Navier's Slip and Evolutionary Navier-Stokes-like Systems with Pressure and Shear-rate Dependent Viscosity

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ABSTRACT. There is compelling experimental evidence for the viscosity of a fluid to depend on the shear rate as well as the mean normal stress (pressure). Moreover, while the viscosity can vary by several orders of magnitude the density suffers very minor variation, when the range of the pressure is sufficiently large, thereby providing justification for considering the fluid as being incompressible while at the same time possessing a viscosity that is dependent on the pressure.

In this article we investigate the mathematical properties of internal unsteady three-dimensional flows of such fluids subject to Navier's slip at the boundary. We establish the long-time existence of a weak solution for large data provided that the viscosity depends on the shear rate and the pressure in a suitably specified manner. This specific relationship however includes the classical Navier-Stokes fluids and power law fluids (with power law index $r - 2$, $r \leq 2$) as special cases. Even for these special cases, the existence results that are being presented are new.

1. INTRODUCTION

1.1. Formulation of the problem and its relevance to mechanics The Navier-Stokes fluid model occupies a central place in fluid mechanics. The equations governing its flows, the Navier-Stokes Equations (NSEs for short) has possibly attracted more attention from mathematical analysts than most other partial

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differential equations of mathematical physics. In mathematical analysis, the question of the well-posedness of the NSEs is regarded as one of the fundamental problems in the theory of partial differential equations. Those fluids, whose behavior cannot be captured by the NSEs are referred to as Non-Newtonian fluids.

NSEs for a homogeneous incompressible fluid take the form:

$$\begin{aligned} &\operatorname{div}\mathbf{v} = 0, \\ &\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) - \operatorname{div}(\nu\mathbf{D}(\mathbf{v})) = -\nabla p + \mathbf{f}, \end{aligned} \tag{1.1}$$

where $\mathbf{v} = (v_1, v_2, v_3)$ is the velocity, p is the pressure, $\mathbf{f}$ is the given specific body force; $\mathbf{D}(\mathbf{v})$ denotes the symmetric part of the velocity gradient $\nabla\mathbf{v}$, which means that $2\mathbf{D}(\mathbf{v}) = \nabla\mathbf{v} + (\nabla\mathbf{v})^T$. The material properties of the fluid are encoded into the viscosity ν and for a Navier-Stokes fluid ν is supposed to be equal to a positive constant value. Consequently, $-\operatorname{div}(\nu\mathbf{D}(\mathbf{v})) = -(\nu/2)\Delta\mathbf{v}$.

Most of the mathematical studies of NSEs consider flows referred to as internal flows when the flow takes place in a bounded domain $\Omega \subset \mathbb{R}^3$ with

$$\mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{1.2}$$

where $\partial\Omega$ denotes the boundary of Ω and $\mathbf{n}$ is the unit normal to the boundary.

Usually, these flows are subject to the no-slip boundary conditions

$$\mathbf{v}_\tau := \mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\mathbf{n} = \mathbf{0} \quad \text{on } (0,T)\times\partial\Omega. \tag{1.3}$$

Note that $\mathbf{v}_\tau = \mathbf{v}$ if (1.2) holds.

Questions concerning the development and prescription of a suitable mathematical description for flows of (incompressible and compressible) fluids within the flow domain and the interactions of the fluid with the boundary were addressed and intensively studied by such scientists as Newton, Euler, Coulomb, Poisson, Navier, Girard, St. Venant and Stokes, amongst many others pioneers of fluid mechanics.

Stokes while developing (1.1)–(1.3) (see [17]) thoroughly discusses the limitations of the following assumptions

(A1) ν being independent of the pressure,

(A2) the velocity adhering to the boundary.

In this paper we relax both the assumptions (A1) and (A2).

Instead of (A1), we consider an incompressible fluid with the viscosity depending on the pressure and the shear rate, i.e., the Cauchy stress $\mathbf{T}$ takes the form

$$\mathbf{T} = -p\mathbf{I} + \nu(p, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}). \tag{1.4}$$

Models of the type (1.4) are used in various engineering areas: elastohydrodynamics, mechanics of granular and visco-elastic materials wherein the bodies are subject to high pressures can serve as appropriate examples. We refer the reader to [8], [9], [5], [11], and [12] for more details related to (1.4) and for a list of references to experimental reports. From these experiments, one can obtain much information concerning the pressure-viscosity relationship and its relevance to the assumptions of incompressibility, as well. The assumptions on the structure of ν specified below include as special cases both the Navier-Stokes fluid (ν is constant) and the fluid with shear-rate dependent viscosity, where

$$\mathbf{T} = -p\mathbf{I} + \nu(|\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}). \tag{1.5}$$

Note that the popular power-law-like fluid $\nu(\mathbf{D}) = |\mathbf{D}|^{r-2}$ or its non-degenerate variant $\nu(\mathbf{D}) = (A + |\mathbf{D}|^2)^{(r-2)/2}$, where $(r-2)$ is the power law index taking values between -1 and ∞ and A is a positive constant, are particular cases of (1.5).

Instead of (A2), we assume that at the fluid-solid boundary, the fluid is capable of slipping described by the Navier's slip boundary conditions:

$$\mathbf{v} \cdot \mathbf{n} = 0 \quad \text{and} \quad (\mathbf{Tn})_\tau + \alpha \mathbf{v}_\tau = \mathbf{0} \quad (\alpha \geq 0), \tag{1.6}$$

where $(\mathbf{Tn})_\tau := \mathbf{Tn} - (\mathbf{Tn} \cdot \mathbf{n})\mathbf{n}$. Note that letting $\alpha \to 0_+$ we obtain the so-called slip boundary conditions (and this case is included into our analysis). Note also, that the limit $\alpha \to +\infty$ formally leads ($(1.6)_2$ is multiplied by $1/\alpha$ first) to the no-slip boundary conditions (1.3). We do not consider this case in the paper.

1.2. The assumptions on the structure of the viscosity and their consequences We assume that the viscosity ν is a C^1-mapping of $\mathbb{R} \times \mathbb{R}_0^+$ into $\mathbb{R}^+$ satisfying for some fixed (but arbitrary) $r \in [1, 2]$ and q, $\alpha \geq 0$ and all $\mathbf{D} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, $\mathbf{B} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $p \in \mathbb{R}$ the following inequalities

$$\begin{aligned} C_1(1 + |\mathbf{D}|^2)^{(r-2)/2}|\mathbf{B}|^2 &\leq \frac{\partial \nu(p, |\mathbf{D}|^2)\mathbf{D}_{ij}}{\partial \mathbf{D}_{kl}} \mathbf{B}_{ij}\mathbf{B}_{kl} \\ &\leq C_2(1 + |\mathbf{D}|^2)^{(r-2)/2}|\mathbf{B}|^2, \end{aligned} \tag{1.7}$$

$$\left| \frac{\partial \nu(p, |\mathbf{D}|^2)}{\partial p} \right| |\mathbf{D}| \leq \gamma_0(1 + \alpha|p|)^{-q}(1 + |\mathbf{D}|^2)^{(r-2)/4}. \tag{1.8}$$

where $\gamma_0 > 0$ is a constant whose value will be restricted in the formulation of the main theorem.

Since $r = 2$ is included in the range of parameters, we see that (1.7) (and naturally also (1.8)) applies to the Navier-Stokes model. Also, if ν is independent of p, then (1.8) is again irrelevant and (1.7) is fulfilled by generalized power-law-like fluids. On the other hand, our assumptions do not permit us to consider any model where the viscosity depends on the pressure only.

Before stating the consequences of the assumptions (1.7)–(1.8), we list some examples fulfilling (1.7) and (1.8) respectively.

Example 1.1. Consider

$$\nu_i(p,|\mathbf{D}|^2) = (1+\gamma_i(p)+|\mathbf{D}|^2)^{(r-2)/2}, \quad i=1,2, \tag{1.9}$$

where $\gamma_i(p)$ have the form ($s \geq 0$)

$$\left.\begin{aligned} \gamma_1(p) &= (1+\alpha^2p^2)^{-s/2}, \\ \gamma_2(p) &= \begin{cases} (1+\exp(\alpha p))^{-s} & \text{if } p>0 \\ 1 & \text{if } p \leq 0 \end{cases} \end{aligned}\right\} \Rightarrow \begin{array}{l} 0 \leq \gamma_i(p) \leq 1, \\ (i=1,2). \end{array} \tag{1.10}$$

We show that the viscosities given by (1.9)–(1.10) satisfy (1.7)–(1.8) for any parameter $r \in (1,2]$ and some q, γ_0 specified below. The fact that the relation (1.7) holds for (1.9) and (1.10) is proved in [9] and we skip a proof of it here. To prove (1.8) we observe that

$$\begin{aligned} &\left|\frac{\partial \nu(p,|\mathbf{D}|^2)}{\partial p}\right| |\mathbf{D}| = \left|\frac{r-2}{2}\right| (1+\gamma_i(p)+|\mathbf{D}|^2)^{(r-4)/2}|\gamma_i'(p)|\,|\mathbf{D}| \\ &= \frac{2-r}{2}(1+|\mathbf{D}|^2)^{(r-2)/4}(1+|\mathbf{D}|^2)^{(2-r)/4}|\gamma_i'(p)|\,|\mathbf{D}|(1+\gamma_i(p)+|\mathbf{D}|^2)^{(r-4)/2} \\ &\leq \frac{2-r}{2}|\gamma_i'(p)|(1+|\mathbf{D}|^2)^{(4-r)/4}(1+|\mathbf{D}|^2)^{(r-2)/4}(1+\gamma_i(p)+|\mathbf{D}|^2)^{(r-4)/2} \\ &\leq \frac{2-r}{2}\frac{|\gamma_i'(p)|}{(1+\gamma_i(p))^{(4-r)/4}}(1+|\mathbf{D}|^2)^{(r-2)/4}. \end{aligned}$$

For γ_1 we have

$$\begin{aligned} \frac{|\gamma_1'(p)|}{(1+\gamma_1(p))^{(4-r)/4}} &\leq \alpha^2 s|p|\frac{(1+\alpha^2p^2)^{(-s-2)/2}}{(1+\alpha^2p^2)^{-(s/2)(4-r)/4}} \\ &\leq \alpha s\frac{(1+\alpha^2p^2)^{(-s-1)/2}}{(1+\alpha^2p^2)^{-(s/2)(4-r)/4}} \\ &\leq \alpha s(1+\alpha^2p^2)^{-rs/8-1/2} \\ &\leq \sqrt{2}\alpha s(1+\alpha|p|)^{-(1+rs/4)}. \end{aligned}$$

Thus (1.8) is satisfied with $q := 1 + rs/4$ and $\gamma_0 := \sqrt{2}\alpha s(2-r)/2$.

For γ_2 we first notice that $\gamma_2'(p) = 0$ if $p \leq 0$. For $p > 0$ we have

$$\begin{aligned} \frac{|\gamma_2'(p)|}{(1+\gamma_2(p))^{(4-r)/4}} &\leq \alpha s\frac{(1+\exp(\alpha p))^{-s}}{(1+\exp(\alpha p))^{-s(4-r)/4}} \\ &\leq \alpha s(1+\exp(\alpha p))^{-rs/4} \leq \alpha s(1+\alpha|p|)^{-1}, \end{aligned}$$

and we can set $q := -1$ and $\gamma_0 := (2-r)/2\alpha s$.

Next, we consider viscosities of the type

$$\nu(p, |\mathbf{D}|^2) = \gamma_p(p)\nu_{\mathbf{D}}(|\mathbf{D}|^2).$$

Example 1.2. Consider

$$\nu(p, |\mathbf{D}|^2) := \frac{\gamma_3(p)}{\sqrt{|\mathbf{D}|^2 + \varepsilon}} + \nu_\infty \tag{1.11}$$

such that for some α, $q \geq 0$ and $\gamma_0 > 0$

$$0 \leq \gamma_3(p) \leq \gamma_\infty \quad \text{and} \quad |\gamma_3'(p)| \leq \gamma_0(1 + \alpha|p|)^{-q}. \tag{1.12}$$

Then the viscosity of the type (1.11) also satisfies the assumption (1.7)–(1.8) with parameters γ_0, q, α, and $r = 2$. Indeed, since

$$\frac{\partial \nu(p, |\mathbf{D}|^2)\mathbf{D}_{ij}}{\partial \mathbf{D}_{k\ell}}\mathbf{B}_{ij}\mathbf{B}_{k\ell} = \nu_\infty|\mathbf{B}|^2 + \frac{\gamma_3(p)}{\sqrt{|\mathbf{D}|^2 + \varepsilon}}|\mathbf{B}|^2 - \mathbf{D}_{ij}\mathbf{D}_{k\ell}\mathbf{B}_{ij}\mathbf{B}_{k\ell}\frac{\gamma_3(p)}{(|\mathbf{D}|^2 + \varepsilon)^{3/2}},$$

we easily conclude that

$$\frac{\partial \nu(p, |\mathbf{D}|^2)\mathbf{D}_{ij}}{\partial \mathbf{D}_{k\ell}}\mathbf{B}_{ij}\mathbf{B}_{k\ell} \geq \nu_\infty|\mathbf{B}|^2,$$

$$\frac{\partial \nu(p, |\mathbf{D}|^2)\mathbf{D}_{ij}}{\partial \mathbf{D}_{k\ell}}\mathbf{B}_{ij}\mathbf{B}_{k\ell} \leq \left(\nu_\infty + \frac{\gamma_\infty}{\varepsilon}\right)|\mathbf{B}|^2.$$

Finally, using (1.12) we have

$$\left|\frac{\partial \nu(p, |\mathbf{D}|^2)}{\partial p}\right| |\mathbf{D}| \leq \gamma_0(1 + \alpha|p|)^{-q}\frac{|\mathbf{D}|}{\sqrt{\varepsilon + |\mathbf{D}|^2}} \leq \gamma_0(1 + \alpha|p|)^{-q}.$$

