Catherine Bandle, Alfred Wagner

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Shape Optimization

Variations of Domains and Applications

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Authors
Prof. Dr. Catherine Bandle
University of Basel
Department of Mathematics and Computer Science
Spiegelgasse 1
4051 Basel
Switzerland
c.bandle@gmx.ch

Priv. Doz. Dr. Alfred Wagner
RWTH Aachen University
Institute for Mathematics
Templergraben 55
52062 Aachen
Germany
wagner@instmath.rwth-aachen.de

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Preface

The description of optimal forms plays an important role in mathematics and applications. In many areas of mathematics, pure and applied, it has recently received a lot of attention.

In this book we consider domain functionals, like energies and eigenvalues of elliptic operators. Assuming the existence of an optimal domain, we are interested in its shape. Global methods like symmetrization and rearrangement inequalities are applicable in certain settings; however, for many interesting problems, no global approach is known.

In the absence of a global approach, one can instead study the effect of local perturbations. In the spirit of calculus one considers one-parameter families of perturbations of a fixed domain. The first derivative with respect to this parameter can be used to derive a necessary condition for a domain to be optimal. The second derivative provides additional information to help to determine local maxima or minima.

In general these derivatives are difficult to analyze. In particular the second derivative, which is crucial to understand stability, is complicated. We develop techniques that enable us to determine its sign in various cases of interest.

The structure of this book can briefly be summarized as follows.

We start with an overview of the main examples that will be treated in this book. They are selected in order to show the breadth of the applicability of our methods. Among them are energies of nonlinear boundary value problems, eigenvalues, and problems of fourth order.

We then describe the class of admissible domain perturbations. It should be emphasized that our technique requires smoothness of the domains. The arguments will in general fail in the case of nonsmooth domains. Nonsmooth analysis for domain variations is not yet available.

We pay special attention to volume and area preserving perturbations. In order to compare the family of perturbed domains, we introduce differential geometric tools. This allows us to compute variational formulas for volume and surface area.

Our functionals are domain and boundary integrals. We present two methods to derive variational formulas. The first is the change of variables method. All quantities will be mapped onto a fixed domain. This requires only the chain rule. The second approach, due to Reynolds, is the moving surface method. It captures the shape by means of a boundary displacement and generally requires additional regularity. The resulting variational formulas coincide and are illustrated with some examples. Among them are problems of optimal control, convolutions, and weighted isoperimetric inequalities.

It becomes apparent that there is a difference between variational formulas for purely geometrical functionals and for those functionals which depend on functions varying with the domain.

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The most involved variational formulas are derived for energies of semilinear elliptic problems. They depend only on the boundary data and the perturbations of the boundary. We discuss these formulas for different boundary value problems.

To find precise estimates for the second variation, we expand it in a suitable basis of functions consisting of eigenfunctions of elliptic eigenvalue problems. In many examples we are able to determine the sign. This method applies to any critical domain. For the ball this system of eigenfunctions restricted to the sphere corresponds to the expansion with respect to spherical harmonics.

The first and second variations allow us to derive some local isoperimetric inequalities. The well-known classical isoperimetric inequality states that among all domains of given volume the ball has the smallest perimeter. We will call an inequality *isoperimetric* if it relates geometrical or physical quantities defined on the same domain and if the equality sign is attained for some optimal domain. We extend this notion to a broader class of functionals such as energies and eigenvalues.

Fourth order problems are studied at the end of the book by means of the moving surface method. For the buckling eigenvalue, we are able to use the variational formulas to prove uniqueness of the optimal domain.

This book attempts to bridge the gap between analysis and geometry. Our exposition is example-driven and draws on problems that are the object of study in the current literature. We present them in a self-contained way, at times pointing out open questions that we hope will lead to further investigation and interesting discoveries.

Domain variation has a long history, which can be traced back to Hadamard (1908). More detailed historical comments are given at the end of the relevant chapters. The book is aimed at readers with basic knowledge of analysis and geometry.

We especially thank Simon Stingelin for producing all the illustrations in this book.

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1 Overview of the basic problems

In this chapter we describe briefly some selected classical problems to which we shall apply domain variations. The theory of domain variations started with Hadamard's formula for the Green's function. For historical reasons we review this formula at the end of this chapter.

1.1 Semilinear elliptic boundary value problems

Let $\Omega \subset \mathbb{R}^n$ be a bounded, smooth domain with the outer normal ν and let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. The boundary value problems taken into consideration are

$$\Delta u + g(u) = 0 \quad \text{in } \Omega,$$

subject to the boundary conditions

$$\begin{aligned} &u = 0, \quad \text{Dirichlet},\\ &\partial_\nu u + \alpha u = 0, \quad \text{Robin},\\ &\partial_\nu u = 0, \quad \text{Neumann}, \end{aligned}$$

where ∂_ν denotes the derivative in the direction of the outer normal and $\alpha \in \mathbb{R}$ is a fixed number.

The solutions are the *Euler–Lagrange equations* corresponding to the energy functional

$$\mathcal{E}(\Omega, u) := \int_\Omega |\nabla u|^2 \, dx - 2 \int_\Omega G(u) \, dx + \alpha \oint_{\partial\Omega} u^2 \, dS.$$

Here G is the primitive of g, i. e., $G' = g$, and dS is the surface element of $\partial\Omega$. For the Dirichlet and Neumann problem set $\alpha = 0$.

If $\Omega \subset \mathbb{R}^2$ is simply connected, the Dirichlet problem with $g = 1$ is a model for the stress of beams with constant cross-section. The torsional rigidity corresponds to $-\mathcal{E}(\Omega, u)$. Saint-Venant conjectured in 1856 that among all domains of given area, the circle has the largest torsional rigidity. This conjecture was later proved by Pólya in 1948.

In this book we shall use the expression *torsion problem* for general boundary value problems with $g(u) = 1$. Special attention will be given to this particular problem with Robin boundary conditions. If α is negative, it is uniquely solvable if and only if α does not coincide with an eigenvalue of the *Steklov problem*

$$\Delta \varphi = 0 \text{ in } \Omega, \quad \partial_\nu \varphi = \mu \varphi \text{ on } \partial\Omega.$$

There is obviously no solution in the case of Neumann boundary conditions.

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In many applications, a domain of prescribed volume is sought, for which the energy is a minimum or maximum. If u is a positive minimizer of $\mathcal{E}(\Omega, u)$ satisfying Dirichlet boundary conditions, rearrangement and symmetrizations are very powerful techniques to study optimal domains (see [98]). For more general problems, in particular for energies whose solutions change sign or satisfy a Robin boundary condition, they do not apply. A new approach to look for optimal domains with Robin boundary conditions is discussed in [29].

1.2 Eigenvalue problems

1.2.1 The problem of vibrating membranes

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Consider the eigenvalue problem

$$\Delta\phi + \lambda\phi = 0 \quad \text{in } \Omega,$$

subject to one of the following boundary conditions:

$$\begin{aligned} \phi = 0, &\quad \text{Dirichlet conditions,} \\ \partial_\nu\phi + \alpha\phi = 0, \alpha \in \mathbb{R}^+, &\quad \text{Robin boundary conditions,} \\ \partial_\nu\phi = 0, &\quad \text{Neumann boundary conditions.} \end{aligned}$$

These problems possess a countable number of positive eigenvalues,

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \lambda_n \to \infty.$$

In the case of Neumann conditions, $\lambda_0 = 0$ is an eigenvalue and the corresponding eigenfunction is $\phi_0 = \text{const}$. The fundamental frequency of the Dirichlet and Robin problems plays a central role in analysis and in physical applications. It can be characterized by the Rayleigh principle

$$\lambda_1^D = \min_{W_0^{1,2}(\Omega)} R(v) := \min_{W_0^{1,2}(\Omega)} \frac{\int_\Omega |\nabla v|^2\,dx}{\int_\Omega v^2\,dx}$$

for the Dirichlet eigenvalue problem and by

$$\lambda_1^R := \min_{W^{1,2}(\Omega)} R(\alpha, v) = \min_{W^{1,2}(\Omega)} \frac{\int_\Omega |\nabla v|^2\,dx + \alpha \oint_{\partial\Omega} v^2\,ds}{\int_\Omega v^2\,dx}$$

for the Robin eigenvalue problem. In both cases the minimum is achieved for the first eigenfunction. As a consequence of the Rayleigh principle together with Harnack's inequality, the first eigenfunction is of constant sign and the lowest eigenvalue is simple.

A physical model in two dimensions is a vibrating membrane which at rest covers the domain Ω. The deflection in the normal direction $u(x,t)$ solves the wave equation $u_{tt} = \Delta u$. If we set $u = e^{i\omega t}\phi(x)$, then ϕ is a solution of the eigenvalue problem described above with $\lambda = \omega^2$. The Dirichlet conditions imply that the membrane is fixed on the boundary, the Robin conditions describe an elastic attachment, and in the case of Neumann conditions the membrane moves freely on the boundary (cf. Figure 1.1).

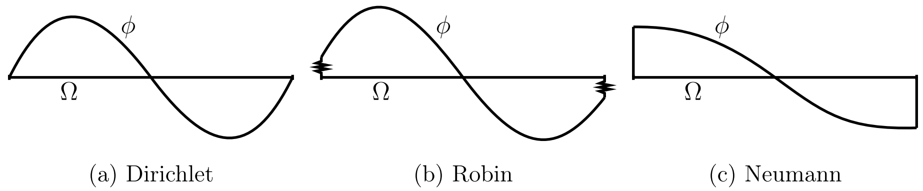

Figure 1.1: Different boundary conditions.

The eigenvalue $\lambda_1^R(\alpha)$ is a concave, monotone increasing function such that $\lambda_1^R(0) = 0$ and $\lim_{\alpha\to\infty} \lambda_1^R(\alpha) = \lambda_1^D$, where λ_1^D corresponds to the first eigenvalue with Dirichlet boundary conditions (see Figure 1.2).

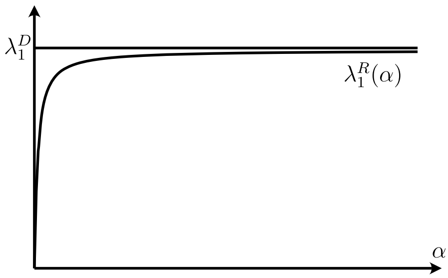

Figure 1.2: Robin eigenvalue for the circle of area 1.

The well-known *Rayleigh–Faber–Krahn inequality* (cf. [99]) states that among all domains of given volume, $\lambda_1^D(\Omega)$ is minimal for the ball. Bossel [25] and Daners [41] have shown that this inequality also holds for $\lambda_1^R(\Omega)$ provided α is positive. Daners and Kennedy [42] proved that the ball is the unique minimal domain.

