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Discretization of the Navier-Stokes Equations with Slip Boundary Condition II

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Abstract—Multilevel methods are a common tool for solving nonlinear systems arising from discretizations of elliptic boundary value problems (see, e.g., [1,2]). Multilevel methods consist of solving the nonlinear problem on a coarse mesh and then performing one or two Newton correction steps on each subsequent mesh thus solving only one or two larger linear systems. In a previous paper by the author [3], a two-level method was proposed for the stationary Navier-Stokes equations with slip boundary condition. Uniqueness of solution to the nonlinear problem (Step 1 of the two-level method) is guaranteed provided that the data of the problem (i.e., the Reynolds number, and the forcing term) is bounded.

In practice, however, the aforementioned bound on the data is rarely satisfied. Consequently, *a priori* error estimates should not rely upon this bound. Such estimates will be established in this report. For the two-level method, the scalings of the meshwidths that guarantee optimal accuracy in the H^1-norm, are equally favorable to those in the uniqueness case. Published by Elsevier Ltd.

Keywords—Navier-Stokes equations, Slip boundary condition, Two-level discretization method, Branches of nonsingular solutions.

1. INTRODUCTION

Consider the incompressible Navier-Stokes equations with slip boundary condition

$$-\operatorname{Re}^{-1}\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \qquad \text{in } \Omega, \tag{1.1a}$$

$$\operatorname{div}\mathbf{u} = 0, \qquad \text{in } \Omega, \tag{1.1b}$$

$$\mathbf{u}\cdot\mathbf{n} = 0, \qquad \text{on } \partial\Omega, \tag{1.1c}$$

$$\mathbf{n}\cdot\Im(\operatorname{Re}^{-1}\mathbf{u},p)\cdot\tau_k = 0, \qquad \text{on } \partial\Omega, \tag{1.1d}$$

where Ω is a simply-connected, bounded polygonal domain in $\mathbb{R}^d$ ($d = 2,3$) whose boundary is piecewise of class C^3, $\mathbf{n} = (n_1,\dots,n_d)$ is the exterior unit normal, $\tau_1,\dots,\tau_{d-1}$ form an orthonormal set of tangent vectors, and $\Im(.,.)$ is the stress tensor defined by

$$\Im(\underline{v},q)_{i,j} = -q\delta_{i,j} + \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}, \qquad 1 \le i,j \le d.$$

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As in the authors' previous paper, the condition $\mathbf{u} \cdot \mathbf{n} = 0$ will be imposed *weakly* on Γ which, physically, implies a zero mean flow on the boundary or roughness of the boundary. We also assume that $\mathbf{f}$ is an L^2 function.

Direct observations and comparisons between numerical simulations and experimental results have shown that the no-slip condition is the correct physical model in fluid flow with *moderate velocities and pressures.* The slip condition applies mainly to free surfaces in free boundary problems (such as the *coating problem* in [4, pp. 9,10; 5]) which are modeled as being stress free (condition (1.1d)).

Equation (1.1) has, in general, more than one solution [3,4]. As shown by the author in [3], uniqueness is guaranteed if the Reynolds number and the external forces satisfy a "small data condition"

$$C\,\mathrm{Re}^2\ |\mathbf{f}|_* < 1, \qquad \text{where } |\mathbf{f}|_* := \sup_{\mathbf{v}\in\mathbf{V}} \frac{\langle \mathbf{f}, \mathbf{v}\rangle_\Omega}{|\mathbf{v}|_1}, \tag{1.2}$$

which, in most real world applications, is *not* satisfied. It is important, without relying on the uniqueness condition, to have an *a priori* result guaranteeing that the proposed multilevel method converges. In order to bypass the *small data* (uniqueness) condition we use the theory of Brezzi, Rappaz and Raviart [7] introduced in Section 2.

In Section 3, we apply the abstract framework of Brezzi, Rappaz and Raviart [7] to the Navier-Stokes equations with slip boundary condition. Using their ideas and also those of Verfürth [4], the definition of nonsingular solutions to the homogeneous Dirichlet problem is extended to those of the problem with slip boundary condition (see Definition 3.1).

Section 4 includes the study of nonsingular solutions to the discrete Navier-Stokes equations with slip boundary condition. We follow Girault and Raviart [8] and Verfürth [4] to give a definition for the nonsingular solution to the discrete problem (see Definition 4.1) assuming sufficient regularity of the true solution (cf. (3.13)). Under the assumption that the discrete problem has a nonsingular solution, we prove that its best approximation in the discrete div-free space is also nonsingular.

In Section 5, we consider a multilevel Newton method proposed by Layton and Lenferink in [1]. In this method we need to solve only a *few* nonlinear equations on the coarsest mesh, and then use this solution to solve *one or two* linear systems on the finer meshes. We will assume that (1.1) has a nonsingular solution and prove that the approximate solution on the coarse mesh is also nonsingular. In addition, we will assume that the coarse mesh approximate solution is sufficiently accurate to guarantee that the solutions on the finer meshes are also nonsingular. Let the mesh-widths be $h_0 > h_1 > \cdots > h_n$ with scalings

$$h_{i+1} = \mathcal{O}\left(h_i^{\mu_i}\right), \qquad \mu_i > 1, \quad i = 0, 1, \dots, n-1.$$

The exponents μ_i are chosen in such a way that the error on the finest mesh is optimal in h_n.

The method is as follows.

ALGORITHM. $(n+1, S)$ Newton.

STEP 1. Construct velocity, pressure, and Lagrange multiplier finite-element spaces: for $0 \le i \le n$ let $\Gamma = \bigcup_{j=1}^{R} \Gamma_j$ with corresponding outward normals $\mathbf{n}_j$ and

$$\begin{aligned} \mathbf{X}^{h_i} &\subset \mathbf{X} := \left(\frac{\mathrm{H}^1(\Omega)}{\mathcal{S}}\right)^d, \\ Y^{h_i} &\subset Y := \mathrm{L}_0^2(\Omega), \\ Z^{h_i} &\subset Z := \prod_{j=1}^{R} \mathrm{H}^{-1/2}(\Gamma_j), \end{aligned}$$

where

$$\mathcal{S} := \operatorname{span}\left\{\mathbf{u}(\underset{\sim}{x}) = \underset{\sim}{\zeta} \times \underset{\sim}{x},\ \underset{\sim}{\zeta} \in \mathbb{R}^3,\ \left|\underset{\sim}{\zeta}\right| = 1,\ \underset{\sim}{\zeta} \text{ is an axis of symmetry of } \Omega\right\}$$

is the space of rigid body rotations of Ω.

STEP 2. Find $\mathbf{u}^{h_0} \in \mathbf{X}^{h_0}$, $p^{h_0} \in Y^{h_0}$, $\rho^{h_0} \in Z^{h_0}$ that satisfy the *nonlinear* problem

$$\begin{aligned} a\left(\mathbf{u}^{h_0}, \mathbf{v}^{h_0}\right) + b_*\left(\mathbf{u}^{h_0}; \mathbf{u}^{h_0}, \mathbf{v}^{h_0}\right) + c\left(\mathbf{v}^{h_0}; p^{h_0}, \rho^{h_0}\right) &= \left\langle \mathbf{f}, \mathbf{v}^{h_0} \right\rangle_\Omega, \\ c\left(\mathbf{u}^{h_0}; q^{h_0}, \sigma^{h_0}\right) &= 0, \end{aligned} \tag{1.3}$$

for all $(\mathbf{v}^{h_0}, q^{h_0}, \sigma^{h_0}) \in (\mathbf{X}^{h_0}, Y^{h_0}, Z^{h_0})$.

STEP 3. For $i = 0, 1, \dots, n-1$:

STEP 3A. Set $\mathbf{u}_0^{h_{i+1}} = \mathbf{u}^{h_i}$.

STEP 3B. For $s = 1, \dots, S$ find $\mathbf{u}_s^{h_{i+1}} \in \mathbf{X}^{h_{i+1}}$, $p_s^{h_{i+1}} \in Y^{h_{i+1}}$, and $\rho_s^{h_{i+1}} \in Z^{h_{i+1}}$ that satisfy the *linear* problem

$$\begin{aligned} a\left(\mathbf{u}_s^{h_{i+1}}, \mathbf{v}^{h_{i+1}}\right) &+ b_*\left(\mathbf{u}_{s-1}^{h_{i+1}}; \mathbf{u}_s^{h_{i+1}}, \mathbf{v}^{h_{i+1}}\right) + b_*\left(\mathbf{u}_s^{h_{i+1}}; \mathbf{u}_{s-1}^{h_{i+1}}, \mathbf{v}^{h_{i+1}}\right) \\ &= \left\langle \mathbf{f}, \mathbf{v}^{h_{i+1}} \right\rangle_\Omega + b_*\left(\mathbf{u}_{s-1}^{h_{i+1}}; \mathbf{u}_{s-1}^{h_{i+1}}, \mathbf{v}^{h_{i+1}}\right) - c\left(\mathbf{v}^{h_{i+1}}; p_s^{h_{i+1}}, \rho_s^{h_{i+1}}\right), \\ c\left(\mathbf{u}_s^{h_{i+1}}; q^{h_{i+1}}, \sigma^{h_{i+1}}\right) &= 0, \end{aligned} \tag{1.4}$$

for all $(\mathbf{v}^{h_{i+1}}, q^{h_{i+1}}, \sigma^{h_{i+1}}) \in (\mathbf{X}^{h_{i+1}}, Y^{h_{i+1}}, Z^{h_{i+1}})$.

