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On the vanishing viscosity limit for the 2D incompressible Navier–Stokes equations with the friction type boundary conditions

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Abstract. The vanishing viscosity limit is considered for the incompressible 2D Navier–Stokes equations in a bounded domain. Motivated by studies of turbulent flow we suppose Navier's friction condition in the tangential direction, i.e. the creation of a vorticity proportional to the tangential velocity. We prove the existence of the regular solutions for the Navier–Stokes equations with smooth compatible data and of the solutions with bounded vorticity for initial vorticity being only bounded. Finally, we establish a uniform L^∞-bound for the vorticity and convergence to the incompressible 2D Euler equations in the inviscid limit.

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1. Introduction

The investigation of the inviscid limit of solutions of the Navier–Stokes equations is a classical issue. Some studies handle the case where the fluid domain has no boundary (the whole space or periodic geometry), see for example [4, 5]. In such cases, we do not encounter particular difficulties caused by the classical no-slip boundary condition $u = 0$, which is responsible for the formation of the turbulent boundary layer. An accurate description of the inviscid limit behaviour in that case is still an open question. A partial answer in the case of a half-space and analytic initial data is given in [12], where the convergence towards the coupled Euler and Prandtl equations is established for a sufficiently small time interval. For early partial results we refer to the work of Fife [6] and references therein. Let us note the special case of the so-called 'free boundary' conditions ($u \cdot \nu = 0$ and $\omega = \operatorname{curl} u = 0$) which is studied in [8], note also the recent works of Temam and Wang [15, 16] on the limit for non-stationary Stokes and Oseen equations.

From a physical point of view, the no-slip boundary condition $u = 0$ is only justified where the molecular viscosity is concerned. But in many cases of practical significance, for example in 2D geophysical models, the interpretation of the viscosity is not clear since it accounts for various turbulent effects at small scale (see Pedlosky [11]). In such cases the no-slip boundary condition is clearly unjustified and various kinds of boundary conditions have been tried.

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We propose studying here the following boundary condition: the tangential component of the viscous stress at the boundary is proportional to the tangential velocity. This condition which was first proposed by Navier [9] is often called the Navier slip boundary condition, see also the comments by Serrin [13]. This condition has also been considered more recently as a physical model for the turbulent boundary layer in the case of a rough surface body (see Parés [10]) where a more general boundary condition related to the modelling of turbulent flows is studied.

The boundary layer scenario which is generally believed to hold for the vanishing viscosity limit with the no-slip boundary condition is the following. If we compare the Navier–Stokes flow with the Euler flow corresponding to the same initial data, the presence of the (small) viscosity term $\mu\Delta u$ and the boundary condition $u = 0$ will drastically modify the flow near the wall in a region with thickness proportional to $\sqrt{\mu}$, the region in which the flow is modified may separate from the boundary, this separation will act as a source of vorticity and its effect may persists in the limit $\mu \to 0$. Our result clearly shows that this does not occur in the case of Navier's slip boundary condition: even if there is some creation of vorticity near the wall, in the limit the flow is well described by Euler equations.

2. The Navier–Stokes equations with Navier's slip boundary conditions

In this section we consider the 2D incompressible Navier–Stokes equations

$$\partial_t u - \mu\Delta u + (u.\nabla)u + \nabla p = f \qquad \text{in } \Omega \times (0, T) \tag{2.1}$$

$$\operatorname{div} u = 0 \qquad \text{in } \Omega \times (0, T) \tag{2.2}$$

$$u(0, \cdot) = u_0 \qquad \text{in } \Omega \tag{2.3}$$

$$u \cdot \nu = 0 \qquad \text{on } \partial\Omega \times (0, T) \tag{2.4}$$

$$2D(u)\nu \cdot \tau + \alpha u \cdot \tau = 0 \qquad \text{on } \partial\Omega \times (0, T) \tag{2.5}$$

where $\mu > 0$ is the coefficient of kinematic viscosity, $\alpha(x)$ is a given positive twice continuously differentiable function defined on $\partial\Omega$, u is the velocity and p is the pressure. $D(u)$ is the rate of strain tensor defined by $D_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$; we will systematically use the notation $A : B$ for $\sum_{i,j} A_{ij}B_{ij}$. We suppose that Ω is a bounded simply connected domain in $\mathbb{R}^2$, $\partial\Omega$ is sufficiently regular and ν and τ are the unit exterior normal and the unit tangent vector, respectively, $\{\nu, \tau\}$ being a direct basis. $f(x, t)$ is a given force and u_0 is an initial velocity.

The existence of a weak solution to (2.1)–(2.5) is straightforward. However, our final goal is to study the limit $\mu \to 0$ and we need some regularity of $\omega = \operatorname{curl} u$. We note that if one fixes vorticity on $\partial\Omega \times (0, T)$ instead of (2.5) then the regularity properties and the vanishing viscosity limit are classical (see, e.g. Lions [8]). Such a boundary condition differs at first sight from (2.5). In order to use ideas from that case we obtain an equivalent form of (2.5) giving a relation between the vorticity ω, the tangential component of the normal viscous stress and the tangential velocity $u \cdot \tau$ on $\partial\Omega \times (0, T)$.

For the mathematical setting of (2.1)–(2.5) we introduce the Hilbert spaces

$$L^2_0(\Omega) = \{z \in L^2(\Omega) : \int_\Omega z \, \mathrm{d}x = 0\}$$

$$V = \{v \in H^1(\Omega)^2 : \text{ div } v = 0 \qquad \text{in } \Omega \text{ and } v \cdot \nu = 0 \text{ on } \partial\Omega\}$$

$$H = \{v \in L^2(\Omega)^2 : \text{ div } v = 0 \qquad \text{in } \Omega \text{ and } v \cdot \nu = 0 \text{ on } \partial\Omega\}$$

$$\mathcal{W} = \{v \in V \cap H^2(\Omega)^2 : 2D(v)\nu \cdot \tau + \alpha v \cdot \tau = 0 \text{ on } \partial\Omega\}.$$

We have the following auxiliary result.

