Discretization of the Navier–Stokes Equations with Slip Boundary Condition

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We propose and analyze a two-level method of discretizing the nonlinear Navier–Stokes equations with slip boundary condition. The slip boundary condition is appropriate for problems that involve free boundaries, flows past chemically reacting walls, and other examples where the usual no-slip condition $\mathbf{u} = 0$ is not valid. The two-level algorithm consists of solving a small nonlinear system of equations on the coarse mesh and then using that solution to solve a larger linear system on the fine mesh. The two-level method exploits the quadratic nonlinearity in the Navier–Stokes equations. Our error estimates show that it has optimal order accuracy, provided that the best approximation to the true solution in the velocity and pressure spaces is bounded above by the data. © 2001 John Wiley & Sons, Inc. Numer Methods Partial Differential Eq 17: 26–42, 2001

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I. INTRODUCTION

Consider the incompressible Navier-Stokes equations with slip boundary condition:

$$-\mathrm{Re}^{-1}\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega, \tag{1.1a}$$

$$\mathrm{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega, \tag{1.1b}$$

$$\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\Omega = \Gamma, \tag{1.1c}$$

$$\mathbf{n}\cdot\Im(\mathrm{Re}^{-1}\mathbf{u}, p)\cdot\boldsymbol{\tau}_k = 0 \quad \text{on} \quad \Gamma \quad \text{for} \quad 1 \le k \le d-1, \tag{1.1d}$$

where Ω is a simply connected, bounded polygonal domain in $\mathbb{R}^d$ (d=2,3) whose boundary, Γ, is piecewise of class C^3, $\mathbf{n} = (n_1, \ldots, n_d)$ is the exterior unit normal, and $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{d-1}$ form an

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orthonormal set of tangent vectors. Here $\Im(.,.)$ is the stress tensor and is given by:

$$\Im(\mathbf{v},q)_{i,j} = -q\delta_{i,j} + \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}, \quad 1 \leq i,j \leq d. \tag{1.2}$$

The condition $\mathbf{u}\cdot\mathbf{n} = 0$ will be imposed *weakly* on Γ, which will give rise to a Lagrange multiplier. As we see in Section IV, this Lagrange multiplier will be approximated by piecewise constants implying a zero mean flow through the boundary or zero mean roughness of the boundary. Herein we assume that $\mathbf{f}$ is an L^2 function.

There has been much discussion regarding the difference between the no-slip ($\mathbf{u} = 0$ on Γ) and slip boundary conditions. The no-slip condition is well established (see, e.g., Gunzburger [1], Girault and Raviart [2] or Galdi [3]) for moderate pressures and velocities from direct observations and comparisons between numerical simulations and experimental results of a large array of complex flow problems. However, as LeRoux [4] reports, early experiments demonstrated that gases at low temperatures slip past solid surfaces. In particular, for sufficiently large Knudsen numbers velocity slip occurs at the wall surface. Such wall slip has also been observed in the flow of nonlinear fluids such as lubricants, hydraulic fracturing fluids, biological fluids, etc.

The slip condition also applies to free surfaces in free boundary problems (such as the $coating\ problem$ in Verfürth [5, pp. 9, 10], Saito and Scriven [7], Silliman and Scriven [8]), which are modeled as being stress free (condition (1.1d)). In free surface problems, the application of the no-slip condition on the fixed part of the boundary (as well as the presence of edges in the flow domain) causes a stress singularity: an infinite velocity gradient appears at the contact lines where the free and rigid surfaces meet. With slip boundary conditions near the solid-free surface contact point, this nonphysical singularity disappears.

A common dilemma in the application of the slip boundary condition concerns the neighborhood in which it should be enforced. For example, consider the interior of a piston (with one end placed at infinity) when we apply pressure. Clearly, the particles on the boundary that are attached to the moving component as well as the fixed part of the boundary of the piston must satisfy a slip boundary condition. The matter that is not so clear is in which neighborhood we should apply it.

In order to present the weak formulation of problem (1.1), we introduce the function spaces: Let $\Gamma = \bigcup_{i=1}^{s} \Gamma_i$, be composed of s pieces, Γ_i, with corresponding outward normals $\mathbf{n}_i$ and

$$\mathcal{S} := \text{span}\{\, \mathbf{u}(\underline{x}) = \underline{\zeta} \times \underline{x},\ \ \underline{\zeta} \in \mathbb{R}^3,\ |\underline{\zeta}| = 1,\ \underline{\zeta} \text{ is an axis of symmetry of } \Omega\,\}, \tag{1.3a}$$

$$X := (\mathrm{H}^1(\Omega))^d/\mathcal{S}, \tag{1.3b}$$

$$Y := \mathrm{L}^2_0(\Omega) = \{w(x) \in \mathrm{L}^2(\Omega),\ \ \int_\Omega w(x)\,dx = 0\}, \tag{1.3c}$$

$$Z := \prod_{i=1}^{s} \mathrm{H}^{-1/2}(\Gamma_i), \tag{1.3d}$$

where $\mathrm{H}^s(D)$ denotes the usual Sobolev space of functions with s derivatives in $\mathrm{L}^2(D)$. In the case where Ω has no axis of symmetry, $\mathcal{S} = \emptyset$. Now the weak formulation of (1.1) is as follows:

Find $(\mathbf{u}, p, \rho) \in (X, Y, Z)$ satisfying:

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) + b(\mathbf{u};\mathbf{u},\mathbf{v}) + c(\mathbf{v};p,\rho) \ &= \langle \mathbf{f},\mathbf{v}\rangle_\Omega, \quad \forall\, \mathbf{v} \in X, \\ c(\mathbf{u};q,\sigma) &= 0, \quad \forall\, q \in Y,\ \sigma \in Z, \end{aligned}$$

where

$$a(\mathbf{u},\mathbf{v}) := \frac{1}{2\text{Re}}\int_\Omega \mathcal{D}(\mathbf{u})\cdot\mathcal{D}(\mathbf{v})\,dx, \tag{1.4a}$$

$$b(\mathbf{u};\mathbf{v},\mathbf{w}) := \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\,dx, \tag{1.4b}$$

$$c(\mathbf{u};p,\rho) := -\int_\Omega p\,\text{div}\mathbf{u}\,dx - \sum_{i=1}^{s}\langle\rho,\mathbf{u}\cdot\mathbf{n}_i\rangle_{\Gamma_i}. \tag{1.4c}$$

Here

$$\mathcal{D}(\mathbf{v})_{i,j} := \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}, \quad 1 \le i,j \le d, \tag{1.5}$$

denotes the *deformation tensor* associated with the velocity field $\mathbf{v}$.

This article considers the approximate solution of (1.1) using a two-level method. Proceeding accordingly, we use two families of partitions $\mathcal{T}^h$ and $\mathcal{T}^H$ ($h << H$) to subdivide our domain into d-simplices with sides of length h and H, respectively, and we denote by (X^h, Y^h, Z^h) and (X^H, Y^H, Z^H) the corresponding finite element spaces. In the sequel, we will assume that these spaces satisfy an inf-sup condition (see Section IV).

The two-level method proceeds as follows:

Step 1. Solve the nonlinear, coarse mesh problem:

Find $(\mathbf{u}^H, p^H, \rho^H) \in (X^H, Y^H, Z^H)$ satisfying:

$$\begin{aligned} a(\mathbf{u}^H,\mathbf{v}^H) + b^*(\mathbf{u}^H;\mathbf{u}^H,\mathbf{v}^H) + c(\mathbf{v}^H;p^H,\rho^H) &= \langle\mathbf{f},\mathbf{v}^H\rangle_\Omega,\\ c(\mathbf{u}^H;q^H,\sigma^H) &= 0, \end{aligned} \tag{1.6}$$

for all $(\mathbf{v}^H, q^H, \sigma^H) \in (X^H, Y^H, Z^H)$, where

$$b^*(\mathbf{u};\mathbf{v},\mathbf{w}) := \frac{1}{2}[b(\mathbf{u};\mathbf{v},\mathbf{w}) - b(\mathbf{u};\mathbf{w},\mathbf{v})] \tag{1.7}$$

Step 2. Solve the following linear, fine mesh problem:

Find $(\mathbf{u}^h, p^h, \rho^h) \in (X^h, Y^h, Z^h)$ satisfying:

$$\begin{aligned} a(\mathbf{u}^h,\mathbf{v}^h) + b^*(\mathbf{u}^h;\mathbf{u}^H,\mathbf{v}^h) + b^*(\mathbf{u}^H;\mathbf{u}^h,\mathbf{v}^h) +&\\ -b^*(\mathbf{u}^H;\mathbf{u}^H,\mathbf{v}^h) + c(\mathbf{v}^h;p^h,\rho^h) &= \langle\mathbf{f},\mathbf{v}^h\rangle_\Omega,\\ c(\mathbf{u}^h;q^h,\sigma^h) &= 0, \end{aligned} \tag{1.8}$$

for all $(\mathbf{v}^h, q^h, \sigma^h) \in (X^h, Y^h, Z^h)$.

In spite of its physical advantages, the slip boundary condition presents the following mathematical difficulties:

(i) The solution to the linearized problem (i.e., problem (1.1) without the term $(\mathbf{u}\cdot\nabla)\mathbf{u}$) is unique up to rigid body rotations of the domain Ω (assuming it has an axis of symmetry),

(ii) Perturbations of the boundary that are not sufficiently smooth result in instability of the solutions of problem (1.1). This problem appears when we consider polygonal domains (Babuška paradox).