Lemma 1.3. *Let ν satisfy* (1.7); *then there exist positive constants $\hat{C}$, $\bar{C}$ such that*

$$\hat{C}(|\mathbf{D}|^r - 1) \leq \nu(p, |\mathbf{D}|^2)\mathbf{D}\cdot\mathbf{D}, \tag{1.13}$$

$$|\nu(p, |\mathbf{D}|^2)\mathbf{D}| \leq \bar{C}(|\mathbf{D}|^{r-1} + 1). \tag{1.14}$$

Proof. See Lemma 1.19 p. 198 in [10]. ▫

The next lemma provides important information regarding the monotonicity of the term $\nu(p, |\mathbf{D}|^2)\mathbf{D}$. For simplicity, we set

$$\mathcal{I}_{\mathbf{D}}(\mathbf{u}, \mathbf{v}) := |\mathbf{D}(\mathbf{u} - \mathbf{v})|^2 \int_0^1 (1 + |\mathbf{D}(\mathbf{v}) + s(\mathbf{D}(\mathbf{u} - \mathbf{v}))|^2)^{(r-2)/2}\,ds, \tag{1.15}$$

$$\mathcal{I}_p(p, q) := |p - q|^2\left(\int_0^1 \gamma^2(q + s(p - q))\,ds\right), \tag{1.16}$$

where

$$\gamma(p) := (1 + \alpha|p|)^{-q} \quad \alpha,\ q \text{ being those in (1.8).} \tag{1.17}$$

Lemma 1.4. *Let* ν *satisfy* (1.7)–(1.8). *Then*

$$\begin{aligned} \frac{C_1}{2}\mathcal{I}_{\mathbf{D}}(\mathbf{u},\mathbf{v}) \le{}& \frac{\gamma_0^2}{2C_1}\mathcal{I}_p(p,q) \\ &+ \Big(\nu(p,|\mathbf{D}(\mathbf{u})|^2)\mathbf{D}(\mathbf{u}) - \nu(q,|\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v})\Big)\cdot\mathbf{D}(\mathbf{u}-\mathbf{v}) \end{aligned} \tag{1.18}$$

and

$$\begin{aligned} &|\nu(p,|\mathbf{D}(\mathbf{u})|^2)\mathbf{D}(\mathbf{u}) - \nu(q,|\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v})| \\ &\le \gamma_0\mathcal{I}_p^{1/2}(p,q) + C_2|\mathbf{D}(\mathbf{u}-\mathbf{v})|\int_0^1 (1+|\mathbf{D}(\mathbf{v})+s(\mathbf{D}(\mathbf{u}-\mathbf{v}))|^2)^{(r-2)/2}\,ds \\ &\le \gamma_0\mathcal{I}_p^{1/2}(p,q) + C_2\mathcal{I}_{\mathbf{D}}^{1/2}(\mathbf{u},\mathbf{v}). \end{aligned} \tag{1.19}$$

Proof. We follow the idea presented in [9] where the same inequality is established for $\gamma(p) \equiv 1$; Lemma 1.4 is thus an easy generalization that we prove here for the sake of completeness.

We set $p_s := q - s(q-p)$, $\mathbf{w}_s = \mathbf{v} - s(\mathbf{v}-\mathbf{u})$. Then

$$\begin{aligned} M(\mathbf{u},\mathbf{v}) &:= (\nu(p,|\mathbf{D}(\mathbf{u})|^2)\mathbf{D}(\mathbf{u}) - \nu(q,|\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}))\cdot\mathbf{D}(\mathbf{u}-\mathbf{v}) \\ &= \int_0^1 \frac{d}{ds}\nu(p_s,|\mathbf{D}(\mathbf{w}_s)|^2)\mathbf{D}(\mathbf{w}_s)\cdot\mathbf{D}(\mathbf{u}-\mathbf{v})\,ds \\ &= \int_0^1 \frac{\partial\nu(p_s,|\mathbf{D}(\mathbf{w}_s)|^2)\mathbf{D}_{k\ell}(\mathbf{w}_s)}{\partial\mathbf{D}_{ij}}\mathbf{D}_{ij}(\mathbf{u}-\mathbf{v})\mathbf{D}_{k\ell}(\mathbf{u}-\mathbf{v})\,ds \\ &\qquad + \int_0^1 \frac{\partial\nu(p_s,|\mathbf{D}(\mathbf{w}_s)|^2)}{\partial p_s}(p-q)\mathbf{D}(\mathbf{w}_s)\cdot\mathbf{D}(\mathbf{u}-\mathbf{v})\,ds. \end{aligned}$$

Using the assumptions (1.7)–(1.8), the fact that $r \le 2$, and applying Hölder's inequality to the second term, one concludes that

$$M(\mathbf{u},\mathbf{v}) \ge C_1\mathcal{I}_{\mathbf{D}}(\mathbf{u},\mathbf{v}) - \gamma_0\mathcal{I}_{\mathbf{D}}^{1/2}(\mathbf{u},\mathbf{v})\mathcal{I}_p^{1/2}(p,q).$$

Young's inequality then completes the proof of (1.18).

The inequality (1.19) can be verified by similar arguments. ◻

1.3. Main results and their relevance to former studies Our objective in this paper is to establish the existence of a solution to the following problem:

$$
(\mathcal{P}) \qquad \begin{cases}
\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) - \operatorname{div}(\nu(p,|\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v})) + \nabla p = \mathbf{f} & \\
 & \text{in } \Omega\times(0,T), \\
\operatorname{div}\mathbf{v} = 0 & \text{in } \Omega\times(0,T), \\
(\nu(p,|\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v})\mathbf{n})_\tau + \alpha\mathbf{v}_\tau = 0 & \text{on } \partial\Omega\times(0,T), \\
\mathbf{v}\cdot\mathbf{n} = 0 & \text{on } \partial\Omega\times(0,T), \\
\displaystyle\int_\Omega p(x,t)\,dx = 0, & \text{a.a. } t\in(0,T), \\
\mathbf{v}(\cdot,0) = \mathbf{v}_0 & \text{in } \Omega.
\end{cases}
$$

The requirement of zero mean value over domains Ω warrants a few comments. It is obvious that if ν is independent of p, then the pressure is determined up to a function of time, and usually in mathematical analysis, in bounded domains, it is fixed so that

$$\int_\Omega p(x,t)\,dx = 0 \quad \text{a.e. in } (0,T).$$

Since only the gradient of the pressure occurs in such systems, this choice has no influence on the solution $\mathbf{v}$ of the problem. This is however not true if ν depends on the pressure. First of all, there is a question as to whether the pressure should be fixed by a condition like $\int_\Omega p(t,x)\,dx = g(t)$, or whether it is uniquely determined just by equations and boundary conditions. The analysis of the spatially-periodic problem suggests that at least in some cases there is an undetermined value of p that has to be fixed (and in spatially-periodic case the choice $\int_\Omega p(t,x)\,dx = 0$ seems to be reasonable). Here, also, we use the same condition (even if it does not seem to be very natural) and we propose another possibility in a forthcoming study.

Before specifying what we mean by a solution to $(\mathcal{P})$ we introduce suitable function spaces. We write that $\Omega \in C^{0,1}$ if $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$ is a bounded open connected set with Lipschitz boundary $\partial\Omega$. If in addition the boundary $\partial\Omega$ is a locally $C^{1,1}$ mapping, then we write $\Omega \in C^{1,1}$.

Let $r \in [1,\infty]$. The Lebesgue spaces $L^r(\Omega)$ equipped with the norm $\|\cdot\|_r$ and the Sobolev spaces $W^{1,r}(\Omega)$ with the norm $\|\cdot\|_{1,r}$ are defined in the standard way. If X is a Banach space, then

$$X^d = \underbrace{X\times X\times\cdots\times X}_{d\text{ times}}.$$

The trace of the Sobolev function u is denoted by $\operatorname{tr} u$; if $\mathbf{u} \in (W^{1,r}(\Omega))^d$, then $\operatorname{tr}\mathbf{u} := (\operatorname{tr} u_1, \dots, \operatorname{tr} u_d)$. For our purpose we introduce the subspaces of vector-valued Sobolev functions which have zero normal part on the boundary. Let $1 \le q \le \infty$. We define

$$\begin{aligned} W^{1,q}_{\mathbf{n}} &:= \overline{\{\mathbf{v} \in (C^\infty(\Omega))^d \cap (C(\bar{\Omega}))^d \mid \operatorname{tr}\mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}}^{\|\cdot\|_{1,q}}, \\ W^{1,q}_{\mathbf{n},\operatorname{div}} &:= \{\mathbf{v} \in W^{1,q}_{\mathbf{n}} \mid \operatorname{div}\mathbf{v} = 0\}, \\ L^q_{\mathbf{n}} &:= \overline{\{\mathbf{v} \in W^{1,q}_{\mathbf{n},\operatorname{div}}\}}^{\|\cdot\|_q}. \end{aligned}$$

We also introduce the following notation for the dual spaces:

$$W^{-1,q'}_{\mathbf{n}} := (W^{1,q}_{\mathbf{n}})^* \quad \text{and} \quad W^{-1,q'}_{\mathbf{n},\operatorname{div}} := (W^{1,q}_{\mathbf{n},\operatorname{div}})^*.$$

All the spaces introduced above are Banach spaces. Moreover, if $1 < q < \infty$, then they are also reflexive and separable. For $r, q \in [1, +\infty]$, we also introduce relevant spaces of Bochner-type, namely,

$$\begin{aligned} X^{r,q} &:= \{\mathbf{u} \in L^r(0,T;W^{1,r}_{\mathbf{n}}) \cap L^q(0,T;L^q(\Omega)^d) \mid \operatorname{tr}\mathbf{u} \in L^2(0,T;(L^2(\partial\Omega))^d)\}, \\ X^{r,q}_{\operatorname{div}} &:= \{\mathbf{u} \in X^{r,q} \mid \operatorname{div}\mathbf{u} = 0\}, \\ Y^{r,q} &:= \{\mathbf{u} \in L^r(0,T;W^{1,r}_{\mathbf{n}}) \mid \operatorname{div}\mathbf{v} \in L^q(0,T;L^q(\Omega)), \operatorname{tr}\mathbf{u} \in L^2(0,T;(L^2(\partial\Omega))^d\}. \end{aligned}$$

Next, we define the notion of a solution to the problem $(\mathcal{P})$.

Definition 1.5. Let $\Omega \in C^{0,1}$, $d = 2, 3$. Let ν satisfy the assumptions (1.7) and (1.8). Let $\mathbf{v}_0 \in L^2_{\mathbf{n},\operatorname{div}}$, $\mathbf{f} \in L^{r'}(0,T;W^{-1,r'}_{\mathbf{n}})$ and $0 < T < \infty$. We say that a couple $(\mathbf{v}, p)$ is a weak solution to the problem $(\mathcal{P})$ if

$$\mathbf{v} \in C(0,T;L^2_{\text{weak}}) \cap L^r(0,T;W^{1,r}_{\mathbf{n},\operatorname{div}}), \tag{1.20}$$

$$\mathbf{v}_{,t} \in (X^{r,s} \cap Y^{r,z})^*, \tag{1.21}$$
$$\text{where } s := \frac{r(d+2)}{r(d+2) - 2(d+1)}, \; z := \frac{r(d+2)}{r(d+2) - 2d},$$

$$p \in L^{(d+2)r/(2d)}(0,T;L^{(d+2)r/(2d)}(\Omega)) \tag{1.22}$$
$$\text{and } \int_\Omega p(x,t) = 0 \quad \text{for a.a. } t \in (0,T),$$

$$\lim_{t\to 0+} \int_\Omega \|\mathbf{v}(t) - \mathbf{v}_0\|_2^2 = 0, \tag{1.23}$$

and the following weak formulation holds

$$
\begin{aligned}
(1.24) \quad \int_0^T \langle \mathbf{v}_{,t}, \boldsymbol{\varphi}\rangle - (\mathbf{v}\otimes\mathbf{v}, \nabla\boldsymbol{\varphi}) + (\nu(p, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}), \mathbf{D}(\boldsymbol{\varphi}))\, dt \\
+ \alpha \int_0^T \int_{\partial\Omega} \mathbf{v}\cdot\boldsymbol{\varphi}\, dS\, dt \\
= \int_0^T (p, \operatorname{div}\boldsymbol{\varphi}) + \langle \mathbf{f}, \boldsymbol{\varphi}\rangle\, dt, \quad \text{for all } \boldsymbol{\varphi} \in X^{r,s} \cap Y^{r,z}.
\end{aligned}
$$

There is a remarkable difference in introducing the pressure between the time independent problem and evolutionary problems in the case of the no-slip boundary condition. While for the stationary problem, the pressure can be easily identified using for example de Rham's theorem, the same tool cannot in general be used for the evolutionary case since time derivatives are not distributions (they usually belong to a dual space of divergenceless functions only).

There is also a telling difference in introducing the pressure between the evolutionary NSEs and the equations that govern the time dependent flows wherein the fluid has a non-constant viscosity. For the NSEs, we can identify the model with the evolutionary Stokes system, where the convective term is included in the right-hand side, and apply the results concerning L^p estimates available for such systems (see [16]). For the models where ν is not constant (and may depend on p or $|\mathbf{D}(\mathbf{v})|^2$) an analogous theory for the generalized Stokes system is not available. In this article we show that the problems wherein the Navier's slip boundary conditions are specified, contrary to no-slip boundary conditions, do not suffer such a deficiency and it is possible to introduce the pressure globally.

Now, we formulate the main theorem of this paper.

Theorem 1.6. *Let* $\Omega \in C^{1,1}$. *Let* ν *satisfy* (1.7)–(1.8) *with the parameters* r, q *such that*

$$2 \geq r > \frac{2(d+1)}{d+2}, \quad q > 1 - \frac{r}{4d}(d+2) \geq 0,$$

and with γ_0 *fulfilling*

$$(1.25) \qquad \gamma_0 < \frac{C_1}{2C_{\text{reg}}(\Omega, 2)(C_1 + C_2)},$$

where $C_{\text{reg}}(\Omega, 2)$ *appears in* (1.32) *below. Let*

$$\mathbf{v}_0 \in L^2_{\mathbf{n},\text{div}} \quad \text{and} \quad \mathbf{f} \in L^{r'}(0, T; W_{\mathbf{n}}^{-1,r'}).$$

Then there exists a weak solution to the problem $(\mathcal{P})$. *Moreover, if* $d = 2$ *and* $r = 2$, *then the weak solution is unique.*

This theorem establishes the first result concerning the long-time existence of weak solutions for large-data to any incompressible fluid model wherein the viscosity depends on the pressure and where flows take place in a bounded container.

