8

Miscellaneous

8.1 The Dirichlet problem for a strongly non-linear equation

Let Ω be a plane domain with a strictly polygonal boundary as in the previous chapters. We keep the same notation as in Section 4.4. We are looking for a function u in Ω which is the solution of

$$-\nabla \cdot \varphi(|\nabla u|)\nabla u + u = f \tag{8,1,1}$$

in Ω, with the Dirichlet boundary condition i.e. $u = 0$ on Γ. In practice the equation $-\nabla \cdot \varphi(|\nabla u|)\,\nabla u = f$ is more often found. The zero-order term in equation (8,1,1) has been added just for technical convenience. See Remark 8.1.8, however. Here f is given in Ω and φ is a positive nondecreasing real function defined on $\mathbb{R}_+ = [0, \infty[$. Therefore we have

$$\varphi(|\nabla u|) \geqslant \varphi(0) > 0$$

and the equation (8,1,1) is elliptic.

Such a problem has been solved by Caccioppoli (1950–51) when Ω is a plane domain with a smooth boundary. This author does not make any assumption on the rate of growth of φ at infinity. In more dimensions, a similar equation has been solved by Ladyzhenskaia and Uralc'eva (1968) under the assumption that the growth of φ is of polynomial type at infinity. This assumption allows one to use an optimization method.

Namely one minimizes a functional related to (8,1,1) in a suitable Sobolev space $\mathring{W}^1_p(\Omega)$. The exponent p is given by the rate of growth of φ at infinity. This method provides a weak solution, while the more classical method of Caccioppoli leads to strong smooth solutions.

Here we want to allow the domain Ω to have corners. The corresponding problem has been solved by Najmi (1978) under the assumption that Ω is convex. The method is very close to Caccioppoli's and requires some smoothness for the solution of the linearized problem. This is the reason why it is assumed that Ω is convex (see Chapters 4 and 5).

For technical purposes it will be convenient to introduce the function

$$a(x)=\varphi(\sqrt{x}), \qquad x\geqslant 0.$$

The function a is positive and nondecreasing. We shall assume in addition that a is C^2. Thus equation (8,1,1) is equivalent to

$$-\nabla\cdot a(|\nabla u|^2)\,\nabla u+u=f \tag{8,1,2}$$

and we shall prove the following result (due to Najmi (1978)):

Theorem 8.1.1 *Let Ω be a plane open and convex set with a strictly polygonal boundary. Let a be any positive non decreasing C^2 function. Then for every $f\in C^{1,\sigma}(\bar{\Omega})$, there exists a unique*

$$u\in W_p^2(\Omega)\cap \mathring{W}_p^1(\Omega)$$

solution of equation (8,1,2) *where $2<p<2/(2-\pi/\omega)$ and ω is the measure of the largest angle of Ω.*

Let us first outline the method of proof. Basically we want to globally invert the non-linear mapping

$$F: u\mapsto -\nabla\cdot a(|\nabla u|^2)\,\nabla u+u$$

between suitably chosen functional spaces. We observe that

$$F(u)=-\sum_{k,l=1}^{2} a_{k,l}(|\nabla u|^2)D_kD_lu+u, \tag{8,1,3}$$

where

$$a_{k,l}(|\nabla u|^2)=a(|\nabla u|^2)\delta_{k,l}+2a'(|\nabla u|^2)D_kuD_lu \tag{8,1,4}$$

for $k, l=1,2$, where $\delta_{k,l}$ is the Kronecker delta. Consequently F is a well-defined mapping from

$$X=W_p^2(\Omega)\cap \mathring{W}_p^1(\Omega) \qquad \text{into } Y=L_p(\Omega), \tag{8,1,5}$$

provided ∇u is continuous for every $u\in W_p^2(\Omega)$. This is achieved by assuming $p>2$ (see Subsection 1.4.5).

A simple criterion for a non-linear mapping to be onto is the following (see e.g. Ambrosetti and Prodi (1973)):

(a) F is *locally invertible* (this can be proved by checking that the Frechet derivative of F is invertible everywhere in X).
(b) F *is proper* i.e. the inverse image of every compact subset of Y is relatively compact in X (this is usually obtained by proving an *a priori* estimate).

In the particular case of F defined by (8,1,3) it will be easy to prove the property (a). Unfortunately we will not be able to prove the full *a priori* estimate which implies the property (b). Thus the above principle will only be a guideline.

Lemma 8.1.2 *F is locally invertible from X into Y provided* $2<p<2/(2-\pi/\omega)$.

Proof The Frechet derivative of F at $u \in X$ is the operator

$$F'(u)\cdot v=\sum_{k,l=1}^{2} D_k[a_{k,l}(|\nabla u|^2)D_l v]+v \tag{8,1,6}$$

where the functions $a_{k,l}$ are defined by (8,1,4). We shall apply Theorem 5.2.1.2 to this operator. Let us check the assumptions of this theorem. (Setting $A=-F'(u)$).

We have

$$\sum_{k,l=1}^{2} a_{k,l}(|\nabla u|^2)\xi_k\xi_l=a(|\nabla u|^2)\,|\boldsymbol{\xi}|^2+2a'(|\nabla u|^2)(\boldsymbol{\xi}\cdot\nabla u)^2\geqslant a(0)\,|\boldsymbol{\xi}|^2.$$

Consequently inequality (5,2,1,1) holds with $\alpha=a(0)$.

It is also easy to check that the coefficients $a_{k,l}(|\nabla u|^2)$ belong to $W_p^1(\Omega)$. Indeed we have

$$\begin{aligned}&D_j a_{k,l}(|\nabla u|^2)\\&\quad=\sum_{i=1}^{2}\{2\delta_{k,l}a'([\nabla u|^2)D_iuD_jD_iu+4a''(|\nabla u|^2)D_iuD_kuD_luD_jD_iu\}\\&\qquad+2a'(|\nabla u|^2)D_luD_jD_ku+2a'(|\nabla u|^2)D_kuD_jD_lu\}.\end{aligned}$$

Each of the terms here is the product of a second derivative of u (i.e. a function belonging to $L_p(\Omega)$) and continuous function of the first derivatives of u (i.e. a bounded function, by the Sobolev imbedding Theorem 1.4.5.2). Consequently we have

$$D_j a_{k,l}(|\nabla u|^2)\in L_p(\Omega),\qquad 1\leqslant j\leqslant 2.$$

Finally inequality (5,2,1,2) is obvious in the particular case under consideration here (we have $a_i=0$, $i\leqslant i\leqslant 2$).

In order to apply Theorem 5.2.1.2, we must calculate $\omega_j(A)$. Since $u\in W_p^2(\Omega)\cap \mathring{W}_p^1(\Omega)$, ∇u is continuous up to the boundary and vanishes at the corners S_j. Accordingly, we have

$$a_{k,l}(|\nabla u(S_j)|^2)=a_{k,l}(0)=a(0)\delta_{k,l}.$$

In the notation of Subsection 5.2.1, this implies that

$$A_j=-a(0)\Delta$$

and $\mathcal{T}_j$ is just the multiplication by $a(0)^{-1/2}$. This implies that

$$\omega_j(A) = \omega_j, \qquad 1 \leqslant j \leqslant N.$$

Theorem 5.2.1.2 shows that $F'(u)$ maps $W_p^2(\Omega) \cap \mathring{W}_p^1(\Omega)$ onto $L_p(\Omega)$, provided

$$\text{Card}\left\{m \in \mathbb{Z} \,\middle|\, -\frac{2}{q} < \frac{m\pi}{\omega_j} < 0\right\} = 0.$$

This is achieved when $2/q < \pi/\omega_j$ for every j when $p < 2/(2 - \pi/\omega)$. ∎

A first step toward the proof of an *a-priori* inequality is the following simple form of the maximum principle (see for instance Protter and Weinberger (1967)).

Lemma 8.1.3 *Let $u \in C^2(\Omega) \cap C^0(\Omega)$ be a solution of the equation*

$$-\sum_{k,l=1}^{2} a_{k,l} D_k D_l u + \sum_{k=1}^{2} a_k D_k u + u = f \qquad \text{in } \Omega$$

where the $a_{k,l}$ and the a_k are continuous functions such that

$$a_{k,l}(x) = a_{l,k}(x), \qquad k, l = 1, 2$$

$$\sum_{k,l=1}^{2} a_{k,l}(x) \xi_k \xi_l \geqslant \alpha\, |\boldsymbol{\xi}|^2$$

for every $\boldsymbol{\xi} \in \mathbb{R}^2$ and $x \in \Omega$ with $\alpha > 0$. Then we have

$$\max_{\bar{\Omega}} |u| \leqslant \max_{\Gamma} |u| + \max_{\bar{\Omega}} |f|. \tag{8,1,7}$$

Proof Let $x_0 \in \bar{\Omega}$ be a point where u reaches its maximum. There are two possible cases: either $x_0 \in \Gamma$, or $x_0 \in \Omega$. If $x_0 \in \Gamma$ we obviously have

$$\max_{\bar{\Omega}} u \leqslant \max_{\Gamma} |u|. \tag{8,1,8}$$

On the other hand, if $x_0 \in \Omega$, the differential equation at x_0 reduces to

$$-\sum_{k,l=1}^{2} a_{k,l}(x_0)(D_k D_l u)(x_0) + u(x_0) = f(x_0).$$

The operator

$$\sum_{k,l=1}^{2} a_{k,l}(x_0) D_k D_l$$

is nothing but the Laplace operator in different coordinates, thus we have

$$\sum_{k,l=1}^{2} a_{k,l}(x_0)(D_k D_l u)(x_0) \leqslant 0.$$

Consequently we have

$$\max_{\bar{\Omega}} u = u(x_0) \leq \max_{\bar{\Omega}} |f|. \tag{8,1,9}$$

Inequalities similar to (8,1,8) (8,1,9) hold for the minimum of u on $\bar{\Omega}$ and this implies (8,1,7). ■

Let us now go back to equation (8,1,2) i.e.

$$F(u) = f,$$

with $u \in W_p^2(\Omega) \cap \mathring{W}_p^1(\Omega)$ and $f \in C^{0,\sigma}(\bar{\Omega})$. The classical interior regularity results imply that $u \in C^{2,\sigma}(\Omega)$ (see for instance Miranda (1970); this can also be easily deduced from results in Section 6.3). In particular, we have

$$u \in C^0(\bar{\Omega}) \cap C^2(\Omega).$$

Consequently Lemma 8.1.3 implies that

$$\max_{\bar{\Omega}} |u| \leq \max_{\bar{\Omega}} |f|. \tag{8,1,10}$$

Next we consider the equations obtained from (8,1,2) by differentiating. Let $v_j = D_j u$, $j = 1, 2$; then we have

$$\begin{aligned} &-\sum_{k,l=1}^{2} a_{k,l}(|\nabla u|^2) D_k D_l v_j \\ &-\sum_{k,l=1}^{2} D_k D_l u \sum_{i=1}^{2} [2\delta_{k,l} a'(|\nabla u|^2) + 4 D_k u D_l u a''(|\nabla u|^2)] D_i u D_i v_j \\ &-2a'(|\nabla u|^2) \sum_{k,l=1}^{2} D_k D_l u [D_l u D_k v_j + D_k u D_l v_j] + v_j = D_j f \qquad \text{in } \Omega. \end{aligned} \tag{8,1,11}$$

Assuming that $f \in C^{1,\sigma}(\bar{\Omega})$ implies that

$$v_j \in C^0(\bar{\Omega}) \cap C^2(\Omega).$$

Applying again Lemma 8.1.3 yields that

$$\max_{\bar{\Omega}} |D_j u| \leq \max_{\Gamma} |D_j u| + \max_{\bar{\Omega}} |D_j f|. \tag{8,1,12}$$

We want to find a bound for $|\nabla u|$ in $\bar{\Omega}$. The last inequality shows that it is enough to find a bound for $|\nabla u| = |\partial u/\partial \nu|$ on Γ. A classical tool is the use of 'barrier functions' (see Oleinik and Radkievitz (1971) for instance).

Lemma 8.1.4 *Let $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ be a solution of the equation*

$$-\sum_{k,l=1}^{2} a_{k,l} D_k D_l u + u = f \qquad \text{in } \Omega \tag{8,1,13}$$

with the boundary condition $u=0$ on Γ. Assume that the $a_{k,l}$ are continuous in Ω such that

$$a_{k,l}(x)=a_{l,k}(x), \qquad k=1,2$$

$$\sum_{k,l=1}^{2} a_{k,l}(x)\xi_k\xi_l \geqslant \alpha\ |\xi|^2$$

for every $\xi\in\mathbb{R}^2$ and $x\in\Omega$, with $\alpha>0$. Then we have

$$\max_{\Gamma}\left|\frac{\partial u}{\partial \nu}\right| \leqslant \frac{\exp d}{\alpha}\max_{\bar{\Omega}}|f|, \tag{8,1,14}$$

where d is the diameter of Ω.

Proof Let us consider one of the sides Γ_j of Ω. We recall that Γ_j is a linear segment and Ω is convex. Let Γ_j be defined by the equation $r=0$ where

$$r=ax_1+bx_2$$

for some real numbers a and b and assume that Ω lies in $r>0$. We define a barrier function w by

$$w(x,y)=C(1-e^{-r}).$$

The function w is nonnegative in $\bar{\Omega}$, provided $C\geqslant 0$. In addition $w=0$ on Γ_j. Consequently we have

$$u-w\leqslant 0 \qquad \text{on } \Gamma \tag{8,1,15}$$

and

$$u-w=0 \qquad \text{on } \Gamma_j. \tag{8,1,16}$$

On the other hand we have

$$-\sum_{k,l=1}^{2} a_{k,l}D_kD_l(u-w)+(u-w)=f-Ce^{-r}\sum_{k,l=1}^{2} a_{k,l}D_krD_lr-w.$$

In addition, it follows from the assumptions on the $a_{k,l}$ that

$$\sum_{k,l=1}^{2} a_{k,l}D_krD_lr\geqslant \alpha\ |\nabla r|^2=\alpha(a^2+b^2).$$

We can assume that $a^2+b^2=1$; consequently we have

$$-\sum_{k,l=1}^{2} a_{k,l}D_kD_l(u-w)+(u-w)\leqslant f-\alpha Ce^{-d},$$

where d is the diameter of Ω. The right-hand side of this last inequality is nonpositive provided

$$C \geqslant \frac{e^d}{\alpha} \max_{\bar{\Omega}} |f|. \tag{8,1,17}$$

It follows that $u-w$ cannot have a strictly positive maximum inside Ω. Together with inequality (8,1,15) this implies that

$$u-w \leqslant 0$$

everywhere in Ω. Then from (8,1,16) it follows that

$$\frac{\partial}{\partial \nu_j}(u-w) \geqslant 0 \qquad \text{on } \Gamma_j.$$

In the same way we show that

$$\frac{\partial}{\partial \nu_j}(-u-w) \geqslant 0 \qquad \text{on } \Gamma_j$$

just by replacing u by $-u$ in the above considerations. This yields

$$\left|\frac{\partial u}{\partial \nu_j}\right| \leqslant \left|\frac{\partial w}{\partial \nu_j}\right| = C. \tag{8,1,18}$$

Summing up inequality (8,1,14) follows from (8,1,17) and (8,1,18). ∎

The above result can be applied to u solution of $F(u)=f$, and with the help of (8,1,10) and (8,1,12) we conclude that

$$\max_{\bar{\Omega}} |u| + \max_{\bar{\Omega}} |\nabla u| \leqslant K\left\{\max_{\bar{\Omega}} |f| + \max_{\bar{\Omega}} |\nabla f|\right\} \tag{8,1,19}$$

provided $u \in W_p^2(\Omega) \cap \mathring{W}_p^1(\Omega)$, $f \in C^{1,\sigma}(\bar{\Omega})$. Here K depends only on α and d.

The above inequality will not be enough for our purpose. Actually we shall need a bound for a Hölder norm of the gradient of u. This will be achieved with the help of a deep regularity result due to Caccioppoli (1950–51). This result concerns the smoothness of u inside Ω; however, it will be easy to extend it into a smoothness result up to the boundary, when Ω is a convex polygon.

Let us recall Caccioppoli's result. A reasonably simple proof can be found in Talenti (1966).

Lemma 8.1.5 *Let Ω be a bounded open subset of $\mathbb{R}^2$. Let*

$$u \in H^2(\Omega) \cap \mathring{H}^1(\Omega)$$

be the solution of

$$-\sum_{k,l=1}^{2} a_{k,l}D_kD_lu+u=f \qquad \textit{in } \Omega$$

where the $a_{k,l}$ are bounded measurable functions in Ω such that

$$a_{k,l}=a_{l,k} \qquad \text{a.e. } \textit{in } \Omega$$

and such that there exists λ_1, λ_2 with $0<\lambda_1\leqslant\lambda_2<+\infty$ such that

$$\lambda_1|\xi|^2\leqslant\sum_{k,l=1}^{2} a_{k,l}\xi_k\xi_l\leqslant\lambda_2|\xi|^2$$

for every $\xi\in\mathbb{R}^2$ and a.e. *in Ω. Assume that $f\in L_p(\Omega)$ with $p>2$. Then for every compact subset K in Ω there exists $\mu\in]0,1[$† which depends only on λ_1, λ_2 and p, and there exists C such that*

$$\|\nabla u\|_{\mu,\infty,K}\leqslant C\{\|f\|_{0,p,\Omega}+\|u\|_{1,2,\Omega}\}. \tag{8,1,20}$$

We shall improve this result as follows.

