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On the asymptotic limit of the Navier–Stokes system on domains with rough boundaries

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Abstract

We study the asymptotic behavior of solutions to the incompressible Navier–Stokes system considered on a sequence of spatial domains, whose boundaries exhibit fast oscillations with amplitude and characteristic wave length proportional to a small parameter. Imposing the complete slip boundary conditions we show that in the asymptotic limit the fluid sticks completely to the boundary provided the oscillations are non-degenerate, meaning not oriented in a single direction.

1. Introduction

A proper choice of boundary conditions plays a significant role in the problems studied in continuum fluid dynamics. In many theoretical studies as well as numerical experiments, the standard well-accepted hypothesis states that a viscous fluid adheres completely to the boundary

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of the physical domain provided the latter is impermeable. If $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ is the Eulerian velocity of the fluid at a time t and a spatial position $\mathbf{x} \in \Omega \subset R^3$, the impermeability of the boundary $\partial\Omega$ means that

$$\mathbf{u}(t, \mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = 0 \quad \text{for any } \mathbf{x} \in \partial\Omega, \tag{1.1}$$

where $\mathbf{n}$ stands for the outer normal vector, while complete adherence can be formulated in terms of the no-slip boundary condition

$$\mathbf{u}(t, \mathbf{x}) = 0 \quad \text{for any } \mathbf{x} \in \partial\Omega. \tag{1.2}$$

Recently, there have been several attempts to give a rigorous mathematical justification of (1.2) based on the concept of rough or rugous boundary (see Casado Díaz et al. [5]). The main idea is to assume that the “real” boundary is never perfectly smooth but contains microscopic asperities of the size significantly smaller than the characteristic length scale of the flow. The “ideal” physical domain Ω is being replaced by a family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ of “rough” domains, where the parameter $\varepsilon > 0$ stands for the amplitude of asperities. Assuming only the impermeability condition (1.1) on $\partial\Omega_\varepsilon$ one can show that the stronger no-stick boundary conditions must be imposed for the limit problem when $\Omega_\varepsilon \to \Omega$ in some sense, provided the distribution of asperities is uniform, more specifically spatially periodic, and “non-degenerate” (see Theorem 1 in [5]). Although such a result can be legitimately viewed as a clear confirmation of (1.2) for viscous fluids, it seems to be at odds with a number of recent mostly numerical studies based on the scale analysis of the boundary layer, where the no-stick boundary conditions (1.2) on a *rough* boundary are replaced by “milder” ones of Navier-type (see Jaeger and Mikelic [6], Mohammadi et al. [10], Basson an Varet [3], among others). From the purely mathematical viewpoint, however, there is absolutely no contradiction, as the Navier-type conditions mentioned above always contain a “friction” term proportional to $\frac{1}{\varepsilon}$, yielding the no-slip condition (1.2) in the asymptotic limit $\varepsilon \to 0$.

In the framework of continuum fluid mechanics, the motion of a viscous incompressible fluid is governed by Navier–Stokes system, specifically, the equation of motion:

$$\partial_t \mathbf{u} + \operatorname{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \operatorname{div}_x \mathbb{S}, \tag{1.3}$$

supplemented with the continuity equation reduced in this particular situation to the incompressibility constraint

$$\operatorname{div}_x \mathbf{u} = 0. \tag{1.4}$$

Here the only state variable is the fluid velocity $\mathbf{u} = \mathbf{u}(t, x)$, while p stands for the pressure or rather the normal stress, and $\mathbb{S}$ denotes the viscous stress tensor. We focus on Newtonian fluids, where

$$\mathbb{S} = \mu\big(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^\perp\big), \tag{1.5}$$

with $\mu > 0$ being the viscosity coefficient.

In order to keep the presentation free of unnecessary technical details, we shall assume that all quantities are periodic with respect to the plain variables (x_1, x_2) with period $(1, 1)$. We point out, however, that no periodicity of the rugous boundary restricted to the unit square $(0, 1)^2$ is *a priori* assumed.

More specifically, system (1.3)–(1.5) will be considered on a family of bounded domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$,

$$\left.\begin{aligned} \Omega_\varepsilon = \bigl\{\mathbf{x} = (x_1, x_2, x_3) \bigm| \mathbf{y} = (x_1, x_2) \in \mathcal{T}^2,\ 0 < x_3 < 1 + \Phi_\varepsilon(x_1, x_2)\bigr\}, \\ \partial\Omega_\varepsilon = B \cup \Gamma_\varepsilon, \\ B = \bigl\{(x_1, x_2, x_3) \bigm| (x_1, x_2) \in \mathcal{T}^2,\ x_3 = 0\bigr\}, \\ \Gamma_\varepsilon = \bigl\{(x_1, x_2, x_3) \bigm| (x_1, x_2) \in \mathcal{T}^2,\ x_3 = 1 + \Phi_\varepsilon(x_1, x_2)\bigr\}, \end{aligned}\right\} \tag{1.6}$$

together with the no-slip boundary conditions

$$\mathbf{u}|B = 0 \tag{1.7}$$

imposed on the "bottom" part and the complete slip (no-stick) boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\Gamma_\varepsilon} = 0, \qquad (\mathbb{S}\mathbf{n}) \times \mathbf{n}|_{\Gamma_\varepsilon} = 0 \tag{1.8}$$

on the "top" Γ_ε. Here the symbol $\mathcal{T}^2 = ((0,1)|_{\{0,1\}})^2$ stands for the two-dimensional torus.

Under the main hypothesis

$$\Phi_\varepsilon \to 0 \quad \text{uniformly on } \mathcal{T}^2,$$

the main objective of the present paper is to identify the limit problem for $\varepsilon \to 0$. Although we do not assume any periodic structure finer than that indicated by the topology of $\mathcal{T}^2$, we focus on the situation when, loosely speaking, the tops Γ_ε are oscillating with "frequency" proportional to $1/\varepsilon$ and "amplitude" ε.

From the mathematical viewpoint, the problem splits in two rather independent tasks: (i) finding the limit system of equations, (ii) identifying the boundary conditions on the target domain

$$\Omega = \mathcal{T}^2 \times (0,1).$$

As for the former problem, one expects, of course, to recover the same system of equations to be satisfied by the limit velocity on the target domain Ω. Indeed any fixed compact set $K \subset \Omega$ will be eventually contained in all Ω_ε for ε small enough; whence the problem reduces to showing weak sequential stability of Navier–Stokes system on any space–time cylinder $(0,T) \times K$. By this we mean that any accumulation point of a sequence $\{\mathbf{u}\}_{\varepsilon>0}$ of solutions, bounded in the associated energy norm, represents another (distributional) solution of (1.3)–(1.5). This is, however, a delicate task as the standard compactness argument based on Lions–Aubin lemma (see Lions [9], Temam [15]) cannot be used in a direct fashion. The main stumbling block here is the fact that we need a piece of "global" information on the pressure terms that may be lost in the asymptotic limit. In order to overcome this difficulty, we use a method based on the existence of a "local" pressure developed recently by Wolf [16].

The problem of identifying the limit boundary conditions was addressed in [5] (see also [1,2] for related results). Very roughly indeed, the rugosity of the boundaries, together with (1.8), result in the no-slip boundary conditions to be satisfied on $\partial\Omega$ by a solution of the limit problem. Such a situation was examined in [5], on condition of periodically distributed asperities on $\partial\Omega_\varepsilon$. It is interesting to note that such a result is completely independent of a particular system of equations and is conditioned only by uniform bounds in a suitable Sobolev space. This kind of behaviour is intimately related to the Mosco convergence of Sobolev spaces and is quite often observed in shape optimization problems (see [4]).