Lemma 2.1. *Suppose $v \in H^2(\Omega)^2$, $v \cdot \nu = 0$ on $\partial\Omega$. Then we have*

$$D(v)\nu \cdot \tau - \tfrac{1}{2}\operatorname{curl} v + \kappa(v \cdot \tau) = 0 \qquad \text{on } \partial\Omega \tag{2.6}$$

where $\operatorname{curl} v = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$ *and κ is the curvature of $\partial\Omega$ given in a standard way by $\frac{\mathrm{d}\tau}{\mathrm{d}s} = -\kappa\nu$.*

Proof. Using the density in $\{v \in H^2(\Omega)^2 : v \cdot \nu = 0 \text{ on } \partial\Omega\}$ of the velocity fields $v \in C^\infty(\bar{\Omega})^2$ such that $v \cdot \nu = 0$ on $\partial\Omega$, and the continuity of the trace operators, it suffices to handle the case where v is a smooth velocity field on $\bar{\Omega}$. Now, after extending $\nu(x)$ to a tubular neighbourhood of $\partial\Omega$, we obtain

$$0 = \frac{\partial}{\partial\tau}(v \cdot \nu) = D(v)\nu \cdot \tau - \frac{1}{2}\operatorname{curl} v + \nabla\nu\tau \cdot \tau(v \cdot \tau) \qquad \text{on } \partial\Omega.$$

After passing to a parametrization of the boundary we find

$$\nabla\nu\tau \cdot \tau = \operatorname{div} \nu = \kappa.$$

This proves the lemma. □

Then lemma 2.1 implies the following.

Lemma 2.2. *There exists a basis $\{v_n\} \subset H^3(\Omega)^2$, for V, which is also an orthonormal basis for H, which satisfies*

$$2D(v_n)\nu \cdot \tau + \alpha v_n \cdot \tau = 0 \qquad \text{on } \partial\Omega. \tag{2.7}$$

Proof. We consider the spectral problem

$$\begin{cases} \Delta^2\psi = -\lambda\Delta\psi & \text{in } \Omega \\ -\Delta\psi = -(2\kappa - \alpha)\nabla\psi \cdot \nu & \text{on } \partial\Omega \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

Its variational form reads:

find $\psi \in V_2 = H^2(\Omega) \cap H_0^1(\Omega)$ and $\lambda \neq 0$ such that

$$\int_\Omega \Delta\psi\,\Delta\varphi\,\mathrm{d}x - \int_{\partial\Omega}(2\kappa - \alpha)\nabla\psi \cdot \nu\nabla\varphi \cdot \nu\,\mathrm{d}S = \lambda\int_\Omega \nabla\psi\nabla\varphi\,\mathrm{d}x \qquad \forall\varphi \in V_2.$$

Obviously the injection $V_2 \hookrightarrow H_0^1(\Omega)$ is dense, continuous and compact. The bilinear form $a(z, \varphi) = \int_\Omega \Delta z\Delta\varphi\,\mathrm{d}x - \int_{\partial\Omega}(2\kappa - \alpha)\nabla z \cdot \nu\nabla\varphi \cdot \nu\,\mathrm{d}S$ is continuous and symmetric on $V_2 \times V_2$. Moreover, using the trace theorem and interpolation between V_2 and $H_0^1(\Omega)$ we find that the form a is $(V_2, H_0^1(\Omega))$-coercive. Now, the variational spectral theory is applicable and (2.8) has a countable set of eigenvalues $\{\lambda_j\}$, with $\lambda_j \to +\infty$ as $j \to +\infty$. The corresponding eigenfunctions $\{\psi_j\}$ form an orthonormal basis for $H_0^1(\Omega)$ and a basis for V_2.

Therefore $\Delta\psi_j \in L^2(\Omega)$ and consequently $\Delta^2\psi_j \in L^2(\Omega)$. Now $\omega_j = -\Delta\psi_j \in H^0(\Omega, \Delta) = \{z \in L^2(\Omega) : \Delta z \in L^2(\Omega)\}$. The theory of weak traces (see, e.g. [2, ch 13]) implies that the zeroth-order trace of a function from $H^0(\Omega, \Delta)$ exists and is an element of $H^{-1/2}(\partial\Omega)$. Since $(2\kappa - \alpha)\nabla\psi_j \cdot \nu \in H^{1/2}(\partial\Omega)$, ω_j is a solution for Dirichlet's problem for the Laplacian, with the forcing term from $L^2(\Omega)$ and the trace from $H^{1/2}(\partial\Omega)$. Hence $\omega_j = -\Delta\psi_j \in H^1(\Omega)$ and after a bootstrap argument we obtain $\omega_j = -\Delta\psi_j \in H^2(\Omega)$.

Every function ω_j satisfies

$$\begin{cases} -\Delta\omega_j = \lambda_j\omega_j & \text{in } \Omega \\ \omega_j = -(2\kappa - \alpha)\nabla\psi_j \cdot \nu & \text{on } \partial\Omega \end{cases} \tag{2.9}$$

and we introduce the velocities v_j by $v_j = \overrightarrow{\text{curl}}\psi_j$. Then $\operatorname{div} v_j = 0$, $\omega_j = \operatorname{curl} v_j$ in Ω and $v_j \cdot \nu = 0$ on $\partial\Omega$. Moreover, after recalling that Ω is simply connected and using lemma 2.1, we discover that there exists a unique pressure field $\pi_j \in H^2(\Omega) \cap L^2_0(\Omega)$ such that

$$\begin{cases} -\Delta v_j + \nabla\pi_j = \lambda_j v_j & \text{in } \Omega \\ \operatorname{div} v_j = 0 & \text{in } \Omega \\ v_j \cdot \nu = 0 & \text{on } \partial\Omega \\ 2D(v_j)\nu \cdot \tau + \alpha v_j \cdot \tau = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

Since $\{\psi_j\}$ is a basis for V_2; $\{v_j\}$ is a basis for V. Obviously, the elements of this basis satisfy (2.7). □

This basic result replaces the density of $\mathcal{V} = \{v \in C_0^\infty(\Omega)^2 : \operatorname{div} v = 0 \text{ in } \Omega\}$ in the corresponding subspace of $H_0^1(\Omega)^2$ from the classical case. We note that $\mathcal{V}$ is not dense in V. Now we extend classical results for the 2D incompressible Navier–Stokes equations to the case of the boundary condition (2.5).