The first problem arises from the fact that the space of rigid rotations is a subspace of the kernel of the Stokes operator. To overcome this problem, we introduce a suitable space (see the definition of the space X above). In addition, we require that $\mathbf{f}$ be orthogonal to all rigid body rotations of Ω.

Verfürth [5, p. 18] shows that, in 2-d, the linearized problem is equivalent to the problem of a simply supported plate with Poisson ratio zero:

$$\begin{aligned} \Delta^2 z &= \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \quad \text{in } \Omega, \\ z = \Delta z - \kappa \frac{\partial z}{\partial \mathbf{n}} &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{1.9}$$

where $\mathbf{u} = \left(\frac{\partial z}{\partial y}, -\frac{\partial z}{\partial x}\right)^T$ and κ is the curvature of the boundary Γ. If our domain is polygonal, then its curvature is piecewise zero, thus we cannot approximate (1.9) by the corresponding problem on a polygon. This is the Babuška paradox.

To combat the second problem, we impose the condition (1.1c) weakly. Accordingly, we introduce a saddle point formulation of condition (1.1c), where the normal stress component appears as a Lagrange multiplier. That is how the Babuška paradox is avoided. The advantage of this approach is that the boundary conditions *need not* be incorporated into the function spaces.

In Section III, we examine the existence of solutions of the continuous problem under the assumption that our flow domain has piecewise Lipschitz continuous boundary. To prove the existence of solutions, we rely on the theory developed in Girault and Raviart [2].

Section IV includes the study of the discrete, *homogeneous* one-level problem following Verfürth [6]. Since the boundary conditions are imposed weakly as a constraint, an additional Lagrange multiplier (associated with the boundary conditions) is introduced into the scheme. Stability of the approximate boundary stresses then requires an additional discrete inf-sup condition (cf. ineq. (4.18)). Establishing this discrete inf-sup condition requires balancing the effects of the additional Lagrange multiplier; Verfürth accomplishes this using velocity bubble functions in the elements with a face on the boundary.

The two-level method, which is the focus of our article, is studied in Section V. The idea of using two-level methods to discretize elliptic boundary value problems was first introduced by J. Xu [9]. Following ideas developed by Layton [10], we show existence and uniqueness of the solution of the initial, *homogeneous* coarse mesh approximation of our algorithm under the assumption that the data is "small" (cf. condition (5.15)). Under a similar condition, we prove the same for the second leg of the algorithm. Finally, our error estimates show that, under the scaling $h \sim (H)^r$, where $r = 3 - \epsilon$ for $\epsilon > 0$ small in 2-d and $r = 5/2$ in 3-d, the solution to the second leg of the algorithm is quasi-optimally accurate in the discrete space *without any iteration*.

II. FUNCTION SPACES

We denote by $\mathrm{H}^k(\Omega)$ the usual $W^{k,2}(\Omega)$ Sobolev space with norm:

$$\|\mathbf{w}\|_{k,\Omega} = \left\{ \sum_{i=1}^{d} \Big(\sum_{|\underline{j}| \le k} \int_\Omega |D^{\underline{j}} w_i|^2 \, dx \Big) \right\}^{1/2}, \quad k \in \mathbb{N},$$

by $|\,.\,|_{k,\Omega}$ the induced seminorm, and by $\mathrm{L}^2(\Omega)$ the space $W^{0,2}(\Omega)$. The dual space of $\mathrm{H}^k(\Omega)$ is $\mathrm{H}^{-k}(\Omega)$ with $\langle . , . \rangle_\Omega$ being the duality pairing. This is an abuse of notation, because $\mathrm{H}^{-k}(\Omega)$ usually denotes the dual of $\mathrm{H}^k_0(\Omega)$. The space $\mathrm{H}^1_0(\Omega)$ consists of all functions in $\mathrm{H}^1(\Omega)$ that vanish on the boundary.

The spaces $\mathrm{H}^{k-1/2}(\Gamma)$ consist of the traces of all functions in $\mathrm{H}^k(\Omega)$. Analogously, we denote by $\mathrm{H}^{-(k-1/2)}(\Gamma)$ the dual space of $\mathrm{H}^{k-1/2}(\Gamma)$ with $\langle . , . \rangle_\Gamma$ being the duality pairing.

The natural norm of the normal component of a function $\mathbf{u} \in X$ on Γ is

$$\|\mathbf{u} \cdot \mathbf{n}\|_\Gamma = \left(\sum_{i=1}^{s} \|\mathbf{u} \cdot \mathbf{n}_i\|^2_{1/2,\Gamma_i} \right)^{1/2}, \tag{2.1}$$

with dual norm $\| \, . \, \|^*_\Gamma$. That is because Ω is polygonal and $\mathbf{u} \in \mathrm{H}^1(\Omega)$ implies $\mathbf{u} \cdot \mathbf{n} \in \mathrm{H}^{-1/2}(\Gamma)$ and *not* in $\mathrm{H}^{1/2}(\Gamma)$, Girault and Raviart [11, Remark 1.1, p. 9].

The *rigid body rotations* of Ω generate the space:

$$\mathcal{S} = \mathrm{span}\{\, \mathbf{u}(\underline{x}) = \underline{\zeta} \times \underline{x},\ \underline{\zeta} \in \mathbb{R}^3,\ |\underline{\zeta}| = 1,\ \underline{\zeta} \text{ is an axis of symmetry of } \Omega \,\}. \tag{2.2}$$

As mentioned in the Introduction, when Ω possesses no axis of symmetry, $\mathcal{S} = \emptyset$. In addition to the spaces $X,\ Y,\ Z$ in (1.3), we introduce the spaces:

$$X_{n,0} : = \{\, \mathbf{u} \in X \mid \mathbf{u} \cdot \mathbf{n}_i = 0 \quad \text{on each } \Gamma_i \,\}, \tag{2.3a}$$
$$V_0 : = \{\mathbf{u} \in X_{n,0} \ : \ \mathrm{div}\mathbf{u} = 0\}. \tag{2.3b}$$

Formulating (1.1) in the quotient space X is necessary, because the solution of the linearized (Stokes) problem is uniquely determined only up to a rigid body rotation of Ω.

Using Green's formula yields the following weak formulation:

Find $\mathbf{u} \in X,\ p \in Y$ with $\mathbf{u} \cdot \mathbf{n}_i|_{\Gamma_i} = 0$ on Γ_i such that

$$\begin{aligned} \frac{1}{2\mathrm{Re}} \int_\Omega \mathcal{D}(\mathbf{u}) \cdot \mathcal{D}(\mathbf{v})\, dx + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v}\, dx - \int_\Omega p \,\mathrm{div}\mathbf{v}\, dx & \\ - \sum_{i=1}^{s} \int_{\Gamma_i} (\mathbf{n}_i \cdot \Im(\mathrm{Re}^{-1}\mathbf{u}, p) \cdot \mathbf{n}_i)\, (\mathbf{v} \cdot \mathbf{n}_i)\, ds &= \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx, \\ - \int_\Omega q \,\mathrm{div}\mathbf{u}\, dx &= 0, \end{aligned} \tag{2.4}$$

for all $\mathbf{v} \in X,\ q \in Y$.

III. ANALYSIS OF THE CONTINUOUS PROBLEM

First we consider the saddle point formulation of the Stokes problem with slip boundary condition. For this we define the Lagrange functional $\mathcal{L} :\ \mathrm{H}^1(\Omega)^d \times \mathrm{L}^2_0(\Omega) \times \prod_{i=1}^{s} \mathrm{H}^{-1/2}(\Gamma_i) \to \mathbb{R}$ as

$$\mathcal{L}(\mathbf{u}, p, \rho) = \frac{1}{4\mathrm{Re}} \int_\Omega |\mathcal{D}(\mathbf{u})|^2\, dx - \int_\Omega \mathbf{f} \cdot \mathbf{u}\, dx - \int_\Omega p \,\mathrm{div}\mathbf{u}\, dx - \sum_{i=1}^{s} \langle \rho, \mathbf{u} \cdot \mathbf{n}_i \rangle_{\Gamma_i}, \tag{3.1}$$

where $\rho \in Z$ can be interpreted to be the *normal stress component* on Γ defined as

$$\rho|_{\Gamma_i} := \mathbf{n}_i \cdot \Im(\mathrm{Re}^{-1}\mathbf{u}, p) \cdot \mathbf{n}_i. \tag{3.2}$$

The saddle point of the Lagrange functional (3.1) is the solution of the Stokes problem with slip boundary condition.

To simplify the notation, we use the forms $a(.\,,.)$, $b(.\,;.\,,.)$ and $c(.\,;.\,,.)$ defined in (1.4); the saddle point problem arising from (3.1) now takes the following form:

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) + c(\mathbf{v};p,\rho) &= \langle \mathbf{f},\mathbf{v}\rangle_\Omega \quad \forall\, \mathbf{v} \in X, \\ c(\mathbf{u};q,\sigma) &= 0 \quad \forall\, q \in Y,\ \sigma \in Z. \end{aligned} \tag{3.3}$$

Analogously, for the Navier–Stokes problem:

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) + b(\mathbf{u};\mathbf{u},\mathbf{v}) + c(\mathbf{v};p,\rho) &= \langle \mathbf{f},\mathbf{v}\rangle_\Omega \quad \forall\, \mathbf{v} \in X, \\ c(\mathbf{u};q,\sigma) &= 0, \quad \forall\, q \in Y,\ \sigma \in Z. \end{aligned} \tag{3.4}$$

Note that the weak formulations (3.4) and (2.4) are equivalent.

