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Some criteria concerning the vorticity and the problem of global regularity for the 3D Navier–Stokes equations

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Abstract We review some results concerning the problem of global-in-time regularity for the initial boundary value problem for the Navier–Stokes equations in three-dimensional domains. In particular, we focus on sufficient conditions on the vorticity field which imply that strong (hence smooth) solutions exist on arbitrary time intervals.

Keywords Navier–Stokes equations · Regularity · Vorticity

Mathematics Subject Classification (2000) 35Q30 · 35B65

1 Introduction

We consider the initial boundary-value problem for the 3D Navier–Stokes equations

$$\begin{cases} u_t + (u \cdot \nabla)\, u - \Delta u + \nabla p = 0 & \text{in } \Omega\times]0,T], \\ \nabla \cdot u = 0 & \text{in } \Omega\times]0,T], \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where the unknowns are the velocity $u = (u^1, u^2, u^3)$ and the pressure p. In order to avoid inessential complications we assume that external force vanishes and that the kinematic viscosity is equal to 1. The physical domain Ω will be either a) the whole-space $\mathbb{R}^3$; b) the half space $\mathbb{R}^3_+$; c) an open and bounded set $\Omega \subset \mathbb{R}^3$ with a smooth

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boundary $\partial\Omega$ (say of class $C^{3,\alpha}$, for some $\alpha > 0$); d) the block $\mathbb{T} =]0, 2\pi[^3$, with periodic boundary conditions. When considering domains with boundary we specify case-by-case the boundary conditions we consider. Most of the results depend critically on them and in some cases we shall be able to deal with the Dirichlet problem, i.e., $u = 0$ on $\partial\Omega\times]0, T]$. In other cases we can obtain results with the so-called "stress-free" boundary conditions:

$$\begin{cases} u \cdot n \;\; = 0 & \text{on } \partial\Omega\times]0, T], \\ \omega \times n = 0 & \text{on } \partial\Omega\times]0, T], \end{cases} \tag{1.2}$$

where $\omega = \nabla \times u = \operatorname{curl} u$ is the vorticity field, while n denotes the exterior unit normal vector. These boundary conditions can be used on a free-surface (see, e.g., Temam [57]). In the presence of a flat boundary they are exactly equal to the classical "Navier's boundary conditions" or slip-without-friction ($u \cdot n = 0$ and $n \cdot \nabla u - (n \cdot \nabla u \cdot n)\, n = 0$, see Serrin [50] and see also the pioneering results of Solonnikov and Ščadilov [55]).

We give now the definition of two main classes of solutions. We use standard notations for Lebesgue L^p and Sobolev H^s spaces. For all function spaces the subscript "$_\sigma$" denotes divergence-free vector fields.

Definition 1.1 (Weak solution à la Leray–Hopf) We say that the vector field u tangential to the boundary $\partial\Omega$ and such that $u \in L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; H^1_\sigma(\Omega))$ is a weak solution to (1.1), with the boundary conditions (1.2), if an energy inequality holds and if

$$\int_0^T\!\!\int_\Omega (-u\,\phi_t + \nabla u \nabla \phi + (u\cdot\nabla)u\,\phi)\;dxdt + \int_0^T\!\!\int_{\partial\Omega} \phi\,\nabla n\, u\, dSdt$$
$$= \int_\Omega u_0(x)\,\phi(x,0)\,dx,$$

for each $\phi \in C^\infty_\sigma([0, T] \times \overline{\Omega})$ such that $\phi(T) = 0$ in Ω, and $\phi \cdot n = 0$ on $\partial\Omega \times [0, T]$.

Remark 1.1 The precise definition of weak solution in the whole space, or in the torus, or even with Dirichlet boundary conditions require small changes. In these cases the definitions are well-known. See, e.g., Galdi [31] and Temam [56] for further details.

Definition 1.2 (Strong solution) We say that a weak solution u is strong in $[0, T]$ if

$$\nabla u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)).$$

We say that a weak solution u is strong in $[0, T_1[$ if it is strong in $[0, T]$, for each $T < T_1$.

