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# Regularity Theory for Mean-Field Game Systems

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# Regularity Theory for Mean-Field Game Systems

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# Preface

This book brings together several recent developments on the regularity theory for mean-field game systems. We detail several classes of methods and present a concise overview of the main techniques developed in the last few years. Most of the forthcoming material deals with simple and computation-friendly examples; this is intended to unveil the main ideas behind the methods rather than focus on the technicalities of particular cases.

The choice of topics presented here reflects the authors' perspective on this fast-growing field of research; it is by no means exhaustive or intended as a complete account of the theory. Rather—and in the best scenario—it serves as an introduction to the material available in scientific papers.

## Book Outline

Mean-field games comprise a wide range of models with distinct properties. Accordingly, no single method addresses existence or regularity issues in all cases. In a restricted number of problems, existence questions on MFGs can be settled through explicit solutions or special transformations. Some of these explicit methods are presented in Chap. 2. Explicit solutions are also essential for the continuation arguments in Chap. 11.

When explicit solutions cannot be found, fixed-point methods, regularization techniques, and continuation arguments provide systematic tools to study the existence of solutions. Usually, a priori bounds are a key ingredient in existence proofs. These bounds are estimates for the size of solutions that are derived before the solution is known to exist. Then, it is often possible to show the existence of the solution. Unless otherwise stated, we work with classical (i.e., $C^\infty$ or at least regular-enough solutions).

We begin our study of a priori bounds for MFGs in Chap. 3, where we examine the Hamilton–Jacobi equation. There, some of the estimates rely only on the optimal control interpretation (see Sect. 3.2) or parabolic regularization effects (see Sect. 3.5). In contrast, other results (see Sect. 3.3 or 3.4) illustrate a subtle interplay between these two mechanisms.

In Chap. 4, we consider transport and Fokker–Planck equations. Both equations preserve mass and positivity. However, the Fokker–Planck equation enjoys strong regularizing properties that we investigate in detail. The chapter ends with a brief discussion of relative entropy inequalities and weak solutions.

A recent development in the theory of solutions of Hamilton–Jacobi equations is the nonlinear adjoint method introduced by L.C. Evans. This method relies on coupling a Hamilton–Jacobi equation with a Fokker–Planck equation. This system resembles (1.1) with $F = 0$. In Chap. 5, we develop the main techniques of this method. The nonlinear adjoint method gives bounds for Hamilton–Jacobi equations that go beyond maximum principle methods. These bounds are obtained by careful integration techniques. In addition to bounds relevant to MFGs, to illustrate the method, we prove semiconcavity estimates and consider the vanishing viscosity problem.

Next, in Chap. 6, we develop techniques that are specific to mean-field games and that combine both equations. These bounds together with the estimates for the Hamilton–Jacobi equation or the Fokker–Planck equation improve earlier results.

Chapter 7 is devoted to stationary models. There, we develop a priori estimates for three different problems. First, we consider MFGs with polynomial dependence on $m$. To get Sobolev regularity, we combine the integral Bernstein estimate in Chap. 3 with the first-order estimates in Chap. 6. Next, we investigate two MFGs with singularities: the congestion problem and the logarithmic nonlinearity.

In Chaps. 8 and 9, we explore time-dependent MFGs. In the first of these two chapters, we consider models without singularities and illustrate two regularity regimes. The first regime corresponds to subquadratic Hamiltonians. In this case, the main tool is the Gagliardo–Nirenberg estimate discussed in Chap. 3. The second regime corresponds to quadratic and superquadratic Hamiltonians. For these, we get the regularity using the nonlinear adjoint method from Chap. 5. Time-dependent MFGs with singularities present substantial challenges and are examined in Chap. 9. There, we investigate logarithmic nonlinearities in the subquadratic setting and the short-time congestion problem.

Chapters 10 and 11 examine MFGs in the nonlocal and local cases, respectively. We use fixed-point methods to get the existence of solutions for nonlocal problems in both first-order and second-order cases. Besides their independent interest, nonlocal MFGs are used later to study local problems through a regularization procedure. Next, in Chap. 11, we present two techniques to prove the existence of solutions to MFGs. First, we discuss the regularization method. Then, we examine continuation arguments for both stationary and time-dependent problems.

## Thanks

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## Bibliographical Notes

Mean-field games were introduced independently and around the same time in the engineering community in [142, 143] and in the mathematics community in [164–167]. Many mathematical aspects of the theory were developed in [174], a course taught by Lions, and several of the techniques in this book can be traced to ideas outlined there.

Before the introduction of MFGs, systems combining a Hamilton–Jacobi equation with a Fokker–Planck or transport equation that resemble MFGs were considered in various settings. For example, the PDE approach to the Aubry–Mather theory [93–95], the problems in [89, 90], and the Benamou–Brenier formulation of the optimal transport problem [29] are forerunners of MFGs. The entropy-penalized scheme in [122] can be reinterpreted as a discrete-time mean-field game.

The goal of this book is to develop the regularity theory for MFGs. These problems have been investigated intensively in the last few years, and we give detailed references at the end of each chapter on the different models and problems. Due to space and time constraints, we cannot discuss the numerous applications of MFGs in engineering and in economics and the many recent results on stochastic methods, numerical analysis, and other MFG models. To make up for these omissions, next, we give a brief bibliography and refer the reader to the books and surveys [30, 61, 121, 133] for more material and references. Also, here, we do not develop the theory of weak solutions to MFGs and instead refer the reader to the following papers [62–64, 68, 98, 195, 196]. Furthermore, we do not discuss numerical methods for MFGs here; for that, see, for example, [1–4, 54, 70, 138–140, 160].

In the engineering community, emerging research includes power grids and energy management [14, 14, 148–150, 179], adaptive control [147, 184] and risk-sensitive or robust control [85, 86, 208, 210], robust MFGs [26, 209], learning [214], and networks [141], among several others [144, 206, 207]. Traffic and crowd models are an important and natural area of application of MFGs [42, 44, 45, 76, 83], as well as related problems on networks and graphs [27, 56, 57].

Some of the first MFG models were motivated by economic growth [167–169]. Subsequently, various problems in economics and finance have been considered in the literature, including socioeconomic models [37, 145, 185], inspection and corruption [153, 156] systemic risk [103], price formation [41, 46, 48, 49, 178], social dynamics [24], consensus [39, 185, 186, 198], and opinion dynamics [23, 36, 201, 202]. In the context of heterogeneous agent models (see [159]) with rational expectations (see [175]), MFGs became a popular modeling tool [5, 6, 176, 187]. An earlier model that predates the emergence of MFGs is the Aiyagari–Bewley–Huggett model [7, 38, 146]. A recent book [133] describes several MFG models in mathematical economics.

MFGs where the agents are subjected to correlated random forces were studied by stochastic methods in [71, 72, 74, 75, 154, 161–163]. The master equation was used to study problems with correlations in [33, 73] and deterministic problems in [105, 124]. An important tool in the study of the $N$ player limit with or without correlations is the theory of nonlinear Markov processes [152]. Some applications of these methods were developed in [155, 157, 158]. Finally, minimax methods were considered in [11–13].

Several authors considered extensions of the original MFG framework. These include finite state mean-field games [21, 99, 113, 126, 128, 131, 132], problems with major and minor agents [183], multi-population models [77, 79, 97], extended MFGs [124, 129, 213], logistic population effects [120], problems with density constraints [180, 199], and obstacle-type problems [116].

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# Contents

# Chapter 1
# Introduction

Lasry and Lions and, more or less simultaneously, Caines, Huang and Malhamé introduced a class of models called mean-field games (MFGs) to study systems with large numbers of identical agents in competition. In these games, the agents are rational and seek to optimize a value function by selecting appropriate controls. The interactions between them are determined by a mean-field coupling that aggregates their individual contributions. Many of these models are formed by a Hamilton–Jacobi equation coupled with a Fokker–Planck equation.

Hamilton–Jacobi and Fokker–Planck equations have been the subject of extensive research. Yet, in MFGs, the coupling between these two equations leads to non-trivial existence, regularity, and uniqueness questions. Here, we focus on the regularity theory for MFGs. For pedagogical reasons, we illustrate our methods with elementary examples. These include the two systems of partial differential equations (PDEs) described next and closely related examples.

We consider a large population of agents. The state of each of them is given by a point $x \in \mathbb{R}^d$ or, in the periodic setting, by a point $x \in \mathbb{T}^d$, where $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is the standard $d$-dimensional torus. We denote by $\mathcal{P}(\mathbb{R}^d)$ (or $\mathcal{P}(\mathbb{T}^d)$) the set of Borel probability measures on $\mathbb{R}^d$ (resp. $\mathbb{T}^d$). The statistical distribution of the agents is described by a probability measure, $m \in \mathcal{P}(\mathbb{R}^d)$ (or $\mathcal{P}(\mathbb{T}^d)$). Each agent has spatial preferences that are determined by a $C^\infty$ function, $V : \mathbb{R}^d \to \mathbb{R}$ (or $V : \mathbb{T}^d \to \mathbb{R}$). Next, we fix a real-valued function, $F$, that encodes the interactions between each agent and the mean field. The domain of $F$ is either $\mathbb{R}^d \times \mathbb{R}^+$ (resp. $\mathbb{T}^d \times \mathbb{R}^+$), the local case, or $\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$ (resp. $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$), the non-local case. In the present discussion, we consider the non-local case. We assume that $F$ is continuous (with respect to the weak convergence in $\mathcal{P}(\mathbb{R}^d)$ or $\mathcal{P}(\mathbb{T}^d)$).

Let $\epsilon \geq 0$. The workhorse of MFG theory is the following system:

$$\begin{cases} -u_t + \frac{|Du|^2}{2} + V(x) = \epsilon\, \Delta u + F(x,m), \\ m_t - \operatorname{div}(mDu) = \epsilon\, \Delta m, \end{cases} \tag{1.1}$$

 
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with initial and terminal conditions

$$\begin{cases} u(x,T) = u_T(x) \\ m(x,0) = m_0(x). \end{cases} \tag{1.2}$$

Here, $m_0$ and $u_T$ are given functions, $m_0 \geq 0$ with $\int_{\mathbb{R}^d} m_0 dx = 1$. Our main goal is to show the existence of solutions, $u, m : \mathbb{R}^d \times [0,T] \to \mathbb{R}$, with $m \geq 0$. Because $m$ is a probability density at $t = 0$, it remains a probability for all positive times (see Sect. 4.1). To avoid technical difficulties, it is common to work with periodic boundary conditions. In this case, the domain of $u$ and $m$ is $\mathbb{T}^d \times [0,T]$. In more general problems, the terminal condition, $u_T$, may depend upon $m(\cdot,T)$, i.e., it has the form $u_T(x, m(\cdot,T))$.

The corresponding stationary MFG is given by

$$\begin{cases} \frac{|Du|^2}{2} + V(x) = \epsilon\, \Delta u + F(x,m) + \overline{H}, \\ -\operatorname{div}(mDu) = \epsilon \Delta m, \end{cases} \tag{1.3}$$

and the solution is a triplet $(u, m, \overline{H})$. In the periodic case, $u, m : \mathbb{T}^d \to \mathbb{R}$ and $\overline{H} \in \mathbb{R}$. We require $m \geq 0$ and $\int_{\mathbb{T}^d} m \mathrm{d}x = 1$.

In (1.1) and (1.3), the equations are coupled through the vector field, $Du$, in the Fokker–Planck equation, and the term $F$ in the Hamilton–Jacobi equation. For $\epsilon = 0$, (1.1) and (1.3) are called first-order or deterministic MFGs. Otherwise, if $\epsilon > 0$, (1.1) and (1.3) determine, respectively, second-order parabolic and elliptic MFGs.

In the rest of this chapter, we present a brief derivation of deterministic MFGs and examine uniqueness.

## 1.1 Derivation of MFG Models

Here, we present a heuristic derivation of a time-dependent deterministic mean-field game that corresponds to $\epsilon = 0$ in (1.1). The case $\epsilon > 0$ is handled in a similar way using stochastic control methods.

### *1.1.1 Optimal Control and Hamilton–Jacobi Equations*

We begin by examining the terminal value deterministic optimal control problem. We fix $T > 0$ and consider an agent whose state is $\mathbf{x}(t) \in \mathbb{R}^d$ for $0 \leq t \leq T$. Let $\mathcal{W} = L^\infty([t,T], \mathbb{R}^d)$. Agents can change their state by choosing a control in $\mathcal{W}$. For each control $\mathbf{v} \in \mathcal{W}$, the state evolves according to

$$\dot{\mathbf{x}}(t) = \mathbf{v}(t). \tag{1.4}$$

We fix a Lagrangian $\tilde{L} : \mathbb{R}^d \times \mathbb{R}^d \times [0, T] \to \mathbb{R}$, with $v \mapsto L(x, v, t)$ uniformly convex. For example,

$$\tilde{L}(x, v, t) = \frac{|v|^2}{2} - V(x) + \tilde{F}(x, t), \tag{1.5}$$

with $\tilde{F} : \mathbb{R}^d \times [0, T] \to \mathbb{R}$ a continuous function bounded from below. Next, we choose a bounded continuous function, $u_T : \mathbb{R}^d \to \mathbb{R}$, called the terminal cost.

Agents have preferences that are encoded in the action functional,

$$J(\mathbf{v}; x, t) = \int_t^T \tilde{L}(\mathbf{x}(s), \mathbf{v}(s), s)ds + u_T(\mathbf{x}(T)),$$

where $\mathbf{x}$ solves (1.4) with the initial condition $\mathbf{x}(t) = x$. Each agent seeks to minimize $J$ among all possible controls in $\mathcal{W}$. The infimum over all controls,

$$u(x, t) = \inf_{\mathbf{v} \in \mathcal{W}} J(\mathbf{v}; x, t), \tag{1.6}$$

is called the value function.

Recall that the Legendre transform, $\tilde{H}$, of $\tilde{L}$ is given by

$$\tilde{H}(x, p, t) = \sup_{v \in \mathbb{R}^d} \left[ -p \cdot v - \tilde{L}(x, v, t) \right]. \tag{1.7}$$

The function $\tilde{H}$ is called the Hamiltonian. By the uniform convexity of $\tilde{L}$ in the second coordinate, the maximum in the previous inequality is achieved at a unique point, $v^*$. For each $(x, t)$, $v^*$ is determined by

$$v^* = -D_p\tilde{H}(x, p, t). \tag{1.8}$$

If $\tilde{L}$ is given by (1.5), then

$$\tilde{H}(x, p, t) = \frac{|p|^2}{2} + V(x) - \tilde{F}(x, t).$$

A classical result in control theory states that if $u \in C^1(\mathbb{R}^d \times [t_0, T])$, then $u$ solves the Hamilton–Jacobi equation,

$$-u_t(x, t) + \tilde{H}(x, D_x u(x, t), t) = 0. \tag{1.9}$$

Further, as we prove next, the optimal control, $\mathbf{v}^*(t)$, is determined in feedback form by

$$\mathbf{v}^*(t) = -D_p\tilde{H}(\mathbf{x}^*(t), D_x u(\mathbf{x}^*(t), t), t). \tag{1.10}$$

In general, the value function is not differentiable. However, it solves (1.9) in a weaker sense—as a viscosity solution. Here, we do not develop this theory. Instead, we show the converse of that statement, namely: if $\tilde{u}$ solves (1.9) and satisfies the terminal condition

$$\tilde{u}(x, T) = u_T(x), \tag{1.11}$$

then $\tilde{u}$ is the value function in (1.6).

**Theorem 1.1 (Verification Theorem).** *Let $\tilde{u} \in C^1(\mathbb{R}^d \times [t_0, T])$ solve (1.9) with the terminal condition (1.11). Let*

$$\mathbf{v}^*(t) = -D_p\tilde{H}(\mathbf{x}^*(t), D_x\tilde{u}(\mathbf{x}^*(t), t), t) \tag{1.12}$$

*and $\mathbf{x}^*(t)$ solve (1.4). Then, $\mathbf{v}^*(t)$ is an optimal control for (1.6) and $\tilde{u}(x,t) = u(x,t)$, where $u$ is the value function in (1.6).*

*Proof.* First, we observe that for any $\mathbf{v}(s)$ and any trajectory $\mathbf{x}$ solving (1.4), we have

$$\tilde{u}(\mathbf{x}(T), T) = \int_t^T (D_x\tilde{u}(\mathbf{x}(s), s) \cdot \mathbf{v}(s) + \tilde{u}_s(\mathbf{x}(s), s))\, ds + \tilde{u}(\mathbf{x}(t), t). \tag{1.13}$$

Because of (1.7), we have

$$(D_x\tilde{u}(\mathbf{x}(s), s) \cdot \mathbf{v}(s) \geq -\tilde{H}(x, D_x\tilde{u}(\mathbf{x}(s), s) - \tilde{L}(\mathbf{x}(s), \mathbf{v}(s)).$$

Furthermore, the previous inequality is an identity for $\mathbf{v} = \mathbf{v}^*$ due to (1.12). By combining (1.13) with (1.7), we get

$$\tilde{u}(x,t) \leq \int_t^T \tilde{L}(\mathbf{x}(s), \mathbf{v}(s), s)ds + u_T(\mathbf{x}(T)). \tag{1.14}$$

Finally, from (1.8), we conclude that the previous inequality is an identity if $\mathbf{v} = \mathbf{v}^*$. $\square$

### 1.1.2 Transport Equation

Let $b : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ be a Lipschitz vector field. Consider a population of agents with dynamics given by

$$\begin{cases} \dot{\mathbf{x}}(t) = b(\mathbf{x}(t), t) & t > 0, \\ \mathbf{x}(0) = x. \end{cases} \tag{1.15}$$

The previous equation induces a flow, $\Phi^t$, in $\mathbb{R}^d$ that maps the initial condition, $x \in \mathbb{R}^d$, at $t = 0$ to the solution of (1.15) at time $t > 0$.

Fix a probability measure, $m_0 \in \mathcal{P}(\mathbb{R}^d)$. For $0 \leq t \leq T$, let $m(\cdot, t)$ be the push-forward by $\Phi^t$ of $m_0$, sometimes denoted by $\Phi^t \sharp m_0$, given by

$$\int_{\mathbb{R}^d} \phi(x) m(x,t) dx = \int_{\mathbb{R}^d} \phi\left(\Phi^t(x)\right) m_0 dx. \tag{1.16}$$

For $0 \leq t \leq T$, $m(\cdot, t)$ is a probability measure. Next, we derive a partial differential equation for $m$.

**Proposition 1.2.** *Let $m$ be determined by (1.16) for some probability measure $m_0 \in \mathcal{P}(\mathbb{R}^d)$. Assume that $b(x,t)$ is Lipschitz continuous in $x$. Let $\Phi^t$ be the flow corresponding to (1.15). Then, $m \in C(\mathbb{R}_0^+, \mathcal{P}(\mathbb{R}^d))$ and*

$$\begin{cases} m_t(x,t) + \operatorname{div}(b(x,t)m(x,t)) = 0, & (x,t) \in \mathbb{R}^d \times [0,T], \\ m(x,0) = m_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.17}$$

*in the distributional sense.*

*Proof.* We recall that $\rho$ solves (1.17) in the distributional sense if

$$-\int_0^T \int_{\mathbb{R}^d} \rho(x,t) \left(\phi_t(x,t) + b(x,t)\phi_x(x,t)\right) dxdt = \int_{\mathbb{R}^d} \rho_0(x)\phi(x,0)dx,$$

for every $\phi \in C_c^\infty(\mathbb{R}^d \times [0,T))$. Differentiating both sides of (1.16) with respect to $t$ gives

$$\int_{\mathbb{R}^d} \phi(x,t) m_t(x,t) dx = \int_{\mathbb{R}^d} \left(b(\Phi^t(x),t) D_x\phi\left(\Phi^t(x),t\right)\right) m_0(x) dx.$$

Therefore,

$$\int_{\mathbb{R}^d} \phi(x,t) m_t(x,t) dx = \int_{\mathbb{R}^d} \left(b(x,t) D_x\phi(x,t)\right) m(x,t) dx,$$

using the definition of $\Phi^t$. To conclude the proof, we integrate the previous identity by parts. □

### *1.1.3 Mean-Field Models*

The mean-field game framework was developed to study systems with an infinite number of rational agents in competition. In these systems, each agent seeks to optimize an individual control problem that depends on statistical information about the whole population. Further, the only information available to the agents is the probability distribution of the agents' states.

Here, the interaction between the mean field and each agent is determined by the running cost. First, we assume that for each time $t$, $m(x,t)$ is a probability density in $\mathbb{R}^d$ that gives the distribution of the agents in the different states. Next, we set

$$\tilde{L}(x,v,t) = L(x,v,m(\cdot,t)).$$

The Lagrangian, $L$, is a real-valued map, $L : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ or $L : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$. The former case is called the local case and the latter the non-local case. In the local case, we interpret $L(x,v,m(\cdot,t))$ as $L(x,v,m(x,t))$. We denote the Legendre transform of $L$ by $H$. Finally, we suppose that each agent seeks to minimize the control problem (1.6). As a result, the value function, $u$, of a representative agent is determined by

$$-u_t + H(x, D_x u, m) = 0.$$

According to Theorem 1.1, if the previous equation has a solution, $u$, the vector field, $b = -D_p H(x, D_x u(x,t), m)$, gives an optimal strategy. Because all agents are rational, they use this strategy. $u$ and $m$ are thus determined by

$$\begin{cases} -u_t + H(x, D_x u, m) = 0 \\ m_t - \operatorname{div}(D_p H m) = 0. \end{cases} \tag{1.18}$$

In addition, if $u_T : \mathbb{R}^d \to \mathbb{R}$ is the terminal value function for the agents and their initial distribution is $m_0 : \mathbb{R}^d \to \mathbb{R}_0^+$ with $\int_{\mathbb{R}^d} m_0 = 1$, we supplement (1.18) with the initial-terminal conditions (1.2).

### 1.1.4 Extensions and Additional Problems

In some applications, deterministic models are unsuitable due to random perturbations. In these cases, stochastic optimal control replaces deterministic control. For the stochastic case, the MFG is given by a second-order, nonlinear parabolic system,

$$\begin{cases} -u_t + H(x, D_x u, D^2 u, m) = 0 \\ m_t - \operatorname{div}(D_p H m) - \sum_{ij} \partial_{ij}(D_{M_{ij}} H m) = 0, \end{cases} \tag{1.19}$$

coupled with the initial-terminal conditions (1.2). The system (1.1) is a particular instance of (1.19). The general theory for systems like (1.19) is not yet completely understood, especially if the dependence on second-order derivatives is nonlinear. However, many important cases can be studied rigorously as we will see here.

In some applications, the initial-terminal problem is replaced by the planning problem: given two probability measures, $m_0$ and $m_T$, we look for a pair $(u, m)$ solving (1.19) under the boundary conditions

$$m(x, 0) = m_0, \qquad m(x, T) = m_T. \tag{1.20}$$

In this book, we consider (1.19) in the whole space, $\mathbb{R}^d$, or with periodic boundary conditions. In this last case, we regard $u$ and $m$ as real-valued functions with domain $\mathbb{T}^d \times [0, T]$. Often, our methods extend in a straightforward way to Dirichlet and Neumann boundary conditions.

In the periodic stationary case, the equation $H(x, D_x u, D^2 u, m) = 0$ may fail to have solutions, or these may not be probability measures. Therefore, we introduce a constant, $\overline{H}$, that represents a long-time average running cost. This constant is known as the effective Hamiltonian in the theory of homogenization. The stationary version of (1.19) becomes

$$\begin{cases} H(x, D_x u, D^2 u, m) = \overline{H} \\ -\operatorname{div}(D_p H m) - \sum_{ij} \partial_{ij}(D_{M_{ij}} H m) = 0, \end{cases} \tag{1.21}$$

where the solution is a triplet, $(u, m, \overline{H})$, with $u, m : \mathbb{T}^d \to \mathbb{R}$, $m \geq 0$, and $\overline{H} \in \mathbb{R}$. The constant $\overline{H}$ is chosen so that $m$ is a probability measure.

### *1.1.5 Uniqueness*

We continue our study of (1.1)–(1.2) by examining the uniqueness of solutions. The fundamental element for the uniqueness is a monotonicity condition introduced by Lasry and Lions. Here, we consider the non-local case, $F : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$, as the local case is analogous. We say that (1.1) satisfies the Lasry–Lions monotonicity condition if

$$\int_{\mathbb{R}^d} [F(x, m_1) - F(x, m_2)] (m_1 - m_2) dx > 0, \tag{1.22}$$

for every $m_1$, $m_2 \in \mathcal{P}(\mathbb{R}^d)$ with $m_1 \neq m_2$.

The next theorem gives the uniqueness of classical solutions to (1.1)–(1.2).

**Theorem 1.3 (Uniqueness of Classical Solutions).** *Assume that the mean-field coupling, $F$, satisfies the Lasry–Lions monotonicity condition. Then, there exists at most one classical solution, $(u, m)$, for (1.1)–(1.2).*

*Proof.* We argue by contradiction. Let $(u_1, m_1)$ and $(u_2, m_2)$ solve (1.1)–(1.2). Define

$$\tilde{u} := u_1 - u_2$$

and

$$\tilde{m} := m_1 - m_2.$$

First, we subtract the equations for $(u_1, m_1)$ and $(u_2, m_2)$ to get

$$\begin{cases} -\tilde{u}_t \ + \ \frac{|Du_1|^2}{2} - \frac{|Du_2|^2}{2} \ = \epsilon \, \Delta\tilde{u} \ + \ F(x, m_1) - F(x, m_2) \\ \tilde{m}_t \ - \ \operatorname{div}(m_1 Du_1) + \ \operatorname{div}(m_2 Du_2) = \epsilon \, \Delta\tilde{m}. \end{cases}$$

Next, we multiply the first equation by $\tilde{m}$ and the second one by $\tilde{u}$. Thereafter, we subtract them and integrate by parts. This leads to the identity

$$\frac{d}{dt}\int_{\mathbb{T}^d} \tilde{u}\tilde{m} = \int_{\mathbb{T}^d} \frac{\tilde{m}}{2}\left(|Du_1|^2 - |Du_2|^2\right) - \int_{\mathbb{T}^d} D\tilde{u}\,(m_1 Du_1 - m_2 Du_2) \qquad (1.23)$$
$$- \int_{\mathbb{T}^d} (F(x, m_1) - F(x, m_2))\,\tilde{m}.$$

Notice that

$$\frac{\tilde{m}}{2}\left(|Du_1|^2 - |Du_2|^2\right) - D\tilde{u}\,(m_1 Du_1 - m_2 Du_2) = -\frac{m_1 + m_2}{2}\,|Du_1 - Du_2|^2\,. \qquad (1.24)$$

Integrating (1.23) over $[0, T]$, using (1.2), (1.24) and the Lasry–Lions monotonicity condition, we obtain

$$\int_{\mathbb{T}^d} (F(x, m_1) - F(x, m_2))\,(m_1 - m_2) + \int_{\mathbb{R}^d} \frac{m_1 + m_2}{2}\,|Du_1 - Du_2|^2 \le 0.$$

Hence, $m_1 = m_2$. Finally, the uniqueness of solutions for (1.1) gives $u_1 = u_2$. □

# Chapter 2
# Explicit Solutions, Special Transformations, and Further Examples

Few mean-field games can be solved explicitly. However, examples for which closed solutions are known illustrate essential features of the theory. Moreover, explicit solutions to MFGs are a key ingredient in the continuation method discussed in Chap. 11.

## 2.1 Explicit Solutions

We begin our study of explicit solutions by considering a first-order quadratic MFG with a logarithmic nonlinearity. While logarithmic nonlinearities pose several technical challenges (see Chaps. 7 and 9), the model considered here can be solved by elementary methods. This game is given by

$$\begin{cases} \frac{|u_x|^2}{2} + V(x) + b(x)u_x = \ln m + \overline{H}, \\ -(m(Du + b(x)))_x = 0, \end{cases} \tag{2.1}$$

with $u, m : \mathbb{T} \to \mathbb{R}$, $m \geq 0$,

$$\int_{\mathbb{T}} mdx = 1,$$

and, for definiteness,

$$\int_{\mathbb{T}} u \, \mathrm{d}x = 0. \tag{2.2}$$

 
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Moreover, we suppose that

$$\int_{\mathbb{T}} b(y)\mathrm{d}y = 0.$$

If $Du + b = 0$, the second equation in (2.1) holds immediately. This suggests that we set

$$u(x) = -\int_0^x b(y)\,\mathrm{d}y + \int_{\mathbb{T}}\int_0^z b(y)\,\mathrm{d}y\,\mathrm{d}z.$$

Using the previous formula in the first equation, we get

$$m(x) = \frac{e^{V(x)-\frac{b^2(x)}{2}}}{\int_{\mathbb{T}} e^{V(y)-\frac{b^2(y)}{2}}\,\mathrm{d}y}.$$

In particular, let $\psi : \mathbb{T}^d \to \mathbb{R}$ be a periodic $C^\infty$ function with $\int_{\mathbb{T}} \psi\,\mathrm{d}x = 0$. Suppose that $b(x) = \psi_x(x)$. Then,

$$u(x) = -\psi(x), \qquad m(x) = \frac{e^{V(x)-\frac{\psi_x^2(x)}{2}}}{\int_{\mathbb{T}} e^{V(y)-\frac{\psi_x^2(y)}{2}}\,\mathrm{d}y}, \quad \text{and} \quad \overline{H} = \ln\left[\int_{\mathbb{T}} e^{V(y)-\frac{\psi_x^2(y)}{2}}\,\mathrm{d}y\right]$$

solves (2.1).

A related problem is the congestion model:

$$\begin{cases} \frac{u_x^2}{2m^{1/2}} + V(x) = \ln m + \overline{H}, \\ -(m^{1/2}u_x)_x = 0. \end{cases} \tag{2.3}$$

It is easy to see that $u(x) = 0$, $m(x) = \frac{e^{V(x)}}{\int_{\mathbb{T}} e^{V(y)}\,\mathrm{d}y}$ and $\overline{H} = \ln \int_{\mathbb{T}} e^{V(x)} dx$ solve (2.3).

## 2.2 The Hopf–Cole Transform

The Hopf–Cole transform is a well-known technique to convert certain nonlinear equations into linear equations. Here, we illustrate an application to MFGs. For $P \in \mathbb{R}^d$, consider the system

$$\begin{cases} -\Delta u + \frac{1}{2}|P + Du|^2 + V(x) = \ln m \\ -\Delta m - \operatorname{div}((P + Du)m) = 0. \end{cases} \tag{2.4}$$

Define $m$ by the Hopf–Cole transform

$$m = e^{\frac{v-u}{2}}, \tag{2.5}$$

where $u$ and $v$ solve

$$\begin{cases} -\Delta u + \frac{1}{2}|P + Du|^2 + V(x) &= \frac{v-u}{2} \\ \Delta v + \frac{1}{2}|P + Dv|^2 + V(x) &= \frac{v-u}{2}. \end{cases} \tag{2.6}$$

By a direct computation, the function, $m$, given by (2.5) solves

$$-\Delta m - \operatorname{div}((P + Du)m) = 0. \tag{2.7}$$

To check this, it is enough to observe that

$$\begin{aligned} -\Delta m &= m\left[\frac{1}{2}\Delta u - \frac{1}{2}\Delta v - \frac{|Du - Dv|^2}{4}\right] \\ &= m\Delta u - \frac{m}{4}\left[|P + Du|^2 - |P + Dv|^2 + |Du - Dv|^2\right] \\ &= (P + Du) \cdot Dm + m\Delta u = \operatorname{div}((P + Du)m). \end{aligned}$$

## 2.3 Gaussian-Quadratic Solutions

Gaussian-quadratic solutions to MFGs are relevant in several applications. In dimension $d \geq 1$, we consider the MFG in $\mathbb{R}^d$ given by

$$\begin{cases} -\Delta u + \frac{1}{2}|Du|^2 + \beta|x|^2 = \ln m + \overline{H} \\ -\Delta m - \operatorname{div}(mDu) = 0. \end{cases} \tag{2.8}$$

We set $m = \mu e^{-u}$ so that the second equation holds trivially. Next, we select

$$u = \alpha|x|^2.$$

Using the ansatz in the first equation of (2.8) gives that $\alpha$ solves

$$2\alpha^2 + \alpha + \beta = 0.$$

If $\beta < 0$, the preceding equation has a solution, $\alpha > 0$. Finally, we determine $\mu$ by the normalization condition, $\int_{\mathbb{R}} m dx = 1$. To find $\overline{H}$, we use the expressions for $u$ and $m$ in the first equation of (2.8).

## 2.4 Interface Formation

In this last example, we describe the formation of interfaces and the breakdown of regularity. For $\lambda \in \mathbb{R}$, we consider the MFG

$$\begin{cases} \frac{|u_x|^2}{2} + \lambda V(x) = m + \overline{H}(\lambda), \\ -(mu_x)_x = 0, \end{cases} \tag{2.9}$$

with periodic conditions; that is, $u, m : \mathbb{T} \to \mathbb{R}$, $m \geq 0$, and $\int_{\mathbb{T}} m \mathrm{d}x = 1$.

First, we attempt to solve (2.9). The second equation in (2.9) implies that $mu_x = c$, for some constant $c$. If $c \neq 0$, then $u_x = \frac{c}{m}$. This is not possible because $\int_{\mathbb{T}} u_x \mathrm{d}x = 0$ and $m > 0$. Therefore, $mu_x = 0$. Accordingly, $u$ is constant in the set $m > 0$. Hence, the second equation holds trivially. Moreover, we gather

$$m(x, \lambda) = \lambda V(x) - \overline{H}(\lambda)$$

on the set $m > 0$. In addition, on the set $m = 0$, the first equation gives

$$\lambda V(x) - \overline{H}(\lambda) \leq 0.$$

Thus,

$$m(x, \lambda) = \left(\lambda V(x) - \overline{H}(\lambda)\right)^+ .$$

The map

$$h \mapsto \int_{\mathbb{T}} (\lambda V(x) - h)^+ \mathrm{d}x$$

is monotone decreasing. Hence, there is a unique value, $\overline{H}(\lambda)$, for which

$$\int_{\mathbb{T}} \left(\lambda V(x) - \overline{H}(\lambda)\right)^+ \mathrm{d}x = 1.$$

If $\lambda$ is small, the condition $\int_{\mathbb{T}} m = 1$ gives

$$\overline{H}(\lambda) = \lambda \int_{\mathbb{T}} V - 1.$$

Thus,

$$m(x, \lambda) = 1 + \lambda \left( V(x) - \int_{\mathbb{T}} V \right).$$

In contrast, for large $|\lambda|$, the condition $m > 0$ fails.

**Fig. 2.1** $\overline{H}(\lambda)$

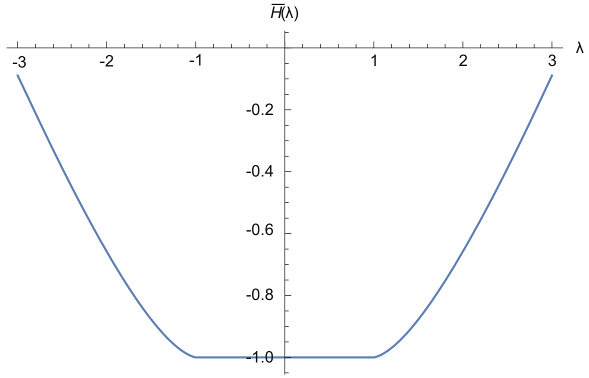
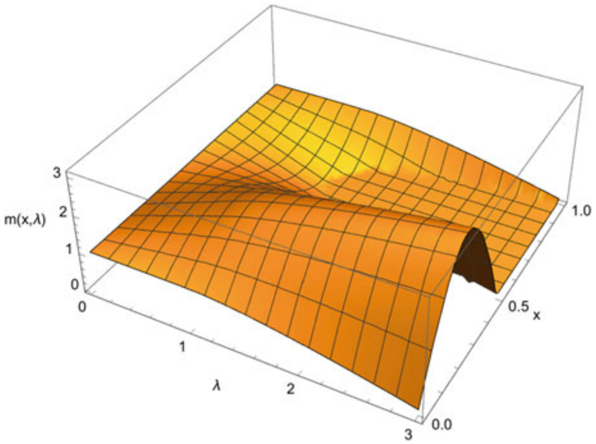



**Fig. 2.2** $m(x, \lambda)$




Because $m(x, \lambda) + \overline{H}(\lambda) - \lambda V(x) = (\overline{H}(\lambda) - \lambda V(x))^+$, we have

$$\frac{|u_x|^2}{2} = (\overline{H}(\lambda) - \lambda V(x))^+.$$

The solution $u$ to the preceding equation can fail to be a classical solution. Furthermore, as we show next, it may admit multiple solutions.

Figure 2.1 illustrates the behavior of $\overline{H}(\lambda)$ for $V(x) = \sin(2\pi x)$, and Fig. 2.2 depicts $m(x, \lambda)$. In general, the solution, $u$, is not unique and may not be differentiable. In Figs. 2.3 and 2.4, we plot two two-periodic solutions $(m, u)$ for $\lambda = 2$.

## 2.5 Bibliographical Notes

The explicit solution in Sect. 2.1 appeared in [8]. The Hopf–Cole transform was introduced in the context of MFGs in [174]. Similar ideas were used in [138] to develop numerical methods and in [66] to show the existence of classical solutions.

**Fig. 2.3** $m(x, 2)$

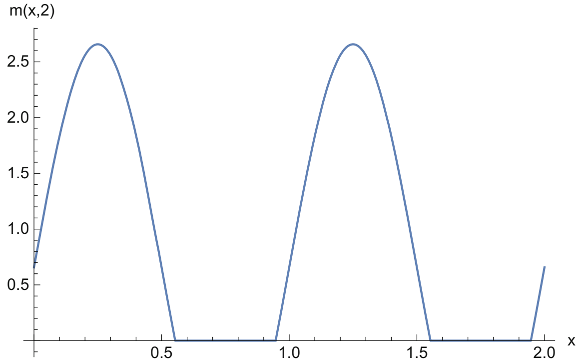
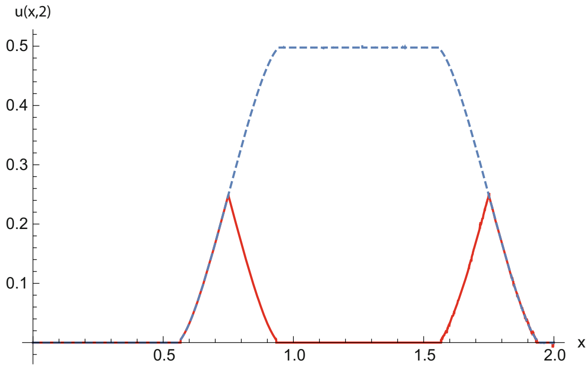



**Fig. 2.4** $u(x, 2)$—two distinct solutions




A remarkable extension of the Hopf–Cole transform was presented in [78]. The Hopf–Cole transform was used in [203] to convert an MFG into a system of Schrödinger equations. Gaussian-quadratic solutions were discussed in [137] and, with more generality, in [15]. Moreover, they have applications in machine learning, in particular, in clustering and non-supervised learning [189, 190]. The $N$-player linear-quadratic counterpart was considered in [17, 18, 197]. Some applications of linear-quadratic MFGs are presented in [22, 31, 32, 145, 184]. The discussion in Sect. 2.3 is inspired by [130]. Explicit examples where MFG partial differential equations are converted into ordinary differential equations were examined in [25, 182]. A further explicit example was studied in [204].

# Chapter 3
# Estimates for the Hamilton–Jacobi Equation

In this chapter, we examine a priori estimates for solutions of Hamilton–Jacobi equations. We are interested in solutions of time-dependent problems, $u : \mathbb{T}^d \times [0,T] \to \mathbb{R}$ or $u : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ of

$$-u_t + H(x, Du) = \epsilon \Delta u. \tag{3.1}$$

For stationary problems, we consider periodic solutions. In this case, a solution is a pair, $(u, \overline{H})$ with $\overline{H} \in \mathbb{R}$ and $u : \mathbb{T}^d \to \mathbb{R}$, satisfying

$$-\epsilon \Delta u + H(x, Du) = \overline{H} \quad \text{in} \quad \mathbb{T}^d. \tag{3.2}$$

The techniques we develop are critical to the study of MFGs. In those games, the Hamilton–Jacobi equation that determines the value function, $u$, depends on the density of the agents, $m$. Often, we have estimates for $m$ in low regularity spaces; Lebesgue or Sobolev spaces are examples. For this reason, a large part of our discussion focuses on integral estimates of $u$.

## 3.1 Comparison Principle

A central tool in the theory of Hamilton–Jacobi equations is the comparison principle stated in the next proposition. In the context of MFGs, the comparison principle is frequently used to get lower bounds for solutions; see, for example, Sect. 6.1 in Chap. 6. Then, upper bounds can be proven by the methods discussed in Sects. 3.2 and 3.4.



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**Proposition 3.1 (Comparison Principle).** *Let $u : \mathbb{T}^d \times [0,T] \to \mathbb{R}$ solve*

$$-u_t + H(x, Du) - \epsilon \Delta u \geq 0 \quad \textit{in} \;\; \mathbb{T}^d \times [0,T), \tag{3.3}$$

*and let $v : \mathbb{T}^d \times [0,T] \to \mathbb{R}$ solve*

$$-v_t + H(x, Dv) - \epsilon \Delta v \leq 0 \quad \textit{in} \;\; \mathbb{T}^d \times [0,T). \tag{3.4}$$

*Suppose that $u \geq v$ at $t = T$. Then, $u \geq v$ in $\mathbb{T}^d \times [0,T)$.*

*Proof.* Let $u^\delta = u + \frac{\delta}{t}$. We have

$$-u_t^\delta + H(x, Du^\delta) - \epsilon \Delta u^\delta > 0 \quad \text{in} \;\; \mathbb{T}^d \times [0,T). \tag{3.5}$$

Subtracting (3.4) from (3.5), we get

$$-(u^\delta - v)_t + H(x, Du^\delta) - H(x, Dv) - \epsilon \Delta(u^\delta - v) > 0 \quad \text{in} \;\; \mathbb{T}^d \times [0,T). \tag{3.6}$$

Consider the function $u^\delta - v$ and let $(x_\delta, t_\delta)$ be a point of minimum of $u^\delta - v$ on $\mathbb{T}^d \times [0,T)$. This minimum is achieved at some point, $t_\delta > 0$. We claim that $t_\delta = T$. If not, at $(x_\delta, t_\delta)$, we have

$$u_t^\delta \geq v_t, \quad Du^\delta = Du, \quad \Delta u^\delta \geq \Delta v.$$

However, the earlier identities and inequality yield a contradiction in (3.6). Accordingly, the minimum of $u^\delta - v$ is attained at $T$. Hence, $u^\delta \geq v$ in $\mathbb{T}^d \times [0,T]$. The conclusion follows by letting $\delta \to 0$. □

In the foregoing theorem, we assumed that $u$ and $v$ are, respectively, classical super and subsolutions; that is, $u$ and $v$ are smooth enough, $u$ satisfies (3.3) and $v$ satisfies (3.4). However, by the theory of viscosity solutions, the comparison principle holds with fewer regularity requirements. The interested reader can find additional material on viscosity solutions in the references at the end of the chapter.

## 3.2 Control Theory Bounds

The Hamilton–Jacobi equations in MFGs are associated with control problems—deterministic control in the first-order case and stochastic control in the second-order case. Here, we consider $C^1$ solutions, $u : \mathbb{R}^d \times [0,T] \to \mathbb{R}$, of the Hamilton–Jacobi equation,

$$-u_t + \frac{|Du|^2}{2} + V(x) = 0, \tag{3.7}$$

with the terminal condition,

$$u(x,T) = u_T(x), \tag{3.8}$$

and investigate the corresponding deterministic control problem.

For convenience, we suppose that $V$ is of class $C^2$ and globally bounded. By Theorem 1.1, a solution of (3.7) is the value function of the control problem

$$u(x,t) = \inf_{\mathbf{x}} \int_t^T \frac{|\dot{\mathbf{x}}(s)|^2}{2} - V(\mathbf{x}(s))ds + u_T(\mathbf{x}(T)), \tag{3.9}$$

where the infimum is taken over all trajectories, $\mathbf{x} \in W^{1,2}([t,T])$, with $\mathbf{x}(t) = x$. In what follows, we prove the existence of optimal trajectories. Then, using the control theory characterization, we obtain various bounds for $u$.

### *3.2.1 Optimal Trajectories*

We begin our study of (3.9) by examining the existence of optimal or minimizing trajectories. Because (3.9) has quadratic growth in $\dot{\mathbf{x}}$, the natural space to look for minimizers is the Sobolev space, $W^{1,2}([t,T])$.