Theorem 2.3. *For a given $f \in H^1(0, T; H)$ and $u_0 \in \mathcal{W}$, there exists a unique function $u \in L^2(0, T; V) \cap L^\infty(0, T; H)$, $\partial_t u \in L^2(0, T; V) \cap L^\infty(0, T; H)$ which satisfies the variational form of (2.1)–(2.5), i.e.*

$$\forall v \in V,\ \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t)v\,\mathrm{d}x + 2\mu\int_\Omega D(u(t)) : D(v)\,\mathrm{d}x + \int_\Omega (u(t).\nabla)u(t)v\,\mathrm{d}x$$
$$+\mu\int_{\partial\Omega}\alpha(u(t)\cdot\tau)(v\cdot\tau)\,\mathrm{d}S = \int_\Omega f(t)v\,\mathrm{d}x \tag{2.11}$$
$$u(0) = u_0. \tag{2.12}$$

Moreover, let in addition $\operatorname{curl} u_0 \in L^\infty(\Omega)$ *and* $\operatorname{curl} f \in L^2(0, T; L^\infty(\Omega))$. *Then $u \in C([0, T]; \mathcal{W})$ and $\omega = \operatorname{curl} u \in C([0, T]; H^1(\Omega)) \cap L^\infty(\Omega \times (0, T))$. Finally, there exists a unique pressure field $p \in C([0, T]; H^1(\Omega) \cap L_0^2(\Omega))$ such that (2.1)–(2.5) holds a.e. on $\Omega \times (0, T)$.*

Proof. The proof is an adaptation of corresponding proofs from [8, 14]. We only give a sketch of it.

We take an $H^2(\Omega)^2$-orthonormal basis $\{v_j\}$ for $\mathcal{W}$. It is also a basis for V by lemma 2.2. For each N we search an approximate solution u_N of (2.11) and (2.12) as follows

$$u_N(t) = \sum_{j=1}^N g_{jN}(t)v_j \tag{2.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_N(t)v_k\,\mathrm{d}x + 2\mu\int_\Omega D(u_N(t)) : D(v_k)\,\mathrm{d}x + \int_\Omega (u_N(t).\nabla)u_N(t)v_k\,\mathrm{d}x$$
$$+\mu\int_{\partial\Omega}\alpha(u_N(t)\cdot\tau)(v_k\cdot\tau)\,\mathrm{d}S = \int_\Omega f(t)v_k\,\mathrm{d}x \quad k = 1, \ldots, N \tag{2.14}$$

$$u_N(0) = \sum_{j=1}^N (u_0, v_j)_{H^2(\Omega)^2} v_j = u_{0N}. \tag{2.15}$$

This is a system of nonlinear ordinary differential equations for the functions $\{g_{jN}\}_{1\leqslant j\leqslant N}$. It contains a quadratic nonlinearity, and has a unique maximal solution $g_{jN} \in C^1[0, T_N[$, for some small time interval $[0, T_N[\subset [0, T]$.

Hence, there is a unique solution $u_N \in C^1([0, T_N[; H^2(\Omega)^2)$ for (2.13)–(2.15), for all N.

In the next step we get the *a priori* estimates. First we have

$$\frac{1}{2}\int_\Omega u_N(t)^2\,\mathrm{d}x + 2\mu\int_0^t \mathrm{d}\vartheta\int_\Omega |D(u_N(\vartheta))|^2\,\mathrm{d}x + \mu\int_0^t \mathrm{d}\vartheta\int_{\partial\Omega}\alpha|u_N(\vartheta)\cdot\tau|^2\,\mathrm{d}S$$
$$\leqslant \frac{1}{2}\int_\Omega u_{0N}^2\,\mathrm{d}x + \|f\|_{L^1(0,t;L^2(\Omega)^2)}\cdot\|u_N\|_{L^\infty(0,t;L^2(\Omega)^2)} \qquad \text{for } t < T_N.$$

Since $\alpha \geqslant 0$ we obtain immediately

$$\|u_N\|_{L^\infty(0,t;L^2(\Omega)^2)} \leqslant \sqrt{2}\|u_{0N}\|_{L^2(\Omega)^2} + 2\|f\|_{L^1(0,T;L^2(\Omega)^2)}, \tag{2.16}$$

from where we deduce that u_N exists on $[0, T]$ and the estimate holds for $t = T$. Next one obtains

$$\sqrt{\mu}\|u_N\|_{L^2(0,T;V)} \leqslant C(\Omega, T)\{\|u_{0N}\|_{L^2(\Omega)^2} + \|f\|_{L^2((0,T)\times\Omega)^2}\}. \tag{2.17}$$

The estimate for $\partial_t u_N$ is analogous to those from Temam [14, theorem 3.5]. We have

$$\int_\Omega\left|\frac{\mathrm{d}}{\mathrm{d}t}u_N(t)\right|^2\,\mathrm{d}x + 2\mu\int_\Omega D(u_N(t)) : D\left(\frac{\mathrm{d}}{\mathrm{d}t}u_N(t)\right)\,\mathrm{d}x + \mu\int_{\partial\Omega}\alpha u_N(t)\cdot\tau\frac{\mathrm{d}u_N(t)}{\mathrm{d}t}\cdot\tau\,\mathrm{d}S$$
$$+\int_\Omega (u_N(t).\nabla)u_N(t)\frac{\mathrm{d}}{\mathrm{d}t}u_N(t)\,\mathrm{d}x = \int_\Omega f(t)\frac{\mathrm{d}}{\mathrm{d}t}u_N(t)\,\mathrm{d}x.$$