The variational formulation and numerical implementation of the stationary Navier-Stokes equations with the "non-standard" boundary conditions (1.2) can be found in Girault [32]. The initial value problem for the Navier–Stokes equations with the above boundary conditions (1.2) poses the same problems as the usual one with more classical boundary conditions: for initial data in $L^2_\sigma(\Omega)$ one can prove global-in-time existence of weak solutions, but their uniqueness or their smoothness represent remarkable open problems. On the other hand, for initial data in $H^1_\sigma(\Omega)$ (and satisfying the boundary conditions of the problem in some sense, see Beirão da Veiga [5], Sect. 3) one can prove local-in-time existence of strong solutions. Moreover, strong solutions are unique and smooth (if the data of the problem are smooth). The proofs can be done by adapting the usual ones concerning Dirichlet boundary conditions in Hopf [36]. See [7], Sect. 2 for further details and also [5] for the Navier's boundary conditions.

In this paper we mainly address the problem of global-in-time existence of smooth solutions, under additional hypotheses on the vorticity and in particular we review and collect recent results on the subject, together with some minor improvements.

We end the introduction by observing that if one is able to control suitable norms of the vorticity, then a unique smooth solution exists. In particular, if one assumes that

$$\omega \in L^\infty(0, T; L^2(\Omega)),$$

then the solution is strong in $[0, T]$. For divergence-free vector fields in $\mathbb{R}^3$ the L^2-norm of the vorticity equals that of ∇u (by using the Biot–Savart law a similar result holds also in bounded domains, see also [7], Lemma 2.7).

Plan of the paper. In this paper we review some of the conditions involving the vorticity which ensure that a weak solution is strong. In Sect. 2 we review some results concerning regularity by means of suitable control of the growth of the vorticity and we prove some new results on a related quantity. Then, in Sect. 3 we present some results with a more "geometric" nature, which are linked with the control of the direction of the vorticity, the direction of the curl of the vorticity and also with a suitable behavior of the helicity.

2 On results based on scaling invariance

In this section we review a few basic results concerning regularity under scaling invariant conditions. The literature about this subject is terrific and we focus just on the results more strictly related with the rest of the paper. Moreover, since we want to stay in the context of classical Sobolev spaces we do not report results in Besov or even more general spaces, even if several results in this framework have been proved in the last years. In addition, we wish to mention that many results concerning regularity if scaling invariant conditions are satisfied only by some components of velocity, vorticity, or pressure's gradient have been proved in the last years due to the contributions of Beirão da Veiga Bae, Cao, Chae, He, Kukavica, Neustupa, Penel, Titi, Zhou, and many others.

It is well-known that weak solutions of the initial value problem belonging to

$$L^r(0, T; L^s(\Omega)) \quad \text{with} \quad \frac{2}{r} + \frac{3}{s} = 1, \quad \text{for} \quad s > 3, \tag{2.1}$$

are unique and space-time smooth. Under this assumption one can easily prove (by using the procedure introduced by Prodi [47] of testing the equations with $-P\Delta u$)

that $\|\nabla u(t)\|_{L^2}$ is uniformly bounded in $[0, T]$, see the review in Galdi [31]. (If condition (2.1) holds locally then one can prove just local space-regularity, see Ohyama [46] and Serrin [51].) In the last 40 years there has been an enormous interest around this condition, but observe that condition (2.1) dates back to a footnote in Leray [41]. At present it seems that there is no clue (apart some logarithmic improvements see Farwig et al. [27,28], Montgomery-Smith [45], Chan and Vasseur [18], Zhou and Lei [63]) to go beyond these exponents. A "justification" for these exponents can be given by recalling that if a pair $\{u, p\}$ solves (1.1) in the whole space, then so does the family $\{u_\lambda, p_\lambda\}_{\lambda>0}$ defined by

$$u_\lambda(x,t) := \lambda\, u(\lambda x, \lambda^2 t), \qquad p_\lambda(x,t) := \lambda^2\, p(\lambda x, \lambda^2 t),$$

and $\|u_\lambda\|_{L^r(0,T;L^s)} = \|u\|_{L^r(0,T;L^s)}$ for r and s as in (2.1). Next, the scaling condition for the gradient of the velocity and for the vorticity is

$$\nabla u \text{ or } \omega \in L^r(0,T;L^s(\Omega)) \quad \text{with} \quad \frac{2}{r}+\frac{3}{s}=2, \quad \text{for} \quad \frac{3}{2}<s<\infty. \tag{2.2}$$

We have the following result.

Theorem 2.1 *Let $u_0 \in H^1_\sigma(\Omega)$ and let u be a weak solution to* (1.1) *in* $[0, T]$. *If condition* (2.2) *is satisfied, then u is a strong, hence smooth, in* $[0, T]$.

For the proof see Beirão da Veiga [2] for the whole space and Ref. [8] for general domains with Dirichlet boundary conditions.

