C. R. Acad. Sci. Paris, t. 330, Série I, p. 1007–1011, 2000
Contrôle optimal/*Optimal Control*
(Problèmes mathématiques de la mécanique/*Mathematical Problems in Mechanics*)

Existence and uniqueness of optimal control to the Navier–Stokes equations

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(Reçu le 14 janvier 2000, accepté après révision le 27 avril 2000)

Abstract. In this Note we establish the existence and uniqueness of solutions for optimal control problems for the 2D Navier–Stokes equations in a 2D domain. Our approach is based on infinite-dimensional optimization; the cost functional is shown to be strictly convex. Generalization to other control problems as well as a gradient algorithm are presented. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Existence et unicité du contrôle optimal des équations de Navier–Stokes

Résumé. *Dans cette Note, nous démontrons l'existence et l'unicité de solutions pour des problèmes de contrôle optimal pour les équations de Navier–Stokes dans un ouvert en dimension deux. Notre approche est fondée sur l'optimisation en dimension infinie ; nous montrons que la fonctionnelle du coût est strictement convexe. Quelques généralisations à d'autres problèmes de contrôle ainsi qu'un algorithme de gradient sont aussi présentés.* © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Dans cette Note, nous étudions le problème du contrôle optimal des équations de Navier–Stokes dans un canal bidimensionnel Ω. Nous nous plaçons dans le cadre de l'optimisation en dimension infinie et nous montrons que la fonctionnelle coût est strictement convexe. Les équations de Navier–Stokes sont écrites sous la forme abstraite :

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au + B(u) = E\Phi, \quad u(0) = u_0,$$

où Φ est le paramètre de contrôle supposé appartenir à un sous-ensemble $\mathcal{X}$ de $\mathrm{L}^2\big(0, T, (\mathrm{L}^2(\Omega))^2\big)$ qui est fermé, borné et convexe. La fonctionnelle de coût est donnée par :

$$J(\Phi) = \frac{1}{2}\int_0^T |\mathcal{C}_1 u|^2\,\mathrm{d}t - \int_0^T\!\!\int_{\partial\Omega}\left(\mathcal{C}_2\nu\,\frac{\partial u}{\partial n}\right)\cdot\vec{r}\,\mathrm{d}\Gamma\,\mathrm{d}t + \frac{\alpha^2}{2}\int_0^T |\Phi|^2\,\mathrm{d}t.$$

Note présentée par Philippe G. CIARLET.

S0764-4442(00)00299-8/FLA

Nous renvoyons le lecteur à la version anglaise pour les notations utilisées, mais nous précisons ici que le nombre α^2 mesure le prix du contrôle, α petit correspondant à un contrôle peu onéreux, α grand correspondant à un contrôle coûteux.

Avant de donner le résultat principal de la Note, nous rappelons que P_m est le projecteur orthogonal sur l'espace engendré par les m premières valeurs propres de l'opérateur de Stokes A et que $Q_m = I - P_m$.

THÉORÈME 1. – *Supposons que $\mathcal{X}$ est un sous-ensemble non vide, fermé, borné et convexe de* $\mathrm{L}^2\big(0,T,(\mathrm{L}^2(\Omega))^2\big)$*. Alors il existe $\alpha_0 = \alpha_0(\mathcal{X},\|u_0\|)$, tel que la fonctionnelle J soit strictement convexe pour $\alpha \geqslant \alpha_0$. De plus, pour $\alpha \geqslant \alpha_0$, il existe une solution unique $\overline{\Phi} \in \mathcal{X}$ et une solution des équations de Navier–Stokes associée $\overline{u}(\overline{\Phi})$ telles que*

$$J(\overline{\Phi}) \leqslant J(\Phi), \quad \forall\, \Phi \in \mathcal{X}.$$

Dans le cas où $\mathcal{C}_1 = Q_m$ et $\mathcal{C}_2 = 0$, pour tout $\alpha > 0$, il existe $m_0 > 0$, $m_0 = m_0(\mathcal{X},\alpha,\|u_0\|)$ tel que les conclusions ci-dessus soient vraies pour $m \geqslant m_0$.

Remarque 1. – L'approche que nous avons suivie s'applique à beaucoup d'autres problèmes de contrôle avec des équations d'état dissipatives non linéaires. Voir aussi ci-dessous le problème de contrôle robuste des équations de Navier–Stokes. D'autres conditions aux limites peuvent être étudiées de la même façon, ainsi que les équations de Navier–Stokes en trois dimensions avec une condition classique de données petites ou de temps petit.

1. Introduction

In this Note we study the optimal and robust control of the Navier–Stokes equations in a 2D channel Ω. We set the problem as an infinite dimensional optimization problem where the cost functional to be minimized describes the features of the flow upon which the control is applied. We establish the existence and uniqueness of the optimal control by showing that the functional is strictly convex. We also propose and study a gradient algorithm for computing the optimal control.

