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The Stokes and Navier-Stokes equations with boundary conditions involving the pressure

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0. Introduction

This paper is concerned with the stationary Stokes and Navier-Stokes equations with non standard boundary conditions: specifically the case where the pressure is given on some part of the boundary will be considered.

To be precise, the flow of a **viscous** incompressible fluid which occupies a bounded domain of $\boldsymbol{R}^3$ is studied. The velocity $\mathbf{u}$ and pressure p are assumed to satisfy in this domain Ω

– either the stationary Stokes equations:

$$-\nu\Delta\mathbf{u}+\nabla p=\mathbf{f} \quad \text{in} \quad \Omega \tag{0.1a}$$

$$\nabla\cdot\mathbf{u}=0 \quad \text{in} \quad \Omega \tag{0.1b}$$

(these equations model the case of a fluid of large viscosity or a slow motion),

– or the stationary Navier-Stokes equations:

$$-\nu\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=\mathbf{f} \quad \text{in} \quad \Omega \tag{0.2a}$$

$$\nabla\cdot\mathbf{u}=0 \quad \text{in} \quad \Omega, \tag{0.2b}$$

in both cases ν denotes the kinematic viscosity of the fluid and $\mathbf{f}$ the density of the external forces.

As regards the boundary conditions, they are assumed to be of three different types:

– the velocity is given on a portion Γ_1 of the boundary of Ω;

– the pressure and the tangential component of the velocity are given on a second portion Γ_2 of the boundary;

– the normal component of the velocity and the tangential component of the vorticity are given on the remainder Γ_3 of the boundary.

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In the case of the Stokes problem (0.1) these boundary conditions read:

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on } \Gamma_1 \tag{0.3a}$$

$$\mathbf{u} \times \mathbf{n} = \mathbf{a} \times \mathbf{n} \quad \text{and} \quad p = p_0 \quad \text{on } \Gamma_2 \tag{0.3b}$$

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n} \quad \text{and} \quad (\nabla \times \mathbf{u}) \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on } \Gamma_3, \tag{0.3c}$$

where $\mathbf{u}_0$, $\mathbf{a}$, p_0, $\mathbf{b}$ and $\mathbf{h}$ are given functions and where $\mathbf{n}$ denotes the unit outward normal to the boundary of Ω.

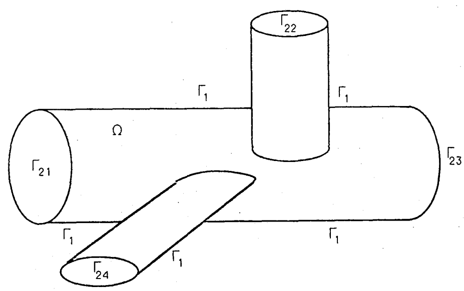

Figure 0.1

In the case of the Navier-Stokes problem (0.2) the boundary conditions read:

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on } \Gamma_1 \tag{0.4a}$$

$$\mathbf{u} \times \mathbf{n} = \mathbf{a} \times \mathbf{n} \quad \text{and} \quad p + (1/2)|\mathbf{u}|^2 = p_0 \quad \text{on } \Gamma_2 \tag{0.4b}$$

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n} \quad \text{and} \quad (\nabla \times \mathbf{u}) \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on } \Gamma_3. \tag{0.4c}$$

Note that in the Navier-Stokes problem the dynamic pressure $p + (1/2)|\mathbf{u}|^2$ plays in (0.4b) the role played by the static pressure p in (0.3b) for the Stokes problem.

The condition (0.3a) (or (0.4a)) on Γ_1 is the classical no-slip boundary condition either with a fixed ($\mathbf{u}_0 = \mathbf{0}$) or a moving boundary (this models the case of a body moving in the flow). In contrast the boundary conditions (0.3b) and (0.3c) (or (0.4b) and (0.4c)) are non standard.

There exists a considerable literature concerning the Stokes and Navier-Stokes equations with a no-slip boundary condition: see the articles [Le1], [Le2], [Le3] by J. Leray and the book [La] by O. Ladyzkenskaya: see also the books [Li], [Ta] and [Te] by J.-L. Lions, L. Tartar and R. Temam, respectively. The present work was announced in [Pi] and [Bè-Co-Mu-Pi]. Other boundary conditions for the Stokes and Navier-Stokes equations (no stress and slip at the boundary) have been studied recently by C. Conca [Co1], [Co2] and R. Verfürth [Ve1], [Ve2].

Applications often give rise to problems where the boundary conditions (0.3a), (0.3b) (or (0.4a), (0.4b)) occur naturally. Specifically the following problems seem to be of particular interest:

EXAMPLE 1. **Flow in a network of pipes.** This example is concerned with a viscous, incompressible fluid which flows in a network of pipes. The domain Ω models this network; Γ_1 denotes the lateral surfaces of the pipes; $\Gamma_2 = \bigcup_i \Gamma_{2i}$ consists of the in-flow and out-flow sections Γ_{2i} of the network where a given pressure seems to be a natural boundary condition to impose; Γ_3 is empty. The lateral surfaces of the pipes are assumed to be rigid and fixed and the no-slip boundary condition is assumed, so $\mathbf{u}_0 = \mathbf{0}$; the flow is assumed to enter and leave the network with a normal velocity, so $\mathbf{a} = \mathbf{0}$. Thus in this example the boundary conditions for the velocity are homogeneous.

EXAMPLE 2. **An obstacle in a pipe.** In this example the flow takes place in a cylindrical pipe but a fixed obstacle is lying in the flow. Here Ω denotes the volume occupied by the fluid, i.e., the cylinder minus the obstacle; Γ_1 consists of the lateral surface of the cylinder and of the boundary of the obstacle; Γ_2 has two parts: the in-flow part Γ_{21} and the out-flow part Γ_{22}; Γ_3 is empty. The no-slip boundary condition is assumed on Γ_1, so $\mathbf{u}_0 = \mathbf{0}$; the in-flow and out-flow are assumed to be normal to the boundary, so $\mathbf{a} = \mathbf{0}$. In this example the boundary conditions on the velocity are again homogeneous.

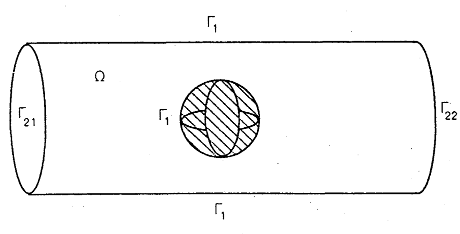

Figure 0.2

EXAMPLE 3. **Modeling the boundary conditions in the large for a flow around and obstacle.** This example models an infinite flow around an obstacle by considering a large box, the boundary of which plays the role of an approximation of infinity; this box can be a big ball, see Figure 0.3. The domain Ω consists of this box minus the obstacle; Γ_1 consists of the boundary of the obstacle which is assumed to be at rest ($\mathbf{u}_0 = \mathbf{0}$); Γ_2 consists of the boundary of the box, where p_0 is chosen to be a given constant, and $\mathbf{a}$ to be the velocity $\mathbf{u}_\infty$ of the fluid at infinity (in general $\mathbf{u}_\infty$ is a constant vector).

It is worthwhile noting that the first two examples are generalizations of Poiseuille flow through a pipe, which models the flow of a viscous incompressible fluid in a finite cylinder: Example 1 deals with the case of several in-flow and out-flow surfaces; Example 2 deals with the case where an obstacle is lying in the flow.

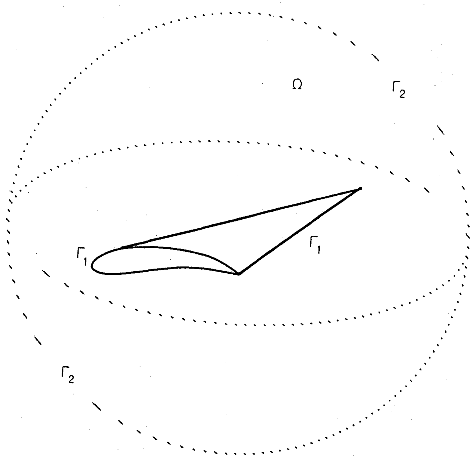

Figure 0.3

The present contribution to the study of problems (0.1), (0.3) (and (0.2), (0.4)) is twofold. Firstly, variational formulations of both problems are given, and these are proved to be equivalent to the boundary value problems considered. Secondly, existence and uniqueness are proved for these variational problems.

The equivalence between the boundary value problems considered above and the variational formulations introduced below in this paper is proved in the classical way, i.e., using test functions and integration by parts. The existence and uniqueness results proved below strongly depend upon the geometry of the problem, i.e., on Ω and on the partition $\{\Gamma_1, \Gamma_2, \Gamma_3\}$ of the boundary. Correspondingly, two "abstract" situations are considered, namely the cases where the bilinear form occurring in the variational formulations is either "V-elliptic" or "V_0-elliptic with V_1 kernel" (see the definitions in Section 1.2). The first case ($a(\cdot,\cdot)$ V-elliptic) is well illustrated by the examples for which Γ_1 is non-empty. The second case ($a(\cdot,\cdot)$ V_0-elliptic with V_1 kernel) is illustrated either by the case where $\Gamma \equiv \Gamma_2$ or by the case where Γ_1 is empty but Ω is simply connected (see Appendix A for details).

The Stokes problem (0.1), (0.3) is studied first, the Navier-Stokes problem being postponed to §2. Only the three dimensional case is considered here, but all the results apply to the two dimensional case by considering functions which do not depend on the third variable and vectors whose third component is zero.

Consider first the Stokes problem. When the bilinear form $a(\cdot,\cdot)$ is assumed to be V-elliptic, an existence and uniqueness result for the velocity is proved using the Lax-Milgram Lemma. When $a(\cdot,\cdot)$ is V_0-elliptic with V_1 kernel, a compatibility condition has to be satisfied by the data in order that a solution exists. When this compatibility condition holds, there is an infinite number of solutions for the velocity; the solution is unique if the fluxes of the velocity on the connected components of Γ_2 are prescribed; the pressure is still uniquely determined up to an additive constant. These results are proved using a variation of Lax-Milgram Lemma.

Consider now the Navier-Stokes problem (0.2), (0.4). Assuming that $a(\cdot,\cdot)$ is V-elliptic the existence and uniqueness of a small solution is proved using the fixed point Theorem of Banach when the data of the problem are small. For possibly large data the existence of a solution is proved using the Galerkin's approximation method for the case where the velocity satisfies homogeneous boundary conditions (i.e., $\mathbf{u}_0 = \mathbf{0}$, $\mathbf{a} = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$), the boundary conditions on the pressure and vorticity being possibly non-homogeneous. If the data $\mathbf{f}$, p_0 and $\mathbf{h}$ of the problem are small compared to the viscosity ν, this solution is unique.

A variation of the boundary conditions (0.3) and (0.4), which allows one to prescribe the fluxes of the velocity on the connected components of Γ_2 is also studied. When $a(\cdot,\cdot)$ is V-elliptic or V_0-elliptic with V_1 kernel, an existence and uniqueness result is proved for the Stokes problem with these boundary conditions, and an existence and uniqueness result in the small is proved for the Navier-Stokes problem when the data are small.

Besides these two main aims (i.e., the variational formulations and the existence and uniqueness results) the paper presents another result concerning problem (0.1), (0.3). Namely, the fluxes of the velocity on the connected components of Γ_2 are proved to be determined directly from the data using particular functions $\omega_1, \omega_2, \dots, \omega_{r-1}$ (where r denotes the number of connected components of Γ_2); these functions depend on the geometry of the problem. Finally, for a numerical study of this kind of boundary-value problems the reader is referred to C. Conca, C. Parés, O. Pironneau and M. Thiriet [Co-Pa-Pi-Th].

1. The Stokes equations with boundary conditions involving the pressure

In this section we will study the Stokes equations with the boundary conditions mentioned in the Introduction. Our aim is to give a variational formulation of this problem and prove a theorem of existence and uniqueness.

1.1. Formulation of the Stokes problem. Let Ω be a connected (not necessarily a simply connected) open set of $\boldsymbol{R}^3$, with a locally Lipschitz boundary Γ (see J. Nečas [Ne] Chapter 1, Section 1.1.3, pp.14–15). We assume that there exist three smooth open subsets of Γ, which we denote by $\{\Gamma_1, \Gamma_2, \Gamma_3\}$ such that:

$$\Gamma_1, \Gamma_2, \Gamma_3 \text{ are of class } \mathcal{C}^{1,1} \tag{1.1a}$$

$$\Gamma_i \cap \Gamma_j = \emptyset \quad \forall i, j = 1, 2, 3; \quad i \neq j \tag{1.1b}$$

$$\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3. \tag{1.1c}$$

We denote by $\Gamma_{21}, \ldots, \Gamma_{2r}$ the connected components of Γ_2, and $\mathbf{n}$ will designate the unit outward normal to Γ. Since Γ is locally Lipschitz, the normal $\mathbf{n}$ is defined almost everywhere on Γ. Moreover, $\mathbf{n}$ is a Lipschitz-function on each connected component of Γ.

In this section, we are interested in studying the (stationary) Stokes equations:

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \tag{1.2a}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.2b}$$

with the following boundary conditions:

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on } \Gamma_1 \tag{1.2c}$$

$$\mathbf{u} \times \mathbf{n} = \mathbf{a} \times \mathbf{n} \quad \text{on } \Gamma_2 \tag{1.2d}$$

$$p = p_0 \quad \text{on } \Gamma_2 \tag{1.2e}$$

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n} \quad \text{on } \Gamma_3 \tag{1.2f}$$

$$(\nabla \times \mathbf{u}) \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on } \Gamma_3. \tag{1.2g}$$

From a physical point of view, we can imagine that the domain Ω is occupied by a Newtonnian viscous incompressible fluid of kinematic viscosity ν. The fluid is subject to the action of an external force of specific density $\mathbf{f}$. The unknowns of problem (1.2) are the velocity $\mathbf{u}$ of the fluid and its pressure p. The constant ν and the functions $\mathbf{f}$, $\mathbf{u}_0$, $\mathbf{a}$, p_0, $\mathbf{b}$ and $\mathbf{h}$ are the data of the problem. We shall assume that they satisfy the following conditions:

$$\nu > 0 \tag{1.3}$$

$$\mathbf{f} \in L^2(\Omega)^3, \quad \nabla \cdot \mathbf{f} \in L^2(\Omega), \quad \nabla \times \mathbf{f} \in L^2(\Omega)^3. \tag{1.4}$$

There exists a function $\mathbf{U}_0 \in H^1(\Omega)^3$ such that:

$$\nabla \cdot \mathbf{U}_0 = 0 \quad \text{in } \Omega \tag{1.5a}$$

$$\mathbf{U}_0 = \mathbf{u}_0 \quad \text{on } \Gamma_1 \tag{1.5b}$$

$$\mathbf{U}_0 \times \mathbf{n} = \mathbf{a} \times \mathbf{n} \quad \text{on } \Gamma_2 \tag{1.5c}$$

$$\mathbf{U}_0 \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n} \quad \text{on } \Gamma_3 \tag{1.5d}$$

$$p_0 \in H^{-1/2}(\Gamma_2)/\boldsymbol{R} \tag{1.6}$$

$$\mathbf{h} \in H^{-1/2}(\Gamma_3)^3. \tag{1.7}$$

It can be noticed that condition (1.5) is an assumption on the regularity and on the continuity of the functions $\mathbf{u}_0$, $\mathbf{a}$ and $\mathbf{b}$. Given $\mathbf{u}_0 \in H^{1/2}(\Gamma_1)^3$, $\mathbf{a} \in H^{1/2}(\Gamma_2)^3$ and $\mathbf{b} \in H^{1/2}(\Gamma_3)^3$, the existence of a function $\mathbf{U}_0$ of $H^1(\Omega)^3$ verifying (1.5) is not evident. The existence of $\mathbf{U}_0$ will not be discussed here; we will simply study problem (1.2) assuming that there exists $\mathbf{U}_0$ verifying (1.5).

