \infty) \to \mathbb{R}$ *is Lipschitz continuous.*

**Proof** We have already proved the representation and the space dependence in Corollary 14.28. We see from the geometrical argument that the space Lipschitz constant is uniform when $t$ varies in any interval $(\tau, \infty)$ with $\tau > T(u_0)$. Therefore, we only need to consider the $t$ dependence. We will use an upper estimate for $v_t$ that we have derived in two forms. Indeed, Theorem 19.2 has established that

$$(t - T_1)v_t - v + x \cdot \nabla v \le 0 \tag{19.15}$$

for $t > T$, while the proof of Theorem 19.4 [145] asserts a similar estimate

$$\frac{1}{2}h\, v_t - v + x \cdot \nabla v \le 0 \tag{19.16}$$

for $h$ small and $T(u_0) < \tau < t \le T_1$. Since $x \cdot \nabla v = r\, \partial v/\partial r$ with $r = |x|$ , this last equation can be written as

$$\frac{d}{dt}\left(e^{-2t/h} v(r_0 e^{(2/h)(t-t_0)}, \vartheta, t)\right) \le 0$$

for $t > t_0 > t_1 + (h/2)$ and $\vartheta$ fixed. Therefore, if $v(r_0, \vartheta, t) = 0$, then,

$$v(r_0 e^{(2/h)(t-t_0)}, \vartheta, t) = 0 \quad \text{for} \quad t > t_0.$$

This gives us the bound $f(\vartheta, t) \le r_0 e^{(2/h)(t-t_0)}$. It follows that

$$f(\vartheta, t) \le f(\vartheta, t_0)\, e^{(2/h)(t-t_0)}.$$

This means that the free boundary grows at most in a linear fashion with respect to $t$ in the radial direction in the time interval $\tau < t \le T_1$. A similar conclusion follows from using estimate (19.15) in the interval $(T_1, \infty)$. ■

In order to prove regularity of the representation of the free boundary as a time function in the sense of Section 14.5, we need some extra assumptions on the initial data that can be summed up as regularity and transversality conditions. Specifically, we assume that $v_0$ is a non-negative and continuous function in $\mathbb{R}^d$ which is positive in a bounded domain $D$ with $C^1$ boundary, and such that

$$v_0 \in C^1(\overline{D}), \qquad \Delta v_0 \ge -K \quad \text{in} \quad D, \tag{19.17}$$

$$v_0 + |\nabla v_0| \ge c > 0 \quad \text{in} \quad (\overline{D}). \tag{19.18}$$

We will refer to these conditions as (R). The last condition means really that $\nabla v_0 \ne 0$ at the boundary of the support of the initial datum. In terms of the representation of the equation as describing the movement of particles, it means that at $t = 0$ the particles near the initial free boundary move with a speed at least $c$. We therefore avoid waiting time situations that are difficult to analyse. Under these initial conditions we can obtain a lower bound for $v_t$ that will be used later to show that the free boundary really expands for all times and at all points.

**Proposition 19.6** *Let $v$ be the pressure of a solution of the PME in $Q$ with initial data satisfying the conditions* (R). *Then, there exist constants $A, B > 0$ such that the inequality*

$$(A-2)v(x,t) + x \cdot \nabla v(x,t) + (At + B)v(x,t) \ge 0 \tag{19.19}$$

*holds in $Q$. The constants $A$ and $B$ depend only on the initial data.*

**Proof** We use a similar strategy as in previous proofs of this section. For small $\varepsilon$ we consider the rescaled function

$$v_\varepsilon(x,t) = \frac{1+A\varepsilon}{(1+\varepsilon)^2} v((1+\varepsilon)x, (1+\varepsilon)t + B\varepsilon). \tag{19.20}$$

If we prove that $v_\varepsilon(x,t) \ge v(x,t)$ for every $\varepsilon > 0$ small, then we can pass to the limit $\varepsilon \to 0$ and get by differentiation of formula (19.20) at $\varepsilon = 0$ the expression we are looking for.

By the maximum principle we need only check that inequality at $t = 0$ since both functions under scrutiny are weak solutions of the PME in $Q$. We have to check that

$$v((1+\varepsilon)x, B\varepsilon) \ge \frac{(1+\varepsilon)^2}{1+A\varepsilon} v_0(x,t).$$

The delicate comparison is not done directly on $v_0$ but on a sequence of smooth approximations and proceeds differently in different regions. We refer to [145], Proposition 3.1 for full details. Let us only remark that the transversality conditions that says that $|\nabla v| \ge c > 0$ near the boundary of the positivity set is used to estimate

$$I_\varepsilon = \frac{1}{\varepsilon}\left(v_\varepsilon(x,t) - v_0(x)\right) = \frac{1}{\varepsilon}\left(\frac{1+A\varepsilon}{(1+\varepsilon)^2} v((1+\varepsilon)x, B\varepsilon) - v(x,0)\right)$$

after a Taylor development for $\varepsilon$ small enough, as

$$I_\varepsilon \ge \frac{1}{2}(A-2)v((1+\varepsilon)x, B\varepsilon) + B\, v_t((1+\varepsilon)x, B\lambda\varepsilon) + x \cdot \nabla v(\xi, 0)$$

where $\lambda \in (0,1)$ and $\xi$ is a point in the segment joining $x$ and $(1+\varepsilon)x$. Combining this with

$$v_t = (m-1)v\Delta v + +|\nabla v|^2 \ge -Cv + c^2,$$

and using the fact that $|\nabla v|$ does not vanish, we can adjust the values of $A$ and $B$ so that $I_\varepsilon > 0$ in an inner neighbourhood of $\partial D$. The precise dependence of $A$ and $B$ on the initial data is described in [145]. ■

**Corollary 19.7** *Under the same conditions, the function*

$$w(s) = v(x(s), t(s))\, e^{(A-2)s}$$

*is non-decreasing along the curves*

$$x(s) = x_0\, e^s, \quad t(s) = \frac{1}{A}\left[(At_0 + B)e^{As} - B\right].$$

*It follows that if the free boundary equation is written as $r = f(\vartheta, t)$, then $f$ satisfies*

$$f(\vartheta, t_1) \ge f\left(\vartheta, t_0 \left(\frac{At_1 + B}{At_0 + B}\right)^{1/A}\right) \tag{19.21}$$

*for all $t_1 > t_0 > T(u_0)$.*

This is an easy exercise for the reader. See Problem 19.3. It also follows that the positivity set advances with a non-zero speed.

**Theorem 19.8** *Under the conditions* (R)*, let us pick a point* $(x_0, t_0) \in Q$ *where* $v > 0$*. Then there exist constants* $A, B, C, \varepsilon$ *depending on* $v_0$*,* $t_0$ *and* $R_1 = d \sup\{d(x_0, y) : y \in D\}$ *such that*

$$v(x,t) \geq v(x_0, t_0)\, e^{-C(t-t_0)} \tag{19.22}$$

*if* $t_0 < t < t_0 + \varepsilon$ *and* $(x,t)$ *lies in the time-like cone*

$$|x - x_0| \leq \frac{R_1}{At_0 + B}\, |t - t_0|. \tag{19.23}$$

It immediately follows from this conical propagation that we can rewrite the free boundary equation $|x| = f(\vartheta, t)$ for $t > T(u_0)$ in the time form.

**Corollary 19.9** *The part of the free boundary* $\Gamma \cap \{t > T(u_0)\}$ *can be written in the form*

$$t = \tau(x)$$

*and* $\tau$ *is a Lipschitz function of* $x$ *for* $x \notin B_{R_0}(0)$*.*

Paper [145] also proves the transversality of $v$ near the free boundary for all later times.

**Theorem 19.10** *Under the conditions* (R)*, there exists a neighbourhood* $S$ *of the free boundary* $\Gamma$ *such that for every time interval* $[t_1, t_2]$ *with* $t_1 > T(u_0)$ *there exists a constant* $C > 0$ *such that*

$$|\nabla v| \geq C$$

*whenever* $(x,t) \in S$*,* $v(x,t) > 0$ *and* $t_1 < t < t_2$*.*

### 19.1.3 $C^{1,\alpha}$ *regularity*

Caffarelli and Wolanski [146] proved that when the initial pressure $v_0 = u_0^{m-1}$ is compactly supported and satisfies the conditions (R), then the interface is actually a $C^{1,\alpha}$ surface. This is a delicate regularity result using the technique of cones of increasing aperture that is well worth mastering, but falls outside of the scope of this text.

## 19.2 Focusing solutions and limited regularity

The focusing solutions of the PME are a family of self-similar solutions of Type II

$$U(x,t) = (T-t)^{\alpha} F(x(T-t)^{-\beta}), \tag{19.24}$$

with the compatibility condition $(m-1)\alpha = 2\beta - 1$, cf. Chapter 16.[1] We want to find a profile $F(\eta)$ $(\eta = x(T-t)^{-\beta})$ that has a hole in the support and

[1]Note the different sign for $\beta$.

is positive in the exterior of the hole. By the properties of the support that we have established in Chapter 14 the hole has to shrink and then vanish in finite time. The standard setting for focusing solutions is when $F$ is radially symmetric, $F(\eta) = F(r)$, $r = |\eta|$, and the hole is a ball $B_{R(t)}(0)$. According to formula (19.24) the radius must have the precise form

$$R(t) = a\,(T-t)^{\beta} \tag{19.25}$$

for some $a > 0$ (the radius of the support of $F$). The main problem is the determination of the exponent or exponents $\beta$ for which a solution with these properties exists (notice that once $\beta$ is known then $\alpha$ follows). The idea of paper [48] is to examine the behaviour of the possible solutions of the ODE that must be satisfied by the profiles $F(r)$:

$$(F^m)'' + \frac{d-1}{r}\,(F^m)' + \alpha F - \beta r F' = 0.$$

The analysis in the corresponding phase plane, cf. Section 16.8 allows us to show that there exists a precise value of the parameter $\beta$, let us call it $\beta_*$ (the *focusing exponent*, it depends on $m$ and $d$), such that a corresponding profile $F$ can be found with the following properties:

(i) $F$ is continuous, non-negative and radially symmetric: $F = F(r)$, $r = |\eta|$;
(ii) $F$ vanishes for $0 < r < a$ and is $C^\infty$ and strictly increasing for $r > a$; $U(x,t)$ given by (19.24) is a weak solution of the PME, and it is even a classical solution in the positivity set, i.e., for $|x| > a(T-t)^{\beta_*}$.

Actually, it is convenient to perform the computations in terms of the pressure variable $v = mu^{m-1}/m-1$, which has a self-similar formula

$$V(x,t) = (T-t)^{2\beta-1} G(x(T-t)^{-\beta}), \tag{19.26}$$

with $G = cF^{m-1}$. The following limit behaviour is also established.

(iii) There exists $c > 0$ such that $G(r)r^{-\varepsilon} \to c$ as $r \to \infty$ if $\varepsilon = (2\beta_* - 1)/\beta_*$.

As a consequence of this property and formula (19.26), the limit profile of the focusing solution is known:

$$\lim_{t\to T} V(x,t) = c|x|^{\varepsilon}. \tag{19.27}$$

We call these profiles found by Aronson and Graveleau the *AG profiles*. We remark that for all $d \ge 1$ a one-parameter family of focusing solutions is obtained; they can be normalized by fixing $a = 1$, or to any other positive value. We indicate the family when needed with the notation $G(\eta; a)$. In the notation of Sections 16.4, 16.8, we have $\varepsilon = -\gamma(m-1)$ so that $\beta = 1/(2-\varepsilon) > 0$.

The main fact proved in [48] about these special solutions is the estimate on the value of the exponent $\beta_*$ and the regularity of $G$.

**Proposition 19.11** *For $d = 1$ we have $\beta_* = 1$ and $V(x,t)$ is Lipschitz continuous. On the contrary, for $d \geq 2$ it turns out that $1/2 < \beta_*(m,d) < 1$ and $V(x,t)$ is only locally Hölder continuous for some Hölder exponent $\varepsilon$ less than 1.*

Let us mention that for $d = 1$ it is well known that a solution with these characteristics corresponds to $\beta_* = 1$, and the solution is in fact the travelling wave, which in terms of the pressure variable says

$$V(x,t) = \frac{m}{m-1} U(x,t)^{m-1} = c(x - c(T-t))_+$$

with a free parameter $c > 0$, see Section 4.3. For $d \geq 2$ the exponent $\beta_*$ does not come from a priori physical or dimensional considerations and is called an *anomalous exponent*; in Zel'dovich's words we have a *self-similarity of the second kind*, a topic that is beautifully explored in Barenblatt's book [63]. See also [515], Chapter 7.

**Corollary 19.12** *It follows from the proposition that $\varepsilon \in (0,1)$ for $d \geq 2$, hence $V$ is not Lipschitz continuous near $x = 0$, $t = T$.*

The graph below presents the plot of a typical focusing solution in the Φ-Ψ-plane introduced in Section 16.5, but now in the case of Type II self-similarity. The free boundary point $P_4$ is the origin of a trajectory that comes into the fourth quadrant and then enters the origin (which means $F(\xi) \to \infty$ along the stable manifold, i.e., for $F(\xi) \sim |\xi|^{-\gamma}$).

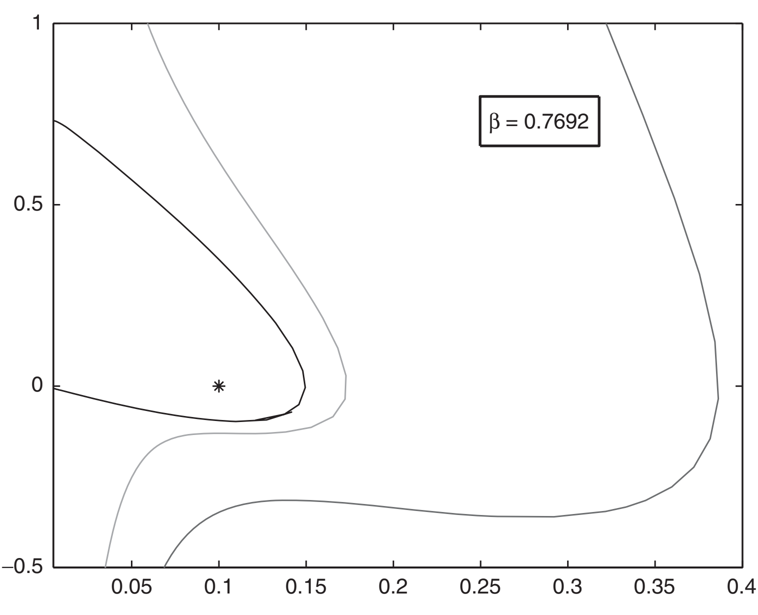


Figure 19.1: The focusing solution in the ΦΨ-plane for $d = 3$, $m = 2$.

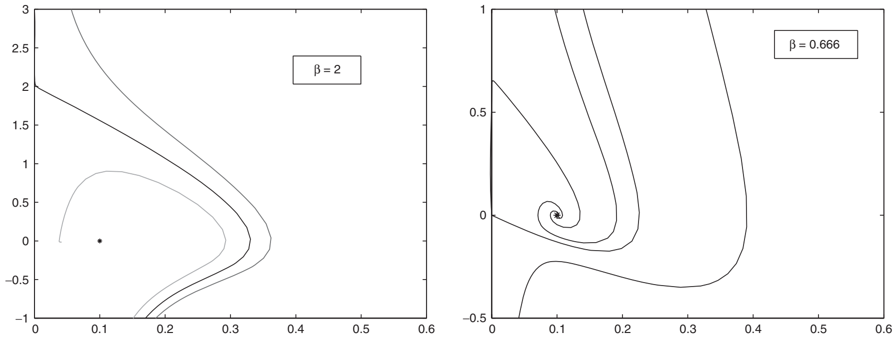


Figure 19.2: Shooting with the wrong $\gamma$ in the $\Phi\Psi$-plane for $d = 3$, $m = 2$.

If $\gamma < 0$ is not carefully chosen the trajectory would end up either in the lower part of the $\Psi$ axis (which means change of sign for the profile), or would spiral around the point $P_2$ (marked with a star, it now lies in the positive part of the $\Phi$-axis). Actually, it is proved that the correct connection is unique.

The shooting argument of the original proof of [48] was simplified in [52], where the focusing profiles are obtained by a continuation method in the parameter $d$ and the implicit function theorem. The method allows us to show that $\beta_*$ is an analytic function of $m$ and $d$, and $F$ is also an analytic function of $r, m, d$ whenever positive. The starting point of the continuation is the known solution in $d = 1$. It is further proved in [47] that $\beta_*(m, d) \to 1/2$ if $m \to \infty$, while it tends to 1 as $m \to 1$, always for $d \geq 2$. The monotonicity of $\beta_*$ as a function of $m$ has been subsequently proved in [41].

### 19.2.1 *Propagation and hole filling. Unbounded speed*

We examine next some of the remarkable consequences of this result when seen from the point of view of mass transport, and explain why the value of $\beta_*$ and the regularity of $G \sim F^{m-1}$ matter for dynamical purposes. It is well-known that the PME can be viewed as a mass conservation law for a density $u$ transported with speed $\mathbf{v}$ in the usual form

$$u_t + \nabla \cdot (u\,\mathbf{v}) = 0. \tag{19.28}$$

In order for $u$ to satisfy the porous medium equation, the *particle speed* must be defined as $\mathbf{v}(x, t) = -mu^{m-1}\nabla u = -\nabla v$, which is known to be a form of the famous Darcy law of flow propagation, Chapter 2. The PME has finite speed of propagation, a fact that has a clear interpretation in the physical models of that chapter, like groundwater infiltration or gas flow in porous media. But, contrary to a popular misconception, that does not mean that the pointwise speed $\mathbf{v}$ of the flow has to be finite everywhere. The boundedness of the particle speed is true in one space dimension but not necessarily in two or more. Let see how this happens in our example. Its free boundary (in other words, the front that separates the empty region from the wet region when we use groundwater infiltration imagery), is given by the surface $\Gamma$ with equation

$$|x| = a(T-t)^{\beta_*}. \tag{19.29}$$

The advance speed of this surface in time is given by the formula

$$\mathbf{v}_f(t) = \beta_* a(T-t)^{\beta_*-1}. \tag{19.30}$$

**Note**

(i) The speed $\mathbf{v}_f$ can be calculated both geometrically, as the value of the normal front speed, and also dynamically, as the limit value of the internal particle speed $\mathbf{v}(x,t)$ as $(x,t) \to \Gamma$. Internal means defined in the wet region, where $V > 0$ and the solution is $C^\infty$; there, $\mathbf{v}$ is given by Darcy's law. This version of Darcy's law is rigourously proved in the pointwise sense for the focusing solutions.

(ii) The front advances towards the origin and it reaches it precisely at $t = T$.

(iii) We come now to a key point in our argument: if $\beta_* < 1$ the speed $\mathbf{v}_f$ tends to infinity as $t \to T$. We conclude that the focusing solutions have a diverging front speed as they approach the focusing time in dimensions $d > 1$ .

(iv) On the contrary, the speed is locally finite in $Q = \mathbb{R}^d \times (0,T)$ away from a neighbourhood of $(x = 0, t = T)$.

### 19.2.2 *Asymptotic convergence*

Angenent and Aronson studied in [25] the question of how generic is the focusing behaviour described by the AG solutions. The answer turns out to be positive for solutions of the PME with radially symmetric initial data.

**Proposition 19.13** *Let $u_0(x)$ be a non-negative, radially symmetric, continuous and compactly supported initial function, which is positive for $r_1 < |x| < r_2$ and zero otherwise. Let $u(x,t)$ be the corresponding solution of the PME. Then there exist $T > 0$ and $a > 0$ such that, as $t \to T$ (with $t < T$), $u(r,t)$ tends to the self-similar solution* (19.24) *with parameter $a$ in the following sense:*

(i) *If $v(r,t)$ is the pressure of the solution, then for each fixed $\eta = x(T-t)^{-\beta^*} \in [0,\infty)$,*

$$\lim_{t\to T} v(\eta(T-t)^{\beta_*}, t)\,(T-t)^{-2\beta_*+1} = G(\eta; a). \tag{19.31}$$

(ii) *The inner interface converges; If $R(t) = \sup\{|x| : v(x,t) = 0\}$ is the radius of the hole of $v$ at time $t$, then*

$$\lim_{t\to T} R(t)/(T-t)^{\beta_*} = a. \tag{19.32}$$

Note that we have changed the statement and notations of [25] for convenience. Subsequent analysis performed in [28, 30] shows that *this stability of the radial* *focusing profiles is lost when non-radial perturbations are admitted.* We will not discuss the question of non-radial focusing solutions which is quite advanced. The reader is referred to the works of Angenent, Aronson and collaborators cited in the bibliography. We just mention that this non-radial instability reminds us of the instabilities of Newtonian viscous flows (Couette flow).

### 19.2.3 *Continuation after the singularity*

The focusing self-similar solution reaches its focusing time with a limit profile which is a power of $|x|$,

$$u(x,T) = c\,|x|^{\varepsilon/(m-1)},$$

where $\varepsilon$ is the exponent of (19.27). The similarity analysis of Section 16.3 allows us to conclude that the solution is self-similar, radially symmetric in the space variable, positive and smooth for $t > T$. See details in [27]. A bit more work allows us to prove that the solutions of Proposition 19.13 have a continuation that is radially symmetric in the space variable, positive and smooth for $t > T$.

### 19.2.4 *Multiple holes*

Interesting variants of the above construction include the case where there are several holes. Generically, they will disappear one by one. Since the configuration is not radially symmetric this case has not been much taken care of except numerically.

A radial case with different holes will imply a central hole plus one or several empty annuli. Considering that the support is still bounded, all these holes will disappear in finite time, but while the central one disappears as a point, the annuli disappear as spheres.

## 19.3 Lipschitz continuity from space to time

In this section we prove the transfer of regularity that happens in the PME from regularity in $x$ into regularity in $t$. This is a general phenomenon that happens in parabolic equations but in the PME it takes the form of transfer of Lipschitz continuity in $x$ into Lipschitz continuity in $t$, which is a quite strong result for an equation that has regularity limitations. A very simple example of this kind of continuity is given by the travelling wave solutions of Section 4.3. But there is another self-similar example that also reflects such regularity and serves as a basis for the proof of the general result.

*Solution with conical data*

We construct a solution $U$ with pressure initial data

$$V_0(x) = |x|, \quad x \in \mathbb{R}^d. \tag{19.33}$$

According to the self-similar analysis of Section 16.3 the unique weak solution of this problem has the form in pressure variable

$$V(x,t) = t\, f(\xi), \quad \xi = x/t. \tag{19.34}$$

the profile function $f \geq 0$ satisfies a second-order ODE that we do not need to write at this point, because the properties are derived as follows: we know from the general theory that $V$ must be non-negative and locally bounded for $t > 0$; this means that $f$ is locally bounded. The initial data must be taken, hence

$$\lim_{\xi\to\infty} \frac{f(\xi)}{\xi} = 1.$$

It is also easy to see that $f$ is an increasing function of $\xi$. The fact that waiting times must be finite implies that $f(0) = a > 0$. This is the minimum value of $f$. Moreover, since $\Delta V_0 \geq 0$ we have $\Delta f \geq 0$ (see Problem 12.7), which means in particular that $V_t \geq |\nabla V|^2 \geq 0$ . This means that $f(\xi) \geq \xi\, f'(\xi) + (f'(\xi))^2$. It follows that $a > 1$. We also have the strict lower bound

$$\begin{aligned} V_t(x,t) \geq \min V_t(x,0) = (m-1)V_0\Delta V_0 + |\nabla V_0|^2 &= (m-1)(d-1)+1 \\ &= k(m,d) > 1. \end{aligned}$$

It follows that $a > 1$.

**Proposition 19.14** *The self-similar solution $V$ is globally Lipschitz continuous.*

**Proof** We have $|\nabla_x V| = f'(\xi) \leq 1$. Since, $V_t \geq |\nabla_x V|^2$, only the upper bound on $V_t = f(\xi) - \xi\, f'(\xi)$ is still needed. Here is the argument: near the point $x_0 = (1, 0, \ldots, 0)$ the solution is $C^\infty$ smooth at time $t_0$ since the data are smooth and the equation is locally uniformly parabolic. This implies that the finite value $V_t(x_0, 0)$ is taken. Hence,

$$\lim_{\xi\to\infty} (f(\xi) - \xi\, f'(\xi)) = V_t(x_0, 0).$$

This means that $V_t$ is globally bounded, $0 < k(m,d) \leq V_t \leq c(m,d)$. Note that $c \geq a > 1$. ■

**Remarks** Scaling allows you to construct a solution with Lipschitz constant $L$ in $x$ and $cL^2$ in $t$. Just define

$$V_L(x,t) = V(Lx, L^2 t) = L^2 t\, f(\xi/L), \quad \xi = x/t.$$

This implies that $\partial_t V_L(x,t) = L^2 \partial_t V(Lx, L^2 t) \geq L^2\, k(m,d)$.

We may now state the general result in a local setting.

**Theorem 19.15** *Let $u$ be a non-negative and bounded solution of the PME that is defined on a cylinder $S = B_{2R}(0) \times (0,T)$. Assume moreover that it is uniformly Lipschitz continuous with respect to the space variable and that*

*$v_t \ge -K$ in $S$. Then, $v$ is also Lipschitz continuous with respect to the time variable in $S' = B_R(0) \times (\tau, T)$ for all $\tau > 0$.*

**Proof** We assume that $v$ is Lipschitz continuous in space in $S$ with constant $L$ and bounded with constant $N$ and will prove that it is Lipschitz continuous in with a constant that depends on $m, d, N, L, R$ and $\tau$. An auxiliary result shows that the solution does not degenerate in a small neighbourhood of a positivity point.

**Lemma 19.16** *Let $(x_0, t_0) \in S'$ and let $\alpha = v(x_0, t_0) > 0$. There exist constants $A$ and $B > 0$ depending only on $d, m, N, L, R$ and $\tau$ such that*

$$\frac{\alpha}{4} \le v(x,t) \le 2\alpha \tag{19.35}$$

*in the cylinder where $|x| \le R$, $|x - x_0| \le A\gamma$ and $0 \le t_0 - t \le B\gamma$; here, $\gamma = \min\{\alpha, \tau\}$.*

**Proof** (i) By assumption we have $|v(x_0,t) - v(x,t)| \le \alpha/2$ if $|x - x_0| \le \alpha/2L$, $|x| \le R$. Hence, $\alpha/2 \le v(x_,t) \le 3\alpha/2$. Moreover, by the assumption on $v_t$ we conclude that

$$v(x,t) \le v(x,t_0) + K(t_0 - t) \le 2\alpha \tag{19.36}$$

if $|x| \le R$, $|x - x_0| \le \alpha/2L$, and $t \in (t_1, t_0)$ with $t_1 = \max\{0, t_0 - \alpha/2K\}$. The upper bound is obtained.