1.2.2 Buckling problem for a clamped plate

In this subsection we are interested in the lowest eigenvalue of the problem

$$\begin{aligned} \Delta^2 u + \Lambda(\Omega)\Delta u &= 0 \quad \text{in } \Omega, \\ u = \partial_\nu u &= 0 \quad \text{in } \partial\Omega. \end{aligned}$$

For its characterization we need the Rayleigh quotient

$$\mathcal{R}(u,\Omega) := \frac{\int_\Omega |\Delta u|^2\,dx}{\int_\Omega |\nabla u|^2\,dx}.$$

Then

$$\Lambda(\Omega) := \inf\{\mathcal{R}(u,\Omega) : u \in W_0^{2,2}(\Omega)\}. \tag{1.2.1}$$

The sign of the first eigenfunction may change depending on Ω. This is in contrast to the eigenfunctions described in the previous section.

The quantity $\Lambda(\Omega)$ is called the *buckling eigenvalue* of Ω. It is well known that there is a discrete spectrum of positive eigenvalues of finite multiplicity and their only accumulation point is ∞. The corresponding eigenfunctions form an orthonormal basis of $W_0^{2,2}(\Omega)$. Some of these statements are found in [125].

In 1951, G. Pólya and G. Szegö formulated the following conjecture (see [98]):

Among all domains Ω of given volume, the ball minimizes $\Lambda(\Omega)$.

Partial results are known; however, the conjecture is still open for general domains. The most general result today is found in [115] and will be presented in this text. A crucial tool aside from the variational formulas is Payne's inequality ([95], see also [56]). He showed that for any bounded domain

$$\lambda_2(\Omega) \le \Lambda(\Omega),$$

where $\lambda_2(\Omega)$ is the second eigenvalue of the Laplacian with Dirichlet boundary conditions, equality holds if and only if Ω is a ball.

1.3 Green's function, Hadamard's formula

The modern theory of shape derivatives or domain variations started with the work of Hadamard [69]. The highlight was his *variational formula for the Green's function.*

The Green's function corresponding to the Laplace operator in Ω with Dirichlet boundary conditions solves

$$\Delta_x G(x,y) = -\delta_y(x) \text{ in } \Omega, \quad G(x,y) = 0 \text{ for } x \in \partial\Omega, y \in \Omega.$$

It can be written in the form

$$G(x,y) = h(x,y) + \gamma(|x-y|), \quad \text{where } \gamma(|x-y|) = \begin{cases} \frac{1}{2\pi}\log|x-y| & \text{if } n = 2, \\ \frac{1}{(n-2)|\partial B_1|}|x-y|^{-2+n} & \text{if } n > 2. \end{cases}$$

Here B_1 is the unit ball in $\mathbb{R}^n$ and $h(x,y)$ is a harmonic function in Ω.

We briefly describe Hadamard's argument. Let Ω_t be a family of domains, the boundary of which is obtained from $\partial\Omega$ by a shift $t\rho$ along its outer normal ν. The parameter t is close to zero and ρ is a smooth function defined on $\partial\Omega$. The Green's function in Ω_t will be denoted by $G(x,y:t)$. Hadamard studied its dependence on small $|t|$ and showed that

$$\lim_{t\to 0} t^{-1}\{G(x,y:t)-G(x,y)\} = \oint_{\partial\Omega} \partial_\nu G(x,z)\partial_\nu G(z,y)\rho(z)\,dS_z.$$

His reasoning was as follows (see Figure 1.3).

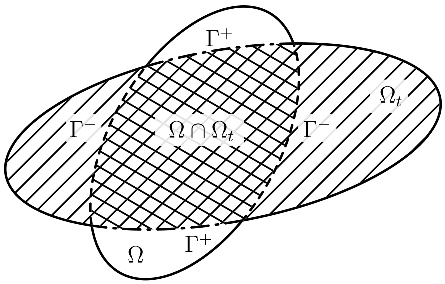

Figure 1.3: Hadamard's domain decomposition.

For $x,y \in \Omega_t \cap \Omega$,

$$\begin{aligned} G(x,y:t)-G(x,y) &= \int_{\Omega_t\cap\Omega} G(x,z:t)\Delta_z G(z,y) - G(x,z)\Delta_z G(z,y:t)\,dz \\ &= \int_{\Gamma^-} G(x,z:t)\partial_\nu G(z,y)\,dS_z - \int_{\Gamma^+} G(x,z)\partial_{\hat\nu} G(z,y:t)\,dS_z, \end{aligned}$$

where $\Gamma^- = \partial\Omega \cap \Omega_t$ and $\Gamma^+ = \partial\Omega_t \cap \Omega$. Here $\hat\nu$ stands for the outer normal of $\Omega_t \cap \Omega$. If the boundaries $\partial\Omega$ and $\partial\Omega_t$ are sufficiently smooth, we have $0 = G(x,z:t) + t\partial_\nu G(x,z:t)\rho(z) + o(t)$ on Γ^-. A similar expansion holds for $G(x,y)$ on Γ^+. Thus,

$$\begin{aligned} G(x,y:t)-G(x,y) = &-\int_{\Gamma^-} t\partial_\nu G(x,z:t)\rho(z)\partial_\nu G(z,y)\,dS_z \\ &+ \int_{\Gamma^+} t\partial_{\hat\nu} G(x,z)\rho(z)\partial_{\hat\nu} G(z,y:t)\,dS_z + o(t^2), \end{aligned}$$

Dividing by t and taking the limit as $t \to 0$ we obtain Hadamard's formula.

In the same memoir Hadamard considered the Green's function for Neumann boundary conditions as well as for the plate problem $\Delta^2 G(x,y) = \delta_y(x)$ in Ω, $G(x,y) = \partial_\nu G(x,y) = 0$ if x on $\partial\Omega$ and y in Ω. He showed that

$$\lim_{t\to 0} t^{-1}\{G(x,y:t) - G(x,y)\} = \oint_{\partial\Omega} \Delta_z G(x,z)\Delta_z G(y,z)\rho(z)\, dS.$$

Later Garabedian and Schiffer [63] gave a rigorous alternative proof of Hadamard's formula. They also computed the second derivative of $G(x,y:t)$ with respect to t. Generalizations to the Green's function of general elliptic boundary value problems are found in [59] and for higher order problems in [96]. The question of smoothness has been studied in [117]. There it is shown that Hadamard's formula holds for $\Omega \in C^{1,1}$, whereas for the second derivative Ω is required to be in $C^{2,\alpha}$, $0 < \alpha < 1$. It should be pointed out that Schiffer [104] transformed Hadamard's formula for plane domains such that it holds without any restriction on the boundary.

2 Basic concepts

The concept of domain perturbation will be introduced. For a bounded smooth domain we define a one-parameter family of smooth diffeomorphisms. This leads to a family of smooth domains which are called “perturbed domains”. In the following we will consider special diffeomorphisms, for example those perturbing the initial domain only in the normal direction (Hadamard perturbations). We show that any domain perturbation can be chosen to be a Hadamard perturbation.

An application of domain variations are the volume and surface functionals. The resulting variational formulas are frequently applied in this text.

We review some differential geometric quantities of surface theory which later appear explicitly in variational formulas.

2.1 Domain perturbations

We will use the following notation:

$$
\begin{aligned}
x := (x_1, x_2, \dots, x_n)&: \quad \text{point in } \mathbb{R}^n\text{, expressed in Cartesian coordinates,}\\
(x \cdot y) \quad \text{or} \quad x \cdot y&: \quad \text{Euclidean scalar product,}\\
|x| = (x \cdot x)^{1/2}&: \quad \text{length of a vector,}\\
dx = \prod_{i=1}^{n} dx_i&: \quad \text{volume element in} \mathbb{R}^n,\\
e_i = (0, \dots, 0, 1, 0, \dots), \quad i = 1, 2, \dots, n&: \quad \text{coordinate axis.}
\end{aligned}
$$

The symbol $o(t)$ stands for expressions in t such that $\frac{o(t)}{t} \to 0$ as $t \to 0$ and $O(t)$ means that $|\frac{O(t)}{t}|$ is bounded as $t \to 0$. We shall always use the Einstein convention that the repeated indices are being summed over.

Throughout this text we assume that $\Omega \subset \mathbb{R}^n$ is a connected bounded smooth domain. The exact smoothness will be determined when needed.

For given $t_0 > 0$ and $t \in (-t_0, t_0)$, let $\{\Omega_t\}_{|t|<t_0}$ be a family of perturbations of the domain $\Omega \subset \mathbb{R}^n$ of the form

$$\Omega_t = \Phi_t(\Omega),$$

where

$$\Phi_t : \overline{\Omega} \to \mathbb{R}^n$$

is smooth in t and x and $\Phi_0(\Omega) = \Omega$. In the literature these perturbations are often called *domain variations*. We will also write $y := \Phi_t(x)$ for short.

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Let

$$\Omega_t := \left\{ y = x + tv(x) + \frac{t^2}{2} w(x) + o(t^2) : x \in \Omega,\ |t| < t_0 \right\}, \tag{2.1.1}$$

where $v = (v_1(x), v_2(x), \dots, v_n(x))$ and $w = (w_1(x), w_2(x), \dots, w_n(x))$ are smooth vector fields. The Jacobi matrix corresponding to the transformation Φ_t is

$$D_{\Phi_t} = I + tD_v + \frac{t^2}{2} D_w + o(t^2), \quad \text{where } (D_v)_{ij} = \partial_i v_j \text{ and } \partial_j = \partial/\partial x_j. \tag{2.1.2}$$

Let $t_0 > 0$ be such that its inverse exists for all $|t| < t_0$ and is of the form

$$(D_{\Phi_t})^{-1} = I - tD_v + \frac{t^2}{2}(2(D_v)^2 - D_w) + o(t^2), \tag{2.1.3}$$

where $(D_v)^2 = \partial_i v_k \partial_k v_j$. Thus, for $t = 0$ we obtain

$$\begin{aligned} &\frac{d}{dt}(D_{\Phi_t}(x))_{ij}\bigg|_{t=0} = \partial_i v_j, \quad \frac{d}{dt}(D_{\Phi_t}(x))^{-1}_{ij}\bigg|_{t=0} = -\partial_i v_j, \\ &\frac{d^2}{dt^2}(D_{\Phi_t}(x))^{-1}_{ij}\bigg|_{t=0} = 2\partial_i v_k \partial_k v_j - \partial_i w_j. \end{aligned} \tag{2.1.4}$$

After the change of coordinates the volume element dy has to be replaced by $J(t)\,dx$, where $J(t) = \det(D_{\Phi_t})$. By Jacobi's formula (see Appendix A.1), we have for small t_0

$$\begin{aligned} J(t) &:= \det\left(I + tD_v + \frac{t^2}{2} D_w + o(t^2) \right) \\ &= 1 + t \operatorname{div} v + \frac{t^2}{2}((\operatorname{div} v)^2 - D_v : D_v + \operatorname{div} w) + o(t^2), \end{aligned} \tag{2.1.5}$$

where

$$D_v : D_v := \partial_i v_j\, \partial_j v_i.$$

Observe that $D_v : D_v$ is the trace of D_v^2. Clearly, $J(t) \neq 0$ if t_0 is small and the map $\Phi_t : \Omega \to \Omega_t$ is therefore a local diffeomorphism by the inverse function theorem. Its inverse will be denoted by Φ_t^{-1}.