STEP 3C. Set $\mathbf{u}^{h_{i+1}} = \mathbf{u}_S^{h_{i+1}}$, $p^{h_{i+1}} = p_S^{h_{i+1}}$, $\rho^{h_{i+1}} = \rho_S^{h_{i+1}}$.

Here

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &:= \frac{1}{2\,\mathrm{Re}} \int_\Omega \mathcal{D}(\mathbf{u}) \cdot \mathcal{D}(\mathbf{v})\, dx, \\ b(\mathbf{u}; \mathbf{v}, \mathbf{w}) &:= \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{w}\, dx, \\ b_*(\mathbf{u}; \mathbf{v}, \mathbf{w}) &:= \frac{1}{2}[b(\mathbf{u}; \mathbf{v}, \mathbf{w}) - b(\mathbf{u}; \mathbf{w}, \mathbf{v})], \\ c(\mathbf{u}; p, \rho) &:= -\int_\Omega p \operatorname{div} \mathbf{u}\, dx - \sum_{j=1}^{R} \langle \rho, \mathbf{u} \cdot \mathbf{n}_j \rangle_{\Gamma_j}, \end{aligned} \tag{1.5}$$

where $\mathcal{D}$ is the *deformation tensor* and is given by

$$\mathcal{D}(\mathbf{v})_{i,j} := \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}, \qquad 1 \le i, \quad j \le d. \tag{1.6}$$

Table 1. Scalings μ_i between mesh levels leading to quasi-optimal approximations for the algorithm $(n+1, 1)$ Newton. Here, ϵ is an arbitrary number in $(0, 1)$ and k is the polynomial degree of the discrete space.

Dimension d, and Norm	Algorithm $(n+1, 1)$ Newton
$d = 2$ H^1-norm	$\mu_0 = 2 + \frac{1-\epsilon}{k}$, $\mu_i = 2 + \frac{1}{k}$ for $i > 0$.
$d = 2$ L^2-norm	$\mu_0 = 2 - \frac{\epsilon+1}{k+1}$, $\mu_i = 2 - \frac{1}{k+1}$ for $i > 0$.
$d = 3$ H^1-norm	$\mu_0 = 2 + \frac{1}{2k}$, $\mu_i = 2 + \frac{1}{k}$ for $i > 0$.
$d = 3$ L^2-norm	$\mu_0 = 2 - \frac{3}{2(k+1)}$, $\mu_i = 2 - \frac{1}{k+1}$ for $i > 0$.

In Section 6, we derive L^2-error estimates for the multilevel method using techniques from the author's previous work [3] and the work of Layton and Lenferink [1]. With the assumption that the coarse mesh velocity error is optimal in $(L^2(\Omega))^d$, the two level method approximation error is optimal in the "graph" norm $(H^1(\Omega)^d \times L^2(\Omega)) \times \prod_{j=1}^{R} H^{-1/2}(\Gamma_j)$ with favorable scalings. With the same scalings, however, the fine mesh velocity error is not optimal in $(L^2(\Omega))^d$. Therefore, we should reduce the scaling if we desire to apply the method recursively and still get optimality in the graph norm. See Table 1 for summary.

Finally, Section 7 includes the derivation of the successive mesh scalings μ_i that yield optimal/quasi-optimal approximations $(\mathbf{u}^{h_i}, p^{h_i}, \rho^{h_i})$, assuming full elliptic regularity of the Stokes problem, velocity elements of polynomial degree k, pressure elements of degree at least $k-1$, and normal stress component elements of polynomial degree $k-1$ on the boundary.

2. NOTATION AND PRELIMINARIES

Let $\mathcal{X}$ and $\tilde{\mathcal{X}}$ be two Banach spaces and Λ a compact interval in $\mathbb{R}$. Given the C^p mapping $(p \geq 1)$

$$F(\lambda, \underline{u}) : \Lambda \times \mathcal{X} \to \tilde{\mathcal{X}}$$

we would like to solve the problem

$$F(\lambda, \underline{u}) = 0. \tag{2.1}$$

DEFINITION 2.1. *Let $(\lambda, \underline{u}(\lambda))$ (with $\lambda \in \Lambda$) be such that*

$$\begin{gathered} \underline{u}(\lambda) \text{ is a continuous function of } \lambda, \\ F(\lambda, \underline{u}(\lambda)) = 0. \end{gathered} \tag{2.2}$$

We call $(\lambda, \underline{u}(\lambda))$ a branch of solutions to equation (2.1).

DEFINITION 2.2. *We call nonsingular the solutions $\underline{u}$ of (2.1) for which*

$$D_{\underline{u}} F(\lambda, \underline{u}(\lambda)) \text{ is an isomorphism from } \mathcal{X} \text{ onto } \tilde{\mathcal{X}}, \qquad \forall \lambda \in \Lambda. \tag{2.3}$$

In other words, Definition 2.2 implies that nonsingular solutions are solutions that are not turning or bifurcation points. We can substitute statement (2.3) with the following (given in [8, Definition 2.1, p. 116]).

DEFINITION 2.3. *Assume that F is a differentiable mapping from $\mathcal{X}$ to $\tilde{\mathcal{X}}$. We say that $\underline{u}$ is a nonsingular solution if there exists a constant $\gamma > 0$, such that for each λ*

$$\|D_{\underline{u}} F(\lambda, \underline{u}(\lambda)) \cdot \underline{v}\|^*_{\tilde{\mathcal{X}}} \geq \gamma \|\underline{v}\|_{\mathcal{X}}, \qquad \forall \underline{v} \in \mathcal{X}, \tag{2.4}$$

where for $\mathcal{L} \in \tilde{\mathcal{X}}$

$$\|\mathcal{L}\|^*_{\tilde{\mathcal{X}}} = \sup_{\underline{w} \in \mathcal{X}} \frac{\langle \mathcal{L}, \underline{w} \rangle_{\mathcal{X}}}{\|\underline{w}\|_{\mathcal{X}}}.$$

We denote by $H^k(\Omega)$ the usual $W^{k,2}(\Omega)$ Sobolev space with norm

$$\|\mathbf{w}\|_k = \left\{ \sum_{i=1}^{d} \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha w_i|^2 \, dx \right) \right\}^{1/2}, \qquad k \in \mathbb{N},$$

by $|.|_k$ the induced seminorm, and by $L^2(\Omega)$ the space $W^{0,2}(\Omega)$. The dual space of $H^k(\Omega)$ is $H^{-k}(\Omega)$ with $\langle .\,, . \rangle_\Omega$ being the duality pairing. This is an abuse of notation since $H^{-k}(\Omega)$ usually denotes the dual of $H^k_0(\Omega)$ which consists of all functions in $H^k(\Omega)$ that vanish on the boundary. The spaces $H^{k-1/2}(\Gamma)$ consist of the traces of all functions in $H^k(\Omega)$. Analogously, $H^{-(k-1/2)}(\Gamma)$ is the dual space of $H^{k-1/2}(\Gamma)$ with $\langle .,. \rangle_\Gamma$ being the duality pairing.

The natural norm of the normal component of a function $\mathbf{u} \in H^1(\Omega)$ on Γ is

$$\|\mathbf{u}\cdot\mathbf{n}\|_\Gamma = \left(\sum_{i=1}^{R} \|\mathbf{u}\cdot\mathbf{n}_i\|^2_{1/2,\Gamma_i}\right)^{1/2}, \tag{2.5}$$

with dual norm $\|.\|^*_\Gamma$. Assuming that Ω is polygonal implies $\mathbf{u}\cdot\mathbf{n} \in H^{-1/2}(\Gamma)$ *not* in $H^{1/2}(\Gamma)$ (see [9, Remark 1.1, p. 9]).