(ii) The lower bound is more delicate. We assert that there exists a small constant $C < 1/2$ such that

$$v(x_0,t) > \frac{\alpha}{4a} \quad \text{if} \ \ 0 \le t_0 - t \le \gamma C. \tag{19.37}$$

If this is not so, using the Lipschitz continuity at such a time $t_1$ we have

$$v(x,t_1) \le \frac{\alpha}{4a} + L|x - x_0|. \tag{19.38}$$

We consider that estimate as a step to get an upper bound for the solution in the domain $S_1 = B_R(x_0) \times (t_1, t_0)$. On the lateral boundary of this domain we also know that $v \le N$. We want to compare $v$ in that domain with a translation of the constructed conical solution

$$\tilde{v}(x,t) = V_{L_1}(x - x_0, t - t_1 + \tau_1),$$

with $L_1 = \max\{2L, N/R\}$ and $\tau_1 \in (0, \tau/2)$ to be chosen. The comparison on the lateral boundary gives $v(x,t) \le N \le \tilde{v}(x,t)$. As for the values at $t = t_1$, $|x - x_0| \le R$, using (19.38) we have $v(x,t) \le \tilde{v}(x,t)$ if

$$L_1^2 k(m,d)\tau_1 \ge \alpha/2a.$$

This gives the estimate for $\tau_1$. The parabolic theory tells us now that $v \le \widetilde{v}$ in $S$, and in particular

$$v(x_0,t_0) \le \widetilde{v}(x_0,t_0) = aL_1^2(t_0 - t_1 + \tau_1) \le aL_1^2\gamma C + \alpha/2k.$$

This quantity would be less that $\alpha$ if $aL_1^2\gamma C < \alpha/2$. We would thus arrive at a contradiction. ■

**Proof of the theorem continued** Once the lemma is proved, we argue as follows: we rescale the pressure as

$$w(x,t) = \frac{1}{\gamma} v(x_0 + \gamma x, t_0 + \gamma t)$$

and then $w$ is bounded in the form

$$\frac{\alpha}{4a\gamma} \le w(x,t) \le \frac{2\alpha}{\gamma}$$

in the cylinder $Q_1$ where $|x| \le A$, $-B \le t \le 0$. If $\alpha \le \tau$ then $\gamma = \alpha$ and we have

$$\frac{1}{4a} \le w(x,t) \le 2$$

in $Q_1$. On the other hand, if $\alpha > \tau$, then $\alpha/\gamma > 1$. Since $\alpha < N$ we have

$$\frac{1}{4a} \le w(x,t) \le \frac{2N}{\gamma}.$$

In both cases the equation satisfied by $w$ in $Q_1$ is uniformly parabolic, so that the standard regularity theory says that there exists a constant $C$ depending on $\tau, m, d, N$ such that $|w_t(0,0)| \le C$. Since $w_t(0,0) = v_t(x_0,t_0)$ the proof follows. ■

It is to be noted that Lipschitz continuity in space is not a property that is necessarily conserved in time for the solutions of the PME as the focusing solutions show. Hence, the assumption that $v$ is Lipschitz continuous in space uniformly for some time is essential.

## 19.4 $C^\infty$ regularity

In his doctoral thesis [347], Koch addresses the problem of higher regularity of the free boundary of non-negative and compactly supported solutions of the PME. A non-degeneracy assumption is made on the data that barely reflects the one used in [145].

**Definition 19.1** *Let $0 < \alpha < 1$. A compactly supported function $u_0 \ge 0$ is called $\alpha$-admissible as an initial datum for the PME if the pressure $v_0 = (m/(m-1))u_0^{m-1}$ is $C^1$ in its support $\Omega$, if $v_0 + |\nabla v_0| \ge c > 0$ in $\Omega$ and if there exists $c_\alpha$ such that*

$$|\nabla v_0(x) - \nabla v_0(y)| \le c_\alpha \left( \frac{|x-y|}{|x-y| + d(x,\partial\Omega) + d(y,\partial\Omega)} \right)^\alpha . \tag{19.39}$$

The problem is transformed into a quasilinear subelliptic problem for which an analogue of the regularity theory of quasilinear parabolic equations is established. The main tools in this delicate analysis are two: on the one hand, the theory of pseudodifferential operators in a non-Euclidean setting; also, Gaussian estimates of the fundamental solution of a linear degenerate parabolic equation obtained from the PME by a transformation of dependent and independent variables that uses the local monotonicity of the solution near the free boundary. In the simplest case the linearization reads

$$Lv = v_t - \sum_{i=1}^{d} \partial_{x_i}(x_d\, \partial_{x_i} v) - \sigma\, \partial_{x_d} v.$$

The study of Gaussian estimates and singular integrals strongly relates the geometric and analytic properties of the equation.

Here is his Theorem 4.

**Theorem 19.17** *Let $U$ be a relatively open subset of the support of a continuous solution of the PME $u \ge 0$, and suppose that $u^{m-1} \in C^{0,1}(U) \cap C^1(U^+)$ and satisfies $\partial_{x_d} u^{m-1} < -c < 0$ in $U^+ = U \cap \{u > 0\}$. Then the following is true:*

(i) *The free boundary of $u$ in $U$ is locally given by $x_d = h(x_1, \ldots, x_{d-1}, t)$ where $h$ is $C^\infty$ smooth. If $K'$ is a compact subset of the domain of $h$, then there exist $R$ and $C$ such that*

$$\sup_{K',\alpha} \alpha!^{-1} \alpha_0!^{-1} R^{|\alpha|} |\partial_{x,t}^{\alpha} h| \le C,$$

*where the supremum is taken over all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_{d-1}, \alpha_0)$.*

(ii) *The pressure $v$ is smooth in $U$ up to the boundary of the support. If $K$ is a compact subset of $U$, then there exist $R$ and $C$ such that*

$$\sup_{(x,t)\in K;\alpha} \alpha!^{-1} \alpha_0!^{-1} R^{|\alpha|} |\partial_{x,t}^{\alpha} v| \le C.$$

This implies that these solutions are analytic in $x$ up to the boundary of the support and they are Gevrey regular in $t$. The assumptions are satisfied by the compactly supported solutions with $\alpha$-admissible data of previous sections.

### 19.4.1 *Eliminating the admissibility condition*

It has been recently proved that the admissibility condition that we are requiring as a pre-condition for the large time regularization to apply is actually satisfied by all solutions with compactly supported initial data after some time $T > 0$, cf. [348].

## 19.5 Further regularity results

Solutions can have regularity coming from special properties of the data or due to the effect of the evolution.

### 19.5.1 *Conservation of initial regularity*

Daskalopoulos and Hamilton [196] study the flow in two space dimensions, $d = 2$; the initial data satisfy appropriate regularity assumptions. Then, the pressure $v = u^{m-1}$ is smooth up to the interface $\Gamma$ for all $t \in (0,T)$, for some $T > 0$. The approach is based on the study of the auxiliary model equation $v_t - x(v_{xx} + v_{yy}) - \nu\, v_x = g$ in the half space $x \geq 0$, which is of interest in itself. By means of a global change of variables, the original free boundary problem is transformed into a fixed boundary problem.

Shmarev [477] studies the Cauchy problem for the PME with absorption $u_t = \Delta u^m - u^p$ in $R^n \times (0,T]$, $d = 1,2,3$, with $m > 1$, $p > 0$, $m + p \geq 2$, $a > 0$, and the work includes as a particular case the PME. He uses a local form of the Lagrangian coordinates, which makes the free boundary stationary after the transformation. He shows that the interface velocity has the form $v = \left[-\frac{m}{m-1}\nabla u^{m-1} + \nabla \Pi\right]\Big|_{\Gamma(t)}$ where $\Pi$ satisfies a suitable elliptic equation. Regularity of the solution and interface is obtained locally in time (this important observation is not clear in the text!).

### 19.5.2 *Concavity results*

There are number of papers where the regularity of the solutions of the Cauchy problem for the PME in the whole space is investigated through the geometrical properties like convexity or concavity. We have reported in Subsection 15.8.4 the conservation of pressure concavity in one space dimension.

Concavity in several space dimensions is studied by Daskalopoulos, Hamilton and Lee [197]. Again, they consider the PME with initial data $u_0$ non-negative, integrable, and compactly supported, but the result is different: they assume that the initial pressure $v_0 = u_0^{m-1}$ is smooth up to the interface and in addition it is root-concave (i.e., $v_0^{1/2}$ is concave) and also satisfies the non-degeneracy condition $|\nabla v_0| > 0$ at $v = 0$. Then the pressure $v = u^{m-1}$ remains $C^\infty$-smooth up to the interface and is root-concave, for all time $0 < t < \infty$ . In particular, the free boundary is $C^\infty$-smooth for all time. The reader should notice that all possible holes are eliminated from scratch by the assumptions.

### 19.5.3 *Eventual concavity*

A very interesting aspect of the phenomenon of increased regularity for large times of the solutions of the PME with compactly supported data is the property of asymptotic concavity that we describe next. More precisely, we prove eventual pressure concavity of the solution profile $u(\cdot,t)$ in the support $\mathcal{S}_u(t)$. We recall that eventual concavity in one space dimension has been described in Subsection 18.7.3.

Lee and Vázquez [365] consider solutions of the PME in the whole space $\mathbb{R}^d$, $d \geq 1$, and assume that the initial data $u_0$ is a non-negative integrable function with support contained in the ball $B(0,R)$ of radius $R$ centred at 0, satisfying the regularity assumption (R) with no assumption of initial concavity or convexity. We know by Theorem 18.1 that if we rescale the solution in the form

$$\widetilde{u}_\lambda(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda\, t) \tag{19.40}$$

with $\beta$ and $\alpha = d\beta$ the standard similarity exponents given by (18.4), then the curve $\widetilde{u}_\lambda(x,t)$ converges as $\lambda \to \infty$ to the ZKB solution in locally in $C^\gamma(Q)$ for some $0 < \gamma < 1$. Two further observations give a clue to a finer large-time behaviour

(i) Koch's results can be used to show that the convergence $\widetilde{u}_\lambda \to \mathcal{U}$ takes place in $C^\infty$ inside the positivity set $\mathcal{P}_{\widetilde{u}_\lambda}$. We already know that $\mathcal{P}_{\widetilde{u}_\lambda}$ converges to $\mathcal{P}_{\mathcal{U}}$, and this convergence extends to uniform bounds for all derivatives up to the boundary.
(ii) The ZKB solution has a space profile that is concave inside the support in a uniform way.

The combination of these facts leads to the following result, Theorem 1.2 of [365].

**Theorem 19.18** *Let $u$ be a solution of the PME in $d$ space dimensions with compactly supported initial data satisfying a non-degeneracy condition as stated below. Then there is $t_c > 0$ such that the pressure $v(x,t)$ is a concave function in $\mathcal{P}(t) = \{x:\ v(x,t) > 0\}$ for $t \ge t_c$. More precisely, for any coordinate direction $x_i$*

$$\lim_{t\to\infty} t\,\frac{\partial^2 v}{\partial x_i^2} = -\beta \tag{19.41}$$

*uniformly in $x \in supp(v)$. Here, $\beta = 1/(d(m-1)+2)$. For the mixed second derivatives the limit is zero.*

In particular, the support of the solution is a convex subset of $\mathbb{R}^d$ which converges to a ball ([365], Corollary 3.5).

**Corollary 19.19** *For all $t \ge t_c$ the level sets $\{x \in \mathbb{R}^d:\ v(x,t) \ge c\}$, $0 < c < \max_x v(x,t)$, are convex subsets of $\mathbb{R}^d$. The function $v(\cdot,t)$ has only one maximum point for all $t > t_c$.*

From symmetry arguments (Aleksandrov's principle) the maximum stays inside the ball that contains the original support.

## 19.6 Various

### 19.6.1 *Precise Hölder regularity*

The focusing solutions are a precise example of the lack of regularity that may happen to solutions of the PME, in the sense that $\nabla(u^{m-1})$ is generally not bounded in the whole of $Q = \mathbb{R}^d \times (0,\infty)$. There is a question of determining the best Hölder exponents for $u$ or for $v = c\,u^{m-1}$, or at least good bounds for

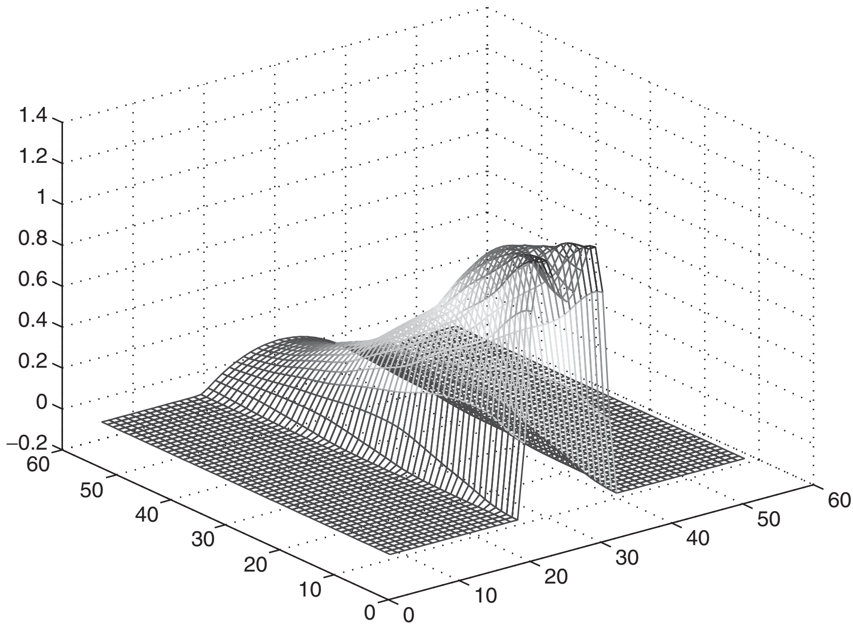


Figure 19.3: Eventual concavity of a solution of the PME in $d = 3$ with $m = 2$.

them. The exponents we get from the focusing solutions are not very explicit and it is not known how optimal they are (in the non-radial case). Different authors have discussed the question of Hölder regularity and derivative estimates.

In his Kentucky Notes [83], Bénilan proves an estimate for the flux $\nabla u^m$ of the solution of the PME in several space dimensions if the following relation holds between $m$ and $d$ holds:

$$(m-1)^2(d-1) < 1. \tag{19.42}$$

The estimate reads

$$|\nabla u^m|^2 \le \frac{cu}{t} \tag{19.43}$$

where

$$c = \frac{m\|u_0\|_\infty^m}{1-(d-1)(m-1)^2}.$$

The conjecture that the flux is always bounded was disproved by Aronson, Gil and Vázquez [47] Theorem 2, who showed that if the focusing profile has a shape $u(x,T) = c\,|x|^\gamma$ (compare with formula (19.27) and put $\varepsilon = (m-1)\gamma$), then for $d \ge 2$ we have

$$\lim_{m\infty} \gamma(m,d) = 0.$$

Hence, $u^m$ cannot be Lipschitz continuous near such a focusing point is $m$ is large.

Hölder estimates of weak solutions with explicit Hölder exponents are obtained by Lu and Jäger, [379] applying the maximum principle. In [379], Section 4, the Hölder continuity of weak solutions of an initial-boundary value problem for general nonlinear reaction-diffusion–convection equations is considered. Under the critical condition on the diffusion function $G$, meas$\{u\colon G'(u) = g(u) = 0\} = 0$, they obtain a Hölder continuous solution $u$ and the sharp regularity estimate $G(u) \in C^{(1)}$ up to boundary. Y. G. Lu, [380] Proves Hölder estimates for equations of the form

$$\partial_t u = \Delta G(u) + \mathrm{div} f(u) + h(u)$$

posed for $x \in \mathbb{R}^d$. The function $G$ is assumed to be $C^2$ smooth and possibly constant on some subset of $\mathbb{R}$ with positive measure, with $G'$ non-negative and the ratio $|GG''/(G')^2|$ bounded by a positive constant $\beta \le (2d)^{-1/2}$. For $d \ge 2$ this implies exponents in the PME equal or less than 2, and closer to 1 as $d$ grows. The result is that $|\nabla G^\alpha(u)| \le M$ with $\alpha > 1 - (1 + \sqrt{1 - 2d\beta^2})/4$.

Moulay and Pierre [392] consider the PME in $Q$ and show that there exists $p > m - 1$ such that $\nabla(u^p) \in L^\infty_{\rm loc}(Q)$ under the preassumption $\nabla(u^r) \in L^{d+2+\varepsilon}_{\rm loc}(Q)$ for any $\varepsilon > 0$ and some $r > 0$, which is satisfied under suitable conditions on $(m, d)$. The proof relies on classical iterative arguments of Moser type.

### 19.6.2 *Fast diffusion flows*

The property of infinite speed of propagation makes all non-negative solutions strictly positive, hence the regularity question disappears for those solutions. The concavity results of Lee and Vázquez can be adapted but now they mean convexity of $u^{m-1}$, which turns out to be a function growing as $|x| \to \infty$ like $C(t)\,|x|^2$ with a very precise coefficient $C(t)$. The condition of initial compact support can be relaxed.

## Notes

**Section 19.1.** We describe the results of paper [145], 1987, on Lipschitz continuity. $C^{1,\alpha}$ regularity is taken from the paper by Caffarelli and Wolanski [146]. The technique of cone opening proposed by Caffarelli has played a major role in the regularity theory of other free boundary problems, like the Stefan problem.

The results have been extended to the evolution $p$-Laplacian equation by Ko [345].

**Section 19.2.** The existence of the focusing was mentioned by J. Graveleau, [273], 1972. Aronson and Graveleau supplied a rigourous proof in [48], 1993. Work on the subject includes [25, 27, 52]. Elongated holes are described in [28, 30]. Focusing for the eikonal equation is discussed in [29], and for the $p$-Laplacian evolution equation in [47] and [259].

The focusing solutions are used in [514] to show that the PME semigroup in $\mathbb{R}^d$ is not contractive in the so-called Wasserstein metrics $d_p$ if $p$ is large enough, including $p = \infty$. It is known that the semigroup is contractive in $d_2$ [152].

**Section 19.3.** The transfer of Lipschitz continuity from space to time follows Aronson and Caffarelli [43] who proved the global result in $d = 1$. We have extended the result to a local version valid in several space dimensions.

Results for general parabolic equations were proved by Kruzhkov [351] and Gilding [262] that imply that $C^\alpha$ regularity in $x$, $0 < \alpha < 1$, implies $C^{\alpha/2}$ regularity in time. The PME behaves better.

**Section 19.4.** Koch's work is a deep work occupying 158 pages that represents major progress in PME theory in the 1990s.

**Section 19.5.** The fact that eventually the space profile of the non-negative solution of the PME has a unique maximum point was shown by Sakaguchi [466].

The eventual concavity results obtained are extended in [365] to the heat equation (where the geometrical property is eventual log-concavity) and fast diffusion (the geometrical property is eventual pressure-convexity). In all cases, the level sets become all convex eventually in time.

The results are true also for the $p$-Laplacian equation (in a somewhat weaker form, cf. [364]. By differentiation, when $d = 1$ this gives information on the zero-set of signed solutions of the PME.

**Section 19.6.** A number of authors have studied the question of continuity and Hölder regularity for more general equations. We mention only some works that have come to our attention:

E. DiBenedetto and V. Vespri, [215] study equations of the form $\beta(u)_t = \Delta u$,

Many authors have studied the GPME with or without lower order terms. Here is a reference: Jäger and Lu [309] consider the one-dimensional Cauchy problem for the general degenerate parabolic equation $u_t + F(u)_x + H(u) = G(u)_{xx}$ with initial datum $u|_{t=0} = u_0(x)$. Under the critical condition $\text{meas}\{u\colon\ g(u) = 0\} = 0$ they obtain the estimate $G(u) \in C^{(1)}$.

In another direction, Ebmeyer [224] studies the PME and the FDE

$$u_t = \Delta(|u|^{m-1}u), \quad m > 0,$$

in a cylinder $\Omega \times (0,T)$ with homogeneous Dirichlet boundary conditions. He proves regularity of $u$ and $|u|^{m-1}u$ in weighted Sobolev spaces and in fractional-order Nikol'skiĭ spaces.

## Problems

### Problem 19.1

(i) Construct a non-negative solution of the PME with infinitely many point singularities corresponding to different focusing points.

(ii) Construct a solution with a singularity supported on a line or on a lower dimensional surface.

(iii) Construct a simple non-compactly supported solution that does not become Lipschitz continuous in pressure after a finite time.

**Problem 19.2** Complete the details of the proof of Corollary 19.3.

**Problem 19.3**

(i) Complete the proof of the comparison part of Proposition 19.6.

(ii) Prove Corollary 19.7.

(iii) Prove Theorem 19.8 and Corollary 19.9.

**Problem 19.4** Solutions with conical data

(i) Construct the solution with conical data of the form

$$V_0(x) = |x|h(\sigma), \quad \sigma = \frac{x}{|x|}, \tag{19.1}$$

with a given bounded function $g : \mathbb{S}^{d-1} \to \mathbb{R}_+$.

(ii) Study the case in $d = 1$ where $h(-1) = 1$ and $h(1) = \varepsilon \in (0, 1)$. Take the limit $\varepsilon \to 0$ to find the travelling wave (4.9).

**Problem 19.5** Prove Bénilan's result (19.43) by writing the equation satisfied by the function $w = u^{m-1/2}$ which has the form

$$w_t = \alpha(w)\Delta w + \beta(w)|\nabla w|^2,$$

for convenient functions $\alpha$ and $\beta$. Write now the equation satisfied by $p = |\nabla w|^2$, which has the form

$$p_t - A\Delta P - B \cdot \nabla p + Cp^2 \le 0.$$

Prove that $c/t$ is a supersolution for that differential inequality.

**Open problem** What is the best Hölder regularity for general non-negative solutions of the PME in the whole space? In other words, what is the worst possible example?

**Open problem** Investigate the analyticity of compactly supported solutions for large times. Study the analyticity of the free boundaries.

**Open problem** Lipschitz continuity should be true under weaker conditions than overflowing the initial ball. For instance, replace ball by convex set.

**Open problem** Prove or disprove the preservation of pressure concavity for the solutions of the PME in several space dimensions.

# 20

# ASYMPTOTIC BEHAVIOUR II. DIRICHLET AND NEUMANN PROBLEMS

This chapter contains a complete study of the large-time behaviour of solutions of the porous medium equation, $u_t = \Delta u^m$ with $m > 1$, posed in a bounded domain of the $n$-dimensional space with homogeneous boundary conditions. Asymptotic profiles are obtained and full proofs of the convergence results are given.

The study of the standard homogeneous Cauchy–Dirichlet problem, CDP, is performed in two sections of very different difficulty: Section 20.1 treats the theory for non-negative solutions, while Section 20.2 covers the general theory without a sign restriction. We use this study to exhibit some of the most common concepts and techniques used in establishing the asymptotic behaviour as $t \to \infty$ of solutions of nonlinear evolution equations. The main ideas involved are rescaling, existence of special solutions, a priori estimates, $\omega$-limits and Lyapunov functionals. The rescaled orbits converge to stationary states which solve a nonlinear elliptic problem. It is worth mentioning that the asymptotic behaviour of the initial Dirichlet problem for the PME is quite different from the classical heat equation, $u_t = \Delta u$, which is just the limit case of the PME when the exponent $m$ tends to 1. We devote some space to point out the main aspects of this comparison with emphasis in the way the nonlinearity is responsible for the new behaviour.

Section 20.3 covers the homogeneous Dirichlet problem posed in a tubular domain. This section contains more recent material and allows us to introduce the concepts of asymptotic simplification and logarithmic scales.

We address the long time behaviour of the homogeneous Neumann problem in Section 20.5. We show stabilization towards the average with exponential decay when the average is not zero, with power decay if the average is zero. This is yet another manifestation of the degeneracy of the equation at the level $u = 0$.

## 20.1 Large-time behaviour of the HDP. Non-negative solutions

We study the large-time behaviour of the solutions of the porous medium equation

$$\partial_t u = \Delta u^m, \quad m > 1. \tag{20.1}$$

Here is the setting: we consider non-negative solutions $u = u(x,t)$ posed in a bounded domain $\Omega \subset \mathbb{R}^n$ for $t \geq 0$. We make no special smoothness assumptions on $\Omega$. We take as initial data any non-trivial function

$$u(\cdot,0)=u_0\in L^1(\Omega),\quad u_0\ge 0, \tag{20.2}$$

with zero boundary data,

$$u=0 \qquad \text{on } \Sigma=\partial\Omega\times(0,\infty). \tag{20.3}$$

We have proved in Chapters 5–8 that this problem has a non-negative weak solution that is unique and depends continuously on the data in the $L^1(\Omega)$-norm. We also know that the solutions generate a semigroup of contractions in the space $X=L^1(\Omega)$, defined by the maps $S_t: u_0\mapsto u(t)$, $t>0$. We will use the abbreviated notation $u(t)$ to denote the function $u(x,t)\in L^1(\Omega)$ for fixed $t>0$ when no confusion arises. Weak solutions enjoy several properties that will appear below.

Our main asymptotic result shows that all non-trivial non-negative solutions have the same long-time behaviour in first approximation; in other words, we have a *universal asymptotic pattern.*

**Theorem 20.1** *There exists a unique self-similar solution of the PME of the form*

$$U(x,t)=t^{-\alpha}\,F(x),\quad \alpha=1/(m-1), \tag{20.4}$$

*such that if $u\ge 0$ is any weak solution of problem* (CD) *we have*

$$\lim_{t\to\infty} t^{\alpha}|u(x,t)-U(x,t)|=\lim_{t\to\infty}|t^{\alpha}\,u(x,t)-F(x)|=0, \tag{20.5}$$

*unless $u$ is trivial, $u\equiv 0$. The convergence is uniform in space and monotone non-decreasing in time. Moreover, the asymptotic profile $F$ is the unique non-negative solution of the stationary problem*

$$\Delta(F^m)+\alpha\,F=0 \qquad \textit{in}\ \ \Omega, \qquad F=0 \quad \textit{on}\ \ \partial\Omega. \tag{20.6}$$

*Explanation*

This result can be explained as follows: the homogeneous boundary condition $u=0$ forces the solutions to decay to zero with a precise rate. Moreover, they forget in first approximation all memory of the initial condition besides the information $u_0\ge 0$, $u_0\not\equiv 0$. The asymptotic shape and size are universal in this class.