Next we will show that $\Phi_t : \Omega \to \Omega_t$ is a bijective map. For this purpose we introduce the function

$$\hat{\Phi}_t(x) = \frac{1}{t}(\Phi_t(x) - x).$$

Lemma 2.1. *Assume that there exists a positive number $c < 1/t_0$ such that*

$$\|D_{\hat{\Phi}_t}\|_{L^\infty(\Omega)} \le c \quad \forall\, |t| < t_0.$$

Then $\Phi_t(x) = \Phi_t(z)$ implies that $x = z$.

Proof. The proof is done by contradiction. We assume that for some fixed t with $|t| < t_0$ there exist two points $x, z \in \Omega$ such that

$$\Phi_t(x) = \Phi_t(z) \quad \text{and} \quad |x - z| > 0.$$

Hence,

$$\begin{aligned} |x - z| &= \left|-t(\hat{\Phi}_t(x) - \hat{\Phi}_t(z))\right| \\ &\le ct_0|x - z|. \end{aligned}$$

The choice $ct_0 < 1$ contradicts $|x - y| > 0$. □

2.2 Geometry of hypersurfaces

2.2.1 Preliminaries

The boundary $\partial\Omega$ will be represented by local coordinates as follows. Let $V \subset \mathbb{R}^n$ be an open set such that

$$V \cap \partial\Omega := \{\tilde{x}(\xi) : \xi \in U \subset \mathbb{R}^{n-1}\}.$$

The metric is given by the *first fundamental form*

$$g_{ij}\, d\xi_i\, d\xi_j := (\tilde{x}_{\xi_i} \cdot \tilde{x}_{\xi_j})\, d\xi_i\, d\xi_j.$$

The inverse of the *metric tensor* g_{ij} will be denoted by g^{ij}. The surface element of $\partial\Omega$ is

$$dS = \sqrt{\det G}\, d\xi \quad \text{with } G = (g_{ij})_{i,j=1,\dots,n-1},$$

where $d\xi = d\xi_1 \cdot \dots \cdot d\xi_{n-1}$. The vectors $\tilde{x}_{\xi_i}$, $i = 1, 2, \dots, n-1$, span the tangent space in a point $\tilde{x}(\xi)$ and ν stands for the unit outer normal at a point $\tilde{x}(\xi) \in \partial\Omega$. In the Euclidean metric this is expressed as $(\nu \cdot \tilde{x}_{\xi_i}) = 0$ for $i = 1, \dots, n-1$ and $(\nu \cdot \nu) = 1$. We shall use the notation $\tilde{\nu}(\xi) := \nu(\tilde{x}(\xi))$.

The integral of a given continuous function $f : \partial\Omega \to \mathbb{R}$ over $V \cap \partial\Omega$ is given by

$$\oint\limits_{V\cap\partial\Omega} f(x)\, dS = \int\limits_U f(\tilde{x}(\xi))\sqrt{\det G(\xi)}\, d\xi = \int\limits_U \tilde{f}(\xi)\sqrt{\det G(\xi)}\, d\xi.$$

Since $\partial\Omega$ is compact, there exists a finite covering $\{V_i\}_{i=1}^{i_0}$ such that $\partial\Omega \subset \cup_{i=1}^{i_0} V_i \cap \partial\Omega$. Let $\{p_i\}_{i=1}^{i_0}$, $p_i \in C_0^\infty(\mathbb{R}^n)$ be a partition of unity subordinate to the covering $\{\partial\Omega \cap V_i\}_{i=0}^{i_0}$, i. e.,

1. $p_i : \mathbb{R}^n \to [0,1]$,
2. $\operatorname{supp}\{p_i\} \subset V_i$,
3. for any $x \in \overline{\Omega}$, $\sum_{i=1}^{i_0} p_i(x) \equiv 1$.

For simplicity we shall write

$$\oint_{\partial\Omega} f(x)\,dS := \sum_{i=1}^{i_0} \oint_{V_i \cap \partial\Omega} p_i(x) f(x)\,dS.$$

Next we measure the deviation of the surface from its tangent space. Assume that $\tilde{x}(\xi)$ is in $C^2(\partial\Omega)$. Then

$$\tilde{x}(\xi) - \tilde{x}(0) = \tilde{x}_{\xi_i}(0)\xi_i + \frac{1}{2}\tilde{x}_{\xi_i\xi_j}(0)\xi_i\xi_j + o(|\xi|^2)$$

and

$$((\tilde{x}(\xi) - \tilde{x}(0)) \cdot \tilde{\nu}(0)) = \frac{1}{2}(\tilde{x}_{\xi_i\xi_j}(0) \cdot \tilde{\nu}(0))\xi_i\xi_j + o(|\xi|^2)$$

The expression

$$L_{ij} := -\tilde{x}_{\xi_i\xi_j} \cdot \tilde{\nu} = \frac{1}{2}(\tilde{\nu}_{\xi_j} \cdot \tilde{x}_{\xi_i} + \tilde{\nu}_{\xi_i} \cdot \tilde{x}_{\xi_j}) \tag{2.2.1}$$

is called the *second fundamental form*. The symmetric matrix $\mathcal{L}$ with elements

$$\mathcal{L}_{ij} = g^{ik} L_{kj} \tag{2.2.2}$$

is called the *Weingarten operator*. Its eigenvalues are the *principal curvatures* κ_i of $\partial\Omega$. Denote by $T(\mathcal{L}) := g^{ik} L_{ki}$ the trace of $\mathcal{L}$.

The *mean curvature* H is the arithmetic mean of its trace

$$H := \frac{1}{n-1} T(\mathcal{L}) = \frac{1}{n-1} g^{ik} L_{ki} = \frac{1}{n-1} \sum_{i=1}^{n-1} \kappa_i. \tag{2.2.3}$$

Finally we like to state Weingarten's equations:

$$\tilde{\nu}_{\xi_i} = \mathcal{L}_{ik}\tilde{x}_{\xi_k} = g^{kj} L_{ji} \tilde{x}_{\xi_k}. \tag{2.2.4}$$

Local orthonormal frame

It is often convenient to use an orthonormal frame, the origin of which lies on the boundary, where the orthonormal axes $\{e_i\}_{i=1}^{n-1}$ span the tangent space and e_n points in the direction of the outer normal (see Figure 2.1). Then for any point $p \in \partial\Omega$ there exist a

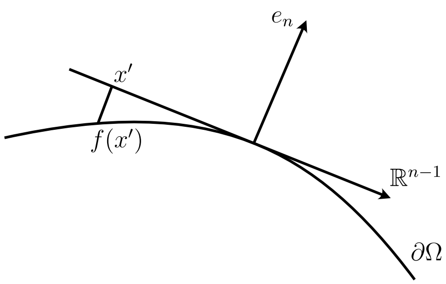

Figure 2.1: Local orthonormal frame.

neighborhood $V = V(0) \subset \mathbb{R}^n$, an open set $U = U(0) \subset \mathbb{R}^{n-1}$, and a suitable translation and rotation such that

$$\begin{aligned}\partial\Omega \cap V = \big\{(x', f(x')) : x' := (x_1, x_2, \dots, x_{n-1}) \subset U(0), f(0) = 0,\\ \partial_j f(0) = 0 \quad \text{for } j \in \{1, \dots, n-1\}\big\}.\end{aligned}$$

In this case $g^{ij}(0) = \delta_{ij}$. Consequently,

$$-\partial_i \partial_j \tilde{x}(x')\big|_{x'=0} = -\partial_i \partial_j f(0) e_n, \quad i, j \in \{1, \dots, n-1\}.$$

Multiplication by $\nu = e_n$ together with the definition of the Weingarten operator $\mathcal{L}$ (see (2.2.2)) yields $\mathcal{L}_{ij} = L_{ij} = -\partial_i \partial_j f(0)$.

Example 2.1. In this example L_{ij} and H will be computes for the sphere ∂B_R. For a suitable subset $U \subset \mathbb{R}^{n-1}$ we have $|\tilde{x}(\xi)| = R$ if $\xi \in U$. Twice differentiation with respect to ξ results in

$$0 = (\tilde{x}_{\xi_i \xi_j} \cdot \tilde{x}) + (\tilde{x}_{\xi_i} \cdot \tilde{x}_{\xi_j}) = (\tilde{x}_{\xi_i \xi_j} \cdot \tilde{\nu})R + g_{ij}.$$

Thus,

$$L_{ij} = -(\tilde{x}_{\xi_i \xi_j} \cdot \tilde{\nu}) = \frac{1}{R} g_{ij} \tag{2.2.5}$$

and

$$H = \frac{1}{n-1} g^{ij} L_{ji} = \frac{1}{(n-1)R} g^{ij} g_{ji} = \frac{1}{R}. \tag{2.2.6}$$

Let $f : \partial\Omega \to \mathbb{R}$ be a smooth function. Then

$$\nabla^\tau \tilde{f} := g^{ij} \frac{\partial \tilde{f}}{\partial \xi_j} \tilde{x}_{\xi_i} \tag{2.2.7}$$

is called the *tangential gradient* of a function $\tilde{f}(\xi) := f(\tilde{x}(\xi))$.

Any vector $v : \partial\Omega \to \mathbb{R}^n$ can be decomposed into $v = c_0\nu + \sum_{i=1}^{n-1} c_i \tilde{x}_{\xi_i}$. Multiplying successively by ν and $\tilde{x}_{\xi_i}$ we find

$$v = (v \cdot \nu)\nu + g^{ij}(v \cdot \tilde{x}_{\xi_i})\tilde{x}_{\xi_j} \quad \text{and} \quad (v \cdot w) = (v \cdot \nu)(w \cdot \nu) + g^{ij}(v \cdot \tilde{x}_{\xi_i})(w \cdot \tilde{x}_{\xi_j}), \tag{2.2.8}$$

where $w : \partial\Omega \to \mathbb{R}^n$. As already emphasized before, we use the Einstein convention and sum over $i, j = 1, 2, \dots, n-1$.

For a smooth vector field $v : \partial\Omega \to \mathbb{R}^n$, which is not necessarily tangent to $\partial\Omega$, we define the *tangential divergence* by

$$\operatorname{div}_{\partial\Omega} v := g^{ij}(\tilde{v}_{\xi_j} \cdot \tilde{x}_{\xi_i}), \quad \text{where } \tilde{v} = v(\tilde{x}(\xi)). \tag{2.2.9}$$

For $v = \nu$ we get

$$\operatorname{div}_{\partial\Omega} \nu = (n-1)H. \tag{2.2.10}$$

This follows immediately from (2.2.1), (2.2.3), and (2.2.9). In particular, for the ball B_R of radius R centered at the origin we have $H = \frac{1}{R}$.

