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Tomás Chacón Rebollo
Roger Lewandowski

Mathematical and Numerical Foundations of Turbulence Models and Applications

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Tomás Chacón Rebollo
Department of Differential Equations
and Numerical Analysis and Institute of
Mathematics (IMUS)
University of Seville
Seville, Spain

Roger Lewandowski
Institute of Mathematical Research
of Rennes, IRMAR - UMR 6625, CNRS
University of Rennes 1
Rennes, France

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To my wife Mame and my children Carmen and Tomás
Tomás Chacón Rebollo

To my wife Sophie and my children Noé, Yann, Sarah, and Marie
Roger Lewandowski

Preface

This book is about turbulence in incompressible fluids.

We have asked people in the street what the word “turbulence” means for them. One woman replied: “Turbulence makes me think of the sea, because it makes one feel what is invisible, what cannot be predicted.” More generally, people answered giving only one word, such as disorder, aircraft, clouds, weather forecast, power, and chemistry. Therefore, turbulence is something that anyone has experienced in one way or another. Mathematicians will answer that turbulence is about fluids, mixing, chaos, and connected scales. It may be a source of inspiration for painters or poets. One may attempt to control it for technological progress. It is however a source of concern because of its impact on environment and human life, the most critical environmental challenge being climate change.

Although understanding turbulence is of primary importance, there is no mathematical definition of it, and many physical mechanisms governing turbulent motions remain unknown. One could say that there is a chance for mankind to understand quantum physics someday, but not turbulence. Nevertheless, it is possible to simulate by means of computers some features of turbulent motions: weather forecasts are rather accurate over 5 days, the mean Gulf Stream path can be calculated, numerical flow simulations around an aircraft wing are in good agreement with experimental data, etc. All these numerical simulations are performed by means of “turbulence models.”

Turbulence models aim to simulate statistical means of turbulent flows or some of their scales. It is however estimated that an accurate computation of all scales of such flows will be possible only by the end of the twenty-first century, if the improvement of the computational resources continues at the same rate.

We do not pretend to give a definition of what turbulence is. Our goal is to provide a comprehensive and innovative presentation of turbulence models, at the crossroads of modeling and mathematical and numerical analysis, including all these aspects in one single book, in complementarity with the other reference manuals in the field.

This book is the synthesis of almost 20 years of thoughts and works about turbulence models, through the meeting of a mathematician with a numerical analyst, leading to a long-term collaboration and friendship. This resulted in several joint research works, which gave us the opportunity to check that the complementarity of these specialities can be quite fruitful. Finally, it led us to the project of jointly writing a book from a comprehensive point of view on one of the most challenging scientific problems, as is the understanding of turbulence: we deliver here what we are able to understand from turbulence.

In mathematics, authors are always listed in alphabetical order, which is the case of this book.

Seville, Spain — Tomás Chacón Rebollo
Rennes, France — Roger Lewandowski
January 2014

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We were first involved in turbulence by Claude Bardos and Olivier Pironneau several decades ago. Our constant discussions with them throughout these years have been crucial in our understanding of the turbulence models, so that our warmest thanks go to them.

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January 2014

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Rennes, France Roger Lewandowski
January 2014

Contents

Chapter 1
Introduction

Understanding turbulence is one of the oldest and most challenging scientific problems. Since the early works of Boussinesq and Reynolds in the late nineteenth century that formalized the basic characteristics of turbulent flows, the analysis of the extremely complex behavior of turbulence has raised the interest of many researchers. The issue of what is “turbulence” is still far from being solved, although some facts can be deduced from observations and experiments. Turbulent flows have a huge impact in human life, from weather forecasting to freshwater supply, energy generation, navigation, biological processes and so on.

Numerical simulation of turbulence is thus of primary importance to improve human life in many ways. Classical fluid mechanics establishes that the motion of a viscous fluid is governed by the Navier–Stokes equations, which in theory should be appropriate to perform numerical simulations of turbulent flows.

However, a turbulent flow is a highly irregular system, characterized by chaotic property changes involving a wide range of scales in nonlinear interaction with each other. These features yield a high computational complexity, which makes today direct numerical simulations of turbulent flows from the Navier–Stokes equations impossible. This is why turbulence models are introduced, in order to reduce this computational complexity.

Besides experiments and physics, mathematical modeling and analysis play a central role in the study of turbulent flows. Mathematics provide a permanent support to build turbulent models for weather forecasting and meteorology, oceanography, climatology, and environmental and industrial applications. Industrial flow softwares (Ansys-Fluent, Comsol, Femap,…) are deeply based upon mathematical and numerical analysis.

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1.1 To Whom the Book Is Addressed

This book is mainly addressed to the research community in mathematical and numerical analysis in fluid mechanics. However, some parts may interest physicists and engineers. It may be used as a general starting guide for PhD students and post-docs in all aspects covered by the book (modeling, mathematical and numerical analysis, numerical simulation) as well as advanced researchers willing to improve their formation in mathematical and numerical analysis of turbulent incompressible flows. A good knowledge in linear algebra and mathematical analysis (including basic aspects of measure theory and functional analysis) is needed.

1.2 Objectives

Our goal is to provide a comprehensive interdisciplinary reference for mathematical modeling, theoretical and numerical analysis, and simulation of 3D incompressible turbulent flows. Turbulence models are intrinsically either continuous or discrete. Theoretical and numerical analysis are strongly interconnected and both share common mathematical bases. Therefore we introduce an integrated approach to model and analyze turbulence models, where modeling and theoretical and numerical analysis are built on common mathematical grounds. In particular, we aim:

1. to take a significant step forward in modeling of 3D incompressible turbulent flows, elaborated as a rigorous and innovative mathematical concept, based on the combination of the fundamental laws of mechanics and physics with measure theory, probabilities, and functional analysis,
2. in resolving some of the mathematical issues raised by the complex nonlinear partial differential equations (PDE) resulting from the modeling process, to identify and implement the most effective mathematical methods to deal with them and to generate a mathematical shell appropriate to such complex systems,
3. to develop basic discretization techniques used in numerical simulations of relevant turbulence models, by means of discretization schemes with complexity reduced to the needs, and to show the practical performances of the models and numerical techniques introduced, which provides a starting guide to the effective numerical solution of turbulence models.

1.3 Structure and General Characteristics

The book is divided into three parts, modeling, analysis, and numerics:

1. Mathematical modeling and derivation of turbulence models, Chaps. 2 to 5
2. Mathematical analysis of the continuous NS-TKE model in a 3D domain, Chaps. 6 to 8

3. Numerical analysis of discrete Smagorinsky and variational multiscale (VMS) models, numerical approximation of the NS-TKE continuous model, and numerical simulation of relevant benchmark flows, Chaps. 9 to 13

Finally, the book ends with an appendix, Chap. A called the "toolbox," devoted to report general mathematical results from functional analysis and measure theory that are used throughout the book.

1. Fluid dynamics and turbulence modeling are often based on heuristic reasonings. We clarify as much as possible the mathematical arguments, in order to reduce and justify the use of heuristic reasonings, in complementarity with other reference manuals in the field. This approach opens up new leads to improve the understanding of mathematical turbulence models based on eddy viscosities, such as the $k - \mathscr{E}$ and large eddy simulations models, and the boundary layer structure by wall laws.

   It is recommended that beginners in fluid mechanics start with Chaps. 2 and 3. However, advanced readers might directly read Chaps. 4 and 5. This part is particularly appropriate for mathematicians and numericians willing to understand how to derive the PDEs of fluid mechanics from basic physical laws.
2. The continuous NS-TKE model couples a Navier–Stokes (NS)-like equation to an equation for the turbulent kinetic energy (TKE), with nonlinear wall laws on the boundary. This model is a by-product of the $k - \mathscr{E}$ model, simpler but which shares the same mathematical features as $k - \mathscr{E}$, such as eddy viscosities and quadratic source terms. The results obtained for NS-TKE can easily be extended to $k - \mathscr{E}$. The mathematical analysis is based on variational and singular perturbation methods, advanced estimate techniques, and compactness methods. Due to the complexity of this system, we have integrated and condensated some of standard PDE's concepts in abstract and generalized packages, hence a new mathematical shell.

   Chapters 6 and 7 deal with steady-state cases. It is recommended to read them one after another. Chapter 8 deals with evolutionary cases and might be read independently.

   Notice that the analysis of continuous LES models is not performed in this book, since it can be found in other books quoted in a thorough bibliography at the end of Chap. 8.
3. Smagorinsky and VMS models are viewed as intrinsic discrete models, specified by finite element schemes, appropriate to fit complex geometric flow configurations, and compatible with industrial en environmental solvers. The asymptotic behavior of their energy balance determines the dissipative structure of the models. The finite element approximation of the NS-TKE continuous model provides an alternative process to simulate turbulent flows from $k - \mathscr{E}$ like models.

   Finally we display practical simulations of some well-known turbulent flows, giving the main source code.

   It is recommended to read Chaps. 9 and 10 before Chaps. 11 and 12. However, Chap. 12 might be read independently.

The analysis techniques used for theoretical and numerical analysis are basically the same: construction of approximated problems by regularization, obtention of stability estimates, and application of the compactness and energy methods.

Each chapter begins with an abstract and an introduction, ends with a bibliography, and is divided into sections and subsections, so that it can be viewed as an independent entity.

However, the global hyperlinks structure of the book connects the chapters and the sections with each other. In particular, a reader who aims to focus on a specific result in a given section can do it by browsing and using the book as a web site, to easily get the connected results, notations, equations, estimates, ..., at different locations of the book.

1.4 Description of Contents

Chapters 2 and 3 are general fluid mechanics chapters, in which we carry out a mathematical derivation of the incompressible Navier–Stokes equations, by means of the mass conservation principle and Newton's law considered in the framework of abstract measure theory. The kinematic of fluids is studied by elementary ordinary differential equations, which highlights the role played by the shear stress and the vorticity during turbulent motions. We formulate the dimensional analysis in an abstract algebraic framework by the introduction of dimensional bases, which allows to properly define the notion of Reynolds numbers. Therefore, we are able to express the Reynolds similarity law through rigorous mathematical definitions and to investigate its area of validity by means of Leray and Fujita–Kato Theorems about global and local solutions to the Navier–Stokes equations. We observe that Leray's solutions to the NSE have long-time averages that satisfy equations involving a Reynolds stress, which entirely justifies by rigorous theorems a steady-state turbulence model.

Chapters 4 and 5 are devoted to the derivation of the models and the main laws of the turbulence. We define the expectation of turbulent fields, also called mean fields, from Fujita–Kato's strong solutions to the NSE by the construction of a probability measure over initial data sets. We express in this framework the Reynolds problem and the Boussinesq closure assumption which yields the concept of eddy viscosity. We introduce the correlation tensors and various definitions of local homogeneous turbulence. We derive the $k - \mathscr{E}$ model from the NSE and discuss the sufficient conditions on the correlations that yield the consistency of this derivation by mathematical theorems. We define local isotropic turbulent flows and prove the Kolmogorov $-5/3$ law for such flows by the appropriate similarity assumptions, once the similarity assumption principle is clearly stated by means of dimensional bases and appropriate scales. This principle is also operated to determine the structure of the boundary layer near any given wall, which yields the wall laws that we include as nonlinear boundary conditions in all the models we study. Wall laws are widely used in engineering applications, ocean–atmosphere dynamics, etc., as a technique to avoid the computation of the very costly boundary layer. Finally we make the connection between the $-5/3$ law and LES models such as the Smagorinsky model.

Chapters 6 and 7 are devoted to the steady-state NS-TKE model by means of variational formulations, in a 3D bounded domain. The difficulties are due to the eddy viscosities, the quadratic source terms, and the wall laws. We first study the NSE with a given eddy viscosity and a wall law, specified by a variational problem $\mathscr{V}\mathscr{P}$, that we approximate by a family $(\mathscr{V}\mathscr{P}_\varepsilon)_{\varepsilon>0}$ derived from a singular perturbation of the incompressibility constraint, called the ε-approximation. Any solution to $\mathscr{V}\mathscr{P}_\varepsilon$ is shown to satisfy stable estimates and $\mathscr{V}\mathscr{P}_\varepsilon \to \mathscr{V}\mathscr{P}$ as $\varepsilon \to 0$ in a sense stated in detail, leading to the abstract notion of convergence of variational problems used throughout these and the following chapters. We pay particular attention to the estimates of the pressure, based on potential vectors. From there, we establish the existence of solutions by linearization and Schauder's fixed-point theorem. Uniqueness cases are studied. We then consider the full NS-TKE model. The TKE equation has a quadratic source term "in L^1," which requires specific estimate techniques, that cannot directly be used because of the wall law. We proceed with a change of variables to derive an equivalent PDE system, to which these techniques apply. The attached variational problem is denoted by $\mathscr{V}\mathscr{P}^\kappa$, the L^1 source term of which is regularized by convolution, resulting in $\mathscr{V}\mathscr{P}^\kappa_n$. We show by linearization that $\mathscr{V}\mathscr{P}^\kappa_n$ admits a solution satisfying stable estimates in appropriate spaces and that $\mathscr{V}\mathscr{P}^\kappa_n \to \mathscr{V}\mathscr{P}^\kappa$ as $n \to \infty$ by the energy method based on energy equalities, hence the existence of a weak solution to the NS-TKE model.

Chapter 8 is devoted to the 3D evolutionary NS-TKE model, whose attached variational problem is denoted by $\mathscr{E}\mathscr{P}^k$. Because of the dimension three, no energy equality occurs, which prevents to use the energy method. We approximate $\mathscr{E}\mathscr{P}^k$ by a Leray-α-like model $\mathscr{E}\mathscr{P}^k_\alpha$ ($\alpha > 0$), in which the transport terms are regularized by convolution, the wall law is truncated, and which satisfies the energy equality. To analyze $\mathscr{E}\mathscr{P}^k_\alpha$, several levels of approximation are necessary. We successively proceed with a truncation of the source term and the ε-approximation, which yields $\mathscr{E}\mathscr{P}^k_{\alpha,\beta}$ and $\mathscr{E}\mathscr{P}^k_{\alpha,\beta,\varepsilon}$, for $\beta, \varepsilon > 0$ two control parameters. We show the existence of solutions to $\mathscr{E}\mathscr{P}^k_{\alpha,\beta,\varepsilon}$ by the Galerkin method and then by the energy method and appropriate estimates:

$$\mathscr{E}\mathscr{P}^k_{\alpha,\beta,\varepsilon} \to \mathscr{E}\mathscr{P}^k_{\alpha,\beta} \;\text{ as }\; \varepsilon \to 0 \;\text{ and }\; \mathscr{E}\mathscr{P}^k_{\alpha,\beta} \to \mathscr{E}\mathscr{P}^k_{\alpha} \;\text{ as }\; \beta \to 0,$$

hence the existence of solutions to $\mathscr{E}\mathscr{P}^k_\alpha$. When $\alpha \to 0$, we obtain a variational problem where the NS part is preserved, but the TKE equation becomes a variational inequality, which is the best we can do at the time these lines are being written.

In all these theoretical chapters, compactness principles have been expressed in integrated packages suitable for this class of PDE systems.

We perform in Chaps. 9 and 10 the numerical analysis of the finite element (FE) discrete Smagorinsky model in steady and evolution regimes, respectively, including wall laws. We face the question whether this model is a good asymptotic approximation of the Navier–Stokes equations, which will become more relevant as the available computational resources allow to decrease the grid size for practical computations. However, we will not prove by this way that LES provides a good approximation to the large scales flow, although it gives partial positive answers. The solution of the FE discrete Smagorinsky model is proven to be weakly convergent to a solution of the Navier–Stokes equations in both the steady and the evolution regimes. However, we only prove that the sub-grid energy asymptotically vanishes in the steady regime. This is related to the lack of energy equality in the 3D evolutionary NSE, an issue similar to that already raised in Chap. 8. Regarding the choice of the discretization schemes, our guideline is to use the simplest scheme that provides appropriate stability estimates, as a model for more complex discretizations. We consider a semi-implicit Euler scheme in time where the eddy diffusion and wall law terms are discretized implicitly.

In Chap. 11 we study the projection-based VMS model. This method provides good predictions of first-order statistics of turbulent flows. It has a simplified structure with respect to residual-based VMS models and equally applies to steady and unsteady flows without further adaptation. Globally, it provides a good compromise between accuracy and computational complexity. Finally, it allows a thorough numerical analysis, parallel to that of Navier–Stokes equations. Our analysis is parallel to that of the FE discrete Smagorinsky model for steady and unsteady flows. We prove stability in natural norms as well as convergence to the NSE. The error estimates for smooth solutions are of optimal order with respect to the polynomial interpolation, in opposition to the Smagorinsky model for which the convergence order is limited by the eddy diffusion term.

Chapter 12 studies the numerical analysis of a regularized NS-TKE model, derived by truncation of the source terms and the eddy viscosities and which is discretized by finite elements. We prove the convergence of the resulting discrete model to the regularized NS-TKE model, either in the steady-state case or the evolutionary case. In the evolutionary case, we consider a time discretization by semi-linearization that decouples the velocity-pressure and TKE boundary value problems. We also prove that the limit TKE equation satisfies a variational inequality instead of an equality, due to a lack of regularity of the velocity. We understand the analysis developed in this chapter as a step toward the analysis of more complex models whose numerical approximation requires the development of new technical tools.

Chapter 13 aims to analyze the numerical performances of the models and numerical techniques that we have studied in the preceding chapters. It is intended to provide a starting guide to the numerical discretization of VMS models for students and researchers interested in the computation of turbulent flows. With this purpose we test the practical performances of several kinds of VMS models, in benchmark steady turbulent flows. Residual-based VMS methods are shown to provide excellent results for at least second-order accurate discretizations, reproducing with good accuracy first- and second-order statistics of the turbulent flow. Projection-based VMS methods provide a good compromise between accuracy and computational complexity, while residual-free bubble-based VMS methods, although yielding an acceptable accuracy, still need further improvements to be handled by non-experienced users.

Chapter 2
Incompressible Navier–Stokes Equations

Abstract We aim to derive the incompressible Navier–Stokes equations from classical mechanics. We define Lagrange and Euler coordinates and the mass density within the framework of measure theory. This yields a mathematical statement that expresses the mass conservation principle, which allows to derive the mass conservation equation. We introduce the incompressible flows and focus on their kinematic, starting with the deformation tensor and the vorticity and then the local deformations of a ball of fluid in an incompressible flow by standard ODEs. We introduce the fluid motion equation for Newtonian fluids through appropriate measures, based on the fundamental law of classical mechanics and the expression of the stress tensor in terms of the deformation tensor. The mass conservation equation coupled to the fluid motion equation yields the incompressible Navier–Stokes equations. This chapter ends with a comprehensive list of boundary conditions associated with the Navier–Stokes equations.

2.1 Introduction

The aim of this chapter is to lay the foundations for basic fluid mechanics, in order to prepare the ground for the mathematical modeling of turbulent flows performed in Chaps. 3, 4, and 5.

We consider a fluid, liquid, or gas, moving in a domain Ω included in $\mathbb{R}^3$. We aim to find a mathematical description of this motion, which is a difficult task since this is a nonlinear physical phenomenon involving many unknowns. The main unknowns are the mass density, the pressure, the velocity, and the temperature, but the list may be longer depending on the particular case being studied.

In this chapter, we derive from physical and mathematical considerations the incompressible Navier–Stokes equations for Newtonian fluids:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} - \nabla \cdot (2\nu D\mathbf{v}) + \nabla p = \mathbf{f}, \tag{2.1}$$

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$$\nabla \cdot \mathbf{v} = 0, \tag{2.2}$$

where $\mathbf{v}$ is the velocity of the flow, $D\mathbf{v} = (1/2)(\nabla\mathbf{v} + \nabla\mathbf{v}^t)$ its deformation tensor, and p its pressure. The momentum equation (2.1) is inherited from Newton's law, while equation (2.2) is the mass conservation equation for incompressible flows.

We define the mass density of the fluid ρ and the velocity $\mathbf{v}$ in Sect. 2.2. To do so, we outline the Lagrangian and Eulerian descriptions of the motion, although we shall only deal with the Euler description after Sect. 2.2. We shall also prove some useful abstract results.

In Sect. 2.3, we obtain the general mass conservation equation satisfied by $\mathbf{v}$ and ρ:

$$\partial_t \rho + \nabla \cdot (\mathbf{v}\rho) = 0. \tag{2.3}$$

Two procedures are developed to derive this equation. One is heuristic and shows the physical features of the mass balance. The other is mathematically rigorous, making use of the abstract results of Sect. 2.2.

We describe in Sect. 2.4 various approximations in the mass conservation equation, which leads to the notion of incompressibility and how equation (2.2) is deduced from equation (2.3) within this framework.

Section 2.5 is concerned with the kinematics of incompressible flows. We study the transformations of an infinitesimal fluid body δV in the flow, over an infinitesimal time period $\delta P = [t, t + \delta T]$. This yields the introduction on the one hand of the deformation tensor $D\mathbf{v}$, which governs the stability of δV over δP, and on the other hand of the vorticity $\boldsymbol{\omega}$, being the angular velocity of δV.

In Sect. 2.6, we perform the analysis of the internal forces acting on the fluid during its motion. The dynamic pressure p is introduced at this stage. This analysis yields the derivation of the momentum equation (2.1) from Newton's law, and finally the incompressible Navier–Stokes equations, presented in their various forms at the end of Sect. 2.6.

A comprehensive list of boundary conditions is presented in Sect. 2.7, describing some examples that are often studied, depending on different flow geometries as well as different approximations.

2.2 General Framework

2.2.1 Aim of the Section

A fluid is a continuous medium that can be continually subdivided into infinitesimal particles of fluid material having a mass. Each particle of fluid sits on an abstract point $\mathbf{x}$ in Ω at time $t \geq 0$. The measure $dm(t, \mathbf{x})$ is the mass of the particle that sits on $\mathbf{x}$ at time t, which will be defined by the end of Sect. 2.2.3.

The measure dm is absolutely continuous with respect to the Lebesgue's measure $d\mathbf{x}$, $dm = \rho d\mathbf{x}$, where $\rho = \rho(t, \mathbf{x})$ is the mass density at a time t and a point $\mathbf{x}$, that defines the mass of fluid per unit volume, expressed in kilograms per meter cubed. The total mass of fluid contained in a fixed subdomain ω of Ω at time t, $m(t, \omega)$, is given by

$$m(t, \omega) = \int_{\omega} \rho(t, \mathbf{x}) d\mathbf{x}, \tag{2.4}$$

provided that $\rho(t, \cdot)$ is locally integrable on Ω, which we shall assume to be the case.

To begin with, we define the Lagrange and Euler coordinate systems. The Lagrange description helps initially, but after this section we shall only use the Euler description. We refer to [7, 8] and [11] for further details about the Lagrange description.

We define the Lagrange and Euler velocities $\mathbf{V}$ and $\mathbf{v}$ in Sect. 2.2.2. The technical lemma 2.1 in Sect. 2.2.3.1 below points out how $\nabla\cdot\mathbf{v}$ is involved in volume variations during the motion. The local volume dv, being a form on the tangent space at a given point $(t, \mathbf{x})$, and the associate mass $dm = \rho dv$ are defined in Sect. 2.2.3.

2.2.2 Euler and Lagrange Coordinates and Velocities

Let us consider a particle of fluid sitting on a point $\mathbf{X} = (X_1, X_2, X_3) \in \Omega$ at time $t = 0$. Assume that this particle moves to a point $\mathbf{x} = (x_1, x_2, x_3) \in \Omega$ at a given time t and that there exists a map

$$F : \mathbb{R}_+ \times \Omega \to \Omega$$

such that

$$\mathbf{x} = F(t, \mathbf{X}) = (F_1(t, \mathbf{X}), F_2(t, \mathbf{X}), F_3(t, \mathbf{X})). \tag{2.5}$$

Assumption 2.1. *We assume that for each fixed $t \geq 0$, the restricted map $F(t, \cdot)$ is a C^1 diffeomorphism on Ω.*

Thus the relation $\mathbf{x} = F(t, \mathbf{X})$ can be inverted to give

$$\mathbf{X} = G(t, \mathbf{x}).$$

We shall say that $\mathbf{X}$ is the Lagrangian coordinate of the particle whereas $\mathbf{x}$ is its Eulerian coordinate.

Definition 2.1. The *Lagrangian velocity* at point $(t, \mathbf{X}) \in \mathbb{R}_+^\star$ of the fluid is the field $\mathbf{V} = \mathbf{V}(t, \mathbf{X})$:

$$\mathbf{V}(t, \mathbf{X}) = \partial_t F(t, \mathbf{X}), \quad \mathbf{V} = (V_1, V_2, V_3), \tag{2.6}$$

where $\partial_t = \partial/\partial t$ is the time derivative. The *Eulerian velocity* at a point $(t, \mathbf{x}) \in \mathbb{R}^\star_+$ is the field $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$:

$$\mathbf{v}(t, \mathbf{x}) = \mathbf{V}(t, G(t, \mathbf{x})), \quad \mathbf{v} = (v_1, v_2, v_3). \tag{2.7}$$

For a better understanding of the Eulerian velocity, let us consider a fixed point $\mathbf{x}$ in Ω and a physical fluid particle going past $\mathbf{x}$ at time t with an Eulerian velocity $\mathbf{v}$. Therefore this particle moves approximately to the point $\mathbf{x} + \mathbf{v}\delta t$ at the time $t + \delta t$ for some small $\delta t > 0$. We note that at a time $t' \neq t$, there is a chance that another physical particle goes past the same point $\mathbf{x}$.

2.2.3 *Volume and Mass*

2.2.3.1 Fundamental Kinematic Relation

We show the fundamental relation (2.12) linking $\nabla_{\mathbf{x}} \cdot \mathbf{v}$ and $\det \nabla_{\mathbf{X}} F$, where

$$\nabla_{\mathbf{X}} F = \left(\frac{\partial F_i}{\partial X_j} \right)_{1 \leq i, j \leq 3}, \quad \nabla_{\mathbf{x}} \cdot \mathbf{v} = \frac{\partial v_i}{\partial x_i}. \tag{2.8}$$

Let δ_{ij} denote the Kronecker tensor:

$$\delta_{ij} = 1 \text{ if } i = j, \quad \delta_{ij} = 0 \text{ if } i \neq j. \tag{2.9}$$

Let ε_{ijk} denote the Levi-Civita tensor that is fully characterized by

$$\varepsilon_{123} = 1; \, \varepsilon_{ijk} \textit{ is antisymmetric against the indices.}$$

On one hand, let $\mathbf{E} = (E_1, E_2, E_3)$ and $\mathbf{F} = (F_1, F_2, F_3)$ be two given vector fields. Then the i$^{\text{th}}$ component of their cross product $\mathbf{E} \times \mathbf{F}$ is given by

$$(\mathbf{E} \times \mathbf{F})_i = \varepsilon_{ijk} E_j F_k, \tag{2.10}$$

where the Einstein summation convention is used. On the other hand, the determinant of any 3×3 matrix $A = (a_{ij})_{1 \leq i, j \leq 3}$ is equal to

$$\det A = \frac{1}{6} \varepsilon_{ijk} \varepsilon_{pqr} a_{ip} a_{jq} a_{ir}, \tag{2.11}$$

Lemma 2.1. *Assume that F defined by (2.5) is of class C^2 on $\mathbb{R}_+ \times \Omega = Q$. Then*

$$\partial_t (\det \nabla_{\mathbf{X}} F) = (\nabla_{\mathbf{x}} \cdot \mathbf{v}) \det \nabla_{\mathbf{X}} F, \tag{2.12}$$

Proof. From (2.11), we have

$$\det \nabla_{\mathbf{X}} F = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}\frac{\partial F_i}{\partial X_p}\frac{\partial F_j}{\partial X_q}\frac{\partial F_k}{\partial X_r}.$$

Therefore

$$\partial_t \det \nabla_{\mathbf{X}} F = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}\left[\frac{\partial^2 F_i}{\partial t \partial X_p}\frac{\partial F_j}{\partial X_q}\frac{\partial F_k}{\partial X_r} + \frac{\partial F_i}{\partial X_p}\frac{\partial^2 F_j}{\partial t \partial X_q}\frac{\partial F_k}{\partial X_r} + \frac{\partial F_i}{\partial X_p}\frac{\partial F_j}{\partial X_q}\frac{\partial^2 F_k}{\partial t \partial X_r}\right]. \tag{2.13}$$

Since F is of class C^2, the Schwarz theorem applies [5] and we have

$$\frac{\partial^2 F}{\partial t \partial X_\ell} = \frac{\partial^2 F}{\partial X_\ell \partial t},$$

regardless of ℓ. Moreover, all indices play the same role in the formula (2.13). Therefore, we may exchange their positions, reorder the terms, and use the antisymmetry of ε_{ijk}, which yields

$$\partial_t \det \nabla_{\mathbf{X}} F = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{pqr}\frac{\partial^2 F_i}{\partial t \partial X_p}\frac{\partial F_j}{\partial X_q}\frac{\partial F_k}{\partial X_r}. \tag{2.14}$$

From the definition (2.6), we have

$$\frac{\partial^2 F}{\partial X_p \partial t} = \frac{\partial V_i}{\partial X_p}$$

at every point $(t, \mathbf{X})$. Moreover, formula (2.7) can also be written as

$$\mathbf{v}(t, F(t, \mathbf{X})) = V(t, \mathbf{X}), \tag{2.15}$$

leading to the following identity:

$$\frac{\partial V_i}{\partial X_p} = \frac{\partial v_i}{\partial x_\alpha}\frac{\partial F_\alpha}{\partial X_p}, \tag{2.16}$$

regardless of i and p. We insert the formula (2.16) in the equality (2.14), leading to

$$\partial_t \det \nabla_{\mathbf{X}} F = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{pqr}\frac{\partial v_i}{\partial x_\alpha}\frac{\partial F_\alpha}{\partial X_p}\frac{\partial F_j}{\partial X_q}\frac{\partial F_k}{\partial X_r}. \tag{2.17}$$

In writing carefully the six nonvanishing terms of ε_{ijk}, which take values in $\{-1, 1\}$, we find that for any fixed indices α, j, and k,

$$\varepsilon_{\alpha jk} \det \nabla_{\mathbf{X}} F = \varepsilon_{pqr} \frac{\partial F_\alpha}{\partial X_p} \frac{\partial F_j}{\partial X_q} \frac{\partial F_k}{\partial X_r}.$$

Then, the equality (2.17) becomes

$$\partial_t \det \nabla_{\mathbf{X}} F = \frac{1}{2} \varepsilon_{\alpha jk} \varepsilon_{ijk} \frac{\partial v_i}{\partial x_\alpha} \det \nabla_{\mathbf{X}} F. \quad (2.18)$$

Using the relation

$$\varepsilon_{\alpha jk} \varepsilon_{ijk} = 2\delta_{\alpha i}, \quad (2.19)$$

we have

$$\frac{1}{2} \varepsilon_{\alpha jk} \varepsilon_{ijk} \frac{\partial v_i}{\partial x_\alpha} = \delta_{\alpha,i} \frac{\partial v_i}{\partial x_\alpha} = \frac{\partial v_i}{\partial x_i} = \nabla_{\mathbf{x}} \cdot \mathbf{v}. \quad (2.20)$$

We combine (2.18) and (2.20), which yields

$$\partial_t \det \nabla_{\mathbf{X}} F = (\nabla_{\mathbf{x}} \cdot \mathbf{v}) \det \nabla_{\mathbf{X}} F,$$

concluding the proof of Lemma 2.1. □

To better understand this result, recall that $\det \nabla_{\mathbf{X}} F$ is involved in the change of variables in integral calculus. Indeed, let $g : \Omega \to \mathbb{R}$ be an integrable function, $\omega_0 \subset\subset \Omega$ be a measurable set, $\omega_t = F(t, \omega_0)$. Then

$$\int_{\omega_t} g(\mathbf{x}) d\mathbf{x} = \int_{\omega_0} g(F(t, \mathbf{X})) |\det \nabla_{\mathbf{X}} F(t, \mathbf{X})| d\mathbf{X}.$$

Roughly speaking, $\det \nabla_{\mathbf{X}} F(t, \cdot)$ measures how the diffeomorphism transforms the Lebesgue measure at time t. Formula (2.12) links its time evolution to $\nabla_{\mathbf{x}} \cdot \mathbf{v}$, which will later appear to be the indicator of the fluid's capacity to perform volume variations, such as compressions or decompressions.

2.2.3.2 Volume Form and Mass Measure

Let $\mathbf{X} \in \Omega$ be a given point and $T_{\mathbf{X}}$ the tangent space at $\mathbf{X}$, which is isomorphic to $\mathbb{R}^3$ in this case [1]. The volume form dv_0 at $\mathbf{X}$ is defined by

$$\forall\, (\zeta_1, \zeta_2, \zeta_3) \in T_{\mathbf{X}}^3, \quad dv_0(\mathbf{X})(\zeta_1, \zeta_2, \zeta_3) = \det(\zeta_1, \zeta_2, \zeta_3). \quad (2.21)$$

Let $t \in \mathbb{R}_+$ be fixed, $\mathbf{x} = F(t, \mathbf{X})$, $T_{t,\mathbf{x}}$ be the tangent space at $(t, \mathbf{x})$, also isomorphic to $\mathbb{R}^3$. Since $F(t, \cdot)$ is a diffeomorphism on Ω,

$$\nabla_{\mathbf{X}} F(t, \mathbf{X}) : T_{\mathbf{X}} \to T_{t,\mathbf{x}}$$

is an isomorphism. Therefore, for each $(\eta_1, \eta_2, \eta_3) \in T_{t,\mathbf{x}}^3$, there exists $(\zeta_1, \zeta_2, \zeta_3) \in T_{\mathbf{X}}^3$ such that

$$(\eta_1, \eta_2, \eta_3) = (\nabla_{\mathbf{X}} F(t, \mathbf{X}) \cdot \zeta_1, \nabla_{\mathbf{X}} F(t, \mathbf{X}) \cdot \zeta_2, \nabla_{\mathbf{X}} F(t, \mathbf{X}) \cdot \zeta_3). \tag{2.22}$$

This allows us to define a local volume form on $T_{t,\mathbf{x}}$ denoted by $dv(t, \mathbf{x})$ and defined by

$$\begin{aligned}\forall\, (\eta_1, \eta_2, \eta_3) \in T_{t,\mathbf{x}}^3,\ dv(t, \mathbf{x})(\eta_1, \eta_2, \eta_3) = det(\eta_1, \eta_2, \eta_3) = \\ \det(\nabla_{\mathbf{X}} F(t, \mathbf{X}) \cdot \zeta_1, \nabla_{\mathbf{X}} F(t, \mathbf{X}) \cdot \zeta_2, \nabla_{\mathbf{X}} F(t, \mathbf{X}) \cdot \zeta_3).\end{aligned} \tag{2.23}$$

The following relation holds true:

$$dv(t, \mathbf{x}) = \det \nabla_{\mathbf{X}} F(t, \mathbf{X}) dv_0(\mathbf{X}), \tag{2.24}$$

following the classical determinant theory [4].

Since parallelepipeds generate a Borel algebra on $T_{t,\mathbf{x}}$, this allows the mass of a particle of fluid that sits on $\mathbf{x}$ at time t, $dm(t, \mathbf{x})$, to be defined as a measure on $T_{\mathbf{X}}$ by the formula

$$dm(t, \mathbf{x}) = \rho(t, \mathbf{x}) dv(t, \mathbf{x}). \tag{2.25}$$

This is in accordance with the definition (2.4) since when t is fixed, $dv(t, \mathbf{x}) = d\mathbf{x}$.

This may seem to be unnecessarily complicated at a first glance. However, we shall see in Sect. 2.3.4 how it simplifies significantly the mathematical derivation of the mass conservation equation (2.26) below.

2.3 Mass Conservation Equation

2.3.1 Aim of the Section

This section is devoted to the derivation of the mass conservation equation

$$\partial_t \rho + \nabla \cdot (\mathbf{v}\rho) = 0, \tag{2.26}$$

satisfied at every $(t, \mathbf{x}) \in \mathbb{R}_+ \times \Omega$, where we have set

$$\forall\, \mathbf{E} = (E_1, E_2, E_3) = \mathbf{E}(t, \mathbf{x}) \in C^1(\mathbb{R}_+ \times \Omega)^3, \quad \nabla \cdot \mathbf{E} = \frac{\partial E_i}{\partial x_i}, \quad \partial_t \rho = \frac{\partial \rho}{\partial t}.$$

From now on, each tensor field depends on $\mathbf{x}$ in space and no longer on $\mathbf{X}$, and we shall no longer explicitly specify $\mathbf{x}$ or $\mathbf{X}$ subscripts when writing derivatives with respect to the space variable.

Equation (2.26) is called a "conservation law," the mass being preserved during the motion. Although it is easy to claim "the mass is preserved," this is difficult to define rigorously.

We derive equation (2.26) using two different procedures. One is heuristic, based on rough approximations that are not rigorously true. Nevertheless, this procedure presents the great advantage that we can simply derive the equation, which enables a better understanding of the physics. This is the aim of Sect. 2.3.2.

The other procedure is based on the relation (2.12) and the definition (2.25) of dm. This allows a rigorous definition of the mass conservation by imposing that the total derivative of dm is equal to zero at each point $(t, \mathbf{x}) \in Q$. Therefore, we must first define the total derivative, which we do in Sect. 2.3.3, that specifies how to derive in time along the flow trajectories. The rest of the program is implemented in Sect. 2.3.4.

Throughout this chapter, we assume

Assumption 2.2. *The fields ρ and $\mathbf{v}$ are of class C^1 on $\mathbb{R}_+ \times \Omega = Q$,*

without stating it systematically.

2.3.2 Heuristical Considerations

Let $\omega \subset\subset \Omega$ be a fixed open set strictly included in Ω, $\Gamma = \partial\omega$ its boundary, $\mathbf{n} = \mathbf{n}(\mathbf{x})$ the outward-pointing unit normal vector at any point $\mathbf{x} \in \Gamma$.

Let $t \in \mathbb{R}_+$, $\delta t > 0$ be an infinitesimal time. We count how much mass of fluid leaves ω through Γ over the time period $[t, t + \delta t]$, a quantity denoted by $\Delta m(t, \delta t, \omega)$. Of course, some mass of fluid might also enter ω through Γ over the same period. This will still be considered as leaving, counted with a nonpositive sign.

The calculation of $\Delta m(t, \delta t, \omega)$ is made in two different ways. We first expand $\Delta m(t, \delta t, \omega)$ around $\delta t = 0$, using the expression (2.4). We secondly carry out a local analysis at Γ to express $\Delta m(t, \delta t, \omega)$ as an integral over Γ, and we use the Stokes formula to transform it into an integral over ω. We obtain two distinct integrals over ω that we equate, thus expressing that the mass budget is balanced and deriving equation (2.26).

Since we require an algebraic loss in mass, the quantity we aim at computing is

$$\Delta m(t, \delta t, \omega) = m(t, \omega) - m(t + \delta t, \omega).$$

Using the Taylor formula and the definition (2.4), we have

$$\begin{aligned} \Delta m(t, \delta t, \omega) &= -\delta t \frac{d}{dt} \int_\omega \rho(t, \mathbf{x}) d\mathbf{x} + O(\delta t^2) \\ &= -\delta t \int_\omega \partial_t \rho(t, \mathbf{x}) d\mathbf{x} + O(\delta t^2). \end{aligned} \tag{2.27}$$

We hold equality (2.27) in reserve for the moment, and we turn to a local analysis at the boundary Γ.

Let $\mathbf{x} \in \Gamma$ be a fixed point, $\delta S \subset \Gamma$ be an infinitesimal part of Γ, whose gravity center is $\mathbf{x}$, and $\mathbf{n}(\mathbf{x})$ be the outward-pointing unit normal vector at $\mathbf{x}$. We assume that the field $(\mathbf{v}, \rho)$ is constant in the vicinity of $(t, \mathbf{x})$ and equal to $(\mathbf{v}(t, \mathbf{x}), \rho(t, \mathbf{x}))$, which is of course not satisfied exactly but is reasonable to first order.

We must characterize the particles that can leave ω through δS over the time period $[t, t + \delta t]$.

From the assumption we made on $\mathbf{v}$ at $(t, \mathbf{x})$, a physical particle of fluid that sits on $\mathbf{y}$, a point near $\mathbf{x}$ at time t, moves to $\mathbf{y} + \mathbf{v}(t, \mathbf{x})\delta t$ at $t + \delta t$. Thus, the particles we are looking for are those contained in the volume

$$\delta V = \mathbf{v}(t, \mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, \delta S\, \delta t$$

at time t. This represents a mass of fluid $\delta m = \rho(t, \mathbf{x})\delta V$. We have to sum δm over $\partial\omega$ to compute the total mass of fluid going out ω over the time period $[t, t + \delta t]$. Hence, we have

$$\Delta m(t, \delta t, \omega) = \delta t \int_{\partial\omega} \rho(t, \mathbf{x})\, \mathbf{v}(t, \mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, dS. \tag{2.28}$$

We apply the Stokes formula (see in [5, 27]) to the right-hand side of (2.28), leading to

$$\Delta m(t, \delta t, \omega) = \delta t \int_\omega \nabla \cdot (\rho(t, \mathbf{x})\, \mathbf{v}(t, \mathbf{x})) d\mathbf{x}. \tag{2.29}$$

We combine the equalities (2.27) and (2.29), divide by $\delta t > 0$, and let it tend to zero. Then we obtain the following relation:

$$\int_\omega [\partial_t \rho(t, \mathbf{x}) d\mathbf{x} + \nabla \cdot (\rho(t, \mathbf{x})\, \mathbf{v}(t, \mathbf{x}))] d\mathbf{x} = 0 \tag{2.30}$$

which holds at every $t \geq 0$ and for every subdomain ω of Ω. As we assume $\mathbf{v}$ and ρ of class C^1, we deduce from the results of integration theory [26] that the relation (2.30) yields the mass conservation equation (2.26). □

2.3.3 Total Derivative

In the previous analysis, we considered that a particle sitting on a point $\mathbf{x}$ at time t moves to the point $\mathbf{x}+\delta\mathbf{x} = \mathbf{x}+\delta t\ \mathbf{v}(t,\mathbf{x})$ at time $t+\delta t$. It is as if the point $\mathbf{x} = \mathbf{x}(t)$ was moving along a trajectory of the ordinary differential equation

$$\mathbf{x}'(t) = \mathbf{v}(t,\mathbf{x}(t)), \tag{2.31}$$

where $\mathbf{x}'(t)$ denotes the standard derivative with respect to t.

Let $\mathbf{E} = \mathbf{E}(t,\mathbf{x})$ be any tensor field of class C^1 on Q. As the point $\mathbf{x}$ also depends on t, so does

$$\mathbf{G}(t) = \mathbf{E}(t,\mathbf{x}(t)).$$

The question arises of how to express the time derivative $\mathbf{G}'(t)$ in terms of $\partial_t\mathbf{E}$ and $\nabla\mathbf{E}$.

Lemma 2.2. *Assume that* $\mathbf{E}$ *is of class* C^1 *on* Q. *We have*

$$\mathbf{G}'(t) = \partial_t\mathbf{E} + \mathbf{v}\cdot\nabla\mathbf{E}. \tag{2.32}$$

The field $\mathbf{G}'(t)$ *is denoted by* $\dfrac{D\mathbf{E}}{Dt}$ *and is called the total derivative of* E.

Proof. We expand $\mathbf{G}$ around $\delta t = 0$,

$$\mathbf{G}(t+\delta t) = \mathbf{E}(t+\delta t,\mathbf{x}(t+\delta t)) = \mathbf{E}(t+\delta t,\mathbf{x}+\delta t\mathbf{v}(t,\mathbf{x}) + o(\delta t)), \tag{2.33}$$

leading to

$$\mathbf{G}(t+\delta t) = \mathbf{G}(t) + \delta t(\partial_t\mathbf{E}(t,\mathbf{x}) + \mathbf{v}(t,\mathbf{x})\cdot\nabla\mathbf{E}(t,\mathbf{x})) + o(\delta t), \tag{2.34}$$

which is valid because $\mathbf{E}$ is of class C^1 on Q. We finally find

$$\mathbf{G}'(t) = \lim_{\delta t\to 0}\frac{G(t+\delta t)-G(t)}{\delta t} = \partial_t\mathbf{E}(t,\mathbf{x}) + \mathbf{v}(t,\mathbf{x})\cdot\nabla\mathbf{E}(t,\mathbf{x}), \tag{2.35}$$

hence the result follows. □

Remark 2.1. The total derivative satisfies the usual derivative rules, namely

$$\frac{D(\mathbf{E}+\mathbf{F})}{Dt} = \frac{D\mathbf{E}}{Dt} + \frac{D\mathbf{F}}{Dt},\quad \frac{D(\mathbf{E}\cdot\mathbf{F})}{Dt} = \frac{D\mathbf{E}}{Dt}\cdot\mathbf{F} + \mathbf{E}\cdot\frac{D\mathbf{F}}{Dt}. \tag{2.36}$$

Remark 2.2. The mass conservation equation (2.26) can be rewritten in terms of a total derivative. Indeed, we have

$$\nabla \cdot (\mathbf{v}\rho) = \mathbf{v} \cdot \nabla\rho + \rho\nabla \cdot \mathbf{v}, \tag{2.37}$$

which allows equation (2.26) to be rewritten as

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla \cdot \mathbf{v}. \tag{2.38}$$

This form of the mass conservation equation may sometimes be useful.

2.3.4 Rigorous Derivation of the Mass Conservation Equation

Let us consider a fluid particle that sits on $\mathbf{x} = \mathbf{x}(t)$ at time t and which moves to $\mathbf{x}(t + \delta t)$ at time $t + \delta t$, almost along a trajectory of the ODE (2.31). Recall that the mass $dm = dm(t, \mathbf{x}) = \rho(t, \mathbf{x})dv(t, \mathbf{x})$ of the particle at $(t, \mathbf{x})$ was defined by (2.25).

The principle of mass conservation is that the mass of the particle remains constant along the trajectory, which is expressed by the following equation:

$$\frac{D(dm)}{Dt} = 0. \tag{2.39}$$

We show in what follows that equation (2.39) is precisely the mass conservation equation (2.26).

According to the definitions (2.25) and (2.32), we have

$$\frac{D(dm)}{Dt} = (\partial_t\rho + \mathbf{v} \cdot \nabla\mathbf{v})dv + \rho\frac{D(dv)}{Dt}. \tag{2.40}$$

We must compute the total derivative $\dfrac{D(dv)}{Dt}$.

We insert the identity (2.12) of Lemma 2.1 in the relation (2.24) that expresses dv in terms of dv_0 and the Jacobian determinant, by noting that $dv_0(\mathbf{X})$ is totally time independent. The point $\mathbf{X}$ denotes the Lagrangian coordinate of the particle, which means its position at time $t = 0$. Therefore, we find the relation

$$\frac{D(dv)}{Dt} = (\nabla \cdot \mathbf{v})\, dv, \tag{2.41}$$

satisfied at every point $(t, \mathbf{x}) \in \mathbb{R}_+ \times \Omega$. We combine equations (2.39), (2.40), and (2.41). Since $dv \neq 0$, we obtain

$$\partial_t\rho + \mathbf{v} \cdot \nabla\rho + \rho\nabla \cdot \mathbf{v} = 0, \tag{2.42}$$

which is the mass conservation equation (2.26), following relation (2.37). $\square$

2.4 Incompressibility

2.4.1 *Basic Definition*

Compressibility and incompressibility are natural physical notions that we all are familiar with from everyday life. Generally speaking, we know that the volume occupied by a fixed mass of gas can be reduced, but that in the case of a liquid, the mass density remains more or less constant during motion. In the first case, we say that the flow is compressible while in the second case it is incompressible, and so the state equation can be written as

$$\rho = \rho(t, \mathbf{x}) = \rho_0 \quad \text{on} \quad Q = \mathbb{R}_+ \times \Omega, \tag{2.43}$$

for some constant $\rho_0 > 0$. In this case, the mass conservation equation (2.26) becomes

$$\nabla \cdot \mathbf{v} = 0. \tag{2.44}$$

The nature of equation (2.44) is kinematic. Moreover, experimental data [10] indicate that there are flow motions, the velocity of which still satisfy equation (2.44), but whose density is not constant. This suggests the global definition:

Definition 2.2. Any fluid flow on Q with $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ as velocity field is incompressible on Q if and only if $\mathbf{v}$ satisfies equation (2.44) at each point $(t, \mathbf{x}) \in Q$.

Incompressible flows preserve the volumes, according to Formula (2.12). Incompressibility refers to the nature of the motion. This is why the term of incompressibility is applied to the flow rather than the physical nature of the fluid. Some gas motions might be considered as incompressible flows, depending on the scales involved. Even if it is more difficult to conceptualize, some liquid motions may be considered as compressible.

In the remainder of the section, we consider the example of oceanic flow which is the typical example of an incompressible flow with a variable density. We then evoke the Mach number, closely linked to the question of compressibility/incompressibility.

2.4.2 *Incompressible Flow with Variable Density: The Example of the Ocean*

The density of the ocean varies by about 2 % around a mean value $\rho_0 = 1035\,\mathrm{Kg.m}^{-3}$ ([10, 21]), so that

$$\left|\frac{D\rho}{\rho}\right| \approx \left|\frac{\rho - \rho_0}{\rho_0}\right| \leq 2.10^{-3}. \tag{2.45}$$

This density change is mainly due to salt, which is not evenly distributed in the water, as well as to temperature variations. As a result, the density ρ of the ocean satisfies a state equation $\rho = \rho(S, \theta)$, where θ denotes the temperature and S the salinity (the mass of salt per unit of volume). This equation is nonlinear and may vary according to the place of study [10]. In some situations, the pressure p may also be involved. Simplified mathematical models use the linearized state equation:

$$\rho = \rho_0 + \alpha_S(S - S_0) + \alpha_\theta(\theta_0 - \theta), \tag{2.46}$$

for $\alpha_\theta > 0$, $\alpha_S > 0$, $S_0 > 0$, and $\theta_0 > 0$ constant.

Nevertheless, the bound (2.45) allows us to consider the ocean's motion as incompressible. To see this, we introduce a typical velocity magnitude U, a typical time magnitude T, and a typical length magnitude L. Those values may change, according to the case we focus on. Their choice usually fixes the parameters for numerical simulations: the time step is related to T while the mesh size is related to L. We assume

$$U = LT^{-1}. \tag{2.47}$$

We examine the magnitude of each term in the mass conservation equation (2.38). We denote by $[\mathbf{E}]$ the magnitude of any given field $\mathbf{E}$. Therefore,

$$[\nabla \cdot \mathbf{v}] = \frac{U}{L} = T^{-1}. \tag{2.48}$$

Similarly,

$$\left[\frac{1}{\rho}\frac{D\rho}{Dt}\right] = T^{-1}\left[\frac{D\rho}{\rho}\right] = 2.10^{-3}T^{-1}. \tag{2.49}$$

Therefore, in equation (2.38), the magnitude of the right-hand side (r.h.s.) differs from that of the left-hand side (l.h.s.) by a coefficient $\varepsilon = 10^{-3}$. This is as if the equation was written as

$$\varepsilon E = F, \tag{2.50}$$

where $O(E) = O(F) = 1$ and $\varepsilon = o(1)$. This is a standard situation in asymptotic analysis [3], and the result is that both E and F must vanish to satisfy equation (2.50), which yields

$$\frac{1}{\rho}\frac{D\rho}{Dt} = \nabla \cdot \mathbf{v} = 0,$$

and hence we can conclude that the flow is incompressible according to the definition 2.2.

2.4.3 *Incompressible Limit*

An observer swimming in the sea with his head underwater may hear the sound of a boat engine apparently close by, although the boat is actually a large distance away. It seems that the speed of the sound in the water is infinite.

The study of sound propagation in fluids [20] led E. Mach to introduce in 1887 the dimensionless number

$$M = \frac{\mathbf{U}}{c},$$

where c is the speed of the sound. This number is now called the Mach number.

It can be shown that when M goes to zero, which implies an infinite speed of sound, then the corresponding limit of the velocity satisfies the incompressible equation (2.44). This is the incompressible limit, which can be derived from asymptotic expansions in the compressible Navier–Stokes equations [22]. An analysis based on physical arguments at small scales yields the same results [2].

Throughout the rest of this book, we assume the following:

Assumption 2.3. *The flow specified by the vector field* $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ *is incompressible, that is,* $\nabla \cdot \mathbf{v} = 0$.

2.5 Kinematic Features of Incompressible Flows

2.5.1 *Aim of the Section*

This aim of this section is to study the transformations of an infinitesimal body of fluid δV during its motion in an incompressible flow, over an infinitesimal time period. We assume the following:

Assumption 2.4. *The body* δV *can be identified to an open set* $\omega \subset \Omega$ *at a given time* t *and to* $\omega_\tau \subset \Omega$ *at time* $t + \tau$, $\tau \in [0, \delta T]$ *for some* $\delta T > 0$. *We also assume that* ω_τ *has a boundary* Γ_τ *of class* C^1 *for all* $\tau \in [0, \delta T]$. *We denote by* $\boldsymbol{n}_\tau$ *the outward-pointing unit normal vector on* Γ_τ.

Recall that during its motion, the total volume δV is constant, thanks to the incompressibility assumption.

The local analysis carried out in Sect.2.5.2 below reveals that the tensor field $\nabla \mathbf{v}(t, \mathbf{x}) = \nabla \mathbf{v}$ defined by

$$\nabla \mathbf{v} = \left(\frac{\partial v_i}{\partial x_j}(t, \mathbf{x}) \right)_{1 \le i, j \le 3} \tag{2.51}$$

governs the first-order transformations of δV over a time period $\delta P = [t, t + \delta T]$, where we assume that over δP, the point $\mathbf{x}$ remains the gravity center of δV, $\delta T = o(1)$ as well as $\mathrm{diam}(\delta V) = o(1)$.

We write $\nabla \mathbf{v}$ at $(t, \mathbf{x})$ in the form

$$\nabla \mathbf{v} = D\mathbf{v} + \nabla^{\mathrm{a}} \mathbf{v}, \tag{2.52}$$

designating by $\nabla^{\mathrm{a}} \mathbf{v}$ the antisymmetrical part of $\nabla \mathbf{v}$ and $D\mathbf{v}$ the symmetric part of $\nabla \mathbf{v}$, namely

$$D\mathbf{v} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^t), \quad \nabla^{\mathrm{a}} \mathbf{v} = \frac{1}{2}(\nabla \mathbf{v} - \nabla \mathbf{v}^t),$$

where A^t denotes the transpose of any matrix A. The tensor $D\mathbf{v}$ is called the deformation tensor.

We show in Sect. 2.5.3 that the spectral analysis of $D\mathbf{v}$ determines in what directions δV remains stable and how it might deform. The incompressibility assumption 2.3 makes the study of stability easy, since the trace of $D\mathbf{v}$ is equal to zero in this case. This analysis also explains why $D\mathbf{v}$ is so important for expressing the internal forces acting on the fluid, performed in Sect. 2.6.

Turning to the tensor $\nabla^{\mathrm{a}} \mathbf{v}$, we show (Lemma 2.3 in Sect. 2.5.4) that it is fully specified through the vorticity vector

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}, \quad \boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3). \tag{2.53}$$

In Sect. 2.5.5, we study the contribution of the vorticity $\boldsymbol{\omega}$ in the transformations of δV.

In the light of the decomposition (2.52), we distinguish three cases:

(i) $D\mathbf{v}$ is large compared to $\boldsymbol{\omega}$, $||\boldsymbol{\omega}|| << ||D\mathbf{v}||$,
(ii) $\boldsymbol{\omega}$ is large compared to $D\mathbf{v}$, $||D\mathbf{v}|| << ||\boldsymbol{\omega}||$,
(iii) they are of the same magnitude, $||D\mathbf{v}|| \approx ||\boldsymbol{\omega}||$,

where by default,

$$\forall\, \mathbf{E} = (E_{ijk\ell\ldots})_{1\le ijk\ell\ldots\le 3}, \quad ||\mathbf{E}|| = ||\mathbf{E}||_2 = (\sum_{1\le ijk\ell\ldots\le 3} E^2_{ijk\ell\ldots})^{\frac{1}{2}},$$

or any equivalent norm. The conclusions of this section are the following.

In case (i), if δV has the form of a football at time t, it is then transformed into a rugby ball.

In case (ii), δV behaves like a rotating solid body, whose angular velocity is $(1/2)\boldsymbol{\omega}$. In regions where (ii) holds, small-scale vortices may be observed.

Case (iii) is more difficult. The transformation of δV might be anything because both effects compensate. To illustrate this, we sketch out in Sect. 2.5.6 the typical example of a shear flow, for which such a compensation occurs.

2.5.2 Local Role of $\nabla \mathbf{v}$ and Fundamental ODE

Let $\mathbf{x}$ be the gravity center of δV at time t and $\mathbf{y} \in \delta V$ be any other point. Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ denote the position of two particles sitting on $\mathbf{x}$ and $\mathbf{y}$ at time t. Let us consider

$$\boldsymbol{\xi} = \boldsymbol{\xi}(t) = \mathbf{x}(t) - \mathbf{y}(t), \tag{2.54}$$

Assume that the particles move to $\mathbf{x} + \delta\mathbf{x}$ and $\mathbf{y} + \delta\mathbf{y}$ respectively at time $t + \delta t$ for some $\delta t > 0$. Let us set $\delta\boldsymbol{\xi} = \delta\mathbf{x} - \delta\mathbf{y}$.

We perform the asymptotic expansions:

$$\delta\mathbf{x} = \mathbf{v}(t, \mathbf{x})\delta t + o(\delta t), \tag{2.55}$$

$$\delta\mathbf{y} = \mathbf{v}(t, \mathbf{y})\delta t + o(\delta t), \tag{2.56}$$

$$\mathbf{v}(t, \mathbf{x}) - \mathbf{v}(t, \mathbf{y}) = \nabla\mathbf{v} \cdot \boldsymbol{\xi} + o(||\boldsymbol{\xi}||), \tag{2.57}$$

where $\nabla\mathbf{v} = \nabla\mathbf{v}(t, \mathbf{x})$. We combine (2.55), (2.56) and (2.57) and we find

$$\frac{\delta\boldsymbol{\xi}}{\delta t} = \nabla\mathbf{v}(t, \mathbf{x}) \cdot \boldsymbol{\xi} + o(\delta t + ||\boldsymbol{\xi}||). \tag{2.58}$$

We take the limit in equation (2.58) as δt goes to 0, which yields the following differential equation:

$$\boldsymbol{\xi}' = \nabla\mathbf{v}(t, \mathbf{x}) \cdot \boldsymbol{\xi} + o(||\boldsymbol{\xi}||). \tag{2.59}$$

This suggests the following local ODE, which corresponds to the first-order term in equation (2.59):

$$\boldsymbol{\xi}'(t + \tau) = \nabla\mathbf{v}(t, \mathbf{x}) \cdot \boldsymbol{\xi}(t + \tau), \tag{2.60}$$

where now $\boldsymbol{\xi} = \boldsymbol{\xi}(t + \tau)$ is only time dependent, t and $\boldsymbol{\xi}(t)$ are given and fixed, $\tau > 0$, and $\boldsymbol{\xi}'$ denotes the derivative with respect to τ.

Although equation (2.60) makes sense around τ=0 from the physical viewpoint, as a linear ODE it possesses a unique global solution defined on $\mathbb{R}$, for any initial datum $\boldsymbol{\xi}$ [5].

However, without any specific information about the matrix $\nabla\mathbf{v}(t, \mathbf{x}) = \nabla\mathbf{v}$, it is difficult to easily picture the overall appearance of the solutions to (2.60). We can only get qualitative stability properties, by using the incompressibility assumption. However, following the decomposition (2.52), it is natural to split the ODE (2.60) into

$$\boldsymbol{\xi}' = D\mathbf{v} \cdot \boldsymbol{\xi}, \tag{2.61}$$

$$\boldsymbol{\xi}' = \nabla^{\mathrm{a}}\mathbf{v} \cdot \boldsymbol{\xi}, \tag{2.62}$$

In what follows, we study (2.61) and (2.62) separately after having first analyzed the stability properties of equation (2.60).

2.5.3 Deformation Tensor

2.5.3.1 Stability

The question is whether $\boldsymbol{\xi}(t+\tau)$ goes to zero as τ goes to infinity, for a given initial data $\boldsymbol{\xi}(t) \neq 0$. If yes, we say that equation (2.60) is stable, otherwise we say it is unstable. We follow the theory developed by A.M. Lyapunov in 1892 (see the details in [13]). To do so, we take the inner product by $\boldsymbol{\xi}$ with both sides of (2.60), which yields

$$(\boldsymbol{\xi}', \boldsymbol{\xi}) = (\nabla\mathbf{v} \cdot \boldsymbol{\xi}, \boldsymbol{\xi}). \tag{2.63}$$

According to a well-known result of linear algebra,

$$(\nabla\mathbf{v} \cdot \boldsymbol{\xi}, \boldsymbol{\xi}) = (D\mathbf{v} \cdot \boldsymbol{\xi}, \boldsymbol{\xi}),$$

regardless of $\boldsymbol{\xi}$. Therefore, equation (2.63) may be written as

$$\frac{1}{2}\frac{d||\boldsymbol{\xi}||^2}{dt} = (D\mathbf{v} \cdot \boldsymbol{\xi}, \boldsymbol{\xi}). \tag{2.64}$$

The inner product $(D\mathbf{v} \cdot \boldsymbol{\xi}, \boldsymbol{\xi})$ is a Lyapunov function for the ODE (2.60) which specifies local stability properties near the point $(t, \mathbf{x})$.

To check the stability properties of the flow, we use the symmetry of $D\mathbf{v}$, which is therefore orthogonal diagonalizable [4]. Moreover, incompressibility yields

$$\mathrm{tr} D\mathbf{v} = 2\nabla \cdot \mathbf{v} = 0.$$

Assume first that $D\mathbf{v} = 0$. Then the velocity is locally constant around $\mathbf{x}$ and particles move along straight lines.

Assume next that $D\mathbf{v} \neq 0$. Because of incompressibility, $D\mathbf{v}$ has at least one strictly negative eigenvalue, denoted by λ_1, and one strictly positive, denoted by λ_2. The ODE is stable along the eigendirection associated with λ_1 and unstable along the one associated with λ_2.

In particular, let $\xi(t) \neq 0$ be an initial datum that is an eigenvector associated with λ_1, then $\xi(t+\tau)$ goes to zero when τ goes to infinity. Let $\xi(t) \neq 0$ be an initial datum that is an eigenvector associated with λ_2, then $||\xi(t+\tau)||$ goes to infinity when τ goes to infinity. If $\lambda_3 \neq 0$, its sign determines if stability holds along a plane or along a line only.

2.5.3.2 When Footballs Become Rugby Balls

Assume that $D\mathbf{v}$ is large compared to $\nabla^{\mathrm{a}}\mathbf{v}$ at $(t, \mathbf{x})$. At that point, the solutions to the fundamental equation (2.60) are very close to those of equation (2.61) that we solve in the eigen-coordinate system of $D\mathbf{v}$.

Let $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ be an orthogonal eigenbasis of the tangent space $T_{t,\mathbf{x}}$ for $D\mathbf{v}$, associated with the eigenvalues λ_1, λ_2, λ_3. From now, we assimilate $T_{t,\mathbf{x}}$ to $\mathbb{R}^3$ for simplicity. We write

$$\boldsymbol{\xi}(t+\tau) = \xi_i(t+\tau)\mathbf{a}_i, \tag{2.65}$$

where each ξ_i satisfies

$$\xi_i' = \lambda_i \xi_i. \tag{2.66}$$

Therefore, the solution to equation (2.61) is

$$\boldsymbol{\xi}(t+\tau) = e^{\lambda_i \tau}\xi_i(t)\mathbf{a}_i. \tag{2.67}$$

To picture what the solution looks like, assume that δV is a ball of radius $r = o(1)$ centered on $\mathbf{x}$ at time t. Then δV instantaneously becomes a rugby ball, an ellipsoid whose axes are defined by the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ and its shape determined by the sign of the eigenvalues.

2.5.4 *Vorticity*

If the body δV were a solid body with $\mathbf{x}$ as its center of gravity, then the velocity of each $\mathbf{y} \in \delta V$ would be expressed by the law [15]:

$$\mathbf{v}(t, \mathbf{y}) = \mathbf{v}(t, \mathbf{x}) - \boldsymbol{\Omega} \times \boldsymbol{\xi}, \tag{2.68}$$

for some angular vector $\boldsymbol{\Omega}$ to be determined.

We rewrite the asymptotic expansion (2.57) using the decomposition (2.52) as follows:

$$\mathbf{v}(t, \mathbf{y}) = \mathbf{v}(t, \mathbf{x}) - D\mathbf{v}(t, \mathbf{x}) \cdot \boldsymbol{\xi} - \nabla^{\mathrm{a}}\mathbf{v}(t, \mathbf{x}) \cdot \boldsymbol{\xi} + o(||\boldsymbol{\xi}||). \tag{2.69}$$

We assume that $D\mathbf{v}$ is negligible against $\nabla^{\mathrm{a}}\mathbf{v}$ at $(t, \mathbf{x})$. Consequently, (2.68) is similar to (2.69), provided that $\nabla^{\mathrm{a}}\mathbf{v} \cdot \boldsymbol{\xi}$ can be written in the form $\boldsymbol{\Omega} \times \boldsymbol{\xi}$.

The appropriate vector field is the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v} = (\omega_1, \omega_2, \omega_3)$, as proved by the following.

Lemma 2.3. *Let*

$$\zeta_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

be the general term of $\nabla^{\mathrm{a}}\mathbf{v}$. *Then the following relation holds:*

$$\zeta_{ij} = -\frac{1}{2}\varepsilon_{ijk}\omega_k, \tag{2.70}$$

for all $1 \leq i, j \leq 3$. *Furthermore,*

$$\nabla^{\mathrm{a}}\mathbf{v}(t,\mathbf{x}) \cdot \boldsymbol{\xi} = \frac{1}{2}\boldsymbol{\omega}(t,\mathbf{x}) \times \boldsymbol{\xi}, \tag{2.71}$$

regardless of $\boldsymbol{\xi}$.

Proof. From now on and if no risk of confusion occurs, we shall write

$$\partial_i = \frac{\partial}{\partial x_i}, \tag{2.72}$$

for every $i = 1, 2, 3$. Following the formula (2.10) and using the antisymmetry of the Levy-Civita tensor, we have

$$\omega_k = \varepsilon_{pqk}\partial_p v_q, \tag{2.73}$$

which yields $\varepsilon_{ijk}\omega_k = \varepsilon_{ijk}\varepsilon_{pqk}\partial_p v_q$. The relation

$$\varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \tag{2.74}$$

shows

$$\varepsilon_{ijk}\omega_k = \partial_i v_j - \partial_j v_i = -2\zeta_{ij},$$

which proves the relation (2.70) as well as the identity (2.71) following (2.10), combined with (2.70). □

As a consequence of the identity (2.71), the expansion (2.69) becomes

$$\mathbf{v}(t,\mathbf{y}) = \mathbf{v}(t,\mathbf{x}) - D\mathbf{v}(t,\mathbf{x}) \cdot \boldsymbol{\xi} - \frac{1}{2}\boldsymbol{\omega}(t,\mathbf{x}) \times \boldsymbol{\xi} + o(||\boldsymbol{\xi}||), \tag{2.75}$$

where we recall that $\boldsymbol{\xi} = \mathbf{x} - \mathbf{y}$.

Remark 2.3. By explaining the formula (2.73), we find

$$\boldsymbol{\omega} = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix},$$

which is the practical expression of the vorticity.

2.5.5 Vortices

When we compare the expansion (2.75) to the expression (2.68) by taking $\boldsymbol{\Omega} = (1/2)\boldsymbol{\omega}$, we observe that the vorticity characterizes the instantaneous rotation of δV, provided $\boldsymbol{\omega}$ is large compared to $D\mathbf{v}$. In this case, the fundamental equation (2.60) is very close to equation (2.62), which we rewrite as

$$\boldsymbol{\xi}' = \frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{\xi}, \tag{2.76}$$

using the relation (2.71), in which $\boldsymbol{\omega} = \boldsymbol{\omega}(t, \mathbf{x})$ for a fixed $(t, \mathbf{x})$. Let us solve equation (2.76).

If $\boldsymbol{\omega} = 0$, then $\boldsymbol{\xi}(t + \tau) = \boldsymbol{\xi}(t)$ regardless of τ. Let us assume that $\boldsymbol{\omega} \neq 0$, and let us consider

$$\mathbf{b}_1 = \frac{\boldsymbol{\omega}}{||\boldsymbol{\omega}||}.$$

Let $\mathbf{b}_2$ and $\mathbf{b}_3$ be such that $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ is an orthogonal basis of $\mathbb{R}^3$ that in particular satisfies

$$\mathbf{b}_1 \times \mathbf{b}_2 = \mathbf{b}_3, \quad \mathbf{b}_2 \times \mathbf{b}_3 = \mathbf{b}_1, \quad \mathbf{b}_3 \times \mathbf{b}_1 = \mathbf{b}_2. \tag{2.77}$$

We write

$$\boldsymbol{\xi} = \tilde{\xi}_i \mathbf{b}_i. \tag{2.78}$$

Using the relations (2.77) and setting $z = \tilde{\xi}_2 + i\tilde{\xi}_3$, we get

$$\tilde{\xi}_1' = 0, \quad z' = i\frac{||\boldsymbol{\omega}||}{2}z, \tag{2.79}$$

which yields

$$\tilde{\xi}_1(t + \tau) = \tilde{\xi}_1(t), \quad z(t + \tau) = e^{i\frac{||\boldsymbol{\omega}||}{2}\tau} z(t). \tag{2.80}$$

Therefore, trajectories rotate around the axis spanned by $\boldsymbol{\omega}$ with a frequency equal to $||\boldsymbol{\omega}||/2$.

The resolution of the ODE (2.76) explains why the vorticity is closely linked to the notion of vortex (also called eddy) which plays a central role in the study of turbulent flows.

However, the fact that $\boldsymbol{\omega} \neq 0$ in a given flow does not imply that an observer will see vortices in the flow, essentially because the analysis carried above is local in time as well as in space and therefore only makes sense for small scales. Moreover, the deformation tensor effects may balance the vorticity effects when $\boldsymbol{\omega}$ and $D(\mathbf{v})$ are of the same magnitude, as shown in the example discussed in Sect. 2.5.6.

The question of defining mathematically a "vortex" as we picture it at large scales is hard. The simplest and popular criterium that is used in practical simulations to locate where there may be vortices is the Q-criterium which says that vortices are located in the set

$$\{(t,\mathbf{x}) \in \mathbb{R}_+ \times \Omega, \quad Q(t,\mathbf{x}) = \frac{1}{2}[|\boldsymbol{\Omega}(t,\mathbf{x})|^2 - |D\mathbf{v}(t,\mathbf{x})|^2] > 0\}. \tag{2.81}$$

2.5.6 A Typical Example of a Shear Flow

In this example, we work in a dimensionless framework for simplicity, dimensional analysis being detailed in Sect. 3.2.

Let $\mathbf{v} = (v_1, v_2, v_3)$ be the stationary vector field defined by

$$\forall\, \mathbf{x} = (x, y, z), \quad v_1(x, y, z) = z, \quad v_2(x, y, z) = v_3(x, y, z) = 0, \tag{2.82}$$

where $\mathbf{x} = x\,\mathbf{e}_1 + y\,\mathbf{e_2} + z\,\mathbf{e}_3$. The field $\mathbf{v}$ satisfies $\nabla\cdot\mathbf{v} = 0$. Basic calculations yield

$$\nabla\mathbf{v} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D\mathbf{v} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\omega} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{2.83}$$

We notice that

$$||\boldsymbol{\omega}||_1 = ||D\mathbf{v}||_1 = 1. \tag{2.84}$$

Let $\boldsymbol{\xi} = (x, y, z)^{\mathrm{T}}$ be given at time t. The solution to equation (2.60), denoted by $\boldsymbol{\xi}_\tau$ at time $t + \tau$ and such that $\boldsymbol{\xi}_0 = \boldsymbol{\xi}$, is equal to

$$\boldsymbol{\xi}_\tau = \begin{pmatrix} x + \tau z \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & \tau \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \boldsymbol{\xi} = R_\tau \cdot \boldsymbol{\xi}. \tag{2.85}$$

For a given $\tau > 0$, the matrix R_τ is the matrix of a transvection [4]. Let us describe carefully the transformation of δV in the flow when $\delta V = B(0, r)$ is a ball centered at the origin. Three cases occur:

(a) Points in $\delta V \cap \{z = 0\}$ remain steady.
(b) Points $\mathbf{x} = (x, y, z) \in \delta V \cap \{z > 0\}$ are such that $x \to \infty$ while y and z remain constant when $\tau \to \infty$.
(c) Points $\mathbf{x} = (x, y, z) \in \delta V \cap \{z < 0\}$ are such that $x \to -\infty$ while y and z also remain constant when $\tau \to \infty$.

The ball δV is sheared in the region $\{\sup(|y|, |z|) \leq 1\}$ and fluid particles contained in δV at time t are mixed in the whole strip, which is a typical process in turbulent flows.

Transformations of bodies totally included in the region $\{z > 0\}$ or $\{z < 0\}$, both stable through R_τ, are sheared along the x-axis.

We study equations (2.61) and (2.62) one by one. Equation (2.62) is already solved by the general formula (2.80). The solution is a rotation whose frequency is equal to $1/2$, around the line spanned by $\boldsymbol{\omega}$, that is, the line $\{x = z = 0\}$.

We turn to the ODE (2.61). The spectrum of $D\mathbf{v}$ is the set $\{-1/2, 1/2, 0\}$. The eigenspace associated with the eigenvalue $-1/2$ is the line $\{x = -z,\ y = 0\}$, spanned by $\mathbf{e}_1 + \mathbf{e}_3$, which is a stability direction according to Sect. 2.5.3.1. The eigenspace associated with the eigenvalue $1/2$ is the line $\{x = z,\ y = 0\}$, spanned by $\mathbf{e}_1 - \mathbf{e}_3$, which is an unstable direction. Finally, the eigenspace associated with the eigenvalue 0 is the y-axis that also coincides with the line spanned by $\boldsymbol{\omega}$. The general solution is then given by

$$\boldsymbol{\xi}_\tau = (x \cosh(\tau) - z \sinh(\tau))\, \mathbf{e}_1 + y\, \mathbf{e}_2 + (-x \sinh(\tau) + z \cosh(\tau))\, \mathbf{e}_3, \tag{2.86}$$

by noting that $\boldsymbol{\omega}$ is neither a stable nor an unstable direction.

The overall impression is that the resolution of (2.61) and (2.62) does not allow the transformations of δV to be pictured in this specific case. This is more easily done by solving the fundamental equation (2.60), which is fortunately straightforward. The solutions are highly unstable, especially when considering bodies initially in both $\{z > 0\}$, $\{z < 0\}$.

Nevertheless, we suspect the fact that $D\mathbf{v}$ has an eigenvalue equal to zero, whose eigenspace is spanned by $\boldsymbol{\omega}$, together with the result (2.84), may explain some of the features of this example. No more can really be said, apart from noting the great importance of shears in turbulent flows.

2.6 The Equation of Motion and the Navier–Stokes Equation

2.6.1 Aim of the Section

We turn to the momentum equation, based on Newton's law:

$$\text{mass} \times \text{acceleration} = \text{total applied forces}, \tag{2.87}$$

which a priori applies over a time period $[t, t + \delta T]$ to a body δV satisfying assumption 2.4.

We start by modeling the total forces applied on $\delta V = \omega_\tau$ at time $t + \tau$. We distinguish two different types of forces:

(i) The body forces, applied at distance on δV, such as gravity, electromagnetic forces, and so on,
(ii) The "internal forces" $F(\tau, \delta V)$, that are those the rest of the fluid applies on δV at time $t + \tau$.

As the internal forces are those that are hard to model, particular attention will be paid to them. The appropriate tool is the stress tensor

$$\boldsymbol{\sigma} = (\sigma_{ij})_{1 \le i,j \le 3} = \boldsymbol{\sigma}(t, \mathbf{x})$$

(see [2, 7, 8, 11, 14]) that is symmetric and such that

$$F(\tau, \delta V) = \int_{\Gamma_\tau} \boldsymbol{\sigma} \cdot \mathbf{n}_\tau, \tag{2.88}$$

which results in an internal force density equal to $(\nabla \cdot \boldsymbol{\sigma})dv$, according to Stokes' formula. Therefore, we must specify $\boldsymbol{\sigma}$ by making some reasonable assumptions. This is the aim of Sect. 2.6.2, where the dynamic pressure is introduced through the relation $p = -(1/3)\mathrm{tr}\boldsymbol{\sigma}$. Furthermore, we introduce the definition of a Newtonian fluid together with the notion of dynamic viscosity μ.

Next we consider a local point of view, just as we did when studying the mass conservation energy in Sect. 2.3. Indeed, we prefer to apply Newton's law to a particle of fluid that sits on $\mathbf{x}$ at time t, rather than to a body δV, which has been useful in finding $\nabla \cdot \boldsymbol{\sigma}$. Once the acceleration of $\mathbf{x}$ at time t has been calculated in Sect. 2.6.3, we present the momentum equation for incompressible flows in as general terms as possible.

We introduce the kinematic viscosity ν in Sect. 2.6.4 and we take the opportunity to present various forms of the incompressible Navier–Stokes equations, each form being useful for theoretical investigations or for practical simulations.

We conclude with Sect. 2.6.5, where we derive the equations satisfied by the vorticity $\boldsymbol{\omega}$ and the pressure p.

2.6.2 Stress Tensor

2.6.2.1 Physical Evidences and Some History

The goal is the determination of $F(\tau, \delta V)$. Before doing any mathematics, we recall some historic physical considerations which manifest such internal forces.

The well-known Archimedes' principle (approximately year 250 B.C, see in [7]) may be stated as follows:

> "Any object, wholly or partially immersed in a fluid, is buoyed up by a force equal to the weight of the fluid displaced by the object."

This law characterizes the internal force in the fluid at rest, which is the hydrostatic pressure.

The famous experience carried out much later by E. Torricelli in 1644 (see in [7]), who constructed the first mercury barometer, has highlighted existence of atmospheric pressure, which varies depending on the weather.

Therefore, the first internal force exerted on any flow that comes to mind is the pressure, although this was initially considered for steady fluids. This led L. Euler to derive in 1757 a momentum equation based on Newton's law. In Euler's equations (see in [2, 7, 8, 14, 22]), which couple the incompressibility equation to the momentum equation, the pressure is the only internal force, which is treated as an unknown of the equation together with the velocity. A fluid governed by Euler's equations is called a perfect fluid. According to legend, D. Bernoulli is supposed to have said a short time after Euler's work:

> "If a perfect fluid would exist, then the birds would not fly."

Indeed, any body moving in a fluid faces a drag that yields an energy dissipation. Moreover, Le Rond d'Alembert [19], shortly after Euler's work, showed that the drag in a perfect fluid is zero, highlighting what was regarded as a paradox at that time. Therefore, something was missing in Euler's model, though it still remains a very exciting mathematical objet.

In the light of this, fluid dynamics was subject to intensive research, especially experimentally. The notion of viscosity that quantifies the concept of drag in flows rapidly emerged, leading in 1822 to the famous model due to Navier [24], who added a term in the Euler equations to model the viscosity effects and the loss of energy by dissipation during the motion. Stokes [28] made a significant contribution (1842–1846), notably in studying the flow around a rigid sphere, that yields the Stokes law.

2.6.2.2 Constitutive Law for Newtonian Flows

The concept of stress tensor, as expressed by (2.88) above, exists for any material that touches continuum mechanics. The stress tensor is often determined by experiments and its expression varies depending on the material under study.

It is convenient to split $\boldsymbol{\sigma}$ as

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mathbb{D}, \tag{2.89}$$

where p is the dynamic pressure and $\mathbb{D}$ the deviatoric part of $\boldsymbol{\sigma}$. We have the fundamental relation

$$p = -\frac{1}{3}\mathrm{tr}\boldsymbol{\sigma}. \tag{2.90}$$

It must be stressed that many experiments indicate that the dynamic pressure agrees with the static pressure [2], so far that a static fluid is considered to be a fluid in motion with a velocity equal to 0.

It remains to specify $\mathbb{D}$, which is responsible of the shear in the flow. The fluids we are interested in are water and air because they are involved in oceanography, meteorology, and climate, which are the applications we have in mind. For both water and air, experiments indicate that $\mathbb{D}$ is a linear function of $\nabla\mathbf{v}$, thus defining Newtonian fluids. The following definition holds.

Definition 2.3. Every fluid whose deviatoric tensor is a linear function of its velocity gradient is called a Newtonian fluid.

In addition to air and water, most organic solvents and mineral oils are also Newtonian fluids. Their main physical property consists in filling the space instantaneously when they are poured into some cavity. In contrast, fluids such as paints, mustard, and ketchup do not behave in the same way and therefore are not Newtonian fluids.

Throughout the book we assume the following:

Assumption 2.5. *The fluid is Newtonian.*

The Newtonian assumption 2.5 leads us to write $\mathbb{D} = (d_{ij})_{1\le i,j\le 3}$ in the form

$$d_{ij} = A_{ijk\ell}\partial_\ell u_k, \tag{2.91}$$

where $(A_{ijk\ell})_{1\le i,j,k,\ell\le 3}$ remains to be specified. We assume that $\mathbb{D} = (d_{ij})_{1\le i,j\le 3}$ is isotropic, which means that it is invariant under coordinates changes. This implies that the tensor $\mathbf{A} = (A_{ijk\ell})_{1\le i,j,k,\ell\le 3}$ is also isotropic. Because of this isotropic assumption 2.3, we know that $\mathbf{A}$ is of the form [12]

$$A_{ijk\ell} = \mu\delta_{ik}\delta_{jl} + \mu'\delta_{i\ell}\delta_{jk} + \mu''\delta_{ij}\delta_{k\ell}, \tag{2.92}$$

where μ, μ', and μ'' are real numbers.

From the incompressibility assumption, we have

$$\delta_{ij}\delta_{k\ell}\partial_\ell u_k = \partial_k u_k = \nabla \cdot \mathbf{v} = 0. \tag{2.93}$$

Therefore, (2.91), (2.92), and (2.93) yield

$$d_{ij} = \mu\delta_{ik}\delta_{jl}\partial_\ell u_k + \mu'\delta_{i\ell}\delta_{jk}\partial_\ell u_k = \mu\partial_j v_i + \mu'\partial_i v_j. \tag{2.94}$$

Furthermore, since $\boldsymbol{\sigma}$ is symmetric, so is $\mathbb{D}$. Therefore we have $\mu = \mu'$ and

$$\mathbb{D} = 2\mu D\mathbf{v}. \tag{2.95}$$

Consequently

$$\boldsymbol{\sigma} = 2\mu D\mathbf{v} - p\mathbf{I}. \tag{2.96}$$

The coefficient μ is the dynamic viscosity, a typical unit of which is Pascal $\times$ seconds. Since viscous effects are known from experiments to be dissipative, we have $\mu > 0$. The dynamic viscosity varies depending on the temperature θ. For air and many other gases, μ satisfies Sutherland's law:

$$\mu = \mu(\theta) = \mu_0 \left(\frac{\theta}{\theta_0}\right)^{\frac{3}{2}} \frac{\theta_0 + C}{\theta + C}, \tag{2.97}$$

where μ_0, θ_0, and C are constants that must be fixed from experiments. For water and many other liquids, μ satisfies the exponential law

$$\mu = \mu(\theta) = \mu_0 e^{-b\theta}, \tag{2.98}$$

for some constants μ_0 and b.

To conclude this subsection, we notice that when we apply the Stokes formula to (2.96), we obtain

$$F(\tau, \delta V) = \int_{\omega_\tau} \nabla \cdot \boldsymbol{\sigma}(t, \mathbf{x}) d\mathbf{x}. \tag{2.99}$$

Therefore, the quantity $\nabla \cdot \boldsymbol{\sigma}$ can be understood as the density function of internal strength. Therefore, we can define the force $d\mathbf{f}_{int}(t, \mathbf{x})$ exerted by the rest of the fluid on a particle at $\mathbf{x} \in \Omega$ as a measure on Ω for a fixed t, by the formula

$$d\mathbf{f}_{int}(t, \mathbf{x}) = (\nabla \cdot \boldsymbol{\sigma}(t, \mathbf{x}))\, dv(t, \mathbf{x}), \tag{2.100}$$

where dv was defined by the formula (2.24). This point of view will be useful in what follows.

2.6.3 The Momentum Equation

We apply Newton's law (2.87) to a given fluid particle that sits on $\mathbf{x}$ at time t. We recall that $\mathbf{v}$ satisfies the regularity assumption 2.2, that is, $\mathbf{v}$ is of class C^1 with respect to t and $\mathbf{x}$.

We assume that the external forces exerted on $\mathbf{x}$ at time t by the fluid can be described by a density function $\mathbf{f}_{ext}(t, \mathbf{x})$. This is completely true for gravity, where $\mathbf{f}_{ext}(t, \mathbf{x}) = \rho(t, \mathbf{x})\mathbf{g}$, $\mathbf{g}$ being the gravitational acceleration. Then the "sum of applied forces" is equal to

$$d\mathbf{f}_{int}(t, \mathbf{x}) + \mathbf{f}_{ext}(t, \mathbf{x}) dv(t, \mathbf{x}) = (\nabla \cdot \boldsymbol{\sigma} + \mathbf{f}_{ext}) dv. \tag{2.101}$$

We aim at computing the acceleration of the particle. This particle moves to $\mathbf{x} + u(t, \mathbf{x})\delta t + o(\delta t)$ at time $t + \delta t$, where its velocity is equal to $\mathbf{v}(t + \delta t, \mathbf{x} + u(t, \mathbf{x})\delta t + o(\delta t))$. Therefore its acceleration, denoted by $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$, is equal to

$$\boldsymbol{\gamma}(t, \mathbf{x}) = \lim_{\delta \to 0} \frac{\mathbf{v}(t + \delta t, \mathbf{x} + u(t, \mathbf{x})\delta t + o(\delta t)) - \mathbf{v}(t, \mathbf{x})}{\delta t} = \frac{D\mathbf{v}}{Dt}(t, \mathbf{x}). \tag{2.102}$$

Following the arguments for proving formula (2.32) in Sect. 2.3.3, we find component by component

$$\boldsymbol{\gamma}_i(t, \mathbf{x}) = \partial_t \mathbf{v}_i + \mathbf{v} \cdot \nabla \mathbf{v}_i = \partial_t \mathbf{v}_i + \mathbf{v}_j \partial_j \mathbf{v}_i. \tag{2.103}$$

The vector, whose coordinates are $(\mathbf{v} \cdot \nabla \mathbf{v}_1, \mathbf{v} \cdot \nabla \mathbf{v}_2, \mathbf{v} \cdot \nabla \mathbf{v}_3)$ and which appears in the expression of γ, is denoted by $(\mathbf{v} \cdot \nabla)\, \mathbf{v}$.

From (2.101), Newton's law applied to our particle at $(t, \mathbf{x})$ yields

$$\rho \boldsymbol{\gamma} dv = (\nabla \cdot \boldsymbol{\sigma} + \mathbf{f}_{ext}) dv. \tag{2.104}$$

We divide each side of this equation by $dv \neq 0$. We find the momentum equation

$$\rho(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\, \mathbf{v}) = \nabla \cdot \boldsymbol{\sigma} + \mathbf{f}_{ext}. \tag{2.105}$$

By using formulas (2.89) and (2.95), this equation becomes

$$\rho(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\, \mathbf{v}) - \nabla \cdot (2\mu D\mathbf{v} - p\mathbf{I}) = \mathbf{f}_{ext}. \tag{2.106}$$

When we combine equation (2.106) with the incompressibility condition (2.44), we get the Navier–Stokes equations in their initial form.

2.6.4 *The Navier–Stokes Equations: Various Forms*

2.6.4.1 Basic Form

It is commonly accepted that any variations of the density ρ are negligible in the momentum equation for incompressible flows ([2, 14]). Therefore, we take $\rho = \rho_0$ in equation (2.106), where ρ_0 is a constant. For instance, $\rho_0 = 1035\,\mathrm{Kg.m}^{-3}$ for the ocean ([10, 21]).

We divide equation (2.106) by ρ_0, and we still denote by p the ratio p/ρ_0, which becomes a "density of pressure per unit of mass and volume." Consistent with usual practice, we still call this new variable "the pressure."

We denote by $\mathbf{f}$ the ratio $\mathbf{f}_{ext}/\rho_0$, still called the "external forcing." Finally, we put

$$\nu = \frac{\mu}{\rho_0}, \tag{2.107}$$

which defines the kinematic viscosity, a typical unit of which is the square meter per second ($m^2 s^{-1}$).

We combine the momentum equation and the mass conservation equation to obtain the main usual form of the incompressible Navier–Stokes equations (NSE in the remainder) while noting

$$\nabla \cdot (p\mathbf{I}) = \nabla p, \tag{2.108}$$

in assuming p to be of class C^1. We find

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} - \nabla \cdot (2\nu D\mathbf{v}) + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0. \end{cases} \tag{2.109}$$

The unknowns are the pressure term $p = p(t, \mathbf{x})$ and the velocity $\mathbf{v}(t, \mathbf{x})$. The external forcing $\mathbf{f} = \mathbf{f}(t, \mathbf{x})$ and the initial value $\mathbf{v}_0 = \mathbf{v}_0(\mathbf{x}) = \mathbf{v}(0, \mathbf{x})$ are given.

Note that the pressure is not a prognostic variable, and so knowledge of its initial value is not required.

2.6.4.2 The Nonlinear Term in Divergence Form

The i^{th} component of the vector $(\mathbf{v} \cdot \nabla)\,\mathbf{v}$ is $v_j \partial_j v_i$. Because of the incompressibility condition, we have

$$v_j \partial_j v_i = \partial_j (v_i v_j). \tag{2.110}$$

The term $\partial_j(v_i v_j)$ is the i^{th} component of the vector $\nabla\cdot(\mathbf{v}\otimes\mathbf{v})$, where $\mathbf{v}\otimes\mathbf{v}$ denotes the tensor $\mathbf{v}\otimes\mathbf{v} = (v_i v_j)_{1\le i,j\le 3}$. This allows the NSE to be written as follows:

$$\begin{cases} \partial_t \mathbf{v} + \nabla\cdot(\mathbf{v}\otimes\mathbf{v}) - \nabla\cdot(2\nu D\mathbf{v}) + \nabla p = \mathbf{f}, \\ \nabla\cdot\mathbf{v} = 0, \end{cases} \tag{2.111}$$

or equivalently

$$\begin{cases} \partial_t \mathbf{v} + \nabla\cdot(\mathbf{v}\otimes\mathbf{v} - 2\nu D\mathbf{v} + p\mathbf{I}) = \mathbf{f}, \\ \nabla\cdot\mathbf{v} = 0. \end{cases} \tag{2.112}$$

This last form might be interesting, especially when $\mathbf{f}$ is a restoring force, $\mathbf{f} = \nabla\cdot V$, such as gravity. Hence, the NSE can be considered as a conservative law of the form

$$\begin{cases} \partial_t \mathbf{v} + \nabla\cdot P(\mathbf{v}, p) = 0, \\ \nabla\cdot\mathbf{v} = 0. \end{cases} \tag{2.113}$$

2.6.4.3 Form with the Vorticity

We note that

$$(\mathbf{v}\cdot\nabla)\,\mathbf{v} = \nabla\mathbf{v}\cdot\mathbf{v}, \tag{2.114}$$

where the r.h.s. above is the product of the matrix $\nabla\mathbf{v}$ by the vector $\mathbf{v}$ (see in [4]). Using the decomposition (2.52) combined with (2.71), we find

$$(\mathbf{v}\cdot\nabla)\,\mathbf{v} = D\mathbf{v}\cdot\mathbf{v} + \frac{1}{2}\boldsymbol{\omega}\times\mathbf{v}, \tag{2.115}$$

leading to

$$(\mathbf{v}\cdot\nabla)\,\mathbf{v} = \nabla\left(\frac{|\mathbf{v}|^2}{2}\right) + \boldsymbol{\omega}\times\mathbf{v}. \tag{2.116}$$

The NSE then take the form

$$\begin{cases} \partial_t \mathbf{v} + \boldsymbol{\omega}\times\mathbf{v} - \nabla\cdot(2\nu D\mathbf{v}) + \nabla\left(p + \dfrac{|\mathbf{v}|^2}{2}\right) = \mathbf{f}, \\ \nabla\cdot\mathbf{v} = 0. \end{cases} \tag{2.117}$$

2.6.4.4 Rotating Fluids

Up to now, we have calculated the acceleration of a particle using formulas (2.102) and (2.103), which require the coordinate system to be Galilean.

In the case of "rotating fluids" such as the atmosphere and the ocean, the acceleration is computed in a local system that turns with the earth, with an angular velocity $\boldsymbol{\Omega}$. Then we have ([10, 15, 21])

$$\boldsymbol{\gamma} = \frac{D\mathbf{v}}{Dt} - 2\boldsymbol{\Omega} \times \mathbf{v}.$$

Therefore, for such flows the NSE become

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} - 2\boldsymbol{\Omega} \times \mathbf{v} - \nabla \cdot (2\nu D\mathbf{v}) + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0, \end{cases} \tag{2.118}$$

which is the form primarily used to model the motion of the ocean. The term $-2\boldsymbol{\Omega} \times \mathbf{v}$ is commonly considered as a force, because any observer in a rotating reference frame feels an eastward deflection, which is of prime importance in meteorology and oceanography.

Although this effect has been known since Galileo, it is called the "Coriolis Force," because of G. Coriolis who formalized it in 1835 [6].

2.6.4.5 Case of a Constant Viscosity

We consider an adiabatic flow, whose viscosity ν remains constant. For the record:

- $\nu = 1.006.10^{-6}\,\mathrm{m^2 s^{-1}}$ for the water at 20° C.
- $\nu = 15.6.10^{-6}\,\mathrm{m^2 s^{-1}}$ for the air at 25° C.

In such case, we have

$$\nabla \cdot (2\nu D\mathbf{v}) = \nu \nabla \cdot (D\mathbf{v}) = \nu(\Delta \mathbf{v} + \nabla(\nabla \cdot \mathbf{v})) = \nu \Delta \mathbf{v}, \tag{2.119}$$

because $D\mathbf{v} = (1/2)(\nabla \mathbf{v} + \nabla \mathbf{v}^t)$ with $\nabla \cdot \mathbf{v} = 0$. Hence, the NSE become

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0. \end{cases} \tag{2.120}$$

2.6.5 Equations for the Vorticity and the Pressure

Throughout this section, we assume that the field $\mathbf{v}$ is of class C^3 and p is of class C^2, with respect to t and $\mathbf{x}$, while the source term $\mathbf{f}$ is of class C^1. This regularity assumption is being made to justify the formal calculus carried out in this section. We also assume that the viscosity ν is constant for simplicity.

2.6.5.1 Vorticity Equation

To find the equation satisfied by $\boldsymbol{\omega}$, we take the curl of the NSE in its form (2.117) when applying (2.119). We check each term carefully.

On one hand, we observe that for any scalar field E, $\nabla \times \nabla E = 0$. On the other hand, the formula (2.74) gives the general rule

$$\nabla \times (\mathbf{E} \times \mathbf{F}) = \nabla \cdot (\mathbf{E} \otimes \mathbf{F}) - \nabla \cdot (\mathbf{F} \otimes \mathbf{E}) = (\mathbf{E} \cdot \nabla)\,\mathbf{F} - (\mathbf{F} \cdot \nabla)\,\mathbf{E} \tag{2.121}$$

satisfied by any vector field $\mathbf{E}$ and $\mathbf{F}$ with free divergence. Due to the regularity assumption, we can write

$$\nabla \cdot \boldsymbol{\omega} = \varepsilon_{ijk} \partial_i \partial_j u_k = -\varepsilon_{jik} \partial_j \partial_i u_k = -\nabla \cdot \boldsymbol{\omega} = 0, \tag{2.122}$$

by using the antisymmetry of the Levy-Civita tensor. Hence, the general rule (2.121) applies to $\mathbf{v}$ and $\boldsymbol{\omega}$. Furthermore, regularity allows the Schwarz theorem to be applied:

$$\nabla \times \Delta \mathbf{v} = \Delta(\nabla \times \mathbf{v}) = \Delta \boldsymbol{\omega}, \quad \nabla \times \partial_t \mathbf{v} = \partial_t (\nabla \times \mathbf{v}) = \partial_t \boldsymbol{\omega}.$$

Accordingly, by taking the curl of (2.117) with ν constant, we find

$$\partial_t \boldsymbol{\omega} + (\mathbf{v} \cdot \nabla)\,\boldsymbol{\omega} - \nu \Delta \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\,\mathbf{v} + \nabla \times \mathbf{f}. \tag{2.123}$$

The term $(\boldsymbol{\omega} \cdot \nabla)\,\mathbf{v}$ is called the vortex-stretching term. It is worth noting that in the two-dimensional case, where things are simplest, this term does not appear in the vorticity equation. This may be of relevance in the study of stratified flows such as large-scale motions in the ocean or in the atmosphere, for example, cyclones and anticyclones, which present some two-dimensional structure.

2.6.5.2 Pressure Equation

We take the divergence of the NSE in its form (2.120) and study each term separately. We have

$$\nabla \cdot (\nabla p) = \partial_i (\partial_i p) = \Delta p. \tag{2.124}$$

Applying the Schwarz theorem together with the incompressibility assumption, we obtain

$$\nabla \cdot \partial_t \mathbf{v} = \partial_t (\nabla \cdot \mathbf{v}) = 0, \quad \nabla \cdot (\Delta \mathbf{v}) = \Delta (\nabla \cdot \mathbf{v}) = 0,$$

which yields

$$\Delta p = \nabla \cdot ((\mathbf{v} \cdot \nabla) \mathbf{v}) + \nabla \cdot \mathbf{f}. \tag{2.125}$$

Furthermore, note that

$$\nabla \cdot ((\mathbf{v} \cdot \nabla) \mathbf{v}) = \partial_i v_j \partial_j v_i = \nabla \mathbf{v} : \nabla \mathbf{v}^{\mathrm{t}}.$$

Equation (2.125) therefore takes the form

$$\Delta p = \nabla \mathbf{v} : \nabla \mathbf{v}^{\mathrm{t}} + \nabla \cdot \mathbf{f}. \tag{2.126}$$

2.7 Boundary Conditions

We derived the NSE from mathematical principles combined with experimental observations. The equations are based on conservation, dynamics, and dissipation principles, which are of course essential features of the flow. However, emphasis must also be given to the role played by the boundary conditions, which are crucial in order to provide a full mathematical description of the flow, which is our main aim.

The boundary conditions describe macroscopic as well as microscopic effects that can be considered as engines of the motion. For example, movements in the air and the sea are essentially due to the heating by the sun, which supplies energy to the sea/air system. This energy is converted into kinetic energy and dissipation. Moreover, the air and sea exchange energy all the time, some of which is dissipated during the transaction.

The energy process sketched above and many others are described through boundary conditions (BC in what follows). They are often hard to model with mathematics, and there are many possible ways of describing the same thing. The choice may vary depending on the specific case under study. However, boundary conditions may also be suggested—and even imposed—by numerical or purely mathematical constraints. Moreover, some boundary conditions are simply mathematical artifacts but relevant for a better understanding of the local nature of the NSE.

In this section, we examine the following BC: periodic BC, the case of a full space, no-slip BC, Navier BC, friction BC, and air/sea interface.

2.7.1 Periodic Boundary Conditions

The periodic BC are certainly the least physical ones of all, but they still remain very popular because they have the great advantage that Fourier analysis can be used to study the NSE, especially when ν is constant. This helps in getting a better understanding of the interaction between small and large scales and the balance between the convection term $(\mathbf{v} \cdot \nabla)\, \mathbf{v}$ and the diffusion term $\nu \Delta \mathbf{v}$ in the NSE, either in the form (2.120) or in the form (2.111).

Let $[0, L]^3$ be a given box, for some $L > 0$. The "set of wave vectors" is defined as the quotient set

$$\mathscr{T}_3 = \frac{2\pi \mathbb{Z}^3}{L}.$$

The domain of study is the torus

$$\mathbb{T}_3 = \frac{[0, L]^3}{\mathscr{T}_3}, \tag{2.127}$$

within which the velocity $\mathbf{v}$ and the pressure p can both be decomposed into Fourier series,

$$\mathbf{v}(t, \mathbf{x}) = \sum_{\mathbf{k} \in \mathscr{T}_3} \hat{\mathbf{v}}_{\mathbf{k}}(t) e^{i \mathbf{k} \cdot \mathbf{x}}, \quad p(t, \mathbf{x}) = \sum_{\mathbf{k} \in \mathscr{T}_3} p_{\mathbf{k}}(t) e^{i \mathbf{k} \cdot \mathbf{x}}, \quad (t, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{T}_3. \tag{2.128}$$

2.7.2 The Full Space

In this case, the flow domain is $\mathbb{R}^3$. It is assumed that the fluid is at rest at infinity, which is not so unreasonable. Rather than forcing $\mathbf{v}$ to be zero at infinity, we impose the integrability condition

$$\forall\, t \in \mathbb{R}_+ \quad \mathbf{v}(t, \cdot) \in L^2(\mathbb{R}^3), \tag{2.129}$$

We emphasize that we do not require p to satisfy any boundary condition.

Remark 2.4. Leray [18] and Oseen [25], who pioneered the mathematical analysis of the NSE, considered this type of BC, with ν constant. However, we impose $\mathbf{v}(t, \cdot) \in L^2(\mathbb{R}^3)$ at all times because of the continuity assumption 2.2 that holds for local time solutions such as those studied by C. Oseen. J. Leray obtained a global time solution to the NSE in this case, which he called a "turbulent solution" (see also Sect. 3.4.2 in Chap. 3), but we do not know if this satisfies assumption 2.2 or not, when $\mathbf{v}_0 \in L^2(\mathbb{R}^3)$ is continuous on $\mathbb{R}^3$. Therefore, the right BC should be "at almost all $t \in \mathbb{R}_+$, $\mathbf{v}(t, \cdot) \in L^2(\mathbb{R}^3)$" as introduced in [18]. The same applies to the other BC below, where "at all" should be replaced by "at almost all."

2.7.3 No-Slip Condition

Let Ω denote the flow domain with a boundary Γ. The three typical cases are:

1. Ω is a half space and Γ is a plane.
2. $\Omega = \mathbb{R}^3 \setminus V$, where V is a bounded smooth set in $\mathbb{R}^3$ and $\Gamma = \partial V$.
3. Ω is a bounded domain in $\mathbb{R}^3$.

In case 1, the plane Γ may be the fixed bottom of an infinite ocean, which has meaning for an observer very deep in the sea.

Case 2 models a body moving in a fluid, such as a plane flying in the air, a fishing net being pulled in the ocean, and many similar examples, in particular the sphere which is the most studied since the initial work by Stokes [28]. The body's velocity is known and denoted by $\mathbf{U}$. We aim to describe the flow structure around the body and to calculate the constrains exerted on it by the fluid. It is more convenient to consider that the body is at rest and that the velocity of the fluid is equal to $-\mathbf{U}$ at infinity.

Case 3 models a flow in a closed cavity, such as fuel in an engine.

The "no-slip condition" is of the form

$$\forall\,(t, \mathbf{x}) \in \mathbb{R}_+ \times \Gamma, \quad \mathbf{v}(t, \mathbf{x}) = 0, \tag{2.130}$$

or more simply

$$\mathbf{v}|_\Gamma = 0. \tag{2.131}$$

Here too, no special condition on the pressure is required at Γ. We sometimes say the "homogeneous Dirichlet BC" instead of "no-slip condition," in line with the terms used in the study of partial differential equations.

The argument that yields the condition (2.131) is based on a microscale observation. Indeed, even if a physical surface Γ may seem very smooth at a macroscale, a closer examination at a microscale reveals many irregularities, which are however very large in comparison with the scale of the fluid at which the NSE hold. Hence, the fluid particles are stuck in the surface's irregularities, leading to the no-slip condition.

2.7.4 Navier Boundary Condition

Although the no-slip condition has been popular for a long time, it has also been very controversial. We may imagine that the fluid slips on the boundary while considering the possibility of friction, for example, a body experiencing drag when it moves in the fluid. The Navier condition represents a balance between slip and friction.

Let $\mathbf{w} = \mathbf{w}(\mathbf{x})$ be any vector field defined on Γ. We introduce the tangential part of $\mathbf{w}(\mathbf{x})$ at $\mathbf{x} \in \Gamma$, denoted by $\mathbf{w}_\tau(\mathbf{x})$:

$$\mathbf{w}_\tau(\mathbf{x}) = \mathbf{w}(\mathbf{x}) - (\mathbf{w}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}))\, \mathbf{n}(\mathbf{x}). \tag{2.132}$$

Let $t \in \mathbb{R}_+$ be a fixed time, $\mathbf{x} \in \Gamma$. We note $\mathbf{n} = \mathbf{n}(x)$, $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ for simplicity. We assume that Γ is not porous, such that no fluid particle crosses Γ, which means $\mathbf{v} = \mathbf{v}_\tau$ almost everywhere on Γ or in other words

$$\mathbf{v} \cdot \mathbf{n}|_\Gamma = 0. \tag{2.133}$$

The calculation of the friction at the boundary is based on the principles introduced in Sect. 2.6.2. Taking the view that the force applied by the fluid on Γ is equal to $\boldsymbol{\sigma} \cdot \mathbf{n}$, we take as friction the corresponding tangential part denoted by $(\boldsymbol{\sigma} \cdot \mathbf{n})_\tau$.

The Navier-slip condition is based on the observation that the fluid is slowed by the frictional force at Γ, resulting in the relation

$$\mathbf{v}_\tau = \mathbf{v} = -\alpha(\boldsymbol{\sigma} \cdot \mathbf{n})_\tau, \tag{2.134}$$

for some $\alpha > 0$. In conclusion, Navier BC are

$$\mathbf{v} \cdot \mathbf{n}|_\Gamma = 0, \quad (\mathbf{v} + \alpha(\boldsymbol{\sigma} \cdot \mathbf{n})_\tau)|_\Gamma = 0, \quad \alpha > 0. \tag{2.135}$$

Note that when α goes to zero, the Navier condition (2.135) converges to the no-slip condition (2.131), at least formally. When α goes to infinity, we find $(\boldsymbol{\sigma} \cdot \mathbf{n})_\tau|_\Gamma = 0$, which is the total slip condition, which only holds in the case of a perfect fluid.

2.7.5 Friction Law

We show in this subsection another way of computing the force exerted by the fluid on a given body V moving with a constant velocity $\mathbf{U}$. Equating the result with $\sigma \cdot \mathbf{n}$ yields another type of BC.

Let G be the center of gravity of V and S its effective area. Assume that at time t, G sits on $\mathbf{x}$. At time $t + \delta t$, G sits on $\mathbf{x} + \mathbf{U}\delta t$. Therefore, the total volume of fluid displaced is equal to $S|\mathbf{U}|\delta t$, the mass of which is equal to $\delta m = \rho S|\mathbf{U}|\delta t$. The momentum carried by the sphere, denoted by $\delta \mathbf{p}_S$, is equal to

$$\delta \mathbf{p}_S = \mathbf{U}\delta m = \rho S \mathbf{U}|\mathbf{U}|\delta t. \tag{2.136}$$

The fluid slides with friction on the body. This suggests that only one part of the momentum of the sphere is transmitted to the fluid. Therefore, the momentum of the displaced fluid is equal to

$$\delta \mathbf{p} = C\rho S \mathbf{U}|\mathbf{U}|\delta t, \tag{2.137}$$

where $C \in]0, 1[$ is a constant that is determined by experiment. Therefore, the force applied on the body by the fluid is equal to

$$\lim_{\delta t \to 0} \frac{\delta \mathbf{p}}{\delta t} = \boldsymbol{\sigma} \cdot n = C\rho S\mathbf{U}|\mathbf{U}|. \tag{2.138}$$

This law was firstly stated by Gauckler [9], then redeveloped by Manning [23], and therefore is often called the Gauckler–Manning law. Engineers also call it the Plotter–Landweber law [16], depending on the context in which they use it. Anyway, this law is in agreement with experiments and is used in numerical simulations (see in [17], for instance).

A natural general BC based on (2.138), which is used, for instance, in the modelization of the ocean–atmosphere interface considered in the next subsection, is

$$\mathbf{v} \cdot \mathbf{n}|_{\Gamma} = 0, \quad (\boldsymbol{\sigma} \cdot n)_{\tau}|_{\Gamma} = C(\mathbf{U}_0 - \mathbf{v}_{\tau})|\mathbf{U}_0 - \mathbf{v}_{\tau}|, \tag{2.139}$$

for some given $\mathbf{U}_0$. Moreover, we shall derive the same law from the turbulence modeling process carried out in Sect. 5.3 in Chap. 5. In this case, we call it the wall law, to which we shall pay attention from Chap. 6.

2.7.6 Ocean–Atmosphere Interface

We conclude this section with the ocean–atmosphere coupling. The usual assumption, known as the rigid lid assumption, is that the interface between the ocean and the atmosphere is a fixed surface, denoted by Γ.

Although this assumption is not very realistic, it is commonly used. Indeed, many highly complicated physical effects occur at the mixing layer between both media. Because of this complexity, we prefer to replace the physical mixing layer by an averaged thin layer called the rigid lid, especially when considering large scales. The energy processes between air and water are then modeled through suitable boundary conditions.

The processes involved in the air/sea coupling are dynamic as well as thermodynamic. We will only briefly outline the dynamic part in this subsection. The BC that we obtain is based on the law (2.139), considering friction between air and water.

For simplicity, we sit on a local earth coordinate frame. Let $\mathbf{k}$ be the vertical unit vector and $(\mathbf{i}, \mathbf{j})$ the unit vectors spanning Γ viewed as a plane in $\mathbb{R}^3$. The coordinates are denoted by (x, y, z).

Note that $\mathbf{k}$ is the outward-pointing unit normal vector $\mathbf{n}^w$ of the ocean at Γ, while $-\mathbf{k}$ is the outward-pointing unit normal vector $\mathbf{n}^a$ of the atmosphere at Γ.

Let $\mathbf{v}^w$ and $\mathbf{v}^a$ denote the water velocity and the air velocity, respectively. We split these into a horizontal part and a vertical part:

$$\mathbf{v}^w = (\mathbf{v}_h^w, w^w), \quad \mathbf{v}^a = (\mathbf{v}_h^a, w^a), \quad \mathbf{v}_h^w = (u^w, v^w), \quad \mathbf{v}_h^a = (u^a, v^a). \tag{2.140}$$

The rigid lid assumption yields

$$w^w|_\Gamma = w^a|_\Gamma = 0. \tag{2.141}$$

This is consistent with the condition (2.133) above, since

$$w^w|_\Gamma = \mathbf{v}^w \cdot \mathbf{n}^w, \quad w^a|_\Gamma = \mathbf{v}^a \cdot \mathbf{n}^a.$$

For the moment, we focus on the ocean. We shall adapt the condition (2.139), where we have to take into account the relative velocity at Γ equal to $\mathbf{v}^a - \mathbf{v}^w$. Note that $|\mathbf{v}^a - \mathbf{v}^w| = |\mathbf{v}_h^a - \mathbf{v}_h^w|$ at Γ due to (2.141).

Let us compute $\boldsymbol{\sigma}^w \cdot \mathbf{n}^w = \boldsymbol{\sigma}^w \cdot \mathbf{k}$, where $\boldsymbol{\sigma}^w = 2\mu_w D(\mathbf{v}^w) - p^w\mathbf{I}$. Applying the basic definitions yields

$$\boldsymbol{\sigma}^w \cdot \mathbf{k} = \begin{pmatrix} \partial_z u^w + \partial_x w^w \\ \partial_z v^w + \partial_y w^w \\ 2\partial_z w^w + p \end{pmatrix}. \tag{2.142}$$

We consider just the two first components of $\boldsymbol{\sigma}^w \cdot \mathbf{k}$. From (2.139), we find that on Γ

$$2\mu_w \frac{\partial \mathbf{v}_h^w}{\partial z} = C(\mathbf{v}^a - \mathbf{v}^w)|\mathbf{v}^a - \mathbf{v}^w|. \tag{2.143}$$

The same analysis holds for the atmosphere by the action and reaction principle, by using $\mathbf{v}^w - \mathbf{v}^a$ instead of $\mathbf{v}^a - \mathbf{v}^w$. Notice that the third component in the relation (2.142) is useless.

In summary, the BC on Γ is given by (2.141), together with

$$2\mu_w \frac{\partial \mathbf{v}_h^w}{\partial z} = C_1(\mathbf{v}^a - \mathbf{v}^w)|\mathbf{v}^a - \mathbf{v}^w|, \quad 2\mu_a \frac{\partial \mathbf{v}_h^a}{\partial z} = C_2(\mathbf{v}^w - \mathbf{v}^a)|\mathbf{v}^w - \mathbf{v}^a|, \tag{2.144}$$

where C_1 and C_2 are two constants that must be fixed from observations [10].

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Chapter 3
Mathematical Basis of Turbulence Modeling

Abstract Dimensional analysis is formalized in an abstract framework, which leads to the introduction of the generalized Reynolds numbers through dimensional bases. We give a rigorous mathematical statement of a version of the Reynolds similarity law, based on the dimensionless form of the Navier–Stokes equations (NSE), which highlights the connection between the Reynolds similarity law and the problem of uniqueness of solutions of the NSE. We review weak solutions *à la Leray* and strong solutions *à la Fujita–Kato*, to discuss the validity of such a law. Furthermore, weak solutions to the NSE are shown to have long-time averages, the equation they satisfy being determined, which allows the introduction of some basic tools of turbulence modeling, such as the Reynolds decomposition and the Reynolds stress.

3.1 Introduction

Having presented the basic ideas of fluid mechanics, we may now start the process of mathematical modeling of turbulent flows, which will be the subject of this and the next two chapters.

One may wonder why this modeling process is necessary. We aim to develop numerical tools to simulate realistic flows and to predict their motion, which will often be turbulent. Because of the structure of the turbulence, any code using the Navier–Stokes equations (NSE) derived in Chap. 2 would be very complex and would require too much computational resources in order to run the simulation. Turbulence models can reduce this complexity, but inevitably introduce a loss of accuracy.

This raises the issue of what is "turbulence," to which there is no real answer, although some facts can be deduced from observations and experiments. For example, while we can observe a wide variety of water flows in nature, two very different types are generally distinguished: calm waters and tumultuous waters,

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which were already depicted by Leonardo da Vinci in 1510 [13] and later studied by Stokes in 1851 [43] and Boussinesq in 1877 [7]. This distinction also holds for many different fluid flows (gas or liquid flows).

History records the contribution of Reynolds in 1883 [43], who introduced a smart experimental device to investigate the nature of the flows. This device is made of a transparent pipe ending in a valve and filled with water in which an ink thread is introduced. The valve is opened which causes a fluid motion. Depending on how the valve is opened, three states of the ink may be observed during the motion:

(i) The ink thread remains straight.
(ii) The ink thread moves following a distorted line.
(iii) The ink mixes with the water and develops eddies of many different sizes.

In case (i), the flow is said to be laminar, in case (ii) transitional, and in case (iii) turbulent. O. Reynolds brought to light a global dimensionless parameter that governs the state of flow, today called the Reynolds number and generally expressed by $Re = UL/\nu$, and that two flows having the same Reynolds number are similar, which is the first similarity law that we shall study in this book.

However, complexity of turbulent flows is due to nonlinear interactions between different scales, and we shall establish that at each scale there is a corresponding Reynolds number, leading to the notion of generalized Reynolds numbers.

The goals of this chapter are:

(a) to properly define the generalized Reynolds numbers and to give a mathematical formulation of the Reynolds similarity law,
(b) to make the link between the Reynolds similarity law and the standard results for the existence of solutions for the NSE, in order to investigate the mathematical validity of the Reynolds similarity law,
(c) to introduce the Reynolds decomposition and the Reynolds stress for long-time averages of global weak solutions to the NSE.

The Reynolds numbers and similarity laws derive from dimensional analysis and the various dimensionless forms of the equations. This is why we specifically give a rigorous mathematical framework to dimensional analysis, which is the aim of Sect. 3.2.

The novelty in Sect. 3.3 is the concept of length–time bases $b = (\lambda, \tau)$, where λ is a given length scale and τ a given time scale. Length–time bases will be used throughout this and the following two chapters. Indeed, this concept provides an appropriate tool for properly defining the generalized Reynolds numbers as well as any law about turbulence based on similarity assumptions, such as the Kolmogorov law or the wall laws studied in Chap. 5.

Section 3.3 concludes with the statement that a given flow satisfies the Reynolds similarity law if the uniqueness of the corresponding dimensionless NSE can be established, which is the case for most laminar flows. However, the situation is less simple for transitional or turbulent flows. We are naturally led to review the existing mathematical results in Sect. 3.4, by confronting the global time turbulent (also weak) solutions *à la Leray*, the uniqueness of which is generally not known, and the local time strong solution *à la Fujita–Kato*, which is unique. Therefore, we can assert at this stage that the Reynolds similarity law makes sense locally in time.

Nevertheless, it is worthwhile asking if there is any connection between the global time turbulent solutions *à la Leray* and the long-time average of turbulent flows considered by Boussinesq [7], Prandtl [33], Reynolds [34], and Stokes [43], where the long-time average $\overline{\mathbf{v}}$ of the velocity $\mathbf{v}$ is formally given by

$$\overline{\mathbf{v}}(\mathbf{x}) = \lim_{T\to\infty} \int_0^T \mathbf{v}(t, \mathbf{x})\, dt.$$

We shall see in Sect. 3.5 that we are able to rigorously prove that the long-time average of turbulent solutions is well defined in appropriate space functions, as well as deriving from the NSE the equations satisfied by $(\overline{\mathbf{v}}, \overline{p})$, by giving a sense to

(i) the Reynolds decomposition $\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}'$,
(ii) the Reynolds stress $\boldsymbol{\sigma}^{(\mathrm{R})} = \overline{\mathbf{v}' \otimes \mathbf{v}'}$,

which allows some of the main elements of the turbulence modeling to appear for the first time in a rigorous mathematical framework.

3.2 Dimensional Analysis

Let us start with a simple example. Let λ be any length scale and e be any energy scale. It is possible to construct from λ and e, a viscosity $\nu_t = \nu_t(\lambda, e)$ by the formula

$$\nu_t = \lambda\sqrt{e}. \tag{3.1}$$

This formula makes sense because it is easily checked that the dimension of $\lambda\sqrt{e}$ is indeed that of a viscosity. The starting point to derive a law like (3.1) is to postulate that $\nu_t = \nu_t(\lambda, e)$ and is of the form

$$\nu_t(\lambda, e) = \lambda^\alpha e^\beta. \tag{3.2}$$

The rest is elementary algebra, noting that the problem might or might not have a solution.

The aim of this section is to build an algebraic tool to perform the dimensional analysis, in particular the similarity laws that occur in turbulence modeling, showing for example how (3.1) might be derived from more deep physical considerations than from a postulate such (3.2), when possible and/or necessary.

3.2.1 Generalities

Turbulent flows are described by physical fields ψ expressed in the mass–time–length system, as shown throughout Chap. 2. We do not take thermal effects into account, nor electromagnetic interactions and possible variations of amount of molecular matter. Each field $\psi = \psi(t, \mathbf{x})$ depends on the time t and the space position $\mathbf{x}$.

The time t and space position $\mathbf{x}$ are defined by their measurements: time is what a clock reads, $|\mathbf{x}|$ is a length, and $\mathbf{x}/|\mathbf{x}|$ indicates a direction. The basic time unit is the second, which is approximately 1/86,400 of a mean solar day. Time and length are related by the speed of light in vacuum, commonly denoted by c, which is a universal physical constant equal to 299,792,458 ms^{-1}, thus defining the meter.

3.2.2 Basic Algebra

Each physical field $\psi = \psi(t, \mathbf{x})$ involved in turbulent flows can be decomposed as

$$\psi = m^{d_m(\psi)} \ell^{d_\ell(\psi)} \tau^{d_\tau(\psi)}, \tag{3.3}$$

where $m = m(t, \mathbf{x})$ is a mass field (expressed in kilograms), $\tau = \tau(t, \mathbf{x})$ a time field (expressed in seconds), and $\ell = \ell(t, \mathbf{x})$ a length field (expressed in meters). In the expression above,

$$\mathbb{D}(\psi) = (d_m(\psi), d_\ell(\psi), d_\tau(\psi)) \in \mathbb{Q}^3, \tag{3.4}$$

is the dimension of ψ. Notice that in particular, $\mathbb{D}(\mathbf{x}) = (0, 1, 0)$, $\mathbb{D}(t) = (0, 0, 1)$. We also use the notation

$$[\psi] = \mathscr{M}^{d_m(\psi)} \mathscr{L}^{d_\ell(\psi)} \mathscr{T}^{d_\tau(\psi)}, \tag{3.5}$$

which is useful in practical calculations. From now on, we denote by $(\mathscr{F}, \times)$ the monoid[1] of all scalar fields related to a given turbulent flow.

Definition 3.1. We say that $\psi \in \mathscr{F}$ is dimensionless if and only if

$$\mathbb{D}(\psi) = (0, 0, 0), \tag{3.6}$$

which is equivalent to

$$\forall \varphi \in \mathscr{F}, \quad \mathbb{D}(\psi\varphi) = \mathbb{D}(\varphi). \tag{3.7}$$

[1] A monoid is an algebraic structure with a single associative binary operation and an identity element.

We list below the main properties of the operator $\mathbb{D}$.

Property 3.1. Products: The operator $\mathbb{D}$ is a morphism from $(\mathscr{F}, \times)$ into the group $(\mathbb{Q}^3, +)$, that is,

$$\forall\, \psi, \varphi \in \mathscr{F}, \quad \mathbb{D}(\psi\varphi) = (d_m(\psi) + d_m(\varphi), d_\ell(\psi) + d_\ell(\varphi), d_\tau(\psi) + d_\tau(\varphi)). \tag{3.8}$$

In particular,

$$\forall\, \psi \in \mathscr{F}, \quad \forall\, p \in \mathbb{Z}, \quad \mathbb{D}(\psi^p) = (p\, d_m(\psi), p\, d_\ell(\psi), p\, d_\tau(\psi)). \tag{3.9}$$

Property 3.2. Sums: The field $\psi + \varphi$ makes sense if and only if $\mathbb{D}(\psi) = \mathbb{D}(\psi)$, and in this case $\mathbb{D}(\psi + \varphi) = \mathbb{D}(\psi) = \mathbb{D}(\varphi)$.

Property 3.3. Derivatives: In what follows, $\psi \in \mathscr{F}$ is given.

$$\mathbb{D}\left(\frac{\partial \psi}{\partial t}\right) = (d_m(\psi), d_\ell(\psi), d_\tau(\psi) - 1), \tag{3.10}$$

$$\forall\, i = 1, 2, 3, \quad \mathbb{D}\left(\frac{\partial \psi}{\partial x_i}\right) = (d_m(\psi), d_\ell(\psi) - 1, d_\tau(\psi)). \tag{3.11}$$

Property 3.4. Integrals:

$$\begin{cases} \qquad\qquad \forall\, V \subset \mathbb{R}^3, \ \text{ s.t. } \overset{\circ}{V} \neq \emptyset, \\ \mathbb{D}\left(\int\int\int_V \psi\, dV\right) = (d_m(\psi), d_\ell(\psi) + 3, d_\tau(\psi)), \end{cases} \tag{3.12}$$

$$\begin{cases} \qquad\qquad \forall\, S \subset \mathbb{R}^3, \ \text{ a nontrivial surface}, \\ \mathbb{D}\left(\int\int_S \psi\, dS\right) = (d_m(\psi), d_\ell(\psi) + 2, d_\tau(\psi)), \end{cases} \tag{3.13}$$

$$\begin{cases} \qquad \forall\, L \subset \mathbb{R}^3, \ \text{ a nontrivial Jordan curve}, \\ \mathbb{D}\left(\int_L \psi\, dl\right) = (d_m(\psi), d_\ell(\psi) + 1, d_\tau(\psi)). \end{cases} \tag{3.14}$$

We also have

$$\forall\, T > 0, \quad \mathbb{D}\left(\int_0^T \psi\, dt\right) = (d_m(\psi), d_\ell(\psi), d_\tau(\psi) + 1). \tag{3.15}$$

3.2.3 Table of Scalar Fields Dimension

We list in the following table the dimension of the main scalar fields involved in fluid mechanics.

Physical field	Dimension $\mathbb{D}$
Mass density ρ	(1, −3, 0)
Dynamic viscosity μ	(1,−1,−1)
Kinematic viscosity ν	(0,2,−1)
Scalar velocity u	(0,1,−1)
Pressure per mass density p	(0,2,−2)
Kinetic energy per mass density $E = (1/2)\lvert\mathbf{v}\rvert^2$	(0, 2, −2)
Dissipation per mass density $\varepsilon = 2\nu\lvert D\mathbf{v}\rvert^2$	(0,2, −3)

Notice that except for ρ and μ, every dimension quoted above is of the form $(0, \alpha, \beta)$. Since ρ and μ will not be involved in the remainder of this chapter, we shall denote from now on $\mathbb{D}(\psi) = (d_\ell(\psi), d_\tau(\psi)) \in \mathbb{Q}^2$. This corresponds to a projection on the dimensional length–time plane.

3.2.4 Dimensional Independence

Definition 3.2. Let $\psi, \varphi \in \mathscr{F}$. We say that ψ and φ are dimensionally independent if and only if

$$\det(\mathbb{D}(\psi), \mathbb{D}(\varphi)) = \det \begin{pmatrix} d_\ell(\psi) & d_\ell(\varphi) \\ d_\tau(\psi) & d_\tau(\varphi) \end{pmatrix} \neq 0. \tag{3.16}$$

Notice that $\det(\mathbb{D}(\psi), \mathbb{D}(\varphi)) \in \mathbb{Q}$. If ψ and φ are scalar dimensionally independent fields, there exists $(p, q) \in \mathbb{Q}^2$ and $(r, s) \in \mathbb{Q}^2$ such that

$$\psi^p \varphi^q \ \text{ is a length,} \qquad \psi^r \varphi^s \ \text{ is a time.}$$

In other words,

$$\mathbb{D}(\psi^p \varphi^q) = (1, 0), \quad \mathbb{D}(\psi^r \varphi^s) = (0, 1). \tag{3.17}$$

More generally, the following holds.

Lemma 3.1. *Let ψ and φ be scalar dimensionally independent fields. Then given any scalar field ξ, there exists a unique $(p, q) \in \mathbb{Q}^2$ such that $\mathbb{D}(\psi^p \varphi^q) = \mathbb{D}(\xi)$.*

Proof. Equation $\mathbb{D}(\psi^p \varphi^q) = \mathbb{D}(\xi)$ is equivalent to the linear system

$$\begin{cases} p\, d_\ell(\psi) + q\, d_\ell(\varphi) = d_\ell(\xi), \\ p\, d_\tau(\psi) + q\, d_\tau(\varphi) = d_\tau(\xi), \end{cases} \tag{3.18}$$

by using the rule (3.8). Since $\det(\mathbb{D}(\psi), \mathbb{D}(\varphi)) \neq 0$, this system has a unique solution (p, q), calculated from the formula

$$
\begin{aligned}
p &= \det(\mathbb{D}(\psi), \mathbb{D}(\varphi))^{-1} \det(\mathbb{D}(\xi), \mathbb{D}(\varphi)) \in \mathbb{Q}, \\
q &= \det(\mathbb{D}(\psi), \mathbb{D}(\varphi))^{-1} \det(\mathbb{D}(\psi), \mathbb{D}(\xi)) \in \mathbb{Q},
\end{aligned}
\tag{3.19}
$$

due to the algebraic field structure of $\mathbb{Q}$. □

Corollary 3.1. *Let ψ_i, $i = 1, 2, 3$ be three scalar fields, and denote $\mathbb{D}(\psi_i) = (\alpha_i, \beta_i)$. If*

$$
rank \begin{pmatrix} \mathbb{D}(\psi_1) \\ \mathbb{D}(\psi_2) \\ \mathbb{D}(\psi_3) \end{pmatrix} = rank \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{pmatrix} = 2.
$$

Then there exists nonzero rational numbers p_i ($i = 1, 2, 3$), such that

$$
\mathbb{D}(\psi_1^{p_1} \psi_2^{p_2} \psi_3^{p_3}) = (0, 0). \tag{3.20}
$$

In other words, let us consider three scalar fields ψ_i, $i = 1, 2, 3$, two of them being dimensionally independent. Then we have that one can form a nontrivial dimensionless field $\psi_1^{p_1} \psi_2^{p_2} \psi_3^{p_3}$. Notice that this dimensionless field is uniquely determined up to an inversion.

3.2.5 Vector and Tensor Dimensional Algebra

Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be the canonical basis of $\mathbb{R}^3$. Each of these vectors is dimensionless,

$$
\forall\, i = 1, 2, 3, \quad \mathbb{D}(\mathbf{e}_i) = (0, 0). \tag{3.21}
$$

In the same way, let Ω be any smooth domain, Γ its boundary, and $\mathbf{n}$ the outward-pointing unit normal vector field on Γ. Then $\mathbf{n}$ is also dimensionless:

$$
\mathbb{D}(\mathbf{n}) = (0, 0). \tag{3.22}
$$

Definition 3.3. Each vector field $\mathbf{w} = (w_1, w_2, w_3) \in \mathscr{F}^3$ we consider from now on satisfies $\mathbb{D}(w_1) = \mathbb{D}(w_2) = \mathbb{D}(w_3)$. Then we set $\mathbb{D}(\mathbf{w}) = \mathbb{D}(w_i)$ ($i = 1, 2, 3$) and denote $\mathbb{D}(\mathbf{w}) = (d_\ell(\mathbf{w}), d_\tau(\mathbf{w}))$.

More generally, each tensor field

$$
\boldsymbol{\psi} = (\psi_{i_1 \ldots i_p})_{1 \le i_k \le n_k} \in \bigotimes_{k=1}^{p} \mathscr{F}^{n_k}
$$

for some integers $n_1, .., n_p$ also verifies $\mathbb{D}(\psi_{i_1 \ldots i_p}) = \mathbb{D}(\psi_{j_1 \ldots j_p})$ for all p-uplets $(i_1, \ldots, i_p)$ and $(j_1, \ldots, j_p)$. Thus, we set $\mathbb{D}(\boldsymbol{\psi}) = \mathbb{D}(\psi_{i_1 \ldots i_p})$, and

$$\mathbb{D}(\boldsymbol{\psi}) = (d_\ell(\boldsymbol{\psi}), d_\tau(\boldsymbol{\psi})). \tag{3.23}$$

The main vector and tensor fields encountered in the formulation of the incompressible NSE are given in the following table.

Physical field	Dimension $\mathbb{D}$
Velocity $\mathbf{v}$	(1,−1)
Deformation tensor $D\mathbf{v}$	(0,−1)
Vorticity $\boldsymbol{\omega}$	(0,−1)
Source term $\mathbf{f}$, force per mass unit	(1, −2)
Stress tensor per mass density $(1/\rho)\boldsymbol{\sigma}$	(2,−2)

3.3 Basic Similarity Setting

It is often said in fluid mechanics that two flows having the same Reynolds numbers and similar geometries share dynamic and kinematic similarity: they are said to be similar. At this stage, this sentence is ambiguous because

(i) the notion of *similarity* must be rigorously defined,
(ii) there are many ways to define the Reynolds number.

We aim to define rigorously these notions and to derive the dimensionless form of the NSE (see Sect. 3.3.2), which is performed by the introduction of dimensional bases.

3.3.1 *General Reynolds Numbers*

We first introduce the notion of a length–time basis $b = (\lambda, \tau)$, λ being a given length and τ a given time. This allows the definition of the b-dimensionless form of any given field $\boldsymbol{\psi} \in \mathscr{F}$ and the Reynolds number associated with b, which we call the generalized Reynolds number. The definition of the classical Reynolds number that could also be called "the large-scale Reynolds number" is postponed to Sect. 3.3.3 below.

3.3.1.1 Length–Time Bases

Real numbers are dimensionless, and we denote by $(t', \mathbf{x}') = (t', x'_1, x'_2, x'_3)$ standard four-uplets in $\mathbb{R} \times \mathbb{R}^3$.

Definition 3.4. A length–time basis is a couple

$$b = (\lambda, \tau), \tag{3.24}$$

where λ is a given constant length and τ a constant time.

Definition 3.5. Let $\boldsymbol{\psi} = \boldsymbol{\psi}(t, \mathbf{x})$ (constant, scalar, vector, tensor...) be any given field defined on a cylinder $Q = [0, T] \times \Omega$. Let $\boldsymbol{\psi}_b$ be the dimensionless field defined by

$$\boldsymbol{\psi}_b(t', \mathbf{x}') = \lambda^{-d_\ell(\boldsymbol{\psi})} \tau^{-d_\tau(\boldsymbol{\psi})} \boldsymbol{\psi}(\tau t', \lambda \mathbf{x}'), \quad (t', \mathbf{x}') \in Q_b = \left[0, \frac{T}{\tau}\right] \times \frac{1}{\lambda} \Omega. \tag{3.25}$$

We say that $\boldsymbol{\psi}_b = \boldsymbol{\psi}_b(t', \mathbf{x}')$ is the b-dimensionless field deduced from $\boldsymbol{\psi}$.

It is easily checked that $\boldsymbol{\psi}_b$ is dimensionless, by using the properties of $\mathbb{D}$ given in item (3.2.2) of Sect. 3.2.

Remark 3.1. According to Lemma 3.1, any couple (ψ, φ) of dimensionally independent constant fields may be used as alternate basis, which is referred to as a dimensional basis.

3.3.1.2 Generalized Reynolds Number

Let us consider a turbulent flow, the kinematic viscosity of which is ν. Following Kolmogorov [23] and Tikhomirov [47], a cascade of length scales λ and time scales τ is involved in the structure of this flow. Let $b = (\lambda, \tau)$ be the length–time basis related to a given scale and

$$V = \lambda \tau^{-1}, \tag{3.26}$$

be the convective associated velocity. The kinematic viscosity ν is a fixed field whose dimension is determined by $d_\ell(\nu) = 2$, $d_\tau(\nu) = -1$. Therefore, according to formula (3.25), the b-dimensionless field ν_b deduced from ν is expressed as

$$\nu_b = \lambda^{-2} \tau \nu = \frac{\nu}{V \lambda}, \tag{3.27}$$

by involving the associated convective velocity V given by (3.26). Let $Re(b)$ be the dimensionless number defined by

$$R(b) = \frac{1}{\nu_b} = \frac{V \lambda}{\nu}. \tag{3.28}$$

We observe that $Re(b)$ is of the same form as the Reynolds number used in classical fluid dynamics (see Sect. 3.3.3 below), which is why we call $Re(b)$ a generalized Reynolds number.

3.3.2 Mathematical Reynolds Similarity

Reynolds similarity is closely linked to the dimensionless form of the NSE, that is, the equations satisfied by $(\mathbf{v}_b, p_b)$ for any given length–time basis $b = (\lambda, \tau)$. Two different dimensionless forms are considered below, depending on how the source term is processed. This allows one to distinguish two formal behaviors of the equations as ν_b goes to zero or to ∞. We then define the mathematical similarity.

3.3.2.1 Dimensionless Form of the NSE

Let $Q = [0, T] \times \Omega$ be any cylinder, ν a kinematic viscosity, $\mathbf{f}$ any source term, and $\mathbf{v}_0 = \mathbf{v}_0(\mathbf{x})$ a given field on Ω. We assume that these data are such that the following NSE:

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = \mathbf{f} & \text{in } Q, \\ \nabla \cdot \mathbf{v} = 0 & \text{in } Q, \\ \mathbf{v} = 0 & \text{on } \Gamma, \\ \mathbf{v} = \mathbf{v}_0 & \text{at } t = 0, \end{cases} \tag{3.29}$$

has a solution $(\mathbf{v}, p)$ that is sufficiently smooth for the needs of the analysis performed in this section.

Lemma 3.2. *Let $b = (\lambda, \tau)$ be any length–time basis and $(\mathbf{v}_b, p_b)$ be the b-dimensionless field deduced from $(\mathbf{v}, p)$. Then $(\mathbf{v}_b, p_b)$ satisfies the following dimensionless NSE,*

$$\begin{cases} \partial_{t'} \mathbf{v}_b + (\mathbf{v}_b \cdot \nabla')\,\mathbf{v}_b - \nu_b \Delta' \mathbf{v}_b + \nabla' p_b = \mathbf{f}_b & \textit{in } Q_b, \\ \nabla' \cdot \mathbf{v}_b = 0 & \textit{in } Q_b, \\ \mathbf{v}_b = 0 & \textit{on } \Gamma_b, \\ \mathbf{v}_b = (\mathbf{v}_0)_b & \textit{at } t = 0. \end{cases} \tag{3.30}$$

Proof. It is easily checked that for any field $\boldsymbol{\psi} = \boldsymbol{\psi}(t, \mathbf{x})$ of class C^1 in time and of class C^2 in space,

$$\partial_t \boldsymbol{\psi}(t, \mathbf{x}) = \lambda^{d_\ell(\boldsymbol{\psi})} \tau^{d_\tau(\boldsymbol{\psi})-1} \partial_{t'} \boldsymbol{\psi}_b(t', \mathbf{x}'), \tag{3.31}$$

$$\nabla \boldsymbol{\psi}(t, \mathbf{x}) = \lambda^{d_\ell(\boldsymbol{\psi})-1} \tau^{d_\tau(\boldsymbol{\psi})} \nabla' \boldsymbol{\psi}_b(t', \mathbf{x}'), \tag{3.32}$$

$$\Delta \boldsymbol{\psi}(t, \mathbf{x}) = \lambda^{d_\ell(\boldsymbol{\psi})-2} \tau^{d_\tau(\boldsymbol{\psi})} \nabla' \boldsymbol{\psi}_b(t', \mathbf{x}'), \tag{3.33}$$

where $(t, \mathbf{x}) = (\tau t', \lambda \mathbf{x}')$. As

$$d_\ell(\mathbf{v}) = 1, \quad d_\tau(\mathbf{v}) = -1, \quad d_\ell(p) = 2, \quad d_\tau(p) = 1,$$

we get from (3.25)

$$\mathbf{v}(t, \mathbf{x}) = \lambda \tau^{-1} \mathbf{v}_b(t', \mathbf{x}'), \quad p(t, \mathbf{x}) = \lambda^2 \tau^{-2} p_b(t', \mathbf{x}'). \tag{3.34}$$

Then (3.30) results from (3.31)–(3.33) applied to $\mathbf{v}$ and p, combined with (3.34). □

3.3.2.2 Further Dimensionless Form

We consider another way to derive a dimensionless form of the NSE, which is based on the determination of other force and pressure scales that occur during the motion.

Starting with the source term, we assume that $\mathbf{f} \in L^1(Q)$ and let F be the average of $\mathbf{f}$, expressed as

$$F = \frac{1}{|Q|} \int \int_Q |\mathbf{f}(t, \mathbf{x})| d\mathbf{x} dt, \tag{3.35}$$

which is of the same dimension as $\mathbf{f}$, i.e., $\mathbb{D}(F) = \mathbb{D}(\mathbf{f}) = (1, -2)$, thus providing a body force scale. Using F, λ, and ν, one can form a dimensionless number $F_{b,\nu}$ given by the formula

$$F_{b,\nu} = \frac{F \lambda^3}{\nu^2}, \tag{3.36}$$

that plays a role in the equations. For large scales, this number shares analogies with the Grashof number (see Eckert and Drake [16]) used to characterize convection in fluid. It is also used by many authors to study various mathematical aspects of turbulence that are linked to scale cascades, such as in Doering and Foias [14], Doering and Gibbon [15], and further references therein.

We now proceed with the pressure, by observing that

$$P_{b,\nu} = \frac{\nu V}{\lambda} \tag{3.37}$$

is a constant field, the dimension of which is a pressure, i.e., $\mathbb{D}(P_{b,\nu}) = \mathbb{D}(p) = (2, -2)$.

In order to get an another dimensionless form of the NSE, we consider the dimensionless fields given by the formula

$$\tilde{p}_{b,\nu}(t', \mathbf{x}') = \frac{p(t, \mathbf{x})}{P_{b,\nu}}, \quad \tilde{\mathbf{f}}(t', \mathbf{x}') = \frac{\mathbf{f}(t, \mathbf{x})}{F}, \tag{3.38}$$

which yields the following dimensionless form of the NSE:

$$\begin{cases} \partial_{t'} \mathbf{v}_b + (\mathbf{v}_b \cdot \nabla')\, \mathbf{v}_b + \nu_b(-\Delta' \mathbf{v}_b + \nabla' \tilde{p}_{b,\nu}) = F_{b,\nu} \nu_b^2 \tilde{\mathbf{f}} & \text{in } Q_b, \\ \nabla' \cdot \mathbf{v}_b = 0 & \text{in } Q_b, \\ \mathbf{v}_b = 0 & \text{on } \Gamma_b, \\ \mathbf{v}_b = (\mathbf{v}_0)_b & \text{at } t = 0, \end{cases} \tag{3.39}$$

that highlights the role played by the number $F_{b,\nu}$ introduced by formula (3.36).

3.3.2.3 Formal Limits

When $\nu_b \to 0$ (or $Re(b) \to \infty$), then the dimensionless NSE (3.30) converges formally to the Euler equations

$$\begin{cases} \partial_{t'} \mathbf{v}_b + (\mathbf{v}_b \cdot \nabla')\, \mathbf{v}_b + \nabla' p_b = \mathbf{f}_b & \text{in } Q_b, \\ \nabla' \cdot \mathbf{v}_b = 0 & \text{in } Q_b, \\ \mathbf{v}_b \cdot \mathbf{n} = 0 & \text{on } \Gamma_b, \\ \mathbf{v}_b = (\mathbf{v}_0)_b & \text{at } t = 0. \end{cases} \tag{3.40}$$

The analytical study of this limit addresses a difficult issue, which has been extensively studied. To summarize, in the whole space and with supplementary hypotheses on the solution of the NSE and Euler equations, such as uniqueness and regularity, then the solution of the NSE (3.30) converges in some sense to the solution of the Euler equations (3.40) when Re_c goes to ∞ (see the review paper Bardos and Titi [4]).

The situation becomes more complicated when Ω has a boundary, giving rise to boundary layers. Only partial results are known for linear cases by taking the limit when $\nu_b \to 0$. These results exhibit additional terms in the Euler equation in the limit, called correctors (see [20] and additional references therein).

When

$$\nu_b \to \infty, \quad F_{b,\nu} \to 0, \quad F_{b,\nu} \nu_b \to \alpha \in \mathbb{R},$$

then the dimensionless NSE (3.39) formally converges to the Stokes equations

$$\begin{cases} -\Delta' \mathbf{v}_b + \nabla' \tilde{p}_{b,\nu} = \alpha\, \tilde{\mathbf{f}} & \text{in } Q_b, \\ \nabla' \cdot \mathbf{v}_b = 0 & \text{in } Q_b, \\ \mathbf{v}_b = 0 & \text{on } \Gamma_b. \end{cases} \tag{3.41}$$

We will not give much attention to a rigorous proof of such convergence. However, it is well known since the works of Stokes [43] that the Stokes equations describe well "slow flows," which are not turbulent and therefore not in the scope of this book.

3.3.2.4 Similar Flows

The pressure in the NSE is defined up to a constant. Therefore, it naturally belongs to quotient spaces (see Sect. 3.4.1 below). We denote by $\tilde{p}$ the class of any p in a suitable quotient space, which does not need to be specified at this stage, since this section deals with formalism. For $i = 1, 2$, let us consider:

(i) $Q^{(i)} = [0, T^{(i)}] \times \Omega^{(i)}$ two cylinders,
(ii) $\nu^{(i)}$ two kinematic viscosities,
(iii) $\mathbf{f}^{(i)}$ two forces per mass unit,
(iv) $\mathbf{v}_0^{(i)} = \mathbf{v}_0^{(i)}(\mathbf{x})$ two velocity fields defined in $\Omega^{(i)}$ ($i = 1, 2$).

Definition 3.6. Let $(\mathbf{v}^{(i)}, \tilde{p}^{(i)})$ be two flows in $Q^{(i)}$, $i = 1, 2$. We say that these two flows are similar if there exist two length–time bases b_1 and b_2, such that

$$Q_{b_1}^{(1)} = Q_{b_2}^{(2)}, \quad (\mathbf{v}_{b_1}^{(1)}, \tilde{p}_{b_1}^{(1)}) = (\mathbf{v}_{b_2}^{(2)}, \ \tilde{p}_{b_2}^{(2)}). \tag{3.42}$$

Let us consider the NSE, for $i = 1, 2$,

$$\begin{cases} \partial_t \mathbf{v}^{(i)} + (\mathbf{v}^{(i)} \cdot \nabla)\, \mathbf{v}^{(i)} - \nu^{(i)} \Delta \mathbf{v}^{(i)} + \nabla p^{(i)} = \mathbf{f}^{(i)} & \text{in } Q^{(i)}, \\ \nabla \cdot \mathbf{v}^{(i)} = 0 & \text{in } Q^{(i)}, \\ \mathbf{v}^{(i)} = 0 & \text{on } \Gamma^{(i)}, \\ \mathbf{v}^{(i)} = \mathbf{v}_0^{(i)} & \text{at } t = 0. \end{cases} \tag{3.43}$$

We assume that the data are such that each of these two NSE have a sufficiently smooth solution $(\mathbf{v}^{(i)}, \tilde{p}^{(i)})$. The similarity hypothesis is stated as follows.

Hypothesis 3.i. If there exist two length–time bases b_1 and b_2 such that

$$Q_{b_1}^{(1)} = Q_{b_2}^{(2)}, \quad (\mathbf{v}_0^{(1)})_{b_1} = (\mathbf{v}_0^{(1)})_{b_2}, \quad \mathbf{f}_{b_1}^{(1)} = \mathbf{f}_{b_2}^{(2)}, \quad \nu_{b_1} = \nu_{b_2},$$

then the two flows $(\mathbf{v}^{(i)}, \tilde{p}^{(i)})$ are similar.

The similarity hypothesis is satisfied if and only if the dimensionless form (3.30) of the NSE has a unique solution. We are left with the question of what are the right conditions imposed on the data to guarantee existence and uniqueness and to fix the correct mathematical framework. This will be discussed in Sect. 3.4 below. Before doing this, we conclude with a discussion of the concept of the Reynolds numbers used by engineers.

3.3.3 The Reynolds Number

Let us consider a flow of viscosity ν in a cylinder Q. In engineering, *the* Reynolds number is defined as the ratio of inertial forces to viscous forces and consequently quantifies the relative importance of these two types of forces for given flow conditions. Roughly speaking, at a given point $(t, \mathbf{x})$, Re is defined by

$$Re = Re(t, \mathbf{x}) = \frac{|(\mathbf{v}(t, \mathbf{x}) \cdot \nabla)\mathbf{v}(t, \mathbf{x})|}{\nu |\Delta \mathbf{v}(t, \mathbf{x})|}, \tag{3.44}$$

assuming $\Delta \mathbf{v}(t, \mathbf{x}) \neq 0$.

Lemma 3.3. *Let $(t, \mathbf{x}) \in Q = [0, T] \times \Omega$, Re be defined by formula (3.44). Then there exists infinitely many length–time bases $b = (\lambda, \tau)$ such that $Re = Re(b)$.*

Proof. Let $b = (\lambda, \tau)$ be any length–time basis. We deduce from formulas (3.31)–(3.33)

$$Re = Re(b) \frac{|(\mathbf{v}_b \cdot \nabla')\mathbf{v}_b|}{|\Delta' \mathbf{v}_b|}, \tag{3.45}$$

at $(t', \mathbf{x}') = (\tau^{-1} t, \lambda^{-1} \mathbf{x})$. Thus, $Re = Re(b)$ if and only if

$$\frac{|(\mathbf{v}_b \cdot \nabla')\mathbf{v}_b|}{|\Delta' \mathbf{v}_b|} = \frac{Re}{Re(b)} = 1. \tag{3.46}$$

The expression for $R(b)$ stated in (3.28), combined with the expression of V (3.26) and (3.46), yields the equation

$$\nu \lambda^{-2} \tau Re = 1, \tag{3.47}$$

which has infinitely many solutions $b = (\lambda, \tau)$ for a given Re. $\square$

In practice, Re is considered as a global constant quantity calculated from a length L defined by the flow geometry and a mean large-scale velocity U, which is deduced from experiments. In this case

$$Re = Re(b) = \frac{UL}{\nu}, \quad b = (L, LU^{-1}). \tag{3.48}$$

The best illustration of this definition is the case of a flow around a sphere of radius L moving at a constant velocity $\mathbf{U}$ in a fluid whose kinematic viscosity is equal to ν, by taking $U = |\mathbf{U}|$.

Further examples of such global Reynolds numbers can be found in [3, 6, 26, 31, 33], for cases of a flow in a pipe, a flow around an aircraft wing, a jet flow, a flow between two plates, the flow behind a grid, oceanic flows, and so on.

Experiments indicate that when Re is low, the flow is smooth and regular; we say that it is laminar. As Re increases, waves and instabilities appear. When Re becomes large enough, there are regions where the flow is chaotic, revealing a wide range of eddies on many different scales; we say that the flow is turbulent in those regions.

The concept of a dimensionless number defined as the ratio of inertial forces to viscous forces was first suggested by Stokes [43] and popularized by Reynolds [34].

3.4 Solutions to the NSE

We aim to state various existence results for the NSE with the no-slip BC, to give a rigorous framework for the Reynolds similarity Hypothesis 3.i, and to make a connection with the mathematical analysis.

As already mentioned, two types of results will be presented: the existence of a global time weak solution *à la Leray* and the existence of a local time solution *à la Fujita–Kato*. We shall study the existence of global weak solutions of turbulence models in future chapters, which is why this notion is more discussed in what follows, although the Reynolds similarity seems to make sense for strong solutions, because of uniqueness.

Throughout this section, we work with the dimensionless form of the NSE. However, we shall refer to the NSE (3.29) rather than (3.30) for simplicity, which means that we write $(\mathbf{v}, p)$ instead of $(\mathbf{v}_b, p_b)$, and ν stands for ν_b.

3.4.1 Functional Spaces

We begin with reminders of the standard functional analysis framework. We assume that Γ is of class C^1 for simplicity.[2] For given $q, p, s \ldots$, we set

$$\mathbf{L}^q(\Omega) = \{\mathbf{w} = (w_1, w_2, w_3);\ w_i \in L^q(\Omega), i = 1, 2, 3\}, \tag{3.49}$$

$$\mathbf{W}^{s,p}(\Omega) = \{\mathbf{w} = (w_1, w_2, w_3);\ w_i \in W^{s,p}(\Omega), i = 1, 2, 3\}. \tag{3.50}$$

We denote by $||\cdot||_{q,p,\Omega}$ the standard $\mathbf{W}^{s,p}(\Omega)$ norm (see [1] and Sect. A.1 in [TB][3]). For any $s > 1/2$, we consider the spaces

$$\mathbf{H}^s(\Omega) = \{\mathbf{w} = (w_1, w_2, w_3);\ w_i \in H^s(\Omega), i = 1, 2, 3\}, \tag{3.51}$$

$$\mathbf{H}^s_0(\Omega) = \{\mathbf{w} \in \mathbf{H}^s(\Omega);\ \gamma_0\mathbf{w} = 0 \text{ on } \Gamma\}. \tag{3.52}$$

[2] Many results explained below also hold for Lipchitz domains, see for instance Tartar [45].

[3] [TB], tool box, Appendix A.

In the definition above, γ_0 is the trace operator, which is defined by

$$\forall\, \varphi \in C^\infty(\overline{\Omega}), \quad \gamma_0 \varphi = \varphi|_\Gamma,$$

that can be extended to $H^s(\Omega)$, when $s > 1/2$, in a continuous operator with values in the space $H^{s-1/2}(\Gamma)$ (see Theorem A.2 in [TB]). When no risk of confusion occurs, we also denote $\gamma_0 \mathbf{w} = \mathbf{w}$. The space $\mathbf{H}_0^1(\Omega)$ is equipped with its standard norm

$$||\mathbf{w}||_{H_0^1(\Omega)} = ||\nabla \mathbf{w}||_{0,2,\Omega},$$

which is a norm equivalent to the $||\cdot||_{1,2,\Omega}$ norm, due to the Poincaré's inequality (see in [8]).

In this chapter we shall make use of the following spaces,

$$\mathscr{V}_{div}(\Omega) = \{\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3), \varphi_i \in \mathscr{D}(\Omega),\ i = 1, 2, 3,\ \nabla \cdot \boldsymbol{\varphi} = 0\}, \tag{3.53}$$

$$\mathbf{V}_{div}(\Omega) = \{\mathbf{w} \in \mathbf{H}_0^1(\Omega),\ \nabla \cdot \mathbf{w} = 0\}, \tag{3.54}$$

$$\mathbf{L}^2_{div,0}(\Omega) = \{\mathbf{w} \in \mathbf{L}^2(\Omega),\ \gamma_n \mathbf{w} = 0 \ \text{ on } \ \Gamma,\ \nabla \cdot \mathbf{w} = 0\}. \tag{3.55}$$

In the definition above, γ_n is the normal trace operator, which is defined by

$$\forall\, \boldsymbol{\varphi} \in C^\infty(\overline{\Omega})^3, \quad \gamma_n \boldsymbol{\varphi} = \boldsymbol{\varphi} \cdot \mathbf{n}|_\Gamma,$$

the vector $\mathbf{n}$ being the outward-pointing unit normal vector to Γ. We know that this operator can be extended to $\mathbf{L}^2_{div}(\Omega)$, in a continuous operator with values in the space $H^{-1/2}(\Gamma)$ (see in [21]), where

$$\mathbf{L}^2_{div}(\Omega) = \{\mathbf{w} \in \mathbf{L}^2(\Omega);\ \nabla \cdot \mathbf{w} \in L^2(\Omega)\}.$$

Modern analysis of the NSE with the no-slip BC[4] by compactness methods is based on the Aubin–Lions lemma, Lemma A.9 in [TB], the application of which is detailed later in Chap. 8, when we will need it for investigating problems with wall laws, introduced in Sect. 5.4.

We simply explain below why the conditions for the application of the Aubin–Lions lemma in the no-slip BC case are fulfilled.

Lemma 3.4. *The following injections hold:*

$$\mathscr{V}_{div}(\Omega) \hookrightarrow \mathbf{V}_{div}(\Omega) \hookrightarrow \mathbf{L}^2_{div,0}(\Omega) \hookrightarrow \mathbf{V}_{div}(\Omega)', \tag{3.56}$$

each space being dense in each other and the injection of $\mathbf{V}_{div}(\Omega)$ *onto* $\mathbf{L}^2_{div,0}(\Omega)$ *being compact.*

[4] BC, boundary conditions.

This result is a consequence of the following statement, often referred to as de Rham theorem, a complete proof of which can be constructed by following the method introduced by Tartar [44, 45] combined with the inequalities in the Sobolev spaces obtained by Nečas [30] (see also details in Girault and Raviart [21]).

Theorem 3.1. *Let* $\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$ *such that*

$$\forall\, \boldsymbol{\varphi} \in \mathscr{V}_{div}(\Omega), \quad \langle \mathbf{F}, \boldsymbol{\varphi} \rangle = 0. \tag{3.57}$$

Then there exists a unique $\tilde{p} \in L^2_0(\Omega)$ *such that*

$$\forall\, p \in \tilde{p}, \quad \mathbf{F} = \nabla p. \tag{3.58}$$

Moreover there exists a constant C*, which only depends on* Ω *and such that*

$$\forall\, \tilde{p} \in L^2_0(\Omega),\ \forall\, p \in \tilde{p}, \quad ||\,\tilde{p}\,||_{L^2_0(\Omega)} \le C||\nabla p||_{-1,2,\Omega}. \tag{3.59}$$

In the statement above, $L^2_0(\Omega)$ is the quotient space of $L^2(\Omega)$ by the constants, equipped with the norm

$$||\,\tilde{p}\,||_{L^2_0(\Omega)} = \inf\{||p||_{0,2,\Omega},\ p \in \tilde{p}\}, \tag{3.60}$$

which is isomorphic to the space

$$\{p \in L^2(\Omega);\ \int_\Omega p(\mathbf{x})\, d\mathbf{x} = 0\}, \tag{3.61}$$

also denoted by $L^2_0(\Omega)$ in the following.

Remark 3.2. Inequality (3.59) can be rephrased as:

$$\inf\{||p||_{0,2,\Omega},\ p \in \tilde{p}\,\} \le C \sup_{\mathbf{w} \in \mathbf{H}^1_0(\Omega)} \frac{(p, \nabla \cdot \mathbf{w})_\Omega}{||\mathbf{w}||_{1,2,\Omega}}. \tag{3.62}$$

This is called an "inf-sup condition" (see also Sect. 9.3.3).

3.4.2 Turbulent Solutions

The first known existence result about the NSE was established by Leray [25], in the case $\Omega = \mathbb{R}^3$, $\mathbf{f} = 0$. In his pioneering paper, J. Leray laid the foundations of modern functional analysis. Taking inspiration from this work, completed later by Hopf [22], we state in this subsection conditions for the existence of a turbulent (weak) solution to the NSE with the no-slip BC.

As this result is well known in the field, we shall not give a detailed proof, which can be found for instance in [12, 17, 19, 24, 28, 29, 45, 46]. However, the techniques that we shall develop in Chap. 8 allow the reader to reconstruct a complete proof of Theorem 3.2 below.

In this subsection, we shall use the Bochner space $L^p([0,T],E)$ and the space of weakly continuous functions valued in E, $C_w([0,T],E)$, E being a given Banach space, $1 \le p \le \infty$. These spaces are described in detail in Sect. A.4.5 in [TB].

3.4.2.1 Turbulent Solution: Definition

The present variational formulation of the NSE is based on projections over spaces of zero divergence fields, such as $\mathbf{V}_{div}(\Omega)$ and $\mathbf{L}^2_{div,0}(\Omega)$. The pressure p, which is not involved in the formulation, is recovered by the de Rham theorem 3.1. In the following, $T > 0$ is a standard time.

Hypothesis 3.i. The standard working hypothesis is $\mathbf{f} \in L^2([0,T],\mathbf{H}^{-1}(\Omega))$ and $\mathbf{v}_0 \in \mathbf{L}^2_{div,0}(\Omega)$.

Definition 3.7. Assume that Hypothesis 3.i holds. We say that $\mathbf{v}$ is a turbulent solution of the NSE (3.29) over $[0,T]$, if

$$\begin{cases} \mathbf{v} \in L^2([0,T],\mathbf{V}_{div}(\Omega)) \cap C_w([0,T],\mathbf{L}^2_{div,0}(\Omega)), \\ \partial_t \mathbf{v} \in L^{4/3}([0,T],\mathbf{V}_{div}(\Omega)'), \end{cases} \tag{3.63}$$

and for all $\mathbf{w} \in L^4([0,T],\mathbf{V}_{div}(\Omega))$,

$$\begin{aligned} \int_0^T \langle \partial_t \mathbf{v}, \mathbf{w} \rangle dt + \int_0^T \int_\Omega ((\mathbf{v}\cdot\nabla)\,\mathbf{v})(t,\mathbf{x}) \cdot \mathbf{w}(t,\mathbf{x})\, d\mathbf{x} dt \\ + \nu \int_0^T \int_\Omega \nabla \mathbf{v}(t,\mathbf{x}) : \nabla \mathbf{w}(t,\mathbf{x})\, d\mathbf{x} dt = \int_0^T \langle \mathbf{f}, \mathbf{w} \rangle dt, \end{aligned} \tag{3.64}$$

and

$$\lim_{t\to 0} ||\mathbf{v}(t,\cdot) - \mathbf{v}_0(\cdot)||_{0,2,\Omega} = 0. \tag{3.65}$$

The reason why we chose $\partial_t \mathbf{v} \in L^{4/3}([0,T],\mathbf{V}_{div}(\Omega)')$ will be clarified below. The condition (3.65) on the initial data is the strong version. We may also use the following lighter one:

$$\text{for all } \boldsymbol{\varphi} \in C^1([0,T],\mathbf{V}_{div}(\Omega)) \text{ such that } \boldsymbol{\varphi}(T,\cdot) = 0,$$

$$\int_0^T \langle \partial_t \mathbf{v}, \boldsymbol{\varphi} \rangle = -\int_\Omega \mathbf{v}_0(\mathbf{x}) \boldsymbol{\varphi}(0,\cdot)\, d\mathbf{x} - \int_0^T \int_\Omega \mathbf{v}(t,\mathbf{x}) \boldsymbol{\varphi}(t,\cdot)\, d\mathbf{x} dt, \tag{3.66}$$

which is usually easier to check (see in Sect. A.4.5 in [TB]).

3.4.2.2 Alternative Formulation

Another equivalent way of defining turbulent solutions proceeds as follows (see in Lions [28]). For simplicity, we denote by (u, v) the duality pairing $\langle L^{p'}(\Omega), L^p(\Omega)\rangle$,

$$(u, v)_\Omega = \int_\Omega u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x}.$$

and we formally define the diffusion and transport operators by

$$a(\mathbf{v}, \mathbf{w}) = \nu(\nabla \mathbf{v}, \nabla \mathbf{w})_\Omega, \quad b(\mathbf{z}; \mathbf{v}, \mathbf{w}) = ((\mathbf{z} \cdot \nabla)\, \mathbf{v}, \mathbf{w})_\Omega. \tag{3.67}$$

The detailed study of these operators is postponed to Sect. 6.2, Chap. 7.

As a result of Lemma A.11 in [TB], any turbulent solution $\mathbf{v}$ to the NSE, such as in Definition 3.7, also satisfies $\forall\, \mathbf{w} \in \mathbf{V}_{div}(\Omega)$,

$$\frac{d}{dt}(\mathbf{v}, \mathbf{w})_\Omega + b(\mathbf{z}; \mathbf{v}, \mathbf{w}) + a(\mathbf{v}, \mathbf{w}) = \langle \mathbf{f}, \mathbf{w} \rangle \quad \text{in } \mathscr{D}'([0, T]), \tag{3.68}$$

which is an equivalent formulation to that expressed by (3.64).

Remark 3.3. The terminology "turbulent solution" was used by Leray [25]. People today generally call them "weak solutions." It is easily checked that any weak (or turbulent) solution to the NSE is a distributional solution to the PDE system (3.29).

Remark 3.4. We shall consider throughout the following chapters alternative variational forms of NSE-like equations and by-products, with other BC than the no-slip BC, such as wall laws (see, e.g., in Chap. 6). In these variational formulations, the pressure is strongly involved, and we shall refer to these as mixed formulations.

3.4.2.3 Existence Results

The existence result, based on the works by Leray [25] and Hopf [22], is the following.

Theorem 3.2. *When Hypothesis 3.i holds, the NSE (3.29) have a turbulent solution* $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ *that satisfies the energy inequality at every* $t \in [0, T]$*:*

$$\frac{d}{2dt} ||\mathbf{v}(t, \cdot)||^2_{0,2,\Omega} + \nu ||\nabla \mathbf{v}(t, \cdot)||^2_{0,2,\Omega} \leq \langle \mathbf{f}, \mathbf{v} \rangle \quad \textit{in } \mathscr{D}'([0, T]). \tag{3.69}$$

The uniqueness of this solution is still an open problem at the time of writing. Similarly, we do not know if the energy inequality (3.69) is an equality. The energy inequality (3.69) also yields

$$\frac{1}{2}||\mathbf{v}(t,\cdot)||^2_{0,2,\Omega} + \nu \int_0^t ||\nabla \mathbf{v}||^2_{0,2,\Omega} \le \frac{1}{2}||\mathbf{v}_0||^2_{0,2,\Omega} + \int_0^t \langle \mathbf{f}, \mathbf{v} \rangle, \tag{3.70}$$

for all $t > 0$. The pressure is recovered from the de Rham theorem, Theorem 3.1, leading to the following statement (see for instance in [17, 28, 45, 46]):

Lemma 3.5. *There exists $\tilde{p} \in \mathscr{D}'([0,T], L^2_0(\Omega))$, such that for any $p \in \tilde{p}$, $(\mathbf{v}, p)$ is a solution of the NSE (3.29) in the sense of distributions.*

The pressure $p \in \tilde{p}$ is considered as a constraint in this kind of formulation. Therefore, p is called a *Lagrange multiplier*. It also can be proved that $p \in L^{5/4}(Q)$ (see for instance in Caffarelli-Kohn-Nirenberg [9]).

3.4.2.4 Estimates and Consequences

The energy inequality (3.70) combined with the inequality (A.37) of Lemma A.18 in [TB] shows that for any $t > 0$,

$$\mathbf{v} \in L^{10/3}(Q_t) \cap L^4([0,t], \mathbf{L}^3(\Omega)) \cap L^{8/3}([0,t], \mathbf{L}^4(\Omega)), \tag{3.71}$$

where $Q_t = [0,t] \times \Omega$. In particular, we have

$$||\mathbf{v}||_{L^{8/3}([0,t],\mathbf{L}^4(\Omega))} \le C, \quad ||\mathbf{v}||_{L^4([0,t],\mathbf{L}^3(\Omega))} \le C', \tag{3.72}$$

where C and C' depend on $||\mathbf{v}_0||_{0,2,\Omega}$, ν and $||\mathbf{f}||_{L^2([0,T],\mathbf{V}_{div}(\Omega)'))}$ (and not on Ω). Estimates (3.72) yield $\mathbf{v} \otimes \mathbf{v} \in L^{4/3}([0,t], L^2(\Omega)^9)$ and

$$||\mathbf{v} \otimes \mathbf{v}||_{L^{4/3}([0,t],L^2(\Omega)^9)} \le C^2. \tag{3.73}$$

Estimate (3.73) enables us to determine the choice $4/3$ as the exponent in Definition 3.7 above. To better understand this, assume that we could take the scalar product of the NSE (3.29) with $\mathbf{v}$ and integrate by parts. We would find that $\mathbf{v}$ satisfies (3.70), where the inequality is actually an equality. We shall refer later on to an *energy equality*. The vector $\mathbf{v}$ would satisfy

$$\mathbf{v} \in L^2([0,T], \mathbf{V}_{div}(\Omega)) \cap L^\infty([0,T], \mathbf{L}^2(\Omega)),$$

hence (3.72) by the inequality (A.37), then (3.73), leading to

$$\nabla \cdot (\mathbf{v} \otimes \mathbf{v}) = (\mathbf{v} \cdot \nabla)\, \mathbf{v} \in L^{4/3}([0,T], \mathbf{H}^{-1}(\Omega)). \tag{3.74}$$

Moreover, as $\mathbf{v} \in L^2([0,T], \mathbf{V}_{div}(\Omega))$, then $-\Delta \mathbf{v} \in L^2([0,T], \mathbf{H}^{-1}(\Omega))$, and because $\mathbf{f} \in L^2([0,T], \mathbf{H}^{-1}(\Omega))$, we find by (3.74),

$$\partial_t \mathbf{v} + \nabla p = \mathbf{f} + \nu \Delta \mathbf{v} - (\mathbf{v} \cdot \nabla)\, \mathbf{v} \in L^{4/3}([0,T], \mathbf{H}^{-1}(\Omega)).$$

We observe that $L^{4/3}([0,T],\mathbf{H}^{-1}(\Omega)) \subset L^{4/3}([0,T],\mathbf{V}_{div}(\Omega)')$, which combined with $\langle \nabla p, \mathbf{w}\rangle = 0$ for all $\mathbf{w} \in \mathbf{V}_{div}(\Omega)$, provided that this duality product makes sense, yields

$$\partial_t \mathbf{v} \in L^{4/3}([0,T],\mathbf{V}_{div}(\Omega)'),$$

which is what we call an a priori estimate. This estimate is currently the best we can get, which forbids taking $\mathbf{v}$ as test in the formulation (3.64), even though this information derives from taking $\mathbf{v}$ as test in the NSE, which is the standard paradox while investigating the NSE. As a consequence, it is not possible to prove that:

(i) the turbulent solution is unique and
(ii) it satisfies the energy equality.

In particular, we cannot conclude that any turbulent (weak) solutions satisfy the Reynolds similarity Hypothesis 3.i introduced in Sect. 3.3.2.

3.4.2.5 Further Regularity

We might ask if the regularity of any turbulent solution increases for more regular data.

Thanks to Helmholtz–Hodge decomposition (see in [21]), we can reduce the problem to the case where $\mathbf{f}$ satisfies $\nabla \cdot \mathbf{f} = 0$. We assume from now that this condition holds, without loss of generality. Therefore, applying regularity results proved by Solonnikov [42] on the evolutionary Stokes equations combined with estimate (3.89), we get

Lemma 3.6. *Assume that*

$$\mathbf{v}_0 \in \mathbf{L}^2_{div,0}(\Omega) \cap \mathbf{W}^{2/5,5/4}(\Omega) \ \textit{and} \ \mathbf{f} \in \mathbf{L}^{5/4}(Q), \tag{3.75}$$

then the turbulent solution $\mathbf{v}$ *verifies in addition*

$$\partial_t \mathbf{v}, \Delta \mathbf{v}, \nabla p \in \mathbf{L}^{5/4}(Q). \tag{3.76}$$

In particular, we deduce from (3.76) that

$$\mathbf{v} \in C([0,T],\mathbf{L}^{5/4}(\Omega)) \ \text{as well as} \ \mathbf{v} \in L^{5/4}([0,T],\mathbf{W}^{2,5/4}(\Omega)). \tag{3.77}$$

This machinery is also well depicted in [9]. Furthermore, following Cattabriga [10], we also conclude

$$\tilde{p} \in L^{5/4}([0,T],W^{1,5/4}(\Omega)/\mathbb{R}), \tag{3.78}$$

However, this regularity enhancement is not sufficient to solve the problem raised above: uniqueness and energy equality.

3.4.3 Global Time Estimate

We consider in this subsection turbulent global time solutions, which are defined in time over all $\mathbb{R}_+$, which will be the starting point for building the long-time average model in Sect. 3.5. We assume throughout this section that

Hypothesis 3.ii. The source term $\mathbf{f} \in \mathbf{H}^{-1}(\Omega) \subset \mathbf{V}_{div}(\Omega)'$ does not depend on t, and we set $F = ||\mathbf{f}||_{-1,2,\Omega}$.

The real number μ denotes the best constant in the Poincaré inequality, written as

$$\forall\, \mathbf{v} \in H_0^1(\Omega) \quad C||\mathbf{v}||_{0,2,\Omega} \leq ||\nabla \mathbf{v}||_{0,2,\Omega}.$$

Energy inequality (3.70) yields $||\mathbf{v}(t,\cdot)||_{0,2,\Omega}$ that is bounded uniformly in t. To be more specific, we prove the following.

Proposition 3.1. *Let* $\mathbf{v}$ *be any turbulent solution to the NSE. Then we have*

$$||\mathbf{v}(t,\cdot)||_{0,2,\Omega}^2 \leq ||\mathbf{v}_0||_{0,2,\Omega}^2\, e^{-\nu\mu t} + \frac{F^2}{\nu^2\mu}(1 - e^{-\nu\mu t}), \tag{3.79}$$

for all $t > 0$.

Proof. Set:

$$W(t) = ||\mathbf{v}(t,\cdot)||_{0,2,\Omega}^2, \quad W(0) = ||\mathbf{v}_0||_{0,2,\Omega}^2. \tag{3.80}$$

Energy inequality (3.69) yields

$$\frac{1}{2}W'(t) + \nu\int_\Omega |\nabla \mathbf{v}|^2 \leq \langle \mathbf{f}, \mathbf{v}\rangle \leq \frac{F^2}{2\nu} + \frac{\nu}{2}\int_\Omega |\nabla \mathbf{v}|^2. \tag{3.81}$$

We apply Poincaré's inequality in the second term of the l.h.s. of (3.81), leading to

$$W'(t) + \nu\mu W(t) \leq \frac{F^2}{\nu}. \tag{3.82}$$

Therefore, W is below the solution of the ordinary differential equation

$$\begin{cases} \lambda'(t) + \nu\mu\lambda(t) = \dfrac{F^2}{\nu}, \\ \lambda(0) = W(0), \end{cases} \tag{3.83}$$

the solution of which is

$$\lambda(t) = W(0)e^{-\nu\mu t} + \frac{F^2}{\nu^2\mu}(1 - e^{-\nu\mu t}), \tag{3.84}$$

hence inequality (3.79). □

As a consequence, we deduce that the turbulent solution is well defined all over $\mathbb{R}_+$, hence can be extended to $L^\infty(\mathbb{R}_+, \mathbf{L}^2_{div}(\Omega))$ as a global time solution. In particular, we have

$$\sup_{t\geq 0} ||\mathbf{v}(t,\cdot)||^2_{0,2,\Omega} \leq \max_{t\geq 0} K(t) = E_\infty, \tag{3.85}$$

where

$$K(t) = ||\mathbf{v}_0||^2_{0,2,\Omega}\, e^{-\nu\mu t} + \frac{F^2}{\nu^2\mu}(1 - e^{-\nu\mu t}). \tag{3.86}$$

We also deduce from (3.81), combined with (3.85), the following inequality:

$$\forall\, t > 0, \quad \frac{1}{t}\int\int_{Q_t} |\nabla\mathbf{v}(s,\mathbf{x})|^2 d\mathbf{x} ds \leq \frac{F^2}{\nu^2} + \frac{||\mathbf{v}_0||^2_{0,2,\Omega}}{\nu t}. \tag{3.87}$$

Moreover, we infer from inequality (A.37) in [TB], $\forall\, t > 0$,

$$\mathbf{v} \in \mathbf{L}^{10/3}(Q_t), \quad ||\mathbf{v}||_{0,10/3,Q_t} \leq C_1 E_\infty^{1/5} ||\nabla\mathbf{v}||^{3/5}_{0,2,Q_t}, \tag{3.88}$$

leading to

$$(\mathbf{v}\cdot\nabla)\,\mathbf{v} \in L^{5/4}(Q_t), \quad ||(\mathbf{v}\cdot\nabla)\,\mathbf{v}||_{0,5/4,Q_t} \leq C_1 E_\infty^{1/5} ||\nabla\mathbf{v}||^{8/5}_{0,2,Q_t}, \tag{3.89}$$

which is the essential information giving a sense to the long-time average introduced in Sect. 3.5, which is the first turbulence model of this book.

Remark 3.5. We say that $(t_0, \mathbf{x}_0)$ is a singularity of the turbulent solution if

$$\lim_{(t,\mathbf{x})\to(t_0,\mathbf{x}_0)} |\mathbf{v}(t,\mathbf{x})| = \infty.$$

Even when $\mathbf{v}_0$ and $\mathbf{f}$ are smooth, it is not known if the turbulent solution has any singularity. This question was studied among others by Serrin [40, 41] and then for "suitable weak solutions" of the NSE, which are weak solutions satisfying a local energy inequality, see Scheffer [35–39], Caffarelli et al. [9], Lin [27], and Choe and Lewis [11].

3.4.4 Strong Solutions

We now state the existence and uniqueness of a strong local time solution to the NSE, according to Fujita and Kato [18], which will serve us in

(i) giving an example of a solution to the NSE that satisfies the Reynolds similarity hypothesis introduced in Sect. 3.3.2,
(ii) preparing the ground for the framework of statistical models we shall introduce in Sect. 4.2.

In this context, $\mathbf{V}_{div}(\Omega) \cap \mathbf{H}^{1/2}(\Omega)$ is the right minimal space for initial data $\mathbf{v}_0$ while $C^{0,\alpha}(\mathbb{R}_+, \mathbf{L}^2(\Omega))$ is the right minimal space for the source term $\mathbf{f}$. However, we shall make use of stronger spaces for the data, because we need very regular solutions in view of Sect. 4.2.

From now on and when no risk of confusion occurs, we denote the class $\tilde{p}$ simply by p. The results of [18] that we need are summarized in the following statement.

Theorem 3.3. *Assume that*

$$\mathbf{v}_0 \in \mathbf{V}_{div}(\Omega) \cap C^{2,\alpha}(\Omega)^3 \ \textit{ and } \ \mathbf{f} \in C^{0,\alpha}(\mathbb{R}_+ \times \Omega) \ \textit{for some} \ \alpha > 0. \tag{3.90}$$

Then there exists $\delta T > 0$ *such that the NSE (3.29) has a unique strong solution* $(\mathbf{v}, p)$ *defined over* $[0, \delta T]$*, which satisfies:*

(i) the fields $\mathbf{v}$*,* $\partial_t \mathbf{v}$*,* $\nabla \mathbf{v}$*,* $\Delta \mathbf{v}$*,* p*,* ∇p *are all Hölder continuous and (3.29) holds in the classical sense,*
(ii) the energy inequality (3.70) is an equality for all $t \in [0, \delta T]$*,*
(iii) the solution is continuous with respect to $\mathbf{v}_0$ *and* $\mathbf{f}$*,*
(iv) the upper bound δT *is governed by*

$$C(\mathbf{v}_0, \mathbf{f}, \nu) = ||\mathbf{v}_0||_{1/2,2,\Omega} + \nu^{-1} ||\mathbf{f}||_{L^\infty(\mathbb{R}_+, \mathbf{L}^2(\Omega))}, \tag{3.91}$$

and

$$\lim_{C(\mathbf{v}_0, \mathbf{f}, \nu) \to \infty} \delta T = 0, \tag{3.92}$$

(v) when $C(\mathbf{v}_0, \mathbf{f}, \nu)$ *is small enough, the strong solution is global in time, which means that one can take* $\delta T = \infty$*.*

The flow considered in item (v) corresponds to a laminar flow while the situation corresponding to formula (3.92) is that of turbulent flows.

In conclusion, the similarity Hypothesis 3.i holds at least for small times, in the case of strong solutions, because of uniqueness. This is in accordance with the modeling process followed in Chap. 2 that conceptually considers local time smooth fields $(\mathbf{v}, p)$ to construct the NSE, as given in Assumptions 2.2 and 2.4.

3.5 Long-Time Average Model

The framework of this section is that of global time turbulent solutions, as investigated in Sect. 3.4.3. We assume that Hypothesis 3.ii holds, which means that the flow is governed by a steady-state source term, and we denote by $\mathbf{v}$ a given weak solution to the NSE, with p the corresponding pressure. We aim in this section to study long-time averages of $\mathbf{v}$ and p, which will establish some principles of turbulence modeling in a rather clear and rigorous mathematical framework, connected with the theoretical results reported above.

Although the source term $\mathbf{f}$ is steady, there is no reason for $(\mathbf{v}, p)$ to become steady. However, we find that at large times t, $(\mathbf{v}, p)$ oscillates around a steady mean flow $(\overline{\mathbf{v}}, \overline{p})$. According to Stokes [43], Boussinesq [7], Reynolds [34], and Prandtl [32], it is worth considering the long-time average of the velocity and the pressure:

$$\overline{\mathbf{v}} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{v}(s,\cdot)\, ds, \quad \overline{p} = \lim_{t\to\infty} \frac{1}{t} \int_0^t p(s,\cdot)\, ds. \tag{3.93}$$

We show in this section that $(\overline{\mathbf{v}}, \overline{p}) \in \mathbf{W}^{2,5/4}(\Omega) \times \mathbf{W}^{1,5/4}(\Omega)/\mathbb{R}$ makes sense in one respect that we shall explain below and verifies the coupled system

$$\begin{cases} (\overline{\mathbf{v}}\cdot\nabla)\,\overline{\mathbf{v}} - \nu\Delta\overline{\mathbf{v}} + \nabla\overline{p} = -\nabla\cdot\boldsymbol{\sigma}^{(\mathrm{R})} + \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\overline{\mathbf{v}} = 0 & \text{in } \Omega, \\ \overline{\mathbf{v}} = 0 & \text{on } \Gamma, \end{cases} \tag{3.94}$$

in the sense of distributions. In system (3.94) above,

$$\boldsymbol{\sigma}^{(\mathrm{R})}(\mathbf{x}) = \overline{\mathbf{v}'(\cdot,\mathbf{x})\otimes\mathbf{v}'(\cdot,\mathbf{x})} = \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} \mathbf{v}'(s,\mathbf{x})\otimes\mathbf{v}'(s,\mathbf{x})\, ds \in L^{5/3}(\Omega)^9, \tag{3.95}$$

for some sequence $(t_n)_{n\in\mathbb{N}}$, that satisfies $t_n \to \infty$ as $n \to \infty$. We call the tensor $\boldsymbol{\sigma}^{(\mathrm{R})}$ a Reynolds stress, which is not solely determined by our analysis below, and $\mathbf{v}' = \mathbf{v} - \overline{\mathbf{v}}$ (which is time dependent), as explained in Sect. 3.5.2 below.

However, the price to pay is that we must assume some regularity for the domain, in order to fulfill the conditions for the application of regularity results for the Stokes problem proved in Amrouche and Girault [2], and more regularity for the source term. More specifically, throughout this section, the hypothesis is

Hypothesis 3.i. The domain Ω is of class $C^{9/4,1}$; $\mathbf{f} \in \mathbf{L}^{5/4}(\Omega) \cap \mathbf{H}^{-1}(\Omega)$ does not depend on t, $\mathbf{v}_0 \in \mathbf{L}^2_{div,0}(\Omega)$.

3.5.1 NSE Long-Time Average

As in Sect. 3.4, we consider dimensionless fields and equations. In this subsection, $\mathbf{v}$ is a given turbulent solution of the NSE, with p its associated pressure. We respect the conditions for the application of the Proposition 3.1.

3.5.1.1 Technical Result

We start with the study of the mean operator M_t over $[0, t]$, for a given fixed time $t > 0$, expressed by

$$M_t(\boldsymbol{\psi}) = \frac{1}{t}\int_0^t \boldsymbol{\psi}(s, \mathbf{x})\, ds, \tag{3.96}$$

$\boldsymbol{\psi} = \boldsymbol{\psi}(t, \mathbf{x})$ being any given field.

Lemma 3.7. *Let $t > 0$, $Q_t = [0, t] \times \Omega$. Assume $\boldsymbol{\psi} \in \mathbf{L}^p(Q_t)$. Then $M_t(\boldsymbol{\psi}) \in \mathbf{L}^p(\Omega)$ and one has*

$$||M_t(\boldsymbol{\psi})||_{0,p,\Omega} \leq \frac{1}{t^{1/p}}||\boldsymbol{\psi}||_{0,p,Q_t}. \tag{3.97}$$

Proof. By the Hölder inequality, we have

$$\left|\frac{1}{t}\int_0^t \boldsymbol{\psi}(s, \mathbf{x})\, ds\right| \leq \frac{1}{t}\int_0^t |\boldsymbol{\psi}(s, \mathbf{x})|^p\, ds. \tag{3.98}$$

Thus (3.97) follows by Fubini's Theorem. □

3.5.1.2 Setting

We study the effect of M_t on $(\mathbf{v}, p)$, in defining

$$\mathbf{V}_t(\mathbf{x}) = M_t(\mathbf{v})(\mathbf{x}), \quad P_t(\mathbf{x}) = M_t(p)(\mathbf{x}). \tag{3.99}$$

We deduce from the NSE that $(\mathbf{V}_t, P_t)$ is solution of the following Stokes problem, at least in the sense of distributions:

$$\begin{cases} -\nu\Delta\mathbf{V}_t + \nabla P_t = -M_t((\mathbf{v}\cdot\nabla)\,\mathbf{v}) + \mathbf{f} + \boldsymbol{\varepsilon}_t & \text{in } Q, \\ \nabla\cdot\mathbf{V}_t = 0 & \text{in } Q, \\ \mathbf{V}_t = 0 & \text{on } \Gamma. \end{cases} \tag{3.100}$$

In system (3.100),

$$\boldsymbol{\varepsilon}_t(\mathbf{x}) = \frac{\mathbf{v}_0(\mathbf{x}) - \mathbf{v}(t,\mathbf{x})}{t}, \tag{3.101}$$

which goes to zero in $\mathbf{L}^2(\Omega)$ when $t \to +\infty$, according to (3.85).

3.5.1.3 Main Result

We aim to take the limit in system (3.100) as $t \to +\infty$. As a result we prove the following.

Theorem 3.4. *When Hypothesis 3.i holds, there exists*

(i) *a sequence* $(t_n)_{n\in\mathbb{N}}$ *that goes to* $+\infty$ *when* $n \to +\infty$,
(ii) $(\overline{\mathbf{v}}, \overline{p}) \in \mathbf{W}^{2,5/4}(\Omega) \times W^{1,5/4}(\Omega)/\mathbb{R}$,
(iii) $\mathbf{F} \in \mathbf{L}^{5/4}(\Omega)$,

such that $(\mathbf{V}_{t_n}, P_{t_n})_{n\in\mathbb{N}}$ *converges to* $(\overline{\mathbf{v}}, \overline{p})$, *weakly in* $\mathbf{W}^{2,5/4}(\Omega) \times W^{1,5/4}(\Omega)/\mathbb{R}$, *that satisfies.*

$$\begin{cases} (\overline{\mathbf{v}}\cdot\nabla)\,\overline{\mathbf{v}} - \nu\Delta\overline{\mathbf{v}} + \nabla\overline{p} = -\mathbf{F} + \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\overline{\mathbf{v}} = 0 & \text{in } \Omega, \\ \overline{\mathbf{v}} = 0 & \text{on } \Gamma, \end{cases} \tag{3.102}$$

in the sense of distributions.

Proof. The proof is divided in three steps. We first find estimates and extract convergent subsequences. We then take the limit in the equations, firstly in the conservation equation and then in the momentum equation.

STEP 1. We first show that the nonlinear term $-M_t((\mathbf{v}\cdot\nabla)\,\mathbf{v})$ is bounded in $\mathbf{L}^{5/4}(\Omega)$. By inequality (3.97) we have

$$||M_t((\mathbf{v}\cdot\nabla)\,\mathbf{v})||_{0,5/4,\Omega} \le \frac{1}{t^{4/5}}||(\mathbf{v}\cdot\nabla)\,\mathbf{v}||_{0,5/4,Q_t}, \tag{3.103}$$

where $Q_t = [0,t]\times\Omega$. Combining this inequality with (3.89) and (3.85), we find

$$||M_t((\mathbf{v}\cdot\nabla)\,\mathbf{v})||^{5/4}_{0,5/4,\Omega} \le C_1^{5/4}E_\infty^{1/4}\left(\frac{1}{t}\int_0^t\int_\Omega |\nabla\mathbf{v}(s,\mathbf{x})|^2 d\mathbf{x}ds\right), \tag{3.104}$$

hence $(M_t((\mathbf{v}\cdot\nabla)\,\mathbf{v}))_{t>0}$ is bounded in $\mathbf{L}^{5/4}(\Omega)$, uniformly in t due to (3.87). Since Ω is of class $C^{1+5/4,1} = C^{9/4,1}$, $\mathbf{f}\in\mathbf{L}^{5/4}(\Omega)$ and

$$(M_t((\mathbf{v}\cdot\nabla)\,\mathbf{v}))_{t>0} \text{ and } (\boldsymbol{\varepsilon}_t)_{t>0} \text{ are bounded in } \mathbf{L}^{5/4}(\Omega), \tag{3.105}$$

the results in [2] apply: there exists a unique solution (V_t, P_t) to system (3.100) that satisfies

$$||\mathbf{V}_t||_{2,5/4,\Omega} + ||P_t||_{W^{1,5/4}(\Omega)/\mathbb{R}} \leq \\ ||M_t((\mathbf{v}\cdot\nabla)\,\mathbf{v})||_{0,5/4,\Omega} + ||\mathbf{f}||_{0,5/4,\Omega} + ||\boldsymbol{\varepsilon}_t||_{0,5/4,\Omega}. \tag{3.106}$$

Because of uniqueness, this solution (V_t, P_t) is indeed that defined by (3.99). Statement (3.105) combined with estimate (3.106) ensures that

$$\begin{cases} (\mathbf{V}_t)_{t>0} & \text{is bounded in } \mathbf{W}^{2,5/4}(\Omega), \\ (P_t)_{t>0} & \text{is bounded in } W^{1,5/4}(\Omega)/\mathbb{R}. \end{cases} \tag{3.107}$$

Therefore, there exist

$$\overline{\mathbf{v}} \in \mathbf{W}^{2,5/4}(\Omega), \quad \overline{p} \in W^{1,5/4}(\Omega)/\mathbb{R}, \quad \mathbf{B} \in \mathbf{L}^{5/4}(\Omega),$$

a sequence $(t_n)_{n\in\mathbb{N}}$ which goes to ∞ as $n \to \infty$, such that

$$\lim_{n\to\infty} \mathbf{V}_{t_n} = \overline{\mathbf{v}} \text{ weakly in } \mathbf{W}^{2,5/4}(\Omega), \tag{3.108}$$

$$\lim_{n\to\infty} P_{t_n} = \overline{p} \text{ weakly in } W^{1,5/4}(\Omega)/\mathbb{R}, \tag{3.109}$$

$$\lim_{n\to\infty} M_{t_n}((\mathbf{v}\cdot\nabla)\,\mathbf{v}) = \mathbf{B} \text{ weakly in } L^{5/4}(\Omega)^9. \tag{3.110}$$

Moreover, $W^{2,5/4}(\Omega) \hookrightarrow W^{1,15/7}(\Omega)$, the injection being compact. Then

$$(\mathbf{V}_{t_n})_{n\in\mathbb{N}} \text{ converges to } \overline{\mathbf{v}} \text{ strongly in } \mathbf{W}^{1,15/7}(\Omega). \tag{3.111}$$

STEP 2. We check that $\nabla\cdot\overline{\mathbf{v}} = 0$ in an appropriate Lebesgue space. To do so, we first prove that $\nabla\cdot V_t = 0$ in $\mathscr{D}'(Q_T)$ regardless of $T > 0$. For any given $\varphi \in \mathscr{D}(Q_T)$, we have

$$\begin{aligned} \langle\nabla\cdot\mathbf{V}_t, \varphi\rangle &= \int\int_Q \nabla\cdot\left(\frac{1}{t}\int_0^t \mathbf{v}(s,\mathbf{x})ds\right)\varphi(t,\mathbf{x})\,dxdt \\ &= -\int\int_Q \left(\int_0^t \mathbf{v}(s,\mathbf{x})ds\right)\cdot\frac{1}{t}\nabla\varphi(t,\mathbf{x})\,dxdt \\ &= \int\int_Q\int_0^t \mathbf{v}(t,\mathbf{x})\cdot\left(\int_0^t \frac{1}{s}\nabla\varphi(s,\mathbf{x})ds\right)dxdt, \end{aligned} \tag{3.112}$$

which holds because $\varphi \in \mathscr{D}(Q_T)$. Moreover, since $\varphi \in \mathscr{D}(Q_T)$, $\forall\, t \in [0,T]$,

$$\int_0^t \frac{1}{s}\nabla\varphi(s,\mathbf{x})ds = \nabla\int_0^t \frac{\varphi(s,\mathbf{x})}{s}ds = \nabla\psi(t,\mathbf{x}). \tag{3.113}$$

Therefore, we deduce from (3.112) and (3.113) that

$$\langle \nabla \cdot \mathbf{V}_t, \varphi \rangle = \langle \mathbf{v}, \nabla \psi \rangle = -\langle \nabla \cdot \mathbf{v}, \psi \rangle = 0, \tag{3.114}$$

and because $\nabla \cdot \mathbf{v} = 0$ that $\langle \nabla \cdot \mathbf{V}_t, \varphi \rangle = 0$. Then

$$\forall\, T > 0, \quad \nabla \cdot \mathbf{V}_t = 0 \ \text{ in } \ \mathscr{D}'(Q_T). \tag{3.115}$$

Furthermore, by setting $\mathbf{V}_0 = \mathbf{v}_0$, we get $\mathbf{V}_t \in C([0, T], \mathbf{L}^2(\Omega))$, so that (3.115) becomes

$$\forall\, t \in [0, T], \quad \nabla \cdot \mathbf{V}_t = 0 \ \text{ in } \ \mathbf{H}^{-1}(\Omega),$$

and in reality in $\mathbf{L}^{15/7}(\Omega)$ by (3.111), and regardless of $T > 0$, which allows us to take the limit as $t_n \to \infty$, leading to $\nabla \cdot \overline{\mathbf{v}} = 0$ in $\mathbf{L}^{15/7}(\Omega)$.

