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Gevrey regularity for Navier–Stokes equations under Lions boundary conditions

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The Navier–Stokes system is considered in a compact Riemannian manifold. Gevrey class regularity is proven under Lions boundary conditions: in 2D for the Rectangle, Cylinder, and Hemisphere, and in 3D for the Rectangle. The cases of the 2D Sphere and 2D and 3D Torus are also revisited.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ be a connected bounded domain located locally on one side of its smooth boundary $\Gamma = \partial\Omega$. The Navier–Stokes system, in $(0, T) \times \Omega$, reads

$$\partial_t u + \langle u \cdot \nabla\rangle u - \nu\Delta u + \nabla p + h = 0, \quad \operatorname{div} u = 0, \quad \mathcal{G}u|_\Gamma = 0, \quad u(0, x) = u_0(x) \qquad (1)$$

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where as usual $u = (u_1, \ldots, u_d)$ and p, defined for $(t, x_1, \ldots, x_d) \in I \times \Omega$, are respectively the unknown velocity field and pressure of the fluid, $\nu > 0$ is the viscosity, the operators ∇ and Δ are respectively the well known gradient and Laplacian in the space variables $(x_1, \ldots, x_d)$, $\langle u \cdot \nabla \rangle v$ stands for $(u \cdot \nabla v_1, \ldots, u \cdot \nabla v_d)$, $\operatorname{div} u := \sum_{i=1}^{d} \partial_{x_i} u_i$ and h is a fixed function. Further, $\mathcal{G}$ is an appropriate linear operator imposing the boundary conditions.

In the case Ω is a compact Riemannian manifold, either with or without boundary, the Navier–Stokes equation reads

$$\partial_t u + \nabla_u^1 u + \nu \Delta_\Omega u + \nabla_\Omega p + h = 0, \quad \operatorname{div} u = 0, \quad \mathcal{G}u|_\Gamma = 0, \quad u(0, x) = u_0(x). \tag{2}$$

That is we just replace the Laplace operator by the Laplace–de Rham operator, the gradient operator by the Riemannian gradient operator, and the nonlinear term by the Levy-Civita connection. Recall that a flat (Euclidean) domain $\Omega \subset \mathbb{R}^d$ can be seen a Riemannian manifold and we have $-\Delta = \Delta_\Omega$, $\nabla = \nabla_\Omega$ and $\langle u \cdot \nabla \rangle v = \nabla_u^1 v$ (see, e.g., [41, Chapter 5]). That is, (2) reads (1) in the Euclidean case. We should say that some authors consider the Navier–Stokes equation on a Riemannian manifold with a slightly different Laplacian operator and sometimes with on more term involving the (Ricci) curvature of the Riemannian manifold. In that case, we also recover (1) in the Euclidean case because the curvature vanishes. Writing the Navier–Stokes as (2), we are following [7,14,20,21,40,41]; for other writings we refer to [10,37].

Often system (2) can be rewritten as an evolutionary system

$$\dot{u} + B(u, u) + Au + h = 0, \quad u(0, x) = u_0(x); \tag{3}$$

where formally $B(u, v) := \Pi \nabla_u^1 v$ and $Au = \nu \Pi \Delta_\Omega u$, and Π is a projection onto a suitable subspace H of divergence free vector fields (cf. [16, Chapter II, Section 3], [39, Section 4], [41, Section 5.5]). Usually $\Pi \nabla = 0$, and we suppose that $h = \Pi h$ (otherwise we have just to take Πh in (3) instead).

The aim of this work is to give some sufficient conditions to guarantee that the solution of system (2) lives in a Gevrey regularity space.

For the case of periodic boundary conditions, that is, for the case $\Omega = \mathbb{T}^d$, the Gevrey regularity has been proven in the pioneering work [17] for the Gevrey class $\mathrm{D}(A^{\frac{1}{2}} \mathrm{e}^{\varphi(t) A^{\frac{1}{2}}})$, provided $u_0 \in \mathrm{D}(A^{\frac{1}{2}})$. Here $\varphi(t) = \min\{\sigma, t\}$, with $\sigma > 0$ fixed depending on the external forcing h. These results have been extended to other Gevrey classes in [35], namely $\mathrm{D}(A^s \mathrm{e}^{\varphi(t) A^{\frac{1}{2}}})$, provided $u_0 \in \mathrm{D}(A^s)$, with $s > \frac{d}{4}$. The first observation is that there is a gap, for the value of s, for $d = 3$. This gap is filled in [5, Theorem 4.1]. Though we consider here bounded domains, we would like to mention the work [6] where unbounded domains $\Omega = \mathbb{R}^l \times \mathbb{T}^m$, $(l, m) \in \mathbb{N}^2 \setminus \{(0, 0)\}$ are considered. Further we refer to [28], where the Gevrey regularity is used to study the level sets of the (scalar) vorticity, in the case $\Omega = \mathbb{T}^2$.

The Gevrey-class regularity is also connected to the analyticity of the solutions, on this connection we refer to [17,18,31,36] for the cases $\Omega \in \{\mathbb{R}^d, \mathbb{T}^d\}$. See also [4] in the setting of Besov spaces and the works [29,30] for (generalized) Euler equations.

For semilinear parabolic equations, with a general nonlinear term, we refer to [15]. We would like to recall the connection between the Gevrey regularity and the reliability of the numerical Galerkin method mentioned in [15, Section 3]. See also [25] and references therein.

From our procedure, for $\Omega = \mathbb{T}^3$, we can recover the Gevrey regularity in $\mathrm{D}(A^s \mathrm{e}^{\varphi(t)A^{\frac{1}{2}}})$, provided $u_0 \in \mathrm{D}(A^s)$, with $s > \frac{1}{2}$. Further for $\Omega = \mathbb{T}^2$, it will follow that the Gevrey regularity holds in $\mathrm{D}(A^s \mathrm{e}^{\varphi(t)A^{\frac{1}{2}}})$, provided $u_0 \in \mathrm{D}(A^s)$, with $s > 0$.

In the case of the Navier–Stokes in the 2D Sphere $\mathbb{S}^2$, from our conditions, we can recover the results obtained in [7], that is to say that the Gevrey regularity holds in $\mathrm{D}(A^s \mathrm{e}^{\varphi(t)A^{\frac{1}{2}}})$, provided $u_0 \in \mathrm{D}(A^s)$, with $s > \frac{1}{2}$.

In the above mentioned cases for Navier–Stokes equations, the bounded manifolds $\mathbb{T}^d$ and $\mathbb{S}^2$ are boundaryless, which means that essentially we have no boundary conditions. Here we consider the case of manifolds with boundary and 4 new results are obtained under Lions boundary conditions, namely, in 2D case for the Rectangle $(0, a) \times (0, b)$, the Cylinder $(0, a) \times b\mathbb{S}^1$, and the Hemisphere $\mathbb{S}^2_+$, and in 3D case for the Rectangle $(0, a) \times (0, b) \times (0, c)$.

For Euler equations, the case of nonempty boundaries is considered in [29,30], where the boundary is Gevrey regular.

By Lions boundary conditions, in two dimensions, we mean the vanishing both of the normal component and of the vorticity of the vector field u at the boundary,

$$u \cdot \mathbf{n} = 0 \quad \text{and} \quad \nabla^{\perp} \cdot u = 0 \quad \text{on} \quad \Gamma,$$

the reason of the terminology (also adopted in [27,39]) is the work done in [33, Section 6.9]. However the terminology is not followed by all authors, for example, in [23, Section 3] they are just called "stress-free boundary conditions". Notice that Lions boundary conditions can be seen as a particular case of (generalized) Navier boundary conditions (cf. [27, Section 1 and Corollary 4.2], cf. [41, system (4.1)–(4.2) and Remark 4.4.1]).

In three dimensions, by Lions boundary conditions we mean the vanishing, at the boundary, of the normal component of the vector field u, and of the tangent component of the vorticity $\operatorname{curl} u$, see [48, Equation (1.4)],

$$u \cdot \mathbf{n} = 0 \quad \text{and} \quad \operatorname{curl} u - ((\operatorname{curl} u) \cdot \mathbf{n})\mathbf{n} = 0 \quad \text{on} \quad \Gamma.$$

The study of Lions and Navier boundary conditions have been addressed by many authors in the last years, either because in some situations they may be more realistic than no-slip boundary conditions or because they are more appropriate in finding a solution for the Euler system as a limit of solutions for the Navier–Stokes system as ν goes to zero (cf. [46,48], [27, Section 8]), or even the possibility to recover the solution under no-slip boundary conditions as a limit of solutions under Navier boundary conditions (cf. [24]), and conversely (cf. [27, Section 9]). We refer also to [2,8,13,19] and references therein.

In both cases of the 2D Rectangle or 2D Cylinder, we obtain that the Gevrey regularity holds in $\mathrm{D}(A^s \mathrm{e}^{\varphi(t)A^{\frac{1}{2}}})$, provided $u_0 \in \mathrm{D}(A^s)$, with $s > 0$. In the cases of the 2D Hemisphere and 3D Rectangle we obtain the analogous result with $s > \frac{1}{2}$ and $s > \frac{1}{4}$, respectively.

The rest of the paper is organized as follows. In Section 2, we give the necessary conditions (as assumptions) for the existence of solutions living in a Gevrey class regularity space. In Section 3, the Gevrey class regularity is proven under the conditions on the sequence of nonrepeated eigenvalues of the Stokes operator. In Section 4, we give the corresponding conditions on the sequence of repeated eigenvalues. In Section 5 we give an extra condition on the bounding of the L^∞-norm by the L^2-norm in eigenprojections which will allow us to derive the results for smaller s than the one given by the general conditions in Sections 3 and 4. In Section 6, we revisit the cases where Ω is the Torus $\mathbb{T}^d$ and the Sphere $\mathbb{S}^2$ and give the 4 new examples mentioned above under Lions boundary conditions.

Note. We will not address here the connection between Gevrey-class regularity and analyticity of the solutions mentioned above, for the cases of boundaryless manifolds $\Omega \in \{\mathbb{R}^d, \mathbb{T}^d\}$ for the Navier–Stokes equations, and also for Euler equations, for the case of Gevrey regular boundaries. Another connection mentioned above that we will not address here, which is important for applications (numerical simulations), is the connection with the reliability of the Galerkin numerical method. Whether these connections can also be made for Navier–Stokes equations under Lions boundary conditions is however an interesting question, which will be (hopefully) answered/addressed elsewhere.

Notation. We write $\mathbb{R}$ and $\mathbb{N}$ for the sets of real numbers and nonnegative integers, respectively, and we define $\mathbb{R}_0 := (0, +\infty)$, and $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$.

Given a Banach space X and an open connected subset $O \subset \mathbb{R}^n$, located on one side of its boundary, let us denote by $L^p(O, X)$, with either $p \in [1, +\infty)$ or $p = \infty$, the Bochner space of measurable functions $f : O \to X$, and such that $|f|_X^p$ is integrable over O, for $p \in [1, +\infty)$, and such that $\operatorname{ess\,sup}_{x \in O} |f(x)|_X < +\infty$, for $p = \infty$. In the case $X = \mathbb{R}$ we recover the usual Lebesgue spaces. In the case $n \geq 2$ we assume that O has a Lipschitz boundary. By $W^{s,p}(O, \mathbb{R})$, for $s \in \mathbb{R}$, denote the usual Sobolev space of order s, see for example [11, Chapter 4]. In the case $p = 2$, as usual, we denote $H^s(O, \mathbb{R}) := W^{s,2}(O, \mathbb{R})$. Recall that $H^0(O, \mathbb{R}) = L^2(O, \mathbb{R})$. For each $s > 0$, we recall also that $H^{-s}(O, \mathbb{R})$ stands for the dual space of $H_0^s(O, \mathbb{R}) =$ closure of $\{f \in C^\infty(O, \mathbb{R}) \mid \operatorname{supp} f \subset O\}$ in $H^s(O, \mathbb{R})$. Notice that $H^{-s}(O, \mathbb{R})$ is a space of distributions.

For a normed space X, we denote by $|\cdot|_X$ the corresponding norm; in the particular case $X = \mathbb{R}$ we denote $|\cdot| := |\cdot|_{\mathbb{R}}$. By X' we denote the dual of X, and by $\langle\cdot,\cdot\rangle_{X',X}$

the duality between X' and X. The dual space is endowed with the usual dual norm: $|f|_{X'} := \sup\{\langle f, x\rangle_{X',X} \mid x \in X \text{ and } |x|_X = 1\}$. In the case that X is a Hilbert space we denote the inner product by $(\cdot,\cdot)_X$.

Given a Riemannian manifold $\Omega = (\Omega, g)$ with Riemannian metric tensor g, we denote by $T\Omega$ the tangent bundle of Ω and by $\mathrm{d}\Omega$ the volume element of Ω. We denote by $H^s(\Omega, \mathbb{R})$ and $H^s(\Omega, T\Omega)$ respectively the Sobolev spaces of functions and vector fields defined in Ω. Recall that if $\Omega = O \subset \mathbb{R}^n$, then $H^s(O, TO) = H^s(O, \mathbb{R}^n) \sim (H^s(O, \mathbb{R}))^n$.

C, C_i, $i = 1, 2, \ldots$, stand for unessential positive constants.

2. Preliminaries

2.1. The evolutionary Navier–Stokes system

Given a d-dimensional compact Riemannian manifold $\Omega = (\Omega, g)$, $d \in \{2, 3\}$, we (suppose we can) write the Navier–Stokes system as an evolutionary system in a suitable closed subspace $H \subseteq \{u \in L^2(\Omega, T\Omega) \mid \operatorname{div} u = 0\}$ of divergence free vector fields

$$\dot{u} + \nu A u + B(u) + h = 0, \quad u(0) = u_0, \tag{4}$$

where $A := \Pi\Delta_\Omega$ is the Stokes operator and $B(u) := B(u, u)$ with $B(u, v) := \Pi\nabla^1_u v$ as a bilinear operator.

Here Π stands for the orthogonal projection in $L^2(\Omega, T\Omega)$ onto H, Δ_Ω stands for the Laplace–de Rham operator, and finally $(u, v) \mapsto \nabla^1_u v$ stands for the Levi-Civita connection (cf. [26, Chapter 3, Section 3.3]).

Recall that, for a domain $\Omega \in \mathbb{R}^d$, we can identify $T\Omega$ with $\mathbb{R}^d$, $\Delta_\Omega = -\Delta$ coincides with the usual Laplacian up to the minus sign, and $\nabla^1_u v = \langle u \cdot \nabla\rangle v$ (see [41, Chapter 5, Sections 5.1 and 5.2], [20, Section 1]).

We consider H, endowed with the norm inherited from $L^2(\Omega, T\Omega)$, as a pivot space, that is, $H = H'$. Let $V \subseteq H$ be another Hilbert space, such that A maps V onto V'. The domain of A, in H, is denoted $\mathrm{D}(A) := \{u \in H \mid Au \in H\}$.

The spaces H, V, and $\mathrm{D}(A)$ will depend on the boundary conditions the fluid will be subjected to. We assume that the inclusion $V \subseteq H$ is dense, continuous, and compact. In this case, $A^{-1} \in \mathcal{L}(H)$ is compact and the eigenvalues of A, repeated accordingly with their multiplicity, form an increasing sequence $(\underline{\lambda}_k)_{k\in\mathbb{N}_0}$,

$$0 < \underline{\lambda}_1 \le \underline{\lambda}_2 \le \underline{\lambda}_3 \le \underline{\lambda}_4 \le \ldots,$$

with $\underline{\lambda}_k$ going to $+\infty$ with k.

Consider also the strictly increasing subsequence $(\lambda_k)_k \in \mathbb{N}_0$ of the distinct (i.e. nonrepeated) eigenvalues

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots;$$

and denote by P_k the orthogonal projection in H onto the eigenspace $P_k H = \{z \in H \mid Az = \lambda_k z\}$, associated with the eigenvalue λ_k,

$$P_k\colon H \to P_k H, \quad v \mapsto P_k v; \tag{5}$$

with $v = P_k v + w$ and $(w,\, z)_H = 0$ for all $z \in P_k H$. Thus, the multiplicity of λ_k coincides with the dimension of the linear space $P_k H$ (which is finite-dimensional, cf. [9, Proposition 4.13]).

We define also the trilinear form

$$b(u,\, v,\, w) := \int\limits_{\Omega} g(\nabla^1_u v,\, w)\, \mathrm{d}\Omega,$$

provided the integral is finite, where $g(\cdot,\, \cdot)$ stands for the scalar product in $T\Omega$ induced by the metric tensor g.

Throughout the paper, we consider the following assumptions:

Assumption 2.1. The spaces V and $\mathrm{D}(A)$ satisfy:

- $V \subset H^1(\Omega,\, T\Omega)$, and $|u|_V := (\langle Au,\, u\rangle_{V',\, V})^{\frac{1}{2}}$ defines a norm equivalent to the one inherited from $H^1(\Omega,\, T\Omega)$;
- $\mathrm{D}(A) \subset H^2(\Omega,\, T\Omega)$, and $|u|_{\mathrm{D}(A)} := |Au|_H$ defines a norm equivalent to the one inherited from $H^2(\Omega,\, T\Omega)$.

Assumption 2.2. The following properties hold for the trilinear form.

- $b(u,\, v,\, w) = -b(u,\, w,\, v)$;
- $|b(u,\, v,\, w)| \le C|u|_{L^\infty(\Omega,\, T\Omega)}|v|_{H^1(\Omega,\, T\Omega)}|w|_{L^2(\Omega,\, T\Omega)}$;
- $|b(u,\, v,\, w)| \le C|u|_{L^2(\Omega,\, T\Omega)}|v|_{H^1(\Omega,\, T\Omega)}|w|_{L^\infty(\Omega,\, T\Omega)}$;
- $|b(u,\, v,\, w)| \le C|u|_{L^4(\Omega,\, T\Omega)}|v|_{H^1(\Omega,\, T\Omega)}|w|_{L^4(\Omega,\, T\Omega)}$.