2.2.2 Projections

Since $\partial\Omega$ is embedded in $\mathbb{R}^n$ we can introduce the projection operator

$$P : \mathbb{R}^n \to T_x\partial\Omega, \quad P(v) := v - (v \cdot \nu)\nu := v^\tau, \quad v = v(x),$$

where $T_x\partial\Omega$ stands for the tangent space of $\partial\Omega$ in x. Consider a function $f : \partial\Omega \to \mathbb{R}$ and let $\hat{f}$ be a C^1-extension to Ω. Such an extension is not uniquely defined. Then the tangential gradient of a scalar function f on $\partial\Omega$ is given by

$$\nabla^\tau f := P(\nabla\hat{f}) = \nabla_x\hat{f} - (\nu \cdot \nabla_x\hat{f})\nu. \tag{2.2.11}$$

Note that the tangential gradient is independent of the extension $\hat{f}$. This definition is equivalent to (2.2.7) in the following sense: We choose any point $x_0 \in \partial\Omega$ and translate and rotate Ω such that

$$x_0 = 0 \quad \text{and} \quad \nu(0) = e_n = (0, \dots, 0, 1).$$

From (2.2.11) we deduce

$$\nabla^\tau f(0) = (\partial_1 f(0), \dots, \partial_{n-1} f(0), 0), \tag{2.2.12}$$

which proves the claim.

The decomposition (2.2.8) shows that $\nabla\hat{f}$ in $\mathbb{R}^n$ can be written as

$$\nabla\hat{f} = (\nabla\hat{f}\cdot\nu)\nu + g^{ij}(\nabla\hat{f}\cdot\tilde{x}_{\xi_i})\tilde{x}_{\xi_j}.$$

Thus,

$$\nabla^\tau\tilde{f} = g^{ij}(0)(\nabla f\cdot\tilde{x}_{\xi_i})\tilde{x}_{\xi_j}, \tag{2.2.13}$$

where we sum over $i, j = 1, \dots, n-1$.

For any smooth vector field $v : \partial\Omega \to \mathbb{R}^n$ we define

$$\operatorname{div}_{\partial\Omega}\hat{v} := \operatorname{div}\hat{v} - (\nu\cdot D_{\hat{v}}\nu) = \partial_i\hat{v}_i - \nu_j\partial_j\hat{v}_i\nu_i, \tag{2.2.14}$$

where $\hat{v} : \Omega \to \mathbb{R}^n$ is any C^1-extension of v and $\hat{v}_i$ is the i-th coordinate with respect to the coordinate system in $\mathbb{R}^n$. Note that tangential differential operators only depend on $v|_{\partial\Omega}$ and not on the extension $\hat{v}$. Definition (2.2.14) is equivalent to (2.2.9).

The next identity will often be used. Let $f : \partial\Omega \to \mathbb{R}$. Then

$$\operatorname{div}_{\partial\Omega}(fv) = \left(v\cdot\nabla^\tau f\right) + f\operatorname{div}_{\partial\Omega} v. \tag{2.2.15}$$

As an application of (2.2.15) we obtain

$$\operatorname{div}_{\partial\Omega}\left[(v\cdot\nu)\nu\right] = \nu\cdot\nabla^\tau(v\cdot\nu) + (v\cdot\nu)\operatorname{div}_{\partial\Omega}\nu.$$

By (2.2.10) this leads to

$$\operatorname{div}_{\partial\Omega}\left[(v\cdot\nu)\nu\right] = (n-1)(v\cdot\nu)H. \tag{2.2.16}$$

The *Laplace–Beltrami operator* on $\partial\Omega$ is defined as

$$\Delta^* := \operatorname{div}_{\partial\Omega}\nabla^\tau. \tag{2.2.17}$$

Its representation in terms of local coordinates is given in Appendix B.1.

We will frequently use integration by parts on $\partial\Omega$. Assume $f \in C^1(\partial\Omega)$ and $v \in C^{0,1}(\partial\Omega, \mathbb{R}^n)$. Then the Gauss theorem on surfaces has the form

$$\oint_{\partial\Omega} f\operatorname{div}_{\partial\Omega} v\,dS = -\oint_{\partial\Omega}\left(v\cdot\nabla^\tau f\right)dS + (n-1)\oint_{\partial\Omega} f(v\cdot\nu)H\,dS. \tag{2.2.18}$$

2.2.3 Reduction to Hadamard perturbations

In this section we first show that the boundary displacement under the perturbation Φ_t can be described by a vector field on $\partial\Omega$ whose first order approximation points in the normal direction.

Definition 2.1. Domain perturbations for which on $\partial\Omega$ the vector field v points in the normal direction, i. e., $v = (v \cdot \nu)\nu$, are called Hadamard perturbations or Hadamard variations.

Note that no restriction is imposed on the second order perturbation w.

To achieve our goal we will proceed as follows:

- For $|t| < t_0$ let Φ_t be a smooth family of diffeomorphisms as described in Section 2.1. We denote by
$$\tilde{\Phi}_t := \Phi_t|_{\partial\Omega} : \partial\Omega \to \partial\Omega_t$$
the restriction of Φ_t to $\partial\Omega$. Furthermore, $\tilde{x}(\xi)$ is a local parametrization of $\partial\Omega$.
- On $\partial\Omega$ we introduce new local coordinates $\hat{\xi} = \varphi_t(\xi)$ such that $\tilde{\Psi}_t = \tilde{\Phi}_t(\xi)$ satisfies
$$\partial_t \tilde{\Psi}_t|_{t=0} = (\partial_t \tilde{\Psi}_t|_{t=0} \cdot \nu)\nu \quad \text{on } \partial\Omega,$$
where $\tilde{\Psi}_t : \partial\Omega \to \partial\Omega_t$ is a Hadamard perturbation.
- We then apply the extension theorem (Theorem A.2) and obtain a diffeomorphism $\Psi_t : \overline{\Omega} \to \overline{\Omega_t}$ such that $\Psi_t|_{\partial\Omega} = \tilde{\Psi}_t$.

Assume that $\partial\Omega \in C^2$ and consider on $\partial\Omega$ a perturbation of the form
$$\tilde{\Phi}_t(\xi) = \tilde{x}(\xi) + t\tilde{v}(\xi) + \frac{t^2}{2}\tilde{w}(\xi) + o(t^2),$$
where $\xi \in U \subset \mathbb{R}^{n-1}$. Here we require $\tilde{v} \in C^1$ and $\tilde{w} \in C^0$. We decompose v into its normal and tangential components. By (2.2.8) we have $v = (v \cdot \nu)\nu + v^\tau$, where, in local coordinates, the tangential part is given by
$$v^\tau = g^{ij}(v \cdot \tilde{x}_{\xi_i})\tilde{x}_{\xi_j}. \tag{2.2.19}$$
We denote by v_k^τ the k-th coordinate of v^τ in $\mathbb{R}^n$.

Next we show that the parametrization φ_t can be chosen such that the perturbation $\tilde{\Phi}_t(\xi)$ can be transformed into a Hadamard perturbation $\tilde{\Psi}_t$.

Theorem 2.1. *Let $\tilde{\Phi}_t : V \cap \partial\Omega \to \partial\Omega_t$ be a diffeomorphism given locally by $\tilde{\Phi}_t(\xi) = \tilde{x}(\xi) + t\tilde{v}(\xi) + \frac{t^2}{2}\tilde{w}(\xi) + o(t^2)$ for all $\xi \in U \subset \mathbb{R}^{n-1}$ and $|t| < t_0$. Then there exists a diffeomorphism $\varphi_t : U \to V$ of the form $\varphi_t(\xi) = \xi + t\eta(\xi) =: \hat{\xi}$ such that (see Figure 2.2)*
$$\Psi_t(\hat{\xi}) := \Phi_t \circ \varphi_t^{-1} = x(\hat{\xi}) + t\rho(\hat{\xi})\nu(\hat{\xi}) + \frac{t^2}{2}z(\hat{\xi}) + o(t^2), \quad \hat{\xi} \in V \subset \partial\Omega,$$
is a Hadamard perturbation. Moreover, the following identities hold:
$$\eta_i(\xi) = g^{ij}(\xi)\tilde{v}_j^\tau(\xi) \quad \textit{and} \quad \tilde{\rho}(\hat{\xi}) = \big(\tilde{v}(\xi) \cdot \tilde{\nu}(\xi)\big)\big|_{\xi=\varphi^{-1}(\hat{\xi})}$$

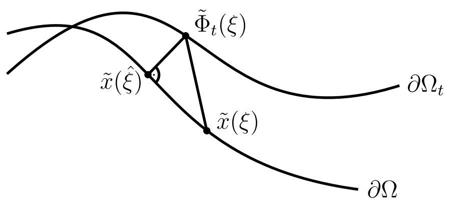

Figure 2.2: Hadamard reduction.

and

$$z(\hat{\xi}) = \tilde{w}(\xi) - \tilde{x}_{\xi_i\xi_j}(\xi)g^{ik}(\xi)\tilde{v}_k^\tau(\xi)g^{jl}(\xi)\tilde{v}_l^\tau(\xi) - 2(\tilde{v}(\xi)\cdot\tilde{\nu}(\xi))_{\xi_i}g^{ik}(\xi)\tilde{v}_k^\tau(\xi)\tilde{\nu}(\xi)$$
$$- 2(\tilde{v}(\xi)\cdot\tilde{\nu}(\xi))g^{ik}(\xi)\tilde{v}_k^\tau(\xi)\tilde{\nu}_{\xi_i}(\xi), \quad \textit{where } \xi = \varphi^{-1}(\hat{\xi}).$$

Proof. 1. We look for $\hat{\xi} = \varphi_t$ of the form $\xi + t\eta(\xi) + o(t)$. Let $\eta_i(\xi)$ be the i-th component of η. Expansions with respect to t lead to

$$x(\varphi_t) = \tilde{x}(\xi) + t\eta_i(\xi)\tilde{x}_{\xi_i}(\xi) + \frac{t^2}{2}[\tilde{x}_{\xi_i\xi_j}(\xi)\eta_i(\xi)\eta_j(\xi)] + o(t^2),$$
$$\rho(\varphi_t) = \tilde{\rho}(\xi) + t\eta_i(\xi)\tilde{\rho}_{\xi_i}(\xi) + o(t),$$
$$\nu(\varphi_t) = \tilde{\nu}(\xi) + t\eta_i(\xi)\tilde{\nu}_{\xi_i}(\xi) + o(t)$$

and

$$z(\varphi_t) = \tilde{z}(\xi + t\eta(\xi)) = \tilde{z}(\xi) + t\eta_i(\xi)\tilde{z}_{\xi_i}(\xi) + o(t).$$