Renardy [15] and Gazzola [7] contributed to the analysis of the appropriate PDE's when ν depends only on p, proving under very strong assumptions existence results for short-time and small-data.

If ν depends on p and $|\mathbf{D}(\mathbf{v})|^2$ as here, Málek et al. [9] and Hron et al. [8] established long-time and large-data existence of weak solution for $q = 0$ and $r \in (3d/(d+2), 2)$, $d = 2, 3$, in a spatially periodic setting. Steady flows subject to no-slip boundary conditions were analyzed in [5].

Theorem 1.6 includes several special cases that deserve separate formulations. We first consider the case $\nu \equiv \nu_0$, where $\nu_0 > 0$ is a constant. Then the problem $(\mathcal{P})$ reduces to the problem $(\mathcal{P}_{ns})$ consisting the NSEs completed by Navier's slip boundary conditions. The Theorem 1.6 then reduces to the following statement.

Corollary 1.7. *Let $\Omega \in C^{1,1}$. Let $\mathbf{v}_0 \in L^2_{\mathbf{n},\mathrm{div}}$ and $\mathbf{f} \in L^2(0,T; W^{-1,2}_{\mathbf{n}})$. Then there exists a weak solution to the problem $(\mathcal{P}_{ns})$. Moreover, if $d = 2$, then the weak solution is unique.*

Let us remark that the existence and regularity of two-dimensional flows described by the Navier-Stokes with Navier's slip boundary condition was established by Clopeau et al. [4] while studying the problem in the limit of vanishing viscosity. The present extension to the three-dimensional setting seems to be new.

If ν depends on the shear rate but not on the pressure, we conclude from Theorem 1.6 the following assertion.

Corollary 1.8. *Let $\Omega \in C^{1,1}$. Let $\nu = \nu(|\mathbf{D}|^2)$ satisfy (1.7). Let $\mathbf{v}_0 \in L^2_{\mathbf{n},\mathrm{div}}$ and $\mathbf{f} \in L^{r'}(0,T; W^{-1,r'}_{\mathbf{n}})$. Let $2 \geq r > 2(d+1)/(d+2)$. Then there exists a weak solution to the problem $(\mathcal{P})$.*

This result can be however strengthened in two directions. Although we do not provide any details, the proofs of these extensions are very similar to the proof of Theorem 1.6.

First, we observe that the upper bound $r \leq 2$ is not needed and we have the following result.

Theorem 1.9. *Let $\Omega \in C^{1,1}$. Let $\nu = \nu(|\mathbf{D}|^2)$ and satisfy (1.7). Let $\mathbf{v}_0 \in L^2_{\mathbf{n},\mathrm{div}}$ and $\mathbf{f} \in L^{r'}(0,T; W^{-1,r'}_{\mathbf{n}})$. Let $r > 2(d+1)/(d+2)$. Then there exists a weak solution to problem $(\mathcal{P})$. Moreover, if $r \geq (d+2)/2$, then the weak solution is unique.*

Theorem 1.9 includes as a special case the Smagorinski model of turbulence (subject to Navier's slip boundary condition) when $r = 3$. Generalized boundary conditions relevant to Smagorinski's model are analyzed by Parés in [14]. Higher differentiability for flows (subject to Navier's boundary conditions) of incompressible fluids with shear-rate dependent viscosity have been obtained by Beirão da Veiga in [1].

The second extension is achieved allowing degenerate/singular behavior of the viscosity as $|\mathbf{D}(\mathbf{v})| \to 0$ or $|\mathbf{D}(\mathbf{v})| \to \infty$ respectively. Note that the assumption (1.7) does not allow us to include the classical power-law fluid model in our analysis. The analysis presented in this paper can be however extended to this case without any significant difficulties. For this reason we just restrict ourselves to the formulation of the result. We assume that instead of (1.7) the following condition

$$C_1|\mathbf{D}|^{r-2}|\mathbf{B}|^2 \leq \frac{\partial \nu(|\mathbf{D}|^2)\mathbf{D}_{ij}}{\partial \mathbf{D}_{k\ell}}\mathbf{B}_{ij}\mathbf{B}_{k\ell} \leq C_2|\mathbf{D}|^{r-2}|\mathbf{B}|^2, \tag{1.26}$$

is valid for certain C_1, $C_2 > 0$ and for all $\mathbf{B}, \mathbf{D} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. Note that the standard power-law model $\nu(|\mathbf{D}|) := \nu_0|\mathbf{D}|^{r-2}$ fulfills the condition (1.26). Then we can establish the following result.

Theorem 1.10. *Let $\Omega \in C^{1,1}$. Let $\nu = \nu(|\mathbf{D}|^2)$ satisfy (1.26) with $r > 2(d+1)/(d+2)$. Let $\mathbf{v}_0 \in L^2_{\mathbf{n},\mathrm{div}}$ and $\mathbf{f} \in L^{r'}(0,T;W^{-1,r'}_{\mathbf{n}})$. Then there exists a weak solution to the problem $(\mathcal{P})$. Moreover, if $r \geq (d+2)/2$, then the weak solution is unique.*

This theorem covers the result established in [6] treating the spatially periodic and slip boundary conditions. Very recently Wolf in [18] extended this result also to the no-slip boundary condition by means of "local" pressure estimates. A drawback of his very interesting study consists in the fact that there is no uniquely defined pressure over $\Omega \times (0,T)$, or even over $\Omega_0 \times (0,T)$, $\Omega_0 \Subset \Omega$. Consequently, it is not at all clear that his approach can be applied to the case where the viscosity depends on the pressure (and the shear rate).

1.4. The structure of the proof and its main ingredients The proof is split into several steps; all of them form particular subsections of Section 2 of this paper.

In Subsection 2.1 we introduce the so-called quasi-compressible approximation problem $(\mathcal{P}^{\varepsilon,\eta})$ that consists of two levels of approximations. In order to have some control on the pressure, we first perturb for all $\varepsilon > 0$, the incompressibility constraint in the following manner:

$$\begin{aligned} -\varepsilon\Delta p + \operatorname{div}\mathbf{v} &= 0 \quad \text{in } \Omega \times (0,T), \\ \frac{\partial p}{\partial \mathbf{n}} &= 0 \quad \text{on } \partial\Omega \times (0,T). \end{aligned} \tag{1.27}$$

In order to preserve apriori estimates we also modify the convective term by a suitable η-approximation.

The proof of the existence of a solution to $(\mathcal{P}^{\varepsilon,\eta})$-approximations for $\varepsilon > 0$, $\eta > 0$ fixed will be carried out via Galerkin approximations incorporating the compactness of velocities, compactness of the pressures based on the observation that

$$\begin{aligned}\varepsilon\|\nabla p^N - \nabla p\|_2^2 &= (\operatorname{div}\mathbf{v}^N - \operatorname{div}\mathbf{v}, p^N - p)\\ &= -(\mathbf{v}^N - \mathbf{v}, \nabla p^N - p)\\ &\to 0 \quad \text{as } \mathbf{v}^N \to \mathbf{v} \text{ strongly in } (L^2(\Omega))^d,\end{aligned}$$

and the "compactness" of the velocity gradient obtained by the method of monotone operators. All this material is presented in Subsection 2.1.

In Subsection 2.2 we will obtain results by taking the limit $\varepsilon \to 0_+$ (i.e., we will obtain the "incompressible" limit). Here the fact that we deal with Navier's boundary condition will play an important role. Note that a similar procedure can also be used for the spatially-periodic problem, but such a procedure does not seem to work in the case of Dirichlet's boundary conditions. In order to let $\varepsilon \to 0_+$ we need to obtain estimates for the pressures that are uniform w.r.t. ε, $\eta > 0$.

This is performed by taking a test function $\boldsymbol{\varphi}$ in the weak formulation of the problem $(\mathcal{P}^{\varepsilon,\eta})$ of the form $\boldsymbol{\varphi} = -\nabla g$ where $g := g^{\varepsilon,\eta}$ solves

$$\begin{aligned}&-\Delta g = |p|^{\alpha-2}p \quad \text{in } \Omega \ (\alpha > 1),\\ &\frac{\partial g}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega, \qquad \int_\Omega g\,dx = 0.\end{aligned} \tag{1.28}$$

Clearly, the term involving the pressure leads to

$$(p, \operatorname{div}\boldsymbol{\varphi}) = (p, -\Delta g) = \|p\|_\alpha^\alpha. \tag{1.29}$$

The task is to control the remaining terms, and the time derivative is the most critical. Using the Helmholtz decomposition

$$\mathbf{v} = \mathbf{v}_{\text{div}} + \nabla g^{\mathbf{v}} \tag{1.30}$$

where $g^{\mathbf{v}}$ is a solution of the auxiliary problem

$$\begin{aligned}&\Delta g^{\mathbf{v}} = \operatorname{div}\mathbf{v} \quad \text{in } \Omega,\\ &\frac{\partial g^{\mathbf{v}}}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega, \qquad \int_\Omega g^{\mathbf{v}}\,dx = 0,\end{aligned} \tag{1.31}$$

we observe, by comparing (1.27) with (1.31), that $g^{\mathbf{v}} = \varepsilon p$. Consequently, we have

$$\begin{aligned}\langle \mathbf{v}_{,t}, \boldsymbol{\varphi}\rangle &= \langle \mathbf{v}_{\text{div},t} + (\nabla g^{\mathbf{v}})_{,t}, -\nabla g\rangle\\ &= -\langle \nabla g^{\mathbf{v}}_{,t}, \nabla g\rangle = \langle -g_{,t}, -\Delta g\rangle\\ &= -\varepsilon\langle p_{,t}, |p|^{\alpha-2}p\rangle = -\varepsilon\frac{d}{dt}\|p\|_\alpha^\alpha \le 0.\end{aligned}$$

Thus, the time derivative acting on the test function ∇g has the correct sign and we separate the pressure from the time derivative. In the analysis of the remaining terms we will apply, in Subsection 2.2 below, the following standard result on the solvability of (1.31). If $\Omega \in C^{1,1}$, then

$$(1.32) \quad \|g^{\mathbf{v}}\|_{2,q} \leq C_{\text{reg}}(\Omega, q)\|\operatorname{div}\mathbf{v}\|_q, \quad \|\mathbf{v}_{\text{div}}\|_{1,q} \leq (C_{\text{reg}}(\Omega, q) + 1)\|\mathbf{v}\|_{1,q},$$
$$(1.33) \quad \|g^{\mathbf{v}}\|_{1,s} \leq C(\Omega, s)\|\mathbf{v}\|_s, \qquad \|\mathbf{v}_{\text{div}}\|_s \leq (C(\Omega, s) + 1)\|\mathbf{v}\|_s,$$

whenever the right hand sides make sense. We wish to remark that this approach of obtaining estimates for the pressures is used several times in the proof of Theorem 1.6.

Finally, in Subsection 2.3, we let $\eta \to 0_+$ and apply the method of $L^\infty(0, T; L^\infty)$ truncation function. The method requires that the convective term $[\nabla\mathbf{v}]\mathbf{v}$ is an integrable function. Since $\mathbf{v} \in L^\infty(0, T; L^2) \cap L^r(0, T; W^{1,r})$, this requirement on the integrability of the convective term leads to the restriction on r that is stated in assertions above: $r \geq 2(d+1)/(d+2)$. For the time dependent models describing flows of fluids where the viscosity ν depends only on $|\mathbf{D}|^2$, this method was first used in [6] for the spatially periodic setting and for slip boundary conditions. For parabolic systems, the method was developed by Boccardo and Murat [2]. We will slightly modify this method and a key role will be played by the assumptions on $\gamma(p)$ and q.

1.5. Trace theorem and Korn's inequality Here we record some important inequalities which will be frequently used in what follows.

Lemma 1.11 (Korn's inequality). *Let $\Omega \in C^{0,1}$ and $q \in (1, \infty)$. Then there exists a positive constant C, depending only on Ω and q, such that for all $\mathbf{v} \in W^{1,q}(\Omega)^d$ which has the trace $\operatorname{tr}\mathbf{v} \in L^2(\partial\Omega)^d$ the following inequality holds*

$$(1.34) \qquad C\|\mathbf{v}\|_{1,q} \leq \|\mathbf{D}(\mathbf{v})\|_q + \|\mathbf{v}\|_{L^2(\partial\Omega)}.$$

Proof. First, we know, see for example the first part of Theorem 1.10, p.196 in [10], that there exists $C'(\Omega, q)$ such that for all $\mathbf{v} \in W^{1,q}(\Omega)^d$

$$(1.35) \qquad \|\mathbf{v}\|_{1,q} \leq C'(\|\mathbf{D}(\mathbf{v})\|_q + \|\mathbf{v}\|_q).$$

Thus, it is enough to show that for all $\mathbf{v} \in W^{1,q}(\Omega)^d$ with $\operatorname{tr}\mathbf{v} \in L^2(\partial\Omega)^d$

$$\|\mathbf{v}\|_q \leq C''(\|\mathbf{D}(\mathbf{v})\|_q + \|\operatorname{tr}\mathbf{v}\|_2)$$

for some positive constant C''. To prove it, we assume the contrary. We take a sequence $\{\mathbf{v}^n\}_{n=1}^\infty$ such that $\|\mathbf{v}^n\|_q = 1$ and

$$1 > n(\|\mathbf{D}(\mathbf{v}^n)\|_q + \|\operatorname{tr}\mathbf{v}^n\|_2).$$

It implies that

$$\|\mathbf{D}(\mathbf{v}^n)\|_q \to 0,$$
$$\|\operatorname{tr}\mathbf{v}^n\|_2 \to 0.$$

With the help of (1.35) we have $\|\mathbf{v}^n\|_{1,q} \leq C' < \infty$. As the space $W^{1,q}(\Omega)^d$ is reflexive ($1 < q < \infty$), we can take a subsequence which is not relabeled such that

$$\mathbf{v}^n \rightharpoonup \mathbf{v} \quad \text{weakly in } W^{1,q}(\Omega)^d.$$

From the fact that we have compact embedding we also have

$$\mathbf{v}^n \to \mathbf{v} \quad \text{strongly in } L^q(\Omega)^d.$$

This convergence then leads to the conclusion that $\|\mathbf{v}\|_q = 1$. On the other hand $\mathbf{v} \in W_0^{1,q}(\Omega)^d$ and $\mathbf{D}(\mathbf{v}) \equiv \mathbf{0}$. Then using of Korn's inequality for functions vanishing on the boundary implies that $\mathbf{v} = \mathbf{0}$, which is a contradiction. ◻

The last lemma gives important information concerning the behavior of some functions on the boundary.