The assumption (1.6) means that p_0 belongs to $H^{-1/2}(\Gamma_2)$ and that p_0 is defined up to an additive constant. The restriction of p_0 to the i-th connected component of Γ_2 will be denoted by p_{0i}, i.e.,

$$p_{0i} = p_0|_{\Gamma_{2i}}, \quad i = 1, \dots, r. \tag{1.8}$$

1.2. Functional framework. Let us introduce the following functional spaces:

$$V = \{\mathbf{v} \in H^1(\Omega)^3 \mid \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1,\ \mathbf{v} \times \mathbf{n} = 0 \text{ on } \Gamma_2, \\ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_3\} \tag{1.9}$$

and

$$V_0 = \left\{\mathbf{v}_0 \in V \mid \int_{\Gamma_{2i}} \mathbf{v}_0 \cdot \mathbf{n} ds = 0 \quad \forall i = 1, \dots r\right\}, \tag{1.10}$$

equipped with the scalar product of $H^1(\Omega)^3$. It is clear that V and V_0 are closed subspaces of $H^1(\Omega)^3$, so they are Hilbert spaces for the scalar product of $H^1(\Omega)^3$.

Let $a(\cdot,\cdot) : H^1(\Omega)^3 \times H^1(\Omega)^3 \to \boldsymbol{R}$ be the following bilinear continuous form on $H^1(\Omega)^3$:

$$a(\mathbf{u}, \mathbf{v}) = \nu \int_\Omega (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) dx \quad \forall \mathbf{u}, \mathbf{v} \in H^1(\Omega)^3. \tag{1.11}$$

Next, we introduce the subspace of V defined by:

$$V_1 = \{\mathbf{w} \in V \mid a(\mathbf{w}, \mathbf{v}_0) = 0 \quad \forall \mathbf{v}_0 \in V_0\}. \tag{1.12}$$

The bilinear form $a(\cdot,\cdot)$ and the spaces V, V_0, and V_1 play a crucial role in the variational formulation of problem (1.2) and in its solution. We shall then study some properties of these spaces and of this bilinear form.

DEFINITION 1.1. We shall say that the bilinear form $a(\cdot,\cdot)$ is *V-elliptic* if there exists a strictly positive constant $\alpha = \alpha(\Omega, \Gamma_1, \Gamma_2, \Gamma_3) > 0$ such that:

$$a(\mathbf{u}, \mathbf{v}) = \nu \int_\Omega |\nabla \times \mathbf{v}|^2 dx \geq \alpha\nu \|\mathbf{v}\|^2_{H^1(\Omega)^3} \quad \forall \mathbf{v} \in V. \qquad \blacksquare \tag{1.13}$$

DEFINITION 1.2. We shall say that the bilinear form $a(\cdot,\cdot)$ is *V_0-elliptic with V_1 kernel* if the following conditions hold:

$a(\cdot,\cdot)$ is V_0-elliptic, i.e., there exists $\alpha_0 = \alpha_0(\Omega, \Gamma_1, \Gamma_2, \Gamma_3) > 0$ such that:

$$a(\mathbf{v}_0, \mathbf{v}_0) = \nu \int_\Omega |\nabla \times \mathbf{v}_0|^2 dx \geq \alpha_0 \nu \|\mathbf{v}_0\|^2_{H^1(\Omega)^3} \quad \forall \mathbf{v}_0 \in V_0 \tag{1.14a}$$

$$V_1 = N(a), \tag{1.14b}$$

where, in (1.14b), $N(a)$ denotes the kernel of $a(\cdot,\cdot)$ in V, i.e.,

$$N(a) = \{\mathbf{w} \in V \mid a(\mathbf{w},\mathbf{v}) = 0 \quad \forall \mathbf{v} \in H^1(\Omega)^3\}. \qquad \blacksquare \tag{1.15a}$$

Let us remark that if either Γ_2 is empty or Γ_2 has only one connected component, then V_0 is equal to V and both notions, V-ellipticity and V_0-ellipticity, coincide. It is also worthwhile to note that if $\mathbf{w}$ is a function in the kernel of $a(\cdot,\cdot)$, then $a(\mathbf{w},\mathbf{w}) = 0$, that is, $\mathbf{w}$ represents an irrotational motion ($\nabla \times \mathbf{w} = \mathbf{0}$ in Ω). Reciprocally, it is obvious that if a function $\mathbf{w}$ verifies $\nabla \times \mathbf{w} = \mathbf{0}$ in Ω, then $\mathbf{w}$ belongs to the kernel of $a(\cdot,\cdot)$. Thus the kernel of $a(\cdot,\cdot)$ in V is merely the following subspace of V,

$$N(a) = \{\mathbf{w} \in V \mid \nabla \times \mathbf{w} = \mathbf{0} \quad \text{in} \quad \Omega\}. \tag{1.15b}$$

On the other hand, we have:

LEMMA 1.3. *If $a(\cdot,\cdot)$ is V-elliptic or if it is V_0-elliptic with V_1 kernel, then V_1 is a topological complement of V_0 in V (i.e., $V = V_0 \oplus V_1$ and V_1 is closed in V) and there exists a basis $\{\omega_1, \ldots, \omega_{r-1}\}$ of V_1, such that*

$$\int_{\Gamma_{2j}} \omega_i \cdot \mathbf{n} ds = \delta_{ij} \quad \forall i,j = 1, \ldots, r-1. \tag{1.16a}$$

$$\int_{\Gamma_{2r}} \omega_i \cdot \mathbf{n} ds = -1 \quad \forall i = 1, \ldots, r-1. \tag{1.16b}$$

PROOF. First, let us prove that V_1 is a topological complement of V_0 in V. Since $a(\cdot,\cdot)$ is continuous in V, V_1 is a closed subspace of V. Let us then prove that $V = V_0 \oplus V_1$. It is clear that $V_0 \cap V_1 = \{0\}$, because if a function $\mathbf{w}$ belongs to $V_0 \cap V_1$, then $a(\mathbf{w},\mathbf{w}) = 0$. Thus $\mathbf{w} = \mathbf{0}$, since $a(\cdot,\cdot)$ is V_0-elliptic. On the other hand, let $\mathbf{v} \in V$ be given. We are going to prove that $\mathbf{v}$ can be split into the following form:

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1,$$

with $\mathbf{v}_0 \in V_0$ and $\mathbf{v}_1 \in V_1$. Effectively, since $a(\cdot,\cdot)$ is V_0-elliptic, this can be done defining $\mathbf{v}_0$ as the unique solution of the following problem:

$$\begin{gathered} \text{Find } \mathbf{v}_0 \in V_0 \text{ such that} \\ a(\mathbf{v}_0,\mathbf{z}) = a(\mathbf{v},\mathbf{z}) \quad \forall \mathbf{z} \in V_0, \end{gathered}$$

and then by setting

$$\mathbf{v}_1 = \mathbf{v} - \mathbf{v}_0.$$

Using the definition of V_1 (see (1.12)) and the definition of $\mathbf{v}_0$ we deduce that $a(\mathbf{v}_1,\mathbf{z}) = 0 \; \forall \mathbf{z} \in V_0$ and therefore, $\mathbf{v}_1 \in V_1$. This ends the proof of the fact that V_1 is a topological complement of V_0 in V.

Now, let us prove that V_1 is a finite-dimensional space. To this end, let us define $\Phi : V_1 \to \boldsymbol{R}^r$ by:

$$\Phi(\mathbf{w}) = \left(\int_{\Gamma_{21}} \mathbf{w} \cdot \mathbf{n} ds, \dots , \int_{\Gamma_{2r}} \mathbf{w} \cdot \mathbf{n} ds \right).$$

If $\mathbf{w} \in V_1$ and $\Phi(\mathbf{w}) = 0$, then $\mathbf{w}$ belongs to V_0 and so, $\mathbf{w} = \mathbf{0}$. Thus Φ is an injective mapping. On the other hand, since V_1 consists of divergence-free functions in Ω, we have:

$$\sum_{i=1}^{r} \int_{\Gamma_{2i}} \mathbf{w} \cdot \mathbf{n} ds = \int_{\Gamma} \mathbf{w} \cdot \mathbf{n} ds = 0 \quad \forall \mathbf{w} \in V_1.$$

The dimension of $\Phi(V_1)$ is therefore less than or equal to $(r-1)$. Since Φ is injective,

$$\dim V_1 \leq r - 1. \tag{1.17}$$

Let $z_1, \dots , z_{r-1}$ be $(r - 1)$ functions of V verifying (1.16), i.e.,

$$\int_{\Gamma_{2j}} \mathbf{z}_i \cdot \mathbf{n} ds = \delta_{ij} \quad \forall i, j = 1, \dots , r - 1, \tag{1.18a}$$

$$\int_{\Gamma_{2r}} \mathbf{z}_i \cdot \mathbf{n} ds = -1 \quad \forall i = 1, \dots , r - 1. \tag{1.18b}$$

To prove the existence of such functions it suffices to observe that the trace map $\mathbf{v} \to \mathbf{v}|_\Gamma$ is continuous from the space of divergence-free functions of $H^1(\Omega)^3$ onto the space of all functions in $H^{1/2}(\Gamma)^3$ whose normal component has a zero mean-value on Γ, and this is true because the boundary of Ω is locally Lipschitz (see V. Girault & P.A. Raviart [Gi-Ra] Lemma I.2.2). For all $i = 1, \dots , r - 1$, we associate to $\mathbf{z}_i$ the function ω_{i0} of V_0, which we define as the unique solution of:

$$\text{Find } \omega_{i0} \in V_0 \tag{1.19a}$$

$$a(\omega_{i0}, \mathbf{v}_0) = -a(\mathbf{z}_i, \mathbf{v}_0) \quad \forall \mathbf{v}_0 \in V_0, \tag{1.19b}$$

then we put

$$\omega_i = \omega_{i0} + \mathbf{z}_i, \quad i = 1, \dots , r - 1. \tag{1.20}$$

Using (1.18) and (1.19), it can be easily verified that the functions ω_i belong to V_1 and that they verify (1.16). On the other hand, (1.16a) implies that they are linearly independent. Thus, it follows from (1.17) that the dimension of V_1 is exactly equal to $(r - 1)$ and that $\{\omega_1, \dots , \omega_{r-1}\}$ is a basis of V_1. This completes the proof of Lemma 1.3. ■

1.3. Model examples. Throughout this paper it will be assumed as a basic hypothesis that one of the two following conditions is verified:

$$a(\cdot,\cdot) \text{ is } V\text{-elliptic} \tag{1.21a}$$

or

$$a(\cdot,\cdot) \text{ is } V_0\text{-elliptic with } V_1 \text{ kernel.} \tag{1.21b}$$

The fact that these assumptions are verified or not in a concrete example depends on the geometry of the problem. In this section, we present three examples where one of the assumptions (1.21a) or the other is verified. The proofs will be given in the Appendix A.

Let Ω be a given region of $\boldsymbol{R}^3$ (with a locally Lipschitz boundary) and let $\{\Gamma_1, \Gamma_2, \Gamma_3\}$ be a partition of its boundary Γ. It is interesting to remark that $a(\cdot,\cdot)$ could be neither V-elliptic nor V_0-elliptic with V_1 kernel. This is the case if, for instance, Ω is not simply-connected, Γ is of class $\mathcal{C}^2$ and Γ is equal to Γ_3. Effectively, in this case, it has been proved by C. Foias & R. Temam [Fo-Te] Lemmas 1.1, 1.2, 1.3, that the dimension of the subspace of all functions $\mathbf{v}$ in V such that $\nabla \times \mathbf{v} = \mathbf{0}$ in Ω is strictly positive (moreover, they proved that it is equal to the number of cuts which should be made to Ω in order to make it simply-connected); the bilinear form cannot thus be V-elliptic. In this case, $a(\cdot,\cdot)$ is neither V_0-elliptic with V_1 kernel, because $\Gamma = \Gamma_3$ and the spaces V and V_0 are the same.

In the three model examples we will assume that Ω or Γ verify one of the following conditions:

$$\Gamma \text{ is of class } \mathcal{C}^{1,1} \tag{1.22a}$$

or

$$\Omega \text{ is a convex polyhedron,} \tag{1.22b}$$

and that the partition $\{\Gamma_1, \Gamma_2, \Gamma_3\}$ of Γ satisfies:

$$\Gamma_2 \text{ and } \Gamma_3 \text{ have no common boundary.} \tag{1.23}$$

The three model examples are:

$$\Gamma_1 \neq \emptyset \tag{1.24a}$$

$$\Gamma = \Gamma_2 \tag{1.24b}$$

$$\Gamma_1 = \emptyset \text{ and } \Omega \text{ simply-connected.} \tag{1.24c}$$

The following Lemma, whose proof is given in Appendix A, allows us to link these three model examples with the hypothesis of coercivity (1.21a) and (1.21b):

LEMMA 1.4. *Assume that Ω and Γ verify one of the conditions* (1.22a) *or* (1.22b) *and that the partition $\{\Gamma_1, \Gamma_2, \Gamma_3\}$ of Γ verifies* (1.23). *If $\Gamma_1 \neq \emptyset$ (case*

(1.24a)), *then* $a(\cdot,\cdot)$ *is* V*-elliptic. If* $\Gamma = \Gamma_2$ (*case* (1.24b)) *or if* $\Gamma_1 = \emptyset$ *and* Ω *is simply-connected* (*case* (1.24c)), *then* $a(\cdot,\cdot)$ *is* V_0*-elliptic with* V_1 *kernel.* ■

Though the three examples of the Introduction verify neither (1.22a) nor (1.22b), it can be proved (see §A.2) that the bilinear form $a(\cdot,\cdot)$ is V-elliptic in all these examples; this follows from the fact that $\Gamma_1 \neq \emptyset$ and from a very simple property of extension. Thus Lemma 1.4 remains valid for the three examples of the Introduction.

1.4. Variational formulation of the problem. Existence and uniqueness results. Let us consider the following variational problem:

$$\text{Find } \mathbf{u} \in H^1(\Omega)^3 \text{ such that} \tag{1.25a}$$

$$(\mathbf{u} - \mathbf{U}_0) \in V \tag{1.25b}$$

$$a(\mathbf{u},\mathbf{v}) = L(\mathbf{v}) \quad \forall \mathbf{v} \in V, \tag{1.25c}$$

where V is the space defined by (1.9), $a(\cdot,\cdot)$ by (1.11), and $L(\cdot)$ is

$$L(\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}dx + \nu\langle \mathbf{h}\times\mathbf{n},\mathbf{v}\rangle_{\Gamma_3} - \langle p_0, \mathbf{v}\cdot\mathbf{n}\rangle_{\Gamma_2} \quad \forall \mathbf{v} \in H^1(\Omega)^3 \tag{1.26}$$

where, in (1.26) and in what follows, if γ is a regular open subset of Γ, we will denote by $\langle\cdot,\cdot\rangle$ the duality bracket between $H^{-1/2}(\gamma)^n$ and $H^{1/2}(\gamma)^n$, $n = 1$ or 3.