Corollary 8.1.6 *Assume that Ω is a convex plane polygon and assume all the hypotheses of Lemma 8.1.5. Then $\nabla u\in C^{0,\mu}(\bar{\Omega})$ and*

$$\|\nabla u\|_{\mu,\infty,\bar{\Omega}}\leqslant C\{\|f\|_{0,p,\Omega}+\|u\|_{1,2,\Omega}\}. \tag{8,1,21}$$

Proof We already know that $\nabla u\in C^{0,\mu}(\Omega)$. We must investigate the behaviour of ∇u near the boundary.

Let us consider one of the sides Γ_j. After rotation and translation we can assume that Γ_j lies on the axis $\{x_2=0\}$ and that Ω lies above the axis. We perform a reflection with respect to $x_2=0$ by setting

$$U(x_1,x_2)=\begin{cases} u(x_1,x_2), & x_2\geqslant 0\\ -u(x_1,-x_2), & x_2<0.\end{cases}$$

Then U is the solution of the equation

$$-\sum_{k,l=1}^{2} A_{k,l}D_kD_lU+U=F \qquad \text{in } \omega, \tag{8,1,22}$$

where F is defined from f in the same way as U was defined from u and where

$$A_{k,l}(x,y)=\begin{cases} a_{k,l}(x_1,x_2), & x_2\geqslant 0\\ (-1)^{k-l}a_{k,l}(x_1,-x_2), & x_2<0.\end{cases} \tag{8,1,23}$$

† $\mu\in]0,\inf(\lambda_1/\lambda_2, 1-2/p)[$.

Here ω denotes the set

$$\omega = \Omega \cup \Gamma_j \cup \check{\Omega}$$

where

$$\check{\Omega} = \{(x_1, -x_2) \mid (x_1, x_2) \in \Omega\}.$$

Now the main remark is that the $A_{k,l}$ are still bounded measurable and that

$$\lambda_1 |\xi|^2 \leq \sum_{k,l=1}^{2} A_{k,l}\xi_k\xi_l \leq \lambda_2 |\xi|^2$$

for every $\xi \in \mathbb{R}^2$ and a.e. in ω. It is also clear that $F \in L_p(\omega)$ and that $U \in H^2(\omega) \cap \mathring{H}^1(\omega)$. Consequently we can apply Lemma 8.1.5 to U. This shows that ∇u is Hölder continuous up to the interior of Γ_j.

Finally let us consider one of the corners S_j. An affine change of coordinates reduces the general case to the case when $S_j = 0$, Γ_j lies on the x_2-axis above zero and Γ_{j+1} lies on the x_1-axis on the right of zero. (Here we have really used the convexity assumption.) Then we perform a double reflection through $\{x_1 = 0\}$ and $\{x_2 = 0\}$ by setting

$$U(x_1, x_2) = \begin{cases} u(x_1, x_2) & x_1, x_2 \geq 0 \\ -u(x_1, -x_2) & x_1 \geq 0, x_2 < 0 \\ -u(-x_1, x_2) & x_1 < 0, x_2 \geq 0 \\ u(-x_1, -x_2) & x_1 < 0, x_2 < 0. \end{cases}$$

In the same way, we define F from f. Then U is solution of (8,1,22) (there is no point here in describing the corresponding $A_{k,l}$ in full detail), where ω denotes now the set

$$\{(x_1, x_2) \mid (\pm x_1, \pm x_2) \in \Omega \cup \Gamma_j \cup \Gamma_{j+1} \cup \{0\}\}.$$

Again, we have $F \in L_p(\omega)$, $U \in H^2(\omega) \cap \mathring{H}^1(\omega)$ and we can define λ_1^* and λ_2^* such that $0 < \lambda_1^* \leq \lambda_2^* < +\infty$ and such that

$$\lambda_1^* |\xi|^2 \leq \sum_{k,l=1} A_{k,l}\xi_k\xi_l \leq \lambda_2^* |\xi|^2$$

for every $\xi \in \mathbb{R}^2$ and a.e. in ω. Applying Lemma 8.1.5 again shows that ∇u is Hölder continuous near S_j. ■

Let us again consider $u \in X$, a solution of $F(u) = f$; thus u is a solution of (8,1,3). We shall apply Corollary 8.1.6 to u. First we have to find λ_1 and λ_2 such that the assumptions hold. We have already shown that we can set

$$\lambda_1 = a(0).$$

On the other hand inequality (8,1,19) provides a bound $(K\|f\|_{1,\infty,\bar{\Omega}})$ for $|\nabla u|$. Consequently, we have

$$\sum_{k,l=1}^{2} a_{k,l}(|\nabla u|^2)\xi_k\xi_l \leqslant \{a(K^2\|f\|_{1,\infty,\bar{\Omega}}) + 2a'(K^2\|f\|_{1,\infty,\bar{\Omega}})K^2\|f\|_{1,\infty,\bar{\Omega}}\}\,|\xi|^2.$$

This gives a value for λ_2.

Summing up, we have proved the following result.

Proposition 8.1.7 *Let* $u \in W_p^2(\Omega) \cap \mathring{W}_p^1(\Omega)$ *be a solution of equation* (8,1,2) *where* Ω *is a plane open and convex set with a strictly polygonal boundary. Assume that* $f \in C^{1,\sigma}(\bar{\Omega})$; *then there exists* $\mu \in]0, 1[$ (μ *depends on* $\|f\|_{1,\infty,\bar{\Omega}}$) such that $\nabla u \in C^{0,\mu}(\bar{\Omega})$, *and there exists a constant* C (C *depends on* $\|f\|_{1,\infty,\bar{\Omega}}$) such that (8,1,21) holds.

Now we have all the preliminary material for proving Theorem 8.1.1.

Proof of Theorem 8.1.1 We introduce the subset $\mathcal{N}$ of $[0, 1]$ defined by the following condition (on $t \in [0, 1]$): there exists $u_t \in X$ which is a solution of

$$F(u_t) = tf.$$

We shall prove that $\mathcal{N}$ is connected. Since $\mathcal{N}$ obviously contains the value $t = 0$, it will follow that $\mathcal{N}$ also contains the value $t = 1$, which is the claim of Theorem 8.1.1 (as far as existence is concerned).

Lemma 8.1.2 implies that $\mathcal{N}$ is open. Then we have to show that $\mathcal{N}$ is closed. Let us consider a sequence t_j, $j = 1, 2, \ldots$, of numbers in $\mathcal{N}$ which converges to some limit t. We have to check that $t \in \mathcal{N}$. Since $t_j \in \mathcal{N}$, there exists a solution $u_{t_j} \in X$ of

$$F(u_{t_j}) = t_j f.$$

From Proposition 8.1.7 the sequence $u_{t_j}, j = 1, 2, \ldots$ is bounded in $C^{1,\mu}(\bar{\Omega})$ for some $\mu \in]0, 1[$. By Ascoli's theorem we know that we can find a subsequence (which we will also denote by $u_{t_j}, j = 1, 2, \ldots$ for simplicity) which converges to some limit u in the topology of $C^1(\bar{\Omega})$. In particular ∇u_{t_j} converges uniformly to ∇u. Consequently

$$a(|\nabla u_{t_j}|^2) \to a(|\nabla u|^2)$$

uniformly.

Since u_{t_j} is a solution of

$$-\nabla \cdot a(|\nabla u_{t_j}|^2)\,\nabla u_{t_j} + u_{t_j} = t_j f \quad \text{in } \Omega,$$

it follows that

$$-\nabla \cdot a(|\nabla u|^2)\,\nabla u + u = tf \qquad \text{in } \Omega$$

in the sense of distributions. By uniform convergence we also have

$$u = 0 \qquad \text{on } \Gamma.$$

On the other hand $u \in C^{1,\mu}(\bar{\Omega})$ since the sequence u_{t_j} is bounded in $C^{1,\mu}(\bar{\Omega})$. Thus the function $a(|\nabla u|^2)$ is also a Hölder continuous. Consequently Theorem 5.2.1.2 implies that u actually belongs to $W^2_p(\Omega)$† since we have assumed that $2 < p < 2/(2 - \pi/\omega)$. In conclusion we have shown that $t \in \mathcal{N}$ (setting $u_t = u$).

Finally the uniqueness of the solution u in X of the equation (8,1,2) is easily checked with the help of the usual monotonicity argument. Indeed let us consider the functional

$$u \mapsto \Psi(u) = \int_\Omega \psi(|\nabla u|^2)\,\mathrm{d}x$$

on X, where $\psi'(t) = a(t)$ for $t \geq 0$. We have also

$$\Psi(u) = \int_\Omega \eta(|\nabla u|)\,\mathrm{d}x$$

where $\eta(t) = \psi(t^2)$. Thus η is a nondecreasing and convex function.

$$\begin{cases} \eta'(t) = 2ta(t^2) \geq 0 \\ \eta''(t) = 2a(t^2) + 4t^2a'(t^2) \geq 2a(0). \end{cases}$$

This implies that Ψ is convex on X and that its Frechet derivative is monotonous. Consequently let u' and u'' be two solutions to the problem (8,1,2); we have

$$\begin{aligned} 0 &= \int_\Omega -\nabla \cdot \{a(|\nabla u'|^2)\,\nabla u' - a(|\nabla u''|^2)\,\nabla u''\}(u' - u'')\,\mathrm{d}x + \int_\Omega |u' - u''|^2\,\mathrm{d}x \\ &\geq \int_\Omega |u' - u''|^2\,\mathrm{d}x. \end{aligned}$$

Thus we have $u' = u''$ and this proves the desired uniqueness. ■

Remark 8.1.8 As we have already mentioned at the beginning of this section, one is more likely to find the equation

$$-\nabla \cdot \varphi(|\nabla u|)\,\nabla u = f \tag{8,1,24}$$

in practical problems. The equation (8,1,1) is obtained by adding a zero-order term for technical convenience. Actually adding this zero-order term has been an important simplification only for the proof of

† Here, we take advantage of the uniqueness of the solution $w \in \mathring{H}^1(\Omega)$ of the equation

$$-\nabla \cdot a(|\nabla u|^2)\,\nabla w + w = tf \qquad \text{in } \Omega.$$

Lemma 8.1.3. A similar maximum principle for the equation without zero-order term can be found in Stampacchia (1965) (Remark 4,4, p. 119). This allows one to show that the result of Theorem 8.1.1 is also valid for equation (8,1,24).

8.2 Some three-dimensional results (an outline)

In Chapters 2 and 3 we proved some smoothness for the solution of an elliptic boundary value problem in a subset Ω of $\mathbb{R}^n$ without any restriction on n. Then, in Chapters 4–7 we assumed that Ω was a plane domain. There we obtained solutions which split into two parts: one regular (in the sense of a suitable Sobolev norm) and the other singular but very explicitly described. Very few similar results are known when Ω is a three-dimensional domain. This subsection is an outline of them. For the sake of simplicity, we shall restrict our purpose to self-adjoint problems (in other words we exclude the oblique boundary conditions).

Let us start with a few remarks about the results proved in Chapter 3. Now we consider Ω a bounded open subset of $\mathbb{R}^3$ whose boundary is a polyhedron. More precisely we assume that the boundary Γ of Ω is the union of a finite number of faces Γ_j, $1 \leqslant j \leqslant N$, each of which is plane. We assume that Ω lies on one side of each of the Γ_j. We denote by $A_{j,k}$ the edge between Γ_j and Γ_k, $\omega_{j,k}$ being the measure of the corresponding angle (inside Ω). Finally we denote by S the set of all the vertices and by A the union of all the edges.

Next we split the set $\{1, 2, \ldots, N\}$ into two non-overlapping subsets $\mathcal{D}$ and $\mathcal{N}$ exactly as we did in Chapter 4. The boundary problem under consideration is the following. Given f we look for u such that

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \gamma_j u = 0 & \text{on } \Gamma_j, \quad j \in \mathcal{D}, \\ \gamma_i \dfrac{\partial u}{\partial \nu_j} = 0 & \text{on } \Gamma_j \quad j \in \mathcal{N}. \end{cases} \tag{8,2,1}$$

If we assume that f belongs to $L_p(\Omega)$, $\frac{6}{5} \leqslant p < \infty$, then the problem has a unique variational solution $u \in H^1(\Omega)$. As in the two-dimensional case the main question is to know what amount of smoothness for u can be derived from the assumption that $f \in W_p^m(\Omega)$ for some integer $m \geqslant 0$.

Theorem 3.1.1.2 implies the existence of a constant C (which does not depend on Ω) such that

$$\sum_{|\alpha|=2} \|D^\alpha u\|_{0,2,\Omega} \leqslant C \, \|f\|_{0,2,\Omega}.$$

Then if we allow C to depend only on the diameter of Ω and if we assume for simplicity that $\mathscr{D}$ is not empty, we have

$$\|u\|_{2,2,\Omega} \leq C\,\|f\|_{0,2,\Omega} \tag{8,2,2}$$

for u satisfying (8,2,1). Indeed the curvature $\mathscr{B}$ is zero on each face Γ_j. This is the basic *a priori* inequality.

Then if $\mathcal{N}$ is empty (a Dirichlet problem) or if $\mathscr{D}$ is empty (a Neumann problem) we have a smoothness result when Ω is convex. Indeed if f belongs to $L_2(\Omega)$, the corresponding solution u belongs to $H^2(\Omega)$.

These are the only results that follow easily from the general statements in Chapter 3. On the other hand, when f is given in $W_p^m(\Omega)$, it follows from Lemma 2.4.1.4 and Theorem 2.5.1.1 that $u \in W_p^{m+2}(\Omega \setminus V)$ for any neighbourhood V of A. We shall now investigate the behaviour of u near an edge and later a vertex (in this latter case we shall also consider domains whose boundary has conical points).

8.2.1 Edges

The basic idea (to be stated rigorously later) is that an edge with measure ω leads to a smooth solution iff an angle with the same measure ω leads to a smooth solution (in the two-dimensional problems). Otherwise if ω is large enough to produce some singular solution in the two-dimensional case, then an edge with the same measure will produce infinitely many singular solutions.

Let us denote by x, y, z the three coordinates and use cylindrical coordinates around the z-axis (i.e. we use polar coordinates in the xOy plane: $x + iy = re^{i\theta}$). In addition let us consider a polyhedron Ω such that one of its edges is on the z-axis; in other words there exist j and k such that

$$A_{j,k} = \{(0, 0, z) \mid a < z < b\} \tag{8,2,1,1}$$

for some real numbers a and b with $a < b$. We also assume that the face Γ_j is contained in the xOz plane.

We introduce two other real numbers a' and b' with $a < a' < b' < b$ and a cut-off function depending only on r such that $\eta(r) = 1$ for $r < \rho$ and $\eta(r) = 0$ for $r > 2\rho$, where ρ is chosen such that

$$K = \{(x, y, z) \mid 0 < r \leq 2\rho,\ 0 < \theta < \omega_{j,k},\ a' \leq z \leq b'\} \subset \Omega.$$

Lemma 8.2.1.1 *Define the function u by*

$$u(x, y, z) = \eta(r) r^{l\pi/\omega_{j,k}} \sin \frac{l\pi\theta}{\omega_{j,k}}\, \varphi(z)$$

where $\varphi \in \mathcal{D}(]a', b'[)$. *Then* $u \in \mathring{H}^1(\Omega)$, $\Delta u \in W_p^m(\Omega)$ *and* $u \notin W_p^{m+2}(\Omega)$ *provided* l *is an integer such that*

$$\max\left\{0, \frac{\omega_{j,k}}{\pi}\left(m-\frac{2}{p}\right)\right\} < l < \frac{\omega_{j,k}}{\pi}\left(m-\frac{2}{p}\right)+2\frac{\omega_{j,k}}{\pi}. \tag{8,2,1,2}$$

The proof of this lemma is very easy. Indeed the support of u is contained in K and one applies Theorem 1.4.5.3.

This lemma produces infinitely many singular solutions, provided the condition (8.2.1.2) on l is not empty, since φ is allowed to span an infinite-dimensional space.

The above functions u fulfil the Dirichlet boundary condition. Similar singular solutions for other (self-adjoint) boundary conditions are easily built: the function

$$u(x, y, z) = \eta(r) r^{l\pi/\omega_{j,k}} \cos \frac{l\pi\theta}{\omega_{j,k}} \varphi(z)$$

fulfils a Neumann boundary condition on Γ_j and Γ_k (and vanishes in a neighbourhood of all the other faces) and the function

$$u(x, y, z) = \eta(r) r^{(l-\frac{1}{2})\pi/\omega_{j,k}} \sin (l-\tfrac{1}{2}) \frac{\pi\theta}{\omega_{j,k}} \varphi(z)$$

fulfils a Dirichlet condition on Γ_j and a Neumann condition on Γ_k (and vanishes near all the other faces).

In particular a nonconvex edge produces infinitely many variational solutions u of the Dirichlet problem (or the Neumann problem) such that $\Delta u \in L_2(\Omega)$, while $u \notin H^2(\Omega)$.

The above lemma has a negative character; let us now turn to a rather obvious regularity result.

