Hamilton-Jacobi equations and Mean field games on networks

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UNIVERSITE BRETAGNE LOIRE / MATHSTIC

UNIVERSITÉ DE RENNES 1

THESE DE DOCTORAT DE

L'UNIVERSITE DE RENNES 1
COMUE UNIVERSITE BRETAGNE LOIRE

ECOLE DOCTORALE N° 601
Mathématiques et Sciences et Technologies
de l'Information et de la Communication
Spécialité : Mathématiques et leurs Interactions

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Équations de Hamilton-Jacobi et jeux à champ moyen sur les réseaux

Thèse présentée et soutenue à l'Université de Rennes 1, le 17 octobre 2018
Unité de recherche : UMR CNRS 6625 Institut de Recherche Mathématique de Rennes (IRMAR)

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Acknowledgements

This dissertation would not have been completed without the help, encouragements and supports from many people who all deserve my sincerest gratitude and appreciation.

In fact, I truly think that no words can express my gratitude to my three advisors, whose personalities, expectations and working styles are very different; however, what they have in common is their dedication in teaching and growing me without which I would not be here writing this acknowledgement.

First of all, my deepest gratitude goes to my doctoral advisor Professor Olivier Ley, who has always been there with me since my master internship, sharing with me his valuable advices, supporting me during the darkest moments of my Ph.D. study and inspiring me by simply being himself. I will never forget the time that he reminded me a thousand times to bring my passport on a travel (I did forget it anyways) or the days when we had to go back and forth a hundred times on a paper and he still patiently bore with me and improved it little by little. I still keep those manuscripts with his hand-writing comments as a reminder of a caring, devoted and respectable advisor.

I wish to express my gratitude most strongly to Professor Yves Achdou for having offered me such an interesting Ph.D. project and more importantly, a chance to discover mathematics in a very different way. To be honest, during my first year, I always felt nervous before every single meeting with him; however throughout the years, I found it more and more interestingly challenging thanks to which I have learned and matured a lot. He provided me the optimal space and also "pressure" to continuously pushing and developing myself and at the same time, a lifeboat when things did not happen as we expected. The most touching moment was when I informed him of my post-doc interview, he replied me with very detailed email instructing me what to do and what not to do before and during the interview. I am, always have been, and will always be, proud to be his student.

A special mention goes to Professor Nicoletta Tchou. If I was allowed to modify the function appearing after her name on the cover of this dissertation, it would be "co-directrice de thèse". Beside being a great advisor, for me she was also the best moderator for our projects and meetings who connected not only ideas, but also the people, us. Additionally, she was usually the one who paid the closest attention to my work and pointed out the most obvious (and silly) mistakes of mine, from which I have learned not only how to do mathematics, but also how to make people understand how we do it in the most comprehensive and appealing manner.

I greatly appreciate Professor Francisco Silva and Professor Claudio Marchi, the two reviewers of this dissertation, who have been so kind to spend time to read, evaluate my works and give me valuable comments. I am also thankful to Professor Pierre Cardaliaguet and Professor Hasnaa Zidani who accepted to be in my thesis committee. Having my thesis committee including these admirable experts in viscosity solutions and mean field games is my great honor.

I am grateful to Institute of Mathematical Research of Rennes (IRMAR, University of Rennes 1) and Center Henri Lebesgue who have financially supported my doctoral study during the past three years. My special thanks to the secretaries Xhenxila Lachambre, Carole Wosiak, Marie-Aude Verger, Florian Rogowski and Chantal Halet who have endlessly supported me even before the first day I was in France till the last. I also acknowledge the partial support from the ANR project HJnet "Hamilton-Jacobi equations on heterogeneous structures and networks" and ANR project "Mean field games". Working in the National Institute of Applied Sciences of Rennes (INSA Rennes) was a wonderful time in my Ph.D. period. I am thankful for the equipment and facilities I fully took advantage of.

I would also like to thank Prof. Boualem Djehiche who offered me the Post-doc position even before when I submitted my thesis, which made me feel much more “secure” and motivated to finish the last phase of my PhD life.

I am truly thankful to my "master" and also my friend Nguyen Quang Huy for the mathematical discussions that we had and career orientations he has shared. A sincere thank goes to Salomé Oudet for her supports during my master internship. I'm grateful to one of my close friends, Nguyen Thi Tuyen for her help and encouragements from my first days in France until now.

A special thanks must go to my PUF close friends: Tran Minh Quan for being friends with another-me-that-no-one-can-see; Truong Kim Tien for being my partner in crime and for the unforgettable EuroTrips; Dang Phuoc Nhat for everything we have been through; and Pham Thi Da Cam for her caring.

Besides, I have also been lucky to meet and make friends with many nice people in Rennes. Thank you, my French friends who share the experience of Ph.D. life and who are many times my saviors: Tangi Migot, Emilie Joannopoulos, Audrey Poterie, and Samuel Corre. Thank you, my mathematics friends, Nguyen Van Trinh, Chau Thi Tuyet Trang, Nguyen Duc Tho, Nguyen Dinh Duong, Nguyen Thi Minh Phuong, Nguyen Thi Hoai Thuong for our friendship, our mathematical discussions and the parties that we can freely talk about anything on earth. To be honest, thank to all of you and what we have together, I sometimes forget that I am living in France. And thank you, my Vietnamese friends in France: Nguyen Hoang Sang, Nguyen Thi Lan Anh, Nguyen Hoai Nam, Nguyen Tran Dan Ngoc, Nguyen Ha Thanh, and many others... for the moral and sometimes physical support.

Last but not least, I would like to express all my gratitude to my family for constant encouragement and unconditional love. Especially, I dedicate this thesis to my wife, who gave up all she had in Vietnam to come with me to France. I cannot forget all the moments when she tried to proofread all my manuscripts, making them much more easily understandable and giving me useful advices even if she is not at all a mathematician. All my stress is gone when I am home, having dinner with her and watching our favorite TV shows/movies. Thank her again for being and bearing with me in this adventure, and the others to come.

Équations de Hamilton-Jacobi et jeux à champ moyen sur les réseaux
Resumé

Cette thèse porte sur l'etude de problèmes des équations de Hamilton-Jacobi-Bellman (HJB) associées à des problèmes de contrôle optimal et jeux à champ moyen sur les réseaux, c'est-à-dire, des ensembles constitués d'arêtes connectées à des sommets. Différentes dynamiques et coûts sont autorisés dans chaque arête du réseau. Sur la Figure 0.1, on présente deux exemples de réseau.

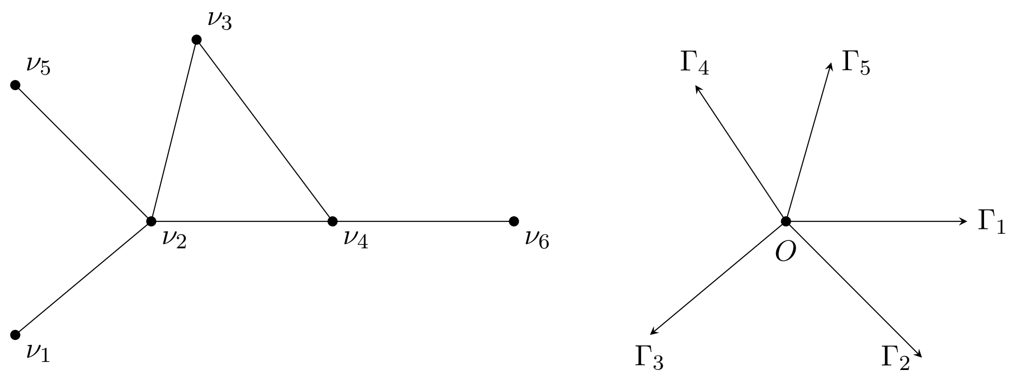

Figure 0.1: À gauche un réseau géneral Γ; à droite, une jonction $\mathcal{G}$.

Dans le chapitre 2, on considère un problème de contrôle optimal sur les réseaux dans l'esprit des travaux d'Achdou, Camilli, Cutrì & Tchou [5] et Imbert, Moneau & Zidani [73]. Plus précisément, nous considérons un problème de contrôle optimal dans lequel nous ajoutons des coûts d'entrée (ou de sortie) aux sommets du réseau et étudions les équations HJB associeés. L'effet des coûts d'entrée/sortie est de rendre discontinue la fonction valeur du problème. Pour simplifier le problème, nous étudions seulement le cas de la jonction, c'est-à-dire, un réseau de la forme $\mathcal{G} = \cup_{i=1}^{N} \Gamma_i$ avec N arêtes Γ_i et un seul sommet O. Nos hypothèses à propos de la dynamique et des coûts sont similaires à ceux faits dans le travail de Achdou, Oudet & Tchou [8], avec des coûts supplémentaires c_i pour entrer dans l'arête Γ_i à partir de O ou d_i pour quitter Γ_i en O. La fonction valeur est continue sur $\mathcal{G}\backslash\{O\}$, mais est en général discontinue en O. Par conséquent, au lieu de considérer la fonction valeur v, nous la remplaçons par la collection $(v_i)_{1\leqslant i\leqslant N}$, où v_i est la restriction de v à l'arête $\Gamma_i\backslash\{O\}$ prolongée par continuité en O. Dans le cas des coûts d'entrée par exemple, notre premier résultat principal est de trouver la relation

entre $\mathsf{v}(O)$, $v_i(O)$ et $v_j(O)+c_j$ pour $1 \leqslant i,j \leqslant N$. Nous en déduisons que les fonctions $(v_i)_{1\leqslant i\leqslant N}$ sont solutions de viscosité du système suivant

$$\begin{aligned} \lambda u_i(x) + H_i\left(x, \frac{du_i}{dx_i}(x)\right) &= 0 && \text{si } x \in \Gamma_i \backslash \{O\}, \\ \lambda u_i(O) + \max\left\{-\lambda \min_{j\neq i}\{u_j(O)+c_j\}, H_i^+\left(O, \frac{du_i}{dx_i}(O)\right), H_O^T\right\} &= 0 && \text{si } x = O, \end{aligned} \tag{0.0.1}$$

où H_i est le hamiltonien correspondant l'arête Γ_i. Le point important est l'expression du hamiltonien en O, qui prend en compte toutes les stratégiés possibles dans voisinage de O. Plus précisément, si la trajectoire est proche de O et appartient à Γ_i alors:

- Le terme $\min_{j\neq i}\{u_j(O)+c_j\}$ prend en compte les situations dans lesquelles la trajectoire entre dans Γ_{i_0} où $u_{i_0}(O)+c_{i_0} = \min_{j\neq i}\{u_j(O)+c_j\}$.
- Le terme $H_i^+\left(O, \frac{du_i}{dx_i}(O)\right)$ prend en compte les situations dans lesquelles la trajectoire ne sort pas de Γ_i.
- Le terme H_O^T prend en compte les situations dans lesquelles la trajectoire reste en O.

La partie la plus importante est consacrée à deux preuves différentes d'un principe de comparaison conduisant à l'unicité d'une solution de viscosité pour (0.0.1) ce qui permet de caratériser la fonction valeur du problèmes: la première utilise des arguments de la théorie du contrôle optimal provenant de Barles, Briani & Chasseigne [19, 20] et [8]; la seconde est inspirée par Lions & Souganidis [86] et utilise des arguments de la théorie des EDP.

Dans le chapitre 3, nous étendons le travail de Camilli & Marchi [32]. Nous étudions des jeux à champ moyen stochastiques (MFG) dans le cas ergodique pour lequel l'espace d'état est un réseau:

$$\begin{cases} -\mu_i\partial^2 v + H(x,\partial v) + \rho = \mathscr{V}[m], & x\in\Gamma_i\backslash\{O\}, i=\overline{1,N}, \\ \mu_i\partial^2 m + \partial(m\partial_p H(x,\partial v)) = 0, & x\in\Gamma_i\backslash\{O\}, i=\overline{1,N}, \\ \sum_{i=1}^N \gamma_i\mu_i\partial_i v(O) = 0, & \\ \sum_{i=1}^N \Big[\mu_i\partial_i m(O) + \partial_p H_i\Big(O,\partial_i v(O)\Big) m_i(O)\Big] = 0, & \\ \partial_i v(\nu_i) = 0,\ \mu_i\partial_i m(\nu_i) + \partial_p H_i\Big(\nu_i,\partial_i v(\nu_i)\Big) m_i(\nu_i) = 0, & i=\overline{1,N}, \\ v_i(O) = v_j(O),\ \dfrac{m_i(O)}{\gamma_i} = \dfrac{m_j(O)}{\gamma_j}, & i,j=\overline{1,N}, \\ \displaystyle\int_{\mathcal{G}} v(x)\,dx = 0, \quad \int_{\mathcal{G}} m(x)\,dx = 1, \quad m\geqslant 0. & \end{cases} \tag{0.0.2}$$

Ici $\{\gamma_i\}$ est un ensemble de constantes positives et $\{\mu_i\}$ est une constante de viscosité correspondant à Γ_i. Commentons le système MFG (0.0.2).

- Les constantes positives (γ_i) sont reliées aux probabilités d'entrée dans les arrêtes du processus stochastique sous-jacent sur le réseau qui décrit la dynamique d'un joueur "moyen".
- Les constantes positives (μ_i) sont les coefficients de diffusion dun processus stochastique dans les arêtes.

- Les EDP de la première ligne sont les équations de Hamilton-Jacobi-Bellman (HJB) ergodiques dans les arêtes, associées au problème de contrôle optimal du joueur typique du problème MFG. Les hypothèses principales sont que nous considèrons des Hamiltoniens sous-quadratiques par rapport au gradient et un couplage $\mathcal{V}$ très général. Ce dernier peut être local, seulement borné inférieurement et strictement croissant (pour obtenir l'unicité).
- Les EDP de la deuxième ligne sont des équations de Fokker-Planck (FP) dans les arêtes qui décrivent la distribution m de l'ensemble de joueurs du problème MFG.
- La troisième ligne est une condition de Kirchhoff pour la fonction valeur du problème de contrôle au sommet.
- La quatrième équation est une condition de transmission pour m au sommet.
- La cinquième ligne traduit les conditions de Kirchhoff pour v aux bords du réseau (qui se réduisent à des conditions de Neumann) et les conditions de transmission pour m aux bords du réseau (qui se réduisent à des conditions de Robin). Ici, dans le cas de la jonction, le bord du réseau est constitué du but des arêtes qui ne contiennent pas O.
- La sixième ligne du système exprime les conditions de continuité pour v et les conditions de saut pour m (ces dernières représentant une des originalité et difficulté du problème).
- La dernière ligne contient des conditions de normalisation pour v et m (qui est une densité de probabilité).

Le premier résultat est de poser le problème et d'expliquer l'obtention du système d'EDP et des conditions de jonctions. Ensuite, nous montrons quelques résultats préliminaires utiles, d'abord sur certains problèmes aux limites linéaires elliptiques équations, puis pour deux d'équations linéaires de Kolmogorov et de Fokker-Planck en dualité. L'existence de solutions faibles est obtenue en appliquant le théorème de Banach-Necas-Babuška à une paire spéciale d'espaces Sobolev appelés V et W ci-dessous et l'alternative de Fredholm. L'unicité vient d'un principe maximum. À l'aide de cas résultats nous prouvons que le système (0.0.2) est bien posé par des arguments de point fixe.

Dans le dernier chapitre, nous considérons le système MFG à horizon fini sur les réseaux suivant:

$$\begin{cases}-\partial_t v-\mu_i\partial^2 v+H\left(x,\partial v\right)=\mathscr{V}[m], & t\in(0,T),\ x\in\Gamma_i\backslash\left\{O\right\},\ i=\overline{1,N},\\ \partial_t m-\mu_i\partial^2 m+\partial\left(m\partial_p H\left(x,\partial v\right)\right)=0, & t\in(0,T),\ x\in\Gamma_i\backslash\left\{O\right\},\ i=\overline{1,N},\\ \sum_{i=1}^{N}\gamma_i\mu_i\partial_i v\left(O,t\right)=0, & t\in(0,T),\\ \sum_{i=1}^{N}\left[\mu_i\partial_i m\left(O,t\right)+\partial_p H_i\Big(O,\partial_i v(O,t)\Big)m_i\left(O,t\right)\right]=0, & t\in(0,T),\\ \partial_i v\left(\nu_i,t\right)=0,\ \mu_i\partial_i m\left(\nu_i,t\right)+\partial_p H_i\Big(\nu_i,\partial_i v(\nu_i,t)\Big)m_i\left(\nu_i,t\right)=0, & t\in(0,T),\ i=\overline{1,N},\\ v_i\left(O,t\right)=v_j\left(O,t\right),\ \dfrac{m_i\left(O,t\right)}{\gamma_i}=\dfrac{m_j\left(O,t\right)}{\gamma_j}, & t\in(0,T),\ i,j=\overline{1,N},\\ v\left(x,T\right)=v_T(x),\ m\left(x,0\right)=m_0(x),\ \displaystyle\int_{\mathcal{G}}m_0(x)dx=1, & x\in\mathcal{G}.\end{cases} \tag{0.0.3}$$

La description et l'interprétation de ce système est semblable au cas du chapitre précédent pour un problème en horizon fini, ce qui se traduit par des équations d'évolution à la place d'un problème stationnaire. Les principales différences dans les hypothèses concernent les équations d'HJB. Nous devons ici considérer des hamiltoniens H qui sont globalement lipschitzien par rapport au gradient (et donc sous-linéaires) et un couplage qui est régularisant (et donc non-local).

Le premier résultat est de pose le problème et d'expliquer l'obtention du système d'EDP aisin que les conditions aux jonctions. Ensuite, nous montrons quelques résultats utiles pour une équation de la chaleur modifiée avec des conditions de Kirchhoff générales et pour des équations de FP avec des conditions de transmission spéciales. Ces résutat utilisent des méthodes de Galerkin type. Notons que les solutions faibles sont définies en utilisant une paire appropriée d'espaces de fonctions Sobolev V et W définis sur le réseau. La principale difficulté dans ce travail est comment d'obtenir la régularité pour la fonction valeur v sur le réseau. L'idée est de dériver l'équation HJB pour v et de prouver une estimation de régularité parabolique pour la solution du problème dérivé. Enfin, nous donnons les preuves des principaux résultats de l'existence et l'unicité du système MFG des EDP en utilisont là aussi des théorèmes de point fixe.

Mots clés: Problèmes de contrôle optimal, équation de Hamilton-Jacobi, équations aux dérivées partielles elliptiques et paraboliques, solutions de viscosité, solutions faibles, réseaux, jeux à champ moyen, condition de Kirchhoff.

Contents

1 Introduction

1.1 General introduction

The aim of this dissertation is to study Hamilton-Jacobi equations associated with optimal control problems and mean field games problems in the case when the state space is a network.

1.1.1 Hamilton-Jacobi equations and optimal control problems

In this section, we recall general optimal control problems, and their connection to Hamilton-Jacobi equations.

First of all, we consider a classical problem in calculus of variation, in which a single agent tries to optimize his path in space time with respect to a fixed cost function. Specifically, suppose that the agent is at $x \in \mathbb{R}^n$ at time $t = 0$ and moves with a given *velocity* (or *dynamic*) f, possibly subject to some random noise. The trajectory of the agent is solution to a *controlled* ordinary differential equation (when $\mu = 0$) or stochastic differential equation (when $\mu > 0$), namely

$$dy\left(s\right) = f\left(y\left(s\right), \alpha\left(s\right)\right) ds + \sqrt{2\mu}\, dW_s, \quad y\left(0\right) = x, \tag{1.1.1}$$

where W_t is a $\mathcal{F}_t$–adapted Wiener process in a reference probability system $(\Omega, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$. The $\mathcal{F}_t$–progressively measurable function $\alpha : [0, +\infty) \to A$ is called the *control* and the set of all admissible control functions is denoted by $\mathcal{T}$. Under mild hypotheses, for a given control α, (1.1.1) has a unique solution $y = y_{x,\alpha}$, which is an absolutely continuous function (when $\mu = 0$) or a continuous simple path (when $\mu > 0$).

Depending on the situation, we will consider several costs associated to a given trajectory. The first one is related to the so-called *infinite horizon* control problem, for which

$$J\left(x, \alpha\right) := \mathbb{E}_x \left[\int_0^\infty \ell\left(y_{x,\alpha}(t), \alpha(t)\right) e^{-\lambda t} dt \right], \tag{1.1.2}$$

with ℓ is a given running cost and λ is a positive constant. The case of *finite horizon* control problem is when one runs the trajectory for a finite range of time $T - t$, starting at x at time t

$$J\left(x, t, \alpha\right) := \mathbb{E}_{x,t} \left[\left\{ \int_t^T \ell\left(y_{x,\alpha}\left(s\right), \alpha(s)\right) ds + v_T\left(y_{x,\alpha}\left(T\right)\right) \right\} \right], \tag{1.1.3}$$

where v_T is called the terminal cost. In the deterministic case $\mu = 0$, there is no expectation.

The optimal control problem is to find an optimal control $\alpha \in \mathcal{T}$ such that the total cost is minimized. In 1950s, to study these problems, Bellman [21] developed a Dynamical Programming approach. The first step is to introduce the *value function* of the problem

$$v(x) = \inf_{\alpha(\cdot)\in\mathcal{T}} J(x, \alpha), \quad \text{or} \quad v(x,t) = \inf_{\alpha(\cdot)\in\mathcal{T}} J(x,t,\alpha), \tag{1.1.4}$$

which is the optimal cost of the optimization problem.

Then the key idea is that v satisfies a functional equation, called the Dynamical Programming Principle (DPP), from which we obtain that v is a solution of a Hamilton-Jacobi-Bellman (HJB) equation, namely

$$\lambda v(x) - \mu\Delta u + H(x, \nabla v(x)) = 0, \tag{1.1.5}$$

in the infinite horizon case, and

$$\begin{cases} -\partial_t v(x,t) - \mu\Delta u + H(x, \partial v) = 0, \\ v(x,T) = v_T(x), \end{cases} \tag{1.1.6}$$

in the finite horizon case. The Hamiltonian H is defined by

$$H(x,p) = \sup_{a\in A}\{-p\cdot f(x,a) - \ell(x,a)\}.$$

We can see that the HJB equations (1.1.5) and (1.1.6) contain all the relevant information to compute the value function and to design the optimal control strategy.

Unfortunately, the value function is not differentiable in general and therefore to make rigorous the previous approach, solutions to the HJB equations need to be considered in a weak sense. In this thesis, we will consider viscosity solutions (Chapter 2) and classical weak solutions (Chapters 3 and Chapter 4).

Weak solutions are well-known and we refer to classical books of Lions [83], Brezis [29] and Evans [49] for details. As far as viscosity solutions are concerned, the history is more recent. From the 1970's, there are considerable breakthroughs dealing with non-smooth value functions. We refer the reader to the books of Aubin & Cellina [12], Aubin & Frankowska [13], Clarke [40, 41, 42] and references therein. At the beginning of the 1980's, Crandall & Lions [46] and Crandall, Evans & Lions [44] introduced viscosity solutions, which appear to be well-adapted to solve PDEs like (1.1.5) and (1.1.6). Main references on the subject are Lions [85], Crandall, Ishii & Lions [45], Barles [17], Bardi & Capuzzo-Dolcetta [14], Fleming & Soner [50], Bardi, Crandall, Evans & Soner [15], Achdou, Barles, Ishii & Litvinov [3].

1.1.2 Mean field games

In this section, we shall give an overview introduction to the mean field games.

Recently, an important research activity on mean field games (MFG for short) has been initiated since the pioneering works [79, 80, 81] of Lasry & Lions. Related ideas have been developed independently in the engineering literature by Huang, Caines & Malhamé, see for example [70, 69, 68]. It aims at studying the asymptotic behavior of stochastic differential games (Nash equilibria) as the number N of agents tends to infinity. Previously, the concept was developed in economic literature under the terminology of heterogeneous agent models, see [10, 23, 71, 77]. In the asymptotic behavior of stochastic differential games, it is assumed that the agents are all identical and that an individual agent can hardly influence the outcome of the game. Moreover, each individual strategy is influenced by some averages of functions of the states of the other agents. In the limit when $N \to +\infty$, a given agent feels the presence of the others through the statistical distribution of the states. Since perturbations of the strategy of a single agent do not influence the statistical states distribution, the latter acts as a parameter in the control problem to be solved by each agent. The delicate question of the passage to the limit is one of the main topics of the book of Carmona & Delarue [38]. When the dynamics of the agents are independent stochastic processes, MFGs naturally lead to a coupled system of two partial differential equations (PDEs for short), a forward in time Kolmogorov or Fokker-Planck (FP) equation and a backward HJB equation. The unknown of this system is a pair of two functions:

- the value function of the stochastic optimal control problem solved by a representative agent. The associated SDE is (1.1.1) with $f = \alpha$, namely

$$dy = \alpha(s)ds + \sqrt{2\mu}\, dW_s,$$

- the density of the distribution of states.

In the infinite horizon limit, one obtains a system of two stationary PDEs. In classical control problems as described in Section 1.1.1, a single agent has his/her own fixed cost to minimize. In MFG, the model is generalized by allowing the cost of the representative agent to also depends on an interaction term between the agents: the cost functional depends on the probability density function m of all agents. More precisely, an typical agent controls the SDE (1.1.1) and we aim to minimize the following cost functional

$$J\left(x,t,\alpha\right) = \mathbb{E}_{x,t}\left[\int_t^T \left(\ell\left(y_{x,\alpha}\left(s\right),\alpha\left(s\right)\right) + \mathscr{V}\left[m\left(\cdot,s\right)\right]\left(y_{x,\alpha}(s)\right)\right)ds + v_T\left(y_{x,\alpha}\left(T\right),T\right)\right].$$

The interaction term $\mathscr{V}$ may have different meanings. If $\mathscr{V}$ is increasing, then the model means that the agent prefers to be away from the other agents, which leads to a repulsive effect. Conversely, a decreasing $\mathscr{V}$ leads to an attractive effect.

Let v be the value function of the problem. From Section 1.1.1, if v is smooth enough, then v is a solution of a viscous HJB equation

$$-\partial_t v - \mu\Delta v + H\left(x,\nabla v\right) = \mathscr{V}\left[m\right], \quad v\left(x,T\right) = v_T\left(x\right). \tag{1.1.7}$$

The equation (1.1.7) is a backward equation since the agents' decisions are based on their goals in the future. The optimal control is heuristically given in feedback form $\alpha^\star\left(x,t\right) = -\partial_p H\left(x,\nabla v\left(x,t\right)\right)$. Now if all agents argue in this way, their repartition will move with a velocity which is due, on the one hand, to the diffusion, and, one the other hand, on the drift term $-\partial_p H\left(x,\nabla v\left(x,t\right)\right)$. This leads to the FP equation

$$-\partial_t m + \mu\Delta m + div\left(\partial_p H\left(x,\nabla v\right)m\right) = 0, \quad m\left(x,0\right) = m_0. \tag{1.1.8}$$

The forward equation (1.1.8) represents where the agents actually end up, based on their initial distribution. To summarize, the non-stationary mean field games equation becomes

$$\begin{cases} -\partial_t v - \mu\Delta v + H\left(x,\nabla v\right) = \mathscr{V}\left[m\right] & \text{in } \mathbb{R}^n \times \left(0,T\right), \\ -\partial_t m + \mu\Delta m + div\left(\partial_p H\left(x,\nabla v\right)m\right) = 0 & \text{in } \mathbb{R}^n \times \left(0,T\right), \\ m\left(0\right) = m_0,\ v\left(x,T\right) = v_T\left(x\right). \end{cases} \tag{1.1.9}$$

Let us discuss the coupling term $\mathscr{V}$ in MFG system (1.1.9). When the coupling $\mathscr{V}$ is such that there is $F : \mathbb{R}^+ \to 0$ with $\mathscr{V}[m](x,t) = F(m(x,t))$, we say that $\mathscr{V}$ is a *local coupling*. But the coupling may also be *nonlocal*, for instance when $\mathscr{V}[m](x,t) = [m(\cdot,t) \star g(\cdot,t)](x)$ where $\star$ stands for the space convolution.

Since the seminal works of Lasry & Lions and Caines, Huang & Malhamé, the subject has been growing quickly. A very nice introduction to the theory of MFGs is supplied in the notes of Cardaliaguet [37] and Achdou [1]. We also refer to the survey paper of Gomes & Saúde [66] and the books of Bensoussan, Frehse & Yam [22], Gomes, Pimentel & Voskanyan [65] and Carmona & Delarue [39] for a general presentation.

1.2 Optimal control problems on networks and Hamilton-Jacobi equations on networks

In this section, we consider optimal control problems and HJ equations on networks and state the results obtained in Chapter 2. There are two main challenges that we have not encountered yet. Firstly, a network is a somewhat complicated space and we shall define the set of admissible controls which make the trajectories remained on networks. Secondly, the Hamiltonian on networks is generally defined branch by branch and so, it may be discontinuous at the vertices, which inhibits us from applying directly the classical techniques to deal with the HJ equations.

The first issue already appears in state constraints control problem and the second one is a natural issue in the case of Hamilton-Jacobi equations with discontinuities in space. We describe now these two cases before tackling the issue of Hamilton-Jacobi equations on networks, which is our main interest in Chapter 2.

1.2.1 State constrained optimal control problems

In optimal control problems with state constraints, we study the trajectories of the controlled dynamical system which are confined in a given set. More specifically, we add the following constraint to the problem (1.1.4):

$$y_{x,\alpha}(t) \in \mathcal{K}, \quad \text{for all } t \in [0,T] \quad \text{in the deterministic case } \mu = 0,$$

and

$$y_{x,\alpha}(t) \in \mathcal{K}, \quad \text{for all } t \in [0,T], \mathbb{P}\text{-a.e. in } \Omega \quad \text{in the stochastic case } \mu > 0,$$

where $\mathcal{K} \subset \mathbb{R}^d$ is a given closed subset. It means the set of admissible controls $\alpha : [0,T] \to A$ has to be restricted to the subset $\mathcal{T}_x$ which keeps the solution of (1.1.1) in $\mathcal{K}$. The value function becomes

$$v(x) = \inf_{\alpha(\cdot)\in\mathcal{T}_x} J(x,\alpha).$$

This induces a serious additional difficulty. For instance in the classical problem (1.1.4), the continuity of the value function is generally easy to obtain by the straightforward computation

$$v(x) - v(y) \leqslant \inf_{\alpha(\cdot)\in\mathcal{T}} J(x,\alpha) - \inf_{\alpha(\cdot)\in\mathcal{T}} J(y,\alpha) \leqslant \sup_{\alpha(\cdot)\in\mathcal{T}} \{J(x,\alpha) - J(y,\alpha)\}, \tag{1.2.1}$$

assuming classical continuity assumptions on J. Now, since $\mathcal{T}$ depends on the starting point of the trajectory, the above computation is not true anymore and continuity of v is more involved. The value function is no longer continuous unless a special controllability assumption is added to modify the dynamics on the boundary of state constraints.

The characterization of the value function as the unique solution of a HJB equation is also another big issue. In [95, 96], Soner characterized the value function of optimal control problems with state constraints as the unique constrained viscosity solutions of the related HJ equation, i.e., viscosity solution "inside $\mathcal{K}$" and viscosity supersolution on the boundary of $\mathcal{K}$. It was then developed by Capuzzo-Dolcetta & Lions [36] and Ishii & Koike [76]. As noted above, we need a special controllability assumption to ensure the continuity of the value function, like the "inward pointing qualification condition (IQ)" in [95]. It means that at each point of $\mathcal{K}$, there exists a control such that the dynamic points inward $\mathcal{K}$.

In such a case, the value function is the unique continuous viscosity solution to an appropriate HJB equation, see [95, 96, 76, 36] and the work of Motta [88]. However, in some cases, we do not have the condition IQ and hence the continuity of the value function is no longer ensured. The "outward pointing qualification condition (OQ)" was therefore introduced by Blanc [24], Frankowska & Plaskacz [54], which assume that every point on the boundary of $\mathcal{K}$ can be reached by a trajectory coming from the interior of $\mathcal{K}$. Under this assumption, one can characterize the value function as the discontinuous bilateral viscosity solution of a HJB equation. There are other works dealing with such kind of problems under weaker conditions, see Bokanowski, Forcadel & Zidani [27], Frankowska & Mazzola [52, 53].

1.2.2 HJB approach for problems with discontinuity in state

This section is concerned with optimal control problems and HJB equations on multi-domains, which means that the control problems and the corresponding Hamiltonians in each domains are different. In the interior of each domain, we have a classical HJB equation like (1.1.5) and (1.1.6). The point is to determine the relevant conditions at the interfaces, as in the state constraints problems at the boundary.

To explain more precisely the problem, we consider a simple example

$$\lambda v(x) + H(x, \nabla v(x)) = 0, \quad x \in \mathbb{R}, \tag{1.2.2}$$

where λ is a positive constant and the Hamiltonian $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$H(x,p) = \begin{cases} H_1(p), & \text{if } x < 0, \\ H_2(p), & \text{if } x \geqslant 0, \end{cases} \tag{1.2.3}$$

where H_1, H_2 are convex and coercive. The Hamiltonian H is definitely discontinuous at $x = 0$ if $H_1 \neq H_2$. In [74], Ishii introduces discontinuous viscosity solutions to solves this problem. A locally bounded function u is a discontinuous viscosity solution of (1.2.2) provided u^* is a subsolution and u_* is a supersolution in the following sense[1]. For all $x \in \mathbb{R}$ and $\varphi \in C^1(\mathbb{R})$ such that x is a maximum (resp. minimum) point of $u^* - \varphi$ (resp. $u_* - \varphi$), then

$$\begin{aligned} \lambda u^*(x) + H_1\left(\varphi'(x)\right) &\leqslant 0 \text{ (resp. } \lambda u_*(x) + H_1\left(\varphi'(x)\right) \geqslant 0), \qquad x < 0, \\ \lambda u^*(x) + H_2\left(\varphi'(x)\right) &\leqslant 0 \text{ (resp. } \lambda u_*(x) + H_2\left(\varphi'(x)\right) \geqslant 0), \qquad x > 0, \end{aligned}$$

and, if $x = 0$ then

$$\begin{aligned} \lambda u^*(x) + \min\left\{H_1\left(\varphi'(0)\right), H_2\left(\varphi'(0)\right)\right\} &\leqslant 0, \qquad x = 0, \\ (\text{resp. } \lambda u_*(x) + \max\left\{H_1\left(\varphi'(0)\right), H_2\left(\varphi'(0)\right)\right\} &\geqslant 0, \qquad x = 0), \end{aligned}$$

Note that the first conditions are classical viscosity inequalities in the open sets $\{x < 0\}$ and $\{x > 0\}$, whereas the last condition is a mixed condition at the interface. Here it means that at least one of the left or right inequality has to hold at $x = 0$.

To see the difficulty to obtain uniqueness, let us try to prove a comparison principle, i.e., to prove that any USC subsolution u is below any LSC supersolution v (uniqueness follows easily from such a result). Assuming by contradiction that $(u - v)(x_0) > 0$ for some x_0, we use the "doubling variable technique" (e.g., [3]), which consists in considering

$$0 < \sup_{x,y\in\mathbb{R}} \phi_\varepsilon(x,y), \quad \text{where} \quad \phi_\varepsilon(x,y) = u(x) - v(y) - \frac{|x-y|^2}{\varepsilon^2}.$$

[1] $u^*(x) = \limsup_{y\to x} u(y)$ is the upper-semicontinuous (USC) envelope of u and $u_*(x) = \liminf_{y\to x} u(y)$ is the lower-semicontinous (LSC) envelope of u.

We assume here for simplicity that the supremum is achieved at $(x_\varepsilon, y_\varepsilon)$. By the choice of the penalization term in ϕ_ε, it follows that x_ε and y_ε tend to the same $\bar{x}$ as $\varepsilon \to 0$.

In the easy case, when $\bar{x} \neq 0$, both x_ε and y_ε are in the same branch for small ε, for instance $x_\varepsilon, y_\varepsilon > 0$. Applying the definition of viscosity solution, we get

$$0 < \lambda u(x_\varepsilon) - \lambda v(y_\varepsilon) = \lambda u(x_\varepsilon) - \lambda v(y_\varepsilon) + H_2\left(\frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2}\right) - H_2\left(\frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2}\right) \leqslant 0,$$

which is a contradiction. But, if $\bar{x} = 0$ (meaning that 0 is a maximum point of $u - v$), then it may happen that x_ε and y_ε are in different branches for all ε, let us say $x_\varepsilon < 0$ and $y_\varepsilon > 0$. In this case, the definition of viscosity solution gives only

$$\lambda u(x_\varepsilon) - \lambda v(y_\varepsilon) + H_1\left(\frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2}\right) - H_2\left(\frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2}\right) \leqslant 0,$$

and we cannot conclude since $H_1\left(\frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2}\right) \neq H_2\left(\frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2}\right)$.

In this thesis, the problem is not specific to this example, but appears when the Hamiltonian is discontinuous with respect to the state variable. Hence, although a comparison principle for viscosity sub and supersolution is proved in [74], we will see later that a viscosity solution in the sense of Ishii is not enough to characterize a value function of an optimal control problem on networks. To overcome this difficulty, it is necessary to impose certain *transmission conditions* on the interfaces, which will lead to new notions of viscosity solutions.

Soravia [97, 98] and Soravia & Garavello [58, 59] were among the first to tackle this issue in the framework of viscosity solutions both from the HJB and optimal control point of view. In [58], the authors study an optimal control with a discontinuous cost with respect to the state variable, which leads to HJB equations with discontinuity in state. However, their value function is not the unique solution of this HJB equation. They can only characterize the minimal and the maximal solution of the HJB equation, using some sub- and super-optimality principles from the optimal control problem. Note that some sub- and super-optimality principles together with additional transmission conditions were later used in a fruitful way in [19, 20, 8] and in Chapter 2 (see below for details).

Some transmission conditions appear in the work of Bressan & Hong [28] about HJB equations in stratified domains. They introduce HJB tangential equations on the interfaces, which allow them to prove a comparison principle using control arguments.

After that, Barles, Briani & Chasseigne [19, 20] solve the problem (1.2.2)-(1.2.3) in general two-domains regional control problems. They consider two transmission conditions on the interface, so-called *singular* and *regular* dynamics, according to the behavior of dynamics on the interface. More precisely, let f_i and A_i be the dynamic and the control set corresponding to Ω_i ($\Omega_1 = (-\infty, 0] = \mathbb{R}^+ e_1$ and $\Omega_1 = [0, +\infty) = \mathbb{R}^+ e_2$). Then the singular case means $f_i(0, a_i) < 0$, while the regular case means $f_i(0, a_i) \geqslant 0$ for all $a_i \in A_i$. Then, there are two different value functions U^- and U^+. The first one is obtained when allowing all kind of controlled strategies (with singular and regular dynamic) while the last one is obtained by forbidding singular dynamics. The authors use the Ishii's notion of solutions and prove that U^- and U^+ are , respectively, the minimal and maximal solution of (1.2.2)-(1.2.3). Hence, they consider additional properties to obtain the characterization result. A comparison principle and a stability results for both value functions are also established. The idea for uniqueness comes from the fact that, in some senses the singular strategies are not encoded in the equations (1.2.2)-(1.2.3), while it is the case for the regular ones.

The article of Rao & Zidani [93] and Rao, Siconolfi & Zidani [92] are also in line with [28]. The problems studied are also close to those considered in [19, 20]. Indeed, the authors are interested

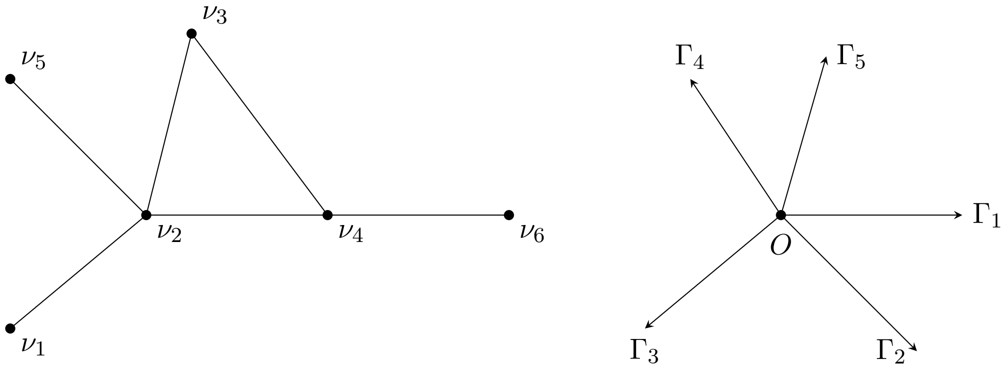

Figure 1.1: On left, a general network Γ; on right, a junction $\mathcal{G}$.

in problems where they consider different HJB equations in each region Ω_i of space instead of studying the equations at the junction. The authors propose a junction condition involving a Hamiltonian H^E which is called *essential Hamiltonian*. The equation with the essential Hamiltonian are stronger for the characterization of sub and supersolutions, but the value function satisfies this equation on the interfaces. Finally, in their work, the comparison principle between USC subsolutions and LSC supersolutions (they are continuous on the interface) is obtained.

Let us also mention some works dealing with HJ equations without using control arguments. In [34, 35], Camilli & Siconolfi propose a theory of HJ equations with Hamiltonians $H(x,p)$ only measurable with respect to the state variable, convex (or quasi-convex in the autonomous case) and coercive respect to p. Another important work is [43] in which Coclite & Risebro study the discontinuous HJ equations that may appear in 3-dimension reconstruction problems from shadows or the "shape from shading". Finally, in [60], Giga & Hamamuki study HJ equations with intermittent source term, which appears for example in training models of crystals.

1.2.3 Hamilton-Jacobi equations and optimal control on networks

In this section, we consider the optimal control problems on networks and the associated HJ equations. A network (or a graph) is a set of items, referred to as vertices (or nodes/crosspoints) which are denoted in this work by ν_i (in general case), or O (in junction case-a network has only one vertex). The connections between them referred to as edges, which are denoted by Γ_α, see example in Figure 1.1.

As explained in Section 1.2.2, to characterize the value function of the optimal control problem on network as the unique viscosity solution of a suitable HJB equation, one needs to find the right transition conditions at the vertices.

One of the first articles about optimal control problems on networks appeared in 2013. Achdou, Camilli, Cutri & Tchou [5] derived the HJB equation associated to an infinite horizon optimal control on a network and proposed a suitable notion of viscosity solution. At the same time, Imbert, Monneau & Zidani [73] proposed an equivalent notion of viscosity solution for studying a Hamilton-Jacobi approach to junction problems and traffic flows. Both [5] and [73] contain first results on comparison principles which were fundamental for several developments that follow. It is also worth mentioning the work by Schieborn & Camilli [94], in which the authors focus on eikonal equations on networks and on a less general notion of viscosity solution. We also refer to the work [33] where Camilli, Marchi & Schieborn study elliptic equations on the edges with Kirchhoff-type conditions at the vertices; after that, they prove the definition of solutions, which is defined in [94], is consistent with the vanishing viscosity method. In the particular case of eikonal equations, Camilli & Marchi [31] establish the equivalence between the definitions given in [5, 73, 94].

Since 2012, several proofs of comparison principles for HJB equations on networks, giving uniqueness of the solution, have been proposed.

1. Following [5], Achdou, Oudet & Tchou [8] prove the comparison principle for a stationary HJB equation arising from an optimal control with infinite horizon, by mixing arguments from the theory of optimal control and PDE techniques. Their proof was inspired by the works of Barles, Briani & Chasseigne [20, 19] on regional optimal control problems with discontinuous dynamics and costs.

2. A different and more general proof, using only arguments from the theory of PDEs was obtained by Imbert & Monneau in [72]. The proof works for quasi-convex Hamiltonians, and for stationary and time-dependent HJB equations. It relies on the construction of suitable *vertex test functions* which are designed to take into account the transition condition at the vertices.

3. A simple and elegant proof, working for non convex Hamiltonians, has been very recently given by Lions & Souganidis [86, 87].

The works [8] and [86, 87] have been particularly influential in our work in Chapter 2 and we will give more details in Section 1.2.4.

1.2.4 Results in Chapter 2: Hamilton-Jacobi equation on networks with switching costs

In Chapter 2, we consider an optimal control problem on a network the setting of which is close to [8]. In addition to this work, we suppose that there are entry (or exit) costs at the boundary of each edge of the network. Our goal is to characterize the value solution as the unique viscosity solution of an appropriate HJB equation.

For simplicity, we only consider a network with only one vertex from which start N semi-infinite straight edges. We call such a network $\mathcal{G}$ a junction. The edges are denoted by $(\Gamma_i)_{i=\overline{1,N}}$. See Figure 1.1 for an example of junction on $\mathbb{R}^2$.

We consider infinite horizon optimal control problems which have different dynamics and running costs in each edge. For $i = \overline{1,N}$, the control sets, dynamics and running cost corresponding to Γ_i are, respectively, denoted by A_i, f_i and ℓ_i. This means that, if a trajectory moves inside Γ_i, we use a control taking values in A_i, the velocity is f_i and we pay the running cost ℓ_i. Interestingly, if we come to O, it is possible to use a control taking values in all A_i.

Precise assumptions are a little technical so we refer the reader to the set of assumptions [H] in Chapter 2 for details. We prefer to give a flavour of the model.

We consider the infinite horizon optimal control problems (1.1.1) on the junction $\mathcal{G}$ with $\mu = 0$. Moving on $\mathcal{G}$, beside paying the cost as in [8], one has to pay the additional entry cost $c_i e^{-\lambda t_{ik}}$. Here c_i is a positive *entry costs* corresponding to Γ_i and $t_{ik} \in K_i$ is the time the trajectory enters $\Gamma_i \backslash \{O\}$. We define a cost functional with entry costs:

$$\mathcal{J}(x,\alpha) = \int_0^{+\infty} \ell\left(y_{x,\alpha}(\zeta), \alpha(\zeta)\right) e^{-\lambda t} dt + \sum_{i=1}^{N} \sum_{k \in K_i \subset \mathbb{N}} c_i e^{-\lambda t_{ik}} \quad \text{(cost functional with entry cost).}$$

Here $\mathcal{T}_x$ is the set of admissible controlled trajectories starting from $x \in \Gamma$ (it depends on x for the same reason as in Section 1.2.1). The value function of the infinite horizon optimal control problem with entry costs is defined by:

$$\mathrm{v}(x) = \inf_{\alpha \in \mathcal{T}_x} \mathcal{J}(x, \alpha) \quad \text{(value function with entry cost).}$$

We only focus on the case with entry costs here, but it is also possible to consider the case with exit costs, see Chapter 2. By the definition of the value function, we are only interested in control laws α such that $J(x, \alpha) < +\infty$. Hence, we only consider the case such that the state cannot switch edges infinitely many times in finite time, otherwise the cost functional is obviously infinite.

As noted above, our definition of the value function is similar to one made in [8] except the additional costs c_i for entering the edge Γ_i at O. This makes the value function possibly discontinuous contrary to [8] where it is continuous. For a better illustration, let us give the following simple example.

Example 1.2.1. Consider the same junction as in problem (1.2.2), $\mathcal{G} = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 = \mathbb{R}^+ e_1$ and $\Gamma_2 = \mathbb{R}^+ e_2$ where $e_1 = -1$ and $e_2 = 1$. The control sets are $A_i = [-1, 1] \times \{i\}$ with $i \in \{1, 2\}$. Set $f_i(x, (a_i, i)) = a_i$ and $\ell_1 \equiv 1, \ell_2(x, (a_2, 2)) = 1 - a_2$. An easy computation gives the explicit formula for the value function u without entry cost,

$$\mathsf{u}(x) = \begin{cases} 0, & \text{in } \Gamma_2, \\ \dfrac{1 - e^{-\lambda |x|}}{\lambda}, & \text{in } \Gamma_1. \end{cases}$$

Note that u is continuous. Now, if we add some positive entry costs c_1 and c_2 in the edges, the value function can again be computed:

1. If $c_2 \geqslant \dfrac{1}{\lambda}$, then
$$\mathsf{v}(x) = \begin{cases} 0 & \text{if } x \in \Gamma_2 \backslash \{O\}, \\ \dfrac{1}{\lambda} & \text{if } x \in \Gamma_1. \end{cases}$$

2. If $c_2 < \dfrac{1}{\lambda}$, then
$$\mathsf{v}(x) = \begin{cases} 0 & \text{if } x \in \Gamma_2 \backslash \{O\}, \\ \dfrac{1 - e^{-\lambda |x|}}{\lambda} + c_2 e^{-\lambda |x|} & \text{if } x \in \Gamma_1. \end{cases}$$

In this case, v is discontinuous, see Figure 1.2. From the above formulas, since $\ell_1 \geqslant \ell_2$, if the trajectory starts at $x \in \Gamma_2$, the bad strategy is moving to O, paying entry cost and entering Γ_1. Hence, c_1 does not appear in the formula for the value function v.

Before stating the main results, let us have a quick look at our assumption $[H]$. In this assumption, the control sets A_i are disjoint; the dynamics are bounded, Lipschitz continuous; the running cost are bounded uniformly continuous. Additionally, we suppose that a controllability assumption holds near O. It means that it is always possible to find a trajectory connecting two points sufficiently close to O. It follows that the restriction $\mathrm{v}|_{\Gamma_i \backslash \{O\}}$ of v is Lipschitz continuous and therefore it may be extended to O in each edge. This extension is denoted by v_i. Note that $v_i(0)$ may not be equal to $v_j(0)$ if $i \neq j$ since, in general, v is not continuous at O.

We are ready to introduce the first theorem.

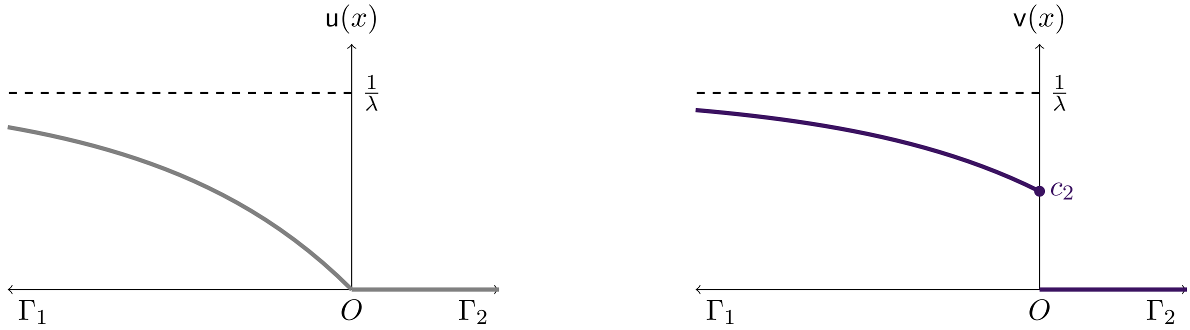

Figure 1.2: Left: The value function u with $\lambda = 1/4$. Right: The value function v with entry cost $c_2 = 2$ and $\lambda = 1/4$.

Theorem 1.2.2 (Theorem 2.2.9 and Lemma 2.2.11)**.** *Under assumption* $[H]$*,*

$$\max_{i=\overline{1,N}} \{v_i(O)\} \leqslant \mathrm{v}(O) = \min\left\{\min_{i=\overline{1,N}} \{v_i(O) + c_i\}, -\frac{H_O^T}{\lambda}\right\}, \tag{1.2.4}$$

where

$$H_O^T = \max_{i=\overline{1,N}} \max_{a_i \in A_i^O} \{-\ell_j(O, a_j)\} = -\min_{i=\overline{1,N}} \min_{a_i \in A_i^O} \{\ell_j(O, a_j)\}. \tag{1.2.5}$$

This theorem makes the link between the value of the original value function v and the values of the extensions v_i at the junction. The equality in (1.2.4) means that if the trajectory begins at O, the optimal strategy is either to stay at O for all time or to enter immediately the edge Γ_i that has the lowest possible cost.

Theorem 1.2.3. *(Theorem 2.6.7) Let* v *be a value function with entry cost. Then* $(v_1, \ldots, v_N)$ *is a viscosity solution of the following Hamilton-Jacobi system*

$$\begin{cases} \lambda u_i(x) + H_i\left(x, \dfrac{du_i}{de_i}(x)\right) = 0 & \text{if } x \in \Gamma_i \backslash \{O\}, i = \overline{1,N}, \\ \lambda u_i(O) + \max\left\{-\lambda \min_{j \neq i} \{u_j(O) + c_j\}, H_i^+\left(O, \dfrac{du_i}{dx_i}(O)\right), H_O^T\right\} = 0 & \text{if } x = O. \end{cases} \tag{1.2.6}$$

Inside every edge $\Gamma_i \backslash \{O\}$, we recognize a classical Hamilton-Jacobi with a Hamiltonian H_i corresponding to the edge Γ_i, namely

$$H_i(x,p) = \max_{a \in A_i} \{-p f_i(x,a) - \ell_i(x,a)\}.$$

Here, the Hamiltonian at O with respect to i is defined by

$$H_i^+(O,p) = \max_{a \in A_i^+} \{-p f_i(x,a) - \ell_i(x,a)\},$$

where $A_i^+ = \{a \in A_i : f_i(O,a) \geqslant 0\}$. At the vertex O, there is a special Hamiltonian or transmission condition, which takes into account all the possible strategies near the vertex:

- The term $\min_{j \neq i} \{u_j(O) + c_j\}$ accounts for situations in which the trajectory enters Γ_{i_0} where $u_{i_0}(O) + c_{i_0} = \min_{j \neq i} \{u_j(O) + c_j\}$.

- The term $H_i^+\left(O, \dfrac{du_i}{dx_i}(O)\right)$ accounts for situations in which the trajectory does not leave Γ_i.
- The term H_O^T accounts for situations in which the trajectory stays at O.

Sketch of proof of Theorem 1.2.3. From (1.2.4) and (1.2.5), it suffices to prove that $v_i(O)$ satisfies

$$\lambda v_i(O) + H_i^+\left(O, \frac{dv_i}{dx_i}(O)\right) \leqslant 0, \quad \text{in the viscosity sense.}$$

Let $a \in A_i^+$ such that $f_i(O,a) > 0$. For all $x \in \Gamma_i$ and near O, using the strong controllability at O, we get

$$\lambda v_i(x) - f_i(x,a)\frac{dv_i}{dx_i}(x,a) - \ell_i(x,a) \leqslant 0, \quad \text{in the viscosity sense,}$$

Let $x \to O$ and the proof for subsolution is done.

The characterization for supersolutions is more difficult. We need to prove the following property (which is Lemma 2.4.3 in Chapter 2): if

$$v_i(O) < \min\left\{\min_{j\neq i}\{v_j(O) + c_j\}, -\frac{H_O^T}{\lambda}\right\}, \tag{1.2.7}$$

then there exist a fixed time $\bar{\tau} > 0$ such that for any $x \in (\Gamma_i \backslash \{O\})$ and near O, we can find an "almost optimal" control which makes the cost functional close to $v(O)$. Moreover, with this control, the trajectory starts at x still remains on $\Gamma_i \backslash \{O\}$ in the time interval $[0, \bar{\tau}]$. Using this property and applying [89, the proof of Lemma 2.2], we deduce the viscosity super-inequality for $v(x)$ and $H_i^+(x,p)$ where $x \in \Gamma_i \backslash \{O\}$ is near O. Let $x \to O$ and the proof for supersolution is done. □

The main result of Chapter 2 is

Theorem 1.2.4 (Comparison Principle, Theorem 2.5.6)**.** *Under the hypothesis* $[H]$*, let* $u = (u_1, \dots, u_N)$ *be a bounded viscosity subsolution of* (1.2.6) *and* $w = (w_1, \dots, w_N)$ *be a bounded viscosity supersolution of* (1.2.6)*; then* $u \leqslant w$ *in* $\mathcal{G}$*, namely* $u_i \leqslant w_i$ *in* Γ_i *for all* $i = \overline{1,N}$.

We provide two different proofs of this theorem, see a sketch of proof below. The first proof is inspired from [8]. The authors focus on optimal control problems with independent dynamics and running costs in the edges, and after that they show that some arguments of [19] can be adapted to yield a simple proof of a comparison principle. The second proof is based on the work of Lions & Souganidis [86] and uses only PDEs tools. More specifically, the authors build a simple but very useful test-function on networks, and adapt it to doubling variable technique.

Let us mention that the existence of a unique viscosity solution follows directly from the comparison principle and Theorem 2.6.7. It is also possible to build a viscosity solution directly from the comparison principle and Perron's method, as in [86].

To prove the comparison principle, we start as in the example in Section 1.2.2. We assume by contradiction that $u_i(x_0) - w_i(x_0) > 0$ for some i and x_0. If the $u_i - w_i$ attains the supremum inside Γ_i, the proof is done by classical arguments. Hence, we only focus on the case

$$u_i(O) - w_i(O) = \max_{x\in\Gamma_i}\{u_i(x) - w_i(x)\} > 0, \tag{1.2.8}$$

for some i.

A sketch of proof of Theorem 1.2.4 inspired by Achdou, Oudet & Tchou [8] . The main idea is that: from (1.2.4), for all j, u_j is a viscosity subsolution of the following equations

$$\begin{cases} \lambda v_j(x) + H_j\left(x, \dfrac{dv_j}{dx_j}(x)\right) & = 0 \quad \text{if } x \in \Gamma_j \backslash \{O\}, \\ \lambda v_i(O) + H_j^+\left(O, \dfrac{dv_j}{dx_i}(O)\right) & = 0 \quad \text{if } x = O. \end{cases} \tag{1.2.9}$$

and the comparison principle for (1.2.9) is proved in [8]. Thus, to obtain the contradiction with (1.2.8), it suffices to prove that w_i is a viscosity supersolution of (1.2.9) with j replaced by i. Now, since $\lambda u_i(O) + H_O^T \leqslant 0$, we have $\lambda w_i(O) + H_O^T < 0$. We now consider the two following cases.

Case 1: If $w_i(O) < \min_{j\neq i}\{w_j(O) + c_j\}$, by (1.2.6), then w is a viscosity supersolution of (1.2.9) and it leads us to a contradiction.

Case 2: If $w_i(O) \geqslant \min_{j\neq i}\{w_j(O) + c_j\}$, then there exists $j_0 \neq i$ such that

$$w_{j_0}(O) + c_{j_0} = \min_{j=\overline{1,N}}\{w_j(O) + c_j\} = \min_{j\neq i}\{w_j(O) + c_j\} \leqslant w_i(O).$$

Therefore, $w_{j_0}(O) < \min_{j\neq j_0}\{w_j(O) + c_j\}$. We also have $\lambda u_i(O) - \lambda \min_{j\neq i}\{u_j(O) + c_j\} \leqslant 0$. Thus $w_{j_0}(O) < u_{j_0}(O)$. Repeating the proof of *Case 1* with j_0, we reach a contradiction. □

A sketch of proof of Theorem 1.2.4 inspired by Lions & Souganidis [86]. As in the example in Section 1.2.2, the key idea of this proof is building an admissible test-function φ_ε such that $u(x) - w(y) - \varphi_\varepsilon(x,y)$ attains a maximum point $(x_\varepsilon, y_\varepsilon) \in \Gamma_i \times \Gamma_i$, then using the viscosity inequalities to obtain the contradiction. We consider the function

$$\begin{aligned} \Psi_{i,\varepsilon} : \Gamma_i \times \Gamma_i &\longrightarrow \mathbb{R} \\ (x,y) &\longrightarrow u_i(x) - w_i(y) - \frac{1}{2\varepsilon}\left[-|x| + |y| + (L+1)\varepsilon\right]^2 - \gamma\left(|x| + |y|\right). \end{aligned}$$

This implies that $\Psi_{i,\varepsilon}$ attains its maximum at $(x,y) \in \Gamma_i \times \Gamma_i$. Using the viscosity inequality

$$u_i(x) - w_i(y) \leqslant H_i\left(y, \frac{-x + y + \delta(\varepsilon)}{\varepsilon} + \gamma\right) - H_i\left(x, \frac{-x + y + \delta(\varepsilon)}{\varepsilon} - \gamma\right). \tag{1.2.10}$$

Let ε tend to 0 and γ tend to 0, we obtain that $u_i(O) - w_i(O) \leqslant 0$, the desired contradiction. □

1.3 Mean field games on networks

The section is devoted to the introduction of our works on infinite horizon (Chapter 3) and finite horizon (Chapter 4) Mean Field Games on general bounded networks. For simplicity, in this section, we only consider the model case of the junction $\mathcal{G}$ with N bounded edges. The other endpoint of Γ_i is denoted by ν_i and the edges Γ_i are oriented from ν_i to O.

Similarly to the problems in Section 1.2.3, what makes the MFG on networks more challenging comparing to classical MFG (Section 1.1.2) is stochastic optimal control problems with state constraints and discontinuity in states. To deal with this, one needs to answer the important question: What is a suitable transition condition at the vertices?

Camilli & Marchi [32] is one of the first articles on infinite horizon MFGs on networks. They consider a particular type of Kirchhoff condition at the vertices for the value function. This condition comes from an assumption which can be informally stated as follows: if, at time τ, the controlled stochastic process X_t associated to a given agent hits O, then the probability that X_{τ^+} belongs to Γ_i is proportional to the diffusion coefficient in Γ_i. Under this assumption, it can be seen that the density of the distribution of states m is continuous at O. In our work, the assumption mentioned above is no longer valid. Therefore, it will be seen later that the value function v satisfies more general Kirchhoff conditions, and accordingly, the density of the distribution of states m is no longer continuous at O; the continuity condition is then replaced by suitable compatibility conditions on the jumps across the vertex. Accordingly, the weak solutions spaces V and W of the uncoupled HJB and FP equations (see (1.3.19)-(1.3.20)) are not the same. To overcome this difficulty, it is essential to consider an isomorphism from V to W, which is then used to build a suitable test-function for each uncoupled equation. See more details in Section 3.1.4.

Finally, under suitable assumptions, we will prove the existence, uniqueness and regularity for the both MFG systems. See Section 3.4 and Section 4.4.1.

1.3.1 Derivation of Mean field games system on networks

In this section, we consider the derivations of both cases: infinite horizon and finite horizon MFG systems on $\mathcal{G}$. The main ideas for both systems are quite similar. However, the technique for finite horizon MFG system is more difficult than the other one. Hence, in this section, we mainly focus on the derivation of finite horizon MFG system.