Hereafter, for simplicity, we will denote

$$\mathcal{B}(u, v) := B(u, v) + B(v, u).$$

Assumption 2.3. There are real numbers $\beta \ge 0$ and $\alpha \in (0, 1)$ such that, for all triples $(n,\, m,\, l) \in \mathbb{N}_0^3$,

$$\begin{cases} (u,\, v,\, w) \in P_n H \times P_m H \times P_l H, \\ (\mathcal{B}(u, v),\, w)_H \ne 0, \end{cases} \quad \text{implies} \quad \lambda_l^\alpha \le \lambda_n^\alpha + \lambda_m^\alpha + \beta.$$

Next, for given $(n,\, m,\, l) \in \mathbb{N}_0^3$, we define the sets

$$\mathcal{F}^{\bullet}_{n,\, m} := \left\{ k \in \mathbb{N}_0 \;\middle|\; \begin{array}{l} (\mathcal{B}(u, v),\, w)_H \ne 0, \\ \text{for some } (u,\, v,\, w) \in P_n H \times P_m H \times P_k H \text{ with } n \le m \end{array} \right\};$$

$$\mathcal{F}^{l}_{n,\bullet} := \left\{ k \in \mathbb{N}_0 \;\middle|\; \begin{array}{l} (\mathcal{B}(u,v),\, w)_H \neq 0, \\ \text{for some } (u,\, v,\, w) \in P_n H \times P_k H \times P_l H \text{ with } n \leq k \end{array} \right\}.$$

Assumption 2.4. There are $C_{\mathcal{F}} \in \mathbb{N}_0$ and $\zeta \in [0, +\infty)$ such that, for all $n \in \mathbb{N}_0$

$$\sup_{(m,\,l)\in\mathbb{N}_0^2} \left\{\mathrm{card}(\mathcal{F}^{\bullet}_{n,\,m}),\, \mathrm{card}(\mathcal{F}^{l}_{n,\,\bullet})\right\} \leq C_{\mathcal{F}} \lambda_n^{\zeta},$$

where $\mathrm{card}(S)$ stands for the cardinality (i.e., the number of elements) of the set S.

Remark 2.5. Assumptions 2.1 and 2.2 are satisfied in well known settings. In contrast, Assumptions 2.3 and 2.4 will be satisfied more seldom and play a key role here to derive the Gevrey class regularity for the solutions of the Navier–Stokes system (4).

2.2. Some auxiliary results

We present now some results that will be useful hereafter.

Proposition 2.6. *For given nonnegative real numbers a, b, and s, with $a + b > 0$ and $s > 0$, it holds*

$$\begin{aligned} &2^{s-1}(a^s + b^s) \leq (a+b)^s \leq a^s + b^s, && \textit{for } 0 < s \leq 1; \\ &a^s + b^s \leq (a+b)^s \leq 2^{s-1}(a^s + b^s), && \textit{for } s \geq 1. \end{aligned}$$

The proof is omitted. It can be done by straightforward computations, for example, by studying the extrema of the functions $(a,b) \mapsto \frac{(a+b)^s}{(a^s+b^s)}$, $s \geq 0$.

Lemma 2.7. *Assumption 2.3 holds only if for all $s > 0$ there exists a nonnegative real number $C_{(s,\,\alpha,\,\beta)} > 0$ depending only on $(s,\, \alpha,\, \beta,\, \lambda_1)$ such that*

$$\begin{cases} (u,\, v,\, w) \in P_n H \times P_m H \times P_l H, \\ (\mathcal{B}(u,v),\, w)_H \neq 0, \end{cases} \quad \textit{implies} \quad \lambda_l^s \leq C_{(s,\,\alpha,\,\beta,\,\lambda_1)}(\lambda_n^s + \lambda_m^s).$$

Proof. From Assumption 2.3, since $(\lambda_k)_{k\in\mathbb{N}_0}$ is an increasing sequence, we have that

$$\lambda_l^\alpha \leq \lambda_n^\alpha + \lambda_m^\alpha + \beta \frac{\lambda_n^\alpha + \lambda_m^\alpha}{2\lambda_1^\alpha} = \left(1 + \frac{\beta}{2\lambda_1^\alpha}\right)(\lambda_n^\alpha + \lambda_m^\alpha).$$

Now for any $s > 0$, it follows that

$$\lambda_l^s \leq \left(1 + \frac{\beta}{2\lambda_1^\alpha}\right)^{\frac{s}{\alpha}} D_{\frac{s}{\alpha}}\, (\lambda_n^s + \lambda_m^s),$$

where the constant $D_{\frac{s}{\alpha}}$ depending only on $\frac{s}{\alpha}$ is given by Proposition 2.6. □

3. Gevrey class regularity

Here we show that, under Assumptions 2.1, 2.2, 2.3, and 2.4, and for suitable data (u_0, h), the solution u of system (4) takes its values $u(t)$ in a Gevrey class regularity space. We follow the arguments in [7,17,35].

3.1. Gevrey spaces and main theorem

Let us set a complete orthonormal system $\{W_k \mid k \in \mathbb{N}_0\}$ of eigenfunctions of the Stokes operator A. That is,

$$AW_k = \underline{\lambda}_k W_k, \text{ for all } k \in \mathbb{N}_0.$$

We recall that any given $u \in H$ can be written in a unique way as $u = \sum_{k\in\mathbb{N}_0} u_k W_k$, with $u_k = (u, W_k)_H \in \mathbb{R}$. Now, given $s \geq 0$ we may define the power A^s of the Stokes operator as

$$A^s u := \sum_{k\in\mathbb{N}_0} \underline{\lambda}_k^s u_k W_k,$$

and we denote its domain by $\mathrm{D}\left(A^s\right) := \{u \in H \mid A^s u \in H\}$.

Analogously we may define the negative powers A^{-s} as

$$A^{-s} u := \sum_{n\in\mathbb{N}_0} \underline{\lambda}_k^{-s} u_k W_k,$$

and $\mathrm{D}\left(A^{-s}\right) := \{u \mid A^{-s}u \in H\}$, more precisely $\mathrm{D}\left(A^{-s}\right)$ is the closure of H in the norm $|u|_{\mathrm{D}(A^{-s})} := \left(\sum_{k\in\mathbb{N}_0} \underline{\lambda}_k^{-2s} u_k^2\right)^{\frac{1}{2}}$.

We recall that for $s = \frac{1}{2}$ we have $\mathrm{D}(A^{\frac{1}{2}}) = V$. For a more complete discussion on the fractional powers of a compact operator we refer to [44, Chapter II, Section 2.1].

Given two more nonnegative real numbers σ and α, we define the Gevrey operator

$$A^s e^{\sigma A^\alpha} u := \sum_{k\in\mathbb{N}_0} e^{\sigma \underline{\lambda}_k^\alpha} \underline{\lambda}_k^s u_k W_k,$$

which domain is the Gevrey space $\mathrm{D}\left(A^s e^{\sigma A^\alpha}\right) := \left\{u \in H \mid A^s e^{\sigma A^\alpha} u \in H\right\}$.

Notice that, for given $s \geq 0$, $\sigma \geq 0$, and $\alpha \geq 0$ the functions in $\{W_k \mid k \in \mathbb{N}_0\}$ are also eigenfunctions for A^s and for $A^s e^{\sigma A^\alpha}$. Indeed for any $k \in \mathbb{N}_0$ it follows that

$$A^s W_k = \underline{\lambda}_k^s W_k \quad \text{and} \quad A^s e^{\sigma A^\alpha} W_k = e^{\sigma \underline{\lambda}_k^\alpha} \underline{\lambda}_k^s W_k.$$

Furthermore the operators A^s and $A^s e^{\sigma A^\alpha}$ are selfadjoint; indeed

$$(A^s u,\, v)_H = \sum_{k\in\mathbb{N}_0} \underline{\lambda}_k^s u_k v_k = (u,\, A^s v)_H,$$
$$(A^s e^{\sigma A^\alpha} u,\, v)_H = \sum_{k\in\mathbb{N}_0} e^{\sigma \underline{\lambda}_k^\alpha} \underline{\lambda}_k^s u_k v_k = (u,\, A^s e^{\sigma A^\alpha} v)_H.$$

Theorem 3.1. *Suppose that the Assumptions 2.1, 2.2, 2.3 and 2.4 hold, and let the strictly increasing sequence of (nonrepeated) eigenvalues $(\lambda_k)_{k\in\mathbb{N}_0}$ of the Stokes operator A satisfy, for some positive real numbers ρ and ξ, the relation*

$$\lambda_k > \rho k^\xi, \quad \textit{for all } k \in \mathbb{N}_0. \tag{6}$$

Further, let us be given $\alpha \in (0,1)$ as in Assumption 2.3, $C_\mathcal{F}$ and $\zeta \geq 0$ as in Assumption 2.4, $\sigma > 0$, $s > \frac{d+2(\xi^{-1}+2\zeta-1)}{4}$, $h \in L^\infty(\mathbb{R}_0, \mathrm{D}(A^{s-\frac{1}{2}} e^{\sigma A^\alpha}))$, $u_0 \in \mathrm{D}(A^s)$, and set $\varphi(t) := \min(\sigma, t)$. Then, there are $T^* > 0$ and a unique solution

$$u \in L^\infty\left((0,T^*), D\left(A^s e^{\varphi(t)A^\alpha}\right)\right) \cap L^2\left((0,T^*), D\left(A^{s+\frac{1}{2}} e^{\varphi(t)A^\alpha}\right)\right), \tag{7}$$

for the Navier–Stokes system (4).

Further, T^* depends on the data $\left(|h|_{L^\infty(\mathbb{R}_0,\, D\left(A^{s-\frac{1}{2}} e^{\sigma A^\alpha}\right)},\, |A^s u_0|_H\right)$ and also on the constants ν, λ_1, d, s, σ, α, β, $C_\mathcal{F}$, ζ, ρ, and ξ.

The proof is given below, in Section 3.3.

3.2. Some preliminary results

We derive some preliminary results that we will need in the proof of Theorem 3.1. Let u solve system (4) and let $\sigma > 0$, $\alpha \in (0, 1)$ and ζ be real numbers as in Theorem 3.1, and set $\varphi(t) = \min(\sigma, t)$. Following the Remark in [17, Section 2.3(iii)], we can see that the function $u^*(t) = e^{\varphi(t)A^\alpha} u(t)$ satisfies $\partial_t u^* = \frac{d\varphi}{dt} A^\alpha e^{\varphi A^\alpha} u + e^{\varphi A^\alpha} \partial_t u$, and denoting $h^*(t) := e^{\varphi(t)A^\alpha} h(t)$, it follows that u^* solves

$$\partial_t u^* + \nu A u^* + e^{\varphi A^\alpha} B(u) + h^* - \frac{d\varphi}{dt} A^\alpha u^* = 0, \tag{8a}$$
$$u^*(0) = u_0. \tag{8b}$$

Now, let $s \geq 0$ be another nonnegative number and multiply (8a) by $A^{2s}u^*$, formally we obtain

$$\begin{aligned}&\left(\partial_t u^*, A^{2s} u^*\right)_H + \nu \left(A u^*, A^{2s} u^*\right)_H \\ &= -\left(e^{\varphi A^\alpha} B(u), A^{2s} u^*\right)_H - \left(h^*, A^{2s} u^*\right)_H + \frac{d\varphi}{dt} \left(A^\alpha u^*, A^{2s} u^*\right)_H.\end{aligned}$$

From the fact that $\left(e^{\varphi A^\alpha} B(u), A^{2s} u^*\right)_H = \left(B(u), A^{2s} e^{\varphi A^\alpha} u^*\right)_H$ and $\left|\frac{d\varphi}{dt}\right| \leq 1$ for all $t \geq 0$, it follows

$$\frac{1}{2}\frac{d}{dt}\left|A^s u^*\right|_H^2 + \nu\left|A^{s+\frac{1}{2}} u^*\right|_H^2$$
$$\leq \left|\left(B(u), A^{2s} e^{\varphi A^\alpha} u^*\right)_H\right| + \left|A^{s-\frac{1}{2}} h^*\right|_H \left|A^{s+\frac{1}{2}} u^*\right|_H + \left|A^{s+\alpha-\frac{1}{2}} u^*\right|_H \left|A^{s+\frac{1}{2}} u^*\right|_H . \quad (9)$$

Now, we find an appropriate bound for the term $\left|\left(B(u), A^{2s} e^{\varphi A^\alpha} u^*\right)_H\right|$. Recall the strictly increasing sequence $(\lambda_k)_{k\in\mathbb{N}_0}$ of all the distinct eigenvalues of the Stokes operator A and the orthogonal projections $P_k : H \to P_k H$ onto the λ_k-eigenspace; see (5) above. We observe that for any $u \in H$, we may write

$$u = \sum_{k\in\mathbb{N}_0} P_k u. \quad (10)$$

Remark 3.2. Given nonnegative real numbers s, α, and σ, $u \in \mathrm{D}\left(A^s e^{\sigma A^\alpha}\right)$, and $l \in \mathbb{N}_0$, we have $P_l(A^s e^{\sigma A^\alpha} u) = \lambda_l^s e^{\sigma \lambda_l^\alpha} P_l u$, and $|u|^2_{\mathrm{D}(A^s e^{\sigma A^\alpha})} = \sum\limits_{k\in\mathbb{N}_0} e^{2\sigma\lambda_k^\alpha} \lambda_k^{2s} \left|P_k u\right|^2$.

From (10) and Assumption 2.2, we may write

$$\left(B(u), A^{2s} e^{\varphi A^\alpha} u^*\right)_H = \sum_{(m,n,l)\in\mathbb{N}_0^3} b\left(P_m u, P_n u, P_l(A^{2s} e^{\varphi A^\alpha} u^*)\right)$$
$$= \frac{1}{2} \sum_{(m,n,l)\in\mathbb{N}_0^3} \left(B(P_m u, P_n u) + B(P_n u, P_m u),\ \lambda_l^{2s} e^{2\varphi\lambda_l^\alpha} P_l u\right)_H ,$$

which leads us to

$$\left(B(u), A^{2s} e^{\varphi A^\alpha} u^*\right)_H$$
$$= - \sum_{\substack{m\in\mathbb{N}_0\\ n<m\\ l\in\mathcal{F}^\bullet_{n,\,m}}} b\left(P_n u, \lambda_l^{2s} e^{2\varphi\lambda_l^\alpha} P_l u, P_m u\right) - \sum_{\substack{m\in\mathbb{N}_0\\ n<m\\ l\in\mathcal{F}^\bullet_{n,\,m}}} b\left(P_m u, \lambda_l^{2s} e^{2\varphi\lambda_l^\alpha} P_l u, P_n u\right)$$
$$- \sum_{\substack{m\in\mathbb{N}_0\\ n=m\\ l\in\mathcal{F}^\bullet_{n,\,m}}} b\left(P_m u, \lambda_l^{2s} e^{2\varphi\lambda_l^\alpha} P_l u, P_n u\right)$$
$$= - \sum_{\substack{m\in\mathbb{N}_0\\ n<m\\ l\in\mathcal{F}^\bullet_{n,\,m}}} b\left(P_n u, \lambda_l^{2s} e^{2\varphi\lambda_l^\alpha} P_l u, P_m u\right) - \sum_{\substack{m\in\mathbb{N}_0\\ n\leq m\\ l\in\mathcal{F}^\bullet_{n,\,m}}} b\left(P_m u, \lambda_l^{2s} e^{2\varphi\lambda_l^\alpha} P_l u, P_n u\right).$$

Hence by Assumptions 2.1, 2.2 and 2.3, we can derive that

$$\left|\left(B(u), A^{2s} e^{\varphi A^\alpha} u^*\right)_H\right|$$
$$\leq 2C \sum_{\substack{m\in\mathbb{N}_0\\ n\leq m\\ l\in\mathcal{F}^\bullet_{n,\,m}}} \left|P_n u\right|_{L^\infty(\Omega,\,T\Omega)} \lambda_l^{2s} e^{2\varphi\lambda_l^\alpha} \left|P_l u\right|_{H^1(\Omega,\,T\Omega)} \left|P_m u\right|_{L^2(\Omega,\,T\Omega)}$$

$$\leq 2C \sum_{\substack{m\in\mathbb{N}_0\\ n\leq m\\ l\in\mathcal{F}^{\bullet}_{n,\,m}}} |P_n u|_{L^\infty(\Omega,\,T\Omega)}\, \lambda_l^{2s} \mathrm{e}^{\varphi(\lambda_l^\alpha+\lambda_n^\alpha+\lambda_m^\alpha+\beta)} \left|A^{\frac{1}{2}}P_l u\right|_{L^2(\Omega,\,T\Omega)} |P_m u|_{L^2(\Omega,\,T\Omega)}$$

$$\leq 2C \sum_{\substack{m\in\mathbb{N}_0\\ n\leq m\\ l\in\mathcal{F}^{\bullet}_{n,\,m}}} \mathrm{e}^{\varphi\beta} |P_n u^*|_{L^\infty(\Omega,\,T\Omega)}\, \lambda_l^{2s+\frac{1}{2}} |P_l u^*|_H |P_m u^*|_H .$$

From a suitable Agmon inequality (cf. [43], Section 2.3), it follows that $|P_n u^*|_{L^\infty(\Omega,\,T\Omega)} \leq C_1 |P_n u^*|_{L^2(\Omega,\,T\Omega)}^{\frac{4-d}{4}} |P_n u^*|_{H^2(\Omega,\,T\Omega)}^{\frac{d}{4}}$ and

$$\left|\left(B(u), A^{2s}e^{\varphi A^\alpha}u^*\right)_H\right| \leq C_2 \mathrm{e}^{\sigma\beta} \sum_{\substack{m\in\mathbb{N}_0\\ n\leq m\\ l\in\mathcal{F}^{\bullet}_{n,\,m}}} \lambda_n^{\frac{d}{4}}\lambda_l^{2s+\frac{1}{2}} |P_n u^*|_H |P_m u^*|_H |P_l u^*|_H. \tag{11}$$

Remark 3.3. Notice that the Agmon inequalities we find in [43, Section 2.3] concern the case Ω is a subset of $\mathbb{R}^d$. However they hold also for a boundaryless manifold $\mathcal{C}$, because we can cover $\mathcal{C}$ by a finite number of charts and use a partition of unity argument. Recall that the Sobolev spaces on a manifold may be defined by means of an atlas of $\mathcal{C}$ (cf. [42, Chapter 4, Section 3]). They hold also for smooth manifolds Ω with smooth boundary $\partial\Omega$ (cf. the discussion after Equation (4.11) in [42, Chapter 4, Section 4]).