We write the Navier–Stokes equations in an abstract form (*see*, e.g., [8,10]):

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au + B(u,u) = E\Phi, \quad u(0) = u_0, \tag{1}$$

where $A = -P\Delta$ is the Stokes operator, and $B(u,v)$ is the bilinear operator $P[(u\cdot\nabla)v]$, P denoting the Leray projector from $(\mathrm{L}^2(\Omega))^2$ onto the space $H = \big\{u \in (\mathrm{L}^2(\Omega))^2;\ \operatorname{div} u = 0,\ u\cdot\vec{n} = 0 \text{ on } \partial\Omega\big\}$, $\vec{n}$ the unit outward normal vector on $\partial\Omega$. We shall also denote by V the space $\{u \in H_0^1(\Omega);\ \operatorname{div} u = 0\}$. The norm and the scalar product of the Hilbert space H and of all spaces L^2 are denoted by $|\cdot|$, $(\cdot,\cdot)$ and the norm of V is denoted by $\|\cdot\|$; E is a bounded operator from $\mathrm{L}^2\big(0,T,(\mathrm{L}^2(\Omega))^2\big)$ into $\mathrm{L}^2(0,T,H)$.

The solution of (1) is denoted by $u(\Phi)$. For $\Phi \in \mathrm{L}^2\big(0,T,(\mathrm{L}^2(\Omega))^2\big)$ and $u_0 \in V$, it is well known that there exists a unique $u(\Phi)$, solution of (1) such that $u(\Phi) \in \mathrm{C}([0,T];V) \cap \mathrm{L}^2(0,T,D(A))$ (*see* [8,10] for instance); so the functional J given below is well-defined. Throughout this note, we assume that the control parameter Φ belongs to a non-empty bounded, closed, convex subset $\mathcal{X}$ of $\mathrm{L}^2\big(0,T,(\mathrm{L}^2(\Omega))^2\big)$. The cost functional J is given by:

$$J(\Phi) = \frac{1}{2}\int_0^T |\mathcal{C}_1 u|^2\,\mathrm{d}t - \int_0^T\!\!\int_{\partial\Omega}\left(\mathcal{C}_2\nu\,\frac{\partial u}{\partial n}\right)\cdot\vec{r}\,\mathrm{d}\Gamma\,\mathrm{d}t + \frac{\alpha^2}{2}\int_0^T |\Phi|^2\,\mathrm{d}t.$$

Note that the dependence of the functional J on the flow $u(\Phi)$ is treated in a fairly general form. For instance $\mathcal{C}_1 = d_1 I_H$ and $\mathcal{C}_2 = 0$ represents the regulation of kinetic energy; $\mathcal{C}_1 = d_2$ Curl and $\mathcal{C}_2 = 0$ represents the regulation of the square of the vorticity; $\mathcal{C}_1 = Q_m = I - P_m$, $m \geqslant 1$, P_m being the orthogonal projection onto the vector space spanned by the first m-eigenvectors of the Stokes operator A, and $\mathcal{C}_2 = 0$ represents the minimization of the high frequency modes of the velocity field u (as in [2]), and finally $\mathcal{C}_1 = 0$ and $\mathcal{C}_2 = d_3 I_{\mathbb{R}^2}$ represents the minimization of the time-average skin friction in the direction $\vec{r} \in \mathbb{R}^2$ integrated over the boundary. The parameter α^2 may be interpreted as the measure of the price of the control, for α small the control is cheap, and for α large, it is expensive.

The main result is given by:

THEOREM 1. – *Assume that $\mathcal{X}$ is a non-empty, closed bounded convex subset of $\mathrm{L}^2\big(0,T,(\mathrm{L}^2(\Omega))^2\big)$ and that u_0 is given in V. Then there exists $\alpha_0 = \alpha_0(\mathcal{X}, \|u_0\|)$ such that for $a \geqslant \alpha_0$ the functional J is strictly convex. Moreover, for $\alpha \geqslant \alpha_0$, there exists a unique optimal control $\overline{\Phi} \in \mathcal{X}$ and an associated $\overline{u}(\overline{\Phi})$ such that*

$$J(\overline{\Phi}) \leqslant J(\Phi), \quad \forall\, \Phi \in \mathcal{X}.$$

Also, if $\mathcal{C}_2 = 0$ and $\mathcal{C}_1 = Q_m$ then, for all $\alpha > 0$, there exists $m_0 = m_0(\mathcal{X}, \alpha, \|u_0\|)$ such that for $m \geqslant m_0$ the conclusions above hold true.