2. This leads to the following expansion of $\tilde{\Psi}_t(\xi) = \Psi_t \circ \varphi_t(\xi)$:

$$\Psi_t \circ \varphi_t(\xi) = \tilde{x}(\xi) + t\eta_i(\xi)\tilde{x}_{\xi_i}(\xi) + \frac{t^2}{2}[\tilde{x}_{\xi_i\xi_j}(\xi)\eta_i(\xi)\eta_j(\xi)]$$
$$+ t\{\tilde{\rho}(\xi)\tilde{\nu}(\xi) + t[\eta_i(\xi)\tilde{\rho}_{\xi_i}(\xi)\tilde{\nu}(\xi) + \tilde{\rho}(\xi)\eta_i(\xi)\tilde{\nu}_{\xi_i}(\xi)]\} + \frac{t^2}{2}\tilde{z}(\xi).$$

For the identity $\tilde{\Phi}_t(\xi) = \tilde{\Psi}_t \circ \varphi_t(\xi) + o(t^2)$ to hold, the following identities up to the order $o(t^2)$ have to be satisfied:

$$t: \quad \tilde{v}(\xi) = \eta_i(\xi)\tilde{x}_{\xi_i}(\xi) + \tilde{\rho}(\xi)\tilde{\nu}(\xi),$$
$$\frac{t^2}{2}: \quad \tilde{w}(\xi) = \tilde{x}_{\xi_i\xi_j}(\xi)\eta_i(\xi)\eta_j(\xi) + 2[\eta_i(\xi)\tilde{\rho}_{\xi_i}(\xi)\tilde{\nu}(\xi) + \tilde{\rho}(\xi)\eta_i(\xi)\tilde{\nu}_{\xi_i}(\xi)] + \tilde{z}(\xi) + o(t^2).$$

If we multiply the first equation by $\tilde{\nu}(\xi)$, we obtain $\tilde{\rho}(\xi) = (\tilde{v}(\xi)\cdot\tilde{\nu}(\xi))$. Then we multiply the same equation by $\tilde{x}_{\xi_j}(\xi)$ and find $\eta_i(\xi) = g^{ij}(\xi)\tilde{v}_j^\tau(\xi)$. Inserting these identities into the second equation we obtain the formula for $z(\hat{\xi})$. The extension theorem (Theorem A.2) then yields a diffeomorphism $\Psi_t : \overline{\Omega} \to \overline{\Omega_t}$ which is a Hadamard perturbation. This establishes the theorem. □

Proposition 2.1. *Let $\tilde{\Phi}_t : \partial\Omega \to \partial\Omega_t$ be a diffeomorphism given locally by $\Phi_t = \tilde{x}(\xi) + t\tilde{v}(\xi) + \frac{t^2}{2}\tilde{w}(\xi)$, where $\xi \in U$ are local coordinates and $|t| < t_0$. Then there exist a parametrization $\varphi_t : U \to U$ with $\varphi_t(\xi) = \xi + t\eta(\xi) + \frac{t^2}{2}\vartheta(\xi)$ and a Hadamard perturbation*

$$\tilde{\Psi}_t(\varphi_t) = \tilde{x}(\varphi_t) + t\tilde{\rho}(\varphi_t)\tilde{\nu}(\varphi_t) + \frac{t^2}{2}(\tilde{z}(\varphi_t) \cdot \tilde{\nu}(\varphi_t))\tilde{\nu}(\varphi_t) + o(t^2), \quad \varphi_t \in U,$$

such that $\tilde{\Phi}_t(\xi) = \tilde{\Psi}_t \circ \varphi_t(\xi) + o(t^2)$ for all $\xi \in U$. As in Theorem 2.1 the following identities hold:

$$\tilde{\rho}(\xi) = \tilde{v}(\xi) \cdot \tilde{\nu}(\xi), \quad \eta_i(\xi) = g^{ij}(\xi)\tilde{v}_j^\tau(\xi).$$

If the coordinates of ϑ are

$$\begin{aligned}\vartheta_i(\xi) = &-g^{ij}(\xi)\tilde{x}_{\xi_j} \cdot \tilde{w}(\xi) + g^{im}(\xi)\big(\tilde{x}_{\xi_m}(\xi) \cdot \tilde{x}_{\xi_r\xi_s}(\xi)\big)g^{rk}(\xi)\tilde{v}_k^\tau(\xi)g^{sl}(\xi)\tilde{v}_l^\tau(\xi)\\ &+ 2(\tilde{v}(\xi) \cdot \tilde{\nu}(\xi))g^{lk}(\xi)\tilde{v}_k^\tau(\xi)\big(\nu_{\xi_l}(\xi) \cdot \tilde{x}_{\xi_m}(\xi)\big)g^{im}(\xi),\end{aligned}$$

then $(\tilde{z}(\varphi_t) \cdot \tilde{\nu}(\varphi_t))\tilde{\nu}(\varphi_t) = (\tilde{z}(\xi) \cdot \tilde{\nu}(\xi))\tilde{\nu}(\xi) + O(t)$, where

$$\begin{aligned}\big(\tilde{z}(\xi) \cdot \tilde{\nu}(\xi)\big) = &\big(\tilde{w}(\xi) \cdot \tilde{\nu}(\xi)\big) + L_{ij}(\xi)g^{ik}(\xi)\tilde{v}_k^\tau(\xi)g^{jl}(\xi)\tilde{v}_l^\tau(\xi)\\ &- 2g^{ik}(\xi)\big(\tilde{v}(\xi) \cdot \tilde{\nu}(\xi)\big)_{\xi_i}\tilde{v}_k^\tau(\xi).\end{aligned}$$

Thus, an additional higher order term in the expansion of $\varphi_t(\xi)$ yields a vector field $\tilde{z}$ which points in the normal direction. More generally the following remark holds.

Remark 2.1. The reduction to Hadamard perturbation is valid for any order of t. This can be seen by applying an induction argument. Assume that

$$\tilde{\Phi}_t(\xi) = \tilde{x}(\xi) + \sum_{k=1}^{N} \frac{t^k}{k!}\tilde{\rho}^{(k)}(\xi)\tilde{\nu}(\xi) + \frac{t^{N+1}}{(N+1)!}\tilde{w}(\xi) + o\big(t^{N+1}\big)$$

for some functions $\tilde{\rho}^{(k)}$. Then the parametrization

$$\varphi_t(\xi) = \xi + \frac{t^{N+1}}{(N+1)!}\eta^{(N+1)}(\xi), \quad \xi \in U \subset \mathbb{R}^{n-1},$$

yields a diffeomorphism

$$\Psi_t(\varphi) = x(\varphi) + \sum_{k=1}^{N+1} \frac{t^k}{k!}\rho^{(k)}(\varphi)\nu(\varphi) + o\big(t^{N+1}\big),$$

where

$$\tilde{\rho}^{(N+1)} = \tilde{w}(\xi)\cdot\tilde{v}(\xi) \quad \text{and} \quad \eta_i^{(N+1)}(\xi) = g^{ij}(\xi)\tilde{w}_j^{\tau}(\xi).$$

Clearly, these computations require higher regularity of the boundary, e. g., $\partial\Omega \in C^{N+2}$.

The following example will play a major role in the sequel.

Example 2.2. Let $\partial\Omega = \partial B_R$ and let $\tilde{x} : U \subset \mathbb{R}^{n-1} \to \mathbb{R}^n$ be any local parametrization such that $|\tilde{x}(\xi)|^2 = R^2$ for all $\xi \in U$. Then

$$\tilde{v}(\xi) = \frac{1}{R}\tilde{x}(\xi), \quad g_{ij}(\xi) = \tilde{x}_{\xi_i}(\xi)\cdot\tilde{x}_{\xi_j}(\xi), \quad L_{ij}(\xi) = -\tilde{v}(\xi)\cdot\tilde{x}_{\xi_i\xi_j}(\xi) = \frac{1}{R}g_{ij}(\xi).$$

For small t_0 we consider the diffeomorphism $\tilde{\Phi}_t : \partial B_R \to \partial\Omega_t$ given locally by its expansion $\tilde{\Phi}_t(\xi) = \tilde{x}(\xi) + t\tilde{v}(\xi) + \frac{t^2}{2}\tilde{w}(\xi) + o(t^2)$ for all $\xi \in U$. By Proposition 2.1 there exist a Hadamard variation $\tilde{\Psi}_t$ and a parametrization φ_t such that $\tilde{\Phi}_t(\xi) = \Psi_t \circ \varphi_t(\xi) + o(t^2)$ for all $\xi \in U$ and

$$\tilde{\Psi}_t(\xi) = \tilde{x}(\xi) + t\tilde{\rho}(\xi)\tilde{v}(\xi) + \frac{t^2}{2}(\tilde{z}(\xi)\cdot\tilde{v}(\xi))\tilde{v}(\xi), \quad \xi \in U, \tag{2.2.20}$$

where

$$\tilde{\rho}(\xi) = \tilde{v}(\xi)\cdot\tilde{v}(\xi)$$

and

$$\tilde{z}(\xi)\cdot\tilde{v}(\xi) = \big(\tilde{w}(\xi)\cdot\tilde{v}(\xi)\big) + \frac{1}{R}g^{kl}(\xi)\tilde{v}_k^{\tau}(\xi)\tilde{v}_l^{\tau}(\xi) - 2g^{jk}(\xi)\tilde{\rho}_{\xi_j}(\xi)\tilde{v}_k^{\tau}(\xi).$$

For later use we will express $\tilde{\Psi}_t$ in terms of Cartesian coordinates. By means of (2.2.7) and (2.2.8) we obtain

$$\begin{aligned}\Psi_t(x) &= x + t\rho(x)\nu(x)\\ &\quad + \frac{t^2}{2}\bigg(\frac{1}{R}\big|v^{\tau}(x)\big|^2 - 2v^{\tau}(x)\cdot\nabla^{\tau}\rho(x) + w(x)\cdot\nu(x)\bigg)\nu(x) + o\big(t^2\big)\end{aligned} \tag{2.2.21}$$

for $x \in \partial B_R$.