To show solvability of problem (3.4), we start by proving the following lemmata. Some of the ideas used in the proof of Lemma 3.2 are due to Verfürth [6, ineq. (2.5), (2.6), p. 701] (also see Verfürth [5, Lemmata 2.1, 2.2, p. 15]).

Lemma 3.1. *Let* $\mathbf{u} \in X_{n,0}$ *with* $\operatorname{div}\mathbf{u} = 0$ *(i.e.* $\mathbf{u} \in V_0$*), and let* $\mathbf{v},\mathbf{w} \in X$. *Then:*

$$b(\mathbf{u};\mathbf{v},\mathbf{w}) = -b(\mathbf{u};\mathbf{w},\mathbf{v}), \tag{3.5}$$

$$b(\mathbf{u};\mathbf{v},\mathbf{v}) = 0. \tag{3.6}$$

Proof. See Girault and Raviart [11, Lemma 2.2, pp. 285]. ∎

Lemma 3.2. *There exists a solution to problem* (2.4) *in* V_0.

Proof. For $\mathbf{u} \in V_0$ problem (2.4) becomes:

Seek $\mathbf{u} \in V_0$ such that:

$$\frac{1}{2\mathrm{Re}} \int_\Omega \mathcal{D}(\mathbf{u}) \cdot \mathcal{D}(\mathbf{v})\, dx + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\, dx = \int_\Omega \mathbf{f}\cdot\mathbf{v}\, dx, \quad \forall\, \mathbf{v} \in V_0. \tag{3.7}$$

With $a(.\,,.)$ and $b(.\,;.\,,.)$ defined in (1.4), Eq. (3.7) takes the form:

$$a(\mathbf{u},\mathbf{v}) + b(\mathbf{u};\mathbf{u},\mathbf{v}) = \langle \mathbf{f},\mathbf{v}\rangle_\Omega, \quad \forall\, \mathbf{v} \in V_0. \tag{3.8}$$

By the theory developed in Girault and Raviart [2, Thm. 1.2, p. 106], there exists a solution of (2.4) in V_0, if we can establish that:

(a) the form $a(.\,,.) + b(.\,;.\,,.)$ is coercive and,

(b) the space V_0 is separable and the form $a(.\,,.) + b(.\,;.\,,.)$ is weakly continuous in V_0.

To show that part (a) is true, we use *Korn's Second Inequality*:

$$\frac{1}{2}\int_\Omega |\mathcal{D}(\mathbf{u})|^2\, dx \geq c_1\|\mathbf{u}\|^2_{1,\Omega} - c_0\|\mathbf{u}\|^2_{0,\Omega}, \quad \forall\, \mathbf{u} \in \mathrm{H}^1(\Omega)^d. \tag{3.9}$$

Inequality (3.9) is proved in Nitsche [12] for domains with Lipschitz boundary, but also holds for polygonal domains. We also make use of the *Poincaré-Morrey Inequality*:

$$c_2\|\mathbf{u}\|^2_{0,\Omega} \leq \int_\Omega |\mathcal{D}(\mathbf{u})|^2\, dx + \sum_{i=1}^{s}\int_{\Gamma_i} |\mathbf{u}\cdot\mathbf{n}|^2\, ds, \quad \forall\, \mathbf{u} \in X, \tag{3.10}$$

Note: Inequality (3.10) holds only because $\mathrm{H}^1(\Omega)$ is modded with $\mathcal{S}$. In general, it does not hold for all $\mathbf{u} \in \mathrm{H}^1(\Omega)$.

The proof of (3.10) can be found in Verfürth [5, Lemmata 2.1, 2.2, p. 15], and it follows from (3.9) by standard arguments. The constants c_0, c_1, and c_2 depend only on Ω. Combining inequalities (3.9) and (3.10), and using the fact that $\mathbf{u}\cdot\mathbf{n}=0$, we obtain

$$\frac{1}{2}\left(\frac{1}{c_1}+\frac{2c_0}{c_1c_2}\right)\int_\Omega |\mathcal{D}(\mathbf{u})|^2\,dx \geq \|\mathbf{u}\|^2_{1,\Omega}, \quad \forall\,\mathbf{u}\in V_0. \tag{3.11}$$

Thus,

$$a(\mathbf{u},\mathbf{u}) \geq \frac{c_1c_2}{\mathrm{Re}(c_2+2c_0)}\|\mathbf{u}\|^2_{1,\Omega}, \quad \forall\,\mathbf{u}\in V_0. \tag{3.12}$$

Equation (3.12) establishes that the bilinear form a(.,.) is coercive in V_0. To show part (b), let $\mathbf{u}\in V_0$, and $\mathbf{u}_l$ be a sequence in V_0 such that $\mathbf{u}_l\to\mathbf{u}$ weakly in V_0 as $l\to\infty$. But from Lemma 3.1:

$$\lim_{l\to\infty} b(\mathbf{u}_l;\mathbf{u}_l,\mathbf{v}) = -\lim_{l\to\infty} b(\mathbf{u}_l;\mathbf{v},\mathbf{u}_l) = -\lim_{l\to\infty}\left(\sum_{j=1}^d u_{l_j}\frac{\partial\mathbf{v}}{\partial x_j},\mathbf{u}_l\right).$$

Since in a Hilbert space $u_l \overset{w}{\to} u$ iff $(u_l,z)\to(u,z)$ for all z in the space (see Kreyszig [13, p. 260]), we have

$$-\lim_{l\to\infty}\left(\sum_{j=1}^d u_{l_j}\frac{\partial\mathbf{v}}{\partial x_j},\mathbf{u}_l\right) = -\lim_{l\to\infty}\left(\sum_{j=1}^d u_j\frac{\partial\mathbf{v}}{\partial x_j},\mathbf{u}_l\right) = b(\mathbf{u};\mathbf{u},\mathbf{v}).$$

Similarly, it is clear that $a(\mathbf{u}_l,\mathbf{v})\to a(\mathbf{u},\mathbf{v})$ as $l\to\infty$. Since V_0 is separable and the forms $a(.\,,.)$ and $b(.\,;.\,,.)$ are continuous (see Lemma 3.3 and Proposition 3.2), the existence theory developed in Girault and Raviart [2] indicates that there exists a solution to problem (2.4) in V_0. ∎

We also prove the following result.

Lemma 3.3. *The bilinear form $a(.\,,.)$ is continuous on $X\times X$:*

$$a(\mathbf{u},\mathbf{v}) \leq 2\mathrm{Re}^{-1}|\mathbf{u}|_1|\mathbf{v}|_1, \tag{3.13}$$

Proof. This follows from the Cauchy–Schwartz inequality. ∎

The main result of this section is as follows.

Proposition 3.4 Existence in the div- and Lagrange multiplier-free space.

The space

$$\mathcal{W} := \{\,\mathbf{u}\in X \mid c(\mathbf{u};p,\rho)=0, \quad \forall\,p\in Y,\ \rho\in Z\} \tag{3.14}$$

is not empty. There are constants $\alpha>0$ and $\beta>0$ such that

$$a(\mathbf{u},\mathbf{u}) \geq \alpha\|\mathbf{u}\|^2_{1,\Omega} \quad \forall\,\mathbf{u}\in\mathcal{W}, \tag{3.15}$$

and

$$\inf_{\substack{0\neq p\in Y\\ 0\neq\rho\in Z}}\ \sup_{0\neq\mathbf{u}\in X}\frac{c(\mathbf{u};p,\rho)}{\|\mathbf{u}\|_{1,\Omega}\left\{\|p\|^2_{0,\Omega}+(\|\rho\|^*_\Gamma)^2\right\}^{1/2}} \geq \beta. \tag{3.16}$$

Proof. This is an easy extension of the proof of Lemma 3.1 in Verfürth [6, p. 702]. ∎

Proposition 3.1 together with the existence theory developed in Girault and Raviart [2, Ch. 4, pp. 104-108] gives solvability of problem (3.4).

To prove uniqueness, we follow Girault and Raviart [2]. Define the quantity:

$$\mathcal{N} := \sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in V_0} \frac{|b^*(\mathbf{u};\mathbf{v},\mathbf{w})|}{|\mathbf{u}|_1\ |\mathbf{v}|_1\ |\mathbf{w}|_1}.$$

Proposition 3.5 Uniqueness of solution for the continuous problem.

Suppose that the data, domain, and Reynolds number satisfy

$$\frac{\mathrm{Re}^2(c_2+2c_0)^2}{(c_1c_2)^2}\,\mathcal{N}\,|\mathbf{f}|_* < 1, \quad \textit{where } |\mathbf{f}|_* := \sup_{\mathbf{v}\in V_0} \frac{\langle \mathbf{f},\mathbf{v}\rangle_\Omega}{|\mathbf{v}|_1}. \tag{3.17}$$

Then the solution to (1.1) *in* $\mathcal{W}$ *is unique.*

Proof. We shall prove Proposition 3.2 by applying the abstract uniqueness result in [2, Thm 1.3, p. 108]. To apply this result, we must verify that $a(.\,,.) + b^*(.\,;.\,,.)$ is coercive in V_0 and that $b^*(.\,;.\,,.)$ is continuous in $\mathcal{W}$.