Taking into account the definitions of $a(\cdot,\cdot)$ and $L(\cdot)$, problem (1.25) becomes:

$$\text{Find } \mathbf{u} \in H^1(\Omega)^3 \tag{1.27a}$$

$$(\mathbf{u} - \mathbf{U}_0) \in V \tag{1.27b}$$

$$\nu\int_\Omega \nabla\times\mathbf{u}\cdot\nabla\times\mathbf{v} = \int_\Omega \mathbf{f}\cdot\mathbf{v} + \nu\langle \mathbf{h}\times\mathbf{n},\mathbf{v}\rangle_{\Gamma_3} - \langle p_0, \mathbf{v}\cdot\mathbf{n}\rangle_{\Gamma_2} \; \forall \mathbf{v} \in V. \tag{1.27c}$$

Since $L(\cdot)$ is a linear form which is continuous in V, the Lax-Milgram Lemma yields:

THEOREM 1.5. *If the bilinear form* $a(\cdot,\cdot)$ *is* V*-elliptic, then the variational problem* (1.25) *has one and only one solution.* ■

Let us recall that in the model example (1.24a) (i.e., the case where $\Gamma_1 \neq \emptyset$), the bilinear form $a(\cdot,\cdot)$ is V-elliptic; we will see in §1.5.1 that the differential interpretation of (1.27) is just the boundary-value problem (1.2).

But the bilinear form $a(\cdot,\cdot)$ is not always V-elliptic. Having seen the model examples (1.24b) and (1.24c), we are led to state in a natural way the following variant of the Lax-Milgram Lemma:

THEOREM 1.6. *Assume that the bilinear form $a(\cdot,\cdot)$ is V_0-elliptic with V_1 kernel and that the linear form $L(\cdot)$ verifies the following condition:*

$$L(\mathbf{w}) = 0 \quad \forall \mathbf{w} \in V_1. \tag{1.28}$$

Let $\mathbf{y}_0$ be the unique solution of the following variational problem:

$$\textit{Find } \mathbf{y}_0 \in H^1(\Omega)^3 \textit{ such that} \tag{1.29a}$$
$$(\mathbf{y}_0 - \mathbf{U}_0) \in V_0 \tag{1.29b}$$
$$a(\mathbf{y}_0, \mathbf{v}_0) = L(\mathbf{v}_0) \quad \forall \mathbf{v}_0 \in V_0. \tag{1.29c}$$

Then for all $\mathbf{w}$ in V_1, the function $\mathbf{u} = \mathbf{y}_0 + \mathbf{w}$ is a solution of (1.25), *and all solutions of problem* (1.25) *can be split into this form. Reciprocally, if problem* (1.25) *has at least one solution, then $L(\cdot)$ verifies the compatibility condition* (1.28). ■

PROOF OF THEOREM 1.6. We begin by showing that $\mathbf{u} = \mathbf{y}_0 + \mathbf{w}$, where $\mathbf{y}_0$ is a solution of (1.29) and $\mathbf{w}$ belongs to V_1. It is obvious that $\mathbf{u} \in H^1(\Omega)^3$ and $(\mathbf{u} - \mathbf{U}_0) \in V$. Let us prove that $\mathbf{u}$ verifies (1.25c). Let $\mathbf{v}$ be any function of V. Using Lemma 1.3, $\mathbf{v}$ can be broken down as follows:

$$\mathbf{v} = \mathbf{z}_0 + \mathbf{z}_1,$$

with $\mathbf{z}_0 \in V_0$, $\mathbf{z}_1 \in V_1$. Since V_1 is the kernel of $a(\cdot,\cdot)$, we have

$$a(\mathbf{u}, \mathbf{v}) = a(\mathbf{y}_0, \mathbf{z}_0).$$

Thus (1.28) and (1.29) imply that

$$a(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}) \quad \forall \mathbf{v} \in V,$$

which finishes the proof of the fact that $\mathbf{u}$ is a solution of (1.25).

Now, we prove that every solution of (1.25) can be written as a sum of the form: $(\mathbf{y}_0 + \mathbf{w})$, with $\mathbf{w} \in V_1$. To do this, let z be a solution of (1.25). By taking the difference between (1.25c) and (1.29c), we deduce:

$$a(\mathbf{z} - \mathbf{y}_0, \mathbf{v}_0) = 0 \quad \forall \mathbf{v}_0 \in V_0.$$

Thus $(\mathbf{z} - \mathbf{y}_0)$ belongs to V_1, and we can conclude that $\mathbf{z} = \mathbf{y}_0 + \mathbf{w}$, with $\mathbf{w} \in V_1$.

Conversely, if problem (1.25) admits a solution $\mathbf{u}$, for $\mathbf{v} = \mathbf{w}$, with $\mathbf{w} \in V_1$, as the test function in (1.25c), it follows easily from the fact that V_1 is the kernel of $a(\cdot,\cdot)$ that $L(\cdot)$ verifies condition (1.28). Theorem 1.6 is therefore proved. ■

According to Lemma 1.4, the model examples (1.24b) ($\Gamma = \Gamma_2$) and (1.24c) ($\Gamma_1 = \emptyset$ and Ω simply-connected) are two situations where the above variant of the Lax-Milgram Lemma applies. We will see in §1.5.2 how the compatibility condition (1.28) and the solution of (1.27) can be interpreted.

1.5. Interpretation of the variational problem and properties of the solution. This section is divided into 3 parts. Section §1.5.1 is devoted to the proof of the equivalence between (1.27) and (1.2) in a sense that we shall define below. In §1.5.2, we use the basis of V_1 (see Lemma 1.3) in order to study the fluxes of the solution of (1.27) through the components Γ_{2j} of Γ_2. Different conclusions are reached according to the ellipticity assumptions fulfilled by $a(\cdot,\cdot)$. If $a(\cdot,\cdot)$ is V-elliptic, we shall derive a formula which yields directly, from the data of the problem, the fluxes on each component of Γ_2. If $a(\cdot,\cdot)$ is V_0-elliptic with V_1 kernel, this formula will be none other than the necessary and sufficient condition for the existence of a solution of the variational problem. In §1.5.3, we introduce a new boundary-value problem which is very similar to (1.2), and we prove a result of existence and uniqueness.

1.5.1. Equivalence between the boundary-value problem and the variational problem. We shall begin by proving:

THEOREM 1.7. *If* $\mathbf{u} \in \mathcal{C}^2(\overline{\Omega})$ *and* $p \in \mathcal{C}^1(\overline{\Omega})$ *are classical solutions of the boundary-value problem* (1.2), *then* $\mathbf{u}$ *is a solution of the variational problem* (1.27).

PROOF. It is done in the usual way. Multiplying equation (1.2a) by $\mathbf{w}$ in V, integrating by parts in Ω, and using (1.2b) and the identity:

$$-\Delta\mathbf{u} = \nabla \times (\nabla \times \mathbf{u}) - \nabla(\nabla \cdot \mathbf{u}), \tag{1.30}$$

we obtain:

$$\nu \int_\Omega (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) dx = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx + \nu \int_\Gamma ((\nabla \times \mathbf{u}) \times \mathbf{n}) \cdot \mathbf{v} ds - \int_\Gamma p\mathbf{v} \cdot \mathbf{n} ds.$$

Thus, taking into account the boundary conditions that verify the functions of V on Γ, we have:

$$\nu \int_\Omega (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) dx = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx + \nu \int_{\Gamma_3} ((\nabla \times \mathbf{u}) \times \mathbf{n}) \cdot \mathbf{v} ds - \int_{\Gamma_2} p\mathbf{v} \cdot \mathbf{n} ds$$

which, in view of the fact that $p = p_0$ on Γ_2 and $(\nabla \times \mathbf{u}) \times \mathbf{n} = \mathbf{h} \times \mathbf{n}$ on Γ_3, implies that $\mathbf{u}$ verifies (1.27c). Furthermore, using the fact that $\mathbf{u}$ is of class $\mathcal{C}^2(\overline{\Omega})$ (which implies that $\mathbf{u} \in H^1(\Omega)^3$) we deduce from (1.2b), (1.2c), (1.2d), (1.2f) and (1.5) that $(\mathbf{u} - \mathbf{U}_0)$ belongs of V. Thus $\mathbf{u}$ is a solution of (1.27). Theorem 1.7 is therefore proved. ■

Reciprocally, we have

THEOREM 1.8. *Assume that the assumptions* (1.4), (1.5), (1.6) *and* (1.7) *are fulfilled. Let* $\mathbf{u}$ *be a solution of the variational problem* (1.27). *Then* $(\nabla \times \mathbf{u})$ *belongs to* $H(\Delta, \Omega)^3$ ($H(\Delta, \Omega) \equiv \{q \in L^2(\Omega) \mid \Delta q \in L^2(\Omega)\}$; *the functions of* $H(\Delta, \Omega)$ *have traces on* Γ *which belong to* $H^{-1/2}(\Gamma)$) *and there exists* $p \in H(\Delta, \Omega)/\boldsymbol{R}$ *such that* $\mathbf{u}$ *and* p *are solutions of the boundary-value problem* (1.2) *in the following sense:*

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \text{ in the sense of distributions in } \Omega \tag{1.31a}$$

$$\nabla \cdot \mathbf{u} = 0 \text{ in the sense of distributions in } \Omega \tag{1.31b}$$

$$\mathbf{u} \text{ satisfies (1.2c), (1.2d), (1.2f) in the sense of } H^1(\Omega)^3 \tag{1.31c}$$

$$p, (\nabla \times \mathbf{u}) \text{ satisfy (1.2e), (1.2g) in the following sense : } \forall \mathbf{v} \in V, \tag{1.31d}$$

$$\begin{aligned}\int_\Omega (-\nu\Delta\mathbf{u} + p) \cdot \mathbf{v} dx - \nu \int_\Omega (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) dx \\ = -\nu\langle \mathbf{h} \times \mathbf{n}, \mathbf{v}\rangle_{\Gamma_3} + \langle p_0, \mathbf{v} \cdot \mathbf{n}\rangle_{\Gamma_2}.\end{aligned}$$

Moreover, if $\nabla \times (\nabla \times \mathbf{u}) \in L^2(\Omega)^3$, *then* (1.31d) *implies that:*

$$p = p_0 \text{ on } \Gamma_2 \text{ in the sense of } H^{-1/2}(\Gamma_2)/\boldsymbol{R} \tag{1.32a}$$

$$(\nabla \times \mathbf{u}) \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \text{ on } \Gamma_3 \text{ in the sense of } H^{-1/2}(\Gamma_2)^3. \tag{1.32b}$$

PROOF. Let $\mathbf{u}$ be a solution of problem (1.27). The fact that $(\mathbf{u} - \mathbf{U}_0)$ belongs to V and that $\mathbf{U}_0$ verifies (1.5) implies that $\mathbf{u}$ satisfies equation (1.2b) in the sense of distributions in Ω, and it also implies that $\mathbf{u}$ verifies the boundary conditions (1.2c), (1.2d), (1.2f) in the sense of the traces of functions of $H^1(\Omega)^3$. This proves (1.31b) and (1.31c).

To prove (1.31a), let us take a divergence-free function $\mathbf{v}$ of $\mathcal{D}(\Omega)^3$ as the test function in (1.27c). Using the definition of distribution derivative, we have:

$$\langle \nu\nabla \times (\nabla \times \mathbf{u}), \mathbf{v}\rangle = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx \quad \forall \mathbf{v} \in \mathcal{D}(\Omega)^3; \ \nabla \cdot \mathbf{v} = 0 \quad \text{in} \quad \Omega,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality bracket between $(\mathcal{D}(\Omega)^3)'$ and $\mathcal{D}(\Omega)^3$. Since the boundary Γ of Ω is locally Lipschitz, it follows from Theorem I.2.3 of V. Girault & P.A. Raviart [Gi-Ra] p.25 (see also R. Temam [Te] Remark I.1.9, p.19) that there exists (a class of functions) $p \in L^2(\Omega)/\boldsymbol{R}$ such that:

$$\nu\nabla \times (\nabla \times \mathbf{u}) + \nabla p = \mathbf{f} \tag{1.33}$$

in the sense of distributions in Ω. This proves (1.31a).

Moreover, applying divergence operator to equation (1.33), we obtain:

$$\Delta p = \nabla \cdot \mathbf{f} \text{ in the sense of distributions in } \Omega. \tag{1.34a}$$

On the other hand, applying rotational operator on both sides of (1.31a), we have:

$$\nu\Delta(\nabla \times \mathbf{u}) = -\nabla \times \mathbf{f} \text{ in the sense of distributions in } \Omega. \tag{1.34b}$$

But $\mathbf{f}$ verifies (1.4), and so, (1.34a) implies that $p \in H(\Delta, \Omega)/\boldsymbol{R}$ and (1.34b) implies that $(\nabla \times \mathbf{u}) \in H(\Delta, \Omega)^3$. This proves the regularity properties of p and of $(\nabla \times \mathbf{u})$ claimed in Theorem 1.8. Moreover, multiplying (1.33) by $\mathbf{v}$ in V, integrating by parts in Ω, and using (1.27c) we are led to:

$$\begin{aligned}\int_\Omega (\nu\nabla \times (\nabla \times \mathbf{u}) + \nabla p) \cdot \mathbf{v}dx - \nu \int_\Omega (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v})dx \\ = -\nu\langle \mathbf{h} \times \mathbf{n}, \mathbf{v}\rangle_{\Gamma_3} + \langle p_0, \mathbf{v} \cdot \mathbf{n}\rangle_{\Gamma_2}\end{aligned} \tag{1.35}$$

for all $\mathbf{v} \in V$, which proves (1.31d).