STEP 3. We now take the limit in the momentum equation. Let $\boldsymbol{\varphi} \in \mathscr{D}(\Omega)^3$. Since $\boldsymbol{\varphi}, \nabla \boldsymbol{\varphi}, \Delta \boldsymbol{\varphi} \in \mathbf{L}^5(\Omega)$, we deduce from (3.108)–(3.110) and the convergence to zero of $(\boldsymbol{\varepsilon}_{t_n})_{n \in \mathbb{N}}$ in all $\mathbf{L}^p(\Omega)$, $p \leq 2$, on the one hand

$$\lim_{n \to \infty} \langle M_{t_n}((\mathbf{v} \cdot \nabla)\, \mathbf{v}), \boldsymbol{\varphi} \rangle = \lim_{n \to \infty} (M_{t_n}((\mathbf{v} \cdot \nabla)\, \mathbf{v}), \boldsymbol{\varphi})_\Omega = (\mathbf{B}, \boldsymbol{\varphi})_\Omega, = \langle \mathbf{B}, \boldsymbol{\varphi} \rangle, \tag{3.116}$$

and on the other hand

$$\lim_{n \to \infty} \langle \boldsymbol{\varepsilon}_{t_n}, \boldsymbol{\varphi} \rangle = \lim_{n \to \infty} (\boldsymbol{\varepsilon}_{t_n}, \boldsymbol{\varphi})_\Omega = 0,$$

$$\lim_{n \to \infty} \langle -\Delta \mathbf{V}_{t_n}, \boldsymbol{\varphi} \rangle = \lim_{n \to \infty} (\mathbf{V}_{t_n}, -\Delta \boldsymbol{\varphi})_\Omega = (\overline{\mathbf{v}}, -\Delta \boldsymbol{\varphi})_\Omega = (-\Delta \overline{\mathbf{v}}, \boldsymbol{\varphi})_\Omega,$$

$$\lim_{n \to \infty} \langle \nabla P_{t_n}, \boldsymbol{\varphi} \rangle = -\lim_{n \to \infty} (P_{t_n}, \nabla \cdot \boldsymbol{\varphi})_\Omega = -(\overline{p}, \nabla \cdot \boldsymbol{\varphi})_\Omega = \langle \nabla \overline{p}, \boldsymbol{\varphi} \rangle,$$

which shows by (3.100) that $(\overline{\mathbf{v}}, \overline{p})$ satisfies in $\mathscr{D}'(\Omega)$,

$$\begin{cases} -\nu \Delta \overline{\mathbf{v}} + \nabla \overline{p} = -\mathbf{B} + \mathbf{f} & \text{in } \ \Omega, \\ \qquad \nabla \cdot \overline{\mathbf{v}} = 0 & \text{in } \ \Omega, \\ \qquad\quad \overline{\mathbf{v}} = 0 & \text{on } \ \Gamma. \end{cases} \tag{3.117}$$

Let $\mathbf{F}$ denote the tensor defined by

$$\mathbf{F} = \mathbf{B} - (\overline{\mathbf{v}} \cdot \nabla)\, \overline{\mathbf{v}} = \mathbf{B} - \nabla \cdot (\overline{\mathbf{v}} \otimes \overline{\mathbf{v}}). \tag{3.118}$$

As $W^{2,5/4}(\Omega) \hookrightarrow L^{15/2}(\Omega)$ and $W^{2,5/4}(\Omega) \hookrightarrow W^{1,15/7}(\Omega)$, we get

$$\nabla \overline{\mathbf{v}} \in \mathbf{L}^{15/7}(\Omega)^3 \ \text{ and } \ \overline{\mathbf{v}} \in \mathbf{L}^{15/2}(\Omega) \ \text{ and then } \ (\overline{\mathbf{v}} \cdot \nabla)\, \overline{\mathbf{v}} \in \mathbf{L}^{15/9}(\Omega) \hookrightarrow \mathbf{L}^{5/4}(\Omega),$$

we deduce that $\mathbf{F} \in \mathbf{L}^{5/4}(\Omega)$. Hence $(\overline{\mathbf{v}}, \overline{p})$ satisfies (3.102) in the sense of distributions. □

We conclude this subsection by formulating problem (3.102) in another way. To do so, we consider the spaces

$$\mathbf{W}^{s,p}_{div}(\Omega) = \{\mathbf{w} \in \mathbf{W}^{s,p}_0(\Omega),\ \nabla \cdot \mathbf{w} = 0\}, \tag{3.119}$$

$s \geq 1, p > 1$, equipped with the usual $\mathbf{W}^{s,p}_0(\Omega)$ norm (see Sect. A.1 in [TB]). According to the definition (3.54), we have $\mathbf{V}_{div}(\Omega) = \mathbf{W}^{1,2}_{div}(\Omega)$. We also know from [2] that $\mathscr{V}_{div}(\Omega)$ expressed by (3.53) is dense in $\mathbf{W}^{1,2}_{div}(\Omega)$ for every $s \geq 1$, $p > 1$. This proves the following:

Corollary 3.2. *The long-time velocity $\overline{\mathbf{v}}$ is a solution to the variational problem:*

$$\textit{For all } \ \mathbf{w} \in \mathbf{W}^{1,5}_{div}(\Omega),$$

$$b(\overline{\mathbf{v}}; \overline{\mathbf{v}}, \mathbf{w}) + a(\overline{\mathbf{v}}, \mathbf{w}) = -(\mathbf{F}, \mathbf{w})_\Omega + (\mathbf{f}, \mathbf{w})_\Omega, \tag{3.120}$$

the operators a and b being defined by (3.67).

Remark 3.6. The proof of Theorem 3.4 contains the proof of the general identity, $\forall\, p \geq 1$, $\forall\, T > 0$, $\forall\, t \in [0, T]$,

$$\forall\, \boldsymbol{\varphi} \in L^1([0, T], \mathbf{W}^{1,p}(\Omega)), \quad \nabla \cdot M_t(\boldsymbol{\varphi}) = M_t(\nabla \cdot \boldsymbol{\varphi}). \tag{3.121}$$

Furthermore, the same reasoning also yields

$$\nabla M_t(\varphi) = M_t(\nabla \varphi), \tag{3.122}$$

which is called the Reynolds rule.

3.5.2 Reynolds Decomposition and Reynolds Stress

This subsection aims to identify the source term $\mathbf{F}$ that appears in system (3.102), to link the results of Theorem 3.4 with the usual approach to modeling turbulence, by introducing the Reynolds decomposition and the Reynolds stress.

3.5.2.1 Problem Statement

Let $\mathbf{v}$ be a given turbulent solution to the NSE and p its associated pressure. We respect the conditions for the application of the Theorem 3.4, which ensures that we can split $(\mathbf{v}, p)$

$$\mathbf{v}(t,\mathbf{x}) = \overline{\mathbf{v}}(\mathbf{x}) + \mathbf{v}'(t,\mathbf{x}), \tag{3.123}$$

$$p(t,\mathbf{x}) = \overline{p}(\mathbf{x}) + p'(t,\mathbf{x}), \tag{3.124}$$

where $(\mathbf{v}', p')$ stands for the fluctuations around the mean field $(\overline{\mathbf{v}}, \overline{p})$. We call the decomposition (3.123)–(3.124) a Reynolds decomposition.

To identify the source term $\mathbf{F}$ in system (3.102), we start from the system (3.100) and notice that, according to the Reynolds rule (3.122),

$$M_t((\mathbf{v}\cdot\nabla)\,\mathbf{v}) = M_t(\nabla\cdot(\mathbf{v}\otimes\mathbf{v})) = \nabla\cdot M_t(\mathbf{v}\otimes\mathbf{v}).$$

We shall find out from the Reynolds decomposition that it suffices to study the convergence of

$$M_t(\mathbf{v}'\otimes\mathbf{v}')(\mathbf{x}) = \frac{1}{t}\int_0^t \mathbf{v}'(s,\mathbf{x})\otimes\mathbf{v}'(s,\mathbf{x})\,ds. \tag{3.125}$$

as $t \to \infty$, which yields what we call a Reynolds stress, denoted by $\boldsymbol{\sigma}^{(\mathrm{R})}$.

Remark 3.7. The definition of $(\overline{\mathbf{v}}, \overline{p})$, and hence the Reynolds decompositions (3.123) and (3.124) and the Reynolds stress that we shall find, depends on the sequence $(t_n)_{n\in\mathbb{N}}$ that appears in Theorem 3.4, and we do not know if the limit of $(\mathbf{V}_t, P_t)_{t>0}$ is solely defined when $t \to \infty$. As a result, we do not know if $\mathbf{F}$ is solely defined too, and even if it were, it is not known if the system (3.102) has a unique solution. All of this implies that without any further information, this analysis will not provide means and decomposition that are intrinsically defined.

3.5.2.2 Main Result

We proceed with the program outlined above. The results are synthesized in the following statement.

Theorem 3.5. *Let $(t_n)_{n\in\mathbb{N}}$ be as in Theorem 3.4 and $\mathbf{F}$ as in (3.102). Then there exists $\boldsymbol{\sigma}^{(\mathrm{R})} \in \mathbf{L}^{5/3}(\Omega)^3$ such that:*

(i) *We can extract from $(M_{t_n}(\mathbf{v}'\otimes\mathbf{v}'))_{n\in\mathbb{N}}$ a subsequence that we denote by $(M_{t_n}(\mathbf{v}'\otimes\mathbf{v}'))_{n\in\mathbb{N}}$, which converges to $\boldsymbol{\sigma}^{(\mathrm{R})}$ weakly in $\mathbf{L}^{5/3}(\Omega)$,*
(ii) $\mathbf{F} = \nabla\cdot\boldsymbol{\sigma}^{(\mathrm{R})}$ *in $\mathscr{D}'(\Omega)^3$,*
(iii) *the following energy balance holds:*

$$\nu||\nabla\overline{\mathbf{v}}||_{0,2,\Omega}^2 + \langle\mathbf{F},\overline{\mathbf{v}}\rangle = (\mathbf{f},\overline{\mathbf{v}})_\Omega, \tag{3.126}$$

(iv) $\mathbf{F}$ *is dissipative, in the sense*

$$\langle\mathbf{F},\overline{\mathbf{v}}\rangle \geq 0. \tag{3.127}$$

Proof. Remember that M_t is defined by (3.96). We derive from (3.123) and (3.124) that

$$V_{t_n} = \overline{\mathbf{v}} + M_{t_n}(\mathbf{v}'), \quad P_{t_n} = \overline{p} + M_{t_n}(p'). \tag{3.128}$$

Therefore we deduce

$$\overline{\mathbf{v}'} = \lim_{n\to\infty} M_{t_n}(\mathbf{v}') = 0, \quad \overline{p'} = \lim_{n\to\infty} M_{t_n}(p') = 0, \tag{3.129}$$

the limit being weak in $\mathbf{W}^{2,5/4}(\Omega)$ and $\mathbf{W}^{1,5/4}(\Omega)/\mathbb{R}$, respectively. In addition $(t_n)_{n\in\mathbb{N}}$ can be chosen such that the convergence of $(M_{t_n}(\mathbf{v}'))_{n\in\mathbb{N}}$ toward 0 is strong in $\mathbf{L}^{15/2}(\Omega)$ because the injection $W^{2,5/4}(\Omega) \hookrightarrow L^{15/2}(\Omega)$ is compact. We now demonstrate each item of the above statement.

Proof of (i). By using decomposition (3.123), we write

$$\mathbf{v}\otimes\mathbf{v} = \overline{\mathbf{v}}\otimes\overline{\mathbf{v}} + \mathbf{v}'\otimes\overline{\mathbf{v}} + \overline{\mathbf{v}}\otimes\mathbf{v}' + \mathbf{v}'\otimes\mathbf{v}', \tag{3.130}$$

leading to

$$M_t(\mathbf{v}\otimes\mathbf{v}) = \overline{\mathbf{v}}\otimes\overline{\mathbf{v}} + M_t(\mathbf{v}')\otimes\overline{\mathbf{v}} + \overline{\mathbf{v}}\otimes M_t(\mathbf{v}') + M_t(\mathbf{v}'\otimes\mathbf{v}'), \tag{3.131}$$

for each $t > 0$. As $\overline{\mathbf{v}}$ and $M_t(\mathbf{v}') \in \mathbf{L}^{15/2}(\Omega)$, we obtain from the Hölder inequality

$$M_t(\mathbf{v}')\otimes\overline{\mathbf{v}} \ \text{ and } \ \overline{\mathbf{v}}\otimes M_t(\mathbf{v}') \in L^{15/4}(\Omega)^9 \hookrightarrow L^{5/3}(\Omega)^9.$$

In particular, (3.129) yields

$$\lim_{n\to\infty} M_{t_n}(\mathbf{v}')\otimes\overline{\mathbf{v}} = \lim_{n\to\infty} \overline{\mathbf{v}}\otimes M_{t_n}(\mathbf{v}') = 0, \tag{3.132}$$

strongly in $L^{5/3}(\Omega)^9$. Moreover, we infer from (3.97), combined with (3.88) and (3.85),

$$||M_t(\mathbf{v}\otimes\mathbf{v})||_{0,5/3,\Omega} \le C_1^{10/3} E_\infty^{2/3} \left(\frac{1}{t} \int_0^t \int_\Omega |\nabla\mathbf{v}|^2 d\mathbf{x} ds \right). \tag{3.133}$$

We are led to rewrite the formula (3.131) in the form of the asymptotic expansion that holds in $L^{5/3}(\Omega)^9$:

$$M_{t_n}(\mathbf{v}\otimes\mathbf{v}) = \overline{\mathbf{v}}\otimes\overline{\mathbf{v}} + M_{t_n}(\mathbf{v}'\otimes\mathbf{v}') + o(1). \tag{3.134}$$

We deduce from the estimate (3.133) that $(M_{t_n}(\mathbf{v}\otimes\mathbf{v}))_{n\in\mathbb{N}}$ is bounded in $\mathbf{L}^{5/3}(\Omega)$. Therefore, we can extract a subsequence (written likewise), which converges weakly in $\mathbf{L}^{5/3}(\Omega)$ to some $\boldsymbol{\vartheta} \in L^{5/3}(\Omega)^9$. The expansion (3.134) shows that the sequence $(M_{t_n}(\mathbf{v}'\otimes\mathbf{v}'))_{n\in\mathbb{N}}$ weakly converges to $\boldsymbol{\sigma}^{(\mathrm{R})} \in L^{5/3}(\Omega)^9$, linked to $\boldsymbol{\vartheta}$ by the relation

$$\boldsymbol{\sigma}^{(\mathrm{R})} = \boldsymbol{\vartheta} - \overline{\mathbf{v}} \otimes \overline{\mathbf{v}}, \tag{3.135}$$

which proves item (i).

Proof of (ii). According to (3.110) and the Reynolds rule (3.122), we note that $\nabla \cdot \boldsymbol{\vartheta} = \mathbf{B} \in L^{5/4}(\Omega)^9$; therefore, (3.118) combined with (3.135) yields $\mathbf{F} = \nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}$.

Proof of (iii). As already quoted, $\overline{\mathbf{v}} \in \mathbf{W}^{2,5/4}(\Omega) \hookrightarrow \mathbf{W}^{1,15/7}(\Omega) \hookrightarrow \mathbf{H}^1(\Omega)$. Moreover, since $\overline{\mathbf{v}} = 0$ on Γ, and $\nabla \cdot \overline{\mathbf{v}} = 0$, then $\overline{\mathbf{v}} \in \mathbf{V}_{div}(\Omega)$. Consequently, we can take $\overline{\mathbf{v}}$ as test in formulation (3.68), which yields

$$\frac{d}{dt}(\mathbf{v}, \overline{\mathbf{v}})_\Omega + b(\mathbf{v}; \mathbf{v}, \overline{\mathbf{v}}) + a(\mathbf{v}, \overline{\mathbf{v}}) = (\mathbf{f}, \overline{\mathbf{v}})_\Omega. \tag{3.136}$$

We integrate (3.136) over $[0, t]$ and divide the result by t, leading to

$$\frac{1}{t}(\mathbf{v}(t,\cdot) - \mathbf{v}_0(\cdot), \overline{\mathbf{v}}(\cdot))_\Omega + (M_t((\mathbf{v}\cdot\nabla)\,\mathbf{v}), \overline{\mathbf{v}})_\Omega + \nu(\nabla \mathbf{V}_t, \nabla\overline{\mathbf{v}})_\Omega = (\mathbf{f}, \overline{\mathbf{v}})_\Omega. \tag{3.137}$$

We take the limit of each term in (3.137). Firstly

$$\frac{1}{t}|(\mathbf{v}(t,\cdot) - \mathbf{v}_0(\cdot), \overline{\mathbf{v}}(\cdot))_\Omega| \le \frac{1}{t}||\mathbf{v}(t,\cdot) - \mathbf{v}_0(\cdot)||_{0,2,\Omega}||\overline{\mathbf{v}}||_{0,2,\Omega}, \tag{3.138}$$

which goes to zero when $t \to \infty$, due to the L^2 uniform bound (3.85). We also have $\overline{\mathbf{v}} \in \mathbf{L}^{15/2}(\Omega)$, and $M_{t_n}((\mathbf{v}\cdot\nabla)\,\mathbf{v})$ converges to $\mathbf{B}$ in $L^{5/4}(\Omega)^9$. Fortunately, we observe that $2/15 + 4/5 = 14/15 < 1$, thus, according to (3.118),

$$\lim_{n\to\infty}(M_{t_n}((\mathbf{v}\cdot\nabla)\,\mathbf{v}), \overline{\mathbf{v}})_\Omega = (\mathbf{B}, \overline{\mathbf{v}})_\Omega = (\mathbf{F}, \overline{\mathbf{v}})_\Omega + ((\overline{\mathbf{v}}\cdot\nabla)\,\overline{\mathbf{v}}, \overline{\mathbf{v}})_\Omega = (\mathbf{F}, \overline{\mathbf{v}})_\Omega, \tag{3.139}$$

since it is easily verified from $\nabla \cdot \overline{\mathbf{v}} = 0$, that $((\overline{\mathbf{v}}\cdot\nabla)\,\overline{\mathbf{v}}, \overline{\mathbf{v}})_\Omega = 0$.

Finally, we deduce from Theorem 3.4 and Sobolev embeddings that $(\nabla \mathbf{V}_{t_n})_{n\in\mathbb{N}}$ converges strongly to $\nabla\overline{\mathbf{v}}$ in $\mathbf{L}^q(\Omega)$ for all $q < 15/2$, in particular for $q = 2$, leading to

$$\lim_{n\to\infty}(\nabla \mathbf{V}_{t_n}, \nabla\overline{\mathbf{v}})_\Omega = (\nabla\overline{\mathbf{v}}, \nabla\overline{\mathbf{v}})_\Omega = ||\nabla\overline{\mathbf{v}}||^2_{0,2,\Omega}, \tag{3.140}$$

hence the energy balance (3.126) follows from (3.137) to (3.140).

Proof of (iv). We start from the energy inequality (3.70) that we divide by t_n, and we let n go to infinity. Using again the strong convergence of $(\nabla \mathbf{V}_{t_n})_{n\in\mathbb{N}}$ to $\nabla\overline{\mathbf{v}}$ in $\mathbf{L}^2(\Omega)$ and the L^2 uniform bound as above, we obtain

$$\nu||\nabla\overline{\mathbf{v}}||_{0,2,\Omega} \le (\mathbf{f}, \overline{\mathbf{v}})_\Omega, \tag{3.141}$$

which combined with (3.126) yields (3.127) and concludes the proof. □

In summary, $(\overline{\mathbf{v}}, \overline{p}) \in \mathbf{W}^{2,5/4}(\Omega) \times \mathbf{W}^{1,5/4}(\Omega)/\mathbb{R}$ satisfies

$$\begin{cases} (\overline{\mathbf{v}} \cdot \nabla)\, \overline{\mathbf{v}} - \nu \Delta \overline{\mathbf{v}} + \nabla \overline{p} = -\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})} + \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \overline{\mathbf{v}} = 0 & \text{in } \Omega, \\ \overline{\mathbf{v}} = 0 & \text{on } \Gamma, \end{cases} \tag{3.142}$$

in the sense of distributions, where in addition $(\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}, \overline{\mathbf{v}})_\Omega \geq 0$.

3.5.3 *Closure Problem*

3.5.3.1 Reynolds Stress: First Episode

The issue is how to calculate in practice values of the mean field components, $\overline{v}_i$, $(i = 1, 2, 3)$, and $\overline{p}$. This cannot be done directly from (3.142), because $\boldsymbol{\sigma}^{(\mathrm{R})}$ is unknown. We have in one respect (see Remark 3.7)

$$\boldsymbol{\sigma}^{(\mathrm{R})} = \overline{\mathbf{v}' \otimes \mathbf{v}'}. \tag{3.143}$$

This is why, following common usage in turbulence modeling, we call $\boldsymbol{\sigma}^{(\mathrm{R})}$ a Reynolds stress. It is a symmetric tensor, as the limit of a sequence of symmetric tensors. In order to derive from system (3.102) a PDE system that fully determines the mean flow $(\overline{\mathbf{v}}, \overline{p})$, we have the following options.

(a) To seek for an equation satisfied by $\boldsymbol{\sigma}^{(\mathrm{R})}$: the method is to combine system (3.142), the Reynolds decomposition (3.123)–(3.124), and the NSE. We then find the system satisfied by $(\mathbf{v}', p')$ that should be resolved to get an equation for $\boldsymbol{\sigma}^{(\mathrm{R})}$.
(b) To postulate a form of $\boldsymbol{\sigma}^{(\mathrm{R})}$ in terms of tractable fields.

Option (a) holds an undeniable theoretical interest (see in Batchelor [5]). Unfortunately, a new unknown tensor $\boldsymbol{\sigma}_3^{(\mathrm{R})}$ appears in the equation satisfied by $\boldsymbol{\sigma}^{(\mathrm{R})}$, where, roughly speaking, $\boldsymbol{\sigma}_3^{(\mathrm{R})} = \overline{\mathbf{v}' \otimes \mathbf{v}' \otimes \mathbf{v}'}$. Then we must derive an equation for $\boldsymbol{\sigma}_3^{(\mathrm{R})}$, in which we find a tensor like $\boldsymbol{\sigma}_4^{(\mathrm{R})} = \overline{\mathbf{v}' \otimes \mathbf{v}' \otimes \mathbf{v}' \otimes \mathbf{v}'}$, and so on. Sooner or later, we have to turn to option (b) concerning one of the $\boldsymbol{\sigma}_k^{(\mathrm{R})}$. This is a *closure problem*, which is the main task in turbulence modeling.

3.5.3.2 Eddy Viscosity and Turbulent Kinetic Energy

To model $\boldsymbol{\sigma}^{(\mathrm{R})}$, we observe:

(i) the dissipative feature of $\boldsymbol{\sigma}^{(\mathrm{R})}$ characterized by (3.127),
(ii) the similarity between system (3.142) and the primitive form of the motion equation of NSE, (2.105) in Sect. 2.6.3,

which yields a strong analogy between $-\boldsymbol{\sigma}^{(\mathrm{R})}$ and the stress tensor $\boldsymbol{\sigma}$. Indeed, we deduce that $-\boldsymbol{\sigma}^{(\mathrm{R})}$ allows the calculation of the forces exerted by the fluctuation on the mean field. Following formula (2.96) in Chap. 2, which models the stress tensor $\boldsymbol{\sigma}$, we are led to postulate the existence of a nonnegative function ν_t, called an eddy viscosity, such that

$$\boldsymbol{\sigma}^{(\mathrm{R})} = -\nu_t D\mathbf{v} + \frac{2}{3}k\mathrm{I}, \tag{3.144}$$

where $D\mathbf{v} = (1/2)(\nabla\mathbf{v} + \nabla\mathbf{v}^t)$ is the deformation tensor and k is the *turbulent kinetic energy*:

$$k = \frac{1}{2}tr\boldsymbol{\sigma}^{(\mathrm{R})} = \frac{1}{2}\overline{|\mathbf{v}'|^2}. \tag{3.145}$$

Equality (3.144) is the *Boussinesq assumption*. Thus, we obtain for $(\overline{\mathbf{v}}, \overline{p})$,

$$\begin{cases} (\overline{\mathbf{v}}\cdot\nabla)\,\overline{\mathbf{v}} - \nabla\cdot((2\nu+\nu_t)D\overline{\mathbf{v}}) + \nabla(\overline{p}+k) = \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\overline{\mathbf{v}} = 0 & \text{in } \Omega, \\ \overline{\mathbf{v}} = 0 & \text{on } \Gamma. \end{cases} \tag{3.146}$$

Since k in this equation is treated as a Lagrange multiplier (see Corollary 3.5 above), it remains to model the eddy viscosity ν_t.

Prandtl introduced in [33] the notion of *mixing length* ℓ. Roughly speaking, ℓ is the mean distance traveled by a ball of fluid, before disappearing because of the turbulent mixing. Prandtl suggested that

$$\nu_t = C\ell^2|D\mathbf{v}|, \tag{3.147}$$

where C is a dimensionless constant. We then have the problem of determining ℓ and the constant C.

The other usual approach starts from the observation that k and ℓ are dimensionally independent (see Definition 3.2). According to Lemma 3.1, we find

$$\mathbb{D}(\ell\sqrt{k}) = \mathbb{D}(\nu_t) = (2, -1), \tag{3.148}$$

which suggests taking

$$\nu_t = C'\ell\sqrt{k}, \tag{3.149}$$

for some dimensionless constant C' bringing the additional issue of the determination of k and the constant C'.

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Chapter 4
The $k - \varepsilon$ Model

Abstract Turbulent flows are considered as random motions, the velocity $\mathbf{v}$ and the pressure p being random variables. After having defined a clear probabilistic framework, we look for equations satisfied by the expectations $\overline{\mathbf{v}}$ and $\overline{p}$ of $\mathbf{v}$ and p, which yield the Reynolds stress $\boldsymbol{\sigma}^{(\mathrm{R})} = \overline{\mathbf{v}' \otimes \mathbf{v}'}$. In keeping with the Boussinesq assumption, we write $\boldsymbol{\sigma}^{(\mathrm{R})}$ in terms of the mean deformation tensor $D\overline{\mathbf{v}}$ and the eddy viscosity ν_t, which must be modeled. The modeling process leads to the assumption that ν_t depends on the turbulent kinetic energy $k = (1/2)\overline{|\mathbf{v}'|^2}$ and the turbulent diffusion $\mathscr{E} = 2\nu\overline{|D\mathbf{v}'|^2}$, addressing the issue of finding equations for k and $\mathscr{E}$ to compute ν_t, and therefore $(\overline{\mathbf{v}}, \overline{p})$. To realize this, hypotheses about turbulence are necessary, such as local homogeneity, expressed by the local invariance of the correlation tensors under translations. The concept of mild homogeneity is introduced, which is the minimal hypothesis about correlations that allows the derivation of the $k - \mathscr{E}$ model carried out in this chapter, using additional standard closure assumptions.

4.1 Introduction

Turbulent flows are chaotic systems, highly sensitive to small changes in data [23], which means that any tiny change in body forces, any external action and/or initial data, might give rise almost instantly to significant changes in the flow features.

To be more specific, let us consider an experiment which measures the velocity (or one of its components) of a turbulent flow N times at a given point. Each measurement is carried out under the same conditions (same initial data, constant temperature, same source). Although advanced technologies allow measurements to be made to high precision, the experiment will yield N different results, because in reality infinitesimal changes occur during each measurement that cannot be controlled.

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However, the statistics in some sense stay the same. Indeed, let $\mathbf{v}_k$ be the result of the kth measurement at a given point $(t, \mathbf{x})$. It is well known that the ensemble average

$$\overline{\mathbf{v}}_n = \frac{1}{N}\sum_{k=1}^{N} \mathbf{v}_k \tag{4.1}$$

converges for large N to a vector denoted by $\overline{\mathbf{v}}$ that, roughly speaking, remains the same for each series of N measurements, reminiscent of the long-time average model considered in Sect. 3.5.

These observations agree with numerical experiments highlighting the enormous difficulty of performing accurate numerical simulations (DNS) of flows with high Reynolds numbers based directly on the NSE and hence the need for turbulence models. There are two major families of turbulence models: the RANS models (Reynolds-averaged Navier–Stokes) and the LES models (Large Eddy Simulation). We focus on RANS models in this chapter.

Following Taylor [24] (see also [2, 21]), we consider the flow $(\mathbf{v}, p) = (\mathbf{v}(t, \mathbf{x}), p(t, \mathbf{x}))$ as a random field $(\mathbf{v}, p) = (\mathbf{v}(t, \mathbf{x}, \omega), p(t, \mathbf{x}, \omega))$, where the new parameter ω belongs to a suitable probabilistic space $(\mathscr{P}, \mu)$. In this framework, the mean field $(\overline{\mathbf{v}}, \overline{p})$ is defined as the expectations,

$$\overline{\mathbf{v}} = E(\mathbf{v}) = \int_{\mathscr{P}} \mathbf{v}\, d\mu(\omega), \quad \overline{p} = E(p) = \int_{\mathscr{P}} p\, d\mu(\omega), \tag{4.2}$$

while the fluctuations are specified by the Reynolds decomposition, as in (3.123) and (3.124):

$$\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}', \quad p = \overline{p} + p'. \tag{4.3}$$

The issue then is to find equations for $(\overline{\mathbf{v}}, \overline{p})$, which will involve the turbulent kinetic energy (TKE) $k = (1/2)\overline{|\mathbf{v}'|^2}$ and the Prandtl mixing length ℓ (cf. Prandtl [22]), which we must also determine.

More fundamentally, we must decide on the right physical assumptions about turbulence to be included in the development of our model. These assumptions are usually based on homogeneity (cf. Batchelor [3]) or local homogeneity on small scales (cf. Kolmogorov [14]), which are related to the correlation tensors that must be carefully defined.

Our experience in fact indicates that instead of ℓ, it is better to introduce the mean dissipation of the fluctuation, $\mathscr{E} = 2\nu\overline{|D\mathbf{v}'|^2}$, related to ℓ by the formula inferred from dimensional analysis,

$$\ell = \frac{k^{3/2}}{\mathscr{E}}, \tag{4.4}$$

and to consider the couple $k - \mathscr{E}$ instead of $k - \ell$. This yields the $k - \mathscr{E}$ model, first developed by Launder and Spalding [16], and known to provide reliable predictions of mean properties for many turbulent flows in engineering applications as well as in oceanography (cf. Burchard [6], Davidson [8], Mohammadi–Pironneau [20], and Wilcox [27]).

The goals of this chapter are:

(a) to give a rigorous definition of the flow as a random field,
(b) to introduce the correlation tensors, to discuss homogeneity, and to introduce mildly homogeneous flows, and
(c) to derive from the NSE the $k - \mathscr{E}$ model, after having clarified the essential closure assumptions that are necessary.

There are different ways of approaching item (a), for example, based on statistical solutions for the NSE introduced by Foias [10–12] (see also [13]). In Sect. 4.2, we start simply, choosing a set of initial data as a probabilistic space, the probability measure of which is the limit of ensemble averages such as (4.1). This is based on strong local time solutions considered in Sect. 3.4.4 and provides a coherent framework in which to compute the expectation of the NSE, including a clear and general definition of the Reynolds stress $\boldsymbol{\sigma}^{(\mathrm{R})} = \overline{\mathbf{v}' \otimes \mathbf{v}'}$ and the notion of eddy viscosity ν_t.

We introduce in Sect. 4.3 the correlation tensors and local homogeneity, expressed by the local invariance of the correlation tensors under the action of translations. Although this definition is the closest to that generally considered in the literature, it is however too restrictive since it applies to only a few turbulent flows, according to the results summarized in Sect. 4.4.4.4 below. Obviously, the range of application of $k - \mathscr{E}$ model is much wider. Hence we wish to determine the minimal mathematical hypothesis to be satisfied by the correlation tensors in order to derive the $k - \mathscr{E}$ model, which results in what we have called mild homogeneity (cf. Sect. 4.3.3).

We shall prove that locally homogeneous flows are mildly homogeneous. We however do not know of any particular example of a mildly homogeneous flow which is not locally homogeneous, although we conjecture that such flows do exist, as suggested by the discussion in Sect. 4.4.4.2.

Section 4.4 is mainly devoted to the derivation of the $k - \mathscr{E}$ model from the NSE. The starting point is the supposition that the eddy viscosity ν_t is a function of k and $\mathscr{E}$, dimensional analysis leading to the formula

$$\nu_t = \nu_t(k, \mathscr{E}) = c_\nu \frac{k^2}{\mathscr{E}}, \tag{4.5}$$

c_ν is a dimensionless constant. In addition to supposing mild homogeneity, we make use of the Boussinesq assumption and the principle that convection by random fields yields diffusion, which leads after a long and technical process to the coupled system of (4.126) and (4.127), known as the $k - \mathscr{E}$ model. The determination of boundary conditions for k and $\mathscr{E}$ is postponed until Chap. 5, together with the introduction of wall laws in Sect. 5.3.3, based upon similarity principles.

4.2 Statistical Model and Mean Equations

4.2.1 Long-Time Average as Statistical Model

We first observe that the long-time average model, studied in Sect. 3.5.1, falls into the class of probabilistic models. Indeed, in this case, the probabilistic space $\mathscr{P}$ is the half line $\mathbb{R}_+$ and

$$\forall\, A \in \mathscr{B}(\mathbb{R}_+), \quad \mu(A) = \lim_{t\to\infty} \frac{1}{t}\lambda(A \cap [0,t]), \tag{4.6}$$

where $\mathscr{B}(\mathbb{R}_+)$ denotes the Borel σ-algebra on $\mathbb{R}_+$ and λ the Lebesgue measure. Formula (4.6) clearly defines a measure, and since

$$\mu(\mathbb{R}_+) = \lim_{t\to\infty} \mu([0,t]) = 1,$$

it is a probability measure, the scope of which can be extended as follows.

Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a measurable function. Then, $f \in L^1(\mu)$ if

$$M_t(f) = \lim_{t\to\infty} \frac{1}{t}\int_0^t f(s)\, ds$$

has a limit in the usual sense, called the long-time average. The long-time average operator can be extended to functions f for which $M_t(f)$ is bounded using the Hahn–Banach theorem. This extension, called the generalized Banach limit, is sometimes used to define large-scale quantities of turbulent flows (see in [13]), but remains ambiguous since it is not unique and has no analytic expression.

We have remarked in Sect. 3.5.1 that the notion of the generalized Banach limit is not necessary in order to define the long-time average of $(\mathbf{v}, p)$, since any turbulent solution to the NSE will have a long-time average, as proved in Theorem 3.4. However, the uniqueness of such an average is unknown, so that a certain ambiguity is also present in this mean field definition, although it appears to be very natural (see also Remark 3.7).

4.2.2 Probabilistic Flows

The probabilistic space is a set $\mathbb{K}$ of initial data. The difficulty is to clearly construct a measure of probability relevant to the usual experimental conditions, by taking the limit of the ensemble averages in the sense of measures. We also must take care to give a sense to the Reynolds rules (4.24)–(4.29), essential in the modeling process that yields the $k - \mathscr{E}$ model.

The spaces and notations are those of Sect. 3.4. We refer to the book by Billingsley [5] for everything that concerns probability theory.

4.2.2.1 Framework

We assume throughout this section that the source term $\mathbf{f}$ in the NSE (3.29) belongs to $C^{0,\alpha}(\mathbb{R}_+ \times \Omega) \cap C(\mathbb{R}_+, \mathbf{L}^2_{div}(\Omega))$, with $\nu > 0$ fixed. In the following model, the uncertainty is in the initial data.

Let $\mathbb{K}$ be a compact subset of $C^{2,\alpha}(\Omega)^3 \cap \mathbf{V}_{div}(\Omega)$ (for some $\alpha > 0$) equipped with the $C^{2,\alpha}(\Omega)^3$ topology and the corresponding Borel σ-algebra, denoted by $\mathscr{B}(\mathbb{K})$. According to Theorem 3.3, as $\mathbf{f} \in C^{0,\alpha}(\mathbb{R}_+ \times \Omega) \cap C(\mathbb{R}_+, \mathbf{L}^2_{div}(\Omega))$, for any $\mathbf{v}_0 \in \mathbb{K}$, there exists $\delta T = \delta T(\mathbf{f}, \nu, \mathbf{v}_0)$ such that the NSE has a unique Hölder continuous solution $(\mathbf{v}, p)$, well defined at each point of $(t, \mathbf{x}) \in [0, \delta T] \times \Omega$. Let

$$\delta T_m = \inf_{\mathbf{v}_0 \in \mathbb{K}} \delta T(\mathbf{f}, \nu, \mathbf{v}_0). \tag{4.7}$$

Following the results of Sect. 3.4.4, in particular the formula (3.91), $\delta T_m > 0$ only depends on $\mathbf{f}$, ν, and diam($\mathbb{K}$).

The model is based on writing the local time solution of the NSE $(\mathbf{v}, p)$, whose initial data is $\mathbf{v}_0 \in \mathbb{K}$ as

$$\forall\, (t, \mathbf{x}) \in [0, \delta T_m] \times \Omega, \quad (\mathbf{v}, p) = (\mathbf{v}(t, \mathbf{x}, \mathbf{v}_0), p(t, \mathbf{x}, \mathbf{v}_0))$$

with $\mathbf{v}_0$ playing the role of ω. Once a probability measure over $\mathbb{K}$ is constructed, we shall be able to specify the mean field as the expectation of $(\mathbf{v}, p)$.

4.2.2.2 Construction of the Probability Measure

The measure that we construct is the limit of the Cesàro means of Dirac measures. Since $C^{2,\alpha}(\Omega)^3$ is a separable space, so is $\mathbb{K}$. Let

$$\tilde{\mathbb{K}} = \{\mathbf{v}_0^{(1)}, \mathbf{v}_0^{(2)}, \ldots, \mathbf{v}_0^{(n)} \ldots\ldots\}, \tag{4.8}$$

be a countable dense set in $\mathbb{K}$, and denote

$$\delta_k = \delta(\mathbf{v}_0^{(k)}), \tag{4.9}$$

the Dirac mass at $\mathbf{v}_0^{(k)}$ in $\mathbb{K}$, expressed by

$$\forall\, A \in \mathscr{B}(\mathbb{K}), \quad \delta_k(A) = 0 \text{ if } \mathbf{v}_0^{(k)} \notin A, \quad \delta_k(A) = 1 \text{ if } \mathbf{v}_0^{(k)} \in A. \tag{4.10}$$

Let μ_n be the Cesàro means of the δ_k's, $k = 1, \ldots, n$,

$$\mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_k, \tag{4.11}$$

thus defining a probability measure on $\mathbb{K}$. Indeed,

$$||\mu_n|| = \mu_n(\mathbf{1}_{\mathbb{K}}) = \int_{\mathbb{K}} \mathbf{1}_{\mathbb{K}} d\mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_k(\mathbb{K}) = 1. \tag{4.12}$$

The sequence $(\mu_n)_{n\in\mathbb{N}}$ being bounded in the sense of the measures, we can extract a subsequence (still denoted by $(\mu_n)_{n\in\mathbb{N}}$), which weakly converges to a measure μ. To be more specific, this means

$$\forall\ f \in C(\mathbb{K}, \mathbb{R}), \quad \lim_{n\to\infty} \int_{\mathbb{K}} f d\mu_n = \int_{\mathbb{K}} f d\mu. \tag{4.13}$$

As $\mathbf{1}_{\mathbb{K}} \in C(\mathbb{K}, \mathbb{R})$, we deduce that $||\mu|| = \mu(\mathbf{1}_{\mathbb{K}}) = 1$. Therefore, μ is a probability measure on K.

Remark 4.1. It is also possible to consider the source term $\mathbf{f}$ as an additional source of uncertainty in the model, which can be included in the description of the probabilistic space. This only involves extra technical difficulties and is not essential for our purpose here.

4.2.2.3 Mean Fields

Let $(t, \mathbf{x}) \in [0, \delta T_m] \times \Omega = Q_m$ be fixed. We denote by $\mathbf{v}(t, \mathbf{x}, \mathbf{v}_0) \in \mathbb{R}^3$ and $p(t, \mathbf{x}, \mathbf{v}_0) \in \mathbb{R}$ the values taken by the strong solution $(\mathbf{v}, p)$ of the NSE (3.29) at $(t, \mathbf{x})$, whose initial data is $\mathbf{v}_0$. This is reasonable since we are dealing with the Hölder continuous solutions. Moreover

$$F_{t,\mathbf{x}} : \begin{cases} \mathbb{K} \longrightarrow \mathbb{R}^4, \\ \mathbf{v}_0 \longrightarrow (\mathbf{v}(t, \mathbf{x}, \mathbf{v}_0), p(t, \mathbf{x}, \mathbf{v}_0)) \end{cases} \tag{4.14}$$

is a continuous map, due to Theorem 3.3. In particular $F_{t,\mathbf{x}} \in L^1((\mathbb{K}, \mu); \mathbb{R}^4)$, which allows us to consider the expectation of $F_{t,\mathbf{x}}$,

$$E(F_{t,\mathbf{x}}) = \int_{\mathbb{K}} F_{t,\mathbf{x}}(\mathbf{v}_0) d\mu(\mathbf{v}_0), \tag{4.15}$$

This defines a mean field at each $(t, \mathbf{x}) \in Q_m$ which we denote from now as

$$\begin{aligned}
\overline{\mathbf{v}}(t,\mathbf{x}) &= \overline{\mathbf{v}(t,\mathbf{x},\mathbf{v}_0)} = \int_{\mathbb{K}} \mathbf{v}(t,\mathbf{x},\mathbf{v}_0)d\mu(\mathbf{v}_0) = E(\mathbf{v}(t,\mathbf{x},\mathbf{v}_0)), \\
\overline{p}(t,\mathbf{x}) &= \overline{p(t,\mathbf{x},\mathbf{v}_0)} = \int_{\mathbb{K}} p(t,\mathbf{x},\mathbf{v}_0)d\mu(\mathbf{v}_0) = E(p(t,\mathbf{x},\mathbf{v}_0)).
\end{aligned} \tag{4.16}$$

Moreover, given any $\mathbf{x} \in \Omega$, let us consider

$$I_{\mathbf{x}} : \begin{cases} \mathbb{K} \longrightarrow \mathbb{R}^3, \\ \mathbf{v}_0 \longrightarrow \mathbf{v}_0(\mathbf{x}). \end{cases} \tag{4.17}$$

Since the $C^{2,\alpha}(\Omega)$ topology is stronger than the uniform convergence topology, this map is continuous, so that $I_{\mathbf{x}} \in L^1((\mathbb{K},\mu);\mathbb{R}^3)$. This allows us to consider the integral

$$E(I_{\mathbf{x}}) = \int_{\mathbb{K}} I_{\mathbf{x}}(\mathbf{v}_0)d\mu(\mathbf{v}_0), \tag{4.18}$$

where we denote

$$\overline{\mathbf{v}_0}(\mathbf{x}) = \int_{\mathbb{K}} \mathbf{v}_0(\mathbf{x})d\mu(\mathbf{v}_0) = E(\mathbf{v}_0). \tag{4.19}$$

We note that

$$\forall\, \mathbf{x} \in \Omega, \quad \overline{\mathbf{v}}(0,\mathbf{x}) = \overline{\mathbf{v}_0}(\mathbf{x}). \tag{4.20}$$

In general, given $\boldsymbol{\psi} \in L^1((\mathbb{K},\mu),\mathbb{R}^n)$, we denote

$$\overline{\boldsymbol{\psi}} = E(\boldsymbol{\psi}) = \int_{\mathbb{K}} \boldsymbol{\psi}(\mathbf{v}_0)d\mu(\mathbf{v}_0) \in \mathbb{R}^n, \tag{4.21}$$

the expectation of $\boldsymbol{\psi}$. By denoting the fluctuation $\boldsymbol{\psi}' = \boldsymbol{\psi} - \overline{\boldsymbol{\psi}}$, the general Reynolds decomposition holds

$$\boldsymbol{\psi} = \overline{\boldsymbol{\psi}} + \boldsymbol{\psi}', \tag{4.22}$$

where $\overline{\boldsymbol{\psi}'} = 0$.

Remark 4.2. The average definition preserves the dimension. That is,

$$\mathbb{D}\overline{\boldsymbol{\psi}} = \mathbb{D}\boldsymbol{\psi}, \tag{4.23}$$

for any field $\boldsymbol{\psi}$.

4.2.3 Mean Equations

We determine in this subsection the PDE system satisfied by the mean field $(\overline{\mathbf{v}}, \overline{p})$. To do so, we establish basic properties of the expectations, such as the Reynolds rules. We then derive the Reynolds stress $\boldsymbol{\sigma}^{(\mathrm{R})}$ from elementary algebraic calculations based on the expectation of the NSE.

4.2.3.1 Reynolds Rules

As $\partial_t \mathbf{v}$, $\nabla \mathbf{v}$, $\nabla\cdot\mathbf{v}$ $\Delta\mathbf{v}$, $\boldsymbol{\omega} = \nabla\times\mathbf{v}$, p and ∇p are all Hölder continuous, the continuity modulus of which is a function of the diameter of $\mathbb{K}$, $\mathbf{f}$, and ν, they all belong to $L^1(\mu)$. Furthermore, the usual results concerning integrals depending on parameters yield, $\forall\ (t, \mathbf{x}) \in Q_m$,

$$\overline{\partial_t \mathbf{v}(t, \mathbf{x}, \mathbf{v}_0)} = \partial_t \overline{\mathbf{v}}(t, \mathbf{x}), \tag{4.24}$$

$$\overline{\nabla \mathbf{v}(t, \mathbf{x}, \mathbf{v}_0)} = \nabla \overline{\mathbf{v}}(t, \mathbf{x}), \tag{4.25}$$

$$\overline{\nabla \cdot \mathbf{v}(t, \mathbf{x}, \mathbf{v}_0)} = \nabla \cdot \overline{\mathbf{v}}(t, \mathbf{x}), \tag{4.26}$$

$$\overline{\Delta \mathbf{v}(t, \mathbf{x}, \mathbf{v}_0)} = \Delta \overline{\mathbf{v}}(t, \mathbf{x}), \tag{4.27}$$

$$\overline{\nabla \times \mathbf{v}(t, \mathbf{x}, \mathbf{v}_0)} = \nabla \times \overline{\mathbf{v}}(t, \mathbf{x}), \tag{4.28}$$

$$\overline{\nabla p(t, \mathbf{x}, \mathbf{v}_0)} = \nabla \overline{p}(t, \mathbf{x}). \tag{4.29}$$

These identities are called *the Reynolds rules*. The first consequence of the Reynolds rules is

Lemma 4.1. *The fluctuation's mean vanishes, i.e.,* $\forall\ (t, \mathbf{x}) \in Q_m$, $\overline{\mathbf{v}'(t, \mathbf{x}, \mathbf{v}_0)} = 0$.

Proof. We infer from the Reynolds decomposition (4.3), $\mathbf{v}' = \mathbf{v} - \overline{\mathbf{v}}$. Therefore $\mathbf{v}' = \mathbf{v}'(t, \mathbf{x}, \mathbf{v}_0) \in \mathbb{R}^3$ is also a random vector at a fixed $(t, \mathbf{x})$. Moreover, we have

$$\overline{\mathbf{v}} = \overline{\overline{\mathbf{v}} + \mathbf{v}'} = \overline{\overline{\mathbf{v}}} + \overline{\mathbf{v}'}, \tag{4.30}$$

by using the linearity of the expectation. We observe that

$$\overline{\overline{\mathbf{v}}}(t, \mathbf{x}) = \int_{\mathbb{K}} \overline{\mathbf{v}}(t, \mathbf{x}) d\mu(\mathbf{v}_0) = \overline{\mathbf{v}}(t, \mathbf{x}) \int_{\mathbb{K}} d\mu(\mathbf{v}_0) = \overline{\mathbf{v}}(t, \mathbf{x}), \tag{4.31}$$

which yields

$$\overline{\mathbf{v}'}(t, \mathbf{x}) = \int_{\mathbb{K}} \mathbf{v}'(t, \mathbf{x}, \mathbf{v}_0) d\mu(\mathbf{v}_0) = 0. \tag{4.32}$$

$\square$

In a general way, for any field $\boldsymbol{\psi}$, we have $\overline{\boldsymbol{\psi}'} = 0$. In particular, $\overline{p'} = 0$, $\overline{\boldsymbol{\omega}'} = 0$, the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ being decomposed as

$$\boldsymbol{\omega} = \overline{\boldsymbol{\omega}} + \boldsymbol{\omega}', \quad \text{where } \overline{\boldsymbol{\omega}} = \nabla \times \overline{\mathbf{v}}, \quad \boldsymbol{\omega}' = \nabla \times \mathbf{v}', \tag{4.33}$$

using the Reynolds rules.

The next lemma makes a connection with Sects. 3.5 and 4.2.1, dedicated to the long-time average model.

Lemma 4.2. *The long-time average measure* $\lim_{t\to\infty} M_t(\cdot)$ *satisfies the Reynolds rules.*

Proof. Spatial rules (4.25)–(4.29) hold as reported in Remark 3.6. Let us verify the time rule (4.24). Since the long-time average is by nature stationary, $\partial_t \overline{\mathbf{v}} = 0$. Furthermore,

$$M_t(\partial_t \mathbf{v})(t, \mathbf{x}) = \frac{\mathbf{v}(t, \mathbf{x}) - \mathbf{v}_0(\mathbf{x})}{t} = -\boldsymbol{\varepsilon}_t(\mathbf{x}), \tag{4.34}$$

[see also (3.101)], which goes to zero when $t \to \infty$. Then, $\overline{\partial_t \mathbf{v}} = 0 = \partial_t \overline{\mathbf{v}}$, at least in $\mathbf{L}^2(\Omega)$, therefore almost everywhere in Ω. □

4.2.3.2 Reynolds Stress: Second Episode

The Reynolds rules (4.24)–(4.29) yield the following system for $(\overline{\mathbf{v}}, \overline{p})$, obtained by taking the expectation of the NSE,

$$\begin{cases} \partial_t \overline{\mathbf{v}} + \nabla \cdot (\overline{\mathbf{v} \otimes \mathbf{v}}) - \nu \Delta \overline{\mathbf{v}} + \nabla \overline{p} = \mathbf{f} & \text{in } Q_m, \\ \nabla \cdot \overline{\mathbf{v}} = 0 & \text{in } Q_m, \\ \overline{\mathbf{v}} = 0 & \text{on } \Gamma, \\ \overline{\mathbf{v}} = \overline{\mathbf{v}_0} & \text{at } t = 0. \end{cases} \tag{4.35}$$

System (4.35) addresses the issue of determining the nonlinear term $\overline{\mathbf{v} \otimes \mathbf{v}}$.

Lemma 4.3. *We have at each* $(t, \mathbf{x}) \in Q_m$,

$$\overline{\mathbf{v} \otimes \mathbf{v}} = \overline{\mathbf{v}} \otimes \overline{\mathbf{v}} + \overline{\mathbf{v}' \otimes \mathbf{v}'}. \tag{4.36}$$

Proof. It results from the Reynolds decomposition that

$$\mathbf{v} \otimes \mathbf{v} = \overline{\mathbf{v}} \otimes \overline{\mathbf{v}} + \mathbf{v}' \otimes \overline{\mathbf{v}} + \overline{\mathbf{v}} \otimes \mathbf{v}' + \mathbf{v}' \otimes \mathbf{v}', \tag{4.37}$$

which is similar to decomposition (3.130). Cancellation of the fluctuation's mean (equality (4.32) in Lemma 4.1) leads to

$$\overline{\overline{\mathbf{v}} \otimes \mathbf{v}'}(t, \mathbf{x}) = \int_{\mathbb{K}} \overline{\mathbf{v}}(t, \mathbf{x}) \otimes \mathbf{v}'(t, \mathbf{x}, \mathbf{v}_0) d\mu(\mathbf{v}_0) = \overline{\mathbf{v}}(t, \mathbf{x}) \otimes \int_{\mathbb{K}} \mathbf{v}'(t, \mathbf{x}, \mathbf{v}_0) d\mu(\mathbf{v}_0) = 0. \tag{4.38}$$

Similarly, $\overline{\mathbf{v}' \otimes \overline{\mathbf{v}}}(t, \mathbf{x}) = 0$, hence (4.36). □

As in Sect. 3.5.3, we define the Reynolds stress $\boldsymbol{\sigma}^{(\mathrm{R})}$ by

$$\boldsymbol{\sigma}^{(\mathrm{R})} = \overline{\mathbf{v}' \otimes \mathbf{v}'}, \tag{4.39}$$

which is the same definition as (3.143). Therefore, system (4.35) becomes

$$\begin{cases} \partial_t \overline{\mathbf{v}} + (\overline{\mathbf{v}} \cdot \nabla)\, \overline{\mathbf{v}} - \nu \Delta \overline{\mathbf{v}} + \nabla \overline{p} = -\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})} + \mathbf{f} & \text{in } Q_m, \\ \nabla \cdot \overline{\mathbf{v}} = 0 & \text{in } Q_m, \\ \overline{\mathbf{v}} = 0 & \text{on } \Gamma, \\ \overline{\mathbf{v}} = \overline{\mathbf{v}_0} & \text{at } t = 0. \end{cases} \tag{4.40}$$

This system is the evolutionary version of system (3.142) in Sect. 3.5.2.2. The issues concerning this system are the same, in particular the Reynolds stress $\boldsymbol{\sigma}^{(\mathrm{R})}$ that needs to be determined. Our guideline throughout this chapter is the Boussinesq assumption, already met in Sect. 3.5.3.2, that assumes that $\boldsymbol{\sigma}^{(\mathrm{R})}$ is related to the mean deformation tensor by the relation,

$$\boldsymbol{\sigma}^{(\mathrm{R})} = -\nu_t D\overline{\mathbf{v}} + \frac{2}{3} k \mathrm{I}, \tag{4.41}$$

where

(i) ν_t is the eddy viscosity,
(ii) $k = \frac{1}{2} tr \boldsymbol{\sigma}^{(\mathrm{R})}$ is the TKE.

This assumption is motivated by the principle that $\boldsymbol{\sigma}^{(\mathrm{R})}$ allows for the calculation of the mean forces exerted by the fluctuations over the mean field.

The main concern is how to model ν_t in terms of mean fields only. It can be expressed in terms of the Prandtl mixing length ℓ and k, according to formula (3.149), or in terms of the deformation tensor following (3.147), leading to the Prandtl–Kolmogorov–Smagorinsky model, which will be discussed at the end of Sect. 5.2.

In either case, k and ℓ are involved, making it necessary to find equations that they satisfy. Dimensional analysis plays a central role in this process, where (ℓ, k) seems to be a natural dimensional basis. Modeling an equation for k proceeds from a natural energy balance. However, finding an equation for ℓ is not easy, and so we substitute it with the mean dissipation $\mathscr{E} = 2\nu \overline{|D\mathbf{v}'|^2}$, the equation for which is modeled from the helicity balance, roughly speaking the energy balance for the vorticity. The mixing length is then computed by the formula (4.4). We will complete the derivation of the $k - \mathscr{E}$ model at the end of Sect. 4.4.

Before continuing the modeling process, we discuss in the next section our hypotheses about turbulence, to make sense of the calculations yielding the model.

4.3 Correlation Tensors and Homogeneity

Homogeneity and isotropy are the standard hypotheses in turbulence modeling. They are designed on the basis of algebraic properties satisfied by the correlation tensors introduced in this section. Section 5.2 discusses various notions of homogeneity and isotropy.

Throughout this section and the rest of this chapter, $(\mathbf{v}, p) = (\mathbf{v}(t, \mathbf{x}, \mathbf{v}_0), p(t, \mathbf{x}, \mathbf{v}_0))$ is a given flow over $Q_m \times \mathbb{K}$, $Q_m = [0, \delta T_m] \times \Omega$. We denote $\mathbf{v} = (v_1, v_2, v_3)$.

4.3.1 *Basic Definition*

A standard issue concerning turbulence is how the flow at any given $(t_1, \mathbf{x}_1) \in Q_m$ is linked to the flow at $(t_2, \mathbf{x}_2) \in Q_m$, if at all. To begin, we recall the notions of independence and covariance in probability theory (cf. Billingsley [5]).

Definition 4.1. Let $\varphi, \psi : \mathbb{K} \to \mathbb{R}$ be two random variables. They are said to be independent if $E(\varphi\psi) = E(\varphi)E(\psi)$, in other words if $\overline{\varphi\psi} = \overline{\varphi}\,\overline{\psi}$.

Definition 4.2. Let $\varphi, \psi : \mathbb{K} \to \mathbb{R}$ be two random variables. The covariance of φ and ψ is defined by $\mathrm{cov}(\varphi, \psi) = E(\varphi\psi) - E(\varphi)E(\psi) = \overline{\varphi\psi} - \overline{\varphi}\,\overline{\psi}$. Roughly speaking, the covariance is a measure of how much φ and ψ change together.

Rather than covariance, we speak about the *correlation tensor* in the study of turbulent flows. The most popular correlation tensor is the tensor $\mathbb{B}_2$ defined by

$$\begin{aligned} &\mathbb{B}_2 = \mathbb{B}_2(M_1, M_2) = (B_{ij}(M_1, M_2))_{1 \le ij \le 3}, \\ &B_{ij}(M_1, M_2) = \overline{v_i(t_1, \mathbf{x}_1, \mathbf{v}_0) v_j(t_2, \mathbf{x}_2, \mathbf{v}_0)} = \\ &\int_{\mathbb{K}} v_i(t_1, \mathbf{x}_1, \mathbf{v}_0) v_j(t_2, \mathbf{x}_2, \mathbf{v}_0) d\mu(\mathbf{v}_0), \end{aligned} \tag{4.42}$$

denoting $M_i = (t_i, \mathbf{x}_i) \in Q_m$ $(i = 1, 2)$. More generally, let $M_1, \dots, M_n \in Q_m$. The correlation tensor $\mathbb{B}_n = \mathbb{B}_n(M_1, \dots, M_n)$ at these points is defined component by component by

$$B_{i_1 \dots i_n}(M_1, \dots, M_n) = \overline{\prod_{k=1}^{n} v_{i_k}(t_k, \mathbf{x}_k, \mathbf{v}_0)} = \int_{\mathbb{K}} \left(\prod_{k=1}^{n} v_{i_k}(t_k, \mathbf{x}_k, \mathbf{v}_0) \right) d\mu(\mathbf{v}_0), \tag{4.43}$$

which is the standard definition (cf. Batchelor [3]).

More general correlation tensors than those given by (4.43) are encountered when we look at the $k - \mathscr{E}$ model. To define these, we need to introduce the set

$$\mathscr{G} = \left\{ \begin{array}{c} v_1, v_2, v_3, p, \partial_j v_i \ (1 \le i, j \le 3)\}, \partial_t v_i \ (1 \le i \le 3), \\ \partial_i p \ (1 \le i \le 3), \partial^2_{ij} v_k \ \ (1 \le i, j, k \le 3) \end{array} \right\}, \tag{4.44}$$

called the complete field family, and

$$\mathscr{H} = \left\{ \begin{array}{c} v'_1, v'_2, v'_3, p, \partial_j v'_i \ (1 \le i, j \le 3)\}, \partial_t v'_i \ (1 \le i \le 3), \\ \partial_i p' \ (1 \le i \le 3), \partial^2_{ij} v'_k \ \ (1 \le i, j, k \le 3) \end{array} \right\}, \tag{4.45}$$

called the fluctuations field family. Each element of

$$\mathscr{F} = \mathscr{G} \cup \mathscr{H} \tag{4.46}$$

is Hölder continuous with respect to $(t, \mathbf{x}) \in Q_m$ and continuous with respect to $\mathbf{v}_0 \in \mathbb{K}$.

4.3.2 Homogeneity

4.3.2.1 Framework

In the following, $D = I \times \omega \subset Q_m$ denotes an open connected subset, such that $I \subset]0, \delta T_m[$ and $\omega \subset\subset \Omega$, which means that $\overline{\omega} \subset \mathring{\Omega}$. Let $M = (t, \mathbf{x}) \in D$, and denote $\tau_t > 0$ and $r_{\mathbf{x}} > 0$ the greatest real numbers such that

$$]t - \tau_t, t + \tau_t[\times B(\mathbf{x}, r_{\mathbf{x}}) \subset D.$$

For simplicity, we also denote

$$(t + \tau, \mathbf{x} + \mathbf{r}) = M + (\tau, \mathbf{r}), \quad (t, \mathbf{x} + \mathbf{r}) = M + \mathbf{r}. \tag{4.47}$$

We introduce in this subsection the concept of homogeneity in D, reflected in the local invariance under spatial translations of the correlation tensors based on the family $\mathscr{F} = \mathscr{G} \cup \mathscr{H}$.

As this concept appears to be too restrictive, we introduce a weaker mathematical concept called mild homogeneity, reflected in a formal property satisfied by second-order fluctuations correlations.

4.3.2.2 Homogeneous Flows

Any $\psi_1, \ldots, \psi_n \in \mathscr{F}$, $M_1, \ldots, M_n \in D$ being given, we set

$$B(\psi_1, \ldots, \psi_n)(M_1, \ldots, M_n) = \overline{\psi_1(M_1) \cdots \psi_n(M_n)}, \tag{4.48}$$

called an n-order correlation.

Definition 4.3. We say that the flow is homogeneous in D, if $\forall\, n \in \mathbb{N}$,

$$\forall\, M_1, \ldots, M_n \in D, \quad \forall\, \psi_1, \ldots, \psi_n \in \mathscr{F}, \quad \forall\, \mathbf{r} \in \mathbb{R}^3 \text{ such that } |\mathbf{r}| \leq \inf_{1 \leq i \leq n} r_{\mathbf{x}_i}, \tag{4.49}$$

we have

$$B(\psi_1, \ldots, \psi_n)(M_1 + \mathbf{r}, \ldots, M_n + \mathbf{r}) = B(\psi_1, \ldots, \psi_n)(M_1, \ldots, M_n), \tag{4.50}$$

B being defined by (4.48).

The following result is straightforward.

Lemma 4.4. *Assume that the flow is homogeneous. Let*

$$\psi_1, \ldots, \psi_n \in \mathscr{F}, \quad M_1, \ldots, M_n \in D, \quad M_i = (t_i, \mathbf{r}_i),$$

such that

$$\forall\, i = 1, \cdots, n, \quad t_i = t.$$

Let $\mathbf{r}_i$ *denote the vector such that* $M_i = M_n + \mathbf{r}_i$. *Then,* $B(\psi_1, \ldots, \psi_n)(M_1, \ldots, M_n)$ *only depends on* t *and* $\mathbf{r}_1, \cdots, \mathbf{r}_{n-1}$.

According to Lemma 4.4, we can denote

$$B(\psi_1, \ldots, \psi_n)(M_1, \ldots, M_n) = B(\psi_1, \ldots, \psi_n)(t; \mathbf{r}_1, \cdots, \mathbf{r}_{n-1}) \tag{4.51}$$

any n-order correlation, defined for $|\mathbf{r}_i| \leq r_{\mathbf{x}_n}$, $i = 1, \cdots, n-1$.

Mean fields of homogeneous flows are characterized in the following result.

Theorem 4.1. *Assume that* $\mathbf{f}$ *satisfies the compatibility condition* $\nabla \mathbf{f} = 0$ *in* D *and the flow is homogeneous in* D. *Then*

(i) $\forall\, \psi \in \mathscr{F}$, $\nabla \overline{\psi} = 0$,
(ii) $\nabla \boldsymbol{\sigma}^{(\mathrm{R})} = 0$ *in* D,
(iii) and we have $\forall\, t \in I$,

$$\overline{\mathbf{v}} = \overline{\mathbf{v}}(t) = \overline{\mathbf{v}}(t_0) + \int_{t_0}^{t} \mathbf{f}(s)\, ds \quad in \quad D, \tag{4.52}$$

by noting $t_0 = \inf I$.

Proof. (i) Take $n = 1$ in (4.50), and consider $\psi \in \mathscr{F}$. We find that

$$\forall\, \mathbf{r} \in \mathbb{R}^3 \text{ such that } |\mathbf{r}| \le r_{\mathbf{x}}, \quad \forall\, M \in D \text{ we have } \overline{\psi}(t, \mathbf{x}+\mathbf{r}) = \overline{\psi}(t, \mathbf{x}). \tag{4.53}$$

Therefore, for any fixed $t \in I$, $\overline{\psi}(t,\cdot)$ is constant in $B(\mathbf{x}, r_{\mathbf{x}})$, $\forall\, \mathbf{x} \in \omega$. As $\overline{\omega}$ is compact and connected, $\overline{\psi}$ is continuous with respect to $\mathbf{x}$, and we deduce that $\overline{\psi}(t,\cdot)$ is constant in ω, hence $\nabla\overline{\psi} = 0$ in D.

(ii) Let us consider

$$\begin{aligned} \tilde{\mathbb{B}}_2 = \tilde{\mathbb{B}}(M_1, M_2) &= (\tilde{B}_{ij}(M_1, M_2))_{1\le ij\le 3}, \\ \tilde{B}_{ij}(M_1, M_2) &= \overline{v_i'(t_1, \mathbf{x}_1, \mathbf{v}_0) v_j'(t_2, \mathbf{x}_2, \mathbf{v}_0)}. \end{aligned} \tag{4.54}$$

For our purpose, it is convenient to set, $\forall\, M = (t, \mathbf{x}) \in D$, $\forall\, \mathbf{r}$ s.t. $|\mathbf{r}| \le r_{\mathbf{x}}$,

$$\tilde{\mathbb{B}}_2(M, M+\mathbf{r}) = \tilde{\mathbb{B}}_2(t, \mathbf{x}, \mathbf{r}). \tag{4.55}$$

From the continuity and derivability properties of the flow, we deduce

$$\lim_{\mathbf{r}\to 0} \tilde{\mathbb{B}}_2(t, \mathbf{x}, \mathbf{r}) = \boldsymbol{\sigma}^{(\mathrm{R})}(t, \mathbf{x}), \quad \lim_{\mathbf{r}\to 0} \frac{\partial \tilde{\mathbb{B}}_2}{\partial \mathbf{x}}(t, \mathbf{x}, \mathbf{r}) = \nabla\boldsymbol{\sigma}^{(\mathrm{R})}(t, \mathbf{x}). \tag{4.56}$$

As the flow is homogeneous, $\tilde{\mathbb{B}}_2$ does not depend on $\mathbf{x}$; therefore,

$$\forall\, (t, \mathbf{x}, \mathbf{r}) \in D \times B(\mathbf{x}, r_{\mathbf{x}}), \quad \frac{\partial \tilde{\mathbb{B}}_2}{\partial \mathbf{x}}(t, \mathbf{x}, \mathbf{r}) = 0, \tag{4.57}$$

hence $\nabla\boldsymbol{\sigma}^{(\mathrm{R})} = 0$ in D.

(iii) We deduce from the above results that the momentum equation in system (4.40) becomes $\partial_t \overline{\mathbf{v}} = \mathbf{f}$, leading to (4.52).

□

Observe that one can take $I =]0, \delta T_m[$. However,

Proposition 4.1. *The only homogeneous flow in Q_m is a flow at rest.*

Proof. Indeed, given a time $t \in]0, \delta T_m[$, it follows from Theorem 4.1 that $\overline{\mathbf{v}}(t,\cdot)$ and $\mathbf{v}'(t,\cdot,\cdot)$ are constant in all Ω. As these fields are continuous, $\overline{\mathbf{v}}(t,\cdot) = 0$ in Ω because of the no-slip boundary condition on Γ. Moreover, since $\mathbf{v}' = \mathbf{v} - \overline{\mathbf{v}}$, then $\mathbf{v}'$ also satisfies the no-slip boundary condition on Γ, hence $\mathbf{v}'(t,\cdot) = 0$ in Ω, which yields $\mathbf{v} = 0$ in Q_m which characterizes a flow at rest. Note that this makes sense only if $\mathbf{f} = 0$, due to the formula (4.52). □

Proposition 4.1 implies that no turbulence model based on homogeneity makes sense throughout Ω, mainly because the flow in the boundary layer near the wall Γ

is not homogeneous. Roughly speaking, the boundary layer is the flow region driven by the friction at the boundary (also called wall). The structure of this boundary layer will be studied in Sect. 5.3.

Specifically, one decomposes Ω into two regions: the boundary layer $\mathscr{BL} \subset \Omega$ in a neighborhood of Γ, characterized by a log law, and $\Omega_c = \Omega \setminus \mathscr{BL}$ being the computational domain, assuming that the flow satisfies some homogeneity property in $D = [0, \delta T_m] \times \Omega_c$. In addition, one has $\Omega = \mathscr{BL} \cup \Omega_c$, $\overset{\circ}{\mathscr{BL}} \cap \Omega_c = \emptyset$. In D, a model such as the $k - \mathscr{E}$ or the Prandtl–Kolmogorov–Smagorinsky model might be used. In this manner, the mean field satisfies on $\partial\Omega_c$ a modeled boundary condition called a wall law, expressed in Sect. 5.3.

Remark 4.3. The usual definitions of homogeneity are used to replace the family $\mathscr{F}$ (expressed by (4.46)) by $\mathscr{G}$ [complete field family (4.44)] in Definition 4.3. Our choice is motivated by the fact that using $\mathscr{F}$ instead of $\mathscr{G}$ allows a complete description of such homogeneous flows (see also item (4.4.4.3) in the Sect. 4.4.4 below). However, in the Sect. 5.2, we use $\mathscr{G}$ in the definition of homogeneity to define isotropic flows. Observe finally that we may also replace $\mathscr{F}$ by $\mathscr{H}$ [fluctuation field family (4.45)] in Definition 4.3, leading to another structure that we shall not consider here.

4.3.3 Mild Homogeneity

Homogeneous flows do exist, but their means are only time dependent in the homogeneity region and do not reflect all turbulent flows that can be encountered in nature. In view of the derivation of the $k - \mathscr{E}$ model, we consider a weaker notion, which appears to be the least restrictive mathematical assumption that allows this derivation, although it is a formal artifact. Standard assumptions on turbulence (ergodicity, Gaussian statistics, isotropy of the fluctuations, etc.) allow similar formal calculations, but with technical complications (see the discussion in Sect. 4.4.4.2).

Definition 4.4. We say that a flow is mildly homogenous in

$$D = I \times \omega, \quad \text{where} \quad I \subset]0, \delta T_m[, \quad \text{and} \quad \omega \subset\subset \Omega,$$

if $\forall\, \psi, \varphi \in \mathscr{H}$ [fluctuations family (4.45)], we have

$$\forall\, M = (t, \mathbf{x}) \in D, \quad \overline{\psi(t,\mathbf{x})\partial_i \varphi(t,\mathbf{x})} = -\overline{\partial_i \psi(t,\mathbf{x})\varphi(t,\mathbf{x})}. \tag{4.58}$$

The following is straightforward.

Lemma 4.5. *Assume a flow is mildly homogeneous. Given any matrix* $A = (A_{ij})_{1\le ij\le 3}$*,* $A_{ij} \in \mathscr{H}$*, any vector field* $\mathbf{w} = (w_1, w_2, w_3)$*,* $w_i \in \mathscr{H}$*, and any scalar field* $q \in \mathscr{H}$*, we have* $\forall\, (t, \mathbf{x}) \in D$*,*

$$\overline{(\nabla \cdot A) \cdot \mathbf{w}} = -\overline{A : \nabla \mathbf{w}}, \quad \overline{\nabla q \cdot \mathbf{w}} = -\overline{q(\nabla \cdot \mathbf{w})}. \tag{4.59}$$

This definition is motivated by:

Lemma 4.6. *Any homogenous flow is mildly homogeneous.*

Proof. We deduce from (4.50) that if a flow is homogeneous, then

$$\forall\, \psi, \varphi \in \mathscr{H}, \quad \forall\, M = (t, \mathbf{x}) \in D, \quad \forall\, \mathbf{r} \in \mathbb{R}^3 \text{ such that } |\mathbf{r}| \leq r_{\mathbf{x}},$$

we have

$$\overline{\psi(t, \mathbf{x} + \mathbf{r})\varphi(t, \mathbf{x})} = \overline{\psi(t, \mathbf{x})\varphi(t, \mathbf{x} - \mathbf{r})}, \tag{4.60}$$

$$\overline{\psi(t, \mathbf{x})\, \varphi(t, \mathbf{x} + \mathbf{r})} = \overline{\psi(t, \mathbf{x} - \mathbf{r})\varphi(t, \mathbf{x})}. \tag{4.61}$$

Furthermore,

$$\overline{\psi(t, \mathbf{x})\partial_i \varphi(t, \mathbf{x})} = \lim_{r \to 0} \frac{1}{2r} \overline{\psi(t, \mathbf{x})\varphi(t, \mathbf{x} + r\mathbf{e}_i) - \psi(t, \mathbf{x})\varphi(t, \mathbf{x} - r\mathbf{e}_i)}. \tag{4.62}$$

Formula (4.58) is then satisfied following (4.62) combined with (4.60) and (4.61). □

Remark 4.4. Formula (4.58) is similar to the usual integration by parts in the case of homogeneous boundary conditions,

$$\forall\, f \in H_0^1(\Omega), \forall\, g \in H_0^1(\Omega), \quad \int_\Omega f\, \partial_i g = -\int_\Omega g\, \partial_i f. \tag{4.63}$$

4.4 Derivation of the $k - \mathscr{E}$ Model

Throughout the rest of this section, we suppose that the flow $(\mathbf{v}, p)$ defined in $Q_m \times \mathbb{K}$ is mildly homogenous in a given region $D = I \times \omega$. Our goal is to derive the famous $k - \mathscr{E}$ model (coupled system (4.126) below) in the considered region D.

As well as the Boussinesq-like assumptions and dimensional analysis, the usual hypotheses about turbulence involved in the original derivation of the $k - \mathscr{E}$ model are ergodicity, Gaussian statistics, and isotropy of the fluctuations [20]. We replace these assumptions by the mild homogeneity assumption and will discuss this choice in Sect. 4.4.4.

Though we have tried to minimize the list of sufficient conditions to derive the model, we do not know if it is the list of necessary conditions. This still remains a grey area about the $k - \mathscr{E}$ model, raising many open mathematical questions.

4.4.1 Turbulent Kinetic Energy Equation

The TKE k was already encountered in Sect. 3.5.3.2. It is defined at $(t, \mathbf{x}) \in Q_m$ by

$$k = \frac{1}{2} tr \boldsymbol{\sigma}^{(\mathrm{R})} = \frac{1}{2} \overline{|\mathbf{v}'|^2} = \frac{1}{2} \int_{\mathbb{K}} |\mathbf{v}'(t, \mathbf{x}, \mathbf{v}_0)|^2 d\mu(\mathbf{v}_0) = \frac{1}{2} E((\mathbf{v} - E(\mathbf{v}))^2), \quad (4.64)$$

where $\boldsymbol{\sigma}^{(\mathrm{R})} = \overline{\mathbf{v}' \otimes \mathbf{v}'}$ is the Reynolds stress already defined by (4.39). The field k is the mean kinetic energy of the flow fluctuations. Apart from the factor $1/2$, it corresponds to the variance of the random vector $\mathbf{v}(t, \mathbf{x}, \mathbf{v}_0)$ at any fixed point $(t, \mathbf{x})$ and is the probabilistic tool that measures how far the flow is from its mean value at $(t, \mathbf{x})$.

The TKE is an essential tool in turbulence modeling, used to measure the intensity of the turbulence. Indeed, if the TKE increases around a given point $(t, \mathbf{x})$, so do the fluctuations, and the flow becomes more turbulent in the neighborhood of $(t, \mathbf{x})$.

We aim at finding an equation for the TKE. It would seem that the minimal requirement to do so is to assume that the turbulence is mildly homogeneous as stated in Definition 4.4. We start with some preliminaries.

Let e denote the kinetic energy of the fluctuation, defined by

$$e = \frac{1}{2} |\mathbf{v}'|^2 = e(t, \mathbf{x}, \mathbf{v}_0), \quad (4.65)$$

satisfying $k = \overline{e}$, so that $e = k + e'$. The strategy to find the TKE equation is

(i) to write the equation satisfied by the fluctuating field $(\mathbf{v}', p')$,
(ii) to infer from that an equation for e,
(iii) to take the expectation of that equation.

To state the main result of this subsection, we first need to introduce the dissipation

$$\varepsilon = 2\nu |D\mathbf{v}|^2. \quad (4.66)$$

Arguing again as in Sect. 4.2.3, item (b), we find

$$\overline{\varepsilon} = 2\nu |D\overline{\mathbf{v}}|^2 + \overline{\varepsilon'}, \quad (4.67)$$

where we have denoted

$$\varepsilon' = 2\nu |D\mathbf{v}'|^2. \quad (4.68)$$

Theorem 4.2. *The TKE k satisfies in the region D the equation*

$$\partial_t k + \overline{\mathbf{v}} \cdot \nabla k + \nabla \cdot \overline{e'\mathbf{v}'} = -\boldsymbol{\sigma}^{(\mathrm{R})} : \nabla \overline{\mathbf{v}} - \overline{\varepsilon'}, \quad (4.69)$$

Proof. We start from the Reynolds decomposition $\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}'$, $p = \overline{p} + p'$. We subtract system (4.40) satisfied by $(\overline{\mathbf{v}}, \overline{p})$ from the NSE, leading to

$$\begin{cases} \partial_t \mathbf{v}' + \mathbf{v} \cdot \nabla \mathbf{v}' + \mathbf{v}' \cdot \nabla \overline{\mathbf{v}} - \nabla \cdot (2\nu D\mathbf{v}') + \nabla p' - \nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})} = 0, \\ \nabla \cdot \mathbf{v}' = 0. \end{cases} \tag{4.70}$$

The dot product of the momentum equation in (4.70) with $\mathbf{v}'$ yields the following equation for e

$$\partial_t e + \mathbf{v} \cdot \nabla e + (\mathbf{v}' \otimes \mathbf{v}') : \nabla \overline{\mathbf{v}} - (\nabla \cdot (2\nu D\mathbf{v}')) \cdot \mathbf{v}' + \nabla p' \cdot \mathbf{v}' - (\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}) \cdot \mathbf{v}' = 0. \tag{4.71}$$

We take the expectation of this equation and study each term one after the other (from right to left in the equation). From $\overline{\mathbf{v}'} = 0$ and since $\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}$ does not depend on $\mathbf{v}_0$, we deduce

$$\overline{(\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}) \cdot \mathbf{v}'} = (\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}) \cdot \overline{\mathbf{v}'} = 0. \tag{4.72}$$

Because $p', \mathbf{v}', \partial_i \mathbf{v}' \in \mathscr{H}$, the mild homogeneity assumption, in particular formula (4.59), yields

$$\overline{\nabla p' \cdot \mathbf{v}'} = -\overline{p' \nabla \cdot \mathbf{v}'} = 0, \tag{4.73}$$

as well as

$$-\overline{(\nabla \cdot (2\nu D\mathbf{v}')) \cdot \mathbf{v}'} = 2\nu \overline{D\mathbf{v}' : \nabla \mathbf{v}'} = 2\nu \overline{D\mathbf{v}' : D\mathbf{v}'} = \overline{\varepsilon'}. \tag{4.74}$$

Furthermore, we also have

$$\overline{(\mathbf{v}' \otimes \mathbf{v}') : \nabla \overline{\mathbf{v}}} = \overline{(\mathbf{v}' \otimes \mathbf{v}')} : \nabla \overline{\mathbf{v}} = \boldsymbol{\sigma}^{(\mathrm{R})} : \nabla \overline{\mathbf{v}}. \tag{4.75}$$

Finally, using the same method as in Sect. 4.2.3.2, which highlights the Reynolds stress, we find

$$\overline{e\mathbf{v}} = k\overline{\mathbf{v}} + \overline{e'\mathbf{v}'}. \tag{4.76}$$

Using $\mathbf{v} \cdot \nabla k = \nabla \cdot (k\mathbf{v})$ since $\nabla \cdot \mathbf{v} = 0$, we deduce from identity (4.76),

$$\overline{\mathbf{v} \cdot \nabla e} = \overline{\mathbf{v}} \cdot \nabla k + \nabla \cdot \overline{e'\mathbf{v}'}. \tag{4.77}$$

Therefore, (4.69) results from the identity $\overline{\partial_t e} = \partial_t k$ [Reynolds rule (4.24)] combined with (4.71)–(4.75) and (4.77). $\square$

4.4.2 The Epsilon Equation

The TKE equation (4.69) raises many issues, such as the determination of the source term $\overline{\varepsilon'} = 2\nu\overline{|D\mathbf{v}'|^2}$, which is the mean dissipation of the fluctuations. From now on, we set

$$\mathscr{E} = \overline{\varepsilon'} = 2\nu\overline{|D\mathbf{v}'|^2}, \tag{4.78}$$

according to common use. The goal of this subsection is to find an equation for $\mathscr{E}$.

We shall often use in the following the formulas (4.58) and (4.59) because of the mild homogeneity assumption. This will not be systematically mentioned.

4.4.2.1 Preliminaries

We have decomposed the vorticity as $\boldsymbol{\omega} = \overline{\boldsymbol{\omega}} + \boldsymbol{\omega}'$ where $\overline{\boldsymbol{\omega}} = \nabla \times \overline{\mathbf{v}}$, $\boldsymbol{\omega}' = \nabla \times \mathbf{v}'$, $\overline{\boldsymbol{\omega}'} = 0$.

Lemma 4.7. *Let $\mathscr{E}$ be the mean dissipation of the fluctuation. Then $\mathscr{E}$ is related to $\boldsymbol{\omega}'$ by the relation*

$$\mathscr{E} = \nu\overline{|\boldsymbol{\omega}'|^2}. \tag{4.79}$$

Proof. As $\mathbf{v}' \in \mathscr{H}$ and $\nabla \cdot \mathbf{v}' = 0$, we have

$$\overline{|D\mathbf{v}'|^2} = -\overline{\mathbf{v}' \cdot \nabla \cdot (D\mathbf{v}')} = -\frac{1}{2}\overline{\mathbf{v}' \cdot \Delta\mathbf{v}'} = \frac{1}{2}\overline{|\nabla\mathbf{v}'|^2}. \tag{4.80}$$

We next use the general formula $\nabla\times\nabla\times\mathbf{w} = \nabla(\nabla\cdot\mathbf{w}) - \Delta\mathbf{w}$, reducing to $\nabla\times\nabla\times\mathbf{w} = -\Delta\mathbf{w}$ for zero divergence fields. We infer from $\boldsymbol{\omega}' \in \mathscr{H}$,

$$\overline{\mathbf{v}' \cdot \nabla \times \nabla \times \mathbf{v}'} = \overline{|\nabla \times \mathbf{v}'|^2} = \overline{|\boldsymbol{\omega}'|^2} = -\overline{\mathbf{v}' \cdot \Delta\mathbf{v}'} = \overline{|\nabla\mathbf{v}'|^2}, \tag{4.81}$$

hence (4.79), due to (4.68) combined with (4.80). □

Formula (4.79) highlights a strong connection between k and $\mathscr{E}$. Indeed, apart from the factor ν, $\mathscr{E}$ measures the variance of the vorticity. The TKE equation (4.69) reveals that turbulence provides energy to fluctuations.

4.4.2.2 Derivation of the Equation

We are now able to determine an equation for $\mathscr{E}$. According to formula (4.79), we first search for the equation satisfied by the vorticity fluctuation $\boldsymbol{\omega}'$.

Lemma 4.8. *The vorticity fluctuation $\boldsymbol{\omega}'$ satisfies in D,*

$$\partial_t \boldsymbol{\omega}' + (\mathbf{v}\cdot\nabla)\,\boldsymbol{\omega}' + (\mathbf{v}'\cdot\nabla)\,\overline{\boldsymbol{\omega}} - (\boldsymbol{\omega}\cdot\nabla)\,\mathbf{v}' - (\boldsymbol{\omega}'\cdot\nabla)\,\overline{\mathbf{v}} - \nu\Delta\boldsymbol{\omega}' = \nabla\times(\nabla\cdot\boldsymbol{\sigma}^{(\mathrm{R})}) \tag{4.82}$$

Proof. We recall that $\boldsymbol{\omega}$ satisfies the equation (see Sect. 2.6.5),

$$\partial_t \boldsymbol{\omega} + (\mathbf{v}\cdot\nabla)\,\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\,\mathbf{v} - \nu\Delta\boldsymbol{\omega} = \nabla\times\mathbf{f}. \tag{4.83}$$

Since $\overline{\boldsymbol{\omega}} = \nabla\times\overline{\mathbf{v}}$, we obtain from the mean NSE (4.40),

$$\partial_t \overline{\boldsymbol{\omega}} + (\overline{\mathbf{v}}\cdot\nabla)\,\overline{\boldsymbol{\omega}} - (\overline{\boldsymbol{\omega}}\cdot\nabla)\,\overline{\mathbf{v}} - \nu\Delta\overline{\boldsymbol{\omega}} = -\nabla\times(\nabla\cdot\boldsymbol{\sigma}^{(\mathrm{R})}) + \nabla\times\mathbf{f}. \tag{4.84}$$

As $\boldsymbol{\omega}' = \boldsymbol{\omega} - \overline{\boldsymbol{\omega}}$, (4.82) is obtained by subtracting (4.84) from (4.83). □

Lemma 4.8 yields:

Lemma 4.9. *The equation satisfied by $\mathscr{E}$ is the following:*

$$\partial_t\mathscr{E} + \overline{\mathbf{v}}\cdot\nabla\mathscr{E} + \nabla\cdot\overline{\nu h'\mathbf{v}'} = 2\nu(\overline{\boldsymbol{\omega}'\otimes\boldsymbol{\omega}'}:\nabla\overline{\mathbf{v}} + \overline{(\boldsymbol{\omega}'\otimes\boldsymbol{\omega}')':\nabla\mathbf{v}'}) - 2\nu^2\overline{|\nabla\boldsymbol{\omega}'|^2}, \tag{4.85}$$

in D.

Proof. Since

$$(\mathbf{v}'\cdot\nabla)\,\overline{\boldsymbol{\omega}} - (\overline{\boldsymbol{\omega}}\cdot\nabla)\,\mathbf{v}' = -\nabla\times(\mathbf{v}'\times\overline{\boldsymbol{\omega}}), \tag{4.86}$$

equation (4.82) becomes

$$\partial_t \boldsymbol{\omega}' + (\mathbf{v}\cdot\nabla)\,\boldsymbol{\omega}' - \nabla\times(\mathbf{v}'\times\overline{\boldsymbol{\omega}}) - (\boldsymbol{\omega}'\cdot\nabla)\,\mathbf{v}' - (\boldsymbol{\omega}'\cdot\nabla)\,\overline{\mathbf{v}} - \nu\Delta\boldsymbol{\omega}' = \nabla\times(\nabla\cdot\boldsymbol{\sigma}^{(\mathrm{R})}). \tag{4.87}$$

Denote

$$h = |\boldsymbol{\omega}'|^2, \quad \text{decomposed as } h = \overline{h} + h'. \tag{4.88}$$

Note that $\mathscr{E} = \nu\overline{h}$. We take the dot product of (4.87) with $\boldsymbol{\omega}'$ which gives

$$\frac{1}{2}\partial_t h + \frac{1}{2}\mathbf{v}\cdot\nabla h - \boldsymbol{\omega}'\cdot\nabla\times(\mathbf{v}'\times\overline{\boldsymbol{\omega}}) = (\boldsymbol{\omega}'\otimes\boldsymbol{\omega}'):\nabla\overline{\mathbf{v}} + \nu\Delta\boldsymbol{\omega}'\cdot\boldsymbol{\omega}' + (\boldsymbol{\omega}'\otimes\boldsymbol{\omega}'):\nabla\mathbf{v}' + \nabla\times(\nabla\cdot\boldsymbol{\sigma}^{(\mathrm{R})})\cdot\boldsymbol{\omega}'. \tag{4.89}$$

We then take the expectation of this equation, and we study each term one after the other, from the simplest to the most complicated, the simplest being

$$\overline{\nabla\times(\nabla\cdot\boldsymbol{\sigma}^{(\mathrm{R})})\cdot\boldsymbol{\omega}'} = \nabla\times(\nabla\cdot\boldsymbol{\sigma}^{(\mathrm{R})})\cdot\overline{\boldsymbol{\omega}'} = 0. \tag{4.90}$$

The next simplest term is the transport term. We have already met similar terms before in the TKE equation and the mean field equation. Skipping the details, we find

$$\overline{\mathbf{v} \cdot \nabla h} = \overline{\mathbf{v}} \cdot \nabla \overline{h} + \nabla \cdot \overline{h' \mathbf{v}'}. \tag{4.91}$$

Moreover, $\boldsymbol{\omega}' \in \mathscr{H}$ yields

$$\overline{\Delta \boldsymbol{\omega}' \cdot \boldsymbol{\omega}'} = -\overline{|\nabla \boldsymbol{\omega}'|^2}. \tag{4.92}$$

Next, from the Reynolds decomposition,

$$\boldsymbol{\omega}' \otimes \boldsymbol{\omega}' = \overline{\boldsymbol{\omega}' \otimes \boldsymbol{\omega}'} + (\boldsymbol{\omega}' \otimes \boldsymbol{\omega}')', \tag{4.93}$$

$\overline{(\boldsymbol{\omega}' \otimes \boldsymbol{\omega}')'} = 0$ and since $\nabla \overline{\mathbf{v}}$ does not depend on $\mathbf{v}_0$, we obtain

$$\overline{(\boldsymbol{\omega}' \otimes \boldsymbol{\omega}') : \nabla \overline{\mathbf{v}}} = \overline{\boldsymbol{\omega}' \otimes \boldsymbol{\omega}'} : \nabla \overline{\mathbf{v}}, \tag{4.94}$$

and since $\overline{\boldsymbol{\omega}' \otimes \boldsymbol{\omega}'}$ does not depend on $\mathbf{v}_0$ either,

$$\overline{(\boldsymbol{\omega}' \otimes \boldsymbol{\omega}') : \nabla \mathbf{v}'} = \overline{(\boldsymbol{\omega}' \otimes \boldsymbol{\omega}')' : \nabla \mathbf{v}'}. \tag{4.95}$$

The last term, $\overline{\boldsymbol{\omega}' \cdot \nabla \times (\mathbf{v}' \times \overline{\boldsymbol{\omega}})}$, is more tricky. Fortunately it vanishes as we now show. Since $\boldsymbol{\omega}', \mathbf{v}' \in \mathscr{H}$ we have

$$\overline{\boldsymbol{\omega}' \cdot \nabla \times (\mathbf{v}' \times \overline{\boldsymbol{\omega}})} = \overline{(\mathbf{v}' \times \overline{\boldsymbol{\omega}}) \cdot \nabla \times \boldsymbol{\omega}'} = \overline{(\mathbf{v}' \times \overline{\boldsymbol{\omega}}) \cdot (\nabla \times \nabla \times \mathbf{v}')}. \tag{4.96}$$

A component-by-component calculation as detailed in (4.98) below shows that the components of $\overline{\boldsymbol{\omega}}$ are constants since $\overline{\boldsymbol{\omega}}$ does not depend on $\mathbf{v}_0$, hence formula (4.58) leads to the result. Moreover, as $\nabla \cdot \mathbf{v}' = 0$, then $\nabla \times \nabla \times \mathbf{v}' = -\Delta \mathbf{v}'$, leading to

$$\overline{\boldsymbol{\omega}' \cdot \nabla \times (\mathbf{v}' \times \overline{\boldsymbol{\omega}})} = -\overline{(\mathbf{v}' \times \overline{\boldsymbol{\omega}}) \cdot \Delta \mathbf{v}'}. \tag{4.97}$$

In writing $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ and since $\overline{\boldsymbol{\omega}}$ does not depend on $\mathbf{v}_0$, we have

$$\begin{aligned} \overline{(\mathbf{v}' \times \overline{\boldsymbol{\omega}}) \cdot \Delta \mathbf{v}'} &= \\ \overline{\omega_3}\, \overline{v_2' \Delta v_1'} - \overline{\omega_2}\, \overline{v_3' \Delta v_1'} + \overline{\omega_1}\, \overline{v_3' \Delta v_2'} - \overline{\omega_3}\, \overline{v_1' \Delta v_2'} + \overline{\omega_2}\, \overline{v_1' \Delta v_3'} - \overline{\omega_1}\, \overline{v_2' \Delta v_3'} &= 0. \end{aligned} \tag{4.98}$$

We conclude the proof by using the identity $\mathscr{E} = \nu \overline{h}$ and combining (4.89)–(4.92), (4.94), (4.95), and (4.98). □

4.4.2.3 Summary of the Equations

Gathering together the k and $\mathscr{E}$ equations yields the system

$$\begin{cases} \partial_t k + \overline{\mathbf{v}} \cdot \nabla k + \nabla \cdot \overline{e'\mathbf{v}'} = -\sigma^{(R)} : \nabla \overline{\mathbf{v}} - \mathscr{E}, \\ \partial_t \mathscr{E} + \overline{\mathbf{v}} \cdot \nabla \mathscr{E} + \nabla \cdot \overline{\nu h' \mathbf{v}'} = 2\nu (\overline{\boldsymbol{\omega}' \otimes \boldsymbol{\omega}'} : \nabla \overline{\mathbf{v}} + \overline{(\boldsymbol{\omega}' \otimes \boldsymbol{\omega}')' : \nabla \mathbf{v}'}) \\ \qquad -2\nu^2 \overline{|\nabla \boldsymbol{\omega}'|^2}, \end{cases} \tag{4.99}$$

where

$$e = \frac{1}{2}|\mathbf{v}|^2, \quad h = |\boldsymbol{\omega}|^2, \quad k = \frac{1}{2}\overline{|\mathbf{v}'|^2}, \quad \mathscr{E} = \nu \overline{|\boldsymbol{\omega}'|^2}. \tag{4.100}$$

These equations have similar structure. Indeed, the tensor $-\overline{\boldsymbol{\omega}' \otimes \boldsymbol{\omega}'}$ in the $\mathscr{E}$-equation plays the same role as $\boldsymbol{\sigma}^{(\mathrm{R})}$ in the k-equation, while $2\nu^2 \overline{|\nabla \boldsymbol{\omega}'|^2}$ in the $\mathscr{E}$-equation plays the same role as $\mathscr{E}$ in the k-equation. However, the term $\overline{(\boldsymbol{\omega}' \otimes \boldsymbol{\omega}')' : \nabla \mathbf{v}'}$ in the $\mathscr{E}$-equation creates a difference between them.

4.4.2.4 Dimensional Analysis

We observe that

$$\mathbb{D}(k) = (2, -2), \quad \mathbb{D}(\mathscr{E}) = (2, -3). \tag{4.101}$$

Therefore, k and $\mathscr{E}$ are dimensionally independent. Thus, according to Lemma 3.1, at each point $(t, \mathbf{x}) \in Q_m$, we can form a length–time dimensional basis, denoted as

$$b(k, \mathscr{E})(t, \mathbf{x}) = (\ell(t, \mathbf{x}), \theta(t, \mathbf{x})) \tag{4.102}$$

where

$$\ell = \ell(t, \mathbf{x}) = \frac{k^{3/2}(t, \mathbf{x})}{\mathscr{E}(t, \mathbf{x})}, \quad \theta = \theta(t, \mathbf{x}) = \frac{k(t, \mathbf{x})}{\mathscr{E}(t, \mathbf{x})}, \tag{4.103}$$

insofar as those quantities are well defined. Usually, it is postulated that ℓ given by formula (4.103) is the Prandtl mixing length, first introduced in Sect. 3.5.3.2. Historically, the practical calculation of the Prandtl mixing length has for a long time been a major concern. No satisfactory equation was found before the work by Launder and Spalding [16], who assumed that ℓ is a function of k and $\mathscr{E}$, expressed by (4.103). Instead of deriving an equation for ℓ, they considered that it is more tractable to find an equation for $\mathscr{E}$, the equation for k involving ℓ, having already been found before by Kolmogorov [25].

4.4.3 Closure Assumptions

System (4.99) cannot be used in that particular form for numerical computations, due to the fluctuating terms $\overline{e'\mathbf{v}'}$, $\overline{\nu h'\mathbf{v}'}$, $\overline{|\nabla\boldsymbol{\omega}'|^2}$. To deal with these, there are two options:

(i) To derive equations for them from the NSE,
(ii) By using suitable physical assumptions, to express them in terms of $\mathbf{v}$, k, and $\mathscr{E}$ to close the system.

We chose option (ii). We first discuss on Boussinesq assumption and then introduce the turbulent diffusion coefficients that allow us to close the equations.

4.4.3.1 Boussinesq Assumption

Remember that our initial aim was to find a PDE system satisfied by the mean field $(\overline{\mathbf{v}}, \overline{p})$, which is subject to the determination of the Reynolds stress $\boldsymbol{\sigma}^{(\mathrm{R})} = \overline{\mathbf{v}' \otimes \mathbf{v}'}$, as already mentioned in Sects. 3.5.3.2 and 4.2.3.

According to Theorem 3.5, which concludes that $\boldsymbol{\sigma}^{(\mathrm{R})}$ is dissipative in the long-time average case, we took into account the Boussinesq assumption that led us to write

$$\boldsymbol{\sigma}^{(\mathrm{R})} = -\nu_t D\mathbf{v} + \frac{2}{3}k\mathrm{I}, \tag{4.104}$$

where $\nu_t > 0$ is the eddy viscosity. We assume that (4.104) still holds in the statistical case. A priori, ν_t is not constant and depends on $(t, \mathbf{x})$. Therefore, system (4.35) becomes

$$\left\{\begin{aligned} \partial_t\overline{\mathbf{v}} + (\overline{\mathbf{v}}\cdot\nabla)\,\overline{\mathbf{v}} - \nabla\cdot((\nu+\nu_t)D\overline{\mathbf{v}}) + \nabla(\overline{p} - (2/3)k) &= \mathbf{f} && \text{in } Q_m,\\ \nabla\cdot\overline{\mathbf{v}} &= 0 && \text{in } Q_m,\\ \overline{\mathbf{v}} &= 0 && \text{on } \Gamma,\\ \overline{\mathbf{v}} &= \overline{\mathbf{v}_0} && \text{at } t = 0, \end{aligned}\right. \tag{4.105}$$

while the k-equation becomes in D,

$$\partial_t k + \overline{\mathbf{v}}\cdot\nabla k + \nabla\cdot\overline{e'\mathbf{v}'} = \nu_t|D\overline{\mathbf{v}}|^2 - \mathscr{E}, \tag{4.106}$$

noting that $k\mathrm{I} : \nabla\overline{\mathbf{v}} = k\,\nabla\cdot\overline{\mathbf{v}} = 0$ and $D\overline{\mathbf{v}} : \nabla\overline{\mathbf{v}} = |D\overline{\mathbf{v}}|^2$.

The important point is the determination of the eddy viscosity ν_t. As already mentioned in Sect. 3.5.3, one possibility is to consider that it is the function of k and ℓ, which we take for granted in the statistical framework of this chapter. Therefore, it is seen from dimensional analysis that

$$\nu_t = \nu_t(k, \ell) = c_\nu\ell\sqrt{k}, \tag{4.107}$$

where $c_v > 0$ is a dimensionless constant that must be fixed according to experimental data. To be more specific,

$$\forall\,(t, \mathbf{x}) \in Q_m, \quad \nu_t(\ell, k)(t, \mathbf{x}) = \nu_t(\ell(t, \mathbf{x}), k(t, \mathbf{x})) = c_v \ell(t, \mathbf{x})\sqrt{k(t, \mathbf{x})}. \quad (4.108)$$

In other words, from (4.103) that expresses ℓ in terms of k and $\mathscr{E}$, we deduce

$$\nu_t = \nu_t(k, \mathscr{E}) = c_v \frac{k^2}{\mathscr{E}}, \tag{4.109}$$

insofar as this quantity makes sense. To close the model, it remains:

(i) to deal with the convection fluctuating terms

$$\nabla \cdot \overline{e'\mathbf{v}'} = \overline{\mathbf{v}' \cdot \nabla e'} \text{ and } \nabla \cdot \overline{\nu h'\mathbf{v}'} = \nu\, \overline{\mathbf{v}' \cdot \nabla h'}$$

in system (4.99),

(ii) to model the r.h.s of the $\mathscr{E}$-equation.

From here on, we shall use dimensional analysis to determine various coefficients in terms of k and $\mathscr{E}$. Dimensional constants appear in this process. They must be fixed from experiment, which will not be systematically mentioned.

4.4.3.2 Turbulent Diffusion Coefficients

Problems of convection by random or periodic fluctuating fields has attracted much attention over the last few decades [1, 4, 7, 15, 18, 28]. It has been rigorously proven for many cases that convection of passive scalars by fluctuating velocity fields yields diffusion of the corresponding means. These results are closely linked to the Boussinesq approximation.

Unfortunately, there is no mathematical evidence that this holds in the case of the $k - \mathscr{E}$ equations, since neither e' nor h' is a passive scalar. In any case, we shall assume that there exist turbulent diffusion coefficients $\mu_{t,k} > 0$ and $\mu_{t,\varepsilon} > 0$ such that

$$\nabla \cdot \overline{e'\mathbf{v}'} = -\nabla \cdot (\mu_{t,k} \nabla \overline{e'}) = -\nabla \cdot (\mu_{t,k} \nabla k), \tag{4.110}$$

$$\nabla \cdot \overline{\nu h'\mathbf{v}'} = -\nabla \cdot (\mu_{t,\varepsilon} \nabla \overline{h'}) = -\nabla \cdot (\mu_{t,\varepsilon} \nabla \mathscr{E}). \tag{4.111}$$

Following the logic outlined in the previous subsections, we set

$$\mu_{t,k} = \mu_{t,k}(k, \mathscr{E}) = c_k \frac{k^2}{\mathscr{E}}, \quad \mu_{t,\varepsilon} = \mu_{t,\varepsilon}(k, \varepsilon) = c_\varepsilon \frac{k^2}{\mathscr{E}}, \tag{4.112}$$

where $c_k > 0$ and $c_\varepsilon > 0$ are dimensionless constants.

From (4.106) combined with closure formulas (4.109) and (4.112), we find the following model equation for k

$$\partial_t k + \overline{\mathbf{v}} \cdot \nabla k - \nabla \cdot \left(c_k \frac{k^2}{\mathscr{E}} \nabla k \right) = c_\nu \frac{k^2}{\mathscr{E}} |D\overline{\mathbf{v}}|^2 - \mathscr{E} \tag{4.113}$$

that holds in the mild homogeneity region D.

4.4.3.3 Closure of $\mathscr{E}$-Equation

To complete the model, we are left with modeling the r.h.s of the $\mathscr{E}$-equation,

$$2\nu(\overline{\boldsymbol{\omega}' \otimes \boldsymbol{\omega}'}) : \nabla\overline{\mathbf{v}} + \overline{(\boldsymbol{\omega}' \otimes \boldsymbol{\omega}')' : \nabla \mathbf{v}'}) - 2\nu^2 \overline{|\nabla \boldsymbol{\omega}'|^2}. \tag{4.114}$$

We consider each term consecutively.

- The tensor

$$\boldsymbol{\sigma}_{\boldsymbol{\omega}}^{(\mathrm{R})} = 2\nu\, \overline{\boldsymbol{\omega}' \otimes \boldsymbol{\omega}'} \tag{4.115}$$

  is a Reynolds-like tensor, so that the Boussinesq assumption applies to it. However, we have

$$\mathscr{E} = \frac{1}{2} tr \boldsymbol{\sigma}_{\boldsymbol{\omega}}^{(\mathrm{R})}, \tag{4.116}$$

  which leads us to assume that it is anti-dissipative, yielding a formula of the form

$$\boldsymbol{\sigma}_{\boldsymbol{\omega}}^{(\mathrm{R})} = \frac{\mathscr{E}}{6} \mathrm{I} + \eta_t D\overline{\mathbf{v}}, \tag{4.117}$$

  for some $\eta_t > 0$. Assuming $\eta_t = \eta_t(k, \varepsilon)$, we infer from dimensional analysis, skipping the details,

$$\eta_t = \eta_t(t, \mathbf{x}) = c_\eta k(t, \mathbf{x}). \tag{4.118}$$

  where $c_\eta > 0$ is a dimensionless constant. Hence, recycling earlier proofs, we come to

$$2\nu\, \overline{\boldsymbol{\omega}' \otimes \boldsymbol{\omega}'} : \nabla\overline{\mathbf{v}} = c_\eta k |D\overline{\mathbf{v}}|^2. \tag{4.119}$$

- The term $\overline{(\boldsymbol{\omega}' \otimes \boldsymbol{\omega}')' : \nabla \mathbf{v}'}$ is a third-order correlation term. At this stage, there is no evidence and no physical reason that this term is either dissipative or anti-dissipative. Nevertheless, to remain consistent with the logic of a closure process based on the Boussinesq assumption, we assume that the tensor $(\boldsymbol{\omega}' \otimes \boldsymbol{\omega}')'$, which is of second order in terms of fluctuations, can be expressed in terms of the first-order fluctuation of the deformation, leading to

$$(\boldsymbol{\omega}' \otimes \boldsymbol{\omega}')' = \frac{1}{3}(|\boldsymbol{\omega}'|^2)'\mathrm{I} + \frac{1}{2}\gamma_t D\mathbf{v}', \tag{4.120}$$

without any assumption on the sign of the coefficient γ_t and apart from that it does not depend on $\mathbf{v}_0$. Therefore, by using calculations described earlier, we obtain

$$2\nu\overline{(\boldsymbol{\omega}' \otimes \boldsymbol{\omega}')' : \nabla\mathbf{v}')} = \gamma_t(2\nu\overline{|D\mathbf{v}'|^2}) = \gamma_t\mathscr{E}. \tag{4.121}$$

Since $\mathbb{D}(\gamma_t) = (0, -1)$, by assuming $\gamma_t = \gamma_t(k, \mathscr{E})$, which is the way to express those coefficients, we are led to set

$$\gamma_t = \gamma_t(t, \mathbf{x}) = \frac{c_\gamma}{\theta(t, \mathbf{x})} = c_\gamma \frac{\mathscr{E}(t, \mathbf{x})}{k(t, \mathbf{x})}, \tag{4.122}$$

where θ was specified by (4.103). Hence

$$2\nu\overline{(\boldsymbol{\omega}' \otimes \boldsymbol{\omega}')' : \nabla\mathbf{v}')} = c_\gamma \frac{\mathscr{E}^2}{k}, \tag{4.123}$$

c_γ being a dimensionless constant whose sign is unspecified.

- We are left with the term $\mathscr{E}_2 = 2\nu^2\overline{|\nabla\boldsymbol{\omega}'|^2}$, which is a dissipative term in the equation for $\mathscr{E}$. As this term closely resembles $\mathscr{E}$, we might be tempted to seek an additional equation for $\mathscr{E}_2$. We shall find an equation similar to the $\mathscr{E}$-equation, including third-order correlation terms, with third-order fluctuating terms, and a dissipative term $\mathscr{E}_3$ needing an additional equation, and so on: this addresses the issue of how many closure equations we need to calculate a mean field $(\overline{\mathbf{v}}, \overline{p})$ as accurately as possible.

A large number of numerical experiments carried out since the work by Launder and Spalding suggest that two closure equations are more than enough and an additional equations do not bring greater accuracy. This is why we may close $\mathscr{E}_2$ using dimensional analysis, by assuming that it only depends on k and $\mathscr{E}$ as we did for turbulent diffusion coefficients. This yields

$$\mathscr{E}_2 = \mathscr{E}_2(k, \mathscr{E}) = c_{\varepsilon_2} \frac{\mathscr{E}^2}{k}, \tag{4.124}$$

where $c_{\varepsilon_2} > 0$ is another dimensionless constant.

In conclusion, by gathering together (4.112), (4.119), (4.123), and (4.124), we find

$$\partial_t \mathscr{E} + \overline{\mathbf{v}} \cdot \nabla \mathscr{E} - \nabla \cdot \left(c_\varepsilon \frac{k^2}{\mathscr{E}} \nabla \mathscr{E} \right) = c_\eta k |D\overline{\mathbf{v}}|^2 - (c_{\varepsilon_2} - c_\gamma) \frac{\mathscr{E}^2}{k}, \tag{4.125}$$

satisfied in D.

4.4.4 *Conclusion and Further Results*

4.4.4.1 Summary of the Model

By synthesizing the previous results, we get the following closed coupled system set in $D = I \times \omega$, where the unknowns are $\overline{\mathbf{v}}$, $\overline{p}$, k, and $\mathscr{E}$,

$$\begin{cases} \partial_t \overline{\mathbf{v}} + (\overline{\mathbf{v}} \cdot \nabla) \overline{\mathbf{v}} - \nabla \cdot \left(\left(\nu + c_\nu \dfrac{k^2}{\mathscr{E}} \right) D\overline{\mathbf{v}} \right) + \nabla(\overline{p} - (2/3)k) = \mathbf{f} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \nabla \cdot \overline{\mathbf{v}} = 0, \\ \partial_t k + \overline{\mathbf{v}} \cdot \nabla k - \nabla \cdot \left(c_k \dfrac{k^2}{\mathscr{E}} \nabla k \right) = c_\nu \dfrac{k^2}{\mathscr{E}} |D\overline{\mathbf{v}}|^2 - \mathscr{E}, \\ \partial_t \mathscr{E} + \overline{\mathbf{v}} \cdot \nabla \mathscr{E} - \nabla \cdot \left(c_\varepsilon \dfrac{k^2}{\mathscr{E}} \nabla \mathscr{E} \right) = c_\eta k |D\overline{\mathbf{v}}|^2 - (c_{\varepsilon_2} - c_\gamma) \dfrac{\mathscr{E}^2}{k}, \end{cases} \tag{4.126}$$

which is known as the $k - \mathscr{E}$ model. Boundary conditions for k and $\mathscr{E}$ will be discussed in Sect. 5.3. Initial data are naturally given by

$$\begin{cases} \overline{\mathbf{v}}|_{t=0} = \overline{\mathbf{v}_0} = E(\mathbf{v}_0), \\ k|_{t=0} = k_0 = \dfrac{1}{2} E(|\mathbf{v}_0 - \overline{\mathbf{v}_0}|^2) = k_0(\mathbf{x}), \\ \mathscr{E}|_{t=0} = \mathscr{E}_0 = \nu E(|\nabla \times \mathbf{v}_0 - \nabla \times \overline{\mathbf{v}_0}|^2) = \mathscr{E}_0(\mathbf{x}). \end{cases} \tag{4.127}$$

We must stress that the nature of the measure μ and the probabilistic space plays no role in the equations. Statistical tools are only involved via the initial conditions, which must be calculated from experimental data.

4.4.4.2 About the Assumptions

Usually, the $k - \mathscr{E}$ model is derived (cf. Mohammadi–Pironneau [20]) by assuming that:

(i) the Boussinesq assumption holds,
(ii) transport of scalar fields by fluctuating vector fields yields turbulent diffusion,
(iii) the eddy viscosity and the turbulent coefficient are all functions of k and $\mathscr{E}$ and can be derived by dimensional analysis,

(iv) additional symmetry properties of turbulent flows hold as well as isotropy of the fluctuations,
(v) turbulence is ergodic,
(vi) turbulent flows are Gaussian, which means that $\mathbf{v}$ has a Gaussian distribution.

We have also assumed in our derivation items (i)–(iii). According to [20], the mild homogeneity assumption yields similar results to (iv)–(vi). One question remains: is a flow which satisfies (iv)–(vi) necessary mildly homogeneous? The answer should be more or less yes, although it is not of primary interest. The class of mild homogeneous flows is definitively a broader class than homogeneous flows or flows satisfying (iv)–(vi).

Furthermore, because of closure assumptions, five dimensionless constants c_I, $I = \nu, k, \varepsilon, \eta, \gamma$ are involved in the system. They are fixed from experimental data corresponding to very simple cases, for which the model can be easily implemented on computers. For example, we find in [20] the following values:

$$c_\nu = c_k = 0.09, \quad c_\varepsilon = 0.07, \quad c_\eta = 0.063, \quad \mu = c_{\varepsilon_2} - c_\gamma = 1.92, \tag{4.128}$$

which are not however universal.

4.4.4.3 Case of Homogeneous Flows

Assume that the turbulence is homogeneous in $D = I \times \omega$. According to Theorem 4.1, $\nabla k = 0$, $\nabla \overline{\mathbf{v}} = 0$, and by an easy extension of the proof, we also have $\nabla \mathscr{E} = 0$, so that the $k - \mathscr{E}$ model reduces to

$$\begin{cases} \partial_t k &= -\mathscr{E}, \\ \partial_t \mathscr{E} &= -\mu \dfrac{\mathscr{E}^2}{k}, \end{cases} \tag{4.129}$$

to which we add initial conditions

$$k_{t=0} = k_0, \quad \mathscr{E}_{t=0} = \mathscr{E}_0. \tag{4.130}$$

Theorem 4.3. *Assume the initial data k_0 and $\mathscr{E}_0$ are constant on Ω, satisfying in addition $k_0, \mathscr{E}_0 > 0$. Then system (4.129) has a unique solution $(k(t), \mathscr{E}(t))$ of class C^1 over a neighborhood of $t = 0$. When $\mu \neq 1$, this solution is given by*

$$k(t) = k_0 \left(1 + (\mu - 1)\frac{\mathscr{E}_0}{k_0} t\right)^{-\frac{1}{\mu-1}}, \quad \mathscr{E}(t) = \mathscr{E}_0 \left(1 + (\mu - 1)\frac{\mathscr{E}_0}{k_0} t\right)^{-\frac{\mu}{\mu-1}}, \tag{4.131}$$

which is defined over $[0, k_0/(1-\mu)\mathscr{E}_0]$ if $\mu < 1$ and over $[0, \infty[$ if $\mu > 1$. When $\mu = 1$, the solution is given by

$$k(t) = k_0 e^{-\frac{\mathscr{E}_0}{k_0}t}, \quad \mathscr{E}(t) = \mathscr{E}_0 e^{-\frac{\mathscr{E}_0}{k_0}t}, \tag{4.132}$$

defined over $[0, \infty[$.

Proof. Since the map $(x, y) \to (-y, -\mu y^2/x)$ is locally Lipschitz continuous on the open set $\mathbb{R}^2 \setminus \{x = 0\}$, we deduce from the Cauchy–Lipschitz theorem that when $k_0, \mathscr{E}_0 > 0$, there exists $t_0 > 0$ such that system (4.129) has a unique solution $(k(t), \mathscr{E}(t))$ of class C^1 over $[0, t_0]$. We search for a solution of the form

$$k(t) = k_0(1 + \lambda t)^{-p}, \quad \mathscr{E}(t) = \mathscr{E}_0(1 + \lambda t)^{-p-1}. \tag{4.133}$$

Inserting this ansatz in (4.129), we obtain the system

$$p\lambda = \frac{\mathscr{E}_0}{k_0}, \quad (p+1)\lambda = \mu\frac{\mathscr{E}_0}{k_0}, \tag{4.134}$$

whose solutions are

$$p = \frac{1}{\mu - 1}, \quad \lambda = (\mu - 1)\frac{\mathscr{E}_0}{k_0}, \tag{4.135}$$

hence (4.131). Formula (4.132) is deduced from (4.131) when $\mu \to 1$. □

To illustrate this point, assume that any event creates a homogeneous turbulence in the domain ω at $t = 0$, consider the flow for $t > 0$, with an eventual homogeneous smooth source term without any fluctuation, and assume $\mu > 1$. Thus far, no factor creates turbulence for $t > 0$, so that turbulence decays at a rate $-(\mu - 1)^{-1}$ following (4.131). There are many experiments that validate a decay law of the form (4.131) for homogeneous turbulence and yield experimental values for μ which confirm that $\mu > 1$ in the considered cases (see in [2]).

4.4.4.4 RANS Models

The $k - \mathscr{E}$ model is widely used in engineering applications, as it provides reliable predictions of mean quantities for many turbulent flows. It has been mainly used in its steady version, and it is understood that it computes a long-time average flow. It is also used to compute transient flows, based on a finite-interval time average. It is the basic model of the RANS models. However, in the $k - \mathscr{E}$ model, the eddy turbulence affects all the flow scales, so that the large eddies are somewhat damped. This is acceptable for a wide class of engineering applications, although many RANS models have been developed to more accurately compute the mixing length, either by refining the modeling of k and $\mathscr{E}$ or by changing $\mathscr{E}$ with a different statistic of the turbulence. This is the case of the Mellor–Yamada model (that uses the variables k and $k\ell$, cf. [19]), $k-\omega$ (that uses the variables k and $\omega = \mathscr{E}/(c_\mu * k)$, cf. [26]), or V2F (that adds an equation for the wall-normal stresses, cf. [9]) models, among others.

All RANS models are formulated through a coupled PDE system that involves the mean velocity $\overline{\mathbf{v}}$, the modified pressure $\overline{p} - (2/3)k$, still denoted by $\overline{p}$, and the statistics of the turbulence $k_1, \cdots, k_n$. The typical PDE system satisfied by $(\mathbf{v}, \overline{p}, k_1, \cdots, k_n)$ is of the form

$$\begin{cases} \partial_t \overline{\mathbf{v}} + (\overline{\mathbf{v}} \cdot \nabla)\, \overline{\mathbf{v}} - \nabla \cdot (\nu_t(k_1, \cdots, k_n) D\overline{\mathbf{v}}) + \nabla \overline{p} = \mathbf{f}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \nabla \cdot \overline{\mathbf{v}} = 0, \\ \partial_t k_j + \overline{\mathbf{v}} \cdot \nabla k_j - \nabla \cdot \big(\mu_{t,j}(k_1, \cdots, k_n) \nabla k_j\big) = \\ \qquad\qquad\qquad\qquad \eta_j(k_1, \cdots, k_n) |D\overline{\mathbf{v}}|^2 - G_j(k_1, \cdots, k_n), \end{cases} \tag{4.136}$$

for some functions ν_t (eddy viscosity), $\mu_{t,j}$ (eddy diffusion of statistic k_j), η_j and G_j, $j = 1, \cdots, n$, that remain to be determined. Most frequently these functions are obtained by dimensional analysis, so they are rational functions of the k_j. We shall say that system (4.136) is an n-order closure system, for $n \geq 2$. It is expected that as the order increases, the accuracy improves. However, models of order larger than $n = 2$ are not frequent, and for many cases, we can take $n = 1$ using the TKE as the only statistic, by providing some convenient mixing length.

All the equations for the k_j in (4.136) have the same structure: on the l.h.s. there appear the material derivative $\partial_t k_j + \overline{\mathbf{v}} \cdot \nabla k_j$ and eddy diffusion $(\nabla \cdot \big(\mu_{t,j} \nabla k_j\big))$ operators, and on the r.h.s. there appear the production $\eta_j |D\overline{\mathbf{v}}|^2$ and dissipation G_j terms, both nonnegative.

The case $n = 1$ requires a closure assumption to link k to $\mathscr{E}$. To do so, we return to formula (4.103) that connects the mixing length ℓ, k and $\mathscr{E}$, by considering ℓ as a given known function. In practical calculations, ℓ is taken to be equal to the local grid size, yielding very accurate results for many turbulent flows (see [8, 17], for instance). The resulting model is usually written in the form

$$\begin{cases} \partial_t \overline{\mathbf{v}} + (\overline{\mathbf{v}} \cdot \nabla)\, \overline{\mathbf{v}} - \nabla \cdot (\nu_t(k) D\overline{\mathbf{v}}) + \nabla(\overline{p}) = \mathbf{f}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \nabla \cdot \overline{\mathbf{v}} = 0, \\ \qquad \partial_t k + \overline{\mathbf{v}} \cdot \nabla k - \nabla \cdot (\mu_t(k) \nabla k) = \nu_t(k) |D\overline{\mathbf{v}}|^2 - \ell^{-1} k \sqrt{k}, \end{cases} \tag{4.137}$$

called the NS-TKE model. In view of the similar structure of each equation for k_j in the n closure model, we will perform the mathematical analysis of the steady-state NS-TKE model in Chap. 7, the evolutionary NS-TKE model in Chap. 8, and its numerical approximation in Chap. 12.

This simplified NS-TKE model, however, retains some of the main mathematical difficulties of the larger-order models. This is especially due to the unboundedness of the eddy coefficients. Moreover, the production terms such as $\nu_t(k)|D\mathbf{v}|^2$ have only L^1 regularity and nonstandard mathematical tools are needed to deal with them.

However, the analysis of the well-posedness of coupling mean velocity-pressure with the $k - \mathscr{E}$ model (4.126) still remains an open problem. In particular, this could be a problem because of eddy coefficients of the form $k^2/\mathscr{E}$ that might blow up or vanish, which cannot be kept under control. Moreover, quadratic source terms such as $k|D\mathbf{v}|^2$ cannot be estimated.

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Chapter 5
Laws of the Turbulence by Similarity Principles

Abstract The physical laws describing turbulence are written down, whose common denominator is that they derive from similarity principles: the $-5/3$ Kolmogorov law for local isotropic turbulent flows, the log law in the turbulent boundary layers, and the wall laws. On the one hand, we investigate local isotropic flows, characterizing their second-order velocity correlation tensors. On the other hand, we analyze the structure of the turbulent boundary layer. In both cases, we derive the appropriate dimensional bases and clearly define appropriate similarity assumptions from physical considerations. The range of validity of these laws is clarified: (i) the inertial range, estimated by the mixing length ℓ, the viscosity ν, and the turbulent dissipation $\mathscr{E}$; (ii) the boundary layer thickness, estimated by ν and the friction velocity $u_\star$. We establish the connection between the $-5/3$ law and subgrid models (SGMs), such as Smagorinsky's model, and provide a detailed analysis of generalized wall laws which express the boundary conditions for the mean velocity at the top of the boundary layer, where SGMs are accurate.

5.1 Introduction

5.1.1 From Richardson to Kolmogorov, via von Kármán

According to Richardson [25] in 1922:

(i) turbulence consists of different eddies,
(ii) an eddy is a localized flow structure,
(iii) large eddies consist of small eddies.

Kolmogorov [13] took up this idea in 1941. He improved it by introducing the concept of energy cascade: the energy of large eddies is transferred to smaller eddies, the energy of which is transferred to even smaller eddies, and so on up to a final eddy size λ_0, known as the Kolmogorov scale, with an associated time

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scale τ_0. Both λ_0 and τ_0 are functions of the viscosity ν and the turbulent dissipation $\mathscr{E}$. Dimensional analysis therefore yields

$$\lambda_0 = \nu^{\frac{3}{4}}\mathscr{E}^{-\frac{1}{4}}, \quad \tau_0 = \nu^{\frac{1}{2}}\mathscr{E}^{-\frac{1}{2}}, \tag{5.1}$$

providing the appropriate length–time basis $b_0 = (\lambda_0, \tau_0)$ to analyze energy transfers. The existence of a universal energy spectrum profile $E = E(k)$ such that

$$\forall\, k \in [k_1, k_2] \subset \left[\frac{2\pi}{\ell}, \frac{2\pi}{\lambda_0}\right], \quad E(k) = C\mathscr{E}^{\frac{2}{3}}k^{-\frac{5}{3}}, \tag{5.2}$$

is known as the $-5/3$ law, where k is the wavenumber, $[k_1, k_2]$ the inertial range, and ℓ the Prandtl mixing length. In obtaining the $-5/3$ law, it must be assumed that the turbulence is isotropic,[1] at least locally, which is credible outside the boundary layer.

von Kármán [12] stated in 1930 that the turbulent boundary layer is governed by ν and the friction velocity $u_\star$, defined by (5.66) below, which yields the length–time basis $b_{bl} = (\lambda_{bl}, \tau_{bl})$, where

$$\lambda_{bl} = \frac{\nu}{u_\star} \quad \tau_{bl} = \frac{\nu}{u_\star^2}. \tag{5.3}$$

Following von Kármán, who first demonstrated the existence of a universal log profile V in the boundary layer, we can write

$$\overline{u}(z) = \frac{u_\star}{\kappa}\left(\log\left(\frac{z}{z_0}\right) + 1\right),$$

where $\overline{u}$ is the tangential mean velocity, considered as the dominant component of the mean velocity in the boundary layer, z is the local distance to the wall, $\kappa \approx 0,41$ is the von Kármán constant, and z_0 is the thickness of the linear viscous sublayer. This result allows the determination of mean velocity's boundary conditions (BC) at the top of the boundary layer[2] called wall laws, expressed by (5.105) below. Moreover, the thickness of the boundary layer is calculated from the formula (5.95). Wall laws, which are similar to the friction law (2.139), are widely used in numerical simulations of turbulent flows.

[1]The mean dissipation $\varepsilon = 2\nu\overline{|D\mathbf{v}|^2}$ is usually used in the $-5/3$ law. For local isotropic turbulence, we show below that this is equivalent to using $\mathscr{E} = 2\nu\overline{|D\mathbf{v}'|^2} = \nu\overline{|\boldsymbol{\omega}'|^2}$, introduced in Sect. 4.4.2, which is consistent with the $k - \mathscr{E}$ model, that yields the calculation of ℓ.

[2]See the discussion that follows the proof of Proposition 4.1.

5.1.2 Similarity Principles

The common denominator of the $-5/3$ and log laws is that both are based on similarity principles. Note that we have already encountered a similarity principle in Sect. 3.3, given by the Reynolds similarity, stated in Definition 3.6. We briefly explain what we mean by a similarity principle. Imagine that we seek the determination of a given mean scalar field Ψ (E or $\overline{u}$ in the present case) in terms of computable and/or experimental available mean quantities $\varphi_1, \cdots, \varphi_n$ (here $\nu, \mathscr{E}$ or $\nu, u_\star$). Roughly speaking, the similarity assumption that reflects the physical properties of Ψ is built in two steps:

STEP 1. We identify from the physics a subfamily $\varphi_{i_1}, \cdots, \varphi_{i_k}$ that governs Ψ in a given range of scales $[s_1, s_2]$, denoting by s the standard control parameter (k, z, r, etc.).

STEP 2. We state that two flows having the same $\varphi_{i_1}, \cdot, \varphi_{i_k}$'s share the same Ψ in $[s_1, s_2]$, which is reflected by : if there are two length–time bases b_1 and b_2 such that

$$[s_1^{(1)}, s_2^{(1)}]_{b_1} = [s_1^{(2)}, s_2^{(2)}]_{b_2} \text{ and } (\varphi_{i_1}^{(1)})_{b_1} = (\varphi_{i_1}^{(2)})_{b_2}, \cdots, (\varphi_{i_k}^{(1)})_{b_1} = (\varphi_{i_k}^{(2)})_{b_2},$$

then

$$\Psi_{b_1} = \Psi_{b_2}.$$

From that, the standard similarity principle goes as follows.

(i) We identify the characteristic scales and the appropriate length–time basis b. We write the b-dimensionless field $\boldsymbol{\psi}_b$ deduced from $\boldsymbol{\psi}$ (see Sect. 3.3.1) as a universal profile V, and determine $\boldsymbol{\psi}$ in terms of $\varphi_1, \cdot, \varphi_n$ and V.

(ii) We use the similarity assumption and perform the right scale analysis to find the length–time basis family that leaves the similarity equation invariant, leading to a functional equation, the solution of which determines the profile V.

5.1.3 Outline of the Chapter

Section 5.2 is devoted to the derivation of the $-5/3$ Kolmogorov law, satisfied by local isotropic flows. We first state a clear definition of isotropic flows, the mathematical properties of which are carefully analyzed. In particular, we determine the general structure of the second-order velocity correlation tensors and we link it to $\mathscr{E}$. We then show the existence of an energy spectrum E. We prove that E satisfies the $-5/3$ law by the similarity assumption that reflects the energy transfers in the inertial range, specified by (5.56). The similarity assumption we introduce is a generalization in our own words of that expressed in the original paper by Kolmogorov [13].[3]

Section 5.3 aims to establish the basic structure of the boundary layer and to derive the wall law. The boundary layer similarity assumption focuses on the eddy viscosity ν_t, which is the unknown profile. We find the log law by analyzing the dominant terms in the mean NSE inside the boundary layer, and we derive the wall law from an asymptotic expansion using the log law. We also obtain boundary conditions for the TKE k and the turbulent dissipation $\mathscr{E}$ at the top of the boundary layer.

We perform in Sect. 5.4 the analysis of general wall laws that occur in various type of boundary layer models. This leads us to compile a list of mathematical properties satisfied by the wall laws, which will be used in all subsequent chapters.

Section 5.5 is devoted to making the connection between Smagorinsky's model and the $-5/3$ law. To achieve this, a cut frequency k_c is fixed in the inertial range $[k_1, k_2]$, assuming that the mean velocity $\overline{\mathbf{v}}$ dissipates its energy in the range $[k_1, k_c]$, while the fluctuation $\mathbf{v}'$ dissipates its energy over $[k_c, k_2]$. After some technical manipulations, we obtain the relation

$$\mathscr{E} = C\delta^2 |D\overline{\mathbf{v}}|^3, \tag{5.4}$$

where, roughly speaking, $\delta = 2\pi/k_c$ signifies the size of the smallest eddies that the model is able to capture in a numerical simulation with grid mesh size of order δ (cf. Lele [20]).

Once this point is reached, the eddy viscosity ν_t is linked to $\mathscr{E}$ and δ by the dimensional relation $\nu_t = \mathscr{E}^{\frac{1}{3}}\delta^{\frac{4}{3}}$. Indeed, in $[k_1, k_c]$, the Kolmogorov scale λ_0 in formula (5.1) is replaced by δ, and ν is replaced by ν_t, following the large-scale modeling principle. We finally get from (5.4)

$$\nu_t = C\delta^2 |D\overline{\mathbf{v}}|,$$

which is often known as Smagorinsky's model [28], which is a subgrid model (SGM). Notice that Prandtl [23] has already introduced a similar formula in 1925 for computing ν_t, as noted in Sect. 3.5.3. This model is sometimes called the Kolmogorov–Prandtl–Smagorinsky model and belongs to the family of large eddy simulation (LES) models.

[3] Kolmogorov has not derived the $-5/3$ law in [13], but the $2/3$ law (5.47) sketched in Sect. 5.2.4. However, the major principles needed to find the $-5/3$ law are those of [13]. This is why the $-5/3$ law is always attributed to Kolmogorov.

5.2 Isotropy and Kolmogorov Law

This section aims to derive the $-5/3$ law for turbulent isotropic flows that specifies the profile of the energy spectrum $E = E(k)$ in the inertial range. To understand this, imagine that each eddy of size r in a turbulent flow is like an elementary wave of wavenumber $k = 2\pi/r$. A turbulent flow can therefore be viewed as a continuum of elementary waves interacting in a nonlinear fashion. To each of these elementary waves there corresponds a wave vector $\mathbf{k} \in \mathbb{R}^3$ that belongs to a 3D torus

$$\Pi = \bigcup_{k=k_{\min}}^{k_{\max}} S_k, \quad \text{where} \quad S_k = \{\mathbf{k} \in \mathbb{R}^3,\ |\mathbf{k}| = k\}.$$

The energy spectrum $E(k)$, if it exists, is the amount of kinetic energy contained in S_k. We prove the existence of the energy spectrum E for isotropic flows in Sect. 5.2.3, by considering the Fourier transform of the second-order velocity correlation tensor $\mathbb{B}_2$ defined by (5.9) below and using the isotropy assumption.

This raises the issue of what isotropy is. We explain in Sect. 5.2.1 that isotropy is reflected by the invariance of the correlation tensors under the action of the orthogonal group $O_3(\mathbb{R})$, expressed below in a local sense. This geometrical property allows for a complete characterization of the second-order velocity correlation tensor $\mathbb{B}_2$, which is stated by Theorem (5.12) in Sect. 5.2.2, where we also make the connection between the turbulent dissipation $\mathscr{E}$ and $\mathbb{B}_2$.

Section5.2.3 is devoted to the proof of the existence of the energy spectrum $E = E(k)$ and the calculation of $\mathscr{E}$ from $E(k)$. The $-5/3$ law is derived in the Sect. 5.2.4. The similarity assumption that yields this law is based on the hypothesis that $E(k)$ is determined by ν and $\mathscr{E}$ for wavenumbers $k \in [2\pi/\ell, 2\pi/\lambda_0]$ and that there exists an interval $[k_1, k_2] \subset\subset [2\pi/\ell, 2\pi/\lambda_0]$, called the inertial range, in which E is determined by $\mathscr{E}$ only.

5.2.1 Definition of Isotropy

5.2.1.1 Background

Let $(\mathbf{v}, p)$ be a flow defined over $Q_m \times \mathbb{K}$ ($\mathbb{K}$ is introduced in Sect. 4.2.2). Recall that $D = I \times \omega \subset Q_m$ denotes an open connected subset, such that $I \subset]0, \delta T_m[$ and $\omega \subset\subset \Omega$. We also recall that for any $M = (t, \mathbf{x}) \in D$, we denote by $\tau_t > 0$ and $r_{\mathbf{x}} > 0$ the greatest real numbers such that

$$]t - \tau_t, t + \tau_t[\times B(\mathbf{x}, r_{\mathbf{x}}) \subset D.$$

For a given $n \geq 2$, we consider

$$M_1, \cdots, M_n \in D, \quad \psi_{i_1 \cdots i_n} \in \mathscr{G}, \quad 1 \le i_k \le 3, \quad k = 1, \cdots, n,$$

where $\mathscr{G}$ is the complete field family defined by (4.44) above. Let $\mathbb{B}_n = \mathbb{B}_n(M_1, \cdots, M_n)$ be the n-order correlation tensor whose components are

$$B_{i_1 \cdots i_n} = B_{i_1 \cdots i_n}(M_1, \cdots, M_n) = \overline{\psi_{i_1}(M_1) \cdots \psi_{i_n}(M_n)}. \tag{5.5}$$

We call $\mathscr{B}_n$ the set of all such n-order correlation tensors.

We assume that the flow is homogeneous in D following Definition 4.3, by exchanging $\mathscr{F}$ [cf. (4.46)] for $\mathscr{G}$ to keep the generality. With a fixed point $M_n = (t_n, \mathbf{x}_n)$, we denote $M_i = (t_i, \mathbf{x}_n + \mathbf{r}_i)$ and we assume that $t_1 = t_2 = \cdots = t_n = t$. Thus, according to Lemma 4.4,

$$\forall\, \mathbb{B}_n \in \mathscr{B}_n, \quad \mathbb{B}_n = \mathbb{B}_n(t; \mathbf{r}_1, \cdots, \mathbf{r}_{n-1}), \tag{5.6}$$

which is well defined for $|\mathbf{r}_i| \le r_{\mathbf{x}_n}, i = 1, \cdots n-1$ and only depends on M_n through the relation $r_{\mathbf{x}_n} = \delta_0$ which we shall fix in what follows.

5.2.1.2 Isotropy

Let $\mathbf{a}_1, \cdots, \mathbf{a}_n \in \mathbb{R}^3$, $\mathbf{a}_i = (a_{i1}, a_{i2}, a_{i3})$. We set

$$(\mathbb{B}_n(t; \mathbf{r}_1, \cdots, \mathbf{r}_{n-1}), (\mathbf{a}_1, \cdots, \mathbf{a}_n)) = a_{1i_1} \cdots a_{ni_n} B_{i_1 \cdots i_n}(t; \mathbf{r}_1, \cdots, \mathbf{r}_{n-1}), \tag{5.7}$$

using the Einstein summation convention. We denote by $O_3(\mathbb{R})$ the orthogonal group, which means that $Q \in O_3(\mathbb{R})$ if and only if $Q\,Q^t = Q^t\,Q = \mathrm{I}$.

Definition 5.1. We say that the flow is isotropic in D if and only if it is homogeneous in D and

$$\begin{aligned} &\forall\, n \ge 2, \quad \forall\, \mathbb{B}_n \in \mathscr{B}_n, \\ &\forall\, Q \in O_3(\mathbb{R}), \quad \forall\, \mathbf{a}_1, \cdots, \mathbf{a}_n \in \mathbb{R}^3, \\ &\forall\, \mathbf{x}_n \in \omega, \quad \forall\, (t; \mathbf{r}_1, \cdots, \mathbf{r}_{n-1}) \in I \times B(0, r_{\mathbf{x}_n})^{n-1}, \end{aligned}$$

then we have

$$\begin{aligned} (\mathbb{B}_n(t; Q\mathbf{r}_1, \cdots, Q\mathbf{r}_{n-1}), (Q\mathbf{a}_1, \cdots, Q\mathbf{a}_n)) = \\ (\mathbb{B}_n(t; \mathbf{r}_1, \cdots, \mathbf{r}_{n-1}), (\mathbf{a}_1, \cdots, \mathbf{a}_n)). \end{aligned} \tag{5.8}$$

5.2.2 The Second-Order Velocity Correlation Tensor

The second-order velocity correlation tensor $\mathbb{B}_2$ was first introduced in Sect. 4.3.1 by the formula (4.42). As its trace in $\mathbf{r} = 0$ is the kinetic energy at a given point, it is easy to understand why it plays a central role in energy transfer investigations. Moreover, it is worth noting that in the case of isotropic flows:

(i) $\mathbb{B}_2$ has a specific structure that can be well identified; in particular it only depends on $r = |\mathbf{r}|$,
(ii) the turbulent dissipation $\mathscr{E}$ can be expressed from the derivative of $\mathbb{B}_2$ at $\mathbf{r} = 0$.

The goal of this subsection is to clarify these points.

5.2.2.1 Isotropy in the Particular Case of $\mathbb{B}_2$

For the simplicity, the time dependence is omitted. Therefore, the second-order velocity correlation tensor is denoted from now by

$$\mathbb{B}_2 = \mathbb{B}_2(\mathbf{r}) = (\overline{v_i(\mathbf{x})v_j(\mathbf{x}+\mathbf{r})})_{1\leq i,j\leq 3} = (B_{ij}(\mathbf{r}))_{1\leq i,j\leq 3}. \tag{5.9}$$

In the expression above, we fix δ_0 once and for all and $\mathbf{x}$ satisfies $d(\mathbf{x}, \partial\omega) \geq \delta_0$ so that $\mathbb{B}_2(\mathbf{r})$ is well defined for $|\mathbf{r}| \leq \delta_0$ and at least of class C^1 with respect to $\mathbf{r}$ (and does not depend on $\mathbf{x}$). In this case, the isotropy hypothesis becomes:

$$\forall\, \mathbf{a}, \mathbf{b} \in \mathbb{R}^3, \quad \forall\, \mathbf{r} \in B(0, \delta_0), \quad \forall\, Q \in O_3(\mathbb{R}),$$
$$(\mathbb{B}_2(Q\mathbf{r})Q\mathbf{b}, Q\mathbf{a}) = (\mathbb{B}_2(\mathbf{r})\mathbf{b}, \mathbf{a}), \tag{5.10}$$

where $(\cdot,\cdot)$ is the usual scalar product on $\mathbb{R}^3$, $\mathbb{B}_2(\mathbf{r})\mathbf{b}$ is the product of the matrix $\mathbb{B}_2(\mathbf{r})$ with the vector $\mathbf{b}$. In the following we set

$$\mathbf{r} = (r_1, r_2, r_3), \quad \mathbf{r}\otimes\mathbf{r} = (r_i r_j)_{1\leq i,j\leq 3}, \quad r = |\mathbf{r}|. \tag{5.11}$$

5.2.2.2 Main Result

The structure of $\mathbb{B}_2$ is given by the following theorem.

Theorem 5.1. *Assume that the flow is isotropic in D. Then there exist two scalar functions $B_d = B_d(r)$ and $B_n = B_n(r)$ of class C^1 on $[0, \delta_0[$ and such that*[4]

[4] B_d stands for the deviatoric component and B_n for the normal component.

$$\forall\, \mathbf{r} \in B(0,\delta_0), \quad \mathbb{B}_2(\mathbf{r}) = (B_d(r) - B_n(r))\frac{\mathbf{r}\otimes\mathbf{r}}{r^2} + B_n(r)\mathrm{I}_3. \tag{5.12}$$

Moreover, B_d and B_n are linked through the following differential relation:

$$\forall\, r \in [0,\delta_0[, \quad rB_d'(r) + 2(B_d(r) - B_n(r)) = 0, \tag{5.13}$$

where $B_d'(r)$ is the derivative of B_d.

Proof. We infer from the isotropy definition (5.10) and the relation $Q^t = Q^{-1}$,

$$\forall\, \mathbf{a},\mathbf{b} \in \mathbb{R}^3, \quad (\mathbb{B}_2(\mathbf{r})\mathbf{b},\mathbf{a}) = (Q^t\mathbb{B}_2(Q\mathbf{r})Q\mathbf{b},\mathbf{a}). \tag{5.14}$$

Hence, we obtain

$$\forall\, \mathbf{r} \in B(0,\delta_0), \quad \forall\, Q \in O_3(\mathbb{R}), \quad \mathbb{B}_2(Q\mathbf{r})Q = Q\mathbb{B}_2(\mathbf{r}). \tag{5.15}$$

Let us take Q of the form

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix}, \quad P \in O_2(\mathbb{R}), \tag{5.16}$$

which satisfies $Q(\mathbf{e}_1) = \mathbf{e}_1$. The matrix $\mathbb{B}_2(r\mathbf{e}_1)$ is partitioned as block matrices:

$$\mathbb{B}_2(r\mathbf{e}_1) = \begin{pmatrix} B_d(r) & \mathbf{y}(r)^t \\ \mathbf{x}(r) & H(r) \end{pmatrix}, \tag{5.17}$$

where $B_d(r) \in \mathbb{R}$, $\mathbf{x}(r), \mathbf{y}(r) \in \mathbb{R}^2$, and $H(r) \in M_2(\mathbb{R})$. We deduce from (5.15)

$$Q\mathbb{B}_2(r\mathbf{e}_1) = \begin{pmatrix} B_d(r) & \mathbf{y}(r)^t \\ P\mathbf{x}(r) & PH(r) \end{pmatrix} = \mathbb{B}_2(r\mathbf{e}_1)Q = \begin{pmatrix} B_d(r) & \mathbf{y}(r)^t P \\ \mathbf{x}(r) & H(r)P \end{pmatrix}. \tag{5.18}$$

Therefore,

$$\forall\, P \in O_2(\mathbb{R}), \qquad P\mathbf{x}(r) = \mathbf{x}(r), \quad \mathbf{y}(r)^t P = \mathbf{y}(r)^t, \quad PH(r) = H(r)P, \tag{5.19}$$

which yields at once $\mathbf{x}(r) = \mathbf{y}(r) = 0$. Moreover, we know from standard algebra that only scalar matrices commute with all matrices in $O_2(\mathbb{R})$, which leads to the existence of a scalar $B_n(r)$ such that $H(r) = B_n(r)\mathrm{I}_2$.

Summarizing, we have

$$\mathbb{B}_2(r\mathbf{e}_1) = \begin{pmatrix} B_d(r) & 0 & 0 \\ 0 & B_n(r) & 0 \\ 0 & 0 & B_n(r) \end{pmatrix} = (B_d(r) - B_n(r))\,\mathbf{e}_1\otimes\mathbf{e}_1 + B_n(r)\mathrm{I}_3. \tag{5.20}$$

Formula (5.12) derives from the isotropy hypothesis (5.15) combined with expression (5.20). Indeed, let $\mathbf{r} \in B(0, \delta_0)$. There exists $Q \in O_3(\mathbb{R})$ such that $\mathbf{r} = rQ(\mathbf{e}_1)$, and we notice that

$$Q(\mathbf{e}_1 \otimes \mathbf{e}_1)Q^{-1} = (Q\mathbf{e}_1) \otimes (Q\mathbf{e}_1) = \frac{\mathbf{r} \otimes \mathbf{r}}{r^2},$$

since $Q^t = Q^{-1}$.

Finally, we find the differential relation (5.13) by taking the divergence of $\mathbf{r}$ in $\mathbb{B}_2$ from expression (5.12) and by observing that the incompressibility of $\mathbf{v}$ yields $\nabla_{\mathbf{r}} \cdot \mathbb{B}_2 = 0$, which makes sense because $\mathbb{B}_2$ is of class C^1 and so are B_d and B_n. □

We conclude from Theorem 5.1 that $\mathbb{B}_2$ only depends on $r = |\mathbf{r}|$, and not $\mathbf{r}$, and is entirely determined by a single real-valued function, following the differential relation (5.13). Moreover, the following relations hold:

$$\begin{aligned} B_d(r) = B_{11}(r, 0, 0), \quad B_n(r) = B_{22}(r, 0, 0) = B_{33}(r, 0, 0), \\ \forall\, i \neq j, \quad B_{ij}(r, 0, 0) = 0. \end{aligned} \tag{5.21}$$

5.2.2.3 Dissipation

Here we aim to establish the link between $\mathbb{B}_2$ and the turbulent dissipation $\mathscr{E}$.

Lemma 5.1. *Assume that the flow is isotropic in D. Then the following identity holds in D:*

$$\mathscr{E} = -\nu \sum_{i,j} \frac{\partial^2 B_{ii}}{\partial r_j^2}(0). \tag{5.22}$$

Proof. Since the flow is homogeneous in D, we deduce from Theorem 4.1 that $\nabla \overline{\mathbf{v}} = 0$, hence by the Reynolds decomposition,

$$D\mathbf{v} = D\mathbf{v}',$$

which holds in D. Therefore,

$$2\nu\overline{|D\mathbf{v}|^2} = \overline{\varepsilon} = 2\nu\overline{|D\mathbf{v}'|^2} = \mathscr{E}, \tag{5.23}$$

which does not depend on $\mathbf{x} \in \omega$ for any fixed time $t \in I$. Moreover, by a proof similar to that which yields (4.80) and because of the homogeneity assumption, Lemma 4.6 (homogeneity yields mild homogeneity), and $\nabla \cdot \mathbf{v} = 0$, we obtain

$$\mathscr{E} = \overline{\varepsilon} = \nu \overline{|\nabla \mathbf{v}|^2} = \nu \sum_{i,j} \overline{\left(\frac{\partial v_i}{\partial x_j} \right)^2}. \tag{5.24}$$

Let $\mathbf{x} \in \omega$ such that $d(\mathbf{x}, \partial\omega) \geq \delta_0$, and $0 \leq h_j < \delta_0$. By a Taylor expansion, we have

$$v_i(\mathbf{x} + h_j \mathbf{e_j}) = v_i(x) + h_j \frac{\partial v_i}{\partial x_j}(\mathbf{x}) + o(h_j), \tag{5.25}$$

which allows us to write

$$\overline{\left(\frac{\partial v_i}{\partial x_j}(\mathbf{x}) \right)^2} = \frac{1}{h_j^2} \left[\overline{v_i(\mathbf{x} + h_j \mathbf{e_j})^2} - 2\overline{v_i(\mathbf{x} + h_j \mathbf{e_j}) v_i(\mathbf{x})} + \overline{v_i(\mathbf{x})^2} \right] + o(h_j). \tag{5.26}$$

We infer from the homogeneity assumption that $\overline{v_i(\mathbf{x} + h_j \mathbf{e_j})^2} = \overline{v_i(\mathbf{x})^2}$, which leads to

$$\overline{\left(\frac{\partial v_i}{\partial x_j}(\mathbf{x}) \right)^2} = \frac{2}{h_j^2} \left[B_{ii}(0) - B_{ii}(h_j \mathbf{e}_j) \right] + o(h_j). \tag{5.27}$$

As we consider smooth solutions, the r.h.s. in (5.27) is finite, so that we necessarily have $\partial_{r_j} B_{ii}(0) = 0$, leading to, for fixed i and j,

$$B_{ii}(h_j \mathbf{e}_j) = B_{ii}(0) + \frac{h_j^2}{2} \frac{\partial^2 B_{ii}}{\partial r_j^2}(0) + o(h_j^2), \tag{5.28}$$

which yields (5.22) by summing up, according to (5.27). □

5.2.3 Energy Spectrum

We assume throughout this subsection and until the end of this section that the flow is isotropic in $D = I \times \omega$. Let

$$\mathbb{E} = \frac{1}{2} tr \mathbb{B}_2|_{\mathbf{r}=0} = \frac{1}{2} \overline{|\mathbf{v}|^2}, \tag{5.29}$$

be the total mean kinetic energy at any $(t, \mathbf{x}) \in D$, which is constant in ω for every $t \in I$, because isotropy implies homogeneity. The goals of this subsection are:

(i) to prove the existence of an energy spectrum $E = E(k)$ that allows for the calculation of $\mathbb{E}$, by considering the Fourier transform of $\mathbb{B}_2$, called the wave energy tensor,
(ii) to make the connection between E and $\mathscr{E}$.

5.2.3.1 Wave Energy Tensor

We extend $\mathbb{B}_2$ by 0 outside $B(0, \delta_0)$, keeping the same notation. Let $\hat{\mathbb{B}}_2$ be its Fourier transform, expressed by

$$\forall\, \mathbf{k} \in \mathbb{R}^3, \quad \hat{\mathbb{B}}_2(\mathbf{k}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \mathbb{B}_2(\mathbf{r}) e^{-i\,\mathbf{k}\cdot\mathbf{r}} d\,\mathbf{r}, \tag{5.30}$$

at a given fixed time $t \in I$, which is implicit. We deduce from the Plancherel formula,

$$\forall\, \mathbf{r} \in \mathbb{R}^3, \quad \mathbb{B}_2(\mathbf{r}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \hat{\mathbb{B}}_2(\mathbf{k}) e^{i\,\mathbf{k}\cdot\mathbf{r}} d\,\mathbf{k}, \tag{5.31}$$

which makes sense since $\mathbb{B}_2 \in L^2(\mathbb{R}^3)^9 \cap L^1(\mathbb{R}^3)^9$.

Theorem 5.2. *The tensor* $\hat{\mathbb{B}}_2$ *is isotropic with respect to* $\mathbf{k}$.

Proof. We must prove that

$$\forall\, Q \in O_3(\mathbb{R}), \quad \forall\, \mathbf{k} \in \mathbb{R}^3, \quad Q^t \hat{\mathbb{B}}_2(Q\mathbf{k})Q = \hat{\mathbb{B}}_2(\mathbf{k}). \tag{5.32}$$

The l.h.s. in (5.32) is equal to

$$I = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} Q^t \mathbb{B}_2(\mathbf{r}) Q e^{-i\,Q\mathbf{k}\cdot\mathbf{r}} d\,\mathbf{r}. \tag{5.33}$$

In this integral we make a change of variables $\mathbf{r} = Q\mathbf{r}'$, by noting that $|\det(\mathrm{Jac} Q)| = 1$ since $Q \in O_3(\mathbb{R})$. Hence,

$$I = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} Q^t \mathbb{B}_2(Q\mathbf{r}') Q e^{-i\,\mathbf{k}\cdot\mathbf{r}'} d\,\mathbf{r}' = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \mathbb{B}_2(\mathbf{r}') e^{-i\,\mathbf{k}\cdot\mathbf{r}'} d\,\mathbf{r}' = \hat{\mathbb{B}}_2(\mathbf{k}), \tag{5.34}$$

where we have used the isotropy of $\mathbb{B}$ in $\mathbf{r}$ and the formula $Q\mathbf{k}\cdot\mathbf{r} = \mathbf{k}\cdot Q^t\mathbf{r} = \mathbf{k}\cdot\mathbf{r}'$, as $Q^t = Q^{-1}$. □

According to the proof of Theorem 5.1, which holds in this case, we deduce the existence of two real-valued functions $\tilde{B}_d$ and $\tilde{B}_n$ of class C^1 such that[5]

$$\forall\, \mathbf{k} \in \mathbb{R}^3, \quad |\mathbf{k}| = k, \quad \hat{\mathbb{B}}_2(\mathbf{k}) = (\tilde{B}_d(k) - \tilde{B}_n(k)) \frac{\mathbf{k} \otimes \mathbf{k}}{k^2} + \tilde{B}_n(k) \mathrm{I}_3. \tag{5.35}$$

[5] k already denotes the TKE, and from now also the wavenumber, $k = |\mathbf{k}|$. This is commonly used in turbulence modeling, although it might sometimes be confusing.

5.2.3.2 Energy Spectrum

The existence of the energy spectrum E is stated as follows.

Theorem 5.3. *There exists a measurable function $E = E(k)$, defined over $\mathbb{R}_+$, the integral of which over $\mathbb{R}$ is finite, and such that*

$$\mathbb{E} = \int_0^\infty E(k)dk, \tag{5.36}$$

where $\mathbb{E}$ is the total kinetic energy specified by (5.29).

Proof. We derive from Plancherel's formula (5.31)

$$\mathbb{E} = \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \hat{B}_{ii}(\mathbf{k})\, d\mathbf{k}. \tag{5.37}$$

Using formula (5.35) yields

$$\hat{B}_{ii}(\mathbf{k}) = \tilde{B}_d(k) + 2\tilde{B}_n(k), \tag{5.38}$$

which combined with Fubini's Theorem leads to

$$\int_{\mathbb{R}^3} \hat{B}_{ii}(\mathbf{k})\, d\mathbf{k} = \int_0^\infty \left(\int_{|\mathbf{k}|=k} \hat{B}_{ii}(\mathbf{k}) d\boldsymbol{\sigma} \right) dk = \int_0^\infty 4\pi k^2 (\tilde{B}_d(k) + 2\tilde{B}_n(k))\, dk, \tag{5.39}$$

by noting $d\boldsymbol{\sigma}$ the standard measure over the sphere $\{|\mathbf{k}| = k\}$. This proves the result, where $E(k)$ is given by

$$E(k) = \left(\frac{k}{2\pi}\right)^2 (\tilde{B}_d(k) + 2\tilde{B}_n(k)). \tag{5.40}$$

Notice that in addition,

$$\mathbb{D}(E) = (3, -2). \tag{5.41}$$

In other words, $[E] = \mathscr{U}^2\mathscr{L}$, where $\mathscr{U}$ denotes the velocity's dimension. □

Remark 5.1. From the physical point of view, $E(k)$ is the amount of kinetic energy in the sphere $S_k = \{|\mathbf{k}| = k\}$. As such, it is expected that $E \geq 0$ in $\mathbb{R}$, and we deduce from (5.36) that $E \in L^1(\mathbb{R}_+)$. Unfortunately, we are not able to prove that $E \geq 0$ from formula (5.40), which remains an open problem.

5.2.3.3 Connection Between E and $\mathscr{E}$

The following statement is one of the key results needed for the derivation of Smagorinsky's model and general SGM.

Lemma 5.2. *The turbulent dissipation $\mathscr{E}$ is deduced from the energy spectrum from the formula*

$$\mathscr{E} = \nu \int_0^\infty k^2 E(k) dk, \tag{5.42}$$

which also states that when $E \geq 0$, then $k^2 E(k) \in L^1(\mathbb{R})$.

Proof. We start from identity (5.22). As

$$B_{ii}(r) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \hat{B}_{ii}(\mathbf{k}) e^{i\,\mathbf{k}\cdot\mathbf{r}} d\mathbf{k}, \tag{5.43}$$

we obtain

$$\frac{\partial^2 B_{ii}}{\partial r_j^2}(0) = -\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} k_j^2 \hat{B}_{ii}(\mathbf{k}) d\mathbf{k}. \tag{5.44}$$

Therefore, according to (5.22) which expresses $\mathscr{E}$ in terms of the second derivatives of the B_{ii}'s, we have

$$\mathscr{E} = \frac{\nu}{(2\pi)^3} \int_{\mathbb{R}^3} |\mathbf{k}|^2 \hat{B}_{ii}(\mathbf{k}) d\mathbf{k}. \tag{5.45}$$

The rest of the proof follows from an identity similar to (5.39) combined with (5.40). In addition, we observe that if $E \geq 0$, then $k^2 E(k) \in L^1(\mathbb{R}_+)$. □

Corollary 5.1. *According to formula (5.23), we also have*

$$\overline{|D\mathbf{v}|^2} = \frac{1}{2} \int_0^\infty k^2 E(k) dk. \tag{5.46}$$

Proof. This is a consequence of formula (5.23). □

5.2.4 Similarity Theory and Law of the −5/3

5.2.4.1 Aim of the Subsection

Kolmogorov [13] proved in 1941 that under suitable similarity and isotropy assumptions, there exists an inertial range $[r_1, r_2]$, where $0 < r_1 < r_2$, such that

$$\forall\, r \in [r_1, r_2], \quad \overline{|\mathbf{v}(x+r) - \mathbf{v}(x)|^2} \sim \mathscr{E}^{2/3} r^{2/3}, \tag{5.47}$$

This law is known as the 2/3 law. He also proved similar laws of this kind in subsequent papers [14, 15] (these papers are also gathered in [31]). The similarity principle set up by Kolmogorov can be adjusted to find the profile $E(k)$ of the energy spectrum, leading to the $-5/3$ law (5.2), that we focus on in this subsection.

In the collective unconscious of fluid scientists, the Reynolds number Re governs the state of a given flow. However, we have seen in Sect. 3.3 that things are much more complicated than that. In particular, Hypothesis 3.i shows that besides the geometry, the kinematic viscosity ν, the initial data $\mathbf{v}_0$, and the source term $\mathbf{f}$ (or similarly any nonhomogeneous boundary condition) should be involved in the statement of the Reynolds similarity principle if it is to make sense.

L. Richardson talked about eddies as localized flow structures, suggesting the existence of large and small eddies. The contribution of L. Prandtl was the introduction of the mixing length ℓ. Rather than eddies, he talked about balls of fluids, ℓ being the distance that a ball of fluid can traverse before being mixed in the flow.

Reading Prandtl's works [23, 24], we are led to understand that ℓ is also more or less the scale of those balls of fluid. Notice that this point of view has strongly influenced the statement of Assumption 2.4, which forms part of the basis of our modeling process deriving the NSE in Chap. 2, or to motivate the Boussinesq assumption, introduced for the first time in Sect. 3.5.3 and used throughout Chap. 4.

The Kolmogorov reference scale is precisely ℓ. Eddies whose size are larger than ℓ are understood to be large eddies, while small eddies refer to those whose size is much smaller than ℓ. Kolmogorov's theory tells us that the physics of small eddies is statistically universal, locally isotropic, driven by ν and $\mathscr{E}$,[6] which constitutes the appropriate dimensional basis to investigate the physics of scales smaller than ℓ.

In the following, we follow the scheme established in Sect. 5.1.2 to derive the $-5/3$ law.

5.2.4.2 Dimensional Framework

As $\mathbb{D}(\mathscr{E}) = (2, -3)$ while $\mathbb{D}(\nu) = (2, -1)$, $\mathscr{E}$ and ν are dimensionally independent. Let us consider the associated length–time basis $b_0 = (\lambda_0, \tau_0)$, determined by

$$\lambda_0 = \nu^{\frac{3}{4}} \mathscr{E}^{-\frac{1}{4}}, \quad \tau_0 = \nu^{\frac{1}{2}} \mathscr{E}^{-\frac{1}{2}}. \tag{5.48}$$

We recall that λ_0 is called the Kolmogorov scale. The important point here is that

$$\mathscr{E}_{b_0} = \nu_{b_0} = 1. \tag{5.49}$$

[6] Or $\overline{\varepsilon}$, what amounts to the same thing.

Moreover, the convective associated velocity is given by

$$\mathrm{v}_0 = \lambda_0 \tau_0^{-1} = \nu^{1/4} \mathscr{E}^{1/4}. \tag{5.50}$$

Now let $b = (\lambda, \tau)$ be any length–time basis and v the associated convective velocity. The dimensionless wavenumber k' is defined by

$$k' = \lambda k. \tag{5.51}$$

Therefore, we have from (5.41)

$$E_b(k') = \frac{1}{\lambda \mathrm{v}^2} E\left(\frac{k'}{\lambda}\right). \tag{5.52}$$

For the simplicity, we write E_0 instead of E_{b_0}. In particular, we have

$$\forall\, k \in \mathbb{R}_+, \quad E(k) = \nu^{\frac{5}{4}} \mathscr{E}^{\frac{1}{4}} E_0(\lambda_0 k), \tag{5.53}$$

and

$$\forall\, b = (\lambda, \tau), \quad \forall\, k' \in \mathbb{R}_+, \quad E_b(k') = \frac{\nu^{\frac{5}{4}} \mathscr{E}^{\frac{1}{4}}}{\lambda \mathrm{v}^2} E_0\left(\frac{\lambda_0}{\lambda} k'\right). \tag{5.54}$$

As the energy spectrum E is entirely determined by ν, $\mathscr{E}$, i.e., that two flows sharing the same ν, $\mathscr{E}$, have the same energy spectrum, then the profile E_0 is universal.

5.2.4.3 Similarity Assumption and the −5/3 Law

We shall suppose that wavenumbers (scales) up to $2\pi/\ell$ correspond to waves—or scales—that describe permanent eddies. We assume that turbulence separates the scales, which is formalized by the following assumption, locally expressed in $D = I \times \omega$.

Assumption 5.1 (Scale separation). *Let ℓ be the Prandtl mixing length. We assume that ℓ is constant in ω at any time $t \in I$ and that*

$$\lambda_0 << \ell. \tag{5.55}$$

This assumption is usually satisfied for high Reynolds number flows, that is, turbulent flows. The main similarity assumption for the derivation of the $-5/3$ law is stated as follows.

Assumption 5.2 (Similarity assumption). *There exists an interval*

$$[k_1, k_2] \subset \left[\frac{2\pi}{\ell}, \frac{2\pi}{\lambda_0}\right], \tag{5.56}$$

called the inertial range, such that $k_1 << k_2$ *and for any* $b_1 = (\lambda_1, \tau_1)$ *and* $b_2 = (\lambda_2, \tau_2)$ *two length-scale basis such that* $\mathscr{E}_{b_1} = \mathscr{E}_{b_2}$*, then* $E_{b_1} = E_{b_2}$ *on* $[\lambda_0 k_1, \lambda_0 k_2]$.

Proposition 5.1. *If Assumptions 5.1 and 5.2 hold, then there exists a constant* C *such that*

$$\forall\, k' \in [\lambda_0 k_1, \lambda_0 k_2] = J_r \quad E_0(k') = C(k')^{-\frac{5}{3}}. \tag{5.57}$$

Corollary 5.2. *The energy spectrum satisfies the* $-5/3$ *law*

$$\forall\, k \in [k_1, k_2], \quad E(k) = C\mathscr{E}^{\frac{2}{3}} k^{-\frac{5}{3}}, \tag{5.58}$$

where C *is a dimensionless constant.*

Proof. Given any dimensionless real number $\alpha > 0$, let us consider the following length–time basis:

$$b^{(\alpha)} = (\alpha^3 \lambda_0, \alpha^2 \tau_0).$$

As

$$\mathscr{E}_{b^{(\alpha)}} = 1 = \mathscr{E}_{b_0},$$

Assumption 5.2 yields

$$\forall\, k' \in J_r, \quad \forall \alpha > 0, \quad E_{b^{(\alpha)}}(k') = E_{b_0}(k'). \tag{5.59}$$

Therefore, we deduce from (5.59) and expression (5.54) that E_0 satisfies the functional equation,

$$\forall\, k' \in J_r, \quad \forall\, \alpha > 0, \quad \frac{1}{\alpha^5} E_0(k') = E_0(\alpha^3 k'), \tag{5.60}$$

whose unique solution is given by

$$\forall\, k' \in J_r, \quad E_0(k') = C(k')^{-\frac{5}{3}}, \quad C = \left(\frac{k_1}{\lambda_0}\right)^{\frac{5}{3}} E_0\left(\frac{k_1}{\lambda_0}\right), \tag{5.61}$$

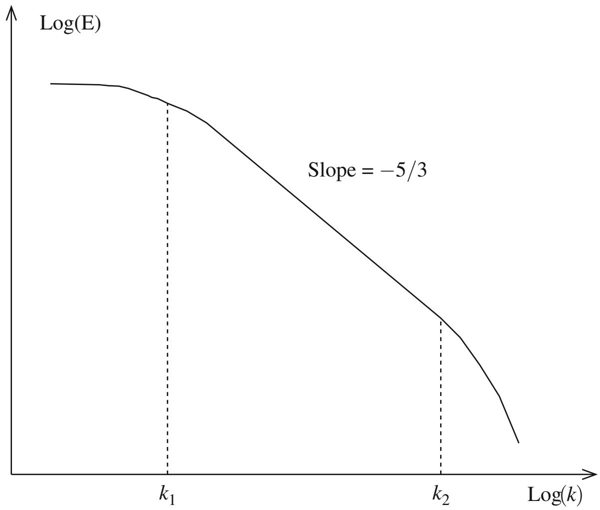

Fig. 5.1 Energy spectrum log–log curve

hence the result. The corollary results from a direct calculation using the expression of λ_0, (5.48), combined with that of E, (5.53). □

The $-5/3$ law is often depicted by the log–log diagram in Fig. 5.1.

Remark 5.2. According to the results of Sect. 4.4.4.3, and for homogeneous flows such that fluctuations are also homogeneous, we find

$$E(k) = C\mathscr{E}_0 \left(1 + (\mu - 1)\frac{\mathscr{E}_0}{k_0} t\right)^{-\frac{\mu}{\mu-1}} k^{-\frac{5}{3}}. \tag{5.62}$$

This formula makes sense only for short time t. Indeed, without any external fluctuating source, the turbulence decays rapidly. We therefore conjecture that the scale separation Assumption 5.1 fails when $t \geq t^\star$, for some $t^\star > 0$ that remains to be determined.

Remark 5.3. The constant C that appears in the $-5/3$ law (5.58) is a universal constant (cf. Lilly [21]). Moreover, k_2^{-1} is of the same order as the Kolmogorov scale λ_0, so that $\lambda_0 k_2 \approx 1$.

5.3 Boundary Layer and Wall Laws

5.3.1 Background

5.3.1.1 Boundary Decomposition

We assume that Ω is a bounded domain of class C^m, $m \geq 1$, and that the no-slip boundary condition holds at the boundary Γ, also called the wall. The flow there is neither homogeneous nor isotropic. Its structure is well identified around a region called the boundary layer and denoted by $\mathscr{BL}$, where shear diffusion processes are dominant because of flow friction at the wall. Note that a turbulent boundary layer which does not detach is very thin compared to the rest of the flow.

Arguing in a similar manner to the proof of the further Theorem 6.1, based on local charts and a unit partition, and when $\mathscr{BL}$ is thin enough:

(i) we can decompose Γ as

$$\Gamma = \bigcup_{i=1}^{n} W_i,$$

(ii) there exist open sets $\mathscr{BL}_{W_i} \subset \Omega$, $i = 1, \cdots, n$, such that

$$\mathscr{BL}_W = \bigcup_{i=1}^{n} \mathscr{BL}_{W_i}, \quad \partial \mathscr{BL}_{W_i} = W_i,$$

(iii) there exist C^m-diffeomorphisms $H_i : \mathscr{BL}_{W_i} \to Q$ where Q is the cylinder

$$Q = \{(x_1, x_2, x_3),\ x_1^2 + x_2^2 < 1,\ |x_3| < 1\},$$

such that

$$H_i(W_i) \subset Q \cap \{z = 0\}, \quad H_i(\mathscr{BL}_{W_i}) \subset Q \cap \{z > 0\}.$$

Let W one of the W_i's, $\mathscr{BL}_W$ the corresponding boundary layer's component. Therefore, we can reduce the investigations to the case where W is of the form

$$\mathscr{BL}_W = S \times \{0 < z < z_1\}, \quad W = S \times \{z = 0\}, \tag{5.63}$$

where $S \subset \mathbb{R}^2$ is a bounded domain of class C^m. For clarity and simplicity in this modeling chapter, we shall examine the structure of the boundary layer in the specific case (5.63) rather than in the general case, which may be described using the diffeomorphisms H_i's and the local charts. We refer to Theorem 6.1 to complete these technical details. However, this case corresponds to the ocean–atmosphere interface considered in Sect. 2.7.6.

5.3.1.2 Outline

The no-slip BC yields $\mathbf{v} = 0$ at W, and according to the usual boundary layer models (see Schlichting [27]), we assume that:

(i) $\overline{\mathbf{v}}$ is parallel to the plane $\{z = 0\}$ in $\mathscr{BL}_W$,
(ii) $\overline{\mathbf{v}}$ is stationary and only depends on z,
(iii) the mean pressure is constant in $\mathscr{BL}_W$,

which translates into

$$\overline{\mathbf{v}} = (\overline{u}, 0, 0) \text{ and } \overline{u} = \overline{u}(z) \text{ in } \mathscr{BL}_W, \quad \nabla\overline{p} = 0 \text{ in } \mathscr{BL}_W, \tag{5.64}$$

which holds if we perform geometrical transformations which do not affect the dynamics.

We aim to derive the profile of $\overline{u}$ and the corresponding boundary condition at $S \times \{z = z_1\}$. The starting assumption is that the flow in $\mathscr{BL}_W$ is governed by ν and the friction velocity $u_\star$, which characterizes the shear stress exerted by the flow over W. After having defined $u_\star$, we introduce the appropriate length–time basis deduced from $(\nu, u_\star)$.

Before following the similarity principle scheme, we model $\mathscr{BL}_W$ by separating the sublayer $S \times [0, z_0]$, where the molecular viscosity effects are dominant, from the sublayer $S \times [z_0, z_1]$, where the eddy viscosity effects are dominant. This allows various simplifications in the mean NSE (4.40) combined with the Boussinesq assumption (4.41). We are led to write $\overline{u} = \overline{u}(z)$ for $z \in [z_0, z_1]$ in the equation for the eddy viscosity $\nu_t = \nu_t(z)$, to which the similarity assumption is applied.

We find that $\overline{u}$ is linear in $[0, z_0]$ and logarithmic in $[z_0, z_1]$, and we calculate z_0 and z_1 as a function of $u_\star$ and ν. We derive the wall law (5.96) from an asymptotic expansion of $\overline{u}$ between 0 and z_1 and using the log law. The wall law is then established for general boundaries. Finally, we derive boundary conditions for k and $\mathscr{E}$ at $z = z_1$.

5.3.2 Boundary Layer Structure

5.3.2.1 Friction Velocity and Dimensional Analysis

The flow exerts a tangential friction over W, calculated by the shear stress

$$\overline{s} = \rho_0 \nu \frac{\partial \overline{u}}{\partial \mathbf{n}}|_{z=0}, \tag{5.65}$$

assumed to be constant over W for simplicity. The friction velocity is defined from the shear stress by

$$u_\star = \sqrt{\frac{|\overline{s}|}{\rho_0}}. \tag{5.66}$$

It is easily verified that the dimension of $u_\star$ is that of a velocity. As ν and $u_\star$, which governs the physics of the flow inside $\mathscr{B}\mathscr{L}_W$, are dimensionally independent, we can consider the associated length–time basis

$$b_{bl} = (\lambda_{bl}, \tau_{bl}), \quad \lambda_{bl} = \frac{\nu}{u_\star} \quad \tau_{bl} = \frac{\nu}{u_\star^2} \tag{5.67}$$

Let

$$L(z') = \overline{u}_{b_{bl}}(z') = \frac{\overline{u}(\lambda_{bl} z')}{u_\star}. \tag{5.68}$$

We assume that the profile L is universal and one has

$$\forall\, z \in [0, z_1], \quad \overline{u}(z) = u_\star L\left(\frac{z}{\lambda_{bl}}\right). \tag{5.69}$$

We must now determine L.

5.3.2.2 Sublayers

The boundary layer's component $\mathscr{B}\mathscr{L}_W$ is divided into two parts:

(i) The viscous part

$$\mathscr{B}\mathscr{L}_{W,v} = S \times [0, z_0],$$

where the dynamics are driven by the molecular viscous shear stress, supposed constant. Therefore, in this viscous sublayer, the mean NSE reduces to

$$\forall\, z \in [0, z_0], \quad \nu \partial_z \overline{u} = C_1, \tag{5.70}$$

for some constant C_1.

(ii) The turbulent part

$$\mathscr{B}\mathscr{L}_{W,t} = S \times [z_0, z_1],$$

where the dynamics are driven by the turbulent shear generated by the Reynolds stress, also assumed constant. In this turbulent sublayer, the mean NSE becomes

$$\forall\, z \in [z_0, z_1], \quad \nu_t \partial_z \overline{u} = C_2, \tag{5.71}$$

where C_2 is also a constant.

We assume that $\partial_z \overline{u} > 0$ in $\mathscr{BL}_W$, which is in agreement with experiment. Moreover, we also assume that $\overline{u}$ is of class C^1. Therefore, by (5.65), (5.66), and because $\mathbf{n} = (0, 0, -1)$ on W,

$$\nu \partial_z \overline{u}|_{z=0} = u_\star^2 = C_1.$$

The interface between the two sublayers is where turbulent and molecular shears are in equilibrium, which implies that since $\overline{u}$ is of class C^1,

$$\nu \partial_z \overline{u} = \nu_t \partial_z \overline{u} \quad \text{on } S \times \{z = z_0\}. \tag{5.72}$$

Therefore, it follows from (5.70) and (5.72), $C_2 = u_\star^2$. In summary, the equations of the boundary layer are

$$\nu \partial_z \overline{u} = u_\star^2 \quad \text{in } \mathscr{BL}_{W,\nu}, \tag{5.73}$$

$$\nu_t \partial_z \overline{u} = u_\star^2 \quad \text{in } \mathscr{BL}_{W,t}. \tag{5.74}$$

As $\overline{u} = 0$ on W, integrating (5.73) over $[0, z_0]$ yields the linear profile,

$$\forall\, z \in [0, z_0], \quad \overline{u}(z) = \frac{u_\star^2}{\nu} z = \frac{u_\star}{\lambda_{bl}} z, \tag{5.75}$$

In terms of the universal profile L, we get

$$\forall\, z' \in \left[0, \frac{z_0}{\lambda_{bl}}\right], \quad L(z') = z'. \tag{5.76}$$

To integrate (5.74), we must determine ν_t. We assume that ν_t is a continuous function of z in $[z_0, z_1]$, and we seek the profile of $\nu_t = \nu_t(z)$, which is achieved using the following similarity assumption, yielding the log law.

5.3.2.3 Similarity Assumption and Log Law

To give meaning to the following similarity assumption, we extend $\nu_t = \nu_t(z)$ to $[0, \infty[$ in a continuous function, still denoted by ν_t. The idea is that ν_t is solely determined by $u_\star$ in the range $[z_0, z_1]$, which plays the same role as the inertial range $[k_1, k_2]$ in the $-5/3$ law.

Assumption 5.3 (Boundary layer similarity assumption). *For any two length-scale bases $b_1 = (\lambda_1, \tau_1)$ and $b_2 = (\lambda_2, \tau_2)$ such that $(u_\star)_{b_1} = (u_\star)_{b_2}$, then $(\nu_t)_{b_1} = (\nu_t)_{b_2}$ on $[z_0/\lambda_{bl}, z_1/\lambda_{bl}] = I_r$.*

Theorem 5.4. *If Assumption 5.3 holds, there exists a dimensionless constant $\kappa \in]0, 1[$ such that*

$$\forall\, z \in [z_0, z_1], \quad \nu_t(z) = \kappa u_\star z, \tag{5.77}$$

and

$$\overline{u}(z) = \frac{u_\star}{\kappa}\left(\log\left(\frac{z}{z_0}\right) + 1\right). \tag{5.78}$$

Moreover,

$$z_0 = \frac{\lambda_{bl}}{\kappa} = \frac{\nu}{\kappa u_\star}, \quad z_1 = z_0 e^{1-\kappa}. \tag{5.79}$$

The dimensionless constant κ is called the von Kármán constant, whose experimental value is about 0.41, and the relation (5.78) is called the log law.

Proof. The proof is divided into three steps. We first demonstrate (5.77), whose direct consequence is the log law (5.78), where $\overline{u}(z_0)$ is the integration constant. We then calculate z_0 and z_1, which yields in the third step the value of $\overline{u}(z_0)$ and therefore (5.78).

STEP 1. We focus on ν_t. We first observe that

$$(u_\star)_{b_{bl}} = 1. \tag{5.80}$$

Let us denote $(\nu_t)_{b_{bl}}(z') = N(z')$, N being another universal profile, that satisfies

$$\forall\, z' \in \left[\frac{z_0}{\lambda_{bl}}, \frac{z_1}{\lambda_{bl}}\right], \quad \nu_t(\lambda_{bl} z') = \nu N(z'). \tag{5.81}$$

Let $b = (\lambda, \tau)$ be any dimensional basis. We deduce from (5.81)

$$(\nu_t)_b(z') = \lambda^{-2}\tau\nu N\left(\frac{\lambda}{\lambda_{bl}} z'\right). \tag{5.82}$$

Let $\alpha > 0$ be any dimensionless number and $b^{(\alpha)}$ be the length–time basis

$$b^{(\alpha)} = (\alpha\lambda_{bl}, \alpha\tau_{bl}). \tag{5.83}$$

A straightforward calculation yields

$$\forall\, \alpha > 0, \quad (u_\star)_{b^{(\alpha)}} = 1 = (u_\star)_{b_{bl}}. \tag{5.84}$$

We deduce from Assumption 5.3 that

$$\forall\, z' \in I_r, \quad \forall\, \alpha > 0, \quad (\nu_t)_{b^{(\alpha)}}(z') = (\nu_t)_{b_{bl}}(z'), \tag{5.85}$$

which combined with (5.82) leads to the functional equation satisfied by N:

$$\forall\, \alpha > 0, \quad \frac{1}{\alpha} N(\alpha z') = N(z'), \tag{5.86}$$

the unique solution of which is

$$\forall\, z' \in \left[\frac{z_0}{\lambda_{bl}}, \frac{z_1}{\lambda_{bl}}\right], \quad N(z') = \kappa z', \quad \kappa = \frac{\lambda_{bl}}{z_0} N\left(\frac{z_0}{\lambda_{bl}}\right), \tag{5.87}$$

which yields

$$\forall\, z \in [z_0, z_1], \quad \nu_t(z) = \kappa u_\star z, \tag{5.88}$$

as claimed by (5.77). Therefore, (5.74) becomes

$$\kappa u_\star z \partial_z \overline{u} = u_\star^2, \tag{5.89}$$

which we integrate over $[z_0, z]$, leading to

$$\overline{u}(z) = \overline{u}(z_0) + \frac{u_\star}{\kappa} \log\left(\frac{z}{z_0}\right). \tag{5.90}$$

The constant $\overline{u}(z_0)$ must be calculated as a function of $u_\star$ and κ. To do so, we first calculate z_0 and z_1.

STEP 2. To carry out the calculation of z_0 and z_1, we start from the equations:

$$\nu_t(z_0)\partial_z \overline{u}(z_0) = \nu \partial_z \overline{u}(z_0), \tag{5.91}$$

$$\overline{u}(z_1) \otimes \overline{u}(z_1) = \nu_t(z_1)\partial_z \overline{u}(z_1). \tag{5.92}$$

Equation (5.91) expresses the equilibrium molecular shear/turbulent stress, and (5.92) the equilibrium turbulent stress/convection. As $\overline{u}$ is of class C^1 and $\partial_z \overline{u} > 0$, we derive from (5.88) combined with (5.91),

$$\kappa u_\star z_0 = \nu, \ \text{ hence } \ z_0 = \frac{\lambda_{bl}}{\kappa} = \frac{\nu}{\kappa u_\star}. \tag{5.93}$$

Finally, let us set $\overline{u}(z) = u_\star f(z)$, the function $f(z)$ being specified by the log law (5.78). The equilibrium relation (5.92) becomes

$$u_\star^2 f(z_1)^2 = \kappa u_\star z_1 f'(z_1), \tag{5.94}$$

whose unique solution, expressed in term of z_0, is

$$z_1 = z_0 e^{1-\kappa}, \tag{5.95}$$

STEP 3. To finish the proof, we must compute $\overline{u}(z_0)$. As $\overline{u}$ is a C^1 function, (5.75) can be used, leading to

$$\overline{u}(z_0) = \frac{u_\star}{\lambda_{bl}} z_0,$$

hence by (5.93), $\overline{u}(z_0) = u_\star/\kappa$, and the log law (5.78) follows from the relation (5.90). □

5.3.3 Wall Law

Once the structure of $\mathscr{BL}$ is known, numerical simulations can be performed in the computational domain $\Omega_c = \Omega \setminus \mathscr{BL}$. We need boundary conditions (BC) at $\Gamma_c = \partial\Omega_c$ for $\overline{\mathbf{v}}$ or k and $\mathscr{E}$. We remain within the framework of Sect. 5.3.1, so that we aim to find BC at $z = z_1$.

5.3.3.1 Velocity's Wall Law

Recall that we have assumed $\partial_z \overline{u} > 0$ in $\mathscr{BL}_W$.

Lemma 5.3. *The following holds at the top of the boundary layer:*

$$\nu_t(z_1)\partial_z\overline{u}(z_1) = \frac{\kappa^2}{(2-\kappa)}\overline{u}(z_1)^2. \tag{5.96}$$

Proof. Expanding $\overline{u}$ between z_1 and 0 in a Taylor series yields

$$0 = \overline{u}(0) = \overline{u}(z_1) - z_1\partial_z\overline{u}(z_1) + o(z_1). \tag{5.97}$$

Experiments show that the boundary layer thickness decreases to zero as Re goes to infinity [27]. In other words,

$$\lim_{Re\to\infty} u_\star = \infty \quad \text{or equivalently} \quad \lim_{Re\to\infty} z_1 = 0. \tag{5.98}$$

It is reasonable therefore to neglect the remaining term in expansion (5.97). Using the relation $\nu_t(z_1) = \kappa u_\star z_1$, we obtain

$$\nu_t(z_1)\partial_z\overline{u}(z_1) = \kappa u_\star \overline{u}(z_1). \tag{5.99}$$

The log law (5.78) allows $u_\star$ to be eliminated in the above relation by writing

$$\overline{u}(z_1) = \frac{u_\star}{\kappa}\left(\log\left(\frac{z_1}{z_0}\right) + 1\right). \tag{5.100}$$

As $z_1/z_0 = e^{1-\kappa}$, we obtain

$$u_\star = \frac{\kappa}{2-\kappa}\overline{u}(z_1), \tag{5.101}$$

hence formula (5.96). □

Remark 5.4. We observe that using (5.99) and (5.101), formula (5.96) also can be written as

$$\nu_t(z_1)\partial_z\overline{u}(z_1) = (2-\kappa)u_\star^2. \tag{5.102}$$

If we remove the assumption $\partial_z\overline{u} > 0$ in $\mathscr{BL}_W$ and $\overline{u}(z_1) \neq 0$, this boundary condition becomes

$$\nu_t(z_1)\partial_z\overline{u}(z_1) = (2-\kappa)u_\star^2\frac{\overline{u}(z_1)}{|\overline{u}(z_1)|}, \tag{5.103}$$

since the friction that the boundary layer exerts on the rest of the fluid has the same orientation as the velocity at its top.

5.3.3.2 General Boundaries

Formula (5.96) is a local expression of the boundary condition satisfied by the mean field at the boundary of the computational domain

$$\Omega_c = \Omega \setminus \mathscr{BL}, \quad \Gamma_c = \partial\Omega_c. \tag{5.104}$$

The generalization from the local expression to a global boundary condition at Γ_c proceeds as follows:

(i) The assumption $\overline{\mathbf{v}} = (v_x, v_y, v_z) = (\overline{u}(z), 0, 0)$ in the local frame of $\mathscr{BL}_W$ becomes $v_z = \overline{\mathbf{v}}\cdot\mathbf{n} = 0$ at Γ_c.
(ii) In the local frame of $\mathscr{BL}_W$, $v_x = \overline{u}(z)$ is the tangential component of $\overline{\mathbf{v}}$ and $\mathbf{n} = (0,0,-1)$ at the top of $\mathscr{BL}_W$. Therefore, according to (5.96) and the motivation behind the relation (5.103) in Remark 5.4, we have

$$\overline{\mathbf{v}}\cdot\mathbf{n} = 0 \;\text{ and }\; -[\nu_t D\overline{\mathbf{v}}\cdot\mathbf{n}]_\tau = C\,\overline{\mathbf{v}}_\tau|\overline{\mathbf{v}}_\tau| = g(\mathbf{v})_\tau \;\text{ on }\; \Gamma_c. \tag{5.105}$$

In the above expression, the subscript τ refers to the tangential component of any vector, decomposed as

$$\mathbf{w} = \mathbf{w}_\tau + (\mathbf{w} \cdot \mathbf{n})\, \mathbf{n}.$$

Moreover $C = C(\mathbf{x})$ is a nonnegative dimensionless function of class C^m that only depends on the geometry of Γ_c.

The boundary condition (5.105) is called a wall law. This wall law has a structure very similar to that of the friction law (2.139). Note that we could be more specific and technical by using the C^m-diffeomorphisms H_i's and the local charts introduced in Sect. 5.3.1 to justify the wall law (5.105). However, this does not bring anything more to the modeling process.

5.3.3.3 Boundary Conditions for k and $\mathscr{E}$

As we have already stated, ν_t is naturally given by the formula

$$\nu_t = C\ell\sqrt{k}, \tag{5.106}$$

for some dimensionless constant C. Recall that we are reasoning in the local frame of W. It is commonly accepted that in the boundary layer, the mixing length at a given point M is of the same order as the distance of M to the wall. Therefore we take $\ell(z) = z$. Combining this expression with $\nu_t(z) = \kappa u_\star z$ in $\mathscr{BL}_{W,t}$ yields a link between k and $u_\star$ in $\mathscr{BL}_{W,t}$ given by

$$k = C u_\star^2 \;\text{ in }\; \mathscr{BL}_{W,t}, \tag{5.107}$$

where we still denote any dimensionless constant by C. Moreover, as ℓ, k, and $\mathscr{E}$ are linked by the relation $\mathscr{E} = \ell^{-1} k^{3/2}$, we find that in $\mathscr{BL}_{W,t}$

$$\mathscr{E} = C\frac{u_\star^3}{z} \;\text{ in }\; \mathscr{BL}_{W,t}. \tag{5.108}$$

We deduce from (5.93) and (5.95),

$$z_1 = \frac{e^{1-\kappa}}{\kappa}\frac{\nu}{u_\star}, \tag{5.109}$$

which leads to the following boundary conditions:

$$k = C_k u_\star^2, \quad \mathscr{E} = C_{\mathscr{E}}\frac{u_\star^4}{\nu} \;\text{ at }\; \Gamma_c. \tag{5.110}$$

5.4 General Wall Laws

5.4.1 Framework

5.4.1.1 Further Examples of Wall Laws

Let L be the dimensionless universal profile defined by (5.68). It is customary in studies of wall laws to write

$$z^+ = z' = \frac{z}{\lambda_{bl}} = \frac{u_\star}{\nu} z, \quad u^+ = (\overline{u})_{b_{bl}} = \frac{\overline{u}}{u_\star}. \tag{5.111}$$

We deduce from Theorem 5.4 and (5.76) that the boundary layer is specified by the equation

$$u^+ = L(z^+), \tag{5.112}$$

where L is the real-valued function:

$$L(z^+) = \begin{cases} z^+ & \text{if} \quad 0 \le z^+ \le \dfrac{1}{\kappa}, \\ \dfrac{1}{\kappa}\left(\log(\kappa z^+) + 1\right) & \text{if } \dfrac{1}{\kappa} \le z^+ \le \dfrac{e^{1-\kappa}}{\kappa}. \end{cases} \tag{5.113}$$

However, the law (5.113) does not take into account the transition zone between the laminar and logarithmic sublayer (called the buffer layer). Spalding [29] proposed a law that takes into account the three sublayers, whose dimensionless profile L is expressed by a single formula defining L^{-1}:

$$z^+ = L^{-1}(u^+) = u^+ + e^{-\kappa C}\left(e^{\kappa u^+} - 1 - (\kappa u^+) - \frac{(\kappa u^+)^2}{2} - \frac{(\kappa u^+)^3}{6}\right) \tag{5.114}$$

Another useful wall law is the Richard law, which also models the three boundary sublayers and expressed by a single formula:

$$L(z^+) = 2,5\,\log(1 + 1,4\,z^+) + 7,8\left(1 - e^{-z^+/11} - \frac{z}{11}e^{-1,33\,z^+}\right). \tag{5.115}$$

5.4.1.2 Problem Setting and Outline

From here on, it is understood that we are investigating mean fields and mean quantities. Therefore, for simplicity and clarity, we drop the overlines; in particular $\mathbf{v}$ stands for $\overline{\mathbf{v}}$, u stands for $\overline{u}$, etc.

For the wall law (5.105) satisfied in the case of the log law, we may ask if general profiles L such as those considered above still yield wall law like

$$\mathbf{v}\cdot\mathbf{n} = 0 \quad \text{and} \quad -[\nu_t D\mathbf{v}\cdot\mathbf{n}]_\tau = g(\mathbf{v})_\tau \text{ on } \Gamma_c, \tag{5.116}$$

where the function $g : \mathbb{R}^3 \to \mathbb{R}^3$ can be determined from the profile L. Following formula (5.103), which does not involve the log law, we are led to seek g in the form

$$g(\mathbf{v}) = \begin{cases} C\dfrac{\mathbf{v}}{|\mathbf{v}|}u_\star^2 & \text{if } |\mathbf{v}| > 0, \\ 0 & \text{if } |\mathbf{v}| = 0, \end{cases} \tag{5.117}$$

for some dimensionless function $C = C(\mathbf{x}) > 0$ of class C^m over Γ_c. This formula raises the question of how to compute $u_\star$ from L.

Within the framework of Sect. 5.3, $\mathscr{BL}_W = S \times [0, z_1]$, the length–time basis is $b_{bl} = (\nu/u_\star, \nu/u_\star^2)$, and there exists a function $L : \mathbb{R}_+ \to \mathbb{R}$ such that $\mathscr{BL}_W$ is entirely determined by the equation $u^+ = L(z^+)$, where u^+ and z^+ are defined by (5.111). Let (z_1^+, u_1^+) be the point

$$z_1^+ = (z_1)_{b_{bl}} = z_1\frac{u_\star}{\nu}, \quad u_1^+ = (u)_{b_{bl}}(z_1^+) = \frac{u(z_1)}{u_\star},$$

that characterizes the top of the boundary layer. The equation $u_1^+ = L(z_1^+)$ becomes

$$\frac{u(z_1)}{u_\star} = L\left(\frac{z_1}{\nu}u_\star\right), \tag{5.118}$$

rewritten for simplicity in the form

$$\frac{v}{u} = L(\lambda u), \tag{5.119}$$

where u stands for $u_\star$, v stands for $u(z_1) = |\mathbf{v}(z_1)|$, and $\lambda = \dfrac{z_1}{\nu}$. The idea is to partially disconnect u and λ and to assume

$$\lim_{u\to 0} u\lambda = \alpha > 0, \tag{5.120}$$

$$\lim_{u\to\infty} \lambda u = +\infty. \tag{5.121}$$

In the following, we aim to solve (5.119) to express u as a function of v for a wall law L, as defined in Definition 5.2 below. This allows us to derive the general structure of the wall laws from (5.117). Although λ actually depends on u, the modeling trick is to decouple it from u in order to solve (5.119). We finally obtain

$$g(\mathbf{v}) = \mathbf{v}H(|\mathbf{v}|), \tag{5.122}$$

where $H : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function that satisfies

$$H(0) = 0 \text{ and } \forall\, v \geq 0, \quad H(v) \leq C(1 + v), \tag{5.123}$$

where the dimensionless constant C only depends on L.