2.3 First and second variation of the volume

We start with the volume $\mathcal{V}(t)$ of Ω_t for small t_0. After a change of variables $x = \Phi_t^{-1}(y)$, it assumes the form

$$\mathcal{V}(t) := |\Omega_t| = \int_\Omega J(t)\,dx. \tag{2.3.1}$$

From (2.1.5) it follows that for small t

$$\mathcal{V}(t) - \mathcal{V}(0) = t\int_\Omega \operatorname{div} v\,dx + \frac{t^2}{2}\int_\Omega \big((\operatorname{div} v)^2 - D_v : D_v + \operatorname{div} w\big)\,dx + o(t^2).$$

This expansion enables us to compute the first variation $\lim_{t\to 0}\frac{\mathcal{V}(t)-\mathcal{V}(0)}{t}$ of $\mathcal{V}(t)$ at the origin. In fact,

$$\dot{\mathcal{V}}(0) = \int_\Omega \operatorname{div} v\,dx = \oint_{\partial\Omega}(v\cdot\nu)\,dS. \tag{2.3.2}$$

The second variation of $\mathcal{V}(t)$ is

$$\ddot{\mathcal{V}}(0) = \int_\Omega \big((\operatorname{div} v)^2 - D_v : D_v + \operatorname{div} w\big)\,dx. \tag{2.3.3}$$

Since

$$\ddot{J}(0) = (\operatorname{div} v)^2 - D_v : D_v + \operatorname{div} w = \partial_j(v_j\partial_i v_i - v_i\partial_i v_j + w_j), \tag{2.3.4}$$

integration by parts implies that

$$\ddot{\mathcal{V}}(0) = \oint_{\partial\Omega}(v\cdot\nu)\operatorname{div} v\,dS - \oint_{\partial\Omega} v_i\partial_i v_j \nu_j\,dS + \oint_{\partial\Omega}(w\cdot\nu)\,dS. \tag{2.3.5}$$

By (2.2.14),

$$\begin{aligned}(v\cdot\nu)\operatorname{div} v - v_i\partial_i v_j\nu_j &= (v\cdot\nu)\operatorname{div}_{\partial\Omega} v + (v\cdot\nu)(\nu\cdot D_v\nu) - (v\cdot D_v\nu)\\ &= (v\cdot\nu)\operatorname{div}_{\partial\Omega} v - \underbrace{\big(v-(v\cdot\nu)\nu\big)}_{v^\tau}\cdot D_v\nu.\end{aligned} \tag{2.3.6}$$

If we insert this identity into (2.3.5), we obtain

$$\ddot{\mathcal{V}}(0) = \oint_{\partial\Omega}\big[(v\cdot\nu)\operatorname{div}_{\partial\Omega} v - (v^\tau\cdot D_v\nu) + (w\cdot\nu)\big]\,dS. \tag{2.3.7}$$

Note that in this second variation only the values of v and w on the boundary and the tangential derivatives of v appear. Therefore, $\ddot{V}(0)$ does not contain derivatives of the perturbation pointing inside the domain.

Throughout this text we shall use the notion of volume preserving perturbation in a broader sense.

Definition 2.2. The perturbation $\Phi_t(x)$ is called volume preserving if $\dot{\mathcal{V}}(0) = 0$ and $\ddot{\mathcal{V}}(0) = 0$.

In this text a domain perturbation will be called of *first order* if it is of the form

$$\Phi_t = x + tv(x) \quad \text{for } |t| < t_0.$$

It will be called of *second order* if

$$\Phi_t = x + tv(x) + \frac{t^2}{2} w(x).$$

Remark 2.2. In the literature different concepts of volume preserving perturbations are considered:

(i) the diffeomorphisms Φ_t for which $|\Omega_t| = |\Omega|$ for all $|t| < t_0$,
(ii) first order diffeomorphisms such that $|\Omega_t| = |\Omega| + o(t)$,
(iii) second order diffeomorphisms such that $|\Omega_t| = |\Omega| + o(t^2)$.

Diffeomorphisms of type (iii) satisfy Definition 2.2. Clearly, a perturbation of type (i) is also a perturbation of type (ii) and (iii), but not necessarily vice versa.

2.3.1 Discussion of $\ddot{\mathcal{V}}(0)$

Our goal is to express the second domain variation of the volume given in (2.3.7) in local coordinates. Using these coordinates we will see that $\ddot{\mathcal{V}}(0)$ depends on the second fundamental form. From (2.2.18) it follows that the first term of $\ddot{\mathcal{V}}(0)$ can be written as

$$\oint_{\partial\Omega} [(v \cdot \nu) \operatorname{div}_{\partial\Omega} v] \, dS = -\oint_{\partial\Omega} (v \cdot \nabla^\tau \rho) \, dS + (n-1) \oint_{\partial\Omega} \rho^2 H \, dS, \quad \rho = (v \cdot \nu).$$

The second term $(v^\tau \cdot D_v \nu)$ in (2.3.7) can be rewritten as

$$(v^\tau \cdot D_v \nu) = v^\tau \cdot \nabla(v \cdot \nu) - (v^\tau \cdot D_\nu v) = v^\tau \cdot \nabla^\tau (v \cdot \nu) - (v^\tau \cdot D_\nu v^\tau).$$

The following formula will be derived in Section 2.4

$$(D_\nu)_{ij} = \partial_i \nu_j = \mathcal{L}_{km} (\tilde{x}_{\xi_m})_i (\tilde{x}_{\xi_k})_j. \tag{2.3.8}$$

This together with (2.2.19) yields

$$(v^\tau \cdot D_\nu v^\tau) = (v \cdot \tilde{x}_{\xi_i}) g^{ij} L_{jl} (v \cdot \tilde{x}_{\xi_l}) = v^\tau \mathcal{L} v^\tau.$$

Thus,

$$(v^\tau \cdot D_v \nu) = v^\tau \cdot \nabla^\tau \rho - v^\tau \cdot \mathcal{L} v^\tau. \tag{2.3.9}$$

This leads to

$$\ddot{\mathcal{V}}(0) = \oint_{\partial\Omega} \left[(n-1)\rho^2 H + (v^\tau \cdot \mathcal{L} v^\tau) - 2(v^\tau \cdot \nabla^\tau \rho) + (w \cdot \nu)\right] dS. \tag{2.3.10}$$

Example 2.3. 1. If $\Omega = B_R$ the above formula (2.3.10) reads as

$$\begin{aligned} \ddot{\mathcal{V}}(0) = & -2 \oint_{\partial B_R} (v \cdot \nabla^\tau (v \cdot \nu))\, dS + \frac{n-1}{R} \oint_{\partial B_R} (v \cdot \nu)^2\, dS \\ & + \frac{1}{R} \oint_{\partial B_R} |v^\tau|^2\, dS + \oint_{\partial B_R} (w \cdot \nu)\, dS. \end{aligned} \tag{2.3.11}$$

For the computation of (2.3.11) we applied the identity $(D_\nu)_{ij} = \frac{1}{R}(\delta_{ij} - \nu_i \nu_j)$ on ∂B_R.

2. For Hadamard perturbations $v^\tau = 0$ we get

$$\ddot{\mathcal{V}}(0) = (n-1) \oint_{\partial\Omega} (v \cdot \nu)^2 H\, dS + \oint_{\partial\Omega} (w \cdot \nu)\, dS. \tag{2.3.12}$$

3. For tangential perturbations $\Phi_t = x + t v^\tau(x) + \frac{t^2}{2} w(x)$ we have $\dot{\mathcal{V}}(0) = 0$ and

$$\ddot{\mathcal{V}}(0) = \oint_{\partial\Omega} \left[v^\tau \cdot \mathcal{L} v^\tau + (w \cdot \nu)\right] dS. \tag{2.3.13}$$

In convex domains the quadratic form $v^\tau \mathcal{L} v^\tau$ is positive.

4. Let $v = (c_1 x_1, c_2 x_2, \dots, c_n x_n)$ and $w = 0$. Then by (2.3.2) and (2.3.3),

$$\dot{\mathcal{V}}(0) = |\Omega| \sum_{i=1}^n c_i, \quad \ddot{\mathcal{V}}(0) = |\Omega| \left(\left(\sum_{i=1}^n c_i \right)^2 - \sum_{i=1}^n c_i^2 \right).$$

5. Consider in a domain in the plane the perturbation $\Phi_t = x + t v(x)$, where $v = (-x_2, x_1)$ is a pure shear. Then $\operatorname{div} v = 0$ and $D_v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Consequently by (2.3.5), $\dot{\mathcal{V}}(0) = 0$ and by (2.3.3), $\ddot{\mathcal{V}}(0) = -\oint_{\partial\Omega} (x \cdot \nu)\, dS = -2|\Omega|$.

6. We replace Φ_t by a Hadamard perturbation (2.2.21) and compute $\ddot{\mathcal{V}}(0)$ for the ball. In (2.3.12) we replace $\rho(x) = (v(x) \cdot \nu)$ by $\rho(\hat{x})$ and $w(x)$ by

$$w(\hat{x}) + \left(\frac{|v^\tau(\hat{x})|^2}{R} - 2v^\tau(\hat{x}) \cdot \nabla^\tau \rho(\hat{x}) \right) \frac{\hat{x}}{R}.$$

Hence,

$$\ddot{\mathcal{V}}(0) = \oint_{\partial B_R} \left(\frac{n-1}{R}\rho^2 - 2(v^\tau \cdot \nabla^\tau \rho) + \frac{|v^\tau|^2}{R} + (w \cdot \nu) \right) dS. \tag{2.3.14}$$

7. The variations of the volume and the area can be used to derive local Steiner formulas for parallel sets Ω_ρ which are known for convex bodies (see for instance [70] for an elementary introduction). For small $|\rho|$ they read as

$$|\Omega_\rho| = |\Omega| + \rho|\partial\Omega| + (n-1)\frac{\rho^2}{2} \oint_{\partial\Omega_\rho} H\, dS + o(\rho^2).$$

Remark 2.3. In general a first order perturbation for which $\dot{\mathcal{V}}(0) = 0$ does not necessarily imply $\ddot{\mathcal{V}}(0) = 0$. It is easy to see that this can always be achieved by adding a term of the second order. In fact, consider the perturbations

$$\Phi_t^{(1)} = x + tv(x) \quad \text{and} \quad \Phi_t^{(2)} = x + \frac{t^2}{2}w(x) + o(t^2),$$
$$\Phi_t = \Phi_t^{(2)} \circ \Phi_t^{(1)} = x + tv(x) + \frac{t^2}{2}w + o(t^2).$$

Clearly, $\Phi_t^{(1)}$ is a first order perturbation, whereas $\Phi_t^{(2)}$ and Φ_t are second order perturbations. Let $\mathcal{V}^{(1)}(t) = |\Phi_t^{(1)}(\Omega)|$, $\mathcal{V}^{(2)}(t) = |\Phi_t^{(2)}(\Omega)|$, and $\mathcal{V}(t) = |\Phi_t(\Omega)|$. The domain variations of $\mathcal{V}(t)$, $\mathcal{V}^{(1)}(t) := \mathcal{V}(\Omega_t^{(1)})$, and $\mathcal{V}^{(2)}(t) := \mathcal{V}(\Omega_t^{(2)})$ are related as follows:

$$\dot{\mathcal{V}}(0) = \dot{\mathcal{V}}^{(1)}(0) \quad \text{and} \quad \ddot{\mathcal{V}}(0) = \ddot{\mathcal{V}}^{(1)}(0) + \ddot{\mathcal{V}}^{(2)}(0).$$

By choosing w appropriately we can achieve $\ddot{\mathcal{V}}(0) = 0$.