The boundary conditions (1.2e) and (1.2g) are implicitly contained in (1.35). To interpret them we must be able to integrate by parts the first term on the left-hand side of (1.35). This is not possible a priori because of the lack of regularity of $\mathbf{u}$ and p. But if we assume that $\nabla \times (\nabla \times \mathbf{u}) \in L^2(\Omega)^3$, using (1.4) and (1.33), we have ∇p in $L^2(\Omega)^3$ (which implies in fact that p belongs to $H^1(\Omega)$), and we can therefore integrate this term by parts. We obtain:

$$-\nu\langle(\nabla \times \mathbf{u}) \times \mathbf{n}, \mathbf{v}\rangle_{\Gamma_3} + \langle p, \mathbf{v} \cdot \mathbf{n}\rangle_{\Gamma_2} = -\nu\langle \mathbf{h} \times \mathbf{n}, \mathbf{v}\rangle_{\Gamma_3} + \langle p_0, \mathbf{v} \cdot \mathbf{n}\rangle_{\Gamma_2} \quad \forall \mathbf{v} \in V.$$

Taking in this expression test functions with compact support in Γ_3, we get (1.32b). Next, we get (1.32a) by taking test functions with compact support in Γ_2 and using the fact that the average of the normal trace of the functions of V is zero on Γ_2. This completes the proof of Theorem 1.8. ■

1.5.2. Computation of the fluxes of the velocity on the connected components of Γ_2. In what follows, we shall denote by F_{2j}^0 the flux of $\mathbf{U}_0$ through Γ_{2j}, i.e.,

$$F_{2j}^0 = \int_{\Gamma_{2j}} \mathbf{U}_0 \cdot \mathbf{n}ds \quad j = 1, \dots, r. \tag{1.36}$$

PROPOSITION 1.9. *Assume that $a(\cdot,\cdot)$ is V-elliptic or V_0-elliptic with V_1 kernel and that the assumptions* (1.3), (1.4), (1.5), (1.6), (1.7) *are fulfilled. If $\mathbf{u}$ is a solution of the variational problem* (1.25) *and if F_{2j} is the flux of $\mathbf{u}$ through Γ_{2j}, which is defined by*:

$$F_{2j} = \int_{\Gamma_{2j}} \mathbf{u} \cdot \mathbf{n}ds, \quad j = 1, \dots, r, \tag{1.37a}$$

then

$$\sum_{j=1}^{r-1}(F_{2j}-F_{2j}^0)a(\omega_i,\omega_j)=L(\omega_i)-a(\mathbf{U}_0,\omega_i)\quad \forall i=1,\dots,r-1, \tag{1.37b}$$

where the functions ω_i are the elements of the basis of V_1, defined in Lemma 1.3. ■

According to the definition of $L(\cdot)$, formula (1.37a) becomes:

$$\sum_{j=1}^{r-1}(F_{2j}-F_{2j}^0)a(\omega_i,\omega_j)$$
$$=\int_\Omega \mathbf{f}\cdot\omega_i dx+\nu\langle\mathbf{h}\times\mathbf{n},\omega_i\rangle_{\Gamma_3}-\langle p_0,\omega_i\cdot\mathbf{n}\rangle_{\Gamma_2}-a(\mathbf{U}_0,\omega_i).$$

PROOF OF PROPOSITION 1.9. Taking $\mathbf{v}=\omega_i$ in (1.25c) we have:

$$a(\mathbf{u},\omega_i)=L(\omega_i)$$

which implies

$$a(\mathbf{u}-\mathbf{U}_0,\omega_i)=L(\omega_i)-a(\mathbf{U}_0,\omega_i).$$

On the other hand, thanks to Lemma 1.3, we can write:

$$\mathbf{u}-\mathbf{U}_0=\mathbf{v}_0+\sum_{j=1}^{r-1}(F_{2j}-F_{2j}^0)\omega_j,$$

with $\mathbf{v}_0\in V_0$. Since $a(\mathbf{v}_0,\omega_i)=0$, combining both identities we get (1.37). Proposition 1.9 is thus proved. ■

If $a(\cdot,\cdot)$ is V-elliptic, it is easy to verify that the coefficients $a(\omega_i,\omega_j)$ which intervene in formula (1.37b) define a symmetric matrix, that is positive definite. Thus, this formula allows the direct computation of the fluxes $F_{21},\dots,F_{2(r-1)}$ of $\mathbf{u}$ through the $(r-1)$ first components of Γ_2. It can be noticed that F_{2r} can be determined from $F_{21},\dots,F_{2(r-1)}$, since the following identity holds:

$$F_{2r}=-\Big(\int_{\Gamma_1}\mathbf{u}_0\cdot\mathbf{n}ds+\sum_{j=1}^{r-1}F_{2j}+\int_{\Gamma_3}\mathbf{b}\cdot\mathbf{n}ds\Big). \tag{1.38}$$

Let us now consider the case where $a(\cdot,\cdot)$ is V_0-elliptic with V_1 kernel. According to Theorem 1.6, for the variational problem (1.25) to admit at least one solution, the data have to verify the compatibility condition (1.28). Using the fact that the functions $\{\omega_1, \ldots, \omega_{r-1}\}$ form a basis of V_1, this condition becomes:

$$L(\omega_i) = 0 \quad \forall i = 1, \ldots, r-1, \tag{1.39a}$$

that is,

$$\int_\Omega \mathbf{f} \cdot \omega_i dx + \nu \langle \mathbf{h} \times \mathbf{n}, \omega_i \rangle_{\Gamma_3} - \langle p_0, \omega_i \cdot \mathbf{n} \rangle = 0 \quad \forall i = 1, \ldots, r-1. \tag{1.39b}$$

On the other hand, since the kernel of $a(\cdot,\cdot)$ is V_1, $a(\mathbf{v}, \omega_i) = 0 \; \forall \mathbf{v} \in H^1(\Omega)^3$, $i = 1, \ldots, r-1$. Thus the term $a(\mathbf{U}_0, \omega_i)$ and the coefficients $a(\omega_i, \omega_j)$ vanish. Therefore, formula (1.37b) reduces just to the compatibility condition (1.39). Moreover, according to Theorem 1.6, when the condition (1.39) is verified, the solutions of the variational problem (1.25) can be written as follows:

$$\mathbf{u} = \mathbf{y}_0 + \mathbf{w}, \tag{1.40}$$

where $\mathbf{w}$ is any function of V_1 and $\mathbf{y}_0$ is the only solution of problem (1.29), i.e.,

$$\text{Find } \mathbf{y}_0 \in H^1(\Omega)^3 \text{ such that} \tag{1.41a}$$

$$(\mathbf{y}_0 - \mathbf{U}_0) \in V_0 \tag{1.41b}$$

$$\begin{aligned} \nu \int_\Omega & \nabla \times \mathbf{y}_0 \cdot \nabla \times \mathbf{v}_0 \\ &= \int_\Omega \mathbf{f} \cdot \mathbf{v}_0 + \nu \langle \mathbf{h} \times \mathbf{n}, \mathbf{v}_0 \rangle_{\Gamma_3} - \langle p_0, \mathbf{v}_0 \cdot \mathbf{n} \rangle_{\Gamma_2} \quad \forall \mathbf{v}_0 \in V_0. \end{aligned} \tag{1.41c}$$

We can remark that it is always possible to choose $\mathbf{w}$ so that the solution $\mathbf{u} = \mathbf{y}_0 + \mathbf{w}$ of (1.25) had prescribed fluxes $F_{21}, \ldots, F_{2r}$ on the connected components of Γ_2 (of course, with F_{2r} verifying (1.38)). In fact, it suffices to choose $\mathbf{w}$ as follows:

$$\mathbf{w} = \sum_{j=1}^{r-1} (F_{2j} - F_{2j}^0) \omega_j. \tag{1.42}$$

1.5.3. The Stokes equations with prescribed fluxes on the connected components of Γ_2. Throughout this section, the bilinear form $a(\cdot,\cdot)$ is assumed to be either V-elliptic or V_0-elliptic with V_1 kernel. Let us consider the following variant of the boundary-value problem (1.2): Find functions $\mathbf{u}$, p and constants $C_1, \ldots, C_r$, defined up to an additive constant (i.e., we are looking for the differences $(C_i - C_r)$ for $i = 1, \dots, r-1$), such that:

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \tag{1.43a}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \tag{1.43b}$$

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on } \Gamma_1 \tag{1.43c}$$

$$\mathbf{u} \times \mathbf{n} = \mathbf{a} \times \mathbf{n} \quad \text{on } \Gamma_2 \tag{1.43d}$$

$$p = \overline{p}_{0i} + C_i \quad \text{on } \Gamma_{2i}, \quad \forall i = 1, \dots, r \tag{1.43e}$$

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n} \quad \text{on } \Gamma_3 \tag{1.43f}$$

$$(\nabla \times \mathbf{u}) \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on } \Gamma_3 \tag{1.43g}$$

$$\int_{\Gamma_{2j}} \mathbf{u} \cdot \mathbf{n} ds = F_{2j}, \quad \forall j = 1, \dots, r, \tag{1.43h}$$

where ν, $\mathbf{f}$, $\mathbf{u}_0$, $\mathbf{a}$, $\overline{p}_0$, $\mathbf{b}$, $\mathbf{h}$ and the fluxes $F_{21}, \dots, F_{2r}$ are given.

To study problem (1.43), we assume that the data of (1.43) fulfill the assumptions (1.3), (1.4), (1.5), (1.7) and we replace (1.6) by:

$$\overline{p}_{0i} \in H^{-1/2}(\Gamma_{2i}) \quad \forall i = 1, \dots, r. \tag{1.44}$$

Integrating (1.43b) by parts in Ω, we see that a necessary condition that (1.43) should possess a solution is that the global flux through Γ is zero, i.e.,

$$\int_{\Gamma_1} \mathbf{u}_0 \cdot \mathbf{n} ds + \sum_{j=1}^{r} F_{2j} + \int_{\Gamma_3} \mathbf{b} \cdot \mathbf{n} ds = 0, \tag{1.45}$$

which we will assume is fulfilled. Indeed, we will suppose that the prescribed fluxes are $F_1, \dots, F_{r-1}$ and that F_{2r} is given in such a way that (1.45) holds.

Using Theorem 1.7 and Proposition 1.9, it is easy to check that if $\mathbf{u} \in \mathcal{C}^2(\overline{\Omega})$, $p \in \mathcal{C}^1(\overline{\Omega})$ and $C_1, \dots, C_r$ is a solution of problem (1.43), then $\mathbf{u}$ and the constants $C_1, \dots, C_r$ are also solutions of the following variational problem:

Find $\mathbf{u} \in H^1(\Omega)^3$ and constants $C_1, \dots, C_r$ (defined up to an additive constant) such that (1.46a)

$$(\mathbf{u} - \mathbf{U}_1) \in V_0 \tag{1.46b}$$

$$a(\mathbf{u}, \mathbf{v}_0) = \overline{L}(\mathbf{v}_0) \quad \forall \mathbf{v}_0 \in V_0 \tag{1.46c}$$

$$C_i - C_r = \overline{L}(\omega_i) - a(\mathbf{U}_0, \omega_i) - \sum_{j=1}^{r-1} (F_{2j} - F_{2j}^0) a(\omega_i, \omega_j) \quad \forall i = 1, \dots, r-1, \tag{1.46d}$$

where V_0 is defined by (1.10) and $\overline{L}(\cdot)$ is defined by:

$$\overline{L}(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx + \nu \langle \mathbf{h} \times \mathbf{n}, \mathbf{v} \rangle_{\Gamma_3} - \langle \overline{p}_0, \mathbf{v} \cdot \mathbf{n} \rangle_{\Gamma_2} \quad \forall \mathbf{v} \in H^1(\Omega)^3, \tag{1.47}$$

and $\mathbf{U}_1 \in H^1(\Omega)^3$ is

$$\mathbf{U}_1 = \mathbf{U}_0 + \sum_{j=1}^{r-1}(F_{2j} - F_{2j}^0)\omega_j. \tag{1.48}$$

Reciprocally, if $\mathbf{u}$ and $C_1, \dots, C_r$ is a solution of (1.46), using the splitting of V into $V_0 \oplus V_1$ (see Lemma 1.3) and defining p_0 by:

$$p_0 = \overline{p}_{0i} + C_i \quad \text{on } \Gamma_{2i} \quad \forall i = 1, \dots, r, \tag{1.49}$$

we can easily verify that $\mathbf{u}$ is a solution of the variational problem (1.25). Thus, according to Theorem 1.8, there exists $p \in H(\Delta, \Omega)/\boldsymbol{R}$ such that $\mathbf{u}$, p, and the constants $C_1, \dots, C_r$ are solutions of (1.43a), ... ,(1.43g) in the sense given by this theorem. On the other hand, the velocity $\mathbf{u}$ satisfies the flux conditions (1.43h), because $(\mathbf{u} - \mathbf{U}_1)$ belongs to V_0 and (1.45) holds true.

The existence and uniqueness of a solution of the variational problem (1.46) is clear: it suffices to note that (1.46) can be broken down into two parts. On one hand, (1.46b) and (1.46c), which give the existence and uniqueness of $\mathbf{u}$ and on the other hand (1.46d), which yields the differences $(C_i - C_r)$. We have thus proved:

THEOREM 1.10. *Assume that $a(\cdot,\cdot)$ is either V-elliptic or V_0-elliptic with V_1 kernel and that the assumptions* (1.3), (1.4), (1.5), (1.7), (1.44) *and* (1.45) *are fulfilled. Then the variational problem* (1.46) *has one and only one solution. The interpretation of this variational problem is none other than the boundary-value problem* (1.43). ■

The three model examples (1.24a) $(\Gamma_1 \neq \emptyset)$, (1.24b) $(\Gamma = \Gamma_2)$ and (1.24c) $(\Gamma_1 = \emptyset, \Omega$ simply connected) are practical situations where Theorem 1.10 applies (see Lemma 1.4). It is interesting to remark that if $a(\cdot,\cdot)$ is V_0-elliptic with V_1 kernel (model example (1.24b) and (1.24c)), then equation (1.46d) of problem (1.46) simply reduces to:

$$C_i - C_r = \overline{L}(\omega_i) \quad \forall i = 1, \dots, r-1. \tag{1.50}$$

2. The Navier-Stokes equations with boundary conditions involving the pressure

In this section, we will extend some of the results of §1 to the case of the Navier-Stokes equations. First, we will consider the case where the velocity verifies homogeneous boundary conditions, but the pressure and the tangential component of the vorticity verify non-homogeneous conditions. This problem is introduced in §2.1. In §2.2, we introduce a variational formulation of this problem and we prove existence of a solution assuming that the bilinear form $a(\cdot,\cdot)$ is V-elliptic. In §2.3, we prove that this solution is unique if the viscosity ν is large enough with respect to the data (or if the data are small enough with respect to ν).

Sections 2.4 and 2.5 are devoted to studying the case of non-homogeneous boundary conditions on the velocity. The corresponding boundary-value problem is introduced there, as well as its variational formulation, and we prove an existence and uniqueness result for a solution under the assumption that $a(\cdot,\cdot)$ is V-elliptic and that the data are small (with respect to ν).

Next, §2.6 is devoted to studying, as in the linear case, a variant of the Navier-Stokes problem where the velocity satisfies prescribed flux-conditions on the connected components of Γ_2. We prove a theorem of existence and uniqueness of a small solution, assuming that $a(\cdot,\cdot)$ is V-elliptic or V_0-elliptic with V_1 kernel, and that the data are small.

2.1. Homogeneous boundary conditions for the velocity. Description of the homogeneous Navier-Stokes problem. Let Ω be a connected open set of $\boldsymbol{R}^3$, with a locally Lipschitz boundary Γ. It is assumed that we can distinguish in Γ three open subsets $\{\Gamma_1, \Gamma_2, \Gamma_3\}$ that verify (1.1). We are interested in studying the Navier-Stokes equations:

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \tag{2.1a}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \tag{2.1b}$$

with the following boundary conditions:

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1 \tag{2.1c}$$

$$\mathbf{u}\times\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_2 \tag{2.1d}$$

$$p + (1/2)|\mathbf{u}|^2 = p_0 \quad \text{on } \Gamma_2 \tag{2.1e}$$

$$\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_3 \tag{2.1f}$$

$$(\nabla\times\mathbf{u})\times\mathbf{n} = \mathbf{h}\times\mathbf{n} \quad \text{on } \Gamma_3, \tag{2.1g}$$

where ν, $\mathbf{f}$, p_0 and $\mathbf{h}$ are given. We assume that they verify (1.3), (1.4), (1.6) and (1.7).