## Remarks

(1) The function $U(x,t)$ is just the separate variables solution that we have announced in Section 4.2, formula 4.2, and constructed in Theorems 5.19, 5.20 as the Friendly Giant.

(2) The result is optimal in the sense that the exponent $\alpha$ in (20.5) cannot be improved and the profile function $F$ is uniquely determined. Of course, we

could improve the result by estimating the error $\varepsilon_u(t) = t^\alpha u(x,t) - F(x)$, see the subsection about rate of convergence below.

(3) $F$ is uniquely determined as a non-negative, weak solution of the elliptic problem, but in fact it is positive and $C^\infty$ smooth in $\Omega$, and continuous up to the boundary.

**Proof** It is divided into several steps.

1. Estimates. The main ingredients are two a priori estimates which are universally valid, i.e., valid (with the same constants) for the whole class of non-negative solutions under consideration. They are the $L^\infty$ bound

$$u(x,t) \le C\, t^{-\alpha}, \quad \alpha = \frac{1}{m-1}, \tag{20.7}$$

and the $u_t$ lower bound,

$$\partial_t u \ge -\frac{u}{(m-1)t}. \tag{20.8}$$

Proofs of these basic facts are given in Sections 5.8 and 8.1.

2. Rescaled orbit and equation. Suggested by the a priori estimate (20.7), we perform the change of variables called in the previous chapter *continuous rescaling* or *time-adapted rescaling*. It is given by

$$u(x,t) = t^{-\alpha}\theta(x,\tau); \quad t = e^\tau, \quad \alpha = \frac{1}{m-1}. \tag{20.9}$$

Then, $\theta$ satisfies the nonlinear reaction–diffusion equation

$$\partial_\tau \theta = \Delta(\theta^m) + \alpha\,\theta, \tag{20.10}$$

which is autonomous, i.e., time does not appear explicitly. Observe that the new time $\tau$ ranges from $-\infty$ to $\infty$. The initial time $t=0$ corresponds to $\tau = -\infty$, but displacing the origin of time $t$ allows us to take any finite initial time for $\tau$, like $\tau_0 = 0$ if the reader feels more comfortable. The location of the time origin does not alter the asymptotic problem and is then a question of convenience; precisely for this reason many authors use a slightly different definition, $t + 1 = e^\tau$, which makes $t = 0$ equivalent to $\tau = 0$. In any case, equation (20.10) would be the same.

We take zero Dirichlet boundary data, in the sense that $\theta^m(\cdot,\tau) \in H_0^1(\Omega)$. The initial data are taken non-negative and integrable in $\Omega$. The possibility of delaying the time origin and the regularity theory allow us to assume that $\theta(x,0)$ is bounded, even continuous.

3. Convergence. The advantage of the new variable is seen when we translate the estimate information in terms of $\theta$. We get:

$$0 \le \theta \le C, \tag{20.11}$$

where $C > 0$ is a universal constant, and

$$\partial_\tau \theta \geq 0. \tag{20.12}$$

They look much simpler! We conclude from this little, but very effective information, that for every $x \in \Omega$ there exists the limit

$$\lim_{\tau \to \infty} \theta(x, \tau) = f(x) \tag{20.13}$$

and this convergence is monotone non-decreasing. This means that the limit is non-trivial, $f(x) \not\equiv 0$. Estimate (20.11) shows that $f$ is bounded. Moreover, by the monotone convergence theorem we have

$$\theta(\cdot, \tau) \to f, \quad \theta^m(\cdot, \tau) \to f^m \tag{20.14}$$

with strong convergence in $L^1(\Omega)$. Since there is a uniform $L^\infty$-bound the convergence takes place in all $L^p(\Omega)$, $p < \infty$ (strong). Uniform convergence (i.e., in $L^\infty$) is also true but needs an extra argument, see point (7) below.

**Note** Convergence is usually obtained from compactness. Here the argument is based on pointwise monotonicity, which is simpler but seldom available.

4. The limit is a stationary solution. Multiply equation (20.10) by any test function $\phi(x) \in C_c^\infty(\Omega)$ and integrate in space, $x \in \Omega$, and time between $\tau_1$ and $\tau_2 = \tau_1 + T$ for a fixed $T > 0$. We have

$$\int_\Omega \theta(\tau_2)\, \phi\, dx - \int_\Omega \theta(\tau_1)\, \phi\, dx = \iint \theta^m\, \Delta\phi\, dx\, d\tau + \alpha \iint \theta\, \phi\, dx\, d\tau,$$

with double integrals in $\Omega \times (\tau_1, \tau_2)$. We keep $T$ fixed and let $\tau_1 \to \infty$. Then $\tau_2$ also goes to infinity, and the left-hand side tends to zero. The right-hand side converges to

$$T \int_\Omega f^m\, \Delta\phi\, dx + \alpha\, T \int_\Omega f\, \phi\, dx.$$

Dividing by $T > 0$ we thus get in the limit $\tau_1 \to \infty$

$$\int_\Omega f^m\, \Delta\phi\, dx + \alpha \int_\Omega f\, \phi\, dx = 0,$$

which is the weak formulation of the equation

$$\Delta f^m + \alpha\, f = 0. \tag{20.15}$$

As a limit of functions in $H_0^1(\Omega)$ we have $f \in H_0^1(\Omega)$. We recall that, as a monotone non-decreasing limit of non-trivial functions, $f$ is non-trivial. We conclude that $f = F$.

**Note** By now the smart reader will have discovered, maybe with some surprise, that we have produced a proof of existence of a solution of the stationary problem (20.15) by a **dynamical method**, via equation (20.10). At the same time, formula

$$U(x,t) = t^{-\alpha}\, F(x)$$

produces a self-similar solution in separated variables of the PME, the Friendly Giant we were looking for.

5. REGULARITY. Suppose we have a bounded solution $g = f^m \in H_0^1(\Omega)$ of equation

$$\Delta g + \alpha\, g^{1/m} = 0. \tag{20.16}$$

By elliptic estimates we then know that $\Delta g$ bounded implies $g \in W^{2,p}(\Omega)$ for all $p < \infty$, hence (Sobolev) $g \in C^{1,\varepsilon}$ for all $\varepsilon < 1$, and by iteration we have $g \in C^{3,\varepsilon}(\Omega)$, in particular it is a classical solution. Following the classical bootstrap argument in regularity theory, we get $g \in C^\infty(\Omega)$. The maximum principle implies that $g$ is strictly positive inside $\Omega$ unless it is identically zero (dynamical approach: start the evolution with data $u_0(x)$ which are continuous and positive at $x_0 \in \Omega$ to conclude that $g(x_0) > 0$). A barrier argument will prove that the solution is continuous up to the boundary. These are exercises in quasilinear parabolic theory that are left to the interested reader, cf. [357]. If $\partial\Omega$ is smooth, then further regularity is obtained at the boundary.

6. UNIQUENESS OF THE STATIONARY SOLUTION. Let us prove that the *non-negative* and *non-trivial* stationary solution is unique. If we have two stationary solutions of (20.6), $F_1$ and $F_2$, we can construct solutions of the PME of the form

$$U_1(x,t) = t^{-\alpha}\, F_1(x), \qquad U_2(x,t) = (t+s)^{-\alpha}\, F_2(x),$$

for some $s > 0$. $U_2$ has initial data $U_2(x,0) = s^{-\alpha} F_2(x)$. Since $U_1(x,0)$ is infinite everywhere we apply comparison to conclude that $U_2(x,t) \le U_1(x,t)$. The technical detail of the proof is as follows: by the $L^1$-dependence theorem for weak solutions of problem (CD) we know that

$$\int_\Omega (U_2(x,t) - U_1(x,t))_+ dx,$$

is decreasing in time. But this integral goes to zero as $t \to 0$ because $U_1(x,t)$ goes pointwise to infinity as $t \to 0$ (use the dominated convergence theorem). We conclude that $(U_2(x,t) - U_1(x,t))_+ = 0$ a.e. in $x$ for every $t > 0$. Using the form of $U_1$ and $U_2$, we get

$$F_2(x) \le \left(\frac{t+s}{t}\right)^{\alpha} F_1(x).$$

Letting $s \to 0$ we get $F_2(x) \le F_1(x)$. The converse inequality is similar.

7. Better convergence. We have established result (20.5) in the sense of $L^1(\Omega)$ convergence. The passage to uniform convergence depends on having better regularity for the solutions, i.e., on a compactness argument. It has been proved in Chapter 7 that uniformly bounded solutions of the PME are $C^\alpha$ continuous in space and time with uniform Hölder exponent and coefficients.

Consider now the second type of rescaling, that we may call *fixed-rate rescaling*

$$u_\lambda(x,t) = \lambda^\alpha u(x,\lambda t). \tag{20.17}$$

For every $\lambda > 0$ the function $u_\lambda$ is still a solution of the PME to which the a priori estimate (20.7) applies. Hence, in a set of the form $\Omega \times (1,2)$ this family is equicontinuous and by Ascoli–Arzelà it converges uniformly along a sequence $\lambda_j \to \infty$. Now, observe that

$$u_\lambda(x,1) = \theta(x,\,\log(\lambda)) \tag{20.18}$$

to conclude that $\theta(x,\,\log(\lambda_j))$ converges uniformly. Since the limit is fixed, $F$, the whole family $\theta(x,\tau)$ converges as $\tau \to \infty$ and (20.5) is proved. A detailed comparison of the two types of rescaling is done in Chapter 18 when studying the asymptotics of the Cauchy problem. This ends the proof of Theorem 20.1. ■

### 20.1.1 *Rate of convergence*

This is a main question in asymptotic theory that can formulated as follows: How fast do the solutions converge to the asymptotic states? Or in other words, to estimate the error that is committed by the first-order asymptotic approximation to a typical solution.

In our case, a simple comparison with the explicit solutions of the form

$$U_s(x,t) = (t+s)^{-\alpha} F(x), \quad s > 0, \tag{20.19}$$

allows us to establish a rate of convergence for suitable data.

**Theorem 20.2** *Assume that the initial data of a solution are positive and satisfy the estimate*

$$u_0(x) \ge \varepsilon\, F(x) \tag{20.20}$$

*for some $\varepsilon > 0$. Then we have the asymptotic estimate:*

$$|t^\alpha u(x,t) - F(x)| \le \frac{C}{t}, \qquad t >> 1, \tag{20.21}$$

*for any $C > \alpha\,\varepsilon^{-1/\alpha}$.*

The proof consists in sandwiching the solution between $U(x,t) = t^{-\alpha}F(x)$ from above and $U(x,t+s)$ for some $s > 0$ from below. We can write the result in the form

$$u(x,t) = t^{-\alpha}\, F(x) + O\,(t^{-\alpha-1}). \tag{20.22}$$

**Remarks**

(1) The question arises: Is the result (20.21) true for general data $u_0 \ge 0$ with some constant $C = C(u_0) < \infty$? The answer is positive if the boundary is regular. Indeed, Aronson and Peletier proved in [49] that when $\partial\Omega$ is compact of class $C^3$, then any solution satisfies an estimate of the form (20.20) *eventually in time*, i.e., for some large time $t_0$ instead of at $t = 0$. In particular, any free boundaries disappear in finite time. However, the result is false for bad boundaries: we have constructed infinitely long boundary waiting times for domains with corners in Subsection 14.3.3.

(2) A simpler condition that does not need a lengthy argument is as follows: if the domain is regular, the solution $g = F^m$ of the elliptic problem will have $C^1$ regularity up to the boundary, and $\partial g/\partial n$ will be continuous and positive on $\partial\Omega$. Then, a sufficient condition on the initial data for the theorem to hold can be formulated as

$$u_0^m \in C^1(\overline{\Omega}), \quad \text{with } \; u_0 > 0 \; \text{ in } \Omega \quad \text{and } \partial_n u_0^m > 0 \text{ on } \; \partial\Omega.$$

(3) Observe finally that an estimate of the form (20.21) with error $O(1/t)$ is optimal in this context, since the explicit solutions (20.19) reach that bound.

*Strict monotonicity*

Once we know that the solutions are smooth and positive in the interior for large times, the problem is no longer degenerate and the strong maximum principle applies to the equation satisfied by $v = \theta_\tau \ge 0$, to show that actually $\theta_\tau > 0$. The convergence is therefore strictly monotone.

### 20.1.2 *Linear versus nonlinear*

The linear equation $u_t = \Delta u$, and its nonlinear counterpart, the PME, have a number of properties in common and also striking differences. Among the first, let us mention that the solutions to both equations, posed in a bounded domain $\Omega$ with homogeneous Dirichlet data and initial data in $L^1(\Omega)$, belong to the space $u \in C([0,\infty) : L^1(\Omega))$. More precisely, both equations generate semigroups of order-preserving contractions in $L^1(\Omega)$ by the rule $S_m(t) : L^1(\Omega) \to L^1(\Omega)$ given by

$$S_m(t)u_0 = u_m(\cdot,t) = u_m(t),$$

where $u_m$ denotes the solution of the Cauchy–Dirichlet problem for the PME (resp. the HE if $m = 1$). Another common property is the fact that the solutions are bounded and continuous for all $t > 0$.

Differences appear at the level of qualitative behaviour and smoothness. While solutions of the HE are $C^\infty$-smooth, solutions of the PME have limited regularity. This is related to the degenerate parabolic character of the equation at the level $u = 0$ which causes the property of finite propagation and the appearance of interfaces, where the local description is rather hyperbolic than parabolic, as we have explained in [502] (the physical idea goes back to the work of [123] in groundwater infiltration). See Chapters 14 and 15.

In any case, when we pass to the limit $m \to 1$ the semigroup $S_m$ tends to $S_1$ in the sense that for fixed $u_0 \in L^1(\Omega)$ the orbit $S_m(t)u_0$ converges to $S_1(t)u_0$ in the norm of $C([0,T] : L^1(\Omega))$ for every $T > 0$ finite. The boundedness of the time interval is crucial. In fact, in the linear case $m = 1$ the large-time behaviour of the solutions of the (CD) problem with non-negative data $u_0 \in L^1(\Omega)$ is given by the formula

$$|\, e^{\lambda_1 t} u_1(x,t) - C\, f_1(x)| \to 0 \quad \text{as } t \to \infty, \tag{20.23}$$

uniformly in $\Omega$. Here, $\lambda_1 > 0$ is the first eigenvalue and $f_1(x) > 0$ the first eigenfunction of the Laplacian in $\Omega$, i.e., a positive solution of (20.15) with $m = 1$. $f_1$ can be normalized to have $L^2$-norm one and then $C$ is given by

$$C(u_0) = \int_\Omega u_0(x)\, f_1(x)\, dx. \tag{20.24}$$

The functional $C(u_0)$ contains the asymptotic information the equation remembers in a first approximation from the initial data. The asymptotic profiles form a linear, one-parameter family of functions, $Cf_1(x)$.

In contrast with this, we see that in the PME case there is one universal model of asymptotic behaviour for non-negative solutions and not a one-dimensional family. Therefore, no dependence on the initial data is felt at the level of first approximation; the corresponding functional $C(u_0)$ is constant. This can be interpreted in terms of the associated diffusion process as saying that the asymptotic behaviour depends only on the boundary outflow, the effect of initial data being reduced in first-order approximation to point out the class of solutions involved (non-negative solutions). In standard dynamical systems terminology, the **basin of attraction** of the profile $F$ for the rescaled flow (20.10) is the whole class of non-negative (non-trivial) data (or solutions).

On the other hand, the asymptotic approximant is an a priori bound for all solutions, an **absolute upper barrier**. This kind of *unilateral obstruction* is impossible in the linear theory. It affects for instance the theory of control, which has been scarcely treated to date for porous medium flows and does not seem to be easy.

If we look at the proof given above we see that it is based on monotonicity. Such a property does not pass to the limit $m \to 1$. Indeed, the corresponding statement (namely, that $u(x,t)e^{\lambda_1 t}$ is non-decreasing in $t$ for all times) is false in the linear case.

### 20.1.3 *On general initial data*

We have chosen initial data with two restrictions: non-negativity and integrability. Assuming for the moment the first one as natural, the fact that the giant assumes infinite initial data may worry some readers. We have explained in Subsection 13.5.2 the theory of initial traces by Dahlberg and Kenig [189] and concluded that the case of the giant is exceptional: if a non-negative weak solution is defined for $t > 0$ and is not the Giant, then it does accept an initial trace in the form of a double measure: there exist two non-negative Radon measures $\mu$ and $\lambda$ in $\Omega$ and in $\partial\Omega$ respectively (depending only on $u$) such that

$$\int_\Omega \delta(x)\, d\mu(x), \quad \int_{\partial\Omega} d\lambda(x) \quad < \infty, \tag{20.25}$$

and for any $\eta \in C^\infty(\mathbb{R}^d)$, $\eta = 0$ on $\Sigma$, we have

$$\lim_{t \searrow 0} \int_\Omega u(x,t)\eta(x)\, dx = \int_\Omega \eta(x)\, d\mu(x) + \int_{\partial\Omega} \frac{\partial \eta}{\partial \nu}\, d\lambda, \tag{20.26}$$

where $\nu$ is the unit outward normal to $\partial\Omega$ and $\delta(x)$ is the function distance to the boundary (in space). However, this generality does not affect the long time theory, since these more general solutions exist and become bounded and smooth for all positive times, falling in this way into our framework for $t > 0$ and producing the same asymptotics.

## 20.2 Asymptotic behaviour for signed solutions

We proceed a step further in mathematical sophistication and eliminate the sign restriction. We study the asymptotic behaviour of weak solutions of the signed PME equation

$$\partial_t u = \Delta(|u|^{m-1}u) \tag{20.27}$$

in the same domain $\Omega \subset \mathbb{R}$ with data $u_0 \in L^1(\Omega)$, without a sign restriction on $u_0$ or $u$. We take zero boundary data, $u(x,t) = 0$ on $\Sigma = \partial\Omega \times (0,\infty)$. This is the general homogeneous Dirichlet problem, which extends the previous section to signed solutions.

We recall that a weak solution of problem HDP is a function $u \in C([0,\infty) : L^1(\Omega))$ such that $|u|^m \in L^2((t_1,t_2) : H_0^1(\Omega))$ for every $0 < t_1 < t_2 < \infty$, equation (20.27) is satisfied in the sense of distributions, and $u(\cdot,t) \to u_0$ in $L^1(\Omega)$ as $t \to 0$. We know that for integrable initial data the weak solution of problem HDP exists, is unique and depends continuously on the data in the $L^1(\Omega)$-norm.

The method of study begins in a way similar to the non-negative case by a rescaled solution and the corresponding equation, plus a priori estimates. The fact that these estimates are poorer implies the need for more sophisticated methods to ensure convergence.

1. Rescaling. We introduce the same type of rescaled function (20.9) which satisfies an analogue of equation (20.10):

$$\partial_\tau \theta = \Delta(|\theta|^{m-1}\theta) + \alpha\,\theta. \tag{20.28}$$

2. Estimates and compactness. Estimate (20.7) is still true in the form

$$|u(x,t)| \le C\,t^{-\alpha},$$

which implies the uniform boundedness of the orbit

$$|\theta(x,\tau)| \le C. \tag{20.29}$$

But there is a marked difference with the non-negative case: the monotonicity estimate (20.8) does not hold anymore, so that we lose the fundamental argument in proving convergence. Estimate (20.8) is replaced by Bénilan and Crandall's second estimate [89]

$$\|\partial_t u(x,t)\|_1 \le \frac{\alpha}{t}\|u_0\|_1,$$

see (8.10), which also implies space regularity, $t\Delta(|u|^{m-1}u) \in L^\infty(0,\infty : L^1(\Omega))$. In terms of $\theta$ we have

$$\|\partial_\tau\theta - \alpha\,\theta\|_1 \le \alpha\,\|\theta_0\|_1, \qquad \|\Delta(|\theta|^{m-1}\theta)\|_1 \le \alpha\,\|\theta_0\|_1. \tag{20.30}$$

We also use the energy estimate; it is obtained by multiplication of (20.28) by $|\theta|^{m-1}\theta$ and integration by parts, which gives

$$\int_{\tau_1}^{\tau_2}\int_\Omega |\nabla(|\theta|^{m-1}\theta)|^2\,dx\,dt + \frac{1}{m+1}\int_\Omega |\theta|^{m+1}(x,t_2)\,dx$$
$$\le \frac{1}{m+1}\int_\Omega |\theta|^{m+1}(x,t_1)\,dx + \alpha\int_{\tau_1}^{\tau_2}\int_\Omega |\theta|^{m+1}\,dx\,dt,$$

which proves that $(|\theta|^m \in L^2(\tau_1,\tau_2 : H_0^1(\Omega))$ uniformly in $\tau_1$ if $\tau_2-\tau_1$ is bounded.

3. Dynamical systems approach. Concept of $\omega$-limit. This consists in viewing the solution as an orbit in a functional space and considering the points to which it accumulates as time goes to infinity.

**Definition** *The positive semi-orbit of a solution $\theta(x,\tau)$ starting at a time $\tau_0$ is the family*

$$\gamma(\theta;\tau_0) = \{\theta(\tau) : \tau \ge \tau_0\},$$

*where $\theta(\tau) = \theta(\cdot,\tau)$ is viewed as an element of a suitable space $X$ of functions in $\Omega$.*

Hopefully, $X$ will be a Banach space or a closed convex subset thereof. With the previous estimates the semi-orbit is a relatively compact subset of $L^1(\Omega)$, which can be taken as $X$. But compactness holds also in $L^p(\Omega)$ for all $p < \infty$

(since the semi-orbit is uniformly bounded in $L^\infty(\Omega)$). Hence, $L^p(\Omega)$ is also a good choice.

In any case, for every sequence $\tau_j \to \infty$ there is a subsequence along which

$$\theta(\tau_j) \to f \qquad \text{in } L^1(\Omega) \text{ strong},$$

and in every $L^p(\Omega)$, $p < \infty$, in the strong topology.

**Definition** *The set of all possible limits of a semi-orbit along sequences $\tau_j \to \infty$ is called the $\omega$-limit of the orbit,*

$$\omega(\theta) = \{f \in L^1(\Omega) : \exists \tau_j \to \infty \text{ and } \theta(\tau_j) \to f\}. \tag{20.31}$$

*The convergence takes place in the topology of the functional space $X$ in question, here any $L^p(\Omega)$, $1 \le p < \infty$ (strong).*

Actually, the regularity theory says that $\theta$ is locally compact in $C(Q)$, $Q = \Omega \times (0, \infty)$, but this result is not needed. An alternative way of writing this definition is

$$\omega(\theta) = \bigcap_{\tau_0 > 0} \overline{\bigcup_{\tau \ge \tau_0} \gamma(\theta; \tau)},$$

where the overline denotes closure. It is well known that the $\omega$-limit is a closed and connected set in $X$. With our estimates the orbits are relatively compact in $C([0, T] : L^1(\Omega))$, $T > 0$. We may consider the shifted orbits

$$\theta_j(\tau) = \theta(\tau + \tau_j)$$

and prove that the solutions $\theta_j(x, \tau)$ converge in $L^\infty(0, T : L^1(\Omega))$ for any fixed $T > 0$ to a function $h = h(x, \tau)$ which is again a weak solution of (20.28) with $h(0) = f \in \omega(\theta)$.

4. The Lyapunov method. In order to identify the limit we need further estimates for very large $\tau$. We proceed as follows. We multiply the equation by $\partial_\tau(|\theta|^{m-1}\theta)$ and integrate by parts to get the identity

$$\frac{d}{d\tau} V(\tau) = -I(\tau) \tag{20.32}$$

with

$$V(\tau) = \frac{1}{2} \int_\Omega |\nabla(|\theta|^{m-1}\theta)|^2 \, dx - \frac{\alpha m}{m+1} \int_\Omega |\theta|^{m+1} \, dx, \tag{20.33}$$

and

$$I(\tau) = m \int_\Omega |\theta|^{m-1} (\theta_\tau)^2 \, dx \ge 0. \tag{20.34}$$

$V$ is called a *Lyapunov functional.* It is clearly well defined on the orbits of our problem. It follows from (20.32) that $I(\tau)$ is integrable and

$$\int_{\tau_1}^{\tau_2} I(\tau)\,d\tau = V(\tau_1) - V(\tau_2),$$

which is bounded uniformly in $\tau_1$ and $\tau_2$ according to the a priori estimates. We conclude that the integral

$$\int_{\tau_1}^{\tau} I(\tau)\,d\tau$$

is convergent as $\tau \to \infty$ and that $V(\tau)$ has a limit as $\tau \to \infty$.