Theorem 8.2.1.2 *Let* $u \in H^1(\Omega)$ *be the solution of the problem* (8,2,1) *with* f *given in* $C^\infty(\bar{\Omega})$. *Then*

$$u \in W_p^m(\Omega \setminus V)$$

where V *is any neighbourhood of* S (*the set of all the vertices*) *provided*

(a) $m - 2/p < \pi/\omega_{j,k}$ *when* j *and* $k \in \mathcal{D}$ *or when* j *and* $k \in \mathcal{N}$.
(b) $m - 2/p < \pi/2\omega_{j,k}$ *when* $j \in \mathcal{D}$ *and* $k \in \mathcal{N}$ *or when* $j \in \mathcal{N}$ *and* $k \in \mathcal{D}$.

Proof We look at the behaviour of u near $A_{j,k}$. After rotation and translation we reduce the general case to the particular case when $A_{j,k}$ is on the Oz axis (i.e. (8,2,1,1) holds and we keep the same notation as in Lemma 8.2.1.1 assuming that Γ_j is in the xOz plane).

It is easy to check that u is regular in the z variable. For this purpose, we replace u by

$$v(x, y, z) = \eta(r)\varphi(z)u(x, y, z),$$

with $\varphi \in \mathscr{D}(]a', b'[)$. We already know that u is regular inside Ω; thus we have

$$\Delta v - \eta\varphi'' u - 2\eta\varphi' \frac{\partial u}{\partial z} \in C^\infty(\bar{\Omega}). \tag{8,2,1,3}$$

Since we start from $u \in H^1(\Omega)$, we have $v \in H^1(\Omega)$

$$\Delta v \in L_2(\Omega).$$

In addition the support of v is contained in K and v fulfils the same boundary conditions as u.

Let τ_h be the translation operator in the direction of z, i.e.

$$\tau_h w(x, y, z) = w(x, y, z + h).$$

The function $(\tau_h v - v)/h$ is a solution of the same boundary conditions as u when h is small enough.

In addition, we have

$$\frac{\tau_h v - v}{h} \in H^1(\Omega).$$

Integrating by parts the integral

$$\int_\Omega (-\Delta + 1) \frac{\tau_h v - v}{h} \frac{\tau_h v - v}{h} \, dx \, dy \, dz$$

yields the following inequality

$$\left\| \frac{\tau_h v - v}{h} \right\|_{1,2,\Omega} \leq \left\| (-\Delta + 1) \frac{\tau_h v - v}{h} \right\|_{-1,2,\Omega}.$$

Taking the limit as $h \searrow 0$, we obtain

$$\left\| \frac{\partial v}{\partial z} \right\|_{1,2,\Omega} \leq \left\| \frac{\partial}{\partial z} (-\Delta + 1) v \right\|_{-1,2,\Omega}$$

and this shows that $\partial v/\partial z \in H^1(\Omega)$. Then letting φ be any function in $\mathscr{D}(]a', b'[)$, we derive that

$$\eta\psi \frac{\partial u}{\partial z} \in H^1(\Omega)$$

for every $\psi \in \mathscr{D}(]a', b'[)$.

Turning back to (8,2,1,3), we have now

$$\Delta v \in H^1(\Omega)$$

and we apply again the same procedure as above to the function

$$\left(\frac{\tau_h - 1}{h}\right)^2 v.$$

By iteration we eventually prove that

$$\eta\psi \frac{\partial^k u}{\partial z^k} \in H^1(\Omega)$$

for every integer $k \geqslant 1$ and every $\psi \in \mathscr{D}(]a', b'[)$. Thus ηu is infinitely differentiable function of z (in $]a', b'[$) with values in $H^1(G)$, where we define G as follows:

$$G = \{(x, y) \mid 0 < r < 2\rho, 0 < \theta < \omega_{j,k}\}.$$

In particular ηu is an infinitely differentiable function with values in $L_p(G)$, for every $p \in]1, \infty[$. Now let us denote by Δ' the Laplace operator in the variables x and y. We have

$$\Delta'(\eta u) = \Delta(\eta u) - \eta D_z^2 u. \tag{8,2,1,4}$$

We already know that u is regular in $\bar{\Omega} \setminus A$; thus we have

$$\Delta(\eta u) \in C^\infty(\bar{K})$$

and consequently $\Delta'(\eta u)$ is infinitely differentiable with values in $L_p(G)$. Applying Theorem 4.4.3.7 shows that ηu is infinitely differentiable with values in $W_p^2(G)$ provided p is such that $2 - 2/p < \pi/\omega_{j,k}$ when j and k belong to the same set ($\mathscr{D}$ or $\mathscr{N}$) and $2 - 2/p < \pi/2\omega_{j,k}$ otherwise.

Going back to (8,2,1,4), we see now that $\Delta'(\eta u)$ is infinitely differentiable with values in $W_p^2(G)$. Then Theorem 5.1.3.5 shows that ηu is infinitely differentiable with values in $W_p^4(G)$ provided p is such that $4 - 2/p < \pi/\omega_{j,k}$ when j and k belong to the same set ($\mathscr{D}$ or $\mathscr{N}$) and $4 - 2/p < \pi/2\omega_{j,k}$ otherwise.

Iterating the above procedure yields the desired result. ■

Remark 8.2.1.3 When the condition (a) (respectively (b)) is violated, the above proof shows that there exist functions $k_l \in C^\infty(]a', b'[)$ such that

$$u(x, y, z) - \sum_{-2+2/p<\lambda_l<0} k_l(z)\mathfrak{S}_l(x, y)$$

is an infinitely differentiable function of z with values in $W_p^2(G)$. (We recall here that a' and b' are any real numbers such that $a < a' < b' < b$ and consequently k_l belongs to $C^\infty(]a, b[)$. In other words $k_l \in C^\infty(A_{j,k})$.)

Here the notation is the following:

$$\lambda_l = \frac{l\pi}{\omega_{j,k}} \qquad \text{when } j \text{ and } k \text{ both belong to } \mathscr{D} \text{ or } \mathscr{N}$$

$$\lambda_l = (l+\tfrac{1}{2})\frac{\pi}{\omega_{j,k}} \qquad \text{otherwise.}$$

In addition

$$\mathfrak{S}_l(x, y) = r^{-\lambda_l}\sin(\lambda_l\theta), \qquad \text{when } j \text{ and } k \text{ belongs to } \mathscr{D}$$

$$\mathfrak{S}_l(x, y) = r^{-\lambda_l}\cos(\lambda_l\theta), \qquad \text{when } j \text{ and } k \text{ belong to } \mathscr{N}$$

$$\mathfrak{S}_l(x, y) = r^{-\lambda_l}\sin(\lambda_l\theta), \qquad \text{when } j \in \mathscr{D} \text{ and } k \in \mathscr{N}$$

when λ_l is not an integer (go back to Definition 5.1.3.4 for the modified definition of $\mathfrak{S}_l$, when λ_l is an integer).

Remark 8.2.1.4 The assumption that $f \in C^\infty(\bar{\Omega})$ in Theorem 8.2.1.2 is not necessary. Assuming only that f belongs to $W_p^{m-2}(\Omega)$ (with $m \geq 2$) leads to the same conclusion; however the proof is much more complicated. It can be found in Kondratiev (1970) when $p=2$ and in Grisvard (1975a) in the general case. Unfortunately we do not know the amount of regularity of the functions k_l (cf. the previous remark) that follows from the assumption that $f \in W_p^{m-2}(\Omega)$.†

Additional results on the behaviour of k_1 when $p=2$ are derived in Grisvard (1982).

Remark 8.2.1.5 As in Section 5.3, one can extend the result of Theorem 8.2.1.2 to the case of an operator with variable coefficients in a domain with a curvilinear edge (a precise definition is left to the taste of the reader). The basic idea is still that the solution belongs to a given Sobolev space iff the two dimensional angles with the same measure as the edge (at any point of the edge) yield regularity (in the corresponding Sobolev space in two variables of course).

8.2.2 Conical points and vertices

We proceed with our investigation of the behaviour of u, the solution of problem (8,2,1), by considering u near one of the vertices. For convenience, we translate this vertex to zero. Thus, in a neighbourhood of 0, Ω

† The particular case when $j \in \mathscr{D}$, $k \in \mathscr{N}$ and $\omega_{j,k} = \pi$ (mixed boundary condition along a 'flat' edge) has been investigated in Eskin (1973). This author has given an explicit formula for the functions k_l involving the data f. Inspection of this formula easily shows that the assumption that f belongs to $H^s(\Omega)$ implies that k_0 belongs to $H^{s+1/2}(A_{j,k})$. Related results can be found in Nikishkin (1979) and Maz'ya and Plamenevskiĭ (1978).

coincides with a cone C whose intersection with the unit sphere S^2 is denoted by G. Thus G is an open subset of the unit sphere whose boundary is the union of a finite number of arcs of great circles.

Here, in order to include cones which have a regular basis, we shall make a more general assumption on G. We shall only assume that ∂G is a curvilinear polygon (a definition similar to Definition 1.4.5.1 can be made here; roughly speaking ∂G is the union of a finite number of curves which cut at angles and G lies on one side of each of these curves only).

As it is natural we introduce the spherical coordinates (r, θ, φ). For the sake of definiteness φ denotes the angle between OM (M the point whose coordinates are x, y and z) and the z-axis; θ denotes the angle between Om (m the projection of M on the xOy plane) and the x-axis.

The basic idea here is that such a cone can produce two kinds of singular solution:

(a) Solutions of the form

$$u(x, y, z) = r^{\lambda}\psi(\theta, \varphi) \tag{8,2,2,1}$$

where ψ is an eigenfunction of the Laplace–Beltrami operator Δ' in G (with the suitable boundary conditions) and λ is related to the eigenvalue. These are singular functions similar to the singular functions $\mathfrak{S}$ of the two-dimensional case (cf. Chapter 4). Such singular functions will arise even when the cone C has a regular basis (i.e. G is smooth). The amount of singularity is related to λ and only a finite number of such functions will be generated outside a given Sobolev space.

(b) Solutions of the form

$$u(x, y, z) = \psi(r)\mathfrak{S}(\theta, \varphi) \tag{8,2,2,2}$$

where ψ is a regular function of r, at least away from zero, while $\mathfrak{S}$ is a singular solution of Δ' on G (again fulfilling suitable boundary conditions which will be made precise later). These are singular functions similar to those produced by an edge (cf. Remark 8.2.1.3). Such singular functions arise only when G has corners (or there is a mixed boundary condition). In addition, ψ spans an infinite-dimensional space and accordingly there are infinitely many such singular solutions.

In order to make the above outline more precise we shall first state a result which shows how the singular solutions (8,2,2,1) arise. For this purpose it is more convenient to consider a domain $\Omega \subset \mathbb{R}^3$ having only conical points corresponding to a regular basis $G \subset S^2$. Exactly as in Chapter 4 we shall start from an *a priori* inequality in $H^2(\Omega)$, which has been derived in Hanna and Smith (1967).

We consider here a bounded open subset Ω of $\mathbb{R}^3$ such that 0 belongs to its boundary Γ. We assume that $\Gamma\backslash\{0\}$ is of class C^2; we assume in

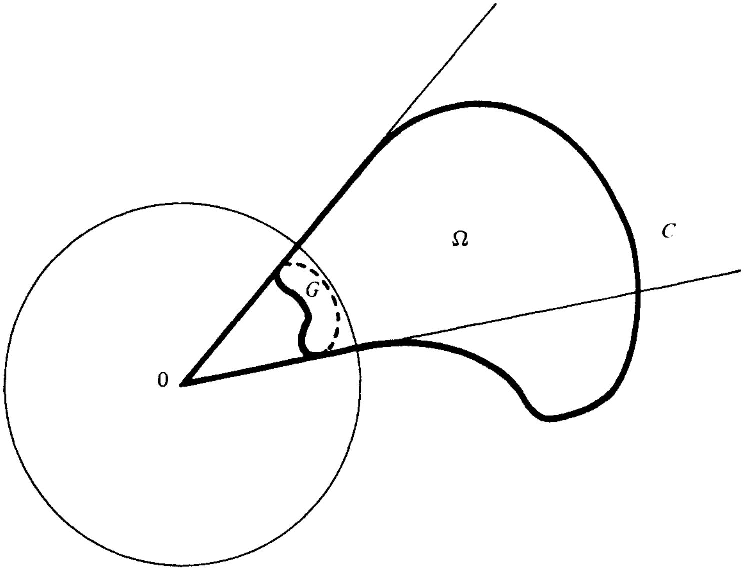

Figure 8.1

addition that there exists a neighbourhood V of 0 such that, in V, Ω coincides with the infinite cone C whose intersection with the unit sphere S^2 is a subset G of S^2 whose boundary ∂G is of class C^2.

Given $f \in L_2(\Omega)$ we look for a solution $u \in H^2(\Omega)$ of

$$\Delta u = f \qquad \text{in } \Omega \tag{8,2,2,3}$$

with either a Dirichlet boundary condition

$$\gamma u = 0 \qquad \text{on } \Gamma \tag{8,2,2,4}$$

or a Neumann boundary condition

$$\gamma \frac{\partial u}{\partial \gamma} = 0 \qquad \text{on } \Gamma \backslash \{0\}. \tag{8,2,2,5}$$

We denote by Δ' the Laplace–Beltrami operator on G with the corresponding boundary condition, i.e. its domain is either $H^2(G) \cap \mathring{H}^1(G)$ for a Dirichlet problem, or

$$\left\{ \varphi \in H^2(G) \,\middle|\, \gamma \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \partial G \right\}$$

for a Neumann problem. In both cases this is a self-adjoint operator in $H = L_2(G)$ whose spectrum is an infinite sequence of real numbers

$-\lambda_l$, $l=1,2,\ldots$ where $\lambda_l \geq 0$, with no limit point. We denote by ψ_l, $l=1,2,\ldots$ the orthonormalized sequence of the related eigenfunctions. Thus we have

$$-\Delta'\psi_l = \lambda_l\psi_l \qquad \text{in } G \tag{8,2,2,6}$$

where $\psi_l \in H^2(G)$ and either $\psi_l \in \mathring{H}^1(G)$ for a Dirichlet problem or $\gamma\, \partial\psi_l/\partial\nu = 0$ on ∂G for a Neuman problem.

The basic *a priori* inequality for the equation (8,2,2,3) with one of the boundary conditions (8,2,2,4) or (8,2,2,5), does not follow from (8,2,2) since Ω is not a polyhedron. We shall prove here the following statement.

Theorem 8.2.2.1 *Assume that $\lambda_l \neq \frac{3}{4}$ for every l; then there exists a constant C such that*

$$\|u\|_{2,2,\Omega} \leq C[\|\Delta u\|_{0,2,\Omega} + \|u\|_{0,2,\Omega}] \tag{8,2,2,7}$$

for every $u \in H^2(\Omega)$ which fulfils either the boundary condition (8,2,2,4) *or the boundary condition* (8,2,2,5).

(Of course one can drop the term $\|u\|_{0,2\Omega}$ in the case of a Dirichlet problem.)

It is clearly seen with the help of a partition of unity that inequality (8,2,2,7) follows from (3,1,1) and a similar inequality in C for functions with bounded support. (In addition, (8,2,2,7) is just (3,1,1) when the cone C is convex.)

Next the proof of the inequality in C relies on the use of weighted spaces similar to those introduced in Subsection 4.3.2. Indeed we denote by $P_2^m(C)$ the space of all the functions u defined in C such that

$$r^{|\alpha|-m}D^\alpha u \in L_2(C)$$

for all $|\alpha| \leq m$.† It is obvious that a function $u \in P_2^m(C)$ which has bounded support also belongs to $H^m(C)$. The converse statement is true up to the addition of a finite-dimensional space. This will be stated in a precise fashion below; however, we must observe at once that a similar statement for the two-dimensional case does not hold (see Kondratiev (1967), Theorem 4.3.2.2, which excludes the case when $p=2$ and the weaker statement in Theorem 7.2.1.1 when $p=2$).

Theorem 8.2.2.2 *Let $u \in H^2(C)$; then $u \in P_2^2(C)$ iff $u(0)=0$.*

The condition $u(0)=0$ is meaningful since every $u \in H^2(C)$ is continuous in $\bar{C}$ (see inclusion (1,4,4,6)). Conversely every $u \in P_2^2(C)$ is locally in H^2 and therefore also continuous in $\bar{C}$ for the same reason.

† $P_2^m(C)$ is equipped with the obvious norm (see Subsection 4.3.2).

Next the condition $u(0)=0$ is necessary for u to belong to $P_2^2(C)$, since this requires u/r^2 to be square integrable near zero and u is continuous there. We just have to prove that the above condition is sufficient. This will be done in two steps.

Lemma 8.2.2.3 *The inequality*

$$\left\{\int_C \frac{|u(x)|^2}{|x|^2}\,dx\right\}^{1/2} \leqslant 2\left\{\int_C |\nabla u(x)|^2\,dx\right\}^{1/2} \tag{8,2,2,8}$$

holds for every $u \in C_c^1(\bar{C})$.

Proof For $u \in C_c^1(\bar{C})$, we have

$$u(r\sigma) = -\int_r^\infty \left\{\frac{d}{dt}\,u(t\sigma)\right\}dt = -\int_r^\infty \sum_{k=1}^3 \sigma_k \frac{\partial u}{\partial x_k}(t\sigma)\,dt$$

for every $\sigma \in G$.