5.4.2 Derivation of Wall Laws

Roughly speaking, a wall-law function is a function $L = L(z^+)$ linear near $z^+ = 0$, whose profile is logarithmic for large values of z^+, as suggested by experiments on turbulent boundary layers. To be more specific

Definition 5.2. A function $L : \mathbb{R}_+ \to R$ is called a wall-law function if $L \in W^{1,\infty}_{loc}(\mathbb{R}_+)$, L is nonnegative and strictly increasing, L' admits a finite number of discontinuities, and

$$\lim_{z^+ \to 0^+} \frac{L(z^+)}{z^+} = C_1, \tag{5.124}$$

$$\lim_{z^+ \to \infty} \frac{L(z^+)}{\log z^+} = C_2, \tag{5.125}$$

where C_1 and C_2 are nonzero constants.

After some elementary but involved calculus, it can be shown that all the three functions L given by (5.113)–(5.115) actually are wall-law functions.

Lemma 5.4. *Let $L : \mathbb{R}_+ \to \mathbb{R}$ be a wall-law function. Then for any $v \in \mathbb{R}_+$, the algebraic equation (5.119) admits a unique solution $u \in \mathbb{R}_+$.*

Proof. Let us rewrite (5.119) as

$$v = F(u), \text{ where } F(u) = u\, L(\lambda u),$$

considering λ as a fixed parameter. As L is strictly increasing and continuous, then F is strictly increasing and continuous. Also, by (5.124), F is continuous at $u = 0$ with $F(0) = 0$. Moreover, by (5.125),

$$\lim_{u \to \infty} F(u) = +\infty.$$

Then F is bijective from $\mathbb{R}_+$ onto $\mathbb{R}_+$. Consequently, (5.119) admits a unique solution $u = F^{-1}(v) = h(v)$. Note that we have $h(0) = 0$. □

Lemma 5.5. *Let $h(v) = u$ be the unique solution of (5.119). Then h is a continuous bijection from $\mathbb{R}_+$ onto $\mathbb{R}_+$. Moreover there exists a constant $C > 0$ such that*

$$|h(v)| \le C\,(1+v), \quad \textit{for all}\;\; v \ge 0. \tag{5.126}$$

Proof. The function $h = F^{-1}$ is clearly bijective and continuous, since F is. Also,

$$\lim_{u\to\infty} \frac{h(v)}{v} = \lim_{u\to\infty} \frac{u}{F(u)} = \lim_{u\to\infty} \frac{1}{L(\lambda u)} = \lim_{u\to\infty} \frac{1}{\log(\lambda u)} \frac{\log(\lambda u)}{L(\lambda u)} = 0,$$

by (5.125) and (5.121). As h is continuous, (5.126) follows. □

Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by

$$g(\mathbf{v}) = \begin{cases} \dfrac{\mathbf{v}}{|\mathbf{v}|}\, h^2(|\mathbf{v}|) & \text{if } |\mathbf{v}| > 0, \\ \quad 0 & \text{if } |\mathbf{v}| = 0. \end{cases} \tag{5.127}$$

The essential properties of g are listed in the following statement.

Lemma 5.6.

(i) $g \in W^{1,\infty}_{loc}(\mathbb{R}^3 \setminus \{0\}) \cap C^0(\mathbb{R}^3)$.
(ii) *g is monotone.*
(iii) *g is positive,*

$$g(\mathbf{v}) \cdot \mathbf{v} \ge 0 \;\; \textit{for all}\;\; \mathbf{v} \in \mathbb{R}^3. \tag{5.128}$$

(iv) *g verifies the growth property*

$$|g(\mathbf{v})| \le C\,(1 + |\mathbf{v}|^2) \;\; \textit{for all}\;\; \mathbf{v} \in \mathbb{R}^3; \tag{5.129}$$

for some constant $C > 0$.
(v) *Finally, ∇g also satisfies*

$$|\nabla g(\mathbf{v})| \le C\,(1 + |\mathbf{v}|) \;\; \textit{a. e. in}\;\; \mathbb{R}^3; \tag{5.130}$$

for some constant $C > 0$.

Proof.

(i) The continuity of g over $\mathbb{R}^\star_+$ follows from the continuity of h in $\mathbb{R}_+$. Furthermore, as $h(v) \to 0$ as $v \to 0$, we infer from (5.127) that $g(\mathbf{v}) \to 0$ as $|\mathbf{v}| \to 0$. Therefore, g is a continuous function over $\mathbb{R}^3$.
To prove that $g \in W^{1,\infty}_{loc}(\mathbb{R}^3 \setminus \{0\})$, we first note that $F \in W^{1,\infty}_{loc}(\mathbb{R}_+)$. Then if $u = h(v)$,

$$h'(v) = \frac{1}{F'(u)} = \frac{1}{L(\lambda u) + \lambda u L'(\lambda u)} \le \frac{1}{L(\lambda u)}, \quad \text{a. e. in } \mathbb{R}_+, \tag{5.131}$$

as L is strictly increasing. Then, as L' admits a finite number of discontinuities, so does h', which is nonnegative. Moreover, we deduce from assumption (5.125) combined with (5.131) and (5.121),

$$\lim_{v\to\infty} h'(v) = 0,$$

and consequently $h' \in L^\infty(\mathbb{R}_+)$. As the application $\mathbf{v} \to |\mathbf{v}|$ belongs to $C^\infty(\mathbb{R}^3 \setminus \{0\})$, we conclude $g \in W^{1,\infty}_{loc}(\mathbb{R}^3 \setminus \{0\})$.

(ii) To prove that g is monotone, we use Lemma A.21 and prove that it is the gradient of a convex function in $\mathbb{R}^3$. Let us define the functions

$$B(v) = \int_0^v h(v')^2\, dv', \quad C(\mathbf{v}) = H(|\mathbf{v}|). \tag{5.132}$$

We claim that

$$g(\mathbf{v}) = \nabla C(\mathbf{v}) \ \text{ for any } \ \mathbf{x} \in \mathbb{R}^3.$$

Indeed, the function C is differentiable and $\nabla C \in W^{1,\infty}_{loc}(\mathbb{R}^3 \setminus \{0\})$, since B is differentiable in $\mathbb{R}$ with $B' \in W^{1,\infty}_{loc}(\mathbb{R})$. Moreover

$$\partial_i C(\mathbf{v}) = \partial_i(|\mathbf{v}|)\, B'(|\mathbf{v}|) = \frac{v_i}{|\mathbf{v}|} h(|\mathbf{v}|)^2 = g_i(\mathbf{v}), \quad i = 1, 2, 3 \ \text{ if } \ \mathbf{v} \in \mathbb{R}^3 \setminus \{0\}.$$

Furthermore, $\nabla C(0) = g(0)$. Indeed,

$$\frac{C(\mathbf{v})}{|\mathbf{v}|} = \frac{1}{|\mathbf{v}|} \int_0^{|\mathbf{v}|} h(v)^2\, dv \le \max_{0 \le v \le |\mathbf{v}|} h(\mathbf{v})^2,$$

and then

$$\lim_{\mathbf{v}\to 0} \frac{C(\mathbf{v})}{|\mathbf{v}|} = 0.$$

The function C is convex, as it is the composition of the functions $\varphi(\mathbf{v}) = |\mathbf{v}|$ and $B(v)$, which are both convex. Indeed,

$$B''(\alpha) = 2\, h(v) h'(v) > 0, \ \text{ a. e. in } \ ]0, +\infty[,$$

as $F^{-1} \in W^{1,\infty}_{loc}(0, +\infty)$, is nonnegative and strictly increasing.

(iii) Property (5.128) follows from the identity $g(\mathbf{v}) \cdot \mathbf{v} = |\mathbf{v}|\, h(|\mathbf{v}|)^2$.

(iv) To determine the growth rates of g at infinity, observe that $|g(\mathbf{v})| = h^2(|\mathbf{v}|)$ if $\mathbf{v} \neq 0$, hence (5.129) follows from (5.126).

(v) We now investigate the growth rates of ∇g. Let $\mathscr{H}(v) = h^2(v)$, so that $g(\mathbf{v}) = \dfrac{\mathbf{v}}{|\mathbf{v}|}\mathscr{H}(|\mathbf{v}|)$. A straightforward calculation yields

$$\partial_j g_i(\mathbf{v}) = \left(\frac{\delta_{ij}}{|\mathbf{v}|} - \frac{v_i v_j}{|\mathbf{v}|^3}\right)\mathscr{H}(|\mathbf{v}|) + \frac{v_i v_j}{|\mathbf{v}|^2}\mathscr{H}'(|\mathbf{v}|). \tag{5.133}$$

We easily deduce from (5.126) that

$$\left|\left(\frac{\delta_{ij}}{|\mathbf{v}|} - \frac{v_i v_j}{|\mathbf{v}|^3}\right)\mathscr{H}(|\mathbf{v}|)\right| \leq C(1 + |\mathbf{v}|). \tag{5.134}$$

It remains to investigate the term involving $\mathscr{H}'(|\mathbf{v}|)$. Since $\mathscr{H}'(v) = 2h(v)h'(v)$, we deduce from the relation $u = h(v) = F^{-1}(v)$,

$$\mathscr{H}'(v) = 2F^{-1}(v)\,\frac{1}{F'(u)} = \frac{2\,u}{L(\lambda u) + \lambda u L'(\lambda u)} \ \text{ a. e. in } \mathbb{R}_+,$$

Thus, as $u = v/L(\lambda u)$, we get

$$\frac{\mathscr{H}'(v)}{v} = \frac{2}{L(\lambda u)\,(L(\lambda u) + \lambda u L'(\lambda u))} \leq \frac{2}{L(\lambda u)^2} \ \text{ a. e. in } \mathbb{R}_+.$$

Then $\displaystyle\lim_{v\to\infty}\frac{\mathscr{H}'(v)}{v} = 0$. Also, $\mathscr{H}'$ is bounded in compact sets as $h \in W^{1,\infty}_{loc}(0, +\infty)$. Consequently there exists a constant $C > 0$ such that

$$|\mathscr{H}'(v)| \leq C\,(1 + v) \ \text{ a. e. in } \mathbb{R}_+. \tag{5.135}$$

We combine (5.133), (5.134), and (5.135), giving

$$|\partial_j g_i(\mathbf{v})| \leq C\,(1 + |\mathbf{v}|),$$

which yields (5.130). □

In order to complete the analysis of wall laws, we prove the following statement.

Lemma 5.7. *The function* $H : \mathbb{R}_+ \to \mathbb{R}$ *defined by*

$$H(v) = \begin{cases} \dfrac{h^2(v)}{v} & \text{if } v \neq 0, \\ 0 & \text{if } v = 0, \end{cases}$$

is a continuous real-valued function.

Proof. It suffices to prove the continuity of H at $v = 0$, since we already know that h is continuous over $\mathbb{R}_+^\star$. Recall that $u = h(v)$, which yields $v = F(u) = u\,L(\lambda u)$, and in the vicinity of $v = 0$, we have

$$H(v) = \frac{u^2}{uL(\lambda u)} = \frac{u}{L(\lambda u)} \sim \frac{u}{L(\alpha)} \to 0 \ \text{ as } \ v \to 0,$$

by (5.120) and $L(\alpha) \neq 0$ since $\alpha > 0$, which completes the proof. □

In conclusion, by combining the result of Lemmas 5.4 and 5.7, we obtain the general wall law

$$\mathbf{v} \cdot \mathbf{n} = 0 \ \text{ and } \ -[\nu_t D\mathbf{v} \cdot \mathbf{n}]_\tau = g(\mathbf{v})_\tau \ \text{ at } \ \Gamma_c, \tag{5.136}$$

where g is of the form $g(\mathbf{v}) = \mathbf{v}H(|\mathbf{v}|)$, H being a continuous function that satisfies $\forall\, v \in \mathbb{R}_+, 0 \leq H(v) \leq C(1 + v)$.

Moreover, we also deduce from (5.110), Lemma 5.4, and (5.5) that there exists a continuous function $k_\Gamma = k_\Gamma(\mathbf{v})$, where $k_\Gamma(\mathbf{v}) = C_k h^2(|\mathbf{v}|)$, such that

$$k = k_\Gamma(\mathbf{v}) \ \text{ on } \ \Gamma, \tag{5.137}$$

which determines the boundary condition for the TKE.

Remark 5.5. Strictly speaking, as already noted in Sect. 5.3.3.2, the functions g and k_Γ should involve a dimensionless function $C = C(\mathbf{x}) \geq 0$, of class C^m on Γ_c, which only depends on the geometry. As this function C does not influence the mathematical structure of the problems studied in the subsequent chapters, we take it as a constant equal to 1 for simplicity.

Remark 5.6. The working assumption (5.120) is in agreement with Theorem 5.4, in particular (5.79), which is not the case of (5.121). However, the results of Lemmas 5.6 and 5.7, based on (5.120) and (5.121), include the case of the basic friction law (5.105) derived from the results of Theorem 5.4, already obtained in Sect. 2.7.5 from other argumentations. There is a paradox, which would require more investigations, however not essential for the rest of the book.

5.5 Large Eddy Simulation and Subgrid Model

In RANS models, all scales of the turbulent flow are modeled through the expectations of the fields and a statistical analysis; hence, the eddy viscosity also applies to large scales. An alternative approach is to resolve large grid scales and to model the subgrid scales. This is the main objective of the so-called LES, which can accurately simulate many flows that RANS cannot. This is the case of transitional flow, flows with large separation or flows past bluff bodies, where the wake is mostly unsteady, for instance. On the other hand, LES is much more expensive than RANS so it is used for specific applications.

The main idea in modeling the effect of the subgrid scales onto the resolved scales is to use the self-similarity of the statistical properties of the turbulence in the inertial range. Smagorinsky [28] was the first to take advantage of this idea in 1963, in the context of numerical simulations of atmospheric weather prediction to run, by taking as eddy viscosity

$$\nu_t = (C_S\, h)^2 |D\overline{\mathbf{v}}|. \tag{5.138}$$

In this formula, h is the grid mesh size, and C_S is a dimensionless constant, whose value is determined to fit experimental results. Deardorff [4] improved and systematized the concept in 1971, when the terminology LES started to appear. Models using eddy viscosities of the form (5.138) are also called SGM.

Modern LES is based on filtering fields by convolution,

$$\overline{\mathbf{v}} = G_\delta \star \mathbf{v}, \tag{5.139}$$

for some appropriate filter G_δ, see in [1, 3, 5–7, 18, 19, 22, 26]. With this in mind, the limit between the resolved and modeled scales must be located in the inertial range, which determines the range the filter parameter δ belongs to, as well as the grid mesh size in a numerical simulation.

We shall not detail the LES modeling process in this book, already described in the numerous books and articles given in the wide bibliography quoted above. It is however striking that there is a strong connection between the modeling framework developed throughout this chapter and the previous one and Smagorinsky's model, as pointed out in the next subsection.

5.5.1 From the $-5/3$ Law to the Smagorinsky Model

In the LES field, SGM's are often related to the $-5/3$ law, although they are used for simulating flows that are not homogeneous nor isotropic. Based on the foregoing discussion, we list below sufficient conditions for the derivation of the basic SGM, Smagorinsky's model, within our framework. As usual we denote $D = I \times \omega$.

(i) There exists $E = E(t, \mathbf{x}, k)$, defined over $I \times \omega \times \mathbb{R}_+$, with $[k] = \mathscr{L}^{-1}$, such that

$$\forall\, (t, \mathbf{x}) \in I \times \omega, \quad \frac{1}{2}\overline{|\mathbf{v}(t, \mathbf{x})|^2} = \int_0^\infty E(t, \mathbf{x}, k)dk. \tag{5.140}$$

(ii) For all $(t, \mathbf{x}) \in D$, there exists $k_1(t, \mathbf{x}), k_2(t, \mathbf{x})$, with

$$0 < \eta_0 \leq k_1(t, \mathbf{x}) \leq \eta_2 << \eta_3 \leq k_2(t, \mathbf{x}) \leq \eta_4, \tag{5.141}$$

where the constants η_i only depend on D, a constant $C > 0$ and such that

$$\forall\, k \in [k_1(t, \mathbf{x}), k_2(t, \mathbf{x})], \quad E(t, \mathbf{x}, k) = C \mathscr{E}(t, \mathbf{x})^{\frac{2}{3}} k^{-\frac{5}{3}}. \tag{5.142}$$

Moreover, we assume that

$$\overline{|D\mathbf{v}(t, \mathbf{x})|^2} = \frac{1}{2} \int_{k_1(t,\mathbf{x})}^{k_2(t,\mathbf{x})} k^2 E(k) dk. \tag{5.143}$$

The functions k_i allow us to define a local inertial range at each $(t, \mathbf{x}) \in Q$. The bound $k_2(t, \mathbf{x})$ refers to a local Kolmogorov scale λ_0, and according to Remark 5.3 and formula (5.48) that initially defines λ_0, we are led to write

$$\forall\, (t, \mathbf{x}) \in I \times \omega, \quad \lambda_0(t, \mathbf{x}) = k_2(t, \mathbf{x})^{-1} = \nu^{\frac{3}{4}} \mathscr{E}(t, \mathbf{x})^{-\frac{1}{4}}. \tag{5.144}$$

Formula (5.143) is based on formula (5.46), rigorously proved for isotropic flows. We assume here that the flow dissipates all the energy in the inertial range, which is consistent with the usual assumptions about turbulence.

We are now in a point to establish the link between the $-5/3$ law and the SGM, based on the assumptions above. Let $\delta > 0$ be any cutoff length, $k_c = \delta^{-1}$, so that

$$\eta_2 \leq k_c \leq \eta_3, \tag{5.145}$$

to ensure that $\forall\, (t, \mathbf{x}) \in I \times \omega$, $k_1(t, \mathbf{x}) < k_c < k_2(t, \mathbf{x})$. The idea is that we cannot simulate all the flow scales in the inertial range, and indeed we only intend to simulate the scales in the range $[k_1, k_c]$. Therefore, δ replaces λ_0, which allows a natural eddy viscosity to be defined through the formula

$$\delta = \nu_t(t, \mathbf{x})^{\frac{3}{4}} \mathscr{E}(t, \mathbf{x})^{-\frac{1}{4}} \ \text{ or } \ \nu_t(t, \mathbf{x}) = \mathscr{E}^{\frac{1}{3}}(t, \mathbf{x}) \delta^{\frac{4}{3}}, \tag{5.146}$$

The goal is to exploit our assumptions, especially the $-5/3$ law, to express $\mathscr{E}$ (without giving the dependence in $(t, \mathbf{x}) \in I \times \omega$ systematically) in terms of δ and $|D\overline{\mathbf{v}}|$ to derive formula (5.154), $\nu_t = C\delta^2 |D\overline{\mathbf{v}}|$.

We obtain from the Reynolds decomposition

$$\overline{|D\mathbf{v}|^2} = |D\overline{\mathbf{v}}|^2 + \overline{|D\mathbf{v}'|^2}, \tag{5.147}$$

where we assume that the mean field scales span the range $[k_1, k_c]$, while the fluctuation scales span the range $[k_c, k_2]$. This means, according to (5.143), $\forall\, (t, \mathbf{x}) \in D$,

$$
\begin{aligned}
|D\overline{\mathbf{v}}(t,\mathbf{x})|^2 &= \frac{1}{2}\int_{k_1(t,\mathbf{x})}^{k_c} k^2 E(t,\mathbf{x},k)dk, \\
\overline{|D\mathbf{v}'(t,\mathbf{x})|^2} &= \frac{1}{2}\int_{k_c}^{k_2(t,\mathbf{x})} k^2 E(t,\mathbf{x},k)dk.
\end{aligned}
\tag{5.148}
$$

We deduce from the $-5/3$ law

$$
|D\overline{\mathbf{v}}|^2 = \frac{3C}{8}\mathscr{E}^{\frac{2}{3}}(k_c^{\frac{4}{3}} - k_1^{\frac{4}{3}}) \approx \frac{3C}{8}\mathscr{E}^{\frac{2}{3}}k_c^{\frac{4}{3}} = \frac{3C}{8}\mathscr{E}^{\frac{2}{3}}\delta^{-\frac{4}{3}} \tag{5.149}
$$

by using $k_1 << k_c$, which yields

$$
\mathscr{E} = C\delta^2|D\overline{\mathbf{v}}|^3, \tag{5.150}
$$

where C is always a generic notation for any nonnegative dimensionless constant. By combining (5.150) and (5.146) we arrive at

$$
\nu_t = C\delta^2|D\overline{\mathbf{v}}| \tag{5.151}
$$

as expected. □

Remarkably, we recover Prandtl's structure for the eddy viscosity, introduced in Sect. 3.5.3, with a mixing length related to the cutoff length by

$$
\ell = \sqrt{C}\,\delta.
$$

In practice a numerical method is required to compute the mean flow. The subgrid scales must also be modeled. The cutoff length is then the grid size h (that can vary in space). This leads to the Smagorinsky model (5.138) in which $\ell = C_S\,h$ is the characteristic length associated with the subgrid scales. A reference value for the constant is $C_S \simeq 0.18$ that may be obtained by a refinement of the preceding statistical analysis. However, in practice, its value is adjusted to better fit the results for each actual flow, taking values in the range $[0.01, 0.2]$.

Near the solid walls, eddy diffusion effects decrease, and the SGM should be adapted. This is usually done either by means of wall laws or by means of "damping functions"that adjust the constant to the distance to the wall, so that (5.138) is changed to (cf. Van Driest [32])

$$
\nu_t = f_\mu(z')\,(C_S\,h)^2|D\overline{\mathbf{v}}|, \quad \text{with} \quad f_\mu(z^+) = 1 - \exp(-z'/\overline{z}), \tag{5.152}
$$

for some dimensional $\overline{z}$ located in the logarithmic layer, typically $\overline{z} \in [20, 30]$.

However, the Smagorinsky model is over-diffusive, as the eddy diffusion affects all resolved flow scales (cf. Zhang et al. [33]). Modern LES models (at least partially) overcome this difficulty, by approximating the filtering by the function G_δ in (5.139). For instance, using asymptotic expansions in the spectral space leads to models such as the Taylor or the rational models (cf. [5, 18]). Some technical difficulties arise: some of these are related to the stability of the models, although the main difficulty is the treatment of the so-called commutation error that only vanishes if the normal stresses vanish on the boundary of the domain. This error may be of the same order as the divergence of the Reynolds stress tensor (cf. [2,10]) and is difficult to control in practice. However, some of these models with suitable numerical discretizations can provide a large improvement in accuracy with respect to the Smagorinsky model (cf. [9]).

5.5.2 *Mathematical and Numerical Analyses of LES Model*

The LES model introduced in the preceding section is intended to model the large scales of the flow, above the cutoff length δ. For this reason it is considered as a continuous model that requires a subsequent numerical discretization to be solved. The equations of this LES model read

$$\begin{cases} \partial_t \overline{\mathbf{v}} + (\overline{\mathbf{v}} \cdot \nabla)\, \overline{\mathbf{v}} - \nabla \cdot ((\nu + \nu_t(\overline{\mathbf{v}}))D\overline{\mathbf{v}}) + \nabla \tilde{p} = \mathbf{f} & \text{in } Q, \\ \nabla \cdot \overline{\mathbf{v}} = 0 & \text{in } Q, \\ \overline{\mathbf{v}} = \overline{\mathbf{v}}_0 & \text{at } t = 0, \end{cases} \tag{5.153}$$

with

$$\nu_t(\overline{\mathbf{v}}) = \ell^2\, |D\overline{\mathbf{v}}|, \tag{5.154}$$

for some subgrid characteristic length $\ell > 0$ associated with the cutoff length by $\ell = C_S\, \delta$. The boundary conditions may include wall laws.

The mathematical and numerical analyses of this model have been the object of a large amount of research. It belongs to the class of models introduced by Ladyzhenskaya [16, 17], whose solutions belong to $H^1([0,T], \mathbf{L}^2(\Omega)) \cap L^3([0,T], \mathbf{W}^3(\Omega))$ for well-suited boundary conditions. This is the case of Dirichlet boundary conditions treated in [16, 17] and of mixed Dirichlet–Navier boundary conditions treated by John in [9]. This regularity allows us to prove uniqueness and, furthermore, well-posedness: the solution depends continuously on the data $\mathbf{f}$ and $\overline{\mathbf{v}}_0$ (and, eventually, of the boundary data). This is coherent with the intuitive idea that the averaged flow should have some additional smoothness with respect to the Navier–Stokes equations. In this sense, LES models are more satisfying from the mathematical point of view than RANS models. Some subsequent work has analyzed Smagorinsky-like models, with variable eddy diffusion that includes dynamic modeling of the C_S constant and wall damping (cf. [30]).

The numerical approximation of the LES model (5.153) considered as a continuous model has to be performed by considering a cutoff length δ independent of the grid size. The question to be faced is assuming that the model indeed governs the behavior of the targeted large-scale flow, whether this large-scale flow is well described by the numerical approximation considered. The ideal objective is to obtain error estimates independent of the Reynolds number of the flow. In John and Layton [11] such estimates are obtained with constants that depend only on δ, for solutions that belong to $L^2([0,T], \mathbf{W}^{1,\infty}(\Omega))$ (with additional regularity with respect to the "natural one"mentioned above). This analysis is extended in [8] to more general LES models.

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Chapter 6
Steady Navier–Stokes Equations with Wall Laws and Fixed Eddy Viscosities

Abstract We consider the Navier–Stokes equations with a given eddy viscosity and a wall law as boundary condition. Once the functional background is properly established, we prove the existence of a weak solution to this problem, obtained by approximations based on a singular perturbation of the incompressibility constraint. We investigate two ways of establishing the existence of approximate solutions: the standard Galerkin method and the linearization method by Schauder's fixed-point theorem. Estimates for the velocity are deduced from energy equalities, whereas estimates for the pressure are derived from appropriate potential vectors. Cases where the solution is unique are also investigated. To achieve our goal, we also develop several theoretical tools that will be used for the analysis of general turbulent models in the following chapters, such as the convergence of families of variational problems or the energy method.

6.1 Introduction

The previous modeling chapters yield several continuous partial differential equation (PDE) systems allowing the calculation of mean fields characterizing turbulent flows, in particular the mean velocity and pressure, denoted from now by $(\mathbf{v}, p)$ instead of $(\overline{\mathbf{v}}, \overline{p})$ for simplicity. We also write Ω and Γ instead of Ω_c and Γ_c. The reference boundary condition (BC) for $\mathbf{v}$ is the general wall law (5.136), which will be a guideline throughout the rest of the book. We are now ready to proceed with the mathematical analysis of the models.

According to the scheme laid out in Sect. 4.4.4.4, we aim to prove the existence of a weak solution to the TKE model (4.137), either in the steady-state case or the evolutionary case, where $\mathbf{v}$ satisfies the wall law (5.136) throughout Γ and where the turbulent kinetic energy k satisfies (5.137) throughout Γ. The case of mixed BC, wall law/no-slip, will be considered from Chap. 9.

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The complete study of the TKE model is technically very complex, since many nonlinear terms interact. This leads us to decouple the difficulties, starting with those due to the convection term and the wall law. We focus in this chapter on the steady-state Navier–Stokes equations (NSE) with a wall law and a given eddy viscosity $\nu_t = \nu_t(\mathbf{x})$:

$$\begin{cases} (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \nabla\cdot[(2\nu+\nu_t)\,D\mathbf{v}] + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega, \\ -\,[(2\nu+\nu_t)\,D\mathbf{v}\cdot\mathbf{n}]_\tau = g(\mathbf{v})_\tau & \text{on } \Gamma, \\ \mathbf{v}\cdot\mathbf{n} = 0 & \text{on } \Gamma. \end{cases} \tag{6.1}$$

The steady-state NSE with various boundary conditions (BC) have been widely studied, but not with nonlinear wall laws nor eddy viscosities. We refer to the thorough presentation by G. Galdi [8] and further references therein. As far as we are aware, system (6.1) has not yet been discussed in the available literature.

Our goal is to lay the mathematical foundations for analyzing turbulence models and to prove that system (6.1) has a weak solution by two different methods, Galerkin or linearization, assuming:

(i) $\nu_t \in L^\infty(\mathbb{R})$ is nonnegative,
(ii) $g = \mathbf{v}H(|\mathbf{v}|) = \mathbf{v}\tilde{H}(\mathbf{v})$ is a wall law derived from (5.127), which satisfies all the properties listed in Lemma 5.6,
(iii) Ω is of class C^m $(m \geq 1)$,[1]
(iv) $\mathbf{f} \in \mathbf{W}(\Omega)'$, where $\mathbf{W}(\Omega)$ is defined by (6.2) just below.

This chapter is organized as follows.

The first task is to properly establish the variational formulation of the NSE (6.1) which yields weak solutions. Unlike the turbulent solutions introduced in Sect. 3.4.2, based on the Leray projector over spaces of free-divergence fields, the weak solutions to the NSE (6.1) considered in this chapter are given by the mixed variational problem (6.12) below, denoted by $\mathscr{VP}$, where not only the velocity $\mathbf{v}$ but also the pressure p is involved. In particular, the velocity space function $\mathbf{W}(\Omega)$ upon which $\mathscr{VP}$ is based can be defined by

$$\mathbf{W}(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) \text{ such that } \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma\}, \tag{6.2}$$

and is studied in detail in Sect. 6.2.1. Note that p will be sought in the space

$$L^2_0(\Omega) = \{p \in L^2(\Omega);\ \int_\Omega p(\mathbf{x})d\mathbf{x} = 0\}.$$

[1] According to a private communication by L. Tartar, the results of this chapter should work for Lipschitz domains.

We then detail how to derive the variational problem $\mathscr{VP}$ from the NSE in Sect. 6.2.3.

The operators involved in $\mathscr{VP}$—*[transport–diffusion]+pressure+wall law*—are carefully analyzed in Sect. 6.3, as well as related compactness properties essential to the analysis carried out in this and the following chapters. These compactness properties are based on standard results of functional analysis and are used at each step of the process developed throughout this and the following chapters. To simplify the presentation as much as possible and to avoid useless duplications, we will introduce integrated theoretical packages, such as the Velocity Extracting Subsequence Principle (VESP) in Sect. 6.3.3, which integrates all the properties satisfied by bounded sequences in $\mathbf{W}(\Omega)$ (existence of weak subsequential limits, compactness in $\mathbf{L}^q(\Omega)$, $q < 6$, properties of the corresponding traces in $\mathbf{H}^{1/2}(\Gamma)$ etc...).

The technical groundwork is completed in Sect. 6.4. In particular, we derive a priori estimates for the velocity and the pressure, which make $\mathscr{VP}$ consistent. The velocity estimate is obtained by the standard energy equality procedure. The derivation of the pressure estimate by suitable potential vector fields is based on the method developed in Bulíček–Málek–Rajagopal [5], where general evolutionary NSE are studied with the linear Navier BC (2.135).

Furthermore, in many cases solutions to nonlinear variational problems are constructed by approximations, which is the case for $\mathscr{VP}$. As is usual in analyses of the incompressible Navier–Stokes equations, the main difficulty arises from the pressure term and the free-divergence constraint. Following [5] again, in Sect. 6.4.3 we approximate the constraint $\nabla \cdot \mathbf{v} = 0$ by the equation

$$-\varepsilon \Delta p + \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \quad \frac{\partial p}{\partial \mathbf{n}}|_\Gamma = 0, \quad \int_\Omega p(\mathbf{x})\, d\mathbf{x}, \tag{6.3}$$

which has a unique solution, given any $\mathbf{v} \in \mathbf{W}(\Omega)$. This allows the pressure to be expressed as a function of the velocity and to introduce approximate variational problems $\mathscr{VP}_\varepsilon$ (see (6.73) below), in which only the velocity is involved. To make the connection between $\mathscr{VP}_\varepsilon$ and $\mathscr{VP}$, we define the convergence of a family (or sequence) of variational problems in Sect. 6.4.5, which will be a very useful concept for all the problems studied in this book.

We then prove that the family $(\mathscr{VP}_\varepsilon)_{\varepsilon>0}$ converges to $\mathscr{VP}$ when $\varepsilon \to 0$. As a result, the existence of a solution to each $\mathscr{VP}_\varepsilon$ yields the existence of a solution to $\mathscr{VP}$. After this, we will focus on $\mathscr{VP}_\varepsilon$ to achieve our goal, in developing two methods:

(i) The standard Galerkin method,
(ii) The linearization method, by Schauder's fixed-point theorem.

The Galerkin method, based on projections on finite-dimensional subspaces of $\mathbf{W}(\Omega)$, is explained in Sect. (6.5). Apart from its undeniable pedagogical interest, this presentation is also a preparation for finite-element analyses carried out from Chap. 9.

The linearization procedure is performed in Sect. 6.6, which is made possible by the structure of the wall law. It is the basis for generating practical algorithms for computing solutions of the nonlinear NSE (6.1), solving linear problems in successive iterations. Section 6.6 also prepares for the analysis of the TKE model carried out in Chap. 7.

To be more specific, given any $\mathbf{z} \in \mathbf{W}(\Omega)$, the linearized NSE at $\mathbf{z}$ are given by the linear PDE system

$$\begin{cases} (\mathbf{z} \cdot \nabla)\, \mathbf{v} - \nabla \cdot [2\nu + \nu_t)\, D\mathbf{v}] + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{v} = 0 & \text{in } \Omega, \\ -\left[(2\nu + \nu_t)\, D\mathbf{v} \cdot \mathbf{n}\right]_\tau = H(|\mathbf{z}|)\mathbf{v}_\tau & \text{on } \Gamma, \\ \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \Gamma. \end{cases} \tag{6.4}$$

The corresponding variational problem is the problem (6.93) denoted by $\mathscr{LP}_{\mathbf{z}}$. To understand how the linear NSE (6.4) yield weak solutions to the NSE (6.1), assume that

(i) $\forall\, \mathbf{z} \in \mathbf{W}(\Omega)$, $\mathscr{LP}_{\mathbf{z}}$ has a unique solution $(\mathbf{v}(\mathbf{z}), p(\mathbf{z})) \in \mathbf{W}(\Omega) \times L^2_0(\Omega)$,
(ii) the application $\mathbf{z} \to \mathbf{v}(\mathbf{z})$ has a fixed point $\mathbf{v}_0$, which means $\mathbf{v}(\mathbf{v}_0) = \mathbf{v}_0$,

therefore, $(\mathbf{v}(\mathbf{v}_0), p(\mathbf{v}_0))$ is a solution to $\mathscr{VP}$. Unfortunately, things become quite difficult and tricky, mainly due to (6.4.iii).[2] Indeed, $H(|\mathbf{z}|)$ may vanish on one part of Γ, leading to a loss of coercivity which does not permit a direct proof of the existence of a solution to $\mathscr{LP}_{\varepsilon,\mathbf{z}}$ by the Lax–Milgram theorem, which is the standard procedure. For this reason, we introduce several interconnected variational problems and proceed to develop the following somewhat circuitous process:

1. We add to (6.4.iii) an extra linear nonnegative term, so that (6.4.iii) is replaced by, for any $\eta > 0$,

$$-\left[(2\nu + \nu_t)\, D\mathbf{v} \cdot \mathbf{n}\right]_\tau = (H(|\mathbf{z}| + \eta)\mathbf{v}_\tau \quad \text{on } \Gamma,$$

   which yields an η regularization to (6.4) whose corresponding variational problem is denoted by $\mathscr{LP}_{\eta,\mathbf{z}}$.
2. We perform in $\mathscr{LP}_{\eta,\mathbf{z}}$ the ε-regularization (6.3), which provides a further linear problem $\mathscr{LP}_{\varepsilon,\eta,\mathbf{z}}$.
3. We show that for a fixed $\mathbf{z} \in \mathbf{W}(\Omega)$ and a fixed $\eta > 0$, the family $(\mathscr{LP}_{\varepsilon,\eta,\mathbf{z}})_{\varepsilon>0}$ converges to $\mathscr{LP}_{\eta,\mathbf{z}}$ when $\varepsilon \to 0$.
4. We prove the existence of a unique solution to $\mathscr{LP}_{\varepsilon,\eta,\mathbf{z}}$ and deduce the existence of a unique solution $(\mathbf{v}_\eta(\mathbf{z}), p_\eta(\mathbf{z}))$ to $\mathscr{LP}_{\eta,\mathbf{z}}$.
5. We prove that the application $\mathbf{z} \to \mathbf{v}_\eta(\mathbf{z})$ fulfills the conditions for the application of Schauder's fixed-point theorem [16] and therefore has a fixed point which

[2] We implicitly number the equations of a given coupled system in the order they are set out with roman numerals.

provides a solution to a variational problem denoted by $\mathscr{V}\mathscr{P}_\eta$, corresponding to the η-regularization of the nonlinear NSE.
6. We show that the family $(\mathscr{V}\mathscr{P}_\eta)_{\eta>0}$ converges to $\mathscr{V}\mathscr{P}$ and conclude.

This chapter concludes with some thoughts concerning uniqueness.

6.2 Variational Formulation

For any given system (S) of PDE there exists a corresponding variational problem $\mathscr{U}\mathscr{P}$, also called its variational formulation. A variational problem involves space functions deduced from a priori estimates and the expression of the equations as dual forms, resulting from integration by parts based on the Stokes formula.[3] Solutions to $\mathscr{U}\mathscr{P}$ are weak solutions to (S). This section is devoted to establishing the variational framework for the study of the NSE (6.1).

We start by analyzing the main functional space $\mathbf{W}(\Omega)$ we shall use in this and the next two chapters, which is the natural space in which to seek the velocity appearing in the NSE (6.1). Roughly speaking, $\mathbf{W}(\Omega)$ is the set of vector fields in $\mathbf{H}^1(\Omega)$, the normal components of which vanish on Γ. We establish the density in $\mathbf{W}(\Omega)$ of C^m fields having zero normal component on Γ, assuming Ω is of class C^m.

We then elaborate the general abstract concept of variational problems and determine which one is associated with the NSE (6.1).

Before proceeding, we recall the functional spaces introduced in Sect. 3.4.1:

(a) $\mathbf{W}^{s,p}(\Omega) = W^{s,p}(\Omega)^3$, $\mathbf{L}^q(\Omega) = W^{0,p}(\Omega)^3 = L^q(\Omega)^3$, denoting by $||\cdot||_{s,p,\Omega}$ the corresponding norm,
(b) $\mathbf{H}^s(\Omega) = W^{s,2}(\Omega)^3$,
(c) the trace operator[4] γ_0 and the normal trace operator γ_n,
(d) the trace spaces $W^{s,p}(\Gamma) = \gamma_0(W^{s+1/p,p}(\Omega))$ $(s > 0)$, denoting by $||\cdot||_{s,p,\Gamma}$ the corresponding norms,
(e) $\mathbf{H}^s(\Gamma) = \gamma_0(\mathbf{H}^{s+1/2})$.

Section A.1 of the Tool Box[5] summarizes the necessary prerequisites concerning Sobolev spaces that are necessary for what follows. The reader is also referred to Adams–Fournier [1] and Tartar [18] for further details about Sobolev spaces.

[3] The Stokes formula may also be referred to as Green's formula.

[4] if $\boldsymbol{\varphi} \in \mathscr{D}(\overline{\Omega})$, $\gamma_0\boldsymbol{\varphi} = \boldsymbol{\varphi}|_\Gamma$, $\gamma_n\boldsymbol{\varphi} = \boldsymbol{\varphi}\cdot\mathbf{n}|_\Gamma$.

[5] Appendix A at the end of the book, also referred to as [TB].

6.2.1 Functional Spaces

6.2.1.1 Definition and Characterization of $\mathbf{W}(\Omega)$

The wall-law condition (6.1.iii) will be integrated into the variational formulation, whereas (6.1.iv), i.e., $\mathbf{v} \cdot \mathbf{n}|_\Gamma = 0$, combined with the proposition 6.1 below, motivates the definition of $\mathbf{W}(\Omega)$.

Normal practice defines the space function as the adherence of the appropriate smooth vector fields space, deduced from the boundary conditions, leading to the definition

$$\mathscr{W}_m(\Omega) = \{\boldsymbol{\varphi} \in \mathscr{C}^m(\overline{\Omega})^3 \text{ such that } \boldsymbol{\varphi} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}, \tag{6.5}$$

$$\mathbf{W}(\Omega) = \overline{\mathscr{W}_m(\Omega)}^{\mathbf{H}^1(\Omega)}, \tag{6.6}$$

where the integer $m \geq 1$ is the regularity order of the domain Ω. Although this definition is unambiguous, it must be completed by a clear characterization of $\mathbf{W}(\Omega)$, otherwise the variational formulation deduced from $\mathbf{W}(\Omega)$ may not yield solutions to the NSE (6.1). Moreover, it seems natural to consider the space

$$\hat{\mathbf{W}}(\Omega) = \{\mathbf{w} \in \mathbf{H}^1(\Omega) \text{ such that } \gamma_n \mathbf{w} = 0 \text{ on } \Gamma\}.$$

According to Lemma A.1 in [TB], the condition $\gamma_n \mathbf{w} = 0$ is meaningful in $L^4(\Gamma)$. In what follows, we show that

Theorem 6.1. $\mathbf{W}(\Omega) = \hat{\mathbf{W}}(\Omega)$.

Proof. Obviously, $\mathbf{W}(\Omega) \subset \hat{\mathbf{W}}(\Omega)$. In order to prove that $\hat{\mathbf{W}}(\Omega) \subset \mathbf{W}(\Omega)$, we must show that any $\mathbf{w} \in \hat{\mathbf{W}}(\Omega)$ is a limit of a sequence in $\mathscr{W}_m(\Omega)$. To do so, we recycle the method detailed in Brézis [4], Chap. IX, based on local charts and a partition of unity.

STEP 1. *Local charts.* Let $Q \subset \mathbb{R}^3$ denote the unit cylinder,

$$Q = \{(x_1, x_2, x_3),\ x_1^2 + x_2^2 < 1,\ |x_3| < 1\},$$

and $Q_+ = Q \cap \{x_3 > 0\}$, $Q_0 = Q \cap \{x_3 = 0\}$. As Γ is compact and Ω is of class C^m, there exist open sets in $\mathbb{R}^3$, $U_1, \cdots, U_k$, such that $\Gamma \subset \bigcup_{j=1}^k U_j$, C^m-diffeomorphisms $H_j : U_j \to \mathbb{R}^3$, $j = 1, \cdots, k$, such that

$$H_j(U_j) = Q, \quad H_j(\Omega \cap U_j) = Q_+, \quad H_j(\Gamma \cap U_j) = Q_0.$$

We denote by $\mathbf{x}' = H_j(\mathbf{x})$ any point in $H_j(U_j)$.

STEP 2. *Partition of unity.* We know that there exists a family of functions $\theta_0, \theta_1, \cdots, \theta_k$, of class C^∞, such that

(i) $0 \leq \theta_j \leq 1$, $\forall\, j = 0, 1, \cdots, k$ and $\sum_{j=0}^{k} \theta_j = 1$ over $\mathbb{R}^3$,

(ii) $\theta_0 \in \mathscr{D}(\Omega)$, $\mathrm{supp}(\theta_j)$ is compact and $\mathrm{supp}(\theta_j) \subset U_j$, $\forall\, j = 1, \cdots, k$.

See for instance [1, 4, 17].

STEP 3. *Smooth approximations.* Let $\mathbf{w} \in \hat{\mathbf{W}}(\Omega)$, which we decompose into

$$\mathbf{w} = \sum_{j=0}^{k} \mathbf{w}_j, \quad \mathbf{w}_j = \theta_j \mathbf{w} \in \hat{\mathbf{W}}(\Omega), \quad j = 0, 1, \cdots, k.$$

We already know that $\mathbf{w}_0 \in \mathbf{H}_0^1(\Omega)$, which leads to the existence of a sequence $(\boldsymbol{\varphi}_0^{(n)})_{n \in \mathbb{N}}$ in $\mathscr{D}(\Omega)^3$ that converges to $\mathbf{w}_0$ for the $\mathbf{H}^1(\Omega)$ norm (see in [4]). We have to approach each $\mathbf{w}_j$ by sequences in $\mathscr{W}_m(\Omega)$.

First, we describe the transformation mapping $H_j : U_j \mapsto Q_+$, which will have the same structure for each open set U_j. By a rigid rotation we can suppose that

$$U_j := \{\mathbf{x} \in \mathbb{R}^3 : \; x_1^2 + x_2^2 < a, \; h(x_1, x_2) < x_3 < a + h(x_1, x_2)\},$$

for some $a > 0$, where for simplicity $h(x_1, x_2)$ represents $h_j(x_1, x_2)$. The boundary corresponds to points such that $h(x_1, x_2) = x_3$, while those $\mathbf{x}$ such that $x_1^2 + x_2^2 < a$ and $x_3 < h(x_1, x_2)$ belong to the complementary of Ω. The parametrization shows that $h \in C^m$ and that $\partial_1 h(0,0) = \partial_2 h(0,0) = 0$. We consider the change of variables $\mathbf{x}' = H(\mathbf{x})$ given by

$$(x_1', x_2', x_3') = (x_1, x_2, x_3 - h(x_1, x_2)),$$

observing that it maps $U_j \cap \Gamma$ on Q_0 and that the Jacobian determinant of H is identically equal to one. The covariant transformation of a vector field $\mathbf{w}_j$ defined on U_j into vector fields $\tilde{\mathbf{w}}_j$ defined on Q (by means of H) is $\forall\, \mathbf{x}' = H_j(\mathbf{x}) \in Q_+ \cup Q_0$,

$$\tilde{w}_{j,1}(\mathbf{x}') = w_{j,1}(H^{-1}(\mathbf{x}')),$$
$$\tilde{w}_{j,2}(\mathbf{x}') = w_{j,2}(H^{-1}(\mathbf{x}')),$$
$$\tilde{w}_{j,3}(\mathbf{x}') = w_{j,3}(H^{-1}(\mathbf{x}')) - \partial_1 h(x_1, x_2) w_{j,1}(H^{-1}(\mathbf{x}')) - \partial_2 h(x_1, x_2) w_{j,2}(H^{-1}(\mathbf{x}')).$$

Observe that if $\mathbf{x}' \in Q_0$, then $\tilde{w}_{j,3}(x_1', x_2', 0) = 0$ (that is, $\tilde{\mathbf{w}}_j(\mathbf{x}')$ is tangential to Q_0) if and only if $\mathbf{w}_j(\mathbf{x})$ is tangential to Γ at $\mathbf{x}$. The outward normal unit vector $\mathbf{n}(\mathbf{x})$ is set to the vector $\tilde{\mathbf{n}}(\mathbf{x}')$ such that $\tilde{n}_1(\mathbf{x}') = \tilde{n}_2(\mathbf{x}') = 0$.

By adapting the technique of [4], we deduce that for each $\tilde{\mathbf{w}}_j$ there exists a sequence $(\tilde{\varphi}_{j,3}^{(n)})_{n \in \mathbb{N}} \in C^\infty(\overline{Q}_+)$ which vanishes in a neighborhood of Q_0 and which converges to $\tilde{w}_{j,3}$ in $H^1(Q_+)$. Similarly, there exists $(\tilde{\varphi}_{j,\alpha}^{(n)})_{n\in\mathbb{N}} \in C^\infty(\overline{Q}_+)$, $\alpha = 1, 2$, which converges to $\tilde{w}_{j,\alpha}$ in $H^1(Q_+)$. Consequently,

$$\tilde{\boldsymbol{\varphi}}_j^{(n)} = (\tilde{\varphi}_{j,1}^{(n)}, \tilde{\varphi}_{j,2}^{(n)}, \tilde{\varphi}_{j,3}^{(n)}) \in C^\infty(\overline{Q_+}), \quad \text{satisfies} \quad \forall\, n \in \mathbb{N}, \ \tilde{\boldsymbol{\varphi}}_j^{(n)} \cdot \tilde{\mathbf{n}} = 0 \ \text{ on } \ Q_0, \tag{6.7}$$

and the sequence $(\tilde{\boldsymbol{\varphi}}_j^{(n)})_{n\in\mathbb{N}}$ converges to $\tilde{\mathbf{w}}_j$ in $\mathbf{H}^1(\Omega)$. Let us consider $\forall\, \mathbf{x} \in U_j$,

$$\begin{aligned}
\boldsymbol{\varphi}_{j,1}(\mathbf{x}) &= \tilde{\boldsymbol{\varphi}}_{j,1}(H(\mathbf{x})) \\
\boldsymbol{\varphi}_{j,2}(\mathbf{x}) &= \tilde{\boldsymbol{\varphi}}_{j,2}(H(\mathbf{x})) \\
\boldsymbol{\varphi}_{j,3}(\mathbf{x}) &= \tilde{\boldsymbol{\varphi}}_{j,3}(H(\mathbf{x})) + \partial_1 h(x_1, x_2)\tilde{\boldsymbol{\varphi}}_{j,1}(H(\mathbf{x})) + \partial_2 h(x_1, x_2)\tilde{\boldsymbol{\varphi}}_{j,2}(H(\mathbf{x}))
\end{aligned}$$

(i.e., the inverse of the covariant transformation) and observe that they all belong to $C^m(\overline{U_j \cap \Omega})$, because H is a C^m-diffeomorphism. Since the support θ_j is compact in U_j, then each $\tilde{\boldsymbol{\varphi}}_j^{(n)}$ can be constructed to be equal to zero in a neighborhood of the top of Q_+, $\{x_3 = 1\} \cap Q_+$, and its lateral boundaries, $\{x_1^2 + x_2^2 = 1, 0 < x_3 < 1\}$. Therefore, the extension of each $\boldsymbol{\varphi}_j^{(n)}$ by zero outside $\overline{U_j \cap \Omega}$ (without changing the notation) is in $C^m(\overline{\Omega})$. Moreover, we infer from (6.7) that $\forall\, j = 1 \cdots k$, $\forall\, n \in \mathbb{N}$, $\boldsymbol{\varphi}_j^{(n)} \cdot \mathbf{n} = 0$ on Γ. Therefore,

$$\boldsymbol{\varphi}^{(n)} = \boldsymbol{\varphi}_0^{(n)} + \sum_{j=1}^{k} \boldsymbol{\varphi}_j^{(n)} \in \mathscr{W}_m(\Omega),$$

and the sequence $(\boldsymbol{\varphi}^{(n)})_{n\in\mathbb{N}}$ converges to $\mathbf{w}$, which ends the proof. □

6.2.1.2 Topology of $\mathbf{W}(\boldsymbol{\Omega})$

The space $\mathbf{W}(\Omega)$ is a closed subspace of $\mathbf{H}^1(\Omega)$ and thus a Hilbert space endowed with the $\mathbf{H}^1(\Omega)$ norm $||\cdot||_{1,2,\Omega}$. According to a variant of Korn's inequality (see in [5, 7, 15] and Sect. A.4.4 in [TB]), this norm is equivalent over $\mathbf{W}(\Omega)$ to the Hilbertian norm

$$\|\mathbf{w}\|_{\mathbf{W}(\Omega)} = (\|D\mathbf{w}\|_{0,2,\Omega}^2 + ||\gamma_0 \mathbf{w}||_{0,2,\Gamma}^2)^{1/2}, \tag{6.8}$$

which derives from the scalar product defined over $\mathbf{W}(\Omega)$,[6]

$$(\mathbf{v}, \mathbf{w}) = (D\mathbf{v}, D\mathbf{w})_\Omega + (\gamma_0 \mathbf{v}, \gamma_0 \mathbf{w})_\Gamma. \tag{6.9}$$

---

[6]Recall that for all measurable sets U whose measure is denoted by λ, $\forall\, u \in L^p(U)$, $\forall v \in L^{p'}(U)$, we denote for simplicity $(u, v)_U = \displaystyle\int_U u(\mathbf{x})v(\mathbf{x})d\lambda(\mathbf{x})$.

In the following sections, we may occasionally write $\mathbf{w}$ instead of $\gamma_0 \mathbf{w}$ when no risk of confusion occurs. We shall use either $||\cdot||_{1,2,\Omega}$ or $||\cdot||_{\mathbf{W}(\Omega)}$.

6.2.1.3 Space for the Pressure

We must determine the appropriate space for the pressure. Following the result of Sect. 6.4.2 below, we will seek the pressure p in $L^2(\Omega)$. We know from Sect. 3.4.1 that the pressure is defined up to a constant and the natural space is therefore the quotient space $L^2(\Omega)/\mathbb{R}$ equipped with the norm defined by (3.60). We know that this space is isomorphic to the space $L^2_0(\Omega)$,

$$L^2_0(\Omega) = \{q \in L^2_0(\Omega) \text{ such that } \int_\Omega q(\mathbf{x})d\mathbf{x} = 0\},$$

which is more convenient to use.

6.2.2 *Generalities Concerning Variational Problems*

Generally speaking, a variational problem $\mathscr{UP}$ is given by:

(i) Two reflexive Banach spaces X_1 and X_2: X_1 is the space of unknowns and X_2 is the space of tests,
(ii) an operator $\mathscr{U} : X_1 \to X_2'$,
(iii) a source term $F \in X_2'$,

and is formulated as

$$\text{Find } \zeta \in X_1 \text{ such that } \forall\, \vartheta \in X_2,$$

$$\langle \mathscr{U}(\zeta), \vartheta \rangle = \langle F, \vartheta \rangle, \tag{6.10}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product. The spaces X_1 and X_2 are Banach spaces for simplicity, but might also be general topological spaces, as in the NS-TKE model investigated in Chap. 7. We say that $\mathscr{UP}$ admits a solution if there exists $\zeta \in X_1$ such that (6.10) is satisfied $\forall\, \vartheta \in X_2$. Note that we may consider a priori solutions which might not exist, in order to derive information about them from formal analysis.

Given any PDE system (S), we can derive a variational problem, whose solutions are weak solutions to (S). On the one hand, the choice of the operator $\mathscr{U}$ is deduced from (S) by a process based on multiplying the equations by the given smooth test functions and performing suitable integrations by parts.

On the other hand, there may be many possible choices for the spaces X_1 and X_2, as long as the duality product in (6.10) makes sense, some spaces however being less appropriate than others. The choice of the most appropriate spaces is generally driven by a priori estimates derived from the equations. In this case, we say that the variational problem is consistent.

Attention must be paid to the fact that there exist variational problems which may not derive from any PDE system. To understand this, let us consider the following simple example. Let $(e_n)_{e\in\mathbb{N}}$ be the sequence of eigenfunctions of the operator $-\Delta$ over $H_0^1(\Omega)$ and $(\lambda_n)_{n\in\mathbb{N}}$ the corresponding sequence of eigenvalues. We know from Brézis [4] that $(e_1, \cdots, e_n, \cdots)$ is a Hilbertian basis[7] of $L^2(\Omega)$ and

$$H_0^1(\Omega) = \{u \in L^2(\Omega);\ \sum_{n=1}^{\infty} \lambda_n u_n^2 < \infty\}, \quad \text{where } \ u_n = (u, e_n)_\Omega.$$

Let $\mathscr{U}$ be the operator defined by

$$\langle \mathscr{U}(u), v\rangle = \sum_{n=1}^{\infty} \frac{2 + n u_n^2}{1 + n u_n^2} \lambda_n u_n v_n. \tag{6.11}$$

Take $X_1 = X_2 = H_0^1(\Omega)$ and as F any form in $H^{-1}(\Omega)$. The resulting variational problem is meaningful. Indeed, we verify by Cauchy–Schwarz inequality that

$$\begin{aligned}|\langle \mathscr{U}(u), v\rangle| \le 2 \sum_{n=1}^{\infty} \lambda_n |u_n||v_n| &\le 2(\sum_{n=1}^{\infty} \lambda_n |u_n|^2)^{\frac{1}{2}} (\sum_{n=1}^{\infty} \lambda_n |v_n|^2)^{\frac{1}{2}} \\ &= 2||u||_{H_0^1(\Omega)} ||v||_{H_0^1(\Omega)},\end{aligned}$$

hence, given any $u \in H_0^1(\Omega)$, $\mathscr{U}(u) \in H^{-1}(\Omega)$. Moreover, this problem is consistent, since if u is any a priori solution, taking $v = u$ as test yields

$$||u||^2_{H_0^1(\Omega)} = \sum_{n=1}^{\infty} \lambda_n u_n^2 \le \langle \mathscr{U}(u), u\rangle = \langle F, u\rangle \le ||F||_{H^{-1}(\Omega)} ||u||_{H_0^1(\Omega)},$$

and therefore $||u||_{H_0^1(\Omega)} \le ||F||_{H^{-1}(\Omega)}$. However there is no straightforward PDE system corresponding to this variational problem.

Remark 6.1. According to concepts introduced by L. Hörmander [11], the operator $\mathscr{U}$ specified by (6.11) might be characterized as a nonlinear pseudo-differential operator.

[7] $\forall\ j \in \mathbb{N}^\star$, $(e_i, e_j)_\Omega = \delta_i^j$, and $\forall\ u \in H_0^1(\Omega)$, $u_n = \sum_{j=1}^{n} (e_j, u)_\Omega e_j \to u$ in $L^2(\Omega)$.

6.2.3 Variational Problem Corresponding to the NSE

The present and two following sections will show that the variational problem associated with the NSE (6.1) is expressed by

$$\text{Find } (\mathbf{v}, p) \in \mathbf{W}(\Omega) \times L^2_0(\Omega) \text{ such that } \forall\, (\mathbf{w}, q) \in \mathbf{W}(\Omega) \times L^2(\Omega),$$

$$\begin{cases} b(\mathbf{v}; \mathbf{v}, \mathbf{w}) + a(\mathbf{v}, \mathbf{w}) + s_v(\mathbf{v}, \mathbf{w}) - (p, \nabla \cdot \mathbf{w})_\Omega + \langle G(\mathbf{v}), \mathbf{w}\rangle = \langle \mathbf{f}, \mathbf{w}\rangle, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\nabla \cdot \mathbf{v}, q)_\Omega = 0, \end{cases} \tag{6.12}$$

where

$$\begin{cases} a(\mathbf{v}, \mathbf{w}) = 2\nu\, (D\mathbf{v}, D\mathbf{w})_\Omega, \\ b(\mathbf{z}; \mathbf{v}, \mathbf{w}) = \dfrac{1}{2}\left[((\mathbf{z} \cdot \nabla)\, \mathbf{v}, \mathbf{w})_\Omega - ((\mathbf{z} \cdot \nabla)\, \mathbf{w}, \mathbf{v})_\Omega\right], \\ s_v(\mathbf{v}, \mathbf{w}) = (\nu_t D\mathbf{v}, D\mathbf{w})_\Omega, \end{cases} \tag{6.13}$$

and in addition,

$$\langle G(\mathbf{v}), \mathbf{w}\rangle = (g(\mathbf{v}), \mathbf{w})_\Gamma = \int_\Gamma g(\mathbf{v}(\mathbf{x})) \cdot \mathbf{w}(\mathbf{x})\, d\Gamma(\mathbf{x}). \tag{6.14}$$

From now on we denote this variational problem by $\mathscr{VP}$. In this case,

(i) $X_1 = \mathbf{W}(\Omega) \times L^2_0(\Omega)$ and $X_2 = \mathbf{W}(\Omega) \times L^2(\Omega)$,
(ii) $\mathscr{U}$ is expressed by

$$\langle \mathscr{U}(\mathbf{v}, p), (\mathbf{w}, q)\rangle = \langle \mathscr{T}(\mathbf{v}), \mathbf{w}\rangle - (p, \nabla \cdot \mathbf{w})_\Omega + \langle G(\mathbf{v}), \mathbf{w}\rangle + (\nabla \cdot \mathbf{v}, q)_\Omega,$$

where $\mathscr{T}$ denotes the transport–diffusion operator,

$$\langle \mathscr{T}(\mathbf{v}), \mathbf{w}\rangle = b(\mathbf{v}; \mathbf{v}, \mathbf{w}) + a(\mathbf{v}, \mathbf{w}) + s_v(\mathbf{v}, \mathbf{w}), \tag{6.15}$$

(iii) the source term F is expressed by

$$\langle F(\mathbf{v}, p), (\mathbf{w}, q)\rangle = \langle \mathbf{f}, \mathbf{w}\rangle.$$

It will be established by the end of Sect. 6.3 that given any $(\mathbf{v}, p) \in X_1$, then $\mathscr{U}(\mathbf{v}, p) \in X_2'$.

Definition 6.1. Any solution to $\mathscr{VP}$ is called a weak solution of the NSE (6.1).

To understand the link between the NSE (6.1) and $\mathscr{VP}$, let us consider a strong solution $(\mathbf{v}, p)$ (if any) of the NSE (6.1), which means that $(\mathbf{v}, p) \in \mathbf{W}(\Omega) \times L^2_0(\Omega)$, $\mathbf{f} \in C^0(\overline{\Omega})$, and

$$(\mathbf{v}, p) \in C^2(\overline{\Omega}) \times C^1(\overline{\Omega}), \quad g(\mathbf{v}) \in C^0(\Gamma), \tag{6.16}$$

so that $(\mathbf{v}, p)$ is a solution of (6.1) in the classical sense. We briefly verify in the following that $(\mathbf{v}, p)$ is also a solution to $\mathscr{VP}$.

We first multiply the incompressibility constraint (6.1.ii) by $q \in L^2(\Omega)$ and integrate over Ω which yields (6.12.ii). We next take the scalar product of (6.1.i) by some $\mathbf{w} \in \mathbf{W}(\Omega)$ and integrate over Ω, considering each term successively.

As $\nabla \cdot \mathbf{v} = 0$, we know from Lemma 6.3 below that $b(\mathbf{v}; \mathbf{v}, \mathbf{w}) = ((\mathbf{v} \cdot \nabla)\, \mathbf{v}, \mathbf{w})$. Moreover, we obtain using Stokes formula

$$- \int_\Omega \nabla \cdot [(2\nu + \nu_t) D\mathbf{v}(\mathbf{x})] \cdot \mathbf{w}(\mathbf{x}) d\mathbf{x} = \\ -((2\nu + \nu_t) D\mathbf{w} \cdot \mathbf{n}, \mathbf{w})_\Gamma + ((2\nu + \nu_t) D\mathbf{v}, \nabla \mathbf{w})_\Omega .$$

Since $\mathbf{w} \cdot n = \mathbf{v} \cdot \mathbf{n} = 0$ on Γ, we have $\mathbf{w}_\tau = \mathbf{w}$, $g(\mathbf{v})_\tau = \mathbf{v}_\tau H(|\mathbf{v}|) = \mathbf{v} H(|\mathbf{v}|)$ and by the wall-law relation (6.1.iii)

$$-((2\nu+\nu_t) D\mathbf{v}\cdot\mathbf{n}, \mathbf{w})_\Gamma = -((2\nu+\nu_t) D\mathbf{v}\cdot\mathbf{n})_\tau, \mathbf{w}_\tau)_\Gamma = (g(\mathbf{v})_\tau, \mathbf{w}_\tau)_\Gamma = (g(\mathbf{v}), \mathbf{w})_\Gamma ,$$

hence

$$-(\nabla \cdot [(2\nu + \nu_t) D\mathbf{v}], \mathbf{w})_\Omega = a(\mathbf{v}, \mathbf{w}) + s_\nu(\mathbf{v}, \mathbf{w}) + \langle G(\mathbf{v}), \mathbf{w} \rangle .$$

Finally, applying Stokes formula once again, we obtain

$$\int_\Omega \nabla p(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}) d\mathbf{x} = \int_\Gamma p(\mathbf{x})\, \mathbf{w}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) d\Gamma(\mathbf{x}) - (p, \nabla \cdot \mathbf{w})_\Omega = -(p, \nabla \cdot \mathbf{w})_\Omega ,$$

since $\mathbf{w} \cdot \mathbf{n} = 0$ on Γ. Therefore, $(\mathbf{v}, p)$ is indeed a weak solution to the NSE (6.1).

Notice that if we assume $(\mathbf{v}, p) \in (\mathbf{H}^2(\Omega) \cap \mathbf{W}(\Omega)) \times (H^1(\Omega) \cap L^2_0(\Omega))$ and $\mathbf{f} \in \mathbf{L}^2(\Omega)$, then equation (6.1.i) makes sense in $\mathbf{L}^2(\Omega)$, (6.1.ii) makes sense in $H^1(\Omega)$, whereas (6.1.iii) makes sense in $\mathbf{H}^{1/2}(\Gamma)$ and (6.1.iv) in $L^4(\Gamma)$. We shall speak of $(\mathbf{v}, p)$ as a mild solution.[8] The next statement makes the connection between mild and weak solutions. We skip the proof, which is standard, especially since Lemma 9.4 below states a similar and more general result.

Lemma 6.1. *Assume that* $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $(\mathbf{v}, p) \in (\mathbf{H}^2(\Omega) \cap \mathbf{W}(\Omega)) \times (H^1(\Omega) \cap L^2_0(\Omega))$. *Then* $(\mathbf{v}, p)$ *is a mild solution of the NSE (6.1) if and only if it is a weak solution.*

We show in the following sections the existence of a weak solution to the NSE (6.1), which leads to the question of regularity, according to Lemma 6.1. Usually, regularity questions concerning weak solutions to the steady-state NSE with the no-slip BC are treated using the general results of Agmon–Douglis–Nirenberg [2] or

[8] Some authors refer to strong solutions rather than mild solutions. In this book, strong solutions are those satisfying (6.16).

the results by Cattabriga [6] on the Stokes problem. Whichever approach we take, we cannot apply these methods because of the nonlinear character of the wall laws, so that this regularity question remains open at the time these lines are being written.

Remark 6.2. In this formalism, the NSE (6.1) can be written as

$$\begin{cases} \mathscr{T}(\mathbf{v}) + \nabla p + G(\mathbf{v}) = \mathbf{f}, \\ \qquad\qquad\qquad\quad \nabla \cdot \mathbf{v} = 0, \end{cases} \tag{6.17}$$

which must be read as

(transport + diffusion + eddy diffusion) + pressure + wall law = *external source*,

incompressibility constraint.

In this form, the condition $\mathbf{v} \cdot \mathbf{n}|_\Gamma = 0$ is not mentioned. It is clarified by the specification of the unknown space $\mathbf{W}(\Omega)$, whose choice is also driven by the a priori estimate carried out in Sect. 6.4.1 below, as already stated.

6.3 Technical Background

This section is devoted to technical considerations, essential for the analysis of $\mathscr{V}\mathscr{P}$, and the proof of the existence of solutions.

The question arises first as to whether $\mathscr{V}\mathscr{P}$ is meaningful. To answer to this question, we discuss the consistency and the properties of the operators a, b, s_ν defined by (6.13); thus, $\mathscr{T} = b + a + s_\nu$, with G defined by (6.14).

Furthermore, whatever method we choose to construct solutions (Galerkin or linearization), we have "to take the limit in the equations," an imprecise statement which may have several meanings. In any cases, this requires compactness properties, based on estimates and standard results of functional analysis, such as the Banach–Alaoglu theorem, Sobolev embedding, and the trace theorems.

The second goal of this section therefore is to prepare the ground for these "limit takings." In particular, we outline what we call the VESP, which is a single package that contains all the compactness properties of a given bounded sequence in $\mathbf{W}(\Omega)$.

6.3.1 Properties of Diffusion and Convective Operators

We begin with the diffusion operators, expressed by the bilinear forms

$$a = a(\mathbf{v}, \mathbf{w}) = 2\nu(D\mathbf{v}, D\mathbf{w})_\Omega, \quad s_\nu = s_\nu(\mathbf{v}, \mathbf{w}) = (\nu_t D\mathbf{v}, D\mathbf{w})_\Omega,$$

whose analysis is the simplest of all.

Lemma 6.2. *Assume $\nu_t \in L^\infty(\mathbb{R}; \mathbb{R}_+)$. The forms $(\mathbf{v}, \mathbf{w}) \to a(\mathbf{v}, \mathbf{w})$, $(\mathbf{v}, \mathbf{w}) \to s_\nu(\mathbf{v}, \mathbf{w})$ are nonnegative bilinear continuous forms on $\mathbf{W}(\Omega)$.*

Proof. These forms are obviously bilinear. By the Cauchy–Schwarz inequality, we find

$$\begin{aligned} \forall\, \mathbf{v}, \mathbf{w} \in \mathbf{W}(\Omega), \quad |a(\mathbf{v}, \mathbf{w})| &\le \nu\, \|D\mathbf{v}\|_{0,2,\Omega}\, \|D\mathbf{w}\|_{0,2,\Omega} \\ &\le \nu ||\mathbf{v}||_{1,2,\Omega} ||\mathbf{w}||_{1,2,\Omega}, \end{aligned} \tag{6.18}$$

hence, the continuity of $a(\mathbf{v}, \mathbf{w})$. Similarly,

$$\begin{aligned} \forall\, \mathbf{v}, \mathbf{w} \in \mathbf{W}(\Omega), \quad |s_\nu(\mathbf{v}, \mathbf{w})| &\le ||\nu_t||_\infty \|D\mathbf{v}\|_{0,2,\Omega} \|D\mathbf{w}\|_{0,2,\Omega} \\ &\le ||\nu_t||_\infty ||\mathbf{v}||_{1,2,\Omega} ||\mathbf{w}||_{1,2,\Omega}, \end{aligned} \tag{6.19}$$

hence the continuity of $s_\nu(\mathbf{v}, \mathbf{w})$. Moreover, since $\nu > 0$ and $\nu_t \ge 0$, we deduce that $\forall\, \mathbf{v} \in \mathbf{W}(\Omega)$, $a(\mathbf{v}, \mathbf{v}), s_\nu(\mathbf{v}, \mathbf{v}) \ge 0$. □

We now investigate the multiple properties of the trilinear form

$$b = b(\mathbf{z}; \mathbf{v}, \mathbf{w}) = \frac{1}{2} \left[((\mathbf{z} \cdot \nabla)\, \mathbf{v}, \mathbf{w})_\Omega - ((\mathbf{z} \cdot \nabla)\, \mathbf{w}, \mathbf{v})_\Omega \right], \tag{6.20}$$

which will be used as the variational nonlinear transport term in the NSE. We also call it convection, hence the terminology.

Lemma 6.3. *The form $(\mathbf{z}, \mathbf{v}, \mathbf{w}) \to b(\mathbf{z}, \mathbf{v}, \mathbf{w})$ verifies the following properties.*

(i) b is trilinear and continuous on $\mathbf{H}^1(\Omega)$, then on $\mathbf{W}(\Omega)$, and in particular,

$$\forall\, \mathbf{z},\ \mathbf{v},\ \mathbf{w} \in \mathbf{H}^1(\Omega), \quad |b(\mathbf{z}; \mathbf{v}, \mathbf{w})| \le C\, \|\mathbf{z}\|_{1,2,\Omega}\, \|\mathbf{v}\|_{1,2,\Omega}\, \|\mathbf{w}\|_{1,2,\Omega}, \tag{6.21}$$

for some constants C only depending on Ω,

(ii) b is antisymmetric,

$$\forall\, \mathbf{z},\ \mathbf{v},\ \mathbf{w} \in \mathbf{H}^1(\Omega), \quad b(\mathbf{z}; \mathbf{v}, \mathbf{w}) = -b(\mathbf{z}; \mathbf{w}, \mathbf{v}), \tag{6.22}$$

(iii) we also have

$$\forall\, \mathbf{z},\ \mathbf{w}\ \in \mathbf{H}^1(\Omega), \quad b(\mathbf{z}; \mathbf{w}, \mathbf{w}) = 0, \tag{6.23}$$

(iv) for any $\mathbf{z} \in \mathbf{W}(\Omega)$ such that $\nabla \cdot \mathbf{z} = 0$ (in $L^2(\Omega)$), we have

$$\forall\, \mathbf{v},\ \mathbf{w}\ \in \mathbf{H}^1(\Omega), \quad b(\mathbf{z}; \mathbf{v}, \mathbf{w}) = ((\mathbf{z} \cdot \nabla)\, \mathbf{v}, \mathbf{w})_\Omega, \tag{6.24}$$

as well as

$$\forall\, \mathbf{w}\ \in \mathbf{H}^1(\Omega), \quad b(\mathbf{z}; \mathbf{z}, \mathbf{w}) = -(\mathbf{z} \otimes \mathbf{z}, \nabla \mathbf{w})_\Omega. \tag{6.25}$$

Proof. (i) We combine the Cauchy–Schwarz inequality with Sobolev embeddings, $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^p(\Omega)$, $p \leq 6$ (recall that $\Omega \subset \mathbb{R}^3$), to obtain

$$\begin{aligned} |((\mathbf{z} \cdot \nabla)\, \mathbf{v}, \mathbf{w})_\Omega| &= \left| \int_\Omega z_j(\mathbf{x}) \partial_j v_i(\mathbf{x}) w_i(\mathbf{x}) d\mathbf{x} \right| \\ &\leq \left(\int_\Omega |\partial_j v_i(\mathbf{x})|^2 d\mathbf{x} \right)^{1/2} \left(\int_\Omega |z_j(\mathbf{x})|^2 |w_i(\mathbf{x})|^2 d\mathbf{x} \right)^{1/2} \\ &\leq C' \, \|\mathbf{z}\|_{0,4,\Omega} \, \|\nabla \mathbf{v}\|_{0,2,\Omega} \, \|\mathbf{w}\|_{0,4,\Omega} \\ &\leq C \, \|\mathbf{z}\|_{1,2,\Omega} \, \|\mathbf{v}\|_{1,2,\Omega} \, \|\mathbf{w}\|_{1,2,\Omega}, \end{aligned}$$

where C is a constant depending only on Ω, using the convention of repeated indexes and $\partial_j = (\partial/\partial x_j)$. The same estimate holds for the term $((\mathbf{z}\cdot\nabla)\, \mathbf{w}, \mathbf{v})_\Omega$, hence (6.21).

(ii) Property (6.22) directly follows from the definition of form b.

(iii) Property (6.23) follows from (6.22).

(iv) We deduce from the Stokes formula

$$\begin{aligned} ((\mathbf{z} \cdot \nabla)\, \mathbf{v}, \mathbf{w})_\Omega &= \int_\Omega z_j(\mathbf{x}) \partial_j v_i(\mathbf{x}) w_i(\mathbf{x}) d\mathbf{x} \\ &= \int_\Gamma v_i(\mathbf{x}) w_i(\mathbf{x}) z_j(\mathbf{x}) n_j(\mathbf{x}) d\Gamma(\mathbf{x}) - \int_\Omega v_i(\mathbf{x}) \partial_j(z_j w_i)(\mathbf{x}) d\mathbf{x} \\ &= (\mathbf{z} \cdot \mathbf{n}, \mathbf{v} \cdot \mathbf{w})_\Gamma - (\nabla \cdot \mathbf{z}, \mathbf{w} \cdot \mathbf{v})_\Omega - ((\mathbf{z} \cdot \nabla)\, \mathbf{w}, \mathbf{v})_\Omega, \end{aligned}$$

which holds for any $\mathbf{z}, \mathbf{v}, \mathbf{w} \in \mathscr{C}^m(\overline{\Omega})^3$, since $m \geq 1$. Then if $\mathbf{z}$ belongs to the space $\mathscr{W}_m(\Omega)$ defined by (6.5), this formula becomes

$$((\mathbf{z} \cdot \nabla)\, \mathbf{v}, \mathbf{w})_\Omega = -(\nabla \cdot \mathbf{z}, \mathbf{w} \cdot \mathbf{v})_\Omega - ((\mathbf{z} \cdot \nabla)\, \mathbf{w}, \mathbf{v})_\Omega. \tag{6.26}$$

We easily deduce from the Cauchy–Schwarz inequality, Sobolev embedding, and trace theorems,

$$\begin{aligned} |(\nabla \cdot \mathbf{z}, \mathbf{w} \cdot \mathbf{v})_\Omega| &\leq C_1 \, \|\mathbf{z}\|_{1,2,\Omega} \, \|\mathbf{v}\|_{1,2,\Omega} \, \|\mathbf{w}\|_{1,2,\Omega}, \\ |((\mathbf{z} \cdot \nabla)\, \mathbf{w}, \mathbf{v})_\Omega| &\leq C_2 \, \|\mathbf{z}\|_{1,2,\Omega} \, \|\mathbf{v}\|_{1,2,\Omega} \, \|\mathbf{w}\|_{1,2,\Omega}, \end{aligned}$$

the constant C_i depending only on Ω. We infer from a standard continuation argument that the equality (6.26) holds for any $\mathbf{z} \in \mathbf{W}(\Omega)$ and $\mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega)$, because $\mathscr{W}_m(\Omega)$ is dense in $\mathbf{W}(\Omega)$ and $\mathscr{C}^m(\overline{\Omega})^3$ is dense in $H^1(\Omega)$, and (6.24) follows if in addition $\nabla \cdot \mathbf{z} = 0$, which holds in $L^2(\Omega)$.

Let $\mathbf{z} \in \mathscr{W}_m(\Omega)$, $\mathbf{w} \in \mathscr{C}^m(\Omega)$, then using the Stokes formula once again yields, as $\mathbf{z} \cdot \mathbf{n}$ vanishes at Γ,

$$(\mathbf{z} \otimes \mathbf{z}, \nabla \mathbf{w})_\Omega = \int_\Omega z_i(\mathbf{x}) z_j(\mathbf{x}) \partial_j v_i(\mathbf{x}) d\mathbf{x} = -((\mathbf{z} \cdot \nabla)\, \mathbf{z}, \mathbf{w})_\Omega - (\nabla \cdot \mathbf{z}, \mathbf{z} \cdot \mathbf{w})_\Omega.$$

Using the same argument as above, we observe that formula (6.25) still holds when $\mathbf{z} \in \mathbf{W}(\Omega)$ and $\mathbf{w} \in \mathbf{H}^1(\Omega)$, in taking $\mathbf{z}$ such that $\nabla \cdot \mathbf{z} = 0$. □

Remark 6.3. At this stage, one may wonder why we take the form b expressed by (6.20) to describe convection instead of $(\mathbf{z}, \mathbf{v}, \mathbf{w}) \to ((\mathbf{z} \cdot \nabla)\, \mathbf{v}, \mathbf{z})_\Omega$. This point will be clarified in Sect. 6.4.4 below.

In conclusion, gathering Lemmas 6.2 and 6.3 yields the following statement:

Lemma 6.4. *Let $\mathscr{T}$ be the transport–diffusion operator expressed by (6.15). Then $\mathscr{T}$ maps continuously $\mathbf{W}(\Omega)$ into $\mathbf{W}(\Omega)'$ and one has*

$$\forall\, \mathbf{v} \in \mathbf{W}(\Omega), \quad ||\mathscr{T}(\mathbf{v})||_{\mathbf{W}(\Omega)'} \le C ||\mathbf{v}||_{\mathbf{W}(\Omega)}(1 + ||\mathbf{v}||_{\mathbf{W}(\Omega)}), \tag{6.27}$$

where the constant C depends only on the data and the domain Ω.

Remark 6.4. Concerning the notations a, b, s_v, historically, the homogeneous Dirichlet problem $-\Delta u = f$, $u|_\Gamma = 0$ was solved by the Lax–Milgram theorem typically set in the form $a(u, v) = (\nabla u, \nabla v)_\Omega = \langle f, v\rangle$. The NSE involves a new trilinear form, logically denoted by b, a notation probably due to J. L. Lions [14].

One may find the notation s_v peculiar. Historically, the first mathematical problem on the NSE involving an eddy viscosity is due to O. Ladyžhenskaya [12], where the eddy viscosity is that of Smagorinsky, hence s_v, in which the subscript v stands for "velocity."

6.3.2 Properties of the Wall-Law Operator

The structure of the boundary term $\langle G(\cdot), \cdot\rangle$ defined by (6.14) (G for "Gamma = Γ"), which models the nonlinear wall law, is rather complicated. Our priority is to show that $\forall\, \mathbf{v} \in \mathbf{W}(\Omega)$, then $G(\mathbf{v}) \in \mathbf{W}(\Omega)'$. As we already know that $\forall\, \mathbf{v} \in \mathbf{W}(\Omega)$, then $\mathscr{T}(\mathbf{v}) \in \mathbf{W}(\Omega)'$, we will be able to conclude that $\mathscr{VP}$ is meaningful.

According to Lemma 5.4, we know that the most general wall-law function $g \in W^{1,\infty}_{loc}(\mathbb{R}^3) \cap C^0(\mathbb{R}^3)$ is of the form

$$g(\mathbf{v}) = \mathbf{v} H(|\mathbf{v}|). \tag{6.28}$$

The relevant properties of g are recalled below:

$$\forall\, \mathbf{v} \in \mathbb{R}^3, \quad 0 \le H(|\mathbf{v}|)| \le C(1 + |\mathbf{v}|), \tag{6.29}$$

denoting by C any generic constant, which leads to

$$\forall\, \mathbf{v} \in \mathbb{R}^3, \quad ,0 \le g(\mathbf{v}) \cdot \mathbf{v}, \tag{6.30}$$

$$|g(\mathbf{v})| \le C(1 + |\mathbf{v}|^2). \tag{6.31}$$

Moreover, we also know that

$$\forall\, \mathbf{v} \in \mathbb{R}^3, \quad |\nabla g(\mathbf{v})| \le C(1 + |\mathbf{v}|), \tag{6.32}$$

$$\forall\, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3, \quad 0 \le (g(\mathbf{v}) - g(\mathbf{w})) \cdot (\mathbf{v} - \mathbf{w}). \tag{6.33}$$

Finally, we shall also assume that H satisfies the technical property,

$$\exists\, C_g > 0,\ \exists\, \alpha \in [0, 1] \text{ such that } \forall\, \mathbf{v} \in \mathbb{R}^3, \quad C_H |\mathbf{v}|^\alpha \le H(|\mathbf{v}|), \tag{6.34}$$

so that we always have

$$\forall\, \mathbf{v} \in \mathbf{W}(\Omega), \quad C_g \int_\Gamma |\mathbf{v}(\mathbf{x})|^{2+\alpha} d\Gamma(\mathbf{x}) \le \langle G(\mathbf{v}), \mathbf{v} \rangle, \tag{6.35}$$

where the constant $C_g > 0$ depends on Γ, C_H, and α. Notice that the most popular case is $H(\mathbf{v}) = C_H |\mathbf{v}|$, which satisfies all these properties.

Lemma 6.5. *Assume that $g \in W^{1,\infty}_{loc}(\mathbb{R}^3) \cap C^0(\mathbb{R}^3)$ such that (6.30), (6.31), (6.32), and (6.33) are satisfied. Then there exists a constant $C > 0$ depending only on Ω such that*

(i) Let $\mathbf{v} \in \mathbf{W}(\Omega)$; then $g(\mathbf{v}) \in \mathbf{L}^2(\Gamma)$ and

$$\|g(\mathbf{v})\|_{0,2,\Gamma} \le C\,(1 + \|\mathbf{v}\|^2_{1,2,\Omega}). \tag{6.36}$$

(ii) For any $\mathbf{v}, \mathbf{w} \in \mathbf{W}(\Omega)$,

$$\|g(\mathbf{v}) - g(\mathbf{w})\|_{0,2,\Gamma} \le C\,(1 + \|\mathbf{v}\|_{1,2,\Omega} + \|\mathbf{w}\|_{1,2,\Omega})\, \|\mathbf{v} - \mathbf{w}\|_{1,2,\Omega}. \tag{6.37}$$

(iii) The functional $G : \mathbf{v} \to G(\mathbf{v})$ defined by $\langle G(\mathbf{v}), \mathbf{w} \rangle = (g(\mathbf{v}), \mathbf{w})_\Gamma$, maps $\mathbf{W}(\Omega)$ in $\mathbf{W}(\Omega)'$ and verifies

$$\|G(\mathbf{v})\|_{\mathbf{W}(\Omega)'} \le C\,(1 + \|\mathbf{v}\|^2_{1,2,\Omega}). \tag{6.38}$$

Moreover, G is positive, which means $\forall\, \mathbf{w} \in \mathbf{W}(\Omega), 0 \le \langle G(\mathbf{w}), \mathbf{w} \rangle$, and it also satisfies the estimate

$$\|G(\mathbf{v}) - G(\mathbf{w})\|_{\mathbf{W}(\Omega)'} \le C\,(1 + \|\mathbf{v}\|_{1,2,\Omega} + \|\mathbf{w}\|_{1,2,\Omega})\, \|\mathbf{v} - \mathbf{w}\|_{1,2,\Omega}. \tag{6.39}$$

(iv) G is continuous and compact.
(v) G is monotone.

Proof. (i) Consider $\mathbf{v} \in \mathbf{W}(\Omega)$. Its trace on Γ, still denoted by $\mathbf{v}$ for simplicity, then belongs to $\mathbf{H}^{1/2}(\Gamma)$ and hence to $\mathbf{L}^4(\Gamma)$, where we have used the trace and Sobolev embedding theorems. Moreover, $||\mathbf{v}||_{0,4,\Gamma} \le C||\mathbf{v}||_{1,2,\Omega}$, hence estimate (6.36) by (6.31).

(ii) As $g \in W^{1,\infty}_{loc}(\mathbb{R}^d) \cap C^0(\mathbb{R}^d)$, the Taylor formula with integral remainder applies (see in [3]), leading to

$$g(\mathbf{v}(\mathbf{x})) - g(\mathbf{w}(\mathbf{x})) = \int_0^1 \nabla g(\theta\, \mathbf{v}(\mathbf{x}) + (1-\theta)\mathbf{w}(\mathbf{x})) \cdot (\mathbf{v}(\mathbf{x}) - \mathbf{w}(\mathbf{x}))\, d\theta.$$

Using estimate (6.32) yields

$$|g(\mathbf{v}(\mathbf{x})) - g(\mathbf{w}(\mathbf{x}))| \le C\,(1 + |\mathbf{v}(\mathbf{x})| + |\mathbf{w}(\mathbf{x})|)\,|\mathbf{v}(\mathbf{x}) - \mathbf{w}(\mathbf{x})|, \qquad (6.40)$$

which in conjunction with the Cauchy–Schwarz inequality leads to

$$\|g(\mathbf{v}) - g(\mathbf{w})\|_{0,2,\Gamma} \le C\,||1 + |\mathbf{v}| + |\mathbf{w}|\,||_{0,4,\Gamma}\,||\mathbf{v} - \mathbf{w}||_{0,4,\Gamma},$$

and hence to (6.37) by combining again the Sobolev embedding trace theorems.

(iii) We start by proving estimate (6.38). Given $\mathbf{w} \in \mathbf{W}(\Omega)$, the duality $\langle G(\mathbf{v}), \mathbf{w}\rangle$ expressed by (6.14) is indeed well defined following (6.36) and satisfies

$$|\langle G(\mathbf{v}), \mathbf{w}\rangle| \le \|g(\mathbf{v})\|_{0,2,\Gamma}\,\|\mathbf{w}\|_{0,2,\Gamma} \le C\,(1 + \|\mathbf{v}\|^2_{1,2,\Omega})\,\|\mathbf{w}\|_{1,2,\Omega}, \qquad (6.41)$$

where we have used (6.36) and the trace theorem again, hence (6.38). The successive constants that appear in the estimates within this proof only depend on Ω.

The positiveness follows directly from (6.30), estimate (6.39) from (6.37). Indeed,

$$\begin{aligned} |\langle G(\mathbf{v}) - G(\mathbf{w}), \mathbf{z}\rangle| &\le \|g(\mathbf{v}) - g(\mathbf{w})\|_{0,2,\Gamma}\;\|\mathbf{z}\|_{0,2,\Gamma} \\ &\le C\;(1 + \|\mathbf{v}\|_{1,2,\Omega} + \|\mathbf{w}\|_{1,2,\Omega})\,\|\mathbf{v} - \mathbf{w}\|_{1,2,\Omega}\;\|\mathbf{z}\|_{1,2,\Omega}, \end{aligned} \qquad (6.42)$$

by using (6.37).

(iv) The continuity of G results from (6.39) which shows that G is Lipschitz over any bounded set in $\mathbf{W}(\Omega)$.

To prove the compactness of G, it is enough to prove that given any sequence $(\mathbf{v}_n)_{n\in\mathbb{N}}$ which weakly converges to some $\mathbf{v} \in \mathbf{W}(\Omega)$, then $(G(\mathbf{v}_n))_{n\in\mathbb{N}}$ converges (strongly) to $G(\mathbf{v})$ in $\mathbf{W}(\Omega)'$ (eventually up to a subsequence). Let $\mathbf{w} \in \mathbf{W}(\Omega)$. We find from Hölder inequality combined with the inequality (6.40),

$$\begin{aligned} |\langle G(\mathbf{v}_n) - G(\mathbf{v}), \mathbf{w}\rangle| &\le C\;||(1 + |\mathbf{v}_n| + |\mathbf{v}|)\,|\mathbf{v}_n - \mathbf{v}|\,||_{0,3/2,\Gamma}\,||\mathbf{w}||_{0,3,\Gamma} \\ &\le C\;||(1 + |\mathbf{v}_n| + |\mathbf{v}|)||_{0,3,\Gamma}\,||\mathbf{v}_n - \mathbf{v}||_{0,3,\Gamma}\,||||\mathbf{w}||_{1,2,\Omega} \\ &\le C'||\mathbf{v}_n - \mathbf{v}||_{0,3,\Gamma}\,||\mathbf{w}||_{1,2,\Omega} \end{aligned} \qquad (6.43)$$

where C' depends on Ω, $||\mathbf{v}||_{0,3,\Gamma}$, and $\sup_{n\in\mathbb{N}}(||\mathbf{v}_n||_{0,3,\Gamma})$. In particular,

$$||G(\mathbf{v}_n) - G(\mathbf{v})||_{\mathbf{W}(\Omega)'} \le C'||\mathbf{v}_n - \mathbf{v}||_{0,3,\Gamma}. \tag{6.44}$$

As $(\mathbf{v}_n)_{n\in\mathbb{N}}$ weakly converges to $\mathbf{v}$ in $\mathbf{W}(\Omega)$, the corresponding traces sequence $(\mathbf{v}_n)_{n\in\mathbb{N}}$ (keeping the same notation) weakly converges to $\mathbf{v}$ in $\mathbf{H}^{1/2}(\Gamma)$. As the embedding $\mathbf{H}^{1/2}(\Gamma) \hookrightarrow \mathbf{L}^3(\Gamma)$ is compact, $||\mathbf{v}_n - \mathbf{v}||^2_{0,3,\Gamma}$ goes to zero when n goes to ∞, eventually up to a subsequence, hence the result by estimate (6.44).

(v) We deduce from (6.33): Given $\mathbf{v},\ \mathbf{w} \in \mathbf{W}(\Omega)$,

$$\langle G(\mathbf{v}) - G(\mathbf{w}), \mathbf{v} - \mathbf{w}\rangle = \int_\Gamma (g(\mathbf{v}(\mathbf{x})) - g(\mathbf{w}(\mathbf{x})))\,(\mathbf{v}(\mathbf{x}) - \mathbf{w}(\mathbf{x}))\,d\Gamma(\mathbf{x}) \ge 0,$$

hence, G is monotone. □

Remark 6.5. The estimates above are expressed in terms of the norm $||\cdot||_{1,2,\Omega}$ for convenience. Because the norm $||\cdot||_{1,2,\Omega}$ is equivalent to the norm $||\cdot||_{\mathbf{W}(\Omega)}$ expressed by (6.8), the same estimates hold for the latter norm, which we shall use in the following sections of this chapter.

6.3.3 Compactness Results: VESP and PSEP

6.3.3.1 VESP: Definition

In the subsequent investigations, we shall often consider a given sequence of velocity fields $(\mathbf{w}_n)_{n\in\mathbb{N}}$ in $\mathbf{W}(\Omega)$, which could for example be a sequence of approximate solutions to our problem. We want to take the limit in the equations, which means that we aim to determine the equations satisfied by the weak subsequential limits of $(\mathbf{w}_n)_{n\in\mathbb{N}}$. The process is more or less always initialized by the same procedure:

(i) we derive from energy equalities an a priori estimate for $(\mathbf{w}_n)_{n\in\mathbb{N}}$, which means $||\mathbf{w}_n|| \le R$ for some R that does not depend on n,
(ii) we extract weak convergent subsequences from $(\mathbf{w}_n)_{n\in\mathbb{N}}$ and apply the standard compactness results of functional analysis.

We aim to gather in a single package all the relevant compactness properties of $(\mathbf{w}_n)_{n\in\mathbb{N}}$, in what we call the "*Velocity Extracting Subsequences Principle*," denoted by VESP.

In addition to the standard Sobolev and trace theorems (see Sects. A.1 and A.2 in [TB]) we shall use the inverse Lebesgue theorem, which is a consequence of the proof of the completeness of L^1 space, as shown in Brézis [4] and stated in Theorem A.10 in [TB].

Let $B_R \subset \mathbf{W}(\Omega)$ be the ball of radius R centered in 0. We know from the Banach–Alaoglu theorem that B_R is weakly compact in $\mathbf{W}(\Omega)$. As $\mathbf{w}_n \in B_R$, according to Sobolev embedding and the trace and the inverse Lebesgue theorems, a subsequence $(\mathbf{w}_{n_j})_{j\in\mathbb{N}}$ can be extracted from $(\mathbf{w}_n)_{n\in\mathbb{N}}$ and there exists $\mathbf{w} \in B_R$ such that

(a) $(\mathbf{w}_{n_j})_{j\in\mathbb{N}}$ weakly converges to $\mathbf{w} \in B_R$ when $j \to \infty$,
(b) $(\mathbf{w}_{n_j})_{j\in\mathbb{N}}$ strongly converges to $\mathbf{w}$ in $\mathbf{L}^p(\Omega)$, $1 \le p < 6$, a.e in Ω, and

$$\forall\, p < 6; \text{ there exists } A_p \in L^p(\Omega) \text{ such that } \forall\, j \in \mathbb{N},\ |\mathbf{w}_{n_j}| \le A_p,$$

a.e in Ω,
(c) $(\gamma_0(\mathbf{w}_{n_j}))_{j\in\mathbb{N}}$ weakly converges to $\gamma_0\mathbf{w}$ in $\mathbf{H}^{1/2}(\Gamma)$, strongly in $\mathbf{L}^q(\Gamma)$, $1 \le p < 4$, a.ein Γ, and

$$\forall q < 4; \text{ there exists } B_q \in L^q(\Gamma) \text{ such that } \forall\, j \in \mathbb{N},\ |\gamma_0(\mathbf{w}_{n_j})| \le B_q$$

a.e in Γ.

Definition 6.2. We will write $(\mathbf{w}_n)_{n\in\mathbb{N}}$ instead of $(\mathbf{w}_{n_j})_{j\in\mathbb{N}}$ for simplicity. The limit $\mathbf{w}$ of the subsequence is called a VESP-limit of $(\mathbf{w}_n)_{n\in\mathbb{N}}$, where VESP stands for *Velocity Extracting Subsequences Principle*. To avoid duplication, in what follows, we shall write "applying the VESP" or "the VESP applies to...," which will include items (a), (b), and (c) without systematically specifying them.

6.3.3.2 Convergence Lemma

Taking the limit of the linear terms is usually straightforward. Difficulties however arise in nonlinear terms. For this reason we formalize the process in a condensed statement.

Lemma 6.6. *Let $(\mathbf{w}_n)_{n\in\mathbb{N}}$ and $(\mathbf{w}_n)_{n\in\mathbb{N}}$ be two bounded sequences in $\mathbf{W}(\Omega)$, to which the VESP has been applied. Let $\mathbf{v}$ and $\mathbf{w}$ be any VESP-limit of these sequences. Moreover assume that $(\mathbf{w}_n)_{n\in\mathbb{N}}$ strongly converges to $\mathbf{w}$. Then*

$$\lim_{n\to\infty} b(\mathbf{v}_n;\mathbf{v}_n,\mathbf{w}_n) = b(\mathbf{v};\mathbf{v},\mathbf{w}), \tag{6.45}$$

$$\lim_{n\to\infty} \langle G(\mathbf{v}_n),\mathbf{w}_n\rangle = \langle G(\mathbf{v}),\mathbf{w}\rangle. \tag{6.46}$$

Proof. We prove each claim one after the other.

STEP 1. *Proof of (6.45)*. Observe that

$$\begin{cases} b(\mathbf{v}_n;\mathbf{v}_n,\mathbf{w}_n) = \dfrac{1}{2}\left(\displaystyle\int_\Omega \mathbf{v}_n\otimes\mathbf{w}_n : \nabla\mathbf{v}_n - \int_\Omega \mathbf{v}_n\otimes\mathbf{v}_n : \nabla\mathbf{w}_n\right), \\ \quad b(\mathbf{v};\mathbf{v},\mathbf{w}) = \dfrac{1}{2}\left(\displaystyle\int_\Omega \mathbf{v}\otimes\mathbf{w} : \nabla\mathbf{v} - \int_\Omega \mathbf{v}\otimes\mathbf{v} : \nabla\mathbf{w}\right). \end{cases} \tag{6.47}$$

From item (b) of the VESP definition, we deduce that

$$\lim_{n\to\infty} \mathbf{v}_n = \mathbf{v} \text{ and } \lim_{n\to\infty} \mathbf{w}_n = \mathbf{w} \text{ in } \mathbf{L}^4(\Omega),$$

hence

$$\lim_{n\to\infty} \mathbf{v}_n \otimes \mathbf{w}_n = \mathbf{v} \otimes \mathbf{w} \text{ and } \lim_{n\to\infty} \mathbf{v}_n \otimes \mathbf{v}_n = \mathbf{v} \otimes \mathbf{v} \text{ in } L^2(\Omega)^9.$$

Therefore, as

$$\lim_{n\to\infty} \nabla \mathbf{v}_n = \nabla \mathbf{v} \text{ and } \lim_{n\to\infty} \nabla \mathbf{w}_n = \nabla \mathbf{w} \text{ weakly in } L^2(\Omega)^9,$$

we deduce (6.45) from (6.47).

STEP 2. *Proof of (6.46).* This is a direct consequence of the compactness of G already proved in Lemma 6.5. □

6.3.3.3 PSEP

We shall also consider pressure sequences bounded in $L^2_0(\Omega)$. We state a similar compactness principle, which is nothing more than weak compactness.

Definition 6.3. Let $(p_\varepsilon)_{\varepsilon>0}$ be a sequence bounded in $L^2_0(\Omega)$. A subsequence, still denoted by $(p_\varepsilon)_{\varepsilon>0}$, can be extracted which weakly converges in $L^2_0(\Omega)$ to some p. To avoid duplication, we shall say from now on that the PESP (Pressure Extracting Subsequences Principle) applies to $(p_\varepsilon)_{\varepsilon>0}$ and p is a PESP-limit.

6.4 A Priori Estimates and Convergence of Variational Problems

This section has three objectives:

1. To prove the consistency of $\mathscr{VP}$ by deriving a priori estimates in $\mathbf{W}(\Omega) \times L^2_0(\Omega)$ satisfied by any a priori solution $(\mathbf{v}, p)$.
2. To construct ε-approximations by a singular perturbation of the incompressibility constraint, which yields variational problems $\mathscr{VP}_\varepsilon$ in which only $\mathbf{v}$ is involved.
3. To elaborate an abstract definition of convergence of families of variational problems, illustrated with the convergence of the family $(\mathscr{VP}_\varepsilon)_{\varepsilon>0}$ toward $\mathscr{VP}$. The convergence of families of variational problems will be used throughout the rest of the book.

This program is subject to the hypothesis:

Hypothesis 6.i.

(i) Ω is of class C^m, $m \geq 1$.
(ii) $\nu_t \in L^\infty(\mathbb{R})$ and is nonnegative.
(iii) $\mathbf{f} \in \mathbf{W}(\Omega)'$.
(iv) The wall law g satisfies (6.30), (6.31), (6.32), in order to use Lemma 6.5. Moreover, the form G satisfies the technical assumption (6.35).

We are now ready to state the main result of this chapter, which will be proved by two different methods:

Theorem 6.2. *When hypothesis 6.i holds, the NSE (6.1) have a weak solution.*

Whatever the method, the solutions to $\mathscr{VP}$ that we construct are limits of solutions to $\mathscr{VP}_\varepsilon$.

6.4.1 A Priori Estimate for the Velocity

Given any variational problem $\mathscr{UP}$, an a priori estimate is an equality of the form $||\zeta||_X \leq C$, satisfied by any a priori solution ζ to $\mathscr{UP}$ and where X is a given Banach space. The constant C depends on the data of the problem. To make the problem $\mathscr{UP}$ consistent, it must be proved that such an estimate holds when $X = X_1$ is the unknown space.

Deriving a priori estimates is based on choosing suitable tests. In the case of the Navier–Stokes equations in general, one takes $\mathbf{v}$ itself, which has already been discussed in Sect. 3.4.2.4 for the no-slip boundary condition. This procedure yields an energy equality, at least formally.

In the case of the NSE (6.1) studied therein, as the transport term does not produce mechanical work and since we are in the steady-state case, the energy equality expresses the balance between the external source contribution and dissipation due to molecular and eddy diffusions as well as friction at the wall. To be more specific,

Proposition 6.1. *Let* $(\mathbf{v}, p)$ *be any weak solution to the NSE (6.1). Then* $\mathbf{v}$ *satisfies the energy equality*

$$\int_\Omega (\nu + \nu_t(\mathbf{x}))|D\mathbf{v}(\mathbf{x})|^2 d\mathbf{x} + \int_\Gamma g(\mathbf{v}(\mathbf{x}))\cdot \mathbf{v}(\mathbf{x}) d\Gamma(\mathbf{x}) = \langle \mathbf{f}, \mathbf{v}\rangle, \tag{6.48}$$

leading to the estimate

$$||\mathbf{v}||_{\mathbf{W}(\Omega)} \leq \left(\frac{||\mathbf{f}||^2_{\mathbf{W}(\Omega)'}}{\inf(\nu, C_g)^2} + \frac{2C_g|\Omega|}{\inf(\nu, C_g)} \right)^{\frac{1}{2}} = C_v. \tag{6.49}$$

Proof. The basic principle is to take $\mathbf{w} = \mathbf{v} \in \mathbf{W}(\Omega)$ in (6.12.i) and to consider each term one after the other. We deduce from (6.12.ii) that $\nabla \cdot \mathbf{v} = 0$ in $\mathscr{D}'(\Omega)$, hence in $L^2(\Omega)$ since $\mathbf{v} \in \mathbf{W}(\Omega)$. Therefore, we obtain $(p, \nabla \cdot \mathbf{v})_\Omega = 0$. Moreover, we also know from item (iii) in Lemma 6.3 that $b(\mathbf{v}; \mathbf{v}, \mathbf{v}) = 0$.[9] By consequence, we have

$$a(\mathbf{v}, \mathbf{v}) + s_\nu(\mathbf{v}, \mathbf{v}) + \langle G(\mathbf{v}), \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle, \tag{6.50}$$

which is indeed the energy equality (6.48). As $\nu_t \geq 0$, combining (6.50) with the inequality (6.35) yields

$$C_g \int_\Gamma |\mathbf{v}(\mathbf{x})|^{2+\alpha} d\mathbf{x} + \nu ||D\mathbf{v}||_{0,2,\Omega} \leq \langle \mathbf{f}, \mathbf{v} \rangle. \tag{6.51}$$

As $\alpha \geq 0$, it is easily verified that

$$\forall\, x \in \mathbb{R}_+, \quad x^2 \leq 1 + x^{2+\alpha}, \tag{6.52}$$

which, when inserted in (6.51) and combined with the definition (6.8) of the $\mathbf{W}(\Omega)$ norm, yields

$$\inf(\nu, C_g) ||\mathbf{v}||^2_{\mathbf{W}(\Omega)} \leq ||\mathbf{v}||^2_{0,2,\Gamma} + \nu ||D\mathbf{v}||^2_{0,2,\Omega} \leq \langle \mathbf{f}, \mathbf{v} \rangle + C_g |\Omega|. \tag{6.53}$$

where $|\Omega| = (1, 1)_\Omega = meas(\Omega)$. By Young's inequality we obtain

$$|\langle \mathbf{f}, \mathbf{v} \rangle| \leq ||\mathbf{f}||_{\mathbf{W}(\Omega)'} ||\mathbf{v}||_{\mathbf{W}(\Omega)} \leq \frac{||\mathbf{f}||^2_{\mathbf{W}(\Omega)'}}{2 \inf(\nu, C_g)} + \frac{1}{2} \inf(\nu, C_g) ||\mathbf{v}||^2_{\mathbf{W}(\Omega)}, \tag{6.54}$$

which proves (6.49) by (6.53). □

6.4.2 A Priori Estimate for the Pressure

The derivation of the a priori estimate for the pressure is much more tricky. We will construct a potential vector $\mathbf{w} \in \mathbf{W}(\Omega)$ such that $||p||^2_{0,2,\Omega} = -\langle p, \nabla \cdot \mathbf{w} \rangle$ and which will serve as test.

Proposition 6.2. *Let* $(\mathbf{v}, p)$ *be any weak solution to the NSE (6.1). Then there exists* $C_p = C_p(\nu, g, \Omega, \mathbf{f}, ||\nu_t||_\infty)$ *such that*

$$||p||_{0,2,\Omega} \leq C_p. \tag{6.55}$$

---

[9]This result will be used throughout this chapter, which will be not systematically mentioned.

Proof. Let $u \in H^2(\Omega)$ be the unique solution to the Neumann problem

$$\begin{cases} -\Delta u = p & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \Gamma, \\ \displaystyle\int_\Omega u(\mathbf{x}) d\mathbf{x} = 0. \end{cases} \tag{6.56}$$

As Ω is of class C^m ($m \geq 1$), we know from standard elliptic theory [9] that

$$||u||_{2,2,\Omega} \leq C||p||_{0,2,\Omega},$$

where C only depends on Ω. Moreover, let $\mathbf{w} = \nabla u$, which satisfies

$$\mathbf{w} \in \mathbf{H}^1(\Omega)^3, \quad \mathbf{w} \cdot \mathbf{n}|_\Gamma = \frac{\partial u}{\partial \mathbf{n}}|_\Gamma = 0.$$

In other words, $\mathbf{w} \in \mathbf{W}(\Omega)$ and there exists a constant depending only on Ω such that

$$||\mathbf{w}||_{\mathbf{W}(\Omega)} \leq C_\Omega ||p||_{0,2,\Omega}. \tag{6.57}$$

We take $\mathbf{w}$ as test in (6.12.i), and since

$$\langle p, \nabla \cdot \mathbf{w} \rangle = (p, \Delta u)_\Omega = -||p||_{0,2,\Omega}^2,$$

we have the following equality,

$$||p||_{0,2,\Omega}^2 = \langle \mathbf{f}, \mathbf{w} \rangle - \langle \mathscr{T}(\mathbf{v}), \mathbf{w} \rangle - \langle G(\mathbf{v}), \mathbf{w} \rangle. \tag{6.58}$$

Let $\rho > 0$, to be fixed later. By using Young's inequality, (6.18), (6.19), (6.21), and (6.38), we deduce from the equality (6.58),

$$||p||_{0,2,\Omega}^2 \leq \frac{5}{2}\rho ||\mathbf{w}||_{\mathbf{W}(\Omega)}^2 + \frac{1}{2\rho}\left(||\mathbf{f}||_{\mathbf{W}(\Omega)'}^2 + C_1 + C_2||\mathbf{v}||^2 + C_3||\mathbf{v}||_{\mathbf{W}(\Omega)}^4 \right), \tag{6.59}$$

where the constants C_i depend on ν, $||\nu_t||_\infty$, g, and Ω and do not need to be explicitly specified.[10] According to the estimate (6.57), we infer from (6.59),

$$\begin{aligned} ||p||_{0,2,\Omega}^2 \leq \frac{5}{2}\rho C_\Omega ||p||_{0,2,\Omega}^2 \\ + \frac{1}{2\rho}\left(||\mathbf{f}||_{\mathbf{W}(\Omega)'}^2 + C_1 + C_2||\mathbf{v}||_{\mathbf{W}}(\Omega)^2 + C_3||\mathbf{v}||_{\mathbf{W}(\Omega)}^4 \right). \end{aligned} \tag{6.60}$$

[10] $||\nu_t||_\infty$ stands for $||\nu_t||_{0,\infty,\Omega}$ for simplicity.

Therefore, by taking $\rho = 1/5C_\Omega$ and using estimate (6.48), we find

$$||p||^2_{0,2,\Omega} \leq \frac{5}{2} C_\Omega \left(||\mathbf{f}||^2_{\mathbf{W}(\Omega)'} + C_1 + C_2 C_\nu^2 + C_3 C_\nu^4 \right) = C_p^2, \tag{6.61}$$

hence estimate (6.55). □

Remark 6.6. The method does not apply to the case of the no-slip boundary condition, even if it holds only on a part of the boundary Γ_D, since we are not able to construct $\mathbf{w}$ so that it vanishes on Γ_D.

6.4.3 Singular Perturbation

The NSE (6.1) are a nonlinear elliptic system, generalizing the definition of Agmon–Douglis–Nirenberg [2]. However this elliptic system degenerates due to the pressure term.

For the no-slip boundary condition, i.e., $\mathbf{v} = 0$ on Γ, this difficulty is bypassed by using the Leray projector over free-divergence vector fields spaces, such as $\mathbf{V}_{div}(\Omega)$ defined by (3.54). Once the equation is projected, one arrives at a nonlinear elliptic problem, whose sole unknown is $\mathbf{v}$ and for which standard methods work. At the end of the process, the pressure is recovered using the De Rham Theorem 3.1.

In our case involving a wall law, the structure of the problem does not allow the De Rham Theorem to be applied. The first idea that comes to mind is to approximate the NSE (6.1) by the family of nonlinear elliptic systems:

$$\begin{cases} (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \nabla\cdot[(2\nu+\nu_t)\,D\mathbf{v}] + \nabla p = \mathbf{f} & \text{in } \Omega, \\ -\varepsilon\Delta p + \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega, \\ -[(2\nu+\nu_t)\,D\mathbf{v}\cdot\mathbf{n}]_\tau = g(\mathbf{v})_\tau & \text{on } \Gamma, \\ \mathbf{v}\cdot\mathbf{n} = 0 & \text{on } \Gamma, \\ \dfrac{\partial p}{\partial \mathbf{n}} = 0 & \text{on } \Gamma, \\ \displaystyle\int_\Omega p(\mathbf{x})d\mathbf{x} = 0. \end{cases} \tag{6.62}$$

At least formally, when $\varepsilon \to 0$, system (6.62) converges to the NSE (6.1). We aim to eliminate p in the system (6.62). Let $\mathbf{v} \in \mathbf{W}(\Omega)$, $\varepsilon > 0$ be fixed, and consider the Neumann problem, which is a subsystem of (6.62):

$$\begin{cases} -\varepsilon\Delta p + \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega, \\ \dfrac{\partial p}{\partial \mathbf{n}} = 0 & \text{on } \Gamma, \\ \displaystyle\int_\Omega p(\mathbf{x})d\mathbf{x} = 0. \end{cases} \tag{6.63}$$

The space suitable for studying this problem is the homogeneous space

$$\overset{\circ}{H}{}^1(\Omega) = \{q \in H^1(\Omega);\ \int_\Omega q(\mathbf{x})d\mathbf{x} = 0\},$$

whose natural norm is $||\nabla q||_{0,2,\Omega}$ by the Poincaré–Wirtinger inequality (see [1]). As $\nabla \cdot \mathbf{v} \in L^2(\Omega)$, we know that Problem (6.63) has a unique solution

$$p \in \overset{\circ}{H}{}^1(\Omega) \cap H^2(\Omega)$$

(Ω is of class C^m, $m \geq 1$, see [4, 9]) which satisfies $\forall\, q \in H^1(\Omega)$

$$\varepsilon(\nabla p, \nabla q)_\Omega + (\nabla \cdot \mathbf{v}, q)_\Omega = 0. \tag{6.64}$$

Therefore, one can define a map

$$P_\varepsilon : \begin{cases} \mathbf{W}(\Omega) \to \overset{\circ}{H}{}^1(\Omega), \\ \quad \mathbf{v} \quad\ \to p, \quad \text{the unique solution to (6.63).} \end{cases} \tag{6.65}$$

The boundary value problem (6.62) becomes

$$\begin{cases} (\mathbf{v} \cdot \nabla)\, \mathbf{v} - \nabla \cdot [(2\nu + \nu_t)\, D\mathbf{v}] + \nabla P_\varepsilon(\mathbf{v}) = \mathbf{f} & \text{in } \ \Omega, \\ \qquad\qquad -[(2\nu + \nu_t)\, D\mathbf{v} \cdot \mathbf{n}]_\tau = g(\mathbf{v})_\tau & \text{on } \ \Gamma, \\ \qquad\qquad\qquad\qquad \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \ \Gamma, \end{cases} \tag{6.66}$$

whose sole unknown is $\mathbf{v}$.

Lemma 6.7. *The map P_ε is a linear continuous map.*

Proof. The map P_ε is obviously linear. Let $\mathbf{v} \in \mathbf{W}(\Omega)$, $p = P_\varepsilon(\mathbf{v})$; we take $p = q$ in the variational formulation (6.64), which yields the energy equality,

$$\varepsilon||\nabla p||^2_{0,2,\Omega} + (p, \nabla \cdot \mathbf{v})_\Omega = 0, \tag{6.67}$$

hence after processing the Stokes formula, recalling that $\partial p/\partial \mathbf{n} = 0$ on Γ,

$$\varepsilon||\nabla p||^2_{0,2,\Omega} - (\nabla p, \mathbf{v})_\Omega = 0, \tag{6.68}$$

and then by the Cauchy–Schwarz and Sobolev inequalities,

$$\varepsilon||\nabla p||_{0,2,\Omega} \leq ||\mathbf{v}||_{0,2,\Omega} \leq C||\mathbf{v}||_{\mathbf{W}(\Omega)}, \tag{6.69}$$

concluding the proof. □

Remark 6.7. We note that estimate (6.69) yields $||\nabla P_\varepsilon(\mathbf{v})||_{0,2,\Omega} = O(||\mathbf{v}||_{\mathbf{W}(\Omega)}/\varepsilon)$; hence, from the Poincaré–Wirtinger inequality, $||P_\varepsilon(\mathbf{v})||_{0,2,\Omega} = O(||\mathbf{v}||_{\mathbf{W}(\Omega)}/\varepsilon)$. Moreover, we also deduce

$$||P_\varepsilon||_{\mathscr{L}(\mathbf{W}(\Omega),L^2_0(\Omega))} = O\left(\frac{1}{\varepsilon}\right) \tag{6.70}$$

Remark 6.8. We also deduce from equality (6.67) that

$$(p, \nabla \cdot \mathbf{v})_\Omega = -\varepsilon||\nabla p||^2_{0,2,\Omega} \leq 0. \tag{6.71}$$

6.4.4 Variational ε-Approximations

The variational problem associated with the boundary value problem (6.66) is

$$\text{Find } \mathbf{v} \in \mathbf{W}(\Omega) \text{ such that } \forall\, \mathbf{w} \in \mathbf{W}(\Omega),$$

$$((\mathbf{v}\cdot\nabla)\,\mathbf{v}, \mathbf{w})_\Omega + a(\mathbf{v}, \mathbf{w}) + s_\nu(\mathbf{v}, \mathbf{w}) - (P_\varepsilon(\mathbf{v}), \nabla\cdot\mathbf{w})_\Omega + \langle G(\mathbf{v}), \mathbf{w}\rangle = \langle \mathbf{f}, \mathbf{w}\rangle. \tag{6.72}$$

Unfortunately, as $\nabla \cdot \mathbf{v}$ does not vanish, $((\mathbf{v} \cdot \nabla)\,\mathbf{v}, \mathbf{v})_\Omega \neq 0$, and we are not able to derive from (6.72) any a priori estimate and hence verify the consistency of this variational problem. However, the form of $\mathscr{VP}$ suggests introducing $\mathscr{VP}_\varepsilon$,

$$\text{Find } \mathbf{v} \in \mathbf{W}(\Omega) \text{ such that } \forall\, \mathbf{w} \in \mathbf{W}(\Omega),$$

$$\langle \mathscr{T}(\mathbf{v}), \mathbf{w}\rangle - (P_\varepsilon(\mathbf{v}), \nabla \cdot \mathbf{w})_\Omega + \langle G(\mathbf{v}), \mathbf{w}\rangle = \langle \mathbf{f}, \mathbf{w}\rangle, \tag{6.73}$$

where $\mathscr{T} = b + a + s_\nu$ is expressed by (6.15). We will see just below that this problem is consistent, since $b(\mathbf{v}; \mathbf{v}, \mathbf{v}) = 0$ given any $\mathbf{v} \in \mathbf{W}(\Omega)$. Moreover, following the calculations carried out in the proof of Lemma 6.3, we find that $\mathscr{VP}_\varepsilon$ is associated with the PDE system:

$$\begin{cases} (\mathbf{v} \cdot \nabla)\,\mathbf{v} + \dfrac{1}{2}\mathbf{v}(\nabla \cdot \mathbf{v}) - \nabla \cdot [(2\nu + \nu_t)\, D\mathbf{v}] + \nabla P_\varepsilon(\mathbf{v}) = \mathbf{f} & \text{in } \Omega, \\ -\,[(2\nu + \nu_t)\, D\mathbf{v} \cdot \mathbf{n}]_\tau = g(\mathbf{v})_\tau & \text{on } \Gamma, \\ \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \Gamma, \end{cases} \tag{6.74}$$

which indeed converges formally to the NSE (6.1) as $\varepsilon \to 0$. Any solution to $\mathscr{VP}_\varepsilon$ is a weak solution to (6.74). By the end of this chapter, we shall have proved the following result.

Theorem 6.3. *Given any $\varepsilon > 0$, the variational problem $\mathscr{VP}_\varepsilon$ expressed by (6.73) admits a solution.*

We will also show in the next subsection that given any solution $\mathbf{v}_\varepsilon$ to $\mathscr{VP}_\varepsilon$, then $(\mathbf{v}_\varepsilon, P_\varepsilon(\mathbf{v}_\varepsilon))_{\varepsilon>0}$ converges to a solution of $\mathscr{VP}$ in a sense that is clarified below. All of this explains why we have preferred to use the form b expressed by (6.20), instead of $(\mathbf{z}, \mathbf{v}, \mathbf{w}) \to ((\mathbf{z} \cdot \nabla)\,\mathbf{v}, \mathbf{z})_\Omega$ (cf. Remark 6.3).

We conclude this section by the following a priori estimates.

Lemma 6.8. *Let $\varepsilon > 0$, $\mathbf{v}_\varepsilon$ be any solution to $\mathscr{VP}_\varepsilon$, and set $p_\varepsilon = P_\varepsilon(\mathbf{v}_\varepsilon)$. Then*

$$||\mathbf{v}_\varepsilon||_{\mathbf{W}(\Omega)} \leq C_v, \tag{6.75}$$

$$||p_\varepsilon||_{0,2,\Omega} \leq C_p, \tag{6.76}$$

where C_v and C_p are the constants defined in (6.49) and (6.76).

Proof. Taking $\mathbf{v}_\varepsilon$ in (6.73) yields

$$a(\mathbf{v}_\varepsilon, \mathbf{v}_\varepsilon) + s_v(\mathbf{v}_\varepsilon, \mathbf{v}_\varepsilon) - (p_\varepsilon, \nabla \cdot \mathbf{v}_\varepsilon)_\Omega + \langle G(\mathbf{v}_\varepsilon), \mathbf{v}_\varepsilon \rangle = \langle \mathbf{f}, \mathbf{v}_\varepsilon \rangle. \tag{6.77}$$

We deduce from (6.71) and (6.77) that

$$a(\mathbf{v}_\varepsilon, \mathbf{v}_\varepsilon) + s_v(\mathbf{v}_\varepsilon, \mathbf{v}_\varepsilon) + \langle G(\mathbf{v}_\varepsilon), \mathbf{v}_\varepsilon \rangle \leq \langle \mathbf{f}, \mathbf{v}_\varepsilon \rangle. \tag{6.78}$$

From there, the rest of the proof proceeds as in the proof of (6.49). Estimate (6.76) follows from the same proof as that of (6.55). □

6.4.5 Convergence of Variational Problems

We aim to prove that the family $(\mathscr{VP}_\varepsilon)_{\varepsilon>0}$ converges to $\mathscr{VP}$ as $\varepsilon \to 0$. We must first elaborate the concept of the convergence of families of variational problems. To prepare the ground for future applications, we state one general abstract definition.

Definition 6.4. Let X_i^ε and Y_i, $i = 1, 2$, $\varepsilon > 0$, be reflexive Banach spaces, such that given any $\varepsilon > 0$, $X_1^\varepsilon \hookrightarrow Y_i$ with dense injection, $Y_2 = \cup_{\varepsilon>0} X_2^\varepsilon$. Let $(\mathscr{UQ}_\varepsilon)_{\varepsilon>0}$ be a given family of variational problems, with unknowns in X_1^ε and tests in X_2^ε. We say that $(\mathscr{UQ}_\varepsilon)_{\varepsilon>0}$ converges to $\mathscr{UQ}$ with unknowns in Y_1 and tests in Y_2 when $\varepsilon \to 0$, if and only if given any $\zeta_\varepsilon \in X_1^\varepsilon$ a priori solution to $\mathscr{UQ}_\varepsilon$, the family $(\zeta_\varepsilon)_{\varepsilon>0}$ is bounded in Y_1 and any weak subsequential limit[11] of $(\zeta_\varepsilon)_{\varepsilon>0}$ in Y_1 is an a priori solution to $\mathscr{UQ}$.

Definition 6.4 can take different forms depending on the particular case under study, and variants might be considered. Moreover, a similar definition can be made for sequences $(\mathscr{UP}_n)_{n\in\mathbb{N}}$.

In the definition above, an a priori solution to $\mathscr{UQ}_\varepsilon$ verifies formally $\mathscr{UQ}_\varepsilon$, which may have no solution. In other words, convergence of families of variational problems does not require the existence of solutions. This idea is well illustrated by Theorem 6.4 below. However, the following is straightforward.

[11] In case of a family $(\zeta_\varepsilon)_{\varepsilon>0}$, subsequential limits are limits of sequences of the form $(\zeta_{\varepsilon_n})_{n\in\mathbb{N}}$, where $\varepsilon_n \to 0$ as $n \to \infty$.

Lemma 6.9. *Assume that $\forall\, \varepsilon > 0$, $\mathscr{UQ}_\varepsilon$ admits a solution and the family $(\mathscr{UQ}_\varepsilon)_{\varepsilon>0}$ converges to $\mathscr{UQ}$. Then $\mathscr{UQ}$ admits a solution.*

Returning to our first issue, we note that the definition of the VESP-limit of a family $(\mathbf{w}_\varepsilon)_{\varepsilon>0}$ is the same as in Definition 6.2, initially stated for sequences. Observe in particular that any weak subsequential limit $\mathbf{w}$ of $(\mathbf{w}_\varepsilon)_{\varepsilon>0}$ in $\mathbf{W}(\Omega)$ is a VESP-limit and conversely. The acronym VESP refers to the additional compactness properties of the particular subsequence $(\mathbf{w}_{\varepsilon_n})_{n\in\mathbb{N}}$ we consider for converging to $\mathbf{w}$. Thus, the convergence result for the family $(\mathscr{VP}_\varepsilon)_{\varepsilon>0}$ to $\mathscr{VP}$ as $\varepsilon \to 0$ will be:

Theorem 6.4. *Let $\varepsilon > 0$, $\mathbf{v}_\varepsilon$ be any a priori solution to $\mathscr{VP}_\varepsilon$, $p_\varepsilon = P_\varepsilon(\mathbf{v}_\varepsilon)$. Let $\mathbf{v}$ be any VESP-limit of the sequence $(\mathbf{v}_\varepsilon)_{\varepsilon>0}$ and p be any PESP-limit of $(p_\varepsilon)_{\varepsilon>0}$. Then $(\mathbf{v}, p)$ is an a priori solution to $\mathscr{VP}$. We still say in this case that the family $(\mathscr{VP}_\varepsilon)_{\varepsilon>0}$ converges to $\mathscr{VP}$ as $\varepsilon \to 0$.*

Proof. By definition, $(\mathbf{v}_\varepsilon, p_\varepsilon)$ is such that for all $\mathbf{w} \in \mathbf{W}(\Omega)$,

$$\langle \mathscr{T}(\mathbf{v}_\varepsilon), \mathbf{w}\rangle - (p_\varepsilon, \nabla\cdot\mathbf{w})_\Omega + \langle G(\mathbf{v}_\varepsilon), \mathbf{w}\rangle = \langle \mathbf{f}, \mathbf{w}\rangle. \tag{6.79}$$

We already know by Lemma 6.8 that $(\mathbf{v}_\varepsilon, p_\varepsilon)_{\varepsilon>0}$ is bounded in $\mathbf{W}(\Omega) \times L^2_0(\Omega)$, so that VESP and PESP limits and $\mathbf{v}$ and p exist. We aim to prove that $(\mathbf{v}, p)$ is a solution to $\mathscr{VP}$. To achieve this goal, we proceed in two steps: we first take the limit in the variational formulation (6.79) to derive equation (6.12.i) in $\mathscr{VP}$ and then in the singular perturbation equation (6.63) to derive equation (6.12.ii).

STEP 1. Let $\mathbf{w} \in \mathbf{W}(\Omega)$. The weak convergence of $(\mathbf{v}_\varepsilon)_{\varepsilon>0}$ gives

$$\left\{\begin{aligned} \lim_{\varepsilon\to 0} a(\mathbf{v}_\varepsilon, \mathbf{w}) &= a(\mathbf{v}, \mathbf{w}),\\ \lim_{\varepsilon\to 0} s_v(\mathbf{v}_\varepsilon, \mathbf{w}) &= s_v(\mathbf{v}, \mathbf{w}),\\ \lim_{\varepsilon\to 0} (p_\varepsilon, \nabla\cdot\mathbf{w})_\Omega &= (p, \nabla\cdot\mathbf{w})_\Omega. \end{aligned}\right. \tag{6.80}$$

Moreover, applying Lemma 6.6, where in this case $(\mathbf{w}_n)_{n\in\mathbb{N}}$ is a constant sequence equal to $\mathbf{w}$, we obtain

$$\lim_{\varepsilon\to 0} b(\mathbf{v}_\varepsilon; \mathbf{v}_\varepsilon, \mathbf{w}) = b(\mathbf{v}; \mathbf{v}, \mathbf{w}), \quad \lim_{\varepsilon\to 0} \langle G(\mathbf{v}_\varepsilon), \mathbf{w}\rangle = \langle G(\mathbf{v}), \mathbf{w}\rangle. \tag{6.81}$$

Consequently, as $(\mathbf{v}_\varepsilon, p_\varepsilon)_{\varepsilon>0}$ satisfies (6.79) for all $\mathbf{w} \in \mathbf{W}(\Omega)$ we deduce from (6.80) and (6.81) that for all $\mathbf{w} \in \mathbf{W}(\Omega)$,

$$\langle \mathscr{T}(\mathbf{v}), \mathbf{w}\rangle - (p, \nabla\cdot\mathbf{w})_\Omega + \langle G(\mathbf{v}), \mathbf{w}\rangle = \langle \mathbf{f}, \mathbf{w}\rangle,$$

which is precisely equation (6.12.i) in $\mathscr{VP}$.

STEP 2. We now take the limit in the singular perturbation equation (6.63), which defines p_ε. Let $q \in \mathscr{D}(\Omega)$. Then we apply once again the Stokes formula in the formulation (6.64),

$$-\varepsilon(p_\varepsilon, \Delta q)_\Omega + (\nabla \cdot \mathbf{v}_\varepsilon, q)_\Omega = 0. \tag{6.82}$$

From the weak convergence of $(\mathbf{v}_\varepsilon)_{\varepsilon>0}$ toward $\mathbf{v}$ in $\mathbf{W}(\Omega)$, we get

$$\lim_{\varepsilon\to 0} (\nabla \cdot \mathbf{v}_\varepsilon, q)_\Omega = (\nabla \cdot \mathbf{v}, q)_\Omega,$$

and from the weak convergence of $(p_\varepsilon)_{\varepsilon>0}$ toward p in $L^2_0(\Omega)$,

$$\lim_{\varepsilon\to 0} (p_\varepsilon, \Delta q)_\Omega = (p, \Delta q)_\Omega, \quad \text{so that} \quad \lim_{\varepsilon\to 0} \varepsilon(p_\varepsilon, \Delta q)_\Omega = 0,$$

which yields by (6.82)

$$\forall\, q \in \mathscr{D}(\Omega), \quad (\nabla \cdot \mathbf{v}, q)_\Omega = 0. \tag{6.83}$$

As $\nabla \cdot \mathbf{v} \in L^2(\Omega)$ and $\mathscr{D}(\Omega)$ is dense in $L^2(\Omega)$, (6.83) holds for every $q \in L^2(\Omega)$. Therefore, the relations (6.80), (6.81), and (6.83) show that $(\mathbf{v}, p)$ is a solution to $\mathscr{VP}$. □

In conclusion, given any $\varepsilon > 0$, it suffices to prove the existence of a solution to $\mathscr{VP}_\varepsilon$ in order to construct a weak solution to the NSE (6.1), which will prove Theorems 6.2 and 6.3 simultaneously.

6.5 Solutions by the Galerkin Method

We show in this section the existence of a solution to $\mathscr{VP}_\varepsilon$ by the Galerkin method, the outline of which is:

(i) to project the variational problem over finite-dimensional spaces of the form $\mathbf{W}_n = span(\mathbf{w}_1, \cdots, \mathbf{w}_n)$, where $(\mathbf{w}_1, \cdots, \mathbf{w}_n, \cdots)$ is a Hilbert basis of $\mathbf{W}(\Omega)$, and to show that the resulting problem $\mathscr{VP}_{n,\varepsilon}$ has a solution $\mathbf{w}_n$ by a standard application of the Brouwer fixed-point theorem, to prove that the sequence $(\mathbf{w}_n)_{n\in\mathbb{N}}$ is bounded in $\mathbf{W}(\Omega)$,

(ii) to take the limit when $n \to \infty$ by showing that for any fixed ε, $(\mathscr{VP}_{n,\varepsilon})_{n\in\mathbb{N}}$ converges to $\mathscr{VP}_\varepsilon$.

6.5.1 Finite-Dimensional Problem

Let $(\mathbf{w}_j)_{j\in\mathbb{N}^\star}$ be a Hilbert basis of $\mathbf{W}(\Omega)$, the existence of which is straightforward, and we set

$$\forall\, n \in \mathbb{N}^\star, \quad \mathbf{W}_n = \text{span}(\mathbf{w}_1, \cdots, \mathbf{w}_n),$$

equipped with the $\mathbf{W}(\Omega)$ Hilbert structure. As $\mathbf{W}_n$ is a finite-dimensional space, we shall identify $\mathbf{W}'_n$ with $\mathbf{W}_n$ when no risk of confusion occurs. Let ε be fixed, and let us consider the variational problem denoted by $\mathscr{V}\mathscr{P}_{n,\varepsilon}$,

$$\text{Find } \mathbf{v} \in \mathbf{W}_n \text{ such that } \forall\, \mathbf{w} \in \mathbf{W}_n,$$

$$\langle \mathscr{T}(\mathbf{v}), \mathbf{w}\rangle - (P_\varepsilon(\mathbf{v}), \nabla \cdot \mathbf{w})_\Omega + \langle G(\mathbf{v}), \mathbf{w}\rangle = \langle \mathbf{f}, \mathbf{w}\rangle. \tag{6.84}$$

Problem $\mathscr{V}\mathscr{P}_{n,\varepsilon}$ differs from $\mathscr{V}\mathscr{P}_\varepsilon$ by the unknown and test spaces, which are both $\mathbf{W}_n$ instead of $\mathbf{W}(\Omega)$.

Theorem 6.5. *Problem $\mathscr{V}\mathscr{P}_{n,\varepsilon}$ has a solution $\mathbf{v}_n \in \mathbf{W}_n$ such that $||\mathbf{v}_n||_{\mathbf{W}(\Omega)} \le C_v$, where C_v is specified by (6.49).*

Proof. Let $\Phi : \mathbf{W}_n \to \mathbf{W}_n$ be the map specified by its dual action:

$$\langle \Phi\mathbf{v}, \mathbf{w}\rangle = \langle \mathscr{T}(\mathbf{v}), \mathbf{w}\rangle - (P_\varepsilon(\mathbf{v}), \nabla \cdot \mathbf{w})_\Omega + \langle G(\mathbf{v}), \mathbf{w}\rangle - \langle \mathbf{f}, \mathbf{w}\rangle. \tag{6.85}$$

Therefore, proving that $\mathscr{V}\mathscr{P}_{n,\varepsilon}$ has a solution is equivalent to proving that the equation $\Phi\mathbf{v} = 0$ has a solution in $\mathbf{W}_n$. To do so, we use Theorem A.5 in [TB], which is a standard variant of Brouwer's theorem, whose conditions for application are:

(i) Φ is continuous,
(ii) $\exists\, \mu > 0$ such that $(\Phi\mathbf{v}, \mathbf{v}) \ge 0$, $\forall\, \mathbf{v} \in \mathbf{W}_n$ satisfying $||\mathbf{v}||_{\mathbf{W}(\Omega)} = \mu$.

Proof of (i). We first observe that

$$\begin{aligned} b(\mathbf{v}_1; \mathbf{v}_1, \mathbf{w}) - b(\mathbf{v}_2; \mathbf{v}_2, \mathbf{w}) = \frac{1}{2}\,&[((\mathbf{v}_1 - \mathbf{v}_2)\cdot\nabla)\,\mathbf{v}_2 + (\mathbf{v}_2\cdot\nabla)\,(\mathbf{v}_1 - \mathbf{v}_2), \mathbf{w})_\Omega \\ &-((\mathbf{v}_1 - \mathbf{v}_2)\cdot\nabla)\,\mathbf{w}, \mathbf{v}_1)_\Omega + ((\mathbf{v}_2\cdot\nabla)\,\mathbf{w}, \mathbf{v}_1 - \mathbf{v}_2)_\Omega]\,, \end{aligned}$$

which leads to

$$\begin{aligned} &|b(\mathbf{v}_1; \mathbf{v}_1, \mathbf{w}) - b(\mathbf{v}_2; \mathbf{v}_2, \mathbf{w})| \le \\ &\qquad C(1 + ||\mathbf{v}_1||_{\mathbf{W}(\Omega)} + ||\mathbf{v}_2||_{\mathbf{W}(\Omega)})||\mathbf{v}_1 - \mathbf{v}_2||_{\mathbf{W}(\Omega)}\, ||\mathbf{w}||_{\mathbf{W}(\Omega)}. \end{aligned}$$

By combining this last inequality with Lemma 6.2 and Lemma 6.7, which ensure the continuity of the contributions due to a, s_v, and P_ε, we finally obtain

$$|(\Phi\mathbf{v}_1 - \Phi\mathbf{v}_2, \mathbf{w})| \le C(1 + ||\mathbf{v}_1||_{\mathbf{W}(\Omega)} + ||\mathbf{v}_2||_{\mathbf{W}(\Omega)})||\mathbf{v}_1 - \mathbf{v}_2||_{\mathbf{W}(\Omega)}\, ||\mathbf{w}||_{\mathbf{W}(\Omega)},$$

and hence by the Riesz representation theorem,

$$||\Phi\mathbf{v}_1 - \Phi\mathbf{v}_2||_{\mathbf{W}_n(\Omega)} \le C(1 + ||\mathbf{v}_1||_{\mathbf{W}(\Omega)} + ||\mathbf{v}_2||_{\mathbf{W}(\Omega)})||\mathbf{v}_1 - \mathbf{v}_2||_{\mathbf{W}(\Omega)}.$$

Therefore, Φ is of class $C^{0,1}$ over each bounded ball centered at the origin, and so continuous over $\mathbf{W}_n$.

Proof of (ii). On the one hand, $b(\mathbf{v};\mathbf{v},\mathbf{v}) = 0$; on the other hand, the inequality (6.71) yields $(P_\varepsilon(\mathbf{v}), \nabla\cdot\mathbf{v})_\Omega \leq 0$. As $0 \leq s_\nu(\mathbf{v},\mathbf{v})$ since $\nu_t \geq 0$, we deduce from (6.35)

$$(\Phi\mathbf{v},\mathbf{v}) \geq \nu||D\mathbf{v}||^2_{0,2,\Omega} + C_g\int_\Gamma |\mathbf{v}(\mathbf{x})|^{2+\alpha} d\Gamma(\mathbf{x}) - \langle\mathbf{f},\mathbf{w}\rangle,$$

which following (6.52) becomes

$$\begin{aligned}(\Phi\mathbf{v},\mathbf{v}) &\geq \nu||D\mathbf{v}||^2_{0,2,\Omega} + C_g||\mathbf{v}||^2_{0,2,\Gamma} - C_g|\Omega| - \langle\mathbf{f},\mathbf{w}\rangle \\ &\geq \inf(\nu, C_g)||\mathbf{v}||^2_{\mathbf{W}(\Omega)} - ||\mathbf{f}||_{\mathbf{W}(\Omega)'}||\mathbf{v}||_{\mathbf{W}(\Omega)} - C_g|\Omega|.\end{aligned} \tag{6.86}$$

As the polynomial of second degree

$$X \to \inf(\nu, C_g)X^2 - ||\mathbf{f}||_{\mathbf{W}(\Omega)'}X - C_g|\Omega|$$

goes to infinity when $X \to \infty$, we deduce from (6.86) that there exists μ , such that $\forall$ $\mathbf{v}$ s.t. $||\mathbf{v}||_{\mathbf{W}(\Omega)} = \mu$, then $(\Phi\mathbf{v},\mathbf{v}) \geq 0$ for $||\mathbf{v}||_{\mathbf{W}(\Omega)} = \mu$, proving item (ii).

In conclusion, there exists $\mathbf{v}_n \in \mathbf{W}_n$ such that $\Phi\mathbf{v}_n = 0$, proving the existence of a solution to $\mathcal{VP}_{n,\varepsilon}$. Moreover, the bound $||\mathbf{v}_n||_{\mathbf{W}(\Omega)} \leq C_\nu$ is derived following the same reasoning as that which yields (6.75). □

6.5.2 *Convergence of the Approximated Problems*

Theorem 6.6. *The sequence of variational problems $(\mathcal{VP}_{n,\varepsilon})_{n\in\mathbb{N}}$ converges to $\mathcal{VP}_\varepsilon$ as $n \to \infty$.*

Proof. Let $\mathbf{v}_n$ denote any solution to $(\mathcal{VP}_{n,\varepsilon})_{n\in\mathbb{N}}$ and $\mathbf{v}$ be any VESP-limit of $(\mathbf{v}_n)_{n\in\mathbb{N}}$. We aim to prove that $\mathbf{v}$ is a solution to $\mathcal{VP}_\varepsilon$.

Let $\mathbf{w} \in \mathbf{W}(\Omega)$, with $\mathbf{w}_n$ its orthogonal projection on $\mathbf{W}_n$. We also can apply the VESP to the sequence $(\mathbf{w}_n)_{n\in\mathbb{N}}$, and here we take the limit and the VESP-limit both equal to $\mathbf{w}$, so that the conditions for the application of Lemma 6.6 are met.

We know that $a(\cdot,\cdot)$, $c(\cdot,\cdot)$, and $\langle\mathbf{f},\cdot\rangle$ are bilinear/linear continuous maps over $\mathbf{W}(\Omega)$; P_ε is linear and continuous over $L^2(\Omega)$. Therefore we easily find that

$$\begin{cases}\lim\limits_{n\to\infty} a(\mathbf{v}_n,\mathbf{w}_n) = a(\mathbf{v},\mathbf{w}), \quad \lim\limits_{n\to\infty} s_\nu(\mathbf{v}_n,\mathbf{w}_n) = s_\nu(\mathbf{v},\mathbf{w}), \\ \lim\limits_{n\to\infty} (P_\varepsilon(\mathbf{v}_n), \nabla\cdot\mathbf{w}_n) = (P_\varepsilon(\mathbf{v}), \nabla\cdot\mathbf{w}), \quad \lim\limits_{n\to\infty} \langle\mathbf{f},\mathbf{w}_n\rangle = \langle\mathbf{f},\mathbf{w}\rangle.\end{cases} \tag{6.87}$$

Moreover, we know from Lemma 6.6,

$$\lim_{n\to\infty} b(\mathbf{v}_n;\mathbf{v}_n,\mathbf{w}_n) = b(\mathbf{v};\mathbf{v},\mathbf{w}), \quad \lim_{n\to\infty} \langle G(\mathbf{v}_n),\mathbf{w}_n\rangle = \langle G(\mathbf{v}),\mathbf{w}\rangle. \tag{6.88}$$

By combining (6.87) and (6.88), we conclude that $\mathbf{v}$ is a solution of $\mathscr{V}\mathscr{P}_\varepsilon$, concluding this proof. $\square$

In conclusion, Theorem 6.3 is proven, since the proof above shows the existence of a solution to the problem $\mathscr{V}\mathscr{P}_\varepsilon$. Therefore, Theorem 6.2 follows from Theorem 6.4 combined with Lemma 6.9.

6.6 Linear Problems

6.6.1 Setting

6.6.1.1 Motivations

It is usually difficult to implement a given nonlinear problem in a numerical code. A standard strategy is to approach the nonlinear problem by a sequence of linear problems, easier to implement. In certain cases, this procedure can be performed by linearization. In particular, we observe that the structure of the wall law allows the NSE (6.1) to be linearized in the form

$$\begin{cases} (\mathbf{z}\cdot\nabla)\,\mathbf{v} - \nabla\cdot[(2\nu+\nu_t)\,D\mathbf{v}] + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega, \\ -\left[(2\nu+\nu_t)\,D\mathbf{v}\cdot\mathbf{n}\right]_\tau = \mathbf{v}_\tau \tilde{H}(\mathbf{z}) & \text{on } \Gamma, \\ \mathbf{v}\cdot\mathbf{n} = 0 & \text{on } \Gamma, \end{cases} \tag{6.89}$$

for a given vector field $\mathbf{z}$, where for simplicity we write $\tilde{H}(\mathbf{z}) = H(|\mathbf{z}|)$. This suggests for example the following explicit numerical scheme,

$$\begin{cases} (\mathbf{v}^{(n)}\cdot\nabla)\,\mathbf{v}^{(n+1)} - \nabla\cdot\left[(2\nu+\nu_t)\,D\mathbf{v}^{(n+1)}\right] + \nabla p^{(n+1)} = \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega, \\ -\left[(2\nu+\nu_t)\,D\mathbf{v}^{(n+1)}\cdot\mathbf{n}\right]_\tau = \mathbf{v}_\tau^{(n+1)} \tilde{H}(\mathbf{v}^{(n)}) & \text{on } \Gamma, \\ \mathbf{v}^{(n+1)}\cdot\mathbf{n} = 0 & \text{on } \Gamma, \end{cases}$$

that may or may not converge (see remark 6.10 below). There are smarter ways to compute numerical solutions from the linearization (6.89), but this is not our goal here.

In our framework, we aim to construct a solution to $\mathscr{V}\mathscr{P}$ from a linearization procedure by the fixed-point theorem, proved by J. Schauder in 1930 (cf. Theorem A.6 in [TB], also proved in [10, 16, 19]). Unfortunately, we are unable to construct directly a weak solution to the linearized NSE and must introduce further approximations to stabilize the system.

We list below the variational problems that can be derived from the linear NSE (6.101) and its ε-approximation one after another, clarifying the reason why some stabilization is necessary to achieve our aim.

6.6.1.2 Linear Wall Law

We first set the linearized wall-law operator, which is defined by

$$\forall\, \mathbf{z} \in \mathbf{W}(\Omega), \quad \langle \mathscr{G}_{\mathbf{z}}(\mathbf{v}), \mathbf{w}\rangle = \int_\Gamma \tilde{H}(\mathbf{z}(\mathbf{x}))\, \mathbf{v}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}) d\Gamma(\mathbf{x}). \tag{6.90}$$

Since $0 \le \tilde{H}(\mathbf{z}) \le C(1 + |\mathbf{z}|)$, the operator $\mathscr{G}_{\mathbf{z}} : \mathbf{W}(\Omega) \to \mathbf{W}(\Omega)'$ is continuous and satisfies

$$||\mathscr{G}_{\mathbf{z}}(\mathbf{v})||_{\mathbf{W}(\Omega)'} \le C(1 + ||\mathbf{z}||_{\mathbf{W}(\Omega)})||\mathbf{v}||_{\mathbf{W}(\Omega)}, \tag{6.91}$$

for some constant C that only depends on Ω. Moreover, $\langle \mathscr{G}_{\mathbf{z}}(\mathbf{v}), \mathbf{v}\rangle \ge 0$. The operator $\mathscr{G}_{\mathbf{z}}$ is the linearization of G in the sense that

$$\forall\, \mathbf{v} \in \mathbf{W}(\Omega), \quad \mathscr{G}_{\mathbf{v}}(\mathbf{v}) = G(\mathbf{v}). \tag{6.92}$$

6.6.1.3 Linearization of $\mathscr{V}\mathscr{P}$

The linear variational problem $\mathscr{L}\mathscr{P}_{\mathbf{z}}$ directly deduced from the linear NSE (6.89) is, for a given $\mathbf{z} \in \mathbf{W}(\Omega)$,

$$\text{Find } (\mathbf{v}, p) \in \mathbf{W}(\Omega) \times L^2_0(\Omega) \text{ such that } \forall\, (\mathbf{w}, q) \in \mathbf{W}(\Omega) \times L^2(\Omega),$$

$$\begin{cases} \langle \mathscr{T}_{\mathbf{z}}(\mathbf{v}), \mathbf{w}\rangle - (p, \nabla \cdot \mathbf{w})_\Omega + \langle \mathscr{G}_{\mathbf{z}}(\mathbf{v}), \mathbf{w}\rangle = \langle \mathbf{f}, \mathbf{w}\rangle. \\ \qquad\qquad\qquad\qquad\qquad\qquad (\nabla \cdot \mathbf{v}, q)_\Omega = 0, \end{cases} \tag{6.93}$$

where we have set[12]

$$\forall\, \mathbf{z} \in \mathbf{W}(\Omega), \quad \langle \mathscr{T}_{\mathbf{z}}(\mathbf{v}), \mathbf{w}\rangle = b(\mathbf{z}; \mathbf{v}, \mathbf{w}) + a(\mathbf{v}, \mathbf{w}) + s_v(\mathbf{v}, \mathbf{w}). \tag{6.94}$$

The operator $\mathscr{T}_{\mathbf{z}}$ is the linearization of $\mathscr{T}$, which means

$$\forall\, \mathbf{v} \in \mathbf{W}(\Omega), \quad \langle \mathscr{T}_{\mathbf{v}}(\mathbf{v}), \mathbf{w}\rangle = \langle \mathscr{T}(\mathbf{v}), \mathbf{w}\rangle, \tag{6.95}$$

so that $\mathscr{L}\mathscr{P}_{\mathbf{z}}$ is the linearization of $\mathscr{V}\mathscr{P}$ at $\mathbf{z} \in \mathbf{W}(\Omega)$.

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[12] a, b and s_v are defined in (6.13) and $\mathscr{T}$ by (6.15).

6.6.1.4 Linearization of $\mathscr{VP}_\varepsilon$

The linearization of $\mathscr{VP}_\varepsilon$ yields the variational problem $\mathscr{LP}_{\varepsilon,\mathbf{z}}$,

$$\text{Find } \mathbf{v} \in \mathbf{W}(\Omega) \text{ such that } \forall\, \mathbf{w} \in \mathbf{W}(\Omega),$$

$$\langle \mathscr{T}_{\mathbf{z}}(\mathbf{v}), \mathbf{w}\rangle - (P_\varepsilon(\mathbf{v}), \nabla\cdot \mathbf{w})_\Omega + \langle \mathscr{G}_{\mathbf{z}}(\mathbf{v}), \mathbf{w}\rangle = \langle \mathbf{f}, \mathbf{w}\rangle. \tag{6.96}$$

Note that it is expected that

$$\lim_{\varepsilon\to 0} \mathscr{LP}_{\varepsilon,\mathbf{z}} = \mathscr{LP}_{\mathbf{z}},$$

in the sense of the convergence of variational problems.

6.6.1.5 η-Regularization

Let us explain why we are not able to prove the existence of a solution to $\mathscr{LP}_{\varepsilon,\mathbf{z}}$, which now becomes

$$\text{find } \mathbf{v} \in \mathbf{W}(\Omega) \text{ such that, } \forall\, \mathbf{w} \in \mathbf{W}(\Omega), \quad A_{\varepsilon,\mathbf{z}}(\mathbf{v}, \mathbf{w}) = \langle \mathbf{f}, \mathbf{w}\rangle, \tag{6.97}$$

where $A_{\varepsilon,\mathbf{z}}$ is the continuous bilinear form over $\mathbf{W}(\Omega)$, which is roughly speaking expressed by

$$A_{\varepsilon,\mathbf{z}} = \mathscr{T}_{\mathbf{z}} + P_\varepsilon + \mathscr{G}_{\mathbf{z}}. \tag{6.98}$$

This falls precisely in the framework of the theorem proved by P. Lax and A. Milgram in 1954 [13] (cf. also in [4]). However, the conditions for the application of the Lax–Milgram theorem stipulate that $A_{\varepsilon,\mathbf{z}}$ must also be coercive, which cannot be proved. Indeed, the best we can get is

$$A_{\varepsilon,\mathbf{z}}(\mathbf{v}, \mathbf{v}) \geq \nu ||D\mathbf{v}||^2_{0,2,\Omega} + \int_\Gamma |\mathbf{v}|^2 \tilde{H}(\mathbf{z}).$$

According to Lemma 5.7, wall laws satisfy $\tilde{H}(0) = 0$. Therefore, if $\mathbf{z}$ vanishes over a subset of Γ, whose surface measure is not equal to zero, $A_{\varepsilon,\mathbf{z}}$ is not coercive and Lax–Migram theorem does not apply.

One possibility that comes to mind is to regularize $A_{\varepsilon,\mathbf{z}}$ by adding a small boundary terms of the form $\eta(\mathbf{v}, \mathbf{w})_\Gamma$ and to introduce the variational problem $\mathscr{LP}_{\varepsilon,\eta,\mathbf{z}}$:

$$\text{Find } \mathbf{v} \in \mathbf{W}(\Omega) \text{ such that } \forall\, \mathbf{w} \in \mathbf{W}(\Omega),$$

$$\langle \mathscr{T}_{\mathbf{z}}(\mathbf{v}), \mathbf{w} \rangle - (P_\varepsilon(\mathbf{v}), \nabla \cdot \mathbf{w})_\Omega + \langle \mathscr{G}_{\mathbf{z}}(\mathbf{v}), \mathbf{w} \rangle + \eta(\mathbf{v}, \mathbf{w})_\Gamma = \langle \mathbf{f}, \mathbf{w} \rangle, \tag{6.99}$$

which will be shown to have a solution in the next subsection.

Note that in terms of PDEs, $\mathscr{LP}_{\varepsilon,\eta,\mathbf{z}}$ at a given $\mathbf{z} \in \mathbf{W}(\Omega)$ and for fixed $\varepsilon, \eta > 0$, is associated with the linear elliptic system:

$$\begin{cases} (\mathbf{z} \cdot \nabla)\, \mathbf{v} + \dfrac{1}{2}\mathbf{v}(\nabla \cdot \mathbf{z}) - \nabla \cdot [(2\nu + \nu_t)\, D\mathbf{v}] + \nabla p = \mathbf{f} & \text{in } \Omega, \\ -\varepsilon \Delta p + \nabla \cdot \mathbf{v} = 0 & \text{in } \Omega, \\ -\left[(2\nu + \nu_t)\, D\mathbf{v} \cdot \mathbf{n}\right]_\tau = \mathbf{v}_\tau(\tilde{H}(\mathbf{z}) + \eta) & \text{on } \Gamma, \\ \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \Gamma, \\ \dfrac{\partial p}{\partial \mathbf{n}} = 0 & \text{on } \Gamma, \\ \displaystyle\int_\Omega p(\mathbf{x}) d\mathbf{x} = 0. \end{cases} \tag{6.100}$$

To derive $\mathscr{LP}_{\varepsilon,\eta,\mathbf{z}}$ from System (6.100), we just have to eliminate the pressure by solving the system (6.63) and to perform the usual steps using the Stokes formula.

6.6.2 *Analysis of the Linearized Problems*

The goals of this subsection are:

(i) to show that $\mathscr{LP}_{\varepsilon,\eta,\mathbf{z}}$ has a unique solution,
(ii) to show that $\lim_{\varepsilon \to 0} \mathscr{LP}_{\varepsilon,\eta,\mathbf{z}} = \mathscr{LP}_{\eta,\mathbf{z}}$, where $\mathscr{LP}_{\eta,\mathbf{z}}$ is the variational problem which is associated with the linearized NSE,

$$\begin{cases} (\mathbf{z} \cdot \nabla)\, \mathbf{v} - \nabla \cdot [2\nu + \nu_t)\, D\mathbf{v}] + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{v} = 0 & \text{in } \Omega, \\ -\left[(2\nu + \nu_t)\, D\mathbf{v} \cdot \mathbf{n}\right]_\tau = \mathbf{v}_\tau(\tilde{H}(\mathbf{z}) + \eta) & \text{on } \Gamma, \\ \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \Gamma, \end{cases} \tag{6.101}$$

which formally converge to the NSE (6.1) as $\eta \to 0$.

6.6.2.1 Existence of Solution to $\mathscr{LP}_{\varepsilon,\eta,\mathbf{z}}$

We assume that $\varepsilon, \eta > 0$ are fixed.

Lemma 6.10. *Problem $\mathscr{LP}_{\varepsilon,\eta,\mathbf{z}}$ has a unique solution $\mathbf{v} \in \mathbf{W}(\Omega)$, which satisfies*

$$||\mathbf{v}||_{\mathbf{W}(\Omega)} \leq \frac{||\mathbf{f}||_{\mathbf{W}(\Omega)'}}{\inf(\nu, \eta)} = C_{\nu,\eta}. \tag{6.102}$$

Proof. Let $A_{\varepsilon,\mathbf{z}}$ be the bilinear form defined by (6.98) and $A_{\varepsilon,\eta,\mathbf{z}}$ be the bilinear form defined by

$$\forall\,(\mathbf{v},\mathbf{w}) \in \mathbf{W}(\Omega)^2, \quad A_{\varepsilon,\eta,\mathbf{z}}(\mathbf{v},\mathbf{w}) = A_{\varepsilon,\mathbf{z}}(\mathbf{v},\mathbf{w}) + \eta(\mathbf{v},\mathbf{w})_\Gamma,$$

so that $\mathscr{LP}_{\varepsilon,\eta,\mathbf{z}}$ becomes: [find $\mathbf{v} \in \mathbf{W}(\Omega)$ such that for all $\mathbf{w} \in \mathbf{W}(\Omega)$,]

$$A_{\varepsilon,\eta,\mathbf{z}}(\mathbf{v},\mathbf{w}) = \langle \mathbf{f},\mathbf{w}\rangle.$$

The previous results show that $A_{\varepsilon,\eta,\mathbf{z}}$ is continuous over $\mathbf{W}(\Omega)$, and according to Lemma 6.2, Lemmas 6.3, (6.70), and (6.91),

$$||A_{\varepsilon,\eta,\mathbf{z}}|| \le C(1 + ||\mathbf{z}||_{\mathbf{W}(\Omega)}) + 2\nu + ||\nu_t||_\infty + \frac{C}{\varepsilon} + \eta.$$

Moreover, we know from:

(i) Lemma 6.3 that $b(\mathbf{z};\mathbf{v},\mathbf{v}) = 0$,
(ii) (6.71) that $(P_\varepsilon(\mathbf{v}), \nabla\cdot\mathbf{w})_\Omega \le 0$,
(iii) hypothesis 6.i that $\nu_t \ge 0$,
(iv) hypothesis 6.i, $\tilde{H} \ge 0$, and (6.90) that $\langle \mathscr{G}_\mathbf{z}(\mathbf{v}),\mathbf{w}\rangle \ge 0$.

We then obtain

$$A_{\varepsilon,\eta,\mathbf{z}}(\mathbf{v},\mathbf{v}) \ge \nu||D\mathbf{v}||^2_{0,2,\Omega} + \eta||\mathbf{v}||^2_{0,2,\Gamma} \ge \inf(\nu,\eta)||\mathbf{v}||^2_{\mathbf{W}(\Omega)}, \tag{6.103}$$

and conclude that $A_{\varepsilon,\eta,\mathbf{z}}$ is coercive. The existence and uniqueness of a solution to $\mathscr{LP}_{\varepsilon,\eta,\mathbf{z}}$ follows from the Lax–Milgram theorem. The estimate (6.102) is a consequence of (6.103), by taking $\mathbf{v}$ as test in $\mathscr{LP}_{\varepsilon,\eta,\mathbf{z}}$. □

(b) **Convergence when $\varepsilon \to 0$.** Let $\mathscr{LP}_{\eta,\mathbf{z}}$ be the problem:

$$\text{Find}(\mathbf{v},p) \in \mathbf{W}(\Omega)\times L^2_0(\Omega) \;\text{ such that }\; \forall\,(\mathbf{w},q) \in \mathbf{W}(\Omega)\times L^2(\Omega),$$

$$\begin{cases} \langle \mathscr{T}_\mathbf{z}(\mathbf{v}),\mathbf{w}\rangle - (p,\nabla\cdot\mathbf{w})_\Omega + \langle \mathscr{G}_\mathbf{z}(\mathbf{v}),\mathbf{w}\rangle + \eta(\mathbf{v},\mathbf{w})_\Gamma = \langle \mathbf{f},\mathbf{w}\rangle, \\ (\nabla\cdot\mathbf{v},q)_\Omega = 0, \end{cases} \tag{6.104}$$

whose solution is the weak solution to the linearized NSE (6.101).

Lemma 6.11. *Given any* $\mathbf{z} \in \mathbf{W}(\Omega)$ *and* $\eta > 0$, *the family* $(\mathscr{LP}_{\varepsilon,\eta,\mathbf{z}})_{\varepsilon>0}$ *converges to* $\mathscr{LP}_{\eta,\mathbf{z}}$ *as* $\varepsilon \to 0$.

Proof. Let $\mathbf{v}_\varepsilon$ be the solution of $\mathscr{LP}_{\varepsilon,\eta,\mathbf{z}}$. We already know that $(\mathbf{v}_\varepsilon)_{\varepsilon>0}$ is bounded in $\mathbf{W}(\Omega)$. Let $p_\varepsilon = P_\varepsilon(\mathbf{v}_\varepsilon)$. To estimate the pressure term, we follow the outline of the proof which provides the inequality (6.61), slightly adapted to the present case. We then obtain

$$||p_\varepsilon||^2_{0,2,\Omega} \leq \frac{5}{2} C_\Omega \left(||\mathbf{f}||^2_{\mathbf{W}(\Omega)'} + C_1 + C_2(\eta^2 + ||\mathbf{z}||^2_{\mathbf{W}(\Omega)}) C^2_{\nu,\eta} + C_3 C^4_{\nu,\eta} \right), \tag{6.105}$$

where the C_i's depend on ν, $||\nu_t||_\infty$, $\tilde{H}$, and Ω. We conclude that $(p_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2_0(\Omega)$, where $p_\varepsilon = P_\varepsilon(\mathbf{v}_\varepsilon)$, in other words

$$||p_\varepsilon||_{0,2,\Omega} \leq C_{p,\eta}(||\mathbf{z}||_{\mathbf{W}(\Omega)}). \tag{6.106}$$

Consequently, the VESP applies to $(\mathbf{v}_\varepsilon)_{\varepsilon>0}$, denoting by $\mathbf{v}$ any VESP-limit, and the PESP applies to $(p_\varepsilon)_{\varepsilon>0}$, denoting by p any PESP-limit. Both satisfy

$$||\mathbf{v}||_{\mathbf{W}(\Omega)} \leq C_{\nu,\eta}, \quad ||p||_{0,2,\Omega} \leq C_{p,\eta}(||\mathbf{z}||_{\mathbf{W}(\Omega)}). \tag{6.107}$$

By reproducing the proof of Theorem 6.4 step by step, we see that $(\mathbf{v}, p)$ is a solution to $\mathscr{LP}_{\eta,\mathbf{z}}$, hence the result. In what follows, we denote by $(\mathbf{v}, p) = (\mathbf{v}_\eta(\mathbf{z}), p_\eta(\mathbf{z}))$ the unique solution to $\mathscr{LP}_{\eta,\mathbf{z}}$. □

Lemma 6.12. *$\mathscr{LP}_{\eta,\mathbf{z}}$ admits a unique solution $(\mathbf{v}_\eta(\mathbf{z}), p_\eta(\mathbf{z}))$ that satisfies estimates (6.107).*

Proof. We already know from Lemma (6.11) that $\mathscr{LP}_{\eta,\mathbf{z}}$ admits a solution satisfying (6.107). It remains to show that the solution is unique.

Let $(\mathbf{v}_1, p_1)$ and $(\mathbf{v}_2, p_2)$ be two solutions, $(\delta\mathbf{v}, \delta p) = (\mathbf{v}_1 - \mathbf{v}_2, p_1 - p_2)$. We have $\forall\ (\mathbf{w}, q) \in \mathbf{W}(\Omega) \times L^2(\Omega)$,

$$\begin{cases} \langle \mathscr{T}_\mathbf{z}(\delta\mathbf{v}), \mathbf{w}\rangle - (\delta p, \nabla \cdot \mathbf{w})_\Omega + \langle \mathscr{G}_\mathbf{z}(\delta\mathbf{v}), \mathbf{w}\rangle + \eta(\delta\mathbf{v}, \mathbf{w})_\Gamma = 0, \\ \hfill (\nabla \cdot \delta\mathbf{v}, q) = 0. \end{cases} \tag{6.108}$$

We first observe that conspicuously $(\delta p, \nabla \cdot \delta\mathbf{v})_\Omega = 0$. Then by taking $\mathbf{w} = \delta\mathbf{v}$ in (6.108) and using $b(\mathbf{z}; \delta\mathbf{v}, \delta\mathbf{v}) = 0$, $s_\nu(\mathbf{v}, \mathbf{v}) \geq 0$ and $\langle \mathscr{G}_\mathbf{z}(\delta\mathbf{v}), \delta\mathbf{v}\rangle \geq 0$, we have

$$\inf(\nu, \eta)(||D\delta\mathbf{v}||^2_{0,2,\Omega} + ||\delta\mathbf{v}||^2_{0,2,\Gamma}) = \inf(\nu, \eta)||\delta\mathbf{v}||^2_{\mathbf{W}(\Omega)} \leq 0,$$

hence $\delta\mathbf{v} = 0$. We also deduce that $(\delta p, \nabla \cdot \mathbf{w})_\Omega = 0$ for all $\mathbf{w} \in \mathbf{W}(\Omega)$ and in particular for all $\mathbf{w} \in \mathscr{D}(\Omega)^3$. Therefore, we deduce from standard results on distributions [17] that δp is constant *a.e.* in Ω, hence equal to zero since it belongs to $L^2_0(\Omega)$. In conclusion, we have $(\delta\mathbf{v}, \delta p) = (0, 0)$, which yields the uniqueness of the solution. □

6.7 Perturbed NSE and Fixed-Point Procedure

6.7.1 Framework and Aim

The linear problems investigated in the previous section suggest introducing the perturbed NSE:

$$\begin{cases} (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \nabla\cdot[(2\nu+\nu_t)\,D\mathbf{v}] + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega, \\ -\,[(2\nu+\nu_t)\,D\mathbf{v}\cdot\mathbf{n}]_\tau = g(\mathbf{v})_\tau + \eta\mathbf{v}_\tau & \text{on } \Gamma, \\ \mathbf{v}\cdot\mathbf{n} = 0 & \text{on } \Gamma, \end{cases} \tag{6.109}$$

whose corresponding variational problem $\mathscr{VP}_\eta$ is

$$\text{Find } (\mathbf{v}, p) \in \mathbf{W}(\Omega)\times L^2_0(\Omega) \text{ such that } \forall\,(\mathbf{w}, q) \in \mathbf{W}(\Omega)\times L^2(\Omega),$$

$$\begin{cases} \langle \mathscr{T}(\mathbf{v}), \mathbf{w}\rangle - (p, \nabla\cdot\mathbf{w})_\Omega + \langle G(\mathbf{v}), \mathbf{w}\rangle + \eta(\mathbf{v}, \mathbf{w})_\Gamma = \langle \mathbf{f}, \mathbf{w}\rangle. \\ (\nabla\cdot\mathbf{v}, q)_\Omega = 0, \end{cases} \tag{6.110}$$

where the operator $\mathscr{T}$ is expressed in formula (6.15). Observe that $\mathscr{LP}_{\eta,\mathbf{z}}$ is the linearized variational problem of $\mathscr{VP}_\eta$, while system (6.101) is the linearization of (6.109) at a given $\mathbf{z}\in\mathbf{W}(\Omega)$.

In this section we prove that for each $\eta > 0$, $\mathscr{VP}_\eta$ has a solution constructed from a fixed point of the application

$$\mathscr{V}_\eta : \begin{cases} \mathbf{W}(\Omega) \to \mathbf{W}(\Omega) \\ \quad\mathbf{z} \quad\;\; \to \mathbf{v}_\eta(\mathbf{z}), \end{cases} \tag{6.111}$$

where $\mathbf{v}_\eta(\mathbf{z})$ is the velocity part of the unique solution $(\mathbf{v}_\eta(\mathbf{z}), p_\eta(\mathbf{z}))$ to $\mathscr{LP}_{\eta,\mathbf{z}}$. Indeed, if $\mathbf{v}$ denotes any fixed point of $\mathscr{V}_\eta$, we observe that $(\mathbf{v}_\eta(\mathbf{v}), p_\eta(\mathbf{v})) = (\mathbf{v}, p_\eta(\mathbf{v}))$ is actually a solution to $\mathscr{VP}_\eta$.

We then show that $(\mathscr{VP}_\eta)_{\eta>0}$ converges to $\mathscr{VP}$ as $\eta\to 0$, in the sense of the convergence of variational problems.

The main goal of this section is thus to prove that $\mathscr{V}_\eta$ fulfills the conditions for the application of Schauder's fixed-point theorem (cf. Theorem A.6 in [TB]). As $\mathbf{W}(\Omega)$ is a separated topological vector space, we must establish that

(i) $\mathscr{V}_\eta$ is continuous,
(ii) there is a convex subset in $\mathbf{W}(\Omega)$, which we take to be a ball B_R in the present case, such that $\mathscr{V}_\eta(B_R)\subset B_R$,
(iii) $\mathscr{V}_\eta(B_R)$ is compact.

The difficult technical point is the compactness of $\mathscr{V}_\eta$. This requires taking the limit in the equations, not only in a weak sense as we have already done many times but also in a strong sense. The procedure we develop is called the energy method, as we prove the convergence of norms by considering energy equalities.

The first step is to show that if $(\mathbf{z}_n)_{n\in\mathbb{N}}$ weakly converges to $\mathbf{z} \in \mathbf{W}(\Omega)$, then $(\mathscr{LP}_{\eta,\mathbf{z}_n})_{n\in\mathbb{N}}$ converges to $\mathscr{LP}_{\eta,\mathbf{z}}$. The second step proceeds with the energy method, which yields the continuity of $\mathscr{V}$. In the third step, we show that $\mathscr{V}_\eta$ has a fixed point and finally takes the limit as $\eta \to 0$ in the last step.

6.7.2 Groundwork

6.7.2.1 Convergence of Linear Problems

Let $\eta > 0$. Recall that for a given $\mathbf{z} \in \mathbf{W}(\Omega)$, $\mathscr{LP}_{\eta,\mathbf{z}}$ is specified by (6.104). As announced, we prove:

Lemma 6.13. *Let $(\mathbf{z}_n)_{n\in\mathbb{N}}$ be weakly convergent to $\mathbf{z}$ in $\mathbf{W}(\Omega)$. Then $(\mathscr{LP}_{\eta,\mathbf{z}_n})_{n\in\mathbb{N}}$ converges to $\mathscr{LP}_{\eta,\mathbf{z}}$.*

Proof. We first apply the VESP to $(\mathbf{z}_n)_{n\in\mathbb{N}}$, and in this case $\mathbf{z}$ is the unique VESP-limit.

Let $(\mathbf{v}_n, p_n)$ denote the solution of $\mathscr{LP}_{\eta,\mathbf{z}_n}$. We deduce from estimate (6.107) that $(\mathbf{v}_n)_{n\in\mathbb{N}}$ is bounded in $\mathbf{W}(\Omega)$ so that the VESP applies to $(\mathbf{v}_n)_{n\in\mathbb{N}}$. Let $\mathbf{v}$ be a VESP-limit. Since $(\mathbf{z}_n)_{n\in\mathbb{N}}$ is weakly convergent, it is a bounded sequence in $\mathbf{W}(\Omega)$.

We must now check that $(p_n)_{n\in\mathbb{N}}$ is bounded. Let us consider

$$R = \sup(||\mathbf{z}_n||, n \in \mathbb{N}).$$

We observe that the expression of the bound $C_{p,\eta}(||\mathbf{z}||_{\mathbf{W}(\Omega)})$ deduced from (6.105) is nondecreasing in $||\mathbf{z}||_{\mathbf{W}(\Omega)}$, leading to

$$||p_n||_{0,2,\Omega} \le C_{p,\eta}(||\mathbf{z}_n||_{\mathbf{W}(\Omega)}) \le C_{p,\eta}(R),$$

hence $(p_n)_{n\in\mathbb{N}}$ is bounded in $L^2_0(\Omega)$, so that the PESP applies to it. Let p be a PESP-limit.

We have to prove that $(\mathbf{v}, p)$ is the solution $\mathscr{LP}_{\eta,\mathbf{z}}$. It has the same convergence properties as (6.80) and (6.81), that is, the convergence

$$b(\mathbf{z}_n; \mathbf{v}_n, \mathbf{w}), a(\mathbf{v}_n, \mathbf{w}), s_v(\mathbf{v}_n, \mathbf{w}), (p_n, \mathbf{w}) \to b(\mathbf{z}; \mathbf{v}, \mathbf{w}), a(\mathbf{v}, \mathbf{w}), s_v(\mathbf{v}, \mathbf{w}), (p, \nabla \cdot \mathbf{w})$$

as $n \to \infty$, respectively, for any $\mathbf{w} \in \mathbf{W}(\Omega)$. Moreover,

$$(\mathbf{v}_n, \mathbf{w})_\Gamma \to (\mathbf{v}, \mathbf{w})_\Gamma, \quad (\nabla \cdot \mathbf{v}_n, q)_\Omega \to (\nabla \cdot \mathbf{v}, q)_\Omega, \ \text{ as } \ n \to \infty.$$

The boundary terms need to be studied more carefully. As $(\mathbf{z}_n)_{n\in\mathbb{N}}$ converges *a.e.* to $\mathbf{z}$ in Γ (writing $\mathbf{z}$ instead of $\gamma_0(\mathbf{z})$ for simplicity), $\tilde{H}$ is continuous, so that $\tilde{H}(\mathbf{z}_n) \to$

$\tilde{H}(\mathbf{z})$ a.e. in Γ. Moreover, according to Sect. 6.3.3.1 where we have defined the VESP, combined with the growth assumption about $\tilde{H}$, we infer that there exists $B_3 \in L^3(\Gamma)$, such that for any $\mathbf{w} \in \mathbf{W}(\Omega)$,

$$\forall\, n \in \mathbb{N}, \quad |\tilde{H}(\mathbf{z}_n)\mathbf{w}| \le C|\mathbf{w}|(1 + |\mathbf{z}_n|) \le C|\mathbf{w}|(1 + B_3) \in L^{3/2}(\Gamma).$$

The Lebesgue theorem shows that $(\tilde{H}(\mathbf{z}_n)\mathbf{w})_{n\in\mathbb{N}}$ converges to $\tilde{H}(\mathbf{z})\mathbf{w}$ in $\mathbf{L}^{3/2}(\Gamma)$. Since $(\mathbf{v}_n)_{n\in\mathbb{N}}$ converges to $\mathbf{v}$ in $\mathbf{L}^3(\Gamma)$ (among others), then

$$\lim_{n\to\infty} \langle \mathscr{G}_{z_n}(\mathbf{v}_n), \mathbf{w}\rangle = \lim_{n\to\infty} (\mathbf{v}_n \tilde{H}(\mathbf{z}_n), \mathbf{w})_\Gamma =$$
$$\lim_{n\to\infty} \int_\Gamma (\tilde{H}(\mathbf{z}_n)\mathbf{w} \cdot \mathbf{v}_n = \int_\Gamma \tilde{H}(\mathbf{z})\mathbf{w} \cdot \mathbf{v} = \langle \mathscr{G}_{\mathbf{z}}(\mathbf{v}), \mathbf{w}\rangle.$$

Summing up, we can conclude that $(\mathbf{v}, p)$ is indeed the solution to $\mathscr{LP}_{\eta,\mathbf{z}}$, ending the proof. □

6.7.2.2 Weak Continuity of $\mathscr{V}_\eta$

Let $\mathbf{z} \in \mathbf{W}(\Omega)$ and $(\mathbf{v}_\eta(\mathbf{z}), p_\eta(\mathbf{z}))$ be the unique solution to $\mathscr{LP}_{\eta,\mathbf{z}}$. Recall that $\mathscr{V}_\eta$ is the application

$$\mathscr{V}_\eta : \begin{cases} \mathbf{W}(\Omega) \to \mathbf{W}(\Omega), \\ \quad \mathbf{z} \quad \to \mathbf{v}_\eta(\mathbf{z}), \end{cases} \tag{6.112}$$

and given any fixed point $\mathbf{v}$ of the application $\mathscr{V}_\eta$, $(\mathbf{v}, p_\eta(\mathbf{v}))$ is a solution to $\mathscr{VP}_\eta$.

Lemma 6.14. *The application $\mathscr{V}_\eta$ is sequentially weakly continuous.*

Proof. Let $(\mathbf{z}_n)_{n\in\mathbb{N}}$ be weakly convergent to $\mathbf{z}$ in $\mathbf{W}(\Omega)$, $\mathbf{v}_n = \mathscr{V}_\eta(\mathbf{z}_n)$. We have to prove that $(\mathbf{v}_n)_{n\in\mathbb{N}}$ converges to $\mathbf{v} = \mathscr{V}_\eta(\mathbf{z})$. We already know from Lemma 6.13 that a subsequence can be extracted from $(\mathbf{v}_n)_{n\in\mathbb{N}}$ that converges to $\mathbf{v}$. But as the solution to $\mathscr{LP}_{\eta,\mathbf{z}}$ is unique, $\mathbf{v}$ is the unique subsequential weak limit to $(\mathbf{v}_n)_{n\in\mathbb{N}}$, hence its weak limit, concluding the proof. □

6.7.3 The Energy Method

We aim to prove the following compactness property, where $\eta > 0$ is fixed.

Lemma 6.15. *Let $(\mathbf{z}_n)_{n\in\mathbb{N}}$ be weakly convergent to $\mathbf{z}$ in $\mathbf{W}(\Omega)$, $\mathbf{v}_n = \mathscr{V}_\eta(\mathbf{z}_n)$. Then $(\mathbf{v}_n)_{n\in N}$ strongly converges in $\mathbf{W}(\Omega)$ to $\mathbf{v} = \mathscr{V}_\eta(\mathbf{z})$.*

Proof. We already know from Lemma 6.14 that $(\mathbf{v}_n)_{n\in N}$ weakly converges in $\mathbf{W}(\Omega)$ to $\mathbf{v} = \mathscr{V}_\eta(\mathbf{z})$. The VESP applies to the sequence $(\mathbf{v}_n)_{n\in\mathbb{N}}$, and we prove in what follows the strong convergence of the extracted subsequence. Once again, the uniqueness argument leads to the conclusion that the whole sequence strongly converges.

To do so, we prove that there is an Hilbertian norm $||\cdot||_{\nu_t,\nu,\eta}$, equivalent to the norm $||\cdot||_{\mathbf{W}(\Omega)}$, such that

$$\lim_{n\to\infty} ||\mathbf{v}_n||_{\nu_t,\nu,\eta} = ||\mathbf{v}||_{\nu_t,\nu,\eta}, \tag{6.113}$$

which will yield the strong convergence of $(\mathbf{v}_n)_{n\in\mathbb{N}}$ to $\mathbf{v}$ in $\mathbf{W}(\Omega)$. The conclusion follows because we know that (see in [4])

(i) the weak convergence of $(\mathbf{v}_n)_{n\in\mathbb{N}}$ toward $\mathbf{v}$, combined with
(ii) the convergence of $(||\mathbf{v}_n||_{\nu_t,\nu,\eta})_{n\in\mathbb{N}}$ toward $||\mathbf{v}||_{\nu_t,\nu,\eta}$,

yields the strong convergence of $(\mathbf{v}_n)_{n\in\mathbb{N}}$ toward $\mathbf{v}$ in $\mathbf{W}(\Omega)$, which is a Hilbert space and hence a uniformly convex Banach space.

It therefore remains for us to prove (6.113). The starting point are the energy equalities, which are obviously satisfied:

$$\begin{aligned}\nu||D\mathbf{v}_n||^2_{0,2,\Omega} + \eta||\mathbf{v}_n||^2_{0,2,\Gamma} + \int_\Omega \nu_t(\mathbf{x})|D\mathbf{v}_n(\mathbf{x})|^2 d\mathbf{x} + \langle \mathcal{G}_{\mathbf{z}_n}(\mathbf{v}_n),\mathbf{v}_n\rangle &= \langle \mathbf{f},\mathbf{v}_n\rangle,\\ \nu||D\mathbf{v}||^2_{0,2,\Omega} + \eta||\mathbf{v}||^2_{0,2,\Gamma} + \int_\Omega \nu_t(\mathbf{x})|D\mathbf{v}(\mathbf{x})|^2 + \langle \mathcal{G}_{\mathbf{z}}(\mathbf{v}),\mathbf{v}\rangle &= \langle \mathbf{f},\mathbf{v}\rangle.\end{aligned}$$

It is already understood that $\langle \mathbf{f},\mathbf{v}_n\rangle \to \langle \mathbf{f},\mathbf{v}\rangle$. We handle the boundary term as in the proof of Lemma 6.13 above. Recall that γ_0 denotes the trace operator. We infer right away from the VESP and the continuity of $\tilde{H}$ that $(\tilde{H}(\gamma_0(\mathbf{z}_n))\gamma_0(\mathbf{v}_n))_{n\in\mathbb{N}}$ converges a.e in Γ to $\tilde{H}(\gamma_0(\mathbf{z}))\gamma_0(\mathbf{v})$ and according to Sect. 6.3.3.1:

(i) there exists $B_{3,v} \in L^3(\Gamma)$ such that $\forall\, n \in \mathbb{N}$, $|\gamma_0(\mathbf{v}_n)| \le B_{3,v}$ a.e. in Γ,
(ii) $0 \le \tilde{H}(\gamma_0(\mathbf{z}_n)) \le C(1 + |\gamma_0(\mathbf{z}_n)|)$ and there exists $B_{3,z} \in L^3(\Gamma)$ such that $\forall\, n \in \mathbb{N}$, we have $|\gamma_0(\mathbf{z}_n)| \le B_{3,z}$ a.e. in Γ,

$$|\tilde{H}(\gamma_0(\mathbf{z}_n))\gamma_0(\mathbf{v}_n)| \le C(1 + B_{3,z})\, B_{3,v} \in L^{3/2}(\Gamma).$$

We therefore have that $(\tilde{H}(\gamma_0(\mathbf{z}_n))\gamma_0(\mathbf{v}_n))_{n\in\mathbb{N}}$ converges to $\tilde{H}(\gamma_0(\mathbf{z}))\gamma_0(\mathbf{v})$ in $L^{3/2}(\Gamma)$, and as $(\gamma_0(\mathbf{v}_n))_{n\in\mathbb{N}}$ converges strongly to $\gamma_0(\mathbf{v})$ in $L^3(\Gamma)$, we obtain (by skipping γ_0 for simplicity)

$$\lim_{n\to\infty}\langle \mathcal{G}_{\mathbf{z}_n}(\mathbf{v}_n),\mathbf{v}_n\rangle = \lim_{n\to\infty}\int_\Gamma \tilde{H}(\mathbf{z}_n)\,|\mathbf{v}_n|^2 = \int_\Gamma \tilde{H}(\mathbf{z})\,|\mathbf{v}|^2 = \langle \mathcal{G}_{\mathbf{z}}(\mathbf{v}),\mathbf{v}\rangle,$$

leading to

$$\lim_{n\to\infty}\left(\nu||D\mathbf{v}_n||^2_{0,2,\Omega}+\eta||\mathbf{v}_n||^2_{0,2,\Gamma}+\int_\Omega \nu_t(\mathbf{x})|D\mathbf{v}_n(\mathbf{x})|^2\right)= \nu||D\mathbf{v}||^2_{0,2,\Omega}+\eta||\mathbf{v}||^2_{0,2,\Gamma}+\int_\Omega \nu_t(\mathbf{x})|D\mathbf{v}(\mathbf{x})|^2. \tag{6.114}$$

Therefore (6.113) holds as promised, where

$$||\mathbf{v}||_{\nu_t,\nu,\eta}=\left(\nu||D\mathbf{v}||^2_{0,2,\Omega}+\eta||\mathbf{v}||^2_{0,2,\Gamma}+\int_\Omega \nu_t(\mathbf{x})|D\mathbf{v}(\mathbf{x})|^2\right)^{\frac{1}{2}}$$

which is indeed a Hilbertian norm equivalent to $||\cdot||_{\mathbf{W}(\Omega)}$, concluding the proof. □

The method developed above, which consists in proving strong convergence through energy equalities, is called the *energy method*.

Corollary 6.1. *The application $\mathscr{V}_\eta$ is continuous over* $\mathbf{W}(\Omega)$.

Proof. This results directly from Lemma 6.15. Indeed, as $\mathbf{W}(\Omega)$ is a Hilbert space, continuity is equivalent to sequential continuity, and every strong convergent sequence is also weakly convergent. □

6.7.4 Fixed-Point Process and Convergence

6.7.4.1 Existence of a Fixed Point

Recall that $\mathscr{V}\mathscr{P}_\eta$ is specified by (6.110).

Lemma 6.16. *The application $\mathscr{V}_\eta$ specified by (6.112) has a fixed point. Therefore, $\mathscr{V}\mathscr{P}_\eta$ admits a solution.*

Proof. It is already known that $\mathscr{V}_\eta$ is continuous. Let $R = C_{\nu,\eta}$, recalling that $||\mathbf{v}_\eta(\mathbf{z})||_{\mathbf{W}(\Omega)} \le C_{\nu,\eta}$ according to estimate (6.107). Let $B_R \subset \mathbf{W}(\Omega)$ be the ball of radius R centered in 0, which is convex. We obviously have $\mathscr{V}_\eta(B_R)\subset B_R$.

It remains to prove that $\mathscr{V}_\eta(B_R)$ is compact. As $\mathscr{V}_\eta(B_R)$ is a closed subset of a metric space, it is enough to show that from any sequence $(\mathbf{v}_n)_{n\in\mathbb{N}}$ in $\mathscr{V}_\eta(B_R)$, a strong convergent subsequence in $\mathbf{W}(\Omega)$ can be extracted.

Let $\mathbf{z}_n \in B_R$ be such that $\mathbf{v}_n = \mathbf{v}_\eta(\mathbf{z}_n)$. The VESP applies to both $(\mathbf{v}_n)_{n\in\mathbb{N}}$ and $(\mathbf{z}_n)_{n\in\mathbb{N}}$, with $\mathbf{v}$ and $\mathbf{z}$ denoting the corresponding VESP-limits. We deduce from Lemma 6.13 that $\mathbf{v}=\mathscr{V}_\eta(\mathbf{z})$ and from Lemma 6.15 that $(\mathbf{v}_n)_{n\in\mathbb{N}}$ strongly converges to $\mathbf{v}$, which proves that $\mathscr{V}_\eta(B_R)$ is compact.

In conclusion, the conditions for the application of Schauder's theorem are fulfilled, ending the proof. □

6.7.4.2 Taking the Limit When η Goes to Zero

We conclude the section with:

Lemma 6.17. *The family* $(\mathscr{V}\mathscr{P}_\eta)_{\eta>0}$ *converges to* $\mathscr{V}\mathscr{P}$ *when* $\eta \to 0$.

Proof. Let $(\mathbf{v}_\eta, p_\eta)$ be any solution to $(\mathscr{V}\mathscr{P}_\eta)_{\eta>0}$. We start by looking for bounds, uniform in η. Taking $\mathbf{w} = \mathbf{v}_\eta$ as test yields

$$a(\mathbf{v}_\eta, \mathbf{v}_\eta) + s_v(\mathbf{v}_\eta, \mathbf{v}_\eta) + \langle G(\mathbf{v}_\eta), \mathbf{v}_\eta\rangle + \eta(\mathbf{v}_\eta, \mathbf{v}_\eta)_\Gamma = \langle \mathbf{f}, \mathbf{v}\rangle, \tag{6.115}$$

hence, as $\eta(\mathbf{v}_\eta, \mathbf{v}_\eta)_\Gamma \geq 0$,

$$a(\mathbf{v}_\eta, \mathbf{v}_\eta) + s_v(\mathbf{v}_\eta, \mathbf{v}_\eta) + \langle G(\mathbf{v}_\eta), \mathbf{v}_\eta\rangle \leq \langle \mathbf{f}, \mathbf{v}\rangle. \tag{6.116}$$

By a similar proof to that of (6.49), we obtain

$$\forall\, \eta > 0, \quad ||\mathbf{v}_\eta||_{\mathbf{W}(\Omega)} \leq C_v \tag{6.117}$$

In summary, $(\mathbf{v}_\eta)_{\eta>0}$ is bounded in $\mathbf{W}(\Omega)$, so that the VESP applies for $\eta \to 0$. Let $\mathbf{v}$ be any VESP-limit.

Customizing the pressure estimate (6.61), we obtain for $\eta \leq 1$,

$$\begin{aligned} ||p_\eta||^2_{0,2,\Omega} &\leq (5/2)C_\Omega \left(||\mathbf{f}||^2_{\mathbf{W}(\Omega)'} + C_1 + C_2(1+\eta^2)C_v^2 + C_3 C_v^4 \right) \\ &\leq (5/2)C_\Omega \left(||\mathbf{f}||^2_{\mathbf{W}(\Omega)'} + C_1 + 2C_2 C_v^2 + C_3 C_v^4 \right) = \tilde{C}_p, \end{aligned} \tag{6.118}$$

which shows that $(p_\eta)_{\eta>0}$ is uniformly bounded in η. Thus the PESP applies to $(p_\eta)_{\eta>0}$ for $\eta \to 0$, and let p be a PESP-limit.

From here, showing that $(\mathbf{v}, p)$ is a solution to $\mathscr{V}\mathscr{P}$ proceeds as in the proof of Theorem 6.4, with the additional term $\eta(\mathbf{v}_\eta, \mathbf{w})$. However, $(\mathbf{v}_\eta)_{\eta>0}$ is bounded in $L^2(\Gamma)$, so that $\eta(\mathbf{v}_\eta, \mathbf{w}) \to 0$ as $\eta \to 0$, for any fixed $\mathbf{w} \in \mathbf{W}(\Omega)$, which concludes the proof. □

Remark 6.9. It can be proved by the energy method that $(\mathbf{v}_\eta)_{\eta>0}$ converges strongly to $\mathbf{v}$ in $\mathbf{W}(\Omega)$.

6.8 Uniqueness

Up to now, we have not discussed the problem of uniqueness. We are unable to prove any uniqueness result for $\mathscr{V}\mathscr{P}$, and then for the NSE (6.1), due once again to the lack of coercivity explained in Sect. 6.6. However, we can prove a uniqueness result for the regularized version $\mathscr{V}\mathscr{P}_\eta$, then for the NSE (6.109), subjected to an additional hypothesis on the data size.

From now $\eta > 0$ is fixed. We recall that $g(\mathbf{v}) = \mathbf{v}\tilde{H}(\mathbf{v}) \in W^{1,\infty}_{loc}(\mathbb{R}^3) \cap C^0(\mathbb{R}^3)$, so that

$$|H(|\mathbf{v}_1|) - H(|\mathbf{v}_2|)| \leq ||\tilde{H}'||_\infty |\mathbf{v}_1 - \mathbf{v}_2|. \tag{6.119}$$

We denote by $T_{0,p,\Gamma}$, $1 \leq p \leq 4$, the best constant such that

$$\forall\, \mathbf{w} \in \mathbf{W}(\Omega), \quad ||\mathbf{w}||_{0,p,\Gamma} \leq C||\mathbf{w}||_{\mathbf{W}(\Omega)}. \tag{6.120}$$

We deduce from estimate (i)) in Lemma 6.3, and the equivalence between $||\cdot||_{1,2,\Omega}$ and $||\cdot||_{\mathbf{W}(\Omega)}$ over $\mathbf{W}(\Omega)$, that

$$\forall\, \mathbf{v}, \mathbf{w}, \mathbf{z} \in \mathbf{W}(\Omega), \quad |b(\mathbf{z}; \mathbf{v}, \mathbf{w})| \leq C||\mathbf{z}||_{\mathbf{W}(\Omega)}||\mathbf{v}||_{\mathbf{W}(\Omega)}||\mathbf{w}||_{\mathbf{W}(\Omega)}, \tag{6.121}$$

where C only depends on Ω. We denote by B_Ω the best constant such that (6.121) holds.

Proposition 6.3. *Assume that*

$$C_\nu(B_\Omega + T_{0,4,\Gamma}||\tilde{H}'||_\infty) < \inf(\nu, \eta), \tag{6.122}$$

then $\mathcal{VP}_\eta$ admits at most one solution.

Proof. Let $(\mathbf{v}_1, p_1)$ and $(\mathbf{v}_2, p_2)$ be two solutions to $\mathcal{VP}_\eta$,

$$(\delta\mathbf{v}, \delta p) = (\mathbf{v}_1 - \mathbf{v}_2, p_1 - p_2).$$

We have $\forall\, (\mathbf{w}, q) \in \mathbf{W}(\Omega) \times L^2(\Omega)$,

$$\begin{cases} \langle \mathcal{T}_{\mathbf{v}_1}(\delta\mathbf{v}), \mathbf{w}\rangle - (\delta p, \nabla\cdot\mathbf{w})_\Omega + \langle \mathcal{G}_{\mathbf{v}_1}(\delta\mathbf{v}), \mathbf{w}\rangle = b(\delta\mathbf{v}; \mathbf{v}_2, \mathbf{w}) - (\mathbf{v}_2\delta\tilde{H}, \mathbf{w})_\Gamma, \\ \qquad\qquad (q, \nabla\cdot\delta\mathbf{v})_\Omega = 0. \end{cases} \tag{6.123}$$

by setting[13]

$$\delta\tilde{H} = \tilde{H}(\mathbf{v}_1) - \tilde{H}(\mathbf{v}_2).$$

As $\nabla\cdot\delta\mathbf{v} = 0$, then $(\delta p, \nabla\cdot\delta\mathbf{v})_\Omega = 0$. We also know that

$$b(\mathbf{v}_1; \delta\mathbf{v}, \delta\mathbf{v}) = 0, \quad s_\nu(\delta\mathbf{v}, \delta\mathbf{v}) \geq 0, \quad \langle \mathcal{G}_{\mathbf{v}_1}(\delta\mathbf{v}), \delta\mathbf{v}\rangle \geq 0.$$

Then by taking $\mathbf{w} = \delta\mathbf{v}$ in (6.123), we obtain

$$\inf(\nu, \eta)||\delta\mathbf{v}||^2_{\mathbf{W}(\Omega)} \leq b(\delta\mathbf{v}; \mathbf{v}_2, \delta\mathbf{v}) - (\mathbf{v}_2(\tilde{H}(\mathbf{v}_1) - \tilde{H}(\mathbf{v}_2)), \delta\mathbf{v})_\Gamma. \tag{6.124}$$

[13] $\mathcal{G}_z$ is expressed by (6.90), $\mathcal{T}_\mathbf{z}$ by (6.94).