5. $\omega$-LIMITS ARE STATIONARY SOLUTIONS. Take now a sequence $\tau_j \to \infty$ as above such that $\theta(x, \tau + \tau_j) \to h(x, \tau)$. Take any test function $\phi(x) \in C_c^\infty(\Omega)$. Integrate equation (20.28) between $\tau_j$ and $\tau_j + T$ for a fixed $T > 0$. We have

$$\int_\Omega \theta(\tau_j + T)\,\phi\,dx - \int_\Omega \theta(\tau_j)\,\phi\,dx = \iint (|\theta|^{m-1}\theta\,\Delta\phi\, + \alpha\,\theta\,\phi)\,dx\,d\tau, \qquad (20.35)$$

with double integral in $(\tau_j, \tau_j + T) \times \Omega$. Let us show that the right-hand side converges. Indeed, for $0 \le s \le T$ we have

$$\begin{aligned}
&\int_\Omega |(|\theta|^{m-1}\theta)(\tau_j + s) - (|\theta|^{m-1}\theta)(\tau_j)|\,dx \\
&\le m \int_{\tau_j}^{\tau_j+s} \int_\Omega |\theta|^{m-1}|\theta_\tau|\,dx\,d\tau \\
&\le C \left( \int_{\tau_j}^{\tau_j+s} \int_\Omega |(\theta^{(m+1)/2})_\tau|^2\,dx\,d\tau \right)^{1/2} \left( \iint |\theta|^{m-1}\,dx\,d\tau \right)^{1/2} \\
&\le C\,T^{1/2}|\Omega|^{1/2} \left( \int_{\tau_j}^{\infty} I(\tau)\,d\tau \right)^{1/2}.
\end{aligned}$$

Since the last integral tends to zero as $\tau_j \to \infty$ we have

$$\int_\Omega |\{(|\theta|^{m-1}\theta)(\tau_j + s) - (|\theta|^{m-1}\theta)(\tau_j)\}\,\Delta\phi|\,dx \to 0,$$

uniformly in $s \in (0, T)$. A similar argument applies to the last integral in (20.35). Hence, the right-hand side converges as $\tau_j \to \infty$ to

$$T \int_\Omega |f|^{m-1} f\,\Delta\phi\,dx + \alpha\,T \int_\Omega f\,\phi\,dx,$$

where we have used the fact that $\theta(x, \tau_j) \to f(x)$. On the other hand, the left-hand side is bounded independently of $T$. Since $T$ is arbitrary we divide by $T > 0$

and let $T \to \infty$ to get

$$\int_\Omega |f|^{m-1} f \, \Delta\phi \, dx + \alpha \int_\Omega f \, \phi \, dx = 0,$$

i.e., $f$ is a solution of the stationary equation, which is just

$$\Delta(|f|^{m-1} f) + \alpha f = 0, \qquad (|f|^m) \in H_0^1(\Omega). \tag{20.36}$$

We easily conclude that $h(x,\tau) = f(x)$ for all $\tau > 0$. The step of better regularity provides for $f \in C(\overline{\Omega})$. Summing up, we have proved the following result.

**Theorem 20.3** *The $\omega$-limit of the rescaled orbit $\theta(x,\tau)$ of a solution of problem* HDP *consists of solutions of the elliptic equation* (20.36).

**Remark** It follows that signed solutions of the HDP for the PME decay like $u = O(t^{-1/(m-1)})$, just like positive solutions. This is in strong disagreement with what happens in the linear case, $m = 1$, where the time rates are given by $e^{-\lambda_i t}$, where $\lambda_i$ are the different eigenvalues of the Laplacian.

### 20.2.1 *Description of the $\omega$-limit in $d = 1$*

Now that we have identified the nature of the elements of the $\omega$-limit of the rescaled orbit, several interesting questions are posed: What do we know about the set of stationary solutions? Is the $\omega$-limit a single element? Is it non-trivial?

Under the restriction $\theta \ge 0$ the answer was clear since there was only one candidate so that the limit element was necessarily unique. In the more general circumstances of changing sign solutions the problem is not so easy. We will investigate here the situation in one space dimension.

**Lemma 20.4** *In one space dimension the set $\mathcal{S}$ of stationary solutions of* (20.36) *is a discrete set. Any solution is composed of a finite number of scaled copies with alternating signs of the positive solution of* (20.15)*, defined in subintervals of the same length.*

**Proof** Without loss of generality we take as $\Omega$ the interval $(0,1)$. In one dimension a function in $H_0^1$ is continuous, hence a solution of (20.36) is composed of different pieces defined in subintervals $I_i = (a_i, b_i)$ of $\Omega$ where it is either positive or negative. In every piece the solution is $C^\infty$ smooth. Moreover, integration of the equation shows that the function

$$h(x) = (|f|^{m-1} f)'$$

must have a limit as $x \to a_i$ or $b_i$. A simple scaling shows that such pieces are just rescaled versions of the unique positive solution of (20.15) defined in $\Omega = (0,1)$,

$$f_i(x) = l_i^\beta \, f((x - a_i)/l_i), \quad \beta = \frac{2}{m-1}, \quad l_i = b_i - a_i. \tag{20.37}$$

Let us examine now the situation at a point where two neighbouring intervals $I_i$ and $I_j$ meet, say, $b_i = a_j$. Then the validity of equation (20.36) in the sense of distributions implies that the derivatives on both sides must coincide

$$(|f|^{m-1}f)'(b_i-) = (|f|^{m-1}f)'(a_j+).$$

It is easy to see from (20.37) that this implies $l_i = l_j$. We conclude that neighbouring intervals must have equal lengths. There is still the possibility that an interval is surrounded by an infinite sequence of intervals whose lengths necessarily go to zero. This is excluded since along this sequence of intervals the derivative $h = (|f|^{m-1}f)'$ necessarily goes to zero, and this contradicts the matching of derivatives at the contact point. ■

**Note** Again we find illustrative the comparison with the case $m = 1$ where we find sine functions with period a fraction of 1.

Next, we discuss the properties of the Lyapunov functional. We recall that the orbit $\{\theta(\tau)\}$ not only has bounded $L^p$-norm and bounded $\|\nabla|\theta|^m\|_2$ norm but also $\Delta\theta^m = \theta^m_{xx}$ is bounded in $L^1(\Omega)$. In one space dimension this implies further regularity.

**Lemma 20.5** *The Lyapunov functional $V$ is continuous in the space*

$$Y = \{f \in L^1(\Omega) :\ (|f|^m)' \in L^2(\Omega),\ (|f|^{m-1}f)'' \in L^1(\Omega)\}.$$

An important conclusion follows.

**Lemma 20.6** (LaSalle's invariance principle) *$V$ is constant on the $\omega$-limit of an orbit.*

**Proof** This is based on the fact that $V$ is decreasing along the orbit, the continuity of $V$ and the definition of $\omega$-limit.

**Lemma 20.7** *For every $f$ in the $\omega$-limit*

$$V(f) = -\frac{\alpha\, m}{2(m+1)} \int_\Omega |f|^{m+1}\, dx. \tag{20.38}$$

Combining this information we conclude that the $\omega$-limit of an orbit must be a connected set in the space $L^1(\Omega)$, composed of stationary solutions such that the integral (20.38) is constant. Inspection of the list of stationary solutions shows that there are only a pair of solutions with the same integral, which differ in the sign. We conclude that the $\omega$-limit is a single element.

**Theorem 20.8** *In one space dimension for every weak solution of problem* HDP *there exists a solution of problem* (20.36) *such that as $t \to \infty$*

$$|t^\alpha u(x,t) - f(x)| \to 0, \tag{20.39}$$

*uniformly in $x \in \Omega$.*

*Infinitely many oscillations. Open problem*

An interesting question is the existence of solutions of the PME which have infinitely many space oscillations and conserve this property in time. In [505] we have constructed solutions in a half line $\{x > 0\}$ with value $u = 0$ at $x = 0$ and infinitely many oscillations near $x = 0$. It is an open problem to construct such solutions so that they satisfy a second boundary condition $u = 0$ at $x = a > 0$.

## 20.3 Asymptotics of the PME in a tubular domain

Our next goal is to study the large-time behaviour of the solutions of the porous medium equation posed in a tube. We have studied the existence of solutions for the HDP in a tube in Subsection 12.8.2 to which we refer for background and notation. The main point is to observe the effect on the previous theory of the combination of finite and infinite dimensions depending on the direction.

1. Estimate from above. We have proved in Theorem 12.22 that for every $u_0 \in L^1_{\rm loc}(\overline{\Omega})$, $u_0 \ge 0$, the solutions of problem THDP satisfy the universal a priori estimate

$$u(x,t) \le F(z)\, t^{-1/(m-1)}, \quad x = (y,z), \tag{20.40}$$

where $F$ is the profile of the Friendly Giant in dimension $d = N - 1$.

2. Comparison from below. In the study of the asymptotic behaviour we need a lower bound. This is obtained by comparing the solution $u(x,t)$ of our problem with the solution of the homogeneous Dirichlet problem (HDP) posed in a sequence of bounded domains

$$\Omega_k = (-k,k) \times D = \{(y,z) : z \in D, |y| < k\}$$

for $k > 0$. As initial data we may take the restriction of $u_0$ to $\Omega_k$. We can apply the previous proposition and show that the solutions of these problems $u_k(x,t)$ satisfy

$$u(x,t) \ge u_k(x,t)$$

and moreover, the family $u_k(x,t)$ is non-decreasing in $k$. As $t \to \infty$ we know that

$$t^\alpha u_k(x,t) \to F_k(x) \tag{20.41}$$

and the convergence is pointwise and monotone, cf. Section 20.1.

3. $\partial_t u$ estimate. We also have an estimate which is true for all problems with zero boundary data, independently of the form of the domain:

$$\partial_t u \ge -\frac{u}{(m-1)t}. \tag{20.42}$$

This estimate means that the function $u(x,t)\, t^\alpha$ is non-decreasing in time.

### 20.3.1 *Basic asymptotic result*

We show that the asymptotic behaviour of any solution in a tube is independent of the longitudinal direction in first approximation. It is given in a first approximation by the *reduced equation* $u_t = \Delta'(u^m)$ posed in the transversal space variables $z = (x_2, \dots, x_N)$. This is a case of *asymptotic simplification* in the sense of [503].

**Theorem 20.9** *Every weak solution $u$ of the Cauchy–Dirichlet problem* (P), *posed in $\Omega$, decays in time like $O(t^{-\alpha})$, with constant that depends on $D$. More precisely, we have the universal behaviour*

$$\lim_{t\to\infty} |t^\alpha u(x,t) - f(z)| = 0 \tag{20.43}$$

*uniformly on compact sets, of the form $[-k,k] \times \overline{D}$, $k > 0$; the asymptotic profile $f$ is the unique non-negative weak solution of the stationary problem in $D$:*

$$\Delta' f^m + \alpha f = 0 \qquad \text{in } \ D, \qquad u = 0 \quad \text{on } \ \partial D. \tag{20.44}$$

**Proof** We proceed in several steps:

1. Rescaled orbit and equation. Suggested by the above a priori estimate we perform the so-called *continuous scaling*, which is given by

$$u(x,t) = t^{-\alpha}\theta(x,\tau); \quad t = e^\tau, \quad \alpha = \frac{1}{m-1}. \tag{20.45}$$

We may repeat the considerations made in Step 3 of the proof of Theorem 20.1. Thus, $\theta$ satisfies the nonlinear reaction–diffusion equation

$$\partial_\tau \theta = \Delta\theta^m + \alpha\,\theta. \tag{20.46}$$

We take zero Dirichlet boundary data, in the sense that $\theta^m \in H_0^1(\Omega)$. The initial data are taken non-negative and integrable in $\Omega$. By a time shift we may assume that $\theta(x,0)$ is bounded, even continuous.

2. Convergence. We now translate the estimate information in terms of $\theta$ to the estimates

$$0 \le \theta \le C, \tag{20.47}$$

where $C > 0$ is a universal constant, and

$$\partial_\tau \theta \ge 0. \tag{20.48}$$

We conclude from this information that for every $x \in \Omega$ there exists the limit

$$\lim_{\tau\to\infty} \theta(x,\tau) = L(x), \tag{20.49}$$

and the convergence is *monotone non-decreasing*. This means that the limit is non-trivial, $L(x) \not\equiv 0$. It must be noted that this limit may depend on $u$, i.e., in principle $L(x) = L_u(x)$. Estimate (20.47) shows that $L$ is bounded, and moreover

$$F_k(x) \le L(x) \le F(z). \tag{20.50}$$

We note that by Levi's monotone convergence Theorem, and using the fact that there is a uniform $L^\infty$-bound, we have

$$\theta(\cdot,\tau) \to L, \quad \theta^m(\cdot,\tau) \to L^m \tag{20.51}$$

with strong convergence in $L^1(\Omega)$, and also in all $L^p(\Omega)$, $p < \infty$ (strong).

**Remark** Uniform convergence (i.e., in $L^\infty$) does not take place because of the 'tail at infinity' that we will study separately.

3. THE LIMIT IS $f$. We only have to prove that the limit of the monotone sequence

$$\lim_{k\to\infty} F_k(x) = F(x),$$

does not depend on $y$. Indeed, if this is so, then $F$ satisfies the limit equation

$$\Delta F(x) + \alpha\, F(x) = \Delta' F(z) + \alpha\, F(z) = 0,$$

in the weak sense in $D$ with zero boundary conditions. By the uniqueness of such solutions for the *reduced problem* we conclude that the limit is the profile $F(z)$. The proof is then finished.

Let us then prove that $F$ does not depend on $y$. This can be done as follows: take a number $a > 0$ and consider the profiles $F_{k,a}$ of the Cauchy–Dirichlet problems posed in the domains

$$\Omega_{k,a} = (-k + a, k + a) \times D.$$

It is clear that $F_{k,a}(y, z) = F_k(y - a, z)$. On the other hand, since $(-k + a, k + a) \subset (-k - a, k + a)$ we have

$$F_{k,a} \le F_{k+a,0} \quad \text{in} \quad \Omega_{k,a}.$$

Hence, passing to the limit when $k \to \infty$:

$$F(y - a, z) \le F(y, z) \quad \forall (y, z), \forall a > 0. \tag{20.52}$$

The same argument applies with $a < 0$. We conclude that $F(y - a, z) = F(y, z)$, hence $F$ is a function of $z$ only. ■

### 20.3.2 *Lateral propagation. Logarithmic speed*

We now come to the more interesting issue, namely, proving that the influence of the infinitely long open tube is felt mostly in the propagation of the free boundary along the longitudinal tube direction. More precisely, we want to show that this propagation takes place with a logarithmic rate, $d(t) \sim \log(t)$.

In order to study the lateral propagation along the tube we assume that the initial data vanish on one of the ends, e.g.,

$$u_0(y, z) = 0 \quad \text{for all} \;\; y \ge d_0. \tag{20.53}$$

By lateral translation we may take any value of $d_0$. The property of finite propagation of the PME implies that the solution $u(x,t)$ will vanish for any positive time $t>0$ past a certain distance $d(t)=\sup\{y : u(y,z,t)>0 \text{ for some } \ y\in D\}$. We want to estimate $d(t)$.

**Theorem 20.10** *Under the above assumptions we have* $d(t)\sim \log(t)$ *as* $t\to\infty$ *in the sense that there are constants* $c_1,c_2>0, K_1,K_2>0$, *such that*

$$c_1\log(t)-K_1\le d(t)\le c_2\log t+K_2 \tag{20.54}$$

*holds for all large* $t$.

**Proof** We work with the rescaled variable $\theta$, for which we know that there is convergence towards $F(z)$ uniformly on compact sets. Moreover, we know that $\theta(y,z,\tau)\le F(z)$ everywhere in $Q$. Assume that $\theta(y,z,0)$ vanishes for $y\ge d_0$. We consider as comparison function the rescaled solution $Z(y,z,t)$ of problem THDP with data

$$\varphi(x,0)=H(x)$$

where $H(x)=f(z)$ when $y\le 0$, and is zero for $y>0$. An easy comparison shows that

$$\theta(y+d_0,z,\tau)\le Z(y,z,\tau),\ \ \forall\tau>0.$$

By the finite propagation property, there is a time $\tau_1$ such that the $Z(\cdot,\tau_1)$ vanishes for $y\ge 1$. This means that $\theta(x,\tau_1)$ vanishes for $y\ge d_0+1$. Besides,

$$\theta(y+d_0+1,z,\tau_1)\le H(x).$$

We can apply the argument inductively to get

$$\theta(y+d_0+n,z,n\tau_1)\le H(x),$$

so that $\theta(\cdot,n\tau_1)$ vanishes for $y\ge d_0+n$. This is equivalent to the upper bound in (20.54).

**Lower bound** The argument is similar but now we use as comparison function the rescaled solution with initial data

$$K(x,L)=(1-\varepsilon)\,F(z)\qquad\text{for all } |y|\le L,$$

and zero otherwise. By the convergence result, $Z(x,\tau)$ evolves on compact sets towards $f(x')$. This, together with the expansion properties of the support and the boundary behaviour imply that there exists $\tau_1>0$ such that

$$Z(y,z,\tau_1)\ge K(L+1,z),$$

so that it does not vanish for $|y|\le L+1$. Hence, by induction

$$Z(y,z,n\tau_1)\ge K(z,L+n).$$

Using again the asymptotic properties for general solutions, there exist $\tau_0, L > 0$ such that

$$\theta(x, \tau_0) \geq K(x, L).$$

Putting both estimates together via the maximum principle we know that there exists $\tau_0$ such that $u$ is positive for all $|y| < L + n$ and $\tau > n\tau_1 + \tau_0$. This gives the lower estimate. ∎

*Travelling wave propagation*

In order to get a more detailed picture of the behaviour near the leading front we use a moving coordinate along the tube axis with $\log(t)$ displacement. Therefore, we write the rescaled variable in the moving frame as

$$u(y, z, t) = t^{-\alpha} v(y - c\tau, z, \tau), \tag{20.55}$$

where $\tau = \log(t + 1)$ as before. This is a shifted variant of the scaling that produced $\theta$, which is the case $c = 0$. The function $v(s, z, \tau)$ satisfies the autonomous equation

$$\partial_\tau v = \Delta v^m + c\,\partial_y v + \alpha v. \tag{20.56}$$

**Travelling waves** Standard TWs are obtained by assuming that $v$ is independent of $z$ and $\tau$. We get the equation for $v = V(s)$

$$(V^m)'' + c\,V' + \alpha\,V = 0. \tag{20.57}$$

Such travelling waves are not going to adjust in an exact way to the behaviour of the solutions of problem THDP but they can serve as super- and subsolutions, and this is exploited in [511]. However, there is a class of travelling waves along the $y$ axis that is not homogeneous along the $x'$ directions. We give here the main points of the analysis of paper [516]. We look for travelling waves for equation (20.56) of the form

$$v(y, z, \tau) = f(y - c\tau, z). \tag{20.58}$$

The TW profile $f$ must then be a solution of the stationary equation

$$\Delta' f^m + f + c\,\partial_y f = 0 \tag{20.59}$$

for $(y, z) \in \Omega$. In our problem setting, we take Dirichlet conditions

$$f = 0 \quad \text{on } \Sigma. \tag{20.60}$$

We want to prove that there exists a value of $c > 0$ such that a travelling wave exists that joins the level $v = 0$ at the right-hand end of the tube $y = \infty$, with the value

$$v(-\infty, z, t) = F(z) \tag{20.61}$$

on the other end, $y = -\infty$. In terms of $f$, this means that at the ends of the tube we have

$$\lim_{y\to-\infty} f(y,z) = F(z), \qquad \lim_{y\to+\infty} f(y,z) = 0, \quad x' \in D. \tag{20.62}$$

The existence of a travelling wave with a certain minimal speed is reflected in the following result.

**Theorem 20.11** *There exists a speed $c_* = c_*(m,\Omega) > 0$ for which Problem* (20.59)–(20.62) *has a weak solution that is continuous and monotone in the $y$ variable. Moreover, the support of the constructed profile $f$ is bounded at the right end of the tube by a free boundary of the form $y = S(z)$. This solution is minimal along all possible TWs travelling to the right end of the tube.*

This reminds us of the famous KPP (Kolmogorov–Petrovski–Piskunov) phenomenon of reaction diffusion equations. The fact that these travelling waves are the desired asymptotic objects is reflected in the following results.

**Theorem 20.12** *Let $v_0(y,z) \ge 0$ have bounded support on the region $y > 0$ and behave as $y \to -\infty$ like*

$$v_0(y,z) \ge \varepsilon F(z) \tag{20.63}$$

*for some $\varepsilon > 0$ and all $y \le y_0$. Then, for all large $\tau$ we have two constants $L_1, L_2$ such that*

$$f(y - c_*\tau + L_2, z) \le v(y,\tau) \le f(y - c_*\tau + L_1, z), \tag{20.64}$$

*where $f$ is the profile of the minimal travelling wave of Theorem* 20.11.

**Theorem 20.13** *The solution with compactly supported initial data travels for large times both to the right and to the left with speed $c_*$. If $y = S_1(z,\tau)$ and $y = S_2(z,\tau)$ are the free boundaries on both sides, we have*

$$\lim_{\tau\to\infty} \frac{S_2(z,\tau)}{\tau} = -\lim_{\tau\to\infty} \frac{S_1(z,\tau)}{\tau} = c_*. \tag{20.65}$$

For the proofs of these results we refer to [516].

## 20.4 Other Dirichlet problems

*The exterior Dirichlet problem*

Let $G \subset \mathbb{R}^d$ be a bounded open set with smooth boundary and let $\Omega = \mathbb{R}^d \setminus \overline{G}$. We do not assume $G$ to be connected, so that it may represent one or several holes in an otherwise homogeneous medium. The goal is to study the large-time behaviour of the solution to the PME in that exterior domain with given boundary data

$$\begin{cases} u_t = \Delta u^m, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = g(x,t), & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{20.66}$$

where $m > 1$. We assume that the initial data $u_0$ are in $L^1(\Omega)$, non-negative in $\Omega$ and compactly supported in $\overline{\Omega}$. The boundary data are non-negative and constant in time for simplicity. We are interested in studying the large time behaviour of the solution and the free boundary, $\Gamma(t) = \partial\{x \in \Omega : u(x,t) > 0\} \setminus \partial\Omega$. Due to the $L^1$-$L^\infty$ regularizing effect, we may assume without loss of generality that $u_0 \in L^\infty(\Omega)$.

The one-dimensional problem has been studied by Atkinson and Peletier [54] for constant boundary data $g > 0$ and by Kamin and Vázquez [326] for $g = 0$. In the first case convergence is proved to a self-similar solution of the Polubarinova type, $U(x,t) = F(x\,t^{-1/2})$ and the interface grows accordingly like $\zeta(t) = O(t^{1/2})$. In the case of homogeneous data, the problem is identified after an antisymmetric extension as the PME equation with changing-sign initial data with zero mass, $\int_{\mathbb{R}} u_0 dx = 0$. The solution is shown to converge uniformly to a self-similar antisymmetric profile, a so-called dipole solution, introduced by Barenblatt and Zel'dovich in [72]. More precisely,

$$\lim_{t\to\infty} t^\alpha |u(x,t) - U_{dip}(x,t)| = 0,$$

uniformly in $\mathbb{R}$, where $U_{dip}(x,t) = t^{-\alpha}F(xt^{-\beta})$ is the dipole solution of Section 4.6.2 with $\alpha = 1/m$ and $\beta = 1/2m$.

The asymptotic behaviour of the solution to the problem in dimensions $d \geq 2$ was studied by Quirós and Vázquez [443] in the case of non-trivial boundary data, $g = g(x) \geq 0$. The analysis leads to a problem of *matched asymptotics* with and inner limit described in the standard variables $(x,t)$ where the solution stabilizes to $H^{1/m}$, where $H(x)$ is a harmonic function in the exterior domain decaying at $x = \infty$ with boundary data $g$ on $\partial G$. The outer behaviour is described in rescaled variables $(\eta, t)$ where

$$\eta = x\,t^{-\beta_1}, \qquad \beta_1 = \frac{m}{d(m-1)+2}, \tag{20.67}$$

and in this frame the asymptotic behaviour is given by a self-similar solution of the PME which is singular at $x = 0$. This outer behaviour allows us to locate the free boundary at a distance

$$|x(t)| \sim C\,t^{\beta_1}. \tag{20.68}$$

for all large times. These formulas hold for $d > 2$. In dimension $d = 2$ the analysis is very sophisticated due to the absence of a suitable self-similar solution to represent the outer behaviour after a scaling by a power of $t$; actually, logarithmic factors appear and the free boundary location is

$$|x(t)| \sim C\, t^{1/2} \,(\log t)^{-(m-1)/(2m)}. \tag{20.69}$$

We see from this analysis that the inner limit is completely determined by $g$ and the matching is needed in order to properly describe the outer limit. Notice finally that for $d = 1$ there was no need to consider the outer and the inner region separately, since there was a global approximation.

We consider now the problem with data $g = 0$ on $\partial G$ for $d \geq 2$. This problem has been formally analysed by King [338] considering radially symmetric solutions; a full analysis is due to Gilding and Goncerzewicz [265] and Brändle et al. [124]. Following the latter work, for $d > 3$ an inner and an outer analysis are performed for $d > 2$ with outer scale

$$\eta = x\, t^{-\beta_0}, \qquad \beta_0 = \frac{1}{d(m-1)+2} \tag{20.70}$$

and this scale marks the location of the free boundary for large times. The reader will notice that it is the same rate as in the free flow in the whole space. i.e., the ZKB solution. In the same direction, it is proved that the solution decays in time like $u = O(t^{-\alpha})$ with $\alpha = d\beta$. Actually, a more precise result says that even though the zero boundary conditions imply that some mass is lost through the inner boundary $\partial G$, the asymptotic limit of the remaining mass is still positive, $0 < M_\infty < M_0$ and the solution $u(x,t)$ converges as $t \to \infty$ to the Barenblatt solution with mass $M_\infty$ in the sense of Theorem 18.1. Paper [124] also computes the loss of mass in terms of $u_0$ and a certain *capacity* of the hole $G$. In order to complete the study we must also consider what happens in the region near the holes (the inner limit). The scaling in this case is simpler: we only have to amplify the solution, keeping the space variable fixed, and then the rescaled function $v_{\rm in}(x,t) = t^\alpha u(x,t)$ converges to a stationary state, $H(x)$ that solves

$$\Delta H = 0, \quad x \in \Omega, \qquad H = 0, \quad x \in \partial\Omega, \qquad H \to C, \quad |x| \to \infty \tag{20.71}$$

for a certain constant $C$ that is adjusted through matching with the Barenblatt function which gives the outer behaviour. The analysis in $d = 2$ implies the presence of some damping factors in $\log(t)$. The main difference with the case $d > 3$ is that now the asymptotic mass is $M_\infty = 0$ like in $d = 1$. But in $d = 1$ there is no need to consider the outer and the inner region separately.