In the present paper, we perform a detailed analysis of this phenomenon, introducing a concept of *measure of rugosity* based on the tools of compensated compactness. More specifically, a rugosity measure will be a parametrized (Young) measure associated to the directions of the normal vectors on Γ_ε. In particular, we relax completely the main assumptions made in [5], namely the uniformity, periodicity, and regularity of oscillations. Rugosity measures, associated with a given direction, vanish on the region with none or mild asperities, while they are strictly positive in the area, where "many" microscopic asperities prevent the fluid from slipping. Accordingly, the kinetic energy being transformed into heat, the velocity vanishes in the asymptotic limit to comply with (1.2). Probably the most interesting example is provided by the boundaries with crystalline structure, where the microscopic asperities of polyhedral type give rise to the no-slip boundary conditions under very mild hypotheses. The situation at a given point of the boundary may become even more complex when the dissipation mechanism is associated to a specific direction yielding a kind of mixed boundary conditions for different components of the velocity field.

The paper is organized as follows. Section 2 contains the standard preliminary material including variational formulation of Navier–Stokes system, with the associated energy estimates and the function spaces framework. Measures of rugosity are introduced in Section 3, together with the necessary technical machinery taken over from the monograph of Pedregal [13]. The main results stated in Theorem 4.1, together with a sample of specific applications related to periodic oscillations, crystalline and self-similar boundaries (see Corollaries 4.1–4.5), are formulated in Section 4. The rest of the paper is devoted to the proof of Theorem 4.1. In Section 5, we recall some known results concerning the strain-preserving extension operators related to Sobolev norms and the associated Korn-type inequalities. Section 6 is devoted to the proof of local compactness for Navier–Stokes system that may be of independent interest. The analysis of the boundary behaviour of solutions completing the proof of the main results is carried on in Section 7.

2. Preliminaries

Eq. (1.4), supplemented with the impermeability condition (1.1) on Γ_ε, can be conveniently recast in terms of a concise variational formulation:

$$\int_{\Omega_\varepsilon} \mathbf{u} \cdot \nabla_x \psi \, \mathrm{d}\mathbf{x} = 0 \quad \text{for any } \psi \in \mathcal{D}\big(\mathcal{T}^2 \times (0,\infty)\big). \tag{2.1}$$

Although a generalized version of Green's theorem holds on Lipschitz domains, relation (2.1) can be used for less regular domains as well as in the case when $\mathbf{u}$ is only an integrable solenoidal function.

Let the symbol $W^{k,p}(\Omega; R^N)$ denote the Sobolev space of functions belonging to the Lebesgue space $L^p(\Omega; R^N)$ and such that all their generalized derivatives up to order k belong to $L^p(\Omega; R^N)$. Furthermore, motivated by (1.7), (1.8), we introduce the spaces

$$W^{1,2}_{\sigma,n}\big(\Omega_\varepsilon; R^3\big) = \big\{\mathbf{v} \in W^{1,2}\big(\Omega_\varepsilon; R^3\big) \;\big|\; \mathbf{v}|_{\{x_3=0\}} = 0,\ \mathbf{v} \text{ satisfies (2.1)}\big\},$$
$$W^{1,2}_{\sigma,0}\big(\Omega; R^3\big) = \big\{\mathbf{v} \in W^{1,2}\big(\Omega; R^3\big) \;\big|\; \mathbf{v}|_{\{x_3=0\}\cup\{x_3=\Phi(x_1,x_2)\}} = 0,\ \mathbf{v} \text{ satisfies (2.1)}\big\}.$$

Definition 2.1. We shall say that $\mathbf{u}$ is a weak solution to Navier–Stokes system (1.3)–(1.5) on the set $(0,T)\times\Omega_\varepsilon$, supplemented with the boundary conditions (1.7), (1.8), together with the initial condition

$$\mathbf{u}(0,\cdot)=\mathbf{u}_0, \tag{2.2}$$

if the following holds:

- $$\mathbf{u}\in L^2\big(0,T;W^{1,2}_{\sigma,n}\big(\Omega_\varepsilon;R^3\big)\big)\cap C_{\mathrm{weak}}\big([0,T];L^2\big(\Omega_\varepsilon;R^3\big)\big); \tag{2.3}$$
- the integral identity

$$\int_0^T\int_{\Omega_\varepsilon}\big(\mathbf{u}\cdot\partial_t\varphi+(\mathbf{u}\otimes\mathbf{u}):\nabla_x\varphi\big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$=\int_0^T\int_{\Omega_\varepsilon}\mu\big(\nabla_x\mathbf{u}+\nabla_x\mathbf{u}^{\perp}\big):\nabla_x\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t-\int_{\Omega_\varepsilon}\mathbf{u}_0\cdot\varphi(0,\cdot)\,\mathrm{d}\mathbf{x} \tag{2.4}$$

  is satisfied for any test function $\varphi\in\mathcal{D}([0,T);W^{1,2}_{\sigma,n}(\Omega_\varepsilon;R^3))$;
- the energy inequality

$$\int_{\Omega_\varepsilon}\frac{1}{2}|\mathbf{u}|^2(\tau)\,\mathrm{d}\mathbf{x}+\frac{\mu}{2}\int_0^\tau\int_{\Omega_\varepsilon}\big|\nabla_x\mathbf{u}+\nabla_x\mathbf{u}^{\perp}\big|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\leqslant\int_{\Omega_\varepsilon}\frac{1}{2}|\mathbf{u}_0|^2\,\mathrm{d}\mathbf{x} \tag{2.5}$$

  holds for a.a. $\tau\in(0,T)$.

In the same fashion, we have

Definition 2.2. We shall say that $\mathbf{u}$ is a weak solution to Navier–Stokes system (1.3)–(1.5) on the set $(0,T)\times\Omega$, supplemented with the boundary condition (1.2), together with the initial condition

$$\mathbf{u}(0,\cdot)=\mathbf{u}_0, \tag{2.6}$$

if the following holds:

- $$\mathbf{u}\in L^2\big(0,T;W^{1,2}_{\sigma,0}\big(\Omega;R^3\big)\big)\cap C_{\mathrm{weak}}\big([0,T];L^2\big(\Omega;R^3\big)\big); \tag{2.7}$$
- the integral identity

$$\int_0^T\int_{\Omega}\big(\mathbf{u}\cdot\partial_t\varphi+(\mathbf{u}\otimes\mathbf{u}):\nabla_x\varphi\big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$=\int_0^T\int_{\Omega}\mu\big(\nabla_x\mathbf{u}+\nabla_x\mathbf{u}^{\perp}\big):\nabla_x\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t-\int_{\Omega}\mathbf{u}_0\cdot\varphi(0,\cdot)\,\mathrm{d}\mathbf{x} \tag{2.8}$$

  is satisfied for any test function $\varphi\in\mathcal{D}([0,T);\mathcal{D}(\Omega;R^3))$ such that $\mathrm{div}_x\,\varphi=0$;

- the energy inequality

$$\int_{\Omega} \frac{1}{2}|\mathbf{u}|^2(\tau)\,\mathrm{d}\mathbf{x} + \frac{\mu}{2}\int_0^{\tau}\int_{\Omega} \left|\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^{\perp}\right|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}t \leqslant \int_{\Omega} \frac{1}{2}|\mathbf{u}_0|^2\,\mathrm{d}\mathbf{x} \tag{2.9}$$

  holds for a.a. $\tau \in (0, T)$.