It follows that

$$\frac{|u(r\sigma)|}{r} \leqslant \frac{1}{r}\int_r^\infty |\nabla u(t\sigma)|\,dt.$$

Integrating, we obtain (8,2,2,8) by applying the second Hardy inequality (see Subsection 1.4.4) with $\alpha = 1$. ∎

Lemma 8.2.2.4 *The inequality*

$$\left\{\int_C \frac{|u(x)|^2}{|x|^4}\,dx\right\}^{1/2} \leqslant 2\int_C \frac{|\nabla u(x)|^2}{|x|^2}\,dx\Big\}^{1/2} \tag{8,2,2,9}$$

holds for every $u \in C_c^1(\bar{C})$ such that $u(0)=0$.

Proof Here we have

$$\begin{aligned} u(r\sigma) &= \int_0^r \left\{\frac{d}{dt}\,u(t\sigma)\right\}dt \\ &= \int_0^r \sum_{k=1}^3 \sigma_k \frac{\partial u}{\partial x_k}(t\sigma)\,dt \end{aligned}$$

for every $\sigma \in G$.

It follows that

$$\frac{|u(r\sigma)|}{r^2} \leqslant \frac{1}{r^2}\int_0^r |\nabla u(t\sigma)\,dt.$$

Integrating, we obtain (8,2,2,9) by applying the first Hardy inequality (see Subsection 1.4.4) with $\alpha = 0$. ∎

Proof of Theorem 8.2.2.2 We consider $u \in C_c^2(\bar{C})$ a dense subspace in $H^2(C)$ (by Theorem 1.4.2.1). We fix $\eta \in \mathscr{D}(\bar{C})$ a cut-off function, such that $\eta(0)=0$, and we apply the previous lemmas to

$$u-u(0)\eta$$

and its first derivatives respectively. We obtain

$$\int_C \frac{|u(x)-u(0)\eta(x)|^2}{|x|^4}\,dx+\int_C \frac{|\nabla(u-u(0)\eta)(x)|^2}{|x|^2}\,dx \leqslant K\int_C \sum_{i,j=1}^{3} |D_iD_j(u-u(0)\eta)(x)|^2\,dx$$

for some constant K. By density (and Sobolev's imbedding theorem) the same inequality holds for every $u \in H^2(C)$. The conclusion follows when $u(0)=0$. ■

Now a first step toward the proof of Theorem 8.2.2.1 is the following preliminary result.

Lemma 8.2.2.5 *There exists a constant K such that*

$$\|u\|_{P_2^2(C)} \leqslant K\,\|\Delta u\|_{0,2,C} \tag{8,2,2,10}$$

for every $u \in P_2^2(C)$ such that either $\gamma u=0$ on ∂C or $\gamma\, \partial u/\partial \nu=0$ on ∂C, provided $\lambda_l \neq \frac{3}{4}$ for every l.

Proof In spherical coordinates, the equation

$$\Delta u=f$$

means

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\Delta' u=f.$$

As in Subsection 4.3.2, we perform the change of variable $r=e^t$, setting

$$v(t,\sigma)=u(e^t\sigma)$$

$$g(t,\sigma)=e^{2t}f(e^t\sigma)$$

for every $t \in \mathbb{R}$ and $\sigma \in G$. We obtain the equation

$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial v}{\partial t}+\Delta' v=g \tag{8,2,2,11}$$

in the infinite cylinder $B=\mathbb{R}\times G$.

If we assume that $\gamma u=0$ on ∂C, then we have $\gamma v=0$ on ∂B; otherwise we assume that $\gamma\, \partial u/\partial\nu = 0$ on ∂C and we have $\gamma\, \partial v/\partial \nu = 0$ on ∂B. Finally the assumption that u belongs to $P_2^2(C)$ implies that

$$e^{-t/2}D^\alpha v \in L_2(B) = L_2(\mathbb{R}; H) \tag{8,2,2,12}$$

for every $|\alpha| \leqslant 2$.

Expanding both sides of equation (8,2,2,11) on the eigenfunctions ψ_l, one obtains

$$v_l''(t) + v_l'(t) - \lambda_l v_l(t) = g_l(t), \tag{8,2,2,13}$$

where

$$v_l(t) = \int_G v(t,\sigma)\psi_l(\sigma)\,d\sigma$$

$$g_l(t) = \int_G g(t,\sigma)\psi_l(\sigma)\,d\sigma.$$

In addition the condition (8,2,2,12) reads as follows:

$$\sum_{l=1}^{\infty} \int_{-\infty}^{+\infty} e^{-t}[|v_l''(t)|^2 + |\lambda_l|^2\,|v_l(t)|^2]\,dt < +\infty.$$

The function g_l is given such that $e^{-t/2}g_l \in L_2(\mathbb{R})$; consequently the equation (8,2,2,13) has a unique solution v_l such that

$$\int_{-\infty}^{+\infty} e^{-t}[|v_l''(t)|^2 + |v_l(t)|^2]\,dt < +\infty$$

iff $\frac{1}{2}$ is not a root of the characteristic equation (of the differential equation)

$$\alpha^2 + \alpha - \lambda_l = 0.$$

This requirement means $\lambda_l \neq \frac{3}{4}$.

In addition it is easy to check that there exists a constant K such that

$$\left\{\int_{-\infty}^{+\infty} e^{-t}\,|v_l(t)|^2\,dt\right\}^{1/2} \leqslant \frac{K}{|\lambda_l|}\left\{\int_{-\infty}^{+\infty} e^{-t}\,|g_l(t)|^2\,dt\right\}^{1/2}$$

when $l \to +\infty$. Summing up, we have

$$\sum_{l=1}^{\infty} \int_{-\infty}^{+\infty} e^{-t}[|v_l''(t)|^2 + |\lambda_l|^2\,|v_l(t)|^2]\,dt \leqslant K'^2 \sum_{l=1}^{\infty} \int_{-\infty}^{+\infty} e^{-t}\,|g_l(t)|^2\,dt$$

for some K' and this implies

$$\sum_{|\alpha|\leqslant 2} \|e^{-t/2}D^\alpha v\|_{0,0,B} \leqslant K''\,\|e^{-t/2}g\|_{0,0,B}$$

by the Bessel identity.

Performing the inverse change of variable $t = \ln r$, one obtains the desired inequality (8,2,2,10). ■

Proof of Theorem 8.2.2.1 Let us assume that $u \in H^2(\Omega)$ and fulfils a Dirichlet boundary condition, then we have $u(0) = 0$ and consequently

$$\eta u \in P_2^2(C)$$

by Theorem 8.2.2.2, where η is a cut-off function such that $\eta \in \mathcal{D}(\bar{\Omega})$, $\eta(x) = 1$ near zero and the support of η is contained in V (the neighbourhood of zero in which Ω coincides with C). Therefore the inequality (8,2,2,7) follows from (8,2,2,10) and from (3,1,1).

When $u \in H^2(\Omega)$ and fulfils a Neumann boundary condition the same procedure leads to the inequality (8,2,2,7) only for those functions $u \in H^2(\Omega)$ such that

$$\gamma \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \Gamma \backslash \{0\}$$

and such that in addition

$$u(0) = 0.$$

This last condition defines a subspace of codimension one in the space

$$V = \left\{u \in H^2(\Omega);\, \gamma \frac{\partial u}{\partial \nu} = 0\right\}$$

in which we want to prove the inequality.

The same technique as in the proof of Theorem 4.3.2.4 allows one to derive the weaker inequality

$$\|u\|_{2,2,\Omega} \leq C\{\|\Delta u\|_{0,2,\Omega} + \|u\|_{0,2,\Omega}\}$$

This completes the proof. ■

From the inequality (8,2,2,7), we shall derive a Fredholm alternative quite similar to the one in Subsection 4.4.1. Let us derive it briefly now. We apply Lemma 4.4.1.1 to the operator $A = \Delta$ considered as an operator from

$$E_1 = \left\{u \in H^2(\Omega;\, \gamma u = 0 \quad \text{on } \Gamma \qquad \left(\text{respectively } \gamma \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma \backslash \{0\}\right)\right\}$$

into $E_2 = L_2(\Omega)$. Thus A has a finite-dimensional kernel and a closed range.

The kernel of A is obviously $\{0\}$ in the case of a Dirichlet problem and the space of the constant functions in the case of a Neumann problem.

The closed range property is far more important. Let us denote by N the orthogonal of the range of A in $L_2(\Omega)$. As in Subsection 4.4.1, it is easily seen that every $v \in N$ is harmonic in Ω and such that $\gamma v = 0$ on $\Gamma\setminus\{0\}$ (respectively $\partial v/\partial \nu = 0$ on $\Gamma\setminus\{0\}$). Since $\Gamma\setminus\{0\}$ is of class C^2 the results in Chapter 2 imply that

$$v \in W_p^2(\Omega \setminus W)$$

for every $p<\infty$ and every neighbourhood W of 0.

Then we look at the behaviour of v in V. In spherical coordinates, we have

$$\frac{\partial^2 v}{\partial r^2}+\frac{2}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\Delta' v = 0.$$

Expanding v on the orthonormal system ψ_l, $l = 1, 2, \ldots$, we obtain

$$v(r, \sigma) = \sum_{l\geq 1} v_l(r)\psi_l(\sigma)$$

for r small enough (say $\leq r_0$), where

$$\sum_{l\geq 1}\int_0^{r_0} |v_l(r)|^2 r^2\, dr \leq \|v\|^2_{0,2,\Omega}. \tag{8,2,2,14}$$

In addition, we have

$$v_l''(r)+\frac{2}{r}v_l'(r)-\frac{\lambda_l}{r^2}v_l(r) = 0, \qquad 0<r<r_0,$$

since v is harmonic. Consequently

$$v_l(r) = a_l r^{\alpha_l} + b_l r^{\beta_l}$$

where a_l and b_l are arbitrary real numbers and

$$\alpha_l = \frac{-1+\sqrt{(1+4\lambda_l)}}{2}, \qquad \beta_l = \frac{-1-\sqrt{(1+4\lambda_l)}}{2}.$$

The condition (8,2,2,14) readily implies that $b_l = 0$ unless $\lambda_l < \frac{3}{4}$. Thus we have

$$v(r, \sigma) = \sum_{\lambda_l<3/4} b_l r^{\beta_l}\psi_l(\sigma) + \sum_{l\geq 1} a_l r^{\alpha_l}\psi_l(\sigma).$$

It is clearly checked that the first sum (which is finite) does not belong to $H^1(\Omega \cap \{r<r_0\})$ while each term of the second sum belongs to this space. More precisely, applying the method of Proposition 4.4.2.2, one shows that this series actually converges in $H^1(\Omega\cap\{r<r_0'\})$ for every $r_0' \in]0, r_0[$. Finally this implies that the dimension of N is the number μ of the

eigenvalues λ_l which are less than $\frac{3}{4}$ in the case of a Dirichlet problem and $\mu+1$ in the case of a Neumann problem (see the proof of Theorem 4.4.3.3).

Summing up, we have sketched the proof of the following result of Kondratiev (1967a).

Theorem 8.2.2.6 *Assume that* $\lambda_l \neq \frac{3}{4}$ *for every* l *and denote by* μ *the number of eigenvalues* λ_l *such that*

$$\lambda_l < \tfrac{3}{4}$$

then the space $\Delta(H^2(\Omega) \cap \mathring{H}^1(\Omega))$ *(respectively* $\Delta\{u \in H^2(\Omega);\ \gamma\ \partial u/\partial\nu = 0$ *on* $\Gamma\backslash\{0\}\})$ *has codimension* μ *(respectively* $\mu+1$*) in* $L_2(\Omega)$.

Consequently we must add μ singular functions to the space $H^2(\Omega)$ in order to describe all the solutions of the equation (8,2,2,3) with f given in $L_2(\Omega)$ under the boundary condition (8,2,2,4) (respectively (8,2,2,5)). Applying the technique of Theorem 4.4.3.7, we set

$$s_l(r\sigma) = r^{\alpha_l}\psi_l(\sigma)\eta(r\sigma)$$

for $\lambda_l < \frac{3}{4}$. These functions are μ functions belonging to

$$H^2(\Omega)\backslash H^1(\Omega)$$

and such that

$$\Delta s_l \in L^2(\Omega)$$

and they fulfil the boundary condition (8,2,2,4) (respectively (8,2,2,5)). Accordingly, we have the following statement.

Corollary 8.2.2.7 *Assume that* $\lambda_l \neq \frac{3}{4}$ *for every* l; *then for every* $f \in L_2(\Omega)$ *there exist constants* c_l *and a function* u *such that*

$$u - \sum_{\lambda_l < 3/4} c_l s_l \in H^2(\Omega),$$

u is solution of the equation (8,2,2,3) *and fulfils the boundary condition* (8,2,2,4) *(respectively* (8,2,2,5) *provided* $\int_\Omega f\,dx = 0$); *u is unique (respectively unique up to the addition of a constant).*

This shows that a conical point produces singular solutions of the form (8,2,2,1).

Let us now discuss the assumption $\lambda_l \neq \frac{3}{4}$, which has been useful in deriving the *a priori* inequality (8,2,2,7). It is easy to check that when G is contained in a hemisphere then the first eigenvalue λ_1 corresponding to the Dirichlet problem is greater than or equal to 1 (cf. for instance Grisvard (1975b)). Accordingly we have shown that u belongs to $H^2(\Omega)$, when the cone C is convex; this was proved in Chapter 3. On the other hand let us consider the particular case when G is a circular cone of angle β. It is shown in Hanna and Smith (1967) that when β increases from π to 2π, the first eigenvalue λ_1 decreases from 1 to 0. Consequently there is one value ($\beta \cong 1.45\pi$) for which the above method of proof is inconclusive (actually, the *a priori* inequality (8,2,2,7) does not hold either).

Curiously the inequality (8,2,2,7) always holds for polyhedral cones, i.e. cones for which ∂G is the union of a finite number of arcs of great circles (cf. inequality (8,2,2)). When the domain of Δ' is contained in $H^2(G)$, the method of proof of Theorem 8.2.2.6 still works. This shows again the regularity of u in $H^2(\Omega)$ when Ω is a convex polyhedron.

So far, we have shown precisely how the singular functions of the form (8,2,2,1) arise. When Ω is a polyhedron several edges meet at the same vertex. Each edge is likely to produce the kind of singular solutions that we have described in Subsection 8.2.1: the functions in Lemma 8.2.1.1 are of the form (8,2,2,2) near each vertex. Unfortunately the precise behaviour of the function ψ in (8,2,2,2) as $r \to 0$ is not yet known.

To conclude this subsection, we give a statement which summarizes the results in Theorem 8.2.1.2 when $m=2$ and an extension to $p \neq 2$ of Corollary 8.2.2.7 when Ω is a polyhedron. The basic assumption is that the angle of the edges are small enough not to produce edge singularities. The proof is very technical and may be found in Grisvard (1975a).

We need some auxiliary notation. We denote by S_i the vertices of Ω, $1 \leq i \leq I$, and by G_i the intersection of the unit sphere centred at S_i with the cone C_i corresponding to S_i. We denote by $\lambda_{i,l}$, $l=1,2,\ldots$ the sequence of the eigenvalues of $-\Delta'$ in G_i with the corresponding boundary conditions.

Theorem 8.2.2.8 *We assume that*

(a) $2-2/p<\pi/\omega_{j,k}$ *when j and $k \in \mathcal{D}$ or when j and $k \in \mathcal{N}$.*
(b) $2-2/p<\pi/2\omega_{j,k}$ *when $j \in \mathcal{D}$ and $k \in \mathcal{N}$ or when $j \in \mathcal{N}$ and $k \in \mathcal{D}$.*

We assume in addition that $\lambda_{i,l} \neq (2-3/p)(3-3/p)$ for every i and every l. Then the space

$$\Delta\left\{u \in W_p^2(\Omega);\ \gamma_j u=0 \text{ on } \Gamma_j,\ j \in \mathcal{D},\ \gamma_j \frac{\partial u}{\partial \nu_j}=0 \text{ on } \Gamma_j,\ j \in \mathcal{N}\right\}$$

has codimension (we assume that $\mathcal{D}$ is not empty for simplicity)

$$\mu=\sum_{i=1}^{I} \operatorname{card}\left\{l \,\middle|\, \lambda_{i,l}<\left(2-\frac{3}{p}\right)\left(3-\frac{3}{p}\right)\right\}$$

in $L_p(\Omega)$.

Consequently the solution u of the problem (8,2,1) with f given in $L_p(\Omega)$ is such that

$$u - \sum_{i=1}^{I} \sum_{\lambda_{i,l} < (2-3/p)(3-3/p)} c_{i,l} s_{i,l} \in W_p^2(\Omega)$$

where $c_{i,l}$ are real constants and

$$s_{i,l}(r_i\sigma_i) = r_i^{\frac{-1+\sqrt{(1+4\lambda_{i,l})}}{2}} \psi_{i,l}(\sigma_i)\eta_i(r_i\sigma_i)$$

in an obvious notation; namely: r_i denotes the distance to S_i, σ_i a point of G_i, $\psi_{i,l}$ is the normalized eigenfunction, corresponding to $-\lambda_{i,l}$, of Δ' with suitable boundary conditions; finally η_i is a cut-off function depending only on r_i such that $\eta_i \in \mathscr{D}(\bar{\Omega})$, $\eta_i = 1$ near S_i and $\eta_i = 0$ outside some neighbourhood of S_i.