2. Proof of Theorem 1

The proof is based on the following lemma:

LEMMA 1. – *Let u be the unique solution of* (1) *with $u(0) = u_0 \in V$, $T > 0$ and $\Phi \in \mathcal{X}$. The mapping $\Phi \mapsto u(\Phi)$ has a Gâteaux derivative $(\mathcal{D}u/\mathcal{D}\Phi)(\Phi') = u'(\Phi')$ in every direction $\Phi' \in \mathrm{L}^2\big(0,T,(\mathrm{L}^2(\Omega))^2\big)$. The Gâteaux derivative $u'(\Phi')$ is the unique solution of the linear evolution equation*:

$$\begin{cases} \dfrac{\mathrm{d}u'}{\mathrm{d}t} + \nu A u' + B'(u,u)u' = E\Phi', \\ u'(0) = 0, \\ u' \in \mathrm{L}^\infty(0,T;\, V) \cap \mathrm{L}^2(0,T;\, D(A)), \end{cases} \tag{2}$$

where $B'(u)v = B(u,v) + B(v,u)$. Furthermore, there exists $\alpha_0 = \alpha_0(\|u_0\|, \nu, T, \mathcal{X})$ such that, for $\alpha \geqslant \alpha_0$, the mapping $\Phi \mapsto J(\Phi)$ is strictly convex and lower semicontinuous.

Proof. – The existence of the Gâteaux derivative as well as its characterization are obtained as in Abergel and Temam [1]. In order to prove the convexity of J, we show that the function $h(\rho) = J(\Phi + \rho\Phi')$ is strictly convex near $\rho = 0$. First, we note that the derivative $h'(\rho)$ of h reads:

$$h'(\rho) = \int_0^T \big(\mathcal{C}_1 u, \mathcal{C}_1 u'\big)\,\mathrm{d}t - \int_0^T \left(\mathcal{C}_2 \nu \frac{\partial u'}{\partial n}, \vec{r}\right)\mathrm{d}t + \alpha^2 \int_9^T \big(E\big(\Phi + \rho\Phi'\big), E\Phi'\big)\,\mathrm{d}t,$$

where u is the solution of (1) corresponding to Φ and u' is the solution of (2). Furthermore, let $u'' = (\mathcal{D}^2 u/\mathcal{D}\Phi^2) \cdot (\Phi', \widehat{\Phi}')$ be the second derivative of u with respect to Φ in the directions Φ' and $\widehat{\Phi}'$, we can show that:

$$\begin{cases} \dfrac{\mathrm{d}u''}{\mathrm{d}t} + \nu A u'' + B(u,u'') + B(u'',u) = -B(\widehat{u}', u') - B(u', \widehat{u}'), \\ u''(0) = 0; \end{cases}$$

where $u' = (\mathcal{D}u/\mathcal{D}\Phi) \cdot (\Phi')$ and $\widehat{u}' = (\mathcal{D}u/\mathcal{D}\Phi) \cdot (\widehat{\Phi}')$.

Now we write

$$h''(\rho) = \int_0^T \big|\mathcal{C}_1 u''\big|^2\,\mathrm{d}t + \int_0^T \big(\mathcal{C}_1 u, \mathcal{C}_1 u''\big)\,\mathrm{d}t - \int_0^T \left(\mathcal{C}_2 \nu \frac{\partial u''}{\partial n}, \vec{r}\right)\mathrm{d}t + \alpha^2 \int_0^T \big|E\Phi'\big|^2\,\mathrm{d}t.$$

Through lengthy calculations described in [6], we can derive a priori estimates on u, u' and u'', which depend on the data ν, R, Ω, the norm of u_0 in V and the diameter of $\mathcal{X}$ in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)^2)$. Using these estimates, we can show that, for α large,

$$h''(0) \geqslant \beta\left|\Phi'\right|^2_{\mathrm{L}^2(0,T,(\mathrm{L}^2(\Omega))^2)} \quad \text{for all } \Phi' \in \mathrm{L}^2\big(0,T,\mathrm{L}^2(\Omega)\big)^2$$

with β a positive constant depending on these data. The proof of the lemma is complete. $\square$

The strict convexity of J and the assumptions on the set $\mathcal{X}$ imply the existence and uniqueness of the minimum of the functional J and hence the optimal control.

In the case $\mathcal{C}_1 = Q_m$ and $\mathcal{C}_2 = 0$ the proof of Lemma 1 applies with the use of

$$|\mathcal{C}_1 u| = |Q_m u| \leqslant \frac{1}{\lambda_m^{1/2}}\|u\|,$$

where λ_m is the m^{th} eigenvalue of the Stokes operator, so that the condition α large may be replaced with λ_m large, i.e., m large, since $\lim\limits_{m\to\infty} \lambda_m = \infty$.

Remark 1. – The approach of this Note applies to many nonlinear control problems for which the cost functional includes a parameter (the price of the control) that can make it strictly convex. For the Navier–Stokes equations themselves we can also consider different boundary conditions such as the channel flow and the dimension three case; in the later case we need the usual assumptions of small initial data or small time.