Lemma 1.12. *Let $\Omega \in C^{0,1}$, $d = 2, 3$, and let $\{\mathbf{v}^i\}_{i=1}^\infty$ be bounded in S, defined for some $1 < q_1, q_2 < \infty$ through*

$$S := \left\{\mathbf{v} \mid \mathbf{v} \in L^\infty(0,T;L^2(\Omega)^d) \cap L^r(0,T;W_{\mathbf{n}}^{1,r}),\ \mathbf{v}_{,t} \in L^{q_1}(0,T;W_{\mathbf{n},\mathrm{div}}^{-1,q_2})\right\}.$$

Let $2 \geq r > 2(d+1)/(d+2)$. Then $\{\operatorname{tr}\mathbf{v}^i\}_{i=1}^\infty$ is precompact in $L^s(0,T;L^s(\partial\Omega)^d)$ for all $s < r(d+2)/d - 2/d$.

Proof. According to [13] there is a continuous *trace* operator tr such that for $k \in \mathbb{R}_+$ and $k > \frac{1}{2}$

$$\operatorname{tr}: W^{k,2}(\Omega)^d \to W^{k-1/2,2}(\partial\Omega)^{d-1}. \tag{1.36}$$

Next, using the Aubin-Lions compactness lemma we observe that for small positive ε_1 and ε_2

$$S \hookrightarrow\hookrightarrow L^r(0,T;W^{1-\varepsilon_1,r}(\Omega)^d) \hookrightarrow L^r(0,T;W^{1/2+\varepsilon_2,2}(\Omega)^d) \quad \text{as } r > \frac{2d}{d+1}.$$

Then (1.36) implies that we can take a subsequence (not relabeled) $\{\mathbf{v}^i\}_{i=1}^\infty$ such that

$$\operatorname{tr}\mathbf{v}^i \to \operatorname{tr}\mathbf{v} \quad \text{strongly in } L^1(0,T;L^1(\partial\Omega)^{d-1}). \tag{1.37}$$

Consider $s \in [2, (r(d+2)-2)/d)$ fixed and arbitrary. We show that

$$\{\operatorname{tr} \mathbf{v}^i\}_{L^{s+\varepsilon}(0,T;L^{s+\varepsilon}(\partial\Omega)^d)} \leq K < \infty \tag{1.38}$$

holds for some positive constant K and sufficiently small $\varepsilon > 0$. The combination of (1.37) and (1.38) completes the proof of Lemma 1.12.

To prove (1.38) we observe for $r \in 2d/(d+1), 2)$ that

$$W^{1,r}(\Omega) \hookrightarrow W^{k,2}(\Omega), \quad k = \frac{dr + 2r - 2d}{2r} > \frac{1}{2}, \tag{1.39}$$

$$W^{\ell,2}(\partial\Omega) \hookrightarrow L^{s+\varepsilon}(\partial\Omega), \quad \ell = \frac{(d-1)(s+\varepsilon-2)}{2(\varepsilon+s)}, \tag{1.40}$$

and for $\ell \leq k - \frac{1}{2}$

$$\|\cdot\|_{\ell+1/2,2} \leq \|\cdot\|_{k,2}^{(\ell+1/2)/k} \|\cdot\|_2^{1-(\ell+1/2)/k}. \tag{1.41}$$

Thus, we have (if $\ell \leq k - \frac{1}{2}$)

$$\begin{aligned}\int_0^T \|\operatorname{tr} \mathbf{v}^i\|_{s+\varepsilon}^{s+\varepsilon}\, dt &\overset{(1.40)}{\leq} C \int_0^T \|\operatorname{tr} \mathbf{v}^i\|_{\ell,2}^{s+\varepsilon}\, dt \overset{(1.36)}{\leq} C \int_0^T \|\mathbf{v}^i\|_{\ell+1/2,2}^{s+\varepsilon}\, dt \\ &\overset{(1.39),(1.41)}{\leq} C \|\mathbf{v}^i\|_{L^\infty(0,T;L^2(\Omega)^d)}^{(1-(\ell+1/2)/k)(s+\varepsilon)} \int_0^T \|\mathbf{v}^i\|_{1,r}^{(s+\varepsilon)(\ell+1/2)/k}\, dt.\end{aligned}$$

Then (1.38) holds provided that $(s+\varepsilon)(\ell+\frac{1}{2})/k \leq r$, $\ell \leq k - \frac{1}{2}$ and $s \geq 2$. These conditions lead to the requirement

$$2 \leq s < \frac{r(d+2)-2}{d}.$$

Such an s exists if

$$r > \frac{2(d+1)}{d+2}. \qquad \square$$

By using Lemma 1.12 we can prove the following result.

Corollary 1.13. *Let* $\Omega \in C^{0,1}$, $d = 2, 3$ *and* $r > 2(d+1)/(d+2)$. *Let* $\{\mathbf{v}^i\}_{i=1}^\infty$ *be bounded in* S. *Then* $\{\operatorname{tr} \mathbf{v}^i\}_{i=1}^\infty$ *is precompact in* $L^2(0,T;L^2(\partial\Omega)^d)$.

2. PROOF OF THE MAIN THEOREM

2.1. ***ε, η approximation*** We start with the quasi-compressible approximation $(\mathcal{P}^{\varepsilon,\eta})$ (for simplicity, we write $(\mathbf{v}, p)$ instead of $(\mathbf{v}^{\varepsilon,\eta}, p^{\varepsilon,\eta})$)

$$
(\mathcal{P}^{\varepsilon,\eta}) \qquad \begin{cases} \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v}_\eta \otimes \mathbf{v}) - \operatorname{div}(\nu(p, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v})) + \nabla p = \mathbf{f} & \\ & \text{in } \Omega \times (0,T), \\ -\varepsilon \Delta p + \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega \times (0,T), \\ (\nu(p, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v})\mathbf{n})_\tau + \alpha \mathbf{v}_\tau = 0 & \text{on } \partial\Omega \times (0,T), \\ \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0,T), \\ \nabla p \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0,T), \\ \int_\Omega p(x,t)\, dx = 0, & \\ \mathbf{v}(x,0) = \mathbf{v}_0. & \end{cases}
$$

The definition of $\mathbf{v}_\eta$ is the following. Let $\mathbf{v} \in W^{1,q}_{\mathbf{n}}$ and $\eta > 0$ be fixed. We define the function φ_η

$$
\varphi_\eta(x) := \begin{cases} 0 & \text{if } \operatorname{dist}(x, \partial\Omega) \le 2\eta \\ 1 & \text{elsewhere.} \end{cases}
$$

Finally, we define $\mathbf{v}_\eta := ((\varphi_\eta \mathbf{v}) * \omega^\eta)_{\operatorname{div}}$ where the symbol $u * \omega^\eta$ is the standard regularization and $(.)_{\operatorname{div}}$ comes from (1.30), noticing that we can use the Helmholtz decomposition as $(\varphi_\eta \mathbf{v}) * \omega^\eta \in (C_0^\infty(\Omega))^d$. Observe that this definition of $\mathbf{v}_\eta$ leads to the identity

$$
\int_\Omega \operatorname{div}(\mathbf{v}_\eta \otimes \mathbf{v}) \cdot \mathbf{v} = 0. \tag{2.1}
$$

Moreover, if $\operatorname{div} \mathbf{v} = 0$, then $\mathbf{v}_\eta \to \mathbf{v}$ in $L^q(0,T; L^q_{\mathbf{n}})$ provided $\mathbf{v} \in L^q(0,T; L^q_{\mathbf{n}})$.

2.1.1. *Galerkin approximation* For a fixed $\mathbf{v} \in W^{1,r}_{\mathbf{n}}$ there exists a unique p solving $\varepsilon \Delta p = \operatorname{div} \mathbf{v}$, $\int_\Omega p\, dx = 0$ with homogeneous Neumann boundary condition. From the regularity theory for the Laplace equation it follows that we can define the mapping $\mathcal{F} : W^{1,r}_{\mathbf{n}} \to W^{2,r}(\Omega)$ $(\hookrightarrow W^{1,2}(\Omega)$ $(r \ge 2d/(d+2)))$ such that $\mathcal{F}(\mathbf{v}) := p$. Moreover, this mapping is continuous.

Thus, the system $(\mathcal{P}^{\varepsilon,\eta})$ can be rewritten in the form

$$
(\mathcal{P}^{\varepsilon,\eta})^* \qquad \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v}_\eta \otimes \mathbf{v}) - \operatorname{div} \nu(\mathcal{F}(\mathbf{v}), |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}) + \nabla \mathcal{F}(\mathbf{v}) = \mathbf{f}
$$

with the same boundary condition on $\mathbf{v}$ as in the problem $(\mathcal{P}^{\varepsilon,\eta})$.

Let $\{\mathbf{w}_j\}_{j=1}^{\infty}$ be a basis of $W_{\mathbf{n}}^{1,r}$. We construct Galerkin approximations $\{\mathbf{v}^N\}_{N=1}^{\infty}$ being of the form

$$\mathbf{v}^N(x,t) := \sum_{i=1}^{N} c_j^N(t)\mathbf{w}_j(x),$$

where $\mathbf{c}^N(t)$ solve the system of ordinary differential equations:

$$\begin{aligned}(2.2)\quad \frac{d}{dt}(\mathbf{v}^N,\mathbf{w}_j) - (\mathbf{v}_\eta^N \otimes \mathbf{v}^N, \nabla \mathbf{w}_j) + (\nu(\mathcal{F}(\mathbf{v}^N), |\mathbf{D}(\mathbf{v}^N)|^2)\mathbf{D}(\mathbf{v}^N), \nabla \mathbf{w}_j) \\ + \alpha \int_{\partial\Omega} \mathbf{v}^n \cdot \mathbf{w}_j \, dS - (\mathcal{F}(\mathbf{v}^N), \operatorname{div} \mathbf{w}_j) \\ = \langle \mathbf{f}, \mathbf{w}_j \rangle \quad \text{for } s = 1,2,\dots,N.\end{aligned}$$

Note that the term $(\mathcal{F}(\mathbf{v}^N), \operatorname{div}\mathbf{w}_j)$ makes sense whenever $r \geq 2d/(d+2)$. Due to the continuity of ν, $\mathcal{F}$ and the definition of $\mathbf{v}_\eta^N$, the local-in-time existence follows from Caratheodory theory. The global-in-time existence will be established by means of apriori estimates proved below.

2.1.2. *Apriori estimates* Testing the first equation in $(\mathcal{P}^{\varepsilon,\eta})$ by p^N and the second one by $\mathbf{v}^N$, integrating over time and using (1.34) leads to the following estimate:

$$\begin{aligned}(2.3)\qquad \sup_{t\in(0,T)} \|\mathbf{v}^N(t)\|_2^2 \\ + \int_0^T \left(\|\mathbf{D}(\mathbf{v}^N)\|_r^r + \varepsilon \|\nabla p^N\|_2^2 + \int_{\partial\Omega} |\mathbf{v}^N|^2 \, dS \right) dt \leq C.\end{aligned}$$

From (2.2) and (2.3) it then follows that ($X_{\text{div}}^{r,s'}$ is defined in Section 1.3)

$$(2.4)\qquad \begin{aligned} &\|\mathbf{v}_{,t}^N\|_{(L^2(0,T;W_{\mathbf{n}}^{1,2}))^*} \leq C(\varepsilon,\eta), \\ &\|\mathbf{v}_{,t}^N\|_{(X_{\text{div}}^{r,s'})^*} \leq C \quad \text{uniformly with respect to } \varepsilon, \eta. \end{aligned}$$

where $s := r(d+2)/(2(d+1)) > 1$ (as $r > 2(d+1)/(d+2)$).