One can also understand the connection between (2.1) and (2.2) by using the Sobolev embedding theorem: if u satisfies condition (2.2) with $s < 3$ then

$$u \in L^r(0,T;L^{s^*}(\Omega)) \quad \text{with} \quad \frac{2}{r}+\frac{3}{s^*}=\frac{2}{r}+3\left[\frac{1}{s}-\frac{1}{3}\right]=1.$$

showing that for $s \geq 3$ condition (2.2) is a "natural extension" of (2.1).

Remark 2.1 Some limiting cases are of (2.1) and (2.2) are of particular interest. For instance the limit case $u \in L^\infty(0,T;L^3(\Omega))$ concerning (2.1) has been addressed only recently by Escauriaza, Seregin, and Šverák [26], by using an approach completely different from energy-type estimate and the proof is based on delicate results on backward uniqueness. Concerning this limit case, see also the result in [43,49]. To my knowledge the corresponding case $\nabla u \in L^\infty(0,T;L^{3/2}(\Omega))$ has not been studied, even if probably it can be handled by suitable changes of the proof in [26].

On the other hand, also the limiting case $\nabla u \in L^2(0,T;L^3(\Omega))$ is of interest, since it does not imply $u \in L^2(0,T;L^\infty(\Omega))$ and some improvements also in BMO spaces can be found in Kozono and Taniuchi [38]. In addition, the case $\nabla u \in L^1(0,T;L^\infty(\Omega))$ (which corresponds to the Beale–Kato–Majda [1] criterion) can be proved in general domains and also with Dirichlet boundary conditions. One can prove that the assumption $\omega \in L^1(0,T;L^\infty(\Omega))$ implies regularity in domains without boundaries or if the boundary value problem is supplemented by the stress-free boundary conditions. This is technically due to the fact that one has to resort to the vorticity equation, see also the discussion in the next section. Finally, related scaling invariant conditions for the pressure have been proved in Berselli and Galdi [13,14] and Kang and Lee [37].

2.1 Some remarks on a condition related to curl ω

Together with ω another relevant physical quantity is the vector field curl ω, which -roughly speaking- measures the amount of rotation of the vorticity, see also the introduction in Sect. 3. We recall that a fundamental identity for divergence-free vector fields, which allows to express the velocity (and its derivatives) in terms of the vorticity is

$$\operatorname{curl}\operatorname{curl} u = -\Delta u. \tag{2.3}$$

Conditions concerning the (scaled) growth of this term in order to have local regularity, have been treated by Gustafson et al. [35], in the spirit of partial regularity results *à la* Caffarelli, Kohn, and Nirenberg. We observe now that the scale-invariance argument implies that regularity should be obtained if

$$\Delta u \in L^r(0, T; L^s(\Omega)) \qquad \text{with} \quad \frac{2}{r} + \frac{3}{s} = 3, \tag{2.4}$$

with the consequent restrictions $1 \leq r \leq +\infty$ and $1 \leq s \leq 3$. Next, if $1 \leq s < 3$, then (2.4) implies (by the usual argument based on Sobolev's embedding theorem) condition (2.2) since

$$\nabla u \in L^r(0, T; L^{s^*}(\Omega)) \qquad \text{with} \quad \frac{2}{r} + \frac{3}{s^*} = \frac{2}{r} + 3\left[\frac{1}{s} - \frac{1}{3}\right] = 2.$$

Consequently in (2.4) the most relevant case (not included in previous results) is $r = 1$ and $s = 3$. We give now a sketch of the proof of a slightly more general result. In the sequel $L^p_w(\mathbb{R}^3) \supset L^p(\mathbb{R}^3)$ denotes the Marcinkiewicz (or weak-L^p) space of measurable functions such that

$$\|f\|_{L^p_w} := \sup_{\sigma>0} \sigma\big(\mu\{x \in \mathbb{R}^3 : \quad |f(x)| > \sigma\}\big)^{1/p} < \infty.$$

See also Kozono and Yamazaki [39] for applications of weak L^n-spaces.

Proposition 2.1 *Let u be a weak solution of (1.1) in $(0, T)$ with $u_0 \in H^1_\sigma(\mathbb{R}^3)$. If*

$$\mathit{curl}\,\omega = \Delta u \in L^1(0, T; L^3_w(\mathbb{R}^3)), \tag{2.5}$$

then u is strong, hence smooth, in $[0, T]$.