Remark 8.2.2.9 An asymptotic expansion near the vertices of the singular part of the solution corresponding to an edge is derived in Grisvard (1982) in the particular case $p = 2$ when the assumption (a) in Theorem 8.2.2.8 is not fulfilled.

8.3 The heat equation

It is well known that one can derive several properties of the heat equation by applying semigroup theory, provided one has a good knowledge of the properties of the Laplace operator (and its resolvent operator). This method has been applied successfully for solving the heat equation with various mixed boundary conditions in regular cylinders. We mean here a cylinder Q of the form

$$Q =]0, T[\times \Omega$$

where Ω is a domain with a smooth boundary in $\mathbb{R}^n$ and $]0, T[$ is an interval in time. Possible references are Lions (1956), Lions and Magenes (1960–63), Krein (1967). One can apply the same kind of method when Ω is a plane polygon and consequently Ω is a cylinder with edges.

However, we shall consider here a different problem. We shall solve a mixed boundary value problem for the heat equation in a domain Q which may not be the Cartesian product of an interval in time by a domain in space. Here we follow work by Sadallah (1976, 1977, 1978). For simplicity, we consider a problem with only one space variable. To be precise, we assume that

$$Q = \{(t, x) \mid 0 < t < T, \varphi_1(t) < x < \varphi_2(t)\},$$

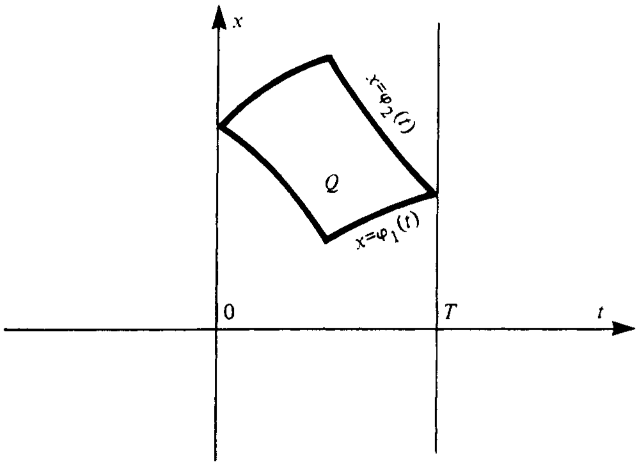

Figure 8.2

where T is a finite positive number, while φ_1 and φ_2 are continuous real-valued functions defined in $[0, T]$ Lipschitz continuous in $]0, T[$, such that

$$\varphi_1(t) < \varphi_2(t)$$

for $t \in]0, T[$. Given $f \in L_2(Q)$ we look for a solution u (as regular as possible) of

$$\begin{cases} D_t u - D_x^2 u = f & \text{in } Q \\ u(0, x) = 0, & \varphi_1(0) < x < \varphi_2(0) \\ u(t, \varphi_1(t)) = u(t, \varphi_2(t)) = 0, & 0 < t < T. \end{cases} \tag{8,3,1}$$

We emphasize that we shall allow φ_1 to coincide with φ_2 for $t = 0$ and for $t = T$. Actually domains of the same kind under a weaker assumption on φ_1 and φ_2 are considered in the works by Baderko (1973, 1975, 1976). This author assumes that φ_1 and φ_2 are only Hölder continuous (with exponent larger than or equal to $\frac{1}{2}$) and solves the heat equation in Q by the potential method introduced by Gevrey (1913). However, in order to apply this method, one must assume that $\varphi_1(0) < \varphi_2(0)$ and that $\varphi_1(T) < \varphi_2(T)$.

Actually the method of Sadallah, which we shall outline here, is a straightforward extension of Chapter 3. We first prove an *a priori* estimate when Q is nice (in a sense to be defined later), and then we take limits in Q in order to reach the kind of domains described above. The *a priori* inequality is proved simply by integration by parts, and again we have very accurate control of the constants involved, as functions of Q; this is why we are able to take limits in Q.

The result we are going to prove here is the following, for which we need the technical assumption

$$\varphi_i'(t)[\varphi_2(t)-\varphi_1(t)]\to 0 \qquad \text{as } t\to 0, \quad i=1,2 \tag{8,3,2}$$

if $\varphi_1(0)=\varphi_2(0)$ and

$$\varphi_i'(t)[\varphi_2(t)-\varphi_1(t)]\to 0 \qquad \text{as } t\to T, \quad i=1,2 \tag{8,3,3}$$

if $\varphi_1(T)=\varphi_2(T)$.

Theorem 8.3.1 *Assume that* (8,3,2) *and* (8,3,3) *hold and that f is given in $L_2(Q)$; then there exists a unique function u which is a solution of*

$$D_t u - D_x^2 u = f \qquad \text{in } Q \tag{8,3,4}$$

such that u, $D_t u$, $D_x u$ and $D_x^2 u$ belong to $L_2(Q)$ and

$$\begin{cases} \gamma u(t,\varphi_i(t))=0, & 0<t<T, \quad i=1,2 \\ \gamma u(0,x)=0, & \varphi_1(0)<x<\varphi_2(0). \end{cases} \tag{8,3,5}$$

We observe that u belongs to $H^1(Q)$ and consequently the trace γu is well defined on ∂Q away from the points $(0, \varphi_i(0))$ and $(T, \varphi_i(T))$, $i=1,2$. Indeed these are the only possible points in a neighbourhood of which the boundary ∂Q may not be Lipschitz.

We emphasize that this is an existence result for a strong solution. The existence and uniqueness of a weak solution with, say,

$$u \quad \text{and} \quad D_x u \in L_2(Q),$$

are easy to derive (see Oleinik and Radkievitz (1971) for instance). Thus Theorem 8.3.1 is mainly a smoothness result.

Let us now carry out some preliminaries. For the time being we consider the simpler case which follows. We replace Q by

$$Q_\alpha = \{(t,x) \mid \alpha<t<T-\alpha,\ \varphi_1(t)<x<\varphi_2(t)\}$$

with $\alpha>0$. Thus we have

$$\begin{cases} \varphi_1(\alpha)<\varphi_2(\alpha) \\ \varphi_1(T-\alpha)<\varphi_2(T-\alpha) \end{cases}$$

and φ_1 and φ_2 are uniformly Lipschitz continuous in $[\alpha, t-\alpha]$. In this case it is very easy to prove the result corresponding to Theorem 8.3.1. Indeed, we can easily find a change of variable ψ mapping Q_α onto the rectangle

$$R_\alpha =]\alpha, T-\alpha[\times]0,1[,$$

which leaves the t variable unchanged. We define ψ as follows:

$$\psi(t, x) = \left\{t, \frac{x-\varphi_1(t)}{\varphi_2(t)-\varphi_1(t)}\right\}.$$

Then we define the functions v and g in R_α by

$$v = u \circ \psi^{-1} \qquad \text{and} \qquad g = f \circ \psi^{-1}.$$

The equation

$$D_t u - D_x^2 u = f$$

in Q_α is equivalent to the following:

$$D_t v + a(t, x) D_x v - b(t) D_x^2 v = g \tag{8,3,6}$$

in R_α, where a and b are defined by

$$a(t, x) = D_t\left(\frac{x-\varphi_1(t)}{\varphi_2(t)-\varphi_1(t)}\right)$$

$$b(t) = \frac{1}{[\varphi_2(t)-\varphi_1(t)]^2}.$$

The mapping ψ is bi-Lipschitz and therefore it preserves the space H^1. In other words u belongs to $H^1(Q_\alpha)$ iff $v \in H^1(R_\alpha)$. The boundary conditions on v which correspond to the boundary conditions on u are the following:

$$\begin{cases} \gamma v(\alpha, x) = 0, & 0 < x < 1 \\ \gamma v(t, 0) = v(t, 1) = 0, & \alpha < t < T-\alpha. \end{cases} \tag{8,3,7}$$

In a first step we consider the simplified equation

$$D_t v - b(t) D_x^2 v = g \qquad \text{in } R_\alpha \tag{8,3,8}$$

with the same boundary conditions on v.

Lemma 8.3.2 *For every $g \in L_2(R_\alpha)$ there exists a unique $v \in H^1(R_\alpha)$, with $D_x^2 v \in L_2(R_\alpha)$, which is a solution of* (8,3,8) *and* (8,3,7).

Proof A simple change of variable (in t) reduces equation (8,3,6) to the heat equation and we can apply some classical results.

Indeed we define the function β as follows:

$$\beta(t) = \int_\alpha^t b(s)\, ds.$$

This is an invertible $C^{1,1}$ mapping from $[\alpha, T-\alpha]$ onto $[0, \beta(T-\alpha)]$. It follows from Lions and Magenes (1968), for instance, that there exists a solution $w \in H^1(R'_\alpha)$ of

$$D_t w - D_x^2 w = g/b$$

in $R'_\alpha =]0, \beta(T-\alpha)[\times]0, 1[$, with

$$\begin{cases} \gamma w(0, x) = 0, & 0<x<1 \\ \gamma w(t, 0) = \gamma w(t, 1) = 0, & 0<t<\beta(T-\alpha) \end{cases}$$

and such that in addition $D_x^2 w \in L_2(R'_\alpha)$.

We obtain the desired function v by setting

$$v(t, x) = w(\beta(t), x). \quad \blacksquare$$

Lemma 8.3.3 *For every $g \in L_2(R_\alpha)$ there exists a unique solution $v \in H^1(R_\alpha)$, with $D_x^2 v \in L_2(R_\alpha)$, of* (8,3,6) *and* (8,3,7).

Proof We denote by A the operator $D_t - b(t)D_x^2$ defined from

$$V = \{v \in H^1(R_\alpha) \mid D_x^2 v \in L_2(R_\alpha),\ v \text{ fulfils } (8,3,7)\}$$

into $L_2(R_\alpha)$. We have shown in Lemma 8.3.2 that A is an isomorphism.

Then it is known (cf. for instance Besov [1969]) that D_x is a compact operator from V into $L_2(R_\alpha)$.† Since a is a bounded function, the operator aD_x is also compact from V into $L_2(R_\alpha)$. Consequently $A + aD_x$ is a Fredholm operator (with index zero) from A into $L_2(R_\alpha)$. Thus the invertibility of $A + aD_x$ will follow from its injectivity.

Accordingly let us consider $v \in V$, a solution of

$$D_t v + aD_x v - bD_x^2 v = 0,$$

in R_α. We perform the inverse change of variable of ψ. Thus we set

$$u = v \circ \psi.$$

It turns out that $u \in H^1(Q_\alpha)$, $D_x^2 u \in L_2(Q_\alpha)$ and

$$D_t u - D_x^2 u = 0 \qquad \text{in } Q_\alpha.$$

In addition u fulfils the homogeneous boundary condition (8,3,5). As

† Actually since R_α is a rectangle, it can easily be shown that there exists a continuous extension operator from $H^{1,2}(R_\alpha)$ into $H^{1,2}(\mathbb{R}^2)$; here for a general plane domain Ω, $H^{1,2}(\Omega)$ is defined by

$$H^{1,2}(\Omega) = \{u \mid u, D_t u, D_x u, D_x^2 u \in L_2(\Omega)\}.$$

Then the compactness of $u \to \varphi D_x u$ from $H^{1,2}(\mathbb{R}^2)$ into $L_2(\mathbb{R}^2)$ is easily checked by Fourier transform, provided $\varphi \in \mathcal{D}(\mathbb{R}^2)$.

usual we calculate the integral:

$$\int_{Q_\alpha} (D_t u - D_x^2 u) u \, dt \, dx.$$

This yields

$$\tfrac{1}{2} \int_{Q_\alpha} |\gamma u|^2 \nu_1 \, ds - \int_{Q_\alpha} \gamma D_x u \gamma u \nu_2 \, ds + \int_{Q_\alpha} |D_x u|^2 \, dt \, dx = 0.$$

All the boundary integrals vanish but

$$\tfrac{1}{2} \int_{\varphi_1(T-\alpha)}^{\varphi_2(T-\alpha)} |\gamma u(T-\alpha, x)|^2 \, dx,$$

which is nonnegative. This yields the inequality

$$\int_{Q_\alpha} |D_x u|^2 \, dt \, dx \leq 0,$$

which implies that u vanishes; this is the desired injectivity. ■

So far we have proved the desired result in the better domains Q_α. Now we shall prove an *a priori* estimate which will allow us to take limits in α.

Lemma 8.3.4 *Let* $]a, b[$ *be a finite real interval. There exists a constant* C (*independent of* a *and* b) *such that*

$$\int_a^b |v(x)|^2 \, dx \leq C(b-a)^2 \int_a^b |v'(x)|^2 \, dx$$

for every $v \in H^1(]a, b[)$ *such that* $\int_a^b v(x) \, dx = 0$.

The proof of this inequality may be found in Nečas (1967) for instance (it is elementary: actually the general case follows from the particular case $]a, b[=]0, 1[$ by an affine change of variable).

We shall apply this inequality later to the function $D_x u$, where u fulfils the assumptions of Theorem 8.3.1. This yields the inequality

$$\int_{\varphi_1(t)}^{\varphi_2(t)} |D_x u(t, x)|^2 \, dx \leq C[\varphi_2(t) - \varphi_1(t)]^2 \int_{\varphi_1(t)}^{\varphi_2(t)} |D_x^2 u(t, x)|^2 \, dx. \qquad (8,3,9)$$

Since we have

$$\int_{\varphi_1(t)}^{\varphi_2(t)} D_x u(t, x) \, dx = \gamma u(t, \varphi_2(t)) - \gamma u(t, \varphi_1(t)) = 0.$$

Lemma 8.3.5 *We assume that* φ_1 *and* φ_2 *fulfil the conditions* (8,3,2) *and*

(8,3,3). *We assume in addition that* $u \in C^2(\bar{Q}_\alpha)$ *and*

$$\begin{cases} u(\alpha, x) = 0, & \varphi_1(\alpha) \leqslant x \leqslant \varphi_2(\alpha) \\ u(t, \varphi_1(t)) = u(t, \varphi_2(t)) = 0, & \alpha \leqslant t \leqslant T - \alpha. \end{cases}$$

There exists a constant K which does not depend on α *and u such that*

$$\|u\|^2 + \|D_t u\|^2 + \|D_x u\|^2 + \|D_x^2 u\|^2 \leqslant K \, \|D_t u - D_x^2 u\|^2 \tag{8,3,10}$$

in the norm of $L_2(Q_\alpha)$.

Proof There are two main steps. First we derive a bound for $D_x u$ by calculating

$$\int_{Q_\alpha} (D_t u - D_x^2 u) u \, dt \, dx.$$

Second we derive a bound for $D_x^2 u$ and $D_t u$ by calculating

$$\int_{Q_\alpha} (D_t u - D_x^2 u)^2 \, dt \, dx$$

as we did in Chapter 3.

Setting $f = D_t u - D_x^2 u$, we have

$$\int_{Q_\alpha} fu \, dt \, dx = \int_{Q_\alpha} |D_x u|^2 \, dt \, dx + \tfrac{1}{2} \int_{\partial Q_\alpha} |u|^2 \nu_1 \, ds - \int_{\partial Q_\alpha} D_x u u \nu_2 \, ds$$
$$\geqslant \int_{Q_\alpha} |D_x u|^2 \, dt \, dx.$$

It follows that

$$\|D_x u\|^2 \leqslant \|f\| \, \|u\|$$

in the norm of $L_2(Q_\alpha)$.

On the other hand, Poincaré's inequality implies that

$$\|u\| \leqslant L \, \|D_x u\|,$$

where $L = \max_{t \in [0,T]} [\varphi_2(t) - \varphi_1(t)]$. It follows that

$$\begin{aligned} &\|u\| \leqslant L^2 \, \|f\| \\ &\|D_x u\| \leqslant L \, \|f\|. \end{aligned} \tag{8,3,11}$$

Then we have

$$\int_{Q_\alpha} f^2 \, dt \, dx = \int_{Q_\alpha} |D_t u|^2 \, dt \, dx + \int_{Q_\alpha} |D_x u|^2 \, dt \, dx$$
$$+ \int_{\partial Q_\alpha} [|D_x u|^2 \nu_1 - 2 D_t u D_x u \nu_2] \, ds. \tag{8,3,12}$$

We shall rewrite the boundary integral making use of the boundary conditions (8,3,5).

On the part of the boundary where $t=\alpha$, we have $\nu_2=0$, $u=0$ and consequently $D_x u=0$. The corresponding boundary integral vanishes. Then on the part of the boundary where $t=T-\alpha$, we have again $\nu_2=0$ and $\nu_1=1$. Accordingly the corresponding boundary integral is nonnegative and we can forget it.