The existence of the weak solutions (for $\Omega = R^3$) in the spirit of Definition 2.2 was established in the seminal paper by Leray [8]. In view of the modern theory based on the concept of Sobolev spaces, the existence of global in time weak solutions for both the no-slip and complete slip boundary conditions can be shown in a standard way provided the initial distribution of the velocity $\mathbf{u}_0 \in L^2(\Omega; R^3)$ satisfies (2.1) (see Ladyzhenskaya [7], Temam [15], among others).

3. Measures of rugosity

Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a family of domains given through (1.6), where

$$\begin{gathered} \Phi_\varepsilon \in W^{1,\infty}(\mathcal{T}^2), \quad \Phi_\varepsilon > 0, \quad \Phi_\varepsilon \to 0 \quad \text{uniformly on } \mathcal{T}^2, \\ \left|\Phi_\varepsilon(\mathbf{y}_1) - \Phi_\varepsilon(\mathbf{y}_2)\right| \leqslant L|\mathbf{y}_1 - \mathbf{y}_2| \quad \text{for any } \mathbf{y}_1, \mathbf{y}_2 \in \mathcal{T}^2, \end{gathered} \tag{3.1}$$

with L independent of ε.

A *measure of rugosity* $\{\mathcal{R}_\mathbf{y}\}_{\mathbf{y}\in\mathcal{T}^2}$ is simply a Young measure associated to the family of gradients $\{\nabla_y \Phi_\varepsilon\}_{\varepsilon>0}$. More specifically, $\{\mathcal{R}_\mathbf{y}\}_{\mathbf{y}\in\mathcal{T}^2}$ is a family of probability measures on R^2 depending measurably on $\mathbf{y}$ such that

$$\int_{R^2} G(\mathbf{y}, \mathbf{Z})\,\mathrm{d}\mathcal{R}_\mathbf{y}(\mathbf{Z}) = \text{weak} \lim_{\varepsilon\to 0} G(\mathbf{y}, \nabla_y \Phi_\varepsilon) \quad \text{for a.a. } \mathbf{y} \in \mathcal{T}^2 \tag{3.2}$$

for any Carathéodory function $G : \mathcal{T}^2 \times R^2 \to R$ (see Theorem 6.2 in Pedregal [13]). Note that such a measure need not be unique.

As the family $\{\Phi_\varepsilon\}_{\varepsilon>0}$ is equi-Lipschitz, there is a bounded set $M \subset R^2$ such that

$$\operatorname{supp}[\mathcal{R}_\mathbf{y}] \subset M \quad \text{for a.a. } \mathbf{y} \in \mathcal{T}^2. \tag{3.3}$$

Furthermore, the quantity $(-\partial_{y_1}\Phi_\varepsilon(\mathbf{y}), -\partial_{y_2}\Phi_\varepsilon(\mathbf{y}), 1)$ represents the outer normal vector to Γ_ε for a.a. $\mathbf{x} = (\mathbf{y}, 1 + \Phi_\varepsilon(\mathbf{y}))$; whence the measure $\mathcal{R}_\mathbf{y}$ characterizes its possible oscillations around the equilibrium position $(0, 0, 1)$. Note that

$$\int_{R^2} \mathbf{Z}\,\mathrm{d}\mathcal{R}_\mathbf{y}(\mathbf{Z}) = (0, 0) \quad \text{for a.a. } \mathbf{y} \in \mathcal{T}^2 \tag{3.4}$$

as, in accordance with (3.1),

$$\partial_{y_1}\Phi_\varepsilon \to 0, \quad \partial_{y_2}\Phi_\varepsilon \to 0 \quad \text{weakly-(*) in } L^\infty(\mathcal{T}^2).$$

Definition 3.1. We shall say that a rugosity measure $\{\mathcal{R}_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{T}^2}$ is non-degenerate at $\mathbf{y} \in \mathcal{T}^2$ if $\mathrm{supp}[\mathcal{R}_{\mathbf{y}}]$ contains two linearly independent vectors in R^2.

4. Main results

Having collected all the preliminary material we are in a position to state the main result of the present paper.

Theorem 4.1. *Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a family of domains defined through* (1.6)*, where Φ_ε satisfies* (3.1)*. Assume that the associated measure of rugosity $\{\mathcal{R}_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{T}^2}$ is non-degenerate at a.a. $\mathbf{y} \in \mathcal{T}^2$ in the sense of Definition* 3.1*. Let $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to Navier–Stokes system* (1.3)–(1.5) *on the set $(0,T)\times\Omega_\varepsilon$ in the sense of Definition* 2.1*, supplemented with the boundary conditions* (1.7), (1.8)*, and an initial datum $\mathbf{u}_0 \in L^2(\mathcal{T}^2\times(0,\infty); R^3)$ independent of ε.*

Then, passing to a subsequence if necessary, we have

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \text{weakly in } L^2\big(0,T; W^{1,2}\big(\mathcal{T}^2\times(0,\infty); R^3\big)\big) \text{ and}$$
$$\text{weakly-}(*) \text{ in } L^\infty\big(0,T; L^2\big(\mathcal{T}^2\times(0,\infty); R^3\big)\big),$$

where $\mathbf{u}$ is a weak solution of Navier–Stokes system on the set $(0,T)\times\Omega = \mathcal{T}^2\times(0,1)$ in the sense of Definition 2.2*, supplemented with the no-slip boundary condition* (1.2) *and the initial datum $\mathbf{u}_0$.*

Remark 4.1. Here the functions $\mathbf{u}_\varepsilon$ have been extended outside Ω_ε in accordance with Proposition 5.1 below. Similarly, the initial datum $\mathbf{u}_0$ is defined on the whole set $\mathcal{T}^2\times(0,\infty)$.

The first application of Theorem 4.1 yields the result proved in [5]:

Corollary 4.1. *Let $\Phi_\varepsilon(\mathbf{y}) = \varepsilon\Phi(\mathbf{y}/\varepsilon)$, $\mathbf{y}\in\mathcal{T}^2$, where $\varepsilon = \varepsilon_n = \frac{1}{n}$, and $\Phi \in W^{1,\infty}(\mathcal{T}^2)$, $\Phi > 0$.*

Then the associated family of domains Ω_ε defined through (1.6) *satisfies the hypotheses of Theorem* 4.1 *provided the mapping*

$$\lambda \in R \mapsto \Phi(\cdot + \lambda\mathbf{e}) \in C\big(\mathcal{T}^2\big) \quad \text{is not constant}$$

for any $\mathbf{e}\in R^2$, $\mathbf{e}\neq 0$.

The second corollary applies to the boundaries with "crystalline" structure:

Corollary 4.2. *Let $\Phi_\varepsilon \in W^{1,\infty}(\mathcal{T}^2)$ be as in* (3.1) *and such that there is a compact set $K \subset R^2$ such that*

$$\nabla_y \Phi_\varepsilon(\mathbf{y}) \in K \quad \text{for a.a. } \mathbf{y}\in\mathcal{T} \text{ and all } \varepsilon > 0.$$

Furthermore, suppose that the segment $[\mathbf{y}_1,\mathbf{y}_2]$ does not contain the point $(0,0)$ for any choice $\mathbf{y}_1,\mathbf{y}_2 \in K$.