We now treat the r.h.s. of (6.124), using the estimate and (6.121), yielding

$$|b(\delta\mathbf{v}; \mathbf{v}_2, \delta\mathbf{v})| \le B_\Omega ||\mathbf{v}_2||_{\mathbf{W}(\Omega)} ||\delta\mathbf{v}||^2_{\mathbf{W}(\Omega)} \le B_\Omega C_\nu ||\delta\mathbf{v}||^2_{\mathbf{W}(\Omega)},$$

which holds according to the estimate (6.117), since $(\mathbf{v}_2, p_2)$ is a solution to $\mathscr{VP}_\eta$. Moreover, by (6.119) and the Cauchy–Schwartz inequality:

$$\begin{aligned} |(\mathbf{v}_2(\tilde{H}(\mathbf{v}_1) - \tilde{H}(\mathbf{v}_2)), \delta\mathbf{v})_\Gamma| &\le ||\tilde{H}'||_\infty ||\mathbf{v}_2||_{0,2,\Gamma} ||\delta\mathbf{v}||^2_{0,4,\Gamma} \\ &\le T_{0,4,\Gamma} ||\tilde{H}'||_\infty C_\nu ||\delta\mathbf{v}||^2_{\mathbf{W}(\Omega)}. \end{aligned}$$

The inequality (6.124) then yields

$$(\inf(\nu, \eta) - C_\nu(B_\Omega + T_{0,4,\Gamma} ||\tilde{H}'||_\infty)||\delta\mathbf{v}||^2_{\mathbf{W}(\Omega)} \le 0,$$

which shows that $||\delta\mathbf{v}||^2_{\mathbf{W}(\Omega)} \le 0$ if (6.122) holds, hence $\mathbf{v}_1 = \mathbf{v}_2$. Therefore (6.12.i) reduces to $(\delta p, \mathbf{w})_\Omega = 0$ for all $\mathbf{w} \in \mathbf{W}(\Omega)$, in particular for all $\mathbf{w} \in \mathscr{D}(\Omega)^3$. Using a well-known result on distributions (see in [17]), we deduce that δp is a.e constant in Ω then equal to zero since it belongs to $L^2_0(\Omega)$.

In conclusion, when condition (6.122) holds, $(\delta\mathbf{v}, \delta p) = (0, 0)$; hence, there is at most one solution to $\mathscr{VP}_\eta$. □

In summary, by combining Lemma 6.16 and Proposition 6.3, we have proved:

Theorem 6.7. *If the hypothesis 6.i and the condition (6.122) hold, then the NSE (6.109) has a unique weak solution.*

Unfortunately, the condition (6.122) is compelling and meaningful only when η is large enough. In particular, for fixed $\mathbf{f}$ and ν, it fails when η goes to zero.

Remark 6.10. We deduce from Theorem 6.7 that when condition (6.122) holds, then given any $\mathbf{v}_0 \in \mathbf{W}(\Omega)$, the sequence $(\mathbf{v}_n, p_n)_{n\in\mathbb{N}}$ specified by

$$\begin{cases} (\mathbf{v}^{(n)} \cdot \nabla)\, \mathbf{v}^{(n+1)} - \nabla \cdot \left[(2\nu + \nu_t)\, D\mathbf{v}^{(n+1)}\right] + \nabla p^{(n+1)} = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{v} = 0 & \text{in } \Omega, \\ -\left[(2\nu + \nu_t)\, D\mathbf{v}^{(n+1)} \cdot \mathbf{n}\right]_\tau = \mathbf{v}_\tau^{(n+1)}(\tilde{H}(\mathbf{v}^{(n)}) + \eta) & \text{on } \Gamma, \\ \mathbf{v}^{(n+1)} \cdot \mathbf{n} = 0 & \text{on } \Gamma, \end{cases}$$

converges to the unique solution of the NSE (6.109).

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Chapter 7
Analysis of the Continuous Steady NS-TKE Model

Abstract We prove the existence of a weak solution to the steady NS-TKE model, which couples the steady Navier–Stokes equations with the equation for the turbulent kinetic energy k. The link includes the eddy viscosities $\nu_t(k,\mathbf{x})$ and $\mu_t(k,\mathbf{x})$, a wall law for the mean velocity $\mathbf{v}$ and $k = k_\Gamma(\mathbf{v})$ on the boundary, and the source term $\nu_t(k)|D\mathbf{v}|^2$ in the TKE equation, which is in L^1. We change the variable k to $\kappa = k - k_\Gamma(\mathbf{v})$, which yields a new variational problem, whose source term is regularized by convolution. We next perform the linearization procedure and use the Schauder fixed-point theorem to prove the existence of solutions to the regularized system. We then use the energy method to take the limit in the sense of the convergence of families of variational problems. Apart from the nonlinearities involved in this problem and a high complexity due to a large number of terms, the main difficulty comes from the quadratic source term $\nu_t(k)|D\mathbf{v}|^2$, which requires sharp estimates and yields a nonstandard variational formulation.

7.1 Introduction

We consider in this chapter the following abstract NS-TKE coupled system with wall laws,

$$\begin{cases} (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \nabla\cdot[(2\nu + \nu_t(k,\mathbf{x}))\,D\mathbf{v}] + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v}\cdot\nabla k - \nabla\cdot(\mu + \mu_t(k,\mathbf{x})\nabla k) + k\,E(k,\mathbf{x}) = \nu_t(k,\mathbf{x})|D\mathbf{v}|^2 & \text{in } \Omega, \\ -[(2\nu + \nu_t(k))\,D\mathbf{v}\cdot\mathbf{n}]_\tau = g(\mathbf{v})_\tau & \text{on } \Gamma, \\ \mathbf{v}\cdot\mathbf{n} = 0 & \text{on } \Gamma, \\ k = k_\Gamma(\mathbf{v}) & \text{on } \Gamma, \end{cases} \tag{7.1}$$

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where the unknowns are the mean velocity $\mathbf{v}$, the mean pressure p, and the turbulent kinetic energy (TKE) denoted by k.

This PDE system derives from the modeling process carried out in Sects. 4.4 and 5.3. In particular, the original system is the system (4.137), the boundary condition for the TKE was derived in (5.137), while the functions ν_t and μ_t are the eddy viscosities. It is a substitute for the $k - \mathscr{E}$ sharing the same mathematical features. Therefore, the results of this chapter can be generalized to the $k-\mathscr{E}$ model, with appropriate assumptions, as well as systems such as (4.136).

The goal of this chapter is to investigate cases for which we are able to prove the existence of weak solutions to the NS-TKE model (7.1).

In addition to the issues raised by the nonlinear term $(\mathbf{v}\cdot\nabla)\,\mathbf{v}$ and the wall law for the velocity, thoroughly investigated in Chap. 6, as well as the transport term $\mathbf{v}\cdot\nabla k$ which will be processed as $(\mathbf{v}\cdot\nabla)\,\mathbf{v}$, this system addresses new issues due to:

(i) the nonlinear boundary condition (7.1.vi) for k,
(ii) the terms involving eddy viscosities, $-\nabla\cdot(\nu_t(k,\mathbf{x})D\mathbf{v})$ and $\nabla\cdot(\mu_t(k,\mathbf{x})\nabla k)$,
(iii) the production term $\nu_t(k,\mathbf{x})|D\mathbf{v}|^2$ in (7.1.iii), which is at best in $L^1(\Omega)$.

(i) The nonlinear boundary condition for k is indeed a serious problem, which has never been studied before. Following standard procedures, we change the variable k to

$$\kappa = k - k_\Gamma(\mathbf{v}),$$

which satisfies homogeneous boundary conditions on Γ, making it possible to use the machinery of elliptic equations (cf. Guilbarg–Trudinger [37]). However, the equation satisfied by κ derived from (7.1.iii) involves a large number of extra terms due to $k_\Gamma(\mathbf{v})$. As a result of these extra terms, we are not able to derive sharp estimates for κ in the case

$$E(k,\mathbf{x}) = \frac{\sqrt{k}}{\ell(\mathbf{x})},$$

although this is the correct expression for the function E in the original NS-TKE model (4.137). For this reason, we must assume that E is a continuous bounded function, as if we would had replaced $k\sqrt{k}$ by $k\sqrt{T_N(k)}$, where T_N is the truncation function expressed by (7.98) below, so that

$$E(k,\mathbf{x}) = \frac{\sqrt{T_N(k)}}{\ell(\mathbf{x})}, \quad \text{where } \forall\,\mathbf{x}\in\Omega,\ \ell(\mathbf{x}) \ge \ell_0 > 0.$$

(ii) There are many steady fluid models in which the viscosity ν is not constant, for instance, the case where ν depends on the temperature, with linear Neumann boundary conditions. This class of problems has widely been studied (see, e.g., [1, 19–24, 33, 44, 47, 49]). It should be noted that some models in these papers also involve a quadratic source term in L^1 in the equation for the temperature. In the current context, nonconstant viscosities are involved in NS-TKE-like models or by-products. The steady-state case has already been discussed when $\mathbf{v}$ satisfies the no-slip boundary condition and k homogeneous boundary conditions. The case where ν_t is a bounded function was studied for instance in [6, 15, 38, 42, 43]. The case of unbounded eddy viscosities is more complicated, and only partial results concerning simplified models are known [35, 36, 41], as well as Lewandowski–Murat in [42], Chap. 5, who discussed the existence of a renormalized solution in a scalar case without pressure.

In a series of papers [3–5, 18], a simplified steady-state NS-TKE model without convection was considered for the coupling of two fluids such as the ocean and the atmosphere, with boundary conditions similar to (2.144) in Sect. 2.7.6, nonlinear BC for the TKEs at the interface, and bounded eddy viscosities.

In this chapter we focus on the case of bounded eddy viscosities, discussing briefly the case of unbounded eddy viscosities toward the end of the chapter. The coupling generated by the eddy viscosities motivates the choice of the linearization method developed in Chap. 6, to investigate the $(\mathbf{v}, p, \kappa)$ system, although the Galerkin method also works. However, the linearization method is the most appropriate for writing practical numerical codes (see the discussion in Sect. 6.6.1).

(iii) Elliptic problems with right-hand side in L^1 and/or measure have been intensively studied.We may distinguish two main trends. One may be attributed to Boccardo and Gallouët and collaborators [2, 9–13, 31, 34, 48]. The concept of solutions in this approach must be brought closer to the notion of entropy solutions, widely used for hyperbolic systems.

The other trend is the renormalized solutions approach. The concept was first elaborated by R.-J. Di Perna and P.-L. Lions for transport and kinetic equations [27–30]. It was then adapted for elliptic problems by P.-L. Lions and F. Murat, although the original paper was never published. This concept has resulted in many papers [7, 8, 16, 17, 25, 26, 39, 40, 45], after the initial course by F. Murat [46] in the University of Seville in 1993.

Unfortunately, neither entropy nor renormalized solutions work in the case of incompressible fluids, because of the pressure and the constraint $\nabla \cdot \mathbf{v} = 0$, even though many ideas behind these concepts can be recycled. We search therefore for a natural variational problem $\mathscr{V}\mathscr{P}^k$ that yields weak solutions to the NS-TKE model (7.1), which is possible when:

(i) ν_t, μ_t, E are nonnegative, continuous, and bounded,
(ii) $k_\Gamma \in W^{1,\infty}(\mathbb{R}^3) \cap C^1(\mathbb{R}^3)$.

The other hypotheses are the same as in Chap. 6, summarized in Hypothesis 6.i. The general outline of this chapter is the following:

(a) writing variational problems, studying the new operators involved, obtaining a priori estimates,
(b) to complement the V-P-ESP of Sect. 6.3.3 by the similar KESP package for $\kappa = k - k_\Gamma(\mathbf{v})$, proving further compactness results,

(c) regularization of the source term, writing and analysis of the corresponding linearized problems,
(d) application of the fixed-point theorem, taking the limit in the equations by the energy method and the concept of convergence of families of variational problems introduced in Sect. 6.4.5.

Items (a), (b), and (d) follow the framework of Chap. 6. Concerning item (c), the usual strategy to prove the existence of solutions to an elliptic system with a source term in L^1 is to regularize it. In this chapter, the source term $\nu_t|D\mathbf{v}|^2$ is regularized by convolution, which means by $\nu_t D(\mathbf{v}\star\rho_n) : D\mathbf{v}$, for some mollifier ρ_n. However, in Chaps. 8 and 12, we will regularize the source term by the truncations $\nu_t T_N(|D\mathbf{v}|^2)$ and $T_N(\nu_t|D\mathbf{v}|^2)$. These choices are motivated by technical reasons which will be clarified by the end of Sect. 7.4.3, in particular in Remark 7.6.

Due to the high complexity of the system, the procedure developed in this chapter is very technical and involves several interconnected variational problems, which may seem confusing at first. The main steps of the procedure are the following, which can be used as a guide while reading:

STEP 1.	Initial TKE Model (7.1).	$\longrightarrow$	$\mathscr{VP}^k$, (7.5):(7.7).
STEP 2.	Changing the variable k by $\kappa = k - k_\Gamma(\mathbf{v})$ in $\mathscr{VP}^k$.	$\longrightarrow$	$\mathscr{VP}^\kappa$, (7.14):(7.16), with extra terms in (7.16.iii).
STEP 3.	Regularization of the source term by convolution, regularized NS-TKE model.	$\longrightarrow$	$\mathscr{VP}^\kappa_n$, (7.58):(7.60).
STEP 4.	η-regularization of the fluid equation.	$\longrightarrow$	$\mathscr{VP}^\kappa_{n,\eta}$, (7.62).
STEP 5.	Linearization and fixed-point procedure.	$\longrightarrow$	Existence of solution to $\mathscr{VP}^\kappa_{n,\eta}$.
STEP 6.	Convergence of $(\mathscr{VP}^\kappa_{n,\eta})_{\eta>0}$ to $\mathscr{VP}^\kappa_n$.	$\longrightarrow$	Existence of solutions to $\mathscr{VP}^\kappa_n$, therefore to $\mathscr{VP}^k_n$.
STEP 7.	Convergence of $(\mathscr{VP}^\kappa_n)_{n\in\mathbb{N}}$ to $\mathscr{VP}^\kappa$.	$\longrightarrow$	Existence of solutions to $\mathscr{VP}^\kappa$, therefore to $\mathscr{VP}^k$.

Observe that the ε-regularization carried out in Sect. 6.4.4 is not necessary to this process, since we can directly use the results of Chap. 6 in Step 5. However, the η-regularization introduced in Sect. 6.6.1 cannot be bypassed.

At the end of the chapter, we prove the maximum principle, which states that if $(\mathbf{v}, p, k)$ is a solution to $\mathscr{VP}^k$ obtained by approximation and $k_\Gamma(\mathbf{v}) \geq 0$ *a.e.* on

Γ, then $k \geq 0$. Then we consider the case of realistic eddy viscosities $\nu_t = \ell\sqrt{k}$, $\mu_t = C_\mu\sqrt{k}$, approximated by $\ell\sqrt{T_N(k)}$ and $C_\mu\ell\sqrt{T_N(k)}$. When we take the limit in the equations as $n \to \infty$, we obtain an inequality for the TKE, the resulting solved PDE system being

$$\begin{cases} (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \nabla\cdot\left[(2\nu + \ell\sqrt{k})\,D\mathbf{v}\right] + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v}\cdot\nabla k - \nabla\cdot\left[(\mu + C_\mu\ell\sqrt{k})\nabla k\right] \geq \ell\sqrt{k}|D\mathbf{v}|^2 - \dfrac{k\sqrt{T_N(k)}}{\ell} & \text{in } \Omega, \\ -\left[(2\nu + \ell\sqrt{k})\,D\mathbf{v}\cdot\mathbf{n}\right]_\tau = g(\mathbf{v})_\tau & \text{on } \Gamma, \\ \mathbf{v}\cdot\mathbf{n} = 0 & \text{on } \Gamma, \\ k = k_\Gamma(\mathbf{v}) & \text{on } \Gamma, \end{cases}$$

where k_Γ is still a $W^{1,\infty}$ function of class C^1 and N is fixed. In the realistic NS-TKE model, $k_\Gamma(\mathbf{v}) = C|\mathbf{v}|^2$, a case that remains open as these lines are written.

7.2 Variational Formulation and Change of Variables

From now on and until stated otherwise, we shall assume that:

Hypothesis 7.i. Hypothesis 6.i holds and in addition:

(i) $\nu_t, \mu_t, E \in L^\infty(\mathbb{R}\times\Omega)$ are continuous with respect to k and also satisfy $\nu_t, \mu_t, E \geq 0$ a.e. in $\mathbb{R}\times\Omega$,
(ii) $k_\Gamma \in W^{1,\infty}(\mathbb{R}^3) \cap C^1(\mathbb{R}^3)$ is a given fixed function.

In this section we:

(a) specify the original variational problem $\mathscr{VP}^k$ associated with the system (7.1),
(b) check that $\mathscr{VP}^k$ is meaningful,
(c) change the variable k to $\kappa = k - k_\Gamma(\mathbf{v})$ which is identically equal to zero on Γ and then write the variational problem $\mathscr{VP}^\kappa$ satisfied by κ,
(d) derive a priori estimates to show that $\mathscr{VP}^\kappa$ and $\mathscr{VP}^k$ are consistent.

7.2.1 Variational Formulation

7.2.1.1 Space Functions

As in Chap. 6, the space of unknowns and tests for $(\mathbf{v}, p)$ is $\mathbf{W}(\Omega)\times L^2_0(\Omega)$. The space of unknowns for $\kappa = k - k_\Gamma(\mathbf{v})$ is $\mathbf{K}_{3/2}(\Omega)$, while the test space is $\mathbf{Q}_3(\Omega)$, where

$$\mathbf{K}_{3/2}(\Omega) = \bigcap_{1\leq q<3/2} W_0^{1,q}(\Omega), \quad \mathbf{Q}_3(\Omega) = \bigcup_{r>3} W_0^{1,r}(\Omega). \tag{7.2}$$

The choice of $\mathbf{K}_{3/2}(\Omega)$, whose topology is not trivial,[1] is motivated by the presence of the source term in L^1 in (7.1.iii), which leads to the application of the Boccardo–Gallouët inequalities [9] (cf. Sect. A.6.2 in [TB] and also in [2, 11]), in seeking estimates for κ in $W_0^{1,q}(\Omega)$, $1 \le q < 3/2$, hence, $\mathbf{K}_{3/2}(\Omega)$. This point will be clarified in Sect. 7.2.4 below.

Observe however that $\mathscr{D}(\Omega)$ is dense in $\mathbf{K}_{3/2}(\Omega)$, which means that given any $\kappa \in \mathbf{K}_{3/2}(\Omega)$, there exists $\kappa_n \in \mathscr{D}(\Omega)$ such that $(\kappa_n)_{n\in\mathbb{N}}$ converges to κ in each $W_0^{1,q}$, $1 \le q < 3/2$. To see this, it is sufficient to use the density of $\mathscr{D}(\Omega)$ in $W_0^{1,q}(\Omega)$ and to argue as in the proof of Lemma 7.4 below.

The space $\mathbf{Q}_3(\Omega)$ is the most restrictive suitable test function space in the list, which motivates its choice. It is clear that $\mathscr{D}(\Omega)$ is dense in $\mathbf{Q}_3(\Omega)$.

7.2.1.2 Variational Formulation

Recall that $a(\mathbf{v},\mathbf{w}) = 2\nu(D\mathbf{v}, D\mathbf{w})_\Omega$ and

$$b(\mathbf{z};\mathbf{v},\mathbf{w}) = \frac{1}{2}\left[((\mathbf{z}\cdot\nabla)\,\mathbf{v},\mathbf{w})_\Omega - ((\mathbf{z}\cdot\nabla)\,\mathbf{w},\mathbf{v})_\Omega\right], \quad \langle G(\mathbf{v}),\mathbf{w}\rangle = (g(\mathbf{v}),\mathbf{w})_\Gamma.$$

Following the discussion in Sect. 6.2.3, we multiply (7.1.iii) by $q \in \mathscr{D}(\Omega)$ and we perform a formal integration by parts using Stokes formula, which leads us to introduce the following operators:

$$\begin{cases} b_e(\mathbf{z};k,l) = \dfrac{1}{2}\left[(\mathbf{z}\cdot\nabla k, l)_\Omega - (\mathbf{z}\cdot\nabla l, k)_\Omega\right], & a_e(k,l) = \mu(\nabla k,\nabla l)_\Omega, \\ s_v(k;\mathbf{v},\mathbf{w}) = (\nu_t(k,\mathbf{x})D\mathbf{v}, D\mathbf{w})_\Omega, & s_e(k;\lambda,l) = (\mu_t(k,\mathbf{x})\nabla\lambda,\nabla l)_\Omega, \\ d(k;\lambda,l) = (\lambda E(k,\mathbf{x}), l)_\Omega, & \mathbb{P}(k,\mathbf{v},\mathbf{x}) = \nu_t(k,\mathbf{x})|D\mathbf{v}|^2. \end{cases}$$

To simplify what follows, we also set

$$\forall\, \mathbf{v},\mathbf{w} \in \mathbf{W}(\Omega), \quad \forall\, k \in \mathbf{K}_{3/2}(\Omega),$$

$$\langle \mathscr{T}^{(k)}(\mathbf{v},k),\mathbf{w}\rangle = b(\mathbf{v};\mathbf{v},\mathbf{w}) + a(\mathbf{v},\mathbf{w}) + s_v(k;\mathbf{v},\mathbf{w}), \tag{7.3}$$

as well as

$$\forall\, \mathbf{v} \in \mathbf{W}(\Omega), \quad \forall\, k \in \mathbf{K}_{3/2}(\Omega), \quad \forall\, l \in \mathbf{Q}_3(\Omega),$$

$$\langle \mathscr{K}^{(k)}(\mathbf{v},k),\mathbf{w}\rangle = b_e(\mathbf{v};k,l) + a_e(k,l) + s_e(k;k,l). \tag{7.4}$$

In other words,

[1] It is a Frechet space (see in Dugundji [32]), whose topology does not need to be specified.

(i) $\mathscr{T}^{(k)}$ and $\mathscr{K}^{(k)}$ gather together transport, diffusion, and turbulent diffusion for $\mathbf{v}$ and k respectively,
(ii) the operator d represents the turbulent diffusion term coming from $\mathscr{E}$ in the original equation for k [cf. (4.126) and (4.137)],
(iii) $\mathbb{P}$ is the quadratic source term.

The variational problem $\mathscr{VP}^k$ associated with the NS-TKE model (7.1) with κ as unknown is

$$\text{Find } (\mathbf{v}, p, \kappa) \in \mathbf{W}(\Omega) \times L^2_0(\Omega) \times \mathbf{K}_{3/2}(\Omega) \text{ such that} \tag{7.5}$$

$$\text{for all } (\mathbf{w}, q, l) \in \mathbf{W}(\Omega) \times L^2(\Omega) \times \mathbf{Q}_3(\Omega), \tag{7.6}$$

$$\begin{cases} \langle \mathscr{T}^{(k)}(\mathbf{v}, k), \mathbf{w}\rangle - (p, \nabla\cdot\mathbf{w})_\Omega + \langle G(\mathbf{v}), \mathbf{w}\rangle = \langle \mathbf{f}, \mathbf{w}\rangle, \\ (\nabla\cdot\mathbf{v}, q)_\Omega = 0, \\ \langle \mathscr{K}^{(k)}(\mathbf{v}, k), \mathbf{w}\rangle + d(k; k, l) = (\mathbb{P}(k, \mathbf{v}, \mathbf{x}), l)_\Omega, \\ k = \kappa + k_\Gamma(\mathbf{v}), \end{cases} \tag{7.7}$$

Any solution to $\mathscr{VP}^k$ is a weak solution to the NS-TKE model (7.1). By generalizing the notion of mild solution of Lemma 6.1 to this case, we obtain:

Lemma 7.1. *Assume that in addition to Hypothesis 7.i,*

$$\mathbf{f} \in \mathbf{L}^2(\Omega), \quad \nu_t, \mu_t \in W^{1,\infty}(\mathbb{R}\times\mathbb{R}), \quad k_\Gamma \in C^2(\mathbb{R}),$$

and

$$(\mathbf{v}, p, \kappa) \in (\mathbf{H}^2(\Omega) \cap \mathbf{W}(\Omega)) \times (H^1(\Omega) \cap L^2_0(\Omega)) \times (H^1_0(\Omega) \times H^2(\Omega)).$$

Let $k = \kappa + k_\Gamma(\mathbf{v})$. *Then* $(\mathbf{v}, p, k)$ *is a mild solution of the NS-TKE model (7.1) if and only if it is a weak solution.*

Sketch of the proof. Observe that as $\nu_t \in W^{1,\infty}(\mathbb{R}\times\mathbb{R})$, when $\mathbf{v} \in \mathbf{H}^2(\Omega)\cap\mathbf{W}(\Omega)$, then $\mathbb{P}(k, \mathbf{v}, \mathbf{x}) \in L^2(\mathbb{R})$. Moreover, as $k_\Gamma \in C^2(\mathbb{R}^3)$ and $\kappa \in H^1_0(\Omega) \times H^2(\Omega)$, we deduce that $k \in H^2(\Omega)$. Finally, a standard calculation shows that

$$\nabla\cdot(\nu_t(k, \mathbf{x})\, D\mathbf{v}) = \nu_t(k, \mathbf{x})\Delta\mathbf{v} + \left(\frac{\partial\nu_t}{\partial k}\nabla k + \frac{\partial\nu_t}{\partial\mathbf{x}}\right) : \nabla\mathbf{v} \in L^2(\Omega).$$

For the same reason, $\nabla\cdot(\mu_t(k, \mathbf{x})\,\nabla k) \in L^2(\Omega)$. The rest of the proof is standard, and we skip the details. □

By the end of this chapter, we shall have proved:

Theorem 7.1. *Assume that Hypothesis 7.i holds. Then* $\mathscr{VP}^k$ *admits a solution.*

Remark 7.1. In the realistic NS-TKE model with wall laws derived in Sects. 4.4 and 5.3, we have

$$\nu_t(k,\mathbf{x}) = C_\nu \ell(\mathbf{x})\sqrt{k}, \quad \mu_t(k,\mathbf{x}) = C_\mu \ell(\mathbf{x})\sqrt{k}, \quad E(k,\mathbf{x}) = \frac{\sqrt{k}}{\ell(\mathbf{x})},$$

when $\ell = \ell(\mathbf{x})$ is a given function. Moreover, in the case of the boundary two layers model, $k_\Gamma(\mathbf{v}) = C_k|\mathbf{v}|^2$. Therefore, items (i) and (ii) of Hypothesis 7.i do not hold in this case, and we are not able to prove an existence result in the realistic case, irrespective of any assumption on ℓ, for technical reasons that will be clear by the end of this chapter. A discussion of currently known results for the realistic case will be presented in Sect. 7.5.3.

7.2.2 *Meaningfulness and Change of Variables*

We first aim to establish that $\mathscr{VP}^k$ is meaningful when Hypothesis 7.i holds. Then we change the variable k to κ and derive from $\mathscr{VP}^k$ the variational problem $\mathscr{VP}^\kappa$ satisfied by $(\mathbf{v}, p, \kappa)$.

7.2.2.1 Meaningfulness

Let $(\mathbf{v}, k) \in \mathbf{W}(\Omega)\times\mathbf{K}_{3/2}(\Omega)$, $l \in \mathbf{Q}_3(\Omega)$ be given. We treat the terms in $\mathscr{VP}^k$ one after another.

(i) As $\nu_t \in L^\infty(\mathbb{R}\times\Omega)$, $\mathbf{x} \to \nu_t(k(\mathbf{x}),\mathbf{x}) \in L^\infty(\Omega)$. Then all the terms in (7.7.i) are well defined, as already shown in Sect. 6.3.

(ii) Because $(\mathbf{v}, k) \in \mathbf{W}(\Omega)\times\mathbf{K}_{3/2}(\Omega)$, then $\mathbf{v} \in \mathbf{L}^6(\Omega)$ and the Hölder inequality implies

$$\mathbf{v}\cdot\nabla k \in L^p(\Omega), \quad \forall\, 1 \le p < 6/5,$$

and $l \in L^\infty(\Omega)$ as $\mathbf{Q}_3(\Omega) \hookrightarrow L^\infty(\Omega)$; hence, $(\mathbf{v}\cdot\nabla k, l)$ is meaningful. Moreover, we deduce from the Sobolev embedding theorem that $k \in L^p(\Omega)$, for any $1 \le p < 3$, leading to

$$k\mathbf{v} \in \mathbf{L}^p(\Omega), \quad \forall\, 1 \le p < 2, \qquad \text{while } \nabla l \in \mathbf{L}^p(\Omega) \text{ for some } p > 3,$$

which allows the product $(\mathbf{v}\cdot\nabla l, k)_\Omega$ to be treated. In sum, $b_e(\mathbf{v}; k, l)$ is actually well defined.

(iii) As

$$\nabla k \in \mathbf{L}^p(\Omega), \quad \forall\, 1 \le p < 3/2, \qquad \text{while } \nabla l \in \mathbf{L}^q(\Omega), \quad \forall\, q > 3,$$

and $\mu_t(k(\cdot),\cdot) \in L^\infty(\Omega)$, then $a_e(k, l)$ and $s_e(k; k, l)$ are well defined.

(iv) The product $l\ E(k(\cdot), \cdot) \in L^\infty(\Omega)$, which makes $d(k; k, l)$ well defined.
(v) Finally, since $\mathbf{v} \in \mathbf{W}(\Omega)$, we deduce that $|D\mathbf{v}|^2 \in L^1(\Omega)$. Moreover, $\nu_t(k(\cdot), \cdot) \in L^\infty(\Omega)$ as well as $l \in L^\infty(\Omega)$, which makes $(\mathbb{P}(k, \mathbf{v}, \mathbf{x}), l)_\Omega$ well defined.

It results from (b):(e) that all the terms in (7.7.iii) are well defined. The processing of the rest of the system is straightforward.

7.2.2.2 Change of Variable

We now write the equation for κ defined by

$$\kappa = k - k_\Gamma(\mathbf{v}) \quad \text{which satisfies} \quad \kappa = 0 \ \text{ on } \ \Gamma.$$

We set

$$\begin{cases} \widetilde{\nu_t}(\kappa, \mathbf{v}, \mathbf{x}) = \nu_t(\kappa + k_\Gamma(\mathbf{v}), \mathbf{x}), \\ \widetilde{\mu_t}(\kappa, \mathbf{v}, \mathbf{x}) = \mu_t(\kappa + k_\Gamma(\mathbf{v}), \mathbf{x}), \\ \tilde{E}(\kappa, \mathbf{v}, \mathbf{x}) = E(\kappa + k_\Gamma(\mathbf{v}), \mathbf{x}). \end{cases}$$

We note that because of Hypothesis 7.i, $\widetilde{\nu_t}, \widetilde{\mu_t}, \tilde{E} \in L^\infty(\mathbb{R} \times \mathbb{R}^3 \times \Omega)$ and are continuous with respect to κ and $\mathbf{v}$. We are led to consider the following operators:

$$\begin{cases} t_v(\lambda, \mathbf{z}; \mathbf{v}, \mathbf{w}) = (\widetilde{\nu_t}(\lambda, \mathbf{z}, \mathbf{x}) D\mathbf{v}, D\mathbf{w})_\Omega, \\ t_e(\lambda, \mathbf{z}; \kappa, l) = (\widetilde{\mu_t}(\lambda, \mathbf{z}, \mathbf{x}) \nabla\kappa, \nabla l)_\Omega, \\ e(\lambda, \mathbf{z}; \kappa, l) = (\kappa \tilde{E}(\lambda, \mathbf{z}, \mathbf{x}), l)_\Omega, \end{cases} \tag{7.8}$$

and we finally put

$$Q(\lambda, \mathbf{z}, \mathbf{x}) = \widetilde{\nu_t}(\lambda, \mathbf{z}, \mathbf{x}) |D\mathbf{z}|^2,$$

and

$$\langle \mathscr{T}^{(\kappa)}(\mathbf{v}, \kappa), \mathbf{w} \rangle = b(\mathbf{v}; \mathbf{v}, \mathbf{w}) + a(\mathbf{v}, \mathbf{w}) + t_v(\kappa, \mathbf{v}; \mathbf{v}, \mathbf{w}), \tag{7.9}$$

$$\langle \mathscr{K}^{(\kappa)}(\mathbf{v}, \kappa), l \rangle = b_e(\mathbf{v}; \kappa, l) + a_e(\kappa, l) + t_e(\kappa, \mathbf{v}; \kappa, l). \tag{7.10}$$

The price to pay is that the nonhomogeneous BC for k generates additional terms in the equation for κ. To make the process as clear as possible, we first define the following operator,

$$\langle \mathscr{K}^{(\kappa)}_{\mathbf{z},\lambda}(\kappa), l \rangle = b_e(\mathbf{z}; \kappa, l) + a_e(\kappa, l) + \ t_e(\lambda, \mathbf{z}; \kappa, l) \tag{7.11}$$

which is the linear operator deduced from $\mathscr{K}^{(\kappa)}$, which means

$$\langle \mathscr{K}^{(\kappa)}_{\kappa,\mathbf{v}}(\kappa), l\rangle = \langle \mathscr{K}^{(\kappa)}(\mathbf{v},\kappa), l\rangle. \tag{7.12}$$

The extra terms due to the change of variables, are expressed by the operator B,

$$B(\lambda, \mathbf{z}; \mathbf{v}, l) = \langle \mathscr{K}^{(\kappa)}_{\mathbf{z},\lambda}(k_\Gamma(\mathbf{v})), l\rangle + e(\lambda, \mathbf{z}; k_\Gamma(\mathbf{v}), l). \tag{7.13}$$

An easy calculation shows that $\mathscr{VP}^k$ is equivalent to the variational problem denoted by $\mathscr{VP}^\kappa$,

$$\text{Find } (\mathbf{v}, p, \kappa) \in \mathbf{W}(\Omega) \times L^2_0(\Omega) \times \mathbf{K}_{3/2}(\Omega) \text{ such that} \tag{7.14}$$

$$\text{for all } (\mathbf{w}, q, l) \in \mathbf{W}(\Omega) \times L^2(\Omega) \times \mathbf{Q}_3(\Omega), \tag{7.15}$$

$$\begin{cases} \langle \mathscr{T}^{(\kappa)}(\mathbf{v},\kappa), \mathbf{w}\rangle - (p, \nabla\cdot\mathbf{w})_\Omega + \langle G(\mathbf{v}), \mathbf{w}\rangle = \langle \mathbf{f}, \mathbf{w}\rangle, \\ (\nabla\cdot\mathbf{v}, q)_\Omega = 0, \\ \langle \mathscr{K}^{(\kappa)}(\mathbf{v},\kappa), l\rangle + e(\kappa, \mathbf{v}; \kappa, l) + B(\kappa, \mathbf{v}; \mathbf{v}, l) = (Q(\kappa, \mathbf{v}, \mathbf{x}), l)_\Omega. \end{cases} \tag{7.16}$$

We will examine $\mathscr{VP}^\kappa$ in order to prove Theorem 7.1.

Remark 7.2. Let $(\mathbf{v}, p, \kappa)$ be a solution to $\mathscr{VP}^\kappa$ such that $\kappa \in H^1_0(\Omega)$. Then according to item a) above, we can substitute the test space $\mathbf{Q}_3(\Omega)$ by the space $H^1_0(\Omega) \cap L^\infty(\Omega)$, which is dense in $\mathbf{Q}_3(\Omega)$. This is meaningful since all the operators involved in $\mathscr{VP}^\kappa$ are bounded in the space $H^1_0(\Omega) \cap L^\infty(\Omega)$, when $\kappa \in H^1_0(\Omega)$.

Remark 7.3. In Chap. 12, we perform the numerical analysis of the NS-TKE model, with mixed boundary conditions. This means that $\Gamma = \Gamma_n \cup \Gamma_D$ and the wall law holds only on Γ_n, while $\mathbf{v} = 0$ and $k = 0$ on Γ_D. For technical reasons arising from Poincaré's inequality, it is possible to deal directly with $\mathscr{VP}^k$ instead of $\mathscr{VP}^\kappa$. In the case where the wall law holds on Γ as a whole, it is essential to deal with $\mathscr{VP}^\kappa$ to get a priori estimates and to prove the existence of weak solutions to the NS-TKE model.

7.2.3 Change of Variable Operator

The aim of this subsection is to analyze the change of variable operator B, which is essential in order to approximate $\mathscr{VP}^\kappa$.

We first notice that $\mathbf{v} \to \mathscr{K}^{(\kappa)}_{\mathbf{z},\lambda}(k_\Gamma(\mathbf{v}))$ [cf. (7.11)] defines an application from $\mathbf{W}(\Omega)$ to $H^{-1}(\Omega)$ for any given $\mathbf{z}$, λ, and that e behaves like a L^2 bilinear product since E is bounded. This suggests that B defines a continuous map from $\mathbf{W}(\Omega)$ to $H^{-1}(\Omega)$.

Before going any further, we must fix some notation. Let

(i) $S_{0,p,\Omega}$ denote the best constant in the inequality, $||\mathbf{z}||_{0,p,\Omega} \leq C||\mathbf{z}||_{\mathbf{W}(\Omega)}$, where $1 \leq p \leq 6$,

(ii) $H_{\mathbf{W}(\Omega)}$ denote the best constant in the inequality $||\nabla\mathbf{z}||_{0,2,\Omega} \leq C||\mathbf{z}||_{\mathbf{W}(\Omega)}$.

Furthermore, observe that as $k_\Gamma \in W^{1,\infty}(\mathbb{R}^3) \cap C^1(\mathbb{R}^3)$,

$$\forall\, \mathbf{v} \in \mathbf{W}(\Omega), \quad k_\Gamma(\mathbf{v}) \in H^1(\Omega) \text{ and } \nabla k_\Gamma(\mathbf{v}) = \nabla\mathbf{v} \cdot k'_\Gamma(\mathbf{v}). \tag{7.17}$$

Moreover, we set[2] $\forall\,(x, y) \in \mathbb{R}^2$,

$$\xi(x) = H_{\mathbf{W}(\Omega)} S_{0,4,\Omega} ||k'_\Gamma||_\infty\, x + ||k_\Gamma||_\infty, \tag{7.18}$$

$$\beta(x, y) = \frac{1}{2} S_{0,4,\Omega}\, y\xi(x) + H_{\mathbf{W}(\Omega)}(\mu + ||\mu_t||_\infty)\, x. \tag{7.19}$$

We start by investigating $\mathscr{K}^{(\kappa)}_{\mathbf{z},\lambda}(k_\Gamma(\mathbf{v}))$:

Lemma 7.2. *Suppose* $(\mathbf{z}, \lambda) \in \mathbf{W}(\Omega) \times W^{1,q}_0(\Omega)$ *is given for some* $q > 1$. *Then*

$$\forall\, \mathbf{v} \in \mathbf{W}(\Omega), \quad \mathscr{K}^{(\kappa)}_{\mathbf{z},\lambda}(k_\Gamma(\mathbf{v})) \in H^{-1}(\Omega),$$

and

$$||\mathscr{K}^{(\kappa)}_{\mathbf{z},\lambda}(k_\Gamma(\mathbf{v}))||_{H^{-1}(\Omega)} \leq \beta(||\mathbf{v}||_{\mathbf{W}(\Omega)}, ||\mathbf{z}||_{\mathbf{W}(\Omega)}). \tag{7.20}$$

Proof. Starting from the general expression of $\mathscr{K}^{(\kappa)}_{\mathbf{z},\lambda}$ given by (7.11), we study each term one after the other.

STEP 1. *Transport operator.* Recall that

$$b_e(\mathbf{z}; k_\Gamma(\mathbf{v}), l) = \frac{1}{2}\left[(\mathbf{z} \cdot \nabla(k_\Gamma(\mathbf{v})), l)_\Omega - (\mathbf{z} \cdot \nabla l, k_\Gamma(\mathbf{v}))_\Omega\right].$$

Therefore, by (7.17) and the Cauchy–Schwarz and Sobolev inequalities, we obtain

$$\begin{aligned} |b_e(\mathbf{z}; k_\Gamma(\mathbf{v}), l)| &\leq \frac{1}{2}(||k'_\Gamma||_\infty ||\mathbf{z}||_{0,4,\Omega} ||l||_{0,4,\Omega} ||\nabla\mathbf{v}||_{0,2,\Omega} \\ &\quad + ||k_\Gamma||_\infty ||\mathbf{z}||_{0,4,\Omega} ||\nabla l||_{0,2,\Omega}) \\ &\leq \frac{1}{2} S_{0,4,\Omega} ||\mathbf{z}||_{\mathbf{W}(\Omega)} \xi(||\mathbf{v}||_{\mathbf{W}(\Omega)}) ||l||_{H^1_0(\Omega)}, \end{aligned} \tag{7.21}$$

where $\xi(x)$ is given by (7.18), and $||l||_{H^1_0(\Omega)} = ||\nabla l||_{0,2,\Omega}$.

[2]Let ψ be any bounded data function of the system, for example, ν_t, μ_t, k_Γ, E, etc. For simplicity the L^∞ norm of ψ is denoted by $||\psi||_\infty$, unless more precision is necessary.

STEP 2. *Diffusion operators.* It is directly deduced from (7.17) and the Cauchy–Schwarz inequality that

$$\begin{aligned} |a_e(k_\Gamma(\mathbf{v}), l)| &\le \mu ||k'_\Gamma||_\infty ||\nabla \mathbf{v}||_{0,2,\Omega} ||\nabla l||_{0,2,\Omega} \\ &\le \mu H_{\mathbf{W}(\Omega)} ||k'_\Gamma||_\infty ||\mathbf{v}||_{\mathbf{W}(\Omega)} ||l||_{H_0^1(\Omega)}. \end{aligned} \tag{7.22}$$

Similarly, we find

$$|t_e(\lambda, \mathbf{z}; k_\Gamma(\mathbf{v}), l)| \le H_{\mathbf{W}(\Omega)} ||\mu_t||_\infty ||k'_\Gamma||_\infty ||\mathbf{v}||_{\mathbf{W}(\Omega)} ||l||_{H_0^1(\Omega)}. \tag{7.23}$$

In conclusion, (7.20) follows from (7.21) to (7.23). □

Corollary 7.1.

$$|B(\lambda, \mathbf{z}; \mathbf{v}, l)| \le \left[\beta(||\mathbf{v}||_{\mathbf{W}(\Omega)}, ||\mathbf{z}||_{\mathbf{W}(\Omega)}) + S_{0,1,\Omega} ||k_\Gamma||_\infty ||K||_\infty\right] ||l||_{H_0^1(\Omega)} \tag{7.24}$$

Proof. As E and k_Γ are bounded, we obtain

$$|e(\lambda, \mathbf{z}; k_\Gamma(\mathbf{v}), l)| \le ||k_\Gamma||_\infty ||E||_\infty ||l||_{0,1,\Omega} \le S_{0,1,\Omega} ||k_\Gamma||_\infty ||E||_\infty ||l||_{H_0^1(\Omega)}, \tag{7.25}$$

hence the result by (7.13) and (7.20). We note in passing that this estimate does not depend on λ. □

Observe that $\beta(x, y)$ expressed by (7.19) is a polynomial function of the form

$$\beta(x, y) = a_1 x + a_2 xy + a_3 y,$$

where the a_i's depend on the data. Therefore, (7.24) yields the simple estimate

$$|B(\lambda, \mathbf{z}; \mathbf{v}, l)| \le C(||\mathbf{v}||_{\mathbf{W}(\Omega)} + ||\mathbf{v}||_{\mathbf{W}(\Omega)} ||\mathbf{z}||_{\mathbf{W}(\Omega)} + ||\mathbf{z}||_{\mathbf{W}(\Omega)} + 1) ||l||_{H_0^1(\Omega)}, \tag{7.26}$$

for some constant C. This is the form that we shall use in what follows.

7.2.4 A Priori Estimates

The energy equality and the *a priori* estimates for $\mathbf{v}$ and p derived in Sects. 6.4.1 and 6.4.2 for $\mathscr{V}\mathscr{P}$ hold for any bounded $\nu_t \ge 0$. Therefore, whether or not ν_t depends on κ or anything else does not change the reasoning. The same proofs hence apply to this case, yielding

Proposition 7.1. *Let* $(\mathbf{v}, p, \kappa)$ *be any solution to* $\mathscr{V}\mathscr{P}^\kappa$. *Then the following energy equality is satisfied:*

$$\int_\Omega (\nu + \widetilde{\nu_t}(\kappa(\mathbf{x}), \mathbf{v}(\mathbf{x}), \mathbf{x}))|D\mathbf{v}(\mathbf{x})|^2 d\mathbf{x} + \int_\Gamma g(\mathbf{v}(\mathbf{x})) \cdot \mathbf{v}(\mathbf{x}) d\Gamma(\mathbf{x}) = \langle \mathbf{f}, \mathbf{v} \rangle. \quad (7.27)$$

The velocity $\mathbf{v}$ *satisfies*

$$||\mathbf{v}||_{\mathbf{W}(\Omega)} \leq \left(\frac{||\mathbf{f}||^2_{\mathbf{W}(\Omega)'}}{\inf(\nu, C_g)} + \frac{2C_g|\Omega|}{\inf(\nu, C_g)} \right)^{\frac{1}{2}} = C_v. \quad (7.28)$$

Moreover, there exists $C_p = C_p(\nu, g, \Omega, \mathbf{f}, ||\nu_t||_\infty)$ *such that*

$$||p||_{0,2,\Omega} \leq C_p. \quad (7.29)$$

Deriving an estimate for κ is much more tricky and can be carried out only for particular solutions to $\mathscr{VP}^\kappa$.

Proposition 7.2. *Let* $(\mathbf{v}, p, \kappa)$ *be any solution to* $\mathscr{VP}^\kappa$ *such that* $\kappa \in H^1_0(\Omega)$*. Then for all* $q < 3/2$*, there exists a constant* $C_{\kappa,q}$ *depending on* ν, $||\nu_t||_\infty$, $\mathbf{f}$, g, μ, $||\mu_t||_\infty$, $||k_\Gamma||_\infty$, $||k'_\Gamma||_\infty$, $||E||_\infty$*, and* Ω*, such that*

$$||\kappa||_{1,q,\Omega} \leq C_{\kappa,q}. \quad (7.30)$$

Proof. The proof proceeds in three steps. We first prove that (7.16.iii) in $\mathscr{VP}^\kappa$ is indeed an equation with a right-hand side in L^1. We then aim to apply the washer inequality, Theorem A.12 in [TB]. To do so, it must be shown that

$$M_n = \int_{n \leq |\kappa| \leq n+1} |\nabla \kappa(\mathbf{x})|^2 \, d\mathbf{x} \quad (7.31)$$

is bounded uniformly in n in order to apply the inequality (A.44), which is carried out by the choice of the suitable test function. Once this suitable test is determined, we estimate M_n.

STEP 1. *R.h.s. in* L^1. The aim is to check that

$$S : \mathbf{x} \to Q(\kappa(\mathbf{x}), \mathbf{v}(\mathbf{x}), \mathbf{x}) = \widetilde{\nu_t}(\kappa(\mathbf{x}), \mathbf{v}(\mathbf{x}), \mathbf{x}))|D\mathbf{v}(\mathbf{x})|^2 \in L^1(\Omega). \quad (7.32)$$

We deduce from estimate (7.28)

$$\int_\Omega S(\mathbf{x}) d\mathbf{x} \leq ||\nu_t||_\infty ||D\mathbf{v}||^2_{0,2,\Omega} \leq ||\nu_t||_\infty C_v^2 = \sigma_1, \quad (7.33)$$

hence the result since $S \geq 0$.

STEP 2. *Determination of the suitable test.* Following Murat [46], we introduce the odd function $H_n \in W^{1,\infty}(\mathbb{R})$ defined by

$$\begin{cases} \forall\, x \in [0, n], & H_n(x) = 0, \\ \forall\, x \in [n, n+1], & H_n(x) = n - 1, \\ \forall\, x \in [n+1, \infty[, & H_n(x) = 1. \end{cases} \tag{7.34}$$

As $H_n \in W^{1,\infty}(\mathbb{R})$ and H_n' has a finite number of discontinuities, we deduce from Stampacchia's theorem (cf. [50] and Theorem A.11) that

$$\forall\, \lambda \in H_0^1(\Omega), \quad H_n(\lambda) \in H_0^1(\Omega), \quad \nabla H_n(\lambda) = H_n'(\lambda)\nabla\lambda.$$

We also notice that because $H_n' = 1$ over $[-1-n, -n] \cup [n, n+1]$, $H_n' = 0$ elsewhere, the following identities hold:

$$\begin{aligned} M_n &= \int_\Omega \nabla(H_n(\kappa(\mathbf{x})) \cdot \nabla\kappa(\mathbf{x}) d\mathbf{x} \\ &= \int_\Omega (H_n'(\kappa(\mathbf{x})))^2 |\nabla\kappa(\mathbf{x})|^2 d\mathbf{x} \\ &= ||\nabla H_n(\kappa)||_{0,2,\Omega}^2. \end{aligned} \tag{7.35}$$

Therefore, according to Remark 7.2 and because $\kappa \in H_0^1(\Omega)$, we can take as test function in (7.16.iii),

$$q = H_n(\kappa) \in H_0^1(\Omega) \cap L^\infty(\Omega), \ \text{ where } \ ||H_n(\kappa)||_{0,\infty,\Omega} \le 1, \tag{7.36}$$

leading to the equality

$$\Sigma = (Q(\kappa, \mathbf{v}, \mathbf{x}), H_n(\kappa))_\Omega, \tag{7.37}$$

where we have set

$$\Sigma = \langle \mathscr{K}^{(\kappa)}(\mathbf{v}, \kappa), H_n(\kappa)\rangle + e(\kappa, \mathbf{v}; \kappa, H_n(\kappa)) + B(\kappa, \mathbf{v}; \mathbf{v}, H_n(\kappa)).$$

We derive from (7.33) and (7.36) the inequality

$$\Sigma \le ||Q(\kappa, \mathbf{v}, \mathbf{x})||_{0,1,\Omega} ||H_n(\kappa)||_{0,\infty,\Omega} \le \sigma_1. \tag{7.38}$$

STEP 3. *Analysis term by term.* We examine each term in Σ one after another. The conditions for the application of Lemma 7.3 below are fulfilled, since $\nabla \cdot \mathbf{v} = 0$ and H_n is the derivative function of a C^1 function. We therefore find

$$b_e(\mathbf{v}; \kappa, H_n(\kappa)) = 0,$$

which, combined with (7.35), yields

$$\langle \mathscr{K}^{(\kappa)}(\mathbf{v}, \kappa), H_n(\kappa)\rangle = \mu M_n + t_e(\kappa, \mathbf{v}; \kappa, H_n(\kappa)) \ge \mu M_n, \tag{7.39}$$

as $t_e(\kappa, \mathbf{v}; \kappa, H_n(\kappa)) \ge 0$. Furthermore, we observe that

$$0 \le e(\kappa, \mathbf{v}; \kappa, H_n(\kappa)) = \int_\Omega \kappa H_n(\kappa) E(\kappa + k_\Gamma(\mathbf{v}), \mathbf{x}) d\mathbf{x}, \tag{7.40}$$

since H_n is odd, so that $H_n(\kappa)\kappa \ge 0$ and also because $E \ge 0$. By now combining (7.37), (7.39), and (7.40), we obtain

$$\mu M_n \le \sigma_1 + |B(\kappa, \mathbf{v}; \mathbf{v}, H_n(\kappa))|. \tag{7.41}$$

We are left with the term generated by the change of variable operator. Using the estimate (7.26) combined with (7.28) and (7.35), we find

$$|B(\kappa, \mathbf{v}; \mathbf{v}, H_n(\kappa))| \le P_2(C_v) M_n^{\frac{1}{2}},$$

P_2 being a second-order polynomial function that does not depend on n, but whose coefficients depend on the various data. Hence using Young's inequality yields

$$\mu M_n \le \sigma_1 + P_2(C_v) M_n^{\frac{1}{2}} \le \sigma_1 + \frac{P_2(C_v)}{2\mu} + \frac{\mu}{2} M_n,$$

leading to

$$M_n \le \frac{\sigma_1}{\mu} + \frac{P_2(C_v)}{2\mu^2}, \tag{7.42}$$

which is actually the desired estimate (7.30), with

$$C_{\kappa,q} = P_q\left(\frac{\sigma_1}{\mu} + \frac{P_2(C_v)}{2\mu^2}\right), \tag{7.43}$$

where P_q is the polynomial function considered in Theorem A.12. This concludes the proof.

□

We now prove a technical lemma, used in the proof of Proposition 7.2.

Lemma 7.3. *Let* $\mathbf{z} \in \mathbf{W}(\Omega)$ *such that* $\nabla \cdot \mathbf{z} = 0$*,* $\lambda \in H_0^1(\Omega)$*,* $G \in C^1(\mathbb{R})$*. Then*

$$b_e(\mathbf{z}; \lambda, G'(\lambda)) = 0. \tag{7.44}$$

Proof. Let us consider $\mathbf{z} \in \mathscr{V}_m(\Omega)$, $\lambda \in \mathscr{D}(\Omega)$. A similar proof to that of Lemma 6.3 yields

$$b_e(\mathbf{z}; \lambda, G'(\lambda)) = (\mathbf{z} \cdot \nabla\lambda, G'(\lambda))_\Omega,$$

which by applying Stokes formula and recalling that $\mathbf{z} \cdot \mathbf{n} = 0$ on Γ leads to

$$
\begin{aligned}
b_e(\mathbf{z};\lambda,G'(\lambda)) &= \int_\Omega \mathbf{v}(\mathbf{x})\cdot\nabla\lambda(\mathbf{x})G'(\lambda(\mathbf{x}))\,d\mathbf{x} \\
&= \int_\Omega \mathbf{v}(\mathbf{x})\cdot\nabla G(\lambda(\mathbf{x}))\,d\mathbf{x} \\
&= -\int_\Omega \nabla\cdot\mathbf{z}(\mathbf{x})G(\lambda(\mathbf{x}))d\mathbf{x}.
\end{aligned}
\tag{7.45}
$$

This holds when $\mathbf{z}\in\mathbf{W}(\Omega)$ and $\lambda\in L^2_0(\Omega)$ by the density of $\mathscr{V}_m(\Omega)$ in $\mathbf{W}(\Omega)$ and $\mathscr{D}(\Omega)$ in $L^2_0(\Omega)$ and the continuation principle, which obviously applies here. We therefore obtain (7.44) when $\nabla\cdot\mathbf{z}=0$. □

Remark 7.4. When $G(0)=0$, the same result holds for any $\mathbf{z}\in\mathbf{H}_1(\Omega)$ such that $\nabla\cdot\mathbf{z}=0$. In other words, the assumption $\mathbf{z}\cdot\mathbf{n}|_\Gamma=0$ is not in fact necessary since if $\lambda\in H^1_0(\Omega)$, then $G(\lambda)\in H^1_0(\Omega)$. Consequently, the boundary term in (7.45) also vanishes in this case.

7.2.5 Extra Estimates

Proposition 7.2 only relates to specific solutions $(\mathbf{v},p,\kappa)$ of $\mathscr{VP}^\kappa$, for which $\kappa\in H^1_0(\Omega)$. This raises the issue of whether any solutions to $\mathscr{VP}^\kappa$ satisfy Estimate (7.30), in other words to determine if $\mathscr{VP}^\kappa$ is consistent.[3]

The point of this is to justify the use of $H_n(\kappa)$ as test in (7.16.iii). However, when $(\mathbf{v},p,\kappa)$ is any solution to $\mathscr{VP}^\kappa$, $H_n(\kappa)\notin\mathbf{Q}_3(\Omega)$, and even though $H_n(\kappa)\in L^\infty(\Omega)$, there is no particular reason that it can be taken as test: the method does not apply—a priori—for any solution to $\mathscr{VP}^\kappa$, so that we do not know if estimate (7.30) always holds and, in particular, we do not know if $\mathscr{VP}^\kappa$ is consistent.

However, the approximate solutions we construct in this chapter will satisfy estimate (7.30). We will say that $\mathscr{VP}^\kappa$ is consistent by approximations.

To prepare the ground, let $S_n\in L^2(\Omega)$ and $\mathscr{UP}^\kappa_n$ denote the variational problem:

$$\text{Find } (\mathbf{v},p,\kappa)\in\mathbf{W}(\Omega)\times L^2_0(\Omega)\times H^1_0(\Omega) \text{ such that} \tag{7.46}$$

$$\text{for all } (\mathbf{w},q,l)\in\mathbf{W}(\Omega)\times L^2(\Omega)\times H^1_0(\Omega), \tag{7.47}$$

$$
\begin{cases}
\langle\mathscr{T}^{(\kappa)}(\mathbf{v},\kappa),\mathbf{w}\rangle-(p,\nabla\cdot\mathbf{w})_\Omega+\langle G(\mathbf{v}),\mathbf{w}\rangle=\langle\mathbf{f},\mathbf{w}\rangle, \\
(\nabla\cdot\mathbf{v},q)_\Omega=0, \\
\langle\mathscr{K}^{(\kappa)}(\mathbf{v},\kappa),l\rangle+e(\kappa,\mathbf{v};\kappa,l)+B(\kappa,\mathbf{v};\mathbf{v},l)=(S_n,l)_\Omega.
\end{cases}
\tag{7.48}
$$

Therefore, the proof of Proposition 7.2 directly gives:

[3] Recall that a variational problem is consistent if any a priori solution belongs to its space of unknowns.

Proposition 7.3. *Assume that the sequence $(S_n)_{n\in\mathbb{N}}$ is bounded in $L^1(\Omega)$, such that*

$$||S_n||_{0,1,\Omega} \leq \sigma_1.$$

Then any solution $(\mathbf{v}_n, p_n, \kappa_n)$ to $\mathscr{UP}_n^\kappa$ satisfies the estimates (7.28), (7.29), (7.30).

7.3 KESP and Compactness

We complete in this section the theoretical background necessary for the analysis of $\mathscr{VP}^\kappa$. We require in particular additional compactness properties and convergence lemmas, following the outline of Sect. 6.3.3.

7.3.1 K Extracting Subsequence Principle (KESP)

We elaborate the KESP compactness package appropriate to $\mathbf{K}_{3/2}(\Omega)$, similar to the VESP of Sect. 6.3.3.

Definition 7.1. Let $(\kappa_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbf{K}_{3/2}(\Omega)$. We say that $(\kappa_n)_{n\in\mathbb{N}}$ is bounded in $\mathbf{K}_{3/2}(\Omega)$ if it is bounded in $W_0^{1,q}(\Omega)$ for any $1 \leq q < 3/2$.

Lemma 7.4. *Let $(\kappa_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbf{K}_{3/2}(\Omega)$. Then there exists $\kappa \in \mathbf{K}_{3/2}(\Omega)$ and a subsequence $(\kappa_{n_j})_{n_j\in\mathbb{N}}$ such that*

(a) for all $q < 3/2$, $(\kappa_{n_j})_{n_j\in\mathbb{N}}$ weakly converges to κ in $W_0^{1,q}(\Omega)$, a.e. *in Ω,*
(b) $(\kappa_{n_j})_{n_j\in\mathbb{N}}$ strongly converges to κ in $L^r(\Omega)$ for all $1 \leq r < 3$.

Proof.

STEP 1. *Proof of (a).* Let $q_0 < 3/2$ be fixed. Since $(\kappa_n)_{n\in\mathbb{N}}$ is bounded in $W_0^{1,q_0}(\Omega)$, we know from standard functional analysis (see [14]) that there exists $\kappa \in W_0^{1,q_0}(\Omega)$ such that we can extract a subsequence from $(\kappa_{n_j})_{n_j\in\mathbb{N}}$ which weakly converges to κ in $W_0^{1,q_0}(\Omega)$. Obviously,

$$\forall\, q < q_0,\ \ \kappa \in W_0^{1,q}(\Omega) \ \text{ and } \ \lim_{j\to\infty} \kappa_{n_j} = \kappa \ \text{ in } \ W_0^{1,q}(\Omega) \ \text{ weak},$$

$$\forall\, 1 \leq r < q_0^\star,\ \lim_{j\to\infty} \kappa_{n_j} = \kappa \ \text{ in } \ L^r(\Omega) \ \text{ strong},$$

by the Sobolev embedding theorem.[4] The inverse Lebesgue Theorem (Theorem A.10 in [TB]) allows us to extract yet another subsequence from $(\kappa_{n_j})_{n_j\in\mathbb{N}}$, which converges a.e. in Ω to κ that we still denote $(\kappa_{n_j})_{n_j\in\mathbb{N}}$.

[4]Recall that $q_0^\star = \dfrac{3-q_0}{3q_0}$.

Let $q_0 < q < 3/2$. Since the sequence $(\kappa_{n_j})_{n_j \in \mathbb{N}}$ is bounded in $W_0^{1,q}(\Omega)$, there exists $\tilde{\kappa} \in W_0^{1,q}(\Omega)$, and a subsequence $(\kappa_{n_{j_k}})_{n_{j_k} \in \mathbb{N}}$ can be extracted which

(i) weakly converges to $\tilde{\kappa}$ in $W_0^{1,q}(\Omega)$,
(ii) strongly in $L^r(\Omega)$, $1 \le r < q^\star$,
(iii) *a.e* in Ω.

Of course, $q_0^\star < q^\star$, and by the uniqueness of the limit in $L^r(\Omega)$ for $1 \le r < q_0^\star$, we infer $\kappa = \tilde{\kappa}$, yielding $\kappa \in W_0^{1,q}(\Omega)$.
Furthermore, we observe that the only possible weak limit in $W_0^{1,q}(\Omega)$ of any subsequence of $(\kappa_{n_j})_{n_j \in \mathbb{N}}$ is κ, which gives the result that $(\kappa_{n_j})_{n_j \in \mathbb{N}}$ weakly converges to κ in $W_0^{1,q}(\Omega)$.

STEP 2. *Proof of (b).* The above reasoning holds $\forall\, q \in]q_0, 3/2[$, hence $\kappa \in \mathbf{K}_{3/2}(\Omega)$.

Finally, let $r < 3$; there exists $q < 3/2$ such that $r < q^\star$. Therefore, by the Sobolev embedding theorem, a subsequence can be extracted from $(\kappa_{n_j})_{n_j \in \mathbb{N}}$, which we still denote $(\kappa_{n_j})_{n_j \in \mathbb{N}}$, such that $(\kappa_{n_j})_{n_j \in \mathbb{N}}$ strongly converges to κ. This holds for any $r < 3$, for the same subsequence, by a uniqueness argument as above. The *a.e.* convergence of $(\kappa_{n_j})_{n_j \in \mathbb{N}}$ has already been derived. □

Definition 7.2. In what follows, we write $(\kappa_{n_j})_{n_j \in \mathbb{N}}$ instead of $(\kappa_n)_{n \in \mathbb{N}}$ for simplicity. We shall say that we apply the *K Extracting Subsequence Principle*, KESP, to the sequence $(\kappa_n)_{n \in \mathbb{N}}$ and that κ is a K-ESP-limit, including all the properties listed in Lemma 7.4.

Remark 7.5. The KESP also applies to bounded sequences in $H_0^1(\Omega)$, where in this case the critical exponent is 6. We shall sometimes refer to this as the "customized KESP."

7.3.2 Convergence Lemma

We prove a series of convergence results, in complement to Lemma 6.6. Let us consider:

(i) $(\mathbf{v}_n)_{n \in \mathbb{N}} \subset \mathbf{W}(\Omega)$ to which the VESP is applied, with $\mathbf{v}$ any V-ESP-limits,
(ii) $(\kappa_n)_{n \in \mathbb{N}} \subset \mathbf{K}_{3/2}(\Omega)$ to which the KESP is applied, with κ any K-ESP-limits,
(iii) $(\mathbf{z}_n)_{n \in \mathbb{N}}$ such that $\forall\, n \in \mathbb{N}$, $\mathbf{z}_n \in \mathbf{L}^6(\Omega)$, which converges to $\mathbf{z} \in \mathbf{L}^6(\Omega)$ in $\mathbf{L}^p(\Omega)$, $1 \le p \le 6$, and *a.e.* in Ω,
(iv) $(\lambda_n)_{n \in \mathbb{N}}$ such that $\forall\, n \in \mathbb{N}$, $\lambda_n \in L^1(\Omega)$, and which converges to $\lambda \in L^1(\Omega)$ *a.e.* in Ω.[5]

[5]The choice of L^1 is to fix ideas and is enough for our purpose. The same result holds in a more general context.

We recall that the operators involved in the next statement are defined by the formulas (7.8) and (7.13).

Lemma 7.5. *We have, for all* $(\mathbf{w}, l) \in \mathbf{W}(\Omega) \times \mathbf{Q}_3(\Omega)$,

$$\lim_{n\to\infty} b_e(\mathbf{z}_n; \kappa_n, l) = b_e(\mathbf{z}; \kappa, l), \tag{7.49}$$

$$\lim_{n\to\infty} t_v(\lambda_n, \mathbf{z}_n; \mathbf{v}_n, \mathbf{w}) = t_v(\lambda, \mathbf{z}; \mathbf{v}, \mathbf{w}), \tag{7.50}$$

$$\lim_{n\to\infty} t_e(\lambda_n, \mathbf{z}_n; \kappa_n, l) = t_e(\lambda, \mathbf{z}; \kappa, l), \tag{7.51}$$

$$\lim_{n\to\infty} e(\lambda_n, \mathbf{z}_n; \kappa_n, l) = e(\lambda, \mathbf{z}; \kappa, l), \tag{7.52}$$

$$\lim_{n\to\infty} B(\lambda_n, \mathbf{z}_n; \mathbf{v}_n, l) = B(\lambda, \mathbf{z}; \mathbf{v}, l). \tag{7.53}$$

Proof. We establish each convergence one after another.

STEP 1. *Proof of (7.49).* We write

$$b_e(\mathbf{z}_n; \kappa_n, l) = \frac{1}{2}((l\, \mathbf{z}_n, \nabla \kappa_n)_\Omega - (\kappa_n \mathbf{z}_n, \nabla l)_\Omega).$$

As $l \in L^\infty(\Omega)$, we observe that

(1) $l\, \mathbf{z}_n \to l\, \mathbf{z}$ strongly in $\mathbf{L}^5(\Omega)$,
(2) since $5/4 < 3/2$, $\nabla \kappa_n \to \nabla \kappa$ weakly in $\mathbf{L}^{5/4}(\Omega)$,

so that

$$(l\, \mathbf{z}_n, \nabla \kappa_n)_\Omega \to (l\, \mathbf{z}, \nabla \kappa)_\Omega \quad \text{when} \quad n \to \infty.$$

Similarly, we infer from the equality $1/6 + 1/3 = 1/2$ that

$$\kappa_n \mathbf{z}_n \to \kappa \mathbf{z} \;\text{ in }\; \mathbf{L}^q(\Omega), \;\; \forall\, 1 \le q < 2.$$

Moreover, there exists $r > 3$ such that $\nabla l \in \mathbf{L}^r(\Omega)$ and, in particular, $r' < 3/2$. Therefore, we have $(\kappa_n \mathbf{z}_n, \nabla l)_\Omega \to (\kappa\, \mathbf{z}, \nabla l)_\Omega)$, hence the result.

STEP 2. *Proof of (7.50)–(7.52).* We write

$$t_v(\lambda_n, \mathbf{z}_n; \mathbf{v}_n, \mathbf{w}) = (\widetilde{\nu_t}(\lambda_n, \mathbf{z}_n) D\mathbf{w}, D\mathbf{v}_n)_\Omega.$$

We know from (iii) and (iv) that $\mathbf{z}_n \to \mathbf{z}$ and $\lambda_n \to \lambda$ a.e. in Ω, so the continuity of $\widetilde{\nu_t}$ leads to $\widetilde{\nu_t}(\lambda_n, \mathbf{z}_n, \mathbf{x}) D\mathbf{w} \to \widetilde{\nu_t}(\lambda, \mathbf{z}, \mathbf{x}) D\mathbf{w}$ a.e. in Ω. Moreover,

$$|\widetilde{\nu_t}(\lambda_n, \mathbf{z}_n, \mathbf{x}) D\mathbf{w}| \le ||\nu_t||_\infty |D\mathbf{w}| \in L^2(\Omega),$$

Hence, we obtain by Lebesgue theorem,

$$(\widetilde{\nu_t}(\lambda_n, \mathbf{z}_n) D\mathbf{w}, D\mathbf{v}_n)_\Omega \to (\widetilde{\nu_t}(\lambda, \mathbf{z}) D\mathbf{w}, D\mathbf{v})_\Omega \quad \text{as} \quad n \to \infty,$$

proving (7.50). Proving (7.51) and (7.52) follows the same procedure, so we shall skip the details.

STEP 3. *Proof of (7.53)*. As k_Γ is continuous and bounded, we easily deduce that

$$k_\Gamma(\mathbf{v}_n) \to k_\Gamma(\mathbf{v}) \ \text{ in } \ L^p(\Omega), \ \forall\, 1 \le p < \infty, \ \text{ and } a.e. \text{ in } \ \Omega.$$

We now prove the weak convergence of $(\nabla(k_\Gamma(\mathbf{v}_n)))_{n\in\mathbb{N}}$ to $\nabla(k_\Gamma(\mathbf{v}))$ in $\mathbf{L}^2(\Omega)$. First, we observe that $\nabla(k_\Gamma(\mathbf{v}_n)) = \nabla\mathbf{v}_n \cdot k'_\Gamma(\mathbf{v}_n)$, so that since k'_Γ is bounded $\nabla(k_\Gamma(\mathbf{v}_n)) \in \mathbf{L}^2(\Omega)$. Let $\mathbf{h} \in \mathbf{L}^2(\Omega)$. We have

$$(\nabla(k_\Gamma(\mathbf{v}_n)), \mathbf{h})_\Omega = (k'_\Gamma(\mathbf{v}_n) \otimes \mathbf{h}, \nabla\mathbf{v}_n)_\Omega.$$

As k'_Γ is also continuous,

$$k'_\Gamma(\mathbf{v}_n) \otimes \mathbf{h} \to k'_\Gamma(\mathbf{v}) \otimes \mathbf{h} \ \text{ a.e. in } \ \Omega, \quad |k'_\Gamma(\mathbf{v}_n) \otimes \mathbf{h}| \le ||k'_\Gamma||_\infty |\mathbf{h}| \in L^2(\Omega).$$

Therefore,

$$k'_\Gamma(\mathbf{v}_n) \otimes \mathbf{h} \to k'_\Gamma(\mathbf{v}) \otimes \mathbf{h} \ \text{ in } \ L^2(\Omega) \ \text{ as } \ n \to \infty,$$

so that the weak convergence in $(\mathbf{v}_n)_{n\in\mathbb{N}}$ in $\mathbf{W}(\Omega)$ leads to

$$(\nabla(k_\Gamma(\mathbf{v}_n)), \mathbf{h})_\Omega = (k'_\Gamma(\mathbf{v}_n) \otimes \mathbf{h}, \nabla\mathbf{v}_n)_\Omega \to (k'_\Gamma(\mathbf{v}) \otimes \mathbf{h}, \nabla\mathbf{v})_\Omega = (\nabla(k_\Gamma(\mathbf{v})), \mathbf{h})_\Omega,$$

which yields in turn the weak convergence of $(\nabla(k_\Gamma(\mathbf{v}_n)))_{n\in\mathbb{N}}$ to $\nabla(k_\Gamma(\mathbf{v}))$ in $\mathbf{L}^2(\Omega)$ as expected and hence the weak convergence of $(k_\Gamma(\mathbf{v}_n))_{n\in\mathbb{N}}$ in $\mathbf{H}^1(\Omega)$, since we already know the L^2 convergence of the sequence. From there, (7.53) derives from (7.49) to (7.52) with $k_\Gamma(\mathbf{v}_n)$ instead of κ_n. Actually, in the proofs above, the fact that κ_n is equal to zero on Γ plays no particular role, and therefore they apply to $k_\Gamma(\mathbf{v}_n)$ too. $\square$

7.4 Regularized NS-TKE Model

The construction of solutions to $\mathscr{V}\mathscr{P}^\kappa$ is based on approximations obtained by regularizing the production term by convolution. In terms of PDEs, this corresponds to the system:

$$\begin{cases} (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \nabla\cdot[(2\nu+\nu_t(k,\mathbf{x}))\,D\mathbf{v}] \;+\nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v}\cdot\nabla k - \nabla\cdot(\mu+\mu_t(k,\mathbf{x})\nabla k) + k\,E(k,\mathbf{x}) = \nu_t(k,\mathbf{x})\,D\mathbf{v}_n : D\mathbf{v} & \text{in } \Omega, \\ -[(2\nu+\nu_t)\,D\mathbf{v}\cdot\mathbf{n}]_\tau = g(\mathbf{v})_\tau & \text{on } \Gamma, \\ \mathbf{v}\cdot\mathbf{n} = 0 & \text{on } \Gamma, \\ k = k_\Gamma(\mathbf{v}) & \text{on } \Gamma, \end{cases} \tag{7.54}$$

where $\mathbf{v}_n = \mathbf{v}\star\rho_n$, for some mollifier ρ_n specified below. We establish in this section the existence of weak solutions of this system.

We first write the variational problem $\mathscr{VP}_n^\kappa$ associated with the regularized NS-TKE model (7.54).

Following the outline of Sect. 6.6, the existence of a solution to $\mathscr{VP}_n^\kappa$ is proved by a linearization procedure and then the Schauder fixed-point theorem. This requires a η-regularization leading to a further variational problem $\mathscr{VP}_{n,\eta}^\kappa$, which is proved to have a solution. We prove that the family $(\mathscr{VP}_{n,\eta}^\kappa)_{\eta>0}$ converges to $\mathscr{VP}_n^\kappa$ when $\eta\to 0$, hence the existence of solution to $\mathscr{VP}_n^\kappa$.

The convergence of the sequence $(\mathscr{VP}_n^\kappa)_{n\in\mathbb{N}}$ is studied in the next Sect. 7.5.

7.4.1 *Construction of the Regularized System*

7.4.1.1 Convolutions

Let $\rho \in C_c^\infty(\mathbb{R}^3)$, with $\mathrm{supp}(\rho)\subset B(0,1)$, $\rho\ge 0$, $||\rho||_{0,1,\mathbb{R}^3} > 0$ and ρ is even, $\rho(-\mathbf{x}) = \rho(\mathbf{x})$. Let

$$\rho_n(\mathbf{x}) = \frac{n^3}{||\rho||_{0,1,\mathbb{R}^3}}\rho(n\,\mathbf{x}).$$

Let $u\in L^p(\Omega)$ $(p\ge 1)$ and $\tilde{u}\in L^p(\mathbb{R}^3)$ be the function defined by

$$\forall\,\mathbf{x}\in\Omega,\;\; \tilde{u}(\mathbf{x}) = u(\mathbf{x}), \quad \forall\mathbf{x}\in\mathbb{R}^3\setminus\Omega,\;\; \tilde{u}(\mathbf{x}) = 0.$$

For simplicity, we still write u instead of $\tilde{u}$, and we consider the convolution product of u by ρ_n,

$$u\star\rho_n(\mathbf{x}) = \int_{\mathbb{R}^3} u(\mathbf{y})\rho_n(\mathbf{x}-\mathbf{y})d\mathbf{y}. \tag{7.55}$$

We know that $u\star\rho_n\in C_c^\infty(\mathbb{R}^3)$, $\mathrm{supp}\,u\star\rho_n\subset\Omega + B(0,1/n)$, and among all the properties satisfied by $u\star\rho_n$, we note that when $p>1$,

$$||u\star\rho_n||_{0,\infty,\mathbb{R}^3} \le n^3\frac{|\Omega|^{\frac{1}{p'}}}{||\rho||_{0,1,\mathbb{R}^3}}||u||_{0,p,\Omega} = n^3 C_\rho ||u||_{0,p,\Omega}. \tag{7.56}$$

Moreover, $(u \star \rho_n)_{n\in\mathbb{N}}$ converges to u in $L^p(\Omega)$ (see in [14]), and we also have

$$||u \star \rho_n||_{0,p,\mathbb{R}^3} \leq ||u||_{0,p,\Omega}. \tag{7.57}$$

As ρ is even, we easily derive from Fubini's theorem

$$\forall\, 1 < p < \infty,\ \forall\, u \in L^p(\Omega),\ \forall\, v \in L^{p'}(\Omega), \quad (u \star \rho_n, v)_\Omega = (u, v \star \rho_n)_\Omega.$$

In the particular case $p = 2$, we deduce that $u \to u \star \rho_n|_\Omega$ defines a continuous self adjoint operator over $L^2(\Omega)$.

Finally, if $\mathbf{z} = (z_1, z_2, z_3)$ is any vector field on Ω, $\mathbf{z}\star\rho_n = (z_1\star\rho_n, z_2\star\rho_n, z_3\star\rho_n)$, and if $A = (a_{ij})_{1\leq ij\leq 3}$ is a second-order tensor field, $A \star \rho_n = (a_{ij} \star \rho_n)_{1\leq ij\leq 3}$.

7.4.1.2 Variational Problem

Let $\mathscr{V}\mathscr{P}_n^\kappa$ denote the following variational problem[6]:

$$\text{Find } (\mathbf{v}, p, \kappa) \in \mathbf{W}(\Omega) \times L^2_0(\Omega) \times H^1_0(\Omega) \text{ such that} \tag{7.58}$$

$$\text{for all } (\mathbf{w}, q, l) \in \mathbf{W}(\Omega) \times L^2(\Omega) \times H^1_0(\Omega), \tag{7.59}$$

$$\begin{cases} \langle \mathscr{T}^{(\kappa)}(\mathbf{v},\kappa), \mathbf{w}\rangle - (p, \nabla\cdot\mathbf{w})_\Omega + \langle G(\mathbf{v}), \mathbf{w}\rangle = \langle \mathbf{f}, \mathbf{w}\rangle, \\ (\nabla\cdot\mathbf{v}, q)_\Omega = 0, \\ \langle \mathscr{K}^{(\kappa)}(\mathbf{v},\kappa), l\rangle + e(\kappa,\mathbf{v};\kappa,l) + B(\kappa,\mathbf{v};\mathbf{v},l) = (Q_n(\kappa,\mathbf{v},\mathbf{x};\mathbf{v}), l)_\Omega, \end{cases} \tag{7.60}$$

where

$$Q_n(\lambda, \mathbf{z}; \mathbf{v}, \mathbf{x}) = \widetilde{\nu_t}(\lambda, \mathbf{z}, \mathbf{x})\, D\mathbf{z}_n : D\mathbf{v}, \quad \mathbf{z}_n = \mathbf{z} \star \rho_n. \tag{7.61}$$

Any solution to $\mathscr{V}\mathscr{P}_n^\kappa$ is a weak solution to the system (7.54). By the end of this section, we shall have proved:

Theorem 7.2. *Let $n \in \mathbb{N}$ be given. Problem $\mathscr{V}\mathscr{P}_n^\kappa$ admits a solution $(\mathbf{v}, p, \kappa)$ which satisfies the fundamental estimates (7.28)–(7.30).*

7.4.2 Linearization of the Regularized System

As in Sect. 6.6, in preparation for the linearization process we must introduce the η-regularized variational problem $\mathscr{V}\mathscr{P}_{n,\eta}^\kappa$ expressed by (7.58), (7.59) and,

[6] $\mathscr{T}^{(\kappa)}$ and $\mathscr{K}^{(\kappa)}$ are defined by (7.9) and (7.10).

$$\begin{cases} \langle \mathscr{T}^{(\kappa)}(\mathbf{v},\kappa),\mathbf{w}\rangle - (p,\nabla\cdot\mathbf{w})_\Omega + \langle G(\mathbf{v}) + \eta\mathbf{v},\mathbf{w}\rangle = \langle \mathbf{f},\mathbf{w}\rangle, \\ (\nabla\cdot\mathbf{v},q)_\Omega = 0, \\ \langle \mathscr{K}^{(\kappa)}(\mathbf{v},\kappa),l\rangle + e(\kappa,\mathbf{v};\kappa,l) + B(\kappa,\mathbf{v};\mathbf{v},l) = (Q_n(\kappa,\mathbf{v},\mathbf{x};\mathbf{v}),l)_\Omega, \end{cases} \tag{7.62}$$

where $\langle \eta\mathbf{v},\mathbf{w}\rangle = \eta(\mathbf{v},\mathbf{w})_\Gamma$. The main results are the following.

Theorem 7.3. *Let $n \in \mathbb{N}$ and $\eta > 0$ be fixed. Then $\mathscr{VP}^\kappa_{n,\eta}$ admits a solution.*

Theorem 7.4. *Let $n \in \mathbb{N}$ be fixed. Then, the family $(\mathscr{VP}^\kappa_{n,\eta})_{\eta>0}$ converges*[7] *to $\mathscr{VP}^\kappa_n$ when $\eta \to 0$.*

Theorem 7.2 is a consequence of Theorems 7.3 and 7.4. To prove these results, we proceed using the fixed-point method with linearization as already outlined. In this section, we initialize the procedure which yields the application whose fixed points provide solutions to $\mathscr{VP}^\kappa_{n,\eta}$.

We will not linearize $\mathscr{VP}^\kappa_{n,\eta}$ as a whole. In order to apply the results of Chap. 6, we proceed in two stages, avoiding the ε-approximations: we first solve the linearized system for $(\mathbf{v}, p)$ which yields a couple $(\mathbf{v}_\eta(\mathbf{z},\lambda), p_\eta(\mathbf{z},\lambda))$ and then solve the linearized equation for κ, using $\mathbf{v}_\eta(\mathbf{z},\lambda)$ in the source term.

7.4.2.1 Linear Equation for $(\mathbf{v}, p)$

Let $(\mathbf{z},\lambda) \in \mathbf{W}(\Omega) \times H^1_0(\Omega)$ be fixed. We consider $\mathscr{T}^{(\kappa)}_{\mathbf{z},\lambda}$ the operator expressed by[8]

$$\langle \mathscr{T}^{(\kappa)}_{\mathbf{z},\lambda}(\mathbf{v}),\mathbf{w}\rangle = b(\mathbf{z};\mathbf{v},\mathbf{w}) + a(\mathbf{v},\mathbf{w}) + t_v(\lambda,\mathbf{z};\mathbf{v},\mathbf{w}) \tag{7.63}$$

which is the linearized operator of $\mathscr{T}^{(\kappa)}$ since $\mathscr{T}^{(\kappa)}(\mathbf{v},\kappa) = \mathscr{T}^{(\kappa)}_{\mathbf{v},\kappa}(\mathbf{v})$. We note that $\mathscr{T}^{(\kappa)}_{\mathbf{z},\lambda}$ is the same operator as $\mathscr{T}_\mathbf{z}$ defined in (6.12), by setting

$$\nu_t(\mathbf{x}) = \widetilde{\nu_t}(\lambda(\mathbf{x}),\mathbf{z}(\mathbf{x}),\mathbf{x}) \in L^\infty(\Omega).$$

Based on this observation, we deduce from Lemma 6.12 that the variational problem[9] $\mathscr{LP}^\kappa_{\eta,\mathbf{z},\lambda}$:

Find $(\mathbf{v}, p) \in \mathbf{W}(\Omega) \times L^2_0(\Omega)$ such that $\forall\, (\mathbf{w}, q) \in \mathbf{W}(\Omega) \times L^2_0(\Omega)$,

$$\begin{cases} \langle \mathscr{T}^{(\kappa)}_{\mathbf{z},\lambda}(\mathbf{v}),\mathbf{w}\rangle - (p,\nabla\cdot\mathbf{w})_\Omega + (\mathscr{G}_\mathbf{z}(\mathbf{v}) + \eta\mathbf{v},\mathbf{w})_\Omega = \langle \mathbf{f},\mathbf{w}\rangle, \\ (\nabla\cdot\mathbf{v},q)_\Omega = 0, \end{cases} \tag{7.64}$$

[7] The convergence of variational problems was first introduced in Sect. 6.4.5.

[8] t_v and t_e are defined by (7.8).

[9] The linear operator $\mathscr{G}_\mathbf{z}$ derived from G is defined by (6.90).

has a unique solution $(\mathbf{v}_\eta, p_\eta) = (\mathbf{v}_\eta(\mathbf{z}, \lambda), p_\eta(\mathbf{z}, \lambda))$, such that

$$||\mathbf{v}_\eta(\mathbf{z}, \lambda)||_{\mathbf{W}(\Omega)} \le C_{v,\eta}, \quad ||p_\eta(\mathbf{z}, \lambda)||_{0,2,\Omega} \le C_{p,\eta}(||\mathbf{z}||_{\mathbf{W}(\Omega)}), \tag{7.65}$$

$C_{v,\eta}$ and $C_{p,\eta}$ are specified by (6.102) and (6.105).

7.4.2.2 Linear Equation for κ

The linear operator for the κ-equation is given by (7.11). Let $\mathscr{KP}^{\kappa}_{\mathbf{z},\lambda,n}$ be the linear problem:

$$\text{Find} \quad \kappa \in H^1_0(\Omega), \ \text{ such that } \ \forall\, l \in H^1_0(\Omega),$$

$$\langle \mathscr{K}^{(\kappa)}_{\mathbf{z},\lambda}(\kappa), l\rangle + e(\lambda, \mathbf{z}; \kappa, l) + B(\lambda, \mathbf{z}; \mathbf{v}_\eta), l) = (Q_n(\lambda, \mathbf{z}, \mathbf{x}; \mathbf{v}_\eta), l)_\Omega, \tag{7.66}$$

where for simplicity we write $\mathbf{v}_\eta = \mathbf{v}_\eta(\mathbf{z}, \lambda)$.

Lemma 7.6. *$\mathscr{KP}^{\kappa}_{\mathbf{z},\lambda,n}$ admits a unique solution.*

Proof. The proof consists in proving that the problem falls within the framework of the Lax–Milgram theorem and that the conditions for its application are fulfilled.

STEP 1. *Implementation.* Let A be the bilinear form

$$A(\kappa, l) = \langle \mathscr{K}^{(\kappa)}_{\mathbf{z},\lambda}(\kappa), l\rangle + e(\lambda, \mathbf{z}; \kappa, l).$$

Then $\mathscr{KP}^{\kappa}_{\mathbf{z},\lambda,n}$ becomes

$$\text{find } \ \kappa \in H^1_0(\Omega), \ \text{ such that } \ \forall\, l \in H^1_0(\Omega),$$

$$A(\kappa, l) = Q_n(\lambda, \mathbf{z}; \mathbf{v}_\eta, \mathbf{x}), l)_\Omega - B(\lambda, \mathbf{z}; \mathbf{v}_\eta, l) = \langle F, l\rangle, \tag{7.67}$$

which falls within the Lax–Milgram framework. We first prove that A is a bilinear continuous operator, then that $F \in H^{-1}(\Omega)$ and finally that A is coercive.

STEP 2. *Continuity of A.* The same arguments as those leading to Lemma 6.4 show that

$$\forall\, \kappa, l \in H^1_0(\Omega), \quad |\langle \mathscr{K}^{(\kappa)}_{\mathbf{z},\lambda}(\kappa), l\rangle| \le (C||\mathbf{z}||_{\mathbf{W}(\Omega)} + \mu + ||\mu_t||_\infty)||\kappa||_{H^1_0(\Omega)}||l||_{H^1_0(\Omega)},$$

where C depends only on Ω. Moreover, by the Cauchy–Schwarz and Poincaré inequalities,

$$|e(\lambda, \mathbf{z}; \kappa, l)| \le ||E||_\infty ||\kappa||_{0,2,\Omega} ||l||_{0,2,\Omega} \le C_p ||E||_\infty ||\kappa||_{H^1_0(\Omega)} ||l||_{H^1_0(\Omega)}.$$

Consequently, A is a continuous bilinear form over $H_0^1(\Omega)$, which satisfies

$$|||A||| \leq C||\mathbf{z}|| + \mu + ||\mu_t||_\infty + C_p||E||_\infty.$$

STEP 3. *Continuity of the r.h.s.* We show that the form F expressed by the r.h.s. of (7.67) is continuous. We first observe that

$$|(Q_n(\lambda, \mathbf{z}; \mathbf{v}_\eta, \mathbf{x}), l)_\Omega \leq ||\nu_t||_\infty ||D\mathbf{z}_n||_{0,\infty,\Omega} ||D\mathbf{v}_\eta||_{0,2,\Omega} ||l||_{H_0^1(\Omega)}.$$

Since $D\mathbf{z}_n = (D\mathbf{z}) \star \rho_n$, we can make use of estimate (7.56) with $u = D\mathbf{z}$, which combined with (7.65) yields

$$|(Q_n(\lambda, \mathbf{z}; \mathbf{v}_\eta(\mathbf{z}, \lambda), \mathbf{x}), l)_\Omega| \leq n^3\, C_\rho ||\nu_t||_\infty C_{\nu,\eta} ||\mathbf{z}||_{\mathbf{W}(\Omega)} ||l||_{H_0^1(\Omega)}. \tag{7.68}$$

Moreover, we deduce from (7.26) combined with (7.65),

$$|B(\lambda, \mathbf{z}; \mathbf{v}, l)| \leq C(C_{\nu,\eta} + (1 + C_{\nu,\eta})||\mathbf{z}||_{\mathbf{W}(\Omega)} + 1)||l||_{H_0^1(\Omega)}. \tag{7.69}$$

Inequalities (7.68) and (7.69) show that $F \in H^{-1}(\Omega)$.

STEP 4. *Coercivity and conclusion.* It remains to verify that A is coercive. By noting that

$$b_e(\mathbf{z}; \kappa, \kappa) = 0, \quad t_e(\lambda, \mathbf{z}; \kappa, \kappa) \geq 0, \quad e(\lambda, \mathbf{z}; \kappa, \kappa) \geq 0,$$

we obtain

$$A(\kappa, \kappa) \geq \mu||\kappa||_{H_0^1(\Omega)},$$

which proves that A is coercive and allows us to conclude that $\mathscr{KP}^\kappa_{\mathbf{z},\lambda,n}$ given by (7.66) has a unique solution $\kappa = \kappa_\eta(\mathbf{z}, \lambda)$.

Moreover, it is easily verified from the above that κ_η satisfies the estimate,

$$||\kappa_\eta(\mathbf{z}, \lambda)||_{H_0^1(\Omega)} \leq C_\kappa \left(\frac{(1 + n^3)||\mathbf{z}||_{\mathbf{W}(\Omega)}}{\eta} + 1 \right), \tag{7.70}$$

where the constant C_κ depends on Ω, C_ρ, ν, $\mathbf{f}$, μ, $||\mu_t||_\infty$, $||k_\Gamma||_\infty$, $||k'_\Gamma||_\infty$, and $||K||_\infty$. □

In the following, $R_{\kappa,n,\eta}(x)$ denotes the first-degree polynomial function

$$R_{\kappa,n,\eta}(x) = C_\kappa \left(\frac{(1 + n^3)x}{\eta} + 1 \right). \tag{7.71}$$

7.4.3 Framework of the Fixed-Point Process

Following the outline of Sect. 6.7, we introduce in this subsection the suitable application whose fixed points provide solutions to $\mathscr{VP}^{\kappa}_{n,\eta}$. We demonstrate its continuity and compactness properties, in order to prepare the ground for the application of the Schauder fixed-point theorem. In the following, $n \in \mathbb{N}$ and $\eta > 0$ are fixed.

7.4.3.1 Implementation and Weak Continuity

Let $\mathscr{E}_\eta$ be the application defined by

$$\mathscr{E}_\eta : \begin{cases} \mathbf{W}(\Omega) \times H^1_0(\Omega) \to & \mathbf{W}(\Omega) \times L^2_0(\Omega) \times H^1_0(\Omega), \\ \qquad (\mathbf{z}, \lambda) & \to (\mathbf{v}_\eta(\mathbf{z}, \lambda), p_\eta(\mathbf{z}, \lambda), \kappa_\eta(\mathbf{z}, \lambda)), \end{cases}$$

where $(\mathbf{v}_\eta(\mathbf{z}, \lambda), p_\eta(\mathbf{z}, \lambda))$ is the solution to $\mathscr{LP}^{\kappa}_{\eta,\mathbf{z},\lambda}$ and $\kappa_\eta(\mathbf{z}, \lambda)$ the solution to $\mathscr{KP}^{\kappa}_{\mathbf{z},\lambda,n}$.

Lemma 7.7. *Let $(\mathbf{z}_m, \lambda_m)_{m\in\mathbb{N}}$ be weakly convergent to $(\mathbf{z}, \lambda)$ in $\mathbf{W}(\Omega) \times H^1_0(\Omega)$, $(\mathbf{v}_m, p_m, \kappa_m) = \mathscr{E}_\eta(\mathbf{z}_m, \lambda_m)$. Then*

$$\lim_{m\to\infty} (\mathbf{v}_m, p_m, \kappa_m) = (\mathbf{v}, p, \kappa) = \mathscr{E}_\eta(\mathbf{z}, \lambda) \;\; \textit{in} \;\; \mathbf{W}(\Omega) \times L^2_0(\Omega) \times H^1_0(\Omega) \;\; \textit{weak.}$$

Moreover, the convergence of $(\mathbf{v}_m)_{m\in\mathbb{N}}$ toward $\mathbf{v}$ is strong in $\mathbf{W}(\Omega)$.

Proof. The proof is based on the energy method developed in Sect. 6.7.3 and is divided into three steps. The various ESP are applied, which allows the limit to be taken in the fluid equation, following the proof of Lemma 6.13.

Particular attention is devoted to taking the limit in the TKE equation, especially in the production term which is a source of difficulty. Note that the strong convergence of $(\mathbf{v}_m)_{m\in\mathbb{N}}$ toward $\mathbf{v}$ is a consequence of this process.

STEP 1. *Application of the extraction subsequences principles.* We start by applying the VESP to $(\mathbf{z}_m)_{m\in\mathbb{N}}$, the customized $H^1_0(\Omega)$ KESP version to $(\lambda_m)_{m\in\mathbb{N}}$, denoting by $\mathbf{z}$ the unique V-ESP-limit of $(\mathbf{z}_m)_{m\in\mathbb{N}}$, while λ is the unique K-ESP-limit of $(\lambda_m)_{m\in\mathbb{N}}$. In particular, these sequences verify items iii) and iv) of Sect. 7.3.2, necessary in order to use Lemma 7.5. Further concerning the ESP, we notice :

(a) According to the estimate (7.65), $(\mathbf{v}_m, p_m)_{m\in\mathbb{N}}$ is bounded in $\mathbf{W}(\Omega) \times L^2_0(\Omega)$ so that the VESP and PESP apply once again, and $\mathbf{v}$ and p denote given ESP-limits.
(b) As $(\mathbf{z}_m)_{m\in\mathbb{N}}$ is bounded in $\mathbf{W}(\Omega)$, so is $(\kappa_m)_{m\in\mathbb{N}}$ in $H^1_0(\Omega)$ by estimate (7.70). The customized KEPS applies to this sequence, and $\kappa \in H^1_0(\Omega)$ denotes a given K-ESP-limit.

We show in what follows that $(\mathbf{v}, p, \kappa) = \mathscr{E}_\eta(\mathbf{z}, \lambda)$. By uniqueness of the limit, it can be concluded that the ESP-limits are weak limits, which means that the full sequence weakly converges, which will conclude the proof.

STEP 2. *Taking the limit in* $\mathscr{LP}^\kappa_{\eta,\mathbf{z},\lambda}$ *expressed by (7.64).* This equation differs from (6.104) investigated in the proof of Lemma 6.13 by the diffusion term, where $s_v(\mathbf{v}_m, \mathbf{w})$ is replaced by $t_v(\lambda_m, \mathbf{z}_m; \mathbf{v}_m, \mathbf{w})$. We know from the convergence Lemma 7.5, item (7.50), that

$$\lim_{m\to\infty} (t_v(\lambda_m, \mathbf{z}_m; \mathbf{v}_m, \mathbf{w}))_{m\in\mathbb{N}} = t_v(\lambda, \mathbf{z}; \mathbf{v}, \mathbf{w}).$$

The other terms are analyzed similarly to the proof of Lemma 6.13. We skip the details and easily conclude

$$\mathbf{v} = \mathbf{v}_\eta(\mathbf{z}, \lambda), \quad p = p_\eta(\mathbf{z}, \lambda).$$

STEP 3. *Taking the limit in* $\mathscr{KP}^\kappa_{\mathbf{z},\lambda,n}$ *expressed by (7.66).* Of course, Lemma 7.5 has been designed in prevision of this proof. It allows the limit to be taken in all the terms of the equation, except in the production term $Q_n(\kappa_m, \mathbf{z}_m; \mathbf{v}_m, \mathbf{x})$ where particular attention must be paid. The key result is reported in:

Lemma 7.8. *The sequence* $(Q_n(\lambda_m, \mathbf{z}_m, \mathbf{x}; \mathbf{v}_m))_{m\in\mathbb{N}}$ *converges to* $Q_n(\lambda, \mathbf{z}, \mathbf{x}; \mathbf{v})$ *in the sense of the* L^1 *weak star topology, so that in particular*

$$\forall\, l \in \mathscr{D}(\Omega), \quad \lim_{m\to\infty} (Q_n(\lambda_m, \mathbf{z}_m; \mathbf{v}_m, \mathbf{x}), l)_\Omega = (Q_n(\lambda, \mathbf{z}; \mathbf{v}, \mathbf{x}), l)_\Omega.$$

The proof of Lemma 7.8 is postponed a little further. This result shows that $(\mathbf{v}, \kappa)$ satisfies (7.66), with tests in $\mathscr{D}(\Omega)$. All the operators involved in (7.66) are in $H^{-1}(\Omega)$, in particular $Q_n(\lambda, \mathbf{z}; \mathbf{v}, \mathbf{x})$ by estimate (7.68). Therefore, $l \in H^1_0(\Omega)$ can be taken as test, since $\mathscr{D}(\Omega)$ is dense in $H^1_0(\Omega)$. Hence we obtain

$$\kappa = \kappa_\eta(\mathbf{z}, \lambda),$$

finishing the proof of Lemma 7.7.

Proof of Lemma 7.8. The proof is divided in three substeps, where the energy method is used once again.

SUBSTEP 3.1. *Separation.* The source Q_n is separated into a product of two terms. To see this, consider the following matrix fields:

$$\Lambda_m = \sqrt{\widetilde{\nu_t}(\lambda_m, \mathbf{z}_m, \mathbf{x})}\, D\mathbf{v}_m, \quad \Lambda = \sqrt{\widetilde{\nu_t}(\lambda, \mathbf{z}, \mathbf{x})} D\mathbf{v}, \tag{7.72}$$

$$\Psi_m = \sqrt{\widetilde{\nu_t}(\lambda_m, \mathbf{z}_m, \mathbf{x})}\, D\mathbf{z}_{m,n}, \quad \Psi = \sqrt{\widetilde{\nu_t}(\lambda, \mathbf{z}, \mathbf{x})}\, D\mathbf{z}_n, \tag{7.73}$$

where $\mathbf{z}_{m,n} = \mathbf{z}_m \star \rho_n$ and, according to the conventions above, $\mathbf{z}_n = \mathbf{z} \star \rho_n$. This allows us to write

$$Q_n(\lambda_m, \mathbf{z}_m; \mathbf{v}_m, \mathbf{x}) = \Lambda_m : \Psi_m, \quad Q_n(\lambda, \mathbf{z}; \mathbf{v}, \mathbf{x}) = \Lambda : \Psi.$$

In order to conclude, it is enough to show that $(\Lambda_m)_{m\in\mathbb{N}}$ strongly converges to Λ in $L^2(\Omega)^9$, whereas $(\Psi_m)_{m\in\mathbb{N}}$ weakly converges to Ψ in Λ in $L^2(\Omega)^9$.

SUBSTEP 3.2. *Weak convergence*. We verify that for subsequences,

$$\lim_{m\to\infty} \Lambda_m = \Lambda \text{ in } L^2(\Omega)^9 \text{ weak}, \quad \lim_{m\to\infty} \Psi_m = \Psi \text{ in } L^2(\Omega)^9 \text{ weak}.$$

The uniqueness of the limits ensures the convergence of the full sequences. The sequence $(\Lambda_m)_{m\in\mathbb{N}}$ is treated following the same arguments as those used to prove (7.50), (7.51), and (7.52) in Lemma 7.5,[10] combined to the L^2 weak convergence of $(D\mathbf{v}_m)_{m\in\mathbb{N}}$ to $D\mathbf{v}$.

Moreover, the properties of the convolution operator listed in Sect. 7.4.1 ensure that $(D\mathbf{z}_{m,n})_{m\in\mathbb{N}}$ weakly converges in $L^2(\Omega)^9$ to $D\mathbf{z}_n$, hence the weak convergence of $(\Psi_m)_{m\in\mathbb{N}}$, again by the arguments used to prove Lemma 7.5 (we also might apply Lemma A.14 in [TB]).

SUBSTEP 3.3. *The energy method*. It remains to show the strong L^2 convergence of $(\Lambda_m)_{m\in\mathbb{N}}$ to Λ. From the above, it suffices to show

$$\lim_{m\to\infty} ||\Lambda_m||_{0,2,\Omega} = ||\Lambda||_{0,2,\Omega}.$$

As we already know from Step 2 that the fluid equation (7.64) is satisfied by $(\mathbf{v}, p, \kappa)$, we can start from the energy equalities, written in the form

$$\begin{aligned} \nu||D\mathbf{v}_m||^2_{0,2,\Omega} + \eta||\mathbf{v}_m||^2_{0,2,\Gamma} + ||\Lambda_m||^2_{0,2,\Omega} &= \langle \mathbf{f}, \mathbf{v}_m\rangle - \langle \mathscr{G}_{\mathbf{z}_m}(\mathbf{v}_m), \mathbf{v}_m\rangle, \\ \nu||D\mathbf{v}||^2_{0,2,\Omega} + \eta||\mathbf{v}||^2_{0,2,\Gamma} + ||\Lambda||^2_{0,2,\Omega} &= \langle \mathbf{f}, \mathbf{v}\rangle - \langle \mathscr{G}_{\mathbf{z}}(\mathbf{v}), \mathbf{v}\rangle. \end{aligned} \tag{7.74}$$

Furthermore

(i) by the proof of Lemma 6.15, it is known that

$$\lim_{m\to\infty} \langle \mathscr{G}_{\mathbf{z}_m}(\mathbf{v}_m), \mathbf{v}_m\rangle = \langle \mathscr{G}_{\mathbf{z}}(\mathbf{v}), \mathbf{v}\rangle, \tag{7.75}$$

(ii) the L^2 weak convergence of $(\Lambda_m)_{m\in\mathbb{N}}$ leads to

$$||\Lambda||^2_{0,2,\Omega} \le \liminf_{m\to\infty} ||\Lambda_m||^2_{0,2,\Omega}, \tag{7.76}$$

(iii) $(\nu||D\mathbf{w}||^2_{0,2,\Omega} + \eta||\mathbf{w}||^2_{0,2,\Gamma})^{1/2}$ is an Hilbertian norm equivalent to $||\mathbf{w}||_{\mathbf{W}(\Omega)}$, so that the weak convergence of $(\mathbf{v}_m)_{m\in\mathbb{N}}$ in $\mathbf{W}(\Omega)$ yields

[10] The inverse Lebesgue Theorem, the Lebesgue theorem, $\widetilde{\nu_t} \in L^\infty(\mathbb{R}\times\mathbb{R}^3\times\Omega)$ and its continuity, the sequences that we consider converge *a.e.* in Ω to their respective limits.

$$\nu||D\mathbf{v}||^2_{0,2,\Omega} + \eta||\mathbf{v}||^2_{0,2,\Gamma} \le \liminf_{m\to\infty}(\nu||D\mathbf{v}_m||^2_{0,2,\Omega} + \eta||\mathbf{v}_m||^2_{0,2,\Gamma}). \tag{7.77}$$

By combining (7.74) and (7.75), we deduce

$$\lim_{m\to\infty}(\nu||D\mathbf{v}_m||^2_{0,2,\Omega} + \eta||\mathbf{v}_m||^2_{0,2,\Gamma} + ||\Lambda_m||^2_{0,2,\Omega}) = \nu||D\mathbf{v}||^2_{0,2,\Omega} + \eta||\mathbf{v}||^2_{0,2,\Gamma} + ||\Lambda||^2_{0,2,\Omega},$$

which we combine with (7.76) and (7.76) to get[11]

$$\lim_{m\to\infty}||\Lambda_m||^2_{0,2,\Omega} = ||\Lambda||^2_{0,2,\Omega},$$
$$\lim_{m\to\infty}(\nu||D\mathbf{v}_m||^2_{0,2,\Omega} + \eta||\mathbf{v}_m||^2_{0,2,\Gamma}) = \nu||D\mathbf{v}||^2_{0,2,\Omega} + \eta||\mathbf{v}||^2_{0,2,\Gamma},$$

since each quantity is nonnegative. By consequence, the convergence of $(\mathbf{v}_m)_{m\in\mathbb{N}}$ and $(\Lambda_m)_{m\in\mathbb{N}}$ to their respective limits is strong in $\mathbf{W}(\Omega)$ and $L^2(\Omega)^9$, respectively, as announced; hence, $(\mathbf{v}, p, \kappa) = \mathscr{E}_\eta(\mathbf{z}, \lambda)$, which concludes the proof. □

Remark 7.6. At this level, we can explain why convolution was chosen to regularize the source term, rather than truncation. This is due to the linearization principle which requires the weak convergence of $(\Psi_m)_{m\in\mathbb{N}}$ expressed by (7.73). This would not work with a truncation, since truncation and weak convergence are not good friends. However, the truncation can be used with a standard Galerkin method, as in Chaps. 8 and 12, since we directly prove the strong convergence of the approximate solutions and use the technical Lemma A.16 in [TB], which links truncature and L^1 strong convergence.

7.4.3.2 Fixed-Point Process

We endow the product space $\mathbf{W}(\Omega) \times H^1_0(\Omega)$ with the norm $||(\mathbf{z}, \lambda)|| = ||\mathbf{z}||_{\mathbf{W}(\Omega)} + ||\lambda||_{H^1_0(\Omega)}$. We focus now on the application

$$\mathscr{Z}_\eta : \begin{cases} \mathbf{W}(\Omega) \times H^1_0(\Omega) \to \mathbf{W}(\Omega) \times H^1_0(\Omega), \\ (\mathbf{z}, \lambda) \to (\mathbf{v}_\eta(\mathbf{z}, \lambda), \kappa_\eta(\mathbf{z}, \lambda)). \end{cases} \tag{7.78}$$

Let $(\mathbf{v}, \kappa)$ be any fixed point of $\mathscr{Z}_\eta$. It is understood that $(\mathbf{v}, p_\eta(\mathbf{v}, \kappa), \kappa)$ is a solution of $\mathscr{VP}^\kappa_{n,\eta}$. The following compactness property holds true.

Lemma 7.9. *Let $(\mathbf{z}_m, \lambda_m)_{m\in\mathbb{N}}$ be weakly convergent to $(\mathbf{z}, \kappa)$ in $\mathbf{W}(\Omega) \times H^1_0(\Omega)$, $(\mathbf{v}_m, \kappa_m) = \mathscr{Z}_\eta(\mathbf{z}_m, \lambda_m)$. Then $(\mathbf{v}_m, \kappa_m)_{m\in N}$ strongly converges in $\mathbf{W}(\Omega) \times H^1_0(\Omega)$ to $(\mathbf{v}, \kappa) = \mathscr{Z}_\eta(\mathbf{z}, \lambda)$.*

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[11]We use the general trivial result: if $a_n, b_n \ge 0$, $a_n + b_n \to a + b$, $\liminf a_n \le a$, $\liminf b_n \le b$, then $a_n \to a$ and $b_n \to b$.

Proof. We already know from the proof of Lemma 7.7 above that $(\mathbf{v}_m)_{m\in\mathbb{N}}$ strongly converges to $\mathbf{v} = \mathbf{v}_\eta(\mathbf{z}, \lambda)$ in $\mathbf{W}(\Omega)$. We also know that $(\kappa_m)_{m\in\mathbb{N}}$ weakly converges in $H_0^1(\Omega)$ to $\kappa = \kappa_\eta(\mathbf{z}, \lambda)$. It remains to prove that this last convergence is strong. To do so, we use the energy method to prove

$$\lim_{m\to\infty} ||\kappa_m||_{H_0^1(\Omega)} = ||\kappa||_{H_0^1(\Omega)}. \tag{7.79}$$

We start from the energy equalities. Recall that κ_m satisfies for all $l \in H_0^1(\Omega)$,

$$\langle \mathscr{K}_{\mathbf{z},\lambda}^{(\kappa)}(\kappa_m), l\rangle + e(\lambda, \mathbf{z}; \kappa_m, l) + B(\lambda, \mathbf{z}; \mathbf{v}_\eta), l) = (Q_n(\lambda, \mathbf{z}, \mathbf{x}; \mathbf{v}_\eta), l)_\Omega. \tag{7.80}$$

We take $l = \kappa_m \in H_0^1(\Omega)$ in (7.80), $l = \kappa$ in the limit equation. Then we use the identities $b_e(\mathbf{z}_m; \kappa_m, \kappa_m) = b_e(\mathbf{z}; \kappa, \kappa) = 0$, which yield the energy equalities

$$\begin{cases} \mu||\kappa_m||^2_{H_0^1(\Omega)} + t_e(\lambda_m, \mathbf{z}_m; \kappa_m, \kappa_m) + e(\lambda_m, \mathbf{z}_m; \kappa_m, \kappa_m) & \\ \qquad\qquad +B(\lambda_m, \mathbf{z}_m; \mathbf{v}_m, \kappa_m) = \displaystyle\int_\Omega \kappa_m \Lambda_m : \Psi_m, & \\ \mu||\kappa_|||^2_{H_0^1(\Omega)} + t_e(\lambda, \mathbf{z}; \kappa, \kappa) + e(\lambda, \mathbf{z}; \kappa, \kappa) + B(\lambda, \mathbf{z}; \mathbf{v}, \kappa) = \displaystyle\int_\Omega \kappa \Lambda : \Psi. & \end{cases}$$

After having invoked the VESP and the customized KESP, arguments similar to those in the proofs of Lemmas 6.6 and 7.5, and taking advantage of the $\mathbf{W}(\Omega)$ strong convergence of $(\mathbf{v}_m)_{m\in\mathbb{N}}$, yields (we skip the details to avoid duplication)

$$\lim_{m\to\infty} e(\lambda_m, \mathbf{z}_m; \kappa_m, \kappa_m) = e(\lambda, \mathbf{z}; \kappa, \kappa), \quad \lim_{m\to\infty} B(\lambda_m, \mathbf{z}_m; \mathbf{v}_m, \kappa_m) = B(\lambda, \mathbf{z}; \mathbf{v}, \kappa).$$

We focus on the production term again, which is always a source of difficulty. It is easy to be convinced from the strong L^2 convergence of $(\Lambda_m)_{m\in\mathbb{N}}$ that[12]

$$\forall\, 1 \le p < 4/3, \quad (\kappa_m \Lambda_m)_{m\in\mathbb{N}} \text{ converges to } \kappa\Lambda \text{ in } L^p(\Omega)^9 \text{ strong,}$$

for instance, for $p = 5/4$. Furthermore, following the same argumentation as that in the proof of the estimate (7.68), we find

$$\forall\, m \in \mathbb{N}, \quad ||\Psi_m||_{0,\infty,\Omega} \le n^3 C_\rho ||\nu_t||_\infty \sup_{m'\in\mathbb{N}} ||D\mathbf{z}_{m'}||_{0,2,\Omega} < \infty,$$

hence $(\Psi_m)_{m\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)^9$ and therefore in particular in $L^5(\Omega)^9$, which allows a subsequence to be extracted that converges weakly in $L^5(\Omega)^9$ to some $\tilde{\Psi}$. As we already know that the full sequence weakly converges to Ψ in $L^2(\Omega)^9$, then we necessarily have $\tilde{\Psi} = \Psi$, and the full sequence weakly converges to Ψ in $L^5(\Omega)^9$. In other words,

[12] Λ_m and Ψ_m are expressed by (7.72) and (7.73).

$$\lim_{m\to\infty}\int_\Omega \kappa_m \Lambda_m : \Psi_m = \int_\Omega \kappa \Lambda : \Psi.$$

The convergence results established above allow us to deduce from the last energy equalities the following convergence,[13]

$$\lim_{m\to\infty}(\mu||\kappa_m||^2_{H^1_0(\Omega)} + t_e(\lambda_m, \mathbf{z}_m; \kappa_m, \kappa_m)) = \mu||\kappa||^2_{H^1_0(\Omega)} + t_e(\lambda, \mathbf{z}; \kappa, \kappa),$$

as expected. Finally, we deduce from the weak convergence of $(\kappa_m)_{m\in\mathbb{N}}$ in $H^1_0(\Omega)$, the *a.e.* convergence of $(\mathbf{z}_m)_{m\in\mathbb{N}}$ and $(\lambda_m)_{m\in\mathbb{N}}$, the continuity of $\widetilde{\mu_t}$ which is bounded,

$$\lim_{m\to\infty}\widetilde{\mu_t}(\lambda_m, \mathbf{z}_m, \mathbf{x})^{1/2}\nabla\kappa_m = \widetilde{\mu_t}(\lambda, \mathbf{z}, \mathbf{x})^{1/2}\nabla\kappa \ \text{ in } \ L^2(\Omega)^3 \ \text{ weak}$$

leading to

$$||\kappa||^2_{H^1_0(\Omega)} \le \liminf_{m\to\infty}||\kappa_m||^2_{H^1_0(\Omega)}, \quad t_e(\lambda, \mathbf{z}; \kappa, \kappa) \le \liminf_{m\to\infty}(t_e(\lambda_m, \mathbf{z}_m; \kappa_m, \kappa_m)),$$

hence, all terms being nonnegative, we obtain (7.79), among other things, concluding the proof. □

Corollary 7.2. *The application $\mathscr{L}_\eta$ defined by (7.78) is continuous.*

7.4.4 Fixed-Point Process and Convergence

7.4.4.1 Proof of Theorem 7.3

To avoid duplication, we summarize in this subsection different parts of the proof, already detailed above in one form or another. The existence of a solution to $\mathscr{V}\mathscr{P}^\kappa_{n,\eta}$,[14] stated by Theorem 7.3, results from:

Lemma 7.10. *Let $\eta > 0$ be fixed. The application $\mathscr{L}_\eta$ specified by (7.78) has a fixed point.*

Proof. We follow the outline of Sect. 6.6, to verify that $\mathscr{L}_\eta$ fulfills the conditions for the application of the Schauder fixed-point theorem, Theorem A.6 in [TB].

The polynomial function $R_{\kappa,n,\eta}$ given by (7.71) provides the bound for κ_η, while $C_{v,\eta}$ is the constant defined by (6.102) which bounds $\mathbf{v}_\eta$. Let R denote the constant

$$R = C_{v,\eta} + R_{\kappa,n,\eta}(C_{v,\eta}).$$

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[13] $t_e(\lambda, \mathbf{z}; \kappa, \kappa) = ||\widetilde{\mu_t}(\lambda, \mathbf{z}, \mathbf{x})^{1/2}\nabla\kappa||^2_{0,2,\Omega}$ according to the expression of t_e given by (7.8).

[14] $\mathscr{V}\mathscr{P}^\kappa_n$ was introduced in Sect. 7.4.1 and is expressed by (7.58)–(7.60), $\mathscr{V}\mathscr{P}^\kappa_{n,\eta}$ by (7.58), (7.59), and (7.62).

Estimates (7.65) and (7.70) yield

$$\mathscr{Z}_\eta(B_R) \subset B_R,$$

with B_R convex in $\mathbf{W}(\Omega) \times H_0^1(\Omega)$. We know from Corollary 7.2 that $\mathscr{Z}_\eta$ is continuous. Furthermore, the same reasoning as in the proof of Lemma 6.16 combined with Lemma 7.9 ensures that $\mathscr{Z}_\eta(B_R)$ is indeed compact. The conditions for the application of the Schauder theorem are thus fulfilled, which concludes the proof. □

7.4.4.2 Proof of Theorem 7.4

According to Lemma 6.9, Theorem 7.4 is established once we have proved:

Lemma 7.11. *Let $n \in \mathbb{N}$ be fixed. The family $(\mathscr{VP}^\kappa_{n,\eta})_{\eta>0}$ converges to $\mathscr{VP}^\kappa_n$ when $\eta \to 0$.*

Proof. Let $(\mathbf{v}_\eta, p_\eta, \kappa_\eta)$ be any solution to $\mathscr{VP}^\kappa_{n,\eta}$, implying that $\forall\, (\mathbf{w}, l) \in \mathbf{W}(\Omega) \times H_0^1(\Omega)$,

$$\begin{cases} \langle \mathscr{T}^{(\kappa)}(\mathbf{v}_\eta, \kappa_\eta), \mathbf{w}\rangle - (p_\eta, \nabla \cdot \mathbf{w})_\Omega + \langle G(\mathbf{v}_\eta) + \eta \mathbf{v}_\eta, \mathbf{w}\rangle = \langle \mathbf{f}, \mathbf{w}\rangle, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\nabla \cdot \mathbf{v}_\eta, q)_\Omega = 0, \\ \langle \mathscr{K}^{(\kappa)}(\mathbf{v}_\eta, \kappa_\eta), l\rangle + e(\kappa_\eta, \mathbf{v}_\eta; \kappa_\eta, l) + B(\kappa_\eta, \mathbf{v}_\eta; \mathbf{v}_\eta, l) = \mathscr{Q}_{n,\eta}, \end{cases} \tag{7.81}$$

by writing

$$\mathscr{Q}_{n,\eta} = (Q_n(\kappa_\eta, \mathbf{v}_\eta; \mathbf{v}_\eta, \mathbf{x}), l)_\Omega.$$

As usual we

(a) derive estimates uniform in η,
(b) apply the ESPs and take the limit.

STEP 1. *Estimates.* Recycling the proof of Lemma 6.17, by replacing $\nu_t(\mathbf{x})$ by $\widetilde{\nu_t}(\kappa_\eta(\mathbf{x}), \mathbf{v}_\eta(\mathbf{x}), \mathbf{x})$ which is nonnegative and in $L^\infty(\Omega)$, it is easily established that

$$||\mathbf{v}_\eta||_{\mathbf{W}(\Omega)} \le C_v, \quad ||p_\eta||_{0,2,\Omega} \le \widetilde{C_p}, \tag{7.82}$$

[see estimates (6.117) and (6.118)] so that the sequence $(\mathbf{v}_\eta, p_\eta)_{\eta>0}$ is bounded in $\mathbf{W}(\Omega) \times L^2_0(\Omega)$.

We now must estimate $||\kappa_\eta||_{H_0^1(\Omega)}$. To do so, taking $l = \kappa_\eta$ in (7.81.iii), we have

$$\mu ||\kappa_\eta||_{H_0^1(\Omega)} \le |B(\kappa_\eta, \mathbf{v}_\eta; \mathbf{v}_\eta, \kappa_\eta)| + |(Q_n(\kappa_\eta, \mathbf{v}_\eta, \mathbf{x}; \mathbf{v}_\eta), \kappa_\eta)_\Omega|, \tag{7.83}$$

since

$$b_e(\mathbf{v}_\eta;\kappa_\eta,\kappa_\eta) = 0, \quad t_e(\kappa_\eta,\mathbf{v}_\eta;\kappa_\eta,\mathbf{v}_\eta) \geq 0, \quad \text{and} \quad e(\kappa_\eta,\mathbf{v}_\eta;\kappa_\eta,\mathbf{v}_\eta) \geq 0.$$

Estimate (7.26) combined with (7.82) leads to

$$|B(\kappa_\eta,\mathbf{v}_\eta;\mathbf{v}_\eta,\kappa_\eta)| \leq P_2(C_v)||\kappa_\eta||_{H_0^1(\Omega)}, \tag{7.84}$$

P_2 being a second-order polynomial function which does not depend on η. A similar reasoning to that providing (7.68) leads to

$$|(Q_n(\kappa_\eta,\mathbf{v}_\eta;\mathbf{v}_\eta,\mathbf{x}),\kappa_\eta)_\Omega| \leq n^3\, C_\rho||\nu_t||_\infty||\mathbf{v}_\eta||^2||\kappa_\eta||_{H_0^1(\Omega)},$$

which yields by (7.82):

$$|(Q_n(\kappa_\eta,\mathbf{v}_\eta;\mathbf{v}_\eta,\mathbf{x}),\kappa_\eta)_\Omega| \leq n^3 C_\rho||\nu_t||_\infty C_v^2||\kappa_\eta||_{H_0^1(\Omega)}. \tag{7.85}$$

In summary, we combine (7.83)–(7.85) to obtain

$$||\kappa_\eta||_{H_0^1(\Omega)} \leq \frac{C}{\mu}((1+n^3)C_v^2 + C_v + 1), \tag{7.86}$$

the constant C being function of the data. The sequence $(\kappa_\eta)_{\eta>0}$ is therefore bounded in $H_0^1(\Omega)$.

STEP 2. *ESP and limit.* From here, we can apply the V-, P-, KESP to the sequences $(\mathbf{v}_\eta)_{\eta>0}$, $(p_\eta)_{\eta>0}$, and $(\kappa_\eta)_{\eta>0}$, and we denote by $\mathbf{v}$, p, and k the V-, P-, K-limits when $\eta \to 0$.

To take the limit in (7.62.i), we proceed as in the proof of Lemma 6.17, enhanced by the convergence of Lemma 7.5, to treat the term $t_v(\kappa_\eta,\mathbf{v}_\eta;\mathbf{v}_\eta,\mathbf{w})$. This shows that $(\mathbf{v},p,k)$ satisfies (7.60.i).

Moreover, arguing as in the proof of Lemma 7.8 above, we observe that $(\mathbf{v}_\eta)_{\eta>0}$ strongly converges to $\mathbf{v}$ in $\mathbf{W}(\Omega)$. Taking the limit in (7.62.ii) to arrive at (7.60.ii) is straightforward.

Finally, the limit in (7.62.iii) is taken by applying the convergence Lemma 7.5, using the strong convergence of $(\mathbf{v}_\eta)_{\eta>0}$, and follows the same procedure as in the proof of the Lemma 7.7.

Hence, $(\mathbf{v},\kappa)$ satisfies (7.60.iii), and in conclusion, $(\mathbf{v},p,\kappa)$ is a solution to $\mathscr{V}\mathscr{P}_n^\kappa$, which completes this proof. □

It is worth noting that $(\mathbf{v},p,k)$, where $k = \kappa + k_\Gamma(\mathbf{v}) \in H^1(\Omega)$, is such that $\forall\,(\mathbf{w},l) \in \mathbf{W}(\Omega) \times H_0^1(\Omega)$,

$$\begin{cases} \langle \mathscr{T}^{(k)}(\mathbf{v},k),\mathbf{w}\rangle - (p,\nabla\cdot\mathbf{w})_\Omega + \langle G(\mathbf{v}),\mathbf{w}\rangle = \langle \mathbf{f},\mathbf{w}\rangle, \\ (\nabla\cdot\mathbf{v},q)_\Omega = 0, \\ \langle \mathscr{K}^{(k)}(\mathbf{v},k),\mathbf{w}\rangle + d(k;k,l) = (\mathbb{P}_n(k,\mathbf{v},\mathbf{x}),l)_\Omega \end{cases} \tag{7.87}$$

by writing

$$\mathbb{P}_n(k, \mathbf{v}, \mathbf{x}) = \nu_t(k, \mathbf{x}) D\mathbf{v}_n : D\mathbf{v}, \quad \mathbf{v}_n = \mathbf{v} \star \rho_n. \tag{7.88}$$

We call $\mathscr{V}\mathscr{P}_n^k$ the variational problem, which is associated with the approximate NS-TKE model (7.54). The results above show the following theorem.

Theorem 7.5. *Let $n \in \mathbb{N}$ be a fixed integer. $\mathscr{V}\mathscr{P}_n^k$ admits a solution $(\mathbf{v}, p, k) \in \mathbf{W}(\Omega) \times L_0^2(\Omega) \times H^1(\Omega)$, such that $k = k_\Gamma(\mathbf{v})$ on Γ. In other words, the approximate NS-TKE model (7.54) has a weak solution.*

Uniqueness remains an open problem, even when the data satisfy additional conditions, such as those considered in Sect. 6.8.

7.5 Convergence to TKE, Extra Properties, and Conclusion

This section starts by proving the convergence of $(\mathscr{V}\mathscr{P}_n^\kappa)_{n\in\mathbb{N}}$ to $\mathscr{V}\mathscr{P}^\kappa$, which will conclude the proof of Theorem 7.1, that is, the existence of a solution to $\mathscr{V}\mathscr{P}^\kappa$ and incidentally the existence of a weak solution to the NS-TKE model (7.1), when Hypothesis 7.i holds.

The maximum principle is then investigated. This consists in proving that when $k_\Gamma(\mathbf{v}) \geq 0$ a.e. on Γ, then given any weak solution to the NS-TKE model (7.1), $(\mathbf{v}, p, k)$, constructed by approximation as we have done above, is such that k is nonnegative on Ω.

To conclude this section we consider the realistic case where $\nu_t = \ell(\mathbf{x})\sqrt{k}$ and $\mu_t = C_\mu \ell(\mathbf{x})\sqrt{k}$, which we approximate by continuous bounded eddy viscosities $(\nu_t^{(n)})_{n\in\mathbb{N}}$, $(\mu_t^{(n)})_{n\in\mathbb{N}}$. We show that the corresponding sequence of variational problems, denoted by $(\mathscr{V}\mathscr{P}^{\kappa,n})_{n\in\mathbb{N}}$, converges to a problem $\mathscr{V}\mathscr{P}_{lim}^\kappa$, in which the fluid equation is preserved, but the TKE equation becomes a variational inequality.

7.5.1 *Convergence Result*

It is worth noting that the solutions to $\mathscr{V}\mathscr{P}^\kappa$ constructed as limits of solutions to $\mathscr{V}\mathscr{P}_n^\kappa$ satisfy the estimates derived in the Sect. 7.2.4. The general result we obtain is summarized in the next statement, including the proof of Theorem 7.1.

Theorem 7.6. *The sequence $(\mathscr{V}\mathscr{P}_n^\kappa)_{n\in\mathbb{N}}$ converges to $\mathscr{V}\mathscr{P}^\kappa$. Moreover, let $(\mathbf{v}, p, \kappa)$ be any solution to $\mathscr{V}\mathscr{P}^\kappa$ constructed as a limit of solutions to $\mathscr{V}\mathscr{P}_n^\kappa$ when $n \to \infty$. Then $(\mathbf{v}, p, \kappa)$ satisfies the estimates (7.28)–(7.30).*

Proof. Let $(\mathbf{v}_n, p_n, \kappa_n) \in \mathbf{W}(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega)$ be any solution to $\mathscr{V}\mathscr{P}_n^\kappa$, which means $\forall\, (\mathbf{w}, l) \in \mathbf{W}(\Omega) \times H_0^1(\Omega)$,

$$\begin{cases} \langle \mathscr{T}^{(\kappa)}(\mathbf{v}_n, \kappa_n), \mathbf{w} \rangle - (p_n, \nabla \cdot \mathbf{w})_\Omega + \langle G(\mathbf{v}_n), \mathbf{w} \rangle = \langle \mathbf{f}, \mathbf{w} \rangle, \\ (\nabla \cdot \mathbf{v}_n, q)_\Omega = 0, \\ \langle \mathscr{K}^{(\kappa)}(\mathbf{v}_n, \kappa_n), l \rangle + e(\kappa_n, \mathbf{v}_n; \kappa_n, l) + B(\kappa_n, \mathbf{v}_n; \mathbf{v}_n, l) = (S_n, l)_\Omega, \end{cases} \tag{7.89}$$

by writing

$$\begin{aligned} S_n = S_n(\mathbf{x}) &= Q_n(\kappa(\mathbf{x}), \mathbf{v}(\mathbf{x}); \mathbf{v}(\mathbf{x}), \mathbf{x}) \\ &= \widetilde{\nu_t}(\kappa_n(\mathbf{x}), \mathbf{v}_n(\mathbf{x}), \mathbf{x})\, D\mathbf{v}_n(\mathbf{x}) : D\mathbf{v}_n \star \rho_n(\mathbf{x}) \in L^2(\Omega). \end{aligned} \tag{7.90}$$

Following the usual basic pattern, the outline of the proof is:

(i) proving that $(\mathbf{v}_n, p_n, \kappa_n)_{n\in\mathbb{N}}$ is bounded in $\mathbf{W}(\Omega) \times L^2_0(\Omega) \times \mathbf{K}_{3/2}(\Omega)$,
(ii) proving that any ESP-limit of $(\mathbf{v}, p, \kappa)$ is a solution to $\mathscr{VP}^\kappa$.

STEP 1. *Estimates.* Based on the foregoing discussion, it is clear that $(\mathbf{v}_n, p_n)_{n\in\mathbb{N}}$ is bounded in $\mathbf{W}(\Omega) \times L^2_0(\Omega)$ and satisfies

$$||\mathbf{v}_n||_{\mathbf{W}(\Omega)} \le C_v, \quad ||p_n|| \le C_p, \tag{7.91}$$

where C_v and C_p are the bounds defined by (7.28) and (7.29).
However, the only currently available bound for $(\kappa_n)_{n\in\mathbb{N}}$ is estimate (7.86), which blows up when $n \to \infty$. To find a bound uniform in n, we remark that $\mathscr{VP}^\kappa_n$ is the same variational problem as $\mathscr{UP}^{(\kappa)}_n$, considered in Sect. 7.2.5, which suggests applying Proposition 7.3. To do so, it is enough to check that $(S_n)_{n\in\mathbb{N}}$ is bounded in $L^1(\Omega)$. The Cauchy–Schwarz inequality and the convolution norms inequality (7.57) and (7.91) then yield

$$\begin{aligned} ||S_n||_{0,1,\Omega} &\le ||\nu_t||_\infty ||D\mathbf{v}_n||_{0,2,\Omega} ||D\mathbf{v}_n \star \rho_n||_{0,2,\Omega} \\ &\le ||\nu_t||_\infty ||D\mathbf{v}_n||^2_{0,2,\Omega} \\ &\le ||\nu_t||_\infty C_v^2 = \sigma_1. \end{aligned}$$

Hence Proposition 7.3 asserts that for all $1 \le q < 3/2$,

$$||\kappa_n||_{1,q,\Omega} \le C_{\kappa,q}, \tag{7.92}$$

where the constant $C_{\kappa,q}$ is defined by (7.43), proving that $(\kappa_n)_{n\in\mathbb{N}}$ is bounded in $\mathbf{K}_{3/2}(\Omega)$.[15]

STEP 2. *ESP and limit.* We apply the V-, P-, KESP to $(\mathbf{v}_n)_{n\in\mathbb{N}}$, $(p_n)_{n\in\mathbb{N}}$, and $(\kappa_n)_{n\in\mathbb{N}}$, this time keeping KESP in its initial version as in Definition 7.1. We denote by $(\mathbf{v}, p, \kappa) \in \mathbf{W}(\Omega) \times L^2_0(\Omega) \times \mathbf{K}_{3/2}(\Omega)$ the V-, P-, KESP-limits, and we are left with the task of proving that $(\mathbf{v}, p, \kappa)$ is indeed a solution to $\mathscr{VP}^\kappa$. Let $(\mathbf{w}, q, l) \in \mathbf{W}(\Omega) \times L^2(\Omega) \times \mathbf{Q}_3(\Omega)$ be given. Arguing as in Lemma 7.11, we find

[15] $\mathbf{K}_{3/2}(\Omega)$ and $\mathbf{Q}_3(\Omega)$ are defined by (7.2).

$$\lim_{n\to\infty}(\langle \mathcal{T}^{(\kappa)}(\mathbf{v}_n,\kappa_n),\mathbf{w}\rangle - (p_n,\nabla\cdot\mathbf{w})_\Omega + \langle G(\mathbf{v}_n),\mathbf{w}\rangle) = \\ \langle \mathcal{T}^{(\kappa)}(\mathbf{v},\kappa),\mathbf{w}\rangle - (p,\nabla\cdot\mathbf{w})_\Omega + \langle G(\mathbf{v}),\mathbf{w}\rangle, \tag{7.93}$$

which proves that (7.16.i) holds. It is easily verified that

$$\lim_{n\to\infty}(\nabla\cdot\mathbf{v}_n,q)_\Omega = (\nabla\cdot\mathbf{v},q)_\Omega, \tag{7.94}$$

hence (7.16.ii).

It remains to take the limit in (7.89.iii). By the convergence Lemma 7.5, we obtain

$$\lim_{n\to\infty}(\langle \mathcal{K}^{(\kappa)}(\mathbf{v}_n,\kappa_n),l\rangle + e(\kappa_n,\mathbf{v}_n;\kappa_n,l) + B(\kappa_n,\mathbf{v}_n;\mathbf{v}_n,l)) = \\ \langle \mathcal{K}^{(\kappa)}(\mathbf{v},\kappa),l\rangle + e(\kappa,\mathbf{v};\kappa,l) + B(\kappa,\mathbf{v};\mathbf{v},l). \tag{7.95}$$

The problem here is to take the limit in the source term $(S_n,l)_\Omega$. Arguing as in the proof of Lemma 7.8, we separate S_n by introducing

$$\Lambda_n = \sqrt{\widetilde{\nu_t}(\kappa_n,\mathbf{v}_n,\mathbf{x})}\,D\mathbf{v}_n, \quad \Psi_n = \sqrt{\widetilde{\nu_t}(\kappa_n,\mathbf{v}_n,\mathbf{x})}\,D\mathbf{v}_n \star \rho_n,$$

so that

$$S_n = \Lambda_n : \Psi_n.$$

In order to conclude, we must prove

$$\lim_{n\to\infty} S_n = |\Lambda|^2 \ \text{ in } \ L^1(\Omega), \ \text{ where } \ \Lambda = \sqrt{\widetilde{\nu_t}(\kappa,\mathbf{v},\mathbf{x})}\,D\mathbf{v} \in L^2(\Omega)^9. \tag{7.96}$$

By reproducing step by step the proof of Lemma 7.8, we establish by the energy method that $(\Lambda_n)_{n\in\mathbb{N}}$ strongly converges to Λ in $L^2(\Omega)^9$. Similarly, we can also easily check that $(D\mathbf{v}_n)_{n\in\mathbb{N}}$ converges strongly in L^2 to $D\mathbf{v}$, which combined with the technical Lemma 7.12 below also yields the strong convergence in L^2 of $(D\mathbf{v}_n \star \rho_n)_{n\in\mathbb{N}}$ toward $D\mathbf{v}$.

It requires little extra effort to show the strong L^2 convergence of $(\Psi_n)_{n\in\mathbb{N}}$ to Ψ, based on the *a.e.* convergence of $(\mathbf{v}_n)_{n\in\mathbb{N}}$ and $(\kappa_n)_{n\in\mathbb{N}}$ to $\mathbf{v}$ and κ, and the fact that $\widetilde{\nu_t}$ is a bounded continuous function. Hence (7.96) holds. To conclude, as $l \in \mathbf{Q}_3(\Omega) \subset L^\infty(\Omega)$, we deduce from the L^1 convergence of $(S_n)_{n\in\mathbb{N}}$,

$$\lim_{n\to\infty}(S_n,l)_\Omega = (Q(\kappa,\mathbf{v};\mathbf{v},\mathbf{x}),l)_\Omega,$$

which combined with (7.114) yields (7.16.iii).

To sum up, any ESP-limit $(\mathbf{v},p,\kappa)$ of $(\mathbf{v}_n,p_n,\kappa_n)_{n\in\mathbb{N}}$ is a solution to $\mathcal{VP}^\kappa$, which concludes this proof as well as the proof of Theorem 7.1. □

It remains to prove the technical lemma that we have used in the above.

Lemma 7.12. *Let $(f_n)_{n\in\mathbb{N}}$ a sequence in $L^2(\Omega)$ which converges to f in $L^2(\Omega)$. Then $(f_n \star \rho_n)_{n\in\mathbb{N}}$ converges to f in $L^2(\Omega)$.*

Proof. We have

$$||f_n \star \rho_n - f||_{0,2,\Omega} \leq ||f - f \star \rho_n||_{0,2,\Omega} + ||(f_n - f) \star \rho_n||_{0,2,\Omega}.$$

On the one hand, it is known that $||f - f \star \rho_n||_{0,2,\Omega} \to 0$ when $n \to \infty$ (see in [14]). Moreover, we also know

$$||(f_n - f) \star \rho_n||_{0,2,\Omega} \leq ||f_n - f||_{0,2,\Omega} \to 0 \text{ as } n \to \infty,$$

hence the result. □

7.5.2 Maximum Principle

Our aim here is to prove that the TKE given by $\mathscr{V}\mathscr{P}^k$ remains nonnegative, which is the least we can expect from an energy.

Before giving any results, we first need to lay out the basic framework for the maximum principle. Let $u \in H_0^1(\Omega)$ that can be decomposed as

$$u = u^+ - u^-, \quad u^+ = \sup(u, 0), \quad u^- = \sup(-u, 0).$$

A famous result by Stampacchia [50] states that both u^+ and u^- are in $H_0^1(\Omega)$ and are orthogonal, which means

$$(u^+, u^-)_\Omega = (\nabla u^+, \nabla u^-)_\Omega = 0 \text{ and } \mathrm{supp}(u^+) \cap \mathrm{supp}(u^+) = \{u = 0\}. \quad (7.97)$$

Moreover, $u \geq 0$ *a.e.* in Ω if and only if $u^- = 0$, and

$$\forall\, a \in \mathbb{R}, \quad \nabla u = 0 \text{ a.e. on } \{x \in \Omega;\, u(x) = a\}.$$

We shall also need the truncation function T_j at height $j > 0$, expressed by

$$T_j(x) = \begin{cases} x & \text{if } |x| \leq j, \\ T_j(x) = \mathrm{sg}(x)\, j & \text{if } |x| > j. \end{cases} \quad (7.98)$$

We are now able to prove:

Theorem 7.7. *Assume that $\forall\, \mathbf{z} \in \mathbb{R}^3$, $k_\Gamma(\mathbf{z}) \geq 0$. Let $(\mathbf{v}, p, k)$ be a weak solution to $\mathscr{V}\mathscr{P}^k$, $(\mathbf{v}, p, \kappa)$ the corresponding solution to $\mathscr{V}\mathscr{P}^\kappa$, such that*

$$(\mathbf{v}, p, \kappa) = \lim_{n\to\infty} (\mathbf{v}_n, p_n, \kappa_n),$$

where $(\mathbf{v}_n, p_n, \kappa_n)$ is a solution to $\mathscr{V}\mathscr{P}^\kappa_n$. Then $k \geq 0$ a.e. in Ω.

Proof. Such a maximum principle is based on taking $-(k^-)$ as test in (7.16.iii). Unfortunately, we do not know whether $k^- \in \mathbf{Q}_3(\Omega)$, and there is no reason justifying its use as a test. Therefore, we must argue by approximation.

The starting point is (7.87.iii) of $\mathscr{V}\mathscr{P}^k_n$, in which we take $l = T_j(-k_n^-)$ as test, for fixed $n, j \in \mathbb{N}$. This is possible since

$$k_n = k_\Gamma(\mathbf{v}_n) \geq 0 \text{ on } \Gamma, \quad T_j \in W^{1,\infty}(\mathbb{R}), \quad T_j(0) = 0 \text{ then } T_j(-k_n^-) \in H_0^1(\Omega),$$

and in addition, $T_j(-k_n^-) \leq 0$. We obtain[16]

$$\langle \mathscr{K}^{(k)}(\mathbf{v}_n, \kappa_n), T_j(-k_n^-)\rangle + d(k_n; k_n, T_j(-k_n^-)) = (\mathbb{P}_n(k_n, \mathbf{v}_n, \mathbf{x}), T_j(-k_n^-))_\Omega. \tag{7.99}$$

We consider each term one after another. We first observe that since T_j is odd and by the properties of the decomposition $k_n = k_n^+ - k_n^-$,

$$\begin{aligned} b_e(\mathbf{v}_n; k_n, T_j(-k_n^-)) &= b_e(\mathbf{v}_n; k_n^+ - k_n^-, -T_j(k_n^-)) \\ &= b_e(\mathbf{v}_n; k_n^-, T_j(k_n^-)) \\ &= 0, \end{aligned} \tag{7.100}$$

where we have used in the last equality the Lemma 7.3 above. Similarly, we obtain

$$\begin{aligned} a_e(k_n, T_j(-k_n^-)) &= a(k_n^+ - k_n^-, -T_j(k_n^-))) \\ &= a_e(k_n^-, T_j(k_n^-)) \\ &= \mu \int T_j'(\mathbf{x})|\nabla k_n^-(\mathbf{x})|^2 d\mathbf{x} \\ &= \mu ||\nabla T_j(k_n^-)||^2_{0,2,\Omega}, \end{aligned} \tag{7.101}$$

since $(T_j'(\mathbf{x}))^2 = T_j'(\mathbf{x})$. We also find by the same reasoning

$$0 \leq s_e(k_n; k_n, T_j(-k_n^-)), \quad 0 \leq d(k_n; k_n, T_j(-k_n^-)),$$

which leads by (7.99)–(7.101) to

$$\mu ||\nabla T_j(k_n^-)||^2_{0,2,\Omega} \leq -(\mathbb{P}_n(k_n, \mathbf{v}_n, \mathbf{x}), T_j(k_n^-))_\Omega. \tag{7.102}$$

We notice that $(T_j(k_n^-))_{n\in\mathbb{N}}$ weakly converges to $T_j(k^-)$ in $H_0^1(\Omega)$ as well as in L^∞ weak star. We also know that $(\mathbb{P}_n(k_n, \mathbf{v}_n, \mathbf{x}))_{n\in\mathbb{N}}$ converges to $\mathbb{P}(k, \mathbf{v}, \mathbf{x})$ in $L^1(\Omega)$ and is nonnegative; hence, as $k^- \geq 0$,

$$\lim_{n\to\infty} -(\mathbb{P}_n(k_n, \mathbf{v}_n, \mathbf{x}), T_j(k_n^-))_\Omega = -(\mathbb{P}(k, \mathbf{v}, \mathbf{x}), T_j(k^-))_\Omega \leq 0.$$

[16] $\mathscr{K}^{(k)}$ is defined by (7.4).

Therefore, (7.102) leads to

$$\mu||\nabla T_j(k^-)||^2_{0,2,\Omega} \leq \liminf_{n\to\infty} \mu||\nabla T_j(k_n^-)||^2_{0,2,\Omega} \leq 0. \tag{7.103}$$

We deduce that $\forall j \in \mathbb{N}$, $T_j(k^-) = 0$ *a.e.* in Ω, which yields $k^- = 0$ *a.e.* in Ω since $k^- \in L^1(\Omega)$; hence, $k \geq 0$ *a.e.* in Ω. □

7.5.3 Unbounded Eddy Viscosities

According to Remark 7.1, Hypothesis 7.i does not cover the realistic NS-TKE model. In this concluding subsection, we prove an extra result to indicate what can be done to approach the realistic model. The case we study is

$$\nu_t(k, \mathbf{x}) = \ell(\mathbf{x})\sqrt{k}, \quad \mu_t(k, \mathbf{x}) = C_\mu \ell(\mathbf{x})\sqrt{k}, \tag{7.104}$$

where $\ell \in L^\infty(\Omega)$ is a nonnegative function, $C_\mu \geq 0$ a given constant, and $k \geq 0$ *a.e.*. We still assume that $E \in L^\infty(\mathbb{R} \times \Omega)$ and $k_\Gamma \in W^{1,\infty}(\mathbb{R}^3) \cap C^1(\mathbb{R}^3)$.

Before starting, it is worth checking that $\mathscr{VP}$ is meaningful in this case, following Sect. 7.2.2. We naturally seek solutions $(\mathbf{v}, p, k)$ in the space $\mathbf{W}(\Omega) \times L^2_0(\Omega) \times \mathbf{K}_{3/2}(\Omega)$. The terms which differ from those investigated in Sect. 7.2.2 are due to the eddy viscosities. We are therefore left with the products $\sqrt{k}\, D\mathbf{v}$ and $\sqrt{k}\, \nabla k$. Hölder's inequality yields

$$\sqrt{k}\, D\mathbf{v} \in \bigcap_{q<3/2} L^q(\Omega)^9, \quad \sqrt{k}\, \nabla k \in \bigcap_{q<6/5} L^q(\Omega), \tag{7.105}$$

which suggests that the right test spaces for the velocity and the TKE are, respectively,

$$\mathbf{X}_3(\Omega) = \mathbf{W}(\Omega) \cap \left(\bigcup_{r>3} \mathbf{W}^{1,r}(\Omega) \right), \quad \mathbf{Q}_6(\Omega) = \bigcup_{r>6} W_0^{1,r}(\Omega). \tag{7.106}$$

From this, it is natural to approximate ν_t and μ_t by the sequences

$$\nu_t^{(n)}(k, \mathbf{x}) = \ell(\mathbf{x})\sqrt{T_n(k)}, \quad \mu_t^{(n)}(k) = C_\mu \ell(\mathbf{x})\sqrt{T_n(k)}, \tag{7.107}$$

where T_n is the truncation function (7.98).

Let $\mathscr{VP}^{\kappa,n}$ denote the variational problem $\mathscr{VP}^\kappa$, whose eddy viscosities derive from $\nu_t^{(n)}$ and $\mu_t^{(n)}$, which are both bounded and continuous functions. The result that holds at this stage is the following.

Theorem 7.8. *The sequence $(\mathscr{VP}^{\kappa,n})_{n\in\mathbb{N}}$ converges to $\mathscr{VP}^\kappa_{lim}$ expressed by*

$$\textit{Find } (\mathbf{v}, p, \kappa) \in \mathbf{W}(\Omega) \times L^2_0(\Omega) \times \mathbf{K}_{3/2}(\Omega) \textit{ such that} \tag{7.108}$$

$$\textit{for all } (\mathbf{w}, q, l) \in \mathbf{X}_3(\Omega) \times L^2(\Omega) \times \mathbf{Q}_6(\Omega) \textit{ with } l \geq 0 \textit{ a.e.} \tag{7.109}$$

$$\begin{cases} \langle \mathscr{T}^{(\kappa)}(\mathbf{v}, \kappa), \mathbf{w}\rangle - (p, \nabla \cdot \mathbf{w})_\Omega + \langle G(\mathbf{v}), \mathbf{w}\rangle = \langle \mathbf{f}, \mathbf{w}\rangle, \\ (\nabla \cdot \mathbf{v}, q)_\Omega = 0, \\ \langle \mathscr{K}^{(\kappa)}(\mathbf{v}, \kappa), l\rangle + e(\kappa, \mathbf{v}; \kappa, l) + B(\kappa, \mathbf{v}; \mathbf{v}, l) \geq (Q(\kappa, \mathbf{v}, \mathbf{x}), l)_\Omega. \end{cases} \tag{7.110}$$

Proof. We know from Theorem 7.6 that for all n, $\mathscr{VP}^{\kappa,n}$ admits a solution $(\mathbf{v}_n, p_n, \kappa_n)$, which means $\forall\, (\mathbf{w}, l) \in \mathbf{W}(\Omega) \times \mathbf{Q}_3(\Omega)$,

$$\begin{cases} \langle \mathscr{T}^{(\kappa,n)}(\mathbf{v}_n, \kappa_n), \mathbf{w}\rangle - (p_n, \nabla \cdot \mathbf{w})_\Omega + \langle G(\mathbf{v}_n), \mathbf{w}\rangle = \langle \mathbf{f}, \mathbf{w}\rangle, \\ (\nabla \cdot \mathbf{v}_n, q)_\Omega = 0, \\ \langle \mathscr{K}^{(\kappa,n)}(\mathbf{v}_n, \kappa_n), l\rangle + e(\kappa_n, \mathbf{v}_n; \kappa_n, l) + B(\kappa_n, \mathbf{v}_n; \mathbf{v}_n, l) = (S_n, l)_\Omega, \end{cases} \tag{7.111}$$

where $\mathscr{T}^{(\kappa,n)}$ and $\mathscr{K}^{(\kappa,n)}$ are obviously defined. We also know from Theorem 7.7 that $k_n = \kappa_n + k_\Gamma(\mathbf{v}_n) \geq 0$.

Moreover, Theorem 7.6 asserts that $(\mathbf{v}_n, p_n, \kappa_n)$ satisfies the uniform estimates (7.28)–(7.30). Let $(\mathbf{v}, p, \kappa)$ be the V-, P-, KESP-limit, and

$$(\mathbf{w}, q, l) \in \mathbf{X}_3(\Omega) \times L^2(\Omega) \times \mathbf{Q}_6(\Omega) \text{ such that } l \geq 0.$$

Skipping the details for clarity, it is not difficult to check that

$$\begin{aligned} \lim_{n\to\infty} (\langle \mathscr{T}^{(\kappa,n)}(\mathbf{v}_n, \kappa_n), \mathbf{w}\rangle - (p_n, \nabla \cdot \mathbf{w})_\Omega + \langle G(\mathbf{v}_n), \mathbf{w}\rangle) = \\ \langle \mathscr{T}^{(\kappa,n)}(\mathbf{v}, \kappa), \mathbf{w}\rangle - (p, \nabla \cdot \mathbf{w})_\Omega + \langle G(\mathbf{v}), \mathbf{w}\rangle, \end{aligned} \tag{7.112}$$

$$\lim_{n\to\infty} (\nabla \cdot \mathbf{v}_n, q)_\Omega = (\nabla \cdot \mathbf{v}, q)_\Omega, \tag{7.113}$$

$$\begin{aligned} \lim_{n\to\infty} (\langle \mathscr{K}^{(\kappa)}(\mathbf{v}_n, \kappa_n), l\rangle + e(\kappa_n, \mathbf{v}_n; \kappa_n, l) + B(\kappa_n, \mathbf{v}_n; \mathbf{v}_n, l)) = \\ \langle \mathscr{K}^{(\kappa)}(\mathbf{v}, \kappa), l\rangle + e(\kappa, \mathbf{v}; \kappa, l) + B(\kappa, \mathbf{v}; \mathbf{v}, l). \end{aligned} \tag{7.114}$$

Therefore, (7.110.i) and (7.110.ii) both hold. Unfortunately, $\mathbf{v}$ cannot be used as test in (7.110.i), since there is no reason that $\mathbf{v} \in \mathbf{X}_3(\Omega)$. The consequence is that $(\mathbf{v}, \kappa)$ does not verify the energy equality (7.27) and the energy method does not apply. However, let Θ_n and Θ denote the functions

$$\begin{aligned} \Theta_n(\mathbf{x}) &= \sqrt{l(\mathbf{x})\tilde{\nu}_t^{(n)}(\kappa_n(\mathbf{x}), \mathbf{v}_n(\mathbf{x}), \mathbf{x})}\, D\mathbf{v}_n(\mathbf{x}), \\ \Theta(\mathbf{x}) &= \sqrt{l(\mathbf{x})\widetilde{\nu_t}(\kappa(\mathbf{x}), \mathbf{v}(\mathbf{x}), \mathbf{x})}\, D\mathbf{v}(\mathbf{x}), \end{aligned}$$

so that

$$(S_n, l)_\Omega = ||\Theta_n||^2_{0,2,\Omega}, \quad (Q(\kappa, \mathbf{v}, \mathbf{x}), l)_\Omega = ||\Theta||^2_{0,2,\Omega}. \tag{7.115}$$

The best we can hope is that $(\Theta_n)_{n\in\mathbb{N}}$ weakly converges to Θ in L^2, leading to

$$||\Theta||^2_{0,2,\Omega} \leq \liminf_{n\to\infty} ||\Theta_n||^2_{0,2,\Omega},$$

hence (7.110.iii) by (7.114) and (7.115). □

In terms of PDEs and to get closer to the original NS-TKE model, we are therefore able to prove the existence of weak solutions to the system

$$\begin{cases} (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \nabla\cdot\Big[(2\nu + \ell\sqrt{k})\,D\mathbf{v}\Big] + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v}\cdot\nabla k - \nabla\cdot\Big[(\mu + C_\mu\ell\sqrt{k})\nabla k\Big] \geq \ell\sqrt{k}|D\mathbf{v}|^2 - \dfrac{k\sqrt{T_N(k)}}{\ell} & \text{in } \Omega, \\ -\Big[(2\nu + \ell\sqrt{k})\,D\mathbf{v}\cdot\mathbf{n}\Big]_\tau = g(\mathbf{v})_\tau & \text{on } \Gamma, \\ \mathbf{v}\cdot\mathbf{n} = 0 & \text{on } \Gamma, \\ k = k_\Gamma(\mathbf{v}) & \text{on } \Gamma, \end{cases}$$

for a fixed N and $k_\Gamma \in W^{1,\infty}(\mathbb{R}^3) \cap C^1\mathbb{R}$, hypotheses that cannot be removed. It seems that the only option to improve the situation would be to establish that the sequence $(\kappa_n)_{n\in\mathbb{N}}$ considered above is bounded in $L^\infty(\Omega)$. Such a result has already been proved in Lederer–Lewandowski [41], but without convection and in the case of periodic boundary conditions. There is also a similar result in Clain–Touzani [20], in the two-dimensional scalar case (without pressure). To the best of our knowledge, the general case considered in this chapter remains open at the time of writing.

Finally, little is known about uniqueness, apart from a simple case for homogeneous boundary conditions, without convection and when ν'_t is small, a result first proved in Brossier–Lewandowski [15] and improved in Bernardi–Chacon–Hecht–Lewandowski [6]. Again, no result analogous to that proved in Sect. 6.8 for the general nonlinear case with wall laws is known.

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Chapter 8
Evolutionary NS-TKE Model

Abstract We study the NS-TKE model with a wall law for the velocity $\mathbf{v}$ and the homogenous boundary condition for the TKE k. The abstract variational framework is specified, and a series of *a priori* narrow estimates is derived. The model is approximated by interconnected approximate Leray-α-like models, in which transport terms are regularized by convolution, the source term, and wall law by truncation. We show that the corresponding families of variational problems admit solutions and converge to one another. In the final step of the process, we obtain an NS-TKE model yielding an inequality for the TKE. This chapter finishes with a thorough bibliographical section on the 3D evolutionary Navier–Stokes equations.

8.1 Introduction

This chapter is devoted to the study of the following evolutionary version of the 3D NS-TKE model:

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \nabla\cdot[2\nu + \nu_t(k,t,\mathbf{x}))\,D\mathbf{v}] + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega, \\ \partial_t k + \mathbf{v}\cdot\nabla k - \nabla\cdot[(\mu + \mu_t(k,t,\mathbf{x}))\nabla k] + k\,E(k,t,\mathbf{x}) = \nu_t(k,t,\mathbf{x})|D\mathbf{v}|^2 & \text{in } \Omega, \\ -[(2\nu + \nu_t)\,D\mathbf{v}\cdot\mathbf{n}]_\tau = g(\mathbf{v})_\tau & \text{on } \Gamma, \\ \mathbf{v}\cdot\mathbf{n} = 0 & \text{on } \Gamma, \\ k = 0 & \text{on } \Gamma, \\ \mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{in } \Omega, \\ k(0,\mathbf{x}) = k_0(\mathbf{x}) & \text{in } \Omega. \end{cases} \tag{8.1}$$

This model involves the 3D incompressible evolutionary Navier–Stokes equations (8.1.i)–(8.1.ii) with an eddy viscosity. The last few decades have seen much intensive study of the general 3D NSE in various forms and with various boundary

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conditions, in particular the case of a constant viscosity $\nu > 0$ and the no-slip boundary condition, the periodic BC, or the case of the entire space $\mathbb{R}^3$. Section 8.8 provides a thorough and detailed bibliography on the field.

Equation (8.1.iii) is a parabolic equation with a right-hand side in L^1, which shares many similarities with the steady-state case studied in Chap. 7. The authors quoted in the bibliography of Chap. 7 have all published work on renormalized and/or entropy solutions to general parabolic equations with r.h.s. in L^1 and/or measure data. It is then easy to generate the corresponding bibliography with MathSciNet, starting from the bibliography of Chap. 7.

However, this bibliography must be completed by quoting Blanchard et al. [46–49] and the extra selection of papers [1, 146, 169, 223, 236, 238, 242]. Unfortunately, neither the concept of renormalized nor of entropy solutions applies to the case of system (8.1), although it is a fruitful source of inspiration.

It is worth stressing that $k = 0$ on Γ within this chapter, unlike Chap. 7 where $k = k_\Gamma(\mathbf{v})$ on Γ, $k_\Gamma \in W^{1,\infty}(\mathbb{R})$.[1] This choice is enforced by very serious technical complications in the unsteady case, since $k = k_\Gamma(\mathbf{v})$ yields a dramatic loss of regularity with respect to time, which prevents us from adapting the technique of changing variable developed in Chap. 7. However, this simplification allows us to include in the analysis the physical case

$$E(k, t, \mathbf{x}) = \sqrt{k}/\ell(t, \mathbf{x}), \quad \ell(t, \mathbf{x}) \geq \ell_0 > 0.$$

As in Chaps. 6 and 7, we will develop a matryoshka-doll strategy to investigate the NS-TKE model (8.1), which means that we will introduce several interconnected problems, such as the following Leray-(α, β) NS-TKE model[2]:

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v}_\alpha \cdot \nabla)\, \mathbf{v} - \nabla \cdot [(2\nu + \nu_t)\, D\mathbf{v}] + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{v} = 0 & \text{in } \Omega, \\ \partial_t k + \mathbf{v}_\alpha \cdot \nabla k - \nabla \cdot [(\mu + \mu_t)\nabla k] + k\, E(k, t, \mathbf{x}) = \mathbb{P}_\beta(\mathbf{v}, k) & \text{in } \Omega, \\ -[(2\nu + \nu_t)\, D\mathbf{v} \cdot \mathbf{n}]_\tau + (1/2)(\mathbf{v}_\alpha \cdot \mathbf{n})\mathbf{v}_\tau = \mathbf{v}_\tau \tilde{H}_\alpha(\mathbf{v}) & \text{on } \Gamma, \\ \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \Gamma, \\ k = 0 & \text{on } \Gamma, \\ \mathbf{v}(0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{in } \Omega, \\ k(0, \mathbf{x}) = k_0(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{8.2}$$

where

(i) $\nu_t = \nu_t(k, t, \mathbf{x})$ and $\mu_t = \mu_t = \mu_t(k, t, \mathbf{x})$ for simplicity,
(ii) $\mathbb{P}_\beta(\mathbf{v}, k) = \nu_t T_{1/\beta}(|D\mathbf{v}|^2)$, where the truncation function T_N $(N = 1/\beta)$ is defined by (7.98),
(iii) We recall that $g(\mathbf{v}) = \mathbf{v}\tilde{H}(\mathbf{v})$, and we put $\tilde{H}_\alpha(\mathbf{v}) = T_{1/\alpha}(\tilde{H}(\mathbf{v}))$,

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[1] Recall that the modeling process of Sect. 5.4.2 yields $k_\Gamma(\mathbf{v}) = C_k|\mathbf{v}|^2$.

[2] Model (8.2) is not a Leray-α model in the usual way. However, it makes sense to use the Leray-α terminology to denote it. See references in item x) of Sect. 8.8.

(iv) $\mathbf{v}_\alpha = \mathbf{v} \star \rho_\alpha$ for some mollifier $\rho_\alpha = \rho_\alpha(\mathbf{x})$. As we do not know whether $\mathbf{v}_\alpha \cdot \mathbf{n} = 0$ on Γ, the term $(1/2)(\mathbf{v}_\alpha \cdot \mathbf{n})\mathbf{v}_\tau$ in (8.3.iv) allows estimates for the velocity to be derived, and vanishes when $\alpha \to 0$.

When $\beta \to 0$, the Leray-(α, β) NS-TKE model converges to the Leray-α NS-TKE model (8.3):

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v}_\alpha \cdot \nabla)\, \mathbf{v} - \nabla \cdot [(2\nu + \nu_t)\, D\mathbf{v}] + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{v} = 0 & \text{in } \Omega, \\ \partial_t k + \mathbf{v}_\alpha \cdot \nabla k - \nabla \cdot [(\mu + \mu_t) \nabla k] + k\, E(k, t, \mathbf{x}) = \nu_t |D\mathbf{v}|^2 & \text{in } \Omega, \\ -[(2\nu + \nu_t)\, D\mathbf{v} \cdot \mathbf{n}]_\tau + (1/2)(\mathbf{v}_\alpha \cdot \mathbf{n})\mathbf{v}_\tau = \mathbf{v}_\tau \tilde{H}_\alpha(\mathbf{v}) & \text{on } \Gamma, \\ \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \Gamma, \\ k = 0 & \text{on } \Gamma, \\ \mathbf{v}(0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{in } \Omega, \\ k(0, \mathbf{x}) = k_0(\mathbf{x}) & \text{in } \Omega. \end{cases} \tag{8.3}$$

Finally, the Leray-α NS-TKE model should converge to the NS-TKE model (8.1) when $\alpha \to 0$. Unfortunately, there is no way to prove such a result. The best we can do is to show the convergence to the NS-TKE inequality model:

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} - \nabla \cdot [(2\nu + \nu_t)\, D\mathbf{v}] + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{v} = 0 & \text{in } \Omega, \\ \partial_t k + \mathbf{v} \cdot \nabla k - \nabla \cdot [(\mu + \mu_t) \nabla k] + k\, E(k, t, \mathbf{x}) \geq \nu_t |D\mathbf{v}|^2 & \text{in } \Omega, \\ -[(2\nu + \nu_t)\, D\mathbf{v} \cdot \mathbf{n}]_\tau = g(\mathbf{v})_\tau & \text{on } \Gamma, \\ \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \Gamma, \\ k = 0 & \text{on } \Gamma, \\ \mathbf{v}(0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{in } \Omega, \\ k(0, \mathbf{x}) = k_0(\mathbf{x}) & \text{in } \Omega. \end{cases} \tag{8.4}$$

All these convergences are expressed in terms of the convergence of variational problems introduced in Sect. 8.5.

The root of the process is the Leray-(α, β)-ε model:

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v}_\alpha \cdot \nabla)\, \mathbf{v} + \dfrac{1}{2}\mathbf{v}(\nabla \cdot \mathbf{v}_\alpha) - \nabla \cdot [(2\nu + \nu_t)\, D\mathbf{v}] + \nabla P_\varepsilon(\mathbf{v}) = \mathbf{f}, \\ \partial_t k + \mathbf{v}_\alpha \cdot \nabla k + \dfrac{1}{2} k(\nabla \cdot \mathbf{v}_\alpha) + \nabla \cdot [(\mu + \mu_t) \nabla k] + k\, E(k, t, \mathbf{x}) = \mathbb{P}_\beta(\mathbf{v}, k), \\ \qquad -[(2\nu + \nu_t)\, D\mathbf{v} \cdot \mathbf{n}]_\tau + \dfrac{1}{2}(\mathbf{v}_\alpha \cdot \mathbf{n})\mathbf{v}_\tau|_\Gamma = \mathbf{v}_\tau \tilde{H}_\alpha(\mathbf{v})), \\ \qquad \mathbf{v} \cdot \mathbf{n}|_\Gamma = 0, \\ \qquad k|_\Gamma = 0, \end{cases}$$

where P_ε is the ε-approximation operator introduced in Sect. 6.4.3. We will show using the Galerkin method that this model admits a solution $(\mathbf{v}, k)$ which satisfies the energy equality $\forall\, t > 0$:

$$\begin{cases} \dfrac{1}{2}||\mathbf{v}(t)||^2_{0,2,\Omega} + \displaystyle\int\int_{Q_t} (2\nu + \nu_t(k(s,\mathbf{x}), s, \mathbf{x})|D\mathbf{v}(s,\mathbf{x})|^2 d\mathbf{x} ds \\ - \displaystyle\int_0^t (P_\varepsilon(\mathbf{v}(s)), \nabla \cdot \mathbf{v}(s))_\Omega ds + \int_0^t \langle G_\alpha(\mathbf{v}(s)), \mathbf{v}(s)\rangle ds = \\ \qquad\qquad\qquad\qquad \dfrac{1}{2}||\mathbf{v}_0||^2_{0,2,\Omega} + \displaystyle\int_0^t \langle \mathbf{f}(s), \mathbf{v}(s)\rangle ds, \end{cases} \tag{8.5}$$

where $Q_t = [0, t] \times \Omega$. This will be achieved by means of the evolutionary version of the energy method, introduced for the first time in Sect. 7.4.3 and which is based on energy equalities of the form (8.5).

We will then prove the convergence of the NS-TKE Leray-(α, β)-ε model to the Leray-(α, β) when $\varepsilon \to 0$, hence showing the existence of a solution, as well as all the other models. NS-TKE Leray-(α, β) and Leray-α are shown to satisfy an energy equality, but not (8.4), which prevents us from proving that inequality (8.4.iii) is in fact an equality.

The chapter is organized as follows. Section 8.2 is devoted to the general functional framework. Sections 8.3 and 8.4 aim to derive from the NS-TKE model the fundamental estimates, essential in defining the variational problems associated with the models. These variational problems are stated in Sect. 8.5. For each of these, we carefully perform a meaningfulness and consistency analysis. Section 8.6 is devoted to the construction of the compactness machinery necessary for taking limits. The proof of the results are finalized in Sect. 8.7.

8.2 Functional Framework

8.2.1 Spaces, Hypotheses, and Operators

8.2.1.1 Spaces

Throughout this chapter,

$$Q = [0, T] \times \Omega. \tag{8.6}$$

We shall use the Bochner spaces $L^p([0, T], X)$, where X is any Banach space and T a time of reference. We refer to the Sect. A.4.5 in [TB] for further details about these spaces[3] (see also in Sobolev [258]). We denote

[3]Section A.4.5 lists general results on parabolic equations that will be used all along this and following chapters.

$$\forall\, u \in L^p([0,T],X), \quad ||u||_{p;X} = \left(\int_0^T ||u(t)||_X\, dt\right)^{\frac{1}{p}},$$

and in the particular case of Sobolev spaces

$$||u||_{p;W^{s,q}(U)} = ||u||_{p;s,q,U}, \quad U = \Omega, \Gamma.$$

Furthermore, let X and Y be two given Banach spaces, $\mathcal{N}_{p,q}(X,Y)$ denotes the space

$$\mathcal{N}_{p,q}(X,Y) = \{u :\ u \in L^p([0,T],X),\ \partial_t u \in L^q([0,T],Y)\}, \tag{8.7}$$

endowed with the norm

$$||u||_{\mathcal{N}_{p,q}(X,Y)} = ||u||_{p;X} + ||\partial_t u||_{q;Y}.$$

The reference space is still

$$\mathbf{W}(\Omega) = \{\mathbf{w} \in \mathbf{H}_1(\Omega) \ \text{ such that } \ \mathbf{w}\cdot\mathbf{n}|_\Gamma = 0\},$$

first introduced and studied in Sect. 6.2.1. Before proceeding, we must discuss which norm over $\mathbf{W}(\Omega)$ to use. In Chaps. 7 and 6, we have used

$$\|\mathbf{w}\|_{\mathbf{W}(\Omega)} = (\|D\mathbf{w}\|^2_{0,2,\Omega} + ||\gamma_0\mathbf{w}||^2_{0,2,\Gamma})^{1/2},$$

appropriate for the steady-state case. Estimates were obtained by the hypothesis

$$C_g||\mathbf{v}||^{2+\alpha}_{0,2,\Gamma} \le \langle G(\mathbf{v}), \mathbf{v}\rangle$$

(see Sect. 6.3.2, satisfied by the friction law $g(\mathbf{v}) = C_w\mathbf{v}|\mathbf{v}|$). This assumption is not necessary in the evolutionary case, and we prefer to work with

$$||\mathbf{w}||_{\mathbf{W}(\Omega)} = (||\mathbf{w}||^2_{0,2,\Omega} + \|D\mathbf{w}\|^2_{0,2,\Omega})^{1/2}, \tag{8.8}$$

which is indeed a norm over $\mathbf{W}(\Omega)$ (see the discussion in item b) of Sect. 6.2.1). The $L^2([0,T],\mathbf{W}(\Omega))$ associated norm is specified by

$$||\mathbf{w}||_{2;\mathbf{W}(\Omega)} = \left(\int_0^T (||\mathbf{w}(t)||^2_{0,2,\Omega} + \|D\mathbf{w}(t)\|^2_{0,2,\Omega})\, dt\right)^{\frac{1}{2}}, \tag{8.9}$$

or alternatively

$$||\mathbf{w}||_{2;\mathbf{W}(\Omega)} = (||\mathbf{w}||^2_{0,2,Q} + ||D\mathbf{v}||^2_{0,2,Q})^{\frac{1}{2}}. \tag{8.10}$$

Following standard use, we take as norm over $L^2([0,T], H_0^1(\Omega))$:

$$||k||_{2;H_0^1(\Omega)} = \left(\int_0^T ||\nabla k(t)||_{0,2,\Omega}^2 dt \right)^{\frac{1}{2}} = ||\nabla k||_{0,2,Q}. \tag{8.11}$$

8.2.1.2 Hypotheses

The hypotheses concerning the data within this chapter are summarized in the following statement:

Hypothesis 8.i.

(i) Ω is of class C^m, $m \geq 1$.
(ii) $\nu_t, \mu_t, E \in L^\infty(\mathbb{R} \times \mathbb{R}_+ \times \Omega)$ are continuous, of class $C^{0,1}_{loc}$ with respect to k, and also satisfy $\nu_t, \mu_t, E \geq 0$ a.e. in $\mathbb{R} \times \Omega$.
(iii) $\mathbf{f} \in L^2_{loc}(\mathbb{R}_+, \mathbf{W}(\Omega)')$.
(iv) The wall law g satisfies (6.30), (6.31), and (6.32), in order to apply Lemma 6.5. By noting $g(\mathbf{v}) = \mathbf{v}\tilde{H}(\mathbf{v})$, we assume that $\tilde{H}$ is $C^{0,1}_{loc}$.

8.2.1.3 Operators

The operators involved in the variational formulation of (8.1) are similar to those introduced in the two previous chapters, the detailed definitions of which are restated below for convenience.

(i) *Transport and diffusion*. In this context, $\mathbf{v} = \mathbf{v}(t) = \mathbf{v}(t,\mathbf{x})$ and $\mathbf{w} = \mathbf{w}(t) = \mathbf{w}(t,\mathbf{x})$ and $k = k(t) = k(t,\mathbf{x})$ and $l = l(t) = l(t,\mathbf{x})$.[4]

$$\begin{cases} a(\mathbf{v},\mathbf{w}) = 2\nu\,(D\mathbf{v}, D\mathbf{w})_\Omega, \\ b(\mathbf{z};\mathbf{v},\mathbf{w}) = \dfrac{1}{2}\left[((\mathbf{z}\cdot\nabla)\,\mathbf{v},\mathbf{w})_\Omega - ((\mathbf{z}\cdot\nabla)\,\mathbf{w},\mathbf{v})_\Omega\right] \end{cases}$$

and

$$\begin{cases} b_e(\mathbf{z};k,l) = \dfrac{1}{2}\left[(\mathbf{z}\cdot\nabla k, l)_\Omega - (\mathbf{z}\cdot\nabla l, k)_\Omega\right], \quad a_e(k,l) = \mu(\nabla k, \nabla l)_\Omega, \\ s_\nu(k;\mathbf{v},\mathbf{w}) = (\nu_t(k,t,\mathbf{x})D\mathbf{v}, D\mathbf{w})_\Omega, \quad s_e(k;\lambda,l) = (\mu_t(k,t,\mathbf{x})\nabla\lambda, \nabla l)_\Omega, \end{cases}$$

Let $\mathscr{T}^{(k)} = \mathscr{T}^{(k)}(\mathbf{v},k)$ and $\mathscr{K}^{(k)} = \mathscr{K}^{(k)}(\mathbf{v},k)$ denote the operators formally defined at any given time t by

[4]The dependence in t and/or $\mathbf{x}$ is mentioned when necessary.

$$\langle \mathscr{T}^{(k)}(\mathbf{v},k),\mathbf{w}) = b(\mathbf{v};\mathbf{v},\mathbf{w}) + a(\mathbf{v},\mathbf{w}) + s_v(k;\mathbf{v},\mathbf{w}), \tag{8.12}$$

$$\langle \mathscr{K}^{(k)}(\mathbf{v},k),l)\rangle = b_e(\mathbf{v};k,l) + a_e(k,l) + s_v(k;k,l). \tag{8.13}$$

(ii) *Source terms.* The wall law operator $G = G(\mathbf{v})$, first expressed by (6.14) (cf. also Sect. 6.3.2) is defined at any given time t by

$$\langle G(\mathbf{v}),\mathbf{w}\rangle = \langle G(\mathbf{v}(t)),\mathbf{w}(t)\rangle = \int_\Gamma \tilde{H}(\mathbf{v})\,\mathbf{v}\cdot\mathbf{w}(t,\mathbf{x})\,d\Gamma(\mathbf{x}). \tag{8.14}$$

The energy dissipation term in the TKE equation (8.1.iii) is denoted by $\mathscr{E} = \mathscr{E}(k)$, where

$$\langle \mathscr{E}(k),l\rangle = \int_\Omega k(t,\mathbf{x})E(k(t,\mathbf{x}),t,\mathbf{x})l(t,\mathbf{x})\,d\mathbf{x}. \tag{8.15}$$

Finally, for simplicity we denote by $\mathbb{P} = \mathbb{P}(\mathbf{v},k)$ the source term,

$$\mathbb{P} = \mathbb{P}(\mathbf{v},k) = \nu_t(k,t,\mathbf{x})|D\mathbf{v}|^2. \tag{8.16}$$

Therefore, the NS-TKE model (8.1) becomes in terms of variational operators:

$$\begin{cases} \partial_t\mathbf{v} + \mathscr{T}^{(k)}(\mathbf{v},k) + \nabla p + G(\mathbf{v}) = \mathbf{f}, \\ \nabla\cdot\mathbf{v} = 0, \\ \partial_t k + \mathscr{K}^{(k)}(\mathbf{v},k) + \mathscr{E}(k) = \mathbb{P}(\mathbf{v},k), \\ \forall\,\mathbf{x}\in\Omega, \quad \mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}), \\ \forall\,\mathbf{x}\in\Omega, \quad k(0,\mathbf{x}) = k_0(\mathbf{x}). \end{cases} \tag{8.17}$$

which integrates the equations and the boundary conditions on Γ.

8.2.2 *Mild Variational Formulation*

We outline an initial meaningful variational problem associated with the system (8.17), which will be convenient for deriving the basic a priori estimates satisfied by the NS-TKE model, essential for the understanding of its global structure. The derivation of this variational problem from the PDE system is based on the Stokes formula, following the outline of Sect. 6.2.3, plus the temporal dimension. We skip the details, which are unimportant for the present discussion.

We will need in what follows the family of spaces:

$$\forall s > \frac{1}{2}, \quad \mathbf{W}_s(\Omega) = \{\mathbf{w}\in\mathbf{W}^{s,2}(\Omega),\ \text{such that}\ \ \gamma_n\mathbf{w} = \mathbf{w}\cdot\mathbf{n}|_\Gamma = 0\}.^{5} \tag{8.18}$$

[5] This space may not be well defined for $s \le 1/2$, see Lions-Magenes [214] or/and Theorem A.2 in [TB].

The first variational formulation of system (8.17) that we consider is denoted by $\mathscr{E}\mathscr{P}_m^k$ and expressed by

$$\begin{aligned} \text{Find: } & \mathbf{v} \in \mathscr{N}_{2,2}(\mathbf{W}_2(\Omega), \mathbf{L}^2(\Omega)) \cap L^\infty([0,T], \mathbf{W}(\Omega)), \\ & p \in L^2([0,T], L_0^2(\Omega)), \\ & k \in \mathscr{N}_{2,2}(H^2(\Omega), L^2(\Omega)) \cap L^\infty([0,T], H_0^1(\Omega)), \end{aligned} \tag{8.19}$$

$$\text{such that } \forall\, \mathbf{w} \in L^2([0,T], \mathbf{W}(\Omega)),\ q \in L^2(Q),\ l \in L^2([0,T], H_0^1(\Omega)), \tag{8.20}$$

$$\begin{cases} \displaystyle\int_0^T (\partial_t \mathbf{v}(t), \mathbf{w}(t))_\Omega dt + \int_0^T \langle \mathscr{T}^{(k)}(\mathbf{v}(t), k(t)), \mathbf{w}(t)\rangle dt \\ \displaystyle -\int_0^T (p(t), \nabla\cdot\mathbf{w}(t))_\Omega dt + \int_0^T \langle G(\mathbf{v}(t)), \mathbf{w}(t)\rangle dt = \int_0^T \langle \mathbf{f}(t), \mathbf{w}(t)\rangle dt, \end{cases} \tag{8.21}$$

$$\int_0^T (\nabla\cdot\mathbf{v}(t), q(t))_\Omega dt = 0, \tag{8.22}$$

$$\begin{cases} \displaystyle\int_0^T (\partial_t k(t), l(t))_\Omega dt + \int_0^T \langle \mathscr{K}^{(k)}(\mathbf{v}(t), k(t)), l(t)\rangle dt \\ \displaystyle +\int_0^T \langle \mathscr{E}(k(t)), l(t)\rangle dt = \int_0^T (\mathbb{P}(\mathbf{v}(t), k(t)), l(t))_\Omega dt, \end{cases} \tag{8.23}$$

$$\begin{cases} \forall\, \boldsymbol{\varphi} \in C^1([0,T], \mathbf{W}(\Omega)) \text{ such that } \boldsymbol{\varphi}(T,\mathbf{x}) = 0, \\ \displaystyle\int_0^T (\partial_t \mathbf{v}(t), \boldsymbol{\varphi}(t))_\Omega dt = -\int_\Omega \boldsymbol{\varphi}(0,\mathbf{x})\cdot\mathbf{v}_0(\mathbf{x})\, d\mathbf{x} - \int\int_Q \frac{\partial\boldsymbol{\varphi}}{\partial t}(s,\mathbf{x})\cdot\mathbf{v}(s,\mathbf{x})\, d\mathbf{x}ds, \end{cases} \tag{8.24}$$

$$\begin{cases} \forall\, \psi \in C^1([0,T], H_0^1(\Omega)) \text{ such that } \psi(T,\mathbf{x}) = 0, \\ \displaystyle\int_0^T \langle \partial_t k(t), \psi(t)\rangle dt = -\int_\Omega \psi(0,\mathbf{x}) k_0(\mathbf{x})\, d\mathbf{x} - \int\int_Q \frac{\partial\psi}{\partial t}(s,\mathbf{x}) k(s,\mathbf{x})\, d\mathbf{x}ds. \end{cases} \tag{8.25}$$

Finally, we know from Lemma A.8 that given any solution $(\mathbf{v}, p, k)$ to $\mathscr{E}\mathscr{P}_m^k$, then $\mathbf{v} \in C([0,T], \mathbf{W}(\Omega))$ and $k \in C([0,T], H_0^1(\Omega))$. Therefore, this formulation must be completed by

$$\mathbf{v}_0 \in \mathbf{W}(\Omega), \quad k_0 \in H_0^1(\Omega). \tag{8.26}$$

Clearly, this formulation is not easy to write down. However, one can easily check that it is meaningful, which means that all the integrals above are meaningful (cf. Sect. 7.2.2).

Remark 8.1. Due to the nature of the unknown and test spaces, following the discussion in Sect. 6.2.3, Lemmas 6.1 and 7.1, we shall say that any solution to $\mathscr{E}\mathscr{P}_m^k$ is a mild solution to the evolutionary NS-TKE model (8.1), the existence of which is not established.

Remark 8.2. Observe that given any solution $(\mathbf{v}, p, k)$ to $\mathscr{EP}^k_m$, then for all $(\mathbf{w}, q, l) \in \mathbf{W}(\Omega) \times L^2(\Omega) \times H^1_0(\Omega)$, it holds in $L^2([0,T])^3$:

$$\left\{\begin{aligned} (\partial_t \mathbf{v}(t), \mathbf{w})_\Omega + \langle \mathscr{T}^{(k)}(\mathbf{v}(t), k(t)), \mathbf{w}\rangle \qquad & \\ -(p(t), \nabla\cdot\mathbf{w})_\Omega + \langle G(\mathbf{v}(t)), \mathbf{w}\rangle &= \langle \mathbf{f}(t), \mathbf{w}\rangle, \\ (\nabla\cdot\mathbf{v}(t), q)_\Omega &= 0, \\ (\partial_t k(t), l)_\Omega + \langle \mathscr{K}^{(k)}(\mathbf{v}(t), k(t)), l\rangle + \langle \mathscr{E}(k(t)), l\rangle &= (\mathbb{P}(\mathbf{v}(t), k(t)), l)_\Omega, \end{aligned}\right. \tag{8.27}$$

which is equivalent to formulations (8.19), (8.21), (8.22), and (8.23). Indeed, generally speaking any $w \in L^p([0,T], E)$ is a limit of simple functions with values in E, for any Banach space E, $1 \le p < \infty$ (cf. Sobolev [258]), which explains this equivalence. This remark lies behind the estimation process developed from Sect. 8.3, until the end of this chapter, and must be kept in mind.

It is not known if the variational problem $\mathscr{EP}^k_m$ is consistent or consistent by approximations, in the sense given in Sect. 7.2.5. However, we will use $\mathscr{EP}^k_m$ to find the narrowest estimates satisfied by the NS-TKE model (8.17), which allows us to determine which spaces are appropriate for formulating variational problems associated with the NS-TKE model.

To be more specific, the space of unknowns $Y_{1,m}$ for $\mathscr{EP}^k_m$ is expressed by (8.19), the space of tests $Y_{2,m}$ by (8.20). We will consider an a priori solution $(\mathbf{v}, p, k) \in Y_{1,m}$ and a list of well-chosen tests $(\mathbf{w}, q, l) \in Y_{2,m}$, one after another. At the end of a long and technical process consisting in many smaller steps, we will find a series of a priori estimates, which will set the spaces $Y_1(Q)$ and $Y_2(Q)$ for the NS-TKE model (8.17), which are specified by (8.91) below. We first treat the fluid equation, then the TKE equation.

8.3 Estimates Derived from the Fluid Equation

We start with the derivation of a priori estimates for the velocity and the direct consequences on the transport–diffusion operator involved in (8.17.i). We estimate the wall-law operator and finally the pressure. This leads us to lay out the optimal variational problem associated with the NSE part of the NS-TKE model. The analysis of the TKE equation (8.17.iii) is postponed until Sect. 8.4.

8.3.1 Estimates for the Velocity

8.3.1.1 A Priori Estimates

Proposition 8.1. *Let* $(\mathbf{v}, p, k)$ *be any solution to* $\mathscr{EP}^k_m$. *Then*

$$||\mathbf{v}||_{\infty;0,2,\Omega} \leq \left(\frac{1}{2\nu}||\mathbf{f}||^2_{2;\mathbf{W}(\Omega)'} + ||\mathbf{v}_0||^2_{0,2,\Omega}\right)^{\frac{1}{2}} = C_{\nu,\infty}(T), \tag{8.28}$$

$$||\mathbf{v}||_{2;\mathbf{W}(\Omega)} \leq \frac{1}{\sqrt{\inf(T^{-1}, 2\nu)}} C_{\nu,\infty}(T) = C_{\nu,2}(T). \tag{8.29}$$

Proof. According to (8.19) and (8.20), we can take $\mathbf{v} \in Y_{1,m} \cap Y_{2,m}$ as test in (8.17.i). We notice that:

(i) Lemma 6.3 shows that $b(\mathbf{v}(t); \mathbf{v}(t), \mathbf{v}(t)) = 0$, $\forall\, t > 0$.
(ii) As $\nabla \cdot \mathbf{v}(t) = 0$, we have $-(p(t), \nabla \cdot \mathbf{v}(t))_\Omega = 0$.
(iii) As we satisfy the conditions for application of Lemma (A.8) in [TB], we have

$$(\partial_t \mathbf{v}, \mathbf{v})_\Omega = \langle \partial_t \mathbf{v}, \mathbf{v}\rangle = \frac{d}{2dt}\int_\Omega |\mathbf{v}(t,\mathbf{x})|^2 d\mathbf{x}. \tag{8.30}$$

All of this provides

$$\frac{d}{2dt}\int_\Omega |\mathbf{v}(t,\mathbf{x})|^2 d\mathbf{x} + \int_\Omega (2\nu + \nu_t(k(t,\mathbf{x}), t, \mathbf{x})|D\mathbf{v}(t,\mathbf{x})|^2 d\mathbf{x} + \langle G(\mathbf{v}), \mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle,$$

which becomes, after integrating over $[0, t]$,

$$\begin{cases} \frac{1}{2}||\mathbf{v}(t)||^2_{0,2,\Omega} + \int\int_{Q_t} (2\nu + \nu_t(k(s,\mathbf{x}), s, \mathbf{x})|D\mathbf{v}(s,\mathbf{x})|^2 ds\, d\mathbf{x} \\ + \int_0^t \langle G(\mathbf{v}(s)), \mathbf{v}(s)\rangle ds = \frac{1}{2}||\mathbf{v}_0||^2_{0,2,\Omega} + \int_0^t \langle \mathbf{f}(s), \mathbf{v}(s)\rangle ds, \end{cases} \tag{8.31}$$

where $Q_t = [0,t] \times \Omega$. As $\langle G(\mathbf{v}), \mathbf{v}\rangle \geq 0$ and $\nu_t \geq 0$, this energy equality combined with the Young inequality leads to

$$||\mathbf{v}(t)||^2_{0,2,\Omega} + 2\nu||D\mathbf{v}||^2_{0,2,Q_t} \leq \frac{1}{2\nu}\int_0^t ||\mathbf{f}(s)||^2_{\mathbf{W}(\Omega)'} ds + ||\mathbf{v}_0||^2_{0,2,\Omega}, \tag{8.32}$$

hence

$$\begin{aligned} ||\mathbf{v}||^2_{\infty;0,2,\Omega} &\leq \frac{1}{2\nu}||\mathbf{f}||^2_{2;\mathbf{W}(\Omega)'} + ||\mathbf{v}_0||^2_{0,2,\Omega}, \\ ||D\mathbf{v}||^2_{0,2,Q} &\leq \frac{1}{2\nu}||\mathbf{f}||^2_{2;\mathbf{W}(\Omega)'} + ||\mathbf{v}_0||^2_{0,2,\Omega}, \end{aligned} \tag{8.33}$$

proving (8.28). As we have

$$||\mathbf{v}||^2_{2;0,2,\Omega} \leq T||\mathbf{v}||^2_{\infty;0,2,\Omega},$$

we find by (8.32) and (8.33):

$$||\mathbf{v}||_{2;\mathbf{W}(\Omega)} \leq \frac{1}{\sqrt{\inf(T^{-1}, 2\nu)}} \left(\frac{1}{2\nu} ||\mathbf{f}||^2_{2;\mathbf{W}(\Omega)'} + ||\mathbf{v}_0||^2_{0,2,\Omega} \right)^{\frac{1}{2}} = C_{\nu,2}(T), \quad (8.34)$$

and (8.29) follows. □

8.3.1.2 The Transport–Diffusion Operator

$\mathscr{T}^{(k)}(\mathbf{v}, k)$, which is defined by formula (8.12) above.

Proposition 8.2. *Let* $(\mathbf{v}, p, k)$ *be any solution to* $\mathscr{E}\mathscr{P}^k_m$. *Then we have* $\mathscr{T}^{(k)}(\mathbf{v}, k) \in L^{4/3}([0, T], \mathbf{W}(\Omega)')$ *and*

$$||\mathscr{T}^{(k)}(\mathbf{v}, k)||_{4/3;\mathbf{W}(\Omega)'} \leq C_{\mathscr{T},4/3}(T), \quad (8.35)$$

in which

$$C_{\mathscr{T},4/3}(T) = (2\nu + ||\nu_t||_\infty)^{\frac{1}{2}} T^{\frac{1}{3}} C_{\nu,2}(T) + \lambda\, C_{\nu,\infty}(T)^{\frac{1}{4}} C_{\nu,2}(T)^{\frac{3}{4}}, \quad (8.36)$$

and λ *is the universal Sobolev constant* $H^1 \hookrightarrow L^6$.

Proof. As $\nabla \cdot \mathbf{v} = 0$, $\nu_t \in L^\infty$, Lemmas 6.2 and 6.3 lead to

$$||\mathscr{T}^{(k)}(\mathbf{v}, k)||_{\mathbf{W}(\Omega)'} \leq (2\nu + ||\nu_t||_\infty)||\mathbf{v}||_{\mathbf{W}(\Omega)} + ||\mathbf{v} \otimes \mathbf{v}||_{0,2,\Omega}. \quad (8.37)$$

Arguing as in Sect. 3.4.2, we combine (8.28) and (8.29) with the interpolation inequality (A.37) of Lemma A.18 in [TB], obtaining

$$||\mathbf{v}||_{8/3;0,4,\Omega} \leq \lambda\, C_{\nu,\infty}(T)^{\frac{1}{4}} C_{\nu,2}(T)^{\frac{3}{4}}, \quad (8.38)$$

which, by the Cauchy–Schwarz inequality, leads to

$$||\mathbf{v} \otimes \mathbf{v}||_{4/3;0,2,\Omega} \leq \lambda^2 C_{\nu,\infty}(T)^{\frac{1}{2}} C_{\nu,2}(T)^{\frac{3}{2}}, \quad (8.39)$$

hence (8.35) by (8.37).

8.3.2 *Improved Estimate for the Wall-Law Operator*

Throughout Chaps. 6 and 7, we have worked with estimate (6.38) satisfied by $G(\mathbf{v})$:

$$||G(\mathbf{v})||_{\mathbf{W}(\Omega)'} \leq C(1 + ||\mathbf{v}||^2_{\mathbf{W}(\Omega)}),$$

which is more than enough to analyze the steady-state case. In the evolutionary case, this estimate combined with (8.29) yields

$$G(\mathbf{v}) \in L^1([0,T], \mathbf{W}(\Omega)').$$

This is unfortunately not enough to derive an estimate for the pressure and therefore must be improved. We shall use $\mathbf{W}_{3/4}(\Omega)$, the general space $\mathbf{W}_p(\Omega)$ being defined by (8.18). Following the interpolation theory carried out in Lions-Magenes [214], we can write

$$\mathbf{W}_{3/4}(\Omega) = [\mathbf{W}(\Omega), \mathbf{L}^2(\Omega)]_{3/4}, \tag{8.40}$$

since $\mathbf{W}(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$ is dense[6] (and compact), so that $\mathbf{W}(\Omega)$ is dense in $\mathbf{W}_{3/4}(\Omega)$. Moreover,

$$\forall\, \mathbf{v} \in \mathbf{W}(\Omega), \quad ||\mathbf{v}||_{3/4,2,\Omega} \le ||\mathbf{v}||_{\mathbf{W}(\Omega)}^{\frac{3}{4}} ||\mathbf{v}||_{0,2,\Omega}^{\frac{1}{4}}. \tag{8.41}$$

We infer in particular that $\mathbf{v} \in L^{8/3}([0,T], \mathbf{W}_{3/4})$, and

$$||\mathbf{v}||_{8/3;3/4,2,\Omega} \le C_{v,\infty}(T)^{\frac{1}{4}} C_{v,2}(T)^{\frac{3}{4}}. \tag{8.42}$$

By the duality principle explained in [214], $\mathbf{W}_{3/4}(\Omega)' \hookrightarrow \mathbf{W}(\Omega)'$ with density. The improvement we need comes from the following extension of Lemma 6.5:

Lemma 8.1. *The operator G maps continuously $\mathbf{W}(\Omega)$ into $\mathbf{W}(\Omega)'$, and one has*

$$\forall\, \mathbf{v} \in \mathbf{W}(\Omega), \quad ||G(\mathbf{v})||_{\mathbf{W}(\Omega)'} \le C(1 + ||\mathbf{v}||_{3/4,2,\Omega}^2), \tag{8.43}$$

where the constant C only depends on Ω.