Lemma 3.4. *Suppose that the Assumptions 2.1, 2.2, 2.3 and 2.4 hold, and let the strictly increasing sequence of (nonrepeated) eigenvalues $(\lambda_k)_{k\in\mathbb{N}_0}$ of the Stokes operator A satisfy (6). Then, for any given $s > \frac{d+2(\xi^{-1}+2\zeta-1)}{4}$, there exists $C_B \in \mathbb{R}_0$ such that*

$$\left|\left(B(u), A^{2s}e^{\varphi A^\alpha}u^*\right)_H\right| \leq C_B |A^s u^*|^2 \left|A^{s+\frac{1}{2}}u^*\right|, \qquad \text{if } 4s \geq d+2\left(\xi^{-1}+2\zeta+1\right);$$

$$\left|\left(B(u), A^{2s}e^{\varphi A^\alpha}u^*\right)_H\right| \leq C_B |A^s u^*|^{\frac{6-\left(d-4s+2\xi^{-1}+4\zeta\right)}{4}} \left|A^{s+\frac{1}{2}}u^*\right|^{\frac{6+d-4s+2\xi^{-1}+4\zeta}{4}}, \qquad \text{if } 4s < d+2\left(\xi^{-1}+2\zeta+1\right).$$

Further, C_B depends on d, s, σ, α, β, $C_{\mathcal{F}}$, ζ, ρ, and ξ.

Proof. From (11), Assumption 2.3 and Lemma 2.7, it follows that

$$\left|\left(B(u), A^{2s}e^{\varphi A^\alpha}u^*\right)_H\right| \leq K \sum_{\substack{m\in\mathbb{N}_0\\ n\leq m\\ l\in\mathcal{F}^{\bullet}_{n,\,m}}} \lambda_n^{\frac{d}{4}}\lambda_l^{s+\frac{1}{2}}\lambda_m^s |P_n u^*|_H |P_m u^*|_H |P_l u^*|_H,$$

with $K = K(s,\,\sigma,\,\alpha,\,\beta,\,\lambda_1)$. Now we notice that for any triple $(m,n,l) \in \mathbb{N}_0^3$ with $n \leq m$ we have that

$$\begin{aligned}l \in \mathcal{F}^{\bullet}_{n,\,m} &\Leftrightarrow (\mathcal{B}(u,v),w)_H \neq 0, \quad \text{for some} \quad (u,v,w) \in P_nH \times P_mH \times P_lH\\ &\Leftrightarrow m \in \mathcal{F}^{l}_{n,\,\bullet}.\end{aligned}$$

Thus, by the Cauchy inequality, we obtain that

$$\begin{aligned}&\left|\left(B(u), A^{2s}e^{\varphi A^{\alpha}}u^*\right)_H\right|^2\\ &\leq K\Bigg(\sum_{\substack{m\in\mathbb{N}_0\\ n\leq m\\ l\in\mathcal{F}^{\bullet}_{n,\,m}}} \lambda_n^{\frac{d}{4}} \left|P_n u^*\right|_H \lambda_m^{2s} \left|P_m u^*\right|_H^2\Bigg)^{\frac12}\Bigg(\sum_{\substack{l\in\mathbb{N}_0\\ n\leq m\\ m\in\mathcal{F}^{l}_{n,\,\bullet}}} \lambda_n^{\frac{d}{4}} \left|P_n u^*\right|_H \lambda_l^{2s+1} \left|P_l u^*\right|_H^2\Bigg)^{\frac12}.\end{aligned}$$

From Assumption 2.4 we obtain

$$\begin{aligned}&\left|\left(B(u), A^{2s}e^{\varphi A^{\alpha}}u^*\right)_H\right|^2\\ &\leq KC_{\mathcal{F}}\left(\sum_{n\in\mathbb{N}_0} \lambda_n^{\frac{d}{4}+\zeta} \left|P_n u^*\right|_H\right)\left(\sum_{m\in\mathbb{N}_0} \lambda_m^{2s} \left|P_m u^*\right|_H^2\right)^{\frac12}\left(\sum_{l\in\mathbb{N}_0} \lambda_l^{2s+1} \left|P_l u^*\right|_H^2\right)^{\frac12}.\end{aligned}$$

Now, again thanks to the Cauchy inequality, for $\gamma \in \mathbb{R}$ we find

$$\begin{aligned}&\left|\left(B(u), A^{2s}e^{\varphi A^{\alpha}}u^*\right)_H\right|\\ &\leq KC_{\mathcal{F}}\left(\sum_{n\in\mathbb{N}_0} \lambda_n^{\frac{d}{2}+2\zeta-2s-\gamma}\right)^{\frac12} \left|A^{s+\frac{\gamma}{2}}u^*\right|_H \left|A^s u^*\right|_H \left|A^{s+\frac12}u^*\right|_H. \end{aligned}\tag{12}$$

Since $s > \frac{d+2\xi^{-1}+4\zeta-2}{4}$, we have $\frac{d}{2} - 2s + 2\zeta < 1 - \xi^{-1}$. We may set $\gamma \in \left(\frac{d}{2} - 2s + \xi^{-1} + 2\zeta, 1\right)$; which implies $\frac{d}{2}-2s+2\zeta-\gamma < -\xi^{-1}$ and $\delta := \left(\frac{d}{2} - 2s + 2\zeta - \gamma\right)\xi < -1$. From (6), it follows that

$$\sum_{n\in\mathbb{N}_0} \lambda_n^{\frac{d}{2}-2s+2\zeta-\gamma} \leq \rho^{\frac{d}{2}-2s+2\zeta-\gamma} \sum_{n\in\mathbb{N}_0} n^{\delta} =: C_{d,s,\rho,\xi,\zeta,\gamma} < +\infty \tag{13}$$

and, choosing in particular $\gamma = \bar{\gamma} := \frac{d-4s+2(\xi^{-1}+2\zeta+1)}{4}$, from (12) and (13), it follows that

$$\left|\left(B(u), A^{2s}e^{\varphi A^{\alpha}}u^*\right)_H\right| \leq KC_{d,s,\rho,\xi,\zeta} \left|A^{s+\frac{\bar{\gamma}}{2}}u^*\right|_H \left|A^s u^*\right|_H \left|A^{s+\frac12}u^*\right|_H.$$

If $\bar{\gamma} \leq 0$, that is if $4s \geq d + 2\left(\xi^{-1} + 2\zeta + 1\right)$, then

$$\left|\left(B(u^*), A^{2s}u\right)_H\right| \leq KC_{d,s,\rho,\xi} \left|A^s u^*\right|_H^2 \left|A^{s+\frac12}u^*\right|_H.$$

If $\bar{\gamma} \in (0,1)$, that is if $d + 2\left(\xi^{-1} + 2\zeta - 1\right) < 4s < d + 2\left(\xi^{-1} + 2\zeta + 1\right)$, then by an interpolation argument (cf. [34], Chapter 1), we can obtain that

$$\begin{aligned}\left|\left(B(u), A^{2s} e^{\varphi A^{\alpha}} u^*\right)_H\right| &\le K C_{d,s,\rho,\xi,\zeta} C_1 \left|A^s u^*\right|_H^{1+(1-\bar{\gamma})} \left|A^{s+\frac{1}{2}} u^*\right|_H^{1+\bar{\gamma}} \\ &= K C_{d,s,\rho,\xi,\zeta} C_1 \left|A^s u^*\right|_H^{\frac{-d+4s-2\xi^{-1}-4\zeta+6}{4}} \left|A^{s+\frac{1}{2}} u^*\right|_H^{\frac{d-4s+2\xi^{-1}+4\zeta+6}{4}},\end{aligned}$$

which completes the proof of the lemma. □

3.3. *Proof of Theorem 3.1*

We look for u in the form $u = e^{-\varphi(t)A^{\alpha}} u^*$ where u^* solves (8). We will use Lemma 3.4, which suggests us to consider two cases.

3.3.1. *The case $4s < d + 2\left(\xi^{-1} + 2\zeta + 1\right)$. Existence*

We start by observing that

$$\left|A^{s+\alpha-\frac{1}{2}} u^*\right|_H \le \lambda_1^{\alpha-\frac{1}{2}} \left|A^s u^*\right|_H, \quad \text{if } \alpha \le \frac{1}{2},$$

and, by an interpolation argument

$$\left|A^{s+\alpha-\frac{1}{2}} u^*\right|_H \le \left|A^s u^*\right|_H^{2(1-\alpha)} \left|A^{s+\frac{1}{2}} u^*\right|_H^{2\alpha-1}, \quad \text{if } \frac{1}{2} < \alpha < 1.$$

Next, from the assumption in the statement of the Theorem we also have $4s > d + 2\left(\xi^{-1} + 2\zeta - 1\right)$, and thus we can find $\frac{6+d-4s+2\xi^{-1}+4\zeta}{4} < 2$. Therefore, we can set $p = \frac{8}{6+d-4s+2\xi^{-1}+4\zeta} > 1$, and q such that $\frac{1}{p} + \frac{1}{q} = 1$, that is, $\frac{1}{q} = \frac{2-(d-4s+2\xi^{-1}+4\zeta)}{8}$.

From (9), Lemma 3.4, and suitable Young inequalities, we derive that

$$\begin{aligned}\frac{d}{dt}\left|A^s u^*\right|_H^2 + \frac{4\nu}{3}\left|A^{s+\frac{1}{2}} u^*\right|_H^2 &\le 2^q \left(\frac{3}{\nu}\right)^{\frac{q}{p}} C_B^q \left|A^s u^*\right|_H^{\left(\frac{-d+4s-2\xi^{-1}-4\zeta+6}{4}\right)q} + \frac{3}{\nu}\left|A^{s-\frac{1}{2}} h^*\right|_H^2 \\ &\quad + \left|A^{s+\alpha-\frac{1}{2}} u^*\right|_H \left|A^{s+\frac{1}{2}} u^*\right|_H. \end{aligned} \tag{14}$$

Notice that in the case $\alpha \in (0, \frac{1}{2}]$ we have

$$\left|A^{s+\alpha-\frac{1}{2}} u^*\right|_H \left|A^{s+\frac{1}{2}} u^*\right|_H \le C_\nu \left|A^s u^*\right|_H^2 + \frac{\nu}{3}\left|A^{s+\frac{1}{2}} u^*\right|_H^2,$$

and in the case $\alpha \in (\frac{1}{2}, 1)$ we have

$$\left|A^{s+\alpha-\frac{1}{2}} u^*\right|_H \left|A^{s+\frac{1}{2}} u^*\right|_H \le C_{\nu,\alpha} \left|A^s u^*\right|_H^{2(1-\alpha)\frac{2}{3-2\alpha}} + \frac{\nu}{3}\left|A^{s+\frac{1}{2}} u^*\right|_H^2$$

$$\leq C_{\nu,\alpha}\left(|A^s u^*|_H + 1\right)^2 + \frac{\nu}{3}\left|A^{s+\frac{1}{2}}u^*\right|_H^2$$
$$\leq 2C_{\nu,\alpha}\left(|A^s u^*|_H^2 + 1\right) + \frac{\nu}{3}\left|A^{s+\frac{1}{2}}u^*\right|_H^2,$$

because $0 < \frac{4(1-\alpha)}{3-2\alpha} < 2$.

Next we observe that $\left(\frac{-d+4s-2\xi^{-1}-4\zeta+6}{4}\right)q = 2+q > 3$, and from Proposition 2.6 it follows $|A^s u^*|_H^{2+q} + 1 \leq \left(|A^s u^*|_H^2 + 1\right)^{\frac{2+q}{2}}$. Therefore, from (14), we can obtain

$$\frac{d}{dt}|A^s u^*|_H^2 + \nu\left|A^{s+\frac{1}{2}}u^*\right|_H^2$$
$$\leq K_1|A^s u^*|_H^{2+q} + \frac{3}{\nu}\left|A^{s-\frac{1}{2}}h^*\right|_H^2 + K_2(|A^s u^*|_H^2 + 1)$$
$$\leq (K_1+K_2)\left(|A^s u^*|_H^2 + 1\right)^{\frac{2+q}{2}} + \frac{3}{\nu}\left|A^{s-\frac{1}{2}}h^*\right|_H^2,$$

with $K_1 + K_2$ depending on ν, λ_1, d, s, σ, α, β, ρ, ξ, ζ, and $C_{\mathcal{F}}$.

Now, setting $K_3 := K_1 + K_2 + \frac{3}{\nu}\left|A^{s-\frac{1}{2}}h^*\right|_{L^\infty((0,\,+\infty),\,H)}^2$, we arrive to

$$\frac{d}{dt}|A^s u^*|_H^2 + \nu\left|A^{s+\frac{1}{2}}u^*\right|_H^2 \leq K_3\left(|A^s u^*|_H^2 + 1\right)^{\frac{2+q}{2}} \tag{15}$$

and, in particular, to

$$\frac{d}{dt}y \leq K_3 y^{\frac{2+q}{2}}, \quad \text{with } y(t) := |A^s u^*(t)|_H^2 + 1,$$

that is, $\frac{d}{dt}y^\gamma \geq \gamma K_3$ with $\gamma := 1 - (\frac{2+q}{2}) = -\frac{q}{2} < 0$. Integrating over the interval $(0, t)$, it follows that $y^\gamma(t) \geq y^\gamma(0) - (\frac{q}{2})K_3 t$. If we set T^* such that $(\frac{q}{2})K_{B,\nu}T^* \leq (\frac{1}{2})y^\gamma(0)$, that is if $T^* \leq \frac{y^\gamma(0)}{qK_{B,\nu}}$, then $y^{-\gamma}(t) \leq 2y^{-\gamma}(0)$, for all $t \in [0, T^*]$. Thus, we obtain

$$|A^s u^*(t)|_H^2 + 1 \leq 4^{\frac{1}{q}}\left(|A^s u(0)|_H^2 + 1\right) \quad \text{for all } t \in [0, T^*],$$

from which, together with $u(0) = u_0 \in \mathrm{D}(A^s)$ and (15), we can conclude that

$$u^* \in L^\infty\left((0,T^*), D(A^s)\right) \cap L^2\left((0,T^*), D(A^{s+\frac{1}{2}})\right) \tag{16}$$

which implies (7).

3.3.2. The case $4s \geq d + 2\left(\xi^{-1} + 2\zeta + 1\right)$. Existence

Using the corresponding inequality from Lemma 3.4, it is straightforward to check that all the arguments from the first case, $4s < d + 2\left(\xi^{-1} + 2\zeta + 1\right)$, can be repeated by taking $p = q = 2$. We will arrive again to the conclusions (16), and (7).

3.3.3. Uniqueness

It remains to check the uniqueness of u. Let v be another solution for (4), and set $\eta = v - u$. We start by noticing that, from (7), with nonnegative $(s, \sigma, \alpha) \in [0, +\infty)^3$, we have in particular that u is a weak solution:

$$u \in L^\infty((0,T^*),H) \cap L^2\left((0,T^*),D\left(A^{\frac{1}{2}}\right)\right).$$

In the case $d = 2$, it is well known that the uniqueness of u will follow from the estimate

$$\begin{aligned} |(B(v) - B(u),\, \eta)_H| = |b(\eta,\, u,\, \eta)| &\le |\eta|_{L^4(\Omega,\, T\Omega)}|\eta|_{L^4(\Omega,\, T\Omega)}|u|_{H^1(\Omega,\, T\Omega)} \\ &\le C|\eta|^2_{H^{\frac{1}{2}}(\Omega,\, T\Omega)}|u|_{H^1(\Omega,\, T\Omega)} \le C_1|\eta|_H|A^{\frac{1}{2}}\eta|_H|A^{\frac{1}{2}}u|_H \end{aligned}$$

(see, e.g., [45, Chapter 3, Section 3.3, Theorem 3.2]).

In the case $d = 3$. Since $s > \frac{d-2}{4} = \frac{1}{4}$, again from (7), we also have that

$$u \in L^\infty((0,T^*),D(A^{s_1})) \subseteq L^\infty\left((0,T^*),H^{2s_1}(\Omega,\, \mathbb{R}^3)\right) \subset L^{r_1}\left((0,T^*),L^{r_2}(\Omega,\, \mathbb{R}^3)\right)$$

with $s_1 < s$ and $s_1 \in (\frac{1}{4}, \frac{1}{2}]$, $r_1 > 1$ and $r_2 = \frac{2d}{d-4s_1} > 3$, by the Sobolev embedding Theorem (cf. [11, Section 4.4, Corollary 4.53]). Now, the uniqueness of u follows from the fact that for r_1 big enough we have that $\frac{2}{r_1} + \frac{d}{r_2} \le 1$, and from [33, Chapter 1, Section 6.8, Theorem 6.9]). □

Remark 3.5. For simplicity we have restricted ourselves to the above formal computations, but those computations will hold for the Galerkin approximations based on the eigenfunctions of A, which means that they can be made rigorous. See, for example, [33, Chapter 1, Section 6.4] and [45, Chapter 3, Section 3].