Using Proposition 3.1, we see that $b^*(\mathbf{w};\mathbf{v},\mathbf{v}) = 0$, for all $\mathbf{w},\ \mathbf{v}\in\mathcal{W}$. Together with Lemma 3.1, we have

$$a(\mathbf{v},\mathbf{v}) + b^*(\mathbf{w};\mathbf{v},\mathbf{v}) \ge \alpha\|\mathbf{v}\|_1^2, \quad \forall\,\mathbf{w},\mathbf{v}\in\mathcal{W}.$$

Following [2, Lemma 2.1, p. 114], it is evident that b^* is continuous with

$$|b^*(\mathbf{w}_1;\mathbf{u},\mathbf{v}) - b^*(\mathbf{w}_2;\mathbf{u},\mathbf{v})| \le \mathcal{N}\,|\mathbf{w}_1-\mathbf{w}_2|_1|\mathbf{u}|_1|\mathbf{v}|_1.$$

Since all requirements of Theorem 1.3 in [2] are satisfied, uniqueness follows. ∎

Condition (3.17) will be always assumed in the sequel.

IV. DISCRETE PROBLEM

Our domain Ω is subdivided into d-simplices with sides of length less than H with $\mathcal{T}^H$ being the corresponding family of partitions. Let $\mathcal{T}^h$ be a family of partitions using d-simplices with sides of length less than h, where $h << H$. We assume that both $\mathcal{T}^\mu\ \ (\mu = h, H)$ satisfy the usual regularity assumptions (see, e.g., Ciarlet [14]):

1. each vertex of Ω is a vertex of a $T\in\mathcal{T}^\mu\ \ (\mu = h, H)$;
2. each $T\in\mathcal{T}^\mu\ \ (\mu = h, H)$ has at least one vertex in the interior of Ω;
3. any two d-simplices $T, T'\in\mathcal{T}^\mu\ \ (\mu = h, H)$ may meet in a vertex, a whole edge, or a whole face;
4. each $T\in\mathcal{T}^h$ and $T\in\mathcal{T}^H$ contain balls with radii $c_0h,\ c_1H$, respectively, and are contained by balls with radii $c_2h,\ c_3H$, respectively.

The constants $c_0, c_1, \ldots$ denote different constants, which are independent of h and H. For each i, denote by $\mathcal{O}_i^h,\ \mathcal{O}_i^H$ the partitions of Γ_i which are induced by $\mathcal{T}^h,\ \mathcal{T}^H$, respectively. Let $\mathcal{P}_k,\ k\ge 0$, be the set of all polynomials in $x_1, \ldots, x_d$ of degree less than or equal to k, and set

$$\mathcal{S}_k^\mu := \{\phi:\Omega\to\mathbb{R}\ |\quad \phi|_T\in\mathcal{P}_k, \quad \forall\ T\in\mathcal{T}^\mu\}, \quad \mu = h, H. \tag{4.1}$$

Note that, if $\mathcal{T}^h$ is obtained from $\mathcal{T}^H$ by refinement, then $\mathcal{S}_k^h \supset \mathcal{S}_k^H$. Along with the space $\tilde{X}^\mu$, the discrete analogs of the spaces Y and Z for $\mu = h, H$ are

$$\mathrm{H}^1(\Omega)^d \supset \tilde{X}^\mu \subset \left\{\mathcal{S}_k^\mu \cap C(\bar{\Omega})\right\}^d, \quad k \geq 1, \tag{4.2}$$

$$Y^\mu \subset \mathcal{S}_l^\mu \cap \mathrm{L}_0^2(\Omega), \quad l \geq 0, \tag{4.3}$$

$$Z^\mu = \{\phi : \Gamma \to \mathbb{R} \mid \quad \forall\, i,\ \phi|_{S_i} \in \mathcal{P}_m, \quad \forall\ S_i \in \mathcal{O}_i^\mu\}, \quad m \geq 0. \tag{4.4}$$

Also let

$$\tilde{X}_0^\mu = \left\{\mathbf{u}^\mu \in \tilde{X}^\mu \mid \quad \mathbf{u}^\mu = 0 \quad \text{on} \quad \Gamma\right\}, \quad \mu = h, H. \tag{4.5}$$

The spaces $\tilde{X}^\mu$ and Y^μ $(\mu = h, H)$ are assumed to satisfy the following properties:
I. There is a constant $\tilde{\beta} > 0$ independent of h and H for which:

$$\inf_{0\neq p^\mu \in Y^\mu} \sup_{0\neq \mathbf{u}^\mu \in \tilde{X}_0^\mu} \frac{\int_\Omega p^\mu \operatorname{div}\mathbf{u}^\mu\, dx}{\|p^\mu\|_{0,\Omega}\|\mathbf{u}^\mu\|_{1,\Omega}} \geq \tilde{\beta}. \tag{4.6}$$

II.

$$\inf_{p^\mu \in Y^\mu} \|p - p^\mu\|_{0,\Omega} \leq c\mu\|p\|_{1,\Omega}, \quad \mu = h, H, \quad \forall\, p \in H^1(\Omega). \tag{4.7}$$

III. There exists a continuous linear operator $\Pi^\mu : \mathrm{H}^1(\Omega)^d \to \tilde{X}^\mu$ for which:

$$\Pi^\mu(\mathrm{H}_0^1(\Omega)^d) \subset \tilde{X}_0^\mu, \tag{4.8}$$

$$\|\mathbf{u} - \Pi^\mu\mathbf{u}\|_{s,\Omega} \leq c\mu^{t-s}\|\mathbf{u}\|_{t,\Omega}, \quad \forall\, \mathbf{u} \in \mathrm{H}^t(\Omega) \quad \text{with } s = 0, 1 \text{ and } t = 1, 2, \tag{4.9}$$

$$\|\mathbf{u} - \Pi^\mu\mathbf{u}\|_{0,\Gamma} \leq c\mu^{1/2}\|\mathbf{u}\|_{1,\Omega}. \tag{4.10}$$

Assumptions II and III are satisfied by the finite element spaces X_0^μ, $\tilde{X}^\mu$, Y^μ and Z^μ (see Ciarlet [14, Chapters 2,3]). Assumption I is the discrete version of the inf-sup condition for the no-slip Stokes problem. The h-independence is necessary for stability of the pressure uniformly in h and optimal error estimates. For other examples of finite element spaces satisfying assumptions I-III, see the appendix of Verfürth [5].

In the theory of Babuška and Brezzi (see [15, 16]), the influence of the constraints $\operatorname{div}\mathbf{u} = 0$ and $\mathbf{u}\cdot\mathbf{n} \overset{w}{=} 0$ (the symbol $\overset{w}{=}$ indicates weak imposition), must be balanced by the velocity space. Assumption I does just that for $\operatorname{div}\mathbf{u} = 0$. The constraint $\mathbf{u}\cdot\mathbf{n} \overset{w}{=} 0$ leads to a second inf-sup condition, which must be satisfied by the finite element spaces. To satisfy the second inf-sup condition, we, following Verfürth, enrich the velocity space as follows.

Consider an element $T^{(j)} \in \mathcal{T}^\mu$ $(\mu = h, H)$, which has a face $S^{(j)}$ on the boundary piece Γ_j. Number the vertices of $T^{(j)}$ so that the vertices on $S^{(j)}$ are numbered first. Let $\xi_1(T^{(j)}), \ldots, \xi_{d+1}(T^{(j)})$ be the barycentric coordinates of $T^{(j)}$ (i.e., the linear functions $\xi_i(T^{(j)})$, $i = 1, \ldots, d+1$, for which each $\xi_i(T^{(j)})$ takes the value 1 on the vertex i and 0 on all other vertices). Define the bubble functions $\mathbf{b}_{(j)}^\mu$ on $T^{(j)}$ for $\mu = h, H$ as follows:

$$\mathbf{b}_{(j)}^\mu = \begin{cases} \mathbf{n}_i \prod_{i=1}^d \xi_i(T^{(j)}), & \text{on } T^{(j)}, \\ 0, & \text{on } \Omega \setminus T^{(j)}. \end{cases} \tag{4.11}$$

Note that, because we multiply the barycentric coordinates together, $\mathbf{b}_{(j)}^\mu$ vanishes on all vertices on $S^{(j)}$. Let

$$\mathcal{B}^\mu = \text{span}\left\{\mathbf{b}_{(j)}^\mu\sigma \mid \quad 1 \leq j \leq s \quad \sigma \in \mathcal{P}_m, \quad S^{(j)} \in \mathcal{O}_j^\mu\right\}, \quad \mu = h, H, \tag{4.12}$$

and define

$$X^\mu := (\tilde{X}^\mu \oplus \mathcal{B}^\mu)/\mathcal{S}\,, \tag{4.13}$$

where $\mathcal{S}$ is the space of rigid body rotations described in Section II (see Eq. (2.2)).

From the above comments on $\mathbf{b}^\mu_{(j)}$, it follows that $\mathcal{B}^\mu \subset (\mathrm{H}^1(\Omega))^d$ as well as that $X^\mu \cap (\mathrm{H}^1_0(\Omega))^d = \tilde{X}^\mu_0$ (since $\mathcal{B}^\mu$ does not vanish on Γ).