**Proposition 3.2.** *Let $V$ be a bounded continuous function. There exists a minimizer, $\boldsymbol{x} \in W^{1,2}([t,T])$, of (3.9).*

*Proof.* Let $\mathbf{x}_n$ be a minimizing sequence for (3.9); that is, a sequence such that

$$u(x,t) = \lim_{n\to\infty} \int_t^T \frac{|\dot{\mathbf{x}}_n(s)|^2}{2} - V(\mathbf{x}_n(s))ds + u_T(\mathbf{x}_n(T)).$$

We have $\sup_n \|\dot{\mathbf{x}}_n\|_{L^2([t,T])} \leq C$. By Poincaré's inequality, we conclude that

$$\sup_n \|\mathbf{x}_n\|_{W^{1,2}([t,T])} < \infty.$$

Next, by Morrey's theorem, the sequence $\mathbf{x}_n$ is equicontinuous and bounded (since $\mathbf{x}_n(t)$ is fixed). Hence, by the Ascoli–Arzelà Theorem, there exists a uniformly convergent subsequence. We can extract a further subsequence that converges weakly in $W^{1,2}$ to a function, $\mathbf{x}$. To prove that $\mathbf{x}$ is a minimum, it is enough to show weakly lower semicontinuity; that is,

$$\liminf_{n\to\infty} \int_t^T \left[\frac{|\dot{\mathbf{x}}_n|^2}{2} - V(\,\mathbf{x}_n)\right] ds + u_T(\mathbf{x}_n(T)) \geq \int_t^T \left[\frac{|\dot{\mathbf{x}}|^2}{2} - V(\mathbf{x})\right] ds + u_T(\mathbf{x}(T)) \tag{3.10}$$

for any sequence $\mathbf{x}_n \rightharpoonup \mathbf{x}$ in $W^{1,2}([t,T])$. By convexity,

$$\int_t^T \left[\frac{|\dot{\mathbf{x}}_n|^2}{2} - V(\mathbf{x}_n)\right] ds + u_T(\mathbf{x}_n(T)) \tag{3.11}$$
$$\geq \int_t^T \left[V(\mathbf{x}) - V(\mathbf{x}_n) + \left[\frac{|\dot{\mathbf{x}}|^2}{2} - V(\mathbf{x})\right] + \dot{\mathbf{x}}(\dot{\mathbf{x}}_n - \dot{\mathbf{x}})\right] ds + u_T(\mathbf{x}_n(T)).$$

Because $\dot{\mathbf{x}}_n \rightharpoonup \dot{\mathbf{x}}$ and $\dot{\mathbf{x}} \in L^2([t,T])$, we have

$$\int_t^T \dot{\mathbf{x}}(\dot{\mathbf{x}}_n - \dot{\mathbf{x}}) \to 0.$$

From the uniform convergence of $\mathbf{x}_n$ to $\mathbf{x}$, we conclude that

$$\int_t^T V(\mathbf{x}_n) - V(\mathbf{x}) \to 0$$

and that

$$u_T(\mathbf{x}_n(T)) \to u_T(\mathbf{x}(T)).$$

Thus, by taking the lim inf in (3.11), we get (3.10). □

The minimizers of (3.9) are solutions to the Euler–Lagrange equation, which is an ordinary differential equation that we derive next.

**Proposition 3.3.** *Let $V$ be a $C^1$ function. Let $\boldsymbol{x} : [t,T] \to \mathbb{R}^d$ be a $W^{1,2}([t,T])$ minimizer for (3.9). Then, $\boldsymbol{x} \in C^2([t,T])$, and it satisfies the Euler–Lagrange equation:*

$$\ddot{\boldsymbol{x}} + D_x V(\boldsymbol{x}) = 0.$$

*Moreover, set $H(p,x) = \frac{|p|^2}{2} + V(x)$. Then, for $\boldsymbol{p} = -\dot{\boldsymbol{x}}$, we have that $(\boldsymbol{x},\boldsymbol{p})$ solves the Hamiltonian dynamics:*

$$\begin{cases} \dot{\boldsymbol{p}} = D_x H(\boldsymbol{p},\boldsymbol{x}) \\ \dot{\boldsymbol{x}} = -D_p H(\boldsymbol{p},\boldsymbol{x}). \end{cases} \tag{3.12}$$

*Proof.* Let $\mathbf{x} : [t,T] \to \mathbb{R}^d$ be a $W^{1,2}([t,T])$ minimizer for (3.9). Fix $\varphi : [0,T] \to \mathbb{R}^d$ of class $C^2$ with compact support on $(t,T)$. Because $\mathbf{x}$ is a minimizer, the function

$$i(\epsilon) = \int_t^T \frac{|\dot{\mathbf{x}} + \epsilon\dot{\varphi}|^2}{2} - V(\mathbf{x} + \epsilon\varphi) + u_T(\mathbf{x}(T))$$

has a minimum at $\epsilon = 0$. Because $i$ is differentiable, we have $i'(0) = 0$. Therefore,

$$\int_t^T [\dot{\mathbf{x}} \cdot \dot{\varphi} - D_x V(\mathbf{x})\varphi]\, ds = 0. \tag{3.13}$$

Next, set

$$\mathbf{p}(t) = p_0 + \int_t^T -D_x V(\mathbf{x}) ds,$$

with $p_0 \in \mathbb{R}^n$ to be chosen later. For each $\varphi \in C_c^2((t,T))$ taking values in $\mathbb{R}^d$, we have

$$\int_t^T \frac{d}{dt}(\mathbf{p} \cdot \varphi) dt = \mathbf{p} \cdot \varphi\big|_t^T = 0.$$

Thus,

$$\int_t^T D_x V(\mathbf{x})\varphi + \mathbf{p} \cdot \dot{\varphi} dt = 0.$$

Using (3.13), we conclude that

$$\int_t^T (\mathbf{p} + \dot{\mathbf{x}}) \cdot \dot{\varphi} dt = 0.$$

Therefore, $\mathbf{p} + \dot{\mathbf{x}}$ is constant. Thus, selecting $p_0$ conveniently, we have

$$\mathbf{p} = -\dot{\mathbf{x}}.$$

Since $\mathbf{p}$ is continuous, the above identity gives $\dot{\mathbf{x}} = -D_p H(\mathbf{p}, \mathbf{x})$, and, for that reason, $\dot{\mathbf{x}}$ is continuous. Moreover, we have

$$\dot{\mathbf{p}} = D_x H(\mathbf{p}, \mathbf{x}),$$

and thus, $\mathbf{p}$ is $C^1$. Further, we have

$$\dot{\mathbf{x}} = -D_p H(\mathbf{p}, \mathbf{x}).$$

Consequently, $\dot{\mathbf{x}}$ is $C^1$. As a result, $\mathbf{x}$ is $C^2$. □

### *3.2.2 Dynamic Programming Principle*

The dynamic programming principle that we prove next is a semigroup property that the value function satisfies.

**Proposition 3.4.** *Let $V$ be a bounded continuous function and $u$ be given by* (3.9). *Then, for any $t'$ with $t < t' < T$, we have*

$$u(x,t) = \inf_{\boldsymbol{x}} \int_t^{t'} \frac{|\dot{\boldsymbol{x}}(s)|^2}{2} - V(\boldsymbol{x}(s))ds + u(\boldsymbol{x}(t'), t'). \tag{3.14}$$

*Proof.* Let

$$\tilde{u}(x,t) = \inf_{\mathbf{x}} \int_t^{t'} \frac{|\dot{\mathbf{x}}(s)|^2}{2} - V(\mathbf{x}(s))ds + u(\mathbf{x}(t'), t'), \tag{3.15}$$

and $u$ be given by (3.9). Take an optimal trajectory, $\mathbf{x}^1$, for (3.15) and select an optimal trajectory, $\mathbf{x}^2$, for $u(\mathbf{x}(t'), t')$. Consider the concatenation of $\mathbf{x}^1$ with $\mathbf{x}^2$ given by

$$\mathbf{x}^3 = \begin{cases} \mathbf{x}^1(s) & t \leq s \leq t' \\ \mathbf{x}^2(s) & t' < s \leq T. \end{cases}$$

We have

$$u(x,t) \leq \int_t^T \frac{|\dot{\mathbf{x}}^3(s)|^2}{2} - V(\mathbf{x}^3(s))ds + u_T(\mathbf{x}^3(T)) = \tilde{u}(x,t).$$

Conversely, let $\mathbf{x}$ be an optimal trajectory in (3.9). Then,

$$u(\mathbf{x}(t'), t') \leq \int_{t'}^T \frac{|\dot{\mathbf{x}}(s)|^2}{2} - V(\mathbf{x}(s))ds + u_T(\mathbf{x}(T)).$$

Consequently,

$$\tilde{u}(x,t) \leq \int_t^{t'} \frac{|\dot{\mathbf{x}}(s)|^2}{2} - V(\mathbf{x}(s))ds + u(\mathbf{x}(t'), t') \leq u(x,t).$$

□

### 3.2.3 Subdifferentials and Superdifferentials of the Value Function

Consider a continuous function, $\psi : \mathbb{R}^d \to \mathbb{R}$. The superdifferential $D_x^+\psi(x)$ of $\psi$ at $x$ is the set of vectors, $p \in \mathbb{R}^d$, such that

$$\limsup_{|v|\to 0} \frac{\psi(x+v) - \psi(x) - p\cdot v}{|v|} \leq 0.$$

Consequently, $p \in D_x^+\psi(x)$ if and only if

$$\psi(x+v) \leq \psi(x) + p\cdot v + o(|v|),$$

as $|v| \to 0$. Similarly, the subdifferential, $D_x^-\psi(x)$, of $\psi$ at $x$ is the set of vectors, $p$, such that

$$\liminf_{|v|\to 0} \frac{\psi(x+v) - \psi(x) - p\cdot v}{|v|} \geq 0.$$

Next, we show that if $\psi$ is differentiable, then

$$D_x^-\psi(x) = D_x^+\psi(x) = \{D_x\psi(x)\}.$$

Therefore, we regard $D^\pm\psi$ as one-sided derivatives.

**Proposition 3.5.** *Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be a continuous function and $x \in \mathbb{R}^d$. If both $D_x^-\psi(x)$ and $D_x^+\psi(x)$ are non-empty, then*

$$D_x^-\psi(x) = D_x^+\psi(x) = \{p\}.$$

*In that case, $\psi$ is differentiable at $x$ with $D_x\psi = p$. Conversely, if $\psi$ is differentiable at $x$, we have*

$$D_x^-\psi(x) = D_x^+\psi(x) = \{D_x\psi(x)\}.$$

*Proof.* Suppose that $D_x^-\psi(x)$ and $D_x^+\psi(x)$ are both non-empty. We claim that these two sets agree and have a single point, $p$. To check this, take $p^- \in D_x^-\psi(x)$ and $p^+ \in D_x^+\psi(x)$. Then,

$$\liminf_{|v|\to 0} \frac{\psi(x+v) - \psi(x) - p^-\cdot v}{|v|} \geq 0,$$

$$\limsup_{|v|\to 0} \frac{\psi(x+v) - \psi(x) - p^+\cdot v}{|v|} \leq 0.$$

Subtracting these two identities, we obtain

$$\liminf_{|v|\to 0} \frac{(p^+ - p^-)\cdot v}{|v|} \geq 0.$$

In particular, by choosing $v = -\epsilon \frac{p^+ - p^-}{|p^- - p^+|}$, we get

$$-|p^- - p^+| \geq 0,$$

so, $p^- = p^+ \equiv p$. Consequently,

$$\lim_{|v| \to 0} \frac{\psi(x+v) - \psi(x) - p \cdot v}{|v|} = 0$$

and thus $D_x \psi = p$.

To prove the converse statement, it suffices to see that if $\psi$ is differentiable, then

$$\psi(x+v) = \psi(x) + D_x \psi(x) \cdot v + o(|v|).$$

□

**Proposition 3.6.** *Let*

$$\psi : \mathbb{R}^d \to \mathbb{R}$$

*be a continuous function. Fix $x_0 \in \mathbb{R}^d$. If $\phi : \mathbb{R}^d \to \mathbb{R}$ is a $C^1$ function such that*

$$\psi(x) - \phi(x)$$

*has a local maximum (resp. minimum) at $x_0$, then*

$$D_x \phi(x_0) \in D_x^+ \psi(x_0) \quad (\textit{resp. } D_x^- \psi(x_0)).$$

*Proof.* Suppose that $\psi(x) - \phi(x)$ has a strict local maximum at 0. Without loss of generality, we can assume that $\psi(0) - \phi(0) = 0$ and $\phi(0) = 0$. So, $\psi(x) - \phi(x) \leq 0$ or, equivalently,

$$\psi(x) \leq p \cdot x + (\phi(x) - p \cdot x).$$

Thus, by setting $p = D_x \phi(0)$ and using

$$\lim_{|x| \to 0} \frac{\phi(x) - p \cdot x}{|x|} = 0,$$

we get $D_x \phi(0) \in D_x^+ \psi(0)$. The case of a minimum is similar. □

**Proposition 3.7.** *Let $u$ be given by (3.9) and let $\boldsymbol{x}$ be a corresponding optimal trajectory. Suppose that $V$ is of class $C^2$. Then, $\boldsymbol{p} = -\dot{\boldsymbol{x}}$ satisfies*

- $\boldsymbol{p}(t') \in D_x^- u(\boldsymbol{x}(t'), t')$ *for* $t < t' \leq T$;
- $\boldsymbol{p}(t') \in D_x^+ u(\boldsymbol{x}(t'), t')$ *for* $t \leq t' < T$.

*Proof.* Let $t < t' \leq T$. By the dynamic programming principle, we have

$$u(x,t) = \int_t^{t'} \frac{|\dot{\mathbf{x}}|^2}{2} - V(\mathbf{x})ds + u(\mathbf{x}(t'), t').$$

Furthermore,

$$u(x,t) \leq \int_t^{t'} \frac{|\dot{\mathbf{x}} + \frac{y}{t'-t}|^2}{2} - V\left(\mathbf{x} + y\frac{s-t}{t'-t}\right) ds + u(\mathbf{x}(t') + y, t').$$

Let

$$\Phi(y) = u(x,t) - \int_t^{t'} \frac{|\dot{\mathbf{x}} + \frac{y}{t'-t}|^2}{2} - V\left(\mathbf{x} + y\frac{s-t}{t'-t}\right) ds.$$

Accordingly,

$$u(\mathbf{x}(t') + y, t') - \Phi(y)$$

has a minimum at $y = 0$. Thus, $D_y\Phi(0) \in D_x^- u(\mathbf{x}(t'), t')$. In addition,

$$D_y\Phi(0) = -\int_t^{t'} \frac{\dot{\mathbf{x}}}{t'-t} - D_xV(\mathbf{x})\frac{s-t}{t'-t} = -\dot{\mathbf{x}}(t') = \mathbf{p}(t')$$

after integrating by parts and using Proposition 3.3.

To prove the second item in the theorem, we use the inequality

$$u(x+y,t) \leq \int_t^{t'} \frac{|\dot{\mathbf{x}} - \frac{y}{t'-t}|^2}{2} - V\left(\mathbf{x} + y\frac{t'-s}{t'-t}\right) ds + u(\mathbf{x}(t'), t').$$

Next, let

$$\Psi(y) = \int_t^{t'} \frac{|\dot{\mathbf{x}} - \frac{y}{t'-t}|^2}{2} - V\left(\mathbf{x} + y\frac{t'-s}{t'-t}\right) ds + u(\mathbf{x}(t'), t').$$

Then the function $u(x+y,t) - \Psi(y)$ has a maximum at $y = 0$. Thus, arguing as before, we get the second part of the theorem. ◻

### 3.2.4 Regularity of the Value Function

A function, $\psi : \mathbb{R}^d \to \mathbb{R}$, is semiconcave if there exists a constant, $C$, such that $\psi - C|x|^2$ is a concave function. Here, we prove that the value function of (3.9) is bounded, Lipschitz, and semiconcave.

**Proposition 3.8.** *Let $u(x,t)$ be given by (3.9). Suppose that $\|V\|_{C^2(\mathbb{R}^d)} \leq C$. Then, there exist constants, $C_0$, $C_1$, and $C_2$, depending only on $u_T$ and $T-t$, such that*

- $|u| \leq C_0$ *for all* $x \in \mathbb{R}^d, 0 \leq t \leq T$.
- $|u(x+y,t) - u(x,t)| \leq C_1|y|$ *for all* $x, y \in \mathbb{R}^d, 0 \leq t \leq T$.
- $u(x+y,t) + u(x-y,t) - 2u(x,t) \leq C_2\left(1 + \frac{1}{T-t}\right)|y|^2$ *for all* $x, y \in \mathbb{R}^d, \quad 0 \leq t < T$.

*Proof.* For the first claim, we take $\mathbf{x}(s) = x$ for $s \in [t,T]$. Therefore,

$$u(x,t) \leq -\int_t^T V(x)ds + u_T(\mathbf{x}(T)) \leq (T-t)c_1 + \|u_T\|_\infty.$$

Furthermore, for any trajectory, $\mathbf{x}$, with $\mathbf{x}(t) = x$, we have

$$\int_t^T \frac{|\dot{\mathbf{x}}(s)|^2}{2} - V(\mathbf{x}(s))ds + u_T(\,\mathbf{x}(T)) \geq -(T-t)\|V\|_\infty - \|u_T\|_\infty.$$

To prove that $u$ is Lipschitz, take $x, y \in \mathbb{R}^d$. Proposition 3.2 gives the existence of an optimal trajectory, $\mathbf{x}^*$, for any initial condition $(x,t)$. Therefore,

$$u(x,t) = \int_t^T \frac{|\dot{\mathbf{x}}^*(s)|^2}{2} - V(\mathbf{x}^*(s))ds + u_T(\mathbf{x}^*(T))$$

and, because $\mathbf{x}^* + y$ is a sub-optimal trajectory, we have

$$u(x+y,t) \leq \int_t^T \frac{|\dot{\mathbf{x}}^*(s)|^2}{2} - V(\mathbf{x}^*(s)+y)ds + u_T(\mathbf{x}^*(T)+y).$$

Then,

$$u(x+y,t) - u(x,t) \leq [(T-t)\|V\|_{C^1} + \|u_T\|_{C^1}]\,|y| \leq (C(T-t)+C)|y|.$$

The previous estimate proves that $u$ is uniformly Lipschitz in $x$.

For the semiconcavity, we take $x, y \in \mathbb{R}^d$ with $|y| \leq 1$, $\mathbf{x}^*$ as above, $\mathbf{y}(s) = y\frac{T-s}{T-t}$,

$$u(x \pm y, t) \leq \int_t^T \frac{|\dot{\mathbf{x}}^*(s) \pm \dot{\mathbf{y}}(s)|^2}{2} - V(\mathbf{x}^*(s) \pm \mathbf{y}(s))ds + u_T(\mathbf{x}^*(T)).$$

Finally, we conclude that

$$u(x + y, t) + u(x - y, t) - 2u(x, t) \leq \frac{|y|^2}{(T-t)} + \|V\|_{C^2}|y|^2 \leq C_2\left(1 + \frac{1}{T-t}\right)|y|^2.$$

□

*Remark 3.9.* If both $V$ and $u_T$ have bounded $C^2$ norms, the semiconcavity estimates in the preceding proposition can be improved and do not depend on $T - t$.

## 3.3 Integral Bernstein Estimate

The integral Bernstein estimate is an important tool for the analysis of second-order stationary Hamilton–Jacobi equations with $L^p$ data. Here, we examine the Hamilton–Jacobi equation,

$$-\Delta u(x) + \frac{|Du(x)|^2}{2} + V(x) = \bar{H}, \tag{3.16}$$

with $V \in L^p$. Our goal is to bound the norm of $Du$ in $L^q$ for some $q > 1$. These estimates are used in Chap. 7 to establish bounds for MFGs in Sobolev spaces. Then, by a bootstrapping argument, they give a priori smoothness for the solutions.

Before stating the main result, we prove two auxiliary estimates.

**Lemma 3.10.** *Let $u \colon \mathbb{T}^d \to \mathbb{R}$ be a $C^3$ function and let $v = |Du|^2$. Suppose that $V$ is $C^1$. Then, there exist positive constants, $c$ and $C$, which do not depend on $u$ or $V$ such that, for every $p > 1$,*

$$-\int_{\mathbb{T}^d} v^p \Delta v \mathrm{d}x \geq \frac{4pc}{(p+1)^2}\left[\left(\int_{\mathbb{T}^d} v^{\frac{(p+1)d}{d-2}}\mathrm{d}x\right)^{\frac{d-2}{d}} - C\left(\int_{\mathbb{T}^d} v^{p+2}\mathrm{d}x\right)^{\frac{p+1}{p+2}}\right]$$

*and*

$$-2\int_{\mathbb{T}^d} DV \cdot Du\, v^p \mathrm{d}x \leq \frac{1}{2}\int_{\mathbb{T}^d} \left|D^2 u\right|^2 v^p \mathrm{d}x + C_p \int_{\mathbb{T}^d} |V|^2 v^p \mathrm{d}x.$$

*Proof.* By integration by parts, we have the identity

$$-\int_{\mathbb{T}^d} v^p \Delta v \mathrm{d}x = \int_{\mathbb{T}^d} p v^{p-1} |Dv|^2 \mathrm{d}x = \frac{4p}{(p+1)^2} \int_{\mathbb{T}^d} |Dv^{\frac{p+1}{2}}|^2 \mathrm{d}x.$$

Next, we use Sobolev's inequality to obtain

$$\int_{\mathbb{T}^d} |Dv^{\frac{p+1}{2}}|^2 \mathrm{d}x + \int_{\mathbb{T}^d} v^{p+1} \mathrm{d}x \geq c \left\| v^{\frac{p+1}{2}} \right\|_{2^*}^2 = c \left( \int_{\mathbb{T}^d} v^{\frac{(p+1)d}{d-2}} \mathrm{d}x \right)^{\frac{d-2}{d}},$$

where $2^* = \frac{2d}{d-2}$ is the Sobolev conjugated exponent of 2. Combining the above two inequalities with

$$\int_{\mathbb{T}^d} v^{p+1} \mathrm{d}x \leq \left( \int_{\mathbb{T}^d} v^{p+2} \mathrm{d}x \right)^{\frac{p+1}{p+2}},$$

we get the first estimate.

For the second inequality, we integrate again by parts to get

$$-\int_{\mathbb{T}^d} DV \cdot Du\, v^p \mathrm{d}x = \int_{\mathbb{T}^d} V \Delta u\, v^p \mathrm{d}x + p \int_{\mathbb{T}^d} V\, v^{p-1} Dv \cdot Du \mathrm{d}x. \tag{3.17}$$

Next, we apply a weighted Cauchy inequality to each of the terms in the right-hand side of the prior identity. First, for any $\kappa > 0$, we have

$$\begin{aligned} \int_{\mathbb{T}^d} V \cdot \Delta u\, v^p \mathrm{d}x &\leq \kappa \int_{\mathbb{T}^d} |\Delta u|^2\, v^p \mathrm{d}x + C_\kappa \int_{\mathbb{T}^d} |V|^2\, v^p \mathrm{d}x \\ &\leq \frac{1}{8} \int_{\mathbb{T}^d} \left|D^2 u\right|^2 v^p \mathrm{d}x + C \int_{\mathbb{T}^d} |V|^2\, v^p \mathrm{d}x, \end{aligned}$$

if we select a small enough $\kappa$. Next, because $v = |Du|^2$ implies $Dv = 2D^2 u Du$, we have

$$p \int_{\mathbb{T}^d} V\, v^{p-1} Dv \cdot Du \mathrm{d}x \leq 2p \int_{\mathbb{T}^d} |V|\, v^p |D^2 u| \mathrm{d}x \leq \frac{1}{8} \int_{\mathbb{T}^d} \left|D^2 u\right|^2 v^p \mathrm{d}x + C_p \int_{\mathbb{T}^d} |V|^2\, v^p \mathrm{d}x.$$

Using the two preceding bounds in (3.17), we get the second estimate. □

**Theorem 3.11.** *Let $u$ be a $C^3$ solution of (3.16). Suppose that $V$ is $C^1$. Then, for any $p > 1$, there exists a constant, $C_p > 0$, that depends only on $|\bar{H}|$, such that*

$$\|Du\|_{L^{\frac{2d(p+1)}{d-2}}(\mathbb{T}^d)} \leq C_p \left( 1 + \|V\|_{L^{\frac{2d(1+p)}{d+2p}}(\mathbb{T}^d)} \right).$$

*Note that $\gamma_p = \frac{2d(1+p)}{d+2p} \to d$ when $p \to \infty$ and that $\gamma_p$ is increasing when $d > 2$.*

*Proof.* We set $v = |Du|^2$. Differentiating (3.16), we have

$$\Delta u_{x_i} = \frac{1}{2} v_{x_i} + V_{x_i}. \tag{3.18}$$

Thus,

$$-\Delta v = -2 \sum_{i,j=1}^{d} \left(u_{x_i x_j}\right)^2 - 2 \sum_{i=1}^{d} u_{x_i} \Delta u_{x_i} = -2 \sum_{i,j=1}^{d} \left(u_{x_i x_j}\right)^2 - 2 \sum_{i=1}^{d} u_{x_i} \left(\frac{1}{2} v_{x_i} + V_{x_i}\right). \tag{3.19}$$

By multiplying (3.19) by $v^p$ and integrating over $\mathbb{T}^d$, we have

$$\begin{aligned} -\int_{\mathbb{T}^d} v^p \Delta v \mathrm{d}x + 2 \int_{\mathbb{T}^d} \left|D^2 u\right|^2 v^p \mathrm{d}x & \\ = -\int_{\mathbb{T}^d} Du \cdot Dv\, v^p \mathrm{d}x - 2 \int_{\mathbb{T}^d} DV \cdot Du\, v^p \mathrm{d}x. & \end{aligned} \tag{3.20}$$

Lemma 3.10 provides bounds for the first term on the left-hand side and the last term on the right-hand side of (3.20).

Further, we observe that for all $\delta > 0$, there exists a constant, $C_\delta > 0$, that does not depend on $u$, such that

$$-\int_{\mathbb{T}^d} Du \cdot Dv\, v^p \mathrm{d}x \leq \delta \int_{\mathbb{T}^d} \left|D^2 u\right|^2 v^p \mathrm{d}x + \frac{C_\delta}{p+1} \int_{\mathbb{T}^d} v^{p+2} \mathrm{d}x \tag{3.21}$$

for every $p > 1$. To check (3.21), we integrate by parts and get

$$\begin{aligned} & -2 \int_{\mathbb{T}^d} v^p Du \cdot Dv \mathrm{d}x = \frac{2}{p+1} \int_{\mathbb{T}^d} v^{p+1} \Delta u \mathrm{d}x \\ & \leq \frac{C}{p+1} \int_{\mathbb{T}^d} v^{p+1} |D^2 u| \mathrm{d}x \leq \delta \int_{\mathbb{T}^d} \left|D^2 u\right|^2 v^p \mathrm{d}x + \frac{C_\delta}{p+1} \int_{\mathbb{T}^d} v^{p+2} \mathrm{d}x, \end{aligned} \tag{3.22}$$

by Young's inequality.

Now, we claim that for any large enough $p > 1$, there exists $C_p > 0$ that does not depend on $u$, such that

$$\left(\int_{\mathbb{T}^d} v^{\frac{d(p+1)}{d-2}} \mathrm{d}x\right)^{\frac{(d-2)}{d(p+1)}} \leq C_p \left(\int_{\mathbb{T}^d} |V|^{2\beta_p} \mathrm{d}x\right)^{\frac{1}{\beta_p}} + C_p, \tag{3.23}$$

where $\beta_p$ is the conjugate exponent of $\frac{d(p+1)}{(d-2)p}$. Further, $\beta_p \to \frac{d}{2}$ when $p \to \infty$.

To prove the previous claim, we combine (3.20) and the first estimate in Lemma 3.10 to get

$$
\begin{aligned}
&c_p\left(\int_{\mathbb{T}^d} v^{\frac{d(p+1)}{d-2}}\mathrm{d}x\right)^{\frac{d-2}{d}} + 2\int_{\mathbb{T}^d}\left|D^2u\right|^2 v^p \mathrm{d}x \leq \qquad (3.24)\\
&c_p\left(\int_{\mathbb{T}^d} v^{p+2}\mathrm{d}x\right)^{\frac{p+1}{p+2}} - \int_{\mathbb{T}^d} Du\cdot Dv\, v^p\mathrm{d}x - 2\int_{\mathbb{T}^d} DV\cdot Du\, v^p\mathrm{d}x,
\end{aligned}
$$

where $c_p := \frac{4p\tilde{C}}{(p+1)^2}$, for some constant $\tilde{C}$. From the second estimate in Lemma 3.10, (3.21), and Young's inequality, ($z^{\frac{p+1}{p+2}} \leq \epsilon z + C_{p,\epsilon}$), we have

$$
\begin{aligned}
&c_p\left(\int_{\mathbb{T}^d} v^{\frac{d(p+1)}{d-2}}\mathrm{d}x\right)^{\frac{d-2}{d}} + \left[2-\left(\frac{1}{2}+\delta\right)\right]\int_{\mathbb{T}^d}\left|D^2u\right|^2 v^p\mathrm{d}x \leq \qquad (3.25)\\
&C_p\int_{\mathbb{T}^d}|V|^2 v^p\mathrm{d}x + \left(\frac{C_\delta}{p+1}+\delta\right)\int_{\mathbb{T}^d} v^{p+2}\mathrm{d}x + C_{p,\delta}.
\end{aligned}
$$

Next, using (3.16), we have

$$
\begin{aligned}
&\int_{\mathbb{T}^d}\left|D^2u\right|^2 v^p\mathrm{d}x \geq \frac{1}{d}\int_{\mathbb{T}^d}|\Delta u|^2 v^p\mathrm{d}x = \frac{1}{d}\int_{\mathbb{T}^d}\left|\frac{v}{2}+V-\overline{H}\right|^2 v^p\mathrm{d}x \qquad (3.26)\\
&\geq \frac{1}{3d}\int_{\mathbb{T}^d} v^2v^p\mathrm{d}x - \frac{1}{d}\int_{\mathbb{T}^d} V^2v^p\mathrm{d}x - \frac{1}{d}C\int v^p\mathrm{d}x\\
&\geq c\int_{\mathbb{T}^d} v^{p+2}\mathrm{d}x - C\int_{\mathbb{T}^d} V^2v^p\mathrm{d}x - C_p,
\end{aligned}
$$

where the second inequality follows from $(a-b-c)^2 \geq \frac{1}{3}a^2 - b^2 - c^2$.

For a small $\delta$ and a large enough $p$, (3.25) and (3.26) yield

$$
\begin{aligned}
\left(\int_{\mathbb{T}^d} v^{\frac{d(p+1)}{d-2}}\mathrm{d}x\right)^{\frac{d-2}{d}} &\leq C_p\int_{\mathbb{T}^d}|V|^2 v^p\mathrm{d}x \quad + \; C_p\\
&\leq C_p\left(\int_{\mathbb{T}^d} v^{\frac{d(p+1)}{d-2}}\mathrm{d}x\right)^{\frac{(d-2)p}{d(p+1)}}\left(\int_{\mathbb{T}^d}|V|^{2\beta_p}\mathrm{d}x\right)^{\frac{1}{\beta_p}} + \; C_p.
\end{aligned}
$$

Hence,

$$
\left(\int_{\mathbb{T}^d} v^{\frac{d(p+1)}{d-2}}\mathrm{d}x\right)^{\frac{(d-2)}{d(p+1)}} \leq C_p\left(\int_{\mathbb{T}^d}|V|^{2\beta_p}\mathrm{d}x\right)^{\frac{1}{\beta_p}} + \; C_p.
$$

This last estimate gives (3.23), and the theorem follows. □

## 3.4 Integral Estimates for HJ Equations

Here, we prove $L^p$ estimates for positive subsolutions of Hamilton–Jacobi equations; that is, we consider functions, $u : \mathbb{T}^d \times [0, T] \to \mathbb{R}$, that satisfy

$$-u_t + \frac{1}{\gamma}|Du|^\gamma + V(x) \leq \Delta u. \tag{3.27}$$

The assumption that $u$ is positive is not critical. Analogous results hold for subsolutions that are bounded from below.

Usually, bounds from below result from the comparison principle in Sect. 3.1. In this section, our goal is to obtain bounds from above. Bounds for solutions of Hamilton–Jacobi equations are essential to get higher regularity. For example, in the next section, we need $L^\infty$ bounds to prove Sobolev regularity of solutions of (3.27).

Two distinct mechanisms give integrability for positive subsolutions of (3.27). The first corresponds to optimal control and is linked to the first-order term. The second is given by stochastic effects and is associated with the Laplacian. In the subquadratic case, $\gamma \leq 2$, diffusion dominates; in the superquadratic case, $\gamma > 2$, the optimal control is the primary source of regularity. Next, we isolate these two effects and prove two estimates using elementary arguments. We begin by considering bounds that are derived through an optimal control technique.

**Proposition 3.12.** *Suppose that $V$ is continuous. Let $p > d+1$. Let $u : \mathbb{T}^d \times [0, T] \to \mathbb{R}$, $u$ of class $C^2$, with $u \geq 0$, solve*

$$-u_t + \frac{1}{\gamma}|Du|^\gamma + V(x,t) \leq 0. \tag{3.28}$$

*Suppose that $u(x, T) = u_T(x)$ is continuous. Then, there exists a constant, $C > 0$, depending only on $\|u_T\|_{L^\infty(\mathbb{T}^d)}$, $\gamma$, $p$, $T$, and $d$ such that*

$$\|u\|_{L^\infty(\mathbb{T}^d \times [0,T])} \leq C + C\|V\|_{L^p(\mathbb{T}^d \times [0,T])}^{\frac{\gamma' p}{d+\gamma' p}}.$$

*Proof.* Let $\gamma'$ be determined by $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$. Fix $y \in \mathbb{R}^d$ and consider the trajectory

$$\mathbf{x}^y(s) = x + \frac{y(s-t)}{T-t}. \tag{3.29}$$

Recall that the Legendre transform of $\frac{|p|^\gamma}{\gamma}$ is $\frac{|v|^{\gamma'}}{\gamma'}$. Consequently, using a version of the representation formula (3.9) for (3.28), we have

$$u(x,t) \leq \int_t^T \left[ \frac{\left|\frac{y}{T-t}\right|^{\gamma'}}{\gamma'} - V\left(x + \frac{y(s-t)}{T-t}, s\right)\right] \mathrm{d}s + u_T(x+y).$$

Next, we average the prior upper bound with the Gaussian kernel

$$\frac{e^{-\frac{|y|^2}{2\sigma^2}}}{(2\pi\sigma^2)^{d/2}}.$$

Because $u_T$ is bounded, $\int_{\mathbb{R}^d} u_T(x+y)\frac{e^{-\frac{|y|^2}{2\sigma^2}}}{(2\pi\sigma^2)^{d/2}} \leq C$. We have

$$\int_t^T \int_{\mathbb{R}^d} \frac{\left|\frac{y}{T-t}\right|^{\gamma'}}{\gamma'} \frac{e^{-\frac{|y|^2}{2\sigma^2}}}{(2\pi\sigma^2)^{d/2}} \mathrm{d}y\mathrm{d}s = C\frac{\sigma^{\gamma'}}{|T-t|^{\gamma'-1}},$$

where $C$ is a constant that does not depend on $\sigma$.

For $p'$ given by $\frac{1}{p} + \frac{1}{p'} = 1$, we have

$$\left\| \frac{e^{-\frac{|y|^2}{2\sigma^2}}}{(2\pi\sigma^2)^{d/2}} \right\|_{L^{p'}(\mathbb{R}^d)} = \frac{C}{\sigma^{d(p'-1)/p'}},$$

for some constant, $C$, independent of $\sigma$. Then, by a change of variables,

$$\begin{aligned} &-\int_t^T \int_{\mathbb{R}^d} V\left(x + \frac{y(s-t)}{T-t}, s\right) \frac{e^{-\frac{|y|^2}{2\sigma^2}}}{(2\pi\sigma^2)^{d/2}} \mathrm{d}y\mathrm{d}s \\ &\leq \frac{C}{\sigma^{d(p'-1)/p'}} \int_t^T \|V(\cdot,s)\|_{L^p(\mathbb{T}^d)} \left|\frac{(s-t)}{T-t}\right|^{-d/p} \mathrm{d}s \\ &\leq \frac{C}{\sigma^{d/p}} \|V(\cdot,s)\|_{L^p(\mathbb{T}^d\times[t,T])} \left(\int_t^T \left|\frac{(s-t)}{T-t}\right|^{-dp'/p} \mathrm{d}s\right)^{1/p'}. \end{aligned} \tag{3.30}$$

For $p > d+1$, we have

$$\left(\int_t^T \left|\frac{(s-t)}{T-t}\right|^{-dp'/p} \mathrm{d}s\right)^{1/p'} = C(T-t)^{1/p'}.$$

Combining these identities gives

$$u(x,t) \leq C\frac{\sigma^{\gamma'}}{|T-t|^{\gamma'-1}} + \frac{C|T-t|^{1/p'}}{\sigma^{d(p'-1)/p'}} \|V\|_{L^p(\mathbb{T}^d\times[t,T])} + C.$$

By minimizing over $\sigma$, we get

$$u(x,t) \leq C + C\|V\|_{L^p(\mathbb{T}^d\times[0,T])}^{\frac{\gamma' p}{d+\gamma' p}}.$$

We note that the constant, $C$, can be chosen uniformly for $0 \leq t \leq T$. □

*Remark 3.13.* The result in the previous proposition can be refined by replacing the sub-optimal trajectories in (3.29) by

$$\mathbf{x}^y(s) = x + y\frac{(s-t)^\kappa}{(T-t)^\kappa}$$

and selecting $\kappa > 0$ conveniently. We leave this approach to the reader who can verify that, in the quadratic case, $\gamma = 2$, it is enough to assume that $V \in L^p(\mathbb{T}^d \times [0,T])$ with $p > d/2 + 1$. For a similar result, obtained by a different method, see Proposition 3.18 below.

*Remark 3.14.* If $V \in L^\infty([0,T], L^p(\mathbb{T}^d))$ with $p > d$, we obtain a similar bound from (3.30). Using the method from the preceding remark, we see that $V \in L^\infty([0,T], L^p(\mathbb{T}^d))$ for $p > \frac{d}{2}$ is enough to ensure $L^\infty$ bounds in the quadratic case, $\gamma = 2$.

**Proposition 3.15.** *Suppose that $V$ is continuous. Let $p > \frac{d}{2} + 1$. Let $u \geq 0$, $u$ of class $C^2$, satisfy (3.27). Suppose that $u(x,T) = u_T(x)$ is continuous. Then, there exists a constant, $C > 0$, that depends only on $\|u_T\|_{L^\infty(\mathbb{T}^d)}$, $\gamma$, $p$, and $d$, such that*

$$\|u\|_{L^\infty(\mathbb{T}^d\times[0,T])} \leq C + C\|V\|_{L^p(\mathbb{T}^d\times[0,T])}.$$

*Proof.* Because $u$ solves (3.27), we have

$$-u_t + V(x,t) \leq \Delta u.$$

By the comparison principle,

$$u \leq w,$$

where $w$ solves

$$-w_t + V(x,t) = \Delta w,$$

with $w(x,T) = u_T(x)$. We write $w = w_0 + w_1$ where $w_0$ solves the previous equation with homogeneous terminal data and $w_1$ solves the homogeneous backward heat equation,

$$-(w_1)_t = \Delta w_1,$$

with $w_1(x,T) = u_T$. Clearly, $w_1$ is bounded by $\|u_T\|_{L^\infty(\mathbb{T}^d)}$.

To bound $w_0$, we use the fundamental solution for the backwards heat equation with homogeneous terminal data to get

$$w_0(x,t) = \int_t^T \int_{\mathbb{R}^d} V(x+y,s) \frac{e^{-\frac{|y|^2}{4(T-s)}}}{(4\pi(T-s))^{d/2}} \mathrm{d}y\mathrm{d}s.$$

We observe that for $p'$ given by $\frac{1}{p} + \frac{1}{p'} = 1$, we have

$$\left\| \sum_{k\in\mathbb{Z}^d} \frac{e^{-\frac{|y+k|^2}{4(T-s)}}}{(4\pi(T-s))^{d/2}} \right\|_{L^{p'}(\mathbb{T}^d)} = \frac{C}{|T-s|^{d(p'-1)/(2p')}}.$$

By Hölder's inequality,

$$\begin{aligned} |w_0(x,t)| &\le C\int_t^T \|V(\cdot,s)\|_{L^p(\mathbb{T}^d)} \frac{1}{|T-s|^{d(p'-1)/(2p')}} \mathrm{d}s \\ &\le C\|V\|_{L^p(\mathbb{T}^d\times[t,T])} \left( \int_t^T \frac{1}{|T-s|^{d(p'-1)/2}} \mathrm{d}s \right)^{1/p'}. \end{aligned} \tag{3.31}$$

If $d(p'-1)/2 < 1$, the preceding integral converges and gives the desired bound. The prior condition is equivalent to $p > \frac{d}{2} + 1$. □

*Remark 3.16.* The boundedness of $u$ also holds if $V \in L^\infty([0,T], L^p(\mathbb{T}^d))$ for $p > \frac{d}{2}$, as we can see from (3.31).

Next, we give an integral identity that is useful in the study of (3.27).

**Lemma 3.17.** *Let $\phi$ be an increasing $C^2$ function. Let $u \ge 0$ solve (3.27). Then,*

$$-\frac{d}{dt}\int_{\mathbb{T}^d} \phi(u)\mathrm{d}x + \int_{\mathbb{T}^d} \left[\phi'(u)|Du|^\gamma + \phi''(u)|Du|^2\right] \mathrm{d}x \le \int_{\mathbb{T}^d} |V(x,t)|\phi'(u)\mathrm{d}x.$$

*Proof.* The proof follows by multiplying (3.27) by $\phi'(u)$. □

Our last result is an application of the foregoing lemma.

**Proposition 3.18.** *Suppose that $V$ is continuous. Fix $p \ge \min\{\frac{d}{2}, \frac{d}{\gamma}\}$. Let $u$ be a $C^2$ function, $u \ge 0$, solving (3.27). Suppose that $u(x,T) = u_T(x)$. Then, there exists a constant, $C_q > 0$, depending only on $\|V\|_{L^\infty([0,T],L^p(\mathbb{T}^d))}$, $\|u_T\|_{L^\infty(\mathbb{T}^d)}$, $p$, $\gamma$, $d$, and $q$ such that*

$$\|u\|_{L^\infty([0,T],L^q(\mathbb{T}^d))} \le C_q$$

*for any $1 \le q < \infty$.*

*Proof.* Let $\lambda > 0$, to be selected later (see Eq. (3.33) for the case $\gamma \leq 2$), and let

$$\psi = e^{\frac{\lambda u}{2}}.$$

Using Lemma 3.17, we have

$$-\frac{d}{dt}\int_{\mathbb{T}^d} \psi^2 \mathrm{d}x + c_\gamma \int_{\mathbb{T}^d} \lambda^{1-\gamma} |D\psi^{\frac{2}{\gamma}}|^\gamma \mathrm{d}x + 4\int_{\mathbb{T}^d} |D\psi|^2 \mathrm{d}x \leq \lambda \int_{\mathbb{T}^d} |V(x,t)|\psi^2 \mathrm{d}x,$$

where $c_\gamma > 0$ is a fixed constant. Integrating the previous identity in $[t, T]$, we have

$$\begin{aligned} \|\psi(\cdot,t)\|_{L^2}^2 + c_\gamma \int_t^T \int_{\mathbb{T}^d} \lambda^{1-\gamma} |D\psi^{\frac{2}{\gamma}}|^\gamma \mathrm{d}x\mathrm{d}s + 4\int_t^T \int_{\mathbb{T}^d} |D\psi|^2 \mathrm{d}x\mathrm{d}s \\ \leq \lambda \int_t^T \int_{\mathbb{T}^d} |V(x,t)|\psi^2 \mathrm{d}x\mathrm{d}s + \|\psi(\cdot,T)\|_{L^2(\mathbb{T}^d)}^2. \end{aligned} \tag{3.32}$$

First, we consider the case $\gamma \leq 2$. By Sobolev's inequality,

$$\|\psi\|_{L^{2^*}}^2 \leq C\int_{\mathbb{T}^d} \psi^2 + |D\psi|^2 \mathrm{d}x,$$

where $2^* = \frac{2d}{d-2}$. Therefore, by Hölder's inequality,

$$\begin{aligned} \int_{\mathbb{T}^d} |V(x,t)|\psi^2 \mathrm{d}x &\leq C\|V\|_{L^{\frac{d}{2}}(\mathbb{T}^d)} \int_{\mathbb{T}^d} \psi^2 + |D\psi|^2 \mathrm{d}x \\ &\leq C\|V\|_{L^{\frac{d}{2}}(\mathbb{T}^d)} \|\psi\|_{L^2(\mathbb{T}^d)}^2 + C\|V\|_{L^{\frac{d}{2}}} \int_{\mathbb{T}^d} |D\psi|^2 \mathrm{d}x. \end{aligned}$$

Applying the prior estimate on the right-hand side of (3.32), we get

$$\begin{aligned} &\|\psi(\cdot,t)\|_{L^2(\mathbb{T}^d)}^2 + 4\int_t^T \int_{\mathbb{T}^d} |D\psi|^2 \mathrm{d}x\mathrm{d}s \\ &\quad\leq \lambda C\|V\|_{L^\infty([0,T],L^{\frac{d}{2}}(\mathbb{T}^d))} \left( (T-t)\|\psi\|_{L^\infty([0,T],L^2(\mathbb{T}^d))}^2 + \int_t^T \int_{\mathbb{T}^d} |D\psi|^2 \mathrm{d}x\mathrm{d}s \right) \\ &\qquad + \|\psi(\cdot,T)\|_{L^2(\mathbb{T}^d)}^2. \end{aligned}$$

Let $0 \leq t^* \leq T$ such that

$$\|\psi(\cdot,t^*)\|_{L^2(\mathbb{T}^d)}^2 = \|\psi\|_{L^\infty([0,T],L^2(\mathbb{T}^d))}^2.$$

Next, we select a small enough $\lambda$ that may depend on $V$, such that

$$\lambda C(1+(T-t))\|V\|_{L^\infty([0,T],L^{\frac{d}{2}}(\mathbb{T}^d))} \leq \frac{1}{2}. \tag{3.33}$$

Then,

$$\frac{1}{2}\|\psi(\cdot,t^*)\|^2_{L^2(\mathbb{T}^d)} + 2\int_{t^*}^T \int_{\mathbb{T}^d} |D\psi|^2 \mathrm{d}x\mathrm{d}s \leq \|\psi(\cdot,T)\|^2_{L^2(\mathbb{T}^d)}.$$

Therefore, $\psi \in L^\infty([0,T],L^2(\mathbb{T}^d)$ and

$$\int_{t^*}^T \int_{\mathbb{T}^d} |D\psi|^2 \mathrm{d}x\mathrm{d}s \leq C.$$

Next, we use $\psi \in L^\infty([0,T],L^2(\mathbb{T}^d))$ to conclude that

$$\sup_{0\leq t\leq T} \int_t^T \int_{\mathbb{T}^d} |D\psi|^2 \mathrm{d}x\mathrm{d}s \leq C.$$

The case $\gamma > 2$ is similar because the Sobolev inequality gives

$$\|\psi^{\frac{2}{\gamma}}\|^\gamma_{L^{\gamma*}(\mathbb{T}^d)} \leq C\int_{\mathbb{T}^d} \psi^2 + |D\psi^{\frac{2}{\gamma}}|^\gamma \mathrm{d}x,$$

where $\gamma^* = \frac{\gamma d}{d-\gamma}$. Next,

$$\begin{aligned}\int_{\mathbb{T}^d} |V(x,t)|\psi^2 \mathrm{d}x &\leq C\|V\|_{L^{d/\gamma}(\mathbb{T}^d)}\|\psi^2\|_{L^{\gamma*/\gamma}(\mathbb{T}^d)} \\ &\leq C\|V\|_{L^{d/\gamma}(\mathbb{T}^d)} \int_{\mathbb{T}^d} \psi^2 + |D\psi^{\frac{2}{\gamma}}|^\gamma \mathrm{d}x.\end{aligned}$$

Then, the proof proceeds as before. □

*Remark 3.19.* The preceding result can be generalized for $V \in L^r([0,T],L^s(\mathbb{T}^d))$. For that, we take $\phi(z) = \frac{z^p}{p}$, for $p > 1$. By Lemma 3.17, we have

$$-\frac{d}{dt}\int_{\mathbb{T}^d} \frac{u^p}{p}\mathrm{d}x + \int_{\mathbb{T}^d} \left[u^{p-1}|Du|^\gamma + (p-1)u^{p-2}|Du|^2\right]\mathrm{d}x = -\int_{\mathbb{T}^d} Vu^{p-1}\mathrm{d}x.$$

Then, through an iterative process, we can get bounds for $u$ in various Lebesgue spaces. Here, we do not pursue this direction and point to the references at the end of the chapter.