2.2. Variational formulation of the homogeneous problem and existence result. Let us consider the following variational problem:

$$\text{Find } \mathbf{u}\in V \text{ such that} \tag{2.2a}$$

$$a(\mathbf{u},\mathbf{v}) + b(\mathbf{u},\mathbf{u},\mathbf{v}) = L(\mathbf{v}) \quad \forall\mathbf{v}\in V, \tag{2.2b}$$

where V, $a(\cdot,\cdot)$ and $L(\cdot)$ are defined by (1.9), (1.11) and (1.26), respectively. With regard to the trilinear form $b(\cdot,\cdot,\cdot)$, it is defined by:

$$b(\cdot,\cdot,\cdot) : H^1(\Omega)^3 \times H^1(\Omega)^3 \times H^1(\Omega)^3 \to \boldsymbol{R} \tag{2.3a}$$

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega ((\nabla\times\mathbf{u})\times\mathbf{v})\cdot\mathbf{w}\,dx \quad \forall\mathbf{u},\mathbf{v},\mathbf{w}\in H^1(\Omega)^3. \tag{2.3b}$$

It is worthwhile noting that if $\mathbf{u}, \mathbf{v}, \mathbf{w} \in H^1(\Omega)^3$, then $\nabla \times \mathbf{u} \in L^2(\Omega)^3$, $\mathbf{v} \in L^4(\Omega)^3$ and $\mathbf{w} \in L^4(\Omega)^3$ because, by the standard Sobolev embeddings, the space $H^1(\Omega)^3$ is included in $L^4(\Omega)^3$, with compact embedding. Therefore, $((\nabla \times \mathbf{u}) \times \mathbf{v}) \cdot \mathbf{w}$ belongs to $L^1(\Omega)^3$ and $b(\cdot,\cdot,\cdot)$ is a well-defined continuous trilinear form in $(H^1(\Omega)^3)^3$. Taking into account the definitions of V, $a(\cdot,\cdot)$, $L(\cdot)$, and $b(\cdot,\cdot,\cdot)$, (2.2) becomes

$$\text{Find } \mathbf{u} \in V \text{ such that} \tag{2.4a}$$

$$\nu \int_\Omega (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) dx + \int_\Omega ((\nabla \times \mathbf{u}) \times \mathbf{u}) \cdot \mathbf{v} dx$$
$$= \int_\Omega \mathbf{f} \cdot \mathbf{v} dx + \nu \langle \mathbf{h} \times \mathbf{n}, \mathbf{v} \rangle_{\Gamma_3} - \langle p_0, \mathbf{v} \cdot \mathbf{n} \rangle_{\Gamma_2} \quad \forall \mathbf{v} \in V. \tag{2.4b}$$

THEOREM 2.1. *If the bilinear form $a(\cdot,\cdot)$ is V-elliptic, then problem (2.2) has at least one solution.* ■

Notice that Theorem 2.1 applies to the model example (1.24a) ($\Gamma_1 \neq \emptyset$). It does not assume any smallness of the data. On the contrary, we will need this assumption to prove the uniqueness result in §2.3 and the existence and uniqueness for the non-homogeneous case in §2.5.

PROOF OF THEOREM 2.1. We use Galerkin's method. V is a closed subspace of $H^1(\Omega)^3$ and it is thus possible to choose a basis $\mathbf{z}_1, \dots, \mathbf{z}_m, \dots$ of V. For every $m \geq 1$, we define an approximate problem by:

$$\text{Find } \alpha_{im} \in \boldsymbol{R} \ (1 \leq i \leq m) \text{ such that } \mathbf{u}_m = \sum_{i=1}^{m} \alpha_{im} \mathbf{z}_i \text{ is a solution of:} \tag{2.5a}$$

$$a(\mathbf{u}_m, \mathbf{z}_i) + b(\mathbf{u}_m, \mathbf{u}_m, \mathbf{z}_i) = L(\mathbf{z}_i) \quad \forall i = 1, \dots, m. \tag{2.5b}$$

To prove that (2.5) admits at least one solution $\mathbf{u}_m$, we will use the following lemma (see J.-L. Lions [Li] Lemma 1.4.3, p.53 or R.Temam [Te] Lemma 2.1.4, p.164):

LEMMA 2.2. *Let Y be a finite-dimensional Hilbert space, with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$. Let P be a continuous mapping from Y into itself such that, for a suitable $\rho > 0$,*

$$(P(\xi), \xi) \geq 0 \quad \forall \xi \ \textit{ such that } \ |\xi| = \rho. \tag{2.6}$$

Then there exists ξ, $|\xi| \leq \rho$, such that $P(\xi) = 0$. ■

Let us consider the space $Y = Y_m$ spanned by $\mathbf{z}_1, \dots, \mathbf{z}_m$, equipped with the scalar product of $H^1(\Omega)^3$, and define $P = P_m$ by

$$(P_m(\xi), \mathbf{z}) = a(\xi, \mathbf{z}) + b(\xi, \xi, \mathbf{z}) - L(\mathbf{z}) \quad \forall \xi, \mathbf{z} \in Y_m.$$

It is clear that P_m is continuous. Let us prove that P_m verifies (2.6). We have:

$$(P_m(\xi), \xi) = a(\xi, \xi) + b(\xi, \xi, \xi) - L(\xi), \tag{2.7}$$

but $((\nabla \times \xi) \times \xi) \cdot \xi = \mathbf{0}$ and hence, $b(\xi, \xi, \xi) = 0$. On the other hand, since $L(\cdot)$ is a continuous linear form on $H^1(\Omega)^3$, there exists a constant C such that

$$|L(\mathbf{v})| \leq C\|\mathbf{v}\|_{H^1(\Omega)^3} \quad \forall \mathbf{v} \in H^1(\Omega)^3. \tag{2.8}$$

Therefore, using the fact that $a(\cdot,\cdot)$ is V-elliptic (see (1.13) for the definition of α), (2.7) and (2.8), we deduce that

$$(P_m(\xi), \xi) \geq \alpha\nu\|\xi\|^2_{H^1(\Omega)^3} - C\|\xi\|_{H^1(\Omega)^3} \quad \forall \xi \in Y_m,$$

which implies that P_m satisfies (2.6) with $\rho = (C/\alpha\nu)$. Then for any $m \geq 1$, there exists $\xi_m \in Y_m$ such that $\|\xi_m\|_{H^1(\Omega)^3} \leq (C/\alpha\nu)$ and $P_m(\xi_m) = 0$. This means that there exists a solution $\mathbf{u}_m$ of (2.5) which verifies:

$$\|\mathbf{u}_m\|_{H^1(\Omega)^3} \leq (C/\alpha\nu).$$

Thus the sequence $\{\mathbf{u}_m\}$ remains bounded in $H^1(\Omega)^3$ as $m \to \infty$. We can then extract a subsequence $\{\mathbf{u}_{m'}\}$ such that:

$$\mathbf{u}_{m'} \longrightarrow \mathbf{u} \text{ in } H^1(\Omega)^3 \text{ weakly as } m' \to \infty, \tag{2.9a}$$

where $\mathbf{u}$ belongs to V (since $\mathbf{z}_1, \ldots, \mathbf{z}_m, \ldots$ is a basis of V). The embedding of $H^1(\Omega)^3$ into $L^4(\Omega)^3$ being compact, we can choose a subsequence so that

$$\mathbf{u}_{m'} \longrightarrow \mathbf{u} \text{ in } L^4(\Omega)^3 \text{ strongly as } m' \to \infty. \tag{2.9b}$$

Let $\mathbf{v}$ in V and select a sequence $\{\mathbf{v}_m\}$ such that $\mathbf{v}_m \in Y_m$ and

$$\mathbf{v}_m \longrightarrow \mathbf{v} \text{ in } V \text{ strongly as } m \to \infty, \tag{2.10a}$$

$$\mathbf{v}_m \longrightarrow \mathbf{v} \text{ in } L^4(\Omega)^3 \text{ strongly as } m \to \infty. \tag{2.10b}$$

According to (2.5b), for every m', we have:

$$a(\mathbf{u}_{m'}, \mathbf{v}_{m'}) + b(\mathbf{u}_{m'}, \mathbf{u}_{m'}, \mathbf{v}_{m'}) = L(\mathbf{v}_{m'}). \tag{2.11}$$

Using (2.9) and (2.10), we pass to the limit in (2.11), and we obtain

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = L(\mathbf{v}) \quad \forall \mathbf{v} \in V, \tag{2.12}$$

which proves that $\mathbf{u}$ satisfies (2.2b). This completes the proof. ■

2.3. A uniqueness theorem. Let us denote by $\|L\|_*$ the norm of L in the dual space of V, i.e.,

$$\|L\|_* = \sup_{\mathbf{v}\in V} \frac{|L(\mathbf{v})|}{\|\mathbf{v}\|_{H^1(\Omega)^3}}. \tag{2.13}$$

In this section, we shall prove:

THEOREM 2.3. *If the bilinear form $a(\cdot,\cdot)$ is V-elliptic, then there exists a positive constant δ such that if*

$$\nu^2 > \delta\|L\|_*, \tag{2.14}$$

then problem (2.2) *has a unique solution.*

PROOF. Let $\mathbf{u}_1$, $\mathbf{u}_2$ be two solutions of (2.2). Taking $\mathbf{v} = \mathbf{u}_1$ in the equation (2.2b) which defines $\mathbf{u}_1$, and noting that $((\nabla \times \mathbf{u}_1) \times \mathbf{u}_1) \cdot \mathbf{u}_1 = 0$, we get

$$\alpha\nu\|\mathbf{u}_1\|^2_{H^1(\Omega)^3} \le \|L\|_*\|\mathbf{u}_1\|_{H^1(\Omega)^3}.$$

Thus, any solution $\mathbf{u}_1$ of (2.2) verifies

$$\|\mathbf{u}_1\|_{H^1(\Omega)^3} \le \|L\|_*/(\alpha\nu). \tag{2.15a}$$

On the other hand, setting $\mathbf{z} = \mathbf{u}_1 - \mathbf{u}_2$, we get

$$a(\mathbf{z},\mathbf{v}) + b(\mathbf{z},\mathbf{u}_1,\mathbf{v}) + b(\mathbf{u}_2,\mathbf{z},\mathbf{v}) = 0 \quad \forall \mathbf{v} \in V. \tag{2.15b}$$

Taking $\mathbf{v} = \mathbf{z}$ in (2.15b), we find

$$\alpha\nu\|\mathbf{z}\|^2_{H^1(\Omega)^3} + b(\mathbf{z},\mathbf{u}_1,\mathbf{z}) + b(\mathbf{u}_2,\mathbf{z},\mathbf{z}) \le 0.$$

But $((\nabla \times \mathbf{u}_2) \times \mathbf{z}) \cdot \mathbf{z} = 0$. Thus $b(\mathbf{z},\mathbf{u}_1,\mathbf{z}) = 0$, which implies

$$\alpha\nu\|\mathbf{z}\|^2_{H^1(\Omega)^3} \le -b(\mathbf{z},\mathbf{u}_1,\mathbf{z}) = -\int_\Omega ((\nabla \times \mathbf{z}) \times \mathbf{u}_1) \cdot \mathbf{z}\, dx,$$

from which, applying Holder's inequality, we obtain

$$\alpha\nu\|\mathbf{z}\|^2_{H^1(\Omega)^3} \le C_0\|\mathbf{u}_1\|_{H^1(\Omega)^3}\|\mathbf{z}\|^2_{H^1(\Omega)^3}, \tag{2.15c}$$

where C_0 is the embedding constant of $H^1(\Omega)^3$ into $L^4(\Omega)^3$. According to (2.15a), we finally conclude that

$$(\alpha\nu - (C_0/\alpha\nu)\|L\|_*)\|\mathbf{z}\|^2_{H^1(\Omega)^3} \le 0,$$

and thus, $\mathbf{z} = \mathbf{0}$ as soon as $\nu^2 > (C_0\|L\|_*/\alpha^2)$. This proves Theorem 2.3 with $\delta = (C_0/\alpha^2)$. ■

2.4. Non-homogeneous boundary conditions for the velocity. Description of the non-homogeneous Navier-Stokes problem. We will now focus our interest in the following problem:

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \tag{2.16a}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega \tag{2.16b}$$

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on } \Gamma_1 \tag{2.16c}$$

$$\mathbf{u}\times\mathbf{n} = \mathbf{a}\times\mathbf{n} \quad \text{on } \Gamma_2 \tag{2.16d}$$

$$p + (1/2)|\mathbf{u}|^2 = p_0 \quad \text{on } \Gamma_2 \tag{2.16e}$$

$$\mathbf{u}\cdot\mathbf{n} = \mathbf{b}\cdot\mathbf{n} \quad \text{on } \Gamma_3 \tag{2.16f}$$

$$(\nabla\times\mathbf{u})\times\mathbf{n} = \mathbf{h}\times\mathbf{n} \quad \text{on } \Gamma_3, \tag{2.16g}$$

where, as in the Stokes problem, ν, $\mathbf{f}$, $\mathbf{u}_0$, $\mathbf{a}$, p_0, $\mathbf{b}$ and $\mathbf{h}$ are given, and $\mathbf{u}$ and p are the unknowns. As in §1, we denote by $\mathbf{U}_0$ an extension of $\mathbf{u}_0$, $\mathbf{a}$ and $\mathbf{b}$ to Ω, and we assume that the data verify (1.3), (1.4), (1.5), (1.6) and (1.7).

To study problem (2.16), we consider its variational formulation:

$$\text{Find } \mathbf{u} \in H^1(\Omega)^3 \text{ such that} \tag{2.17a}$$

$$(\mathbf{u} - \mathbf{U}_0) \in V \tag{2.17b}$$

$$a(\mathbf{u},\mathbf{v}) + b(\mathbf{u},\mathbf{u},\mathbf{v}) = L(\mathbf{v}) \quad \forall \mathbf{v} \in V, \tag{2.17c}$$

which means due to the definitions of $a(\cdot,\cdot)$, $L(\cdot)$ and $b(\cdot,\cdot,\cdot)$ (see (1.11), (1.26) and (2.3)):

$$\text{Find } \mathbf{u} \in H^1(\Omega)^3 \text{ such that} \tag{2.18a}$$

$$(\mathbf{u} - \mathbf{U}_0) \in V \tag{2.18b}$$

$$\begin{aligned} &\nu\int_\Omega (\nabla\times\mathbf{u})\cdot(\nabla\times\mathbf{v})dx + \int_\Omega ((\nabla\times\mathbf{u})\times\mathbf{u})\cdot\mathbf{v}dx \\ &\quad = \int_\Omega \mathbf{f}\cdot\mathbf{v}dx + \nu\langle\mathbf{h}\times\mathbf{n},\mathbf{v}\rangle_{\Gamma_3} - \langle p_0, \mathbf{v}\cdot\mathbf{n}\rangle_{\Gamma_2} \quad \forall\mathbf{v}\in V, \end{aligned} \tag{2.18c}$$

where V is defined by (1.9).