Using (1.7), (2.3), (2.4), Corollary 1.13, Korn's inequality and Aubin-Lions compactness lemma we have

$$\mathbf{v}^N_{,t} \rightharpoonup \mathbf{v}_{,t} \qquad \text{weakly in } (X^{r,2})^*, \tag{2.5}$$

$$\mathbf{v}^N \rightharpoonup^* \mathbf{v} \qquad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)^d), \tag{2.6}$$

$$\mathbf{v}^N \rightharpoonup \mathbf{v} \qquad \text{weakly in } L^r(0,T;W^{1,r}_{\mathbf{n}}), \tag{2.7}$$

$$\operatorname{tr}\mathbf{v}^N \to \operatorname{tr}\mathbf{v} \qquad \text{strongly in } L^2(0,T;L^2(\partial\Omega)), \tag{2.8}$$

$$\mathbf{v}^N \to \mathbf{v} \qquad \text{strongly in } L^z(0,T;L^z(\Omega)^d) \text{ for } z \in [1, r(d+2)/d), \tag{2.9}$$

$$p^N \rightharpoonup p \qquad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{2.10}$$

$$\nu(p^N, |\mathbf{D}(\mathbf{v}^N)|^2)\mathbf{D}(\mathbf{v}^N) \rightharpoonup \overline{\nu\mathbf{D}} \qquad \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)^{d\times d}). \tag{2.11}$$

We now let $N \to \infty$ in the Galerkin approximation, to get for all $\psi \in L^2(0,T;W^{1,2}(\Omega))$, $\boldsymbol{\varphi} \in X^{r,2}$ that

$$\int_0^T \varepsilon(\nabla p, \nabla\psi)\,d\tau = \int_0^T (\mathbf{v}, \nabla\psi)\,d\tau \tag{2.12}$$

$$\int_0^T \left\{\langle \mathbf{v}_{,t}, \boldsymbol{\varphi}\rangle - (\mathbf{v}^\eta \otimes \mathbf{v}, \nabla\boldsymbol{\varphi}) + (\overline{\nu\mathbf{D}}, \mathbf{D}(\boldsymbol{\varphi})) + \alpha\int_{\partial\Omega} \mathbf{v}\cdot\boldsymbol{\varphi}\,dS\right\} d\tau = \int_0^T \{(p, \operatorname{div}\boldsymbol{\varphi}) + \langle \mathbf{f}, \boldsymbol{\varphi}\rangle\}\,d\tau\,. \tag{2.13}$$

To get the strong convergence of the pressure we can compute

$$\begin{aligned}
&\int_0^T \|\nabla(p^N - p)\|_2^2\,dt \\
&\quad = \int_0^T (\nabla p^N, \nabla p^N) - (\nabla p^N, \nabla p) - (\nabla p, \nabla(p^N - p))\,dt \\
&\quad = \int_0^T -(\nabla p, \nabla(p^n - p)) - (\nabla p^N, \nabla p) + \frac{1}{\varepsilon}(\mathbf{v}^N, \nabla p^N)\,dt \\
&\quad \xrightarrow{(2.9)} \int_0^T -(\nabla p, \nabla p) + \frac{1}{\varepsilon}(\mathbf{v}, \nabla p)\,dt \overset{(2.12)}{=} 0,
\end{aligned}$$

where we have used the strong convergence of $\mathbf{v}^N$ stated in (2.9). Hence

$$p^N \to p \quad \text{strongly in } L^2(0,T;W^{1,2}(\Omega)).$$

By using (1.18) we have for all $\boldsymbol{\varphi} \in X^{r,2}$ (after using the fact that $\gamma \leq 1$)

$$0 \leq \int_0^T \Big((\nu(p^N, |\mathbf{D}(\mathbf{v}^N)|^2)\mathbf{D}(\mathbf{v}^N) - \nu(p, |\mathbf{D}(\boldsymbol{\varphi})|^2)\mathbf{D}(\boldsymbol{\varphi}), \mathbf{D}(\mathbf{v}^N - \boldsymbol{\varphi})) + \frac{\gamma_0^2}{2C_1} \|p^N - p\|_2^2 \Big)\, dt.$$

We use the Galerkin approximation (2.2) to replace the term

$$(\nu(p^N, |\mathbf{D}(\mathbf{v}^N)|^2)\mathbf{D}(\mathbf{v}^N), \mathbf{D}(\mathbf{v}^N))$$

and take the limit as $N \to \infty$. Using the strong convergence of the pressure, (2.5)–(2.9) and (2.13) we conclude that

$$0 \leq \int_0^T (\overline{\nu \mathbf{D}} - \nu(p, |\mathbf{D}(\boldsymbol{\varphi})|^2)\mathbf{D}(\boldsymbol{\varphi}), \mathbf{D}(\mathbf{v} - \boldsymbol{\varphi}))\, d\tau.$$

A possible choice $\boldsymbol{\varphi} := \mathbf{v} \pm \lambda \mathbf{u}$ then implies (after using the standard Minty method) that

$$\overline{\nu \mathbf{D}} = \nu(p, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}) \quad \text{a.e. in } \Omega \times (0,T).$$

The solvability of the problem $(\mathcal{P}^{\varepsilon,\eta})$ is complete.

2.2. Limit $\varepsilon \to 0$ In this subsection we will establish the existence of a weak solution to the following problem $(\mathcal{P}^{\eta})$ (we write for simplicity $(\mathbf{v}, p)$ instead $(\mathbf{v}^{\eta}, p^{\eta})$)

$$(\mathcal{P}^{\eta}) \qquad \begin{cases} \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v}_{\eta} \otimes \mathbf{v}) - \operatorname{div}(\nu(p, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v})) + \nabla p = \mathbf{f} & \\ & \text{in } \Omega \times (0,T), \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega \times (0,T), \\ (\nu(p, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v})\mathbf{n})_{\tau} + \alpha \mathbf{v}_{\tau} = 0 & \text{on } \partial\Omega \times (0,T), \\ \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0,T), \\ \displaystyle\int_{\Omega} p(x,t)\, dx = 0, & \\ \mathbf{v}(x,0) = \mathbf{v}_0. & \end{cases}$$

To prove the existence of the solution to $(\mathcal{P}^{\eta})$ we use the problem $(\mathcal{P}^{\varepsilon,\eta})$ and we take the limit in ε.

Here, we abbreviate the solutions $(\mathbf{v}^{\varepsilon,\eta}, p^{\varepsilon,\eta})$ of $(\mathcal{P}^{\varepsilon,\eta})$ by $(\mathbf{v}^{\varepsilon}, p^{\varepsilon})$. After using weak lower semicontinuity of all terms (that are independent of ε) in (2.3) and in (2.4), we have

$$(2.14) \quad \sup_{t\in(0,T)} \|\mathbf{v}^{\varepsilon}(t)\|_2^2 + \|\mathbf{v}^{\varepsilon}_{,t}\|_{(X^{r,s'}_{\mathrm{div}})^*} + \int_0^T \left(\|\mathbf{D}(\mathbf{v}^{\varepsilon})\|_r^r + \int_{\partial\Omega} |\mathbf{v}^{\varepsilon}|^2 \, dS \right) dt \le C,$$

with $s := r(d+2)/(2(d+1)) > 1$.

We also need estimates on the pressure p^{ε} that are uniform w.r.t $\varepsilon > 0$. To do it, we consider g^{ε} solving the following homogeneous Neumann problem for the Laplace equation:

$$(2.15) \qquad \Delta g^{\varepsilon}(t) = p^{\varepsilon}(t) \text{ in } \Omega, \quad \int_{\Omega} g^{\varepsilon}(t)\, dx = 0, \quad \frac{\partial g^{\varepsilon}}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega.$$

Note that $\|g^{\varepsilon}(t)\|_{2,2} \le C_{\mathrm{reg}}(\Omega, 2)\|p^{\varepsilon}(t)\|_2$ for a.a. $t \in (0,T)$. Taking $\boldsymbol{\varphi} := \nabla g^{\varepsilon}$ in (2.13) we obtain

$$\int_0^t \|p^{\varepsilon}\|_2^2 \, dt = \sum_{i=1}^{5} I_j,$$

where (we use just standard Hölder's, Young's and embedding inequalities to estimate $I_1, \dots, I_4$)

$$I_1 := \alpha \int_0^t \int_{\partial\Omega} \mathbf{v}^{\varepsilon} \cdot \nabla g^{\varepsilon} \, dS \, d\tau \le C + \frac{1}{8} \int_0^t \|p^{\varepsilon}\|_2^2,$$

$$I_2 := -\int_0^t (\mathbf{v}^{\varepsilon}_{\eta} \otimes \mathbf{v}^{\varepsilon}, \nabla^2 g^{\varepsilon}) \, d\tau \le C(\eta) + \frac{1}{8} \int_0^t \|p^{\varepsilon}\|_2^2,$$

$$I_3 := -\int_0^t \langle \mathbf{f}, \nabla g^{\varepsilon} \rangle \le C + \frac{1}{8} \int_0^t \|p^{\varepsilon}\|_2^2,$$

$$I_4 := \int_0^t (\nu(p^{\varepsilon}, |\mathbf{D}(\mathbf{v}^{\varepsilon})|^2)\mathbf{D}(\mathbf{v}^{\varepsilon}), \nabla^2 g^{\varepsilon}) \, d\tau \le C + \frac{1}{8} \int_0^t \|p^{\varepsilon}\|_2^2 \, d\tau,$$

$$I_5 := \int_0^t \langle \mathbf{v}^{\varepsilon}_{,t}, \nabla g^{\varepsilon} \rangle \, d\tau.$$

Next, we will show that $I_5 \le 0$. Note first that I_5 is well defined. Assume for a moment that p^{ε} and $\mathbf{v}^{\varepsilon}$ are smooth. Then the function g^{ε} introduced in (2.15) is smooth as well. To the velocity $\mathbf{v}^{\varepsilon}$ we apply the Helmholtz decomposition (1.30) to obtain

$$I_5 = \int_0^t \langle \mathbf{v}^{\varepsilon}_{\mathrm{div},t} + \nabla g^{\mathbf{v}^{\varepsilon}}_{,t}, \nabla g^{\varepsilon} \rangle \, d\tau$$

$$\overset{\mathrm{div}\, \mathbf{v}^{\varepsilon}_{\mathrm{div}} = 0}{=} \int_0^t \langle \nabla g^{\mathbf{v}^{\varepsilon}}_{,t}, \nabla g^{\varepsilon} \rangle \, d\tau \overset{\nabla \varphi^{\varepsilon}\cdot \mathbf{n} = 0|_{\partial\Omega}}{=} -\int_0^t (g^{\mathbf{v}^{\varepsilon}}_{,t}, \Delta g^{\varepsilon}) \, d\tau.$$

By using the definitions of $g^{\mathbf{v}^\varepsilon}$, g^ε and the equation (2.12) we have

$$\frac{1}{\varepsilon}\Delta g^{\mathbf{v}^\varepsilon} = \frac{1}{\varepsilon}\operatorname{div}\mathbf{v}^\varepsilon = \Delta p^\varepsilon, \quad \Delta g^\varepsilon = p^\varepsilon.$$

The uniqueness of the solution to the Laplace equation then gives

$$\frac{1}{\varepsilon}g^{\mathbf{v}^\varepsilon} = \Delta\varphi^\varepsilon = p^\varepsilon.$$

Hence,

$$2I_5 = -2\varepsilon\int_0^t (p^\varepsilon_{,t}, p^\varepsilon)\,d\tau = -\varepsilon\|p^\varepsilon(t)\|_2^2 + \varepsilon\|p^\varepsilon(0)\|_2^2 = -\varepsilon\|p^\varepsilon(t)\|_2^2 \le 0,$$

where we used the fact that $p^\varepsilon(0) = 0$ ($\operatorname{div}\mathbf{v}^\varepsilon(0) = 0$). Thus, for smooth functions I_5 is non-positive. By means of the density of smooth functions we conclude that the same holds also for the original couple $(\mathbf{v}^\varepsilon, p^\varepsilon)$. To summarize, the above computation implies that

$$\|p^\varepsilon\|_{L^2(0,T;L^2(\Omega))} \le C(\eta). \tag{2.16}$$

Consequently,

$$\|\mathbf{v}^\varepsilon_{,t}\|_{(Y^{r,2}\cap X^{r,s})^*} \le C(\eta). \tag{2.17}$$

It then follows from (2.14), (2.16) and (2.17), Aubin-Lions lemma, Korn's inequality, and Corollary 1.13 that

$$\mathbf{v}^\varepsilon_{,t} \rightharpoonup \mathbf{v}_{,t} \quad \text{weakly in } (X^{r,s'}_{\operatorname{div}})^*, \tag{2.18}$$

$$\mathbf{v}^\varepsilon \rightharpoonup^* \mathbf{v} \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)^d), \tag{2.19}$$

$$\mathbf{v}^\varepsilon \rightharpoonup \mathbf{v} \quad \text{weakly in } L^r(0,T;W^{1,r}_{\mathbf{n}}), \tag{2.20}$$

$$\operatorname{tr}\mathbf{v}^\varepsilon \to \operatorname{tr}\mathbf{v} \quad \text{strongly in } L^2(0,T;L^2(\partial\Omega)), \tag{2.21}$$

$$\mathbf{v}^\varepsilon \to \mathbf{v} \quad \text{strongly in } L^z(0,T;L^z(\Omega)^d) \text{ for } z \in [1, r(d+2)/d), \tag{2.22}$$

$$p^\varepsilon \rightharpoonup p \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \tag{2.23}$$

$$\mathbf{v}^\varepsilon_{,t} \rightharpoonup \mathbf{v}_{,t} \quad \text{weakly in } (Y^{r,2}\cap X^{r,s}))^*, \tag{2.24}$$

and

$$\nu(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)\mathbf{D}(\mathbf{v}^\varepsilon) \rightharpoonup \overline{\nu\mathbf{D}} \quad \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)^{d\times d}). \tag{2.25}$$

Moreover, after taking the limit in the first equation of $(\mathcal{P}^{\varepsilon,\eta})$ we have

$$(\operatorname{div}\mathbf{v}, \varphi) = 0 \quad \forall\, \varphi \in C^\infty(\Omega),$$

which implies that $\operatorname{div}\mathbf{v} = 0$ a.e. in $\Omega \times (0,T)$.

Strong convergence is sufficient to take the limit in the case of the convective term ($r > 2d/(d+2)$) and the boundary term. For taking the limit in the linear terms it is enough to use the weak convergence established above.