Proof. We observe that

(a) $\gamma_0 : \mathbf{W}_{3/4}(\Omega) \to \mathbf{H}^{1/4}(\Gamma)$ is a continuous map,

(b) $\mathbf{H}^{1/4}(\Gamma) \hookrightarrow \mathbf{L}^{\frac{8}{3}}(\Gamma)$.

Let $\mathbf{w} \in \mathbf{W}(\Omega)$. As $|g(\mathbf{v})| \le C(1 + |\mathbf{v}|^2)$, Hölder's inequality yields

$$|\langle G(\mathbf{v}), \mathbf{w}\rangle| \le C(||\mathbf{w}||_{0,1,\Gamma} + ||\mathbf{v}||_{0,8/3,\Gamma}^2 ||\mathbf{w}||_{0,4,\Gamma}),$$

hence the result by the trace Theorem. □

In the present context, according to (8.29) and (8.42), we find by easy calculation

$$||G(\mathbf{v})||_{4/3;\mathbf{W}(\Omega)'} \le C(T + C_{v,\infty}(T)^{\frac{2}{3}} C_{v,2}(T)^2)^{\frac{3}{4}} = C_{G,4/3}(T). \tag{8.44}$$

We are now ready to estimate the pressure.

[6] Observe that $\mathscr{D}(\Omega)^3 \subset \mathbf{W}(\Omega)$ and $\mathscr{D}(\Omega)^3$ is dense in $\mathbf{L}^2(\Omega)$.

8.3.3 Estimate for the Pressure

We find below a universal bound for the pressure in $L^{4/3}(Q)$. We follow the method of Bulíček-Málek-Rajagopal [55], already introduced in Sect. (6.4.2) on the steady-state case.

Proposition 8.3. *Let* $(\mathbf{v}, p, k)$ *be any solution to* $\mathscr{EP}_m^k$. *Then*

$$||p||_{0,4/3,Q} \leq C(T^3||\mathbf{f}||_{2;\mathbf{W}(\Omega)'}^{\frac{4}{3}} + C_{\mathscr{F},4/3}(T) + C_{G,4/3}(T)) = C_{p,4/3}(T), \quad (8.45)$$

where $C_{\mathscr{F},4/3}(T)$ *is specified by (8.36) and* $C_{G,4/3}(T)$ *by (8.44).*

Proof. Let $1 < \theta \leq 2$, to be fixed later, and let us consider the PDE system:

$$\begin{cases} -\Delta u = p|p|^{\theta-2} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \Gamma, \\ \displaystyle\int_\Omega u(\mathbf{x})\, d\mathbf{x} = 0. \end{cases} \quad (8.46)$$

As $1 < \theta \leq 2$, for almost all $t \in [0, T]$, $p = p(t) \in L^\theta(\Omega)$, hence $p|p|^{\theta-2} \in L^{\theta'}(\Omega)$ ($\theta' = \theta/(1-\theta)$). We deduce from standard elliptic theory [159] that for almost all $t \in [0, T]$, problem (8.46) has a unique solution $u = u(t) \in W^{2,\theta'}(\Omega)$, such that

$$||u||_{2,\theta'} \leq C||p|p|^{\theta-2}||_{0,\theta',\Omega} = C||p||_{0,\theta,\Omega}^{\theta-1}, \quad (8.47)$$

where C depends only on Ω. Let

$$\mathbf{w} = \nabla u, \ \text{ so that } \ \mathbf{w}\cdot\mathbf{n} = 0 \ \text{ on } \ \Gamma \ \text{ and } \ ||\mathbf{w}||_{1,\theta',\Omega} \leq C||p||_{0,\theta,\Omega}^{\theta-1}, \quad (8.48)$$

in particular, $\mathbf{w} = \mathbf{w}(t) \in \mathbf{W}_{\theta'}(\Omega)$ (cf. (8.18)).

We take $\mathbf{w}$ as test in (8.17.i). This choice will be validated by the end of the proof, once θ has been fixed. We first investigate formally the pressure and time derivative terms. We notice that

$$(\nabla p, \mathbf{w})_\Omega = -(p, \nabla\cdot\mathbf{w}) = (p, -\Delta u)_\Omega = ||p||_{0,\theta,\Omega}^\theta. \quad (8.49)$$

Moreover, as $\nabla\cdot\mathbf{v} = 0$, $\mathbf{v} \in \mathbf{W}_2(\Omega)$, $\partial_t\mathbf{v} \in \mathbf{L}^2(\Omega)$, we have $\partial_t\mathbf{v} \in \mathbf{L}^2_{div,0}(\Omega)$, where the space $\mathbf{L}^2_{div,0}(\Omega)$ is defined by (3.55). This means

$$\nabla\cdot(\partial_t\mathbf{v}) = 0, \quad \partial_t\mathbf{v}\cdot\mathbf{n} = 0 \ \text{ on } \ \Gamma,$$

leading to

$$(\partial_t\mathbf{v}, \mathbf{w})_\Omega = (\partial_t\mathbf{v}, \nabla u)_\Omega = 0. \quad (8.50)$$

Therefore, (8.49) and (8.50) give

$$||p||_{0,\theta,\Omega}^{\theta} = \langle \mathbf{f}, \mathbf{w} \rangle - \langle \mathscr{T}^{(k)}(\mathbf{v}, k), \mathbf{w} \rangle - \langle G(\mathbf{v}), \mathbf{w} \rangle. \tag{8.51}$$

Estimates (8.35) and (8.44) suggest taking $\theta = 4/3$. As $(\mathbf{v}, p, k)$ is a solution to $\mathscr{E}\mathscr{P}_m^k$, $p \in L^2(Q)$, and in particular $p \in L^{4/3}(Q)$. By consequence, we infer from (8.48) that $\mathbf{w} \in L^4([0,T], \mathbf{W}_4(\Omega))$, which validates its choice as test in (8.17.i) in view of the definition (8.20) of the space of tests.

As $\mathbf{W}_4(\Omega) \hookrightarrow \mathbf{W}(\Omega)$, the injection being dense,[7] we deduce $\mathbf{W}(\Omega)' \hookrightarrow \mathbf{W}_4(\Omega)'$, the injection being dense; hence (8.51) combined with (8.48) yields

$$\begin{aligned} ||p||_{0,4/3,\Omega}^{\frac{4}{3}} &\leq C\left[||\mathbf{f}||_{\mathbf{W}(\Omega)'} + ||\mathscr{T}^{(k)}(\mathbf{v}, k)||_{\mathbf{W}(\Omega)'} + ||G(\mathbf{v})||_{\mathbf{W}(\Omega)'}\right] ||\mathbf{w}||_{1,4,\Omega}, \\ &\leq C\left[||\mathbf{f}||_{\mathbf{W}(\Omega)'} + ||\mathscr{T}^{(k)}(\mathbf{v}, k)||_{\mathbf{W}(\Omega)'} + ||G(\mathbf{v})||_{\mathbf{W}(\Omega)'}\right] ||p||_{0,4/3,\Omega}^{\frac{4}{3}-1} \end{aligned}$$

hence, for almost all $t \in [0, T]$,

$$||p(t)||_{0,4/3,\Omega} \leq C(||\mathbf{f}(t)||_{\mathbf{W}(\Omega)'} + ||\mathscr{T}^{(k)}(\mathbf{v}(t), k(t))||_{\mathbf{W}(\Omega)'} + ||G(\mathbf{v}(t))||_{\mathbf{W}(\Omega)'}),$$

leading to

$$||p||_{4/3;0,4/3,\Omega} = ||p||_{0,4/3,Q} \leq C(T^3||\mathbf{f}||_{2;\mathbf{W}(\Omega)'}^{\frac{4}{3}} + C_{\mathscr{F},4/3}(T) + C_{G,4/3}(T)), \tag{8.53}$$

as claimed. □

8.3.4 NSE with Wall Law

We are now in a position to find the best theoretical spaces of unknowns and tests for the fluid part of the NS-TKE model, which are the NSE with wall law and eddy viscosity. The last issue is the regularity of $\partial_t \mathbf{v}$. We deduce from (8.53)

$$\nabla p \in L^{\frac{4}{3}}([0,T], \mathbf{W}_4(\Omega)'), \tag{8.54}$$

and as $\partial_t \mathbf{v} = \mathbf{f} - \mathscr{T}^{(k)}(\mathbf{v}) - \nabla p - G(\mathbf{v})$, (8.35) and (8.44), combined with (8.54), yield

[7] To check this, observe that $\mathbf{W}_4(\Omega) \hookrightarrow \mathscr{W}_0(\Omega)$, with dense injection, by recalling that

$$\mathscr{W}_m(\Omega) = \{\boldsymbol{\varphi} \in \mathscr{C}^m(\overline{\Omega})^3 \text{ such that } \boldsymbol{\varphi} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}, \tag{8.52}$$

for some $m \geq 1$, the domain Ω being of class C^m. Then adapt the proof of Theorem 6.1.

$$\partial_t \mathbf{v} \in L^{\frac{4}{3}}([0,T], \mathbf{W}_4(\Omega)')$$

and the following holds:

$$||\partial_t \mathbf{v}||_{4/3;\mathbf{W}_4(\Omega)'} \le C(C_{\mathscr{F},4/3}(T) + C_{p,4/3}(T) + C_{G,4/3}(T) + T^{\frac{1}{4}}||\mathbf{f}||_{\mathbf{W}(\Omega)'}), \quad (8.55)$$

for some constant $C > 0$ that only depends on Ω. The conclusion is that

(i) we take as the space of unknowns for Eq. (8.17.i)

$$\left[\mathscr{N}_{2,4/3}(\mathbf{W}(\Omega), \mathbf{W}_4(\Omega)') \cap L^\infty([0,T], \mathbf{L}^2(\Omega))\right] \times L^{\frac{4}{3}}(Q),$$

where the spaces $\mathscr{N}_{p,q}$ are defined by (8.7),

(ii) we would like to take $L^4([0,T], \mathbf{W}_4(\Omega))$ as the space of tests for Eq. (8.17.i),

(iii) the most appropriate space of tests for Eq. (8.17.ii) remains $L^2(Q)$.

Remark 8.3. Although $L^4([0,T], \mathbf{W}_4(\Omega))$ is the best space of tests for (8.17.i), we will take $L^\infty([0,T], \mathscr{W}_m(\Omega))$ as the tests space in the following. This choice is motivated by issues involving compactness properties investigated in Lemma 8.8. Of course, we could deal with $L^4([0,T], \mathbf{W}_4(\Omega))$, but at the price of extra technical complications, not necessarily relevant.

This suggests considering the evolutionary NSE system with wall law, where the eddy viscosity $\nu_t = \nu_t(t,\mathbf{x}) \in L^\infty(\mathbb{R}_+ \times \Omega)$ is fixed:

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \nabla\cdot[(2\nu + \nu_t(t,\mathbf{x}))\,D\mathbf{v}] + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega, \\ -\left[(2\nu + \nu_t)\,D\mathbf{v}\cdot\mathbf{n}\right]_\tau = g(\mathbf{v})_\tau & \text{on } \Gamma, \\ \mathbf{v}\cdot\mathbf{n} = 0 & \text{on } \Gamma, \\ \mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{in } \Omega. \end{cases} \quad (8.56)$$

According to the notations of Sect. 6.2.3, and the above, the variational problem associated with the NSE (8.56) denoted by $\mathscr{EP}$ is naturally

$$\begin{cases} \textit{Unknown space:} \\ \left[\mathscr{N}_{2,4/3}(\mathbf{W}(\Omega), \mathbf{W}_4(\Omega)') \cap L^\infty([0,T], \mathbf{L}^2(\Omega))\right] \times L^{\frac{4}{3}}(Q) = X_1(Q), \\ \textit{Test space: } L^\infty([0,T], \mathscr{W}_m(\Omega)) \times L^2(Q) = X_2(Q), \\ \textit{Initial data: } \mathbf{v}_0 \in \mathbf{L}^2(\Omega), \end{cases} \quad (8.57)$$

$$\begin{cases} \partial_t \mathbf{v} + \mathscr{T}(\mathbf{v}) + \nabla p + G(\mathbf{v}) = \mathbf{f}, \\ \nabla\cdot\mathbf{v} = 0, \end{cases} \quad (8.58)$$

where $\mathscr{T} = b + a + s_\nu$, the function k being fixed in s_ν, measurable and finite a.e. in Ω, and where $\mathbf{v} = \mathbf{v}(t,\mathbf{x})$ and $p = p(t,\mathbf{x})$ are the unknowns. In other words, $\mathscr{EP}$ is expressed by

(1) find $(\mathbf{v}, p) \in X_1(Q)$ such that for all $(\mathbf{w}, q) \in X_2(Q)$, (8.21) and (8.22) hold with $\mathscr{T}$ instead of $\mathscr{T}^{(k)}$ and $\langle \partial_t \mathbf{v}, \mathbf{w} \rangle$ instead of $(\partial_t \mathbf{v}, \mathbf{w})_\Omega$,
(2) $\forall\, \boldsymbol{\varphi} \in C^1([0, T], \mathscr{W}_m(\Omega))$ such that $\boldsymbol{\varphi}(T, \mathbf{x}) = 0$, (8.24) holds.

The variational problem $\mathscr{EP}$ is meaningful, since all the a priori estimates above still hold if we replace $\mathscr{T}$ by $\mathscr{T}^{(k)}$. We do not know if it is consistent. However, we will see that it is consistent by approximations, which means that it admits a solution constructed by approximations that satisfies estimates (8.28), (8.29), and (8.45). In particular, we will have proven by the end of this chapter:

Theorem 8.1. *Assume that $\nu_t = \nu_t(t, \mathbf{x}) \in L^\infty(\mathbb{R}_+ \times \Omega)$ and that items (i), (iii), and (iv) of hypothesis 8.i hold. Then Problem $\mathscr{EP}$ admits a solution $(\mathbf{v}, p)$, which satisfies in addition $\mathbf{v} \in C_w([0, T], L^2(\Omega))$.*[8]

Uniqueness is an open problem, as well as the issue of the energy equality, as discussed in Remark 8.7 below.

8.4 Analysis of the TKE Equation

We focus in this section on the TKE equation (8.17.iii). In particular, we aim to find estimates for k and determine, for this equation, what are the most appropriate theoretical spaces for the unknowns and tests.

To start with, observe that estimate (8.29) provides

$$||\mathbb{P}(\mathbf{v}, k)||_{0,1,Q} = \int\int_Q \nu_t(k(t, \mathbf{x}), t, \mathbf{x})|D\mathbf{v}(t, \mathbf{x})|^2 d\mathbf{x}dt \le ||\nu_t||_\infty C_{\nu,2}^2 = \sigma_1, \tag{8.59}$$

so that $\mathbb{P}(\mathbf{v}, k) \in L^1(Q)$ and (8.17.iii) is indeed a parabolic equation with a r.h.s. in L^1. The main result of this section is the following:

Proposition 8.4. *Let $(\mathbf{v}, p, k)$ be any solution to $\mathscr{EP}_m^k$. Then for all $1 \le q < 5/4$, there exists a constant $C_{k,q}(T)$ such that*

$$||k||_{q;1,q,\Omega} \le C_{k,q}(T), \tag{8.60}$$

where $C_{k,q}(T)$ depends on σ_1, $||k_0||_{0,1,\Omega}$, μ, and q.

The proof will be completed by the end of this section. It is based on the evolutionary version of the Boccardo–Gallouët Theorem (cf. Theorem A.13 in [TB]). The procedure leading to apply this theorem requires that we:

(a) find a bound for k in $L^\infty([0, T], L^1(\Omega))$,
(b) prove that the washers

[8] cf. definition A.4.

$$M_n^{(e)} = \int \int_{\{n \le |k| \le n+1\}} |\nabla k(t, \mathbf{x})|^2 d\mathbf{x} dt \tag{8.61}$$

are bounded uniformly in n.

As in the case of Proposition 7.2, the proof is based on the choice of the correct test functions.

8.4.1 $L^\infty([0, T], L^1(\Omega))$ Bound

This subsection is devoted to proving the estimate (8.77) below. Ideally, it should be derived by taking $sg(k) = k/|k|$ as test function in Eq. (8.17.iii), since

$$(\partial_t k, sg(k))_\Omega = \frac{d}{dt} \int_\Omega |k(t, \mathbf{x})| d\mathbf{x},$$

providing the wanted $||k||_{\infty;0,1,\Omega}$ bound. However, the derivative of $x \to sg(x)$ is the Heaviside function, which does not allow the manipulation of $\nabla(sg(k))$ in the equations.

To get around this problem, we proceed with approximations as in [209, 210]. Let $\varepsilon > 0$ and $\varphi_\varepsilon \in W^{1,\infty}(\mathbb{R})$ be the odd function defined on $\mathbb{R}$ by

$$\forall\ 0 \le x \le \varepsilon,\ \ \varphi_\varepsilon(x) = \frac{x}{\varepsilon}, \quad \forall\ x \ge \varepsilon,\ \ \varphi_\varepsilon(x) = 1,$$

which approximates $sg(x)$ a.e. in $\mathbb{R}$. To be more specific, the family $(\varphi_\varepsilon)_{\varepsilon > 0}$ satisfies

$$\forall\ x \ne 0, \quad \lim_{\varepsilon \to 0} \varphi_\varepsilon(x) = sg(x). \tag{8.62}$$

Moreover, we have

$$\forall\ \varepsilon > 0, \quad ||\varphi_\varepsilon||_\infty = 1. \tag{8.63}$$

Notice also that $\varphi_\varepsilon' \ge 0$, and φ_ε' has a finite number of discontinuities. However it is worthwhile noticing that

$$||\varphi_\varepsilon'||_\infty = O\left(\frac{1}{\varepsilon}\right). \tag{8.64}$$

We denote by ψ_ε the primitive function of φ_ε that vanishes at 0:

$$\psi_\varepsilon(x) = \int_0^x \varphi_\varepsilon(x')\, dx', \tag{8.65}$$

which is the even function specified by

$$\forall \ 0 \le x \le \varepsilon, \ \psi_\varepsilon(x) = \frac{x^2}{2\varepsilon}, \quad \forall \, x \ge \varepsilon, \ \ \psi_\varepsilon(x) = x - \frac{\varepsilon}{2}. \tag{8.66}$$

This an approximation to $|x|$, which means

$$\forall \, x \in \mathbb{R}, \quad \lim_{\varepsilon \to 0} \psi_\varepsilon(x) = |x|, \tag{8.67}$$

that satisfies in addition

$$0 \le \psi_\varepsilon(x) \le |x| + \frac{\varepsilon}{2}. \tag{8.68}$$

We aim to take $\varphi_\varepsilon(k)$ as test in (8.17.iii). Indeed, $\varphi_\varepsilon \in W^{1,\infty}(\mathbb{R})$, $\varphi(0) = 0$, and φ_ε' has a finite number of discontinuities. Therefore by the Stampacchia Theorem (cf. Theorem A.11), as $k \in L^\infty([0, T], H_0^1(\Omega))$, for almost all $t \in [0, T]$, $\varphi_\varepsilon(k(t)) \in H_0^1(\Omega)$ and according to (8.64),

$$||\varphi_\varepsilon(k(t))||_{H_0^1(\Omega)} \le \frac{C}{\varepsilon} ||k(t)||_{H_0^1(\Omega)},$$

leading to $\varphi_\varepsilon(k) \in L^\infty([0, T], H_0^1(\Omega))$; hence, $\varphi_\varepsilon(k)$ belongs to the test space for (8.17.iii) (cf. the definition of the test space (8.20)). Taking it as test leads to, for almost all $t \in [0, T]$,

$$\begin{aligned} \langle \partial_t k, \varphi_\varepsilon(k) \rangle + b_e(\mathbf{v}; k, \varphi_\varepsilon(k)) + a_e(k, \varphi_\varepsilon(k)) + t_e(k, \mathbf{v}; k, \varphi_\varepsilon(k)) \\ + \langle \mathscr{E}(k), \varphi_\varepsilon(k) \rangle = (\mathbb{P}(\mathbf{v}, k), \varphi_\varepsilon(k))_\Omega. \end{aligned} \tag{8.69}$$

We study each term of this identity consecutively.

(i) *Evolutionary term.* We obtain using (8.65)

$$\langle \partial_t k, \varphi_\varepsilon(k) \rangle = \frac{d}{dt} \int_\Omega \psi_\varepsilon(k)(t, \mathbf{x}) d\mathbf{x}. \tag{8.70}$$

(ii) *Transport.* By Lemma 7.3, we find

$$b_e(\mathbf{v}; k, \varphi_\varepsilon(k)) = 0. \tag{8.71}$$

(iii) *Diffusion.* As $\varphi_\varepsilon' \ge 0$,

$$a_e(k, \varphi_\varepsilon(k)) + t_e(k, \mathbf{v}; k, \varphi_\varepsilon(k)) = \int_\Omega (\mu + \mu_t(k, t, \mathbf{x}) \varphi_\varepsilon'(k) |\nabla k|^2 \ge 0. \tag{8.72}$$

(iv) *Energy dissipation term*

$$\langle \mathscr{E}(k), \varphi_\varepsilon(k) \rangle = \int_\Omega k\varphi_\varepsilon(k) E(k, t, \mathbf{x}) \geq 0, \tag{8.73}$$

since $E \geq 0$ and φ_ε is odd.

(v) *Source term*. By (8.63) and as $||\varphi_\varepsilon||_\infty = 1$, we have

$$|(\mathbb{P}(\mathbf{v}, k), \varphi_\varepsilon(k))_\Omega| \leq \int_\Omega \mathbb{P}(\mathbf{v}, k) d\mathbf{x}. \tag{8.74}$$

Gathering (8.70), (8.71), (8.72), (8.73), and (8.74), $\mathscr{K}^{(k)} = b_e + a_e + t_e$, we deduce

$$\frac{d}{dt} \int_\Omega \psi_\varepsilon(k)(t, \mathbf{x})) d\mathbf{x} \leq \int_\Omega \mathbb{P}(\mathbf{v}, k) d\mathbf{x}, \tag{8.75}$$

leading to $\forall\, t \in [0, T]$:

$$||\psi_\varepsilon(k(t))||_{0,1,\infty} \leq \sigma_1 + ||\psi_\varepsilon(k_0||_{0,1,\Omega} \leq \sigma_1 + ||k_0||_{0,1,\Omega} + \frac{\varepsilon}{2}, \tag{8.76}$$

where we also have used (8.59) and (8.68). So, as $k \in L^1(Q)$ and $\psi_\varepsilon \geq 0$, we derive from Fatou's Lemma,

$$||k(t)||_{0,1,\infty} \leq \lim_{n \to \infty} ||\psi_\varepsilon(k(t))||_{0,1,\infty},$$

and then by taking the limit as $\varepsilon \to 0$ in (8.76), we find

$$\forall\, t \in [0, T], \quad ||k(t)||_{0,1,\infty} \leq \sigma_1 + ||k_0||_{0,1,\Omega} = C_{k,\infty,1}(T), \tag{8.77}$$

concluding indeed that $k \in L^\infty([0, T], L^1(\Omega))$, its norm being bounded by $C_{k,\infty,1}(T)$.

8.4.2 *Washer Bounds*

We first derive the estimate (8.86) below satisfied by the washers $M_n^{(e)}$ (8.61). We then draw conclusions about the estimates satisfied by k in $L^p([0, T], W_0^{1,p}(\Omega))$ for all $p < 5/4$, as well as the most appropriate spaces of unknowns and tests for equation (8.17.iii).

8.4.2.1 Estimate of $M_n^{(e)}$

We proceed as in the proof of Proposition 7.30, by taking $H_n(k)$ as test in Eq. (8.17.iii), the function H_n being expressed by (7.34). Following the outline of Sect. 8.4.1, we deduce that $H_n(k)$ is a possible test function.

Let $\tilde{K}_n$ be the primitive function of H_n that vanishes in 0, which is a nonnegative even function that does not need to be specified. However, the following property of $\tilde{K}_n$ will be useful:

$$\forall\, x \in \mathbb{R}, \quad 0 \le \tilde{K}_n(x) \le |x|. \tag{8.78}$$

Taking $H_n(k)$ yields the same identity as (8.69), the terms of which are considered one after another:

(i) *Evolutionary term.* We obtain

$$\langle \partial_t k, H_n(k)\rangle = \frac{d}{dt}\int_\Omega \tilde{K}_n(k(t,\mathbf{x}))d\mathbf{x}. \tag{8.79}$$

(ii) *Transport.* By Lemma 7.3, we find

$$b_e(\mathbf{v}; k, H_n(k)) = 0. \tag{8.80}$$

(iii) *Diffusion.* As $H' \ge 0$:

$$t_e(k,\mathbf{v}; k, H_n(k)) = \int_\Omega \mu_t(k,t,\mathbf{x})H_n'(k)|\nabla k|^2 \ge 0. \tag{8.81}$$

Furthermore, according to (7.35),

$$a_e(k, H_n(k)) = \mu \int_{\{n\le |k|\le n+1\}} |\nabla k(t,\mathbf{x})|^2 d\mathbf{x}. \tag{8.82}$$

(iv) *Energy dissipation term*

$$\langle \mathscr{E}(k), H_n(k)\rangle = \int_\Omega k H_n(k) E(k,t,\mathbf{x}) \ge 0, \tag{8.83}$$

since $E \ge 0$ and H_n is odd, so that $kH_n(k) \ge 0$.

(v) *Source term.* Since $|H_n| \le 1$, we have

$$|(\mathbb{P}(\mathbf{v},k), H_n(k))_\Omega| \le \int_\Omega \mathbb{P}(\mathbf{v},k). \tag{8.84}$$

Combining (8.79), (8.80), (8.81), (8.82), (8.83), and (8.84), we obtain

$$\frac{d}{dt}\int_\Omega \tilde{K}_n(k(t,\mathbf{x}))d\mathbf{x} + \mu\int_{\{n\le|k|\le n+1\}} |\nabla k(t,\mathbf{x})|^2 d\mathbf{x} \le \int_\Omega \mathbb{P}(\mathbf{v},k)d\mathbf{x},$$

that we integrate over $[0, T]$ to give

$$\int_{\Omega} \tilde{K}_n(k(T,\mathbf{x}))d\mathbf{x} + \mu M_n^{(e)} \leq \sigma_1 + ||k_0||_{0,1,\Omega}, \tag{8.85}$$

where $M_n^{(e)}$ is expressed by (8.61), and (8.78) provides $0 \leq \tilde{K}_n(k_0) \leq |k_0|$. As $\tilde{K}_n \geq 0$, we have in particular

$$M_n^{(e)} \leq \frac{C_{k,\infty,1}(T)}{\mu}. \tag{8.86}$$

8.4.2.2 Conclusion

According to (8.77), (8.86), and inequality (A.49), Theorem A.13 in [TB], we can conclude that

$$\forall\, 1 \leq q < \frac{5}{4}, \quad ||k||_{q;1,q,\Omega} \leq Q_q\left(\frac{C_{k,\infty,1}(T)}{\mu}, C_{k,\infty,1}(T)\right) = C_{k,q}(T), \tag{8.87}$$

where Q_q is a polynomial function whose degree and coefficients only depend on Ω and q, hence Proposition 8.4. The explicit expression of Q_q is complicated and not essential. However, it can be proved that

$$\lim_{q\to 5/4} Q_q\left(\frac{C_{k,\infty,1}(T)}{\mu}, C_{k,\infty,1}(T)\right) = \infty.$$

For the purpose of estimating $\partial_t k$, let us look at the other terms in the equations. We deduce from (8.87) that

$$\mathscr{K}^{(k)}(\mathbf{v},k) + \mathscr{E}(k) \in \bigcap_{s<5/4} L^s([0,T], W^{-1,s}(\Omega)),$$

while as $\mathbb{P}(\mathbf{v},k) \in L^1([0,T]\times\Omega)$,

$$\mathbb{P}(\mathbf{v},k) \in \bigcap_{s<3/2} L^s([0,T], W^{-1,s}(\Omega)).$$

In conclusion,

$$\partial_t k = \mathbb{P}(\mathbf{v},k) - \mathscr{K}^{(k)}(\mathbf{v},k) - \mathscr{E}(k) \in \bigcap_{s<5/4} L^s([0,T], W^{-1,s}(\Omega)),$$

which yields

$$k \in \bigcap_{s<5/4} \mathscr{N}_{5/4,5/4}(W_0^{1,s}(\Omega), W^{-1,s}(\Omega)), \tag{8.88}$$

the spaces $\mathscr{N}_{p,q}(E, F)$ being generally expressed by (8.7). When we combine this result with estimate (8.77), we are led to introducing

$$\mathbb{K}_{5/4}(Q) = [\bigcap_{s<5/4} \mathscr{N}_{5/4,5/4}(W_0^{1,s}(\Omega), W^{-1,s}(\Omega))] \cap L^\infty([0,T], L^1(\Omega)), \quad (8.89)$$

as the most appropriate space of unknowns for the TKE equation (8.17.iii).

Remark 8.4. The result above indicates that the most appropriate space of tests for (8.17.iii) is the space

$$\mathbb{T}_5(Q) = \bigcup_{r>5} L^r([0,T], W_0^{1,r}(\Omega)). \quad (8.90)$$

However, for the reasons given in Remark 8.3, we shall take $L^\infty([0,T], \mathscr{D}(\Omega))$ instead of $\mathbb{T}_5(Q)$.

8.5 Matryoshka Dolls

8.5.1 NS-TKE Inequality Model

According to the conclusions of Sects. 8.3.4 and 8.4.2, the natural meaningful variational problem associated with the NS-TKE model (8.1), denoted by $\mathscr{EP}^k$, is the following[9]:

$$\begin{cases} \textit{Unknown space: } X_1(Q) \times \mathbb{K}_{5/4}(Q) = Y_1(Q), \\ \textit{Test space: } X_2(Q) \times L^\infty([0,T], \mathscr{D}(\Omega)) = Y_2(Q), \\ \textit{Initial data: } \mathbf{v}_0 \in \mathbf{L}^2(\Omega), \quad k_0 \in L^1(\Omega), \end{cases} \quad (8.91)$$

$$\begin{cases} \partial_t \mathbf{v} + \mathscr{T}^{(k)}(\mathbf{v}, k) + \nabla p + G(\mathbf{v}) = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0, \\ \partial_t k + \mathscr{K}^{(k)}(\mathbf{v}, k) + \mathscr{E}(k) = \mathbb{P}(\mathbf{v}, k), \end{cases} \quad (8.92)$$

where the spaces $X_1(Q)$ and $X_2(Q)$, initially introduced in (8.57), are given by

$$\begin{aligned} X_1(Q) &= \left[\mathscr{N}_{2,4/3}(\mathbf{W}(\Omega), \mathbf{W}_4(\Omega)') \cap L^\infty([0,T], \mathbf{L}^2(\Omega))\right] \times L^{\frac{4}{3}}(Q), \\ X_2(Q) &= L^\infty([0,T], \mathscr{W}_m(\Omega)) \times L^2(Q). \end{aligned} \quad (8.93)$$

[9]Hypothesis 8.i holds until the end of the chapter.

Problem $\mathscr{EP}^k$ means:

(1) find $(\mathbf{v}, p, k) \in Y_1$ such that for all $(\mathbf{w}, q, l) \in Y_2(Q)$, (8.21), (8.22) and (8.23) hold, where $(\partial_t \mathbf{v}(t), \mathbf{w}(t))_\Omega$ and $(\partial_t k(t), l(t))_\Omega$ are replaced by $\langle \partial_t \mathbf{v}(t), \mathbf{w}(t)\rangle$ and $\langle \partial_t k(t), l(t)\rangle$,
(2) $\forall\, \boldsymbol{\varphi} \in C^1([0,T], \mathscr{W}_m(\Omega))$ such that $\boldsymbol{\varphi}(T, \mathbf{x}) = 0$, (8.24) holds,
(3) $\forall\, \psi \in C^1([0,T], \mathscr{D}(\Omega))$ such that $\psi(T, \mathbf{x}) = 0$, (8.25) holds.

Unfortunately, we are not able to prove the existence of a solution to the problem $\mathscr{EP}^k$. Indeed, as the spaces of velocity unknowns and tests, namely

$$\mathscr{N}_{2,4/3}(\mathbf{W}(\Omega), \mathbf{W}_4(\Omega)') \cap L^\infty([0,T], \mathbf{L}^2(\Omega)) \text{ and } L^\infty([0,T], \mathscr{W}_m(\Omega)),$$

are unrelated, given any a priori solution $(\mathbf{v}, p, k)$ to $\mathscr{EP}^k$, we cannot take $\mathbf{v}$ as test in (8.92.i), so that no energy equality occurs. This prevents us from proceeding with the energy method developed in Chaps. 6 and 7 (cf. Sect. 6.7.3) to take the limit in the quadratic source term $\mathbb{P}(\mathbf{v}, k)$ of equation (8.92.iii).

This is the same difficulty as we encountered in Sect. 7.5.3 on the steady-state case with unbounded eddy viscosities. Based on that example, we introduce the variational problem $\mathscr{IP}^k$ expressed by (8.91) and

$$\begin{cases} \partial_t \mathbf{v} + \mathscr{T}^{(k)}(\mathbf{v}, k) + \nabla p + G(\mathbf{v}) = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0, \\ \partial_t k + \mathscr{K}^{(k)}(\mathbf{v}, k) + \mathscr{E}(k) \geq \mathbb{P}(\mathbf{v}, k), \end{cases} \tag{8.94}$$

where (8.94.iii) means

$$\forall\, l \in L^\infty([0,T], \mathscr{D}(\Omega)) \text{ such that } l \geq 0 \quad \text{a.e. in} \quad \Omega,$$

$$\begin{cases} \displaystyle\int_0^T \langle \partial_t k(t), l(t)\rangle dt + \int_0^T \langle \mathscr{K}^{(k)}(\mathbf{v}(t), k(t)), l(t)\rangle dt \\ \displaystyle + \int_0^T \langle \mathscr{E}(k(t)), l(t)\rangle dt \geq \int_0^T (\mathbb{P}(\mathbf{v}(t), k(t)), l(t))_\Omega dt. \end{cases} \tag{8.95}$$

Observe that $\mathscr{IP}^k$ is the variational problem associated with the PDE system (8.4). By the end of this chapter, we will have proved the following result:

Theorem 8.2. *Assume that hypothesis 8.i holds. Then Problem $\mathscr{IP}^k$ admits a solution $(\mathbf{v}, p, k)$.*

In the case where ν_t does not depend on k, $\mathscr{IP}^k$ reduces to $\mathscr{EP}$ given by (8.57) and (8.58). In consequence, Theorem 8.1 is a corollary of Theorem 8.2.

We list in the following subsections the families of variational problems interconnected with each other, deduced from successive regularization procedures, the end of the chain being $\mathscr{IP}^k$. The outline is:

(i) Introduction of the regularized nonlinear operators by convolution and truncature,
(ii) Implementation of the evolutionary version of the ε-approximations first introduced in Sects. 6.4.3 and 6.4.4,
(iii) Setting up the different regularized variational problems, checking their coherence and consistency.

8.5.2 Regularization Process

We aim to regularize each operator that poses a problem: transport, quadratic source term, pressure, and wall law.

As in Sect. 7.4.1, we consider $\rho \in C_c^\infty(\mathbb{R}^3)$, with

$$\text{supp}(\rho) \subset [-1,1] \times B(0,1), \quad \rho \geq 0, \quad ||\rho||_{0,1,\mathbb{R}^3} > 0,$$

and ρ is even, $\rho(-\mathbf{x}) = \rho(\mathbf{x})$. Let $\alpha > 0$, and

$$\rho_\alpha(t,\mathbf{x}) = \alpha^{-3} ||\rho||^{-1}_{0,1,\mathbb{R}^3} \rho\left(\frac{\mathbf{x}}{\alpha}\right).$$

- *Transport terms*. Let $\alpha > 0$ and b_α and $b_{e,\alpha}$ be the regularized transport operators defined by

$$b_\alpha(\mathbf{z};\mathbf{v},\mathbf{w}) = \frac{1}{2}\left[((\mathbf{z} \star \rho_\alpha \cdot \nabla)\,\mathbf{v},\mathbf{w})_\Omega - ((\mathbf{z} \star \rho_\alpha \cdot \nabla)\,\mathbf{w},\mathbf{v})_\Omega\right], \quad (8.96)$$

$$b_{e,\alpha}(\mathbf{z};k,l) = \frac{1}{2}\left[(\mathbf{z} \star \rho_\alpha \cdot \nabla k, l)_\Omega - (\mathbf{z} \star \rho_\alpha \cdot \nabla l, k)_\Omega\right]. \quad (8.97)$$

Observe that

$$b_\alpha(\mathbf{z};\mathbf{v},\mathbf{w}) = b(\mathbf{z}_\alpha;\mathbf{v},\mathbf{w}), \quad b_{e,\alpha}(\mathbf{z};k,l) = b_e(\mathbf{z}_\alpha;k,l), \quad \mathbf{z}_\alpha = \mathbf{z} \star \rho_\alpha.$$

Let $\mathscr{T}_\alpha^{(k)}(\mathbf{v},k)$ and $\mathscr{K}_\alpha^{(k)}(\mathbf{v},k)$ denote the operators:

$$\langle \mathscr{T}_\alpha^{(k)}(\mathbf{v},k),\mathbf{w}) = b_\alpha(\mathbf{v};\mathbf{v},\mathbf{w}) + a(\mathbf{v},\mathbf{w}) + s_v(k;\mathbf{v},\mathbf{w}), \quad (8.98)$$

$$\langle \mathscr{K}_\alpha^{(k)}(\mathbf{v},k),l)\rangle = b_{e,\alpha}(\mathbf{v};k,l) + a_e(k,l) + s_v(k;k,l). \quad (8.99)$$

- *Quadratic source term*. We set

$$\mathbb{P}_\beta = \mathbb{P}_\beta(k,t,\mathbf{x}) = \nu_t(k,t,\mathbf{x}) T_{1/\beta}(|D\mathbf{v}|^2), \quad (8.100)$$

where the truncation function T_N ($N = 1/\beta$) was first defined by (7.98), specifically $T_N(x) = x$ if $|x| \le N$, else $T_N(x) = Nx/|x|$.

- ε-*approximation*. The operator P_ε was first defined by (6.65). We recall that, given any $\mathbf{v} \in \mathbf{W}(\Omega)$, $p = P_\varepsilon(\mathbf{v})$ is the unique solution to the problem:

$$-\varepsilon \Delta p + \nabla \cdot \mathbf{v} = 0, \quad \frac{\partial p}{\partial \mathbf{n}}|_\Gamma = 0, \quad \int_\Omega p(\mathbf{x}) d\mathbf{x} = 0. \tag{8.101}$$

- *Wall law*. The wall law is regularized by the operator G_α,

$$\langle G_\alpha(\mathbf{v}), \mathbf{w} \rangle = (\mathbf{v} \tilde{H}_\alpha(\mathbf{v}), \mathbf{w})_\Gamma, \quad \tilde{H}_\alpha(\mathbf{v}) = T_{1/\alpha}(\tilde{H}(\mathbf{v})), \tag{8.102}$$

by recalling that $g(\mathbf{v}) = \mathbf{v}\tilde{H}(\mathbf{v})$.

8.5.3 Leray-$(\boldsymbol{\alpha}, \boldsymbol{\beta})$-$\boldsymbol{\varepsilon}$ NS-TKE Model

The Leray-(α, β)-ε NS-TKE model is a model which only involves $(\mathbf{v}, k)$. Its major focus is its dynamical system structure, which allows us to prove it has a solution by means of ordinary differential equations over finite subspaces of $\mathbf{W}(\Omega)$, through the application of the parabolic version of the Galerkin method (cf. Sect. 6.5).

The model is first expressed by its associated variational problem $\mathscr{EP}^k_{\alpha,\beta,\varepsilon}$. To start with, we introduce the space

$$Z_1(Q) = G_{2,\nu}(Q) \times G_{2,k}(Q), \tag{8.103}$$

where

$$\begin{aligned} G_{2,\nu}(Q) &= \mathscr{N}_{2,2}(\mathbf{W}(\Omega), \mathbf{W}(\Omega)') \cap L^\infty([0,T], \mathbf{L}^2(\Omega)), \\ G_{2,k}(Q) &= \mathscr{N}_{2,2}(H^1_0(\Omega), H^{-1}(\Omega)) \cap L^\infty([0,T], L^2(\Omega)). \end{aligned} \tag{8.104}$$

Given any $\alpha, \beta, \varepsilon > 0$, let $\mathscr{EP}^k_{\alpha,\beta,\varepsilon}$ be the variational problem,

$$\begin{cases} \textit{Unknown space: } Z_1(Q), \\ \textit{Test space: } L^2([0,T], \mathbf{W}(\Omega) \times H^1_0(\Omega)) = Z_2(Q), \\ \textit{Initial data: } \mathbf{v}_0 \in \mathbf{L}^2(\Omega), \quad k_0 \in L^2(\Omega), \end{cases} \tag{8.105}$$

$$\begin{cases} \partial_t \mathbf{v} + \mathscr{T}^{(k)}_\alpha(\mathbf{v}, k) + \nabla P_\varepsilon(\mathbf{v}) + G_\alpha(\mathbf{v}) = \mathbf{f}, \\ \qquad\qquad \partial_t k + \mathscr{K}^{(k)}_\alpha(\mathbf{v}, k) + \mathscr{E}(k) = \mathbb{P}_\beta(\mathbf{v}, k), \end{cases} \tag{8.106}$$

which is associated with the PDE system, which we write in a simplified way:

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v}_\alpha \cdot \nabla)\, \mathbf{v} + \dfrac{1}{2}\mathbf{v}(\nabla \cdot \mathbf{v}_\alpha) - \nabla \cdot [(2\nu + \nu_t)\, D\mathbf{v}] + \nabla P_\varepsilon(\mathbf{v}) = \mathbf{f}, \\ \partial_t k + \mathbf{v}_\alpha \cdot \nabla k + \dfrac{1}{2}k(\nabla \cdot \mathbf{v}_\alpha) + \nabla \cdot [(\mu + \mu_t)\nabla k] + k\, E(k, t, \mathbf{x}) = \nu_t T_{1/\beta}(|D\mathbf{v}|^2), \\ \qquad - [(2\nu + \nu_t)\, D\mathbf{v} \cdot \mathbf{n}]_\tau + \dfrac{1}{2}(\mathbf{v}_\alpha \cdot \mathbf{n})\mathbf{v}_\tau|_\Gamma = \mathbf{v}_\tau T_{1/\alpha}(\tilde{H}(\mathbf{v})), \\ \qquad \mathbf{v} \cdot \mathbf{n}|_\Gamma = 0, \\ \qquad k|_\Gamma = 0, \end{cases}$$

with the corresponding initial data, and where $\mathbf{v}_\alpha = \mathbf{v} \star \rho_\alpha$. Problem $\mathscr{E}\mathscr{P}^k_{\alpha,\beta,\varepsilon}$ is specified by:

(1) (8.21) holds with $\langle \partial_t \mathbf{v}(t), \mathbf{w}(t)\rangle$ instead of $(\partial_t \mathbf{v}(t), \mathbf{w}(t))_\Omega$, b_α instead of b, $P_\varepsilon(\mathbf{v}(t))$ instead of $p(t)$, and G_α instead of G,
(2) (8.23) holds with $\langle \partial_t k(t), l(t)\rangle$ instead of $(\partial_t k(t), l(t))_\Omega$, $b_{e,\alpha}$ instead of b_e, and $\mathbb{P}(\mathbf{v}(t), k(t))$ is replaced by $\mathbb{P}_\beta(\mathbf{v}(t), k(t))$,
(3) (8.24) and (8.25) hold.

It is worthwhile noting that Remark 8.2 applies to $\mathscr{E}\mathscr{P}^k_{\alpha,\beta,\varepsilon}$ as well as to all the following variational problems, by replacing when needed $L^2([0, T])^3$ by the appropriate $L^p([0, T]) \times L^q([0, T]) \cdots$ space.

Lemma 8.2. *Let $\alpha, \beta, \varepsilon > 0$ be given. Then problem $\mathscr{E}\mathscr{P}^k_{\alpha,\beta,\varepsilon}$ is meaningful and consistent, and any a priori solution $(\mathbf{v}, k)$ satisfies the energy equality, $\forall\, t > 0$,*

$$\begin{cases} \dfrac{1}{2}||\mathbf{v}(t)||^2_{0,2,\Omega} + \displaystyle\int\int_{Q_t} (2\nu + \nu_t(k(s, \mathbf{x}), s, \mathbf{x})|D\mathbf{v}(s, \mathbf{x})|^2 d\mathbf{x}ds \\ -\displaystyle\int_0^t (P_\varepsilon(\mathbf{v}(s)), \nabla \cdot \mathbf{v}(s))_\Omega ds + \int_0^t \langle G_\alpha(\mathbf{v}(s)), \mathbf{v}(s)\rangle ds = \\ \qquad \dfrac{1}{2}||\mathbf{v}_0||^2_{0,2,\Omega} + \displaystyle\int_0^t \langle \mathbf{f}(s), \mathbf{v}(s)\rangle ds, \end{cases} \tag{8.107}$$

where $Q_t = [0, t] \times \Omega$.

Proof. Let $(\mathbf{v}, k) \in Z_1(Q)$ denote any a priori solution to $\mathscr{E}\mathscr{P}^k_{\alpha,\beta,\varepsilon}$. Once the meaningfulness and the consistency are proven, the energy equality (8.107) is straightforward since:

(a) as $\mathbf{v} \in L^2([0, T], \mathbf{W}(\Omega))$, it can be taken as test in (8.106.i),
(b) we have by Lemma 6.3,

$$b_\alpha(\mathbf{v}; \mathbf{v}; \mathbf{v}) = b(\mathbf{v}_\alpha; \mathbf{v}, \mathbf{v}) = 0,$$

(c) $\mathbf{v}_0 \in \mathbf{L}^2(\Omega)$,
(d) Lemma A.8 in [TB] applies, and we can integrate the resulting equation with respect to time over $[0, T]$.

We first prove the meaningfulness, then the consistency, step by step. Let $(\mathbf{w}, l) \in Z_2(T)$ denote any given test.

STEP 1. *Meaningfulness of (8.106.i).* As $\mathbf{f} \in L^2([0,T], \mathbf{W}(\Omega)')$, we must prove that all terms in the l.h.s. of (8.106.i) belong to $L^2([0,T], \mathbf{W}(\Omega)') = \left[L^2([0,T], \mathbf{W}(\Omega))\right]'$. We consider each term separately:

- *Transport term.* We first observe that the inequality

$$||\mathbf{v}_\alpha||_{0,\infty,\Omega} \leq C\alpha^{-3}||\mathbf{v}||_{0,2,\Omega}$$

combined with $\mathbf{v} \in L^\infty([0,T], \mathbf{L}^2(\Omega))$ yields $\mathbf{v}_\alpha \in \mathbf{L}^\infty(Q)$ and

$$||\mathbf{v}_\alpha||_{0,\infty,Q} \leq C\alpha^{-3}||\mathbf{v}||_{\infty;0,2,\Omega}. \tag{8.108}$$

Based on this fact and

$$|b(\mathbf{v}_\alpha(t), \mathbf{v}(t), \mathbf{w}(t))| \leq \frac{1}{2}||\mathbf{v}_\alpha||_{0,\infty,\Omega}\left[||\mathbf{v}(t)||_{\mathbf{W}(\Omega)}||\mathbf{w}(t)||_{0,2,\Omega} + ||\mathbf{w}(t)||_{\mathbf{W}(\Omega)}||\mathbf{v}(t)||_{0,2,\Omega}\right], \tag{8.109}$$

the following holds for almost all $t \in [0,T]$:

$$|b(\mathbf{v}_\alpha(t), \mathbf{v}(t), \mathbf{w}(t))| \leq \frac{C}{2\alpha^3}||\mathbf{v}||_{\infty;0,2,\Omega}||\mathbf{v}(t)||_{\mathbf{W}(\Omega)}||\mathbf{w}(t)||_{\mathbf{W}(\Omega)}. \tag{8.110}$$

Therefore, by the Cauchy–Schwarz and Sobolev inequalities

$$\left|\int_0^T b(\mathbf{v}_\alpha(t), \mathbf{v}(t), \mathbf{w}(t))dt\right| \leq \frac{C}{2\alpha^3}||\mathbf{v}||_{\infty;0,2,\Omega}||\mathbf{v}||_{2;\mathbf{W}(\Omega)}||\mathbf{w}||_{2;\mathbf{W}(\Omega)}, \tag{8.111}$$

where C only depends on Ω.
- *Diffusion term.* We obviously have

$$\int_0^T (a(\mathbf{v}(t), \mathbf{w}(t)) + s_\nu(\mathbf{v}(t), \mathbf{w}(t))dt \leq (2\nu + ||\nu_t||_\infty)||\mathbf{v}||_{2;\mathbf{W}(\Omega)}||\mathbf{w}||_{2;\mathbf{W}(\Omega)}.$$

Thus, by (8.111), we have $\mathscr{T}_\alpha^{(k)}(\mathbf{v}, k) \in L^2([0,T], \mathbf{W}(\Omega)')$ and

$$||\mathscr{T}_\alpha^{(k)}(\mathbf{v}, k)||_{2;\mathbf{W}(\Omega)'} \leq C\left(2\nu + ||\nu_t||_\infty + \frac{1}{\alpha^3}||\mathbf{v}||_{\infty;0,2,\Omega}\right)||\mathbf{v}||_{2;\mathbf{W}(\Omega)}. \tag{8.112}$$

- *Pressure term.* We deduce from estimate 6.70,

$$\left|\int_0^T (P_\varepsilon(\mathbf{v}(t), \nabla\cdot\mathbf{w}(t))_\Omega dt\right| \leq \frac{C}{\varepsilon}||\mathbf{v}||_{2;\mathbf{W}(\Omega)}||\mathbf{w}||_{2;\mathbf{W}(\Omega)}, \tag{8.113}$$

so that $\nabla P_\varepsilon(\mathbf{v}) \in L^2([0,T], \mathbf{W}(\Omega)')$ and

$$||\nabla P_\varepsilon(\mathbf{v})||_{2;\mathbf{W}(\Omega)'} \leq \frac{C}{\varepsilon}||\mathbf{v}||_{2;\mathbf{W}(\Omega)}. \tag{8.114}$$

- *Wall law*. As $||\tilde{H}_\alpha||_\infty \leq 1/\alpha$, then $G_\alpha(\mathbf{v}) \in L^2([0,T], \mathbf{W}(\Omega)')$ and we have

$$||G_\alpha(\mathbf{v})||_{2;\mathbf{W}(\Omega)'} \leq \frac{C}{\alpha}||\mathbf{v}||_{2;\mathbf{W}(\Omega)}, \tag{8.115}$$

which concludes the proof.

STEP 2. *Meaningfulness of (8.106.ii).* We must prove that all terms in the l.h.s. and the source term of (8.106.ii) belong to $L^2([0,T], H^{-1}(\Omega)) = \left[L^2([0,T], H_0^1(\Omega))\right]'$. The same technique as in Step 1 provides

$$||\mathscr{K}_\alpha^{(k)}(\mathbf{v},k)||_{2;\mathbf{W}(\Omega)'} \leq C\left(\mu + ||\mu_t||_\infty + \frac{1}{\alpha^3}||\mathbf{v}||_{\infty;0,2,\Omega}\right)||k||_{2;H_0^1(\Omega)}. \tag{8.116}$$

Furthermore, as $E \in L^\infty(\mathbb{R} \times \mathbb{R}_+ \times \Omega)$,

$$||\mathscr{E}(k)||_{2;\mathbf{W}(\Omega)'} \leq ||E||_\infty ||k||_{0,2,Q}. \tag{8.117}$$

Finally, $\mathbb{P}_\beta(\mathbf{v},k) \in L^\infty(Q)$ and

$$||\mathbb{P}_\beta(\mathbf{v},k)||_{0,\infty,Q} \leq \beta^{-1}||\nu_t||_\infty, \tag{8.118}$$

in particular $\mathbb{P}_\beta(\mathbf{v},k) \in L^2([0,T], H^{-1}(\Omega))$.

STEP 3. *Consistency*. We start with proving the consistency of (8.106.i). Combining the energy equality (8.107) with the inequality (6.71), namely $(P_\varepsilon(\mathbf{v}), \nabla\cdot\mathbf{v}) \leq 0$, yields

$$\left\{\begin{aligned} &\frac{1}{2}||\mathbf{v}(t)||_{0,2,\Omega}^2 + \int\int_{Q_t}(2\nu + \nu_t(k(s,\mathbf{x}),s,\mathbf{x})|D\mathbf{v}(s,\mathbf{x})|^2 ds d\mathbf{x} \\ &\qquad + \int_0^t \langle G_\alpha(\mathbf{v}(s)),\mathbf{v}(s)\rangle ds \leq \frac{1}{2}||\mathbf{v}_0||_{0,2,\Omega}^2 + \int_0^t \langle \mathbf{f}(s),\mathbf{v}(s)\rangle ds. \end{aligned}\right. \tag{8.119}$$

From here, following the proof of Proposition 8.1, we deduce from (8.119) that $\mathbf{v}$ satisfies estimates (8.28) and (8.29). It is worth noting that this estimate depends neither on ε nor on α. We easily deduce from Step 1 that

$$||\partial_t\mathbf{v}||_{2;\mathbf{W}(\Omega)'} \leq C\left(1 + \frac{1}{\varepsilon} + \frac{1}{\alpha^3}\right), \tag{8.120}$$

where C depends on T, ν, $||\nu_t||_\infty$, $||\mathbf{v}_0||_{0,2,\Omega}$, and $||\mathbf{f}||_{2;\mathbf{W}(\Omega)'}$, which yields $\mathbf{v} \in G_{2,\nu}(Q)$ (cf. (8.104)), thereby proving the consistency of (8.106.i).

We now prove the consistency of (8.106.ii). In view of the nature of the spaces $G_{2,k}(Q)$ and $Z_2(Q)$ (cf. (8.105)), we can take k as test in (8.106.ii) and use Lemma A.8, $b_{e,\alpha}(\mathbf{v}; k, k) = 0$ which obviously holds, $\mu_t \geq 0$. This standard procedure provides

$$\frac{1}{2}||k||^2_{\infty;0,2,\Omega} + \mu||\nabla k||^2_{0,2,Q} \leq ||\nu_t||_\infty \beta^{-1}||k||_{0,1,Q} + \frac{1}{2}||k_0||^2_{0,2,\Omega}, \quad (8.121)$$

Using the Sobolev and Young inequalities gives

$$||k||^2_{\infty;0,2,\Omega} + \mu||\nabla k||^2_{0,2,Q} \leq \frac{C}{\mu\beta} + ||k_0||^2_{0,2,\Omega}. \quad (8.122)$$

Hence, by a process similar to that used in the proof of Proposition 8.1,

$$||k||_{\infty;0,2,\Omega} \leq \left(\frac{C}{\mu\beta} + ||k_0||^2_{0,2,\Omega}\right)^{\frac{1}{2}} = C^{(\beta)}_{k,\infty,2}(T), \quad (8.123)$$

$$||k||_{2;H^1_0(\Omega)} \leq \frac{1}{\sqrt{\inf(T^{-1}, 2\mu)}} C^{(\beta)}_{k,\infty,2}(T) = C^{(\beta)}_{k,2}(T). \quad (8.124)$$

Finally, we deduce from the previous estimates that

$$||\partial_t k||_{2;H^{-1}(\Omega)} \leq C\left(1 + \frac{1}{\alpha^3} + \frac{1}{\beta}\right), \quad (8.125)$$

where C depends on the data, which proves $k \in G_{2,k}(Q)$, hence the consistency of (8.106.ii), thereby concluding the proof.

□

When $\varepsilon \to 0$, the Leray-α-ε NS-TKE model converges to the Leray-α NS-TKE model, formulated in the following.

8.5.4 Leray-(α, β) NS-TKE Model

The variational problem $\mathscr{EP}^k_{\alpha,\beta}$ associated with the Leray-(α, β) NS-TKE model (8.2), and whose unknown is $(\mathbf{v}, p, k)$, is the following:

$$\begin{cases} \textit{Unknowns} : G_{2,\nu}(Q) \times L^2([0,T], L^2_0(\Omega)) \times G_{2,k}(Q) = U_1(Q), \\ \textit{Tests} : L^2([0,T], \mathbf{W}(\Omega)) \times L^2(Q) \times L^2([0,T], H^1_0(\Omega)) = U_2(Q), \\ \textit{Initial data} : \mathbf{v}_0 \in \mathbf{L}^2(\Omega), \quad k_0 \in L^2(\Omega), \end{cases} \quad (8.126)$$

$$\begin{cases} \partial_t \mathbf{v} + \mathscr{T}_\alpha^{(k)}(\mathbf{v}, k) + \nabla p + G_\alpha(\mathbf{v}) = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0, \\ \partial_t k + \mathscr{K}_\alpha^{(k)}(\mathbf{v}, k) + \mathscr{E}(k) = \mathbb{P}_\beta(\mathbf{v}, k), \end{cases} \tag{8.127}$$

recalling that $G_{2,\nu}(Q)$ and $G_{2,k}(Q)$ are defined by (8.104). This means that

(1) (8.21) holds with $\langle \partial_t \mathbf{v}(t), \mathbf{w}(t) \rangle$ instead of $(\partial_t \mathbf{v}(t), \mathbf{w}(t))_\Omega$, b_α instead of b, G_α instead of G,
(2) (8.22) holds,
(3) (8.23) holds with $\langle \partial_t k(t), l(t) \rangle$ instead of $(\partial_t k(t), l(t))_\Omega$, $b_{e,\alpha}$ instead of b_e, and $\mathbb{P}(\mathbf{v}(t), k(t))$ is replaced by $\mathbb{P}_\beta(\mathbf{v}(t), k(t))$,
(4) (8.24) and (8.25) hold.

By combining the proof of Proposition 8.2 with that of Proposition 8.3 in taking $\theta = 2$, we deduce that $\mathscr{EP}^k_{\alpha,\beta}$ is meaningful and consistent. At this level, the estimates for $||p||_{0,2,Q}$ and $||\partial_t \mathbf{v}||_{2,\mathbf{W}(\Omega)'}$, which do not need to be specified, depend on α (and not on β) and blow up when $\alpha \to 0$. We just need to note:

$$||p||_{0,2,Q} \le C_{p,2,\alpha}, \quad ||\partial_t \mathbf{v}||_{2;\mathbf{W}(\Omega)'} \le C_{\partial_t v,2,\alpha}. \tag{8.128}$$

However, the solutions of $\mathscr{EP}^k_{\alpha,\beta}$ also satisfy estimates uniform in α and β, which are the same as those derived in Sects. 8.3.1, 8.3.3 and 8.4. To be more specific:

Lemma 8.3. *Let $(\mathbf{v}, p, k) \in U_1(Q)$ (cf. (8.126)) be any a priori solution to $\mathscr{EP}^k_{\alpha,\beta}$. Then*

(i) $\mathbf{v}$ satisfies estimates (8.28) and (8.29),
(ii) p satisfies estimate (8.45) and $\partial_t \mathbf{v}$ satisfies (8.55),
(iii) k satisfies estimates (8.60) and (8.77), as well as (8.88).

Moreover, $(\mathbf{v}, k)$ satisfies the energy equality, $\forall\, t > 0$,

$$\begin{cases} \frac{1}{2}||\mathbf{v}(t)||^2_{0,2,\Omega} + \int\int_{Q_t} (2\nu + \nu_t(k(s,\mathbf{x}), s, \mathbf{x})|D\mathbf{v}(s,\mathbf{x})|^2 ds d\mathbf{x} \\ \qquad + \int_0^t \langle G_\alpha(\mathbf{v}(s)), \mathbf{v}(s)\rangle ds = \frac{1}{2}||\mathbf{v}_0||^2_{0,2,\Omega} + \int_0^t \langle \mathbf{f}(s), \mathbf{v}(s)\rangle ds, \end{cases} \tag{8.129}$$

Proof. The energy equality (8.129) is obviously satisfied in this case. We prove each item separately.

(i) Let $(\mathbf{v}, p, k) \in U_1(Q)$ be any a priori solution to $\mathscr{EP}^k_{\alpha,\beta}$. Reproducing the analysis of $\mathscr{EP}^k_{\alpha,\beta,\varepsilon}$, shows that $\mathbf{v}$ satisfies estimates (8.28) and (8.29).
(ii) We know from inequality (7.57) that $||\mathbf{v}_\alpha||_{0,p,\Omega} \le ||\mathbf{v}||_{0,p,\Omega}$. Consequently, adjusting inequality (8.37) leads to

$$||\mathscr{T}_\alpha^{(k)}(\mathbf{v}, k)||_{\mathbf{W}(\Omega)'} \le ||\mathscr{T}^{(k)}(\mathbf{v}, k)||_{\mathbf{W}(\Omega)'}.$$

Furthermore, as $0 \leq T_{1/\alpha}(\tilde{H}(\mathbf{v})) \leq \tilde{H}(\mathbf{v})$, we also obtain

$$||G_\alpha(\mathbf{v})||_{\mathbf{W}(\Omega)'} \leq ||G(\mathbf{v})||_{\mathbf{W}(\Omega)'}.$$

From here, the proof of Proposition 8.3 also applies to this case, and we deduce that p and $\partial_t \mathbf{v}$ indeed satisfy estimates (8.45) and (8.55).[10]

(iii) As $\forall\, x > 0$, $T_{1/\beta}(x) \leq x$, we observe

$$||\mathbb{P}_\beta(\mathbf{v}, k)||_{0,1,Q} \leq ||\mathbb{P}(\mathbf{v}, k)||_{0,1,Q} \leq \sigma_1, \tag{8.130}$$

where σ_1 is defined by (8.59). Then the procedure developed in Sect. 8.4 can also be applied to this case, and in addition Ω is bounded and therefore k_0 also belongs to $L^1(\Omega)$. The only difference is the issue of (8.71) and (8.80). Fortunately, as $\nabla \cdot \mathbf{v} = 0$ we have $\nabla \cdot \mathbf{v}_\alpha = 0$, giving

$$\forall\, k, l \in H_0^1(\Omega), \quad b_{e,\alpha}(\mathbf{v}; k, l) = (\mathbf{v}_\alpha \cdot \nabla k, l)_\Omega.$$

According to Remark 7.4, Lemma 7.3 still applies in this case, even if we do not know whether or not $\mathbf{v}_\alpha \cdot \mathbf{n}|_\Gamma = 0$, hence (8.71) and (8.80) with $b_{e,\alpha}$ instead of b, and therefore (8.60) and (8.77). The remainder of the proof is straightforward. □

8.5.5 Leray-α NS-TKE Model and Results

Let $\mathscr{VP}_\alpha^\kappa$ be the variational problem associated with the Leray-α model (8.3) and specified by

$$\begin{cases} \textit{Unknowns: } G_{2,\nu}(Q) \times L^2([0,T], L_0^2(\Omega)) \times \mathbb{K}_{5/4}(Q) = V_1(Q), \\ \textit{Tests: } L^2([0,T], \mathbf{W}(\Omega)) \times L^2(Q) \times L^\infty([0,T], \mathscr{D}(\Omega)) = V_2(Q), \\ \textit{Initial data: } \mathbf{v}_0 \in \mathbf{L}^2(\Omega), \quad k_0 \in L^1(\Omega), \end{cases} \tag{8.131}$$

$$\begin{cases} \partial_t \mathbf{v} + \mathscr{T}_\alpha^{(k)}(\mathbf{v}, k) + \nabla p + G_\alpha(\mathbf{v}) = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0, \\ \partial_t k + \mathscr{K}_\alpha^{(k)}(\mathbf{v}, k) + \mathscr{E}(k) = \mathbb{P}(\mathbf{v}, k), \end{cases} \tag{8.132}$$

where $\mathbb{K}_{5/4}(Q)$ and $G_{2,\nu}(Q)$ are defined by (8.89) and (8.104), respectively. In this problem,

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[10] Strictly speaking, we cannot directly write equality (8.50) in this case, and we should proceed by approximation, which would lead us into extra technical issues before arriving at the same result. Therefore we skip the details.

(1) (8.21) holds with $\langle \partial_t \mathbf{v}(t), \mathbf{w}(t)\rangle$ instead of $(\partial_t \mathbf{v}(t), \mathbf{w}(t))_\Omega$, b_α instead of b, G_α instead of G,
(2) (8.22) holds,
(3) (8.23) holds with $\langle \partial_t k(t), l(t)\rangle$ instead of $(\partial_t k(t), l(t))_\Omega$, $b_{e,\alpha}$ instead of b_e,
(4) (8.24) holds and (8.25) holds $\forall\, \psi \in C^1([0,T], \mathscr{D}(\Omega))$ such that $\psi(T, \mathbf{x}) = 0$.

We summarize the properties satisfied by $\mathscr{E}\mathscr{P}^k_\alpha$ in the next statement, deduced from the argumentation above.

Lemma 8.4. *Problem $\mathscr{E}\mathscr{P}^k_\alpha$ is meaningful and consistent. Moreover, let $(\mathbf{v}, p, k)$ be any a priori solution to $\mathscr{E}\mathscr{P}^k_\alpha$. Then*

(i) $\mathbf{v}$ satisfies estimates (8.28) and (8.29),
(ii) p and $\partial_t \mathbf{v}$ satisfy estimate (8.128),
(iii) p satisfies estimate (8.45), and $\partial_t \mathbf{v}$ satisfies (8.55),
(iv) k satisfies estimates (8.60) and (8.77), as well as (8.88).

Finally, $(\mathbf{v}, k)$ satisfies the energy equality (8.129).

By the end of this chapter, we will have proven the following results.

Theorem 8.3. *Given any $\alpha, \beta, \varepsilon > 0$, problem $\mathscr{E}\mathscr{P}^k_{\alpha,\beta,\varepsilon}$ admits a solution.*

Theorem 8.4. *Given any $\alpha, \beta > 0$, the family $(\mathscr{E}\mathscr{P}^k_{\alpha,\beta,\varepsilon})_{\varepsilon>0}$ converges to $\mathscr{E}\mathscr{P}^k_{\alpha,\beta}$, which therefore admits a solution.*

Theorem 8.5. *Given any $\alpha > 0$, the family $(\mathscr{E}\mathscr{P}^k_{\alpha,\beta})_{\beta>0}$ converges to $\mathscr{E}\mathscr{P}^k_\alpha$, which therefore admits a solution.*

Theorem 8.6. *The family $(\mathscr{E}\mathscr{P}^k_\alpha)_{\alpha>0}$ converges to $\mathscr{I}\mathscr{P}^k$.*

Observe that Theorems 8.5 and 8.6 yield Theorem 8.2.

8.6 Compactness Machinery

As in Chaps. 6 and 7, the proofs of the theorems rely on general compactness results. In this section, we aim in particular to

(i) outline the appropriate space sequences for the purpose of the Aubin–Lions Lemma,
(ii) describe the various extracting subsequence principle packages, similar to those introduced in Sects. 6.3.3 and 7.3,
(iii) establish some convergence lemmas.

8.6.1 Aubin–Lions Lemma Framework

Evolutionary equations involve compactness principles which are a little bit more tricky than in the steady-state case and require the Aubin–Lions Lemma [14, 213, 214], specified by Lemma A.9 in [TB].

This raises the issue of finding the appropriate space sequences that fulfill the condition for the application of Aubin–Lions Lemma, which will be determined by all the estimates established from Sect. 8.3.1.

We first consider the case of the equation satisfied by the velocity $\mathbf{v}$, then the TKE equation.

- *Velocity equation.* Whatever the variational problem, the issue of the nonlinear wall-law boundary condition must be addressed, which needs extra compactness properties on Γ, motivating the choices in what follows.
  We already have faced this difficulty in Sect. 8.3.2, when seeking estimates for $||G(\mathbf{v})||_{\mathbf{W}(\Omega)}$. To deal with it, we introduced the space $\mathbf{W}_{3/4}(\Omega)$, first defined by (8.18) and then characterized as an interpolation space by (8.40). The results from Lions-Magenes [214] already mentioned in Sect. 8.3.2, combined with the identification of the Hilbert space $\mathbf{W}_{3/4}(\Omega)$ with its dual space, yield the sequence

$$\mathbf{W}(\Omega) \hookrightarrow \mathbf{W}_{3/4}(\Omega) \hookrightarrow \mathbf{W}(\Omega)', \tag{8.133}$$

  each space being dense in the next. Moreover, as $\mathbf{W}(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$ and the embedding is compact, then we deduce from (8.41) that $\mathbf{W}(\Omega) \hookrightarrow \mathbf{W}_{3/4}(\Omega)$ is also compact.
  However, although sequence (8.133) is appropriate for demonstrating Theorems 8.3, 8.4 and 8.5, it is not appropriate for proving Theorem 8.6, due to estimate (8.55) involving $\mathbf{W}_4(\Omega)$. We have already noted in the proof of Proposition (8.3) that $\mathbf{W}(\Omega)' \hookrightarrow \mathbf{W}_4(\Omega)'$, with dense injection. Consequently, we will use the sequence

$$\mathbf{W}(\Omega) \hookrightarrow \mathbf{W}_{3/4}(\Omega) \hookrightarrow \mathbf{W}_4(\Omega)', \tag{8.134}$$

  for proving Theorem 8.6.
- *TKE equation.* From this point of view, the TKE equation is much more simple. According to the above, the standard sequence

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \tag{8.135}$$

is appropriate for the proof of Theorems 8.3 and 8.4. Moreover, any given $q_0 < 5/4$ close enough from $5/4$ provides a sequence

$$W_0^{1,q_0}(\Omega) \hookrightarrow L^{q_0}(\Omega) \hookrightarrow W_0^{-1,q_0'}(\Omega), \tag{8.136}$$

that deals with the proof of Theorems 8.5 and 8.6.

8.6.2 Evolutionary Extracting Subsequence Principles

8.6.2.1 Evolutionary Velocity Extracting Subsequence Principle

Lemma 8.5. *Let $p = 4/3, 2$ and $(\mathbf{v}_n)_{n\in\mathbf{N}}$ be a sequence bounded in the spaces $\mathcal{N}_{2,p}(\mathbf{W}(\Omega), \mathbf{W}(\Omega)')$ and $L^\infty([0,T], \mathbf{L}^2(\Omega))$. Then there exists*

$$\mathbf{v} \in \mathcal{N}_{2,p}(\mathbf{W}(\Omega), \mathbf{W}(\Omega)') \cap L^\infty([0,T], \mathbf{L}^2(\Omega)) = G_{p,v}(Q) \tag{8.137}$$

such that from the sequence $(\mathbf{v}_n)_{n\in\mathbf{N}}$, we can extract a subsequence $(\mathbf{v}_{n_k})_{k\in\mathbf{N}}$ which converges to $\mathbf{v}$:

(i) weakly in $\in L^2([0,T], \mathbf{W}(\Omega))$,
(ii) weakly star in $L^\infty([0,T], \mathbf{L}^2(\Omega))$,
(iii) strongly in $L^2([0,T], \mathbf{W}_{3/4}(\Omega))$,
(iv) strongly in $\mathbf{L}^p(Q)$, $1 \le p < 10/3$, a.e. in Q,
(v) moreover, $(\gamma_0\mathbf{v}_{n_k})_{k\in\mathbf{N}}$ strongly converges to $\gamma_0\mathbf{v}$ in $\mathbf{L}^2([0,T]\times\Gamma)$, a.e. in $[0,T]\times\Gamma$, and there exists $F_2 \in \mathbf{L}^2([0,T]\times\Gamma)$ such that $|\gamma_0\mathbf{v}_{n_k}| \le F_2$.

Moreover, $(\mathbf{v}_{n_k})_{k\in\mathbf{N}}$ may be chosen such that $(\partial_t\mathbf{v}_{n_k})_{k\in\mathbf{N}}$ weakly converges to $\partial_t\mathbf{v}$ in $L^p([0,T], \mathbf{W}(\Omega)')$.

Proof. We treat each item separately.

(i) Holds since $L^2([0,T], \mathbf{W}(\Omega))$ is a reflexive Banach space.
(ii) Holds since $L^\infty([0,T], \mathbf{L}^2(\Omega)) = (L^1([0,T], \mathbf{L}^2(\Omega)))'$.
(iii) Follows from Aubin–Lions' Lemma discussed above.
(iv) As $(\mathbf{v}_n)_{n\in\mathbf{N}}$ is bounded in $L^2([0,T], \mathbf{W}(\Omega))$, it is also bounded in the space $L^2([0,T], \mathbf{L}^6(\Omega))$ as well as in $L^\infty([0,T], \mathbf{L}^2(\Omega))$. We infer from the interpolation inequality (A.37) that it is bounded in $\mathbf{L}^{10/3}(Q)$.

Item (iii) ensures it is compact in $\mathbf{L}^{4/3}(Q)$. Consequently, we deduce from the Hölder inequality the compactness of $(\mathbf{v}_n)_{n\in\mathbf{N}}$ in $\mathbf{L}^p(Q)$, $1 \le p < 10/3$, since Q is bounded.

The a.e. convergence results from the inverse Lebesgue Theorem (cf. Theorem A.10).

(v) We recall that $\gamma_0 : \mathbf{W}_{3/4}(\Omega) \to \mathbf{H}^{1/4}(\Gamma)$ is a continuous map, $\mathbf{H}^{1/4}(\Gamma) \hookrightarrow \mathbf{L}^2(\Gamma)$, and $L^2([0,T], \mathbf{L}^2(\Gamma)) = \mathbf{L}^2([0,T]\times\Gamma)$, hence item iv).

In order to conclude, observe that $L^p([0,T], \mathbf{W}(\Omega)')$ is also a reflexive space. Therefore, since $(\partial_t \mathbf{v}_n)_{n\in\mathbf{N}}$ is bounded in $L^p([0,T], \mathbf{W}(\Omega)')$, $(\mathbf{v}_{n_k})_{k\in\mathbf{N}}$ may be chosen such that $(\partial_t \mathbf{v}_{n_k})_{k\in\mathbf{N}}$ weakly converges to some $\mathbf{g}$ in $L^p([0,T], \mathbf{W}(\Omega)')$. Lemma A.6 in [TB] combined with the results above yields $g = \partial_t \mathbf{v}$. □

When we use Lemma 8.5, we shall refer to it as the VESP, $\mathbf{v}$ as the the V-ESP-limit, and the subsequence $(\mathbf{v}_{n_k})_{k\in\mathbf{N}}$ will be denoted by $(\mathbf{v}_n)_{n\in\mathbf{N}}$ for simplicity, including items (i) to (v). The same applies for families $(\mathbf{v}_\varepsilon)_{\varepsilon>0}$.

8.6.2.2 Evolutionary K Extracting Subsequence Principle

Let $J_{q,k}(Q)$ denote the space:

$$J_{q,k}(Q) = \mathcal{N}_{q,q}(H_0^1(\Omega), H^{-1}(\Omega)) \cap L^\infty([0,T], L^1(\Omega)). \tag{8.138}$$

Lemma 8.6. *Let $(k_n)_{n\in\mathbf{N}}$ be a sequence bounded in each $J_{q,k}(Q)$, $1 \le q < 5/4$. Then there exists*

$$k \in \mathbb{K}_{5/4}(Q) = \bigcap_{q<5/4} J_{q,k}(Q),$$

(cf. also (8.89)) such that from $(k_n)_{n\in\mathbf{N}}$, we can extract a subsequence $(k_{n_j})_{j\in\mathbf{N}}$ which converges to k:

(i) weakly in each $\mathcal{N}_{q,q}(W_0^{1,q}(\Omega), W^{-1,q}(\Omega))$, $1 \le q < 5/4$,
(ii) strongly in each $L^q(Q)$, $1 \le q < 29/14$, a.e. in Q.

Moreover, $(k_{n_j})_{j\in\mathbf{N}}$ may be chosen such that $(\partial_t k_{n_j})_{j\in\mathbf{N}}$ weakly converges to k in each $W^{-1,q}$, $1 \le q < 5/4$.

Proof. Observe that each space $\mathcal{N}_{q,q}(W_0^{1,q}(\Omega), W^{-1,q}(\Omega))$ is a reflexive Banach space, and if $q_1 \le q_2$,

$$\mathcal{N}_{q_2,q_2}(W_0^{1,q_2}(\Omega), W^{-1,q_2}(\Omega)) \subset \mathcal{N}_{q_1,q_1}(W_0^{1,q_1}(\Omega), W^{-1,q_1}(\Omega)).$$

Consequently, item (i) follows using reasoning similar to that of Lemma 7.4. Item ii) derives from the Aubin–Lions Lemma applied to $\mathcal{N}_{q,q}(W_0^{1,q}(\Omega), W^{-1,q}(\Omega))$, $q < 5/4$, then using the uniqueness of the limit. This yields compactness in $L^{5/4}(\Omega)$.

We obtain the exponent 29/14 by observing that $W_0^{1,5/4}(\Omega) \hookrightarrow L^{15/7}(\Omega)$. Then the Hölder inequality allows the identification of the interpolate spaces between the spaces $L^{5/4}([0,T], L^{15/7}(\Omega))$ and $L^\infty([0,T], L^1(\Omega))$ following the same technique as that which leads to the interpolation inequality (A.37). We find that

$$L^{5/4}([0,T], L^{15/7}(\Omega)) \cap L^\infty([0,T], L^1(\Omega)) \hookrightarrow L^{\frac{29}{14}}(Q).$$

We skip the technical details.

The a.e. convergence is a consequence of the inverse Lebesgue Theorem. The statement concerning $(\partial_t k_n)_{n\in\mathbf{N}}$ relies on the same argumentation as in the proof of Lemma 8.5, and we again skip the technical details. □

As for the V-ESP, we shall speak of the K-ESP-limit and write $(k_n)_{n\in\mathbf{N}}$ instead of $(k_{n_j})_{j\in\mathbf{N}}$. We will also use the same acronym when $\mathbb{K}_{5/4}(Q)$ is replaced by the space $G_{2,k}(Q)$ (cf. (8.104)), as in the case of the regularized systems, and when adjusting the exponent in item (ii), strong compactness being held in $L^q(Q)$, $\forall q < 10/3$.

8.6.2.3 P Extracting Subsequence Principle

Let $1 < q \le 2$. In order to homogenize notation, we will refer to as the P-ESP-limit of any sequence (or family) $(p_n)_{n\in\mathbf{N}}$, any weak subsequential limit p.

8.6.3 *Convergence Lemmas*

Lemma 8.7. *Let $\alpha > 0$ be fixed, $(\mathbf{v}_n, k_n)_{n\in\mathbf{N}}$ be bounded in $Z_1(Q)$ (cf. (8.103)) to which the ESP's apply, and $(\mathbf{v}, k)$ be any V-K-ESP-limit, in the sense of Lemma 8.5 for the case $p = 2$, in the sense of Lemma 8.6 for the case $G_{2,k}(Q)$ (cf. (8.104)).*

Let $(\mathbf{w}_n, k_n)_{n\in\mathbf{N}}$ be a sequence in $L^2([0,T], \mathbf{W}(\Omega) \times H_0^1(\Omega))$ which converges to $(\mathbf{w}, k) \in L^2([0,T], \mathbf{W}(\Omega) \times H_0^1(\Omega))$.

Then we have

$$\lim_{n\to\infty} \int_0^T \langle \mathscr{T}_\alpha^{(k)}(\mathbf{v}_n(t), k_n(t)), \mathbf{w}_n(t)\rangle \, dt = \int_0^T \langle \mathscr{T}_\alpha^{(k)}(\mathbf{v}(t), k(t)), \mathbf{w}(t)\rangle dt, \tag{8.139}$$

$$\lim_{n\to\infty} \int_0^T \langle \mathscr{K}_\alpha^{(k)}(\mathbf{v}_n(t), k_n(t)), l_n(t)\rangle \, dt = \int_0^T \langle \mathscr{K}_\alpha^{(k)}(\mathbf{v}(t), k(t)), l(t)\rangle dt, \tag{8.140}$$

as well as

$$\lim_{n\to\infty} \int_0^T \langle G_\alpha(\mathbf{v}_n), \mathbf{w}_n\rangle dt = \int_0^T \langle G_\alpha(\mathbf{v}), \mathbf{w}\rangle dt, \tag{8.141}$$

$$\lim_{n\to\infty} \int_0^T \langle \mathscr{E}(k_n(t)), l_n(t)\rangle dt = \langle \mathscr{E}(k(t)), l(t)\rangle dt. \tag{8.142}$$

Proof. These results rely on the basic principles developed in Chaps. 6 and 7, in particular Sects. 6.3.3 and 7.3.2. We prove each item one after another:

- (8.139) and (8.140). Let $\mathbf{v}_{n,\alpha} = \mathbf{v}_n \star \rho_\alpha$. Note that:

(i) From

$$||\mathbf{v}_{n,\alpha} - \mathbf{v}_\alpha||_{0,2,\Omega} \leq ||\mathbf{v}_n - \mathbf{v}||_{0,2,\Omega}$$

and the $L^2(Q)$ strong convergence of $(\mathbf{v}_n)_{n\in\mathbf{N}}$ to $\mathbf{v}$, we deduce the $\mathbf{L}^2(Q)$ strong convergence of $(\mathbf{v}_{n,\alpha})_{n\in\mathbf{N}}$ to $\mathbf{v}_\alpha$.

(ii) As $(\mathbf{v}_n)_{n\in\mathbf{N}}$ is bounded in $Z_1(Q)$, and therefore in $L^\infty([0,T],\mathbf{L}^2(\Omega))$, the inequality

$$||\mathbf{v}_{n,\alpha}||_{0,\infty,\Omega} \leq C\alpha^{-3}||\mathbf{v}_n||_{0,2,\Omega}$$

ensures that $(\mathbf{v}_{n,\alpha})_{n\in\mathbf{N}}$ is bounded in $\mathbf{L}^\infty(Q)$.

Consequently, given any $1 \leq p < \infty$, the sequence $(\mathbf{v}_{n,\alpha})_{n\in\mathbf{N}}$ converges to $\mathbf{v}_\alpha$ in $\mathbf{L}^p(Q)$, in particular in $\mathbf{L}^6(Q)$.
Item (iv) in Lemma 8.5 indicates that $(\mathbf{v}_n)_{n\in\mathbf{N}}$ converges to $\mathbf{v}$ in $\mathbf{L}^3(Q)$ which ensures that

$$\lim_{n\to\infty}(\mathbf{v}_n \otimes \mathbf{v}_{n,\alpha}) = \mathbf{v}\otimes\mathbf{v}_\alpha \ \text{ in } \ L^2(Q)^9,$$

and in view of the assumption on $(\mathbf{w}_n)_{n\in\mathbf{N}}$,

$$\lim_{n\to\infty}\int\int_Q \mathbf{v}_n \otimes \mathbf{v}_{n,\alpha} : \nabla\mathbf{w}_n = \int\int_Q \mathbf{v}\otimes\mathbf{v}_\alpha : \nabla\mathbf{w}.$$

Similarly, as $(\mathbf{v}_{n,\alpha})_{n\in\mathbf{N}}$ is bounded in $\mathbf{L}^\infty(Q)$ and converges to $\mathbf{v}_\alpha$ in $\mathbf{L}^6(Q)$, and as $(\mathbf{w}_n)_{n\in\mathbf{N}}$ converges to $\mathbf{w}$ in $L^2([0,T],\mathbf{W}(\Omega))$, we deduce from the inverse Lebesgue Theorem, then the Lebesgue Theorem, that for a subsequence,

$$\lim_{n\to\infty}(\mathbf{w}_n \otimes \mathbf{v}_{n,\alpha}) = \mathbf{w}\otimes\mathbf{v}_\alpha \ \text{ in } \ L^2(Q)^9,$$

and then for the whole sequence by the uniqueness of the limit, hence

$$\lim_{n\to\infty}\int\int_Q \mathbf{w}_n \otimes \mathbf{v}_{n,\alpha} : \nabla\mathbf{v}_n = \int\int_Q \mathbf{w}\otimes\mathbf{v}_\alpha : \nabla\mathbf{v}.$$

In conclusion, this shows that

$$\lim_{n\to\infty}\int_0^T b_\alpha(\mathbf{v}_n(t);\mathbf{v}_n(t),\mathbf{w}_n(t))dt = \int_0^T b_\alpha(\mathbf{v}(t);\mathbf{v}(t),\mathbf{w}(t))dt. \quad (8.143)$$

The same argumentation also yields

$$\lim_{n\to\infty}\int_0^T b_{\alpha,e}(\mathbf{v}_n(t);k_n(t),l_n(t))dt = \int_0^T b_\alpha(\mathbf{v}(t);k(t),l(t))dt. \quad (8.144)$$

Furthermore, as

(a) ν_t and μ_t are continuous and bounded,
(b) $k_n \to k$ a.e. in Q,
(c) $(D\mathbf{w}_n)_{n\in\mathbf{N}}$ and $(\nabla l_n)_{n\in\mathbf{N}}$ converge to $D\mathbf{w}$ and ∇l in $L^2(Q)^9$ and $L^2(Q)^3$ strong,

we deduce

$$\lim_{n\to\infty} \nu_t(k_n, t, \mathbf{x}) D\mathbf{w}_n = \nu_t(k, t, \mathbf{x}) D\mathbf{w}, \quad \lim_{n\to\infty} \mu_t(k_n, t, \mathbf{x}) \nabla l_n = \mu_t(k, t, \mathbf{x}) \nabla l,$$

in $L^2(Q)^9$ and $L^2(Q)^3$ strong, respectively. Hence as $(D\mathbf{v}_n)_{n\in\mathbf{N}}$ and $(\nabla l_n)_{n\in\mathbf{N}}$ converge to $D\mathbf{v}$ and ∇l in $L^2(Q)^9$ and $L^2(Q)^3$ weak, respectively, we obtain

$$\lim_{n\to\infty} \int_0^T s_v(k_n(t); \mathbf{v}_n(t), \mathbf{w}_n(t)) dt = \int_0^T s_v(k(t); \mathbf{v}(t), \mathbf{w}(t)) dt, \tag{8.145}$$

$$\lim_{n\to\infty} \int_0^T s_e(k_n(t); k_n(t), l_n(t)) dt = \int_0^T s_e(k(t); k(t), l(t)) dt. \tag{8.146}$$

A similar but in fact simpler argumentation leads to

$$\lim_{n\to\infty} \int_0^T a(\mathbf{v}_n(t), \mathbf{w}_n(t)) dt = \int_0^T a(\mathbf{v}(t), \mathbf{w}(t)) dt, \tag{8.147}$$

$$\lim_{n\to\infty} \int_0^T a_e(k_n(t), l_n(t)) dt = \int_0^T a_e(k(t), l(t)) dt. \tag{8.148}$$

concluding this point.

- (8.141) and (8.142). On the one hand, we know from the V-ESP that $\gamma_0(\mathbf{v}_n) \to \gamma_0(\mathbf{v})$ in $L^2(\Gamma)$ strong and a.e. as $n \to \infty$. We deduce by the usual argumentation that as $\tilde{H}_\alpha$ is bounded and continuous, then

$$\lim_{n\to\infty} \gamma_0(\mathbf{v}_n) \tilde{H}_\alpha(\gamma_0(\mathbf{v}_n)) = \gamma_0(\mathbf{v}) \tilde{H}_\alpha(\gamma_0(\mathbf{v})) \text{ in } L^2(\Gamma) \text{ strong.}$$

On the other hand, since $(\mathbf{w}_n)_{n\in\mathbf{N}}$ is convergent in $L^2([0,T], \mathbf{W}(\Omega))$, we deduce by the trace Theorem that it is bounded in $L^2([0,T] \times \Gamma)$. Therefore, up to a subsequence, $\gamma_0(\mathbf{w}_n) \to \gamma_0(\mathbf{w})$ in $L^2(\Gamma)$ weak, hence (8.141).
Item (8.142) derives from similar considerations since the function E is continuous and bounded. $\square$

Remark 8.5. The same result holds if instead of applying the K-ESP on $(k_n)_{n\in\mathbf{N}}$ in the sense of Lemma 8.6 for the case $G_{2,k}(Q)$, we apply the K-ESP in its original formulation, by means of the space $\mathbb{K}_{5/4}(Q)$.

Lemma 8.8. *Let* $(\mathbf{v}^{(\alpha)}, k^{(\alpha)})_{\alpha>0}$ *be a bounded family in* $G_{4/3,\nu}(Q) \times \mathbb{K}_{5/4}(Q)$ *(cf. (8.89) and (8.137)), to which the ESP's apply when* $\alpha \to 0$, *and* $(\mathbf{v}, k)$ *be any*

V-K-ESP-limit, in the sense of Lemma 8.5 for the case $p = 4/3$, and Lemma 8.6 for the case of $\mathbb{K}_{5/4}(Q)$.

Then, given any $\mathbf{w} \in L^\infty([0,T], \mathscr{W}_m(\Omega))$ and $l \in L^\infty([0,T], \mathscr{D}(\Omega))$, we have

$$\lim_{\alpha\to 0} \int_0^T \langle \mathscr{T}_\alpha^{(k)}(\mathbf{v}^{(\alpha)}(t), k^{(\alpha)}(t)), \mathbf{w}(t)\rangle\, dt = \int_0^T \langle \mathscr{T}^{(k)}(\mathbf{v}(t), k(t)), \mathbf{w}(t)\rangle dt, \tag{8.149}$$

$$\lim_{\alpha\to 0} \int_0^T \langle \mathscr{K}_\alpha^{(k)}(\mathbf{v}^{(\alpha)}(t), k^{(\alpha)}(t)), l(t)\rangle\, dt = \int_0^T \langle \mathscr{K}^{(k)}(\mathbf{v}(t), k(t)), l(t)\rangle dt, \tag{8.150}$$

as well as

$$\lim_{\alpha\to 0} \int_0^T \langle G_\alpha(\mathbf{v}^{(\alpha)}), \mathbf{w}\rangle dt = \int_0^T \langle G(\mathbf{v}), \mathbf{w}\rangle dt, \tag{8.151}$$

$$\lim_{\alpha\to 0} \int_0^T \langle \mathscr{E}(k^{(\alpha)}(t)), l(t)\rangle dt = \langle \mathscr{E}(k(t)), l(t)\rangle dt. \tag{8.152}$$

Proof. The proof is similar to the previous ones but simpler due to the regularity of the chosen tests.

- (8.149) and (8.150). With minor changes, the convergence of the diffusion terms is treated in a similar fashion to those of (8.139) and (8.140). We focus on the transport terms. We claim:

$$\lim_{\alpha\to 0} \mathbf{v}_\alpha^{(\alpha)} = \mathbf{v} \;\text{ in }\; \mathbf{L}^3(Q). \tag{8.153}$$

  Indeed,

$$||\mathbf{v}_\alpha^{(\alpha)} - \mathbf{v}||_{0,3,\Omega} \le ||\mathbf{v} - \mathbf{v}_\alpha||_{0,3,\Omega} + ||(\mathbf{v}_\alpha - \mathbf{v}) \star \rho_\alpha||_{0,3,\Omega} \le 2||\mathbf{v} - \mathbf{v}_\alpha||_{0,3,\Omega}. \tag{8.154}$$

  Let $\varphi_\alpha(t) = ||\mathbf{v}(t) - \mathbf{v}_\alpha(t)||_{0,3,\Omega}$. On the one hand $(\varphi_\alpha(t))_{\alpha>0}$ converges to 0 a.e. in $[0,T]$. On the other hand, $|\varphi_\alpha(t)| \le 2||\mathbf{v}||_{0,3,\Omega} \in L^3([0,T])$, hence (8.153) by (8.154) and the Lebesgue Theorem.

  Therefore, as $\mathbf{w} \in \mathbf{L}^\infty(Q)$ and $\nabla\mathbf{w} \in L^\infty(Q)^9$, $\mathbf{v}_\alpha \to \mathbf{v}$ in $\mathbf{L}^p(Q)$, $p < 10/3$, $\nabla\mathbf{v}_\alpha \to \nabla\mathbf{v}$ in $L^2(Q)^9$ weak, we find

$$\lim_{\alpha\to 0} \int_0^T b_\alpha(\mathbf{v}_\alpha; \mathbf{v}_\alpha, \mathbf{w})\, dt = \lim_{\alpha\to 0} \int_0^T b(\mathbf{v}_\alpha^{(\alpha)}; \mathbf{v}_\alpha, \mathbf{w})\, dt = \int_0^T b(\mathbf{v}; \mathbf{v}, \mathbf{w})\, dt.$$

  By a similar argument, we also have

$$\lim_{\alpha\to 0} \int_0^T b_{e,\alpha}(\mathbf{v}_\alpha; k_\alpha, l)\, dt = \lim_{\alpha\to 0} \int_0^T b_e(\mathbf{v}_\alpha^{(\alpha)}; k_\alpha, l)\, dt = \int_0^T b_e(\mathbf{v}; k, l)\, dt,$$

  which concludes this point.

- (8.151) and (8.152). As $\tilde{H}$ is continuous and $0 \le \tilde{H}(\mathbf{z}) \le C(1 + |\mathbf{z}|)$, we have,

$$\lim_{\alpha\to 0} \tilde{H}(\gamma_0(\mathbf{v}_\alpha)) = \tilde{H}(\gamma_0(\mathbf{v}_\alpha)) \;\text{ in }\; L^2(\Gamma) \;\text{ strong.}$$

We deduce from Lemma A.16:

$$\lim_{\alpha\to 0} \tilde{H}_\alpha(\gamma_0(\mathbf{v}_\alpha)) = \lim_{\alpha\to 0} T_{1/\alpha}\tilde{H}(\gamma_0(\mathbf{v}_\alpha)) = \tilde{H}(\gamma_0(\mathbf{v})) \;\text{ in }\; L^2(\Gamma) \;\text{ strong,}$$

therefore

$$\lim_{\alpha\to 0} \gamma_0(\mathbf{v}_\alpha)\tilde{H}_\alpha(\gamma_0(\mathbf{v}_\alpha)) = \gamma_0(\mathbf{v})\tilde{H}(\gamma_0(\mathbf{v})) \;\text{ in }\; \mathbf{L}^1(\Gamma),$$

hence (8.151) as $\gamma_0\mathbf{w} \in \mathbf{L}^\infty(\Gamma)$. Convergence (8.152) is straightforward in view of the properties satisfied by E. □

8.6.4 The Energy Method

The energy method relies on energy equalities such as (8.107), in which we take the limit to determine the convergence of the dissipation term, in order to be able to take the limit in the TKE equation source term. In the evolutionary case, the energy equality starts as, for all t,

$$\frac{1}{2}||\mathbf{v}_n(t)||^2_{0,2,\Omega} + \int\int_{Q_t} (2\nu + \nu_t(k_n, t, \mathbf{x}))|D\mathbf{v}_n|^2 + \dots$$

We aim to prove that the second term in the above expression converges to the corresponding limit. The issue is the convergence of the first term, namely, $||\mathbf{v}_n(t)||^2_{0,2,\Omega}$. To deal with this term, it is tempting to integrate the energy equality with respect to t over $[0, T]$, which in turn leads us to consider $||\mathbf{v}_n(t)||^2_{0,2,Q}$, which is known to converge.

However, this operation also introduces the factor $(T-t)$ throughout the integrals in the resulting equalities. As $(T - t)$ vanishes at $t = T$, this factor might be responsible for a degree of degeneracy. To avoid this, we introduce another reference time $T' > T$ and consider the solution over $[0, T']$. We then perform the same procedure over $[0, T']$. As $T' - t \ge T' - T > 0$ over $[0, T]$, we are in this way able to retrieve enough information to conclude over $[0, T]$.

This method is consistent since, as we will see later, we can construct solutions over $[0, T']$ whatever the value of $T' < \infty$. For instance, we can take $T' = T + 1$. To be more specific, in the case of Problem $\mathscr{EP}^k_{\alpha,\beta,\varepsilon}$, we state the following, where throughout we set $Q' = [0, T'] \times \Omega$.

Lemma 8.9. *Let $\alpha, \beta, \varepsilon > 0$ be fixed, $T' > T$ another reference time, $(\mathbf{v}_n, k_n)_{n\in\mathbf{N}}$ a sequence in $Z_1(Q')$ (cf. definition (8.103)) such that for each $n \in \mathbf{N}$, $(\mathbf{v}_n, k_n)$ is a solution to $\mathscr{EP}^k_{\alpha,\beta,\varepsilon}$ over Q', recalling that $\mathscr{EP}^k_{\alpha,\beta,\varepsilon}$ is specified by (8.105) and (8.106).*

Then $(\mathbf{v}_n, k_n)_{n\in\mathbf{N}}$ is bounded in $Z_1(Q')$ and let $(\mathbf{v}, k)$ be any V-K-ESP-limit over $[0, T']$, in the sense of Lemma 8.5 for the case $p = 2$, in the sense of Lemma 8.6 for the case $G_{2,k}(Q)$. Then $(\mathbf{v}_n)_{n\in\mathbf{N}}$ strongly converges to $\mathbf{v}$ in $L^2([0, T], \mathbf{W}(\Omega))$ and $(\mathbf{v}, k)$ is a solution to $\mathscr{EP}^k_{\alpha,\beta,\varepsilon}$ over Q.

Proof. The proof is rather technical and divided into 5 steps:

STEP 1. *Initialization.* By Lemma 8.7 with T' instead of T, we deduce that $(\mathbf{v}, k)$ satisfies the fluid equation (8.106.i) over Q', the convergence of the pressure term being deduced from the continuity properties of the operator P_ε. Lemma 8.7 also allows us to take the limit in the l.h.s. of the TKE equation (8.106.ii) over Q'. We must focus on the source term $\mathbb{P}(\mathbf{v}_n, k_n)$.

Furthermore, the energy equality (8.107) is satisfied by each $(\mathbf{v}_n, k_n)$ as well as by $(\mathbf{v}, k)$, for all $t > 0$. The energy equality satisfied by $(\mathbf{v}_n, k_n)$ is written in the form

$$EC_n(t)+\int_0^t D_n(s)ds-\int_0^t Pr_n(s)ds+\int_0^t BC_n(s)ds = EC(0)+\int_0^t F_n(s)ds \tag{8.155}$$

while the energy equality satisfied by $(\mathbf{v}, k)$ is written in the form

$$EC(t) + \int_0^t D(s)ds - \int_0^t Pr(s)ds + \int_0^t BC(s)ds = EC(0) + \int_0^t F(s)ds. \tag{8.156}$$

The energy method involves proving the convergence of $(D\mathbf{v}_n)_{n\in\mathbf{N}}$ to $D\mathbf{v}$ in $L^2(Q)^9$ strong by means of (8.155) and (8.156).

STEP 2. *Fields to which the energy method applies.* In order to conclude, we will prove that the sequence $(\lambda_n)_{n\in\mathbf{N}}$ expressed by

$$\Lambda_n(t, \mathbf{x}) = \left((T' - t)(2\nu + \nu_t(k_n(t, \mathbf{x}), t, \mathbf{x}))\right)^{\frac{1}{2}} D\mathbf{v}_n(t, \mathbf{x}) \tag{8.157}$$

converges to

$$\Lambda(t, \mathbf{x}) = \left((T' - t)(2\nu + \nu_t(k(t, \mathbf{x}), t, \mathbf{x})\right)^{\frac{1}{2}} D\mathbf{v}(t, \mathbf{x}) \tag{8.158}$$

in $L^2(Q')$ strong. It is already understood that $(\Lambda_n)_{n\in\mathbf{N}}$ converges to Λ in $L^2(Q')^9$ weak. We will infer from (8.155) and (8.156) that

$$\lim_{n\to\infty} ||\Lambda_n||_{0,2,Q'} = ||\Lambda||_{0,2,Q'}, \tag{8.159}$$

which is sufficient to conclude that the convergence is indeed strong.

STEP 3. *Processing of energy equalities.* The method is based on the simple identity

$$\int_0^{T'} \left(\int_0^t \psi(s)ds \right) dt = \int_0^{T'} (T'-t)\psi(t)dt. \tag{8.160}$$

Therefore, integrating (8.155) and (8.156) over $[0, T']$ provides identities of the form

$$\frac{1}{2}||\mathbf{v}_n||^2_{0,2,Q'} + ||\Lambda_n||^2_{0,2,Q'} - I_{1,n} + I_{2,n} = C_0 + I_{3,n}, \tag{8.161}$$

$$\frac{1}{2}||\mathbf{v}||^2_{0,2,Q'} + ||\Lambda||^2_{0,2,Q'} - I_1 + I_2 = C_0 + I_3. \tag{8.162}$$

From the V-ESP, we already know that

$$\lim_{n\to\infty} ||\mathbf{v}_n||^2_{0,2,Q'} = ||\mathbf{v}||^2_{0,2,Q'}. \tag{8.163}$$

Therefore, (8.159) is subject to

$$\lim_{n\to\infty} I_{j,n} \to I_j, \quad j = 1, 2, 3.$$

We check each convergence individually.

- The term $I_{1,n}$ is specified by

$$I_{1,n} = \int \int_{Q'} (T'-t) P_\varepsilon(\mathbf{v}_n(t,\mathbf{x})) \nabla \cdot \mathbf{v}_n(t,\mathbf{x}) d\mathbf{x} dt.$$

As P_ε is a continuous operator over $\mathbf{L}^2(Q')$ and $\mathbf{v}_n \to \mathbf{v}$ in $\mathbf{L}^2(Q')$ strong by the V-ESP, then

$$\lim_{n\to\infty} (T'-t) P_\varepsilon(\mathbf{v}_n) = (T'-t) P_\varepsilon(\mathbf{v}) \text{ in } L^2(Q') \text{ strong.}$$

From the weak L^2 convergence of $(\nabla \cdot \mathbf{v}_n)_{n\in\mathbf{N}}$ to $\nabla \cdot \mathbf{v}$, we obtain

$$\lim_{n\to\infty} I_{1,n} = I_1 = \int \int_{Q'} (T'-t) P_\varepsilon(\mathbf{v}(t,\mathbf{x})) \nabla \cdot \mathbf{v}(t,\mathbf{x}) d\mathbf{x} dt. \tag{8.164}$$

- The term $I_{2,n}$ is specified by

$$I_{2,n} = \int_0^{T'} \int_\Gamma (T'-t) \mathbf{v}_n(t,\mathbf{x}) \cdot \mathbf{v}_n(t,\mathbf{x}) \tilde{H}_\alpha(\mathbf{v}_n(t,\mathbf{x})) d\Gamma(\mathbf{x}) dt. \tag{8.165}$$

As $\tilde{H}_\alpha$ is continuous and bounded, $(\gamma_0(\mathbf{v}_n))_{n\in\mathbb{N}}$ converges to $\mathbf{v}$ a.e. in Q' (cf. item (v) in the V-ESP Lemma 8.5); then

$$\lim_{n\to\infty}\left[\gamma_0(\mathbf{v}_n))\tilde{H}_\alpha(\gamma_0(\mathbf{v}_n))\right] = \gamma_0(\mathbf{v}))\tilde{H}_\alpha(\gamma_0(\mathbf{v})) \quad \text{a.e. in} \quad Q',$$

which combined with

$$|\gamma_0(\mathbf{v}))\tilde{H}_\alpha(\gamma_0(\mathbf{v})| \le \frac{1}{\alpha}F_2 \in L^2(Q'),$$

leads to

$$\lim_{n\to\infty}\left[\gamma_0(\mathbf{v}_n))\tilde{H}_\alpha(\gamma_0(\mathbf{v}_n))\right] = \gamma_0(\mathbf{v}))\tilde{H}_\alpha(\gamma_0(\mathbf{v}) \quad \text{in} \quad L^2(Q'),$$

by the Lebesgue Theorem. Consequently, as $(T'-t)\gamma_0(\mathbf{v}_n) \to (T'-t)\gamma_0(\mathbf{v}_n)$ in $L^2(Q')$, we have, according to (8.165),

$$\lim_{n\to\infty} I_{2,n} = I_2 = \int_0^{T'}\int_\Gamma (T'-t)\mathbf{v}(t,\mathbf{x})\cdot\mathbf{v}(t,\mathbf{x})\tilde{H}_\alpha(\mathbf{v}(t,\mathbf{x}))d\Gamma(\mathbf{x})dt. \quad (8.166)$$

- The term $I_{3,n}$ is specified by

$$I_{3,n} = \int_0^{T'}(T'-t)\langle \mathbf{f}(t), \mathbf{v}_n(t)\rangle dt. \quad (8.167)$$

The weak convergence of $(\mathbf{v}_n)_{n\in\mathbb{N}}$ to $\mathbf{v}$ in $L^2([0,T'],\mathbf{W}(\Omega))$ combined with $\mathbf{f} \in L^2([0,T'],\mathbf{W}(\Omega)')$ (Hypothesis 8.i supposes $\mathbf{f} \in L^2_{loc}(\mathbb{R}_+,\mathbf{W}(\Omega)')$) yields

$$\lim_{n\to\infty} I_{3,n} = I_3 = \int_0^{T'}(T'-t)\langle \mathbf{f}(t), \mathbf{v}(t)\rangle dt, \quad (8.168)$$

concluding this part of the proof.

STEP 4. *Strong convergence.* Let g_n and g be the functions defined by

$$g_n(t,\mathbf{x}) = ((T'-t)(2\nu + \nu_t(k_n(t,\mathbf{x}),t,\mathbf{x})))^{\frac{1}{2}},$$
$$g(t,\mathbf{x}) = ((T'-t)(2\nu + \nu_t(k(t,\mathbf{x}),t,\mathbf{x})))^{\frac{1}{2}},$$

which both satisfy $\forall\,(t,\mathbf{x}) \in Q$,

$$(2\nu(T'-T))^{\frac{1}{2}} \le g(t,\mathbf{x}), g_n(t,\mathbf{x}) \le ((T'-T)(2\nu + ||\nu_t||_\infty))^{\frac{1}{2}}. \quad (8.169)$$

Observe on the one hand that

$$D\mathbf{v}_n = g_n^{-1}\Lambda_n, \quad D\mathbf{v} = g^{-1}\Lambda. \quad (8.170)$$

On the other hand, we know from the K-ESP that $(k_n)_{n\in\mathbf{N}}$ converges a.e. to k. Hence, as ν_t is continuous, we have

$$\begin{aligned} &\lim_{n\to\infty} g_n^{-1} = g^{-1} \text{ a.e. in } Q, \\ &\forall\, n \in \mathbf{N}, \quad ((T'-T)(2\nu + ||\nu_t||_\infty))^{-\frac{1}{2}} \le g_n^{-1} \le (2\nu(T'-T))^{-\frac{1}{2}}. \end{aligned} \tag{8.171}$$

On the other hand, as $(\Lambda_n)_{n\in\mathbf{N}}$ strongly converges to Λ in $L^2(Q')^9$, then also in $L^2(Q)^9$, we infer from the inverse Lebesgue Theorem that up to a subsequence,

$$\begin{aligned} &\lim_{n\to\infty} \Lambda_n = \Lambda \text{ a.e. in } Q, \\ &\exists\, G \in L^2(Q) \text{ such that } \forall\, n \in \mathbf{N}, \quad |\Lambda_n| \le G. \end{aligned} \tag{8.172}$$

As a result,

$$\begin{aligned} &\lim_{n\to\infty} D\mathbf{v}_n = D\mathbf{v} \text{ a.e. in } Q, \\ &\forall\, n \in \mathbf{N}, \quad |D\mathbf{v}_n| \le (2\nu(T'-T))^{-\frac{1}{2}} G \in L^2(Q). \end{aligned} \tag{8.173}$$

In consequence, $(D\mathbf{v}_n)_{n\in\mathbf{N}}$ converges in $L^2(Q)^9$ strong. As $(\mathbf{v}_n)_{n\in\mathbf{N}}$ is already known to converge in $\mathbf{L}^2(Q)$, we deduce the strong convergence of $(\mathbf{v}_n)_{n\in\mathbf{N}}$ in $L^2([0,T], \mathbf{W}(\Omega))$ as claimed.

STEP 5. *Conclusion.* It remains to take the limit in the equations. Lemma 8.7 and the continuity property of the operator P_ε already mentioned allow the limit to be taken in the fluid equation (8.105.i) as well as in the l.h.s. of the TKE equation (8.105.ii).

Furthermore, as ν_t is continuous and bounded, $(k_n)_{n\in\mathbf{N}}$ converges a.e. to k in Q, and in view of

(i) $T_{1/\beta}$ is continuous, $T_{1/\beta} \le \beta^{-1}$
(ii) $|D\mathbf{v}_n|^2 \to |D\mathbf{v}|^2$ a.e. in Q,

we have

$$\lim_{n\to\infty} \mathbb{P}_\beta(\mathbf{v}_n, k_n) = \mathbb{P}_\beta(\mathbf{v}, k) \text{ in } L^p(Q) \text{ strong, } 1 \le p < \infty. \tag{8.174}$$

hence the possibility of taking the limit in the TKE equation (8.106.ii) and we can conclude that $(\mathbf{v}, k)$ is indeed a solution to Problem $\mathscr{E}\mathscr{P}^k_{\alpha,\beta,\varepsilon}$. □

8.7 Proof of the Main Results

This section is devoted to proving the results stated at the end of Sect. 8.5 and to concluding the chapter.

We first prove Theorem 8.3, namely the existence of a solution to $\mathscr{EP}^k_{\alpha,\beta,\varepsilon}$ (cf. (8.105) and (8.106)) for any given $\alpha, \beta, \varepsilon > 0$.

We proceed using the Galerkin method, which aims to approach $\mathscr{EP}^k_{\alpha,\beta,\varepsilon}$ by a sequence of ordinary differential equations (ODEs) set on finite dimensional spaces, and then take the limit.

This method is subject to finding a suitable Hilbert basis $(\mathbf{z}_1, \cdots, \mathbf{z}_n, \cdots)$ on $\mathbf{L}^2(\Omega)$, consisting of fields in $\mathbf{W}(\Omega)$, constructed as eigenvectors of a given compact operator, deduced from an appropriate PDE system.

The Hilbert basis $(q_1, \cdots, q_n, \cdots)$ over $L^2(\Omega)$ that we shall use consists of the standard sequence of eigenfunctions in $H^1_0(\Omega)$ of $-\Delta$.

Once this framework is established, we can write the ODEs obtained by the projection of $\mathscr{EP}^k_{\alpha,\beta,\varepsilon}$ over $\mathrm{span}\{(\mathbf{z}_1, q_1), \cdot, (\mathbf{z}_n, q_n)\}$. These ODEs are shown to satisfy the conditions for the application of the Cauchy–Lipschitz Theorem, the existence of a global solution being ensured by the $L^\infty([0,T], \mathbf{L}^2(\Omega))$ estimate.

We finally take the limit when $n \to \infty$ by using the results of Sects. 8.5 and 8.6. The last delicate task will be to take the limit as $\varepsilon \to 0$, which required finding an estimate for the pressure term which is uniform in ε.

8.7.1 Special Basis

As already stated, $(q_j)_{j \in \mathbf{N}^\star}$ is the Hilbert basis of $L^2(\Omega)$ constructed from the spectral decomposition of $-\Delta$, where in particular $q_j \in H^1_0(\Omega)$, $j = 1, \cdots$. We set

$$H_n = \mathrm{span}\{q_1, \cdots, q_n\}. \tag{8.175}$$

To construct a Hilbert basis in $\mathbf{L}^2(\Omega)$ consisting of fields in $\mathbf{W}(\Omega)$, we consider the problem

$$\begin{cases} -\nabla \cdot (D\mathbf{w}) + \nabla r = \mathbf{f} & \text{in } \Omega, \\ -\Delta r + \nabla \cdot \mathbf{w} = g & \text{in } \Omega, \\ \mathbf{w} \cdot \mathbf{n} = 0 & \text{on } \Gamma, \\ -(D\mathbf{w} \cdot \mathbf{n})_\tau = \mathbf{w}_\tau & \text{on } \Gamma \\ \dfrac{\partial r}{\partial \mathbf{n}} = 0 & \text{on } \Gamma, \\ \displaystyle\int_\Omega r(\mathbf{x}) d\mathbf{x} = 0. \end{cases} \tag{8.176}$$

We infer from the results of Chap. 6 that given any

$$(\mathbf{f}, g) \in \mathbf{L}^2(\Omega) \times L^2_0(\Omega),$$

then (8.176) has a unique weak solution:

$$(\mathbf{w}, r) = A(\mathbf{f}, g) \in \mathbf{W}(\Omega) \times L^2_0(\Omega).$$

The arguments developed in Chap. 6 also show that A is a linear compact operator over $\mathbf{L}^2(\Omega) \times L^2_0(\Omega)$. By consequence, there exist:

(i) $(\lambda_n)_{n\in\mathbb{N}}$ such that $0 < \lambda_1 \leq \lambda_2 \cdots$ and $\lambda_n \to \infty$ as $n \to \infty$,
(ii) $(\mathbf{z}_n, r_n) \in \mathbf{W}(\Omega) \times L^2(\Omega)$ such that $((\mathbf{z}_1, r_1) \cdots, (\mathbf{z}_n, r_n), \cdots)$ is a Hilbert basis of $\mathbf{L}^2(\Omega) \times L^2_0(\Omega)$ and such that for all $n \geq 1$,

$$\begin{cases} -\nabla \cdot (D\mathbf{z}_n) + \nabla r_n = \lambda_n \mathbf{z}_n & \text{in } \Omega, \\ -\Delta r_n + \nabla \cdot \mathbf{z}_n = \lambda_n r_n & \text{in } \Omega, \\ \mathbf{z}_n \cdot \mathbf{n} = 0 & \text{on } \Gamma, \\ (D\mathbf{z}_n \cdot \mathbf{n})_\tau = (\mathbf{z}_n)_\tau & \text{on } \Gamma \\ \dfrac{\partial r_n}{\partial \mathbf{n}} = 0 & \text{on } \Gamma, \\ \displaystyle\int_\Omega r_n(\mathbf{x}) d\mathbf{x} = 0. \end{cases} \tag{8.177}$$

Moreover, the family $(\mathbf{z}_n, r_n)$ is also orthogonal in $\mathbf{W}(\Omega) \times L^2_0(\Omega)$, when this space is endowed with the scalar product

$$(D\mathbf{w}, D\mathbf{z})_\Omega + (\mathbf{w}, \mathbf{z})_\Gamma + (\nabla r, \nabla q).$$

According to the terminology introduced by J.-L. Lions, we will call the basis $(\mathbf{z}_n, r_n)_{n\in\mathbb{N}}$ a special basis. In the following we set

$$\mathbf{W}_n = \text{span}\{\mathbf{z}_1, \cdots, \mathbf{z}_n\}. \tag{8.178}$$

Remark 8.6. There are many other ways of constructing a Hilbert basis of $\mathbf{L}^2(\Omega)$ consisting of fields in $\mathbf{W}(\Omega)$. However, the way taken by Problem (8.176) is particularly suited to the analysis of the NS-TKE model.

8.7.2 Ordinary Differential Equations

We seek $(\mathbf{v}, k)$ in the form

$$\mathbf{v} = \sum_{j=1}^n g_{j,n}(t)\mathbf{z}_j, \quad k = \sum_{j=1}^n h_{j,n}(t) q_j, \tag{8.179}$$

and we introduce the following vectors in $\mathbb{R}^n$:

$$V = V(t) = \begin{pmatrix} g_{1,n}(t) \\ \vdots \\ g_{n,n}(t) \end{pmatrix}, \quad K = K(t) = \begin{pmatrix} h_{1,n}(t) \\ \vdots \\ h_{n,n}(t) \end{pmatrix}, \tag{8.180}$$

and it is expected that $V(t)$ and $K(t)$ are of class C^1. We aim to find an ODE satisfied by $(V(t), K(t))$, derived from $\mathscr{E}\mathscr{P}^k_{\alpha,\beta,\varepsilon}$. We start by fixing the initial data. Let

$$\mathbf{v}_{0,n} = \sum_{j=1}^{n} (\mathbf{v}_0, \mathbf{w}_j)_\Omega \mathbf{z}_j, \quad k_{0,n} = \sum_{j=1}^{n} (k_0, l_j)_\Omega q_j, \tag{8.181}$$

and let V_0 and K_0 denote the vectors

$$V_0 = \begin{pmatrix} (\mathbf{v}_0, \mathbf{w}_1)_\Omega \\ \vdots \\ (\mathbf{v}_0, \mathbf{w}_n)_\Omega \end{pmatrix}, \quad K_0 = \begin{pmatrix} (k_0, l_1)_\Omega \\ \vdots \\ (k_0, l_n)_\Omega \end{pmatrix}. \tag{8.182}$$

The variational problem $\mathscr{V}\mathscr{P}_n$ we consider here is at any given time t[11]:

Find $(\mathbf{v}, k)$ of the form (8.179) such that for all $(\mathbf{w}, l) \in \mathbf{W}_n \times H_n$,

$$\begin{cases} \dfrac{d}{dt}(\mathbf{v}, \mathbf{w})_\Omega + \langle \mathscr{T}^{(k)}_\alpha(\mathbf{v}, k), \mathbf{w}\rangle - \langle P_\varepsilon(\mathbf{v}), \nabla\cdot\mathbf{w}\rangle + \langle G_\alpha(\mathbf{v}), \mathbf{w}\rangle = \langle \mathbf{f}, \mathbf{w}\rangle, \\ \dfrac{d}{dt}(k, l)_\Omega + \langle \mathscr{K}^{(k)}_\alpha(\mathbf{v}, k), l\rangle + \langle \mathscr{E}(k), l\rangle = (\mathbb{P}_\beta(\mathbf{v}, k), l)_\Omega, \end{cases} \tag{8.183}$$

with in addition

$$V(0) = V_0, \quad K(0) = K_0. \tag{8.184}$$

In particular, it is equivalent to taking $(\mathbf{w}_1, l_1)$, $(\mathbf{w}_2, l_2)\cdots(\mathbf{w}_n, l_n)$ as successive tests which, using the fact that the basis $\{(\mathbf{z}_1, q_1), (\mathbf{z}_2, l_2)\cdots(\mathbf{z}_n, q_n)\}$ is orthonormal in $\mathbf{L}^2(\Omega) \times L^2(\Omega)$, yield the differential system satisfied by $(V(t), K(t))$ in the following form:

$$\begin{cases} \dfrac{d}{dt}V(t) + B_\alpha(V(t)) + A\cdot V(t) + S_v(K(t), t)\cdot V(t) \\ \qquad\qquad - \tilde{P}_\varepsilon \cdot V(t) + V(t)\cdot \tilde{G}_\alpha(V(t)) = F(t)\cdot V(t), \\ \dfrac{d}{dt}K(t) + B_{e,\alpha}(V(t), K(t)) + A_e\cdot K(t) + S_{v,e}(K(t), t)\cdot K(t) \\ \qquad\qquad + \tilde{E}(K(t), t)\cdot K(t) = \tilde{\mathbb{P}}_\beta(V(t), K(t), t), \end{cases} \tag{8.185}$$

which can also be rewritten in the form

$$\begin{cases} u'(t) = \Psi(u(t), t)), \\ u(0) = u_0. \end{cases} \tag{8.186}$$

[11] We refer to Remark 8.2 to make the link between this and earlier formulations.

by setting

$$u(t) = \begin{pmatrix} V(t) \\ K(t) \end{pmatrix}, \quad u_0 = \begin{pmatrix} V_0 \\ K_0 \end{pmatrix}.$$

It is not essential to write down the detailed expression of the nonlinear functional Ψ, which can be obtained by a simple but long and technical calculation; this is left as an exercise for the reader.

It is enough to recall that according to Assumption 8.i, ν_t, μ_t, E are continuous and are of class $C^{0,1}_{loc}$ with respect to k, and $\tilde{H}$ is of class $C^{0,1}_{loc}$ with respect to k. Moreover the quadratic terms are also of class $C^{0,1}_{loc}$ with respect to V and K, and the truncation function is of class $C^{0,1}$.

In conclusion, it can be said that the ODE system (8.186) fulfills the conditions for the application of the Cauchy–Lipschitz Theorem. Therefore, there exists a time T_n such that (8.186) admits a unique solution of class C^1 over $[0, T_n[$. In other words, $\mathscr{VP}_n$ expressed by (8.183) admits a unique solution $(\mathbf{v}, k) \in C^1([0, T_n[, \mathbf{W}_n \times H_n)$.

At this stage, nothing allows us to say that $T_n > T$. However, this is a classic issue raised by the Galerkin method involving parabolic equations (cf. Lions [213]). Following the usual procedure, we take $(\mathbf{v}, k) \in \mathbf{W}_n \times H_n$ as test in $\mathscr{VP}_n$, and then integrate with respect to time over $[0, t]$, for any given $t \in [0, T_n[$. From there, we can follow the reasoning of Proposition 8.1 and the proof of Lemma 8.2 line by line, to obtain

$$||\mathbf{v}(t)||_{0,2,\Omega} \leq C_{v,\infty}(T), \quad ||k(t)||_{0,2,\Omega} \leq C^{(\beta)}_{k,\infty,2}(T) \tag{8.187}$$

where $C_{v,\infty}(T)$ and $C^{(\beta)}_{k,\infty,2}(T)$ are specified by (8.28) and (8.123) respectively. Note that (8.187) holds whatever the value of $T > 0$, since $\mathbf{f} \in L^2_{loc}(\mathbb{R}_+, \mathbf{W}(\Omega)')$. Observe also that

$$||\mathbf{v}(t)||^2_{0,2,\Omega} = \sum_{j=1}^{n} g_{j,n}(t)^2, \quad ||k||^2_{0,2,\Omega} = \sum_{j=1}^{n} h_{j,n}(t)^2. \tag{8.188}$$

All this allows us to conclude that $||u(t)||$ is bounded over $[0, T]$ whatever the value of $T < \infty$. The standard ODE theory states that under this condition, $u(t)$ can be extended as a global solution of (8.186) over $[0, T]$, of class C^1 with respect to t.

8.7.3 Taking the First Limit

We notice that the solution $(\mathbf{v}, k)$ of (8.186) is the unique solution to the variational problem $\mathscr{EP}^k_{\alpha,\beta,\varepsilon,n}$:

$$\begin{cases} \textit{Unknown space: } C^1([0,T],\mathbf{W}_n \times H_n), \\ \textit{Test space: } L^2([0,T],\mathbf{W}_n \times H_n) = Z_{2,n}(Q), \\ \textit{Initial data: } \mathbf{v}_{0,n} \text{ and } k_{0,n}, \end{cases} \tag{8.189}$$

$$\begin{cases} \partial_t \mathbf{v} + \mathscr{T}_\alpha^{(k)}(\mathbf{v},k) + \nabla P_\varepsilon(\mathbf{v}) + G_\alpha(\mathbf{v}) = \mathbf{f}, \\ \qquad\qquad \partial_t k + \mathscr{K}_\alpha^{(k)}(\mathbf{v},k) + \mathscr{E}(k) = \mathbb{P}_\beta(\mathbf{v},k), \end{cases} \tag{8.190}$$

which holds in a manner similar to that formulated in Sect. 8.2.2. The proof of Theorem 8.3 will be completed once we have shown:

Lemma 8.10. *Let $\alpha, \beta, \varepsilon > 0$ be fixed. The sequence $(\mathscr{EP}^k_{\alpha,\beta,\varepsilon,n})_{n\in\mathbf{N}}$ converges to $\mathscr{EP}^k_{\alpha,\beta,\varepsilon}$ as $n \to \infty$.*[12]

Proof. Given any $n \in \mathbf{N}^\star$, let $(\mathbf{v}_n, k_n)$ denote the solution to $\mathscr{EP}^k_{\alpha,\beta,\varepsilon,n}$. The proof of Lemma 8.2 still applies in this case, leading to

(i) $(\mathbf{v}_n)_{n\in\mathbf{N}}$ satisfies estimates (8.28) and (8.29) and $(\partial_t \mathbf{v}_n)_{n\in\mathbf{N}}$ satisfies estimate (8.120),
(ii) $(k_n)_{n\in\mathbf{N}}$ satisfies estimates (8.123) and (8.124) and $(\partial_t k_n)_{n\in\mathbf{N}}$ satisfies estimate (8.125).

From here, the compactness machinery built in Sect. 8.6.2 can be set in motion, in particular we can apply Lemmas 8.5 and 8.6. Let $(\mathbf{v}, k)$ be any V-K-ESP-limit of $(\mathbf{v}_n, k_n)_{n\in\mathbf{N}}$, in the sense of Lemma 8.5 for the case $p = 2$ and Lemma 8.6 for the case of $G_{2,k}(Q)$ (cf. (8.104)). Our goal is to prove that $(\mathbf{v}, k)$ is indeed a solution to $\mathscr{EP}^k_{\alpha,\beta,\varepsilon}$. For simplicity, we write (8.106) in the form:

$$\begin{cases} \partial_t \mathbf{v}_n + \mathscr{O}_{\alpha,\varepsilon}(\mathbf{v}_n, k_n) = \mathbf{f}, \\ \ \partial_t k_n + \mathscr{W}_\alpha(\mathbf{v}_n, k_n) = \mathbb{P}_\beta(\mathbf{v}_n, k_n), \end{cases} \tag{8.191}$$

Let $(\mathbf{w}, k) \in L^2([0,T], \mathbf{W}(\Omega) \times H^1_0(\Omega)) = Z_2(Q)$,

$$(\mathbf{w}_n, l_n) = Pr_n(\mathbf{w}, l),$$

where Pr_n denotes the orthogonal projector from $Z_2(Q)$ onto $Z_{2,n}(Q) = \mathbf{W}_n \times H_n$. Note that according to Lemma 8.11 below,

$$\lim_{n\to\infty} (\mathbf{w}_n, k_n) = (\mathbf{w}, k) \text{ in } Z_2(Q) \text{ strong.}$$

Therefore, we directly have, in view of the V- and KESP properties,

[12] Recall that $\mathscr{EP}^k_{\alpha,\beta,\varepsilon}$ is specified by (8.105) and (8.106).

$$\begin{aligned}
\lim_{n\to\infty}\int_0^T \langle \partial_t \mathbf{v}_n(t), \mathbf{w}_n(t)\rangle dt &= \int_0^T \langle \partial_t \mathbf{v}(t), \mathbf{w}(t)\rangle dt, \\
\lim_{n\to\infty}\int_0^T \langle \partial_t k_n(t), l_n(t)\rangle dt &= \int_0^T \langle \partial_t k(t), l(t)\rangle dt, \\
\lim_{n\to\infty}\int_0^t \langle \mathbf{f}(t), \mathbf{w}_n(t)\rangle dt &= \int_0^T \langle \mathbf{f}(t), \mathbf{w}(t)\rangle dt.
\end{aligned} \tag{8.192}$$

Furthermore, by Lemma 8.7 combined with the continuity property of the operator P_ε, we also have

$$\begin{aligned}
\lim_{n\to\infty}\int_0^T \langle \mathscr{O}_{\alpha,\varepsilon}(\mathbf{v}_n(t), k_n(t)), \mathbf{w}_n(t)\rangle dt &= \int_0^T \langle \mathscr{O}_{\alpha,\varepsilon}(\mathbf{v}(t), k(t)), \mathbf{w}(t)\rangle dt, \\
\lim_{n\to\infty}\int_0^T \langle \mathscr{W}_{\alpha}(\mathbf{v}_n(t), k_n(t)), l_n(t)\rangle dt &= \int_0^T \langle \mathscr{W}_{\alpha}(\mathbf{v}(t), k(t)), l(t)\rangle dt.
\end{aligned} \tag{8.193}$$

Finally,

$$\lim_{n\to\infty}\int_0^T (P_\varepsilon(\mathbf{v}_n(t)), \nabla\cdot\mathbf{w}_n(t))_\Omega dt = \int_0^T (P_\varepsilon(\mathbf{v}(t)), \nabla\cdot\mathbf{w}(t))_\Omega dt. \tag{8.194}$$

It can be concluded that at this level, we can take the limit in (8.191.i), so that $(\mathbf{v}, k) \in Z_2(Q)$ (cf. (8.105)) satisfies the fluid equation (8.106.i) of $\mathscr{EP}^k_{\alpha,\beta,\varepsilon}$ and the energy equality (8.107).

Moreover, according to the above, we can take the limit in all terms of (8.191.ii), except in the source term that has not yet been discussed. Obviously, each $(\mathbf{v}_n, k_n)$ verifies the energy equality (8.107) as well. Therefore, we are definitely in the same situation as in Lemma 8.9: the energy method applies, providing the convergence result (8.174) that leads to

$$\lim_{n\to\infty}\int_0^T (\mathbb{P}_\beta(\mathbf{v}_n(t), k_n(t)), l_n(t))_\Omega dt = \int_0^T (\mathbb{P}_\beta(\mathbf{v}(t), k(t)), l(t))_\Omega dt. \tag{8.195}$$

This point allows us to establish that $(\mathbf{v}, k)$ also satisfies the TKE equation (8.106.ii) of $\mathscr{EP}^k_{\alpha,\beta,\varepsilon}$.

To complete the proof, we must show that $(\mathbf{v}_0, k_0)$ is indeed the initial data of the resulting problem. In view of the regularity of $(\mathbf{v}_n, k_n)$, which is in particular of class C^1 with respect to time, we have $\forall\, n \in \mathbf{N}$:

$$\begin{cases}
\qquad\qquad \forall\, \boldsymbol{\varphi} \in C^1([0,T], \mathbf{W}(\Omega)) \ \text{ such that } \ \boldsymbol{\varphi}(T, \mathbf{x}) = 0, \\
\displaystyle\int_0^T (\partial_t \mathbf{v}_n(t), \boldsymbol{\varphi}(t))_\Omega dt = -\int_\Omega \boldsymbol{\varphi}(0,\mathbf{x})\cdot\mathbf{v}_{0,n}(\mathbf{x})\, d\mathbf{x} - \iint_Q \frac{\partial\boldsymbol{\varphi}}{\partial t}(s,\mathbf{x})\cdot\mathbf{v}_n(s,\mathbf{x})\, d\mathbf{x}ds,
\end{cases}$$

$$\begin{cases} \forall \psi \in C^1([0,T], H_0^1(\Omega)) \text{ such that } \psi(T,\mathbf{x}) = 0, \\ \int_0^T \langle \partial_t k_n(t), \psi(t) \rangle dt = -\int_\Omega \psi(0,\mathbf{x})\, k_{0,n}(\mathbf{x})\, d\mathbf{x} - \int\int_Q \frac{\partial \psi}{\partial t}(s,\mathbf{x})\, k_n(s,\mathbf{x})\, d\mathbf{x}ds, \end{cases}$$

where $(\mathbf{v}_{0,n}, k_{0,n})$ is defined by (3.65). We deduce that $(\mathbf{v}, k)$ also satisfies (8.24) and (8.25) by:

(i) the $L^2(\Omega)$ strong convergence of $(\mathbf{v}_{0,n}, k_{0,n})$ to $(\mathbf{v}_0, k_0)$,
(ii) the $L^2([0,T], \mathbf{W}(\Omega) \times H^{-1}(\Omega))$ weak convergence of $(\partial_t \mathbf{v}_n, \partial_t k_n)_{n\in\mathbf{N}}$ to $(\partial_t \mathbf{v}, \partial_t k)$,
(iii) the $L^2(Q)$ strong convergence of $(\mathbf{v}_n, k_n)$ to $(\mathbf{v}, k)$,

concluding the proof. □

It remains for us to prove:

Lemma 8.11. *Let H be any Hilbert space, endowed with the Hilbert basis $(e_1, \cdot, e_n, \cdots)$, $v = v(t) \in L^2([0,T], H)$, and consider*

$$v_n = \sum_{k=1}^{n} (e_k, v) e_k.$$

Then $(v_n)_{n\in\mathbf{N}}$ converges towards v in $L^2([0,T], H)$ when $n \to \infty$.

Proof. Let $\varphi_n(t) = ||v_n(t) - v(t)||$. We have to prove that $(\varphi_n)_{n\in\mathbf{N}}$ converges to 0 in $L^2([0,T])$. We note that for any fixed t, $(v_n(t))_{n\in\mathbf{N}}$ converges to $v(t)$ in H, since $v_n(t)$ is the orthogonal projection of $v(t)$ over H_n. Therefore $(\varphi_n)_{n\in\mathbf{N}}$ simply converges to 0. Moreover, $||v_n|| \le ||v||$, so that $|\varphi(t)| \le 2||v(t)|| \in L^2([0,T])$, hence the result by Lebesgue's Theorem. □

8.7.4 Taking the Second Limit

This subsection is devoted to proving Theorem 8.4, namely the convergence of $(\mathscr{EP}^k_{\alpha,\beta,\varepsilon})_{\varepsilon>0}$ to $\mathscr{EP}^k_{\alpha,\beta}$ when $\varepsilon \to 0$, where $\mathscr{EP}^k_{\alpha,\beta}$ is specified by (8.126) and (8.127).

Given any $\alpha, \beta > 0$, let $(\mathbf{v}_\varepsilon, k_\varepsilon)_{\varepsilon>0}$ denote any solution to $\mathscr{EP}^k_{\alpha,\beta,\varepsilon}$,

$$p_\varepsilon = P_\varepsilon(\mathbf{v}_\varepsilon).$$

We have to prove that:

(i) $(\mathbf{v}_\varepsilon, p_\varepsilon, k_\varepsilon)_{\varepsilon>0}$ is bounded in $U_1(Q)$ (cf. (8.126)),
(ii) every V-P-K-ESP-limit of $(\mathbf{v}_\varepsilon, p_\varepsilon, k_\varepsilon)_{\varepsilon>0}$ is a solution to $\mathscr{EP}^k_{\alpha,\beta}$.

We verify each item individually.

8.7.4.1 Estimates

According to the proof of Lemma 8.2, we already know that $(\mathbf{v}_\varepsilon)_{\varepsilon>0}$ satisfies estimates (8.28) and (8.29), while $(k_\varepsilon)_{\varepsilon>0}$ satisfies estimates (8.123) and (8.124), and $(\partial_t k_\varepsilon)_{E>0}$ satisfies (8.125). All these estimates are uniform in ε. In particular, we already know that $(k_\varepsilon)_{\varepsilon>0}$ is bounded in the space $G_{2,k}(T)$ (cf. (8.104)).

Unfortunately, the only available estimate on $(p_\varepsilon)_{\varepsilon>0}$ is given by (8.114) and is of order $(1/\varepsilon)$, that on $(\partial_t \mathbf{v}_\varepsilon)_{\varepsilon>0}$, given by (8.120), being of order $1/\varepsilon$ as well. Therefore, to achieve our goal, we must find a bound uniform in ε to control $(p_\varepsilon)_{\varepsilon>0}$ in $L^2([0,T], L^2_0(\Omega))$, then $(\partial_t \mathbf{v}_\varepsilon)_{\varepsilon>0}$ in $L^2([0,T], \mathbf{W}(\Omega)')$, finally $(\mathbf{v}_\varepsilon)_{\varepsilon>0}$ in $G_{2,v}(T)$. This we now do.

The procedure to derive an estimate for $(p_\varepsilon)_{\varepsilon>0}$ relies on the method developed in Sect. 8.3.3. Let u_ε denote the unique solution to the Neumann problem

$$- \Delta u_\varepsilon = p_\varepsilon, \quad \frac{\partial u_\varepsilon}{\partial \mathbf{n}}|_\Gamma = 0, \quad \int_\Omega u_\varepsilon(x)\, d\mathbf{x} = 0. \tag{8.196}$$

We also denote $\mathbf{w}_\varepsilon = \nabla u_\varepsilon \in \mathbf{W}(\Omega)$, which is known to verify at almost all $t \in [0,T]$:

$$||\mathbf{w}_\varepsilon||_{\mathbf{W}(\Omega)} \leq C_\Omega ||p_\varepsilon||_{0,2,\Omega}, \tag{8.197}$$

$$\langle p_\varepsilon, \nabla \cdot \mathbf{w}_\varepsilon \rangle = -||p_\varepsilon||^2_{0,2,\Omega}. \tag{8.198}$$

Estimate (6.69), combined with the Poincaré–Wirtinger inequality, leads to

$$\varepsilon ||p_\varepsilon||_{0,2,\Omega} \leq C||\mathbf{v}_\varepsilon||_{0,2,\Omega} \leq C'||\mathbf{v}_\varepsilon||_{\mathbf{W}(\Omega)}, \tag{8.199}$$

which combined with (8.197) ensures that $\mathbf{w}_\varepsilon \in L^2([0,T], \mathbf{W}(\Omega))$, so that it can be taken as test in the fluid equation (8.106.i) of $\mathscr{EP}^k_{\alpha,\beta,\varepsilon}$.

Now comes the trick that makes things work. We use the Helmholz decomposition (see in [92, 160]), which allows us to decompose the velocity $\mathbf{v}_\varepsilon$ as

$$\mathbf{v}_\varepsilon = \mathbf{v}_{div,\varepsilon} + \nabla v_\varepsilon,$$

where $\nabla \cdot \mathbf{v}_{div,\varepsilon} = 0$ and $\mathbf{v}_{div,\varepsilon} \cdot \mathbf{n} = 0$ on Γ, so that

$$\nabla \cdot \mathbf{v}_\varepsilon = \Delta v_\varepsilon \ \text{ in } \ \Omega, \quad \frac{\partial v_\varepsilon}{\partial \mathbf{n}} = 0 \ \text{ on } \ \Gamma, \quad \int_\Omega v_\varepsilon(t,\mathbf{x})\, d\mathbf{x} = 0.$$

We conclude from the definition of P_ε, specified by (8.101), that $v_\varepsilon = \varepsilon p_\varepsilon$. We infer from all of this[13]

[13]The integrations by parts result from the identities $\nabla \cdot (\partial_t \mathbf{v}_{div,\varepsilon}) = 0$ and $\partial_t \mathbf{v}_{div,\varepsilon} \cdot \mathbf{n} = 0$ on Γ, and roughly speaking (8.200) should proceed by approximation, which is technical, and the details are not essential for our purpose here.

$$\begin{aligned}\langle \partial_t \mathbf{v}_\varepsilon, \mathbf{w}_\varepsilon \rangle &= \langle \partial_t \mathbf{v}_{div,\varepsilon} + \partial_t \nabla v_\varepsilon, \nabla u_\varepsilon \rangle \\ &= \langle \partial_t \nabla v, \nabla u \rangle = -\langle \partial_t v_\varepsilon, \Delta u_\varepsilon \rangle \\ &= \varepsilon^2 \langle \partial_t p_\varepsilon, p_\varepsilon \rangle = \varepsilon^2 \frac{d}{2dt} ||p_\varepsilon||^2_{0,2,\Omega}.\end{aligned} \tag{8.200}$$

Combining (8.198) with (8.200) shows that taking $\mathbf{w}_\varepsilon$ as test in (8.106.i) provides the equality, satisfied at almost all $t \in [0, T]$:

$$\varepsilon^2 \frac{d}{2dt} ||p_\varepsilon||^2_{0,2,\Omega} + ||p_\varepsilon||^2_{0,2,\Omega} = \langle \mathbf{f}, \mathbf{w}_\varepsilon \rangle - \langle \mathscr{T}^{(k)}_\alpha(\mathbf{v}_\varepsilon, k_\varepsilon), \mathbf{w}_\varepsilon \rangle + \langle G_\alpha(\mathbf{v}_\varepsilon), \mathbf{w}_\varepsilon \rangle,$$

leading to

$$\begin{aligned}&\frac{1}{2}\varepsilon^2 ||p_\varepsilon(T)||^2_{0,2,\Omega} + ||p_\varepsilon||^2_{0,2,Q} = \\ &\qquad \int_0^T \langle \mathbf{f}(t), \mathbf{w}_\varepsilon(t) \rangle dt - \int_0^T \langle \mathscr{T}^{(k)}_\alpha(\mathbf{v}_\varepsilon(t), k_\varepsilon(t)), \mathbf{w}_\varepsilon(t) \rangle dt \\ &\qquad\qquad + \int_0^T \langle G_\alpha(\mathbf{v}_\varepsilon(t)), \mathbf{w}_\varepsilon(t) \rangle dt + \frac{1}{2}\varepsilon^2 ||p_\varepsilon(0)||^2_{0,2,\Omega}.\end{aligned} \tag{8.201}$$

We deduce from inequality (8.199)

$$\frac{1}{2}\varepsilon^2 ||p_\varepsilon(0)||^2_{0,2,\Omega} = \frac{1}{2}\varepsilon^2 ||P_\varepsilon(\mathbf{v}_0)||^2_{0,2,\Omega} \le \frac{C}{2} ||\mathbf{v}_0||^2_{0,2,\Omega}, \tag{8.202}$$

which we combine with (8.112), (8.115), (8.29), (8.197), (8.201), and Young's inequality. This allows us to see that there exists a constant $C'_{p,2,\alpha}(T)$ that does not depend either on ε nor on β and such that

$$\forall\, \varepsilon > 0, \quad ||p_\varepsilon||^2_{0,2,Q} \le C'_{p,2,\alpha}(T). \tag{8.203}$$

Following argumentation that has been used many times before, we deduce from the estimates that have been collected within this subsection that there exists $C'_{\partial_t v,2,\alpha}(T)$ such that

$$\forall\, \varepsilon > 0, \quad ||\partial_t \mathbf{v}_\varepsilon||^2_{0,2,Q} \le C'_{\partial_t v,2,\alpha}(T), \tag{8.204}$$

hence $(\mathbf{v}_\varepsilon, p_\varepsilon, k_\varepsilon)_{\varepsilon>0}$ is indeed bounded in $U_1(Q)$; we denote by $(\mathbf{v}, p, k)$ any V-P-K-ESP-limit, in the sense of Lemma 8.5 for the case $p = 2$ and Lemma 8.6 for the case of $G_{2,k}(Q)$ (cf. (8.104)), and the $L^2(Q)$ weak convergence for the pressure.

8.7.4.2 Taking the Limit

The goal is to show that $(\mathbf{v}, p, k)$ is a solution to $\mathscr{EP}^k_{\alpha,\beta}$ by the energy method developed in Sect. 8.6.4. This is achieved point by point as in the proof of Lemma 8.10, apart from a few details that we shall focus on.

It is understood that $(\mathbf{v}, p, k)$ satisfies the fluid equation by (8.127.i) of $\mathscr{EP}^k_{\alpha,\beta}$ and that we can take the limit of all terms of the l.h.s. of TKE equation (8.127.iii) (cf. Lemma 8.7). Moreover, given any $\varepsilon > 0$, $(\mathbf{v}_\varepsilon, p_\varepsilon, k_\varepsilon)_{\varepsilon>0}$ satisfies the energy equality (8.129). To complete the proof, it is sufficient to prove the following:

(i) $\mathbf{v}$ satisfies the constraint equation (8.127.ii) of $\mathscr{EP}^k_{\alpha,\beta}$, and therefore, the energy equality (8.107) holds,

(ii) the following convergence holds:

$$\forall\, t \in [0, T], \quad \lim_{\varepsilon\to 0} \int_0^t (p_\varepsilon(s), \nabla\cdot\mathbf{v}_\varepsilon(s))_\Omega ds = 0. \tag{8.205}$$

(i) It is sufficient to recycle step 2 of the proof of Theorem 6.4, performing an additional integration with respect to time. It is then straightforward to arrive at the energy equality.

(ii) As $p_\varepsilon \to p$ in $L^2(Q)$ weak, then $\nabla p_\varepsilon \to \nabla p$ in $L^2([0, T], \mathbf{W}(\Omega)')$ weak as $\varepsilon \to 0$. As $\mathbf{v}_\varepsilon \to \mathbf{v}$ in $L^2(Q)$ strong and $\mathbf{v}_\varepsilon(s) \in \mathbf{W}(\Omega)$ for almost all $s \in [0, T]$, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_0^t (p_\varepsilon(s), \nabla\cdot\mathbf{v}_\varepsilon(s))_\Omega ds &= -\lim_{\varepsilon\to 0} \int_0^t \langle\nabla p_\varepsilon(s), \mathbf{v}_\varepsilon(s)\rangle ds \\ &= -\lim_{\varepsilon\to 0} \int_0^t \langle\nabla p_\varepsilon(s), \mathbf{v}_\varepsilon(s)\rangle ds \\ &= -\int_0^t \langle\nabla p(s), \mathbf{v}(s)\rangle ds \\ &= \int_0^t \langle p(s), \nabla\cdot\mathbf{v}(s)\rangle ds = 0,\end{aligned}$$

by (8.127.ii). Finally, we can take the limit in the formulations of the initial conditions (8.24) and (8.25) using the same reasoning, which concludes the proof of Theorem 8.4. □

8.7.5 Final Proofs

It remains for us to finalize the proofs of Theorems 8.5 and 8.6. Their demonstration makes full use of the techniques and results developed throughout this chapter, and only a few more details are required to complete the proofs

8.7.5.1 About Theorem 8.5

We have to show the convergence of $\mathscr{E}\mathscr{P}^k_{\alpha,\beta}$ to $\mathscr{E}\mathscr{P}^k_{\alpha}$ (8.131) and (8.132), as $\beta \to 0$,

Let $\alpha > 0$ be fixed and $(\mathbf{v}_\beta, p_\beta, k_\beta)$ be any solution to $\mathscr{E}\mathscr{P}^k_{\alpha,\beta}$. We already know from estimate (8.128), combined with those listed in Lemma 8.3, that:

(i) $(\mathbf{v}_\beta, p_\beta)_{\beta>0}$ is bounded in the space $G_{2,\nu}(Q) \times L^2([0,T], L^2_0(\Omega))$, where $G_{2,\nu}(Q)$ is defined by (8.104),
(ii) $(k_\beta)_{\beta>0}$ is bounded in the space $\mathbb{K}_{5/4}(Q)$, defined by (8.89).

Let $(\mathbf{v}, p, k)$ be any V-P-K-ESP-limit, in the sense of Lemma 8.5 for the case $p = 2$ and Lemma 8.6 for the case of $\mathbb{K}_{5/4}(Q)$, and the $L^2(Q)$ weak convergence for the pressure.

As above, $(\mathbf{v}, p, k)$ satisfies the fluid equation (8.132.i) of $\mathscr{E}\mathscr{P}^k_{\alpha}$, the constraint (8.132.ii), and we can take the limit in all terms in the l.h.s. of the TKE equation (8.132.iii).

Moreover, $(\mathbf{v}_\beta, p_\beta, k_\beta)$ satisfies the energy equality (8.107) as well as $(\mathbf{v}, p, k)$. Therefore, the energy method applies, and by the same proof as that of Lemma 8.9, we deduce:

$$\lim_{\beta\to 0} |D\mathbf{v}_\beta|^2 = |D\mathbf{v}|^2 \;\text{ in }\; L^1(Q) \;\text{ strong}, \tag{8.206}$$

hence by Lemma A.16 in [TB],

$$\lim_{\beta\to 0} T_{1/\beta}(|D\mathbf{v}_\beta|^2) = |D\mathbf{v}|^2 \;\text{ in }\; L^1(Q) \;\text{ strong}, \tag{8.207}$$

and as ν_t is bounded and continuous,

$$\mathbb{P}_\beta(\mathbf{v}_\beta, k_\beta) = \mathbb{P}(\mathbf{v}, k) \;\text{ in }\; L^1(Q) \;\text{ strong}, \tag{8.208}$$

where we recall that the source terms are defined in (8.16) and (8.100). Therefore, we can also take the limit in the TKE equation.

Finally, in the same way, we can also take the limit in (8.24) as well as in (8.25), $\forall\, \psi \in C^1([0,T], \mathscr{D}(\Omega))$ such that $\psi(T, \mathbf{x}) = 0$, which concludes this point. □

8.7.5.2 About Theorem 8.6

The last task is to prove that $(\mathscr{E}\mathscr{P}^k_{\alpha})_{\alpha>0}$ converges to the NS-TKE inequality model $\mathscr{I}\mathscr{P}^k$ (8.91)–(8.94).

Let $(\mathbf{v}_\alpha, p_\alpha, k_\alpha)$ be any solution to $\mathscr{E}\mathscr{P}^k_{\alpha}$. According to Lemma 8.4, the family $(\mathbf{v}_\alpha, p_\alpha, k_\alpha)_{\alpha>0}$ is bounded in the space $Y_1(Q)$ (cf.(8.91)). Let $(\mathbf{v}, p, k)$ be any V-P-K-ESP-limit, in the sense of Lemma 8.5 for the case $p = 4/3$ and Lemma 8.6 for the case of $\mathbb{K}_{5/4}(Q)$, and the $L^{4/3}(Q)$ weak convergence for the pressure. It must be proven that $(\mathbf{v}, p, k)$ is a solution to $\mathscr{I}\mathscr{P}^k$.

According to Lemma 8.8, $(\mathbf{v}, p, k)$ satisfies the fluid and constraint equations (8.94.i) and (8.94.ii). Moreover, we can take the limit in all terms of the l.h.s. of the TKE equation (8.94.iii). Unfortunately, the energy equality is not necessarily satisfied in the limit, so that we cannot apply the energy method. The only thing that can be said is the following. Let

$$\Lambda_\alpha = \sqrt{\nu_t(k_\alpha, t, \mathbf{x})} D\mathbf{v}_\alpha, \quad \Lambda = \sqrt{\nu_t(k, t, \mathbf{x})} D\mathbf{v}.$$

Then as $(\Lambda_\alpha)_{\alpha>0}$ converges to Λ in $L^2(\Omega)^9$ weak, which is a Hilbert space, then

$$||\Lambda||_{0,2,\Omega} \leq \liminf_{\alpha \to 0} ||\Lambda_\alpha||_{0,2,\Omega},$$

hence the variational inequality (8.94.iii). Checking the initial data is straightforward. □

Remark 8.7. When the wall-law function satisfies an extra convexity assumption, it is possible to show that in the case of $\mathscr{IP}^k$, $(\mathbf{v}, k)$ satisfies the energy inequality, $\forall\, t > 0$:

$$\begin{cases} \dfrac{1}{2}||\mathbf{v}(t)||^2_{0,2,\Omega} + \displaystyle\int\int_{Q_t} (2\nu + \nu_t(k(s,\mathbf{x}), s, \mathbf{x})|D\mathbf{v}(s,\mathbf{x})|^2 ds\, d\mathbf{x} \\ + \displaystyle\int_0^t \int_\Gamma g(\mathbf{v}(s,\mathbf{x})) \cdot \mathbf{v}(s,\mathbf{x}) d\Gamma(\mathbf{x}) ds \leq \dfrac{1}{2}||\mathbf{v}_0||^2_{0,2,\Omega} + \displaystyle\int_0^t \langle \mathbf{f}(s), \mathbf{v}(s)\rangle ds, \end{cases} \tag{8.209}$$

and we do not know whether this inequality is an equality.

Remark 8.8. Starting as in the proof of Theorem 7.7 and by means of Gronwall's Lemma, it is possible to show that if $k_0 \geq 0$ a.e. in Ω, then $k \geq 0$ a.e. in Q for all the variational problems considered above.

Remark 8.9. It is also possible to show that $\mathscr{EP}^k_\alpha$ has a solution when $E(k, t, \mathbf{x}) = |k|^{1/2}/\ell(\mathbf{x})$ and $\mu_t = C_\mu \ell \sqrt{k}$. The case $\nu_t = C_\nu \ell \sqrt{k}$ can only be considered in the framework of the inequality NS-TKE model $\mathscr{IP}^k$.

Remark 8.10. This analysis can be extended to the $k - \mathscr{E}$ model under suitable assumptions and more generally to models of the form (4.136).

8.8 Bibliographical Section

We conclude the theoretical analysis of the NS-TKE continuous model by a thorough bibliography for the 3D Naviers–Stokes equations, which have attracted a particular interest in the last 15 years, especially since the release of the Clay Mathematics Institute millennium problem (cf. Fefferman [116]). For instance, at the time of writing, typing “Navier–Stokes" as key word on the data basis MathSciNet yields 7720 matches, while “turbulence" yields 3818 matches.

This bibliography completes that of Chap. 3. Mainly based on articles from last 15 years, it focuses on 3D evolutionary NSE and some related mathematical LES models, such as Leray-alpha, Bardina and deconvolution models.

Articles and books on Euler equations, 2D NSE, geophysical equations such as the primitive equations, or climate turbulent models are not considered in the list below. We have also not mentioned the connections between the NSE and the homogenization theory, or fluid-structure interaction.

This list starts with a list of recent books. Then it is organized in thematic sections, which may be interconnected. Particular attention has been paid to ensure all trends are mentioned, explaining the selection made.

(i) Books:

Bardos-Nicolaenko [19], Berselli-Iliescu-Layton [40], Boyer-Fabrie [52], Cannone [61], Constantin-Foias [83], Doering-Gibbon [102], Coron [86], E. Feireisl [117], Foias-Manley-Rosa-Temam [129], Fois-Temam [267], Galdi [134], John [190], Layton-Rebholz [204], Lemarié-Rieusset [206], Lions [216], Majda-Bertozzi [221], Málek-Nečas-Rokyta-Ružička [223], Robinson [243], Ruelle [249], Temam [265, 266]

(ii) Attractors, dynamical systems:

Babin-Nicolaenko [16], Bartuccelli-Constantin-Doering-Gibbon-Gisselfält [20], Cheskidov-Foias [78], Constantin [81, 82], Constantin-Foias-Temam [84], Dung-Nicolaenko [100], Debussche-Temam [95], Foias [125], Foias-Jolly-Kukavica-Titi [128], Foias-Saut [130], Foias-Temam [131, 132], Gibbon-Titi [157], Layton [198], Hoff-Ziane [175], Lewandowski-Preaux [211], Miranville [230], J. Málek-Nečas [222], Miranville-Wang [231], Pinto de Moura-Robinson-Sánchez-Gabites [237], Ruelle-Takens [248], Titi [262–264].

(iii) Uniqueness results and related:

Berselli-Romito [44] Chemin [67, 69], Chemin-Gallagher [72], Danchin [88], Farwig-Taniuchi [115], Feireisl-Jin-Novotný [118], Gala [133], Galdi [135], Gallagher [142], Gallagher-Ibrahim-Majdoub [145], Iftimie [183], Kukavica-Vlad [192], Lemarié-Rieusset, [207] Lions-Masmoudi [217], Marchand-Paicu [225], Okamoto [233], Sinai-Arnold [252],

(iv) General results from harmonic analysis:

Bahouri-Gallagher [17], Cannone-Meyer [62], Cannone-Planchon-Schonbek [65], Chemin [68], Chemin-Gallagher [71, 72], Chemin-Lerner [74], Gallagher [141], Gallagher-Iftimie-Planchon [144], Giga-Inui-Mahalov [149], Giga-Miyakawa [148], Hmidi-Keraani [173], Iftimie [184], Kukavica-Vicol [192], Kozono-Taniuchi [196], Meyer [227], Seregin [256], Yoneda [269]

(v) Regularity issues, singularities, suitable weak solutions:
Avrin-Babin-Mahalov-Nicolaenko [15], Amrouche-Seloula [11], Amrouche-Rodríguez [12, 13], Beirão da Veiga [22–25, 27, 29], Beirão da Veiga-Kaplický-Ružička [30], Beirão da Veiga-Berselli [31], Berselli [34–39], Berselli-Galdi [41], Cao [56], Chemin [70], Chemin-Gallagher-Paicu [73], Farwig-Galdi-Sohr [113], Gibbon [153, 155], Gibbon-Doering [99], Gallagher-Koch-Planchon [143], Giga-Miura [147], Guillén-Tierra [162], Hou [178], Hou-Lei-Li [180], Hou-Shi-Wang [181], Jia-Šverák [189], Kukavica-Ziane [193–195] Mahalov-Nicolaenko-Seregin [220], Málek-Nečas-Pokorny-Schonbek [224], Rusin-Šverák [250], Seregin [253–256], Seregin-Šverák [257], Šverák [259, 260], Titi [261]

(vi) Navier boundary conditions:
Amrouche-Nečasová-Raudin [8], Amrouche-Penel-Seloula [10], Beirão da Veiga [26, 28], Berselli [31–33], Casado-Luna-Suárez [60], Chen-Qiang [76], Bulíček-Málek-Rajagopal [54, 55], Hron-Le Roux-Málek-Rajagopal [182], Gie-Kelliher [158], Iftimie-Sueur [185], Hoff [174], Iftimie-Raugel-Sell [186], Masmoudi-Rousset [226], Neustupa-Penel [232]

(vii) Exterior and unbounded domains, flows around obstacles:
Alliot-Amrouche [110], Amrouche-Hoang [9], Bulíček-Majdoub-Málek [53], Deuring [97], Farwig-Galdi-Kyed [111], Farwig-Komo [112], Farwig-Kozono-Sohr [114], Galdi-Maremonti-Zhou [136], Galdi-Kyed [137], Galdi-Silvestre [138, 139], Galdi [140], Han [171], Hillairet-Wittwer [172], Razafison [239],

(viii) Energy spectrum, Kolmogorov cascade, scales:
Bartuccelli-Doering-Gibbon-Malham [21], Chen-Glimm [75], Cheskidov-Shvydkoy-Friedlander [79], Constantin-Doering-Titi [66], Dascaliuc-Foias-Jolly [90], Dascaliuc-Grujić [91], Dunca-Neda-Rebholz [108], Dunca-Kohler-Neda-Rebholz [109], Doering-Foias [101], Doering-Gibbon [103], Doering-Titi [104], Gibbon [154], Goto [163], Holm-Tronci [176], Hou-Hu-Hussain [179], Iliescu-Wang [187].

(ix) NSE and control theory:
Bewley-Temam-Ziane [45], Colin-Fabrie [80], Coron [85], Coron-Guerrero [87], De Los Reyes-Griesse [98], Fernández-Cara [119, 120], Fernández-Cara-Guerrero- Imanuvilov-Yu-Puel [121, 122], Guerrero-Imanuvilov-Yu-Puel [164], Lions-Zuazua [215], Liu [218], Mahalov-Titi-Leibovich [219].

(x) Leray-alpha, Bardina, deconvolution, and related mathematical LES models:
Ali [3–5], Berselli-Galdi-Iliescu-Layton [42], Berselli-Lewandowski [43], Borggaard-Iliescu [51], Cao-Lunasin-Titi [58], Cao-Holm-Titi [59], Dunca [105, 106], Dunca-Epshteyn [107], Foias-Holm-Titi [126, 127], Gibbon-Holm [156], Geurts [165], Geurts-Holm [166–168], Holm [177], Ilyin-Lunasin-Titi [188], Larios-Titi [197], Layton-Lewandowski [199–203], Layton-Rebholz-Sussman [205], Levant-Ramos-Titi [208], Lewandowski [210], Rebholz [240], Rebholz-Sussman [241], Vishik-Titi-Chepyzhov [268]

(xi) Stochastic NSE, Leray-alpha, deconvolution, and related models:
Albeverio-Debussche-Xu [2], Brzeźniak-Peszat [50], Carelli-Prohl [57], Chen-Gao-Guo [77], Da Prato-Debussche [89], Debussche [93], Debussche-Odasso [94], Deugoue-Sango [96] Flandoli-Mahalov [123], Gao-Sun [152], Glatt Holtz-Ziane [161], Gunzburger-Labovsky [151], Mikulevicius [228], Mikulevicius-Rozovskii [229], Kim [191], Röckner-Zhang [244], Röckner-Zhang-Zhang [245], Romito [246], Romito-Xu [247], Sango [251],

(xii) Non-Newtonian fluids:
Amrouche-Cioranescu [6], Amrouche-Girault [7], Cioranescu-Girault [63], Cioranescu-Girault-Glowinski-Scott [64], Girault-Scott [150], Guo-Guo [170] Friz-Guillén-Rojas [124], Linshiz-Titi [212], Paicu-Raugel-Rekalo [234], Paicu-Vicol [235],

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Chapter 9
Finite Element Approximation of the Steady Smagorinsky Model

Abstract This chapter is devoted to the numerical approximation of the Smagorinsky model, in steady regime. We consider this model as a regularization of Navier–Stokes equations that includes the modeling of eddy diffusion effects by means of a discrete viscosity. We introduce Lagrange finite element spaces adapted to approximate the slip condition. We prove stability and strong convergence for solutions with the natural minimal regularity. We moreover study the asymptotic energy balance and in particular prove that the subgrid energy associated to the eddy diffusion asymptotically vanishes. We analyze the approximation of laminar flow by the SM, by means of error estimates for smooth solutions. These show a lack of optimality due to the Smagorinsky modeling of eddy viscosity.

9.1 Introduction

The LES models usually are considered as a continuous models, independent of the numerical approximation considered. They are intended to model the large scales of the flow, above a given cutoff length δ within the inertial range. The effect of the subgrid scales on these large scales is assumed to be modeled by the eddy diffusion terms. The statistical scale-similarity properties are used to model the eddy diffusion terms that in principle should affect a range of small scales of the resolved flow (the "subfilter" scales) and not the entire scale spectrum.

As we mentioned in Chap. 5, a thorough mathematical and numerical analysis of a large class of LES models has been performed: The LES models have unique weak solutions smoother than those of Navier–Stokes equations and are well-posed problems. This allows to perform a thorough numerical analysis of standard numerical approximations, dealing with stability, uniform well-posedness, and error estimates (cf. [40–43]). In this framework, the relevant question that should be answered is up to what extent the numerical solution of the LES model approaches the targeted large scales of the turbulent flow. It is possible to rigorously derive

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averaged models that exhibit a dissipative behavior, such as the long-time average one obtained in Sect. 3.5. However, up to the knowledge of the authors, there are no proofs at the present day of the convergence of the solution provided by a LES model to some kind of average of the solution provided by the Navier–Stokes equations.

Also, in practice, some relationship between the cutoff length δ and the grid size h must be set. Indeed, if $\delta >> h$, then the numerical solution will solve scales smaller than the modeled ones, so an unuseful computational effort is being made. If $\delta << h$ a large discretization error to approximate the large-scale flow is being made. So usually a good choice corresponds to $h \simeq \delta$. We may write this relation as $\delta \simeq h$, in the sense that in practice a discretization of the LES model with a grid size h would give a good approximation of the large-scale flow with characteristic length scales $\delta = h$.

In this context we face the question of whether the discretized model (5.153) when $\delta \simeq h$ is a good approximation of the Navier–Stokes equations. This is the basic question that should be positive answered to validate both the LES approach and its numerical discretization. This question will become more and more relevant as the available computational resources allow to decrease h for practical computations. The analysis of whether the LES modeling provides a good approximation to the large-scale flow is not solved in this way, but this gives some basic validation of the LES approach.

This is the approach we focus in this and the next chapters. We perform in Chaps. 9 and 10 the numerical analysis of the finite element approximation of the Smagorinsky model (SM in the sequel) in steady and evolution regimes, respectively, including wall laws (cf. [48]). The SM is considered as a viscous numerical approximation of Navier–Stokes equations where the underlying assumption is again that the effect of the subgrid scales on the resolved scales is modeled by the eddy diffusion terms and that the resolved scales are an approximation of the mean turbulent flow. We shall prove that indeed it provides a solution that converges to a weak solution of Navier–Stokes equations. So, the effects of the eddy diffusion on the large-scale flow disappear if all scales are resolved.

Our choice of finite element approximations is based upon their ability to fit complex geometric flow configurations. For this reason finite elements are widely used in many flow solvers for industrial applications. This is the case, for instance, of the Ansys-Fluent, Comsol, and Femap platforms (cf. [1, 29, 47]), among others. We focus on continuous Lagrange finite elements which are well adapted to partial differential equations of second order and are frequently used in the industrial solvers mentioned above. More general finite elements or different kinds of discretizations (spectral or finite volumes) would introduce an unnecessary complexity. In addition, finite element discretizations allow to construct Galerkin approximations of the variational problems, thus allowing to use all the mathematical analysis procedures, in particular the functional analysis tools, already introduced in the preceding chapters.

In our model we include wall-law boundary conditions (BC) to take into account turbulent boundary layers along solid walls. The use of wall laws is commonly used in engineering applications to avoid the computation of boundary layers generated by solid walls. So we have considered convenient to include the analysis of this kind of boundary conditions in our analysis. In addition, to be somewhat realistic, we include Dirichlet BC on the inflow part of the boundary. Dirichlet BC typically model inflow boundaries, such as an air intake. To avoid unnecessary complexities, we just impose homogeneous Dirichlet BC and do not include outflow conditions. However, the analysis of this rather simple kind of mixed boundary conditions requires the construction of dense spaces of smooth functions to prove that the limit functions indeed are solutions in the sense of distributions. This kind of result does not appear in the literature, up to the knowledge of the authors. Instead we have proved that a family of convenient spaces of finite element functions provides an internal approximation with $W^{1,\infty}$ regularity, for polyhedric domains. This regularity is sufficient to pass to the limit in the discrete problems. General Dirichlet and outflow BCs may be taken into account by standard techniques that we describe in Sect. 9.8.2. The approximation of mixed BC in more general domains may be performed by well-known techniques that fit into our general functional framework of approximation of mixed problems, but are much more involved technically, in particular in what concerns the building of dense spaces of smooth functions. We outline these techniques in Sect. 9.8.3.

The present chapter is devoted to the analysis of the steady Smagorinsky model by the finite element method. The analysis of steady models of turbulence makes sense beyond a pure technical exercise. Indeed, from the modeling point of view, we proved in Sect. 3.5 that the asymptotic limit of the averaged time statistics of the evolution flow satisfies the dissipative steady problem (3.102). Moreover, from the practical point of view, in many engineering applications steady flows are targeted, as they provide reliable predictions of the average properties of interest (drag, lift, shear, ...). Although the SM in practice is over-diffusive for most flows, its analysis is the basis for more complex turbulence models. This is the reason we include it in this book.

From the analytical point of view we use several techniques already introduced in Chap. 6:

- The Galerkin method: to approximate infinite-dimensional variational problems by sets of algebraic equations in finite dimension (the finite element method is a particular Galerkin method).
- Linearization of nonlinear non-compact terms: to obtain well-posed problems.
- Compactness: to prove existence of solutions of finite- and infinite-dimensional problems. This allows to prove the convergence of the approximations to a weak solution.
- Energy method: to prove the strong convergence of the approximated solutions.

In addition we need to adapt the use of these techniques to a finite element discretization. We thus construct the finite element spaces, adapted to mixed wall laws, and prove that these indeed provide Galerkin approximations of the infinite-dimensional spaces used in the variational problems. Furthermore, these spaces play the role of dense spaces of smooth functions. This is performed in

Sects. 9.3.1–9.3.3. In this way in this chapter, so as in Chaps. 10–12, all devoted to numerical approximations, we exclusively just focus our attention on numerical analysis aspects.

The chapter is structured as follows. In Sect. 9.2 we introduce the Navier–Stokes equations wall-law problem that we intend to approximate. In Sect. 9.3 we introduce the Lagrange finite element spaces and their adaptation to the approximation of slip BC and incompressible flows. Section 9.4 is devoted to the interpretation of the variational formulation for Navier–Stokes equations introduced in Sect. 9.2. In Sect. 9.5 we introduce the SM finite element discretization. Section 9.6 performs the numerical analysis of the approximation of steady turbulent flows by the SM using finite element discretizations. We prove stability and strong convergence for solutions with the natural minimal regularity. We moreover prove that the subgrid energy associated to the eddy diffusion vanishes asymptotically as $h \to 0$. In Sect. 9.7 we perform the error analysis for laminar flow. These show a lack of optimality due to the modeling of eddy viscosity. Finally Sect. 9.8 introduces some complements to the analysis performed in the chapter that deal on alternative space discretizations, improved modeling of eddy viscosity in the SM, numerical treatment of slip BC for general domains, and mathematical justification of wall laws.

9.2 Navier–Stokes Equations with Mixed Boundary Conditions

In this section we introduce a mixed boundary value problem for the Navier–Stokes equations that includes wall-law BC in combination with homogeneous Dirichlet BC. This will be the limit problem to be reached by the SM that we shall consider in the rest of the chapter.

Let us consider a polyhedric bounded domain $\Omega \subset \mathbb{R}^d$ (for brevity we name "polyhedric" a polygonal domain when $d = 2$ and a polyhedric domain when $d = 3$). We split the boundary of Ω as $\partial\Omega = \Gamma = \overline{\Gamma_n} \cup \overline{\Gamma_D}$ where Γ_D and Γ_n are disjoint measurable open subsets of Γ with positive measure. We set wall-law BC on Γ_n and Dirichlet BC on Γ_D. We consider the following boundary value problem for the Navier–Stokes equations:

Find a velocity field $\mathbf{v} : \Omega \to \mathbb{R}^d$ and a pressure $p : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - \nabla \cdot [2\nu\, D\mathbf{v}] + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{v} = 0 & \text{in } \Omega, \\ -[2\nu\, D\mathbf{v} \cdot \mathbf{n}]_{\boldsymbol{\tau}} = g(\mathbf{v})_{\boldsymbol{\tau}} & \text{on } \Gamma_n, \\ \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \Gamma_n, \\ \mathbf{v} = 0 & \text{on } \Gamma_D, \end{cases} \tag{9.1}$$

where $\mathbf{n}$ is the normal to $\partial\Omega$, the subscript $_\tau$ represents the tangential component with respect to $\partial\Omega$ defined as $\mathbf{v}_\tau = \mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\mathbf{n}$, and $g : \mathbb{R}^d \to \mathbb{R}^d$ is a given function. As we are considering polyhedric domains, then the normal and the tangent vectors to $\partial\Omega$ are well defined, but on the edges of its faces.

In a more general context, this problem may be set on domains with Lipschitz continuous boundary. For homogeneous Dirichlet BC in the whole Γ its numerical analysis has been extensively studied. Let us mention, for instance, the basic books of Girault and Raviart [34] and Temam [49], where in particular it is proved that it admits a solution $(\mathbf{v}, p)$ that belongs to $\mathbf{H}^1(\Omega) \times L^2_0(\Omega)$, whose norm is bounded in terms of the data $\mathbf{f}$. However the treatment of slip condition on general Lipschitz domains is quite involved technically (see Sect. 9.8.3), and this complexity is largely increased when dealing with mixed boundary conditions as we are considering. For this reason we assume that Ω is polyhedric.

To give a variational formulation to problem (9.1), let us denote by γ_0 and by γ_n the trace and the normal trace operators on Γ. The operator γ_0 is linear and bounded from $H^1(\Omega)$ onto $H^{1/2}(\Gamma_D)$ (see Theorem A.2), while the operator γ_n is linear and bounded from $\mathbf{H}^1(\Omega)$ onto $L^4(\Gamma)$, and $\gamma_n \mathbf{w} = (\gamma_0 \mathbf{w}) \cdot \mathbf{n}$ a. e. on Γ if $\mathbf{w} \in \mathbf{H}^1(\Omega)$ (see Lemma A.1). Let us consider the space

$$\mathbf{W}_D(\Omega) = \{\mathbf{w} \in \mathbf{H}^1(\Omega) \;\text{ such that }\; \gamma_0\mathbf{w} = 0 \;\text{ on }\; \Gamma_D,\; \gamma_n\mathbf{w} = 0 \;\text{ on }\; \Gamma_n\}. \quad (9.2)$$

This is a closed subspace of $\mathbf{H}^1(\Omega)$, and thus a Hilbert space endowed with the $\mathbf{H}^1(\Omega)$ norm (see Sect. A.2). This norm is equivalent to the norm

$$\|\mathbf{w}\|_{\mathbf{W}_D(\Omega)} = \|D\mathbf{w}\|_{0,2,\Omega},$$

thanks to the Korn inequality (cf. Sect. A.4.4).

We shall consider weak solutions of (9.1), defined as follows.

Definition 9.1. Let $\mathbf{f} \in \mathbf{W}_D(\Omega)'$. A pair $(\mathbf{v}, p) \in \mathbf{W}_D(\Omega) \times L^2_0(\Omega)$ is a weak solution of the boundary value (9.1) for the Navier–Stokes equations if it satisfies

$$\mathscr{VP}\begin{cases} b(\mathbf{v};\mathbf{v},\mathbf{w}) + a(\mathbf{v},\mathbf{w}) - (p, \nabla\cdot\mathbf{w})_\Omega + \langle G(\mathbf{v}), \mathbf{w}\rangle = \langle \mathbf{f}, \mathbf{w}\rangle, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\nabla\cdot\mathbf{v}, q)_\Omega = 0, \end{cases} \quad (9.3)$$

for any $(\mathbf{w}, q) \in \mathbf{W}_D(\Omega) \times L^2_0(\Omega)$, where $\langle\cdot,\cdot\rangle$ denotes the duality product between $\mathbf{W}_D(\Omega)'$ and $\mathbf{W}_D(\Omega)$ and the forms a, b, and G are given by (6.13) and (6.14).

The properties of forms a, b, and G are respectively stated in Lemmas 6.2, 6.3, and 6.5. Green's integration formulas used in the proof of Lemma 6.3 that yield (6.24) and (6.25) actually hold if Ω is Lipschitz (cf. [34], Lemma 1.4) so in particular if it is polyhedric as we consider here.

A solution $(\mathbf{v}, p)$ of the variational problem (9.3) satisfies the Navier–Stokes equations (9.1) in a convenient sense, where each equation takes place in a specific Sobolev space. To prove this result some density result by smooth functions is needed. We do not know any density result on $\mathbf{W}_D(\Omega)$ of C^m functions, similar to the density of $\mathscr{C}^m(\overline{\Omega})^3$ in $\mathbf{W}(\Omega)$ proved in Sect. 6.2. Instead we use the density of the finite element approximations that we use to approximate the variational problem (9.3) that we introduce in the next section. So we report this proof to Sect. 9.4.

9.3 Mixed Finite Element Approximations

We approximate $\mathbf{W}_D(\Omega) \times L^2_0(\Omega)$ by a family of pairs of finite element spaces

$$(\mathbf{W}_h, M_h) \subset [\mathbf{W}^{1,\infty}(\Omega) \cap \mathbf{C}^0(\Omega)] \times [W^{1,\infty}(\Omega) \cap C^0(\Omega)],$$

associated to a family of triangulations $(\mathscr{T}_h)_{h>0}$ of Ω. This section introduces the construction of finite element *internal approximation* of $\mathbf{W}_D(\Omega) \times L^2_0(\Omega)$, in the following sense:

Definition 9.2. Let B be a separable Banach space. An internal approximation of B is a family $(B_h)_{h>0}$ of subspaces of finite dimension of B such that for any $b \in B$,

$$\lim_{h \to 0} d_B(b, B_h) = 0.$$

The concept of internal approximation is a slight extension of the concept of Hilbert basis, introduced in Sect. 6.5. Moreover, the solvability of the pressure for incompressible flows follows if the discrete velocity and pressure spaces satisfy a *uniform discrete inf-sup condition*: There exists a constant $\alpha > 0$ independent of h such that

$$\alpha \|q_h\|_{0,2,\Omega} \le \sup_{\mathbf{w}_h \in \mathbf{W}_h} \frac{(\nabla \cdot \mathbf{w}_h, q_h)_\Omega}{\|\mathbf{w}_h\|_{1,2,\Omega}}, \quad \forall q_h \in M_h, \quad \forall h > 0. \tag{9.4}$$

We proceed in several steps: Lagrange finite element approximations of $H^1(\Omega)$ (Sect. 9.3.1), approximation of the velocity space $\mathbf{W}_D(\Omega)$ by interpolation of the slip boundary condition (Sect. 9.3.2), and approximation of the velocity–pressure spaces by means of mixed finite elements (Sect. 9.3.3).

9.3.1 Lagrange Finite Element Spaces

This section introduces the basic aspects of the interpolation by finite element functions that we use to build our finite element approximation of $\mathbf{W}_D(\Omega) \times L^2_0(\Omega)$. It is based upon the book of Bernardi et al. [5]. Other relevant references on finite element approximation of PDEs are the works of Brenner and Scott [8], Ciarlet [24], and Ern and Guermond [31], among others.

The finite element spaces are internal approximations of Sobolev spaces formed by piecewise polynomial functions. To introduce definition of these spaces, let us recall that we assume that Ω is a bounded polyhedric domain.

Definition 9.3. A triangulation of Ω is a finite collection of compact subsets of Ω, either polygons if $d = 3$ or polyhedra if $d = 3$, $(K_i)_{i=1}^n$ such that:

1. $\overline{\Omega} = \bigcup_{i=1}^{n} K_i$;
2. $\overset{\circ}{K}_i \cap \overset{\circ}{K}_j = \emptyset$ for $i \neq j$;
3. The intersection of ∂K_i and ∂K_j is either the empty set, a vertex, a side, or a face.

The parameter h stands for the largest diameter of the elements of $\mathscr{T}_h$:

$$h = \max_{K \in \mathscr{T}_h} h_K, \quad h_K = \mathrm{diam}(K).$$

The parameter h is called the grid size. The family of triangulations is assumed to run on a sequence of sizes decreasing to zero. As this sequence in general is not predetermined, the notation $(\mathscr{T}_h)_{h>0}$ is used to denote all these possible sequences of triangulations with grid size decreasing to zero.

Definition 9.4. The family of triangulations $(\mathscr{T}_h)_{h>0}$ is regular if there exists a constant $C > 0$ such that

$$h_K \leq C\, \rho_K, \;\text{ for all }\; K \in \mathscr{T}_h, \;\text{ for all }\; h > 0, \tag{9.5}$$

where ρ_K is the largest diameter of all balls included in K. The smallest possible constant C in (9.5) is called the aspect ratio of the family of triangulations $(\mathscr{T}_h)_{h>0}$.

We consider triangular grids formed by triangles when $d = 2$ and by tetrahedra when $d = 3$ or quadrilateral grids formed by quadrilaterals when $d = 2$ and by parallelepipeds when $d = 3$. We shall denote by $P_l(K)$ the set of polynomials of degree smaller than or equal to l defined on K and by $Q_l(K)$ the set of polynomials on each variable $x_1, \cdots, x_d$ defined on K. The space $P_l(K)$ is spanned by the functions

$$p(\mathbf{x}) = \sum_{0 \leq \alpha_1 + \cdots + \alpha_d \leq l} c_{\alpha_1, \cdots, \alpha_d}\, x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \;\text{ for any }\; \mathbf{x} = (x_1, \cdots, x_d) \in \Omega,$$

where the $\alpha_i \geq 0$ are integer numbers and the $c_{\alpha_1, \cdots, \alpha_d}$ are real numbers. Also, $Q_l(K)$ is spanned by the functions

$$q(\mathbf{x}) = \sum_{\alpha_1=0}^{l} \cdots \sum_{\alpha_d=0}^{l} c_{\alpha_1, \cdots, \alpha_d}\, x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \;\text{ for any }\; \mathbf{x} \in \Omega.$$

The space $P_l(K)$ is well suited to build triangular of tetrahedral finite elements, while $Q_l(K)$ is well suited to build quadrilateral or parallelepipedic finite elements (cf. [5] for more details).

To simplify the notation, we shall denote by $R_l(K)$ either $P_l(K)$ for triangular finite element spaces or $Q_l(K)$ for quadrilateral finite element spaces.

Definition 9.5. The family of triangulations $(\mathscr{T}_h)_{h>0}$ is affine-equivalent to a reference element K^* if for any $K \in \mathscr{T}_h$, there exists a non-singular affine mapping F_K bijective from K^* onto K.

We assume that for either triangular or quadrilateral finite elements the family of triangulations is affine-equivalent.

We define the C^0 finite element space

$$V_h^{(l)}(\mathscr{T}_h) = \{v_h \in C^0(\overline{\Omega}) \text{ such that } v_{h_{|K}} \in R_l(K), \text{ for all } K \in \mathscr{T}_h\}. \tag{9.6}$$

The space $V_h^{(l)}(\mathscr{T}_h)$ is a subset of $W^{1,\infty}(\Omega)$ and then of $H^1(\Omega)$. We shall not need finite element spaces formed by smoother functions.

For each space $V_h^{(l)}(\mathscr{T}_h)$ there exists a set of interpolation nodes $\mathscr{A}_h \subset \overline{\Omega}$ such that the Lagrange interpolation problem

$$(P) \begin{cases} \text{Given the real values } (v_\alpha)_{\alpha \in \mathscr{A}_h}, \\ \text{find } v_h \in V_h^{(l)}(\mathscr{T}_h) \text{ such that } v_h(\alpha) = v_\alpha \text{ for all } \alpha \in \mathscr{A}_h, \end{cases} \tag{9.7}$$

admits a unique solution. In particular, there exists a unique function $\lambda_\alpha \in V_h^{(l)}(\mathscr{T}_h)$ such that

$$\lambda_\alpha(\beta) = \delta_{\alpha\beta}, \text{ for all } \beta \in \mathscr{A}_h. \tag{9.8}$$

Consequently, the solution of problem (P) is

$$v_h(\mathbf{x}) = \sum_{\alpha \in \mathscr{A}_h} v_\alpha \, \lambda_\alpha(\mathbf{x}), \text{ for all } \mathbf{x} \in \overline{\Omega}. \tag{9.9}$$

This allows to define the Lagrange interpolation operator $\Pi_h : C^0(\overline{\Omega}) \mapsto V_h^{(l)}(\mathscr{T}_h)$ by

$$\Pi_h v(\mathbf{x}) = \sum_{\alpha \in \mathscr{A}_h} v(\alpha) \, \lambda_\alpha(\mathbf{x}), \text{ for all } \mathbf{x} \in \overline{\Omega}, \tag{9.10}$$

for $v \in C^0(\overline{\Omega})$. Note that $\Pi_h v$ is the only element of V_h that satisfies $\Pi_h v(\alpha) = v(\alpha), \text{ for all } \alpha \in \mathscr{A}_h$.

The finite element spaces $V_h^{(l)}(\mathscr{T}_h)$ are called affine-equivalent. A more general class is the one formed by the isoparametric finite element spaces, for which F_K is not necessarily an affine transformation and then K is not necessarily polyhedric. This class includes general quadrilateral elements, prismatic elements with nonparallel bases, and elements with curved faces, among others (cf. [24]). Isoparametric finite elements are especially well suited to discretize partial differential equations on domains with curved faces. We do not consider here this situation, to avoid nonessential complexities that have been treated elsewhere (cf. for instance [24,31]).

The finite element approximation theory ensures that any function of $W^{1,q}(\Omega)$, $1 \leq q < +\infty$, may be approximated by functions of $V_h^{(l)}(\mathscr{T}_h)$:

Theorem 9.1. *Consider a regular family of triangulations of Ω, $(\mathscr{T}_h)_{h>0}$, and an integer $l \geq 1$. There exists a linear interpolation operator $\Pi_h : L^1(\Omega) \to V_h^{(l)}(\mathscr{T}_h)$, such that, for $1 \leq q < +\infty$,*

(i) For any $v \in L^q(\Omega)$,

$$\|\Pi_h(v)\|_{0,q,\Omega} \leq C\, \|v\|_{0,q,\Omega}. \tag{9.11}$$

For any $v \in W^{1,q}(\Omega)$,

$$\|\Pi_h(v)\|_{1,q,\Omega} \leq C\, \|v\|_{1,q,\Omega}, \tag{9.12}$$

where C is a constant depending only on q, Ω, d, and the aspect ratio of the family of triangulations $(\mathscr{T}_h)_{h>0}$.

(ii) For any $v \in W^{1,q}(\Omega)$,

$$\lim_{h\to 0} \|v - \Pi_h(v)\|_{1,q,\Omega} = 0. \tag{9.13}$$

(iii) If $v \in W_0^{1,q}(\Omega)$, then $\Pi_h(v) \in V_h^{(l)}(\mathscr{T}_h) \cap W_0^{1,q}(\Omega)$.

(iv) Let $m = 0$ or $m = 1$. If $v \in W^{k,r}(\Omega)$, for some integer k such that $m + 1 \leq k \leq l + 1$ and $1 \leq r \leq +\infty$, then the following error estimate holds:

$$\|v - \Pi_h(v)\|_{m,q,\Omega} \leq C\, h^{k-m+\frac{d}{q}-\frac{d}{r}} |v|_{k,r,\Omega}, \tag{9.14}$$

where C is a constant depending only on m, k, q, r, Ω, d, and the aspect ratio of the family of triangulations $(\mathscr{T}_h)_{h>0}$.
In addition, for any $K \in \mathscr{T}_h$,

$$\|v - \Pi_h(v)\|_{m,q,K} \leq C\, h^{k-m+\frac{d}{q}-\frac{d}{r}} |v|_{k,r,\Delta_K}, \tag{9.15}$$

where Δ_K is the union of all elements of $\mathscr{T}_h$ that intersect K.

(v) The Lagrange interpolation operator Π_h defined by (9.10) satisfies items (i), (ii), (iii), and (iv) for functions v that additionally belong to $C^0(\overline{\Omega})$.

The restriction $m \leq 1$ in (iv) applies because C^0 finite elements may only approximate functions in $L^2(\Omega)$ and $H^1(\Omega)$ norms. Also, the restriction $k \leq l + 1$

means that the error estimates (9.14) and (9.15) cannot be improved by increasing the regularity of the function v beyond this limit: If $v \in W^{k,r}(\Omega)$ for some $k \geq l+2$, then both error estimates hold with $k = l + 1$. We shall particularly be interested in the case $q = r = 2$. In this case, estimate (9.14) reads

$$\|v - \Pi_h(v)\|_{m,2,\Omega} \leq C\, h^{k-m} |v|_{k,2,\Omega}, \quad m = 0, 1. \tag{9.16}$$

By Theorem 9.1, the family $(V_h^{(l)}(\mathscr{T}_h))_{h>0}$ is an internal approximation of $H^1(\Omega)$ and also of $L^2(\Omega)$ if $l \geq 1$.

Remark 9.1. The interpolation operator Π_h may be constructed in several ways. Usually it is constructed by Lagrange interpolation of nodal values which are calculated as local means of the function to be interpolated. These means are needed because in general the functions of $W^{1,q}(\Omega)$ are not continuous and then have no nodal values. We may mention today the classical Clément interpolator (cf. [25]), which verifies properties (i), (ii), and (iv) but not property (iii), and the more recent of Bernardi et al. (cf. [5]) and Scott and Zhang (cf. [45]) that satisfy all three properties.

Let us describe the BMR (Bernardi–Maday–Rapetti) interpolation operator for triangular elements that we shall use in the sequel. It is an adaptation of the Lagrange interpolation operator to function with only L^1 regularity, in the form

$$\tilde{\Pi}_h(v) = \sum_{\alpha \in \mathscr{A}_h} \pi_\alpha v(\alpha)\, \lambda_\alpha(\mathbf{x}), \ \text{ for all } \ \mathbf{x} \in \overline{\Omega}, \tag{9.17}$$

where π_α is a regularization operator, by local L^2 projection. Concretely, to each node α of $\mathscr{A}_h$, we associate either an element or an edge (in 2D) or a side (in 3D) of some element of the triangulation that contains the node α, that we denote K_α in any case. We then define $\pi_\alpha : L^1(K_\alpha) \mapsto P_l(K_\alpha)$ by

$$\int_{K_\alpha} (v - \pi_\alpha v)\, q = 0, \ \text{ for all } \ q \in P_l(K_\alpha). \tag{9.18}$$

Some useful relationships between norms of finite element functions are given by the inverse inequalities, stated as follows (cf. [5]):

Theorem 9.2. *Let q_1, q_2 be two real numbers such that $1 \leq q_1, q_2 \leq +\infty$. Let k_1, k_2 be to nonnegative integer numbers. Assume that $k_2 \leq k_1$ and $k_2 - \frac{d}{q_2} \leq k_1 - \frac{d}{q_1}$. For any nonnegative integer l there exists a constant $C > 0$ such that*

$$|v|_{k_1,q_1,K} \leq C\, \rho_K^{k_2-k_1-\frac{d}{q_2}}\, h_K^{\frac{d}{q_1}}\, |v|_{k_2,q_2,K}, \quad \forall v \in R_l(K). \tag{9.19}$$

If in addition the family of triangulations $(\mathscr{T}_h)_{h>0}$ is regular, then for all $\mathscr{T}_h$

$$|v|_{k_1,q_1,K} \le C\, h_K^{k_2-k_1-\frac{d}{q_2}+\frac{d}{q_1}} |v|_{k_2,q_2,K}, \quad \forall K \in \mathscr{T}_h, \quad \forall v \in R_l(K) \qquad (9.20)$$

where the constant C only depends on q_1, q_2, k_1, k_2, d, l, and the aspect ratio of the family of triangulations.

9.3.2 Finite Element Approximation of Slip Condition for Polyhedric Domains

The approximation of the slip boundary condition for domain with curved boundaries in a naive way may lead to impose a wrong boundary condition: Assume for instance that $d = 2$ and Ω is a circle. Assume that Ω is approximated by a polygonal domain Ω_h with boundary Γ_h. Consider some triangulation $\mathscr{T}_h$ of Ω_h. Let us approximate the velocity field $\mathbf{w}$ by a finite element velocity $\mathbf{w}_h$ of space $V_h^{(1)}(\mathscr{T}_h)$, formed by piecewise affine functions. Imposing the slip condition $\mathbf{w}_h \cdot \mathbf{n} = 0$ on each edge of $\mathscr{T}_h$ leads to $\mathbf{w}_h = 0$ at each grid node located on Γ_h because $\mathbf{w}_h$ is continuous. For piecewise affine finite elements, this yields $\mathbf{w}_h = 0$ on Γ_h. Then, if $\mathbf{w}_h$ converges to some $\mathbf{w}$ in $\mathbf{H}^1(\Omega)$ as $h \to 0$, we conclude that $\mathbf{w} = 0$ on Γ.

There exist well-established techniques to solve this difficulty, introduced by R. Verfürth. For instance, the slip condition may be considered as a restriction and implemented through a saddle-point problem approach (cf. [52, 53]). Another possible remedy is to use isoparametric finite elements to fit the curved parts of the boundary (cf. [51]). However these solutions are too involved technically for the purposes of this book, so we consider polyhedric domains. For these domains the abovementioned "naive" way of imposing the slip boundary condition works, as we shall prove in this section.

Let us assume that Ω is a polyhedric domain. We shall consider triangulations $\mathscr{T}_h$ that exactly match Γ_D and Γ_n, in the following sense:

Definition 9.6. A triangulation $\mathscr{T}_h$ of $\overline{\Omega}$ is admissible if both $\overline{\Gamma_D}$ and $\overline{\Gamma_n}$ may be split as the union of whole edges ($d = 2$) or faces ($d = 3$) of some elements of $\mathscr{T}_h$.

The boundary Γ may be split as the union of the sides (in 2D) or faces (in 3D) $\Sigma_1, \cdots, \Sigma_m$ that we assume to be closed sets of Ω, in such a way that $\overline{\Gamma_n} = \cup_{i=1}^{k-1} \Sigma_i$, $\overline{\Gamma_D} = \cup_{i=k}^{m} \Sigma_i$ for some integer $k \in 2, \cdots, m$. We approximate $\mathbf{W}_D(\Omega)$ by the discrete spaces defined by

$$\mathbf{W}_h = \{\mathbf{w}_h \in V_h^d \ \text{ s. t. } \ \mathbf{w}_h = 0 \ \text{ on } \ \overline{\Gamma_D},\, \mathbf{w}_h \cdot \mathbf{n}_i = 0 \ \text{ on } \ \Sigma_i,\, i = 1, \cdots, k-1\} \qquad (9.21)$$

where V_h is a Lagrange finite element space constructed on an admissible triangulation of a polyhedric domain Ω and $\mathbf{n}_i$ denotes the normal to Σ_i, outer to Ω. Note that the conditions $\mathbf{w}_h \cdot \mathbf{n}_i = 0$ on Σ_i, $i = 1, \cdots, k-1$ and $\mathbf{w}_h = 0$ on $\overline{\Gamma_D}$ imply $\gamma_n \mathbf{w}_h = \mathbf{w}_h \cdot \mathbf{n} = 0$ a. e. in Γ. Then $\mathbf{W}_h$ is a subspace of $\mathbf{W}_D(\Omega)$.