The solution to the Stokes problem is unique up to rigid body rotations of the domain Ω. That is because the Stokes operator is symmetric and is unaffected by the skew symmetric spin tensor which describes rigid body rotations. Therefore, we define the space of rigid body rotations as

$$\mathcal{S} = \operatorname{span}\left\{\mathbf{u}(\underline{x}) = \underline{\zeta}\times\underline{x},\ \underline{\zeta}\in\mathbb{R}^3,\ |\underline{\zeta}| = 1,\ \underline{\zeta} \text{ is an axis of symmetry of } \Omega\right\}, \tag{2.6}$$

and define the velocity space to be the set of all functions with entries in $H^1(\Omega)$ modulo $\mathcal{S}$. Thus,

$$\begin{aligned}
\mathbf{X} &:= \left(H^1(\Omega)\right)^d/\mathcal{S},\\
\mathbf{X}_{n,0} &:= \left\{\mathbf{u}\in\mathbf{X} \mid \mathbf{u}\cdot\mathbf{n} = 0,\ \text{on } \Gamma\right\},\\
Y &:= L^2_0(\Omega),\\
Z &:= \prod_{j=1}^{R} H^{-1/2}(\Gamma_j),\\
\mathbf{V} &:= \{\mathbf{u}\in\mathbf{X}_{n,0} \mid \operatorname{div}\mathbf{u} = 0\}.
\end{aligned}$$

3. NONSINGULAR SOLUTIONS OF THE CONTINUOUS PROBLEM

Consider the weak formulation of the Navier-Stokes equations with slip boundary condition

$$\begin{aligned}
a(\mathbf{u},\mathbf{v}) + b(\mathbf{u};\mathbf{u},\mathbf{v}) + c(\mathbf{v};p,\rho) &= \langle \mathbf{f},\mathbf{v}\rangle_\Omega, && \forall\,\mathbf{v}\in\mathbf{X},\\
c(\mathbf{u};q,\sigma) &= 0, && \forall\,q\in Y,\quad \sigma\in Z,
\end{aligned} \tag{3.1}$$

where $a(.,.)$, $b(.;.,.)$, and $c(.;.,.)$ are defined in (1.5). We can rewrite this as

$$\begin{aligned}
a(\mathbf{u},\mathbf{v}) + c(\mathbf{v};p,\rho) &= \langle \mathbf{f} - (\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v}\rangle_\Omega, && \forall\,\mathbf{v}\in\mathbf{X},\\
c(\mathbf{u};q,\sigma) &= 0, && \forall\,q\in Y,\quad \sigma\in Z.
\end{aligned} \tag{3.2}$$

Let

$$\mathcal{X} := \left(H^1(\Omega)\right)^d/\mathcal{S}\times L^2_0(\Omega)\times\prod_{j=i}^{R} H^{-1/2}(\Gamma_j)$$

and define the affine operator $T : H^{-1}(\Omega)^d \to \mathcal{X}$ to be the solution operator of the Stokes problem with slip boundary condition; that is given a function $\mathbf{g}\in(H^{-1}(\Omega))^d$ the operator T yields the *unique* (mod $\mathcal{S}$) solution $\underline{u} = (\mathbf{u},p,\rho)$ to the problem

$$\begin{aligned}
a(\mathbf{u},\mathbf{v}) + c(\mathbf{v};p,\rho) &= \langle \mathbf{g},\mathbf{v}\rangle_\Omega, && \forall\,\mathbf{v}\in\mathbf{X},\\
c(\mathbf{u};q,\sigma) &= 0, && \forall\,q\in Y,\quad \sigma\in Z.
\end{aligned} \tag{3.3}$$

Let $G : (H^1(\Omega))^d \to (L^{3/2}(\Omega))^d$ be defined as

$$G(\mathbf{u}) = \mathbf{f} - (\mathbf{u}\cdot\nabla)\mathbf{u}. \tag{3.4}$$

Clearly, $T(\lambda G(\mathbf{u}))$ (with $\lambda = \mathrm{Re}$) is the unique solution to problem (3.1). On the other hand, the solution to (3.1) is also a solution to

$$F(\lambda, \underline{u}) = \underline{u} - T(\lambda G(\mathbf{u})) = 0. \tag{3.5}$$

Assuming that $\underline{u}$ is a *nonsingular* solution to (3.2) implies that for any $\underline{w} \in \mathcal{X}$ there exists a unique $\underline{v} \in \mathcal{X}$ satisfying

$$D_{\underline{u}}F(\lambda, \underline{u}) \cdot \underline{v} = \underline{w}, \tag{3.6}$$

where $\underline{v} = (\mathbf{v}, p_1, \rho_1)$, $\underline{w} = (\mathbf{w}, p_2, \rho_2)$. A direct calculation shows that

$$D_{\underline{u}}F(\lambda, \underline{u}) \cdot \underline{v} = \underline{v} - T\left(\lambda\left(u_i \frac{\partial \mathbf{v}}{\partial x_i} + v_i \frac{\partial \mathbf{u}}{\partial x_i}\right)\right).$$

Problem (3.6) now becomes: for any $\underline{w}$ find a unique $\underline{v}$ satisfying

$$\begin{aligned} \lambda a(\mathbf{v} - \mathbf{w}, \mathbf{z}) + c(\mathbf{z}; \lambda(p_1 - p_2), \lambda(\rho_1 - \rho_2)) &= \left\langle -\lambda\left(u_i \frac{\partial \mathbf{v}}{\partial x_i} + v_i \frac{\partial \mathbf{u}}{\partial x_i}\right), \mathbf{z}\right\rangle_\Omega, \\ c(\mathbf{v} - \mathbf{w}; q, \sigma) &= 0, \end{aligned} \tag{3.7}$$

for all $\mathbf{z} \in \mathbf{X}$, $q \in Y$, $\sigma \in Z$. Using [4] we see that for every $\tilde{\mathbf{f}} \in (\mathrm{H}^{-1}(\Omega))^d$ there exists a unique solution $\underline{r}$ to (3.3), provided the form $c(.;.,.)$ satisfies an appropriate LBB-condition. Let $\underline{r} = \underline{w}$ in (3.7). Adding equations (3.3) for $\underline{w}$ to (3.7) yields the following: *$\underline{u}$ is a nonsingular solution of problem (3.2) if and only if for each $\tilde{\mathbf{f}} \in (H^{-1}(\Omega))^d$ the following problem:*

$$\begin{aligned} a(\mathbf{v}, \mathbf{z}) + c(\mathbf{z}; p_1, \rho_1) + \left\langle\left(u_i \frac{\partial \mathbf{v}}{\partial x_i} + v_i \frac{\partial \mathbf{u}}{\partial x_i}\right), \mathbf{z}\right\rangle_\Omega &= \left\langle \tilde{\mathbf{f}}, \mathbf{z}\right\rangle_\Omega, \\ c(\mathbf{v}; q, \sigma) &= 0, \end{aligned} \tag{3.8}$$

has a unique solution $\underline{v} \in (\mathbf{X}, Y, Z)$, for all $\mathbf{z} \in \mathbf{X}$, $q \in Y$, $\sigma \in Z$. In the div-free space $\mathbf{V}$ problem (3.8) becomes

$$a(\mathbf{v}, \mathbf{z}) + b_*(\mathbf{u}; \mathbf{v}, \mathbf{z}) + b_*(\mathbf{v}; \mathbf{u}, \mathbf{z}) = \left\langle \tilde{\mathbf{f}}, \mathbf{z}\right\rangle_\Omega, \qquad \forall \mathbf{z} \in \mathbf{V}. \tag{3.9}$$

REMARK 3.1. In [3, Lemma 3.2], the author proved that for the Navier-Stokes equations with slip boundary condition $b(\mathbf{u}; \mathbf{v}, \mathbf{w}) = b_*(\mathbf{u}; \mathbf{v}, \mathbf{w})$ for $\mathbf{u} \in \mathbf{V}$ and $\mathbf{v}, \mathbf{w} \in \mathbf{X}$.

Using the abstract elliptic theory of Lax-Milgram [10] we arrive at the following definition.

DEFINITION 3.1. *The function $\underline{u} = (\mathbf{u}, 0, 0)$, $\mathbf{u} \in \mathbf{V}$ is a nonsingular solution to equation (1.1) if and only if there exists a constant $\gamma > 0$, such that*

$$\sup_{\mathbf{z} \in \mathbf{V}} \frac{A(\mathbf{u}, \mathbf{v}, \mathbf{z})}{\|\mathbf{z}\|_{\mathbf{V}}} \geq \gamma \|\mathbf{v}\|_{1,\Omega}, \qquad \forall \mathbf{v} \in \mathbf{V}, \tag{3.10}$$

where

$$A(\mathbf{u}, \mathbf{v}, \mathbf{z}) = a(\mathbf{v}, \mathbf{z}) + b_*(\mathbf{u}; \mathbf{v}, \mathbf{z}) + b_*(\mathbf{v}; \mathbf{u}, \mathbf{z}). \tag{3.11}$$

REMARK 3.2. To the knowledge of the author, proving that solutions to the Navier-Stokes equations with slip boundary condition are, in general, nonsingular is an open problem.