The mixed finite element approximation of our problem is given by

Find $\mathbf{u}^\mu \in X^\mu,\ p^\mu \in Y^\mu, \rho^\mu \in Z^\mu$ such that:

$$\begin{aligned} a(\mathbf{u}^\mu,\mathbf{v}^\mu) + b^*(\mathbf{u}^\mu;\mathbf{u}^\mu,\mathbf{v}^\mu) + c(\mathbf{v}^\mu;p^\mu,\rho^\mu) &= \langle \mathbf{f},\mathbf{v}^\mu\rangle_\Omega, \quad \forall\, \mathbf{v}^\mu \in X^\mu,\\ c(\mathbf{u}^\mu;q^\mu,\sigma^\mu) &= 0, \quad \forall\, q^\mu \in Y^\mu,\ \sigma^\mu \in Z^\mu, \end{aligned}$$

where

$$b^*(\mathbf{u};\mathbf{v},\mathbf{w}) = \frac{1}{2}\left[b(\mathbf{u};\mathbf{v},\mathbf{w}) - b(\mathbf{u};\mathbf{w},\mathbf{v})\right]. \tag{4.14}$$

The existence and uniqueness of a solution to the discrete linearized problem is studied in Verfürth [5, pp. 706–708]. The discrete nonlinear problem, however, requires more care because we have to show ellipticity of the form $a(\,.\,,.) + b^*(\,.\,;.\,,.)$.

We start by introducing the discrete divergence, Lagrange multiplier-free space V^μ:

$$V^\mu := \{\mathbf{u}^\mu \in X^\mu \mid \quad c(\mathbf{u}^\mu;p^\mu,\rho^\mu) = 0, \quad \forall\ p^\mu \in Y^\mu,\ \rho^\mu \in Z^\mu\}\,. \tag{4.15}$$

Also, let

$$A(\mathbf{w};\mathbf{u},\mathbf{v}) = a(\mathbf{u},\mathbf{v}) + b^*(\mathbf{w};\mathbf{u},\mathbf{v}). \tag{4.16}$$

Ellipticity of the form $A(\,.\,;.\,,.)$ follows from (3.12) and Lemma 3.1. Now, continuity of $b^*(\,.\,;.\,,.)$ in V^μ follows from Holder's inequality and the Sobolev imbedding theorem. Accordingly, we define the quantity:

$$\mathcal{N}^\mu := \sup_{\mathbf{u}^\mu,\mathbf{v}^\mu,\mathbf{w}^\mu \in V^\mu} \frac{|b^*(\mathbf{u}^\mu;\mathbf{v}^\mu,\mathbf{w}^\mu)|}{|\mathbf{u}^\mu|_1\ |\mathbf{v}^\mu|_1\ |\mathbf{w}^\mu|_1}. \tag{4.17}$$

The following lemma of Verfürth proves solvability of the discrete problem.

Lemma 4.1 Discrete inf-sup condition.

There is a constant $\beta > 0$ independent of μ, such that

$$\inf_{\substack{0\neq p^\mu \in Y^\mu\\ 0\neq \rho^\mu \in Z^\mu}} \ \sup_{0\neq \mathbf{u}^\mu \in X^\mu} \frac{c(\mathbf{u}^\mu;p^\mu,\rho^\mu)}{\|\mathbf{u}^\mu\|_{1,\Omega^\mu}\left\{\|p^\mu\|^2_{0,\Omega^\mu} + (\|\rho^\mu\|^*_\Gamma)^2\right\}^{1/2}} \geq \beta. \tag{4.18}$$

Proof. See Verfürth [5, p. 707]. ■

The proof given by Verfürth does *not* use specific examples of the spaces X^μ, Y^μ. It is assumed, however, that Z^μ is the space of piecewise constants along the boundary. A specific example of spaces that satisfy the above inf-sup condition (4.18) are the spaces that correspond to the "augmented" MINI-element in Verfürth [5, Appendix, p. 95].

V. ALGORITHM

First recall that the two-level method proceeds as follows:

With a choice of $X^{\mu}, Y^{\mu}, Z^{\mu}$, with $\mu = h,\ H$ such that condition (4.18) is satisfied:
Step 1. Solve the nonlinear, coarse mesh problem:
Find $(\mathbf{u}^H, p^H, \rho^H) \in (X^H, Y^H, Z^H)$ satisfying:

$$\begin{aligned} a(\mathbf{u}^H, \mathbf{v}^H) + b^*(\mathbf{u}^H; \mathbf{u}^H, \mathbf{v}^H) + c(\mathbf{v}^H; p^H, \rho^H) &= \langle \mathbf{f}, \mathbf{v}^H \rangle_\Omega, \\ c(\mathbf{u}^H; q^H, \sigma^H) &= 0, \end{aligned} \tag{5.1}$$

for all $(\mathbf{v}^H, q^H, \sigma^H) \in (X^H, Y^H, Z^H)$.
Step 2. Solve the following linear, fine mesh problem:
Find $(\mathbf{u}^h, p^h, \rho^h) \in (X^h, Y^h, Z^h)$ satisfying:

$$\begin{aligned} a(\mathbf{u}^h, \mathbf{v}^h) + b^*(\mathbf{u}^h; \mathbf{u}^H, \mathbf{v}^h) + b^*(\mathbf{u}^H; \mathbf{u}^h, \mathbf{v}^h) + & \\ -b^*(\mathbf{u}^H; \mathbf{u}^H, \mathbf{v}^h) + c(\mathbf{v}^h; p^h, \rho^h) &= \langle \mathbf{f}, \mathbf{v}^h \rangle_\Omega, \\ c(\mathbf{u}^h; q^h, \sigma^h) &= 0, \end{aligned} \tag{5.2}$$

for all $(\mathbf{v}^h, q^h, \sigma^h) \in (X^h, Y^h, Z^h)$.
To develop error estimates for (5.1), (5.2), we start with two identities.

Lemma 5.1. *Let* $\mathbf{u}, \mathbf{u}^H, \mathbf{v} \in \mathrm{H}^1(\Omega)^d$. *Then*

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = -(\mathbf{u}^H \cdot \nabla)\mathbf{u}^H + (\mathbf{u} \cdot \nabla)\mathbf{u}^H + (\mathbf{u}^H \cdot \nabla)\mathbf{u} + \left[(\mathbf{u} - \mathbf{u}^H) \cdot \nabla\right](\mathbf{u} - \mathbf{u}^H), \tag{5.3}$$

thus,

$$\begin{aligned} b^*(\mathbf{u}; \mathbf{u}, \mathbf{v}) =& b^*(\mathbf{u}; \mathbf{u}^H, \mathbf{v}) + b^*(\mathbf{u}^H; \mathbf{u}, \mathbf{v}) \\ &- b^*(\mathbf{u}^H; \mathbf{u}^H, \mathbf{v}) + b^*(\mathbf{u} - \mathbf{u}^H; \mathbf{u} - \mathbf{u}^H, \mathbf{v}). \end{aligned} \tag{5.4}$$

Proof. See Layton [10, p. 35]. ■

Using Lemma 5.1, in (3.4) yields

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b^*(\mathbf{u}; \mathbf{u}^H, \mathbf{v}) - b^*(\mathbf{u}^H; \mathbf{u}, \mathbf{v}) - b^*(\mathbf{u}^H; \mathbf{u}^H, \mathbf{v}) & \\ + b^*(\mathbf{u} - \mathbf{u}^H; \mathbf{u} - \mathbf{u}^H, \mathbf{v}) + c(\mathbf{v}; p, \rho) &= \langle \mathbf{f}, \mathbf{v} \rangle_\Omega, \quad \forall\, \mathbf{v} \in X, \\ c(\mathbf{u}; q, \sigma) &= 0, \quad \forall\, q \in Y,\ \sigma \in Z. \end{aligned} \tag{5.5}$$

Equation (5.2) can be rewritten in V^h (defined in (4.15)) as:
Find $\mathbf{u}^h \in V^h$ satisfying:

$$a(\mathbf{u}^h, \mathbf{v}^h) + b^*(\mathbf{u}^h; \mathbf{u}^H, \mathbf{v}^h) + b^*(\mathbf{u}^H; \mathbf{u}^h, \mathbf{v}^h) - b^*(\mathbf{u}^H; \mathbf{u}^H, \mathbf{v}^h) = \langle \mathbf{f}, \mathbf{v}^h \rangle_\Omega, \tag{5.6}$$

for all $\mathbf{v}^h \in V^h$. Now Lemma 4.1 guarantees that problems (5.6) and (5.2) are equivalent.
Let $\mathbf{w}^h \in V^h$. Define:

$$\boldsymbol{\phi}^h = \mathbf{u}^h - \mathbf{w}^h, \tag{5.7}$$

$$\boldsymbol{\eta}^h = \mathbf{u} - \mathbf{w}^h, \tag{5.8}$$

Let $\mathbf{v} = \mathbf{v}^h$ in Eq. (5.5); subtracting (5.6) from it yields:

$$\begin{aligned} a(\mathbf{u} - \mathbf{u}^h, \mathbf{v}^h) + b^*(\mathbf{u} - \mathbf{u}^h; \mathbf{u}^H, \mathbf{v}^h) + b^*(\mathbf{u}^H; \mathbf{u} - \mathbf{u}^h, \mathbf{v}^h) + & \\ + b^*(\mathbf{u} - \mathbf{u}^H; \mathbf{u} - \mathbf{u}^H, \mathbf{v}^h) + c(\mathbf{v}^h; p - p^h, \rho - \rho^h) &= 0. \end{aligned}$$

Subtracting and adding $\mathbf{w}^h$ in $\mathbf{u}-\mathbf{u}^h$ results:

$$\begin{aligned} a(\boldsymbol{\eta}^h,\mathbf{v}^h)+b^*(\boldsymbol{\eta}^h;\mathbf{u}^H,\mathbf{v}^h)+b^*(\mathbf{u}^H;\boldsymbol{\eta}^h,\mathbf{v}^h)+ \\ +b^*(\mathbf{u}-\mathbf{u}^H;\mathbf{u}-\mathbf{u}^H,\mathbf{v}^h)+c(\mathbf{v}^h;p-p^h,\rho-\rho^h)= \\ =a(\boldsymbol{\phi}^h,\mathbf{v}^h)+b^*(\boldsymbol{\phi}^h;\mathbf{u}^H,\mathbf{v}^h)+b^*(\mathbf{u}^H;\boldsymbol{\phi}^h,\mathbf{v}^h), \end{aligned} \tag{5.9}$$

which holds for any $(\mathbf{v}^h,p^h,\rho^h)\in V^h\times Y^h\times Z^h$.