To take the limit of the term containing the viscosity we need almost everywhere pointwise convergence for the velocity gradients $\nabla\mathbf{v}^\varepsilon$ and pressures p^ε. We use the inequality (1.18) from Lemma 1.4 for functions $(\mathbf{v}, p)$ and $(\mathbf{v}^\varepsilon, p^\varepsilon)$. We also simplify the formulas (1.18)–(1.19) by using the fact that $0 \le \gamma(p) \le 1$. We have

$$(2.26)\quad \frac{C_1}{2}\int_\Omega \mathcal{I}_{\mathbf{D}}(\mathbf{v}^\varepsilon, \mathbf{v})\,dx \le \frac{\gamma_0^2}{2C_1}\|p^\varepsilon - p\|_2^2 + \int_\Omega (\nu(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)\mathbf{D}(\mathbf{v}^\varepsilon) - \nu(p, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}))\cdot \mathbf{D}(\mathbf{v}^\varepsilon - \mathbf{v})\,dx.$$

Using the weak formulation (2.13) of $(\mathcal{P}^{\varepsilon,\eta})$ with $\boldsymbol{\varphi} = \mathbf{v}^\varepsilon - \mathbf{v}$ to replace the term

$$(\nu(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)\mathbf{D}(\mathbf{v}^\varepsilon), \mathbf{D}(\mathbf{v}^\varepsilon - \mathbf{v}))$$

in (2.26), we conclude that

$$(2.27)\qquad \frac{C_1}{2}\int_0^T\int_\Omega \mathcal{I}_{\mathbf{D}}(\mathbf{v}^\varepsilon, \mathbf{v})\,dx\,d\tau \le \frac{\gamma_0^2}{2C_1}\|p^\varepsilon - p\|_2^2\,d\tau + f(\varepsilon)$$

where $f(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Next, we consider g^ε solving

$$\begin{aligned} \Delta g^\varepsilon(t) &= p^\varepsilon(t) - p(t) && \text{in } \Omega,\\ \frac{\partial g^\varepsilon(t)}{\partial \mathbf{n}} &= 0 && \text{on } \partial\Omega,\\ \int_\Omega g^\varepsilon(t)\,dx &= 0 && \text{for a. a. } t \in (0,T). \end{aligned}$$

Note that (2.23) implies that

$$(2.28)\qquad g^\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^2(0,T; W^{2,2}(\Omega)).$$

Inserting $\boldsymbol{\varphi} = \nabla g^\varepsilon$ into (2.13) leads to

$$\begin{aligned}(2.29)\qquad \int_0^T \|p^\varepsilon - p\|_2^2\,dt &= \int_0^T (p^\varepsilon, p^\varepsilon - p) - (p, p^\varepsilon - p)\,dt\\ &= \sum_{i=1}^{5} J_i - \underbrace{\int_0^T (p, p^\varepsilon - p)\,dt}_{\to 0},\end{aligned}$$

where

$$(2.30)\qquad J_1 := \int_0^T \langle \mathbf{v}^\varepsilon_{,t}, \nabla g^\varepsilon\rangle\,dt,$$

$$(2.31)\qquad J_2 := -\int_0^T (\mathbf{v}^\varepsilon_\eta \otimes \mathbf{v}^\varepsilon, \nabla^2 g^\varepsilon)\,dt,$$

$$(2.32)\qquad J_3 := \alpha \int_0^T \int_{\partial\Omega} \mathbf{v}^\varepsilon \cdot \nabla g^\varepsilon\,dS\,dt,$$

$$(2.33)\qquad J_4 := -\int_0^T \langle \mathbf{f}, \nabla g^\varepsilon\rangle\,dt,$$

$$(2.34)\qquad J_5 := \int_0^T (\nu(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)\mathbf{D}(\mathbf{v}^\varepsilon), \nabla^2 g^\varepsilon)\,dt.$$

To estimate J_1 we split $g^\varepsilon = g_1^\varepsilon - g$ where

$$\Delta g_1^\varepsilon(t) = p^\varepsilon(t) \text{ in } \Omega, \qquad \frac{\partial g_1^\varepsilon(t)}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega, \qquad \int_\Omega g_1^\varepsilon\,dx = 0,$$

$$\Delta g_1(t) = p(t) \text{ in } \Omega, \qquad \frac{\partial g_1(t)}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega, \qquad \int_\Omega g_1\,dx = 0.$$

Then

$$\begin{aligned}(2.35)\quad \limsup_{\varepsilon\to 0} J_1 &= \limsup_{\varepsilon\to 0}\left(\int_0^T \langle \mathbf{v}^\varepsilon_{,t}, \nabla g_1^\varepsilon - \nabla g_1\rangle\,dt\right)\\ &= \underbrace{\limsup_{\varepsilon\to 0}\left(\int_0^T \langle \mathbf{v}^\varepsilon_{,t}, \nabla g_1^\varepsilon\rangle\,dt\right)}_{\le 0} - \underbrace{\lim_{\varepsilon\to 0}\left(\int_0^T \langle \mathbf{v}^\varepsilon_{,t}, \nabla g_1\rangle\,dt\right)}_{\xrightarrow{\operatorname{div}\mathbf{v}=0} 0} \le 0.\end{aligned}$$

Regarding J_2, J_3 and J_4 we easily conclude that

$$(2.36)\qquad \lim_{\varepsilon\to 0} J_2 = 0 \quad \text{by (2.22), (2.28)},$$

$$(2.37)\qquad \lim_{\varepsilon\to 0} J_3 = 0 \quad \text{by (2.21), (2.28)},$$

$$(2.38)\qquad \lim_{\varepsilon\to 0} J_4 = 0 \quad \text{by (2.28)}.$$

Finally, applying (1.19) from Lemma 1.4 to the term J_5 we obtain that

$$
\begin{aligned}
(2.39)\qquad J_5 &= \int_0^T (\nu(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)\mathbf{D}(\mathbf{v}^\varepsilon) - \nu(p, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}), \nabla^2 g^\varepsilon)\, dt \\
&\quad + \underbrace{\int_0^T (\nu(p, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}), \nabla^2 g^\varepsilon)\, dt}_{\to 0} \\
&\overset{(1.19)+\text{Hölder}}{\leq} \int_0^T \left(C_2 C_{\mathrm{reg}}(\Omega, 2) \left(\int_\Omega \mathcal{I}_{\mathbf{D}}(\mathbf{v}^\varepsilon, \mathbf{v})\, dx \right)^{1/2} \|p^\varepsilon - p\|_2 \right. \\
&\quad \left. + \gamma_0 C_{\mathrm{reg}}(\Omega, 2) \|p^\varepsilon - p\|_2^2 \right) dt + f(\varepsilon) \\
&\overset{\text{Young}}{\leq} \left(\gamma_0 C_{\mathrm{reg}}(\Omega, 2) + \frac{1 - \gamma_0 C_{\mathrm{reg}}(\Omega, 2)}{2} \right) \int_0^T \|p^\varepsilon - p\|_2^2\, dt \\
&\quad + \frac{C_2^2 C_{\mathrm{reg}}^2(\Omega, 2)}{2(1 - \gamma_0 C_{\mathrm{reg}}(\Omega, 2))} \int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}(\mathbf{v}^\varepsilon, \mathbf{v})\, dx\, dt + f(\varepsilon).
\end{aligned}
$$

Combining (2.29)–(2.39) then leads to

$$
\begin{aligned}
(2.40)\qquad & \int_0^T \|p^\varepsilon - p\|_2^2\, dt \\
& \leq C f(\varepsilon) + \frac{C_2^2 C_{\mathrm{reg}}^2(\Omega, 2)}{(1 - \gamma_0 C_{\mathrm{reg}}(\Omega, 2))^2} \int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}(\mathbf{v}^\varepsilon, \mathbf{v})\, dx\, dt.
\end{aligned}
$$

Substituting (2.40) into (2.27), and using Young's inequality we obtain

$$
\begin{aligned}
& \frac{C_1}{2} \int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}(\mathbf{v}^\varepsilon, \mathbf{v})\, dx\, dt \\
& \leq C f(\varepsilon) + \frac{\gamma_0^2}{2C_1} \frac{C_2^2 C_{\mathrm{reg}}^2(\Omega, 2)}{(1 - \gamma_0 C_{\mathrm{reg}}(\Omega, 2))^2} \int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}(\mathbf{v}^\varepsilon, \mathbf{v})\, dx\, dt
\end{aligned}
$$

and by using the assumption concerning γ_0, namely

$$
\gamma_0 \leq \frac{C_1}{2C_{\mathrm{reg}}(\Omega, 2)(C_1 + C_2)},
$$

we know that

$$
\frac{C_1}{2} > \frac{\gamma_0^2}{2C_1} \frac{C_2^2 C_{\mathrm{reg}}^2(\Omega, 2)}{(1 - \gamma_0 C_{\mathrm{reg}}(\Omega, 2))^2}.
$$

Therefore,

$$\int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}(\mathbf{v}^\varepsilon, \mathbf{v})\, dx\, dt \to 0 \tag{2.41}$$

and consequently using (2.40)–(2.41) we come to the conclusion that

$$\int_0^T \|p^\varepsilon - p\|_2^2\, dt \to 0 \quad \text{as } \varepsilon \to 0. \tag{2.42}$$

Finally, we will show that (2.41) implies a.e. convergence of $\mathbf{D}(\mathbf{v}^\varepsilon)$. As

$$\int_0^T \|\nabla \mathbf{v}^\varepsilon\|_r^r \le C,$$

we have

$$\begin{aligned}
(2.43)\quad & \int_0^T \int_\Omega |\mathbf{D}(\mathbf{v}^\varepsilon - \mathbf{v})|\, dx\, d\tau \\
&\le \int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}^{1/2}(\mathbf{v}^\varepsilon, \mathbf{v}) \Big(\int_0^1 (1 + |\mathbf{D}(\mathbf{v}^\varepsilon) - s(\mathbf{D}(\mathbf{v}^\varepsilon - \mathbf{v}))|^2)^{(r-2)/2} \Big)^{-1/2} dx\, dt \\
&\le C \int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}^{1/2}(\mathbf{v}^\varepsilon, \mathbf{v})(1 + |\mathbf{D}(\mathbf{v}^\varepsilon)| + |\mathbf{D}(\mathbf{v})|)^{(2-r)/2}\, dx\, dt \\
&\overset{\text{Hölder}}{\le} C \Big(\int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}(\mathbf{v}^\varepsilon, \mathbf{v})\, dx\, dt \Big)^{1/2} \Big(\int_0^T (1 + \|\nabla \mathbf{v}^\varepsilon\|_{2-r}^{2-r} + \|\nabla \mathbf{v}\|_{2-r}^{2-r}\, dt) \Big)^{1/2} \\
&\le C \Big(\int_0^T \int_\Omega \mathcal{I}_{\mathbf{D}}(\mathbf{v}^\varepsilon, \mathbf{v})\, dx\, dt \Big)^{1/2} \xrightarrow{(2.41)} 0
\end{aligned}$$

and we conclude at least for a subsequence that

$$\mathbf{D}(\mathbf{v}^\varepsilon) \to \mathbf{D}(\mathbf{v}) \quad \text{a.e. in } \Omega \times (0, T). \tag{2.44}$$

Hence, we can use Vitali's convergence theorem, (2.44) and (2.42) to conclude that

$$\overline{\nu \mathbf{D}} = \nu(p, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}), \quad \text{a.e. in } \Omega \times (0, T).$$

This completes the solvability of the problem $(\mathcal{P}^\eta)$.

2.3. Limit $\eta \to 0$ Considering the (unique) solutions $(\mathbf{v}^\eta, p^\eta)$ of the problem $(\mathcal{P}^\eta)$ that have been constructed, we summarize first the estimates that are uniform w.r.t. $\eta > 0$. Then by letting $\eta \to 0_+$ we aim to show that the limit functions solve the problem $(\mathcal{P})$.

Note that the estimates (2.14) are uniform w.r.t. $\eta > 0$ and consequently hold also for $\mathbf{v}^\eta$. This is not however true for the estimate (2.16) and we have to modify the argument slightly (skipping however the details).

Consider for $\sigma = r(d+2)/(2d)$ the problem

$$\Delta g^\eta(t) = |p^\eta(t)|^{\sigma-2}p^\eta(t) - \frac{1}{|\Omega|}\int_\Omega |p^\eta|^{\sigma-2}(t)p^\eta(t)\,dx \quad \text{in } \Omega,$$

$$\frac{\partial g^\eta(t)}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega, \qquad \int_\Omega g^\eta(t)\,dx = 0.$$

From the L^p-theory of the Neumann problem for the Laplace equation we have

$$\int_0^T \|\varphi^\eta\|_{2,\sigma'}^{\sigma'}\,dt \le C\int_0^T \|p^\eta\|_s^s.$$

Inserting $\boldsymbol{\varphi} = \nabla g^\eta$ into the weak formulation of the problem $(\mathcal{P}^\eta)$ and proceeding step by step as in the preceding subsection, we conclude that

$$\int_0^T \|p^\eta\|_\sigma^\sigma\,dt \le C. \tag{2.45}$$

Consequently (as in the previous subsection)

$$\|\mathbf{v}_{,t}^\eta\|_{(X^{r,s'}\cap Y^{r,\sigma'})^*} \le C, \tag{2.46}$$

where $\sigma' = \sigma/(\sigma-1)$, $s' = s/(s-1)$ and $s = r(d+2)/(2(d+1))$.

The uniform estimates (2.14), (2.45) and (2.46) again imply that (after using a suitable subsequence)

$$\mathbf{v}^\eta \rightharpoonup^* \mathbf{v} \qquad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)^d), \tag{2.47}$$

$$\mathbf{v}^\eta \rightharpoonup \mathbf{v} \qquad \text{weakly in } L^r(0,T;W_{\mathbf{n}}^{1,r}), \tag{2.48}$$

$$\operatorname{tr}\mathbf{v}^\eta \to \operatorname{tr}\mathbf{v} \qquad \text{strongly in } L^2(0,T;L^2(\partial\Omega)), \tag{2.49}$$

$$\mathbf{v}^\eta \to \mathbf{v} \qquad \text{strongly in } L^z(0,T;L^z(\Omega)^d) \text{ for } z \in [1, r(d+2)/d), \tag{2.50}$$

$$p^\eta \rightharpoonup p \qquad \text{weakly in } L^s(0,T;L^s(\Omega)), \tag{2.51}$$

$$\mathbf{v}_{,t}^\eta \rightharpoonup \mathbf{v}_{,t} \qquad \text{weakly in } (X^{r,s'}\cap Y^{r,\sigma'})^*, \tag{2.52}$$

and

$$\nu(p^\eta, |\mathbf{D}(\mathbf{v}^\eta)|^2)\mathbf{D}(\mathbf{v}^\eta) \rightharpoonup \overline{\nu\mathbf{D}} \quad \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)^{d\times d}). \tag{2.53}$$

To end the proof it remains to show the a.e. convergence of $\mathbf{D}(\mathbf{v}^\eta)$ and p^η. Then using all of weak convergences shown above and Vitali's theorem completes the proof of our theorem.

We will follow the approach described in [6]. Taking a sequence $\eta_j \to 0$ as $j \to \infty$, we set $\{\mathbf{v}^j\}_{j=1}^\infty := \{\mathbf{v}^{\eta_j}\}_{j=1}^\infty$.