Proof The proof is done by multiplying the equations by $-\Delta u$, performing integration by parts, and by using usual manipulations together with the following generalized

Hölder inequality: there exists $C > 0$ such that for all $v \in H^1(\mathbb{R}^3)$, $w \in L^2(\mathbb{R}^3)$, and $z \in L^3_w(\mathbb{R}^3)$, then

$$\left| \int_{\mathbb{R}^3} v\, w\, z\, dx \right| \leq C \|v\|_{L^{6,2}} \|w\|_{L^{2,2}} \|z\|_{L^3_w},$$

where $L^{p,q} = L^{p,q}(\mathbb{R}^3)$ denotes the Lorentz space (recall that $L^{p,\infty} = L^p_w$ and $L^{p,p} = L^p$). Then, by using the integral representation

$$z(x) = \frac{1}{12\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \cdot \nabla z(y)\, dy,$$

of z in terms of its gradient, the Hardy–Littlewood–Sobolev inequality, and the "General Marcinkiewicz Interpolation" it follows that there exists $C > 0$ such that

$$\|z\|_{L^{6,2}} \leq C \|\nabla z\|_{L^2},$$

for all $z \in H^1(\mathbb{R}^3)$. Hence, one gets

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2_{L^2} + \nu \|\Delta u\|^2_{L^2} \leq \left| \int_{\mathbb{R}^3} (u \cdot \nabla) u\, \Delta u\, dx \right| \leq C \|\Delta u\|_{L^3_w} \|\nabla u\|^2_{L^2}.$$

Then, Gronwall's lemma implies that

$$\|\nabla u(t)\|^2_{L^2} \leq \|\nabla u_0\|_{L^2} \mathrm{Exp}\left[2C \int_0^T \|\Delta u(\tau)\|_{L^3_w}\, d\tau \right],$$

showing uniform boundedness for $\|\nabla u(t)\|_{L^2}$ provided that (2.5) holds. Missing details to make all calculations completely rigorous by means of unique continuation for strong solutions can be easily filled by the reader. □

3 On some "geometric" conditions

In this section we review some sufficient conditions for regularity, which are based on suitable geometric behavior of the solutions. A basic underlying idea from the mathematical point of view is

> …try to control of the growth of the vorticity by estimating the vortex-stretching term with some extra assumptions. These assumptions are suggested by the 2D situation (in which global regularity is known) and are of geometric nature.

Geometric attempts to understand turbulent flows are for instance those based on knowledge of the behavior of the vorticity's direction, started with Constantin [19] and Constantin and Fefferman [21]. The guidance from the physical point of view comes from the fact that elongated "small structures" supporting large values of $|\omega|$ have been observed by several authors, see, e.g., Vincent and Meneguzzi [59]. In addition, coherent vortices exist in the form of randomly oriented thin tubes see She, Jackson, and Orszag [52]. Experimentally [60] vortex sheets deform curling up like potato-chips, most strongly when rolling up into tubes. In addition, a non-trivial collision of two vortex tubes has been conjectured as a possible mechanism for the creation of singularities (a mechanism different from those leading to *squirt* or to *self similar* singularities). In the 2D case these singularities cannot occur since the vorticity is always perpendicular to the plane of motion and hence $\angle(\widehat{\omega}(x,t),\widehat{\omega}(y,t)) \equiv 0$ for all couples $x, y \in \mathbb{R}^2$. Hereafter $\angle(a,b)$ denotes the angle between the vectors a and b, while for each non-null vector v

$$\widehat{v} := \frac{v}{|v|}.$$

Early investigations in [21] concern the regularity, if $\angle(\widehat{\omega}(x,t),\widehat{\omega}(y,t)) \to 0$ fast-enough, when $x \to y$, while the known best results are the following.

Theorem 3.1 (See [3,6]) *Let* $\Omega = \mathbb{R}^3, u_0 \in H^1_\sigma(\Omega)$ *and let* u *be a weak solution to (1.1) in* $[0,T]$. *In addition, suppose either that there exists* $\beta \in [1/2, 1]$ *and* $g \in L^a(0,T; L^b(\Omega))$, *with*

$$\frac{2}{a} + \frac{3}{b} = \beta - \frac{1}{2} \quad \textit{for} \quad a \in \left[\frac{4}{2\beta-1}, \infty\right],$$

such that

$$\sin \angle(\widehat{\omega}(x,t),\widehat{\omega}(y,t)) \leq g(t,x)|x-y|^\beta, \qquad \textit{a.e. } x, y \in \Omega, \ \textit{a.e. } t \in]0,T[, \tag{3.1}$$

or either that there exists $\beta \in]0, 1/2]$ *such that*

$$\sin \angle(\widehat{\omega}(x,t),\widehat{\omega}(y,t)) \leq c|x-y|^\beta, \qquad \textit{a.e. } x, y \in \Omega, \ \textit{a.e. } t \in]0,T[, \tag{3.2}$$

and that

$$\omega \in L^2(0,T; L^s(\Omega)), \qquad \textit{with } s = \frac{3}{\beta+1}.$$

Then, u *is strong, hence smooth, in* $[0,T]$.