## 3.5 Gagliardo–Nirenberg Estimates

In the analysis of Hamilton–Jacobi equations in MFGs, the following line of reasoning is frequently used: first, the comparison principle gives lower bounds for the solutions; second, the results in the previous sections provide upper bounds; finally, the methods outlined next give the regularity of solutions. These methods are based on the Gagliardo–Nirenberg inequality. We consider

$$-u_t + \frac{1}{\gamma}|Du|^{\gamma} + V(x) = \Delta u, \tag{3.34}$$

with $1 < \gamma < 2$. In this range of parameters, the nonlinearity $|Du|^{\gamma}$ can be regarded as a perturbation of the heat equation. Then, the Gagliardo–Nirenberg inequality together with the earlier bounds in $L^{\infty}$ and the regularity of the heat equation give estimates for the solutions in Sobolev spaces.

**Proposition 3.20.** *Let $u \in W^{2,p}(\mathbb{T}^d)$. Then, for $1 < r < \infty$, there exists a constant, $C > 0$, such that*

$$\|Du\|_{L^{\gamma r}([0,T],L^{\gamma p}(\mathbb{T}^d))} \leq C \left\|D^2u\right\|^{\frac{1}{2}}_{L^r([0,T],L^p(\mathbb{T}^d))} \|u\|^{\frac{1}{2}}_{L^{\infty}(\mathbb{T}^d\times[0,T])}.$$

*Proof.* The Gagliardo–Nirenberg inequality gives

$$\|Du\|_{L^{2r}([0,T],L^{2p}(\mathbb{T}^d))} \leq C \left\|D^2u\right\|^{\frac{1}{2}}_{L^r([0,T],L^p(\mathbb{T}^d))} \|u\|^{\frac{1}{2}}_{L^{\infty}(\mathbb{T}^d\times[0,T])}.$$

Because $1 < \gamma < 2$, we have

$$\|Du\|_{L^{\gamma r}([0,T],L^{\gamma p}(\mathbb{T}^d))} \leq C \|Du\|_{L^{2r}([0,T],L^{2p}(\mathbb{T}^d))}.$$

Therefore, by combining these two inequalities, we conclude the proof. □

Next, we recall a standard result for the heat equation.

**Lemma 3.21.** *Let $u$ be a solution of (3.34) with $1 < \gamma < 2$. Then, for $1 < r, p < \infty$, there exists a constant $C > 0$ such that*

$$\|u_t\|_{L^r\left([0,T],L^p(\mathbb{T}^d)\right)}, \left\|D^2u\right\|_{L^r\left([0,T],L^p(\mathbb{T}^d)\right)} \leq C\|V\|_{L^r\left([0,T],L^p(\mathbb{T}^d)\right)} + C\|Du\|^{\gamma}_{L^{\gamma r}\left([0,T],L^{\gamma p}(\mathbb{T}^d)\right)}.$$

By combining Proposition 3.20 with Lemma 3.21, we get the following Hessian integrability estimate:

**Proposition 3.22.** *Let $u$ be a solution of (3.34) with $1 < \gamma < 2$. Fix $1 < r, p < \infty$. Then, there exists a constant, $C > 0$, such that*

$$\left\|D^2u\right\|_{L^r\left([0,T],L^p(\mathbb{T}^d)\right)} \leq C\|V\|_{L^r\left([0,T],L^p(\mathbb{T}^d)\right)} + C\|u\|^{\frac{\gamma}{2-\gamma}}_{L^{\infty}(\mathbb{T}^d\times[0,T]))}.$$

*Proof.* We start by combining Lemma 3.21 with Proposition 3.20 to obtain

$$\left\|D^2u\right\|_{L^r\left([0,T],L^p(\mathbb{T}^d)\right)} \leq C\left\|D^2u\right\|^{\frac{\gamma}{2}}_{L^r\left([0,T],L^p(\mathbb{T}^d)\right)}\|u\|^{\frac{\gamma}{2}}_{L^\infty(\mathbb{T}^d\times[0,T]))} + C\|V\|_{L^r\left([0,T],L^p(\mathbb{T}^d)\right)}.$$

Because $1<\gamma<2$, a straightforward application of a weighted Young's inequality yields the result. □

**Theorem 3.23.** *Let $u$ be a solution of (3.34) with $1<\gamma<2$ and assume that $V\in L^r(0,T;L^p(\mathbb{T}^d))$ for $1<r,\ p<\infty$. Then, there exists a constant, $C>0$, such that*

$$\|Du\|_{L^{\gamma r}\left([0,T],L^{\gamma p}(\mathbb{T}^d)\right)} \leq C + C\|V\|_{L^r([0,T],L^p(\mathbb{T}^d))} + C\|u\|^{\frac{2}{2-\gamma}}_{L^\infty(\mathbb{T}^d\times[0,T])}.$$

*Proof.* By combining Propositions 3.20 and 3.22, we have

$$\|Du\|_{L^{\gamma r}\left([0,T],L^{\gamma p}(\mathbb{T}^d)\right)} \leq C\|V\|^{\frac{1}{2}}_{L^r\left([0,T],L^p(\mathbb{T}^d)\right)}\|u\|^{\frac{1}{2}}_{L^\infty(\mathbb{T}^d\times[0,T])} + C\|u\|^{\frac{2}{2-\gamma}}_{L^\infty(\mathbb{T}^d\times[0,T])}.$$

Notice that $\|V\|_{L^r\left([0,T],L^p(\mathbb{T}^d)\right)} < C$ for some $C>0$. In addition,

$$\frac{1}{2} < \frac{2}{2-\gamma},$$

for $1<\gamma<2$. The result follows from a weighted Young's inequality. □

By the estimates in Sect. 3.4, positive solutions of (3.34) are bounded in $L^\infty(\mathbb{T}^d\times[0,T])$, provided that $V\in L^p(\mathbb{T}^d\times[0,T])$ for $p$ satisfying

$$p > \frac{d}{2} + 1. \tag{3.35}$$

As a result, the foregoing Theorem gives uniform estimates for positive solutions of (3.27) that we state next.

**Corollary 3.24.** *Suppose that $u_T\geq 0$, $V\leq 0$, and $V\in L^p(\mathbb{T}^d\times[0,T])$ for some $p$ satisfying (3.35). Let $u$ solve (3.34). Then, there exists a constant, $C$, such that*

$$\|Du\|_{L^{\gamma p}(\mathbb{T}^d\times[0,T])} \leq C.$$

## 3.6 Bibliographical Notes

Hamilton–Jacobi equations arise in various contexts, including classical mechanics [10, 108], control theory [28, 87, 101, 177], and front propagation [43, 106, 188, 200]. Maximum and comparison principles are essential to the study of elliptic and parabolic equations, see [47, 107]. In Hamilton–Jacobi equations, comparison properties are at the heart of the theory of viscosity solutions [16, 19, 59, 80, 102, 112, 151, 172]. A classical introduction to calculus of variations is [40]. A more contemporary approach is described in [81] and [82]. Minimizers of calculus of variation problems with Sobolev potentials are examined in [100]. Further bounds on Hamilton–Jacobi equations using control theory methods are discussed in [16] and [59] for first-order equations, and in [102] for second-order equations. S. Bernstein introduced certain a priori estimates for the Dirichlet problems in [34, 35]. Some of these estimates were generalized for other problems. In particular, the integral Bernstein estimates considered here were developed in [173]. Different authors considered integral estimates for Hamilton–Jacobi equations and second-order elliptic problems. The exponential transform in Sect. 3.4 appeared in [151]. The estimate mentioned in Remark 3.13 was developed in [193]. The result in Remark 3.19 was discussed in [68]. Here, we did not study the Hölder estimates for superquadratic problems and refer the reader to [20, 58, 60, 65, 69]. The statement and proof of the Gagliardo–Nirenberg inequality can be found in [104]. The methods described in the previous section can be improved to include the quadratic case [9].

# Chapter 4
# Estimates for the Transport and Fokker–Planck Equations

In this chapter, we turn our attention to the second equation in the MFG system, the transport equation,

$$m_t(x,t) + \operatorname{div}(b(x,t)m(x,t)) = 0 \quad \text{in } \mathbb{T}^d \times [0,T], \tag{4.1}$$

or the Fokker–Planck equation,

$$m_t(x,t) + \operatorname{div}(b(x,t)m(x,t)) = \Delta m(x,t), \quad \text{in } \mathbb{T}^d \times [0,T], \tag{4.2}$$

where $b : \mathbb{T}^d \times [0,T] \to \mathbb{R}^d$ is a smooth vector field. Both (4.1) and (4.2) are equipped with the initial condition

$$m(x,0) = m_0(x). \tag{4.3}$$

We assume that $m_0 \geq 0$ with $\int m_0 = 1$; that is, $m_0$ is a probability measure. As before, we assume $m_0$ to be of class $C^\infty$ to simplify the presentation. Except for the discussion of weak solutions, a solution to (4.1) or (4.2) is a positive $C^\infty$ function, $m$. The choice of $\mathbb{T}^d$ as the spatial domain is of minor importance; many of our results extend to bounded domains with Dirichlet or Neumann boundary conditions or to $\mathbb{R}^d$ if we assume enough decay of the solution.

Our primary goal is to understand integrability and regularity properties of (4.1) and (4.2) in terms of the vector field, $b$.

## 4.1 Mass Conservation and Positivity of Solutions

In this section, we examine two properties of solutions to (4.1) and (4.2), namely positivity and mass conservation.



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SpringerBriefs in Mathematics, DOI 10.1007/978-3-319-38934-9_4

**Proposition 4.1.** *Let $m$ solve either (4.1) or (4.2) with the initial condition (4.3). Then,*

$$\int_{\mathbb{T}^d} m(x,t)\mathrm{d}x = 1$$

*for all $t \geq 0$.*

*Proof.* If $m$ solves either (4.1) or (4.2), integration by parts yields

$$\frac{d}{dt}\int_{\mathbb{T}^d} m(x,t)\mathrm{d}x = -\int_{\mathbb{T}^d} \operatorname{div}(bm)\mathrm{d}x = 0.$$

For solutions of (4.2), we combine the former computation with

$$\int_{\mathbb{T}^d} \Delta m(x,t)\mathrm{d}x = \int_{\mathbb{T}^d} \operatorname{div}(\nabla m(x,t))\mathrm{d}x = 0.$$

□

*Remark 4.2.* Proposition 4.1 holds in bounded domains with homogeneous Neumann boundary conditions if $b$ is orthogonal to the boundary. Those boundary conditions encode a zero net flow through the boundary. For Dirichlet boundary conditions, we have

$$\int_{\mathbb{T}^d} m(x,t)\mathrm{d}x \leq 1.$$

**Proposition 4.3.** *The transport equation (4.1) and the Fokker–Planck equation (4.2) preserve positivity: if $m_0 \geq 0$ and $m$ solves either (4.1) or (4.2), then $m(x,t) \geq 0,\ \forall (x,t) \in \mathbb{T}^d \times [0,T]$.*

*Proof.* The proof is based on a duality argument. We take $s \in [0,T]$ and consider the adjoint equation to (4.2):

$$\begin{cases} v_t(x,t) + b \cdot Dv(x,t) = -\Delta v(x,t),\ \forall (x,t) \in \mathbb{T}^d \times [0,s] \\ v(x,s) = \phi(x), \end{cases} \tag{4.4}$$

where $\phi \in C^\infty(\mathbb{T}^d)$, $\phi(x) > 0$, $\forall x \in \mathbb{T}^d$.

First, we claim that, by the comparison principle, $v(x,t) > 0,\ \forall (x,t) \in \mathbb{T}^d \times [0,s]$. Second, we multiply (4.2) by $m$ and add (4.4) multiplied by $v$. Integrating the resulting expression in $\mathbb{T}^d$, we get

$$\frac{d}{dt}\int_{\mathbb{T}^d} m(x,t)v(x,t)\mathrm{d}x = 0,$$

after integration by parts. Next, integrating in $[0, s]$, we have

$$\int_{\mathbb{T}^d} m(x,s)\phi(x)\mathrm{d}x = \int_{\mathbb{T}^d} v(x,0)m_0(x)\mathrm{d}x > 0.$$

Finally, since the previous identity holds for any positive $\phi$, we have $m(x,s) \geq 0$. □

## 4.2 Regularizing Effects of the Fokker–Planck Equation

To investigate the regularizing effects of the Fokker–Planck equation (4.2), we record two useful identities.

**Proposition 4.4.** *Let $m$ be a smooth solution of (4.2) with $m > 0$ and assume that $\phi \in C^2(\mathbb{R})$. Then,*

$$\frac{d}{dt}\int_{\mathbb{T}^d} \phi(m)\mathrm{d}x + \int_{\mathbb{T}^d} \operatorname{div}(b)\left(m\phi'(m) - \phi(m)\right)\mathrm{d}x = -\int_{\mathbb{T}^d} \phi''(m)\left|Dm\right|^2 \mathrm{d}x, \tag{4.5}$$

*or, equivalently,*

$$\frac{d}{dt}\int_{\mathbb{T}^d} \phi(m)\mathrm{d}x - \int_{\mathbb{T}^d} m\phi''(m)Dm\cdot b\mathrm{d}x = -\int_{\mathbb{T}^d} \phi''(m)\left|Dm\right|^2 \mathrm{d}x. \tag{4.6}$$

*Proof.* The two identities follow by multiplying (4.2) by $\phi'(m)$ and integrating by parts. □

Next, we record some consequences of the preceding result.

**Proposition 4.5.** *Let $m$ be a smooth solution of (4.2) with $m > 0$. Then, there exist $C > 0$ and $c > 0$, such that*

$$\frac{d}{dt}\int_{\mathbb{T}^d} \frac{1}{m(x,t)}\mathrm{d}x \leq C\int_{\mathbb{T}^d} \frac{|b|^2}{m}\mathrm{d}x - c\int_{\mathbb{T}^d} \frac{|Dm|^2}{m^3}\mathrm{d}x, \tag{4.7}$$

$$\frac{d}{dt}\int_{\mathbb{T}^d} \ln m(x,t)\mathrm{d}x \geq -C\int_{\mathbb{T}^d} |b|^2\mathrm{d}x + c\int_{\mathbb{T}^d} |D\ln m|^2\mathrm{d}x, \tag{4.8}$$

*and*

$$\frac{d}{dt}\int_{\mathbb{T}^d} m\ln m\mathrm{d}x \leq \int_{\mathbb{T}^d} |b(x,t)|\,|Dm|\,\mathrm{d}x - \int_{\mathbb{T}^d} \frac{|Dm|^2}{m}\mathrm{d}x. \tag{4.9}$$

*Proof.* For the first two assertions, we take, respectively, $\phi(z) \equiv \frac{1}{z^2}$ and $\phi(z) \equiv \frac{1}{z}$ in Proposition (4.4) to get

$$\frac{d}{dt}\int_{\mathbb{T}^d}\frac{1}{m(x,t)}\mathrm{d}x = 2\int_{\mathbb{T}^d} bm^{-2}Dm\mathrm{d}x - 2\int_{\mathbb{T}^d} m^{-3}|Dm|^2\mathrm{d}x$$

and

$$\frac{d}{dt}\int_{\mathbb{T}^d}\ln m(x,t)\mathrm{d}x = -\int_{\mathbb{T}^d} bm^{-1}Dm\mathrm{d}x + \int_{\mathbb{T}^d} m^{-2}|Dm|^2\mathrm{d}x.$$

Then, we use a weighted Cauchy inequality to get the results.

For the last assertion, we use $\phi(z) = z\ln z$ to get

$$\frac{d}{dt}\int_{\mathbb{T}^d} m(x,t)\ln m(x,t)\mathrm{d}x \le \int_{\mathbb{T}^d} |b(x,t)|\,|Dm|\,\mathrm{d}x - \int_{\mathbb{T}^d}\frac{|Dm|^2}{m}\mathrm{d}x.$$

□

**Corollary 4.6.** *Let $m$ be a smooth solution of (4.2) with $m > 0$, $m(x,0) = m_0$, $\int_{\mathbb{T}^d} m_0(x)\mathrm{d}x = 1$, and $m_0 > \gamma > 0$. Then, there exist constants, $C$, $C_\gamma > 0$, such that*

$$\int_0^T\int_{\mathbb{T}^d}|D\ln m|^2\mathrm{d}x\mathrm{d}t \le C\int_0^T\int_{\mathbb{T}^d}|b|^2\mathrm{d}x\mathrm{d}t + C_\gamma. \tag{4.10}$$

*Proof.* Because $m_0$ is bounded from below by a positive quantity, the integral

$$\int_{\mathbb{T}^d}\ln m(x,0)\mathrm{d}x$$

is bounded from below; moreover, Jensen's inequality bounds it from above. Hence,

$$\left|\int_{\mathbb{T}^d}\ln m(x,0)\mathrm{d}x\right| \le C.$$

Next, an additional application of Jensen's inequality gives

$$\int_{\mathbb{T}^d}\ln m(x,T)\mathrm{d}x \le 0.$$

Finally, we integrate (4.8) in $[0,T]$ and use the earlier bounds to get (4.10). □

**Corollary 4.7.** *Let $m$ be a smooth solution of (4.2) with $m > 0$, $m(x,0) = m_0$, $\int_{\mathbb{T}^d} m_0(x)\mathrm{d}x = 1$, $m_0 > 0$. Then,*

$$\int_{\mathbb{T}^d} m(x,T)\ln m(x,T)\mathrm{d}x + \int_0^T\int_{\mathbb{T}^d}\frac{|Dm|^2}{2m}\mathrm{d}x\mathrm{d}t$$
$$\le \frac{1}{2}\int_0^T\int_{\mathbb{T}^d}|b|^2 m\mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}^d} m(x,0)\ln m(x,0)\mathrm{d}x.$$

*Proof.* First, we integrate (4.9) in $[0, T]$. The result follows from the estimate

$$\int_0^T \int_{\mathbb{T}^d} |b||Dm| \le \int_0^T \int_{\mathbb{T}^d} \frac{|b|^2 m}{2} + \frac{|Dm|^2}{2m}.$$

□

## 4.3 Fokker–Planck Equation with Singular Initial Conditions

Let $\rho$ solve (4.2) with a Dirac delta as the initial condition; that is, $\rho$ solves

$$\begin{cases} \rho_t(x,t) + \operatorname{div}(b(x,t)\rho(x,t)) = \Delta\rho(x,t), \ \forall (x,t) \in \mathbb{T}^d \times [0,T] \\ \rho(x,0) = \delta_{x_0}, \end{cases} \tag{4.11}$$

in the sense of distributions. If $b$ is regular, then $\rho$ is a function for $t > 0$ (see Remark 4.9). In the next proposition, we give integral estimates on the derivatives of $\rho$.

**Proposition 4.8.** *Let $\rho$ solve (4.11). Then, for any $0 < \alpha < 1$, there exists a constant $C > 0$ that does not depend on the solution, such that*

$$\int_0^T \int_{\mathbb{T}^d} |D(\rho^{\frac{\alpha}{2}})|^2 \mathrm{d}x dt \le C + C \int_0^T \int_{\mathbb{T}^d} |b|^2 \rho^\alpha(x,t) \mathrm{d}x dt.$$

*Proof.* Using Proposition (4.4) for $m = \rho$ and $\phi(z) \equiv z^\alpha$ and integrating on $[0, T]$, we obtain

$$\begin{aligned} c_\alpha \int_0^T \int_{\mathbb{T}^d} |D(\rho^{\frac{\alpha}{2}})|^2 \mathrm{d}x dt =& \frac{1}{\alpha} \int_{\mathbb{T}^d} (\rho^\alpha(x,T) - \rho^\alpha(x,0)) \mathrm{d}x \\ &+ (1-\alpha) \int_0^T \int_{\mathbb{T}^d} \rho^{\alpha-1} b \cdot D\rho \mathrm{d}x dt \\ \le& C + \varepsilon \int_0^T \int_{\mathbb{T}^d} |D(\rho^{\frac{\alpha}{2}})|^2 \mathrm{d}x dt + C_\varepsilon \int_0^T \int_{\mathbb{T}^d} |b|^2 \rho^\alpha \mathrm{d}x dt \end{aligned}$$

for any $\varepsilon > 0$, where $c_\alpha = \frac{4(1-\alpha)}{\alpha^2}$. Here, we use

$$\int_{\mathbb{T}^d} \rho^\alpha(x,0) \mathrm{d}x \le 1, \qquad \int_{\mathbb{T}^d} \rho^\alpha(x,T) \mathrm{d}x \le 1,$$

and Jensen's inequality. Taking a small enough $\varepsilon$ yields the result. □

*Remark 4.9.* To justify rigorously the above computations, it suffices to argue by approximation. For that, we consider a family of solutions, $\rho^\varepsilon$, of (5.2) with the initial value $\eta_\varepsilon$, where $\eta_\varepsilon\colon \mathbb{T}^d \to \mathbb{R}$ is a family of smooth, compactly supported functions with $\int_{\mathbb{T}^d} \eta_\varepsilon(x)\mathrm{d}x = 1$ and $\eta_\varepsilon \rightharpoonup \delta_{x_0}$, as $\varepsilon \to 0$. Then, we carry out the preceding computations with $\rho^\varepsilon$ and, finally, we let $\varepsilon \to 0$.

## 4.4 Iterative Estimates for the Fokker–Planck Equation

Because (4.2) preserves mass and positivity of the initial conditions, solutions with initial data in $L^1(\mathbb{T}^d)$ satisfy $m(\cdot, t) \in L^1(\mathbb{T}^d)$ for every $t \in [0, T]$. However, in many applications, we need a higher regularity for $m$, and it is critical to ensure that

$$m(\cdot, t) \in L^p(\mathbb{T}^d),$$

for some $p > 1$. Here, we study the integrability of solutions of the Fokker–Planck equation using an iterative argument.

First, we consider solutions to (4.2) that satisfy the estimate

$$\int_0^T \int_{\mathbb{T}^d} |\operatorname{div} b(x,t)|^2 \, m\mathrm{d}x\mathrm{d}t \leq C.$$

Often, the previous estimate holds in MFGs (see Sects. 6.4 and 6.5 in Chap. 6). Next, we investigate estimates for $m$ that depend polynomially on $L^p$ norms of the drift term. $b$. Because the cases $p < \infty$ and $p = \infty$ require distinct treatment, these are presented separately.

### *4.4.1 Regularity by Estimates on the Divergence of the Drift*

In this section, we consider how to use estimates on $\operatorname{div} b$ to get a higher integrability for the solutions of (4.2)–(4.3). Before we proceed, we recall the following version of the Poincaré inequality:

**Proposition 4.10.** *For any $0 < r < 2$, there exists a constant, $C_r > 0$, such that, for any $f \in W^{1,2}(\mathbb{T}^d)$, we have*

$$\left(\int_{\mathbb{T}^d} |f|^{2^*} \mathrm{d}x\right)^{1/2^*} \leq C_r \left[\left(\int_{\mathbb{T}^d} |Df|^2 \mathrm{d}x\right)^{1/2} + \left(\int_{\mathbb{T}^d} |f|^r \mathrm{d}x\right)^{1/r}\right].$$

*Proof.* The proof uses the compactness argument of the Poincaré inequality. By the Sobolev theorem, it suffices to show that

$$\left(\int_{\mathbb{T}^d} |f|^2 \mathrm{d}x\right)^{1/2} \leq C_r \left[\left(\int_{\mathbb{T}^d} |Df|^2 \mathrm{d}x\right)^{1/2} + \left(\int_{\mathbb{T}^d} |f|^r \mathrm{d}x\right)^{1/r}\right].$$

If the inequality in the statement is not true, then we can find a sequence, $f_n$, such that

$$\left(\int_{\mathbb{T}^d} |f_n|^2 \mathrm{d}x\right)^{1/2} = 1$$

and

$$\left[\left(\int_{\mathbb{T}^d} |Df_n|^2 \mathrm{d}x\right)^{1/2} + \left(\int_{\mathbb{T}^d} |f_n|^r \mathrm{d}x\right)^{1/r}\right] < \frac{1}{n}.$$

By the Rellich–Kondrachov theorem, we can extract a subsequence such that $f_n \to \tilde{f}$, strongly in $L^2$, and therefore

$$\left(\int_{\mathbb{T}^d} |\tilde{f}|^2 \mathrm{d}x\right)^{1/2} = 1.$$

However, $D\tilde{f} = 0$ and, by Fatou's Lemma,

$$\int_{\mathbb{T}^d} |\tilde{f}|^r \mathrm{d}x \leq \liminf_{n\to\infty} \int_{\mathbb{T}^d} |f_n|^r \mathrm{d}x = 0.$$

Hence, $\tilde{f} = 0$, which is a contradiction. □

**Theorem 4.11.** *Let $m$ be a solution of* (4.2)–(4.3) *and assume that*

$$\int_0^T \int_{\mathbb{T}^d} |\operatorname{div} b(x,t)|^2 \, m \mathrm{d}x \mathrm{d}t < \tilde{C} \tag{4.12}$$

*for some $\tilde{C} > 0$. Then, there exists a constant, $C_r > 0$, that depends only on $\tilde{C}$, $r$ and the dimension $d$ such that*

1. *if $d > 2$,*

$$\|m^\epsilon\|_{L^\infty([0,T],L^r(\mathbb{T}^d))} \leq C_r$$

   *for $1 \leq r < \frac{2^*}{2}$;*
2. *if $d \leq 2$,*

$$\|m^\epsilon\|_{L^\infty([0,T],L^r(\mathbb{T}^d))} \leq C_r$$

   *for any $1 \leq r < \infty$.*

*Proof.* To prove the Theorem, we argue by induction. For that, we define an increasing sequence, $\beta_n$, for which we prove

$$\|m(\cdot,t)\|_{L^{1+\beta_n}(\mathbb{T}^d)} \leq C$$

for some $C = C(n) > 0$. Set $\beta_0 = 0$; thus, we have

$$\|m(\cdot,t)\|_{L^{1+\beta_0}(\mathbb{T}^d)} = 1 \leq C$$

for every $t \in [0,T]$.

We begin with the case $d > 2$ and define $\beta_{n+1}$ by

$$\beta_{n+1} = \frac{2}{d}(\beta_n + 1).$$

Because $\beta_n$ is the $n$th partial sum of the geometric series with term $\dfrac{2^n}{d^n}$ and $\beta_1 = \frac{2}{d}$, we get

$$\lim_{n\to\infty} \beta_n = \frac{2}{d-2} = \frac{2^*}{2} - 1.$$

Next, we set

$$q_n = \frac{2^*}{2}(\beta_{n+1} + 1) = \frac{d}{d-2}(\beta_{n+1} + 1).$$

Because $\beta_n < \frac{2}{d-2}$, we have

$$q_n > \frac{d}{d-2}\beta_{n+1} + \beta_{n+1} + 1 > 2\beta_{n+1} + 1.$$

Hence, Hölder's inequality implies that

$$\|m(\cdot,t)\|_{L^{2\beta_{n+1}+1}(\mathbb{T}^d)} \leq \|m(\cdot,t)\|^{1-\lambda_n}_{L^{1+\beta_n}(\mathbb{T}^d)} \|m(\cdot,t)\|^{\lambda_n}_{L^{q_n}(\mathbb{T}^d)},$$

where $0 < \lambda_n < 1$ is defined via

$$\frac{\lambda_n}{q_n} + \frac{1-\lambda_n}{1+\beta_n} = \frac{1}{2\beta_{n+1}+1}. \tag{4.13}$$

The previous equation gives

$$\lambda_n = \frac{q_n}{q_n - \beta_n - 1}\,\frac{2\beta_{n+1} - \beta_n}{1 + 2\beta_{n+1}} = \frac{\beta_{n+1} + 1}{1 + 2\beta_{n+1}}.$$

Because $\|m(\cdot,t)\|_{L^{1+\beta_n}(\mathbb{T}^d)} \leq C$, it follows that

$$\|m\|^{2\beta_{n+1}+1}_{L^{2\beta_{n+1}+1}(\mathbb{T}^d)} \leq C\|m\|^{\lambda_n(2\beta_{n+1}+1)}_{L^{q_n}(\mathbb{T}^d)} = C\|m\|^{\beta_{n+1}+1}_{L^{q_n}(\mathbb{T}^d)}. \tag{4.14}$$

Next, using $\phi(z) = z^{\beta+1}$, with $\beta > 0$, Proposition 4.4 gives

$$\frac{d}{dt}\int_{\mathbb{T}^d} m^{\beta+1}\mathrm{d}x = -\beta\int_{\mathbb{T}^d} \operatorname{div}(b(x,t))m^{\beta+1}\mathrm{d}x - \beta(\beta+1)\int_{\mathbb{T}^d} m^{\beta-1}|D_x m|^2\mathrm{d}x.$$

Integrating the previous equation on $[0,\tau]$, we get

$$\begin{aligned}\int_{\mathbb{T}^d} m^{\beta+1}(x,\tau)\mathrm{d}x = \int_{\mathbb{T}^d} m^{\beta+1}(x,0)\mathrm{d}x - \beta\int_0^\tau\int_{\mathbb{T}^d} \operatorname{div}(b(x,t))m^{\beta+1}\mathrm{d}x\mathrm{d}t \\ - \frac{4\beta}{\beta+1}\int_0^\tau\int_{\mathbb{T}^d} |D_x m^{\frac{\beta+1}{2}}|^2\mathrm{d}x\mathrm{d}t.\end{aligned}$$

By rearranging the terms, we have

$$\begin{aligned}\int_{\mathbb{T}^d} m^{\beta+1}(x,\tau)\mathrm{d}x + \frac{4\beta}{\beta+1}\int_0^\tau\int_{\mathbb{T}^d} |D_x m^{\frac{\beta+1}{2}}|^2\mathrm{d}x\mathrm{d}t \\ = \int_{\mathbb{T}^d} m^{\beta+1}(x,0)\mathrm{d}x - \beta\int_0^\tau\int_{\mathbb{T}^d} \operatorname{div}(b(x,t))m^{\beta+1}\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{4.15}$$

In addition,

$$\begin{aligned}\int_{\mathbb{T}^d} |\operatorname{div}(b(x,t))m^{\beta+1}|\mathrm{d}x &\le \left(\int_{\mathbb{T}^d} |\operatorname{div}(b(x,t))|^2 m\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{T}^d} m^{2\beta+1}\mathrm{d}x\right)^{1/2} \\ &\le C_\delta\left(\int_{\mathbb{T}^d} |\operatorname{div}(b(x,t))|^2 m\mathrm{d}x\right) + \delta\left(\int_{\mathbb{T}^d} m^{2\beta+1}\mathrm{d}x\right),\end{aligned} \tag{4.16}$$

where all integrals are evaluated at a fixed time, $t$.

In (4.15), we set $\beta = \beta_{n+1}$. Accordingly, (4.14) and (4.16) imply that

$$\begin{aligned}\int_{\mathbb{T}^d} m^{\beta_{n+1}+1}(x,\tau)\mathrm{d}x + \frac{4\beta_{n+1}}{\beta_{n+1}+1}\int_0^\tau\int_{\mathbb{T}^d} |D_x m^{\frac{\beta_{n+1}+1}{2}}(x,t)|^2\mathrm{d}x\mathrm{d}t \\ \le \int_{\mathbb{T}^d} m^{\beta_{n+1}+1}(x,0)\mathrm{d}x + C_\delta\int_0^\tau\int_{\mathbb{T}^d} |\operatorname{div}(b(x,t))|^2 m\mathrm{d}x\mathrm{d}t \\ + \delta\int_0^\tau \|m\|_{L^{q_n}(\mathbb{T}^d)}^{\beta_{n+1}+1}\mathrm{d}t\end{aligned} \tag{4.17}$$

for any $\tau \in [0,T]$.

Because $q_n = \frac{2^*}{2}(\beta_{n+1}+1)$ and $\int_{\mathbb{T}^d} m(x,t)\mathrm{d}x = 1$, for all $0 \le t \le T$, Proposition 4.10 gives

$$\|m\|_{L^{q_n}(\mathbb{T}^d)}^{\beta_{n+1}+1} = \|m^{\frac{\beta_{n+1}+1}{2}}\|_{L^{2^*}(\mathbb{T}^d)}^2 \leq C + C\int_{\mathbb{T}^d} |Dm^{\frac{\beta_{n+1}+1}{2}}(x,t)|^2 \mathrm{d}x.$$

Combining the previous bound with (4.17) and choosing a small enough $\delta$, we conclude that there exists $\delta_1 > 0$ such that

$$\int_{\mathbb{T}^d} m^{\beta_{n+1}+1}(x,\tau)\mathrm{d}x + \delta_1 \int_0^\tau \|m\|_{q_n}^{\beta_{n+1}+1}\mathrm{d}t$$
$$\leq C + C\int_{\mathbb{T}^d} m^{\beta_{n+1}+1}(x,0)\mathrm{d}x + C\int_0^\tau \int_{\mathbb{T}^d} |\operatorname{div}(b(x,t))|^2 m \mathrm{d}x\mathrm{d}t.$$

Because of (4.12), the last term on the right-hand side is bounded; this concludes the proof of the first assertion of the Theorem.

Now, we consider the case when $d = 2$. As before, we define $\beta_n$ inductively. We start with $\beta_0 = 0$. Next, we fix $p > 1$ and set

$$\beta_{n+1} := \frac{p-1}{p}(\beta_n + 1).$$

Then, $\beta_n$ is the $n$th partial sum of the geometric series with term

$$\frac{(p-1)^n}{p^n}.$$

Therefore,

$$\lim_{n\to\infty} \beta_n = p - 1.$$

Let

$$q_n = p(\beta_{n+1} + 1).$$

For $\lambda_n$ as in (4.13), we obtain

$$\|m\|_{L^{2\beta_{n+1}+1}(\mathbb{T}^d)} \leq \|m\|_{L^{1+\beta_n}(\mathbb{T}^d)}^{1-\lambda_n} \|m\|_{L^{q_n}(\mathbb{T}^d)}^{\lambda_n}.$$

Hence, (4.13) leads to

$$\lambda_n(2\beta_{n+1} + 1) = 1 + \beta_{n+1}.$$

Because $\|m\|_{L^{1+\beta_n}(\mathbb{T}^d)} \leq C$, it follows that

$$\int_{\mathbb{T}^d} m^{2\beta_{n+1}+1}\mathrm{d}x = \|m\|_{L^{2\beta_{n+1}+1}(\mathbb{T}^d)}^{2\beta_{n+1}+1} \leq C\|m\|_{L^{q_n}(\mathbb{T}^d)}^{\lambda_n(2\beta_{n+1}+1)} = C\|m\|_{L^{q_n}(\mathbb{T}^d)}^{1+\beta_{n+1}}. \tag{4.18}$$

By gathering (4.15), (4.16), and (4.18), we obtain

$$\begin{aligned}&\int_{\mathbb{T}^d} m^{\beta_{n+1}+1}(x,\tau)\mathrm{d}x + \frac{4\beta_{n+1}}{\beta_{n+1}+1}\int_0^\tau\int_{\mathbb{T}^d}|D_x m^{\frac{\beta_{n+1}+1}{2}}(x,t)|^2\mathrm{d}x\mathrm{d}t\\&\le \int_{\mathbb{T}^d} m^{\beta_{n+1}+1}(x,0)\mathrm{d}x + C_\delta\int_0^\tau\int_{\mathbb{T}^d}|\operatorname{div}(b(x,t))|^2\mathrm{d}x\mathrm{d}t + \delta\int_0^\tau \|m\|^{1+\beta_{n+1}}_{L^{q_n}(\mathbb{T}^d)}\mathrm{d}t\end{aligned} \tag{4.19}$$

for any $\tau \in [0,T]$. As before, Proposition 4.10 gives

$$\|m\|^{1+\beta_{n+1}}_{L^{q_n}(\mathbb{T}^d)} = \|m^{\frac{\beta_{n+1}+1}{2}}\|^2_{L^{\frac{2q_n}{\beta_{n+1}+1}}(\mathbb{T}^d)} \le C\int_{\mathbb{T}^d}|D_x m^{\frac{\beta_{n+1}+1}{2}}|^2\mathrm{d}x + C. \tag{4.20}$$

In light of (4.19) and (4.20), we choose a small enough $\delta > 0$. Then, for some $\delta_1 > 0$, we have

$$\begin{aligned}&\int_{\mathbb{T}^d} m^{\beta_{n+1}+1}(x,\tau)\mathrm{d}x + \delta_1\int_0^\tau \|m\|^{\beta_{n+1}+1}_{L^{q_n}(\mathbb{T}^d)}\mathrm{d}t\\&\le C + C\int_{\mathbb{T}^d} m^{\beta_{n+1}+1}(x,0)\mathrm{d}x + C\int_0^\tau\int_{\mathbb{T}^d}|\operatorname{div}(b(x,t))|^2\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Finally, as before, (4.12) provides an upper bound for the last term on the right-hand side. This reasoning concludes the proof of the Theorem. □

### 4.4.2 *Polynomial Estimates for the Fokker–Planck Equation, $p < \infty$*

Next, we investigate a priori bounds for $L^p$ norms of the solutions of (4.2) that are polynomial in norms of the drift, $b$. These bounds are essential in the study of regularity of MFGs. Here, we consider the case when $p < \infty$. This case is used for the study of subquadratic MFGs in Chap. 8. The case $p = \infty$ is examined in the next section and used in the study of superquadratic MFGs.

**Proposition 4.12.** *Let $m$ be a smooth solution of (4.2) and assume that $\beta > 1$. Then, there exist non-negative constants, $C$ and $c$, such that*

$$\frac{d}{dt}\int_{\mathbb{T}^d} m^\beta(x,t)\mathrm{d}x \le C\left\|b^2\right\|_{L^p(\mathbb{T}^d)}\left\|m^\beta\right\|_{L^q(\mathbb{T}^d)} - c\int_{\mathbb{T}^d}\left|D\left(m^{\frac{\beta}{2}}\right)\right|^2\mathrm{d}x,$$

*where*

$$1 \le p, q \le \infty \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

*Proof.* Let $\phi(z) \equiv z^\beta$. According to Proposition (4.4), we have

$$\frac{d}{dt}\int_{\mathbb{T}^d} m^\beta \,\mathrm{d}x = \beta(\beta-1)\int_{\mathbb{T}^d} m^{\beta-1} b\cdot Dm\mathrm{d}x - \beta(\beta-1)\int_{\mathbb{T}^d} m^{\beta-2}|Dm|^2\mathrm{d}x$$
$$\leq C\int_{\mathbb{T}^d}|b|^2 m^\beta \mathrm{d}x - c\int_{\mathbb{T}^d} m^{\beta-2}|Dm|^2\mathrm{d}x.$$

The result follows from Hölder's inequality. □

Next, to proceed with our study, we define the sequence $(\beta_n)_{n\in\mathbb{N}}$ as

$$\beta_{n+1} := \theta\beta_n \tag{4.21}$$

for some $\theta > 1$ to be fixed later. We choose $\beta_0 > 1$ to be any number such that

$$\int_{\mathbb{T}^d} m^{\beta_0}(x,t)\mathrm{d}x < \infty$$

for every $t \in [0,T]$.

*Remark 4.13.* In some applications in MFGs, we have a priori bounds on $\int_0^T\int_{\mathbb{T}^d}|\operatorname{div} b|^2 m$. Hence, by the results in the preceding section, $\beta_0$ can be chosen as close to $\frac{d}{d-2}$ as desired.

For convenience, we take

$$1 < q < \frac{d}{d-2}. \tag{4.22}$$

Next, we choose $0 \leq \kappa \leq 1$ such that

$$\frac{1}{q\beta_{n+1}} = \frac{\kappa}{\beta_n} + \frac{2(1-\kappa)}{2^*\beta_{n+1}}.$$

From the previous identity, we have

$$\kappa = \frac{d+2q-dq}{q\left[(\theta-1)d+2\right]}. \tag{4.23}$$

For later reference, we record in the next Lemma an elementary inequality.

**Lemma 4.14.** *Let $m : \mathbb{T}^d \to \mathbb{R}$ be a smooth, non-negative function with $\int_{\mathbb{T}^d} m\mathrm{d}x = 1$. Then, there exists a constant, $C > 0$, such that*

$$\left\|m^{\beta_{n+1}}\right\|_{L^q(\mathbb{T}^d)} \leq C\left(\int_{\mathbb{T}^d} m^{\beta_n}\mathrm{d}x\right)^{\theta\kappa}\left[1+\left(\int_{\mathbb{T}^d}\left|D\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2\mathrm{d}x\right)^{1-\kappa}\right].$$

*Proof.* First, we notice that Hölder's inequality gives

$$\left(\int_{\mathbb{T}^d} m^{q\beta_{n+1}}\mathrm{d}x\right)^{\frac{1}{q\beta_{n+1}}} \leq \left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\frac{\kappa}{\beta_n}} \left(\int_{\mathbb{T}^d} m^{\frac{2^*\beta_{n+1}}{2}}\right)^{\frac{2(1-\kappa)}{2^*\beta_{n+1}}}.$$

Therefore,

$$\left\|m^{\beta_{n+1}}\right\|_{L^q(\mathbb{T}^d)} \leq \left(\int_{\mathbb{T}^d} m^{\beta_n}\mathrm{d}x\right)^{\theta\kappa} \left(\int_{\mathbb{T}^d} m^{\frac{2^*\beta_{n+1}}{2}}\mathrm{d}x\right)^{\frac{2(1-\kappa)}{2^*}}. \tag{4.24}$$

Because $\int_{\mathbb{T}^d} m\mathrm{d}x = 1$, Proposition 4.10 gives

$$\left\|m^{\frac{\beta_{n+1}}{2}}\right\|^{2(1-\kappa)}_{L^{2^*}(\mathbb{T}^d)} \leq C\left(\int_{\mathbb{T}^d}\left|D\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2\mathrm{d}x\right)^{1-\kappa} + C. \tag{4.25}$$

The Lemma follows by combining (4.24) with (4.25). □

**Proposition 4.15.** *Let $m$ be a solution of* (4.2)*. Let $\kappa$ and $q$ be as in* (4.23) *and* (4.22)*. Define $p,\ r > 1$ by*

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \textit{and} \quad r\kappa = 1. \tag{4.26}$$

*Then, there exists a constant, $C > 0$, such that*

$$\frac{d}{dt}\int_{\mathbb{T}^d} m^{\beta_{n+1}}\mathrm{d}x \leq C\left[1 + \left\||b|^2\right\|^r_{L^p(\mathbb{T}^d)}\left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\theta}\right].$$

*Remark 4.16.* Elementary computations show that for $p > \frac{d}{2}$ and $r > \frac{2p}{2p-d}$, there exists $\theta, q > 1$ such that (4.22), (4.23), and (4.26) hold simultaneously.