We take $t = 0$ and after noticing that $u_N(0) \in \mathcal{W}$ implies

$$2\mu\int_\Omega D(u_N(0)) : D\left(\frac{\mathrm{d}}{\mathrm{d}t}u_N(0)\right)\,\mathrm{d}x + \mu\int_{\partial\Omega}\alpha u_N(0)\cdot\tau\frac{\mathrm{d}u_N(0)}{\mathrm{d}t}\cdot\tau\,\mathrm{d}S$$
$$= -\mu\int_\Omega \Delta u_N(0)\frac{\mathrm{d}u_N(0)}{\mathrm{d}t}\,\mathrm{d}x$$

we obtain

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}u_N(0)\right\|_{L^2(\Omega)^2} \leqslant C\max\{1, \mu\}\{1 + \|f(0)\|_{L^2(\Omega)^2} + \|u_{0N}\|^2_{H^2(\Omega)^2}\} \tag{2.18}$$

where C does not depend on μ, f and u_0.

What follows is the same as in the classical case of the no-slip condition: we differentiate (2.14) with respect to t and obtain the following estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left|\frac{\mathrm{d}}{\mathrm{d}t}u_N(t)\right|^2\,\mathrm{d}x + 2\mu\int_\Omega\left|D\left(\frac{\mathrm{d}}{\mathrm{d}t}u_N(t)\right)\right|^2\,\mathrm{d}x + \mu\int_{\partial\Omega}\alpha\left|\frac{\mathrm{d}u_N(t)}{\mathrm{d}t}\cdot\tau\right|^2\,\mathrm{d}S$$
$$\leqslant \left\|\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right\|_{L^2(\Omega)^2}\left\|\frac{\mathrm{d}u_N(t)}{\mathrm{d}t}\right\|_{L^2(\Omega)^2} + C\left\|\frac{\mathrm{d}u_N(t)}{\mathrm{d}t}\right\|^2_{H^{1/2}(\Omega)^2}\|\nabla u_N(t)\|_{L^2(\Omega)^4}. \tag{2.19}$$

Using Korn's inequality, interpolation between Sobolev spaces, (2.17), (2.18) and Gronwall's inequality we obtain

$$\frac{\mathrm{d}u_N(t)}{\mathrm{d}t}\text{belongs to a bounded set of } L^\infty(0, T; H)\cap L^2(0, T; V). \tag{2.20}$$

In complete analogy with the classical situation, estimates (2.16), (2.17) and (2.20) are sufficient for passing to the limit.

The existence and uniqueness of the corresponding pressure field is standard.

It remains to prove that $\omega = \operatorname{curl} u \in C([0,T]; H^1(\Omega)) \cap L^\infty(\Omega \times (0,T))$ and $u \in C([0,T]; H^2(\Omega)^2)$.

First, using the equation for $\partial_t u$ we find that $\partial_t u \in L^2(0,T; V)$ and $\partial_{tt} u \in L^2(0,T; V')$. Consequently, $\partial_t u \in C([0,T]; H)$. Let β be a positive number such that

$$2\mu \int_\Omega D(v) : D(v)\,\mathrm{d}x + \beta \int_\Omega |v|^2\,\mathrm{d}x + 2\mu \int_{\partial\Omega} \kappa |v\cdot\tau|^2\,\mathrm{d}S \geqslant \int_\Omega |v|^2\,\mathrm{d}x \text{ for } \forall v \in V.$$

Furthermore, let $\Phi = -(u.\nabla)u - \partial_t u + f + \beta u$. Then $u \in C^{1/2}([0,T]; H^1(\Omega)^2)$ and $\partial_t u \in C([0,T]; H)$ imply $\Phi \in C([0,T]; L^q(\Omega)^2)$, $\forall q < 2$, and $g = \left(\kappa - \frac{\alpha}{2}\right) u \cdot \tau \in C^{1/2}([0,T]; H^{1/2}(\partial\Omega)^2)$.

Therefore, for every $t \in [0,T]$, u is the unique variational solution for the following Stokes problem

$$\begin{cases} -\mu\Delta u + \nabla p + \beta u = \Phi & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u\cdot\nu = 0 & \text{on } \partial\Omega \\ D(u)\nu\cdot\tau + \kappa u\cdot\tau = g & \text{on } \partial\Omega. \end{cases} \tag{2.21}$$

We wish to establish L^q-estimates for the second derivatives of u. It turns out to be much simpler than in the classical case of the no-slip condition on $\partial\Omega$.

First, using the elliptic theory we obtain the unique solvability of the auxiliary problem

$$\begin{cases} -\mu\triangle w + \beta w = \operatorname{curl}\, \Phi & \text{on } \Omega \\ w = 2g & \text{on } \partial\Omega \end{cases} \tag{2.22}$$

in $C([0,T]; W^{1,q}(\Omega))$.

Next we consider the problem

$$\begin{cases} -\triangle\psi = w & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega \end{cases}$$

with w given by (2.22). Then the theory from Grisvard [7] implies the existence of a unique solution $\psi \in C([0,T]; W^{3,q}(\Omega))$. Let us introduce v by $v = \overrightarrow{\operatorname{curl}}\psi$. Then $v \in C([0,T]; W^{2,q}(\Omega)^2)$, $\operatorname{div} v = 0$ and $\operatorname{curl} v = w$ in Ω. Furthermore, $v\cdot\nu = 0$ and $D(v)\nu\cdot\tau + \kappa v\cdot\tau = g$ on $\partial\Omega$. Finally, we find that v is a variational solution for (2.21) and, consequently, $v = u$ on $\Omega \times [0,T]$.

Hence $u \in C([0,T]; W^{2,q}(\Omega)^2)$, $\forall q \in (1,2)$. Applying a standard bootstrap argument we get $u \in C([0,T]; V \cap H^2(\Omega)^2)$, $\omega = \operatorname{curl} u \in C([0,T]; H^1(\Omega))$ and $\partial_t\omega \in L^2(\Omega \times (0,T))$.