Consider a real valued function $a \in PC(\mathcal{G})$, where $PC(\mathcal{G})$ contains all piecewise functions on $\mathcal{G}$ which are continuous except at O where it can be extended by continuity. Let us consider the linear partial differential operator:

$$\mathcal{L}u\,(x) = \mathcal{L}_i u\,(x) := \mu_i \partial^2 u\,(x) + a_i\,(x)\,\partial u\,(x)\,, \quad \text{if } x \in \Gamma_i, \tag{1.3.1}$$

with domain

$$D\,(\mathcal{L}) := \left\{ u \in C^2\,(\mathcal{G}) : \sum_{i=1}^{N} \gamma_i \mu_i \partial_i u\,(O) = 0, \;\; \text{for all } i = \overline{1,N} \right\}. \tag{1.3.2}$$

Freidlin and Sheu proved in [55] that the operator $\mathcal{L}$ is the infinitesimal generator of a Feller-Markov process on $\mathcal{G}$ with continuous sample paths. The operators $\mathcal{L}_i$ and the transmission conditions at the vertices

$$\sum_{i=1}^{N} \gamma_i \mu_i \partial_i u(O) = 0 \tag{1.3.3}$$

define such a process in a unique way, see also [56, Theorem 3.1]. The process can be written (X_t, i_t) where $X_t \in \Gamma_{i_t}$. Moreover, there exist

1. a one dimensional Wiener process W_t,
2. continuous non-decreasing processes $\ell_{i,t}$, $i = \overline{1,N}$, which are measurable with respect to the σ-field generated by (X_t, i_t),
3. continuous non-increasing processes $h_{i,t}$, $i = \overline{1,N}$, which are measurable with respect to the σ-field generated by (X_t, i_t), such that

$$\begin{aligned} dx_t = \sqrt{2\mu_{i_t}} dW_t + a_{i_t}(x_t)dt + d\ell_{i,t} + dh_{i,t}, \\ \ell_{i,t} \text{ increases only when } X_t = O, \\ h_{i,t} \text{ decreases only when } X_t = \nu_i, \end{aligned} \tag{1.3.4}$$

and for all function $v \in C^{2,1}(\mathcal{G} \times [0,T])$ such that

$$\sum_{i=1}^{N} p_{ia}\partial_i v(O,t) = 0, \quad \partial_i v(\nu_i,t) = 0, \quad i = \overline{1,N}, t \in [0,T],$$

the process

$$M_t = v(X_t,t) - \int_0^t \big(\partial_t v(X_s,s) + \mu_{i_s}\partial^2 v(X_s,s) + a_{i_s}(X_s,s)\partial v(X_s,s)\big)\, ds \tag{1.3.5}$$

is a martingale, namely

$$\mathbb{E}(M_t|X_s) = M_s, \quad \text{for all } 0 \leqslant s < t \leqslant T. \tag{1.3.6}$$

The goal is to derive the boundary value problem satisfied by the law of the stochastic process X_t. Since the derivation here is formal, we assume that the law of the stochastic process X_t is a measure which is absolutely continuous with respect to the Lebesgue measure on $\mathcal{G}$ and regular enough so that the following computations make sense. Let $m(x,t)$ be its density. We have

$$\mathbb{E}[v(X_t,t)] = \int_{\mathcal{G}} v(x,t)\, m(x,t)\, dx, \quad \text{for all } v \in PC(\mathcal{G} \times [0,T]). \tag{1.3.7}$$

Consider $u \in C^{2,1}(\mathcal{G} \times [0,T])$ such that for all $t \in [0,T]$, $u(\cdot,t) \in D(\mathcal{L})$. Then from (1.3.5)–(1.3.6), we see that

$$\mathbb{E}[u(X_t,t)] = \mathbb{E}[u(X_0,0)] + \mathbb{E}\left[\int_0^t \big(\partial_t u(X_s,s) + \mu_{i_s}\partial^2 u(X_s,s) + a_{i_s}(X_t,t)\partial u(X_t,t)\big)\right].$$

Taking the time-derivative of each member of (1.3.7), we obtain

$$\int_{\mathcal{G}} \partial_t(um)(x,t)\, dx = \mathbb{E}\big(\partial_t u(X_t,t) + \mu_{i_s}\partial^2 u(X_t,t) + a_{i_s}(X_t,t)\partial u(X_t,t)\big).$$

Using again (1.3.7), we get

$$\int_{\mathcal{G}} \big(\mu\partial^2 u(x,t) + a(x,t)\partial u(x,t)\big)\, m(x,t)dx = \int_{\mathcal{G}} u(x,t)\partial_t m(x,t)\, dx.$$

By integration by parts, we get

$$\begin{aligned}
0 \quad = \quad & \sum_{i=1}^{N}\int_{\Gamma_i} \big(\partial_t m(x,t) - \mu_i\partial^2 m(x,t) + \partial(am)(x,t)\big)\, u(x,t)dx \\
& + \sum_{i=1}^{N} \left[a|_{\Gamma_i}(\nu_i,t)m|_{\Gamma_i}(\nu_i,t) - \mu_i\partial_i m(\nu_i,t)\right] u|_{\Gamma_i}(\nu_i,t) \\
& - \sum_{i=1}^{N} \left[a|_{\Gamma_i}(O,t)m|_{\Gamma_i}(O,t) + \mu_i\partial_i m(O,t)\right] u|_{\Gamma_i}(O,t) \\
& - \sum_{i=1}^{N} \mu_i m|_{\Gamma_i}(O,t)\partial_i u(O,t).
\end{aligned} \tag{1.3.8}$$

We choose first, for $i = \overline{1, N}$, a smooth function u which is compactly supported in $(\Gamma_i \backslash \{O, \nu_i\}) \times [0, T]$. Hence $u|_{\Gamma_i}(\nu_i, t) = u|_{\Gamma_i}(O, t) = 0$ and $\partial_i u(\nu_i, t) = \partial_i u(O, t) = 0$ for $i = \overline{1, N}$. Notice that $u(\cdot, t) \in D(\mathcal{L})$. It follows that m satisfies

$$\left(\partial_t m - \mu_i \partial^2 m + \partial\left(ma\right)\right)(x, t) = 0, \quad \text{for } x \in \Gamma_i \backslash \{O\},\ t \in (0, T),\ i = \overline{1, N}. \tag{1.3.9}$$

For a smooth function $\chi : [0, T] \to \mathbb{R}$ compactly supported in $(0, T)$, we may choose for every $i \in \{1, \ldots, N\}$, a smooth function u such that $u(\nu_j, t) = \chi(t)\delta_{i,j}$ and $u(O, t) = 0$ for all $t \in [0, T]$, $j = \overline{1, N}$ and $\partial_j u(O, t) = 0$ for all $t \in [0, T]$, $j = \overline{1, N}$, we infer a boundary condition for m

$$a|_{\Gamma_i}(\nu_i, t) m|_{\Gamma_i}(\nu_i, t) - \mu_i \partial_i m(\nu_i, t) = 0, \qquad i = \overline{1, N},\ t \in (0, T).$$

Next, we choose a smooth function u such that $\partial_j u(O, t) = 0$ for all $t \in [0, T]$, $j = \overline{1, N}$, we infer a condition for m at O:

$$\sum_{i=1}^{N} a|_{\Gamma_i}(O, t) m|_{\Gamma_i}(O, t) + \mu_i \partial_i m(O, t) = 0, \qquad t \in (0, T).$$

Finally, for a smooth function $\chi : [0, T] \to \mathbb{R}$ compactly supported in $(0, T)$, we choose u such that

- $u(\cdot, t) \in D(\mathcal{L})$
- $\partial_i u(O, t) = \chi(t)/p_i$, $\partial_j u(O, t) = -\chi(t)/p_j$, $\partial_k u(O, t) = 0$ if $k \neq i, j$.

Using such a test-function in (1.3.8) yields a jump condition for m,

$$\frac{m|_{\Gamma_i}(O, t)}{\gamma_i} = \frac{m|_{\Gamma_j}(O, t)}{\gamma_j}, \quad i, j = \overline{1, N},\ t \in (0, T),$$

in which $\gamma_i = p_i/\mu_i$.

Summarizing, we get the following boundary value problem for m:

$$\begin{cases} \partial_t m - \mu_i \partial^2 m + \partial\left(ma\right) = 0, & (x, t) \in (\Gamma_i \backslash \{O\}) \times (0, T),\ i = \overline{1, N}, \\ \displaystyle\sum_{i=1}^{N} \mu_i \partial_i m\left(O, t\right) + a|_{\Gamma_i}(O, t) m|_{\Gamma_i}\left(O, t\right) = 0, & t \in (0, T), \\ \mu_i \partial_i m\left(\nu_i, t\right) - a|_{\Gamma_i}(\nu_i, t) m|_{\Gamma_i}\left(\nu_i, t\right) = 0, & t \in (0, T), i = \overline{1, N}, \\ \dfrac{m|_{\Gamma_i}(O, t)}{\gamma_i} = \dfrac{m|_{\Gamma_j}(O, t)}{\gamma_j}, & t \in (0, T),\ i, j = \overline{1, N}, \\ m(x, 0) = m_0(x), & x \in \mathcal{G}. \end{cases} \tag{1.3.10}$$

Consider a continuum of indistinguishable agents moving on the network $\mathcal{G}$. Under suitable assumptions, the theory of MFGs asserts that the distribution of states is absolutely continuous with respect to Lebesgue measure on $\mathcal{G}$. Hereafter, m stands for the density of the distribution of states: $m \geqslant 0$ and $\int_{\mathcal{G}} m(x, t) dx = 1$ for $t \in [0, T]$.

The state of a representative agent at time t is a time-continuous controlled stochastic process X_t in $\mathcal{G}$, as defined previously, where the control is the drift a_t, supposed to be of the form $a_t = a(X_t, t)$.

For a representative agent, the optimal control problem is of the form

$$v\left(x, t\right) = \inf_{a} \mathbb{E}_{xt} \left[\int_t^T \left(L\left(X_s, a_s\right) + \mathscr{V}\left[m(\cdot, t)\right]\left(X_s\right)\right) ds + v_T\left(X_T\right) \right], \tag{1.3.11}$$

where $\mathbb{E}_{xt}$ stands for the expectation conditioned by the event $X_t = x$. The functions and operators involved in (1.3.11) will be described below.

Let us assume that there is an optimal feedback law, i.e., a function $a^\star$ defined on $\mathcal{G} \times [0,T]$ which is sufficiently regular in the edges of $\mathcal{G}$ such that the optimal control at time t is given by $a_t^\star = a^\star(X_t, t)$. An informal way to describe the behavior of the process at the vertices is as follows: if X_t hits O, then it enters Γ_i, $i = \overline{1,N}$ with probability $p_i > 0$.

Let us discuss the ingredients in (1.3.11). The running cost depends separately on the control and on the distribution of states. The contribution of the distribution of states involves the coupling cost operator $\mathscr{V}$, which may be either nonlocal and regularizing, i.e., $\mathscr{V} : \mathcal{P}(\mathcal{G}) \to \mathcal{C}^2(\mathcal{G})$ for example, or local, i.e. $\mathscr{V}[m](x) = F(m(x))$ where $F : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function.

The contribution of the control involves the Lagrangian L, i.e., a real valued function defined on $(\cup_{i\in\mathcal{A}}\Gamma_i\backslash\mathcal{V}) \times \mathbb{R}$. If $x \in \Gamma_i\backslash\mathcal{V}$ and $a \in \mathbb{R}$, $L(x,a) = L_i(x,a)$, where L_i is a continuous real valued function defined on $\Gamma_i \times \mathbb{R}$. We assume that $\lim_{|a|\to\infty} \inf_{x\in\Gamma_i} \frac{L_i(x,a)}{|a|} = +\infty$. The last one is the terminal cost v_T. Further assumptions on L, $\mathscr{V}$ and v_T will be made below.

Under suitable assumptions, Ito calculus as in [56, 55] and the dynamic programming principle lead to the following HJB equation on $\mathcal{G}$, more precisely the following boundary value problem:

$$\begin{cases} -\partial_t v - \mu_i \partial^2 v + H(x, \partial v) = \mathscr{V}[m(\cdot,t)](x), & \text{in } (\Gamma_i\backslash\{O\}) \times (0,T), i = \overline{1,N}, \\ v|_{\Gamma_i}(O,t) = v|_{\Gamma_j}(O,t) & t \in (0,T)\, i,j = \overline{1,N}, \\ \sum_{i=1}^{N} \gamma_i \mu_i \partial_i v(O,t) = 0, & t \in (0,T), \\ \partial_i v(\nu_i, t) = 0, & t \in (0,T), i = \overline{1,N}, \\ v(x,T) = v_T(x), & x \in \mathcal{G}. \end{cases} \tag{1.3.12}$$

We refer to [79, 81] for the interpretation of the value function v. Let us comment the different equations in (1.3.12):

1. The Hamiltonian H is a real valued function defined on $\left(\cup_{i=1}^N \Gamma_i\backslash\{O\}\right) \times \mathbb{R}$. For $x \in \Gamma_i\backslash\{O\}$ and $p \in \mathbb{R}$,
$$H(x,p) = \sup_a \{-ap - L_i(x,a)\},$$
The Hamiltonians $H|_{\Gamma_i\times\mathbb{R}}$ are supposed to be C^1 and coercive with respect to p uniformly in x.

2. The second condition means in particular that v is continuous at the vertices.

3. The third equation in (1.3.12) is a Kirchhoff transmission condition (or Neumann boundary condition if $\nu_i \in \partial\mathcal{G}$); it is the consequence of the assumption on the behavior of X_s at vertices.

If (1.3.11) has a smooth solution, it provides a feedback law for the optimal control problem, i.e.,
$$a^\star(x,t) = -\partial_p H(x, \partial v(x,t)).$$
According to the previous part, the density $m(x,t)$ of the law of the optimal stochastic process X_t satisfies (1.3.10) (where a is replaced by $a^\star$). Finally, replacing $a^\star(x,t)$ by the value

$-\partial_p H(x, \partial v(x,t))$, we obtain the system

$$\begin{cases}-\partial_t v-\mu_i \partial^2 v+H(x,\partial v)=\mathscr{V}[m], & t\in(0,T),\ x\in\Gamma_i\backslash\{O\},\ i=\overline{1,N},\\ \partial_t m-\mu_i\partial^2 m+\partial\left(m\partial_p H(x,\partial v)\right)=0, & t\in(0,T),\ x\in\Gamma_i\backslash\{O\},\ i=\overline{1,N},\\ \sum_{i=1}^N \gamma_i\mu_i\partial_i v(O,t)=0, & t\in(0,T),\\ \sum_{i=1}^N\Big[\mu_i\partial_i m(O,t)+\partial_p H_i\Big(O,\partial_i v(O,t)\Big)m_i(O,t)\Big]=0, & t\in(0,T),\\ \partial_i v(\nu_i,t)=0,\ \mu_i\partial_i m(\nu_i,t)+\partial_p H_i\Big(\nu_i,\partial_i v(\nu_i,t)\Big)m_i(\nu_i,t)=0, & t\in(0,T),\ i=\overline{1,N},\\ v_i(O,t)=v_j(O,t),\ \dfrac{m_i(O,t)}{\gamma_i}=\dfrac{m_j(O,t)}{\gamma_j}, & t\in(0,T),\ i,j=\overline{1,N},\\ v(x,T)=v_T(x),\ m(x,0)=m_0(x),\ \displaystyle\int_{\mathcal{G}} m_0(x)dx=1, & x\in\mathcal{G}.\end{cases} \tag{1.3.13}$$

At a vertex O, the transmission conditions for both v and m consist of N linear relations, which is the appropriate number of relations to have a well posed problem. If $\nu_i\in\partial\mathcal{G}$, there is of course only one condition.

To end this part, we introduce the infinite horizon MFG system. Since the idea and the technique for derivation of this system are similar to (even simpler than) the finite case, we shall not explain them in details.

The first goal is also to derive the boundary value problem satisfying the law of the stochastic process X_t. Consider the invariant measure associated with the process X_t and assume that it is absolutely continuous with respect to the Lebesgue measure on $\mathcal{G}$. Let m be its density:

$$\mathbb{E}[u(X_t)]:=\int_{\mathcal{G}} u(x)m(x)dx, \quad \text{for all } u\in PC(\mathcal{G}). \tag{1.3.14}$$

Taking the time-derivative of each member of (1.3.14), choosing appropriate test-functions $u\in D(\mathcal{L})\subset PC(\mathcal{G})$ step by step as in the finite case, we get the following boundary value problem for m

$$\begin{cases}-\mu_i\partial^2 m-\partial(bm)=0, & \text{in } \Gamma_i\backslash\{O\},\\ \dfrac{m_i(O)}{\gamma_i}=\dfrac{m_j(O)}{\gamma_j}, & i,j=\overline{1,N},\\ \sum_{i=1}^N\left[b(O)\,m_{\Gamma_i}(O)+\mu_i\partial_i m(O)\right]=0, & \\ \mu_i\partial_i m(\nu_i)+\partial_p H_i\Big(\nu_i,\partial_i v(\nu_i)\Big)m_i(\nu_i)=0, & i=\overline{1,N},\end{cases} \tag{1.3.15}$$

In the ergodic case, we aim to minimize the average cost

$$\rho:=\inf_{a_s}\liminf_{T\to+\infty}\frac{1}{T}\mathbb{E}_x\left[\int_0^T L(X_s,a_s)+\mathscr{V}[m(\cdot,s)(X_s)]\,ds\right], \tag{1.3.16}$$

where $\mathbb{E}_x$ stands for the expectation conditioned by the event $X_0=x$. The functions and operators involved in (1.3.16) were described in the derivation for finite horizon MFG system above. Under suitable assumptions, the Ito calculus and the dynamic programming principle lead to the following ergodic HJ equation on $\mathcal{G}$

$$\begin{cases} -\mu_i \partial^2 v + H(x, \partial v) + \rho = f, & \text{if } x \in \Gamma_i \backslash \{O\}, \\ v_i(O) = v_j(O), & i, j = \overline{1, N}, \\ \sum_{i=1}^{N} \gamma_i \mu_i \partial_i v(O) = 0, \partial_i v(\nu_i) = 0, & i = \overline{1, N}. \end{cases} \tag{1.3.17}$$

If (1.3.16) has a smooth solution, then it provides a feedback law for the optimal control problem, i.e., $a^\star(x) = -\partial_p H(x, \partial v(x))$. At the MFG equilibrium, m is the density of the invariant measure associated with the optimal control feedback law, so it satisfies (1.3.15), where a is replaced by $a^\star$. To summarize, we get the following system

$$\begin{cases} -\mu_i \partial^2 v + H(x, \partial v) + \rho = \mathscr{V}[m], & x \in \Gamma_i \backslash \{O\}, i = \overline{1, N}, \\ \mu_i \partial^2 m + \partial(m \partial_p H(x, \partial v)) = 0, & x \in \Gamma_i \backslash \{O\}, i = \overline{1, N}, \\ \sum_{i=1}^{N} \gamma_i \mu_i \partial_i v(O) = 0, & \\ \sum_{i=1}^{N} \left[\mu_i \partial_i m(O) + \partial_p H_i \Big(O, \partial_i v(O)\Big) m_i(O) \right] = 0, & \\ \partial_i v(\nu_i) = 0, \mu_i \partial_i m(\nu_i) + \partial_p H_i \Big(\nu_i, \partial_i v(\nu_i)\Big) m_i(\nu_i) = 0, & i = \overline{1, N} \\ v_i(O) = v_j(O), \dfrac{m_i(O)}{\gamma_i} = \dfrac{m_j(O)}{\gamma_j}, & i, j = \overline{1, N}, \\ \int_{\mathcal{G}} v(x)\, dx = 0, \quad \int_{\mathcal{G}} m(x)\, dx = 1, \quad m \geqslant 0. & \end{cases} \tag{1.3.18}$$

1.3.2 Results in Chapter 3: A Class of Infinite Horizon Mean Field Games on Networks

After obtaining the transmission conditions at the O for both the value function and the density, we shall define a weak solution for MFG system in suitable Sobolev spaces on networks as follows:

$$V := H^1(\mathcal{G}) = \left\{ v \in C(\mathcal{G}) : v_i \in H^1(\Gamma_i) \text{ for } i = \overline{1, N} \right\}, \tag{1.3.19}$$

$$W := \left\{ w \in PC(\mathcal{G}) : w_i \in H^1(\Gamma_i) \text{ and } \frac{w_i(O)}{\gamma_i} = \frac{w_j(O)}{\gamma_j} \text{ for all } i, j = \overline{1, N} \right\}. \tag{1.3.20}$$

These functions spaces are indeed suitable since if we multiply the first equation in (1.3.18) with $w \in W$, integrate over $\mathcal{G}$ and use integration by part for each Γ_i, one gets

$$\sum_{i=1}^{N} \int_{\Gamma_i} (\mu_i \partial v \partial w + H_i(x, \partial v) w + \rho w)\, dx - \sum_{i=1}^{N} \mu_i w_i(\nu_i) \partial_i v(\nu_i) + \sum_{i=1}^{N} \mu_i w_i(O) \partial_i v(O) = 0. \tag{1.3.21}$$

The second term is 0 because of Neumann boundary condition. By the property of the functions space W, the last term becomes

$$\sum_{i=1}^{N} \mu_i \gamma_i \frac{w_i(O)}{\gamma_i} \partial_i v(O) = \frac{w_1(O)}{\gamma_1} \sum_{i=1}^{N} \mu_i \gamma_i \partial_i v(O),$$

and it also vanishes by the Kirchhoff condition. Thus, from (1.3.21), one gets

$$\sum_{i=1}^{N}\int_{\Gamma_i}\left(\mu_i\partial v\partial w + H_i\left(x,\partial v\right)w + \rho w\right)dx = 0. \tag{1.3.22}$$

Similarly, multiply the second equation in (1.3.18) with $u \in V$, integrate over $\mathcal{G}$, use integration by part for each Γ_i, apply the boundary conditions, transition conditions for m and the continuity of u, one gets

$$\sum_{i=1}^{N}\int_{\Gamma_i}\left(\mu_i\partial m\partial w + \partial_p H_i\left(x,\partial v\right)w\partial u\right)dx = 0. \tag{1.3.23}$$

These computations motivate the following definition of weak solutions for the MFG system on networks.

Definition 1.3.1. The triple $(v, m, \rho) \in V \times W \times \mathbb{R}$ is a weak solution of (1.3.18) if (v, ρ) satisfies (1.3.22) for all $w \in W$ and m satisfies (1.3.23) for all $u \in V$.

We introduce our assumption in Chapter 3

(Hamiltonian) We assume that

$$H_i \in C^1\left(\Gamma_i \times \mathbb{R}\right); \tag{1.3.24}$$

$$H_i\left(x,\cdot\right) \text{ is convex in } p \text{ for each } x \in \Gamma_i; \tag{1.3.25}$$

$$H_i\left(x,p\right) \geqslant C_0\left|p\right|^q - C_1 \text{ for } \left(x,p\right) \in \Gamma_i \times \mathbb{R}; \tag{1.3.26}$$

$$\left|\partial_p H_i\left(x,p\right)\right| \leqslant C_2\left(\left|p\right|^{q-1} + 1\right) \text{ for } \left(x,p\right) \in \Gamma_i \times \mathbb{R}. \tag{1.3.27}$$

for some constants C_0, C_1 and C_2.

(Coupling term) We assume that

$$\mathscr{V}\left[\tilde{m}\right](x) = F\left(m\left(x\right)\right) \text{ with } F \in C\left(\left[0,+\infty\right);\mathbb{R}\right), \tag{1.3.28}$$

for all $\tilde{m}$ which are absolutely continuous with respect to the Lebesgue measure and such that $d\tilde{m}\left(x\right) = m\left(x\right)dx$. We shall also suppose that F is bounded from below, i.e., there exists a positive constant M such that

$$F\left(r\right) \geqslant -M, \quad \text{for all } r \in \left[0,+\infty\right). \tag{1.3.29}$$

Theorem 1.3.2. *Under assumption* (1.3.24)-(1.3.29)*, there exists a weak solution of* (1.3.18) $(v, m, \rho) \in V \times W \times \mathbb{R}$*. Moreover,* $v \in C^2(\mathcal{G})$ *and* $m_i \in C^1(\Gamma_i)$ *for all* i*. Finally, the uniqueness of* (1.3.18) *holds under assumption strictly increasing of* F*.*

Let us give a sketch of proof for this theorem. First of all, we study the well-posedness for weak solution of the uncoupled HJB and FP equations in suitable Sobolev spaces V and W. More precisely, we study some linear boundary value problems with elliptic equations (Step 1), then on a pair of linear Kolmogorov and Fokker-Planck equations in duality (Step 1 and Step 2). By and large, the existence of weak solutions is obtained by applying Banach-Necas-Babuška theorem to the Sobolev spaces V and W and Fredholm's alternative. Uniqueness comes from a maximum principle. In [32], the authors apply Lax-Milgram lemma instead of Banach-Necas-Babuška theorem to get existence and uniqueness. However, it is impossible to apply Lax-Milgram lemma in our work since our solution space and the test-function space are not the same, which results from the "jump condition". Using Step 1, we obtain the well-posedness for classical solution of the HJB (Step 3) and apply it to prove the well-posedness for classical solution of the ergodic problem (Step 4). Then, using the well-posedness for weak solution of the uncoupled ergodic equation and FP equation, we establish the existence result for the MFG system by a fixed point argument and a truncation technique (Step 5). Uniqueness is proved when the coupling is increasing, i.e., the function F in (1.3.18) is increasing for local couplings (in the case of nonlocal coupling, the increasing condition is replaced by (1.3.42)). Finally, classical arguments will then lead to the regularity and uniqueness of the solutions.

Step 1: [Section 3.2.1, Section 3.2.2 and a part of Section 3.2.3] Consider a Kolmogorov's equation, for $b \in PC(\Gamma)$, $g \in W'$ and $\lambda \geqslant 0$:

$$\begin{cases} -\mu_i \partial^2 v + b\partial v + \lambda v = g, & \text{in } \Gamma_i \backslash \{O\}, \\ v_i(O) = v_j(O), & i,j = \overline{1,N}, \\ \sum_{i=1}^{N} \gamma_i \mu_i \partial_i v(O) = 0. & \end{cases} \tag{1.3.30}$$

A weak solution of (1.3.30) is a function $v \in V$ such that

$$A^\star(v,w) := \sum_{i=1}^{N} \int_{\Gamma_i} (\mu_i \partial v \partial w + b\partial v w)\, dx = \int_{\mathcal{G}} g w dx, \quad \text{for all } w \in W.$$

If $\lambda > 0$, (1.3.30) has a unique weak solution. If $\lambda = 0$ and $g = 0$, the set of solutions of (1.3.30) is the set of constant functions on Γ.

Step 2: [Section 3.2.3] Let $\lambda_0 > 0$ and $h \in V'$, we consider the following problem

$$\begin{cases} \lambda_0 m - \mu_i \partial^2 m - \partial(bm) = h, & \text{in } \Gamma_i \backslash \{O\}, \\ \dfrac{m_i(O)}{\gamma_i} = \dfrac{m_j(O)}{\gamma_j}, & i,j = \overline{1,N}, \\ \sum_{i=1}^{N} \left[b(O) m_{\Gamma_i}(O) + \mu_i \partial_i m(O) \right] = 0, & \\ \mu_i \partial_i m(\nu_i) + \partial_p H_i\Big(\nu_i, \partial_i v(\nu_i)\Big) m_i(\nu_i) = 0, & i = \overline{1,N}, \end{cases} \tag{1.3.31}$$

with

$$m \geqslant 0, \quad \int_{\mathcal{G}} m dx = 1. \tag{1.3.32}$$

A weak solution of (1.3.31) is a function $m \in W$ such that

$$A_{\lambda_0}(m,v) := \sum_{i=1}^{N} \int_{\Gamma_i} \left[\lambda_0 m v + (\mu_i \partial m + bm)\partial v \right] dx = \int_{\Gamma} h v dx, \quad \text{for all } v \in V.$$

For λ_0 large enough, we prove that there exists a constant C such that

$$\inf_{w \in W} \sup_{v \in V} \frac{A_{\lambda_0}(w,v)}{\|v\|_V \|w\|_W} \geqslant C, \quad \inf_{v \in V} \sup_{w \in W} \frac{A_{\lambda_0}(w,v)}{\|w\|_W \|v\|_V} \geqslant C,$$

by Babuska's lemma (or inf sup lemma), we obtain the well-posedness for (1.3.31).

Next, we prove the well-posedness for Fokker-Planck equation on $\mathcal{G}$, namely (1.3.31) with $\lambda_0 = 0$. Step 2 allows us to define a linear operator:

$$\mathcal{T} : L^2(\mathcal{G}) \longrightarrow W \hookrightarrow L^2(\mathcal{G}), \quad \mathcal{T}(\overline{m}) = m,$$

where m is the solution of (1.3.31)–(1.3.32) with $h = \lambda_0 \overline{m}$. Using the uniformly estimate for (1.3.31) and applying the fixed point theory, there exists a solution $\bar{m}$ for FP equation. Next, from Step 1, the set of solutions of (1.3.30), with $\lambda_0 = 0$ and $g = 0$, is the set of constant functions on $\mathcal{G}$. Hence, applying the Fredholm alternative, the set of solutions of FP equation is 1-dimensional. By the normalization condition (1.3.32), we obtain the uniqueness for FP equation. Moreover, by the comparison principle, $\bar{m}$ is strictly positive.

Step 3: [Section 3.3.1] We study the Hamilton-Jacobi equation for all $\lambda > 0$

$$\begin{cases} -\mu_i \partial^2 u_\lambda + H(x, \partial u_\lambda) + \lambda u_\lambda = f, & \text{in } \Gamma_i \backslash \{O\}, \\ u_{i\lambda}(O) = u_{j\lambda}(O), & i, j = \overline{1, N}, \\ \sum_{i=1}^{N} \gamma_i \mu_i \partial_i v(O) = 0, \partial_i v(\nu_i) = 0, & i = \overline{1, N}. \end{cases} \tag{1.3.33}$$

We first deal with the bounded Hamiltonian. From Step 1, this allows us to define a linear operator

$$\mathcal{T} : V \longrightarrow V, \quad \mathcal{T}(u) = v,$$

with $b = 0$ and $g = H(x, \partial u)$. Using the fixed point theory, we obtain the existence of (1.3.33). The uniqueness is a consequence of the comparison principle. Now, to deal with quadratic Hamiltonian, we use the truncation technique combining with the previous process (with bounded Hamiltonian). This step is adapted the classical proof of Boccardo, Murat & Puel [26].

Step 4: [Section 3.3.2] We are ready to solve the ergodic problem

$$\begin{cases} -\mu_i \partial^2 v + H(x, \partial v) + \rho = f, & \text{if } x \in \Gamma_i \backslash \{O\}, \\ v_i(O) = v_j(O), & i, j = \overline{1, N}, \\ \sum_{i=1}^{N} \gamma_i \mu_i \partial_i v(O) = 0, \partial_i v(\nu_i) = 0, & i = \overline{1, N}, \end{cases} \tag{1.3.34}$$

and

$$\int_{\mathcal{G}} v dx = 0. \tag{1.3.35}$$

From Step 3 and the quadratic Hamiltonian (1.3.27), we can obtain some uniform estimate for (1.3.33). Hence, the existence of (1.3.34) is deduced by letting λ to 0. The uniqueness results from the maximum principle of the classical equations on 1 dimension and the normalization condition (1.3.35).

Step 5: [Section 3.4] In the last step, using the well-posedness for weak solution of the uncoupled FP equation (Step 2) and ergodic equation (Step 4), we establish the existence for the MFG system with the bounded coupling term F. Then using the truncation technique based on the energy estimate, the existence with the general coupling term F is obtained. Finally, the uniqueness is deduced by using the increasing coupling term.

1.3.3 Results in Chapter 4: A Class of Finite Horizon Mean Field Games on Networks

Camilli & Marchi [32] introduce the MFG systems on networks in the finite time horizon case (1.3.13) from two different points of view: either as the characterization of a Pareto equilibrium for dynamic games with a large number of indistinguishable players or as the optimality conditions for optimal control problems whose dynamic is governed by a PDE. They also show that two models lead to the same transition conditions. However, they only study for the infinite horizon MFG system (1.3.18) without jump condition of density m.

The first result in Chapter 4 also explains where the MFG system (1.3.13) comes from, see Section 1.3.1 or Sections 4.1.2–4.1.3.

Let us introduce our assumptions in Chapter 4.

(Hamiltonian) We assume that

$$
\begin{aligned}
&H_i \in C^1(\Gamma_i \times \mathbb{R}), && && (1.3.36)\\
&H_i(x,\cdot) \text{ is convex in } p, && \text{for any } x \in \Gamma_i, && (1.3.37)\\
&H_i(x,p) \leqslant C_0(|p|+1), && \text{for any } (x,p) \in \Gamma_i \times \mathbb{R}, && (1.3.38)\\
&|\partial_p H_i(x,p)| \leqslant C_0, && \text{for any } (x,p) \in \Gamma_i \times \mathbb{R}, && (1.3.39)\\
&|\partial_x H_i(x,p)| \leqslant C_0(|p|+1), && \text{for any } (x,p) \in \Gamma_i \times \mathbb{R}, && (1.3.40)
\end{aligned}
$$

for a constant C_0 independent of i.

(Coupling operator) We assume that $\mathscr{V}$ is a continuous map from $L^2(\mathcal{G})$ to $L^2(\mathcal{G})$, such that for all $m \in L^2(\mathcal{G})$,

$$\|\mathscr{V}[m]\|_{L^2(\mathcal{G})} \leqslant C(\|m\|_{L^2(\mathcal{G})}+1). \tag{1.3.41}$$

Note that such an assumption is satisfied by local operators of the form $\mathscr{V}[m](x) = F(m(x))$ where F is a Lipschitz-continuous function.

(Initial and terminal data) $v_T \in H^1(\mathcal{G})$ and $m_0 \in L^2(\mathcal{G})$.

(Stronger assumption for Hamiltonian and coupling term) For $i = \overline{1,N}$, $\partial_p H_i(x,p)$ is Lipschitz continuous on $\Gamma_i \times \mathbb{R}$ and the coupling $\mathscr{V}$ maps the topological dual of W to $H_b^1(\Gamma)$; more precisely, $\mathscr{V}$ defines a Lipschitz map from W' to $H_b^1(\Gamma)$. Note that such an assumption is not satisfied by local operators.

The above set of assumptions, except the last one, is referred to as [A], will be the running assumptions hereafter. We will also say that the coupling $\mathscr{V}$ is strictly increasing if, for any $m_1, m_2 \in \mathcal{M} \cap L^2(\mathcal{G})$,

$$\int_{\mathcal{G}} (m_1 - m_2)(\mathscr{V}[m_1] - \mathscr{V}[m_2])dx \geqslant 0 \tag{1.3.42}$$

and equality implies $m_1 = m_2$.

Definition 1.3.3. (solutions of the MFG system) A weak solution of the MFG system (1.3.13) is a pair (v, m) such that

$$
\begin{aligned}
&v \in L^2\left(0,T;H^2(\mathcal{G})\right) \cap C([0,T];V),\ \partial_t v \in L^2\left(0,T;L^2(\mathcal{G})\right),\\
&m \in L^2(0,T;W) \cap C((0,T];L^2(\mathcal{G}) \cap \mathcal{M}),\ \partial_t m \in L^2\left(0,T;V'\right),
\end{aligned}
$$

and v satisfies

$$\begin{cases} -\displaystyle\sum_{i=1}^{N}\int_{\Gamma_i}\left[\partial_t v\left(x,t\right)\mathsf{w}\left(x\right)+\mu_i\partial v\left(x,t\right)\partial\mathsf{w}\left(x,t\right)+H\left(x,\partial v\left(x,t\right)\right)\mathsf{w}\left(x\right)\right]dx \\ \qquad\qquad\qquad = \displaystyle\int_{\mathcal{G}}\mathscr{V}[m(\cdot,t)](x)\mathsf{w}\left(x\right)dx, \quad \text{for all } \mathsf{w}\in W, \text{ a.e. } t\in(0,T), \\ v(x,T)=v_T(x) \quad \text{for a.e. } x\in\mathcal{G}, \end{cases}$$

and m satisfies

$$\begin{cases} \displaystyle\sum_{i=1}^{N}\int_{\Gamma_i}\left[\partial_t m\left(x,t\right)\mathsf{v}\left(x\right)dx+\mu_i\partial m\left(x,t\right)\partial\mathsf{v}\left(x\right)+\partial_p H\left(x,\partial v\left(x,t\right)\right)m\left(x,t\right)\partial\mathsf{v}\left(x\right)\right]dx \\ \qquad\qquad\qquad = 0, \quad \text{for all } \mathsf{v}\in V, \text{ a.e. } t\in(0,T), \\ m(x,0)=m_0(x) \quad \text{for a.e. } x\in\mathcal{G}, \end{cases}$$

where V and W are introduced in Section 1.3.2.

We ready to introduce the main theorem of Chapter 4.

Theorem 1.3.4 (Theorem 4.1.11)**.** *Under assumptions [A],*

(i) (Existence) There exists a weak solution (v,m) of (1.3.13).

(ii) (Uniqueness) If $\mathscr{V}$ is strictly increasing, then the solution is unique.

(iii) (Regularity) If $\mathscr{V}$ satisfies furthermore the stronger assumptions for coupling term then $v\in C^{2,1}(\mathcal{G}\times[0,T])$.
Moreover, if the Hamiltonian H_i satisfies the stronger assumptions for Hamiltonians, and if $m_0\in W$, then $m\in C([0,T];W)\cap W^{1,2}(0,T;L^2(\mathcal{G}))\cap L^2(0,T;H^2_b(\mathcal{G}))$.

Let us give the idea of the proof for Theorem 1.3.4. We first study a modified heat equation and FP equation on $\mathcal{G}$, whose existence results are obtained by using the Galerkin's method to construct solutions of certain finite-dimensional approximations to these equations. The uniqueness is a direct consequence of the energy estimate. Next, we shall establish the existence result for weak solutions of MFG system (1.3.13) by a fixed point argument. Uniqueness will also be proved for strictly increasing couplings.

Step 1: [Section 4.2] Consider a modified heat equation on $\mathcal{G}$ with general Kirchhoff condition

$$\begin{cases} -\partial_t v-\mu_i\partial^2 v=h, & \text{in } (\Gamma_i\backslash\{O\})\times(0,T), i=\overline{1,N}, \\ v_i(O,t)=v_j(O,t), & t\in(0,T),\ i,j=\overline{1,N}, \\ \displaystyle\sum_{i=1}^{N}\gamma_i\mu_i\partial_i v\left(O,t\right)=0, & t\in(0,T), \\ \partial_i v\left(\nu_i,t\right)=0, & t\in(0,T), i=\overline{1,N}, \\ v(x,T)=v_T(x), & x\in\mathcal{G}, \end{cases} \tag{1.3.43}$$

for $h\in L^2(0,T;W')$ and $v_T\in L^2(\mathcal{G})$. A weak solution of (1.3.43) is a function $v\in L^2(0,T;V)\cap C([0,t];L^2(\mathcal{G}))$ such that $\partial_t v\in L^2(0,T;W')$ and

$$\begin{cases} -\left\langle\partial_t v\left(t\right),w\right\rangle_{W',W}+\displaystyle\sum_{i=1}^{N}\int_{\Gamma_i}\mu_i\partial v\partial w dx=\left\langle h,w\right\rangle_{W',W} \quad \text{for all } w\in W \text{ and a.e. } t\in(0,T), \\ v\left(x,T\right)=v_T(x). \end{cases} \tag{1.3.44}$$

We use the Galerkin's method, namely we construct solutions of some finite-dimensional approximations to (1.3.44), to prove the existence of (1.3.44). Moreover, if $v_T \in V$, we get more regularity for v: $v \in L^2(0,T;H^2(\mathcal{G})) \cap C([0,T];V)$ and $\partial_t v \in L^2(0,T;L^2(\mathcal{G}))$. The uniqueness and stability is a direct consequence of the energy estimate.

Step 2: [Section 4.3] In this step, we study a boundary value problem including a Fokker-Planck equation

$$\begin{cases} \partial_t m - \mu_i \partial^2 m - \partial(bm) = 0, & \text{in } (\Gamma_i \backslash \{O\}) \times (0,T), \ i = \overline{1,N}, \\ \dfrac{m_i(O,t)}{\gamma_i} = \dfrac{m_j(O,t)}{\gamma_j}, & t \in (0,T), \ i,j = \overline{1,N}, \\ \displaystyle\sum_{i=1}^{N} \mu_i \partial_i m(O,t) - b_i(O,t) m_i(O,t) = 0, & t \in (0,T), \\ \displaystyle\sum_{i \in \mathcal{A}_i} \mu_i \partial_i m(\nu_i,t) + b_i(\nu_i,t) m_i(\nu_i,t) = 0, & t \in (0,T), \ i = \overline{1,N}, \\ m(x,0) = m_0(x), & x \in \mathcal{G}, \end{cases} \tag{1.3.45}$$

where $b \in PC(\mathcal{G} \times [0,T])$ and $m_0 \in L^2(\mathcal{G})$. A weak solution of (1.3.45) is a function $m \in L^2(0,T;W) \cap C([0,T];L^2(\mathcal{G}))$ such that $\partial_t m \in L^2(0,T;V')$ and

$$\begin{cases} \langle \partial_t m, v \rangle_{V',V} + \displaystyle\sum_{i=1}^{N} \int_{\mathcal{G}} \mu_i \partial m \partial v dx + \int_{\mathcal{G}} bm \partial v dx = 0 & \text{for all } v \in V \text{ and a.e. } t \in (0,T), \\ m(\cdot,0) = m_0. \end{cases} \tag{1.3.46}$$

Similarly to Step 1, the existence is deduced from the Galerkin's method and the uniqueness and stability is a consequence of the energy estimate. If b_i is Lipschitz continuous on Γ_i, we get more regularity for m: $m \in L^2(0,T;H_b^2(\mathcal{G})) \cap C([0,T];W)$ and $\partial_t m \in L^2(0,T;L^2(\mathcal{G}))$.

Step 3: [Section 4.4.1] We consider the boundary value problem including a HJ equation on $\mathcal{G}$, namely (1.3.43) with $h = f - H(x,\partial v)$, where $f \in L^2(0,T;L^2(\mathcal{G}))$ and the Hamiltonian H satisfies the running assumption [A]. We first work on bounded H. From Step 1, this allows us to define a linear operator

$$\mathcal{T} : L^2(0,T;V) \longrightarrow L^2(0,T;V), \quad \mathcal{T}(u) = v,$$

with $h = f - H(x,\partial u)$. The existence is obtained by using the fixed point theory,. The uniqueness is a consequence of the comparison principle. Next, to deal with sublinear Hamiltonian, we use the truncation technique combining with the previous process (with bounded Hamiltonian).

Step 4: [Section 4.4.2] The main difficulty in this work is how to obtain the regularity for the boundary value problem including a HJ equation on $\mathcal{G}$. Let us explain formally the idea. We derive the HJB equation for v and prove some regularity estimate for the solution of the derived problem. More precisely, $u := \partial v$ satisfies the following PDE

$$-\partial_t u - \mu_i \partial^2 u + \partial(H(x,u)) = \partial f,$$

with terminal condition $u(x,T) = \partial v_T(x)$. From the Kirchhoff conditions in the uncoupled HJ equations, we obtain a transition condition of Dirichlet type for u,

$$\sum_{i=1}^{N} \mu_i \gamma_i u|_{\Gamma_i}(O,t) = 0, \quad t \in (0,T).$$

Now, by extending continuously the HJB uncoupled until the vertex O in the branches Γ_i and Γ_j, $i,j=\overline{1,N}$ and using the continuity condition of v, one gets

$$-\mu_i\partial^2 v|_{\Gamma_i}+H|_{\Gamma_i}(\nu_i,\partial v|_{\Gamma_i}(\nu_i,t))-f|_{\Gamma_i}(\nu_i,t)=-\mu_j\partial^2 v|_{\Gamma_j}+H|_{\Gamma_j}(\nu_i,\partial v|_{\Gamma_j}(\nu_i,t))-f|_{\Gamma_j}(\nu_i,t).$$

This gives a second transition condition for u

$$\mu_i\partial u|_{\Gamma_i}(\nu_i,t)-H|_{\Gamma_i}(\nu_i,u|_{\Gamma_i}(\nu_i,t))+f|_{\Gamma_i}(\nu_i,t)=\mu_j\partial u|_{\Gamma_j}(\nu_i,t)-H|_{\Gamma_j}(\nu_i,u|_{\Gamma_j}(\nu_i,t))+f|_{\Gamma_j}(\nu_i,t).$$

Hence, we shall study the following nonlinear boundary value problem for $u=\partial v$,

$$\begin{cases}-\partial_t u-\mu_i\partial^2 u+\partial_p H(x,u)\,\partial u=G(x,t)\,, & (\Gamma_i\backslash\{O\})\times(0,T)\,,\ i=\overline{1,N},\\ \displaystyle\sum_{i\in\mathcal{A}_i}\gamma_i\mu_i u|_{\Gamma_i}(O,t)=0, & t\in(0,T)\,,\\ \mu_i\partial u|_{\Gamma_i}(O,t)-H|_{\Gamma_i}(O,u|_{\Gamma_i}(O,t))+f|_{\Gamma_i}(O,t) & \\ \quad=\mu_j\partial u|_{\Gamma_j}(O,t)-H|_{\Gamma_j}(O,u|_{\Gamma_j}(O,t))+f|_{\Gamma_j}(O,t)\,, & t\in(0,T)\,,\ i,j=\overline{1,N},\\ u(x,T)=u_T(x)\,, & x\in\mathcal{G},\end{cases}\tag{1.3.47}$$

where $G\in L^2\left(0,T;L^2(\mathcal{G})\right)$ and $u_T\in H_b^1(\mathcal{G})$. After obtaining the existence, uniqueness and regularity for (1.3.47) (by using also the Galerkin's method and energy estimate), the regularity of the HJB equations will follow by proving $u=\partial v$ in case $G=\partial f-\partial_x H(x,\partial v)$ and $u_T=\partial v_T$.

Step 5: [Section 4.5] In the last step, using Step 2 and Step 3 and the fixed point theory, we can obtain the existence for weak solution (v,m) of MFG system (1.3.13). The uniqueness is a consequence of the increasing coupling term. Finally, we get the regularity of (1.3.13) from Step 4 and the stronger assumptions for Hamiltonian and coupling term.

We end by comparing the results in the stationary and the non-stationary case. Assumptions in the non-stationary case are more restrictive. We only focus on the more basic assumptions, globally Lipschitz Hamiltonian, instead of subquadratic ones in the stationary case, and rather strong assumptions on the coupling cost. This will allow us to concentrate on the difficulties induced by the Kirchhoff conditions. Therefore, this work should be seen as a first and necessary step to deal with more difficult situations, for example with quadratic or subquadratic Hamiltonians. We believe that treating such cases will possible by combining the results contained in the present work with methods that can be found in [78, 82].

2 Hamilton-Jacobi equations for optimal control on networks with entry or exit costs

Abstract: We consider an optimal control on networks in the spirit of the works of Achdou, Camilli, Cutrì & Tchou [5] and Imbert, Monneau & Zidani [73]. The main new feature is that there are entry (or exit) costs at the edges of the network leading to a possible discontinuous value function. We characterize the value function as the unique viscosity solution of a new Hamilton-Jacobi system. The uniqueness is a consequence of a comparison principle for which we give two different proofs, one with arguments from the theory of optimal control inspired by Achdou, Oudet & Tchou [8] and one based on partial differential equations techniques inspired by a recent work of Lions & Souganidis [87].

2.1 Introduction

A network (or a graph) is a set of items, referred to as vertices or nodes, which are connected by edges (see Figure 2.1 for example). Recently, several research projects have been devoted to dynamical systems and differential equations on networks, in general or more particularly in connection with problems of data transmission or traffic management (see for example Garavello & Piccoli [57] and Engel, Fijavvz, Nagelm & Sikolya [47]).

An optimal control problem is an optimization problem where an agent tries to minimize a cost which depends on the solution of a controlled ordinary differential equation (ODE). The ODE is controlled in the sense that it depends on a function called the control. The goal is to find the best control in order to minimize the given cost. In many situations, the optimal value of the problem as a function of the initial state (and possibly of the initial time when the horizon of the problem is finite) is a viscosity solution of a Hamilton-Jacobi-Bellman partial differential equation (HJB equation). Under appropriate conditions, the HJB equation has a unique viscosity solution characterizing by this way the value function. Moreover, the optimal control may be recovered from the solution of the HJB equation, at least if the latter is smooth enough.

The first articles about optimal control problems in which the set of admissible states is a network (therefore the state variable is a continuous one) appeared in 2012: in [5], Achdou, Camilli, Cutrì & Tchou derived the HJB equation associated to an infinite horizon optimal control on a network and proposed a suitable notion of viscosity solution. Obviously, the main difficulties arise at the vertices where the network does not have a regular differential structure. As a result, the new admissible test-functions whose restriction to each edge is C^1 are applied. Independently and at the same time, Imbert, Monneau & Zidani [73] proposed an equivalent notion of viscosity solution for studying a Hamilton-Jacobi approach to junction problems and traffic flows. Both [5] and [73] contain first results on comparison principles which were improved

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later. It is also worth mentioning the work by Schieborn & Camilli [94], in which the authors focus on eikonal equations on networks and on a less general notion of viscosity solution. In the particular case of eikonal equations, Camilli & Marchi established in [31] the equivalence between the definitions given in [5, 73, 94].

Since 2012, several proofs of comparison principles for HJB equations on networks, giving uniqueness of the solution, have been proposed.

1. In [8], Achdou, Oudet & Tchou give a proof of a comparison principle for a stationary HJB equation arising from an optimal control with infinite horizon, (therefore the Hamiltonian is convex) by mixing arguments from the theory of optimal control and PDE techniques. Such a proof was much inspired by works of Barles, Briani & Chasseigne [20, 19], on regional optimal control problems in $\mathbb{R}^d$, (with discontinuous dynamics and costs).

2. A different and more general proof, using only arguments from the theory of PDEs was obtained by Imbert & Monneau in [72]. The proof works for quasi-convex Hamiltonians, and for stationary and time-dependent HJB equations. It relies on the construction of suitable *vertex test functions.*

3. A very simple and elegant proof, working for non convex Hamiltonians, has been very recently given by Lions & Souganidis [86, 87].

The goal of this paper is to consider an optimal control problem on a network in which there are entry (or exit) costs at each edge of the network and to study the related HJB equations. The effect of the entry/exit costs is to make the value function of the problem discontinuous. Discontinuous solutions of Hamilton-Jacobi equation have been studied by various authors, see for example Barles [16], Frankowska & Mazzola [52], and in particular Graber, Hermosilla & Zidani [67] for different HJB equations on networks with discontinuous solutions.

To simplify the problem, we first study the case of junction, i.e., a network of the form $\mathcal{G} = \cup_{i=1}^{N}\Gamma_i$ with N edges Γ_i (Γ_i is the closed half line $\mathbb{R}^+ e_i$) and only one vertex O, where $\{O\} = \cap_{i=1}^{N}\Gamma_i$. This can be easily generalized into networks with an arbitrary number of vertices. In the case of the junction described above, our assumptions about the dynamics and the running costs are similar to those made in [8], except that additional costs c_i for entering the edge Γ_i at O or d_i for exiting Γ_i at O are added in the cost functional. Accordingly, the value function is continuous on $\mathcal{G}$, but is in general discontinuous at the vertex O. Hence, instead of considering the value function v, we split it into the collection $(v_i)_{1\leqslant i\leqslant N}$, where v_i is continuous function defined on the edge Γ_i. More precisely,

$$v_i(x) = \begin{cases} \mathsf{v}(x) & \text{if } x \in \Gamma_i \backslash \{O\}, \\ \lim\limits_{\delta\to 0^+} \mathsf{v}(\delta e_i) & \text{if } x = O. \end{cases}$$

Our approach is therefore reminiscent of optimal switching problems (impulsional control): in the present case the switches can only occur at the vertex O. Note that our assumptions will ensure that $\mathsf{v}|_{\Gamma_i\backslash\{O\}}$ is Lipschitz continuous near O and that $\lim_{\delta\to 0^+}\mathsf{v}(\delta e_i)$ does exist. In the case of entry costs for example, our first main result will be to find the relation between $\mathsf{v}(O)$, $v_i(O)$ and $v_j(O) + c_j$ for $i, j = \overline{1, N}$.

This will show that the functions $(v_i)_{1\leqslant i\leqslant N}$ are (suitably defined) viscosity solutions of the following system

$$\begin{aligned} \lambda u_i(x) + H_i\left(x, \frac{du_i}{dx_i}(x)\right) &= 0 && \text{if } x \in \Gamma_i \backslash \{O\}, \\ \lambda u_i(O) + \max\left\{-\lambda \min_{j\neq i}\{u_j(O) + c_j\}, H_i^+\left(O, \frac{du_i}{dx_i}(O)\right), H_O^T\right\} &= 0 && \text{if } x = O. \end{aligned} \tag{2.1.1}$$

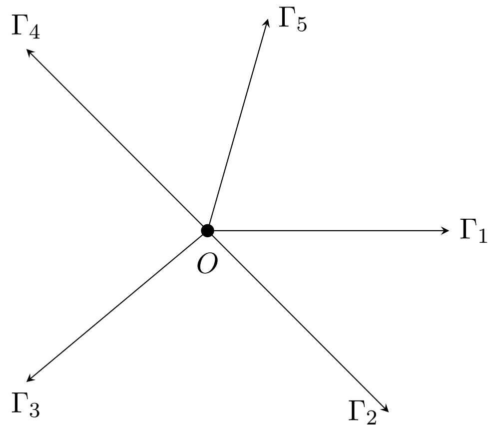

Figure 2.1: The junction $\mathcal{G}$ with 5 edges on $\mathbb{R}^2$

Here H_i is the Hamiltonian corresponding to edge Γ_i. At vertex O, the definition of the Hamiltonian has to be particular, in order to consider all the possibilities when x is close to O. More specifically, if x is close to O and belongs to Γ_i then:

- The term $\min_{j\neq i}\{u_j(O)+c_j\}$ accounts for situations in which the trajectory enters Γ_{i_0} where $u_{i_0}(O)+c_{i_0}=\min_{j\neq i}\{u_j(O)+c_j\}$.
- The term $H_i^+\left(O,\dfrac{du_i}{dx_i}(O)\right)$ accounts for situations in which the trajectory does not leave Γ_i.
- The term H_O^T accounts for situations in which the trajectory stays at O.

The most important part of the paper will be devoted to two different proofs of a comparison principle leading to the well-posedness of (2.1.1): the first one uses arguments from optimal control theory coming from [19, 20] and [8]; the second one is inspired by Lions & Souganidis [86] and uses arguments from the theory of PDEs.

The paper is organized as follows: Section 2.2 deals with the optimal control problems with entry and exit costs: we give a simple example in which the value function is discontinuous at the vertex O, and also prove results on the structure of the value function near O. In Section 2.3, the new system of (2.1.1) is defined and a suitable notion of viscosity solutions is proposed. In Section 2.4, we prove our value functions are viscosity solutions of the above mentioned system. In Section 2.5, some properties of viscosity sub and supersolution are given and used to obtain the comparison principle. Finally, optimal control problems with entry costs which may be zero and related HJB equations are considered in Section 2.6.

2.2 Optimal control problem on junction with entry/exit costs

2.2.1 The geometry

We consider the model case of the junction in $\mathbb{R}^d$ with N semi-infinite straight edges, $N>1$. The edges are denoted by $(\Gamma_i)_{i=\overline{1,N}}$ where Γ_i is the closed half-line $\mathbb{R}^+e_i$. The vectors e_i are two by two distinct unit vectors in $\mathbb{R}^d$. The half-lines Γ_i are glued at the vertex O to form the junction $\mathcal{G}$

$$\mathcal{G} = \bigcup_{i=1}^{N} \Gamma_i.$$

The geodetic distance $d(x,y)$ between two points x, y of $\mathcal{G}$ is

$$d(x,y) = \begin{cases} |x-y| & \text{if } x, y \text{ belong to the same egde } \Gamma_i, \\ |x| + |y| & \text{if } x, y \text{ belong to different edges } \Gamma_i \text{ and } \Gamma_j. \end{cases}$$

2.2.2 The optimal control problem

We consider infinite horizon optimal control problems which have different dynamic and running costs for each and every edge. For $i = \overline{1, N}$,

- the set of control on Γ_i is denoted by A_i
- the system is driven by a dynamics f_i
- there is a running cost ℓ_i.

Our main assumptions, referred to as $[H]$ hereafter, are as follows:

$[H0]$ (**Control sets**) Let A be a metric space (one can take $A = \mathbb{R}^d$). For $i = \overline{1, N}$, A_i is a nonempty compact subset of A and the sets A_i are disjoint.

$[H1]$ (**Dynamics**) For $i = \overline{1, N}$, the function $f_i : \Gamma_i \times A_i \to \mathbb{R}$ is continuous and bounded by M. Moreover, there exists $L > 0$ such that

$$|f_i(x,a) - f_i(y,a)| \leqslant L|x-y| \qquad \text{for all } x, y \in \Gamma_i, a \in A_i.$$

Hereafter, we will use the notation $F_i(x)$ for the set $\{f_i(x,a)e_i : a \in A_i\}$.

$[H2]$ (**Running costs**) For $i = \overline{1, N}$, the function $\ell_i : \Gamma_i \times A_i \to \mathbb{R}$ is a continuous function bounded by $M > 0$. There exists a modulus of continuity ω such that

$$|\ell_i(x,a) - \ell_i(y,a)| \leqslant \omega(|x-y|) \qquad \text{for all } x, y \in \Gamma_i, a \in A_i.$$

$[H3]$ (**Convexity of dynamic and costs**) For $x \in \Gamma_i$, the following set

$$\mathrm{FL}_i(x) = \{(f_i(x,a)e_i, \ell_i(x,a)) : a \in A_i\}$$

is non-empty, closed and convex.

$[H4]$ (**Strong controllability**) There exists a real number $\delta > 0$ such that

$$[-\delta e_i, \delta e_i] \subset F_i(O) = \{f_i(O,a)e_i : a \in A_i\}.$$

Remark 2.2.1. The assumption that the sets A_i are disjoint is not restrictive. Indeed, if A_i are not disjoint, then we define $\tilde{A}_i = A_i \times \{i\}$ and $\tilde{f}_i(x,\tilde{a}) = f_i(x,a), \tilde{\ell}_i(x,\tilde{a}) = \ell_i(x,a)$ with $\tilde{a} = (a,i)$ with $a \in A_i$. The assumption $[H3]$ is made to avoid the use of relaxed control. With assumption $[H4]$, one gets that the Hamiltonian which will appear later is coercive for x close to the O. Moreover, $[H4]$ is an important assumption to prove Lemma 2.2.7 and Lemma 2.5.3.

Let

$$\mathcal{M} = \left\{(x,a) : x \in \mathcal{G}, a \in A_i \text{ if } x \in \Gamma_i \backslash \{O\}, \text{ and } a \in \cup_{i=1}^N A_i \text{ if } x = O\right\}.$$

Then $\mathcal{M}$ is closed. We also define the function on $\mathcal{M}$ by

$$\text{for all } (x,a) \in \mathcal{M}, \quad f(x,a) = \begin{cases} f_i(x,a)\, e_i & \text{if } x \in \Gamma_i \backslash \{O\} \text{ and } a \in A_i, \\ f_i(O,a)\, e_i & \text{if } x = O \text{ and } a \in A_i. \end{cases}$$

The function f is continuous on $\mathcal{M}$ since the sets A_i are disjoint.

Definition 2.2.2 (*The speed set and the admissible control set*)**.** The set $\tilde{F}(x)$ which contains all the "possible speeds" at x is defined by

$$\tilde{F}(x) = \begin{cases} F_i(x) & \text{if } x \in \Gamma_i \backslash (O), \\ \bigcup_{i=1}^N F_i(O) & \text{if } x = O. \end{cases}$$

For $x \in \mathcal{G}$, the set of admissible trajectories starting from x is

$$Y_x = \left\{ y_x \in Lip\left(\mathbb{R}^+; \mathcal{G}\right) \;\middle|\; \begin{array}{ll} \dot{y}_x(t) \in \tilde{F}(y_x(t)) & \text{for a.e. } t > 0 \\ y_x(0) = x. & \end{array} \right\}.$$

According to [8, Theorem 1.2], a solution y_x can be associated with several control laws. We introduce the set of admissible controlled trajectories starting from x

$$\mathcal{T}_x = \left\{(y_x, \alpha) \in L^\infty_{loc}\left(\mathbb{R}^+; \mathcal{M}\right) : y_x \in Lip\left(\mathbb{R}^+; \mathcal{G}\right) \text{ and } y_x(t) = x + \int_0^t f(y_x(s), \alpha(s))\, ds \right\}.$$

Notice that, if $(y_x, \alpha) \in \mathcal{T}_x$ then $y_x \in Y_x$. Hereafter, we will denote y_x by $y_{x,\alpha}$ if $(y_x, \alpha) \in \mathcal{T}_x$. For any $y_{x,\alpha}$, we can define the closed set $T_O = \{t \in \mathbb{R}^+ : y_{x,\alpha}(t) = O\}$ and the open set T_i in $\mathbb{R}^+ = [0, +\infty)$ by $T_i = \{t \in \mathbb{R}^+ : y_{x,\alpha}(t) \in \Gamma_i \backslash \{O\}\}$. The set T_i is a countable union of disjoint open intervals

$$T_i = \bigcup_{k \in K_i \subset \mathbb{N}} T_{ik} = \begin{cases} [0, \eta_{i0}) \cup \bigcup_{k \in K_i \subset \mathbb{N}^\star} (t_{ik}, \eta_{ik}) & \text{if } x \in \Gamma_i \backslash \{O\}, \\ \bigcup_{k \in K_i \subset \mathbb{N}^\star} (t_{ik}, \eta_{ik}) & \text{if } x \notin \Gamma_i \backslash \{O\}, \end{cases}$$

where $K_i = \{1, \ldots, n\}$ if the trajectory $y_{x,\alpha}$ enters Γ_i n times and $K_i = \mathbb{N}$ if the trajectory $y_{x,\alpha}$ enters Γ_i infinite times.

Remark 2.2.3. From the above definition, one can see that t_{ik} is an entry time in $\Gamma_i \backslash \{O\}$ and η_{ik} is an exit time from $\Gamma_i \backslash \{O\}$. Hence

$$y_{x,\alpha}(t_{ik}) = y_{x,\alpha}(\eta_{ik}) = O.$$

Let $C = \{c_1, c_2, \ldots, c_N\}$ be a set of ***entry costs*** and $D = \{d_1, d_2, \ldots, d_N\}$ be a set of ***exit costs***. We underline that, except in Section 2.6, entry and exist costs are positive.

In the sequel, we define two different *cost functionals* (the first one corresponds to the case when there is a cost for entering the edges and the second one corresponds to the case when there is a cost for exiting the edges):

Definition 2.2.4 (The cost functionals and value functions with entry/exit costs). The costs associated to trajectory $(y_{x,\alpha},\alpha)\in\mathcal{T}_x$ are defined by

$$J(x;(y_{x,\alpha},\alpha))=\int_0^{+\infty}\ell(y_{x,\alpha}(t),\alpha(t))\,e^{-\lambda t}dt+\sum_{i=1}^{N}\sum_{k\in K_i}c_i e^{-\lambda t_{ik}}\quad\text{(cost functional with entry cost)},$$

and

$$\hat{J}(x;(y_{x,\alpha},\alpha))=\int_0^{+\infty}\ell(y_{x,\alpha}(t),\alpha(t))\,e^{-\lambda t}dt+\sum_{i=1}^{N}\sum_{k\in K_i}d_i e^{-\lambda \eta_{ik}}\quad\text{(cost functional with exit cost)},$$

where the running cost $\ell:\mathcal{M}\to\mathbb{R}$ is

$$\ell(x,a)=\begin{cases}\ell_i(x,a) & \text{if } x\in\Gamma_i\backslash\{O\}\text{ and } a\in A_i,\\ \ell_i(O,a) & \text{if } x=0\text{ and } a\in A_i.\end{cases}$$

Hereafter, to simplify the notation, we will use $J(x,\alpha)$ and $\hat{J}(x,\alpha)$ instead of $J(x;(y_{x,\alpha},\alpha))$ and $\hat{J}(x;(y_{x,\alpha},\alpha))$, respectively.

The value functions of the infinite horizon optimal control problem are defined by:

$$\mathsf{v}(x)=\inf_{(y_{x,\alpha},\alpha)\in\mathcal{T}_x}J(x;(y_{x,\alpha},\alpha))\quad\text{(value function with entry cost)},$$

and

$$\hat{\mathsf{v}}(x)=\inf_{(y_{x,\alpha},\alpha)\in\mathcal{T}_x}\hat{J}(x;(y_{x,\alpha},\alpha))\quad\text{(value function with exit cost)}.$$

Remark 2.2.5. By the definition of the value function, we are mainly interested in a control law α such that $J(x,\alpha)<+\infty$. In such a case, if $|K_i|=+\infty$, then we can order $\{t_{ik},\eta_{ik}:\ k\in\mathbb{N}\}$ such that

$$t_{i1}<\eta_{i1}<t_{i2}<\eta_{i2}<\ldots<t_{ik}<\eta_{ik}<\ldots,$$

and

$$\lim_{k\to\infty}t_{ik}=\lim_{k\to\infty}\eta_{ik}=+\infty.$$

Indeed, assuming if $\lim_{k\to\infty}t_{ik}=\bar{t}<+\infty$, then

$$J(x,\alpha)\geqslant -\frac{M}{\lambda}+\sum_{k=1}^{+\infty}e^{-\lambda t_{ik}}c_i=-\frac{M}{\lambda}+c_i\sum_{k=1}^{+\infty}e^{-\lambda t_{ik}}=+\infty,$$

in contradiction with $J(x,\alpha)<+\infty$. This means that the state cannot switch edges infinitely many times in finite time, otherwise the cost functional is obviously infinite.