*Proof.* By Proposition 4.12, we have

$$\frac{d}{dt}\int_{\mathbb{T}^d} m^{\beta_{n+1}}\mathrm{d}x \leq \left\||b|^2\right\|_{L^p(\mathbb{T}^d)}\left\|m^{\beta_{n+1}}\right\|_{L^q(\mathbb{T}^d)} - c\int_{\mathbb{T}^d}\left|D\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2.$$

Because of Lemma 4.14, the above inequality becomes

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}^d} m^{\beta_{n+1}} \leq{}& C\left\||b|^2\right\|_{L^p(\mathbb{T}^d)}\left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\theta\kappa}\left[1 + \left(\int_{\mathbb{T}^d}\left|D\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2\right)^{1-\kappa}\right]\\ &- c\int_{\mathbb{T}^d}\left|D\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2.\end{aligned}$$

Let $r'$ be given by

$$\frac{1}{r}+\frac{1}{r'}=1.$$

Taking into account that $r=\frac{1}{\kappa}$, we have

$$r'(1-\kappa)\;=\;1.$$

Hence, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}^d} m^{\beta_{n+1}} &\leq C\left\||b|^2\right\|_{L^p(\mathbb{T}^d)}\left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\theta\kappa}\left(\int_{\mathbb{T}^d}\left|D\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2\right)^{1-\kappa}\\ &\quad + C\left\||b|^2\right\|_{L^p(\mathbb{T}^d)}\left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\theta\kappa} - c\int_{\mathbb{T}^d}\left|D\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2\\ &\leq C\left\||b|^2\right\|_{L^p(\mathbb{T}^d)}\left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\theta\kappa} + C\left\||b|^2\right\|^r_{L^p(\mathbb{T}^d)}\left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\theta}\\ &\quad + \delta\int_{\mathbb{T}^d}\left|D\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2 - c\int_{\mathbb{T}^d}\left|D\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2,\end{aligned}$$

where the last inequality follows from Young's inequality weighted by $\delta$. By choosing a small enough $\delta$ and using that $\kappa<1$, the result follows. □

**Proposition 4.17.** *Let $m$ be a solution of* (4.2) *and let $r$ and $p$ be as in Remark 4.16. Then, there exist constants, $C>0$ and $\theta>1$, such that*

$$\int_{\mathbb{T}^d} m^{\beta_n} \leq C + C\left\||b|^2\right\|^{r_n}_{L^r([0,T],L^p(\mathbb{T}^d))},$$

*where $\beta_n$ is given by* (4.21) *and*

$$r_n := r\frac{\theta^n-1}{\theta-1}.$$

*Proof.* We argue by induction in $n$. For $n=1$, Proposition 4.15 yields

$$\frac{d}{dt}\int_{\mathbb{T}^d} m^{\beta_1}(x,t)\mathrm{d}x \leq C\left[1+\left\||b|^2\right\|^r_{L^p(\mathbb{T}^d)}\left(\int_{\mathbb{T}^d} m^{\beta_0}(x,t)\mathrm{d}x\right)^{\theta}\right].$$

Because

$$\int_{\mathbb{T}^d} m^{\beta_0}(x,t)\mathrm{d}x \leq C,$$

we have

$$\int_{\mathbb{T}^d} m^{\beta_1}(x,\tau)\mathrm{d}x \le C\left(1+\int_0^\tau \left\| |b|^2 \right\|^r_{L^p(\mathbb{T}^d)} \mathrm{d}t\right) \le C\left(1+\left\| |b|^2 \right\|^r_{L^r(0,T;L^p(\mathbb{T}^d))}\right)$$

for $\tau \in (0,T]$. The result holds for $n=1$.

For $\tau \in (0,T]$, the induction hypothesis implies that

$$\int_0^\tau \frac{d}{dt}\int_{\mathbb{T}^d} m^{\beta_{n+1}}\mathrm{d}x\mathrm{d}t \le C + C\int_0^\tau \left\| |b|^2 \right\|^r_{L^p(\mathbb{T}^d)} \left(1+\left\| |b|^2 \right\|^{r_n}_{L^p(\mathbb{T}^d)}\right)^\theta \mathrm{d}t.$$

Therefore,

$$\begin{aligned}\int_{\mathbb{T}^d} m^{\beta_{n+1}}(x,\tau)\mathrm{d}x &\le C\int_0^\tau \left\| |b|^2 \right\|^r_{L^p(\mathbb{T}^d)} \left\| |b|^2 \right\|^{r_n\theta}_{L^r(0,T;L^p(\mathbb{T}^d))} \mathrm{d}t + C\int_0^\tau \left\| |b|^2 \right\|^r_{L^p(\mathbb{T}^d)} \mathrm{d}t + C\\ &\le C\left(1+\left\| |b|^2 \right\|^{r+r_n\theta}_{L^r(0,T;L^p(\mathbb{T}^d))}\right).\end{aligned}$$

□

### 4.4.3 Polynomial Estimates for the Fokker–Planck Equation, $p=\infty$

We end this section by obtaining estimates for $m$ in $L^\infty([0,T],L^p(\mathbb{T}^d))$ that are polynomial in the $L^\infty$-norm of the vector field, $b$.

As previously, fix $\beta_0>1$ for which

$$\int_{\mathbb{T}^d} m^{\beta_0}(x,t)\mathrm{d}x < \infty.$$

Let $(\beta_n)_{n\in\mathbb{N}}$ be as in (4.21). We consider here only the case when $d>2$ as the case $d=2$ is similar.

**Lemma 4.18.** *Let $m:\mathbb{T}^d\to\mathbb{R}$, $d>2$, be a smooth, non-negative function. Then,*

$$\int_{\mathbb{T}^d} m^{\beta_{n+1}}(\tau,x)\mathrm{d}x \le \left(\int_{\mathbb{T}^d} m^{\beta_n}(x,\tau)\,\mathrm{d}x\right)^{\theta\kappa}\left(\int_{\mathbb{T}^d} m^{\frac{2^*\beta_{n+1}}{2}}(x,\tau)\,\mathrm{d}x\right)^{\frac{2(1-\kappa)}{2^*}},$$

*where $\kappa$ is given by*

$$\kappa = \frac{2}{d(\theta-1)+2}. \tag{4.27}$$

*Proof.* Hölder's inequality yields

$$\left(\int_{\mathbb{T}^d} m^{\beta_{n+1}}\right)^{\frac{1}{\beta_{n+1}}} \leq \left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\frac{\kappa}{\beta_n}} \left(\int_{\mathbb{T}^d} m^{\frac{2^*}{2}\beta_{n+1}}\right)^{\frac{(1-\kappa)}{\frac{2^*}{2}\beta_{n+1}}}$$

provided that

$$\frac{1}{\theta\beta_n} = \frac{\kappa}{\beta_n} + \frac{2(1-\kappa)}{2^*\theta\beta_n}. \tag{4.28}$$

The statement follows by rearranging the exponents. Solving (4.28) for $\kappa$ gives (4.27) with $0 \leq \kappa \leq 1$. □

**Proposition 4.19.** *Let $m : \mathbb{T}^d \to \mathbb{R}$, $d > 2$, be a smooth non-negative function with $\int_{\mathbb{T}^d} m\mathrm{d}x = 1$. Then, there exists a constant, $C > 0$, such that*

$$\int_{\mathbb{T}^d} m^{\beta_{n+1}}\mathrm{d}x \leq \left[\int_{\mathbb{T}^d} m^{\beta_n}\mathrm{d}x\right]^{\theta\kappa} C\left[1 + \left(\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2\mathrm{d}x\right)^{(1-\kappa)}\right]$$

*for $\kappa$ as in* (4.27).

*Proof.* The result follows by combining Proposition 4.10 with Lemma 4.18. □

Next, we produce an upper bound for

$$\frac{d}{dt}\|m\|^{\beta_{n+1}}_{L^{\beta_{n+1}}(\mathbb{T}^d)}.$$

**Proposition 4.20.** *Let $m$ be a solution of* (4.2) *with $d > 2$. Let $\kappa$ be given by* (4.27) *and $r = \frac{1}{\kappa}$. Then,*

$$\frac{d}{dt}\int_{\mathbb{T}^d} m^{\beta_{n+1}}\mathrm{d}x \leq C + C\left\||b|^2\right\|^r_{L^\infty(\mathbb{T}^d)}\left(\int_{\mathbb{T}^d} m^{\beta_n}\mathrm{d}x\right)^\theta. \tag{4.29}$$

*Proof.* By Proposition 4.4, we have

$$\frac{d}{dt}\int_{\mathbb{T}^d} m^{\beta_{n+1}}(x,t)\mathrm{d}x \leq C\left\||b|^2\right\|_{L^\infty(\mathbb{T}^d)}\int_{\mathbb{T}^d} m^{\beta_{n+1}}(x,t)\mathrm{d}x - c\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2\mathrm{d}x.$$

Using Proposition 4.19 in the previous inequality gives

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}^d} m^{\beta_{n+1}} &\leq C\left\||b|^2\right\|_{L^\infty(\mathbb{T}^d)}\left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\theta\kappa}\left[C\left(\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_n+1}{2}}\right)\right|^2 \mathrm{d}x\right)^{(1-\kappa)}+C\right]\\ &\quad -c\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_n+1}{2}}\right)\right|^2\\ &\leq C\left\||b|^2\right\|_{L^\infty(\mathbb{T}^d)}\left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\theta\kappa}\left(\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_n+1}{2}}\right)\right|^2\right)^{(1-\kappa)}\\ &\quad + C\left\||b|^2\right\|_{L^\infty(\mathbb{T}^d)}\left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\theta\kappa} - c\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_n+1}{2}}\right)\right|^2\\ &\leq C\left\||b|^2\right\|_{L^\infty(\mathbb{T}^d)}\left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\theta\kappa} + C\left\||b|^2\right\|_{\infty}^{r}\left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\theta}\\ &\leq C + C\left\||b|^2\right\|_{L^\infty(\mathbb{T}^d)}^{r}\left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\theta},\end{aligned}$$

where we used Young's inequality weighted by $\varepsilon$ for the conjugate exponents, $r$ and $r'$, given by

$$r' = \frac{1}{1-\kappa} \quad \text{and} \quad r = \frac{1}{\kappa}.$$

□

**Corollary 4.21.** *Let $m$ be a solution of (4.2)–(4.3), $d > 2$ and assume that*

$$m \in L^\infty\left([0,T], L^{\beta_0}\left(\mathbb{T}^d\right)\right)$$

*for some $\beta_0 \geq 1$. Consider the sequence $(\beta_n)_{n\in\mathbb{N}}$ given by (4.21) for $\theta > 1$, and let $r$ be as in Proposition 4.20. Then,*

$$\int_{\mathbb{T}^d} m^{\beta_{n+1}}(\tau,x)\mathrm{d}x \leq C + C\left\||b|^2\right\|_{L^\infty(\mathbb{T}^d\times[0,T])}^{r_{n+1}},$$

*where $(r_n)_{n\in\mathbb{N}}$ is given by*

$$r_n \;=\; r\left(\frac{\theta^n-1}{\theta-1}\right).$$

*Proof.* We use an induction argument. Integrating (4.29) on $(0,\tau)$ and rearranging the exponents, we get

$$\int_{\mathbb{T}^d} m^{\beta_{n+1}}(\tau,x)\,\mathrm{d}x \leq C\left\||b|^2\right\|_{L^\infty(\mathbb{T}^d\times[0,T])}^{r}\int_0^\tau\left(\int_{\mathbb{T}^d} m^{\beta_n}\mathrm{d}x\right)^{\theta}\mathrm{d}t + C. \tag{4.30}$$

First, we check the statement for $n = 0$. In this case, we have

$$\int_{\mathbb{T}^d} m^{\beta_0}(\tau, x)\,\mathrm{d}x \le C.$$

Consider the induction hypothesis:

$$\int_{\mathbb{T}^d} m^{\beta_n}\mathrm{d}x \le C + C\left\||b|^2\right\|^{r_n}_{L^\infty(\mathbb{T}^d\times[0,T])}.$$

Hence,

$$\begin{aligned}\int_{\mathbb{T}^d} m^{\beta_{n+1}}(\tau, x)\,\mathrm{d}x &\le C\left\||b|^2\right\|^{r}_{L^\infty(\mathbb{T}^d\times[0,T])} + C\left\||b|^2\right\|^{r+r_n\theta}_{L^\infty(\mathbb{T}^d\times[0,T])} + C\\ &\le C\left\||b|^2\right\|^{r+r_n\theta}_{L^\infty(\mathbb{T}^d\times[0,T])} + C,\end{aligned}$$

where we have used a weighted Young's inequality. Therefore,

$$r_{n+1} \;=\; r + r_n\theta.$$

Finally, we have

$$r + r\frac{\theta^n - 1}{\theta - 1}\theta = \frac{r\theta - r + r\theta^{n+1} - r\theta}{\theta - 1} = r\frac{\theta^{n+1} - 1}{\theta - 1} = r_{n+1},$$

which completes the proof. □

## 4.5 Relative Entropy

Let $b : \mathbb{T}^d \times [0, T] \to \mathbb{R}^d$ be a smooth vector field. Let $m$ solve

$$m_t + \operatorname{div}(bm) = \Delta m. \tag{4.31}$$

Suppose that we have bounds on $L^p$ norms of $m$. Often, this is the case in MFGs (see, for example, the first-order and second-order estimates in Chap. 6). Here, we investigate some consequences of this integrability to other solutions to the Fokker–Planck equation.

Let $\rho$ solve

$$\rho_t + \operatorname{div}(b\rho) = \Delta\rho, \tag{4.32}$$

and suppose that

$$\rho = \theta m, \tag{4.33}$$

for some function $\theta : \mathbb{T}^d \times [0, T] \to \mathbb{R}$.

The relative entropy between $\rho$ and $m$ is the integral

$$\int_{\mathbb{T}^d} \theta \ln \theta m \mathrm{d}x.$$

More generally, for a convex function $\phi$, the $\phi$-entropy is

$$\int_{\mathbb{T}^d} \phi(\theta) m \mathrm{d}x.$$

Next, we derive a PDE for $\theta$ and $\phi(\theta)$. Then, we examine the integrability of $\theta$ with respect to the measure, $m$, and get further integrability for $\rho$.

**Lemma 4.22.** *Let $m$ be a solution to (4.31) with $m > 0$. Let $(\rho, \theta)$ solve (4.32)–(4.33). Then,*

$$\theta_t + b \cdot D\theta - 2\frac{Dm}{m} D\theta = \Delta\theta. \tag{4.34}$$

*Furthermore, for any convex function, $\phi(z)$, we have*

$$(\phi(\theta))_t + \left(b - \frac{2Dm}{m}\right) D\left(\phi(\theta)\right) = \Delta\left(\phi(\theta)\right) - \phi''(\theta)\left|D\theta\right|^2. \tag{4.35}$$

*In particular, for $\phi(z) = z^p$ and $p > 1$, we have*

$$(\theta^p)_t + \left(b - \frac{2Dm}{m}\right) D\left(\theta^p\right) = \Delta\left(\theta^p\right) - p(p-1)\theta^{p-2}\left|D\theta\right|^2. \tag{4.36}$$

*Proof.* From (4.33), we have

$$\rho_t = \theta_t m + \theta m_t,$$

$$\operatorname{div}(b \cdot \rho) = \operatorname{div}(b \cdot m)\,\theta + bmD\theta,$$

and

$$\Delta\rho = \Delta\theta m + \Delta m \theta + 2D\theta Dm.$$

From these three identities, we readily obtain (4.34). To establish (4.35), we multiply (4.34) by $\phi'(\theta)$. The identity (4.36) corresponds to $\phi(\theta) = \theta^p$. □

**Lemma 4.23.** *Let $m$ solve (4.31) and $(\rho, \theta)$ solve (4.32)–(4.33). Then,*

$$\frac{d}{dt}\int_{\mathbb{T}^d} \theta^p m \mathrm{d}x = -C_p \int_{\mathbb{T}^d} \left|D\theta^{\frac{p}{2}}\right|^2 m \mathrm{d}x, \tag{4.37}$$

*for some constant, $C_p > 0$.*

*Proof.* Multiplying (4.36) by $m$, (4.31) by $\theta^p$ and adding these expressions, we have

$$\begin{aligned}\frac{d}{dt}(\theta^p m) &= \left(-b + \frac{2Dm}{m}\right) D(\theta^p)\, m - \operatorname{div}(b\cdot m)\,\theta^p \\ &\quad + \Delta(\theta^p)\, m + \theta^p \Delta m - p(p-1)\theta^{p-2}|D\theta|^2 m \\ &= -\operatorname{div}(b\cdot\theta^p m) + \Delta(\theta^p m) - C_p\left|D\theta^{\frac{p}{2}}\right|^2 m.\end{aligned}$$

By integrating over $\mathbb{T}^d$, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}^d}\theta^p m\mathrm{d}x &= \int_{\mathbb{T}^d}\Delta(\theta^p m)\,\mathrm{d}x - \int_{\mathbb{T}^d}\operatorname{div}(b\cdot(\theta^p m))\,\mathrm{d}x \\ &\quad - C_p\int_{\mathbb{T}^d}\left|D\theta^{\frac{p}{2}}\right|^2 m\mathrm{d}x = -C_p\int_{\mathbb{T}^d}\left|D\theta^{\frac{p}{2}}\right|^2 m\mathrm{d}x.\end{aligned}$$

□

**Lemma 4.24.** *Let $m$ solve (4.31), with $m > 0$, and let $\rho$ solve (4.32). Assume that (4.33) holds. Then,*

$$\int_{\mathbb{T}^d}\rho^q\mathrm{d}x = \int_{\mathbb{T}^d}\theta^q m^q\mathrm{d}x \le C\int_{\mathbb{T}^d}\theta^p m\mathrm{d}x + C\int_{\mathbb{T}^d} m^r\mathrm{d}x,$$

*where*

$$q = \frac{pr}{r+p-1} > 1 \tag{4.38}$$

*and*

$$r > 1. \tag{4.39}$$

*Proof.* Let $0 < a < 1$. Then,

$$\theta^q m^q = \theta^q m^{a+q-a}.$$

Next, we notice that

$$\theta^q m^q \le \theta^p m + m^r, \tag{4.40}$$

where

$$p = \frac{q}{a} \tag{4.41}$$

and

$$r = \frac{q-a}{1-a}. \tag{4.42}$$

According to (4.42), we have that $r > 1$. By combining (4.41) and (4.42), we get (4.38). Integrating (4.40) over $\mathbb{T}^d$ yields the result. □

**Corollary 4.25.** *Let $m$ solve (4.31), with $m > 0$, and let $\rho$ solve (4.32). Suppose that $p$, $q$, and $r$ satisfy (4.41)–(4.42). Assume that*

$$\int_{\mathbb{T}^d} \theta^p(x,0)m(x,0)\mathrm{d}x \le C$$

*and $m \in L^\infty([0,T], L^r(\mathbb{T}^d))$. Then,*

$$\int_{\mathbb{T}^d} \rho^q \mathrm{d}x \le C,$$

*on $0 \le t \le T$.*

## 4.6 Weak Solutions

If the vector field, $b$, has low regularity, the Fokker–Planck equation may not have $C^2$ solutions. To study (4.2), we thus need to consider weak solutions. We say that $m : \mathbb{T}^d \times [0,T] \to \mathbb{R}_0^+$, $m \in L^1(\mathbb{T}^d \times [0,T])$, is a weak solution of (4.2) if $m|b|^2 \in L^1$ and

$$\int_0^T \int_{\mathbb{T}^d} m(-\phi_t + bD\phi - \Delta\phi)\mathrm{d}x\mathrm{d}t = \int_{\mathbb{T}^d} m_0(x)\phi(x,0)\mathrm{d}x. \tag{4.43}$$

For (4.43) to be well defined, it is enough that $m|b| \in L^1$. However, the additional integrability requirement $m|b|^2 \in L^1$ makes it possible to obtain further properties. Here, we investigate the uniqueness of solutions.

To establish the uniqueness of weak solutions, we introduce the approximate dual problem. We fix a vector field, $\tilde{b} : \mathbb{T}^d \times [0,T] \to \mathbb{R}^d$, and consider the PDE

$$-v_t + \tilde{b}Dv = \Delta v + \psi, \tag{4.44}$$

together with the terminal condition $v(x,T)=0$. If $\tilde{b}$ is $C^\infty$, the previous equation admits a solution. By the maximum principle, if $\psi$ is bounded, so is $v$.

**Proposition 4.26.** *Let $v$ solve (4.44) and let $m$ be a weak solution of (4.2). Suppose that $\psi\in L^\infty$. Then,*

$$\int_0^T\int_{\mathbb{T}^d} m|Dv|^2\mathrm{d}x\mathrm{d}t\leq C\|\psi\|_\infty^2\left(1+\int_0^T\int_{\mathbb{T}^d} m|b-\tilde{b}|^2\mathrm{d}x\mathrm{d}t\right).$$

*Proof.* Let $v$ solve (4.44). Because $m$ is a weak solution of (4.2), we have

$$\int_0^T\int_{\mathbb{T}^d} m(-(v^2)_t+bD(v^2)-\Delta(v^2))\mathrm{d}x\mathrm{d}t=\int_{\mathbb{T}^d} m_0(x)(v(x,0))^2\mathrm{d}x.$$

Thus,

$$\frac{1}{2}\int_{\mathbb{T}^d} m_0(x)(v(x,0))^2\mathrm{d}x+\int_0^T\int_{\mathbb{T}^d} -mv\psi-vm(b-\tilde{b})\,Dv+m|Dv|^2dx\mathrm{d}t=0.$$

Because $m_0\geq 0$ and $\|v\|_\infty\leq C\|\psi\|_\infty$, the result follows from a weighted Cauchy inequality. □

**Proposition 4.27.** *Let $b\in L^2(\mathbb{T}^d\times[0,T])$. Then, there exists at most one weak solution of the Fokker–Planck equation (4.2) with the initial condition (4.3).*

*Proof.* Let $m_1$ and $m_2$ be two weak solutions to (4.2) with the initial condition (4.3). Consider a sequence, $b_\epsilon$, of smooth vector fields converging to $b$ in $L^2$ with respect to the measure $1+m_1+m_2$; that is,

$$\int_0^T\int_{\mathbb{T}^d} |b_\epsilon-b|^2(1+m_1+m_2)dx\mathrm{d}t\to 0.$$

Denote by $m$ either $m_1$ or $m_2$. Because $m$ is a weak solution of (4.2) and $v^\epsilon$ solves (4.44), we get

$$\int_0^T\int_{\mathbb{T}^d} m\psi+m(b-b_\epsilon)Dv^\epsilon=\int_{\mathbb{T}^d} m_0(x)v^\epsilon(x,0)\mathrm{d}x.$$

By the estimates in the preceding proposition, we have

$$\int_0^T\int_{\mathbb{T}^d} m(b-b_\epsilon)Dv^\epsilon\to 0.$$

The above discussion thus gives

$$\int_0^T \int_{\mathbb{T}^d} (m_1 - m_2)\psi = 0$$

for any $\psi$. Accordingly, $m_1 = m_2$. $\square$

## 4.7 Bibliographical Notes

The singular initial condition for the Fokker–Planck equation was used in [130, 135] to establish $L^\infty$ bounds for Hamilton–Jacobi equations. The iterative estimates in Sect. 4.4 follow [134] and [135]. The books [191] and [192] discuss applications of the transport equation and the Fokker–Planck equation in mathematical biology. The book [192] includes a discussion of relative entropy and some applications. Our discussion of weak solutions is inspired by [195, 196].

# Chapter 5
# The Nonlinear Adjoint Method

The nonlinear adjoint method was introduced by L.C. Evans as a tool to study Hamilton–Jacobi equations. This method draws on earlier research on Aubry–Mather and weak KAM theories and has many applications including the vanishing viscosity limit, the infinity Laplacian, non-convex Hamilton–Jacobi equations, and MFGs. We have already encountered a duality argument in the proof of Proposition 4.3. Here, we consider the Hamilton–Jacobi equation,

$$-u_t + \frac{|Du|^2}{2} + V(x,t) = \Delta u, \tag{5.1}$$

develop the nonlinear adjoint method, and derive several estimates for the solution, $u$. These estimates highlight multiple regularity mechanisms and generalize some of the earlier results.

We assume that $V \in C^\infty(\mathbb{T}^d \times [0,T])$. For $x_0 \in \mathbb{T}^d$, we introduce the adjoint variable, $\rho$, as the solution of

$$\begin{cases} \rho_t - \operatorname{div}(Du(x)\rho) = \Delta\rho, \\ \rho(x,0) = \delta_{x_0}. \end{cases} \tag{5.2}$$

The above equation is equivalent to (4.11) for $b = -Du$. Hence, from the results in the previous chapter, we have $\rho \geq 0$ and $\int_{\mathbb{T}^d} \rho(x,t)\mathrm{d}x = 1$. Our first goal is to derive a representation formula for solutions of (5.1) in terms of $\rho$. This is the central idea in the nonlinear adjoint method.

 
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SpringerBriefs in Mathematics, DOI 10.1007/978-3-319-38934-9_5

## 5.1 Representation of Solutions and Lipschitz Bounds

An essential tool for the study of first-order PDEs is the method of characteristics. This method does not extend in a straightforward way to second-order equations. A possible extension uses backward–forward stochastic differential equations. Our approach uses the adjoint equation (5.2) to represent solutions of (5.1) by integrals with respect to $\rho$. The results and methods here complement the comparison principle in Sect. 3.1 and the Lipschitz and semiconcavity bounds in Sect. 3.2.

**Proposition 5.1.** *Let $u$ and $\rho$ solve (5.1) and (5.2), respectively. Then, for any $T > 0$,*

$$u(x_0, 0) = \int_0^T \int_{\mathbb{T}^d} \left[\frac{|Du|^2}{2} - V(x,t)\right] \rho(x,t) \mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}^d} u_T(x)\rho(x,T)\mathrm{d}x.$$

*Proof.* We multiply (5.1) by $\rho$ and integrate by parts using (5.2). □

Next, we consider a distribution, $\tilde{\rho}$, solving

$$\begin{cases} \tilde{\rho}_t - \operatorname{div}(b(x,t)\tilde{\rho}) = \Delta\tilde{\rho}, \\ \tilde{\rho}(x,0) = \delta_{x_0}. \end{cases} \tag{5.3}$$

In the following proposition, we give an upper bound for $u$.

**Proposition 5.2.** *Let $u$ and $\tilde{\rho}$ solve (5.1) and (5.3), respectively. Then,*

$$u(x_0, 0) \le \int_0^T \int_{\mathbb{T}^d} \left[\frac{|b|^2}{2} - V(x,t)\right] \tilde{\rho}(x,t) \mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}^d} u_T(x)\tilde{\rho}(x,T)\mathrm{d}x. \tag{5.4}$$

*Proof.* We multiply (5.1) by $\tilde{\rho}$ and integrate by parts using (5.3) to get

$$u(x_0, 0) = \int_0^T \int_{\mathbb{T}^d} \left[b \cdot Du - \frac{|Du|^2}{2} - V(x,t)\right] \tilde{\rho}(x,t) \mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}^d} u_T(x)\tilde{\rho}(x,T)\mathrm{d}x.$$

The result follows from Cauchy's inequality. □

The bound in the preceding proposition results from a stochastic control interpretation of (5.1). By Proposition 5.1, the vector field $b = Du$ gives an equality in (5.4), whereas any other vector field gives an inequality. We thus interpret $b$ as the optimal drift; the reader can compare this result with Theorem 1.1.

*Remark 5.3.* If $V$ is bounded, Proposition 5.1 gives

$$u(x,t) \ge -\|V\|_{L^\infty}(T-t) - \|u_T\|_{L^\infty}.$$

At the same time, Proposition 5.2 for $b = 0$ gives

$$u(x,t) \geq \|V\|_{L^\infty}(T-t) + \|u_T\|_{L^\infty}.$$

These two bounds can also be proved by the Comparison Principle, Proposition 3.1.

Next, we give a representation formula for the derivatives of $u$:

**Proposition 5.4.** *Let $u$ and $\rho$ solve (5.1) and (5.2), respectively. Then,*

$$D_{x_i}u(x_0,0) = -\int_0^T \int_{\mathbb{T}^d} D_{x_i}V(x,t)\rho(x,t)\mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}^d} D_{x_i}u_T(x)\rho(x,T)\mathrm{d}x.$$

*Proof.* The proof follows by differentiating (5.1) and integrating with respect to $\rho$. □

## 5.2 Conserved Quantities

Conserved quantities are essential features in Hamiltonian mechanics. Here, we show that some of these quantities have a counterpart in the adjoint method. Moreover, in MFGs, such quantities play an essential role in the analysis of the long-time behavior. This matter is examined in Sect. 6.7.

First, we consider a $C^1$ Hamiltonian, $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. The Hamiltonian dynamics associated with $H$ is given by the following ordinary differential equation:

$$\begin{cases} \dot{\mathbf{x}} = -D_p H(\mathbf{x},\mathbf{p}) \\ \dot{\mathbf{p}} = D_x H(\mathbf{x},\mathbf{p}). \end{cases} \tag{5.5}$$

We say that a $C^1$ function, $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, is a conserved quantity by (5.5) if

$$\frac{d}{dt}F(\mathbf{x},\mathbf{p}) = 0.$$

For example, is the Hamiltonian, $H$, is the total energy of the system and is a conserved quantity because

$$\frac{d}{dt}H(\mathbf{x},\mathbf{p}) = D_x H(\mathbf{x},\mathbf{p})\dot{\mathbf{x}} + D_p H(\mathbf{x},\mathbf{p})\dot{\mathbf{p}} = 0$$

using (5.5).

Next, we define the Poisson bracket between two $C^1$ functions, $F, G : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, as

$$\{F, G\} = \sum_{i=1}^{d} \frac{\partial F}{\partial x_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial x_i}.$$

A straightforward computation shows that $F$ is conserved by (5.5) if and only if $\{F, H\} = 0$. Accordingly, $H$ is conserved because $\{H, H\} = 0$.

**Proposition 5.5.** *Let $u \in C^2(\mathbb{T}^d \times [0, T])$ solve*

$$-u_t + H(x, D_x u) = 0,$$

*and suppose that $\rho$ solves the adjoint equation*

$$\rho_t - \operatorname{div}(D_p H(x, Du)\rho) = 0.$$

*Then,*

$$\frac{d}{dt}\int_{\mathbb{R}^d} H(x, Du)\ \rho \mathrm{d}x = 0.$$

*Furthermore, if $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfies $\{H, F\} = 0$, then*

$$\frac{d}{dt}\int_{\mathbb{R}^d} F(x, Du)\ \rho \mathrm{d}x = 0.$$

*Proof.* We have

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d} F\ \rho &= \int_{\mathbb{R}^d} F\rho_t + D_p F(x, Du) Du_t \rho \mathrm{d}x \\
&= \int_{\mathbb{R}^d} F \operatorname{div}(D_p H \rho) + D_p F(x, Du) D(H(x, Du))\rho \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \{H, F\}\rho \mathrm{d}x
\end{aligned}$$

after integrating by parts. □

## 5.3 The Vanishing Viscosity Convergence Rate

We consider the following equation on $\mathbb{T}^d \times [0, T]$ :

$$\begin{cases} -u_t^\epsilon + \frac{|Du^\epsilon|^2}{2} + V(x) = \epsilon \Delta u^\epsilon, \\ u^\epsilon(x, T) = u_T^\epsilon(x), \end{cases} \tag{5.6}$$

where we assume that $V$ and $u_T$ are $C^1$ functions. The limit $\epsilon \to 0$ in the preceding equation is called the vanishing viscosity limit, and its study was one of the first applications of the nonlinear adjoint method. Our aim is to investigate the convergence rate of the solution, $u^\epsilon$, as $\epsilon \to 0$.

We assume that $u^\epsilon$ is differentiable in $\epsilon$ and let $w^\epsilon = \frac{\partial}{\partial \epsilon} u^\epsilon$. Then, differentiating (5.6) in $\epsilon$, we get the following equation for $w^\epsilon$:

$$\begin{cases} -w_t^\epsilon + Du^\epsilon Dw^\epsilon = \epsilon \Delta w^\epsilon + \Delta u^\epsilon, \\ w^\epsilon(x,T) = 0. \end{cases} \tag{5.7}$$

Next, we use the adjoint method to prove that $w^\epsilon$ is $O(\epsilon^{-1/2})$. Once we establish this bound, we get

$$|u^{\epsilon_1} - u^{\epsilon_2}| \le C\left(\frac{1}{\sqrt{\epsilon_1}} + \frac{1}{\sqrt{\epsilon_2}}\right)|\epsilon_1 - \epsilon_2|,$$

which gives the convergence rate as $\epsilon \to 0$.

Let $\rho^\epsilon$ be the adjoint variable defined as before by

$$\begin{cases} \rho_t^\epsilon - \operatorname{div}(Du^\epsilon(x)\rho^\epsilon) = \epsilon \Delta \rho^\epsilon, \\ \rho^\epsilon(x,s) = \delta_{x_0}. \end{cases} \tag{5.8}$$

By the discussion in Sect. 4.1 and in Sect. 5.1 (see Remark 5.3 and Proposition 5.4), we have

$$|u^\epsilon|, |Du^\epsilon| \le C, \quad \rho^\epsilon \ge 0, \int_{\mathbb{T}^d} \rho^\epsilon(x,t)\mathrm{d}x = 1. \tag{5.9}$$

We begin with an auxiliary result.

**Lemma 5.6.** *Let $u^\epsilon$ be a classical solution of* (5.6) *and $\rho^\epsilon$ solve the adjoint equation* (5.8)*. Then, there exists a constant, $C > 0$, such that*

$$\epsilon \int_s^T \int_{\mathbb{T}^d} |D^2 u^\epsilon|^2 \rho^\epsilon \mathrm{d}x\mathrm{d}t \le C.$$

*Proof.* Let $v^\epsilon = \frac{|Du^\epsilon|^2}{2}$. Differentiating the first equation in (5.6) and multiplying it by $Du^\epsilon$, we get

$$-v_t^\epsilon + Du^\epsilon \cdot Dv^\epsilon + DVDu^\epsilon = \epsilon \Delta v^\epsilon - \epsilon |D^2 u^\epsilon|^2.$$

Next, we multiply the previous identity by $\rho^\epsilon$, integrate by parts in $x$ and $t$, and use (5.8). Accordingly, we get

$$\epsilon \int_s^T \int_{\mathbb{T}^d} |D^2 u^\epsilon|^2 \rho^\epsilon \mathrm{d}x\mathrm{d}t \quad = -\int_s^T \int_{\mathbb{T}^d} DVDu^\epsilon \rho^\epsilon \mathrm{d}x + \int_{\mathbb{T}^d} \rho^\epsilon(x,T) v^\epsilon(x,T)\mathrm{d}x - v^\epsilon(x,0).$$

Consequently, by (5.9),

$$\epsilon \int_s^T \int_{\mathbb{T}^d} |D^2 u^\epsilon|^2 \rho^\epsilon \mathrm{d}x\mathrm{d}t \leq C.$$

□

*Remark 5.7.* The proof of the previous lemma does not depend on the convexity of the Hamiltonian, $H(x,p) = \frac{|p|^2}{2}$, in (5.6). Similar bounds therefore hold for nonconvex Hamiltonians. In the convex case, including (5.6), we can prove a stronger estimate. This estimate is discussed in Theorem 5.9, and the next theorem can be improved accordingly.

**Theorem 5.8.** *Let $w^\epsilon$ be as before. Then, there exists a constant, $C > 0$, such that*

$$\sup_{[0,T]\times\mathbb{T}^d} |w^\epsilon| \leq C\left(\frac{T}{\epsilon}\right)^{\frac{1}{2}}.$$

*Consequently, $u^\epsilon$ converges as $\epsilon \to 0$ and*

$$\sup_{[0,T]\times\mathbb{T}^d} |u^\epsilon - u| \leq C\,(T\epsilon)^{\frac{1}{2}},$$

*where $u := \lim_{\epsilon\to 0} u^\epsilon$.*

*Proof.* Multiply (5.7) by $\rho^\epsilon$ and integrate by parts to get

$$\begin{aligned} w^\epsilon(x,s) &= \int_s^T \int_{\mathbb{T}^d} \Delta u^\epsilon \rho^\epsilon \mathrm{d}x\mathrm{d}t \\ &\leq \left(\int_s^T \int_{\mathbb{T}^d} |\Delta u^\epsilon|^2 \rho^\epsilon \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} \left(\int_s^T \int_{\mathbb{T}^d} \rho^\epsilon \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} \leq C\left(\frac{T}{\epsilon}\right)^{\frac{1}{2}}, \end{aligned}$$

where we used Hölder's inequality and Lemma 5.6. Furthermore,

$$|u^\epsilon - u| \leq \int_0^\epsilon |w^\epsilon| d\epsilon \leq C\,(T\epsilon)^{\frac{1}{2}}.$$

□

## 5.4 Semiconcavity Estimates

Consider the Hamilton–Jacobi equation:

$$\begin{cases} -u_t + \frac{|Du|^2}{2} + V(x) = \epsilon \Delta u, \\ u(x,T) = u_T(x). \end{cases} \tag{5.10}$$

In general, it is not possible to get bounds for the second derivatives of $u$ that are uniform in $\epsilon$. However, in this instance, solutions are semiconcave; that is, second-derivatives satisfy unilateral bounds. Our estimates extend the ones in Sect. 3.2.4.

**Theorem 5.9.** *Assume that $V$ and $u_T$ are $C^2$. Let $(u,m)$ solve (5.10). Then, there exists a constant, $C > 0$, such that*

$$D^2u(x,t) \leq CI, \quad (x,t) \in [0,T] \times \mathbb{T}^d.$$

*Without the assumption that $u_T$ is $C^2$, we have the bound*

$$D^2u(x,t) \leq C\left(T - t + \frac{1}{T-t}\right)I, \quad (x,t) \in [0,T) \times \mathbb{T}^d.$$

*Remark 5.10.* The constants in the previous theorem do not depend on $\epsilon$.

*Proof.* Because the viscosity coefficient, $\epsilon$, plays no role in the proof, we set $\epsilon = 1$. Take any vector, $\xi \in \mathbb{R}^d$ with $|\xi| = 1$. It is enough to prove that $\xi^T D^2 u \xi = u_{\xi\xi} \leq C$ for all $(x,t) \in [0,T] \times \mathbb{T}^d$. Let $w = u_{\xi\xi}$. Differentiating (5.10) twice in the direction of $\xi$ gives

$$-w_t + DuDw + |Du_\xi|^2 + V_{\xi\xi} = \Delta w.$$

Integrating the previous identity with respect to the adjoint variable, $\rho$, we obtain

$$w(x_0,s) + \int_s^T \int_{\mathbb{T}^d} |Du_\xi|^2 \rho \mathrm{d}x\mathrm{d}t = -\int_s^T \int_{\mathbb{T}^d} V_{\xi\xi} \rho \mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}^d} (u_T)_{\xi\xi}(x)\rho(x,T)\mathrm{d}x.$$

Because $\int_s^T \int_{\mathbb{T}^d} |Du_\xi|^2 \rho \mathrm{d}x\mathrm{d}t \geq 0$, we get $w(x_0,s) \leq C$.

For the second bound, let $\tilde{w} = \left(\frac{T-t}{T-s}\right)^2 w$. Then,

$$-\tilde{w}_t - 2\frac{T-t}{(T-s)^2} w + DuD\tilde{w} + \left(\frac{T-t}{T-s}\right)^2 |Du_\xi|^2 + \left(\frac{T-t}{T-s}\right)^2 V_{\xi\xi} = \Delta \tilde{w}.$$

Hence, taking into account that $|w| = |\xi Du_\xi| \leq |Du_\xi|$, we have

$$\begin{aligned}
&w(x_0, s) + \int_s^T \int_{\mathbb{T}^d} \left(\frac{T-t}{T-s}\right)^2 |w|^2 \rho \mathrm{d}x\mathrm{d}t \\
&\leq \int_s^T \int_{\mathbb{T}^d} |V_{\xi\xi}| \rho \mathrm{d}x\mathrm{d}t + \int_s^T \int_{\mathbb{T}^d} 2\frac{T-t}{(T-s)^2} |w| \rho \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Thus,

$$\begin{aligned}
&w(x_0, s) + \int_s^T \int_{\mathbb{T}^d} \left(\frac{T-t}{T-s}\right)^2 |w|^2 \rho \mathrm{d}x\mathrm{d}t \\
&\leq C(T-s) + \int_s^T \int_{\mathbb{T}^d} \left(\frac{T-t}{T-s}\right)^2 |w|^2 \rho \mathrm{d}x\mathrm{d}t + \int_s^T \int_{\mathbb{T}^d} \frac{1}{(T-s)^2} \rho \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Consequently,

$$w(x_0, s) \leq C\left(T - s + \frac{1}{T-s}\right).$$

□

## 5.5 Lipschitz Regularity for the Heat Equation

Next, we give Lipschitz estimates for solutions of the heat equation. In MFGs, these estimates are used to prove the regularity of solutions of

$$-u_t + \frac{|Du|^\gamma}{\gamma} = \Delta u + F(m)$$

once we know that $|Du|^\gamma$ and $F(m)$ have enough integrability.

**Theorem 5.11.** *Let u be a solution of*

$$\begin{cases} u_t(x,t) + \Delta u(x,t) = f & \text{in} \quad \mathbb{T}^d \times [0,T], \\ u(x,T) = u_T(x) & \text{in} \quad \mathbb{T}^d, \end{cases} \tag{5.11}$$

*with* $u_T \in W^{1,\infty}(\mathbb{T}^d)$ *and* $f \in L^a(\mathbb{T}^d \times [0,T])$ *with* $a > d+2$. *Then,*

$$\|Du\|_{L^\infty(\mathbb{T}^d \times [0,T])} \leq C\|f\|_{L^a(\mathbb{T}^d \times [0,T])}.$$

*Proof.* Consider the adjoint equation to (5.11); that is,

$$\rho_t(x,t) - \Delta \rho(x,t) = 0 \tag{5.12}$$

equipped with initial condition

$$\rho(\cdot,\tau)=\delta_{x_0}, \tag{5.13}$$

for arbitrary $\tau\in[0,T)$ and $x_0\in\mathbb{T}^d$. Select $\nu$ with

$$0<\nu<1. \tag{5.14}$$

Proposition 4.8, with $b=0$ yields

$$\int_0^\tau\int_{\mathbb{T}^d}|D\rho^{\nu/2}|^2\mathrm{d}x\,\mathrm{d}t\le\frac{\nu}{4(1-\nu)}<C, \tag{5.15}$$

where the last inequality follows from the fact that $0<\nu<1$.

Next, we fix a unit vector, $\xi\in\mathbb{R}^d$. Accordingly, we have

$$u_\xi(x_0,\tau)=\int_{\mathbb{T}^d}(u_T)_\xi\rho(x,T)\mathrm{d}x+\int_0^\tau\int_{\mathbb{T}^d}f_\xi\rho(x,t)\mathrm{d}x\mathrm{d}t. \tag{5.16}$$

Clearly,

$$\left|\int_{\mathbb{T}^d}(u_T)_\xi(x)\rho(x,T)\right|\mathrm{d}x\le\|u_T\|_{W^{1,\infty}(\mathbb{T}^d)}.$$

Hence, it remains to bound

$$\int_0^\tau\int_{\mathbb{T}^d}f_\xi\rho(x,t)\mathrm{d}x\mathrm{d}t.$$

For $0<\nu<1$, we have

$$\begin{aligned}\left|\int_0^\tau\int_{\mathbb{T}^d}f\rho_\xi\right|\le&\int_0^\tau\int_{\mathbb{T}^d}|f|\rho^{1-\frac{\nu}{2}}|\rho^{\frac{\nu}{2}-1}D\rho|\\ \le&\|f\|_{L^a(\mathbb{T}^d\times[0,\tau])}\|\rho^{1-\frac{\nu}{2}}\|_{L^b(\mathbb{T}^d\times[0,\tau])}\|D\rho^{\frac{\nu}{2}}\|_{L^2(\mathbb{T}^d\times[0,\tau])},\end{aligned}$$

with

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{2},\quad a,b\ge1. \tag{5.17}$$

In light of (5.15), it suffices to estimate

$$\|\rho^{1-\frac{\nu}{2}}\|_{L^b(\mathbb{T}^d\times[0,\tau])}.$$

For that, assume that there exists $\kappa$ such that

$$1-\kappa+\frac{2\kappa}{2^*\nu}=\frac{\kappa}{\nu}, \qquad 0\leq\kappa\leq 1. \tag{5.18}$$

Set

$$b=\frac{\nu}{\kappa(1-\frac{\nu}{2})}. \tag{5.19}$$

By Hölder's inequality, we have

$$\left(\int_{\mathbb{T}^d}\rho^{b(1-\frac{\nu}{2})}\right)^{\frac{1}{b(1-\frac{\nu}{2})}}=\left(\int_{\mathbb{T}^d}\rho^{b(1-\frac{\nu}{2})}\right)^{\frac{\kappa}{\nu}}\leq\left(\int_{\mathbb{T}^d}\rho\right)^{1-\kappa}\left(\int_{\mathbb{T}^d}\rho^{\frac{2^*\nu}{2}}\right)^{\frac{2\kappa}{2^*\nu}}.$$

Sobolev's inequality yields

$$\left(\int_{\mathbb{T}^d}\rho^{\frac{2^*\nu}{2}}\right)^{\frac{2}{2^*}}\leq C+C\int_{\mathbb{T}^d}|D\rho^{\frac{\nu}{2}}|^2.$$

Hence,

$$\int_{\mathbb{T}^d}\rho^{b(1-\frac{\nu}{2})}\leq C+C\int_{\mathbb{T}^d}|D\rho^{\frac{\nu}{2}}|^2.$$

The above inequality implies that

$$\int_0^\tau\int_{\mathbb{T}^d}\rho^{b(1-\frac{\nu}{2})}\leq C+C\int_0^\tau\int_{\mathbb{T}^d}|D\rho^{\frac{\nu}{2}}|^2\leq C.$$

The previous inequality concludes the proof once we check the existence of $\nu$, $\kappa$, and $b$ such that (5.14), (5.17), (5.18), and (5.19) hold. Elementary computations show the existence of those numbers for $a>d+2$. □

## 5.6 Irregular Potentials

Here, we prove Lipschitz estimates for the solutions of (5.1), assuming only suitable integrability on $V$. As usual, we work with classical (smooth) solutions and regular potentials, $V$. However, the bounds that we obtain depend only on the integrability properties of $V$.