The Lagrange interpolation operator Π_h on V_h defined by (9.10) preserves both the no-slip and the slip condition, for continuous velocities:

Lemma 9.1. *Assume that the triangulation $\mathscr{T}_h$ is admissible in the sense of Definition 9.6. Let $\mathbf{w} \in \mathbf{W}_D(\Omega) \cap \mathbf{C}^0(\overline{\Omega})$. Then $\Pi_h \mathbf{w} \in \mathbf{W}_D(\Omega)$.*

Proof. Let F be an edge (when $d = 2$) or face (when $d = 3$) of some element of $\mathscr{T}_h$. Assume $F \subset \overline{\Gamma_n}$. Let $\mathbf{x} \in F$. As $\mathbf{n}$ is constant on F,

$$(\Pi_h \mathbf{w})(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = \left(\sum_{\alpha \in \mathscr{A}_h \cap F} \mathbf{w}(\alpha)\, \lambda_\alpha(\mathbf{x}) \right) \cdot \mathbf{n}_{|F} \tag{9.22}$$
$$= \sum_{\alpha \in \mathscr{A}_h \cap F} \mathbf{w}(\alpha) \cdot \mathbf{n}(\alpha)\, \lambda_\alpha(\mathbf{x}) = 0,$$

where we have used that $\mathbf{w}(\alpha) \cdot \mathbf{n}(\alpha) = 0$ if $\alpha \in F$. Consequently, $(\Pi_h \mathbf{w}) \cdot \mathbf{n} = 0$ on Γ_n. Also, if $F \subset \overline{\Gamma_D}$, then $(\Pi_h \mathbf{w})(\mathbf{x}) = \left(\sum_{\alpha \in \mathscr{A}_h \cap F} \mathbf{w}(\alpha)\, \lambda_\alpha(\mathbf{x}) \right) = 0$ because $\mathbf{w} = 0$ on Γ_D. As $\mathscr{T}_h$ is admissible, then $\overline{\Gamma_D}$ is the union of whole edges or faces of elements of $\mathscr{T}_h$. Thus, $\Pi_h \mathbf{w} = 0$ on Γ_D. By construction, $\Pi_h \mathbf{w} \in \mathbf{H}^1(\Omega)$. We conclude that $\Pi_h \mathbf{w} \in \mathbf{W}_D(\Omega)$. □

This allows to prove that the family of spaces $(\mathbf{W}_h)_{h>0}$ is an internal approximation of $\mathbf{W}_h$ if a convenient space of smooth functions is dense in $\mathbf{W}_D(\Omega)$:

Theorem 9.3. *Assume that the set*

$$\mathscr{W}(\overline{\Omega}, \Gamma_D) = \{\varphi \in \mathscr{D}(\overline{\Omega}) \ \textit{such that} \ \varphi = 0 \ \textit{in a neighborhood of} \ \Gamma_D, \ \varphi \cdot \mathbf{n} = 0 \ \textit{on} \ \Gamma_n\}$$

is dense in $\mathbf{W}_D(\Omega)$. Consider a regular family of admissible triangulations of Ω, $(\mathscr{T}_h)_{h>0}$. Then, the family of spaces $(\mathbf{W}_h)_{h>0}$ is an internal approximation of $\mathbf{W}_h$.

Proof. Let $\mathbf{w} \in \mathbf{W}_D(\Omega)$. As $\mathscr{W}(\overline{\Omega}, \Gamma_D)$ is dense in $\mathbf{W}_D(\Omega)$, for any $\varepsilon > 0$ there exists $\mathbf{z}_\varepsilon \in \mathscr{W}(\overline{\Omega}, \Gamma_D)^d$ such that $\|\mathbf{z}_\varepsilon - \mathbf{w}\|_{1,2,\Omega} < \varepsilon/2$. Due to Theorem 9.1, there exists $h_\varepsilon > 0$ such that if $0 < h < h_\varepsilon$, $\|\Pi_h(\mathbf{z}_\varepsilon) - \mathbf{z}_\varepsilon\|_{1,2,\Omega} < \varepsilon/2$. Then $\|\Pi_h(\mathbf{z}_\varepsilon) - \mathbf{w}\|_{1,2,\Omega} < \varepsilon$ if $0 < h < h_\varepsilon$. So, $\lim_{h\to 0} d_{1,2,\Omega}(\mathbf{w}, \mathbf{W}_h) = 0$. Also, by Lemma 9.1, $\Pi_h \mathbf{z}_\varepsilon \in \mathbf{W}_D(\Omega)$. The conclusion follows. □

This result is based upon the density of $\mathscr{W}(\overline{\Omega}, \Gamma_D)$ in $\mathbf{W}_D(\Omega)$ which, as far as we know, only can be proved for very particular domains (See Sect. A.3). We may go around this difficulty for polyhedric domains by means of an adaptation of the BMR interpolation operator, $\overrightarrow{\Pi}_h : \mathbf{H}^1(\Omega) \mapsto V_h^d$ of the form

$$\overrightarrow{\Pi}_h \mathbf{v}(\mathbf{x}) = \sum_{\alpha \in \mathscr{A}_h} \overrightarrow{\pi}_\alpha \mathbf{v}(\alpha)\, \lambda_\alpha(\mathbf{x}), \ \text{ for any } \ \mathbf{x} \in \overline{\Omega} \tag{9.23}$$

Note that $\overrightarrow{\Pi}_h \mathbf{v}$ is the standard Lagrange interpolate of a smoothed function that takes on the regularized values $\overrightarrow{\pi}_\alpha \mathbf{v}(\alpha)$ at the nodes $\alpha \in \mathscr{A}_h$. Then it holds (cf. Sect. A.3.1)

Lemma 9.2. *Assume that Ω is polyhedric and that Γ_D can be split as the union of whole sides of $\partial\Omega$. Then there exists an operator $\overrightarrow{\Pi}_h$ of the form (9.23) such that if $\mathbf{v} \in \mathbf{W}_D(\Omega)$, then $\overrightarrow{\Pi}_h \mathbf{v}$ belongs to the discrete space $\mathbf{W}_h$ defined by (9.21). Furthermore, the family of spaces $(\mathbf{W}_h)_{h>0}$ is an internal approximation of $\mathbf{W}_D(\Omega)$.*

9.3.3 Mixed Approximations of Incompressible Flows

Mixed formulations of incompressible flow equations are built with pairs of finite element spaces $(\mathbf{W}_h, M_h)$ that approximate $\mathbf{W}_D(\Omega) \times L^2_0(\Omega)$, where the last space is the quotient space $L^2_0(\Omega) = L^2(\Omega)/\mathbb{R}$. As we mentioned in the introduction of Sect. 9.3, the solvability of the pressure for incompressible flows follows if the discrete velocity and pressure spaces satisfy the uniform discrete inf-sup condition (9.4).

Let us describe some pairs of spaces satisfying the discrete inf-sup conditions. The Taylor–Hood pairs of finite element spaces corresponds to the setting $\mathbf{W}_h = \mathbf{V}_{0h}^{(m+1)}$, $M_h = M_h^{(m)}$ with

$$\mathbf{V}_{0h}^{(l)} = V_h^{(l)}(\mathscr{T}_h)^d \cap \mathbf{H}^1_0(\Omega), \quad M_h^{(m)} = V_h^{(m)}(\mathscr{T}_h) \cap L^2_0(\Omega). \tag{9.24}$$

The family $((\mathbf{V}_{0h}^{(m+1)}, M_h^{(m)}))_{h>0}$ for $m \geq 1$ satisfies the uniform discrete inf-sup condition if the family of triangulations $(\mathscr{T}_h)_{h>0}$ is regular (cf. [9, 10]). However, the space $\mathbf{V}_{0h}^{(m+1)}$ does not approximate $\mathbf{W}_D(\Omega)$, and for this reason we replace it by the space $\mathbf{W}_h = \mathbf{V}_h^{(m+1)}$, given by

$$\mathbf{V}_h^{(l)} = \{\mathbf{w}_h \in V_h^{(l)}(\mathscr{T}_h)^d \mid \mathbf{w}_h = 0 \text{ on } \overline{\Gamma_D},\ \mathbf{w}_h \cdot \mathbf{n} = 0 \text{ on } \Sigma_i, i = 1, \cdots, k-1\}. \tag{9.25}$$

As $\mathbf{V}_{0h}^{(m+1)} \subset \mathbf{V}_h^{(m+1)}$, then trivially the family $((\mathbf{V}_h^{(m+1)}, M_h^{(m)}))_{h>0}$ for $m \geq 1$ also satisfies the uniform discrete inf-sup condition. However, this pair of elements is rather costly and elements with less degrees of freedom in velocity are preferable. This is the case of the mini-element, where the velocity space is enriched with bubble finite element functions. Let us define this space, for simplicity for $l = 1$. To each element $K \in \mathscr{T}_h$ we associate the bubble function b_K defined as some polynomial function such that

$$b_K = 0 \text{ on } \partial K, \text{ and } b_K = 1 \text{ at the barycenter of } K.$$

We then define the bubble space $\mathbb{B}_h = span(b_K \ \text{ for } \ K \in \mathcal{T}_h)$ and set

$$\begin{cases} \mathbf{V}_h^{(1)} = \{\mathbf{w}_h \in (V_h^{(1)}(\mathcal{T}_h) \oplus \mathbb{B}_h)^d \mid \mathbf{w}_h \cdot \mathbf{n} = 0 \ \text{ on } \ \Gamma_n, \ \mathbf{w}_h = 0 \ \text{ on } \ \Gamma_D\}, \\ M_h^{(1)} = V_h^{(1)}(\mathcal{T}_h) \cap L_0^2(\Omega). \end{cases}$$

Then the family of spaces $((\mathbf{V}_h^{(1)}, M_h^{(1)}))_{h>0}$ also satisfies the discrete inf-sup condition if the family of triangulations $(\mathcal{T}_h)_{h>0}$ is regular. For a higher degree of interpolation of pressures, a space of bubble finite elements with larger dimension is needed to stabilize the incompressibility restriction.

A general review of pairs of spaces satisfying the inf-sup condition for incompressible flows, and also in the more general context of approximation of PDEs by mixed methods, may be found in Brezzi and Fortin [10].

We need a special projection of the velocity field that weakly preserves the divergence. This is possible if the family of pairs of finite element spaces satisfy the inf-sup condition.

Definition 9.7. Let $\mathbf{v} \in \mathbf{W}_D(\Omega)$. The discrete Stokes projection of $\mathbf{v}$ on $\mathbf{W}_h$ is the velocity component $\mathbf{v}_h$ of the following problem: Find $(\mathbf{v}_h, p_h) \in \mathbf{W}_h \times M_h$ that verifies, for all $(\mathbf{w}_h, q_h) \in \mathbf{W}_h \times M_h$

$$\begin{cases} (\nabla \mathbf{v}_h, \nabla \mathbf{w}_h)_\Omega - (p_h, \nabla \cdot \mathbf{w}_h)_\Omega = (\nabla \mathbf{v}, \nabla \mathbf{w}_h)_\Omega, \\ (\nabla \cdot \mathbf{v}_h, q_h)_\Omega = (\nabla \cdot \mathbf{v}, q_h)_\Omega. \end{cases} \tag{9.26}$$

This problem fits into the class of *saddle-point problems* studied in Sect. A.7. By Theorem A.17, it admits a unique solution if the pair of spaces $(\mathbf{W}_h, M_h)$ satisfies the inf-sup condition. Moreover,

Lemma 9.3. *Assume that the family of spaces $((\mathbf{W}_h, M_h))_{h>0}$ satisfies the discrete inf-sup condition (9.4). Then the following error bound holds:*

$$\|\mathbf{v} - \mathbf{v}_h\|_{1,2,\Omega} \le C \inf_{\mathbf{w}_h \in \mathbf{W}_h} \|\mathbf{v} - \mathbf{w}_h\|_{1,2,\Omega} \tag{9.27}$$

for some constant $C > 0$ independent of h.

Proof. Let $\mathbf{z}_h \in \mathbf{W}_h$. Define $\mathbf{f} \in \mathbf{W}_D(\Omega)'$, $g \in L_0^2(\Omega)'$ by

$$\langle \mathbf{f}, \mathbf{w} \rangle_{\mathbf{W}_D(\Omega)} = (\nabla(\mathbf{v} - \mathbf{z}_h), \nabla \mathbf{w})_\Omega, \quad \langle g, q \rangle_{L_0^2(\Omega)'} = (\nabla \cdot (\mathbf{v} - \mathbf{z}_h), q)_\Omega,$$

$\forall \mathbf{w} \in \mathbf{W}_D(\Omega), \ q \in M$. Then the pair $(\mathbf{e}_h = \mathbf{v}_h - \mathbf{z}_h, p_h) \in \mathbf{W}_h \times M_h$ verifies, for all $(\mathbf{w}_h, q_h) \in \mathbf{W}_h \times M_h$,

$$\begin{cases} (\nabla \mathbf{e}_h, \nabla \mathbf{w}_h)_\Omega - (p_h, \nabla \cdot \mathbf{w}_h)_\Omega = \langle \mathbf{f}, \mathbf{w}_h \rangle_{\mathbf{W}_D(\Omega)}, \\ (\nabla \cdot \mathbf{e}_h, q_h)_\Omega = \langle g, q_h \rangle_{L_0^2(\Omega)'}. \end{cases}$$

By Theorem A.17 and estimate (A.65),

$$\|\mathbf{e}_h\|_{1,2,\Omega} + \|p_h\|_{0,2,\Omega} \le C\left(\|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'} + \|g\|_{L^2_0(\Omega)'}\right) \le C\,\|\mathbf{v}-\mathbf{z}_h\|_{1,2,\Omega}.$$

Consequently, $\|\mathbf{v}-\mathbf{v}_h\|_{1,2,\Omega} \le C\,\|\mathbf{v}-\mathbf{z}_h\|_{1,2,\Omega}$ for all $\mathbf{z}_h \in \mathbf{W}_h$. The conclusion follows. □

Usually, the constant C depends on d, Ω, and also on the aspect ratio of the family of triangulations, as the constant α appearing in the discrete inf-sup condition does.

9.4 Interpretation of Variational Problem

We are now in a position to specify in which sense the solutions of the variational problem (9.3) satisfy (9.1).

Lemma 9.4. *Let $(\mathbf{v}, p) \in \mathbf{W}_D(\Omega) \times L^2_0(\Omega)$ be a solution of the variational problem (9.3). Then, the first and second equations in the Navier–Stokes equations (9.1) respectively hold in $\mathbf{H}^{-1}(\Omega)$ and $L^2(\Omega)$, and the BC hold in the following senses:*

$$\gamma_0\,\mathbf{v} = 0 \ \ in \ \ \mathbf{H}^{1/2}(\Gamma_D), \qquad \gamma_n\,\mathbf{v} = 0 \ \ in \ \ L^4(\Gamma).$$

In addition, if $\mathbf{v} \in \mathbf{H}^2(\Omega)$ and $p \in H^1(\Omega)$, then the condition

$$-\left[2\,\nu\,D\mathbf{v}\cdot\mathbf{n}\right]_{\boldsymbol{\tau}} = g(\mathbf{v})_{\boldsymbol{\tau}}$$

holds in $L^2(\Gamma_n)^{d-1}$.

Proof. Let $\mathbf{w} \in \mathbf{W}_h$. Then, as $\mathbf{w}$ is continuous on Ω and C^∞ on any element K of the grid $\mathscr{T}_h$,

$$\begin{aligned}\int_\Omega \nabla\cdot\mathbf{w}(\mathbf{x})\,d\mathbf{x} &= \sum_{K\in\mathscr{T}_h}\int_K \nabla\cdot\mathbf{w}(\mathbf{x})\,d\mathbf{x} = \sum_{K\in\mathscr{T}_h}\int_{\partial K} \mathbf{w}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x})\,d(\partial K)(\mathbf{x})\\ &= \sum_{K\in\mathscr{T}_h}\int_{\partial K\cap\Gamma} \mathbf{w}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x})\,d(\partial K)(\mathbf{x}) = \int_\Gamma \mathbf{w}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x})\,d\Gamma(\mathbf{x}) = \mathbf{0},\end{aligned} \tag{9.28}$$

and then $\displaystyle\int_\Omega \nabla\cdot\mathbf{w}(\mathbf{x})\,d\mathbf{x} = 0$ for $\mathbf{w} \in \mathbf{W}_D(\Omega)$ by density. Thus $\nabla\cdot\mathbf{v} \in \mathbf{L}^2_0(\Omega)$, and the second equation in (9.3) implies $\nabla\cdot\mathbf{v} = 0$ in $L^2(\Omega)$. Also, integration by parts yields

$$b(\mathbf{v};\mathbf{v},\mathbf{w}) = (\mathbf{v}\cdot\nabla\mathbf{v},\mathbf{w})_\Omega = (\nabla\cdot(\mathbf{v}\otimes\mathbf{v}),\mathbf{w})_\Omega \ \text{ for all } \ \mathbf{v},\,\mathbf{w} \in \mathbf{W}_h.$$

By density this identity also holds if $\mathbf{v}, \mathbf{w} \in \mathbf{W}_D(\Omega)$, similar to the proof of Lemma 6.6. Let $\mathbf{w} \in \mathscr{D}(\Omega)^d$. Observe that

$$\langle \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - \nabla \cdot (2\,\nu\, D\mathbf{v}) + \nabla p, \mathbf{w}\rangle_{\mathbf{H}^1(\Omega)} = b(\mathbf{v}; \mathbf{v}, \mathbf{w}) + a(\mathbf{v}, \mathbf{w}) - (p, \nabla \cdot \mathbf{w})_\Omega.$$

Then, by (9.3), the identity

$$\nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - \nabla \cdot (2\,\nu\, D\mathbf{v}) + \nabla p = \mathbf{f} \tag{9.29}$$

holds in $\mathbf{H}^{-1}(\Omega)$. Also, as $\mathbf{v} \in \mathbf{W}_D(\Omega)$, by (A.10), $\gamma_0 \mathbf{v} = 0$ in $\mathbf{H}^{1/2}(\Gamma_D)$, and $\gamma_n \mathbf{v} = (\gamma_0 \mathbf{v}) \cdot \mathbf{n} = 0$ in $L^4(\Gamma)$ by Lemma A.1.

Next, assume $\mathbf{v} \in \mathbf{H}^2(\Omega)$, $p \in H^1(\Omega)$, $\mathbf{w} \in \mathbf{W}_h$. Then an integration by parts similar to (9.28) yields

$$(D\mathbf{v} \cdot \mathbf{n}, \mathbf{w})_{\Gamma_n} = (\nabla \cdot D\mathbf{v}, \mathbf{w})_\Omega + (D\mathbf{v}, D\mathbf{w})_\Omega \tag{9.30}$$

As $(D\mathbf{v} \cdot \mathbf{n}, \mathbf{w})_{\Gamma_n} = ([D\mathbf{v} \cdot \mathbf{n}]_{\boldsymbol{\tau}}, \mathbf{w}_{\boldsymbol{\tau}})_{\Gamma_n}$, from (9.3), we obtain

$$(\mathbf{v} \cdot \nabla \mathbf{v} - 2\,\nu\, \nabla \cdot D\mathbf{v} + \nabla p, \mathbf{w})_\Omega + ([2\,\nu D\mathbf{v} \cdot \mathbf{n}]_{\boldsymbol{\tau}}, \mathbf{w}_{\boldsymbol{\tau}})_{\Gamma_n} + \langle G(\mathbf{v}), \mathbf{w}\rangle = \langle \mathbf{f}, \mathbf{w}\rangle.$$

As now the identity (9.29) holds in $\mathbf{L}^2(\Omega)$, from the preceding identity, we obtain

$$([2\,\nu D\mathbf{v} \cdot \mathbf{n} + g(\mathbf{v})]_{\boldsymbol{\tau}}, \mathbf{w}_{\boldsymbol{\tau}})_{\Gamma_n} = 0 \;\text{ for all }\; \mathbf{w} \in \mathbf{W}_h. \tag{9.31}$$

As $(\mathbf{W}_h)_{h>0}$ is an internal approximation of $\mathbf{W}_h$, then for any function $\mathbf{w} \in \mathbf{W}_D(\Omega)$ there exists a sequence $(\mathbf{w}_h)_{h>0}$ such that $\mathbf{w}_h \in \mathbf{W}_h$ and

$$\lim_{h\to 0} \|\gamma_0(\mathbf{w}) - \gamma_0(\mathbf{w}_h)\|_{1/2,\Gamma} = 0.$$

Then (9.31) holds for any $\mathbf{w} \in \mathbf{W}_D(\Omega)$. As $D\mathbf{v}_{|\Gamma_n} \in \mathbf{H}^{1/2}(\Gamma_n)$ and $g(\mathbf{v}) \in \mathbf{L}^2(\Gamma_n)$ (see Lemma 6.5), we deduce that $-[2\,\nu D\mathbf{v} \cdot \mathbf{n}]_{\boldsymbol{\tau}} = g(\mathbf{v})_{\boldsymbol{\tau}}$ holds in $L^2(\Gamma_n)^{d-1}$. $\square$

9.5 Discretization

To approximate problem (9.1) we consider mixed formulations in an abstract framework: We consider a family of pairs of finite-dimensional spaces $(\mathbf{W}_h, M_h) \subset \mathbf{W}_D(\Omega) \times L^2_0(\Omega)$ that satisfies the following hypotheses:

Hypothesis 9.i. The family of pairs of spaces $((\mathbf{W}_h, M_h))_{h>0}$ is an internal approximation of $\mathbf{W}_D(\Omega) \times L^2_0(\Omega)$, in the sense of Definition 9.2: For all $(\mathbf{w}, p) \in \mathbf{W}_D(\Omega) \times L^2_0(\Omega)$ there exists a sequence $((\mathbf{v}_h, p_h))_{h>0}$ such that $(\mathbf{v}_h, p_h) \in \mathbf{W}_h \times M_h$ and

$$\lim_{h\to 0}(\|\mathbf{v} - \mathbf{v}_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega}) = 0.$$

Hypothesis 9.ii. The family of pairs of spaces $((\mathbf{W}_{0h}, M_h))_{h>0}$, where $\mathbf{W}_{0h} = \mathbf{W}_h \cap H^1_0(\Omega)$, satisfies the uniform discrete inf-sup condition (9.4): There exists a constant $\alpha > 0$ independent of h such that

$$\alpha \|q_h\|_{0,2,\Omega} \le \sup_{\mathbf{w}_h \in \mathbf{W}_h} \frac{(\nabla \cdot \mathbf{w}_h, q_h)_\Omega}{\|\mathbf{w}_h\|_{1,2,\Omega}}, \quad \forall q_h \in M_h, \quad \forall h > 0.$$

The first hypothesis guarantees that the finite element spaces approximate the velocity–pressure space $\mathbf{W}_D(\Omega) \times L^2_0(\Omega)$ in convenient norms to retrieve the variational problem (9.1) when the grid size tends to zero. The second one ensures the stability of discretization of the pressure. The pairs of Lagrange finite element spaces defined in Sect. 9.3.3 satisfy those two hypotheses when the domain Ω is polyhedric. There exist other possible spaces that satisfy them, by adaptation of more general finite element or spectral spaces (cf. [3, 4, 10, 20]) to the strong setting of the slip boundary condition that we consider here. For general Lipschitz domains, the weak setting of slip BC described in Sect. 9.8.3 is well suited.

We approximate the weak formulation (9.3) of the boundary value problem (9.1) for Navier–Stokes equations by the SM, stated as

$(\mathscr{V}\mathscr{P})_h$ Find $(\mathbf{v}_h, p_h) \in \mathbf{W}_h \times M_h$ such that for all $(\mathbf{w}_h, q_h) \in \mathbf{W}_h \times M_h$,

$$\begin{cases} b(\mathbf{v}_h; \mathbf{v}_h, \mathbf{w}_h) + a(\mathbf{v}_h, \mathbf{w}_h) + c(\mathbf{v}_h; \mathbf{w}_h) - (p_h, \nabla \cdot \mathbf{w}_h)_\Omega \\ \qquad\qquad + \langle G(\mathbf{v}_h), \mathbf{w}_h \rangle = \langle \mathbf{f}, \mathbf{w}_h \rangle, \\ \qquad\qquad (\nabla \cdot \mathbf{v}_h, q_h)_\Omega = 0; \end{cases} \tag{9.32}$$

where the form c is defined by

$$c(\mathbf{v}; \mathbf{w}) = (\nu_t(\mathbf{v})\, D\mathbf{v}, D\mathbf{w})_\Omega, \tag{9.33}$$

where in its turn the eddy viscosity is given by (5.138). In practice, for irregular grids, the mixing length is identified with a constant times the local grid size,

$$\nu_t(\mathbf{v})(\mathbf{x}) = C_S^2\, h_K^2\, |D(\mathbf{v}_{|_K})(\mathbf{x})| \text{ if } \mathbf{x} \in K. \tag{9.34}$$

9.6 Stability and Convergence Analysis

In this section we analyze the discretization (9.32) of the steady SM. We prove that the problem (9.32) admits a solution $(\mathbf{v}_h, p_h)$ which is uniformly bounded in $\mathbf{H}^1(\Omega) \times L^2(\Omega)$. This result will be the key to next prove the convergence to a solution of the Navier–Stokes equations (9.3) by a compactness argument. The next section shall deal with the derivation of optimal error estimates for diffusion-dominated flows.

Our analysis needs some properties of the eddy viscosity ν_t and the form c that we state next.

Lemma 9.5. *There exists a constant $C > 0$ depending only on the aspect ratio of the family of triangulations $(\mathscr{T}_h)_{h>0}$ such that*

$$\|\nu_t(\mathbf{v}_h)\|_{0,\infty,\Omega} \le C\, h^{2-d/2}\, \|D(\mathbf{v}_h)\|_{0,2,\Omega}, \ \textit{for all}\ \mathbf{v}_h \in \mathbf{W}_h. \tag{9.35}$$

Proof. Consider $\mathbf{v}_h \in \mathbf{W}_h$. As $\nabla \mathbf{v}_h$ is piecewise continuous, there exists $K \in \mathscr{T}_h$ such that

$$\|\nu_t(\mathbf{v}_h)\|_{0,\infty,\Omega} = \|\nu_t(\mathbf{v}_h)\|_{0,\infty,K} \le C_S^2\, h_K^2\, \|D(\mathbf{v}_h)\|_{0,\infty,K}.$$

By the inverse estimate (9.20), $\|\nabla \mathbf{v}_h\|_{0,\infty,K} \le C\, h_K^{-d/2}\, \|\nabla \mathbf{v}_h\|_{0,2,K}$ for some constant $C > 0$ depending only on the aspect ratio of the family of triangulations. Then,

$$\|\nu_t(\mathbf{v}_h)\|_{0,\infty,\Omega} \le C C_S^2\, h_K^{2-d/2}\, \|D(\mathbf{v}_h)\|_{0,2,K} \le C C_S^2\, h^{2-d/2}\, \|D(\mathbf{v}_h)\|_{0,2,\Omega}.$$

□

From this result we deduce:

Lemma 9.6. *The form c defined by (9.33) satisfies the following properties:*

(i) c is nonnegative, in the sense that

$$c(\mathbf{v}; \mathbf{v}) \ge 0, \ \textit{for all}\ \mathbf{v} \in H^1(\Omega).$$

(ii) Assume that the family of triangulations $(\mathscr{T}_h)_{h>0}$ is regular. Then, for any $\mathbf{v}_h,\ \mathbf{w}_h \in \mathbf{W}_h$,

$$|c(\mathbf{v}_h; \mathbf{w}_h)| \le C\, h^{2-d/2}\, \|D(\mathbf{v}_h)\|_{0,2,\Omega}^2 \|D(\mathbf{w}_h)\|_{0,2,\Omega}, \tag{9.36}$$

for some constant $C > 0$ depending only on d, Ω, and the aspect ratio of the family of triangulations.

(iii) Assume that the family of triangulations $(\mathscr{T}_h)_{h>0}$ is regular. Let $(\mathbf{v}_h)_{h>0}$ and $(\mathbf{w}_h)_{h>0}$ be two sequences such that $\mathbf{v}_h$, $\mathbf{w}_h \in \mathbf{W}_h$. Then, if both sequences are bounded in $\mathbf{H}^1(\Omega)^d$,

$$\lim_{h\to 0} c(\mathbf{v}_h;\mathbf{w}_h) = 0. \tag{9.37}$$

Proof.

(i) Let $\mathbf{v} \in \mathbf{H}^1(\Omega)$. Then,

$$c(\mathbf{v};\mathbf{v}) = \int_\Omega \nu_t(\mathbf{v})\,|D\mathbf{v}|^2\,d\mathbf{x} \geq 0.$$

(ii) By estimate (9.35),

$$\begin{aligned}|c(\mathbf{v}_h;\mathbf{w}_h)| &\leq \|\nu_t(\mathbf{v}_h)\|_{0,\infty,\Omega}\,\|D(\mathbf{v}_h)\|_{0,2,\Omega}\,\|D(\mathbf{w}_h)\|_{0,2,\Omega}\\ &\leq C\,h^{2-d/2}\,\|D(\mathbf{v}_h)\|^2_{0,2,\Omega}\,\|D(\mathbf{w}_h)\|_{0,2,\Omega}.\end{aligned}$$

(iii) Statement (9.37) directly follows from (9.36).

□

Problem (9.32) is a set of nonlinear equations in finite dimension. These nonlinearities appear as a result of several effects: the convection operator, the eddy viscosity, and the wall-law BC. To deal with these nonlinearities, we prove the existence of solution via Brouwer's fixed-point theorem (Theorem A.5).

Theorem 9.4. *Let $(\mathscr{T}_h)_{h>0}$ be a regular family of triangulations of the domain Ω. Let $((\mathbf{W}_h, M_h))_{h>0}$ be a family of pairs of finite element spaces satisfying Hypotheses 9.i and 9.ii. Then for any $\mathbf{f} \in \mathbf{W}_D(\Omega)'$ the discrete SM (9.32) admits at least a solution that satisfies the estimates:*

$$\|D(\mathbf{v}_h)\|_{0,2,\Omega} \leq \frac{1}{\nu}\,\|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'}, \tag{9.38}$$

$$\|p_h\|_{0,2,\Omega} \leq \frac{C}{\nu^2}\,(1+h^{2-d/2})\,\|\mathbf{f}\|^2_{\mathbf{W}_D(\Omega)'} + C\,\frac{1}{\nu}(1+\nu)\,\|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'}, \tag{9.39}$$

where $C > 0$ is a constant depending only on d, Ω, and the aspect ratio of the family of triangulations.

Proof. We prove the existence of solution in two steps.

STEP 1: *Existence of the velocity*. Let us define the mapping $\Phi_h : \mathbf{W}_h \to \mathbf{W}_h'$ as follows: Given $\mathbf{z}_h \in \mathbf{W}_h$,

$$\langle \Phi_h(\mathbf{z}_h), \mathbf{w}_h\rangle = b(\mathbf{z}_h;\mathbf{z}_h,\mathbf{w}_h) + a(\mathbf{z}_h,\mathbf{w}_h) + c(\mathbf{z}_h;\mathbf{w}_h) + \langle G(\mathbf{z}_h),\mathbf{w}_h\rangle - \langle \mathbf{f},\mathbf{w}_h\rangle,$$

for any $\mathbf{w}_h \in \mathbf{W}_h$. This equation has a unique solution as its r.h.s. defines a linear functional on $\mathbf{W}_h$. This functional is necessarily continuous as $\mathbf{W}_h$ is a space of finite dimension.

Let us next prove that Φ_h is continuous. Let us consider a base $(\varphi_i)_{i=1}^N$ of $\mathbf{W}_h$, and the associated dual base of $\mathbf{W}_h'$, $(\varphi_i^*)_{i=1}^N$, characterized by $\langle \varphi_i^*, \varphi_j \rangle = \delta_{ij}$, for all $i, j = 1, \cdots, N$. Then, $\Phi(\mathbf{z}_h) = \sum_{i=1}^{N} \lambda_i(\mathbf{z}_h)\, \varphi_i^*$ with $\lambda_i(\mathbf{z}_h) = \langle \Phi(\mathbf{z}_h), \varphi_i \rangle$.

Any $\mathbf{z}_h \in \mathbf{W}_h$ may be expressed as $\mathbf{z}_h = \sum_{k=1}^{N} \mathbf{z}_k\, \varphi_k$ for some $\mathbf{z}_1, \cdots, \mathbf{z}_N \in \mathbb{R}^d$. Then,

$$
\begin{aligned}
\lambda_i(\mathbf{z}_h) = & \sum_{k,l=1}^{N} b(\varphi_k; \varphi_l, \varphi_i)\, \mathbf{z}_k\, \mathbf{z}_l + \sum_{k=1}^{N} a(\varphi_k, \varphi_i)\, \mathbf{z}_k \\
& + \sum_{k=1}^{N} \int_\Omega \sigma(\mathbf{z}_1, \cdots, \mathbf{z}_N, \mathbf{x}) D(\varphi_k)(\mathbf{x}) : \nabla D(\varphi_i)(\mathbf{x})\, d\mathbf{x} \\
& + \int_{\Gamma_n} \tau(\mathbf{z}_1, \cdots, \mathbf{z}_N, \mathbf{x}) \varphi_i(\mathbf{x})\, d\Gamma_n(\mathbf{x}) - \langle f, \varphi_i \rangle,
\end{aligned}
$$

where the functions σ and τ are defined by

$$
\sigma(\mathbf{z}_1, \cdots, \mathbf{z}_N, \mathbf{x}) = \nu_t(\mathbf{z}_h)(\mathbf{x}) = C_S\, h_K\, |\sum_{k=1}^{N} \mathbf{z}_k\, D(\varphi_k)(\mathbf{x})| \ \text{ if } \ \mathbf{x} \in K,
$$

$$
\tau(\mathbf{z}_1, \cdots, \mathbf{z}_N, \mathbf{x}) = g\left(\sum_{k=1}^{N} \mathbf{z}_k\, \varphi_k(\mathbf{x}) \right) \ \text{ for } \ \mathbf{x} \in \Gamma_n.
$$

As σ and τ are continuous functions, then Φ is a continuous mapping from $\mathbf{W}_h$ on $\mathbf{W}_h'$. Consider the subspace Z_h of $\mathbf{W}_h$ defined by

$$
Z_h = \{\mathbf{w}_h \in \mathbf{W}_h \ \text{ such that } \ (\nabla \cdot \mathbf{w}_h, q_h) = 0, \ \text{ for all } \ q_h \in M_h\}.
$$

Z_h is a nonempty closed subspace of $\mathbf{H}^1(\Omega)$. Then it is a Hilbert space endowed with the $\mathbf{H}^1(\Omega)$ norm. Let $\mathbf{z}_h \in Z_h$. As $b(\mathbf{z}_h; \mathbf{z}_h, \mathbf{z}_h) = 0$ and c and G are nonnegative,

$$
\begin{aligned}
\langle \Phi_h(\mathbf{z}_h), \mathbf{z}_h \rangle & \geq a(\mathbf{z}_h, \mathbf{z}_h) - \langle \mathbf{f}, \mathbf{z}_h \rangle \geq \nu \, \|D(\mathbf{z}_h)\|_{0,2,\Omega}^2 - \langle \mathbf{f}, \mathbf{z}_h \rangle \\
& \geq \nu \, \|D(\mathbf{z}_h)\|_{0,2,\Omega}^2 - \frac{\nu}{2} \, \|D(\mathbf{z}_h)\|_{0,2,\Omega}^2 - \frac{2}{\nu} \, \|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'}^2
\end{aligned}
$$

$$\geq \frac{\nu}{2} \|D(\mathbf{z}_h)\|_{0,2,\Omega}^2 - \frac{2}{\nu} \|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'}^2,$$

by Young's inequality. Let $\|D(\mathbf{z}_h)\|_{0,2,\Omega} = \frac{2}{\nu} \|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'}$. Then, $\langle \Phi_h(\mathbf{z}_h), \mathbf{z}_h \rangle_{H^1(\Omega)} \geq 0$. Consequently, by Theorem A.5, the equation

$$b(\mathbf{v}_h; \mathbf{v}_h, \mathbf{w}_h) + a(\mathbf{v}_h, \mathbf{w}_h) + c(\mathbf{v}_h; \mathbf{w}_h) + \langle G(\mathbf{v}_h), \mathbf{w}_h \rangle = \langle \mathbf{f}, \mathbf{w}_h \rangle \quad \forall \mathbf{w}_h \in Z_h \quad (9.40)$$

admits a solution $\mathbf{v}_h \in Z_h$ such that $\|D(\mathbf{v}_h)\|_{0,2,\Omega} \leq \frac{2}{\nu} \|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'}$. To obtain the estimate (9.38), set $\mathbf{w}_h = \mathbf{v}_h$ in (9.32). This yields

$$\nu \|D(\mathbf{v}_h)\|_{0,2,\Omega}^2 + c(\mathbf{v}_h; \mathbf{v}_h) + \langle G(\mathbf{v}_h), \mathbf{v}_h \rangle = \langle \mathbf{f}, \mathbf{v}_h \rangle$$

Then, $\nu \|D(\mathbf{v}_h)\|_{0,2,\Omega}^2 \leq \|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'} \|D(\mathbf{v}_h)\|_{0,2,\Omega}$, and (9.38) follows.

STEP 2: *Existence of the pressure.* As $\mathbf{v}_h$ is a solution of (9.40), then $\Phi_h(\mathbf{v}_h) \in Z_h^{\perp}$. By Lemma A.20, there exists a discrete pressure $p_h \in M_h$ such that $(\mathbf{v}_h, p_h)$ is a solution of (9.32). The estimate for the norm of the pressure is obtained via the discrete inf-sup condition (estimate (A.60) in Lemma A.20),

$$\|p_h\|_{0,2,\Omega} \leq \alpha^{-1} \|\Phi_h\|_{\mathbf{W}_h'},$$

for some constant $\alpha > 0$. By estimates (6.18), (6.21), (6.41), and (9.36),

$$\langle \Phi_h(\mathbf{v}_h), \mathbf{w}_h \rangle \leq C \left[(1 + \nu + (1 + h^{2-d/2}) \|D(\mathbf{v}_h)\|_{0,2,\Omega}) \|D(\mathbf{v}_h)\|_{0,2,\Omega} \right] \|\mathbf{w}_h\|_{1,2,\Omega}$$
$$+ \|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'} \|\mathbf{w}_h\|_{1,2,\Omega}.$$

Then, the pressure estimate (9.39) follows from the velocity estimate (9.38).

□

We next prove the convergence of the solution provided by method (9.32) to a weak solution of the Navier–Stokes boundary value problem model (9.1).

Theorem 9.5. *Under the hypotheses of Theorem 9.4, the sequence of discrete variational problems $(\mathcal{VP})_{h>0}$ converges to the variational problem $\mathcal{VP}$. More specifically, the sequence $((\mathbf{v}_h, p_h))_{h>0}$ provided by the SM (9.32) contains a subsequence strongly convergent in $\mathbf{H}^1(\Omega)^d \times L^2(\Omega)$ to a weak solution $(\mathbf{v}, p) \in \mathbf{W}_D(\Omega) \times L^2(\Omega)$ of the steady Navier–Stokes equation (9.1). If this solution is unique, then the whole sequence converges to it.*

Proof. By estimates (9.38) and (9.39), the sequence $((\mathbf{v}_h, p_h))_{h>0}$ is bounded in the space $\mathbf{W}_D(\Omega) \times L_0^2(\Omega)$. As this is a Hilbert space, then this sequence contains a subsequence that we denote in the same way weakly convergent in $\mathbf{W}_D(\Omega) \times L_0^2(\Omega)$ to some pair $(\mathbf{v}, p)$. As the injection of $H^1(\Omega)$ in $L^q(\Omega)$ is compact for $1 \leq q <$

$q^* = \frac{2d}{d-2}$, we may assume that the subsequence is strongly convergent in $\mathbf{L}^q(\Omega)$ for $1 \le q < q^*$, and so in particular in $\mathbf{L}^4(\Omega)$

Also, the operator G is compact from $\mathbf{H}^1(\Omega)$ to $\mathbf{H}^1(\Omega)'$. Then we may assume that the sequence $(G(\mathbf{v}_h))_{h>0}$ is strongly convergent in $\mathbf{H}^1(\Omega)'$.

Let $(\mathbf{w}, q) \in \mathbf{W}_D(\Omega) \times L^2_0(\Omega)$. By Hypothesis 9.i, there exists a sequence $((\mathbf{w}_h, q_h))_{h>0}$ such that $(\mathbf{w}_h, q_h) \in \mathbf{W}_h \times M_h$ which is strongly convergent in $\mathbf{H}^1(\Omega) \times L^2(\Omega)$ to $(\mathbf{w}, q)$.

As a is bilinear and continuous, $\lim\limits_{h\to 0} a(\mathbf{v}_h, \mathbf{w}_h) = a(\mathbf{v}, \mathbf{w})$. Next, as the sequences $(\mathbf{v}_h)_{h>0}$ and $(\mathbf{w}_h)_{h>0}$ are bounded in $\mathbf{H}^1(\Omega)$, by Lemma 9.6, $\lim\limits_{h\to 0} c(\mathbf{v}_h; \mathbf{w}_h) = 0$. Also,

$$
\begin{aligned}
|(\mathbf{v}_h \cdot \nabla \mathbf{v}_h, \mathbf{w}_h)_\Omega - (\mathbf{v} \cdot \nabla \mathbf{v}, \mathbf{w})_\Omega| &\le |((\mathbf{v}_h - \mathbf{v}) \cdot \nabla \mathbf{v}_h, \mathbf{w}_h)_\Omega| + |(\mathbf{v} \cdot \nabla(\mathbf{v}_h - \mathbf{v}), \mathbf{w})_\Omega| \\
&\quad + |(\mathbf{v} \cdot \nabla \mathbf{v}_h, \mathbf{w}_h - \mathbf{w})_\Omega| \le \|\mathbf{v}_h - \mathbf{v}\|_{0,4,\Omega} \, \|\nabla \mathbf{v}_h\|_{0,2,\Omega} \, \|\mathbf{w}_h\|_{0,4,\Omega} \\
+ \sum_{i,j=1}^{d} |(\partial_j (v_{hi} - v_i), v_j w_i)_\Omega| &+ \|\mathbf{v}\|_{0,4,\Omega} \, \|\nabla \mathbf{v}_h\|_{0,2,\Omega} \, \|\mathbf{w}_h - \mathbf{w}\|_{0,4,\Omega},
\end{aligned}
$$

where we denote $\mathbf{v}_h = (v_{h1}, \cdots, v_{hd})$. All terms in the r.h.s. of the last inequality vanish in the limit because $(\mathbf{v}_h)_{h>0}$ is strongly convergent in $\mathbf{L}^4(\Omega)$, $(\partial_i v_{hi},)_{h>0}$ is weakly convergent in $L^2(\Omega)$, and $(\mathbf{w}_h)_{h>0}$ is strongly convergent in $\mathbf{H}^1(\Omega)$. Then, $\lim\limits_{h\to 0} (\mathbf{v}_h \cdot \nabla \mathbf{v}_h, \mathbf{w}_h)_\Omega = (\mathbf{v} \cdot \nabla \mathbf{v}, \mathbf{w})_\Omega$. Similarly, $\lim\limits_{h\to 0} (\mathbf{v}_h \cdot \nabla \mathbf{w}_h, \mathbf{v}_h)_\Omega = (\mathbf{v} \cdot \nabla \mathbf{w}, \mathbf{v})_\Omega$, and then

$$
\lim_{h\to 0} b(\mathbf{v}_h \cdot \nabla \mathbf{v}_h, \mathbf{w}_h) = b(\mathbf{v} \cdot \nabla \mathbf{v}, \mathbf{w}).
$$

Also, as $(\nabla \cdot \mathbf{v}_h)_{h>0}$ is weakly convergent in $L^2(\Omega)$ to $\nabla \cdot \mathbf{v}_h$ and $(q_h)_{h>0}$ is strongly convergent in $L^2(\Omega)$ to q,

$$
\lim_{h\to 0} (\nabla \cdot \mathbf{v}_h, q_h)_\Omega = (\nabla \cdot \mathbf{v}, q)_\Omega.
$$

Consequently, the pair $(\mathbf{v}, q)$ is a weak solution of Navier–Stokes equations (9.3). To prove the strong convergence of the velocities, set $\mathbf{w}_h = \mathbf{v}_h$ in (9.32). Then by (6.23),

$$
2\nu \, \|D(\mathbf{v}_h)\|^2_{0,2,\Omega} = \langle \mathbf{f}, \mathbf{v}_h \rangle - c(\mathbf{v}_h; \mathbf{v}_h) - \langle G(\mathbf{v}_h), \mathbf{v}_h \rangle.
$$

By Lemma 9.6 (iii), $\lim\limits_{h\to 0} c(\mathbf{v}_h; \mathbf{v}_h) = 0$. Also, as $(G(\mathbf{v}_h))_{h>0}$ is strongly convergent in $\mathbf{H}^1(\Omega)'$ (up to a subsequence), and then $\lim\limits_{h\to 0} \langle G(\mathbf{v}_h), \mathbf{v}_h \rangle = \langle G(\mathbf{v}), \mathbf{v} \rangle$. Consequently,

$$
\lim_{h\to 0} 2\nu \, \|D(\mathbf{v}_h)\|^2_{0,2,\Omega} = \langle \mathbf{f}, \mathbf{v} \rangle - \langle G(\mathbf{v}), \mathbf{v} \rangle = 2\nu \, \|D\mathbf{v}\|^2_{0,2,\Omega},
$$

where the last equality occurs because $(\mathbf{v}, q)$ is a weak solution of Navier–Stokes equations (9.3). As $\mathbf{W}_D(\Omega)$ is a Hilbert space and $(\mathbf{v}_h)_{h>0}$ is weakly convergent to $\mathbf{v}$, this proves the strong convergence.

To prove the strong convergence of the pressures, we use the discrete inf-sup condition to estimate $\| p_h - p\|_{0,2,\Omega}$. There exists a sequence $(P_h)_{h>0}$ such that $P_h \in M_h$ for all $h > 0$ which is strongly convergent in $L^2_0(\Omega)$ to p. Let $\mathbf{w}_h \in \mathbf{W}_{0h}$. As $\mathbf{W}_{0h} \subset \mathbf{H}^1_0(\Omega) \subset \mathbf{W}_D(\Omega)$, then (9.32) holds with $\mathbf{w} = \mathbf{w}_h$. Let us write $(p, \nabla\cdot\mathbf{w}) = (P_h, \nabla \cdot \mathbf{w}) + (p - P_h, \nabla \cdot \mathbf{w})$ in (9.3) and subtract (9.3) with $\mathbf{w} = \mathbf{w}_h \in \mathbf{W}_{0h}$ from (9.32). This yields

$$
\begin{aligned}
(p_h - P_h, \nabla \cdot \mathbf{w}_h) = b(\mathbf{v}_h; \mathbf{v}_h, \mathbf{w}_h) - b(\mathbf{v}; \mathbf{v}, \mathbf{w}_h) + a(\mathbf{v}_h - \mathbf{v}, \mathbf{w}_h) + c(\mathbf{v}_h; \mathbf{w}_h) \\
+ \langle G(\mathbf{v}_h) - G(\mathbf{v}), \mathbf{w}_h\rangle + (p - P_h, \nabla \cdot \mathbf{w}_h).
\end{aligned}
$$

As

$$
\begin{aligned}
b(\mathbf{v}_h; \mathbf{v}_h, \mathbf{w}_h) - b(\mathbf{v}; \mathbf{v}, \mathbf{w}_h) &= b(\mathbf{v}_h; \mathbf{v}_h - \mathbf{v}, \mathbf{w}_h) + b(\mathbf{v}_h - \mathbf{v}; \mathbf{v}, \mathbf{w}_h) \\
&\leq C\, \|D(\mathbf{v}_h - \mathbf{v})\|_{0,2,\Omega}\, (\|D(\mathbf{v}_h)\|_{0,2,\Omega} + \|D\mathbf{v}\|_{0,2,\Omega}),
\end{aligned}
$$

using (9.36) and the continuity of a we deduce

$$
\begin{aligned}
(p_h - P_h, \nabla \cdot \mathbf{w}_h) \leq C\, [(\|D(\mathbf{v}_h)\|_{0,2,\Omega} + \|D\mathbf{v}\|_{0,2,\Omega} + \nu)\, \|D(\mathbf{v}_h - \mathbf{v})\|_{0,2,\Omega} \\
+ h^{2-d/2}\, \|D(\mathbf{v}_h)\|^2_{0,2,\Omega} + \|G(\mathbf{v}_h) - G(\mathbf{v})\|_{\mathbf{W}_D(\Omega)'} + \|p - P_h\|_{0,2,\Omega}\,]\|\mathbf{w}_h\|_{1,2,\Omega}.
\end{aligned}
$$

As G is continuous and $\mathbf{v}_h \to \mathbf{v}$ strongly in $\mathbf{H}^1(\Omega)$, then $\lim\limits_{h\to 0} \|G(\mathbf{v}_h) - G(\mathbf{v})\|_{\mathbf{W}_D(\Omega)'} = 0$. Consequently, by Hypothesis 9.ii, $\lim\limits_{h\to 0} \|p_h - P_h\|_{0,2,\Omega} = 0$. Then p_h strongly converges to p in $L^2(\Omega)$.

It remains to prove that if the Navier–Stokes equations (9.1) admit a unique solution $(\mathbf{v}, p_h)$, then the whole sequence $((\mathbf{v}_h, p))_{h>0}$ converges to it. This is a standard result that holds when compactness arguments are used, which is proved by reductio ad absurdum: Assume that the whole sequence does not converge to $(\mathbf{v}, p_h)$. Then there exists a subsequence of $((\mathbf{v}_h, p_h))_{h>0}$ that lies outside some ball of $\mathbf{W}_D(\Omega) \times L^2_0(\Omega)$ with center $(\mathbf{v}, p)$. Then the preceding compactness argument proves that a subsequence of this subsequence would converge to the unique solution $(\mathbf{v}, p)$, what is absurd. □

9.6.1 Asymptotic Energy Balance

The proof of Theorem 9.5 contains as a subproduct the asymptotic energy balance of SM (9.32). Indeed, let us respectively define the deformation energy E_D, the boundary friction energy E_F, and the subgrid eddy dissipation energy E_S by

$$E_D\mathbf{v} = 2\,\nu\,\|D\mathbf{v}\|_{0,2,\Omega}^2,$$

$$E_F(\mathbf{v}) = \langle G(\mathbf{v}), \mathbf{v}\rangle = \int_{\Gamma_n} g(\mathbf{v}(\mathbf{x}))\cdot\mathbf{v}(\mathbf{x})\,d\mathbf{x},$$

$$E_S(\mathbf{v}) = c(\mathbf{v}_h, \mathbf{v}_h) = C_S^2 \sum_{K\in\mathcal{T}_h} h_K^2 \int_K |D(\mathbf{v}_h)(\mathbf{x})|^3\,d\mathbf{x}.$$

Then it holds

Corollary 9.1. *Let $(\mathbf{v}_h)_{h>0}$ be a subsequence of solutions of SM (9.32) strongly convergent in $\mathbf{W}_D(\Omega)$ to a solution $\mathbf{v}$ of Navier–Stokes equations (9.1). Then*

$$\lim_{h\to 0} E_D(\mathbf{v}_h) = E_D\mathbf{v}, \quad \lim_{h\to 0} E_F(\mathbf{v}_h) = E_F(\mathbf{v}), \quad \lim_{h\to 0} E_S(\mathbf{v}_h) = 0.$$

This is, the total deformation and the boundary friction energies separately converge, while the total energy balance is asymptotically maintained:

$$\lim_{h\to 0}\left[E_D(\mathbf{v}_h) + E_S(\mathbf{v}_h) + E_F(\mathbf{v}_h)\right] = E_D\mathbf{v} + E_F(\mathbf{v}).$$

Observe that due to the incompressibility the pressure deformation energy plays no role in the energy balance.

9.7 Error Estimates

In this section we derive error estimates for discretization (9.32) for diffusion-dominated flows. The order of convergence of these estimates is limited to $O(h^{2-d/2})$, due to the penalty nature of the eddy diffusion term.

Similar error estimates may be obtained in a more general framework, when the solution of Navier–Stokes equations is located in a branch of non-singular Reynolds numbers, in the sense that at that Reynolds number there are no bifurcations to more complex flows (See Remark 9.3). Roughly speaking, the mathematical concept of non-singular flow is closer to the physical concept of laminar flow than just the diffusion-dominated one. However, we consider here the diffusion-dominated regime (much more restrictive than non-singular flow) as we are interested in determining the accuracy of SM for smooth solutions and which is the impact of the eddy diffusion term in the accuracy order.

We start by setting conditions that ensure the uniqueness of solutions of Navier–Stokes equations (9.1). As b is a bounded trilinear form, the quantity

$$\beta = \sup_{\mathbf{z},\mathbf{v},\mathbf{w}\in\mathbf{W}_D(\Omega)} \frac{b(\mathbf{z};\mathbf{v},\mathbf{w})}{\|D(\mathbf{z})\|_{0,2,\Omega}\,\|D\mathbf{v}\|_{0,2,\Omega}\,\|D\mathbf{w}\|_{0,2,\Omega}}.$$

is finite.

Theorem 9.6. *Assume that*

$$\nu^2 > \beta \, \|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'}. \tag{9.41}$$

Then the weak solution of Navier–Stokes equations (9.1) is unique.

Proof. Consider two solutions $\mathbf{z}$, $\mathbf{v} \in \mathbf{W}_D(\Omega)$ of (9.1). Let $\mathbf{e} = \mathbf{v} - \mathbf{z}$ and subtract the equations satisfied by $\mathbf{z}$ and $\mathbf{v}$ with $\mathbf{w} = \mathbf{e}$. Then

$$\nu \|D\mathbf{e}\|_{0,2,\Omega}^2 + \langle G(\mathbf{v}) - G(\mathbf{z}), \mathbf{e}\rangle = b(\mathbf{v}; \mathbf{v}, \mathbf{e}) - b(\mathbf{z}; \mathbf{z}, \mathbf{e})$$
$$= b(\mathbf{v}; \mathbf{v}, \mathbf{e}) \mp b(\mathbf{z}; \mathbf{v}, \mathbf{e}) - b(\mathbf{z}; \mathbf{z}, \mathbf{e}) = b(\mathbf{e}; \mathbf{v}, \mathbf{e}) \le \beta \, \|D\mathbf{v}\|_{0,2,\Omega} \, \|D\mathbf{e}\|_{0,2,\Omega}^2.$$

By the monotonicity of G, $\langle G(\mathbf{v}) - G(\mathbf{z}), \mathbf{e}\rangle \ge 0$. Using estimate (9.38),

$$\nu \|D\mathbf{e}\|_{0,2,\Omega}^2 \le \frac{1}{\nu} \beta \, \|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'} \, \|D\mathbf{e}\|_{0,2,\Omega}^2.$$

Then condition (9.41) implies $\mathbf{e} = 0$. □

Remark 9.2. The condition (9.41) means that the flow is diffusion-dominated: The diffusion is large enough to balance the convection effects relative to the data $\mathbf{f}$.

To state the error estimates result, let us denote the distance between some $\mathbf{w} \in \mathbf{W}_D(\Omega)$ and the space $\mathbf{W}_h$ by

$$d_{1,2,\Omega}(\mathbf{w}, \mathbf{W}_h) = \inf_{\mathbf{z}_h \in \mathbf{W}_h} \|\mathbf{w} - \mathbf{z}_h\|_{1,2,\Omega}.$$

Observe that there exists $\mathbf{w}_h \in \mathbf{W}_h$ such that $\|\mathbf{w} - \mathbf{w}_h\|_{1,2,\Omega} = d_{1,2,\Omega}(\mathbf{w}, \mathbf{W}_h)$ because $\mathbf{W}_h$ is a closed subspace of $\mathbf{W}_D(\Omega)$. In fact, $\mathbf{w}_h$ is the orthogonal projection of $\mathbf{w}$ on $\mathbf{W}_D(\Omega)$ with respect to the scalar product induced by the norm $\|D(\cdot)\|_{0,2,\Omega}$. Denote similarly the distance between some $q \in L_0^2(\Omega)$ and M_h by

$$d_{0,2,\Omega}(q, M_h) = \inf_{r_h \in M_h} \|q - r_h\|_{0,2,\Omega}.$$

Again, there exists $q_h \in M_h$ such that $\|q - q_h\|_{0,2,\Omega} = d_{0,2,\Omega}(q, M_h)$.

Theorem 9.7. *Under the hypotheses of Theorem 9.4, assume that the data* $\mathbf{f}$ *satisfy the estimate (9.41). Then the following error estimates for the solution of method (9.32) hold:*

$$\|D(\mathbf{v} - \mathbf{v}_h)\|_{0,2,\Omega} \le C \left[h^{2-2/d} + d_{1,2,\Omega}(\mathbf{v}, \mathbf{W}_h) + d_{0,2,\Omega}(p, M_h) \right]; \tag{9.42}$$
$$\|q - q_h\|_{0,2,\Omega} \le C \left[h^{2-2/d} + d_{1,2,\Omega}(\mathbf{v}, \mathbf{W}_h) + d_{0,2,\Omega}(p, M_h) \right], \tag{9.43}$$

for some constant C independent of h.

Proof. Let $\overline{\mathbf{v}}_h \in \mathbf{W}_h$ be the Stokes projection of $\mathbf{v}$ on $\mathbf{W}_h$ introduced in Sect. 9.3.3. Also, let $\overline{p}_h \in M_h$ such that $\| p - \overline{p}_h\|_{0,2,\Omega} = d_{0,2,\Omega}(q, M_h)$. From (9.3),

$$b(\overline{\mathbf{v}}_h; \overline{\mathbf{v}}_h, \mathbf{w}) + a(\overline{\mathbf{v}}_h, \mathbf{w}) + \langle G(\overline{\mathbf{v}}_h), \mathbf{w}\rangle - (\overline{p}_h, \nabla\cdot\mathbf{w})_\Omega = \langle \mathbf{f}, \mathbf{w}\rangle + \langle \varepsilon_h, \mathbf{w}\rangle, \tag{9.44}$$

for all $\mathbf{w} \in \mathbf{W}_D(\Omega)$, where $\varepsilon_h \in \mathbf{W}_D(\Omega)'$ is the consistency error, defined by

$$\begin{aligned}\langle \varepsilon_h, \mathbf{w}\rangle &= b(\overline{\mathbf{v}}_h; \overline{\mathbf{v}}_h, \mathbf{w}) - b(\mathbf{v}; \mathbf{v}, \mathbf{w}) + a(\overline{\mathbf{v}}_h - \mathbf{v}, \mathbf{w}) + \langle G(\overline{\mathbf{v}}_h) - G(\mathbf{v}), \mathbf{w}\rangle \\ &\quad + (p - \overline{p}_h, \nabla\cdot\mathbf{w}).\end{aligned}$$

Set $\mathbf{e}_h = \overline{\mathbf{v}}_h - \mathbf{v}_h$, $\lambda_h = \overline{p}_h - p_h$. Setting $\mathbf{w} = \mathbf{w}_h \in \mathbf{W}_h$ and subtracting (9.44) from (9.3) we obtain the error equation:

$$\begin{aligned}b(\mathbf{e}_h; \mathbf{e}_h, \mathbf{w}_h) + a(\mathbf{e}_h, \mathbf{w}_h) + \langle G(\overline{\mathbf{v}}_h) - G(\mathbf{v}_h), \mathbf{w}_h\rangle - (\lambda_h, \nabla\cdot\mathbf{w}_h)_\Omega \\ = \langle \varepsilon_h, \mathbf{w}_h\rangle - b(\mathbf{v}_h; \mathbf{e}_h, \mathbf{w}_h) - b(\mathbf{e}_h; \mathbf{v}_h, \mathbf{w}_h) - c(\mathbf{v}_h; \mathbf{w}_h),\end{aligned} \tag{9.45}$$

where we have used the identity $b(\overline{\mathbf{v}}_h; \overline{\mathbf{v}}_h, \mathbf{w}_h) - b(\mathbf{v}_h; \mathbf{v}_h, \mathbf{w}_h) = b(\mathbf{e}_h; \mathbf{e}_h, \mathbf{w}_h) + b(\mathbf{v}_h; \mathbf{e}_h, \mathbf{w}_h) + b(\mathbf{e}_h; \mathbf{v}_h, \mathbf{w}_h)$. Set $\mathbf{w}_h = \mathbf{e}_h$. As $(\nabla\cdot\overline{\mathbf{v}}_h, \lambda_h) = 0$, then $(\nabla\cdot\mathbf{e}_h, \lambda_h) = 0$. Using this identity, (9.36) and $\langle G(\overline{\mathbf{v}}_h) - G(\mathbf{v}_h), \mathbf{e}_h\rangle \ge 0$ by the monotonicity of the form G, we deduce

$$\begin{aligned}\nu\|D(\mathbf{e}_h)\|^2_{0,2,\Omega} &\le \langle \varepsilon_h, \mathbf{e}_h\rangle - b(\mathbf{e}_h; \mathbf{v}_h, \mathbf{e}_h) - c(\mathbf{v}_h; \mathbf{e}_h) \le \|\varepsilon_h\|_{\mathbf{W}_D(\Omega)'}\|D(\mathbf{e}_h)\|_{0,2,\Omega} \\ &\quad + \|b\|\,\|D(\mathbf{e}_h)\|^2_{0,2,\Omega}\,\|D(\mathbf{v}_h)\|_{0,2,\Omega} + |c(\mathbf{v}_h; \mathbf{e}_h)| \\ &\le \|\varepsilon_h\|_{\mathbf{W}_D(\Omega)'}\|D(\mathbf{e}_h)\|_{0,2,\Omega} + \frac{1}{2\,\nu}\|b\|\ \|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'}\|D(\mathbf{e}_h)\|^2_{0,2,\Omega} \\ &\quad + |c(\mathbf{v}_h; \mathbf{e}_h)|.\end{aligned}$$

By (9.41), $\delta = 2\,\nu - \dfrac{1}{2\,\nu}\|b\|\ \|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'} > 0$. Using Young's inequality,

$$\delta\,\|D(\mathbf{e}_h)\|^2_{0,2,\Omega} \le \frac{2}{\delta}\,\|\varepsilon_h\|^2_{\mathbf{W}_D(\Omega)'} + \frac{\delta}{2}\,\|D(\mathbf{e}_h)\|^2_{0,2,\Omega} + |c(\mathbf{v}_h; \mathbf{e}_h)|.$$

Then,

$$\delta\|D(\mathbf{e}_h)\|^2_{0,2,\Omega} \le \frac{4}{\delta}\,\|\varepsilon_h\|^2_{\mathbf{W}_D(\Omega)'} + 2\,|c(\mathbf{v}_h; \mathbf{e}_h)|. \tag{9.46}$$

Also, using (9.36) and (9.38),

$$\begin{aligned}|c(\mathbf{v}_h; \mathbf{e}_h)| &\le C\,h^{2-d/2}\,\|D(\mathbf{v}_h)\|^2_{0,2,\Omega}\,\|D(\mathbf{e}_h)\|_{0,2,\Omega} \\ &\le C\,h^{2-d/2}\,\|\mathbf{f}\|^2_{\mathbf{W}_D(\Omega)'}\,\|D(\mathbf{e}_h)\|_{0,2,\Omega}.\end{aligned} \tag{9.47}$$

Combining (9.47) with (9.46), and using again Young's inequality,

$$\delta \,\|D(\mathbf{e}_h)\|_{0,2,\Omega}^2 \leq \frac{8}{\delta}\,\|\varepsilon_h\|_{\mathbf{W}_D(\Omega)'}^2 + \frac{C}{\delta}\,h^{2(2-d/2)}\,\|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'}^4.$$

To estimate ε_h, consider that

$$\begin{aligned}|b(\overline{\mathbf{v}}_h;\overline{\mathbf{v}}_h,\mathbf{w}) - b(\mathbf{v};\mathbf{v},\mathbf{w})| &\leq |b(\overline{\mathbf{v}}_h;\overline{\mathbf{v}}_h-\mathbf{v},\mathbf{w}) + b(\overline{\mathbf{v}}_h-\mathbf{v};\mathbf{v},\mathbf{w})|\\ &\leq C\,(\|D\mathbf{v}\|_{0,2,\Omega} + \|D(\overline{\mathbf{v}}_h)\|_{0,2,\Omega})\|D(\overline{\mathbf{v}}_h-\mathbf{w})\|_{0,2,\Omega}\,\|D\mathbf{w}\|_{0,2,\Omega};\end{aligned}$$

$$\begin{aligned}|a(\overline{\mathbf{v}}_h-\mathbf{v},\mathbf{w})| &\leq C\,\|D(\overline{\mathbf{v}}_h-\mathbf{w})\|_{0,2,\Omega}\,\|D\mathbf{w}\|_{0,2,\Omega},\\ |(\overline{p}_h-p,\nabla\cdot\mathbf{w})| &\leq C\,\|p-\overline{p}_h\|_{0,2,\Omega}\,\|D\mathbf{w}\|_{0,2,\Omega}.\end{aligned}$$

Then, using estimate (6.39),

$$\|\varepsilon_h\|_{\mathbf{W}_D(\Omega)'} \leq C\,(d_{1,2,\Omega}(\mathbf{v},\mathbf{W}_h) + d_{0,2,\Omega}(p,M_h)). \tag{9.48}$$

Now (9.42) follows using $\|D(\mathbf{v}-\mathbf{v}_h)\|_{0,2,\Omega} \leq \|D(\mathbf{v}-\overline{\mathbf{v}}_h)\|_{0,2,\Omega} + \|D(\mathbf{e}_h)\|_{0,2,\Omega}$.

To obtain the estimates for the pressure error, from the error equation (9.45) by similar arguments, we deduce

$$(\lambda_h,\mathbf{w}_h) \leq C\,(\|D(\mathbf{e}_h)\|_{0,2,\Omega}^2 + \|D(\mathbf{e}_h)\|_{0,2,\Omega} + \|\varepsilon_h\|_{\mathbf{W}_D(\Omega)'} + h^{2-d/2})\,\|D(\mathbf{w}_h)\|_{0,2,\Omega}.$$

Then (9.43) follows by the discrete inf-sup condition (Hypothesis 9.ii) and estimates (9.42) and (9.48). □

Corollary 9.2. *Under the hypotheses of Theorem 9.7, assume in addition that the family of pairs of spaces $((\mathbf{W}_h, M_h))_{h>0}$ satisfies the optimal interpolation error estimates (9.14) stated in Theorem 9.1. Then the solution $(\mathbf{v}_h, p_h)$ of the discrete SM (9.32) satisfies the error estimates*

$$\|D(\mathbf{v}-\mathbf{v}_h)\|_{0,2,\Omega} \leq C\,h^{2-2/d}, \quad \|p-p_h\|_{0,2,\Omega} \leq C\,h^{2-2/d}, \tag{9.49}$$

for some constant C independent of h.

Remark 9.3. Similar estimates may be obtained in a more general framework that does not require the flow to be diffusion-dominated. This is the approximation of branches of non-singular solutions of nonlinear equations by means of the implicit function theorem, due to Brezzi et al. (cf. [11–13]). In this context the Navier–Stokes problems appear as a family of problems dependent on a parameter: the Reynolds number Re. A solution of Navier–Stokes equations is called non-singular at a given Re if the derivative of the operator at this solution generates an isomorphism from $\mathbf{W}_D(\Omega) \times L_0^2(\Omega)$ onto its dual space. On a branch of non-singular solutions bifurcating situations (multiplicity of solutions, in particular) cannot occur. The diffusion-dominated flows are a particular class of non-singular solutions. This theory is quite involved technically and we have preferred not to apply it here to shorten our analysis. The orders of accuracy for velocity and pressure would be the same as those given by Theorem 9.7, as these are limited by the eddy viscosity term.

Remark 9.4. The convergence order of method (9.32) is limited by the penalty-like term introduced by the modeling of turbulent diffusion in the SM. This order is optimal (first order) only for piecewise affine finite elements when $d = 2$, but is limited to $1/2$ when $d = 3$. There is no interest in increasing the accuracy of the interpolation as this would require a larger computational effort without increasing the accuracy of the numerical solution. This low convergence order appears linked to the diffusive nature of the SM that extends the eddy diffusion to all wavenumbers.

9.8 Further Remarks

9.8.1 Space Discretizations

We have considered mixed discretizations of the steady SM in order to avoid nonessential technical difficulties. More complex discretizations may be considered, in order to decrease the computational time and memory requirements of the solvers.

Diminishing the computational time may be achieved by means of stabilized solvers that allow to circumvent the discrete inf-sup condition and to use of equal-order interpolation for velocity and pressure. In addition, stabilized methods provide some control of the spurious instabilities due to the discretization of operator terms such as convection, rotation, or reaction, whenever these are dominant at the discrete level.

Stabilized discretizations were introduced by Huges, Franca, and co-workers (cf. [30, 39]). These discretizations are based upon "augmented" discretizations of the flow equations that include additional terms in the standard Galerkin discretization. In the case of the steady subgrid eddy viscosity model (9.1), these discretizations read

Obtain $(\mathbf{v}_h, p_h) \in \mathbf{W}_h \times N_h$ such that for all $(\mathbf{w}_h, q_h) \in \mathbf{W}_h \times N_h$,

$$\left\{ \begin{aligned} & b(\mathbf{v}_h; \mathbf{v}_h, \mathbf{w}_h) + a(\mathbf{v}_h, \mathbf{w}_h) + c(\mathbf{v}_h; \mathbf{w}_h) + \langle G(\mathbf{v}_h), \mathbf{w}_h \rangle \\ & -(p_h, \nabla \cdot \mathbf{w}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, q_h)_\Omega + s_h((\mathbf{v}_h, p_h); (\mathbf{w}_h, q_h)) \\ & \qquad = \langle \mathbf{f}, \mathbf{w}_h \rangle + \langle \mathbf{F}_h, (\mathbf{w}_h, q_h) \rangle, \end{aligned} \right. \tag{9.50}$$

where $s_h(\cdot, \cdot)$ is a stabilizing term and $\mathbf{F}_h$ is a term that keeps the consistency of the method, given by

$$s_h((\mathbf{v}_h, p_h); (\mathbf{w}_h, q_h)) = -\sum_{K \in \mathcal{T}_h} \tau_K \, (N(\mathbf{v}_h; \mathbf{v}_h, p_h), M(\mathbf{v}_h; \mathbf{w}_h, q_h))_K,$$
$$+ \sum_{K \in \mathcal{T}_h} \sigma_K \, (\nabla \cdot \mathbf{v}_h, \nabla \cdot \mathbf{w}_h))_K;$$
$$\langle \mathbf{F}_h, (\mathbf{w}_h, q_h) \rangle = -\sum_{K \in \mathcal{T}_h} \tau_K \, (\mathbf{f}, M(\mathbf{v}_h; \mathbf{w}_h, q_h))_K,$$

where

$$N(\mathbf{v}_h; \mathbf{w}_h, p_h) = \mathbf{v}_h \cdot \nabla \mathbf{w}_h - \nu \Delta \mathbf{w}_h + \nabla p_h,$$
$$M(\mathbf{v}_h; \mathbf{w}_h, q_h) = -\mathbf{v}_h \cdot \nabla \mathbf{w}_h + \varepsilon \, \nu \Delta \mathbf{w}_h - \nabla q_h, \tag{9.51}$$

and the τ_K and σ_K are the so-called stabilization coefficients, respectively corresponding to momentum conservation and continuity equations, and ε is a parameter. The cases $\varepsilon = 1$, $\varepsilon = 0$, and $\varepsilon = -1$ respectively correspond to the Galerkin/least squares (GALS), streamline upwind/Petrov–Galerkin (SUPG) (cf. [14]), and adjoint-stabilized (also called uncommon stabilized, cf. [39]) methods. The main interest of these discretizations is that no additional degrees of freedom in velocity are needed to achieve the stability of the pressure discretization. For instance, $\mathbf{W}_h$ and M_h may be set as $\mathbf{W}_h = \mathbf{V}_h^{(m)}$, $N_h = M_h^{(m)}$ for $m \geq 1$, with the notations introduced in Sect. 9.3.3. This yields an optimal-order approximation, so that the error estimates of Theorems 9.7 and 11.2 hold for this discretization.

The stabilizing coefficients are designed by asymptotic scaling arguments (cf. [2]) applied to the framework of stabilized methods (cf. [38, 46]). Respectively, τ_K and σ_K are discrete approximations of the inverse advection operator $(L_a)^{-1}$, $L_a = \partial_t + \tilde{\mathbf{v}}_h \cdot \nabla - \nu \, \Delta$, and of the continuity operator $\nabla \cdot (L_a)^{-1} \, \nabla$ on element K. Their expression follows the structure

$$\tau_K = (C_1 \, \Delta t + C_2 \, V_K \, h_K^{-1} + C_3 \, \nu \, h_K^{-2})^{-1}, \quad \sigma_K = C_4 \, h_K^{-2} \, \tau_K,$$

where V_K is some mean velocity on element K and C_1, C_2, C_3, and C_4 are constants to be determined by comparison with well-known results.

A variant of the preceding methods is the orthogonal subscales (OSS) method, introduced by Codina (cf. [26, 28]), that is obtained by setting $\varepsilon = -1$ in (9.51) and replacing the operators N and M that appear in the expression of the stabilizing term s_h respectively by $(I - P_h)N$ and $(I - P_h)M$ where I is the identity operator and P_h is the L^2 projection operator on the velocity space. This method is oriented to the modeling of turbulence (cf. [27]).

All the preceding methods are residual based, in the sense that the stabilizing terms are products of the residual by some convenient test function. As a consequence, the exact solution exactly satisfies the discrete equations, whenever it is smooth enough. An alternative is provided by the projection-stabilized methods that have a simpler structure, but that are only approximately consistent (cf. [7, 17, 22, 44]). In particular the penalty term-by-term stabilized method provides a separate stabilization of each single operator term that could lead to unstable discretizations (e.g., convection, pressure gradient, etc.). The discretization of the subgrid eddy viscosity model (9.1) by this method reads as follows:

Obtain $(\mathbf{v}_h, p_h) \in \mathbf{W}_h \times N_h$ such that for all $(\mathbf{w}_h, q_h) \in \mathbf{W}_h \times N_h$,

$$\begin{cases} b(\mathbf{v}_h; \mathbf{v}_h, \mathbf{w}_h) + a(\mathbf{v}_h, \mathbf{w}_h) + c(\mathbf{v}_h; \mathbf{w}_h) + \langle G(\mathbf{v}_h), \mathbf{w}_h \rangle \\ -(p_h, \nabla \cdot \mathbf{w}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, q_h)_\Omega + s_{conv,h}(\mathbf{v}_h, \mathbf{w}_h) + s_{pres,h}(p_h, q_h) = \\ \langle \mathbf{f}, \mathbf{w}_h \rangle \quad , \end{cases} \tag{9.52}$$

with

$$s_{conv,h}(\mathbf{v}_h, \mathbf{w}_h) = \sum_{K \in \mathscr{T}_h} \tau_{conv,K} \left((I - \sigma_h)(\mathbf{v}_h \cdot \nabla \mathbf{v}_h), (I - \sigma_h)(\mathbf{v}_h \cdot \nabla \mathbf{w}_h) \right)_K,$$

$$s_{pres,h}(p_h, q_h) = \sum_{K \in \mathscr{T}_h} \tau_{pres,K} \left((I - \sigma_h)(\nabla p_h), (I - \sigma_h)(\nabla q_h) \right)_K,$$

where $\tau_{conv,K}$ and $\tau_{press,K}$ respectively are stabilization coefficients for convection and pressure and σ_h is an interpolation or projection operator on an auxiliary large-scale finite element velocity space. Typically, spaces $\mathbf{W}_h$ and N_h are set as $\mathbf{W}_h = \mathbf{V}_h^{(m)}$, $N_h = M_h^{(m)}$ for $m \geq 2$, and σ_h as an interpolation operator of piecewise continuous functions on space $\mathbf{V}_h^{(m-1)}$ or on the velocity space $\mathbf{V}_h^{(m)}$.

A further simplification arises when $\sigma_h = 0$ that corresponds to the pure penalty stabilized method. This method is simpler to implement, but its order of accuracy is reduced to one (cf. [21]).

9.8.2 Treatment of General Dirichlet and Outflow BCs

In practical situations, inflow and outflow BCs should be taken into account. The boundary of Ω is split into $\Gamma = \Gamma_{in} \cup \Gamma_{out} \cup \Gamma_n$, and typically the BC imposed are (cf. Glowinski [35], Sects. I.2 and III.15.4)

$$\begin{cases} \mathbf{v} = \mathbf{v}_{in} & \text{on } \Gamma_{in}, \\ \mathbf{n} \cdot [2\,\nu\, D\mathbf{v}] = 0 & \text{on } \Gamma_{out}, \\ \mathbf{n} \cdot [2\,\nu\, D\mathbf{v}]_{\boldsymbol{\tau}} = g(\mathbf{v})_{\boldsymbol{\tau}}, \quad \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \Gamma_n, \end{cases} \tag{9.53}$$

where $\mathbf{v}_{in}$ is given. The Dirichlet BC above on Γ_{in} may be reduced to a homogeneous Dirichlet BC by replacing the velocity $\mathbf{v}$ by the new unknown $\tilde{\mathbf{v}} = \mathbf{v} - \mathbf{v}_{in}$.

The BC on Γ_{out} expresses a theoretical absence of stresses between the flow inside and outside the computational domain. This is not a physical BC that is accurate only if the flow across Γ_{out} is irrotational. It is directly taken into account in our variational problem (9.3) of Navier–Stokes equations, as this still applies if the space $\mathbf{W}_D(\Omega)$ is replaced by

$$\mathbf{W}_D(\Omega) = \{\mathbf{w} \in \mathbf{H}^1(\Omega) \text{ such that } \gamma_n \mathbf{w} = 0 \text{ on } \Gamma_n,\ \gamma_0 \mathbf{w} = 0 \text{ on } \Gamma_{in}\}. \quad (9.54)$$

If $\mathbf{v}_{in}$ is small enough, the preceding stability and error analysis also applies. Performing the discretization of these BC is rather straightforward. Indeed, the discrete space $\mathbf{W}_h$ defined by (9.21) must be replaced by

$$\mathbf{W}_h = \{\mathbf{w}_h \in V_h^d \text{ such that } \mathbf{w}_h \cdot \mathbf{n} = 0 \text{ on } \Sigma_i, i = 1, \cdots, k-1,\ \mathbf{w}_h = 0 \text{ on } \overline{\Gamma_{in}}\},$$

and the discretization (9.32) still applies, excepting that now we look for a solution $\mathbf{v}_h \in \tilde{\mathbf{v}} + \mathbf{W}_h$. Note that in this space the unknowns located on Γ_{out} are not blocked and then must be solved through the discretization.

9.8.3 Weak Discretization of the Slip BC

An alternative way to discretize the slip condition is to consider it as a restriction that is formulated as a saddle-point problem and discretized by mixed methods. This technique is introduced and analyzed in Verfürth [51–53]. It provides conformal finite element approximations, as only the zero trace condition on Γ_D must be satisfied by the discrete space.

An additional difficulty arises when the domain is not polyhedric. In this case if it is approximated by polyhedric domains, the approach of [51–53] only provides first-order approximations, when Γ is smooth enough. To increase the accuracy of the approximations, isoparametric finite elements are used.

The analysis of the SM introduced in this chapter may be adapted to this kind of discretization of the slip condition, with some technical work. This is based upon the basic structure of nonlinear saddle-point problems that share both formulations.

9.8.4 Improved Eddy Viscosity Modeling

The over-diffusive effects of SM near the solid walls may be avoided by means of the use of wall-law functions, as we have done in this chapter. There is an alternative treatment, which is the use of a wall viscosity-damping constant C_S (which indeed is no longer a constant). Let us mention, for instance, the Van-Driest damping introduced in Sect. 5.5.1 (cf. [50]):

$$\nu_t(\mathbf{v}_h)(\mathbf{x}) = C_S^2(\mathbf{x})\, h_K^2\, |D(\mathbf{v}_h)(\mathbf{x})|, \text{ for } \mathbf{x} \in K, \quad (9.55)$$

where $C_S(\mathbf{x}) = C_S\left(1 - e^{-(z^+/\overline{z})}\right)$. Here, we recall that z^+ is the dimensional distance to the wall defined in Sect. 5.4, and $\overline{z}$ is a constant distance located inside the logarithmic layer that sets the intensity of the damping. The analysis of SM with Van-Driest damping closely follows the preceding one, using that the function $C_s(\mathbf{x})$ is bounded and C^∞.

Another improvement of the SM is yield by the Germano dynamic in time adjustment of the constant C_S, to better fit the dissipation balance of the flow (cf. [32, 33]). This adjustment is based upon the extrapolation of the information on the resolved fields at two-scale levels to compute a (theoretical) optimal value for $C_S = C_S(\mathbf{x}, t)$, at each point $\mathbf{x}$ and time step t. However, in practice $C_S(\mathbf{x}, t)$ may take negative values, which is interpreted as "backscatter," i.e., inverse transfer of energy from small to larger eddies. To enforce the boundedness of $C_S(\mathbf{x}, t)$, the "clipping" procedure is enforced. However this ad hoc approach does not yield a smooth $C_S(\mathbf{x}, t)$. A positive, bounded and smooth function $C_S(\mathbf{x}, t)$ was proposed in Borgaard et al. [6].

Again the preceding analysis may be extended to this case if the function $C_S(\mathbf{x}, t)$ is assumed to be positive, bounded and smooth.

9.8.5 Mathematical Justification of SM

A formal justification of SM is found in a series of papers by Hou and co-workers (cf. [36, 37]). This justification is based upon a two-scale expansion of the Navier–Stokes flow, in the form

$$\begin{cases} \mathbf{v}^\varepsilon(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}, t) + \mathbf{w}(\theta, \mathbf{z}, t, \sigma), \\ p^\varepsilon(\mathbf{x}, t) = p(\mathbf{x}, t) + q(\theta, \mathbf{z}, t, \sigma), \end{cases} \tag{9.56}$$

where $\theta = \theta(\mathbf{x}, t)$ are the Lagrangian coordinates of the flow and $\mathbf{z} = \dfrac{\theta}{\varepsilon}$ and $\sigma = \dfrac{t}{\varepsilon}$ are the fast variables. $\mathbf{w}$ and q are assumed to be periodic with respect to $\mathbf{z}$ and to have zero space–time mean in $(\mathbf{z}, \sigma)$ (in a sense similar to the long-time average introduced in Sect. 3.5). The ansatz (9.56) corresponds to initial conditions of the form

$$\mathbf{v}^\varepsilon(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}) + \mathbf{w}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}).$$

This initial condition, apparently with a two-scale structure, in fact is a re-parametrization of a function with infinitely many scales. Indeed, the Fourier expansion of a function $\mathbf{v}$ may be split as

$$\mathbf{v}(\mathbf{x}) = \sum_{\mathbf{k}\in Z^n} \hat{\mathbf{v}}_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}} = \mathbf{V}(\mathbf{x}) + \mathbf{W}\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right),$$

where for some integer $K \geq 1$, the functions $\mathbf{V}$ and $\mathbf{W}$ are defined by

$$\mathbf{V}(\mathbf{x}) = \sum_{\mathbf{k}\in Z^n,\, |\mathbf{k}|\leq K/2} \hat{\mathbf{v}}_{\mathbf{k}}\, e^{2\pi i\, \mathbf{k}\cdot\mathbf{x}},$$

$$\begin{aligned}\mathbf{W}(\mathbf{x}) &= \sum_{\mathbf{k}\in Z^n,\, |\mathbf{k}|>K/2} \hat{\mathbf{v}}_{\mathbf{k}}\, e^{2\pi i\, \mathbf{k}\cdot\mathbf{x}} = \sum_{\mathbf{k}^{(s)}\neq 0,\, |\mathbf{k}^{(l)}|\leq K/2} \hat{\mathbf{v}}_{K\mathbf{k}^{(s)}+\mathbf{k}^{(l)}}\, e^{2\pi i\, (K\mathbf{k}^{(s)}+\mathbf{k}^{(l)})\cdot\mathbf{x}} \\ &= \sum_{\mathbf{k}^{(s)}\neq 0} \hat{\mathbf{v}}^{(s)}(\mathbf{k}^{(s)}, \mathbf{x})\, e^{2\pi i\, \mathbf{k}^{(s)}\cdot\mathbf{x}/\varepsilon} = \mathbf{W}\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right), \quad \text{where } \varepsilon = 1/K.\end{aligned}$$

By means of formal homogenization techniques, the mean field $(\mathbf{v}, p)$ and the flow perturbation $(\mathbf{w}, q)$ are shown, up to the first order in ε, to be the solution of a coupled system of PDEs in micro- and macroscales of the form

$$\begin{cases} \partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} - \nu\,\Delta\mathbf{v} + \nabla p + \mathscr{R} = \mathbf{f}\ , \\ \nabla\cdot\mathbf{v} = 0\ , \\ \partial_t\theta + \mathbf{v}\cdot\nabla\theta = 0\ , \\ \mathbf{v}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}), \quad \theta(\mathbf{x},0) = \mathbf{x}\ ; \end{cases} \tag{9.57}$$

$$\begin{cases} \partial_\tau \mathbf{u} + \mathbf{u}\cdot\nabla_{\mathbf{z}}\mathbf{u} - \dfrac{\nu}{\varepsilon}\nabla_{\mathbf{z}}\cdot(B\nabla_{\mathbf{z}}\mathbf{u}) + B\nabla_{\mathbf{z}} q = 0\ , \\ \nabla_{\mathbf{z}}\cdot\mathbf{u} = 0\ , \\ \mathbf{u}(0) = A\,\mathbf{w}(0) = 0\ ; \end{cases} \tag{9.58}$$

where $\mathscr{R} = \langle\mathbf{w}\otimes\mathbf{w}\rangle$ is the Reynolds stress tensor, $\langle\cdot\rangle$ denotes the space–time mean, $A = \nabla\theta$, $B = AA^t$ (A and B are square matrices of dimension d), and $\mathbf{u} = A\mathbf{w}$. The tensor $\mathscr{R}$ is expressed in terms of the mean flow as follows:

$$\mathscr{R} = A^{-1}\tilde{\mathscr{R}}A^{-t}, \quad \text{where } \tilde{\mathscr{R}} = \langle\mathbf{u}\otimes\mathbf{u}\rangle.$$

By Rivlin–Ericksen's theorem (cf. [23]), as $\tilde{\mathscr{R}}$ is symmetric, it should have the structure

$$\tilde{\mathscr{R}} = a_0 + a_1\, B + a_2\, B^2,$$

where the coefficients a_0, a_1, a_2 are functions of the invariants of B: $Tr(B)$, $Tr(B^2)$, and $Tr(B^3)$. Assume that the characteristic time of the subgrid scales is τ. As $A = \nabla_{\mathbf{x}}\theta = I - \tau\,\nabla\mathbf{v} + O(\tau^2)$, then $B = I - 2\tau\, D\mathbf{v} + O(\tau^2)$, and

$$\mathscr{R} = \alpha\, I - \beta\,\tau\, D\mathbf{v} + O(\tau^2) \ \text{ for some } \alpha,\, \beta \in \mathbb{R}.$$

The coefficient β is a function of the invariants of $D\mathbf{v}$. As $Tr(D\mathbf{v}) = 0$, neglecting $Tr(D\mathbf{v}^3)$ this yields β as a function of $Tr(D\mathbf{v}^2) = |D\mathbf{v}|^2$. The time scale of the subgrid scales for Navier–Stokes flow is yield by dimensional analysis as

$$\tau = \begin{cases} \dfrac{h_K^2}{\sqrt{\nu}} & \text{if } Re_K \leq R_0 \ \ \text{(diffusion-dominated flow),} \\ \dfrac{h_K^2}{\nu Re_K} & \text{if } Re_K \geq R_0 \ \text{(convection-dominated flow);} \end{cases}$$

where $Re_K = \frac{V_K h_K}{\nu}$ is the element Reynolds number, constructed with some norm V_K of the velocity $\mathbf{v}$ on the element K. Then, τ is of order h_K^2. This yields the Smagorinsky modeling for the Reynolds stress tensor:

$$\mathscr{R} \simeq \alpha\, I - C_S\, h_K^2\, |D\mathbf{v}|\, D\mathbf{v}.$$

9.8.6 Mathematical Justification of Wall Laws

A linear version of the wall-law BC appearing in the Navier–Stokes problem (9.1) has been mathematically justified for walls with large rugosity (in some specific sense) and laminar flows. This is the so-called Navier BC:

$$\mathbf{n} \cdot [2\,\nu\, D\mathbf{v}]_{\boldsymbol{\tau}} = 0 \ \text{ on } \ \Gamma_n. \tag{9.59}$$

It has been proved to be asymptotically equivalent to the no-slip boundary condition $\mathbf{v} = 0$ on Γ_n if the boundary Γ_n is perturbed by a periodic rugosity. Concretely, if the period of the rugosity is large enough with respect to its amplitude and both tend to zero, the BC (9.59), besides the slip BC ($\mathbf{v} \cdot \mathbf{n} = 0$ on Γ_n) is asymptotically equivalent as $\varepsilon \to 0$ to the no-slip boundary condition ($\mathbf{v} = 0$ on Γ_n) (cf. Casado et al. [19]). The situation is different from that of the wall-law BC for turbulent flows:

$$\mathbf{n} \cdot [(2\,\nu + \nu_t(\mathbf{v}))D\mathbf{v}]_{\boldsymbol{\tau}} = g(\mathbf{v})_{\boldsymbol{\tau}} \ \text{ on } \ \Gamma_n, \tag{9.60}$$

where the term $g(\mathbf{v})$ models the turbulent stress generated by the boundary layer, and where Γ_n is a fictitious boundary located inside the flow, and not a mean boundary with respect to the rugosity. See also [15, 16, 18] for the analysis of related rugous walls in more general situations.

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Chapter 10
Finite Element Approximation of Evolution Smagorinsky Model

Abstract In this chapter we deal with the numerical approximation of the unsteady Navier–Stokes equations in turbulent regime by means of the SM. As in the steady case, we shall consider this model as intrinsically discrete. We consider a semi-implicit discretization in time by the Euler method as a model time discretization. We analyze stability, error, and well-posedness for all flow regimes and study the asymptotic error balance.

10.1 Introduction

The numerical approximation of unsteady Navier–Stokes equations faces many technical difficulties. Specific discretization techniques should be used to treat the incompressibility restriction in a context of time discretization, to ensure the stability of the pressure discretization. Also, the stability of the velocity discretization needs the use of implicit or semi-implicit discretizations of the nonlinear convection term to avoid small time steps. A good accuracy needs the use of high-order solvers that should be specifically designed to meet the stability restrictions mentioned above, while keeping a (relatively) low computational complexity.

The simplest methods are built by combining time discretizations obtained by extrapolations to the Navier–Stokes equations of the standard methods for solving ordinary differential equations, with mixed discretizations in space. This is the case in particular of the implicit and explicit Euler and Crank–Nicolson discretizations in time, combined with mixed discretizations in space. A further improvement is to extrapolate the construction of Runge–Kutta methods, giving rise among others to the fractionary-step methods. The Crank–Nicolson scheme was studied for instance by Heywood and Rannacher [27] and Temam [46]. A survey of the fractionary-step methods can be found in Gresho and Sani [14]. A study of the application of Crank–Nicolson and fractionary step θ-scheme discretizations to the numerical solution of incompressible Navier–Stokes equations can be found in the book by Turek [47].

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The preceding methods essentially apply to diffusion-dominated flows. As the Reynolds number increases the convection term become dominant in the discrete equations, and the discrete velocity develops spurious instabilities that worsen as the Reynolds number increases. The method of characteristics provides a remedy to this problem. It is based upon a time discretization that transforms the material derivative on a time derivative, along the flow lines. It was introduced by Pironneau in [34]. Several extensions to high-order discretizations and complex flow equations have been performed (cf. [1, 4, 5, 35–37, 39]). Another treatment to the convection-dominance effect is provided by the stabilized methods, as was mentioned in Sect. 9.8.1.

All the preceding methods lead to the solution of coupled velocity–pressure linear systems, usually with very large size. Projection methods, introduced by Chorin and Temam (cf. [9, 20, 21, 45]), are used to decrease the very large size of the linear systems that result after discretization. These methods decouple the computation of velocity and pressure at each time step. The pressure satisfies a Poisson equation for which appropriate boundary conditions should be provided. Several improvements have subsequently taken place to extend them to high-order discretizations and more complex flow equations (cf. [22, 24, 32, 38, 43]).

The preceding considerations apply to the discretization of the SM that we consider here. However, additional difficulties arise due to the additional nonlinearities and the modeling of subgrid effects. Methods with high-order accuracy are needed to decrease the error due to the numerical discretization below the subgrid terms. In some numerical experiments, the effect of the subgrid models is completely or partially masked by the numerical error when second-order accurate methods are employed. Some analyses based upon similarity hypothesis indicate that an eighth order of accuracy is needed to obtain a negligible numerical diffusion face to eddy diffusion (cf. Sagaut [40], Chap. 8). However, in practice second-order methods present a dependency with respect to the subgrid model used. This is shown, for instance, in the benchmark tests presented for laminar flow problems in Turek [47] and for the numerical solution of LES models by John [28]. In particular, this is the case of the (implicit) Crank–Nicolson scheme, as we observe in several numerical tests presented in Chap. 13.

In addition, the time discretization of wall laws should be performed with care to preserve their dissipative nature with the purpose of ensuring stability. Both requirements are met by means of implicit method, as we shall consider here. However implicit discretizations lead to nonlinear algebraic systems of equations that need specific solvers. We study the solution of these problems by fixed-point algorithms in Sect. 11.6.

The standard numerical analysis of these discretization techniques for the unsteady Navier–Stokes equations proves their stability in the "natural" $L^2((0,T),\ \mathbf{H}^1(\Omega))$, and $L^\infty((0,T), \mathbf{L}^2(\Omega))$ norms. This guarantees the weak convergence of the numerical approximations to a weak solution of the Navier–Stokes equations. However, the low regularity of the weak solution avoids to prove the strong convergence of the numerical approximations and limits the energy balance to an inequality, as was already remarked in Chap. 8. In the case of the SM, these difficulties are increased by the presence of the nonlinearities due to the eddy diffusion term in the SM and the wall-law term.

In this context our strategy is to analyze the easiest space-time discretization that contains the main difficulties, as a model for more complex discretizations. Actually, we consider a semi-implicit Euler scheme in time combined with the discretization in space by Lagrange finite elements, already introduced in Chap. 9. We do not consider any velocity–pressure decoupling strategy to avoid nonessential difficulties. Our analysis may be applied with some care to more general discretizations, such as Crank–Nicolson scheme, or fractional step methods. Its application to other kind of discretizations needs the combination of their specific analysis techniques to our treatment of the eddy diffusion and wall-law terms.

In our analysis we prove the stability of the implicit Euler discretization of SM in $L^2((0,T),\mathbf{H}^1(\Omega))$, and $L^\infty((0,T),\mathbf{L}^2(\Omega))$ norms and the weak convergence in these spaces to a weak solution of the Navier–Stokes equations including wall laws:

$$\begin{cases} \partial_t \mathbf{v} + \nabla\cdot(\mathbf{v}\otimes\mathbf{v}) - \nabla\cdot(\nu\, D\mathbf{v}) + \nabla p = \mathbf{f} & \text{in } \Omega\times(0,T); \\ \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega\times(0,T); \\ [\nu\, D\mathbf{v}\cdot\mathbf{n}]_\tau = g(\mathbf{v})_\tau & \text{on } \Gamma_n\times(0,T); \\ \mathbf{v}\cdot\mathbf{n} = 0 & \text{on } \Gamma_n\times(0,T); \\ \mathbf{v} = 0 & \text{on } \Gamma_D\times(0,T); \\ \mathbf{v}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}) & \text{in } \Omega. \end{cases} \tag{10.1}$$

The effects of the eddy diffusion on the large-scale flow disappear in a weak sense if all scales are resolved.

Our analysis is based upon the compactness method. We obtain estimates of time and space derivatives of the velocity to pass to the limit in the nonlinear, viscosity (laminar and eddy), and wall-law terms. In addition to the $L^2((0,T),\mathbf{H}^1(\Omega))$, and $L^\infty((0,T),\mathbf{L}^2(\Omega))$ estimates, we obtain estimates for a fractional time derivative of the velocity in Nikolskii spaces (defined in Sect. A.4.5). We subsequently derive estimates for the primitive in time of the pressure in $L^\infty((0,T),L^2(\Omega))$. The analysis of improved regularity of weak solutions of Navier–Stokes equations (cf. Sect. 3.4.2) is based upon the use of specific test functions that depend in a nonlinear way on the velocity. The extension of this analysis to the numerical discretizations still has not been done, up to the knowledge of the authors.

We only prove weak convergence to a solution of Navier–Stokes equations. The estimates that we obtain, much as in the case of the standard analysis of Navier–Stokes equations, do not allow to obtain a solution smooth enough to be used as test function in the variational formulation. Then we cannot prove strong convergence. This is a standard drawback at the present moment that seriously affects the analysis of Navier–Stokes equations (cf. Chap. 8).

We perform our error estimate analysis for general flow regimes and not just for convection-dominated flows as in the steady case. As in the steady case, we shall prove that the order of convergence is suboptimal, with respect to the accuracy of the finite element discretization due to the eddy viscosity term.

We study the well-posedness of the discrete problems. We prove that each actual discrete problem is well posed. However the uniform continuity with respect to the discretization parameters would hold only if the discrete solutions are bounded in $L^2((0,T),\mathbf{W}^3(\Omega))$.

We also analyze the energy balance. The lack of regularity of the solution is again an obstacle to prove that the dissipated subgrid energy vanishes in the limit $(h,\Delta t) \to (0,0)$. Moreover, it also avoids to pass to the limit in the wall-law term. We are able to obtain an upper bound for the energy balance in the case of the Manning law, but not for nonlinear wall laws. The eddy diffusion term is of low help to obtain better estimates and does not asymptotically vanish in a strong sense.

From the analytical point of view we use some variants of the compactness method introduced in Chap. 8. The global strategy to prove convergence is the same, based upon estimates of the space and time derivatives of the velocity in convenient norms. We here use a compactness lemma for space-time functions based upon the estimates of a fractional time derivative of the velocity. This is a flexible tool well adapted to the more restrictive framework of finite element approximations. We also face the lack of a density result of smooth functions in the velocity space, similarly to the steady-state case treated in Chap. 9. We overcome it in the same way, using the density of the finite element spaces in the velocity space.

The chapter is structured as follows: In Sect. 10.2 we state the weak formulation of problem (10.1) that we shall consider. We prove that smooth solutions of this weak formulation indeed are solutions of (10.1) in a strong sense. Section 10.3 is devoted to introduce the discretization that we consider: implicit Euler in time and Lagrange finite elements in space. In Sect. 10.4 we perform the stability and convergence analysis, while the error analysis is performed in Sect. 10.5. We respectively study the asymptotic energy balance and the well-posedness of the discrete problems in Sects. 10.6 and 10.7. We finally address some further remarks in Sect. 10.8 concerning alternative time discretizations, the numerical analysis of the LES–Smagorinsky model, and the suitability of solutions of Navier–Stokes equations, which are solutions that satisfy a local energy estimate.

10.2 Weak Formulation of SM

In this section we give a variational formulation to the mixed boundary value problem (10.1) for the Navier–Stokes equations. We shall approximate this formulation in Sect. 10.3 by the SM–finite element method.

A serious technical difficulty to analyze the unsteady Navier–Stokes and related equations is the obtention of estimates of the pressure in $L^p(Q)$ norms, where we recall that $Q = \Omega \times (0,T)$. This is usually done for the continuous problem by means of test functions that have a nonlinear dependence on the pressure (cf. Bulícek et al. [7], for instance). This is hard to adapt to Galerkin discretizations, where the test functions belong to linear spaces. A way of avoiding this difficulty is to use spaces of free-divergence functions, and then the pressure disappears from the weak formulation (cf. Glowinski [13], Lions [31], Temam [46]). However here we are interested in approximating the pressure, as it plays a relevant physical role in many practical applications. We shall overcome these difficulties by replacing the pressure by its primitive in time as an unknown. We shall prove in a rather natural way that this time primitive of the pressure has $L^\infty((0,T), L^2(\Omega))$ regularity.

To state the weak formulation of problem (10.1), let us introduce the space of free-divergence functions:

$$\mathbf{W}_{Div}(\Omega) = \{\mathbf{w} \in \mathbf{W}_D(\Omega) \;\text{ s. t. }\; \nabla \cdot \mathbf{w} = 0 \;\text{ a.e. in }\; Q\}.$$

The space $\mathbf{W}_{Div}(\Omega)$ is a closed subspace of $\mathbf{W}_D(\Omega)$, and then it is a Hilbert space endowed with the $\mathbf{H}^1(\Omega)$ norm.

Definition 10.1. Let $\mathbf{f} \in L^2(\mathbf{W}_D(\Omega)')$, $\mathbf{v}_0 \in \mathbf{W}_D(\Omega)'$. A pair $(\mathbf{v}, p) \in \mathscr{D}'(Q)^d \times \mathscr{D}'(Q)$ is a weak solution of the Navier–Stokes problem (10.1) if for all $\mathbf{v} \in L^2(\mathbf{W}_{Div}(\Omega)) \cap L^\infty(\mathbf{L}^2)$, there exists $P \in L^2(L^2)$ such that $p = \partial_t P$, and for all $\mathbf{w} \in \mathbf{W}_D(\Omega)$, $\varphi \in \mathscr{D}([0,T])$ such that $\varphi(T) = 0$:

$$\mathscr{V}\mathscr{P}\begin{cases} -\displaystyle\int_0^T (\mathbf{v}(t), \mathbf{w})_\Omega \varphi'(t)\,dt - \langle \mathbf{v}_0, \mathbf{w}\rangle\,\varphi(0) \\ +\displaystyle\int_0^T [b(\mathbf{v}(t); \mathbf{v}(t), \mathbf{w})\,dt + a(\mathbf{v}(t), \mathbf{w}) + \langle G(\mathbf{v}(t)), \mathbf{w}\rangle]\;\varphi(t)\,dt \\ +\displaystyle\int_0^T (P(t), \nabla \cdot \mathbf{w})_\Omega \varphi'(t)\,dt = \int_0^T \langle \mathbf{f}(t), \mathbf{w}\rangle \varphi(t)\,dt. \end{cases} \tag{10.2}$$

This definition makes sense because due to the regularity asked for $\mathbf{v}$ and P, all terms in (10.2) are integrable in $(0,T)$. The weak solutions given by this definition are solutions of the Navier–Stokes equations in the following sense.

Lemma 10.1. *Let $(\mathbf{v}, p) \in \mathscr{D}'(Q)^d \times \mathscr{D}'(Q)$ be a weak solution of the Navier–Stokes problem (10.1). Then*

(i) The equations

$$\partial_t \mathbf{v} + \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - \nabla \cdot (\nu\, D\mathbf{v}) + \nabla p = \mathbf{f} \;\; \textit{and} \;\; \nabla \cdot \mathbf{v} = 0 \tag{10.3}$$

respectively hold in $\mathscr{D}'(Q)^d$ and in $L^2(Q)$.

(ii)

$$\mathbf{v} \in C^0([0,T], \mathbf{W}_{Div}(\Omega)') \;\; \textit{and} \;\; \mathbf{v}(0) = \mathbf{v}_0 \;\; \textit{in} \;\; \mathbf{W}_{Div}(\Omega)'.$$

(iii)

$$\gamma_0 \mathbf{v} = 0 \;\; \textit{in} \;\; L^2(\mathbf{H}^{1/2}(\Gamma_D)), \quad \gamma_n \mathbf{v} = 0 \;\; \textit{in} \;\; L^2(\mathbf{L}^4(\Gamma)).$$

(iv) If $\mathbf{v} \in L^2(\mathbf{H}^2)$*,* $\partial_t \mathbf{v} \in L^2(\mathbf{L}^2)$*, and* $p \in L^2(\mathbf{H}^1)$*, then*

$$-[\nu \cdot D\mathbf{v} \cdot \mathbf{n}]_\tau = g(\mathbf{v})_\tau \quad in \;\; L^1(L^{3/2}(\Gamma_n))^{d-1}.$$

Proof.

(i) As $\mathbf{v} \in L^1(Q)$, then $\mathbf{v}$ generates a distribution, and

$$\langle \partial_t \mathbf{v}, \mathbf{w} \otimes \varphi \rangle_{\mathscr{D}(Q)} = -\int_Q \mathbf{v}(\mathbf{x}, t)\, \partial_t (\mathbf{w}(\mathbf{x})\varphi(t))\, d\mathbf{x}\, dt = -\int_0^T (\mathbf{v}(t), \mathbf{w})_\Omega\, \varphi'(t)\, dt,$$

for all $\mathbf{w} \in \mathscr{D}(\Omega)^d$, $\varphi \in \mathscr{D}(0, T)$. Similarly, as $P \in L^1(Q)$,

$$\langle \nabla(\partial_t P), \mathbf{w} \otimes \varphi \rangle_{\mathscr{D}(Q)} = \int_0^T (P(t), \nabla \cdot \mathbf{w})_\Omega\, \varphi'(t)\, dt,$$

Then, integrating by parts and using $\langle G(\mathbf{v}(t)), \mathbf{w} \rangle = 0$ and $\nabla \cdot \mathbf{v} = 0$ a.e. in Q, (10.2) implies

$$\langle \partial_t \mathbf{v} + \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - \nabla \cdot (\nu\, D\mathbf{v}) + \nabla p - \mathbf{f}, \mathbf{w} \otimes \varphi \rangle_{\mathscr{D}(Q)} = 0$$

for all $\mathbf{w} \in \mathscr{D}(\Omega)^d$, $\varphi \in \mathscr{D}(0, T)$. By Lemma A.11, we deduce (10.3). Also, as $\mathbf{v} \in L^2(\mathbf{W}_{Div}(\Omega))$, then $\nabla \cdot \mathbf{v} = 0$ in $L^2(Q)$.

(ii) Let $\Phi(t) \in \mathbf{W}_D(\Omega)'$ defined a.e. in $(0, T)$ by

$$\langle \Phi(t), \mathbf{z} \rangle = b(\mathbf{v}(t); \mathbf{v}(t), \mathbf{z}) + a(\mathbf{v}(t), \mathbf{z}) + \langle G(\mathbf{v}(t)), \mathbf{z} \rangle - \langle \mathbf{f}(t), \mathbf{z} \rangle.$$

By estimates (6.18), (6.21), and (6.38), there exists a constant $C > 0$ such that

$$\|\Phi(t)\|_{\mathbf{W}_D(\Omega)'} \le C\, (\|D(\mathbf{v}(t))\|_{0,2,\Omega}^2 + \|D(\mathbf{v}(t))\|_{0,2,\Omega} + \|\mathbf{f}(t)\|_{\mathbf{W}_D(\Omega)}).$$

Then $\Phi \in L^1(\mathbf{W}_D(\Omega)')$. From (10.2) we deduce that for all $\mathbf{w} \in \mathbf{W}_{Div}(\Omega)$, $\varphi \in \mathscr{D}(0, T)$,

$$\int_0^T \langle \mathbf{v}(t), \mathbf{w} \rangle_{\mathbf{W}_{Div}(\Omega)}\, \varphi'(t)\, dt = \int_0^T \langle \Phi(t), \mathbf{w} \rangle_{\mathbf{W}_{Div}(\Omega)}\, \varphi(t)\, dt.$$

By Lemma A.6 (ii) this implies $\partial_t \mathbf{v} = -\Phi \in L^1(\mathbf{W}_{Div}(\Omega)')$, and $\mathbf{v} \in C^0([0, T], \mathbf{W}_{Div}(\Omega)')$. Also, by Lemma A.6 (iii), if $\varphi \in \mathscr{D}([0, T])$ is such that $\varphi(T) = 0$, then

$$\int_0^T \langle \partial_t \mathbf{v}(t), \mathbf{w} \rangle_{\mathbf{W}_{Div}(\Omega)}\, \varphi(t)\, dt = -\langle \mathbf{v}(0), \mathbf{w} \rangle_{\mathbf{W}_{Div}(\Omega)}\, \varphi(0) - \int_0^T (\mathbf{v}(t), \mathbf{w})_\Omega\, \varphi'(t)\, dt.$$

As $\mathbf{v}_0 \in \mathbf{W}_D(\Omega)' \hookrightarrow \mathbf{W}_{Div}(\Omega)'$, by (10.2), it follows

$$\int_0^T \langle \partial_t \mathbf{v}(t) + \Phi(t), \mathbf{w} \rangle_{\mathbf{W}_{Div}(\Omega)} \varphi(t)\, dt + \langle \mathbf{v}_0 - \mathbf{v}(0), \mathbf{w} \rangle_{\mathbf{W}_{Div}(\Omega)} \varphi(0) = 0,$$

and so $\langle \mathbf{v}_0 - \mathbf{v}(0), \mathbf{w} \rangle_{\mathbf{W}_{Div}(\Omega)} = 0$ for all $\mathbf{w} \in \mathbf{W}_{Div}(\Omega)$. We conclude that $\mathbf{v}(0) = \mathbf{v}_0$ in $\mathbf{W}_{Div}(\Omega)'$.

(iii) As γ_0 is a linear mapping from $H^1(\Omega)$ to $H^{1/2}(\Gamma_D)$, then $\gamma_0 \mathbf{v} \in L^2(H^{1/2}(\Gamma_D))$. As $\mathbf{v} \in L^2(\mathbf{W}_D(\Omega))$, then $\gamma_0 \mathbf{v} = 0$ in $L^2(H^{1/2}(\Gamma_D))$. Similarly, $\gamma_n \mathbf{v} = 0$ in $L^2(\mathbf{L}^4(\Gamma_n))$.

(iv) Assume $\mathbf{v} \in L^2(\mathbf{H}^2)$, $\partial_t \mathbf{v} \in L^2(\mathbf{L}^2)$, $p \in L^2(\mathbf{H}^1)$. The Green's formula (9.30) implies

$$\int_0^T \int_{\Gamma_n} [\nu\, D\mathbf{v}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) - g(\mathbf{v}(\mathbf{x}, t))]_{\boldsymbol{\tau}} \cdot \mathbf{w}_{\boldsymbol{\tau}}(\mathbf{x}) \varphi(t)\, d\Gamma_n(\mathbf{x})\, dt = 0, \quad (10.4)$$

for all $\mathbf{w} \in \mathbf{W}_D(\Omega)$, $\varphi \in \mathscr{D}(0, T)$. By (6.36), $g(\mathbf{v}) \in L^1(\mathbf{L}^{3/2}(\Gamma_n))$. As $\gamma_0(D\mathbf{v}) \in L^2(\mathbf{H}^{1/2}(\Gamma))$, then $[\nu\, D\mathbf{v} \cdot \mathbf{n} - g(\mathbf{v})] \in L^1(\mathbf{L}^{3/2}(\Gamma_n))$. Consequently, by (10.4), Lemma A.11 implies $[\nu\, D\mathbf{v} \cdot \mathbf{n} - g(\mathbf{v})]_{\boldsymbol{\tau}} = 0$ in $L^1(L^{3/2}(\Gamma_n)^{d-1})$. $\square$

With some more technical work it is possible to prove that $\mathbf{v}$ is weakly continuous from $[0, T]$ into $\mathbf{L}^2(\Omega))$ (i.e., the functions $t \in [0, T] \mapsto (\mathbf{v}(t), \mathbf{w})_\Omega$ are continuous, for any $\mathbf{w} \in \mathbf{L}^2(\Omega))$. Then the initial condition $\mathbf{v}(0) = \mathbf{v}_0$ holds in $\mathbf{L}^2(\Omega)$ (cf. Simon [44]).

10.3 Space-Time Discretization

We set in this section a full discretization of the unsteady Navier–Stokes equations (10.2) by the SM in the context of a finite element discretization in space.

Consider a family of couples of velocity–pressure finite element spaces $((\mathbf{W}_h, M_h))_{h>0}$ that satisfies Hypotheses 9.i and 9.ii (stated in Sect. 9.5). Let N be a positive integer and define the time step $\Delta t = T/N$ and the discrete times of solution $t_n = n\Delta t$, $n = 0, 1, \cdots, N$. We obtain the approximations $\mathbf{v}_h^n$, p_h^n to $\mathbf{v}(t_n, \cdot)$ and $p(t_n, \cdot)$ by:

- *Initialization.* Set

$$\mathbf{v}_h^0 = \mathbf{v}_{0h}. \quad (10.5)$$

- *Iteration.* For $n = 0, 1, \cdots, N-1$: Assume known $\mathbf{v}_h^n \in \mathbf{W}_h$.
  Obtain $\mathbf{v}_h^{n+1} \in \mathbf{W}_h$, $p_h^{n+1} \in M_h$ solution of the variational problem
  For all $\mathbf{w}_h \in \mathbf{W}_h$, $q_h \in M_h$,

$$(\mathscr{V}\mathscr{P})_h \begin{cases} (\dfrac{\mathbf{v}_h^{n+1}-\mathbf{v}_h^n}{\Delta t}, \mathbf{w}_h)_\Omega + b(\mathbf{v}_h^n; \mathbf{v}_h^{n+1}, \mathbf{w}_h) + a(\mathbf{v}_h^{n+1}, \mathbf{w}_h) \\ +c(\mathbf{v}_h^{n+1}; \mathbf{w}_h) + \langle G(\mathbf{v}_h^{n+1}), \mathbf{w}_h\rangle - (p_h^{n+1}, \nabla\cdot\mathbf{w}_h)_\Omega = \langle \mathbf{f}^{n+1}, \mathbf{w}_h\rangle, \\ \qquad (\nabla\cdot\mathbf{v}_h^{n+1}, q_h)_\Omega = 0, \end{cases} \tag{10.6}$$

where for brevity the symbol $<\cdot,\cdot>$ denotes the duality $<\cdot,\cdot>_{\mathbf{W}_D(\Omega)}$, $\mathbf{f}^{n+1}$ is the average value of $\mathbf{f}$ in $[t_n, t_{n+1}]$, and $\mathbf{v}_{0h}$ is some interpolate of $\mathbf{v}_0$ on $\mathbf{W}_h$. If $\mathbf{v}_0 \in \mathbf{H}^1(\Omega)'$, such $\mathbf{v}_{0h}$ may be obtained for instance by means of the discrete $\mathbf{L}^2(\Omega)$ Riesz projection on $\mathbf{W}_h$:

$$(\mathbf{v}_{0h}, \mathbf{w}_h)_\Omega = \langle \mathbf{v}_0, \mathbf{w}_h\rangle_{\mathbf{H}^1(\Omega)}, \quad \text{for all } \mathbf{w}_h \in \mathbf{W}_h. \tag{10.7}$$

In method (10.6) the discretization of the convection term is semi-implicit, while that of the remaining terms is implicit. This allows to achieve the stability of the scheme in $L^\infty(\mathbf{L}^2)$ and $L^2(\mathbf{H}^1)$ norms. These stability properties also are shared by the θ-scheme, defined as

Obtain $\mathbf{v}_h^{n+1} \in \mathbf{W}_h$, $p_h^{n+1} \in M_h$ such that for all $\mathbf{w}_h \in \mathbf{W}_h$, $q_h \in M_h$,

$$\begin{cases} (\dfrac{\mathbf{v}_h^{n+1}-\mathbf{v}_h^n}{\Delta t}, \mathbf{w}_h)_\Omega + b(\mathbf{v}_h^{n+\varepsilon\theta}; \mathbf{v}_h^{n+\theta}, \mathbf{w}_h) + a(\mathbf{v}_h^{n+\theta}, \mathbf{w}_h) + c(\mathbf{v}_h^{n+\theta}, \mathbf{w}_h) \\ \qquad +\langle G(\mathbf{v}_h^{n+\theta}), \mathbf{w}_h\rangle - (p_h^{n+\theta}, \nabla\cdot\mathbf{w}_h)_\Omega = \langle \mathbf{f}^{n+\theta}, \mathbf{w}_h\rangle, \\ \qquad (\nabla\cdot\mathbf{v}_h^{n+\theta}, q_h)_\Omega = 0, \end{cases} \tag{10.8}$$

where $0 \le \theta \le 1$, $\varepsilon = 0$ or 1, and

$$\mathbf{v}_h^{n+\theta} = \theta\mathbf{v}_h^{n+1} + (1-\theta)\mathbf{v}_h^n,\ p_h^{n+\theta} = \theta p_h^{n+1} + (1-\theta)p_h^n,\ \mathbf{f}^{n+\theta} = \theta\mathbf{f}^{n+1} + (1-\theta)\mathbf{f}^n.$$

The choice $\varepsilon = 1$, $\theta = 1/2$ corresponds to the Crank–Nicolson scheme, which is second-order accurate in time. When $\varepsilon = 1$, for any θ this is a fully implicit scheme; in particular $\theta = 1$ corresponds to the fully implicit Euler scheme. The implicit discretization of the convection term yields a nonlinear algebraic system of equations which is quite costly to be solved from the computational point of view. It is preferable in practice to replace it by a semi-implicit discretization that corresponds to $\varepsilon = 0$. This discretization is much less costly. In exchange, the second-order accuracy is lost, although $\theta = 1/2$ still provides a better accuracy than the semi-implicit Euler scheme (10.6). This modified Crank–Nicolson scheme is frequently used in turbulence simulation, as it provides a good compromise between accuracy and computational complexity, while keeping the numerical diffusion levels below the subgrid terms. For simplicity, we shall however focus our analysis of the semi-explicit Euler scheme (10.6), as a model for the analysis of method (10.8).

We next prove the stability of method (10.6) and the convergence of its solution to a weak solution of the unsteady Navier–Stokes equations (10.2). To state these results, we shall consider the following discrete functions:

- $\mathbf{v}_h : [0, T] \to \mathbf{W}_h$ is the piecewise linear in time function that takes on the value $\mathbf{v}_h^n$ at $t = t_n = n\Delta t$:

$$\mathbf{v}_h(t) := \frac{t_{n+1} - t}{\Delta t}\mathbf{v}_h^n + \frac{t - t_n}{\Delta t}\mathbf{v}_h^{n+1}.$$

- $\tilde{p}_h : (0, T) \to M_h$ is the piecewise constant in time function that takes on the value p_h^n in the time interval (t_n, t_{n+1}). This function is defined a.e. in $[0, T]$.
- $P_h : [0, T] \to M_h$ is the primitive of the discrete pressure function $\tilde{p}_h$:

$$P_h(t) := \int_0^t \tilde{p}_h(s)\, ds.$$

- $\tilde{\mathbf{v}}_h : (-\Delta t, T) \to \mathbf{W}_h$ is the piecewise constant function that takes on the value $\mathbf{v}_h^{n+1}$ on (t_n, t_{n+1}), and $\tilde{\mathbf{v}}_h(t) = \mathbf{v}_h^0$ in $(-\Delta t, 0)$. This function is defined a.e. in $(-\Delta t, T)$.
- $\tilde{\mathbf{v}}_h^- : (0, T) \to \mathbf{W}_h$ is the piecewise constant function that takes on the value $\mathbf{v}_h^n$ on (t_n, t_{n+1}). This function is defined a.e. in $(0, T)$.

For simplicity of notation we do not make explicit the dependence of these functions upon Δt.

10.4 Stability and Convergence Analysis

In this section we prove that the discretization (10.6) provides a solution that converges in a convenient sense to a weak solution of Navier–Stokes equations (10.2). This is based upon the proof of the stability of this discretization: Its solutions are uniformly bounded in $(h, \Delta t)$ in appropriate norms to ensure convergence. We use the Nikolskii spaces $N^{s,p}(0, T; B)$, defined in Sect. A.4.5, to bound a fractional time derivative of the velocity. This is needed to ensure compactness in appropriate space-time spaces to pass to the limit in discretization (10.6).

We start by stating the stability result.

Theorem 10.1. *Assume that the family of grids $(\mathcal{T}_h)_{h>0}$ is regular. Assume that $\mathbf{v}_0 \in \mathbf{H}^1(\Omega)'$, $\mathbf{f} \in L^2(\mathbf{W}_D(\Omega)')^d$. Let $((\mathbf{W}_h, M_h))_{h>0}$ be a family of pairs of finite element spaces satisfying Hypotheses 9.i and 9.ii. Then problem (10.6) admits a unique solution. Moreover this solution satisfies the following estimates:*

$$\|\mathbf{v}_h\|_{L^\infty(\mathbf{L}^2)} + \sqrt{\nu}\,\|\mathbf{v}_h\|_{L^2(\mathbf{H}^1)} + h_{\min}\,\|D(\mathbf{v}_h)\|_{L^3(\mathbf{L}^3)}^{3/2} \tag{10.9}$$

$$+\int_0^T < G(\tilde{\mathbf{v}}_h(t), \tilde{\mathbf{v}}_h(t)) > \, dt \le C_1 \left(\|\mathbf{v}_{0h}\|_{0,2,\Omega} + \frac{1}{\sqrt{\nu}} \|\mathbf{f}\|_{L^2(\mathbf{W}_D(\Omega)')} \right),$$

$$\|\mathbf{v}_h\|_{N^{1/4,2}(\mathbf{L}^2)} \le C_2, \tag{10.10}$$

and

$$\| P_h \|_{L^\infty(L^2)} \le C_2, \tag{10.11}$$

for some constant $C_1 > 0$ *independent of* h, Δt *and* ν, *and some constant* $C_2 > 0$ *independent of* h *and* Δt, *where* $h_{\min} = \min_{K \in \mathcal{T}_h} h_K$.

Proof. We proceed by steps.

STEP 1. *Existence and solution of discrete problem.* Problem (10.6) can be written as:
Find $\mathbf{v}_h^{n+1} \in \mathbf{W}_h$, $p_h^{n+1} \in M_h$ such that for all $\mathbf{w}_h \in \mathbf{W}_h$, $q_h \in M_h$,

$$\begin{cases} b(\mathbf{v}_h^n; \mathbf{v}_h^{n+1}, \mathbf{w}_h) + \tilde{a}(\mathbf{v}_h^{n+1}, \mathbf{w}_h) + c(\mathbf{v}_h^{n+1}; \mathbf{w}_h) + \langle G(\mathbf{v}_h^{n+1}), \mathbf{w}_h \rangle \\ \qquad\qquad -(p_h^{n+1}, \nabla \cdot \mathbf{w}_h)_\Omega = \langle \tilde{\mathbf{f}}^{n+1}, \mathbf{w}_h \rangle, \\ \qquad\qquad (\nabla \cdot \mathbf{v}_h^{n+1}, q_h)_\Omega = 0, \end{cases}$$

where $\tilde{a}(\mathbf{v}, \mathbf{w}) = \dfrac{1}{\Delta t} (\mathbf{v}, \mathbf{w})_\Omega + a(\mathbf{v}, \mathbf{w})$ and $\tilde{\mathbf{f}}^{n+1} = \mathbf{f}^{n+1} + \dfrac{\mathbf{v}_h^n}{\Delta t}$. This problem fits into the same functional framework as the steady SM (9.32), because $\tilde{a}$ is an inner product on space $\mathbf{W}_D(\Omega)$ that generates a norm equivalent to the $H^1(\Omega)^d$ norm. Then the existence of a solution follows from Brouwer's fixed-point theorem, by Steps 1 and 2 of Theorem 9.4.

STEP 2. *Velocity estimates.* To obtain estimate (10.9), observe that

$$2(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n, \mathbf{v}_h^{n+1})_\Omega = \|\mathbf{v}_h^{n+1}\|_{0,2,\Omega}^2 - \|\mathbf{v}_h^n\|_{0,2,\Omega}^2 + \|\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\|_{0,2,\Omega}^2.$$

Then, setting $\mathbf{w}_h = \mathbf{v}_h^{n+1}$ and $q_h = p_h^{n+1}$ in (10.6) yields

$$\frac{1}{2}\|\mathbf{v}_h^{n+1}\|_{0,2,\Omega}^2 + \frac{1}{2}\|\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\|_{0,2,\Omega}^2 + \Delta t\, \nu \|D(\mathbf{v}_h^{n+1})\|_{0,2,\Omega}^2 + \left\langle G(\mathbf{v}_h^{n+1}), \mathbf{v}_h^{n+1} \right\rangle$$

$$+ C_S^2\, h_{\min}^2\, \Delta t\, \|D(\mathbf{v}_h^{n+1})\|_{0,3,\Omega}^3 \le \frac{1}{2}\|\mathbf{v}_h^n\|_{0,2,\Omega}^2 + \Delta t < \mathbf{f}^{n+1}, \mathbf{v}_h^{n+1} > . \tag{10.12}$$

Using Young's inequality,

$$\|\mathbf{v}_h^{n+1}\|_{0,2,\Omega}^2 + \|\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\|_{0,2,\Omega}^2 + \Delta t\, \nu \|D(\mathbf{v}_h^{n+1})\|_{0,2,\Omega}^2 + 2 < G(\mathbf{v}_h^{n+1}), \mathbf{v}_h^{n+1} >$$

$$+ 2\, C_S^2\, h_{\min}^2\, \Delta t\, \|D(\mathbf{v}_h^{n+1})\|_{0,3,\Omega}^3 \le \|\mathbf{v}_h^n\|_{0,2,\Omega}^2 + 4 \Delta t\, \nu^{-1} \|\mathbf{f}^{n+1}\|_{\mathbf{W}_D(\Omega)'}^2. \tag{10.13}$$

Summing estimates (10.13) for $n = 0, 1, \cdots, k$ for some $k \leq N-1$,

$$\|\mathbf{v}_h^{k+1}\|_{0,2,\Omega}^2 + \sum_{n=0}^{k} \|\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\|_{0,2,\Omega}^2 + \nu\,\Delta t \sum_{n=0}^{k} \|D(\mathbf{v}_h^{k+1})\|_{0,2,\Omega}^2 +$$

$$+2\,\Delta t \sum_{n=0}^{k} < G(\mathbf{v}_h^{k+1}), \mathbf{v}_h^{k+1} > +2\,C_S^2\,h_{\min}^2\,\Delta t \sum_{n=0}^{k} \|D(\mathbf{v}_h^{k+1})\|_{0,3,\Omega}^3$$

$$\leq \|\mathbf{v}_{0h}\|_{0,2,\Omega}^2 + 4\Delta t\,\nu^{-1} \sum_{n=0}^{k} \|\mathbf{f}^{n+1}\|_{\mathbf{W}_D(\Omega)'}^2. \tag{10.14}$$

This yields estimate (10.9), as $\sum_{n=0}^{N-1} \Delta t\,\|\mathbf{f}^{n+1}\|_{\mathbf{W}_D(\Omega)'}^2 \leq \|\mathbf{f}\|_{L^2(\mathbf{W}_D(\Omega)')}^2$, and

$$\|\mathbf{v}_h\|_{L^\infty(\mathbf{L}^2)} = \max_{n=0,1,\cdots,N} \|\mathbf{v}_h^n\|_{0,2,\Omega}, \quad \|\mathbf{v}_h\|_{L^2(\mathbf{H}^1)}^2 \leq C\,\Delta t \sum_{n=0}^{N} \|D(\mathbf{v}_h^n)\|_{0,2,\Omega}^2,$$

$$\|D(\mathbf{v}_h)\|_{L^3(\mathbf{L}^3)}^3 \leq C\,\Delta t \sum_{n=0}^{N} \|D(\mathbf{v}_h^n)\|_{0,3,\Omega}^3,$$

for some constant $C > 0$ independent of h and Δt.

STEP 3. *Uniqueness of solution of discrete problem*. The uniqueness of solutions is a consequence of the well-posedness of the discrete problem (See Theorem 10.4).

STEP 4. *Velocity time increment estimates.* Let us restate problem (10.6) as

$$\begin{cases} (\partial_t \mathbf{v}_h(t), \mathbf{w}_h) + b(\tilde{\mathbf{v}}_h(t-\Delta t); \tilde{\mathbf{v}}_h(t), \mathbf{w}_h) + a(\tilde{\mathbf{v}}_h(t), \mathbf{w}_h) + c(\tilde{\mathbf{v}}_h(t); \mathbf{w}_h) \\ \quad +\langle G(\tilde{\mathbf{v}}_h(t)), \mathbf{w}_h\rangle - (\tilde{p}_h(t), \nabla\cdot\mathbf{w}_h)_\Omega = \langle\tilde{\mathbf{f}}_h(t), \mathbf{w}_h\rangle \\ \qquad (\nabla\cdot\tilde{\mathbf{v}}_h(t), q_h)_\Omega = 0, \end{cases} \tag{10.15}$$

$a.e.$ in $[0, T]$. Let us integrate (10.15) in $(t, t+\delta)$ for $t \in [0, T-\delta]$,

$$(\tau_\delta \mathbf{v}_h(t), \mathbf{w}_h)_\Omega = \int_t^{t+\delta} \langle \mathscr{F}_h(s), \mathbf{w}_h\rangle\, ds + \int_t^{t+\delta} (\tilde{p}_h(s), \nabla\cdot\mathbf{w}_h)_\Omega\, dt, \tag{10.16}$$

where $\tau_\delta \mathbf{v}_h(t) = \mathbf{v}_h(t+\delta) - \mathbf{v}_h(t)$, and $\mathscr{F}_h(s) \in \mathbf{W}_D(\Omega)'$ is defined a.e. in $(0, T)$ by

$$\langle \mathscr{F}_h(s), \mathbf{w}\rangle = -b(\tilde{\mathbf{v}}_h(s-\Delta t); \tilde{\mathbf{v}}_h(s), \mathbf{w}) - a(\tilde{\mathbf{v}}_h(s), \mathbf{w}) - c(\tilde{\mathbf{v}}_h(s); \mathbf{w})$$
$$- \langle G(\tilde{\mathbf{v}}_h(s)), \mathbf{w}\rangle + \langle\tilde{\mathbf{f}}_h(s), \mathbf{w}\rangle, \quad \text{for all } \mathbf{w} \in \mathbf{W}_D(\Omega).$$

Setting $\mathbf{w}_h = \tau_\delta \mathbf{v}_h(t)$ and integrating from 0 to $T - \delta$,

$$\int_0^{T-\delta} \|\tau_\delta \mathbf{v}_h(t)\|_{0,2,\Omega}^2 \, dt = \int_0^{T-\delta} \int_t^{t+\delta} \langle \mathscr{F}_h(s), \tau_\delta \mathbf{v}_h(t) \rangle \, ds \, dt, \tag{10.17}$$

where we have used that $(\nabla \cdot \tau_\delta \mathbf{v}_h(t), \tilde{p}_h(s)) = 0$, a.e. for $t, s \in (0, T)$ as $\tilde{p}_h(s) \in M_h$. Estimates (6.18), (6.21), (6.38), and (9.36) yield

$$\|\mathscr{F}_h(s)\|_{\mathbf{W}_D(\Omega)'} \le C \left[\|\tilde{\mathbf{v}}_h(s - \Delta t)\|_{1,2,\Omega}^2 + (1 + C\, h^2) \|D(\tilde{\mathbf{v}}_h(s))\|_{0,2,\Omega}^2 \right.$$
$$\left. +1 + \|D(\tilde{\mathbf{v}}_h(s))\|_{0,2,\Omega} + \|\tilde{\mathbf{f}}_h(s)\|_{\mathbf{W}_D(\Omega)'} \right].$$

Due to estimate (10.9), this implies that $\mathscr{F}_h \in L^1(\mathbf{W}_D(\Omega)')$ and

$$\|\mathscr{F}_h\|_{L^1(\mathbf{W}_D(\Omega)')} \le C \tag{10.18}$$

for some constant $C > 0$ independent of h and Δt. Now, we use Fubini's theorem to estimate the r.h.s. of (10.17), as follows:

$$\begin{aligned}
\left| \int_0^{T-\delta} \|\tau_\delta \mathbf{v}_h(t)\|_{0,2,\Omega}^2 \, dt \right| &= \left| \int_0^T \int_{s-\delta}^{s} \langle \mathscr{F}_h(s), \widetilde{\tau_\delta \mathbf{v}_h}(t) \rangle \, dt \, ds \right| \\
&\le \int_0^T \|\mathscr{F}_h(s)\|_{\mathbf{W}_D(\Omega)'} \int_{s-\delta}^{s} \|D(\widetilde{\tau_\delta \mathbf{v}_h}(t))\|_{0,2,\Omega} \, dt \, ds \\
&\le \int_0^T \|\mathscr{F}_h(s)\|_{\mathbf{W}_D(\Omega)'} \, \delta^{1/2} \left(\int_{s-\delta}^{s} \|D(\widetilde{\tau_\delta \mathbf{v}_h}(t))\|_{0,2,\Omega}^2 \, dt \right)^{1/2} ds \\
&\le \delta^{1/2} \|\mathscr{F}_h\|_{L^1(\mathbf{W}_D(\Omega)')} \|D(\tau_\delta \mathbf{v}_h)\|_{L^2(\mathbf{H}^1)} \le C \delta^{1/2} \|\mathbf{v}_h\|_{L^2(\mathbf{H}^1)} \le C\, \delta^{1/2},
\end{aligned} \tag{10.19}$$

for some constant C independent of h, where $\tilde{v}$ denotes the extension by zero outside $[0, T - \delta]$ of a function v. The last estimate follows from (10.9). Estimate (10.19) yields (10.10).

STEP 5. *Estimate of the primitive of the pressure.* Setting $\mathbf{w}_h \in \mathbf{W}_h \cap \mathbf{H}_0^1(\Omega)$, (10.15) yields

$$\begin{aligned}
(P_h(t), \mathbf{w}_h)_\Omega &= (\mathbf{v}_h(t) - \mathbf{v}_{0h}, \mathbf{w}_h)_\Omega - \int_0^t \langle \mathscr{F}_h(s), \mathbf{w}_h \rangle \, ds \\
&\le C \left(\|\mathbf{v}_h\|_{L^\infty(\mathbf{L}^2)} + \|\mathbf{v}_{0h}\|_{0,2,\Omega} + \|\mathscr{F}\|_{L^1(\mathbf{W}_D(\Omega)')} \right) \|\mathbf{w}_h\|_{1,2,\Omega} \\
&\le C \, \|\mathbf{w}_h\|_{1,2,\Omega},
\end{aligned}$$

where the last estimate follows from estimates (10.9) and (10.18). Then, by the inf-sup condition (Hypothesis 9.ii), estimate (10.11) follows. □

Remark 10.1. The stability of the θ-scheme (10.8), when $\theta \geq 1/2$, follows from the identity

$$(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n, \mathbf{v}_h^{n+\theta})_\Omega = \frac{1}{2}\,\|\mathbf{v}_h^{n+1}\|_{0,2,\Omega}^2 - \frac{1}{2}\,\|\mathbf{v}_h^n\|_{0,2,\Omega}^2 + \left(\theta - \frac{1}{2}\right)\|\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\|_{0,2,\Omega}^2.$$

Remark 10.2. There are high technical difficulties to obtain uniformly bounds for the discrete pressures in a Banach space of space-time functions. Indeed, we use the inf-sup condition (9.4). From (10.6), estimates (6.38), (6.18), (6.21), and (9.36) yield

$$\begin{aligned}(\nabla\cdot\mathbf{w}_h, p_h^{n+1})_\Omega \leq \Big[&\frac{1}{\Delta t}\|\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\|_{0,2,\Omega} + C\,\|D(\mathbf{v}_h^n)\|_{0,2,\Omega}\|D(\mathbf{v}_h^{n+1})\|_{0,2,\Omega} \\ &+ (\nu + C)\|D(\mathbf{v}_h^{n+1})\|_{0,2,\Omega} + C\,h^{2-d/2}\|D(\mathbf{v}_h^{n+1})\|_{0,2,\Omega}^2 + \|\mathbf{f}^{n+1}\|_{\mathbf{W}_D(\Omega)'}\Big]\|\mathbf{w}_h\|_{1,2,\Omega}.\end{aligned} \tag{10.20}$$

Then, by (9.4)

$$\begin{aligned}\|p_h^{n+1}\|_{0,2,\Omega} \leq C\Big[&\frac{1}{\Delta t}\|\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\|_{0,2,\Omega} + \|D(\mathbf{v}_h^n)\|_{0,2,\Omega}^2 + \|D(\mathbf{v}_h^{n+1})\|_{0,2,\Omega}^2 \\ &+ \|D(\mathbf{v}_h^{n+1})\|_{0,2,\Omega} + \|\mathbf{f}^{n+1}\|_{\mathbf{W}_D(\Omega)'}\Big].\end{aligned} \tag{10.21}$$

Consequently, from (10.9) and (10.14), we may uniformly bound the quantity

$$\sqrt{\Delta t}\sum_{n=0}^{N} \Delta t\,\|p_h^{n+1}\|_{0,2,\Omega}.$$

Then $\|\tilde{p}_h\|_{L^1(L^2)}$ is only bounded as $\frac{1}{\sqrt{\Delta t}}$.

To prove the convergence we need some preliminary results.

Lemma 10.2. *Let $\mathbf{z} \in L^\infty(\mathbf{L}^2) \cap L^2(\mathbf{L}^4)$. Then $\mathbf{z} \in \mathbf{L}^3(Q)$ and*

$$\|\mathbf{z}\|_{0,3,Q} \leq \|\mathbf{z}\|_{L^\infty(\mathbf{L}^2)}^{1/3}\,\|\mathbf{z}\|_{L^2(\mathbf{L}^4)}^{2/3} \tag{10.22}$$

Proof. Let $r \in [2,4]$. By Hölder inequality,

$$\|\mathbf{z}(t)\|_{0,r,\Omega}^r \leq \|\mathbf{z}(t)\|_{0,2,\Omega}^{2\theta}\,\|\mathbf{z}(t)\|_{0,4,\Omega}^{4(1-\theta)} \leq \|\mathbf{z}\|_{L^\infty(\mathbf{L}^2)}^{2\theta}\,\|\mathbf{z}(t)\|_{0,4,\Omega}^{4(1-\theta)},\quad \text{a.e. in } (0,T)$$

where $r = 2\theta + 4(1-\theta)$. Setting $r = 3$ we obtain $\theta = 1/2$. Integrating in time the above inequality yields (10.22). □

This result is proved similarly to Lemma A.18, but we include it here for the reader's convenience.

Lemma 10.3. *Assume that the sequence* $\{\tilde{\mathbf{v}}_h^-\}_{h>0} \subset C^0(\mathbf{W}_h)$ *strongly converges to* $\mathbf{v}$ *in* $L^3(Q)$ *and that* $\{\mathbf{w}_h\}_{h>0} \subset \mathbf{W}_h$ *strongly converges to* $\mathbf{w} \in \mathbf{W}_D(\Omega)$ *in* $\mathbf{W}_D(\Omega)$. *Let* $\varphi \in \mathcal{D}([0,T])$. *Then* $\tilde{v}_{ih}^-(\mathbf{x},t)\, w_{jh}(\mathbf{x})\, \varphi(t)$ *strongly converges to* $\tilde{v}_i(\mathbf{x},t)\, w_j(\mathbf{x})\, \varphi(t)$ *in* $L^2(Q)$, $i,j = 1,\cdots,d$, *where we denote* $\tilde{\mathbf{v}}_h^- = (\tilde{v}_{1h}^-,\cdots,\tilde{v}_{dh}^-)$, $\mathbf{w}_h = (w_{1h},\cdots,w_{dh})$.

Proof. We use the triangle and Hölder's inequalities and Sobolev's injections:

$$
\begin{aligned}
&\|\tilde{v}_{ih}^-\, w_{jh}\, \varphi - \tilde{v}_i^-\, w_j\, \varphi\|_{0,2,Q} \le \|\tilde{v}_{ih}^-\,(w_{jh} - w_j)\,\varphi\|_{0,6,Q} + \|(\tilde{v}_{ih}^- - \tilde{v}_i^-)\, w_j\, \varphi\|_{0,2,Q} \\
&\le C_1\, \left(\|\tilde{v}_{ih}^-\|_{0,3,Q}\, \|w_{jh} - w_j\|_{0,6,\Omega} + \|\tilde{v}_{ih}^- - \tilde{v}_i^-\|_{0,3,Q}\, \|w_j\|_{0,2,\Omega} \right) \|\varphi\|_{0,\infty,(0,T)} \\
&\le C_2\, \left(\|w_{jh} - w_j\|_{1,2,\Omega} + \|\tilde{v}_{ih}^- - \tilde{v}_i^-\|_{0,3,Q} \right),
\end{aligned}
$$

where C_1 and C_2 are constants that do not depend on h and Δt. □

We are now in a position to state the.

Theorem 10.2. *Assume that*

- *the family of triangulations* $(\mathcal{T}_h)_{h>0}$ *is regular;*
- *hypotheses 9.i and 9.ii hold;*
- $\mathbf{f} \in L^2(\mathbf{W}_D(\Omega)')$ *and* $\mathbf{v}_0 \in \mathbf{L}^2(\Omega)$*;*
- $\mathbf{v}_{0h}$ *is given by (10.7).*

Then the sequence of discrete variational problems $(\mathcal{VP})_{h>0}$ *converges to the variational problem* $\mathcal{VP}$. *More specifically, the sequence* $((\mathbf{v}_h, p_h))_{h>0}$ *contains a subsequence* $((\mathbf{v}_{h'}, p_{h'}))_{h'>0}$*that is weakly convergent in* $L^2(\mathbf{H}^1) \times H^{-1}(\mathbf{L}^2)$ *to a weak solution* $(\mathbf{v}, p)$ *of the unsteady Navier–Stokes equations (10.1). Moreover* $(\mathbf{v}_{h'})_{h'>0}$ *is weakly-* convergent in* $L^\infty(\mathbf{L}^2)$ *to* $\mathbf{v}$, *strongly in* $L^2(\mathbf{H}^s)$ *for* $0 \le s < 1$, *and the primitives in time of the pressures* $(p_{h'})_{h'>0}$ *are weakly-* convergent in* $L^\infty(L^2)$ *to a primitive in time of the pressure* p.

If the solution of the problem (10.2) is unique, then the whole sequence converges to it.

Proof. We proceed by steps.

STEP 1. *Extraction of convergent subsequences.* Observe that by (10.7),

$$
\|\mathbf{v}_{0h}\|_{\mathbf{L}^2(\Omega)} \le \|\mathbf{v}_0\|_{\mathbf{L}^2(\Omega)}. \tag{10.23}
$$

Then, by estimates (10.9) and (10.10), $\mathbf{v}_h$ is uniformly bounded in $L^2(\mathbf{H}^1)$, in $L^\infty(\mathbf{L}^2)$, and in $N^{1/4,2}(\mathbf{L}^2)$. By Theorem A.1, the injection $H^1(\Omega) \hookrightarrow H^s(\Omega)$ is compact for $0 \le s < 1$. Applying Lemma A.10 with $X = \mathbf{H}^1(\Omega)$, $E = \mathbf{H}^s(\Omega)$, and $Y = \mathbf{L}^2(\Omega)$, it follows that the sequence $(\mathbf{v}_h)_{h>0}$ is compact in $L^2(\mathbf{H}^s)$ for $0 \le s < 1$.

By estimate (10.11), the sequence $(P_h)_{h>0}$ is uniformly bounded in $L^2(L^2)$. Then the sequence $((\mathbf{v}_h, P_h))_{h>0}$ contains a subsequence (that we still denote in the same way) such that $(\mathbf{v}_h)_{h>0}$ is strongly convergent in $L^2(\mathbf{H}^s)$ to some $\mathbf{v}$, for any $0 \le s < 1$, weakly in $L^2(\mathbf{H}^1)$ and weakly-* in $L^\infty(\mathbf{L}^2)$, and $(P_h)_{h>0}$ is weakly-* convergent in $L^\infty(L^2)$ to some P. We prove in the sequel that the pair $(\mathbf{v}, \partial_t P)$ is a weak solution of Navier–Stokes equations (10.2) in the sense of Definition (10.1).

Also, by (10.9) the sequence $\tilde{\mathbf{v}}_h$ is uniformly bounded in $L^2(\mathbf{H}^1)$ and in $L^\infty(\mathbf{L}^2)$. Then, it contains a subsequence (that we may assume to be a subsequence of the preceding one) weakly convergent in $L^2(\mathbf{H}^1)$ and weakly-* convergent in $L^\infty(\mathbf{L}^2)$ to some $\tilde{\mathbf{v}}$.

Both limit functions $\mathbf{v}$ and $\tilde{\mathbf{v}}$ are equal. Indeed,

$$
\begin{aligned}
\|\mathbf{v}_h - \tilde{\mathbf{v}}_h\|^2_{L^2(\mathbf{L}^2)} &= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left\| \frac{t_{n+1}-t}{\Delta t}\mathbf{v}_h^n + \frac{t-t_n}{\Delta t}\mathbf{v}_h^{n+1} - \mathbf{v}_h^{n+1}\right\|^2_{0,2,\Omega} dt \\
&\le \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \frac{t_{n+1}-t}{\Delta t} \|\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\|^2_{0,2,\Omega}\, dt \\
&\le \Delta t \sum_{n=0}^{N-1} \|\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\|^2_{0,2,\Omega} \le \Delta t\, \|\mathbf{v}_0\|_{1,2,\Omega} + \frac{\Delta t}{2\nu}\|\mathbf{f}\|^2_{L^2(\mathbf{W}_D(\Omega))}.
\end{aligned}
$$

This implies that $\tilde{\mathbf{v}}_h$ strongly converges to $\mathbf{v}$ in $L^2(\mathbf{H}^s)$, $0 \le s < 1$. Similarly, $\tilde{\mathbf{v}}_h^-$ contains a subsequence strongly convergent to $\mathbf{v}$ $L^2(\mathbf{H}^s)$, $0 \le s < 1$. Indeed,

$$
\begin{aligned}
\|\mathbf{v}_h - \tilde{\mathbf{v}}_h^-\|^2_{L^2(\mathbf{L}^2)} &\le \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left\| \frac{t_{n+1}-t}{\Delta t}\mathbf{v}_h^n + \frac{t-t_n}{\Delta t}\mathbf{v}_h^{n+1} - \mathbf{v}_h^{n}\right\|^2_{0,2,\Omega} dt \\
&\le \Delta t\, \|\mathbf{v}_0\|_{1,2,\Omega} + \frac{\Delta t}{2\nu}\|\mathbf{f}\|^2_{L^2(\mathbf{W}_D(\Omega))}.
\end{aligned}
$$

As the injection of $H^1(\Omega)$ in $L^r(\Omega)$ is compact if $1 \le r < 2d/(d-2)$, by the same argument we may assume that $\tilde{\mathbf{v}}_h$ and $\tilde{\mathbf{v}}_h^-$ strongly converge to $\mathbf{v}$ in $L^2(\mathbf{L}^r)$, $1 \le r < 2d/(d-2)$.

STEP 2. *Limit of the momentum conservation equation.* To pass to the limit in the momentum conservation equation in (10.15) we reformulate it as

$$
\begin{aligned}
&- \int_0^T (\mathbf{v}_h(t), \mathbf{w}_h)_\Omega \varphi'(t)\, dt - (\mathbf{v}_{h0}, \mathbf{w}_h)_\Omega\, \varphi(0) + \int_0^T b(\tilde{\mathbf{v}}_h^-(t); \tilde{\mathbf{v}}_h(t), \mathbf{w}_h)\, \varphi(t)\, dt \\
&+ \int_0^T a(\tilde{\mathbf{v}}_h(t), \mathbf{w}_h)\, \varphi(t)\, dt + \int_0^T c(\tilde{\mathbf{v}}_h(t); \mathbf{w}_h)\, \varphi(t)\, dt \\
&+ \int_0^T \langle G(\tilde{\mathbf{v}}_h(t)), \mathbf{w}_h\rangle\, \varphi(t)\, dt + \int_0^T (P_h(t), \nabla\cdot\mathbf{w}_h)_\Omega \varphi'(t)\, dt \\
&= \int_0^T \langle \tilde{\mathbf{f}}_h(t), \mathbf{w}_h\rangle\, \varphi(t)\, dt, \quad \text{for all } \mathbf{w} \in \mathbf{W}_h,
\end{aligned} \tag{10.24}
$$

for any function $\varphi \in \mathscr{D}([0, T])$ such that $\varphi(T) = 0$.

Time derivative term. Let $\mathbf{w} \in \mathbf{W}_D(\Omega)$. By Hypothesis 9.i, there exists a sequence $(\mathbf{w}_h)_{h>0}$ such that $\mathbf{w}_h \in \mathbf{W}_h$ that converges to $\mathbf{w}$ in $\mathbf{W}_h$ in $\mathbf{W}_D(\Omega)$. By Lemma 10.2, the sequences $\tilde{\mathbf{v}}_h^-$ and $\tilde{\mathbf{v}}_h$ strongly converge to $\mathbf{v}$ in $L^3(Q_T)$. Then

$$\lim_{(h,\Delta t)\to 0} \int_0^T (\mathbf{v}_h(t), \mathbf{w}_h)_\Omega \varphi'(t)\, dt = \int_0^T (\mathbf{v}(t), \mathbf{w})_\Omega \varphi'(t)\, dt.$$

Convection term. To pass to the limit in the convection term, observe that by the Green's formula (6.26),

$$(\tilde{\mathbf{v}}_h^-(t)\cdot\nabla\mathbf{w}_h, \tilde{\mathbf{v}}_h(t))_\Omega = -(\tilde{\mathbf{v}}_h^-(t)\cdot\nabla\tilde{\mathbf{v}}_h(t), \mathbf{w}_h)_\Omega - (\nabla\cdot\tilde{\mathbf{v}}_h^-(t), \mathbf{w}_h\cdot\tilde{\mathbf{v}}_h(t)) \text{ a.e. in } (0,T),$$

and then

$$b(\tilde{\mathbf{v}}_h^-(t); \tilde{\mathbf{v}}_h(t), \mathbf{w}_h) = (\tilde{\mathbf{v}}_h^-(t)\cdot\nabla\tilde{\mathbf{v}}_h(t), \mathbf{w}_h)_\Omega - \frac{1}{2}(\nabla\cdot\tilde{\mathbf{v}}_h^-(t), \mathbf{w}_h\cdot\tilde{\mathbf{v}}_h(t)) \text{ a.e. in } (0,T).$$

By Lemma 10.3, as $\nabla\tilde{\mathbf{v}}_h(t)$ weakly converges to $\nabla\mathbf{v}$ in $L^2(Q)$,

$$\lim_{(h,\Delta t)\to 0} \int_0^T (\tilde{\mathbf{v}}_h^-(t)\cdot\nabla\tilde{\mathbf{v}}_h(t), \mathbf{w}_h)_\Omega \varphi(t)\, dt = \int_0^T (\mathbf{v}(t)\cdot\nabla\mathbf{v}(t), \mathbf{w})_\Omega \varphi(t)\, dt,$$

and similarly

$$\lim_{(h,\Delta t)\to 0} \int_0^T (\nabla\cdot\tilde{\mathbf{v}}_h^-(t), \mathbf{w}_h\cdot\tilde{\mathbf{v}}_h(t))\varphi(t)\, dt = \int_0^T (\nabla\cdot\mathbf{v}(t), \mathbf{w}\cdot\mathbf{v}(t))_\Omega \varphi(t)\, dt.$$

Consequently,

$$\lim_{(h,\Delta t)\to 0} \int_0^T b(\tilde{\mathbf{v}}_h^-(t); \tilde{\mathbf{v}}_h(t), \mathbf{w}_h)_\Omega \varphi(t)\, dt = \int_0^T b(\mathbf{v}(t); \mathbf{v}(t), \mathbf{w})_\Omega \varphi(t)\, dt.$$

Diffusion terms. As $\tilde{\mathbf{v}}_h(t)$ is weakly convergent to $\mathbf{v}$ in $L^2(\mathbf{H}^1)$,

$$\lim_{(h,\Delta t)\to 0} \int_0^T a_0(\tilde{\mathbf{v}}_h(t), \mathbf{w}_h)\, \varphi(t)\, dt = \int_0^T a_0(\mathbf{v}(t), \mathbf{w})\, \varphi(t)\, dt.$$

Next, by (9.36),

$$\begin{aligned}
&\left| \int_0^T c(\tilde{\mathbf{v}}_h(t); \tilde{\mathbf{v}}_h(t), \mathbf{w}_h)\, \varphi(t)\, dt \right| \\
&\qquad \le C\, h^{2-d/2} \int_0^T \|D(\mathbf{v}_h(t))\|_{0,2,\Omega}^2 \|D(\mathbf{w}_h)\|_{0,2,\Omega}\, |\varphi(t)|\, dt \\
&\qquad \le C\, h^{2-d/2}\, \|D(\mathbf{v}_h)\|_{L^2(\mathbf{L}^2)}^2 \|D(\mathbf{w}_h)\|_{0,2,\Omega}\, \|\varphi\|_{0,\infty,(0,T)},
\end{aligned}$$

and then

$$\lim_{(h,\Delta t)\to 0} \int_0^T c(\tilde{\mathbf{v}}_h(t); \tilde{\mathbf{v}}_h(t), \mathbf{w}_h)\, \varphi(t)\, dt = 0.$$

Wall-law term. Observe that for any $1/2 < s < 1$ the trace application from $H^s(\Omega)$ onto $H^{s-1/2}(\Gamma)$ is continuous and by Sobolev injection (Theorem A.1), $H^{s-1/2}(\Gamma)$ is continuously embedded into $L^{2(d-1)/(d-2s)}(\Gamma)$. Then we may assume, up to a subsequence, that the sequence $(\mathbf{v}_h)_{h>0}$ is strongly convergent to $\mathbf{v}$ in $L^2(0,T;\mathbf{L}^p(\Gamma_n))$ for any $1 \le p < 4$ for either $d = 2$ or $d = 3$. We bound

$$\left| \int_0^T \langle G(\mathbf{v}_h(t)), \mathbf{w}_h\rangle\, \varphi(t)\, dt - \int_0^T \langle G(\mathbf{v}(t)), \mathbf{w}\rangle\, \varphi(t)\, dt \right| \le$$
$$\left| \int_0^T (g(\mathbf{v}_h(t)) - g(\mathbf{v}(t)), \mathbf{w}_h)_{\Gamma_n}\, \varphi(t)\, dt \right| + \left| \int_0^T \langle G(\mathbf{v}(t)), \mathbf{w}_h - \mathbf{w}\rangle\, \varphi(t)\, dt \right|$$

Due to estimate (6.37),

$$\begin{aligned}
&\left| \int_0^T (g(\mathbf{v}_h(t)) - g(\mathbf{v}(t)), \mathbf{w}_h)_{\Gamma_n}\, \varphi(t)\, dt \right| \\
&\le \|\varphi\|_{0,\infty,(0,T)} \int_0^T \|g(\mathbf{v}_h(t)) - g(\mathbf{v}(t))\|_{0,3/2,\Gamma_n}\, \|\mathbf{w}_h\|_{0,3,\Gamma_n}\, dt \\
&\le C\, \|\varphi\|_{0,\infty,(0,T)}\, \|\mathbf{w}_h\|_{0,3,\Gamma_n} \int_0^T (1 + \|\mathbf{v}_h(t)\|_{0,3,\Gamma_n} + \|\mathbf{v}(t)\|_{0,3,\Gamma_n})\, \|\mathbf{v}_h(t) - \mathbf{v}(t)\|_{0,3,\Gamma_n}\, dt \\
&\le C\, \|\varphi\|_{0,\infty}\, \|\mathbf{w}_h\|_{0,3,\Gamma_n}\, (\sqrt{T} + \|\mathbf{v}_h(t)\|_{L^2(\mathbf{H}^1)} + \|\mathbf{v}(t)\|_{L^2(\mathbf{H}^1)})\, \|\mathbf{v}_h(t) - \mathbf{v}(t)\|_{L^2(\mathbf{L}^3(\Gamma_n))}.
\end{aligned}$$

Also, by (6.38),

$$\begin{aligned}
&\left| \int_0^T \langle G(\mathbf{v}(t)), \mathbf{w}_h - \mathbf{w}\rangle\, \varphi(t)\, dt \right| \le \\
&C\, \|\mathbf{w}_h - \mathbf{w}\|_{1,2,\Omega}\, \|\varphi\|_{0,\infty,(0,T)} \int_0^T \left(1 + \|\mathbf{v}(t)\|^2_{1,2,\Omega}\right) dt \\
&\le C \left(T + \|\mathbf{v}\|^2_{L^2(H^1)}\right) \|\mathbf{w}_h - \mathbf{w}\|_{1,2,\Omega}\, \|\varphi\|_{0,\infty,(0,T)}.
\end{aligned}$$

Consequently,

$$\lim_{(h,\Delta t)\to 0} \int_0^T \langle G(\mathbf{v}_h(t)), \mathbf{w}_h\rangle\, \varphi(t)\, dt = \int_0^T \langle G(\mathbf{v}(t)), \mathbf{w}\rangle\, \varphi(t)\, dt.$$

Pressure term. Observe that the product $\nabla\cdot\mathbf{w}_h(\mathbf{x})\, \varphi'(t)$ is strongly convergent in $L^2(L^2)$ to $\nabla\cdot\mathbf{w}(\mathbf{x})\, \varphi'(t)$. As $(P_h)_{h>0}$ is weakly-* convergent in $L^\infty(L^2)$ to P,

$$\lim_{(h,\Delta t)\to 0} \int_0^T (P_h, \nabla \cdot \mathbf{w}_h(\mathbf{x}))_\Omega \, \varphi'(t) \, dt = \int_0^T (P, \nabla \cdot \mathbf{w}(\mathbf{x}))_\Omega \, \varphi'(t) \, dt.$$

Also, as $\tilde{\mathbf{f}}_h$ strongly converges to $\mathbf{f}$ in $L^2(\mathbf{W}_D(\Omega)')$,

$$\lim_{(h,\Delta t)\to 0} \int_0^T \langle \tilde{\mathbf{f}}_h(t), \mathbf{w}_h \rangle \varphi(t) \, dt = \int_0^T \langle \mathbf{f}(t), \mathbf{w}_h \rangle \varphi(t) \, dt.$$

Further, as $\mathbf{v}_0$ is given by (10.7) and $\mathbf{w}_h$ converges to $\mathbf{w}$ in $\mathbf{L}^2(\Omega)$,

$$\lim_{h\to 0} (\mathbf{v}_{0h}, \mathbf{w}_h)_\Omega = \lim_{h\to 0} (\mathbf{v}_0, \mathbf{w}_h)_\Omega = (\mathbf{v}_0, \mathbf{w})_\Omega = \langle \mathbf{v}_0, \mathbf{w} \rangle,$$

where the last equality holds because $\mathbf{v}_0 \in \mathbf{L}^2(\Omega) \hookrightarrow \mathbf{W}_D(\Omega)'$.

STEP 3. *Limit of the continuity equation.* To pass to the limit in the continuity equation, let us consider some function $q \in L^2_0(\Omega)$, and some interpolate $q_h \in M_h$, strongly convergent in $L^2_0(\Omega)$ to q. As $\nabla\cdot\mathbf{v}_h$ weakly converges to $\nabla\cdot\mathbf{v}$ in $L^2(L^2)$,

$$\int_0^T (\nabla \cdot \mathbf{v}(t), q)_\Omega \, \varphi(t) \, dt = \lim_{(h,\Delta t)\to 0} \int_0^T (\nabla \cdot \mathbf{v}_h(t), q_h)_\Omega \, \varphi(t) \, dt.$$

Consequently,

$$\int_0^T (\nabla \cdot \mathbf{v}(t), q)_\Omega \, \varphi(t) \, dt = 0, \quad \forall q \in L^2_0(\Omega), \ \forall \varphi \in \mathscr{D}(0,T). \tag{10.25}$$

As $\nabla \cdot \mathbf{v}_h$ weakly converges to $\nabla \cdot \mathbf{v}$ in $L^2(L^2)$,

$$\begin{aligned} \int_0^T (\nabla \cdot \mathbf{v}(t), 1)_\Omega \, \varphi(t) \, dt &= \lim_{(h,\Delta t)\to 0} \int_0^T \left(\int_\Omega \nabla \cdot \mathbf{v}_h(\mathbf{x},t) \, d\mathbf{x} \right) \varphi(t) \, dt \\ &= \int_0^T \left(\int_\Gamma \mathbf{v}_h(\mathbf{x},t) \cdot \mathbf{n}(\mathbf{x}) \, d\Gamma(\mathbf{x}) \right) \varphi(t) \, dt = 0, \end{aligned}$$

because $\mathbf{v}_h \cdot \mathbf{n} = 0$ on Γ as $\mathbf{v}_h \in \mathbf{W}_h$. Thus $(\nabla \cdot \mathbf{v}(t), 1)_\Omega = 0$ a.e. in $(0,T)$, and (10.25) holds for all $q \in L^2(\Omega)$. Then, by Lemma A.11, we deduce that

$$\nabla \cdot \mathbf{v} = 0 \quad \text{a.e. in} \quad \Omega \times (0,T).$$

STEP 4. *Conclusion.* As a consequence of the preceding analysis, $\mathbf{v}$ belongs to $L^2(\mathbf{W}_{Div}(\Omega)) \cap L^\infty(\mathbf{L}^2)$, P belongs to $L^2(L^2)$, and the pair $(\mathbf{v}, P)$ satisfies (10.2). Thus, the pair $(\mathbf{v}, \partial_t P)$ is a weak solution of the Navier–Stokes problem (10.1) in the sense of Definition 10.1. As P_h weakly converges to P in $L^2(L^2)$, then $p_h = \partial_t P_h$ weakly converges to $p = \partial_t P$ in $H^{-1}(L^2)$.

If the solution of Navier–Stokes equations (10.2) is unique, then the whole sequence converges to it, as this proof is based upon a compactness argument. This is proved by reductio ad absurdum, similarly to the analogous statement in Theorem 9.5. □

10.5 Error Estimates

We next prove error estimates for the approximation of the unsteady Navier–Stokes equations by the discrete SM (10.5) and (10.6). We obtain these estimates for general flow regimes and not just for convection-dominated flows as in the steady case.

To state this result, we need a few technical results. We start with a discrete version of the Gronwall Lemma.

Lemma 10.4. *Let* $(\alpha_n)_{n=0}^N$, $(\beta_n)_{n=0}^N$ *be two sequences of nonnegative real numbers such that*

$$(1 - C_n\, \Delta t)\alpha_{n+1} \le (1 + D_n\, \Delta t)\, \alpha_n + \beta_n, \;\; \textit{for} \;\; n = 0, 1, \cdots, N-1$$

for two sequences of nonnegative numbers $(C_n)_{n=0}^N$, $(D_n)_{n=0}^N$.

Assume $\Delta t \le 1/(2\, \max(C_1, \cdots, C_N))$*; then*

$$\max_{n=0,1,\cdots,N} \alpha_n \le \exp(2\, \Delta t S_N)\, \alpha_0 + 2\, \exp(2\, \Delta t S_{N-1}) \sum_{n=0}^{N-1} \beta_n \tag{10.26}$$

where $S_N = \sum_{n=0}^{N-1}(C_n + D_n)$*. In particular, if* $C_n = D_n = C$ *for* $n = 0, \cdots, N-1$ *and* $\Delta t < 1/(2C)$*, then*

$$\max_{n=0,1,\cdots,N} \alpha_n \le e^{4CT}\, \alpha_0 + 2\, e^{4CT} \sum_{n=0}^{N-1} \beta_n \tag{10.27}$$

Proof. As $\Delta t \le 1/(2\max(C_1, \cdots, C_N))$, $1 - C_n\, \Delta t \ge 1/2$ for all $n = 0, 1, \cdots, N-1$. Then, denoting $E_n = C_n + D_n$,

$$\begin{aligned}
\alpha_{n+1} &\le \frac{1 + D_n\, \Delta t}{1 - C_n\, \Delta t}\alpha_n + \frac{1}{1 - C_n\, \Delta t}\beta_n \le \left(1 + \frac{E_n}{1 - C_n\, \Delta t}\Delta t\right)\alpha_n + 2\,\beta_n \\
&\le (1 + 2E_n \Delta t)\, \alpha_n + 2\,\beta_n \le e^{2E_n \Delta t}\, \alpha_n + 2\,\beta_n \\
&\le e^{2(E_n + E_{n-1})\Delta t}\, \alpha_{n-1} + 2\,[\beta_n + e^{2E_{n-1}\Delta t}\, \beta_{n-1}]
\end{aligned}$$

$$\begin{aligned}
&\vdots \\
&\le e^{2(E_n+E_{n-1}+\cdots+E_0)\Delta t}\,\alpha_0+2\left[\beta_n+e^{2E_{n-1}\Delta t}\,\beta_{n-1}+\cdots+e^{2(E_{n-1}+\cdots+E_0)\Delta t}\,\beta_0\right] \\
&\le e^{2(E_n+E_{n-1}+\cdots+E_0)\Delta t}\,\alpha_0+2\,e^{2(E_{n-1}+\cdots+E_0)\Delta t}\sum_{k=0}^{n}\beta_k .
\end{aligned}$$

Thus, (10.26) follows. From here, (10.27) follows considering that $N\,\Delta t = T$. □

We also need some properties of the form c:

Lemma 10.5. *The form c defined by (9.33) is monotone on $\mathbf{W}^{1,3}(\Omega)$, and there exist two constants C_1, $C_2 > 0$ such that*

$$c(\mathbf{w};\mathbf{w}-\mathbf{u})-c(\mathbf{u};\mathbf{w}-\mathbf{u}) \ge C_1\,h_{\min}^2\,\|D(\mathbf{w}-\mathbf{u})\|_{0,3,\Omega}^3 \tag{10.28}$$

for all $\mathbf{w},\ \mathbf{u}\in\mathbf{W}^{1,3}(\Omega)$,

$$\begin{aligned}
|c(\mathbf{w}_h;\mathbf{z}_h)-c(\mathbf{u}_h;\mathbf{z}_h)| \le C_2\,h^{2-d/2}\,(\|D(\mathbf{w}_h)\|_{0,2,\Omega}+\|D(\mathbf{u}_h)\|_{0,2,\Omega})\|D(\mathbf{z}_h)\|_{0,2,\Omega} \\
\times\,\|D(\mathbf{w}_h-\mathbf{u}_h)\|_{0,2,\Omega},\ \textit{for all}\ \mathbf{u}_h,\,\mathbf{w}_h,\,\mathbf{z}_h\in\mathbf{W}_h. \quad (10.29)
\end{aligned}$$

Proof. Let us define the functional

$$J(\mathbf{w})=\frac{2}{3}\int_\Omega \Pi(\mathbf{x})\,|D(\mathbf{w}(\mathbf{x}))|^3\,d\mathbf{x}=\frac{2}{3}\int_\Omega \Pi(\mathbf{x})\left(\sum_{i,j=1}^{d}|d_{ij}(\mathbf{w}(\mathbf{x}))|^2\right)^{3/2}d\mathbf{x},$$

for $\mathbf{w}\in\mathbf{W}^{1,3}(\Omega)$, with $d_{ij}(\mathbf{w})=\frac{1}{2}(\partial_i\mathbf{w}_j+\partial_j\mathbf{w}_i)$, where Π is the piecewise constant function defined by

$$\Pi(\mathbf{x})=C_S^2\,h_K^2\ \text{ for any }\ \mathbf{x}\in K.$$

The functional J is the cube of a weighted $L^3(\Omega)$ norm of $D\mathbf{w}$. Then it is a convex and twice Gâteaux-differentiable functional. Its Gâteaux derivative is given by

$$\begin{aligned}
\langle J'(\mathbf{w}),\mathbf{z}\rangle &= \int_\Omega \Pi(\mathbf{x})\left(\sum_{i,j=1}^{d}|d_{ij}(\mathbf{w}(\mathbf{x}))|^2\right)^{1/2}\sum_{i,j=1}^{d} d_{ij}(\mathbf{w}(\mathbf{x}))d_{ij}(\mathbf{z}(\mathbf{x}))\,d\mathbf{x} \\
&= \int_\Omega \Pi(\mathbf{x})\,|D(\mathbf{w}(\mathbf{x}))|\,D(\mathbf{w}(\mathbf{x})):D(\mathbf{z}(\mathbf{x}))\,d\mathbf{x}=c(\mathbf{w};\mathbf{z}).
\end{aligned}$$

As J is convex, then J' is monotone, by Lemma A.21.

Next, consider $\mathbf{w},\mathbf{u},\mathbf{z}\in\mathbf{W}^{1,3}(\Omega)$. Let $\mathbf{w}^t=t\mathbf{w}+(1-t)\mathbf{u}$. Then,

$$c(\mathbf{w};\mathbf{z}) - c(\mathbf{u};\mathbf{z}) = \langle J'(\mathbf{w}),\mathbf{z}\rangle - \langle J'(\mathbf{u}),\mathbf{z}\rangle = \left\langle \int_0^1 \frac{d}{dt} J'(\mathbf{w}^t), \mathbf{z}\, dt \right\rangle$$
$$= \int_\Omega \Pi(\mathbf{x}) \left(\int_0^1 \frac{D(\mathbf{w}^t(\mathbf{x})) : D((\mathbf{w}-\mathbf{u})(\mathbf{x}))}{2|D(\mathbf{w}^t(\mathbf{x}))|}\, D(\mathbf{w}^t(\mathbf{x})) : D(\mathbf{z}(\mathbf{x}))\, dt \right) d\mathbf{x}$$
$$+ \int_\Omega \Pi(\mathbf{x}) \int_0^1 |D(\mathbf{w}^t(\mathbf{x}))|\, D((\mathbf{w}-\mathbf{u})(\mathbf{x})) : D(\mathbf{z}(\mathbf{x}))\, dt\, d\mathbf{x}. \tag{10.30}$$

Let $\mathbf{z} = \mathbf{w} - \mathbf{u}$, $\mathbf{z}^t = t\mathbf{w} + (1-t)\mathbf{u}$; then (10.30) yields

$$c(\mathbf{w};\mathbf{z}) - c(\mathbf{u};\mathbf{z}) \geq \int_\Omega \Pi(\mathbf{x}) \left(\int_0^1 |D(\mathbf{z}^t(\mathbf{x}))|\, dt \right) |D(\mathbf{z}(\mathbf{x}))|^2\, d\mathbf{x}. \tag{10.31}$$

By the finite-dimensional equivalence of norms,

$$\int_0^1 |D(\mathbf{z}^t(\mathbf{x}))|\, dt \geq C \int_0^1 \sum_{i,j=1}^d |d_{ij}(\mathbf{z}^t(\mathbf{x}))|\, dt \geq \frac{C}{4} \sum_{i,j=1}^d |d_{ij}(\mathbf{z}(\mathbf{x}))| \geq C'\, |D(\mathbf{z}(\mathbf{x}))|,$$

where we have used that for $a, b \in \mathbb{R}$, $\displaystyle\int_0^1 |t\, a + (1-t)\, b|\, dt \geq \frac{1}{4}\, |a-b|$. Then from (10.31) we deduce (10.28) by

$$c(\mathbf{w};\mathbf{z}) - c(\mathbf{u};\mathbf{z}) \geq \frac{1}{2} \int_\Omega \Pi(\mathbf{x})\, |D(\mathbf{z}(\mathbf{x}))|^3\, d\mathbf{x} \geq \frac{1}{2}\, C_S^2\, h_{\min}\, \|D(\mathbf{z})\|_{0,3,\Omega}.$$

To prove (10.29), let $\mathbf{u}_h, \mathbf{w}_h, \mathbf{z}_h \in \mathbf{W}_h$; from (10.30) we obtain

$$|c(\mathbf{w};\mathbf{z}) - c(\mathbf{u};\mathbf{z})| \leq$$
$$\leq C_S^2\, \|D(\mathbf{w}_h - \mathbf{u}_h)\|_{0,\infty,\Omega} \sum_{K\in\mathcal{T}_h} h_K^2 \left\| \int_0^1 |D(\mathbf{w}_h^t)|\, dt \right\|_{0,2,K} \|D(\mathbf{z}_h)\|_{0,2,K}$$
$$\leq C\, \|D(\mathbf{w}_h - \mathbf{u}_h)\|_{0,2,\Omega} \sum_{K\in\mathcal{T}_h} h_K^{2-d/2} \left\| \int_0^1 |D(\mathbf{w}_h^t)|\, dt \right\|_{0,2,K} \|D(\mathbf{z}_h)\|_{0,2,K}$$
$$\leq C\, h^{2-d/2}\, \|D(\mathbf{w}_h - \mathbf{u}_h)\|_{0,2,\Omega}\, \left(\|D(\mathbf{u}_h)\|_{0,2,\Omega} + \|D(\mathbf{u}_h)\|_{0,2,\Omega} \right)\, \|D(\mathbf{z}_h)\|_{0,2,\Omega}$$

for some constant C, using the inverse inequalities (9.20). We thus obtain (10.29). □

We shall use the following notation:

$$d_{l^p(B)}(\mathbf{v}, \mathbf{W}_h) = \left[\sum_{n=1}^N \Delta t\, d_B(\mathbf{v}(t_n), \mathbf{W}_h)^p \right]^{1/p},\ d_{l^\infty(B)}(\mathbf{v}, \mathbf{W}_h) = \max_{n=1,\cdots,N} d_B(\mathbf{v}(t_n), \mathbf{W}_h),$$

where B is a Banach space such that $\mathbf{W}_h \subset B$.

Theorem 10.3. *Assume that the family of triangulations* $(\mathscr{T}_h)_{h>0}$ *is uniformly regular, that the data verify* $\mathbf{f} \in C^0(L^2)$, $\partial_t \mathbf{f} \in L^2(\mathbf{W}_D(\Omega)')$, $\mathbf{v}_0 \in \mathbf{W}^{1,3}(\Omega)$, *and that the unsteady Navier–Stokes equations (10.2) admit a solution* $(\mathbf{v}, p) \in C^0(\mathbf{W}^{1,3}) \times C^0(L^2)$ *such that* $\partial_t^2 \mathbf{v} \in L^2(\mathbf{L}^2)$. *Then the sequence* $((\mathbf{v}_h, p_h))_{h>0}$ *given by the discrete unsteady SM (10.5) and (10.6) satisfies the error estimates:*

$$\|\mathbf{v} - \mathbf{v}_h\|_{l^\infty(\mathbf{L}^2)} + \|\mathbf{v} - \mathbf{v}_h\|_{l^2(\mathbf{H}^1)} \leq M(h, \Delta t) + d_{l^\infty(\mathbf{L}^2)}(\mathbf{v}, \mathbf{W}_h), \quad (10.32)$$

$$\|P - P_h\|_{l^\infty(L^2)} \leq M(h, \Delta t) + C\, d_{l^\infty(L^2)}(P, M_h), \quad (10.33)$$

where $P(\mathbf{x}, t) = \int_0^t p(\mathbf{x}, s)\, ds, \quad P_h(\mathbf{x}, t) = \int_0^t \tilde{p}_h(\mathbf{x}, s)\, ds,$

$$M(h, \Delta t) = C\ [d_{0,2,\Omega}(\mathbf{v}_0, \mathbf{W}_h) + \frac{1}{\Delta t}\, d_{l^2(\mathbf{L}^2)}(\mathbf{v}, \mathbf{W}_h) + d_{l^2(\mathbf{H}^1)}(\mathbf{v}, \mathbf{W}_h) + d_{l^2(L^2)}(p, M_h) + h^{2-d/2} + \Delta t], \quad (10.34)$$

and C *is a constant independent of* h *and* Δt.

Proof. We proceed by steps.

STEP 1. *Error equation.* As $\nabla \cdot \mathbf{v}(t) = 0\ \forall t \in [0, T]$, by Lemma 9.3, the Stokes projection $\overline{\mathbf{v}}_h^n$ of $\mathbf{v}(t_n)$ on $\mathbf{W}_h$ satisfies $\|D(\mathbf{v}(t_n) - \overline{\mathbf{v}}_h^n)\|_{0,2,\Omega} \leq C\, d_{1,2,\Omega}(\mathbf{v}(t_n), \mathbf{W}_h)$ for some constant $C > 0$ independent of h and Δt, and $(\nabla \cdot \overline{\mathbf{v}}_h^n, q_h)_\Omega = 0$ for all $q_h \in M_h$. Also, let $\overline{p}_h^n \in M_h$ such that $\|p(t_n) - \overline{p}_h^n\|_{0,2,\Omega} = d_{0,2,\Omega}(p(t_n), M_h)$.

We shall denote along this proof by C the constants that will appear in the estimates, independent of h and Δt, but eventually depending on the parameters T and ν, the data $\mathbf{f}$ and $\mathbf{v}_0$, and the solution $\mathbf{v}$.

Define the errors in velocity and pressure by $\mathbf{e}_h^n = \mathbf{v}_h^n - \overline{\mathbf{v}}_h^n$, $\lambda_h^n = p_h^n - \overline{p}_h^n$. As $\partial_t^2 \mathbf{v} \in L^2(\mathbf{L}^2)$, then $\partial_t \mathbf{v} \in C^0([0, T], \mathbf{L}^2)$. As $\mathbf{f} \in C^0(\mathbf{L}^2)$, $(\mathbf{v}, p) \in C^0(\mathbf{W}^{1,3}) \times C^0(L^2)$, then the unsteady Navier–Stokes equations (10.2) yield

$$\begin{cases} (\partial_t \mathbf{v}(t), \mathbf{w})_\Omega + b(\mathbf{v}(t); \mathbf{v}(t), \mathbf{w}) + a(\mathbf{v}(t), \mathbf{w}) - (p(t), \nabla \cdot \mathbf{w})_\Omega \\ \qquad + \langle G(\mathbf{v}(t)), \mathbf{w} \rangle = \langle \mathbf{f}(t), \mathbf{w} \rangle \\ \qquad (\nabla \cdot \mathbf{v}(t), q)_\Omega = 0, \\ \qquad \mathbf{v}(0) = \mathbf{v}_0, \end{cases} \quad (10.35)$$

for all $\mathbf{w} \in \mathbf{W}_D(\Omega)$, $q \in L_0^2(\Omega)$, for all $t \in [0, T]$. Subtracting term by term (10.35) at $t = t_{n+1}$ from (10.6) we obtain, for all $\mathbf{w}_h \in \mathbf{W}_h$, $q_h \in M_h$,

$$\begin{cases} \left(\dfrac{\mathbf{e}_h^{n+1} - \mathbf{e}_h^n}{\Delta t}, \mathbf{w}_h \right)_\Omega + b(\mathbf{v}_h^n; \mathbf{v}_h^{n+1}, \mathbf{w}_h) - b(\overline{\mathbf{v}}_h^n; \overline{\mathbf{v}}_h^{n+1}, \mathbf{w}_h) + a(\mathbf{e}_h^{n+1}, \mathbf{w}_h) \\ \qquad + c(\mathbf{v}_h^{n+1}, \mathbf{w}_h) - c(\overline{\mathbf{v}}_h^{n+1}, \mathbf{w}_h) - (\lambda_h^{n+1}, \nabla \cdot \mathbf{w}_h)_\Omega \\ \qquad + \langle G(\mathbf{v}_h^{n+1}) - G(\overline{\mathbf{v}}_h^{n+1}), \mathbf{w}_h \rangle = \langle \varepsilon_h^{n+1}, \mathbf{w}_h \rangle, \\ \qquad (\nabla \cdot \mathbf{e}_h^{n+1}, q_h)_\Omega = 0, \end{cases}$$

(10.36)

where $\varepsilon_h^{n+1} \in \mathbf{W}_D(\Omega)'$ is the consistency error, defined by

$$\langle \varepsilon_h^{n+1}, \mathbf{w}\rangle = \left(\partial_t \mathbf{v}(t_{n+1}) - \frac{\overline{\mathbf{v}}_h^{n+1} - \overline{\mathbf{v}}_h^n}{\Delta t}, \mathbf{w}\right)_\Omega \tag{10.37}$$
$$+ b(\mathbf{v}(t_{n+1}); \mathbf{v}(t_{n+1}), \mathbf{w}) - b(\overline{\mathbf{v}}_h^n; \overline{\mathbf{v}}_h^{n+1}, \mathbf{w})$$
$$+ a(\mathbf{v}(t_{n+1}) - \overline{\mathbf{v}}_h^{n+1}, \mathbf{w}) - c(\overline{\mathbf{v}}_h^{n+1}, \mathbf{w})$$
$$- (p(t_{n+1}) - \overline{p}_h^{n+1}, \nabla \cdot \mathbf{w})_\Omega + \langle G(\mathbf{v}(t_{n+1})) - G(\overline{\mathbf{v}}_h^{n+1}), \mathbf{w}\rangle$$
$$+ \langle \mathbf{f}^{n+1} - \mathbf{f}(t_{n+1}), \mathbf{w}\rangle.$$

STEP 2. *Velocity estimate.* Set $\mathbf{w}_h = \mathbf{e}_h^{n+1}$ in (10.36). Using

$$2\,(\mathbf{e}_h^{n+1} - \mathbf{e}_h^n, \mathbf{e}_h^{n+1})_\Omega = \|\mathbf{e}_h^{n+1}\|_{0,2,\Omega}^2 - \|\mathbf{e}_h^n\|_{0,2,\Omega}^2 + \|\mathbf{e}_h^{n+1} - \mathbf{e}_h^n\|_{0,2,\Omega},$$

$$b(\mathbf{v}_h^n; \mathbf{v}_h^{n+1}, \mathbf{e}_h^{n+1}) - b(\overline{\mathbf{v}}_h^n; \overline{\mathbf{v}}_h^{n+1}, \mathbf{e}_h^{n+1}) = b(\mathbf{e}_h^n; \overline{\mathbf{v}}_h^{n+1}, \mathbf{e}_h^{n+1})$$

and the monotonicity of G and (10.28), we deduce

$$\|\mathbf{e}_h^{n+1}\|_{0,2,\Omega}^2 - \|\mathbf{e}_h^n\|_{0,2,\Omega}^2 + \|\mathbf{e}_h^{n+1} - \mathbf{e}_h^n\|_{0,2,\Omega}^2 + 2\,\nu\,\Delta t\|D(\mathbf{e}_h^{n+1})\|_{0,2,\Omega}^2$$
$$+ 2\,C_S^2\,h_{\min}^2\,\nu\,\Delta t\|D(\mathbf{e}_h^{n+1})\|_{0,3,\Omega}^3 \le -2\,\Delta t\, b(\mathbf{e}_h^n; \overline{\mathbf{v}}_h^{n+1}, \mathbf{e}_h^{n+1}) + 2\,\Delta t\,\langle \varepsilon_h^{n+1}, \mathbf{e}_h^{n+1}\rangle$$
$$\le C\,\Delta t\,\|\mathbf{e}_h^n\|_{0,2,\Omega}\|D(\overline{\mathbf{v}}_h^{n+1})\|_{0,3,\Omega}\|D(\mathbf{e}_h^{n+1})\|_{0,2,\Omega} + \frac{\nu}{2}\,\Delta t\|D(\mathbf{e}_h^{n+1})\|_{0,2,\Omega}^2$$
$$+ 2\nu^{-1}\,\Delta t\|\varepsilon_h^{n+1}\|_{\mathbf{W}_D(\Omega)'} ^2 \tag{10.38}$$
$$\le C\,\nu^{-1}\Delta t\,\|\mathbf{e}_h^n\|_{0,2,\Omega}^2 + \nu\,\Delta t\|D(\mathbf{e}_h^{n+1})\|_{0,2,\Omega}^2 + 2\nu^{-1}\,\Delta t\|\varepsilon_h^{n+1}\|_{\mathbf{W}_D(\Omega)'}^2,$$

where we have applied Young's inequality and that, as $\mathbf{v} \in C^0(\mathbf{W}^{1,3})$ [11, 12],

$$\|D(\overline{\mathbf{v}}_h^n)\|_{0,3,\Omega} \le C\,\|D(\mathbf{v}(t_n))\|_{0,3,\Omega} \le C\,\|\mathbf{v}\|_{L^\infty(\mathbf{W}^{1,3})}, \quad \text{for all } n = 0, 1, \cdots, N.$$

Then,

$$\|\mathbf{e}_h^{n+1}\|_{0,2,\Omega}^2 + \|\mathbf{e}_h^{n+1} - \mathbf{e}_h^n\|_{0,2,\Omega}^2 + \nu\,\Delta t\|D(\mathbf{e}_h^{n+1})\|_{0,2,\Omega}^2 \tag{10.39}$$
$$\le (1 + C\,\nu^{-1}\Delta t)\|\mathbf{e}_h^n\|_{0,2,\Omega}^2 + 2\,\nu^{-1}\Delta t\,\|\varepsilon_h^{n+1}\|_{\mathbf{W}_D(\Omega)'}^2.$$

We now apply the discrete Gronwall Lemma 10.4 with

$$\alpha_n = \|\mathbf{e}_h^n\|_{0,2,\Omega}^2, \quad \beta_n = 2\,\nu^{-1}\Delta t\,\|\varepsilon_h^{n+1}\|_{\mathbf{W}_D(\Omega)'}^2 \tag{10.40}$$

to deduce

$$\max_{n=0,1,\cdots,N} \|\mathbf{e}_h^n\|_{0,2,\Omega} \le D := C \left[\|\mathbf{e}_h^0\|_{0,2,\Omega} + \left(\sum_{n=0}^{N-1} \Delta t \; \|\varepsilon_h^{n+1}\|^2_{\mathbf{W}_D(\Omega)'} \right)^{1/2} \right], \tag{10.41}$$

where C is a function of ν and T that increases to $+\infty$ as ν decreases to 0 or as T increases to $+\infty$. Summing with respect to n in (10.39), we obtain

$$\begin{aligned}
&\|\mathbf{e}_h^N\|^2_{0,2,\Omega} + \sum_{n=0}^{N-1} \|\mathbf{e}_h^{n+1} - \mathbf{e}_h^n\|^2_{0,2,\Omega} + \nu \sum_{n=0}^{N-1} \Delta t \|D(\mathbf{e}_h^{n+1})\|^2_{0,2,\Omega} \\
&\le \|\mathbf{e}_h^0\|^2_{0,2,\Omega} + \sum_{n=0}^{N-1} \Delta t \; \|\mathbf{e}_h^{n+1}\|^2_{0,2,\Omega} + \sum_{n=0}^{N-1} \Delta t \; \|\varepsilon_h^{n+1}\|^2_{\mathbf{W}_D(\Omega)'} \\
&\le \max_{n=0,\cdots,N} \|\mathbf{e}_h^n\|^2_{0,2,\Omega} \left(1 + \sum_{n=0}^{N-1} \Delta t \right) + \sum_{n=0}^{N-1} \Delta t \; \|\varepsilon_h^{n+1}\|^2_{\mathbf{W}_D(\Omega)'} \le C \; D^2.
\end{aligned} \tag{10.42}$$

STEP 3. *Pressure estimate.* From (10.36),

$$\begin{aligned}
(\lambda_h^{n+1}, \nabla \cdot \mathbf{w}_h) &= \left(\frac{\mathbf{e}_h^{n+1} - \mathbf{e}_h^n}{\Delta t}, \mathbf{w}_h \right)_\Omega + b(\mathbf{v}_h^n; \mathbf{e}_h^{n+1}, \mathbf{w}_h) - b(\mathbf{e}_h^n; \overline{\mathbf{v}}_h^{n+1}, \mathbf{w}_h) \\
&+ a(\mathbf{e}_h^{n+1}, \mathbf{w}_h) + c(\mathbf{v}_h^{n+1}, \mathbf{w}_h) - c(\overline{\mathbf{v}}_h^{n+1}, \mathbf{w}_h) + \langle G(\mathbf{v}_h^{n+1}) - G(\overline{\mathbf{v}}_h^{n+1}), \mathbf{w}_h \rangle \\
&- \langle \varepsilon_h^{n+1}, \mathbf{w}_h \rangle.
\end{aligned}$$

Let $\Lambda_h^n = \sum_{k=0}^{n} \Delta t \lambda_h^{n+1} = \sum_{k=0}^{n} \Delta t \; (p_h^n - \overline{p}_h^n)$. Then, by estimates (6.21), (6.39), (9.36), and (10.29),

$$\begin{aligned}
(\Lambda_h^{n+1}, \nabla \cdot \mathbf{w}_h) &= (\mathbf{e}_h^{n+1} - \mathbf{e}_h^0, \mathbf{w}_h)_\Omega + \sum_{k=0}^{n} \Delta t \left[b(\mathbf{v}_h^n; \mathbf{e}_h^{n+1}, \mathbf{w}_h) - b(\mathbf{e}_h^n; \overline{\mathbf{v}}_h^{n+1}, \mathbf{w}_h) \right] \\
&+ \sum_{k=0}^{n} \Delta t \; a(\mathbf{e}_h^{n+1}, \mathbf{w}_h) + \sum_{k=0}^{n} \Delta t \left[c(\mathbf{v}_h^{n+1}, \mathbf{w}_h) - c(\overline{\mathbf{v}}_h^{n+1}, \mathbf{w}_h) \right] \\
&+ \sum_{k=0}^{n} \Delta t \; \langle G(\mathbf{v}_h^{n+1}) - G(\overline{\mathbf{v}}_h^{n+1}), \mathbf{w}_h \rangle - \sum_{k=0}^{n} \Delta t \langle \varepsilon_h^{n+1}, \mathbf{w}_h \rangle.
\end{aligned}$$

From here, using (10.9),

$$
\begin{aligned}
&\frac{(\Lambda_h^{n+1}, \nabla \cdot \mathbf{w}_h)}{\|\mathbf{w}_h\|_{1,2,\Omega}} \le \|\mathbf{e}_h^{n+1}\|_{0,2,\Omega} + \|\mathbf{e}_h^0\|_{0,2,\Omega} \\
&+ C \sum_{k=0}^{n} \Delta t \, \left(\|\mathbf{v}_h^k\|_{0,2,\Omega} \|D(\mathbf{e}_h^{k+1})\|_{0,2,\Omega} + \|\mathbf{e}_h^n\|_{0,2,\Omega} \|D(\overline{\mathbf{v}}_h^{k+1})\|_{0,2,\Omega} \right) \\
&+ C\, h^{2-d/2} \sum_{k=0}^{n} \Delta t \, \left(\|D(\mathbf{v}_h^{k+1})\|_{0,2,\Omega}^2 + \|D(\overline{\mathbf{v}}_h^{k+1})\|_{0,2,\Omega}^2 \right) \\
&+ C\, h^{2-d/2} \sum_{k=0}^{n} \Delta t \, \left(\|D(\mathbf{v}_h^{k+1})\|_{0,2,\Omega} + \|D(\overline{\mathbf{v}}_h^{k+1})\|_{0,2,\Omega} \right) \|D(\mathbf{e}_h^{k+1})\|_{0,2,\Omega} \\
&+ \sum_{k=0}^{n} \Delta t \, \|\varepsilon_h^{k+1}\|_{\mathbf{W}_D(\Omega)'} \\
&\le \|\mathbf{e}_h^{k+1}\|_{0,2,\Omega} + \|\mathbf{e}_h^0\|_{0,2,\Omega} \qquad (10.43) \\
&+ C\, \|\mathbf{v}_h\|_{l^\infty(L^2)} \sum_{k=0}^{n} \Delta t \, \|D(\mathbf{e}_h^{k+1})\|_{0,2,\Omega} + T \sup_{k=0,1,\cdots,N} \|D(\overline{\mathbf{v}}_h^k)\|_{0,2,\Omega} \, \|\mathbf{e}_h^k\|_{0,2,\Omega} \\
&+ C\, h^{2-d/2} + C\, h^{2-d/2} \sum_{k=0}^{n} \Delta t \, \|D(\mathbf{e}_h^{k+1})\|_{0,2,\Omega} + \sum_{k=0}^{n} \Delta t \, \|\varepsilon_h^{k+1}\|_{\mathbf{W}_D(\Omega)'}.
\end{aligned}
$$

As $\mathbf{v} \in C^0(\mathbf{H}^1)$, then

$$
\begin{aligned}
\|D(\overline{\mathbf{v}}_h^n)\|_{0,2,\Omega} &\le \|D(\overline{\mathbf{v}}_h^n - \mathbf{v}(t_n))\|_{0,2,\Omega} + \|D(\mathbf{v}(t_n))\|_{0,2,\Omega} \\
&\le C\, d_{1,2,\Omega}(\mathbf{v}(t_n), \mathbf{W}_h) + \|D(\mathbf{v}(t_n))\|_{0,2,\Omega} \le (C+1)\, \|D(\mathbf{v}(t_n))\|_{0,2,\Omega},
\end{aligned}
$$

and so

$$
\sup_{n=0,1,\cdots,N} \|D(\overline{\mathbf{v}}_h^n)\|_{0,2,\Omega} \le C. \qquad (10.44)
$$

Then, using the inf-sup condition (9.4) and estimate (10.9),

$$
\begin{aligned}
\|\Lambda_h^{n+1}\|_{0,2,\Omega} &\le C \left(\sup_{k=0,1,\cdots,N} \|\mathbf{e}_h^k\|_{0,2,\Omega} + \left(\sum_{k=0}^{N} \Delta t \, \|D(\mathbf{e}_h^{k+1})\|_{0,2,\Omega}^2 \right)^{1/2} + h^{2-d/2} \right) \\
&+ C \left(\sum_{k=0}^{N} \Delta t \, \|\varepsilon_h^{k+1}\|_{\mathbf{W}_D(\Omega)'}^2 \right)^{1/2} \le C\, (D + h^{2-d/2}). \qquad (10.45)
\end{aligned}
$$

STEP 4. *Consistency error estimate.* Let us write $\langle \varepsilon_h^{n+1}, \mathbf{w}\rangle = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7$, with

$$\varepsilon_1 = \left(\partial_t \mathbf{v}(t_{n+1}) - \frac{\overline{\mathbf{v}}_h^{n+1} - \overline{\mathbf{v}}_h^n}{\Delta t}, \mathbf{w}\right)_\Omega, \ \varepsilon_2 = b(\mathbf{v}(t_{n+1}); \mathbf{v}(t_{n+1}), \mathbf{w}) - b(\overline{\mathbf{v}}_h^n; \overline{\mathbf{v}}_h^{n+1}, \mathbf{w}),$$

$$\varepsilon_3 = a(\mathbf{v}(t_{n+1}) - \overline{\mathbf{v}}_h^{n+1}, \mathbf{w}),\ \varepsilon_4 = -c(\overline{\mathbf{v}}_h^{n+1}, \mathbf{w}),\ \varepsilon_5 = (\overline{p}_h^{n+1} - p(t_{n+1}), \nabla\cdot\mathbf{w})_\Omega,$$

$$\varepsilon_6 = \langle G(\mathbf{v}(t_{n+1})) - G(\overline{\mathbf{v}}_h^{n+1}), \mathbf{w}\rangle,\ \varepsilon_7 = \langle \mathbf{f}^{n+1} - \mathbf{f}(t_{n+1}), \mathbf{w}\rangle$$

We shall denote $\overline{\mathbf{e}}_h^n = \overline{\mathbf{v}}_h^n - \mathbf{v}(t_n)$.
Estimate of ε_1. We split $\varepsilon_1 = \varepsilon_{11} + \varepsilon_{12}$, with

$$\varepsilon_{11} = \left(\frac{\mathbf{v}(t_{n+1}) - \mathbf{v}(t_n)}{\Delta t} - \frac{\overline{\mathbf{v}}_h^{n+1} - \overline{\mathbf{v}}_h^n}{\Delta t}, \mathbf{w}\right)_\Omega, \ \varepsilon_{12} = \left(\partial_t \mathbf{v}(t_{n+1}) - \frac{\mathbf{v}(t_{n+1}) - \mathbf{v}(t_n)}{\Delta t}, \mathbf{w}\right)_\Omega$$

We estimate ε_{11} by

$$|\varepsilon_{11}| \le \frac{1}{\Delta t} \left(\|\overline{\mathbf{e}}_h^{n+1}\|_{0,2,\Omega} + \|\overline{\mathbf{e}}_h^n\|_{0,2,\Omega}\right) \|\mathbf{w}\|_{0,2,\Omega}. \tag{10.46}$$

To bound ε_{12}, observe that

$$\mathbf{v}(t_{n+1}) - \mathbf{v}(t_n) - \Delta t\ \partial_t \mathbf{v}(t_{n+1}) = \int_{t_n}^{t_{n+1}} \int_{t_{n+1}}^{s} \partial_t^2 \mathbf{v}(t)\, dt\, ds.$$

Then,

$$\begin{aligned} |\varepsilon_{12}| &\le \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \int_{t_n}^{t_{n+1}} \|\partial_t^2 \mathbf{v}(t)\|_{0,2,\Omega}\, dt\, ds\, \|\mathbf{w}\|_{0,2,\Omega} \\ &\le C\, \sqrt{\Delta t} \left(\int_{t_n}^{t_{n+1}} \|\partial_t^2 \mathbf{v}(t)\|_{0,2,\Omega}^2\, dt\right)^{1/2} \|\mathbf{w}\|_{0,2,\Omega}. \end{aligned} \tag{10.47}$$

Estimate of ε_2. As $\nabla\cdot\mathbf{v}(t) = \nabla\cdot\overline{\mathbf{v}}(t) = 0$, by Lemma 6.3 (iv), we deduce $b(\mathbf{v}(t_{n+1}); \mathbf{v}(t_{n+1}), \mathbf{w}) = (\mathbf{v}(t_{n+1})\cdot\nabla\mathbf{v}(t_{n+1}), \mathbf{w})$, $b(\overline{\mathbf{v}}_h^n; \overline{\mathbf{v}}_h^{n+1}, \mathbf{w}) = (\overline{\mathbf{v}}_h^n \cdot \nabla\overline{\mathbf{v}}_h^{n+1}, \mathbf{w})$. Then by Hölder inequality and Sobolev injections,

$$\begin{aligned} |\varepsilon_2| &\le |((\mathbf{v}(t_{n+1}) - \mathbf{v}(t_n))\cdot\nabla\mathbf{v}(t_{n+1}), \mathbf{w})| + |(\mathbf{v}(t_n)\cdot\nabla\overline{\mathbf{e}}_h^{n+1}, \mathbf{w})| + |(\overline{\mathbf{e}}_h^n \cdot \nabla\overline{\mathbf{v}}_h^{n+1}, \mathbf{w})| \\ &\le C\, \|\mathbf{v}(t_{n+1}) - \mathbf{v}(t_n)\|_{0,2,\Omega}\, \|D(\mathbf{v}(t_{n+1}))\|_{0,3,\Omega}\, \|D\mathbf{w}\|_{0,2,\Omega} \\ &+ \left[\|D(\mathbf{v}(t_n))\|_{0,2,\Omega}\, \|D(\overline{\mathbf{e}}_h^{n+1})\|_{0,2,\Omega} + \|D(\overline{\mathbf{e}}_h^n)\|_{0,2,\Omega}\, \|D(\overline{\mathbf{v}}_h^{n+1})\|_{0,2,\Omega}\right] \|D\mathbf{w}\|_{0,2,\Omega} \\ &\le C\left[\int_{t_n}^{t_{n+1}} \|\partial_t \mathbf{v}(t)\|_{0,2,\Omega}\, dt + \|D(\overline{\mathbf{e}}_h^n)\|_{0,2,\Omega} + \|D(\overline{\mathbf{e}}_h^{n+1})\|_{0,2,\Omega}\right] \|D\mathbf{w}\|_{0,2,\Omega} \end{aligned} \tag{10.48}$$

where we have used that $\mathbf{v} \in C^0(\mathbf{W}^{1,3})$.