Remark 3.1 It is clear that one can assume (3.1) or (3.2) to hold just where the vorticity at both points x and y is larger than some constant $K > 0$. In addition, the hypotheses may be relaxed by assuming that they are satisfied merely for $|x-y| < \delta$, for some constant $\delta > 0$. The same observations apply to all the results if this section. Moreover, local results for *suitable weak solutions* have been proved by Chae et al. [17]. Related conditions have been also explored by Grujić and Ruzmaikina [33,34] and by Zhou [61,62], while similar criteria concerning vorticity's direction has been also proved for the Euler equations, see Constantin et al. [22] and Deng et al. [24].

Proof (Sketch of the proof of Theorem 3.1) We give a sketch of the proof, in the case in which the vorticity is always non-zero so $\widehat{\omega}$ is well-defined at each point (the general case can be treated by a suitable splitting argument). We give the idea of the main steps in the most significant case: $\beta = 1/2$, $g \equiv 1$, and $s = 2$.

This result is mainly based on an a suitable integral representation formula derived from (2.3). In fact, the Biot-Savart law reads (for each fixed t)

$$u(x) = -\frac{1}{4\pi}\int\limits_{\mathbb{R}^3}\left(\nabla\frac{1}{|y|}\right)\times\omega(x+y)\,dy. \tag{3.3}$$

By differentiating (3.3) we obtain the following expression for the *strain matrix*

$$S[\omega](x) := \frac{1}{2}\left[\nabla u(x) + (\nabla u(x))^*\right] = \frac{3}{4\pi}\,P.V.\int\limits_{\mathbb{R}^3} M(\hat{y},\omega(x+y))\frac{dy}{|y|^3}.$$

where the matrix $M(\hat{y},\omega) := \frac{1}{2}\left[\hat{y}\otimes(\hat{y}\times\omega) + (\hat{y}\times\omega)\otimes\hat{y}\right]$ defines a proper singular operator, since its mean value on the unit sphere is zero, when ω is held fixed. With this formula we can give a representation for the *stretching rate*:

$$\alpha(x) := S(x)\widehat{\omega}(x)\cdot\widehat{\omega}(x) = \frac{3}{4\pi}\,P.V.\int\limits_{\mathbb{R}^3} D(\hat{y},\widehat{\omega}(x+y),\widehat{\omega}(x)|\omega(x+y)|\frac{dy}{|y|^3},$$

where $D(v_1, v_2, v_3) := (v_1\cdot v_3)\,\mathrm{Determinant}(v_1, v_2, v_3)$, see [21].

Taking the curl of (1.1), multiplying by ω and integrating by parts one obtains

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|^2_{L^2} + \|\nabla\omega(t)\|^2_{L^2} = \int\limits_{\mathbb{R}^3} S(x,t)\,\omega(x,t)\cdot\omega(x,t)\,dx. \tag{3.4}$$

Without extra-assumptions the right-hand side is estimated by $\frac{1}{2}\|\nabla\omega(t)\|^2_{L^2} + c\|\omega(t)\|^6_{L^2}$, which is not suitable to obtain estimates on large time-intervals. On the other hand, by using the representation formulæ and from (3.1) with $\beta = 1/2$ one gets (for each t)

$$\begin{aligned} S(x)\omega(x)\cdot\omega(x) &= \frac{3}{4\pi}|\omega(x)|^2\,P.V.\int\limits_{\mathbb{R}^3} D(\hat{y},\widehat{\omega}(x+y),\widehat{\omega}(x))|\omega(x+y)|\,\frac{dy}{|y|^3},\\ &\le C|\omega(x)|^2\int\limits_{\mathbb{R}^3}|\omega(x+y)|\,\frac{dy}{|y|^{5/2}}. \end{aligned}$$

Next, the Hardy–Littlewood–Sobolev inequality implies that

$$\left\| \int_{\mathbb{R}^3} |\omega(x+y)| \frac{dy}{|y|^{5/2}} \right\|_{L^3} \leq c\, \|\omega\|_{L^2},$$