2.5. An existence and uniqueness result for the non-homogeneous Navier-Stokes problem. In this section, we prove that if either the data are small enough or if the fluid viscosity is sufficiently large, then there exists at least one solution of (2.17). The solution we construct is in a small neighbourhood of

$\mathbf{U}_0$. Moreover, this is the unique solution of (2.17). Let us begin by fixing some notations:

$$M = \sup_{\mathbf{u},\mathbf{v}\in H^1(\Omega)^3} \frac{a(\mathbf{u},\mathbf{v})}{\|\mathbf{u}\|_{H^1(\Omega)^3}\|\mathbf{v}\|_{H^1(\Omega)^3}} \tag{2.19}$$

$$K = \sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in H^1(\Omega)^3} \frac{|b(\mathbf{u},\mathbf{v},\mathbf{w})|}{\|\mathbf{u}\|_{H^1(\Omega)^3}\|\mathbf{v}\|_{H^1(\Omega)^3}\|\mathbf{w}\|_{H^1(\Omega)^3}}. \tag{2.20}$$

Observe that M is well-defined, since $a(\cdot,\cdot)$ is a continuous bilinear form in $H^1(\Omega)^3$. K is also well-defined, because $b(\cdot,\cdot,\cdot)$ is a continuous trilinear form in $(H^1(\Omega)^3)^3$. On the other hand, $K \le C_0^2$, where C_0 is the embedding constant of $H^1(\Omega)^3$ into $L^4(\Omega)^3$. Set

$$A = \|L\|_* + \|\mathbf{U}_0\|_{H^1(\Omega)^3}(M + 2K\|\mathbf{U}_0\|_{H^1(\Omega)^3}), \tag{2.21}$$

and denote by $B(\mathbf{U}_0, 2A/\alpha\nu)$ the ball of $H^1(\Omega)^3$ centered at $\mathbf{U}_0$, with radius $(2A/\alpha\nu)$, i.e.,

$$B(\mathbf{U}_0, 2A/\alpha\nu) = \{\mathbf{v}\in H^1(\Omega)^3 \mid \|\mathbf{v}-\mathbf{U}_0\|_{H^1(\Omega)^3} \le 2A/\alpha\nu\}. \tag{2.22}$$

THEOREM 2.4. *Assume that the bilinear form $a(\cdot,\cdot)$ is V-elliptic and that the data of problem* (2.17) *verify the following condition*

$$\nu^2 > (8AK/\alpha^2), \tag{2.23}$$

where α, A and K are defined by (1.13), (2.21), (2.20), *respectively. Then problem* (2.17) *has a unique solution, which belongs to $B(\mathbf{U}_0, 2A/\alpha\nu)$.* ■

Observe that Theorem 2.4 applies to the model example (1.24a) ($\Gamma_1 \neq \emptyset$).

PROOF OF THEOREM 2.4. Let $G : B(\mathbf{U}_0, 2A/\alpha\nu) \to H^1(\Omega)^3$ be the mapping defined as $G(\mathbf{v}) = \mathbf{u}$, where $\mathbf{u}$ is the unique solution of the following Stokes problem:

$$\text{Find } \mathbf{u}\in H^1(\Omega)^3 \text{ such that} \tag{2.24a}$$

$$(\mathbf{u}-\mathbf{U}_0)\in V \tag{2.24b}$$

$$a(\mathbf{u},\mathbf{z}) = L(\mathbf{z}) - b(\mathbf{v},\mathbf{v},\mathbf{z}) \quad \forall \mathbf{z}\in V. \tag{2.24c}$$

The proof consists in showing that G has a unique fixed point in $B(\mathbf{U}_0, 2A/\alpha\nu)$. Let us first prove that G maps $B(\mathbf{U}_0, 2A/\alpha\nu)$ into itself.

Choosing $\mathbf{z} = (\mathbf{u}-\mathbf{U}_0)$ in (2.24c), we obtain

$$a(\mathbf{u}-\mathbf{U}_0, \mathbf{u}-\mathbf{U}_0) = L(\mathbf{u}-\mathbf{U}_0) - a(\mathbf{U}_0, \mathbf{u}-\mathbf{U}_0) - b(\mathbf{v},\mathbf{v},\mathbf{u}-\mathbf{U}_0),$$

which, in view of (1.13), (2.13), (2.19) and (2.20), implies

$$\alpha\nu\|\mathbf{u}-\mathbf{U}_0\|^2 \leq (\|L\|_* + M\|\mathbf{U}_0\| + K\|\mathbf{v}\|^2)\|\mathbf{u}-\mathbf{U}_0\| \tag{2.25a}$$

where, in (2.25a) and in what follows, we denote by $\|\cdot\|$ the norm of $H^1(\Omega)^3$. But $\|\mathbf{v}\|^2 \leq 2\|\mathbf{v}-\mathbf{U}_0\|^2 + \|\mathbf{U}_0\|^2$, hence (2.25a), (2.21) imply that:

$$\alpha\nu\|\mathbf{u}-\mathbf{U}_0\| \leq A + 2K\|\mathbf{v}-\mathbf{U}_0\|^2. \tag{2.25b}$$

Therefore, if $\mathbf{v}$ belongs to $B(\mathbf{U}_0, 2A/\alpha\nu)$, by using hypothesis (2.23), we deduce

$$\alpha\nu\|\mathbf{u}-\mathbf{U}_0\| \leq 2A,$$

that is, $\mathbf{u} \in B(\mathbf{U}_0, 2A/\alpha\nu)$.

Let us prove that G is strictly contracting. To this end, let $\mathbf{v}_1$ and $\mathbf{v}_2$ be given in $\dot{B}(\mathbf{U}_0, 2A/\alpha\nu)$. Set

$$\mathbf{u}_1 = G(\mathbf{v}_1) \quad \text{and} \quad \mathbf{u}_2 = G(\mathbf{v}_2).$$

Since $\mathbf{u}_1$ is the unique solution of (2.24) for $\mathbf{v} = \mathbf{v}_i$, $i = 1, 2$, it follows by subtraction that

$$a(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{z}) = -b(\mathbf{v}_1, \mathbf{v}_1, \mathbf{z}) + b(\mathbf{v}_2, \mathbf{v}_2, \mathbf{z}) \quad \forall \mathbf{z} \in V.$$

Choosing $\mathbf{z} = \mathbf{u}_1 - \mathbf{u}_2$ in this expression, we obtain

$$a(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) = -b(\mathbf{v}_1, \mathbf{v}_1, \mathbf{u}_1 - \mathbf{u}_2) + b(\mathbf{v}_2, \mathbf{v}_2, \mathbf{u}_1 - \mathbf{u}_2),$$

that is,

$$a(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) = -b(\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_1, \mathbf{u}_1 - \mathbf{u}_2) + b(\mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_1, \mathbf{u}_1 - \mathbf{u}_2),$$

from which, using (1.13) and (2.20), we deduce

$$\alpha\nu\|\mathbf{u}_1 - \mathbf{u}_2\|^2 \leq K(\|\mathbf{v}_1\| + \|\mathbf{v}_2\|)\|\mathbf{v}_1 - \mathbf{v}_2\|\,\|\mathbf{u}_1 - \mathbf{u}_2\|.$$

But $\|\mathbf{v}_i\| \leq (2A/\alpha\nu) + \|\mathbf{U}_0\|$, $i = 1, 2$, then we have

$$\alpha\nu\|\mathbf{u}_1 - \mathbf{u}_2\|^2 \leq K((4A/\alpha\nu) + 2\|\mathbf{U}_0\|)\|\mathbf{v}_1 - \mathbf{v}_2\|. \tag{2.26}$$

On the other hand, hypothesis (2.23) implies on one side that

$$K(4A/\alpha\nu) < \alpha\nu/2, \tag{2.27a}$$

and on the other side (since according to (2.21), $\|\mathbf{U}_0\|^2 < (A/2K)$) that

$$2K\|\mathbf{U}_0\| < \alpha\nu/2. \tag{2.27b}$$

Combining (2.26) with (2.27), we finally conclude that

$$\|\mathbf{u}_1 - \mathbf{u}_2\| < \|\mathbf{v}_1 - \mathbf{v}_2\|,$$

which proves that G is a strict contraction. Thus G admits a unique fixed point in $B(\mathbf{U}_0, 2A/\alpha\nu)$. This fixed point is a solution of problem (2.17).

Now, let us prove uniqueness. To do this, let $\mathbf{u}_1$ be the solution of (2.17) obtained by the above fixed point process, and let $\mathbf{u}_2$ be any other solution of (2.17). If $\mathbf{z} = \mathbf{u}_1 - \mathbf{u}_2$, we get by subtraction

$$\mathbf{z} \in V \tag{2.28a}$$

$$a(\mathbf{z}, \mathbf{v}) + b(\mathbf{z}, \mathbf{u}_1, \mathbf{v}) + b(\mathbf{u}_2, \mathbf{z}, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in V. \tag{2.28b}$$

Following similar arguments to that of the proof of Theorem 2.3, we can easily deduce that

$$\alpha\nu\|\mathbf{z}\|^2 \leq K\|\mathbf{u}_1\|\,\|\mathbf{z}\|^2.$$

But $\|\mathbf{u}_1\| \leq (2A/\alpha\nu) + \|\mathbf{U}_0\|$, and hence we obtain:

$$\alpha\nu\|\mathbf{z}\|^2 \leq ((2AK/\alpha\nu) + K\|\mathbf{U}_0\|)\|\mathbf{z}\|^2,$$

from which, using (2.27), we are led to:

$$\alpha\nu\|\mathbf{z}\|^2 \leq (\alpha\nu/2)\|\mathbf{z}\|^2,$$

which implies that $\mathbf{z} = \mathbf{0}$. This completes the proof of Theorem 2.4. ■

2.6. The Navier-Stokes equations with prescribed fluxes on the connected components of Γ_2. Throughout this section it will be assumed that the bilinear form $a(\cdot,\cdot)$ is V-elliptic or V_0-elliptic with V_1 kernel. As for the linear case (see §1.5.3), we will also study the following variant of the boundary-value problem (2.1): Find $\mathbf{u}$, p and constants $C_1, \ldots, C_r$ defined up to an additive constant such that:

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \tag{2.29a}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega \tag{2.29b}$$

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on } \Gamma_1 \tag{2.29c}$$

$$\mathbf{u}\times\mathbf{n} = \mathbf{a}\times\mathbf{n} \quad \text{on } \Gamma_2 \tag{2.29d}$$

$$p + (1/2)|\mathbf{u}|^2 = \overline{p}_{0i} + C_i \quad \text{on } \Gamma_{2i}, \quad \forall i = 1, \ldots, r \tag{2.29e}$$

$$\mathbf{u}\cdot\mathbf{n} = \mathbf{b}\cdot\mathbf{n} \quad \text{on } \Gamma_3 \tag{2.29f}$$

$$(\nabla\times\mathbf{u})\times\mathbf{n} = \mathbf{h}\times\mathbf{n} \quad \text{on } \Gamma_3 \tag{2.29g}$$

$$\int_{\Gamma_{2j}} \mathbf{u} \cdot \mathbf{n} ds = F_{2j} \quad \forall j = 1, \dots, r, \tag{2.29h}$$

where ν, $\mathbf{f}$, $\mathbf{u}_0$, $\mathbf{a}$, $\overline{p}_0$, $\mathbf{b}$, $\mathbf{h}$ and the fluxes $F_{21}, \dots, F_{2r}$ are problem's data. We assume that these data verify (1.3), (1.4), (1.5), (1.7); we replace (1.6) by (1.44) and we assume that (1.45) is verified. Setting

$$p_{0i} = \overline{p}_{0i} + C_i, \quad i = 1, \dots, r, \tag{2.30}$$

and using the splitting of V into $V_0 \oplus V_1$, and the basis $\{\omega_1, \dots, \omega_{r-1}\}$ of V_1 (see Lemma 1.3), we see that it is a straightforward matter to show that the variational formulation of the boundary-value problem (2.29) is:

Find $\mathbf{u} \in H^1(\Omega)^3$ and constants $C_1, \dots, C_r$ (defined up to an additive constant) such that (2.31a)

$$(\mathbf{u} - \mathbf{U}_1) \in V_0 \tag{2.31b}$$

$$a(\mathbf{u}, \mathbf{v}_0) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}_0) = \overline{L}(\mathbf{v}_0) \quad \forall \mathbf{v}_0 \in V_0 \tag{2.31c}$$

$$C_i - C_r = \overline{L}(\omega_i) - b(\mathbf{u}, \mathbf{u}, \omega_i) - a(\mathbf{U}_0, \omega_i) - \sum_{j=1}^{r-1} (F_{2j} - F_{2j}^0) a(\omega_i, \omega_j), \tag{2.31d}$$

for all $i = 1, \dots, r-1$, where V_0, $\overline{L}(\cdot)$ are defined by (1.10), (1.47) respectively, and $\mathbf{U}_1 \in H^1(\Omega)^3$ is defined by (1.48), i.e.,

$$\mathbf{U}_1 = \mathbf{U}_0 + \sum_{j=1}^{r-1} (F_{2j} - F_{2j}^0) \omega_j. \tag{2.32}$$

As regards existence and uniqueness of a solution of this problem, we shall see that if either ν is sufficiently large with respect to the data or if the data are small enough with respect to ν, then problem (2.31) admits one and only one solution. The velocity associated to this solution is in a small neighbourhood of $\mathbf{U}_1$ in $H^1(\Omega)^3$.

Let $\overline{A}$ be defined by

$$\overline{A} = \|\overline{L}\|_* + \|\mathbf{U}_1\|_{H^1(\Omega)^3} (M + 2K \|\mathbf{U}_1\|_{H^1(\Omega)^3}), \tag{2.33}$$

where the constants M, K are defined by (2.19), (2.20), and $\|\overline{L}\|_*$ designates the norm of $\overline{L}$ in the dual of V, i.e.,

$$\|\overline{L}\|_* = \sup_{\mathbf{v}_0 \in V_0} \frac{|\overline{L}(\mathbf{v}_0)|}{\|\mathbf{v}_0\|_{H^1(\Omega)^3}}. \tag{2.34}$$

THEOREM 2.5. *Assume that the bilinear form $a(\cdot,\cdot)$ is V_0-elliptic with V_1 kernel and that the following condition holds*:

$$\nu^2 > (8\overline{A}K/\alpha_0^2), \tag{2.35}$$

where α_0, K *and* $\overline{A}$ *are defined by* (1.14a), (2.20) *and* (2.33), *respectively. Then problem* (2.31) *has a unique solution. The corresponding velocity belongs to* $B(\mathbf{U}_1, 2\overline{A}/\alpha_0\nu)$. ■

PROOF. The proof is almost identical to that of Theorem 2.4. First, observe that problem (2.31) can be split up into two simpler problems. On one hand, (2.31b,c), which only involves $\mathbf{u}$, and on the other hand, (2.31d) which yields the values of the differences $(C_i - C_r)$, once $\mathbf{u}$ is known. We are thus led to solve the following variational problem:

$$\text{Find } \mathbf{u} \in H^1(\Omega)^3 \text{ such that} \tag{2.36a}$$

$$(\mathbf{u} - \mathbf{U}_1) \in V_0 \tag{2.36b}$$

$$a(\mathbf{u}, \mathbf{v}_0) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}_0) = \overline{L}(\mathbf{v}_0) \quad \forall \mathbf{v}_0 \in V_0. \tag{2.36c}$$

To prove Theorem 2.5, it is then sufficient to follow step by step the proof of Theorem 2.4, but replacing V by V_0, L by $\overline{L}$, $\mathbf{U}_0$ by $\mathbf{U}_1$, A by $\overline{A}$, and α by α_0. Doing that we conclude that if (2.35) is verified, then (2.36) admits a unique solution and that this solution belongs to $B(\mathbf{U}_1, 2\overline{A}/\alpha_0\nu)$. Once the existence of $\mathbf{u}$ has been proved, formula (2.31d) defines the constants $C_1, \ldots, C_r$ up to an additive constant. This finishes the proof of Theorem 2.5. ■

According to Lemma 1.4, the three model examples (1.24a) ($\Gamma_1 \neq \emptyset$), (1.24b) ($\Gamma = \Gamma_2$) and (1.24c) ($\Gamma_1 = \emptyset$, Ω simply connected) are concrete situations where Theorem 2.5 applies. As in the linear case, it is interesting to note that if $a(\cdot,\cdot)$ is V_0-elliptic with V_1 kernel (cases (1.24b) and (1.24c)), the equation (2.31d) reduces to:

$$C_i - C_r = \overline{L}(\omega_i) - b(\mathbf{u}, \mathbf{u}, \omega_i) \quad \forall i = 1, \ldots, r-1. \quad ■ \tag{2.37}$$