Following Girault and Raviart [2], we define the following quantities, which are all finite and bounded uniformly in h and H:

$$\mathcal{N}:=\sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in V}\frac{|b^*(\mathbf{u};\mathbf{v},\mathbf{w})|}{|\mathbf{u}|_1\,|\mathbf{v}|_1\,|\mathbf{w}|_1}, \tag{5.10}$$

$$\mathcal{N}^H:=\sup_{\mathbf{u}^H,\mathbf{v}^H,\mathbf{w}^H\in V^H}\frac{|b^*(\mathbf{u}^H;\mathbf{v}^H,\mathbf{w}^H)|}{|\mathbf{u}^H|_1\,|\mathbf{v}^H|_1\,|\mathbf{w}^H|_1}, \tag{5.11}$$

$$\mathcal{N}_H^h:=\sup_{\substack{\mathbf{u}^h,\mathbf{w}^h\in V^h\\ \mathbf{v}^H\in V^H}}\frac{|b^*(\mathbf{u}^h;\mathbf{v}^H,\mathbf{w}^h)|}{|\mathbf{u}^h|_1\,|\mathbf{v}^H|_1\,|\mathbf{w}^h|_1}, \tag{5.12}$$

$$\mathcal{N}_h^H:=\sup_{\substack{\mathbf{u}^H,\mathbf{v}^H\in V^H\\ \mathbf{w}^h\in V^h}}\frac{|b^*(\mathbf{u}^H;\mathbf{v}^H,\mathbf{w}^h)|}{|\mathbf{u}^H|_1\,|\mathbf{v}^H|_1\,|\mathbf{w}^h|_1}. \tag{5.13}$$

Since the inf-sup condition (4.18) holds (replacing h with H), as well as assumptions II and III, $\mathcal{N}^H\to\mathcal{N}$ as $H\to 0$ and $\mathcal{N}_H^h$, $\mathcal{N}_h^H\to\mathcal{N}$ as $h,H\to 0$ following the arguments in Girault and Raviart [2, pp. 123–125].

Lemma 5.2 Existence and uniqueness of Step 1.
Solutions to (5.1) *exist in* V^H *and satisfy:*

$$|\mathbf{u}^H|_{1,\Omega}\le\alpha^{-1}|\mathbf{f}|_*, \tag{5.14}$$

where $\alpha=\frac{(c_1c_2)}{\mathrm{Re}(c_2+2c_0)}$. *Now suppose that*

$$\alpha^{-2}\mathcal{N}^H\,|\mathbf{f}|_*<1; \tag{5.15}$$

then the solution to (5.1) *is unique.*

Proof. Existence follows from the theory developed in Verfürth [6]. Since $\mathbf{u}^H\in V^H$, Eq. (5.1) with $\mathbf{v}^H=\mathbf{u}^H$ yields:

$$\alpha|\mathbf{u}^H|_{1,\Omega}^2\le|\mathbf{f}|_*|\mathbf{u}^H|_1\Rightarrow|\mathbf{u}^H|_{1,\Omega}\le\alpha^{-1}|\mathbf{f}|_*,$$

as required.

To prove uniqueness, we proceed along the same lines as Layton [10, Lemma 2.4, p. 36]. Let $\mathbf{u}_1^H$ and $\mathbf{u}_2^H$ be two solutions to problem (5.1). Define $\mathbf{d}^H$ to be their difference. Then

$$\begin{aligned}\alpha|\mathbf{d}^H|_{1,\Omega}^2&\le a(\mathbf{d}^H,\mathbf{d}^H)+b^*(\mathbf{u}_1^H;\mathbf{d}^H,\mathbf{d}^H)\\ &=a(\mathbf{u}_1^H,\mathbf{d}^H)+b^*(\mathbf{u}_1^H;\mathbf{u}_1^H,\mathbf{d}^H)-(a(\mathbf{u}_2^H,\mathbf{d}^H)+b^*(\mathbf{u}_1^H;\mathbf{u}_2^H,\mathbf{d}^H).\end{aligned}$$

Adding and subtracting $b^*(\mathbf{u}_2^H;\mathbf{u}_2^H,\mathbf{d}^H)$ inside the parenthesis yields

$$\begin{aligned}\alpha|\mathbf{d}^H|_{1,\Omega}^2&\le b^*(\mathbf{u}_2^H;\mathbf{u}_2^H,\mathbf{d}^H)-b^*(\mathbf{u}_1^H;\mathbf{u}_2^H,\mathbf{d}^H)\\ &=-b^*(\mathbf{d}^H;\mathbf{u}_2^H,\mathbf{d}^H)\le\mathcal{N}^H|\mathbf{d}^H|_{1,\Omega}^2|\mathbf{u}_2^H|_{1,\Omega}\le\mathcal{N}^H\alpha^{-1}|\mathbf{f}|_*|\mathbf{d}^H|_{1,\Omega}^2.\end{aligned}$$

It follows that $|\mathbf{d}^H|_1^2=0$ provided condition (5.15) holds. ∎

Lemma 5.3 Existence and uniqueness of Step 2.
Given a solution $\mathbf{u}^H$ *to problem* (5.1)*, let*

$$\alpha^{-2}\,\mathcal{N}_H^h\,|\mathbf{f}|_* < 1. \tag{5.16}$$

Then the solution $\mathbf{u}^h$ *to problem* (5.2) *exists uniquely in* V^h *and satisfies:*

$$|\mathbf{u}^h|_1 \le \left[\alpha - \mathcal{N}_H^h \alpha^{-1}|\mathbf{f}|_*\right]^{-1}\left[1+\mathcal{N}_h^H\alpha^{-2}|\mathbf{f}|_*\right]|\mathbf{f}|_*. \tag{5.17}$$

Proof. We start looking for solutions in V^h. Using Eq. (5.6) together with (5.10)–(5.13) we have

$$\begin{aligned}\alpha|\mathbf{u}^h|_1^2 - \mathcal{N}_H^h|\mathbf{u}^h|_1^2|\mathbf{u}^H|_1 &\le a(\mathbf{u}^h,\mathbf{u}^h) + b^*(\mathbf{u}^h;\mathbf{u}^H,\mathbf{u}^h)\\ &= \langle \mathbf{f},\mathbf{u}^H\rangle_\Omega + b^*(\mathbf{u}^H;\mathbf{u}^H,\mathbf{u}^h) \le |\mathbf{f}|_*|\mathbf{u}^h|_1 + \mathcal{N}_h^H|\mathbf{u}^H|_1^2|\mathbf{u}^h|_1.\end{aligned} \tag{5.18}$$

Using (5.14) yields the bound (5.17). Note that (5.18) is an *a priori* bound. Since (5.2) is linear in $\mathbf{u}^h$, existence and uniqueness in V^h follow. ■

Theorem 5.4 Convergence of the Two-Level Method.
Suppose that condition (5.16) *holds. Let*

$$\theta := \alpha - \mathcal{N}_H^h\alpha^{-1}|\mathbf{f}|_*. \tag{5.19}$$

Then, for $\epsilon > 0$ *small enough, the error satisfies*

$$\begin{aligned}&|\mathbf{u}-\mathbf{u}^h|_{1,\Omega} + \left\{\|p-p^h\|_{0,\Omega}^2 + (\|\rho-\rho^h\|_\Gamma^*)^2\right\}^{1/2} \le\\ &\le \mathcal{K}_1 \inf_{\mathbf{v}^h\in X^h}|\mathbf{u}-\mathbf{v}^h|_{1,\Omega} + \begin{cases}\theta^{-1}C_2(\epsilon)H^{-\epsilon}|\mathbf{u}-\mathbf{u}^H|_{1,\Omega}|\mathbf{u}-\mathbf{u}^H|_{0,\Omega} & \text{in 2-d}\\ CH^{-1/2}|\mathbf{u}-\mathbf{u}^H|_{1,\Omega}|\mathbf{u}-\mathbf{u}^H|_{0,\Omega} & \text{in 3-d}\end{cases}\\ &+\left[1+(\beta^{-1}+\theta^{-1})(1+C_3h)^{1/2}\right]\inf_{\substack{0\ne q^h\in Y^h\\ 0\ne\sigma^h\in Z^h}}\left\{\|p-q^h\|_{0,\Omega}^2 + (\|\rho-\sigma^h\|_\Gamma^*)^2\right\}^{1/2},\end{aligned} \tag{5.20}$$

where

$$\mathcal{K}_1 = \left[1+2(\theta\mathrm{Re})^{-1} + 2C(\beta\mathrm{Re})^{-1}\right] + \left[C_1(d,\Omega)\theta^{-1} + 4\beta^{-1}C(d,\Omega)\right]\alpha^{-1}|\mathbf{f}|_*.$$