Let $k \in \mathbb{N}$ be an arbitrary and fixed. We set for $i \in \mathbb{N}$

$$\begin{aligned}(2.54)\quad g^{ki} &:= |\nabla \mathbf{v}^k|^r + |\nabla \mathbf{v}^i|^r \\ &+ (|\nu(p^k, |\mathbf{D}(\mathbf{v}^k)|^2)\mathbf{D}(\mathbf{v}^k)| + |\nu(p^i, |\mathbf{D}(\mathbf{v}^i)|^2)\mathbf{D}(\mathbf{v}^i)|)(|\mathbf{D}(\mathbf{v}^k)| + |\mathbf{D}(\mathbf{v}^i)|).\end{aligned}$$

It follows from apriori estimates that

$$0 \le \int_0^T \int_\Omega g^{ki}\, dx\, dt \le K$$

with some constant K, $1 \le K < \infty$. We shall prove the following property:

$$(2.55)\qquad \left|\ \begin{array}{l} \text{for every } \varepsilon^* > 0 \text{ there is } L \le \varepsilon^*/K \text{ and there is } \{\mathbf{v}^\ell\}_{\ell=1}^\infty \subset \\ \{\mathbf{v}^j\}_{j=1}^\infty \text{ and sets } E^{k\ell} := \{(x,t) \in \Omega \times (0,T) \mid L^2 \le \\ |\mathbf{v}^\ell(x,t) - \mathbf{v}^k(x,t)| < L\} \text{ such that} \\ \qquad\qquad \displaystyle\int E^{k\ell} g^{k\ell}\, dx\, dt \le \varepsilon^*. \end{array}\right.$$

To see this, we fix $\varepsilon^* \in (0,1)$, set $L_1 := \varepsilon^*/K$ and take $N \in \mathbb{N}$ such that $N > K/\varepsilon^*$. We define iteratively $L_i := L_{i-1}^2$ for $i = 2, 3, \dots, N$ and we set

$$E_i^{kj} := \{(x,t) \in \Omega \times (0,T) \mid L_i^2 \le |\mathbf{v}^j(x,t) - \mathbf{v}^k(x,t)| < L_i\} \quad (i = 1, \dots, N).$$

For j fixed, E_i^{kj} are mutually disjoint. Consequently

$$\sum_{i=1}^N \int_{E_i^{kj}} g^{kj}\, dx\, dt \le K.$$

As $N\varepsilon^* > K$, for each $j \in \mathbb{N}$ there is $i_0(j) \in \{1, \dots, N\}$ such that

$$\int_{E_{i_0(j)}^{kj}} g^{kj}\, dx\, dt \le \varepsilon^*.$$

As $i_0(j)$ are taken from a finite set of indices, we can take a subsequence $\{\mathbf{v}^\ell\}_{\ell=1}^\infty \subset \{\mathbf{v}^j\}_{j=1}^\infty$ such that $i_0(\ell) = i_0^*$ for each ℓ. The property (2.55) is then proved by setting $L := L_{i_0^*}$ and $E^{k\ell} := E_{i_0^*}^{k\ell}$. □

Let $\varepsilon^* > 0$ be arbitrary but fixed. We find the sequence $\{\mathbf{v}^\ell\}_{\ell=1}^\infty$ and L satisfying (2.55). Then we define $\mathbf{u}^{k\ell}$ as

$$(2.56)\qquad \mathbf{u}^{k\ell} := (\mathbf{v}^k - \mathbf{v}^\ell)\left(1 - \min\left(\frac{|\mathbf{v}^k - \mathbf{v}^\ell|}{L}, 1\right)\right),$$

and the sets $Q^{k\ell}$, Q as

$$Q^{k\ell} := \{(x,t) : |\mathbf{v}^k - \mathbf{v}^\ell| < L\}, \quad Q := (0,T) \times \Omega.$$

By using (2.47)–(2.50) and the fact that $\|\mathbf{u}^{k\ell}(t)\|_\infty \le C$ we have

$$\mathbf{u}^{k\ell} \overset{\ell\to\infty}{\rightharpoonup} \mathbf{u}^k \qquad \text{weakly in } L^r(0,T;W^{1,r}_{\mathbf{n}}), \tag{2.57}$$

$$\mathbf{u}^{k\ell} \xrightarrow{\ell\to\infty} \mathbf{u}^k \qquad \text{strongly in } L^s(0,T;L^s(\Omega)^d) \quad \forall\, s < \infty, \tag{2.58}$$

$$\operatorname{tr}\mathbf{u}^{k\ell} \xrightarrow{\ell\to\infty} \operatorname{tr}\mathbf{u}^k \quad \text{strongly in } L^2(0,T;L^2(\partial\Omega)^d), \tag{2.59}$$

where

$$\mathbf{u}^k := (\mathbf{v}^k - \mathbf{v})\left(1 - \min\left(\frac{|\mathbf{v}^k - \mathbf{v}|}{L}, 1\right)\right).$$

Let us compute the divergence of $\mathbf{u}^{k\ell}$:

$$\operatorname{div}\mathbf{u}^{k\ell} = -\frac{1}{L}(\mathbf{v}^k - \mathbf{v}^\ell) \cdot \nabla|\mathbf{v}^\ell - \mathbf{v}^k|\chi_{Q^{k\ell}}$$

where χ_U denotes the characteristic function of the set U. Hence, the L^r-norm of $\operatorname{div}\mathbf{u}^{k\ell}$ can be bounded by means of the property (2.55) as follows:

$$\begin{aligned}\int_0^T \|\operatorname{div}\mathbf{u}^{k\ell}\|_r^r &\le \int_Q \frac{|\mathbf{v}^k - \mathbf{v}^\ell|^r}{L^r}|\nabla(\mathbf{v}^k - \mathbf{v}^\ell)|^r \chi_{Q^{k\ell}}\,dx\,dt \\ &= \int_{Q^{k\ell}\setminus E^{k\ell}} \cdots\,dx\,dt + \int_{E^{k\ell}} \cdots\,dx\,dt \\ &\le C(L^r + \varepsilon^*) \le C\varepsilon^*.\end{aligned} \tag{2.60}$$

The Helmholtz decomposition then gives $\mathbf{u}^{k\ell} = \mathbf{u}^{k\ell}_{\mathrm{div}} + \nabla g^{\mathbf{u}^{k\ell}}$ whereas (2.60) implies that

$$\int_0^T \|g^{\mathbf{u}^{k\ell}}\|_{2,r}^r\,dt \le C\varepsilon^*. \tag{2.61}$$

Moreover, (2.58) and the $W^{1,p}$-theory for the Laplace equation imply that

$$\mathbf{u}^{k\ell}_{\mathrm{div}} \xrightarrow{\ell\to\infty} \mathbf{u}^k_{\mathrm{div}} \quad \text{strongly in } L^s(0,T;L^s(\Omega)^d),\ \forall\, s < \infty. \tag{2.62}$$

Next, for $m \in \mathbb{N}$ we introduce the notation

$$U^m := \nu(p^m, |\mathbf{D}(\mathbf{v}^m)|^2)\mathbf{D}(\mathbf{v}^m) \in L^{r'}(\Omega \times (0,T)).$$

The integration of (1.18) over $Q^{k\ell}$ (with $\mathbf{u} := \mathbf{v}^k$, $\mathbf{v} := \mathbf{v}^\ell$, $p := p^k$, $q := p^\ell$) leads to

$$\begin{aligned}(2.63)\quad \frac{C_1}{2}\int_{Q^{k\ell}} \mathcal{I}_{\mathbf{D}}(\mathbf{v}^k,\mathbf{v}^\ell)\,dx\,dt &\le \frac{\gamma_0^2}{2C_1}\int_{Q^{k\ell}} \mathcal{I}_p(p^k,p^\ell)\,dx\,dt + \int_{Q^{k\ell}} (U^k - U^\ell)\mathbf{D}(\mathbf{v}^k - \mathbf{v}^\ell)\,dx\,dt \\ &=: Y_1 + Y_2.\end{aligned}$$

A simple calculation gives

$$\begin{aligned}(2.64)\qquad Y_2 &= \int_Q (U^k - U^\ell)\mathbf{D}(\mathbf{u}^{k\ell})\,dx\,dt \\ &\quad + \int_{Q^{k\ell}} (U^k - U^\ell)\mathbf{D}\left((\mathbf{v}^k - \mathbf{v}^\ell)\frac{|\mathbf{v}^k - \mathbf{v}^\ell|}{L}\right)dx\,dt \\ &= \int_Q (U^k - U^\ell)\mathbf{D}(\mathbf{u}^{k\ell}_{\mathrm{div}})\,dx\,dt + \int_Q (U^k - U^\ell)\mathbf{D}(\nabla g^{\mathbf{u}^{k\ell}})\,dx\,dt \\ &\quad + \int_{Q^{k\ell}} (U^k - U^\ell)\mathbf{D}\left((\mathbf{v}^k - \mathbf{v}^\ell)\frac{|\mathbf{v}^k - \mathbf{v}^\ell|}{L}\right)dx\,dt \\ &\overset{(2.55),\,(2.61)}{\le} \int_Q (U^k - U^\ell)\mathbf{D}(\mathbf{u}^{k\ell}_{\mathrm{div}})\,dx\,dt + C\varepsilon^* := Y_3 + C\varepsilon^*.\end{aligned}$$

Taking $\boldsymbol{\varphi} = \mathbf{u}^{k\ell}_{\mathrm{div}}$ as the test function in the weak formulation of the problem $(\mathcal{P}^\eta)$, we observe that Y_3 can be expressed in the form $Y_3 = \sum_{i=1}^3 I_i$, where

$$\begin{aligned} I_1 &= -\int_0^T \langle \mathbf{v}^k_{,t} - \mathbf{v}^\ell_{,t}, \mathbf{u}^{k\ell}_{\mathrm{div}}\rangle \overset{\mathrm{div}\,\mathbf{v}^k - \mathbf{v}^\ell = 0}{=} -\int_0^T \langle \mathbf{v}^k_{,t} - \mathbf{v}^\ell_{,t}, \mathbf{u}^{k\ell}\rangle \le {}^1 0, \\ I_2 &= -\int_Q (\mathbf{v}^k_{\eta(k)}\nabla\mathbf{v}^k - \mathbf{v}^\ell_{\eta(\ell)}\nabla\mathbf{v}^\ell)(\mathbf{u}^{k\ell}) \le C\|\mathbf{u}^{k\ell}\|_{L^s(0,T;L^s)} \\ &\qquad\qquad (s \text{ sufficiently large}), \\ I_3 &= -\int_{(0,T)\times\partial\Omega} (\mathbf{v}^k - \mathbf{v}^\ell)\mathbf{u}^{k\ell}\,dS\,dt \le C\|\mathbf{u}^{kl}\|_{L^2(0,T;L^2(\partial\Omega))}\end{aligned}$$

[1]Here we have to note that the second equality in the estimate for I_1 is only formal (the duality does not make good sense). To prove the estimate for I_1 rigorously we set $\mathbf{w} := \mathbf{v}^k - \mathbf{v}^\ell$. From the density of smooth function we can find a sequence $\mathbf{w}^n \in C^\infty(0,T;C^\infty_{\mathbf{n},\mathrm{div}}(\Omega))$, $\mathbf{w}^n(0) = 0$ such that

$$\int_0^T \langle \mathbf{v}^k_{,t} - \mathbf{v}^\ell_{,t}, \mathbf{u}^{k\ell}_{\mathrm{div}}\rangle\,dt = \lim_{n\to\infty}\int_0^T \left\langle \mathbf{w}^n_{,t}, \left(\mathbf{w}^n\left(1 - \min\left(\frac{|\mathbf{w}^n|}{L}, 1\right)\right)\right)_{\mathrm{div}}\right\rangle dt.$$

Regarding Y_1 introduced in (2.63) we need to estimate $\mathcal{I}_p(p^k, p^\ell)$. For this purpose we consider $g^{k\ell}$ as a solution of the Neumann problem

$$\Delta g^{k\ell}(t) = \frac{\mathcal{I}_p(p^k,p^\ell)(t)}{p^k(t)-p^\ell(t)} - \frac{1}{|\Omega|}\int_\Omega \frac{\mathcal{I}_p(p^k,p^\ell)(t)}{p^k(t)-p^\ell(t)}\,dx \quad \text{in } \Omega \text{ for a.a. } t \in (0,T), \tag{2.65}$$

$$\frac{\partial g^{k\ell}(t)}{\partial \mathbf{n}} = 0 \quad \text{on } \Omega, \tag{2.66}$$

$$\int_\Omega g^{k\ell}(t)\,dx = 0 \quad \text{for a.a. } t \in (0,T). \tag{2.67}$$

Since $\gamma^2(p) \le \gamma(p)$ we have

$$\begin{aligned}
&\int_0^T \|\nabla^2 g^{k\ell}\|_2^2 \\
&\quad \le 2C_{\mathrm{reg}}(2,\Omega)\int_Q |p^k - p^\ell|^2\Big(\int_0^1 \gamma^2(p^\ell + s(p^k-p^\ell))\,ds\Big)^2\,dx\,dt \\
&\quad \le 2C_{\mathrm{reg}}(2,\Omega)\int_Q \mathcal{I}_p(p^k,p^\ell)\,dx\,dt \le C.
\end{aligned}$$

Next, we incorporate the stronger assumption following from (1.8), namely

$$\gamma(p) \le (1+\alpha|p|)^{-q},$$

and use the inequality (see the proof of Lemma 1.19, p. 201 in [10] for details)

$$\int_0^1 \Big(1+\alpha\,|p^\ell + s(p^k-p^\ell)|\Big)^{-q}\,ds \le C|p^k - p^\ell|^{-q}$$