Finally, ω is the unique solution of the parabolic problem

$$\begin{cases} \partial_t\omega - \mu\Delta\omega + (u.\nabla)\omega = \operatorname{curl} f & \text{in } \Omega \times (0,T) \\ \omega(0) = \operatorname{curl} u_0 & \text{in } \Omega \\ \omega = (2\kappa - \alpha) u\cdot\tau & \text{on } \partial\Omega \times (0,T) \end{cases} \tag{2.23}$$

for a given $u \in C([0,T]; V \cap H^2(\Omega)^2)$.

The maximum principle for the smooth solutions to the scalar parabolic equations is classical; here we need an L^∞-estimate for the mild solution ω for (2.23). Let us give a brief direct estimate.

Let $\omega_1 = \omega - \int_0^t \|\operatorname{curl} f(\vartheta)\|_{L^\infty(\Omega)}\,\mathrm{d}\vartheta - C_0$, where $C_0 = \|\operatorname{curl} u_0\|_{L^\infty(\Omega)} + \|2\kappa - \alpha\|_{L^\infty(\Omega)}\|u\cdot\tau\|_{L^\infty(\partial\Omega\times(0,T))}$. Then ω_1 satisfies equation (2.23) but with the forcing term $g_1 =$

$\operatorname{curl} f - \|\operatorname{curl} f(t)\|_{L^\infty(\Omega)} \leqslant 0$ (a.e.) on $\Omega \times (0, T)$ instead of curl f and with the trace $h_1 = (2\kappa - \alpha)u \cdot \tau - C_0 - \int_0^t \|\operatorname{curl} f(\vartheta)\|_{L^\infty(\Omega)} \,\mathrm{d}\vartheta \leqslant 0$ (a.e.) on $\partial\Omega \times (0, T)$.

Using the standard maximum principle we get $\omega_1(t) \leqslant 0$ (a.e.) on Ω for all $t \in [0, T]$. Hence,

$$\omega(t) \leqslant \int_0^t \|\operatorname{curl} f(\vartheta)\|_{L^\infty(\Omega)} \,\mathrm{d}\vartheta + \|\operatorname{curl} u_0\|_{L^\infty(\Omega)} + \|2\kappa - \alpha\|_{L^\infty(\partial\Omega)} \|u \cdot \tau\|_{L^\infty(\partial\Omega \times (0,T))}.$$

The lower bound is proved analogously. Therefore, we have obtained the estimate

$$\begin{aligned} \|\omega\|_{L^\infty(\Omega \times (0,T))} \leqslant & \int_0^T \|\operatorname{curl} f(t)\|_{L^\infty(\Omega)} \mathrm{d}t + \|\operatorname{curl} u_0\|_{L^\infty(\Omega)} \\ & + \|2\kappa - \alpha\|_{L^\infty(\partial\Omega)} \|u \cdot \tau\|_{L^\infty(\partial\Omega \times (0,T))}. \end{aligned} \tag{2.24}$$

This proves the theorem. □

Remark. It should be emphasized that estimate (2.16) is independent of μ. Estimate (2.24) will be transformed into an estimate independent of μ in section 3.

3. Vanishing viscosity limit

As already mentioned in the introduction, passing to the vanishing viscosity limit is in general possible only for incompressible 2D flows without boundaries. The presence of a boundary condition complicates the situation significantly, since it acts as a source of vorticity. This can be avoided by setting the so-called 'free boundary' conditions $\omega = 0$ and $u \cdot \nu = 0$ at the boundary, but in many situations it is not realistic. Our boundary conditions are simpler than the classical no-slip ones since we have some information about the creation of vorticity at the boundary. Nonetheless, the boundary creates a vorticity proportional to the size of the tangential velocity and we loose most of the estimates from the 'free boundary condition' case. Noticeably, we no longer control $\sqrt{\mu}\nabla\omega^\mu$ and the idea is to control the L^∞-norm of the vorticity by interpolation and using the estimate (2.16).

We now establish the basic *a priori* estimates for solutions of (2.11), (2.12) required for the ensuing passing to the limit $\mu \to 0$.

Proposition 3.1. *Let $f \in H^1(0, T; H)$, with $\operatorname{curl} f \in L^2(0, T; L^\infty(\Omega))$, and $u_0 \in \mathcal{W}$, $\operatorname{curl} u_0 \in L^\infty(\Omega)$. Let $u^\mu \in L^\infty(0, T; V)$ be the solution of (2.11), (2.12) and let $\omega^\mu = \operatorname{curl} u^\mu \in L^\infty(\Omega \times (0, T))$ be the corresponding vorticity field. Then $\forall\mu > 0$ and we have*

$$\|u^\mu\|_{L^\infty(0,T;L^2(\Omega)^2)} + \sqrt{\mu}\|u^\mu\|_{L^2(0,T;V)} \leqslant C\{\|u_0\|_{L^2(\Omega)^2} + \|f\|_{L^2((0,T)\times\Omega)^2}\} \tag{3.1}$$

$$\|\omega^\mu\|_{L^\infty(\Omega\times(0,T))} \leqslant CE \tag{3.2}$$

$$\|\partial_t\omega^\mu\|_{L^2(0,T;(H^2(\Omega)\cap H_0^1(\Omega))')} \leqslant C(E + E^2) \tag{3.3}$$

where

$$E = \|u_0\|_{L^2(\Omega)^2} + \|\operatorname{curl} u_0\|_{L^\infty(\Omega)} + \|\operatorname{curl} f\|_{L^2(0,T;L^\infty(\Omega))} + \|f\|_{L^2(\Omega\times(0,T))^2}$$

and C depends only on Ω and T.

Proof. First, the estimate (3.1) is an obvious consequence of (2.16).