*Problems in quadrants and sectors*

Bonafede et al. [120] study the qualitative properties of the solutions of the Cauchy problem for the PME $u_t = \Delta u^m$, $m > 1$, posed in $D = {\mathbb{R}^d}_k \times (0,\infty)$, where the space domain is a quadrant

$${\mathbb{R}^d}_k = \mathbb{R}^d \cap \{x_1, \ldots, x_k > 0\}, \quad 1 \leq k \leq d,\ d \geq 1,$$

with initial data $u(x,0) = u_0(x)$ on $\mathbb{R}^d{}_k$, and Dirichlet data $u(x,t) = 0$ on $\partial\mathbb{R}^d{}_k \times (0,\infty)$. Assuming that $u_0$ is compactly supported and setting

$$\zeta(t) = \sup\{|x| : \ x \in \operatorname{supp}(u(\cdot,t))\},$$

the following bounds of $\zeta(t)$ are found:

$$\gamma_1 \mu_k(0)^{\sigma(m-1)} t^\sigma \le \zeta(t) \le \gamma_2 \mu_k(0)^{\sigma(m-1)} t^\sigma$$

for all $t > T_0$, where $\gamma_1$, $\gamma_2$ are positive constants, $\mu_k(0)$ is the moment of $u_0(x)$, $\sigma = 1/((d+k)(m-1)+2)$, and $T_0$ is a sufficiently large constant.

This result generalizes the asymptotics for the half line $I = (0,\infty)$ with $u = 0$ at $x = 0$ that has been proved by Kamin and Vázquez [326] (this is the case $d = k = 1$) using the dipole solution and the problem in a half space (case $d = k > 1$) studied by Hulshof and Vázquez [299]. In those cases the asymptotic estimate holds from below for the whole interface along cones non-tangential with the boundary and stabilization is proved towards a self-similar solution (a dipole). Similar results should be true for the case of general quadrants.

Andreucci et al. study in [18] the large time behaviour of solutions to the Neumann problem for quasilinear second order degenerate parabolic equations in domains with non-compact boundary.

*GPME*

The case of superslow diffusion is studied by Kersner, Galaktionov and Vázquez in [248], see also Chapter 3 of the book [255]. The equation is $u_t = \Delta(e^{-u})$, with Dirichlet boundary conditions and stabilization to a separate variables solutions is true for the new variable $v = e^{-1/u}$, also used in the Cauchy problem described in Subsection 18.10.3.

*Eventual concavity*

The question of eventual concavity of the solutions of the homogeneous Dirichlet problem in a bounded domain is studied by Lee and Vázquez in [366]. The precise result says that $u^{(m-1)/2}$ is eventually concave in the space variable if the domain is bounded and convex. As a consequence, all superlevel sets are convex subsets of $\Omega$.

**Theorem 20.14** *Let $\Omega$ be a convex bounded domain and let $u_0$ be a non-negative and integrable initial function. Then, if $u(x,t)$ is the solution of the PME there is a time $T(u_0,\Omega,m)$ such that for $t \ge T$ the function $u^{(m-1)/2}$ is strictly concave in the space variable.*

## 20.5 Asymptotics of the Neumann problem

We consider now the solutions of the homogeneous Neumann problem, HNP, for the PME $\partial_t u = \Delta(|u|^{m-1}u)$ posed in a bounded domain $\Omega \in \mathbb{R}^d$. This problem has been studied in Chapter 11. The initial and boundary conditions are given by (11.2) and (11.3). We assume that $u_0 \in L^1(\Omega)$, and no sign assumption is made. Existence and uniqueness of a weak energy solution is proved in Section 11.2. The uniform continuity of the solutions is proved in Chapter 7. Let us also recall the mass conservation law,

$$M(u_0) = \int_\omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx$$

that plays a role in what follows.

We are interested here in the asymptotic behaviour. The main result says that solutions of this problem stabilize towards a homogeneous state, which is given by the average of the initial data

$$a(u_0) = \frac{1}{|\Omega|}\int_\Omega u_0(x)\,dx. \tag{20.72}$$

We want to prove that result with a rate of convergence. The rate depends on the value of $a(u_0)$.

### 20.5.1 *Case of zero mass*

The most peculiar case corresponds to average zero (i.e., mass zero), i.e., $\int u_0\,dx = 0$. This case has been already studied in Chapter 11. We recall the result of Theorem 11.9.

**Theorem 20.15** *For solutions with mass $M = 0$ the $L^\infty$ bound takes the form*

$$|u(x,t)| \le C_1(m,\Omega)\,t^{-1/(m-1)}. \tag{20.73}$$

*The result is sharp.*

Therefore, the approach to equilibrium takes place with a power rate in time. The sharpness of the result is shown by exhibiting changing sign solutions that have the separate variables form

$$u(x,t) = t^{-1/(m-1)}F(x).$$

Existence of an infinite family of such solutions is proved in [8], and we refer the reader to that paper.

### 20.5.2 *Case of non-zero mass*

In this case there is an exponential rate of stabilization. We need to prove two facts: first the stabilization and then the rate. Here is the full result.

**Theorem 20.16** *Let $u$ be a solution of problem* HNP *with $u_0 \in L^1(\Omega)$. As $t \to \infty$ we have*

$$\lim_{t\to\infty} |u(x,t) - a(u_0)| = 0. \tag{20.74}$$

*The limit is uniform. Moreover, if $a(u_0) \neq 0$, then there exist constants $\sigma_0 > 0$ depending on $m$ and $d$, and $K = K(u_0)$ such that*

$$\|u(x,t) - a(u_0)\|_\infty \le K\, e^{-\sigma t} \tag{20.75}$$

*with $\sigma = \sigma_0\, |a(u_0)|^{m-1}$.*

**Proof** (i) In the first step we prove convergence towards the average. By the property of $L^1$-contraction we may assume that $u_0$ is bounded and smooth. Since the case $M = 0$ is already known, and the equation is symmetric under sign change, we may assume that the mass is positive, $M > 0$.

We use some of the dynamical systems ideas of Section 20.2. It is clear that the orbit $u(t)$ is bounded and continuous with a modulus of continuity, hence compact. In order to study the $\omega$-set of the orbit $u$, we argue as follows. Along any sequence $t_n \to \infty$ we may take a subsequence such that $u(x, t_{n_k} + t)$ converges on compact time intervals to a continuous and bounded function $w(x,t)$. Conservation of mass implies in the limit that

$$\int_\Omega w(x,t)\, dx = M(u_0) \quad \forall t > 0.$$

In order to analyse what is $w$, we use the energy, $E(u(t)) = \int_\Omega |u|^{m+1}(t)\, dx$ as a Lyapunov function. Arguing in the usual way, we get for every $h > 0$

$$\int_\Omega |u(t_n)|^{m+1}\, dx - \int_\Omega |u(t_n + h)|^{m+1}\, dx = \int_{t_n}^{t_n+h} \int_\Omega |\nabla |u|^m|^2\, dxdt, \tag{20.76}$$

which proves that $E(u(t))$ is a non-increasing function of time, and that

$$I(t) = \int_t^\infty \int_\Omega |\nabla |u|^m|^2\, dxdt < \infty,$$

so that it goes to zero as $t \to \infty$. We now take a smooth test function $\zeta(x,t)$ with support in $\Omega \times [0,h]$ and use the equation to write

$$\int_\Omega u(x, t_n + t)\, \zeta_t\, dx = \int_{t_n}^{t_n+h} \int_\Omega \nabla(|u|^{m-1}u) \cdot \nabla\zeta\, dxdt.$$

We obtain the estimate

$$|\int_\Omega u(t_n + h)\zeta_t\, dx| \le C\, h^{1/2} I(t)^{1/2}.$$

This proves that the limit $\int w(x,t)\zeta_t\, dx = 0$ for every test function $\zeta$. It follows that $w$ does not depend on time. But $w$ satisfies the PME, hence

$$\int_\Omega \nabla |w|^{m-1} w \Delta\zeta\, dx = 0$$

for every test function of the Neumann problem, which implies that $w$ is a constant. Then, conservation of mass fixes the constant. It follows that the $\omega$-limit has only one point, and this point must be $a(u_0)$.

(ii) THE RATE. The scaling $u(x,t) = a\,\widetilde{u}(x, a^{m-1}t)$ allows us to work in the case where $a(u_0) = 1$. The uniform convergence of the previous step implies that for all $t$ large enough, $u(x,t) > 0$, hence the solution is smooth and classical and the equation non-degenerate. We can then use linearized analysis to show that $u - a(u_0)$ decays like the solutions of the heat equation with zero mass. See Problem 20.6. ■

*More general equations*

We can consider the GPME equation $\partial_t u = \Delta\Phi(u)$ instead of the PME. Suppose that $\Phi$ is $C^2$ smooth and $\beta = \Phi^{-1}$ satisfies condition (7.6). The initial and boundary conditions are given by (11.2) and (11.3). We assume that $u_0 \in L^\infty(\Omega)$, and no sign assumption is made. Existence and uniqueness of a weak energy solution is proved in Section 11.2. The uniform continuity of the solutions is proved in Chapter 7. Under such conditions that stabilization theorem can be proved and the exponential rate proved when $M \neq 0$.

*Nonlinear boundary conditions*

Mazón and Toledo [386] study the large-time behaviour of solutions of the filtration equation in bounded domains, namely $u_t = \Delta\Phi(u)$ in $\Omega \times (0,\infty)$, $-\partial\Phi(u)/\partial\eta \in \beta(u)$ on $\partial\Omega \times (0,\infty)$, $u(x,0) = u_0(x)$ in $\Omega$. Equations of this sort arise in many applications. The main tools used here are the accretivity results of Bénilan and the invariance principle of C. Dafermos about lower semicontinuous Lyapunov functionals.

Igbida proves [304] that: (i) given $u_0 \in L^1(\Omega)$, the corresponding generalized solution $u$ to the problem converges to a stationary solution $\underline{u}$ as $t \to +\infty$. Since (IBVP) may have several steady states, the remainder of the paper is devoted to characterizing $\underline{u}$ in terms of $u_0$ under additional assumptions on $\varphi$ and $\gamma$ (more specifically, $\gamma(r) = \alpha r$, $r \in \mathbb{R}$, for some $\alpha \in [0,+\infty]$). In particular, $\underline{u}$ can be characterized in terms of the solution to an elliptic problem closely related to the so-called ‘mesa problem’ for the porous medium equation.

## 20.6 Asymptotics on compact manifolds

There is no difficulty in adapting the arguments of the Neumann problem to the large time behaviour of the PME on a compact manifold without boundary. Theorems 20.15 and 20.16 are true. The constant in the estimate of the former one gives information on the manifold.

## Notes

**Section 20.1.** The material of this section is taken from the survey paper [511] where further information is given. Thus, the monotonicity method can be modified into a partial monotonicity argument, as suggested in the book [469], Chapter II.4.1.

**Section 20.2.** Also taken from [511]. The Lyapunov argument follows essentially Langlais and Phillips [358]. The technique applies to equations of the form $u_t = \Delta u^m + f(x,u)$, with suitable assumptions on $f$.

**Section 20.3.** This section is a novel study that allows to introduce the concepts of asymptotic simplification and logarithmic scales. The material is developed in more detail in [511] and [516].

Note that the rescaled solutions with $\tau = \log(t)$ fall into the category of eternal solutions for equation (20.10), while the original $u$ was defined only forward in time.

**Section 20.5.** The homogeneous Neumann problem was studied by Alikakos and Rostamian [8], 1981. Our proofs are considerably simplified in our version.

**Section 20.6.** This subject seems to be new.

## Problems

**Problem 20.1** Complete the propagation analysis of Subsection 20.3.2.

**Problem 20.2** Use the asymptotic behaviour of Theorem 20.1 to prove that the possible waiting times of a solution of the HDP for the PME at points of $\Omega$ are necessarily finite. Prove by comparison that the same is true for every non-negative solution of the PME defined in $Q = \Omega \times (0,\infty)$, independently of the boundary conditions.

**Problem 20.3**$^*$ Study the HDP in one half of a tubular domain, i.e., $\Omega = S \times (0,\infty)$.

**Problem 20.4**$^*$ Study the HDP in a tubular domain with changing sign data.

**Problem 20.5** Study the HDP in a tube of higher codimension, i.e., $\Omega = S \times \mathbb{R}^k$, where $S \subset \mathbb{R}^{d-k}$ and $k > 1$.

**Problem 20.6** Complete the linearized analysis at the end of Theorem 20.16.

*Hint*: See [8], page 767.

**Problem 20.7** Justify the assertions of Subsection 20.5.2 and Section 20.6.

# COMPLEMENTS

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# 21

# FURTHER APPLICATIONS

This chapter complements Chapter 2. We present a collection of new examples taken from different branches of science. We should bear in mind that in many of the applications the PME or the GPME is only a first approximation to a more accurate physical model and it is acceptable in the whole or a part of the region of interest.

The first examples come from fluid dynamics, starting by the well-known model of viscous droplets spreading by gravity. We cover then topics from underground flows important for instance in water management or oil recovery. We give attention also to models of plasma physics. Limits of particle models are also treated. nonlinear diffusion is also a tool in modelling semiconductors and in image processing.

## 21.1 Thin liquid film spreading under gravity

The unsteady creeping motion of a thin sheet of a very viscous liquid as it advances over an insulating flat bed is an illuminating example of fluid obeying a nonlinear diffusion equation in a conveniently simplified representation. Suppose that a film of viscous fluid spreads over a horizontal plane. If the film is thin enough we can simplify the Navier–Stokes equations using the approximation of lubrication theory. In order to simplify the presentation we assume that the problem depends on two space dimensions $(x, y)$, the $x$ axis goes in the direction of the horizontal bed and $y$ points in the vertical direction; the third dimension is forgotten by symmetry. We write the equations for velocity $\mathbf{v} = (u, v)$ and pressure $p$ as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}, \qquad \frac{\partial p}{\partial y} = -\rho\, g. \tag{21.1}$$

The first is conservation of mass for an incompressible fluid, the other two the simplified Navier–Stokes dynamics, forgetting inertial terms. We will put the constant viscosity $\mu$, density $\rho$ and gravity $g$ equal to 1. These equations hold in the region occupied by the fluid, $0 \le y \le h(x, t)$. We also neglect surface tension, so that the approximate equations on the liquid surface $y = h(x, t)$ are

$$p = 0, \qquad \frac{\partial u}{\partial y} = 0. \tag{21.2}$$

We also have the kinematical conditions on the free surface

$$v = \frac{\partial h}{\partial t} + u\,\frac{\partial h}{\partial x} \tag{21.3}$$

plus the no-slip conditions at $y = 0$

$$u = v = 0. \tag{21.4}$$

This set of equations can integrated as follows: the last two equations of (21.1) together with the boundary conditions give

$$p(x, y, t) = h(x, t) - y, \quad u(x, y, t) = -\frac{1}{2}\frac{\partial h}{\partial x}\,y(y - 2h).$$

The conservation of mass can be modified into the form

$$\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}\int_0^h u\,dy,$$

obtained by expressing the variation of fluid content in a small portion of fluid contained between $x$ and $x + dx$. Evaluation of the last integral gives

$$\frac{\partial h}{\partial t} = \frac{1}{3}\frac{\partial}{\partial x}\left(h^3\,\frac{\partial h}{\partial x}\right). \tag{21.5}$$

This is the PME with $m = 4$. See Buckmaster [135] who studies the effect of an insulating gently sloping boundary on a stratified fluid. These motions are called *gravity currents*; Marino et al. [382] describe a plane viscous gravity current of silicone oil on a glass substrate, see also Smith [481]. Lacey, Ockendon and Tayler [355] and Gratton and Vigo [272] focus on the waiting time aspect of the solutions. Waiting times are viewed in this respect as a form of *metastability* since the fluid seems to be stationary when we only look at its leading front, but in fact it ends up moving. According to the surprising result of Aronso, Caffarelli and Vázquez [45], it can even start moving abruptly (cf. the corner point, Section 15.5.2).

### 21.1.1 *Higher order models for thin films*

If we take into account the influence of surface tension as the main effect in the dynamics of the fluid droplet instead of gravity, we arrive at a fourth-order equation of the form

$$h_t + (f(h)\,h_{xxx})_x = 0, \quad f(u) = |u|^n, \tag{21.6}$$

and its several dimensional analogues like $h_t + \nabla\cdot(f(h)\nabla\Delta h) = 0$. There are other applications: the equation with $f(h) = |h|$ models a thin neck of fluid in the Hele–Shaw cell, and $h(x, t)$ is the local thickness of the film or neck.

The properties of this equation are very different from the PME: to begin with, the maximum principle applies only in special circumstances. Pioneering work is due to Bernis and Friedman [102]. Source type solutions were produced in Bernis, Peletier and Williams [104]. Research has been very active but many open problems still remain open. This is a large research direction that falls outside of the scope of the present text but connects with many of its problems.

### 21.1.2 *Related application*

A related subject concerns the lubrication equation applied to the modelling of elastic journal bearings. A reference is Conway and Lee [173]. See also [403].

## 21.2 The equations of unsaturated filtration

The Darcy law appears as the basic dynamical relation in a number of problems in the theory of filtration of underground fluids that lead to equations similar to the PME. We present here the application to modelling flow in unsaturated media. We still assume that we have a homogeneous, isotropic and rigid porous medium partially filled with fluid. The flow can be described in terms of two main quantities, the *volumetric moisture content*, represented by a variable $\theta$ with values in the interval $[0,1]$, and the *velocity*, represented by $\mathbf{V}$, which are functions of space $x$ and time $t$ (the last is a vector). These quantities are related by the following laws:

(i) *Continuity equation*:

$$\frac{\partial \theta}{\partial t} + \nabla \cdot \mathbf{V} = 0. \tag{21.7}$$

(ii) *Darcy's law*: this now takes the form

$$\mathbf{V} = -K(\theta)\, \nabla \varphi. \tag{21.8}$$

This introduces the *hydraulic conductivity* which is a monotone function of $\theta$ whose form is given by the experimental evidence. The other quantity is $\varphi$, the *total potential*, that replaces the usual pressure of Darcy's law as presented in Section 2.1.

Ignoring a number of side effects (chemical, osmotic and thermal), the potential can be expressed as

$$\varphi = \Psi + z, \tag{21.9}$$

where $\Psi$ is the hydrostatic potential and $z$ (the vertical coordinate) represents the gravitational potential. The hydrostatic potential represents the effect of capillary suction. The filtration theory assumes that there is a definite empirical relationship between $\Psi$ and $\theta$. This relation can be very complicated due to hysteresis effects. For many porous media the relationships between $K$, $\Psi$ and $\theta$ can be expressed in the form

$$K(\theta) = K_0 \theta^n, \quad K(\theta)\frac{d\Psi}{d\theta} = D_0 \theta^{m-1},$$

where $K_0, D_0, m$ and $n$ are positive constants. This leads to the filtration equation

$$\frac{\partial \theta}{\partial t} = c_1 \Delta \theta^m + c_2 \frac{\partial}{\partial z} \theta^n. \tag{21.10}$$

In case of horizontal flow, $\theta = \theta(x, y, t)$, the last component is neglected and we get the porous medium equation.

A realistic assumption for the application to partially saturated flows makes $\theta$ a function of $\Psi$, $\theta = F(\Psi)$, where $F$ is a continuous function, which is strictly increasing for negative potentials and constant $F = \overline{\theta}$ for $\Psi \geq 0$. We then get the equation

$$\frac{\partial F(\Psi)}{\partial t} = \nabla \cdot (D(\Psi) \nabla \Psi) + \frac{\partial}{\partial z} D(\Psi) \tag{21.11}$$

in the unsaturated region (with $D(\Psi) = K(F(\Psi))$), while the equation becomes elliptic in the saturated region

$$\Delta \Psi = 0. \tag{21.12}$$

Such a mixed-type equation is more general than the ones considered in this book, and gives an idea of the many directions in which the theory of nonlinear parabolic equations has developed under the influence of the flow problems of the real world.

References: Richards [449], Bear [76].

**More on soil mechanics** Applications of the PME are described by Philip [433] and Parlange, Braddock and Chu [417].

## 21.3 Immiscible fluids. Oil equations

We discuss now the equations that arise in the study of the flow of two immiscible fluids, e.g., water and oil, in a porous medium. We assume that the flow is non-turbulent and incompressible and the medium is isotropic and homogeneous. The basic variables are the saturations of the two fluids, $s_1$ and $s_2$, which have values between 0 and 1 and satisfy $s_1 + s_2 = 1$; the velocities $v_1$, $v_2$ of the two fluids and the pressures $p_1$ and $p_2$.

The basic equations are:

(i) The conservation of mass

$$\partial_t(m\,\rho_1\, s_1) + \operatorname{div}(\rho_1\, \mathbf{v_1}) = 0, \qquad \partial_t(m\,\rho_2\, s_2) + \operatorname{div}(\rho_2\, \mathbf{v_2}) = 0. \tag{21.13}$$

Since $s_1 + s_2 = 1$, we can put $s = s_1$ as the main unknown and then $s_2 = 1 - s$. The porosity $m$ and the densities $\rho_1$, $\rho_2$ are positive and constant. Cancelling them by scaling we get

$$\partial_t s + \operatorname{div}(\mathbf{u_1}) = 0, \qquad \partial_t(1 - s) + \operatorname{div}(\mathbf{u_2}) = 0. \tag{21.14}$$

(ii) Darcy's law reads

$$\mathbf{u_1} = -\frac{K}{\mu_1} f_1(\nabla p_1 + \rho_1\, g\, \nabla z), \qquad \mathbf{u_2} = -\frac{K}{\mu_2} f_2(\nabla p_2 + \rho_2\, g\, \nabla z), \quad (21.15)$$

where $\mu_i$ is dynamic viscosity, $K$ is the absolute permeability and the $f_i$ are the relative permeabilities; the latter depend on $s$ in a very marked way that is derived from experimental data.

(iii) There is a relation between the pressures due to capillary effects that takes the form

$$p_2 - p_1 = p_c(s). \quad (21.16)$$

Again the function $p_c$ is given by the modelling. In the classical Muskat–Leverett model $f_1(s)$, $f_2$ and $p_c$ are universal functions.

We now disregard the gravity effects and assume that the net flow $\mathbf{u_1} + \mathbf{u_2} = 0$, as happens in imbibition processes. In that case the equations can be combined into a unique equation for $s$

$$\partial_t s = \Delta\Phi(s), \quad (21.17)$$

where $\Phi$ is a monotone non-decreasing function, so that (21.18) is a form the GPME. The precise form of $\Phi$ is

$$\Phi(s) = \frac{K}{\mu_2} \int_0^s F(\xi) f_2(\xi) |p_c'(\xi)|\, d\xi \quad (21.18)$$

with

$$F(s) = \frac{f_1(s)}{f_1(s) + \mu f_2(s)}, \qquad \mu = \frac{\mu_1}{\mu_2}. \quad (21.19)$$

As a consequence of the form of $f_i(s)$ and $p_c$ the function $\Phi$ is very degenerate: it vanishes identically for $0 \le s \le s_*$ and is also flat for $s^* \le s \le 1$

**Comments** These equations are very important for the industry of oil recovery. The realistic models are of course quite complicated. Thus, when the assumption of velocity balance is eliminated the equation becomes a system where convection effects are usually dominant and the theory is quite different. See Barenblatt et al. [67], Bear [76, 77], Gagneux et al. [244], Peaceman [421]; see also [507]. The inhomogeneity of the medium is also very important. Van Duyn [220] points out the possibility of hysteresis effects. non-equilibrium capillary effects have been studied by Barenblatt and coworkers [69] and also by Cuesta et al. [184].

## 21.4 Boundary layer theory

The porous medium equation with $m = 2$ appears in boundary layer theory when the Prandtl equations for flow past a plate are translated to the von Mises variables $x$ (distance downstream) and $\psi$ (stream function), and the downstream function is set to zero, cf. [404, 472]. The following Prandtl boundary-layer equations for the steady two-dimensional flow past a fixed wall are considered:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+U\frac{\partial U}{\partial x}, \tag{21.20}$$

where $u, v$ are the velocity components along and perpendicular to the wall, and $\nu$ is the kinematic viscosity. $U(x)$ is the velocity at the edge of the boundary layer, which is assumed to have the form $U(x)=U_0(1+x)^m$, where the constants $U_0>0$ and $m\geq 0$. The no-slip boundary condition will be satisfied if $u=v=0$ at $y=0$, and the asymptotic condition at the edge of the boundary layer requires $u(x,y)\to U(x)$ for $y\to\infty$ uniformly in $x$. At the line $x=0$, which must not correspond to a leading edge, the following initial condition is prescribed: $u(0,y)=u_0(y)$, $0<y<\infty$.

When the equations are written in terms of the stream function $\psi$ such that $\psi_x=-v$, $\psi_y=u$, they take the Blasius form

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\frac{\partial^3\psi}{\partial y^3} \tag{21.21}$$

for $0<y<\infty$, and $x>0$, which is a third-order equation. Additional conditions are

$$\psi=\frac{\partial\psi}{\partial y}=0 \quad \text{at} \quad y=0; \qquad \psi=y+O(1) \quad \text{as } y\to 0.$$

Equation (21.21) is transformed into a PME as follows: we consider $u=\partial\psi/\partial y$ as a function of $x$ and $\psi$; classical calculus rules give

$$\frac{\partial^2\psi}{\partial y^2}=u\frac{\partial u}{\partial\psi}, \qquad \frac{\partial^2\psi}{\partial x\partial y}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial\psi}\frac{\partial\psi}{\partial x}, \qquad \frac{\partial^3\psi}{\partial y^3}=u\frac{\partial}{\partial\psi}\left(u\frac{\partial}{\partial y}\right)$$

and this leads to the PME equation

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial\psi}\left(u\frac{\partial u}{\partial\psi}\right) \tag{21.22}$$

where the distance downstream $x$ acts as time and $\psi$ as space variable.

## 21.5 Spread of magma in volcanos

Ockendon et al. [404] explain the application to model the horizontal spreading of highly fissurized volcanos, which can be considered as shallow porous media through which magma flows from below. The upper surface of the volcano moves normal to itself with a speed proportional to the magma incoming flux rate. If $h$ is the height of the volcano surface above some reference level, the magma pressure is approximately hydrostatic, the velocity is given by Darcy's law, hence approximately $\sim -\nabla h$ and Boussinesq equation is finally obtained

$$\partial_t=\nabla\cdot(h\nabla h).$$

## 21.6 Signed solutions in groundwater flow

The porous medium equation with sign changes has been proposed to describe the mixing of fresh and salt groundwater due to mechanical dispersion. The unknown function $u$ denotes the velocity of the fluids, which may take positive as well as negative values since fresh and salt water can flow in opposite directions before mixing. When we take into account the diffusion of the fluids in the direction orthogonal to the initial discontinuity plane, labelled as $x = 0$, the equations governing the fluid can be simplified into

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left((a + b|u|)\,\frac{\partial u}{\partial x}\right), \tag{21.23}$$

where $u$ is the rescaled fluid discharge (or mass flow rate), and $a, b > 0$. The reference is de Josselin and van Duijn [312].