The role of w becomes apparent if we consider rotations. Consider a rotation in the (x_1, x_2)-plane. The polar angle θ_1 (see the representation of x in spherical coordinates in Appendix B.1.1) is replaced by $\theta_1 + t$. Then for small $|t|$, x is transformed into $x + t(-x_2, x_1, 0, \dots, 0) + \frac{t^2}{2}(-x_1, -x_2, 0, \dots, 0) + o(t^2)$. In this case,

$$\Phi_t^{(1)} = x + t(-x_2, x_1, 0, \dots, 0), \quad \Phi_t^{(2)} = x + \frac{t^2}{2}(-x_1, -x_2, 0, \dots, 0) + o(t^2) \quad \text{and}$$
$$\Phi_t = x + t(-x_2, x_1, 0, \dots, 0) + \frac{t^2}{2}(-x_1, -x_2, 0, \dots, 0) + o(t^2).$$

Hence, $\operatorname{div} v = 0$ implies that

$$\dot{\mathcal{V}}_1(0) = \dot{\mathcal{V}}_2(0) = \dot{\mathcal{V}}(0) = 0.$$

Moreover,

$$D_v = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

From (2.3.5) it follows that

$$\ddot{\mathcal{V}}_1(0) = 2 \oint_{\partial\Omega} (\nu_1 x_1 + \nu_2 x_2)\, dS = -\ddot{\mathcal{V}}_2(0) \neq 0.$$

Obviously, $\ddot{\mathcal{V}}(0) = 0$.

In conclusion, the second order variation of the rotated domain vanishes only if in the expansion of the rotation, second order terms are taken into account.

As an example consider the ellipse $E = (4\cos(\theta_1), 2\sin(\theta_1))$. The transformations of E under the perturbations $\Phi_t^{(1)}$ and Φ_t for $t = 0.5$ are pictured in Figure 2.3.

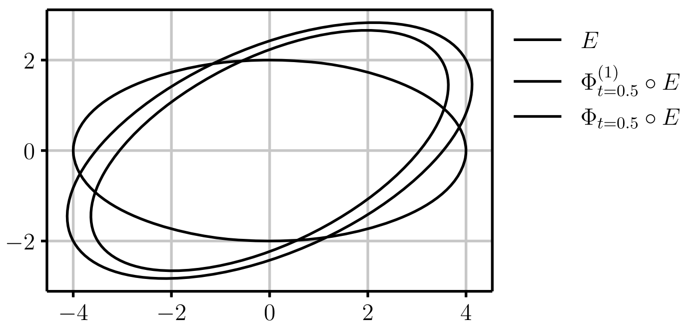

Figure 2.3: Ellipses under first and second order rotation.

This example allows a generalization.

Proposition 2.2. *Let Ω be convex and let $\Phi_t(x) = x + tv^\tau(x)$ be a first order tangential perturbation. Then $\dot{\mathcal{V}}(0) = 0$ and $\ddot{\mathcal{V}}(0) > 0$.*

Proof. The first statement is immediate (see (2.3.2)). Clearly, if $\Phi_t = x + tv^\tau$, then for convex domains $\Omega \subset \Omega_t$ and $|\Omega_t| > |\Omega|$. Because $\mathcal{L}$ is positive for convex domains this is in accordance with formula (2.3.13). □

Note that Proposition 2.2 is not necessarily true if $\Phi_t(x) = x + tv^\tau + \frac{t^2}{2}w$.

2.3.2 First and second variation of the surface area perimeter

We now compute the perimeter or the surface area $\mathcal{S}(t) := |\partial\Omega_t|$ of Ω_t. If the boundary $\partial\Omega$ is represented locally by $\tilde{x}(\xi)$, then in view of (2.1.1), a local parametrization of $\partial\Omega_t$ is given by

$$\tilde{y}(\xi) = \tilde{x}(\xi) + t\tilde{v}(\xi) + \frac{t^2}{2}\tilde{w}(\xi) + o(t^2).$$

As before we write $\tilde{v}(\xi) = v(\tilde{x}(\xi))$ and $\tilde{w}(\xi) = w(\tilde{x}(\xi))$. In order to compute the metric on $\partial\Omega_t$ we introduce the following notation:

$$\begin{aligned}
&g_{ij} := (\tilde{x}_{\xi_i} \cdot \tilde{x}_{\xi_j}), \quad G := (g_{ij})_{i,j=1,2,\dots,n-1},\\
&a_{ij} := (\tilde{x}_{\xi_i} \cdot \tilde{v}_{\xi_j}) + (\tilde{x}_{\xi_j} \cdot \tilde{v}_{\xi_i}), \quad A := (a_{ij})_{i,j=1,\dots,n-1},\\
&b_{ij} := 2(\tilde{v}_{\xi_i} \cdot \tilde{v}_{\xi_j}) + (\tilde{w}_{\xi_i} \cdot \tilde{x}_{\xi_j}) + (\tilde{w}_{\xi_j} \cdot \tilde{x}_{\xi_i}), \quad B := (b_{ij})_{i,j=1,\dots,n-1}.
\end{aligned}$$

Then the metric g_{ij}^t on $\partial\Omega_t$ is given by

$$g_{ij}^t := (y_{\xi_i} \cdot y_{\xi_j}) = g_{ij} + ta_{ij} + \frac{t^2}{2}b_{ij} + o(t^2), \quad G^t := (g_{ij}^t)_{i,j=1,\dots,n}. \tag{2.3.15}$$

The surface element on $\partial\Omega_t$ is

$$dS(t) = \sqrt{\det G^t}\,d\xi = \sqrt{\det G}\left\{\det\left(I + tG^{-1}A + \frac{t^2}{2}G^{-1}B + o(t^2)\right)\right\}^{\frac{1}{2}} d\xi,$$

where $d\xi = d\xi_1 \dots d\xi_{n-1}$. It can therefore be written as

$$dS(t) = m(t)dS, \tag{2.3.16}$$

where dS is the surface element of $\partial\Omega$.

Let us write for short $T(C) := \operatorname{trace}(C)$. By Jacobi's formula (see Section A.1),

$$\begin{aligned}
&\det\left(I + tG^{-1}A + \frac{t^2}{2}G^{-1}B + o(t^2)\right)\\
&\quad = 1 + tT(G^{-1}A) + \frac{t^2}{2}[(T(G^{-1}A))^2 - G^{-1}A : G^{-1}A + T(G^{-1}B)] + o(t^2).
\end{aligned}$$

Then for small t we get

$$\begin{aligned}
m(t) &= \sqrt{\det\left(I + tG^{-1}A + \frac{t^2}{2}G^{-1}B\right) + o(t^2)}\\
&= 1 + \frac{t}{2}T(G^{-1}A) + \frac{t^2}{2}\left(\frac{(T(G^{-1}A))^2}{4} - \frac{G^{-1}A : G^{-1}A}{2} + \frac{T(G^{-1}B)}{2}\right)\\
&\quad + o(t^2).
\end{aligned} \tag{2.3.17}$$

Let us write for short

$$\partial_i^* \tilde{v} := g^{ij} \tilde{v}_{\xi_j}. \tag{2.3.18}$$

With this notation we have by (2.2.9) and (2.3.15)

$$\operatorname{div}_{\partial\Omega} \tilde{v} = (\partial_i^* \tilde{v} \cdot \tilde{x}_{\xi_i}), \quad g^{ij} a_{jk} = (\partial_i^* \tilde{x} \cdot \tilde{v}_{\xi_k}) + (\partial_i^* \tilde{v} \cdot \tilde{x}_{\xi_k}).$$

A straightforward computation leads to

$$\begin{aligned} T(G^{-1}A) &= 2 \operatorname{div}_{\partial\Omega} \tilde{v}, \\ G^{-1}A : G^{-1}A &= g^{ij} a_{jk} g^{kl} a_{li} \\ &= (\partial_i^* \tilde{v} \cdot \tilde{x}_{\xi_k})(\partial_k^* \tilde{v} \cdot \tilde{x}_{\xi_i}) + 2(\partial_i^* \tilde{v} \cdot \tilde{x}_{\xi_k})(\tilde{v}_{\xi_i} \cdot \partial_k^* x) + (\tilde{v}_{\xi_k} \cdot \partial_i^* x)(\tilde{v}_{\xi_i} \cdot \partial_k^* x) \\ &= 2(\partial_i^* \tilde{v} \cdot \tilde{x}_{\xi_k})(\partial_k^* \tilde{v} \cdot \tilde{x}_{\xi_i}) + 2(\partial_i^* \tilde{v} \cdot \tilde{x}_{\xi_k})(\tilde{v}_{\xi_i} \cdot \partial_k^* \tilde{x}), \\ T(G^{-1}B) &= 2(\tilde{v}_{\xi_i} \cdot \partial_i^* \tilde{v}) + 2 \operatorname{div}_{\partial\Omega} \tilde{w}. \end{aligned}$$

Decomposing $v = v^\tau + (v \cdot \nu)\nu$ and applying (2.2.10), we obtain

$$\dot{m}(0) = \operatorname{div}_{\partial\Omega} v^\tau + (n-1)H(v \cdot \nu). \tag{2.3.19}$$

Consequently, from (2.3.16) and (2.3.19) it follows that at $t = 0$ the first variation of the surface area $\mathcal{S}(t)$ of $\partial\Omega_t$ is

$$\dot{\mathcal{S}}(0) = \oint_{\partial\Omega} \dot{m}(0)\, dS = (n-1) \oint_{\partial\Omega} H(v \cdot \nu)\, dS. \tag{2.3.20}$$

This together with the first variation of the volume implies the following lemma.

Lemma 2.2. *For volume preserving perturbations the surface area is critical, i. e.,* $\dot{\mathcal{S}}(0) = 0$, *if and only if* H *is constant.*

Proof. Let

$$\overline{H} := \frac{1}{|\partial\Omega|} \oint_{\partial\Omega} H\, dS$$

be the mean value of H. From (2.3.2) we have $\oint_{\partial\Omega} (v \cdot \nu)\, dS = 0$. Hence,

$$0 = \oint_{\partial\Omega} \{H - \overline{H}\}(v \cdot \nu)\, dS.$$

Choose $(v \cdot \nu) = H - \overline{H}$. Then

$$\oint_{\partial\Omega} (H - \overline{H})^2\, dS = 0.$$

The assertion is now obvious. □

By Alexandrov's uniqueness theorem [1] any closed connected C^2 surface of constant mean curvature is a sphere.

We mention an identity which will frequently be used in the sequel. From (2.3.19) and (2.2.18) it follows that

$$\oint_{\partial\Omega} f(x)\dot{m}(0)\,dS = -\oint_{\partial\Omega}(v^\tau\cdot\nabla f)\,dS + (n-1)\oint_{\partial\Omega} fH(v\cdot\nu)\,dS. \tag{2.3.21}$$

Next we shall compute $\ddot{m}(0)$.