**Theorem 5.12.** *Let $u$ solve (5.1). Fix $r>d$. Suppose that $V\geq 0$. For any $\tilde{\delta}>0$, there exists a constant, $C>0$, such that*

$$\operatorname{Lip}(u)\leq C+C\,\|V\|^2_{L^\infty([0,T],L^r(\mathbb{T}^d))}+\tilde{\delta}\,\|u\|_{L^\infty(\mathbb{T}^d\times[0,T])}\,.$$

*Proof.* First, integrating (5.1), we get

$$\int_0^T \int_{\mathbb{T}^d} |Du|^2 \mathrm{d}x\mathrm{d}t \leq C\|V\|_{L^\infty([0,T],L^r(\mathbb{T}^d))} + C\|u\|_{L^\infty(\mathbb{T}^d\times[0,T])}.$$

Next, the adjoint equation (5.2) gives the following representation formula for $u$:

$$u(x_0,0) = \int_0^T \int_{\mathbb{T}^d} \left(\frac{|Du|^2}{2} + V\right)\rho \mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}^d} u(x,T)\rho(x,T)\mathrm{d}x. \tag{5.20}$$

The positivity of $V$ thus ensures that

$$\int_0^T \int_{\mathbb{T}^d} |Du|^2 \rho \mathrm{d}x\mathrm{d}t \leq C\|u\|_{L^\infty(\mathbb{T}^d\times[0,T])}.$$

Let $0 < \nu < 1$. Combining the preceding estimate with (5.2), we get

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^d} |D\rho^{\frac{\nu}{2}}|^2 \mathrm{d}x\mathrm{d}t &\leq C\int_0^T \int_{\mathbb{T}^d} |Du|^2\rho + |Du|^2 + C\mathrm{d}x\mathrm{d}t\\ &\leq C + C\|V\|_{L^\infty([0,T],L^r(\mathbb{T}^d))} + C\|u\|_{L^\infty(\mathbb{T}^d\times[0,T])}.\end{aligned}$$

Now, we set $v := D_{x_i}u$ and observe that $v$ solves

$$-v_t + Du\cdot Dv - \Delta v = -D_{x_i}(V).$$

Integrating with respect to $\rho$, we get

$$v(x_0,\tau) = -\int_\tau^T \int_{\mathbb{T}^d} D_{x_i}(V)\rho \mathrm{d}x\mathrm{d}t \; + \int_{\mathbb{T}^d} v(x,T)\rho(x,T)\mathrm{d}x.$$

Therefore, we have

$$|v(x_0,\tau)| \leq C + \int_0^T \left|\int_{\mathbb{T}^d} D_{x_i}(V)\rho \mathrm{d}x\right| \mathrm{d}t. \tag{5.21}$$

Next, we estimate the last term on the right-hand side of (5.21). We have

$$\int_{T^d} D_{x_i}(V)\rho \mathrm{d}x = -\int_{\mathbb{T}^d} VD_{x_i}(\rho)\mathrm{d}x = \frac{2}{\nu}\int_{\mathbb{T}^d} V\rho^{1-\nu/2}D_{x_i}(\rho^{\nu/2})\mathrm{d}x.$$

Thus,

$$\int_0^T \left|\int_{\mathbb{T}^d} D_{x_i}(V)\rho \mathrm{d}x\right| \mathrm{d}t \leq C\int_0^T \int_{\mathbb{T}^d} V^2\rho^{2-\nu}\mathrm{d}x\mathrm{d}t + C\int_0^T \int_{\mathbb{T}^d} |D(\rho^{\nu/2})|^2 \mathrm{d}x\mathrm{d}t.$$

We bound the first term of the previous inequality as follows:

$$\int_{\mathbb{T}^d} V^2\rho^{2-\nu}\mathrm{d}x \le \|V^2\|_{L^{\frac{r}{2}}(\mathbb{T}^d)}\|\rho^{2-\nu}\|_{L^{\frac{r}{r-2}}(\mathbb{T}^d)} = \|V\|^2_{L^r(\mathbb{T}^d)}\|\rho\|^{2-\nu}_{L^{\frac{(2-\nu)r}{r-2}}(\mathbb{T}^d)}.$$

Because $d < r$, we can select $\nu$ close to 1 such that

$$\frac{(2-\nu)r}{r-2} \le \frac{2^*\nu}{2} = \frac{\nu d}{d-2}$$

(in dimension $d = 2$, we replace $2^*$ by a large enough $p$). Then, by Sobolev's inequality,

$$\|\rho\|_{L^{\frac{2^*\nu}{2}}(\mathbb{T}^d)} \le C\|D(\rho^{\nu/2})\|^{\frac{2}{\nu}}_{L^2(\mathbb{T}^d)} + C. \tag{5.22}$$

Using Hölder's inequality, we get

$$\|\rho\|_{L^{\frac{(2-\nu)r}{r-2}}(\mathbb{T}^d)} \le \|\rho\|^{1-\theta_1}_{L^1(\mathbb{T}^d)}\|\rho\|^{\theta_1}_{L^{\frac{2^*\nu}{2}}(\mathbb{T}^d)} \le C\|D(\rho^{\nu/2})\|^{\frac{2\theta_1}{\nu}}_{L^2(\mathbb{T}^d)} + C, \tag{5.23}$$

where $\theta_1$ is defined by

$$\frac{r-2}{(2-\nu)r} = \frac{1-\theta_1}{1} + \frac{2\theta_1}{2^*\nu}.$$

As $\nu \to 1$, we have $\theta_1 \to \frac{d}{r}$. Moreover, for $\nu > 1 - \frac{1}{d} + \frac{1}{r}$, we have

$$\frac{(2-\nu)\theta_1}{\nu} < 1.$$

Then, Young's inequality yields

$$\begin{aligned}
\int_0^T \left|\int_{\mathbb{T}^d} D_{x_i}(V)\rho\mathrm{d}x\right| \mathrm{d}t &\le C\,\|V\|^2_{L^\infty([0,T],L^r(\mathbb{T}^d))} + \delta_1\int_0^T\left(\int_{\mathbb{T}^d}|D(\rho^{\nu/2})|^2\right)^{(2-\nu)\theta_1/\nu} + C_{\delta_1}\\
&\le C\,\|V\|^2_{L^\infty([0,T],L^r(\mathbb{T}^d))} + \delta_1\left(\int_0^T\int_{\mathbb{T}^d}|D(\rho^{\nu/2})|^2\right)^{(2-\nu)\theta_1/\nu} + C_{\delta_1}\\
&\le C\,\|V\|^2_{L^\infty([0,T],L^r(\mathbb{T}^d))} + \tilde{\delta}\,\|u\|_{L^\infty(\mathbb{T}^d\times[0,T])} + C.
\end{aligned} \tag{5.24}$$

Hence,

$$\mathrm{Lip}(u) \le C + C\,\|V\|^2_{L^\infty([0,T],L^r(\mathbb{T}^d))} + \tilde{\delta}\,\|u\|_{L^\infty(\mathbb{T}^d\times[0,T])}.$$

□

*Remark 5.13.* The above theorem holds in a more general setting. Namely, it is enough to assume that $u$ solves

$$-u_t + H(x, Du) + V = \Delta u,$$

where $V$, as before, is bounded in $L^\infty([0,T], L^r(\mathbb{T}^d)$, $H\colon \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfies for some constants, $c, C > 0$,

$$|D_p H(x,p)|^2 \leq C|p|^2 + C, \tag{5.25}$$

$$D_p H(x,p)p - H(x,p) \geq c|p|^2, \tag{5.26}$$

and

$$|D_x H(x,p)| \leq C + \psi(x)|p|^\beta,\ 0 \leq \beta < 2, \quad \text{for some } \psi \in L^{\frac{2r}{2-\beta}}(\mathbb{T}^d). \tag{5.27}$$

## 5.7 The Hopf–Cole Transform

In this last section, we apply the Hopf–Cole transform and use the results in the preceding section to get lower bounds on solutions of the Fokker–Planck equation.

**Theorem 5.14.** *Let* $m : \mathbb{T}^d \times [0,T] \to \mathbb{R}$, $m > 0$, *solve*

$$\begin{cases} m_t + \operatorname{div}(bm) = \Delta m, \\ m(x,0) = m_0(x), \end{cases}$$

*where* $m_0$ *is a given smooth function with* $m_0 > k_0 > 0$, $\|Dm_0\|_\infty \leq C$ *and* $\|b\|_{L^\infty} \leq C$, $\|\operatorname{div}(b)\|_{L^r} \leq C$, *for some* $r > d$. *Assume that we have the bound* $\left|\int_{\mathbb{T}^d} \ln m(x,T)\mathrm{d}x\right| \leq C$. *Then, m is bounded from below.*

*Proof.* We use the Hopf–Cole transform, $w(x,t) = -\ln m(x, T-t)$. Then, $w$ solves the Hamilton–Jacobi equation,

$$-w_t + |Dw|^2 + b \cdot Dw - \operatorname{div}(b) = \Delta w, \qquad w_T = \ln m_0.$$

Integrating the equation in $t$ and $x$ and using $\left|\int_{\mathbb{T}^d} \ln m(x,T)\mathrm{d}x\right| \leq C$, we get $\|Dw\|_{L^2} \leq C$. Finally, we use Theorem 5.12 and Remark 5.13 to conclude $\|D\ln m\|_\infty \leq C$, which implies the result. □

## 5.8 Bibliographical Notes

The nonlinear adjoint method was introduced in [91] as a tool to study the vanishing viscosity problem for non-convex Hamiltonians. The earlier work [170] used related ideas to investigate the $L^1$ stability of Hamilton–Jacobi equations. Here, the discussion in Sects. 5.1–5.4 is partially based on [91]. The results on conserved quantities were explored in [50] to develop the Aubry–Mather theory for non-convex Hamiltonians. In the context of MFGs, the adjoint method was used to get Lipschitz regularity for Hamilton–Jacobi equations with $L^p$ potentials [129, 130, 135]. In Sect. 5.6, we follow [135]. The nonlinear adjoint method is currently an essential tool for the analysis of Hamilton–Jacobi equations. Some of its applications include stationary Hamilton–Jacobi equations [211], long-time behavior of Hamilton–Jacobi equations [53, 181], the infinity Laplacian problem [96], convergence of numerical schemes [52], the Aubry–Mather theory for non-convex problems [50], non-convex Hamilton–Jacobi equations [92], obstacle and weakly coupled systems [51]. Some of the techniques in the Aubry–Mather theory [93–95] and its extensions [55, 89, 90, 109–111] are precursors to the adjoint method. The adjoint equation also appears in optimal transport [29].

# Chapter 6
# Estimates for MFGs

In the absence of special transformations or explicit solutions, the analysis of MFGs often relies on a priori bounds. Here, we investigate estimates that are commonly used. We begin by using the maximum principle to obtain one-sided bounds. Next, we consider energy-type estimates that give additional bounds. These two techniques extend to a broad class of mean-field game problems. Equally important are the consequences of these bounds when combined with earlier results. In Sect. 6.3, we develop some of these aspects. In the remainder of this chapter, we discuss other methods that rely on the particular structure of the problems. First, we present a second-order estimate that is used frequently in the periodic setting. Next, we consider a technique that gives Lipschitz bounds for stationary first-order MFGs. Subsequently, we examine energy conservation principles. Finally, we prove estimates for the Fokker–Planck equation that depend on uniform ellipticity or parabolicity of the MFG system.

One of the problems we consider is the periodic stationary MFG,

$$\begin{cases} -\epsilon\Delta u + \frac{|Du|^2}{2} + V(x) = F(m) + \overline{H} \\ -\epsilon\Delta m - \operatorname{div}(mDu) = 0, \end{cases} \tag{6.1}$$

where the unknowns are $u : \mathbb{T}^d \to \mathbb{R}$, $m : \mathbb{T}^d \to \mathbb{R}$, with $m \geq 0$ and $\int m = 1$, and $\overline{H} \in \mathbb{R}$. The other problem we examine is the time-dependent MFG,

$$\begin{cases} -u_t - \epsilon\Delta u + \frac{|Du|^2}{2} + V(x) = F(m) \\ m_t - \epsilon\Delta m - \operatorname{div}(mDu) = 0, \end{cases} \tag{6.2}$$

where $T > 0$, and the unknowns are $u : \mathbb{T}^d \times [0, T] \to \mathbb{R}$, $m : \mathbb{T}^d \times [0, T] \to \mathbb{R}$, with $m \geq 0$, together with the initial-terminal conditions



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$$\begin{cases} u(x,T) = u_T(x) \\ m(x,0) = m_0(x). \end{cases} \tag{6.3}$$

Here, $m_0 > 0$ with $\int m_0 = 1$.

We suppose that $V : \mathbb{T}^d \to \mathbb{R}$ is a $C^\infty$ function. The function $F$ that encodes the interactions between each agent and the mean field is either a real-valued function, $F : \mathbb{R}^+ \to \mathbb{R}$ (or $F : \mathbb{R}_0^+ \to \mathbb{R}$), or a function defined on the space of probability measures, $F : \mathcal{P}(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d)$. The former is called the local case, and the latter is the non-local case. For local problems, we suppose $F(m)$ that is $C^\infty$ in the set $m > 0$. For some of the estimates, we require that $F$ satisfies the following property:

$$\int_{\mathbb{T}^d} F(m) \leq C + \frac{1}{2}\int_{\mathbb{T}^d} mF(m). \tag{6.4}$$

Here, we assume that all functions are $C^\infty$. In particular, $u_T, m_0 \in C^\infty$. Moreover, we suppose that $m$ and $m_0$ are strictly positive. We say that $(u, m, \overline{H})$ or $(u, m)$ is a classical solution of, respectively, (6.1) or (6.2)–(6.3), if $u$ and $m$ are $C^\infty$, $m > 0$, $(u, m)$ solves (6.1) or (6.2), and, in the time-dependent case, (6.3) holds.

## 6.1 Maximum Principle Bounds

If $F \geq 0$, maximum principle techniques give important bounds for the solutions of (6.1) and (6.2). For stationary problems, the maximum principle implies one-sided bounds on $\overline{H}$. For time-dependent problems, the maximum principle yields lower bounds for $u$.

**Proposition 6.1.** *Let $u$ be a classical solution of* (6.1). *Suppose that $F \geq 0$. Then,*

$$\overline{H} \leq \sup_{\mathbb{T}^d} V.$$

*Proof.* Because $u$ is periodic, it achieves a minimum at a point, $x_0$. At this point, $Du(x_0) = 0$ and $\Delta u \geq 0$. Consequently,

$$V(x_0) \geq \overline{H} + F(m) \geq \overline{H}.$$

Hence, $\overline{H} \leq \sup V$. □

In the case of time-dependent problems, we obtain bounds from below for the solution, $u$.

**Proposition 6.2.** *Let $u$ be a classical solution of* (6.2) *and $F \geq 0$. Then, $u$ is bounded from below.*

*Proof.* Since $F \geq 0$, we have

$$-u_t - \epsilon \Delta u + \frac{|Du|^2}{2} \geq -\|V\|_{L^\infty(\mathbb{T}^d \times [0,T])}.$$

The function $v(x,t) = -\|u_T\|_\infty - (T-t)\|V\|_{L^\infty(\mathbb{T}^d \times [0,T])}$ is therefore a subsolution. Hence, by the Comparison Principle given in Proposition 3.1, we have

$$u(x,t) \geq -\|u_T\|_{L^\infty(\mathbb{T}^d)} - (T-t)\|V\|_{L^\infty(\mathbb{T}^d \times [0,T])}.$$ □

## 6.2 First-Order Estimates

First-order or energy estimates bound integral norms of classical solutions of first- or second-order MFGs. These estimates are obtained by the multiplier method. In the two examples considered here, we multiply the Hamilton–Jacobi equation and the Fokker–Planck equation by suitable functions of $m$ and $u$, respectively. The estimates follow by integration by parts and elementary inequalities. While the proofs are straightforward, these estimates are essential in the theory of MFGs. In the next section, we examine some consequences of these results. A more general technique is developed in Sect. 6.8.

**Proposition 6.3.** *There exists a constant, $C$, such that, for any classical solution, $(u, m, \overline{H})$, of (6.1), we have*

$$\int_{\mathbb{T}^d} \frac{|Du|^2}{2}(1+m) + \frac{1}{2}F(m)m\mathrm{d}x \leq C. \tag{6.5}$$

*Proof.* Multiply the first equation in (6.1) by $(m-1)$ and the second equation by $-u$. Adding the resulting expressions and integrating by parts gives

$$\int_{\mathbb{T}^d} \frac{|Du|^2}{2}(1+m) + mF(m)\mathrm{d}x = \int_{\mathbb{T}^d} V(m-1) + F(m)\mathrm{d}x. \tag{6.6}$$

Using (6.4) in the preceding identity gives (6.5). □

Next, we obtain a bound for $\overline{H}$.

**Corollary 6.4.** *Let $(u, m, \overline{H})$ be a classical solution of (6.1). Suppose that $F \geq 0$. Then, there exists a constant, $C$, not depending on the particular solution, such that*

$$|\overline{H}| \leq C.$$

*Proof.* By Proposition 6.3, we have $\frac{|Du|^2}{2} \in L^1$. As a result of (6.4), we have $F(m) \in L^1$. Therefore, integrating the first equation of (6.1), we obtain the bound for $\overline{H}$. □

*Remark 6.5.* An important case not covered by the previous Corollary is $F(m) = \ln m$. Here, Proposition 6.3 combined with (6.6) provides the bound

$$\int_{\mathbb{T}^d} \ln m \mathrm{d}x \le C.$$

Because $m \mapsto m \ln m$ is bounded from below, (6.6) also gives the opposite bound. Therefore, by integrating the first equation in (6.1), we conclude that $\overline{H}$ is bounded.

Now, we focus our attention on the time-dependent problem (6.2) and prove bounds to (6.5).

**Proposition 6.6.** *There exists a constant, $C > 0$, such that, for any classical solution, $(u, m)$, of (6.2), we have*

$$\int_{\mathbb{T}^d} \int_0^T (m + m_0) \frac{|Du|^2}{2} + mF(m) \mathrm{d}t \mathrm{d}x \le C.$$

*Proof.* Multiply the first equation in (6.2) by $(m - m_0)$ and the second equation by $(u_T - u)$. Adding the resulting expressions and integrating in $\mathbb{T}^d \times [0, T]$ gives

$$\begin{aligned} 0 = & \int_{\mathbb{T}^d} \int_0^T [(m - m_0)(u_T - u)_t + (u_T - u)(m - m_0)_t] \, \mathrm{d}t \mathrm{d}x \\ & + \int_{\mathbb{T}^d} \int_0^T [(\epsilon(m - m_0)\Delta(u_T - u) - \epsilon(u_T - u)\Delta(m - m_0)] \, \mathrm{d}t \mathrm{d}x \\ & + \int_{\mathbb{T}^d} \int_0^T [-\epsilon(m - m_0)\Delta u_T - \epsilon(u_T - u)\Delta m_0] \, \mathrm{d}t \mathrm{d}x \\ & + \int_{\mathbb{T}^d} \int_0^T \left[ (m - m_0) \frac{|Du|^2}{2} + u \operatorname{div}(mDu) \right] \mathrm{d}t \mathrm{d}x \\ & + \int_{\mathbb{T}^d} \int_0^T [-u_T \operatorname{div}(mDu) + (m - m_0)V(x)] \, \mathrm{d}t \mathrm{d}x \\ & + \int_{\mathbb{T}^d} \int_0^T (m_0 - m)F(m) \mathrm{d}t \mathrm{d}x. \end{aligned} \tag{6.7}$$

Using the boundary conditions, we have

$$\int_{\mathbb{T}^d} \int_0^T [(m - m_0)(u_T - u)_t + (u_T - u)(m - m_0)_t] \, \mathrm{d}t \mathrm{d}x = 0,$$

and

$$\int_{\mathbb{T}^d} \int_0^T [\epsilon(m - m_0)\Delta(u_T - u) - \epsilon(u_T - u)\Delta(m - m_0)] \, \mathrm{d}t \mathrm{d}x = 0.$$

Because $m$ and $m_0$ are probability measures and $u_T$ is of class $C^\infty$, there exists a positive constant, $C$, such that

$$\left|\int_{\mathbb{T}^d}\int_0^T [-\epsilon(m-m_0)\Delta u_T - \epsilon u_T\Delta m_0]\,\mathrm{d}t\mathrm{d}x\right| \leq C.$$

Furthermore, for any $\delta > 0$ there exists a constant, $C > 0$, such that

$$\left|\int_{\mathbb{T}^d}\int_0^T \epsilon u\Delta m_0 \mathrm{d}t\mathrm{d}x\right| \leq \delta\int_{\mathbb{T}^d}\int_0^T |Du|^2\mathrm{d}t\mathrm{d}x + C.$$

Similarly, we have

$$\left|\int_{\mathbb{T}^d}\int_0^T -u_T\,\mathrm{div}(mDu)\mathrm{d}t\mathrm{d}x\right| \leq \delta\int_{\mathbb{T}^d}\int_0^T |Du|^2 m\mathrm{d}t\mathrm{d}x + C.$$

Finally, we get

$$\left|\int_{\mathbb{T}^d}\int_0^T (m-m_0)V(x)\mathrm{d}t\mathrm{d}x\right| \leq C.$$

Using the preceding identities and estimates in (6.7), selecting $\delta = \frac{1}{4}$, and integrating by parts, we obtain

$$\int_{\mathbb{T}^d}\int_0^T \left[(m+m_0)\frac{|Du|^2}{4} + mF(m)\right]\mathrm{d}t\mathrm{d}x \leq C + \int_{\mathbb{T}^d}\int_0^T m_0F(m)\mathrm{d}t\mathrm{d}x.$$

The statement follows by using (6.4) in the previous estimate. □

## 6.3 Additional Estimates for Solutions of the Fokker–Plank Equation

Here, we examine uniformly parabolic MFG. To simplify, we set $\epsilon = 1$ in (6.2). Our result combines the first-order estimates with the regularity for the Fokker–Planck equation to get estimates for derivatives of $m$.

**Proposition 6.7.** *Let $(u,m)$ solve (6.2). Then, there exists a positive constant, $C$, independent of the solution such that*

$$\int_0^T\int_{\mathbb{T}^d} |D\ln m|^2 + |Dm^{1/2}|^2\mathrm{d}x\mathrm{d}t \leq C.$$

*Proof.* The result follows by combining Proposition 6.6 with Corollaries 4.6 and 4.7. □

**Corollary 6.8.** *Let* $(u, m)$ *solve* (6.2)*. Suppose that* $F(m) = m^\alpha$ *for some* $\alpha > 0$*. Then, there exists a positive constant, C, independent of the solution such that*

$$\int_0^T \int_{\mathbb{T}^d} |Dm|^q \mathrm{d}x\mathrm{d}t \le C,$$

*where* $q = 2\frac{1+\alpha}{2+\alpha}$*.*

*Proof.* By Hölder's inequality, for any $s \ge 0$ and $\frac{1}{r} + \frac{1}{r'} = 1$, we have

$$\int_0^T \int_{\mathbb{T}^d} |Dm|^q \mathrm{d}x\mathrm{d}t \le \left( \int_0^T \int_{\mathbb{T}^d} \frac{|Dm|^{qr}}{m^{sr}} \mathrm{d}x\mathrm{d}t \right)^{1/r} \left( \int_0^T \int_{\mathbb{T}^d} m^{sr'} \mathrm{d}x\mathrm{d}t \right)^{1/r'} .$$

By Propositions 6.7 and 6.6, we have bounds for, respectively, $\int_0^T \int_{\mathbb{T}^d} \frac{|Dm|^2}{m}$ and $\int_0^T \int_{\mathbb{T}^d} m^{\alpha+1}$. We therefore select

$$\begin{cases} qr = 2 \\ sr = 1 \\ sr' = \alpha + 1. \end{cases} \tag{6.8}$$

Solving (6.8) gives $q = 2\frac{1+\alpha}{2+\alpha}$. □

## 6.4 Second-Order Estimates

Now, we discuss a second-order estimate for mean-field games that gives further regularity for the solutions. Remarkably, these second-order estimates are valid for classical solutions of first-order mean-field games.

### *6.4.1 Stationary Problems*

For stationary problems, the second-order estimates provide a bound on the Hessian of $u$ with respect to the measure, $m$.

**Proposition 6.9.** *Let* $(u, m)$ *be a classical solution of* (6.1) *with* $V \in C^2$*. Assume that* $F(m)$ *is local. Then, there exists a constant,* $C > 0$*, that does not depend on the solution* $(u, m)$ *such that*

$$\int_{\mathbb{T}^d} |D^2u|^2 m + F'(m)|Dm|^2 \mathrm{d}x \leq C.$$

*Proof.* By applying the operator $\Delta$ to the first equation of (6.1), we obtain

$$-\epsilon\Delta^2 u + \Delta V + |D^2u|^2 + Du \cdot D\Delta u - \operatorname{div}(F'(m)Dm) = 0.$$

Integrating with respect to $m$ and using

$$\int_{Tt^d} \left(\epsilon\Delta^2 u + Du \cdot D\Delta u\right) m\mathrm{d}x = -\int_{\mathbb{T}^d} \Delta u \left(-\epsilon\Delta m - \operatorname{div}(Dum)\right) \mathrm{d}x = 0,$$

we get

$$\int_{\mathbb{T}^d} F'(m)|Dm|^2 \mathrm{d}x + \int_{\mathbb{T}^d} |D^2u|^2 m\mathrm{d}x \leq \int_{\mathbb{T}^d} |\Delta V| m\mathrm{d}x \leq C.$$

□

### 6.4.2 *Time-Dependent Problems*

The next proposition is the counterpart of Proposition 6.9 for time-dependent MFGs.

**Proposition 6.10.** *Let* $(u, m)$ *be a classical solution of* (6.2) *with* $m_0, u_T, V \in C^2$. *Assume that* $F(m)$ *is local. Then, there exists a constant,* $C > 0$, *that is independent of the solution,* $(u, m)$, *such that*

$$\int_0^T \int_{\mathbb{T}^d} |D^2u|^2 m + F'(m)|Dm|^2 \mathrm{d}x\mathrm{d}t \leq C + C\|u(\cdot, 0)\|_{L^1(\mathbb{T}^d)}.$$

*Proof.* Applying the operator $\Delta$ to the first equation of (6.2), we have

$$-(\Delta u)_t - \epsilon\Delta^2 u + \Delta V + |D^2u|^2 + Du \cdot D\Delta u - \operatorname{div}(F'(m)Dm) = 0.$$

Multiplying by $m$, integrating in $x$ and $t$, using integration by parts and the equation for $m$, we get

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^d} |D^2u|^2 m + F'(m)|Dm|^2 \mathrm{d}x\mathrm{d}t \\
&\quad = -\int_0^T \int_{\mathbb{T}^d} \Delta V m\mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}^d} m(x, T)\Delta u(x, T)\mathrm{d}x - \int_{\mathbb{T}^d} \Delta m(x, 0) u(x, 0)\mathrm{d}x.
\end{aligned}$$

Consequently,

$$\int_0^T \int_{\mathbb{T}^d} |D^2u|^2 m + F'(m)|Dm|^2 \mathrm{d}x\mathrm{d}t \leq C + C\|u(\cdot, 0)\|_{L^1(\mathbb{T}^d)}.$$

□

*Remark 6.11.* If $F \geq 0$, the boundedness of $\|u(\cdot, 0)\|_{L^1(\mathbb{T}^d)}$ follows from the lower bounds in Proposition 6.2 combined with the bound

$$\int_{\mathbb{T}^d} u(x, 0)\mathrm{d}x \leq C$$

from Proposition 6.6.

## 6.5 Some Consequences of Second-Order Estimates

Next, we continue our study of the parabolic case with $\epsilon = 1$ and combine the previous estimates with the regularity results for the Fokker–Planck equation obtained earlier.

**Proposition 6.12.** *Let $(u, m)$ solve (6.2)–(6.3) with $\epsilon = 1$. Suppose that $F(m) = m^\alpha$ for some $\alpha > 0$. Then, $D(mDu) \in L^1(\mathbb{T}^d \times [0, T])$.*

*Proof.* We have

$$D_{x_i}(mD_{x_j}u) = D_{x_i}mD_{x_j}u + mD^2_{x_ix_j}u.$$

Next,

$$|D_{x_i}mD_{x_j}u| \leq \frac{|Dm|}{m^{1/2}}m^{1/2}|Du| \leq \frac{|Dm|^2}{2m} + \frac{m|Du|^2}{2}.$$

The expression on the right-hand side is in $L^1(\mathbb{T}^d \times [0, T])$ due to Propositions 6.7 and 6.6. Finally, we have

$$|mD^2_{x_ix_j}u| \leq \frac{m}{2} + \frac{m|D^2u|^2}{2}.$$

Because $m$ is a probability measure and because of Proposition 6.10, the right-hand side in the previous bound is also integrable. □

According to the preceding estimate $m_t - \Delta m \in L^1$.

## 6.6 The Evans Method for the Evans–Aronsson Problem

The Evans–Aronsson problem consists of minimizing the integral functional,

$$\int_{\mathbb{T}^d} e^{\frac{|Du|^2}{2}+V(x)}\mathrm{d}x, \tag{6.9}$$

among all functions, $u \in W^{1,\infty}(\mathbb{T}^d)$. A smooth enough minimizer solves the Euler–Lagrange equation,

$$-\operatorname{div}\left(e^{\frac{|Du|^2}{2}+V(x)}Du\right) = 0. \tag{6.10}$$

Because the functional (6.9) is convex, any solution to the Euler–Lagrange equation is a minimizer. Here, we prove a priori Lipschitz bounds for any solution, $u$, of (6.10). Because of this estimate, the methods we develop later give the existence of a solution for (6.10) and, consequently, of a minimizer for (6.9).

Remarkably, (6.10) can be written as an MFG. For that, we set

$$m = e^{\frac{|Du|^2}{2}+V(x)-\overline{H}},$$

where $\overline{H}$ is chosen such that $\int_{\mathbb{T}^d} m = 1$. Thus,

$$\begin{cases} \frac{|Du|^2}{2} + V(x) = \ln m + \overline{H} \\ -\operatorname{div}(mDu) = 0. \end{cases} \tag{6.11}$$

Because the function $m \mapsto m\ln m$ is bounded from below, Proposition 6.3 gives $Du \in L^2$. In addition, by Remark 6.5, we have that $\overline{H}$ is bounded. In the next proposition, we prove our main result, the Lipschitz regularity for $u$.

**Proposition 6.13.** *Let* $(u, m, \overline{H})$ *solve* (6.11). *Then, there exists a constant,* $C > 0$, *independent of the solution such that* $\|Du\|_{L^\infty} \leq C$.

*Proof.* We begin by multiplying the second equation in (6.11) by $\operatorname{div}(m^p Du)$. After that, we integrate on $\mathbb{T}^d$, integrate by parts, and apply the identity

$$\sum_{i,j}\int_{\mathbb{T}^d}(mu_{x_i})_{x_i}(m^p u_{x_j})_{x_j} = \sum_{i,j}\int_{\mathbb{T}^d}(mu_{x_i})_{x_j}(m^p u_{x_j})_{x_i}$$

to get

$$\int_{\mathbb{T}^d} m^{p+1}\sum_{i,j}|D_{x_ix_j}u|^2 + pm^{p-1}|Dm\cdot Du|^2 + (p+1)m^p D^2_{x_ix_j}uD_{x_i}uD_{x_j}m = 0. \tag{6.12}$$

Next, we differentiate the first equation in (6.11) and get

$$\sum_i D_{x_i}uD^2_{x_ix_j}u + D_{x_j}V = \frac{D_{x_j}m}{m}.$$

Next, we multiply the preceding identity by $m^p m_{x_j}$ to conclude that

$$\int_{\mathbb{T}^d} m^p \sum_{i,j} D^2_{x_ix_j}uD_{x_i}uD_{x_j}m = \int_{\mathbb{T}^d} m^{p-1}|Dm|^2 - \int_{\mathbb{T}^d} m^p \sum_j D_{x_j}mD_{x_j}V.$$

Combining the prior identity with (6.12) gives

$$\int_{\mathbb{T}^d} m^{p+1} \sum_{i,j} |D_{x_ix_j}u|^2 + pm^p|Dm\cdot Du|^2 + (p+1)\int_{\mathbb{T}^d} m^{p-1}|Dm|^2$$
$$= (p+1)\int_{\mathbb{T}^d} m^p \sum_j D_{x_j}mD_{x_j}V \le \frac{p+1}{2}\int_{\mathbb{T}^d} m^{p-1}|Dm|^2 + C(p+1)\int_{\mathbb{T}^d} m^{p+1}$$

for some positive constant, $C$, independent of $p$. Accordingly, we have the estimate

$$\int_{\mathbb{T}^d} m^{p-1}|Dm|^2 \le C\int_{\mathbb{T}^d} m^{p+1}. \tag{6.13}$$

By Sobolev's theorem, we have

$$\left[\int_{\mathbb{T}^d} m^{\frac{2^*(p+1)}{2}}\right]^{\frac{1}{2^*}} \le C\left[\int_{\mathbb{T}^d} m^{p+1} + \int_{\mathbb{T}^d} |Dm^{\frac{p+1}{2}}|^2\right]^{\frac{1}{2}} \le C(1+|p|)\left[\int_{\mathbb{T}^d} m^{p+1}\right]^{\frac{1}{2}}.$$

Thus,

$$\|m\|_{L^{\frac{2^*(p+1)}{2}}} \le C(1+|p|)^{\frac{2}{p+1}}\|m\|_{L^{p+1}}. \tag{6.14}$$

To finish the proof, we use Moser's iteration method. First, we define the sequence $p_n = \theta^n$ for some $1 < \theta < \frac{2^*}{2}$. Because $m \ge 0$ and $\int m = 1$, we have $\|m\|_{p_0} = 1$. Suppose that

$$\frac{1}{\theta} = \alpha + \frac{1-\alpha}{2^*/2}.$$

Then,

$$\|m\|_{p_{n+1}} \le \|m\|^{\alpha}_{p_n}\|m\|^{1-\alpha}_{2^*p_n/2}.$$

By (6.14), we obtain

$$\|m\|_{p_{n+1}} \leq C|p_n|^{(1-\alpha)\frac{2}{p_n}} \|m\|_{p_n}^{\alpha} \|m\|_{p_n}^{(1-\alpha)} \leq C|p_n|^{(1-\alpha)\frac{2}{p_n}} \|m\|_{p_n}.$$

By induction, we get

$$\|m\|_{p_{n+1}} \leq C\Theta_n,$$

where $\ln \Theta_n \leq \sum_{j=1}^{n} \frac{C+\ln p_j}{p_j}$. Because the previous series is convergent, we have $\|m\|_{L^q} \leq C$ for all $1 \leq q < \infty$; that is, $C$ is independent of $q$. Hence, $m \in L^\infty$. Thus, the first equation in (6.11) gives $Du \in L^\infty$. □

## 6.7 An Energy Conservation Principle

Here, we give a conservation of energy principle for time-dependent MFGs. This energy conservation principle is essential to the study of the long-time limit of MFGs.

Let $\Phi : \mathbb{R} \to \mathbb{R}$ be an increasing function. We consider the time-dependent mean-field game

$$\begin{cases} -u_t + H(x, Du) = \epsilon \Delta u + \Phi'(m), \\ m_t - \operatorname{div}(D_p H(x, Du)m) = \epsilon \Delta m. \end{cases} \tag{6.15}$$

Then, we have

**Proposition 6.14.** *Let* $(u, m)$ *solve* (6.15). *Then,*

$$\frac{d}{dt} \int_{\mathbb{T}^d} Hm - \Phi(m) + \epsilon DuDm = 0.$$

*Proof.* We have

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{T}^d} Hm - \Phi(m) + \epsilon DuDm \\ &\quad = \int_{\mathbb{T}^d} (H - \Phi'(m))m_t + D_p H D_x u_t m + \epsilon (Du_t Dm + DuDm_t) \\ &\quad = \int_{\mathbb{T}^d} (H - \Phi'(m) - u_t - \epsilon \Delta u)m_t = 0. \end{aligned}$$

□

While the energy conservation provides strong estimates for solutions to (6.15), its application is somewhat restricted as it depends on the specific form of the equations. For example, no energy conservation principle is known for general MFGs.

## 6.8 Porreta's Cross Estimates

We end this chapter with a class of estimates that involve the solution $(u, m)$ to (6.2). These estimates build upon the idea that $\phi(u)$ and $\psi(m)$ are approximate solutions to (6.2). Therefore, using a multiplier method, we gain control over several integral quantities. These estimates give compactness for approximate solutions of (6.2). We begin our discussion with an auxiliary identity.

**Lemma 6.15.** *Let $\epsilon = 1$ and $V = 0$. Assume that $(u, m)$ solves (6.2)–(6.3) and suppose that $\phi$, $\psi \in C^2$. Then,*

$$\int_0^T \int_{\mathbb{T}^d} \Big[\Delta(\phi(u_T)) - DuD(\phi(u_T)) + F(m)\phi'(u) + \left[-\phi''(u) + \frac{\phi'(u)}{2}\right] |Du|^2\Big](\psi(m) - \psi(m_0))\mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_{\mathbb{T}^d} \Big[\operatorname{div}((\psi(m_0)Du) + \Delta(\psi(m_0)). - \psi''(m)|Dm|^2 - (\psi(m) - \psi'(m)m)\Delta u\Big](\phi(u) - \phi(u_T))\mathrm{d}x\mathrm{d}t.$$

*Proof.* We fix two convex increasing functions, $\phi$ and $\psi$. To begin with, we multiply the Hamilton–Jacobi equation in (6.2) by $\phi'(u)$ to get

$$-(\phi(u))_t - \Delta(\phi(u)) + DuD(\phi(u)) = F(m)\phi'(u) + \left[-\phi''(u) + \frac{\phi'(u)}{2}\right] |Du|^2. \tag{6.16}$$

Next, we multiply the Fokker–Planck equation by $\psi'(m)$ and get

$$(\psi(m))_t - \operatorname{div}(\psi(m)Du) - \Delta(\psi(m)) = -\psi''(m)|Dm|^2 - (\psi(m) - \psi'(m)m)\Delta u. \tag{6.17}$$

Now, to cancel the boundary conditions, we rewrite (6.16) and (6.17) as

$$-(\phi(u) - \phi(u_T))_t - \Delta(\phi(u) - \phi(u_T)) + DuD(\phi(u) - \phi(u_T)) \tag{6.18}$$
$$= \Delta(\phi(u_T)) - DuD(\phi(u_T)) + F(m)\phi'(u) + \left[-\phi''(u) + \frac{\phi'(u)}{2}\right] |Du|^2$$

and

$$(\psi(m)-\psi(m_0))_t - \operatorname{div}((\psi(m)-\psi(m_0))Du) - \Delta(\psi(m)-\psi(m_0)) \tag{6.19}$$
$$= \operatorname{div}(\psi(m_0)Du) + \Delta(\psi(m_0)) - \psi''(m)|Dm|^2 - (\psi(m)-\psi'(m)m)\Delta u.$$

Next, we multiply (6.18) by $\psi(m)-\psi(m_0)$ and (6.19) by $\phi(u)-\phi(u_T)$. Finally, we subtract the resulting expressions and integrate in $\mathbb{T}^d\times[0,T]$. These operations give the desired identity. □

**Corollary 6.16.** *Let $\epsilon = 1$ and $V = 0$. Assume that $m_0$ is of class $C^1$ and bounded from below, $m_0 \geq \kappa_0 > 0$, $u_T > 1$ is $C^2$, $F$ is non-decreasing and non-negative. Suppose that $u_T \geq 1$. Then, for any solution $(u,m)$ to (6.2)–(6.3) and $r \geq 1$, we have*

$$\int_0^T\int_{\mathbb{T}^d} mF(m)u^{r-1}dxdt + \int_0^T\int_{\mathbb{T}^d} u^{r-1}|Du|^2(m+m_0)\mathrm{d}x\mathrm{d}t \leq$$
$$C\int_0^T\int_{\mathbb{T}^d} u^{r-1}\mathrm{d}x\mathrm{d}t + c_r,$$

*where the constants $c_r, C$ depend only on $r, F, \|m_0\|_{C^1}$ and $\|u_T\|_{C^2}$.*

*Remark 6.17.* Proposition 6.6 is a particular case of the previous result.

*Remark 6.18.* The condition $u_T \geq 1$ simplifies the statement because $u \geq 1$. A similar result holds if $u_T$ is bounded from below. Because $F$ is non-negative, $u$ is bounded from below. The same technique gives an estimate for $(u+k)^r$ for some constant, $k$.

*Proof.* Let $\phi(u) = u^r$ and $\psi(m) = m$ in Lemma 6.15. Then,

$$\int_0^T\int_{\mathbb{T}^d}\left[\Delta(\phi(u_T)) - DuD(\phi(u_T)) + F(m)ru^{r-1} + \left[-\phi''(u)+\frac{\phi'(u)}{2}\right]|Du|^2\right](m-m_0)\mathrm{d}x\mathrm{d}t$$
$$= \int_0^T\int_{\mathbb{T}^d}[\operatorname{div}(m_0Du)+\Delta(m_0)]\,(\phi(u)-\phi(u_T))\mathrm{d}x\mathrm{d}t$$
$$= \int_0^T\int_{\mathbb{T}^d} -\phi'(u)m_0|Du|^2 + m_0DuD\left(\phi(u_T)\right) - D(m_0)D\left(\phi(u)\right) - m_0\Delta\left(\phi(u_T)\right)\mathrm{d}x\mathrm{d}t.$$

After some cancellations, we gather

$$\int_0^T\int_{\mathbb{T}^d} F(m)ru^{r-1}(m-m_0)dxdt + \int_0^T\int_{\mathbb{T}^d}\frac{\phi'(u)}{2}|Du|^2(m+m_0)\mathrm{d}x\mathrm{d}t$$
$$= \int_0^T\int_{\mathbb{T}^d}\phi''(u)(m-m_0)|Du|^2 + mDuD\left(\phi(u_T)\right) - D(m_0)D\left(\phi(u)\right)$$
$$- m\Delta\left(\phi(u_T)\right)\mathrm{d}x\mathrm{d}t.$$

Since $F$ is non-decreasing and non-negative, we have

$$F(m)m_0 \leq \frac{1}{2}mF(m)\mathbb{1}_{m\geq 2m_0} + m_0F(2m_0)\mathbb{1}_{m\leq 2m_0} \leq \frac{1}{2}mF(m) + C(F, \|m_0\|_\infty).$$

Thus,

$$\begin{aligned}
&\int_0^T\int_{\mathbb{T}^d}\frac{1}{2}mF(m)u^{r-1}dxdt + \frac{1}{2}\int_0^T\int_{\mathbb{T}^d}u^{r-1}|Du|^2(m+m_0)\mathrm{d}x\mathrm{d}t\\
&\leq (r-1)\int_0^T\int_{\mathbb{T}^d}u^{r-2}(m-m_0)|Du|^2\mathrm{d}x\mathrm{d}t - \int_0^T\int_{\mathbb{T}^d}u^{r-1}D(m_0)Du\mathrm{d}x\mathrm{d}t\\
&+\frac{1}{r}\int_0^T\int_{\mathbb{T}^d}\left(mDuD\left(\phi(u_T)\right) - m\Delta\left(\phi(u_T)\right)\right)\mathrm{d}x\mathrm{d}t + C(F,\|m_0\|_\infty)\int_0^T\int_{\mathbb{T}^d}u^{r-1}\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

For the first three terms on the right-hand side of the above equality, we have

$$\begin{aligned}
&(r-1)\int_0^T\int_{\mathbb{T}^d}u^{r-2}(m-m_0)|Du|^2\mathrm{d}x\mathrm{d}t \leq \delta\int_0^T\int_{\mathbb{T}^d}(m+m_0)u^{r-1}|Du|^2\mathrm{d}x\mathrm{d}t\\
&+c_\delta(r)\int_0^T\int_{\mathbb{T}^d}(m+m_0)|Du|^2\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

with $c_\delta(1) = 0$,

$$\begin{aligned}
-\int_0^T\int_{\mathbb{T}^d}u^{r-1}D(m_0)Du\mathrm{d}x\mathrm{d}t &\leq \delta\int_0^T\int_{\mathbb{T}^d}u^{r-1}|Du|^2\mathrm{d}x\mathrm{d}t\\
&+C(\delta,\|m_0\|_{C^1})\int_0^T\int_{\mathbb{T}^d}u^{r-1}\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

and

$$\int_0^T\int_{\mathbb{T}^d}\left[mDuD\left(\phi(u_T)\right) - m\Delta\left(\phi(u_T)\right)\right]\mathrm{d}x\mathrm{d}t \leq C(\delta,\|u_T\|_{C^2}) + \delta\int_0^T\int_{\mathbb{T}^d}m|Du|^2\mathrm{d}x\mathrm{d}t.$$

Hence,

$$\begin{aligned}
&\int_0^T\int_{\mathbb{T}^d}mF(m)u^{r-1}\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{T}^d}u^{r-1}|Du|^2(m+m_0)\mathrm{d}x\mathrm{d}t \leq\\
&c_r\int_0^T\int_{\mathbb{T}^d}(m+m_0)|Du|^2\mathrm{d}x\mathrm{d}t + C(F,\|m_0\|_{C^1},\|u_T\|_{C^2})\left(1+\int_0^T\int_{\mathbb{T}^d}u^{r-1}\mathrm{d}x\mathrm{d}t\right).
\end{aligned}$$

To end the proof, we apply Proposition 6.6. □

**Corollary 6.19.** *Let $\epsilon = 1$ and $V = 0$; assume that $m_0$ is in $C^1$ and bounded from below, $m_0 \geq \kappa_0 > 0$, $u_T$ is $C^2$, $F$ is non-decreasing and non-negative. Then, for any solution $(u, m)$ to (6.2)–(6.3) and $r \geq 1$, we have*

$$\int_{\mathbb{T}^d} (u(x,t))^r \mathrm{d}x + \int_t^T \int_{\mathbb{T}^d} mF(m)u^{r-1} dxdt + \int_t^T \int_{\mathbb{T}^d} u^{r-1}|Du|^2(m+1)\mathrm{d}x\mathrm{d}t \leq C_r.$$

*Proof.* Integrating (6.16) for $\phi(u) = u^r$, we get

$$\int_{\mathbb{T}^d} (u(x,t))^r \mathrm{d}x = \int_{\mathbb{T}^d} u_T^r \mathrm{d}x + \int_t^T \int_{\mathbb{T}^d} rF(m)u^{r-1} \mathrm{d}x\mathrm{d}s +$$
$$\int_t^T \int_{\mathbb{T}^d} \left[-r(r-1)u^{r-2} - \frac{r}{2}u^{r-1}\right] |Du|^2 \mathrm{d}x\mathrm{d}s.$$

Thus, from Corollary 6.16 and $F(m) \leq F(1) + mF(m)$, we get

$$\int_{\mathbb{T}^d} (u(x,t))^r \mathrm{d}x \leq C + C_r \int_t^T \int_{\mathbb{T}^d} u^r \mathrm{d}x\mathrm{d}s.$$

By Gronwall's inequality,

$$\sup_{[0,T]} \int_{\mathbb{T}^d} (u(x,t))^r \mathrm{d}x \leq C_r.$$

Corollary 6.16 concludes the proof. □

Next, we organize the identity Lemma 6.15 in a more convenient form.