Appendix A

A.1. Analysis of the examples of Section 1.3. Proof of Lemma 1.4. Let Ω be a connected bounded open set of $\boldsymbol{R}^3$, with a locally Lipschitz boundary Γ. We assume that one of the following conditions hold:

$$\Gamma \text{ is of class } \mathcal{C}^{1,1}, \tag{A.1}$$

or

$$\Omega \text{ is a convex polyhedron.} \tag{A.2}$$

Additionally, we assume that there exist three smooth open subsets $\{\Gamma_1, \Gamma_2, \Gamma_3\}$ of Γ, which verify conditions (1.1) and

$$\Gamma_2 \text{ and } \Gamma_3 \text{ have no common boundary.} \tag{A.3}$$

Define

$$U = \{\mathbf{v} \in L^2(\Omega)^3 \mid \nabla \times \mathbf{v} \in L^2(\Omega)^3,\ \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_1 \cup \Gamma_2,$$
$$\mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_1 \cup \Gamma_3\}. \tag{A.4}$$

Let us recall that if $\mathbf{v}$ is a function of $L^2(\Omega)^3$ such that $\nabla \times \mathbf{v} \in L^2(\Omega)^3$ and $\nabla \cdot \mathbf{v} \in L^2(\Omega)$, then $\mathbf{v} \times \mathbf{n}$ is well defined in $H^{-1/2}(\Gamma)^3$ and $\mathbf{v} \cdot \mathbf{n}$ is well-defined in $H^{-1/2}(\Gamma)$ (see V. Girault & P.A. Raviart [Gi-Ra] Chapter 1); definition (A.4) is thus meaningful. Moreover, U is a Hilbert space with the scalar product:

$$(\mathbf{v}, \mathbf{w})_U = \int_\Omega \mathbf{v} \cdot \mathbf{w} dx + \int_\Omega (\nabla \times \mathbf{v}) \cdot (\nabla \times \mathbf{w}) dx \quad \forall \mathbf{v}, \mathbf{w} \in U. \tag{A.5a}$$

The norm in U will be denoted $\|\cdot\|_U$, i.e.,

$$\|\mathbf{v}\|_U = \{\|\mathbf{v}\|^2_{L^2(\Omega)^3} + \|\nabla \times \mathbf{v}\|^2_{L^2(\Omega)^3}\}^{1/2}. \tag{A.5b}$$

Observe that if $\mathbf{v}$ belongs to U, then $\mathbf{v}$ is zero on Γ_1, since $\mathbf{v} \times \mathbf{n}$ and $\mathbf{v} \cdot \mathbf{n}$ vanish on Γ_1. Furthermore, if V is the space defined by (1.9), and equipped with the norm of $H^1(\Omega)^3$, then $V \subset U$, with continuous embedding. Actually, we have:

THEOREM A.1. *Assume that Ω and Γ verify* (A.1) *or* (A.2) *and that the partition $\{\Gamma_1, \Gamma_2, \Gamma_3\}$ of Γ verify conditions* (1.1) *and* (A.3). *Thus $U = V$, algebraically and topologically.* ■

PROOF. We will use a localization method and Theorems I.3.7, I.3.8 and I.3.9 from V. Girault & P.A. Raviart [Gi-Ra] pp.52–55. First, note that in order to prove this theorem, it is enough to prove that U is included in V with continuous embedding.

Using the fact that Γ_2 and Γ_3 have no common boundary (see (A.3)), we can cover $\overline{\Omega}$ by open balls $\{\theta_i\}_{i=1}^{\ell}$ so that, if $(\theta_i \cap \Omega)$ meets Γ, then the part $(\theta_i \cap \Omega)$ meeting Γ is either included in $(\Gamma_1 \cup \Gamma_2)$ or in $(\Gamma_1 \cup \Gamma_3)$, i.e.,

$$(\theta_i \cap \overline{\Omega}) \cap \Gamma \neq \emptyset \Rightarrow \begin{array}{c} (\theta_i \cap \overline{\Omega}) \cap \Gamma \subset (\Gamma_1 \cup \Gamma_2) \\ \text{or} \\ (\theta_i \cap \overline{\Omega}) \cap \Gamma \subset (\Gamma_1 \cup \Gamma_3), \end{array} \tag{A.6}$$

for all $i = 1, \dots, \ell$. Next, we introduce a partition of unity $\{\alpha_i\}_{i=1}^{\ell}$ subordinate to the open cover $\{\theta_i\}_{i=1}^{\ell}$, i.e.,

$$\alpha_i \in C_0^\infty(\theta_i) \quad \forall i = 1, \dots, \ell. \tag{A.7a}$$

$$0 \leq \alpha_i(x) \leq 1 \quad \forall i = 1, \dots, \ell, \quad \forall x \in \overline{\Omega} \tag{A.7b}$$

$$\sum_{i=1}^{\ell} \alpha_i(x) = 1 \quad \forall x \in \overline{\Omega}. \tag{A.7c}$$

For any function $\mathbf{v}$ of U, we write:

$$\mathbf{v} = \sum_{i=1}^{\ell} \alpha_i \mathbf{v}, \tag{A.7d}$$

and clearly the proof of the fact that U is included in V with continuous embedding simply reduces to prove that for any function $\mathbf{v}$ in U, the function $\alpha_i \mathbf{v}$ belongs to $H^1(\Omega)^3$ and that $\|\alpha_i \mathbf{v}\|_{H^1(\Omega)^3} \leq C\|\mathbf{v}\|_U$. This latter estimate will be first proved for the case (A.1) and next for (A.2):

(i) **Γ is of class $\mathcal{C}^{1,1}$.** In this case, since Γ is smooth and α_i has compact support in θ_i, it is not restrictive to assume that the function $\alpha_i \mathbf{v}$ is defined in an (open) subset θ_i' of $(\theta_i \cap \Omega)$ such that

$$\partial\theta_i' \text{ is of class } \mathcal{C}^{1,1} \tag{A.8a}$$

$$\operatorname{supp}(\alpha_i) \cap \Omega \subset \theta_i', \tag{A.8b}$$

and that θ_i' verifies property (A.6), as it does $(\theta_i \cap \Omega)$ (see Figure A.1).

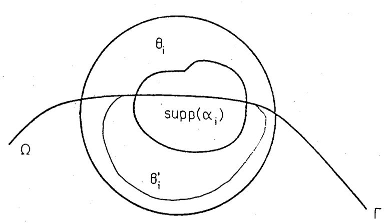

Figure A.1

Therefore, using (A.6), (A.7a), (A.8), and the boundary condition verified by the elements of U, we deduce that $\alpha_i \mathbf{v}$ verifies on $\partial\theta_i'$ one of the following conditions:

$$\begin{aligned} &\alpha_i \mathbf{v} = \mathbf{0} \quad \text{on a subregion } A_i \text{ of } \partial\theta_i' \\ &(\alpha_i \mathbf{v}) \times \mathbf{n} = \mathbf{0} \quad \text{on } \partial\theta_i' \backslash A_i. \end{aligned} \tag{A.9a}$$

$$\alpha_i \mathbf{v} = \mathbf{0} \quad \text{on a subregion } A_i \text{ of } \partial\theta_i'$$
$$(\alpha_i \mathbf{v}) \cdot \mathbf{n} = 0 \quad \text{on } \partial\theta_i' \backslash A_i. \tag{A.9b}$$

Thus,

$$(\alpha_i \mathbf{v}) \times \mathbf{n} = \mathbf{0} \quad \text{on } \partial\theta_i' \tag{A.10a}$$

or

$$(\alpha_i \mathbf{v}) \cdot \mathbf{n} = 0 \quad \text{on } \partial\theta_i'. \tag{A.10b}$$

On the other hand, it is clear that $\alpha_i \mathbf{v}$ belongs to $L^2(\theta_i')^3$, $\nabla \cdot (\alpha_i \mathbf{v})$ belongs to $L^2(\theta_i')$, and that $\nabla \times (\alpha_i \mathbf{v})$ belongs to $L^2(\theta_i')^3$, and then using Theorem I.3.7 of [Gi-Ra] p.52 (if $\alpha_i \mathbf{v}$ verifies (A.10a)) or Theorem I.3.8 of the same reference p.54 (if $\alpha_i \mathbf{v}$ verifies (A.10b)), we see that it follows that $\alpha_i \mathbf{v}$ belongs to $H^1(\theta_i')^3$ and there exists a constant $C_i = C_i(\theta_i')$ such that

$$\|\alpha_i \mathbf{v}\|_{H^1(\theta_i')^3} \leq C_i \{\|\alpha_i \mathbf{v}\|^2_{L^2(\theta_i')^3} + \|\nabla \cdot (\alpha_i \mathbf{v})\|^2_{L^2(\theta_i')} + \|\nabla \times (\alpha_i \mathbf{v})\|^2_{L^2(\theta_i')^3}\}^{1/2}.$$

Since this estimate holds for all $i = 1, \dots, \ell$, it follows that there exists a constant $C = C(\Omega, \{\theta_i\}, \{\alpha_i\})$ such that

$$\|\alpha_i \mathbf{v}\|_{H^1(\theta_i')^3} \leq C \{\|\mathbf{v}\|^2_{L^2(\Omega)^3} + \|\nabla \cdot \mathbf{v}\|^2_{L^2(\Omega)} + \|\nabla \times \mathbf{v}\|^2_{L^2(\Omega)^3}\}^{1/2} \quad \forall i = 1, \dots, \ell,$$

which implies that $\mathbf{v} \in H^1(\Omega)^3$. Since $\nabla \cdot \mathbf{v} = 0$ in Ω, we have:

$$\|\alpha_i \mathbf{v}\|_{H^1(\Omega)^3} \leq C \{\|\mathbf{v}\|^2_{L^2(\Omega)^3} + \|\nabla \times \mathbf{v}\|^2_{L^2(\Omega)^3}\}^{1/2} \quad \forall i = 1, \dots, \ell.$$

This completes the proof of the theorem, if Γ is of class $\mathcal{C}^{1,1}$. ■

(ii) **Ω is a convex polyhedron.** In this case, the proof of Theorem A.1 is very similar to that of the case (i). The only difference arises in how the open sets θ_i' are constructed and how Theorem I.3.8 of [Gi-Ra] is applied.

In this case, we define θ_i' as any convex polyhedron included in $(\Omega \cap \theta_i)$, which verifies the following condition (see Figure A.2):

$$\text{supp}\,(\alpha_i) \cap \Omega \subset \theta_i'. \tag{A.11}$$

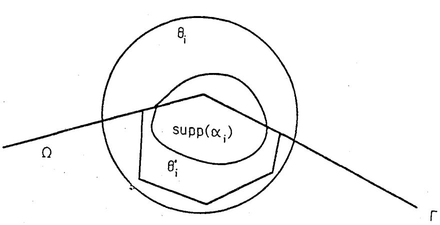

Figure A.2

We note that θ_i' verifies property (A.6). Now, it is enough to use Theorem I.3.9 of [Gi-Ra] p.55 instead of Theorem I.3.8 to conclude that Theorem I.3.7 still applies in this case. This completes the proof of Theorem A.1. ■

PROOF OF LEMMA 1.4. Now, using Theorem A.1, we shall prove Lemma 1.4 by studying successively each one of the three model examples.

MODEL EXAMPLE 1: $\Gamma_1 \neq \emptyset$ (see (1.24a)). In this example, we must prove that the bilinear form $a(\cdot,\cdot)$ is V-elliptic. First, we prove that $a(\cdot,\cdot)$ induces a norm in V.

LEMMA A.2. *Let Ω be a bounded connected open subset of $\boldsymbol{R}^3$, with a locally Lipschitz boundary Γ. Assume that the partition $\{\Gamma_1, \Gamma_2, \Gamma_3\}$ of Γ verifies* (1.1) *and that*

$$\Gamma_1 \neq \emptyset. \tag{A.12}$$

Then the map defined by

$$\mathbf{v} \in V \to \|\nabla \times \mathbf{v}\|_{L^2(\Omega)^3} = \left\{ \int_\Omega |\nabla \times \mathbf{v}|^2 dx \right\}^{1/2}, \tag{A.13}$$

is a norm in V. ■

PROOF. It is clear that (A.13) defines a semi-norm in V. Therefore, to prove Lemma A.2, it suffices to show that if $\mathbf{v} \in V$, $\nabla \times \mathbf{v} = \mathbf{0}$ in Ω, then $\mathbf{v} = \mathbf{0}$.

Using the method introduced by C. Foias & R. Temam [Fo-Te], first, we construct a simply-connected open set Ω° by introducing in Ω a finite number of smooth cuts. More precisely, let $\Sigma_1, \dots, \Sigma_N$ be N two-dimensional $\mathcal{C}^\infty$ manifolds so that $\Sigma_i \cap \Sigma_j = \emptyset$ for $i \neq j$ and such that the open set $\Omega^\circ \equiv \Omega \backslash \Sigma$ is simply-connected, where $\Sigma \equiv \bigcup_{i=1}^N \Sigma_i$ (see Figure A.3). It is worth mentioning that the boundary $(\Sigma \cup \Gamma)$ of Ω° is locally Lipschitz in the sense of R. Temam [Te] p.2.

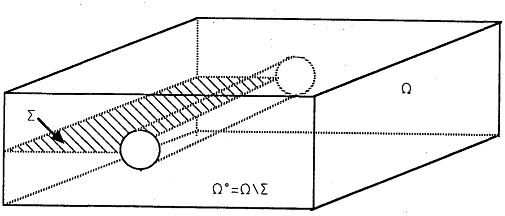

Figure A.3

Let $\mathbf{v} \in V$, $\nabla \times \mathbf{v} = \mathbf{0}$ in Ω. Since $\mathbf{v}$ is a rotational-free function, $\mathbf{v}$ is locally a gradient and more precisely, there exists a unique class q° in $H^1(\Omega^\circ)/\boldsymbol{R}$ such that

$$\mathbf{v} = \nabla q^\circ \quad \text{in} \quad \Omega. \tag{A.14}$$

Since $\mathbf{v}$ is a divergence-free function in Ω (and so, in Ω° too) we have:

$$\Delta q^\circ = \nabla \cdot \mathbf{v} = 0 \quad \text{in} \quad \Omega^\circ. \tag{A.15a}$$

Then q° is a C^∞ function in Ω° (q° is a multi-valued function in Ω, since it can take different values on either side of Σ).