Proof. Consider equation (5.9):

$$\begin{aligned}&a(\boldsymbol{\eta}^h,\mathbf{v}^h) + b^*(\boldsymbol{\eta}^h;\mathbf{u}^H,\mathbf{v}^h) + b^*(\mathbf{u}^H;\boldsymbol{\eta}^h,\mathbf{v}^h) +\\ &\quad + b^*(\mathbf{u}-\mathbf{u}^H;\mathbf{u}-\mathbf{u}^H,\mathbf{v}^h) + c(\mathbf{v}^h;p-p^h,\rho-\rho^h) =\\ &\quad = a(\boldsymbol{\phi}^h,\mathbf{v}^h) + b^*(\boldsymbol{\phi}^h;\mathbf{u}^H,\mathbf{v}^h) + b^*(\mathbf{u}^H;\boldsymbol{\phi}^h,\mathbf{v}^h).\end{aligned}$$

Let $\mathbf{v}^h = \boldsymbol{\phi}^h$; then on the right-hand side:

$$b^*(\mathbf{u}^H;\boldsymbol{\phi}^h,\boldsymbol{\phi}^h) = 0,$$

by Lemma 3.1. Thus, we have

$$a(\boldsymbol{\phi}^h,\boldsymbol{\phi}^h) + b^*(\boldsymbol{\phi}^h;\mathbf{u}^H,\boldsymbol{\phi}^h) \ge \alpha|\boldsymbol{\phi}^h|_1^2 - \mathcal{N}_H^h|\boldsymbol{\phi}^h|_1^2|\mathbf{u}^H|_1, \tag{5.21}$$

using aforementioned bounds. The left-hand side of Eq. (5.9) is as follows:

$$\begin{aligned}&a(\boldsymbol{\eta}^h,\boldsymbol{\phi}^h) + b^*(\boldsymbol{\eta}^h;\mathbf{u}^H,\boldsymbol{\phi}^h)\\ &\quad + b^*(\mathbf{u}^H;\boldsymbol{\eta}^h,\boldsymbol{\phi}^h) + b^*(\mathbf{u}-\mathbf{u}^H;\mathbf{u}-\mathbf{u}^H,\boldsymbol{\phi}^h) + c(\underline{\phi}^h;p-q^h,\rho-\sigma^h).\end{aligned}$$

The three middle terms of the above expression can be bounded as in Temam [17, Lemma 2.1, p. 12] (with $m_1 = m_3 = 1,\ m_2 = 0$). Thus, if $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in (\mathrm{H}^1(\Omega)^d)^3$, then

$$b^*(\mathbf{u}; \mathbf{v}, \mathbf{w}) \le C(d, \Omega)|\mathbf{u}|_1|\mathbf{v}|_1|\mathbf{w}|_1. \tag{5.22}$$

For the discrete case, however, we can improve this estimate by using the Sobolev Imbedding Theorem and an inverse inequality for quasi-uniform meshes as follows: since (in 2-d) $X^\mu \subset \mathrm{H}^{1+\epsilon}(\Omega)$ for any $\epsilon > 0$ small enough we have, for $\mu_j = H$ or h:

$$b^*(\mathbf{u}_{\mu_1}, \mathbf{v}_{\mu_2}, \mathbf{w}_{\mu_3}) \le C(\epsilon)\mu_1^{-\epsilon}\|\mathbf{u}_{\mu_1}\|_{0,\Omega}\|\mathbf{v}_{\mu_2}\|_{1,\Omega}\|\mathbf{w}_{\mu_3}\|_{1,\Omega}. \tag{5.23}$$

In 3-d, we have

$$b^*(\mathbf{u}_{\mu_1}, \mathbf{v}_{\mu_2}, \mathbf{w}_{\mu_3}) \le C\mu_1^{-1/2}\|\mathbf{u}_{\mu_1}\|_{0,\Omega}\|\mathbf{v}_{\mu_2}\|_{1,\Omega}\|\mathbf{w}_{\mu_3}\|_{1,\Omega}. \tag{5.24}$$

To get a bound for $c(.;.,.)$, we use approximation results and inverse estimates for finite elements. Thus,

$$\begin{aligned} c(\mathbf{u}^h; p^h, \rho^h) &= -\int_\Omega p^h \operatorname{div}\mathbf{u}^h\, dx - \sum_{i=1}^{s}\langle \rho^h, \mathbf{u}^h \cdot \mathbf{n}_i\rangle_{\Gamma_i} \\ &\le \|p^h\|_{0,\Omega}|\mathbf{u}^h|_{1,\Omega} + \|\rho^h\|_\Gamma^*\sum_{i=1}^{s}\|\mathbf{u}^h\cdot\mathbf{n}_i\|_{0,\Gamma_i} \\ &\le \left\{\|p^h\|_{0,\Omega}^2 + (\|\rho^h\|_\Gamma^*)^2\right\}^{1/2}(1 + Ch)^{1/2}\|\mathbf{u}^h\|_{1,\Omega}. \end{aligned} \tag{5.25}$$

Using inequalities (3.13) and (5.21)-(5.25) in (5.9) yields (in 2-d):

$$\begin{aligned} \alpha|\boldsymbol{\phi}^h|_{1,\Omega}^2 - &\mathcal{N}_H^h|\boldsymbol{\phi}^h|_{1,\Omega}^2|\mathbf{u}^H|_{1,\Omega} \le \\ &\le 2\mathrm{Re}^{-1}|\boldsymbol{\eta}^h|_{1,\Omega}|\boldsymbol{\phi}^h|_{1,\Omega} + C_1(d,\Omega)|\boldsymbol{\eta}^h|_{1,\Omega}|\mathbf{u}^H|_{1,\Omega}|\boldsymbol{\phi}^h|_{1,\Omega} + \\ &\quad + \left\{\|p - q^h\|_{0,\Omega}^2 + (\|\rho - \sigma^h\|_\Gamma^*)^2\right\}^{1/2}(1 + C_3)h^{1/2}|\boldsymbol{\phi}^h|_{1,\Omega} + \\ &\quad + C_2(\epsilon)H^{-\epsilon}|\mathbf{u} - \mathbf{u}^H|_{1,\Omega}|\mathbf{u} - \mathbf{u}^H|_{0,\Omega}|\boldsymbol{\phi}^h|_{1,\Omega}. \end{aligned}$$

Using the bound (5.14) and canceling out terms yields (in 2-d):

$$\begin{aligned} |\boldsymbol{\phi}^h|_{1,\Omega} \le\ & 2(\theta\mathrm{Re})^{-1}|\boldsymbol{\eta}^h|_{1,\Omega} + \theta^{-1}C_1(d,\Omega)\alpha^{-1}|\underline{f}|_*|\boldsymbol{\eta}^h|_{1,\Omega} \\ &+ \theta^{-1}C_2(\epsilon)H^{-\epsilon}|\mathbf{u} - \mathbf{u}^H|_{1,\Omega}|\mathbf{u} - \mathbf{u}^H|_{0,\Omega} \\ &+ \theta^{-1}(1 + C_3h)^{1/2}\left\{\|p - q^h\|_{0,\Omega}^2 + (\|\rho - \sigma^h\|_\Gamma^*)^2\right\}^{1/2}. \end{aligned}$$

Substitution of $\boldsymbol{\phi}^h = \mathbf{u}^h - \mathbf{w}^h$ and $\boldsymbol{\eta}^h = \mathbf{u} - \mathbf{w}^h$ and the use of the triangle inequality give (in 2-d):

$$\begin{aligned} |\mathbf{u} - \mathbf{u}^h|_{1,\Omega} \le\ & [1 + 2(\theta\mathrm{Re})^{-1}]\inf_{\mathbf{w}^h\in V^h}|\mathbf{u} - \mathbf{w}^h|_{1,\Omega} \\ &+ \theta^{-1}C_1(d,\Omega)\alpha^{-1}|\underline{f}|_*\inf_{\mathbf{w}^h\in V^h}|\mathbf{u} - \mathbf{w}^h|_{1,\Omega} \\ &+ \theta^{-1}C_2(\epsilon)H^{-\epsilon}|\mathbf{u} - \mathbf{u}^H|_{1,\Omega}|\mathbf{u} - \mathbf{u}^H|_{0,\Omega} \\ &+ \theta^{-1}(1 + C_3h)^{1/2}\inf_{\substack{0\ne q^h\in Y^h \\ 0\ne\sigma^h\in Z^h}}\left\{\|p - q^h\|_{0,\Omega}^2 + (\|\rho - \sigma^h\|_\Gamma^*)^2\right\}^{1/2}. \end{aligned} \tag{5.26}$$

To get a bound for the pressure, we proceed as in Girault and Raviart [2, pp. 60–61]. Let $\mathbf{u}^h$ be the solution to (5.2) and $\mathbf{u}$ be the solution to (3.4). Then

$$\begin{aligned} c(\mathbf{v}^h; p^h, \rho^h) = a(\mathbf{u}-\mathbf{u}^h, \mathbf{v}^h) + b^*(\mathbf{u}; \mathbf{u}, \mathbf{v}^h) - b^*(\mathbf{u}^h; \mathbf{u}^H, \mathbf{v}^h) \\ - b^*(\mathbf{u}^H; \mathbf{u}^h, \mathbf{v}^h) + b^*(\mathbf{u}^H; \mathbf{u}^H, \mathbf{v}^h) + c(\mathbf{v}^h; p, \rho). \end{aligned} \tag{5.27}$$