Then the associated family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ satisfies the hypotheses of Theorem 4.1.

Remark 4.2. Typically, the set K is finite for domains with "crystalline" structure.

The next application may be viewed as a stability result with respect to small perturbations.

Corollary 4.3. *Let $\Phi_\varepsilon = \Psi_\varepsilon + H_\varepsilon$, where Ψ_ε satisfies the hypotheses of Theorem* 4.1 *and*

$$H_\varepsilon \to 0 \quad \textit{weakly-(*) in } W^{1,\infty}(\mathcal{T}^2) \textit{ and strongly in } W^{1,1}(\mathcal{T}^2).$$

Then the conclusion of Theorem 4.1 *holds for the family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ associated to $\{\Phi_\varepsilon\}_{\varepsilon>0}$.*

The following result seems to be closest to the intuitive understanding of rugosity. We set

$$\operatorname{osc}[\mathbf{w}](\mathbf{y}) = \liminf_{r\to 0} \frac{1}{|B_r(\mathbf{y})|}\left(\liminf_{\varepsilon\to 0} \int\limits_{B_r(\mathbf{y})} |\mathbf{w}\cdot\nabla_y \Phi_\varepsilon|\,\mathrm{d}z\right), \quad |\mathbf{w}| = 1,$$

where $B_r(\mathbf{y})$ is the ball of radius r centered at $\mathbf{y}$. Loosely speaking, osc measure the oscillations of the normal vector in the direction $\mathbf{w}$.

Corollary 4.4. *Let $\{\Phi_\varepsilon\}_\varepsilon$ be as in* (3.1) *and such that*

$$\operatorname{osc}[\mathbf{w}](\mathbf{y}) \geqslant c_w > 0 \quad \textit{for all } \mathbf{w}, \quad |\mathbf{w}| = 1,$$

for a.a. $\mathbf{y} \in \mathcal{T}^2$.

Then the associated family of domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$ satisfies the hypotheses of Theorem 4.1.

The last example concerns the boundaries with "self-similar" structure. Consider a function $\mathcal{A} \in W^{1,\infty}(R^2)$ such that

$$\begin{gathered}\operatorname{supp}[\mathcal{A}] \subset \{\mathbf{y} = (y^1, y^2) \in R^2 \mid -1 < y^j < 1,\ j = 1,2\},\\ 0 \leqslant \mathcal{A}(\mathbf{y}) \leqslant 1 \quad \text{for all } \mathbf{y} \in R^2, \quad \sup_{\mathbf{y}\in R^2} \mathcal{A}(\mathbf{y}) = 1.\end{gathered} \tag{4.1}$$

Corollary 4.5. *Let $\Phi_\varepsilon \in W^{1,\infty}(\mathcal{T}^2)$ be given as*

$$\Phi_\varepsilon(\mathbf{y}) = \sum_{i=1}^{m_\varepsilon} \delta_i \mathcal{A}\left(\frac{\mathbf{y} - \mathbf{y}_i}{\delta_i}\right), \quad 0 < \delta_i \leqslant \varepsilon,$$

where $\mathcal{A}$ satisfies (4.1). *Furthermore, assume that*

$$\max_{m=1,2} |y_i^m - y_j^m| \geqslant (\delta_i + \delta_j) \quad \textit{for } i \neq j,$$

and

$$\textit{for any } \mathbf{y} \in \mathcal{T}_2 \textit{ there exists } k \quad \textit{such that} \quad \max_{m=1,2} |y^m - y_k^m| \leqslant \delta_k.$$

Then the associated family of domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$ satisfies the hypotheses of Theorem 4.1.

The rest of the paper is devoted to the proof of Theorem 4.1 and Corollaries 4.1–4.5.

5. Korn's inequality

The so-called second Korn inequality reads

$$\|\mathbf{v}\|^2_{W^{1,2}(V;R^3)} \leqslant c(V)\left(\|\mathbf{v}\|^2_{L^2(V;R^3)} + \int_V \left|\nabla_x \mathbf{v} + \nabla_x \mathbf{v}^{\perp}\right|^2 \mathrm{d}\mathbf{x}\right) \tag{5.1}$$

for any $\mathbf{v} \in W^{1,2}(V; R^3)$. Its validity is closely related to the geometrical properties of the domain V and so is the optimal value of the constant c (the proof for domains with Lipschitz boundary can be found in the monograph by Nečas [11, Chapter 3, Theorem 7.9]).

The following result is due to Nitsche [12, Lemma 4].

Proposition 5.1. *Let Ω_ε be given by* (1.6)*, where Φ_ε satisfies* (3.1).

Then there exists an extension operator E_{Ω_ε},

$$E_{\Omega_\varepsilon} : W^{1,2}\big(\Omega_\varepsilon; R^3\big) \to W^{1,2}\big(\mathcal{T}^2 \times R; R^3\big), \qquad E_{\Omega_\varepsilon}[\mathbf{v}]|_{\Omega} = \mathbf{v},$$

such that a Korn-type inequality

$$\big\|E_{\Omega_\varepsilon}[\mathbf{v}]\big\|^2_{W^{1,2}(\mathcal{T}^2 \text{ times } R;R^3)} \leqslant c(L)\left(\|\mathbf{v}\|^2_{L^2(\Omega_\varepsilon;R^3)} + \int_{\Omega\varepsilon} \left|\nabla_x \mathbf{v} + \nabla_x \mathbf{v}^{\perp}\right|^2 \mathrm{d}\mathbf{x}\right) \tag{5.2}$$

holds for all $\mathbf{v} \in W^{1,2}(\Omega_\varepsilon; R^3)$, where L is the Lipschitz constant of the function Φ_ε.

In virtue of Proposition 5.1, the functions $\mathbf{u}_\varepsilon$ may extended to the "half-space" domain $\mathcal{T}^2 \times (0, \infty)$ independent of ε as stated in Theorem 4.1.

6. Local sequential stability

Our aim is to show that the solution set $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ is weakly sequentially stable with respect to the natural topology induced through the energy *a priori* estimates. More precisely, we shall show that any weak limit $\mathbf{u}$ of $\{\mathbf{u}_\varepsilon\}_\varepsilon$ satisfies the integral identity (2.8). Such a result may be of independent interest in applications, whenever the boundary of the physical domain is not fixed.

Let us start with a variant of a remarkable result by Wolf [16, Theorem 2.5]:

Lemma 6.1. *Let $\Omega = \mathcal{T}^2 \times (0, 1)$. Assume that $\mathbf{v} \in C_{\text{weak}}(0, T; L^2(\Omega; R^3))$, $\mathbb{Q} \in L^q(0, T; L^2(\Omega; R^{3\times 3}))$, $q \geqslant 1$, satisfy the integral identity*

$$\int_0^T \int_\Omega \mathbf{v} \cdot \partial_t \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t + \int_0^T \int_\Omega \mathbb{Q} : \nabla_x \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t = 0$$

for any $\varphi \in \mathcal{D}(0, T; \mathcal{D}(\Omega; R^3))$, $\mathrm{div}_x \varphi = 0$. Furthermore, let $\mathrm{div}_x \mathbf{v} = 0$ in $\mathcal{D}'((0, T) \times \Omega)$.