Finally, by using Hölder's inequality, the interpolation $L^3 = [L^2, L^6]_{1/2}$, and the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ one readily gets

$$\left| \int_{\mathbb{R}^3} S(x)\, \omega(x) \cdot \omega(x)\, dx \right| \leq \frac{1}{2} \|\nabla\omega\|^2_{L^2} + C\|\omega\|^4_{L^2}. \tag{3.5}$$

Hence, one obtains the differential inequality

$$\frac{d}{dt}\|\omega(t)\|^2_{L^2} + \|\nabla\omega(t)\|^2_{L^2} \leq 2C\|\omega(t)\|^4_{L^2}, \tag{3.6}$$

which (by standard arguments) implies a uniform bound for $\|\omega(t)\|_{L^2}$ in $[0, T]$. □

Recently the same result has been extended to the boundary value problem in the half-space case $\mathbb{R}^3_+$ with the Navier's boundary conditions (1.2), see Beirão da Veiga [4]. In this specific case the Navier's boundary conditions imply that one can still explicitly reconstruct u from ω by solving (for each fixed t) the system

$$\begin{cases} -\Delta u = \operatorname{curl} \omega & \text{in } \mathbb{R}^3_+, \\ \frac{\partial u^1}{\partial x_3} = \frac{\partial u^2}{\partial x_3} = u^3 = 0 & \text{on } \{x_3 = 0\}. \end{cases} \tag{3.7}$$

They are three *independent* Poisson problems for the various components of u, which can be solved *separately* by using the Green function for the Poisson problem with Dirichlet and Neumann boundary conditions in the half-space.

Remark 3.2 The same observation, with other related systems, has been also recently used to show existence of very-weak solutions for the stationary Navier–Stokes equations with Navier's boundary conditions in Ref. [10].

Theorem 3.2 (See [4]) *Let $\Omega = \mathbb{R}^3_+$ and let u be a weak solution to* (1.1) and (1.2) *in* $[0, T]$. *Suppose that $u_0 \in H^1_\sigma(\Omega)$ such that $u^3 \in H^1_0(\Omega)$. Let the other hypotheses of Theorem 3.1 be satisfied. Then, u is strong, hence smooth in* $[0, T]$.

The proof follows the same approach of that in the full-space, even if there are several technical points to be improved. In particular the explicit solution of system (3.7) allows to prove the same inequality as in (3.5) for the vortex stretching term. Moreover, one of the main observations is that the vorticity equation can be used when considering the problem with Navier's boundary conditions. In fact, in this case (a flat boundary), conditions (1.2) imply that

$$\omega^1 = \omega^2 = \frac{\partial\, \omega^3}{\partial\, x_3} = 0 \qquad \text{on } \{x_3 = 0\}.$$

This allows to test the vorticity equation with the vorticity itself and to derive the Gauss-Green identity

$$-\int_{\mathbb{R}^3_+} \Delta\omega\,\omega\,dx = \int_{\mathbb{R}^3_+} |\nabla\omega|^2\,dx.$$

Remark 3.3 Interesting enough, in the case of Dirichlet boundary conditions, the lack of the latter identity concerning the linear term (and not the nonlinear term) is source of problems. Partial integration of $-\Delta\,\omega\cdot\,\omega$ gives rise to the non-vanishing boundary integral

$$\iint_{\{x_3=0\}} \frac{\partial\omega}{\partial x_3}\cdot\omega\,dx_1dx_2,$$

and a suitable extra-assumption seems necessary. See Beirão da Veiga [4] for further details. Recall also that -generally speaking- the vorticity equation cannot be used in energy estimates for the Dirichlet problem, since one does not have correct behavior of the boundary values of ω. See also the results of Rautmann [48]. On the contrary, the vortex-stretching term can be treated also with the boundary condition $u = 0$, more or less in the same way as in the full or in the half-space.

Next, the same results have been proved in a smooth and bounded domain with the boundary conditions (1.2) and the tools used in the half-space need substantial technical improvements.

Theorem 3.3 (See [7]) *Let $\Omega \subset \mathbb{R}^3$ be an open and bounded set with a boundary $\partial\Omega$, of class $C^{3,\alpha}$, for some $\alpha > 0$. Suppose that $u_0 \in H^1(\Omega)$, $\nabla\cdot u_0 = 0$, and u is a weak solution to* (1.1) and (1.2) *in* $[0, T]$. *Let the other hypotheses of* Theorem 3.1 *be satisfied. Then, u is strong, hence smooth, in* $[0, T]$.