The following example shows that the value function with entry costs is possibly discontinuous (The same holds for the value function with exit costs).

Example 2.2.6. Consider the network $\mathcal{G}=\Gamma_1\cup\Gamma_2$ where $\Gamma_1=\mathbb{R}^+e_1=(-\infty,0]$ and $\Gamma_2=\mathbb{R}^+e_2=[0,+\infty)$. The control sets are $A_i=[-1,1]\times\{i\}$ with $i\in\{1,2\}$. Set

$$(f(x,a),\ \ell(x,a))=\begin{cases}(f_i(x,(a_i,i))\,e_i,\ \ell_i(x,(a_i,i))) & \text{if } x\in\Gamma_i\backslash\{O\}\ \text{and } a=(a_i,i)\in A_i,\\ (f_i(O,(a_i,i))\,e_i,\ \ell_i(O,(a_i,i))) & \text{if } x=O\text{ and } a=(a_i,i)\in A_i,\end{cases}$$

where $f_i(x,(a_i,i)) = a_i$ and $\ell_1 \equiv 1$, $\ell_2(x,(a_2,2)) = 1 - a_2$. For $x \in \Gamma_2 \backslash \{O\}$, then $\mathsf{v}(x) = v_2(x) = 0$ with optimal strategy consists in choosing $\alpha(t) \equiv (1,2)$. For $x \in \Gamma_1$, we can check that $\mathsf{v}(x) = \min\left\{\frac{1}{\lambda}, \frac{1-e^{-\lambda|x|}}{\lambda} + c_2 e^{-\lambda|x|}\right\}$. More precisely, for all $x \in \Gamma_1$, we have

$$\mathsf{v}(x) = \begin{cases} \frac{1}{\lambda} & \text{if } c_2 \geqslant \frac{1}{\lambda}, \text{ with the optimal control } \alpha(t) \equiv (-1,1), \\ \frac{1-e^{-\lambda|x|}}{\lambda} + c_2 e^{-\lambda|x|} & \text{if } c_2 < \frac{1}{\lambda}, \text{ with the optimal control } \alpha(t) = \begin{cases} (1,1) & \text{if } t \leqslant |x|, \\ (1,2) & \text{if } t \geqslant |x|. \end{cases} \end{cases}$$

Summarizing, we have the two following cases

1. If $c_2 \geqslant 1/\lambda$, then
$$\mathsf{v}(x) = \begin{cases} 0 & \text{if } x \in \Gamma_2 \backslash \{O\}, \\ \frac{1}{\lambda} & \text{if } x \in \Gamma_1. \end{cases}$$
The graph of the value function with entry costs $c_2 \geqslant 1/\lambda = 1$ is plotted in the left of Figure 2.2.

2. If $c_2 < 1/\lambda$, then
$$\mathsf{v}(x) = \begin{cases} 0 & \text{if } x \in \Gamma_2 \backslash \{O\}, \\ \frac{1-e^{-\lambda|x|}}{\lambda} + c_2 e^{-\lambda|x|} & \text{if } x \in \Gamma_1. \end{cases}$$
The graph of the value function with entry costs $c_2 = 1/2 < 1 = 1/\lambda$ is plotted in the right of Figure 2.2.

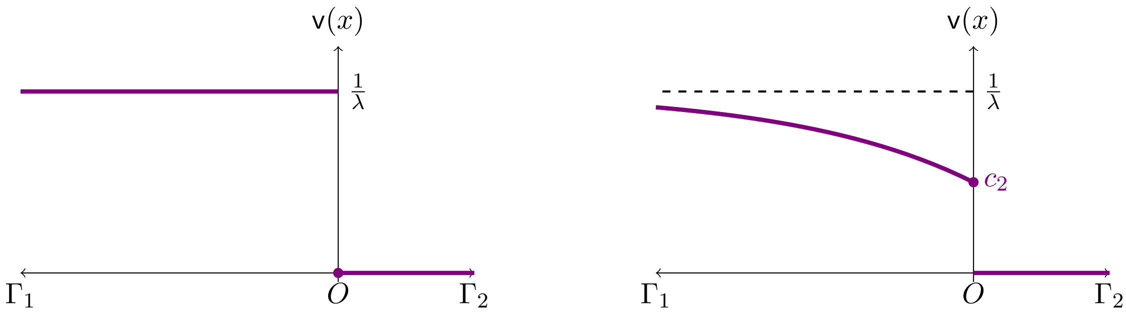

Figure 2.2: Left: The value function v with entry cost $c_2 \geqslant 1/\lambda = 4$. Right: The value function v with entry cost $c_2 = 2 < 1/\lambda = 4$.

Lemma 2.2.7. *Under assumptions [H1] and [H4], there exist two positive numbers r_0 and C such that for all $x_1, x_2 \in B(O, r_0) \cap \mathcal{G}$, there exists $\left(y_{x_1,\alpha_{x_1,x_2}}, \alpha_{x_1,x_2}\right) \in \mathcal{T}_{x_1}$ and $\tau_{x_1,x_2} \leqslant Cd(x_1,x_2)$ such that $y_{x_1}(\tau_{x_1,x_2}) = x_2$.*

Proof of Lemma 2.2.7. This proof is classical. It is sufficient to consider the case when x_1 and x_2 belong to same edge Γ_i, since in the other cases, we will use O as a connecting point between x_1

and x_2. According to Assumption $[H4]$, there exists $a \in A_i$ such that $f_i(O,a) = \delta$. Additionally, by the Lipschitz continuity of f_i,

$$|f_i(O,a) - f_i(x,a)| \leqslant L\,|x|\,,$$

hence, if we choose $r_0 := \delta/2L > 0$, then $f_i(x,a) \geqslant \delta/2$ for all $x \in B(O,r_0) \cap \Gamma_i$. Let x_1, x_2 be in $B(O,r_0) \cap \Gamma_i$ with $|x_1| < |x_2|$: there exist a control law α and $\tau_{x_1,x_2} > 0$ such that $\alpha(t) = a$ if $0 \leqslant t \leqslant \tau_{x_1,x_2}$ and $y_{x_1,\alpha}(\tau_{x_1,x_2}) = x_2$. Moreover, since the velocity $f_i(y_{x_1,\alpha}(t), \alpha(t))$ is always greater than $\delta/2$ when $t \leqslant \tau_{x_1,x_2}$, then $\tau_{x_1,x_2} \leqslant 2/\delta d(x_1,x_2)$. If $|x_1| > |x_2|$, the proof is achieved by replacing $a \in A_i$ by $\overline{a} \in A_i$ such that $f_i(O,\overline{a}) = -\delta$ and applying the same argument as above. □

2.2.3 Some properties of value function at the vertex

Lemma 2.2.8. *Under assumption $[H]$, $\mathsf{v}|_{\Gamma_i\backslash\{O\}}$ and $\hat{\mathsf{v}}|_{\Gamma_i\backslash\{O\}}$ are continuous for any $i = \overline{1,N}$. Moreover, there exists $\varepsilon > 0$ such that $\mathsf{v}|_{\Gamma_i\backslash\{O\}}$ and $\hat{\mathsf{v}}|_{\Gamma_i\backslash\{O\}}$ are Lipschitz continuous in $(\Gamma_i\backslash\{O\}) \cap B(O,\varepsilon)$. Hence, it is possible to extend $\mathsf{v}|_{\Gamma_i\backslash\{O\}}$ and $\hat{\mathsf{v}}|_{\Gamma_i\backslash\{O\}}$ at O into Lipschitz continuous functions in $\Gamma_i \cap B(O,\varepsilon)$. Hereafter, v_i and $\hat{v}_i$ denote these extensions.*

Proof of Lemma 2.2.8. The proof of continuity inside the edge is classical by using $[H4]$, see [4] for more details. The proof of Lipschitz continuity is a consequence of Lemma 2.2.7. Indeed, for x, y belong to $\Gamma_i \cap B(0,\varepsilon)$, by Lemma 2.2.7 and the definition of value function, we have

$$\mathsf{v}(x) - \mathsf{v}(z) = v_i(x) - v_i(z) \leqslant \int_0^{\tau_{x,z}} \ell_i\left(y_{x,\alpha_{x,z}}(t), \alpha_{x,z}(t)\right) e^{-\lambda t} dt + v_i(z)\left(e^{-\lambda \tau_{x,z}} - 1\right).$$

Since ℓ_i is bounded by M from $[H2]$, v_i is bounded in $\Gamma_i \cap B(O,\varepsilon)$ and $e^{-\lambda\tau_{x,z}} - 1$ is bounded by $\tau_{x,y}$, there exists a constant $\overline{C}$ such that

$$v_i(x) - v_i(z) \leqslant \overline{C}\tau_{x,z} \leqslant \overline{C}C\,|x - z|\,.$$

The last inequality follows from the Lemma 2.2.7. The inequality $v_i(z) - v_i(x) \leqslant \overline{C}C\,|x-z|$ is obtained in a similar way. The proof is done. □

Let us define the tangential Hamiltonian H_O^T at vertex O by

$$H_O^T = \max_{i=\overline{1,N}} \max_{a_i \in A_i^O} \{-\ell_j(O,a_j)\} = -\min_{i=\overline{1,N}} \min_{a_i \in A_i^O} \{\ell_j(O,a_j)\}, \tag{2.2.1}$$

where $A_i^O = \{a_i \in A_i : f_i(O,a_i) = 0\}$. The relation between the values $\mathsf{v}(O)$, $v_i(O)$ and H_O^T will be given in the next theorem. Hereafter, the proofs of the results will be supplied only for the value function with entry costs v, the proofs concerning the value function with exit costs $\hat{\mathsf{v}}$ are totally similar.

Theorem 2.2.9. *Under assumption $[H]$, the value functions v and $\hat{\mathsf{v}}$ satisfy*

$$\mathsf{v}(O) = \min\left\{\min_{i=\overline{1,N}}\{v_i(O) + c_i\}, -\frac{H_O^T}{\lambda}\right\},$$

and

$$\hat{\mathsf{v}}(O) = \min\left\{\min_{i=\overline{1,N}}\{\hat{v}_i(O)\}, -\frac{H_O^T}{\lambda}\right\}.$$

Remark 2.2.10. Theorem 2.2.9 gives us the characterization of the value function at vertex O.

The proof of Theorem 2.2.9, makes use of Lemma 2.2.11 and Lemma 2.2.12 below.

Lemma 2.2.11 (Value functions v and $\hat{\mathsf{v}}$ at O)**.** *Under assumption* $[H]$*, then*

$$\max_{i=\overline{1,N}} \{v_i(O)\} \leqslant \mathsf{v}(O) \leqslant \min_{i=\overline{1,N}} \{v_i(O) + c_i\},$$

and

$$\max_{i=\overline{1,N}} \{\hat{v}_i(O) - d_i\} \leqslant \hat{\mathsf{v}}(O) \leqslant \min_{i=\overline{1,N}} \{\hat{v}_i(O)\}.$$

Proof of Lemma 2.2.11. We divide the proof into two parts.

Prove that $\max_{i=\overline{1,N}} \{v_i(O)\} \leqslant \mathsf{v}(O)$. First, we fix $i \in \{1, \ldots, N\}$ and any control law $\overline{\alpha}$ such that $(y_{O,\bar{\alpha}}, \bar{\alpha}) \in \mathcal{T}_O$. Let $x \in \Gamma_i \backslash \{O\}$ such that $|x|$ is small. From Lemma 2.2.7, there exists a control law $\alpha_{x,O}$ connecting x and O and we consider

$$\alpha(s) = \begin{cases} \alpha_{x,O}(s) & \text{if } s \leqslant \tau_{x,O}, \\ \bar{\alpha}(s - \tau_{x,O}) & \text{if } s > \tau_{x,O}. \end{cases}$$

It means that the trajectory goes from x to O with the control law $\alpha_{x,O}$ and then proceeds with the control law $\bar{\alpha}$. Therefore

$$\mathsf{v}(x) = v_i(x) \leqslant J(x, \alpha) = \int_0^{\tau_{x,O}} \ell_i(y_{x,\alpha}(s), \alpha(s)) e^{-\lambda s} ds + e^{-\lambda \tau_{x,O}} J(O, \bar{\alpha}).$$

Since $\overline{\alpha}$ is chosen arbitrarily and ℓ_i is bounded by M, we get

$$v_i(x) \leqslant M\tau_{x,O} + e^{-\lambda \tau_{x,O}} \mathsf{v}(O).$$

Let x tend to O then $\tau_{x,O}$ tend to 0 from Lemma 2.2.7. Therefore, $v_i(O) \leqslant \mathsf{v}(O)$. Since the above inequality holds for $i = \overline{1,N}$, we obtain that

$$\max_{i=\overline{1,N}} \{v_i(O)\} \leqslant \mathsf{v}(O).$$

Prove that $\mathsf{v}(O) \leqslant \min_{i=\overline{1,N}} \{v_i(O) + c_i\}$. For $i = \overline{1,N}$; we claim that $\mathsf{v}(O) \leqslant v_i(O) + c_i$. Consider $x \in \Gamma_i \backslash \{O\}$ with $|x|$ small enough and any control law $\bar{\alpha}_x$ such that $(y_{x,\bar{\alpha}_x}, \bar{\alpha}_x) \in \mathcal{T}_x$. From Lemma 2.2.7, there exists a control law $\alpha_{O,x}$ connecting O and x and we consider

$$\alpha(s) = \begin{cases} \alpha_{O,x}(s) & \text{if } s \leqslant \tau_{O,x}, \\ \bar{\alpha}_x(s - \tau_{O,x}) & \text{if } s > \tau_{O,x}. \end{cases}$$

It means that the trajectory goes from O to x using the control law $\alpha_{O,x}$ then proceeds with the control law $\bar{\alpha}_x$. Therefore

$$\mathsf{v}(O) \leqslant J(O, \alpha) = c_i + \int_0^{\tau_{O,x}} \ell_i(y_{O,\alpha}(s), \alpha(s)) e^{-\lambda s} ds + e^{-\lambda \tau_{O,x}} J(x, \bar{\alpha}_x).$$

Since $\overline{\alpha}_x$ is chosen arbitrarily and ℓ_i is bounded by M, we get

$$\mathsf{v}(O) \leqslant c_i + M\tau_{O,x} + e^{-\lambda \tau_{O,x}} v_i(x)$$

Let x tend to O then $\tau_{O,x}$ tends to 0 from Lemma 2.2.7, then $\mathsf{v}(O) \leqslant c_i + v_i(O)$. Since the above inequality holds for $i = \overline{1,N}$, we obtain that

$$\mathsf{v}(O) \leqslant \min_{i=\overline{1,N}} \{v_i(O) + c_i\}.$$

□

Lemma 2.2.12. *The value functions* v *and* $\hat{\mathsf{v}}$ *satisfy*

$$\mathsf{v}(O), \hat{\mathsf{v}}(O) \leqslant -\frac{H_O^T}{\lambda} \tag{2.2.2}$$

where H_O^T *is defined in* (2.2.1).

Proof of Lemma 2.2.12. From (2.2.1), there exists $j \in \{1, \dots, N\}$ and $a_j \in A_j^O$ such that

$$H_O^T = -\min_{i=\overline{1,N}} \min_{a_i \in A_i^O} \{\ell_i(O, a_i)\} = -\ell_j(O, a_j)$$

Let the control law α be defined by $\alpha(s) \equiv a_j$ for all s, then

$$\mathsf{v}(O) \leqslant J(O, \alpha) = \int_0^{+\infty} \ell_j(O, a_j) e^{-\lambda s} ds = \frac{\ell_j(O, a_j)}{\lambda} = -\frac{H_O^T}{\lambda}.$$

□

We are ready to prove Theorem 2.2.9.

Proof of Theorem 2.2.9. According to Lemma 2.2.11 and Lemma 2.2.12,

$$\mathsf{v}(O) \leqslant \min\left\{\min_{i=\overline{1,N}} \{v_i(O) + c_i\}, -\frac{H_O^T}{\lambda}\right\}.$$

Assuming that

$$\mathsf{v}(O) < \min_{i=\overline{1,N}} \{v_i(O) + c_i\}, \tag{2.2.3}$$

it is sufficient to prove that $\mathsf{v}(O) = -\dfrac{H_O^T}{\lambda}$. By (2.2.3), there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that $\varepsilon_n \to 0$ and

$$\mathsf{v}(O) + \varepsilon_n < \min_{i=\overline{1,N}} \{v_i(O) + c_i\} \quad \text{for all } n \in \mathbb{N}.$$

On the other hand, there exists an ε_n-optimal control α_n, $\mathsf{v}(O) + \varepsilon_n > J(O, \alpha_n)$. Let us define the first time that the trajectory y_{O,α_n} leaves O

$$t_n := \inf_{i=\overline{1,N}} T_i^n,$$

where T_i^n is the set of times t for which $y_{O,\alpha_n}(t)$ belongs to $\Gamma_i \backslash \{O\}$. Notice that t_n is possibly $+\infty$, in which case $y_{O,\alpha_n}(s) = O$ for all $s \in [0, +\infty)$. Extracting a subsequence if necessary, we may assume that t_n tends to $\bar{t} \in [0, +\infty]$ when ε_n tends to 0.

If there exists a subsequence of $\{t_n\}_{n\in\mathbb{N}}$ (which is still noted $\{t_n\}_{n\in\mathbb{N}}$) such that $t_n = +\infty$ for all $n \in \mathbb{N}$, then for a.e. $s \in [0, +\infty)$

$$\begin{cases} f(y_{O,\alpha_n}(s), \alpha_n(s)) & = f(O, \alpha_n(s)) = 0, \\ \ell(y_{O,\alpha_n}(s), \alpha_n(s)) & = \ell(O, \alpha_n(s)). \end{cases}$$

In this case, $\alpha_n(s) \in \cup_{i=1}^N A_i^O$ for a.e. $s \in [0, +\infty)$. Therefore, for a.e. $s \in [0, +\infty)$

$$\ell(y_{O,\alpha_n}(s), \alpha_n(s)) = \ell(O, \alpha_n(s)) \geqslant -H_O^T,$$

and

$$\mathsf{v}(O)+\varepsilon_n > J(O,\alpha_n) = \int_0^{+\infty} \ell(O,\alpha_n(s))\, e^{-\lambda s} ds \geqslant \int_0^{+\infty} \left(-H_O^T\right) e^{-\lambda s} ds = -\frac{H_O^T}{\lambda}.$$

By letting n tend to ∞, we get $\mathsf{v}(O) \geqslant -H_O^T/\lambda$. On the other hand, since $\mathsf{v}(O) \leqslant -H_O^T/\lambda$ by Lemma 2.2.12, this implies that $\mathsf{v}(O) = -H_O^T/\lambda$.

Let us now assume that $0 \leqslant t_n < +\infty$ for all n large enough. Then, for a fixed n and for any positive $\delta \leqslant \delta_n$ where δ_n small enough, $y_{O,\alpha_n}(s)$ still belongs to some $\Gamma_{i(n)}\backslash\{O\}$ for all $s \in (t_n, t_n+\delta]$. We have

$$\begin{aligned}
\mathsf{v}(O)+\varepsilon_n &> J(O,\alpha_n) \\
&= \int_0^{t_n} \ell(y_{O,\alpha_n}(s),\alpha_n(s))\, e^{-\lambda s} ds + c_{i(n)} e^{-\lambda t_n} + \int_{t_n}^{t_n+\delta} \ell_{i(n)}(y_{O,\alpha_n}(s),\alpha_n(s))\, e^{-\lambda s} ds \\
&\quad + e^{-\lambda(t_n+\delta)} J(y_{O,\alpha_n}(t_n+\delta), \alpha_n(\cdot + t_n+\delta)) \\
&\geqslant \int_0^{t_n} \ell(y_{O,\alpha_n}(s),\alpha_n(s))\, e^{-\lambda s} ds + c_{i(n)} e^{-\lambda t_n} + \int_{t_n}^{t_n+\delta} \ell_{i(n)}(y_{O,\alpha_n}(s),\alpha_n(s))\, e^{-\lambda s} ds \\
&\quad + e^{-\lambda(t_n+\delta)} \mathsf{v}(y_{O,\alpha_n}(t_n+\delta)) \\
&= \int_0^{t_n} \ell(y_{O,\alpha_n}(s),\alpha_n(s))\, e^{-\lambda s} ds + c_{i(n)} e^{-\lambda t_n} + \int_{t_n}^{t_n+\delta} \ell_{i(n)}(y_{O,\alpha_n}(s),\alpha_n(s))\, e^{-\lambda s} ds \\
&\quad + e^{-\lambda(t_n+\delta)} v_{i(n)}(y_{O,\alpha_n}(t_n+\delta)).
\end{aligned}$$

By letting δ tend to 0,

$$\mathsf{v}(O)+\varepsilon_n \geqslant \int_0^{t_n} \ell(y_{O,\alpha_n}(s),\alpha_n(s))\, e^{-\lambda s} ds + c_{i(n)} e^{-\lambda t_n} + e^{-\lambda t_n} v_{i(n)}(O).$$

Note that $y_{O,\alpha_n}(s) = O$ for all $s \in [0,t_n]$, i.e., $f(O,\alpha_n(s)) = 0$ a.e. $s \in [0,t_n)$. Hence

$$\begin{aligned}
\mathsf{v}(O)+\varepsilon_n &\geqslant \int_0^{t_n} \ell(O,\alpha_n(s))\, e^{-\lambda s} ds + c_{i(n)} e^{-\lambda t_n} + e^{-\lambda t_n} v_{i(n)}(O) \\
&\geqslant \int_0^{t_n} \left(-H_O^T\right) e^{-\lambda s} ds + c_{i(n)} e^{-\lambda t_n} + e^{-\lambda t_n} v_{i(n)}(O) \\
&= \frac{1-e^{-\lambda t_n}}{\lambda} \left(-H_O^T\right) + c_{i(n)} e^{-\lambda t_n} + e^{-\lambda t_n} v_{i(n)}(O).
\end{aligned}$$

Choose a subsequence $\{\varepsilon_{n_k}\}_{k\in\mathbb{N}}$ of $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that for some $i_0 \in \{1,\dots,N\}$, $c_{i(n_k)} = c_{i_0}$ for all k. By letting k tend to ∞, recall that $\lim_{k\to\infty} t_{n_k} = \bar{t}$, we have three possible cases

1. If $\bar{t} = +\infty$, then $\mathsf{v}(O) \geqslant -\dfrac{H_O^T}{\lambda}$. By Lemma 2.2.12, we obtain $\mathsf{v}(O) = -\dfrac{H_O^T}{\lambda}$.
2. If $\bar{t} = 0$, then $\mathsf{v}(O) \geqslant c_{i_0} + v_{i_0}(O)$. By (2.2.3), we obtain a contradiction.
3. If $\bar{t} \in (0,+\infty)$, then $\mathsf{v}(O) \geqslant \dfrac{1-e^{-\lambda \bar{t}}}{\lambda}\left(-H_O^T\right) + \left[c_{i_0} + v_{i_0}(O)\right] e^{-\lambda \bar{t}}$. By (2.2.3), $c_{i_0} + v_{i_0}(O) > \mathsf{v}(O)$, so
$$\mathsf{v}(O) > \frac{1-e^{-\lambda \bar{t}}}{\lambda}\left(-H_O^T\right) + \mathsf{v}(O)\, e^{-\lambda \bar{t}}.$$
This yields $\mathsf{v}(O) > -H_O^T/\lambda$, and finally obtain a contradiction by Lemma 2.2.12.

□

2.3 The Hamilton-Jacobi systems. Viscosity solutions

2.3.1 Test-functions

Definition 2.3.1. A function $\varphi : \Gamma_1 \times \ldots \times \Gamma_N \to \mathbb{R}^N$ is an admissible test-function if there exists $(\varphi_i)_{i=\overline{1,N}}$, $\varphi_i \in C^1(\Gamma_i)$, such that $\varphi(x_1, \ldots, x_N) = (\varphi_1(x_1), \ldots, \varphi_N(x_N))$. The set of admissible test-function is denoted by $\mathcal{R}(\mathcal{G})$.

2.3.2 Definition of viscosity solution

Definition 2.3.2 (Hamiltonian)**.** We define the Hamiltonian $H_i : \Gamma_i \times \mathbb{R} \to \mathbb{R}$ by

$$H_i(x,p) = \max_{a\in A_i} \{-pf_i(x,a) - \ell_i(x,a)\}$$

and the Hamiltonian $H_i^+(O,\cdot) : \mathbb{R} \to \mathbb{R}$ by

$$H_i^+(O,p) = \max_{a\in A_i^+} \{-pf_i(O,a) - \ell_i(O,a)\},$$

where $A_i^+ = \{a_i \in A_i : f_i(O,a_i) \geqslant 0\}$. Recall that *the tangential Hamiltonian at* O, H_O^T, has been defined in (2.2.1).

We now introduce the Hamilton-Jacobi system for the case with entry costs

$$\begin{aligned} \lambda u_i(x) + H_i\left(x, \frac{du_i}{dx_i}(x)\right) &= 0 && \text{if } x \in \Gamma_i\backslash\{O\}, \\ \lambda u_i(O) + \max\left\{-\lambda \min_{j\neq i}\{u_j(O)+c_j\}, H_i^+\left(O, \frac{du_i}{dx_i}(O)\right), H_O^T\right\} &= 0 && \text{if } x = O, \end{aligned} \tag{2.3.1}$$

for all $i = \overline{1,N}$ and the Hamilton-Jacobi system with exit costs

$$\begin{aligned} \lambda \hat{u}_i(x) + H_i\left(x, \frac{d\hat{u}_i}{dx_i}(x)\right) &= 0 && \text{if } x \in \Gamma_i\backslash\{O\}, \\ \lambda \hat{u}_i(O) + \max\left\{-\lambda \min_{j\neq i}\{\hat{u}_j(O)+d_i\}, H_i^+\left(O, \frac{d\hat{u}_i}{dx_i}(O)\right), H_O^T - \lambda d_i\right\} &= 0 && \text{if } x = O, \end{aligned} \tag{2.3.2}$$

for all $i = \overline{1,N}$ and their viscosity solutions.

Definition 2.3.3 (Viscosity solution with entry costs)**.**

- A function $u := (u_1, \ldots, u_N)$ where $u_i \in USC(\Gamma_i;\mathbb{R})$ for all $i = \overline{1,N}$, is called a *viscosity subsolution* of (2.3.1) if for any $(\varphi_1, \ldots, \varphi_N) \in \mathcal{R}(\mathcal{G})$, any $i = \overline{1,N}$ and any $x_i \in \Gamma_i$ such that $u_i - \varphi_i$ has a *local maximum point* on Γ_i at x_i, then

$$\begin{aligned} \lambda u_i(x_i) + H_i\left(x, \frac{d\varphi_i}{dx_i}(x_i)\right) &\leqslant 0 && \text{if } x_i \in \Gamma_i\backslash\{O\}, \\ \lambda u_i(O) + \max\left\{-\lambda \min_{j\neq i}\{u_j(O)+c_j\}, H_i^+\left(O, \frac{d\varphi_i}{dx_i}(O)\right), H_O^T\right\} &\leqslant 0 && \text{if } x_i = O. \end{aligned}$$

- A function $u := (u_1, \ldots, u_N)$ where $u_i \in LSC(\Gamma_i;\mathbb{R})$ for all $i = \overline{1,N}$, is called a *viscosity supersolution* of (2.3.1) if for any $(\varphi_1, \ldots, \varphi_N) \in \mathcal{R}(\mathcal{G})$, any $i = \overline{1,N}$ and any $x_i \in \Gamma_i$ such that $u_i - \varphi_i$ has a *local minimum point* on Γ_i at x_i, then

$$\begin{aligned} \lambda u_i(x_i) + H_i\left(x_i, \frac{d\varphi_i}{dx_i}(x_i)\right) &\geqslant 0 && \text{if } x_i \in \Gamma_i\backslash\{O\}, \\ \lambda u_i(O) + \max\left\{-\lambda \min_{j\neq i}\{u_j(O)+c_j\}, H_i^+\left(O, \frac{d\varphi_i}{dx_i}(O)\right), H_O^T\right\} &\geqslant 0 && \text{if } x_i = O. \end{aligned}$$

• A functions $u := (u_1, \dots, u_N)$ where $u_i \in C(\Gamma_i; \mathbb{R})$ for all $i = \overline{1, N}$, is called a *viscosity solution* of (2.3.1) if it is both a viscosity subsolution and a viscosity supersolution of (2.3.1).

Definition 2.3.4 (Viscosity solution with exit costs)**.**
• A function $\hat{u} := (\hat{u}_1, \dots, \hat{u}_N)$ where $\hat{u}_i \in USC(\Gamma_i; \mathbb{R})$ for all $i = \overline{1, N}$, is called a *viscosity subsolution* of (2.3.2) if for any $(\psi_1, \dots, \psi_N) \in \mathcal{R}(\mathcal{G})$, any $i = \overline{1, N}$ and any $y_i \in \Gamma_i$ such that $\hat{u}_i - \psi_i$ has a *local maximum point* on Γ_i at y_i, then

$$\begin{aligned}
\lambda \hat{u}_i(y_i) + H_i\left(y_i, \frac{d\psi_i}{dx_i}(y_i)\right) &\leqslant 0 && \text{if } y_i \in \Gamma_i \backslash \{O\}, \\
\lambda \hat{u}_i(O) + \max\left\{-\lambda \min_{j \neq i}\{\hat{u}_j(O)\} - \lambda d_i, H_i^+\left(O, \frac{d\psi_i}{dx_i}(O)\right), H_O^T - \lambda d_i\right\} &\leqslant 0 && \text{if } y_i = O.
\end{aligned}$$

• A function $\hat{u} := (\hat{u}_1, \dots, \hat{u}_N)$ where $\hat{u}_i \in LSC(\Gamma_i; \mathbb{R})$ for all $i = \overline{1, N}$, is called a *viscosity supersolution* of (2.3.2) if for any $(\psi_1, \dots, \psi_N) \in \mathcal{R}(\mathcal{G})$, any $i = \overline{1, N}$ and any $y_i \in \Gamma_i$ such that $u_i - \psi_i$ has a *local minimum point* on Γ_i at y_i, then

$$\begin{aligned}
\lambda \hat{u}_i(y_i) + H_i\left(y_i, \frac{d\psi_i}{dx_i}(y_i)\right) &\geqslant 0 && \text{if } y_i \in \Gamma_i \backslash \{O\}, \\
\lambda \hat{u}_i(O) + \max\left\{-\lambda \min_{j \neq i}\{\hat{u}_j(O)\} - \lambda d_i, H_i^+\left(O, \frac{d\psi_i}{dx_i}(O)\right), H_O^T - \lambda d_i\right\} &\geqslant 0 && \text{if } y_i = O.
\end{aligned}$$

• A functions $\hat{u} := (\hat{u}_1, \dots, \hat{u}_N)$ where $\hat{u}_i \in C(\Gamma_i; \mathbb{R})$ for all $i = \overline{1, N}$, is called a *viscosity solution* of (2.3.2) if it is both a viscosity subsolution and a viscosity supersolution of (2.3.2).

Remark 2.3.5. This notion of viscosity solution is consitent with the one of [8]. It can be seen in Section 2.6 when all the switching costs are zero, our definition and the one of [8] coincide.

2.4 Connections between the value functions and the Hamilton-Jacobi systems.

Let v be the value function of the optimal control problem with entry costs and $\hat{\mathsf{v}}$ be a value function of the optimal control problem with exit costs. Recall that $v_i, \hat{v}_i : \Gamma_i \to \mathbb{R}$ are defined in Lemma 2.2.8 by

$$\begin{cases} v_i(x) = \mathsf{v}(x) & \text{if } x \in \Gamma_i \backslash \{O\}, \\ v_i(O) = \lim_{\Gamma_i \backslash \{O\} \ni x \to O} \mathsf{v}(x), \end{cases} \quad \text{and} \quad \begin{cases} \hat{v}_i(x) = \hat{\mathsf{v}}(x) & \text{if } x \in \Gamma_i \backslash \{O\}, \\ \hat{v}_i(O) = \lim_{\Gamma_i \backslash \{O\} \ni x \to O} \hat{\mathsf{v}}(x). \end{cases}$$

We wish to prove that $v := (v_1, v_2, \dots, v_N)$ and $\hat{v} := (\hat{v}_1, \dots, \hat{v}_N)$ are respectively viscosity solutions of (2.3.1) and (2.3.2). In fact, since $\mathcal{G} \backslash \{O\}$ is a finite union of open intervals in which the classical theory can be applied, we obtain that v_i and $\hat{v}_i$ are viscosity solutions of

$$\lambda u(x) + H_i(x, Du(x)) = 0 \quad \text{in } \Gamma_i \backslash \{O\}.$$

Therefore, we can restrict ourselves to prove the following theorem.

Theorem 2.4.1. *For $i = \overline{1, N}$, the function v_i satisfies*

$$\lambda v_i(O) + \max\left\{-\lambda \min_{j \neq i}\{v_j(O) + c_j\}, H_i^+\left(O, \frac{dv_i}{dx_i}(O)\right), H_O^T\right\} = 0$$

in the viscosity sense. The function $\hat{v}_i$ *satisfies*

$$\lambda \hat{v}_i(O) + \max\left\{-\lambda \min_{j\neq i}\{\hat{v}_j(O) + d_i\}, H_i^+\left(O, \frac{d\hat{v}_i}{dx_i}(O)\right), H_O^T - \lambda d_i\right\} = 0$$

in the viscosity sense.

The proof of Theorem 2.4.1 follows from Lemmas 2.4.2 and 2.4.5 below. We focus on v_i since the proof for $\hat{v}_i$ is similar.

Lemma 2.4.2. *For* $i = \overline{1,N}$*, the function* v_i *is a viscosity subsolution of* (2.3.1) *at* O*.*

Proof of Lemma 2.4.2. From Theorem 2.2.9,

$$\lambda v_i(O) + \max\left\{-\lambda \min_{j\neq i}\{v_j(O) + c_j\}, H_O^T\right\} \leqslant 0.$$

It is thus sufficient to prove that

$$\lambda v_i(O) + H_i^+\left(O, \frac{dv_i}{dx_i}(O)\right) \leqslant 0$$

in the viscosity sense. Let $a_i \in A_i$ be such that $f_i(O, a_i) > 0$. Setting $\alpha(t) \equiv a_i$ then $(y_{x,\alpha}, \alpha) \in \mathcal{T}_x$ for all $x \in \Gamma_i$. Moreover, for all $x \in \Gamma_i \backslash \{O\}$, $y_{x,\alpha}(t) \in \Gamma_i \backslash \{O\}$ (the trajectory cannot approach O since the speed pushes it away from O for $y_{x,\alpha} \in \Gamma_i \cap B(O,r)$). Note that it is not sufficient to choose $a_i \in A_i$ such that $f(O, a_i) = 0$ since it can lead to $f(x, a_i) < 0$ for all $x \in \Gamma_i \backslash \{O\}$. Next, for $\tau > 0$ fixed and any $x \in \Gamma_i$, if we choose

$$\alpha_x(t) = \begin{cases} \alpha(t) = a_i & 0 \leqslant t \leqslant \tau, \\ \hat{\alpha}(t - \tau) & t \geqslant \tau, \end{cases} \tag{2.4.1}$$

where $\hat{\alpha}$ is arbitrary, then $y_{x.\alpha_x}(t) \in \Gamma_i \backslash \{O\}$ for all $t \in [0, \tau]$. It yields

$$v_i(x) \leqslant J(x, \alpha_x) = \int_0^\tau \ell_i(y_{x,\alpha}(s), a_i) e^{-\lambda s} ds + e^{-\lambda \tau} J(y_{x,\alpha}(\tau), \hat{\alpha}).$$

Since this holds for any $\hat{\alpha}$ (α_x is arbitrary for $t > \tau$), we deduce that

$$v_i(x) \leqslant \int_0^\tau \ell_i(y_{x,\alpha_x}(s), a_i) e^{-\lambda s} ds + e^{-\lambda \tau} v_i(y_{x,\alpha_x}(\tau)). \tag{2.4.2}$$

Since $f_i(\cdot, a)$ is Lipschitz continuous by $[H1]$, we also have for all $t \in [0, \tau]$,

$$\begin{aligned} |y_{x,\alpha_x}(t) - y_{O,\alpha_O}(t)| &= \left| x + \int_0^t f_i(y_{x,\alpha}(s), a_i) e_i ds - \int_0^t f_i(y_{O,\alpha}(s), a_i) e_i ds \right| \\ &\leqslant |x| + L \int_0^t |y_{x,\alpha}(s) - y_{O,\alpha}(s)| ds, \end{aligned}$$

where α_0 satisfies (2.4.1) with $x = O$. According to Grönwall's inequality,

$$|y_{x,\alpha_x}(t) - y_{O,\alpha_O}(t)| \leqslant |x| e^{Lt},$$

for $t \in [0, \tau]$, yielding that $y_{x,\alpha_x}(t)$ tends to $y_{O,\alpha_O}(t)$ when x tends to O. Hence, from (2.4.2), by letting $x \to O$, we obtain

$$v_i(O) \leqslant \int_0^\tau \ell_i\left(y_{O,\alpha_O}(s), a_i\right) e^{-\lambda s} ds + e^{-\lambda\tau} v_i\left(y_{O,\alpha_O}(\tau)\right).$$

Let φ be a function in $C^1(\Gamma_i)$ such that $0 = v_i(O) - \varphi(O) = \max_{\Gamma_i}(v_i - \varphi)$. This yields

$$\frac{\varphi(O) - \varphi\left(y_{O,\alpha_O}(\tau)\right)}{\tau} \leqslant \frac{1}{\tau}\int_0^\tau \ell_i\left(y_{O,\alpha_O}(s), a_i\right) e^{-\lambda s} ds + \frac{\left(e^{-\lambda\tau} - 1\right) v_i\left(y_{O,\alpha_O}(\tau)\right)}{\tau}.$$

By letting τ tend to 0, we obtain that

$$-f_i(O, a_i)\frac{d\varphi}{dx_i}(O) \leqslant \ell_i(O, a_i) - \lambda v_i(O).$$

Hence,

$$\lambda v_i(O) + \sup_{a \in A_i : f_i(O,a) > 0}\left\{-f_i(O,a)\frac{dv_i}{dx_i}(O) - \ell_i(O,a)\right\} \leqslant 0$$

in the viscosity sense. Finally, from Corollary 2.7.2 in Appendix, we have

$$\sup_{a \in A_i : f_i(O,a) > 0}\left\{-f_i(O,a)\frac{d\varphi_i}{dx_i}(O) - \ell_i(O,a)\right\} = \max_{a \in A_i : f_i(O,a) \geqslant 0}\left\{-f_i(O,a)\frac{d\varphi_i}{dx_i}(O) - \ell_i(O,a)\right\}.$$

The proof is complete. □

Lemma 2.4.3. *If*

$$v_i(O) < \min\left\{\min_{j \neq i}\{v_j(O) + c_j\}, -\frac{H_O^T}{\lambda}\right\}, \tag{2.4.3}$$

then there exist $\bar{\tau} > 0, r > 0$ and $\varepsilon_0 > 0$ such that for any $x \in (\Gamma_i \backslash \{O\}) \cap B(O, r)$, any $\varepsilon < \varepsilon_0$ and any $\varepsilon-$optimal control law $\alpha_{\varepsilon,x}$ for x,

$$y_{x,\alpha_{\varepsilon,x}}(s) \in \Gamma_i \backslash \{O\}, \quad \textit{for all } s \in [0, \bar{\tau}].$$

Remark 2.4.4. Roughly speaking, this lemma takes into account the case $\lambda v_i + H_i^+\left(x, \frac{dv_i}{dx_i}(O)\right) \leqslant 0$, i.e., the situation when the trajectory does not leave Γ_i, see introduction.

Proof of Lemma 2.4.3. Suppose by contradiction that there exist sequences $\{\varepsilon_n\}, \{\tau_n\} \subset \mathbb{R}^+$ and $\{x_n\} \subset \Gamma_i \backslash \{O\}$ such that $\varepsilon_n \searrow 0$, $x_n \to O, \tau_n \searrow 0$ and a control law α_n such that α_n is ε_n-optimal control law and $y_{x_n,\alpha_n}(\tau_n) = O$. This implies that

$$v_i(x_n) + \varepsilon_n > J(x_n, \alpha_n) = \int_0^{\tau_n} \ell\left(y_{x_n,\alpha_n}(s), \alpha_n(s)\right) e^{-\lambda s} ds + e^{-\lambda\tau_n} J\left(O, \alpha_n(\cdot + \tau_n)\right). \tag{2.4.4}$$

Since ℓ is bounded by M by $[H1]$, then $v_i(x_n) + \varepsilon_n \geqslant -\tau_n M + e^{-\lambda\tau_n}\mathsf{v}(O)$. By letting n tend to ∞, we obtain

$$v_i(O) \geqslant \mathsf{v}(O). \tag{2.4.5}$$

From (2.4.3), it follows that

$$\min\left\{\min_{j \neq i}\{v_j(O) + c_j\}, -\frac{H_O^T}{\lambda}\right\} > \mathsf{v}(O).$$

However, $\mathsf{v}(O) = \min\left\{\min_j\{v_j(O) + c_j\}, -H_O^T/\lambda\right\}$ by Theorem 2.2.9. Therefore, $\mathsf{v}(O) = v_i(O) + c_i > v_i(O)$, which is a contradiction with (2.4.5). □

Lemma 2.4.5. *The function v_i is a viscosity supersolution of (2.3.1) at O.*

Proof of Lemma 2.4.5. We adapt the proof of Oudet [89] and start by assuming that

$$v_i(O) < \min\left\{\min_{j\neq i}\{v_j(O)+c_j\}, -\frac{H_O^T}{\lambda}\right\}.$$

We need to prove that

$$\lambda v_i(O) + H_i^+\left(O, \frac{dv_i}{dx_i}(O)\right) \geqslant 0$$

in the viscosity sense. Let $\varphi \in C^1(\Gamma_i)$ be such that

$$0 = v_i(O) - \varphi(O) \leqslant v_i(x) - \varphi(x) \quad \text{for all } x \in \Gamma_i, \tag{2.4.6}$$

and $\{x_\varepsilon\} \subset \Gamma_i \backslash \{O\}$ be any sequence such that x_ε tends to O when ε tends to 0. From the dynamic programming principle and Lemma 2.4.3, there exists $\bar{\tau}$ such that for any $\varepsilon > 0$, there exists $(y_\varepsilon, \alpha_\varepsilon) := (y_{x_\varepsilon,\alpha_\varepsilon}, \alpha_\varepsilon) \in \mathcal{T}_{x_\varepsilon}$ such that $y_\varepsilon(\tau) \in \Gamma_i \backslash \{O\}$ for any $\tau \in [0, \bar{\tau}]$ and

$$v_i(x_\varepsilon) + \varepsilon \geqslant \int_0^\tau \ell_i(y_\varepsilon(s), \alpha_\varepsilon(s)) e^{-\lambda s} ds + e^{-\lambda\tau} v_i(y_\varepsilon(\tau)).$$

Then, according to (2.4.6)

$$\begin{aligned} v_i(x_\varepsilon) - v_i(O) + \varepsilon \geqslant & \int_0^\tau \ell_i(y_\varepsilon(s), \alpha_\varepsilon(s)) e^{-\lambda s} ds + e^{-\lambda\tau}[\varphi(y_\varepsilon(\tau)) - \varphi(O)] \\ & - v_i(O)\left(1 - e^{-\lambda\tau}\right). \end{aligned} \tag{2.4.7}$$

Next,

$$\begin{cases} \int_0^\tau \ell_i(y_\varepsilon(s), \alpha_\varepsilon(s)) e^{-\lambda s} ds & = \int_0^\tau \ell_i(y_\varepsilon(s), \alpha_\varepsilon(s)) ds + o(\tau), \\ [\varphi(y_\varepsilon(\tau)) - \varphi(O)] e^{-\lambda\tau} & = \varphi(y_\varepsilon(\tau)) - \varphi(O) + \tau o_\varepsilon(1) + o(\tau), \end{cases}$$

and

$$\begin{cases} v_i(x_\varepsilon) - v_i(O) & = o_\varepsilon(1), \\ v_i(O)\left(1 - e^{-\lambda\tau}\right) & = o(\tau) + \tau\lambda v_i(O), \end{cases}$$

where the notation $o_\varepsilon(1)$ is used for a quantity which is independent on τ and tends to 0 as ε tends to 0. For a positive integer k, the notation $o(\tau^k)$ is used for a quantity that is independent on ε and such that $o(\tau^k)/\tau^k \to 0$ as $\tau \to 0$. Finally, $\mathcal{O}(\tau^k)$ stands for a quantity independent on ε such that $\mathcal{O}(\tau^k)/\tau^k$ remains bounded as $\tau \to 0$. From (2.4.7), we obtain that

$$\tau\lambda v_i(O) \geqslant \int_0^\tau \ell_i(y_\varepsilon(s), \alpha_\varepsilon(s)) ds + \varphi(y_\varepsilon(\tau)) - \varphi(O) + \tau o_\varepsilon(1) + o(\tau) + o_\varepsilon(1). \tag{2.4.8}$$

Since $y_\varepsilon(\tau) \in \Gamma_i$ for all ε, one has

$$\varphi(y_\varepsilon(\tau)) - \varphi(x_\varepsilon) = \int_0^\tau \frac{d\varphi}{dx_i}(y_\varepsilon(s)) \dot{y}_\varepsilon(s) ds = \int_0^\tau \frac{d\varphi}{dx_i}(y_\varepsilon(s)) f_i(y_\varepsilon(s), \alpha_\varepsilon(s)) ds.$$

Hence, from (2.4.8)

$$\tau\lambda v_i(O) - \int_0^\tau \left[\ell_i(y_\varepsilon(s), \alpha_\varepsilon(s)) + \frac{d\varphi}{dx_i}(y_\varepsilon(s)) f_i(y_\varepsilon(s), \alpha_\varepsilon(s))\right] ds \geqslant \tau o_\varepsilon(1) + o(\tau) + o_\varepsilon(1). \tag{2.4.9}$$

Moreover, $\varphi(x_\varepsilon) - \varphi(O) = o_\varepsilon(1)$ and that $\dfrac{d\varphi}{dx_i}(y_\varepsilon(s)) = \dfrac{d\varphi}{dx_i}(O) + o_\varepsilon(1) + \mathcal{O}(s)$. Thus

$$\lambda v_i(O) - \frac{1}{\tau}\int_0^\tau \left[\ell_i(y_\varepsilon(s), \alpha_\varepsilon(s)) + \frac{d\varphi}{dx_i}(O) f_i(y_\varepsilon(s), \alpha_\varepsilon(s))\right] ds \geqslant o_\varepsilon(1) + \frac{o(\tau)}{\tau} + \frac{o_\varepsilon(1)}{\tau}. \tag{2.4.10}$$

Let $\varepsilon_n \to 0$ as $n \to \infty$ and $\tau_m \to 0$ as $m \to \infty$ such that

$$(a_{mn}, b_{mn}) := \left(\frac{1}{\tau_m}\int_0^{\tau_m} f_i(y_{\varepsilon_n}(s), \alpha_{\varepsilon_n}(s)) e_i ds, \frac{1}{\tau_m}\int_0^{\tau_m} \ell_i(y_{\varepsilon_n}(s), \alpha_{\varepsilon_n}(s)) ds\right) \longrightarrow (a,b) \in \mathbb{R}e_i \times \mathbb{R}$$

as $n, m \to \infty$. By $[H1]$ and $[H2]$

$$\begin{cases} f_i(y_{\varepsilon_n}(s), \alpha_{\varepsilon_n}(s)) e_i &= f_i(O, \alpha_{\varepsilon_n}(s)) + L|y_{\varepsilon_n}(s)| = f_i(O, \alpha_{\varepsilon_n}(s)) e_i + o_n(1) + o_m(1), \\ \ell_i(y_{\varepsilon_n}(s), \alpha_{\varepsilon_n}(s)) e_i &= \ell_i(O, \alpha_{\varepsilon_n}(s)) + \omega(|y_{\varepsilon_n}(s)|) = \ell_i(O, \alpha_{\varepsilon_n}(s)) e_i + o_n(1) + o_m(1). \end{cases}$$

It follows that

$$\begin{aligned}(a_{mn}, b_{mn}) &= \left(\frac{1}{\tau_m}\int_0^{\tau_m} f_i(O, \alpha_{\varepsilon_n}(s)) e_i ds, \frac{1}{\tau_m}\int_0^{\tau_m} \ell_i(O, \alpha_{\varepsilon_n}(s)) ds\right) + o_n(1) + o_m(1) \\ &\in \mathrm{FL}_i(O) + o_n(1) + o_m(1),\end{aligned}$$

since $\mathrm{FL}_i(O)$ is closed and convex. Sending $n, m \to \infty$, we obtain $(a,b) \in \mathrm{FL}_i(O)$ so there exists $\overline{a} \in A_i$ such that

$$\lim_{m,n\to\infty}\left(\frac{1}{\tau_m}\int_0^{\tau_m} f_i(y_{\varepsilon_n}(s), \alpha_{\varepsilon_n}(s)) e_i ds, \frac{1}{\tau_m}\int_0^{\tau_m} \ell_i(y_{\varepsilon_n}(s), \alpha_{\varepsilon_n}(s)) ds\right) = (f_i(O, \overline{a}) e_i, \ell_i(O, \overline{a})). \tag{2.4.11}$$

On the other hand, from Lemma 2.4.3, $y_{\varepsilon_n}(s) \in \Gamma_i \backslash \{O\}$ for all $s \in [0, \tau_m]$. This yields

$$y_{\varepsilon_n}(\tau_m) = \left[\int_0^{\tau_n} f_i(y_{\varepsilon_n}(s), \alpha_{\varepsilon_n}(s)) ds\right] e_i + x_{\varepsilon_n}.$$

Since $|y_{\varepsilon_n}(\tau_m)| > 0$, then

$$\frac{1}{\tau_m}\int_0^{\tau_m} f_i(y_{\varepsilon_n}(s), \alpha_{\varepsilon_n}(s)) ds \geqslant -\frac{|x_{\varepsilon_n}|}{\tau_m}.$$

Let ε_n tend to 0, then let τ_m tend to 0, one gets $f_i(O, \overline{a}) \geqslant 0$, so $\overline{a} \in A_i^+$. Hence, from (2.4.10) and (2.4.11), replacing ε by ε_n and τ by τ_m, let ε_n tend to 0, then let τ_m tend to 0, we finally obtain

$$\lambda v_i(O) + \max_{a \in A_i^+}\left\{-f_i(O,a)\frac{d\varphi}{dx_i}(O) - \ell_i(O,a)\right\} \geqslant \lambda v_i(O) + \left[-f_i(O, \overline{a})\frac{d\varphi}{dx_i}(O) - \ell_i(O, \overline{a})\right] \geqslant 0.$$

□

2.5 Comparison Principle and Uniqueness

Inspired by [19, 20], we begin by proving some properties of sub and super viscosity solutions of (2.3.1). The following three lemmas are reminiscent of Lemma 3.4, Theorem 3.1 and Lemma 3.5 in [8].

Lemma 2.5.1. *Let $w = (w_1, \ldots, w_N)$ be a viscosity supersolution of (2.3.1). Let $x \in \Gamma_i \backslash \{O\}$ and assume that*

$$w_i(O) < \min\left\{\min_{j\neq i}\{w_j(O) + c_j\}, -\frac{H_O^T}{\lambda}\right\}. \tag{2.5.1}$$

Then for all $t > 0$,

$$w_i(x) \geqslant \inf_{\alpha_i(\cdot),\theta_i}\left(\int_0^{t\wedge\theta_i} \ell_i\left(y_x^i(s), \alpha_i(s)\right) e^{-\lambda s} ds + w_i\left(y_x^i(t\wedge\theta_i)\right) e^{-\lambda(t\wedge\theta_i)}\right),$$

where $\alpha_i \in L^\infty(0,\infty;A_i)$, y_x^i is the solution of $y_x^i(t) = x + \left[\int_0^t f_i\left(y_x^i(s), \alpha_i(s)\right) ds\right] e_i$ and θ_i satisfies $y_x^i(\theta_i) = 0$ and θ_i lies in $[\tau_i, \overline{\tau_i}]$, where τ_i is the exit time of y_x^i from $\Gamma_i \backslash \{O\}$ and $\overline{\tau_i}$ is the exit time of y_x^i from Γ_i.

Proof of Lemma 2.5.1. According to (2.5.1), the function w_i is a viscosity supersolution of the following problem in Γ_i

$$\begin{cases} \lambda w_i(x) + H_i\left(x, \dfrac{dw_i}{dx_i}(x)\right) & = 0 \quad \text{if } x \in \Gamma_i \backslash \{O\}, \\ \lambda w_i(O) + H_i^+\left(O, \dfrac{dw_i}{dx_i}(O)\right) & = 0 \quad \text{if } x = O. \end{cases} \tag{2.5.2}$$

Hence, we can apply the result in [8, Lemma 3.4]. We refer to [19] for a detailed proof. The main point of that proof uses the results of Blanc [24, 25] on minimal supersolutions of exit time control problems. ☐

Lemma 2.5.2 (Super-optimality)**.** *Under assumption [H], let $w = (w_1, \ldots, w_N)$ be a viscosity supersolution of (2.3.1) that satisfies (2.5.1); then there exists a sequence $\{\eta_k\}_{k\in\mathbb{N}}$ of strictly positive real numbers such that $\lim_{k\to\infty} \eta_k = \eta > 0$ and a sequence $x_k \in \Gamma_i \backslash \{O\}$ such that $\lim_{k\to\infty} x_k = O, \lim_{k\to\infty} w_i(x_k) = w_i(O)$ and for each k, there exists a control law α_i^k such that the corresponding trajectory $y_{x_k}(s) \in \Gamma_i$ for all $s \in [0, \eta_k]$ and*

$$w_i(x_k) \geqslant \int_0^{\eta_k} \ell_i\left(y_{x_k}(s), \alpha_i^k(s)\right) e^{-\lambda s} ds + w_i\left(y_{x_k}(\eta_k)\right) e^{-\lambda \eta_k}.$$

Proof of Lemma 2.5.2. According to (2.5.1) $\hat{w}_i(O) < -H_O^T/\lambda$. Hence, this proof is complete by applying the proof of in [8, Theorem 3.1]. ☐

Lemma 2.5.3. *Under assumption [H], let $u = (u_1, \ldots, u_N)$ be a viscosity subsolution of (2.3.1). Then u_i is Lipschitz continuous in $B(O,r) \cap \Gamma_i$. Therefore, there exists a test function $\varphi_i \in C^1(\Gamma_i)$ which touches u_i from above at O.*

Proof of Lemma 2.5.3. Since u is a viscosity subsolution of (2.3.1), u_i is a viscosity subsolution of (2.5.2). Recal that $H_i(x,\cdot)$ is coercive for any $x \in \Gamma_i \cap B(O,r)$, we can apply the proof in [8, Lemma 3.2], which is based on arguments due to Ishii and contained in [75]. ☐

Lemma 2.5.4 (Sub-optimality)**.** *Under assumption [H], let $u = (u_1, \ldots, u_N)$ be a viscosity subsolution of (2.3.1). Consider $i = \overline{1,N}, x \in \Gamma_i \backslash \{O\}$ and $\alpha_i \in L^\infty(0,\infty;A_i)$. Let $T > 0$ be such that $y_x(t) = x + \left[\int_0^t f_i(y_x(s), \alpha_i(s)) ds\right] e_i$ belongs to Γ_i for any $t \in [0,T]$, then*

$$u_i(x) \leqslant \int_0^T \ell_i(y_x(s), \alpha_i(s)) e^{-\lambda s} ds + u_i(y_x(T)) e^{-\lambda T}.$$

Proof of Lemma 2.5.4. Since u is a viscosity subsolution of (2.3.1), u_i is a viscosity subsolution of (2.5.2) and satisfies $u_i(O) \leqslant -H_O^T/\lambda$. Hence, we can apply the proof in [8, Lemma 3.5]. ☐

Remark 2.5.5. Under assumption $[H]$, Lemmas 2.5.1, 2.5.2, 2.5.3 and 2.5.4 hold for vicosity sub and supersolution $\hat{u}$ and $\hat{w}$ respectively, of the exit cost control problem if (2.5.1) replaced by

$$\hat{w}_i(O) < \min\left\{\min_{j \neq i}\{\hat{w}_j(O)\} + d_i, -\frac{H_O^T}{\lambda} + d_i\right\}.$$

Theorem 2.5.6 (Comparison Principle)**.** *Under assumption $[H]$, let u be a bounded viscosity subsolution of (2.3.1) and w be a bounded viscosity supersolution of (2.3.1); then $u \leqslant w$ componentwise. This theorem also holds for viscosity sub and supersolution $\hat{u}$ and $\hat{w}$, respectively, of the exit cost control problem (2.3.2).*

We give two proofs of Theorem 2.5.6. The first one is inspired by [8] and uses the previously stated lemmas. The second one uses the elegant arguments proposed in [86].

Proof of Theorem 2.5.6 inspired by [8] . We focus on u and w, the arguments used for the comparison of $\hat{u}$ and $\hat{w}$ are totally similar. Suppose by contradiction that there exists $x \in \Gamma_i$ such that $u_i(x) - w_i(x) > 0$. By classical comparison arguments for the boundary value problem, see [18], $\sup_{\partial\Gamma_i}\{u_i - v_i\}^+ \geqslant \sup_{\Gamma_i}\{u_i - v_i\}^+$, so we have

$$u_i(O) - w_i(O) = \max_{x \in \Gamma_i}\{u_i(x) - w_i(x)\} > 0.$$

By definition of viscosity subsolution

$$\lambda u_i(O) + H_O^T \leqslant 0. \tag{2.5.3}$$

This implies $\lambda w_i(O) + H_O^T < 0$. We now consider the two following cases.

Case 1: If $w_i(O) < \min_{j \neq i}\{w_j(O) + c_j\}$, from Lemma 2.5.2 (using the same notations),

$$w_i(x_k) \geqslant \int_0^{\eta_k} \ell_i\left(y_{x_k}(s), \alpha_i^k(s)\right) e^{-\lambda s} ds + w_i(y_{x_k}(\eta_k)) e^{-\lambda \eta_k}.$$

Moreover, according to Lemma 2.5.4, we also have

$$u_i(x_k) \leqslant \int_0^{\eta_k} \ell_i\left(y_{x_k}(s), \alpha_i^k(s)\right) e^{-\lambda s} ds + u_i(y_{x_k}(\eta_k)) e^{-\lambda \eta_k}.$$

This yields

$$u_i(x_k) - w_i(x_k) \leqslant [u_i(y_{x_k}(\eta_k)) - w_i(y_{x_k}(\eta_k))] e^{-\lambda \eta_k} \leqslant [u_i(O) - w_i(O)] e^{-\lambda \eta_k}.$$

By letting k tend to ∞, one gets

$$u_i(O) - w_i(O) \leqslant [u_i(O) - w_i(O)] e^{-\lambda \eta}.$$

This implies that $u_i(O) - w_i(O) \leqslant 0$ and leads to a contradiction.

Case 2: If $w_i(O) \geqslant \min_{j \neq i} \{w_j(O) + c_j\}$, then there exists $j_0 \neq i$ such that

$$w_{j_0}(O) + c_{j_0} = \min_{j=\overline{1,N}} \{w_j(O) + c_j\} = \min_{j \neq i} \{w_j(O) + c_j\} \leqslant w_i(O),$$

because $c_i > 0$. Since c_{j_0} is positive

$$w_{j_0}(O) < \min_{j \neq j_0} \{w_j(O) + c_j\}. \tag{2.5.4}$$

Next, by Lemma 2.5.3, there exists a test function φ_i in $C^1(J_i)$ that touches u_i from above at O, it yields

$$\begin{aligned} & \lambda u_i(O) - \lambda \min_{j \neq i} \{u_j(O) + c_j\} \\ \leqslant \quad & \lambda u_i(O) + \max \left\{ -\lambda \min_{j \neq i} \{u_j(O) + c_j\}, H_i^+\left(O, \frac{d\varphi_i}{dx_i}(O)\right), H_0^T \right\} \\ \leqslant \quad & 0. \end{aligned}$$

Therefore

$$w_{j_0}(O) + c_{j_0} \leqslant w_i(O) < u_i(O) \leqslant \min_{j \neq i} \{u_j(O) + c_j\} \leqslant u_{j_0}(O) + c_{j_0}.$$

Thus

$$w_{j_0}(O) < u_{j_0}(O). \tag{2.5.5}$$

Replacing index i by j_0 in (2.5.3), we get

$$\lambda w_{j_0}(O) + H_O^T < 0. \tag{2.5.6}$$

By (2.5.4) and (2.5.6), (2.5.1) holds true. Repeating the proof of *Case 1* with j_0, we reach a contradiction with (2.5.5). It ends the proof.

□

The comparison principle can also be obtained alternatively, using the arguments which were very recently proposed by Lions & Souganidis in [86]. This new proof is self-combined and the arguments do not rely at all on optimal control theory, but are deeply connected to the ideas used by Soner [95, 96] and Capuzzo-Dolcetta & Lions [36] for proving comparison principles for state-constrained Hamilton-Jacobi equations.

Proof of Theorem 2.5.6 inspired by [86]. We start as in first proof. We argue by contradiction without loss of generality, assuming that there exists i such that

$$u_i(O) - w_i(O) = \max_{\Gamma_i} \{u_i(x) - w_i(x)\} > 0.$$

Therefore $w_i(O) < -H_O^T/\lambda$. We now consider the two following cases.