Estimates of ε_3, ε_4 *and* ε_5. We directly have

$$|\varepsilon_3| \le C\, \|D(\overline{\mathbf{e}}_h^{n+1})\|_{0,2,\Omega}\, \|D\mathbf{w}\|_{0,2,\Omega}; \tag{10.49}$$

$$\begin{aligned} |\varepsilon_4| &\le C\, h^{2-d/2}\, \|D(\overline{\mathbf{v}}_h^{n+1})\|_{0,2,\Omega}^2\, \|D\mathbf{w}\|_{0,2,\Omega} \\ &\le C\, h^{2-d/2}\, \|D\mathbf{w}\|_{0,2,\Omega}, \end{aligned} \tag{10.50}$$

where we have used (9.36) and (10.44);

$$|\varepsilon_5| \le C\, \|\overline{p}_h^{n+1} - p(t_{n+1})\|_{0,2,\Omega}\, \|D\mathbf{w}\|_{0,2,\Omega}. \tag{10.51}$$

Estimate of ε_6.

$$\begin{aligned} |\varepsilon_6| &\le C\, (1 + \|\mathbf{v}(t_{n+1})\|_{1,2,\Omega}^2 + \|\overline{\mathbf{v}}_h^{n+1}\|_{1,2,\Omega}^2)\, \|D(\overline{\mathbf{e}}_h^{n+1})\|_{0,2,\Omega}\, \|D\mathbf{w}\|_{0,2,\Omega} \\ &\le C\, \|D(\overline{\mathbf{e}}_h^{n+1})\|_{0,2,\Omega}\, \|D\mathbf{w}\|_{0,2,\Omega}, \end{aligned} \tag{10.52}$$

where we have used (6.39) and (10.44).
Estimate of ε_7. Observe that

$$\mathbf{f}^{n+1} - \mathbf{f}(t_{n+1}) = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} (\mathbf{f}(t) - \mathbf{f}(t_{n+1}))\, dt = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \int_{t_{n+1}}^{t} \partial_t \mathbf{f}(s)\, dt,$$

and then

$$\begin{aligned} |\varepsilon_7| &\le \int_{t_n}^{t_{n+1}} \|\partial_t \mathbf{f}(t)\|_{\mathbf{W}_D(\Omega)'}\, dt\, \|D\mathbf{w}\|_{0,2,\Omega} \\ &\le \sqrt{\Delta t} \left(\int_{t_n}^{t_{n+1}} \|\partial_t \mathbf{f}(t)\|_{\mathbf{W}_D(\Omega)'}^2\, dt \right)^{1/2} \|D\mathbf{w}\|_{0,2,\Omega}. \end{aligned} \tag{10.53}$$

Consequently,

$$\begin{aligned} \sum_{n=0}^{N-1} \Delta t \|\varepsilon_h^{n+1}\|_{\mathbf{W}_D(\Omega)'}^2 &\le C \left[(\Delta t)^{-2} \sum_{n=0}^{N-1} \Delta t \|\overline{\mathbf{e}}_h^{n+1}\|_{0,2,\Omega}^2 + \sum_{n=0}^{N-1} \Delta t \|D(\overline{\mathbf{e}}_h^{n+1})\|_{0,2,\Omega}^2 \right] \\ &+ C \left[\sum_{n=0}^{N-1} \Delta t \|p_h^{n+1} - p(t_{n+1})\|_{0,2,\Omega}^2 + h^{2(2-d/2)} \sum_{n=0}^{N-1} \Delta t \right] \\ &+ C \left[(\Delta t)^2 (\, \|\partial_t \mathbf{v}\|_{l^2(\boldsymbol{L}^2)}^2 \|\partial_t^2 \mathbf{v}\|_{l^2(\boldsymbol{L}^2)}^2 + \|\partial_t \mathbf{f}\|_{l^2(\mathbf{W}_D(\Omega)')}^2\,) \right] \\ &\le C \left((\Delta t)^{-2}\, d_{l^2(\boldsymbol{L}^2)}(\mathbf{v}, \mathbf{W}_h)^2 + d_{l^2(\boldsymbol{H}^1)}(\mathbf{v}, \mathbf{W}_h)^2 + d_{l^2(L^2)}(p, M_h)^2 \right) \\ &+ C\, \left(h^{2(2-d/2)} + (\Delta t)^2 \right). \end{aligned} \tag{10.54}$$

STEP 5. *Conclusion.* Using (10.54),

$$D \le C\, M(h, \Delta t) \tag{10.55}$$

where $M(h, \Delta t)$ is defined by (10.34). Let $\overline{P}_h^n = \sum_{k=0}^{n} \Delta t\, \overline{p}_h^k$. Then

$$\max_{n=0,1,\cdots,N} \| P(t_n) - \overline{P}_h^n \|_{0,2,\Omega} \le C\, d_{l^\infty(L^2)}(P, M_h).$$

Also,

$$\max_{n=0,1,\cdots,N} \| \overline{\mathbf{e}}_h^n \|_{0,2,\Omega} \le C\, d_{l^\infty(\boldsymbol{L}^2)}(\mathbf{v}, \mathbf{W}_h),$$

$$\sum_{n=0}^{N-1} \Delta t\, \| D(\overline{\mathbf{e}}_h^n) \|_{1,2,\Omega} \le C\, d_{l^2(\boldsymbol{H}^1)}(\mathbf{v}, \mathbf{W}_h).$$

Combining these estimates with estimates (10.41), (10.42), (10.45), and (10.55), we deduce (10.32) and (10.33). □

Corollary 10.1. *Under the hypotheses of Theorem 10.3, assume in addition that the family of pairs of spaces $((\mathbf{W}_h, M_h))_{h>0}$ satisfies the optimal error estimates (9.14) stated in Theorem 9.1. Assume $\mathbf{v}_0 \in \mathbf{W}^{1,3}(\Omega)$, $\mathbf{v} \in L^2(\mathbf{H}^2) \cap L^\infty(\mathbf{W}_D(\Omega))$, $p \in L^2(\mathbf{H}^1)$. Then the solution $(\mathbf{v}_h, p_h)$ of the discrete unsteady SM (10.5) and (10.6) satisfies the error estimates*

$$\|\mathbf{v} - \mathbf{v}_h\|_{l^\infty(L^2)} + \|\mathbf{v} - \mathbf{v}_h\|_{l^2(H^1)} + \|P - \tilde{P}_h\|_{l^\infty(L^2)} \le C\, (\frac{h^2}{\Delta t} + \Delta t + h^{2-d/2}) \tag{10.56}$$

for some constant C independent of h and Δt.

Proof. Estimate (10.56) follows from (10.32) and (10.33) combined with the finite element error estimate (9.14), observing that as $p \in L^2(\mathbf{H}^1)$, then $P \in L^\infty(\mathbf{H}^1)$. □

Remark 10.3. The best choice in estimate (10.56) corresponds to $\Delta t = O(h)$, as this keeps the error stemming from the finite element discretization below the error due to the subgrid term. This yields a convergence order of $h^{2-d/2}$.

Estimate (10.56) applies to the primitive in time of the pressure and not to the pressure itself. We believe that the pressure possibly strongly converges in a weaker non-Hilbertian norm, $l^1(L^1)$, although its proof faces hard technical difficulties, as it requires the use of test functions with nonlinear dependence with respect to the pressure.

10.6 Asymptotic Energy Balance

In the evolution case there are no results that prove that weak solutions of the Smagorinsky model verify an asymptotic energy identity as occurs for the steady SM (see Sect. 9.6.1), to the best of the knowledge of the authors to date. This is due to the low regularity of the weak solution, which has serious consequences: The energy dissipated by the eddy diffusion cannot be proved to vanish in the limit, also it is not possible to pass to the limit in the term corresponding to the energy dissipated at the wall, and moreover the weak solution cannot be taken as a test function in the weak formulation (10.2), so even without turbulence modeling, it is not possible to prove strong convergence. We instead can prove an energy inequality, related to the dissipative nature of the approximation (10.6), for some simplified wall laws. Indeed, assume that the wall law is given by the Glaucker–Manning law (2.138):

$$g(\mathbf{v}) = c_f\,|\mathbf{v}|\,\mathbf{v},$$

where $c_f > 0$ is a friction coefficient. Then the following holds:

Lemma 10.6. *Let* $\mathbf{v} \in L^2(\mathbf{W}_{Div}(\Omega)) \cap L^\infty(\mathbf{L}^2)$ *a weak solution [together with some pressure* $p \in \mathscr{D}'(Q)$*] of problem (10.2) that is obtained as a weak limit of some sequence* $(\mathbf{v}_h)_{h>0}$ *in the terms stated in Theorem 10.2. Then*

$$\frac{1}{2}\|\mathbf{v}(t)\|_{0,2,\Omega}^2 + \nu \int_0^t \|D(\mathbf{v}(s))\|_{0,2,\Omega}^2\,ds + \int_0^t \int_{\Gamma_n} \langle G(\mathbf{v}(s)), \mathbf{v}(s)\rangle\,ds$$
$$\leq \frac{1}{2}\|\mathbf{v}_0\|_{0,2,\Omega}^2 + \int_0^t \langle \mathbf{f}(s), \mathbf{v}(s)\rangle\,ds, \quad (10.57)$$

for almost every $t \in [0, T]$.

Proof. We start from estimate (10.58). Using that

$$< \mathbf{f}^{n+1}, \mathbf{v}_h^{n+1} >= \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} < \mathbf{f}(t), \tilde{\mathbf{v}}_h(t) > dt,$$

we deduce

$$\frac{1}{2}\|\tilde{\mathbf{v}}_h(t)\|_{0,2,\Omega}^2 + \nu \int_{t_n}^t \|D(\tilde{\mathbf{v}}_h(s))\|_{0,2,\Omega}^2\,ds + \int_{t_n}^t \langle G(\tilde{\mathbf{v}}_h(s)), \tilde{\mathbf{v}}_h(s)\rangle\,ds$$
$$\leq \frac{1}{2}\|\mathbf{v}_h^{n+1}\|_{0,2,\Omega}^2 + \nu\,\Delta t\,\|D(\mathbf{v}_h^{n+1})\|_{0,2,\Omega}^2 + \Delta t\,\left\langle G(\mathbf{v}_h^{n+1}), \mathbf{v}_h^{n+1}\right\rangle$$
$$\leq \frac{1}{2}\|\mathbf{v}_h^{n}\|_{0,2,\Omega}^2 + \int_{t_n}^{t_{n+1}} < \mathbf{f}(s), \tilde{\mathbf{v}}_h(s) > ds \quad \text{for all } t \in (t_n, t_{n+1}). \quad (10.58)$$

Then, summing up in n from $n = 0$ to $n = k - 1$ for $k = 2, \cdots, N$, and using (10.23),

$$\frac{1}{2}\|\tilde{\mathbf{v}}_h(t)\|_{0,2,\Omega}^2 + \nu \int_0^t \|D(\tilde{\mathbf{v}}_h(s))\|_{0,2,\Omega}^2\, ds + \int_0^t \langle G(\tilde{\mathbf{v}}_h(s)), \tilde{\mathbf{v}}_h(s)\rangle\, ds$$
$$\leq \frac{1}{2}\|\mathbf{v}_0\|_{0,2,\Omega}^2 + \int_0^{t_k(t)} < \mathbf{f}(s), \tilde{\mathbf{v}}_h(s) > ds$$
$$\leq \frac{1}{2}\|\mathbf{v}_0\|_{0,2,\Omega}^2 + \int_0^{t} < \mathbf{f}(s), \tilde{\mathbf{v}}_h(s) > ds + C\sqrt{\Delta t}. \tag{10.59}$$

where $t_k(t) = \min(t_n \,|\, t_n \leq t)$, for almost all $t \in [0, T]$, and the last inequality is obtained as follows:

$$\left| \int_0^{t_k(t)} < \mathbf{f}(s), \tilde{\mathbf{v}}_h(s) > ds - \int_0^{t} < \mathbf{f}(s), \tilde{\mathbf{v}}_h(s) > ds \right|$$
$$\leq \|\mathbf{f}\|_{L^1(t_k(t)-\Delta t, t_k(t)), \mathbf{L}^2)} \|\tilde{\mathbf{v}}_h\|_{L^\infty(\mathbf{L}^2)} \leq \sqrt{\Delta t}\, \|\mathbf{f}\|_{L^2(t_k(t)-\Delta t, t_k(t)), \mathbf{L}^2)} \|\tilde{\mathbf{v}}_h\|_{L^\infty(\mathbf{L}^2)}$$
$$\leq C\sqrt{\Delta t}.$$

Observe that

$$\int_0^t \langle G(\mathbf{z}(s)), \mathbf{z}(s)\rangle\, ds = c_f\, \|\mathbf{z}\|_{L^3((0,t),\mathbf{L}^3(\Gamma_n))}^3 \quad \text{for any } \mathbf{z} \in L^3((0,t), \mathbf{L}^3(\Gamma_n)).$$

Then by (10.9) the sequence $(\tilde{\mathbf{v}}_h)_{h>0}$ is bounded in $L^3((0,T), \mathbf{L}^3(\Gamma_n))$, and from the proof of Theorem 10.2 we know that it strongly converges to $\mathbf{v}$ in $L^2((0,T), \mathbf{L}^2(\Gamma_n))$. So we may assume that $\tilde{\mathbf{v}}_h$ weakly converges to $\mathbf{v}$ in $L^3((0,T), \mathbf{L}^3(\Gamma_n))$. Using now the weak-* lower semicontinuity of the $\|\cdot\|_{0,\infty,\Omega}$ norm and the weak lower semicontinuity of the norm in reflexive spaces (cf. Brézis [6], Chap. 3), we deduce

$$\frac{1}{2}\|\mathbf{v}(t)\|_{0,2,\Omega}^2 + \nu \int_0^t \|D(\mathbf{v}(s))\|_{0,2,\Omega}^2\, ds + \int_0^t \langle G(\mathbf{v}(t)), \mathbf{v}(t)\rangle\, ds$$
$$\leq \liminf_{(h,\Delta t)\to 0} \left(\frac{1}{2}\|\tilde{\mathbf{v}}_h(t)\|_{0,2,\Omega}^2 + \nu \int_0^t \|D(\tilde{\mathbf{v}}_h(t))\|_{0,2,\Omega}^2\, ds + \int_0^t \langle G(\tilde{\mathbf{v}}_h(t)), \tilde{\mathbf{v}}_h(t)\rangle\, ds \right)$$

which, combined to (10.59), proves (10.57). □

In this proof the subgrid dissipation energy term,

$$E_S(\tilde{\mathbf{v}}_h) = C_S^2 \int_0^T \sum_{K\in\mathscr{T}_h} h_K^2\, \|D(\tilde{\mathbf{v}}_h(t))\|_{0,3,K}^3\, dt$$

has been treated only using that it is positive. By estimate (10.9), it is uniformly bounded with respect to h and Δt. However, the stability $L^\infty(\mathbf{L}^2)$ and $L^2(\mathbf{H}^1)$ estimates, combined with inverse inequalities (as was used in the steady case), are not sufficient to prove that $E_S(\mathbf{v}_h)$ asymptotically vanishes: As

$$\begin{cases} \|D(\tilde{\mathbf{v}}_h(t))\|_{0,3,K} \leq C\, h_K^{-1-d/6}\, \|\tilde{\mathbf{v}}_h(t)\|_{0,2,K}, \\ \|D(\tilde{\mathbf{v}}_h(t))\|_{0,3,K} \leq C\, h_K^{-d/6}\, \|D(\tilde{\mathbf{v}}_h(t))\|_{0,2,K}, \end{cases} \tag{10.60}$$

we deduce

$$E_S(\tilde{\mathbf{v}}_h) \leq C_S^2\, h_{\min}^{1-d/2} \|\tilde{\mathbf{v}}_h\|_{L^\infty(\mathbf{L}^2)}\, \|\tilde{\mathbf{v}}_h\|_{L^2(\mathbf{H}^1)}.$$

An eddy viscosity of order h^α with $\alpha > 1 + d/2$ instead of $\alpha = 2$ would ensure that $E_S(\tilde{\mathbf{v}}_h)$ asymptotically vanishes.

So in principle weak solutions of Navier–Stokes equations obtained by the SM–Galerkin approximation could bear some asymptotic concentration of subgrid energy, letting the energy inequality (10.57) to be a strict inequality. However this would not happen if the $\mathbf{v}_h$ are bounded in stronger norms, in particular if they are uniformly bounded in $L^\infty(\mathbf{L}^\infty)$.

10.7 Well-Posedness

The well-posedness is one of the criteria proposed in Guermond et al. [23] to consider mathematically acceptable a LES model. This well-posedness requires uniqueness and continuous dependence of the solution with respect to data in convenient norms. For continuous LES models (5.153), both properties are essentially based upon the $L^3(\mathbf{W}^{1,3}(\Omega))$ regularity of the solution, obtained in its turn thanks to the presence of the SM term with a fixed cutoff length δ (cf. John [28], Chap. 6). However this regularity deteriorates as the cutoff parameter δ tends to zero, to the "standard"regularity $L^2(\mathbf{H}^1(\Omega))$ for Navier–Stokes equations, which has not been proved to ensure the well-posedness.

In the discrete setting, the well-posedness is a step further with respect to stability. It requires uniqueness of solutions for the discrete problems at each time step and uniform dependence of the solutions with respect to the data. Uniqueness of solutions may be simply reached by considering explicit methods, but then the stability faces severe restrictions on the time steps. Implicit and semi-implicit time discretizations take advantage of the dissipative nature of the discrete SM method, yielding uniqueness of solutions and stability in $L^\infty(\mathbf{L}^2(\Omega))$ and $L^2(\mathbf{H}^1(\Omega))$ norms (Theorem 10.1). It is worth to analyze the parallelisms between the well-posedness of implicit and semi-implicit time discretizations and that of LES models:

Theorem 10.4. *Let* $(\mathbf{v}_h^n)_{n=0}^N$, $(\mathbf{z}_h^n)_{n=0}^N$ *be the solutions of the discrete model (10.6) respectively corresponding to data* $(\mathbf{v}_{0h}, \mathbf{f})$ *and* $(\mathbf{z}_{0h}, g)$. *There exists a constant* $C > 0$ *independent of* h, Δt, *and* ν *such that*

$$\max_{0\le n\le N} \|\mathbf{v}_h^n - \mathbf{z}_h^n\|_{0,2,\Omega}^2 \le C M(\mathbf{v}_h) \left(\|\mathbf{v}_{0h} - \mathbf{z}_{0h}\|_{0,2,\Omega}^2 + \frac{1}{\nu} \sum_{n=1}^N \|\mathbf{f}^n - \mathbf{g}^n\|_{\mathbf{W}_D(\Omega)'}^2 \right) \tag{10.61}$$

where $M(\mathbf{v}_h) = \Delta t \sum_{n=1}^N \|D(\mathbf{v}_h^n)\|_{0,3,\Omega}^2$.

Proof. We proceed similarly to the obtention of error estimates. The difference $\mathbf{d}_h^n = \mathbf{v}_h^n - \mathbf{z}_h^n$ satisfies the same problem (10.36) as the error $\mathbf{e}_h^n$ but with ε_h^{n+1} replaced by $\mathbf{f}^{n+1} - \mathbf{g}^{n+1}$. Similarly to (10.38) we deduce

$$\begin{aligned}
&\|\mathbf{d}_h^{n+1}\|_{0,2,\Omega}^2 - \|\mathbf{d}_h^n\|_{0,2,\Omega}^2 + \|\mathbf{d}_h^{n+1} - \mathbf{d}_h^n\|_{0,2,\Omega}^2 + 2\,\nu\,\Delta t \|D(\mathbf{d}_h^{n+1})\|_{0,2,\Omega}^2 \\
&+ 2\,C_S^2\,h_{\min}^2\,\nu\,\Delta t \|D(\mathbf{d}_h^{n+1})\|_{0,3,\Omega}^3 \\
&\le C\,\Delta t\,\|\mathbf{d}_h^n\|_{0,2,\Omega} \|D(\mathbf{v}_h^{n+1})\|_{0,3,\Omega} \|D(\mathbf{d}_h^{n+1})\|_{0,2,\Omega} + \frac{\nu}{2}\,\Delta t \|D(\mathbf{d}_h^{n+1})\|_{0,2,\Omega}^2 \\
&+ 2\nu^{-1}\,\Delta t \|\mathbf{f}^{n+1} - \mathbf{g}^{n+1}\|_{\mathbf{W}_D(\Omega)'}^2 \\
&\le C^2\,\Delta t\,\|\mathbf{d}_h^n\|_{0,2,\Omega}^2 \|D(\mathbf{v}_h^{n+1})\|_{0,3,\Omega}^2 + \frac{\nu}{2}\,\Delta t \|D(\mathbf{d}_h^{n+1})\|_{0,2,\Omega}^2 \\
&+ 2\nu^{-1}\,\Delta t \|\mathbf{f}^{n+1} - \mathbf{g}^{n+1}\|_{\mathbf{W}_D(\Omega)'}^2
\end{aligned}$$

Thus,

$$\|\mathbf{d}_h^{n+1}\|_{0,2,\Omega}^2 \le \|\mathbf{d}_h^n\|_{0,2,\Omega}^2 \left(1 + C^2\,\Delta t\,\|D(\mathbf{v}_h^{n+1})\|_{0,3,\Omega}^2\right) + 2\nu^{-1}\,\Delta t \|\mathbf{f}^{n+1} - \mathbf{g}^{n+1}\|_{\mathbf{W}_D(\Omega)'}^2$$

From the discrete Gronwall Lemma 10.4, (10.61) follows. □

Consequently, the discrete problems are well posed, but the uniformity with respect to the discretization parameters would require that the continuity constant $M(\mathbf{v}_h)$ is uniformly bounded. The $L^2(\mathbf{H}^1)$ stability bounds provided by (10.9) and the inverse estimate (9.19) yield an estimate for $M(\mathbf{v}_h)$ by means of

$$M(\mathbf{v}_h) \le C_2\,\nu^{-2}\,h_{\min}^{-d/3}\Delta t \sum_{n=1}^N \|D(\mathbf{v}_h^n)\|_{0,2,\Omega}^2 \le C_3\,\nu^{-2}\,h_{\min}^{-d/3}.$$

We may also use the estimate for $\|D(\mathbf{v}_h)\|_{L^3(\mathbf{L}^3)}$ in (10.9):

$$M(\mathbf{v}_h) \le T^{1/3}\,\|D(\mathbf{v}_h)\|_{L^3(\mathbf{L}^3)}^2 \le C\,\nu^{-2/3}\,h_{\min}^{-4/3}.$$

Thus, the SM terms provides a worse asymptotic estimate for $M(\mathbf{v}_h)$ than the one provided by the laminar viscosity term, although for practical situations it is a better one: ν can take values below 10^{-6} while h hardly can be smaller than 10^{-3}.

10.8 Further Remarks

10.8.1 Time Discretizations

Decreasing the memory requirements may be achieved by means of the projection algorithms. These are fractional step methods, which decompose the global operator of a PDE as the sum of partial operators and solve these partial operators in successive time substeps of the method (cf. Yanenko [48] or Glowinski [13], Chap. II). The first projection algorithm was introduced by Chorin and Temam (cf. [9, 45]). This splits the computation of velocity and pressure by introducing a projection step of the velocity onto the space of free-divergence functions. For the solution of Navier–Stokes equations (10.1) by the SM, this scheme may be schematically described as follows: Given $(\mathbf{v}^n, p^n)$, compute $(\mathbf{v}^{n+1}, p^{n+1})$ by

$$\begin{cases} \dfrac{\tilde{\mathbf{v}}^{n+1} - \mathbf{v}^n}{\Delta t} + \tilde{\mathbf{v}}^{n+1} \cdot \nabla \tilde{\mathbf{v}}^{n+1} - \nabla \cdot (\nu_t D(\tilde{\mathbf{v}}^{n+1})) = \mathbf{f}^{n+1} & \text{in} \quad \Omega \,, \\ \Delta t \nabla p^{n+1} + (\mathbf{v}^{n+1} - \tilde{\mathbf{v}}^{n+1}) = 0, \quad \nabla \cdot \mathbf{v}^{n+1} = 0 & \text{in} \quad \Omega \,. \end{cases}$$

The boundary conditions on the velocity are set in the first step. The second step is the projection of the intermediate velocity $\tilde{\mathbf{v}}^{n+1}$ onto the space of free-divergence functions. The main advantage of projection methods is that the computation of velocity and pressure is decoupled. This provides a reduced requirement of computer memory to store velocity and pressure. There are, however, some drawbacks: the velocity $\mathbf{v}^{n+1}$ does not verify the boundary conditions, while the velocity $\tilde{\mathbf{v}}^{n+1}$ is not divergence-free. Also, in practice artificial boundary conditions (usually zero normal derivative) on the pressure p^{n+1} need to be introduced to compute it.

This method has been extended to high-order in time discretizations for Stokes equations (cf. Guermond et al. [24] and references therein for a complete survey). Its extension to high-order solvers for Navier–Stokes equations is much more complex technically, due to the discretization of the nonlinear convection operator (cf. Guermond [16, 17], Shen [42]).

High accuracy in time may also be achieved by means of Runge–Kutta solvers, particularly of IMEX (implicit–explicit) kind. In Gresho et al. [15] and Kay et al. [30] a solver based upon the trapezoid rule is introduced. The convection velocity is discretized by an explicit method that uses the two preceding time steps. To solve the Navier–Stokes equation (10.1) by the SM, this method is schematically described as follows: Given $(\mathbf{v}^n, p^n)$, compute $(\mathbf{v}^{n+1}, p^{n+1})$ by

$$\begin{cases} \dfrac{2}{\Delta t}\mathbf{v}^{n+1} + \mathbf{w}^{n+1}\cdot\nabla\mathbf{v}^{n+1} - \nabla\cdot(\nu_t D(\mathbf{v}^{n+1})) + \nabla p^{n+1} = \mathbf{f}^{n+1} \text{ in } \quad \Omega, \\ \qquad\qquad\qquad\qquad \nabla\cdot\mathbf{v}^{n+1} \qquad\qquad\qquad\qquad\quad = \;\; 0 \;\; \text{in} \quad \Omega, \end{cases} \tag{10.62}$$

plus boundary conditions, where

$$\mathbf{w}^{n+1} = 2\,\mathbf{v}^n - \mathbf{v}^{n-1},$$

$$\mathbf{f}^{n+1} = \frac{2}{\Delta t}\mathbf{v}^n + \nabla\cdot(\nu_t D(\mathbf{v}^{n1})) - \mathbf{v}^n\cdot\nabla\mathbf{v}^n - \nabla p^n.$$

This provides a second-order solver in time. It may be adapted in time using an auxiliary explicit solver for the estimation of the time step that achieves large savings of computational time. The extension of this solver to higher order in time is quite involved technically and requires storing the velocity computed in several time steps.

10.8.2 Approximation of LES–Smagorinsky Model by Mixed Methods

The preceding analysis can be extended to that of the Euler+Galerkin approximation of the LES–Smagorinsky model (5.153) and (5.154). Method (10.6) is changed into a similar one, with the only replacement of the form c by

$$\hat{c}(\mathbf{v};\mathbf{w}) = C_S^2\,\delta^2\,(|D\mathbf{v}|\,D\mathbf{v}, D\mathbf{w})_\Omega :$$

Obtain $\mathbf{v}_h^{n+1} \in \mathbf{W}_h,\ p_h^{n+1} \in M_h$ such that for all $\mathbf{w}_h \in \mathbf{W}_h,\ q_h \in M_h$,

$$\begin{cases} (\dfrac{\mathbf{v}_h^{n+1} - \mathbf{v}_h^n}{\Delta t}, \mathbf{w}_h)_\Omega + b(\mathbf{v}_h^n;\mathbf{v}_h^{n+1},\mathbf{w}_h) + a(\mathbf{v}_h^{n+1},\mathbf{w}_h) + \hat{c}(\mathbf{v}_h^{n+1};\mathbf{w}_h) \\ \qquad + \langle G(\mathbf{v}_h^{n+1}),\mathbf{w}_h\rangle - (p_h^{n+1},\nabla\cdot\mathbf{w}_h)_\Omega = \langle \mathbf{f}^{n+1},\mathbf{w}_h\rangle, \\ \qquad\qquad (\nabla\cdot\mathbf{v}_h^{n+1}, q_h)_\Omega = 0, \end{cases} \tag{10.63}$$

Estimate (10.14) yields the additional stability of this approximation in $L^3(\mathbf{W}^{1,3})$, as it holds with $h_{\min}$ changed into δ. This allows to prove the weak convergence of the sequence $(\mathbf{v}_h)_{h>0}$ to a weak solution of model (5.153) and (5.154) in $L^3(\mathbf{W}^{1,3})$. This is the well-known regularity of the weak solution of the LES–Smagorinsky model. In general, the solutions of regularization of the Navier–Stokes equations with hyperviscosity of the form

$$\nu_t(\mathbf{v}) = C\,|D\mathbf{v}|^{p-1}, \quad \text{with} \quad p > 1$$

have $L^{p+1}(\mathbf{H}^{p+1})$ regularity.

Moreover, the $L^3(\mathbf{W}^{1,3})$ regularity allows to prove the strong convergence in $L^\infty(\mathbf{L}^2)$. Indeed, the problems (10.63) are uniformly well posed in $L^\infty(\mathbf{L}^2)$ norm, as in this case the continuity constant $M(\mathbf{v}_h)$ that appears in Theorem 10.4 is uniformly bounded.

A thorough analysis of the approximation of LES models by mixed methods can be found in John [28, 29] and Parés [33].

10.8.3 Suitable Weak Solutions

Suitable weak solutions of Navier–Stokes equations are weak of solutions $(\mathbf{v}, p) \in L^2(\mathbf{H}^1) \times \mathscr{D}'(L^2)$ that are not only globally but locally dissipative, in the sense that they satisfy a local energy balance,

$$\partial_t \left(\frac{1}{2}|\mathbf{v}|^2\right) + \nabla \cdot \left(\left(\frac{1}{2}|\mathbf{v}|^2 + p\right) \mathbf{v}\right) - \nu \, \frac{1}{2} \, \Delta(|\mathbf{v}|^2) + \nu \, |\nabla \mathbf{v}|^2 - \mathbf{f} \cdot \mathbf{v} \leq 0 \quad (10.64)$$

in the sense of distributions. This concept was introduced by Sheffer (cf. [41]). Suitable weak solutions have a partial regularity, proved by Cafarelli et al. (cf. [8]), that bounds the Hausdorff measure of the set of singular points of the solution (See Sect. 3.4.3). Solutions of Navier–Stokes equations constructed by regularization procedures, such as adding an hyperviscosity term, as in the case of Smagorinsky models (cf. [3]) or by regularization of the convection operator (as the Leray turbulent solutions or those provided by the α-models) (cf. [2, 10]) are suitable. Guermond proved in [18, 19] that weak solutions obtained through Galerkin approximations also are suitable for periodic and homogeneous Dirichlet boundary conditions, whenever the approximation spaces satisfy a commutation property: Space $\mathbf{W}_h$ satisfies the discrete commutator property if there is a bounded linear operator $P_h : \mathbf{H}^1(\Omega) \mapsto \mathbf{W}_h$ such that for all $\varphi \in \mathbf{W}^{2,\infty}(\Omega)$ and all $\mathbf{v}_h \in \mathbf{W}_h$, it holds

$$\|\varphi \, \mathbf{v}_h - P_h(\varphi \, \mathbf{v}_h)\|_{l,2,\Omega} \leq C \, h^{1+m-l} \, \|\mathbf{v}_h\|_{m,2,\Omega} \, \|\varphi\|_{m+1,\infty,\Omega} \quad \text{for} \quad 0 \leq l \leq m \leq 1.$$

When P_h is a projection (i.e., it is invariant on $\mathbf{W}_h$), this provides an estimate for the commutator $[\varphi, P_h] = \varphi \circ P_h - P_h \circ \varphi$. This property is needed to handle the use of nonlinear test functions within the Galerkin formulation.

The concept of suitable weak solutions has generated some idea to design new LES models. In [25, 26] Guermond and co-workers study a LES model based upon the idea of adding a numerical viscosity proportional to the default to equilibrium in the local energy equation.

Although the boundary conditions that we are considering here are not periodic, neither homogeneous Dirichlet, it is very likely that the Galerkin solutions that we obtain indeed are suitable. At the present time there is not, up to the best knowledge of the authors, any study of the suitability of Galerkin solutions for mixed boundary conditions.

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Chapter 11
A Projection-Based Variational Multiscale Model

Abstract In this chapter we study a projection-based VMS model. In this model the subgrid effects are modeled by an eddy diffusion term that acts only on a range of small resolved scales. We prove stability and perform a convergence analysis to the Navier–Stokes equations, including wall laws, in steady and unsteady regimes. We analyze the asymptotic convergence balance. We finally prove that this method attempts optimal accuracy for smooth flows.

11.1 Introduction

The eddy diffusion in Smagorinsky model affects all the flow scales, and as a consequence, the large resolved scales are overdamped. This yields results with low accuracy, unuseful for most flows of practical interest. This difficulty is solved by the advanced LES models, as in these models the eddy viscosity affects only a short range of resolved small scales. This property is achieved by the spectral functional models, the Taylor and rational LES models (cf. [10, 16, 42]) and also the α-models, studied in Chap. 8.

However, in LES models the averaged flow in general does not satisfy the boundary conditions. This generates the so-called commutation error, as we mentioned in Sect. 5.5.1. Controlling the errors due to commutation requires that the boundary layer must be solved in the numerical simulation. This may be quite costly in terms of computational effort (cf. [31]).

Both difficulties are solved by the variational multiscale (VMS) methods. The VMS procedure was introduced in 1998 by Hughes et al. in [24] for multiscale modeling in continuum mechanics and was subsequently applied to flow problems and turbulence modeling (cf. [25–27]).

In its original version, the VMS modeling provides separate equations for large and small scales, with coupling terms. The small scales are driven by the residual associated to the large scales. The large-scale flow is searched for in a

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finite-dimensional space (usually a finite element space or similar), while the small scales live in an infinite-dimensional space. The "closure" problem in this case is to provide an approximate solution to the unresolved small-scale flow in terms of the resolved flow. A first approach to solve this problem was to only include eddy viscosity in the small resolved scale equations, to model the dissipative effects of the small unresolved scales.

The main advantage of this combined model is that the eddy diffusion only affects the small resolved scales, thus avoiding over-diffusive effects. The discrete model includes a set of PDE for the large scales and another set of PDE for the small resolved scales. This approach was studied by Huges and co-workers in [21, 26–28] by means of spectral discretizations and static/dynamic eddy viscosity modeling, with quite satisfying results for homogeneous isotropic flows and equilibrium and non-equilibrium turbulent channel flows. Good numerical results were also obtained with the static approach by other investigators (cf. Collis [15], Jeanmart and Winckelmans [30], and Ramakrishnan and Collis [40]). Farhat and Koobus applied this approach to compressible flows with finite volume solvers on unstructured meshes, and applied it to several relevant test cases and industrial flows, with quite satisfying results (cf. [35]). A review of this approach with many references to relevant literature was published by Gravemeier [17].

A simplification of the VMS procedure consists in adding to the Galerkin weak formulation an eddy diffusion term that only affects a range (the "subfilter" scales) of small resolved scales. This range is determined by a projection operator on the space of large resolved scales that filters out the largest scales. For this reason this is called the "projection-based" VMS method. This kind of projection-based methods only need a grid and a projection operator on an underlying coarser grid to be implemented. A variant of such methods consists in filtering the small scales of the deformation tensor to construct the eddy diffusion term. This method has been studied by John and co-workers (cf. [32, 34]). Combined with second-order discretizations in time, projection-based VMS methods provide accurate results for first-order averages of equilibrium turbulent flows.

An alternative approach, which was subsequently developed, is the "residual-based" VMS turbulence modeling. The basic procedure is to keep all terms in the residual-driven structure of the resolved flow equations and to perform an approximated analytical element-wise solution of the small-scale flow. This procedure, which does not make use of the statistical theory of equilibrium turbulence, was independently introduced by Codina (cf. [11, 12]) and Hughes and co-workers (cf. [3, 8]).

Several refinements of the projection-based VMS models have been performed since their introduction. Weakly enforced Dirichlet boundary conditions were introduced to obtain accurate results for wall-bounded turbulent flows without a high resolution of the boundary layers (cf. [1–5, 37]). The VMS model has been combined with other numerical techniques to solve complex coupled fluid–structure problems. Isogeometric analysis has been used as a geometric modeling and simulation framework (cf. [1,22,29]). Arbitrary Lagrangian–Eulerian (cf. [23]) and Deforming-Spatial-Domain/Stabilized Space-Time (cf. [43, 45, 46]) versions of the residual-based VMS method have been applied to fluid–structure coupled problems (cf. [6,43–47]).

In the projection-based VMS model introduced by Codina, only the orthogonal projection of the residual on the mean scales space is included. This is the so-called orthogonal subscales VMS model. One of the relevant features of the OSS model is that it introduces a numerical diffusion on the large scales which is asymptotically equivalent, as the Reynolds number increases, to the eddy viscosity dissipated by the unresolved scales (cf. [13,20,39]). However, no eddy viscosity modeling is required by the residual-based VMS models.

Here we shall study the projection-based VMS model. This is a method that provides good predictions of first-order statistics of several relevant turbulent flows, as we shall present in Chap. 13. It has a simplified structure with respect to residual-based VMS models and equally applies to steady and unsteady flows without further adaptation. Globally, it provides a good compromise between accuracy and computational complexity. Finally, it allows a thorough numerical analysis, parallel to that of Navier–Stokes equations.

Our analysis is parallel to that of the SM for steady and unsteady flows. We prove stability in natural norms, so as weak convergence to a weak solution. The asymptotic energy balance is similar: In the steady case the subgrid eddy energy asymptotically vanishes. In the evolution case this only occurs for solutions with some additional regularity, although less than the regularity needed by the evolution SM. The error estimates for smooth solutions are of optimal order with respect to the polynomial interpolation, in opposition to the SM for which the convergence order is limited by the eddy diffusion term.

The analysis of more complex VMS methods, in particular of residual-based methods, requires further adaptations of the analysis that we present here. The subgrid terms have a very complex structure that includes convective interactions between large and small scales, thus setting serious technical problems just to prove stability. This field of analysis is today in progress.

This chapter is organized as follows. In Sect. 11.2 we describe the derivation of the projection-based VMS model, so as the version studied in [32,34]. Section 11.3 is devoted to the analysis of the steady projection-based VMS model. We prove stability and perform the error analysis. This analysis is extended in Sect. 11.4 to the unsteady version of the model. Section 11.5 studies the asymptotic energy balance. Section 11.6 describes some numerical techniques to solve the nonlinear algebraic problems associated to the SM. Finally in Sect. 11.7 we give some additional remarks addressing the derivation of residual-based and residual-free bubble-based VMS models.

11.2 Model Statement

To describe the VMS model setting, we decompose the spaces $\mathbf{W}_D(\Omega)$ and $M = L_0^2(\Omega)$ as

$$\mathbf{W}_D(\Omega) = \tilde{\mathbf{W}}_h \oplus \mathbf{W}', \quad M = \tilde{M}_h \oplus M',$$

where $\tilde{\mathbf{W}}_h$ and $\tilde{M}_h$, respectively, are the large-scale finite-dimensional spaces for velocity and pressure and $\mathbf{W}'$ and M' are the small-scale complementary spaces. The sum is assumed to be direct (i.e., $\tilde{\mathbf{W}}_h \cap \mathbf{W}' = \{0\}$, $\tilde{M}_h \cap M' = \{0\}$) to ensure a proper separation between large and small scales.

For the sake of simplicity, we shall assume homogeneous wall laws, i.e., $G(\mathbf{v}) = 0$, and later shall discuss how to model the nonhomogeneous case within the VMS procedure.

The solution of Navier–Stokes equations (9.1) is decomposed into

$$(\mathbf{v}, p) = (\tilde{\mathbf{v}}_h, \tilde{p}_h) + (\mathbf{v}', p'), \text{ with } (\tilde{\mathbf{v}}_h, \tilde{p}_h) \in \tilde{\mathbf{W}}_h \times \tilde{M}_h, \quad (\mathbf{v}', p') \in \mathbf{W}' \times M'. \tag{11.1}$$

The pairs $(\tilde{\mathbf{v}}_h, \tilde{p}_h)$ and $(\mathbf{v}', p')$ satisfy the following set of coupled equations:

$$\langle N(\mathbf{v}; \tilde{\mathbf{v}}_h, \tilde{p}_h), (\tilde{\mathbf{w}}_h, \tilde{q}_h)\rangle = -\langle R(\mathbf{v}; \mathbf{v}', p'), (\tilde{\mathbf{w}}_h, \tilde{q}_h)\rangle, \tag{11.2}$$

for all $(\tilde{\mathbf{w}}_h, \tilde{q}_h) \in \tilde{\mathbf{W}}_h \times \tilde{M}_h$, and

$$\langle N(\mathbf{v}; \mathbf{v}', p'), (\mathbf{w}', q')\rangle = -\langle R(\mathbf{v}; \tilde{\mathbf{v}}_h, \tilde{p}_h), (\mathbf{w}', q')\rangle, \tag{11.3}$$

for all $(\mathbf{w}', q') \in \mathbf{W}' \times M'$, where

$$\langle N(\mathbf{v}; \mathbf{z}, r), (\mathbf{w}, q)\rangle = \begin{bmatrix} \frac{d}{dt}(\mathbf{z}, \mathbf{w})_\Omega + b(\mathbf{v}; \mathbf{z}, \mathbf{w}) + a(\mathbf{z}, \mathbf{w}) - (r, \nabla \cdot \mathbf{w}) \\ (\nabla \cdot \mathbf{z}, q) \end{bmatrix}$$

and

$$\mathbb{R}(\mathbf{v}; \mathbf{z}, r) = N(\mathbf{v}; \mathbf{z}, r) - \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}$$

is the residual associated to the linearized Navier–Stokes equations. In the coupled set of (11.2) and (11.3), the large scales are driven by the residual associated to the small scales $R(\mathbf{v}, \mathbf{v}', p')$ while the small scales are driven by the residual associated to the large scales $R(\mathbf{v}; \tilde{\mathbf{v}}_h, \tilde{p}_h)$.

The VMS–Smagorinsky modeling is a discretization of this set of macro-microscale equations, based upon the following procedure:

(i) Approximate the small-scale spaces $\mathbf{W}'$ and M' by finite-dimensional subspaces of *small resolved scales* $\mathbf{W}'_h$ and M'_h, respectively. Then

$\mathbf{W}' = \mathbf{W}'_h \oplus \mathbf{W}''$, $M' = M'_h \oplus M''$, where $\mathbf{W}''$ and M'' are complementary spaces of *small unresolved scales* of infinite dimension. This yields the unique decompositions

$$\begin{aligned} \mathbf{w} &= \tilde{\mathbf{w}}_h + \mathbf{w}'_h + \mathbf{w}'' \quad \text{for all } \ \mathbf{w} \in \mathbf{W}_D(\Omega), \\ q &= \tilde{q}_h + q'_h + q'' \quad \text{for all } \ q \in M, \end{aligned}$$

with obvious notations.

(ii) Neglect the interactions between large and small unresolved scales. It is assumed that the interaction of large–small unresolved scales is weak whenever the latter lay inside the inertial spectrum.

(iii) Model the action of small unresolved scales on small resolved scales in problem (11.3) by the eddy viscosity procedure.

Let us decompose

$$\mathbf{v} = \tilde{\mathbf{v}}_h + \mathbf{v}'_h + \mathbf{v}'' \in \tilde{\mathbf{W}}_h \oplus \mathbf{W}'_h \oplus \mathbf{W}'', \quad p = \tilde{p}_h + p'_h + p'' \in \tilde{M}_h \oplus M'_h \oplus M''.$$

Denote

$$\mathbf{W}_h = \tilde{\mathbf{W}}_h \oplus \mathbf{W}'_h, \ \ M_h = \tilde{M}_h \oplus M'_h, \ \ \mathbf{v}_h = \tilde{\mathbf{v}}_h + \mathbf{v}'_h \in \mathbf{W}_h, \ \ p_h = \tilde{p}_h + p'_h \in M_h. \tag{11.4}$$

Equation (11.2) for $(\tilde{\mathbf{v}}_h, \tilde{p}_h)$ is modeled as follows: By (ii) we approximate

$$b(\mathbf{v}; \tilde{\mathbf{v}}_h, \tilde{\mathbf{w}}_h) \simeq b(\mathbf{v}_h; \tilde{\mathbf{v}}_h, \tilde{\mathbf{w}}_h), \quad \langle R(\mathbf{v}, \mathbf{v}', p'), (\tilde{\mathbf{w}}_h, \tilde{q}_h)\rangle \simeq \langle R(\mathbf{v}_h, \mathbf{v}'_h, p'_h), (\tilde{\mathbf{w}}_h, \tilde{q}_h)\rangle,$$

where $\mathbf{v}_h$ is given by (11.4). This suggests the following modeled equation for $(\tilde{\mathbf{v}}_h, \tilde{p}_h)$:

$$\langle N(\mathbf{v}_h; \tilde{\mathbf{v}}_h, \tilde{p}_h), (\tilde{\mathbf{w}}_h, \tilde{q}_h)\rangle = -\langle R(\mathbf{v}_h; \mathbf{v}'_h, p'_h), (\tilde{\mathbf{w}}_h, \tilde{q}_h)\rangle, \tag{11.5}$$

for all $(\tilde{\mathbf{w}}_h, \tilde{q}_h) \in \tilde{\mathbf{W}}_h \times \tilde{M}_h$.

The modeling of (11.3) is more involved. Set $\mathbf{w}' = \mathbf{w}'_h \in \mathbf{W}'_h$. As

$$b(\mathbf{v}; \mathbf{v}', \mathbf{w}'_h) = b(\mathbf{v}_h; \mathbf{v}'_h, \mathbf{w}'_h) + b(\mathbf{v}_h; \mathbf{v}'', \mathbf{w}'_h) + b(\mathbf{v}''; \mathbf{v}'_h, \mathbf{w}'_h) + b(\mathbf{v}''; \mathbf{v}'', \mathbf{w}'_h)$$

and

$$\langle R(\mathbf{v}; \tilde{\mathbf{v}}_h, \tilde{p}_h), (\mathbf{w}'_h, q')\rangle = \langle R(\mathbf{v}_h; \tilde{\mathbf{v}}_h, \tilde{p}_h), (\mathbf{w}'_h, q')\rangle + b(\mathbf{v}''; \tilde{\mathbf{v}}_h, \mathbf{w}'_h),$$

(11.3) becomes

$$\left\langle N(\mathbf{v}_h; \mathbf{v}'_h, p'_h) + \begin{bmatrix} T(\mathbf{v}_h, \mathbf{v}'', p'') \\ 0 \end{bmatrix}, (\mathbf{w}'_h, q'_h) \right\rangle = -\langle R(\mathbf{v}_h; \tilde{\mathbf{v}}_h, \tilde{p}_h), (\mathbf{w}'_h, q'_h)\rangle \\ -b(\mathbf{v}''; \tilde{\mathbf{v}}_h, \mathbf{w}'_h), \tag{11.6}$$

for all $(\mathbf{w}'_h, q'_h) \in \mathbf{W}'_h \times M'_h$, where

$$\langle T(\mathbf{v}_h, \mathbf{v}'', p''), \mathbf{w}'_h\rangle = \frac{d}{dt}(\mathbf{v}'', \mathbf{w}'_h)_\Omega + b(\mathbf{v}_h; \mathbf{v}'', \mathbf{w}'_h) + b(\mathbf{v}''; \mathbf{v}_h, \mathbf{w}'_h) + b(\mathbf{v}''; \mathbf{v}'', \mathbf{w}'_h) \\ + a(\mathbf{v}'', \mathbf{w}'_h) - (p'', \nabla \cdot \mathbf{w}'_h)_\Omega$$

Applying (ii) and (iii), we approximate $\langle T(\mathbf{v}_h, \mathbf{v}'', p''), \mathbf{w}'_h\rangle \simeq c'(\mathbf{v}_h, \mathbf{w}_h)$, where

$$c'(\mathbf{v}_h; \mathbf{w}_h) = c(\mathbf{v}'_h; \mathbf{w}'_h) = (\nu_t(\mathbf{v}'_h)\, D(\mathbf{v}'_h), D(\mathbf{w}'_h)). \tag{11.7}$$

Thus, we deduce the following modeled equation for $(\mathbf{v}'_h, p'_h)$:

$$\left\langle N(\mathbf{v}_h; \mathbf{v}'_h, p'_h), (\mathbf{w}'_h, q'_h)\right\rangle + \begin{bmatrix} c'(\mathbf{v}_h, \mathbf{w}_h) \\ 0 \end{bmatrix} = -\left\langle R(\mathbf{v}_h; \tilde{\mathbf{v}}_h, \tilde{p}_h), (\mathbf{w}'_h, q'_h)\right\rangle, \tag{11.8}$$

for all $(\mathbf{w}'_h, q'_h) \in \mathbf{W}'_h \times M'_h$. Summing up problems (11.5) and (11.8) this model may be simplified to a single problem for the unknown $(\mathbf{v}_h, p_h) \in \mathbf{W}_h \times M_h$ defined by (11.4). This problem reads as follows:

$$\langle N(\mathbf{v}_h; \mathbf{v}_h, p_h), (\mathbf{w}_h, q_h)\rangle + \begin{bmatrix} c'(\mathbf{v}_h, \mathbf{w}_h) \\ 0 \end{bmatrix} = \begin{bmatrix} \langle \mathbf{f}, \mathbf{w}_h\rangle \\ 0 \end{bmatrix}, \tag{11.9}$$

for all $(\mathbf{w}_h, q_h) \in \mathbf{W}_h \times M_h$. For nonhomogeneous wall laws, as these apply to the mean flow, we identify the mean flow with the resolved flow and generalize (11.9) to (in detailed form)

$$\begin{cases} \dfrac{d}{dt}(\mathbf{v}_h, \mathbf{w}_h)_\Omega + b(\mathbf{v}_h; \mathbf{v}_h, \mathbf{w}_h) + a(\mathbf{v}_h, \mathbf{w}_h) - (p_h, \nabla \cdot \mathbf{w}_h)_\Omega + c'(\mathbf{v}_h; \mathbf{w}_h) \\ \qquad\qquad + \langle G(\mathbf{v}_h), \mathbf{w}_h\rangle = \langle \mathbf{f}, \mathbf{w}_h\rangle, \\ \qquad\qquad (\nabla \cdot \mathbf{v}_h, q_h)_\Omega = 0, \end{cases} \tag{11.10}$$

for all $(\mathbf{w}_h, q_h) \in \mathbf{W}_h \times M_h$. This is the VMS–Smagorinsky model that we shall analyze in this chapter.

To specify the form c', let us consider the restriction operator $\tilde{\Pi}_h : \mathbf{W}_h \to \tilde{\mathbf{W}}_h$ defined by

$$\forall \mathbf{w}_h \in \mathbf{W}_h,\ \tilde{\Pi}_h \mathbf{w}_h = \tilde{\mathbf{w}}_h, \ \text{ such that } \ \mathbf{w}_h = \tilde{\mathbf{w}}_h + \mathbf{w}'_h \in \tilde{\mathbf{W}}_h \oplus \mathbf{W}'_h.$$

As the sum $\tilde{\mathbf{W}}_h \oplus \mathbf{W}'_h$ is direct, then the decomposition $\mathbf{w}_h = \tilde{\mathbf{w}}_h + \mathbf{w}'_h$ is unique. Then $\mathbf{w}'_h = (I - \tilde{\Pi}_h)\mathbf{w}_h$ and $c'(\mathbf{v}_h, \mathbf{w}_h)$ is well defined as a function of $\mathbf{v}_h$ and $\mathbf{w}_h$. In fact, this modified eddy viscosity term is the only difference between the VMS–Smagorinsky models (11.10) and the standard SM (9.32). With this definition of the eddy viscosity term, the turbulent diffusion only depends on the small scales of the flow. Note that the spaces of small resolved scales $\tilde{\mathbf{W}}_h$ and $\tilde{M}_h$ do not need

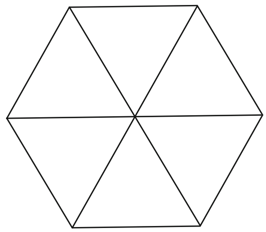

Fig. 11.1 Grid τ_H with $H = 2h$

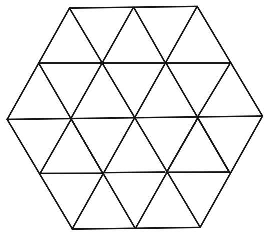

Fig. 11.2 Grid τ_h

to be explicitly constructed to compute discretization (11.10). Instead, only the restriction operator $\tilde{\Pi}_h$ is needed.

In the terminology of VMS methods, method (9.32) is the *small–small* setting of eddy viscosity (cf. [26]). Another possibility is to set the turbulent diffusion as a function of the whole velocity field (the *large–small* setting of eddy viscosity in VMS methods):

$$c'(\mathbf{v}_h; \mathbf{w}_h) = (\nu_t(\mathbf{v}_h)\, D(\mathbf{v}_h'), D(\mathbf{w}_h')). \tag{11.11}$$

In the context of finite element discretizations, space $\tilde{\mathbf{W}}_h$ may be formed either by polynomials of degree smaller than those of $\mathbf{W}_h$:

$$\mathbf{W}_h = V_h^{(l)}(\mathscr{T}_h), \quad \tilde{\mathbf{W}}_h = V_h^{(k)}(\mathscr{T}_h), \text{ with } 1 \le k < l, \tag{11.12}$$

or by polynomials of the same degree constructed on a coarser grid:

$$\mathbf{W}_h = V_h^{(l)}(\mathscr{T}_h), \quad \tilde{\mathbf{W}}_h = V_H^{(l)}(\mathscr{T}_H), \tag{11.13}$$

where $\mathscr{T}_H$ is a grid coarser than $\mathscr{T}_h$, typically $H = O(h)$. The restriction operator $\tilde{\Pi}_h$ should be a stable interpolation or projection operator (Figs. 11.1 and 11.2).

The above VMS–Smagorinsky models, in its two versions, provide more accurate results than the SM, even including several improvements, such as Van-Driest damping of over-diffusive wall effects and dynamic adjustment of the constant C_S.

In particular, for periodic turbulent flows, the $k^{-5/3}$ Kolmogorov law is recovered for larger wavenumbers. This last effect is due to the correct damping of the energy dissipated by the unresolved scales (cf. [25, 27]). For wall-bounded turbulent flows, the small–small setting provides slightly more accurate results than the large–small setting, although both provide a better accuracy than SM with the mentioned improvements (cf. [26]).

The role of the high-frequency components $(I - \tilde{\Pi}_h)\mathbf{v}_h$ that appear in the eddy diffusion term of model (11.15) is to absorb the energy consumed in the formation of small eddies in the inertial range. So the basic grid to build space $\mathbf{W}_h$ should be fine enough to ensure that space $\mathbf{W}_h$ covers the large scales and an initial segment of the inertial range. This segment is accurately solved by model (11.15).

In Berselli et al. [7], Chap. 11, the eddy diffusion term in model (11.15) is reformulated as an eddy diffusion acting on the large scales of the deformation tensor. This applies when the restriction operator is the elliptic projection $\tilde{\Pi}_h$ on space $\tilde{\mathbf{W}}_h$, defined for any $\mathbf{v} \in \mathbf{W}_D(\Omega)$ by $\tilde{\Pi}_h \mathbf{v} \in \tilde{\mathbf{W}}_h$,

$$(D(\tilde{\Pi}_h \mathbf{v}), D(\mathbf{w}_h))_\Omega = (D\mathbf{v}, D(\mathbf{w}_h))_\Omega, \text{ for all } \mathbf{w}_h \in \tilde{\mathbf{W}}_h.$$

Indeed, let us define the space

$$L_h = D(\tilde{\mathbf{W}}_h) = \{d_h \in L^2(\Omega)^{d\times d} \text{ such that } d_h = D(\mathbf{w}_h) \text{ for some } \mathbf{w}_h \in \tilde{\mathbf{W}}_h\}.$$

L_h is a finite-dimensional space of deformations. Denote by σ_h the $L^2(\Omega)^{d\times d}$ orthogonal projection on L_h. Then $\sigma_h(D\mathbf{v}) = D(\tilde{\Pi}_h(\mathbf{v}))$, for all $\mathbf{v} \in \mathbf{W}_D(\Omega)$, because

$$(\sigma_h(D\mathbf{v}), D(\mathbf{w}_h))_\Omega = (D\mathbf{v}, D(\mathbf{w}_h))_\Omega = (D(\tilde{\Pi}_h(\mathbf{v})), D(\mathbf{w}_h))_\Omega, \text{ for all } \mathbf{w}_h \in \tilde{\mathbf{W}}_h.$$

This allows to rewrite the term c' given by either (11.7) or (11.11) as

$$c'(\mathbf{v}_h, \mathbf{w}_h) = (\nu_t\, (I - \sigma_h)(D(\mathbf{v}_h)), (I - \sigma_h)(D(\mathbf{w}_h))_\Omega, \tag{11.14}$$

where ν_t stands for either $\nu_t(\mathbf{v}_h)$ or $\nu_t(\mathbf{v}'_h)$. Then, to compute the eddy diffusion in VMS–Smagorinsky models, filtering the small-scale component of the velocity is equivalent to filtering the small-scale component of the deformation tensor. This allows in practice to replace the eddy diffusion terms (11.7) or (11.11) by (11.14), which is simpler to implement in some cases (for piecewise affine discretizations, for instance). However this assumes that $\tilde{\Pi}_h$ is the elliptic projection operator. Methods (11.9)–(11.14) are usually called projection based-VMS method (Fig. 11.3).

The projection-based VMS model make apparent three families of resolved and subgrid scales, as mentioned in Sagaut [42], Chap. 8. These three categories are the:

1. Subgrid scales, which are those not included in the numerical simulation and whose effects on the resolved scales have to be modeled

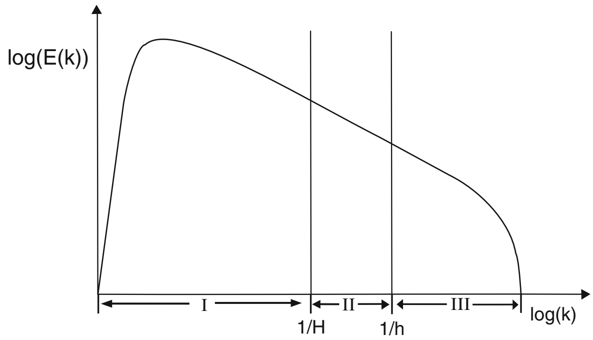

Fig. 11.3 Energy spectrum for VMS models: large resolved scales (I), subfilter scales (II), and subgrid scales (III)

2. Subfilter scales, which are of a size less than the effective filter cutoff length, which are scales resolved in the usual sense, but whose dynamics is strongly affected by the subgrid model
3. Physically resolved scales, which are those of a size greater than the effective filter cutoff length, whose dynamics is perfectly captured by the simulation

For projection VMS methods, the cutoff length is the grid size associated to the resolved scales space $\mathbf{W}_h$. The subfilter scales are the component $(\mathbf{w}'_h, p'_h) \in \mathbf{W}'_h \times M'_h$ of the discrete solution $(\mathbf{v}_h, p_h)$.

11.3 Stability and Error Analysis

In this section we perform the numerical analysis of the steady version of the VMS–Smagorinsky models (11.10): Obtain $(\mathbf{w}_h, q_h) \in \mathbf{W}_h \times M_h$ such that

$$(\mathscr{VP})_h \left\{ \begin{aligned} b(\mathbf{v}_h; \mathbf{v}_h, \mathbf{w}_h) + a(\mathbf{v}_h, \mathbf{w}_h) - (p_h, \nabla \cdot \mathbf{w}_h)_\Omega + c'(\mathbf{v}_h; \mathbf{w}_h) \\ + \langle G(\mathbf{v}_h), \mathbf{w}_h \rangle = \langle \mathbf{f}, \mathbf{w}_h \rangle, \\ (\nabla \cdot \mathbf{v}_h, q_h)_\Omega = 0; \end{aligned} \right. \tag{11.15}$$

for all $(\mathbf{w}_h, q_h) \in \mathbf{W}_h \times M_h$.

Let us assume that the interpolation operator $\tilde{\Pi}_h$ is uniformly stable in $\mathbf{W}_D(\Omega)$ norm, and of optimal order of accuracy: There exists a constant $C > 0$ independent of h such that

$$\|D(\tilde{\Pi}_h \mathbf{w}_h)\|_{0,2,\Omega} \leq C \, \|D(\mathbf{w}_h)\|_{0,2,\Omega} \quad \forall \mathbf{w}_h \in \mathbf{W}_h, \tag{11.16}$$

$$\|D((I - \tilde{\Pi}_h)\mathbf{w})\|_{0,2,\Omega} \leq C \, d_{1,2,\Omega}(\mathbf{w}, \mathbf{W}_h) \quad \forall \mathbf{w} \in \mathbf{W}_D(\Omega). \tag{11.17}$$

Several operators $\tilde{\Pi}_h$ verifying these properties are reported in Remark 9.1. It may also be the Oswald quasi-interpolation operator that applies to piecewise polynomial functions, either continuous or not (cf. [38]), and is simpler to compute.

We may now state the stability and accuracy properties of model (11.15):

Theorem 11.1. *Under the hypotheses of Theorem 9.4, assume also that (11.16) holds. Then the VMS–Smagorinsky models (11.15) admits a solution $(\mathbf{v}_h, p_h)$ that satisfies estimates (9.38) and (9.39).*

Moreover the family of discrete variational problems $(\mathscr{VP})_h$ converges to the variational problem $\mathscr{VP}$ (9.3) in the terms stated in Theorem 9.5.

Proof. The proof of the stability estimates (9.38) and (9.39) is the same as in Theorem 9.4 because both forms c' defined by (11.7) and (11.11) also satisfy the estimate (9.36) due to (11.16). □

Theorem 11.2. *Under the hypotheses of Theorem 9.4, assume that (11.16) and (11.17) hold and that the data $\mathbf{f}$ satisfy the estimate (9.41). Then the following error estimates for the solution of method (11.15) hold for h small enough:*

- *If the form c' is defined by (11.7),*

$$\|D(\mathbf{v}-\mathbf{v}_h)\|_{0,2,\Omega} \le C\,[\,d_{1,2,\Omega}(\mathbf{v},\mathbf{W}_h) + d_{0,2,\Omega}(p,M_h) \qquad h^{2-d/2} d_{1,2,\Omega}(\mathbf{v},\tilde{\mathbf{W}}_h)^2\,]; \tag{11.18}$$

$$\|q-q_h\|_{0,2,\Omega} \le C\,[\,d_{1,2,\Omega}(\mathbf{v},\mathbf{W}_h) + d_{0,2,\Omega}(p,M_h) + h^{2-d/2} d_{1,2,\Omega}(\mathbf{v},\tilde{\mathbf{W}}_h)^2\,]. \tag{11.19}$$

  for some $C > 0$ independent of h.
- *If the form c' is defined by (11.11),*

$$\|D(\mathbf{v}-\mathbf{v}_h)\|_{0,2,\Omega} \le C\,[\,d_{1,2,\Omega}(\mathbf{v},\mathbf{W}_h) + d_{0,2,\Omega}(p,M_h) + h^{2-d/2} d_{1,2,\Omega}(\mathbf{v},\tilde{\mathbf{W}}_h)\,]; \tag{11.20}$$

$$\|q-q_h\|_{0,2,\Omega} \le C\,[\,d_{1,2,\Omega}(\mathbf{v},\mathbf{W}_h) + d_{0,2,\Omega}(p,M_h), + h^{2-d/2} d_{1,2,\Omega}(\mathbf{v},\tilde{\mathbf{W}}_h)\,]. \tag{11.21}$$

  for some constant C independent of h.

Proof. The proof of the error estimates is the same of the proof of Theorem 9.4 up to identity (9.46) that now reads

$$\delta\|D(\mathbf{e}_h)\|_{0,2,\Omega}^2 \le \frac{4}{\delta}\,\|\varepsilon_h\|_{\mathbf{W}_D(\Omega)'}^2 + 2\,|c'(\mathbf{v}_h;\mathbf{e}_h)|. \tag{11.22}$$

The term $c'(\mathbf{v}_h;\mathbf{e}_h)$ is the only difference between this equality and (9.45).

- Assume that c' is defined by (11.7). Then

$$\begin{aligned}|c'(\mathbf{v}_h;\mathbf{e}_h)| &\le C\,\|\nu_t(\mathbf{v}_h')\|_{0,\infty,\Omega}\,\|D(\mathbf{v}_h')\|_{0,2,\Omega}\,\|D(\mathbf{e}_h')\|_{0,2,\Omega}\\ &\le C\,h^{2-d/2}\,\|D(\mathbf{v}_h')\|^2_{0,2,\Omega}\,\|D(\mathbf{e}_h')\|_{0,2,\Omega}.\end{aligned} \tag{11.23}$$

Set $\overline{\mathbf{v}}_h' = (I-\tilde{\Pi}_h)(\overline{\mathbf{v}}_h)$, where $\overline{\mathbf{v}}_h$ is the Stokes projection of $\mathbf{v}$ on $\mathbf{W}_h$. Then, using (11.16) and (11.17),

$$\begin{aligned}\|D(\mathbf{v}_h')\|_{0,2,\Omega} &\le \|D(\mathbf{e}_h')\|_{0,2,\Omega} + \|D(\overline{\mathbf{v}}_h'-\mathbf{v}')\|_{0,2,\Omega} + \|D(\mathbf{v}')\|_{0,2,\Omega}\\ &\le C\,\|D(\mathbf{e}_h)\|_{0,2,\Omega} + C\,\|D(\overline{\mathbf{v}}_h-\mathbf{v})\|_{0,2,\Omega} + C\,d_{1,2,\Omega}(\mathbf{v},\tilde{\mathbf{W}}_h)\\ &\le C\,\|D(\mathbf{e}_h)\|_{0,2,\Omega} + C\,d_{1,2,\Omega}(\mathbf{v},\mathbf{W}_h) + C\,d_{1,2,\Omega}(\mathbf{v},\tilde{\mathbf{W}}_h)\end{aligned} \tag{11.24}$$

Inserting the last inequality in (11.23),

$$\begin{aligned}|c'(\mathbf{v}_h;\mathbf{e}_h)| &\le C\,h^{2-d/2}\,\|D(\mathbf{e}_h)\|^3_{0,2,\Omega}\\ &\quad + C\,h^{2-d/2}(d_{1,2,\Omega}(\mathbf{v},\mathbf{W}_h) + d_{1,2,\Omega}(\mathbf{v},\tilde{\mathbf{W}}_h))^2\|D(\mathbf{e}_h)\|_{0,2,\Omega}.\end{aligned} \tag{11.25}$$

Observe that

$$\|D(\mathbf{e}_h)\|_{0,2,\Omega} \le \|D(\mathbf{v}_h)\|_{0,2,\Omega} + \|D(\overline{\mathbf{v}}_h)\|_{0,2,\Omega} \le C\,(\|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'} + \|D\mathbf{v}\|_{0,2,\Omega}) \le C,$$

and then

$$\|D(\mathbf{e}_h)\|^3_{0,2,\Omega} \le C\,\|D(\mathbf{e}_h)\|^2_{0,2,\Omega} \tag{11.26}$$

Combining and (11.25) with (11.26) and (11.22),

$$\begin{aligned}(\delta - C\,h^{2-d/2})\|D(\mathbf{e}_h)\|^2_{0,2,\Omega} &\le \frac{4}{\delta}\,\|\varepsilon_h\|^2_{\mathbf{W}_D(\Omega)'}\\ &+C\,h^{2-d/2}(d_{1,2,\Omega}(\mathbf{v},\mathbf{W}_h) + d_{1,2,\Omega}(\mathbf{v},\tilde{\mathbf{W}}_h))^2\|D(\mathbf{e}_h)\|_{0,2,\Omega}.\end{aligned}$$

For small enough h, say $h < \left(\dfrac{\delta}{4C}\right)^{1/(2-d/2)}$, using Young's inequality, we deduce

$$\frac{\delta}{4}\|D(\mathbf{e}_h)\|^2_{0,2,\Omega} \le \frac{4}{\delta}\,\|\varepsilon_h\|^2_{\mathbf{W}_D(\Omega)'} + \frac{C}{\delta}\,h^{2(2-d/2)}(d_{1,2,\Omega}(\mathbf{v},\mathbf{W}_h) + d_{1,2,\Omega}(\mathbf{v},\tilde{\mathbf{W}}_h))^4.$$

Combining this estimate with (9.48), and assuming $d_{1,2,\Omega}(\mathbf{v}, \mathbf{W}_h) \leq 1$ for h small enough,

$$\|D(\mathbf{e}_h)\|_{0,2,\Omega} \leq \frac{C}{\delta}(d_{1,2,\Omega}(\mathbf{v}, \mathbf{W}_h) + h^{(2-d/2)} d_{1,2,\Omega}(\mathbf{v}, \tilde{\mathbf{W}}_h)^2 + d_{0,2,\Omega}(p, M_h)).$$

- Assume that c' is defined by (11.11). Then

$$\begin{aligned} |c'(\mathbf{v}_h; \mathbf{e}_h)| &\leq C\, \|\nu_t(\mathbf{v}_h)\|_{0,\infty,\Omega}\, \|D(\mathbf{v}'_h)\|_{0,2,\Omega}\, \|D(\mathbf{e}'_h)\|_{0,2,\Omega} \\ &\leq C\, h^{2-d/2}\, \|D(\mathbf{v}_h)\|_{0,2,\Omega}\, \|D(\mathbf{v}'_h)\|_{0,2,\Omega}\, \|D(\mathbf{e}_h)\|_{0,2,\Omega} \\ &\leq C\, h^{2-d/2}\, \|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'} \|D(\mathbf{v}'_h)\|_{0,2,\Omega}\, \|D(\mathbf{e}_h)\|_{0,2,\Omega} \\ &\leq C\, h^{2-d/2}\, \|D(\mathbf{e}_h)\|^2_{0,2,\Omega} \\ &\quad + C\, h^{2-d/2}(d_{1,2,\Omega}(\mathbf{v}, \mathbf{W}_h) + d_{1,2,\Omega}(\mathbf{v}, \tilde{\mathbf{W}}_h)) \|D(\mathbf{e}_h)\|_{0,2,\Omega} \end{aligned} \tag{11.27}$$

where the last estimate follows from (11.24). The remaining of the proof is that of Theorem 9.4.

□

Remark 11.1. To obtain optimal estimates when the form c' is defined by (11.7), spaces $\mathbf{W}_h$ and $\tilde{\mathbf{W}}_h$ must be chosen in such a way that the term $h^{(2-d/2)} d_{1,2,\Omega}(\mathbf{v}, \tilde{\mathbf{W}}_h)^2$ is at least of the same order as the term $d_{1,2,\Omega}(\mathbf{v}, \mathbf{W}_h)$ with respect to h. If spaces $\mathbf{W}_h$ and $\tilde{\mathbf{W}}_h$ are given by (11.12) with $H = 2h$, this occurs if

$$2 - \frac{d}{2} + 2k \geq l, \tag{11.28}$$

so it is enough to take k as the integer part of $l/2$ if $d = 2$ and of $(l+1)/2$ if $d = 3$ (and always $k \geq 1$) to achieve optimal convergence. If spaces $\mathbf{W}_h$ and $\tilde{\mathbf{W}}_h$ are given by (11.13), then the optimal order estimates directly hold, as in this case $d_{1,2,\Omega}(\mathbf{v}, \mathbf{W}_h)$ and $d_{1,2,\Omega}(\mathbf{v}, \tilde{\mathbf{W}}_h)$ are both of order h^l.

Also for the form c' defined by (11.11), the optimal error estimates hold if $h^{(2-d/2)} d_{1,2,\Omega}(\mathbf{v}, \tilde{\mathbf{W}}_h)$ is at least of the same order as the term $d_{1,2,\Omega}(\mathbf{v}, \mathbf{W}_h)$. Again this occurs if the spaces are set according to (11.13). However, for the setting (11.12), $d_{1,2,\Omega}(\mathbf{v}, \tilde{\mathbf{W}}_h)$ is of order h^k, so we should have $l = k + 2 - d/2$. This is achieved with $k = l - 1$ when $d = 2$, but when $d = 3$ the choice $k = l$ makes vanish the eddy diffusion term in model (11.15). The best choice is $k = l - 1$, yielding a suboptimal method, of order $h^{l-1/2}$.

11.4 Unsteady Projection-Based VMS Model

We perform in this section the analysis of the unsteady version of the VMS–Smagorinsky models introduced in Sect. 11.1. We use the notations introduced in the preceding sections.

We consider as a model problem the semi-implicit Euler VMS–SM discretization of the unsteady Navier–Stokes equations (10.2):

- *Initialization.* Set

$$\mathbf{v}_h^0 = \mathbf{v}_{0h}. \tag{11.29}$$

- *Iteration.* For $n = 0, 1, \cdots, N-1$: Assume known $\mathbf{v}_h^n \in \mathbf{W}_h$.
  $(\mathscr{V}\mathscr{P})_h$ Obtain $\mathbf{v}_h^{n+1} \in \mathbf{W}_h$, $p_h^{n+1} \in M_h$ such that for all $\mathbf{w}_h \in \mathbf{W}_h$, $q_h \in M_h$,

$$\begin{cases} (\dfrac{\mathbf{v}_h^{n+1} - \mathbf{v}_h^n}{\Delta t}, \mathbf{w}_h)_\Omega + b(\mathbf{v}_h^n; \mathbf{v}_h^{n+1}, \mathbf{w}_h) + a(\mathbf{v}_h^{n+1}, \mathbf{w}_h) + c'(\mathbf{v}_h^{n+1}; \mathbf{w}_h) \\ \qquad + \langle G(\mathbf{v}_h^{n+1}), \mathbf{w}_h \rangle - (p_h^{n+1}, \nabla \cdot \mathbf{w}_h)_\Omega = \langle \mathbf{f}^{n+1}, \mathbf{w}_h \rangle, \\ \qquad (\nabla \cdot \mathbf{v}_h^{n+1}, q_h)_\Omega = 0, \end{cases} \tag{11.30}$$

where the form c' is defined by (11.7). The same analysis that follows applies to the θ-scheme (10.8) time discretization of VMS–Smagorinsky model. We shall not consider here the form c' defined by (11.11) because this form is not monotone. The monotonicity of the form c' given by (11.7) follows because it is the gradient of a convex functional, similarly to the monotonicity of the form c proved in Lemma 10.5. We state this result as follows:

Lemma 11.1. *The form c defined by (11.7) is monotone and satisfies the estimates*

$$|c(\mathbf{w};\mathbf{z}) - c(\mathbf{u};\mathbf{z})| \le C\, h^{2-d/2} (\|D(\mathbf{w}')\|_{0,2,\Omega} + \|D(\mathbf{u}')\|_{0,2,\Omega}) \|D(\mathbf{z}')\|_{0,2,\Omega} \times \|D(\mathbf{w}' - \mathbf{u}')\|_{0,2,\Omega}, \tag{11.31}$$

for some constant C independent of h, $\mathbf{w}$, $\mathbf{u}$, and $\mathbf{z}$, where $\mathbf{w}' = (I - \tilde{\Pi}_h)\mathbf{w}$, $\mathbf{u}' = (I - \tilde{\Pi}_h)\mathbf{u}$, $\mathbf{z}' = (I - \tilde{\Pi}_h)\mathbf{z}$.

Proof. Observe that $c(\mathbf{w};\mathbf{z}) = c(\mathbf{w}';\mathbf{z}')$, where c is the form defined by (9.33). Then the monotonicity of c implies that of d. Also, (11.31) directly follows from (10.29). □

For this form c' discretization (11.30) is stable and convergent. We state this result without proof as it is very close of the proofs stated in Sect. 10.4.

Theorem 11.3. *The discrete variational problem $(\mathscr{V}\mathscr{P})_h$ (11.30) admits a unique solution that satisfies estimates (10.9)–(10.11).*

Moreover the sequence of discrete variational problems $(\mathscr{V}\mathscr{P})_h$ converges to the solution of the variational problem $\mathscr{V}\mathscr{P}$ (10.2) in the terms stated in Theorem 10.2.

We next estate the error estimates for method (11.29) and (11.30):

Theorem 11.4. *Under the hypotheses of Theorem 10.3, the sequence* $\{(\mathbf{v}_h, p_h)\}_{h>0}$ *given by the discrete unsteady VMS–Smagorinsky models (11.29) and (11.30) satisfies the error estimates*

$$\|\mathbf{v}-\mathbf{v}_h\|_{l^\infty(\mathbf{L}^2)} + \|\mathbf{v}-\mathbf{v}_h\|_{l^2(\mathbf{H}^1)} \le M'(h,\Delta t) + d_{l^\infty(\mathbf{L}^2)}(\mathbf{v},\mathbf{W}_h), \quad (11.32)$$

$$\|P-P_h\|_{l^\infty(L^2)} \le M'(h,\Delta t) + C\, d_{l^\infty(L^2)}(P,M_h), \quad (11.33)$$

where

$$M'(h,\Delta t) = C\,[\, d_{0,2,\Omega}(\mathbf{v}(0),\mathbf{W}_h) + \frac{1}{\Delta t}\, d_{l^2(L^2)}(\mathbf{v},\mathbf{W}_h) + d_{l^2(\boldsymbol{H}^1)}(\mathbf{v},\mathbf{W}_h)$$
$$+ d_{l^2(L^2)}(p,M_h) + h^{2-d/2}\,(d_{l^4(\boldsymbol{H}^1)}(\mathbf{v},\mathbf{W}_h)^2 + d_{l^4(\boldsymbol{H}^1)}(\mathbf{v},\tilde{\mathbf{W}}_h)^2) + \Delta t\,],$$

and C *is a constant independent of* h *and* Δt, *increasing with* T.

Proof. The proof is that of Theorem 10.3, using Lemma 11.31, excepting the estimate of the component ε_4 of the consistency error, that now is defined (we explicit the dependence upon the time step n) as

$$\varepsilon_4^{n+1} = -c'(\overline{\mathbf{v}}_h^{n+1},\mathbf{w}) = -c(\overline{\mathbf{v}}_h^{n+1'},\mathbf{w}') \text{ for any } \mathbf{w}\in \mathbf{W}_D(\Omega).$$

Denote $\overline{\mathbf{e}}_h^{n+1'} = (I-\tilde{\Pi}_h)\overline{\mathbf{e}}_h^{n+1}$ and $\mathbf{v}(t_{n+1})' = (I-\tilde{\Pi}_h)\mathbf{v}(t_{n+1})$. Then

$$\|D(\overline{\mathbf{v}}_h^{n+1'})\|_{0,2,\Omega} \le \|D(\overline{\mathbf{e}}_h^{n+1'})\|_{0,2,\Omega} + \|D(\mathbf{v}(t_{n+1})')\|_{0,2,\Omega}$$
$$\le C\,\|D(\overline{\mathbf{e}}_h^{n+1})\|_{0,2,\Omega} + \|D(\mathbf{v}(t_{n+1})')\|_{0,2,\Omega}.$$

Combining this estimate with (9.36) we deduce

$$|\varepsilon_4^{n+1}| \le C\,h^{2-d/2}\,\|D(\overline{\mathbf{v}}_h^{n+1'})\|_{0,2,\Omega}^2\,\|D(\mathbf{w}')\|_{0,2,\Omega}$$
$$\le C\,h^{2-d/2}\,\left(\|D(\overline{\mathbf{e}}_h^{n+1})\|_{0,2,\Omega}^2 + \|D(\mathbf{v}(t_{n+1})')\|_{0,2,\Omega}^2\right)\,\|D(\mathbf{w}')\|_{0,2,\Omega}$$

Then,

$$\sum_{n=0}^{N-1}\Delta t\,\|\varepsilon_4^{n+1}\|_{\mathbf{W}_D(\Omega)'}^2 \le C\,h^{2(2-d/2)}\,\left(\|D(\overline{\mathbf{e}}_h^{n+1})\|_{0,2,\Omega}^4 + \|D(\mathbf{v}(t_{n+1})')\|_{0,2,\Omega}^4\right)$$
$$\le C\,h^{2(2-d/2)}\,(d_{l^4(\boldsymbol{H}^1)}(\mathbf{v},\mathbf{W}_h)^4 + d_{l^4(\boldsymbol{H}^1)}(\mathbf{v},\tilde{\mathbf{W}}_h)^4).$$

Replacing the estimate of ε_4 in the proof of Theorem 10.3 by this estimate, we conclude the error estimates (11.32) and (11.33). $\square$

Remark 11.2. Due to the finite element interpolation estimates (9.14), the error estimates (11.32) and (11.33) would be of optimal order in space if the term

$$h^{2-d/2}\,(d_{l^4(\boldsymbol{H}^1)}(\mathbf{v},\mathbf{W}_h)^2 + d_{l^4(\boldsymbol{H}^1)}(\mathbf{v},\tilde{\mathbf{W}}_h)^2)$$

is at least of the same order as the term $d_{l^2(\boldsymbol{H}^1)}(\mathbf{v},\mathbf{W}_h)$ for smooth enough $\mathbf{v}$. If spaces $\mathbf{W}_h$ and $\tilde{\mathbf{W}}_h$ are given by (11.12), this occurs if

$$2-\frac{d}{2}+2k \geq l, \tag{11.34}$$

so it is enough to take k as the integer part of $l/2$ when $d=2$ and as that of $(l+1)/2$ when $d=3$ (and always $k\geq 1$) to achieve optimal convergence. If spaces $\mathbf{W}_h$ and $\tilde{\mathbf{W}}_h$ are given by (11.13), then (11.34) directly holds.

In any case, estimates (11.32) and (11.33) are of first order in time. Thus the optimal choice of the finite element spaces corresponds to the lowest possible computational cost that achieves an overall first order in space and time. This corresponds to $k=1, l=2$ when $d=2$ and to $k=1, l=3$ when $d=3$ if spaces $\mathbf{W}_h$ and $\tilde{\mathbf{W}}_h$ are given by (11.12). If these are given by (11.13) with $H=O(h)$, then this optimal choice corresponds to $l=1$. This is the less costly choice.

11.5 Asymptotic Energy Balance

In the steady case the asymptotic energy balance for the VMS–Smagorinsky models (9.32) and (11.15) is similar to the one for the SM (9.32). Corollary 9.1 still holds, with the subgrid dissipation energy E_S replaced by

$$E'_S(\mathbf{v}) = C_S^2 \sum_{K\in\mathscr{T}_h} h_K^2 \int_K |D(\mathbf{v}'_h)(\mathbf{x})|^3\,d\mathbf{x}.$$

In particular, E'_S asymptotically vanishes as $h\to 0$. In the unsteady case we still cannot prove that the sub-grid dissipation energy asymptotically vanishes, although this needs less smoothness than the SM. Indeed, estimates (10.60), applied to $\tilde{\mathbf{v}}'_h$, yield

$$h_K^2\,\|D(\tilde{\mathbf{v}}'_h(t))\|_{0,3,K} \leq C\,h_K^{1-d/2}\,\|\tilde{\mathbf{v}}'_h(t)\|_{0,2,K}\,\|D(\tilde{\mathbf{v}}'_h(t))\|_{0,2,K},$$

By the local inverse estimates (9.19),

$$\|\tilde{\mathbf{v}}'_h(t)\|_{0,2,K} \leq C_r\,h^{d(1/2-1/r)}\,\|\tilde{\mathbf{v}}'_h(t)\|_{0,r,K}.$$

Then,

$$h_K^2 \, \|D(\tilde{\mathbf{v}}_h(t))\|_{0,3,K} \le C \, h_K^{1-d/r} \, \|\tilde{\mathbf{v}}'_h(t)\|_{0,r,K}\,, \; \|D(\tilde{\mathbf{v}}'_h(t))\|_{0,2,K}\,,$$

Consequently, the subgrid energy,

$$E'_S(\mathbf{v}) = C_S^2 \int_0^T \sum_{K\in\mathscr{T}_h} h_K^2 \int_K |D(\tilde{\mathbf{v}}'_h)(\mathbf{x},t)|^3 \, d\mathbf{x} \, dt$$

asymptotically vanishes if $\tilde{\mathbf{v}}_h \in L^\infty(\mathbf{L}^r)$ for some $r > d$.

11.6 Solution of Discrete Problems by Linearization

The VMS–Smagorinsky models (9.32) and (11.15) are sets of nonlinear algebraic equations whose practical solution is not straightforward and needs special techniques to be implemented. The difficulties arise from to the nonlinearities that appear in the equations: the convection, the turbulent diffusion, and the wall-law terms. We propose in this section a linearization technique, well suited to jointly deal with both difficulties to solve the algebraic nonlinear problems equivalent to either (9.32) or (11.15), which is similar to that carried out in Sect. 6.6.

Let us recall that according to Lemmas 5.4 and 5.7, $g(\mathbf{v}) = \mathbf{v}H(|\mathbf{v}|)$, where H is a continuous function that satisfies $0 \le H(|\mathbf{v}|) \le C_g(1+|\mathbf{v}|)$. Given some $\mathbf{z}_h \in \mathbf{W}_h$, let us consider the functional $\mathscr{G}_{\mathbf{z}_h} : \mathbf{W}_h \to \mathbf{W}'_h$ by

$$\langle \mathscr{G}_{\mathbf{z}_h}(\mathbf{v}_h), \mathbf{w}_h \rangle = \int_{\Gamma_n} \mathbf{v}_h(\mathbf{x}) \cdot \mathbf{w}_h(\mathbf{x}) \, H(|\mathbf{z}_h(\mathbf{x})|) \, d\Gamma_n(\mathbf{x}),$$

which was defined by (6.90) in Sect. 6.6. Note that, in some sense, $\mathscr{G}$ is a linearization of G, as $G(\mathbf{v}_h) = \mathscr{G}_{\mathbf{v}_h}(\mathbf{v}_h)$. It is well defined as $\mathbf{z}_h$ is a continuous function and then $H(|\mathbf{z}_h|)$ is bounded on $\overline{\Omega}$. More concretely,

$$\max_{\mathbf{x}\in\overline{\Omega}} H(|\mathbf{z}_h(\mathbf{x})|) \le M(\|\mathbf{z}_h\|_{0,\infty,\Omega}), \quad \text{where} \quad M(r) = \max_{x\in[0,r]} H(x). \tag{11.35}$$

Moreover, $\mathscr{G}_{\mathbf{z}_h}$ is linear and positive: $\langle \mathscr{G}_{\mathbf{z}_h}(\mathbf{v}_h), \mathbf{v}_h \rangle \ge 0$.

Also, given $\mathbf{z} \in \mathbf{W}_D(\Omega)$, let us define the form $\mathscr{A}_{\mathbf{z}} : \mathbf{W}_D(\Omega) \times \mathbf{W}_D(\Omega) \mapsto \mathbb{R}$ by

$$\mathscr{A}_{\mathbf{z}}(\mathbf{v}, \mathbf{w}) = (\nu_t(\mathbf{z}) \, D\mathbf{v}, D\mathbf{w})_\Omega \quad \text{for any} \quad \mathbf{v}, \, \mathbf{w} \in \mathbf{W}_D(\Omega).$$

Then $\mathscr{A}_{\mathbf{z}}$ is bilinear, bounded, and nonnegative, for any $\mathbf{z} \in \mathbf{W}_D(\Omega)$. Moreover, $\mathscr{A}_{\mathbf{v}}(\mathbf{v}, \mathbf{w}) = c(\mathbf{v}, \mathbf{w})$. A similar definition applies to the alternative forms c' defined by (11.7) and (11.11).

Let us set the linearized problem.

Given $\mathbf{z}_h \in \mathbf{W}_h$, find $(\hat{\mathbf{v}}_h, \hat{p}_h) \in \mathbf{W}_h \times M_h$ such that for all $(\mathbf{w}_h, q_h) \in \mathbf{W}_h \times M_h$,

$$\begin{cases} b(\mathbf{z}_h; \hat{\mathbf{v}}_h, \mathbf{w}_h) + a(\hat{\mathbf{v}}_h, \mathbf{w}_h) + \mathscr{A}_{\mathbf{z}_h}(\hat{\mathbf{v}}_h, \mathbf{w}_h) + \langle \mathscr{G}_{\mathbf{z}_h}(\hat{\mathbf{v}}_h), \mathbf{w}_h \rangle \\ \qquad\qquad -(\hat{p}_h, \nabla \cdot \mathbf{w}_h)_\Omega = \langle \mathbf{f}, \mathbf{w}_h \rangle, \\ \qquad\qquad (\nabla \cdot \hat{\mathbf{v}}_h, q_h)_\Omega = 0. \end{cases} \tag{11.36}$$

Lemma 11.2. *Problem (11.36) admits a unique solution that satisfies the estimates*

$$\| D(\hat{\mathbf{v}}_h) \|_{0,2,\Omega} \le \frac{1}{2\,\nu} \, \|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'}, \tag{11.37}$$

$$\| \hat{p}_h \|_{0,2,\Omega} \le C \left(\frac{M(\|\mathbf{z}_h\|_{0,\infty,\Omega})}{2\,\nu} + \| D(\mathbf{z}_h) \|_{0,2,\Omega} \right) \|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'} \tag{11.38}$$

$$+ \frac{C}{\nu^2} \, (1 + h^{2-d/2}) \, \|\mathbf{f}\|^2_{\mathbf{W}_D(\Omega)'}. \tag{11.39}$$

Moreover, the mapping $T : \mathbf{W}_h \to \mathbf{W}_h$ given by $T(\mathbf{z}_h) = \hat{\mathbf{v}}_h$ admits a fixed point, which is a solution of problem (9.32).

Proof. Problem (11.36) is equivalent to a square linear system of dimension $dim(\mathbf{W}_h) + dim(M_h)$. Indeed, let $\{\psi_i\}_{i=1}^K$ and $\{\sigma_j\}_{j=1}^L$ respectively denote a vector base of $\mathbf{W}_h$ and of M_h, with $K = dim(\mathbf{W}_h)$, $L = dim(M_h)$. Assume

$$\hat{\mathbf{v}}_h = \sum_{i=1}^K v_i \, \psi_i, \quad \hat{p}_h = \sum_{k=1}^L p_k \, \sigma_k, \quad \text{with } \; v_i, \; p_k \in \mathbb{R}.$$

Then problem (11.36) may be written as a linear system $A\mathbf{x} = \mathbf{b}$, where A is a square matrix of dimension $K + L$ and $\mathbf{x}$ and $\mathbf{b}$ are vectors of dimension $K + L$, defined by

$$A = \begin{pmatrix} M & B \\ \mathrm{B}^t & O \end{pmatrix}, \quad \mathbf{x} = (v_1, \cdots, v_K; p_1, \cdots, p_L)^t, \quad \mathbf{b} = (\langle \mathbf{f}, \psi_1 \rangle, \cdots, \langle \mathbf{f}, \psi_K \rangle; 0, \cdots, 0)^t,$$

with M and B the matrices of dimensions $K \times K$ and $K \times L$ defined by

$$M_{ij} = b(\mathbf{z}_h; \psi_j, \psi_i) + a(\psi_j, \psi_i) + \mathscr{A}_{\mathbf{z}_h}(\psi_j, \psi_i) + \langle \mathscr{G}_{\mathbf{z}_h}(\psi_j), \psi_i \rangle, \quad i, j = 1, \cdots, K;$$

$$B_{ik} = -(\nabla \cdot \psi_i, \sigma_k), \quad i = 1, \cdots, K, \; k = 1, \cdots, L.$$

Then, to prove the existence of solution of problem (11.36) is equivalent to prove its uniqueness. To do this, it is enough to prove estimates (11.37) and (11.38) assuming that there exists a solution.

Let us assume that $(\hat{\mathbf{v}}_h, \hat{p}_h)$ is a solution of (11.36). Observe that

$$b(\mathbf{z}_h; \mathbf{w}_h, \mathbf{w}_h) + a(\mathbf{w}_h, \mathbf{w}_h) + \mathscr{A}_{\mathbf{z}_h}(\mathbf{w}_h, \mathbf{w}_h) + \langle \mathscr{G}_{\mathbf{z}_h}(\mathbf{w}_h), \mathbf{w}_h \rangle$$
$$\ge a(\mathbf{w}_h, \mathbf{w}_h) \ge 2\,\nu \, \| D(\mathbf{w}_h) \|_{0,2,\Omega}, \; \forall \mathbf{w}_h \in \mathbf{W}_h.$$

Then estimate (11.37) follows. Estimate (11.38) is obtained similarly to (9.39), using (6.21) and taking into account that

$$\begin{aligned}\langle \mathscr{G}_{\mathbf{z}_h}(\mathbf{v}_h), \mathbf{w}_h\rangle &\le M(\|\mathbf{z}_h\|_{0,\infty,\Omega})\, \|\mathbf{v}_h\|_{0,2,\Gamma_n}\, \|\mathbf{w}_h\|_{0,2,\Gamma_n} \\ &\le C\, M(\|\mathbf{z}_h\|_{0,\infty,\Omega})\, \|D(\mathbf{v}_h)\|_{0,2,\Omega}\, \|D(\mathbf{w}_h)\|_{0,2,\Omega}.\end{aligned}$$

Let us now consider the mapping $\mathbf{z}_h \in \mathbf{W}_h \rightarrow \hat{\mathbf{v}}_h \in \mathbf{W}_h$. As $\mathbf{W}_h$ is a normed space of finite dimension, then all norms on $\mathbf{W}_h$ are equivalent. Thus, there exists a constant $k_h > 0$ such that $\|\mathbf{z}_h\|_{0,\infty,\Omega} \le k_h\, \|D(\mathbf{z}_h)\|_{0,2,\Omega}$, for all $\mathbf{z}_h \in \mathbf{W}_h$. Taking $\mathbf{z}_h \in \mathbf{W}_h$ such that $\|D(\mathbf{z}_h)\|_{0,2,\Omega} \le \frac{1}{2\,\nu}\, \|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'}$, by estimate (11.37), T transforms the ball B of $\mathbf{W}_h$ of center 0 and radius $R = \frac{1}{2\,\nu}\, \|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'}$ into a part of itself. Also, T is continuous by estimate (11.37), as it is linear. Then, by Brouwer's fixed-point theorem (Theorem A.5), T admits a fixed point that by construction is a solution of the discrete SM (9.32) or (11.15). □

Remark 11.3. The solution of (9.32) may be obtained by successive approximations of the transformation T. Indeed, by (11.38), the pressure $\hat{p}_h$ satisfies the estimate

$$\begin{aligned}\|\hat{p}_h\|_{0,2,\Omega} \le\; & \frac{C}{\nu^2}\,(1 + h^{2-d/2})\, \|\mathbf{f}\|^2_{\mathbf{W}_D(\Omega)'} \\ & + C\,\Big(\frac{M\left(k_h\, \frac{1}{\nu}\, \|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'}\right)}{\nu} + \frac{\|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'}}{\nu}\Big)\, \|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'}.\end{aligned}$$

Then, the sequence of iterates $\{T^n \mathbf{z}_h^0\}_{n\ge 0}$ of some initial $\mathbf{z}_h^0 \in \mathbf{W}_h$ is bounded in $\mathbf{W}_h$, and the sequence of associated pressures, say, $\{\hat{p}_h^n\}_{n\ge 0}$, is bounded in M_h. Thus, the sequence $\{(T^n\mathbf{z}_h^0, \hat{p}_h^n)\}_{n\ge 0}$ admits a convergent subsequence in $\mathbf{W}_h \times M_h$ to some $(\mathbf{v}_h, p_h)$, which is a solution of (9.32). The full sequence converges if the solution of this problem is unique, as is the case of the discrete problems (10.6) and (11.30).

However for large Reynolds numbers this procedure may provide a very slow convergence to the solution of the SM (9.32) or (11.15). More sophisticated methods may then be applied, such as Newton's method, based upon an approximation of the operator by its tangent operator.

There is another way to afford the solution of the VMS–Smagorinsky models (9.32) and (11.15). The wall-law operator is a monotone operator, as it is the infimum of a convex energy functional (the function C defined by (5.132) in the proof of Lemma 5.4). However, the global SM operator is not symmetric, and then its solution is not the infimum of an energy functional. This difficulty may be overcome for the unsteady SM by a suitable time discretization based upon the method of characteristics that recovers a symmetric operator at each time step. This procedure may be adapted to the solution of the steady VMS–Smagorinsky models by a pseudo-time approach, but it is less efficient that the linearization procedure presented above.

11.7 Further Remarks

In this section we include some additional information that complements the modeling and analysis performed previously. We give the main hints of the subjects considered, without going into the details. The interested readers may consult the references included.

11.7.1 Residual-Free Bubble-Based VMS Turbulence Modeling

A function $v \in H^1(\Omega)$ is called a bubble function with respect to the grid $\mathcal{T}_h$ if $v_{|K} \in H_0^1(K)$, for any element $K \in \mathcal{T}_h$. Consider a linear elliptic or parabolic PDE:

$$L(v) = g \text{ in } \Omega, \quad v = 0 \text{ on } \Gamma. \tag{11.40}$$

A function $v \in H^1(\Omega)$ is called a residual-free bubble function for (11.40) if v is a bubble function and solves (11.40) on any element $K \in \mathcal{T}_h$. In such a case, $v_{|K} \in H^1(K)$ is the solution of

$$L(v_{|K}) = g_{|K} \text{ in } K, \quad v = 0 \text{ on } \partial K, \text{ for any } K \in \mathcal{T}_h.$$

Assume that $g_{|K}$ belongs to a finite-dimensional subset, say W_K, of $H^{-1}(K)$. Let $\{\varphi_1^{(K)}, \cdots, \varphi_{N_K}^{(K)}\}$ be a base of W_K. Let $z_i^{(K)}$ be the solution of the problem:

$$L(z_i^{(K)}) = \varphi_i^{(K)} \text{ in } K, \quad z_i^{(K)} = 0 \text{ on } \partial K.$$

Assume $g_{|K} = \sum_{i=1}^{N_K} f_i^{(K)} \varphi_i^{(K)}$. Then $v_{|K} = \sum_{i=1}^{N} f_i \, z_i^{(K)}$. So, pre-computing the $z_i^{(K)}$ allows to directly obtain a residual-free bubble solution of (11.40).

This idea may be used to approximately solve problem (11.3) with the two-scale decomposition (11.1). Indeed, approximate $R(\mathbf{v}; \tilde{\mathbf{v}}_h, \tilde{p}_h)$ by $R_h(\mathbf{v}_h; \tilde{\mathbf{v}}_h, \tilde{p}_h)$ defined as

$$\langle R_h(\mathbf{v}_h; \tilde{\mathbf{v}}_h, \tilde{p}_h), (\mathbf{w}, q)\rangle = \left\langle N(\mathbf{v}_h; \tilde{\mathbf{v}}_h, \tilde{p}_h) - \begin{bmatrix} \mathbf{f}_h \\ 0 \end{bmatrix}, (\mathbf{w}, q)\right\rangle,$$

where $\mathbf{f}_h$ is some suitable finite element interpolate of $\mathbf{f}$. Observe that $R_h(\mathbf{v}_h; \tilde{\mathbf{v}}_h, \tilde{p}_h)$ belongs to a finite-dimensional subspace of $\mathbf{W}' \times M'$. Then, problem (11.3) is approximated by $(\mathbf{v}', p') \in \mathbf{W}' \times M'$ such that

$$\left\langle N(\mathbf{v}_h; \mathbf{v}', p'), (\mathbf{w}', q')\right\rangle = -\langle R_h(\mathbf{v}_h; \tilde{\mathbf{v}}_h, \tilde{p}_h), (\mathbf{w}, q)\rangle \text{ for any } (\mathbf{w}', q') \in \mathbf{W}' \times M'. \tag{11.41}$$

This problem fits in the general framework of problem (11.40) and then in principle may be locally exactly solved by bubble functions. However, several practical problems arise: On one hand, a convenient time discretization must be performed, in order to solve (11.41) in a coupled form with (11.2). On another hand, the velocity $\mathbf{v}_h$ is changing from a time step to another, making necessary further simplifications to avoid huge memory requirements.

The use of bubble functions to stabilize the discretization of Navier–Stokes equations was introduced by Russo [41]. The use of bubble functions (as polynomials of high order) to approximate the small-scale flow in VMS methods was introduced in the pioneering paper of Hughes et al. [25]. Subsequent improvements of this idea are due to Gravemeier et al. (cf. [18, 19]), Collis (cf. [14]), and John and Kindle (cf. [33]), among others.

Bubble functions can only approximate functions in L^p norm for $1 \leq p < +\infty$ and not in H^1 norm. Indeed, a bounded sequence of bubble functions in $H^1(\Omega)$ necessarily weakly converges to zero in this space as the grid size decreases to zero (cf. [9]). Then using residual free bubbles to approximate the small-scale flow may provide a good balance of the kinetic energy held by the small-scale flow, but not a good approximation of the small-scale flow itself.

11.7.2 Residual-Based VMS Turbulence Modeling

The residual-based VMS turbulence modeling is based upon two basic procedures: Keep all terms in the resolved flow equations (11.2) and parameterize the unresolved flow in terms of the resolved flow by solving (11.3) by an approximated analytical procedure. Indeed, if (11.3) are exactly solved, then the unresolved flow is obtained as a functional of the residual associated to the resolved flow,

$$(\mathbf{v}', p') = \mathscr{F}(R(\mathbf{v}; \tilde{\mathbf{v}}_h, \tilde{p}_h))),$$

which is approximated by some analytical expression,

$$(\mathbf{v}', p') \simeq (\mathbf{v}'_h, p'_h) = \mathscr{F}_h(R(\tilde{\mathbf{v}}_h; \tilde{\mathbf{v}}_h, \tilde{p}_h))), \tag{11.42}$$

where $\mathbf{v}'_h$ and p'_h are functions of $\tilde{\mathbf{v}}_h$ and $\tilde{p}_h$. Inserting this expression into the resolved flow equations (11.2), we obtain the basic structure of the residual-based VMS turbulence model:

$$\langle N(\tilde{\mathbf{v}}_h + \mathbf{v}'_h; \tilde{\mathbf{v}}_h + \mathbf{v}'_h, \tilde{p}_h + p'_h), (\tilde{\mathbf{w}}_h, \tilde{q}_h)\rangle = \langle \mathbf{f}, \tilde{\mathbf{w}}_h\rangle, \tag{11.43}$$

for all $(\tilde{\mathbf{w}}_h, \tilde{q}_h) \in \tilde{\mathbf{W}}_h \times \tilde{M}_h$. The convection term includes the terms $b(\mathbf{v}'_h; \mathbf{v}'_h, \mathbf{w}_h)$ (Reynolds stress) and $b(\tilde{\mathbf{v}}_h; \mathbf{v}'_h, \mathbf{w}_h) + b(\mathbf{v}'_h; \tilde{\mathbf{v}}_h, \mathbf{w}_h)$ (cross stress) that do not appear in any of the stabilized methods mentioned in Sect. 9.8.1.

The deduction of the parameterization of the unresolved scales (11.42) is usually performed by a diagonalization procedure:

$$\mathscr{F}_h(R(\tilde{\mathbf{v}}_h; \tilde{\mathbf{v}}_h, \tilde{p}_h))) = \boldsymbol{\tau}\ P_h\left(R(\tilde{\mathbf{v}}_h; \tilde{\mathbf{v}}_h, \tilde{p}_h))\right), \tag{11.44}$$

where $\boldsymbol{\tau} = \boldsymbol{\tau}(\tilde{\mathbf{v}}_h; \tilde{\mathbf{v}}_h, \tilde{p}_h)$ is a matrix that takes a constant value on each grid element and P_h is some stable projection operator onto the mean flow space $\tilde{\mathbf{W}}_h \times \tilde{M}_h$. In Huges, Calo et al. [8] P_h is the identity, while in Codina [11] P_h is the orthogonal L^2 projection operator onto the mean flow space (this is the "orthogonal subscales" VMS method). Also, usually $\boldsymbol{\tau}$ is a diagonal matrix,

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{v,K}\, I_{3\times 3} & \\ & \tau_{p,K} \end{bmatrix}, \quad \text{on element } K, \tag{11.45}$$

where τ_K and σ_K are the stabilized coefficients mentioned in Sect. 9.8.1.

Some simplifications are applied to the modeled equation (11.43) to transform its structure into a structure similar to that of the stabilized method (9.50), with some additional stabilizing terms due to the cross and Reynolds stresses.

11.7.3 An Alternative Subgrid Model

The analysis of the Smagorinsky and projection-based VMS models performed in the last chapters may be extended to the discrete version of the model studied by Layton and Lewandowski in [36]. In this model the eddy diffusion is given by

$$\nu_t = \nu_t(\mathbf{v}_h) = C_{LL}\, h_K\, |\mathbf{v}_h - \overline{\mathbf{v}_h}|, \tag{11.46}$$

where $\overline{\mathbf{v}_h}$ is some local average of $\mathbf{v}_h$ and C_{LL} a (theoretically) universal constant. In the finite element context the averaging procedure may be achieved by means of some filtering projector Π_h on the resolved large-scale space: $\overline{\mathbf{v}_h} = \Pi_h(\mathbf{v}_h)$. We may consider either the pure subgrid version of the model (9.32) or the projection-based version (11.10), with the eddy diffusion given by (11.46). Standard finite element error estimates yield [cf. (9.15)]

$$\|\mathbf{v}_h - \overline{\mathbf{v}_h}\|_{0,\infty,K} \le C\, h_K\, \|\nabla \mathbf{v}_h\|_{0,\infty,\Delta_K}.$$

Then Lemma 9.5 also applies and thus the rest of the analysis.

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Chapter 12
Numerical Approximation of NS-TKE Model

Abstract In this chapter we perform the numerical analysis of finite element approximations of the NS-TKE model. We consider truncated eddy viscosities and production term so as a smooth friction boundary condition for the TKE. In the steady case we prove stability and strong convergence to a weak solution. In the evolution case we consider a semi-implicit Euler scheme that decouples velocity and TKE. We prove the stability of the scheme and weak convergence to a limit problem in which the TKE only verifies a variational inequality.

12.1 Introduction

In this chapter we perform the numerical analysis of finite element approximations of the NS-TKE model for both steady and evolution flows. As we remarked in Chap. 7, the analysis of the NS-TKE models presents hard additional difficulties with respect to the Navier–Stokes equations, due to the combination of nonlinear effects that generates a low regularity of the solutions. For this reason we analyze a modified model with several regularizations, and our conclusions are only partial. In particular for the unsteady model we only are able to prove that the TKE satisfies a variational inequality instead of the targeted PDE equation.

Let us recall that the main mathematical difficulties of analysis of NS-TKE model are that the source term for the TKE (the production term) has just $L^1(\Omega)$ regularity and then the equation for the TKE does not admit a Hilbertian formulation in $H^1(\Omega)$. The numerical analysis of NS-TKE models performed up to date has applied to simplified TKE equations in order to treat this difficulty. In [2–4] Bernardi et al. study a model for two-coupled turbulent fluids and its numerical approximation, where the TKE equation only contains eddy diffusion (with bounded eddy viscosities) and production term. This equation is reformulated by transposition and is found to admit a solution with $H^{1/2}$ regularity. However the transposition procedure does not apply whenever the convection operator is present.

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In [12] Chacón et al. solve the same coupled model for large eddy viscosities by a linearization procedure. The procedure is proved to converge for large eddy viscosities, assuming that the velocities are smooth enough. An analysis of an adaptive strategy to apply in separated sub-domains an NS-TKE model and the Navier–Stokes equations is studied in Bernardi et al. [5]. The determination of the sub-domains is based upon an a posteriori error analysis that applies to smooth flows.

The numerical analysis of the finite element approximation of elliptic equations with r.h.s. in L^1 only has been performed for linear diffusion equations (cf. Casado et al. [11]). The main reason is that the extension of the Boccardo–Gallouet estimates only holds if the numerical scheme satisfies a discrete maximum principle. For convection–diffusion equations there exist a few finite element schemes that satisfy this principle, all of them based upon adding shock-capturing terms to the discretization (cf. [10, 14, 32]). However, the extension of numerical analysis of these schemes to the solution of equations with L^1 r.h.s. has not been performed yet. For finite volume discretizations there exist more schemes satisfying the maximum principle, whose analysis has been performed in some cases. We address some comments on this issue in Sect. 12.4.2.

We thus focus our analysis on the numerical approximation of a regularized model by truncation of unbounded terms, in particular the production term for the TKE equation. This regularization by truncation is more appropriate to numerical discretizations than the regularization by convolution used in Chap. 7.

We consider space and time discretizations for the NS-TKE model which are natural extensions of those introduced in Chaps. 9 and 10 for steady and evolution Smagorinsky models. For unsteady flow we consider a time discretization by semi-linearization that decouples the velocity-pressure and TKE boundary value problems. The friction boundary condition for the TKE is formulated as the restriction to the boundary of a smooth distributed function. We consider mixed Dirichlet-wall law boundary conditions, what allows to simplify the analysis of Chap. 7, particularly to treat the friction boundary condition. In the evolution case we assume that the boundary value for the TKE reduces to a constant, and we only are able to prove that the limit TKE satisfies a variational inequality, due to a lack of regularity.

We understand the analysis performed in this chapter as a step toward the analysis of more complex models, whose theoretical analysis has been performed in Chaps. 7 and 8, so as in some works (see, e.g., [1, 6, 7, 15, 19]).

The chapter is organized as follows. We at first study the Lagrange finite element discretization of the steady NS-TKE model (Sect. 12.2), proving its stability and convergence to the continuous model (Sect. 12.2.3). We next study the discretization of the unsteady NS-TKE model in Sect. 12.3. We introduce a weak formulation (Sect. 12.3.1) and the numerical discretization (Sect. 12.3.2) and study the stability and convergence of this discretization (Sect. 12.3.3).

12.2 Steady NS-TKE Model

In this section we perform the analysis of the finite element discretization of the steady NS-TKE model stated in Chap. 7. We introduce the discretization and prove its stability and the strong convergence to a weak solution of the model.

12.2.1 Statement of Steady Model Equations

Let us recall the steady NS-TKE model:

Find a velocity field $\mathbf{v} : \overline{\Omega} \to \mathbb{R}^d$, a pressure $p : \Omega \to \mathbb{R}$ and a TKE variable $k : \overline{\Omega} \to \mathbb{R}$ such that

$$\left\{\begin{array}{rll} \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - \nabla \cdot (\tilde{\nu}_t(k)\, D\mathbf{v}) + \nabla p = \mathbf{f} & & \text{in } \Omega; \\ \nabla \cdot \mathbf{v} = 0 & & \text{in } \Omega; \\ \nabla \cdot (\mathbf{v}k) - \nabla \cdot (\tilde{\mu}_t(k)\, \nabla k) + \dfrac{k^{3/2}}{\ell} = P(k, \mathbf{v}) & & \text{in } \Omega; \\ -[\mathbf{n} \cdot (\tilde{\nu}_t(k))\, D\mathbf{v}]_{\boldsymbol{\tau}} = g(\mathbf{v})_{\boldsymbol{\tau}}, \quad \mathbf{v} \cdot \mathbf{n} = 0, & k = k_\Gamma(\mathbf{v}) & \text{on } \Gamma_n; \\ \mathbf{v} = 0, & k = 0 & \text{on } \Gamma_D. \end{array}\right. \tag{12.1}$$

Here

$$\left\{\begin{array}{l} \tilde{\nu}_t(\kappa) = 2\nu + \nu_t(\kappa), \ \ \nu_t(\kappa) = \ell\, \sqrt{\kappa}, \ \text{ and} \\ \tilde{\mu}_t(\kappa) = \mu + \mu_t(\kappa), \ \ \mu_t(\kappa) = C_\mu\, \ell\, \sqrt{\kappa}, \end{array}\right. \tag{12.2}$$

where ν_t and μ_t respectively are the eddy viscosity and the eddy diffusion for k, and

$$P(k, \mathbf{v}) = \tilde{\nu}_t(k)\, |D\mathbf{v}|^2, \quad k_\Gamma(\mathbf{v}) = C\, |\mathbf{v}|^2 \tag{12.3}$$

are the production and TKE boundary terms in the realistic NS-TKE model.

We shall perform the analysis of system (12.1) with some simplifications. We assume that the functions ν_t, μ_t, and k_Γ satisfy:

Hypothesis 12.i. It holds

- $\tilde{\nu}_t \in W^{1,\infty}(\mathbb{R})$, $\tilde{\mu}_t \in C^0(\mathbb{R})$, $\quad \underline{\mu} \le \tilde{\nu}_t(\kappa) \le \overline{\nu}$, $\quad \underline{\mu} \le \tilde{\mu}_t(\kappa) \le \overline{\mu} \quad \forall \kappa \in \mathbb{R}$, for some positive constants $\underline{\mu}, \overline{\nu}, \overline{\mu}$.
- $k_\Gamma \in W^{1,\infty}(\mathbb{R}^3)$, $k_\Gamma(0) = 0$.

This hypothesis is somewhat more restrictive than Hypothesis 7.i. However it is reasonable if $\mathbf{v}$ and k are bounded and k is nonnegative, as in this case $\mathbf{v}$ and k satisfy a system like (12.1) where the functions ν_t, μ_t defined by (12.2) and k_Γ defined by (12.3) are replaced by truncated approximations $T_L(\nu_t)$, $T_L(\mu_t)$, and $T_L(k_\Gamma)$, for L large enough, where we recall that the truncation function T_L is defined by

$$T_L(x) = \begin{cases} x \text{ if } |x| \le L, \\ sg(x)\, L \text{ if } |x| > L. \end{cases}$$

We consider three-dimensional flows, as this is the more relevant case. Let us recall the definition of the spaces $\mathbf{K}_{3/2}(\Omega)$ and $\mathbf{Q}_3(\Omega)$:

$$\mathbf{K}_{3/2}(\Omega) = \bigcap_{1\le q<3/d} W_0^{1,q}(\Omega), \quad \mathbf{Q}_3(\Omega) = \bigcup_{r>3} W_0^{1,r}(\Omega).$$

We consider the following weak formulation of problem (12.1):

Find $(\mathbf{v}, p, k_0) \in \mathbf{W}_D(\Omega) \times L_0^2(\Omega) \times \mathbf{K}_{3/2}(\Omega)$ such that

$$(\mathscr{V}\mathscr{P}) \begin{cases} b(\mathbf{v};\mathbf{v},\mathbf{w}) + s_v(k;\mathbf{v},\mathbf{w}) - (p, \nabla\cdot\mathbf{w})_\Omega + \langle G(\mathbf{v}),\mathbf{w}\rangle = \langle \mathbf{f},\mathbf{w}\rangle, \\ (\nabla\cdot\mathbf{v}, q)_\Omega = 0, \\ b_k(\mathbf{v};k,l) + s_k(k;k,l) + d(k;l) = (P(k,\mathbf{v}),l), \\ k = k_0 + k_\Gamma(\mathbf{v}), \end{cases} \tag{12.4}$$

for all $(\mathbf{w}, q, l) \in \mathbf{W}_D(\Omega) \times L_0^2(\Omega) \times \mathbf{Q}_3(\Omega)$, where (we recall)

$$b_k(\mathbf{z};\mathbf{v},\mathbf{w}) = \frac{1}{2}\left[(\mathbf{z}\cdot\nabla\mathbf{v},\mathbf{w})_\Omega - (\mathbf{z}\cdot\nabla\mathbf{w},\mathbf{v})_\Omega\right], \tag{12.5}$$

$$s_v(\kappa;\mathbf{z},\mathbf{w}) = (\tilde{\nu}_t(\kappa)\, D\mathbf{z}, D\mathbf{w})_\Omega, \tag{12.6}$$

$$b_k(\mathbf{z};\kappa,l) = \frac{1}{2}\left[(\mathbf{z}\cdot\nabla\kappa,l)_\Omega - (\mathbf{z}\cdot\nabla l,\kappa)_\Omega\right], \tag{12.7}$$

$$s_k(\kappa;\lambda,l) = (\tilde{\mu}_t(\kappa)\,\nabla\lambda,\nabla l)_\Omega, \quad d(\kappa;l) = \frac{1}{\ell}\,(\sqrt{|\kappa|}\,\kappa, l)_\Omega, \tag{12.8}$$

$$P(\kappa,\mathbf{z}) = \tilde{\nu}_t(\kappa)\,|D\mathbf{z}|^2, \tag{12.9}$$

for all $\kappa,\ l,\ \lambda \in H^1(\Omega),\ \mathbf{v},\mathbf{w},\mathbf{z} \in \mathbf{H}^1(\Omega)$. As ν_t and μ_t are bounded functions, then s_v and s_k are well defined. As $H^1(\Omega) \hookrightarrow L^6(\Omega)$, the dissipation term $d(k;l)$ is well defined if $k,\ l \in H^1(\Omega)$. This term has been treated to take into account possible negative values of the variable k. This treatment is justified by Theorem 7.7: the solution k of the TKE equation in problem (12.4) is nonnegative.

The solutions of problem (12.4) satisfy problem (12.1) in the following sense:

Lemma 12.1. *Let $(\mathbf{v}, p, k) \in \mathbf{W}_D(\Omega) \times L_0^2(\Omega) \times \mathbf{K}_{3/2}$ be a solution of the variational problem (12.4) such that k is nonnegative. Then the first, second, and third equations in (12.1) respectively hold in $\mathbf{H}^{-1}(\Omega)$, $L^2(\Omega)$, and $W^{-1,r'}(\Omega)$ for* $r > 3$. *Moreover, the boundary conditions hold in the following senses:*

$$\gamma_0\,\mathbf{v} = 0 \ \text{ in } \ \mathbf{H}^{1/2}(\Gamma_D), \quad \gamma_n\,\mathbf{v} = 0 \ \text{ in } \ L^2(\mathbf{L}^4(\Gamma)),$$

$$\gamma_0\, k = \gamma_0\,(k_\Gamma(\mathbf{v})) \ \text{ in } \ H^{1/2}(\Gamma), \quad \gamma_0\, k = 0 \ \text{ in } \ H^{1/2}(\Gamma_D), \tag{12.10}$$

In addition, if $\mathbf{v} \in \mathbf{H}^2(\Omega)$*, then the condition*

$$-\left[\mathbf{n} \cdot (\tilde{\nu}_t(k))\, D\mathbf{v}\right]_{\boldsymbol{\tau}} = g(\mathbf{v})_{\boldsymbol{\tau}}$$

holds in $L^2(\Gamma_n)^{d-1}$.

Proof. The proof of Lemma 9.4 also applies for the equations and boundary conditions satisfied by the velocity in problem (12.1). The only difference is that the laminar viscosity ν here is replaced by $\tilde{\nu}_t$. As this is a continuous bounded function, all steps of the proof still hold.

Also, all terms in the equation for k in problem (12.4) define bounded linear functionals on $W_0^{1,r}(\Omega)$ for $r > d$. Indeed, by Sobolev injections $\mathbf{v} \cdot \nabla k \in L^r(\Omega)$ for $r < 6/5$, and $\sqrt{|k|}\, k \in L^r(\Omega)$ for $r < 2$. As $\tilde{\nu}_t$ is bounded, $\tilde{\nu}_t(k)\nabla k \in L^q(\Omega)$ for $q < 3/2$. Moreover, $P(k, \mathbf{v}) \in L^1(\Omega)$ and $W_0^{1,r}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ for $r > 3$. Then the equation for k holds in $W^{-1,q}(\Omega)$ for $q < 3/2$.

Finally, $k_\Gamma \in H^1(\Omega)$ because $k_\Gamma \in W^{1,\infty}(\mathbb{R})$. Moreover $\gamma_0 k = \gamma_0 k_\Gamma$ and $k_\Gamma(0) = 0$. Then the conditions (12.10) hold. □

12.2.2 Discretization

In this section we approximate problem (12.1) by mixed formulations. We use pairs of finite element spaces $(\mathbf{W}_h, M_h) \subset \mathbf{H}^1(\Omega, \Gamma_D) \times L_0^2(\Omega)$, associated to a family of admissible triangulations $(\mathcal{T}_h)_{h>0}$ of Ω in the sense of Definition 9.4. We assume that the family of pairs of spaces $((\mathbf{W}_h, M_h))_{h>0}$ satisfies Hypotheses 9.i and 9.ii, stated in Sect. 9.5. In addition we consider finite element spaces $K_h \subset H^1(\Omega)$, $K_{0h} = K_h \cap H_0^1(\Omega)$ to approximate the TKE. We shall assume

Hypothesis 12.ii. The families of finite element spaces $(K_h)_{h>0}$, $(K_{0h})_{h>0}$ respectively are internal approximations of $H^1(\Omega)$ and $H_0^1(\Omega)$.

The Lagrange finite element spaces defined in Sect. 9.3.1 satisfy this hypothesis.

Let us finally consider the Lagrange interpolation operator $\Pi_h : C^0(\overline{\Omega}) \to K_h$ defined by (9.10) (or any other satisfying the estimates yield by Theorem (9.1)).

With these ingredients, we set the following finite element approximation of problem (12.1):

$$\text{Given } L > 0, \text{ find } (\mathbf{v}_h, p_h, k_{0h}) \in \mathbf{W}_h \times M_h \times K_{0h} \text{ such that}$$

$$(\mathcal{VP})_h \begin{cases} b(\mathbf{v}_h; \mathbf{v}_h, \mathbf{w}_h) + s_v(k_h; \mathbf{v}_h, \mathbf{w}_h) - (p_h, \nabla \cdot \mathbf{w}_h)_\Omega + \langle G(\mathbf{v}_h), \mathbf{w}_h \rangle \\ \hfill = \langle \mathbf{f}, \mathbf{w}_h \rangle, \\ \hfill (\nabla \cdot \mathbf{v}_h, q_h)_\Omega = 0, \\ b_k(\mathbf{v}_h; k_h, l_h) + s_k(k_h; k_h, l_h) + d(k_h; l_h) = (P_L(\mathbf{v}_h, k_h), l_h), \\ \hfill k_h = k_{0h} + \Pi_h(k_\Gamma(\mathbf{v}_h)) \end{cases} \tag{12.11}$$

for all $(\mathbf{w}_h, q_h, l_h) \in \mathbf{W}_h \times M_h \times K_{0h}$, where $P_L = T_L(P)$. The solutions of problem (12.11) depend on the truncation parameter L. However we omit this dependency for brevity. Also, although there exist discretizations that ensure the positiveness of k, we have preferred to use the Galerkin method, much simpler to analyze, again to avoid technical complexities out of the scope of the book. We address some comments on these discretizations in Sect. 12.4.2.

We next state the existence of solutions of problem (12.11), which is obtained simultaneously with its stability, by means of Brouwer's fixed point theorem:

Theorem 12.1. *Let $(\mathscr{T}_h)_{h>0}$ be a regular family of triangulations of the domain Ω. Let $((\mathbf{W}_h, M_h, K_h))_{h>0}$ be a family of pairs of finite element spaces satisfying Hypotheses 9.i, 9.ii, and 12.ii. Assume that the functions $\tilde{\nu}_t$, $\tilde{\mu}_t$, and k_Γ satisfy Hypothesis 12.i. Then for any $\mathbf{f} \in \mathbf{W}_D(\Omega)'$ the discrete variational problem (12.11) admits at least a solution that satisfies the estimates*

$$\|D\mathbf{v}_h\|_{0,2,\Omega} \le \frac{1}{2\nu} \|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'}, \tag{12.12}$$

$$\|p_h\|_{0,2,\Omega} \le \frac{C}{\nu^2} \|\mathbf{f}\|^2_{\mathbf{W}_D(\Omega)'} + C\,\frac{1}{\nu}(1+\nu)\,\|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'}, \tag{12.13}$$

$$\|\nabla k_h\|_{0,2,\Omega} \le \frac{C}{\mu}\left(\frac{1}{\mu} + L + \frac{\mu + \overline{\nu} + 1}{\nu}\,\|f\|_{\mathbf{W}_D(\Omega)'}\right), \tag{12.14}$$

where $C > 0$ is a constant depending only on d, Ω and the aspect ratio of the family of triangulations.

Proof. Let us define the mapping $\mathscr{F} : \mathbf{W}_h \times K_h \mapsto \mathbf{W}_h \times K_h$ as follows: Given $(\mathbf{u}_h, \kappa_h) \in \mathbf{W}_h \times K_h$, $\mathscr{F}(\mathbf{u}_h, \kappa_h) = (\mathbf{z}_h, \lambda_h) \in \mathbf{W}_h \times K_h$ is defined in two steps by

STEP 1. The velocity $\mathbf{z}_h$, besides an associated pressure $r_h \in M_h$, is the solution of the linear problem

$$\begin{cases} b(\mathbf{u}_h; \mathbf{z}_h, \mathbf{w}_h) + s_\nu(\kappa_h; \mathbf{z}_h, \mathbf{w}_h) - (r_h, \nabla\cdot\mathbf{w}_h)_\Omega + \langle \mathscr{G}_{\mathbf{u}_h}(\mathbf{z}_h), \mathbf{w}_h\rangle = \langle \mathbf{f}, \mathbf{w}_h\rangle, \\ \qquad (\nabla\cdot\mathbf{z}_h, q_h)_\Omega = 0, \end{cases}$$

for all $(\mathbf{w}_h, q_h) \in \mathbf{W}_h \times M_h$. This problem admits a unique solution, similarly to problem (11.36).

STEP 2. The energy λ_h is the solution of the linear problem

$$b_k(\mathbf{z}_h; \lambda_h, l_h) + s_k(\kappa_h; \lambda_h, l_h) + (\sqrt{|\kappa_h|}\,\lambda_h, l_h) = (P_L(\mathbf{z}_h, \kappa_h), l_h), \tag{12.15}$$

for all $l_h \in K_{0h}$, with $\lambda_h = \lambda_{0h} + \Pi_h(k_\Gamma(\mathbf{z}_h))$ for some $\lambda_{0h} \in K_{0h}$. This problem also admits a unique solution. Indeed, it is equivalent to

$$A(\mathbf{z}_h; \lambda_{0h}, l_h) = \langle Q(\mathbf{z}_h, \kappa_h), l_h\rangle, \quad \text{for all } \ l_h \in K_{0h}, \tag{12.16}$$

with

$$A(\mathbf{z}_h; \lambda_{0h}, l_h) = b_k(\mathbf{z}_h; \lambda_{0h}, l_h) + s_k(\kappa_h; \lambda_{0h}, l_h) + (\sqrt{|\kappa_h|}\, \lambda_{0h}, l_h)_\Omega,$$

$$\langle Q(\mathbf{z}_h, \kappa_h), l_h \rangle = (P_L(\mathbf{z}_h, \kappa_h), l_h) - A(\mathbf{z}_h; \sigma_h, l_h),$$

where for brevity we denote $\sigma_h = \Pi_h(k_\Gamma(\mathbf{z}_h))$. Problem (12.16) admits a unique solution as the bilinear form $A(\mathbf{z}_h; \cdot, \cdot)$ is coercive on K_{0h}:

$$A(\mathbf{z}_h; l_h, l_h) \geq \mu\, \|\nabla l_h\|^2_{0,2,\Omega} \geq C_P^2\, \mu\, \|l_h\|^2_{1,2,\Omega},$$

where C_P is the constant of the Poincaré inequality on $H_0^1(\Omega)$.
The mapping $\mathscr{F}$ is continuous as the preceding procedures yield $\mathbf{z}_h$ and λ_h from $\mathbf{u}_h$ and κ_h as the composition of continuous mappings: square root, absolute value, truncation, μ_t, ν_t, k_Γ, sums, products, and quotients by nonzero divisors.

STEP 3. We next prove that there exists a nonempty compact set S of $\mathbf{W}_h \times K_h$ (endowed with the $\mathbf{H}^1(\Omega) \times H^1(\Omega)$ norm) which is mapped into a part of itself by the mapping $\mathscr{F}$. Then, by Brouwer's Theorem (Theorem A.4), $\mathscr{F}$ admits a fixed point, which is a solution of problem (12.11). To prove it, set $\mathbf{w}_h = \mathbf{z}_h$ and $q_h = r_h$ in problem (12.15). Similarly to the proof of Lemma 11.2 we deduce the estimate

$$\|D\mathbf{z}_h\|_{0,2,\Omega} \leq \frac{1}{2\nu}\, \|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'}. \tag{12.17}$$

Also, setting $l_h = \lambda_{0h}$ in (12.16) we deduce

$$\mu\, \|\nabla \lambda_{0h}\|^2_{0,2,\Omega} = \langle Q(\mathbf{z}_h, \kappa), \lambda_{0h} \rangle. \tag{12.18}$$

The r.h.s. of this equality is bounded as follows. By the Poincaré and Young's inequalities,

$$|(P_L(\mathbf{z}_h, \kappa_h), l_h)| \leq L\, \|l_h\|_{0,1,\Omega} \leq C\, L\, \|l_h\|_{0,2,\Omega} \leq \frac{C}{\mu}\, L^2 + \frac{\mu}{5}\|\nabla l_h\|_{0,2,\Omega}. \tag{12.19}$$

Integrating by parts and using that $k_\Gamma \in W^{1,\infty}(\mathbb{R})$,

$$|b_k(\mathbf{z}_h; \sigma_h, l_{0h})| = |-(\mathbf{z}_h \cdot \nabla l_{0h}, \sigma_h)_\Omega - \frac{1}{2}\, (\nabla \cdot \mathbf{z}_h, \sigma_h\, l_{0h})_\Omega|$$

$$\leq\ C\, \|D\mathbf{z}_h\|_{1,2,\Omega}\, \|l_{0h}\|_{1,2,\Omega} \leq C\, \frac{1}{\mu}\, \|D\mathbf{z}_h\|^2_{0,2,\Omega} + \frac{\mu}{5}\, \|\nabla l_h\|^2_{0,2,\Omega}. \tag{12.20}$$

As ν_t is bounded from above,

$$s_k(\kappa_h;\sigma_h,l_h) \le \overline{\nu}\,\|\nabla\sigma_h\|_{0,2,\Omega}\,\|\nabla l_h\|_{0,2,\Omega} \le \frac{5}{4}\,\frac{\overline{\nu}^2}{\mu}\,\|\sigma_h\|^2_{1,2,\Omega} + \frac{\mu}{5}\,\|\nabla l_h\|^2_{0,2,\Omega}$$
$$\le C\,\frac{\overline{\nu}^2}{\mu}\,\|D\mathbf{z}_h\|^2_{0,2,\Omega} + \frac{\mu}{5}\,\|\nabla l_h\|^2_{0,2,\Omega}, \tag{12.21}$$

where we have used that, as Π_h is stable in H^1 norm and $k_\Gamma \in W^{1,\infty}(\mathbb{R})$,

$$\|\sigma_h\|_{1,2,\Omega} \le C\|k_\Gamma(\mathbf{z}_h)\|_{1,2,\Omega} \le C\,\|D\mathbf{z}_h\|_{0,2,\Omega}. \tag{12.22}$$

Moreover, by Hölder's inequality,

$$(\sqrt{|\kappa_h|}\,\sigma_h,l_h)_\Omega \le C\,\|\sqrt{|\kappa_h|}\|_{0,2,\Omega}\,\|l_h\|_{0,2,\Omega} \le C\,\|\kappa_h\|^{1/2}_{0,1,\Omega}\,\|\nabla l_h\|_{0,2,\Omega}$$
$$\le C\,\frac{1}{\mu}\,\|\kappa_h\|_{0,1,\Omega} + \frac{\mu}{5}\,\|\nabla l_h\|^2_{0,2,\Omega}$$
$$\le C\,\frac{1}{\mu}\,\|\nabla\kappa_h\|_{0,2,\Omega} + \frac{\mu}{5}\,\|\nabla l_h\|^2_{0,2,\Omega}, \tag{12.23}$$

where we have used that the $H^1(\Omega)$ norm and seminorm are equivalent on the space $V_{\Gamma_D} = \{\kappa \in H^1(\Omega) \text{ s.t. } \kappa_{|\Gamma_D=0}\}$. Combining (12.18), (12.19), (12.20), (12.21), and (12.23) and setting $l_{0h} = \lambda_{0h}$,

$$\mu\,\|\nabla\lambda_{0h}\|^2_{0,2,\Omega} \le C\,(\frac{1}{\mu}\,L^2 + \frac{1+\overline{\nu}^2}{\mu}\,\|D\mathbf{z}_h\|^2_{0,2,\Omega} + \frac{1}{\mu}\,\|\nabla\kappa_h\|_{0,2,\Omega}\,), \tag{12.24}$$

As $\lambda_h = \lambda_{0h} + \sigma_h$ combining (12.24) with (12.17) and (12.22) we deduce

$$\mu^2\,\|\nabla\lambda_h\|^2_{0,2,\Omega} \le C\,\left(A + \|\nabla\kappa_h\|_{0,2,\Omega}\right), \tag{12.25}$$

where $A = \left(L^2 + \dfrac{\mu^2+\overline{\nu}^2+1}{\nu^2}\,\|f\|^2_{\mathbf{W}_D(\Omega)'}\right)$. Assume now that $\|\nabla\kappa_h\|^2_{0,2,\Omega} \le R$ for some $R \ge 0$. Then $\|\nabla\lambda_h\|^2_{0,2,\Omega} \le \dfrac{1}{\mu^2}\,C\,(A+R)$. Consequently, $\|\nabla\lambda_h\|_{0,2,\Omega} \le R$ when $C\,(A+R) \le \mu^2\,R^2$. This occurs for R large enough, in particular when $R = \hat{R} = C\,\left(\dfrac{1}{\mu^2} + \dfrac{\sqrt{A}}{\mu}\right)$. Let us define now the set

$$S {=} \{(\mathbf{u}_h,\kappa_h) \in \mathbf{W}_h \times K_h \ \text{ s.t. } \|D\mathbf{u}_h\|_{0,2,\Omega} \le \frac{1}{2\nu}\,\|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'},\ \|\nabla\kappa_h\|_{0,2,\Omega} \le \hat{R}\,\}.$$

Then S is nonempty and compact (it is bounded and closed in a space of finite dimension). We have proved that $\mathscr{F}(S) \subset S$.

This proves that problem (12.11) admits a solution $(\mathbf{v}_h, p_h, k_h)$ such that $\mathbf{v}_h$ and k_h respectively satisfy estimates (12.12) and (12.14). The estimate for the pressure (12.13) is obtained as in Theorem 9.4 from the one for the velocity. □

12.2.3 Stability and Convergence Analysis

In this section we prove the convergence of (a subsequence of) the sequence of solutions $\{(\mathbf{v}_h, p_h, k_h)\}_{h>0}$ of the discrete variational problems (12.11) to a solution of the variational problem (12.5) when the production term for the TKE is truncated:

$$\text{Find } (\mathbf{v}, p, k_0) \in \mathbf{W}_D(\Omega) \times L^2_0(\Omega) \times H^1_0(\Omega) \text{ such that}$$

$$(\mathscr{V}\mathscr{P})_L \left\{ \begin{aligned} b(\mathbf{v};\mathbf{v},\mathbf{w}) + s_\nu(k;\mathbf{v},\mathbf{w}) - (p, \nabla\cdot\mathbf{w})_\Omega + \langle G(\mathbf{v}),\mathbf{w}\rangle &= \langle \mathbf{f},\mathbf{w}\rangle, \\ (\nabla\cdot\mathbf{v}, q)_\Omega &= 0, \\ b_k(k;\mathbf{v},l) + s_k(k;k,l) + d(k;l) &= (P_L(k,\mathbf{v}), l), \\ k &= k_0 + k_\Gamma(\mathbf{v}), \end{aligned} \right. \tag{12.26}$$

for all $(\mathbf{w}, q, l) \in \mathbf{W}_D(\Omega) \times L^2_0(\Omega) \times H^1_0(\Omega)$. We apply the compactness method, that only yields the convergence of a subsequence. The convergence of the full sequence will follow in particular if model (12.26) admits a unique solution. As a sub-product of our analysis, we deduce a partial well-posedness result for model (12.26): It admits a solution which is bounded (in convenient norms) by the (convenient) norm of the data.

We shall need the following standard approximation property for the space of discrete TKE:

Hypothesis 12.iii.

- For all $l \in \mathscr{D}(\Omega)$, there exists a sequence $(l_h)_{h>0}$ such that $l_h \in K_{0h}$ and $\lim_{h\to 0} \|l - l_h\|_{1,\infty,\Omega} = 0$.

These properties are satisfied by the Lagrange finite element spaces introduced in Sect. 9.3.1.

Theorem 12.2. *Under the hypotheses of Theorem 12.1, assume also that Hypothesis 12.iii holds. Then the sequence of variational problems $(\mathscr{V}\mathscr{P})_h$ (12.11) converges to the variational problem $(\mathscr{V}\mathscr{P})_L$ (12.26). More specifically, the sequence $\{(\mathbf{v}_h, p_h, k_{0h})\}_{h>0}$ of solutions of the discrete problem (12.11) contains a subsequence that is strongly convergent in $\mathbf{W}_D(\Omega) \times L^2_0(\Omega) \times H^1(\Omega)$ to a solution $(\mathbf{v}, p, k)$ of the truncated problem (12.26). This solution satisfies the estimates*

$$\|D\mathbf{v}\|_{0,2,\Omega} \le \frac{1}{2\nu} \|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'}, \tag{12.27}$$

$$\|p\|_{0,2,\Omega} \leq \frac{C}{\nu^2} \|\mathbf{f}\|^2_{\mathbf{W}_D(\Omega)'} + C\,\frac{1}{\nu}(1+\nu)\,\|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'}. \tag{12.28}$$

$$\|\nabla k\|_{0,2,\Omega} \leq \frac{C}{\mu}\left(\frac{1}{\mu} + L + \frac{\mu+\overline{\nu}+1}{\nu}\,\|f\|_{\mathbf{W}_D(\Omega)'}\right), \tag{12.29}$$

for some constant $C > 0$. If the solution of problem (12.26) is unique, then the whole sequence converges to it.

Proof. We proceed by steps.

STEP 1. *Extraction of convergent subsequences.* As $\mathbf{W}_D(\Omega) \times L^2_0(\Omega) \times H^1(\Omega)$ is a Hilbert space, there exists a subsequence (that we denote in the same way) of $((\mathbf{v}_h, p_h, k_{0h}))_{h>0}$ that weakly converges to some $(\mathbf{v}, p, k)$ in that space. As $H^1(\Omega) \hookrightarrow L^r(\Omega)$ for $r < 6$, we may assume that $(\mathbf{v}_h)_{h>0}$ strongly converges in $\mathbf{L}^r(\Omega)$ and that $(k_h)_{h>0}$ converges strongly in $L^r(\Omega)$ for $1 \leq r < 6$ and a.e. in Ω.

STEP 2. *Limit of momentum conservation equation.* Let $\mathbf{w} \in \mathbf{W}_D(\Omega)$. By Hypothesis 9.i there exists a sequence $(\mathbf{w}_h)_{h>0}$ with $\mathbf{w}_h \in \mathbf{W}_h$ strongly convergent in $\mathbf{W}_D(\Omega)$ to $\mathbf{w}$, such that $\mathbf{w}_h \in \mathbf{W}_h$ for all $h > 0$.

To pass to the limit in the eddy diffusion term, let us prove that $\tilde{\nu}_t(k_h)\,D\mathbf{w}_h$ strongly converges to $\tilde{\nu}_t(k)\,D\mathbf{w}$ in $L^2(\Omega)^{d\times d}$. Indeed, as $\nu_t \in W^{1,\infty}(\mathbb{R})$,

$$\begin{aligned}
&\|\tilde{\nu}_t(k_h)\,D\mathbf{w}_h - \tilde{\nu}_t(k)\,D\mathbf{w}\|_{0,2,\Omega} \leq \|[\tilde{\nu}_t(k_h) - \tilde{\nu}_t(k)]\,D\mathbf{w}_h\|_{0,2,\Omega} + \\
&\|\tilde{\nu}_t(k)\,[D\mathbf{w}_h - D\mathbf{w}]\|_{0,2,\Omega} \leq \|\tilde{\nu}_t'\|_{0,\infty,\mathbb{R}}\,\|k_h - k\|_{0,2,\Omega}\|D\mathbf{w}_h\|_{0,2,\Omega} + \\
&\|\tilde{\nu}_t\|_{0,\infty,\mathbb{R}}\,\|D\mathbf{w}_h - D\mathbf{w}\|_{0,2,\Omega} \leq C\;[\|k_h - k\|_{0,2,\Omega} + \|D\mathbf{w}_h - D\mathbf{w}\|_{0,2,\Omega}]
\end{aligned}$$

Thus, $\tilde{\nu}_t(k_h)\,D\mathbf{w}_h$ strongly converges to $\tilde{\nu}_t(k)\,D\mathbf{w}$ in $L^2(\Omega)^{d\times d}$. As $D\mathbf{v}_h$ weakly converges to $D\mathbf{v}$ in $L^2(\Omega)^{d\times d}$, we conclude that

$$\lim_{h\to 0}(\tilde{\nu}_t(k_h)\,D\mathbf{v}_h, D\mathbf{w}_h)_\Omega = (\tilde{\nu}_t(k)\,D\mathbf{v}, D\mathbf{w})_\Omega, \quad \text{for all } \mathbf{w} \in \mathbf{W}_D(\Omega).$$

All the remaining terms in the momentum conservation equation pass to the limit as in the proof of Theorem 9.5. Then the pair $(\mathbf{v}, p)$ satisfies the momentum conservation equation, for all $\mathbf{w} \in \mathbf{W}_D(\Omega)$.

STEP 3. *Limit of deformation energy.* Here we prove that (up to a subsequence)

$$\lim_{h\to 0}\|\sqrt{\tilde{\nu}_t(k_h)}\,D\mathbf{v}_h\|_{0,2,\Omega} = \|\sqrt{\tilde{\nu}_t(k)}\,D\mathbf{v}\|_{0,2,\Omega}. \tag{12.30}$$

Indeed, set $\mathbf{w}_h = \mathbf{v}_h$ in (12.11). Then by the compactness of G as an operator form $H^1(\Omega)$ onto $[H^1(\Omega)]'$, there exists a subsequence of $(\mathbf{v}_h)_{h>0}$, still denoted in the same way, such that

$$
\begin{aligned}
\lim_{h\to 0} \| \sqrt{\tilde{\nu}_t(k_h)}\, D\mathbf{v}_h \|^2_{0,2,\Omega} &= \lim_{h\to 0} \left(\langle \mathbf{f}, \mathbf{v}_h \rangle - \langle G(\mathbf{v}_h), \mathbf{v}_h \rangle \right) \\
&= \langle \mathbf{f}, \mathbf{v} \rangle - \langle G(\mathbf{v}), \mathbf{v} \rangle = \| \sqrt{\tilde{\nu}_t(k)}\, D\mathbf{v} \|^2_{0,2,\Omega},
\end{aligned}
$$

where the last equality follows setting $\mathbf{w} = \mathbf{v}$ in (12.26). It follows that $\sqrt{\tilde{\nu}_t(k_h)}\, D\mathbf{v}_h$ converges to $\sqrt{\tilde{\nu}_t(k)}\, D\mathbf{v}$ in $L^2(\Omega)^{d\times d}$. Indeed, $\sqrt{\tilde{\nu}_t(k_h)}\, D\mathbf{v}_h$ is bounded in $L^2(\Omega)^{d\times d}$ and then (up to a subsequence) it weakly converges in this space to some function ρ. To identify ρ, consider $\varphi \in L^2(\Omega)^{d\times d}$. Then $\sqrt{\tilde{\nu}_t(k_h)}\, \varphi$ converges a.e. to $\sqrt{\tilde{\nu}_t(k)}\, \varphi$ and is uniformly bounded by some function of $L^2(\Omega)^{d\times d}$. Then it strongly converges in this space. Consequently,

$$
\begin{aligned}
\lim_{h\to 0} \int_\Omega \sqrt{\tilde{\nu}_t(k_h(\mathbf{x}))}\, D\mathbf{v}_h(\mathbf{x}) : \varphi(\mathbf{x})\, d\mathbf{x} &= \int_\Omega \sqrt{\tilde{\nu}_t(k(\mathbf{x}))}\, D\mathbf{v}(\mathbf{x}) : \varphi(\mathbf{x})\, d\mathbf{x} \\
&= \int_\Omega \rho(\mathbf{x}) : \varphi(\mathbf{x})\, d\mathbf{x}
\end{aligned}
$$

Thus, $\rho = \sqrt{\tilde{\nu}_t(k)}\, D\mathbf{v}$. As the limit is unique, the whole sequence converges to it. By (12.30), we deduce that $\sqrt{\tilde{\nu}_t(k_h)}\, D\mathbf{v}_h$ strongly converges to $\sqrt{\tilde{\nu}_t(k)}\, D\mathbf{v}$ in $L^2(\Omega)^{d\times d}$.

STEP 4. *Strong convergence of velocity and pressure.* As $\mathbf{v}_h$ weakly converges to $\mathbf{v}$ in $\mathbf{W}_D(\Omega)$, the strong convergence of $\mathbf{v}_h$ to $\mathbf{v}$ in this space will follow if we prove that $\|D\mathbf{v}_h\|_{0,2,\Omega}$ converges to $\|D\mathbf{v}_h\|_{0,2,\Omega}$:

$$
\begin{aligned}
\left| \|D\mathbf{v}_h\|^2_{0,2,\Omega} - \|D\mathbf{v}\|^2_{0,2,\Omega} \right| &\le \frac{1}{\nu} \int_\Omega \tilde{\nu}_t(k_h(\mathbf{x})) \left| |D\mathbf{v}_h(\mathbf{x})|^2 - |D\mathbf{v}(\mathbf{x})|^2 \right| d\mathbf{x} \\
&\le \frac{1}{\nu} \int_\Omega \left| \tilde{\nu}_t(k_h(\mathbf{x}))\, |D\mathbf{v}_h(\mathbf{x})|^2 - \tilde{\nu}_t(k(\mathbf{x}))\, |D\mathbf{v}(\mathbf{x})|^2 \right| d\mathbf{x} \qquad (12.31) \\
&+ \frac{1}{\nu} \int_\Omega \left| \tilde{\nu}_t(k(\mathbf{x}))\, |D\mathbf{v}(\mathbf{x})|^2 - \tilde{\nu}_t(k_h(\mathbf{x}))\, |D\mathbf{v}(\mathbf{x})|^2 \right| d\mathbf{x}. \qquad (12.32)
\end{aligned}
$$

The term in (12.31) tends to zero due to Step 3. Also, $\tilde{\nu}_t(k_h(\mathbf{x}))\, |D\mathbf{v}(\mathbf{x})|^2$ converges a.e. to $\tilde{\nu}_t(k(\mathbf{x}))\, |D\mathbf{v}(\mathbf{x})|^2$ in Ω and is uniformly bounded by some function of $L^1(\Omega)$. Then the term in (12.32) also tends to zero.

The strong convergence of the pressure is proved similarly to that in Theorem 9.5.

STEP 5. *Limit of the TKE equation.* The convection and diffusion terms in the TKE in problem (12.11) pass to the limit as the corresponding terms in the momentum conservation equation.

Dissipation term. Let us consider the real function $\phi(\alpha) = \sqrt{|\alpha|}\,\alpha$. As $\nabla \phi(k_h) = \frac{3}{2} \sqrt{|k_h|}\, \nabla k_h$,

$$
\|\nabla \phi(k_h)\|_{0,1,\Omega} \le \frac{3}{2} \|k_h\|^{1/2}_{0,1,\Omega}\, \|\nabla k_h\|_{0,2,\Omega}.
$$

Then $\phi(k_h)$ is bounded in $W^{1,1}(\Omega)$, and as it converges a.e. to $\phi(k)$ in Ω, by Sobolev injections it is strongly convergent to $\phi(k)$ in $L^{3/2}(\Omega)$, up to a subsequence. Let $l \in \mathscr{D}(\Omega)$. By Hypothesis 12.iii, there exists a sequence $(l_h)_{h>0}$ such that $l_h \in K_{0h}$ strongly converges to l in $W^{1,3}(\Omega)$. Then

$$\lim_{h\to 0}(\sqrt{|k_h|}\,k_h, l_h)_\Omega = (\sqrt{|k|}\,k, l)_\Omega.$$

Production term. Denote $g_h = \tilde{\nu}_t(k_h)\,|D\mathbf{v}_h|^2$, $g = \tilde{\nu}_t(k)\,|D\mathbf{v}|^2$. By Step 3, g_h strongly converges to g in $L^1(\Omega)$. As $|T_L'| \le 1$ a.e. in $\mathbb{R}$, then $|T_L(g_h) - T_L(g)| \le |g_h - g|$, and $T_L(g_h)$ strongly converges to $T_L(g)$ in $L^1(\Omega)$. Thus,

$$\lim_{h\to 0}(P_L(k_h, \mathbf{v}_h), l_h)_\Omega = (P_L(k, \mathbf{v}), l)_\Omega.$$

We deduce that the equation for the TKE in model (12.26) holds for all $l \in \mathscr{D}(\Omega)$. But the forms $b(k; \mathbf{v}, \cdot)$ and $s_k(k; k, \cdot)$ belong to $H^{-1}(\Omega)$, $\sqrt{|k|}\,k \in L^4(\Omega)$ and $P_L(k, \mathbf{v}) \in L^\infty(\Omega)$. Then (12.26) holds for all $l \in H_0^1(\Omega)$ as $\mathscr{D}(\Omega)$ is dense in $H_0^1(\Omega)$.

STEP 6. *Dirichlet boundary condition for TKE.* As $k_\Gamma \in W^{1,\infty}(\mathbb{R})$, $k_\Gamma(\mathbf{v}_h)$ strongly converges to $k_\Gamma(\mathbf{v})$ in $H^1(\Omega)$. Also, due to the H^1 stability of the interpolation operator Π_h,

$$\begin{aligned}
&\|\Pi_h(k_\Gamma(\mathbf{v}_h)) - k_\Gamma(\mathbf{v})\|_{1,2,\Omega} \\
&\le \|\Pi_h(k_\Gamma(\mathbf{v}_h)) - \Pi_h(k_\Gamma(\mathbf{v}))\|_{1,2,\Omega} + \|\Pi_h(k_\Gamma(\mathbf{v})) - k_\Gamma(\mathbf{v})\|_{1,2,\Omega} \\
&\le C\,\|k_\Gamma(\mathbf{v}_h) - k_\Gamma(\mathbf{v})\|_{1,2,\Omega} + \|\Pi_h(k_\Gamma(\mathbf{v})) - k_\Gamma(\mathbf{v})\|_{1,2,\Omega}.
\end{aligned}$$

Then $\Pi_h(k_\Gamma(\mathbf{v}_h))$ converges to $k_\Gamma(\mathbf{v})$ in $H^1(\Omega)$. As $k_{bh} = k_h - \Pi_h(k_\Gamma(\mathbf{v}_h))$, we deduce that k_{bh} converges to $k_0 = k - k_\Gamma(\mathbf{v})$ in $H^1(\Omega)$. Finally, $k_0 \in H_0^1(\Omega)$ because each k_{bh} belongs to $H_0^1(\Omega)$. □

12.3 Unsteady NS-TKE Model

In this section we perform the finite element approximation of the unsteady TKE introduced in Chap. 8. We prove its stability and give some partial convergence results. Our purpose is to stress the difficulties that arise in the unsteady case to pass to the limit in the production term for the TKE. The main technical difficulty, once more, is the lack of regularity of the velocity. This prevents to prove the strong convergence of the velocity and consequently to pass to the limit in the production term for the TKE. As a consequence, only a super-solution of the TKE equation is obtained.

The lack of regularity of the velocity also prevents to obtain estimates for the time derivative of the lifting of the boundary conditions for the TKE $k = k_\Gamma$. To avoid this difficulty we shall assume $k_\Gamma = constant$ in our analysis, which models a constant generation of TKE on the whole boundary Γ. Our analysis may be extended by standard techniques to the case where k_Γ is a smooth function that does not vary in time, although we restrict ourselves to $k_\Gamma = constant$ to avoid unnecessary complexities.

12.3.1 Statement of Unsteady Model Equations

In this section we state a weak formulation for the unsteady first-order viscosity model that we shall use in our analysis:

Find a velocity field $\mathbf{v} : \overline{\Omega} \times [0, T] \to \mathbb{R}^d$, a pressure $p : \Omega \times (0, T) \to \mathbb{R}$, and a TKE variable $k : \overline{\Omega} \times [0, T] \to \mathbb{R}$ such that

$$\begin{cases} \partial_t \mathbf{v} + \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - \nabla \cdot (\tilde{\nu}_t(k)\, D\mathbf{v}) + \nabla p = \mathbf{f} & \text{in } \Omega \times (0,T); \\ \nabla \cdot \mathbf{v} = 0 & \text{in } \Omega \times (0,T); \\ \partial_t k + \nabla \cdot (\mathbf{v}k) - \nabla \cdot (\tilde{\mu}_t(k)\, \nabla k) + \dfrac{k^{3/2}}{\ell} = P_L(k, \mathbf{v}) & \text{in } \Omega \times (0,T); \\ \mathbf{v}(0) = \mathbf{v}_0, \ k(0) = k_0, & \text{in } \Omega, \\ -(\tilde{\nu}_t(k)\, \mathbf{n} \cdot D\mathbf{v})_{\boldsymbol{\tau}} = g(\mathbf{v})_{\boldsymbol{\tau}}, \ \mathbf{v} \cdot \mathbf{n} = 0, \ k = k_\Gamma(\mathbf{v}) & \text{on } \Gamma_n \times (0,T); \\ \mathbf{v} = 0, \ k = 0 & \text{on } \Gamma_D \times (0,T). \end{cases} \tag{12.33}$$

We shall consider weak solutions of system (12.33), defined as follows.

Definition 12.1. Let $\mathbf{f} \in L^2(\mathbf{W}_D(\Omega)')$, $\mathbf{v}_0 \in \mathbf{W}_D(\Omega)'$, $k_0 \in H^{-1}(\Omega)$. A triplet $(\mathbf{v}, p, k) \in \mathscr{D}'(Q_T)^d \times \mathscr{D}'(Q_T) \times \mathscr{D}'(Q_T)$ is a weak solution of problem (12.33) if it satisfies the variational problem

$(\mathscr{V}\mathscr{P})$ Find $\mathbf{v} \in L^2(\mathbf{W}_{Div}(\Omega)) \cap L^\infty(\mathbf{L}^2)$, $R \in L^\infty(L^2)$ such that $p = \partial_t R$, $k \in L^2(H^1) \cap L^\infty(L^2)$, such that for all $\mathbf{w} \in \mathbf{W}_D(\Omega)$, $l \in H_0^1(\Omega)$, φ, $\psi \in \mathscr{D}([0,T])$ such that $\varphi(T) = 0$, $\psi(T) = 0$,

$$\begin{cases} -\displaystyle\int_0^T (\mathbf{v}(t), \mathbf{w})_\Omega \varphi'(t)\, dt - \langle \mathbf{v}_0, \mathbf{w} \rangle\, \varphi(0) \\ +\displaystyle\int_0^T [b(\mathbf{v}(t); \mathbf{v}(t), \mathbf{w})\, dt + s_v(k(t); \mathbf{v}(t), \mathbf{w}) + \langle G(\mathbf{v}(t)), \mathbf{w} \rangle]\, \varphi(t)\, dt \\ +\displaystyle\int_0^T (R(t), \nabla \cdot \mathbf{w})_\Omega \varphi'(t)\, dt = \int_0^T \langle \mathbf{f}(t), \mathbf{w} \rangle \varphi(t)\, dt; \end{cases} \tag{12.34}$$

$$\begin{cases} -\displaystyle\int_0^T (k(t), l)_\Omega \psi'(t)\,dt - \langle k_0, l\rangle\, \psi(0) \\ +\displaystyle\int_0^T \left[b_k(\mathbf{v}(t); k(t), l)\,dt + s_k(k(t), l) + d(k(t), l)\right]\, \psi(t)\,dt \\ = \displaystyle\int_0^T \left(P_L(k(t), v(t)), l\right)_\Omega\, \psi(t)\,dt; \end{cases} \tag{12.35}$$

$$\gamma_0\,(k(t)) = \gamma_0\,(k_\Gamma(t))\,, \quad \text{in } \; H^{1/2}(\Gamma), \;\; \text{a.e. in } \; (0, T). \tag{12.36}$$

This definition makes sense because due to the regularity asked for $\mathbf{v}$, R, and k, all terms in (12.34) and (12.35) are integrable in $(0, T)$. The weak solutions given by this definition are solutions of model (12.33) in the following sense.

Lemma 12.2. *Let $(\mathbf{v}, p, k) \in \mathscr{D}'(Q_T)^d \times \mathscr{D}'(Q_T) \times \mathscr{D}'(Q_T)$ be a weak solution of model (12.33) such that k is nonnegative. Then*

(i) The equations

$$\partial_t \mathbf{v} + \nabla\cdot(\mathbf{v}\otimes\mathbf{v}) - \nabla\cdot(\nu_t(k)\, D\mathbf{v}) + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{v} = 0, \tag{12.37}$$

$$\partial_t k + \nabla\cdot(\mathbf{v}\, k) - \nabla\cdot(\mu_t(k)\,\nabla k) + \frac{k^{3/2}}{\ell} = P_L(k, v) \tag{12.38}$$

respectively hold in $\mathscr{D}'(Q_T)^d$, in $L^2(Q_T)$, and in $\mathscr{D}'(Q_T)$.

(ii)

$$\mathbf{v} \in C^0([0,T], \mathbf{W}_{Div}(\Omega)') \;\; \textit{and} \;\; \mathbf{v}(0) = \mathbf{v}_0 \;\; \textit{in} \;\; \mathbf{W}_{Div}(\Omega)',$$

$$k \in C^0([0,T], H^{-1}(\Omega)) \;\; \textit{and} \;\; k(0) = k_0 \;\; \textit{in} \;\; H^{-1}(\Omega),$$

(iii)

$$\gamma_0\mathbf{v} = 0 \;\; \textit{in} \;\; L^2(\mathbf{H}^{1/2}(\Gamma_D)), \quad \gamma_n\mathbf{v} = 0 \;\; \textit{in} \;\; L^2(L^4(\Gamma_n)),$$

$$\gamma_0\, k = \gamma_0\,(k_\Gamma(\mathbf{v})) \;\; \textit{in} \;\; L^2(H^{1/2}(\Gamma_n)), \;\; \gamma_0\, k = 0 \;\; \textit{in} \;\; L^2(H^{1/2}(\Gamma_D)).$$

(iv) If $\mathbf{v} \in L^2(\mathbf{H}^2)$, $\partial_t\mathbf{v} \in L^2(\mathbf{L}^2)$ and $R \in L^2(\mathbf{H}^1)$, then $-(\tilde{\nu}_t(k)\,\mathbf{n}\cdot D\mathbf{v})_\tau = g(\mathbf{v})_\tau$ in $L^1(L^{3/2}(\Gamma_n))^{d-1}$.

The proof of this lemma is similar to that of Lemma 10.1; we omit it for brevity.

12.3.2 Discretization

We perform a first-order discretization in time combined with a finite element discretization in space, similarly to that introduced in Sect. 10.3. We linearize the eddy diffusion terms in order to obtain a linear approximation of them.

Consider a positive integer number N and define the time step $\Delta t = T/N$ and the discrete times of solution $t_n = n\Delta t$, $n = 0, 1, \cdots, N$. We obtain the approximations $\mathbf{v}_h^n$, p_h^n, and k_h^n to $\mathbf{v}(t_n, \cdot)$, $p(t_n, \cdot)$, and $k(t_n, \cdot)$ by

- *Initialization.* Assume that $\mathbf{v}_0 \in \mathbf{L}^2(\Omega)$ and $k_0 \in L^2(\Omega)$. Set

$$\mathbf{v}_h^0 = \mathbf{v}_{0h}, \quad k_h^0 = k_{0h}, \tag{12.39}$$

where $\mathbf{v}_{0h} \in \mathbf{W}_h$ and $k_{0h} \in K_h$ respectively are interpolates of $\mathbf{v}_0$ and k_0 that satisfy

$$\lim_{h\to 0} \|\mathbf{v}_{0h} - \mathbf{v}_0\|_{0,2,\Omega} = 0, \quad \lim_{h\to 0} \|k_{0h} - k\|_{0,2,\Omega} = 0. \tag{12.40}$$

For instance, $\mathbf{v}_{0h}$ may be given by (10.7) and similarly k_{0h}.

- *Iteration.* For $n = 0, 1, \cdots, N-1$: Assume known $\mathbf{v}_h^n \in \mathbf{W}_h$, $k_h^n \in K_h$. The new iterates are solution of the variational problem $(\mathcal{VP})_h$. Obtain $\mathbf{v}_h^{n+1} \in \mathbf{W}_h$, $p_h^{n+1} \in M_h$, and $k_h^{n+1} \in K_h$ such that for all $\mathbf{w}_h \in \mathbf{W}_h$, $q_h \in M_h$, $l_h \in K_{0h}$,

$$\begin{cases} (\dfrac{\mathbf{v}_h^{n+1} - \mathbf{v}_h^n}{\Delta t}, \mathbf{w}_h)_\Omega + b(\mathbf{v}_h^n; \mathbf{v}_h^{n+1}, \mathbf{w}_h) + s_v(k_h^n; \mathbf{v}_h^{n+1}, \mathbf{w}_h) \\ \qquad + \langle G(\mathbf{v}_h^{n+1}), \mathbf{w}_h\rangle - (p_h^{n+1}, \nabla \cdot \mathbf{w}_h)_\Omega = \langle \mathbf{f}^{n+1}, \mathbf{w}_h\rangle, \\ \qquad\qquad (\nabla \cdot \mathbf{v}_h^{n+1}, q_h)_\Omega = 0, \end{cases} \tag{12.41}$$

$$\begin{cases} (\dfrac{k_h^{n+1} - k_h^n}{\Delta t}, l_h)_\Omega + b_k(\mathbf{v}_h^n; k_h^{n+1}, l_h) + s_k(k_h^n; k_h^{n+1}, l_h) + d(k_h^{n+1}; l_h) \\ \qquad\qquad = (P_L(k_h^n, \mathbf{v}_h^{n+1}), l_h)_\Omega, \end{cases} \tag{12.42}$$

$$k_h^{n+1} = k_{0h}^{n+1} + k_\Gamma \ \text{ for some } \ k_{0h}^{n+1} \in K_{0h} \tag{12.43}$$

where $\mathbf{f}^{n+1}$ is the average value of $\mathbf{f}$ in $[t_n, t_{n+1}]$ and we assume k_Γ constant.

Note that the equations for $(\mathbf{v}_h^{n+1}, p_h^{n+1})$ and k_h^{n+1} are decoupled. This yields discrete problems with smaller size. We pass from a global problem with $d+2$ unknowns to two problems, one with $d+1$ unknowns and another with 1 unknown. This allows an easier use of parallel solvers. The problem for k_h^{n+1} is linear, while the only nonlinearity present in the problem for $(\mathbf{v}_h^{n+1}, p_h^{n+1})$ is the wall-law term that can be treated by linearization (See Sect. 11.6). The eddy viscosities have been linearized in both problems.

Systems (12.41), (12.42), and (12.43) admit a solution. This may be proved by a simple linearization process, similarly to that used in the proof of Theorem 12.1, that we do not detail for simplicity.

More complex discretizations may be considered, by increasing the discretization order in time, by linearizing the wall-law term, or by upwinding the convection term to stabilize the convection-dominance effects. Here we have considered a relatively simple discretization whose analysis, however, contains the main difficulties that the presence of the closure terms adds to the yet complex discretization of Navier–Stokes equations.

12.3.3 Stability and Convergence Analysis

Let us add the following discrete functions to those introduced in Sect. 10.3:

- $k_h : [0, T] \to K_h$ is the piecewise linear in time function that takes on the value k_h^n at $t = t_n$.
- $k_{0h} : [0, T] \to K_{0h}$ is the piecewise linear in time function that takes on the value k_{0h}^n at $t = t_n$.
- $\sigma_h : [0, T] \to K_h$ is the piecewise linear in time function that takes on the value $\sigma_h^n = \Pi_h(k_\Gamma)$ at $t = t_n$.
- $\tilde{k}_h : (-\Delta t, T) \to K_h$ is the piecewise constant function that takes on the value k_h^{n+1} on (t_n, t_{n+1}), and $\tilde{k}_h(t) = k_h^0$ in $(-\Delta t, 0)$. This function is defined a.e. in $(-\Delta t, T)$.
- $\tilde{k}_h^- : (0, T) \to K_h$ is the piecewise constant function that takes on the value k_h^n on (t_n, t_{n+1}). This function is defined a.e. in $(0, T)$.
- $\tilde{k}_{bh} : (0, T) \to K_{0h}$ is the piecewise constant function that takes on the value k_{0h}^n on (t_n, t_{n+1}). This function is defined a.e. in $(0, T)$.

Again to simplify an already complex notation we do not make explicit the dependence of these functions upon Δt. We shall work with the following reformulation of problems (12.41), (12.42), and (12.43):

$$
\begin{aligned}
&- \int_0^T (\mathbf{v}_h(t), \mathbf{w}_h)_\Omega \varphi'(t)\, dt - (\mathbf{v}_{0h}, \mathbf{w}_h)_\Omega\, \varphi(0) + \int_0^T b(\tilde{\mathbf{v}}_h^-(t); \tilde{\mathbf{v}}_h(t), \mathbf{w}_h)\, \varphi(t)\, dt \\
&+ \int_0^T s_v(\tilde{k}_h^-(t); \tilde{\mathbf{v}}_h(t), \mathbf{w}_h)\, \varphi(t)\, dt + \int_0^T \langle G(\tilde{\mathbf{v}}_h(t)), \mathbf{w}_h\rangle\, \varphi(t)\, dt \\
&+ \int_0^T (P_h(t), \nabla\cdot \mathbf{w}_h)_\Omega \varphi'(t)\, dt = \int_0^T \langle \tilde{\mathbf{f}}_h(t), \mathbf{w}_h\rangle\, \varphi(t)\, dt, \qquad (12.44)
\end{aligned}
$$

for all $\mathbf{w}_h \in \mathbf{W}_h$ and for all $\varphi \in \mathscr{D}([0, T])$ such that $\varphi(T) = 0$;

$$
\begin{aligned}
&- \int_0^T (k_h(t), l_h)_\Omega \psi'(t)\, dt - (k_{0h}, l_h)_\Omega\, \psi(0) + \int_0^T b_k(\tilde{\mathbf{v}}_h^-(t); \tilde{k}_h(t), l_h)\, \varphi(t)\, dt \\
&+ \int_0^T s_k(\tilde{k}_h^-; \tilde{k}_h(t), l_h)\, \psi(t)\, dt + \int_0^T d(\tilde{k}_h(t); l_h)\, \psi(t)\, dt \\
&= \int_0^T \Big(P_L(\tilde{k}_h^-(t), \tilde{\mathbf{v}}_h^-(t)), l_h\Big)_\Omega\, \psi(t)\, dt, \qquad (12.45)
\end{aligned}
$$

for all $l_h \in K_{0h}$ and for all $\psi \in \mathscr{D}([0,T])$ such that $\psi(T) = 0$,

$$\tilde{k}_h(t) = \tilde{k}_{bh}(t) + k_\Gamma \quad \text{in} \quad H^{1/2}(\Gamma) \quad \text{a.e. in} \quad (0,T). \tag{12.46}$$

The stability of problems (12.41), (12.42), and (12.43) is stated next.

Theorem 12.3. *Assume that the family of grids* $\{\mathscr{T}_h\}_{h>0}$ *is regular. Assume that* $\mathbf{v}_0 \in \mathbf{L}^2(\Omega)$, $k_0 \in L^2(\Omega)$, $\mathbf{f} \in L^2(\mathbf{W}_D(\Omega)')$. *Let* $((\mathbf{W}_h, M_h, K_h))_{h>0}$ *be a family of pairs of finite element spaces satisfying Hypotheses 9.i, 9.ii, and 12.ii. Assume that the functions* $\tilde{\nu}_t$ *and* $\tilde{\mu}_t$ *satisfy Hypothesis 12.i. Then the discrete variational problem* $(\mathscr{V}\mathscr{P})_h$ *(12.41), (12.42), and (12.43) admit a solution that satisfies the following estimates:*

$$\|\mathbf{v}_h\|_{L^\infty(\mathbf{L}^2)} + \sqrt{\nu}\,\|\mathbf{v}_h\|_{L^2(\mathbf{H}^1)} \le C_1 \left(\|\mathbf{v}_{0h}\|_{0,2,\Omega} + \frac{1}{\sqrt{\nu}} \|\mathbf{f}\|_{L^2(\mathbf{W}_D(\Omega)')} \right), \tag{12.47}$$

$$\|\mathbf{v}_h\|_{N^{1/4,2}(\mathbf{L}^2)} \le C_2, \quad \|P_h\|_{L^\infty(L^2)} \le C_2, \tag{12.48}$$

$$\begin{aligned} \|k_h\|_{L^\infty(\mathbf{L}^2)} + \|k_h\|_{L^2(\mathbf{H}^1)} \le{}& \|k_{0h}\|_{0,2,\Omega} + |k_\Gamma| \\ &+ C_2 \left(\sqrt{T} + \|\mathbf{v}_{0h}\|_{0,2,\Omega} + \|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'} \right), \end{aligned} \tag{12.49}$$

and

$$\|k_h\|_{N^{1/4,2}(L^2)}\, dt \le C_2, \tag{12.50}$$

for some constant $C_1 > 0$ *independent of* h, Δt, ν, *and* μ *and some constant* $C_2 > 0$ *independent of* h *and* Δt.

Proof. Estimates (12.47) and (12.48) are derived as in Theorem 10.1, using that by Hypothesis 12.i $\tilde{\mu}_t$ is bounded from below by μ and bounded. To obtain the estimates for the TKE, we proceed by steps.

STEP 1. *Estimates for TKE.* Set $l_h = k_{0h}^{n+1}$ in (12.42). We obtain

$$\begin{aligned} &\left(\frac{k_{0h}^{n+1} - k_{0h}^n}{\Delta t}, k_{0h}^{n+1} \right)_\Omega + b_k(\mathbf{v}_h^n; k_{0h}^{n+1}, k_{0h}^{n+1}) + s_k(k_h^n; k_{0h}^{n+1}, k_{0h}^{n+1}) \\ &+ (\sqrt{|k_h^{n+1}|}\, k_{0h}^{n+1}, k_{0h}^{n+1})_\Omega = (P_L(k_h^n, \mathbf{v}_h^{n+1}), k_{0h}^{n+1})_\Omega - \left(\frac{\sigma_h^{n+1} - \sigma_h^n}{\Delta t}, k_{0h}^{n+1} \right)_\Omega \\ &- b_k(\mathbf{v}_h^n; \sigma_h^{n+1}, k_{0h}^{n+1}) - s_k(k_h^n; \sigma_h^{n+1}, k_{0h}^{n+1}) - (\sqrt{|k_h^{n+1}|}\, \sigma_h^{n+1}, k_{0h}^{n+1})_\Omega. \end{aligned} \tag{12.51}$$

where using a notation similar to the one used in the steady case we denote $\sigma_h^n = k_\Gamma$. Then $\sigma_h^{n+1} - \sigma_h^n = 0$ and all remaining terms in the r.h.s. of (12.51) are bounded as in the proof of Theorem 12.1 by (12.19), (12.20), (12.21), and (12.23). Adapting these estimates, (12.51) yields

$$\|k_{0h}^{n+1}\|_{0,2,\Omega}^2 - \|k_{0h}^n\|_{0,2,\Omega}^2 + \|k_{0h}^{n+1} - k_{0h}^n\|_{0,2,\Omega}^2 + \frac{\mu}{5}\Delta t \|\nabla k_{0h}^{n+1}\|_{0,2,\Omega}^2$$
$$\leq C\,\Delta t \left(\frac{1}{\mu} L^2 + \frac{1+\overline{\nu}^2}{\nu} \|D\mathbf{v}_h^{n+1}\|_{0,2,\Omega}^2 + \frac{1}{\mu} \|\nabla k_{0h}^{n+1}\|_{0,2,\Omega} \right). \quad (12.52)$$

We bound

$$C\ \Delta t \frac{1}{\mu} \|\nabla k_{0h}^{n+1}\|_{0,2,\Omega} \leq \frac{C\ \Delta t}{\mu^3} + \frac{\mu}{10}\Delta t \|\nabla k_{0h}^{n+1}\|_{0,2,\Omega}^2.$$

Inserting this estimate in (12.52) and summing up for $n = 0, 1, \cdots, k$, for $k \leq N-1$,

$$\|k_{0h}^{k+1}\|_{0,2,\Omega}^2 + \sum_{n=0}^{k} \|k_{0h}^{n+1} - k_{0h}^n\|_{0,2,\Omega}^2 + \frac{\mu}{10}\Delta t \sum_{n=0}^{k} \|\nabla k_{0h}^{n+1}\|_{0,2,\Omega}^2$$
$$\leq \|k_{0h}^0\|_{0,2,\Omega}^2 + \frac{C}{\mu}(n+1)\Delta t\, L^2 + C\,\frac{1+\overline{\nu}^2}{\nu}\Delta t \sum_{n=0}^{k} \|D\mathbf{v}_h^{n+1}\|_{0,2,\Omega}^2$$
$$+ \frac{C}{\mu^3}(n+1)\Delta t.$$

Using (10.14) that also holds here, we deduce

$$\|k_{0h}^{k+1}\|_{0,2,\Omega}^2 + \sum_{n=0}^{k} \|k_{0h}^{n+1} - k_{0h}^n\|_{0,2,\Omega}^2 + \frac{\mu}{10}\Delta t \sum_{n=0}^{k} \|\nabla k_{0h}^{n+1}\|_{0,2,\Omega}^2$$
$$\leq \|k_{0h}^0\|_{0,2,\Omega}^2 + \frac{C}{\mu}\left(L^2 + \frac{1}{\mu^2}\right) T$$
$$+ C\left(\frac{1+\overline{\nu}^2}{\nu^2} + \frac{1}{\mu}\right)\left[\|\mathbf{v}_{0h}\|_{0,2,\Omega}^2 + \frac{1}{\nu}\|\mathbf{f}\|_{\mathbf{W}_D(\Omega)'}^2\right].$$

Then

$$\|k_h^{k+1}\|_{0,2,\Omega}^2 + \sum_{n=0}^{k} \|k_h^{n+1} - k_h^n\|_{0,2,\Omega}^2 + \frac{\mu}{4}\Delta t \sum_{n=0}^{k} \|\nabla k_h^{n+1}\|_{0,2,\Omega}^2$$

$$\le \|k_{0h}\|_{0,2,\Omega}^2 + \frac{C}{\mu}\left(L^2 + \frac{1}{\mu^2}\right) T + |k_\Gamma|^2$$
$$+C\left(1 + \frac{1+\overline{\nu}^2}{\nu^2} + \frac{1}{\mu}\right)\left[\|\mathbf{v}_{0h}\|_{0,2,\Omega}^2 + \frac{1}{\nu}\|\mathbf{f}\|^2_{\mathbf{W}_D(\Omega)'}\right]. \tag{12.53}$$

Estimate (12.49) follows.

STEP 2. *Estimates for time increment of TKE.* Integrating the equation for k_h^{n+1} in (12.42) in $(t, t+\delta)$ yields

$$(\tau_\delta k_{0h}(t), l_h)_\Omega = \int_t^{t+\delta} \langle \mathscr{K}(s), l_h\rangle_{H_0^1(\Omega)}\, ds, \tag{12.54}$$

for all $l_h \in K_{0h}$, where $\mathscr{K}(t) \in H^{-1}(\Omega)$ is defined by

$$\langle \mathscr{K}(t), l\rangle_{H_0^1(\Omega)} = -b_k(\mathbf{v}_h(t); \tilde{k}_h(t), l) - s_k(\tilde{k}_h^-(t), l) - d(\tilde{k}_h(t), l)$$
$$+\left(P_L(\tilde{k}_h^-(t), \tilde{\mathbf{v}}_h^-(t)), l\right)_\Omega, \quad \text{for all } l \in H_0^1(\Omega).$$

The form $\mathscr{K}$ is bounded by

$$\|\mathscr{K}(t)\|_{H^{-1}(\Omega)} \le C\left(\|\mathbf{v}(t)\|_{1,2,\Omega}\,\|\tilde{k}_h(t)\|_{1,2,\Omega} + \overline{\mu}\,\|\tilde{k}_h(t)\|^2_{1,2,\Omega}\right.$$
$$\left. +\ \|\tilde{k}_h(t)\|^{3/2}_{0,2,\Omega} + L\right).$$

Then, using (12.47) and (12.49),

$$\|\mathscr{K}\|_{L^1(H^{-1})} \le C. \tag{12.55}$$

Setting $l_h = \tau_\delta k_{0h}(t)$ in (12.54) and applying Fubini's Theorem,

$$\int_0^{T-\delta} \|\tau_\delta k_{0h}(t)\|^2_{0,2,\Omega}\, dt = \int_0^{T-\delta}\int_t^{t+\delta} \langle \mathscr{K}(s), \tau_\delta k_{0h}(t)\rangle_{H_0^1(\Omega)}\, ds\, dt$$
$$+ \int_0^{T-\delta} (\tau_\delta \sigma_h(t), \tau_\delta k_{0h}(t))_\Omega\, dt \le \int_0^T \int_{s-\delta}^s \langle \mathscr{K}(s), \widetilde{\tau_\delta k_{0h}}(t)\rangle_{H_0^1(\Omega)}\, dt\, ds$$
$$+\frac{1}{2}\int_0^{T-\delta} \|\tau_\delta \sigma_h(t)\|^2_{0,2,\Omega}\, dt + \frac{1}{2}\int_0^{T-\delta} \|\tau_\delta k_{0h}(t)\|^2_{0,2,\Omega}\, dt.$$

Thus

$$\int_0^{T-\delta} \|\tau_\delta k_{0h}(t)\|^2_{0,2,\Omega}\, dt \le 2\left(\int_0^T \|\mathscr{K}(s)\|_{H^{-1}(\Omega)}\, ds\right)\left(\int_{s-\delta}^s \|\widetilde{\tau_\delta k_{0h}}(t)\|_{1,2,\Omega}\, dt\right)$$

$$+\delta^{1/2} \leq 2\,\|\mathscr{K}\|_{L^1(H^{-1})}\,\delta^{1/2}\left(\int_{s-\delta}^{s}\|\widetilde{\tau_\delta k_{0h}}(t)\|^2_{1,2,\Omega}\,dt\right)^{1/2}+\delta^{1/2}$$
$$\leq C\,\delta\|k_{0h}\|_{L^2(H^1)}+\delta^{1/2} \leq C\,\delta^{1/2}.$$

As $k_h = k_{0h} + k_\Gamma$ then $\tau_\delta k_h = \tau_\delta k_{0h}$, and estimate (12.50) follows. □

The convergence analysis for discretization (12.11) faces the difficulty of proving the strong convergence of the velocity in $L^2(\mathbf{H}^1)$. This is required to pass to the limit in the production term for the TKE. Proving such strong convergence would require to use the continuous velocity as test function in the weak formulations (12.34) and (12.35), and this would require $\partial_t \mathbf{v} \in L^2(\mathbf{L}^2)$. We are thus lead to a partial convergence result: The velocity indeed satisfies the targeted limit equations in model (12.11), while the TKE just satisfies a variational inequality.

Theorem 12.4. *Under the hypotheses of Theorem 12.3, assume in addition that Hypothesis 12.iii holds. Then the sequence of solutions $(\mathbf{v}_h, p_h, k_h)_{h>0}$ provided by the discrete variational problems $(\mathscr{V}\mathscr{P})_h$ (12.41) and (12.42) contains a subsequence $((\mathbf{v}_{h'}, p_{h'}, k_{h'}))_{h'>0}$ that is weakly convergent in $L^2(\mathbf{H}^1) \times H^{-1}(\mathbf{L}^2) \times L^2(H^1)$ to a weak super-solution $(\mathbf{v}, p, k)$ of problem (12.33) as $(h, \Delta t) \to 0$. Such super-solution satisfies the variational problem $(\mathscr{V}\mathscr{P})'$ formed by (12.34), (12.36), and*

$$\begin{cases} -\displaystyle\int_0^T (k(t), l)_\Omega \psi'(t)\,dt - (k_0, l)_\Omega\,\psi(0) \\ +\displaystyle\int_0^T [b_k(\mathbf{v}(t); k(t), l)\,dt + s_k(k(t), l) + d(k(t), l)]\ \psi(t)\,dt \\ \geq \displaystyle\int_0^T (P_L(k(t), v(t), l)_\Omega\ \psi(t)\,dt; \\ \textit{for all } l \in \mathscr{D}(\Omega) \textit{ and all } \psi \in \mathscr{D}(0,T) \textit{ such that} \\ l \geq 0 \textit{ in } \Omega,\ \psi \geq 0 \textit{ in } (0,T),\ \psi(T) = 0. \end{cases} \tag{12.56}$$

Moreover $(\mathbf{v}_{h'})_{h'>0}$ is weakly- convergent in $L^\infty(\mathbf{L}^2)$ to $\mathbf{v}$, strongly in $L^2(\mathbf{H}^s)$ for $0 \leq s < 1$; the primitives in time of the pressures $(p_{h'})_{h'>0}$ are weakly-* convergent in $L^\infty(L^2)$ to a primitive in time of the pressure p; and $(k_{h'})_{h'>0}$ is weakly-* convergent in $L^\infty(L^2)$ to k, strongly in $L^2(H^s)$ for $0 \leq s < 1$.*

Proof. We proceed as in the proof of Theorem 10.2.

STEP 1. *Extraction of convergent subsequences.* Due to estimates (12.47), (12.48), and (12.49) and the assumption $k_0 \in H^1(\Omega)$, arguing as in Step 1 of the proof of Theorem 10.2, we deduce that the sequence $((\mathbf{v}_h, P_h, k_h))_{h>0}$ contains a subsequence (that we still denote in the same way) such that $(\mathbf{v}_h)_{h>0}$ is strongly convergent in $L^2(\mathbf{H}^s)$ to some $\mathbf{v}$, for any $0 \leq s < 1$, weakly in $L^2(\mathbf{H}^1)$, and weakly-* in $L^\infty(\mathbf{L}^2)$; $(P_h)_{h>0}$ is weakly-* convergent in $L^\infty(L^2)$ to some R, and $(k_h)_{h>0}$ is strongly convergent in $L^2(H^s)$ to some k, for any $0 \leq s < 1$, weakly in $L^2(H^1)$, and weakly-* in $L^\infty(L^2)$.

Moreover, $\tilde{\mathbf{v}}_h$ and $\tilde{\mathbf{v}}_h^-$ strongly converge in $L^2(\mathbf{L}^r)$ to $\mathbf{v}$ and $\tilde{k}_h$ and $\tilde{k}_h^-$ strongly converge in $L^2(L^r)$, $1 \le r < 6$ to k. We may assume in addition that $\tilde{k}_h$ converges a.e. in Q_T to k.

We prove that the pair $(\mathbf{v}, \partial_t R, k)$ is a weak solution of problems (12.34)–(12.56).

STEP 2. *Limit of momentum conservation and continuity equations.* All terms of problem (12.44) pass to the limit as in the proof of Theorem 10.2, excepting the eddy diffusion term. To analyze this limit, let $\mathbf{w} \in \mathbf{W}_D(\Omega)$, $\varphi \in \mathscr{D}(0,T)$. By Hypothesis 12.iii, there exists $\mathbf{w}_h \in \mathbf{W}_h$ that converges to $\mathbf{w}$ in $\mathbf{W}$. Arguing as in Step 3 of Theorem 12.2 we deduce

$$\|\tilde{\nu}_t(\tilde{k}_h) D\mathbf{w}_h - \tilde{\nu}_t(k) D\mathbf{w}\|_{0,2,Q_T} \le C\left[\|\tilde{k}_h - k\|_{0,2,Q_T} + \|D\mathbf{w}_h - D\mathbf{w}\|_{0,2,Q_T}\right].$$

As $\varphi \in L^\infty(0,T)$, then $\tilde{\nu}_t(\tilde{k}_h)(t,\mathbf{x})\, D\mathbf{w}_h(\mathbf{x})\,\varphi(t)$ strongly converges to $\tilde{\nu}_t(k)(t,\mathbf{x})\, D\mathbf{w}(\mathbf{x})\,\varphi(t)$ in $L^2(Q_T)^{d\times d}$. As $D\mathbf{v}_h$ weakly converges to $D\mathbf{v}$ in $L^2(Q_T)^{d\times d}$, we conclude that

$$\lim_{(h,\Delta t)\to 0} \int_0^T s_v(\tilde{k}_h(t); \mathbf{v}_h(t), \mathbf{w}_h)\,\varphi(t)\,dt = \int_0^T s_v(k(t); \mathbf{v}(t), \mathbf{w})\,\varphi(t)\,dt.$$

STEP 3. *Limit of equation for TKE.* The time derivative and the convection terms in (12.45) converge as the corresponding terms for the momentum equation in Step 2 of Theorem 10.2. The eddy diffusion term converges as in Step 1 above. To analyze the remaining terms, let $\psi \in \mathscr{D}(0,T)$, $l \in \mathscr{D}(\Omega)$. Consider a sequence $(l_h)_{h>0}$ such that $l_h \in K_{0h}$, strongly convergent in $W^{1,\infty}(\Omega)$ to l.