A quantity used in our error estimates is

$$\mathcal{K} = \max\left\{\|\mathbf{f}\|_{0,\Omega}, \sup_{\lambda \in \Lambda} \|\mathbf{u}(\lambda)\|_{2,\Omega}\right\} < \infty. \tag{3.12}$$

Note that we have used the H^2-norm of $\mathbf{u}(\lambda)$. Using the regularity estimate

$$\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega} \leq C\|\mathbf{f}\|_{0,\Omega}, \tag{3.13}$$

for the Stokes problem and a bootstrapping argument, one can show that $\mathbf{u}(\lambda) \in \mathrm{H}^2(\Omega)$, provided that Γ is of class $\mathcal{C}^2$, [11]), or $d = 2$, Ω is convex and Γ is Lipschitz continuous [12]. Even though it is unclear whether $\mathbf{u}(\lambda) \in \mathrm{H}^2(\Omega)$ for polygonal domains in 3-D, we will assume that (3.13) holds.

4. NONSINGULAR SOLUTIONS OF THE DISCRETE PROBLEM

Our domain Ω is subdivided into d-simplices with sides of length less than h with $\mathcal{T}^h$ being the corresponding family of partitions. We will assume that $\mathcal{T}^h$ satisfies the usual regularity assumptions (see, e.g., [13]). For each j, denote by $\mathcal{O}_j^h$ the partition of Γ_j which is induced by $\mathcal{T}^h$. Let $\mathcal{P}_k$, $k \geq 0$, be the set of all polynomials in $x_1, \dots, x_d$ of degree less than or equal to k and set

$$\mathcal{S}_k^h := \left\{\phi : \Omega \to \mathbb{R} \mid \phi|_T \in \mathcal{P}_k,\ \forall \mathcal{T} \in \mathcal{T}^h\right\}. \tag{4.1}$$

As seen in [4] or [3] the boundary conditions induce an *additional* Lagrange multiplier. In order to stabilize the effect of the new Lagrange multiplier Verfürth [4] inserts bubble functions, $(\mathcal{B}^h)$ for the elements with a face on the boundary, into the velocity space. To this end, define the spaces $\tilde{\mathbf{X}}^h$ and $\tilde{\mathbf{X}}_0^h$ as

$$\tilde{\mathbf{X}}^h = \left\{\mathcal{S}_k^h \cap C\left(\bar{\Omega}\right)\right\}^d,$$
$$\tilde{\mathbf{X}}_0^h = \left\{\mathbf{u}^h \in \tilde{\mathbf{X}}^h \mid \mathbf{u}^h = 0 \text{ on } \Gamma\right\}.$$

Consider an element $K^{(j)} \in \mathcal{T}^h$ which has a face $\partial K^{(j)}$ on the boundary piece Γ_j. Number the vertices of $K^{(j)}$ so that the vertices on $\partial K^{(j)}$ are numbered first. Let $\xi_1(K^{(j)}), \dots, \xi_{d+1}(K^{(j)})$ be the barycentric coordinates of $K^{(j)}$. Define the bubble functions $\mathbf{b}_{(j)}^h$ on $K^{(j)}$ as follows:

$$\mathbf{b}_{(j)}^h = \begin{cases} \mathbf{n}_i \prod_{i=1}^{d} \xi_i\left(K^{(j)}\right), & \text{on } K^{(j)}, \\ 0, & \text{on } \Omega \setminus K^{(j)}. \end{cases} \tag{4.2}$$

Let

$$\mathcal{B}^h = \operatorname{span}\left\{\mathbf{b}_{(j)}^h \sigma \mid 1 \leq j \leq s,\ \sigma \in \mathcal{P}_m,\ \partial K^{(j)} \in \mathcal{O}_j^h\right\}. \tag{4.3}$$

Then the discrete analogs of the spaces $\mathbf{X}$, Y, Z, and $\mathbf{V}$ are

$$\begin{aligned}
&\mathbf{X}^h \subset \left(\tilde{\mathbf{X}}^h \oplus \mathcal{B}^h\right)\big/\mathcal{S}, && k \geq 1,\\
&Y^h \subset \mathcal{S}_l^h \cap L_0^2(\Omega), && l \geq 0,\\
&Z^h = \left\{\phi : \Gamma \to \mathbb{R} \mid \phi|_{\partial K^{(j)}} \in \mathcal{P}_m,\ \forall \partial K^{(j)} \in \mathcal{O}_j^h,\ 1 \leq j \leq s\right\}, && m \geq 0,\\
&\mathbf{V}^h = \left\{\mathbf{u}^h \in \mathbf{X}^h \mid c\left(\mathbf{u}^h, q^h, \sigma^h\right) = 0,\ \forall q^h \in Y^h,\ \sigma^h \in Z^h\right\}.
\end{aligned}$$

The spaces $\tilde{\mathbf{X}}_0^h$ and Y^h are assumed to satisfy the following properties.

I. There is a constant $\tilde{\beta} > 0$ independent of h for which

$$\inf_{0 \neq p^h \in Y^h} \sup_{0 \neq \mathbf{u}^h \in \tilde{\mathbf{X}}_0^h} \frac{\int_\Omega p^h \operatorname{div} \mathbf{u}^h \, dx}{\|p^h\|_{0,\Omega} \|\mathbf{u}^h\|_{1,\Omega}} \geq \tilde{\beta}. \tag{4.4a}$$

II.

$$\inf_{p^h \in Y^h} \left\|p - p^h\right\|_{0,\Omega} \leq ch\|p\|_{1,\Omega}, \qquad \forall p \in \mathrm{H}^1(\Omega). \tag{4.4b}$$

III. There exists a continuous linear operator $\Pi^h : \mathrm{H}^1(\Omega)^d \to \tilde{\mathbf{X}}^h$ for which

$$\Pi^h\left(\mathrm{H}_0^1(\Omega)^d\right) \subset \tilde{\mathbf{X}}_0^h, \tag{4.4c}$$

$$\left\|\mathbf{u} - \Pi^h \mathbf{u}\right\|_{s,\Omega} \leq ch^{t-s}\|\mathbf{u}\|_{t,\Omega}, \qquad \forall \mathbf{u} \in \left(\mathrm{H}^t(\Omega)\right)^d, \quad s = 0, 1, \quad t = 1, 2, \tag{4.4d}$$

$$\left\|\mathbf{u} - \Pi^h \mathbf{u}\right\|_{0,\Gamma} \leq ch^{1/2}\|\mathbf{u}\|_{1,\Omega}. \tag{4.4e}$$

We have to select discrete spaces that satisfy the above assumptions as well as satisfy an additional LBB-condition

$$\inf_{\substack{0\neq p^h\in Y^h\\ 0\neq\rho^h\in Z^h}} \sup_{0\neq \mathbf{u}^h\in\mathbf{X}^h} \frac{c\left(\mathbf{u}^h; p^h, \rho^h\right)}{\|\mathbf{u}^h\|_{1,\Omega}\left\{\|p^h\|_{0,\Omega}^2 + \left(\|\rho^h\|_\Gamma^*\right)^2\right\}^{1/2}} \geq \beta, \tag{4.5}$$

for $\beta > 0$ independent of h, that balances the new Lagrange multiplier. Examples of spaces that satisfy Assumptions I–III as well as condition (4.5) include, among others, the *MINI*-element of Arnold, Brezzi and Fortin [14, pp. 337–344], see [15], and the Taylor-Hood element, see [16].

The discrete Stokes problem with slip boundary condition is as follows: find $\underline{u}^h = (\mathbf{u}^h, p^h, \rho^h)$ satisfying:

$$\begin{aligned} a\left(\mathbf{u}^h, \mathbf{v}^h\right) + b_*\left(\mathbf{u}^h; \mathbf{u}^h, \mathbf{v}^h\right) + c\left(\mathbf{v}^h; p^h, \rho^h\right) &= \langle \mathbf{z}, \mathbf{v}^h\rangle_\Omega \\ c\left(\mathbf{u}^h; q^h, \sigma^h\right) &= 0, \end{aligned} \tag{4.6}$$

for all $\mathbf{v}^h \in \mathbf{X}^h$, $q^h \in Y^h$, $\sigma^h \in Z^h$. Defining $F(\lambda, \underline{u}^h(\lambda))$ as in the last section, we say that

$$\underline{u}^h \text{ is a } \textit{nonsingular} \text{ solution to } F\left(\lambda, \underline{u}^h(\lambda)\right) = 0 \iff \left(D_{\mathbf{u}^h} F\left(\lambda, \underline{u}^h\right)\right)^{-1} \text{ exists.}$$

We can also give the following definition.