Case 1: If $w_i(O) < \min_{j \neq i} \{w_j(O) + c_j\}$, then w_i is a viscosity supersolution of (2.5.2). Recall that by Lemma 2.5.3, there exists a positive number L such that for $i = \overline{1,N}$, u_i is Lipschitz continuous with Lipschitz constant L in $\Gamma_i \cap B(0,r)$. We consider the function

$$\begin{aligned} \Psi_{i,\varepsilon} : \Gamma_i \times \Gamma_i &\longrightarrow \mathbb{R} \\ (x,y) &\longrightarrow u_i(x) - w_i(y) - \frac{1}{2\varepsilon}\left[-|x| + |y| + \delta(\varepsilon)\right]^2 - \gamma(|x| + |y|), \end{aligned}$$

where $\delta(\varepsilon) = (L+1)\varepsilon$ and $\gamma \in (0, 1/2)$. It is clear that $\Psi_{i,\varepsilon}$ attains its maximum $M_{\varepsilon,\gamma}$ at $(x_{\varepsilon,\gamma}, y_{\varepsilon,\gamma}) \in \Gamma_i \times \Gamma_i$. By classical techniques, we check that $x_{\varepsilon,\gamma}, y_{\varepsilon,\gamma} \to O$ and that $(x_{\varepsilon,\gamma} - y_{\varepsilon,\gamma})^2/\varepsilon \to 0$ as $\varepsilon \to 0$. Indeed, one has

$$\begin{aligned} & u_i(x_{\varepsilon,\gamma}) - w_i(y_{\varepsilon,\gamma}) - \frac{[-|x_{\varepsilon,\gamma}| + |y_{\varepsilon,\gamma}| + \delta(\varepsilon)]^2}{2\varepsilon} - \gamma(|x_{\varepsilon,\gamma}| + |y_{\varepsilon,\gamma}|) \\ \geqslant \; & \max_{\Gamma_i} \{u_i(x) - w_i(x) - 2\gamma|x|\} - \frac{\delta^2(\varepsilon)}{2\varepsilon} \qquad (2.5.7) \\ \geqslant \; & u_i(O) - w_i(O) - \frac{(L+1)^2}{2}\varepsilon. \qquad (2.5.8) \end{aligned}$$

Since $u_i(O) - v_i(O) > 0$, the term in (2.5.8) is positive when ε is small enough. We also deduce from the above inequality and from the boundedness of u_i and w_i that, maybe after the extraction of a subsequence, $x_{\varepsilon,\gamma}, y_{\varepsilon,\gamma} \to x_\gamma$ as $\varepsilon \to 0$, for some $x_\gamma \in \Gamma_i$. From (2.5.7),

$$u_i(x_{\varepsilon,\gamma}) - w_i(y_{\varepsilon,\gamma}) - \frac{(|x_{\varepsilon,\gamma}| - |y_{\varepsilon,\gamma}|)^2}{2\varepsilon} - \frac{(-|x_{\varepsilon,\gamma}| + |y_{\varepsilon,\gamma}|)\delta(\varepsilon)}{\varepsilon} \geqslant \max_{\Gamma_i}\{u_i(x) - w_i(x) - 2\gamma|x|\}.$$

Taking the lim sup on both sides of this inequality when $\varepsilon \to 0$,

$$\begin{aligned} u_i(x_\gamma) - w_i(x_\gamma) - 2\gamma|x_\gamma| &\geqslant \max_{\Gamma_i}\{u_i(x) - w_i(x) - 2\gamma|x|\} + \limsup_{\varepsilon\to0} \frac{(|x_{\varepsilon,\gamma}| - |y_{\varepsilon,\gamma}|)^2}{2\varepsilon} \\ &\geqslant u_i(O) - w_i(O) + \limsup_{\varepsilon\to0} \frac{(|x_{\varepsilon,\gamma}| - |y_{\varepsilon,\gamma}|)^2}{2\varepsilon} \\ &\geqslant u_i(O) - w_i(O) + \liminf_{\varepsilon\to0} \frac{(|x_{\varepsilon,\gamma}| - |y_{\varepsilon,\gamma}|)^2}{2\varepsilon} \\ &\geqslant u_i(O) - w_i(O). \end{aligned}$$

Recalling that $u_i(O) - w_i(O) = \max_{\Gamma_i}(u_i - w_i)$, we obtain from the inequalities above that $x_\gamma = O$ and that

$$\lim_{\varepsilon\to0} \frac{(|x_{\varepsilon,\gamma}| - |y_{\varepsilon,\gamma}|)^2}{2\varepsilon} = 0. \qquad (2.5.9)$$

We claim that if $\varepsilon > 0$, then $x_{\varepsilon,\gamma} \neq O$. Indeed, assume by contradiction that $x_{\varepsilon,\gamma} = O$:

1. if $y_{\varepsilon,\gamma} > 0$, then

$$\begin{aligned} M_{\varepsilon,\gamma} &= u_i(O) - w_i(y_{\varepsilon,\gamma}) - \frac{1}{2\varepsilon}[|y_{\varepsilon,\gamma}| + \delta(\varepsilon)]^2 - \gamma|y_{\varepsilon,\gamma}| \\ &\geqslant u_i(y_{\varepsilon,\gamma}) - w_i(y_{\varepsilon,\gamma}) - \frac{\delta^2(\varepsilon)}{2\varepsilon} - 2\gamma|y_{\varepsilon,\gamma}|. \end{aligned}$$

   Since u_i is Lipschitz continuous in $B(O,r) \cap \Gamma_i$, we see that for ε small enough

$$L|y_{\varepsilon.\gamma}| \geqslant u_i(O) - u_i(y_{\varepsilon,\gamma}) \geqslant \frac{|y_{\varepsilon,\gamma}|^2}{2\varepsilon} + \frac{|y_{\varepsilon,\gamma}|\delta(\varepsilon)}{\varepsilon} - \gamma|y_{\varepsilon,\gamma}| \geqslant \frac{|y_{\varepsilon,\gamma}|\delta(\varepsilon)}{\varepsilon} - \gamma|y_{\varepsilon,\gamma}|.$$

   Therefore, if $y_{\varepsilon,\gamma} \neq O$, then $L \geqslant L+1-\gamma$ which gives a contradiction since $\gamma \in (0, 1/2)$.

2. Otherwise, if $y_{\varepsilon,\gamma} = O$, then

$$M_{\varepsilon,\gamma} = u_i(O) - w_i(O) - \frac{\delta^2(\varepsilon)}{2\varepsilon} \geqslant u_i(\varepsilon e_i) - w_i(O) - \frac{1}{2\varepsilon}[-\varepsilon + \delta(\varepsilon)]^2 - \gamma\varepsilon.$$

Since u_i is Lipschitz continuous in $B(O,r)\cap\Gamma_i$, we see that for ε small enough,

$$L\varepsilon \geqslant u_i(O) - u_i(\varepsilon e_i) \geqslant \frac{|y_{\varepsilon.\gamma}|^2}{2\varepsilon} + \frac{|y_{\varepsilon.\gamma}|\,\delta(\varepsilon)}{\varepsilon} - 2\gamma|y_{\varepsilon.\gamma}| \geqslant \frac{|y_{\varepsilon.\gamma}|\,\delta(\varepsilon)}{\varepsilon} - 2\gamma|y_{\varepsilon.\gamma}|.$$

This implies that $L \geqslant -1/2 + L + 1 - \gamma$, which gives a contradiction since $\gamma \in (0,1/2)$. Therefore the claim is proved. It follows that we can apply the viscosity inequality for u_i at $x_{\varepsilon,\gamma}$. Moreover, notice that the viscosity supersolution inequality (2.5.2) holds also for $y_{\varepsilon,\gamma} = 0$ since $H_i(O,p) \leqslant H_i^+(O,p)$ for any p. Therefore

$$\begin{aligned}
u_i(x_{\varepsilon,\gamma}) + H_i\left(x_{\varepsilon,\gamma}, \frac{-x_{\varepsilon,\gamma} + y_{\varepsilon,\gamma} + \delta(\varepsilon)}{\varepsilon} + \gamma\right) \leqslant 0,\\
w_i(y_{\varepsilon,\gamma}) + H_i\left(y_{\varepsilon,\gamma}, \frac{-x_{\varepsilon,\gamma} + y_{\varepsilon,\gamma} + \delta(\varepsilon)}{\varepsilon} - \gamma\right) \geqslant 0.
\end{aligned}$$

Subtracting the two inequalities,

$$u_i(x_{\varepsilon,\gamma}) - w_i(y_{\varepsilon,\gamma}) \leqslant H_i\left(y_{\varepsilon,\gamma}, \frac{-x_{\varepsilon,\gamma} + y_{\varepsilon,\gamma} + \delta(\varepsilon)}{\varepsilon} + \gamma\right) - H_i\left(x_{\varepsilon,\gamma}, \frac{-x_{\varepsilon,\gamma} + y_{\varepsilon,\gamma} + \delta(\varepsilon)}{\varepsilon} - \gamma\right). \tag{2.5.10}$$

Using $[H1]$ and $[H2]$, it is easy to see that there exists $\overline{M}_i > 0$ such that for any $x,y \in \Gamma_i$ and $p,q \in \mathbb{R}$

$$\begin{aligned}
|H_i(x,p) - H_i(y,q)| &\leqslant |H_i(x,p) - H_i(y,p)| + |H_i(y,p) - H_i(y,q)|\\
&\leqslant \overline{M}_i|x-y|(1+|p|) + \overline{M}_i|p-q|.
\end{aligned}$$

It yields

$$\begin{aligned}
u_i(x_{\varepsilon,\gamma}) - w_i(y_{\varepsilon,\gamma}) &\leqslant \overline{M}_i\left[|x_{\varepsilon,\gamma} - y_{\varepsilon,\gamma}|\left(1 + \left|\frac{-x_{\varepsilon,\gamma} + y_{\varepsilon,\gamma} + \delta(\varepsilon)}{\varepsilon} - \gamma\right|\right) + 2|\gamma|\right]\\
&\leqslant \overline{M}_i\left[|x_{\varepsilon,\gamma} - y_{\varepsilon,\gamma}|\left(\gamma + 1 + \frac{\delta(\varepsilon)}{\varepsilon}\right) + \frac{|x_{\varepsilon,\gamma} - y_{\varepsilon,\gamma}|^2}{\varepsilon} + 2|\gamma|\right].
\end{aligned}$$

Applying (2.5.9), let ε tend to 0 and γ tend to 0, we obtain that $u_i(O) - w_i(O) \leqslant 0$, the desired contradiction.

Case 2: $w_i(O) \geqslant \min_{j\neq i}\{w_j(O) + c_j\} = w_{j_0}(O) + c_{j_0}$. Using the same arguments as in *Case 2* of the first proof, we get

$$w_{j_0} < \min\left\{\min_{j\neq j_0}\{w_j(O) + c_j\}, -\frac{H_O^T}{\lambda}\right\}$$

and $w_{j_0}(O) < u_{j_0}(O)$. Repeating *Case 1*, replacing the index i by j_0, implies that $w_{j_0}(O) \geqslant u_{j_0}(O)$, the desired contradiction.

□

Corollary 2.5.7 (Uniqueness). If v is the value function (with entry costs) and $(v_1,\ldots,v_N)$ is defined by

$$v_i(x) = \begin{cases} \mathsf{v}(x) & \text{if } x \in \Gamma_i\backslash\{O\},\\ \lim_{\delta\to 0^+}\mathsf{v}(\delta e_i) & \text{if } x = O,\end{cases}$$

then $(v_1, \dots, v_N)$ is the unique bounded viscosity solution of (2.3.1).
Similarly, if $\hat{\mathsf{v}}$ is the value function (with exit costs) and $(\hat{v}_1, \dots, \hat{v}_N)$ is defined by

$$\hat{v}_i(x) = \begin{cases} \hat{\mathsf{v}}(x), & \text{if } x \in \Gamma_i \backslash \{O\}, \\ \lim_{\delta \to 0^+} \hat{\mathsf{v}}(\delta e_i), & \text{if } x = O, \end{cases}$$

then $(\hat{v}_1, \dots, \hat{v}_N)$ is the unique bounded viscosity solution of (2.3.2).

Remark 2.5.8. From Corollary 2.5.7, we see that in order to characterize the original value function with entry costs, we need to solve first the Hamilton-Jacobi system (2.3.1) and find the unique viscosity solution $(v_1, \dots, v_N)$. The original value function v with entry costs satisfies

$$\mathsf{v}(x) = \begin{cases} v_i(x), & \text{if } x \in \Gamma_i \backslash \{O\}, \\ \min\left\{\min_{i=\overline{1,N}} \{v_i(O) + c_i\}, -H_O^T/\lambda\right\}, & \text{if } x = O. \end{cases}$$

The characterization of $\mathsf{v}(O)$ follows from Theorem 2.2.9. The characterization of the original value function with exit costs $\hat{\mathsf{v}}$ is similar.

2.6 A more general optimal control problem

In what follows, we generalize the control problem studied in the previous sections by allowing some of the entry (or exit) costs to be zero. The situation can be viewed as intermediary between the one studied in [8] when all the entry (or exit) costs were zero, and that studied above when all the entry or exit costs were positive. Accordingly, every result presented below will mainly be obtained by combining the arguments proposed above with those used in [8]. Hence, we will present the results and omit the proofs.

To be more specific, we consider the optimal control problems with non-negative entry cost $\overline{C} = \{\bar{c}_1, \dots \bar{c}_m, \bar{c}_{m+1}, \dots \bar{c}_N\}$ where $\bar{c}_i = 0$ if $i \leqslant m$ and $\bar{c}_i > 0$ if $i > m$, keeping all the assumptions and definitions of Section 2.2 unchanged. The value function associated to $\overline{C}$ will be denoted by V. Similarly to Lemma 2.2.8, $\mathsf{V}|_{\Gamma_i \backslash \{O\}}$ is continuous and Lipschitz continuous near O: therefore, it is possible to extend $\mathsf{V}|_{\Gamma_i \backslash \{O\}}$ at O. This extension will be noted $\mathcal{V}_i$. Moreover, one can check that $\mathcal{V}_i(O) = \mathcal{V}_j(O)$ for all $i, j \leqslant m$, which means that $\mathsf{V}|_{\cup_{i=1}^m \Gamma_i}$ is a continuous function which will be noted $\mathcal{V}_c$ hereafter.

Combining the arguments in [8] and in Section 2.2 leads us to the following theorem.

Theorem 2.6.1. *The value function* V *satisfies*

$$\max_{i=m+1,N} \{\mathcal{V}_i(O)\} \leqslant \mathsf{V}(O) = \mathcal{V}_c(O) \leqslant \min\left\{\min_{i=m+1,N} \{\mathcal{V}_i(O) + \bar{c}_i\}, -\frac{H_O^T}{\lambda}\right\}.$$

Remark 2.6.2. In the case when $\bar{c}_i = 0$ for $i = \overline{1,N}$, V is continuous on $\mathcal{G}$ and it is exactly the value function of the problem studied in [8].

We now define a set of admissible test-function and the Hamilton-Jacobi equation that will characterize V.

Definition 2.6.3. A function $\varphi : (\cup_{i=1}^m \Gamma_i) \times \Gamma_{m+1} \times \ldots \times \Gamma_N \to \mathbb{R}^{N-m+1}$ of the form

$$\varphi(x_c, x_{m+1}, \dots, x_N) = (\varphi_c(x_c), \varphi_{m+1}(x_{m+1}), \dots, \varphi_N(x_N))$$

is an admissible test-function if

- φ_c is continuous and for $i \leqslant m$, $\varphi_c|_{\Gamma_i}$ belongs to $C^1(\Gamma_i)$,
- for $i > m$, φ_i belongs to $C^1(\Gamma_i)$,
- the space of admissible test-function is noted $R(\mathcal{G})$.

Definition 2.6.4. A function $U = (U_c, U_{m+1}, \dots, U_N)$ where $U_c \in USC\left(\cup_{j=1}^m \Gamma_j; \mathbb{R}\right), U_i \in USC(\Gamma_i; \mathbb{R})$ is called a *viscosity subsolution* of the Hamilton-Jacobi system if for any test-function $(\varphi_c, \varphi_{m+1}, \dots, \varphi_N) \in R(\mathcal{G})$:

1. if $U_c - \varphi_c$ has a local maximum at $x_c \in \cup_{j=1}^m \Gamma_j$ and if
   - $x_c \in \Gamma_j \backslash \{O\}$ for some $j \leqslant m$, then
   $$\lambda U_c(x_c) + H_j\left(x, \frac{d\varphi_c}{dx_j}(x_c)\right) \leqslant 0,$$
   - $x_c = O$, then
   $$\lambda U_c(O) + \max\left\{-\lambda \min_{j>m}\{U_j(O) + \bar{c}_j\}, \max_{j\leqslant m}\left\{H_j^+\left(O, \frac{d\varphi_c}{dx_j}(O)\right)\right\}, H_O^T\right\} \leqslant 0;$$
2. if $U_i - \varphi_i$ has a *local maximum point* at $x_i \in \Gamma_i$ for $i > m$, and if
   - $x_i \in \Gamma_i \backslash \{O\}$, then
   $$\lambda U_i(x_i) + H_i\left(x, \frac{d\varphi_i}{dx_i}(x_i)\right) \leqslant 0,$$
   - $x_i = O$, then
   $$\lambda U_i(O) + \max\left\{-\lambda \min_{j>m, j\neq i}\{U_j(O) + \bar{c}_j\}, -\lambda U_c(O), H_i^+\left(O, \frac{d\varphi_i}{dx_i}(O)\right), H_O^T\right\} \leqslant 0.$$

A function $U = (U_c, U_{m+1}, \dots, U_N)$ where $U_c \in LSC\left(\cup_{j=1}^m \Gamma_j; \mathbb{R}\right), U_i \in LSC(\Gamma_i; \mathbb{R})$ is called a *viscosity supersolution* of the Hamilton-Jacobi system if for any $(\varphi_c, \varphi_{m+1}, \dots, \varphi_N) \in R(\mathcal{G})$:

1. if $U_c - \varphi_c$ has a local maximum at $x_c \in \cup_{j=1}^m \Gamma_j$ and if
   - $x_c \in \Gamma_j \backslash \{O\}$ for some $j \leqslant m$, then
   $$\lambda U_c(x_c) + H_j\left(x, \frac{d\varphi_c}{dx_j}(x_c)\right) \geqslant 0,$$
   - $x_c = O$, then
   $$\lambda U_c(O) + \max\left\{-\lambda \min_{j>m}\{U_j(O) + \bar{c}_j\}, \max_{j\leqslant m}\left\{H_j^+\left(O, \frac{d\varphi_c}{dx_j}(O)\right)\right\}, H_O^T\right\} \geqslant 0;$$
2. if $U_i - \varphi_i$ has a *local minimum point* at $x_i \in \Gamma_i$ for $i > m$, and if
   - $x_i \in \Gamma_i \backslash \{O\}$, then
   $$\lambda U_i(x_i) + H_i\left(x, \frac{d\varphi_i}{dx_i}(x_i)\right) \geqslant 0,$$

- $x_i = O$ for $i > m$ then

$$\lambda U_i(O) + \max\left\{-\lambda \min_{j>m, j\neq i}\{U_j(O) + \bar{c}_j\}, -\lambda U_c(O), H_i^+\left(O, \frac{d\varphi_i}{dx_i}(O)\right), H_O^T\right\} \geqslant 0.$$

A function $U = (U_c, U_1, \ldots, U_m)$ where $U_c \in C(\cup_{j\leqslant m}\Gamma_j; \mathbb{R})$ and $U_i \in C(\Gamma_i; \mathbb{R})$ for all $i > m$ is called a *viscosity solution* of the Hamilton-Jacobi system if it is both a viscosity subsolution and a viscosity supersolution of the Hamilton-Jacobi system.

Remark 2.6.5. The term $-\lambda H_C(O)$ in the above definition accounts for the situation in which the trajectory enters $\cup_{j=1}^m \Gamma j$. The term $\max_{j\leqslant m}\left\{H_j^+\left(O, \frac{d\varphi_c}{dx_j}(O)\right)\right\}$ accounts for the situation in which the trajectory enters Γ_{i_0} where $H_{i_0}^+\left(O, \frac{d\varphi_c}{dx_j}(O)\right) = \max_{j\leqslant m}\left\{H_j^+\left(O, \frac{d\varphi_c}{dx_j}(O)\right)\right\}$.

Remark 2.6.6. In the case when $\bar{c}_i = 0$ for $i = \overline{1, N}$, i,e., $m = N$, the term $-\lambda \min_{j>m} U_j(O) + \bar{c}_j$ vanishes. This implies that

$$\begin{aligned}\max\left\{-\lambda \min_{j>m}\{U_j(O) + \bar{c}_j\}, \max_{j\leqslant m}\left\{H_j^+\left(O, \frac{d\varphi_c}{dx_j}(O)\right)\right\}, H_O^T\right\} &= \max_{j=\overline{1,N}}\left\{H_j^+\left(O, \frac{d\varphi_c}{dx_j}(O)\right)\right\} \\ &= H_O\left(\frac{d\varphi_c}{dx_1}(O), \ldots, \frac{d\varphi_c}{dx_N}(O)\right).\end{aligned}$$

where $H_O(p_1, \ldots, p_N)$ is defined in [8, page 6]. This means that, in the case when all the entry costs $\bar{c}_j$ vanish, we recover the notion of viscosity solution proposed in [8].

We now study the relation between the value function V and the Hamilton-Jacobi system.

Theorem 2.6.7. *Let V be the value function corresponding to the entry costs $\overline{C}$, then the vector-valued function $(\mathcal{V}_c, \mathcal{V}_{m+1}, \ldots, \mathcal{V}_N)$ is a viscosity solution of the Hamilton-Jacobi system.*

Let us state the comparison principle for the Hamilton-Jacobi system.

Theorem 2.6.8. *Let $U = (U_c, U_{m+1}, \ldots, U_N)$ and $W = (W_c, W_{m+1}, \ldots, W_N)$ be a bounded viscosity subsolution and a viscosity supersolution, respectively, of the Hamilton-Jacobi system. The following holds: $U \leqslant W$ in $\mathcal{G}$, i.e., $U_c \leqslant W_c$ on $\cup_{j=1}^m \Gamma_j$, and $U_i \leqslant W_i$ in Γ_i for all $i > m$.*

Proof of Theorem 2.6.8. Suppose by contradiction that there exists $i \in \{1, \ldots, N\}$ and $x \in \Gamma_i$ such that

$$\begin{cases} U_c(x) - W_c(x) > 0, & \text{if } i \leqslant m, \\ U_i(x) - W_i(x) > 0, & \text{if } i > m, \end{cases}$$

then

$$\begin{cases} U_c(O) - W_c(O) = \max_{\cup_{j=1}^m \Gamma_j}\{U_c - W_c\} > 0, & \text{if } i \leqslant m, \\ U_i(O) - W_i(O) = \max_{\Gamma_i}\{U_i - W_i\} > 0, & \text{if } i > m, \end{cases}$$

since the case where the positive maximum is achieved outside the junction leads to a contradition by classical comparison results.

Case 1: $U_c(O) - W_c(O) = \max_{\cup_{i=1}^m \Gamma_i}(U_c - W_c) > 0$

Sub-case 1-a: $W_c(O) < \min_{j>m}\{W_j(O)+\bar{c}_j\}$. Since $W_c(O) < U_c(O) \leqslant -H_O^T/\lambda$, the function W_c is a viscosity supersolution of

$$\begin{cases} \lambda W_c(x) + H_i\left(x, \dfrac{dW_c}{dx_i}(x)\right) & = 0 \quad \text{if } i \leqslant m, x \in \Gamma_i \backslash \{O\}, \\ \lambda W_c(O) + H_c\left(\dfrac{dW_c}{dx_1}(O), \ldots, \dfrac{dW_c}{dx_m}(O)\right) & = 0 \quad \text{if } x = O. \end{cases}$$

where $H_c(p_1, \ldots, p_m) = \max_{i \leqslant m} H_i^+(O, p_i)$. Applying Lemma 2.7.3 in the Appendix, we obtain that $U_c(O) \leqslant W_c(O)$ in contradiction with the assumption.

Sub-case 1-b: $W_c(O) \geqslant \min_{j>m}\{W_j(O)+\bar{c}_j\} = W_{i_0}(O)+\bar{c}_{i_0}$. Since $\bar{c}_{i_0} > 0$, we first see that $W_{i_0}(O) < \min\left\{\min_{j>m}\{W_j(O)+\bar{c}_j\}, W_c(O), -H_O^T/\lambda\right\}$. Hence, W_{i_0} is a viscosity supersolution of (2.5.2) replacing i by i_0. Moreover, since

$$U_{i_0}(O) + \bar{c}_{i_0} \geqslant \min_{j>m}(U_j(O)+\bar{c}_j) \geqslant U_c(O) > W_c(O) > W_{i_0}(O) + \bar{c}_{i_0},$$

then $U_{i_0}(O) > W_{i_0}(O)$. Applying the same argument as *Case 1* in the second proof of Theorem 2.5.6 replacing i by i_0, we obtain that $U_{i_0}(O) \leqslant W_{i_0}(O)$, which is contradictory.

Case 2: $U_i(O) - W_i(O) = \max\limits_{\Gamma_i}(U_i - W_i) > 0$ for some $i > m$. Using the definition of viscosity subsolutions and *Case 1*, we see that $W_i(O) < U_i(O) \leqslant U_c(O) \leqslant W_c(O)$.

Sub-case 2-a: $W_i(O) < \min_{j>m}\{W_j(O)+\bar{c}_j\}$. Since $U_i(O) < -H_O^T/\lambda$, we first see that $W_i(O) < \min\left\{\min_{j>m}\{W_j(O)+\bar{c}_j\}, W_c(O), -H_O^T/\lambda\right\}$. Hence, W_i is a viscosity supersolution of (2.5.2). Applying the same argument as in *Case 1* in the second proof of Theorem 2.5.6, we see that $U_i(O) \leqslant W_i(O)$, which is contradictory.

Sub-case 2-b: $W_i(O) \geqslant \min_{j>m}\{W_j(O)+\bar{c}_j\} = W_{i_0}(O)+\bar{c}_{i_0}$. Since $\bar{c}_{i_0} > 0$, we can check that $W_{i_0}(O) < \min\left\{\min_{j>m}\{W_j(O)+\bar{c}_j\}, W_c(O), -H_O^T/\lambda\right\}$. Hence, W_{i_0} is a viscosity supersolution of (2.5.2) replacing i by i_0. Moreover, since

$$U_{i_0}(O) + \bar{c}_{i_0} \geqslant \min_{j>m}(U_j(O)+\bar{c}_j) \geqslant U_c(O) > W_i(O) > W_{i_0}(O) + \bar{c}_{i_0},$$

then $U_{i_0}(O) > W_{i_0}(O)$. Applying the same argument as *Case 1* in the second proof of Theorem 2.5.6 replacing i by i_0, we obtain that $U_{i_0}(O) \leqslant W_{i_0}(O)$ which is contradictory.

□

2.7 Appendix

Lemma 2.7.1. *For any $a \in A_i^+$, there exists a sequence $\{a_n\}$ such that $a_n \in A_i$ and*

$$\begin{aligned} f_i(O, a_n) &\geqslant \frac{\delta}{n} > 0, \\ |f_i(O,a_n) - f_i(O,a)| &\leqslant \frac{2M}{n}, \\ |\ell_i(O,a_n) - \ell_i(O,a)| &\leqslant \frac{2M}{n}. \end{aligned}$$

Proof of Lemma 2.7.1. From assumption $[H4]$, there exists $a_\delta \in A_i$ such that $f_i(O, a_\delta) = \delta$. Since $\mathrm{FL}_i(O)$ is convex (by assumption $[H3]$), for any $n \in \mathbb{N}, a \in A_i^+$

$$\frac{1}{n}\left(f_i(O, a_\delta)e_i, \ell_i(O, a_\delta)\right) + \left(1 - \frac{1}{n}\right)\left(f_i(O, a), \ell_i(O, a)e_i\right) \in \mathrm{FL}_i(O).$$

Then, there exists a sequence $\{a_n\}$ such that $a_n \in A_i$ and

$$\frac{1}{n}\left(f_i(O, a_\delta), \ell_i(O, a_\delta)\right) + \left(1 - \frac{1}{n}\right)\left(f_i(O, a), \ell_i(O, a)\right) = \left(f_i(O, a_n), \ell_i(O, a_n)\right) \in \mathrm{FL}_i(O). \tag{2.7.1}$$

Notice that $f_i(O, a) \geqslant 0$ since $a \in A_i^+$, this yields

$$f_i(O, a_n) \geqslant \frac{f_i(O, a_\delta)}{n} = \frac{\delta}{n} > 0.$$

From (2.7.1), we also have

$$|f_i(O, a_n) - f_i(O, a)| = \frac{1}{n}|f_i(O, a_\delta) - f_i(O, a)| \leqslant \frac{2M}{n},$$

and

$$|\ell_i(O, a_n) - \ell_i(O, a)| = \frac{1}{n}|\ell_i(O, a_\delta) - \ell_i(O, a)| \leqslant \frac{2M}{n}.$$

□

We can state the following corollary of Lemma 2.7.1:

Corollary 2.7.2. For $i = \overline{1, N}$ and $p_i \in \mathbb{R}$,

$$\max_{a \in A_i \text{ s.t. } f_i(O,a) \geqslant 0} \left\{-f_i(O, a)p_i - \ell_i(O, a)\right\} = \sup_{a \in A_i \text{ s.t. } f_i(O,a) > 0} \left\{-f_i(O, a)p_i - \ell_i(O, a)\right\}.$$

Lemma 2.7.3. *If U_c and W_c are respectively viscosity sub and supersolution of*

$$\begin{aligned} \lambda U_c(x) + H_i\left(x, \frac{dU_c}{dx_i}(x)\right) &\leqslant 0 \text{ if } x \in \Gamma_i \backslash \{O\}, \\ \lambda U_c(O) + H_c\left(\frac{dU_c}{dx_1}(O), \dots, \frac{dU_c}{dx_m}(O)\right) &\leqslant 0 \text{ if } x = O, \end{aligned}$$

and

$$\begin{aligned} \lambda W_c(x) + H_i\left(x, \frac{dW_c}{dx_i}(x)\right) &\geqslant 0 \text{ if } x \in \Gamma_i \backslash \{O\}, \\ \lambda W_c(O) + H_c\left(\frac{dW_c}{dx_1}(O), \dots, \frac{dW_c}{dx_m}(O)\right) &\geqslant 0 \text{ if } x = O, \end{aligned}$$

then $U_c(x) \leqslant W_c(x)$ for all $x \in \bigcup_{i=1}^m \Gamma_i$.

Proof of Lemma 2.7.3. Assume that there exists $\hat{x} \in \Gamma_i$ where $1 \leqslant i \leqslant m$ and $U_c(\hat{x}) - W_c(\hat{x}) > 0$. By classical comparison principle for the boundary problem on Γ_i, one gets

$$U_c(O) - W_c(O) = \max_{\Gamma_i}\{U_c(x) - W_c(x)\} > 0.$$

Applying again classical comparison principle for the boundary problem for each edge Γ_j

$$U_c(O) - W_c(O) = \max_{\bigcup_{i=1}^m \Gamma_i} \{U_c(x) - W_c(x)\} > 0.$$

For $j = \overline{1,N}$, we consider the function

$$\begin{aligned} \Psi_{j,\varepsilon,\gamma} : \Gamma_j \times \Gamma_j &\longrightarrow \mathbb{R} \\ (x,y) &\longrightarrow U_c(x) - W_c(y) - \frac{1}{2\varepsilon}\left[-|x| + |y| + \delta(\varepsilon)\right]^2 - \gamma(|x| + |y|), \end{aligned}$$

where $\delta(\varepsilon) = (L+1)\varepsilon$, $\gamma \in \left(0, \frac{1}{2}\right)$.
The function $\Psi_{j,\varepsilon}$ attains its maximum at $(x_{j,\varepsilon,\gamma}, y_{j,\varepsilon,\gamma}) \in \Gamma_j \times \Gamma_j$. Applying the same argument as in the second proof of Theorem 2.5.6, we have $x_{j,\varepsilon,\gamma}, y_{j,\varepsilon,\gamma} \to O$ and $\frac{(x_{j,\varepsilon,\gamma} - y_{j,\varepsilon,\gamma})^2}{\varepsilon} \to 0$ as $\varepsilon \to 0$. Moreover, for any $j = \overline{1,m}$, $x_{j,\varepsilon,\gamma} \neq O$. We claim that $y_{j,\varepsilon,\gamma}$ must be O for ε small enough . Indeed, if there exists a sequence ε_n such that $y_{j,\varepsilon_n,\gamma} \in \Gamma_j \backslash \{O\}$, then applying viscosity inequalities, we have

$$\begin{aligned} U_c(x_{j,\varepsilon_n,\gamma}) + H_j\left(x_{j,\varepsilon_n,\gamma}, \frac{-x_{j,\varepsilon_n,\gamma} + y_{j,\varepsilon_n,\gamma} + \delta(\varepsilon_n)}{\varepsilon_n} + \gamma\right) &\leqslant 0, \\ W_c(y_{j,\varepsilon_n,\gamma}) + H_j\left(y_{j,\varepsilon_n,\gamma}, \frac{-x_{j,\varepsilon_n,\gamma} + y_{j,\varepsilon_n,\gamma} + \delta(\varepsilon_n)}{\varepsilon_n} - \gamma\right) &\geqslant 0. \end{aligned}$$

Subtracting the two inequalities and using (2.5.10) with H_j, we obtain

$$U_c(x_{j,\varepsilon_n,\gamma}) - W_c(y_{j,\varepsilon_n,\gamma}) \leqslant \overline{M}_j |x_{j,\varepsilon_n,\gamma} - y_{j,\varepsilon_n,\gamma}| \left(1 + \left|\frac{-x_{j,\varepsilon_n,\gamma} + y_{j,\varepsilon_n,\gamma} + \delta(\varepsilon_n)}{\varepsilon_n} - \gamma\right|\right) + \overline{M}_j 2\gamma.$$

Recall that we already have $\frac{(x_{j,\varepsilon_n,\gamma} - y_{j,\varepsilon_n,\gamma})^2}{\varepsilon_n} \to 0$ as $n \to \infty$. Let n tend to ∞ and γ tend to 0 then we obtain $U_c(O) - W_c(O) \leqslant 0$. It leads us to a contradiction. So this claim is proved.
Define the function $\Psi : \bigcup_{j=1}^m \Gamma_j \to \mathbb{R}$ by

$$\Psi|_{\Gamma_i}(y) = \frac{1}{2\varepsilon}\sum_{j \neq i}\left\{\left[-|x_{i,\varepsilon,\gamma}| + \delta(\varepsilon)\right]^2 - \gamma|x_{i,\varepsilon,\gamma}|\right\} + \frac{1}{2\varepsilon}\left[-|x_{i,\varepsilon,\gamma}| + |y| + \delta(\varepsilon)\right]^2 + \gamma(-|x_{i,\varepsilon,\gamma}| + |y|).$$

We can see that Ψ is continuous on $\bigcup_{j=1}^m \Gamma_j$ and belongs to $C^1(\Gamma_j)$ for $j = \overline{1,m}$. Moreover, for $j = \overline{1,m}$ and for ε small enough, $y_{j,\varepsilon,\gamma}$=O then the function $\Psi + W_c$ has a minimum point at O. It yields

$$\lambda W_c(O) + H_c\left(\frac{-\overline{x}_{1,\varepsilon,\gamma} + \delta(\varepsilon)}{\varepsilon}, \ldots, \frac{-\overline{x}_{m,\varepsilon,\gamma} + \delta(\varepsilon)}{\varepsilon}\right) \geqslant 0.$$

By definition of H_c, there exists $j_0 \in \{1, \ldots, m\}$ such that

$$\lambda W_c(O) + H_{j_0}^+\left(O, \frac{-\overline{x}_{j_0,\varepsilon,\gamma} + \delta(\varepsilon)}{\varepsilon}\right) \geqslant 0.$$

This implies

$$\lambda W_c(O) + H_{j_0}\left(O, \frac{-\overline{x}_{j_0,\varepsilon,\gamma} + \delta(\varepsilon)}{\varepsilon}\right) \geqslant 0$$

On the other hand, since $x_{j_0,\varepsilon,\gamma} \in \Gamma_{j_0} \backslash \{O\}$, we have

$$\lambda U_c\left(\overline{x}_{j_0,\varepsilon,\gamma}\right) + H_{j_0}\left(x_{j_0,\varepsilon,\gamma}, \frac{-\overline{x}_{j_0,\varepsilon,\gamma} + \delta\left(\varepsilon\right)}{\varepsilon}\right) \leqslant 0.$$

Subtracting the two inequalities and using properties of Hamiltonian H_{j_0}, let ε tend to 0 then γ tend to 0, we obtain that $U_c\left(O\right) - W_c\left(O\right) \leqslant 0$, which is contradictory. □

3 A Class of Mean Field Games on Networks. Part One: the Ergodic Case

Abstract: We consider stochastic mean field games for which the state space is a network. In the ergodic case, they are described by a system coupling a Hamilton-Jacobi-Bellman equation and a Fokker-Planck equation, whose unknowns are the invariant measure m, a value function v, and the ergodic constant ρ. The function v is continuous and satisfies general Kirchhoff conditions at the vertices. The invariant measure m satisfies dual transmission conditions: in particular, m is discontinuous across the vertices in general, and the values of m on each side of the vertices satisfy special compatibility conditions. Existence and uniqueness are proven, under suitable assumptions.

3.1 Introduction and main results

Recently, an important research activity on mean field games (MFGs for short) has been initiated since the pioneering works [79, 80, 81] of Lasry & Lions (related ideas have been developed independently in the engineering literature by Huang, Caines & Malhamé, see for example [70, 69, 68]): it aims at studying the asymptotic behavior of stochastic differential games (Nash equilibria) as the number N of agents tends to infinity. In these models, it is assumed that the agents are all identical and that an individual agent can hardly influence the outcome of the game. Moreover, each individual strategy is influenced by some averages of functions of the states of the other agents. In the limit when $N \to +\infty$, a given agent feels the presence of the others through the statistical distribution of the states. Since perturbations of the strategy of a single agent do not influence the statistical states distribution, the latter acts as a parameter in the control problem to be solved by each agent. The delicate question of the passage to the limit is one of the main topics of the book of Carmona & Delarue, [38]. When the dynamics of the agents are independent stochastic processes, MFGs naturally lead to a coupled system of two partial differential equations (PDEs for short), a forward in time Kolmogorov or Fokker-Planck (FP) equation and a backward Hamilton-Jacobi-Bellman (HJB) equation. The unknown of this system is a pair of functions: the value function of the stochastic optimal control problem solved by a representative agent and the density of the distribution of states. In the infinite horizon limit, one obtains a system of two stationary PDEs.

A very nice introduction to the theory of MFGs is supplied in the notes of Cardaliaguet [37]. Theoretical results on the existence of classical solutions to the previously mentioned system of PDEs can be found in [79, 80, 81, 62, 64, 63]. Weak solutions have been studied in [81, 90, 91, 9]. The numerical approximation of these systems of PDEs has been discussed in [6, 2, 9].

A network (or a graph) is a set of items, referred to as vertices (or nodes/crosspoints), with connections between them referred to as edges. In the recent years, there has been an increasing

[0]this chapter is submitted and a preprint is available: Yves Achdou, Manh-Khang Dao, Olivier Ley and Nicoletta Tchou, *A Class of Infinite Horizon Mean Field Games on Networks*, hal-01802120.

interest in the investigation of dynamical systems and differential equations on networks, in particular in connection with problems of data transmission and traffic management (see for example [57, 47, 51]). The literature on optimal control in which the state variable takes its values on a network is recent: deterministic control problems and related Hamilton-Jacobi equations were studied in [5, 73, 8, 72, 86, 87]. Stochastic processes on networks and related Kirchhoff conditions at the vertices were studied in [56, 55].

The present work is devoted to infinite horizon stochastic mean field games taking place on networks. The most important difficulty will be to deal with the transition conditions at the vertices. The latter are obtained from the theory of stochastic control in [56, 55], see Section 3.1.3 below. In [32], the first article on MFGs on networks, Camilli & Marchi consider a particular type of Kirchhoff condition at the vertices for the value function: this condition comes from an assumption which can be informally stated as follows: consider a vertex ν of the network and assume that it is the intersection of p edges $\Gamma_1, \dots, \Gamma_p$, ; if, at time τ, the controlled stochastic process X_t associated to a given agent hits ν, then the probability that X_{τ^+} belongs to Γ_i is proportional to the diffusion coefficient in Γ_i. Under this assumption, it can be seen that the density of the distribution of states is continuous at the vertices of the network. In the present work, the above mentioned assumption is not made any longer. Therefore, it will be seen below that the value function satisfies more general Kirchhoff conditions, and accordingly, that the density of the distribution of states is no longer continuous at the vertices; the continuity condition is then replaced by suitable compatibility conditions on the jumps across the vertex. Moreover, as it will be explained in Remark 3.1.14 below, more general assumptions on the coupling costs will be made. Mean field games on networks with finite horizon will be considered in Chapter 4.

After obtaining the transmission conditions at the vertices for both the value function and the density, we shall prove existence and uniqueness of weak solutions of the uncoupled HJB and FP equations (in suitable Sobolev spaces). We have chosen to work with weak solutions because it is a convenient way to deal with existence and uniqueness in the stationary regime, but also because it is difficult to avoid it in the nonstationary case, see Chapter 4 for finite horizon MFGs. Classical arguments will then lead to the regularity of the solutions. Next, we shall establish the existence result for the MFG system by a fixed point argument and a truncation technique. Uniqueness will also be proved under suitable assumptions.

The present work is organized as follows: the remainder of Section 3.1 is devoted to setting the problem and obtaining the system of PDEs and the transmission conditions at the vertices. Section 3.2 contains useful results, first about some linear boundary value problems with elliptic equations, then on a pair of linear Kolmogorov and Fokker-Planck equations in duality. By and large, the existence of weak solutions is obtained by applying Banach-Necas-Babuška theorem to a special pair of Sobolev spaces referred to as V and W below and Fredholm's alternative, and uniqueness comes from a maximum principle. Section 3.3 is devoted to the HJB equation associated with an ergodic problem. Finally, the proofs of the main results of existence and uniqueness for the MFG system of PDEs are completed in Section 3.1.

3.1.1 Networks and function spaces

The geometry

A bounded network Γ (or a bounded connected graph) is a connected subset of $\mathbb{R}^n$ made of a finite number of bounded non-intersecting straight segments, referred to as edges, which connect nodes referred to as vertices. The finite collection of vertices and the finite set of closed edges are respectively denoted by $\mathcal{V} := \{\nu_i, i \in I\}$ and $\mathcal{E} := \{\Gamma_\alpha, \alpha \in \mathcal{A}\}$, where I and A are finite sets of indices contained in $\mathbb{N}$. We assume that for $\alpha, \beta \in \mathcal{A}$, if $\alpha \neq \beta$, then $\Gamma_\alpha \cap \Gamma_\beta$ is either empty or made of a single vertex. The length of Γ_α is denoted by ℓ_α. Given $\nu_i \in \mathcal{V}$, the set of indices of edges that are adjacent to the vertex ν_i is denoted by $\mathcal{A}_i = \{\alpha \in \mathcal{A} : \nu_i \in \Gamma_\alpha\}$. A vertex ν_i is named a *boundary vertex* if $\sharp(\mathcal{A}_i) = 1$, otherwise it is named a *transition vertex*. The set containing all the boundary vertices is named the *boundary* of the network and is denoted by $\partial\Gamma$ hereafter.

The edges $\Gamma_\alpha \in \mathcal{E}$ are oriented in an arbitrary manner. In most of what follows, we shall make the following arbitrary choice that an edge $\Gamma_\alpha \in \mathcal{E}$ connecting two vertices ν_i and ν_j, with $i < j$ is oriented from ν_i toward ν_j: this induces a natural parametrization $\pi_\alpha : [0, \ell_\alpha] \to \Gamma_\alpha = [\nu_i, \nu_j]$:

$$\pi_\alpha(y) = \frac{\ell_\alpha - y}{\ell_\alpha}\nu_i + \frac{y}{\ell_\alpha}\nu_j \quad \text{for } y \in [0, \ell_\alpha]. \tag{3.1.1}$$

For a function $v : \Gamma \to \mathbb{R}$ and $\alpha \in \mathcal{A}$, we define $v_\alpha : (0, \ell_\alpha) \to \mathbb{R}$ by

$$v_\alpha(x) := v|_{\Gamma_\alpha} \circ \pi_\alpha(x), \quad \text{for all } x \in (0, \ell_\alpha).$$

Remark 3.1.1. In what precedes, the edges have been arbitrarily oriented from the vertex with the smaller index toward the vertex with the larger one. Other choices are of course possible. In particular, by possibly dividing a single edge into two, adding thereby new artificial vertices, it is always possible to assume that for all vertices $\nu_i \in \mathcal{V}$,

$$\text{either } \pi_\alpha(\nu_i) = 0, \text{ for all } \alpha \in \mathcal{A}_i \text{ or } \pi_\alpha(\nu_i) = \ell_\alpha, \text{ for all } \alpha \in \mathcal{A}_i. \tag{3.1.2}$$

This idea was used by Von Below in [99]: some edges of Γ are cut into two by adding artificial vertices so that the new oriented network $\overline{\Gamma}$ has the property (3.1.2), see Figure 3.1 for an example.

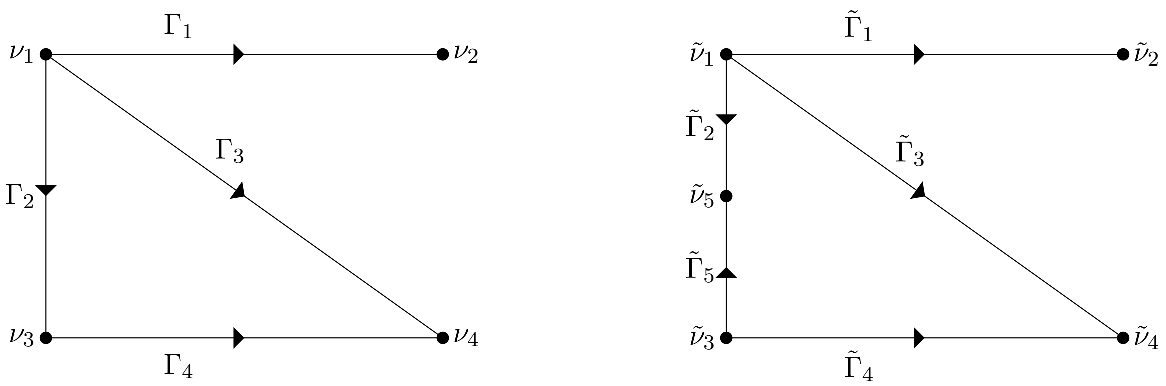

Figure 3.1: Left: the network Γ in which the edges are oriented toward the vertex with larger index (4 vertices and 4 edges). Right: a new network $\tilde{\Gamma}$ obtained by adding an artificial vertex (5 vertices and 5 edges): the oriented edges sharing a given vertex ν either have all their starting point equal ν, or have all their terminal point equal ν.

In Sections 3.1.2 and 3.1.3 below, especially when dealing with stochastic calculus, it will be convenient to assume that property (3.1.2) holds. In the remaining part of the paper, it will be convenient to work with the original network, i.e. without the additional artificial vertices and with the orientation of the edges that has been chosen initially.

Function spaces

The set of continuous functions on Γ is denoted by $C(\Gamma)$ and we set

$$PC(\Gamma) := \left\{ v : \Gamma \to \mathbb{R} \ : \ \text{for all } \alpha \in \mathcal{A}, \left| \begin{array}{l} v_\alpha \in C(0,\ell_\alpha) \\ v_\alpha \text{ can be extended by continuity to } [0,\ell_\alpha]. \end{array} \right. \right\}.$$

By the definition of piecewise continuous functions $v \in PC(\Gamma)$, for all $\alpha \in \mathcal{A}$, it is possible to extend $v|_{\Gamma_\alpha}$ by continuity at the endpoints of Γ_α: if $\Gamma_\alpha = [\nu_i, \nu_j]$, we set

$$v|_{\Gamma_\alpha}(x) = \begin{cases} v_\alpha\left(\pi_\alpha^{-1}(x)\right), & \text{if } x \in \Gamma_\alpha \backslash \mathcal{V}, \\ v_\alpha(0) := \lim\limits_{y \to 0^+} v_\alpha(y), & \text{if } x = \nu_i, \\ v_\alpha(\ell_\alpha) := \lim\limits_{y \to \ell_\alpha^-} v_\alpha(y), & \text{if } x = \nu_j. \end{cases} \tag{3.1.3}$$

For $m \in \mathbb{N}$, the space of m-times continuously differentiable functions on Γ is defined by

$$C^m(\Gamma) := \{ v \in C(\Gamma) : v_\alpha \in C^m([0,\ell_\alpha]) \text{ for all } \alpha \in \mathcal{A} \},$$

and is endowed with the norm

$$\|v\|_{C^m(\Gamma)} := \sum_{\alpha \in \mathcal{A}} \sum_{k \leqslant m} \left\| \partial^k v_\alpha \right\|_{L^\infty(0,\ell_\alpha)}.$$

For $\sigma \in (0,1)$, the space $C^{m,\sigma}(\Gamma)$, contains the functions $v \in C^m(\Gamma)$ such that $\partial^m v_\alpha \in C^{0,\sigma}([0,\ell_\alpha])$ for all $\alpha \in \mathcal{A}$; it is endowed with the norm

$$\|v\|_{C^{m,\sigma}(\Gamma)} := \|v\|_{C^m(\Gamma)} + \sup_{\alpha \in \mathcal{A}} \sup_{\substack{y \neq z \\ y,z \in [0,\ell_\alpha]}} \frac{|\partial^m v_\alpha(y) - \partial^m v_\alpha(z)|}{|y-z|^\sigma}.$$

For a positive integer m and a function $v \in C^m(\Gamma)$, we set for $k \leqslant m$

$$\partial^k v(x) = \partial^k v_\alpha\left(\pi_\alpha^{-1}(x)\right) \text{ if } x \in \Gamma_\alpha \backslash \mathcal{V}. \tag{3.1.4}$$

Notice that $v \in C^k(\Gamma)$ is continuous on Γ but that the derivatives $\partial^l v$, $0 < l \leqslant k$ are not defined at the vertices. For a vertex ν, we define $\partial_\alpha v(\nu)$ as the *outward* directional derivative of $v|_{\Gamma_\alpha}$ at ν as follows:

$$\partial_\alpha v(\nu) := \begin{cases} \lim\limits_{h \to 0^+} \dfrac{v_\alpha(0) - v_\alpha(h)}{h}, & \text{if } \nu = \pi_\alpha(0), \\ \lim\limits_{h \to 0^+} \dfrac{v_\alpha(\ell_\alpha) - v_\alpha(\ell_\alpha - h)}{h}, & \text{if } \nu = \pi_\alpha(\ell_\alpha). \end{cases} \tag{3.1.5}$$

For all $i \in I$ and $\alpha \in \mathcal{A}_i$, setting

$$n_{i\alpha} = \begin{cases} 1 & \text{if } \nu_i = \pi_\alpha(\ell_\alpha), \\ -1 & \text{if } \nu_i = \pi_\alpha(0), \end{cases} \tag{3.1.6}$$

we have

$$\partial_\alpha v(\nu_i) = n_{i\alpha}\, \partial v|_{\Gamma_\alpha}(\nu_i) = n_{i\alpha}\, \partial v_\alpha(\pi_\alpha^{-1}(\nu_i)). \tag{3.1.7}$$

Remark 3.1.2. Note that in (3.1.5), changing the orientation of the edge does not change the value of $\partial_\alpha v(\nu)$.

If for all $\alpha \in \mathcal{A}$, v_α is Lebesgue-integrable on $(0, \ell_\alpha)$, then the integral of v on Γ is defined by

$$\int_\Gamma v(x)\,dx = \sum_{\alpha \in \mathcal{A}} \int_0^{\ell_\alpha} v_\alpha(y)\,dy.$$

For $p \in [1, \infty]$,

$$L^p(\Gamma) := \{v : v_\alpha \in L^p(0, \ell_\alpha) \text{ for all } \alpha \in \mathcal{A}\} = \{v : v|_{\Gamma_\alpha} \in L^p(\Gamma_\alpha) \text{ for all } \alpha \in \mathcal{A}\},$$

is endowed with the norm

$$\|v\|_{L^p(\Gamma)} := \left(\sum_{\alpha \in \mathcal{A}} \|v_\alpha\|^p_{L^p(0,\ell_\alpha)}\right)^{\frac{1}{p}}, \text{ if } 1 \leqslant p < +\infty, \text{ and } \max_{\alpha \in \mathcal{A}} \|v_\alpha\|_{L^\infty(0,\ell_\alpha)}, \text{ if } p = \infty.$$

We shall also need to deal with functions on Γ whose restrictions to the edges are weakly-differentiable: we shall use the same notations for the weak derivatives. Let us introduce Sobolev spaces on Γ.

Definition 3.1.3. For any integer $s \geqslant 1$ and any real number $p \geqslant 1$, the Sobolev space $W^{s,p}(\Gamma)$ is defined as follows:

$$W^{s,p}(\Gamma) := \{v \in C(\Gamma) : v_\alpha \in W^{s,p}(0, \ell_\alpha) \text{ for all } \alpha \in \mathcal{A}\},$$

and endowed with the norm

$$\|v\|_{W^{s,p}(\Gamma)} = \left(\sum_{k=1}^{s} \sum_{\alpha \in \mathcal{A}} \left\|\partial^k v_\alpha\right\|^p_{L^p(0,\ell_\alpha)} + \|v\|^p_{L^p(\Gamma)}\right)^{\frac{1}{p}}.$$

We also set $H^s(\Gamma) = W^{s,2}(\Gamma)$.

3.1.2 A class of stochastic processes on Γ

After rescaling the edges, it may be assumed that $\ell_\alpha = 1$ for all $\alpha \in \mathcal{A}$. Let $\mu_\alpha, \alpha \in \mathcal{A}$ and $p_{i\alpha}, i \in I, \alpha \in \mathcal{A}_i$ be positive constants such that $\sum_{\alpha \in \mathcal{A}_i} p_{i\alpha} = 1$. Consider also a real valued function $a \in PC(\Gamma)$.
As in Remark 3.1.1, we make the assumption (3.1.2) by possibly adding artificial nodes: if ν_i is such an artificial node, then $\sharp(\mathcal{A}_i) = 2$, and we assume that $p_{i\alpha} = 1/2$ for $\alpha \in \mathcal{A}_i$. The diffusion parameter μ has the same value on the two sides of an artificial vertex. Similarly, the function a does not have jumps across an artificial vertex.
Let us consider the linear differential operator:

$$\mathcal{L}u(x) = \mathcal{L}_\alpha u(x) := \mu_\alpha \partial^2 u(x) + a|_{\Gamma_\alpha}(x)\,\partial u(x), \quad \text{if } x \in \Gamma_\alpha, \tag{3.1.8}$$

with domain

$$D(\mathcal{L}) := \left\{u \in C^2(\Gamma) : \sum_{\alpha \in \mathcal{A}_i} p_{i\alpha} \partial_\alpha u(\nu_i) = 0, \text{ for all } i \in I\right\}. \tag{3.1.9}$$

Remark 3.1.4. Note that in the definition of $D(\mathcal{L})$, the condition at boundary vertices boils down to a Neumann condition.

Freidlin & Sheu proved in [55] that

1. The operator $\mathcal{L}$ is the infinitesimal generator of a Feller-Markov process on Γ with continuous sample paths. The operators $\mathcal{L}_\alpha$ and the transmission conditions at the vertices

$$\sum_{\alpha\in\mathcal{A}_i} p_{i\alpha}\partial_\alpha u(\nu_i) = 0 \tag{3.1.10}$$

define such a process in a unique way, see also [56, Theorem 3.1]. The process can be written (X_t, α_t) where $X_t \in \Gamma_{\alpha_t}$. If $X_t = \nu_i$, $i \in I$, α_t is arbitrarily chosen as the smallest index in $\mathcal{A}_i$. Setting $x_t = \pi_{\alpha_t}(X_t)$ defines the process x_t with values in $[0, 1]$.

2. There exist
   a) a one dimensional Wiener process W_t,
   b) continuous non-decreasing processes $\ell_{i,t}$, $i \in I$, which are measurable with respect to the σ-field generated by (X_t, α_t),
   c) continuous non-increasing processes $h_{i,t}$, $i \in I$, which are measurable with respect to the σ-field generated by (X_t, α_t),

   such that

$$\begin{aligned} dx_t &= \sqrt{\mu_{\alpha_t}}dW_t + a_{\alpha_t}(x_t)dt + d\ell_{i,t} + dh_{i,t}, \\ \ell_{i,t} &\text{ increases only when } X_t = \nu_i \text{ and } x_t = 0, \\ h_{i,t} &\text{ decreases only when } X_t = \nu_i \text{ and } x_t = 1. \end{aligned} \tag{3.1.11}$$

3. The following Ito formula holds: for any real valued function $u \in C^2(\Gamma)$:

$$\begin{aligned} u(X_t) =& u(X_0) \\ &+ \sum_{\alpha\in\mathcal{A}} \int_0^t \mathbb{1}_{\{X_s\in\Gamma_\alpha\backslash\mathcal{V}\}} \left(\mu_\alpha\partial^2 u(X_s) + a(X_s)\partial u(X_s)ds + \sqrt{2\mu_\alpha}\partial u(X_s)dW_s\right) \\ &+ \sum_{i\in I}\sum_{\alpha\in\mathcal{A}_i} p_{i\alpha}\partial_\alpha u(\nu_i)(\ell_{i,t} + h_{i,t}). \end{aligned} \tag{3.1.12}$$

Remark 3.1.5. The assumption that all the edges have unit length is not restrictive, because we can always rescale the constants μ_α and the piecewise continuous function a. The Ito formula in (3.1.12) holds when this assumption is not satisfied.

Consider the invariant measure associated with the process X_t. We may assume that it is absolutely continuous with respect to the Lebesgue measure on Γ. Let m be its density:

$$\mathbb{E}\left[u\left(X_t\right)\right] := \int_\Gamma u\left(x\right) m\left(x\right) dx, \qquad \text{for all } u \in PC(\Gamma). \tag{3.1.13}$$

We focus on functions $u \in D\left(\mathcal{L}\right)$. Taking the time-derivative of each member of (3.1.13), Ito's formula (3.1.12) and (3.1.10) lead to:

$$\mathbb{E}\left[\mathbb{1}_{\{X_t\notin\mathcal{V}\}}\left(a\partial u(X_t) + \mu\partial^2 u(X_t)\right)\right] = 0.$$

This implies that

$$\int_\Gamma \left(a(x)\partial u(x) + \mu\partial^2 u(x)\right) m(x)dx = 0. \tag{3.1.14}$$

Since for $\alpha \in \mathcal{A}$, any smooth function on Γ compactly supported in $\Gamma_\alpha \backslash \mathcal{V}$ clearly belongs to $D(\mathcal{L})$, (3.1.14) implies that m satisfies

$$-\mu_\alpha \partial^2 m + \partial(ma) = 0 \tag{3.1.15}$$

in the sense of distributions in the edges $\Gamma_\alpha \backslash \mathcal{V}$, $\alpha \in \mathcal{A}$. This implies that there exists a real number c_α such that

$$-\mu_\alpha \partial m|_{\Gamma_\alpha} = -m|_{\Gamma_\alpha} a|_{\Gamma_\alpha} + c_\alpha. \tag{3.1.16}$$

So $m|_{\Gamma_\alpha}$ is C^1 regular, and (3.1.16) is true pointwise. Using this information and (3.1.14), we find that, for all $u \in D(\mathcal{L})$,

$$\sum_{i\in I}\sum_{\alpha\in\mathcal{A}_i} \mu_\alpha m|_{\Gamma_\alpha}(\nu_i)\,\partial_\alpha u(\nu_i) + \sum_{\beta\in\mathcal{A}}\int_{\Gamma_\beta} \partial u|_{\Gamma_\beta}(x)\Big(-\mu_\beta \partial m|_{\Gamma_\beta}(x) + a|_{\Gamma_\beta}(x) m|_{\Gamma_\beta}(x)\Big)dx = 0.$$

This and (3.1.16) imply that

$$-\sum_{i\in I}\sum_{\alpha\in\mathcal{A}_i} \mu_\alpha m|_{\Gamma_\alpha}(\nu_i)\,\partial_\alpha u(\nu_i) + \sum_{\beta\in\mathcal{A}} c_\beta \int_{\Gamma_\beta} \partial u|_{\Gamma_\beta}(x)dx = 0. \tag{3.1.17}$$

For all $i \in I$, it is possible to choose a function $u \in D(\mathcal{L})$ such that

1. $u(\nu_j) = \delta_{i,j}$ for all $j \in I$;
2. $\partial_\alpha u(\nu_j) = 0$ for all $j \in I$ and $\alpha \in \mathcal{A}_j$.

Using such a test-function in (3.1.17) implies that for all $i \in I$,

$$0 = \sum_{\beta\in\mathcal{A}} c_\beta \int_{\Gamma_\beta} \partial u|_{\Gamma_\beta}(x)dx = \sum_{j\in I}\sum_{\alpha\in\mathcal{A}_j} c_\alpha n_{j,\alpha} u|_{\Gamma_\alpha}(\nu_j) = \sum_{\alpha\in\mathcal{A}_i} n_{i\alpha} c_\alpha, \tag{3.1.18}$$

where $n_{i\alpha}$ is defined in (3.1.6).

For all $i \in I$ and $\alpha, \beta \in \mathcal{A}_i$, it is possible to choose a function $u \in D(\mathcal{L})$ such that

1. u takes the same value at each vertex of Γ, which implies that $\int_{\Gamma_\delta} \partial u|_{\Gamma_\delta}(x)dx = 0$ for all $\delta \in \mathcal{A}$;
2. $\partial_\alpha u(\nu_i) = 1/p_{i\alpha}$, $\partial_\beta u(\nu_i) = -1/p_{i\beta}$ and all the other first order directional derivatives of u at the vertices are 0.

Using such a test-function in (3.1.17) yields

$$\frac{m|_{\Gamma_\alpha}(\nu_i)}{\gamma_{i\alpha}} = \frac{m|_{\Gamma_\beta}(\nu_i)}{\gamma_{i\beta}}, \quad \text{for all } \alpha, \beta \in \mathcal{A}_i, \nu_i \in \mathcal{V},$$

in which

$$\gamma_{i\alpha} = \frac{p_{i\alpha}}{\mu_\alpha}, \quad \text{for all } i \in I, \alpha \in \mathcal{A}_i. \tag{3.1.19}$$

Next, for $i \in I$, multiplying (3.1.16) at $x = \nu_i$ by $n_{i\alpha}$ for all $\alpha \in \mathcal{A}_i$, then summing over all $\alpha \in \mathcal{A}_i$, we get

$$\sum_{\alpha\in\mathcal{A}_i} \mu_\alpha \partial_\alpha m(\nu_i) - n_{i\alpha}\Big(m|_{\Gamma_\alpha}(\nu_i)\, a|_{\Gamma_\alpha}(\nu_i) - c_\alpha\Big) = 0,$$

and using (3.1.18), we obtain that

$$\sum_{\alpha\in\mathcal{A}_i}\mu_\alpha\partial_\alpha m\left(\nu_i\right)-n_{i\alpha}a|_{\Gamma_\alpha}\left(\nu_i\right)m|_{\Gamma_\alpha}\left(\nu_i\right)=0,\quad\text{for all } i\in I. \tag{3.1.20}$$

Summarizing, we get the following boundary value problem for m (recalling that the coefficients $n_{i\alpha}$ are defined in (3.1.6)):

$$\begin{cases} -\mu_\alpha\partial^2 m+\partial\left(ma\right)=0, & x\in\left(\Gamma_\alpha\backslash\mathcal{V}\right),\ \alpha\in\mathcal{A},\\ \displaystyle\sum_{\alpha\in\mathcal{A}_i}\mu_\alpha\partial_\alpha m\left(\nu_i\right)-n_{i\alpha}a|_{\Gamma_\alpha}(\nu_i)m|_{\Gamma_\alpha}\left(\nu_i\right)=0, & \nu_i\in\mathcal{V},\\ \dfrac{m|_{\Gamma_\alpha}\left(\nu_i\right)}{\gamma_{i\alpha}}=\dfrac{m|_{\Gamma_\beta}\left(\nu_i\right)}{\gamma_{i\beta}}, & \alpha,\beta\in\mathcal{A}_i,\nu_i\in\mathcal{V}.\end{cases} \tag{3.1.21}$$

3.1.3 Formal derivation of the MFG system on Γ

Consider a continuum of indistinguishable agents moving on the network Γ. The set of Borel probability measure on Γ is denoted by $\mathcal{P}\left(\Gamma\right)$. Under suitable assumptions, the theory of MFGs asserts that the distribution of states is absolutely continuous with respect to Lebesgue measure on Γ. Hereafter, m stands for the density of the distribution of states: $m\geqslant 0$ and $\int_\Gamma m(x)dx=1$.

The state of a representative agent at time t is a time-continuous controlled stochastic process X_t in Γ, as defined in Section 3.1.2, where the control is the drift a_t, supposed to be of the form $a_t=a(X_t)$. The function $X\mapsto a(X)$ is the feedback.

For a representative agent, the optimal control problem is of the form:

$$\rho:=\inf_{a_s}\liminf_{T\to+\infty}\frac{1}{T}\mathbb{E}_x\left[\int_0^T L\left(X_s,a_s\right)+\mathscr{V}\left[m\left(\cdot,s\right)\left(X_s\right)\right]ds\right], \tag{3.1.22}$$

where $\mathbb{E}_x$ stands for the expectation conditioned by the event $X_0=x$. The functions and operators involved in (3.1.22) will be described below.

Let us assume that there is an optimal feedback law, i.e. a function $a^\star$ defined on Γ which is sufficiently regular in the edges of the network, such that the optimal control at time t is given by $a_t^\star=a^\star(X_t)$. Then, almost surely if $X_t\in\Gamma_\alpha\backslash\mathcal{V}$,

$$d\pi_\alpha^{-1}(X_t)=a_\alpha^\star(\pi_\alpha^{-1}(X_t))dt+\sqrt{2\mu_\alpha}dW_t.$$

An informal way to describe the behavior of the process at the vertices is as follows: if X_t hits $\nu_i\in\mathcal{V}$, then it enters Γ_α, $\alpha\in\mathcal{A}_i$ with probability $p_{i\alpha}>0$.