4. Considering repeated eigenvalues

In some cases it will be more convenient to work with the sequence $(\underline{\lambda}_k)_{k\in\mathbb{N}_0}$ of repeated eigenvalues. In that case we have to adjust our assumptions to obtain the corresponding version of the Theorem 3.1. Consider the system of eigenfunctions $\{W_k \mid k \in \mathbb{N}_0\}$.

Assumption 4.1. There are real numbers $\alpha \in (0, 1)$ and $\beta \ge 0$, such that for all triples $(n,\, m,\, l) \in \mathbb{N}_0^3$

$$(\mathcal{B}(W_n, W_m),\, W_l)_H \ne 0, \quad \text{implies} \quad \underline{\lambda}_l^\alpha \le \underline{\lambda}_n^\alpha + \underline{\lambda}_m^\alpha + \beta.$$

For given $(n,\, m,\, l) \in \mathbb{N}_0^3$, we define the sets

$$\begin{aligned} \underline{\mathcal{F}}^{\bullet}_{n,\,m} &:= \{k \in \mathbb{N}_0 \mid (\mathcal{B}(W_n, W_m),\, W_k)_H \ne 0,\ \text{with } n \le m\}\,; \\ \underline{\mathcal{F}}^{l}_{n,\,\bullet} &:= \{k \in \mathbb{N}_0 \mid (\mathcal{B}(W_n, W_k),\, W_l)_H \ne 0,\ \text{with } n \le k\}\,. \end{aligned}$$

Assumption 4.2. There are $C_{\mathcal{F}} \in \mathbb{N}_0$ and $\zeta \in [0, +\infty)$, such that for all $n \in \mathbb{N}_0$ we have

$$\sup_{(m,l)\in\mathbb{N}_0^2} \left\{ \operatorname{card}(\underline{\mathcal{F}}^{\bullet}_{n,\,m}), \operatorname{card}(\underline{\mathcal{F}}^{l}_{n,\,\bullet}) \right\} \le C_{\mathcal{F}} \underline{\lambda}_n^{\zeta}.$$

Theorem 4.3. *Suppose that the Assumptions 2.1, 2.2, 4.1 and 4.2 hold, and let the increasing sequence of (repeated) eigenvalues $(\underline{\lambda}_k)_{k\in\mathbb{N}_0}$ of the Stokes operator A satisfy, for some positive real numbers ρ, ξ,*

$$\underline{\lambda}_k > \rho k^{\xi}, \quad \textit{for all } k \in \mathbb{N}_0.$$

Further, let us be given $\alpha \in (0, 1)$ as in Assumption 4.1, $C_{\mathcal{F}}$ and $\zeta \ge 0$ as in Assumption 4.2, $s > \frac{d+2(\xi^{-1}+2\zeta-1)}{4}$, $\sigma > 0$, $h \in L^\infty(\mathbb{R}_0, \mathrm{D}(A^{s-\frac{1}{2}} e^{\sigma A^\alpha}))$, $u_0 \in \mathrm{D}(A^s)$, and set $\varphi(t) := \min(\sigma, t)$. Then, there are $T^* > 0$ and a unique solution

$$u \in L^\infty\left((0, T^*), D\left(A^s e^{\varphi(t)A^\alpha}\right)\right) \cap L^2\left((0, T^*), D\left(A^{s+\frac{1}{2}} e^{\varphi(t)A^\alpha}\right)\right)$$

for the Navier–Stokes system (4).

Further, T^* depends on the data $\left(|h|_{L^\infty(\mathbb{R}_0,\, D\left(A^{s-\frac{1}{2}} e^{\sigma A^\alpha}\right)}, |A^s u_0|_H\right)$ and also on the constants ν, d, s, σ, α, ρ, ξ, ζ, and $C_{\mathcal{F}}$.

The proof can be done following line by line that of Theorem 3.1.

5. Bounding the L^∞-norm by the L^2-norm for eigenprojections

The following observation will enable us to take a smaller s in Theorems 3.1 and 4.3.

If we can find a bound $|P_n u|_{L^\infty(\Omega,\, T\Omega)} \le C\lambda_n^{\theta} |P_n u|_H$ with $\theta < \frac{d}{4}$ and C independent of n, then we can take θ in the place of $\frac{d}{4}$ in (11). As a corollary, we can replace d by 4θ in Theorem 3.1, provided s satisfies $s \ge 0$ in the case $d = 2$ and $s > \frac{1}{4}$ in the case $d = 3$, in order to guarantee the uniqueness of the solution. The analogous conclusion holds for Theorem 4.3, if we can find a bound $|W_n|_{L^\infty(\Omega,\, T\Omega)} \le C\lambda_n^{\theta}$.

6. Examples

We start by revisiting the cases where Ω is the Torus $\mathbb{T}^d$ and the Sphere $\mathbb{S}^2$. Then we give 4 new examples under Lions boundary conditions. Namely the cases of Hemisphere and Cylinder in 2D, and the case of the Rectangle both 2D and 3D.

Remark 6.1. In some situations like in the case of general Navier boundary conditions it may be useful to split the Stokes operator $\Pi\Delta$ as $\Pi\Delta = A + C$ (cf. [41, Chapter 4, Section 4.2]), or it may be interesting to consider an additional linear external forcing (like a Coriolis forcing as in [7]). In these cases we will have the system

$$\dot{u} + B(u, u) + Au + Cu + h = 0, \quad u(0, x) = u_0(x),$$

instead of (3). Notice that Theorems 3.1 and 4.3 will hold in these cases provided we have the estimate $(Cu, A^{2s}u)_{V',V} \leq C_1|A^s u|_H|A^{s+\frac{1}{2}}u|_H$. A better estimate holds in the case of the two-dimensional Navier–Stokes equation under the action of a Coriolis force $\tilde{C}u$: from [7, Lemma 1], for $s > \frac{1}{2}$ it holds $(Cu, A^{2s}u)_{V',V} = (A^{s+\frac{1}{2}}Cu, A^{s-\frac{1}{2}}u)_H \leq C_1|A^s|_H|A^{s-\frac{1}{2}}|_H$, with $Cu := \Pi\tilde{C}u$.

6.1. 2D and 3D Torus

We consider the torus $\mathbb{T}^d = \Pi_{i=1}^d \mathbb{S}^1 \sim (0, 2\pi]^d$, $d \in \{2, 3\}$. This corresponds to the case where we take periodic boundary conditions in $\mathbb{R}^d$ with period 2π in each direction x_i, $i \in \{1, \ldots, d\}$. We also assume that the average $\int_{\mathbb{T}^2} u(t)\, \mathrm{d}\mathbb{T}^d$ vanishes for (a.e.) $t \geq 0$ (cf. [16, Chapter II, eq. (2.5)], [1, Section 2.1]). In this case the Navier–Stokes system can be rewritten as an evolutionary equation in the space of divergence free and zero averaged vector fields $H = \{u \in L^2(\mathbb{T}^d, T\mathbb{T}^d) \sim L^2(\mathbb{T}^d, \mathbb{R}^d) \mid \operatorname{div} u = 0 \text{ and } \int_{\mathbb{T}^2} u\, \mathrm{d}\mathbb{T}^d = 0\}$, with the spaces V and $\mathrm{D}(A)$, defined in Section 2.1 given by $V = H \cap H^1(\mathbb{T}^d, T\mathbb{T}^d)$ and $\mathrm{D}(A) = H \cap H^2(\mathbb{T}^d, T\mathbb{T}^d)$.

We will show that in this case we can take $\alpha = \frac{1}{2}$, $\xi = \frac{2}{d}$, and $\zeta = 0$ in Theorem 4.3, and $\theta = 0$ in Section 5. That is, we can take $s > \frac{d-2}{4}$, in Theorem 4.3.

To simplify the writing we will denote the usual Euclidean scalar product $(u, v)_{\mathbb{R}^d}$ in $\mathbb{R}^d$ by $u \cdot v := \sum_{i=1}^d u_i v_i$. It is well known that a vector field can be written as

$$u = \sum_{k \in \mathbb{Z}^d \setminus \{0_d\}} u_k \mathrm{e}^{\mathbf{i} k \cdot x},$$

where 0_d stands for the zero element $(0, \ldots, 0) \in \mathbb{R}^d$, $\mathbf{i} \sim 0 + 1\mathbf{i}$ is the imaginary complex unit, and the coefficients satisfy $k \cdot u_k = 0$ and $u_{-k} = \overline{u_k}$, where the overline stands for the complex conjugate. The condition $k \cdot u_k = 0$ comes from the divergence free condition, and $u_{-k} = \overline{u_k}$ comes from the fact that u is a function with (real) values in $\mathbb{R}^3$. Thus

$$u = \sum_{k \in \mathbb{Z}^d;\ k > 0_d} \mathrm{Re}(u_k) \cos(k \cdot x) - \mathrm{Im}(u_k) \sin(k \cdot x),$$

where $k > 0_d$ is understood in the lexicographical order, that is either $k_1 > 0$, or $k_1 = 0$ and $k_2 > 0$, or $(k_1, k_{d-1}) = (0, 0)$ and $k_d > 0$, and that a basis of vector fields in H is given by

$$\mathcal{W} = \{w_k^j \cos(k \cdot x),\ w_k^j \sin(k \cdot x) \mid k \in \mathbb{Z}^d,\ k > 0_d \text{ and } j \in \{1, d-1\}\}$$

where for each $k \in \mathbb{Z}^d$, $k > 0_d$, $\{w_k^1, w_k^{d-1}\}$ is a basis for the orthogonal space $\{k\}^\perp$ of $\{k\}$, in $\mathbb{R}^d$. That is, $\operatorname{span}\{w_k^1, w_k^{d-1}\} = \{k\}^\perp$ (cf. [41, Chapter 6, Section 1] for the case $d = 2$).

Moreover we may choose the vectors w_k^j so that the basis above is orthonormal, that is, we can write

$$u = \sum_{\substack{k\in\mathbb{Z}^d;\ k>0_d\\ j\in\{1,d-1\}}} u_{k,j}^c w_k^j \cos(k\cdot x) + u_{k,j}^s w_k^j \sin(k\cdot x).$$

Since the cardinality of $\{k \in \mathbb{Z}^d \mid k > 0_d\} \times \{1,\, d-1\}$ is equal to that of $\mathbb{N}_0$ we could write the previous sum as $u = \sum_{\hat{k}\in\mathbb{N}_0} u_{\hat{k}} W_{\hat{k}}$, as in the preceding text (cf. Section 3). However we can check the Assumptions 2.1, 2.2, 4.1, and 4.2 without performing that writing explicitly.

Checking Assumptions 2.1 and 2.2. Assumptions 2.1 and 2.2 (cf. [43, Section 2.3]); are well known to hold under periodic boundary conditions.

Checking Assumptions 4.1 and 4.2. We proceed as follows: first we obtain

$$\begin{aligned}
B(w_n^j \cos(n\cdot x),\, w_m^i \cos(m\cdot x)) &= -\Pi(w_n^j\cdot m)\cos(n\cdot x)\sin(m\cdot x)w_m^i,\\
B(w_n^j \cos(n\cdot x),\, w_m^i \sin(m\cdot x)) &= \Pi(w_n^j\cdot m)\cos(n\cdot x)\cos(m\cdot x)w_m^i,\\
B(w_n^j \sin(n\cdot x),\, w_m^i \cos(m\cdot x)) &= -\Pi(w_n^j\cdot m)\sin(n\cdot x)\sin(m\cdot x)w_m^i,\\
B(w_n^j \sin(n\cdot x),\, w_m^i \sin(m\cdot x)) &= \Pi(w_n^j\cdot m)\sin(n\cdot x)\cos(m\cdot x)w_m^i,
\end{aligned}$$

from which we can find that

$$\begin{aligned}
&\mathcal{B}(w_n^j \cos(n\cdot x),\, w_m^i \cos(m\cdot x))\\
&= -\Pi w_m^i(w_n^j\cdot m)\cos(n\cdot x)\sin(m\cdot x) - \Pi w_n^j(w_m^j\cdot n)\cos(m\cdot x)\sin(n\cdot x)\\
&= \frac{1}{2}\Pi(-w_m^i(w_n^j\cdot m) - w_n^j(w_m^j\cdot n))\sin((m+n)\cdot x)\\
&\quad + \frac{1}{2}\Pi(-w_m^i(w_n^j\cdot m) + w_n^j(w_m^j\cdot n))\sin((m-n)\cdot x),
\end{aligned}$$

then, it is straightforward to check that $\mathcal{B}(w_n^j\cos(n\cdot x), w_m^i\cos(m\cdot x))$ is orthogonal in $L^2(\mathbb{T}^d,\, T\mathbb{T}^d)$ to all the elements in $\mathcal{W}$ except those in

$$\{w_{m+n}^j \sin\left((m+n)\cdot x\right),\, w_{[m-n]}^j \sin\left(([m-n])\cdot x\right) \mid j\in\{1,\, d-1\}\},$$

where we denote

$$[m-n] = \begin{cases} m-n & \text{if } m-n>0_d\\ n-m & \text{if } n-m>0_d \text{ or } n-m=0_d \end{cases}.$$

In other words, we can conclude that $(\mathcal{B}(w_n^j\cos(n\cdot x), w_m^i\cos(m\cdot x)),\, v)_H \neq 0$ only if

$$v \in \operatorname{span}\{w^j_{m+n} \sin((m+n)\cdot x),\, w^j_{[m-n]} \sin(([m-n])\cdot x) \mid j \in \{1,\, d-1\}\}.$$

Analogously, we can conclude that $(\mathcal{B}(w^j_n \sin(n\cdot x), w^i_m \sin(m\cdot x)),\, v)_H \neq 0$ only if

$$v \in \operatorname{span}\{w^j_{m+n} \sin((m+n)\cdot x),\, w^j_{[m-n]} \sin(([m-n])\cdot x) \mid j \in \{1,\, d-1\}\}.$$

Besides that $(\mathcal{B}(w^j_n \sin(n\cdot x), w^i_m \cos(m\cdot x)),\, v)_H \neq 0$ only if

$$v \in \operatorname{span}\{w^j_{m+n} \cos((m+n)\cdot x),\, w^j_{[m-n]} \cos(([m-n])\cdot x) \mid j \in \{1,\, d-1\}\};$$

and that $(\mathcal{B}(w^j_n \cos(n\cdot x), w^i_m \sin(m\cdot x)),\, v)_H \neq 0$ only if

$$v \in \operatorname{span}\{w^j_{m+n} \cos((m+n)\cdot x),\, w^j_{[m-n]} \cos(([m-n])\cdot x) \mid j \in \{1,\, d-1\}\}.$$

Notice that for given $\hat{n}$, $\hat{m}$, and $\hat{l}$, given in $\mathbb{N}_0$ we have

$$\begin{aligned}\operatorname{card}(\underline{\mathcal{F}}^{\bullet}_{\hat{n},\,\hat{m}}) &:= \left\{\hat{k} \in \mathbb{N}_0 \mid (\mathcal{B}(W_{\hat{n}}, W_{\hat{m}}),\, W_{\hat{k}})_H \neq 0\right\},\\ \operatorname{card}(\underline{\mathcal{F}}^{\hat{l}}_{\hat{n},\,\bullet}) &:= \left\{\hat{k} \in \mathbb{N}_0 \mid (\mathcal{B}(W_{\hat{n}}, W_{\hat{k}}),\, W_{\hat{l}})_H \neq 0\right\},\end{aligned}$$

which allow us to conclude that $\operatorname{card}(\underline{\mathcal{F}}^{\bullet}_{\hat{n},\,\hat{m}}) \leq 2\max\{\dim(\{n+m\}^\perp), \dim(\{[m-n]\}^\perp)\} \leq 2(d-1)$, and necessarily $\operatorname{card}(\underline{\mathcal{F}}^{\hat{l}}_{\hat{n},\,\bullet}) \leq 2(d-1)$, because $l > 0_d$ satisfies $l \in \{k+n, [k-n]\}$ only if $k \in \{[l-n], l+n\}$. That is, we can take $C_{\mathcal{F}} = 2(d-1)$ and $\zeta = 0$ in Assumption 4.2.

Assumption 4.1 follows from the fact that the eigenvalue associated to $w^j_n \sin(n\cdot x)$ and $w^j_n \cos(n\cdot x)$, is given by $|n|^2_{\mathbb{R}^d} = n\cdot n$, and by the triangle inequality, $|n \pm m|_{\mathbb{R}^d} \leq |n|_{\mathbb{R}^d} + |m|_{\mathbb{R}^d}$, which implies that Assumption 4.1 holds with $\alpha = \frac{1}{2}$ and $\beta = 0$.

Looking for the value θ in Section 5. From $|w^j_n \sin(n\cdot x)|^2_{L^\infty(\mathbb{T}^d,\, T\mathbb{T}^d)} \leq |w^j_n|^2$ and $|w^j_n \sin(n\cdot x)|_{L^2(\mathbb{T}^d,\, T\mathbb{T}^d)} = 1$, we can conclude that $|w^j_n|^2 \leq 2^d\pi^{-d}$ and $|w^j_n \sin(n\cdot x)|^2_{L^\infty(\mathbb{T}^d,\, T\mathbb{T}^d)} \leq 2^d\pi^{-d}$, and similarly $|w^j_n \cos(n\cdot x)|^2_{L^\infty(\mathbb{T}^d,\, T\mathbb{T}^d)} \leq 2^d\pi^{-d}$. Hence, we can take $\theta = 0$ in Section 5.