Then there exist $p_r \in L^q(0,T;L^2(\Omega))$, $p_h \in L^\infty(0,T;L^2(\Omega))$,

$$\Delta_x p_h = 0 \quad \textit{in } \mathcal{D}'\big((0,T)\times\Omega\big), \qquad \int\limits_\Omega p_h \,\mathrm{d}\mathbf{x} = 0, \tag{6.1}$$

satisfying

$$\begin{aligned}&\int\limits_0^T\int\limits_\Omega \mathbf{v}\cdot\partial_t\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int\limits_0^T\int\limits_\Omega \mathbb{Q} : \nabla_x\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\quad = \int\limits_0^T\int\limits_\Omega p_r \operatorname{div}_x\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int\limits_0^T\int\limits_\Omega p_h \partial_t \operatorname{div}_x\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\end{aligned}$$

for any $\varphi \in \mathcal{D}(0,T;\mathcal{D}(\Omega;R^3))$. *In addition,*

$$\begin{aligned}\|p_r\|_{L^q(0,T;L^2(\Omega))} &\leqslant P\|\mathbb{Q}\|_{L^q(0,T;L^2(\Omega;R^{3\times3}))}, \\ \|p_h\|_{L^\infty(0,T;L^2(\Omega))} &\leqslant P\big(\|\mathbf{v}\|_{L^\infty(0,T;L^2(\Omega;R^3))} + \|\mathbb{Q}\|_{L^q(0,T;L^2(\Omega;R^{3\times3}))}\big),\end{aligned} \tag{6.2}$$

where the constant P *depends solely on* q, T, *and* Ω.

Proof. The "regular" component p_r of the pressure is determined as

$$p_r(t) = \Delta_x P_r(t) \quad \text{for a.a. } t \in (0,T),$$

where $P_r \in W_0^{2,2}(\Omega)$ is the unique solution of the elliptic problem

$$\int\limits_\Omega \Delta_x P_r \Delta_x \psi \,\mathrm{d}\mathbf{x} = -\int\limits_\Omega \mathbb{Q} : \nabla_x^2\psi \,\mathrm{d}\mathbf{x} \quad \text{for any } \psi \in W_0^{2,2}(\Omega). \tag{6.3}$$

Note that the bilinear form on the left-hand side of (6.3) is a scalar product on the Hilbert space $W_0^{2,2}(\Omega)$, while the quantity on the right-hand side represents a continuous bilinear form on the same space. In particular, p_r satisfies the estimate claimed in (6.2).

On the other hand, we have

$$\int\limits_0^T\Big[\int\limits_\Omega \big(\mathbf{v} - \mathbf{v}(0)\big)\cdot\psi \,\mathrm{d}\mathbf{x}\Big]\partial_t\eta \,\mathrm{d}t - \int\limits_0^T\Big[\int\limits_\Omega I[\mathbb{Q}] : \nabla_x\psi \,\mathrm{d}\mathbf{x}\Big]\partial_t\eta \,\mathrm{d}t = 0 \tag{6.4}$$

for any $\eta \in \mathcal{D}(0,T)$, $\psi \in \mathcal{D}(\Omega;R^3)$, $\operatorname{div}_x\psi = 0$, where we have set

$$I[\mathbb{Q}](\tau,\cdot) = \int\limits_0^\tau \mathbb{Q}(t,\cdot)\,\mathrm{d}t.$$

By virtue of Lemma 2.2.1 in [14], there exists a function $p = p(t, \cdot)$ such that

$$\int_{\Omega} \big(\mathbf{v} - \mathbf{v}(0)\big) \cdot \psi \, \mathrm{d}\mathbf{x} - \int_{\Omega} I[\mathbb{Q}] : \nabla_x \psi \, \mathrm{d}\mathbf{x} = \int_{\Omega} p \, \mathrm{div}_x \, \psi \, \mathrm{d}\mathbf{x} \tag{6.5}$$

for a.a. $t \in (0, T)$ and all $\psi \in \mathcal{D}(\Omega)$. Moreover,

$$\int_{\Omega} p \, \mathrm{d}\mathbf{x} = 0$$

and

$$\|p\|_{L^2(\Omega)} \leqslant c\big(\|\mathbf{v} - \mathbf{v}_0\|_{L^2(\Omega; R^3)} + \big\|I[\mathbb{Q}]\big\|_{L^2(\Omega; R^{3\times 3})}\big).$$

Thus

$$\int_0^T \int_{\Omega} \mathbf{v} \cdot \partial_t \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t + \int_0^T \int_{\Omega} \mathbb{Q} : \nabla_x \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t = \int_0^T \int_{\Omega} p \partial_t \, \mathrm{div}_x \, \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t$$

for any $\varphi \in \mathcal{D}((0, T) \times \Omega; R^3)$.

Setting

$$p_h(\tau) = p(\tau) - \int_0^{\tau} p_r(t) \, \mathrm{d}t$$

we have to show that $p_h(\tau, \cdot)$ is a harmonic function in $\mathbf{x}$ for a.a. $\tau \in (0, T)$. To this end, it is enough to take $\psi = \nabla_x \eta$, $\eta \in \mathcal{D}(\Omega)$, in (6.5) and to compare the resulting expression with (6.3). $\square$

We are in a position to state the local stability property of bounded families of solutions to Navier–Stokes system. Note that such a task reduces essentially to the "weak" compactness property of the convective term stated in what follows.

Proposition 6.1. *Let* $\{\mathbf{u}_\varepsilon\}_{n=1}^{\infty}$ *be a family of vector fields defined on a cylinder* $(0, T) \times \Omega$*, with* $\Omega = \mathcal{T}^2 \times (0, 1)$*, satisfying the integral identity* (2.8) *for any test function* $\varphi \in \mathcal{D}(0, T; \mathcal{D}(\Omega; R^3))$ *such that* $\mathrm{div}_x \, \varphi = 0$*. Furthermore, assume that*

$$\left.\begin{aligned} \mathbf{u}_\varepsilon \in C_{\mathrm{weak}}\big(0, T; L^2\big(\Omega; R^3\big)\big) \cap L^2\big(0, T; W^{1,2}\big(\Omega; R^3\big)\big), \quad \mathrm{div}_x \, \mathbf{u}_\varepsilon = 0, \\ \operatorname*{ess\,sup}_{t \in (0,T)} \int_{\Omega} \big|\mathbf{u}_\varepsilon(t)\big|^2 \, \mathrm{d}\mathbf{x} + \int_0^T \int_{\Omega} |\nabla_x \mathbf{u}_\varepsilon|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \leqslant E_\infty, \end{aligned}\right\} \tag{6.6}$$

where the constant E_∞ *is independent of* $\varepsilon > 0$.

Then, passing to a subsequence if necessary, we have

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \textit{weakly in } L^2\big(0,T; W^{1,2}(\Omega, R^3)\big) \textit{ and weakly-}(*) \textit{ in } L^\infty\big(0,T; L^2(\Omega; R^3)\big),$$

and

$$\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \overline{\mathbf{u} \otimes \mathbf{u}} \quad \textit{weakly in } L^{\frac{3}{2}}\big(0,T; L^{\frac{3}{2}}(\Omega; R^{3\times 3})\big),$$

where

$$\int_0^T \int_\Omega \overline{\mathbf{u} \otimes \mathbf{u}} : \nabla_x \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t = \int_0^T \int_\Omega (\mathbf{u} \otimes \mathbf{u}) : \nabla_x \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \tag{6.7}$$

for any $\varphi \in \mathcal{D}(0,T; \mathcal{D}(\Omega; R^3))$, $\mathrm{div}_x \, \varphi = 0$.