Remark 2. – The gradient algorithm which is recalled below converges globally to the unique solution of the optimal control problem.

3. Identification of the gradient and the gradient algorithm

The main result of this section is the following:

THEOREM 2. – *For sufficiently large α, the gradient of the cost functional $J(\Phi)$ is given by*:

$$\frac{\mathrm{D}J}{\mathrm{D}\Phi}(\Phi) = E^*\widetilde{u} + \alpha^2\Phi,$$

where $\widetilde{u}$ is found from the solution $(u,\widetilde{u})$ of the coupled system:

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu Au + B(u,u) = E\Phi,$$

$$-\frac{\mathrm{d}\widetilde{u}}{\mathrm{d}t} + \nu A^*\widetilde{u} + B'(u)^*\widetilde{u} = \mathcal{C}_1^*\mathcal{C}_1 u, \tag{3}$$

$$u \in V, \quad \widetilde{u}(t) \in V_r = \big\{v \in (H^1(\Omega))^2;\ \operatorname{div} u = 0,\ v = \mathcal{C}_2^* r \textit{ on } \partial\Omega\big\}, \tag{4}$$

$$u(0) = u_0, \quad \widetilde{u}(T) = 0.$$

Proof. – The lemma follows as in [1] and [5].

The gradient algorithm. – We start with $\Phi^0 = 0$ on $[0,T]$.
Iteration k:
(i) find u^k from (3) with $\Phi = \Phi^k$ and initial condition u_0;
(ii) find $\widetilde{u}^k$ from (4) with $u = u^k$;
(iii) write $\Phi^{k+1} = \Phi^k - \beta_k \frac{\mathrm{D}J}{\mathrm{D}\Phi}(\Phi^k)$, where $0 < C_1 \leqslant \beta_k \leqslant C_2 < 1$.
It can be shown that these sequences converge as $k \to \infty$. For a numerical implementation of this algorithm (*see* [3] and [4]).

4. Robust control problem

The cost functional contains an additional term due to disturbances and it is given by:

$$J(\Psi,\Phi)=\frac{1}{2}\int_0^T|\mathcal{C}_1u|^2\,\mathrm{d}t-\int_0^T\!\!\int_{\partial\Omega}\left(\mathcal{C}_2\nu\frac{\partial u}{\partial n}\right)\cdot\vec{r}\,\mathrm{d}\Gamma\,\mathrm{d}t+\frac{1}{2}\int_0^T\!\!\int_{\Omega}\left[\alpha^2|\Phi|^2-\gamma^2|\Psi|^2\right]\mathrm{d}t.$$

Here u is the unique solution of the Navier–Stokes equations written in an abstract form

$$\frac{\mathrm{d}u}{\mathrm{d}t}+Au+B(u,u)=E_1\Phi+E_2\Psi.$$

As in the case of optimal control, we assume that

$$\Phi\in\mathcal{X}\subset\mathrm{L}^2\big(0,T,(\mathrm{L}^2(\Omega)^2)\big)\quad\text{and}\quad\Psi\in\mathcal{Y}\subset\mathrm{L}^2\big(0,T,(\mathrm{L}^2(\Omega))^2\big).$$

The main result is as follows (*see* [5] for details).

THEOREM 3. – *Assume that $\mathcal{X}$ and $\mathcal{Y}$ are non-empty, closed, bounded convex subsets of $\mathrm{L}^2\big(0,T,(\mathrm{L}^2(\Omega))^2\big)$. Then there exist $\gamma_0=\gamma_0(\mathcal{X},\mathcal{Y},\|u_0\|)$ and $\alpha_0=\alpha_0(\mathcal{X},\mathcal{Y},\|u_0\|)$ such that for $\gamma\geqslant\gamma_0$ and $\alpha\geqslant\alpha_0$, $\Psi\mapsto J(\Psi,\Phi)$ is strictly concave upper semicontinuous and $\Phi\mapsto J(\Psi,\Phi)$ is strictly convex lower semicontinuous. For $\gamma\geqslant\gamma_0$ and $\alpha\geqslant\alpha_0$, there exists moreover a unique saddle point $(\overline{\Psi},\overline{\Phi})$ on $\mathcal{X}\times\mathcal{Y}$ and an associated $\overline{u}=u(\overline{\Psi},\overline{\Phi})$ such that*

$$J\big(\Psi,\overline{\Phi}\big)\leqslant J\big(\overline{\Psi},\overline{\Phi}\big)\leqslant J\big(\overline{\Psi},\Phi\big),\quad\forall\,(\Psi,\Phi)\in\mathcal{Y}\times\mathcal{X}.$$

Proof. – We use a priori estimates and an existence result of a saddle point in infinite dimension (as, e.g., in [7], Chapter 7). For more details we refer to [5].

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