DEFINITION 4.1. *The function $\underline{u}^h = (\mathbf{u}, 0, 0)$, $\mathbf{u}^h \in \mathbf{V}^h$ is a nonsingular solution to the discrete Navier-Stokes equation with slip boundary condition if and only if there exists a constant $\gamma^h > 0$, such that*

$$\sup_{\mathbf{z}^h\in\mathbf{V}^h} \frac{A\left(\mathbf{u}^h, \mathbf{v}^h, \mathbf{z}^h\right)}{\|\mathbf{z}^h\|_{\mathbf{V}^h}} \geq \gamma^h \left\|\mathbf{v}^h\right\|_{1,\Omega}, \qquad \forall\, \mathbf{v}^h \in \mathbf{V}^h, \tag{4.7}$$

where $A(.,.,.)$ is as in equation (3.11).

Assume that $\mathbf{u}(\lambda)$ is a nonsingular solution to (3.1) and let $\tilde{\mathbf{u}}^h(\lambda)$ be the best approximation to $\mathbf{u}(\lambda)$ in $\mathbf{X}^h$ with respect to the H^1-norm. It is proven by Verfürth [4, Lemma 6.2] that for sufficiently small h, $(\tilde{\mathbf{u}}^h, \tilde{p}^h, \tilde{\rho}^h)$ is a nonsingular solution to (4.6). We can also show that $\tilde{\underline{u}}^h := (\tilde{\mathbf{u}}^h, \tilde{p}^h, \tilde{\rho}^h)$ is a nonsingular solution to $F(\lambda, \underline{u}^h(\lambda)) = 0$, thereby satisfying (4.7), see [8, Lemma 3.3, p. 130].

Following Remark 3.1 in [8, p. 130], we see that (4.7) is satisfied by functions "near" $\tilde{\mathbf{u}}^h$. That is, there exist two constants $\tilde{\gamma}_* > 0$ and $\tilde{\delta} > 0$, such that

$$\sup_{\mathbf{w}^h\in\mathbf{V}^h} \frac{A\left(\mathbf{y}^h, \mathbf{v}^h, \mathbf{w}^h\right)}{\|\mathbf{w}^h\|_{1,\Omega}} \geq \tilde{\gamma}_* \left\|\mathbf{v}^h\right\|_{1,\Omega}, \qquad \forall\, \mathbf{v}^h \in \mathbf{V}^h, \tag{4.8}$$

for all $\mathbf{y}^h \in \mathbf{V}^h$, such that $|\mathbf{y}^h - \tilde{\mathbf{u}}^h|_{1,\Omega} \leq \tilde{\delta}$.

Consider a different mesh-width, say H, that is larger than h and define the discrete spaces $\mathbf{X}^H \subset \mathbf{X}^h$, $Y^H \subset Y^h$, $Z^H \subset Z^h$, $\mathbf{V}^H \subset \mathbf{V}^h$ in the same way as the spaces $\mathbf{X}^h$, Y^h, Z^h, $\mathbf{V}^h$. We can prove that (4.8) not only holds for all $\mathbf{y}^h \in \mathbf{V}^h$ close to $\tilde{\mathbf{u}}^h$ but also for elements in $\mathbf{V}^H$ that are "near" $\tilde{\mathbf{u}}^h$. The argument proceeds the same way as the proof of Remark 3.1 in [8].

COROLLARY 4.1. *Under Assumptions I–III, $\mathbf{y}^H \in \mathbf{V}^H$ is also a nonsingular solution to $F(\lambda, \underline{u}^h(\lambda)) = 0$ if there exist two constants $\tilde{\gamma}_* > 0$ and $\tilde{\delta} > 0$, such that*

$$\sup_{\mathbf{w}^h\in\mathbf{V}^h} \frac{A\left(\mathbf{y}^H, \mathbf{v}^h, \mathbf{w}^h\right)}{\|\mathbf{w}^h\|_{1,\Omega}} \geq \tilde{\gamma}_* \left\|\mathbf{v}^h\right\|_{1,\Omega}, \qquad \forall\, \mathbf{v}^h \in \mathbf{V}^h, \tag{4.9}$$

for all $\mathbf{y}^H \in \mathbf{V}^H$ with $|\mathbf{y}^H - \tilde{\mathbf{u}}^h|_{1,\Omega} \leq \tilde{\delta}$.

PROOF. Adding and subtracting terms in equation (3.11) yields

$$A\left(\mathbf{y}^H, \mathbf{v}^h, \mathbf{w}^h\right) = A\left(\tilde{\mathbf{u}}^h, \mathbf{v}^h, \mathbf{w}^h\right) + b_*\left(\mathbf{y}^H - \tilde{\mathbf{u}}^h; \mathbf{v}^h, \mathbf{w}^h\right) + b_*\left(\mathbf{v}^h; \mathbf{y}^H - \tilde{\mathbf{u}}^h, \mathbf{w}^h\right).$$

Following Girault and Raviart [8, p. 122], we define the quantities

$$\mathcal{N} = \sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{V}} \frac{|b_*(\mathbf{u};\mathbf{v},\mathbf{w})|}{|\mathbf{u}|_1|\mathbf{v}|_1|\mathbf{w}|_1},$$

$$\mathcal{N}^h = \sup_{\mathbf{u}^h,\mathbf{v}^h,\mathbf{w}^h\in\mathbf{V}^h} \frac{\left|b_*\left(\mathbf{u}^h;\mathbf{v}^h,\mathbf{w}^h\right)\right|}{\left|\mathbf{u}^h\right|_1\left|\mathbf{v}^h\right|_1\left|\mathbf{w}^h\right|_1},$$

$$\mathcal{N}_H^h = \sup_{\substack{\mathbf{u}^h,\mathbf{w}^h\in\mathbf{V}^h\\ \mathbf{v}^H\in V^H}} \frac{\left|b_*\left(\mathbf{u}^h;\mathbf{v}^H,\mathbf{w}^h\right)\right|}{\left|\mathbf{u}^h\right|_1\left|\mathbf{v}^H\right|_1\left|\mathbf{w}^h\right|_1},$$

$$\mathcal{N}_h^H = \sup_{\substack{\mathbf{u}^h,\mathbf{w}^h\in\mathbf{V}^h\\ \mathbf{v}^H\in V^H}} \frac{\left|b_*\left(\mathbf{v}^H;\mathbf{u}^h,\mathbf{w}^h\right)\right|}{\left|\mathbf{u}^h\right|_1\left|\mathbf{v}^H\right|_1\left|\mathbf{w}^h\right|_1}.$$

If Assumptions I–III are satisfied then from [8, Lemma 3.1, p. 123]

$$\lim_{h\to 0} \mathcal{N}^h = \mathcal{N}.$$

Using the same techniques we can show that

$$\lim_{h,H\to 0} \mathcal{N}_H^h = \lim_{h,H\to 0} \mathcal{N}_h^H = \mathcal{N}.$$

A straightforward calculation yields the estimate

$$\sup_{\mathbf{w}^h\in\mathbf{V}^h} \frac{A\left(\mathbf{y}^H,\mathbf{v}^h,\mathbf{w}^h\right)}{\left\|\mathbf{w}^h\right\|_{1,\Omega}} \geq \left(\gamma_* - \left[\mathcal{N}_h^H + \mathcal{N}_H^h\right]\left|\mathbf{y}^H - \tilde{\mathbf{u}}^h\right|_{1,\Omega}\right)\left\|\mathbf{v}^h\right\|_{1,\Omega}, \tag{4.10}$$

for all $\mathbf{v}^h \in \mathbf{V}^h$ and $\mathbf{y}^H \in \mathbf{V}^H$ with $|\mathbf{y}^H - \tilde{\mathbf{u}}^h|_{1,\Omega} \leq \tilde{\delta}$. Thus, with the choice

$$\tilde{\delta} \leq \frac{\gamma_*}{\mathcal{N}_h^H + \mathcal{N}_H^h},$$

our claim is proven. ∎

5. ERROR ESTIMATES ON THE TWO-LEVEL METHOD

In this section we are concerned with the error in the approximate solution to problem (1.1) obtained by using a two-level method. We use two partitions $\mathcal{T}^{h_0}$, $\mathcal{T}^{h_1}$ to subdivide our domain. Let the mesh-widths be $H = h_0$, $h = h_1$ with scalings

$$h = \mathcal{O}\left(H^\mu\right).$$

The exponent μ is chosen in such a way that the error on the fine mesh is optimal in h. The method is as follows.