Let us discuss the ingredients in (3.1.22): the running cost depends separately on the control and on the distribution of states. The contribution of the distribution of states involves the coupling cost operator, which can either be nonlocal, i.e. $V:\mathcal{P}\left(\Gamma\right)\to\mathcal{C}^2(\Gamma)$, or local, i.e. $V[m](x)=F(m(x))$ assuming that m is absolutely continuous with respect to the Lebesgue measure, where $F:\mathbb{R}^+\to\mathbb{R}$ is a continuous function.

The contribution of the control involves the Lagrangian L, i.e. a real valued function defined on $(\cup_{\alpha\in\mathcal{A}}\Gamma_\alpha\backslash\mathcal{V})\times\mathbb{R}$. If $x\in\Gamma_\alpha\backslash\mathcal{V}$ and $a\in\mathbb{R}$, $L(x,a)=L_\alpha(\pi_\alpha^{-1}(x),a)$, where L_α is a continuous real valued function defined on $[0,\ell_\alpha]\times\mathbb{R}$. We assume that $\lim_{|a|\to\infty}\inf_{y\in\Gamma_\alpha}\frac{L_\alpha(y,a)}{|a|}=+\infty$. Further assumptions on L and V will be made below.

Under suitable assumptions, the Ito calculus recalled in Section 3.1.2 and the dynamic programming principle lead to the following ergodic Hamilton-Jacobi equation on Γ, more precisely the following boundary value problem:

$$\begin{cases} -\mu_\alpha \partial^2 v + H\left(x, \partial v\right) + \rho = \mathscr{V}\left[m\right](x), & x \in \left(\Gamma_\alpha \backslash \mathcal{V}\right), \alpha \in \mathcal{A}, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v\left(\nu_i\right) = 0, & \nu_i \in \mathcal{V}, \\ v|_{\Gamma_\alpha}\left(\nu_i\right) = v|_{\Gamma_\beta}\left(\nu_i\right), & \alpha, \beta \in \mathcal{A}_i, \nu_i \in \mathcal{V}, \\ \displaystyle\int_\Gamma v(x) dx = 0. \end{cases} \tag{3.1.23}$$

We refer to [79, 81] for the interpretation of the value function v and the ergodic cost ρ.

Let us comment the different equations in (3.1.23):

1. The Hamiltonian H is a real valued function defined on $\left(\cup_{\alpha \in \mathcal{A}} \Gamma_\alpha \backslash \mathcal{V}\right) \times \mathbb{R}$. For $x \in \Gamma_\alpha \backslash \mathcal{V}$ and $p \in \mathbb{R}$,
$$H\left(x, p\right) = \sup_a \left\{-ap - L_\alpha\left(\pi_\alpha^{-1}(x), a\right)\right\},$$
The Hamiltonian is supposed to be C^1 and coercive with respect to p uniformly in x.

2. The second equation in (3.1.23) is a Kirchhoff transmission condition (or Neumann boundary condition if $\nu_i \in \partial\Gamma$); it is the consequence of the assumption on the behavior of X_s at vertices. It involves the positive constants $\gamma_{i\alpha}$ defined in (3.1.19).

3. The third condition means in particular that v is continuous at the vertices.

4. The fourth equation is a normalization condition.

If (3.1.23) has a smooth solution, then it provides a feedback law for the optimal control problem, i.e.
$$a^\star(x) = -\partial_p H\left(x, \partial v\left(x\right)\right).$$
At the MFG equilibrium, m is the density of the invariant measure associated with the optimal feedback law, so, according to Section 3.1.2, it satisfies (3.1.21) (where a is replaced by $a^\star$). Finally, replacing $a^\star(x)$ by the value $-\partial_p H\left(x, \partial v\left(x\right)\right)$, we get the system

$$\begin{cases} -\mu_\alpha \partial^2 v + H\left(x, \partial v\right) + \rho = \mathscr{V}\left(\left[m\right]\right), & x \in \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ \mu_\alpha \partial^2 m + \partial\left(m \partial_p H\left(x, \partial v\right)\right) = 0, & x \in \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial v_\alpha\left(\nu_i\right) = 0, & \nu_i \in \mathcal{V}, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} \left[\mu_\alpha \partial_\alpha m\left(\nu_i\right) + n_{i\alpha} \partial_p H_\alpha\Big(\nu_i, \partial v|_{\Gamma_\alpha}(\nu_i)\Big) m|_{\Gamma_\alpha}\left(\nu_i\right)\right] = 0, & \nu_i \in \mathcal{V}, \\ v|_{\Gamma_\alpha}\left(\nu_i\right) = v|_{\Gamma_\beta}\left(\nu_i\right), & \alpha, \beta \in \mathcal{A}_i, \nu_i \in \mathcal{V}, \\ \dfrac{m|_{\Gamma_\alpha}\left(\nu_i\right)}{\gamma_{i\alpha}} = \dfrac{m|_{\Gamma_\beta}\left(\nu_i\right)}{\gamma_{i\beta}}, & \alpha, \beta \in \mathcal{A}_i, \nu_i \in \mathcal{V}, \\ \displaystyle\int_\Gamma v\left(x\right) dx = 0, \quad \int_\Gamma m\left(x\right) dx = 1, \quad m \geqslant 0. \end{cases} \tag{3.1.24}$$

At a vertex ν_i, $i \in I$, the transmission conditions for both v and m consist of $d_{\nu_i} = \sharp(\mathcal{A}_i)$ linear relations, which is the appropriate number of relations to have a well posed problem. If $\nu_i \in \partial\Gamma$, there is of course only one Neumann like condition for v and for m.

Remark 3.1.6. In [32], the authors assume that $\gamma_{i\alpha} = \gamma_{i\beta}$ for all $i \in I$, $\alpha, \beta \in \mathcal{A}_i$. Therefore, the density m does not have jumps across the transition vertices.

3.1.4 Assumptions and main results

Assumptions

Let $(\mu_\alpha)_{\alpha\in\mathcal{A}}$ be a family of positive numbers, and for each $i \in I$ let $(\gamma_{i\alpha})_{\alpha\in\mathcal{A}_i}$ be a family of positive numbers such that $\sum_{\alpha\in\mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha = 1$.
Consider the Hamiltonian $H : \Gamma \times \mathbb{R} \to \mathbb{R}$, with $H|_{\Gamma_\alpha} : \Gamma_\alpha \times \mathbb{R} \to \mathbb{R}$. We assume that, for some positive constants C_0, C_1, C_2 and $q \in (1, 2]$,

$$H_\alpha \in C^1\left([0,\ell_\alpha] \times \mathbb{R}\right); \tag{3.1.25}$$

$$H_\alpha(x,\cdot) \text{ is convex in } p \text{ for each } x \in [0,\ell_\alpha]; \tag{3.1.26}$$

$$H_\alpha(x,p) \geqslant C_0|p|^q - C_1 \text{ for } (x,p) \in [0,\ell_\alpha] \times \mathbb{R}; \tag{3.1.27}$$

$$|\partial_p H_\alpha(x,p)| \leqslant C_2\left(|p|^{q-1} + 1\right) \text{ for } (x,p) \in [0,\ell_\alpha] \times \mathbb{R}. \tag{3.1.28}$$

Remark 3.1.7. From (3.1.28), there exists a positive constant C_q such that

$$|H_\alpha(x,p)| \leqslant C_q\left(|p|^q + 1\right), \quad \text{for all } (x,p) \in [0,\ell_\alpha] \times \mathbb{R}. \tag{3.1.29}$$

Below, we shall focus on local coupling operators $\mathscr{V}$, namely

$$\mathscr{V}[\tilde{m}](x) = F(m(x)) \text{ with } F \in C([0,+\infty);\mathbb{R}), \tag{3.1.30}$$

for all $\tilde{m}$ which are absolutely continuous with respect to the Lebesgue measure and such that $d\tilde{m}(x) = m(x)\,dx$. We shall also suppose that F is bounded from below, i.e., there exists a positive constant M such that

$$F(r) \geqslant -M, \quad \text{for all } r \in [0,+\infty). \tag{3.1.31}$$

Function spaces related to the Kirchhoff conditions

Let us introduce two function spaces on Γ, which will be the key ingredients in order to build weak solutions of (3.1.24).

Definition 3.1.8. We define two Sobolev spaces:

$$V := H^1(\Gamma), \tag{3.1.32}$$

see definition 3.1.3, and

$$W := \left\{ w : \Gamma \to \mathbb{R} : \; w_\alpha \in H^1(0,\ell_\alpha) \text{ for all } \alpha \in \mathcal{A}, \text{ and } \frac{w|_{\Gamma_\alpha}(\nu_i)}{\gamma_{i\alpha}} = \frac{w|_{\Gamma_\beta}(\nu_i)}{\gamma_{i\beta}} \text{ for all } i \in I,\ \alpha,\beta \in \mathcal{A}_i \right\} \tag{3.1.33}$$

which is also a Hilbert space, endowed with the norm $\|w\|_W = \left(\sum_{\alpha\in\mathcal{A}} \|w_\alpha\|^2_{H^1(0,\ell_\alpha)}\right)^{\frac{1}{2}}$.

Definition 3.1.9. Let the function $\psi \in W$ and $\phi \in PC(\Gamma)$ be defined as follows:

$$\begin{cases} \psi_\alpha \text{ is affine on } (0,\ell_\alpha), \\ \psi|_{\Gamma_\alpha}(\nu_i) = \gamma_{i\alpha}, \text{ if } \alpha \in \mathcal{A}_i, \\ \psi \text{ is constant on the edges } \Gamma_\alpha \text{ which touch the boundary of } \Gamma. \end{cases} \tag{3.1.34}$$

$$\begin{cases} \phi_\alpha \text{ is affine on } (0,\ell_\alpha), \\ \phi|_{\Gamma_\alpha}(\nu_i) = \dfrac{1}{\gamma_{i\alpha}}, \text{ if } \alpha \in \mathcal{A}_i, \\ \phi \text{ is constant on the edges } \Gamma_\alpha \text{ which touch the boundary of } \Gamma. \end{cases} \tag{3.1.35}$$

Note that both functions ψ, ϕ are positive and bounded. We set $\overline{\psi} = \max_\Gamma \psi$, $\underline{\psi} = \min_\Gamma \psi$, $\overline{\phi} = \max_\Gamma \phi$, $\underline{\phi} = \min_\Gamma \phi$.

Remark 3.1.10. One can see that $v \in V \longmapsto v\psi$ is an isomorphism from V onto W and $w \in W \longmapsto w\phi$ is the inverse isomorphism.

Definition 3.1.11. Let the function space $\mathcal{W} \subset W$ be defined as follows:

$$\mathcal{W} := \left\{ m : \Gamma \to \mathbb{R} : m_\alpha \in C^1([0,\ell_\alpha]) \text{ and } \frac{m|_{\Gamma_\alpha}(\nu_i)}{\gamma_{i\alpha}} = \frac{m|_{\Gamma_\beta}(\nu_i)}{\gamma_{i\beta}} \text{ for all } i \in I, \alpha, \beta \in \mathcal{A}_i \right\}. \tag{3.1.36}$$

Main result

Definition 3.1.12. A solution of the Mean Field Games system (3.1.24) is a triple $(v,\rho,m) \in C^2(\Gamma) \times \mathbb{R} \times \mathcal{W}$ such that (v,ρ) is a classical solution of

$$\begin{cases} -\mu_\alpha \partial^2 v + H(x,\partial v) + \rho = F(m), & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ \sum\limits_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha \partial_\alpha v(\nu_i) = 0, & \text{if } \nu_i \in \mathcal{V}, \end{cases} \tag{3.1.37}$$

(note that v is continuous at the vertices from the definition of $C^2(\Gamma)$), and m satisfies

$$\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \left[\mu_\alpha \partial m \partial u + \partial\left(m \partial_p H(x,\partial v)\right) u\right] dx = 0, \quad \text{for all } u \in V, \tag{3.1.38}$$

where V is defined in (3.1.32).

We are ready to state the main result:

Theorem 3.1.13. *If assumptions (3.1.25)-(3.1.28) and (3.1.30)-(3.1.31) are satisfied, then there exists a solution $(v,m,\rho) \in C^2(\Gamma) \times \mathcal{W} \times \mathbb{R}$ of (3.1.24). If F is locally Lipschitz continuous, then $v \in C^{2,1}(\Gamma)$. Moreover if F is strictly increasing, then the solution is unique.*

Remark 3.1.14. The proof of the existence result in [32] is valid only in the case when the coupling cost F is bounded.

Remark 3.1.15. The existence result in Theorem 3.1.13 holds if we assume that the coupling operator V is non local and regularizing, i.e., V is a continuous map from $\mathcal{P}$ to a bounded subset of $\mathcal{F}$, with

$$\mathcal{F} := \left\{ f : \Gamma \to \mathbb{R} : \ f|_{\Gamma_\alpha} \in C^{0,\sigma}(\Gamma_\alpha) \right\}.$$

The proof, omitted in what follows, is similar to that of Theorem 3.4.1 below.

3.2 Preliminary: A class of linear boundary value problems

This section contains elementary results on the solvability of some linear boundary value problems on Γ. To the best of our knowledge, these results are not available in the literature.

3.2.1 A first class of problems

We recall that the constants μ_α and $\gamma_{i,\alpha}$ are defined in Section 3.1.2. Let λ be a positive number. We start with very simple linear boundary value problems, in which the only difficulty is the Kirchhoff condition:

$$\begin{cases} -\mu_\alpha \partial^2 v + \lambda v = f, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i), & \alpha, \beta \in \mathcal{A}_i,\ i \in I, \\ \sum\limits_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v\,(\nu_i) = 0, & i \in I, \end{cases} \tag{3.2.1}$$

where $f \in W'$, W' is the topological dual of W.

Remark 3.2.1. We have already noticed that, if $\nu_i \in \partial\Gamma$, the last condition in (3.2.1) boils down to a standard Neumann boundary condition $\partial_\alpha v\,(\nu_i) = 0$, in which α is the unique element of $\mathcal{A}_i$. Otherwise, if $\nu_i \in \mathcal{V} \backslash \partial\Gamma$, the last condition in (3.2.1) is the Kirchhoff condition discussed above.

Definition 3.2.2. A weak solution of (3.2.1) is a function $v \in V$ such that

$$\mathscr{B}_\lambda\,(v, w) = \langle f, w \rangle_{W', W}\,, \quad \text{for all } w \in W, \tag{3.2.2}$$

where $\mathscr{B}_\lambda : V \times W \to \mathbb{R}$ is the bilinear form defined as follows:

$$\mathscr{B}_\lambda\,(v, w) = \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} (\mu_\alpha \partial v \partial w + \lambda v w)\, dx.$$

Remark 3.2.3. Formally, (3.2.2) is obtained by testing the first line of (3.2.1) by $w \in W$, integrating by part the left hand side on each Γ_α and summing over $\alpha \in \mathcal{A}$. There is no contribution from the vertices, because of the Kirchhoff condition and the transmission condition satisfied by the elements of W.

Remark 3.2.4. By using classical elliptic regularity (see [61]), or by using the fact that Γ_α are line segments, i.e. one dimensional sets and solving the ODE, we see that if v is a weak solution of (3.2.1) with $f \in PC\,(\Gamma)$, then $v \in C^2\,(\Gamma)$.

Let us first study the homogeneous case, i.e. $f = 0$.

Lemma 3.2.5. *The following linear boundary value problem*

$$\begin{cases} -\partial^2 v + \lambda v = 0, & \textit{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i), & \alpha, \beta \in \mathcal{A}_i,\ i \in I, \\ \sum\limits_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v\,(\nu_i) = 0, & i \in I, \end{cases} \tag{3.2.3}$$

has a unique solution: $v = 0$.

Proof. Let $\mathcal{I}_i := \{k \in I : \ k \neq i;\ \nu_k \in \Gamma_\alpha \text{ for some } \alpha \in \mathcal{A}_i\}$ be the set of indices of the vertices which are connected to ν_i. By Remark 3.1.1, it is not restrictive to assume (in the remainder of the proof) that for all $k \in \mathcal{I}_i$, $\Gamma_\alpha = \Gamma_{\alpha_{ik}} = [\nu_i, \nu_k]$ is oriented from ν_i to ν_k.

For $k \in \mathcal{I}_i$, $\Gamma_\alpha = [\nu_i, \nu_k]$, using the parametrization (3.1.1), the linear ODE (3.2.3) in the branch Γ_α is

$$-v''_\alpha\,(y) + \lambda v_\alpha\,(y) = 0, \quad \text{in } (0, \ell_\alpha)\,,$$

whose solution is

$$v_\alpha(y) = \zeta_\alpha \cosh\left(\sqrt{\lambda} y\right) + \xi_\alpha \sinh\left(\sqrt{\lambda} y\right), \tag{3.2.4}$$

with

$$\begin{cases} \zeta_\alpha = v_\alpha(0) = v(\nu_i), \\ \zeta_\alpha \cosh\left(\sqrt{\lambda}\ell_\alpha\right) + \xi_\alpha \sinh\left(\sqrt{\lambda}\ell_\alpha\right) = v_\alpha(\ell_\alpha) = v(\nu_k). \end{cases}$$

It follows that $\partial_\alpha v(\nu_i) = \sqrt{\lambda}\xi_\alpha = \dfrac{\sqrt{\lambda}}{\sinh\left(\sqrt{\lambda}\ell_\alpha\right)}\left[v(\nu_k) - v(\nu_i)\cosh\left(\sqrt{\lambda}\ell_\alpha\right)\right]$. Hence, the transmission condition in (3.2.3) becomes: for all $i \in I$,

$$\begin{aligned} 0 = \sum_{\alpha\in\mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha \partial_\alpha v(\nu_i) &= \sum_{k\in\mathcal{I}_i} \frac{\sqrt{\lambda}\gamma_{i\alpha_{ik}}\mu_{\alpha_{ik}}}{\sinh\left(\sqrt{\lambda}\ell_{\alpha_{ik}}\right)}\left[v(\nu_k) - v(\nu_i)\cosh\left(\sqrt{\lambda}\ell_{\alpha_{ik}}\right)\right] \\ &= \sum_{k\in\mathcal{I}_i} \frac{-\sqrt{\lambda}\gamma_{i\alpha_{ik}}\mu_{\alpha_{ik}}\cosh\left(\sqrt{\lambda}\ell_{\alpha_{ik}}\right)}{\sinh\left(\sqrt{\lambda}\ell_{\alpha_{ik}}\right)} v(\nu_i) + \sum_{k\in\mathcal{I}_i} \frac{\sqrt{\lambda}\gamma_{i\alpha_{ik}}\mu_{\alpha_{ik}}}{\sinh\left(\sqrt{\lambda}\ell_{\alpha_{ik}}\right)} v(\nu_k). \end{aligned}$$

Therefore, we obtain a system of linear equations of the form $MU = 0$ with $M = (M_{ij})_{1\leqslant i,j\leqslant N}$, $N = \sharp(I)$, and $U = (v(\nu_1), \dots, v(\nu_N))^T$, where M is defined by

$$\begin{cases} M_{ii} &= \displaystyle\sum_{k\in\mathcal{I}_i} \gamma_{i\alpha_{ik}}\mu_{\alpha_{ik}} \frac{\cosh\left(\sqrt{\lambda}\ell_{\alpha_{ik}}\right)}{\sinh\left(\sqrt{\lambda}\ell_{\alpha_{ik}}\right)} > 0, \\ M_{ik} &= \dfrac{-\gamma_{i\alpha_{ik}}\mu_{\alpha_{ik}}}{\sinh\left(\sqrt{\lambda}\ell_{\alpha_{ik}}\right)} \leqslant 0, \quad k \in \mathcal{I}_i, \\ M_{ik} &= 0, \quad k \notin \mathcal{I}_i. \end{cases}$$

For all $i \in I$, since $\cosh\left(\sqrt{\lambda}\ell_{\alpha_{ik}}\right) > 1$ for all $k \in \mathcal{I}_i$, the sum of the entries on each row is positive and M is diagonal dominant. Hence, M is invertible and the system has a unique solution $U = 0$. Finally, by solving the ODE in each branch Γ_β with $v_\beta(0) = v_\beta(\ell_\beta) = 0$, we obtain that $v = 0$ on Γ. ☐

Let us now study the non-homogeneous problems (3.2.1).

Lemma 3.2.6. *For any f in W', (3.2.1) has a unique weak solution v in V, see Definition 3.1.8. Moreover, there exists a constant C such that $\|v\|_V \leqslant C\|f\|_{W'}$.*

Proof. First of all, we claim that for $\lambda_0 > 0$ large enough and any $f \in W'$, the problem

$$\mathscr{B}_\lambda(v, w) + \lambda_0(v, w) = \langle f, w\rangle_{W',W} \tag{3.2.5}$$

has a unique solution $v \in V$. Let us prove the claim. Let $v \in V$, then $\hat{w} := v\psi$ belongs to W, where ψ is given by Definition 3.1.9. Let us set $\overline{\partial\psi} := \max_\Gamma|\partial\psi|$ and $\underline{\psi} := \min_\Gamma \psi > 0$, ($\partial\psi$ is bounded, see Definition 3.1.9); we get

$$\begin{aligned} \mathscr{B}_\lambda(v, \hat{w}) + \lambda_0(v, \hat{w}) &= \sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} (\mu_\alpha \partial v \partial\hat{w} + \lambda v\hat{w} + \lambda_0 v\hat{w})\, dx \\ &= \sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \left[\mu_\alpha|\partial v|^2\psi + \mu_\alpha(v\partial v)\partial\psi + (\lambda+\lambda_0)v^2\psi\right] dx \\ &\geqslant \sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \left[\mu_\alpha|\partial v|^2\underline{\psi} - \mu_\alpha|v||\partial v|\overline{\partial\psi} + (\lambda+\lambda_0)v^2\underline{\psi}\right] dx \\ &\geqslant \sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \left[\frac{\mu_\alpha\underline{\psi}}{2}|\partial v|^2 + \left(\lambda_0\underline{\psi} - \frac{\mu_\alpha\overline{\partial\psi}^2}{2\underline{\psi}}\right)v^2\right] dx. \end{aligned} \tag{3.2.6}$$

When $\lambda_0 \geqslant \dfrac{\mu_\alpha}{2} + \dfrac{\mu_\alpha \overline{\partial\psi}^2}{2\underline{\psi}^2}$ for all $\alpha \in \mathcal{A}$, we obtain that

$$\mathscr{B}_\lambda(v,\hat{w}) + \lambda_0 (v,\hat{w}) \geqslant \frac{\underline{\mu}\,\underline{\psi}}{2} \|v\|_V^2 \geqslant \frac{\underline{\mu}\,\underline{\psi}}{2C_\psi} \|v\|_V \|\hat{w}\|_W,$$

using the fact that, from Remark 3.1.10, there exists a positive constant C_ψ such that $\|v\psi\|_W \leqslant C_\psi \|v\|_V$ for all $v \in V$. This yields

$$\inf_{v\in V} \sup_{w\in W} \frac{\mathscr{B}_\lambda(v,w) + \lambda_0 (v,w)}{\|v\|_V \|w\|_W} \geqslant \frac{\underline{\mu}\,\underline{\psi}}{2C_\psi}. \tag{3.2.7}$$

Using a similar argument for any $w \in W$ and $\hat{v} = w\phi$, where ϕ is given in Definition 3.1.9, we obtain that for λ_0 large enough, there exist a positive constant C_ϕ such that

$$\inf_{w\in W} \sup_{v\in V} \frac{\mathscr{B}_\lambda(v,w) + \lambda_0 (v,w)}{\|w\|_W \|v\|_V} \geqslant \frac{\underline{\mu}\,\underline{\phi}}{2C_\phi}. \tag{3.2.8}$$

From (3.2.7) and (3.2.8), by the Banach-Necas-Babuška lemma (see [48] or [30]), for λ_0 large enough, there exists a positive constant C such that for any $f \in W'$, there exists a unique solution $v \in V$ of (3.2.5) and $\|v\|_V \leqslant C\|f\|_{W'}$. Hence, our claim is proved.

Now, we fix λ_0 large enough and we define the continuous linear operator $\overline{R}_{\lambda_0} : W' \to V$ where $\overline{R}_{\lambda_0}(f) = v$ is the unique solution of (3.2.5). Since the injection $\mathcal{I}$ from V to W' is compact, then $\mathcal{I} \circ \overline{R}_{\lambda_0}$ is a compact operator from W' into W'. By the Fredholm alternative (see [61]), one of the following assertions holds:

$$\text{There exists } \overline{v} \in W' \backslash \{0\} \text{ such that } \left(Id - \lambda_0\left(\mathcal{I} \circ \overline{R}_{\lambda_0}\right)\right)\overline{v} = 0. \tag{3.2.9}$$

$$\text{For any } g \in W', \text{ there exists a unique } \overline{v} \in W' \text{ such that } \left(Id - \lambda_0\left(\mathcal{I} \circ \overline{R}_{\lambda_0}\right)\right)\overline{v} = g. \tag{3.2.10}$$

We claim that (3.2.10) holds. Indeed, assume by contradiction that (3.2.9) holds. Then there exists $\overline{v} \neq 0$ such that

$$\begin{cases} \overline{v} \in V, \\ \mathcal{I} \circ \overline{R}_{\lambda_0} \overline{v} = \dfrac{\overline{v}}{\lambda_0}. \end{cases}$$

Therefore

$$\begin{cases} \overline{v} \in V, \\ \mathscr{B}_\lambda\left(\dfrac{\overline{v}}{\lambda_0}, w\right) + \lambda_0\left(\dfrac{\overline{v}}{\lambda_0}, w\right) = (\overline{v}, w), \quad \text{for all } w \in W. \end{cases}$$

This yields that $\mathscr{B}_\lambda(\overline{v}, w) = 0$ for all $w \in W$ and by Lemma 3.2.5, we get that $\overline{v} = 0$, which leads us to a contradiction. Hence, our claim is proved.

Next, from (3.2.10), we can check that for any $g \in V \subset W'$, there exists a unique $\overline{v} \in V$ such that $\left(Id - \lambda_0\left(\mathcal{I} \circ \overline{R}_{\lambda_0}\right)\right)\overline{v} = g$. It allows us to define

$$\left(Id - \lambda_0\left(\mathcal{I} \circ \overline{R}_{\lambda_0}\right)\right)^{-1}|_V : V \longrightarrow V \subset W'.$$

Let us consider the operator $\mathcal{T} := \left(Id - \lambda_0\left(\mathcal{I} \circ \overline{R}_{\lambda_0}\right)\right)^{-1}|_V \circ \overline{R}_{\lambda_0} : W' \longrightarrow V$. We claim that for any $f \in W'$, $\mathcal{T}(f)$ is a solution of (3.2.1). Indeed, set $g := \overline{R}_{\lambda_0}(f)$ and $\overline{v} := \mathcal{T}(f) = \left(Id - \lambda_0\left(\mathcal{I} \circ \overline{R}_{\lambda_0}\right)\right)^{-1}|_V(g)$. Then

$$\mathscr{B}_\lambda(g,w) + \lambda_0 (g,w) = (f,w), \quad \text{for all } w \in W,$$

and $\overline{R}_{\lambda_0}(\overline{v}) = (\overline{v} - g)/\lambda_0$. Therefore,

$$\mathscr{B}_\lambda(\overline{v}, w) = \mathscr{B}_\lambda(g, w) + \lambda_0(g, w) = (f, w), \quad \text{for all } w \in W.$$

Hence, our claim is proved and (3.2.1) has a unique weak solution.

Finally, since $\left(Id - \lambda_0\left(\mathcal{I} \circ \overline{R}_{\lambda_0}\right)\right)|_V : V \longrightarrow V$ is injective (by uniqueness for (3.2.3)), onto (by Fredholm alternative) and continuous, we get that

$$\left(Id - \lambda_0\left(\mathcal{I} \circ \overline{R}_{\lambda_0}\right)\right)^{-1}|_V = \left(\left(Id - \lambda_0\left(\mathcal{I} \circ \overline{R}_{\lambda_0}\right)\right)|_V\right)^{-1}$$

is also continuous by the open mapping theorem. This yields that $\left(Id - \lambda_0\left(\mathcal{I} \circ \overline{R}_{\lambda_0}\right)\right)^{-1}|_V \circ \overline{R}_{\lambda_0}$ is continuous from W' into V. Hence, there exists a positive constant C such that for any $f \in W'$, there exists a unique weak solution v of (3.2.1) and $\|v\|_V \leqslant C\|f\|_{W'}$. The proof of Lemma 3.2.6 is complete.

□

3.2.2 The Kolmogorov equation

This subsection is devoted to the following boundary value problem including a Kolmogorov equation

$$\begin{cases} -\mu_\alpha \partial^2 v + b\partial v = 0, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i), & \alpha, \beta \in \mathcal{A}_i, i \in I, \\ \sum\limits_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha \partial_\alpha v(\nu_i) = 0, & i \in I, \end{cases} \tag{3.2.11}$$

for $b \in PC(\Gamma)$.

Definition 3.2.7. A weak solution of (3.2.11) is a function $v \in V$ such that

$$\mathscr{A}^\star(v, w) = 0, \quad \text{for all } w \in W,$$

where $\mathscr{A}^\star : V \times W \to \mathbb{R}$ is the bilinear form defined by

$$\mathscr{A}^\star(v, w) := \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} (\mu_\alpha \partial v \partial w + b\partial v w)\, dx.$$

Remark 3.2.8. By using the fact that Γ_α are line segments, i.e. one dimensional and solving the ODE, we see that if v is a weak solution of (3.2.11), then $v \in C^2(\Gamma)$.

The uniqueness of solutions of (3.2.11), up to the addition of constants, is obtained by using a maximum principle:

Lemma 3.2.9. *For $b \in PC(\Gamma)$, the set of solutions of (3.2.11) is the set of constant functions on Γ.*

Proof of Lemma 3.2.9. First of all, any constant function on Γ is a solution of (3.2.11). Now let v be a solution of (3.2.11) then $v \in C^2(\Gamma)$ by Remark 3.2.8. Assume that the maximum of v over Γ is achieved in Γ_α; by the maximum principle, it is achieved at some endpoint ν_i of Γ_α. Without loss of generality, using Remark 3.1.1, we can assume that $\pi_\beta(\nu_i) = 0$ for all $\beta \in \mathcal{A}_i$. We have $\partial_\beta v(\nu_i) \geqslant 0$ for all $\beta \in \mathcal{A}_i$ because ν_i is the maximum point of v. Since all the coefficients $\gamma_{i\beta}, \mu_\beta$ are positive, by the Kirchhoff condition if ν_i is a transition vertex, or by the Neumann boundary condition if ν_i is a boundary vertex, we infer that $\partial_\beta v(\nu_i) = 0$ for all $\beta \in \mathcal{A}_i$. This implies that ∂v_β is a solution of the first order linear homogeneous ODE

$$\begin{cases} -\mu_\beta u' + b_\beta u = 0, & \text{on } [0, \ell_\beta], \\ u(0) = 0. \end{cases}$$

Therefore, $\partial v_\beta \equiv 0$ and v is constant on Γ_β for all $\beta \in \mathcal{A}_i$. We can propagate this argument, starting from the vertices connected to ν_i. Since the network Γ is connected and v is continuous, we obtain that v is constant on Γ. □

3.2.3 The dual Fokker-Planck equation

This paragraph is devoted to the dual boundary value problem of (3.2.11); it involves a Fokker-Planck equation:

$$\begin{cases} -\mu_\alpha \partial^2 m - \partial(bm) = 0, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ \dfrac{m|_{\Gamma_\alpha}(\nu_i)}{\gamma_{i\alpha}} = \dfrac{m|_{\Gamma_\beta}(\nu_i)}{\gamma_{i\beta}}, & \alpha, \beta \in \mathcal{A}_i,\ i \in I, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} \left[n_{i\alpha} b|_{\Gamma_\alpha}(\nu_i)\, m|_{\Gamma_\alpha}(\nu_i) + \mu_\alpha \partial_\alpha m(\nu_i) \right] = 0, & i \in I, \end{cases} \tag{3.2.12}$$

where $b \in PC(\Gamma)$, with

$$m \geqslant 0, \quad \int_\Gamma m dx = 1. \tag{3.2.13}$$

First of all, let λ_0 be a nonnegative constant; for all $h \in V'$, we now introduce the modified boundary value problem

$$\begin{cases} \lambda_0 m - \mu_\alpha \partial^2 m - \partial(bm) = h, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ \dfrac{m|_{\Gamma_\alpha}(\nu_i)}{\gamma_{i\alpha}} = \dfrac{m|_{\Gamma_\beta}(\nu_i)}{\gamma_{i\beta}}, & \alpha, \beta \in \mathcal{A}_i,\ i \in I, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} \left[n_{i\alpha} b(\nu_i)\, m|_{\Gamma_\alpha}(\nu_i) + \mu_\alpha \partial_\alpha m(\nu_i) \right] = 0, & i \in I. \end{cases} \tag{3.2.14}$$

Definition 3.2.10. A weak solution of (3.2.14) is a function $m \in W$ such that

$$\mathscr{A}_{\lambda_0}(m, v) = \langle h, v \rangle_{V', V}, \quad \text{for all } v \in V,$$

where $\mathscr{A}_{\lambda_0} : W \times V \to \mathbb{R}$ is the bilinear form defined by

$$\mathscr{A}_{\lambda_0}(m, v) = \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \left[\lambda_0 m v + (\mu_\alpha \partial m + bm) \partial v \right] dx.$$

Definition 3.2.11. A weak solution of (3.2.12) is a function $m \in W$ such that

$$\mathscr{A}_0(m, v) := \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} (\mu_\alpha \partial m + bm) \partial v dx = 0, \quad \text{for all } v \in V. \tag{3.2.15}$$

Remark 3.2.12. Formally, to get (3.2.15), we multiply the first line of (3.2.12) by $v \in V$, integrate by part, sum over $\alpha \in \mathcal{A}$ and use the third line of (3.2.12) to see that there is no contribution from the vertices.

Theorem 3.2.13. *For any $b \in PC(\Gamma)$,*

- *(Existence) There exists a solution $\widehat{m} \in W$ of (3.2.12)-(3.2.13) satisfying*
$$\|\widehat{m}\|_W \leqslant C, \quad 0 \leqslant \widehat{m} \leqslant C, \tag{3.2.16}$$
where the constant C depends only on $\|b\|_\infty$ and $\{\mu_\alpha\}_{\alpha \in A}$. Moreover, $\widehat{m}_\alpha \in C^1(0, \ell_\alpha)$ for all $\alpha \in \mathcal{A}$. Hence, $\widehat{m} \in \mathcal{W}$.
- *(Uniqueness) $\widehat{m}$ is the unique solution of (3.2.12)-(3.2.13).*
- *(Strictly positive solution) $\widehat{m}$ is strictly positive.*

Proof of existence in Theorem 3.2.13. We divide the proof of existence into three steps:

Step 1: Let λ_0 be a large positive constant that will be chosen later. We claim that for $\overline{m} \in L^2(\Gamma)$ and $h := \lambda_0 \overline{m} \in L^2(\Gamma) \subset V'$, (3.2.14) has a unique solution $m \in W$. This allows us to define a linear operator as follows:
$$\mathcal{T} : L^2(\Gamma) \longrightarrow W,$$
$$\overline{m} \longmapsto m,$$
where m is the solution of (3.2.14) with $h = \lambda_0 \overline{m}$. We are going to prove that $\mathcal{T}$ is well-defined and continuous, i.e, for all $\overline{m} \in L^2(\Gamma)$, (3.2.14) has a unique solution that depends continuously on $\overline{m}$. For $w \in W$, set $\widehat{v} := w\phi \in V$ where ϕ is given by Definition 3.1.9. We have
$$\begin{aligned}
\mathscr{A}_{\lambda_0}(w, \widehat{v}) &= \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \left[\lambda_0 \phi w^2 + (\mu_\alpha \partial w + bw)\, \partial(w\phi)\right] dx \\
&= \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \left[(\lambda_0 \phi + b\partial\phi)\, w^2 + (\mu_\alpha \partial\phi + b\phi)\, w\partial w + \mu_\alpha \phi\, (\partial w)^2\right] dx.
\end{aligned}$$
It follows that when λ_0 is large enough (larger than a constant that only depends on b, ϕ and μ_α), $\mathscr{A}_{\lambda_0}(w, \widehat{v}) \geqslant \widehat{C}_{\lambda_0} \|w\|_W^2$ for some positive constant $\widehat{C}_{\lambda_0}$. Moreover, by Remark 3.1.10, there exists a positive constant C_ϕ such that for all $w \in W$, we have $\|w\phi\|_V \leqslant C_\phi \|w\|_W$. This yields
$$\inf_{w \in W} \sup_{v \in V} \frac{\mathscr{A}_{\lambda_0}(w, v)}{\|v\|_V \|w\|_W} \geqslant \frac{\widehat{C}_{\lambda_0}}{C_\phi}.$$
Using similar arguments, for λ_0 large enough, there exist two positive constants C_{λ_0} and C_ψ such that
$$\inf_{v \in V} \sup_{w \in W} \frac{\mathscr{A}_{\lambda_0}(w, v)}{\|w\|_W \|v\|_V} \geqslant \frac{C_{\lambda_0}}{C_\psi}.$$
From Banach-Necas-Babuška lemma (see [48] or [30]), there exists a constant $\overline{C}$ such that for all $\overline{m} \in L^2(\Gamma)$, there exists a unique solution m of (3.2.14) with $h = \lambda_0 \overline{m}$ and $\|m\|_W \leqslant \overline{C} \|\overline{m}\|_{L^2(\Gamma)}$. Hence, the map $\mathcal{T}$ is well-defined and continuous from $L^2(\Gamma)$ to W.

Step 2: Let K be the set defined by
$$K := \left\{ m \in L^2(\Gamma) : m \geqslant 0 \text{ and } \int_\Gamma m dx = 1 \right\}.$$
We claim that $\mathcal{T}(K) \subset K$ which means $\int_\Gamma m = 1$ and $m \geqslant 0$. Indeed, using $v = 1$ as a test function in (3.2.14), we have $\int_\Gamma m dx = \int_\Gamma \overline{m} dx = 1$. Next, let
$$m^-(x) = \begin{cases} 0 & \text{if } m(x) \geqslant 0, \\ -m(x) & \text{if } m(x) < 0. \end{cases}$$

Notice that $m^- \in W$ and $m^-\phi \in V$, where ϕ is given by Definition 3.1.9. Using $m^-\phi$ as a test function in (3.2.14) yields

$$\sum_{\alpha\in\mathcal{A}} -\int_{\Gamma_\alpha} \left[(\lambda_0\phi + b\partial\phi)(m^-)^2 + \mu_\alpha(\partial m^-)^2\phi + (\mu_\alpha\partial\phi + b\phi)\, m^-\partial m^-\right] dx = \int_\Gamma \lambda_0 \overline{m} m^-\phi dx.$$

We can see that the right hand side is non-negative. Moreover, for λ_0 large enough (larger than the same constant as above, which only depends on b, ϕ and μ_α), the left hand side is non-positive. This implies that $m^- = 0$, and hence $m \geqslant 0$. Therefore, the claim is proved.

Step 3: We claim that $\mathcal{T}$ has a fixed point. Let us now focus on the case when $\overline{m} \in K$. Using $m\phi$ as a test function in (3.2.14) yields

$$\sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \left[(\lambda_0\phi + b\partial\phi)\, m^2 + \mu_\alpha (\partial m)^2 \phi + (\mu_\alpha\partial\phi + b\phi)\, m\, (\partial m)\right] dx = \int_\Gamma \lambda_0 \overline{m} m\phi dx. \quad (3.2.17)$$

Since $H^1(0,\ell_\alpha)$ is continuously embedded in $L^\infty(0,\ell_\alpha)$, there exists a positive constant C (independent of $\overline{m} \in K$) such that

$$\int_\Gamma \overline{m} m\phi dx \leqslant \int_\Gamma \overline{m} dx \, \|m\|_{L^\infty(\Gamma)} \overline{\phi} = \|m\|_{L^\infty(\Gamma)} \overline{\phi} \leqslant C \|m\|_W .$$

Hence, from (3.2.17), for λ_0 large enough, there exists a positive constant C_1 such that $C_1 \|m\|_W^2 \leqslant \lambda_0 C \|m\|_W$. Thus

$$\|m\|_W \leqslant \frac{\lambda_0 C}{C_1}. \quad (3.2.18)$$

Therefore, $\mathcal{T}(K)$ is bounded in W. Since the bounded subsets of W are relatively compact in $L^2(\Gamma)$, $\overline{\mathcal{T}(K)}$ is compact in $L^2(\Gamma)$. Moreover, we can see that K is closed and convex in $L^2(\Gamma)$. By Schauder fixed point theorem, see [61, Corollary 11.2], $\mathcal{T}$ has a fixed point $\widehat{m} \in K$ which is also a solution of (3.2.12) and $\|\widehat{m}\|_W \leqslant \lambda_0 C/C_1$.

Finally, from the PDE in (3.2.12), for all $\alpha \in \mathcal{A}$, we have $(\widehat{m}'_\alpha + b_\alpha\widehat{m}_\alpha)' = 0$ on $(0,\ell_\alpha)$. This implies that there exists a constant C_α such that

$$\widehat{m}'_\alpha + b_\alpha \widehat{m}_\alpha = C_\alpha, \quad \text{for all } x \in (0,\ell_\alpha). \quad (3.2.19)$$

It follows that

$$\widehat{m}'_\alpha \in C([0,\ell_\alpha]), \quad \text{for all } \alpha \in \mathcal{A}. \quad (3.2.20)$$

Hence $\widehat{m}_\alpha \in C^1([0,\ell_\alpha])$ for all $\alpha \in \mathcal{A}$. Thus, $\widehat{m} \in \mathcal{W}$. □

Remark 3.2.14. Let $m \in W$ be a solution of (3.2.12). If $b, \partial b \in PC(\Gamma)$, standard arguments yield that $m_\alpha \in C^2(0,\ell_\alpha)$ for all $\alpha \in \mathcal{A}$. Moreover, by Theorem 3.2.13, there exists a constant C which depends only on $\|b\|_\infty, \{\|\partial b_\alpha\|_\infty\}_{\alpha\in\mathcal{A}}$ and μ_α such that $\|m_\alpha\|_{C^2(0,\ell_\alpha)} \leqslant C$ for all $\alpha \in \mathcal{A}$.

Proof of the positivity in Theorem 3.2.13. From (3.2.13), $\widehat{m}$ is non-negative on Γ. Assume by contradiction that there exists $x_0 \in \Gamma_\alpha$ for some $\alpha \in \mathcal{A}$ such that $\widehat{m}|_{\Gamma_\alpha}(x_0) = 0$. Therefore, the minimum of $\widehat{m}$ over Γ is achieved at $x_0 \in \Gamma_\alpha$. If $x_0 \in \Gamma_\alpha\backslash\mathcal{V}$, then $\partial\widehat{m}(x_0) = 0$. In (3.2.19), we thus have $C_\alpha = 0$, and hence $\widehat{m}_\alpha$ satisfies

$$\widehat{m}'_\alpha + b_\alpha\widehat{m}_\alpha = 0, \quad \text{on } [0,\ell_\alpha],$$

with $\widehat{m}_\alpha\left(\pi_\alpha^{-1}(x_0)\right) = 0$. It follows that $\widehat{m}_\alpha \equiv 0$ and $\widehat{m}|_{\Gamma_\alpha}(\nu_i) = \widehat{m}|_{\Gamma_\alpha}(\nu_j) = 0$ if $\Gamma_\alpha = [\nu_i, \nu_j]$.

Therefore, it is enough to consider $x_0 \in \mathcal{V}$.
Now, from Remark 3.1.1, we may assume without loss of generality that $x_0 = \nu_i$ and $\pi_\beta(\nu_i) = 0$ for all $\beta \in \mathcal{A}_i$. We have the following two cases.

Case 1: if $x_0 = \nu_i$ is a transition vertex, then, since $\widehat{m}$ belongs to W, we get

$$\widehat{m}|_{\Gamma_\beta}(\nu_i) = \frac{\gamma_{i\beta}}{\gamma_{i\alpha}} \widehat{m}|_{\Gamma_\alpha}(\nu_i) = 0, \quad \text{for all } \beta \in \mathcal{A}_i. \tag{3.2.21}$$

This yields that ν_i is also a minimum point of $\widehat{m}|_{\Gamma_\beta}$ for all $\beta \in \mathcal{A}_i$. Thus $\partial_\beta \widehat{m}(\nu_i) \leqslant 0$ for all $\beta \in \mathcal{A}_i$. From the transmission condition in (3.2.12) which has a classical meaning thanks to (3.2.20), $\partial_\beta \widehat{m}(\nu_i) = 0$, since all the coefficients μ_β are positive. From (3.2.19), for all $\beta \in \mathcal{A}_i$, we have

$$C_\beta = \widehat{m}'_\beta(0) + b_\beta(0)\widehat{m}_\beta(0) = 0.$$

Therefore, $\widehat{m}'_\beta(y) + b_\beta(y)\widehat{m}_\beta(y) = 0$, for all $y \in [0, \ell_\beta]$ with $\widehat{m}_\beta(0) = 0$. This implies that $\widehat{m}_\beta \equiv 0$ for all $\beta \in \mathcal{A}_i$. We can propagate the arguments from the vertices connected to ν_i. Since Γ is connected, we obtain that $\widehat{m} \equiv 0$ on Γ.

Case 2: if $x_0 = \nu_i$ is a boundary vertex, then the Robin condition in (3.2.12) implies that $\partial_\alpha \widehat{m}(\nu_i) = 0$ since μ_α is positive. From (3.2.19), we have $C_\alpha = 0$. Therefore, $\widehat{m}'_\alpha(y) + b_\alpha(y)\widehat{m}_\alpha(y) = 0$, for all $y \in [0, \ell_\alpha]$ with $\widehat{m}_\alpha(0) = 0$. This implies that $\widehat{m}(\nu_j) = 0$ where ν_j is the other endpoint of Γ_α. We are back to Case 1, so $\widehat{m} \equiv 0$ on Γ.

Finally, we have found that $\widehat{m} \equiv 0$ on Γ, in contradiction with $\int_\Gamma \widehat{m} dx = 1$. □

Now we prove uniqueness for (3.2.12)-(3.2.13).

Proof of uniqueness in Theorem 3.2.13. The proof of uniqueness is similar to the argument in [32, Proposition 13]. As in the proof of Lemma 3.2.6, we can prove that for λ_0 large enough, there exists a constant C such that for any $f \in V'$, there exists a unique $w \in W$ which satisfies

$$\mathscr{A}_{\lambda_0}(w, v) = \langle f, v \rangle_{V', V} \text{ for all } v \in V. \tag{3.2.22}$$

and $\|w\|_W \leqslant C \|f\|_{V'}$. This allows us to define the continuous linear operator

$$\begin{aligned} S_{\lambda_0} : L^2(\Gamma) &\longrightarrow W, \\ f &\longmapsto w, \end{aligned}$$

where w is a solution of (3.2.22). Then we define $R_{\lambda_0} = \mathcal{J} \circ S_{\lambda_0}$ where $\mathcal{J}$ is the injection from W in $L^2(\Gamma)$, which is compact. Obviously, R_{λ_0} is a compact operator from $L^2(\Gamma)$ into $L^2(\Gamma)$. Moreover, $m \in W$ is a solution of (3.2.12) if and only if $m \in \ker(Id - \lambda_0 R_{\lambda_0})$. By Fredholm alternative, see [29], $\dim\ker(Id - \lambda_0 R_{\lambda_0}) = \dim\ker\left(Id - \lambda_0 R^\star_{\lambda_0}\right)$.

In order to characterize $R^\star_{\lambda_0}$, we now consider the following boundary value problem for $g \in L^2(\Gamma) \subset W'$:

$$\begin{cases} \lambda_0 v - \mu_\alpha \partial^2 v + b\partial v = g, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i) & \alpha, \beta \in \mathcal{A}_i, \ i \in I, \\ \sum\limits_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha \partial_\alpha v(\nu_i) = 0, & i \in I. \end{cases} \tag{3.2.23}$$

A weak solution of (3.2.23) is a function $v \in V$ such that

$$\mathscr{T}_{\lambda_0}(v, w) := \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} (\lambda_0 v w + \mu_\alpha \partial v \partial w + b w \partial v) dx = \int_\Gamma g w dx, \quad \text{for all } w \in W.$$

Using similar arguments as in the proof of existence in Theorem 3.2.13, we see that for λ_0 large enough and all $g \in L^2(\Gamma)$, there exists a unique solution $v \in V$ of (3.2.23). Moreover, there exists a constant C such that $\|v\|_V \leqslant C\|g\|_{L^2(\Gamma)}$ for all $g \in L^2(\Gamma)$. This allows us to define a continuous operator

$$\begin{aligned} \mathcal{T}_{\lambda_0} : L^2(\Gamma) &\longrightarrow V, \\ g &\longmapsto v. \end{aligned}$$

Then we define $\tilde{R}_{\lambda_0} = \mathcal{I} \circ \mathcal{T}_{\lambda_0}$ where $\mathcal{I}$ is the injection from V in $L^2(\Gamma)$, which is compact. Therefore, $\tilde{R}_{\lambda_0}$ is a compact operator from $L^2(\Gamma)$ into $L^2(\Gamma)$. For any $g \in L^2(\Gamma)$, set $v = \mathcal{T}_{\lambda_0} g$ which is the unique solution of (3.2.23). Noticing that $\mathscr{T}_{\lambda_0}(v,w) = \mathcal{A}_{\lambda_0}(w,v)$ for all $v \in V, w \in W$, we obtain that

$$(g, R_{\lambda_0} f)_{L^2(\Gamma)} = (g, \mathcal{J} \circ S_{\lambda_0} f)_{L^2(\Gamma)} = \mathscr{T}_{\lambda_0}(v, S_{\lambda_0} f) = \mathscr{A}_{\lambda_0}(S_{\lambda_0} f, v) = (f,v)_{L^2(\Gamma)} = (f, \tilde{R}_{\lambda_0} g)_{L^2(\Gamma)}.$$

Thus $R^\star_{\lambda_0} = \tilde{R}_{\lambda_0}$. But $\ker\left(Id - \lambda_0 \tilde{R}_{\lambda_0}\right)$ is the set of solutions of (3.2.11), which, from Lemma 3.2.9, consists of constant functions on Γ. This implies that $\dim\ker\left(Id - \lambda_0 R^\star_{\lambda_0}\right) = 1$ and then that

$$\dim\ker\left(Id - \lambda_0 R_{\lambda_0}\right) = \dim\ker\left(Id - \lambda_0 R^\star_{\lambda_0}\right) = 1.$$

Finally, since the solutions m of (3.2.12) are in $\ker\left(Id - \lambda_0 R_{\lambda_0}\right)$ and satisfy in addition the normalization condition $\int_\Gamma m dx = 1$, we obtain the desired uniqueness property in Theorem 3.2.13. □

3.3 Hamilton-Jacobi equation and the ergodic problem

3.3.1 The Hamilton-Jacobi equation

This section is devoted to the following boundary value problem including a Hamilton-Jacobi equation:

$$\begin{cases} -\mu_\alpha \partial^2 v + H(x, \partial v) + \lambda v = 0, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in A, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i), & \alpha, \beta \in \mathcal{A}_i,\ i \in I, \\ \sum\limits_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v(\nu_i) = 0, & i \in I, \end{cases} \tag{3.3.1}$$

where λ is a positive constant and the Hamiltonian $H : \Gamma \times \mathbb{R} \to \mathbb{R}$ is defined in Section 3.1, except that, in (3.3.1) and the whole Section 3.3.1 below, the Hamiltonian contains the coupling term, i.e, $H(x, \partial v)$ in (3.3.1) plays the role of $H(x, \partial v) - F(m(x))$ in (3.1.24).

Definition 3.3.1.
- A classical solution of (3.3.1) is a function $v \in C^2(\Gamma)$ which satisfies (3.3.1) pointwise.
- A weak solution of (3.3.1) is a function $v \in V$ such that

$$\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \left(\mu_\alpha \partial v \partial w + H(x, \partial v) w + \lambda v w\right) dx = 0 \quad \text{for all } w \in W.$$

Proposition 3.3.2. *Assume that*

$$H_\alpha \in C([0, \ell_\alpha] \times \mathbb{R}), \tag{3.3.2}$$

$$|H(x,p)| \leqslant C_2 \left(1 + |p|^2\right) \text{ for all } x \in \Gamma, p \in \mathbb{R}, \tag{3.3.3}$$

where C_2 is a positive constant. There exists a classical solution v of (3.3.1). Moreover, if H_α is locally Lipschitz with respect to both variables for all $\alpha \in \mathcal{A}$, then the solution v belongs to $C^{2,1}(\Gamma)$.

Remark 3.3.3. Assume (3.3.2) and that $v \in H^2(\Gamma) \subset V$ is a weak solution of (3.3.1). From the compact embedding of $H^2([0,\ell_\alpha])$ into $C^{1,\sigma}([0,\ell_\alpha])$ for all $\sigma \in (0,1/2)$, we get $v \in C^{1,\sigma}(\Gamma)$. Therefore, from the PDE in (3.3.1)

$$\mu_\alpha \partial^2 v_\alpha(\cdot) = H_\alpha(\cdot, \partial v_\alpha(\cdot)) + \lambda v_\alpha(\cdot) \in C([0,\ell_\alpha]).$$

It follows that v is a classical solution of (3.3.1).

Remark 3.3.4. Assume now that H is locally Lipschitz continuous and that $v \in H^2(\Gamma) \subset V$ is a weak solution of (3.3.1). From Remark 3.3.3, $v \in C^{1,\sigma}(\Gamma)$ for $\sigma \in (0,1/2)$ and the function $-\lambda v_\alpha - H_\alpha(\cdot, \partial v_\alpha)$ belongs to $C^{0,\sigma}([0,\ell_\alpha])$. Then, from the first line of (3.3.1), $v \in C^{2,\sigma}(\Gamma)$. This implies that $\partial v_\alpha \in \text{Lip}[0,\ell_\alpha]$ and using the PDE again, we see that $v \in C^{2,1}(\Gamma)$.

Let us start with the case when H is a bounded Hamiltonian.

Lemma 3.3.5. *Assume (3.3.2) and for some $C_H > 0$,*

$$|H(x,p)| \leqslant C_H, \quad \textit{for all } (x,p) \in \Gamma \times \mathbb{R}. \tag{3.3.4}$$

There exists a classical solution v of (3.3.1). Moreover, if H_α is locally Lipschitz in $[0,\ell_\alpha] \times \mathbb{R}$ for all $\alpha \in \mathcal{A}$ then the solution v belongs to $C^{2,1}(\Gamma)$.

Proof of Lemma 3.3.5. For any $u \in V$, from Lemma 3.2.6, the following boundary value problem:

$$\begin{cases} -\mu_\alpha \partial^2 v + \lambda v = -H(x, \partial u), & \text{if } x \in \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i), & \alpha, \beta \in \mathcal{A}_i, \ i \in I, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v(\nu_i) = 0, & i \in I, \end{cases} \tag{3.3.5}$$

has a unique weak solution $v \in V$. This allows us to define the map $\mathcal{T}$

$$\begin{aligned} \mathcal{T} : V &\longrightarrow V, \\ u &\longmapsto v. \end{aligned}$$

Moreover, from Lemma 3.2.6, there exists a constant C such that

$$\|v\|_V \leqslant C \|H(x,\partial u)\|_{L^2(\Gamma)} \leqslant C C_H |\Gamma|^{1/2}, \tag{3.3.6}$$

where $|\Gamma| = \Sigma_{\alpha\in\mathcal{A}} \ell_\alpha$. Therefore, from the PDE in (3.3.5),

$$\underline{\mu} \|\partial^2 v\|_{L^2(\Gamma)} \leqslant \lambda \|v\|_{L^2(\Gamma)} + \|H(x,\partial u)\|_{L^2(\Gamma)} \leqslant \lambda \|v\|_V + C_H |\Gamma|^{1/2} \leqslant (\lambda C + 1) C_H |\Gamma|^{1/2}, \tag{3.3.7}$$

where $\underline{\mu} := \min_{\alpha\in\mathcal{A}} \mu_\alpha$. From (3.3.6) and (3.3.7), $\mathcal{T}(V)$ is a bounded subset of $H^2(\Gamma)$ defined in Definition 3.1.3. From the compact embedding of $H^2(\Gamma)$ into V, we deduce that $\overline{\mathcal{T}(V)}$ is a compact subset of V.

Next, we claim that $\mathcal{T}$ is continuous from V to V. Assuming that

$$\begin{cases} u_n \to u, & \text{in } V, \\ v_n = \mathcal{T}(u_n), & \text{for all } n, \\ v = \mathcal{T}(u), \end{cases} \tag{3.3.8}$$

we need to prove that $v_n \to v$ in V. Since $\{v_n\}$ is uniformly bounded in $H^2(\Gamma)$, then, up to the extraction of a subsequence, $v_n \to \widehat{v}$ in $C^{1,\sigma}(\Gamma)$ for some $\sigma \in (0, 1/2)$. From (3.3.8), we have that $\partial u_n \to \partial u$ in $L^2(\Gamma_\alpha)$ for all $\alpha \in \mathcal{A}$. This yields that, up to another extraction of a subsequence, $\partial u_n \to \partial u$ almost everywhere in Γ_α. Thus $H(x, \partial u_n) \to H(x, \partial u)$ in $L^2(\Gamma_\alpha)$ by Lebesgue dominated convergence theorem. Hence, $\widehat{v}$ is a weak solution of (3.3.5). Since the latter is unique, $\widehat{v} = v$ and we can conclude that the whole sequence v_n converges to v. The claim is proved.

From Schauder fixed point theorem, see [61, Corollary 11.2], $\mathcal{T}$ admits a fixed point which is a weak solution of (3.3.1). Moreover, recalling that $v \in H^2(\Gamma)$, we obtain that v is a classical solution of (3.3.1) from Remark 3.3.3.

Assume now that H is locally Lipschitz. Since $v_\alpha \in H^2(0, \ell_\alpha)$ for all $\alpha \in \mathcal{A}$, we may use Remark 3.3.4 and obtain that $v \in C^{2,1}(\Gamma)$. □

Lemma 3.3.6. *If $v, u \in C^2(\Gamma)$ satisfy*

$$\begin{cases} -\mu_\alpha \partial^2 v + H(x, \partial v) + \lambda v \geqslant -\mu_\alpha \partial^2 u + H(x, \partial u) + \lambda u, & \text{if } x \in \Gamma_\alpha \backslash \mathcal{V}, \alpha \in A, \\ \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v(\nu_i) \leqslant \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha u(\nu_i), & \text{if } \nu_i \in \mathcal{V}, \end{cases} \tag{3.3.9}$$

then $v \geqslant u$.

Proof of Lemma 3.3.6. The proof is reminiscent of an argument in [33]. Suppose by contradiction that $\delta := \max_\Gamma \{u - v\} > 0$. Let $x_0 \in \Gamma_\alpha$ be a maximum point of $u - v$. It suffices to consider the case when $x_0 \in \mathcal{V}$, since if $x_0 \in \Gamma \backslash \mathcal{V}$, then

$$u(x_0) > v(x_0), \quad \partial u(x_0) = \partial v(x_0), \quad \partial^2 u(x_0) \leqslant \partial^2 v(x_0),$$

and we obtain a contradiction with the first line of (3.3.9).
Now consider the case when $x_0 = \nu_i \in \mathcal{V}$; from Remark 3.1.1, we can assume without restriction that $\pi_\alpha(0) = \nu_i$. Since $u - v$ achieves its maximum over Γ at ν_i, we obtain that

$$\partial_\beta u(\nu_i) \geqslant \partial_\beta v(\nu_i), \quad \text{for all } \beta \in \mathcal{A}_i.$$

From Kirchhoff conditions in (3.3.9), this implies that

$$\partial_\beta u(\nu_i) = \partial_\beta v(\nu_i), \quad \text{for all } \beta \in \mathcal{A}_i.$$

It follows that $\partial v_\alpha(0) = \partial u_\alpha(0)$. Using the first line of (3.3.9), we get that

$$-\mu_\alpha \left[\partial^2 v_\alpha(0) - \partial^2 u_\alpha(0)\right] \geqslant \underbrace{H_\alpha(0, \partial u_\alpha(0)) - H_\alpha(0, \partial v_\alpha(0))}_{=0} + \lambda(u_\alpha(0) - v_\alpha(0)) > 0.$$

Therefore, $u_\alpha - v_\alpha$ is locally strictly convex in $[0, \ell_\alpha]$ near 0 and its first order derivative vanishes at 0. This contradicts the fact that ν_i is the maximum point of $u - v$. □

We now turn to Proposition 3.3.2.

Proof of Proposition 3.3.2. We adapt the classical proof of Boccardo, Murat and Puel in [26]. First of all, we truncate the Hamiltonian as follows:

$$H_n(x, p) = \begin{cases} H(x, p), & \text{if } |p| \leqslant n, \\ H\left(x, \dfrac{p}{|p|} n\right), & \text{if } |p| > n. \end{cases}$$

By Lemma 3.3.5, for all $n \in \mathbb{N}$, since $H_n(x,p)$ is continuous and bounded by $C_2\left(1+n^2\right)$, there exists a classical solution $v_n \in C^2(\Gamma)$ for the following boundary value problem

$$\begin{cases} -\mu_\alpha \partial^2 v + H_n(x, \partial v) + \lambda v = 0, & x \in \Gamma_\alpha \backslash \mathcal{V}, \alpha \in A, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i), & \text{for all } \alpha, \beta \in \mathcal{A}_i,\ i \in I, \\ \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha \partial_\alpha v(\nu_i) = 0, & i \in I. \end{cases} \tag{3.3.10}$$

We wish to pass to the limit as n tend to $+\infty$; we first need to estimate v_n uniformly in n, successively in $L^\infty(\Gamma)$, $H^1(\Gamma)$ and $H^2(\Gamma)$.

Estimate in $L^\infty(\Gamma)$. Since $|H_n(x,p)| \leqslant c\left(1+|p|^2\right)$ for all x, p, then $\varphi = -c/\lambda$ and $\overline{\varphi} = c/\lambda$ are respectively a sub and supersolution of (3.3.10). Therefore, from Lemma 3.3.6, we obtain $|\lambda v_n| \leqslant c$.