**Lemma 6.20.** *Let $\epsilon = 1$ and $V = 0$. Let $(u, m)$ solve (6.2)–(6.3). Then,*

$$\int_0^T \int_{\mathbb{T}^d} F(m)\phi'(u)\psi(m)\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} \frac{\phi'(u)}{2}|Du|^2(\psi(m) + \psi(m_0))\mathrm{d}x\mathrm{d}t$$
$$+ \int_0^T \int_{\mathbb{T}^d} \psi''(m)|Dm|^2\phi(u)\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} m\psi''(m)DmDu\phi(u)\mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_{\mathbb{T}^d} F(m)\phi'(u)\psi(m_0)\mathrm{d}x\mathrm{d}t +$$
$$\int_0^T \int_{\mathbb{T}^d} \left[\phi''(u)(\psi(m) - \psi(m_0)) + (\psi(m) - \psi'(m)m)\phi'(u)\right] |Du|^2 \mathrm{d}x\mathrm{d}t$$
$$\int_{\mathbb{T}^d} (\psi(m_0) - \psi(m_T))\,\phi(u_T)\mathrm{d}x + \int_0^T \int_{\mathbb{T}^d} \Delta(\psi(m_0))\phi(u)\mathrm{d}x\mathrm{d}t.$$

*Proof.* From Lemma 6.15, after integrating by parts, we have

$$\int_0^T \int_{\mathbb{T}^d} F(m)\phi'(u)\psi(m)\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} \frac{\phi'(u)}{2}|Du|^2(\psi(m) + \psi(m_0))\mathrm{d}x\mathrm{d}t$$
$$+ \int_0^T \int_{\mathbb{T}^d} \left[\psi''(m)|Dm|^2 + (\psi(m) - \psi'(m)m)\Delta u\right](\phi(u) - \phi(u_T))\mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_{\mathbb{T}^d} F(m)\phi'(u)\psi(m_0)\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} \phi''(u)|Du|^2(\psi(m) - \psi(m_0))\mathrm{d}x\mathrm{d}t$$
$$+ \int_0^T \int_{\mathbb{T}^d} (-\operatorname{div}(\psi(m)Du) - \Delta(\psi(m)))\,\phi(u_T) + \Delta(\psi(m_0))\phi(u)\mathrm{d}x\mathrm{d}t.$$

Next, using Eq. (6.17), we obtain

$$\int_0^T \int_{\mathbb{T}^d} F(m)\phi'(u)\psi(m)\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} \frac{\phi'(u)}{2}|Du|^2(\psi(m) + \psi(m_0))\mathrm{d}x\mathrm{d}t$$
$$+ \int_0^T \int_{\mathbb{T}^d} \left[\psi''(m)|Dm|^2 + (\psi(m) - \psi'(m)m)\Delta u\right]\phi(u)\mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_{\mathbb{T}^d} F(m)\phi'(u)\psi(m_0)\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} \phi''(u)|Du|^2(\psi(m) - \psi(m_0))\mathrm{d}x\mathrm{d}t$$
$$+ \int_0^T \int_{\mathbb{T}^d} \left[(-\psi(m))_t\,\phi(u_T) + \Delta(\psi(m_0))\phi(u)\right]\mathrm{d}x\mathrm{d}t.$$

Furthermore,

$$\int_0^T \int_{\mathbb{T}^d} (\psi(m) - \psi'(m)m)\phi(u)\Delta u\mathrm{d}x\mathrm{d}t = \int_0^T \int_{\mathbb{T}^d} m\psi''(m)DmDu\phi(u)\mathrm{d}x\mathrm{d}t$$
$$- \int_0^T \int_{\mathbb{T}^d} (\psi(m) - \psi'(m)m)\phi'(u)|Du|^2\mathrm{d}x\mathrm{d}t.$$

Hence, we have the claim. □

**Corollary 6.21.** *Let $\epsilon = 1$ and $V = 0$. Assume that $m_0$ is of class $C^1$ and bounded from below, $m_0 \geq \kappa_0 > 0$, and that $u_T$ is of class $C^2$. In addition, suppose that $F$ is non-decreasing and non-negative. Then, there exist $\lambda, \sigma > 0$ such that, for any solution $(u, m)$ of (6.2)–(6.3), we have*

$$\int_0^T \int_{\mathbb{T}^d} F(m)e^{\lambda u}m^{1+\sigma} + e^{\lambda u}|Du|^2(m^{1+\sigma} + 1) + m^{\sigma-1}|Dm|^2e^{\lambda u}\mathrm{d}x\mathrm{d}t \leq C.$$

*Proof.* Let $\phi(u) = \lambda^{-1}e^{\lambda u}$ and $\psi(m) = m^{1+\sigma}$. Then, from the identity in Lemma 6.20, we get

$$\begin{aligned}
&\int_0^T\int_{\mathbb{T}^d} F(m)e^{\lambda u}m^{1+\sigma}\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{T}^d} \frac{e^{\lambda u}}{2}|Du|^2(m^{1+\sigma}+m_0^{1+\sigma})\mathrm{d}x\mathrm{d}t+\\
&\lambda^{-1}\sigma(1+\sigma)\int_0^T\int_{\mathbb{T}^d} m^{\sigma-1}|Dm|^2e^{\lambda u}d x\mathrm{d}t + \lambda^{-1}\sigma(1+\sigma)\int_0^T\int_{\mathbb{T}^d} m^{\sigma}DmDue^{\lambda u}\mathrm{d}x\mathrm{d}t\\
&= \int_0^T\int_{\mathbb{T}^d} F(m)e^{\lambda u}m_0^{1+\sigma}\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{T}^d}\left[(\lambda-\sigma)m^{1+\sigma}-\lambda m_0^{1+\sigma}\right]e^{\lambda u}|Du|^2\mathrm{d}x\mathrm{d}t+\\
&\lambda^{-1}\int_{\mathbb{T}^d}\left(m_0^{1+\sigma}-m_T^{1+\sigma}\right)e^{\lambda u_T}\mathrm{d}x - \int_0^T\int_{\mathbb{T}^d}\lambda^{-1}e^{\lambda u}\Delta(m_0^{1+\sigma})\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{6.20}$$

Next, we take $\sigma < \lambda < \frac{1}{4}$. Then, using

$$F(m)m_0^{1+\sigma} \le \delta F(m)m^{1+\sigma} + C(\delta, F, m_0)$$

and weighted Cauchy inequalities, we can absorb the first two terms of the right-hand side into the first two terms of the left-hand side. Accordingly, we have

$$\begin{aligned}
&\frac{1}{2}\int_0^T\int_{\mathbb{T}^d} F(m)e^{\lambda u}m^{1+\sigma}\mathrm{d}x\mathrm{d}t + \frac{1}{4}\int_0^T\int_{\mathbb{T}^d} e^{\lambda u}|Du|^2(m^{1+\sigma}+m_0^{1+\sigma})\mathrm{d}x\mathrm{d}t+\\
&\lambda^{-1}\sigma(1+\sigma)\int_0^T\int_{\mathbb{T}^d} m^{\sigma-1}|Dm|^2e^{\lambda u}\mathrm{d}x\mathrm{d}t + \lambda^{-1}\sigma(1+\sigma)\int_0^T\int_{\mathbb{T}^d} m^{\sigma}DmDue^{\lambda u}\mathrm{d}x\mathrm{d}t \le\\
&\lambda^{-1}\int_{\mathbb{T}^d}\left(m_0^{1+\sigma}-m_T^{1+\sigma}\right)e^{\lambda u_T}\mathrm{d}x + C\int_0^T\int_{\mathbb{T}^d} e^{\lambda u}\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Choosing a small enough $\sigma$ such that $\lambda^{-1}\sigma(1+\sigma) < 1$, we can further absorb the fourth term on the left-hand side of the above inequality:

$$\begin{aligned}
&\int_0^T\int_{\mathbb{T}^d} F(m)e^{\lambda u}m^{1+\sigma}\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{T}^d} e^{\lambda u}|Du|^2(m^{1+\sigma}+m_0^{1+\sigma})\mathrm{d}x\mathrm{d}t+\\
&\lambda^{-1}\sigma(1+\sigma)\int_0^T\int_{\mathbb{T}^d} m^{\sigma-1}|Dm|^2e^{\lambda u}\mathrm{d}x\mathrm{d}t \le C + C\int_0^T\int_{\mathbb{T}^d} e^{\lambda u}\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

To finish the proof, we need to estimate $\int_0^T\int_{\mathbb{T}^d} e^{\lambda u}\mathrm{d}x\mathrm{d}t$.

First, we consider the case when $\sigma = 0$. Accordingly, the preceding equation gives

$$\int_0^T\int_{\mathbb{T}^d} F(m)e^{\lambda u}m\mathrm{d}x\mathrm{d}t \le C + C\int_0^T\int_{\mathbb{T}^d} e^{\lambda u}\mathrm{d}x\mathrm{d}t. \tag{6.21}$$

On the other hand, integrating (6.16), we obtain

$$\int_{\mathbb{T}^d} e^{\lambda u(x,t)} \mathrm{d}x - \int_{\mathbb{T}^d} e^{\lambda u_T} \mathrm{d}x + \lambda\left(\lambda + \frac{1}{2}\right) \int_t^T \int_{\mathbb{T}^d} |Du|^2 e^{\lambda u} \mathrm{d}x\mathrm{d}s$$
$$= \lambda \int_t^T \int_{\mathbb{T}^d} F(m) e^{\lambda u} \mathrm{d}x\mathrm{d}s. \tag{6.22}$$

Combining the foregoing estimates, we get

$$\int_{\mathbb{T}^d} e^{\lambda u(x,t)} \mathrm{d}x + \int_0^T \int_{\mathbb{T}^d} |D(e^{\frac{\lambda}{2} u})|^2 \mathrm{d}x\mathrm{d}t \leq C + C \int_0^T \int_{\mathbb{T}^d} e^{\lambda u} \mathrm{d}x\mathrm{d}s. \tag{6.23}$$

Let $v = e^{\frac{\lambda}{2} u(x,t)}$. By the Gagliardo–Nirenberg inequality, we have

$$\int_0^T \int_{\mathbb{T}^d} v^{2+4/d} \mathrm{d}x\mathrm{d}t \leq \sup_{[0,T]} \left( \int_{\mathbb{T}^d} |v(x,t)|^2 \right)^{\frac{2}{d}} \left( \int_0^T \int_{\mathbb{T}^d} |Dv|^2 \mathrm{d}x\mathrm{d}t \right).$$

Thus,

$$\int_0^T \int_{\mathbb{T}^d} e^{\lambda(1+2/d)u} \mathrm{d}x\mathrm{d}t \leq C \left( 1 + \left( \int_0^T \int_{\mathbb{T}^d} e^{\lambda u} \mathrm{d}x\mathrm{d}t \right)^{1+2/d} \right).$$

Using Hölder's inequality

$$\left( \int_0^T \int_{\mathbb{T}^d} e^{\lambda u} \mathrm{d}x\mathrm{d}t \right)^{1+\frac{2}{d}} \leq C(Te^{\lambda N})^{1+\frac{2}{d}} + C|\{u > N\}|^{\frac{2}{d+2}} \int_0^T \int_{\mathbb{T}^d} e^{\lambda(1+2/d)u} \mathrm{d}x\mathrm{d}t.$$

Since $|\{u > N\}| \leq \frac{\|u\|_1}{N} \leq \frac{C}{N}$ by Corollary 6.19, choosing a large enough $N$, gives

$$\int_0^T \int_{\mathbb{T}^d} e^{\lambda u} \mathrm{d}x\mathrm{d}t \leq C.$$

To end the proof, we observe that the case $\sigma > 0$ is immediate because we have

$$\int_0^T \int_{\mathbb{T}^d} e^{\lambda u} \mathrm{d}x\mathrm{d}t \leq C.$$

□

## 6.9 Bibliographical Notes

The estimates in Sects. 6.1–6.4 appeared first in [164–166]. A version of the second-order estimates was introduced in [90] for stationary problems. In the context of the Aubry–Mather theory, similar bounds appeared in [93, 110, 111]. The discussion in Sect. 6.6 is a simplified version of the argument in [89]. The results in Sects. 6.5 and 6.8 are taken from [195] (also see [196]). The energy conservation identity was used in [66] and [67] in the study of the long-time convergence of mean-field games.

Other bounds for stationary MFGs that rely only on elementary methods can be found in [127, 129] and [213]. The methods considered here can be generalized to many other cases including obstacle-type problems [116], weakly coupled MFGs [115], and multi-population models [77].

# Chapter 7
# A Priori Bounds for Stationary Models

We draw upon our earlier results to study stationary MFGs. Here, we illustrate various techniques in three models. First, we use the Bernstein estimates given in Theorem 3.11, to obtain Sobolev estimates for the value function. Next, we consider a congestion problem and show, through a remarkable identity, that $m > 0$. Finally, we examine an MFG with a logarithmic nonlinearity. This model presents substantial challenges since the logarithm is not bounded from below. However, a clever integration by parts argument gives the necessary bounds for its study.

## 7.1 The Bernstein Method

We fix a $C^2$ potential, $V : \mathbb{T}^d \to \mathbb{R}$, and look for a solution, $(u, m, \overline{H})$, of the MFG

$$\begin{cases} -\Delta u(x) + \frac{|Du(x)|^2}{2} + V(x) = \bar{H} + m^\alpha, \\ -\Delta m - \operatorname{div}(mDu) = 0, \\ \int u\mathrm{d}x = 0, \ \int m\mathrm{d}x = 1, \end{cases} \tag{7.1}$$

with $u, m : \mathbb{T}^d \to \mathbb{R}$ and $\overline{H} \in \mathbb{R}$. Here, we establish a preliminary result, namely that $u \in W^{1,q}$ for any $q > 1$. For that, we combine the integral Bernstein estimates in Chap. 3 with the first-order estimates in Chap. 6. In Chap. 11, we use these estimates to prove the existence of a classical solution of (7.1).

**Theorem 7.1.** *Let $(u, m, \overline{H})$ solve (7.1) and $0 < \alpha < \frac{1}{d-1}$. Suppose that $u, m \in C^2(\mathbb{T}^d)$. Then, for every $q > 1$, there exists a constant, $C_q > 0$, that depends only on $\|V\|_{L^{1+\frac{1}{\alpha}}(\mathbb{T}^d)}$, such that $\|Du\|_{L^q(\mathbb{T}^d)} \leq C_q$.*



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SpringerBriefs in Mathematics, DOI 10.1007/978-3-319-38934-9_7

*Proof.* To prove the claim, we use Theorem 3.11 with $V$ replaced by $V(x) - m^\alpha$. By Proposition 6.3 and Corollary 6.4, we have

$$|\bar{H}| \le C, \ \|m^\alpha\|_{L^{1+\frac{1}{\alpha}}(\mathbb{T}^d)} \le C.$$

Because $d < 1 + \frac{1}{\alpha}$, we have that for $p$ large $\gamma_p < 1 + \frac{1}{\alpha}$. Thus, Theorem 3.11 gives

$$\|Du\|_{L^q(\mathbb{T}^d)} \le C_q \text{ for every } q > 1. \qquad \square$$

## 7.2 A MFG with Congestion

The second problem we examine in this chapter is the MFG with congestion given by

$$\begin{cases} u - \Delta u + \frac{|Du|^2}{2m^\alpha} + V(x) = 0 \\ m - \Delta m - \operatorname{div}\left(m^{1-\alpha} Du\right) = 1, \end{cases} \tag{7.2}$$

where $u, m \in C^2(\mathbb{T}^d)$ and $m \ge 0$. In addition, we suppose that $V \in C(\mathbb{T}^d)$ and $0 < \alpha < 1$. We use the particular structure of (7.2) to prove that $\frac{1}{m}$ is bounded.

First, we state an auxiliary Lemma:

**Lemma 7.2.** *There exists a constant, $C := C(\|V\|_\infty) \ge 0$, such that, for any classical solution, $(u, m)$, of (7.2), we have*

$$\|u\|_{L^\infty(\mathbb{T}^d)} \le C. \tag{7.3}$$

*Furthermore, any solution, $m$, to the second equation in (7.2) is a probability density; that is, $m \ge 0$ on $\mathbb{T}^d$ and $\|m\|_{L^1(\mathbb{T}^d)} = 1$.*

*Proof.* To get the $L^\infty$ bound, we evaluate the first equation in (7.2) at a point of maximum of $u$ (resp., minimum). At that point $Du = 0$, $\Delta u \le 0$ (resp., $\ge 0$) and $V$ is bounded on $\mathbb{T}^d$. Thus, (7.3) follows. If we argue as in Proposition 4.3, then, $m$ is non-negative. Furthermore, it has a total mass of 1 by integrating the second equation in (7.2). $\square$

In the next proposition, we improve the previous lemma and prove that $m$ is strictly positive.

**Proposition 7.3.** *There exists a constant, $C := C(\|V\|_\infty) \ge 0$, such that for any classical solution, $(u, m)$, of (7.2), we have*

$$\left\|\frac{1}{m}\right\|_{L^\infty(\mathbb{T}^d)} \le C.$$

*Proof.* Let $r > \alpha$. For the proof, we first establish the identity:

$$\begin{aligned}&\int_{\mathbb{T}^d} \frac{1}{(r+1-\alpha)m^{r+1-\alpha}}\,\mathrm{d}x + \int_{\mathbb{T}^d} \frac{|Du|^2}{2rm^{r+\alpha}}\,\mathrm{d}x + \int_{\mathbb{T}^d} \frac{|Dm|^2}{m^{r+2-\alpha}}\,\mathrm{d}x \\ &= \int_{\mathbb{T}^d} \left[-\frac{V}{rm^r} - \frac{u}{rm^r} + \frac{1}{(r+1-\alpha)m^{r-\alpha}}\right]\mathrm{d}x.\end{aligned} \tag{7.4}$$

To prove the above, we subtract the second equation of (7.2) divided by $(r+1-\alpha)m^{r+1-\alpha}$ from the first equation of (7.2) divided by $rm^r$. Then,

$$\begin{aligned}&\int_{\mathbb{T}^d} \left[u - \Delta u + \frac{|Du|^2}{2m^\alpha} + V\right]\cdot \frac{1}{rm^r}\,\mathrm{d}x \\ &- \int_{\mathbb{T}^d} \left[m - \Delta m - \operatorname{div}\left(m^{1-\alpha}Du\right)\right]\cdot \frac{1}{(r+1-\alpha)m^{r+1-\alpha}}\,\mathrm{d}x \\ &= -\int_{\mathbb{T}^d} \frac{1}{(r+1-\alpha)m^{r+1-\alpha}}\,\mathrm{d}x.\end{aligned} \tag{7.5}$$

Next, we observe that

$$\int_{\mathbb{T}^d} \frac{\Delta u}{rm^r}\,\mathrm{d}x = \int_{\mathbb{T}^d} \frac{Du\cdot Dm}{m^{r+1}}\,\mathrm{d}x$$

and

$$\int_{\mathbb{T}^d} \frac{\operatorname{div}\left(m^{1-\alpha}Du\right)}{(r+1-\alpha)m^{r+1-\alpha}}\,\mathrm{d}x = \int_{\mathbb{T}^d} \frac{Du\cdot Dm}{m^{r+1}}\,\mathrm{d}x.$$

Hence,

$$\int_{\mathbb{T}^d} \frac{\Delta u}{rm^r}\,\mathrm{d}x - \int_{\mathbb{T}^d} \frac{\operatorname{div}\left(m^{1-\alpha}Du\right)}{(r+1-\alpha)m^{r+1-\alpha}}\,\mathrm{d}x = 0. \tag{7.6}$$

Therefore, (7.5) is reduced to (7.4).

Now, we note that Lemma 7.2 combined with (7.4) gives

$$\begin{aligned}&\int_{\mathbb{T}^d} \frac{1}{(r+1-\alpha)m^{r+1-\alpha}}\,\mathrm{d}x + \int_{\mathbb{T}^d} \frac{|Du|^2}{2rm^{r+\alpha}}\,\mathrm{d}x + \int_{\mathbb{T}^d} \frac{|Dm|^2}{m^{r+2-\alpha}}\,\mathrm{d}x \\ &\leq \int_{\mathbb{T}^d} \frac{C}{rm^r}\,\mathrm{d}x + \int_{\mathbb{T}^d} \frac{C}{(r-\alpha)m^{r-\alpha}}\,\mathrm{d}x.\end{aligned}$$

By Young's inequality for $\alpha \in [0, 1)$, we have

$$\frac{C}{rm^r} \leq \frac{1}{4(r+1-\alpha)m^{r+1-\alpha}} + C_r^1$$

and

$$\frac{C}{(r-\alpha)m^{r-\alpha}} \leq \frac{1}{4(r-\alpha)m^{r+1-\alpha}} + C_r^2$$

with

$$C_r^1 := \frac{(1-\alpha)4^{\frac{r}{1-\alpha}} C^{\frac{r+1-\alpha}{1-\alpha}}}{r(r+1-\alpha)}, \; C_r^2 := \frac{4^{r-\alpha} C^{r+1-\alpha}(r-\alpha)^{r-\alpha-1}}{(r+1-\alpha)^{r+1-\alpha}}.$$

Therefore,

$$\frac{1}{r+1-\alpha} \int_{\mathbb{T}^d} \frac{1}{m^{r-\alpha+1}} \leq 2(C_r^1 + C_r^2).$$

Thus, we get

$$\left\| \frac{1}{m} \right\|_{L^{r+1-\alpha}(\mathbb{T}^d)} \leq \left[ 2(r+1-\alpha)(C_r^1 + C_r^2) \right]^{\frac{1}{r+1-\alpha}} =: C_\alpha(r).$$

Finally, we check that, for any $r_0 > \alpha$, there exists $C_\alpha$ for which

$$C_\alpha(r) \leq C_\alpha, \quad \text{for all } r \in [r_0, \infty). \qquad \square$$

## 7.3 Logarithmic Nonlinearity

Because $\ln m$ is not bounded from below, MFGs with logarithmic nonlinearities present substantial challenges. Here, we collect some estimates to overcome these issues. For $(u, m, \overline{H})$, $u, m : \mathbb{T}^d \to \mathbb{R}$ and $\overline{H} \in \mathbb{R}$, we consider the problem

$$\begin{cases} -\Delta u(x) + \frac{|Du(x)|^2}{2} + V(x) = \bar{H} + \ln m, \\ -\Delta m - \operatorname{div}(mDu) = 0, \\ \int u \mathrm{d}x = 0, \; \int m \mathrm{d}x = 1. \end{cases} \tag{7.7}$$

**Proposition 7.4.** *Let $(u, m)$ solve (7.7). Then, there exists a constant, $C > 0$, that depends only on $\|V\|_{L^\infty(\mathbb{T}^d)}$ and $|\bar{H}|$, such that $\|\ln m\|_{H^1(\mathbb{T}^d)} \leq C$.*

*Proof.* Integrating the first equation in (7.7), we get

$$\int_{\mathbb{T}^d} \frac{|Du|^2}{2} \mathrm{d}x = -\int_{\mathbb{T}^d} V \mathrm{d}x + \bar{H} + \int \ln m \mathrm{d}x.$$

Using Jensen's inequality, we have

$$0 \geq \int_{\mathbb{T}^d} \ln m \mathrm{d}x \geq -C + \int_{\mathbb{T}^d} \frac{|Du|^2}{2} \mathrm{d}x \geq -C.$$

Therefore,

$$\int_{\mathbb{T}^d} \frac{|Du|^2}{2} \mathrm{d}x, \left| \int_{\mathbb{T}^d} \ln m \mathrm{d}x \right| \leq C.$$

Multiplying the second equation in (7.7) by $\frac{1}{m}$, integrating by parts and using the above, we get

$$\int_{\mathbb{T}^d} |D \ln m|^2 \mathrm{d}x \leq C \int_{\mathbb{T}^d} |Du|^2 \mathrm{d}x \leq C. \tag{7.8}$$

Finally, the foregoing bound and the Poincaré inequality give

$$\int_{\mathbb{T}^d} |\ln m|^2 \mathrm{d}x \leq C \left[ \left( \int_{\mathbb{T}^d} \ln m \mathrm{d}x \right)^2 + \int_{\mathbb{T}^d} |D \ln m|^2 \mathrm{d}x \right] \leq C.$$

□

*Remark 7.5.* The estimate (7.8) is a stationary version of the result in Proposition 4.5.

The previous proposition can be improved as follows.

**Proposition 7.6.** *Let* $(u, m, \overline{H})$ *solve* (7.7)*. Then, for every* $1 \leq p < \infty$*, there exists a constant,* $C_p > 0$*, that does not depend on the solution, such that*

$$\| |\ln m|^p \|_{H^1(\mathbb{T}^d)} \leq C_p.$$

*Proof.* We prove by induction that $f_k = |\ln m|^{\frac{k+1}{2}} \in H^1(\mathbb{T}^d)$ for any $k \in \mathbb{N}$. The case $k = 1$ is given by Proposition 7.4. Let $l \geq 1$ and suppose that $\|f_k\|_{H^1(\mathbb{T}^d)} \leq C_l$ for all $k \leq l$. Then, we have

$$\|Df_k\|_{L^2}^2 = \int_{\mathbb{T}^d} \frac{|\ln m|^{k-1}}{m^2} |Dm|^2 \mathrm{d}x \leq C_l^2$$

and

$$\|f_k\|_{L^2}^2 = \int_{\mathbb{T}^d} |\ln m|^{k+1} \mathrm{d}x \leq C_l^2.$$

Next, we show that $f_{l+1} \in H^1(\mathbb{T}^d)$. Let $F_l(z) = \int_1^z \frac{|\ln y|^l}{y^2} \mathrm{d}y$. Multiplying the second equation of (7.7) by $F_l(m)$ and integrating by parts, we get

$$\int_{\mathbb{T}^d} \frac{|\ln m|^l}{m^2} |Dm|^2 \mathrm{d}x = -\int_{\mathbb{T}^d} \frac{|\ln m|^l}{m} DmDu \mathrm{d}x \leq \frac{1}{2} \int_{\mathbb{T}^d} \frac{|\ln m|^l}{m^2} |Dm|^2 \mathrm{d}x + \frac{1}{2} \int_{\mathbb{T}^d} |\ln m|^l |Du|^2 \mathrm{d}x.$$

Thus,

$$\int_{\mathbb{T}^d} \frac{|\ln m|^l}{m^2} |Dm|^2 \mathrm{d}x \leq \int_{\mathbb{T}^d} |\ln m|^l |Du|^2 \mathrm{d}x. \tag{7.9}$$

From the first equation of (7.7), we infer that

$$\frac{|Du|^2}{2} \leq C + |\ln m| + \Delta u.$$

Therefore, multiplying by $|\ln m|^l$ and integrating gives

$$\int_{\mathbb{T}^d} |\ln m|^l |Du|^2 \mathrm{d}x \leq C \int_{\mathbb{T}^d} |\ln m|^l \mathrm{d}x + C \int_{\mathbb{T}^d} |\ln m|^{l+1} \mathrm{d}x + \int_{\mathbb{T}^d} \Delta u |\ln m|^l \mathrm{d}x.$$

Integrating by parts the last term yields

$$\int_{\mathbb{T}^d} |\ln m|^l |Du|^2 \mathrm{d}x \leq C \int_{\mathbb{T}^d} |\ln m|^l \mathrm{d}x + C \int_{\mathbb{T}^d} |\ln m|^{l+1} \mathrm{d}x - \int_{\mathbb{T}^d} Du |\ln m|^{l-1} \frac{Dm}{m} \operatorname{sgn}(\ln m) \mathrm{d}x$$

from the last term. This integration by parts is valid because, for any smooth function $f$, the identity $D(|f|^p) = p|f|^{p-2} \operatorname{sgn}(f) Df$ holds both a.e. and in the sense of distributions. Accordingly,

$$\begin{aligned}
\int_{\mathbb{T}^d} |\ln m|^l |Du|^2 \mathrm{d}x &\leq C \int_{\mathbb{T}^d} |\ln m|^l \mathrm{d}x + C \int_{\mathbb{T}^d} |\ln m|^{l+1} \mathrm{d}x \\
&\quad + C \int_{\mathbb{T}^d} |\ln m|^{l-1} |Du|^2 \mathrm{d}x + C \int_{\mathbb{T}^d} |\ln m|^{l-1} \frac{|Dm|^2}{m^2} \mathrm{d}x \\
&\leq C + C \int_{\mathbb{T}^d} |\ln m|^{l+1} \mathrm{d}x \\
&\quad + C \int_{\mathbb{T}^d} [\epsilon |\ln m|^l + C(\epsilon)] |Du|^2 \mathrm{d}x + C \int_{\mathbb{T}^d} |\ln m|^{l-1} \frac{|Dm|^2}{m^2} \mathrm{d}x,
\end{aligned}$$

which yields

$$\int_{\mathbb{T}^d} |\ln m|^l |Du|^2 \mathrm{d}x \leq C \int_{\mathbb{T}^d} |\ln m|^{l+1} \mathrm{d}x + C \int_{\mathbb{T}^d} |\ln m|^{l-1} \frac{|Dm|^2}{m^2} \mathrm{d}x + C. \tag{7.10}$$

Combining (7.9) and (7.10), we get

$$\int_{\mathbb{T}^d} \frac{|\ln m|^l}{m^2} |Dm|^2 \mathrm{d}x \leq C \int_{\mathbb{T}^d} |\ln m|^{l+1} \mathrm{d}x + C \int_{\mathbb{T}^d} |\ln m|^{l-1} \frac{|Dm|^2}{m^2} \mathrm{d}x + C.$$

Consequently,

$$\|Df_{l+1}\|_{L^2}^2 \leq C\|f_l\|_{L^2}^2 + C\|Df_l\|_{L^2}^2 + C \leq C_{l+1}.$$

Since $|\ln m|^{l+1} = f_l^2 \in L^1$, we have $f_{l+1} = |\ln m|^{\frac{l+1}{2}} \in L^1$. By Poincaré's inequality,

$$\|f_{l+1}\|_{L^2}^2 \leq \|f_{l+1}\|_{L^1}^2 + C\|Df_{l+1}\|_{L^2}^2 \leq C_{l+1};$$

this concludes the proof. □

## 7.4 Bibliographical Notes

The paper [164] introduced the first a priori estimates for stationary MFGs. Subsequently, several other estimates were developed in [127, 129] and [130]. The integral Bernstein method was introduced in [173] in the context of Hamilton–Jacobi equations. It was then used in an MFG in [77]. Our presentation follows [194], where this method is explored for more refined estimates.

For MFGs with congestion, the bound for $\frac{1}{m}$ considered here was obtained in [114]. Because this estimate depends in a crucial way on a cancellation between the Hamilton–Jacobi equation and the Fokker–Planck equation, general Hamiltonians cannot be studied with this method. In this case, the existence of solutions remains an open problem.

First-order MFGs with logarithmic nonlinearities were studied in the context of the Aubry–Mather theory in [90]. Then, a one-dimensional problem was investigated in [125]. Second-order MFGs with logarithmic nonlinearities were first studied in [130]. These results were subsequently improved in [194].

# Chapter 8
# A Priori Bounds for Time-Dependent Models

We continue our study of the regularity of MFGs by considering the time-dependent problem

$$\begin{cases} -u_t + \frac{1}{\gamma}|Du|^\gamma = \Delta u + m^\alpha & \text{in} \quad \mathbb{T}^d \times [0,T], \\ m_t - \operatorname{div}(|Du|^{\gamma-1} m) = \Delta m & \text{in} \quad \mathbb{T}^d \times [0,T], \end{cases} \tag{8.1}$$

where $1 < \gamma \leq 2$ and $\alpha > 0$. For $\gamma < 2$, we are in the subquadratic case; for $\gamma = 2$ the quadratic case. In the first instance, the non-linearity $|Du|^\gamma$ acts as a perturbation of the heat equation and the main regularity tool is the Gagliardo–Nirenberg inequality. In the second instance, the Hopf–Cole transformation gives an explicit way to study (8.1). However, this transformation cannot be used to superquadratic problems. As a consequence, here, we use a technique that extends for superquadratic problems, $\gamma > 2$, based on the nonlinear adjoint method. In the next chapter, we investigate two time-dependent problems with singularities—the logarithmic nonlinearity and the congestion problem—for which different methods are required.

To get bounds for (8.1), we combine the estimates for Fokker–Planck equations with estimates for Hamilton–Jacobi equations according to the strategy that we outline next. First, we fix two function spaces, $X$ and $Y$, which are typically Lebesgue or Sobolev spaces. Next, we use the regularity of the Fokker–Planck equation to show that

$$\|m^\alpha\|_X \leq C_1 + C_1 \|Du\|_Y^{\theta_1},$$

where $C_1 > 0$ is a constant and $\theta_1 > 0$. Subsequently, we apply bounds for the Hamilton–Jacobi equation to get an estimate of the form:

$$\|Du\|_Y \leq C_2 + C_2 \|m^\alpha\|_X^{\theta_2},$$



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where $C_2 > 0$ and $\theta_2 > 0$. Finally, we combine the prior inequalities to obtain

$$\|Du\|_Y \leq C + C\,\|Du\|_Y^{\theta_1\theta_2}. \tag{8.2}$$

If $\theta_1\theta_2 < 1$, we have an estimate for $Du$ in $Y$. The choice of the spaces, $X$ and $Y$, and of the corresponding estimates depends on $\gamma$ and is distinct for subquadratic and quadratic problems.

## 8.1 Subquadratic Hamiltonians

To study subquadratic Hamiltonians, we combine the polynomial estimates for Fokker–Planck equations in Proposition 4.17, the integral bounds for Hamilton–Jacobi equations in Proposition 3.15, and the bounds given by the Gagliardo–Nirenberg inequality in Theorem 3.23. Because our goal is to illustrate the main ideas in the simplest possible way, the next result is far from optimal. We refer the reader to Remark 8.2 and to the bibliographical notes at the end of the chapter for sharper results.

**Theorem 8.1.** *Let $(u,m)$ solve (8.1) with $1 < \gamma < 2$. Suppose that*

$$0 < \alpha < \frac{2-\gamma}{2d(\gamma-1)}.$$

*Then, for any $1 < s < \infty$, there exists a constant, $C_s$, such that*

$$\|m\|_{L^s(\mathbb{T}^d\times[0,T])} + \|Du\|_{L^s(\mathbb{T}^d\times[0,T])} \leq C_s.$$

*Proof.* By Theorem 3.23 with $r = p = \frac{\gamma-1}{\gamma}\bar{r}$, we have

$$\|Du\|_{L^{(\gamma-1)\bar{r}}(\mathbb{T}^d\times[0,T])} \leq C\|m^\alpha\|_{L^{(\gamma-1)\bar{r}/\gamma}(\mathbb{T}^d\times[0,T])} + C\|u\|_{L^\infty(\mathbb{T}^d\times[0,T])}^{2/(2-\gamma)}, \tag{8.3}$$

provided that

$$\frac{(\gamma-1)}{\gamma}\bar{r} > 1. \tag{8.4}$$

Proposition 3.15, with $p = \frac{\gamma-1}{\gamma}\bar{r}$, gives

$$\|u\|_{L^\infty(\mathbb{T}^d\times[0,T])} \leq C + C\|m^\alpha\|_{L^{(\gamma-1)\bar{r}/\gamma}(\mathbb{T}^d\times[0,T])} = C + C\|m\|^\alpha_{L^{\alpha(\gamma-1)\bar{r}/\gamma}(\mathbb{T}^d\times[0,T])} \tag{8.5}$$

if

$$\frac{(\gamma-1)}{\gamma}\bar{r} > \frac{d}{2} + 1. \tag{8.6}$$

Finally, Proposition 4.17 with $r = p = \frac{\bar{r}}{2}$, $\beta_0 = 1$, and $n = 1$ implies that

$$\|m\|^{\theta}_{L^{\theta}(\mathbb{T}^d\times[0,T])} \leq C + C\|Du\|^{(\gamma-1)\bar{r}}_{L^{(\gamma-1)\bar{r}}(\mathbb{T}^d\times[0,T])}, \tag{8.7}$$

provided that Remark 4.16 holds, and

$$\frac{\bar{r}}{2} > \frac{d}{2} \qquad \text{and} \qquad \frac{\bar{r}}{2} > \frac{\bar{r}}{\bar{r}-d}, \tag{8.8}$$

or, equivalently,

$$\bar{r} > d + 2, \tag{8.9}$$

and conditions (4.23) and (4.26) are met such that

$$\theta = \frac{2(\bar{r}/2 - 1)}{d}. \tag{8.10}$$

Suppose that $\theta$ satisfies

$$\theta \geq \alpha\frac{\gamma-1}{\gamma}\bar{r}. \tag{8.11}$$

Then, we get

$$\|Du\|_{L^{(\gamma-1)\bar{r}}(\mathbb{T}^d\times[0,T])} \leq C + C\|Du\|^{(\gamma-1)\bar{r}\alpha/\theta}_{L^{(\gamma-1)\bar{r}}(\mathbb{T}^d\times[0,T])} + C\|Du\|^{2(\gamma-1)\bar{r}\alpha/(\theta(2-\gamma))}_{L^{(\gamma-1)\bar{r}}(\mathbb{T}^d\times[0,T])}. \tag{8.12}$$

Therefore, if

$$\frac{2(\gamma-1)\bar{r}\alpha}{\theta(2-\gamma)} < 1, \tag{8.13}$$

we have

$$\|Du\|_{L^{(\gamma-1)\bar{r}}(\mathbb{T}^d\times[0,T])} \leq C.$$

Elementary computations show that for $0 < \alpha < \frac{2-\gamma}{2d(\gamma-1)}$, there exist $\bar{r} > 1$ and $\theta > 1$ such that (8.6), (8.9), (8.10), (8.11), and (8.13) hold simultaneously. Then, the iterative estimates in Proposition 4.17 give $m \in L^s$ for any $1 < s < \infty$. Thus, by Proposition 3.15, we have $Du \in L^s$ for any $1 < s < \infty$. □

*Remark 8.2.* The preceding result can be improved by taking $\beta_0 > 1$. For that, we can use the first-order estimates in Sect. 6.2 of Chap. 6 or the second-order estimates in Sect. 6.4 of the same chapter combined with the iterative estimates in Sect. 4.4.1 of Chap. 4. We refer the reader to the bibliographical notes for additional results.

## 8.2 Quadratic Hamiltonians

Here, we consider $\gamma = 2$ in (8.1) and get bounds for the $L^\infty$ norm of $Du$.

**Theorem 8.3.** *Let $(u, m)$ solve (8.1) with $\gamma = 2$. Suppose that $d > 2$ and*

$$0 < \alpha < \frac{3}{4d} - \frac{1}{2d^2}.$$

*Then, there exists a constant, $C$, that does not depend on the solution, such that $\|Du\|_{L^\infty(\mathbb{T}^d\times[0,T])} \leq C$.*

*Proof.* From Theorem 5.12, we have

$$\|Du\|_{L^\infty} \leq C + C\|m^\alpha\|^2_{L^\infty([0,T],L^{\bar{r}}(\mathbb{T}^d))} + C\|u\|_{L^\infty(\mathbb{T}^d\times[0,T])}, \tag{8.14}$$

provided that

$$\bar{r} > d. \tag{8.15}$$

Next, by Remark 3.16, we have

$$\|u\|_{L^\infty(\mathbb{T}^d\times[0,T])} \leq C + C\|m^\alpha\|_{L^\infty([0,T],L^{\bar{r}}(\mathbb{T}^d))} \leq C + C\|m\|^\alpha_{L^\infty([0,T],L^{\alpha\bar{r}}(\mathbb{T}^d))},$$

provided that $\bar{r} > \frac{d}{2}$, which holds by (8.15). Finally, Corollary 4.21 with $\beta_0 = 1$ and $n = 0$ gives

$$\|m\|^\theta_{L^\infty([0,T],L^\theta(\mathbb{T}^d))} \leq C + C\,\|Du\|^{2\bar{r}}_{L^\infty(\mathbb{T}^d\times[0,T])}, \tag{8.16}$$

with

$$\bar{r} = \frac{2 + d(\theta - 1)}{2} \qquad \text{and} \qquad \theta > 1. \tag{8.17}$$

Suppose that

$$\theta \geq \alpha\bar{r}. \tag{8.18}$$

Combining the previous estimates, we get

$$\|Du\|_{L^\infty(\mathbb{T}^d\times[0,T])} \leq C + C\|m\|^{2\alpha}_{L^\infty([0,T],L^{\alpha\bar{r}}(\mathbb{T}^d))} \leq C + C\|Du\|^{\frac{4\alpha\bar{r}}{\theta}}_{L^\infty(\mathbb{T}^d\times[0,T])}.$$

Thus, if

$$4\alpha\bar{r} < \theta, \tag{8.19}$$

we get $Du \in L^\infty(\mathbb{T}^d \times [0,T])$.

Elementary computations show that there exist $r > 1$ and $\theta > 1$ satisfying (8.15), (8.17), (8.18), and (8.19) if $\alpha < \frac{3}{4d} - \frac{1}{2d^2}$ for $d > 2$. □

*Remark 8.4.* As in the preceding section, the previous result can be improved by selecting $\beta_0 > 1$. Moreover, the methods in this section can be used for superquadratic Hamiltonians; that is, $\gamma > 2$. We refer the reader to the bibliographical notes for additional results.

## 8.3 Bibliographical Notes

The techniques used in this chapter were developed in the thesis [193] and in the papers [134] and [135] for subquadratic and superquadratic problems, respectively. For unbounded domains, see [118]. An application of these methods to the forward–forward MFG problem can be found in [119].

# Chapter 9
# A Priori Bounds for Models with Singularities

Here, we discuss two problems—an MFG with a logarithmic nonlinearity and an MFG with congestion effects. Stationary versions of these two problems were considered in Chap. 7. However, the techniques for time-dependent problems are substantially different from the ones used in the stationary case.

## 9.1 Logarithmic Nonlinearities

We begin our study of MFGs with singularities by examining the system

$$\begin{cases} -u_t(x,t) + \frac{1}{2}|Du(x,t)|^\gamma = \Delta u(x,t) + \ln m(x,t), & (x,t) \in \mathbb{T}^d \times [0,T], \\ m_t(x,t) - \operatorname{div}(m|Du|^{\gamma-2}Du) = \Delta m(x,t), & (x,t) \in \mathbb{T}^d \times [0,T], \end{cases} \tag{9.1}$$

with initial-terminal boundary conditions

$$\begin{cases} u(x,T) = u_T(x), & x \in \mathbb{T}^d, \\ m(x,0) = m_0(x), & x \in \mathbb{T}^d, \end{cases} \tag{9.2}$$

where $T > 0$ is a fixed terminal instant. The Hamiltonian associated with (9.1) is

$$\frac{|p|^\gamma}{\gamma} - \ln m.$$

The corresponding Lagrangian given by the Legendre transform [see (1.7)] is thus

$$\frac{|v|^{\gamma'}}{\gamma'} + \ln m,$$



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with $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$. Hence, areas of low density are highly desirable from the point of view of agents. This effect should therefore force the density, $m$, to be bounded from below. Here, however, the primary mechanism for regularity is given by the diffusion that overcomes the nonlinearity, $|p|^\gamma$, if $\gamma$ is close to 1 and prevents low-density regions.

**Lemma 9.1.** *Let $(u, m)$ be a classical solution of (9.1)–(9.2) with $m > 0$. Then,*

$$\frac{d}{dt}\left[\ln\left(\int_{\mathbb{T}^d} \frac{1}{m}\mathrm{d}x\right)\right] \leq C\left\||Du|^2\right\|_{L^\infty(\mathbb{T}^d)}^{\gamma-1} + C.$$

*Proof.* Proposition 4.5 ensures that, for some constants, $C > 0$ and $c > 0$, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}^d} \frac{1}{m}dt &\leq C\int_{\mathbb{T}^d} \frac{|Du|^{2(\gamma-1)}}{m}\mathrm{d}x - c\int_{\mathbb{T}^d} \frac{|Dm|^2}{m^3}\mathrm{d}x \\ &\leq C\,\|Du\|_{L^\infty(\mathbb{T}^d)}^{2(\gamma-1)}\int_{\mathbb{T}^d} \frac{1}{m}\mathrm{d}x.\end{aligned}$$

Hence, we conclude that

$$\frac{d}{dt}\left[\ln\left(\int_{\mathbb{T}^d} \frac{1}{m}\mathrm{d}x\right)\right] \leq C\,\|Du\|_{L^\infty(\mathbb{T}^d)}^{2(\gamma-1)}.$$

□

**Lemma 9.2.** *Assume that $m : \mathbb{T}^d \to \mathbb{R}^+$ is integrable with $\int_{\mathbb{T}^d} m = 1$. Then, for $p > 1$, there exists a constant, $C > 0$, such that*

$$\int_{\mathbb{T}^d} |\ln m|^p \mathrm{d}x \leq C + C\int_{m\leq 1}\left(\ln\frac{1}{m}\right)^p \mathrm{d}x.$$

*Proof.* For $p > 1$, we have

$$\int_{\mathbb{T}^d} |\ln m|^p \mathrm{d}x = \int_{m\leq 1}\left(\ln\frac{1}{m}\right)^p \mathrm{d}x + \int_{m>1}(\ln m)^p \mathrm{d}x.$$

Because $\ln z$ is sublinear for $z > 1$, we get

$$\ln m \leq C_\delta m^\delta,$$

for every $\delta > 0$, provided that $m > 1$. Hence, we infer that

$$\int_{\mathbb{T}^d} |\ln m|^p \mathrm{d}x \leq \int_{m\leq 1}\left(\ln\frac{1}{m}\right)^p \mathrm{d}x + C$$

and conclude the proof. □

Next, we present an auxiliary lemma:

**Lemma 9.3.** *There exists* $0 < A = A(p)$*, such that* $(\ln z)^p$ *is a concave function of* $z$ *for* $z > \frac{1}{A}$.