On the part of the boundary where $x=\varphi_i(t)$, $i=1,2$, we have

$$u(t,\varphi_i(t))=0.$$

Differentiating with respect to t we obtain

$$D_t u=-\varphi_i'(t)D_x u.$$

Consequently the corresponding boundary integral is

$$\mathcal{J}=-\int_\alpha^{T-\alpha}|D_x u(t,\varphi_1(t))|^2\,\varphi_1'(t)\,dt+\int_\alpha^{T-\alpha}|D_x u(t,\varphi_2(t))|^2\,\varphi_2'(t)\,dt. \tag{8,3,13}$$

We convert this boundary integral into a surface integral by setting

$$\begin{aligned}[D_x u(t,\varphi_1(t))]^2&=-\frac{\varphi_2(t)-x}{\varphi_2(t)-\varphi_1(t)}[D_x u(t,x)]^2\Big|_{x=\varphi_1(t)}^{x=\varphi_2(t)}\\ &=-\int_{\varphi_1(t)}^{\varphi_2(t)}\frac{\partial}{\partial x}\left\{\frac{\varphi_2(t)-x}{\varphi_2(t)-\varphi_1(t)}[D_x u(t,x)]^2\right\}dx\\ &=-\int_{\varphi_1(t)}^{\varphi_2(t)}\frac{\varphi_2(t)-x}{\varphi_2(t)-\varphi_1(t)}2D_x u(t,x)D_x^2u(t,x)\,dx\\ &\quad+\int_{\varphi_1(t)}^{\varphi_2(t)}\frac{1}{\varphi_2(t)-\varphi_1(t)}[D_x u(t,x)]^2\,dx\end{aligned}$$

and consequently

$$\begin{aligned}|D_x u(t,\varphi_1(t))|^2&\le 2\int_{\varphi_1(t)}^{\varphi_2(t)}|D_x u(t,x)|\,|D_x^2u(t,x)|\,dx\\ &\quad+\int_{\varphi_1(t)}^{\varphi_2(t)}\frac{1}{\varphi_2(t)-\varphi_1(t)}|D_x u(t,x)|^2\,dx.\end{aligned}$$

A similar inequality holds for $D_x u(t,\varphi_2(t))$ and this yields

$$\begin{aligned}|\mathcal{J}|&\le 2\int_{Q_\alpha}|D_x u|\,|D_x^2u|\,[|\varphi_1'(t)|+|\varphi_2'(t)|]\,dt\,dx\\ &\quad+\int_{Q_\alpha}\frac{|\varphi_1'(t)|+|\varphi_2'(t)|}{\varphi_2(t)-\varphi_1(t)}|D_x u|^2\,dt\,dx.\end{aligned}$$

It follows that

$$|\mathscr{I}| \leqslant \tfrac{1}{2}\|D_x^2 u\|^2 + \int_{Q_\alpha} |D_x u|^2 \left\{ 2[|\varphi_1'(t)| + |\varphi_2'(t)|]^2 + \frac{|\varphi_1'(t)| + |\varphi_2'(t)|}{\varphi_2(t) - \varphi_1(t)} \right\} dt\, dx.$$

With inequality (8,3,9) this yields

$$|\mathscr{I}| \leqslant \tfrac{1}{2}\|D_x^2 u\|^2 + M_\varepsilon \|D_x u\|^2 + N_\varepsilon \|D_x^2 u\|^2,$$

where

$$M_\varepsilon = \underset{t \in]\varepsilon, T-\varepsilon]}{\text{l.u.b.}} \left\{ 2[|\varphi_1'(t)| + |\varphi_2'(t)|]^2 + \frac{|\varphi_1'(t)| + |\varphi_2'(t)|}{\varphi_2(t) - \varphi_1(t)} \right\}$$

and

$$N_\varepsilon = \underset{t \in]0,\varepsilon] \cup [T-\varepsilon, T[}{\text{l.u.b.}} \{ 2[|\varphi_1'(t)| + |\varphi_2'(t)|]^2 [\varphi_2(t) - \varphi_1(t)|^2 + [|\varphi_1'(t)| + |\varphi_2'(t)|][\varphi_2(t) - \varphi_1(t)] \}$$

for every $\varepsilon > 0$.

Going back to (8,3,12) we have

$$\|D_t u\|^2 + \|D_x^2 u\|^2 \leqslant \|f\|^2 + |\mathscr{I}| \leqslant \|f\|^2 + (\tfrac{1}{2} + N_\varepsilon)\|D_x^2 u\|^2 + M_\varepsilon \|D_x u\|^2;$$

with (8,3,11), this yields

$$\|D_t u\|^2 + \|D_x^2 u\|^2 \leqslant (1 + L^2 M_\varepsilon)\|f\|^2 + (\tfrac{1}{2} + N_\varepsilon)\|D_x^2 u\|^2.$$

Finally we take advantage of the assumptions (8,3,2) and (8,3,3). Thus choosing ε small enough, we have $N_\varepsilon \leqslant \frac{1}{4}$ and consequently

$$\|D_t u\|^2 + \tfrac{1}{4}\|D_x^2 u\|^2 \leqslant (1 + L^2 M_\varepsilon)\} \|f\|^2.$$

Summing up, we have

$$\|u\|^2 + \|D_x u\|^2 + \|D_x^2 u\|^2 + \|D_t u\|^2 \leqslant \{L^4 + L^2 + 5(1 + L^2 M_\varepsilon)\} \|f\|^2$$

provided $N_\varepsilon \leqslant \frac{1}{4}$. ■

We shall need an extension of inequality (8,3,10) to functions with less regularity. This requires a density lemma:

Lemma 8.3.6 *Every $u \in H^1(Q_\alpha)$ such that*

$$\begin{cases} D_x^2 u \in L_2(Q_\alpha) \\ \gamma u(t, \varphi_1(t)) = \gamma u(t, \varphi_2(t)) = 0, & \alpha < t < T - \alpha \\ \gamma u(\alpha, x) = 0, & \varphi_1(\alpha) < x < \varphi_2(\alpha) \end{cases}$$

can be approximated by a sequence u_n, $n = 1, 2, \ldots$ of functions belonging *to* $C^2(\bar{Q}_\alpha)$ *such that*

$$\begin{cases} u_n(t, \varphi_1(t)) = u_n(t, \varphi_2(t)) = 0, & \alpha < t < T-\alpha \\ u_n(\alpha, x) = 0, & \varphi_1(\alpha) < x < \varphi_2(\alpha). \end{cases}$$

The convergence is such that

$$\|u_n - u\|_{1,2,Q_\alpha} + \|D_x^2(u_n - u)\|_{0,2,Q_\alpha} \to 0$$

as $n \to +\infty$.

Proof One easily replaces Q_α by R_α with the help of the change of variable ψ defined above. Then the proof in the case of R_α is just an exercise. ■

This implies clearly that the inequality (8,3,10) holds for every $u \in H^1(Q_\alpha)$ which fulfils the assumptions in Lemma 8.3.6. We are now able to prove Theorem 8.3.1.

Proof of Theorem 8.3.1 By Lemma 8.3.3, there exists for each $\alpha > 0$ (small enough), a unique

$$u_\alpha \in H^1(Q_\alpha)$$

such that $D_x^2 u_\alpha \in L_2(Q_\alpha)$ and

$$\begin{cases} D_t u_\alpha - D_x^2 u_\alpha = f & \text{in } Q_\alpha \\ u_\alpha(\alpha, x) = 0, & \varphi_1(\alpha) < x < \varphi_2(\alpha) \\ u_\alpha(t, \varphi_1(t)) = u_\alpha(t, \varphi_2(t)) = 0, & \alpha < t < T-\alpha. \end{cases}$$

In addition the inequality (8,3,10) holds for u_α and accordingly we have

$$\|u_\alpha\|_{1,2,Q_\alpha}^2 + \|D_x^2 u_\alpha\|_{0,2,Q_\alpha}^2 < K \|f\|_{0,2,Q}^2 .$$

We consider a sequence $\alpha_n \to 0$, as $n \to +\infty$. The related sequences of functions

$$u_{\alpha_n}, \qquad D_t u_{\alpha_n}, \qquad D_x u_{\alpha_n} \qquad \text{and} \qquad D_x^2 u_{\alpha_n},$$

$n = 1, 2, \ldots$, are bounded in $L_2(Q)$. By replacing α_n by a suitable subsequence (that we denote again by α_n, $n = 1, 2, \ldots$ for simplicity), there exist functions

$$u, \quad v_1, \quad v_2 \quad \text{and} \quad w$$

in $L_2(Q)$ such that

$$\tilde{u}_{\alpha_n} \to u \qquad \widetilde{D_t u_{\alpha_n}} \to v_1 \qquad \widetilde{D_x u_{\alpha_n}} \to v_2 \qquad \widetilde{D_x^2 u_{\alpha_n}} \to w$$

weakly in $L_2(Q)$ as $n \to +\infty$.

The remainder of the proof is similar to the proof of Theorem 3.2.1.2. We have clearly

$$v_1 = D_t u, \qquad v_2 = D_x u \qquad \text{and} \qquad w = D_x^2 u$$

in the sense of distributions in Q and consequently we have

$$D_t u - D_x^2 u = f \qquad \text{in } Q.$$

Finally let $\varphi \in \mathscr{D}(\mathbb{R})$ be such that

$$\begin{cases} \varphi(t) = 1 & \text{for } t \leqslant T - \varepsilon \\ \varphi(t) = 0 & \text{for } t \geqslant T - \varepsilon/2. \end{cases}$$

with $\varepsilon > 0$. We have clearly

$$\varphi \tilde{u}_\alpha \in \mathring{H}^1(Q)$$

and $\varphi \tilde{u}_\alpha$ remains bounded in $\mathring{H}^1(Q)$. Thus, taking the limit, we have

$$\varphi u \in \mathring{H}^1(Q).$$

This implies that

$$\begin{aligned} &(\gamma u)(0, x) = 0, & &\varphi_1(0) < x < \varphi_2(0) \\ &(\gamma u)(t, \varphi_1(t)) = (\gamma u)(t, \varphi_2(t)) = 0, & &0 < t < T - \varepsilon. \end{aligned}$$

Since the above boundary conditions hold for every $\varepsilon > 0$ we have proved the existence of a function u having the properties listed in Theorem 8.3.1. As we have already observed the uniqueness of u is classical. ■

Remark 8.3.7 Inspection of the identity (8,3,12) shows that the condition (8,3,2) is useless when φ_2 is nondecreasing and φ_1 is nonincreasing near zero. In the same way, the condition (8,3,3) is useless when φ_2 is nondecreasing and φ_1 is nonincreasing near T.

Remark 8.3.8 The works by Sadallah mentioned above include similar results for the heat equation in more space variables and for the equation

$$D_t u + (-1)^m D_x^{2m} u = f,$$

m an arbitrary positive integer (such an equation is also studied in the works by Baderko mentioned above).

The domains considered in Theorem 8.3.1 include all the convex polygons but not all the polygons. However, it is very easy to derive a similar result for any polygon.

Corollary 8.3.9 *Let Q be a plane open domain with a polygonal bound-*

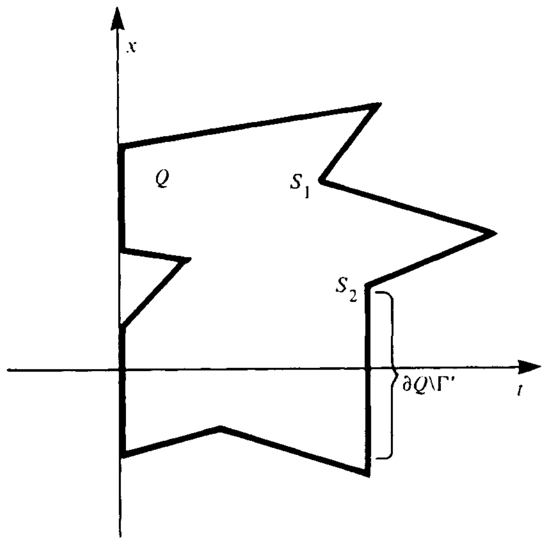

Figure 8.3

ary. Then the operator $D_t - D_x^2$ is a one-to-one operator from

$$V = \{u \in H^1(Q);\ D_x u \in L_2(Q),\ \gamma u = 0 \text{ on } \Gamma'\}$$

into $L_2(Q)$, where Γ' is the part of the boundary ∂Q where $\nu_1 < 1$. In addition the image of $D_t - D_x^2$ is closed and has finite codimension μ, the number of corners $S_j = (t_j, x_j)$, with the following property: There exists a neighbourhood U_j of S_j such that

$$t \geqslant t_j$$

for every point $(t, x) \in U_j \cap \partial Q$.

The proof consists in applying Theorem 8.3.1 in each polygon Q_i of a covering of Q such that

(a) Q_i is a convex open subset of Q with a polygonal boundary
(b) $Q_i \cap Q_j = \varnothing$ for $i \neq j$
(c) $\bar{Q} = \bigcup_{i=1}^{I} \bar{Q}_i$.

The details can be found in Sadallah (1976).

Remark 8.3.10 Similar results for the operator $D_t + (-1)^m D_x^{2m}$ are derived in Sadallah (1983).

8.4 The numerical solution of elliptic problems with singularities

In this section, we take for granted that the reader is familiar with the finite element method for solving elliptic boundary problems in domains with smooth boundaries (cf. for instance Ciarlet (1978)). The analysis of the finite element method usually relies on the assumption that the solution of the given problem is regular enough. However, the implementation of this method is very often done on problems in polygonal domains which prevent the solution from being smooth everywhere.

As we saw in previous chapters, the presence of corners leads to singular behaviour of the solution only near the corners. This singular behaviour occurs even when the data of the problem are very smooth. It strongly affects the accuracy of the finite element method throughout the whole domain. We shall outline here the two main procedures which have been proposed to overcome this difficulty. The first is based on mesh refinements and has been analysed by several authors; see Babuska and Kellogg (1972), Babuska *et al.* (1979), Raugel (1978), Schatz and Wahlbin (1978–79), Thatcher (1975) for instance. This method may be applied to most of the practical problems since it requires only a qualitative knowledge of the behaviour of the solution near the corners (see details in Subsection 8.4.1). The second consists in augmenting the space of trial functions in which one looks for the approximate solution. This is done by adding some of the singular solutions of the problem to the usual spaces of piecewise polynomial functions (cf., for instance, Fix *et al.* (1976), Babuska and Kellogg (1972), Lelievre (1976b), Djaoua (1977) and Ladeveze and Peyret (1974)). This procedure requires a very accurate knowledge of the singular solutions and consequently it can be applied only to special problems (see details in Subsection 8.4.2).

Since the purpose of this section is only to illustrate the procedures mentioned above, we shall consider only the simplest model problem; namely we shall consider the Dirichlet problem for the Laplace equation in a plane polygon with only one nonconvex corner. We shall approximate its solution by means of a Galerkin method using trial functions which are piecewise first-order polynomials for simplicity.

Some slightly different approaches to the singularity problems, using integral equations, may be found in Wendland *et al.* (1979).

8.4.1 Weighted spaces and mesh refinements

Let us again fix some notation which we keep consistent with that of Chapter 4 (see Section 4.1). Accordingly Ω is a plane domain with a polygonal boundary Γ, the union of a finite number N of linear segments $\bar{\Gamma}_j$ numbered according to the positive orientation. We denote by ω_j the

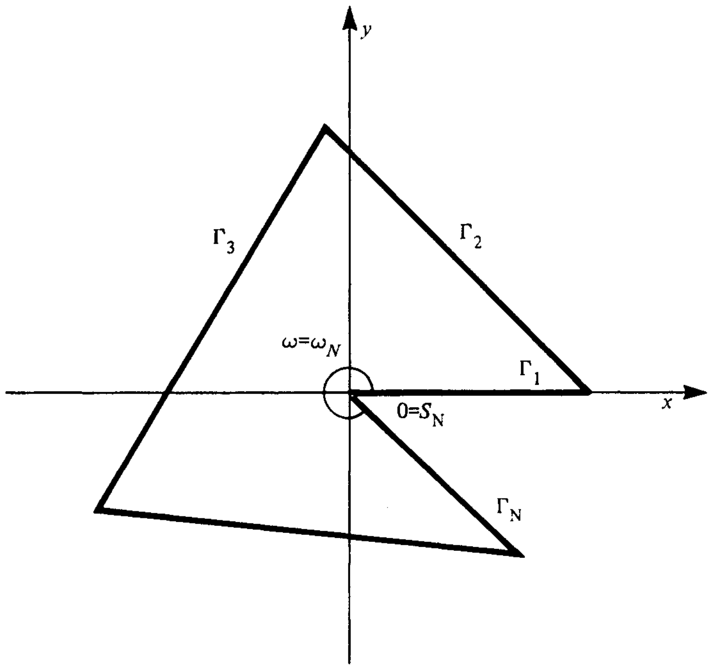

Figure 8.4

angle between Γ_j and Γ_{j+1} and we assume that $\omega_j < \pi$ for every j but $j = N$. For simplicity we assume that S_N, the corner point between Γ_N and Γ_1, has been translated to the origin. In addition we assume that Γ_1 is included in the positive abscissa axis (Ox) while Γ_N is supported by the half line whose angle with Ox is ω_N (counted counterclockwise). For further simplicity we set $\omega = \omega_N$.

Given $f \in L_2(\Omega)$ we look for a solution $u \in \mathring{H}^1(\Omega)$ of

$$-\Delta u = f \qquad \text{in } \Omega. \tag{8,4,1,1}$$

We know (Chapter 4) that there exists a unique solution u and in addition there exists a unique number λ such that

$$u - \lambda r^{\pi/\omega} \sin \frac{\pi\theta}{\omega} \in H^2(\Omega). \tag{8,4,1,2}$$

In theory this solution u is obtained by applying the variational method of Lemma 2.2.1.1. We set

$$V = \mathring{H}^1(\Omega)$$

and

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \qquad u, v \in V.$$

Then u is the unique element of V such that

$$a(u, v) = \int_\Omega fv \, dx \tag{8,4,1,3}$$

for every $v \in V$.