Asymptotic behavior of the (repeated) eigenvalues. From [16, Chapter II, Section 6] we know that the asymptotic behavior of the (repeated) eigenvalues of the Stokes operator in $\mathbb{T}^d$ satisfy $\lambda_k \sim \lambda_1 k^{\frac{2}{d}}$ and more precisely

$$\lim_{k\to+\infty} \frac{\lambda_k}{\lambda_1 k^{\frac{2}{d}}} =: q > 0;$$

then in particular there is $k_0 \in \mathbb{N}_0$ such that $\frac{\lambda_k}{\lambda_1 k^{\frac{2}{d}}} \geq \frac{q}{2}$ for all $k > k_0$, which implies that for all $k \in \mathbb{N}_0$ we have $\lambda_k > \rho k^{\frac{2}{d}}$ if $\rho < \lambda_1 \min\{\frac{q}{2}, q_0\}$, with $q_0 := \min_{k\leq k_0} \frac{\lambda_k}{\lambda_1 k^{\frac{2}{d}}}$. That is, we can take $\xi = \frac{2}{d}$ in Theorem 4.3.

Conclusion. Taking into account Section 5, we conclude that Theorem 4.3 holds with $\alpha = \frac{1}{2}$ and $s > \frac{d-2}{4}$. This covers the results in [17,35] where an analogous procedure is used. Our results also agree with those in [5] in the Hilbert setting. The procedure used in [5] is different and the borderline case $s = \frac{1}{4}$ in 3D is also considered. Finally, recall that we have used the strict inequality $s > \frac{1}{4}$ to prove the uniqueness of the solution, for example.

6.2. 2D Sphere

Let $\Omega = \mathbb{S}^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ be the two-dimensional sphere with the Riemannian metric induced from the usual Euclidean metric in $\mathbb{R}^3$.

In this case we can write the Navier–Stokes system as an evolutionary equation in the space $H := \{u \in L^2(\Omega, T\Omega) \mid \nabla \cdot u = 0\} \cap \{\nabla^\perp \psi \mid \psi \in H^1(\mathbb{S}^2, \mathbb{R})\}$, with $V := H \cap H^1(\Omega, T\Omega)$ and $\mathrm{D}(A) := H \cap H^2(\Omega, T\Omega)$ (cf. [41, Section 5.6], [7, Section 2]).

Remark 6.2. Notice that in [7, Section 2] and [41, Section 5.6] the definitions and notations of the curl of a function f are different; in the former reference it is denoted $\operatorname{Curl} f$ and in the latter $\nabla^\perp f$; however we can show that $\operatorname{Curl} f = -\nabla^\perp f$.

In this case we will use Theorem 3.1 and Section 5, and show that there we can take $\theta = \frac{1}{4}$, $\xi = 2$, $\zeta = \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $s > \frac{1}{2}$. In particular we recover the result in [7].

The complete system of eigenfunctions and respective eigenvalues for $A = -\nu \Pi \Delta$, in H, is presented in [7, Section 2], and it is given by

$$\{Z_n^m(\vartheta, \phi) = \lambda_n^{-\frac{1}{2}} \nabla^\perp Y_n^m(\vartheta, \phi) \mid n \in \mathbb{N}_0, m \in \mathbb{Z}, \text{ and } |m| \leq n\}, \tag{17}$$

where $\vartheta \in (0, \pi)$, $\phi \in (0, 2\pi)$, and for each $Y_n^m(\vartheta, \phi) := C_n^m e^{im\phi} P_n^m(\cos \vartheta)$ is a normalized eigenfunctions of the Laplacian in $L^2(\mathbb{S}^2, \mathbb{R})$ associated with the eigenvalue $\lambda_n = n(n+1)$, with $C_n^m := \left(\frac{2n+1}{4\pi} \frac{(n-|m|)!}{(n-|m|)!}\right)^{\frac{1}{2}}$ and P_n^m is the Ferrers' associated Legendre function of the first kind

$$P_n^m(x) := \frac{\left(1 - x^2\right)^{\frac{m}{2}}}{2^n n!} \frac{\mathrm{d}^{n+m}(x^2 - 1)^n}{\mathrm{d}x^{n+m}}, \quad P_n^{-m}(x) := P_n^m(x) \tag{18}$$

for $m \in \{k \in \mathbb{N} : k \leq n\}$, defined for $|x| \leq 1$. For further details on these functions we refer to [47, Chapter XV, Section 15.5].

For any $(u, v, w) \in P_n H \times P_m H \times P_l H$, there are scalar functions (ψ_u, ψ_v, ψ_w) called stream functions associated with (u, v, w) respectively such that

$$u = -\nabla^\perp \psi_u, \quad v = -\nabla^\perp \psi_v, \quad w = -\nabla^\perp \psi_w$$

where

$$\psi_u = \sum_{|i|\le n} \psi_u^i Y_n^i, \quad \psi_v = \sum_{|j|\le m} \psi_v^j Y_m^j, \quad \psi_w = \sum_{|k|\le l} \psi_w^k Y_l^k,$$

and $\psi_u^i = \overline{\psi_u^{-i}}$, $\psi_v^j = \overline{\psi_v^{-j}}$, and $\psi_w^k = \overline{\psi_w^{-k}}$.

Checking Assumptions 2.1 and 2.2. Assumptions 2.1 and the estimates in Assumption 2.2 follow straightforwardly. For the skew-symmetry property $b(u, v, w) = -b(u, w, v)$ we refer to [3, Section 8, Equation (59)].

Checking Assumptions 2.3 and 2.4. Following [7, Section 3, Lemma 2] (cf. [3, Section 9, Equation (90)]), for eigenfunctions $u \in P_n H$, $v \in P_m H$, and $w \in P_l H$ we obtain

$$\begin{aligned}
|(\mathcal{B}(u,v),w)_H| &= \left|\left(\Pi\left(\Delta\psi_v \nabla\psi_u\right), \nabla^\perp\psi_w\right)_H + \left(\Pi\left(\Delta\psi_u \nabla\psi_v\right), \nabla^\perp\psi_w\right)_H\right| \\
&= \left|\left(\Pi\left(\sum_{|j|\le m} \psi_v^j \Delta Y_m^j \sum_{|i|\le n} \psi_u^i \nabla Y_n^i\right), \sum_{|k|\le l} \psi_w^l \nabla^\perp Y_l^k\right)_H\right. \\
&\quad \left. + \left(\Pi\left(\sum_{|j|\le m} \psi_v^j \Delta Y_m^j \sum_{|i|\le n} \psi_u^i \nabla Y_n^i\right), \sum_{|k|\le l} \psi_w^l \nabla^\perp Y_l^k\right)_H\right| \\
&= \left|\sum_{|i|\le n}\sum_{|j|\le m}\sum_{|k|\le l} \psi_u^i\psi_v^j\psi_w^l \left(\Delta Y_m^j \nabla Y_n^i, \nabla^\perp Y_l^k\right)_H\right. \\
&\quad \left. + \sum_{|i|\le n}\sum_{|j|\le m}\sum_{|k|\le l} \psi_u^i\psi_v^j\psi_w^l \left(\Delta Y_m^j \nabla Y_n^i, \nabla^\perp Y_l^k\right)_H\right|
\end{aligned}$$

An explicit expression for the scalar product $\left(\Delta Y_m^j \nabla Y_n^i, \nabla^\perp Y_l^k\right)_H$ is given in [14, Theorem 5.3], that expression involves the so-called Wigner-3j symbols. For this symbols we refer also to [12, Section 3.7] and [38, Section 2]. From that expression in [14, Theorem 5.3], recalling that the Wigner-3j symbol $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ vanishes unless all the conditions

i. $m_1 + m_2 + m_3 = 0$,
ii. $j_1 + j_2 + j_3$ is an integer (if $m_1 = m_2 = m_3 = 0$, $j_1 + j_2 + j_3$ is an even integer),
iii. $|m_k| \le j_k$, and
iv. $|j_1 - j_2| \le j_3 \le j_1 + j_2$

are satisfied, we conclude that $(u, v, w) \in P_n H \times P_m H \times P_l H$ and $(\mathcal{B}(u,v),w)_H \ne 0$ only if $m - n \le l < m + n$ and $m + n + l$ is odd (cf. [14, Corollary 5.4]).

Therefore, we obtain that necessarily $\mathrm{card}(\mathcal{F}_{n,m}^{\bullet}) \le 2n$ and $\mathrm{card}(\mathcal{F}_{n,\bullet}^{l}) \le 2n$, that is, Assumption 2.4 holds for $C_{\mathcal{F}} = 2$ and $\zeta = \frac{1}{2}$.

For $(u, v, w) \in P_nH \times P_mH \times P_lH$ and $(\mathcal{B}(u, v), w)_H \neq 0$ we have $l \in [m-n, m+n]$, then $\lambda_l < \lambda_{n+m}$, and from $\lambda_{n+m} = (n+m)(n+m+1) = (n+m)^2+n+m = (n+m+\frac{1}{2})^2-\frac{1}{4}$ we find $\lambda_l^{\frac{1}{2}} < \lambda_{n+m}^{\frac{1}{2}} \leq n+m+\frac{1}{2} < \lambda_n^{\frac{1}{2}} + \lambda_m^{\frac{1}{2}} + \frac{1}{2}$, and it follows that Assumption 2.3 holds with $\alpha = \frac{1}{2}$.

The parameters $\boldsymbol{\theta}$ and $\boldsymbol{\xi}$. From [7, Section 3, Lemma 2], we can take $\theta = \frac{1}{4}$ in Section 5, and from $\lambda_k = k(k+1) > k^2$ it follows that (6) holds with $\xi = 2$.

Conclusion. Taking into account Section 5, we conclude that Theorem 3.1 holds with $\alpha = \frac{1}{2}$ and $s > \frac{1}{2}$. This agrees with the results in [7].

6.3. 2D Hemisphere

Let Ω be the Hemisphere $\mathbb{S}^2_+ = \left\{(x_1, x_2, x_3) \in \mathbb{S}^2 \mid x_3 > 0\right\}$. On the boundary $\partial\mathbb{S}^2_+$ of $\mathbb{S}^2_+$, $\partial\mathbb{S}^2_+ = \left\{(x_1, x_2, x_3) \in \mathbb{S}^2 \mid x_3 = 0\right\}$, we impose the Lions boundary conditions, that is, we consider the evolutionary Navier–Stokes equation in $H := \{u \in L^2(\Omega, T\Omega) \mid \nabla \cdot u = 0 \text{ and } g(u, \mathbf{n}) = 0\} \cap \{\nabla^\perp \psi \mid \psi \in H^1(\mathbb{S}^2_+, \mathbb{R})\}$, with $V := H \cap H^1(\Omega, T\Omega)$ and $\mathrm{D}(A) := V \cap \{u \in H^2(\Omega, T\Omega) \mid \nabla^\perp \cdot u = 0 \text{ on } \partial\mathbb{S}^2_+\}$ (cf. [41, Sections 5.5 and 6.4]).

In this case we will use Theorem 3.1 and Section 5, and show that as in the case of the Sphere, in Section 6.2, there we can take $\theta = \frac{1}{4}$, $\xi = 2$, $\zeta = \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $s > \frac{1}{2}$.

In spherical coordinates $\mathbb{S}^2 \sim (\vartheta, \phi) \in [0, \pi] \times [0, 2\pi)$ the Hemisphere corresponds to $\mathbb{S}^2_+ \sim (\vartheta, \phi) \in [0, \frac{\pi}{2}) \times [0, 2\pi)$. It turns out that from the system (17) we can construct a complete system in H formed by eigenfunctions of A, which is

$$\left\{ Z_n^m(\vartheta, \phi)|_{\vartheta \in [0, \frac{\pi}{2})} = \lambda_n^{-\frac{1}{2}} \nabla^\perp Y_n^m(\vartheta, \phi)|_{\vartheta \in [0, \frac{\pi}{2})} \,\middle|\, \begin{array}{l} n \in \mathbb{N}_0, \ m \in \mathbb{Z}, \\ |m| \leq n, \ |m| + n \text{ is odd} \end{array} \right\} \tag{19}$$

(cf. [41, Proposition 6.4.2]). Let us show that the system is complete. For $Z_n^m(\vartheta, \phi)$ in (17) we have that $\nabla^\perp \cdot Z_n^m(\vartheta, \phi) = \lambda_n^{-\frac{1}{2}} \Delta Y_n^m(\vartheta, \phi) = \lambda_n^{\frac{1}{2}} Y_n^m(\vartheta, \phi)$, and if $|m| + n$ is odd we have that $Y_n^m(\frac{\pi}{2}, \phi) = 0$, that is, $Z_n^m(\vartheta, \phi)|_{\vartheta \in [0, \frac{\pi}{2})}$ is in $\mathrm{D}(A)$. Further we have that for $\vartheta_1 \in [0, \frac{\pi}{2}]$,

$$\begin{aligned} Z_n^m(\tfrac{\pi}{2} - \vartheta_1, \phi) &= Z_n^m(\tfrac{\pi}{2} + \vartheta_1, \phi), && \text{if } |m| + n \text{ is odd;} \\ Z_n^m(\tfrac{\pi}{2} - \vartheta_1, \phi) &= -Z_n^m(\tfrac{\pi}{2} + \vartheta_1, \phi), && \text{if } |m| + n \text{ is even.} \end{aligned}$$

Notice that from (18), we can see that $P_n^m(-x) = -P_n^m(x)$ if $|m| + n$ is odd, and $P_n^m(-x) = P_n^m(x)$ if $|m| + n$ is even.

To show that (19) is complete in H, it is sufficient to show that the family of stream functions $\{Y_n^m(\vartheta, \phi)|_{\vartheta \in [0, \frac{\pi}{2})} \mid n \in \mathbb{N}_0, \ m \in \mathbb{Z}, \ |m| \leq n, \text{ and } |m| + n \text{ is odd}\}$ form a complete system in $L^2(\mathbb{S}^2_+, \mathbb{R})$. Let $g(\vartheta, \phi)$ be a function defined on the Hemisphere $[0, \frac{\pi}{2}) \times [0, 2\pi)$; we extend it to a function $\tilde{g}$ defined on the Sphere as follows

$$\tilde{g}(\vartheta,\,\phi)=\begin{cases} g(\vartheta,\,\phi) & \text{if}\quad \vartheta\in[0,\,\frac{\pi}{2}),\\ -g(\pi-\vartheta,\,\phi) & \text{if}\quad \vartheta\in(\frac{\pi}{2},\,\pi]. \end{cases}$$

We know that we can write $\tilde{g}=\sum_{(n,m)\in\mathcal{S}}(\tilde{g},Y_n^m)_{L^2(\mathbb{S}^2,\,\mathbb{R})}Y_n^m$ where $\mathcal{S}:=\{(n,\,m)\in\mathbb{Z}^2 \mid n\in\mathbb{N} \text{ and } |m|\le n\}$.

By using spherical coordinates, we find for even $|m|+n$

$$\int_{\frac{\pi}{2}}^{\pi}\tilde{g}(\vartheta,\phi)Y_n^m(\vartheta,\phi)\sin(\vartheta)\mathrm{d}\vartheta=\int_0^{\frac{\pi}{2}}\tilde{g}(\tfrac{\pi}{2}+\vartheta_1,\phi)Y_n^m(\tfrac{\pi}{2}+\vartheta_1,\phi)\sin(\tfrac{\pi}{2}+\vartheta_1)\mathrm{d}\vartheta_1$$

$$=\int_0^{\frac{\pi}{2}}-g(\tfrac{\pi}{2}-\vartheta_1,\phi)Y_n^m(\tfrac{\pi}{2}-\vartheta_1,\phi)\sin(\tfrac{\pi}{2}-\vartheta_1)\mathrm{d}\vartheta_1$$

$$=-\int_{\frac{\pi}{2}}^{0}g(\vartheta_2,\phi)Y_n^m(\vartheta_2,\phi)\sin(\vartheta_2)(-\mathrm{d}\vartheta_2)=-\int_0^{\frac{\pi}{2}}g(\vartheta_2,\phi)Y_n^m(\vartheta_2,\phi)\sin(\vartheta_2)\mathrm{d}\vartheta_2,$$

which implies $\int_0^{\pi}g(\vartheta,\phi)Y_n^m(\vartheta,\phi)\sin(\vartheta)\mathrm{d}\vartheta=0$. For even $|m|+n$, it follows $(\tilde{g},Y_n^m)_{L^2(\mathbb{S}^2,\,\mathbb{R})}=\int_0^{2\pi}d\phi\int_0^{\pi}\tilde{g}(\vartheta,\phi)Y_n^m(\vartheta,\phi)\sin(\vartheta)\mathrm{d}\vartheta=0$, that is, $\tilde{g}=\sum_{(n,m)\in\mathcal{S}_+}(\tilde{g},Y_n^m)_{L^2(\mathbb{S}^2,\,\mathbb{R})}Y_n^m$, with $\mathcal{S}_+:=\{(n,m)\in\mathcal{S}\mid |m|+n \text{ is odd}\}$, and

$$g=\tilde{g}|_{\vartheta\in[0,\,\frac{\pi}{2})}=\sum_{(n,m)\in\mathcal{S}_+}(\tilde{g},Y_n^m)_{L^2(\mathbb{S}^2,\,\mathbb{R})}Y_n^m|_{\vartheta\in[0,\,\frac{\pi}{2})},$$

which shows that the set $\{Y_n^m(\vartheta,\phi)|_{\vartheta\in[0,\,\frac{\pi}{2})}\mid(n,\,m)\in\mathcal{S}\}$ is complete in $L^2(\mathbb{S}^2_+,\,\mathbb{R})$.