*Proof.* By elementary computations, we have

$$[(\ln z)^p]'' = \frac{p\,(\ln z)^{p-2}}{z^2}\,[p - 1 - \ln z]\,.$$

For $z > e^{p-1}$, we get

$$[(\ln z)^p]'' < 0.$$

Hence, by setting $A = A(p) := e^{1-p}$, we prove the lemma. □

**Lemma 9.4.** *Let* $(u, m)$ *solve* (9.1)–(9.2)*. Then, there exists a constant,* $C > 0$*, such that*

$$\int_{m\leq 1}\left(\ln\frac{1}{m(x,\tau)}\right)^p \mathrm{d}x \leq C + C\,\|Du\|^{2p(\gamma-1)}_{L^\infty(\mathbb{T}^d\times[0,T])}\,.$$

*Proof.* First, we observe that

$$\int_{m\leq 1}\left(\ln\frac{1}{m}\right)^p \mathrm{d}x = \int_{A\leq m\leq 1}\left(\ln\frac{1}{m}\right)^p \mathrm{d}x + \int_{m<A}\left(\ln\frac{1}{m}\right)^p \mathrm{d}x,$$

for every $0 < A < 1$. By choosing $A$ as in Lemma 9.3, we get

$$\int_{A\leq m\leq 1}\left(\ln\frac{1}{m}\right)^p \mathrm{d}x \leq C\max_{A\leq m\leq 1}\left|\ln\frac{1}{m}\right|^p \leq C.$$

Now, we set $\Psi(z) := (\ln z)^p$ for $z \geq \frac{1}{A}$ and extend $\Psi$ continuously and linearly for $z < \frac{1}{A}$. We can do this so that $\Psi$ is globally concave and increasing.

Therefore, Jensen's inequality implies that

$$\frac{1}{|\{m\leq A\}|}\int_{m\leq A}\left(\ln\frac{1}{m}\right)^p \mathrm{d}x = \frac{1}{|\{m\leq A\}|}\int_{m\leq A}\Psi\left(\frac{1}{m}\right)\mathrm{d}x \leq \Psi\left(\frac{1}{|\{m\leq A\}|}\int_{m<A}\frac{1}{m}\right).$$

Because $\Psi$ is increasing, we have that

$$\frac{1}{|\{m \leq A\}|} \int_{m \leq A} \left( \ln \frac{1}{m} \right)^p \mathrm{d}x \leq \Psi \left( \frac{1}{|\{m \leq A\}|} \int_{\mathbb{T}^d} \frac{1}{m} \right).$$

In the sequel, we consider two cases: either

$$\frac{1}{|\{m \leq A\}|} \int_{\mathbb{T}^d} \frac{1}{m} < \frac{1}{A},$$

or

$$\frac{1}{|\{m \leq A\}|} \int_{\mathbb{T}^d} \frac{1}{m} > \frac{1}{A}.$$

The former inequality yields

$$\int_{m \leq A} \left( \ln \frac{1}{m} \right)^p \mathrm{d}x \leq \Psi \left( \frac{1}{A} \right),$$

whereas the latter implies that

$$\Psi \left( \frac{1}{|\{m \leq A\}|} \int_{\mathbb{T}^d} \frac{1}{m} \right) = \left[ \ln \left( \frac{1}{|\{m \leq A\}|} \right) + \ln \left( \int_{\mathbb{T}^d} \frac{1}{m} \right) \right]^p.$$

Because $|\{m \leq A\}| \leq 1$, we conclude that

$$\begin{aligned}\int_{m \leq A} \left( \ln \frac{1}{m} \right)^p \mathrm{d}x \leq \Psi \left( \frac{1}{A} \right) &+ C_p \left[ \ln \left( \int_{\mathbb{T}^d} \frac{1}{m} \right) \right]^p \\ &+ C_p \, |\{m \leq A\}| \left[ \ln \left( \frac{1}{|\{m \leq A\}|} \right) \right]^p.\end{aligned}$$

Since

$$\frac{1}{|\{m \leq A\}|} \geq 1,$$

we have

$$\ln \left( \frac{1}{|\{m \leq A\}|} \right) \leq C_\delta \left( \frac{1}{|\{m \leq A\}|} \right)^\delta,$$

for every $\delta > 0$. Set $\delta := \frac{1}{p}$; then,

$$\int_{m\leq A}\left(\ln\frac{1}{m}\right)^p \mathrm{d}x \leq C + C\left[\ln\left(\int_{\mathbb{T}^d}\frac{1}{m}\right)\right]^p + C\frac{|\{m\leq A\}|}{|\{m\leq A\}|},$$

i.e.,

$$\int_{m\leq A}\left(\ln\frac{1}{m}\right)^p \mathrm{d}x \leq C + C\left[\ln\left(\int_{\mathbb{T}^d}\frac{1}{m}\right)\right]^p;$$

this, in light of Lemma 9.1, concludes the proof. □

We close this section with the main theorem that gives a bound for $\ln m$ in terms of norms of $Du$.

**Theorem 9.5.** *Let $(u,m)$ solve (9.1)–(9.2). Then, for every $p>1$,*

$$\|\ln m\|_{L^\infty(0,T;L^p(\mathbb{T}^d))} \leq C + C\|Du\|^{2(\gamma-1)}_{L^\infty(\mathbb{T}^d\times[0,T])}.$$

*Proof.* The claim results from combining Lemmas 9.2 and 9.4 to conclude that

$$\int_{\mathbb{T}^d}|\ln m|^p \mathrm{d}x \leq C + C\left\||Du|^2\right\|^{p(\gamma-1)}_{L^\infty(\mathbb{T}^d\times[0,T])}.$$

□

## 9.2 Congestion Models: Local Existence

Here, we consider the MFG congestion model given by

$$\begin{cases} -u_t - \Delta u + \frac{m^\alpha}{\gamma}\left(\frac{|Du|}{m^\alpha}\right)^\gamma = 0, \\ m_t - \Delta m - \operatorname{div}\left(\frac{Du}{m^\alpha}\left(\frac{|Du|}{m^\alpha}\right)^{\gamma-2} m\right) = 0, \end{cases} \tag{9.3}$$

together with the initial-terminal conditions

$$u(x,T) = u_T(x),\ m(x,0) = m_0(x). \tag{9.4}$$

As before, we work in the spatially periodic setting. The unknowns in (9.3) are $u\colon \mathbb{T}^d\times[0,T]\to\mathbb{R}$ and $m\colon\mathbb{T}^d\times[0,T]\to\mathbb{R}^+$. We assume that $u_T, m_0\colon\mathbb{T}^d\to\mathbb{R}$ are given $C^\infty$ functions with $m_0>0$. The terminal cost, $u_T\colon\mathbb{T}^d\to\mathbb{R}$, is globally bounded with bounded derivatives of all orders. The initial distribution, $m_0$, is a $C^\infty$ probability density: $\int_{\mathbb{T}^d} m_0(x)\mathrm{d}x = 1$. Finally, we suppose that $1 \leq \gamma < 2$ and, as previously, $0 < \alpha < 1$.

The specific form of the Hamiltonian in (9.3) is motivated by the following consideration. With congestion, agents face difficulties in moving at high speed in high-density areas. Hence, it is natural to consider the Lagrangian,

$$L(x, v, m) = m^\alpha \frac{|v|^{\gamma'}}{\gamma'},$$

with $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$. The corresponding Hamiltonian is

$$H(x, p, m) = \frac{m^\alpha}{\gamma} \left(\frac{|p|}{m^\alpha}\right)^\gamma.$$

### 9.2.1 Estimates for Arbitrary Terminal Time

We begin our study of (9.3) by proving estimates that are valid for all terminal times, $T > 0$. Unfortunately, these estimates are not strong enough to prove the existence of solutions. Therefore, in the next section, we consider the short-time problem. There, we examine a local-in-time estimate that gives a bound for $\frac{1}{m}$ provided that $T$ is small enough.

The next two results are straightforward consequences of the comparison principle in Proposition 3.1.

**Proposition 9.6.** *For any $C^\infty$ solution, $(u, m)$, of (9.3)–(9.4), we have*

$$\|m(\cdot, t)\|_{L^1(\mathbb{T}^d)} = 1,$$

*and $u \geq -\|u_T\|_{L^\infty(\mathbb{T}^d)}$, for all $0 \leq t \leq T$.*

*Proof.* Proposition 4.1 gives $\int_{\mathbb{T}^d} m(x, t)\mathrm{d}x = 1$ for all $t \geq 0$. Moreover, by Proposition 4.3, we have $m \geq 0$ for all $t \geq 0$.

The lower bound on $u$ results from the comparison principle in Proposition 3.1 with $v = \min u_T$. □

**Proposition 9.7.** *For any $C^\infty$ solution, $(u, m)$, of (9.3)–(9.4), $u \leq \|u_T\|_{L^\infty(\mathbb{T}^d)}$.*

*Proof.* Because $\frac{m^\alpha}{\gamma}\left(\frac{|Du|}{m^\alpha}\right)^\gamma \geq 0$, we have

$$-u_t - \Delta u \leq 0.$$

Then, Proposition 3.1 gives $u \leq \max u_T$. □

The two preceding propositions give the following corollary.

**Corollary 9.8.** *For any $C^\infty$ solution $(u, m)$ of (9.3)–(9.4), $\|u\|_{L^\infty(\mathbb{T}^d)} \leq \|u_T\|_{L^\infty(\mathbb{T}^d)}$.*

**Proposition 9.9.** *There exists a constant, $C := C(\|u_T\|_{L^\infty(\mathbb{T}^d)}, T)$, such that, for any $C^\infty$ solution, $(u, m)$, of (9.3)–(9.4), we have*

$$\int_0^T \int_{\mathbb{T}^d} \frac{|Du|^\gamma}{m^{\bar{\alpha}}} \mathrm{d}x dt \leq C, \tag{9.5}$$

*where*

$$\bar{\alpha} = (\gamma - 1)\alpha < 1. \tag{9.6}$$

*Proof.* We integrate the first equation in (9.3) with respect to $x$ and $t$. Then, we use the bounds on $u$ from the previous Corollary to get

$$\int_t^T \int_{\mathbb{T}^d} \frac{m^\alpha}{\gamma} \left(\frac{|Du|}{m^\alpha}\right)^\gamma \mathrm{d}x\mathrm{d}s = \int_{\mathbb{T}^d} u(x, T)\mathrm{d}x - \int_{\mathbb{T}^d} u(x, 0)\mathrm{d}x \leq C,$$

Thus, (9.5) follows. □

**Proposition 9.10.** *There exists a constant, $C := C(\|u_T\|_{L^\infty(\mathbb{T}^d)}, T)$, such that, for any $C^\infty$ solution, $(u, m)$, of (9.3)–(9.4), we have*

$$\int_0^T \int_{\mathbb{T}^d} |Du|^\gamma m^{1-\bar{\alpha}} \mathrm{d}x dt \leq C,$$

*where $\bar{\alpha}$ is given by (9.6).*

*Proof.* We multiply the first equation in (9.3) by $m$ and subtract the second equation multiplied by $u$. Then, integration by parts yields

$$\int_0^t \int_{\mathbb{T}^d} m^{1+\alpha} \left[\frac{Du}{m^\alpha} \cdot \frac{Du}{m^\alpha} \left(\frac{|Du|}{m^\alpha}\right)^{\gamma-2} - \frac{1}{\gamma}\left(\frac{|Du|}{m^\alpha}\right)^\gamma\right] \mathrm{d}x ds = \int_{\mathbb{T}^d} u(x, 0) m_0(x)\mathrm{d}x$$
$$- \int_{\mathbb{T}^d} u(x, t) m(x, t)\mathrm{d}x \leq 2\|u\|_{L^\infty(\mathbb{T}^d \times [0,T])}.$$

The claim in the statement follows from Corollary 9.8 and the identity $p \cdot p|p|^{\gamma-2} - \frac{1}{\gamma}|p|^\gamma = \frac{1}{\gamma'}|p|^\gamma$. □

**Proposition 9.11.** *There exist constants, $c_r, C_r := C_r(\alpha, T) > 0$, that have polynomial growth in $r$, such that, for any $C^\infty$ solution, $(u, m)$, of (9.3)–(9.4) and $r > 1$,*

$$\int_{\mathbb{T}^d} \frac{1}{m^r(x,t)} \mathrm{d}x + c_r \int_0^t \int_{\mathbb{T}^d} \left| D \frac{1}{m^{r/2}} \right|^2 \mathrm{d}x ds + \int_0^t \int_{\mathbb{T}^d} \frac{|Du|^\gamma}{m^{r+\bar{\alpha}}} \mathrm{d}x ds$$
$$\leq C_r + C_r \int_0^t \int_{\mathbb{T}^d} \frac{1}{m^q} \mathrm{d}x \mathrm{d}s$$

*for all* $0 \leq t \leq T$*, where* $\bar{\alpha}$ *is given by* (9.6) *and*

$$q = r + \frac{2\bar{\alpha}}{2-\gamma}. \tag{9.7}$$

*Proof.* By adding a constant to $u_T$, we can assume, without loss of generality, that $u \leq -1$. We fix $r > 1$, multiply the first equation in (9.3) by $\frac{1}{m^r}$ and add it to the second equation multiplied by $r\frac{u}{m^{r+1}}$. After integrating by parts, we obtain

$$-\int_{\mathbb{T}^d} \left(\frac{u}{m^r}\right)_t \mathrm{d}x - \int_{\mathbb{T}^d} r(r+1) \frac{u|Dm|^2}{m^{r+2}} \mathrm{d}x + \int_{\mathbb{T}^d} m^\alpha \frac{\frac{1}{\gamma}\left(\frac{|Du|}{m^\alpha}\right)^\gamma + r\frac{Du}{m^\alpha} \cdot \frac{Du}{m^\alpha}\left(\frac{|Du|}{m^\alpha}\right)^{\gamma-2}}{m^r} \mathrm{d}x$$
$$-\int_{\mathbb{T}^d} r(r+1) \frac{u\frac{Du}{m^\alpha}\left(\frac{|Du|}{m^\alpha}\right)^{\gamma-2} \cdot Dm}{m^{r+1}} \mathrm{d}x = 0.$$

We integrate the preceding identity in $t$ and use $-u \geq 1$, $|u| \leq C$, to get

$$\int_{\mathbb{T}^d} \frac{1}{m^r(x,t)} \mathrm{d}x + r(r+1) \int_0^t \int_{\mathbb{T}^d} \frac{|Dm|^2}{m^{r+2}} \mathrm{d}x \mathrm{d}s + \left(\frac{1}{\gamma} + r\right) \int_0^t \int_{\mathbb{T}^d} \frac{|Du|^\gamma}{m^{r+\bar{\alpha}}} \mathrm{d}x \mathrm{d}s \leq$$
$$r(r+1)C \int_0^t \int_{\mathbb{T}^d} |Du|^{\gamma-1} \frac{|Dm|}{m^{r+\bar{\alpha}+1}} \mathrm{d}x \mathrm{d}s + C \int_{\mathbb{T}^d} \frac{1}{m^r(x,0)} \mathrm{d}x.$$

The required estimate follows from the inequality

$$|Du|^{\gamma-1} \frac{|Dm|}{m^{r+\bar{\alpha}+1}} \leq \epsilon \frac{|Du|^\gamma}{m^{r+\bar{\alpha}}} + \epsilon \frac{|Dm|^2}{m^{r+2}} + C_\epsilon \frac{1}{m^q},$$

where $q$ is given by (9.7). □

### 9.2.2 Short-Time Estimates

In this section, we establish estimates for $C^\infty$ solutions of (9.3)–(9.4) for small values of $T$. The key idea is to use the estimate in Proposition 9.11 to control the growth of $\frac{1}{m}$. Because $q > r$ in (9.7), we can achieve bounds only for small $T$. We begin with the following bound on $\frac{1}{m}$.

**Theorem 9.12.** *There exist* $r_0 > 0$, *a time*, $t_1(r) > 0$, *and constants*, $C = C(r, \gamma, \alpha) > 0$ *and* $\delta > 0$, *such that, for any* $C^\infty$ *solution*, $(u, m)$, *of* (9.3)–(9.4) *and* $r \geq r_0$,

$$\int_{\mathbb{T}^d} \frac{1}{m^r(x,t)} \mathrm{d}x \leq C \left[ 1 + \frac{1}{(t_1 - t)^\delta} \right], \ \forall t < t_1.$$

*Proof.* Let $2^* = \frac{2d}{d-2}$ be the Sobolev conjugate exponent to 2. We choose a sufficiently large $r_0$ such that

$$\frac{2^*}{2} r = \frac{dr}{d-2} > q = r + \frac{2\bar{\alpha}}{2-\gamma}$$

for $r \geq r_0$. Let $\lambda > 0$ be such that

$$\frac{2^*}{2} r\lambda + r(1-\lambda) = q;$$

that is,

$$\lambda = \frac{\bar{\alpha}(d-2)}{(2-\gamma)r} < 1$$

for $r \geq r_0$. We set $\bar{\lambda} = \frac{2^*}{2}\lambda = \frac{\bar{\alpha}d}{(2-\gamma)r}$. If $r_0$ is large enough, we have $\bar{\lambda} < 1$ and

$$\beta = \frac{1-\lambda}{1-\bar{\lambda}} > 1$$

for all $r \geq r_0$. Then, using Hölder's and Young's inequalities, we get

$$\begin{aligned}
\int_{\mathbb{T}^d} \frac{1}{m^q} \mathrm{d}x &\leq \left( \int_{\mathbb{T}^d} \frac{1}{m^{\frac{2^*}{2} r}} \mathrm{d}x \right)^\lambda \left( \int_{\mathbb{T}^d} \frac{1}{m^r} \mathrm{d}x \right)^{1-\lambda} \\
&= \left[ \left( \int_{\mathbb{T}^d} \frac{1}{m^{\frac{2^*}{2} r}} \mathrm{d}x \right)^{2/2^*} \right]^{\bar{\lambda}} \left[ \left( \int_{\mathbb{T}^d} \frac{1}{m^r} \mathrm{d}x \right)^\beta \right]^{1-\bar{\lambda}} \\
&\leq \varepsilon \bar{\lambda} \left( \int_{\mathbb{T}^d} \frac{1}{m^{\frac{2^*}{2} r}} \mathrm{d}x \right)^{2/2^*} + \frac{1}{\varepsilon^\tau} (1-\bar{\lambda}) \left( \int_{\mathbb{T}^d} \frac{1}{m^r} \mathrm{d}x \right)^\beta
\end{aligned}$$

for any $\varepsilon > 0$ and some exponent $\tau > 0$. From Sobolev's inequality,

$$\int_{\mathbb{T}^d} \frac{|Dm|^2}{m^{r+2}} \mathrm{d}x = \frac{4}{r^2} \int_{\mathbb{T}^d} \left| D\left( \frac{1}{m^{r/2}} \right) \right|^2 \mathrm{d}x \geq c \frac{4}{r^2} \left( \int_{\mathbb{T}^d} \frac{1}{m^{\frac{2^*}{2} r}} \mathrm{d}x \right)^{2/2^*} - \frac{4}{r^2} \int_{\mathbb{T}^d} \frac{1}{m^r} \mathrm{d}x.$$

By combining Proposition 9.11 and the above inequalities with the estimate

$$\int_{\mathbb{T}^d} \frac{1}{m^r} \mathrm{d}x \le \varepsilon \left( \int_{\mathbb{T}^d} \frac{1}{m^r} \mathrm{d}x \right)^{\beta} + C_\varepsilon, \ \forall \varepsilon > 0,$$

we get

$$\int_{\mathbb{T}^d} \frac{1}{m^r(x,t)} \mathrm{d}x \le C + C \int_0^t \left( \int_{\mathbb{T}^d} \frac{1}{m^r} \mathrm{d}x \right)^{\beta} dt, \ \forall t \in [0,T]. \tag{9.8}$$

Let $h(t) = \int_{\mathbb{T}^d} \frac{1}{m(x,t)^r} \mathrm{d}x$ and $H(t) = \int\limits_0^t h^\beta(s) ds$. Then, the previous inequality reads

$$h(t) \le C_{r,\gamma,T} + C_{r,\gamma,T} H(t).$$

Thus,

$$\dot{H}(t) = h^\beta(t) \le C_{r,\alpha,\gamma,T}(1 + H(t))^\beta. \tag{9.9}$$

Integrating (9.9) and taking into account that $H(0) = 0$, we get

$$(1 + H(t))^{1-\beta} \ge 1 - (\beta - 1) C_{r,\gamma,T} t.$$

Accordingly,

$$H(t) \le \frac{1}{\left[1 - (\beta - 1) C_{r,\gamma,T} t\right]^{\frac{1}{\beta-1}}} \qquad \text{for all } t < t_1(r) := \frac{1}{(\beta - 1) C_{r,\gamma,T}}.$$

Consequently,

$$\int_{\mathbb{T}^d} \frac{1}{m(x,t)^r} \mathrm{d}x = h(t) \le C_{r,\gamma,T} + C_{r,\gamma,T} H(t) \le C + \frac{C}{(t_1 - t)^{\frac{1}{\beta-1}}}, \ t < t_1.$$

□

**Corollary 9.13.** *Let $r_0$ and $t_1(r)$ be as in Theorem 9.12. For $r > r_0$, let $t \le t_1(r) \equiv t_1$. Then, there exist constants, $C_r$ and $\delta_r$, such that, for any $C^\infty$ solution, $(u, m)$, of* (9.3)–(9.4)*,*

$$\int_0^t \int_{\mathbb{T}^d} \left| D \frac{1}{m^{r/2}} \right|^2 \mathrm{d}x\mathrm{d}t \le C_r + \frac{C_r}{(t_1 - t)^{\delta_r}}, \quad \forall\, t < t_1. \tag{9.10}$$

Iterating the estimates from Proposition 9.11, we get uniform bounds in $r$, as we prove next.

**Proposition 9.14.** *There exist $r_1 > 0$ and constants, $C_t = C_t(r, \gamma, \alpha) > 0$ and $\beta_r > 1$, such that, for any $C^\infty$ solution, $(u, m)$, to (9.3)–(9.4) and $r \geq r_1$,*

$$\left\| \frac{1}{m} \right\|_{L^\infty([0,t]\times\mathbb{T}^d)} \leq C_t \left( 1 + \left\| \frac{1}{m} \right\|^{\beta_r}_{L^\infty([0,t],L^r(\mathbb{T}^d))} \right).$$

*Remark 9.15.* We observe that the previous result is not a local result. If we establish bounds for $\frac{1}{m}$ in $L^\infty([0,t], L^r(\mathbb{T}^d))$ for some $t > 0$, we get $\frac{1}{m} \in L^\infty$ $([0,t] \times \mathbb{T}^d)$.

*Proof.* For $r > 1$, choose $\theta_n > 0$ such that

$$r^{n+1} + \delta = (1-\theta_n) r^n + \theta_n \frac{2^*}{2} r^{n+1},$$

where $\delta = \frac{2\bar{\alpha}}{2-\gamma}$; that is, $\theta_n = \frac{1 - \frac{1}{r} + \frac{\delta}{r^{n+1}}}{\frac{2^*}{2} - \frac{1}{r}} > 0$. Set $\lambda_n = \frac{2^*}{2}\theta_n$ and $\beta_n = \frac{1-\theta_n}{1-\lambda_n}$. Then, there exists $r_1 > 1$ such that, for any $r \geq r_1$ and any $n \geq 1$, we have $\lambda_n < 1$. We fix a time, $t$. As in the previous proposition, using a weighted Hölder's inequality, we have

$$\int_{\mathbb{T}^d} \frac{1}{m^{r^{n+1}+\delta}} \mathrm{d}x \leq \left[ \left( \int_{\mathbb{T}^d} \frac{1}{m^{\frac{2^*}{2} r^{n+1}}} \mathrm{d}x \right)^{2/2^*} \right]^{\lambda_n} \left[ \left( \int_{\mathbb{T}^d} \frac{1}{m^{r^n}} \mathrm{d}x \right)^{\beta_n} \right]^{1-\lambda_n}$$

$$\leq \varepsilon\lambda_n \left( \int_{\mathbb{T}^d} \frac{1}{m^{\frac{2^*}{2} r^{n+1}}} \mathrm{d}x \right)^{2/2^*} + \frac{1}{\varepsilon^\tau}(1-\lambda_n) \left( \int_{\mathbb{T}^d} \frac{1}{m^{r^n}} \mathrm{d}x \right)^{\beta_n},$$

where $\varepsilon > 0$ and $\tau > 0$ is a suitable exponent. Next, Proposition 9.11 and Sobolev's inequality imply that

$$\int_{\mathbb{T}^d} \frac{1}{m^{r^{n+1}}(x,t)} \mathrm{d}x + \int_0^t \left( \int_{\mathbb{T}^d} \frac{1}{m^{\frac{2^*}{2} r^{n+1}}(x,s)} \mathrm{d}x \right)^{2/2^*} ds$$
$$\leq C_{r^{n+1}} + C_{r^{n+1}} \int_0^t \int_{\mathbb{T}^d} \frac{1}{m^{r^{n+1}+\delta}(x,s)} \mathrm{d}x\mathrm{d}s.$$

From these two inequalities, we conclude that

$$\int_{\mathbb{T}^d} \frac{1}{m^{r^{n+1}}(x,t)} \mathrm{d}x \leq C_{r^{n+1}} + C_{r^{n+1}} \int_0^t \left( \int_{\mathbb{T}^d} \frac{1}{m^{r^n}(x,s)} \mathrm{d}x \right)^{\beta_n} \mathrm{d}s.$$

Define

$$A_n(t) = \max_{[0,t]} \int_{\mathbb{T}^d} \frac{1}{m^{r^n}(x,\cdot)} \mathrm{d}x.$$

From the above estimate,

$$1 + A_{n+1}(t) \leq \max\{1,t\} C_n (1 + A_n(t))^{\beta_n},$$

where $C_n = O(r^{nk})$ for some $k > 1$. Proceeding inductively, we get

$$(1 + A_{n+1}(t))^{\frac{1}{\beta_1 \cdot \ldots \cdot \beta_n}} \leq C_t^{\sum_{i=1}^n \frac{1}{\beta_1 \cdot \ldots \cdot \beta_i}} r^{\sum_{i=1}^n \frac{ik}{\beta_1 \cdot \ldots \cdot \beta_i}} (1 + A_1).$$

Note that

$$\beta_n = \frac{1-\theta_n}{1-\lambda_n} = r\left(1 + \frac{(\frac{2^*}{2}r - 1)\delta}{r^{n+1}(\frac{2^*}{2} - 1 - \frac{2^*}{2}\frac{\delta}{r^n})}\right) := r(1 + q_n).$$

Because

$$\lim_{n\to\infty} r^n \frac{(\frac{2^*}{2}r - 1)\delta}{r^{n+1}(\frac{2^*}{2} - 1 - \frac{2^*}{2}\frac{\delta}{r^n})} = \frac{\frac{2^*}{2}\delta}{\frac{2^*}{2} - 1},$$

we have $q_n = O(r^{-n}) > 0$, the series $\sum_{i=1}^\infty \frac{ik}{\beta_1 \cdot \ldots \cdot \beta_i}$, $\sum_{i=1}^\infty \frac{1}{\beta_1 \cdot \ldots \cdot \beta_i}$, and the infinite product $\prod_{i=1}^\infty (1 + q_i)$ converges. From this, we get

$$\left\| \frac{1}{m} \right\|_{L^\infty([0,t], L^{r^{n+1}}(\mathbb{T}^d))} \leq C_t \left(1 + \left\| \frac{1}{m} \right\|_{L^\infty([0,t], L^r(\mathbb{T}^d))}^{\beta_r}\right),$$

for some constants, $C_t > 0$ and $\beta_r = \prod_{i=1}^\infty (1 + q_i) > 1$, that do not depend on the solution. By letting $n \to \infty$, we obtain the result. □

The results of Theorem 9.12, Proposition 9.14, and Corollary 9.13 prove the following:

**Theorem 9.16.** *There exist a time, $T_0 > 0$, and a constant, $C = C(\gamma, \alpha) > 0$, such that, for any $C^\infty$ solution, $(u, m)$, of (9.3)–(9.4), we have*

$$\left\| \frac{1}{m} \right\|_{L^\infty([0,T_0]\times\mathbb{T}^d)} \leq C.$$

## 9.3 Bibliographical Notes

Time-dependent MFGs with logarithmic nonlinearity were first studied in [117]. MFGs with congestion were introduced in [174]. The existence of classical solutions for stationary MFGs with quadratic Hamiltonians was proven in [114]. The existence of solutions for time-dependent problems is known only for the short-time problem. Weak solutions of congestion problems were investigated in [136]. Here, in our approach to the congestion problem, we follow [123].

# Chapter 10
# Non-local Mean-Field Games: Existence

MFGs where the Hamilton–Jacobi equation depends on the distribution of players in a non-local way make up an important group of problems. In many examples, this dependence is given by regularizing convolution operators. We split the discussion of non-local problems into two cases. First, we consider first-order MFGs. Here, semiconcavity bounds and the optimal control characterization of the Hamilton–Jacobi equation are the main tools. Next, we examine second-order MFGs. Here, the regularizing effects of parabolic equations and the $L^2$ stability of the Fokker–Planck equation are the main ingredients of the proof.

## 10.1 First-Order, Non-local Mean-Field Games

We denote by $\mathcal{P}_1(\mathbb{R}^d)$ the set of Borel probability measures in $\mathbb{R}^d$ with a finite first moment. The 1-Wasserstein distance between two probability measures, $\theta_1$ and $\theta_2$, with finite first moments is

$$d_1(\theta_1, \theta_2) = \inf_{\pi} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| \mathrm{d}\pi(x, y),$$

where the infimum is taken over the set, $\Pi(\theta_1, \theta_2)$, of all probability measures, $\pi$, in $\mathbb{R}^d \times \mathbb{R}^d$ whose first marginal is $\theta_1$ and whose second marginal is $\theta_2$. The 1-Wasserstein distance makes $\mathcal{P}_1(\mathbb{R}^d)$ a metric space.

We define the norm, $\|\cdot\|_{C^2}$, as

$$\|g\|_{C^2} = \sup_{x \in \mathbb{R}^d} \left[ |g(x)| + |D_x g(x)| + |D^2_{xx} g(x)| \right],$$

for any $g \in C^2(\mathbb{R}^d)$.



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Fix $F : \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}$, and suppose that the map $m \mapsto F(\cdot, m)$, $m \in \mathcal{P}_1(\mathbb{R}^d)$, is continuous from $\mathcal{P}_1$ to $C^2(\mathbb{R}^d)$. Next, select initial and terminal conditions, $m_0 \in \mathcal{P}_1(\mathbb{R}^d)$ and $u_T \in C^2(\mathbb{R}^d)$. We consider the MFG

$$\begin{cases} -u_t + \frac{|Du|^2}{2} = F(x, m), \\ m_t - \operatorname{div}(mDu) = 0, \end{cases} \tag{10.1}$$

with the initial-terminal conditions

$$\begin{cases} u(x, T) = u_T(x), \\ m(x, 0) = m_0(x). \end{cases} \tag{10.2}$$

Because first-order Hamilton–Jacobi equations may fail to have $C^1$ solutions, we look for a solution, $(u, m)$, with $u : \mathbb{R}^d \times [0, T] \to \mathbb{R}$, $u$ bounded and locally Lipschitz, and $m \in C([0, T], \mathcal{P}_1(\mathbb{R}^d))$.

Fix $m \in C([0, T], \mathcal{P}_1(\mathbb{R}^d))$. In (10.1), the Hamiltonian, $H(x, p) = \frac{|p|^2}{2} - F(x, m)$, is convex in $p$. For that reason, we say that $u$ is a viscosity solution of the first equation in (10.1) if

$$u(x, t) = \inf_{\mathbf{x}} \int_t^T \left[ \frac{|\dot{\mathbf{x}}(s)|^2}{2} + F(\mathbf{x}(s), m(\cdot, s)) \right] \mathrm{d}s + u_T(\mathbf{x}(T)),$$

where the infimum is taken over all absolutely continuous trajectories, $\mathbf{x}$, with $\mathbf{x}(t) = x$. Though this is not the usual definition of a viscosity solution, it is equivalent to the usual one in this case. We refer the reader to the end of Chap. 3 for bibliographical references.

We say that $(u, m)$ solves (10.1)–(10.2) if $u$ is a viscosity solution of the first equation in (10.1), $m \in C([0, T], \mathcal{P}_1(\mathbb{R}^d))$ is a solution in the sense of distributions of the second equation, and (10.2) holds.

**Theorem 10.1.** *Assume that $m_0$ is absolutely continuous with respect to the Lebesgue measure and that there exists a constant, $C > 0$, such that*

$$\sup_{m \in \mathcal{P}_1(\mathbb{R}^d)} \|F(\cdot, m)\|_{C^2} \leq C \tag{10.3}$$

*and*

$$\sup_{x \in \mathbb{R}^d} \|F(x, m) - F(x, \bar{m})\|_{C^1} \leq C d_1(m, \bar{m}), \ \forall m, \bar{m} \in \mathcal{P}_1(\mathbb{R}^d). \tag{10.4}$$

*Then, there is a solution, $(u, m)$, of (10.1)–(10.2) such that $u$ is a Lipschitz continuous and semiconcave viscosity solution of the Hamilton–Jacobi equation,*

$$-u_t + \frac{|Du|^2}{2} = F(x, m),$$

*and* $m \in C([0,T],\mathcal{P}_1(\mathbb{R}^d))$ *is a weak solution of the transport equation*

$$m_t - \operatorname{div}(mDu) = 0.$$

*Proof.* We use a fixed-point argument. Let $\Psi : \mathcal{P}_1(\mathbb{R}^d) \mapsto C(\mathbb{R}^d)$ be as follows: for $m^1 \in \mathcal{P}_1(\mathbb{R}^d)$, we define $\Psi(m^1)$ as the solution of the optimal control problem

$$u^1(x,t) = \inf_{\mathbf{x}} \int_t^T \frac{|\dot{\mathbf{x}}(s)|^2}{2} + F(\mathbf{x}(s), m^1(s))\mathrm{d}s + u_T(\mathbf{x}(T)), \tag{10.5}$$

where the infimum is taken over all absolutely continuous trajectories, $\mathbf{x}$, with $\mathbf{x}(t) = x$. Then, $u^1$ is the viscosity solution to

$$-u_t^1 + \frac{\left|Du^1\right|^2}{2} = F(x, m^1), \tag{10.6}$$

with the terminal condition $u^1(x,T) = u_T(x)$.

We proceed with the analysis of the operator, $\Psi$. Fix $m^1 \in \mathcal{P}_1(\mathbb{R}^d)$ and let $u^1 = \Psi(m^1)$. By Proposition 3.8, $u^1$ is uniformly bounded, Lipschitz, and locally uniformly semiconcave on $[0,T)$.

Though viscosity solutions may fail to be differentiable, by semiconcavity, they are differentiable almost everywhere. The Hamiltonian corresponding to (10.6) is $\frac{|p|^2}{2} - F(x, m^1)$. The corresponding Hamiltonian dynamics is

$$\begin{cases} \dot{\mathbf{x}} = -\mathbf{p}, \\ \dot{\mathbf{p}} = -D_x F(\mathbf{x}, m^1), \end{cases} \tag{10.7}$$

Consequently, if $x \in \mathbb{R}^d$ is a point of differentiability of $u^1(x,0)$, the solution of (10.7) with the initial conditions

$$\mathbf{x}(0) = x, \qquad \mathbf{p}(0) = D_x u^1(x,0),$$

is an optimal trajectory for (10.5). Moreover, $u^1$ is differentiable at $(\mathbf{x}(t), t)$, with $\mathbf{p}(t) = D_x u^1(\mathbf{x}(t), t)$ for $0 < t < T$.

Let $(\Phi^1(x,t,s), \Phi^2(x,t,s)) = (\mathbf{x}(s), \mathbf{p}(s))$ be the flow defined for almost every $(x,t)$ through (10.7), satisfying

$$\begin{cases} \partial_s \Phi^1(x,t,s) = -\Phi^2(x,t,s), \\ \partial_s \Phi^2(x,t,s) = -D_x F(\Phi^1(x,t,s), m^1), \\ \Phi^1(x,t,t) = x, \quad \Phi^2(x,t,t) = Du^1(x,t). \end{cases} \tag{10.8}$$

Note that $\Phi^2(x,t,s) = Du^1(\Phi^1(x,t,s),s)$. Furthermore, $\Phi^1$ satisfies

$$|\Phi^1(x,t,s') - \Phi^1(x,t,s)| \leq C|s-s'|.$$

Taking into account that $m^1(0)$ is absolutely continuous, we define

$$m^2(t) = \Phi^1(\cdot,0,t)\sharp m^1(0).$$

Finally, we define the operator, $U\colon C([0,T],\mathcal{P}_1) \to C([0,T],\mathcal{P}_1)$, by $U(m^1) = m^2$.

Next, we check that $U$ is continuous. Consider a sequence, $m_n^1$, in $C([0,T],\mathcal{P}_1)$. Suppose that $m_n^1 \to m^1$ in $C([0,T],\mathcal{P}_1)$, for some $m^1 \in \mathcal{P}_1$. The property (10.4) ensures the stability of the control problem (10.5); thus, $u_n^1 \to u^1$ a.e. Because $u_n^1$ and $u^1$ are Lipschitz continuous and semiconcave, we have $Du_n^1 \to Du^1$ a.e. The latter convergence combined with (10.4) and the stability of the ODE (10.7) give $\|\Phi_n(x,0,\cdot) - \Phi(x,0,\cdot)\|_{C([0,T])} \to 0$ for a.e. $x$, where $\Phi_n = (\Phi_n^1, \Phi_n^2)$ and $\Phi = (\Phi^1, \Phi^2)$. Since

$$d_1(m_n^2(s), m^2(s)) \leq \int_{\mathbb{R}^d} |\Phi_n^1(x,0,s) - \Phi^1(x,0,s)|\mathrm{d}m_0(x),$$

the Dominated Convergence Theorem implies that $m_n^2 \to m^2$ in $C([0,T],\mathcal{P}_1)$. Hence, $U$ is continuous. Consequently, by Schauder's Fixed-point Theorem, $U$ has a fixed point, $m$. Moreover, $u = \Psi(m)$ is a viscosity solution to

$$-u_t + \frac{|Du|^2}{2} = F(x,m).$$

Thus, $m = U(m) = \Phi^1(\cdot,0,t)\sharp m(0)$, where $D_s\Phi^1(x,0,s) = -\Phi^2(x,0,s) = -D_xu(\Phi^1(x,0,s),s)$. By the argument outlined in Sect. 1.1.2, $m$ is a weak solution of

$$m_t - \operatorname{div}(Dum) = 0. \qquad \square$$

## 10.2 Second-Order, Non-local Mean-Field Games

To study non-local, second-order MFGs, we use a fixed-point argument. In contrast to first-order equations, second-order equations have strong regularizing properties. Hence, our proof uses a distinct argument from the first-order case. This argument relies on the $L^2$ stability of the Fokker–Planck equation.

**Theorem 10.2.** *Suppose that*

1. $F : \mathbb{R}^d \times L^2(\mathbb{T}^d) \to \mathbb{R}$ *is uniformly bounded,*

2. *There exists a constant,* $C > 0$, *such that*

$$|F(x,m_1) - F(x,m_2)| \leq C\|m_1 - m_2\|_{L^2(\mathbb{T}^d)},$$

*$\forall m_1, m_2 \in L^2(\mathbb{T}^d)$, and*

$$\|F(\cdot, m)\|_{C^2},\ \|u_T\|_{C^2} \leq C,\ \forall m \in L^2(\mathbb{T}^d).$$

*3. $m_0$ is continuous, $\int m_0 = 1$, $m_0 \geq 0$.*

*Then, there exists a solution , $(u, m)$, of*

$$\begin{cases} -u_t + \frac{|Du|^2}{2} = F(x, m) + \Delta u, \\ m_t - \operatorname{div}(mDu) = \Delta m, \\ m(x, 0) = m_0(x), \quad u(x, T) = u_T(x), \end{cases} \tag{10.9}$$

*with $m \in C([0, T], L^2(\mathbb{T}^d))$ and $u \in C^{1,2}([0, T] \times \mathbb{R}^d)$.*

*Proof.* Let $m \in C([0, T], L^2(\mathbb{T}^d))$. Then, there exists a unique solution, $U \in C^{1,2}([0, T] \times \mathbb{T}^d)$, of

$$-U_t + \frac{|DU|^2}{2} = \Delta U + F(x, m), \qquad U(x, T) = u_T(x).$$

Furthermore, $U$ is globally bounded, Lipschitz continuous, and locally semiconcave on $[0, T)$ with uniform bounds depending on the assumptions on $F$ and $u_T$. Next, we define the map, $\Phi : C([0, T], L^2(\mathbb{T}^d)) \to C([0, T], L^2(\mathbb{T}^d))$, as $\Phi(m) = \tilde{m}$, where $\tilde{m}$ solves the Fokker–Planck equation:

$$\tilde{m}_t - \operatorname{div}(DU\tilde{m}) = \Delta\tilde{m}, \qquad \tilde{m}(x, 0) = m_0(x).$$

**Lemma 10.3.** *The map $\Phi : C([0, T], L^2(\mathbb{T}^d)) \to C([0, T], L^2(\mathbb{T}^d))$ is continuous.*

*Proof.* Let $m_n \to m$ in $C([0, T], L^2(\mathbb{T}^d))$. By the continuity of $F$ and the stability of solutions of Hamilton–Jacobi equations, we conclude that $U_n \to U$ uniformly on $[0, T] \times \mathbb{R}^d$. Next, the uniform semiconcavity of $U$ and $U_n$ gives $DU_n \to DU$ a.e. We denote $w^n = \tilde{m}_n - \tilde{m}$. Then,

$$w_t^n - \operatorname{div}(DU_n w^n) - \operatorname{div}\left(\tilde{m}(DU_n - DU)\right) = \Delta w^n, \qquad w_n(x, 0) = 0.$$

Multiplying the preceding PDE by $w_n$ and integrating on $\mathbb{R}^d$, we get

$$\frac{d}{dt}\|w^n(\cdot, t)\|^2_{L^2(\mathbb{R}^d)} = 2\int_{\mathbb{R}^d} \left[-|Dw^n|^2 - w^n Dw^n \cdot DU_n - \tilde{m}Dw^n \cdot (DU_n - DU)\right] \mathrm{d}x.$$

Therefore, using Cauchy's inequality,

$$\frac{d}{dt}\|w^n(\cdot, t)\|^2_{L^2(\mathbb{R}^d)} \leq C\|DU_n\|^2_\infty \|w^n(\cdot, t)\|^2_{L^2(\mathbb{R}^d)} + C\|\tilde{m}(DU_n - DU)\|^2_{L^2(\mathbb{R}^d)}.$$

Thus, by Gronwall's inequality,

$$\|w^n(\cdot,t)\|^2_{L^2(\mathbb{R}^d)} \leq C\|\tilde{m}(DU_n - DU)\|^2_{L^2([0,t]\times\mathbb{R}^d)}.$$

Further, by the Dominated Convergence Theorem, the right-hand side in the prior estimate converges to 0. Thus, $w^n \to 0$ in $C([0,T], L^2(\mathbb{T}^d))$; that is, $\Phi(m_n) \to \Phi(m)$. □

**Proposition 10.4.** *The set* $K = \overline{\Phi\left(C([0,T], L^2(\mathbb{T}^d))\right)}$ *is compact.*

*Proof.* It is enough to prove that, for any sequence, $m_n \in C([0,T], L^2(\mathbb{T}^d))$, we have that $\Phi(m_n)$ has a convergent subsequence. Because the sequence $U_n = \Phi(m_n)$ is equibounded and equiLipschitz, the Arzelá–Ascoli theorem gives that $\{U_n\}$ has a convergent subsequence. Moreover, because $U_n$ is uniformly semiconcave, $DU_n$ converges a.e. Therefore, arguing as in the previous lemma, we conclude that, through the same subsequence, $\Phi(m_n)$ converges in $C([0,T], L^2(\mathbb{T}^d))$. □

Finally, using Schauder's fixed-point theorem, there exists $m \in K$, with $\Phi(m) = m$, and so

$$\begin{cases} -u_t + \frac{|Du|^2}{2} = F(m) + \Delta u, \; u(x,T) = u_T(x) \\ m_t - \operatorname{div}(mDu) = \Delta m, \; m(x,0) = m_0(x). \end{cases}$$

□

## 10.3 Bibliographical Notes

The use of fixed-point methods to prove the existence of solutions was first discussed in [174]. The proof of Theorem 10.1 is a variation of the fixed-point argument from [61]. In the previous reference, the proof uses the quadratic structure of the Hamiltonian. Here, we adapt the ideas from [124], and our proof extends to a wide class of Hamiltonians and avoids measurable selection arguments. The book [212] is a standard reference for optimal transport and Wasserstein distance. The proof of Theorem 10.2 uses only PDE arguments and does not require the use of stochastic differential equations as in [61].