Proof. To begin with, it is easy to check that the quantities

$$\mathbf{v} = \mathbf{u}_\varepsilon, \qquad \mathbb{Q} = \mu\big(\nabla_x \mathbf{u}_\varepsilon + \nabla_x \mathbf{u}_\varepsilon^\perp\big) - \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$$

satisfy the hypotheses of Lemma 6.1 with $q = \frac{4}{3}$. Consequently, one can find the functions $p_{r,\varepsilon}$, $p_{h,\varepsilon}$ such that the integral identity

$$\begin{aligned}
&\int_0^T \int_\Omega \mathbf{u}_\varepsilon \cdot \partial_t \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t + \int_0^T \int_\Omega (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\
&\quad = \int_0^T \int_\Omega \mu\big(\nabla_x \mathbf{u}_\varepsilon + \nabla_x \mathbf{u}_\varepsilon^\perp\big) : \nabla_x \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t + \int_0^T \int_\Omega p_{r,\varepsilon} \, \mathrm{div}_x \, \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\
&\qquad + \int_0^T \int_\Omega p_{h,\varepsilon} \partial_t \, \mathrm{div}_x \, \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t
\end{aligned} \tag{6.8}$$

holds for any $\varphi \in \mathcal{D}(0,T; \mathcal{D}(B; R^3))$.

Moreover, in accordance with (6.1), (6.2), we can suppose that

$$\|p_{r,\varepsilon}\|_{L^{\frac{4}{3}}(0,T;L^2(V))} + \|p_{h,\varepsilon}\|_{L^\infty(0,T;C^2(\overline{V}))} \leqslant E_\infty(V) \tag{6.9}$$

for any open set $V \subset \overline{V} \subset \Omega$.

Passing to a subsequence as the case may be, we have

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \text{weakly in } L^2\big(0,T; W^{1,2}(\Omega, R^3)\big) \text{ and weakly-}(*) \text{ in } L^\infty\big(0,T; L^2(\Omega; R^3)\big),$$

$$p_{r,\varepsilon} \to p_r \quad \text{weakly in } L^{\frac{4}{3}}\big(0,T; L^2(\Omega)\big),$$

$$p_{h,\varepsilon} \to p_h \quad \text{weakly-}(*) \text{ in } L^\infty\big(0,T; L^2(\Omega; R^3)\big),$$

and

$$\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \overline{\mathbf{u}\otimes\mathbf{u}} \quad \text{weakly in } L^{\frac{3}{2}}\big(0,T; L^{\frac{3}{2}}\big(\Omega; R^{3\times 3}\big)\big).$$

Consequently, in order to complete the proof, it is enough to show that (6.7) holds for any $\varphi \in \mathcal{D}(0,T;\mathcal{D}(\Omega;R^3))$, $\operatorname{div}_x \varphi = 0$.

From (6.8) we deduce that

$$\mathbf{u}_\varepsilon + \nabla_x p_{h,\varepsilon} \to \mathbf{u} + \nabla_x p_h \quad \text{in } C_{\text{weak}}\big([0,T]; L^2\big(V; R^3\big)\big), \quad V \subset \overline{V} \subset \Omega;$$

whence, in the spirit of Lions–Aubin argument,

$$\int_0^T \int_\Omega \varphi |\mathbf{u}_\varepsilon + \nabla_x p_{h,\varepsilon}|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \to \int_0^T \int_\Omega \varphi |\mathbf{u} + \nabla_x p_h|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \quad \text{for any } \varphi \in \mathcal{D}\big((0,T)\times\Omega\big),$$

in other words,

$$\mathbf{u}_\varepsilon + \nabla_x p_{h,\varepsilon} \to \mathbf{u} + \nabla_x p_h \quad \text{(strongly) in } L^2\big(0,T; L^2\big(V; R^3\big)\big), \quad V \subset \overline{V} \subset B. \tag{6.10}$$

Thus

$$\begin{aligned}
\int_0^T \int_\Omega \overline{\mathbf{u}\otimes\mathbf{u}} : \nabla_x \varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t &= \lim_{\varepsilon\to 0} \int_0^T \int_\Omega (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= \lim_{\varepsilon\to 0} \int_0^T \int_\Omega \big((\mathbf{u}_\varepsilon + \nabla_x p_{h,\varepsilon}) \otimes \mathbf{u}_\varepsilon\big) : \nabla_x \varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad - \lim_{\varepsilon\to 0} \int_0^T \int_\Omega \big(\nabla_x p_{h,\varepsilon} \otimes (\mathbf{u}_\varepsilon + \nabla_x p_{h,\varepsilon})\big) : \nabla_x \varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad + \lim_{\varepsilon\to 0} \int_0^T \int_\Omega (\nabla_x p_{h,\varepsilon} \otimes \nabla_x p_{h,\varepsilon}) : \nabla_x \varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= \int_0^T \int_\Omega (\mathbf{u}\otimes\mathbf{u}) : \nabla_x \varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t
\end{aligned}$$

whenever $\varphi \in \mathcal{D}(0,T;\mathcal{D}(\Omega;R^3))$, $\operatorname{div}_x \varphi = 0$. Indeed

$$\int_0^T \int_\Omega (\nabla_x p \otimes \nabla_x p) : \nabla_x \varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = -\int_0^T \int_\Omega \left(\frac{1}{2}\nabla_x |\nabla_x p|^2 \cdot \varphi + \Delta_x p \nabla_x p \cdot \varphi\right) \mathrm{d}\mathbf{x}\,\mathrm{d}t = 0$$

for $p = p_{h,\varepsilon}$, p_h as both $p_{h,\varepsilon}$, p_h are harmonic functions in Ω. $\square$

7. Proof of the main results

7.1. Proof of Theorem 4.1

Step 1. Let $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ be a family of solutions to Navier–Stokes system satisfying the hypotheses of Theorem 4.1. Combining the energy inequality (2.5) together with Proposition 5.1 one can extend the functions $\mathbf{u}_\varepsilon$ to the "half-space" $\mathcal{T}^2 \times (0,\infty)$ in such a way that

$$\mathbf{u}_\varepsilon \in L^2\big(0,T; W_0^{1,2}\big(V;R^3\big)\big) \cap L^\infty\big(0,T;L^2\big(V;R^3\big)\big),$$

$$\{\mathbf{u}_\varepsilon\}_{\varepsilon>0} \quad \text{bounded in } L^2\big(0,T; W_0^{1,2}\big(V;R^3\big)\big) \cap L^\infty\big(0,T;L^2\big(V;R^3\big)\big), \tag{7.1}$$

where $V = \mathcal{T}^2 \times (0,R)$, and

$$R > 1 + \sup_\varepsilon \|\Phi_\varepsilon\|_{C(\mathcal{T}^2)}.$$

Consequently, passing to a suitable subsequence as the case may be, we can assume that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \text{weakly in } L^2\big(0,T; W_0^{1,2}\big(V;R^3\big)\big) \text{ and weakly-(*) in } L^\infty\big(0,T;L^2\big(V;R^3\big)\big). \tag{7.2}$$

Furthermore, by virtue of Proposition 6.1, the velocity field $\mathbf{u}$ satisfies the integral identity (2.8) for any test function $\varphi \in \mathcal{D}(0,T;\mathcal{D}(\Omega;R^3))$ such that $\operatorname{div}_x \varphi = 0$.