ALGORITHM. Newton $(2,1)$.

STEP I. Solve the nonlinear, coarse mesh problem: find $\mathbf{u}^H \in \mathbf{X}^H$, $p^H \in Y^H$, $\rho^H \in Z^H$ *satisfying*

$$\begin{aligned} a\left(\mathbf{u}^H,\mathbf{v}^H\right) + b_*\left(\mathbf{u}^H;\mathbf{u}^H,\mathbf{v}^H\right) + c\left(\mathbf{v}^H;p^H,\rho^H\right) &= \left\langle \mathbf{f},\mathbf{v}^H\right\rangle_\Omega, \\ c\left(\mathbf{u}^H;q^H,\sigma^H\right) &= 0, \end{aligned} \tag{5.1}$$

for all $(\mathbf{v}^H,q^H,\sigma^H) \in (\mathbf{X}^H,Y^H,Z^H)$.

STEP II. Solve the following linear, fine mesh problem: find $\mathbf{u}^h \in \mathbf{X}^h$, $p^h \in Y^h$, $\rho^h \in Z^h$ *satisfying*

$$A\left(\mathbf{u}^H,\mathbf{u}^h,\mathbf{v}^h\right) + c\left(\mathbf{v}^h;p^h,\rho^h\right) - b_*\left(\mathbf{u}^H;\mathbf{u}^H,\mathbf{v}^h\right) = \left\langle \mathbf{f},\mathbf{v}^h\right\rangle_\Omega, \tag{5.2}$$

for all $(\mathbf{v}^h,q^h,\sigma^h) \in (\mathbf{X}^h,Y^h,Z^h)$.

In the sequel, we will use the following bound on the trilinear form $b_*(.\,;.\,,.)$:

$$|b_*(\mathbf{u},\mathbf{v},\mathbf{w})| \leq C|\mathbf{u}|_1|\mathbf{v}|_1|\mathbf{w}|_1,$$

proven in [17, Lemma 2.1].

THEOREM 5.1. H^1-ERROR ESTIMATES.

Suppose that $(\mathbf{u}, p, \rho)$ is a nonsingular solution of (3.1) and its best approximation in $\mathbf{X}^H$ is $(\tilde{\mathbf{u}}^H, \tilde{p}^H, \tilde{\rho}^H)$. Assume that $\mathbf{X}^\mu \subset X$, $Y^\mu \subset Y$, $Z^\mu \subset Z$, for $\mu = h, H$ and Assumptions I–III hold.

(i) *There are positive constants $\tilde{H}$ and α, such that for all H with $0 < H \leq \tilde{H}$, there are unique $\mathbf{u}^H \in \{\mathbf{v} \in \mathbf{X}^H \mid |\mathbf{v} - \tilde{\mathbf{u}}^H|_1 < \alpha\}$, $p^H \in Y^H$ and $\rho^H \in Z^H$ satisfying (5.1). Furthermore, the error estimate*

$$\left|\mathbf{u} - \mathbf{u}^H\right|_{1,\Omega} + \left\|p - p^H\right\|_{0,\Omega} + \left\|\rho - \rho^H\right\|_\Gamma^* + h^{1/2}\left(\sum_{j=1}^R \left\|\rho - \rho^H\right\|_{0,\Gamma_j}^2\right)^{1/2} \leq C(\mathbf{u}(\lambda), \mathbf{f})H \quad (5.3)$$

holds for the MINI element.

(ii) *Assume that H is small enough and that the bound (5.3) holds. Then there exist unique $(\mathbf{u}^h, p^h, \rho^h) \in (\mathbf{X}^h, Y^h, Z^h)$ satisfying (5.2). Moreover, for $\epsilon > 0$ the error satisfies*

$$\begin{aligned}
&\left|\mathbf{u} - \mathbf{u}^h\right|_{1,\Omega} + \left\{\left\|p - p^h\right\|_{0,\Omega}^2 + \left(\left\|\rho - \rho^h\right\|_\Gamma^*\right)^2\right\}^{1/2} \\
&\quad \leq C \inf_{\mathbf{v}^h \in \mathbf{X}^h} \left|\mathbf{u} - \mathbf{v}^h\right|_{1,\Omega} \\
&\quad + C\begin{cases} \left|\mathbf{u} - \mathbf{u}^H\right|_{1,\Omega}^{1+\epsilon} \left\|\mathbf{u} - \mathbf{u}^H\right\|_{0,\Omega}^{1-\epsilon}, & \text{in 2-D}, \\ \left|\mathbf{u} - \mathbf{u}^H\right|_{1,\Omega}^{1(1/2)} \left\|\mathbf{u} - \mathbf{u}^H\right\|_{0,\Omega}^{1/2}, & \text{in 3-D}, \end{cases} \\
&\quad + \left[1 + C(1 + Ch)^{1/2}\right] \inf_{\substack{0 \neq q^h \in Y^h \\ 0 \neq \sigma^h \in Z^h}} \left\{\left\|p - q^h\right\|_{0,\Omega}^2 + \left(\left\|\rho - \sigma^h\right\|_\Gamma^*\right)^2\right\}^{1/2}.
\end{aligned} \quad (5.4)$$

PROOF.

(i) See [4, Theorem 6.3, p. 51].

(ii) Let $\mathbf{w}^h \in \mathbf{V}^h$. Define

$$\phi^h = \mathbf{u}^h - \mathbf{w}^h, \quad (5.5)$$

$$\eta^h = \mathbf{u} - \mathbf{w}^h, \quad (5.6)$$

and let $\mathbf{v}^h = \mathbf{w}^h \in \mathbf{V}^h$ in equation (5.2). Subtracting (5.2) from (3.1) with $\mathbf{v} = \mathbf{v}^h$ and using the relationship

$$\begin{aligned}
b^*(\mathbf{u}; \mathbf{u}, \mathbf{v}) &= b^*\left(\mathbf{u}; \mathbf{u}^H, \mathbf{v}\right) + b^*\left(\mathbf{u}^H; \mathbf{u}, \mathbf{v}\right) \\
&\quad - b^*\left(\mathbf{u}^H; \mathbf{u}^H, \mathbf{v}\right) + b^*\left(\mathbf{u} - \mathbf{u}^H; \mathbf{u} - \mathbf{u}^H, \mathbf{v}\right),
\end{aligned} \quad (5.7)$$

Layton [18, p. 35], yields

$$A\left(\mathbf{u}^H, \mathbf{u} - \mathbf{u}^h, \mathbf{w}^h\right) + b_*\left(\mathbf{u} - \mathbf{u}^H; \mathbf{u} - \mathbf{u}^H, \mathbf{w}^h\right) + c\left(\mathbf{w}^h; p - p^h, \rho - \rho^h\right) = 0. \quad (5.8)$$

Subtracting and adding $\mathbf{w}^h$ in $\mathbf{u} - \mathbf{u}^h$ results

$$\begin{aligned}
&A\left(\mathbf{u}^H, \eta^h, \mathbf{w}^h\right) + b_*\left(\mathbf{u} - \mathbf{u}^H; \mathbf{u} - \mathbf{u}^H, \mathbf{w}^h\right) \\
&\quad + c\left(\mathbf{w}^h; p - q^h, \rho - \sigma^h\right) = A\left(\mathbf{u}^H, \phi^h, \mathbf{w}^h\right),
\end{aligned} \quad (5.9)$$

which holds for any $(\mathbf{w}^h, q^h, \sigma^h) \in \mathbf{V}^h \times Y^h \times Z^h$. In inequality (4.10) let $\mathbf{v}^h = \phi^h$ and $\mathbf{y}^H = \mathbf{u}^H$. Assuming $\mathbf{u}^H$ is "close" to $\tilde{\mathbf{u}}^h$, Corollary 4.1 gives

$$\tilde{\gamma}_* \left\|\phi^h\right\|_{1,\Omega} \leq \sup_{\mathbf{w}^h \in \mathbf{V}^h} \frac{A\left(\mathbf{u}^H, \phi^h, \mathbf{w}^h\right)}{\left\|\mathbf{w}^h\right\|_{1,\Omega}}. \quad (5.10)$$