## 21.7 Limits of kinetic and radiation models

There is much interest in the diffusive models obtained in the limit of Boltzmann kinetic equations. Maybe the simplest model is Carleman's 1D model with two types of particles [147], that leads to diffusion equations of fast diffusion type. The mathematics of kinetic equations are reviewed in Perthame [428]. These models have an interest in plasma physics. A classical reference for radiative transfer is Chandrasekhar [160].

### 21.7.1 *Carleman's model*

A very popular fast diffusion model was proposed by Carleman to study the diffusive limit of kinetic equations [147]. He considered just two types of particles in a one-dimensional setting moving with speeds $c$ and $-c$. If the densities are $u$ and $v$ respectively you can write their simple dynamics as

$$\begin{cases} \partial_t u + c\,\partial_x u = k(u,v)(v-u) \\ \partial_t v - c\,\partial_x v = k(u,v)(u-v), \end{cases} \tag{21.24}$$

for some interaction kernel $k(u,v) \ge 0$. In a typical case we have $k = (u+v)^\alpha c^2$. Now write the equations for the joint density $\rho_c = u+v$ and the flux $j_c = (u - v)c$:

$$\begin{cases} \partial_t \rho_c + \partial_x j_c = 0 \\ c^{-2}\partial_t j_c + \partial_x \rho_c = -2\rho_c^\alpha j_c. \end{cases} \tag{21.25}$$

We now pass to the limit $c \to \infty$ to obtain to first order in powers or $\varepsilon = 1/c$:

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{\rho^\alpha}\frac{\partial \rho}{\partial x}\right), \tag{21.26}$$

which is the FDE with $m = 1 - \alpha$. Taking negative $\alpha$ the PME can be obtained. There is wide literature on the topic with rigorous justifications of the limit process (so-called diffusive limit), cf. e.g. [354, 376, 387, 468].

### 21.7.2 *Rosseland model*

This is an example of approximation of a transfer equation with complex behaviour by a nonlinear diffusion equation. The radiative transfer equation is a kinetic equation of the form

$$\partial_t f + \frac{1}{\varepsilon} v \cdot \nabla_x f + \frac{1}{\varepsilon^2}\, \sigma(\overline{f})(f - \overline{f}) = 0 \tag{21.27}$$

for a scalar density $f(x, v, t)$ that depends on space $x$ and velocity $v \in \mathbb{S}^{d-1}$ and evolves with time; $\overline{f}$ denotes the angular average of $f$, i.e., the average in velocity space

$$\overline{f} = \frac{1}{|\mathbb{S}^{d-1}|} \int_{(S)^{d-1}} f(x, v, t)\, dv.$$

It is used to describe the absorption and emission of photons in a hot medium by Bardos et al. [58]. Function $\sigma$ measures the opacity of the medium. The small parameter $\varepsilon$ is inserted to indicate the correct scaling of the equation; it represents the mean free path of the particles. Initial conditions are given, $f(x, v, 0)$, an integrable, non-negative and compactly supported function.

Since it is a transport equation the maximum speed of propagation is finite, given actually by $1/\varepsilon$. But the opacity term in $\sigma$ makes the average behaviour less fast. the limit $\varepsilon \to 0$, is studied in [58]. Working with a formal expansion in $\varepsilon$

$$f = f_0(x, v, t) + \varepsilon f_1(x, v, t) + \varepsilon^2 f_2(x, v, t) + \ldots$$

substituting into the equation and identifying powers of $\varepsilon$ we get

$$f_0 = f_0(x, t), \quad v \cdot \nabla_x f_0 + \sigma(f_0) f_1 = 0. \quad \partial_t f_0 + v \cdot \nabla_x f_1 = 0.$$

Finally, the GPME is obtained for $u = f_0$ in the form

$$u_t = \Delta F(u), \quad \text{with} \quad F'(u) = \frac{1}{d\,\sigma(u)} \tag{21.28}$$

(we recall that here $d$ is the space dimension). This is the diffusive limit, called in this case the *Rosseland approximation*.

It is proved in [59] that the unique solution of the scaled radiative transfer equation converges as $\epsilon \downarrow 0$ to the unique solution of the degenerate parabolic equation (21.28) with the same initial boundary conditions, thus justifying the Rosseland approximation. No monotonicity assumptions are imposed on the cross-section $\sigma$. Semigroup techniques, a priori $L^\infty$-bounds and compactness arguments are used in the proofs.

The standard conditions on the opacity are: $\sigma$ is continuous for $u > 0$ with $\sigma(u) \geq \sigma_R > 0$ for $0 < u < R$ and a 'possible blow-up at $u = 0$ so that

$$\int_0^1 \frac{du}{u\sigma(u)} < \infty, \qquad \int_1^\infty \frac{du}{u\sigma(u)} = \infty.$$

We know that under these conditions the solutions of the GPME have finite propagation. It is proved in [430] that the finite propagation property is also true before the limit, i.e., for $\varepsilon > 0$, by means of the construction of finite travelling waves.

### 21.7.3 *Marshak waves*

The equation is

$$(\theta + k\theta^n)_t = \Delta(\theta^m)$$

with parameters $n, m > 1$. It describes the transport of photons in a star-like medium. Refer to Marshak's paper in 1958 [384], cf. also [59, 360]. Asymptotic behaviour like the power case (i.e., like PME) is shown in [115] and [415] for bounded and unbounded domains respectively.

## 21.8 The PME as the limit of particle models

The porous medium equation models time evolution of the particle density of an ideal gas flow in a homogeneous porous medium, i.e. from a macroscopic point of view. It is natural to try to explain this macroscopic evolution from a microscopic point of view, dealing with large systems of interacting particles. Such an explanation can be produced by deriving the macroscopic dynamics as the limit dynamics for the microscopic evolution, as the number of particles tends to infinity, using certain rescaling procedures.

Following Oelshläger [410] we derive Boussinesq's equation as the limit dynamics of large systems of particles whose evolution is deterministic, governed by a system of ordinary differential equations of gradient form, depending on a scaling parameter $\beta \in (0, 1)$. In particular, we consider the system

$$\frac{dx_N^k}{dt} = -\frac{1}{N} \sum_{m=1, m\neq k}^{N} \nabla V_N(d_{k,m}(t)), \qquad d_{k.m} = x_N^k(t) - x_N^m(t),$$

for $k = 1, \ldots N$. The interaction potential $V_N$ is defined on $\mathbb{R}^d$ by scaling of some fixed function $V_1$:

$$V_N(x) = \lambda^d \, V_1(\lambda\, x), \quad \text{with } \lambda = N^{\beta/d}, \quad \text{and some } \beta \in (0, 1).$$

We are interested in the bulk behaviour of the whole population, hence we focus on the empirical process

$$t \mapsto x_N(t) = \frac{1}{N}\sum_{k=1}^{N} \delta_{x_N^k(t)},$$

where $\delta_a$ is Dirac's delta measure concentrated at $a \in \mathbb{R}^d$. Note that this is a measure valued process.

Depending on the scaling parameter $\beta$ we obtain in the limit $N \to \infty$ different versions of the porous medium equation as limit dynamics. The general result is as follows:

**Theorem 21.1** *Suppose that $x_N(0)$ converges as $N \to \infty$ to a function $p_0(x)$. Then $x_N(t)$ converges to a function $p(x,t)$ which solves a porous medium equation with initial data $p_0$.*

For $\beta \in (0,1)$ we get Boussinesq's equation as the limit dynamics. If $\beta = 1$ and the space dimension $d = 1$ then the so-called hydrodynamic limit of a one-dimensional particle system is obtained.

## 21.9 Diffusive coagulation-fragmentation models

This is a very interesting case of diffusive limit of particle dynamics. Diffusive coagulation-fragmentation equations describe the dynamics of a system of a large number of clusters undergoing in a process of binary coagulation and fragmentation events and diffusing in space by Brownian motion. The type of equations is

$$\frac{\partial f}{\partial t} - d\Delta_x f = Q(f) \text{ in } (0,+\infty) \times \Omega \times \mathbb{R}^d$$

where $f = (f_i(x,t))_{i\geq 1}$ represents the density of the different clusters. The equation is posed for $x \in \Omega$, an open bounded subset of $\mathbb{R}^d$, with Neumann boundary conditions and given initial data $f_i(x,0) \geq 0$. The reaction term $Q(f) = (Q_i(f))_{i\geq 1}$ accounts for the binary coagulation and fragmentation reactions and is given by a quadratic expression in the $f_i$'s involving as coefficients the coagulation and fragmentation rates as coefficients.

Escobedo, Laurençot and Mischler study these models in [226] and pass to the continuous limit (so-called fast reaction limit) by inserting a small parameter $\varepsilon > 0$ into $Q$ and letting $\varepsilon \to 0$. In this way they arrive at an equation for the local mass $\rho(x,t)$ of the form

$$\partial_t \rho = \Delta\Sigma(\rho), \tag{21.29}$$

where $\Sigma$ is a monotone function. In other words, in the diffusive limit the GPME is obtained. We refer to the paper for further details.

## 21.10 Diffusion in semiconductors

Generalized or simple porous medium equations, as well as systems, arise in the study the concentration-dependent diffusion of dopant impurities into semiconductors. In particular, Schwendeman [474] examines the two-dimensional diffusion in the vicinity of a mask; numerical solutions are obtained for dopant diffusion with fixed-total-concentration and with constant-surface-concentration, as well as power series solutions. An analytical study is performed by King [334, 335, 336]. In [339] the model equation is

$$\partial_t c = \partial_x (D(c, c_s)\, \partial_x c), \tag{21.30}$$

where $c$ is the concentration of a dopant in the semiconductor, and the equation applies in a region $s(t) < x < \infty$, where the surface $x = s(t)$ is supposed to be known. In this equation the diffusivity $D$ depends on the local concentration $c$ and $c_s \equiv c(s(t), t)$.

## 21.11 Contrast enhancement in image processing

Computer vision has become in recent decades a mathematical discipline which relies on the differential-geometric approach. More specifically, an appropriate technique of image processing consists of formulating a partial differential equation of evolution type for the image intensity, $I(x, y)$. This function, also called the *grey level*, takes values in the interval $0 \le I \le 1$ and is defined on a two-dimensional image domain, $\Omega$. PDEs are used as a tool to modify this function so that the quality of the image is improved. Two main aspects of such improvement are noise removal and edge enhancement.

The usual evolution model based on the heat equation (which amounts to convolution with a family of Gaussian kernels) has many advantages since the properties of the equation are well known, but it leads to image blurring. A critique of this model was done by Perona and Malik [427], 1990, who suggested the use of nonlinear diffusion models. The nonlinearity is created by the law relating the image intensity flux to the image intensity. The models could be degenerate or singular. The following anisotropic diffusion model

$$I_t = \nabla \cdot (g(|\nabla I|)\, \nabla I), \tag{21.31}$$

was proposed in [427]. They suggested choices for $g$ of the form $g(s) = C/(1 + s/K)^{1+\alpha}$ which would dampen the diffusion at the points of large gradients, thus allowing us to preserve edges. It was demonstrated that it produces an effect of enhancement of image edges that has a strong interest in the application to processing and recognition of images. There is a very important mathematical problem related to this model. Working in 1D, calling $I_x = u$ and putting $\phi(u) = g(|u|)\, u$ the equation becomes $I_t = \phi'(I_x) I_{xx}$ so that

$$u_t = (\phi'(u) u_x)_x = \phi'(u) u_{xx} + \phi''(u)(u_x)^2. \tag{21.32}$$

This is formally a GPME but the equation ceases to be forward parabolic at points where $\phi$ is not increasing. The Perona–Malik model has had a deep influence, and many other models have been proposed, see [12, 411].

We will focus here in the model proposed by Malladi and Sethian [381] which leads (after proper scaling) to the following equation for the image intensity:

$$I_t = (1 + |\nabla I|^2)^{1/2}\,\kappa \tag{21.33}$$

where $\kappa$ denotes the curvature of the surface $z = I(x, y)$. The equation represents movement by curvature (curvature flow) and can be written as

$$I_t = \frac{(1 + I_y^2)I_{xx} - 2I_xI_yI_{xy} + (1 + I_x^2)I_{yy}}{1 + I_x^2 + I_y^2}. \tag{21.34}$$

We can also consider the more general flow given by equation

$$I_t = \frac{(1 + I_y^2)I_{xx} - 2I_xI_yI_{xy} + (1 + I_x^2)I_{yy}}{(1 + I_x^2 + I_y^2)^{1+\alpha}}. \tag{21.35}$$

Along with the former case $\alpha = 0$, the case $\alpha = 1$ has also attracted the attention of researchers (Beltrami flow, cf. Sochen et al. [483]).

The asymptotic and numerical treatment of these models shows the enhancement of the intensity contrasts by formation of regions of large intensity gradients, i.e., the normal component of the image intensity gradient becomes quite large. This phenomenon allowed [65] to suggest the existence of a *boundary layer* where large gradients concentrate and to focus on this boundary layer where a further simplification of the model is possible. Arguing locally around a sharp gradient point and choosing the $x$-axis as the direction normal to the boundary layer or front, we may disregard the effect of $y$ derivatives with respect to the $x$ derivatives in (21.35). In this way we get the *reduced equation*, which is just the one-dimensional version of (21.35)

$$I_t = \frac{I_{xx}}{(1 + I_x^2)^{1+\alpha}}, \tag{21.36}$$

where we have neglected $I_y, I_{yy}$. The mathematical problem consists in solving this equation with suitable boundary data, namely, $I = 0$ on the left-hand side of the contour and $u = 1$ on the right-hand side (be that a finite or an infinite distance). As initial conditions we take

$$I(x, 0) = I_0(x),$$

satisfying $0 < I_0 < 1$ and $I_0' > 0$ in an interval $J = (a, b)$ and constant values otherwise, zero to the left, 1 to the right. As was pointed out in [65], the phenomenon of *gradient enhancement* takes place in this model (in a proper setting) for all $\alpha \geq 0$: the spatial gradient of the solutions, $I_x$, increases with time, and its support shrinks. It is proved in [70] that this is best solved in terms of the GPME satisfied by $u = I_x$ by formulating a singular free boundary problem. Convergence to a sharp front is even proved in finite time in the interval $0 > \alpha > -1/2$. The conditions on $I_0$ can be relaxed: the less stringent size restriction $0 \leq I_0 \leq 1$ with lack of monotonicity is treated in [13].

## 21.12 Stochastic models. PME with noise

The model studied by Da Prato and Röckner [192] concerns the porous medium equation with a noisy perturbation,

$$\partial_t u = \Delta(|u|^{m-1}u) + \sum_{j=1}^{\infty} f_j \frac{dB_j}{dt}. \tag{21.37}$$

Here $m > 1$ and the last term represents a random noise, where the $B_j$'s are mutually independent standard Brownian motions. By adding an additional viscosity term, $\varepsilon\,\Delta u$ with $\varepsilon > 0$, martingale solutions are obtained. This term is intended to make the equation non-degenerate. A different approach for this model is followed by Kim [333].

A different topic is pursued by Benachour et al. [75] who study the porous media equation in the form:

$$u_t = \frac{1}{2}(u^{2n+1})_{xx}, \tag{21.38}$$

with $u(0,\cdot) = \mu$. Here $m > 0$ and $\mu$ is a probability measure. They associate with equation (21.38) the following stochastic differential equation:

$$X_t = X_0 + \int_0^t u^n(s, X_s)\, dB_s, \quad t > 0, \tag{21.39}$$

subject to the constraint

$$P(X_t \in dx) = u(t,x)dx. \tag{21.40}$$

Here $(B_t; t \geq 0)$ is a Brownian motion on $\mathbb{R}$ and $\mu$ is the probability density of $X_0$. It follows from an application of Itô's formula that if $(X_t, \mu)$ is a solution of (21.39)–(21.40), then $u$ is a solution of (21.38). Moreover, the stochastic model of (21.38) represented by (21.39)–(21.40) appears when we consider a system of $N$ particles with interaction and let $N \to \infty$.

The authors prove an existence and uniqueness theorem of weak solutions of Problem (21.39)–(21.40) for a large class of initial data $\mu$. In particular, under some conditions on $\mu$ they prove that equation (21.38) has a unique strictly positive, smooth solution $u$ and that $u^m$ is Lipschitz continuous. In this case, equation (21.39)–(21.40) has a unique strong solution.

## 21.13 General filtration equations

A great variety of diffusion and heat propagation problems can be described by equations (or systems of equations) that include terms accounting for convection and absorption/reaction. Diffusion is modelled by a second-order elliptic operator, possibly nonlinear; convection appears as a first order term; finally, absorption/reaction is reflected in a zero-order term. The equations take the forms like

$$\frac{\partial u}{\partial t} = \Delta A(u) + C \cdot \nabla B(u) + F(u) + f \tag{21.41}$$

where $A$ is a monotone increasing function. Putting $C = F = 0$ we find the standard filtration equation. Therefore, equations like (21.41) are called filtration equations or equations of porous medium type if $A$ is not linear. A more general form can be

$$\frac{\partial F(u)}{\partial t} = \nabla \mathbf{A}(x, t, u, Du) + B(x, t, u, Du), \tag{21.42}$$

where $\mathbf{A} = (A_1, \cdot, A_d)$ is a vector function. This form can be vectorial and includes equations of $p$-Laplacian type.

## 21.14 Other

Newman and Sagan discuss models for the interstellar diffusion of galactic civilizations in the context of population dynamics. They propose diffusion modelled by the PME [398].

# Appendix

## BASIC FACTS

This appendix contains auxiliary information that has been mentioned or used in the text. Some of the sections have an independent interest because they contain developments of topics of text. Thus, the question of non-contractivity of the PME in various norms discussed in Section A.11 is an interesting and quite open problem.

### A.1 Notations and basic facts

#### A.1.1 *Points and sets*

We will use notations that are rather standard in PDE texts, like [229, 261] or equivalent, which we assume known to the reader. As usual, $\mathbb{R}$ is the real line, $(a, b)$ denotes an open interval, $[a, b]$ a closed one, and $\mathbb{R}_+ = (0, \infty)$. We denote the space dimension by $d = 1, 2, \ldots$, according to physics usage. Points in $\mathbb{R}^d$ for $d > 1$ are denoted by $x = (x_1, \ldots, x_d)$. For vectors $\mathbf{u}$ and $\mathbf{v} \in \mathbb{R}^d$ the scalar product is denoted by $\mathbf{u} \cdot \mathbf{v}$, and sometimes by $\langle \mathbf{u}, \mathbf{v} \rangle$; $\mathbf{e_i}$ denotes the unitary vector in the positive $i$-th direction. We denote by $B_R(x)$ the open ball of radius $R$ in $\mathbb{R}^d$ centred at $x \in \mathbb{R}^d$. The set of parts of $X$ is denoted by $\mathcal{P}(X) = 2^X$.

For a subset $E$ of a metric space, $\overline{E}$ denotes its closure and $\partial E$ its boundary. We denote the Lebesgue measure by $dx = dx_1 \cdots dx_d$ and the Lebesgue measure of a measurable set $E \subset \mathbb{R}^d$ by $|E|$ or $\text{meas}(E)$. The measure (volume) of the unit ball is given by

$$\omega_d = \frac{2\pi^{d/2}}{d\Gamma(d/2)},$$

where $\Gamma$ is Euler's gamma function. $\mathbb{S}^{d-1}$ denotes the unit sphere $\{x : |x| = 1\}$ in $\mathbb{R}^d$. Its element of area is denoted by $dS$ or $d\sigma$. Its total area (i.e., its $(d-1)$-dimensional measure) is $d\,\omega_d$.

We usually denote by $\Omega \subset \mathbb{R}^d$ the domain where the spatial variable lives. A regular domain is a domain whose boundary $\Gamma = \partial\Omega$ is locally a $C^{k,\alpha}$ hypersurface for some $k \geq 1$ and $\alpha \in (0, 1)$. Typically, $\Gamma \in C^{2,\alpha}$. But we will also consider the generality of domains with a Lipschitz boundary, which means that $\Gamma$ can be viewed locally as the graph of a Lipschitz function after an appropriate rotation of the coordinate axes, and in addition $\Omega$ is locally on one side of $\Gamma$, cf. [277]. This generality allows for domains with corners which are found in some applications. Unless mentioned to the contrary, the boundary will be assumed to be $C^k$ regular with $k \geq 2$. For $x \in \Omega$ we define the distance to the boundary as

$$d(x, \partial\Omega) = \sup\{d(x, y) : y \in \partial\Omega\}.$$

For a compact set $K \subset \Omega$ we define

$$d(K, \partial\Omega) = \inf\{d(x, \partial\Omega) : x \in K\}.$$

These distances are always positive.

We will often deal with space-time domains. $Q$ is the cylinder $\Omega \times \mathbb{R}_+$ and for $0 < T < \infty$ we write $Q_T = \Omega \times (0, T)$ and $Q^T = \Omega \times (T, \infty)$. The lateral boundary of $Q$ is denoted by $\Sigma = \partial\Omega \times [0, \infty)$, while $\Sigma_T = \partial\Omega \times [0, T]$.

### A.1.2 *Functions*

The characteristic function of a set $E$ is denoted by $\chi_E$: its value is 1 for $x \in E$, 0 otherwise. We sometimes use the notations $\mathrm{Dom}(f) = D(f)$ and $\mathrm{Im}(f) = R(f)$ to denote the domain and range of a function respectively. If the domain is a set $E$ we may write $\mathrm{Im}(f) = f(E)$.

The symbols $(s)_+$, $(s)^+$ mean $\max\{s, 0\}$, i.e. the positive part of the number $s$, and $(s)_- = (s)^- = \max\{-s, 0\}$, the negative part. For a function we have

$$f^+ = \max\{f, 0\} = f\chi_{\{f \geq 0\}}, \quad f^- = \max\{-f, 0\} = -f\chi_{\{f \leq 0\}}$$

so that $f = f^+ - f^-$. Sometimes, $f_+$ and $f_-$ are used for convenience. The function sign, better called $\mathrm{sign}_0$, is defined as

$$\mathrm{sign}_0(s) = 1 \text{ for } s > 0, \quad \mathrm{sign}_0(s) = 0 \ \text{ for } s = 0, \quad \mathrm{sign}_0(s) = -1 \ \text{ for } s < 0.$$

Note that, strictly speaking, *sign* is a multivalued operator, cf. Section A.3. The function $\mathrm{sign}_0^+$ is defined as

$$\mathrm{sign}_0^+(s) = 1 \text{ for } s > 0, \quad \mathrm{sign}_0^+(s) = 0 \ \text{ for } s \leq 0,$$

and

$$\mathrm{sign}_0^-(s) = -1 \text{ for } s < 0, \quad \mathrm{sign}_0^+(s) = 0 \ \text{ for } s \geq 0.$$

We have $\mathrm{sign}_0(s) = \mathrm{sign}_0^+(s) + \mathrm{sign}_0^-(s)$. We will often write $\mathrm{sign}(s)$ instead of $\mathrm{sign}_0(s)$ if no confusion arises.

There will also be frequent use of *cut-off functions*. The basic cut-off function is a function $\zeta(x) \in C^\infty(\mathbb{R}^d)$ which satisfies the following conditions: $0 \leq \zeta \leq 1$, $\zeta(x) = 1$ if and only if $|x| \leq 1$, and $\zeta(x) = 0$ if $|x| \geq 2$. We will use its scalings: $\zeta_r(x) = \zeta(x/r)$ for $r > 0$.

If $f$ is a one-dimensional function, the expression $\lim_{x\to a-} f(x)$ means $\lim f(x)$ as $x \to a$ with $x < a$. Similar meaning for $\lim_{x\to a+} f(x)$.

We use the notations $O$ and $o$ in the sense of Landau.

A.1.3 *Integrals and derivatives*

Integrals without limits are understood to extend to the whole domain under consideration, $\Omega$, $Q$ or $Q_T$, depending on the context. We use different notations for partial derivatives, like $u_t = \partial_t u = \partial u/\partial t$ and so on, the first being most common in the literature, the second one being convenient to avoid confusion with subindexes. Especially in regularity theory, we use the notation $D^\alpha u$ where $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index, to denote the derivative of order $|\alpha| = \sum_i \alpha_i$ which is taken $\alpha_i$ times with respect to the variable $x_i$. We usually write $\nabla u$, sometimes $\nabla_x u$, for the spatial gradient of a function. We also use the symbol $⨍$ to denote average, see Section 7.1.

A.1.4 *Functional spaces*

$C(\Omega)$, $C^k(\Omega)$ and $C^\infty(\Omega)$ denote the spaces of continuous, $k$-times differentiable and infinitely differentiable functions in $\Omega$, $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$ denotes the $C^\infty$-smooth functions with compact support in $\Omega$ and $\mathcal{D}'(\Omega)$ the space of distributions. We use $C_0(\Omega)$ for continuous functions that vanish on the boundary. For $0 < \alpha < 1$, $C^\alpha(\overline{\Omega})$ is the Banach space of functions which are uniformly Hölder continuous in $\Omega$. In case they are only uniformly continuous in the interior we get the space $C^\alpha(\Omega)$ which is not a normed space, but a metric space. Functions with Hölder continuous derivatives form the spaces $C^{k,\alpha}(\overline{\Omega})$ and $C^{k,\alpha}(\Omega)$. When $\alpha = 1$ we get the Lipschitz spaces, like $Lip\,(\Omega)$. Note that the notation $C^1(\overline{\Omega})$ for that space becomes inconsistent in that case, since the symbol is already in use for functions with one continuous derivative. Hence, $Lip\,(\Omega)$ is sometimes denoted as $C^{0,1}(\overline{\Omega})$. The concept of modulus of continuity will be introduced in Section 7.5.1.