Proceeding as above for the extension $\tilde{g}$ and for odd $|m|+n$ we have $(\tilde{g},Y_n^m)_{L^2(\mathbb{S}^2,\,\mathbb{R})}=2(g,Y_n^m|_{\vartheta\in[0,\,\frac{\pi}{2})})_{L^2(\mathbb{S}^2_+,\,\mathbb{R})}$, and also $Y_n^m=\tilde{h}$ with $h=Y_n^m|_{\vartheta\in[0,\,\frac{\pi}{2})}$. In particular we conclude that the family $\{Y_n^m(\vartheta,\phi)|_{\vartheta\in[0,\,\frac{\pi}{2})}\mid(n,\,m)\in\mathcal{S}\}$ is orthogonal in $L^2(\mathbb{S}^2_+,\,\mathbb{R})$ and then it forms a basis in $L^2(\mathbb{S}^2_+,\,\mathbb{R})$.

As a consequence we can conclude that the family (19) form a complete system in H. Notice that for $n=0$, Y_0^0 is a constant function, and the vector field $\nabla^\perp Y_0^0\in L^2(\mathbb{S}^2,\,T\mathbb{S}^2)$ vanishes. From the fact that $(Y_n^m,Y_n^m)_{L^2(\mathbb{S}^2,\,\mathbb{R})}=2(Y_n^m|_{\vartheta\in[0,\,\frac{\pi}{2})},Y_n^m|_{\vartheta\in[0,\,\frac{\pi}{2})})_{L^2(\mathbb{S}^2_+,\,\mathbb{R})}$, we can normalize that system as

$$\left\{\sqrt{2}Z_n^m(\vartheta,\phi)|_{\vartheta\in[0,\,\frac{\pi}{2})}=\lambda_n^{-\frac{1}{2}}\nabla^\perp Y_n^m(\vartheta,\phi)|_{\vartheta\in[0,\,\frac{\pi}{2})}\;\middle|\;\begin{array}{l} n\in\mathbb{N}_0,\ m\in\mathbb{Z},\\ |m|\le n,\ |m|+n \text{ is odd}\end{array}\right\}\tag{20}$$

Conclusion. We can follow the arguments in the case of the Sphere, in Section 6.2, to conclude that Theorem 3.1 holds with $\alpha=\frac{1}{2}$ and $s>\frac{1}{2}$.

6.4. 2D Rectangle

Let Ω be the two-dimensional Rectangle $\Omega = (0, a) \times (0, b) \subset \mathbb{R}^2$. On the boundary we impose the Lions boundary conditions, that is, we consider the evolutionary Navier–Stokes equation in $H := \{u \in L^2(\Omega, \mathbb{R}^2) \mid \nabla \cdot u = 0 \text{ and } u \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$, with $V := H \cap H^1(\Omega, \mathbb{R}^2)$ and $\mathrm{D}(A) := V \cap \{u \in H^2(\Omega, \mathbb{R}^2) \mid \nabla^\perp \cdot u = 0 \text{ on } \partial\Omega\}$ (cf. [39] and [41, Section 6.3]).

We will show that in this case we can take $\alpha = \frac{1}{2}$, $\xi = 1$, and $\zeta = 0$ in Theorem 4.3, and $\theta = 0$ in Section 5. That is, we can take $s > 0$, in Theorem 4.3.

The complete system of eigenfunctions $\{Y_{(k_1,\, k_2)} \mid (k_1,\, k_2) \in \mathbb{N}_0^2\}$ and respective eigenvalues $\{\lambda_{(k_1,\, k_2)} \mid (k_1,\, k_2) \in \mathbb{N}_0^2\}$ of A, can be found in [39, Sections 2.2 and 2.3], they are given by

$$Y_{(k_1,k_2)} := \begin{pmatrix} -\frac{k_2\pi}{b} \sin\left(\frac{k_1\pi x_1}{a}\right) \cos\left(\frac{k_2\pi x_2}{b}\right) \\ \frac{k_1\pi}{a} \cos\left(\frac{k_1\pi x_1}{a}\right) \sin\left(\frac{k_2\pi x_2}{b}\right) \end{pmatrix}, \quad \lambda_{(k_1,\, k_2)} := \pi^2 \left(\frac{k_1^2}{a^2} + \frac{k_2^2}{b^2}\right). \tag{21}$$

Though, the above systems are indexed over $\mathbb{N}_0^2$, like in Section 6.1, we can check the Assumptions 2.1, 2.2, 4.1, and 4.2 without rewriting the families as indexed over $\mathbb{N}_0$.

We may normalize the family (21), obtaining the system $\{W_{(k_1,k_2)} \mid (k_1,\, k_2) \in \mathbb{N}_0^2\}$, with

$$W_{(k_1,k_2)} := 2(ab\lambda_{(k_1,\, k_2)})^{-\frac{1}{2}} \begin{pmatrix} -\frac{k_2\pi}{b} \sin\left(\frac{k_1\pi x_1}{a}\right) \cos\left(\frac{k_2\pi x_2}{b}\right) \\ \frac{k_1\pi}{a} \cos\left(\frac{k_1\pi x_1}{a}\right) \sin\left(\frac{k_2\pi x_2}{b}\right) \end{pmatrix}. \tag{22}$$

Checking Assumptions 2.1 and 2.2. We refer to [39] and [43, Section 2.3].

Checking Assumptions 4.1 and 4.2. From [39, equation (6.1)], we can derive that $\left(\mathcal{B}(Y_{(n_1,\, n_2)}, Y_{(m_1,\, m_2)}),\, Y_{(l_1,\, l_2)}\right)_H \neq 0$ only if

$$l_1 = |n_1 \pm m_1| \text{ and } l_2 = |n_2 \pm m_2|,$$

from which we can conclude that for $\hat{n}$, $\hat{m}$, and $\hat{l}$, given in $\mathbb{N}_0$ we have that $\mathrm{card}(\underline{\mathcal{F}}^{\bullet}_{\hat{n},\, \hat{m}}) \leq 4$ and $\mathrm{card}(\underline{\mathcal{F}}^{\hat{l}}_{\hat{n},\, \bullet}) \leq 4$. That is, Assumption 4.2 holds with $\zeta = 0$. We also see that necessarily $\lambda_{(l_1,\, l_2)} \leq \lambda_{(n_1+m_1,\, n_2+m_2)}$; noticing that $(k_1,\, k_2) \mapsto \lambda_{(k_1,\, k_2)}$ is a scalar product, we conclude that $\lambda^{\frac{1}{2}}_{(l_1,\, l_2)} \leq \lambda^{\frac{1}{2}}_{(n_1,\, n_2)} + \lambda^{\frac{1}{2}}_{(m_1,\, m_2)}$, that is, Assumption 4.1 holds with $\alpha = \frac{1}{2}$.

Looking for the value $\boldsymbol{\theta}$ in Section 5. We have that

$$\begin{aligned} |W_{(k_1,\, k_2)}|^2_{L^\infty(\Omega,\, \mathbb{R}^2)} &= \max_{(x_1,\, x_2) \in \Omega} |W_{(k_1,\, k_2)}(x_1,\, x_2)|^2_{\mathbb{R}^2} \leq 4(ab\lambda_{(k_1,\, k_2)})^{-1}\left(\tfrac{k_2^2\pi^2}{b^2} + \tfrac{k_1^2\pi^2}{a^2}\right) \\ &= 4(ab)^{-1}, \end{aligned}$$

that is, we can take $\theta = 0$.

Asymptotic behavior of the (repeated) eigenvalues. We recall that for an open simply connected domain $\Omega \subset \mathbb{R}^2$, under Lions boundary conditions, the eigenvalues of the Stokes operator $A : \mathrm{D}(A) \to H$ are those of the Dirichlet Laplacian $\Delta H^2(\Omega, \mathbb{R}) \cap H^1_0(\Omega, \mathbb{R}) \to L^2(\Omega, \mathbb{R})$, that is, $Au = \lambda u$ if, and only if, $\Delta\nabla^\perp u = \lambda\nabla^\perp u$. Thus, from [32, Corollary 1] we have that we can take $\rho < \frac{2\pi}{ab}$ and $\xi = 1$ in Theorem 4.3.

For the sake of completeness we would like also to refer to the results in [22], and references therein, for the case of no-slip boundary conditions.

Conclusion. Taking into account Section 5, we conclude that Theorem 4.3 holds with $\alpha = \frac{1}{2}$ and $s > 0$.

6.5. 2D Cylinder

Let Ω be a two-dimensional Cylinder $\Omega = \left(\frac{a}{2\pi}\mathbb{S}^1\right) \times (0, b) \sim (0, a) \times (0, b)$. On the boundary $(0, a) \times \{0, b\}$ we impose the Lions boundary conditions, that is, we consider the evolutionary Navier–Stokes equation in $H := \{u \in L^2(\Omega, \mathbb{R}^2) \mid \nabla\cdot u = 0 \text{ and } u\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}$, with $V := H \cap H^1(\Omega, \mathbb{R}^2)$ and $\mathrm{D}(A) := V \cap \{u \in H^2(\Omega, \mathbb{R}^2) \mid \nabla^\perp\cdot u = 0 \text{ on } \partial\Omega\}$. We can see the domain Ω as an infinite channel $\mathbb{R} \times (0, b)$ where we take a-periodic boundary conditions on the infinite direction $x_1 \in \mathbb{R}$ and Lions boundary conditions on the boundary $\mathbb{R} \times \{0, b\}$.

We will show that in this case we can take $\alpha = \frac{1}{2}$, $\xi = 1$, and $\zeta = 0$ in Theorem 4.3, and $\theta = 0$ in Section 5. That is, we can take $s > 0$, in Theorem 4.3.

A complete system of orthogonal eigenfunctions of A $\{Y^\varsigma_n, Y^\varkappa_m \mid n \in \mathbb{N}_0^2, m \in \mathbb{N} \times \mathbb{N}_0\}$, and corresponding eigenvalues $\{\lambda^\varsigma_n, \lambda^\varkappa_m \mid n \in \mathbb{N}_0^2, m \in \mathbb{N} \times \mathbb{N}_0\}$, are given by

$$\begin{aligned} Y^\varsigma_k = Y^\varsigma_{(k_1,k_2)} &= \begin{pmatrix} -\frac{k_2\pi}{b}\sin\left(\frac{2k_1\pi x_1}{a}\right)\cos\left(\frac{k_2\pi x_2}{b}\right) \\ \frac{2k_1\pi}{a}\cos\left(\frac{2k_1\pi x_1}{a}\right)\sin\left(\frac{k_2\pi x_2}{b}\right) \end{pmatrix}, \\ Y^\varkappa_k = Y^\varkappa_{(k_1,k_2)} &= \begin{pmatrix} -\frac{k_2\pi}{b}\cos\left(\frac{2k_1\pi x_1}{a}\right)\cos\left(\frac{k_2\pi x_2}{b}\right) \\ -\frac{2k_1\pi}{a}\sin\left(\frac{2k_1\pi x_1}{a}\right)\sin\left(\frac{k_2\pi x_2}{b}\right) \end{pmatrix}, \end{aligned} \tag{23}$$

and $\lambda^\varsigma_{(k_1,k_2)} = \lambda^\varkappa_{(k_1,k_2)} = \lambda_{(k_1,k_2)} := \pi^2\left(\frac{(2k_1)^2}{a^2} + \frac{k_2^2}{b^2}\right)$.

Remark 6.3. Notice that $Y^\varsigma_n = \nabla^\perp\psi^\varsigma_n$, $Y^\varkappa_m = \nabla^\perp\psi^\varkappa_n$, with $\psi^\varsigma_n := \sin\left(\frac{2k_1\pi x_1}{a}\right)\sin\left(\frac{k_2\pi x_2}{b}\right)$ and $\psi^\varkappa_n := \cos\left(\frac{2k_1\pi x_1}{a}\right)\sin\left(\frac{k_2\pi x_2}{b}\right)$; notice also that the set of stream functions $\{Y^\varsigma_n, Y^\varkappa_m \mid n \in \mathbb{N}_0^2, m \in \mathbb{N} \times \mathbb{N}_0\}, \mid n \in \mathbb{N}_0^2, m \in \mathbb{N} \times \mathbb{N}_0\}$ is an orthogonal and complete, in $L^2(\Omega, \mathbb{R}^2)$, system of eigenfunctions of the Laplacian in $\Omega \sim (0, a) \times (0, b)$.

We may normalize the family, obtaining the normalized system $\{W^\varsigma_n, W^\varkappa_m \mid n \in \mathbb{N}_0^2, m \in \mathbb{N} \times \mathbb{N}_0\}$, with

$$W^\varsigma_k := 2(ab\lambda_k)^{-\frac{1}{2}}Y^\varsigma_k, \quad W^\varkappa_k := 2(ab\lambda_k)^{-\frac{1}{2}}Y^\varkappa_k. \tag{24}$$

Now we check our assumptions, proceeding as in the case of the Rectangle, in Section 6.4.

Checking Assumptions 2.1 and 2.2. The assumptions follow straightforwardly.

Checking Assumptions 4.1 and 4.2. From the discussion following Corollary 5.6.3 in [41] we can conclude that $\nabla^\perp\cdot\mathcal{B}(u,v)=(\nabla\Delta^{-1}\nabla^\perp\cdot v,\,\nabla^\perp\nabla^\perp\cdot u)_{\mathbb{R}^2}+(\nabla\Delta^{-1}\nabla^\perp\cdot u,\,\nabla^\perp\nabla^\perp\cdot v)_{\mathbb{R}^2}$. If u and v are eigenfunctions from (23) with associated eigenvalues λ_u and λ_v, and associated eigenfunctions ψ_u and ψ_v, we obtain

$$\begin{aligned}\nabla^\perp\cdot\mathcal{B}(u,v)&=\lambda_u(\nabla\psi_v,\,\nabla^\perp\psi_u)_{\mathbb{R}^2}+\lambda_v(\nabla\psi_u,\,\nabla^\perp\psi_v)_{\mathbb{R}^2}\\&=(\lambda_u-\lambda_v)(\nabla^\perp\psi_u,\,\nabla\psi_v)_{\mathbb{R}^2}.\end{aligned}\tag{25}$$

From straightforward computations, we find the following expressions

$$\begin{aligned}&(\nabla^\perp\psi_n^\varsigma,\,\nabla\psi_m^\varsigma)_{\mathbb{R}^2}\\&=\begin{pmatrix}-\frac{n_2\pi}{b}\sin\left(\frac{2n_1\pi x_1}{a}\right)\cos\left(\frac{n_2\pi x_2}{b}\right)\\ \frac{2n_1\pi}{a}\cos\left(\frac{2n_1\pi x_1}{a}\right)\sin\left(\frac{n_2\pi x_2}{b}\right)\end{pmatrix}\cdot\begin{pmatrix}\frac{2m_1\pi}{a}\cos\left(\frac{2m_1\pi x_1}{a}\right)\sin\left(\frac{m_2\pi x_2}{b}\right)\\ \frac{m_2\pi}{b}\sin\left(\frac{2m_1\pi x_1}{a}\right)\cos\left(\frac{m_2\pi x_2}{b}\right)\end{pmatrix}\\&=-\frac{2\pi^2n_2m_1}{ab}\sin\left(\frac{2n_1\pi x_1}{a}\right)\cos\left(\frac{n_2\pi x_2}{b}\right)\cos\left(\frac{2m_1\pi x_1}{a}\right)\sin\left(\frac{m_2\pi x_2}{b}\right)\\&\quad+\frac{2\pi^2n_1m_2}{ab}\cos\left(\frac{2n_1\pi x_1}{a}\right)\sin\left(\frac{n_2\pi x_2}{b}\right)\sin\left(\frac{2m_1\pi x_1}{a}\right)\cos\left(\frac{m_2\pi x_2}{b}\right);\end{aligned}$$

$$\begin{aligned}&(\nabla^\perp\psi_n^\varsigma,\,\nabla\psi_m^\varkappa)_{\mathbb{R}^2}\\&=\begin{pmatrix}-\frac{n_2\pi}{b}\sin\left(\frac{2n_1\pi x_1}{a}\right)\cos\left(\frac{n_2\pi x_2}{b}\right)\\ \frac{2n_1\pi}{a}\cos\left(\frac{2n_1\pi x_1}{a}\right)\sin\left(\frac{n_2\pi x_2}{b}\right)\end{pmatrix}\cdot\begin{pmatrix}-\frac{2m_1\pi}{a}\sin\left(\frac{2m_1\pi x_1}{a}\right)\sin\left(\frac{m_2\pi x_2}{b}\right)\\ \frac{m_2\pi}{b}\cos\left(\frac{2m_1\pi x_1}{a}\right)\cos\left(\frac{m_2\pi x_2}{b}\right)\end{pmatrix}\\&=\frac{2\pi^2n_2m_1}{ab}\sin\left(\frac{2n_1\pi x_1}{a}\right)\cos\left(\frac{n_2\pi x_2}{b}\right)\sin\left(\frac{2m_1\pi x_1}{a}\right)\sin\left(\frac{m_2\pi x_2}{b}\right)\\&\quad+\frac{2\pi^2n_1m_2}{ab}\cos\left(\frac{2n_1\pi x_1}{a}\right)\sin\left(\frac{n_2\pi x_2}{b}\right)\cos\left(\frac{2m_1\pi x_1}{a}\right)\cos\left(\frac{m_2\pi x_2}{b}\right);\end{aligned}$$

$$\begin{aligned}&(\nabla^\perp\psi_n^\varkappa,\,\nabla\psi_m^\varkappa)_{\mathbb{R}^2}\\&=\begin{pmatrix}-\frac{n_2\pi}{b}\cos\left(\frac{2n_1\pi x_1}{a}\right)\cos\left(\frac{n_2\pi x_2}{b}\right)\\ -\frac{2n_1\pi}{a}\sin\left(\frac{2n_1\pi x_1}{a}\right)\sin\left(\frac{n_2\pi x_2}{b}\right)\end{pmatrix}\cdot\begin{pmatrix}-\frac{2m_1\pi}{a}\sin\left(\frac{2m_1\pi x_1}{a}\right)\sin\left(\frac{m_2\pi x_2}{b}\right)\\ \frac{m_2\pi}{b}\cos\left(\frac{2m_1\pi x_1}{a}\right)\cos\left(\frac{m_2\pi x_2}{b}\right)\end{pmatrix}\\&=\frac{2\pi^2n_2m_1}{ab}\cos\left(\frac{2n_1\pi x_1}{a}\right)\cos\left(\frac{n_2\pi x_2}{b}\right)\sin\left(\frac{2m_1\pi x_1}{a}\right)\sin\left(\frac{m_2\pi x_2}{b}\right)\\&\quad-\frac{2\pi^2n_1m_2}{ab}\sin\left(\frac{2n_1\pi x_1}{a}\right)\sin\left(\frac{n_2\pi x_2}{b}\right)\cos\left(\frac{2m_1\pi x_1}{a}\right)\cos\left(\frac{m_2\pi x_2}{b}\right).\end{aligned}$$