Using equation (5.9) now furnishes

$$
\begin{aligned}
\tilde{\gamma}_* \left\|\phi^h\right\|_{1,\Omega} \leq \sup_{\mathbf{w}^h \in \mathbf{V}^h} & \frac{1}{\left\|\mathbf{w}^h\right\|_{1,\Omega}} \left[A\left(\mathbf{u}^H, \boldsymbol{\eta}^h, \mathbf{w}^h\right)\right. \\
&\left. + b_*\left(\mathbf{u}-\mathbf{u}^H; \mathbf{u}-\mathbf{u}^H, \mathbf{w}^h\right) + c\left(\mathbf{w}^h; p-q^h, \rho-\sigma^h\right)\right].
\end{aligned} \tag{5.11}
$$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X}$, we have

$$
a(\mathbf{u}, \mathbf{v}) \leq 2\,\mathrm{Re}^{-1}\, |\mathbf{u}|_1 |\mathbf{v}|_1, \tag{5.12}
$$

$$
b_*(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq \begin{cases} C_\epsilon \|\mathbf{u}\|_{0,\Omega}^{1-\epsilon} \|\mathbf{u}\|_{1,\Omega}^{\epsilon} \|\mathbf{v}\|_{1,\Omega} \|\mathbf{w}\|_{1,\Omega}, & \text{for 2-D and } \epsilon > 0, \\ C \|\mathbf{u}\|_{0,\Omega}^{1/2} \|\mathbf{u}\|_{1,\Omega}^{1/2} \|\mathbf{v}\|_{1,\Omega} \|\mathbf{w}\|_{1,\Omega}, & \text{for 3-D}, \end{cases} \tag{5.13}
$$

using [17, inequality (2.20), p. 11] or [1, Lemma 2.1]. We also use the bound

$$
c\left(\mathbf{u}^h; p^h, \rho^h\right) \leq \left\{ \left\|p^h\right\|_{0,\Omega}^2 + \left(\left\|\rho^h\right\|_\Gamma^*\right)^2 \right\}^{1/2} (1+Ch)^{1/2} \left\|\mathbf{u}^h\right\|_{1,\Omega}, \tag{5.14}
$$

proven in [3, pp. 20,21]. Using bounds (5.12)–(5.14) in (5.11) yields

$$
\begin{aligned}
\tilde{\gamma}_* \left\|\mathbf{u}^h - \mathbf{w}^h\right\|_{1,\Omega} &\leq 2\,\mathrm{Re}^{-1} \left|\mathbf{u}-\mathbf{w}^h\right|_{1,\Omega} + C\left|\mathbf{u}-\mathbf{w}^h\right|_{1,\Omega} \left|\mathbf{u}^H\right|_{1,\Omega} \\
&+ C \begin{cases} \left|\mathbf{u}-\mathbf{u}^H\right|_{1,\Omega}^{1+\epsilon} \left\|\mathbf{u}-\mathbf{u}^H\right\|_{0,\Omega}^{1-\epsilon}, & \text{in 2-D}, \\ \left|\mathbf{u}-\mathbf{u}^H\right|_{1,\Omega}^{1(1/2)} \left\|\mathbf{u}-\mathbf{u}^H\right\|_{0,\Omega}^{1/2}, & \text{in 3-D}, \end{cases} \\
&+ \left\{ \left\|p-q^h\right\|_{0,\Omega}^2 + \left(\left\|\rho-\sigma^h\right\|_\Gamma^*\right)^2 \right\}^{1/2} (1+Ch)^{1/2}.
\end{aligned} \tag{5.15}
$$

Using the bound

$$
\left|\mathbf{u}^H\right|_1 \leq \frac{\mathrm{Re}(c_2 + 2c_0)}{c_1 c_2} |\mathbf{f}|_*, \tag{5.16}
$$

proven in [3, p. 17], the triangle inequality and taking the infimum over all $\mathbf{w}^h \in \mathbf{V}^h$ and all q^h and σ^h in Y^h and Z^h, respectively, yields the bound

$$
\begin{aligned}
\left\|\mathbf{u}-\mathbf{u}^h\right\|_{1,\Omega} &\leq C \inf_{\mathbf{w}^h \in \mathbf{V}^h} \left|\mathbf{u}-\mathbf{w}^h\right|_{1,\Omega} \\
&+ C\tilde{\gamma}_*^{-1} \begin{cases} \left|\mathbf{u}-\mathbf{u}^H\right|_{1,\Omega}^{1+\epsilon} \left\|\mathbf{u}-\mathbf{u}^H\right\|_{0,\Omega}^{1-\epsilon}, & \text{in 2-D}, \\ \left|\mathbf{u}-\mathbf{u}^H\right|_{1,\Omega}^{1(1/2)} \left\|\mathbf{u}-\mathbf{u}^H\right\|_{0,\Omega}^{1/2}, & \text{in 3-D}, \end{cases} \\
&+ \inf_{\substack{0 \neq q^h \in Y^h \\ 0 \neq \sigma^h \in Z^h}} \left\{ \left\|p-q^h\right\|_{0,\Omega}^2 + \left(\left\|\rho-\sigma^h\right\|_\Gamma^*\right)^2 \right\}^{1/2} \tilde{\gamma}_*^{-1}(1+Ch)^{1/2}.
\end{aligned} \tag{5.17}
$$

Using the bound for the error in pressure and normal stress component derived in [3, inequality (5.35), p. 24]

$$
\begin{aligned}
\left\{ \left\|p-p^h\right\|_{0,\Omega}^2 + \left(\left\|\rho-\rho^h\right\|_\Gamma^*\right)^2 \right\}^{1/2} &\leq C \inf_{\mathbf{w}^h \in \mathbf{V}^h} \left|\mathbf{u}-\mathbf{w}^h\right|_1 \\
+ \left[1 + \beta^{-1}(1+Ch)^{1/2}\right] &\inf_{\substack{0 \neq p^h \in Y^h \\ 0 \neq \rho^h \in Z^h}} \left\{ \left\|p-q^h\right\|_{0,\Omega}^2 + \left(\left\|\rho-\sigma^h\right\|_\Gamma^*\right)^2 \right\}^{1/2},
\end{aligned} \tag{5.18}
$$

and adding it to inequality (5.17) yields the required error estimate. ∎

6. L^2-ERROR ESTIMATES

To obtain an L^2-norm estimate for the error $\mathbf{u}-\mathbf{u}^h$ we must use a duality argument. Together with equation (5.2), consider the following problem: given $\mathbf{z} \in L^2(\Omega)$, find $\psi \in \mathbf{X}$, $\xi \in Y$, and $\theta \in Z$ that satisfy

$$\begin{aligned} A\left(\mathbf{u}^H; \mathbf{v}, \psi\right) + c(\mathbf{v}; \xi, \theta) &= \langle \mathbf{z}, \mathbf{v} \rangle, \\ c(\psi; q, \sigma) &= 0, \end{aligned} \tag{6.1}$$

for all $\mathbf{v} \in \mathbf{X}$, $q \in Y$, $\sigma \in Z$. The duality proof requires a regularity result for the above equation, and therefore we make the following assumption:

$$\|\psi\|_2 + \|\xi\|_1 + \sum_{j=1}^{R} \|\theta\|_{1/2,\Gamma_j} \le C\|\mathbf{z}\|_0. \tag{6.2}$$

This assumption holds when $\mathbf{u}^H$ is sufficiently close to $\mathbf{u}$, $\underline{u} = (\mathbf{u}, p, \rho)$ is a nonsingular solution to problem (3.1) (enabling the use of Definition 3.1), and $\partial\Omega$ is Lipschitz continuous. With all the above assumptions, problem (6.1) has a unique solution. Inequality (6.2) is derived from inequality (5.12) in [4, p. 41] and the fact that (ψ, ξ, θ) is a solution to the Stokes problem with slip boundary condition and right-hand side $\langle \mathbf{z}, \mathbf{v}\rangle - b_*(\mathbf{u}^H; \mathbf{v}, \psi) - b_*(\mathbf{v}; \mathbf{u}^H, \psi)$.