For $1 \le p \le \infty$ we denote the usual Lebesgue spaces by $L^p(\Omega)$ with norm $\|\cdot\|_p$, while $H^1(\Omega)$ and $H^1_0(\Omega)$ are the usual Sobolev spaces; the subscript *loc* refers to local spaces. A general reference for Sobolev spaces is [4]. When dealing with functions in Sobolev spaces, derivatives mean distributional derivatives. As a rule, we will identify Lebesgue measurable real functions defined in $\Omega$ up to a set of measure zero. We will abridge the expression almost everywhere in the usual form as a.e. Embedding and compactness theorems (Sobolev embeddings and the like) are assumed as defined for instance in [4, 229, 372]. Let us recall the Rellich–Kondrachov theorem: *Let $\Omega$ be a bounded domain with $C^1$ boundary. Then,*

$$p < d \Longrightarrow W^{1,p}(\Omega) \subset L^q(\Omega) \textit{ for all } q \in [1, p^*),\ \tfrac{1}{p^*} = \tfrac{1}{p} - \tfrac{1}{d};$$

$$p = d \Longrightarrow W^{1,p}(\Omega) \subset L^q(\Omega) \textit{ for all } q \in [1, +\infty);$$

$$p > d \Longrightarrow W^{1,p}(\Omega) \subset C(\overline{\Omega}).$$

*All these injections are compact. In particular, $W^{1,p}(\Omega) \subset L^p(\Omega)$ with compact injection for all $p \ge 1$. In $\Omega = \mathbb{R}^d$, the above injections are compact in local topology (convergence on compact subsets).*

Similar statements apply to functions defined in $Q$, $Q_T$ or their closures. $C^{2,1}(Q)$ denotes those functions being twice differentiable in the space variables and once in time. For a function $u(x,t)$, we use the abbreviated notation $u(t)$ to denote the function-valued map $t \mapsto u(\cdot,t)$.

We will use frequently classes of non-negative solutions. In that sense, $L^p(\Omega)_+$ denotes the set of functions $f \in L^p(\Omega)$ such that $f \geq 0$. We will sometimes use weighted spaces, like $L^1_\delta(\Omega)$ in Section 6.6. The space $H^{-1}(\Omega)$ is described and used in Section 6.7.

Spaces of vector valued functions are used in the abstract settings, especially in Chapter 10. Care must be taken with some subtleties when the values are taken in an infinite dimensional metric space $X$. Thus, not all absolutely continuous functions $\mathbb{R} \to X$ are differentiable everywhere. Cf. in this respect the appendix of [128]. Let us only mention that given a measure space $(\Omega, \mathcal{P}, \mu)$, a Banach space $X$ has the Radon–Nikodým property with respect to $\mu$ if for every bounded variation, countably additive $\mu$-continuous vector measure $\nu$ valued in $X$, there is a Bochner integrable function $g : \Omega \to X$ such that $\nu(E) = \int_E g\, d\mu$ for every $\mu$ measurable set $E$. In that case, every absolutely continuous function $f : [a,b] \to X$ is also a.e. differentiable. By default $\mu$ is the Lebesgue measure in $\mathbb{R}^d$. Every reflexive Banach space is R-N, but $L^1(\Omega)$ and $L^\infty(\Omega)$ are not.

### A.1.5 *Some integrals and constants*

We list some of the integrals that enter the calculation of the best constants in the smoothing effect.

(i) Euler's gamma function is defined as

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\, dt, \quad p > 0.$$

We have $\Gamma(p) = (p-1)\,\Gamma(p-1)$, and $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$. As $p \to \infty$ we have

$$\Gamma(p) \sim (p/e)^p (2\pi p)^{1/2}.$$

(ii) Euler's beta function is defined for $p, q > 0$ as

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\, dt = 2\int_0^1 s^{2p-1}(1-s^2)^{q-1}\, ds.$$

We have $B(p,q) = B(q,p)$ and the basic relation

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

as well as the equivalent expressions with parameter $r > 0$

$$B(p,q) = r\int_0^1 s^{rp-1}(1-s^r)^{q-1}\, ds = r\int_0^\infty \frac{x^{rq-1}}{(1+x^r)^{p+q}}\, dx.$$

These expressions are usually found for the value $r = 2$.

### A.1.6 *Various*

We will devote the next sections to developing a number of less standard topics that are needed or convenient to read the book. Other notations and concepts are explained in the text as they occur.

## A.2 Nonlinear operators

The theory of nonlinear operators in a Banach space is a main tool of the theory developed in Chapter 10. We recall that in its more general nonlinear and possibly multivalued version, an operator $A$ in a Banach space $X$ is a map $A$ from a subset of $D(A) \subset X$ into the set of parts of $X$, $\mathcal{P}(X)$. We write $A(x)$ or $Ax$ for the image of $x$ (it is a subset of $X$). We always take as $D(A)$ the essential domain, $D(A) = \{x : A(x) \neq \emptyset\}$. We denote by $R(A)$ the range of $A$, a subset of $X$:

$$R(A) = \bigcup \{A(x) : x \in D(A)\}.$$

For $x \in D(A)$ we denote by $A^o x$ the element with minimal norm in $Ax$, and we have $D(A^o) = D(A)$.

In this generality, it is often convenient to identify the operator with its graph, $\Gamma(A)$, a subset of $X \times X$. We say that an operator $B$ extends an operator $A$ if $\Gamma(A) \subset \Gamma(B)$. This is an order relation. Thus, $A^o$ is extended by $A$. An operator is closed if and only if its graph is a closed subset of $X \times X$. We say that $B$ if the closure of $A$ iff $\Gamma(B)$ is the closure of $\Gamma(A)$. The sum of two operators is defined as

$$(A + B)x = Ax + Bx = \{z = y_1 + y_2, y_1 \in Ax_1, y_2 \in Bx\}$$

on the domain $D(A + B) = D(A) \cap D(B)$. There is no problem is defining $\lambda A$ for $\lambda \in \mathbb{R}$. The composition $A \circ B = AB$ is defined as

$$(A \circ B)x = \{z \in A(y) : y \in B(x)\}$$

on the domain where that definition is not empty, $D(A \circ B) = \{x : B(x) \cap D(A) \neq \emptyset\}$.

The inverse $A^{-1}$ is easily understood in the sense of graphs, just changing the order of domain and image

$$y \in A^{-1}(x) \quad \text{iff} \quad x \in A(y).$$

The ease in defining inverses is one of the strong points of using multivalued operators. Generally speaking, the inverse $A^{-1}$ of also a multivalued operator, but there are cases in which $A$ is multivalued and $A^{-1}$ is single valued.

We refer to Chapter 10 for the definitions of monotone and accretive operators and their variants. Use is made in that chapter of integrals of vector-valued maps $f \in L^1(0, T : X)$ where $X$ is a Banach space. The integral is understood in the sense of Bochner with respect to Lebesgue measure in $(0, T) \subset \mathbb{R}$; it means that the functions are strong measurable and

$$\int \|f(t)\|_X \, dt < \infty,$$

cf. [529]. In this setting an absolutely continuous function need not be differentiable a.e., so this condition has to be added when needed.

We point out that the family of resolvent operators associated to an operator $A$ is defined as

$$J_\lambda(A) = (I + \lambda A)^{-1}.$$

They are in principle multivalued operators, that come from solving equations of the form $u + \lambda Au \ni f$ which is equivalent to $u \in J_\lambda f$. Note that $D(J_\lambda) = R(I + \lambda A)$. For monotone or accretive operators, as defined in Chapter 10, the resolvent is a single-valued (non-strictly) contractive map.

More on accretive definitions in Subsection 10.2.3. Monotone operators in Hilbert spaces are treated in Section 10.1. A very important class of maximal monotone operators is given by the subdifferentials of proper convex functions, that have been defined at the end of Section 10.1.

## A.3 Maximal monotone graphs

We will study nonlinear parabolic equations like $u_t = \Delta\varphi(u) + f$, and their elliptic counterparts, like $-\Delta v + \beta(v) = f$. To simplify, we may assume that $\varphi$ is a continuous and monotone increasing function of its argument $u \in \mathbb{R}$, and then $\beta$ is its inverse function. Making the requirement of parabolicity on the first equation leads to the condition $\varphi'(s) > 0$ for all $s$. However, the second equation, which is used to solve the first, does not need such a strong requirement. It is then possible and useful to consider a greater generality in which $\varphi$ and $\beta$ can be any *maximal monotone graph* in $\mathbb{R}^2$.

For the concept and applications of maximal monotone graph (m.m.g. for short) we may refer the reader to Brezis' treatise [128], which covers the much more general theory of maximal monotone operators in Hilbert spaces. Let us remark that this generality has been introduced into nonlinear analysis because of its interest in modelling a number of physical applications, most notably to formulate variational inequalities.

Here is a summary of the main facts that we need: a m.m.g. $\varphi$ in $\mathbb{R}^2$ is the natural generalization of the concept of monotone non-decreasing real function to treat in an efficient way the cases where there are discontinuities; since we are dealing with monotone functions, they must be jump discontinuities. We want to fill in these 'gaps' for the benefit of obtaining existence of solutions of the equations where $\varphi$ appears. Then, the function must become multivalued and contain vertical segments (corresponding to the jumps). The multivalued function $\varphi$ is defined in a maximal interval $D(\varphi)$ which is not necessarily $\mathbb{R}$, and can be open or closed on either end. If one of the ends of $D(\varphi)$ is finite and not included in $D(\varphi)$, then there is a vertical asymptote at this end; if it is included, there is a semi-infinite vertical segment in the graph. Typical maximal monotone graphs appearing in the nonlinear ODEs and PDEs of Mathematical Physics are the *sign* function

$$\operatorname{sign}(s) = \begin{cases} 1 & \text{for } s > 0, \\ -1 & \text{for } s < 0, \\ [-1,1] & \text{for } s = 0; \end{cases}$$

its positive part, denoted by $\operatorname{sign}^+(s)$, where we modify the sign so that $\operatorname{sign}^+(s) = 0$ for $s < 0$ and $\operatorname{sign}^+(0) = [0,1]$; the Stefan graph, defined by $H(s) = c\,s + L\operatorname{sign}^+(s)$ with constants $c, L > 0$; and the angle graph, $A(s) = 0$ for $s \geq 0$, $A(0) = (-\infty, 0]$, which is defined in $D(A) = [0, \infty)$.

One of the main advantages of this generality, which will be used here, is the fact that the inverse of a m.m.g. is again a m.m.g.; actually, both graphs are symmetric with respect to the main bisectrix in $\mathbb{R}^2$.

The standard and somewhat awkward notation when using multi-valued operators is set inclusion, so that when $(a, b)$ is a point in the graph $\varphi$ we write $b \in \varphi(a)$ instead of $b = \varphi(a)$, since generally $\varphi(a)$ is not a singleton.

### A.3.1 *Comparison of maximal monotone graphs*

In the study of the filtration equation (GPME) we will be interested in comparing the concentrations of solutions of two equations with different nonlinearities $\varphi$. This final goal will be prepared with a result for elliptic equations. We introduce the following concepts.

**Definition A.1** *We say that a maximal monotone graph $\varphi_1$ is weaker than another one $\varphi_2$, and we write $\varphi_1 \prec \varphi_2$, if they have the same domains, $D(\varphi_1) = D(\varphi_2)$, and there is a contraction $\gamma : \mathbb{R} \to \mathbb{R}$ such that*

$$\varphi_1 = \gamma \circ \varphi_2. \tag{A.1}$$

By contraction we mean $|\gamma(a) - \gamma(b)| \leq |a - b|$. This implies in particular $\varphi_1$ must have horizonal points (or horizontal intervals) at the same values of the argument as $\varphi_2$, and maybe some more. We also assume that $\varphi_1$ does not accept vertical intervals (i.e., it is one-valued). Note that for smooth graphs condition (A.1) just means that

$$\varphi_1'(s) \leq \varphi_2'(s), \quad \text{for every } s \in D(\varphi_2), \tag{A.2}$$

which is easier to remember or to manipulate. We will see that $\varphi'$ is interpreted as the diffusivity in many parabolic problems, so that relation (A.2) can be phrased as: $\varphi_1$ *is less diffusive than* $\varphi_2$. This explains why it will be important in the evolution analysis.

In the development of the corresponding elliptic theory we will need to rephrase this condition in terms of the inverse graphs $\beta_i$ entering the equations of the form

$$-\Delta v + \beta(v) \ni f.$$

It then means that there is a contraction $\gamma : \mathbb{R} \to \mathbb{R}$ such that

$$\beta_2 = \beta_1 \circ \gamma. \tag{A.3}$$

To be precise, we also have to specify the relation of the domains, $D(\beta_1) = \gamma(D(\beta_2))$. But, as a general rule, we will prefer to stick to comparisons of diffusivities, $\varphi = \beta^{-1}$.

## A.4 Measures

In the study of the initial value problem we use Radon measures as initial data. We recall that a Radon measure $\mu$ is in principle defined as a (real-valued) linear map on $C_c(\Omega)$, [455, 473], where $\Omega$ will be for us an open subset of $\mathbb{R}^d$. The Riesz theorem allows to associate to a Radon measure a Borel measure, which is a real-valued map on sets, that we will also denote by $\mu$. Actually, the measure is Borel regular and locally finite. Note the alternative notations for integrals with respect to a measure: $\int f(x)\,\mu(dx) = \int f(x) d\mu(x)$. Both appear in the literature. The family of Borel subsets of $X$ is denoted by $\mathcal{B}(X)$.

The space of Radon measures on a separable metric space (or more generally a locally compact Hausdorff space) $X$ is denoted by $\mathcal{M}(X)$, the subset of positive measures by $\mathcal{M}^+(X)$, the subset of finite measures by $\mathcal{M}_b(X)$. Given a measure $\mu \in \mathcal{M}$, we denote by $\mu = \mu^+ - \mu^-$ the Hahn–Jordan decomposition of $\mu$ into non-negative measures. We denote by $\mathcal{P}(X)$ is the family of all *probability measures*, non-negative measures with total mass 1.

*Convergence of measures*

The natural convergence in $\mathcal{M}(X)$ is defined by the rule that $\mu_n \to \mu$ iff

$$\lim_{n\to\infty} \int_X f(x)\, d\mu_n(x) = \int f(x)\, d\mu(x)$$

for every $f \in C_c(X)$. Technically, this is the *weak-* convergence*, its topology is described as $\sigma(\mathcal{M}(X); C_c(X))$, and it is also referred to as *vague convergence*. In the weak-* topology the usual compactness statement applies: bounded families contain convergent subsequences. The problem is that the limit measure can be *defective*, i.e., can have less mass than the limit of the masses in the convergent family, the explanation is that some mass can go to infinity or to the boundary. The problem is avoided by *weak convergence*, where we take test functions $f \in C_b(X)$, the set of all bounded and continuous functions on $X$, the topology being denoted by $\sigma(\mathcal{M}(X); C_b(X))$. Weak convergence of measures, also called *narrow convergence* in probability theory, is stricter than vague convergence and the total mass is conserved in the limit. It coincides with weak-* convergence is $X$ is compact.

In general, weak convergence needs some extra property. This is well-known in probability. A family of probability measures $\mu_i$ on a metric space $M$ is said to be *tight* if for every $\varepsilon > 0$ there exists a compact $K$ such that

$$\sup_i \mu_i(M \setminus K) \le \varepsilon.$$

Prokhorov's theorem gives a criterion for weak convergence: *Suppose that a sequence of probability measures on a space $M$ is tight. Then there exists a subsequence $\mu_{n_k}$ which converges weakly to a limit probability measure $\mu$.*

The same result holds if probability measures are replaced by non-negative Radon measures with finite and fixed total mass.

*BV functions*

We will also need the space of functions of bounded variation, $BV(\Omega)$: it consists of the functions $f \in L^1(\Omega)$ whose distributional gradient $Du$ is a Radon measure with bounded variation defined as

$$\|Du\| = \sup\left\{\int_\Omega u \operatorname{div}\phi\, dx,\ \phi = (\phi_1, \dots, \phi_d) \in C_c^1(\Omega : \mathbb{R}^n),\ |\phi(x)| \le 1 \text{ a.e.}\right\}.$$

It is a Banach space normed by $\|u\|_{BV} = \|u\|_1 + \|Du\|$. We have $W^{1,1}(\Omega) \subset BV(\Omega)$ and in fact it is the natural closure of $W^{1,1}(\Omega)$ in the sense that bounded sequences in $W^{1,1}(\Omega)$ converge in the weak-* topology of $BV(\Omega)$ after passing to a subsequence.

## A.5 Marcinkiewicz spaces

Different classes of functional spaces are natural in the study of symmetrization, for instance the Lebesgue spaces $L^p(\Omega)$. Also the Marcinkiewicz spaces play a role. The Marcinkiewicz space $M^p(\mathbb{R}^d)$, $1 < p < \infty$, is defined as set of $f \in L^1_{loc}(\mathbb{R}^d)$ such that

$$\int_K |f(x)|\, dx \le C|K|^{(p-1)/p}, \tag{A.4}$$

for all subsets $K$ of finite measure, cf. [87]. The minimal $C$ in (A.4) gives a norm in this space, i.e.,

$$\|f\|_{M^p(\mathbb{R}^d)} = \sup\left\{\operatorname{meas}(K)^{-(p-1)/p} \int_K |f|\, dx :\ K \subset \mathbb{R}^d,\ \operatorname{meas}(K) > 0\right\}. \tag{A.5}$$

Since functions in $L^p(\mathbb{R}^d)$ satisfy inequality (A.4) with $C = \|f\|_{L^p}$ (by Hölder's inequality), we conclude that $L^p(\mathbb{R}^d) \subset M^p(\mathbb{R}^d)$ and $\|f\|_{M^p} \le \|f\|_{L^p}$. The Marcinkiewicz space is a particular case of Lorentz space, precisely $L^{p,\infty}(\mathbb{R}^d)$, and is also called *weak $L^p$ space.*

Marcinkiewicz spaces will be important in our study of symmetrization, tied to the idea of 'worst case strategy' that plays an important role in our study of smoothing effects, [515]. They appear also in potential theory.

## A.6 Some ideas of potential theory

Potential theory is usually done in dimensions $d \geq 3$, while dimensions $d = 1, 2$ are a bit special and need a different treatment. Therefore, we restrict our considerations to $d \geq 3$ in a first stage. Consider the fundamental solution of the Laplace equation on $\mathbb{R}^d$, $d \geq 3$:

$$E_d(x) = \frac{1}{(d-2)d\omega_d |x|^{d-2}}. \tag{A.6}$$

The Newtonian potential of an $L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$ is defined by convolution with $E_d$:

$$N(f(x)) = (f \star E_d)(x) = \int_{\mathbb{R}^d} f(y) E_d(x-y)\, dy. \tag{A.7}$$

It is known that the map $f \mapsto N(f)$ sends $L^1(\mathbb{R}^d)$ into the Marcinkiewicz space $M^q(\mathbb{R}^d) = L^{q,\infty}(\mathbb{R}^d)$ with $p = d/(d-2)$, and $L^p(\mathbb{R}^d)$ into $C_b(\mathbb{R}^d)$ if $p > d/2$. We have

$$-\Delta N(f) = f.$$

In the case of a bounded subdomain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, we use the Green function with zero boundary conditions, $G = G_\Omega(x)$, to define $\mathcal{G}f \in W_0^{1,1}(\Omega)$ by

$$(\mathcal{G}f)(x) = \int_{\mathbb{R}^d} f(y) G_\Omega(x, y)\, dy \tag{A.8}$$

and then $-\Delta \mathcal{G}(f) = f$. Clearly, $0 \leq G_\Omega(x, y) \leq E_d(x-y)$ if $d \geq 3$.

## A.7 A lemma from measure theory

We show here a version of the result that says that a continuous function can not have derivatives that are measures supported in sets where the function takes a discrete set of values.

**Lemma A.1** *Let $u(x)$ be a continuous function in a domain $Q$ of $\mathbb{R}^n$ and let $t$ be one of the coordinates. If we assume that $u_t$ is a bounded Radon measure and $u_t \in L^1_{loc}(\{u \neq 0\})$, then $u_t$ is an integrable function.*

**Proof** It is immediate to see that the measure $\mu = u_t$ can be split into the

$$\mu = f + \mu_0,$$

where $f$ is the restriction of $u_t$ to the open set $\{u \neq 0\}$, hence an $L^1_{\rm loc}$ function by assumption, and $\mu_0$ is the restriction to the closed set $K := \{u = 0\}$, a measure in principle. We also have

$$\|f\|_{L^1} + \|\mu_0\|_{\mathcal{M}} = \|\mu\|_{\mathcal{M}} < \infty.$$

We want to prove that $\mu_0 = 0$. In order to do that, we select a function $p = p_\varepsilon \in C^1(\mathbb{R})$ such that $0 \le p'_\varepsilon(s) \le 1$ and

$$p_\varepsilon(s) = s \quad \text{for } |s| \le \varepsilon, \quad p_\varepsilon(s) = 2\varepsilon \operatorname{sign}(s) \quad \text{for } |s| \ge 3\varepsilon.$$

It is clear that $p(u) = u$ on the set $K_\varepsilon = \{|u| < \varepsilon\}$, a neighbourhood of $K$, so that $p(u)_t$ restricted to $K$ is just $\mu_0$. Moreover, it can be easily proved by approximation that

$$p_\varepsilon(u)_t = p'_\varepsilon(u) f + \mu_0.$$

Take now a test function $\eta \in C_c^1(Q)$. We have $\langle p_\varepsilon(u)_t, \eta\rangle = -\int p_\varepsilon(u)\eta_t \, dxdt$, hence

$$|\langle p_\varepsilon(u)_t, \eta\rangle| = \left|\int p_\varepsilon(u)\eta_t \, dx\right| \le \varepsilon \|\eta_t\|_{L^1} = O(\varepsilon).$$

On the other hand, if $G_\varepsilon = \{-\varepsilon < u < 0\} \cup \{0 < u < \varepsilon\}$ we get

$$\left|\int p'_\varepsilon(u) f \eta \, dx\right| \le C \int_{G_{3\varepsilon}} |f| \, dx,$$

which goes to zero as $\varepsilon \to 0$ since $G_\varepsilon$ tends to the empty set. Therefore,

$$|\langle \mu_0, \eta\rangle| \le O(\varepsilon) + o(1),$$

and we conclude that $\langle \mu_0, \eta\rangle = 0$. Since $\eta \in C_c^1(Q)$ is arbitrary, we get $\mu_0 = 0$. ■

## A.8 Results for semiharmonic functions

The theory of the Cauchy problem for the PME exploits at several places the fact that non-negative solutions satisfy an estimate of the form

$$\Delta v(x,t) \ge -c(t), \quad v = u^{m-1}.$$

This property is technically called *semi-subharmonicity* of the pressure, and appears also in other nonlinear theories. It has some consequences for the size of the solution, a fact that we explore here. Some of them have been used in the proofs. Here is the technical result that we use in Chapter 9, Lemma 9.9.

**Lemma A.2** *Let $g$ be any non-negative, smooth, bounded and integrable function in $\mathbb{R}^d$ such that*

$$\Delta(g^p) \ge -K \tag{A.9}$$

*for some $p$ and $K > 0$. Then, $g \in L^\infty(\mathbb{R}^d)$ and $\|g\|_\infty$ depends only on $p$,$K$, $d$ and $\|g\|_1$ in the form*

$$\|g\|_\infty \le C(p,d)\, \|g\|_1^\rho\, K^\sigma, \tag{A.10}$$

*with $\rho = 2/(2p+d)$ and $\sigma = d/(2p+d)$.*

**Proof** Let $f(x) = g^p$. Then, $\Delta f \geq -K$. Therefore, the function

$$F(x) = f(x) + \frac{K}{2d}|x - x_0|^2 \tag{A.11}$$

is subharmonic in $\mathbb{R}^d$ for every $x_0 \in \mathbb{R}^d$. Then, for every $R > 0$ we have

$$F(x_0) \leq \fint_B F(x)dx, \tag{A.12}$$

where $B = B_R(x_0)$ and $\fint_B$ denotes average on $B$. The argument will continue in a different way for $p > 1$ and for $0 < p \leq 1$.

(i) In the latter case, $0 < p < 1$, we can use (A.12) to estimate $f$ at an arbitrary point $x_0$ as follows:

$$\begin{aligned} g^p(x_0) &\leq \fint_B g^p(x)dx + \frac{K}{2d}\fint_B |x - x_0|^2 dx \leq \left(\fint_B g\,dx\right)^p + \frac{K\,R^2}{2(d+2)} \\ &\leq \|g\|_1^p \left(\omega_d\, R^d\right)^{-p} + \frac{K}{2(d+2)}\,R^2. \end{aligned} \tag{A.13}$$

($\omega_d$ denotes the volume of the unit ball). Minimization of the last expression with respect to $R > 0$ gives

$$g^p(x_0) \leq C\,\|g\|_1^{\frac{2p}{pd+2}}\,K^{\frac{pd}{pd+2}},$$

which is equivalent to (A.10).

(ii) For $p > 1$ we modify the calculation as follows: we pick a point $x_0$ of maximum for $g$ and estimate $g(x_0)$ as follows:

$$\begin{aligned} g^p(x_0) &\leq \textstyle\fint_B g^p(x)\,dx + \frac{K}{2d}\fint_B |x|^2 dx \leq g^{p-1}(x_0)\fint_B g\,dx + \frac{K\,R^2}{2(d+2)} \\ &\leq g^{p-1}(x_0)\|g\|_1 \frac{1}{\omega_d\,R^d} + \frac{K\,R^2}{2(d+2)}. \end{aligned}$$

putting $y = g(x_0)$, we can write this expression in the form

$$y^p \leq Ay^{p-1} + B \quad \text{with} \quad A = c_1\|g\|_1 R^{-d},\ B = c_2 K R^2,$$

which after an elementary calculation gives

$$y \leq A + B^{1/p} = c_1\|g\|_1 R^{-d} + (c_2 K R^2)^{1/p}.$$

Minimization of this expression in $R$ gives (A.10).

(iii) Dimensional analysis shows that the exponents in formula (A.10) are correct. Actually, we only need to prove the formula for $\|g\|_1 = 1$, $R = 1$. ■

There is a local version of this result that we need in Chapters 12 and 18.