Thus, if we denote $\varsigma_c^{z,\,l}:=\sin\left(\frac{l\pi z}{c}\right)$ and $\varkappa_c^{z,\,l}:=\cos\left(\frac{l\pi z}{c}\right)$; $m\wedge n=m_1n_2-m_2n_1$ and $m\vee n=m_1n_2+m_2n_1$, we obtain

$$\begin{aligned}&(\nabla^\perp\psi_n^\varsigma,\,\nabla\psi_m^\varsigma)_{\mathbb{R}^2}\\&=-\frac{\pi^2n_2m_1}{2ab}\left(\varsigma_a^{x_1,\,2(n_1+m_1)}+\varsigma_a^{x_1,\,2(n_1-m_1)}\right)\left(\varsigma_b^{x_2,\,n_2+m_2}-\varsigma_b^{x_2,\,n_2-m_2}\right)\\&\quad+\frac{\pi^2n_1m_2}{2ab}\left(\varsigma_a^{x_1,\,2(n_1+m_1)}-\varsigma_a^{x_1,\,2(n_1-m_1)}\right)\left(\varsigma_b^{x_2,\,m_2+n_2}+\varsigma_b^{x_2,\,n_2-m_2}\right)\\&=-\frac{\pi^2m\wedge n}{2ab}\varsigma_a^{x_1,\,2(n_1+m_1)}\varsigma_b^{x_2,\,n_2+m_2}+\frac{\pi^2m\vee n}{2ab}\varsigma_a^{x_1,\,2(n_1+m_1)}\varsigma_b^{x_2,\,n_2-m_2}\\&\quad-\frac{\pi^2m\vee n}{2ab}\varsigma_a^{x_1,\,2(n_1-m_1)}\varsigma_b^{x_2,\,n_2+m_2}+\frac{\pi^2m\wedge n}{2ab}\varsigma_a^{x_1,\,2(n_1-m_1)}\varsigma_b^{x_2,\,n_2-m_2};\end{aligned}\tag{26}$$

$$\begin{aligned}
&(\nabla\psi_n^{\varsigma},\, \nabla^\perp\psi_m^{\varkappa})_{\mathbb{R}^2}\\
&= +\tfrac{\pi^2 n_2 m_1}{2ab}\left(\varkappa_a^{x_1,\,2(n_1-m_1)} - \varkappa_a^{x_1,\,2(n_1+m_1)}\right)\left(\varsigma_b^{x_2,\,n_2+m_2} - \varsigma_b^{x_2,\,n_2-,m_2}\right)\\
&\quad + \tfrac{\pi^2 n_1 m_2}{2ab}\left(\varkappa_a^{x_1,\,2(n_1+m_1)} + \varkappa_a^{x_1,\,2(n_1-m_1)}\right)\left(\varsigma_b^{x_2,\,n_2+m_2} + \varsigma_b^{x_2,\,n_2-m_2}\right)\\
&= -\tfrac{\pi^2 m\wedge n}{2ab}\varkappa_a^{x_1,\,2(n_1+m_1)}\varsigma_b^{x_2,\,n_2+m_2} + \tfrac{\pi^2 m\vee n}{2ab}\varkappa_a^{x_1,\,2(n_1+m_1)}\varsigma_b^{x_2,\,n_2-m_2}\\
&\quad + \tfrac{\pi^2 m\vee n}{2ab}\varkappa_a^{x_1,\,2(n_1-m_1)}\varsigma_b^{x_2,\,n_2+m_2} - \tfrac{\pi^2 m\wedge n}{2ab}\varkappa_a^{x_1,\,2(n_1-m_1)}\varsigma_b^{x_2,\,n_2-m_2};
\end{aligned} \tag{27}$$

$$\begin{aligned}
&(\nabla\psi_n^{\varkappa},\, \nabla^\perp\psi_m^{\varkappa})_{\mathbb{R}^2}\\
&= \tfrac{\pi^2 n_2 m_1}{2ab}\left(\varsigma_a^{x_1,\,2(n_1+m_1)} - \varsigma_a^{x_1,\,2(n_1-m_1)}\right)\left(\varsigma_b^{x_2,\,n_2+m_2} - \varsigma_b^{x_2,\,n_2-m_2}\right)\\
&\quad - \tfrac{\pi^2 n_1 m_2}{2ab}\left(\varsigma_a^{x_1,\,2(n_1+m_1)} + \varsigma_a^{x_1,\,2(n_1-m_1)}\right)\left(\varsigma_b^{x_2,\,m_2+n_2} + \varsigma_b^{x_2,\,n_2-m_2}\right)\\
&= \tfrac{\pi^2 m\wedge n}{2ab}\varsigma_a^{x_1,\,2(n_1+m_1)}\varsigma_b^{x_2,\,n_2+m_2} - \tfrac{\pi^2 m\vee n}{2ab}\varsigma_a^{x_1,\,2(n_1+m_1)}\varsigma_b^{x_2,\,n_2-m_2}\\
&\quad - \tfrac{\pi^2 m\vee n}{2ab}\varsigma_a^{x_1,\,2(n_1-m_1)}\varsigma_b^{x_2,\,n_2+m_2} + \tfrac{\pi^2 m\wedge n}{2ab}\varsigma_a^{x_1,\,2(n_1-m_1)}\varsigma_b^{x_2,\,n_2-m_2}.
\end{aligned} \tag{28}$$

Hence from $u = \nabla^\perp\psi_u = \nabla^\perp\Delta^{-1}\nabla^\perp\cdot u$, (25), (26), (27) and (28), we obtain

$$\begin{aligned}
&\mathcal{B}(Y_n^{\varsigma},\, Y_m^{\varsigma})\\
&= \tfrac{\lambda_n-\lambda_m}{\lambda_{m(++)n}}\tfrac{\pi^2 n\wedge m}{2ab}Y^{\varsigma}_{n(++)m} + \tfrac{\lambda_n-\lambda_m}{\lambda_{n(+-)m}}\tfrac{\pi^2 n\vee m}{2ab}\operatorname{sign}(n_2-m_2)Y^{\varsigma}_{n(+-)m}\\
&\quad - \tfrac{\lambda_n-\lambda_m}{\lambda_{n(-+)m}}\tfrac{\pi^2 n\vee m}{2ab}\operatorname{sign}(n_1-m_1)Y^{\varsigma}_{n(-+)m} - \tfrac{\lambda_n-\lambda_m}{\lambda_{m(--)n}}\tfrac{\pi^2 n\wedge m}{2ab}\operatorname{sign}(n_1-m_1)\operatorname{sign}(n_2-m_2)Y^{\varsigma}_{n(--)m};\\
&\mathcal{B}(Y_n^{\varsigma}, Y_m^{\varkappa})\\
&= \tfrac{\lambda_n-\lambda_m}{\lambda_{n(++)m}}\tfrac{\pi^2 n\wedge m}{2ab}Y^{\varkappa}_{n(++)m} + \tfrac{\lambda_n-\lambda_m}{\lambda_{n(+-)m}}\tfrac{\pi^2 n\vee m}{2ab}\operatorname{sign}(n_2-m_2)Y^{\varkappa}_{n(+-)m}\\
&\quad + \tfrac{\lambda_n-\lambda_m}{\lambda_{n(-+)n}}\tfrac{\pi^2 n\vee m}{2ab}Y^{\varkappa}_{n(-+)m} + \tfrac{\lambda_n-\lambda_m}{\lambda_{n(--)m}}\tfrac{\pi^2 n\wedge m}{2ab}\operatorname{sign}(n_2-m_2)Y^{\varkappa}_{n(+-)m};\\
&\mathcal{B}(Y_n^{\varkappa}, Y_m^{\varkappa})\\
&= -\tfrac{\lambda_n-\lambda_m}{\lambda_{n(++)m}}\tfrac{\pi^2 n\wedge m}{2ab}Y^{\varsigma}_{n(++)m} - \tfrac{\lambda_n-\lambda_m}{\lambda_{n(+-)m}}\tfrac{\pi^2 n\vee m}{2ab}\operatorname{sign}(n_2-m_2)Y^{\varsigma}_{n(+-)m}\\
&\quad - \tfrac{\lambda_n-\lambda_m}{\lambda_{n(-+)n}}\tfrac{\pi^2 n\vee m}{2ab}\operatorname{sign}(n_1-m_1)Y^{\varsigma}_{n(-+)m} - \tfrac{\lambda_n-\lambda_m}{\lambda_{n(--)m}}\tfrac{\pi^2 n\wedge m}{2ab}\operatorname{sign}(n_1-m_1)\operatorname{sign}(n_2-m_2)Y^{\varsigma}_{n(--)m};
\end{aligned} \tag{29}$$

where $n(\star_1\star_2)m := (|n_1\star_1 m_1|,\, |n_2\star_2 m_2|) \in \mathbb{N}^2$, with $\{\star_1,\, \star_2\} \in \{-,\, +\}^2$ and for $k_1 \in \mathbb{N}$, $Y^{\varsigma}_{(k_1,\,0)} := Y^{\varkappa}_{(k_1,\,0)} := 0$. Notice that these are expressions similar to that obtained for the case of the Rectangle in [39, equation (6.1)]. Notice also that in [39, Section 2.3] the eigenvalues are negative and here they are positive, this is because in [39] it is considered the usual Laplacian Δ in $(0,\, a)\times(0,\, b)$ and here (cf. the discussion following Equation (2)) we consider the Laplace–de Rham operator $\Delta_\Omega = -\Delta = A$.

From (29), and (24), we conclude that for $\hat n$, $\hat m$, and $\hat l$, given in $\mathbb{N}_0$, we have $\operatorname{card}(\underline{\mathcal{F}}^{\bullet}_{\hat n,\,\hat m}) \le 4$ and $\operatorname{card}(\underline{\mathcal{F}}^{\hat l}_{\hat n,\,\bullet}) \le 4$. That is, Assumption 4.2 holds with $\zeta = 0$. We also see that necessarily $\lambda_{(l_1,\,l_2)} \le \lambda_{(n_1+m_1,\,n_2+m_2)}$; from which we can conclude that $\lambda^{\frac12}_{(l_1,\,l_2)} \le \lambda^{\frac12}_{(n_1,\,n_2)} + \lambda^{\frac12}_{(m_1,\,m_2)}$, that is, Assumption 4.1 holds with $\alpha = \frac12$.

Looking for the value θ in Section 5. We can take $\theta = 0$, because, proceeding as in the case of the Rectangle, in Section 6.4, we obtain

$$|W_k^{\varsigma}|^2_{L^\infty(\Omega,\,\mathbb{R}^2)} \le 4(ab)^{-1}, \quad |W_k^{\varkappa}|^2_{L^\infty(\Omega,\,\mathbb{R}^2)} \le 4(ab)^{-1}.$$

Asymptotic behavior of the (repeated) eigenvalues. Notice that the family $\{\lambda_k \mid k \in N_0^2\} = \{\lambda_k^{\varsigma} \mid k \in \mathbb{N}_0^2\} = \{\lambda_k^{\varkappa} \mid k \in \mathbb{N}_0^2\}$ is a subset of $\{\lambda_k^R \mid k \in \mathbb{N}_0^2\}$ where λ_k^R are the eigenvalues of the Dirichlet Laplacian on the Rectangle $R := (0,\,a) \times (0,\,b)$, in Section 6.4. Hence, ordering the families as $\{\lambda_k \mid k \in \mathbb{N}_0^2\} = \{\tilde\lambda_n \mid n \in \mathbb{N}_0\}$ and $\{\lambda_k^R \mid k \in \mathbb{N}_0^2\} = \{\tilde\lambda_n^R \mid n \in \mathbb{N}_0\}$ such that $\tilde\lambda_n \le \tilde\lambda_{n+1}$ and $\tilde\lambda_n^R \le \tilde\lambda_{n+1}^R$, we can conclude that $(\tilde\lambda_n)_{n\in\mathbb{N}_0}$ is a subsequence of $(\tilde\lambda_n^R)_{n\in\mathbb{N}_0}$. Now we already know that

$$\tilde\lambda_n^R \ge \frac{2\pi}{ab} n,$$

which implies $\tilde\lambda_n \ge \tilde\lambda_n^R \ge \frac{2\pi}{ab} n$ for all $n \in \mathbb{N}_0$. The family $\{\lambda_k \mid k \in \mathbb{N}_0^2\}$ is repeated twice $\lambda_k = \lambda_k^{\varsigma} = \lambda_k^{\varkappa}$ for $k \in \mathbb{N}_0^2$. Then for the ordered families, we can write

$$\tilde\lambda_n^{\varsigma} \ge \tilde\lambda_n^R \ge \frac{2\pi}{ab} n \text{ and } \tilde\lambda_n^{\varkappa} \ge \tilde\lambda_n^R \ge \frac{2\pi}{ab} n \text{ for all } n \in \mathbb{N}_0.$$

Finally the family of eigenvalues $\{\tilde\lambda_n^{\varkappa,\,0} := \lambda^{\varkappa}_{(0,\,n)} \mid n \in \mathbb{N}_0\}$ satisfies

$$\tilde\lambda_n^{\varkappa,\,0} = \frac{\pi^2}{b^2} n^2 \ge \frac{\pi^2}{b^2} n \text{ for all } n \in \mathbb{N}_0.$$

In particular ordering the set $\{\tilde\lambda_n^{\varsigma},\, \tilde\lambda_n^{\varkappa},\, \tilde\lambda_n^{\varkappa,\,0} \mid n \in \mathbb{N}_0\}$, in a nondecreasing way, we obtain the sequence of repeated eigenvalues $(\underline{\lambda}_n)_{n\in\mathbb{N}_0}$ in the case of the Cylinder. Moreover setting $\varrho = \min\{\frac{2\pi}{ab},\, \frac{\pi^2}{b^2}\}$, from Lemma 6.4 below, we can conclude that $\underline{\lambda}_m \ge \varrho_1 m$ for all $m \in \mathbb{N}_0$, for a suitable $\varrho_1 > 0$. Thus, we can take $\rho < \varrho_1$ and $\xi = 1$ in Theorem 4.3.

Conclusion. Taking into account Section 5, we conclude that Theorem 4.3 holds with $\alpha = \frac{1}{2}$ and $s > 0$.

Lemma 6.4. *Let us be given $\varrho > 0$, $\xi \ge 0$, $N \in \mathbb{N}_0$, and N nondecreasing sequences $(\underline{\lambda}_{i,n})_{n\in\mathbb{N}_0}$, $i \in \{1, 2, \dots, N\}$. Let $(\underline{\lambda}_n^*)_{n\in\mathbb{N}_0}$ be the nondecreasing sequence we obtain by collecting all the sequences so that if λ appears μ_i times in the sequence $(\underline{\lambda}_{i,n})_{n\in\mathbb{N}_0}$, then it appears $\sum_{i=1}^{N} \mu_i$ times in the sequence $(\underline{\lambda}_n^*)_{n\in\mathbb{N}_0}$. If for each $i \in \{1, 2, \dots, N\}$ we have $\underline{\lambda}_{i,n} \ge \varrho n^{\xi}$, then there exists $\varrho_1 > 0$ such that $\underline{\lambda}_n^* \ge \varrho_1 n^{\xi}$.*

Proof. We observe that there are at most Nn elements in the set $\{\underline{\lambda}_n^* \mid n \in \mathbb{N}_0\}$ that are not bigger than $\varrho(n+1)^{\xi}$, that is, $\underline{\lambda}_{Nn+1}^* \ge \varrho(n+1)^{\xi}$. Hence, since $\underline{\lambda}_{N(n+1)}^* \ge \underline{\lambda}_{Nn+j}^* \ge \underline{\lambda}_{Nn+1}^*$, for $j \in \{1, 2, \dots, N\}$, we can conclude that for $m \ge N+1$, $\underline{\lambda}_m^* \ge \varrho\lfloor\frac{m+N-1}{N}\rfloor^{\xi}$

where, for a positive real number r, $\lfloor r \rfloor$ stands for the biggest integer below r, that is $\lfloor r \rfloor \in \mathbb{N}$ and $r = \lfloor r \rfloor + r_1$ with $r_1 \in [0, 1)$. In particular, from $\lfloor \frac{m+N-1}{N} \rfloor \geq \frac{m-1}{N} = \frac{m-1}{Nm} m$, we find $\underline{\lambda}_m^* \geq \varrho(\frac{m-1}{Nm})^\xi m^\xi \geq \frac{\varrho}{(N+1)^\xi} m^\xi$, for $m \geq N+1$. So for $\varrho_0 := \min_{n \in \{1,2,\dots,N\}} \{\frac{\underline{\lambda}_n^*}{n^\xi}\}$ and $\varrho_1 := \min\{\frac{\varrho}{(N+1)^\xi}, \varrho_0\}$, we have that $\underline{\lambda}_m \geq \varrho_1 m^\xi$, for all $m \in \mathbb{N}_0$. □

6.6. 3D Rectangle

Let Ω be the three-dimensional Rectangle $\Omega = (0,a) \times (0,b) \times (0,c) \subset \mathbb{R}^3$. On the boundary we impose the Lions boundary conditions, that is, we consider the evolutionary Navier–Stokes equation in $H := \{u \in L^2(\Omega, \mathbb{R}^2) \mid \nabla \cdot u = 0 \text{ and } u \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$, with $V := H \cap H^1(\Omega, \mathbb{R}^2)$ and $\mathrm{D}(A) := V \cap \{u \in H^2(\Omega, \mathbb{R}^2) \mid \operatorname{curl} u = ((\operatorname{curl} u) \cdot \mathbf{n})\mathbf{n} \text{ on } \partial\Omega\}$ (cf. [48, Equation (1.4)]).