Finally, by virtue of weak lower semi-continuity of convex functionals, we check easily that the limit velocity field $\mathbf{u}$ satisfies the energy inequality (2.9) on Ω.

Consequently, in order to complete the proof of Theorem 4.1, we have only to show that $\mathbf{u}$ satisfies the no-slip boundary condition (1.2). This will be done in the next step.

Step 2. To begin with, it is easy to check that $\mathbf{u}$ satisfies (2.1), or, equivalently,

$$u_3 = \mathbf{u}\cdot\mathbf{n} = 0 \quad \text{a.a. on } \{x_3 = 1\}. \tag{7.3}$$

Furthermore, introducing the mollified quantities

$$\mathbf{u}_\varepsilon^\delta(\tau,\cdot) = \int_0^T \kappa_\delta(\tau - t)\mathbf{u}_\varepsilon(t,\cdot)\,\mathrm{d}t,$$

where $\kappa_\delta \in \mathcal{D}(R)$ is a suitable family of regularizing kernels, one can see that

$$\mathbf{u}_\varepsilon^\delta \to \mathbf{u}^\delta \quad \text{as } \varepsilon \to 0 \text{ in } C_{\mathrm{loc,weak}}\big((0,T); W_0^{1,2}\big(V;R^3\big)\big)$$

and

$$\mathbf{u}^\delta(t) \to \mathbf{u}(t) \quad \text{for } \delta \to 0 \text{ in } W_0^{1,2}\big(V;R^3\big) \text{ for a.a. } t \in (0,T).$$

Consequently, it is enough to show that

$$\mathbf{v}|_\Omega \in W_0^{1,2}\big(\Omega;R^3\big), \tag{7.4}$$

whenever

$$\mathbf{v}_\varepsilon \to \mathbf{v} \quad \text{weakly in } W_0^{1,2}\big(V; R^3\big), \qquad \mathbf{v}_\varepsilon|_{\Omega_\varepsilon} \in W^{1,2}_{\sigma,n}\big(\Omega_\varepsilon; R^3\big). \tag{7.5}$$

To begin with, similarly to the above, it is easy to check that

$$v_3 = \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{a.a. on } \{x_3 = 1\}. \tag{7.6}$$

Moreover, we report the following crucial observation.

Lemma 7.1. *Assume that* $\{\mathbf{v}_\varepsilon\}_{\varepsilon>0}$ *satisfies* (7.5). *Let* $\{\mathcal{R}_\mathbf{y}\}_{\mathbf{y}\in\mathcal{T}^2}$ *be the measure of rugosity associated to the family* Φ_ε.

Then we have

$$\big[v^1(\mathbf{y},1), v^2(\mathbf{y},1)\big] \cdot \int_{R^2} D(\mathbf{Z})\mathbf{Z}\,\mathrm{d}\mathcal{R}_\mathbf{y}(\mathbf{Z}) = 0 \quad \text{for all } D \in C\big(R^2\big) \tag{7.7}$$

for a.a. $\mathbf{y} \in \mathcal{T}^2$.

Proof. In accordance with (7.5) we have

$$\begin{aligned} 0 &= \int_{\Gamma_\varepsilon} \psi D(\nabla_y \Phi_\varepsilon)\mathbf{n} \cdot \mathbf{v}_\varepsilon \,\mathrm{d}\sigma \\ &= \int_{\mathcal{T}^2} \psi(\mathbf{y}) D\big(\nabla_y \Phi_\varepsilon(\mathbf{y})\big)\nabla_x \Phi_\varepsilon(\mathbf{y}) \cdot \big[v_\varepsilon^1, v_\varepsilon^2\big]\big(\mathbf{y}, 1+\Phi_\varepsilon(\mathbf{y})\big)\,\mathrm{d}\mathbf{y} \\ &\quad - \int_{\mathcal{T}^2} \psi(\mathbf{y}) D\big(\nabla_y \Phi_\varepsilon(\mathbf{y})\big) v_\varepsilon^3\big(\mathbf{y}, 1+\Phi_\varepsilon(\mathbf{y})\big)\,\mathrm{d}\mathbf{y} \end{aligned} \tag{7.8}$$

for any $\psi \in \mathcal{D}(\mathcal{T}^2)$. Here, in order to define ψ and $D(\nabla_y \Phi_\varepsilon)$ on Γ_ε, we have identified $\mathbf{y} \in \mathcal{T}^2 \approx (\mathbf{y}, 1+\Phi_\varepsilon(\mathbf{y})) \in \Gamma_\varepsilon$.

On the other hand, for any smooth $\mathbf{w}$, one has

$$\mathbf{w}\big(\mathbf{y}, 1+\Phi_\varepsilon(\mathbf{y})\big) - \mathbf{w}(\mathbf{y}, 1) = \int_1^{1+\Phi_\varepsilon(\mathbf{y})} \partial_{x_3}\mathbf{w}(\mathbf{y}, z)\,\mathrm{d}z;$$

whence

$$\int_{\mathcal{T}^2} \big|\mathbf{w}\big(\mathbf{y}, 1+\Phi_\varepsilon(\mathbf{y})\big) - \mathbf{w}(\mathbf{y},1)\big|\,\mathrm{d}\mathbf{y} \leqslant \int_{\mathcal{T}^2} \int_1^{1+\|\Phi_\varepsilon\|_{L^\infty(\mathcal{T}^2)}} |\nabla_x \mathbf{w}|\,\mathrm{d}y. \tag{7.9}$$

Estimate (7.9) can be verified for any function $w \in W_0^{1,2}(V; R^3)$, in particular, for $\mathbf{w} = \mathbf{v}_\varepsilon$, via approximation by smooth functions.

Thus, in accordance with (7.8), we get

$$\lim_{\varepsilon\to 0}\int_{\mathcal{T}^2} \psi(\mathbf{y}) D\big(\nabla_y \Phi_\varepsilon(\mathbf{y})\big)\nabla_x \Phi_\varepsilon(\mathbf{y}) \cdot \big[v_\varepsilon^1, v_\varepsilon^2\big](\mathbf{y}, 1)\,\mathrm{d}\mathbf{y}$$

$$= \lim_{\varepsilon\to 0}\int_{\mathcal{T}^2} \psi(\mathbf{y}) D\big(\nabla_y \Phi_\varepsilon(\mathbf{y})\big) v_\varepsilon^3(\mathbf{y}, 1)\,\mathrm{d}\mathbf{y} \quad \text{for any } \psi \in \mathcal{D}\big(\mathcal{T}^2\big).$$

Finally, as

$$\mathbf{v}_\varepsilon(\cdot, 1) \to \mathbf{v}(\cdot, 1) \quad \text{in } L^2\big(\mathcal{T}^2; R^3\big) \quad \text{and} \quad v^3(\cdot, 1) = \mathbf{v}\cdot\mathbf{n} = 0,$$

we obtain

$$\int_{\mathcal{T}^2} \psi(\mathbf{y})\bigg(\big[v^1, v^2\big](\mathbf{y}) \cdot \int_{R^2} D(\mathbf{Z})\mathbf{Z}\,\mathrm{d}\mathcal{R}_{\mathbf{y}}(\mathbf{Z})\bigg)\,\mathrm{d}\mathbf{y} = 0$$

for any $\psi \in \mathcal{D}(\mathcal{T}^2)$ and all $D \in C^2(R^2)$. Consequently, relation (7.7) holds for any $\mathbf{y} \in \mathcal{T}^2$—a Lebesgue point of the mapping

$$\mathbf{y} \mapsto \big[v^1(\mathbf{y}, 1), v^2(\mathbf{y}, 1)\big]\mathcal{R}_{\mathbf{y}} \in L^2\big(\mathcal{T}^2; \mathcal{M}^+\big(R^2\big) \times \mathcal{M}^+\big(R^2\big)\big). \qquad \square$$