In particular, adaption of the previous results to this case uses the fact that

$$u(x) = \int_\Omega G(x, y)\,\omega(y)\,dy,$$

for a single Green matrix. In order get explicit and precise expressions, similar to (3.3), one has to localize the problem, a not trivial and quite technical matter, cf. [7], Sec. 3. To do this we used in a substantial way the results of Solonnikov [53,54], where the author constructs global Green's matrices for a large class of boundary value problems and systems of partial differential equations, by means of localization and Green matrices for the half-space.

In addition, also the integrations by parts require some care and under the stress-free boundary conditions one gets

$$
\begin{aligned}
&-\int\limits_{\Omega} \Delta\omega \cdot \omega \, dx \\
&\quad = \int\limits_{\Omega} |\nabla\omega|^2 \, dx - \int\limits_{\partial\Omega} (\varepsilon_{1jk}\,\varepsilon_{1\beta\gamma} + \varepsilon_{2jk}\,\varepsilon_{2\beta\gamma} + \varepsilon_{3jk}\,\varepsilon_{3\beta\gamma})\,\omega_j\,\omega_\beta\,\partial_k n_\gamma \, dS,
\end{aligned}
$$

where ε_{ijk} is the totally anti-symmetric Ricci tensor and summation over repeated indices is assumed. Consequently the surface integrals represents a "*low order term*" and it can be controlled by standard trace theorems.

From the point of view of applications the behavior of the vorticity is of crucial importance also in the numerical simulation of turbulent flows. In fact, large variations of the vorticity's direction may be used in order to detect regions of intense vortex activity. This has been also used to propose a *selective Smagorinsky model* in order to design advanced Large Eddy Simulation methods for turbulent flows, see Cottet et al. [23] and Berselli et al. [15]. Consequently it is of interest to understand if regularity holds when the vorticity's direction is not smooth (say Hölder continuous as in the previous theorems), but it satisfies just a "*small cone property*".

Theorem 3.4 (See [9]) *Let* $\Omega = \mathbb{R}^3$, $u_0 \in H^1_\sigma(\Omega)$, *and let* u *be a weak solution to* (1.1) *in* $[0, T]$. *There exists a constant* $\mathcal{C}_1 = \mathcal{C}_1(\|u_0\|_{H^1}) > 0$ *such that if*

$$
\sin \angle(\widehat{\omega}(x,t), \widehat{\omega}(y,t)) \leq \mathcal{C}_1, \qquad a.e\ x,\ y \in \mathbb{R}^3 \quad a.e.\ t \in]0, T[, \tag{3.8}
$$

then u *is strong, hence smooth, in* $[0, T]$ *(An explicit expression of* $\mathcal{C}_1$ *in terms of the data of the problem can be given).*

This result is proved more or less by the same approach of Theorem 3.1. If one assumes that the condition (3.8) is satisfied just for $|x - y| < \delta$, then one can estimate the value of $\mathcal{C}_1$ in terms also of δ. Consequently large variations of the vorticity's direction are allowed if one compares points far away each other on some vortex line, by making several (small) intermediate steps.

Next, motivated also by some physical guessing described in [9] results concerning the direction of curl ω have been also proved. Studying this quantity amounts of taking into consideration the rolling of lines tangent to the vorticity's direction. In fact, this quantity measures the amount of rotation and plays a fundamental role in the analysis of several physical phenomena. We recall some examples: formation of fractal structures in turbulent flows, the catalytic surface oxidation, the Belousov–Zhabotinsky chemical reaction, aggregating colonies of slime mold, and recent applications also to super-fluids, see Eltsov et al. [25]. Fiber rotation (with strong analogies with 3D Euler equations) of the heart tissue are also suspected to be one of the main causes in cardiac arrhythmia and fibrillation, see Fenton and Karma [29].