By using $\operatorname{div}\mathbf{w}^n = 0$ we have

$$\begin{aligned}
&= \lim_{n\to\infty}\int_0^T \Big\langle \mathbf{w}^n_{,t}, \mathbf{w}^n\Big(1-\min\Big(\frac{|\mathbf{w}^n|}{L},1\Big)\Big)\Big\rangle\,dt \\
&= \lim_{n\to\infty}\int_0^T\int_\Omega \frac{1}{2}|\mathbf{w}^n|^2_{,t}\Big(1-\min\Big(\frac{|\mathbf{w}^n|}{L},1\Big)\Big)\,dx\,dt \\
&= \lim_{n\to\infty}\int_0^T\int_\Omega F^n(x,t)_{,t}\,dx\,dt \\
&= \lim_{n\to\infty}\int_\Omega F^n(x,T) - F^n(x,0)\,dx \overset{F^n(\cdot,0)=0}{=} \lim_{n\to\infty}\int_\Omega F^n(x,T)\,dx \ge 0,
\end{aligned}$$

where F^n is defined as

$$F^n(x,t) := \begin{cases} |\mathbf{w}^n|^2\Big(1-\dfrac{2}{3}\dfrac{|\mathbf{w}^n|}{L}\Big) & \text{if } |\mathbf{w}^n| < L, \\ \dfrac{1}{3}L^2 & \text{if } |\mathbf{w}^n| \ge L. \end{cases}$$

to conclude from (2.65) and (2.51) that for all $\rho \le (d+2)r/(2d(1-2q))$

$$\int_0^T \|\nabla^2 g^{k\ell}\|_\rho^\rho \, dt \le C \int_Q |p^k - p^\ell|^{\rho(1-2q)} \, dx \, dt \le C. \tag{2.68}$$

Note that from the assumption on q we can also find some $\rho > 2$ satisfying (2.68). Also, it follows from (2.68) and the trace theorem that

$$\int_0^T \int_{\partial\Omega} |\operatorname{tr} \nabla g^{k\ell}|^2 \, dS \, dt \le C. \tag{2.69}$$

At this juncture, we subtract the weak formulation for $(\mathcal{P}^{\eta(k)})$ from that for $(\mathcal{P}^{\eta(\ell)})$ and take $\nabla g^{k\ell}$ as a test function in the resulting equation, and obtain

$$\int_Q \mathcal{I}_p(p^k, p^\ell) \, dx \, dt = \sum_{a=1}^{4} I_a, \tag{2.70}$$

where

$$I_1 = -\int_Q (\mathbf{v}^k_{\eta(k)} \otimes \mathbf{v}^k - \mathbf{v}^\ell_{\eta(\ell)} \otimes \mathbf{v}^\ell, \nabla^2 g^{k\ell}) \, dx \, dt \le C\|\mathbf{v}^k - \mathbf{v}^\ell\|_{L^s(0,T;L^s)}$$

$$\text{for some } s < r\frac{d+2}{d} \text{ as } q > 1 - \frac{r}{4d}(d+2) \text{ by (2.68)},$$

$$I_2 = \int_0^T \langle \mathbf{v}^k_{,t} - \mathbf{v}^\ell_{,t}, \nabla g^{k\ell} \rangle \, dt \overset{\operatorname{div}\mathbf{v}^k - \mathbf{v}^\ell = 0}{=} 0,$$

$$I_3 = \alpha \int_{(0,T)\times\partial\Omega} (\mathbf{v}^k - \mathbf{v}^\ell)(\nabla g^{k\ell}) \, dS \, dt \overset{(2.69)}{\le} C\|\mathbf{v}^k - \mathbf{v}^\ell\|_{L^2(0,T;L^2(\partial\Omega))},$$

$$I_4 = \int_Q (U^k - U^\ell)\nabla^2 g^{k\ell} \overset{(1.19)}{\le} 2\gamma_0 C_{\mathrm{reg}}(2,\Omega) \int_Q \mathcal{I}_p(p^k, p^\ell) \, dx \, dt$$

$$+ C_2 \int_{Q\setminus Q^{k\ell}} J \, dx \, dt + C_2 \int_{Q^{k\ell}} J \, dx \, dt,$$

where the symbol J stands for

$$J := \left| \mathbf{D}(\mathbf{v}^k - \mathbf{v}^\ell) \int_0^1 (1 + |\mathbf{D}(\mathbf{v}^k + s(\mathbf{v}^\ell - \mathbf{v}^k))|^2)^{(r-2)/2} \, ds \right| |\nabla^2 g^{k\ell}|.$$

Since, for some $\alpha > 0$, we have

$$\int_{Q\setminus Q^{k\ell}} J\,dx\,dt \le \int_{Q\setminus Q^{k\ell}} C(1 + |\mathbf{D}(\mathbf{u}^k)| + |\mathbf{D}(\mathbf{u}^\ell)|)^{r-1}|\nabla^2 g^{k\ell}|\,dx\,dt$$
$$\le C(1 + \|\nabla \mathbf{v}^k\|_{L^r(Q)} + \|\nabla \mathbf{v}^\ell\|_{L^r(Q)})^{r-1}\|\nabla^2 g^{k\ell}\|_{L^s(Q)}\,|(Q\setminus Q^{k\ell})|^{\alpha(r,s)}$$
$$\overset{(2.68)\text{ with } s>2}{\le} C|(Q\setminus Q^{k\ell})|^\alpha,$$

$$\text{(2.71)} \quad \int_{Q^{k\ell}} J\,dx\,dt \le 2C_{\mathrm{reg}}(2,\Omega)\Big(\int_{Q^{k\ell}} \mathcal{I}_{\mathbf{D}}(\mathbf{v}^k,\mathbf{v}^\ell)\Big)^{1/2}\Big(\int_{Q^{k\ell}} \mathcal{I}_p(p^k,p^\ell)\Big)^{1/2}$$
$$\le \frac{2C_{\mathrm{reg}}^2(\Omega,2)C_2}{1-2\gamma_0 C_{\mathrm{reg}}(\Omega,2)}\int_{Q^{k\ell}} \mathcal{I}_{\mathbf{D}}(\mathbf{v}^k,\mathbf{v}^\ell)\,dx\,dt$$
$$+ \frac{1-2\gamma_0 C_{\mathrm{reg}}(\Omega,2)}{2C_2}\int_{Q^{k\ell}} \mathcal{I}_p(p^k,p^\ell)\,dx\,dt,$$

we conclude from (2.70) and (2.71) that

$$\text{(2.72)} \quad \int_Q \mathcal{I}_p(p^k,p^\ell)\,dx\,dt \le \frac{4C_{\mathrm{reg}}^2(\Omega,2)C_2^2}{(1-2\gamma_0 C_{\mathrm{reg}}(\Omega,2))^2}\int_{Q^{k\ell}} \mathcal{I}_{\mathbf{D}}(\mathbf{v}^k,\mathbf{v}^\ell)\,dx\,dt$$
$$+ C(\|\mathbf{v}^k-\mathbf{v}^\ell\|_{L^s(Q)} + \|\operatorname{tr}(\mathbf{v}^k-\mathbf{v}^\ell)\|_{L^2((0,T)\times\partial\Omega)} + |Q\setminus Q^{k\ell}|^\alpha)$$

with some $1 \le s < r(d+2)/d$ and some suitable positive α.

Finally, proceeding as in the previous subsection ($\varepsilon \to 0$), combining the estimates (2.63) with (2.64), (2.72) and using the assumption regarding γ_0, we obtain the following inequality (with suitable $s < r(d+2)/d$ and some $\alpha > 0$)

$$\text{(2.73)} \quad \int_{Q^{k\ell}} \mathcal{I}_{\mathbf{D}}(\mathbf{v}^k,\mathbf{v}^\ell)\,dx\,dt + \int_Q \mathcal{I}_p(p^k,p^\ell)\,dx\,dt$$
$$\le C(\varepsilon^* + \|\mathbf{v}^k-\mathbf{v}^\ell\|_{L^s(Q)} + \|\operatorname{tr}(\mathbf{v}^k-\mathbf{v}^\ell)\|_{L^2((0,T)\times\partial\Omega)} + |Q\setminus Q^{k\ell}|)^\alpha.$$

In order to establish the pointwise convergence of the velocity gradient we observe that for small $\beta > 0$

$$\text{(2.74)} \quad \int_Q |\mathbf{D}(\mathbf{v}^k-\mathbf{v})|^{1+\beta}\,dx\,dt \le \liminf_{\ell\to\infty}\int_Q |\mathbf{D}(\mathbf{v}^k-\mathbf{v}^\ell)|^{1+\beta}\,dx\,dt$$
$$= \liminf_{\ell\to\infty}\Big(\int_{Q^{k\ell}} \cdots\,dx\,dt + \int_{Q\setminus Q^{k\ell}} \cdots\,dx\,dt\Big) \le$$

$$\overset{(2.73)}{\leq} C \limsup_{\ell \to \infty} \left(\varepsilon^* + |Q \setminus Q^{k\ell}| + \int_Q |\mathbf{v}^k - \mathbf{v}^\ell|^s \, dx \, dt + \int_0^T \int_{\partial\Omega} |\mathbf{v}^k - \mathbf{v}^\ell|^2 \, dS \, dt \right)^\alpha$$

$$= C\left(\varepsilon^* + \|\mathbf{v}^k - \mathbf{v}\|^s_{L^s(0,T;L^s)} + \|\mathbf{v}^k - \mathbf{v}\|^2_{L^2(0,T;L^2(\partial\Omega)^d)} + |Q \setminus Q^k|\right)^\alpha$$

for some $\alpha > 0$ and some suitable s. The symbol Q^k denotes the set

$$Q^k := \{(x,t) : |\mathbf{v}^k(x,t) - \mathbf{v}(x,t)| < L\}.$$

Taking the limit w.r.t. k in (2.74) leads to

$$0 \leq \limsup_{k \to \infty} \int_0^T \int_\Omega |\mathbf{D}(\mathbf{v}^k - \mathbf{v})|^{1+\beta} \, dx \, dt \leq C\varepsilon^*. \tag{2.75}$$

As ε^* is arbitrarily small, (2.75) implies that $\mathbf{D}(\mathbf{v}^k) \to \mathbf{D}(\mathbf{v})$ almost everywhere in $\Omega \times (0,T)$. The same conclusion is true also for pointwise convergence of the pressure p^k. Hence, using Vitali's lemma then completes the proof of the existence of a weak solution to the problem $(\mathcal{P})$.

The uniqueness of weak solutions in two dimension for $r = 2$ is proved in [3].

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References

[1] H. Beirão da Veiga, *On the regularity of flows with Ladyzhenskaya shear-dependent viscosity and slip or nonslip boundary conditions*, Comm. Pure Appl. Math. **58** (2005), 552–577, http://dx.doi.org/10.1002/cpa.20036. MR 2119869 (2005k:35329)

[2] L. Boccardo and F. Murat, *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*, Nonlinear Analysis **19** (1992), 581–597, http://dx.doi.org/10.1016/0362-546X(92)90023-8. MR 1183665 (93h:35061)

[3] M. Bulíček, J. Málek, and D. Pražák, *On the dimension of the attractor for a class of fluids with pressure dependent viscosities*, Commun. Pure Appl. Anal. **4** (2005), 805–822. MR 2172723 (2006e:37141)

[4] T. Clopeau, A. Mikelić, and R. Robert, *On the vanishing viscosity limit for the* 2D *incompressible Navier-Stokes equations with the friction type boundary conditions*, Nonlinearity **11** (1998), 1625–1636, http://dx.doi.org/10.1088/0951-7715/11/6/011. MR 1660366 (99g:35102)

[5] M. FRANTA, J. MÁLEK, and K. R. RAJAGOPAL, *On steady flows of fluids with pressure- and shear-dependent viscosities*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **461** (2005), 651–670. MR 2121929 (2005k:76007)

[6] J. FREHSE, J. MÁLEK, and M. STEINHAUER, *On existence results for fluids with shear dependent viscosity—unsteady flows*, Proc. Partial Differential Equations, Praha, 1998, Chapman & Hall/CRC Res. Notes Math., vol. 406, Chapman & Hall/CRC, Boca Raton, FL, 2000, pp. 121–129. MR 1713880 (2000e:76013)

[7] F. GAZZOLA, *A note on the evolution of Navier-Stokes equations with a pressure-dependent viscosity*, Z. Angew. Math. Phys. **48** (1997), 760–773. MR 1478411 (98m:35156)

[8] J. HRON, J. MÁLEK, and K. R. RAJAGOPAL, *Simple flows of fluids with pressure dependent viscosities*, Proceedings of The Royal Society London A: Mathematical, Physical and Engineering Sciences **477** (2001), 277–302.

[9] J. MÁLEK, J. NEČAS, and K. R. RAJAGOPAL, *Global analysis of the flows of fluids with pressure-dependent viscosities*, Arch. Ration. Mech. Anal. **165** (2002), 243–269. MR 1941479 (2004h:76003)

[10] J. MÁLEK, J. NEČAS, M. ROKYTA, and M. RŮŽIČKA, *Weak and Measure-valued Solutions to Evolutionary PDEs*, Chapman & Hall, London, 1996, ISBN 0-412-57750-X.

[11] J. MÁLEK and K. R. RAJAGOPAL, *Mathematical issues concerning the Navier-Stokes equations and some of its generalizations*, Evolutionary Equations. Vol. II, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2005, pp. 371–459.

[12] ———, *Mathematical Properties of the Equations Governing the Flow of Fluids with Pressure and Shear Rate Dependent Viscosities (chapter 5)*, Handbook of Mathematical Fluid Dynamics, vol. 4, Elsevier B.V., 2007.

[13] J. NEČAS, *Les méthodes directes en théorie des équations elliptiques*, Masson et Cie, Éditeurs, Paris, 1967. MR 0227584 (37 #3168) (French)

[14] C. PARÉS, *Existence, uniqueness and regularity of solution of the equations of a turbulence model for incompressible fluids*, Appl. Anal. **43** (1992), 245–296. MR 1284321 (95e:35160)

[15] M. RENARDY, *Some remarks on the Navier-Stokes equations with a pressure-dependent viscosity*, Comm. Partial Differential Equations **11** (1986), 779–793. MR 837931 (87h:35272)

[16] V. A. SOLONNIKOV, *Estimates for solutions of a non-stationary linearized system of Navier-Stokes equations*, Trudy Mat. Inst. Steklov. **70** (1964), 213–317. MR 0171094 (30 #1325) (Russian)

[17] G. G. STOKES, *On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids*, Trans. Cambridge Phil. Soc. **8** (1845), 287–305.

[18] J. WOLF, *Existence of weak solutions to the equations of nonstationary motion of non-newtonian fluids with shear-dependent viscosity*, J. Math. Fluid Mech. (2007) (to appear).

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