We will show that in this case we can take $\alpha = \frac{1}{2}$, $\xi = \frac{2}{3}$, and $\zeta = 0$ in Theorem 4.3, and $\theta = 0$ in Section 5. That is, we can take $s > \frac{1}{4}$, in Theorem 4.3.

It is know that, for $r > 0$,

$$\{\sin(\tfrac{k\pi x_1}{r}) \mid k \in \mathbb{N}_0\} \quad \text{and} \quad \{\cos(\tfrac{k\pi x_1}{r}) \mid k \in \mathbb{N}\}$$

are two complete systems in $L^2((0,r), \mathbb{R})$. Hence defining for $x \in \Omega$ and $k \in \mathbb{N}^3$,

$$\begin{aligned}
\psi_{1,k}(x) &:= \sin(\tfrac{k_1\pi x_1}{a}) \cos(\tfrac{k_2\pi x_2}{b}) \cos(\tfrac{k_3\pi x_3}{c}),\\
\psi_{2,k}(x) &:= \cos(\tfrac{k_1\pi x_1}{a}) \sin(\tfrac{k_2\pi x_2}{b}) \cos(\tfrac{k_3\pi x_3}{c}),\\
\psi_{3,k}(x) &:= \cos(\tfrac{k_1\pi x_1}{a}) \cos(\tfrac{k_2\pi x_2}{b}) \sin(\tfrac{k_3\pi x_3}{c}),
\end{aligned}$$

we have that the systems

$$\{\psi_{1,k}(x) \mid k \in \mathbb{N}^3,\ k_1 \neq 0\},\ \{\psi_{2,k}(x) \mid k \in \mathbb{N}^3,\ k_2 \neq 0\},\ \text{and}\ \{\psi_{3,k}(x) \mid k \in \mathbb{N}^3,\ k_3 \neq 0\}$$

are complete orthogonal basis in $L^2(\Omega, \mathbb{R})$. That is, we can write a vector field $u = (u_1, u_2, u_3) \in L^2(\Omega, \mathbb{R}^3)$ as

$$u_1 = \sum_{k \in \mathbb{N}_0 \times \mathbb{N} \times \mathbb{N}} u_{1,k}\psi_{1,k}, \quad u_2 = \sum_{k \in \mathbb{N} \times \mathbb{N}_0 \times \mathbb{N}} u_{2,k}\psi_{2,k}(x), \quad u_3 = \sum_{k \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}_0} u_{3,k}\psi_{3,k}(x).$$

To simplify the computations we set $u_{i,k} := 0$ if $k_i = 0$. A divergence free vector field satisfies

$$0 = \nabla \cdot u = \sum_{k \in \mathbb{N}} \pi \left(\frac{k_1 u_{1,k}}{a} + \frac{k_2 u_{2,k}}{b} + \frac{k_3 u_{3,k}}{c} \right) \cos(\tfrac{k_1\pi x_1}{a}) \cos(\tfrac{k_2\pi x_2}{b}) \cos(\tfrac{k_3\pi x_3}{c}),$$

that is,

$$(k, u_k)_* := \frac{k_1 u_{1,k}}{a} + \frac{k_2 u_{2,k}}{b} + \frac{k_3 u_{3,k}}{c} = 0.$$

Notice that $(\cdot,\cdot)_*$ defines a scalar product in $\mathbb{R}^3$. It follows that a complete orthonormal system for the divergence free vector fields is given by

$$\left\{ Y^{j(k),k} := \begin{pmatrix} w_1^{j(k),k}\psi_{1,k} \\ w_2^{j(k),k}\psi_{2,k} \\ w_3^{j(k),k}\psi_{3,k} \end{pmatrix} \middle| \, k \in \mathbb{N}^3 \setminus \{(0,0,0)\} \text{ and } w^{j(k),k} \in \{k\}_0^{\perp_*} \right\},$$

where $\{w^{j(k),k} \mid j(k) \in \{1,2\}\} \subset \{k\}_0^{\perp_*} = \{z \in \mathbb{R}^3 \setminus \{(0,0,0)\} \mid (z,k)_* = 0, \text{ and } z_i = 0$ if $k_i = 0\}$ is linearly independent.

Remark 6.5. Notice that we can rescale the $w^{j(k),k}$s so that the functions are normalized. Notice also that $j(k)$ depends on k, namely $\{k\}_0^{\perp_*}$ has two vectors if, and only if, $k \in \mathbb{N}_0^3$, and in the case k has only one vanishing component then $\{k\}_0^{\perp_*}$ has just one element, and the set $\{k\}_0^{\perp_*}$ is empty if k has two vanishing components (in this case $[\psi_{1,k} \quad \psi_{2,k} \quad \psi_{3,k}]^\top$ is a gradient).

Now we check that $Y^{j(k),k} \in \mathrm{D}(A)$. Indeed, that $Y^{j(k),k} \cdot \mathbf{n} = 0$ on the boundary is clear, and for $\operatorname{curl} Y^{j(k),k}$ we find

$$\operatorname{curl} Y^{j(k),k} = -\pi \begin{pmatrix} \left(\frac{k_2}{b} w_3^{j(k),k} - \frac{k_3}{c} w_2^{j(k),k}\right) \cos(\frac{k_1\pi x_1}{a}) \sin(\frac{k_2\pi x_2}{b}) \sin(\frac{k_3\pi x_3}{c}) \\ \left(\frac{k_3}{c} w_1^{j(k),k} - \frac{k_1}{a} w_3^{j(k),k}\right) \sin(\frac{k_1\pi x_1}{a}) \cos(\frac{k_2\pi x_2}{b}) \sin(\frac{k_3\pi x_3}{c}) \\ \left(\frac{k_1}{a} w_2^{j(k),k} - \frac{k_2}{b} w_1^{j(k),k}\right) \sin(\frac{k_1\pi x_1}{a}) \sin(\frac{k_2\pi x_2}{b}) \cos(\frac{k_3\pi x_3}{c}) \end{pmatrix},$$

that is, $\operatorname{curl} Y^{j(k),k}$ is normal to the boundary.

Checking Assumptions 2.1 and 2.2. Again this is standard [43, Section 2.3].

Checking Assumptions 4.1 and 4.2. Starting by writing

$$\begin{aligned} &\left(Y^{j(n),n} \cdot \nabla\right) Y^{j(m),m} \\ &= \pi \begin{pmatrix} w_1^{j(m),m} Y^{j(n),n} \cdot \begin{pmatrix} \frac{m_1}{a} \cos(\frac{m_1\pi x_1}{a}) \cos(\frac{m_2\pi x_2}{b}) \cos(\frac{m_3\pi x_3}{c}) \\ -\frac{m_2}{b} \sin(\frac{m_1\pi x_1}{a}) \sin(\frac{m_2\pi x_2}{b}) \cos(\frac{m_3\pi x_3}{c}) \\ -\frac{m_3}{c} \sin(\frac{m_1\pi x_1}{a}) \cos(\frac{m_2\pi x_2}{b}) \sin(\frac{m_3\pi x_3}{c}) \end{pmatrix} \\ w_2^{j(m),m} Y^{j(n),n} \cdot \begin{pmatrix} -\frac{m_1}{a} \sin(\frac{m_1\pi x_1}{a}) \sin(\frac{m_2\pi x_2}{b}) \cos(\frac{m_3\pi x_3}{c}) \\ \frac{m_2}{b} \cos(\frac{m_1\pi x_1}{a}) \cos(\frac{m_2\pi x_2}{b}) \cos(\frac{m_3\pi x_3}{c}) \\ -\frac{m_3}{c} \cos(\frac{m_1\pi x_1}{a}) \sin(\frac{m_2\pi x_2}{b}) \sin(\frac{m_3\pi x_3}{c}) \end{pmatrix} \\ w_3^{j(m),m} Y^{j(n),n} \cdot \begin{pmatrix} -\frac{m_1}{a} \sin(\frac{m_1\pi x_1}{a}) \cos(\frac{m_2\pi x_2}{b}) \sin(\frac{m_3\pi x_3}{c}) \\ -\frac{m_2}{b} \cos(\frac{m_1\pi x_1}{a}) \sin(\frac{m_2\pi x_2}{b}) \sin(\frac{m_3\pi x_3}{c}) \\ \frac{m_3}{c} \cos(\frac{m_1\pi x_1}{a}) \cos(\frac{m_2\pi x_2}{b}) \cos(\frac{m_3\pi x_3}{c}) \end{pmatrix} \end{pmatrix}, \end{aligned}$$

it is not hard to check that the components of $\left(Y^{j(n),n}\cdot\nabla\right)Y^{j(m),m}+\left(Y^{j(n),n}\cdot\nabla\right)Y^{j(m),m}$ can be written as

$$\left(\left(Y^{j(n),n}\cdot\nabla\right)Y^{j(m),m}+\left(Y^{j(m),m}\cdot\nabla\right)Y^{j(n),n}\right)_1=\sum_{\substack{k=(n(\star_1\star_2\star_3)m)^+\\ \{\star_1,\star_2,\star_3\}\in\{-,+\}}}\beta_{1,k}\psi_{1,k};\tag{30a}$$

$$\left(\left(Y^{j(n),n}\cdot\nabla\right)Y^{j(m),m}+\left(Y^{j(m),m}\cdot\nabla\right)Y^{j(n),n}\right)_2=\sum_{\substack{k=(n(\star_1\star_2\star_3)m)^+\\ \{\star_1,\star_2,\star_3\}\in\{-,+\}}}\beta_{2,k}\psi_{2,k};\tag{30b}$$

$$\left(\left(Y^{j(n),n}\cdot\nabla\right)Y^{j(m),m}+\left(Y^{j(m),m}\cdot\nabla\right)Y^{j(n),n}\right)_3=\sum_{\substack{k=(n(\star_1\star_2\star_3)m)^+\\ \{\star_1,\star_2,\star_3\}\in\{-,+\}}}\beta_{3,k}\psi_{3,k};\tag{30c}$$

for suitable constants $\beta_{i,k}\in\mathbb{R}$. Using the orthogonality of the system $\{\psi_{i,k}\mid k\in\mathbb{N}^3,\ k_i\neq 0\}$ (for a fixed $i\in\{1,2,3\}$), it follows that

$$\begin{aligned}&\left(\mathcal{B}(Y^{j(n),n},Y^{j(m),m}),\,Y^{j(l),l}\right)_{L^2(\Omega,\mathbb{R}^3)}\\&=\left(\left(Y^{j(n),n}\cdot\nabla\right)Y^{j(m),m}+\left(Y^{j(m),m}\cdot\nabla\right)Y^{j(n),n},\,Y^{j(l),l}\right)_{L^2(\Omega,\mathbb{R}^3)}\neq 0\end{aligned}$$

only if

$$l\in\left\{(n(\star_1\star_2\star_3)m)^+\;\middle|\;\{\star_1,\star_2,\star_3\}\subseteq\{-,+\}\right\}=:S_{n,m},$$

from which we conclude that for $\hat{n}$, $\hat{m}$, and $\hat{l}$, given in $\mathbb{N}_0$, we have $\operatorname{card}(\underline{\mathcal{F}}^{\bullet}_{\hat{n},\,\hat{m}})\leq\operatorname{card}(S_{n,m})\leq 8\max_{z\in S_{n,m}}\{\dim(\{z\}_0^{\perp_*})\}\leq 16$, and similarly we can conclude that $\operatorname{card}(\underline{\mathcal{F}}^{\hat{l}}_{\hat{n},\,\bullet})\leq 16$. That is, Assumption 4.2 holds with $\zeta=0$. The eigenvalue associated to $Y^{j(k),k}$ is given by $\lambda_k=\pi^2\left(\frac{k_1^2}{a^2}+\frac{k_2^2}{b^2}+\frac{k_3^2}{c^2}\right)$ from which we can conclude that Assumption 4.1 holds with $\alpha=\frac{1}{2}$.

Looking for the value θ in Section 5. We can find that

$$\begin{aligned}|Y^{j(k),k}|^2_{L^\infty(\Omega,\mathbb{R}^3)}\leq|w^{j(k),k}|^2&\leq\max_{i\in\{1,2,3\}}|\psi_{i,k}|^{-2}_{L^2(\Omega,\mathbb{R}^3)}\sum_{i=1}^{3}(w_i^{j(k),k})^2|\psi_{i,k}|^2_{L^2(\Omega,\mathbb{R}^3)}\\&=\left(\tfrac{8}{abc}\right)|Y^{j(k),k}|^2_{L^2(\Omega,\mathbb{R}^3)}=8(abc)^{-1},\end{aligned}$$

that is, we can take $\theta=0$.

Asymptotic behavior of the (repeated) eigenvalues. Let us order the repeated eigenvalues $\lambda^{j(k),k}$, corresponding to the family of eigenfunctions $W_n=Y^{j(k),k}$, arriving to the sequence $\underline{\lambda}_n$. Notice that it is clear that $Y^{j(k),k}$ is an eigenfunction of the Lions Laplacian, and since it is in $\mathrm{D}(A)$ it is also an eigenfunction of the Lions Stokes operator A. Now, let $\underline{\lambda}_n^0$ be the sequence of the (repeated) eigenvalues of the (scalar) Dirichlet Laplacian $-\Delta$ in the rectangle Ω. From [32, Corollary 1] we have that

$$\underline{\lambda}_n^0 \geq Cn^{\frac{2}{3}}, \quad \text{for all} \quad n \in \mathbb{N}_0.$$

Recall that $\underline{\lambda}_n^0 = \pi^2\left(\frac{k_1^2}{a^2}+\frac{k_2^2}{b^2}+\frac{k_3^2}{c^2}\right)$ for some $k \in \mathbb{N}_0^3$. Repeating each eigenvalue $\underline{\lambda}_n^0$ twice and reordering we obtain a sequence we denote by $\underline{\lambda}_n^{0,2}$. It follows that $\underline{\lambda}_{2n}^{0,2} \geq Cn^{\frac{2}{3}}$ and $\underline{\lambda}_{2n-1}^{0,2} \geq Cn^{\frac{2}{3}}$, from which we can conclude that $\underline{\lambda}_n^{0,2} \geq \frac{C}{2^{\frac{2}{3}}}n^{\frac{2}{3}}$ if n is even and $\underline{\lambda}_n^{0,2} \geq \frac{C}{2^{\frac{2}{3}}}(n+1)^{\frac{2}{3}}$ if n is odd. Hence

$$\underline{\lambda}_n^{0,2} \geq \frac{C}{2^{\frac{2}{3}}}n^{\frac{2}{3}}, \quad \text{for all} \quad n \in \mathbb{N}_0.$$

Let us add to $\lambda_n^{0,2}$ the families

$$\left\{ \lambda_k^{i,1} = \pi^2\left(\frac{k_1^2}{a^2}+\frac{k_2^2}{b^2}+\frac{k_3^2}{c^2}\right) \,\middle|\, k \in \mathbb{N}_0^3,\ k_i = 0 \right\}, \quad i \in \{1,2,3\}$$

and denote by $\underline{\lambda}_n^*$ the family we obtain after reordering.

Reordering the family $\{\lambda_k^{i,1}\}$, for each $i \in \{1,2,3\}$, we obtain a family $\{\underline{\lambda}_n^{i,1}\}$, which from [32, Corollary 1] (for a $2D$ Rectangle) satisfies

$$\underline{\lambda}_n^{i,1} \geq C_1 n^{\frac{2}{2}} \geq C_1 n^{\frac{2}{3}}, \quad \text{for all} \quad n \in \mathbb{N}_0, \quad \text{and for fixed} \quad i \in \{1,2,3\}.$$

Observe that the repeated eigenvalues $\underline{\lambda}_n$ associated to the eigenfunctions $Y^{j(k),k}$ is a subsequence of the ordered collection $\{\underline{\lambda}_n^*\} = \{\underline{\lambda}_n^{0,2}, \underline{\lambda}_n^{i,1}\}$, therefore we have $\underline{\lambda}_n \geq \underline{\lambda}_n^*$, and from Lemma 6.4 (with $\varrho = \min\{\frac{C}{2^{\frac{2}{3}}}, C_1\}$) it follows that $\underline{\lambda}_n \geq \varrho_1 n^{\frac{2}{3}}$, for a suitable $\varrho_1 > 0$. Thus, we can take $\rho < \varrho_1$ and $\xi = \frac{2}{3}$ in Theorem 4.3.

Conclusion. Taking into account Section 5, we conclude that Theorem 4.3 holds with $\alpha = \frac{1}{2}$ and $s > \frac{1}{4}$.

Remark 6.6. As we have seen, we did not need to know explicitly $\mathcal{B}(Y^{j(n),n}, Y^{j(m),m})$ to check the Assumption 4.2. Though the exact expression can be found by direct computations (by finding explicitly the $\beta_{i,k}$s in (30) and then projection onto H), since it is quite long and not needed, we do not present it here.

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