Theorem 3.5 (See [9]) *Let* $\Omega = \mathbb{R}^3$, $u_0 \in H^1_\sigma(\Omega) \cap W^{2,6/5}(\Omega)$, *and let* u *be a weak solution to* (1.1) *in* $[0, T]$. *There exists a constant* $\mathcal{D}_1 = \mathcal{D}_1(\|u_0\|_{H^1}) > 0$ *such that if*

$$
\sin \angle(\widehat{curl\ \omega}(x,t), \widehat{curl\ \omega}(y,t)) \leq \mathcal{D}_1, \qquad a.e\ x,\ y \in \mathbb{R}^3 \quad a.e.\ t \in]0, T[,
$$

then u is strong, hence smooth, in $[0, T]$ *(An explicit expression of* $\mathcal{D}_1$ *in terms of the data can be given).*

Also this result is based on the usual energy estimates and on some integral transformations classical in potential theory. The main idea is to resort to the rotational formulation of the Navier–Stokes equation

$$u_t + \omega \times u - \Delta u + \nabla \left(p + \frac{1}{2}|u|^2 \right) = 0,$$

to use the representation $u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\operatorname{curl}\omega(x+y)}{|y|}\,dy$ and to use as test function $-\Delta u = \operatorname{curl}\omega$. Next, standard manipulation are used to prove the result.

3.1 On helicity and regularity

To end our overview we show a result concerning the mutual interaction of (direction of) velocity and vorticity. Many results (most of them are recalled in the previous section, see also the review in Constantin [20] and Chae [16]) have been devoted to understand if the scenario with alignment of vorticity's direction can bound the growth of vorticity and consequently destroy a possible mechanism of creation of singularities generating by vortex-tube collision.

A different path, not explored before, is that in the 2D case there is another specific feature not linked with vorticity at neighboring points, but on velocity–vorticity interaction. For plane flows velocity and vorticity are perpendicular and hence the *helicity* $H(t) := \int_\Omega u(x,t)\cdot\omega(x,t)\,dx$ is identically zero. Our main intent is to understand if a "2D-type condition" $u \perp \omega$, may imply global-in-time regularity for the solutions of the three-dimensional problem. To support this claim, observe also that in the axial symmetric case without swirl (*i.e.,* $u^\theta = 0$) the velocity field is $u = (u^r, 0, u^z)$, where θ denotes the azimuthal angle. In this case, there are several results concerning the well-posedness for large times, dating back to Ladyženskaya [40] and Ukhovskii and Iudovich [58]. Well-known calculations show that (similarly to $\mathbb{R}^2$) the vorticity vector has only the azimuthal component different from zero

$$\omega = (0, \omega^\theta, 0) = (0, u^r_z - u^z_r, 0)$$

For related results see also the paper by Mahalov et al. [42].

We also recall that interesting geometrical properties connecting helicity and knot's theory have been proved by Moffatt [44], while recent results on the helicity are also those by Foias et al. [30].

Theorem 3.6 (See [11]) *Let us assume that* $\Omega = \mathbb{T}$ *(with periodic boundary conditions) and let u be a weak solution to* (1.1) *in* $[0, T]$, *with* $u_0 \in H^4_\sigma(\mathbb{T})$. *If there exists* $C > 0$ *such that*

$$|u(x,t)\cdot\omega(y,t)| \le C\,|x-y|\,|u(x,t)|\,|\omega(y,t)|, \qquad a.e.\ x, y \in \Omega,\ a.e.\ t \in]0, T[, \tag{3.9}$$

then u is strong, hence regular, in $[0, T]$.

The proof of this theorem is not based on singular integrals, potential theory, or representation formulæ, but is based on a much simpler approximation by finite differences. We approximate ∇u in the "vortex stretching term" from the right-hand-side of the vorticity equation by means of finite differences. In this way we obtain (in coordinates)

$$\frac{\partial \omega^i(x,t)}{\partial t} + u^j(x,t)\frac{\partial \omega^i(x,t)}{\partial x_j} - \frac{\partial^2 \omega^i(x,t)}{\partial x_l^2}$$
$$= \omega^j(x,t)\frac{u^i(x+K\,e_j,t) - u^i(x,t)}{K} + \mathrm{R},$$

where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$ and R is a quantity expressed by means of the Taylor's series expansion with Lagrange's remainder. Next, one uses the usual energy-type estimates obtained by testing with ω: by using local existence of smooth solutions one can estimate the term concerning the remainder and by using hypothesis (3.9) one can estimate the first term on the right-hand side as follows

$$\left| \int_{\mathbb{T}} \omega_j(x,t)\frac{u_i(x+K\,e_j,t) - u_i(x,t)}{K}\omega_i(x,t)\,dx \right|$$
$$\leq C\int_{\mathbb{T}} |\omega(x,t)|\,|u_j(x+K\,e_i,t)|\,|\omega(x,t)|.$$

Next, usual Sobolev inequalities can be used to derive the same inequality as in (3.6).

To conclude we observe that the same approach (but with more technicalities) can be used to study the problem in a bounded and smooth domain with the Navier's boundary conditions, see [12].

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