The Galerkin method for approximating u consists in replacing the space V by a finite-dimensional subspace V_h of V in the above setting. Thus we consider u_h the unique element of V_h such that

$$a(u_h, v) = \int_\Omega fv \, dx \tag{8,4,1,4}$$

for every $v \in V_h$. This new problem is equivalent to a set of n linear algebraic equations with n unknowns, n being the dimension of V_h. On the one hand one expects to solve explicitly this set of equations with the minimal amount of calculations. On the other hand one expects that u_h significantly approximates u; in other words one wants the error

$$\|u - u_h\|$$

to be small for some suitable norm. Satisfying both requirements depends strongly on the choice of V_h.

The basic tool for estimating the error is Céa's lemma (see Theorem 2.4.1 in Ciarlet (1978)):

$$\|u - u_h\|_V \leq \frac{M}{\alpha} \inf_{v \in V_h} \|u - v\|_V, \tag{8,4,1,5}$$

where α denotes the coerciveness constant (cf. Lemma 2.2.1.1) while M is the constant such that

$$|a(u, v)| \leq M \, \|u\|_V \, \|v\|_V$$

for every u and v in V.

In the finite element method V_h is built with the help of a triangulation $\mathcal{T}_h$ over $\bar{\Omega}$; $\mathcal{T}_h$ is a set of closed triangles (we assume that the triangles are not 'degenerate' i.e. their interiors are not empty) such that

(a) $\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$;

(b) for each distinct $K_1, K_2 \in \mathcal{T}_h$, one has

$$\mathring{K}_1 \cap \mathring{K}_2 = \varnothing$$

(c) any edge of any triangle K_1 is either a subset of the boundary Γ or an edge of another triangle K_2 in the triangulation.

The number h related to the triangulation $\mathcal{T}_h$ is defined by

$$h = \max_{K \in \mathcal{T}_h} h_K,$$

where h_K is the diameter of K. This number h is supposed to vary and approach zero. While h varies we assume that the corresponding family of triangulations is regular, i.e. that there exists a constant σ such that

$$\frac{h_K}{\rho_K} \leq \sigma$$

for every K in $\mathcal{T}_h$, where ρ_K is the interior diameter of K. In other words, ρ_K is the diameter of the biggest disc included in K.

Once a family of such triangulations has been chosen the simplest choice of a related family of spaces V_h is as follows: the functions belonging to V_h are the continuous functions on $\bar{\Omega}$ which vanish on Γ and whose restrictions to each $K \in \mathcal{T}_h$ are 'linear' (i.e. affine).

In order to take advantage of Céa's lemma we need an estimate for

$$\inf_{v \in V_h} \|u - v\|_V = d_V(u; V_h).$$

The classical result is that there exists a constant C such that

$$d_V(u; V_h) \leq Ch^k \|u\|_{k+1,2,\Omega} \tag{8,4,1,6}$$

provided $u \in H^{k+1}(\Omega)$, $k = 1, 2$ (see Section 3.2 in Ciarlet (1978)). Extrapolating this inequality to non-integral values of k, $1 \leq k \leq 2$, and taking in account (8,4,1,2), one expects here the estimate

$$d_V(u; V_h) = O(h^{\pi/\omega - \varepsilon})$$

for every $\varepsilon > 0$. Indeed, by Theorem 1.4.5.3, we have

$$u \in H^{1+(\pi/\omega)-\varepsilon}(\Omega) \setminus H^{1+(\pi/\omega)}(\Omega)$$

for every $\varepsilon > 0$, if λ does not vanish. Even choosing higher-order finite element spaces leads to the same limitation of the asymptotic rate of convergence of the error as $h \to 0$. However, as we shall show now, the above inequality (8,4,1,6) does not yield the best estimate of the asymptotic rate of convergence provided some additional assumptions are made on $\mathcal{T}_h$.

Indeed, the property (8,4,1,2) prevents u from belonging to $H^2(\Omega)$ when λ does not vanish. Nevertheless it allows u to belong to a weighted space corresponding to the second order of differentiation. For this purpose let us set a new definition.

Definition 8.4.1.1 *For α a nonnegative real number, we denote by $H^{2,\alpha}(\Omega)$ the space of all functions $u \in H^1(\Omega)$ such that in addition*

$$r^\alpha D^\beta u \in L_2(\Omega)$$

for every β such that $|\beta| = 2$.

We observe that for $\omega \in]\pi, 2\pi[$ we have $u \in H^{2,\alpha}(\Omega)$ for every α such that

$$\alpha > 1 - \pi/\omega \tag{8,4,1,7}$$

Some preliminary properties of those spaces will be useful.

Lemma 8.4.1.2 *We equip $H^{2,\alpha}(\Omega)$ with the norm*

$$u \to \left\{ \|u\|^2_{1,2,\Omega} + \sum_{|\beta|=2} \|r^\alpha D^\beta u\|^2_{0,2,\Omega} \right\}.$$

Then the natural imbedding of $H^{2,\alpha}(\Omega)$ into $H^1(\Omega)$ is compact for $\alpha < 1$. In addition $H^{2,\alpha}(\Omega)$ is continuously imbedded in $C^0(\bar{\Omega})$.

Proof A mere application of Hölder's inequality shows that

$$H^{2,\alpha}(\Omega) \subset W^2_p(\Omega)$$

for every p such that $1 < p < 2/(\alpha + 1)$. Furthermore the corresponding imbedding is continuous.

It follows that $H^{2,\alpha}(\Omega)$ is continuously imbedded in $C^0(\bar{\Omega})$ by Theorem 1.4.5.2, provided $\alpha < 1$. The compactness of the imbedding of $H^{2,\alpha}(\Omega)$ into $H^1(\Omega)$ is a consequence of Theorem 1.4.3.2. ■

Lemma 8.4.1.3 *Let $P_1(\Omega)$ be the space of the first-order polynomials restricted to Ω. Then there exists a constant C such that*

$$\inf_{p \in P_1(\Omega)} \|u - p\|^2_{H^{2,\alpha}(\Omega)} \leq C^2 \sum_{|\beta|=2} \|r^\alpha D^\beta u\|^2_{0,2,\Omega} \tag{8,4,1,8}$$

for every $u \in H^{2,\alpha}(\Omega)$.

It is worth observing the similarity of this lemma with Theorem 3.1.1 in Ciarlet (1978).

Proof A first step is the proof of the following inequality

$$\|v\|^2_{H^{2,\alpha}(\Omega)} \leq C^2 \sum_{|\beta|=2} \|r^\alpha D^\beta v\|^2_{0,2,\Omega} \tag{8,4,1,9}$$

for every $v \in P_1(\Omega)^\perp$ the orthogonal of $P_1(\Omega)$ in $H^{2,\alpha}(\Omega)$.

Indeed if (8,4,1,9) does not hold, there exists a sequence v_n, $n = 1, 2, \dots$ of functions in $P_1(\Omega)^\perp$ such that

$$\|v_n\|_{H^{2,\alpha}(\Omega)} = 1 \tag{8,4,1,10}$$

for every n, while

$$r^\alpha D^\beta v_n \to 0, \qquad |\beta| = 2 \tag{8,4,1,11}$$

in $L_2(\Omega)$ as $n \to +\infty$.

The compactness of the imbedding of $H^{2,\alpha}(\Omega)$ into $H^1(\Omega)$ (Lemma 8.4.1.2) implies that there exists a subsequence which is strongly convergent in $H^1(\Omega)$. Again we denote this subsequence by v_n, $n = 1, 2, \dots$ and thus there exists $v \in H^1(\Omega)$ such that

$$v_n \to v$$

in $H^1(\Omega)$ as $n \to +\infty$.

Next, by the very definition of the norm in $H^{2,\alpha}(\Omega)$, v_n, $n = 1, 2, \dots$ is a Cauchy sequence in $H^{2,\alpha}(\Omega)$. Indeed we have

$$\|v_n - v_m\|^2_{H^{2,\alpha}(\Omega)} = \|v_n - v_m\|^2_{1,2,\Omega} + \sum_{|\beta|=2} \|r^\alpha D^\beta (v_n - v_m)\|^2_{0,2,\Omega}$$

and both terms on the right-hand side converge to zero as n and $m \to +\infty$. Accordingly we have

$$v \in H^{2,\alpha}(\Omega)$$

and

$$v_n \to v$$

in the norm of $H^{2,\alpha}(\Omega)$. It follows that $v \in P_1(\Omega)^\perp$ since $v_n \in P_1(\Omega)^\perp$ for every n, and furthermore (8,4,1,11) implies that

$$D^\beta v = 0, \qquad |\beta| = 2.$$

It follows that $v \in P_1(\Omega) \cap P_1(\Omega)^\perp$, i.e. $v = 0$. This contradicts (8,4,1,10), which implies that

$$\|v\|_{H^{2,\alpha}(\Omega)} = 1.$$

Now we complete the proof by observing that (8,4,1,8) follows from (8,4,1,9) by setting

$$v = u - p,$$

where p is the orthogonal projection of u onto $P_1(\Omega)$. ■

From now on we denote by $\hat{K}$ the model triangle whose vertices are

$(0,0)$, $(0,1)$ and $(1,0)$. For any function $u \in H^{2,\alpha}(\hat{K})$, we denote by $\hat{\Pi}u$ the first order interpolating polynomial i.e.

$$\hat{\Pi}u \in P_1(\hat{K})$$

and

$$\hat{\Pi}u = u \qquad \text{at } (0,0), (0,1) \text{ and } (1,0).$$

This makes sense since u is continuous by Lemma 8.4.1.2. Then for every $p \in P_1(\hat{K})$ we have

$$u - \hat{\Pi}u = (1-\hat{\Pi})(u-p).$$

Both the identity operator and $\hat{\Pi}$ are continuous from $H^{2,\alpha}(\hat{K})$ into $H^1(\hat{K})$. Consequently there exists a constant $\hat{C}$ such that

$$\|1-\hat{\Pi}\|_{H^{2,\alpha}(\hat{K})\to H^1(\hat{K})} \leq \hat{C}$$

and thus we have

$$\|u-\hat{\Pi}u\|_{1,2,\hat{K}} = \|(1-\hat{\Pi})(u-p)\|_{1,2,\hat{K}} \leq \hat{C}\,\|u-p\|_{H^{2,\alpha}(\Omega)}.$$

Taking the infimum in p it follows from (8,4,1,8) that

$$\|u-\hat{\Pi}u\|^2_{1,2,\hat{K}} \leq \hat{C}^2 \sum_{|\beta|=2} \|r^\alpha D^\beta u\|^2_{0,2,\hat{K}} \qquad (8,4,1,12)$$

for every $u \in H^{2,\alpha}(\hat{K})$.

The above inequality is fundamental in the sequel. We shall need a similar inequality on an arbitrary triangle. For this purpose let us consider a triangle K whose vertices are a, b, c with

$$a = (a_1, a_2), \qquad b = (a_1+b_1, a_2+b_2), \qquad c = (a_1+c_1, a_2+c_2).$$

The triangle K is the image of the model triangle $\hat{K}$ under the affine mapping

$$\Phi_K : x \to a + T_K x,$$

where the matrix of T_K is

$$\begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix}.$$

We have already introduced above the numbers h_K (diameter of K) and ρ_K (radius of the biggest circle contained in K). We can estimate T_K with the help of these numbers: obviously we have

$$\|T_K\| \leq \sqrt{2}\, h_K \qquad \text{and} \qquad \|T_K^{-1}\| \leq \frac{\sqrt{2}}{\rho_K}. \qquad (8,4,1,13)$$

For $u \in H^{2,\alpha}(K)$ we denote by $\Pi_K u$ the first-order interpolating polynomial, i.e.

$$\Pi_K u \in P_1(K)$$

and

$$\Pi_K u = u \qquad \text{at the vertices of } K.$$

We have the following estimate:

Lemma 8.4.1.4 *There exists a constant C independent of the triangle K such that*

$$\|\nabla(u - \Pi_K u)\|_{0,2,K}^2 \leqslant C\,\|T_K^{-1}\|^{2+2\alpha}\,\|T_K\|^4 \sum_{|\beta|=2} \int_K d(x,a)^{2\alpha}\,|D^\beta u(x)|^2\,dx \quad (8,4,1,14)$$

for every $u \in H^{2,\alpha}(K)$.

Proof We set $\hat{u} = u \circ \Phi_K$. Obviously we have $\hat{u} \in H^{2,\alpha}(\hat{K})$, and in addition

$$(\Pi_K u) \circ \Phi_K = \hat{\Pi}\hat{u}.$$

Then we can apply inequality (8,4,1,12) to $\hat{u}$: this yields

$$\|\nabla(\hat{u} - \hat{\Pi}\hat{u})\|_{0,2,\hat{K}}^2 \leqslant \hat{C}^2 \sum_{|\beta|=2} \|r^\alpha D^\beta \hat{u}\|_{0,2,\hat{K}}^2,$$

or equivalently

$$\int_{\hat{K}} |\nabla[(u - \Pi_K u) \circ \Phi_K](\hat{x})|^2\,d\hat{x} \leqslant \hat{C}^2 \sum_{|\beta|=2} \int_{\hat{K}} |r(\hat{x})^\alpha D^\beta[u \circ \Phi_K](\hat{x})|^2\,d\hat{x}.$$

Next applying the chain rule for differentiation we get

$$\int_{\hat{K}} |T_K[\nabla(u - \Pi_K u)] \circ \Phi_K(\hat{x})|^2\,d\hat{x} \leqslant \hat{C}^2 \sum_{|\beta|=2} \int_{\hat{K}} |r(\hat{x})^\alpha T_K^2 D^\beta u \circ \Phi_K(\hat{x})|^2\,d\hat{x}.$$

Finally we perform the obvious change of variable, setting $x = \Phi_K(\hat{x})$. Thus we obtain

$$\int_K |\nabla(u - \Pi_K u)|^2\,dx \leqslant \hat{C}^2\,\|T_K^{-1}\|^2\,\|T_K\|^4 \sum_{|\beta|=2} \int_K |r(\Phi_K^{-1}(x))^\alpha D^\beta u(x)|^2\,dx.$$

The desired inequality follows since we have

$$r(\Phi_K^{-1}(x)) = d(\Phi_K^{-1}(x);\ \Phi_K^{-1}(a)) \leqslant \|T_K^{-1}\|\,d(x,a). \quad \blacksquare$$

Remark 8.4.1.5 We shall use inequality (8,4,1,14) only in two particular cases. First, when $\alpha=0$, we get the inequality (already proved in Ciarlet (1978)):

$$\|\nabla(u-\Pi_K u)\|_{0,2,K}^2 \leq C^2 \|T_K^{-1}\|^2 \|T_K\|^4 \sum_{|\beta|=2} \|D^\beta u\|_{0,2,K}^2. \tag{8,4,1,15}$$

Second when $\alpha<1$ and $a=0$ (i.e. one of the vertices of K is the origin) one gets the weighted inequality

$$\|\nabla(u-\Pi_K u)\|_{0,2,K}^2 \leq C^2 \|T_K^{-1}\|^{2+2\alpha} \|T_K\|^4 \sum_{|\beta|=2} \|r^\alpha D^\beta u\|_{0,2,K}^2. \tag{8,4,1,16}$$

The next statement is an easy consequence of these preliminaries. We consider a triangulation over $\bar{\Omega}$ as above and Π_h, the interpolation operator, defined as follows for every $u\in H^{2,\alpha}(\Omega)$:

(a) $\Pi_h u|_K \in P_1(K)$ for every $K\in\mathcal{T}_h$.
(b) $\Pi_h u = u$ at any vertex of any $K\in\mathcal{T}_h$.

Theorem 8.4.1.6 *We assume that the family of triangulations $\mathcal{T}_h$ satisfies the following conditions as $h\to 0$; there exists σ such that*

(a) $\max_{K\in\mathcal{T}_h} h_K/\rho_K \leq \sigma$ *for every* h;
(b) $h_K \leq \sigma h^{1/(1-\alpha)}$ *for every* $K\in\mathcal{T}_h$ *such that one of the corners of K is at* 0;
(c) $h_K \leq \sigma h \inf_K r^\alpha$ *for every* $K\in\mathcal{T}_h$ *with no corner at* 0.

Then there exists a constant C such that

$$\|u-\Pi_h u\|_{1,2,\Omega} \leq Ch\,\|u\|_{H^{2,\alpha}(\Omega)} \tag{8,4,1,17}$$

for every $h>0$ and every $u\in H^{2,\alpha}(\Omega)$, provided $\alpha<1$.

Proof We observe that for every $k\in\mathcal{T}_h$ the restriction $\Pi_h u|_K$ of $\Pi_h u$ to K is just $\Pi_K(u|_K)$, where $u|_K$ is the restriction of u to K. Thus we can apply one of the inequalities (8,4,1,15), (8,4,1,16) to $u|_K$.