Our second step concerns (3.2). We now proceed by reconsidering the bound (2.24). It reads

$$\|\omega^\mu\|_{L^\infty(\Omega\times(0,T))} \leqslant A + B\|u^\mu \cdot \tau\|_{L^\infty(\partial\Omega\times(0,T))}$$

where

$$A = \|\operatorname{curl} u_0\|_{L^\infty(\Omega)} + \sqrt{T}\|\operatorname{curl} f\|_{L^2(0,T;L^\infty(\Omega))}$$

and

$$B = 2\|\kappa\|_{L^\infty(\partial\Omega)} + \|\alpha\|_{L^\infty(\partial\Omega)}.$$

On the other hand, $u^\mu = \overrightarrow{\operatorname{curl}}\psi^\mu$ and $\omega^\mu = -\Delta\psi^\mu$, where $\psi^\mu \in W^{2,q}(\Omega)\cap H_0^1(\Omega)$ is the corresponding stream function. Hence

$$\|\Delta\psi^\mu\|_{L^\infty(\Omega\times(0,T))} \leqslant A + B\left\|\frac{\partial\psi^\mu}{\partial\nu}\right\|_{L^\infty(\partial\Omega\times(0,T))}. \tag{3.4}$$

Let $q \in (2, +\infty)$ be a given number. Then we have

$$\begin{aligned}\left\|\frac{\partial\psi^\mu(t)}{\partial\nu}\right\|_{L^\infty(\partial\Omega)} &\leqslant C\|\nabla\psi^\mu(t)\|_{C(\overline{\Omega})^2}\\ &\leqslant C\|\nabla\psi^\mu(t)\|^\theta_{L^2(\Omega)^2}\|\psi^\mu(t)\|^{1-\theta}_{W^{2,q}(\Omega)}\end{aligned} \tag{3.5}$$

where $\theta = \frac{1}{2}\frac{q-2}{q-1}$. We refer to [1, ch 5], for the proof of this classical interpolation inequality. Moreover,

$$\|\nabla\psi^\mu(t)\|_{L^2(\Omega)^2} = \|u^\mu(t)\|_{L^2(\Omega)^2} \tag{3.6}$$

and

$$\|\psi^\mu(t)\|_{W^{2,q}(\Omega)} \leqslant C(\Omega, q)\|\Delta\psi^\mu(t)\|_{L^q(\Omega)}. \tag{3.7}$$

Inserting (3.6) and (3.7) into (3.5) gives

$$\left\|\frac{\partial\psi^\mu(t)}{\partial\nu}\right\|_{L^\infty(\partial\Omega)} \leqslant C\|u^\mu(t)\|^\theta_{L^2(\Omega)^2}\|\Delta\psi^\mu(t)\|^{1-\theta}_{L^\infty(\Omega)} \qquad \text{(a.e.) on } (0,T) \tag{3.8}$$

and after substituting (3.8) into (3.4) we obtain

$$\|\Delta\psi^\mu\|_{L^\infty(\Omega\times(0,T))} \leqslant \frac{1}{\theta}A + (CB)^{1/\theta}\|u^\mu\|_{L^\infty(0,T;L^2(\Omega)^2)}.$$

This proves (3.2).

It now remains to prove (3.3). We use the equation for the vorticity ω^μ, which gives

$$\partial_t\omega^\mu = \mu\Delta\omega^\mu - \operatorname{div}(u^\mu\omega^\mu) + \operatorname{curl} f \tag{3.9}$$

and (3.3) follows from (3.1), (3.2) and (3.9). This completes the proof. □

The uniform estimates obtained in proposition 3.1 are weaker than corresponding ones in the case of the so-called 'free boundary condition' $\omega = 0$, $u\cdot\nu = 0$ on $\partial\Omega$ (see [8]), since we have no information on the behaviour of $\nabla\omega^\mu$. However, they are sufficient for passing to the limit and we have the following result.

Theorem 3.2. *Let* $f \in H^1(0, T; H)$, *with* $\operatorname{curl} f \in L^2(0, T; L^\infty(\Omega))$, *and* $u_0 \in \mathcal{W}$, $\operatorname{curl} u_0 \in L^\infty(\Omega)$. *Let* $u^\mu \in L^\infty(0, T; V)$ *be the corresponding solution for (2.11), (2.12) and* $\omega^\mu = \operatorname{curl} u^\mu \in L^\infty(\Omega \times (0, T))$. *Then we have*

$$u^\mu \to u \qquad \text{in } L^q(0, T; W^{\alpha,q'}(\Omega)^2) \qquad \text{for } \alpha \in (0, 1), q, q' \in (1, +\infty)$$
$$u^\mu \rightharpoonup u \qquad \text{in } L^2(0, T; V) \qquad \text{weakly}$$
$$\omega^\mu \overset{\star}{\rightharpoonup} \omega = \operatorname{curl} u \qquad \text{in } L^\infty(\Omega \times (0, T)) \qquad \text{weak-}\star$$

as $\mu \to 0$, *where* $\{u, \omega\}$ *is the unique solution to the 2D incompressible Euler system*

$$\partial_t \omega + \operatorname{div}(u\omega) = \operatorname{curl} f \qquad \text{in } \Omega \times (0, T)$$
$$\operatorname{div} u = 0 \qquad \text{in } \Omega \times (0, T)$$
$$\operatorname{curl} u = \omega \qquad \text{in } \Omega \times (0, T)$$
$$u \cdot \nu = 0 \qquad \text{on } \partial\Omega \times (0, T)$$
$$\omega(0) = \operatorname{curl} u_0 \qquad \text{in } \Omega.$$

Proof. Using estimates (3.1)–(3.3) we deduce that the sequence $\{u^\mu\}$ is relatively strongly compact in $L^q(0, T; W^{\alpha,q'}(\Omega)^2)$, $1 < q, q' < +\infty$, $0 < \alpha < 1$. Therefore after passing to a suitable subsequence in a familiar fashion we obtain the desired convergences towards a velocity field $u \in L^2(0, T; V) \cap L^q(0, T; W^{\alpha,q'}(\Omega)^2)$, such that $\omega = \operatorname{curl} u \in L^\infty(\Omega \times (0, T))$. Uniqueness (and consequently the convergence of the whole sequence) is a consequence of Yudovich's uniqueness theorem for the 2D incompressible Euler equations (see [17]). $\square$

4. The case of non-smooth data

It is of importance to consider the case of a non-smooth initial vorticity $\omega_0 = \operatorname{curl} u_0$. In this direction we have the following result.