Using Lemma 5.1, we see that the right-hand side of (5.27) is equal to

$$\begin{aligned} &a(\mathbf{u}-\mathbf{u}^h, \mathbf{v}^h) + b^*(\mathbf{u}-\mathbf{u}^h; \mathbf{u}^H, \mathbf{v}^h) \\ &\quad + b^*(\mathbf{u}^H; \mathbf{u}-\mathbf{u}^h, \mathbf{v}^h) + b^*(\mathbf{u}-\mathbf{u}^H; \mathbf{u}-\mathbf{u}^H, \mathbf{v}^h) + c(\mathbf{v}^h; p, \rho). \end{aligned} \tag{5.28}$$

Add $c(\mathbf{v}^h; -q^h, -\sigma^h)$ to both sides of (5.27). Since $c(.\,;.\,,.)$ is *not* trilinear, but really bilinear, we get

$$\begin{aligned} c(\mathbf{v}^h; p^h - q^h, \rho^h - \sigma^h) = a(\mathbf{u}-\mathbf{u}^h, \mathbf{v}^h) + b^*(\mathbf{u}-\mathbf{u}^h; \mathbf{u}^H, \mathbf{v}^h) \\ + b^*(\mathbf{u}^H; \mathbf{u}-\mathbf{u}^h, \mathbf{v}^h) + b^*(\mathbf{u}-\mathbf{u}^H; \mathbf{u}-\mathbf{u}^H, \mathbf{v}^h) \\ + c(\mathbf{v}^h; p - q^h, \rho - \sigma^h). \end{aligned} \tag{5.29}$$

Dividing both sides by $|\mathbf{v}^h|_{1,\Omega}$ and using the inf-sup condition gives (for 2-d):

$$\begin{aligned} &\left\{\|p^h - q^h\|_{0,\Omega}^2 + (\|\rho^h - \sigma^h\|_\Gamma^*)^2\right\}^{1/2} \\ &\quad \le 2(\beta \mathrm{Re})^{-1}|\mathbf{u}-\mathbf{u}^h|_{1,\Omega} + 2\beta^{-1}(C(d,\Omega)+C)|\mathbf{u}-\mathbf{u}^h|_{1,\Omega}|\mathbf{u}^H|_{1,\Omega} \\ &\quad + \beta^{-1}\left\{\|p - q^h\|_{0,\Omega}^2 + (\|\rho - \sigma^h\|_\Gamma^*)^2\right\}^{1/2}(1+Ch)^{1/2} \\ &\quad + C(\epsilon)H^{-\epsilon}|\mathbf{u}-\mathbf{u}^H|_{1,\Omega}|\mathbf{u}-\mathbf{u}^H|_{0,\Omega}. \end{aligned} \tag{5.30}$$

In the 3-d case, we let $\epsilon = 1/2$.

We clearly see by adding and subtracting in (5.30) that

$$\begin{aligned} &\left\{\|p^h - q^h\|_{0,\Omega}^2 + (\|\rho^h - \sigma^h\|_\Gamma^*)^2\right\}^{1/2} \\ &\quad \ge \left[(\|p-p^h\|_{0,\Omega} - \|p-q^h\|_{0,\Omega})^2 + (\|\rho-\rho^h\|_\Gamma^* - \|\rho-\sigma^h\|_\Gamma^*)^2\right]^{1/2} \\ &\quad \ge \left|(\|p-p^h\|_{0,\Omega}^2 + (\|\rho-\rho^h\|_\Gamma^*)^2)^{1/2} - (\|p-q^h\|_{0,\Omega}^2 + (\|\rho-\sigma^h\|_\Gamma^*)^2)^{1/2}\right| \\ &\quad \ge \left\{\|p-p^h\|_{0,\Omega}^2 + (\|\rho-\rho^h\|_\Gamma^*)^2\right\}^{1/2} - \left\{\|p-q^h\|_{0,\Omega}^2 + (\|\rho-\sigma^h\|_\Gamma^*)^2\right\}^{1/2}. \end{aligned} \tag{5.31}$$

Combining (5.30) and (5.31) yields the bound:

$$\begin{aligned} &\left\{\|p-p^h\|_{0,\Omega}^2 + (\|\rho-\rho^h\|_\Gamma^*)^2\right\}^{1/2} \\ &\quad \le C\left[2(\beta \mathrm{Re})^{-1} + 2\beta^{-1}(C(d,\Omega)+C)\alpha^{-1}|\mathbf{f}|_*\right] \inf_{\mathbf{v}^h \in V^h} |\mathbf{u}-\mathbf{v}^h|_1 \\ &\quad + \left[1 + \beta^{-1}(1+C_3 h)^{1/2}\right] \inf_{\substack{0 \ne p^h \in Y^h \\ 0 \ne \rho^h \in Z^h}} \left\{\|p-q^h\|_{0,\Omega}^2 + (\|\rho-\sigma^h\|_\Gamma^*)^2\right\}^{1/2} \\ &\quad + C(\epsilon)H^{-\epsilon}|\mathbf{u}-\mathbf{u}^H|_{1,\Omega}|\mathbf{u}-\mathbf{u}^H|_{0,\Omega}. \end{aligned} \tag{5.32}$$

Adding (5.25) and (5.32), and letting the infimum be over X^h (because of the inf-sup condition), yields our result. ∎

VI. REMARKS

One significant question concerns the degree of coarseness and fineness of our two meshes so that our approximation to the true solution is optimal. Suppose that the condition:

$$\inf_{\mathbf{v}^h \in X^h} |\mathbf{u} - \mathbf{v}^h|_{1,\Omega} + \inf_{\substack{0 \neq p^h \in Y^h \\ 0 \neq \rho^h \in Z^h}} \left\{ \|p - p^h\|_{0,\Omega}^2 + (\|\rho - \rho^h\|_\Gamma^*)^2 \right\}^{1/2} \leq Ch\|\mathbf{f}\|_0, \tag{6.1}$$

holds for the chosen finite element spaces. Assuming that we use linears in the first step, we have

$$\|\mathbf{u} - \mathbf{u}^H\|_{1,\Omega} \leq CH \max\{\|\mathbf{f}\|_{0,\Omega}, \|\mathbf{u}\|_{2,\Omega}\}.$$

Using the duality argument of Aubin [18] and Nitsche [19] yields

$$\|\mathbf{u} - \mathbf{u}^H\|_{0,\Omega} \leq CH^2 \max\{\|\mathbf{f}\|_{0,\Omega}, \|\mathbf{u}\|_{2,\Omega}\},$$

since we match the boundary (Ω is polygonal). Now, balancing error terms on the right-hand side of (5.20) suggests that a scaling given by $h \sim H^r$, where $r = 3 - \epsilon$ in 2-d and $r = 5/2$ in 3-d, ensures optimal order accuracy.

Note that apart from the velocity and pressure, the mixed method employed yields additional information on an important physical quantity, the normal stress component.

Also note that we can apply our algorithm to the first step (equation (5.1)) to further reduce the number of nonlinear equations needed to be solved.

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References

1. M. D. Gunzburger, ‘‘Finite element methods for viscous incompressible flows, A guide to theory, practice, and algorithms’’, Academic Press, New York, 1989.
2. V. Girault, P. A. Raviart, Finite element approximation of the Navier–Stokes equations, Lecture Notes in Mathematics v. 749, Springer–Verlag, Berlin, Heidelberg, New York, 1979.
3. G. P. Galdi, Introduction to the mathematical theory of the Navier–Stokes equations, Vol. 2, Tracts in Natural Philosophy, Vol. 39, Springer, Berlin, 1998.
4. C. LeRoux, Second grade fluids with slip boundary conditions, Ph.D. Thesis, University of Pretoria, Pretoria, May 1997.
5. R. Verfürth, Finite element approximation of stationary Navier–Stokes equations with slip boundary condition, Habilitationsschrift, Report No. 75, Univ. Bochum, 1986.
6. R. Verfürth, Finite element approximation of incompressible Navier–Stokes equations with slip boundary condition, Numer Math 50 (1987), 697–721.
7. S. Saito, L. E. Scriven, Study of coating flow by the finite element method, J. Comp. Phys. 42 (1981), 53–76.
8. J. Silliman, L. E. Scriven, Separating flow near a static contact line: slip at a wall and shape of a free surface, J Comp Phys 34 (1980), 287–313.
9. J. Xu, Two grid discretization techniques for linear and non-linear PDE's, SIAM J Numer Anal 5 (1996), 1759–1777.
10. W. J. Layton, A two-level discretization method for the Navier-Stokes equations, Comp Math Appl 26 (1993), 33–38.

11. V. Girault, P. A. Raviart, "Finite element methods for Navier–Stokes equations," Theory and algorithms, Springer–Verlag, New York, 1986.
12. J. A. Nitsche, On Korn's second inequality, RAIRO Anal Numer, 15 (1981), 237–248.
13. E. Kryeszig, Introductory functional analysis with applications, Wiley, New York, 1978.
14. P. G. Ciarlet, The finite element method for elliptic problems, North–Holland, Amsterdam, 1978.
15. I. Babuška, The finite element method with Lagrange multipliers, Numer Math 20 (1973), 179–192.
16. F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrange multipliers, RAIRO Anal Numer 8 (1974), 129–151.
17. R. Temam, Navier–Stokes equations and non-linear functional analysis, CBMS-NSF Conf Ser Vol 41, SIAM, Philadelphia, 1983.
18. J. P. Aubin, Behavior of the error of the approximate solutions of boundary value problems for linear elliptic operators by Galerkin's and finite difference methods, Ann Sc Norm Super Pisa 21 (1967), 599–637.
19. J. A. Nitsche, Ein Kriterium für die Quasi-Optimalität des Ritzschen Verfahrens, Numer Math 11 (1968), 346–348.