STEP 4. *Initial condition.* By (12.40),

$$\lim_{h\to 0} (k_{0h}, l_h)_\Omega\, \psi(0) = (k, l)_\Omega\, \psi(0).$$

STEP 5. *Dissipation term.* Consider the function $\phi(\alpha) = \sqrt{|\alpha|}\,\alpha$. Then $\phi(\tilde{k}_h)$ is bounded in $W^{1,1}(Q_T)$. Indeed,

$$\int_\Omega |\nabla\phi(\tilde{k}_h(t))|\,d\mathbf{x} \le \frac{3}{2}\,\|\tilde{k}_h(t)\|_{0,2,\Omega}^{1/2}\,\|\nabla\tilde{k}_h(t)\|_{0,2,\Omega}^{1/2} \le C\,\|\nabla\tilde{k}_h(t)\|_{0,2,\Omega}^{1/2} \text{ a.e. in } (0,T),$$

and then

$$\|\nabla\phi(\tilde{k}_h)\|_{0,1,Q_T} \le C\sqrt{T}\,\|\nabla\tilde{k}_h\|_{0,2,Q_T} \le C.$$

As the injection of $W^{1,1}(Q_T)$ in $L^{3/2}(Q_T)$ is compact, we deduce that there exists a subsequence $\phi(\tilde{k}_h)$ (that we denote in the same way) convergent in $L^1(Q_T)$ to $\phi(k)$. As $l_h(\mathbf{x})\,\psi(t)$ converges in $L^\infty(Q_T)$,

$$\lim_{(h,\Delta t)\to 0} \int_0^T d(\tilde{k}_h(t); l_h)\, \psi(t)\, dt = \int_0^T d(k(t); l)\, \psi(t)\, dt.$$

STEP 6. *Production term.* For this term only an inequality passes to the limit. As the sequence $P_L(\tilde{k}_h, \mathbf{v}_h)$ is bounded in $L^\infty(Q_T)$, we may assume that it is weakly-* convergent to some P in this space. Let $\psi \in \mathscr{D}(0,T)$, $l \in \mathscr{D}(\Omega)$ such that $\psi \geq 0$ in $(0,T)$, $l \geq 0$ in Ω. We next prove that

$$\int_{Q_T} P_L(k, \mathbf{v})(t, \mathbf{x})\, l(\mathbf{x})\, \psi(t)\, d\mathbf{x}\, dt \leq \int_{Q_T} P(t, \mathbf{x})\, l(\mathbf{x})\, \psi(t)\, d\mathbf{x}\, dt. \tag{12.57}$$

Then the inequality in (12.56) follows. To prove it, let us denote the sequence $(\mathbf{v}_h)_{h>0}$ by $(\mathbf{v}_{h_n})_{n\geq 0}$. As $\mathbf{v}_{h_n}$ weakly converges in $L^2(\mathbf{H}^1)$ to $\mathbf{v}$, by Mazur's Theorem (Theorem A.8), there exists a sequence of convex combinations of some $\mathbf{v}_{h_n}$:

$$\mathbf{w}_h^{(n)} = \sum_{j=n}^{M_n} \alpha_j^{(n)}\, \mathbf{v}_{h_j}$$

which is strongly convergent to $\mathbf{v}$ in $L^2(\mathbf{H}^1)$. By the Lebesgue Theorem, it follows that $P_L(\tilde{k}_{h_n}, \mathbf{w}_h^{(n)})$ is strongly convergent to $P_L(k, \mathbf{v})$ in $L^1(Q_T)$. Due to the convexity of T_L on $\mathbb{R}^+$ and that of the norm,

$$P_L(\tilde{k}_{h_n}, \mathbf{w}_h^{(n)}) \leq \sum_{j=n}^{M_n} \alpha_j^{(n)}\, P_L(\tilde{k}_{h_n}, \mathbf{v}_{h_j}). \tag{12.58}$$

By Egorov's Theorem (Theorem A.9), as $\tilde{\nu}_t(\tilde{k}_{h_n})$ converges a.e. to $\tilde{\nu}_t(k)$, it is quasi-uniformly convergent: For any $\varepsilon > 0$ there exists a measurable set $Q_\varepsilon \subset Q_T$ such that $|Q_T \setminus Q_\varepsilon| < \varepsilon$, and $\tilde{\nu}_t(\tilde{k}_{h_n})$ is uniformly convergent to $\tilde{\nu}_t(k)$ in Q_ε. As

$$\begin{aligned}
\| P_L(\tilde{k}_{h_m}, \mathbf{v}_{h_n}) - P_L(k, \mathbf{v}_{h_n}) \|_{0,1,Q_T} &\leq \| (\tilde{\nu}_t(\tilde{k}_{h_m}) - \tilde{\nu}_t(k))\, |\nabla \mathbf{v}_{h_n}|^2 \|_{0,1,Q_\varepsilon} + \| L \|_{0,1,Q_T \setminus Q_\varepsilon} \\
&\leq \| \tilde{\nu}_t(\tilde{k}_{h_m}) - \tilde{\nu}_t(k) \|_{0,\infty,Q_\varepsilon}\, \| \mathbf{v}_{h_n} \|_{L^2(H^1)} + L\, \varepsilon \\
&\leq C\, \| \tilde{\nu}_t(\tilde{k}_{h_m}) - \tilde{\nu}_t(k) \|_{0,\infty,Q_\varepsilon} + L\, \varepsilon
\end{aligned}$$

we deduce that for all $\varepsilon > 0$, there exists a m_ε such that

$$\max_{m \geq m_\varepsilon} \max_{n \geq 1} \| P_L(\tilde{k}_{h_m}, \mathbf{v}_{h_n}) - P_L(k, \mathbf{v}_{h_n}) \|_{0,1,Q_T} < \varepsilon \tag{12.59}$$

Denote $\sigma(t, \mathbf{x}) = l(\mathbf{x})\, \psi(t)$. Observe that (we omit the integration variables for brevity)

$$\left|\int_{Q_T}\left(\sum_{j=n}^{M_n}\alpha_j^{(n)}\,P_L(\tilde{k}_{h_n},\mathbf{v}_{h_j})\sigma - P\,\sigma\right)\right| \le \sum_{j=n}^{M_n}\alpha_j^{(n)}\int_{Q_T}\left|P_L(\tilde{k}_{h_n},\mathbf{v}_{h_j}) - P_L(k,\mathbf{v}_{h_j})\right|\sigma$$

$$+\sum_{j=n}^{M_n}\alpha_j^{(n)}\int_{Q_T}\left|P_L(k,\mathbf{v}_{h_j}) - P_L(\tilde{k}_{h_j},\mathbf{v}_{h_j})\right|\sigma + \sum_{j=n}^{M_n}\alpha_j^{(n)}\left|\int_{Q_T}\left(P_L(\tilde{k}_{h_j},\mathbf{v}_{h_j})\sigma - P\,\sigma\right)\right|$$

$$\le 2\max_{m\ge n}\max_{l\ge 1}\|P_L(\tilde{k}_{h_m},\mathbf{v}_{h_l}) - P_L(k,\mathbf{v}_{h_l})\|_{0,1,Q_T}\,\|\sigma\|_{0,\infty,Q_T}$$

$$+\max_{j\ge n}\left|\int_{Q_T}\left(P_L(\tilde{k}_{h_j},\mathbf{v}_{h_j})\sigma - P\,\sigma\right)\right|.$$

As $P_L(\tilde{k}_{h_j},\mathbf{v}_{h_j})$ converges to P weak-* in $L^\infty(Q_T)$, using (12.59), we deduce

$$\lim_{n\to\infty}\int_{Q_T}\sum_{j=n}^{M_n}\alpha_j^{(n)}\,P_L(\tilde{k}_{h_n},\mathbf{v}_{h_j})\,\sigma = \int_{Q_T}P\,\sigma.$$

STEP 7. *Conclusion.* Using (12.58) and $\sigma \ge 0$ in Q_T, we conclude (12.57). Then $(\mathbf{v}, p, k)$ is a solution of the variational problem $(\mathcal{VP})'$. □

12.4 Further Remarks

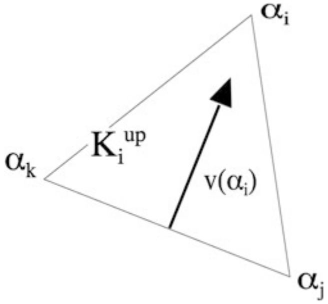
12.4.1 Numerical Schemes Satisfying the Discrete Maximum Principle

To obtain positive numerical kinetic energies, the equations for k in models (12.1) and (12.33) are solved by means of numerical schemes that satisfy the discrete maximum principle: The numerical solution is positive if the boundary data k_Γ and the r.h.s. P are positive. Although the Galerkin method satisfies the maximum principle for small enough grid size h, this is usually achieved for extremely small h, out of practical interest. The main difficulty arises when boundary or internal layers occur, typically with length scales much smaller than the grid size, that generate spurious solutions in the Galerkin solution. Instead stabilized methods are used. These methods damp these oscillations, partially or totally, by means of adding some kind of artificial diffusion. This artificial diffusion takes into account the information provided upwind by the flow and usually is called "upwind method." There are many proposals of stabilized discretizations; an extensive review can be found in Ross et al. [31].

To describe some of these methods, let us consider the linear convection–diffusion–reaction equation

$$\mathbf{v}\cdot\nabla k - \nu\,\Delta k + \alpha\,k = P \;\text{ in }\; \Omega, \quad k = k_\Gamma \;\text{ on }\; \Gamma,$$

Fig. 12.1 Upwind element



with $\nu > 0$, $\alpha > 0$, where we assume that the convection velocity $\mathbf{v}$ is divergence-free. The Galerkin approximation on K_h of this problem is : find $k_h \in \Pi_h(k_\Gamma) + K_{0h}$ such that

$$a(k_h, l_h) = (P, l_h)_\Omega \quad \forall l_h \in K_{0h}, \tag{12.60}$$

where $a(k,l) = (\mathbf{v} \cdot \nabla k, l)_\Omega + \nu\,(\nabla k, \nabla l)_\Omega + \alpha\,(k,l)_\Omega$.

The method of Tabata (cf. [32]), one of the first upwind finite element methods, assigns an upwind element K_i^{up} to each internal vertex α_i. This upwind element K_i^{up} is any element one of whose vertex is α_i, such that $\mathbf{v}(\alpha_i)$ points from K_i^{up} toward α_i; see Fig. 12.1.

The discrete problem is obtained from (12.60) using the following approximations:

$$(\mathbf{v} \cdot \nabla k_h, \varphi_i)_\Omega \simeq (\mathbf{v}(\alpha_i) \cdot \nabla k_{h|_{K_i^{up}}}, \varphi_i)_\Omega, \quad (P, \varphi_i)_\Omega \simeq (P(\alpha_i), \varphi_i)_\Omega$$

for all internal node α_i, where φ_i denotes the Lagrange basis function of K_{0h} associated to the node α_i. This method is positive if α is large enough with respect to ν, in the sense that the associated matrix is an M-matrix: Its diagonal entries are positive, and its off-diagonal entries are nonpositive. This ensures the maximum principle. Moreover, it is proved to provide convergent approximations in $H^1(\Omega)$ norm. In exchange, it is not exactly consistent, in the sense that the exact solution of the convection–diffusion equation does not satisfy the discrete problem. For this reason it provides only a first-order accurate approximation, with rather high levels of numerical diffusion.

An extensively used method that provides to a great extent accurate and oscillation-free solutions is the Streamline Upwind Petrov–Galerkin (SUPG) method, introduced by Brooks and Hughes (cf. [9]). It reads: find $k_h \in \Pi_h(k_\Gamma) + K_{0h}$ such that

$$a(k_h, \lambda_h)_\Omega + (R_h(k_h), \tau\, \mathbf{u} \cdot \nabla \lambda_h)_\Omega = (P, \lambda_h)_\Omega, \quad \text{for all } \lambda_h \in K_h$$

where $R_h(k_h)$ is the residual, defined element-wise by

$$R_h(k_h)_{|_K} = \mathbf{u} \cdot \nabla \left(k_{h_{|_K}}\right) - \nu\, \Delta \left(k_{h_{|_K}}\right) - \alpha\, k_{h_{|_K}} - f_{|_K} \text{ on each } K \in \mathcal{T}_h,$$

and $\tau \in L^\infty(\Omega)$ is a nonnegative stabilization parameter, whose adjustment is crucial to obtain a good accuracy of the method (cf. John and Knobloch [22, 23]). The SUPG method, however, does not preclude small nonphysical oscillations localized in narrow regions along sharp layers. To damp these oscillations, artificial crosswind diffusion in the neighborhood of layers is added to the SUPG formulation. This procedure is usually called "discontinuity (or shock) capturing." The artificial diffusion in these methods typically depends on the discrete solution k_h. Thus, the resulting methods are nonlinear (even if the original problem is linear). This is necessary to achieve methods with second-order accuracy, as linear methods satisfying the maximum principle are at most of first order. This is stated by the well-known Godunov Theorem (cf. Toro [33]).

All these methods provide some damping of spurious oscillations due to sharp layers, but few of them really satisfy the maximum principle. Let us mention, for instance, the stabilized method of Burman and Ern (cf. [10]) that develops a complete analysis of stability and convergence and the crosswind dissipation method of Codina (cf. [14]). In its turn, the Mizukami and Hughes method (cf. [30]) provides an M-matrix and quite accurate numerical results, but no convergence analysis is available. This situation is somewhat improved for the PSI (Positive Stream-wise Implicit) method (cf. [16]): It provides an M-matrix, and some partial convergence results for the diffusion-dominated regime hold (cf. [13]).

An alternative technique is provided by the flux correction schemes, oriented to bound the oscillations in regions of high gradients. These techniques were introduced for finite difference discretizations (cf. [8, 34]) and later were applied to finite element discretizations (cf. [24–29]).

12.4.2 Approximation of Elliptic Equations with r.h.s. in L^1

The equation for the TKE in model (12.1) has only $L^1(\Omega)$ regularity. For this reason the TKE has only $W^{1,q}(\Omega)$ regularity, for $1 \le q < 3/2$. This is proved by means of the interpolation estimates stated in Theorem A.12.

This analysis has been extended in Casado et al. [11] to piecewise affine finite element discretizations of elliptic equations of the form

$$-\nabla \cdot (A(\mathbf{x})\, \nabla k) = P \text{ in } \Omega,$$

where $A \in L^\infty(\Omega)^{d\times d}$ is a uniformly positive-defined diffusion matrix. This analysis holds when the matrix resulting from the discretization is an M-matrix. This property seems to be essential to obtain uniform estimates of the quantities $M(k_h)$ defined by (A.42).

Also, Gallouët and Herbin have proved a similar result for the Poisson equation with measure data in [18]. In this case this analysis is an adaptation to piecewise finite element discretizations of a general analysis for finite volume approximations of convection–diffusion equations without coerciveness, performed in [17]. These techniques were subsequently extended to nonlinear transport–diffusion equations (cf. [20, 21]). Finite volume discretizations provide a more favorable framework to build discretizations that yield M-matrices, due to the possibility of limiting the flux between adjacent cells.

The extension of the analysis of [11, 18] to convection–diffusion equations, thus, requires discretizations that yield an M-matrix. However, very few finite element discretizations provide an M-matrix. Moreover, the convergence analysis of most of these has not been performed even for linear convection–diffusion equations. Further, the extension to the evolutionary convection–diffusion equations of the analysis of [11] or [18] has not been realized yet.

This is an open subject of research whose resolution would lead to the extension of the analysis performed in this chapter for the Galerkin approximation of the TKE equation to methods that ensure its positiveness.

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Chapter 13
Numerical Experiments

Abstract This chapter is devoted to analyze the numerical performances of the models and numerical techniques that we have studied in the preceding chapters. It is intended to provide a starting guide to the numerical discretization of VMS models for students and researchers interested in the computation of turbulent flows. With this purpose we test the practical performances of the VMS models, in some relevant benchmark turbulent flows.

13.1 Introduction

This chapter is devoted to analyze the numerical performances of the models and numerical techniques that we have studied in the preceding chapters. It is intended to provide a starting guide to the numerical discretization of VMS models for students and researchers interested in the computation of turbulent flows. With this purpose we test the basic aspects of the implementation of the discrete models, so as the practical performances of the VMS models, in commonly used turbulent flows.

RANS methods are used in many engineering applications and its performances are found in a wide literature. We have preferred to address the rather new VMS methods and to analyze their practical performances in view of their use in engineering applications. Specifically, our numerical experiments are aimed to the following targets:

- To introduce the main features of some relevant turbulent flows commonly used in the testing of turbulence models
- To determine the ability of the projection-based VMS model studied in Chap. 9 to accurately solve relevant turbulent flows
- To compare the performances of the various VMS models tested, providing some hints of the more adequate models for specific applications

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We restrict our tests to steady flows as the best documented test flows are steady. However, in our tests the steady states are mostly reached through unsteady discretizations. Residual-based VMS methods provide excellent results for at least second-order accurate discretizations, reproducing with high accuracy first- and second-order statistics. Projection-based VMS methods provide a good compromise between accuracy and computational complexity, while residual-free bubble-based VMS methods, although yielding acceptable accuracy, still need further improvements to be handled by non-experienced users.

From the point of view of engineering applications, all VMS methods, as LES methods, are much more costly than RANS models. Thus their use in real applications for the next future is to be restricted to geometries with low complexity and specific flows for which a high accuracy is needed.

13.2 The 2D Backward Step Flow

The backward step flow is the flow generated by a sudden expansion of a wall in a wall-bounded flow (see Fig. 13.1). The incoming flow is parabolic, a large vortex is formed behind the step front, and as the Reynolds number increases, secondary vortices are formed downwind in both the upper and lower walls. It is a simplified case of flows in ducts that separate due to sudden variation of the section geometry, of large importance in hydraulic and aeronautic engineering.

The backward step flow is well documented from both the experimental and numerical approaches. In Armaly et al. [2] a thorough experimental study is reported: The flow is laminar and reaches a steady state roughly up to $Re \simeq 1.000$. Then it becomes transitional up to $Re \simeq 5.000$. For larger values of the Reynolds number, the flow is fully turbulent, but reaches an almost-steady state. The flow remains 2D up to $Re \simeq 400$ and becomes 3D beyond this value, characterized by the formation of a secondary vortex in the wall opposite to the step. Seemingly, a tertiary vortex appears in the lower wall for larger Re, and progressively a quaternary vortex appears in alternate walls as Re increases (cf. [2]).

We shall restrict ourselves to test the accuracy in the computation of the main reattachment length, which is the length of the main recirculating region behind the step. This is preferable because there is a rather large uncertainty in the actual

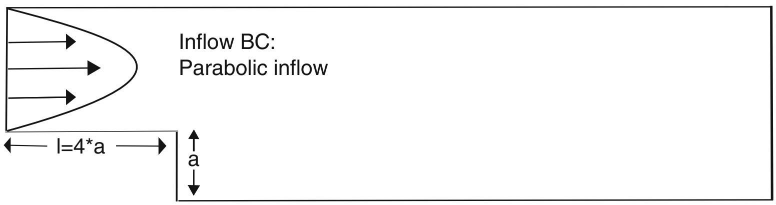

Fig. 13.1 Geometry of the 2D backward step flow

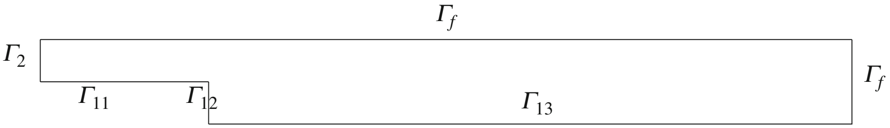

Fig. 13.2 Computational domain for Test 1

location of the other reattachment lengths. We refer to [2] and [16] for a more detailed description of the difficulties linked to this problem.

This test examines the ability of the projection-based VMS models to accurately compute the steady turbulent 2D backward step flow. We have compared the solution provided by the single Smagorinsky model (9.32) with those provided by the VMS-subgrid eddy viscosity model (11.15) with the following settings of the eddy viscosity term:

- **VMS-S model:** The small–small VMS-Smagorinsky setting, given by (11.7)
- **VMS-B model:** The projection-based VMS method introduced in Berselli et al. [6] given by (11.14), replacing the $L^2(\Omega)^{d\times d}$ orthogonal projection σ_h on space L_h by an interpolation operator on a coarser finite element space

The interest of considering the VMS-B model is that in practice the operator σ_h is rather difficult to compute, as this requires building space L_h. Instead it is easier to compute σ_h as a local average operator. In this case this model does not coincide with the VMS-S model, so it is worth to compare their performances. Also, the Smagorinsky model here is considered as a reference model that, surprisingly, yields satisfying results.

In addition, we include the modeling of the boundary layer turbulence by means of wall laws. This yields good results without further adaptations, as the flow recirculates and then the boundary layer thickness is approximately constant, so approximating the line $z^+ = constant$ by a straight line is accurate. However this is not the case for flows along large plates, where the boundary layer thickness increases in the stream-wise direction, and the use of wall law requires specific adaptations.

The actual geometry that we have used is represented in Fig. 13.2. The relevant geometrical data are the step height a and the step length $l = 4 * a$. The length of the computational domain is about 20 times the step height. The boundary of the domain is decomposed into $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_f$. Wall-law conditions are imposed on Γ_1, parabolic Dirichlet boundary conditions are imposed on Γ_2, and free normal stress is imposed on Γ_f:

$$\begin{cases} \mathbf{v} = \mathbf{V}_{Parabolic} & \text{in } \Gamma_2, \\ \mathbf{v}\cdot\mathbf{n} = 0, [\mathbf{n}\cdot\mu_t(\mathbf{v})D(\mathbf{v})]_\tau = g(\mathbf{v})_\tau & \text{in } \Gamma_1, \\ \mathbf{n}\cdot[\mu_t(\mathbf{v})D(\mathbf{v}) + p\,I] = 0 & \text{in } \Gamma_f. \end{cases} \tag{13.1}$$

We have set the logarithmic wall law. The corresponding function L is given by (5.113), where we set $z^+ = 25$. The Reynolds number for this flow is computed by $Re = \dfrac{V_{max}\, a}{\nu}$, where V_{max} is the maximum of the parabolic profile $\mathbf{V}_{Parabolic}$. The discrete space for velocities has been modified to take into account the free normal stress condition on Γ_f. It is defined by

$$\mathbf{W}_h = \{\mathbf{w}_h \in V_h^d \ \text{ such that } \ \mathbf{w}_h \cdot \mathbf{n} = 0 \ \text{ on } \ \overline{\Gamma_{1k}},\ k = 1, 2, 3,\ \mathbf{w}_h = 0 \ \text{ on } \ \overline{\Gamma_2}\}, \tag{13.2}$$

where V_h is a Lagrange finite element space constructed on an admissible triangulation of Ω and $\Gamma_{1k},\ k = 1, 2, 3$ are the three segments of straight line contained in Γ_1.

For all three models we have preferred to use a stabilized discretization due to the relative computational complexity of the flow. We use the projection term-by-term stabilized discretization (9.52) with $\mathbb{P}_2$ polynomial discretization of velocity and pressure to achieve a good accuracy. We also set a search for the steady state through a semi-implicit evolution approach. In each time step we thus solve the problem

Obtain $(\mathbf{v}_h^{n+1}, p_h^{n+1}) \in \mathbf{W}_h \times N_h$ such that for all $(\mathbf{w}_h, q_h) \in \mathbf{W}_h \times N_h$,

$$\begin{cases} (\dfrac{\mathbf{v}_h^{n+1} - \mathbf{v}_h^n}{\Delta t}, \mathbf{w}_h)_\Omega + b(\mathbf{v}_h^n; \mathbf{v}_h^{n+1}, \mathbf{w}_h) + a(\mathbf{v}_h^{n+1}, \mathbf{w}_h) + d(\mathbf{v}_h^{n+1}; \mathbf{w}_h) \\ +\langle G(\mathbf{v}_h^{n+1}), \mathbf{w}_h\rangle - (p_h^{n+1}, \nabla \cdot \mathbf{w}_h)_\Omega + s_{conv,h}(\mathbf{v}_h^{n+1}, \mathbf{w}_h) = \langle \mathbf{f}, \mathbf{w}_h\rangle, \\ \qquad\qquad (\nabla \cdot \mathbf{v}_h^{n+1}, q_h)_\Omega + s_{pres,h}(p_h^{n+1}, q_h) = 0, \end{cases} \tag{13.3}$$

where $d(\mathbf{v}_h, \mathbf{w}_h)$ denotes the eddy diffusion terms defined by either (9.33), (11.7), or (11.14), $\mathbf{W}_h = \mathbf{V}_h^{(2)}$, $N_h = M_h^{(2)}$. The stabilizing terms are defined by

$$s_{conv,h}(\mathbf{v}_h, \mathbf{w}_h) = \sum_{K \in \mathcal{T}_h} \tau_{conv,K}\ ((I - \sigma_h)(\mathbf{v}_h \cdot \nabla \mathbf{v}_h), (I - \sigma_h)(\mathbf{v}_h \cdot \nabla \mathbf{w}_h))_K, \tag{13.4}$$

$$s_{pres,h}(p_h, q_h) = \sum_{K \in \mathcal{T}_h} \tau_{pres,K}\ ((I - \sigma_h)(\nabla p_h), (I - \sigma_h)(\nabla q_h))_K, \tag{13.5}$$

where σ_h is the Oswald quasi-interpolant operator on $\mathbf{V}_h^{(1)}$ (cf. [35]). This is basically a Lagrange interpolation operator (9.10) that acts on piecewise continuous functions v, using as nodal values $v(\alpha)$ the average of v on all elements of the grid that share the node α. The convective stabilization coefficient, following Henao et al. [26] and Chacón [8], is given by

$$\tau_{conv,K} = C_{conv}\, \frac{1}{\xi(Pe_K^{(1)}) + Pe_K^{(1)}\, \xi(Pe_K^{(2)})}, \qquad \tau_{press,K} = C_{press}\, h_K^2,$$

where C_{conv} and C_{pres} are numerical constants, and

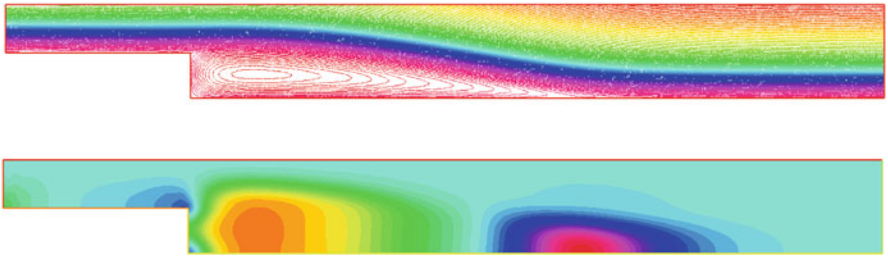

Fig. 13.3 Test 1: Smagorinsky model. Streamline function and pressure map for Re= 10000

$$Pe_K^{(1)} = \frac{6\,\gamma\,\Delta t\,\nu}{h_K^2}, \quad Pe_K^{(2)} = \frac{\|\mathbf{v}_h^n\|_{0,2,K}\,h_K}{3\nu}, \quad \xi(x) = \max\{1, x\}, \quad \gamma \in (0, 1).$$

The numbers $Pe_K^{(1)}$ and $Pe_K^{(2)}$ respectively are the transient and steady Péclet numbers. The stability of the discretization (13.3) is largely dependent on the presence of the transient Péclet number in the convection stabilization coefficients.

The test stop for the time-stepping procedure has been set to

$$\frac{\|\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\|_{0,2,\Omega}}{\|\mathbf{v}_h^n\|_{0,2,\Omega}} < 10^{-4}.$$

Good results are obtained with a mesh width $h = a/10$, provided the time-stepping procedure converges to the steady state. This does not occur for the transitional regime, for Reynolds numbers roughly ranging from $Re = 1000$ to $Re = 50000$. Figure 13.3 shows the streamlines and the pressure map provided by the Smagorinsky model with $Re = 10.000$ for a grid with mesh width $h = a/20$. The solution presents a high quality; no diffusive effects arise close to the wall boundaries; thus the use of wall laws avoids the excessive diffusivity of Smagorinsky model close to the walls. Also, no instabilities due to convection-dominance occur; thus the stabilization provided by the convection stabilization term is effective. Finally, no oscillations appear in the pressure field, so the pressure stabilization also is effective. Similar results are obtained with the VMS-S and VMS-B models.

Tables 13.1, 13.2, and 13.3 present the computed length of the main vortex formed in the step flow. This length has been computed as a dimensional length by $X_r = x_r/a$, where x_r is the reattachment length. The length x_r has been computed as the distance from the step front to the point $\mathbf{x}_r$ of the lower wall determined by the condition $\partial_x v_1(\mathbf{x}_r) = 0$. It is computed by bisection. The experimental values have been reported by Armaly et al. [2]. These have been calculated for turbulent flows by averaging several experiments with the same Reynolds number, and for this reason the values reported are considered only as approximated.

Table 13.1 Test 1: Comparison of reattachment points for laminar regime

Reynolds number	VMS-S	VMS-B	Smagorinsky	Experimental
100	2.86	2.86	2.76	3
500	9.18	9.45	10.61	10

Computed with $\gamma = 0.5,\ \Delta t = 10^2,\ h = a/20$

Table 13.2 Test 1: Comparison of reattachment points for turbulent regime

Reynolds number	VMS-S	VMS-B	Smagorinsky	Experimental
5000	7.29	7.29	7.29	$\simeq 7.25$
6000	7.58	7.58	7.58	$\simeq 7$
10000	8.45	8.45	8.45	$\simeq 8$

Computed with $\gamma = 0.5,\ \Delta t = 10^2,\ h = a/20$

Table 13.3 Test 1: Comparison of reattachment points for turbulent regime

Reynolds number	VMS-S	VMS-B	Smagorinsky	Experimental
5000	7.47	7.90	Not converged	$\simeq 7.25$
6000	7.88	8.29	Not converged	$\simeq 7$
10000	8.02	8.85	Not converged	$\simeq 8$

Computed with $\gamma = 0,\ \Delta t = 10^2,\ h = a/20$

Table 13.1 corresponds to laminar flow. In this case no-slip boundary conditions have been used on Γ_1 instead of wall laws, and the point $\mathbf{x_r}$ has been located at a distance of 10^{-3} from the wall. No steady solution is obtained for Reynolds numbers ranging from $Re = 1000$ to $Re \lessapprox 5000$. Tables 13.2 and 13.3 correspond to turbulent flow, computed with wall laws. Only in Table 13.2 the convection stabilization takes into account the time step. We observe that the Smagorinsky model is more sensitive to this effect. However, whenever it reaches a steady solution, the computed X_r is as accurate as those provided by the VMS-S and VMS-B models, and all three are quite accurate with respect to the experimental values.

We conclude that all tested subgrid-eddy viscosity models are able to accurately compute the 2D backward step flow, whenever wall-law boundary conditions are used, and the numerical model settings allow to reach a steady solution.

13.3 The 3D Cavity Flow

The cavity flow is one of the most studied problems in computational fluid dynamics. It consists in computing the flow induced in a cavity by an external flow, parallel to its lid. The 2D lid-driven cavity flow is an excellent test for new numerical models and methods because of the simplicity of the geometry and of the boundary conditions. Yet it keeps the relevant difficulty of correctly solving the inertial effects. The main effect is to generate a vortex induced by the velocity

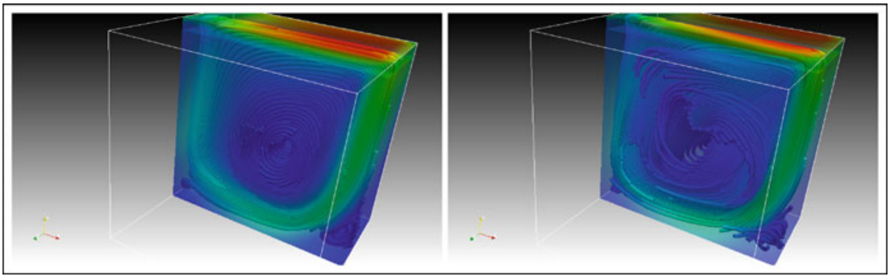

Fig. 13.4 3D cavity flow: streamlines at $Re = 3200$ (*left*) and $Re = 7500$ (*right*); results for VMS-S method

imposed at the lid and the counter-rotating vortex that appear near the bottom corners as the Reynolds number increases. This flow physically seems to exist up to Re $\simeq$ 800, due to the appearance of 3D instabilities for larger Re (cf. Albensoeder et al. [1]). Beyond this value numerically a steady solution seems to exist for Re up to 7.500, approximately. Bifurcations to periodic unsteady solutions, obtained by DNS simulations, are reported by many authors (cf. Auteri et al. [3], Peng et al. [37], and Tiesinga et al. [41], among others). However, these unsteady solutions could be due to insufficient space resolution. Indeed, Erturk reports steady solutions up to Re= 20.000 for fine enough grids (cf. [15]). In this context, computing a turbulent 2D lid-driven cavity flow has a purely academic value. It is useful for testing models by comparison to other models, so as numerical techniques.

The 3D cavity flow presents some genuine 3D features, even at relatively low Reynolds numbers. One of the most remarkable is the formation of the Taylor–Görtler-like (TGL) vortices (see Fig. 13.4), which are small counter-rotating vortices that develop near the inferior corners of the cavity. Similarly to the 2D case, these vortices appear as a consequence of the formation of a large vortex in the center of the cavity that separates from the lateral walls, but now have a genuine 3D structure (cf. [40]). Zang et al. [42] report the results of the simulation of the 3D cavity flow by a LES model in a finite volume method, using the dynamic procedure of Germano et al. (cf. [24]). Based on experimental experiences performed in Prasad and Koseff [36], they describe the flow at Re $=$ 3200 to be essentially laminar, although an inherent unsteadiness may occur. For Re $=$ 7500, a transitional stage is reached, since the flow becomes unstable near the downstream eddies at Reynolds numbers higher than about 6000. When Re $=$ 10000, the flow becomes fully turbulent. Thus, laminar, transient, and turbulent regimes are covered by the choice of these three cases.

The primary goal of the simulation of the 3D cavity flow is to obtain a bounded kinetic energy as time increases, during the complete simulation time needed to reach a stable equilibrium (cf. [32]). This may look a simple requirement, but some turbulence models violate it. Indeed, Iliescu et al. [28] report the results obtained with three subgrid-scale models for $Re = 10^4$: the Smagorinsky model, a traditional Taylor LES model of Clark et al. [11], and two variants of a new rational LES model

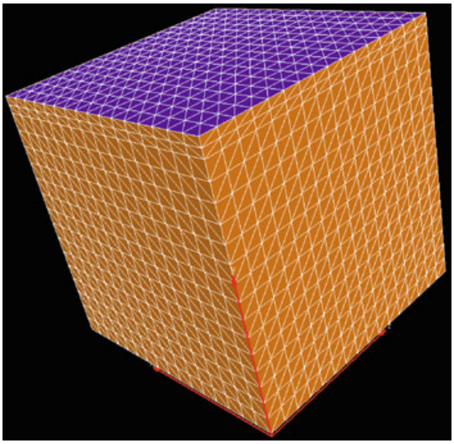

Fig. 13.5 3D cavity flow: mesh with $16 \times 16\times$ cells

developed in Galdi and Layton [18]. It is shown that the Taylor LES model produces an energy blowup in finite time. The two rational LES models did not cause an energy blowup, but exhibited important oscillations. The standard Smagorinsky model (with the Smagorinsky constant $C_S = 0.1$) turned out to be notably more diffusive, as expected.

In the present test we simulate the 3D cavity flow at Reynolds numbers ($Re = 1/\nu$) 3200, 7500, and 10000, following the work of Chacón et al. [10]. This work compares the solution provided by model (13.3) to the eddy viscosity modeling given by the VMS-S and VMS-B models, so as to the standard Smagorinsky model (SM). In all three cases a static modeling of the Smagorinsky constant $C_S = 0.1$ is used. Dirichlet boundary conditions are used: On the top boundary layer a unit horizontal velocity is prescribed, while no-slip boundary conditions on the rest of the boundary are set.

Relatively coarse basic meshes are used, with the purpose of obtaining accurate results with a low computational cost. The basic grid consists of a $16 \times 16 \times 16$ partition of the unit cube with 26.112 tetrahedra, where in addition the grid line corresponding to the top boundary layer is refined, in order to handle large velocity gradients (Fig. 13.5). This provides a large improvement in the accuracy of the numerical results. Three-dimensional $P2$ finite elements for velocity and pressure are used.

A semi-implicit Crank–Nicolson scheme for the temporal discretization is used. The use of at least second-order accurate discretizations in space and time has been found as being essential to obtain accurate solutions of nonlinear flow equations. The use of, for instance, a semi-implicit Euler scheme yields excessive numerical diffusion (cf. [9]). In addition, the Crank–Nicolson scheme provides a good compromise between accuracy and computational cost. In particular, it is less expensive in terms of storage requirements with respect to two-step schemes (e.g., Adams-Bashforth method) that could achieve a second-order accuracy in time. The following linearized system is solved at each time step:

Obtain $(\overline{\mathbf{u}}_h^{n+1}, p_h^{n+1}) \in \mathbf{W}_h \times N_h$ such that for all $(\overline{\mathbf{v}}_h, q_h) \in \mathbf{W}_h \times N_h$,

$$\begin{cases} \left(\dfrac{\overline{\mathbf{u}}_h^{n+1} - \overline{\mathbf{u}}_h^n}{\Delta t}, \overline{\mathbf{v}}_h\right)_\Omega + b(\overline{\mathbf{u}}_h^n, \overline{\mathbf{u}}_h^{n+\theta}, \overline{\mathbf{v}}_h) + a(\overline{\mathbf{u}}_h^{n+\theta}, \overline{\mathbf{v}}_h) + \hat{c}(\overline{\mathbf{u}}_h^{n+\theta}, \overline{\mathbf{v}}_h) \\ \qquad -(p_h^{n+\theta}, \nabla \cdot \overline{\mathbf{v}}_h)_\Omega + \hat{s}_{conv,h}(\overline{\mathbf{u}}_h^{n+\theta}, \overline{\mathbf{v}}_h) = 0, \\ \\ \qquad (\nabla \cdot \overline{\mathbf{u}}_h^{n+\theta}, q_h)_\Omega + \tilde{s}_{pres,h}(p_h^{n+\theta}, q_h) = 0, \end{cases} \tag{13.6}$$

where

$$\overline{\mathbf{u}}_h^{n+\theta} = \theta \overline{\mathbf{u}}_h^{n+1} + (1-\theta)\overline{\mathbf{u}}_h^n, \quad p_h^{n+\theta} = \theta p_h^{n+1} + (1-\theta) p_h^n, \quad \theta = 1/2.$$

Here, the form $\hat{c}$ denotes the linearized (with respect to the convection velocity at a previous time step $\overline{\mathbf{u}}_h^n$) eddy diffusion term defined by either (11.7), (11.11), or (11.14), $\mathbf{W}_h = [V_h^2(\Omega) \cap H_0^1(\Omega)]^d$, and $N_h = V_h^2(\Omega)$. We do not include any modeling of the boundary layers as this is not really needed to obtain good results; it is enough to somewhat decrease the grid spacing in the wall-normal direction just for the first cell.

The stabilizing term $s_{conv,h}$ is given, as in the 2D case, by (13.4), while $\tilde{s}_{pres,h}$ is defined by

$$\tilde{s}_{pres,h}(p_h^{n+\theta}, q_h) = \sum_{K \in \mathscr{T}_h} \tau_{pres,K} \, (\nabla p_h, \nabla q_h)_K, \tag{13.7}$$

This definition of $\tilde{s}_{pres,h}$ is due to the fact that the pressure stabilizing term $s_{pres,h}$ given by (13.5) generates small numerical instabilities during the computation. The stabilization coefficients are set, following Codina [13], by

$$\tau_{p,K}^n = \tau_{\nu,K}^n = \left[\left(c_1 \frac{\nu + \tilde{\nu}_T^n}{h_k^2}\right) + \left(c_2 \frac{U_K^n}{h_k}\right)\right]^{-1},$$

where $U_K^n = \|\overline{\mathbf{u}}_h^n\|_{0,2,K}/|K|^{1/2}$ and $\tilde{\nu}_T^n = (C_S h_K)^2 U_K^{*,n}$, with:

- $U_K^{*,n} = \|D(\overline{\mathbf{u}}_h^n)\|_{0,2,K}/|K|^{1/2}$ for the SM model;
- $U_K^{*,n} = \|D(\Pi_h^* \overline{\mathbf{u}}_h^n)\|_{0,2,K}/|K|^{1/2}$ for the VMS-S model;
- $U_K^{*,n} = \|\tilde{\Pi}_h^* D(\overline{\mathbf{u}}_h^n)\|_{0,2,K}/|K|^{1/2}$ for the VMS-B model;

and $c_1 = 16$ (as we are using quadratic elements), $c_2 = \sqrt{c_1}$. The linear system issued from problem (13.6) is implemented on a FreeFem++ code (cf. [25]), and the corresponding sparse matrix is treated by a GMRES (generalized minimal residual) solver (cf. [38]).

It is crucial to discretize the convection term by an antisymmetric form, such as $b(\overline{\mathbf{u}}_h^n, \cdot, \cdot)$, in order to obtain a good stability in time. This ensures the conservation of the kinetic energy in the absence of diffusive effects and source terms.

The numerical results exhibit effectively the formation of three-dimensional TGL corner vortices at the cavity end walls that interact with the primary circulation vortex, thus influencing the distribution of momentum within the entire cavity (cf. Fig. 13.4). In the case $Re = 3200$, in accordance to Prasad and Koseff [36], it is possible to discern these vortices as organized structures, while for higher Re, increasing turbulent effects cause the breakdown of these organized structures, resulting in a "weaker" flow when compared with the pure two-dimensional flow (see, e.g., the numerical simulations of Ghia et al. [19]), in which it is not possible to discern at all the presence of TGL vortices. This suggests that the high-frequency turbulent fluctuations become dominant, and they partially destroy the integrity (or coherence) of the TGL vortices.

A characteristic time scale T_{cav} of the 3D cavity flow is the time it takes for a fluid particle located on the center of the top boundary to turn and (approximately) reach again that position (cf. Zang et al. [42]). This time scale is roughly estimated to be about ten time units for the current calculation. Within this time period (cf. [20]), the flow is expected to develop to full extent, including a subsequent relaxation time. Using this information, to obtain quasi-steady results, the following strategy is followed: initially, the simulation is run for five time scales T_{cav}. Afterwards, statistics are collected for another five time scales T_{cav}.

The time evolution of the computed values of the total kinetic energy,

$$E_{kin}(\overline{\mathbf{u}}_h^n) = \frac{1}{2}\int_{\Omega} \overline{\mathbf{u}}_h^n \cdot \overline{\mathbf{u}}_h^n d\,\mathbf{x}$$

for $Re = 3200$, $Re = 7500$, $Re = 10000$ is shown in Fig. 13.8. The flows become roughly stationary at $t \simeq 5\,T_{cav}$, as expected. Method SM introduces the highest values of the total kinetic energy for all three Reynolds numbers considered. Both VMS-S and VMS-B methods introduce similar levels of numerical viscosity, although the VMS-B method is slightly less viscous for all cases.

Those results compare well to those reported for some VMS methods with several real grid levels. For instance, Gravemeier compares in [20] the performances of two VMS models using several nested refined meshes (based on the residual-free bubble method), called three-level method and two-level method, with the ones of the adjoint-stabilized finite element method (see Sect. 9.8.1) and of the Smagorinsky model on a PSPG (pressure-stabilizing/Petrov–Galerkin) finite element discretization. These results are in good agreement with the ones obtained in Gravemeier [20] with a similar number of degrees of freedom. In particular, the SM method acts close to the Smagorinsky model (SM) of Gravemeier, solved with a pressure-stabilizing/Petrov–Galerkin discretization. Also, the energy curves for the VMS-S and VMS-B methods are located between the energy curves of the three-level (3L) and two-level (2L) methods of Gravemeier (cf. [21, 22]).

Also, the mean velocities $\langle u_1 \rangle$ and $\langle u_3 \rangle$ are computed through discrete time averages, according to

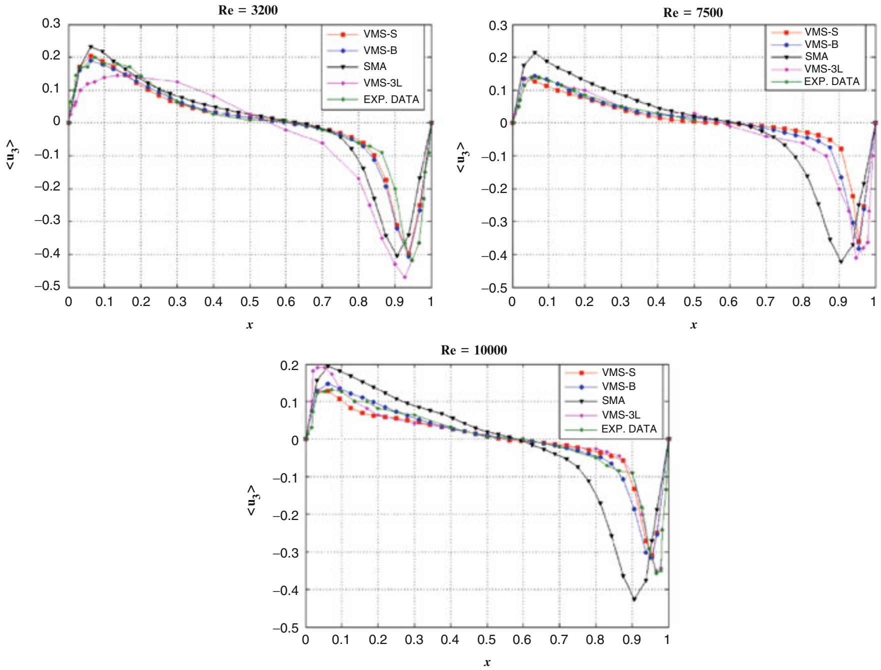

Fig. 13.6 $\langle u_3 \rangle$ on the horizontal centerline ($z = 0.5$) of the midplane $y = 0.5$ for $Re = 3200$, $Re = 7500$, $Re = 10000$

$$\langle u_i \rangle(\mathbf{x}) = \frac{1}{N/2} \sum_{n=N/2}^{N-1} u_i(\mathbf{x}, t_n), \quad i = 1, 3, \quad N = \text{time step number} = 1000.$$

Figure 13.6 shows the mean velocities $\langle u_1 \rangle$ and $\langle u_3 \rangle$ on the centerline $z = 0.5$ of the transversal midplane $y = 0.5$. Both the VMS-S model and the VMS-B model show a good agreement with the experimental data of Prasad and Koseff [36], even with the coarse basic discretization at hand. Both methods also compare well to the VMS-3L method of Gravemeier [20]. A similar accuracy is found on the centerline $x = 0.5$ of the transversal midplane (see Fig. 13.7). The SM model is the one that presents the largest distance from the experimental curves.

Also, Table 13.4 shows a discrete normalized L^2-norm of the deviations between the computed and experimental mean velocities at the transversal midplane $y = 0.5$, $e_0^{\langle u_1 \rangle}$, $e_0^{\langle u_3 \rangle}$, computed by

$$e_0^{\langle u_1 \rangle} = \left[\frac{\displaystyle\int_{z=0}^{z=1} |(\langle u_1 \rangle_h - \langle u_1 \rangle_{exp})(0.5, 0.5, z)|^2 dz}{\displaystyle\int_{z=0}^{z=1} |\langle u_1 \rangle_{exp}(0.5, 0.5, z)|^2 dz} \right]^{1/2}. \tag{13.8}$$

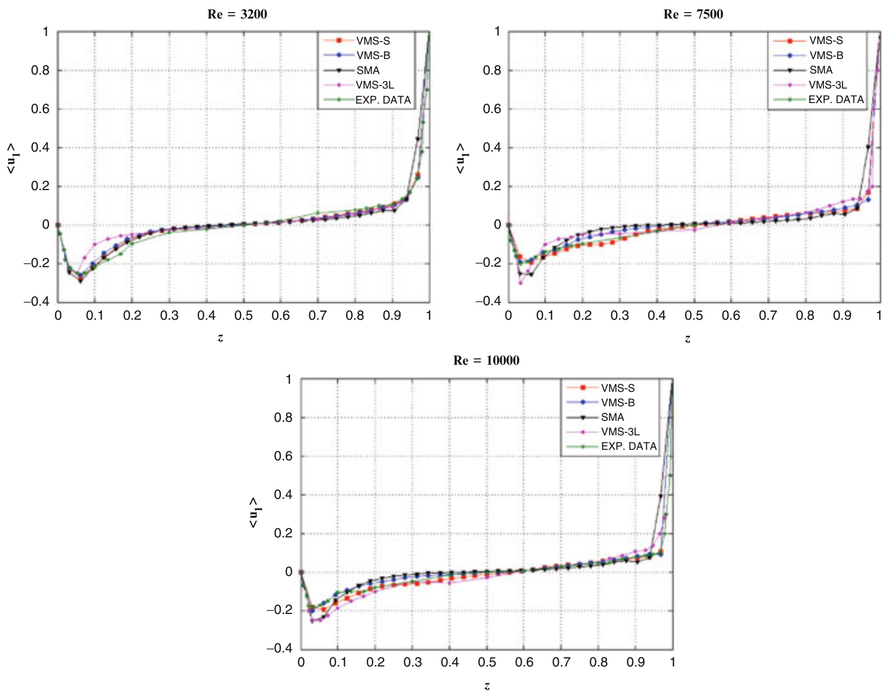

Fig. 13.7 $\langle u_1 \rangle$ on the vertical centerline ($x = 0.5$) of the midplane $y = 0.5$ for $Re = 3200$, $Re = 7500$, $Re = 10000$

Table 13.4 L^2-norm of the deviation from the experimental profiles for the mean velocities

	$e_0^{\langle u_1 \rangle}$			$e_0^{\langle u_3 \rangle}$		
Methods	$Re = 3200$	$Re = 7500$	$Re = 10000$	$Re = 3200$	$Re = 7500$	$Re = 10000$
VMS-S	0.1497	0.1247	0.2689	0.2186	0.1706	0.2858
VMS-B	0.1677	0.1832	0.2231	0.2399	0.1572	0.2691
SM	0.1790	0.2627	0.4231	0.2870	0.2066	0.4147
VMS-3L	0.2434	0.3529	0.2962	0.6522	0.1428	0.2153

$$e_0^{\langle u_3 \rangle} = \left[\frac{\displaystyle\int_{x=0}^{x=1} |(\langle u_3 \rangle_h - \langle u_3 \rangle_{exp})(x, 0.5, 0.5)|^2 dx}{\displaystyle\int_{x=0}^{x=1} |\langle u_3 \rangle_{exp}(x, 0.5, 0.5)|^2 dx} \right]^{1/2} . \tag{13.9}$$

We may observe that the errors due to the SM method deteriorate as Re increases. The errors due to all VMS methods remain within the same levels, between 15% and 30%.

However, larger errors appear for the root-mean-square values and the cross components of the Reynolds stress tensor. These deviations are shown for $Re = 10.000$

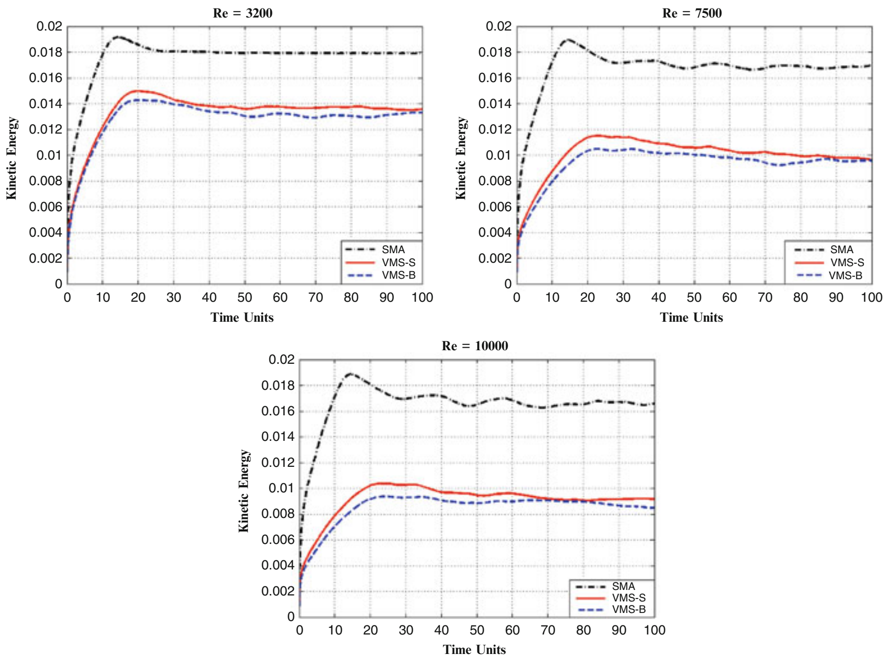

Fig. 13.8 Temporal evolution of the total kinetic energy for $Re = 3200$ (*top left*), $Re = 7500$ (*top right*), and $Re = 10000$ (*bottom*)

in Fig. 13.9 and in Table 13.5. This table presents the quantitative normalized discrete L^2-norm of the errors. One could think that this could be the consequence of a low accuracy, but mis-predictions of various peaks of these curves may also be found in the numerical results of Zang et al. [42], achieved with a four-times finer discretization in every coordinate direction. Thus, these errors are possible due to an incomplete modeling.

As a conclusion, in the case of the 3D cavity flow, both VMS-S and VMS-B methods act in practice with the accuracy/inaccuracies of a residual-free bubble-based VMS method with three grid levels. This is not surprising as both the VMS-S and VMS-B methods indeed are methods with three grid levels: resolved large scales, resolved small scales, and unresolved scales, whose action on the resolved scales is modeled through the subgrid eddy turbulence projection term.

13.4 Turbulent Channel Flow

The third numerical test flow that we consider is a steady turbulent channel flow. The 3D channel flow is one of the most popular test problems for the investigation of wall-bounded turbulent flows. The flow domain is the slice of the 3D space bounded

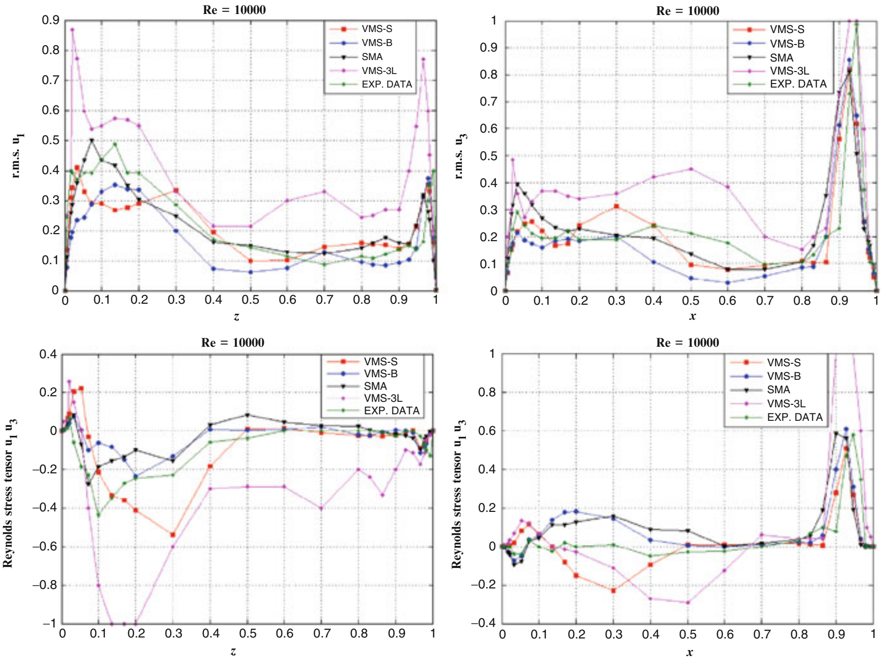

Fig. 13.9 $\sqrt{\langle \tilde{u}_1^2 \rangle}$ (*top left*, factor 10), $\sqrt{\langle \tilde{u}_3^2 \rangle}$ (*top right*, factor 10), and $\langle \tilde{u}_1 \tilde{u}_3 \rangle$ (*bottom*, factor 500) on the centerlines $x = 0.5$ (*left*) and $z = 0.5$ (*right*) of the midplane $y = 0.5$ for $Re = 10000$

Table 13.5 L^2-norm of the deviation for the r.m.s. and the components of the Reynolds stress tensor

	$e_0^{\sqrt{\langle \tilde{u}_1^2 \rangle}}$	$e_0^{\sqrt{\langle \tilde{u}_3^2 \rangle}}$	$e_0^{\langle \tilde{u}_1 \tilde{u}_3 \rangle (z)}$	$e_0^{\langle \tilde{u}_1 \tilde{u}_3 \rangle (x)}$
Methods	$Re = 10000$			
VMS-S	0.3226	0.4094	0.9429	0.9093
VMS-B	0.3493	0.4749	0.6859	0.8976
SM	0.2424	0.5109	0.6195	1.1532
VMS-3L	0.7808	0.6975	2.3415	1.8805

by two parallel walls. A direction parallel to the walls is chosen to be the stream-wise direction of the flow. In practice, homogeneous Dirichlet boundary conditions ($\mathbf{v} = 0$) are imposed at the walls, while periodic boundary conditions are set in the stream-wise and crosswise directions. The flow is driven by a constant pressure gradient source term $\mathbf{f} = (f_p, 0, 0)$.

The parameter that characterizes this flow is the wall friction Reynolds number, defined as $Re_\tau = \dfrac{u_\star \delta}{\nu}$, where $u_\star$ is the wall friction velocity and δ is the half-width of the channel. The parameter Re_τ may be set to a given value by specifying f_p, thanks to the relation $u_\star = \sqrt{f_p \delta}$, attempted at the steady state (cf. [14]). As a benchmark, the accurate direct numerical simulation (DNS) results of Moser, Kim, and Mansour [34] for quasi-steady flow are frequently used.

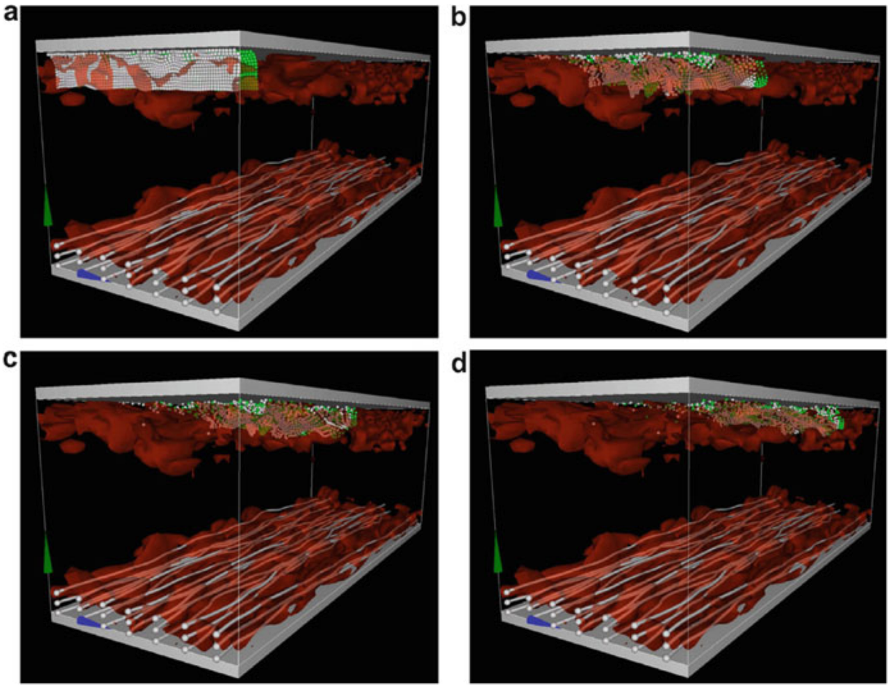

Fig. 13.10 Turbulent channel flow at $Re_\tau = 395$. Iso-surfaces and streamlines of stream-wise velocity at increasing times (from 13.10a–d). A set of points (in white) is launched at initial time (Fig. 13.10a) from the inflow boundary (from Bazilevs et al. [4])

In practice reaching a steady state for the turbulent channel flow from a naive initialization is extremely computer consuming and there exist several strategies to shorten the time to reach a (quasi) steady state. These strategies are based upon perturbing the flow by unstable modes in the inflow boundary and solving the flow in several stages on progressively finer grids, among others. A purely steady state is hardly reachable as this flow is quite sensitive to perturbations. Instead of steady quantities, average quantities on significant time intervals, once the flow is well developed, are used.

Figure 13.10 displays several snapshots of the developed turbulent channel flow at $Re_\tau = 395$, computed with a residual-based VMS method by Bazilevs et al. [4] (that we describe below). Fine details of the flow are observed. A set of points is launched in a vertical plane contained in the inflow boundary, close to the upper wall. Points located close to the wall travel at smaller velocities, while a strong mixing takes place close to the outflow boundary.

The turbulent channel flow was pioneered as a LES test problem by Moin and Kim (cf. [33]) and extensively used to test several versions of LES models (cf. Carati et al. [7], Galdi and Layton [18], Sagaut [39], and Iliescu et al. [27], among others). It has also been used to test VMS models. We shall next describe the results obtained with projection-based, residual-free bubble-based (following John and Kindle [31]) and residual-based VMS (following Bazilevs et al. [4]) methods. The basic aspects of these three methods were respectively introduced in Sects. 11.1, 11.7.1, and 11.7.2.

13.4.1 Residual-Free Bubble-Based vs Projection-Based VMS Method

John and Kindle perform in [31] a comparison between the projection-based VMS method (11.9)–(11.14) and several versions of the residual-free bubble-based VMS method, applied to the numerical simulation of turbulent channel flow.

The projection-based method (11.9)–(11.14) is discretized by means of a Crank–Nicolson scheme, similar to (13.6) with $\theta = 1/2$, but without stabilizing terms. Hexahedral grids with Q_2 finite elements for velocity are used: The velocity space is

$$\mathbf{W}_h = \{\mathbf{w}_h \in V_h^{(2)}(\mathscr{T}_h)^d \mid \mathbf{w}_h \text{ is } 2\pi\text{-periodic in } x_1 \text{ and } \pi\text{-periodic in } x_3, \\ \mathbf{w}_h = 0 \ \text{ on } x_2 = 0,\ x_2 = 2\ \}. \quad (13.10)$$

where $V_h^{(l)}(\mathscr{T}_h)^d$ is defined by (9.6). Also, P_1 discontinuous finite elements are used for pressure: The pressure space is

$$M_h = \{v_h \in L^2(\Omega) \ \text{such that}\ v_{h_{|K}} \in P_1(K), \ \text{for all}\ K \in \mathscr{T}_h\}. \quad (13.11)$$

This pair of spaces $(\mathbf{W}_h, M_h)$ is inf-sup stable and is considered to be the most stable and best performing element for finite element discretizations of the Navier–Stokes equations fulfilling the inf-sup condition (cf. Fortin [17], Gresho and Sani [23]). This space–time discretization provides an overall second-order accuracy, which as we already mentioned is needed to obtain good discretization error levels for incompressible flow simulations (cf. also John [30]). The space L_h for discrete deformations is chosen either as the P_0 or P_1 discontinuous finite element spaces, to obtain a good computational efficiency. The operator σ_h is the L^2 orthogonal projection on L_h.

The grids are uniform in the stream-wise and span-wise directions, while in the wall-normal directions the nodes are concentrated close to the walls, following the formula

$$y_i = 1 - \cos\left(\frac{i\pi}{N}\right), \quad i = 0, 1, \cdots, N,$$

where N is the number of mesh layers in the wall-normal direction. Finally, the eddy viscosity ν_t is given either by

$$\nu_t = C_S\, \delta^2\, \|D(\mathbf{v}_h)\|, \quad \text{(large–small setting)}, \tag{13.12}$$

or

$$\nu_t = C_S\, \delta^2\, \|(I - \sigma_h)(D(\mathbf{v}_h))\|, \quad \text{(small–small setting)}. \tag{13.13}$$

The cutoff parameter δ in practice takes the value $\delta = h_K$ on each element K of the grid. The projection term (11.14) is treated in an implicit way within the Crank–Nicolson overall scheme to obtain a globally stable scheme.

The residual-free bubble-based VMS method starts from a Crank–Nicolson discretization of the two-level problems (11.2)–(11.3). The computation and storage of the residual-free bubble functions is quite computationally consuming, and several simplifications to solve the small-scale equations are made. The small-scale pressure is not solved, but modeled as in the residual-based VMS methods, via (11.44)–(11.45).

$$\tilde{p}_h^n = -\tau_{p,K}\, \nabla \cdot \tilde{\mathbf{v}}_h^n$$

The coefficient $\tau_{p,K}$ is computed either by a constant value, advised for inf-sup stable finite elements in isotropic meshes (cf. [5]),

$$\tau_{p,K} = 1/2, \tag{13.14}$$

or by a choice for stabilized methods with equal-order interpolation for velocity and pressure (cf. [12]):

$$\tau_{p,K} = \frac{1}{2}\sqrt{\nu^2 + \left(\frac{\tilde{h}_K\, \|\mathbf{v}_h\|_{0,2,K}}{2}\right)^2}, \tag{13.15}$$

where $\tilde{h}_K$ is twice the minimal length of an edge of element K. This modeling of the small-scale pressure has several advantages. On one hand, this makes appear in the large-scale equation (11.2) the diffusive grad-div term:

$$(\tau_{p,K} \nabla \cdot \tilde{\mathbf{v}}_h^n, \nabla \cdot \mathbf{w}_h). \tag{13.16}$$

On the other hand, this allows to eliminate the incompressibility restriction for the small-scale resolved flow. Thus, the small-scale equations (11.3) are approximated by a system of convection–diffusion equations. In these equations a subgrid eddy turbulence term is added to model the effect of the small unresolved scales on the small resolved scales. This system is discretized by an Euler-like method where, to avoid the storage requirements, the effect of the small resolved scales in the previous step is eliminated. However these equations are strongly convection-dominated,

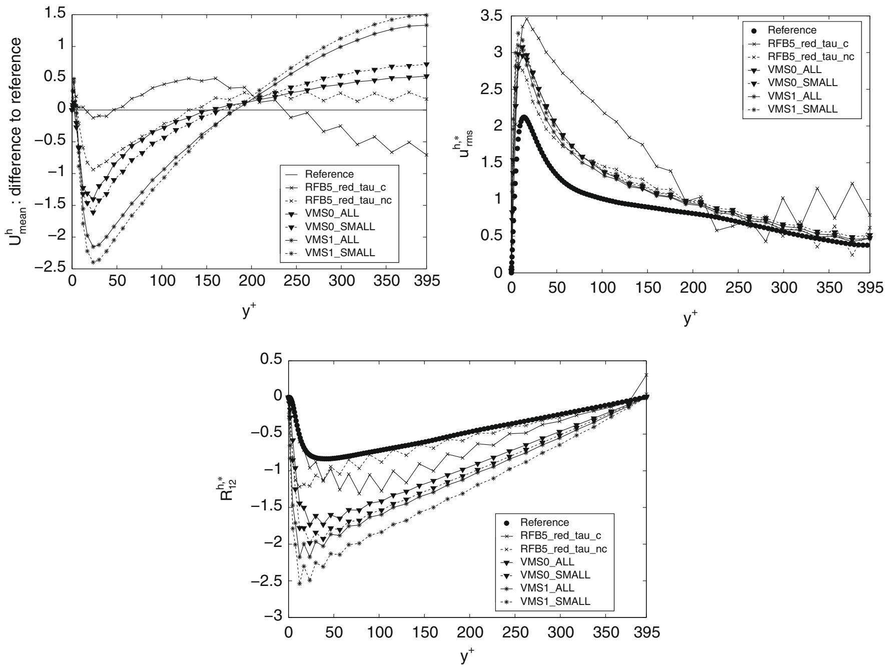

Fig. 13.11 Turbulent channel flow at $Re_\tau = 395$: projection-based and residual-free bubble-based VMS methods. Deviation with respect to mean stream-wise velocity $< u_1 >$ (*top left*), random-mean-square friction velocity (*top right*), and $< u_1 u_2 >$ (from John and Kindle [31])

and then the constant in the eddy turbulence term is increased to high values (typically $C_S = 1$). Moreover, the small-scale space $\mathbf{W}'$ is approximated by a finite-dimensional residual-free bubble space $\mathbf{W}^h_{bub}$ of Q_1 finite elements, constructed on a finer grid. Finally, the discrete velocity and pressure spaces for the large resolved scales, as for the above-referenced projection-based VMS method, are given by (13.10) and (13.11). The overall method is called "reduced residual-free bubble-based VMS method". In [31] several versions of the bubble-based VMS method are presented, but here we retain this one as it yields the best results among them.

Figure 13.11 displays the results obtained in [31] for $Re_\tau = 395$ with a grid corresponding to $N_y = 32$ and 8112 cells. First- and second-order statistics of the flow are presented, computed as a time average once the flow has attempted a quasi-steady state, similarly to the cavity flow above.

For the projection-based VMS methods, the abbreviation VMSn stands for the choice $L_h = P_n$, $n = 1, 2$, while ALL and SMALL respectively stand for the (13.12) and (13.13) settings of ν_t. We observe that both settings of the eddy diffusion provide similar results. However, a larger dependency with respect to the choice of the space L_h appears, the better ones corresponding to the P_0 finite element space, possibly due to a larger numerical diffusion. In any case, the second-order statistics are overpredicted, particularly close to the wall.

For the residual-free bubble-based VMS methods, the abbreviation RFBn_red stands for the reduced residual-free bubble-based VMS method with $n \times n \times n$ meshes for approximating the local solutions of the small-scale equation. Also, tau_c and tau_nc respectively stand for the constant and nonconstant settings of the stabilization parameter given by (13.14) and (13.15). We observe that in general the use of the nonconstant $\tau_{p,K}$ yields better results than the constant one and also of all presented projection-based VMS methods. However, the second-order statistics still are overpredicted close to the wall. This appears to be a limitation of the model rather than being due to numerical inaccuracy. It should be noted also that for coarser grids, the projection-based VMS methods yield more accurate results than the bubble-based ones.

From the computational point of view, the use of bubble-based VMS methods is quite involved. Several simplifications and modeling procedures are needed, but the more questionable point is the usefulness of the small-scale equations. Indeed, it is not clear whether the numerical diffusion needed to stabilize the convection-dominated convection–diffusion problems for the small scales hides the eddy viscosity in principle due to the model. In their turn, projection-based VMS methods are simpler to work out, with reduced modeling issues.

13.4.2 Residual-Based VMS Method

In [4] Hughes and co-workers present the application of the residual-based VMS method (see Sect. 11.7.2) to the numerical simulation of turbulent channel flow at $Re_\tau = 395$. The domain is a rectangular box $\Omega = (0, 2\pi) \times (-1, 1) \times (0, 2/3\,\pi)$. Velocity and pressure are discretized by NURBS (non-uniform rational basis splines), which are tensor products of spline functions. Consider an interval $I = [a, b] \subset \mathbb{R}$ and a discretization support $\delta = \{a = x_0 < x_1 < \cdots < x_n = b\}$. Let us define the splines space, for $k \geq 0$, $l \geq k + 1$,

$$Z_l^{(k)}(\delta) = \{z_h \in C^k(I) \,|\, z_{h|_{[x_{i-1},x_i]}} \in \mathbb{P}_l([x_{i-1}, x_i]),\ i = 1, \cdots, n\}.$$

Consider now three discretization supports δ_1, δ_2, and δ_3 respectively of the intervals $[0, 2\pi]$, $[-1, 1]$, and $[0, 2/3\,\pi]$. Denote $\Delta = \delta_1 \times \delta_2 \times \delta_3$ and define the NURBS space

$$V_l^{(k)}(\Delta) = \{v_h \in C^k(\overline{\Omega}) \,|\, v_h(x_1, x_2, x_3) = z_1(x_1)\, z_2(x_2)\, z_3(x_3) \quad \text{for some } z_i \in Z_l^{(k)}(\delta_i),\ i = 1, 2, 3\}.$$

The velocity and pressure spaces are given by

$$\tilde{\mathbf{W}}_h = \tilde{\mathbf{W}}_l^{(k)}(\Delta) = \{\mathbf{w}_h \in [V_l^{(k)}(\Delta)]^3 \mid \mathbf{w}_h \text{ is } 2\pi\text{-periodic in } x_1 \text{ and } 2/3\,\pi\text{-periodic in } x_3,$$
$$\mathbf{w}_h = 0 \text{ on } x_2 = -1,\ x_2 = 1\ \}, \qquad (13.17)$$

$$\tilde{\mathbf{M}}_h = \tilde{M}_l^{(k)}(\Delta) = \tilde{\mathbf{V}}_l^{(k)}(\Delta) \cap L_0^2(\Omega). \qquad (13.18)$$

Equal-order interpolation is used in velocity and pressure, as the residual-based VMS method in particular provides a stabilization of the pressure discretization. This happens because after inserting the modeling of small scales yielded by (11.44), the large-scale equation (11.43) contains a term similar to the term (13.7), which is a weighted Laplacian of the pressure. In the tests, C^0-continuous linear, C^1-continuous quadratic, and C^2-cubic NURBS (that respectively correspond to spaces $\tilde{\mathbf{W}}_l^{(k)}(\Delta) \times \tilde{M}_l^{(k)}(\Delta)$ with $k = 0, l = 1$; $k = 1, l = 2$; and $k = 2, l = 3$) are used. For these spaces, in the stream-wise and span-wise directions, the number of basis functions is equal to the number of elements in these directions (the number n in the case of the support δ). In the wall-normal direction, the number of basis functions is $n + l$. Also, a generalized second-order α-method is used for time discretization (cf. Jahnsen et al. [29]). This method requires the solution of a residual equation at each time step, which is performed by a multistage quasi-Newton method.

Figure 13.12 displays the results for C^0-continuous linear, C^1-continuous quadratic, and C^2-cubic NURBS for a grid with 64^3 elements, compared to a direct numerical simulation. The results presented are the mean stream-wise velocity and the root-mean-square velocity fluctuations. We observe that the linear NURBS accuracy provides an overprediction of some of the second-order statistics, while the quadratic and cubic provide very impressive results: Both first- and second-order statistics are quite close to the DNS ones, and in particular those provided by the cubic NURBS are almost indistinguishable from these.

13.5 Conclusion

We have presented in this chapter the results provided by projection-based, residual-free bubble-based, and residual-based VMS methods, for three relevant tests cases.

From the point of view of accuracy, the residual-based VMS method with at least quadratic NURBS provides very accurate results to compute mean quantities so as second-order statistics, while the remaining methods are accurate for the mean flow, but present important inaccuracies for the second-order statistics. In any case, an overall second-order accuracy in space and time is needed to avoid large inaccuracies.

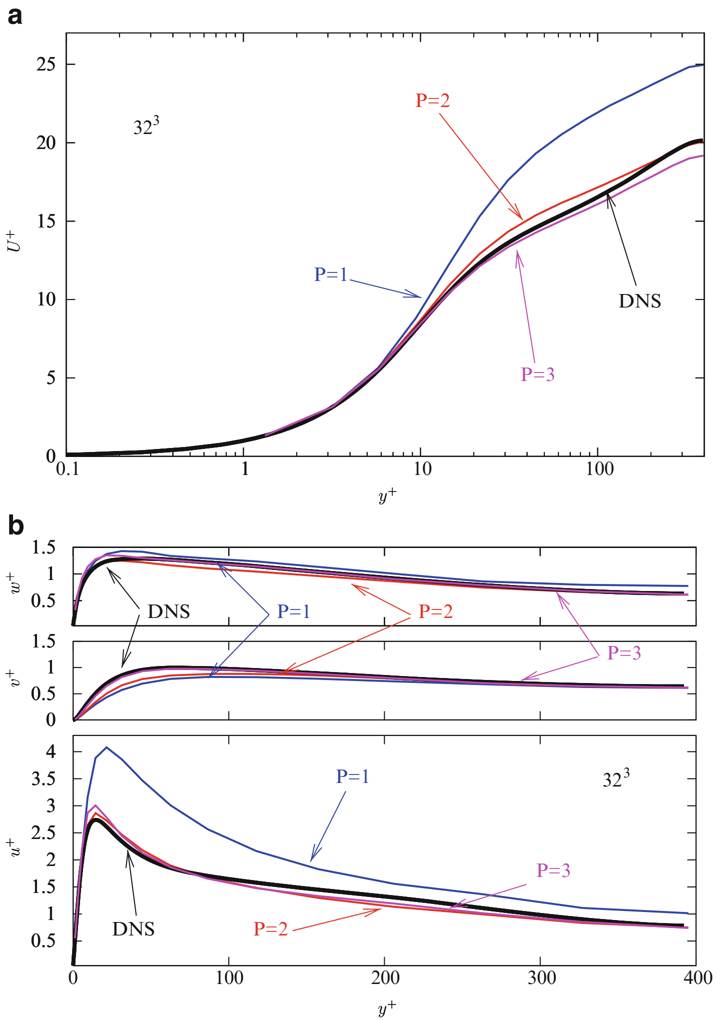

Fig. 13.12 Turbulent channel flow at $Re_\tau = 395$. (**a**) Mean stream-wise velocity and (**b**) root-mean-square velocity fluctuations (from Bazilevs et al. [4])

From the computational point of view, the use of bubble-based VMS methods is quite involved, and the usefulness of the small-scale equations is doubtful. Also, the structure of residual-based VMS methods is quite complex, requiring a large programming effort to make them run. Projection-based VMS methods are simpler to work out, with reduced modeling issues.

From the point of view of modeling, the additional advantage introduced by residual-based VMS models is to keep all inertial interactions between large and small resolved scales. The action of the small unresolved scales on the resolved ones is taken into account through a relatively simple diagonalization procedure (see Eq. (11.44)), where a good definition of the stabilization coefficients is crucial. Projection-based and bubble finite element-based VMS methods introduce subgrid eddy diffusion terms that appear to be rather inefficient to approximate the second-order statistics.

From the point of view of engineering applications, we must consider that all VMS methods are LES methods and thus much more costly than RANS models. Thus their use in real applications needs large computational improvements and for the next future is to be restricted to geometries with low complexity and specific flows for which a high accuracy is needed.

In sum, bubble-based VMS would need further improvement to be used by non-experienced researchers. Residual-based VMS methods provide the best choice when a good accuracy in the computation of second-order statistics is needed, while projection-based VMS methods provide a compromise between accuracy and computational complexity.

13.6 FreeFem++ Numerical Code for VMS-A and VMS-B Models

This section presents the numerical code used to compute the 3D lid-driven cavity flow by method (13.6). It is written in FreeFem++ language (see [25]). FreeFem++ is a high-level integrated development environment for the numerical solution of 2D and 3D partial differential equations by the finite element method. It includes a high-level language close to the mathematical notation that allows to program complex discretizations with a highly reduced number of unknowns. It also includes built-in grid builders, post-processing of results, and several solvers of linear systems.

We display the VMS-S model and include as comments the modifications for VMS-B and SM models. This code has been made by Dr. Samuele Rubino, in the framework of his PhD thesis.

```
 //LOAD LIBRARIES
load "msh3"
load "Element_P1dc1"

//BUILD MESH
int nn=16;
mesh Th2=readmesh("Th.msh");//2D MESH
int[int] rdown=[0,5],rup=[0,5],rmid=[1,5,2,5,3,6,4,5];
real zmin=0.,zmax=1.;
mesh3 Th=buildlayers(Th2,nn,zbound=[zmin,zmax],
reffacelow=rdown,reffaceup=rup,reffacemid=rmid);
//3D MESH
```

```
//SET NUMERICAL SIMULATION
real nu0=1./3200.;//Re=3200, Re=7500 and Re=10000
real CS=0.1;
real cc1=16.;
real cc2=sqrt(cc1);
real epspen=1.e-10;
real dt=0.1;
real dtt=1./dt;
real t=0.;
real theta=0.5;//theta = 0.5 Crank-Nicolson scheme
int nIter=1000;
int initialIt=0;

//DEFINE FE SPACES
fespace Vh3P2(Th,[P23d,P23d,P23d,P23d]);
fespace Vh3P1dc(Th,[P1dc3d,P1dc3d,P1dc3d,P1dc3d]);
fespace Vh3P1(Th,[P13d,P13d,P13d,P13d]);
fespace Vh3P0(Th,[P03d,P03d,P03d,P03d]);
fespace VhP2(Th,P23d);
fespace VhP1dc(Th,P1dc3d);
fespace VhP1(Th,P13d);
fespace VhP0(Th,P03d);
fespace Vh2(Th2,P2);

//INITIALIZATION
Vh3P2 [u1,u2,u3,p];
VhP2 u1tmp,u2tmp,u3tmp;
{
u1tmp = 0.;
u2tmp = 0.;
u3tmp = 0.;
}
VhP0 tKcod,tKTcod,tau,mk,hT=hTriangle;
real hs=hT[].min;

//DEFINE MACRO
macro Grad(u) [dx(u),dy(u),dz(u)]//EOM
macro div(u1,u2,u3) (dx(u1)+dy(u2)+dz(u3))//EOM
macro eps(u1,u2,u3,v1,v2,v3) (4.*(dx(u1)*dx(v1)
+dy(u2)*dy(v2)+dz(u3)*dz(v3)) +2.*((dx(u2)
+dy(u1))*(dx(v2)+dy(v1))+(dx(u3)+dz(u1))*(dx(v3)+dz
  (v1))
+(dy(u3)+dz(u2))*(dy(v3)+dz(v2))))//EOM
macro Neps(u1,u2,u3) (sqrt(4.*(dx(u1)^2+dy(u2)^2
```

```
+dz(u3)^2)+2.*((dx(u2)+dy(u1))^2+(dx(u3)+dz(u1))^2
+(dy(u3)+dz(u2))^2)))//EOM

//TIME-INDEPENDENT VARIATIONAL FORMULATION
varf vNS([uu1,uu2,uu3,pp],[v1,v2,v3,q])=
int3d(Th)(dtt*(uu1*v1+uu2*v2+uu3*v3)
+theta*(nu0/2.)*eps(uu1,uu2,uu3,v1,v2,v3)
-theta*div(v1,v2,v3)*pp+theta*div(uu1,uu2,uu3)*q
+epspen*pp*q);

//BUILD TIME-INDEPENDENT MATRIX
cout << "-----------------------------------------"
<< endl;
cout << "Starting iteration number " << initialIt
<< " Time = " << t << endl;
cout << "-----------------------------------------"
<< endl;

cout << "Build A Fix." << endl;

matrix Af = vNS(Vh3P2,Vh3P2);

real Ekin=0.;
cout << "Kinetic Energy = " << Ekin << endl;

cout << "-----------------------------------------"
<< endl;
cout << "End of iteration number " << initialIt
<< " Time = " << t << endl;
cout << "-----------------------------------------"
<< endl;

//ELEMENTS VOLUME
varf med(unused,v)=int3d(Th)(1.*v);
real[int] medk=med(0,VhP0);
mk[]=sqrt(medk);

//INTERPOLATION MATRIX
matrix DX3,DY3,DZ3;
{
matrix DXYZ3u1,DXYZ3u2,DXYZ3u3,DXYZ3p;

int[int] c0 = [0,-1,-1,-1];
int[int] c1 = [-1,1,-1,-1];
int[int] c2 = [-1,-1,2,-1];
```

```
int[int] c3 = [-1,-1,-1,3];

DXYZ3u1 = interpolate(Vh3P1dc,Vh3P2,U2Vc=c0,op=1);
DXYZ3u2 = interpolate(Vh3P1dc,Vh3P2,U2Vc=c1,op=1);
DXYZ3u3 = interpolate(Vh3P1dc,Vh3P2,U2Vc=c2,op=1);
DXYZ3p  = interpolate(Vh3P1dc,Vh3P2,U2Vc=c3,op=1);
DX3     = DXYZ3u1 + DXYZ3u2 + DXYZ3u3 + DXYZ3p;

DXYZ3u1 = interpolate(Vh3P1dc,Vh3P2,U2Vc=c0,op=2);
DXYZ3u2 = interpolate(Vh3P1dc,Vh3P2,U2Vc=c1,op=2);
DXYZ3u3 = interpolate(Vh3P1dc,Vh3P2,U2Vc=c2,op=2);
DXYZ3p  = interpolate(Vh3P1dc,Vh3P2,U2Vc=c3,op=2);
DY3     = DXYZ3u1 + DXYZ3u2 + DXYZ3u3 + DXYZ3p;

DXYZ3u1 = interpolate(Vh3P1dc,Vh3P2,U2Vc=c0,op=3);
DXYZ3u2 = interpolate(Vh3P1dc,Vh3P2,U2Vc=c1,op=3);
DXYZ3u3 = interpolate(Vh3P1dc,Vh3P2,U2Vc=c2,op=3);
DXYZ3p  = interpolate(Vh3P1dc,Vh3P2,U2Vc=c3,op=3);
DZ3     = DXYZ3u1 + DXYZ3u2 + DXYZ3u3 + DXYZ3p;
}

int[int] cs2=[0];
matrix Dxu1 = interpolate(VhP1dc,Vh3P2,U2Vc=cs2,op=1);
matrix Dyu1 = interpolate(VhP1dc,Vh3P2,U2Vc=cs2,op=2);
matrix Dzu1 = interpolate(VhP1dc,Vh3P2,U2Vc=cs2,op=3);

cs2=[1];
matrix Dxu2 = interpolate(VhP1dc,Vh3P2,U2Vc=cs2,op=1);
matrix Dyu2 = interpolate(VhP1dc,Vh3P2,U2Vc=cs2,op=2);
matrix Dzu2 = interpolate(VhP1dc,Vh3P2,U2Vc=cs2,op=3);

cs2=[2];
matrix Dxu3 = interpolate(VhP1dc,Vh3P2,U2Vc=cs2,op=1);
matrix Dyu3 = interpolate(VhP1dc,Vh3P2,U2Vc=cs2,op=2);
matrix Dzu3 = interpolate(VhP1dc,Vh3P2,U2Vc=cs2,op=3);

cs2=[3];
matrix Dxp = interpolate(VhP1dc,Vh3P2,U2Vc=cs2,op=1);
matrix Dyp = interpolate(VhP1dc,Vh3P2,U2Vc=cs2,op=2);
matrix Dzp = interpolate(VhP1dc,Vh3P2,U2Vc=cs2,op=3);

matrix IPh;
matrix IPhId;
matrix I3P2;
```

```
// VMS-S MODEL
{
matrix Id,Idh;
matrix Id3;
{
VhP2 fAux2 = 1.;
VhP1dc fAux1dc=1.;
Id = fAux2[];
Idh = fAux1dc[];
Id3 = [[Id,0,0,0],[0,Id,0,0],[0,0,Id,0],[0,0,0,Id]];
}
matrix PIg = interpolate(VhP1,VhP1dc);
matrix IPg = interpolate(VhP1dc,VhP1);
matrix IPPIg = IPg*PIg;
IPh = Idh + (-1.)*IPPIg;

matrix PI = interpolate(VhP1,VhP2);
matrix IP = interpolate(VhP2,VhP1);
matrix IPPI = IP*PI;
IPhId = Id + (-1.)*IPPI;

matrix PI3 = interpolate(Vh3P1,Vh3P2);
matrix IP3 = interpolate(Vh3P2,Vh3P1);
matrix IPPI3 = IP3*PI3;
I3P2 = Id3 + (-1.)*IPPI3;
}

/* VMS-B MODEL
{
matrix Idh;
matrix Id3;
{
VhP1dc fAux1dc=1.;
Idh = fAux1dc[];
Id3 = [[Idh,0,0,0],[0,Idh,0,0],[0,0,Idh,0],[0,0,0,
         Idh]];
}
matrix PIg = interpolate(VhP1,VhP1dc);
matrix IPg = interpolate(VhP1dc,VhP1);
matrix IPPIg = IPg*PIg;
IPh = Idh + (-1.)*IPPIg;

matrix PI = interpolate(VhP0,VhP1dc);
matrix IP = interpolate(VhP1dc,VhP0);
matrix IPPI = IP*PI;
```

```
IPhId = Idh + (-1.)*IPPI;

matrix PI3 = interpolate(Vh3P0,Vh3P1dc);
matrix IP3 = interpolate(Vh3P1dc,Vh3P0);
matrix IPPI3 = IP3*PI3;
I3P2 = Id3 + (-1.)*IPPI3;
}
*/

/*  SM MODEL
{
matrix Idh;
{
VhP1dc fAux1dc=1.;
Idh = fAux1dc[];
}
matrix PIg = interpolate(VhP1,VhP1dc);
matrix IPg = interpolate(VhP1dc,VhP1);
matrix IPPIg = IPg*PIg;
IPh = Idh + (-1.)*IPPIg;
}
*/

matrix DDx = IPh*Dxp;
matrix DDy = IPh*Dyp;
matrix DDz = IPh*Dzp;

matrix DXun = interpolate(VhP1dc,VhP2,op=1);
matrix DYun = interpolate(VhP1dc,VhP2,op=2);
matrix DZun = interpolate(VhP1dc,VhP2,op=3);

VhP1dc u1dcX,u1dcY,u1dcZ,u2dcX,u2dcY,u2dcZ,u3dcX,
u3dcY,u3dcZ;

//START LOOP IN TIME
for(int i=1;i<=nIter;++i)
{
t+=dt;
cout << "-----------------------------------------"
<< endl;
cout << "Starting iteration number " << i
<< " Time = " << t << endl;
cout << "-----------------------------------------"
<< endl;
```

```
//STABILIZATION COEFFICIENTS
varf tauK(unused,v)=int3d(Th)((u1^2+u2^2+u3^2)*v);

//   VMS MODELS
varf tauKT(unused,v)=int3d(Th)(((u1dcX)^2+(u2dcY)^2
+(u3dcZ)^2+0.5*((u2dcX+u1dcY)^2+(u3dcX+u1dcZ)^2
+(u3dcY+u2dcZ)^2))*v);
/*   SM MODEL
varf tauKT(unused,v)=int3d(Th)((dx(u1)^2+dy(u2)^2
+dz(u3)^2+0.5*((dx(u2)+dy(u1))^2+(dx(u3)+dz(u1))^2
+(dy(u3)+dz(u2))^2))*v);
*/

real[int] tK=tauK(0,VhP0);
real[int] tKT=tauKT(0,VhP0);
tKcod[]=sqrt(tK);
tKTcod[]=sqrt(tKT);
tau=((cc1*(nu0+((CS*hs)^2)*(tKTcod/mk))/hs^2)
+ (cc2*(tKcod/mk)/hs))^(-1.);

//VMS-Smagorinsky MATRIX
Vh3P1dc [u1p,u2p,u3p,pprev];

varf VMSSma([u1l,u2l,u3l,unused],[v1l,v2l,v3l,ql]) =
int3d(Th)(theta*((CS*hs)^2)*sqrt((u1dcX)^2+(u2dcY)^2
+(u3dcZ)^2+0.5*((u2dcX+u1dcY)^2+(u3dcX+u1dcZ)^2
+(u3dcY+u2dcZ)^2))*(u1l*v1l+u2l*v2l+u3l*v3l));
varf VMSSmaN([unused1,unused2,unused3,unusedp],
[v1l,v2l,v3l,ql]) =
int3d(Th)(-theta*((CS*hs)^2)*sqrt((u1dcX)^2+(u2dcY)^2
+(u3dcZ)^2+0.5*((u2dcX+u1dcY)^2+(u3dcX+u1dcZ)^2
+(u3dcY+u2dcZ)^2))*(u1p*v1l+u2p*v2l+u3p*v3l));

cout << "Build A VMS-Smagorinsky" << endl;
matrix M = VMSSma(Vh3P1dc,Vh3P1dc);

matrix Sma;
matrix DXX;
matrix DYY;
matrix DZZ;
real[int] bMX(Vh3P2.ndof);
real[int] bMY(Vh3P2.ndof);
real[int] bMZ(Vh3P2.ndof);

//  VMS-S MODEL
```

```
{
matrix Maux;

Maux = DX3*I3P2;
DXX  = (Maux')*M;
DXX  = DXX*Maux;
u1p[]= Maux*u1[];
real[int] bM = VMSSmaN(0,Vh3P1dc);
bMX = (Maux')*bM;

Maux = DY3*I3P2;
DYY  = (Maux')*M;
DYY  = DYY*Maux;
u1p[]= Maux*u1[];
bM = VMSSmaN(0,Vh3P1dc);
bMY = (Maux')*bM;

Maux = DZ3*I3P2;
DZZ  = (Maux')*M;
DZZ  = DZZ*Maux;
u1p[]= Maux*u1[];
bM = VMSSmaN(0,Vh3P1dc);
bMZ = (Maux')*bM;
}

/*  VMS-B MODEL
{
matrix Maux;

Maux = I3P2*DX3;
DXX  = (Maux')*M;
DXX  = DXX*Maux;
u1p[]= Maux*u1[];
real[int] bM = VMSSmaN(0,Vh3P1dc);
bMX = (Maux')*bM;

Maux = I3P2*DY3;
DYY  = (Maux')*M;
DYY  = DYY*Maux;
u1p[]= Maux*u1[];
bM = VMSSmaN(0,Vh3P1dc);
bMY = (Maux')*bM;

Maux = I3P2*DZ3;
DZZ  = (Maux')*M;
```

```
DZZ  = DZZ*Maux;
u1p[]= Maux*u1[];
bM = VMSSmaN(0,Vh3P1dc);
bMZ = (Maux')*bM;
}
*/

Sma = DXX + DYY + DZZ;

//CONVECTION STABILIZATION MATRIX
VhP1dc fu1dc,fu2dc,fu3dc;

varf termConv(uu1,v1)=int3d(Th)(theta*tau*uu1*v1);
varf termConvN1(unused,v1)=
int3d(Th)(-theta*tau*fu1dc*v1);
varf termConvN2(unused,v1)=
int3d(Th)(-theta*tau*fu2dc*v1);
varf termConvN3(unused,v1)=
int3d(Th)(-theta*tau*fu3dc*v1);

cout << "Build A Conv.-Stab." << endl;
matrix TermC=termConv(VhP1dc,VhP1dc);

matrix ES;

VhP1dc u1dc = u1;
VhP1dc u2dc = u2;
VhP1dc u3dc = u3;

matrix U1dc = u1dc[];
matrix U2dc = u2dc[];
matrix U3dc = u3dc[];

//FIRST COMPONENT
matrix ESu1;
real[int] bESu1(Vh3P2.ndof);
{
matrix DF1=U1dc*Dxu1;
matrix DF2=U2dc*Dyu1;
matrix DF3=U3dc*Dzu1;
matrix DF=DF1+DF2+DF3;
matrix E=IPh*DF;
matrix EE=TermC*E;
ESu1=(E')*EE;
fu1dc[]=E*u1[];
```

```
real[int] TermCN1=termConvN1(0,VhP1dc);
bESu1=(E')*TermCN1;
}

//SECOND COMPONENT
matrix ESu2;
real[int] bESu2(Vh3P2.ndof);
{
matrix DF1=U1dc*Dxu2;
matrix DF2=U2dc*Dyu2;
matrix DF3=U3dc*Dzu2;
matrix DF=DF1+DF2+DF3;
matrix E=IPh*DF;
matrix EE=TermC*E;
ESu2=(E')*EE;
fu2dc[]=E*u1[];
real[int] TermCN2=termConvN2(0,VhP1dc);
bESu2=(E')*TermCN2;
}

//THIRD COMPONENT
matrix ESu3;
real[int] bESu3(Vh3P2.ndof);
{
matrix DF1=U1dc*Dxu3;
matrix DF2=U2dc*Dyu3;
matrix DF3=U3dc*Dzu3;
matrix DF=DF1+DF2+DF3;
matrix E=IPh*DF;
matrix EE=TermC*E;
ESu3=(E')*EE;
fu3dc[]=E*u1[];
real[int] TermCN3=termConvN3(0,VhP1dc);
bESu3=(E')*TermCN3;
}

ES=ESu1+ESu2+ESu3;

//BUILD TIME-DEPENDENT MATRIX
varf vNSp([uu1,uu2,uu3,pp],[v1,v2,v3,q])=
int3d(Th)(theta*0.5*(([u1,u2,u3]'*Grad(uu1))*v1
+([u1,u2,u3]'*Grad(uu2))*v2+([u1,u2,u3]'*Grad(uu3))
       *v3)
-theta*0.5*(([u1,u2,u3]'*Grad(v1))*uu1
+([u1,u2,u3]'*Grad(v2))*uu2+([u1,u2,u3]'*Grad(v3))
```

```
               *uu3)
//+theta*(((CS*hTriangle)^2)/2.)*Neps(u1,u2,u3)
*(Grad(uu1)'*Grad(v1)+Grad(uu2)'*Grad(v2)
+Grad(uu3)'*Grad(v3)) SM MODEL
+theta*tau*(Grad(pp)'*Grad(q))
)
+on(6,uu1=1.,uu2=0.,uu3=0.)
+on(5,uu1=0.,uu2=0.,uu3=0.)
+int3d(Th)(dtt*(u1*v1+u2*v2+u3*v3)
-theta*0.5*(([u1,u2,u3]'*Grad(u1))*v1
+([u1,u2,u3]'*Grad(u2))*v2+([u1,u2,u3]'*Grad(u3))*v3)
+theta*0.5*(([u1,u2,u3]'*Grad(v1))*u1
+([u1,u2,u3]'*Grad(v2))*u2+([u1,u2,u3]'*Grad(v3))*u3)
-theta*(nu0/2.)*eps(u1,u2,u3,v1,v2,v3)
//-theta*(((CS*hTriangle)^2)/2.)*Neps(u1,u2,u3)
*(Grad(u1)'*Grad(v1)+Grad(u2)'*Grad(v2)
+Grad(u3)'*Grad(v3)) SM MODEL
-theta*tau*(Grad(p)'*Grad(q))
+theta*div(v1,v2,v3)*p-theta*div(u1,u2,u3)*q
)
;

cout << "Build A Var." << endl;
matrix Av = vNSp(Vh3P2,Vh3P2);

cout << "Build A Fin" << endl;
matrix A = Af + Sma + ES + Av;
//matrix A = Af + ES + Av; SM MODEL

cout << "Factorize A Fin." << endl;
set(A,solver=GMRES);
real[int] bv = vNSp(0,Vh3P2);

//   VMS MODELS
real[int] b = bv + bMX;
b = b + bMY;
b = b + bMZ;
b = b + bESu1;
b = b + bESu2;
b = b + bESu3;

/*   SM MODEL
real[int] b = bv + bESu1;
b = b + bESu2;
b = b + bESu3;
```

```
*/

u1[]= A^-1 * b;

//ACTUALIZATION (ONLY FOR VMS MODELS)
u1tmp = u1;
u2tmp = u2;
u3tmp = u3;

//   VMS-S MODEL
VhP2 udcAux;

udcAux[] = IPhId*u1tmp[];
u1dcX[]  = DXun*udcAux[];
u1dcY[]  = DYun*udcAux[];
u1dcZ[]  = DZun*udcAux[];

udcAux[] = IPhId*u2tmp[];
u2dcX[]  = DXun*udcAux[];
u2dcY[]  = DYun*udcAux[];
u2dcZ[]  = DZun*udcAux[];

udcAux[] = IPhId*u3tmp[];
u3dcX[]  = DXun*udcAux[];
u3dcY[]  = DYun*udcAux[];
u3dcZ[]  = DZun*udcAux[];

/*   VMS-B MODEL
VhP1dc udcAux;

udcAux[] = DXun*u1tmp[];
u1dcX[]  = IPhId*udcAux[];
udcAux[] = DYun*u1tmp[];
u1dcY[]  = IPhId*udcAux[];
udcAux[] = DZun*u1tmp[];
u1dcZ[]  = IPhId*udcAux[];

udcAux[] = DXun*u2tmp[];
u2dcX[]  = IPhId*udcAux[];
udcAux[] = DYun*u2tmp[];
u2dcY[]  = IPhId*udcAux[];
udcAux[] = DZun*u2tmp[];
u2dcZ[]  = IPhId*udcAux[];

udcAux[] = DXun*u3tmp[];
```

```
u3dcX[]  = IPhId*udcAux[];
udcAux[] = DYun*u3tmp[];
u3dcY[]  = IPhId*udcAux[];
udcAux[] = DZun*u3tmp[];
u3dcZ[]  = IPhId*udcAux[];
*/

Ekin=int3d(Th)(0.5*(u1^2+u2^2+u3^2));
cout << "Kinetic Energy = " << Ekin << endl;

cout << "----------------------------------------"
<< endl;
cout << "End of iteration number " << i
<< " Time = " << t << endl;
cout << "----------------------------------------"
<< endl;
}

//END LOOP IN TIME
```

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Appendix A
Tool Box

This appendix contains a detailed description of the concepts and results that are used as the mathematical basics of the book, together with some references where these may be found. In some cases we present the results as they appear in the references. In other cases we adapt them to simplify their use within the book. We also include some new auxiliary results, rather than proving them in the main body of the book, to simplify its reading.

A.1 Sobolev Embedding Theorem

Definition A.1. A bounded domain $\Omega \subset \mathbb{R}^d$ is called Lipschitz if for every $\mathbf{x}_0 \in \Gamma$ there exists a system of local coordinates $\{x_1, \cdots, x_n\}$, obtained from the original one via a rigid motion, such that $\mathbf{x}_0$ is the origin in this system of coordinates, and a Lipschitz continuous function $\phi : \mathbb{R}^{d-1} \mapsto \mathbb{R}$, such that

$$\begin{cases} \Omega \cap C(r,h) = \{\mathbf{x} = (\mathbf{x}', x_d) \in C(r,h) \ \text{ s. t. } \ \mathbf{x}' \in Q_r \ \text{ and } \ x_d > \phi(\mathbf{x}')\}, \\ \Gamma \cap C(r,h) = \{\mathbf{x} = (\mathbf{x}', \dots, x_d) \in C(r,h); \ \text{ s. t. } \ \mathbf{x}' \in Q_r \ \text{ and } \ x_d = \phi(\mathbf{x}')\}, \end{cases}$$

where $\mathbf{x}' = (x_1, x_2, \dots, x_{d-1})$, Q_r denotes the cube $(-r, r)^{d-1} \subset \mathbb{R}^{d-1}$ and $C(r,h)$ denotes the cylinder $Q_r \times (-h, h) \subset \mathbb{R}^d$, for some $h > 0$, $r > 0$.

As Γ is compact, there exist a finite number of local coordinates that parameterize the whole Γ. Observe that if Ω is Lipschitz, then it is locally located on one side of its boundary.

We work with the full scale of Sobolev spaces, in particular of fractionary order. Sobolev spaces provide the framework for the modern analysis of partial differential equations. Let us respectively denote by $C^k(\Omega)$ and $C^\infty(\Omega)$ the space of functions k times and indefinitely differentiable on Ω. Let us define the spaces of smooth functions

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$$\mathscr{D}(\Omega) = \{w \in C^\infty(\Omega) \ \text{ such that } \ \mathrm{supp}(w) \subset \Omega\}, \tag{A.1}$$

where $\mathrm{supp}(w) = \overline{\{\mathbf{x} \in \Omega \ \text{ such that } \ w(\mathbf{x}) \neq 0\}}$ denotes the support of w, and

$$\mathscr{D}(\overline{\Omega}) = \mathscr{D}(\mathbb{R}^n)_{|\overline{\Omega}} = \{w_{|\overline{\Omega}} \ \text{ such that } \ w \in \mathscr{D}(\mathbb{R}^n)\}. \tag{A.2}$$

Consider a multi-index $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$, where $\mathbb{N}$ denotes the set of the natural numbers.

Definition A.2. We say that a function $g \in L^1(\Omega)$ is the derivative of order α in the sense of distributions of some function $f \in L^1(\Omega)$ and denote $g = D^\alpha f = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} f$, if

$$\int_\Omega g(\mathbf{x})\, \varphi(\mathbf{x})\, d\mathbf{x} = (-1)^{|\alpha|} \int_\Omega f(\mathbf{x})\, D^\alpha \varphi(\mathbf{x})\, d\mathbf{x}, \ \text{ for all } \ \varphi \in \mathscr{D}(\Omega),$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

The function $D^\alpha f$ is called the D^α derivative of f in the sense of distributions. This definition is consistent as if there exists $D^\alpha f$ in the classical sense and belongs to $L^1(\Omega)$, then it coincides with the D^α derivative of f in the sense of distributions. It is understood that $D^0 f = f$.

We are now in a position to define the Sobolev spaces of integer order $k \geq 0$. Let $1 \leq p \leq +\infty$. Then we define

$$W^{k,p}(\Omega) = \{w \in L^p(\Omega) \ \text{ s. t. } \ D^\alpha w \in L^p(\Omega), \ \text{ for all } \ \alpha \in \mathbb{N}^n, \ |\alpha| \leq k\}. \tag{A.3}$$

This is a Banach space endowed with the norm

$$\|w\|_{k,p,\Omega} = \left(\sum_{|\alpha| \leq k} \|D^\alpha w\|_{L^p(\Omega)}^p \right)^{1/p}, \tag{A.4}$$

where

$$\|w\|_{L^p(\Omega)} = \left(\int |w(\mathbf{x})|^p \, d\mathbf{x} \right)^{1/p} \ \text{ if } \ 1 \leq p < +\infty,$$
$$\|w\|_{L^p(\Omega)} = \mathop{essup}_{\mathbf{x} \in \Omega} |w(\mathbf{x})| \ \text{ if } \ p = +\infty.$$

Here, the essential supremum *essup* is defined as

$$\mathop{essup}_{\mathbf{x} \in \Omega} w(\mathbf{x}) = \inf\{a \in \mathbb{R} \ \text{ s. t. } \ \mu(\{\mathbf{x} \in \Omega \ \text{ s. t. } \ w(\mathbf{x}) > a\}) > 0\},$$

where μ denotes the Lebesgue measure. For the sake of consistency of notation, we denote $\|w\|_{0,p,\Omega}$ the $L^p(\Omega)$ norm of w. We next define the seminorm, for $0 < s < 1$,

$$|w|_{s,p,\Omega} = \left(\int_\Omega \int_\Omega \frac{|w(\mathbf{x}) - w(\mathbf{y})|^p}{|x - y|^{d+ps}} \, d\mathbf{x} \, d\mathbf{y} \right)^{1/p}, \tag{A.5}$$

and the Sobolev spaces of fractional order $k + s > 0$, with $k \geq 0$ integer and $0 < s < 1$,

$$W^{k+s,p}(\Omega) = \{w \in W^{k,p}(\Omega) \text{ s. t. } \sup_{|\alpha|=k} |D^\alpha w|_{s,p,\Omega} < +\infty\}. \tag{A.6}$$

This is a Banach space endowed with the norm

$$|w|_{s,p,\Omega} = |w|_{s,p,\Omega} + \sup_{|\alpha|=k} |D^\alpha w|_{s,p,\Omega}. \tag{A.7}$$

When Ω is Lipschitz, the set $\mathscr{D}(\overline{\Omega})$ is dense in $W^{k+s,p}(\Omega)$ for finite p. We shall also denote

$$W_0^{k+s,p}(\Omega) = \overline{\mathscr{D}(\Omega)}^{W^{k+s,p}(\Omega)}, \tag{A.8}$$

i.e., $W_0^{k+s,p}(\Omega)$ is the closure of $\mathscr{D}(\Omega)$ in the norm of $W^{k+s,p}(\Omega)$. When $p = 2$ it is customary to denote $H^{k+s}(\Omega) = W^{k+s,2}(\Omega)$, $H_0^{k+s}(\Omega) = W_0^{k+s,2}(\Omega)$.

The Sobolev embedding theorem is a fundamental result for the modern analysis of partial differential equations, where these equations are considered in a weak sense and the solutions are searched for in Sobolev spaces.

Given two Banach spaces X and Y, we use the notation $X \hookrightarrow Y$ to indicate that the space X is continuously embedded in Y.

Theorem A.1. *Let Ω be an open Lipschitz-continuous bounded domain of $\mathbb{R}^d$ and let $p \in (0, +\infty)$, and n, $m \in \mathbb{N}$ with $n < m$. The following embeddings hold algebraically and topologically:*

$$W^{m,p}(\Omega) \hookrightarrow \begin{cases} W^{n,q}(\Omega) \ \textit{for all} \ q \in [1, q^*) \ \textit{if} \ \dfrac{1}{q^*} = \dfrac{1}{p} - \dfrac{m-n}{d} \geq 0, \\ \\ C^n(\overline{\Omega}) \qquad \textit{if} \qquad \dfrac{1}{p} < \dfrac{m-n}{d}. \end{cases} \tag{A.9}$$

If $q^* < +\infty$, the first embedding also holds for $q = q^*$. Moreover, the embedding is compact if $q \in [1, q^*)$.

Those results may be found in Adams [1], Brézis [10], and Grisvard [16], for instance.

A.2 Trace Spaces and Normal Component Trace

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Assume that the boundary of Ω is decomposed into $\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_n}$, where Γ_D and Γ_n are disjoint measurable subsets of Γ with positive measure. We intend to impose no-slip boundary conditions on Γ_D and slip boundary conditions on Γ_n.

For a function $\varphi \in \mathscr{D}(\overline{\Omega})$ its restriction $\varphi_{|\Gamma}$ to Γ is defined by simply taking its pointwise values: $\varphi_{|\Gamma}(\mathbf{x}) = \varphi(\mathbf{x})$ for all $\mathbf{x} \in \Gamma$. This mapping may be extended by density to a linear bounded mapping defined on Sobolev spaces that could be restricted to a part of Γ (Cf. Adams [1], Grisvard [16]):

Theorem A.2. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz bounded domain and $1 < p < +\infty$. Then there exists a unique linear bounded trace mapping $\gamma_0 : W^{1,p}(\Omega) \mapsto W^{1-1/p,p}(\Gamma)$ such that $\gamma_0\varphi = \varphi_{|\Gamma}$ if $\varphi \in \mathscr{D}(\overline{\Omega})$, and*

$$\|\gamma_0 w\|_{1-1/p,p,\Gamma} \leq C_p \, \|w\|_{1,p,\Omega}, \quad \textit{for all} \;\; w \in W^{1,p}(\Omega),$$

for some constant $C_p > 0$.

Given $\mathbf{w} \in \mathscr{D}(\overline{\Omega})^d$, let us consider the normal trace of $\mathbf{w}$ on Γ, defined by $(\mathbf{w} \cdot \mathbf{n})(\mathbf{x}) = \mathbf{w}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$, for all $\mathbf{x} \in \Gamma$, where $\mathbf{n}(\mathbf{x})$ is the normal to Γ at point $\mathbf{x}$, outward to Ω. As Ω is Lipschitz, then the normal $\mathbf{n}$ exists a.e. on Γ. The normal trace may be extended by density to a mapping defined on $\mathbf{H}^1(\Omega)$:

Lemma A.1. *Let Ω be a bounded Lipschitz domain. Then there exists a linear bounded mapping $\gamma_n : \mathbf{H}^1(\Omega) \mapsto L^q(\Gamma), 1 \leq q \leq 4$, such that $\gamma_n \mathbf{w} = \mathbf{w} \cdot \mathbf{n}$ a.e. on Γ if $\mathbf{w} \in \mathscr{D}(\overline{\Omega})^d$. Moreover, for all $\mathbf{w} \in \mathbf{H}^1(\Omega)$, $\gamma_n \mathbf{w} = (\gamma_0 \mathbf{w}) \cdot \mathbf{n}$ a.e. in Γ.*

Proof. By Sobolev injections (Theorem A.1), $H^{1/2}(\Gamma)$ is continuously injected into $L^q(\Gamma)$. Also, $\mathbf{n} \in L^\infty(\Gamma)$, as Ω is Lipschitz. Then for $\mathbf{w} \in \mathscr{D}(\overline{\Omega})^d$,

$$\|\mathbf{w} \cdot \mathbf{n}\|_{0,q,\Gamma} \leq \|\mathbf{w}\|_{0,q,\Gamma} \, \|\mathbf{n}\|_{0,\infty,\Gamma} \leq C \, \|\mathbf{w}\|_{1/2,2,\Gamma} \leq C' \, \|\mathbf{w}\|_{1,2,\Omega},$$

for some constants C and C' independent of $\mathbf{w}$. As $\mathscr{D}(\overline{\Omega})^d$ is dense in $\mathbf{H}^1(\Omega)$, which is a Banach space, then the normal trace extends to a linear bounded mapping from $\mathbf{H}^1(\Omega)$ into $L^q(\Gamma)$. Moreover, let $\mathbf{w} \in \mathbf{H}^1(\Omega)$. Let $\mathbf{w}_k$ be a sequence of functions of $\mathscr{D}(\overline{\Omega})^d$ that converge to $\mathbf{w}$ in $\mathbf{H}^1(\Omega)$. Then, $\gamma_0 \mathbf{w}_k \to \gamma_0 \mathbf{w}$ a.e. in Γ. As $(\gamma_0 \mathbf{w}_k) \cdot \mathbf{n} \to \gamma_n \mathbf{w}$ in $L^q(\Gamma)$, we deduce that $\gamma_n \mathbf{w} = (\gamma_0 \mathbf{w}) \cdot \mathbf{n}$ a.e. in Γ. □

A.3 Density Results

Assume that $|\Gamma_D| > 0$. Let us consider the space

$$\mathbf{W}_D(\Omega) = \{\mathbf{w} \in \mathbf{H}^1(\Omega) \;\text{ such that }\; \gamma_0 \mathbf{w} = 0 \;\text{ on }\; \Gamma_D, \; \gamma_n \mathbf{w} = 0 \;\text{ on }\; \Gamma_n\}. \tag{A.10}$$

By Theorem A.2 (i) and Lemma A.1, $\mathbf{W}_D(\Omega)$ is a closed subspace of $\mathbf{H}^1(\Omega)$. Moreover, by Korn's inequality (Cf. Sect. A.4.4), the seminorm $\|D(\mathbf{v})\|_{0,2,\Omega}$ is a norm on $\mathbf{W}_D(\Omega)$ which is equivalent to the $\mathbf{H}^1(\Omega)$.

For theoretical and numerical analysis purposes, it is convenient to establish the density in $\mathbf{W}_D(\Omega)$ of some space of smooth functions. A "candidate"space to be dense in $\mathbf{W}_D(\Omega)$ is

$$\mathscr{W}(\Omega, \Gamma_D) = \{\varphi \in \mathscr{D}(\overline{\Omega}, \Gamma_D)^d \text{ such that } \varphi \cdot \mathbf{n} = 0 \text{ on } \Gamma_n\}, \tag{A.11}$$

where

$$\mathscr{D}(\overline{\Omega}, \Gamma_D) = \{\varphi \in \mathscr{D}(\overline{\Omega}) \text{ s.t. } \varphi = 0 \text{ in a neighborhood of } \overline{\Gamma_D}\}. \tag{A.12}$$

This is inspired by some results that prove the density of the set $\mathscr{D}(\overline{\Omega}, \Gamma_D)$ in

$$H^1(\Omega, \Gamma_D) = \{\mathbf{w} \in H^1(\Omega) \text{ s.t. } \gamma_0\mathbf{w} = 0 \text{ on } \overline{\Gamma_D}\}$$

whenever $\overline{\Gamma_D} \cap \overline{\Gamma_n}$ is smooth in a convenient sense. For instance, it holds if $\overline{\Gamma_D} \cap \overline{\Gamma_n}$ has a finite number of connected component (Cf. Bernard [4]). Also, in Mitrea et al. [22], a similar density result is proved in Besov spaces if $\overline{\Gamma_D}$ and $\overline{\Gamma_n}$ are locally on one side of $\overline{\Gamma_D} \cap \overline{\Gamma_n}$.

However, these results up to our knowledge have not been extended to the density of $\mathscr{W}(\Omega, \Gamma_D)$ in $\mathbf{W}_D(\Omega)$ for general Lipschitz domains when the measure of Γ_D is positive, due to the possible existence of jumps of the normal $\mathbf{n}$.

A simple proof applies if Γ_n is contained in a straight line when $d = 2$ or in a plane when $d = 3$. Indeed, let us assume that the system of coordinates, if needed, is modified by rigid motion in such a way that the normal to Γ_n is $\mathbf{n}_{|\Gamma_n} = (0, \cdots, 0, 1)$. Then, denoting $\varphi = (\varphi_1, \cdots, \varphi_d)$,

$$\begin{aligned} \mathscr{W}(\Omega, \Gamma_D) &= \{\varphi \in \mathscr{D}(\overline{\Omega}, \Gamma_D)^d \text{ s. t. } \varphi_d = 0 \text{ on } \Gamma\} \\ &= \mathscr{D}(\overline{\Omega}, \Gamma_D)^{d-1} \times \left[\mathscr{D}(\overline{\Omega}, \Gamma_D) \cap H_0^1(\Omega)\right] \end{aligned}$$

As $\overline{\Gamma_D} \cap \overline{\Gamma_n}$ has a finite number of connected components, then (Cf. [4]), $\overline{\mathscr{D}(\overline{\Omega}, \Gamma_D)}^{H^1(\Omega)} = H^1(\Omega, \Gamma_D)$. Also, as $\overline{\mathscr{D}(\overline{\Omega}, \Gamma_D) \cap H_0^1(\Omega)}^{H^1(\Omega)} = H_0^1(\Omega)$, we deduce that

$$\mathbf{W}_D(\Omega) = H^1(\Omega, \Gamma_D)^{d-1} \times H_0^1(\Omega). \tag{A.13}$$

When the measure of Γ_D is zero, only slip boundary conditions are applied. The relevant space for the velocities is

$$\mathbf{W}(\Omega) = \{\mathbf{w} \in \mathbf{H}^1(\Omega) \text{ such that } \gamma_n\mathbf{w} = 0 \text{ on } \Gamma\}.$$

When Ω is smooth, space $\mathbf{W}(\Omega)$ may be cast as $\mathbf{W}(\Omega) = \overline{\mathscr{W}(\Omega)}^{\mathbf{H}^1(\Omega)}$ where

$$\mathscr{W}(\Omega) = \{\varphi \in \mathscr{D}(\overline{\Omega})^d \text{ such that } \varphi \cdot \mathbf{n} = 0 \text{ on } \Gamma\}.$$

This result is proved in Sect. 6.2. In this case, the seminorm $\|D(\mathbf{v})\|_{0,2,\Omega}$ is no longer equivalent to the $\mathbf{H}^1(\Omega)$ norm on $\mathbf{W}(\Omega)$.

A.3.1 Polyhedric Domains

Assume that Ω is polyhedric and that Γ_D and Γ_n are segments of straight lines when $d = 2$ or polyhedric when $d = 3$. Assume also that Γ_D is admissible in the sense that it can be split as the union of whole sides of Ω. In this case the standard finite element spaces are an internal approximation of $\mathbf{W}_D(\Omega)$. We use the concepts and notation introduced in Sect. 9.3.1.

Let us denote $\Sigma_k, \cdots, \Sigma_m$, $1 \le k \le m$ the sides of Ω that we assume to be closed $(d-1)-$dimensional sets. Assume that $\overline{\Gamma_D} = \cup_{i=k}^m \Sigma_i$. Consider a node $\alpha \in \mathscr{A}_h$. Define the vector regularization operator $\vec{\pi}_\alpha$ as follows:

1. If α lies in the interior of Ω, let K_α be any element of $\mathscr{T}_h$ that contains α.
2. Else if either α lies in some Σ_i, $i = k, \cdots, m$ or in the interior of some Σ_i, $i = 1, \cdots, k-1$, let K_α be a side (in 2D) or face (in 3D) of some element of $\mathscr{T}_h$, such that K_α is contained in $\overline{\Sigma_i}$ and contains α.

   The vector regularization operator $\vec{\pi}_\alpha : \mathbf{L}^1(K_\alpha) \mapsto P_l^d$ is defined as in (9.18), component-wise:

$$\vec{\pi}_\alpha \mathbf{v} = (\pi_\alpha v_1, \cdots, \pi_\alpha v_d)^t. \tag{A.14}$$

3. Else α must belong to the intersection of the boundaries of (at least) two Σ_i, $i = 1, \cdots, k-1$, say Σ_1 and Σ_2. We associate to α two sets $K_{\alpha,1} \subset \Sigma_1$ and $K_{\alpha,2} \subset \Sigma_2$, both of them either a side (in 2D) or a face (in 3D) of some element of $\mathscr{T}_h$ and such that both contain α (See Fig. A.1). Additionally, in 3D we associate one more set $K_{\alpha,3}$ defined as either $K_{\alpha,1}$ or $K_{\alpha,2}$. Let us respectively denote by $\mathbf{n}_1$ and $\mathbf{n}_2$ the normal vectors to Σ_1 and Σ_2. Denote $\mathbf{n}_3 = \mathbf{n}_1 \times \mathbf{n}_2$ (vector product of $\mathbf{n}_1$ and $\mathbf{n}_2$), $\mathbf{m}_2 = \mathbf{n}_1 \times \mathbf{n}_3$. Then $\{\mathbf{n}_1, \mathbf{m}_2, \mathbf{n}_3\}$ is an orthogonal basis of $\mathbb{R}^3$ (in 2D, $\mathbf{n}_3 = 0$). There exist two nonzero constants $a, b \in \mathbb{R}$, such that $\mathbf{n}_2 = a\,\mathbf{n}_1 + b\,\mathbf{m}_2$. We then set

$$\begin{aligned}\vec{\pi}_\alpha \mathbf{v} = {} & \pi_\alpha^{(1)}(\mathbf{v}\cdot\mathbf{n}_1)\,\mathbf{n}_1 + \frac{1}{b}\left[\pi_\alpha^{(2)}(\mathbf{v}\cdot\mathbf{n}_2) - a\,\pi_\alpha^{(1)}(\mathbf{v}\cdot\mathbf{n}_1)\right]\mathbf{m}_2 \\ & + \pi_\alpha^{(3)}(\mathbf{v}\cdot\mathbf{n}_3)\,\mathbf{n}_3, \quad \text{for } \mathbf{v} \in \mathbf{H}^1(\Delta_\alpha)\end{aligned} \tag{A.15}$$

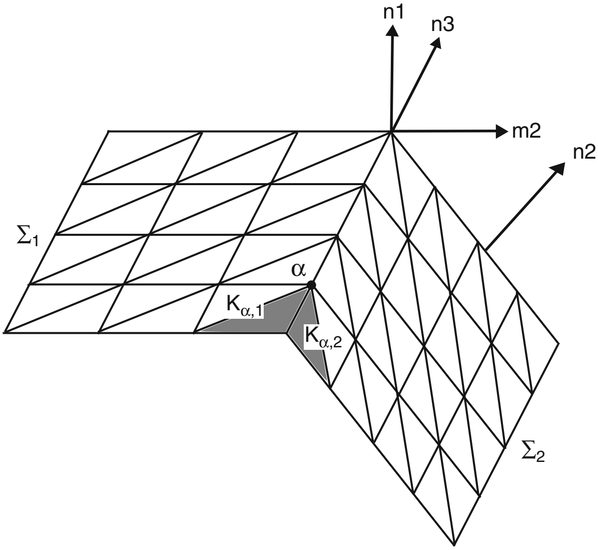

Fig. A.1 Setting of interpolation sets when the normal $\mathbf{n}$ jumps

where $\pi_\alpha^{(1)}$, $\pi_\alpha^{(2)}$, and $\pi_\alpha^{(3)}$ are the regularization operators defined by (9.18) with K_α respectively equal to $K_{\alpha,1}$, $K_{\alpha,2}$ and $K_{\alpha,3}$ and Δ_α is the union of the elements of $\mathscr{T}_h$ that contain α.

Note that in the last case we assume $\mathbf{v} \in \mathbf{H}^1(\Delta_\alpha)$, as then by Sobolev injection $(\mathbf{v}\cdot\mathbf{n})_{|K_{\alpha,i}} \in L^4(K_{\alpha,i})$, $i = 1, 2$, and so $\vec{\pi}_\alpha \mathbf{v}$ is well defined. We are now in a position to define the adapted vector BMR interpolation operator $\vec{\Pi}_h : \mathbf{H}^1(\Omega) \mapsto V_h^d$ by

$$\vec{\Pi}_h \mathbf{v}(\mathbf{x}) = \sum_{\alpha\in\mathscr{A}_h} \vec{\pi}_\alpha \mathbf{v}(\alpha)\, \lambda_\alpha(\mathbf{x}), \quad \text{for any } \mathbf{x} \in \overline{\Omega}. \tag{A.16}$$

Note that $\vec{\Pi}_h \mathbf{v}$ is the standard Lagrange interpolate of a smoothed function that takes on the regularized values $\vec{\pi}_\alpha \mathbf{v}(\alpha)$ at the nodes $\alpha \in \mathscr{A}_h$. We then have

Lemma A.2. *Assume that Γ_D and the triangulation $\mathscr{T}_h$ are admissible. Let $\mathbf{v} \in \mathbf{W}_D(\Omega)$. Then $\vec{\Pi}_h \mathbf{v}$ belongs to the discrete space $\mathbf{W}_h$ defined by (9.21).*

Proof. Let $\alpha \in \mathscr{A}_h$.

(i) If α lies in the closure of some Σ_i, $i = k, \cdots, n$, then $K_\alpha \subset \overline{\Gamma_D}$, and by (9.18) we deduce $\vec{\pi}_\alpha \mathbf{v}(\alpha) = 0$. As $\mathscr{T}_h$ is admissible, from (A.16), we obtain

$$\vec{\Pi}_h \mathbf{v}(\mathbf{x}) = \sum_{\alpha\in\mathscr{A}_h\cap\overline{\Gamma_D}} \vec{\pi}_\alpha \mathbf{v}(\alpha)\, \lambda_\alpha(\mathbf{x}), \quad \text{for any } \mathbf{x} \in \overline{\Gamma_D}$$

Consequently, $\vec{\Pi}_h \mathbf{v} = 0$ on $\overline{\Gamma_D}$.

(ii) If α lies in the interior of some Σ_i, $i = 1, \cdots, k-1$, from (9.18), we deduce

$$\int_{K_\alpha} (\mathbf{v}\cdot\mathbf{n} - (\vec{\pi}_\alpha \mathbf{v})\cdot\mathbf{n})\, q = 0, \quad \text{for all } \; q \in P_l(K_\alpha),$$

where $\mathbf{n}$ is the normal to K_α. As $\mathbf{n}$ is constant, then, $(\vec{\pi}_\alpha \mathbf{v})\cdot\mathbf{n} \in P_l(K_\alpha)$ and consequently $\vec{\pi}_\alpha \mathbf{v}\cdot\mathbf{n} = 0$ and, in particular, $(\vec{\pi}_\alpha \mathbf{v}(\alpha))\cdot\mathbf{n} = 0$.

(iii) Assume now that α lies in the intersection of the boundaries of two Σ_i, $i = 1, \cdots, k-1$, say Σ_1 and Σ_2. Let us denote by $\mathbf{n}_i$ the normal to Σ_i, $i = 1, 2$. From (A.15) we readily deduce

$$(\vec{\pi}_\alpha \mathbf{v})\cdot\mathbf{n}_1 = \pi_\alpha^{(1)}(\mathbf{v}\cdot\mathbf{n}_1), \quad (\vec{\pi}_\alpha \mathbf{v})\cdot\mathbf{n}_2 = \pi_\alpha^{(2)}(\mathbf{v}\cdot\mathbf{n}_2)$$

As $\mathbf{v} \in \mathbf{W}_D(\Omega)$, then $\mathbf{v}\cdot\mathbf{n}_i = 0$ a.e. on $K_{\alpha,i}$, $i = 1, 2$. Thus, $\pi_\alpha^{(i)}(\mathbf{v}\cdot\mathbf{n}_i)(\alpha) = 0$, reasoning as in the item 2 above. As a consequence, $(\vec{\pi}_\alpha \mathbf{v}(\alpha))\cdot\mathbf{n}_i = 0$, $i = 1, 2$.

In sum, items 1, 2, and 3 above prove that for any Σ_i, $i = 1, \cdots, m$ and for any $\alpha \in \mathscr{A}_h \cup \overline{\Sigma_i}$ it holds $(\vec{\pi}_\alpha \mathbf{v}(\alpha))\cdot\mathbf{n}_i = 0$, where $\mathbf{n}_i$ denotes the normal to Σ_i. Here we use that by item 1 if $i = k, \cdots, m$, then $\vec{\pi}_\alpha \mathbf{v}(\alpha) = 0$. As $\mathscr{T}_h$ is admissible, we obtain

$$\vec{\Pi}_h \mathbf{v}(\mathbf{x})\cdot\mathbf{n}_i = \sum_{\alpha\in\mathscr{A}_h\cap\overline{\Sigma_i}} (\vec{\pi}_\alpha \mathbf{v}(\alpha))\cdot\mathbf{n}_i \;\lambda_\alpha(\mathbf{x}), \quad \text{for any } \; \mathbf{x} \in \overline{\Sigma_i}$$

Consequently, $(\vec{\Pi}_h \mathbf{v})\cdot\mathbf{n}_i = 0$ on Σ_i, $i = 1, \cdots, m$.

We are now in a position to prove

Theorem A.3. *Assume that Γ_D is admissible. Consider a regular family of admissible triangulations of Ω, $(\mathscr{T}_h)_{h>0}$. Then*

(i) There exists a constant $C > 0$ such that for any $\mathbf{v} \in \mathbf{H}^2(\Omega)$,

$$\|\mathbf{v} - \vec{\Pi}_h \mathbf{v}\|_{1,2,\Omega} \le C\, h\, \|\mathbf{v}\|_{2,2,\Omega}. \tag{A.17}$$

(ii) The family of spaces $(\mathbf{W}_h)_{h>0}$ is an internal approximation of $\mathbf{W}_D(\Omega)$.

Proof. (i) Our proof is an adaptation of that of Theorem 3.10 in Bernardi et al. [6]. We use two basic technical results stated in [6]:

- The stability of the local regularization operator π_α, stated in Lemma 3.1 of [6]: Let $1 \le p \le +\infty$. Then there exists a constant $C = C(p, \Omega)$ such that

$$\|\pi_\alpha v\|_{0,p,K_\alpha} \le C\, \|v\|_{0,p,K_\alpha}, \quad \text{for any } \; v \in L^p(K_\alpha). \tag{A.18}$$

- The local error estimate stated in Lemma 3.6 of [6]: Let $1 \le p \le +\infty$. Then there exists a constant $C = C(p, \Omega)$ such that,

$$\inf_{q \in P_1} \{ \|v - q\|_{0,p,\Delta_K} + h_K \, \|\nabla(v - q)\|_{0,p,\Delta_K} \} \le C \, h_K^l \, \|v\|_{l,p,\Delta_K} \quad \text{(A.19)}$$

  for any $v \in W^{l,p}(\Delta_K)$, $l = 1, 2$.

We proceed by steps. We use the notation introduced in the definition of operator $\overrightarrow{\Pi}_h$.

Step 1: Local stability of $\overrightarrow{\Pi}_h$. Let $K \in \mathscr{T}_h$. Let $\mathbf{v} \in \mathbf{H}^1(\Omega)$. Then

$$\overrightarrow{\Pi}_h \mathbf{v}(\mathbf{x}) = \sum_{\alpha \in \mathscr{A}_h \cap \overline{K}} \overrightarrow{\pi}_\alpha \mathbf{v}(\alpha) \; \lambda_\alpha(\mathbf{x}), \quad \text{for any } \; \mathbf{x} \in \overline{K}.$$

Thus

$$\|\overrightarrow{\Pi}_h \mathbf{v}\|_{0,2,K} \le \sum_{\alpha \in \mathscr{A}_h \cap \overline{K}} \|\overrightarrow{\pi}_\alpha \mathbf{v}(\alpha)\|_{0,\infty,K_\alpha} \, \|\lambda_\alpha\|_{0,2,K} \quad \text{(A.20)}$$

$$\le C \sum_{\alpha \in \mathscr{A}_h \cap \overline{K}} h_K^{2/d} \, h_{K_\alpha}^{-2/d} \, \|\overrightarrow{\pi}_\alpha \mathbf{v}(\alpha)\|_{0,2,K_\alpha} \le C \sum_{\alpha \in \mathscr{A}_h \cap \overline{K}} \|\overrightarrow{\pi}_\alpha \mathbf{v}(\alpha)\|_{0,2,K_\alpha}$$

where we have used the estimates

$$\|\overrightarrow{\pi}_\alpha \mathbf{v}(\alpha)\|_{0,\infty,K_\alpha} \le C \, h_{K_\alpha}^{-d/2} \, \|\overrightarrow{\pi}_\alpha \mathbf{v}(\alpha)\|_{0,2,K}, \quad \|\lambda_\alpha\|_{0,2,K} \le C \, h_K^{d/2}, \quad h_K^{2/d} \, h_{K_\alpha}^{-2/d} \le C.$$

The first one is deduced from (9.20) and the second one by change of variables to the reference element, while the third one holds because the triangulation $\mathscr{T}_h$ is regular. If $\overrightarrow{\pi}_\alpha$ is defined by (A.14), then (A.18) yields

$$\|\overrightarrow{\pi}_\alpha \mathbf{v}(\alpha)\|_{0,2,K_\alpha} \le C \, \|\mathbf{v}\|_{0,2,K_\alpha}.$$

If $\overrightarrow{\pi}_\alpha$ is defined by (A.15), then also using (A.18),

$$\|\overrightarrow{\pi}_\alpha \mathbf{v}(\alpha)\|_{0,2,K_\alpha} \le C \sum_{i=1}^{3} \|\pi_\alpha^{(i)}(\mathbf{v} \cdot \mathbf{n}_i)\|_{0,2,K_\alpha} \le C \sum_{i=1}^{3} \|\mathbf{v} \cdot \mathbf{n}_i\|_{0,2,K_\alpha} \le C \, \|\mathbf{v}\|_{0,2,K_\alpha}.$$

Combining the last two estimates with (A.20) we deduce

$$\|\overrightarrow{\Pi}_h \mathbf{v}\|_{0,2,K} \le C \, \|\mathbf{v}\|_{0,2,\Delta_K}, \quad \text{(A.21)}$$

where Δ_K denotes the union of the elements of $\mathscr{T}_h$ that are neighbors of K.

Step 2: Invariance of $\overrightarrow{\Pi}_h$ on P_l^d. If $\overrightarrow{\pi}_\alpha$ is defined by (A.14), then $\overrightarrow{\pi}_\alpha(\mathbf{q}) = \mathbf{q}$ for any $\mathbf{q} \in P_l^d$ as a consequence of (9.18).

If $\overrightarrow{\pi}_\alpha$ is defined by (A.15), let $\mathbf{q} \in P_l^d$. As $\mathbf{q}\cdot\mathbf{n}_i \in P_l$, then from (9.18), $\pi_\alpha^{(i)}(\mathbf{q}\cdot\mathbf{n}_i) = \mathbf{q}\cdot\mathbf{n}_i$, $i = 1, 2, 3$. Thus, as $\mathbf{m}_2 = \dfrac{1}{b}\,(\mathbf{n}_2 - a\,\mathbf{n}_1)$,

$$\begin{aligned}\overrightarrow{\pi}_\alpha(\mathbf{q}) &= \mathbf{q}\cdot\mathbf{n}_1\,\mathbf{n}_1 + \frac{1}{b}\,[\mathbf{q}\cdot\mathbf{n}_2 - a\,\mathbf{q}\cdot\mathbf{n}_1]\,\mathbf{m}_2 + \mathbf{q}\cdot\mathbf{n}_3\,\mathbf{n}_3\\ &= \mathbf{q}\cdot\mathbf{n}_1\,\mathbf{n}_1 + \mathbf{q}\cdot\mathbf{m}_2\,\mathbf{m}_2 + \mathbf{q}\cdot\mathbf{n}_3\,\mathbf{n}_3 = \mathbf{q}\end{aligned}$$

As $\sum_{\alpha\in\mathscr{A}_h} \lambda_\alpha(\mathbf{x}) = 1$ for all $\mathbf{x} \in \Omega$, we conclude that $\overrightarrow{\Pi}_h$ is invariant on P_l^d.

Step 3: Conclusion. Let $\mathbf{v} \in \mathbf{H}^l(\Omega)$, $l = 1, 2$. Using Step 2, for any $\mathbf{q} \in P_l^d$,

$$\begin{aligned}\|\nabla(\mathbf{v} - \overrightarrow{\Pi}_h\mathbf{v})\|_{0,2,K} &\le \|\nabla(\mathbf{v}-\mathbf{q})\|_{0,2,K} + \|\nabla\,\overrightarrow{\Pi}_h(\mathbf{v}-\mathbf{q})\|_{0,2,K}\\ &\le \|\nabla(\mathbf{v}-\mathbf{q})\|_{0,2,K} + C\,h_K^{-1}\,\|\overrightarrow{\Pi}_h(\mathbf{v}-\mathbf{q})\|_{0,2,K}\\ &\le \|\nabla(\mathbf{v}-\mathbf{q})\|_{0,2,K} + C\,h_K^{-1}\,\|\mathbf{v}-\mathbf{q}\|_{0,2,\Delta_K}\\ &\le C\,h_K^{l-1}\,\|\mathbf{v}\|_{l,2,\Delta_K}, \quad \text{for } l = 1, 2\end{aligned}$$

where we have used the inverse estimates (9.20), the stability of $\overrightarrow{\Pi}_h$, and estimate (A.19) with $l = 2$. Squaring, summing up in K, and considering that Δ_K has a bounded number of elements due to the regularity of the family of triangulations yield

$$\|\nabla(\mathbf{v} - \overrightarrow{\Pi}_h\mathbf{v})\|_{0,2,\Omega} \le C\,h^{l-1}\,\|\mathbf{v}\|_{l,2,\Omega} \tag{A.22}$$

A similar argument shows that

$$\|\mathbf{v} - \overrightarrow{\Pi}_h\mathbf{v}\|_{0,2,\Omega} \le C\,h^{l}\,\|\mathbf{v}\|_{l,2,\Omega}. \tag{A.23}$$

The conclusion follows.

(ii) Let us at first notice that space $\mathbf{W}_h$ is a subspace of $\mathbf{W}_D(\Omega)$. Also, that when $l = 1$ estimates (A.22) and (A.23) read

$$\|\mathbf{v} - \overrightarrow{\Pi}_h\mathbf{v}\|_{1,2,\Omega} \le C\,\|\mathbf{v}\|_{1,2,\Omega} \quad \text{for any } \mathbf{v} \in \mathbf{H}^1(\Omega),$$

from which we deduce the stability of operator $\overrightarrow{\Pi}_h$ on $\mathbf{H}^1(\Omega)$:

$$\|\overrightarrow{\Pi}_h\mathbf{v}\|_{1,2,\Omega} \le \|\mathbf{v} - \overrightarrow{\Pi}_h\mathbf{v}\|_{1,2,\Omega} + \|\mathbf{v}\|_{1,2,\Omega} \le C\,\|\mathbf{v}\|_{1,2,\Omega}.$$

Let $\mathbf{v} \in \mathbf{W}_D(\Omega)$. As $\mathscr{D}(\overline{\Omega})^d$ is dense in $\mathbf{H}^1(\Omega)$, there exists a sequence $\{\mathbf{v}_n\}_{n\geq 1} \subset \mathscr{D}(\overline{\Omega})^d$ that converges in $\mathbf{H}^1(\Omega)$ to $\mathbf{v}$. Let $\varepsilon > 0$. Choose $n_\varepsilon \geq 1$ such that $C\,\|\mathbf{v}_{n_\varepsilon} - \mathbf{v}\|_{1,2,\Omega} < \varepsilon/2$. From (i) we know that there exists $h_\varepsilon > 0$ such that if $0 < h < h_\varepsilon$, then $\|\overrightarrow{\Pi}_h \mathbf{v}_{n_\varepsilon} - \mathbf{v}_{n_\varepsilon}\|_{1,2,\Omega} < \varepsilon/2$. Then, using the stability in of $\overrightarrow{\Pi}_h$ on $\mathbf{H}^1(\Omega)$,

$$
\begin{aligned}
\|\overrightarrow{\Pi}_h \mathbf{v} - \mathbf{v}\|_{1,2,\Omega} &\leq \|\overrightarrow{\Pi}_h (\mathbf{v} - \mathbf{v}_{n_\varepsilon})\|_{1,2,\Omega} + \|\overrightarrow{\Pi}_h \mathbf{v}_{n_\varepsilon} - \mathbf{v}_{n_\varepsilon}\|_{1,2,\Omega} + \|\mathbf{v}_{n_\varepsilon} - \mathbf{v}\|_{1,2,\Omega} \\
&\leq \|\overrightarrow{\Pi}_h \mathbf{v}_{n_\varepsilon} - \mathbf{v}_n\|_{1,2,\Omega} + C\,\|\mathbf{v}_{n_\varepsilon} - \mathbf{v}\|_{1,2,\Omega} \\
&\leq \|\overrightarrow{\Pi}_h \mathbf{v}_{n_\varepsilon} - \mathbf{v}_n\|_{1,2,\Omega} + \varepsilon/2 < \varepsilon \qquad \text{if } \ 0 < h < h_\varepsilon.
\end{aligned}
$$

As by Lemma A.2 $\overrightarrow{\Pi}_h \mathbf{v} \in \mathbf{W}_h$, we conclude that $\lim\limits_{h\to o} d_{1,2,\Omega}(\mathbf{v}, \mathbf{W}_h) = 0$.

This proves Lemma 9.2

A.4 Some Results from Functional Analysis

In this section we recall some concepts and results of functional analysis that support both our theoretical and numerical analysis.

A.4.1 Fixed-Point Theorems

Many results proved in this book are based on fixed point theorems, in particular:

(i) the Brouwer fixed-point theorem (Cf. [12]),
(ii) the Shauder fixed-point theorem (Cf. [27]).

Those results are detailed in the book by E. Zeidler [33].

Theorem A.4. *(Brouwer) Let B be a normed space of finite dimension. Let $K \subset B$ be a nonempty, closed, and convex set. Let $\Phi : X \to X$ a continuous function such that $\Phi(K) \subset K$. Then Φ admits a fixed point $x \in K$: $\Phi(x) = x$.*

The following variant of the Brouwer's theorem is also proved in [30].

Theorem A.5. *Let H be a Hilbert space of finite dimension endowed with an inner product $(\cdot,\cdot)_H$ and $\Phi : H \to H'$ a continuous function. Assume that there exists $\mu > 0$ such that*

$$
\langle \Phi(y), y \rangle_H \geq 0, \quad \forall y \in H \ \ \text{such that} \ \ \|y\|_H = \mu.
$$

Then there exists some $x \in H$ such that $\Phi(x) = 0$ and $\|x\|_H \leq \mu$.

Theorem A.6 (Schauder). *Let E be a separated topological vector space, $K \subset E$ be a convex subset, and $\mathscr{F} : K \to K$ be a continuous function on K, equipped with the topology inherited from that of E. Assume that $\mathscr{F}(K)$ is a compact subset of K. Then $\mathscr{F}$ has a fixed point, that is, there exists $u \in K$ such that $\mathscr{F}(u) = u$.*

A.4.2 Monotonicity Theorem

Existence results sometimes are proven by using the monotonicity of some operators involved in the equations. The following general statement may be found in [19] and [24].

Theorem A.7. *Let W be a reflexive and separable Banach space with norm $||\cdot||$, and let W' be its dual space. The duality pairing between W and W' is denoted by $\langle \cdot, \cdot \rangle$. Let $A, B : W \to W'$ be two operators such that:*

1. *A is bounded,*
2. *the function $t \in \mathbb{R} \to \langle A(u + tv), w \rangle$ is continuous for all $u, v, w \in W$,*
3. *A is monotone, $\langle A(u - v), u - v \rangle \geq 0$ for all $u, v \in W$,*
4. *$\dfrac{\langle Aw, w \rangle}{||w||} \to \infty$ as $||w|| \to \infty$,*
5. *$\langle Bw, w \rangle = 0$ for all $w \in W$.*
6. *B is continuous from W to W' weak star.*

Then, the operator $A + B$ is surjective.

A.4.3 L^p Continuity of Composed Functions

We here state a result on the continuity of mappings transforming functions of $L^p(\Omega)$ into functions of $L^q(\Omega)$.

Definition A.3. Let $G : \Omega \to \Omega$ be a function. The Nemytskii functional associated to g is the transformation that associates to a function v defined on Ω the function $Nv : \Omega \to \mathbb{R}$ defined by $Nv(\mathbf{x}) = G(v(\mathbf{x}))$, $\forall \mathbf{x} \in \Omega$.

Lemma A.3. *Let $G : \Omega \to \Omega$ be a continuous function. Let $1 \leq p, q < +\infty$. Assume that the Nemytskii functional associated to G satisfies $N(L^p(\Omega)) \subset L^q(\Omega)$. Then N is continuous from $L^p(\Omega)$ on $L^q(\Omega)$ for the strong topologies of these spaces.*

The proof of this result may be found, in a more general context, in [18].

A.4.4 Korn Inequality

We use the following equivalence of norms:

Lemma A.4. *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^d$. Assume that Γ_D is an open subset of $\partial\Omega$ with positive Lebesgue measure. Then the seminorm $\|D(\mathbf{w})\|^2_{0,2,\Omega}$ defines a norm on $\mathbf{H}^1(\Omega,\Gamma_D)$ which is equivalent to the standard $\mathbf{H}^1(\Omega)$ norm.*

This is a consequence of the first Korn inequality: There exists a constant $C > 0$ depending only on Ω and Γ_D such that

$$\|\nabla\mathbf{v}\|_{0,2,\Omega} \le C\,\|D(\mathbf{v})\|_{0,2,\Omega} \;\text{ for any }\; \mathbf{v}\in\mathbf{H}^1(\Omega,\Gamma_D). \tag{A.24}$$

In its turn, Korn's first inequality is a particular case of the following general result, due to Brenner (Cf. [7]):

Lemma A.5. *Let Φ be a seminorm on $\mathbf{H}^1(\Omega)$ satisfying*

(i) There exists a constant C_1 such that

$$\Phi(\mathbf{v}) \le C\,\|\mathbf{v}\|_{1,2,\Omega} \;\textit{ for any }\; \mathbf{v}\in\mathbf{H}^1(\Omega).$$

(ii) The condition $\Phi(\mathbf{m}) = 0$ for a rigid body motion holds if and only if $\mathbf{m}$ is a constant vector.

Then there exists a constant C_2 depending only on Ω such that

$$\|\nabla\mathbf{v}\|_{0,2,\Omega} \le C_2\left(\|D(\mathbf{v})\|^2_{0,2,\Omega} + \Phi(\mathbf{v})^2\right)^{1/2} \;\textit{ for any }\; \mathbf{v}\in\mathbf{H}^1(\Omega).$$

Inequality (A.24) corresponds to $\Phi(\mathbf{v}) = \|\gamma_0\mathbf{v}\|_{0,2,\Gamma}$.

Alternative proofs may be found for instance in Malek et al. [21] or in Horgan [17]. In [17] several applications of Korn's inequalities in continuum mechanics are developed.

A.4.5 Some Basic Results for Parabolic Equations

A.4.5.1 Bochner Integral and Time Derivative

We list what we need to know about the Bochner integral and the time derivative. Let E be a Banach space. According to Bochner [9] (see also in [20]) we can establish a theory of measurability and integration of functions from $[0,T]$ in E and define the space $L^1([0,T],E)$, the Bochner integral, satisfying $\forall\, f\in L^1([0,T],E)$:

$$||f(\cdot)||_E \in L^1(\mathbb{R}) \text{ and } \left|\left|\int_0^T f(t)dt\right|\right|_E \leq \int_0^T ||f(t)||_E dt,$$
$$\forall\, \varphi \in E', \quad \langle \varphi, f(\cdot)\rangle \in L^1(\mathbb{R}) \text{ and } \langle \varphi, \int_0^T f(t)dt\rangle = \int_0^T \langle \varphi, f(t)\rangle dt.$$

This allows to define the space $L^p([0,T],E)$, $1 \leq p < \infty$, the norm of which is given by

$$||f||_{L^p([0,T],E)} = \left(\int_0^T ||f(t)||_E^p dt\right)^{\frac{1}{p}},$$

the norm of the space $L^\infty([0,T],E)$ being $essup_{t\in[0,T} ||f(t)||_E$. Moreover, when $1 < p < \infty$, then the dual space of $L^p([0,T],E)$ is expressed by

$$(L^p([0,T],E))' = L^{p'}([0,T],E'), \text{ with } \frac{1}{p} + \frac{1}{p'} = 1.$$

For a general Banach space E, we shall write

$$||f||_{L^p([0,T],E)} = ||f||_{p;E}, \tag{A.25}$$

and in the particular case $E = W^{s,q}(\Omega)$ is a given Sobolev space, we shall write the norm of the space $L^p([0,T],W^{s,q}(\Omega))$ as

$$||f||_{L^p([0,T],W^{s,q}(\Omega))} = ||f||_{p;s,q,\Omega}. \tag{A.26}$$

Finally, let $f : [0,T] \to E$ be measurable and $g : [0,T] \to E$ be Bochner integrable. We say that g is the time derivative of f, and we put $g = \partial_t f$, if and only if there exists $f_0 \in E$ be such that

$$\forall\, t \in [0,T], \quad f(t) = f_0 + \int_0^t g(s)ds. \tag{A.27}$$

A.4.5.2 Time Derivative

The following result will be useful to give a sense to the initial conditions for parabolic equations. It may be found in [30]:

Lemma A.6. *Let E be a Banach space. Let v, $g \in L^1(0,T;E)$. Then the following three conditions are equivalent:*

(i) f *is a.e. in* $(0,T)$ *equal to a primitive of* g, *i.e.,*

$$f(t) = \xi + \int_0^T g(s)\,ds, \quad \textit{a.e. in} \quad (0,T), \quad \textit{for some} \quad \xi \in E.$$

(ii) For each $\varphi \in \mathscr{D}(0,T)$,

$$\int_0^T f(t)\,\varphi'(t)\,dt = -\int_0^T g(t)\,\varphi(t)\,dt.$$

(iii) For each $\eta \in E'$,

$$\frac{d}{dt}\langle f, \eta\rangle_{E'} = \langle g, \eta\rangle_{E'} \quad \textit{in} \quad \mathscr{D}'(0,T).$$

If these conditions are satisfied, then f *is a.e. in* $(0,T)$ *equal to a function of* $C^0([0,T],E)$, *and we set* $g = \partial_t f$.

Moreover, we deduce from the general properties of the Bochner integral that

$$\begin{cases} g = \partial_t f \in L^1([0,T],E) \text{ and } f(0) = f_0 \text{ if and only if} \\ \qquad \forall\, \varphi \in C^1([0,T],E') \text{ such that } \varphi(T) = 0, \\ \displaystyle\int_0^T \langle \varphi(t), g(t)\rangle dt = -\langle \varphi(0), f_0\rangle - \int_0^T \langle \varphi'(t), f(t)\rangle dt. \end{cases} \tag{A.28}$$

A.4.5.3 Continuity

There are many cases in which the space E involved in Lemma A.6 is a dual space and not a Lebesgue or a Sobolev space, although we look for continuity of trajectories induced by the solutions of parabolic equation in such spaces. It may happened that it is not possible to get continuity, but weak continuity holds, this notion being explained in what follows.

Definition A.4. Let E be a Banach space, $v \in L^\infty([0,T],E)$. We say that v is weakly continuous with values in E, if for every $t \in [0,T]$, $v(t) \in E$ and if $\forall\, \boldsymbol{\eta} \in E'$, the function $t \to \langle \boldsymbol{\eta}, v\rangle$ is a continuous function of t over $[0,T]$. We denote $C_w([0,T],E)$ the space of all weakly continuous functions with values in E.

The proof of the following useful results can be found in [30].

Lemma A.7. *Let* X *and* Y *be two reflexive Banach spaces,* $X \hookrightarrow Y$, *the injection being continuous and dense. Let* $v \in L^\infty([0,T],X) \cap C_w([0,T],Y)$. *Then,* $v \in C_w([0,T],X)$.

Lemma A.8. *Let $V \hookrightarrow H \hookrightarrow V'$ be reflexive Banach spaces, the injections being continuous and dense. Let $u \in L^2([0,T],V)$ such that $\partial_t u \in L^2([0,T],V')$. Then u is a.e. equal to a continuous function $[0,T] \to H$ and*

$$\frac{1}{2}\frac{d}{dt}||u||_H^2 = \langle \partial_t u, u \rangle.$$

A.4.5.4 Compactness Results

The compactness of approximations of nonlinear parabolic equations related to Navier–Stokes equations follows from Aubin–Lions compactness theorems (Cf. [19]).

Lemma A.9 (Aubin–Lions). *Let $E \hookrightarrow F \hookrightarrow G$ be Banach spaces dense in each other, the injection $E \hookrightarrow F$ being compact. Let $1 \le p,q < \infty$, and consider*

$$\mathscr{N}_{p,q}(E,G) = \{\mathbf{v} : \ \mathbf{v} \in L^p([0,T],E), \ \partial_t \mathbf{v} \in L^q([0,T],G)\}, \tag{A.29}$$

equipped with the norm

$$||\mathbf{v}||_{\mathscr{N}_{p,q}(E,G)} = ||\mathbf{v}||_{p;E} + ||\partial_t \mathbf{v}||_{q;G}. \tag{A.30}$$

Then the injection $\mathscr{N}_{p,q}(E,G) \hookrightarrow L^p([0,T],F)$ is compact.

We shall also use a result due to Simon (Cf. [28]), which is an alternative version to the Aubon–Lions lemma. To state it, let us consider Nikolskii spaces, which are subspaces of $L^p([0,T],E)$, where $[0,T]$ is an interval of $\mathbb{R}$ and E is a Banach space. The Nikolskii space of order $r \in [0,1]$ and exponent $p \in [0,+\infty]$ is defined as

$$N^{r,p}(0,T;E) = \{ f \in L^p(0,T;E) \ \text{ such that } \ \| f \|_{\tilde{N}^{r,p}} < +\infty \},$$

with

$$\| f \|_{\tilde{N}^{r,p}} = \sup_{\delta > 0} \frac{1}{\delta^r} \| \tau_\delta f \|_{L^p(0,T-\delta;E)},$$

where $\tau_\delta f(t) = f(t+\delta) - f(t)$, $0 \le t \le T - \delta$. Space $N^{r,p}(0,T;E)$, endowed with the norm

$$\| f \|_{N^{r,p}(0,T;B)} = \| f \|_{L^p(0,T;B)} + \| f \|_{\tilde{N}^{r,p}}$$

is a Banach space. We may think of $N^{r,p}(0,T;E)$ as being formed by functions whose fractional derivative in time of order r belongs to $L^p(0,T;E)$.

Simon's theorem is stated as follows:

Lemma A.10. *Let E, F, G be Banach spaces such that $E \hookrightarrow F \hookrightarrow G$ where the injection $E \hookrightarrow F$ is compact. Then the injection*

$$L^p(0,T;E) \cap N^{r,p}(0,T;G) \hookrightarrow L^p(0,T;F) \ \textit{with} \ 0 < r < 1, \ 1 \leq p \leq +\infty$$

is compact.

Whenever there is no source of confusion, we shall denote $N^{s,p}(0,T;G)$ by $N^{s,p}(G)$.

A.4.5.5 Density Result

We shall also need a specific result for distributions defined in Q_T. Let us recall that a linear functional f on $\mathscr{D}(Q_T)$ is a distribution if for any sequence $\{\sigma_n\}_{n\geq 1} \subset \mathscr{D}(Q_T)$ that converges uniformly to zero, as well as all their derivatives, such that all the functions σ_n have their support contained in some compact subset of Q_T, independent of the index n, then $\lim\limits_{n\to\infty} f(\sigma_n) = 0$.

In particular, if $f \in L^1(Q_T)$, then it generates the distribution T_f defined by

$$\langle T_f, \sigma \rangle_{\mathscr{D}(Q_T)} = \int_{Q_T} f(\mathbf{x},t)\, \sigma(\mathbf{x},t)\, d\mathbf{x}\, dt, \quad \text{for all } \sigma \in \mathscr{D}(Q_T).$$

Usually T_f is identified with f.

The space of distributions on Q_T is denoted $\mathscr{D}'(Q_T)$. The following result will be useful to give a weak sense to parabolic equations:

Lemma A.11. *Let $\Omega \subset \mathbb{R}^d$ be an open set. Then $\mathscr{D}(\Omega) \otimes \mathscr{D}(0,T)$ is sequentially dense in $\mathscr{D}(Q_T)$, with $Q_T = \Omega \times (0,T)$, in the sense that for any $\sigma \in \mathscr{D}(Q_T)$, there exists two sequences $(w_n)_{n\geq 1} \subset \mathscr{D}(\Omega)$, $(\varphi_n)_{n\geq 1} \subset \mathscr{D}(0,T)$ such that σ and all functions $(w_n \otimes \varphi_n)(\mathbf{x},t) = w_n(\mathbf{x})\varphi_n(t)$ have their supports contained in a compact subset K of Q_T, and*

$$\lim_{n\to\infty} \|\partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} \partial_t^{\beta} (\sigma - w_n\varphi_n)\|_{C^0(\overline{\Omega})} = 0,$$

for any multi-index $(\alpha_1, \cdots, \alpha_d, \beta) \in \mathbb{N}^{d+1}$. As a consequence, if for some distribution $f \in \mathscr{D}'(Q_T)$ it holds

$$\langle f, w \otimes \varphi \rangle_{\mathscr{D}(Q_T)} = 0 \ \textit{for all} \ w \in \mathscr{D}(\Omega), \ \varphi \in \mathscr{D}(0,T),$$

then $f = 0$ in $\mathscr{D}'(Q_T)$.

This result is a particular case of Theorem 39.2 in [31].

A.4.6 The Mazur Theorem

Convex sets that are closed for the weak topology also in Hilbert spaces are strongly closed. The Banach–Saks theorem in Hilbert spaces states that

Theorem A.8. *Let H be a Hilbert space. Given in H a sequence $(v_n)_{n\geq 0}$ which converges weakly to an element v, we can select a subsequence $(v_{n_k})_{k\geq 0}$ such that the arithmetic means*

$$u_k = \frac{v_{n_1} + v_{n_2} + \cdots + v_{n_k}}{k}$$

converge in strongly to v. In addition, it is possible to choose $n_k \geq k$ for all $k \geq 1$

This result may be found, for instance, in Renardy et al. [26].

A.4.7 The Egorov Theorem

Egorov's theorem (also named Severini–Egorov theorem) establishes a condition for the uniform convergence of a pointwise convergent sequence of measurable functions.

Theorem A.9. *Let (X, B, μ) be a measure space and let E be a measurable set with $\mu(E) > 0$. Let $(f_n)_{n\geq 1}$ be a sequence of measurable functions on E such that each f_n is finite almost everywhere in E and converges almost everywhere in E to a finite limit. Then for every $\varepsilon > 0$, there exists a subset E_ε of E with $\mu(E \setminus E_\varepsilon) < \varepsilon$ such that $(f_n)_{n\geq 1}$ converges uniformly on E_ε .*

If $X = \mathbb{R}^n$ and B is either the class of Borel sets or the class of Lebesgue measurable sets, then the set E_ε can be chosen to be a closed set.

This result may be found, for instance, in Weeden and Zygmund [32]. Here we use it when X is a Sobolev space of real functions defined on some open set $\Omega \subset \mathbb{R}^d$ and μ is the Lebesgue measure.

A.5 General Results from Measure Theory

A.5.1 Inverse Lebesgue Theorem

We very often use in the analysis of the models what we call inverse Lebesgue theorem, which is in fact a consequence of the proof of the completeness of L^1 space. A detailed proof might be found in [10].

Theorem A.10. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^1(\Omega)$ which converges to f in $L^1(\Omega)$. Then from the sequence $(f_n)_{n\in\mathbb{N}}$ we can extract a subsequence, still written as $(f_n)_{n\in\mathbb{N}}$, which converges to f almost everywhere and such that there exists $G \in L^1(\Omega)$ such that*

$$\forall\, n \in \mathbb{N}, \quad |f_n| \le G. \tag{A.31}$$

A.5.2 Application of Inverse Lebesgue Theorem

We assume $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ bounded and smooth.

A.5.2.1 Simple Product

Let $r > 1$ and $s > 1$ such that

$$\frac{1}{p} = \frac{1}{r} + \frac{1}{s} < 1. \tag{A.32}$$

Lemma A.12. *Let $(f_n)_{n\in\mathbb{N}}$ which converges to f in $L^r(\Omega)$ weak and $(g_n)_{n\in\mathbb{N}}$ which converges to g in $L^s(\Omega)$. Then $(f_n g_n)_{n\in\mathbb{N}}$ converges to fg in $L^p(\Omega)$ weak.*

Proof. Let $u \in L^{p'}(\Omega)$, and notice that

$$(f_n g_n, u)_\Omega = (f_n, g_n u)_\Omega,$$

in observing that $g_n u \in L^{r'}(\Omega)$. We apply Lebesgue inverse theorem A.10 to the sequence $(g_n)_{n\in\mathbb{N}}$, in extracting a subsequence, still written as $(g_n)_{n\in\mathbb{N}}$, which converges *a.e.* to g and satisfies $|g_n| \le H \in L^s(\Omega)$. Therefore, $(ug_n)_{n\in\mathbb{N}}$ converges *a.e.* to ug and $|ug_n| \le |u|H \in L^{r'}(\Omega)$. Therefore, we deduce from the usual Lebesgue Theorem that $(ug_n)_{n\in\mathbb{N}}$ converges to ug in $L^{r'}(\Omega)$. Since $(f_n)_{n\in\mathbb{N}}$ converges to f in $L^r(\Omega)$,

$$\lim_{n\to\infty} (f_n, g_n u)_\Omega = (f, ug)_\Omega = (fg, u)_\Omega = \lim_{n\to\infty} (f_n g_n, u)_\Omega,$$

for all $u \in L^{p'}(\Omega)$, hence the weak convergence in $L^p(\Omega)$ of $(f_n g_n)_{n\in\mathbb{N}}$ to fg. As the limit is unique, the whole sequence converges. □

Remark A.1. If in addition $(f_n)_{n\in\mathbb{N}}$ converges to f in $L^r(\Omega)$, then $(f_n g_n)_{n\in\mathbb{N}}$ goes to fg in $L^p(\Omega)$.

A.5.2.2 Composed Product

Let $r > 1$, $s > 1$, and $\alpha > 0$ such that

$$\frac{1}{p} = \frac{1}{r} + \frac{\alpha}{s} < 1. \tag{A.33}$$

Let $G : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\forall\, x \in \mathbb{R}, \quad |G(x)| \le C(1 + |x|^\alpha), \tag{A.34}$$

for some constant $C > 0$.

Lemma A.13. *Let $(f_n)_{n\in\mathbb{N}}$ which converges to f in $L^r(\Omega)$ weak and $(g_n)_{n\in\mathbb{N}}$ which converges to g in $L^s(\Omega)$. Then $(f_n G(g_n))_{n\in\mathbb{N}}$ converges to $f G(g)$ in $L^p(\Omega)$ weak.*

Proof. According to Lemma A.12, we only have to prove that $(G(g_n))_{n\in\mathbb{N}}$ converges to $G(g)$ in $L^{s/\alpha}(\Omega)$. We apply again inverse Lebesgue Theorem A.10 to the sequence $(g_n)_{n\in\mathbb{N}}$, in extracting a subsequence, still written as $(g_n)_{n\in\mathbb{N}}$, which converges *a.e.* to g and satisfies $|g_n| \le H \in L^s(\Omega)$. We deduce from the continuity of G that $(G(g_n))_{n\in\mathbb{N}}$ converges *a.e.* to $G(g)$. Moreover, by (A.34), we get

$$|G(g_n)| \le C(1 + |g_n|^\alpha) \le C(1 + |H|^\alpha) \in L^{\frac{s}{\alpha}}(\Omega),$$

since we have assumed Ω bounded. The conclusion follows from Lebesgue Theorem. As the limit is unique, the whole sequence converges. $\square$

A.5.2.3 Composed Product: Bounded Case

Let $r > 1$, $s \ge 1$, and $G : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function.

Lemma A.14. *Let $(f_n)_{n\in\mathbb{N}}$ that converges to f in $L^r(\Omega)$ weak and $(g_n)_{n\in\mathbb{N}}$ that converges to g in $L^s(\Omega)$. Then $(f_n G(g_n))_{n\in\mathbb{N}}$ converges to $f G(g)$ in $L^r(\Omega)$ weak.*

Proof. Let $u \in L^{r'}(\Omega)$, and recall that

$$(f_n G(g_n), u)_\Omega = (f_n, u\, G(g_n))_\Omega,$$

in observing that $u\, G(g_n) \in L^{r'}(\Omega)$ since $G(g_n) \in L^\infty(\Omega)$. We apply inverse Lebesgue Theorem A.10 to the sequence $(g_n)_{n\in\mathbb{N}}$, in extracting a subsequence, still written as $(g_n)_{n\in\mathbb{N}}$, which converges *a.e.* to g and satisfies $|g_n| \le H \in L^s(\Omega)$. Since G is continuous, $(uG(g_n))_{n\in\mathbb{N}}$ converges to $uG(g)$ *a.e.* in Ω, and we also have $|u\, G(g_n)| \le ||G||_\infty |u| \in L^{r'}(\Omega)$. Therefore, we deduce from Lebesgue theorem that $(uG(g_n))_{n\in\mathbb{N}}$ converges to $uG(g)$ in $L^{r'}(\Omega)$; hence

$$\lim_{n\to\infty}(f_n, u\,G(g_n))_\Omega = (f, u\,G(g))_\Omega = (f\;G(g), u)_\Omega = \lim_{n\to\infty}(f_n G(g_n), u)_\Omega,$$

which proves the result, and as the limit is unique, the whole sequence converges. □

By a direct application of Lebesgue theorem, we also have

Corollary A.1. *If in addition $(f_n)_{n\in\mathbb{N}}$ converges to f in $L^r(\Omega)$, then $(f_n G(g_n))_{n\in\mathbb{N}}$ converges to $f\,G(g)$ in $L^r(\Omega)$.*

Applying Lemma A.14 with $f = 1$ yields

Corollary A.2. *The sequence $(G(g_n))_{n\in\mathbb{N}}$ converges to $G(g)$ in $L^\alpha(\Omega)$ for all $\alpha \in [1,\infty[$.*

We finish this lemma's serie with the following one, whose proof is also based on inverse Lebesgue Theorem. We skip the details of the proof for simplicity.

Lemma A.15. *Let the sequence $(f_n)_{n\in\mathbb{N}}$ converge to f in $L^1(\Omega)$, $(g_n)_{n\in\mathbb{N}}$ converge to g in $L^s(\Omega)$, $s > 1$. Then $(f_n G(g_n))_{n\in\mathbb{N}}$ converges to $f\,G(g)$ in $L^\infty(\Omega)$ weak star, which means that $\forall\, l \in L^1(\Omega)$, $\lim_{n\to\infty}(f_n G(g_n), l)_\Omega = (f\,G(g), l)_\Omega$.*

A.5.2.4 Truncature Convergence

Let T_n be the truncation function at height $n > 0$ and defined by

$$T_n(x) = \begin{cases} x & \text{if } |x| \le n, \\ T_n(x) = \mathrm{sg}(x)\,n & \text{if } |x| > n, \end{cases} \tag{A.35}$$

Lemma A.16. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence that converges to f in $L^1(\Omega)$, where $f_n \ge 0$ a.e. for all n. Then $(T_n(f_n))_{n\in N}$ converges to f in $L^1(\Omega)$.*

Proof. Applying the inverse Lebesgue theorem A.10, we know that besides extracting a subsequence, $(f_n)_{n\in\mathbb{N}}$ converges to f a.e. in Ω and that there exists $H \in L^1(\Omega)$ such that $|f_n|^2 \le H$. Notice that $f \ge 0$ a.e. We deduce that $(T_n(f_n))_{n\in\mathbb{N}}$ converges to f a.e. in Ω and $T_n(f_n) \le H$. Indeed, let $\tilde{\Omega} \subset \Omega$ the domain over which $(f_n)_{n\in\mathbb{N}}$ converges to f and

$$\tilde{\tilde{\Omega}} = \tilde{\Omega} \cap \{f \ne \infty\}.$$

Since $f \in L^1(\Omega)$, then $\mathrm{meas}(\Omega \setminus \tilde{\tilde{\Omega}}) = 0$. Let $\mathbf{x} \in \tilde{\tilde{\Omega}}$. Since $f(\mathbf{x}) \ne \infty$, there exists $n_1 \in \mathbb{N}$ such that $f(\mathbf{x}) \le n_1 - 1$. Since $(f_n(\mathbf{x}))_{n\in N}$ converges to $f(\mathbf{x})$, there exists $n_2 \in \mathbb{N}$ such that $\forall\, n \ge n_2$, $f_n(\mathbf{x}) \le f(\mathbf{x}) + 1/2$. Therefore,

$$\forall\, n \ge \sup(n_1, n_2), \quad f_n(\mathbf{x}) \le f(x) + 1/2 \le n_1 - 1/2 < nj,$$

and for those integers $T_n(f_n(\mathbf{x})) = f_n(\mathbf{x})$ converges to $f(\mathbf{x})$, hence the *a.e.* convergence of $(T_n(f_n(\mathbf{x})))_{n\in\mathbb{N}}$ to f. Finally, we have $T_n(f_n) \leq f_n \leq H$. The conclusion follows from Lebesgue theorem, and as the limit is unique, the whole sequence converges, what ends the proof. □

Remark A.1. The assumption about the sign of f_n in Lemma A.16 can be removed, in writting

$$f_n = f_n^+ - f_n^-, \quad f_n^+ = \frac{|f| + f}{2}, \quad f_n^- = \frac{|f| - f}{2}.$$

A.5.3 Stampacchia Theorem

The two following results are due to Stampacchia [29].

Lemma A.17. *Let $u \in H_0^1(\Omega)$, $a \in \mathbb{R}$. Assume that*

$$meas\{x \in \Omega;\ u(x) = a\} \neq 0.$$

Then $\nabla u = 0$ a.e. on the set $\{x \in \Omega;\ u(x) = a\}$.

Theorem A.11. *Let $G \in W^{1,\infty}(\mathbb{R})$ such that G' has a finite number of discontinuities, and $G(0) = 0$. Let $u \in H_0^1(\Omega)$. Then $G(u) \in H_0^1(\Omega)$ and the following holds*

$$\nabla(G(u)) = G'(u)\nabla u, \tag{A.36}$$

a.e. in Ω.

A.6 Interpolation Inequalities

A.6.1 Basic Interpolation Result

Let $t > 0$, $Q_t = [0, t] \times \Omega$. Let $u \in L^\infty([0, t], L^2(\Omega)) \cap L^2([0, t], L^6(\Omega))$, and let

$$n_{p,q}(t) = ||u||_{L^p([0,t],L^q(\Omega))},$$

when it is well defined.

Lemma A.18. *It holds*

$$\forall\, r \in [2, 6], \quad n_{\frac{4r}{3(r-2)},r}(t) \leq n_{\infty,2}^{\frac{6-r}{2r}}(t)\, n_{2,6}^{\frac{3(r-2)}{2r}}(t). \tag{A.37}$$

Proof. Let $r \in [2, 6]$ and write $r = 2\theta + 6(1-\theta)$. By Hölder inequality one has

$$\int_\Omega |u|^r \le \left(\int_\Omega |u|^2\right)^\theta \left(\int_\Omega |u|^6\right)^{1-\theta} \le n_{\infty,2}^{2\theta} ||u||_{0,6,\Omega}^{6(1-\theta)}. \tag{A.38}$$

Writing $\theta = \dfrac{6-r}{4}$ yields

$$\left(\int_\Omega |u|^r\right)^{\frac{4}{3(r-2)}} \le n_{\infty,2}^{\frac{2(6-r)}{3(r-2)}} ||u||_{0,6,\Omega}^2, \tag{A.39}$$

that is,

$$||u||_{0,r,\Omega}^{\frac{4r}{3(r-2)}} \le n_{\infty,2}^{\frac{2(6-r)}{3(r-2)}} ||u||_{0,6,\Omega}^2. \tag{A.40}$$

Inequality (A.37) results from (A.40) after integrating with respect to the time and an standard algebraic calculation. □

A.6.2 Washer Inequalities

We state the washer inequalities proved by Boccardo–Gallouët [8], in the form that is used for our applications, and for any dimension $d \ge 2$. These inequalities are also known as the Boccardo–Gallouët inequalities.

A.6.2.1 The Steady-State Case

Let $u \in H_0^1(\Omega)$, and for any $n \in \mathbb{N}$,

$$B_n^s(u) = \{\mathbf{x} \in \Omega;\ n \le |u(\mathbf{x})| < n+1\}. \tag{A.41}$$

$$M_s(u) = \sup_{n \in \mathbb{N}} \left(\int_{B_n^s(u)} |\nabla u|^2\right). \tag{A.42}$$

Theorem A.12. *For all q such that*

$$1 \le q < \frac{d}{d-1}, \tag{A.43}$$

there exists a polynomial $P_q \in \mathbb{R}[X]$, whose coefficients are all nonnegative and the degree depends on q, and such that

$$\forall\, u \in H_0^1(\Omega), \quad ||u||_{1,q,\Omega} \le P_q(M_s(u)). \tag{A.44}$$

A.6.2.2 The Evolutionary Case

Let

$$u \in L^2([0,T], H_0^1(\Omega)) \cap L^\infty([0,T], L^1(\Omega)),$$

and for any $n \in \mathbb{N}$,

$$B_n^e(u) = \{(t,\mathbf{x}) \in [0,T] \times \Omega;\; n \le |u(t,\mathbf{x})| < n+1\}, \tag{A.45}$$

$$M_e(u) = \sup_{n \in N} \left(\int_0^T \int_{B_n^e(u)} |\nabla u|^2 \right), \tag{A.46}$$

Theorem A.13. *For all q such that*

$$1 \le q < \frac{d+2}{d+1}, \tag{A.47}$$

there exists a polynomial $Q_q \in \mathbb{R}[X,Y]$, whose coefficients are all nonnegative and the degree depends on q, such that

$$\forall\, u \in L^2([0,T], H_0^1(\Omega)) \cap L^\infty([0,T], L^1(\Omega)), \tag{A.48}$$

$$||u||_{q;1,q,\Omega} \le Q_q(M_e(u), ||u||_{\infty;0,1,\Omega}). \tag{A.49}$$

A.7 Convex Functionals and Saddle Point Problems

Some of the PDEs that we consider may be cast as minimization problems with restrictions (linked to the incompressibility or to boundary conditions). This kind of problems may be set in an abstract framework as saddle point problems, associated to optimization problems with restrictions.

Definition A.5. Let X be a Hilbert space. A functional $J : X \mapsto \mathbb{R}$ is called convex if

$$J(\theta u + (1-\theta) w) \le \theta\, J(u) + (1-\theta)\, J(w), \quad \text{for all } \; u,\, w \in X, \quad \theta \in [0,1].$$

J is called strictly convex if the inequality is strict when $\theta \in (0,1)$.

Definition A.6. Given $u, v \in X$, the functional J is Gâteaux-differentiable at u in the direction w if the function

$$\phi(\theta) = J(u + \theta\, w), \quad \text{for } \; \theta \in \mathbb{R}$$

is derivable at $\theta = 0$. In this case $\phi'(0)$ is called the directional derivative of J at u in the direction of w and is denoted by $J'(u)(w)$.

The functional J is Gâteaux-differentiable at u if the directional derivative $J'(u)w$ exists at any $w \in X$, and the mapping $J'(u) : w \in X \mapsto J'(u)w$ belongs to X'. In this case, $\langle J'(u), w\rangle_X = J'(u)(w)$, where the symbol $\langle \cdot, \cdot \rangle_X$ denotes the duality between X' and X.

Also, given some additional $z \in X$, the functional J is twice Gâteaux-differentiable at u in the direction (z, w) if there exist $J'(u + \theta z)(w)$ when $|\theta| < \varepsilon$, for some $\varepsilon > 0$, and the function

$$\sigma(\theta) = \langle J'(u + \theta z), w\rangle_X$$

is differentiable at $\theta = 0$. In this case $\sigma'(0)$ is called the second derivative of J at u in the direction (z, w) and is denoted by $J''(u)(z, w)$.

Theorem A.14. *Let U be a convex nonempty subset of X. Let $J : U \mapsto \mathbb{R}$ a convex Gâteaux-differentiable functional. Then $v \in U$ solves the minimization problem*

$$J(v) = \inf_{w \in U} J(w) \tag{A.50}$$

if and only if

$$\langle J'(v), w - v\rangle_X \geq 0 \ \textit{for any} \ w \in U. \tag{A.51}$$

If U in addition is closed, then (A.50) admits at least one solution $v \in U$. If J is strictly convex, then (A.50) admits at most one solution in U.

Problem (A.50) is called a *constrained optimization* problem. We are interested in solving problem (A.50) when J is a strictly convex Gâteaux-differentiable functional , and

$$U = \{w \in X \ \text{ such that } \ b(w, q) = \langle g, q\rangle_Q \quad \forall q \in Q\}, \tag{A.52}$$

where Q is a Hilbert space, $b : X \times Q \mapsto \mathbb{R}$ is a bilinear bounded form, and $g \in Q'$. Then U is closed and convex. By Theorem A.14 problem (A.50) admits a unique solution if U is nonempty.

Problem (A.50) may be solved by duality techniques. These techniques have the main advantage of eliminating the restrictions by means of an auxiliary variable, the *Lagrange multiplier*, as follows: Define the Lagrangian

$$\mathscr{L}(w, q) = J(w) + b(w, q) - \langle g, q\rangle_Q \ \text{ for any } \ (w, q) \in X \times Q. \tag{A.53}$$

Definition A.7. A pair $(v, p) \in X \times Q$ is a saddle point of $\mathscr{L}$ on $X \times Q$ if

$$\text{for any } \ q \in Q, \quad \mathscr{L}(v, q) \leq \mathscr{L}(v, p) \leq \mathscr{L}(w, p), \qquad \text{for any } \ w \in X. \tag{A.54}$$

The associated "primal"problem is

$$F(v) = \min_{w \in X} F(w), \quad \text{where} \quad F(w) = \sup_{q \in Q} \mathscr{L}(w, q),$$

and the associated "dual"problem is

$$G(p) = \max_{q \in Q} G(q), \quad \text{where} \quad G(q) = \inf_{w \in X} \mathscr{L}(w, q).$$

The following result shows the equivalence between both formulations (Cf. [2]):

Theorem A.15. *The couple* (v, p) *is a saddle point of* $\mathscr{L}$ *over* $X \times Q$ *if and only if* v *is a solution of the primal problem and* p *is a solution of the dual problem. Moreover,*

$$F(v) = G(p) = \min_{w \in X} \left(\sup_{q \in Q} \mathscr{L}(w, q) \right) = \max_{q \in Q} \left(\inf_{w \in X} \mathscr{L}(w, q) \right).$$

The solution p of the dual problem is called the Lagrange multiplier associated to the restriction. The associated saddle point is characterized as follows:

Theorem A.16. *Assume that* $J : X \mapsto \mathbb{R}$ *is a strictly convex Gâteaux-differentiable functional and that* U *is given by (A.52) where* $b : X \times Q \mapsto \mathbb{R}$ *is a bounded bilinear form. Then* $(v, q) \in X \times Q$ *is a saddle point of the Lagrangian*

$$\mathscr{L}(w, q) = J(w) + b(w, q) - \langle g, q \rangle_Q \tag{A.55}$$

if and only if it satisfies the optimality conditions

$$\begin{cases} \langle J'(v), w \rangle_X + b(w, p) = 0, & \textit{for any } w \in X \\ \qquad\qquad b(v, q) = \langle g, q \rangle_Q, & \textit{for any } q \in Q. \end{cases} \tag{A.56}$$

Proof. Assume that (v, p) is a saddle point of $\mathscr{L}$. The first inequality in (A.54) is equivalent to

$$b(v, q) - \langle g, q \rangle_Q \leq b(v, p) - \langle g, p \rangle_Q \quad \text{for any } q \in Q.$$

Then, taking $q = p \pm r$ for any $r \in Q$, we deduce that first inequality in (A.54) is equivalent to $b(v, r) = \langle g, r \rangle_Q$, $\forall r \in Q$. Also, the second inequality in (A.54) means that v solves the problem $J_p(v) = \inf_{w \in X} J_p(w)$, with $J_p(w) = J(w) - b(w, p) + \langle g, p \rangle_Q$. J_p is Gâteaux-differentiable and strictly convex. By Theorem A.14 this problem admits a unique solution, characterized by $\langle J_p'(v), w \rangle_X = \langle J'(v), w \rangle_X + b(w, p) = 0$, for any $w \in X$. $\square$

The saddle point equations (A.56) admit a solution if the pair of spaces (X, Q) satisfies a compatibility condition, called the *inf-sup condition*:

Definition A.8. The pair of spaces (X, Q) satisfies the inf-sup condition if

$$\alpha = \inf_{q\in Q} \sup_{w\in X} \frac{b(w,q)}{\|w\|_X \|q\|_Q} > 0. \tag{A.57}$$

In (A.57) the sup is extended to the nonzero functions of X and Q, but we omit it for brevity. Usually, this condition is used as

$$\alpha \|q\|_Q \leq \sup_{w\in X} \frac{b(w,q)}{\|w\|_X}, \quad \text{for all } q \in Q. \tag{A.58}$$

Let us consider the spaces

$$Z = \{z \in X \text{ such that } b(z,q) = 0, \text{ for all } q \in Q.\}$$

and the topological complement of Z:

$$Z^\perp = \{\sigma \in X' \text{ such that } \langle\sigma, z\rangle_X = 0, \text{ for all } z \in Z\}.$$

We shall use the following result (Cf. [23]):

Lemma A.19. *Let G be a subspace of a normed space B. Then G is closed if and only if $(G^\perp)^\perp = G$.*

Then we have

Lemma A.20. *For any $\sigma \in Z^\perp$ there exists some $p \in Q$ such that*

$$b(w,p) = \langle\sigma, w\rangle_X \text{ for all } w \in X. \tag{A.59}$$

Such p is unique if the pair of spaces (X, Q) satisfies the inf-sup condition (A.58). Moreover, in this case

$$\|p\|_Q \leq \alpha^{-1} \|\sigma\|_{X'}. \tag{A.60}$$

Proof. Define the operator $\mathscr{G} : Q \to X'$ by

$$\forall q \in Q, \quad \langle\mathscr{G}(q), z\rangle_X = b(z,q), \quad \forall z \in X.$$

Observe that $Z = Im(\mathscr{G})^\perp$. As b is continuous, $Im(\mathscr{G})$ is closed, and by Lemma A.19, $Z^\perp = Im(\mathscr{G})$. Then, given $\sigma \in Z^\perp$, there exists $p \in Q$ verifying (A.59). Under condition (A.58), p is trivially unique. Estimate (A.60) follows from (A.58) and (A.59)

We thus deduce

Corollary A.3. *Assume that the hypotheses of Theorem A.16 hold. Assume also that the inf-sup condition (A.58) holds. Then the saddle point equations (A.56) admit a unique solution and, consequently, the Lagrangian (A.55) admits a unique saddle point on $X \times Q$.*

The inf-sup condition ensures in particular that the dual problem admits a unique solution. Note that by (A.51) , $G(q) = \mathscr{L}(v_q, q)$, where $v_q \in X$ is the solution of

$$\langle J'(v_q), w\rangle_X + b(w, q) = \langle g, q\rangle_Q, \quad \text{for any } \; w \in X.$$

The dual problem may be solved by descent methods. These methods determine a sequence $\{p_n\}_{n\geq 0} \subset Q$ that approximate p, in such a way that

$$G(p_{n+1}) = \sup_{\gamma\in\mathbb{R}} G(p_n + \gamma\, d_n)$$

where d_n is a descent direction. In particular, the method of Uzawa corresponds to the use of the standard gradient method, $d_n = \nabla G(p_n)$, where $\nabla G(p_n) \in Q$ is defined by $(\nabla G(p_n), r) = \langle G'(p_n), r\rangle_Q$, for any $r \in Q$ (Cf. [2]).

In addition we need some relationship between convexity and monotonicity.

Lemma A.21. *Let J be a functional defined on a Hilbert space X. Assume that J is Gâteaux-differentiable on X. Then J' is monotone if and only if J is convex.*

This result may be found in Ekeland [15]. Also, the analysis of convex functionals including the relationships between convexity and differentiability may be found for instance in [14] and [15].

A.8 Approximation of Linear Saddle Point Problems

Let us consider the linear saddle point problem

$$\begin{cases} a(v,w) + b(w,p) = \langle f, w\rangle_X, & \text{for any } \; w \in X \\ \qquad\qquad\; b(v,q) = \langle g, q\rangle_Q, & \text{for any } \; q \in Q, \end{cases} \tag{A.61}$$

where $f \in X'$, $g \in Q'$, and

- The form a is bilinear and bounded.
- The form a is coercive: There exists $\beta > 0$ such that

$$a(w,w) \geq \beta\, \|w\|_X^2, \quad \text{for all } \; w \in X. \tag{A.62}$$

- The form $b : X \times Q \mapsto \mathbb{R}$ is bilinear and bounded.
- The pair of spaces (X, Q) satisfies the inf-sup condition (A.58).

Problem (A.61) coincides with the saddle point problem (A.56) if $J(w) = \frac{1}{2}a(w,w) - \langle f, w\rangle_X$ where a is a scalar product on X. However, in general the form a needs not to be symmetric, and then problem (A.61) is not equivalent to an optimization problem.

The standard analysis of the approximation of problem (A.61) was introduced by Brezzi (Cf. [11]) and Babuška (Cf. [3]). Under the above conditions, this problem is proved to be well posed: It admits a unique solution that continuously depends on the data. Let us next consider two families of approximating subspaces of X and Q, respectively, $(X_h)_{h>0}$ and $(Q_h)_{h>0}$, where $h > 0$ denotes the approximation parameter. We set the approximated problems as follows:

Find $(v_h, p_h) \in X_h \times Q_h$ such that

$$\begin{cases} a(v_h, w_h) + b(w_h, p_h) = \langle f, w_h\rangle_X, & \text{for all } w_h \in X_h, \\ b(v_h, q_h) = \langle g, q\rangle_Q, & \text{for all } q_h \in Q. \end{cases} \tag{A.63}$$

Discretization (A.63) is usually referred as a *mixed* approximation, as both v and p are approximated. It is stable if the discrete spaces X_h and Q_h satisfy a compatibility condition:

Definition A.9. The family of pairs of spaces $(X_h, Q_h)_{h>0}$ satisfies the uniform discrete inf-sup condition if there exists a constant $\gamma > 0$ independent of h such that

$$\gamma \|q_h\|_Q \le \sup_{w_h \in X_h} \frac{b(w_h, q_h)}{\|w_h\|_X}, \quad \forall q_h \in Q_h, \quad \forall h > 0. \tag{A.64}$$

If a particular pair of spaces (X_h, Q_h) satisfies (A.64) with a constant γ possibly depending on h, we say that the pair (X_h, Q_h) satisfies a discrete inf-sup condition.

Under the uniform discrete inf-sup condition, the following stability and approximation result holds (Cf. [11]):

Theorem A.17. *Assume that the family of pairs of spaces $\{(X_h, Q_h)\}_{h>0}$ satisfies the uniform discrete inf-sup condition (A.9). Then problem (A.63) admits a unique solution that satisfies the estimates*

$$\begin{cases} \|v_h\|_X + \|p_h\|_Q \le C\, \left(\|f\|_{X'} + \|g\|_{Q'}\right), \\ \|v - v_h\|_X + \|p - p_h\|_Q \le C\, \left(d_X(v, X_h) + d_Q(p, Q_h)\right), \end{cases} \tag{A.65}$$

for some constant $C > 0$ independent of h.

Here, we have denoted by $d_E(b, E_h)$ the distance in a normed space E from some $b \in E$ to some closed subspace $E_h \subset E$:

$$d_E(b, E_h) = \inf_{b_h \in E_h} \|b - b_h\|_E.$$

Estimate (A.65) reduces the obtention of error estimates for the approximation of the saddle point problem (A.61) by (A.63) to an approximation problem: Spaces X and Q should correctly be approximated by X_h and Q_h to obtain a convergent approximation:

Definition A.10. Let B be a separable Banach space. An internal approximation of B is a family $\{B_h\}_{h>0}$ of subspaces of finite dimension of B such that for any $b \in B$:

$$\lim_{h\to 0} d_B(b, B_h) = 0.$$

Thus, if $\{X_h \times Q_h\}_{h>0}$ is an internal approximation of $X \times Q$ satisfying the uniform discrete inf-sup condition, by (A.65) the mixed approximation (A.63) is convergent, in the sense that

$$\lim_{h\to 0}(\|v - v_h\|_X + \|p - p_h\|_Q) = 0.$$

The discrete inf-sup condition ensures in particular that the set of elements of X_h satisfying the restriction $b(z_h, q_h) = 0$ for $q_h \in Q_h$ is not empty.

Lemma A.22. *Assume that the pair of spaces (X_h, Q_h) satisfies an inf-sup condition (A.64) for some constant $\gamma > 0$. Then*

$$dim(X_h) = dim(Q_h) + dim(Z_h), \tag{A.66}$$

where

$$Z_h = \{z_h \in X_h \ \ such\ that\ \ b(z_h, q_h) = 0,\ \ for\ all\ \ q_h \in Q_h.\}$$

Proof. Let us identify Q_h with its topological dual and define the discrete operator $\mathscr{D}_h : X_h \to Q_h$ by

$$\forall w_h \in X_h, \quad (\mathscr{D}_h(w_h), q_h)_{Q_h} = b(w_h, q_h), \quad \forall q_h \in Q_h,$$

where $(\cdot,\cdot)_{Q_h}$ denotes the inner product in Q_h. The operator $\mathscr{D}_h$ is surjective under condition (A.64). Indeed, if $b(w_h, q_h) = 0$, for all $w_h \in X_h$, then by condition (A.64), $q_h = 0$. Then $Im(\mathscr{D}_h)^{\perp} = \{0\}$ and $Im(\mathscr{D}_h) = Q_h$. Consequently,

$$\dim(X_h) = \dim(Q_h) + \dim(Ker(\mathscr{D}_h)).$$

The conclusion follows because $Ker(\mathscr{D}_h) = Z_h$. $\square$

Consequently, if $\dim(X_h) > \dim(Q_h)$, space Z_h is not trivial. In general, the discrete inf-sup condition holds if space X_h is rich enough in degrees of freedom with respect to space Q_h.

By Lemma A.20, the discrete inf-sup condition ensures the unique solvability of the discrete Lagrange multiplier p_h. This does not require that the constant γ appearing in the discrete inf-sup condition (A.64) is independent of h. Asking γ to be independent of h allows to obtain uniform bounds of the discrete pressures in $L^2(\Omega)$ norm. For some discretizations, the parameter γ is not independent on the discretization parameter h. This ensures the solvability of the Lagrange multiplier at the discrete level, but the estimate (A.60) for its norm deteriorates as $h \to 0$. This effect is counterbalanced if the accuracy of the discretization is large enough, as this ensures the convergence of the multiplier. This is the situation, for instance, for spectral discretizations of incompressible flows, where the Lagrange multiplier is the pressure (Cf. [5, 25], Cf. also [13] for an introduction to spectral methods for PDEs).

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