Theorem 4.1. *Let* $f \in L^2(0, T; H)$, *with* $\operatorname{curl} f \in L^2(0, T; L^\infty(\Omega))$, *and* $u_0 \in V$, $\operatorname{curl} u_0 \in L^\infty(\Omega)$. *Then there exists a unique solution* $u^\mu \in L^2(0, T; V)$ *with* $\partial_t u^\mu \in L^2(0, T; V')$ *and* $\omega^\mu = \operatorname{curl} u^\mu \in L^\infty(\Omega \times (0, T))$ *for the variational problem (2.11), (2.12). Furthermore,* $\{u^\mu, \omega^\mu\}$ *satisfies estimates (3.1)–(3.3).*

In order to prove theorem 4.1 we need the following lemma.

Lemma 4.2. *Let* $u_0 \in V$ *and* $\omega_0 = \operatorname{curl} u_0 \in L^\infty(\Omega)$. *Then there exists a sequence* $\{u_0^\delta\} \subset \mathcal{W}$, *with* $\omega_0^\delta = \operatorname{curl} u_0^\delta \in L^\infty(\Omega)$ *and satisfying*

$$\omega_0^\delta \to \omega_0 \qquad \text{in } L^2(\Omega) \qquad \text{and} \qquad \|\omega_0^\delta\|_{L^\infty(\Omega)} \leqslant C\|\omega_0\|_{L^\infty(\Omega)}. \tag{4.1}$$

Proof of lemma 4.2. First, we recall some well known facts.

We denote by $d(x) = d(x, \partial\Omega)$ the distance between the point $x \in \bar{\Omega}$ and the boundary $\partial\Omega$. Then d is uniformly Lipschitz continuous on $\bar{\Omega}$. For $\delta > 0$ we define $U_\delta = \{x \in \bar{\Omega} : d(x) < \delta\}$. For δ small enough the orthogonal projection $r : U_\delta \to \partial\Omega$ is a well-defined continuously differentiable mapping (it is enough to take δ such that U_δ is in a tubular neighbourhood of $\partial\Omega$, see e.g. [3]).

We extend $\omega_0 = \operatorname{curl} u_0$ by zero outside Ω and consider functions $\zeta_\delta \in C_0^\infty(\Omega)$ such that $0 \leqslant \zeta_\delta \leqslant 1$ and $\zeta_\delta = 1$ in a neighbourhood of $\Omega \setminus U_\delta$. Let J_δ be a mollifier.

Now we show that for any given $\delta > 0$, there is a unique function $G_\delta \in L^\infty(\partial\Omega)$ such that for

$$\omega_0^\delta = \zeta_\delta J_\delta * \omega_0 + (1 - \zeta_\delta(x))\mathrm{e}^{-d(x)/\delta} G_\delta(r(x)) \tag{4.2}$$

the overdetermined system

$$\begin{cases} \operatorname{div} u_0^\delta = 0 & \text{in } \Omega; \qquad \operatorname{curl} u_0^\delta = \omega_0^\delta \qquad \text{in } \Omega \\ u_0^\delta \cdot \nu = 0 & \text{on } \partial\Omega; \qquad \omega_0^\delta = (2\kappa - \alpha) u_0^\delta \cdot \tau \qquad \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

has a unique solution.

Indeed for any given $G \in L^\infty(\partial\Omega)$ we define $\Psi(G) \in L^\infty(\partial\Omega)$ by $\Psi(G) = (2\kappa - \alpha)u^\delta \cdot \tau$, where u^δ is the velocity field corresponding to the vorticity $\omega^\delta = \zeta_\delta J_\delta * \omega_0 + (1 - \zeta_\delta(x))\mathrm{e}^{-d(x)/\delta} G(r(x))$.

We now show that for δ small enough $\Psi : L^\infty(\partial\Omega) \to L^\infty(\partial\Omega)$ is a Lipschitz contractive mapping.

It should be noted that, since $\omega^\delta \in L^\infty(\Omega)$, we have $u^\delta \in W^{1,q}(\Omega)^2$ and $u^\delta \cdot \tau \in W^{1-1/q,q}(\partial\Omega) \subset L^\infty(\partial\Omega), \forall q \in (2, +\infty)$. Moreover, for $(G_1, G_2) \in L^\infty(\partial\Omega)^2$ we have

$$\begin{aligned} \|\Psi(G_1) - \Psi(G_2)\|_{L^\infty(\partial\Omega)} &\leqslant C\|\omega_1^\delta - \omega_2^\delta\|_{L^q(\Omega)} \leqslant C\|(G_1 - G_2)\mathrm{e}^{-d(x)/\delta}\|_{L^q(U_\delta)} \\ &\leqslant C\delta^{1/q}\|G_1 - G_2\|_{L^\infty(\partial\Omega)} \qquad q > 2. \end{aligned}$$

Thus, for δ small enough Ψ has a unique fixed point $G_\delta \in L^\infty(\partial\Omega)$. Of course $G_\delta \in W^{1-1/q,q}(\partial\Omega)$, for all $q \in (2, +\infty)$, so that $G_\delta \in C^{0,\alpha}(\partial\Omega)$, for all $\alpha \in (0, 1)$. Therefore, $\omega_0^\delta \in C^{0,\alpha}(\bar\Omega)$ and $u_0^\delta \in C^{1,\alpha}(\bar\Omega)^2$, so that $G_\delta \in C^1(\partial\Omega)$ and, obviously, $\omega_0^\delta \in H^1(\Omega)$.