**Lemma A.3** *Let $g$ be any non-negative, smooth, bounded function in the ball $B_2 = B_{2R}(a) \subset \mathbb{R}^d$, and assume that $g \in L^1(B_2)$ and*

$$\Delta(g^p) \geq -K \tag{A.14}$$

*for some $p$ and $K > 0$. Then $g \in L^\infty(B_1)$ with $B_1 = B_R(a))$ and $\|g\|_\infty$ depends only on $s$, $K$, $d$, $R$ and $\|g\|_{L^1(B_2)}$. More precisely,*

(i) *We have*

$$\|g\|_{L^\infty(B_R(0))} \leq C(p,d)\left(\|g\|_{L^1(B_{2R}(0))}R^{-d} + K^{1/p}R^{2/p}\right). \tag{A.15}$$

(ii) *If $\|g\|_1$ is very small compared with $R$ and $K$ the estimate takes the form*

$$\|g\|_{L^\infty(B_1)} \leq C(p,d)\,\|g\|^{\rho}_{L^1(B_2)}\,K^{\sigma}, \tag{A.16}$$

*with $\rho = 2/(2p+d)$ and $\sigma = d/(2p+d)$. The smallness condition is*

$$\|g\|^p_{L^1(B_2)} \leq cKR^{dp+2}. \tag{A.17}$$

**Proof** If $0 < p \leq 1$ it is very similar to the previous one replacing $\mathbb{R}^d$ by $B_{2R}(a)$. Indeed, part (i) can be repeated to get for every $x_0$ with $|x_0| \leq r$, with $0 < r < R$, integrating in $B = B_r(x_0)$ to get:

$$g^p(x_0) \leq \omega_d^{-p} r^{-dp}\|g\|^p_{L^1_{B_2(0)}} + \frac{Kr^2}{2(d+2)}.$$

Minimization in $0 \leq r \leq R$ gives a bound for $g(x_0)$. In particular, when $\|g\|_1$ is small enough the minimum takes place for $0 < r < R$ and we obtain the stated result in this case by the same calculation as in the previous lemma. The smallness condition $r_{\min} < R$ is implied by (A.17).

(ii) When $p > 1$ the technique of proof has to be changed. Actually, the result is implied by Theorem 9.20 of Gilbarg-Trudinger [261], which use Aleksandrov's maximum principle. ■

## A.9 Three notes on the Giant and elliptic problems

We review here some approaches to the construction of the Giant, i.e., the positive self-similar solution of the PME with separated variables form, $u(x,t) = t^{-\alpha}f(x)$. As we have said, it is equivalent to solving the nonlinear elliptic problem $\Delta f^m + \alpha f = 0$, with $f = 0$ on $\partial\Omega$. As we have said in Section 5.9, it is best written in the form (5.70)

$$\Delta g + \frac{1}{m-1}g^{\frac{1}{m}} = 0, \quad g \in H^1_0(\Omega). \tag{A.18}$$

with $\alpha = 1/(m-1)$. Up to a constant, it is the same as (4.6). See also (20.16). We can view this equation as a **nonlinear eigenvalue problem**.

### A.9.1 *Nonlinear elliptic approach. Calculus of variations*

For experts in elliptic equations, the typical approach to solving the semilinear elliptic equation (A.18) is to view the solution $g$ as a **critical point** of the functional

$$J(g) = \frac{1}{2}\int_\Omega |\nabla g|^2\,dx - \frac{\alpha m}{m+1}\int_\Omega |g|^{\frac{m+1}{m}}\,dx, \tag{A.19}$$

defined in $H_0^1(\Omega)$.

**Theorem A.4** *The positive solution of* (A.18) *is the minimum of* $J$ *in* $H_0^1(\Omega)$.

**Proof** (i) $J$ is well defined in $H_0^1(\Omega)$: simply observe that $1 + 1/m < 2$ and use Sobolev embeddings.

(ii) $J$ is bounded from below in $H_0^1(\Omega)$: in fact, using Poincaré's inequality we get

$$J(g) \to \infty \qquad \text{as} \quad \|g\|_{H_0^1} \to \infty.$$

(iii) The infimum is negative, hence it cannot correspond to the trivial function. Take a family of functions of the form $g_s(x) = s\,g_1(x)$ with some $g_1 \in H_0^1(\Omega)$, $g_1 \geq 0$. Then

$$J(g_s) = A\,s^2 - B\,s^{\frac{m+1}{m}}$$

for some positive $A, B$. Hence $J(g_s) < 0$ for some $s$ near 0.

(iv) Along any minimizing sequence there is convergence in $H_0^1(\Omega)$ and the infimum is taken, hence it is a minimum.

Observe first that $J(g_n)$ converges to $J_{\min}$. Then $|\nabla g_n|$ is uniformly bounded in $L^2(\Omega)$, hence $g_n$ converges weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$ to some $g \in H_0^1(\Omega)$. In the limit we have by the standard argument of lower semi-continuity of the integral of the gradient square:

$$J_{\min} \geq \frac{1}{2}\int_\Omega |\nabla g|^2\,dx - \frac{\alpha\,m}{m+1}\int_\Omega |g|^{\frac{m+1}{m}}\,dx.$$

Note that $(m+1)/m < 2$. But $J_{\min}$ is the minimum, hence there must be equality. This implies that

$$\lim_{n\to\infty}\int_\Omega |\nabla g_n|^2\,dx = \int_\Omega |\nabla g|^2\,dx,$$

which means that $g_n \to g$ in $H_0^1(\Omega)$ [Explanation: We are using the lemma: if $f_n \to f$ weakly in $L^2(\Omega)$ and $\|f_n\|_2 \to \|f\|_2$, then the convergence is strong. The proof consists of writing the difference

$$\int_\Omega |f_n - f|^2\,dx = \int_\Omega f_n^2\,dx + \int_\Omega f^2\,dx - 2\int_\Omega f_n f\,dx,$$

and taking limits.]

(v) The minimum satisfies equation (A.18).

Let $g$ be the minimum. Consider the family $g_\varepsilon = g + \varepsilon\phi$ where $\phi \in C_c^\infty(\Omega)$ is any non-negative test function and $\varepsilon$ is a real number. Write $J(g_\varepsilon) - J(g) \geq 0$ as

$$\frac{1}{\varepsilon}(J(g_\varepsilon) - J(g)) = \int_\Omega \nabla g \cdot \nabla\phi\, dx + \frac{\varepsilon}{2}\int_\Omega |\nabla\phi|^2\, dx - \alpha \int_\Omega \tilde{g}_\varepsilon^{1/m}\,\phi\, dx,$$

where $\tilde{g}_\varepsilon(x)$ is a value between $g(x)$ and $g(x) + \varepsilon\phi(x)$ (mean value theorem). Take now $\varepsilon > 0$ and pass to the limit $\varepsilon \to 0$ to get

$$\int_\Omega \nabla g \cdot \nabla\phi\, dx - \alpha\int_\Omega g^{1/m}\,\phi\, dx \geq 0.$$

When $\varepsilon < 0$ we get the converse inequality. Therefore, (A.18) holds in the sense of distributions (This classical calculation is called in the calculus of variations 'obtaining the Euler–Lagrange equation'.)

(vi) Any solution of (A.18) satisfies $\int |\nabla g|^2\, dx = \alpha \int |g|^{(m+1)/m}\, dx$, hence

$$J(g) = -\frac{\alpha(m-1)}{2(m+1)}\int_\Omega |g|^{\frac{m+1}{m}}\, dx. \tag{A.20}$$

The absolute minimum corresponds therefore to the maximal stationary solution which is the positive one.

(vii) The uniqueness of the positive solution in this kind of 'nonlinear eigenvalue problems' is a well-known result in the calculus of variations. It comes from a general result of functional analysis, Krein–Rutman's theorem. ■

**Note** The constant $\alpha > 0$ in (20.15), (A.18) plays no role since it can be given any value after a rescaling. Indeed, if $g$ is a solution of (A.18) and we put

$$G(x) = \lambda\, g(x), \qquad \lambda = \alpha^{\frac{m}{m-1}}, \tag{A.21}$$

then $G$ satisfies $\Delta G + G^{1/m} = 0$. This is a curious property of some nonlinear problems, that linear eigenvalue problems do *not* have.

### A.9.2 *Another dynamical proof of existence*

We construct the Giant, i.e., a positive self-similar solution of the separated variables form, $u(x,t) = t^{-\alpha} f(x)$, by a different method, based also on the properties of the evolution. As we have said, it is equivalent to solving the nonlinear elliptic problem

$$\Delta f^m + \alpha\, f = 0,$$

with $f = 0$ on $\partial\Omega$. The idea is to take a sequence of solutions with data

$$u_{0n}(x) = n, \tag{A.22}$$

we obtain a unique weak solution $u_n(x,t) \ge 0$ of the PME. (Note: the reader may prefer to take $u_{0n}(x) = n\phi(x)$, where $\phi$ is a nice smooth and positive function in $\Omega$ that vanishes on the boundary. He is welcome.) The family $\{u_n\}_n$ is monotone increasing in $n$ (maximum principle). There exists a limit

$$U(x,t) = \lim_{n\to\infty} u_n(x,t),$$

and this limit is finite, since it satisfies the universal estimate $U(x,t) \le C\,t^{-\alpha}$. The scaling transformation

$$(\mathcal{T}_k u)(x,t) = ku(x, k^{m-1}t)$$

produces out of a solution of equation (20.1) with data $u_0(x)$ another solution of the same equation with initial data $(\mathcal{T}_k u)(x,0) = ku_0(x)$. It thus transforms $u_n$ into $u_{nk}$. In the limit it transforms $U$ again into $U$. Therefore, $U$ is scaling invariant:

$$k\,U(x, k^{m-1}t) = U(x,t)$$

for all $x \in \Omega$ and $k, t > 0$. In other words, setting $k^{m-1}t = 1$,

$$U(x,t) = t^{-\alpha}U(x,1) = t^{-\alpha}\, f(x).$$

It is clear that $g = f^m \in H_0^1(\Omega)$ is a positive and bounded solution of the nonlinear eigenvalue problem (A.18).

### A.9.3 *Another construction of the Giant*

The giant can also be obtained as the limit of the so-called fundamental solutions, i.e., the solutions $u_c(x,t)$ of the problem with initial data

$$u_c(x,0) = c\,\delta_a(x), \tag{A.23}$$

where $a$ is any point in $\Omega$ and $\delta_a(x)$ is Dirac's delta function with singularity located at $a$. Such solutions exist in the weak sense and the data are taken as initial traces in the sense of bounded measures. It can be proved that

$$\lim_{c\to\infty} u_c(x,t) = U(x,t). \tag{A.24}$$

The convergence to the Giant has been justified for the similar situation occurring for the equation of diffusion-absorption

$$u_t = \Delta u^m - u^p, \tag{A.25}$$

with $0 \le q \le m$, cf. [162], Theorem 7.1. It is then enough to take $p = 1$ and make a change of variables $v = u\,e^t$ with corresponding scaling of time to obtain the desired result for the PME. Note that depending on the equation other types of limit may occur. A classification of the four different types of limits of the fundamental solutions as $c \to \infty$ which are possible for nonlinear heat equations has been performed by Vázquez and Véron in [518].

## A.10 Optimality of the asymptotic convergence for the PME

We devote this section to a first exploration of the sharpness of the convergence rates in Theorem 18.1 for general classes of initial data. This is taken from [509], pp. 91–93.

*Counterexample*

*Given any decreasing function $\rho(t) \to 0$, there exists a solution of the Cauchy problem with integrable and non-negative initial data of mass $M > 0$ such that*

$$\mathit{lim\ sup}_{t\to\infty} \frac{(u(0,t) - \mathcal{U}(0,t;M))\, t^{\alpha}}{\rho(t)} = \infty. \tag{A.26}$$

*Moreover, we can also get*

$$\mathit{lim\ sup}_{t\to\infty} \frac{\|u(t) - \mathcal{U}(t;M)\|_1}{\rho(t)} = \infty. \tag{A.27}$$

*We can also ask the solution to be radially symmetric with respect to the space variable.*

### Construction

(i) We recall that the proof need only be done for $M = 1$ since the scaling transformation

$$\widehat{u}_c(x,t) = c^{m-1}\, u(x, c\,t) \tag{A.28}$$

reduces a solution of mass $M > 0$ to a solution of mass 1 if $c = M^{-1/(m-1)}$. We take an initial function of the form

$$u_0(x) = \sum_{k=1}^{\infty} c_k\, \chi_k(x - a_k),$$

where $\chi_k(x)$ is the characteristic function of the ball of radius $r_k$ centred at 0. The sequences $a_k, c_k$ and $r_k$ have to be determined in a suitable way. In the first place, we impose the conditions $c_k, r_k \ge 0$ and $c_k\, r_k^n = 2^{-k}/\omega$ (where $\omega$ is the volume of the ball of radius 1). Then, $M = \omega \sum_1^{\infty} c_k\, r_k^n = 1\,.$

(ii) We construct solutions $u_k$ with initial data of the form

$$u_k(x,0) = \sum_1^{k} c_i\, \chi_i(x - a_i), \tag{A.29}$$

and we proceed to choose $c_k$ and $a_k$ in an iterative way. In any case the mass of $u_k$ is $M_k = 1 - 2^{-k}$, and we observe that (by the main convergence result) for every $\varepsilon > 0$ there must be a time $t_k(\varepsilon)$ (which depends also on the precise choice of the initial data) such that

$$t^{\alpha} |u_k(0,t) - \mathcal{U}(0,t;M_k)| \le \varepsilon$$

for all $t \geq t_k(\varepsilon)$. We now recall that $\mathcal{U}(0,t;M) = c\,M^{2\beta}\,t^{-\alpha}$, so that the difference between $t^\alpha \mathcal{U}(0,t;M)$ and $t^\alpha \mathcal{U}(0,t;M')$ is constant in time, and in fact it can be estimated as larger than

$$t^\alpha\,(\mathcal{U}(0,t;M) - \mathcal{U}(0,t;M')) \geq k_1\,(M - M')$$

with the same constant $k_1 > 0$ for all $1 \geq M > M' \geq 1/2$.

(iii) The iterative construction of the $u_k$ starts as follows. We may take $c_1$ as we like, e.g., $c_1 = 1$, then $r_1 = (2\omega)^{-1/n}$, and find the solution $u_1(x,t)$ with data $u_1(x,0) = c_1\,\chi_1(x)$. Its mass is $M_1 = 1/2$ for all times. As said above, for sufficiently large times we have

$$t^\alpha |u_1(0,t) - \mathcal{U}(0,t;M_1)| \leq \varepsilon.$$

We can also find $t_1$ such that $\rho(t_1) < (1/2)k_1\,(M - M_1) = k_1/4$. Using the estimate for the difference of source-type solutions and the triangular inequality, and taking $\varepsilon$ small enough ($\varepsilon \leq k_1/4$), we get for all $t \geq t_1$

$$\begin{aligned} &t^\alpha |u_1(0,t) - \mathcal{U}(0,t;1)| \\ &\quad \geq t^\alpha |\mathcal{U}(0,t;1) - \mathcal{U}(0,t;M_1)| - t^\alpha |u_1(0,t) - \mathcal{U}(0,t;M_1)| \qquad \text{(A.30)} \\ &\quad \geq k_1(1 - M_1) - \varepsilon \geq k_1/4 \geq \rho(t_1) \geq \rho(t). \qquad \text{(A.31)} \end{aligned}$$

(iv) Iteration step. Assuming that we have constructed $u_2, \ldots, u_{k-1}$ by solving the equation with data (A.29), we proceed to choose $c_k$, and $a_k$ and construct $u_k$ as follows. We can take any $c_k > 0$, and then find $a_k$ large enough so that the support of the solution $v_k$ with initial data $v_k(x,0) = \chi_k(x - a_k)$ does not intersect the support of $u_{k-1}$ until a time $t_k > 2\,t_{k-1}$ (and we can even estimate how far $a_k$ must be located for large $t_k$ because we have a precise control of the support of $u_{k-1}$ for large times, thanks to Theorem 18.8). Then, it is immediate to see that

$$u_k(x,t) = u_{k-1}(x,t) + v_k(x,t)$$

for all $x \in \mathbb{R}^n$ and $0 \leq t \leq t_k$ (i.e., superposition holds as long as the supports are disjoint). Indeed, this means that for all $0 \leq t \leq t_{k-1}$ we also have $u_k(0,t) = u_{k-2}(0,t)$, and by iteration we conclude that

$$u_k(0,t) = u_j(0,t) \quad \text{for all } 1 \leq j < k \text{ and } 0 \leq t \leq t_{j+1}.$$

We now remark that $t_k$ can be delayed as much as we like (on the condition of taking $a_k$ far away). If we choose $t_k$ large enough, the main asymptotic theorem implies the behaviour

$$t_k^\alpha u_k(0,t_k) = t_k^\alpha u_{k-1}(0,t_k) \sim t_k^\alpha \mathcal{U}(0,t_k;M_{k-1}).$$

We want the error to be less than $k_1(1 - M_k)/2 = 2^{-(k+1)}k_1$. We also suggest to wait until $\rho(t_k) \leq 2^{-(2k+1)}\,k_1$. Using again the triangle inequality:

$|u_k - \mathcal{U}(M)| \ge |\mathcal{U}(M) - \mathcal{U}(M_{k-1})| - |u_k - \mathcal{U}(M_{k-1})|$ with $M = 1$, we get

$$t_k^\alpha |u_k(0,t_k) - \mathcal{U}(0,t_k;1)| \ge 2^{-(k+1)}k_1 \ge 2^k \rho(t_k).$$

(v) In the final step we take the limit

$$u(x,t) = \lim_{k\to\infty} u_k(x,t).$$

By what was said before we may conclude that for $t \le t_k$ we have $u(0,t) = u_k(0,t)$, so that

$$\lim_{n\to\infty} t_k^\alpha \frac{u(0,t_k) - \mathcal{U}(0,t_k;1)}{\rho(t_k)} = \infty.$$

This concludes the proof of the $L^\infty$-estimate.

(vi) The construction can be easily modified so that the data $u_0$ are radially symmetric by defining $\chi_k$ to be the characteristic function of the annulus $A_k = \{x : a_k \le |x| \le a_k + r_k\}$ and imposing that $c_k$ times the volume of $A_k$ to equal $2^{-k}$. The construction is repeated with the same attention to be given to $a_k$, i.e., to the far location of the $A_k$.

(vii) For the $L^1$ part we just observe that, taking $t_k$ large enough we have at time $t = t_k$ and in a very large ball $B_k$ (as large as we please by the iteration construction) the equality $u = u_k$ and the approximation

$$\|u_k - \mathcal{U}(x,t_k;1)\| \ge \|u_k - \mathcal{U}(x,t_k;1)\|_{L^1(\{x\notin B_k\})} \ge 2^{-k},$$

since the mass of $u$ contained outside this ball is known ($2^{-k}$), and that of $\mathcal{U}$ is zero there. The result follows if the $t_k$ have been chosen as before, $\rho(t_k)2^k \to 0$. ■

## A.11 Non-contractivity of the PME flow in $L^p$ spaces

We complete here the analysis started in Section 4.5.2 about lack of contractivity of the PME flow in different $L^p$ spaces. This is the result that comes out of the blow-up example.

**Theorem A.5** *The PME flow posed in the whole space is not contractive in the spaces $L^p(\mathbb{R}^d)$ if $m \ge 2$ and $p_c < p \le \infty$ with*

$$p_c = 2 + \frac{d(m-1)}{2}. \tag{A.32}$$

*Neither is the flow in a bounded domain with homogeneous Dirichlet or Neumann boundary conditions.*

**Proof** We divide the proof in several steps for more clarity. We start with the Cauchy problem.

*1. Case $L^\infty$*

(i) We have seen in formula (4.48) an example of two non-negative solutions whose pressures are ordered and differ by a constant $C(t)$ for every fixed $t$ and this constant grows with time. When $m = 2$ pressure and density are proportional, so that this example shows a particular case of increase of the norm

$$d_\infty(U_2, U_1, t) := \|U_2(\cdot, t) - U_1(\cdot, t)\|_\infty.$$

(ii) It can be objected that the example is constructed in the class of growing solutions and not in a more natural class, like bounded solutions. Such an objection is easily overcome by continuity. We construct two increasing families of solutions $u_{in}(x, t)$, $i = 1, 2$, with non-negative data $u_{in}(x, 0) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ so that $u_{in}(x, 0) \uparrow U_i(x, 0)$ a.e. The functions $u_{in}(x, 0)$ can be even smooth and compactly supported. If we perform the construction in a suitable way we have

$$d_\infty(u_{2n}, u_{1n}, 0) \leq d_\infty(U_2, U_1, 0) = C(0).$$

On the other hand, we know by the local regularity theory that the families of solutions $u_{1n}$, $u_{2n}$ are locally uniformly Hölder continuous as long as they are locally bounded, and this happens for $0 < t < T - \varepsilon$. We conclude that $u_{in}(x, t) \to U_i(x, t)$ locally uniformly, so that given $\varepsilon > 0$ we have

$$d_\infty(u_{2n}, u_{1n}, t) \geq d_\infty(U_2, U_1, t) - \varepsilon = C(t) - \varepsilon.$$

for all large $n$ and $0 < t < T - \varepsilon$. Since $C(t) > C(0)$ the proof is complete in this case.

(iii) The situation for $m > 2$ is even better since $U_i = aV_i^\gamma$ with $\gamma = 1/(m-1) < 1$. We now observe that given $C > 0$, the function

$$f(v) = (v + C)^\gamma - v^\gamma,$$

is decreasing in $v$ for $0 \leq v < \infty$. This implies that, since $V_2 = V_1 + C(t)$, for every fixed time the maximum of the difference

$$U_2 - U_1 = a(V_2^\gamma - V_1^\gamma)$$

is taken at the minimum value of $V_1$, i.e., at $x = 0$. But then we have

$$d_\infty(U_2, U_1, t) = U_2(0, t) - U_1(0, t) = a(C_2^{1/(m-1)} - C_1^{1/(m-1)})/(T - t)^\alpha,$$

and this goes to infinity as $t \to T$.

The adaptation to the class of bounded and integrable solutions is done as before.

*2. Case $L^p$, p large*

(i) We modify the argument of Case 1 with the problem posed in the whole space by defining $U_2$ as a modification of $U_1$ in the two available parameters, $T$ and $C$. In this case we take $V_1$ as blow-up solution with $C=0$ and $T=1$

$$V_1(x,t) = \frac{K\,|x|^2}{1-t}$$

where $K$ is an unimportant universal constant, and

$$V_2(x,t) = \frac{C(T-t)^{2\beta} + K\,|x|^2}{T-t}$$

where $C>0$ and $T=1+\varepsilon$ with $\varepsilon$ small. Let us define

$$N_p(U_2,U_1,t) = \int_{\mathbb{R}^d} (U_2(x,t) - U_1(x,t))_+^p\,dx$$

for all $0<t<1$. We are going to prove that this quantity increases with time for all large $p$.

(ii) In order to estimate it at $t=0$ we calculate the point where both initial functions are equal as

$$x^2(0) \approx C/K\varepsilon,$$

where we have used the fact that $T=1+\varepsilon\approx 1$. In this interval the maximum value of the difference $U_2(x,0)-U_1(x,0)$ is of order $C^\gamma$, $\gamma = 1/(m-1)$, and we have

$$N_p(U_2,U_1,0) \le cC^{p\gamma}x(0)^d = c\,C^{p\gamma+d/2}\varepsilon^{-d/2}.$$

(iii) We now estimate $N_p$ at a time $t_1 = 1-\tau$. We have the values of the pressure at the origin

$$V_2(0,t_1) = \frac{C}{(\tau+\varepsilon)^{\alpha(m-1)}}, \quad V_1(0,t) = 0.$$

Moreover, $V_1(x,t_1) \le V_2(0,t_1)/2$ in a ball of radius

$$x^2(t_1) \approx \frac{C(\tau+\varepsilon)^{2\beta}\tau}{\tau+\varepsilon}$$

so that

$$\begin{aligned} N_p(U_2,U_1,t_1) &\ge c\frac{C^{p\gamma}}{(\tau+\varepsilon)^{p\alpha}}\,\frac{C^{d/2}(\tau+\varepsilon)^{d\beta}\tau^{d/2}}{(\tau+\varepsilon)^{d/2}} \\ &= c\,C^{p\gamma+d/2}\,\frac{(\tau+\varepsilon)^{d\beta}\tau^{d/2}}{(\tau+\varepsilon)^{pd\beta+n/2}}. \end{aligned}$$

We therefore need the inequality

$$\frac{(\tau+\varepsilon)^{(p-1)d\beta+d/2}}{\tau^{d/2}\varepsilon^{d/2}} \to 0.$$

This holds for $p-1>1/2\beta$, i.e., $p>2+(d(m-1)/2)$ with $\tau\sim\varepsilon$.

In order to revert to the comparison of $L^p$ norms we replace $U_2$ by the solution $U_3$ with initial data

$$U_3(0)=\max\{U_1(0),U_2(0)\}.$$

Then, we have

$$\|U_3(0)-U_1(0)\|_p=N_p(U_2,U_1,0).$$

On the other hand, the observation that $U_3\geq U_2$ and $U_3\geq U_1$ leads to the inequality

$$\|U_3(t_1)-U_1(t_1)\|_p\geq N_p(U_2,U_1,t_1)>N_p(U_2,U_1,0).$$

The proof of increase of the $L^p$ norm of the difference is thus complete, modulo approximation with bounded solutions if we want to prove the result in that class.

*3. The Dirichlet data*

The approximation process that we have mentioned before can be done with solutions of Dirichlet problems or Neumann in expanding balls. We conclude that for some of these balls there is an example of non-contraction for the same $m$ and $p$ as in the Cauchy problem. Since the PME is invariant under scaling the result is true for all balls.

For the case of a general domain replace the balls of radius $R\to\infty$ by scaled copies of the domain and argue in the same way as before. ■

*Open problems*

(1) What is the best bound for $p_c$ in the above result. Is $p_c=2$? Is $p_c=1$?
(2) Extend the result to the range $1<m<2$.

### A.11.1 *Other contractivity properties*

(i) Contractivity in $H^{-1}(\Omega)$ is discussed in Sections 6.7 and 10.1.4. It applies to the GPME.
(ii) The PME semigroup in $\mathbb{R}$ is contractive with respect to all Wasserstein distances defined in Section 10.4. The focusing solutions are used in [514] to show that the PME semigroup in $\mathbb{R}^d$ is not contractive in these Wasserstein

metrics $d_p$ if $p$ is large enough, including $p = \infty$. However, the semigroup is contractive in the case $p = 2$ [152]. Thus has been used very elegantly by Toscani in proving sharp asymptotics [494].

See also [158] where the asymptotic complexity of the patterns of the GPME is studied.

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