Let Γ_1' be a non-empty connected open subset of Γ_1. Since $\mathbf{v}$ vanishes on Γ_1, we can deduce that

$$q^\circ = 0^\circ \quad \text{on} \quad \Gamma_1' \tag{A.15b}$$
$$\partial q^\circ/\partial n = 0 \quad \text{on} \quad \Gamma_1'. \tag{A.15c}$$

The classical continuation principle for analytic functions and the fact that Ω° is connected imply that $q^\circ = 0^\circ$ in Ω. Then $\mathbf{v} = \mathbf{0}$ and the proof of Lemma A.2 is thus finished. ■

Now, let us prove that $a(\cdot,\cdot)$ induces in V a norm, which is equivalent to the $H^1(\Omega)^3$-norm. First, it can be observed that Theorem A.1 implies the existence of a constant $C_1 = C_1(\Omega)$ such that

$$\|\mathbf{v}\|_{H^1(\Omega)^3} \le C_1\{\|\mathbf{v}\|^2_{L^2(\Omega)^3} + \|\nabla \times \mathbf{v}\|^2_{L^2(\Omega)^3}\}^{1/2} \quad \forall \mathbf{v} \in V. \tag{A.16}$$

Therefore, to prove the V-ellipticity of $a(\cdot,\cdot)$, it suffices to show that there exists a constant $C_2 = C_2(\Omega)$ such that

$$\|\mathbf{v}\|_{L^2(\Omega)^3} \le C_2\|\nabla \times \mathbf{v}\|_{L^2(\Omega)^3} \quad \forall \mathbf{v} \in V. \tag{A.17}$$

To prove this, we use (A.16) and show that the negation of (A.17) leads to a contradiction. If (A.17) does not hold, we can find a sequence $\{\mathbf{v}_n\}$ in V such that:

$$\|\mathbf{v}_n\|_{L^2(\Omega)^3} = 1 \quad \forall n \tag{A.18a}$$
$$\|\nabla \times \mathbf{v}_n\|_{L^2(\Omega)^3} \le 1/n \quad \forall n. \tag{A.18b}$$

Using (A.16), we see that (A.18) implies that the sequence $\{\mathbf{v}_n\}$ remains bounded in V, as $n \to \infty$. It is then possible to extract a subsequence $\{\mathbf{v}_{n'}\}$, and show that there exists $\mathbf{v} \in V$ such that:

$$\mathbf{v}_{n'} \longrightarrow \mathbf{v} \text{ in } H^1(\Omega)^3 \text{ weakly and } L^2(\Omega)^3 \text{ strongly, as } n' \to \infty. \tag{A.19}$$

But (A.18b) implies that $\nabla \times \mathbf{v} = \mathbf{0}$ in Ω. Thus, since $\mathbf{v} \in V$, Lemma A.2 allows us to conclude that $\mathbf{v} = \mathbf{0}$ and hence, (A.18a) is clearly in contradiction to (A.19). This finishes the proof of Lemma 1.4, if Γ_1 is non-empty (model example (1.24a)).

■

MODEL EXAMPLE 2: $\Gamma = \Gamma_2$ (see (1.24b)). As in the previous example, the region Ω can either be simply-connected or not. The boundary Γ of Ω coincides with Γ_2 and the connected components of Γ_2 are then the same as that of Γ. Lemmas II.3 and II.4 of J.M. Domínguez [Do] (or Lemma 2.2 and Theorem 2.4 of A. Bendali, J.M. Domínguez and S. Gallic [Be-Do-Ga]) can be applied to this situation, and they show that the subspace V_1 of V is characterized by

$$V_1 = \{\mathbf{v} \in V \mid \nabla \times \mathbf{v} = \mathbf{0} \text{ in } \Omega\},$$

(then V_1 is the kernel of $a(\cdot,\cdot)$ in V) and that the map

$$\mathbf{v} \in V \longrightarrow \left\{\|\nabla \times \mathbf{v}\|^2_{L^2(\Omega)^3} + \sum_{i=1}^{r} \left|\int_{\Gamma_{2i}} \mathbf{v}\cdot\mathbf{n}ds\right|^2\right\}^{1/2}$$

is a norm of V, equivalent to the norm of $H^1(\Omega)^3$. Thus, the map $\mathbf{v} \in V_0 \rightarrow \|\nabla \times \mathbf{v}\|_{L^2(\Omega)^3}$ is a norm in V_0, and it is equivalent to the norm of $H^1(\Omega)^3$. In this case, the bilinear form $a(\cdot,\cdot)$ is therefore V_0-elliptic with V_1 kernel. This completes the proof of Lemma 1.4 in the case $\Gamma = \Gamma_2$ (model example (1.24b)). ■

MODEL EXAMPLE 3: $\Gamma_1 = \emptyset$, Ω simply connected (see (1.24c)). In this case, we must additionally prove that $a(\cdot,\cdot)$ is V_0-elliptic with V_1 kernel. First, we prove that $a(\cdot,\cdot)$ induces a norm in the space V_0.

LEMMA A.3. *Let Ω be a connected bounded open set of $\boldsymbol{R}^3$, with a locally Lipschitz boundary Γ. If Ω is simply connected, then the map defined by:*

$$\mathbf{v} \in V_0 \rightarrow \|\nabla \times \mathbf{v}\|_{L^2(\Omega)^3} \tag{A.20}$$

is a norm in V_0.

PROOF. Since $\|\nabla \times \mathbf{v}\|_{L^2(\Omega)^3}$ defines a semi-norm in V_0, it suffices to prove that if $\mathbf{v} \in V_0$ and $\nabla \times \mathbf{v} = \mathbf{0}$ in Ω, then $\mathbf{v} = \mathbf{0}$.

Let $\mathbf{v} \in V_0$ be such that $\nabla \times \mathbf{v} = \mathbf{0}$ in Ω. Since Ω is simply connected, it follows from Theorem I.2.9 of [Gi-Ra] p.31 that there exists a unique class q° in $H^1(\Omega)^3/\boldsymbol{R}$ such that

$$\mathbf{v} = \nabla q^\circ \quad \text{in} \quad \Omega. \tag{A.21}$$

Indeed, q° is in $H^2(\Omega)/\boldsymbol{R}$, because $\mathbf{v}$ is in $H^1(\Omega)^3$. Combining (A.21) with the fact that $\mathbf{v}$ is a divergence-free function in Ω, we have:

$$\Delta q^\circ = 0 \quad \text{in} \quad \Omega. \tag{A.22a}$$

But, $\mathbf{v}\cdot\mathbf{n} = 0$ on $(\Gamma_1 \cup \Gamma_3)$ and $\mathbf{v}\cdot\mathbf{n}$ averages to zero on each connected component of Γ_2, then

$$\partial q^\circ/\partial n = 0 \quad \text{on} \quad (\Gamma_1 \cup \Gamma_3) \tag{A.22b}$$

$$\int_{\Gamma_{2i}} (\partial q^\circ / \partial n) ds = 0 \quad \forall i = 1, \ldots, r. \tag{A.22c}$$

On the other hand, $\mathbf{v} \times \mathbf{n} = 0$ on Γ_2, so we have

$$q^\circ = d_i = constant \quad \text{on } \Gamma_{2i}, \quad \forall i = 1, \ldots, r. \tag{A.22d}$$

Let us multiply (A.22a) by q° and integrate by parts in Ω. Using (A.22b), (A22c), (A.22d), we obtain

$$\|\nabla q^\circ\|^2_{L^2(\Omega)^3} = \int_\Gamma (\partial q^\circ / \partial n) q^\circ ds = \sum_{i=1}^{r} d_i \int_{\Gamma_{2i}} (\partial q^\circ / \partial n) ds = 0,$$

which implies that $\nabla q^\circ = \mathbf{0}$, that is, $\mathbf{v} = \mathbf{0}$. Lemma A.3 is therefore proved. ■

To prove that (if (A.3) and (A.1) or (A.2) are fulfilled, then) (A.20) defines a norm in V_0, equivalent to the $H^1(\Omega)^3$-norm (which in fact proves that $a(\cdot, \cdot)$ is V_0-elliptic), it suffices to proceed exactly as in the model example (1.24a), but replacing V by V_0.

Now, let us prove that V_1 is the kernel of $a(\cdot, \cdot)$. To this end, for each $k = 1, \ldots, r-1$, we define q_k of $H^1(\Omega)^3$ as the unique solution of the following boundary-value problem in Ω:

$$\Delta q_k = 0 \quad \text{in } \Omega \tag{A.23a}$$

$$\partial q_k / \partial n = 0 \quad \text{on } \Gamma_3 \tag{A.23b}$$

$$q_k = 1 \quad \text{on } \Gamma_{2k} \tag{A.23c}$$

$$q_k = 0 \quad \text{on } \Gamma_{2j}, \quad \forall j \neq k \tag{A.23d}$$

$$q_k = 1 \quad \text{on } \Gamma_{2r}, \tag{A.23e}$$

and we put

$$\mathbf{z}_k = \nabla q_k. \tag{A.24}$$

The function $\mathbf{z}_k$ belongs to $L^2(\Omega)^3$ and it is a divergence and rotational free function in Ω. Moreover, the boundary conditions of q_k imply that $\mathbf{z}_k \times \mathbf{n} = \mathbf{0}$ on Γ_2 and that $\mathbf{z} \cdot \mathbf{n} = 0$ on Γ_3. Therefore, the function $\mathbf{z}_k$ belongs to the space U, defined by (A.4). According to Theorem A.1, $\mathbf{z}_k$ belongs to V. Since $\nabla \times \mathbf{z}_k = \mathbf{0}$, then $\mathbf{z}_k$ belongs to V_1.

Let us prove that the functions $\mathbf{z}_k$ are linearly independent. Effectively, if $\sum_{k=1}^{r-1} \lambda_k \mathbf{z}_k = \mathbf{0}$ for some $\lambda_k \in \boldsymbol{R}$, then $\sum_{k=1}^{r-1} \lambda_k q_k = c = constant$ in Ω, since Ω is connected. Then we deduce from (A.23c,d) that $\lambda_k = c$ for $k = 1, \ldots, r-1$, and from (A.23e) that $c = 0$ (if $r > 1$).

Now, let $\phi : V_1 \to \boldsymbol{R}^r$ be the mapping defined as follows:

$$\phi(\mathbf{w}) = \left(\int_{\Gamma_{2i}} \mathbf{w} \cdot \mathbf{n} ds, \ldots, \int_{\Gamma_{2r}} \mathbf{w} \cdot \mathbf{n} ds \right).$$

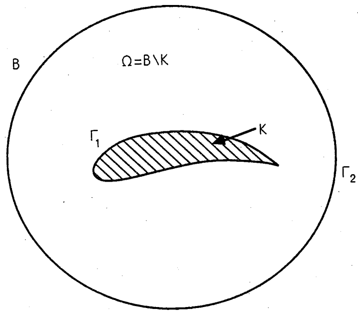

Figure A.4

This mapping is injective because, if $\phi(\mathbf{w}) = \mathbf{0}$, then $\mathbf{w} \in V_0$, and hence $a(\mathbf{w}, \mathbf{w}) = 0$. This implies that $\mathbf{w} = \mathbf{0}$, since $a(\cdot, \cdot)$ is V_0-elliptic. On the other hand, in V_1 there are only divergence-free functions, so we have

$$\sum_{i=1}^{r} (\phi(\mathbf{w}))_i = 0 \quad \forall \mathbf{w} \in V_1.$$

Since ϕ is injective, the dimension of V_1 is then less than or equal to $(r-1)$.

Thereby, $(r-1)$ linearly independent rotational-free functions of V_1 have been constructed. It follows that any element of V_1 is then a rotational-free function. This proves that V_1 is the kernel of $a(\cdot, \cdot)$ in V (see (1.15)) and completes the proof of Lemma 1.4 in the case where Γ_1 is non-empty and Ω is simply connected (model example (1.24c)). ■

A.2. Analysis of the examples of the Introduction. In this section, we prove that in the three examples of the Introduction, the bilinear form $a(\cdot, \cdot)$ is V-elliptic. This result is not merely an application of the theorems of the above section, because in these examples, the region Ω is neither a convex polyhedron nor of class $\mathcal{C}^{1,1}$. We will be led to the above framework by a technical trick.

A.2.1. Example 3 of the Introduction. In this example, Ω is an open ball B from which we remove a closed set K (the obstacle), with a boundary which is not necessarily smooth (see Figure A.4).

The boundary Γ of Ω has two components:

$$\Gamma_1 = \partial K \qquad \Gamma_2 = \partial B.$$

To prove that $a(\cdot, \cdot)$ is V-elliptic, it suffices to identify the space $V = V(\Omega)$, which is defined by:

$$V(\Omega) = \{\mathbf{v} \in H^1(\Omega)^3 \mid \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1, \ \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_2\},$$

with the space $\widetilde{V}(B)$ (of all extensions by zero inside K of functions $\mathbf{v} \in V(B)$) defined by

$$\widetilde{V}(B) = \{\widetilde{\mathbf{v}} \in H^1(B)^3 \mid \nabla \cdot \widetilde{\mathbf{v}} = 0 \text{ in } B,\ \widetilde{\mathbf{v}} = \mathbf{0} \text{ in } K,\ \widetilde{\mathbf{v}} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_2\}.$$

Since the boundary of B is smooth, Lemma 1.4 applies to $V(B)$ and then there exists $\alpha > 0$ such that

$$a(\mathbf{v}, \mathbf{v}) \geq \alpha \|\mathbf{v}\|^2_{H^1(B)^3} \quad \forall \mathbf{v} \in V(B).$$

But, for all $\mathbf{v} \in V(\Omega)$, $\|\mathbf{v}\|^2_{H^1(B)^3} = \|\widetilde{\mathbf{v}}\|^2_{H^1(B)^3}$, which proves the $V(\Omega)$-ellipticity of $a(\cdot, \cdot)$. ■

A.2.2. Example 1 of the Introduction. In this example, the region Ω represents a network of pipes, Γ_1 denotes the lateral surfaces of the pipes, and Γ_2 the in-flow and out-flow sections. We introduce a smooth region B (not necessarily a ball) with a $\mathcal{C}^{1,1}$ boundary such that $\Omega \subset B$ and $\Gamma_2 \subset \partial B$ (see Figure A.5).

To prove that $a(\cdot, \cdot)$ is $V(\Omega)$-elliptic, it suffices to identify $V(\Omega)$ with the subspace $\widetilde{V}(B)$ of $V(B)$, defined by:

$$\widetilde{V}(B) = \{\widetilde{\mathbf{v}} \in H^1(B)^3 \mid \nabla \cdot \widetilde{\mathbf{v}} = 0 \text{ in } B,\ \widetilde{\mathbf{v}} = \mathbf{0} \text{ in } (B \backslash \Omega),\ \widetilde{\mathbf{v}} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_2\},$$

and next, to apply Lemma 1.4 to $V(B)$ as before. ■

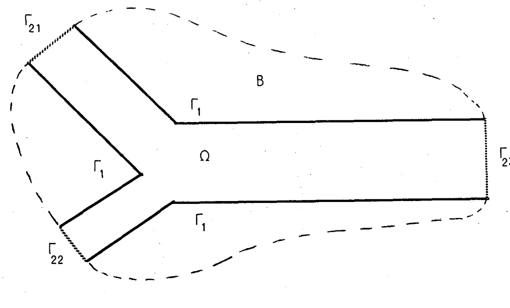

Figure A.5

A.2.3. Example 2 of the Introduction. It is treated by mixing the ingredients used in the previous examples: the functions of $V(\Omega)$ are extended by zero inside the obstacle and into $(B \backslash \Omega)$, where B is a $\mathcal{C}^{1,1}$ open set such that $\Omega \subset B$ and $\Gamma_2 \subset \partial B$ (see Figure A.6).

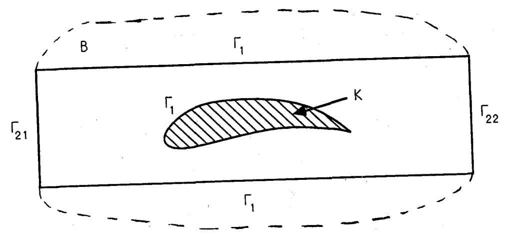

Figure A.6

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