From (4.2) we straightforwardly obtain

$$\|\omega_0^\delta\|_{L^\infty(\Omega)} \leqslant \|\omega_0\|_{L^\infty(\Omega)} + \|G_\delta\|_{L^\infty(\partial\Omega)}.$$

Furthermore, we have

$$\begin{aligned} \|G_\delta\|_{L^\infty(\Omega)} &\leqslant C\|u_0^\delta \cdot \tau\|_{L^\infty(\partial\Omega)} \leqslant C\|u_0^\delta\|_{W^{1,q}(\Omega)^2} \leqslant C\|\omega_0^\delta\|_{L^q(\Omega)} \\ &\leqslant C\{\|\omega_0\|_{L^\infty(\Omega)} + \delta^{1/q}\|G_\delta\|_{L^\infty(\partial\Omega)}\}. \end{aligned}$$

Thus for δ small enough we obtain

$$\|G_\delta\|_{L^\infty(\partial\Omega)} \leqslant C\|\omega_0\|_{L^\infty(\Omega)}.$$

The strong L^2-convergence of ω_0^δ towards ω_0 is obvious. □

Proof of theorem 4.1. We introduce f^δ, the mollification in time variable of f, as the forcing term. Now taking u_0^δ as the initial velocity and ω_0^δ as the initial vorticity, we have $u_0^\delta \in \mathcal{W}$. Thus, theorem 2.3 gives a unique solution $u^{\delta,\mu} \in C([0, T]; \mathcal{W})$, $\partial_t u^{\delta,\mu} \in L^2(0, T; V)$ and $\omega^{\delta,\mu} = \operatorname{curl} u^{\delta,\mu} \in C([0, T]; H^1(\Omega)) \cap L^\infty(\Omega \times (0, T))$ for the variational problem (2.11), (2.12). From proposition 3.1 we know that it satisfies estimates (3.1)–(3.3). By a simple transposition argument (3.3) yields

$$\|\partial_t u^{\delta,\mu}\|_{L^2(0,T;V')} + \|\partial_t \psi^{\delta,\mu}\|_{L^2(\Omega\times(0,T))} \leqslant C(E_\delta + E_\delta^2), \tag{4.4}$$

where

$$E_\delta = \|u_0^\delta\|_{L^2(\Omega)^2} + \|\operatorname{curl} u_0^\delta\|_{L^\infty(\Omega)} + \|\operatorname{curl} f^\delta\|_{L^2(0,T;L^\infty(\Omega))} + \|f^\delta\|_{L^2(\Omega\times(0,T))^2}.$$

Since (3.2) implies that $\|\omega^{\delta,\mu}\|_{L^\infty(\Omega\times(0,T))}$ remains bounded as $\delta \to 0$, we deduce that the sequence $\{u^{\delta,\mu}\}$ is relatively compact in $L^\ell(0,T;W^{s,q}(\Omega)^2)$, $1 < \ell, q < +\infty$, $0 < s < 1$. Therefore, after taking a subsequence if needed, we have

$$
\begin{aligned}
&u^{\delta,\mu} \to u^\mu \qquad \text{in } L^\ell(0,T;W^{s,q}(\Omega)^2) \qquad \text{strongly}\\
&u^{\delta,\mu} \rightharpoonup u^\mu \qquad \text{in } L^2(0,T;V) \qquad \text{weakly}\\
&\partial_t u^{\delta,\mu} \rightharpoonup \partial_t u^\mu \qquad \text{in } L^2(0,T;V') \qquad \text{weakly}\\
&\omega^{\delta,\mu} \overset{\star}{\rightharpoonup} \omega^\mu = \operatorname{curl} u^\mu \qquad \text{in } L^\infty(\Omega\times(0,T)) \qquad \text{weak-}\star
\end{aligned}
$$

where $u^\mu \in L^\infty(0,T;W^{1,q}(\Omega)^2)$, $\partial_t u^\mu \in L^2(0,T;V')$, $\omega^\mu \in L^\infty(\Omega\times(0,T))$, and $\partial_t\omega^\mu \in L^2(0,T;(H^2(\Omega)\cap H^1_0(\Omega))')$. u^μ obviously satisfies estimates (3.1)–(3.3) and it is the unique solution for (2.11), (2.12), so that the whole sequence converges and the theorem is proved. $\square$

A direct consequence of theorem 4.1 is that the estimates (3.1)–(3.3) also hold for the case of non-smooth data. Therefore, under the assumptions on the data from theorem 4.1, it is possible to pass to the zero viscosity limit and we have the following.

Theorem 4.3. *Let $f \in L^2(0,T;H)$, with $\operatorname{curl} f \in L^2(0,T;L^\infty(\Omega))$, and $u_0 \in V$, $\operatorname{curl} u_0 \in L^\infty(\Omega)$. Let $u^\mu \in L^\infty(0,T;V)$ be the corresponding solution for (2.11), (2.12) and $\omega^\mu = \operatorname{curl} u^\mu$. Then we have*

$$
\begin{aligned}
&u^\mu \to u \qquad \textit{in } L^q(0,T;W^{\alpha,q'}(\Omega)^2) \qquad \textit{for } \alpha\in(0,1),\ q,q'\in(1,+\infty)\\
&u^\mu \rightharpoonup u \qquad \textit{in } L^2(0,T;V) \qquad \textit{weakly}\\
&\omega^\mu \overset{\star}{\rightharpoonup} \omega = \operatorname{curl} u \qquad \textit{in } L^\infty(\Omega\times(0,T)) \qquad \textit{weak-}\star
\end{aligned}
$$

as $\mu \to 0$, where $\{u,\omega\}$ is the unique solution to the 2D incompressible Euler system

$$
\begin{aligned}
&\partial_t\omega + \operatorname{div}(u\omega) = \operatorname{curl} f \qquad \textit{in } \Omega\times(0,T)\\
&\operatorname{div} u = 0 \qquad \textit{in } \Omega\times(0,T)\\
&\operatorname{curl} u = \omega \qquad \textit{in } \Omega\times(0,T)\\
&u\cdot\nu = 0 \qquad \textit{on } \partial\Omega\times(0,T)\\
&\omega(0) = \operatorname{curl} u_0 \qquad \textit{in } \Omega.
\end{aligned}
$$

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