Estimate in V. For a positive constant K to be chosen later, we introduce $w_n := e^{Kv_n^2} v_n \psi \in W$, where ψ is given in Definition 3.1.9. Using w_n as a test function in (3.3.10) leads to

$$\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} (\mu_\alpha \partial v_n \partial w_n + \lambda v_n w_n)\, dx = -\int_\Gamma H_n(x, \partial v_n)\, w_n dx.$$

Since $|H_n(x,p)| \leqslant c\left(1+p^2\right)$, we have

$$\begin{aligned} &\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} e^{Kv_n^2} \left[(\mu_\alpha \psi)(\partial v_n)^2 + (\mu_\alpha 2K\psi)\, v_n^2 (\partial v_n)^2 + (\mu_\alpha \partial\psi)\, v_n \partial v_n + \lambda \psi v_n^2\right] dx \\ &\leqslant \int_\Gamma e^{Kv_n^2} |H_n(x, \partial v_n)|\, |v_n \psi|\, dx \\ &\leqslant \int_\Gamma c e^{Kv_n^2} \psi\, |v_n|\, dx + \int_\Gamma c\psi e^{Kv_n^2} |v_n|\, \psi\, (\partial v_n)^2\, dx \\ &\leqslant \int_\Gamma e^{Kv_n^2} \left(\lambda \psi v_n^2 + \psi \frac{c^2}{4\lambda}\right) dx + \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} e^{Kv_n^2} \left[\frac{\mu_\alpha}{2} \psi (\partial v_n)^2 + \frac{c^2}{2\mu_\alpha} \psi (\partial v_n)^2 v_n^2\right] dx, \end{aligned}$$

where we have used Young inequalities. Since $\lambda > 0$ and $\psi > 0$, we deduce that

$$\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} e^{Kv_n^2} \left[\left(\frac{\mu_\alpha}{2}\psi\right)(\partial v_n)^2 + 2\psi\left(\mu_\alpha K - \frac{c^2}{4\mu_\alpha}\right) v_n^2 (\partial v_n)^2 + (\mu_\alpha \partial \psi)\, v_n \partial v_n\right] dx \leqslant \frac{c^2}{4\lambda} \int_\Gamma e^{Kv_n^2} \psi dx. \tag{3.3.11}$$

Next, choosing $K > (1 + c^2/4\underline{\mu})/\underline{\mu}$ yields that

$$\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} e^{Kv_n^2} \left[\frac{\mu_\alpha}{2}\psi (\partial v_n)^2 + 2\psi v_n^2 (\partial v_n)^2 + (\mu_\alpha \partial\psi)\, v_n \partial v_n\right] dx \leqslant C$$

for a positive constant C independent of n, because v_n is bounded by c/λ. Since ψ is bounded from below by a positive number and $\partial\psi$ is piecewise constant on Γ, we infer that

$$\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} e^{Kv_n^2} v_n^2 (\partial v_n)^2 \leqslant \widetilde{C},$$

where $\widetilde{C}$ is a positive constant independent on n. Using this information and (3.3.11) again, we obtain that $\int_\Gamma (\partial v_n)^2$ is bounded uniformly in n. There exists a constant $\overline{C}$ such that $\|v_n\|_V \leqslant \overline{C}$ for all n.

Estimate in $H^2(\Gamma)$. From the PDE in (3.3.10) and (3.3.3), we have

$$\underline{\mu}\left|\partial^2 v_n\right| \leqslant c + c\left|\partial v_n\right|^2 + \lambda\left|v_n\right|, \quad \text{for all } \alpha \in \mathcal{A}.$$

Thus $\partial^2 v_n$ is uniformly bounded in $L^1(\Gamma)$. This and the previous estimate on $\|\partial v_n\|_{L^2(\Gamma)}$ yield that ∂v_n is uniformly bounded in $L^\infty(\Gamma)$, from the continuous embedding of $W^{1,1}(0,\ell_\alpha)$ into $C([0,\ell_\alpha])$. Therefore, from (3.3.10), we get that $\partial^2 v_n$ is uniformly bounded in $L^\infty(\Gamma)$. This implies in particular that v_n is uniformly bounded in $W^{2,\infty}(\Gamma)$.

Hence, for any $\sigma \in (0,1)$, up to the extraction of a subsequence, there exists $v \in V$ such that $v_n \to v$ in $C^{1,\sigma}(\Gamma)$. This yields that $H_n(x,\partial v_n) \to H(x,\partial v)$ for all $x \in \Gamma$. By Lebesgue's Dominated Convergence Theorem, we obtain that v is a weak solution of (3.3.1), and since $v \in C^{1,\sigma}(\Gamma)$, by Remark 3.3.3, v is a classical solution of (3.3.1).

Assume now that H is locally Lipschitz. We may use Remark 3.3.4 and obtain that $v \in C^{2,1}(\Gamma)$. The proof is complete. $\square$

3.3.2 The ergodic problem

For $f \in PC(\Gamma)$, we wish to prove the existence of $(v,\rho) \in C^2(\Gamma) \times \mathbb{R}$ such that

$$\begin{cases} -\mu_\alpha \partial^2 v + H(x,\partial v) + \rho = f(x), & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i), & \alpha,\beta \in \mathcal{A}_i,\ i \in I, \\ \sum\limits_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha \partial_\alpha v(\nu_i) = 0, & i \in I, \end{cases} \tag{3.3.12}$$

with the normalization condition

$$\int_\Gamma v dx = 0. \tag{3.3.13}$$

Theorem 3.3.7. *Assume (3.1.25)-(3.1.27). There exists a unique couple $(v,\rho) \in C^2(\Gamma) \times \mathbb{R}$ satisfying (3.3.12)-(3.3.13), with $|\rho| \leqslant \max_{x\in\Gamma}|H(x,0) - f(x)|$. There exists a constant $\overline{C}$ which only depends upon $\|f\|_{L^\infty(\Gamma)}, \mu_\alpha$ and the constants in (3.1.27) such that*

$$\|v\|_{C^2(\Gamma)} \leqslant \overline{C}. \tag{3.3.14}$$

Moreover, for some $\sigma \in (0,1)$, if $f_\alpha \in C^{0,\sigma}([0,\ell_\alpha])$ for all $\alpha \in \mathcal{A}$, then $(v,\rho) \in C^{2,\sigma}(\Gamma) \times \mathbb{R}$; there exists a constant $\overline{C}$ which only depends upon $\|f_\alpha\|_{C^{0,\sigma}([0,\ell_\alpha])}, \mu_\alpha$ and the constants in (3.1.27) such that

$$\|v\|_{C^{2.\sigma}(\Gamma)} \leqslant \overline{C}. \tag{3.3.15}$$

Proof of existence in Theorem 3.3.7. By Proposition 3.3.2, for any $\lambda > 0$, the following boundary value problem

$$\begin{cases} -\mu_\alpha \partial^2 v + H(x,\partial v) + \lambda v = f, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i), & \alpha,\beta \in \mathcal{A}_i,\ i \in I, \\ \sum\limits_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha \partial_\alpha v(\nu_i) = 0, & i \in I, \end{cases} \tag{3.3.16}$$

has a unique solution $v_\lambda \in C^2(\Gamma)$. Set $C := \max_\Gamma |f(\cdot) - H(\cdot,0)|$. The constant functions $\underline{\varphi} := -C/\lambda$ and $\overline{\varphi} = C/\lambda$ are respectively sub and supersolution of (3.3.16). By Lemma 3.3.6,

$$-C \leqslant \lambda v_\lambda(x) \leqslant C, \quad \text{for all } x \in \Gamma. \tag{3.3.17}$$

Next, set $u_\lambda := v_\lambda - \min_\Gamma v_\lambda$. We see that u_λ is the unique classical solution of

$$\begin{cases} -\mu_\alpha \partial^2 u_\lambda + H(x, \partial u_\lambda) + \lambda u_\lambda + \lambda \min_\Gamma v_\lambda = f, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ u|_{\Gamma_\alpha}(\nu_i) = u|_{\Gamma_\beta}(\nu_i), & \alpha, \beta \in \mathcal{A}_i,\ i \in I, \\ \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha u_\lambda(\nu_i) = 0, & i \in I. \end{cases} \tag{3.3.18}$$

Before passing to the limit as λ tends 0, we need to estimate u_λ in $C^2(\Gamma)$ uniformly with respect to λ. We do this in two steps:

Step 1: Estimate of $\|\partial u_\lambda\|_{L^q(\Gamma)}$. Using ψ as a test-function in (3.3.18), see Definition 3.1.9, and recalling that $\lambda u_\lambda + \lambda \min_\Gamma v_\lambda = \lambda v_\lambda$, we see that

$$\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \mu_\alpha \partial u_\lambda \partial \psi dx + \int_\Gamma (H(x, \partial u_\lambda) + \lambda v_\lambda) \psi dx = \int_\Gamma f \psi dx.$$

From (3.1.27) and (3.3.17),

$$\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \mu_\alpha \partial u_\lambda \partial \psi dx + \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} C_0 |\partial u_\lambda|^q \psi dx \leqslant \int_\Gamma (f + C + C_1) \psi dx.$$

On the other hand, since $q > 1$, $\psi \geqslant \underline{\psi} > 0$ and $\partial \psi$ is bounded, there exists a large enough positive constant C' such that

$$\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \mu_\alpha \partial u_\lambda \partial \psi dx + \frac{1}{2} \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} C_0 |\partial u_\lambda|^q \psi dx + C' > 0, \quad \text{for all } \lambda > 0.$$

Subtracting the two inequalities, we get

$$\frac{C_0}{2} \underline{\psi} \int_\Gamma |\partial u_\lambda|^q dx \leqslant \int_\Gamma (f + C + C_1) \psi dx + C'.$$

Hence, for all $\lambda > 0$, we have

$$\|\partial u_\lambda\|_{L^q(\Gamma)} \leqslant \widetilde{C}, \tag{3.3.19}$$

where $\widetilde{C} := \left[\left(2 \int_\Gamma (|f| + C + C_1) \psi dx + 2C'\right) / (C_0 \underline{\psi})\right]^{1/q}$.

Step 2: Estimate of $\|u_\lambda\|_{C^2(\Gamma)}$. Since $u_\lambda = v_\lambda - \min_\Gamma v_\lambda$, there exists $\alpha \in \mathcal{A}$ and $x_\lambda \in \Gamma_\alpha$ such that $u_\lambda(x_\lambda) = 0$. For all $\lambda > 0$ and $x \in \Gamma_\alpha$, we have

$$|u_\lambda(x)| = |u_\lambda(x) - u_\lambda(x_\lambda)| \leqslant \int_\Gamma |\partial u_\lambda| dx \leqslant \|\partial u_\lambda\|_{L^q(\Gamma)} |\Gamma|^{q/(q-1)}.$$

From (3.3.19) and the latter inequality, we deduce

$$\|u_\lambda|_{\Gamma_\alpha}\|_{L^\infty(\Gamma_\alpha)} \leqslant \widetilde{C} |\Gamma|^{q/(q-1)}.$$

Let ν_i be a transition vertex which belongs to $\partial \Gamma_\alpha$. For all $\beta \in \mathcal{A}_i$, $y \in \Gamma_\beta$,

$$|u_\lambda(y)| \leqslant |u_\lambda(y) - u_\lambda(\nu_i)| + |u_\lambda(\nu_i)| \leqslant 2\widetilde{C} |\Gamma|^{q/(q-1)}.$$

Since the network is connected and the number of edges is finite, repeating the argument as many times as necessary, we obtain that there exists $M \in \mathbb{N}$ such that

$$\|u_\lambda\|_{L^\infty(\Gamma)} \leqslant M\widetilde{C} |\Gamma|^{q/(q-1)}.$$

This bound is uniform with respect to $\lambda \in (0,1]$. Next, from (3.3.18) and (3.1.29), we get

$$\underline{\mu}\left|\partial^2 u_\lambda\right| \leqslant \left|H\left(x, \partial u_\lambda\right)\right| + \left|\lambda v_\lambda\right| + |f| \leqslant C_q\left(1 + \left|\partial u_\lambda\right|^q\right) + C + \|f\|_{L^\infty(\Gamma)}.$$

Hence, from (3.3.19), $\partial^2 u_\lambda$ is bounded in $L^1(\Gamma)$ uniformly with respect to $\lambda \in (0,1]$. From the continuous embedding of $W^{1,1}(0,\ell_\alpha)$ in $C([0,\ell_\alpha])$, we infer that $\partial u_\lambda|_{\Gamma_\alpha}$ is bounded in $C(\Gamma_\alpha)$ uniformly with respect to $\lambda \in (0,1]$. From the equation (3.3.18) and (3.3.17), this implies that u_λ is bounded in $C^2(\Gamma)$ uniformly with respect to $\lambda \in (0,1]$.

After the extraction of a subsequence, we may assume that when $\lambda \to 0^+$, the sequence u_λ converges to some function $v \in C^{1,1}(\Gamma)$ and that $\lambda \min v_\lambda$ converges to some constant ρ. Notice that v still satisfies the Kirchhoff conditions since $\partial u_\lambda|_{\Gamma_\alpha}(\nu_i) \to \partial v|_{\Gamma_\alpha}(\nu_i)$ as $\lambda \to 0^+$. Passing to the limit in (3.3.18), we get that the couple (v,ρ) satisfies (3.3.12) in the weak sense, then in the classical sense by using an argument similar to Remark 3.3.3. Adding a constant to v, we also get (3.3.13).

Furthermore, if for some $\sigma \in (0,1)$, $f|_{\Gamma_\alpha} \in C^{0,\sigma}(\Gamma_\alpha)$ for all $\alpha \in \mathcal{A}$, a bootstrap argument using the Lipschitz continuity of H on the bounded subsets of $\Gamma \times \mathbb{R}$ shows that u_λ is bounded in $C^{2,\sigma}(\Gamma)$ uniformly with respect to $\lambda \in (0,1]$. After a further extraction of a subsequence if necessary, we obtain (3.3.15).

□

Proof of uniqueness in Theorem 3.3.7. Assume that there exist two solutions (v,ρ) and $(\tilde{v},\tilde{\rho})$ of (3.3.12)-(3.3.13). First of all, we claim that $\rho = \tilde{\rho}$. By symmetry, it suffices to prove that $\rho \geqslant \tilde{\rho}$. Let x_0 be a maximum point of $e := \tilde{v} - v$. Using similar arguments as in the proof of Lemma 3.3.6, with λv and λu respectively replaced by ρ and $\tilde{\rho}$, we get $\rho \geqslant \tilde{\rho}$ and the claim is proved.

We now prove the uniqueness of v. Since H_α belongs to $C^1(\Gamma_\alpha \times \mathbb{R})$ for all $\alpha \in \mathcal{A}$, then e is a solution of

$$\mu_\alpha \partial^2 e_\alpha - \left[\int_0^1 \partial_p H_\alpha\left(y, \theta \partial v_\alpha + (1-\theta)\partial \tilde{v}_\alpha\right) d\theta\right] \partial e_\alpha = 0, \quad \text{in } (0,\ell_\alpha),$$

with the same transition and boundary condition as in (3.3.12). By Lemma 3.2.9, e is a constant function on Γ. Moreover, from (3.3.13) , we know that $\int_\Gamma e dx = 0$. This yields that $e = 0$ on Γ. Hence, (3.3.12)-(3.3.13) has a unique solution. □

Remark 3.3.8. Since there exists a unique solution of (3.3.12)-(3.3.13), we conclude that the whole sequence $(u_\lambda, \lambda v_\lambda)$ in the proof of Theorem 3.3.7 converges to (v,ρ) as $\lambda \to 0$.

3.4 Proof of the main result

We first prove Theorem 3.1.13 when F is bounded.

Theorem 3.4.1. *Assume (3.1.25)-(3.1.28), (3.1.30) and that F is bounded. There exists a solution $(v,m,\rho) \in C^2(\Gamma) \times \mathcal{W} \times \mathbb{R}$ to the mean field games system (3.1.24). If F is locally Lipschitz continuous, then $v \in C^{2,1}(\Gamma)$. If furthermore F is strictly increasing, then the solution is unique.*

Proof of existence in Theorem 3.4.1. We adapt the proof of Camilli & Marchi in [32, Theorem 1]. For $\sigma \in (0,1/2)$ let us introduce the space

$$\mathcal{M}_\sigma = \left\{m : m_\alpha \in C^{0,\sigma}([0,\ell_\alpha]) \text{ and } \frac{m|_{\Gamma_\alpha}(\nu_i)}{\gamma_{i\alpha}} = \frac{m|_{\Gamma_\beta}(\nu_i)}{\gamma_{i\beta}} \text{ for all } i \in I \text{ and } \alpha,\beta \in \mathcal{A}_i\right\}$$

which, endowed with the norm

$$\|m\|_{\mathcal{M}_\sigma} = \|m\|_{L^\infty(\Gamma)} + \max_{\alpha\in\mathcal{A}} \sup_{y,z\in[0,\ell_\alpha],y\neq z} \frac{|m_\alpha(y) - m_\alpha(z)|}{|y-z|^\sigma},$$

is a Banach space. Now consider the set

$$\mathcal{K} = \left\{ m \in \mathcal{M}_\sigma : m \geqslant 0 \text{ and } \int_\Gamma m dx = 1 \right\}$$

and observe that $\mathcal{K}$ is a closed and convex subset of $\mathcal{M}_\sigma$. We define a map $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ as follows: given $m \in \mathcal{K}$, set $f = F(m)$. By Theorem 3.3.7, (3.3.12)-(3.3.13) has a unique solution $(v,\rho) \in C^2(\Gamma) \times \mathbb{R}$. Next, for v given, we solve (3.2.12)-(3.2.13) with $b(\cdot) = \partial_p H(\cdot, \partial v(\cdot)) \in PC(\Gamma)$. By Theorem 3.2.13, there exists a unique solution $\overline{m} \in \mathcal{K} \cap W$ of (3.2.12)-(3.2.13). We set $\mathcal{T}(m) = \overline{m}$; we claim that $\mathcal{T}$ is continuous and has a precompact image. We proceed in several steps:

$\mathcal{T}$ is continuous. Let $m_n, m \in \mathcal{K}$ be such that $\|m_n - m\|_{\mathcal{M}_\sigma} \to 0$ as $n \to +\infty$; set $\overline{m}_n = \mathcal{T}(m_n), \overline{m} = \mathcal{T}(m)$. We need to prove that $\overline{m}_n \to \overline{m}$ in $\mathcal{M}_\sigma$. Let $(v_n,\rho_n), (v,\rho)$ be the solutions of (3.3.12)-(3.3.13) corresponding respectively to $f = F(m_n)$ and $f = F(m)$. Using estimate (3.3.14), we see that up to the extraction of a subsequence, we may assume that $(v_n,\rho_n) \to (\overline{v},\overline{\rho})$ in $C^1(\Gamma)\times\mathbb{R}$. Since $F(m_n)|_{\Gamma_\alpha} \to F(m)|_{\Gamma_\alpha}$ in $C(\Gamma_\alpha)$, $H_\alpha(y,(\partial v_n)_\alpha) \to H_\alpha(y,\partial\overline{v}_\alpha)$ in $C([0,\ell_\alpha])$, and since it is possible to pass to the limit in the transmission and boundary conditions thanks to the C^1-convergence, we obtain that $(\overline{v},\overline{\rho})$ is a weak (and strong by Remark 3.3.3) solution of (3.3.12)-(3.3.13). By uniqueness, $(\overline{v},\overline{\rho}) = (v,\rho)$ and the whole sequence (v_n,ρ_n) converges.

Next, $\overline{m}_n = \mathcal{T}(m_n), \overline{m} = \mathcal{T}(m)$ are respectively the solutions of (3.2.12)-(3.2.13) corresponding to $b = \partial_p H(x,\partial v_n)$ and $b = \partial_p H(x,\partial v)$. From the estimate (3.2.16), since $\partial_p H(x,\partial v_n)$ is uniformly bounded in $L^\infty(\Gamma)$, we see that $\overline{m}_n$ is uniformly bounded in W. Therefore, up to the extraction of subsequence, we have

$$\begin{cases} \overline{m}_n \rightharpoonup \widehat{m} & \text{in } W, \\ \overline{m}_n \to \widehat{m} & \text{in } \mathcal{M}_\sigma, \end{cases}$$

because W is compactly embedded in $\mathcal{M}_\sigma$ for $\sigma \in (0,1/2)$. It is easy to pass to the limit and find that $\widehat{m}$ is a solution of (3.2.12)-(3.2.13) with $b = \partial_p H(x,\partial v)$. From Theorem 3.2.13, we obtain that $\overline{m} = \widehat{m}$, and hence the whole sequence $\overline{m}_n$ converges to $\overline{m}$.

The image of $\mathcal{T}$ is precompact. Since $F \in C^0(\mathbb{R}^+;\mathbb{R})$ is a uniformly bounded function, we see that $F(m)$ is bounded in $L^\infty(\Gamma)$ uniformly with respect to $m \in \mathcal{K}$. From Theorem 3.3.7, there exists a constant $\overline{C}$ such that for all $m \in \mathcal{K}$, the unique solution v of (3.3.12)-(3.3.13) with $f = F(m)$ satisfies $\|v\|_{C^2(\Gamma)} \leqslant \overline{C}$. From Theorem 3.2.13, we obtain that $\overline{m} = \mathcal{T}(m)$ is bounded in W by a constant independent of m. Since W is compactly embedded in $\mathcal{M}_\sigma$, for $\sigma \in (0,1/2)$ we deduce that $\mathcal{T}$ has a precompact image.

End of the proof. We can apply Schauder fixed point theorem (see [61, Corollary 11.2]) to conclude that the map $\mathcal{T}$ admits a fixed point m. By Theorem 3.2.13, we get $m \in \mathcal{W}$. Hence, there exists a solution $(v,m,\rho) \in C^2(\Gamma)\times\mathcal{W}\times\mathbb{R}$ to the mean field games system (3.1.24). If F is locally Lipschitz continuous, then $v \in C^{2,1}(\Gamma)$ from the final part of Theorem 3.3.7. □

Proof of uniqueness in Theorem 3.4.1. We assume that F is strictly increasing and that there exist two solutions (v_1,m_1,ρ_1) and (v_2,m_2,ρ_2) of (3.1.24). We set $\overline{v} = v_1 - v_2, \overline{m} = m_1 - m_2$

and $\overline{\rho} = \rho_1 - \rho_2$ and write the equations for $\overline{v}, \overline{m}$ and $\overline{\rho}$

$$\begin{cases} -\mu_\alpha \partial^2 \overline{v} + H(x, \partial v_1) - H(x, \partial v_2) + \overline{\rho} - (F(m_1) - F(m_2)) = 0, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \\ -\mu_\alpha \partial^2 \overline{m} - \partial (m_1 \partial_p H(x, \partial v_1)) + \partial (m_2 \partial_p H(x, \partial v_2)) = 0, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \\ \overline{v}|_{\Gamma_\alpha}(\nu_i) = \overline{v}|_{\Gamma_\beta}(\nu_i), \quad \dfrac{\overline{m}|_{\Gamma_\alpha}(\nu_i)}{\gamma_{i\alpha}} = \dfrac{\overline{m}|_{\Gamma_\beta}(\nu_i)}{\gamma_{i\beta}}, & \alpha, \beta \in \mathcal{A}_i, i \in I, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha \overline{v}(\nu_i) = 0, & i \in I, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} n_{i\alpha} \left[m_1|_{\Gamma_\alpha}(\nu_i) \partial_p H(\nu_i, \partial v_1|_{\Gamma_\alpha}(\nu_i)) - m_2|_{\Gamma_\alpha}(\nu_i) \partial_p H(\nu_i, \partial v_2|_{\Gamma_\alpha}(\nu_i)) \right] & \\ \displaystyle+ \sum_{\alpha \in \mathcal{A}_i} \mu_\alpha \partial_\alpha \overline{m}(\nu_i) = 0, & i \in I, \\ \displaystyle\int_\Gamma \overline{v} dx = 0, \quad \int_\Gamma \overline{m} dx = 0. & \end{cases} \tag{3.4.1}$$

Multiplying the equation for $\overline{v}$ by $\overline{m}$ and integrating over Γ_α, we get

$$\int_{\Gamma_\alpha} \mu_\alpha \partial \overline{v} \partial \overline{m} + [H(x, \partial v_1) - H(x, \partial v_2) + \overline{\rho} - (F(m_1) - F(m_2))] \overline{m} dx - [\mu_\alpha \overline{m}_\alpha \partial \overline{v}_\alpha]_0^{\ell_\alpha} = 0. \tag{3.4.2}$$

Multiplying the equation for $\overline{m}$ by $\overline{v}$ and integrating over Γ_α, we get

$$\begin{aligned} &\int_{\Gamma_\alpha} \mu_\alpha \partial \overline{v} \partial \overline{m} + [m_1 \partial_p H(x, \partial v_1) - m_2 \partial_p H(x, \partial v_2)] \partial \overline{v} dx \\ &- \Big[\overline{v}|_{\Gamma_\alpha} (\mu_\alpha \partial \overline{m}|_{\Gamma_\alpha} + m_1|_{\Gamma_\alpha} \partial_p H(x, \partial v_1|_{\Gamma_\alpha}) - m_2|_{\Gamma_\alpha} \partial_p H(x, \partial v_2|_{\Gamma_\alpha})) \Big]_0^{\ell_\alpha} = 0. \end{aligned} \tag{3.4.3}$$

Subtracting (3.4.2) to (3.4.3), summing over $\alpha \in \mathcal{A}$, assembling the terms corresponding to a same vertex ν_i and taking into account the transmission and the normalization condition for $\overline{v}$ and $\overline{m}$, we obtain

$$\begin{aligned} 0 = &\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} (m_1 - m_2) [F(m_1) - F(m_2)] dx \\ &+ \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} m_1 [H(x, \partial v_2) - H(x, \partial v_1) + \partial_p H(x, \partial v_1) \partial \overline{v}] dx \\ &+ \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} m_2 [H(x, \partial v_1) - H(x, \partial v_2) - \partial_p H(x, \partial v_2) \partial \overline{v}] dx. \end{aligned}$$

Since F is strictly monotone then the first sum is non-negative. Moreover, by the convexity of H and the positivity of m_1, m_2, the last two sums are non-negative. Therefore, we have that $m_1 = m_2$. From Theorem 3.3.7, we finally obtain $v_1 = v_2$ and $\rho_1 = \rho_2$. ☐

Proof of Theorem 3.1.13 for a general coupling F. We only need to modify the proof of existence.

We now truncate the coupling function as follows:

$$F_n(r) = \begin{cases} F(r), & \text{if } |r| \leqslant n, \\ F\left(\dfrac{r}{|r|} n\right), & \text{if } |r| \geqslant n. \end{cases}$$

Then F_n is continuous, bounded below by $-M$ as in (3.1.31) and bounded above by some constant C_n. By Theorem 3.4.1, for all $n \in \mathbb{N}$, there exists a unique solution $(v_n, m_n, \rho_n) \in C^2(\Gamma) \times \mathcal{W} \times \mathbb{R}$ of the mean field game system (3.1.24) where F is replaced by F_n. We wish to pass to the limit as $n \to +\infty$. We proceed in several steps:

Step 1: ρ_n is bounded from below. Multiplying the HJB equation in (3.1.24) by m_n and the Fokker-Planck equation in (3.1.24) by v_n, using integration by parts and the transmission conditions, we obtain that

$$\sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \mu_\alpha \partial v_n \partial m_n dx + \int_\Gamma H(x, \partial v_n) m_n dx + \rho_n = \int_\Gamma F_n(m_n) m_n dx, \tag{3.4.4}$$

and

$$\sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \mu_\alpha \partial v_n \partial m_n dx + \int_\Gamma \partial_p H(x, \partial v_n) m_n \partial v_n dx = 0. \tag{3.4.5}$$

Subtracting the two equations, we obtain

$$\rho_n = \int_\Gamma F_n(m_n) m_n dx + \int_\Gamma [\partial_p H(x, \partial v_n) \partial v_n - H(x, \partial v_n)] m_n dx. \tag{3.4.6}$$

In what follows, the constant C may vary from line to line but remains independent of n. From (3.1.26), we see that $\partial_p H(x, \partial v_n) \partial v_n - H(x, \partial v_n) \geqslant -H(x, 0) \geqslant -C$. Therefore

$$\rho_n \geqslant \int_\Gamma F_n(m_n) m_n dx - C \int_\Gamma m_n dx = \int_\Gamma F_n(m_n) m_n dx - C. \tag{3.4.7}$$

Hence, since $F_n + M \geqslant 0$ and $\int_\Gamma m_n dx = 1$, we get that ρ_n is bounded from below by $-M - C$ independently of n.

Step 2: ρ_n and $\int_\Gamma F_n(m_n) dx$ are uniformly bounded. By Theorem 3.2.13, there exists a positive solution $w \in W$ of (3.2.12)-(3.2.13) with $b = 0$. It yields

$$\begin{cases} \displaystyle\sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \mu_\alpha \partial w \partial u dx = 0, & \text{for all } u \in V, \\ \int_\Gamma w dx = 1. \end{cases}$$

Multiplying the HJB equation of (3.1.24) by w, using integration by parts and the Kirchhoff condition, we get

$$\underbrace{\sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \mu_\alpha \partial v_n \partial w dx}_{=0} + \int_\Gamma H(x, \partial v_n) w dx + \rho_n \underbrace{\int_\Gamma w dx}_{=1} = \int_\Gamma F_n(m_n) w dx.$$

This implies, using (3.1.27), (3.2.16) and $F_n + M \geqslant 0$,

$$\begin{aligned} \rho_n &= \int_\Gamma F_n(m_n) w dx - \int_\Gamma H(x, \partial v_n) w dx \\ &\leqslant \|w\|_{L^\infty(\Gamma)} \int_\Gamma (F_n(m_n) + M) dx - M - \int_\Gamma (C_0 |\partial v_n|^q - C_1) w dx \\ &\leqslant C \int_\Gamma F_n(m_n) dx + C - \int_\Gamma C_0 |\partial v_n|^q w dx. \end{aligned} \tag{3.4.8}$$

Thus, by (3.4.7), we have

$$-M-C\leqslant\int_\Gamma F_n\left(m_n\right)m_n dx-C\leqslant\rho_n\leqslant C\int_\Gamma F_n\left(m_n\right)dx+C.\tag{3.4.9}$$

Let $K>0$ be a constant to be chosen later. We have

$$\begin{aligned}\int_\Gamma F_n\left(m_n\right)dx&\leqslant\int_{m_n\geqslant K}(F_n\left(m_n\right)+M)dx+\int_{m_n\leqslant K}F_n\left(m_n\right)dx\\&\leqslant\frac{1}{K}\int_{m_n\geqslant K}\left[F_n\left(m_n\right)+M\right]m_n dx+\sup_{0\leqslant r\leqslant K}F_n(r)\int_{m_n\leqslant K}dx\\&\leqslant\frac{1}{K}\int_{m_n\geqslant K}\left[F_n\left(m_n\right)+M\right]m_n dx+\sup_{0\leqslant r\leqslant K}F(r)\int_{m_n\leqslant K}dx \quad (3.4.10)\\&\leqslant\frac{1}{K}\int_\Gamma F_n\left(m_n\right)m_n dx+\frac{M}{K}+C_K, \quad (3.4.11)\end{aligned}$$

where $C_K:=|\sup_{0\leqslant r\leqslant K}F(r)|$ is independent of n. Choosing $K=2C$ where C is the constant in (3.4.9), we get by combining (3.4.11) with (3.4.9) that

$$\int_\Gamma F_n(m_n)m_n\leqslant C.$$

Using (3.4.11) again, we obtain

$$\int_\Gamma F_n(m_n)dx\leqslant C.$$

Hence, from (3.4.9), we conclude that $|\rho_n|+|\int_\Gamma F_n(m_n)dx|\leqslant C$.

Step 3: Prove that $F_n\left(m_n\right)$ is uniformly integrable and v_n and m_n are uniformly bounded respectively in $C^1\left(\Gamma\right)$ and W. Let E be a measurable with $|E|=\eta$. By (3.4.10) with Γ is replaced by E, we have

$$\begin{aligned}\int_E F_n(m_n)dx&\leqslant\frac{1}{K}\int_{E\cap\{m_n\geqslant K\}}[F_n\left(m_n\right)+M]m_n dx+\sup_{0\leqslant r\leqslant K}F(r)\int_{E\cap\{m_n\leqslant K\}}dx\\&\leqslant\frac{C+M}{K}+C_K\eta,\end{aligned}$$

where the last inequality comes from $\int_E F_n(m_n)m_n dx\leqslant C$ and $C_K=|\sup_{0\leqslant r\leqslant K}F(r)|$. Therefore, for all $\varepsilon>0$, we may choose K such that $(C+M)/K\leqslant\varepsilon/2$ and then η such that $C_K\eta\leqslant\varepsilon/2$ and get

$$\int_E F_n\left(m_n\right)dx\leqslant\varepsilon,\quad\text{for all } E \text{ which satisfies } |E|\leqslant\eta,$$

which proves the uniform integrability of $\{F_n(m_n)\}_n$.

Next, since ρ_n and $\int_\Gamma F_n\left(m_n\right)dx$ are uniformly bounded, we infer from (3.4.8) that ∂v_n is uniformly bounded in $L^q\left(\Gamma\right)$. Since by the condition $\int_\Gamma v_n dx=0$, there exists $\overline{x}_n$ such that $v_n(\overline{x}_n)=0$, we infer from the latter bound that v_n is uniformly bounded in $L^\infty(\Gamma)$. Using the HJB equation in (3.1.24) and Remark 3.1.7, we get

$$\mu_\alpha|\partial^2 v_n|\leqslant|H(x,\partial v_n)|+|F_n(m_n)|+|\rho_n|\leqslant C_q(|\partial v_n|^q+1)+|F_n(m_n)|+|\rho_n|.$$

We obtain that $\partial^2 v_n$ is uniformly bounded in $L^1\left(\Gamma\right)$, which implies that v_n is uniformly bounded in $C^1(\Gamma)$. Therefore the sequence of functions $C_q(|\partial v_n|^q+1)+|F_n(m_n)|+|\rho_n|$ is uniformly integrable, and so is $\partial^2 v_n$. This implies that ∂v_n is equicontinuous. Hence, $\{v_n\}$ is relatively compact in $C^1(\Gamma)$ by Arzelà-Ascoli's theorem. Finally, from the Fokker-Planck equation and Theorem 3.2.13, since $\partial_p H(x, \partial v_n)$ is uniformly bounded in $L^\infty(\Gamma)$, we obtain that m_n is uniformly bounded in W.

Step 4: Passage to the limit

From Step 1 and 2, since $\{\rho_n\}$ is uniformly bounded, there exists $\rho \in \mathbb{R}$ such that $\rho_n \to \rho$ up to the extraction of subsequence. From Step 3, there exists $m \in W$ such that $m_n \rightharpoonup m$ in W and $m_n \to m$ almost everywhere, up to the extraction of subsequence. Also from Step 3, since $F_n(m_n)$ is uniformly integrable, from Vitali theorem, we have

$$\lim_{n\to\infty} \int_\Gamma F_n(m_n)\, \tilde{w} dx = \int_\Gamma F(m)\, \tilde{w} dx, \quad \text{for all } \tilde{w} \in W.$$

From Step 3, up to the extraction of subsequence, there exists $v \in C^1(\Gamma)$ such that $v_n \to v$ in $C^1(\Gamma)$. Hence, (v, ρ, m) satisfies the weak form of the MFG system:

$$\sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \mu_\alpha \partial v \partial \tilde{w} dx + \int_\Gamma (H(x, \partial v) + \rho)\, \tilde{w} dx = \int_\Gamma F(m)\, \tilde{w} dx, \quad \text{for all } \tilde{w} \in W,$$

and

$$\sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \mu_\alpha \partial m \partial \tilde{v} dx + \int_\Gamma \partial_p H(x, \partial v)\, m \partial \tilde{v} dx = 0, \quad \text{for all } \tilde{v} \in V.$$

Finally, we prove the regularity for the solution of (3.1.24). Since $m \in W$, we get $F(m) \in C^{0,\sigma}(\Gamma)$ for some constant $\sigma \in (0, 1/2)$. By Theorem 3.3.7, $v \in C^2$ ($v \in C^{2,\sigma}(\Gamma)$ if F is locally Lipschitz continuous). Then, by Theorem 3.2.13, we get $m \in \mathcal{W}$. We also obtain that v satisfy the Kirchhoff condition and transition condition in (3.1.24). The proof is done. $\square$

4 A Class of Mean Field Games on Networks. Part two: Finite Horizon Games

Abstract: We consider stochastic mean field games for which the state space is a network. In the non-stationary case, they are described by a system coupling a Hamilton-Jacobi-Bellman equation and a Fokker-Planck equation, whose unknowns are an measure m and a value function v. The function v is continuous and satisfies general Kirchhoff conditions at the vertices. The measure m satisfies dual transmission conditions: in particular, m is discontinuous across the vertices in general, and the values of m on each side of the vertices satisfy special compatibility conditions. Existence and uniqueness are proven, under suitable assumptions.

4.1 Introduction and main results

The present work is devoted to finite horizon stochastic mean field games taking place on networks. The most important difficulty will be to deal with the transition conditions at the vertices. The latter are obtained from the theory of stochastic control in [56, 55], see Section 4.1.3 below. In [32], the first article on MFGs on networks, Camilli & Marchi consider a particular type of Kirchhoff condition at the vertices for the value function: this condition comes from an assumption which can be informally stated as follows: consider a vertex ν of the network and assume that it is the intersection of p edges $\Gamma_1, \dots, \Gamma_p$, ; if, at time τ, the controlled stochastic process X_t associated to a given agent hits ν, then the probability that X_{τ^+} belongs to Γ_i is proportional to the diffusion coefficient in Γ_i. Under this assumption, it can be seen that the density of the distribution of states is continuous at the vertices of the network. In the present work, the above mentioned assumption is not made any longer. Therefore, it will be seen below that the value function satisfies more general Kirchhoff conditions, and accordingly, that the density of the distribution of states is no longer continuous at the vertices; the continuity condition is then replaced by suitable compatibility conditions on the jumps across the vertices. A complete study of the system of differential equations arising in infinite horizon mean field games on networks with at most quadratic Hamiltonians and very general coupling costs has been supplied in a previous work, see [7].

In the present work, we focus on the more basic case, namely finite horizon MFG with globally Lipschitz Hamiltonian with rather strong assumptions on the coupling cost. This will allow us to concentrate on the difficulties induced by the Kirchhoff conditions. Therefore, this work should be seen as a first and necessary step in order to deal with more difficult situations, for example with quadratic or subquadratic Hamiltonians. We believe that treating such cases will possible by combining the results contained in the present work with methods that can be found in [78, 82].

[0]this chapter is a work in preparation: Yves Achdou, Manh-Khang Dao, Olivier Ley and Nicoletta Tchou, *A Class of Finite Horizon Mean Field Games on Networks*.

After obtaining the transmission conditions at the vertices for both the value function and the density, we shall prove existence and uniqueness of weak solutions of the uncoupled HJB and FP equations (in suitable space-time Sobolev spaces), and regularity results.

The present work is organized as follows: the remainder of Section 4.1 is devoted to setting the problem and obtaining the system of partial differential equations and the transmission conditions at the vertices. Section 4.2 contains useful results on a modified heat equation in the network with general Kirchhoff conditions. Section 4.3 is devoted to the Fokker-Planck equation. Weak solutions are defined by using a special pair of Sobolev spaces of functions defined on the network referred to as V and W below. Section 4.4 is devoted to the HJB equation supplemented with the Kirchhoff conditions: it addresses the main difficulty of the work, consisting of obtaining regularity results for the weak solution (note that, to the best of our knowledge, such results for networks and general Kirchhoff conditions do not exist in the literature). Finally, the proofs of the main results of existence and uniqueness for the MFG system of partial differential equations are completed in Section 4.5.

4.1.1 Networks and function spaces

The geometry

A bounded network Γ (or a bounded connected graph) is a connected subset of $\mathbb{R}^n$ made of a finite number of bounded non-intersecting straight segments, referred to as edges, which connect nodes referred to as vertices. The finite collection of vertices and the finite set of closed edges are respectively denoted by $\mathcal{V} := \{\nu_i, i \in I\}$ and $\mathcal{E} := \{\Gamma_\alpha, \alpha \in \mathcal{A}\}$, where I and $\mathcal{A}$ are finite sets of indices contained in $\mathbb{N}$. We assume that for $\alpha, \beta \in \mathcal{A}$, if $\alpha \neq \beta$, then $\Gamma_\alpha \cap \Gamma_\beta$ is either empty or made of a single vertex. The length of Γ_α is denoted by ℓ_α. Given $\nu_i \in \mathcal{V}$, the set of indices of edges that are adjacent to the vertex ν_i is denoted by $\mathcal{A}_i = \{\alpha \in \mathcal{A} : \nu_i \in \Gamma_\alpha\}$. A vertex ν_i is named a *boundary vertex* if $\sharp(\mathcal{A}_i) = 1$, otherwise it is named a *transition vertex*. The set containing all the boundary vertices is named the *boundary* of the network and is denoted by $\partial\Gamma$ hereafter.

The edges $\Gamma_\alpha \in \mathcal{E}$ are oriented in an arbitrary manner. In most of what follows, we shall make the following arbitrary choice that an edge $\Gamma_\alpha \in \mathcal{E}$ connecting two vertices ν_i and ν_j, with $i < j$ is oriented from ν_i toward ν_j: this induces a natural parametrization $\pi_\alpha : [0, \ell_\alpha] \to \Gamma_\alpha = [\nu_i, \nu_j]$:

$$\pi_\alpha(y) = \frac{\ell_\alpha - y}{\ell_\alpha}\nu_i + \frac{y}{\ell_\alpha}\nu_j \quad \text{for } y \in [0, \ell_\alpha]. \tag{4.1.1}$$

For a function $v : \Gamma \to \mathbb{R}$ and $\alpha \in \mathcal{A}$, we define $v_\alpha : (0, \ell_\alpha) \to \mathbb{R}$ by

$$v_\alpha(x) := v|_{\Gamma_\alpha} \circ \pi_\alpha(x), \qquad \text{for all } x \in (0, \ell_\alpha).$$

Remark 4.1.1. In what precedes, the edges have been arbitrarily oriented from the vertex with the smaller index toward the vertex with the larger one. Other choices are of course possible. In particular, by possibly dividing a single edge into two, adding thereby new artificial vertices, it is always possible to assume that for all vertices $\nu_i \in \mathcal{V}$,

$$\text{either } \pi_\alpha(0) = \nu_i, \text{ for all } \alpha \in \mathcal{A}_i \text{ or } \pi_\alpha(\ell_\alpha) = \nu_i, \text{ for all } \alpha \in \mathcal{A}_i. \tag{4.1.2}$$

This idea was used by Von Below in [99]: some edges of Γ are cut into two by adding artificial vertices so that the new oriented network $\overline{\Gamma}$ has the property (4.1.2), see Figure 4.1 for an example.

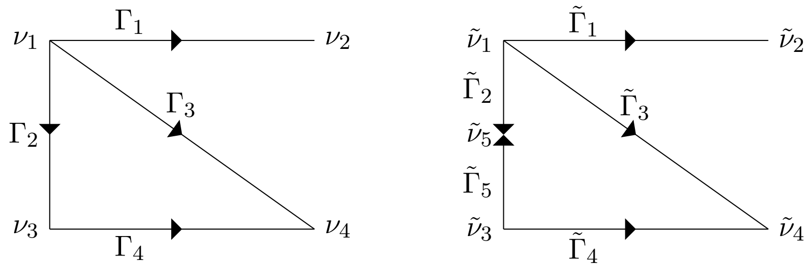

Figure 4.1: Left: the network Γ in which the edges are oriented toward the vertex with larger index (4 vertices and 4 edges). Right: a new network $\tilde\Gamma$ obtained by adding an artificial vertex (5 vertices and 5 edges): the oriented edges sharing a given vertex ν either have all their starting point equal ν, or have all their terminal point equal ν.

Function spaces related to the space variable

The set of continuous functions on Γ is denoted by $C(\Gamma)$ and we set

$$PC(\Gamma) := \left\{ v : \Gamma \to \mathbb{R} \ : \ \text{for all } \alpha \in \mathcal{A}, \ \middle| \ \begin{array}{l} v_\alpha \in C(0,\ell_\alpha) \\ v_\alpha \text{ can be extended by continuity to } [0,\ell_\alpha]. \end{array} \right\}.$$

By the definition of piecewise continuous functions $v \in PC(\Gamma)$, for all $\alpha \in \mathcal{A}$, it is possible to extend $v|_{\Gamma_\alpha}$ by continuity at the endpoints of Γ_α: if $\Gamma_\alpha = [\nu_i, \nu_j]$, we set

$$v|_{\Gamma_\alpha}(x) = \begin{cases} v_\alpha\left(\pi_\alpha^{-1}(x)\right), & \text{if } x \in \Gamma_\alpha \backslash \mathcal{V}, \\ v_\alpha(0) := \lim\limits_{y\to 0^+} v_\alpha(y), & \text{if } x = \nu_i, \\ v_\alpha(\ell_\alpha) := \lim\limits_{y\to \ell_\alpha^-} v_\alpha(y), & \text{if } x = \nu_j. \end{cases} \tag{4.1.3}$$

For $m \in \mathbb{N}$, the space of m-times continuously differentiable functions on Γ is defined by

$$C^m(\Gamma) := \{ v \in C(\Gamma) : v_\alpha \in C^m([0,\ell_\alpha]) \text{ for all } \alpha \in \mathcal{A} \},$$

and is endowed with the norm $\|v\|_{C^m(\Gamma)} := \max_{\alpha\in\mathcal{A}} \max_{0\leqslant k\leqslant m} \|\partial^k v_\alpha\|_{L^\infty(0,\ell_\alpha)}$. For $\sigma \in (0,1)$, the space $C^{m,\sigma}(\Gamma)$, contains the functions $v \in C^m(\Gamma)$ such that $\partial^m v_\alpha \in C^{0,\sigma}([0,\ell_\alpha])$ for all $\alpha \in \mathcal{A}$; it is endowed with the norm

$$\|v\|_{C^{m,\sigma}(\Gamma)} := \|v\|_{C^m(\Gamma)} + \sup_{\alpha\in\mathcal{A}} \sup_{\substack{y\neq z \\ y,z\in[0,\ell_\alpha]}} \frac{|\partial^m v_\alpha(y) - \partial^m v_\alpha(z)|}{|y-z|^\sigma}.$$

For a positive integer m and a function $v \in C^m(\Gamma)$, we set for $k \leqslant m$,

$$\partial^k v(x) = \partial^k v_\alpha\left(\pi_\alpha^{-1}(x)\right) \text{ if } x \in \Gamma_\alpha \backslash \mathcal{V}. \tag{4.1.4}$$

Notice that $v \in C^k(\Gamma)$ is continuous on Γ but that the derivatives $\partial^l v$, $0 < l \leqslant k$ are not defined at the vertices. For a vertex ν, we define $\partial_\alpha v(\nu)$ as the *outward* directional derivative of $v|_{\Gamma_\alpha}$ at ν as follows:

$$\partial_\alpha v(\nu) := \begin{cases} \lim\limits_{h\to 0^+} \dfrac{v_\alpha(0) - v_\alpha(h)}{h}, & \text{if } \nu = \pi_\alpha(0), \\ \lim\limits_{h\to 0^+} \dfrac{v_\alpha(\ell_\alpha) - v_\alpha(\ell_\alpha - h)}{h}, & \text{if } \nu = \pi_\alpha(\ell_\alpha). \end{cases} \tag{4.1.5}$$

For all $i \in I$ and $\alpha \in \mathcal{A}_i$, setting

$$n_{i\alpha} = \begin{cases} 1 & \text{if } \nu_i = \pi_\alpha(\ell_\alpha), \\ -1 & \text{if } \nu_i = \pi_\alpha(0), \end{cases} \tag{4.1.6}$$

we have

$$\partial_\alpha v(\nu_i) = n_{i\alpha}\, \partial v|_{\Gamma_\alpha}(\nu_i) = n_{i\alpha}\, \partial v_\alpha(\pi_\alpha^{-1}(\nu_i)). \tag{4.1.7}$$

Remark 4.1.2. Changing the orientation of the edge does not change the value of $\partial_\alpha v(\nu)$ in (4.1.5).

We say that v is Lebesgue-integrable on Γ_α if v_α is Lebesgue-integrable on $(0, \ell_\alpha)$. In this case, for all $x_1, x_2 \in \Gamma_\alpha$,

$$\int_{[x_1,x_2]} v\,(x)\, dx := \int_{\pi_\alpha^{-1}(x_1)}^{\pi_\alpha^{-1}(x_2)} v_\alpha\,(y)\, dy. \tag{4.1.8}$$

When v is Lebesgue-integrable on Γ_α for all $\alpha \in \mathcal{A}$, we say that v is Lebesgue-integrable on Γ and we define

$$\int_\Gamma v\,(x)\, dx := \sum_{\alpha \in \mathcal{A}} \int_0^{\ell_\alpha} v_\alpha\,(y)\, dy.$$

The space $L^p(\Gamma) = \{v : v|_{\Gamma_\alpha} \in L^p(\Gamma_\alpha) \text{ for all } \alpha \in \mathcal{A}\}$, $p \in [1, \infty]$, is endowed with the norm $\|v\|_{L^p(\Gamma)} := \left(\sum_{\alpha \in \mathcal{A}} \|v_\alpha\|^p_{L^p(0,\ell_\alpha)}\right)^{\frac{1}{p}}$ if $1 \leqslant p < \infty$, and $\max_{\alpha \in \mathcal{A}} \|v_\alpha\|_{L^\infty(0,\ell_\alpha)}$ if $p = +\infty$. We shall also need to deal with functions on Γ whose restrictions to the edges are weakly-differentiable: we shall use the same notations for the weak derivatives.

Definition 4.1.3. For any integer $s \geqslant 1$ and any real number $p \geqslant 1$, the Sobolev space $W_b^{s,p}(\Gamma)$ is defined as follows

$$W_b^{s,p}(\Gamma) := \left\{v : \Gamma \to \mathbb{R} \text{ s.t. } v_\alpha \in W^{s,p}(0, \ell_\alpha) \text{ for all } \alpha \in \mathcal{A}\right\},$$

and endowed with the norm

$$\|v\|_{W_b^{s,p}(\Gamma)} = \left(\sum_{k=1}^{s} \sum_{\alpha \in \mathcal{A}} \left\|\partial^k v_\alpha\right\|^p_{L^p(0,\ell_\alpha)} + \|v\|^p_{L^p(\Gamma)}\right)^{\frac{1}{p}}.$$

For $s \in \mathbb{N}\backslash\{0\}$, we also set $H_b^s(\Gamma) = W_b^{s,2}(\Gamma)$ and $H^s(\Gamma) = C(\Gamma) \cap H_b^s(\Gamma)$.

Finally, when dealing with probability distributions in mean field games, we will often use the set $\mathcal{M}$ of probability densities, i.e., $m \in L^1(\Gamma)$, $m \geqslant 0$ and $\int_\Gamma m(x)dx = 1$.

Some space-time function spaces

The space of continuous real valued functions on $\Gamma \times [0, T]$ is denoted by $C(\Gamma \times [0, T])$.
Let $PC(\Gamma \times [0, T])$ be the space of the functions $v : \Gamma \times [0, T] \to \mathbb{R}$ such that

1. for all $t \in [0, T]$, $v(\cdot, t)$ belongs to $PC(\Gamma)$
2. for all $\alpha \in \mathcal{A}$, $v|_{\Gamma_\alpha \times [0,T]}$ is continuous on $\Gamma_\alpha \times [0, T]$;

For a function $v \in PC(\Gamma \times [0,T])$, $\alpha \in \mathcal{A}$, we set $v_\alpha(x,t) = v|_{\Gamma_\alpha \times [0,t]}(\pi_\alpha(x),t)$ for all $(x,t) \in [0,\ell_\alpha] \times [0,T]$.

For two nonnegative integers m and n, let $C^{m,n}(\Gamma \times [0,T])$ be the space of continuous real valued functions v on $\Gamma \times [0,T]$ such that for all $\alpha \in \mathcal{A}$, $v|_{\Gamma_\alpha \times [0,T]} \in C^{m,n}(\Gamma_\alpha \times [0,T])$. For $\sigma \in (0,1)$, $\tau \in (0,1)$, we define in the same manner $C^{m+\sigma,n+\tau}(\Gamma \times [0,T])$

Useful results on continuous and compact embeddings of space-time function spaces are given in Appendix 4.6.

4.1.2 A class of stochastic processes on Γ

After rescaling the edges, it may be assumed that $\ell_\alpha = 1$ for all $\alpha \in \mathcal{A}$. Let $\mu_\alpha, \alpha \in \mathcal{A}$ and $p_{i\alpha}, i \in I, \alpha \in \mathcal{A}_i$ be positive constants such that $\sum_{\alpha \in \mathcal{A}_i} p_{i\alpha} = 1$. Consider also a real valued function $a \in PC(\Gamma \times [0,T])$, such that, for all $\alpha \in \mathcal{A}$, $t \in [0,T]$, $a|_{\Gamma_\alpha}(\cdot,t)$ belongs to $C^1(\Gamma_\alpha)$.

As in Remark 4.1.1, we make the assumption (4.1.2) by possibly adding artificial nodes: if ν_i is such an artificial node, then $\sharp(\mathcal{A}_i) = 2$, and we assume that $p_{i\alpha} = 1/2$ for $\alpha \in \mathcal{A}_i$. The diffusion parameter μ has the same value on the two sides of an artificial vertex. Similarly, the function a does not have jumps across an artificial vertex.

Consider a Brownian motion (W_t) defined on the real line. Following Freidlin & Sheu ([55]), we know that there exists a unique Markov process on Γ with continuous sample paths that can be written (X_t, α_t) where $X_t \in \Gamma_{\alpha_t}$ (if $X_t = \nu_i$, $i \in I$, α_t is arbitrarily chosen as the smallest index in $\mathcal{A}_i$) such that, defining the process $x_t = \pi_{\alpha_t}(X_t)$ with values in $[0,1]$,

- $$dx_t = \sqrt{2\mu_{\alpha_t}} dW_t + a_{\alpha_t}(x_t,t)dt + d\ell_{i,t} + dh_{i,t}, \tag{4.1.9}$$
- $\ell_{i,t}$ is continuous non-decreasing process (measurable with respect to the σ-field generated by (X_t,α_t)) which increases only when $X_t = \nu_i$ and $x_t = 0$,
- $h_{i,t}$ is continuous non-increasing process (measurable with respect to the σ-field generated by (X_t,α_t)) which decreases only when $X_t = \nu_i$ and $x_t = 1$,

and for all function $v \in C^{2,1}(\Gamma \times [0,T])$ such that

$$\sum_{\alpha \in \mathcal{A}_i} p_{i\alpha} \partial_\alpha v\,(\nu_i,t) = 0, \quad \text{for all } i \in I, t \in [0,T], \tag{4.1.10}$$

the process

$$M_t = v(X_t,t) - \int_0^t \Big(\partial_t v\,(X_s,s) + \mu_{\alpha_s} \partial^2 v\,(X_s,s) + a|_{\Gamma_{\alpha_s}}(X_s,s)\, \partial v\,(X_s,s) \Big) ds \tag{4.1.11}$$

is a martingale, i.e.,

$$\mathbb{E}(M_t | X_s) = M_s, \quad \text{for all } 0 \leqslant s < t \leqslant T. \tag{4.1.12}$$

For what follows, it will be convenient to set

$$D := \left\{ u \in C^2(\Gamma) : \sum_{\alpha \in \mathcal{A}_i} p_{i\alpha} \partial_\alpha u\,(\nu_i) = 0, \quad \text{for all } i \in I \right\}. \tag{4.1.13}$$

Remark 4.1.4. Note that in (4.1.10), the condition at boundary vertices boils down to a Neumann condition.

Remark 4.1.5. The assumption that all the edges have unit length is not restrictive, because we can always rescale the constants μ_α and the piecewise continuous function a.

The goal is to derive the boundary value problem satisfied by the law of the stochastic process X_t. Since the derivation here is formal, we assume that the law of the stochastic process X_t is a measure which is absolutely continuous with respect to the Lebesgue measure on Γ and regular enough so that the following computations make sense. Let $m(x,t)$ be its density. We have

$$\mathbb{E}\left[v\left(X_t,t\right)\right] = \int_\Gamma v\left(x,t\right) m\left(x,t\right) dx, \qquad \text{for all } v \in PC(\Gamma \times [0,T]). \tag{4.1.14}$$

Consider $u \in C^{2,1}(\Gamma \times [0,T])$ such that for all $t \in [0,T]$, $u(\cdot,t) \in D$. Then, from (4.1.11)-(4.1.12), we see that

$$\mathbb{E}\left[u\left(X_t,t\right)\right] = \mathbb{E}\left[u\left(X_0,0\right)\right] + \mathbb{E}\left[\int_0^t \Big(\partial_t u\left(X_s,s\right) + \mu_{\alpha_s}\partial^2 u\left(X_s,s\right) + a|_{\Gamma_{\alpha_s}}\left(X_s,s\right)\partial u\left(X_s,s\right)\Big) ds\right]. \tag{4.1.15}$$

Taking the time-derivative of each member of (4.1.15), we obtain

$$\int_\Gamma \partial_t(um)(x,t)dx = \mathbb{E}\Big(\partial_t u\left(X_t,t\right) + \mu_{\alpha_t}\partial^2 u\left(X_t,t\right) + a|_{\Gamma_{\alpha_t}}\left(X_t,t\right)\partial u\left(X_t,t\right)\Big).$$

Using again (4.1.14), we get

$$\int_\Gamma \left(\mu\partial^2 u(x,t) + a(x,t)\partial u(x,t)\right) m(x,t)dx = \int_\Gamma u(x,t)\partial_t m(x,t)dx. \tag{4.1.16}$$

By integration by parts, recalling (4.1.2), we get

$$\begin{aligned} 0 \;=\; & \sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \left(\partial_t m(x,t) - \mu_\alpha\partial^2 m(x,t) + \partial(am)(x,t)\right) u(x,t)dx \\ & - \sum_{i\in I}\sum_{\alpha\in\mathcal{A}_i} \left[n_{i\alpha} a|_{\Gamma_\alpha}(\nu_i,t) m|_{\Gamma_\alpha}(\nu_i,t) - \mu_\alpha\partial_\alpha m(\nu_i,t)\right] u|_{\Gamma_\alpha}(\nu_i,t) \\ & - \sum_{i\in I}\sum_{\alpha\in\mathcal{A}_i} \mu_\alpha m|_{\Gamma_\alpha}(\nu_i,t)\partial_\alpha u(\nu_i,t), \end{aligned} \tag{4.1.17}$$

where $n_{i\alpha}$ is defined in (4.1.6).

We choose first, for every $\alpha \in \mathcal{A}$, a smooth function u which is compactly supported in $(\Gamma_\alpha\backslash\mathcal{V}) \times [0,T]$. Hence $u|_{\Gamma_\beta}(\nu_i,t) = 0$ and $\partial_\beta u(\nu_i,t) = 0$ for all $i \in I, \beta \in \mathcal{A}_i$. Notice that $u(\cdot,t) \in D$. It follows that m satisfies

$$\left(\partial_t m - \mu_\alpha\partial^2 m + \partial\left(ma\right)\right)(x,t) = 0, \quad \text{for } x \in \Gamma_\alpha\backslash\mathcal{V},\ t \in (0,T),\ \alpha \in \mathcal{A}. \tag{4.1.18}$$

For a smooth function $\chi : [0,T] \to \mathbb{R}$ compactly supported in $(0,T)$, we may choose for every $i \in I$, a smooth function u such that $u(\nu_j,t) = \chi(t)\delta_{i,j}$ for all $t \in [0,T]$, $j \in I$ and $\partial_\alpha u(\nu_j,t) = 0$ for all $t \in [0,T]$, $j \in I$ and $\alpha \in \mathcal{A}_j$, we infer a condition for m at the vertices,

$$\sum_{\alpha\in\mathcal{A}_i} n_{i\alpha} a|_{\Gamma_\alpha}(\nu_i,t) m|_{\Gamma_\alpha}(\nu_i,t) - \mu_\alpha\partial_\alpha m(\nu_i,t) = 0 \qquad \text{for all } i \in I,\ t \in (0,T). \tag{4.1.19}$$

This condition is called a transmission condition if ν_i is a transition vertex and reduces to a Robin boundary condition when ν_i is a boundary vertex.

Finally, for a smooth function $\chi : [0,T] \to \mathbb{R}$ compactly supported in $(0,T)$, for every transition vertex $\nu_i \in \mathcal{V}\backslash\partial\Gamma$ and $\alpha, \beta \in \mathcal{A}_i$, we choose u such that

- $u(\cdot,t) \in D$
- $\partial_\alpha u(\nu_i,t) = \chi(t)/p_{i\alpha}$, $\partial_\beta u(\nu_i) = -\chi(t)/p_{i\beta}$, $\partial_\gamma u(\nu_i) = 0$ if $\gamma \in \mathcal{A}_i \backslash \{\alpha,\beta\}$
- The directional derivatives of u at the vertices $\nu \neq \nu_i$ are 0.

Using such a test-function in (4.1.17) yields a jump condition for m,

$$\frac{m|_{\Gamma_\alpha}(\nu_i,t)}{\gamma_{i\alpha}} = \frac{m|_{\Gamma_\beta}(\nu_i,t)}{\gamma_{i\beta}}, \quad \text{for all } \alpha,\beta \in \mathcal{A}_i, \nu_i \in \mathcal{V}, t \in (0,T),$$

in which

$$\gamma_{i\alpha} = \frac{p_{i\alpha}}{\mu_\alpha}, \quad \text{for all } i \in I, \alpha \in \mathcal{A}_i. \tag{4.1.20}$$

Summarizing, we get the following boundary value problem for m (recall that the coefficients $n_{i\alpha}$ are defined in (4.1.6)):

$$\begin{cases} \partial_t m - \mu_\alpha \partial^2 m + \partial(ma) = 0, & (x,t) \in (\Gamma_\alpha \backslash \mathcal{V}) \times (0,T),\ \alpha \in \mathcal{A}, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} \mu_\alpha \partial_\alpha m(\nu_i,t) - n_{i\alpha} a|_{\Gamma_\alpha}(\nu_i) m|_{\Gamma_\alpha}(\nu_i,t) = 0, & t \in (0,T), \nu_i \in \mathcal{V}, \\ \dfrac{m|_{\Gamma_\alpha}(\nu_i,t)}{\gamma_{i\alpha}} = \dfrac{m|_{\Gamma_\beta}(\nu_i,t)}{\gamma_{i\beta}}, & t \in (0,T),\ \alpha,\beta \in \mathcal{A}_i,\ \nu_i \in \mathcal{V}, \\ m(x,0) = m_0(x), & x \in \Gamma. \end{cases} \tag{4.1.21}$$

4.1.3 Formal derivation of the MFG system on Γ

Consider a continuum of indistinguishable agents moving on the network Γ. Under suitable assumptions, the theory of MFGs asserts that the distribution of states is absolutely continuous with respect to Lebesgue measure on Γ. Hereafter, m stands for the density of the distribution of states: $m \geqslant 0$ and $\int_\Gamma m(x,t)dx = 1$ for $t \in [0,T]$.

The state of a representative agent at time t is a time-continuous controlled stochastic process X_t in Γ, as defined in Section 4.1.2, where the control is the drift a_t, supposed to be of the form $a_t = a(X_t,t)$.

For a representative agent, the optimal control problem is of the form

$$v(x,t) = \inf_a \mathbb{E}_{xt}\left[\int_t^T \left(L(X_s,a_s) + \mathscr{V}[m(\cdot,t)](X_s)\right) ds + v_T(X_T)\right], \tag{4.1.22}$$

where $\mathbb{E}_{xt}$ stands for the expectation conditioned by the event $X_t = x$. The functions and operators involved in (4.1.22) will be described below.

Let us assume that there is an optimal feedback law, i.e. a function $a^\star$ defined on $\Gamma \times [0,T]$ which is sufficiently regular in the edges of the network, such that the optimal control at time t is given by $a_t^\star = a^\star(X_t,t)$. Then, almost surely if $X_t \in \Gamma_\alpha \backslash \mathcal{V}$,

$$d\pi_\alpha^{-1}(X_t) = a_\alpha^\star(\pi_\alpha^{-1}(X_t),t)dt + \sqrt{2\mu_\alpha}dW_t.$$

An informal way to describe the behavior of the process at the vertices is as follows: if X_t hits $\nu_i \in \mathcal{V}$, then it enters Γ_α, $\alpha \in \mathcal{A}_i$ with probability $p_{i\alpha} > 0$, ($p_{i\alpha}$ was introduced in Section 4.1.2).

Let us discuss the ingredients in (4.1.22). The running cost depends separately on the control and on the distribution of states. The contribution of the distribution of states involves the coupling cost operator $\mathscr{V}$, which may be either nonlocal and regularizing, i.e., $\mathscr{V}:\mathcal{P}(\Gamma)\to\mathcal{C}^2(\Gamma)$ for example, or local, i.e. $\mathscr{V}[m](x)=F(m(x))$ where $F:\mathbb{R}^+\to\mathbb{R}$ is a continuous function.