If the measure $\mathcal{R}_{\mathbf{y}}$ is non-degenerate, that means, if $\operatorname{supp}[\mathcal{R}_{\mathbf{y}}]$ is not contained in a 1D subspace of R^2, it is easy to check there exist two functions $D_i \in C(R^2)$, $i = 1, 2$, such that the vectors

$$\int_{R^2} D_i(\mathbf{Z})\mathbf{Z}\,\mathrm{d}\mathcal{R}_{\mathbf{y}}(\mathbf{Z}), \quad i = 1, 2,$$

form a basis in R^2. Thus, by virtue of (7.7), both v_1 and v_2 must vanish at $\mathbf{y}$, in other words, relation (7.5) yields (7.4) whenever $\mathcal{R}_{\mathbf{y}}$ is non-degenerate for a.a. $\mathbf{y} \in \mathcal{T}^2$. Theorem 4.1 has been proved.

7.2. *Proof of Corollaries 4.1–4.5*

(i) In order to show Corollary 4.1, observe first that

$$\nabla_y \Phi_\varepsilon(\mathbf{y}) = \nabla_y \Phi(\mathbf{y}/\varepsilon) \quad \text{for a.a. } \mathbf{y} \in \mathcal{T}^2.$$

As a consequence of the Riemann–Lebesgue lemma, we get that

$$G(\nabla_y \Phi_\varepsilon) \to \int_{\mathcal{T}^2} G(\nabla_y \Phi)\,\mathrm{d}\mathbf{y} \quad \text{weakly-(*) in } L^\infty\big(\mathcal{T}^2\big)$$

for any $G \in C(R^2)$. In particular, the rugosity measure is homogeneous, that means,

$$\mathcal{R}_{\mathbf{y}_1} = \mathcal{R}_{\mathbf{y}_2} \quad \text{for } \mathbf{y}_1, \mathbf{y}_2 \in \mathcal{T}^2.$$

Arguing by contradiction we assume that $\mathcal{R}$ is degenerate, say,

$$\operatorname{supp}[\mathcal{R}] \subset \left\{[0, Z_2] \;\middle|\; Z_2 \in R\right\}.$$

Taking $G(\mathbf{Z}) = Z_1$, $\mathbf{Z} = (Z_1, Z_2)$, we get

$$\int_{\mathcal{T}^2} |\partial_{y_1} \Phi|^2 \,\mathrm{d}\mathbf{y} = 0,$$

that means, Φ depends only on y_2 in contrast with the hypotheses of Corollary 4.1.

(ii) Under the hypotheses of Corollary 4.2, we have

$$\operatorname{supp}[\mathcal{R}_{\mathbf{y}}] \subset K \quad \text{for a.a. } \mathbf{y} \in \mathcal{T}^2;$$

whence, as the center of gravity of $\mathcal{R}_{\mathbf{y}}$ is the point $(0, 0)$, the rugosity measure must be non-degenerate.

(iii) The proof of Corollary 4.3 is straightforward.

(iv) As the quantity osc is continuous with respect to $\mathbf{w}$, it is easy to observe the hypotheses of Corollary 4.4 imply

$$\operatorname{osc}[w](\mathbf{y}) > c > 0 \quad \text{for all } \mathbf{w}, \quad |\mathbf{w}| = 1, \tag{7.10}$$

for a.a. $\mathbf{y} \in \mathcal{T}^2$.

On the other hand, we can assume

$$|\mathbf{w} \cdot \nabla_y \Phi_\varepsilon| \to \chi_w \quad \text{weakly-(*) in } L^\infty\big(\mathcal{T}^2\big),$$

where, by virtue of (7.10),

$$\chi_w(\mathbf{y}) > c \quad \text{for a.a. } \mathbf{y} \in \mathcal{T}^2.$$

Since

$$\chi_w(\mathbf{y}) = \int_{R^2} |\mathbf{w} \cdot \mathbf{Z}| \,\mathrm{d}\mathcal{R}_{\mathbf{y}}(\mathbf{Z}) \quad \text{for a.a. } \mathbf{y} \in \mathcal{T}^2$$

we conclude that

$$\int_{R^2} |\mathbf{w} \cdot \mathbf{Z}| \,\mathrm{d}\mathcal{R}_{\mathbf{y}}(\mathbf{Z}) \geqslant c \quad \text{for a.a. } \mathbf{y} \in \mathcal{T}^2;$$

whence, necessarily,

$$\int_{R^2} |\mathbf{w}\cdot\mathbf{Z}|\,\mathrm{d}\mathcal{R}_{\mathbf{y}}(\mathbf{Z}) \geqslant c \quad \text{for all } \mathbf{w}, \quad |\mathbf{w}| = 1,$$

for a.a. $\mathbf{y} \in \mathcal{T}^2$. Consequently, $\{\mathcal{R}_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{T}^2}$ is non-degenerate at a.a. $\mathbf{y}$ in the sense of Definition 3.1.

(v) Under the hypotheses of Corollary 4.5, we have

$$\nabla_y \Phi_\varepsilon(\mathbf{y}) = \sum_{i=1}^{m_\varepsilon} \nabla_y \mathcal{A}\left(\frac{\mathbf{y}-\mathbf{y}_i}{\delta_i}\right).$$

On the other hand, in accordance with (4.1), there exist two open balls $V_1, V_2 \subset R^2$ such that

$$\overline{V}_k \cap -\overline{V}_j = \emptyset \quad \text{for } k, j = 1, 2, \tag{7.11}$$

and

$$\operatorname{meas}\bigl\{\mathbf{y} \in R^2 \bigm| \nabla_y \mathcal{A}(\mathbf{y}) \in V_k\bigr\} = m_k > 0 \quad \text{for } k = 1, 2.$$

At this stage, it is convenient to introduce a norm d on R^2 as

$$d(\mathbf{y}) = \max_{k=1,2} \bigl|y^k\bigr| \quad \text{for all } \mathbf{y} = \bigl(y^1, y^2\bigr) \in R^2.$$

Consequently, for any square $Q \subset \mathcal{T}^2$ we have

$$\operatorname{meas}\bigl\{\mathbf{y} \in Q \bigm| \nabla_y \mathcal{A}(\mathbf{y}) \in V_k\bigr\} \geqslant m_k \sum_{\{i \mid \mathbf{y}_i \in Q,\ \operatorname{dist}_d(\mathbf{y}_i, \partial Q) \geqslant \delta_i\}} (\delta_i)^2 \geqslant m_k(1-4\varepsilon)\operatorname{meas}[Q].$$

Thus we have

$$\operatorname{supp}[\mathcal{R}_{\mathbf{y}}] \cap \overline{V}_k \neq \emptyset \quad \text{for } k = 1, 2 \text{ and a.a. } \mathbf{y} \in \mathcal{T}^2;$$

whence, by virtue of (7.11), the family $\{\mathcal{R}_{\mathbf{y}}\}_{\mathbf{y}\in\mathcal{T}^2}$ meets the hypotheses of Theorem 4.1.

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