# Chapter 11
# Local Mean-Field Games: Existence

In this last chapter, we address the existence problem for local mean-field games. First, we illustrate the bootstrapping technique and put together the previous estimates. Thanks to this technique, we show that solutions of stationary MFGs are bounded a priori in all Sobolev spaces. This is an essential step for the two existence methods developed next. The first method is a regularization procedure in which we perturb the original local MFG into a non-local problem. By the results of the preceding chapter, this non-local problem admits a solution. Then, a limiting procedure gives the existence of a solution. The second method we consider is the continuation method. The key idea is to deform the original MFG into a problem that can be solved explicitly. Then, a topological argument shows that it is possible to deform the solution of the latter MFG into the former. This argument uses both the earlier bounds and an infinite dimensional version of the implicit function theorem.

## 11.1 Bootstrapping Regularity

Next, we develop a way to prove the regularity of solutions of stationary MFGs called the bootstrapping method. This method is based on the observation that if the solutions of MFGs are regular enough, the equations give further regularity immediately. Moreover, this process can be iterated indefinitely. Here, we illustrate this idea in the stationary case. The time-dependent case is similar.

For illustration, we consider (7.1) with $\alpha$ as in Theorem 7.1. For every $q > 1$, Theorem 7.1 ensures that

$$\|Du\|_{L^q(\mathbb{T}^d)} \leq C_q,$$

for some constant, $C_q$. Next, we show that this regularity for $u$ immediately implies that $u$ and $m$ are a priori bounded in all Sobolev spaces and, hence, are a priori bounded in $C^\infty$.

 
D.A. Gomes et al., *Regularity Theory for Mean-Field Game Systems*,
SpringerBriefs in Mathematics, DOI 10.1007/978-3-319-38934-9_11

**Proposition 11.1.** *Let* $(u, m, \overline{H})$ *solve* (7.1) *with* $u$ *and* $m$ *in* $C^\infty(\mathbb{T}^d)$*, and let* $m > 0$*. Then, there exists a constant,* $C > 0$*, such that*

$$\|\ln m\|_{W^{1,q}(\mathbb{T}^d)} \leq C.$$

*Proof.* Because of Theorem 7.1, standard elliptic regularity theory applied to the first equation in (7.1) yields

$$\|u\|_{W^{2,q}(\mathbb{T}^d)} \leq C_q$$

for every $q > 1$. Therefore, Morrey's Embedding Theorem implies that $u \in C^{1,\beta}(\mathbb{T}^d)$ for some $\beta \in (0,1)$. Next, set $w = -2\ln m$. Straightforward computations show that $w$ satisfies

$$-\Delta w + \frac{1}{2}|Dw|^2 - Du \cdot Dw + 2\,\mathrm{div}(Du) = 0.$$

Thus, Theorem 3.11 gives

$$\|Dw\|_{L^{\frac{2d(p+1)}{d-2}}(\mathbb{T}^d)} \leq C_p\left(C + \|Du \cdot Dw\|_{L^{\frac{2d(1+p)}{d+2p}}(\mathbb{T}^d)} + \|\,\mathrm{div}(Du)\|_{L^{\frac{2d(1+p)}{d+2p}}(\mathbb{T}^d)}\right).$$

Since $Dw \in L^2$ and $\frac{2d(p+1)}{d-2} > \frac{2d(1+p)}{d+2p}$, we get $Dw \in L^q$ for any $q \geq 1$ by using an interpolation argument in the preceding estimate. Hence, $\ln m$ is a Hölder continuous function. Because $\int_{\mathbb{T}^d} m dx = 1$, $m$ is bounded from above and from below. Consequently, $\|\ln m\|_{L^q(\mathbb{T}^d)}$ is a priori bounded by some universal constant that depends only on $q$. ◻

**Proposition 11.2.** *Let* $(u, m, \overline{H})$ *solve* (7.1) *with* $u$ *and* $m$ *in* $C^\infty(\mathbb{T}^d)$*, and let* $m > 0$*. For any* $k \geq 1$ *and* $q > 1$*, there exists a constant,* $C_{k,q} > 0$*, such that*

$$\left\|D^k u\right\|_{L^q(\mathbb{T}^d)}, \left\|D^k m\right\|_{L^q(\mathbb{T}^d)} \leq C_{k,q}.$$

*Proof.* The preceding results give

$$\|u\|_{W^{2,a}(\mathbb{T}^d)} \leq C_a$$

for every $1 < a < \infty$. Also, Proposition 11.1 gives

$$\|\ln m\|_{W^{1,a}(\mathbb{T}^d)} \leq C$$

for any $1 < a < \infty$. By differentiating the first equation in (7.1), we obtain

$$-D_x\Delta u = D_x g(m) - D^2 u Du. \tag{11.1}$$

Finally, we observe that the right-hand side of (11.1) is bounded in $L^a(\mathbb{T}^d)$. Thus,

$$\|u\|_{W^{3,a}(\mathbb{T}^d)} \leq C_{3,a},$$

which leads to

$$\|m\|_{W^{2,a}(\mathbb{T}^d)} \leq C_{2,a}.$$

The proof proceeds by iterating this procedure up to order $k$. □

*Remark 11.3.* Because bootstrapping arguments are very similar, we do not discuss here the other stationary models nor the time-dependent cases and refer the reader to the bibliography.

## 11.2 Regularization Methods

Frequently, to investigate the existence of solutions of a partial differential equation, we introduce a regularized version of that PDE. Usually, this new problem is well understood or easier to study, and the existence of solutions is straightforward. Then, a limiting procedure together with a compactness argument gives a solution to the original problem.

Here, we illustrate the regularization method in the time-dependent MFG given by (8.1). We introduce the regularized non-linearity,

$$g_\epsilon(m) := \eta_\epsilon * g\left(\eta_\epsilon * m\right),$$

where $\eta_\epsilon$ is a symmetric standard mollifier and the convolution in the previous definition is in the variable $x$ only. The regularized system is

$$\begin{cases} -u_t^\epsilon + \frac{1}{\gamma}\left|Du^\epsilon\right|^\gamma = \Delta u^\epsilon + g_\epsilon(m^\epsilon), & \text{in} \quad \mathbb{T}^d \times [0,T], \\ m_t^\epsilon - \operatorname{div}(\left|Du^\epsilon\right|^{\gamma-1} m^\epsilon) = \Delta m^\epsilon, & \text{in} \quad \mathbb{T}^d \times [0,T], \end{cases} \tag{11.2}$$

with the initial-terminal boundary conditions

$$u^\epsilon(x,T) = u_T(x) \quad \text{and} \quad m^\epsilon(x,0) = m_0(x).$$

Remarkably, (11.2) satisfies the same a priori estimates as (8.1). In particular, $u^\epsilon$ and $m^\epsilon$ are bounded in all Sobolev space with $\epsilon$ independent bounds. Hence, up to a subsequence, $(u^\epsilon, m^\epsilon) \to (u,m)$, and $(u,m)$ solves (8.1). Moreover, $(u,m)$ inherits the regularity of the limiting sequence. Thus, $(u,m)$ is in any Sobolev space and is of class $C^\infty(\mathbb{T}^d \times [0,T])$.

For example, here, we examine the first-order estimates and the second-order estimates from Propositions 6.6 and 6.10, respectively. For $g(m) = m^\alpha$, Proposition 6.6 means that there exists $C > 0$ not depending on $\epsilon$ such that

$$\int_0^T \int_{\mathbb{T}^d} (m^\epsilon + m_0)\frac{\left|Du^\epsilon\right|^2}{2} + m^\epsilon\left(\eta_\epsilon * (\eta_\epsilon * m^\epsilon)^\alpha\right) \mathrm{d}x\mathrm{d}t \leq C.$$

Because $\eta_\epsilon$ is symmetric, we obtain

$$\int_0^T \int_{\mathbb{T}^d} (m^\epsilon + m_0)\, \frac{|Du^\epsilon|^2}{2} + (\eta_\epsilon * m^\epsilon)^{\alpha+1}\, \mathrm{d}x\mathrm{d}t \;\leq\; C.$$

Therefore, in the case of polynomial nonlinearities, the mean-field coupling for the regularized problem, $g_\epsilon(m^\epsilon)$, is in $L^{\alpha+1}(\mathbb{T}^d)$, uniformly in $\epsilon$.

A similar reasoning applies to the second-order estimates, namely, Proposition 6.10. We have

$$\int_{\mathbb{T}^d} |D^2 u^\epsilon|^2 m + \alpha(\eta_\epsilon * m^\epsilon)^{\alpha-1}|D(\eta_\epsilon * m^\epsilon)|^2 \mathrm{d}x \leq C.$$

To get the previous estimate, we follow the proof of Proposition 6.10. The only difference is the term

$$\alpha \int_0^T \int_{\mathbb{T}^d} \operatorname{div}\left[\eta_\epsilon * \left((\eta_\epsilon * m^\epsilon)^{\alpha-1} D(\eta_\epsilon * m^\epsilon)\right)\right] m^\epsilon \mathrm{d}x\mathrm{d}t \tag{11.3}$$

that we address as follows. Integrating by parts in (11.3), we get

$$\begin{aligned} \alpha \int_0^T \int_{\mathbb{T}^d} &\operatorname{div}\left(\eta_\epsilon * ((\eta_\epsilon * m^\epsilon)^{\alpha-1} D(\eta_\epsilon * m^\epsilon)\right) m^\epsilon \mathrm{d}x\mathrm{d}t \\ &= -\alpha \int_0^T \int_{\mathbb{T}^d} (\eta_\epsilon * m^\epsilon)^{\alpha-1} |D(\eta_\epsilon * m^\epsilon)|^2 \mathrm{d}x\mathrm{d}t, \end{aligned}$$

using the symmetry of $\eta_\epsilon$. The previous computation thus ensures the second-order estimates in Proposition 6.10 hold for the regularized problem.

## 11.3 Continuation Method: Stationary Problems

The regularization methods examined earlier depend on the particular structure of the problem. In some cases, it may be difficult to construct a regularized problem with strong enough bounds. An alternative is the continuation method. Here, we illustrate it by proving the existence of smooth solutions of

$$\begin{cases} -\Delta u + \frac{|Du|^2}{2} + V(x) = \overline{H} + g(m), \\ -\Delta m - \operatorname{div}(Dum) = 0, \\ \int_{\mathbb{T}^d} u = 0, \quad \int_{\mathbb{T}^d} m = 1. \end{cases} \tag{11.4}$$

As usual, in the previous equation, the unknowns are the smooth functions, $u\colon \mathbb{T}^d \to \mathbb{R}$, $m\colon \mathbb{T}^d \to \mathbb{R}^+$, and the constant, $\overline{H} \in \mathbb{R}$.

First, for $0 \leq \lambda \leq 1$, we consider the family of problems

$$\begin{cases} -\Delta m_\lambda - \operatorname{div}(Du_\lambda m_\lambda) = 0, \\ \Delta u_\lambda - \frac{|Du_\lambda|^2}{2} - \lambda V + \overline{H}_\lambda + g(m_\lambda) = 0, \\ \int_{\mathbb{T}^d} u_\lambda = 0, \quad \int_{\mathbb{T}^d} m_\lambda = 1. \end{cases} \tag{11.5}$$

For $\lambda = 1$, (11.5) is (11.4).

Next, we set

$$\dot{H}^k(\mathbb{T}^d, \mathbb{R}) = \left\{ f \in H^k(\mathbb{T}^d, \mathbb{R}) : \int_{\mathbb{T}^d} f \mathrm{d}x = 0 \right\}$$

and consider the Hilbert space, $F^k = \dot{H}^k(\mathbb{T}^d, \mathbb{R}) \times H^k(\mathbb{T}^d, \mathbb{R}) \times \mathbb{R}$, with the norm

$$\|w\|^2_{F^k} = \|\psi\|^2_{\dot{H}^k(\mathbb{T}^d,\mathbb{R})} + \|f\|^2_{H^k(\mathbb{T}^d,\mathbb{R})} + |h|^2$$

for $w = (\psi, f, h) \in F^k$. By Sobolev's Embedding Theorem for $k > \frac{d}{2}$, we have $H^k(\mathbb{T}^d, \mathbb{R}) \subset C^{0,\beta}(\mathbb{T}^d, \mathbb{R})$ for some $\beta \in (0, 1)$. Thanks to this embedding, we define the space, $H^k_+(\mathbb{T}^d, \mathbb{R})$, for $k > \frac{d}{2}$ as the set of (everywhere) positive functions in $H^k(\mathbb{T}^d, \mathbb{R})$. For any $k > \frac{d}{2}$, we let

$$F^k_+ = \dot{H}^k(\mathbb{T}^d, \mathbb{R}) \times H^k_+(\mathbb{T}^d, \mathbb{R}) \times \mathbb{R}.$$

Finally, we recall that a classical solution to (11.5) is a tuple, $(u_\lambda, m_\lambda, \overline{H}_\lambda) \in \bigcap_{k \geq 0} F^k_+$.

**Theorem 11.4.** *Assume that $g$, $V \in C^\infty(\mathbb{T}^d)$ with $g'(z) > 0$ for $z \in (0, +\infty)$ and that we have the a priori estimate for any solution of* (11.5)*:*

$$|\overline{H}| + \left\| \frac{1}{m_\lambda} \right\|_{L^\infty(\mathbb{T}^d)} + \|u_\lambda\|_{W^{k,p}(\mathbb{T}^d)} + \|m_\lambda\|_{W^{k,p}(\mathbb{T}^d)} \leq C_{k,p}.$$

*Then, there exists a classical solution to* (11.4).

*Proof.* For large enough $k$, we define $E: \mathbb{R} \times F^k_+ \to F^{k-2}$ by

$$E(\lambda, u, m, \overline{H}) = \begin{pmatrix} -\Delta m - \operatorname{div}(Dum) \\ \Delta u - \frac{|Du|^2}{2} - \lambda V + \overline{H} + g(m) \\ -\int_{\mathbb{T}^d} m + 1 \end{pmatrix}.$$

Then, (11.5) is equivalent to $E(\lambda, v_\lambda) = 0$, where $v_\lambda = (u_\lambda, m_\lambda, \overline{H}_\lambda)$. The partial derivative of $E$ in the second variable at the point $v_\lambda = (u_\lambda, m_\lambda, \overline{H}_\lambda)$,

$$\mathcal{L}_\lambda = D_2 E(\lambda, v_\lambda): F^k \to F^{k-2},$$

is

$$\mathcal{L}_\lambda(w)(x) = \begin{pmatrix} -\Delta f(x) - \operatorname{div}(Du_\lambda f(x) + m_\lambda D\psi) \\ \Delta\psi(x) - Du_\lambda D\psi + g'(m_\lambda(x))f(x) + h \\ -\int_{\mathbb{T}^d} f \end{pmatrix},$$

where $w = (\psi, f, h) \in F^k$. In principle, $\mathcal{L}_\lambda$ is only defined as a linear map on $F^k$ for a large enough $k$. However, by inspection of the coefficients, it is easy to see that it admits a unique extension to $F^k$ for any $k > 1$.

We define the set

$$\Lambda := \{\, \lambda \mid \quad 0 \le \lambda \le 1, \text{ (11.5) has a classical solution } (u_\lambda, m_\lambda, \overline{H}_\lambda) \,\}.$$

Note that $0 \in \Lambda$ as $(u_0, m_0, \overline{H}_0) \equiv (0, 1, -g(1))$ is a solution to (11.5) for $\lambda = 0$. Our goal is to prove $\Lambda = [0, 1]$. The a priori bounds in the statement mean that $\Lambda$ is a closed set. To prove that $\Lambda$ is open, we show that $\mathcal{L}_\lambda$ is invertible and apply the implicit function theorem. To prove invertibility, we use arguments related to the ones in the proof of the Lax–Milgram theorem and the structure of $\mathcal{L}_\lambda$. Let $F = F^1$. For $w_1, w_2 \in F$ with smooth components, we define

$$B_\lambda[w_1, w_2] = \int_{\mathbb{T}^d} w_2 \cdot \mathcal{L}_\lambda(w_1).$$

For smooth $w_1, w_2$, integration by parts gives

$$\begin{aligned} B_\lambda[w_1, w_2] = \int_{\mathbb{T}^d} [&m_\lambda D\psi_1 \cdot D\psi_2 + f_1 Du_\lambda D\psi_2 - f_2 Du_\lambda D\psi_1 \\ &+ g'(m_\lambda) f_1 f_2 + Df_1 D\psi_2 - Df_2 D\psi_1 + h_1 f_2 - h_2 f_1]. \end{aligned} \tag{11.6}$$

This last expression is well defined on $F \times F$. Thus, it defines a bilinear form $B_\lambda : F \times F \to \mathbb{R}$.

*Claim 11.5.* $B_\lambda$ is bounded, i.e.,

$$|B_\lambda[w_1, w_2]| \le C\|w_1\|_F \|w_2\|_F.$$

To prove the claim, we use Holder's inequality on each summand.

*Claim 11.6.* There exists a linear bounded mapping, $A : F \to F$, such that $B_\lambda[w_1, w_2] = (Aw_1, w_2)_F$.

This claim follows from Claim 11.5 and the Riesz Representation Theorem.

*Claim 11.7.* There exists a positive constant, $c$, such that $\|Aw\|_F \ge c\|w\|_F$ for all $w \in F$.

If the previous claim were false, then there would exist a sequence, $w_n \in F$, with $\|w_n\|_F = 1$ such that $Aw_n \to 0$. Let $w_n = (\psi_n, f_n, h_n)$. Then,

$$\int_{\mathbb{T}^d} m_\lambda |D\psi_n|^2 + g'(m_\lambda) f_n^2 = B_\lambda[w_n, w_n] \to 0. \tag{11.7}$$

By combining the a priori estimates on $\frac{1}{m_\lambda}$ with the fact that $g$ is strictly increasing and smooth, we have $g'(m_\lambda) > \delta > 0$. Then, (11.7) implies that $\psi_n \to 0$ in $\dot{H}_0^1$ and $f_n \to 0$ in $L^2$. Taking $\check{w}_n = (f_n - \int f_n, 0, 0) \in F$, we get

$$\int_{\mathbb{T}^d} [|Df_n|^2 + m_\lambda D\psi_n \cdot Df_n + f_n Du_\lambda Df_n] = B[w_n, \check{w}_n] = (Aw_n, \check{w}_n),$$

Therefore,

$$\frac{1}{2}\|Df_n\|^2_{L^2(\mathbb{T}^d)} - C\left(\|D\psi_n\|^2_{L^2(\mathbb{T}^d)} + \|f_n\|^2_{L^2(\mathbb{T}^d)}\right) \le (Aw_n, \check{w}_n) \to 0,$$

where the constant, $C$, depends only on $u_\lambda$. Because $D\psi_n, f_n \to 0$ in $L^2$, we have $f_n \to 0$ in $H^1(\mathbb{T}^d)$. Finally, we take $\check{w} = (0, 1, 0)$. Accordingly, we get

$$\int_{\mathbb{T}^d} [-Du_\lambda D\psi_n + g'(m_\lambda) f_n] + h_n = B[w_n, \check{w}] = (Aw_n, \check{w}) \to 0.$$

Because $D\psi_n, f_n \to 0$ in $L^2$, we have $h_n \to 0$. Hence, $\|w_n\|_F \to 0$, which contradicts $\|w_n\|_F = 1$.

*Claim 11.8.* $R(A)$ is closed in $F$.

This claim follows from the preceding one.

*Claim 11.9.* $R(A) = F$.

By contradiction, suppose that $R(A) \neq F$. Then, because $R(A)$ is closed in $F$, there exists a vector, $w \neq 0$, with $w \perp R(A)$. Let $w = (\psi, f, h)$. Then,

$$0 = (Aw, w) = B_\lambda[w, w] \ge \int_{\mathbb{T}^d} \theta |D\psi|^2 + \delta |f|^2.$$

Therefore, $\psi = 0$ and $f = 0$. Next, we choose $\bar{w} = (0, 1, 0)$. Similarly, we have $h = B_\lambda[\bar{w}, w] = (A\bar{w}, w) = 0$. Thus, $w = 0$, and, consequently, $R(A) = F$.

*Claim 11.10.* For any $w_0 \in F^0$, there exists a unique $w \in F$ such that $B_\lambda[w, \tilde{w}] = (w_0, \tilde{w})_{F^0}$ for all $\tilde{w} \in F$. Consequently, $w$ is the unique weak solution of the equation $\mathcal{L}_\lambda(w) = w_0$. Moreover, $w \in F^2$ and $\mathcal{L}_\lambda(w) = w_0$ in the sense of $F^2$.

Consider the functional $\tilde{w} \mapsto (w_0, \tilde{w})_{F^0}$ on $F$. By the Riesz Representation Theorem, there exists $\omega \in F$ such that $(w_0, \tilde{w})_{F^0} = (\omega, \tilde{w})_F$. Taking $w = A^{-1}\omega$, we get

$$B[w, \tilde{w}] = (Aw, \tilde{w})_F = (\omega, \tilde{w})_F = (w_0, \tilde{w})_{F^0}.$$

Therefore, $f$ is a weak solution to

$$-\Delta f - \operatorname{div}(m_\lambda D\psi + fDu_\lambda) = \psi_0$$

and $\psi$ is a weak solution to

$$\Delta\psi - Du_\lambda D\psi + g'(m_\lambda)f + h = f_0.$$

Standard results from the regularity theory for elliptic equations combined with bootstrapping arguments give $w = (\psi, f, h) \in F^2$. Thus, $\mathcal{L}_\lambda(w) = w_0$.

Consequently, $\mathcal{L}_\lambda$ is a bijective operator from $F^2$ to $F^0$. Then, $\mathcal{L}_\lambda$ is injective as an operator from $F^k$ to $F^{k-2}$ for any $k \geq 2$. To prove that it is also surjective, take any $w_0 \in F^{k-2}$. Then, there exists $w \in F^2$ such that $\mathcal{L}_\lambda(w) = w_0$. Returning again to elliptic regularity and bootstrapping arguments, we conclude that $w \in F^k$. Hence, $\mathcal{L}_\lambda : F^k \to F^{k-2}$ is surjective and, therefore, also bijective.

*Claim 11.11.* $\mathcal{L}_\lambda$ is an isomorphism from $F^k$ to $F^{k-2}$ for any $k \geq 2$.

Because $\mathcal{L}_\lambda : F^k \to F^{k-2}$ is bijective, we just need to check that it is also bounded. The boundedness follows directly from bounds on $u_\lambda$ and $m_\lambda$ and the smoothness of $V$ and $g$.

*Claim 11.12.* The set $\Lambda$ is open.

We choose $k > d/2 + 1$ so that $H^{k-1}(\mathbb{T}^d, \mathbb{R})$ is an algebra. For each $\lambda_0 \in \Lambda$, the partial derivative, $\mathcal{L} = D_2E(\lambda_0, v_{\lambda_0}) : F^k \to F^{k-2}$, is an isometry. By the Implicit Function Theorem, there exists a unique solution $v_\lambda \in F^k_+$ to $E(\lambda, v_\lambda) = 0$, in some neighborhood, $U$, of $\lambda_0$. Since $H^{k-1}(\mathbb{T}^d, \mathbb{R})$ is an algebra, bootstrapping arguments yield that $u_\lambda$ and $m_\lambda$ are smooth. Therefore, $v_\lambda$ is a classical solution to (11.4). Hence, $U \subset \Lambda$, which in turn proves that $\Lambda$ is open.

We have proven that $\Lambda$ is both open and closed; hence, $\Lambda = [0, 1]$. This argument ends the proof of the theorem. □

## 11.4 Continuation Method: Time-Dependent Problems

The continuation method can also be used for time-dependent problems. Here, we examine the model

$$\begin{cases} -u_t - \Delta u + \frac{|Du|^2}{2} = V(x, m), \\ m_t - \Delta m - \operatorname{div}(Du\, m) = 0, \\ u(x, T) = u_T(x),\ m(x, 0) = m_0(x), \end{cases} \tag{11.8}$$

where $u_T\colon \mathbb{T}^d \to \mathbb{R}$, $m_0\colon \mathbb{T}^d \to \mathbb{R}^+$ and $V\colon \mathbb{T}^d \times [0,+\infty) \to \mathbb{R}$ are given smooth functions, with $V(x,z)$ non-decreasing in $z$. As usual, the unknowns are the smooth functions $u\colon \mathbb{T}^d \times [0,T] \to \mathbb{R}$ and $m\colon \mathbb{T}^d \times [0,T] \to \mathbb{R}^+$.

We introduce the problem

$$\begin{cases} -u_t - \Delta u + \frac{|Du|^2}{2} = (1-\lambda)V(x,m) + \lambda \arctan(m), \\ m_t - \Delta m - \operatorname{div}(Du\, m) = 0, \\ u(x,T) = (1-\lambda)u_T := \Psi_\lambda,\ m(x,0) = (1-\lambda)m_0 + \lambda := m_{0,\lambda}, \end{cases} \tag{11.9}$$

where $0 \le \lambda \le 1$. For convenience, we set

$$V_\lambda(x,m) = (1-\lambda)V(x,m) + \lambda \arctan(m).$$

We assume that the following a priori bounds hold for any solution, $(u_\lambda, m_\lambda)$, of (11.9).

$$\left\| \frac{1}{m} \right\|_{L^\infty(\mathbb{T}^d\times[0,T]))} + \|D_t^k D_x^l m_\lambda\|_{L^p(\mathbb{T}^d\times[0,T])} + \|D_t^k D_x^l u_\lambda\|_{L^p(\mathbb{T}^d\times[0,T])} \le C_{k,l,p} \tag{11.10}$$

for any $k, l \in \mathbb{N}$ and $1 \le p < \infty$.

**Theorem 11.13.** *Assume that the a priori bound* (11.10) *holds for any solution,* $(u_\lambda, m_\lambda)$, *of* (11.9) *and that* $u_T, m_0, V$ *are as above. Then, there exists a smooth solution,* $(u,m)$, *of* (11.13).

When $\lambda = 1$, (11.9) has a unique solution, namely $u = \frac{\pi}{4}(T-t)$, $m = 1$. As in the preceding section, our goal is to prove that the set, $\Lambda$, of values $0 \le \lambda \le 1$ for which (11.9) admits a solution is relatively open and closed. Therefore, $\Lambda = [0,1]$ and, thus, (11.13) has a solution.

For $k \ge -1$, we set

$$F^k([0,T],\mathbb{T}^d) = \cap_{2k_1+k_2=k} H^{k_1}([0,T], H^{k_2}(\mathbb{T}^d)),$$

where the intersection is taken over all integers, $k_1 \ge 0, k_2 \ge -1$. The space $F^k([0,T],\mathbb{T}^d)$ is a Banach space endowed with the norm

$$\|f\|_{F^k([0,T],\mathbb{T}^d)} = \sum_{2k_1+k_2=k} \|f\|_{H^{k_1}([0,T],H^{k_2}(\mathbb{T}^d))} \,.$$

Moreover, there exists $\tilde{k}_d$, depending only on the dimension $d$, such that, for $k \ge \tilde{k}_d$, $F^{k-2}$ is an algebra. Let $k \ge \tilde{k}_d$ and consider the operator

$$\mathcal{M}_\lambda\colon F^k([0,T],\mathbb{T}^d) \times F^k([0,T],\mathbb{T}^d) \to F^{k-2}([0,T],\mathbb{T}^d) \times F^{k-2}([0,T],\mathbb{T}^d) \\ \times H^{k-1}(\mathbb{T}^d) \times H^{k-1}(\mathbb{T}^d)$$

given by

$$\mathcal{M}_\lambda \begin{bmatrix} u \\ m \end{bmatrix} = \begin{bmatrix} m_t - \Delta m - \operatorname{div}(Dum) \\ u_t + \Delta u - \frac{|Du|^2}{2} + V_\lambda(x, m) \\ m(x, 0) - m_{0,\lambda}(x) \\ u(x, T) - \Psi_\lambda(x) \end{bmatrix}.$$

Then, (11.9) is equivalent to

$$\mathcal{M}_\lambda \begin{bmatrix} u \\ m \end{bmatrix} = 0, \tag{11.11}$$

and (11.13) then reads as $\mathcal{M}_0 \begin{bmatrix} u \\ m \end{bmatrix} = 0$. Moreover, as we remarked before, $\mathcal{M}_1 \begin{bmatrix} u \\ m \end{bmatrix} = 0$ has only the trivial solution $u = \frac{\pi}{4}(T - t)$ and $m = 1$. We consider the linearized operator, $\mathcal{L}$, given by

$$\mathcal{L}_\lambda \begin{bmatrix} v \\ f \end{bmatrix} = \lim_{\varepsilon \to 0} \frac{\mathcal{M}_\lambda \begin{bmatrix} u + \varepsilon v \\ m + \varepsilon f \end{bmatrix} - \mathcal{M}_\lambda \begin{bmatrix} u \\ m \end{bmatrix}}{\varepsilon} = \begin{bmatrix} f_t - \Delta f - \operatorname{div}\left[ Duf + mDv \right] \\ \\ v_t + \Delta v - DuDv + D_z V_\lambda f \\ \\ f(x, 0) \\ \\ v(x, T) \end{bmatrix}.$$

Note that $\mathcal{L}_\lambda : F^k([0, T], \mathbb{T}^d) \times F^k([0, T], \mathbb{T}^d) \to F^{k-2}([0, T], \mathbb{T}^d) \times F^{k-2}([0, T], \mathbb{T}^d) \times H^{k-1}(\mathbb{T}^d) \times H^{k-1}(\mathbb{T}^d)$ is a bounded linear operator for all large enough $k$. However, if $u$ and $m$ are $C^\infty$ solutions to (11.9), then $\mathcal{L}_\lambda$ admits a unique extension as a bounded linear operator $\mathcal{L}_\lambda : F^k([0, T], \mathbb{T}^d) \times F^k([0, T], \mathbb{T}^d) \to F^{k-2}([0, T], \mathbb{T}^d) \times F^{k-2}([0, T], \mathbb{T}^d) \times H^{k-1}(\mathbb{T}^d) \times H^{k-1}(\mathbb{T}^d)$, for all $k \geq 1$.

The form $\langle \cdot, \cdot \rangle$ denotes the scalar product on $L^2(\mathbb{T}^d)$. To apply the inverse function theorem, we need to prove that the linear operator, $\mathcal{L}_\lambda$, is invertible. For this, we begin by showing that the equation $\mathcal{L}_\lambda w = W$ has a unique weak solution in the sense of the following definition.

**Definition 11.14.** For $h, g \in L^2([0, T], L^2(\mathbb{T}^d)), A, B \in L^2(\mathbb{T}^d)$, set

$$W(x, t) = \begin{bmatrix} h(x, t) \\ g(x, t) \\ A(x) \\ B(x) \end{bmatrix}.$$

A function, $w = \begin{bmatrix} v \\ f \end{bmatrix}$, with

$$v, f \in L^2([0,T], H^1(\mathbb{T}^d)) \text{ and } v_t, f_t \in L^2([0,T], H^{-1}(\mathbb{T}^d)), \text{ that is } v, f \in F^1([0,T], \mathbb{T}^d), \tag{11.12}$$

is called a weak solution of $\mathcal{L}_\lambda w = W$ if:

1. for any $\bar{v}, \bar{f} \in H^1(\mathbb{T}^d)$ and for a.e. $t$, $0 \le t \le T_0$, we have

$$\begin{cases} \langle f_t, \bar{f} \rangle + \left\langle Df + Du_\lambda f + mDv, D\bar{f} \right\rangle = \langle h, \bar{f} \rangle \\ \langle v_t, \bar{v} \rangle - \langle Dv, D\bar{v} \rangle - \langle Du_\lambda \cdot Dv - D_z V_\lambda f, \bar{v} \rangle = \langle g, \bar{v} \rangle. \end{cases} \tag{11.13}$$

2. $f(x,0) = A(x), \quad v(x,T) = B(x)$.

*Remark 11.15.* Note that (11.12) implies that $v, f \in C([0,T], L^2(\mathbb{T}^d))$ (see, e.g., [88], Section 5.9.2, Theorem 3). Therefore, the traces $f(x,0)$ and $v(x,T)$ are well defined.

**Lemma 11.16 (Uniqueness of Weak Solutions).** *Let $(u_\lambda, m_\lambda)$ be a $C^\infty$ solution to (11.9) and the a priori bounds (11.10) hold. Then, there exists at most one weak solution to the equation $\mathcal{L}_\lambda w = W$ in the sense of Definition 11.14.*

*Proof.* Since the equation $\mathcal{L}_\lambda w = W$ is linear, it is enough to prove that $\mathcal{L}_\lambda w = 0$ has only the trivial solution $w = 0$. For this, we take $\bar{f} = v, \bar{v} = f$ in (11.13). Adding both equations and integrating in time, we obtain

$$\int_0^{T_0} \int_{\mathbb{T}^d} m_\lambda |Dv|^2 + D_z V_\lambda f^2 \mathrm{d}x\mathrm{d}t = 0.$$

Thus, we get $f = 0, Dv = 0$. Consequently, $v \equiv v(t)$. Next, by looking at the second equation in (11.13), for $\bar{v} = v(t)$, we obtain

$$\frac{d}{dt} \langle v, v \rangle = 0.$$

Using the boundary conditions for $v$, we conclude that $v = 0$, therefore, that $w = 0$. □

To prove the existence of weak solutions, we use the Galerkin approximation method. We consider a sequence of $C^\infty$ functions, $e_k = e_k(x)$, $k \in \mathbb{N}$, such that $\{ e_k \}_{k=1}^\infty$ is an orthogonal basis of $H^1(\mathbb{T}^d)$ and an orthonormal basis of $L^2(\mathbb{T}^d)$. We construct a sequence of finite dimensional approximations to weak solutions of (11.9) as follows. Let $v_N, f_N \colon [0,T] \to H^1(\mathbb{T}^d)$ be given by

$$f_N(t) = \sum_{k=1}^N A_N^k(t) e_k, \quad v_N(t) = \sum_{k=1}^N B_N^k(t) e_k.$$

Next, we show that we can select the coefficients, $A_N^k, B_N^k$, such that

$$\begin{cases} \langle f_N', e_k\rangle + \langle Df_N + f_N Du_\lambda + m_\lambda Dv_N, De_k\rangle = \langle h, e_k\rangle, \\ \langle v_N', e_k\rangle - \langle Dv_N, De_k\rangle - \langle Du_\lambda \cdot Dv_N - D_z V_\lambda f_N, e_k\rangle = \langle g, e_k\rangle, \end{cases} \tag{11.14}$$

and

$$A_N^k(0) = \langle A, e_k\rangle,\ B_N^k(T) = \langle B, e_k\rangle, \qquad k = 1, 2, \ldots, N. \tag{11.15}$$

First, we observe that (11.14) is equivalent to:

$$\begin{cases} \dot{A}_N^k + \sum_{l=1}^N \langle De_l + e_l Du_\lambda, De_k\rangle A_N^l + \sum_{l=1}^N \langle m_\lambda \cdot De_l, De_k\rangle B_N^l = \langle h, e_k\rangle, \\ \dot{B}_N^k - \sum_{l=1}^N \langle De_l + Du_\lambda \cdot De_l, De_k\rangle B_N^l - \sum_{l=1}^N \langle e_l D_z V_\lambda, e_k\rangle A_N^l = \langle g, e_k\rangle. \end{cases} \tag{11.16}$$

Second, because (11.16) is a linear system of ODEs, the only difficulty in proving the existence of solutions concerns the boundary conditions (11.15). Existence is not immediate because half of the boundary conditions are given at the initial time, whereas the other half are given at the terminal time. From the standard theory of ordinary differential equations, the initial value problem for (11.16) (that is, with $A_N^k(0)$ and $B_N^k(0)$ prescribed) has a unique solution. Hence, to prove the existence of solutions to (11.16), it is enough to show the existence of solutions for the corresponding homogeneous problem:

$$\begin{cases} \dot{A}_N^k + \sum_{l=1}^N \langle De_l + e_l Du_\lambda, De_k\rangle A_N^l + \sum_{l=1}^N \langle m \cdot De_l, De_k\rangle B_N^l = 0, \\ \dot{B}_N^k - \sum_{l=1}^N \langle De_l + Du_\lambda \cdot De_l, De_k\rangle B_N^l - \sum_{l=1}^N \langle e_l D_z V_\lambda, e_k\rangle A_N^l = 0, \end{cases} \tag{11.17}$$

with arbitrary $\tilde{A}_N^k(0)$ and $\tilde{B}_N^k(T)$, $1 \leq k \leq N$. Indeed, any solution to (11.16)–(11.15), $(A, B)$ can be written as a sum of a particular solution to (11.16), $(\bar{A}, \bar{B})$, for instance, with

$$\bar{A}_N^k(0) = 0,\ \bar{B}_N^k(0) = 0,\ k = 1, 2, \ldots, N,$$

and a solution, $(\tilde{A}, \tilde{B})$, to (11.17) with suitable initial and terminal conditions such that (11.15) holds for $(A, B) = (\bar{A} + \tilde{A}, \bar{B} + \tilde{B})$.

Finally, we regard the solution of the initial value problem for the homogeneous system corresponding to (11.16) as a linear operator on $\mathbb{R}^{2N}$:

$$(A_N(0), B_N(0)) \mapsto (A_N(0), B_N(T)). \tag{11.18}$$

We need to prove that this mapping is surjective. Because (11.18) is a linear mapping from $\mathbb{R}^{2N}$ to $\mathbb{R}^{2N}$, surjectivity is equivalent to injectivity. Therefore, it suffices to prove that the homogeneous system of ODEs corresponding to (11.16), subject to initial-terminal conditions, $A_N(0) = B_N(T) = 0$, has only the trivial solution $A_N = B_N \equiv 0$. Let $f_N, v_N$ solve (11.14) with $h = g \equiv 0, A = B \equiv 0$. From (11.14), we obtain (11.13) for $f = \bar{v} = f_N, v = \bar{f} = v_N$. Using the same argument as in Lemma 11.16, we conclude that $f_N = v_N \equiv 0$.

Next, we provide energy estimates for these approximations to ensure the weak convergence of approximate solutions through some subsequence.

**Lemma 11.17.** *Let $(u_\lambda, m_\lambda)$ be a $C^\infty$ solution to (11.9) and let the a priori bounds in (11.10) hold. Then, there exists a constant, $C$, such that, for any $C^\infty$ solution $(u_\lambda, m_\lambda)$ to (11.9), we have*

$$\max_{0\leq t\leq T} \|(f_N, v_N)\|_{(L^2(\mathbb{T}^d))^2} + \|(f_N, v_N)\|_{(L^2(0,T;H^1(\mathbb{T}^d)))^2} + \|(f_N', v_N')\|_{(L^2(0,T;H^{-1}(\mathbb{T}^d)))^2}$$
$$\leq C\left(\|h\|_{L^2(0,T;L^2(\mathbb{T}^d))} + \|g\|_{L^2(0,T;L^2(\mathbb{T}^d))} + \|A\|_{L^2(\mathbb{T}^d)} + \|B\|_{L^2(\mathbb{T}^d)}\right).$$

**Lemma 11.18 (Existence of Weak Solutions).** *Let $(u_\lambda, m_\lambda)$ be a $C^\infty$ solution to (11.9) and let the a priori bounds (11.10) hold. Then, there exists a weak solution of $\mathcal{L}_\lambda w = W$ in the sense of (11.13).*

*Proof.* By the energy estimates, there exist subsequences of $v_N, f_N$ and functions $v, f \in L^2(0, T_0; H^1(\mathbb{T}^d))$, with $v' = v_t, f' = f_t \in L^2(0, T_0; H^{-1}(\mathbb{T}^d))$, such that

$$\begin{cases} v_N \rightharpoonup v,\ f_N \rightharpoonup f, \text{ weakly in } L^2(0, T_0; H^1(\mathbb{T}^d)) \\ v_N' \rightharpoonup v',\ f_N' \rightharpoonup f', \text{ weakly in } L^2(0, T_0; H^{-1}(\mathbb{T}^d)). \end{cases}$$

For fixed $N_0$, let $\bar{v}, \bar{f} \in \operatorname{span}\{e_k : 1 \leq k \leq N_0\}$ with $\|\bar{v}\|_{L^2(0,T_0;H^1(\mathbb{T}^d))}, \|\bar{f}\|_{L^2(0,T_0;H^1(\mathbb{T}^d))} \leq 1$. According to the definition of $v_N, f_N$, (11.13) holds with $v = v_N$ and $f = f_N$ for every $N \geq N_0$. Weak convergence then implies (11.13) for $v, f$ and any $\bar{v}, \bar{f} \in \operatorname{span}\{e_k : 1 \leq k \leq N_0\}$. The above convergence implies that $v_N \rightharpoonup v$, $f_N \rightharpoonup f$ also in $C(0, T_0; L^2(\mathbb{T}^d))$. Therefore, the initial and terminal conditions on $f, v$ hold as well. Since $\cup_{N\geq 1} \operatorname{span}\{e_k : 1 \leq k \leq N\}$ is dense in $L^2(0, T_0; H^1(\mathbb{T}^d))$, the proof is complete. □

**Lemma 11.19 (Higher Regularity).** *Let $(u_\lambda, m_\lambda)$ be a $C^\infty$ solution to (11.9) and let the a priori bounds in (11.10) hold. Assume that $A, B \in H^{k+1}(\mathbb{T}^d), h, g \in F^{2k}([0, T], \mathbb{T}^d)$ and let $W = [h, g, A, B]^t$. Then, for any weak solution, $w = [f, v]^t$, of $\mathcal{L}_\lambda w = W$, we have $v, f \in F^{2k+2}([0, T], \mathbb{T}^d)$.*

The proof follows from the following result on the regularizing properties of the heat equation and a bootstrap argument.

**Lemma 11.20.** *Let $\tilde{h} \in H^{k_1}([0, T], H^{k_2}(\mathbb{T}^d))$, $\tilde{g} \in H^{2k_1+k_2+1}(\mathbb{T}^d)$ for some $k_1, k_2 \geq 0$, and let $\tilde{u} \in F^1([0, T], \mathbb{T}^d)$ be a weak solution of the heat equation*

$$\begin{cases} \tilde{u}_t - \Delta\tilde{u} = \tilde{h} \\ \tilde{u}(x, 0) = \tilde{g}(x). \end{cases}$$

*Then, $\tilde{u} \in H^{k_1}([0, T], H^{k_2+2}(\mathbb{T}^d)) \cap H^{k_1+1}([0, T], H^{k_2}(\mathbb{T}^d))$.*

*Proof of Theorem 11.13*. The bounds (11.10) and the Arzela–Ascoli Theorem imply that $\Lambda$ is a closed subset of the interval $[0, 1]$. We prove that it is also open. Let $\lambda_0 \in \Lambda$. Using (11.10), we see that the operator

$$\mathcal{L}_{\lambda_0}: F^{2k} \times F^{2k} \to F^{2k-2} \times F^{2k-2} \times H^{2k-1} \times H^{2k-1}$$

is bounded for every $k \geq 1$. Using Lemmas 11.16, 11.18, and 11.19, we conclude that $\mathcal{L}_{\lambda_0}$ is bijective. It is thus also invertible. We choose a large enough $k$ and $l = \lfloor \frac{2k}{3} \rfloor$ such that $H^l([0,T], H^l(\mathbb{T}^d))$ is an algebra. By the inverse function theorem, there is a neighborhood $U$ of $\lambda_0$ where the equation $\mathcal{M}_\lambda \begin{bmatrix} u \\ m \end{bmatrix} = 0$ has a unique solution, $(u_\lambda, m_\lambda)$, in $F^{2k}([0,T], \mathbb{T}^d) \times F^{2k}([0,T], \mathbb{T}^d)$. Then, $u_\lambda, m_\lambda \in H^l([0,T], H^l(\mathbb{T}^d))$. The inverse function theorem implies that the mapping $\lambda \mapsto (u_\lambda, m_\lambda)$ is continuous. Hence, we can assume that in the neighborhood, $U$, $m_\lambda$ is bounded away from zero. This observation together with the fact that $H^l([0,T], H^l(\mathbb{T}^d))$ is an algebra allows us to use the regularity theory and a bootstrap argument to conclude that $(u_\lambda, m_\lambda)$ are $C^\infty$. Accordingly, $U \subset \Lambda$. Consequently, we have proved that $\Lambda$ is an open set in $[0, 1]$. Because $1 \in \Lambda$, we know that $\Lambda \neq \emptyset$. Therefore, $\Lambda = [0, 1]$. In particular, $0 \in \Lambda$. □

## 11.5 Bibliographical Notes

The book [171] gives a systematic account on regularization and compactness methods in partial differential equations. A comprehensive account of the main ideas behind bootstrapping methods can be found in [205]. In MFGs, the regularization examined in Sect. 11.2 was introduced in [134] and [135]. The construction of solutions for time-dependent MFGs in [117] also relies on a regularization argument. In the context of weak solutions, other regularizations were proposed in [196] and [195]. The continuation method is a well-known technique for elliptic and parabolic equations. In elliptic equations, the continuation argument is usually set up in $C^{2,\alpha}$ spaces using Schauder estimates; see, for example, [107]. In contrast, for MFGs, it is more convenient to work with Sobolev spaces. A proof of the implicit and inverse function theorems in Banach spaces can be found in [84]. For the application of the continuation methods to stationary MFGs, see [114, 129, 130]. For time-dependent MFGs, this method has been used in [123]. For the Galerkin method, see, for example, [88].

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