If one of the vertices of K is 0 we make use of (8,4,1,16); this yields

$$\|\nabla(u-\Pi_h u)\|_{0,2,K}^2 \leq C^2 \|T_K^{-1}\|^{2+2\alpha} \|T_K\|^4 \sum_{|\beta|=2} \|r^\alpha D^\beta u\|_{0,2,K}^2.$$

On the other hand if no vertex of K is 0 we make use of (8,4,1,15). Thus we get

$$\begin{aligned}\|\nabla(u-\Pi_h u)\|_{0,2,K}^2 &\leq C^2 \|T_K^{-1}\|^2 \|T_K\|^4 \sum_{|\beta|=2} \|D^\beta u\|_{0,2,K}^2\\ &\leq C^2 \|T_K^{-1}\|^2 \|T_K\|^4 \left(\inf_K r^{2\alpha}\right)^{-1} \sum_{|\beta|=2} \|r^\alpha D^\beta u\|_{0,2,K}^2.\end{aligned}$$

In both cases the inequalities (8,4,1,13) and the assumptions (a)–(c) in the statement of Theorem 8.4.1.6 imply the following inequality:

$$\|\nabla(u - \Pi_h u)\|^2_{0,2,K} \leq C^2 h^2 \sum_{|\beta|=2} \|r^\alpha D^\beta u\|^2_{0,2,K}$$

with possibly another value for the constant C (yet independent of K and u). Inequality (8,4,1,17) follows by addition (over $K \in \mathcal{T}_h$) and with the help of Poincaré's inequality (cf. Theorem 1.4.3.4). ■

Corollary 8.4.1.7 *If the triangulation $\mathcal{T}_h$ fulfils the conditions in Theorem* 8.4.1.6, *then there exists a constant C which depends on neither u nor h such that*

$$\|u - u_h\|_{1,2,\Omega} \leq Ch \|u\|_{H^{2,\alpha}(\Omega)} \tag{8,4,1,18}$$

(*we recall that u and u_h are defined by* (8,4,1,1) *and* (8,4,1,4) *respectively*).

This follows obviously from inequality (8,4,1,17) and Céa's lemma (inequality (8,4,1,5)).

This result shows that one can expect the same asymptotic rate of convergence (as $h \to 0$) of the error (in the norm of $H^1(\Omega)$) as in the regular case provided the mesh is refined in a suitable way near 0. In addition it is also shown in Babuska *et al.* (1979) that this is the best asymptotic rate of convergence that one can expect for spaces V_h such that the dimension does not grow faster than $0(h^{-2})$ as $h \to 0$. This can also be derived from the asymptotic estimates of diameters in El-Kolli (1971).

In practice one has to make sure that meshes refined in the above way do exist. In Raugel (1978), the following method is proposed:

First step: divide Ω into 'big' triangles;

Second step: divide each of the big triangles which have no vertex at zero in the usual uniform way (i.e. divide each side into n subsegments of the same length and proceed).

Third step: divide each of the big triangles which have a vertex at zero, according to the ratios

$$\left(\frac{i}{n}\right)^{1/(1-\alpha)}, \qquad 1 \leq i \leq n$$

along the sides which end at zero. Divide the third side in the usual way (see figure) and proceed as usual.

With such a procedure the dimension of V_h is equivalent to n^2 (as $n = h^{-1}$ goes to infinity).

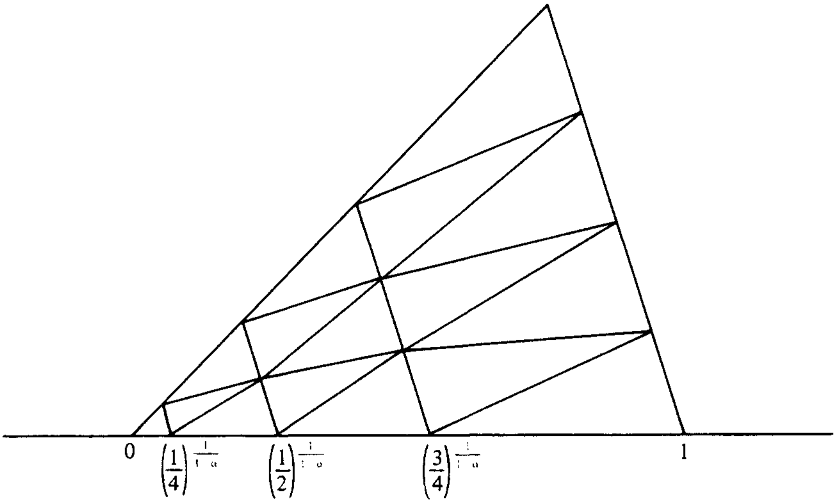

Figure 8.5

Remark 8.4.1.8 It is important to emphasize that, once a refined mesh has been chosen, the algebra of the approximate problem (8,4,1,4) is unchanged with respect to the regular case. In other words the stiffness matrix has the same structure as in the case when all the triangles have the same order of magnitude.

Remark 8.4.1.9 We observe that all the previous analysis has been done starting from the knowledge that u belongs to some weighted space $H^{2,\alpha}(\Omega)$. We never used the explicit form of the singularity given by (8,4,1,2). This is the reason why such an analysis may be carried out for problems which are much more general than the model problem (8,4,1,1).

Remark 8.4.1.10 No attempt is made in the current literature about the numerical treatment of singular problems to estimate the constant C in inequality (8,4,1,18). In practice, the implementation is done with a chosen $h>0$ and the order of magnitude of C is as important as the asymptotic rate of convergence in h for estimating the error.

8.4.2 Augmenting the space of trial functions

Another method for overcoming the polluting effect of the corners on the finite element method has been proposed. Instead of refining the mesh one keeps a regular mesh all over Ω, i.e. we consider triangulations $\mathcal{T}_h$ such that there exists σ_1 and σ_2 such that

$$\sigma_1 h \leqslant \rho_k \leqslant h_K \leqslant \sigma_2 h$$

for every $K \in \mathcal{T}_h$. However, we shall consider a bigger space V_h than before.

To be precise let us denote by E_h the space of the continuous functions u on $\bar{\Omega}$ whose restriction to each $K \in \mathcal{T}_h$ belongs to $P_1(K)$ and which vanish on Γ. We also introduce a cut-off function η, of r, which is identically 1 near zero and which vanishes near all the Γ_j except Γ_1 and Γ_N (notation of Subsection 8.4.1). Then we define V_h as the direct sum of E_h and the one-dimensional space generated by the function

$$r, \theta \to \eta(r) r^{\pi/\omega} \sin \frac{\pi\theta}{\omega} = \eta(r) u_s(r, \theta).$$

In other words we set

$$V_h = E_h \oplus \mathbb{R}\{\eta u_s\}.$$

With this new space V_h we consider the approximate problem (8,4,1,4). Again we can apply Cea's lemma and we get

$$\|u - u_h\|_{1,2,\Omega} \leq \frac{M}{\alpha} \inf_{v \in V_h} \|u - v\|_{1,2,\Omega}.$$

From (8,4,1,2) we know that

$$u = w + \lambda \eta u_s,$$

where $w \in H^2(\Omega)$. Thus we have

$$\inf_{v \in V_h} \|u - v\|_{1,2,\Omega} = \inf_{\varphi \in E_h} \|w - \varphi\|_{1,2,\Omega}$$

and from the results in Section 3.2 of Ciarlet (1978), we conclude that there exists a constant C such that

$$\inf_{\varphi \in E_h} \|w - \varphi\|_{1,2,\Omega} \leq Ch \|w\|_{2,2,\Omega}.$$

Summing up we have

$$\|u - u_h\|_{1,2,\Omega} \leq Ch \|w\|_{2,2,\Omega} \tag{8,4,2,1}$$

with, possibly, another value for C. With such a choice of V_h, we have obtained the same asymptotic rate of convergence as in the regular case (and again here the dimension of V_h is of the order h^{-2} as in Subsection 8.4.1).

Adding the one-dimensional space generated by ηu_s to E_h, the usual space of trial functions, disturbs the sparseness of the matrix of problem (8,4,1,4). Therefore it is advisable to choose the support of η small enough to be contained in those $K \in \mathcal{T}_h$ which have one corner at 0. Accordingly an accurate study of the asymptotic error estimate might be done with a cut-off function η depending also on h. Some details can be found in Destuynder and Djaoua (1979) and Lelievre (1976b).

8.4.3 Calculating the stress intensity factor

The coefficient λ of the singular part of the solution in (8,4,1,2) is often called the 'stress intensity factor'. Indeed in mechanical problems the stresses are given by the gradient of the solution u. Here we have

$$u = \lambda r^{\pi/\omega} \sin \frac{\pi\theta}{\omega} + w,$$

where $w \in H^2(\Omega)$. Actually it was shown in Chapter 4 that there exists $p_0 > 2$ such that $w \in W_p^2(\Omega)$ for every $f \in L_p(\Omega)$ provided $p < p_0$.† Consequently the gradient of w is bounded and the unbounded part of ∇u is

$$\lambda \nabla r^{\pi/\omega} \sin \frac{\pi\theta}{\omega}.$$

In most practical problems one is mainly interested in calculating (approximately) λ rather than the whole solution u. When one works with the method outlined in Subsection 8.4.2, one can show an error estimate of the form

$$|\lambda - \lambda_h| \leq C\sqrt{h}\, \|w\|_{2,2,\Omega},$$

where λ_h is defined by

$$u_h - \lambda_h \eta u_S \in E_h.$$

This is proved in Djaoua (1983) in the particular case of a crack. This shows poor convergence of λ_h toward λ.

In addition it is not clear how to compute λ when one uses only mesh refinements as outlined in Subsection 8.4.1.

Several approaches have been proposed in Schatz and Wahlbin (1978–79), Destuynder and Djaoua (1979) and finally in Destuynder *et al.* (1981, 1983) for the particular case of a crack. Here we shall rely on a very simple method devised in Bellout and Moussaoui (1981). We rely on the results in Section 4.4. There we introduced (Subsection 4.4.1) the space N_2 of all functions v in $L_2(\Omega)$ which are orthogonal to the image of $H^2(\Omega) \cap \mathring{H}^1(\Omega)$ by Δ. We saw that every v belonging to N_2 is harmonic in Ω and vanishes on the boundary Γ in the weak sense. In addition we have proven (in Subsection 4.4.2) that N_2 is one-dimensional and generated by one function σ such that

$$\sigma - r^{-\pi/\omega} \sin \frac{\pi\theta}{\omega} \in H^1(\Omega).$$

† To be precise we have

$$p_0 = \min\left\{\left(1 - \frac{\pi}{\omega_N}\right)^{-1};\ \min_{\substack{j=1,2,\ldots,N-1\\ \omega_j > \pi/2}} \left(1 - \frac{\pi}{2\omega_j}\right)^{-1}\right\}.$$

This suggests that λ is proportional to the scalar product

$$\int_\Omega f\sigma \, dx.$$

Unfortunately the function σ is not known explicitly because Ω is too general a domain. Consequently we are unable to calculate the proportionality ratio above. To make this calculation feasible we replace Ω by the simpler domain

$$D_\rho = \{re^{i\theta}; 0 < r < \rho, 0 < \theta < \omega\},$$

where ρ is chosen small enough such that

$$\rho < \min_{j=2,\dots,N-1} d(0, \bar{\Gamma}_j).$$

In addition we assume that η vanishes for $r \geq \rho/2$. We can now state the basic preliminary result of this subsection.

Lemma 8.4.3.1 *Assume that $u \in \mathring{H}^1(\Omega)$ and that*

$$u - \lambda r^{\pi/\omega} \sin \frac{\pi\theta}{\omega} \in H^2(\Omega);$$

then

$$\lambda = \frac{1}{\pi} \int_\Omega \Delta(\eta u) r^{-\pi/\omega} \sin \frac{\pi\theta}{\omega} \, dx. \tag{8,4,3,1}$$

Proof We use the same method as in Lemma 4.4.4.10.

We denote by w the difference

$$u - \lambda r^{\pi/\omega} \sin \frac{\pi\theta}{\omega} = u - \lambda u_S.$$

First we shall calculate the integral

$$I_1 = \int_\Omega \Delta(\eta w) r^{-\pi/\omega} \sin \frac{\pi\theta}{\omega} \, dx$$

by applying Theorem 1.5.3.6. For this purpose we introduce a second cut-off function η_1 which is equal to one on the support of η and which vanishes for $r \geq \rho$. Thus we have

$$I_1 = \int_\Omega \Delta(\eta w) \eta_1 r^{-\pi/\omega} \sin \frac{\pi\theta}{\omega} \, dx.$$

It is clear that $\eta w \in W_q^2(\Omega)$ for $q \leqslant 2$ and that the function

$$r,\theta \rightarrow \eta_1 r^{-\pi/\omega} \sin \frac{\pi\theta}{\omega} = \psi(r, \theta)$$

belongs to $D(A; L_p(\Omega))$ for $p < 2\omega/\pi$. In addition we have

$$\eta w(S_j) = 0$$

for every j. Thus (1,5,3,6) yields

$$I_1 = \int_\Omega \eta w \Delta\left(\eta_1 r^{-\pi/\omega} \sin \frac{\pi\theta}{\omega}\right) \mathrm{d}x = 0,$$

since the traces of all the involved functions are zero on the boundary and the function $r^{-\pi/\omega} \sin \pi\theta/\omega$ is harmonic on the support of η.

In order to calculate

$$I_2 = \int_\Omega \Delta\left(\eta r^{\pi/\omega} \sin \frac{\pi\theta}{\omega}\right) r^{-\pi/\omega} \sin \frac{\pi\theta}{\omega} \mathrm{d}x,$$

we apply the Green formula on the subdomain

$$\Omega_\varepsilon = \Omega \backslash \{r < \varepsilon\}$$

and we take the limit as $\varepsilon \rightarrow 0$. This yields

$$\begin{aligned} I_2 &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} \Delta\left(\eta r^{\pi/\omega} \sin \frac{\pi\theta}{\omega}\right) r^{-\pi/\omega} \sin \frac{\pi\theta}{\omega} \mathrm{d}x \\ &= \lim_{\varepsilon \rightarrow 0} \varepsilon \int_0^\omega \left\{\frac{\pi}{\omega} \varepsilon^{\pi/\omega - 1} \sin \frac{\pi\theta}{\omega} \varepsilon^{-\pi/\omega} \sin \frac{\pi\theta}{\omega} \right. \\ &\quad \left. + \varepsilon^{\pi/\omega} \sin \frac{\pi\theta}{\omega} \frac{\pi}{\omega} \varepsilon^{-\pi/\omega - 1} \sin \frac{\pi\theta}{\omega}\right\} \mathrm{d}\theta \\ &= \frac{2\pi}{\omega} \int_0^\omega \left[\sin \frac{\pi\theta}{\omega}\right]^2 \mathrm{d}\theta = \pi. \end{aligned}$$

Summing up, we have proved that

$$\int_\Omega \Delta(\eta u) r^{-\pi/\omega} \sin \frac{\pi\theta}{\omega} \mathrm{d}x = I_1 + \lambda I_2 = \pi\lambda. \quad \blacksquare$$

Let us set

$$\Psi = r^{-\pi/\omega} \sin \frac{\pi\theta}{\omega}.$$

Integrating by parts we derive from identity (8,4,3,1) that

$$\begin{aligned}\lambda &= \frac{1}{\pi}\int_\Omega [\eta f + 2\nabla\eta\cdot\nabla u + (\Delta\eta)u]\Psi\,dx\\ &= \frac{1}{\pi}\int_\Omega [\eta f\Psi + (\Delta\eta)u\Psi - 2u\,\operatorname{div}(\Psi\nabla\eta)]\,dx\\ &= \frac{1}{\pi}\int_\Omega \eta f\Psi\,dx - \frac{1}{\pi}\int_\Omega\{\Psi\Delta\eta + 2\nabla\eta\cdot\nabla\Psi\}u\,dx \qquad (8,4,3,2)\end{aligned}$$

due to the fact that $\nabla\eta$ vanishes for small r and for $r\geq\rho/2$.

This suggests that we define an approximate value for λ by setting

$$\lambda_h = \frac{1}{\pi}\int_\Omega \eta f\Psi\,dx - \frac{1}{\pi}\int_\Omega\{\Psi\Delta\eta + 2\nabla\eta\cdot\nabla\Psi\}u_h\,dx, \qquad (8,4,3,3)$$

where u_h is the solution of (8,4,1,4). It is easy to estimate $\lambda-\lambda_h$.

Theorem 8.4.3.2 *Let u be the solution of the problem* (8,4,1,1), *u_h the solution of the approximate problem* (8,4,1,4) *and λ such that* (8,4,1,2) *holds. Assume λ_h is defined by* (8,4,3,3) *and that the triangulations $\mathcal{T}_h$ are regular. Then there exists a constant C* (*which does not depend on h or f*) *such that*

$$|\lambda-\lambda_h|\leq Ch\,\|f\|_{0,2,\Omega}.$$

Proof Clearly it follows from (8,4,3,2) and (8,4,3,3) that

$$|\lambda-\lambda_h|\leq K\,\|u-u_h\|_{0,2,\Omega} \qquad (8,4,3,4)$$

for some constant K. Then as we have shown in Section 8.4.1 it follows from (8,4,1,6) that

$$\|u-u_h\|_{1,2,\Omega}=O(\sqrt{h})$$

at least. In addition one has $u\in H^{3/2}(\Omega)$ at least (assuming $\omega<2\pi$). Therefore the Aubin–Nitsche trick (cf. Ciarlet (1978)) implies that

$$\|u-u_h\|_{0,2,\Omega}=O(h). \qquad (8,4,3,5)$$

The result obviously follows from (8,4,3,4) and (8,4,3,5). ■

Remark 8.4.3.3 Let us emphasize that no mesh refinement is needed in Theorem 8.4.3.2.