The contribution of the control involves the Lagrangian L, i.e., a real valued function defined on $(\cup_{\alpha\in\mathcal{A}}\Gamma_\alpha\backslash\mathcal{V})\times\mathbb{R}$. If $x\in\Gamma_\alpha\backslash\mathcal{V}$ and $a\in\mathbb{R}$, $L(x,a)=L_\alpha(\pi_\alpha^{-1}(x),a)$, where L_α is a continuous real valued function defined on $[0,\ell_\alpha]\times\mathbb{R}$. We assume that $\lim_{|a|\to\infty}\inf_{y\in\Gamma_\alpha}\frac{L_\alpha(y,a)}{|a|}=+\infty$. The last one is the terminal cost v_T. Further assumptions on L, $\mathscr{V}$ and v_T will be made below.

Under suitable assumptions, Ito calculus as in [56, 55] and the dynamic programming principle lead to the following HJB equation on Γ, more precisely the following boundary value problem:

$$\begin{cases}-\partial_t v-\mu_\alpha\partial^2 v+H(x,\partial v)=\mathscr{V}[m(\cdot,t)](x), & \text{in } (\Gamma_\alpha\backslash\mathcal{V})\times(0,T),\alpha\in A,\\ v|_{\Gamma_\alpha}(\nu_i,t)=v|_{\Gamma_\beta}(\nu_i,t) & \text{for all } \nu_i\in\mathcal{V},t\in(0,T)\,\alpha,\beta\in\mathcal{A}_i,\\ \displaystyle\sum_{\alpha\in\mathcal{A}_i}\gamma_{i\alpha}\mu_\alpha\partial_\alpha v(\nu_i,t)=0, & \text{if } (\nu_i,t)\in\mathcal{V}\times(0,T),\\ v(x,T)=v_T(x).\end{cases}\tag{4.1.23}$$

We refer to [79, 81] for the interpretation of the value function v. Let us comment the different equations in (4.1.23):

1. The Hamiltonian H is a real valued function defined on $(\cup_{\alpha\in\mathcal{A}}\Gamma_\alpha\backslash\mathcal{V})\times\mathbb{R}$. For $x\in\Gamma_\alpha\backslash\mathcal{V}$ and $p\in\mathbb{R}$,
$$H(x,p)=\sup_a\left\{-ap-L_\alpha\left(\pi_\alpha^{-1}(x),a\right)\right\}.$$
The Hamiltonians $H|_{\Gamma_\alpha\times\mathbb{R}}$ are supposed to be C^1 and coercive with respect to p uniformly in x.

2. The second condition means in particular that v is continuous at the vertices.

3. The third equation in (4.1.23) is a Kirchhoff transmission condition (or Neumann boundary condition if $\nu_i\in\partial\Gamma$); it is the consequence of the assumption on the behavior of X_s at vertices. It involves the positive constants $\gamma_{i\alpha}$ defined in (4.1.20).

If (4.1.22) has a smooth solution, it provides a feedback law for the optimal control problem, i.e.,
$$a^\star(x,t)=-\partial_p H(x,\partial v(x,t)).$$
According to Section 4.1.2, the density $m(x,t)$ of the law of the optimal stochastic process X_t satisfies (4.1.21) (where a is replaced by $a^\star$). Finally, replacing $a^\star(x,t)$ by the value $-\partial_p H(x,\partial v(x,t))$, we obtain the system

$$\begin{cases}-\partial_t v-\mu_\alpha\partial^2 v+H(x,\partial v)=\mathscr{V}[m(\cdot,t)](x), & (x,t)\in(\Gamma_\alpha\backslash\mathcal{V})\times(0,T),\alpha\in A,\\ \partial_t m-\mu_\alpha\partial^2 m-\partial(m\partial_p H(x,\partial v))=0, & (x,t)\in(\Gamma_\alpha\backslash\mathcal{V})\times(0,T),\alpha\in A,\\ \displaystyle\sum_{\alpha\in\mathcal{A}_i}\gamma_{i\alpha}\mu_\alpha\partial_\alpha v(\nu_i,t)=0, & (\nu_i,t)\in\mathcal{V}\times(0,T),\\ \displaystyle\sum_{\alpha\in\mathcal{A}_i}\mu_\alpha\partial_\alpha m(\nu_i,t)+n_{i\alpha}\partial_p H^\alpha(\nu_i,\partial v|_{\Gamma_\alpha}(\nu_i,t))\,m|_{\Gamma_\alpha}(\nu_i,t)=0, & (\nu_i,t)\in\mathcal{V}\times(0,T),\\ v|_{\Gamma_\alpha}(\nu_i,t)=v|_{\Gamma_\beta}(\nu_i,t),\ \dfrac{m|_{\Gamma_\alpha}(\nu_i,t)}{\gamma_{i\alpha}}=\dfrac{m|_{\Gamma_\beta}(\nu_i,t)}{\gamma_{i\beta}}, & \alpha,\beta\in\mathcal{A}_i,(\nu_i,t)\in\mathcal{V}\times(0,T),\\ v(x,T)=v_T(x),\ m(x,0)=m_0(x) & x\in\Gamma,\end{cases}\tag{4.1.24}$$

where $H^\alpha := H|_{\Gamma_\alpha\times\mathbb{R}}$. At a vertex ν_i, $i \in I$, the transmission conditions for both v and m consist of $d_{\nu_i} = \sharp(\mathcal{A}_i)$ linear relations, which is the appropriate number of relations to have a well posed problem. If $\nu_i \in \partial\Gamma$, there is of course only one condition.

4.1.4 Assumptions and main results

Before giving the precise definition of solutions of the MFG system (4.1.24) and stating our result, we need to introduce some suitable functions spaces.

Function spaces related to the Kirchhoff conditions

The following function spaces will be the key ingredients in order to build weak solutions of (4.1.24).

Definition 4.1.6. We define two Sobolev spaces: $V := H^1(\Gamma)$, and

$$W := \left\{ w : \Gamma \to \mathbb{R} : \; w \in H_b^1(\Gamma) \; \text{ and } \; \frac{w|_{\Gamma_\alpha}(\nu_i)}{\gamma_{i\alpha}} = \frac{w|_{\Gamma_\beta}(\nu_i)}{\gamma_{i\beta}} \text{ for all } i \in I, \; \alpha, \beta \in \mathcal{A}_i \right\}, \quad (4.1.25)$$

which is a subspace of $H_b^1(\Gamma)$.

Definition 4.1.7. Let the function $\varphi \in W$ be defined as follows:

$$\begin{cases} \varphi_\alpha \text{ is affine on } (0, \ell_\alpha), \\ \varphi|_{\Gamma_\alpha}(\nu_i) = \gamma_{i\alpha}, \text{ if } \alpha \in \mathcal{A}_i, \\ \varphi \text{ is constant on the edges } \Gamma_\alpha \text{ which touch the boundary of } \Gamma. \end{cases} \quad (4.1.26)$$

Note that φ is positive and bounded. We set $\overline{\varphi} = \max_\Gamma \varphi$, $\underline{\varphi} = \min_\Gamma \varphi$.

Remark 4.1.8. One can see that $v \in V \longmapsto v\varphi$ is an isomorphism from V onto W and $w \in W \longmapsto w\varphi^{-1}$ is the inverse isomorphism.

Definition 4.1.9. Let the function space $\mathcal{W} \subset W$ be defined as follows:

$$\mathcal{W} := \left\{ m : \Gamma \to \mathbb{R} : m_\alpha \in C^1([0, \ell_\alpha]) \text{ and } \frac{m|_{\Gamma_\alpha}(\nu_i)}{\gamma_{i\alpha}} = \frac{m|_{\Gamma_\beta}(\nu_i)}{\gamma_{i\beta}} \text{ for all } i \in I, \alpha, \beta \in \mathcal{A}_i \right\}. \quad (4.1.27)$$

Running assumptions

(Diffusion constants) $(\mu_\alpha)_{\alpha\in\mathcal{A}}$ is a family of positive numbers.

(Jump coefficients) $(\gamma_{i\alpha})_{\alpha\in\mathcal{A}_i}$ is a family of positive numbers such that $\sum_{\alpha\in\mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha = 1$.

(Hamiltonian) The Hamiltonian H is defined by the collection $H^\alpha := H|_{\Gamma_\alpha\times\mathbb{R}}$, $\alpha \in \mathcal{A}$: we assume that

$$H^\alpha \in C^1(\Gamma_\alpha \times \mathbb{R}), \quad (4.1.28)$$
$$H^\alpha(x, \cdot) \text{ is convex in } p, \qquad \text{for any } x \in \Gamma_\alpha, \quad (4.1.29)$$
$$H^\alpha(x, p) \leqslant C_0(|p| + 1), \qquad \text{for any } (x, p) \in \Gamma_\alpha \times \mathbb{R}, \quad (4.1.30)$$
$$|\partial_p H^\alpha(x, p)| \leqslant C_0, \qquad \text{for any } (x, p) \in \Gamma_\alpha \times \mathbb{R}, \quad (4.1.31)$$
$$|\partial_x H^\alpha(x, p)| \leqslant C_0(|p| + 1), \qquad \text{for any } (x, p) \in \Gamma_\alpha \times \mathbb{R}, \quad (4.1.32)$$

for a constant C_0 independent of α.

(Coupling operator) We assume that $\mathscr{V}$ is a continuous map from $L^2(\Gamma)$ to $L^2(\Gamma)$, such that for all $m \in L^2(\Gamma)$,

$$\|\mathscr{V}[m]\|_{L^2(\Gamma)} \leqslant C(\|m\|_{L^2(\Gamma)} + 1). \tag{4.1.33}$$

Note that such an assumption is satisfied by local operators of the form $\mathscr{V}[m](x) = F(m(x))$ where F is a Lipschitz continuous function.

(Initial and terminal data) $v_T \in H^1(\Gamma)$ and $m_0 \in L^2(\Gamma) \cap \mathcal{M}$.

The above set of assumptions, referred to as (H), will be the running assumptions hereafter. We will use the following notation: $\underline{\mu} := \min_{\alpha\in\mathcal{A}} \mu_\alpha > 0$ and $\overline{\mu} := \max_{\alpha\in\mathcal{A}} \mu_\alpha$.

We will also say that the coupling $\mathscr{V}$ is strictly increasing if, for any $m_1, m_2 \in \mathcal{M} \cap L^2(\Gamma)$,

$$\int_\Gamma (m_1 - m_2)(\mathscr{V}[m_1] - \mathscr{V}[m_2])dx \geqslant 0$$

and equality implies $m_1 = m_2$.

Stronger assumptions on the coupling operator

We will sometimes need to strengthen the assumptions on the coupling operator, namely that $\mathscr{V}$ has the following smoothing properties:

$\mathscr{V}$ maps the topological dual of W to $H^1_b(\Gamma)$; more precisely, $\mathscr{V}$ defines a Lipschitz map from W' to $H^1_b(\Gamma)$.

Note that such an assumption is not satisfied by local operators.

Definition of solutions and main result

Definition 4.1.10. (solutions of the MFG system) A weak solution of the Mean field games system (4.1.24) is a pair (v, m) such that

$$\begin{aligned} &v \in L^2\left(0,T;H^2(\Gamma)\right) \cap C([0,T];V),\ \partial_t v \in L^2\left(0,T;L^2(\Gamma)\right),\\ &m \in L^2(0,T;W) \cap C([0,T];L^2(\Gamma)\cap\mathcal{M}),\ \partial_t m \in L^2\left(0,T;V'\right), \end{aligned}$$

v satisfies

$$\begin{cases} \displaystyle\sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \left[-\partial_t v(x,t)\,\mathsf{w}(x) + \mu_\alpha \partial v(x,t)\,\partial \mathsf{w}(x) + H(x,\partial v(x,t))\,\mathsf{w}(x)\right] dx \\ \displaystyle\qquad\qquad = \int_\Gamma \mathscr{V}[m(\cdot,t)](x)\mathsf{w}(x)\,dx, \quad \text{for all } \mathsf{w} \in W, \text{ a.e. } t \in (0,T), \\ v(x,T) = v_T(x) \quad \text{for a.e. } x \in \Gamma, \end{cases}$$

and m satisfies

$$\begin{cases} \displaystyle\sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \left[\partial_t m(x,t)\,\mathsf{v}(x)\,dx + \mu_\alpha \partial m(x,t)\,\partial \mathsf{v}(x) + \partial_p H(x,\partial v(x,t))\,m(x,t)\,\partial \mathsf{v}(x)\right] dx \\ \qquad\qquad = 0, \quad \text{for all } \mathsf{v} \in V, \text{ a.e. } t \in (0,T), \\ m(x,0) = m_0(x) \quad \text{for a.e. } x \in \Gamma, \end{cases}$$

where V and W are introduced in Definition 4.1.6.

We are ready to state the main result:

Theorem 4.1.11. *Under assumptions (H),*

(i) (Existence) There exists a weak solution (v, m) *of* (4.1.24).

(ii) (Uniqueness) If $\mathscr{V}$ *is strictly increasing, then the solution is unique.*

(iii) (Regularity) If $\mathscr{V}$ *satisfies furthermore the stronger assumptions made in Section 4.1.4 and if* $v_T \in C^{2+\eta}(\Gamma) \cap D$ *for some* $\eta \in (0,1)$ *(*D *is given in* (4.1.13)*), then* $v \in C^{2,1}(\Gamma \times [0,T])$*. Moreover, if for all* $\alpha \in A$, $\partial_p H^\alpha(x,p)$ *is a Lipschitz function defined in* $\Gamma_\alpha \times \mathbb{R}$*, and if* $m_0 \in W$*, then* $m \in C([0,T]; W) \cap W^{1,2}(0,T; L^2(\Gamma)) \cap L^2(0,T; H_b^2(\Gamma))$.

4.2 Preliminary: a modified heat equation on the network with general Kirchhoff conditions

This section contains results on the solvability of some linear boundary value problems with terminal condition, that will be useful in what follows. Consider

$$\begin{cases} -\partial_t v - \mu_\alpha \partial^2 v = h, & \text{in } (\Gamma_\alpha \backslash \mathcal{V}) \times (0,T), \alpha \in \mathcal{A}, \\ v|_{\Gamma_\alpha}(\nu_i, t) = v|_{\Gamma_\beta}(\nu_i, t), & t \in (0,T), \ \alpha, \beta \in \mathcal{A}_i, \nu_i \in \mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v(\nu_i, t) = 0, & t \in (0,T), \nu_i \in \mathcal{V}, \\ v(x,T) = v_T(x), & x \in \Gamma, \end{cases} \tag{4.2.1}$$

where $h \in L^2(0,T; W')$ and $v_T \in L^2(\Gamma)$.

Definition 4.2.1. If $v_T \in L^2(\Gamma)$ and $h \in L^2(0,T; W')$, a weak solution of (4.2.1) is a function $v \in L^2(0,T; V) \cap C([0,T]; L^2(\Gamma))$ such that $\partial_t v \in L^2(0,T; W')$ and

$$\begin{cases} -\langle \partial_t v(t), w \rangle_{W',W} + \mathscr{B}(v(\cdot,t), w) = \langle h(t), w \rangle_{W',W} & \text{for all } w \in W \text{ and a.e. } t \in (0,T), \\ v(x,T) = v_T(x), \end{cases} \tag{4.2.2}$$

where $\mathscr{B} : V \times W \to \mathbb{R}$ is the bilinear form defined as follows:

$$\mathscr{B}(v,w) := \int_\Gamma \mu \partial v \partial w dx = \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \mu_\alpha \partial v \partial w dx.$$

We use the Galerkin's method (see [49]), i.e., we construct solutions of some finite-dimensional approximations to (4.2.1).

Recall that φ has been defined in Definition 4.1.7. We notice first that the symmetric bilinear form $\widehat{\mathscr{B}}(u,v) := \int_\Gamma \mu \varphi \partial u \partial v$ is such that $(u,v) \mapsto (u,v)_{L^2(\Gamma)} + \widehat{\mathscr{B}}(u,v)$ is an inner product in V equivalent to the standard inner product in V, namely $(u,v)_V = (u,v)_{L^2(\Gamma)} + \int_\Gamma \partial u \partial v$. Therefore, by standard Fredholm's theory, there exist

- a non decreasing sequence of nonnegative real numbers $(\lambda_k)_{k=1}^\infty$, that tends to $+\infty$ as $k \to \infty$,
- a Hilbert basis $(\mathsf{v}_k)_{k=1}^\infty$ of $L^2(\Gamma)$, which is also a a total sequence of V (and orthogonal if V is endowed with the scalar product $(u,v)_{L^2(\Gamma)} + \widehat{\mathscr{B}}(u,v)$),

such that

$$\widehat{\mathscr{B}}(\mathsf{v}_k, v) = \lambda_k(\mathsf{v}_k, v)_{L^2(\Gamma)} \quad \text{for all } v \in V. \tag{4.2.3}$$

Note that

$$\int_\Gamma \mu \partial \mathsf{v}_k \partial \mathsf{v}_\ell \varphi dx = \begin{cases} \lambda_k & \text{if } k = \ell, \\ 0 & \text{if } k \neq \ell. \end{cases}$$

Note also that v_k is a weak solution of

$$\begin{cases} -\mu_\alpha \partial \left(\varphi \partial \mathsf{v}_k\right) = \lambda_k \mathsf{v}_k, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ \mathsf{v}_k|_{\Gamma_\alpha} \left(\nu_i\right) = \mathsf{v}_k|_{\Gamma_\beta} \left(\nu_i\right), & \alpha, \beta \in \mathcal{A}_i, \\ \sum\limits_{\alpha \in \mathcal{A}} \gamma_{i\alpha} \mu_\alpha \partial_\alpha \mathsf{v}_k \left(\nu_i\right) = 0, & \nu_i \in \mathcal{V}, \end{cases} \tag{4.2.4}$$

which implies that $\mathsf{v}_k \in C^2(\Gamma)$.

Finally, by Remark 4.1.8, the sequence $(\varphi \mathsf{v}_k)_{k=1}^\infty$ is a total family in W (but is not orthogonal if W is endowed with the standard inner product).

Lemma 4.2.2. *For any positive integer n, there exist n absolutely continuous functions $y_k^n : [0, T] \to \mathbb{R}$, $k = 1, \dots, n$, and a function $v_n : [0, T] \to L^2(\Gamma)$ of the form*

$$v_n\left(x, t\right) = \sum_{k=1}^n y_k^n\left(t\right) \mathsf{v}_k(x), \tag{4.2.5}$$

such that

$$y_k^n\left(T\right) = \int_\Gamma v_T \mathsf{v}_k dx, \quad \textit{for } k = 1, \dots, n, \tag{4.2.6}$$

and

$$-\frac{d}{dt}(v_n, \mathsf{v}_k\varphi)_{L^2(\Gamma)} + \mathscr{B}\left(v_n, \mathsf{v}_k\varphi\right) = \langle h(t), \mathsf{v}_k\varphi\rangle, \quad \textit{for a.a. } t \in \left(0, T\right), \ \textit{for all } k = 1, \dots, n. \tag{4.2.7}$$

Proof of Lemma 4.2.2. For $n \geqslant 1$, we consider the symmetric n by n matrix M_n defined by

$$\left(M_n\right)_{k\ell} = \int_\Gamma \mathsf{v}_k \mathsf{v}_\ell \varphi dx.$$

Since φ is positive and $(\mathsf{v}_k)_{k=1}^\infty$ is a Hilbert basis of $L^2\left(\Gamma\right)$, we can check that M_n is a positive definite matrix and there exist two constants c, C independent of n such that

$$c\left|\xi\right|^2 \leqslant \sum_{k,\ell=1}^n \left(M_n\right)_{k\ell} \xi_k \xi_\ell \leqslant C\left|\xi\right|^2, \quad \text{for all } \xi \in \mathbb{R}^n. \tag{4.2.8}$$

Looking for v_n of the form (4.2.5), and setting $Y = (y_1^n, \dots, y_n^n)^T$, $\dot{Y} = \left(\frac{d}{dt} y_1^n, \dots, \frac{d}{dt} y_n^n\right)^T$, (4.2.7) implies that we have to solve the following system of differential equations

$$-M_n \dot{Y} + BY = F_n, \quad Y(T) = \left(\int_\Gamma v_T \mathsf{v}_1, \dots, \int_\Gamma v_T \mathsf{v}_n\right)^T,$$

where $B_{k\ell} = \mathscr{B}\left(\mathsf{v}_\ell, \mathsf{v}_k\varphi\right)$ and $F_n(t) = \left(\langle h(t), \mathsf{v}_1\varphi\rangle, \dots, \langle h(t), \mathsf{v}_n\varphi\rangle\right)^T$. Since the matrix M_n is invertible, the ODE system has a unique absolutely continuous solution. The lemma is proved. $\square$

We propose to send n to $+\infty$ and show that a subsequence of $\{v_n\}$ converges to a solution of (4.2.1). Hence, we need some uniform estimates for $\{v_n\}$.

Lemma 4.2.3. *There exists a constant C depending only on Γ, $(\mu_\alpha)_{\alpha\in\mathcal{A}}$, T and φ such that*

$$\|v_n\|_{L^\infty(0,T;L^2(\Gamma))} + \|v_n\|_{L^2(0,T;V)} + \|\partial_t v_n\|_{L^2(0,T;W')} \leqslant C\left(\|h\|_{L^2(0,T;W')} + \|v_T\|_{L^2(\Gamma)}\right).$$

Proof of Lemma 4.2.3. Multiplying (4.2.7) by $y_k^n(t)\,e^{\lambda t}$ for a positive constant λ to be chosen later, summing for $k=1,\dots,n$ and using the formula (4.2.5) for v_n, we get

$$-\int_\Gamma \partial_t v_n v_n e^{\lambda t}\varphi dx + \int_\Gamma \mu\partial v_n \partial\left(v_n e^{\lambda t}\varphi\right)dx = e^{\lambda t}\langle h(t), v_n\varphi\rangle_{W',W},$$

and

$$-\int_\Gamma\left[\partial_t\left(\frac{v_n^2}{2}e^{\lambda t}\right) - \frac{\lambda}{2}v_n^2 e^{\lambda t}\right]\varphi dx + \int_\Gamma \mu\left(\partial v_n\right)^2 e^{\lambda t}\varphi dx + \int_\Gamma \mu\partial v_n v_n e^{\lambda t}\partial\varphi dx = e^{\lambda t}\langle h(t), v_n\varphi\rangle.$$

Integrating both sides from s to T, we obtain

$$\begin{aligned}
&\int_\Gamma\left(\frac{v_n^2(x,s)}{2}e^{\lambda s} - \frac{v_n^2(x,T)}{2}e^{\lambda T}\right)\varphi dx + \frac{\lambda}{2}\int_s^T\int_\Gamma v_n^2 e^{\lambda t}\varphi dxdt\\
&+\int_s^T\int_\Gamma \mu(\partial v_n)^2 e^{\lambda t}\varphi dxdt + \int_s^T\int_\Gamma \mu\partial v_n v_n e^{\lambda t}\partial\varphi dxdt\\
=\ &\int_s^T e^{\lambda t}\langle h(t), v_n(t)\varphi\rangle dt\\
\leqslant\ & C\int_s^T e^{\lambda t}\|h(t)\|_{W'}\|v_n(t)\|_V dt\\
\leqslant\ & \frac{1}{2}\int_s^T\int_\Gamma \mu\left((\partial v_n)^2 + v_n^2\right)e^{\lambda t}\varphi dxdt + \frac{C^2}{2\underline{\mu}}\int_s^T e^{\lambda t}\|h(t)\|_{W'}^2 dt,
\end{aligned}$$

where C is positive constant depending on φ, because of Remark 4.1.8. Therefore,

$$\begin{aligned}
&e^{\lambda s}\int_\Gamma \frac{v_n^2(x,s)}{2}\varphi dx + \frac{1}{4}\int_s^T\int_\Gamma \mu(\partial v_n)^2\varphi e^{\lambda t}dxdt + \left(\frac{\lambda}{2} - \frac{\overline{\mu}}{2} - \overline{\mu}\frac{\|\partial\varphi\|_{L^\infty(\Gamma)}^2}{\underline{\varphi}^2}\right)\int_s^T\int_\Gamma v_n^2 e^{\lambda t}\varphi dxdt\\
\leqslant\ & e^{\lambda T}\int_\Gamma \frac{v_n^2(x,T)}{2}\varphi dx + \frac{C^2}{2\underline{\mu}}e^{\lambda T}\int_s^T \|h(t)\|_{W'}^2 dt.
\end{aligned}$$

Choosing $\lambda \geqslant 1/2 + \overline{\mu} + 2\overline{\mu}||\partial\varphi||_{L^\infty(\Gamma)}^2/\underline{\varphi}^2$ and noticing that $\int_\Gamma v_n^2(x,T)\varphi dx$ is bounded by $\overline{\varphi}\int_\Gamma v_T^2 dx$ from (4.2.6), it follows that

$$\begin{aligned}
&\int_\Gamma v_n^2(x,s)\varphi dx + \int_s^T\int_\Gamma v_n^2\varphi dxdt + \int_s^T\int_\Gamma \mu(\partial v_n)^2\varphi dxdt\\
\leqslant\ & 2e^{\lambda T}\left(\frac{C^2}{\underline{\mu}}\|h\|_{L^2(0,T;W')}^2 + \overline{\varphi}\int_\Gamma v_T^2 dx\right).
\end{aligned}\tag{4.2.9}$$

Estimate of v_n in $L^\infty\left(0,T;L^2(\Gamma)\right)$ and $L^2(0,T;V)$. From (4.2.9), it is straightforward to see that

$$\|v_n\|_{L^\infty(0,T;L^2(\Gamma))} + \|v_n\|_{L^2(0,T;V)} \leqslant C\left(\|h\|_{L^2(0,T;W')} + \|v_T\|_{L^2(\Gamma)}\right),\tag{4.2.10}$$

for some constant C depending only on $(\mu_\alpha)_{\alpha\in\mathcal{A}}$, φ and T.

Estimate $\partial_t v_n$ in $L^2(0,T;W')$. Consider the closed subspace G_1 of W defined by $G_1 = \left\{w \in W : \int_\Gamma \mathsf{v}_k w dx = 0 \text{ for all } k \leqslant n\right\}$. It has a finite co-dimension equal to n. Consider also the n-dimensional subspace $G_2 = \operatorname{span}\{\mathsf{v}_1\varphi, \dots, \mathsf{v}_n\varphi\}$ of W. The invertibility of the matrix M_n introduced in the proof of Lemma 4.2.2 implies that $G_1 \cap G_2 = \{0\}$. This implies that $W = G_1 \oplus G_2$. For $w \in W$, we can write w of the form $w = w_n + \hat{w}_n$, where $w_n \in G_2$ and $\hat{w}_n \in G_1$. Hence, for a.e. $t \in [0,T]$, from (4.2.5) and (4.2.7), one gets

$$\langle \partial_t v_n(t), w\rangle_{W',W} = \frac{d}{dt}\left(\int_\Gamma v_n w dx\right) = \frac{d}{dt}\left(\int_\Gamma v_n w_n dx\right) = -\langle h(t), w_n\rangle_{W',W} + \int_\Gamma \mu \partial v_n \partial w_n dx. \tag{4.2.11}$$

Since there exists a constant C independent of n such that $\|w_n\|_W \leqslant C\|w\|_W$, it follows that

$$\|\partial_t v_n(t)\|_{W'} \leqslant C\left(\|h(t)\|_{W'} + \overline{\mu}\,\|v_n(t)\|_V\right),$$

for almost every t, and therefore, from (4.2.10), we obtain

$$\|\partial_t v_n(t)\|^2_{L^2(0,T;W')} \leqslant C\left(\|h\|^2_{L^2(0,T;W')} + \|v_T\|^2_{L^2(\Gamma)}\right),$$

for a constant C independent of n. □

Theorem 4.2.4. *There exists a unique solution v of (4.2.1), which satisfies*

$$\|v\|_{L^\infty(0,T;L^2(\Gamma))} + \|v\|_{L^2(0,T;V)} + \|\partial_t v\|_{L^2(0,T;W')} \leqslant C\left(\|h\|_{L^2(0,T;W')} + \|v_T\|_{L^2(\Gamma)}\right), \tag{4.2.12}$$

where C is a constant that depends only on Γ, $(\mu_\alpha)_{\alpha\in\mathcal{A}}$, T and φ.

Proof of Theorem 4.2.4. From Lemma 4.2.3, the sequence $(v_n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;V)$ and the sequence $(\partial_t v_n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;W')$. Hence, up to the extraction of a subsequence, there exists a function v such that $v \in L^2(0,T;V)$, $\partial_t v \in L^2(0,T;W')$ and

$$\begin{cases} v_n \rightharpoonup v & \text{weakly in } L^2(0,T;V), \\ \partial_t v_n \rightharpoonup \partial_t v & \text{weakly in } L^2(0,T;W'). \end{cases} \tag{4.2.13}$$

Fix an integer N and choose a function $\overline{v} \in C^1([0,T];V)$ having the form

$$\overline{v}(t) = \sum_{k=1}^N d_k(t)\,\mathsf{v}_k, \tag{4.2.14}$$

where $d_1, \dots, d_N$ are given real valued C^1 functions defined in $[0,T]$. For all $n \geqslant N$, multiplying (4.2.7) by $d_k(t)$, summing for $k = 1, \dots, n$ and integrating over $(0,T)$ leads to

$$-\int_0^T\int_\Gamma \partial_t v_n \overline{v}\varphi dx dt + \int_0^T\int_\Gamma \mu\partial v_n \partial(\overline{v}\varphi)\,dx dt = \int_0^T \langle h, \overline{v}\varphi\rangle dt. \tag{4.2.15}$$

Letting $n \to +\infty$, we obtain from (4.2.13) that

$$-\int_0^T \langle \partial_t v, \overline{v}\varphi\rangle dt + \int_0^T\int_\Gamma \mu\partial v\partial(\overline{v}\varphi)\,dx dt = \int_0^T \langle h, \overline{v}\varphi\rangle dt. \tag{4.2.16}$$

Since the functions of the form (4.2.14) are dense in $L^2(0,T;V)$, (4.2.16) holds for all test function $\overline{v} \in L^2(0,T;V)$. Recalling the isomorphism $\overline{v} \in V \mapsto \overline{v}\varphi \in W$ (see Remark 4.1.8), we obtain that, for all $w \in W$ and $\psi \in C_c^1(0,T)$,

$$-\int_0^T \langle \partial_t v, w\rangle \psi dt + \int_0^T \int_\Gamma \mu \partial v \partial w \psi dx dt = \int_0^T \langle h, w\rangle \psi dt.$$

This implies that, for a.e. $t \in (0,T)$,

$$-\langle \partial_t v, w\rangle + \mathscr{B}(v,w) = \langle h, w\rangle \quad \text{for all } w \in W.$$

Using Theorem 3.1 in [84] (or the same argument as in [49, pages 287-288]), we see that $v \in C([0,T]; L^2_\varphi(\Gamma))$, where $L^2_\varphi(\Gamma) = \{w : \Gamma \to \mathbb{R} \ : \ \int_\Gamma w^2 \varphi dx < +\infty\}$, and since φ is bounded from below and from above by positive numbers, $L^2_\varphi(\Gamma) = L^2(\Gamma)$ with equivalent norms. Moreover,

$$\max_{0 \leqslant t \leqslant T} \|v(\cdot,t)\|_{L^2(\Gamma)} \leqslant C\left(\|\partial_t v\|_{L^2(0,T;W')} + \|v\|_{L^2(0,T;V)}\right).$$

We are now going to prove $v(T) = v_T$. For all $\overline{v} \in C^1([0,T];V)$ of the form (4.2.14) and such that $\overline{v}(0) = 0$, we deduce from (4.2.15) and (4.2.16) that

$$\begin{aligned} &-\int_0^T \int_\Gamma \partial_t \overline{v} v_n \varphi dx dt - \int_\Gamma \overline{v}(T) v_n(T) \varphi dx + \int_0^T \int_\Gamma \mu \partial v_n \partial(\overline{v}\varphi) dx dt \\ = &-\int_0^T \int_\Gamma \partial_t \overline{v} v \varphi dx dt - \int_\Gamma \overline{v}(T) v(T) \varphi dx + \int_0^T \int_\Gamma \mu \partial v \partial(\overline{v}\varphi) dx dt. \end{aligned}$$

We know that $v_n(T) \to v_T$ in $L^2(\Gamma)$. Then, using (4.2.13), we obtain

$$\int_\Gamma \overline{v}(T) v_T \varphi dx = \int_\Gamma \overline{v}(T) v(T) \varphi dx.$$

Since the functions of the form $\sum_{k=1}^N d_k(T)\mathsf{v}_k$ are dense in $L^2(\Gamma)$, we conclude that $v(T) = v_T$.

In order to prove the energy estimate (4.2.12), we use $ve^{\lambda t}\varphi$ as a test function in (4.2.2) and apply similar arguments as in the proof of Lemma 4.2.3 for λ large enough, we get (4.2.12).

Finally, if $h = 0$ and $v_T = 0$, by the energy estimate for v in (4.2.12), we deduce that $v = 0$. Uniqueness is proved. □

Theorem 4.2.5. *If $v_T \in V$ and $h \in L^2(\Gamma \times (0,T))$, then the unique solution v of (4.2.1) satisfies $v \in L^2\left(0,T;H^2(\Gamma)\right) \cap C([0,T];V)$ and $\partial_t v \in L^2\left(0,T;L^2(\Gamma)\right)$. Moreover,*

$$\|v\|_{L^\infty(0,T;V)} + \|v\|_{L^2(0,T;H^2(\Gamma))} + \|\partial_t v\|_{L^2(0,T;L^2(\Gamma))} \leqslant C\left(\|h\|_{L^2(0,T;L^2(\Gamma))} + \|v_T\|_V\right), \quad (4.2.17)$$

for a positive constant C that depends only on Γ, $(\mu_\alpha)_{\alpha \in \mathcal{A}}$, T and φ.

Proof of Theorem 4.2.5. It is enough to prove estimate (4.2.17) for v_n.

Multiplying (4.2.7) by $-\frac{d}{dt}y_k^n$, summing for $k = 1, \dots, n$ and using (4.2.5) leads to

$$\int_\Gamma (\partial_t v_n)^2 \varphi dx - \int_\Gamma \mu \partial v_n \partial(\partial_t v_n \varphi) dx = -\int_\Gamma h \partial_t v_n \varphi dx,$$

hence

$$\int_\Gamma (\partial_t v_n)^2 \varphi dx - \int_\Gamma \mu \partial_t \frac{(\partial v_n)^2}{2} \varphi dx - \int_\Gamma \mu \partial v_n \partial_t v_n \partial \varphi dx = -\int_\Gamma h \partial_t v_n \varphi dx.$$

Multiplying by $e^{\lambda t}$ where λ will chosen later, and taking the integral from s to T, we obtain

$$\begin{aligned}
& \int_s^T \int_\Gamma (\partial_t v_n)^2 e^{\lambda t} \varphi dx dt - \int_\Gamma \frac{\mu}{2} \left[(\partial v_n(T))^2 e^{\lambda T} - (\partial v_n(s))^2 e^{\lambda s} \right] \varphi dx \\
& + \lambda \int_s^T \int_\Gamma \frac{\mu}{2} (\partial v_n)^2 e^{\lambda t} \varphi dx dt - \int_s^T \int_\Gamma \mu \partial v_n \partial_t v_n e^{\lambda t} \partial \varphi dx dt \\
= & - \int_s^T \int_\Gamma h \partial_t v_n e^{\lambda t} \varphi dx dt \\
\leqslant & \frac{1}{2} \int_s^T \int_\Gamma h^2 e^{\lambda t} \varphi dx dt + \frac{1}{2} \int_s^T \int_\Gamma (\partial_t v_n)^2 e^{\lambda t} \varphi dx dt. \qquad (4.2.18)
\end{aligned}$$

Let us deal with the term $\int_\Gamma (\partial v_n(x,T))^2 \varphi dx$. From (4.2.6),

$$\begin{aligned}
\int_\Gamma \mu (\partial v_n(x,T))^2 \varphi dx &= \sum_{k=1}^n \lambda_k \left(\int_\Gamma v_T \mathsf{v}_k dx \right)^2 \\
&\leqslant \sum_{k=1}^\infty \lambda_k \left(\int_\Gamma v_T \mathsf{v}_k dx \right)^2 = \int_\Gamma \mu (\partial v_T(x))^2 \varphi dx \\
&\leqslant \overline{\mu} \int_\Gamma (\partial v_T(x))^2 \varphi dx.
\end{aligned}$$

Then, choosing $\lambda = 2\overline{\mu}^2 ||\partial\varphi||^2_{L^\infty(\Gamma)} / (\underline{\varphi}^2 \underline{\mu})$, we obtain that

$$\int_\Gamma 2\mu (\partial v_n(x,s))^2 \varphi dx + \int_s^T \int_\Gamma (\partial_t v_n)^2 \varphi dx dt \leqslant 2e^{\lambda T} \overline{\varphi} \left(\|h\|^2_{L^2(\Gamma \times (0,T))} + \overline{\mu} \int_\Gamma (\partial v_T)^2 dx \right). \quad (4.2.19)$$

Estimate of ∂v_n in $L^\infty\left(0,T;L^2(\Gamma)\right)$ and $\partial_t v_n$ in $L^2(\Gamma \times (0,T))$. From (4.2.19), it is straightforward to see that

$$\|\partial v_n\|_{L^\infty(0,T;L^2(\Gamma))} + \|\partial_t v_n\|_{L^2(\Gamma \times (0,T))} \leqslant C \left(\|h\|_{L^2(\Gamma \times (0,T))} + \|\partial v_T\|_{L^2(\Gamma)} \right)$$

for some constant C depending only on Γ, μ, T and φ.

Estimate of $\partial^2 v_n$ in $L^2(\Gamma \times (0,T))$. Finally, using the PDE in (4.2.1), we can see that $\partial^2 v_n$ belongs to $L^2(\Gamma \times (0,T))$ and is bounded by $C\left(\|h\|_{L^2(\Gamma \times (0,T))} + \|v_T\|_V\right)$, hence v_n is bounded in $L^2\left(0,T;H^2(\Gamma)\right)$ by the same quantity. The Kirchhoff conditions (which boil down to Neumann conditions at $\partial\Gamma$) are therefore satisfied in a strong sense for almost all t.
Using Theorem 3.1 in [84] (or a similar argument as [49] pages 287-288), we see that v in $C([0,T];V)$.

□

4.3 The Fokker-Planck equation

This paragraph is devoted to a boundary value problem including a Fokker-Planck equation

$$\begin{cases}
\partial_t m - \mu_\alpha \partial^2 m - \partial(bm) = 0, & \text{in } (\Gamma_\alpha \backslash \mathcal{V}) \times (0,T), \ \alpha \in \mathcal{A}, \\
\dfrac{m|_{\Gamma_\alpha}(\nu_i,t)}{\gamma_{i\alpha}} = \dfrac{m|_{\Gamma_\beta}(\nu_i,t)}{\gamma_{i\beta}}, & t \in (0,T), \ \alpha,\beta \in \mathcal{A}_i, \ \nu_i \in \mathcal{V} \backslash \partial\Gamma, \\
\sum_{\alpha \in \mathcal{A}_i} \mu_\alpha \partial_\alpha m(\nu_i,t) + n_{i\alpha} b(\nu_i,t) m|_{\Gamma_\alpha}(\nu_i,t) = 0, & t \in (0,T), \ \nu_i \in \mathcal{V}, \\
m(x,0) = m_0(x), & x \in \Gamma,
\end{cases} \qquad (4.3.1)$$

where $b \in PC(\Gamma \times [0,T])$ and $m_0 \in L^2(\Gamma)$.

Definition 4.3.1. A weak solution of (4.3.1) is a function $m \in L^2(0,T;W) \cap C([0,T];L^2(\Gamma))$ such that $\partial_t m \in L^2(0,T;V')$ and

$$\begin{cases} \langle \partial_t m, v\rangle_{V',V} + \mathscr{A}(m,v) = 0 & \text{for all } v \in V \text{ and a.e. } t \in (0,T), \\ m(\cdot,0) = m_0, \end{cases} \tag{4.3.2}$$

where $\mathscr{A} : W \times V \to \mathbb{R}$ is the bilinear form

$$\mathscr{A}(v,w) = \int_\Gamma \mu \partial m \partial v dx + \int_\Gamma bm\partial v dx.$$

Using similar arguments as in Section 4.2, in particular a Galerkin method, we obtain the following result, the proof of which is omitted.

Theorem 4.3.2. *If $b \in L^\infty(\Gamma \times (0,T))$ and $m_0 \in L^2(\Gamma)$, there exists a unique function $m \in L^2(0,T;W) \cap C([0,T];L^2(\Gamma))$ such that $\partial_t m \in L^2(0,T;V')$ and (4.3.2). Moreover, there exists a constant C which depends on $(\mu_\alpha)_{\alpha\in\mathcal{A}}$, $\|b\|_\infty$, T and φ, such that*

$$\|m\|_{L^2(0,T;W)} + \|m\|_{L^\infty(0,T;L^2(\Gamma))} + \|\partial_t m\|_{L^2(0,T;V')} \leqslant C\|m_0\|_{L^2(\Gamma)}. \tag{4.3.3}$$

Remark 4.3.3. If $m_0 \in \mathcal{M}$, which will be the case when solving the MFG system (4.1.24), then $m(\cdot,t) \in \mathcal{M}$ for all $t \in [0,T]$. Indeed, we use $v \equiv 1 \in V$ as a test-function for (4.3.1). Since $\partial v = 0$, integrating (4.3.2) from 0 to t, we get $\int_0^t \int_\Gamma \partial_t m(x,s)dxds = 0$. This implies that

$$\int_\Gamma m(x,t)dx = \int_\Gamma m_0(x)dx = 1, \quad \text{for all } t \in (0,T].$$

Setting $m^- = -\mathbb{1}_{\{m<0\}}m$, we can also use $v = \varphi^{-1}m^- e^{-\lambda t}$ as a test-function for $\lambda \in \mathbb{R}_+$. Indeed, the latter function belongs to $L^2(0,T;V)$. Taking λ large enough and using similar arguments as for the energy estimate (4.3.3) yield that $m^- = 0$, i.e., $m \geqslant 0$.

We end this section by stating a stability result, which will be useful in the proof of the main Theorem.

Lemma 4.3.4. *Let $m_{0\varepsilon}, b_\varepsilon$ be sequences of functions satisfying*

$$m_{0\varepsilon} \longrightarrow m_0 \text{ in } L^2(\Gamma), \qquad b_\varepsilon \longrightarrow b \text{ in } L^2(\Gamma \times (0,T)),$$

and for some positive number K independent of ε, $\|b\|_{L^\infty(\Gamma\times(0,T))} \leqslant K$, $\|b_\varepsilon\|_{L^\infty(\Gamma\times(0,T))} \leqslant K$. Let m_ε (respectively m) be the solution of (4.3.2) corresponding to the datum $m_{0\varepsilon}$ (resp. m_0) and the coefficient b_ε (resp. b). The sequence (m_ε) converges to m in $L^2(0,T;W) \cap L^\infty(0,T;L^2(\Gamma))$, and the sequence $(\partial_t m_\varepsilon)$ converges to $(\partial_t m)$ in $L^2(0,T;V')$.

Proof of Lemma 4.3.4. Taking $(m_\varepsilon - m)e^{-\lambda t}\varphi^{-1}$ as a test-function in the versions of (4.3.2) satisfied by m_ε and m, subtracting, we obtain that

$$\begin{aligned} &\int_\Gamma \left[\frac{1}{2}\partial_t\left((m_\varepsilon - m)^2 e^{-\lambda t}\right) + \frac{\lambda}{2}(m_\varepsilon - m)^2 e^{-\lambda t}\right]\varphi^{-1}dx + \int_\Gamma \mu(\partial(m_\varepsilon - m))^2 e^{-\lambda t}\varphi^{-1}dx \\ &+ \int_\Gamma \mu(m_\varepsilon - m)\partial(m_\varepsilon - m)e^{-\lambda t}\partial(\varphi^{-1})dx + \int_\Gamma (b_\varepsilon m_\varepsilon - bm)\partial(m_\varepsilon - m)e^{-\lambda t}\varphi^{-1}dx \\ &+ \int_\Gamma (b_\varepsilon m_\varepsilon - bm)(m_\varepsilon - m)e^{-\lambda t}\partial(\varphi^{-1})dx = 0. \end{aligned} \tag{4.3.4}$$

There exists a positive constant K such that $\|b_\varepsilon\|_\infty, \|b\|_\infty \leqslant K$ for all ε. Hence, there exists a positive constant C (in fact it varies from one line to the other in what follows) such that

$$\begin{aligned}
&\int_\Gamma \left[\frac{1}{2}\partial_t\left((m_\varepsilon - m)^2 e^{-\lambda t}\right) + \frac{\lambda}{2}(m_\varepsilon - m)^2 e^{-\lambda t}\right]\varphi^{-1}dx + \int_\Gamma \mu(\partial(m_\varepsilon - m))^2 e^{-\lambda t}\varphi^{-1}dx \\
\leqslant\ & C\int_\Gamma \left(|m_\varepsilon - m|^2 + |m_\varepsilon - m|\,|\partial(m_\varepsilon - m)| + |m|\,|b_\varepsilon - b|\,(|\partial(m_\varepsilon - m)| + |m_\varepsilon - m|)\right)e^{-\lambda t}\varphi^{-1}dx \\
\leqslant\ & C\int_\Gamma \left(|m_\varepsilon - m|^2 + |b_\varepsilon - b|^2 m^2\right)e^{-\lambda t}\varphi^{-1}dx + \int_\Gamma \frac{\mu}{2}(\partial(m_\varepsilon - m))^2 e^{-\lambda t}\varphi^{-1}dx.
\end{aligned}$$

The assumptions on the coefficents b_ε and b imply in fact that $b_\varepsilon \to b$ in $L^p(\Gamma \times (0,T))$ for all $1 \leqslant p < \infty$. On the other hand, we know that $m \in L^q(\Gamma \times (0,T))$ for all $1 \leqslant q < \infty$. From the latter observation with $p = q = 4$, we see that the quantity $\int_0^T \int_\Gamma \left(|b_\varepsilon - b|^2 m^2\right)e^{-\lambda t}\varphi^{-1}dxdt$ tends to 0 as $\varepsilon \to 0$ uniformly in $\lambda > 0$. We write

$$\int_0^T \int_\Gamma \left(|b_\varepsilon - b|^2 m^2\right)e^{-\lambda t}\varphi^{-1}dxdt = o_\varepsilon(1).$$

Choosing λ large enough and integrating the latter inequality from 0 to $t \in [0,T]$, we obtain

$$\|m_\varepsilon - m\|_{L^2(0,T;W)} + \|m_\varepsilon - m\|_{L^\infty(0,T;L^2(\Gamma))} \leqslant o_\varepsilon(1) + C\|m_{0\varepsilon} - m_0\|_{L^2(\Gamma)}.$$

Subtracting the two versions of (4.3.2) and using the latter estimate also yields

$$\|\partial_t m_\varepsilon - \partial_t m\|_{L^2(0,T;V')} \leqslant o_\varepsilon(1) + C\|m_{0\varepsilon} - m_0\|_{L^2(\Gamma)},$$

which achieves the proof. □

4.4 The Hamilton-Jacobi equation

This section is devoted to the following boundary value problem including a Hamilton-Jacobi equation

$$\begin{cases}
-\partial_t v - \mu_\alpha \partial^2 v + H(x, \partial v) = f, & \text{in } (\Gamma_\alpha \backslash \mathcal{V}) \times (0,T),\ \alpha \in \mathcal{A}, \\
v|_{\Gamma_\alpha}(\nu_i, t) = v|_{\Gamma_\beta}(\nu_i, t) & t \in (0,T),\ \alpha, \beta \in \mathcal{A}_i,\ \nu_i \in \mathcal{V}, \\
\sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha \partial_\alpha v(\nu_i, t) = 0, & t \in (0,T),\ \nu_i \in \mathcal{V}, \\
v(x,T) = v_T(x), & x \in \Gamma,
\end{cases} \tag{4.4.1}$$

where $f \in L^2(\Gamma \times (0,T))$, $v_T \in V$ and the Hamiltonian $H : \Gamma \times \mathbb{R} \to \mathbb{R}$ satisfies the running assumptions (H).

Definition 4.4.1. For $f \in L^2(\Gamma \times (0,T))$ and $v_T \in V$, a weak solution of (4.4.1) is a function $v \in L^2\left(0,T; H^2(\Gamma)\right) \cap C([0,T]; V)$ such that $\partial_t v \in L^2(\Gamma \times (0,T))$ and

$$\int_\Gamma (-\partial_t v w + \mu \partial v \partial w + H(x, \partial v) w)\,dx = \int_\Gamma f w dx \quad \text{for all } w \in W, \text{ a.a. } t \in (0,T), \tag{4.4.2}$$

$$v(x,T) = v_T(x). \tag{4.4.3}$$

We start by proving existence and uniqueness of a weak solution for (4.4.1). Next, further regularity for the solution will be obtained under stronger assumptions.

4.4.1 Existence and uniqueness for the Hamilton-Jacobi equation

Theorem 4.4.2. *Under the running assumptions (H), if $f \in L^2(\Gamma \times (0,T))$, then the boundary value problem (4.4.1) has a unique weak solution.*

Uniqueness is a direct consequence of the following proposition.

Proposition 4.4.3. *(Comparison principle) Under the same assumptions as in Theorem 4.4.2, let v and $\hat{v}$ be respectively weak sub- and super-solution of (4.4.1), i.e., $v, \hat{v} \in L^2\left(0,T;H^2(\Gamma)\right)$ and $\partial_t v, \partial_t \hat{v} \in L^2(\Gamma \times (0,T))$ such that*

$$\begin{cases} \displaystyle\int_\Gamma \left(-\partial_t v w + \mu \partial v \partial w + H(x,\partial v) w\right) dx \leqslant \int_\Gamma f w dx, \\ \displaystyle\int_\Gamma \left(-\partial_t \hat{v} w + \mu \partial \hat{v} \partial w + H(x,\partial \hat{v}) w\right) dx \geqslant \int_\Gamma f w dx, \\ v(x,T) \leqslant v_T(x) \leqslant \hat{v}(x,T) \qquad \text{for a.a. } x \in \Gamma. \end{cases} \quad \text{for all } w \in W,\ w \geqslant 0,\ \text{a.a. } t \in (0,T),$$

Then $v \leqslant \hat{v}$ in $\Gamma \times (0,T)$.

Proof of Proposition 4.4.3. Setting $\overline{v} = v - \hat{v}$, we have, for all $w \in W$ such that $w \geqslant 0$ and for a.a $t \in (0,T)$:

$$\int_\Gamma -\partial_t \overline{v} w + \mu \partial \overline{v} \partial w + \left(H(x,\partial v) - H(x,\partial \hat{v})\right) w dx \leqslant 0,$$

and $\overline{v}(x,T) \leqslant 0$ for all $x \in \Gamma$. Set $\overline{v}^+ = \overline{v}\, \mathbb{1}_{\{\overline{v}>0\}}$ and $w = \overline{v}^+ e^{\lambda t}\varphi$. We have

$$\begin{aligned} &-\int_\Gamma \partial_t \left(\frac{(\overline{v}^+)^2}{2} e^{\lambda t}\right) \varphi dx + \int_\Gamma \frac{\lambda}{2} (\overline{v}^+)^2 e^{\lambda t} \varphi dx + \int_\Gamma \mu \partial \overline{v}^+ \partial(\overline{v}^+ \varphi) e^{\lambda t} dx \\ &+ \int_\Gamma \left[H(x,\partial v) - H(x,\partial \hat{v})\right] \overline{v}^+ \varphi e^{\lambda t} dx \leqslant 0. \end{aligned}$$

Integrating from 0 to T, we get

$$\begin{aligned} &\int_\Gamma \left(\frac{\overline{v}^+(0)^2}{2} - \frac{\overline{v}^+(T)^2}{2} e^{\lambda T}\right) \varphi dx + \int_0^T \int_\Gamma \frac{\lambda}{2} (\overline{v}^+)^2 e^{\lambda t} \varphi dx dt \\ &+ \int_0^T \int_\Gamma \mu (\partial \overline{v}^+)^2 \varphi e^{\lambda t} dx dt + \int_0^T \int_\Gamma \mu \partial \overline{v}^+ \overline{v}^+ \partial \varphi e^{\lambda t} dx dt \\ &+ \int_0^T \int_\Gamma \left[H(x,\partial v) - H(x,\partial \hat{v})\right] \overline{v}^+ \varphi e^{\lambda t} dx dt \leqslant 0. \end{aligned}$$

From (4.1.31), $|H(x,\partial v) - H(x,\partial \hat{v})| \leqslant C_0 |\partial \overline{v}|$. Hence, since $\overline{v}^+(T) = 0$ and $|\partial \overline{v}| \overline{v}^+ = |\partial \overline{v}^+| \overline{v}^+$ almost everywhere, we get

$$\int_0^T \int_\Gamma \left(\frac{\lambda}{2} (\overline{v}^+)^2 + \mu (\partial \overline{v}^+)^2\right) e^{\lambda t} \varphi dx dt - \int_0^T \int_\Gamma \left(\mu |\partial \varphi| + C_0 \varphi\right) |\partial \overline{v}^+| \overline{v}^+ e^{\lambda t} dx dt \leqslant 0. \tag{4.4.4}$$

For λ large enough, the first term in the left hand side is not smaller than the second term. This implies that $\overline{v}^+ = 0$. $\square$

Now we prove Theorem 4.4.3. We start with a bounded Hamiltonian H.

Proof of existence in Theorem 4.4.3 when H is bounded by C_H. Take $\overline{v} \in L^2(0,T;V)$ and $f \in L^2(\Gamma \times (0,T))$. From Theorem 4.2.4 and Theorem 4.2.5 with $h = f - H(x, \partial\overline{v})$ and $v_T \in V$, the following boundary value problem

$$\begin{cases} -\partial_t v - \mu_\alpha \partial^2 v = f - H(x, \partial \overline{v}), & \text{in } (\Gamma_\alpha \backslash \mathcal{V}) \times (0,T), \ \alpha \in \mathcal{A}, \\ v|_{\Gamma_\alpha}(\nu_i, t) = v|_{\Gamma_\beta}(\nu_i, t), & t \in (0,T), \ \alpha, \beta \in \mathcal{A}_i, \ \nu_i \in \mathcal{A}_i, \\ \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v(\nu_i, t) = 0, & t \in \times (0,T), \ \nu_i \in \mathcal{V}, \\ v(x,T) = v_T(x), & x \in \Gamma, \end{cases} \tag{4.4.5}$$

has a unique weak solution $v \in L^2\left(0,T;H^2(\Gamma)\right) \cap C([0,T];V) \cap W^{1,2}\left(0,T;L^2(\Gamma)\right)$. This allows us to define the map $\mathcal{T}$:

$$\begin{aligned} \mathcal{T} : L^2(0,T;V) &\longrightarrow L^2(0,T;V), \\ \overline{v} &\longmapsto v. \end{aligned}$$

From (4.1.31), $\overline{v} \longmapsto H(x, \partial\overline{v})$ is continuous from $L^2(0,T;V)$ into $L^2(\Gamma \times (0,T))$. Using again Theorem 4.2.5, we have that T is continuous from $L^2(0,T;V)$ to $L^2(0,T;V)$. Moreover, there exists a constant C depending only on C_H,Γ,$(\mu_\alpha)_{\alpha\in\mathcal{A}}$, f, T, φ and v_T such that

$$\|\partial_t v\|_{L^2(\Gamma\times(0,T))} + \|v\|_{L^2(0,T;H^2(\Gamma))} \leqslant C. \tag{4.4.6}$$

Therefore, from Aubin-Lions theorem (see Lemma 4.6.1), we obtain that $\mathcal{T}\left(L^2(0,T;V)\right)$ is relatively compact in $L^2(0,T;V)$. By Schauder fixed point theorem, see [61, Corollary 11.2], the operator $\mathcal{T}$ admits a fixed point which is a weak solution of (4.4.1). □

Proof of existence in Theorem 4.4.2 in the general case. Now we truncate the Hamiltonian as follows

$$H_n(x,p) = \begin{cases} H(x,p) & \text{if } |p| \leqslant n, \\ H\left(x, \dfrac{p}{|p|} n\right) & \text{if } |p| > n. \end{cases}$$

From the previous proof for bounded Hamiltonians, for all n, there exists a solution $v_n \in L^2\left(0,T;H^2(\Gamma)\right) \cap C([0,T];V) \cap W^{1,2}\left(0,T;L^2(\Gamma)\right)$ of (4.4.1), where H is replaced by H_n. We propose to send n to $+\infty$ and to show a subsequence of $\{v_n\}$ converges to a solution of (4.4.1). Hence, we need some uniform estimates for $\{v_n\}$. As in the proof of Proposition 4.4.3, using $-v_n e^{\lambda t}\varphi$ as a test-function, integrating from 0 to T and noticing that H is sublinear, see (4.1.30), we obtain

$$\begin{aligned} &\int_\Gamma \left[\frac{v_n^2(x,0)}{2} - \frac{v_n^2(x,T)}{2} e^{\lambda T}\right] \varphi dx + \int_0^T \int_\Gamma \left[\frac{\lambda}{2} v_n^2 e^{\lambda t}\varphi + \mu |\partial v_n|^2 e^{\lambda t}\varphi + \mu \partial v_n v_n e^{\lambda t} \partial\varphi\right] dx dt \\ = \ & -\int_0^T \int_\Gamma H_n(x, \partial v_n) v_n e^{\lambda t} \varphi dx dt + \int_0^T \int_\Gamma f v_n e^{\lambda t}\varphi dx dt \\ \leqslant \ & C_0 \int_0^T \int_\Gamma (1 + |\partial v_n|) |v_n| e^{\lambda t}\varphi dx dt + \frac{1}{2}\int_0^T \int_\Gamma f^2 e^{\lambda t}\varphi dx dt + \frac{1}{2}\int_0^T \int_\Gamma v_n^2 e^{\lambda t}\varphi dx dt. \end{aligned}$$

In the following lines, the constant C above will vary from line to line and will depend only on $(\mu_\alpha)_{\alpha\in\mathcal{A}}, C_H, T$ and φ. Taking λ large enough leads to the following estimate:

$$\|v_n\|_{L^2(0,T;V)} \leqslant C\left(\|f\|_{L^2(0,T;L^2(\Gamma))} + \|v_T\|_{L^2(\Gamma)} + 1\right), \tag{4.4.7}$$

and thus, from (4.1.30) again, we also obtain

$$\int_0^T \int_\Gamma |H_n(x, \partial v_n)|^2 \, dxdt \leqslant \int_0^T \int_\Gamma C_0^2 (|\partial v_n| + 1)^2 \, dxdt \leqslant \int_0^T \int_\Gamma 2C_0^2 \left(|\partial v_n|^2 + 1\right) dxdt$$
$$\leqslant C \left(\|f\|^2_{L^2(0,T;L^2(\Gamma))} + \|v_T\|^2_{L^2(\Gamma)} + 1 \right).$$

Therefore, $\{H_n(x, \partial v_n) - f\}$ is uniformly bounded in $L^2\left(0,T;L^2(\Gamma)\right)$. From Theorem 4.2.5, we obtain that $(v_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^2\left(0,T;H^2(\Gamma)\right) \cap C([0,T];V) \cap W^{1,2}\left(0,T;L^2(\Gamma)\right)$. By the Aubin-Lions theorem (see Lemma 4.6.1), $(v_n)_n$ is relatively compact in $L^2(0,T;V)$ (and bounded in $C([0,T];V)$). Hence, up to the extraction of a subsequence, there exists $v \in L^2(0,T;V) \cap W^{1,2}\left(0,T;L^2(\Gamma)\right)$ such that

$$v_n \to v, \quad \text{in } L^2(0,T;V) \quad \text{(strongly)}, \qquad \partial_t v_n \rightharpoonup \partial_t v, \quad \text{in } L^2(\Gamma \times (0,T)) \quad \text{(weakly)}. \tag{4.4.8}$$

Hence, $H_n(x, \partial v_n) \to H(x, \partial v)$ a.e. in $\Gamma \times (0,T)$. Note also that we can apply Lebesgue dominated convergence theorem to $H_n(x, \partial v_n)$ because $H_n(x, \partial v_n) \leqslant H(x, \partial v_n) \leqslant C_0(1+|\partial v_n|)$. Therefore, $H_n(x, \partial v_n) \to H(x, \partial v)$ in $L^2(\Gamma \times (0,T))$. Thus, it is possible to pass to the limit in the weak formulation satisfied by v_n and obtain that for all $w \in W$, $\chi \in C_c(0,T)$,

$$\int_0^T \chi(t) \left(-\int_\Gamma \partial_t v w dx + \int_\Gamma \partial v \partial w dx + \int_\Gamma H(x, \partial v) \, w dx \right) dt = \int_0^T \chi(t) \left(\int_\Gamma f w dx \right) dt.$$

Therefore, v satisfies (4.4.2).

From Theorem 4.2.4, $v_n(T) = v_T$ for all n. Since for all $\alpha \in \mathcal{A}$, $(v_n)_n$ tends to v in $L^2(\Gamma_\alpha \times (0,T))$ strongly and in $W^{1,2}(\Gamma_\alpha \times (0,T))$ weakly, $v_n|_{\Gamma_\alpha \times \{t=T\}}$ converges to $v|_{\Gamma_\alpha \times \{t=T\}}$ in $L^2(\Gamma_\alpha)$ strongly. Passing to the limit in the latter identity, we get (4.4.3). We have proven that v is a weak solution of (4.4.1). □

We end the section with a stability result for the Hamilton-Jacobi equation.

Lemma 4.4.4. *Let $(v_{T\varepsilon})_\varepsilon, (f_\varepsilon)_\varepsilon$ be sequences of functions satisfying*

$$v_{T\varepsilon} \longrightarrow v_T \ \text{in } V, \qquad f_\varepsilon \longrightarrow f \ \text{in } L^2(\Gamma \times (0,T)).$$

Let v_ε be the weak solution of (4.4.1) with data $v_{T\varepsilon}, f_\varepsilon$, then $(v_\varepsilon)_\varepsilon$ converges in $L^2\left(0,T;H^2(\Gamma)\right) \cap C([0,T];V) \cap W^{1,2}\left(0,T;L^2(\Gamma)\right)$ to the weak solution v of (4.4.1) with data v_T, f.

Proof of Lemma 4.4.4. Subtracting the two PDEs for v_ε and v, multiplying by $(v_\varepsilon - v)\, e^{\lambda t}\varphi^{-1}$, taking the integral on $\Gamma \times (0,T)$ and using similar computations as in the proof of Proposition 4.4.3, we obtain

$$\|v_\varepsilon - v\|_{L^2(0,T;V)} \leqslant C \left(\|f_\varepsilon - f\|_{L^2(\Gamma\times(0,T))} + \|v_{T\varepsilon} - v_T\|_{L^2(\Gamma)} \right),$$

for λ large enough and C independent of ε. This proves the convergence of v_ϵ to v in $L^2(0,T;V)$. Then, the convergence in $L^2\left(0,T;H^2(\Gamma)\right) \cap C([0,T];V) \cap W^{1,2}\left(0,T;L^2(\Gamma)\right)$ results from the assumption that H is Lipschitz with respect to its second argument, and from stability results for the linear boundary value problem (4.2.1) which are obtained with similar arguments as in the proof of Theorem 4.2.5. □

4.4.2 Regularity for the Hamilton-Jacobi equation

In this section, we prove further regularity for the solution of (4.4.1).

Theorem 4.4.5. *We suppose that the assumptions of Theorem 4.4.2 hold and that, in addition, $v_T \in H^2(\Gamma)$ satisfies the Kirchhoff conditions given by the third equation in (4.4.1), $f \in PC(\Gamma \times [0,T]) \cap L^2(0,T;H_b^1(\Omega))$ and $\partial_t f \in L^2(0,T;H_b^1(\Gamma))$.*

Then, the unique solution v of (4.4.1) satisfies $v \in L^2\left(0,T;H^3\left(\Gamma\right)\right)$ and $\partial_t v \in L^2\left(0,T;H^1\left(\Gamma\right)\right)$. Moreover, there exists a constant C depending only on $\|v_T\|_{H^2(\Gamma)}$, $(\mu_\alpha)_{\alpha\in\mathcal{A}}$, H and f such that

$$\|v\|_{L^2(0,T;H^3(\Gamma))} + \|\partial_t v\|_{L^2(0,T;H^1(\Gamma))} \leqslant C. \tag{4.4.9}$$

If, in addition, there exists $\eta \in (0,1)$ such that $v_T \in C^{2+\eta}(\Gamma)$ then there exists $\tau \in (0,1)$ such $v \in C^{2+\tau,1+\frac{\tau}{2}}(\Gamma \times [0,T])$, and v is a classical solution of (4.4.1).