THEOREM 6.1. L^2-ERROR ESTIMATES.
Let $\mathbf{u}^H \in \mathbf{V}^H$, and let $\mathbf{u}^h \in \mathbf{X}^h$, $p^h \in Y^h$, $\rho^h \in Z^h$ satisfy (5.2). Let the regularity condition (6.2) hold and assume that $\mathbf{X}^\mu \subset \mathbf{X}$, $Y^\mu \subset Y$, $Z^\mu \subset Z$, for $\mu = h, H$, and Assumptions I–III hold. Then the L^2-error $\|\mathbf{u}-\mathbf{u}^h\|_0$ satisfies the following:

$$\begin{aligned} \left\|\mathbf{u}-\mathbf{u}^h\right\|_0 &\le \left[Ch + Ch(1+Ch)^{1/2}\right]\left|\mathbf{u}-\mathbf{u}^h\right|_1 \\ &+ Ch(1+Ch)^{1/2}\left\{\left\|p-q^h\right\|_0^2 + \left\|\rho-\sigma^h\right\|_{-1/2,\Gamma}^2\right\}^{1/2} \\ &+ C(Ch+C)\begin{cases} \left|\mathbf{u}-\mathbf{u}^H\right|_{1,\Omega}^{1+\epsilon}\left\|\mathbf{u}-\mathbf{u}^H\right\|_{0,\Omega}^{1-\epsilon}, & \text{in 2-D}, \\ \left|\mathbf{u}-\mathbf{u}^H\right|_{1,\Omega}^{1(1/2)}\left\|\mathbf{u}-\mathbf{u}^H\right\|_{0,\Omega}^{1/2}, & \text{in 3-D}. \end{cases} \end{aligned} \tag{6.3}$$

PROOF. We put $\mathbf{z} = \mathbf{v} = \mathbf{u}-\mathbf{u}^h$ and let $\psi \in \mathbf{V}$ in (6.1) to yield

$$\left\|\mathbf{u}-\mathbf{u}^h\right\|_0^2 = A\left(\mathbf{u}^H; \mathbf{u}-\mathbf{u}^h, \psi\right) + c\left(\mathbf{u}-\mathbf{u}^h; \xi, \theta\right). \tag{6.4}$$

Let $\psi^h \in \mathbf{V}^h$ be the best approximation of ψ in the space $\mathbf{V}^h$ with respect to the H^1-seminorm. By Assumption III, (6.2), and with $\mathbf{u}$, $\mathbf{u}^h$ replaced by ψ, ψ^h, we have

$$\left|\psi-\psi^h\right|_1 \le Ch\left\|\mathbf{u}-\mathbf{u}^h\right\|_0. \tag{6.5}$$

Take $\xi^h \in Y^h$ and $\theta^h \in Z^h$ so that

$$\left\|\xi-\xi^h\right\|_0 = \inf_{q^h\in Y^h}\left\|\xi-q^h\right\|_0$$

and

$$\left\|\theta-\theta^h\right\|_\Gamma^* = \inf_{\sigma^h\in Z^h}\left\|\theta-\sigma^h\right\|_\Gamma^*.$$

By Assumption II and (6.2) we have

$$\left\|\xi-\xi^h\right\|_0 + \left\|\theta-\theta^h\right\|_\Gamma^* \le C'h\left\|\mathbf{u}-\mathbf{u}^h\right\|_0. \tag{6.6}$$

The bound on $\|\theta - \theta^h\|_\Gamma^*$ comes from an easy extension of inequality (5.12) in [4] and inequality (6.2). Equation (6.4) and (5.8) with $\tilde{\mathbf{w}}^h$ replaced by ψ^h yields

$$\begin{aligned} \left\|\mathbf{u}-\mathbf{u}^h\right\|_0^2 = A\left(\mathbf{u}^H;\mathbf{u}-\mathbf{u}^h,\psi-\psi^h\right) + c\left(\mathbf{u}-\mathbf{u}^h;\xi-\xi^h,\theta-\theta^h\right) \\ +c\left(\psi-\psi^h;p-q^h,\rho-\sigma^h\right) - b_*\left(\mathbf{u}-\mathbf{u}^H;\mathbf{u}-\mathbf{u}^H,\psi-\psi^h\right) \\ +b_*\left(\mathbf{u}-\mathbf{u}^H;\mathbf{u}-\mathbf{u}^H,\psi\right), \end{aligned} \tag{6.7}$$

since $\mathbf{V}^h \subset \mathbf{V}$. Using (3.11), (5.12), (5.14), (6.5), and (6.6) we obtain the following bound:

$$\begin{aligned} \left\|\mathbf{u}-\mathbf{u}^h\right\|_0^2 &\leq \frac{2}{\mathrm{Re}}\left|\mathbf{u}-\mathbf{u}^h\right|_1\left|\psi-\psi^h\right|_1 + C\left|\mathbf{u}^H\right|_1\left|\mathbf{u}-\mathbf{u}^h\right|_1\left|\psi-\psi^h\right|_1 \\ &+\left\{\left\|\xi-\xi^h\right\|_0^2+\left(\left\|\theta-\theta^h\right\|_\Gamma^*\right)^2\right\}^{1/2}(1+Ch)^{1/2}\left|\mathbf{u}-\mathbf{u}^h\right|_1 \\ &+\left\{\left\|p-q^h\right\|_0^2+\left(\left\|\rho-\sigma^h\right\|_\Gamma^*\right)^2\right\}^{1/2}(1+Ch)^{1/2}\left|\psi-\psi^h\right|_1 \\ &+C\left(\left|\psi-\psi^h\right|_1+|\psi|_1\right)\begin{cases} \left|\mathbf{u}-\mathbf{u}^H\right|_{1,\Omega}^{1+\epsilon}\left\|\mathbf{u}-\mathbf{u}^H\right\|_{0,\Omega}^{1-\epsilon}, & \text{in 2-D,} \\ \left|\mathbf{u}-\mathbf{u}^H\right|_{1,\Omega}^{1(1/2)}\left\|\mathbf{u}-\mathbf{u}^H\right\|_{0,\Omega}^{1/2}, & \text{in 3-D,} \end{cases} \\ &\leq\left(\frac{2C}{\mathrm{Re}}+C\,\mathrm{Re}|\mathbf{f}|_*\right)h\left|\mathbf{u}-\mathbf{u}^h\right|_1\left\|\mathbf{u}-\mathbf{u}^h\right\|_0 \\ &+(C+Ch)(1+Ch)^{1/2}\left|\mathbf{u}-\mathbf{u}^h\right|_1\left\|\mathbf{u}-\mathbf{u}^h\right\|_0 \\ &+\left\{\left\|p-q^h\right\|_0^2+\left(\left\|\rho-\sigma^h\right\|_\Gamma^*\right)^2\right\}^{1/2}(1+Ch)^{1/2}Ch\left\|\mathbf{u}-\mathbf{u}^h\right\|_0 \\ &+C(Ch+C)\left\|\mathbf{u}-\mathbf{u}^h\right\|_0\begin{cases} \left|\mathbf{u}-\mathbf{u}^H\right|_{1,\Omega}^{1+\epsilon}\left\|\mathbf{u}-\mathbf{u}^H\right\|_{0,\Omega}^{1-\epsilon}, & \text{in 2-D,} \\ \left|\mathbf{u}-\mathbf{u}^H\right|_{1,\Omega}^{1(1/2)}\left\|\mathbf{u}-\mathbf{u}^H\right\|_{0,\Omega}^{1/2}, & \text{in 3-D.} \end{cases} \end{aligned} \tag{6.8}$$

Now the error estimate (6.3) follows. ∎

7. SCALINGS FOR THE MULTILEVEL METHOD

Suppose that the condition

$$\inf_{\mathbf{u}^{h_i}\in\mathbf{X}^{h_i}}\left|\mathbf{u}-\mathbf{u}^{h_i}\right|_{1,\Omega} + \inf_{\substack{0\neq p^{h_i}\in Y^{h_i}\\ 0\neq\rho^{h_i}\in Z^{h_i}}}\left\{\left\|p-p^{h_i}\right\|_{0,\Omega}^2+\left(\left\|\rho-\rho^{h_i}\right\|_\Gamma^*\right)^2\right\}^{1/2} \leq Ch_i^k\|\mathbf{f}\|_0, \tag{7.1}$$

$i > 0$, holds for velocity elements of polynomial degree k, pressure elements of degree at least $k-1$, and normal stress component elements of polynomial degree $k-1$ on the boundary. Assuming that we also use velocity elements of polynomial degree k in the first step, we have

$$\left\|\mathbf{u}-\mathbf{u}^{h_0}\right\|_{1,\Omega} \leq Ch_0^k \max\left\{\|\mathbf{f}\|_{0,\Omega},\|\mathbf{u}\|_{2,\Omega}\right\}.$$

Using the duality argument of Aubin [19] and Nitsche [20] yields

$$\left\|\mathbf{u}-\mathbf{u}^{h_0}\right\|_{0,\Omega} \leq Ch_0^{k+1}\max\left\{\|\mathbf{f}\|_{0,\Omega},\|\mathbf{u}\|_{2,\Omega}\right\},$$

since we match the boundary (Ω is polygonal). Now, balancing error terms on the right-hand side of (5.4) suggests that a scaling given by $h_1 = O(h_0^{\mu_0})$, where $\mu_0 = 2 + (1-\epsilon)/k$ in 2-D and $\mu_0 = 2 + 1/2k$ in 3-D, ensures optimal order accuracy. Similarly we obtain scalings μ_0 for the L^2 error which are listed in Table 1. The scalings for the multilevel method can be furnished similarly using inequalities (5.4) and (6.3) recursively.

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