The main idea to prove Theorem 4.4.5 is to differentiate (4.4.1) with respect to the space variable and to prove some regularity properties for the derived equation. Let us explain formally our method. Assuming the solution v of (4.4.1) is in $C^{2,1}(\Gamma \times (0,T))$ and taking the space-derivative of (4.4.1) on $(\Gamma_\alpha \backslash \mathcal{V}) \times (0,T)$, we have

$$-\partial_t \partial v - \mu_\alpha \partial^3 v + \partial\left(H(x,\partial v)\right) = \partial f.$$

Therefore, $u = \partial v$ satisfies the following PDE

$$-\partial_t u - \mu_\alpha \partial^2 u + \partial\left(H\left(x,u\right)\right) = \partial f,$$

with terminal condition $u(x,T) = \partial v_T(x)$. From the Kirchhoff conditions in (4.4.1) and Remark 4.1.1, we obtain a condition for u of Dirichlet type, namely

$$\sum_{\alpha\in\mathcal{A}_i} \mu_\alpha \gamma_{i\alpha} n_{i\alpha} u|_{\Gamma_\alpha}(\nu_i,t) = 0, \quad t \in (0,T),\ \nu_i \in \mathcal{V}.$$

Note that the latter condition is an homogeneous Dirichlet condition at the boundary vertices of Γ.

Now, by extending continuously the PDEs in (4.4.1) until the vertex ν_i in the branchs Γ_α and Γ_β, $\alpha,\beta \in \mathcal{A}_i$, and using the continuity condition in (4.4.1), one gets

$$-\mu_\alpha \partial^2 v|_{\Gamma_\alpha} + H^\alpha\left(\nu_i, \partial v|_{\Gamma_\alpha}(\nu_i,t)\right) - f|_{\Gamma_\alpha}(\nu_i,t) = -\mu_\beta \partial^2 v|_{\Gamma_\beta} + H^\beta(\nu_i, \partial v|_{\Gamma_\beta}(\nu_i,t)) - f|_{\Gamma_\beta}(\nu_i,t).$$

This gives a second transmission condition for u at $\nu_i \in \mathcal{V}\backslash\partial\Gamma$ of Robin type, namely

$$\begin{aligned} &\mu_\alpha \partial u|_{\Gamma_\alpha}\left(\nu_i,t\right) - H^\alpha(\nu_i, u|_{\Gamma_\alpha}(\nu_i,t)) + f|_{\Gamma_\alpha}\left(\nu_i,t\right) \\ =&\mu_\beta \partial u|_{\Gamma_\beta}\left(\nu_i,t\right) - H^\beta(\nu_i, u|_{\Gamma_\beta}(\nu_i,t)) + f|_{\Gamma_\beta}(\nu_i,t), \end{aligned} \tag{4.4.10}$$

which is equivalent to

$$\begin{aligned} &\mu_\alpha n_{i\alpha} \partial_\alpha u\left(\nu_i,t\right) - H^\alpha(\nu_i, u|_{\Gamma_\alpha}(\nu_i,t)) + f|_{\Gamma_\alpha}\left(\nu_i,t\right) \\ =&\mu_\beta n_{i\beta} \partial_\beta u\left(\nu_i,t\right) - H^\beta(\nu_i, u|_{\Gamma_\beta}(\nu_i,t)) + f|_{\Gamma_\beta}(\nu_i,t). \end{aligned} \tag{4.4.11}$$

Hence, we shall study the following nonlinear boundary value problem for $u = \partial v$,

$$\begin{cases} -\partial_t u - \mu_\alpha \partial^2 u + \partial\left(H\left(x,u\right)\right) = \partial f\left(x,t\right), & \left(x,t\right) \in \left(\Gamma_\alpha \backslash \mathcal{V}\right) \times \left(0,T\right),\ \alpha \in \mathcal{A}, \\ \displaystyle\sum_{\alpha\in\mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha n_{i\alpha} u|_{\Gamma_\alpha}\left(\nu_i,t\right) = 0, & t \in \left(0,T\right),\ \nu_i \in \mathcal{V}, \\ \mu_\alpha n_{i\alpha} \partial_\alpha u\left(\nu_i,t\right) - H^\alpha(\nu_i, u|_{\Gamma_\alpha}(\nu_i,t)) + f|_{\Gamma_\alpha}\left(\nu_i,t\right) & \\ \quad = \mu_\beta n_{i\beta} \partial_\beta u\left(\nu_i,t\right) - H^\beta(\nu_i, u|_{\Gamma_\beta}(\nu_i,t)) + f|_{\Gamma_\beta}\left(\nu_i,t\right), & t \in \left(0,T\right),\ \alpha,\beta \in \mathcal{A}_i,\ \nu_i \in \mathcal{V}\backslash\partial\Gamma, \\ u\left(x,T\right) = u_T\left(x\right), & x \in \Gamma, \end{cases} \tag{4.4.12}$$

where $\partial f \in L^2(\Gamma \times (0,T))$ and $u_T \in F$ defined in (4.4.13) below. Theorem 4.4.5 will follow by choosing $u_T = \partial v_T$.

In order to define the weak solutions of (4.4.12), we need the following subspaces of $H_b^1(\Gamma)$.

Definition 4.4.6. We define the Sobolev spaces

$$F := \left\{ u \in H_b^1(\Gamma) \text{ and } \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha n_{i\alpha} u|_{\Gamma_\alpha}(\nu_i) = 0 \text{ for all } \nu_i \in \mathcal{V} \right\}, \tag{4.4.13}$$

$$E := \left\{ \mathsf{e} \in H_b^1(\Gamma) \text{ and } \sum_{\alpha \in \mathcal{A}_i} n_{i\alpha} \mathsf{e}|_{\Gamma_\alpha}(\nu_i) = 0 \text{ for all } \nu_i \in \mathcal{V} \right\}. \tag{4.4.14}$$

Definition 4.4.7. Let the function ψ be defined as follows:

$$\begin{cases} \psi_\alpha \text{ is affine on } (0, \ell_\alpha), \\ \psi|_{\Gamma_\alpha}(\nu_i) = \mu_\alpha \gamma_{i\alpha}, \text{ if } \nu_i \in \mathcal{V} \backslash \partial\Gamma, \ \alpha \in \mathcal{A}_i, \\ \psi \text{ is constant on the edges } \Gamma_\alpha \text{ which touch the boundary of } \Gamma. \end{cases} \tag{4.4.15}$$

Note that ψ is positive and bounded. The map $f \longmapsto f\psi$ is an isomorphism from F onto E.

Definition 4.4.8. A weak solution of (4.4.12) is a function $u \in L^2(0,T;F)$ such that $\partial_t u \in L^2(0,T;E')$, $u(\cdot,T) = u_T$ and

$$-\langle \partial_t u, \mathsf{e} \rangle_{E',E} + \int_\Gamma \Big(\mu \partial u \partial \mathsf{e} - (H(x,u))\, \partial \mathsf{e} \Big) dx = -\int_\Gamma f \partial \mathsf{e} dx, \quad \text{for all } e \in E, \text{ a.a } t \in (0,T), \tag{4.4.16}$$

Remark 4.4.9. Note that if u is regular enough, then (4.4.16) can also be written

$$\begin{aligned} & -\langle \partial_t u, \mathsf{e} \rangle_{E',E} + \int_\Gamma \Big(\mu \partial u \partial \mathsf{e} + \partial (H(x,u))\, \mathsf{e} \Big) dx \\ & -\sum_{i \in I} \sum_{\alpha \in \mathcal{A}_i} n_{i\alpha} \left[H^\alpha(\nu_i, u|_{\Gamma_\alpha}(\nu_i,t)) - f|_{\Gamma_\alpha}(\nu_i,t) \right] \mathsf{e}|_{\Gamma_\alpha}(\nu_i) \\ = & \int_\Gamma (\partial f) \mathsf{e} dx, \qquad \text{for all } e \in E, \text{ a.a } t \in (0,T). \end{aligned} \tag{4.4.17}$$

Remark 4.4.10. To explain formally the definition of weak solutions, let us use $\mathsf{e} \in E$ as a test-function in the PDE in (4.4.12). After an integration by parts, we get

$$\int_\Gamma (-\partial_t u \mathsf{e} + \mu \partial u \partial \mathsf{e} + \partial (H(x,u))\, \mathsf{e})\, dx - \sum_{i \in I} \sum_{\alpha \in \mathcal{A}_i} n_{i\alpha} \mu_\alpha \partial u|_{\Gamma_\alpha}(\nu_i,t) \mathsf{e}|_{\Gamma_\alpha}(\nu_i) = \int_\Gamma (\partial f) \mathsf{e} dx,$$

where $n_{i\alpha}$ is defined in (4.1.6). On the one hand, from the second transmission condition, there exists a function $c_i : (0,T) \to \mathbb{R}$ such that $\mu_\alpha \partial u|_{\Gamma_\alpha}(\nu_i,t) - H^\alpha(\nu_i, u|_{\Gamma_\alpha}(\nu_i,t)) + f|_{\Gamma_\alpha}(\nu_i,t) = c_i(t)$ for all $\alpha \in \mathcal{A}_i$. It follows that

$$\begin{aligned} & -\sum_{i \in I} \sum_{\alpha \in \mathcal{A}_i} n_{i\alpha} \mu_\alpha \partial u|_{\Gamma_\alpha}(\nu_i,t) \mathsf{e}|_{\Gamma_\alpha}(\nu_i) \\ = & -\sum_{i \in I} c_i(t) \sum_{\alpha \in \mathcal{A}_i} n_{i\alpha} \mathsf{e}|_{\Gamma_\alpha}(\nu_i) + \sum_{i \in I} \sum_{\alpha \in \mathcal{A}_i} n_{i\alpha} \left[-H^\alpha(\nu_i, u|_{\Gamma_\alpha}(\nu_i,t)) + f|_{\Gamma_\alpha}(\nu_i,t) \right] \mathsf{e}|_{\Gamma_\alpha}(\nu_i) \\ = & \sum_{i \in I} \sum_{\alpha \in \mathcal{A}_i} n_{i\alpha} \left[-H^\alpha(\nu_i, u|_{\Gamma_\alpha}(\nu_i,t)) + f|_{\Gamma_\alpha}(\nu_i,t) \right] \mathsf{e}|_{\Gamma_\alpha}(\nu_i), \end{aligned}$$

because $\mathsf{e} \in E$. Then we may use the Remark 4.4.9 and obtain (4.4.16).

We start by proving the following result about (4.4.12) and then give the proof of Theorem 4.4.5.

Theorem 4.4.11. *Under the running assumptions, if $u_T \in F$, $f \in C(\Gamma\times[0,T]) \cap L^2(0,T;H_b^1(\Gamma)$ and $\partial_t f \in L^2(0,T;H_b^1(\Gamma))$, then (4.4.12) has a unique weak solution u. Moreover, there exists a constant C depending only on Γ, T, ψ, $\|u_T\|_F$, $\|\partial f\|_{L^2(\Gamma\times(0,T))}$, $\|f\|_{C(\Gamma\times[0,T])}$ and $\|\partial_t f\|_{L^2(0,T;H_b^1(\Gamma))}$ such that*

$$\|u\|_{L^2(0,T;H_b^2(\Gamma))} + \|u\|_{C([0,T];F)} + \|\partial_t u\|_{L^2(\Gamma\times(0,T))} \leqslant C. \tag{4.4.18}$$

Remark 4.4.12. Theorem 4.4.11 implies that $u(\cdot,t) \in C^1(\Gamma_\alpha)$ for all $\alpha \in \mathcal{A}$ for a.e. t. Hence, the transmission conditions for u hold in a classical sense for a.e. $t \in [0,T]$.

We use the Galerkin's method to construct solutions of certain finite-dimension approximations to (4.4.12).

We notice first that the symmetric bilinear form $\breve{\mathscr{B}}(u,v) := \int_\Gamma \mu\psi^{-1}\partial u\partial v$ is such that $(u,v) \mapsto (u,v)_{L^2(\Gamma)} + \breve{\mathscr{B}}(u,v)$ is an inner product in E equivalent to the standard inner product in E, namely $(u,v)_E = (u,v)_{L^2(\Gamma)} + \int_\Gamma \partial u\partial v$. Therefore, by standard Fredholm's theory, there exist

- a non decreasing sequence of nonnegative real numbers $(\lambda_k)_{k=1}^\infty$, that tends to $+\infty$ as $k \to \infty$
- A Hilbert basis $(\mathsf{e}_k)_{k=1}^\infty$ of $L^2(\Gamma)$, which is also a a total sequence of E (and orthogonal if E is endowed with the scalar product $(u,v)_{L^2(\Gamma)} + \breve{\mathscr{B}}(u,v)$),

such that

$$\breve{\mathscr{B}}(\mathsf{e}_k, e) = \lambda_k(\mathsf{e}_k, e)_{L^2(\Gamma)}, \quad \text{for all } e \in E. \tag{4.4.19}$$

Note that

$$\int_\Gamma \mu\partial\mathsf{e}_k\partial\mathsf{e}_\ell\psi^{-1}dx = \begin{cases} \lambda_k & \text{if } k=\ell, \\ 0 & \text{if } k \neq \ell. \end{cases}$$

Note also that e_k is a weak solution of

$$\begin{cases} -\mu_\alpha\partial\left(\psi^{-1}\partial\mathsf{e}_k\right) = \lambda_k\mathsf{e}_k & \text{in } \Gamma_\alpha\backslash\mathcal{V}, \alpha\in\mathcal{A}, \\ \dfrac{\partial_\alpha\mathsf{e}_k(\nu_i)}{\gamma_{i\alpha}} = \dfrac{\partial_\beta\mathsf{e}_k(\nu_i)}{\gamma_{i\beta}} & \text{for all } \alpha,\beta\in\mathcal{A}_i, \\ \sum_{\alpha\in\mathcal{A}_i} n_{i\alpha}e_k|_{\Gamma_\alpha}(\nu_i) = 0 & \text{if } \nu_i\in\mathcal{V}. \end{cases} \tag{4.4.20}$$

which implies that $\mathsf{e}_k|_{\Gamma_\alpha} \in C^2(\Gamma_\alpha)$ for all $\alpha \in \mathcal{A}$.

Finally, the sequence $(\mathsf{f}_k)_{k=1}^\infty$ given by $\mathsf{f}_k = \psi^{-1}\mathsf{e}_k$ is a total family in F (but is not orthogonal).

Lemma 4.4.13. *Under the assumptions made in Theorem 4.4.11, for any positive integer n, there exist n absolutely continuous functions $y_k^n : [0,T] \to \mathbb{R}$, $k = 1,\dots,n$, and a function $u_n : [0,T] \to L^2(\Gamma)$ of the form*

$$u_n(t) = \sum_{k=1}^n y_k^n(t)\,\mathsf{f}_k, \tag{4.4.21}$$

such that for all $k = 1,\dots,n$,

$$y_k^n(T) = \int_\Gamma u_T\mathsf{f}_k\psi^2dx, \tag{4.4.22}$$

and

$$-\frac{d}{dt}(u_n,\mathsf{f}_k\psi)_{L^2(\Gamma)} + \int_\Gamma\Big(\mu\partial u_n - H(x,u_n)\Big)\partial(\mathsf{f}_k\psi)\,dx = -\int_\Gamma f\partial(\mathsf{f}_k\psi)dx. \tag{4.4.23}$$

Proof of Lemma 4.4.13. The proof follows the same lines as the one of Lemma 4.2.2 but it is more technical since we obtain a system of nonlinear differential equations. For $n \geqslant 1$, we consider the symmetric n by n matrix M_n defined by

$$(M_n)_{k\ell} = \int_\Gamma \mathsf{f}_k \mathsf{f}_\ell \psi dx.$$

Since ψ is positive and bounded and since $(\psi \mathsf{f}_k)_{k=1}^\infty$ is a Hilbert basis of $L^2(\Gamma)$, we can check that M_n is a positive definite matrix and there exist two constants c, C independent of n such that

$$c|\xi|^2 \leqslant \sum_{k,\ell=1}^n (M_n)_{k\ell}\xi_k\xi_\ell \leqslant C|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^n. \tag{4.4.24}$$

Looking for u_n of the form (4.4.21) and setting $Y = (y_1^n, \dots, y_n^n)^T$, $\dot{Y} = \left(\frac{d}{dt}y_1^n, \dots, \frac{d}{dt}y_n^n\right)^T$, (4.2.7) implies that we have to solve the following a system of ODEs:

$$\begin{cases} -M_n\dot{Y}(t) + BY(t) + \mathcal{H}(Y)(t) = G(t), & t \in [0,T] \\ Y(T) = \left(\int_\Gamma u_T \mathsf{f}_1 \psi^2 dx, \cdots, \int_\Gamma u_T \mathsf{f}_n \psi^2 dx\right)^T, \end{cases} \tag{4.4.25}$$

where

- $B_{k\ell} = \int_\Gamma \mu \partial \mathsf{f}_\ell \partial(\psi \mathsf{f}_k) dx$
- $\mathcal{H}_i(Y) = -\int_\Gamma H(x, Y^T F)\partial(\mathsf{f}_i\psi)dx$ with $F = (\mathsf{f}_1, \cdots, \mathsf{f}_n)^T$ and $Y^T F = \sum_\ell y_\ell^n \mathsf{f}_\ell = u_n$
- $G_i(t) = -\int_\Gamma f(x,t)\partial(\mathsf{f}_i\psi)dx$ for all $i \in 1, \cdots, n$.

Since the matrix M is invertible and the function $\mathcal{H}$ is Lipschitz continuous by (4.1.31), the system (4.4.25) has a unique global solution. This ends the proof of the lemma. $\square$

We start by giving some estimates for the approximation u_n.

Lemma 4.4.14. *Under the assumptions made in Theorem 4.4.11, there exists a constant C depending only on Γ, T, ψ, $\|u_T\|_F$, $\|\partial f\|_{L^2(\Gamma\times(0,T))}$ $\|f\|_{C(\Gamma\times[0,T])}$ and $\|\partial_t f\|_{L^2(0,T;H_b^1(\Gamma))}$ such that*

$$\|u_n\|_{L^\infty(0,T;F)} + \|u_n\|_{L^2\left(0,T;H_b^2(\Gamma)\right)} + \|\partial_t u_n\|_{L^2(\Gamma\times(0,T))} \leqslant C.$$

Proof of Lemma 4.4.14. We divide the proof into two steps:

Step 1: Uniform estimates of u_n in $L^\infty(0,T;L^2(\Gamma))$, $L^2(0,T;F)$ and $W^{1,2}(0,T;E')$. Multiplying (4.4.23) by $y_k^n(t)\,\mathsf{f}_k e^{\lambda t}\psi$ where λ is a positive constant to be chosen later, summing for $k = 1, \dots, n$ and using (4.4.21), we get

$$-\int_\Gamma \partial_t u_n u_n e^{\lambda t}\psi dx + \int_\Gamma \Big(\mu\partial u_n - H(x,u_n)\Big)\partial\left(u_n e^{\lambda t}\psi\right)dx = -\int_\Gamma f\partial(u_n\psi e^{\lambda t})dx.$$

In the following lines, C will be a constant that may vary from lines to lines. Since H satisfies (4.1.30) and f is bounded, there exists a constant C such that

$$\begin{aligned} &-\int_\Gamma \left[\partial_t\left(\frac{u_n^2}{2}e^{\lambda t}\right) - \frac{\lambda}{2}u_n^2 e^{\lambda t}\right]\psi dx + \int_\Gamma \mu|\partial u_n|^2 e^{\lambda t}\psi dx - C\int_\Gamma |u_n|\left(|u_n| + |\partial u_n|\right)e^{\lambda t}dx \\ \leqslant &C\int_\Gamma (|u_n| + |\partial u_n|)e^{\lambda t}dx. \end{aligned} \tag{4.4.26}$$

The desired estimate on u_n is obtained from the previous inequality in a similar way as in the proof of Lemma 4.2.3, by taking λ large enough.

Step 2: Uniform estimates of u_n in $L^\infty(0,T;F)\cap L^2(0,T;H_b^2(\Gamma))$ and of $\partial_t u_n$ in $L^2(\Gamma\times(0,T))$. Multiplying (4.4.23) by $\partial_t y_k^n(t)\,\mathsf{f}_k e^{\lambda t}\psi$ where λ is a positive constant to be chosen later, integrating by part the term containing H and f (all the integration by parts are justified) summing for $k=1,\dots,n$ and using (4.4.21), we obtain that

$$
\begin{aligned}
&-\int_\Gamma (\partial_t u_n)^2 e^{\lambda t}\psi dx + \int_\Gamma \mu\partial u_n \partial\left(\partial_t u_n e^{\lambda t}\psi\right)dx + \int_\Gamma \partial\left(H\left(x,u_n\right)\right)\partial_t u_n e^{\lambda t}\psi dx \\
&-\sum_{i\in I}\sum_{\alpha\in\mathcal{A}_i} n_{i\alpha}\left[H^\alpha(\nu_i, u_n|_{\Gamma_\alpha}(\nu_i,t)) - f|_{\Gamma_\alpha}(\nu_i,t)\right]\partial_t u_n|_{\Gamma_\alpha}(\nu_i,t)\,\psi|_{\Gamma_\alpha}(\nu_i)\,e^{\lambda t} = \int_\Gamma \partial f\partial_t u_n\psi e^{\lambda t}dx.
\end{aligned}
\tag{4.4.27}
$$

Note that from (4.1.31) and (4.1.32),

$$
\left|\partial\left(H\left(x,u_n\right)\right)\right| \leqslant C_0(1+|u_n|+|\partial u_n|) \tag{4.4.28}
$$

so, from Step 1, this function is bounded in $L^2(\Gamma\times(0,T))$ by a constant. Moreover,

$$
\int_s^T\int_\Gamma \partial f\partial_t u_n\psi e^{\lambda t}dxdt \leqslant C\left(\int_s^T\int_\Gamma(\partial f)^2 e^{\lambda t}dxdt\right)^{\frac12}\left(\int_s^T\int_\Gamma(\partial_t u_n)^2 e^{\lambda t}\psi dxdt\right)^{\frac12}, \tag{4.4.29}
$$

and we can also estimate the term $\int_\Gamma \mu\partial u_n\partial\left(\partial_t u_n e^{\lambda t}\psi\right)dx$ as in the proof of Theorem 4.2.5. Therefore, the only new difficulty with respect to the proof of Theorem 4.2.5 consists of obtaining a bound for the term

$$
\sum_{i\in I}\sum_{\alpha\in\mathcal{A}_i} n_{i\alpha}\left[H^\alpha\left(\nu_i,u_n|_{\Gamma_\alpha}(\nu_i,t)\right) - f|_{\Gamma_\alpha}(\nu_i,t)\right]\partial_t u_n|_{\Gamma_\alpha}(\nu_i,t)\,e^{\lambda t}\psi|_{\Gamma_\alpha}(\nu_i).
$$

Let $\mathcal{J}_{i\alpha}(p)$ be the primitive function of $p\mapsto H^\alpha(\nu_i,p)$ such that $\mathcal{J}_{i\alpha}(0)=0$:

$$
H^\alpha\left(\nu_i,u_n|_{\Gamma_\alpha}(\nu_i,s)\right)\partial_t u_n|_{\Gamma_\alpha}(\nu_i,s) = \frac{d}{dt}\mathcal{J}_{i\alpha}(u_n|_{\Gamma_\alpha}(\nu_i,s)).
$$

We can then write

$$
\begin{aligned}
&-\int_s^T\left(n_{i\alpha}H^\alpha\left(\nu_i,u_n|_{\Gamma_\alpha}(\nu_i,t)\right)\partial_t u_n|_{\Gamma_\alpha}(\nu_i,t)\,e^{\lambda t}\psi|_{\Gamma_\alpha}(\nu_i)\right)dt \\
=&n_{i\alpha}\psi|_{\Gamma_\alpha}(\nu_i)\left(-\mathcal{J}_{i\alpha}\left(u_n|_{\Gamma_\alpha}(\nu_i,T)\right)e^{\lambda T} + \mathcal{J}_{i\alpha}\left(u_n|_{\Gamma_\alpha}(\nu_i,s)\right)e^{\lambda s} + \lambda\int_s^T\mathcal{J}_{i\alpha}\left(u_n|_{\Gamma_\alpha}(\nu_i,t)\right)e^{\lambda t}dt\right).
\end{aligned}
$$

Since $H^\alpha(x,\cdot)$ is sublinear, see (4.1.30), $|\mathcal{J}_{i\alpha}(p)|$ is subquadratic, i.e., $|\mathcal{J}_{i\alpha}(p)|\leqslant C(1+p^2)$, for a constant C independent of α and i. This implies that

$$
\begin{aligned}
&\left|\int_s^T\left(n_{i\alpha}H^\alpha\left(\nu_i,u_n|_{\Gamma_\alpha}(\nu_i,t)\right)\partial_t u_n|_{\Gamma_\alpha}(\nu_i,t)\,e^{\lambda t}\psi|_{\Gamma_\alpha}(\nu_i)\right)dt\right| \\
\leqslant& C\left(e^{\lambda T} + u_n^2|_{\Gamma_\alpha}(\nu_i,T)\,e^{\lambda T} + u_n^2|_{\Gamma_\alpha}(\nu_i,s)\,e^{\lambda s}\right) + C\lambda\int_0^T\left(1+u_n^2|_{\Gamma_\alpha}(\nu_i,t)\right)e^{\lambda t}dt.
\end{aligned}
$$

Note that, from Step 1 and the stability of the trace, $\lambda\int_s^T\left(1+u_n^2|_{\Gamma_\alpha}(\nu_i,t)\right)e^{\lambda t}dt\leqslant C\lambda e^{\lambda T}$. To summarize

$$
\begin{aligned}
&\left|\int_s^T\left(n_{i\alpha}H^\alpha\left(\nu_i,u_n|_{\Gamma_\alpha}(\nu_i,t)\right)\partial_t u_n|_{\Gamma_\alpha}(\nu_i,t)\,e^{\lambda t}\psi|_{\Gamma_\alpha}(\nu_i)\right)dt\right| \\
\leqslant& C\left(u_n^2|_{\Gamma_\alpha}(\nu_i,T)\,e^{\lambda T} + u_n^2|_{\Gamma_\alpha}(\nu_i,s)^2\,e^{\lambda s}\right) + \tilde{C}(\lambda).
\end{aligned}
\tag{4.4.30}
$$

Similarly, using the fact that $f \in C(\Gamma \times [0,T])$ and $\partial_t f|_{\Gamma_\alpha}(\nu_i, \cdot) \in L^2(0,T)$, and integrating by part, we see that

$$\begin{aligned}
&\left|\int_s^T f|_{\Gamma_\alpha}(\nu_i,t)\,\partial_t u_n|_{\Gamma_\alpha}(\nu_i,t)\,e^{\lambda t}dt\right| \\
= &\left|(f|_{\Gamma_\alpha}u_n)|_{\Gamma_\alpha}(\nu_i,T)\,e^{\lambda T} - (f|_{\Gamma_\alpha}u_n)|_{\Gamma_\alpha}(\nu_i,t)\,e^{\lambda s} - \int_s^T \left(\lambda f|_{\Gamma_\alpha}(\nu_i,t) + \partial_t f|_{\Gamma_\alpha}(\nu_i,t)\right)u_n|_{\Gamma_\alpha}(\nu_i,t)\,e^{\lambda t}dt\right| \\
\leqslant\ &C\left(|u_n|_{\Gamma_\alpha}(\nu_i,T)|e^{\lambda T} + |u_n|_{\Gamma_\alpha}(\nu_i,s)|e^{\lambda s} + \lambda\int_s^T |u_n|_{\Gamma_\alpha}(\nu_i,t)|\,e^{\lambda t}dt\right) \\
&+\frac{1}{2}\int_s^T u_n^2|_{\Gamma_\alpha}(\nu_i,t)\,e^{\lambda t}dt + \frac{1}{2}\int_s^T \left(\partial_t f|_{\Gamma_\alpha}(\nu_i,t)\right)^2 e^{\lambda t}dt.
\end{aligned}$$

From Step 1 and the assumptions on f, the last three terms in the right hand side of the latter estimate are bounded by a constant depending on λ, but not on n. To summarize,

$$\left|\int_s^T f|_{\Gamma_\alpha}(\nu_i,t)\,\partial_t u_n|_{\Gamma_\alpha}(\nu_i,t)\,e^{\lambda t}dt\right| \leqslant C\left(|u_n|_{\Gamma_\alpha}(\nu_i,T)|e^{\lambda T} + |u_n|_{\Gamma_\alpha}(\nu_i,s)|e^{\lambda s}\right) + \tilde{C}(\lambda). \tag{4.4.31}$$

To conclude from (4.4.30) and (4.4.31), we use the following estimates

$$\begin{cases} |u_n|_{\Gamma_\alpha}(\nu_i,t)| \leqslant C\left(\displaystyle\int_{\Gamma_\alpha}|u_n(x,t)|\,dx + \int_{\Gamma_\alpha}|\partial u_n(x,t)|\,dx\right), \\ u_n^2|_{\Gamma_\alpha}(\nu_i,t) \leqslant C\left(\displaystyle\int_{\Gamma_\alpha}u_n^2(x,t)\,dx + \int_{\Gamma_\alpha}|u_n\partial u_n(x,t)|\,dx\right), \end{cases} \tag{4.4.32}$$

for $t = s$ and $t = T$.

Then proceeding as in the proof of Theorem 4.2.5 and combining (4.4.27), (4.4.28), (4.4.29), (4.4.30) and (4.4.31) with (4.4.32), we find the desired estimates by taking λ large enough.

Let us end the proof by proving (4.4.32). The function $\phi = u_n|_{\Gamma_\alpha}(\cdot,t)$ is in $H^1(\Gamma_\alpha)$. By the continuous embedding $H^1(\Gamma_\alpha) \hookrightarrow C(\Gamma_\alpha)$, we can define ϕ in the pointwise sense (and even at two endpoints of any edges, see (4.1.3)). For all $\alpha \in \mathcal{A}$ and $x, y \in \Gamma_\alpha$, we have $\phi(x) = \phi(y) + \int_{[y,x]}\partial\phi(\xi)d\xi$. It follows

$$|\Gamma_\alpha|\phi(x) = \int_{\Gamma_\alpha}\phi(x)dy = \int_{\Gamma_\alpha}\phi(y)dy + \int_{\Gamma_\alpha}\int_{[y,x]}\partial\phi(\xi)d\xi dy \leqslant \int_{\Gamma_\alpha}|\phi(\xi)|d\xi + |\Gamma_\alpha|\int_{\Gamma_\alpha}|\partial\phi(\xi)|d\xi,$$

which gives the first estimate setting $x = \nu_i$. The second estimate is obtained in the same way replacing ϕ by ϕ^2 and using the fact that $W^{1,1}(\Gamma_\alpha)ss$ is continuously embedded in $C(\Gamma_\alpha)$. □

Proof of Theorem 4.4.11. From Lemma 4.4.14, up to the extraction of a subsequence, there exists $u \in L^2\left(0,T;H_b^2(\Gamma)\right) \cap W^{1,2}(\Gamma \times (0,T))$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{in } L^2\left(0,T;F \cap H_b^2(\Gamma)\right), \\ \partial_t u_n \rightharpoonup \partial_t u, & \text{in } L^2\left(\Gamma \times (0,T)\right). \end{cases} \tag{4.4.33}$$

Moreover, by Aubin-Lions Theorem (see Lemma 4.6.1),

$$L^2\left(0,T;F \cap H_b^2(\Gamma)\right) \cap W^{1,2}\left(0,T;L^2(\Gamma)\right) \overset{\text{compact}}{\hookrightarrow} L^2\left(0,T;F\right),$$

so up to the extraction of a subsequence, we may assume that $u_n \to u$ in $L^2(0,T;F)$ and almost everywhere. Moreover, from the compactness of the trace operator from $W^{1,2}(\Gamma_\alpha \times (0,T))$ to $L^2(\partial\Gamma_\alpha \times (0,T))$, $u_n|_{\partial\Gamma_\alpha\times(0,T)} \to u|_{\partial\Gamma_\alpha\times(0,T)}$ in $L^2(\partial\Gamma_\alpha \times (0,T))$ and for almost every $t \in (0,T)$. Similarly, $u_n|_{\Gamma_\alpha\times\{t=T\}} \to u|_{\Gamma_\alpha\times\{t=T\}}$ in $L^2(\Gamma_\alpha)$ and almost everywhere in Γ_α. Then, using the Lipschitz continuity of H with respect to its second argument, and similar arguments as in the proof of Theorem 4.2.4, we obtain the existence of a solution of (4.4.12) satisfying (4.4.18) by letting $n \to +\infty$. Since $H^2(\Gamma_\alpha) \subset C^{1+\sigma}(\Gamma_\alpha)$ for some $\sigma \in (0,1/2)$, $u(\cdot,t) \in C^{1+\sigma}(\Gamma_\alpha)$ for all $\alpha \in \mathcal{A}$ and a.a. t.

Finally, the proof of uniqueness is a consequence of the energy estimate (4.4.18) for u. □

Next, we want to prove that, if u is the solution of (4.4.12) and v is the solution (4.4.1), then $\partial v = u$. It means that we have to define a primitive function on the network Γ.

Definition 4.4.15. Let $x \in \Gamma_{\alpha_0} = [\nu_{i_0}, \nu_{i_1}]$ and $y \in \Gamma_{\alpha_m} = [\nu_{i_m}, \nu_{i_{m+1}}]$. We denote the set of paths joining from x to y by $\overrightarrow{xy}$. More precisely, if $\mathcal{L} \in \overrightarrow{xy}$, we can write $\mathcal{L}$ under the form

$$\mathcal{L} = x \to \nu_{i_1} \to \nu_{i_2} \to \ldots \to \nu_{i_m} \to y,$$

with $\nu_{i_k} \in \mathcal{V}$ and $[\nu_{i_k}, \nu_{i_{k+1}}] = \Gamma_{\alpha_k}$. The integral of a function ϕ on $\mathcal{L}$ is defined by

$$\int_{\mathcal{L}} \phi(\xi)\, d\xi = \int_{[x,\nu_{i_1}]} \phi(\xi)\, d\xi + \sum_{k=1}^{m} \int_{[\nu_{i_k},\nu_{i_{k+1}}]} \phi(\xi)\, d\xi + \int_{[\nu_{i_m},y]} \phi(\xi)\, d\xi, \tag{4.4.34}$$

recalling that the integrals on a segment are defined in (4.1.8).

Lemma 4.4.16. *Let u be the unique solution of (4.4.12) with $u_T = \partial v_T$. Then for all $x,y \in \Gamma$ and a.e. $t \in [0,T]$,*

$$\int_{\mathcal{L}_1} u(\zeta,t)\, d\zeta = \int_{\mathcal{L}_2} u(\zeta,t)\, d\zeta, \quad \text{for all } \mathcal{L}_1, \mathcal{L}_2 \in \overrightarrow{xy}.$$

This means that the integral of u from x to y does not depend on the path. Hence, for any $\mathcal{L} \in \overrightarrow{xy}$, we can define

$$\int_{\overrightarrow{xy}} u(\zeta,t)\, d\zeta := \int_{\mathcal{L}} u(\zeta,t)\, d\zeta.$$

Proof of Lemma 4.4.16. First, it is sufficient to prove $\int_{\mathcal{L}} u(\zeta,t)\, d\zeta = 0$ for all $\mathcal{L} \in \overrightarrow{xx}$. Secondly, if a given edge is browsed twice in opposite senses, the two related contributions to the integral sum to zero. It follows that, without loss of generality, we only need to consider loops in $\overrightarrow{xx}$ such that all the complete edges that it contains are browsed once only. It is also easy to see that we can focus on the case when $x \in \mathcal{V}$. To summarize, we only need to prove that

$$\int_{\mathcal{L}} u(\zeta,t)\, d\zeta = 0$$

when $\nu_{i_0} \in \mathcal{V}\backslash\partial\Gamma$ and $\mathcal{L} = \nu_{i_0} \to \nu_{i_1} \to \ldots \to \nu_{i_m} \to \nu_{i_0}$, where $\nu_{i_k} \neq \nu_{i_\ell}$ for $k \neq l$.

The following conditions

1. $\mathsf{e}|_{\Gamma_\alpha} = 0$ on each edge Γ_α not contained in $\mathcal{L}$
2. for all $k = 0,\ldots m-1$, $\mathsf{e}|_{\Gamma_{\alpha_k}} = 1_{i_k<i_{k+1}} - 1_{i_k>i_{k+1}}$ if Γ_{α_k} is the edge joining ν_{i_k} and $\nu_{i_{k+1}}$
3. $\mathsf{e}|_{\Gamma_{\alpha_m}} = 1_{i_m<i_0} - 1_{i_m>i_0}$ if Γ_{α_m} is the edge joining ν_{i_m} and ν_{i_0}

define a unique function $\mathsf{e} \in E$ which takes at most two values on $\mathcal{L}$, namely ± 1.

From Definition 4.4.15, we have

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{L}} u(\zeta,t)\,dt &= \sum_{k=0}^{m} \frac{d}{dt}\int_{[\nu_{i_k},\nu_{i_{k+1}}]} u(\zeta,t)\,d\zeta + \frac{d}{dt}\int_{[\nu_{i_m},\nu_{i_0}]} u(\zeta,t)\,d\zeta \\ &= \frac{d}{dt}\int_{\Gamma} u(\zeta,t)\,\mathsf{e}(\zeta)\,d\zeta = \int_{\Gamma} \partial_t u(\zeta,t)\,\mathsf{e}(\zeta)\,d\zeta. \end{aligned}$$

Then, using Definition 4.4.8, Remark 4.4.9 and Remark 4.4.10 yields that

$$\begin{aligned} &\frac{d}{dt}\int_{\mathcal{L}} u(\zeta,t)\,d\zeta \\ =& \sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \left[-\mu_\alpha \partial^2 u(\zeta,t) + \partial H(\zeta, u(\zeta,t)) - \partial f(\zeta,t)\right] \mathsf{e}(\zeta)\,d\zeta \\ =& \sum_{k=0}^{m} \int_{\Gamma_{\alpha_k}} \left[-\mu_{\alpha_k} \partial^2 u(\zeta,t) + \partial H(\zeta, u(\zeta,t)) - \partial f(\zeta,t)\right] \mathsf{e}(\zeta)\,d\zeta \\ =& \sum_{k=0}^{m} \mathsf{e}|_{\Gamma_{\alpha_k}}(\nu_i) \left(\begin{array}{l} n_{i_{k+1}\alpha_k}\left(-\mu_{\alpha_k}\partial u|_{\Gamma_{\alpha_k}}(\nu_{i_{k+1}},t) + H^\alpha\left(\nu_{i_{k+1}}, u|_{\Gamma_{\alpha_k}}(\nu_{i_{k+1}},t)\right) - f(\nu_{i_{k+1}},t)\right) \\ +n_{i_k\alpha_k}\left(-\mu_{\alpha_k}\partial u|_{\Gamma_{\alpha_k}}(\nu_{i_k},t) + H^\alpha\left(\nu_{i_k}, u|_{\Gamma_{\alpha_k}}(\nu_{i_k},t)\right) - f(\nu_{i_k},t)\right) \end{array} \right), \end{aligned}$$

where we have set $i_{m+1} = i_0$. Now using (4.4.10) (which is satisfied for a.e. t from the regularity of u) and the fact that $\mathsf{e} \in E$, we conclude that

$$\frac{d}{dt}\int_{\mathcal{L}} u(\zeta,t)\,d\zeta = 0. \tag{4.4.35}$$

Hence

$$\int_{\mathcal{L}} u(\zeta,t)\,d\zeta = \int_{\mathcal{L}} u(\zeta,T)\,d\zeta = \int_{\mathcal{L}} u_T(\zeta)\,d\zeta = \int_{\mathcal{L}} \partial v_T(\zeta)\,d\zeta = 0,$$

where the last identity comes from the assumption that $v_T \in V$ (the continuity of v_T).

□

Lemma 4.4.17. *If $u_T = \partial v_T \in F$, then the weak solution u of (4.4.12) satisfies $u = \partial v$ where v is the unique solution of (4.4.1).*

Proof of Lemma 4.4.17. For simplicity, we write the proof in the case when $\partial\Gamma \neq \varnothing$. The proof is similar in the other case.

Let us fix some vertex $\nu_k \in \partial\Gamma$. From standard regularity results for Hamilton-Jacobi equation with homogeneous Neumann condition, we know that that there exists ω, a closed neighborhood of $\{\nu_k\}$ in Γ made of a single straigt line segment and containing no other vertices of Γ than ν_k, such that $v|_{\omega\times(0,T)} \in L^2(0,T;H^3(\omega)) \cap C([0,T];H^2(\omega) \cap W^{1,2}(0,T;H^1(\omega))$. Hence, v satisfies the Hamilton-Jacobi equation at almost every point of $\omega \times (0,T)$. Moreover the equation

$$\partial_t v(\nu_k,t) + \mu\partial^2 v(\nu_k,t) - H(\nu_k,0) + f(\nu_k,t) = 0 \tag{4.4.36}$$

holds for almost every $t \in (0,T)$ and in $L^2(0,T)$.

For every $x \in \Gamma$ and $t \in [0,T]$, we define

$$\hat{v}(x,t) = v(\nu_k,t) + \int_{\overrightarrow{\nu_k x}} u(\zeta,t)\,d\zeta. \tag{4.4.37}$$

Remark 4.4.18. If $\partial\Gamma = \varnothing$, then the proof should be modified by replacing ν_k by a point $\nu \in \Gamma\backslash\mathcal{V}$ and by using local regularity results for the HJB equation in (4.4.1).

We claim that $\hat{v}$ is a solution of (4.4.1).

First, $\hat{v}(\cdot, t)$ is continuous on Γ. Indeed, $\hat{v}(y,t) - \hat{v}(x,t) = \int_{\overrightarrow{xy}} u(\zeta,t)d\zeta$. On the other hand, $u \in C([0,T];F) \subset L^\infty(\Gamma \times [0,T])$. It follows that $|\hat{v}(y,t) - \hat{v}(x,t)| \leqslant ||u||_{L^\infty(\Gamma\times[0,T])}\text{dist}(x,y)$ which implies that $\hat{v}(\cdot,t)$ is continuous on Γ.

Next, from the terminal conditions for u,

$$\hat{v}(x,T) = v(\nu_k,T) + \int_{\overrightarrow{\nu_k x}} u(\zeta,T)\,d\zeta = v_T(\nu_k) + \int_{\overrightarrow{\nu_k x}} \partial v_T(\zeta)\,d\zeta = v_T(x),$$

where the last identity follows from the continuity of v_T on Γ.

Let us check the Kirchhoff condition for $\hat{v}$. Take $\nu_i \in \mathcal{V}$ and $\alpha \in \mathcal{A}_i$. From (4.1.7), for a.e. $t \in (0,T)$, $\partial_\alpha \hat{v}(\nu_i,t) = n_{i\alpha}\partial\hat{v}|_{\Gamma_\alpha}(\nu_i,t)$ and from (4.4.37), $\partial\hat{v}|_{\Gamma_\alpha}(\nu_i,t) = u|_{\Gamma_\alpha}(\nu_i,t)$. Since $u(\cdot,t) \in F$, we get

$$\sum_{\alpha\in\mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha\partial_\alpha\hat{v}(\nu_i,t) = \sum_{\alpha\in\mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha n_{i\alpha}u|_{\Gamma_\alpha}(\nu_i,t) = 0,$$

which is exactly the Kirchhoff condition for $\hat{v}$ at ν_i.

There remains to prove $\hat{v}$ solves the Hamilton-Jacobi equation in $\Gamma\backslash\mathcal{V}$: Take $x \in \Gamma_\alpha\backslash\mathcal{V}$ for some $\alpha \in \mathcal{A}$ and consider a path $\overrightarrow{\nu_k x} \ni \mathcal{L} = \nu_{i_0} \to \cdots \to \nu_{i_m} \to x$, where $i_0 = k$ and $\nu_{i_m} \in \Gamma_\alpha$. Let $\nu_{i_{m+1}}$ be the other endpoint of Γ_α. We proceed as in the proof of Lemma 4.4.16: the following conditions

1. $\mathsf{e}|_{\Gamma_\alpha} = 0$ on each edge Γ_α not contained in $\mathcal{L}$
2. for all $j = 0,\dots m$, $\mathsf{e}|_{\Gamma_j} = 1_{i_j<i_{j+1}} - 1_{i_j>i_{j+1}}$ if Γ_j is the edge joining ν_{i_j} and $\nu_{i_{j+1}}$

define a unique piecewise constant function e which takes at most two values on $\mathcal{L}$, namely ±1. Note that e does not belong to E because $\mathsf{e}(\nu_k) \neq 0$, but that e satisfies $\sum_{\alpha\in\mathcal{A}_i} n_{i\alpha}\mathsf{e}|_{\Gamma_\alpha}(\nu_i) = 0$ for all $\nu_i \in \mathcal{V}\backslash\partial\Gamma$.

Using this function, a similar computation as in the proof of Lemma 4.4.16 implies that, for almost every $t \in (0,T)$,

$$\begin{aligned}\partial_t\hat{v}(x,t) - \partial_t v(\nu_k,t) = &-\mu_\alpha\partial u|_{\Gamma_\alpha}(x,t) + H(x,u|_{\Gamma_\alpha}(x,t)) - f(x,t)\\ &+ \mu_\alpha\partial_2 v(\nu_k,t) - H(\nu_k,0) + f(\nu_k,t).\end{aligned}$$

Then, using (4.4.36) and the fact that $\partial\hat{v} = u$, the latter identity yields that for almost every $(x,t) \in (0,T)\times\Gamma$,

$$\partial_t\hat{v}(x,t) + \mu_\alpha\partial^2\hat{v}(x,t) - H(x,\partial\hat{v}(x,t)) + f(x,t) = 0.$$

We have proven that $\hat{v}$ is a solution of (4.4.1). Since v is the unique solution of (4.4.1), we conclude that $v = \hat{v}$ and $\partial v = u$. ☐

We are now ready to give the proof of Theorem 4.4.5.

Proof of Theorem 4.4.5. Since $\partial v = u$ by Lemma 4.4.17 and u satisfies (4.4.18) by Theorem 4.4.11, we obtain that $v \in L^2(0,T;H^3(\Gamma))$ and $\partial_t v \in L^2(0,T;H^1(\Gamma))$ and (4.4.9) holds.

Therefore, using an interpolation result combined with Sobolev embeddings, see [11] or Lemma 4.6.2 in the Appendix, $v \in C^{1+\sigma,\sigma/2}(\Gamma\times[0,T])$ for some $0 < \sigma < 1$.

Finally, we know that since $f \in W^{1,2}(0,T,H_b^1(\Gamma))$, $f|_{\Gamma_\alpha\times[0,T]} \in C^{\eta,\eta}(\Gamma_\alpha \times [0,T])$ for all $\eta \in (0,1/2)$. If $f \in C^{\eta,\frac{\eta}{2}}(\Gamma_\alpha \times [0,T])$ for some $\eta \in (0,1/2)$, we claim that $v \in C^{2,1}(\Gamma \times [0,T])$. This is a direct consequence of a theorem of Von Below, see the main theorem in [99], for the (modified) heat equation

$$-\partial_t w - \mu_\alpha \partial^2 w = g(x,t), \tag{4.4.38}$$

with the same Kirchhoff conditions as in (4.4.1): Note that if the terminal Cauchy condition for w is $w(\cdot, t = T) = v_T$ and if $g = f - H(x,\partial v)$, then $w = v$. Now $g = f - H(x,\partial v) \in C^{\tau,\frac{\tau}{2}}(\Gamma_\alpha \times [0,T])$, where $1/2 > \tau = \min(\sigma,\eta) > 0$. Using the result in [99], we obtain that $v = w \in C^{2+\tau,1+\tau/2}(\Gamma_\alpha \times [0,T])$, then that v is a classical solution of (4.4.1). □

4.5 Existence, uniqueness and regularity for the MFG system (Proof of Theorem 4.1.11)

Proof of existence in Theorem 4.1.11. First of all, given m_0 and v_T, let us construct the map $\mathcal{T}$ from $L^2(0,T;V)$ to itself as follows.

Given $v \in L^2(0,T;V)$, we first define m as the weak solution of (4.3.1) with initial data m_0 and $b = H_p(x,\partial v)$. We know that $m \in L^2(0,T;W) \cap C([0,T];L^2(\Gamma)) \cap W^{1,2}(0,T;V')$.

We claim that if $v_n \to v$ in $L^2(0,T;V)$ then $H_p(\cdot,\partial v_n)$ tends to $H_p(\cdot,\partial v)$ in $L^2(\Gamma\times(0,T))$. To prove the claim, we argue by contradiction: assume that there exist a positive number ϵ and a subsequence $v_{\phi(n)}$ such that $\|H_p(\cdot,\partial v_{\phi(n)}) - H_p(\cdot,\partial v)\|_{L^2(\Gamma\times(0,T))} > \epsilon$. Then since $\partial v_{\phi(n)}$ tends to ∂v in $L^2(\Gamma\times(0,T))$, we can extract another subsequence $v_{\psi(n)}$ from $v_{\phi(n)}$ such that $\partial v_{\psi(n)}$ tends to ∂v almost every where in $\Gamma\times(0,T)$. From the continuity of H_p, we deduce that $H_p(\cdot,\partial v_{\psi(n)})$ tends to $H_p(\cdot,\partial v)$ almost everywhere in $\Gamma\times(0,T)$. Since there exists a positive constant C_0 such that $\|H_p(\cdot,\partial v_{\psi(n)})\|_\infty \leqslant C_0$, $\|H_p(\cdot,\partial v)\|_\infty \leqslant C_0$, Lebesgue dominated convergence theorem ensures that $H_p(\cdot,\partial v_{\psi(n)})$ tends to $H_p(\cdot,\partial v)$ in $L^2(\Gamma\times(0,T))$, which is the desired contradiction.

To summarize, $H_p(\cdot,\partial v_n)$ tends to $H_p(\cdot,\partial v)$ in $L^2(\Gamma\times(0,T))$ on the one hand, and for a positive constant C_0, $\|H_p(\cdot,\partial v_n)\|_\infty \leqslant C_0$, $\|H_p(\cdot,\partial v)\|_\infty \leqslant C_0$. Using Lemma 4.3.4, we see that m_n, the weak solution of (4.3.1) with initial data m_0 and $b = H_p(x,\partial v_n)$ converges to m in $L^2(0,T;W) \cap L^\infty\left(0,T;L^2(\Gamma)\right) \cap W^{1,2}(0,T;V')$. Hence, the map $v \mapsto m$ is continuous from $L^2(0,T;V)$ to $L^2(0,T;W) \cap L^\infty\left(0,T;L^2(\Gamma)\right) \cap W^{1,2}(0,T;V')$. Moreover, the a priori estimate (4.3.3) holds uniformly with respect to v.

Then, knowing m, we construct $\mathcal{T}(v) \equiv \tilde v$ as the unique weak solution of (4.4.1) with $f(x,t) = \mathscr{V}[m(\cdot,t)](x)$. Note that $m \mapsto f$ is continuous and locally bounded from $L^2(\Gamma\times(0,T))$ to $L^2(\Gamma\times(0,T))$. Then Lemma 4.4.4 ensures that the map $m \to \tilde v$ is continuous from $L^2(\Gamma\times(0,T))$ to $L^2\left(0,T;H^2(\Gamma)\right) \cap L^\infty(0,T;V) \cap W^{1,2}(0,T;L^2(\Gamma))$. From Aubin-Lions theorem, see Lemma 4.6.1, $m \to \tilde v$ maps bounded sets of $L^2(\Gamma\times(0,T))$ to relatively compact sets of $L^2(0,T;V)$.

Therefore, the map $\mathcal{T} : v \mapsto \tilde v$ is continuous from $L^2(0,T;V)$ to $L^2(0,T;V)$ and has a relatively compact image. Finally, we apply Schauder fixed point theorem [61, Corollary 11.2] and conclude that the map $\mathcal{T}$ admits a fixed point v. We know that $v \in L^2\left(0,T;H^2(\Gamma)\right) \cap L^\infty(0,T;V) \cap W^{1,2}(0,T;L^2(\Gamma))$ and $m \in L^2(0,T;W) \cap L^\infty\left(0,T;L^2(\Gamma)\right) \cap W^{1,2}(0,T;V'(\Gamma))$.

Hence, there exists a weak solution (v,m) to the mean field games system (4.1.24). □

Proof of uniqueness in Theorem 4.1.11. We assume that there exist two solutions (v_1,m_1) and

(v_2, m_2) of (4.1.24). We set $\overline{v} = v_1 - v_2$ and $\overline{m} = m_1 - m_2$ and write the system for $\overline{v}, \overline{m}$

$$\begin{cases} -\partial_t \overline{v} - \mu_\alpha \partial^2 \overline{v} + H(x, \partial v_1) - H(x, \partial v_2) - (\mathscr{V}[m_1] - \mathscr{V}[m_2]) = 0, & x \in \Gamma_\alpha \backslash \mathcal{V},\ t \in (0,T), \\ \partial_t \overline{m} - \mu_\alpha \partial^2 \overline{m} - \partial(m_1 \partial_p H(x, \partial m_1) - m_2 \partial_p H(x, \partial m_2)) = 0 & x \in \Gamma_\alpha \backslash \mathcal{V},\ t \in (0,T), \\ \overline{v}|_{\Gamma_\alpha}(\nu_i, t) = \overline{v}|_{\Gamma_\beta}(\nu_i, t), \ \dfrac{\overline{m}|_{\Gamma_\alpha}(\nu_i, t)}{\gamma_{i\alpha}} = \dfrac{\overline{m}|_{\Gamma_\beta}(\nu_i, t)}{\gamma_{i\beta}}, & \alpha, \beta \in \mathcal{A}_i,\ \nu_i \in \mathcal{V}, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha \overline{v}(\nu_i, t) = 0, & \nu_i \in \mathcal{V},\ t \in (0,T), \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} n_{i\alpha} [m_1|_{\Gamma_\alpha}(\nu_i) \partial_p H^\alpha(\nu_i, \partial v_1|_{\Gamma_\alpha}(\nu_i, t)) - m_2|_{\Gamma_\alpha}(\nu_i) \partial_p H^\alpha(\nu_i, \partial v_2|_{\Gamma_\alpha}(\nu_i, t))] & \\ \qquad + \displaystyle\sum_{\alpha \in \mathcal{A}_i} \mu_\alpha \partial_\alpha \overline{m}(\nu_i, t) = 0, & \nu_i \in \mathcal{V},\ t \in (0,T), \\ \overline{v}(x, T) = 0, \ \overline{m}(x, 0) = 0. & \end{cases}$$

Testing by $\overline{m}$ the boundary value problem satified by $\overline{u}$, testing by $\overline{u}$ the boundary value problem satified by $\overline{m}$, subtracting, we obtain

$$\begin{aligned} &\int_0^T \int_\Gamma (m_1 - m_2)(\mathscr{V}[m_1] - \mathscr{V}[m_2)] \, dx dt + \int_0^T \int_\Gamma \partial_t (\overline{m}\ \overline{v}) \, dx dt \\ &+ \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} m_1 [H(x, \partial v_2) - H(x, \partial v_1) - \partial_p H(x, \partial v_1) \partial \overline{v}] \, dx \\ &+ \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} m_2 [H(x, \partial v_1) - H(x, \partial v_2) + \partial_p H(x, \partial v_1) \partial \overline{v}] \, dx = 0. \end{aligned}$$

Since $\mathscr{V}$ is strictly monotone, the first sum is nonnegative. Moreover,

$$\int_0^T \int_\Gamma \partial_t (\overline{m}\ \overline{v}) \, dx dt = \int_\Gamma [\overline{m}(x,T)\ \overline{v}(x,T) - \overline{m}(x,0)\ \overline{v}(x,0)] dx = 0,$$

since $\overline{v}(x,T) = 0$ and $\overline{m}(x,0) = 0$. From the convexity of H and the fact that m_1, m_2 are nonnegative, the last two sums are nonnegative. Therefore, all the terms are zero and thanks again to the fact that $\mathscr{V}$ is strictly increasing, we obtain $m_1 = m_2$. From Lemma 4.4.2, we finally obtain $v_1 = v_2$. □

Proof of regularity in Theorem 4.1.11. We make the stronger assumptions written in Section 4.1.4 on the coupling operator $\mathscr{V}$. We know that $\mathscr{V}[m] \in W^{1,2}(0,T; H_b^1(\Gamma)) \cap PC(\Gamma \times [0,T])$. Assuming also that $v_T \in V$ and $\partial v_T \in F$, we can apply the regularity result in Theorem 4.4.5: $v \in L^2\left(0,T; H^3(\Gamma)\right) \cap W^{1,2}\left(0,T; H^1(\Gamma)\right)$.

Moreover, since $\mathscr{V}[m] \in W^{1,2}(0,T, H_b^1(\Gamma))$, we know that $(\mathscr{V}[m])|_{\Gamma_\alpha \times [0,T]} \in C^{\sigma,\sigma/2}(\Gamma_\alpha \times [0,T])$ for all $0 < \sigma < 1/2$. If $v_T \in C^{2+\eta} \cap D$ for some $\eta \in (0,1)$ (D is defined in (4.1.13)), then from Theorem 4.4.5, $v \in C^{2+\tau,1+\tau/2}(\Gamma \times [0,T])$ for some $\tau \in (0,1)$ and the boundary value problem for v is satisfied in a classical sense.

In turn, if for all $\alpha \in \mathcal{A}$, $\partial_p H^\alpha(x,p)$ is a Lipschitz function defined in $\Gamma_\alpha \times \mathbb{R}$, and if $m_0 \in W$, then we can use the latter regularity of v and arguments similar to those contained in the proof of Theorem 4.2.5 and prove that $m \in C([0,T]; W) \cap W^{1,2}(0,T; L^2(\Gamma)) \cap L^2(0,T; H_b^2(\Gamma))$. □

4.6 Appendix: Some continuous and compact embeddings

Lemma 4.6.1. *(Aubin-Lions Lemma, see [83]) Let X_0,X and X_1 be function spaces, (X_0 and X_1 are reflexive). Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. Consider some real numbers $1 < p, q < +\infty$. Then the following set*

$$\{v : (0,T) \mapsto X_0 : \ v \in L^p\left(0,T;X_0\right), \ \partial_t v \in L^q\left(0,T;X_1\right)\}$$

is compactly embedded in $L^p\left(0,T;X\right)$.

Lemma 4.6.2. *(Amann, see [11]) Let $\phi : [a,b] \times [0,T] \to \mathbb{R}$ such that $\phi \in L^2(0,T;H^2(a,b))$ and $\partial_t \phi \in L^2(0,T;L^2(a,b))$. Then $\phi \in C^s(0,T;H^1(a,b))$ for some $s \in (0,1/2)$.*

This result is a consequence of the general result [11, Theorem 1.1] taking into account [11, Remark 7.4]. More precisely, we have

$$E_1 := H^2(a,b) \overset{\text{compact}}{\hookrightarrow} E := H^1(a,b) \hookrightarrow E_0 := L^2(a,b).$$

Let $r_0 = r_1 = r = 2$, $\sigma_0 = 0$, $\sigma_1 = 2$ and $\sigma = 1$. For any $\nu \in (0,1)$, we define

$$\frac{1}{r_\nu} = \frac{1}{r_0} + \frac{1-\nu}{r_1}, \quad \sigma_\nu := (1-\nu)s_0 + \nu s_1.$$

This implies that $r_\nu = 2$ and $\sigma_\nu = 2\nu$. Therefore, if $\nu \in (1/2,1)$, then the following inequality is satisfied

$$\sigma - 1/r < \sigma_\nu - 1/r_\nu < \sigma_1 - 1/r_1.$$

Hence, we infer from [11, Remark 7.4]

$$E_1 \hookrightarrow (E_0,E_1)_{\nu,1} \hookrightarrow (E_0.E_1)_{\nu,r_\nu} = W^{\sigma_\nu,r_\nu}(a,b) \hookrightarrow E,$$

where $(E_0,E_1)_{\nu,1}$, $(E_0.E_1)_{\nu,r_\nu}$ are interpolation spaces. This is precisely the assumption allowing to apply [11, Theorem 1.1], which gives the result of Lemma 4.6.2.

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UNIVERSITE BRETAGNE LOIRE / MATHSTIC

UNIVERSITÉ DE RENNES 1

Titre : Équation de Hamilton-Jacobi et jeux à champ moyen sur les réseaux

Mots clés : Problèmes de contrôle optimal, équation de Hamilton-Jacobi, jeux à champ moyen, réseaux

Résumé : Cette thèse porte d'une part sur l'é ude d'é uations de Hamilton-Jacobi-Bellman (HJB) associées à des problèmes de contrôle optimal et d'autre part de jeux à champ moyen (MFG) avec la particularité qu'on se place sur des réseaux, pour lesquels on autorise différentes dynamiques et coûts dans chaque arête.

Dans la première partie, on considère un problème de contrôle optimal sur les réseaux dans lesquels on rajoute des coûts d'entré (ou de sortie) aux sommets conduisant à une é entuelle discontinuité de la fonction valeur. Celle-ci est caracté isée comme l'unique solution de viscosité d'une éq ation Hamilton-Jacobi (HJ) pour laquelle une condition de jonction adéquate est établie.

La deuxième partie concerne les MFG stochastiques sur les réseaux dans le cas ergodique. Ils sont décrits par un système couplant une équation de HJB et une équation de Fokker-Planck, dont les inconnues sont: la densité m qui est en général discontinue aux sommets et satisfait deux conditions de transmission aux sommets; la fonction valeur v qui est continue et satisfait des conditions de Kirchhoff aux sommets et enfin la constante ergodique ρ. L'existence et l'unicité sont prouvé s pour des Hamiltoniens sous-quadratiques et des couplages bornés inférieurement généraux.

Enfin, dans une dernière partie, nous étudions le même problème non stationnaire pour des Hamiltoniens sous-linéaires et un couplage régularisant. La principale difficulté supplémentaire par rapport au cas stationnaire est d'établir la ré ularité. Notre approche consiste à é udier la solution de l'équ tion de HJ dé ivé pour gagner de la régularité sur la solution de l'équ tion initiale.

Title : Hamilton-Jacobi equations and Mean field games on networks

Keywords : Optimal control problems, Hamilton-Jacobi equation, Mean Field Games, networks

Abstract: The dissertation focuses on the study of Hamilton-Jacobi-Bellman (HJB) equations associated with optimal control problems and mean field games (MFG) problems in the case when the state space is a network. Different dynamics and running costs are allowed in each edge of the network.

In the first part, we consider an optimal control on networks in which there are entry (or exit) costs at the edges of the network leading to a possible discontinuous value function. The value function is characterized as the unique viscosity solution of a Hamilton-Jacobi (HJ) equation for which an adequate junction condition is established.

The second part is about stochastic MFG for which the state space is a network in the ergodic case. They are described by a system coupling a HJB equation and a Fokker-Planck equation, whose unknowns are: the density m which is in general discontinuous at the vertices and satisfies dual transmission condition; the value function v is continuous and satisfies general Kirchhoff conditions at the vertices and the ergodic constant ρ. Existence and uniqueness are proven for subquadratic Hamiltonian and the general coupling term which is bounded from below.

Finally, in the last part, we study non-stationary stochastic MFG on networks. The transition conditions for v and m are similar to the ones given in second part. We prove the existence and uniqueness of a weak solution for sublinear Hamiltonian and bounded non-local regularizing coupling term. The main additional difficulty compared to the stationary case is to establish the regularity of the system. Our approach is to study the solution of the derived HJ equation to gain regularity over the initial equation.