From (2.3.17) and (2.3.18) we get

$$\begin{aligned}\ddot{m}(0) = {}&(\operatorname{div}_{\partial\Omega}\tilde{v})^2 - (\partial_i^*\tilde{v}\cdot\tilde{x}_{\xi_k})(\partial_k^*\tilde{v}\cdot\tilde{x}_{\xi_i})\\ &- (\partial_i^*\tilde{v}\cdot\tilde{x}_{\xi_k})(\tilde{v}_{\xi_i}\cdot\partial_k^*\tilde{x}) + (\partial_j^*\tilde{v}\cdot\tilde{v}_{\xi_j}) + \operatorname{div}_{\partial\Omega}\tilde{w}.\end{aligned} \tag{2.3.22}$$

Hence, (2.3.16) implies that the second variation of the surface area perimeter at $t=0$ is (cf. Section 2.2.1)

$$\begin{aligned}\ddot{\mathcal{S}}(0) &= \oint_{\partial\Omega}\ddot{m}(0)\,dS = \sum_{i=1}^{i_0}\oint_{V_i\cap\partial\Omega}\ddot{m}(0)\,dS\\ &= \sum_{i=1}^{i_0}\int_{U_i} p_i(\tilde{x})[(\operatorname{div}_{\partial\Omega}\tilde{v})^2 - (\partial_j^*\tilde{v}\cdot\tilde{x}_{\xi_k})(\partial_k^*\tilde{v}\cdot\tilde{x}_{\xi_j}) - (\partial_j^*\tilde{v}\cdot\tilde{x}_{\xi_k})(\tilde{v}_{\xi_j}\cdot\partial_k^*\tilde{x})]dS\\ &\quad + \sum_{i=1}^{i_0}\int_{U_i} p_i(\tilde{x})(\partial_j^*\tilde{v}\cdot\tilde{v}_{\xi_j})\,dS + (n-1)\sum_{i=1}^{i_0}\int_{U_i} p_i(x)(\tilde{w}\cdot\tilde{\nu})H\,dS.\end{aligned} \tag{2.3.23}$$

In short, we have

$$\begin{aligned}\ddot{\mathcal{S}}(0) = {}&\oint_{\partial\Omega}(\operatorname{div}_{\partial\Omega}\tilde{v})^2 - (\partial_i^*\tilde{v}\cdot\tilde{x}_{\xi_k})(\partial_k^*\tilde{v}\cdot\tilde{x}_{\xi_i}) - (\partial_i^*\tilde{v}\cdot\tilde{x}_{\xi_k})(\tilde{v}_{\xi_i}\cdot\partial_k^*\tilde{x}) + (\partial_j^*\tilde{v}\cdot\tilde{v}_{\xi_j})\,dS\\ &+ \oint_{\partial\Omega}(\operatorname{div}_{\partial\Omega}\tilde{w})\,dS.\end{aligned}$$

Example 2.4. 1. We start with the computation of $\ddot{\mathcal{S}}(0)$ if $\partial\Omega$ is the sphere ∂B_R and Φ_t is a Hadamard perturbation such that $v=\rho\nu$. Then the following formulas are valid:

$$\begin{aligned}\operatorname{div}_{\partial\Omega}\tilde{v} &= \frac{n-1}{R}\rho,\\ \tilde{v}_{\xi_i}\cdot\partial_k^*\tilde{x} &= g^{kl}((\rho_{\xi_i}\tilde{\nu}+\rho\tilde{\nu}_{\xi_i})\cdot\tilde{x}_{\xi_l}) = \rho g^{kl}(\tilde{\nu}_{\xi_i}\cdot\tilde{x}_{\xi_l}) = -\rho g^{kl}(\tilde{\nu}\cdot\tilde{x}_{\xi_l\xi_i})\\ &= \rho g^{kl}L_{li} = \frac{\rho}{R}g^{kl}g_{li} = \frac{\rho}{R}\delta_{ki},\\ \partial_i^*\tilde{v}\cdot\tilde{x}_{\xi_k} &= g^{ij}(\tilde{v}_{\xi_j}\cdot\tilde{x}_{\xi_k}) = \rho g^{ij}(\tilde{\nu}_{\xi_j}\cdot\tilde{x}_{\xi_k}) = -\rho g^{ij}(\tilde{\nu}\cdot\tilde{x}_{\xi_k\xi_j}) = \frac{\rho}{R}\delta_{ik}.\end{aligned}$$

Moreover,

$$\partial_j^* \tilde{v} \cdot \tilde{v}_{\xi_j} = g^{jk} \tilde{v}_{\xi_k} \cdot \tilde{v}_{\xi_j} = g^{jk} (\rho_{\xi_k} \tilde{\nu} + \rho \tilde{\nu}_{\xi_k}) \cdot (\rho_{\xi_j} \tilde{\nu} + \rho \tilde{\nu}_{\xi_j}) = g^{jk} \rho_{\xi_k} \rho_{\xi_j} + \rho^2 g^{jk} \tilde{\nu}_{\xi_k} \cdot \tilde{\nu}_{\xi_j}.$$

For the last term we use (2.2.8), take into account that $(\tilde{\nu}_{\xi_k} \cdot \tilde{\nu}) = 0$, and apply (2.2.5):

$$\begin{aligned}
\rho^2 g^{jk} \tilde{\nu}_{\xi_k} \cdot \tilde{\nu}_{\xi_j} &= \rho^2 g^{jk} (\tilde{\nu}_{\xi_k} \cdot \tilde{\nu})(\tilde{\nu}_{\xi_j} \cdot \tilde{\nu}) + \rho^2 g^{jk} g^{mn} (\tilde{\nu}_{\xi_k} \cdot \tilde{x}_{\xi_m})(\tilde{\nu}_{\xi_j} \cdot \tilde{x}_{\xi_n}) \\
&= \rho^2 g^{jk} g^{mn} (\tilde{\nu}_{\xi_k} \cdot \tilde{x}_{\xi_m})(\tilde{\nu}_{\xi_j} \cdot \tilde{x}_{\xi_n}) \\
&= \rho^2 g^{jk} g^{mn} L_{km} L_{jn} = \frac{\rho^2}{R^2} g^{jk} g^{mn} g_{km} g_{jn} = \frac{\rho^2}{R^2} \delta_{kn} \delta_{kn} \\
&= (n-1) \frac{\rho^2}{R^2}.
\end{aligned}$$

Hence,

$$\partial_j^* \tilde{v} \cdot \tilde{v}_{\xi_j} = g^{ik} \rho_{\xi_k} \rho_{\xi_j} + (n-1) \frac{\rho^2}{R^2}.$$

Then (2.3.22) takes the form

$$\ddot{m}(0) = g^{jk} \rho_{\xi_k} \rho_{\xi_j} + (n-1)(n-2) \frac{\rho^2}{R^2} + \operatorname{div}_{\partial\Omega} \tilde{w}.$$

Hence,

$$\ddot{\mathcal{S}}(0) = \oint_{\partial B_R} \left(g^{ij} \rho_{\xi_i} \rho_{\xi_j} + \frac{(n-1)(n-2)}{R^2} \rho^2 + \frac{n-1}{R} (w \cdot \nu) \right) dS. \tag{2.3.24}$$

2. Let Ω be a smooth domain in the plane. Its boundary curve $(x_1(s), x_2(s))$, $s \in I$, is parametrized by arc length. Then the tangent vector $\dot{x}(s)$ satisfies $|\dot{x}(s)| = 1$; thus, $g^{11} = 1$. Moreover, $\kappa := \ddot{x}(s) = H$ is the curvature of the curve in $x(s)$. For any smooth vector field v we set $\tilde{v}(s) := v(x(s))$. We need to compute the terms in (2.3.22). From (2.2.9) we deduce

$$\left(\operatorname{div}_{\partial\Omega} \tilde{v}(x(s))\right)^2 = \left(\partial_s \tilde{v}(s) \cdot \dot{x}(s)\right)^2.$$

Since

$$\begin{aligned}
&\partial_i^* \tilde{v} \cdot \tilde{x}_{\xi_k} = g^{11}(s) \partial_s \tilde{v}(s) \cdot \dot{\tilde{x}}(s) = \partial_s \tilde{v}(s) \cdot \dot{\tilde{x}}(s), \\
&\tilde{v}_{\xi_i} \cdot \partial_k^* x = \partial_s \tilde{v}(s) \cdot \dot{\tilde{x}}(s),
\end{aligned}$$

and

$$\partial_j^* \tilde{v} \cdot \tilde{v}_{\xi_j} = \left(\partial_s \tilde{v}(s)\right)^2,$$

we obtain

$$\ddot{\mathcal{S}}(0) = \int_I \left(\partial_s \tilde{v}(s)\right)^2 - \left(\partial_s \tilde{v}(s) \cdot \dot{x}(s)\right)^2 ds + \int_I \tilde{\kappa}(s)\left(\tilde{w}(s) \cdot \tilde{\nu}(s)\right) ds.$$

If we decompose $v = \rho\nu + v^\tau \dot{x}(s)$ into its normal and tangential components, we obtain

$$\partial_s \tilde{v}(s) = \partial_s \tilde{\rho}(s)\tilde{\nu}(s) + \tilde{\rho}(s)\partial_s \tilde{\nu}(s) + \partial_s \tilde{v}^\tau(s)\dot{x}(s) + \tilde{v}^\tau(s)\ddot{x}(s).$$

By Frenet's formula,

$$\partial_s \tilde{\nu}(s) = \tilde{\kappa}(s)\dot{x}(s) \quad \text{and} \quad \ddot{x}(s) = -\tilde{\kappa}(s)\tilde{\nu}(s).$$

Hence,

$$\partial_s \tilde{v}(s) = \partial_s \tilde{\rho}(s)\tilde{\nu}(s) + \tilde{\kappa}(s)\tilde{\rho}(s)\dot{x}(s) + \partial_s \tilde{v}^\tau(s)\dot{x}(s) - \tilde{\kappa}(s)\tilde{v}^\tau(s)\tilde{\nu}(s).$$

The first integrand for $\ddot{\mathcal{S}}(0)$ can be written as

$$\left(\partial_s \tilde{v}(s)\right)^2 - \left(\partial_s \tilde{v}(s) \cdot \dot{x}(s)\right)^2 = \left(\partial_s \tilde{\rho}(s) - \tilde{\kappa}(s)\tilde{v}^\tau(s)\right)^2.$$

Hence,

$$\ddot{\mathcal{S}}(0) = \int_I \left(\partial_s \tilde{\rho}(s) - \tilde{\kappa}(s)\tilde{v}^\tau(s)\right)^2 ds + \int_I \tilde{\kappa}(s)\left(\tilde{w}(s) \cdot \tilde{\nu}(s)\right) ds.$$

3. Consider a stretch in one direction $v = (x_1, 0, \dots, 0)$. Then $\operatorname{div}_{\partial\Omega} = g^{ij}\frac{\partial x_1}{\partial \xi_j}\frac{\partial x_1}{\partial \xi_j}$, $g^{ij}a_{jk} = 2g^{ij}\frac{x_1}{\partial \xi_j}\frac{\partial x_1}{\partial \xi_k}$. Without loss of generality we may assume that at a fixed point on $\partial\Omega$, $g^{ij} = \delta_{ij}$,

$$\ddot{m}(0) = 2\sum_{i \neq j}\left(\frac{\partial x_1}{\partial \xi_i}\right)^2\left(\frac{\partial x_1}{\partial \xi_j}\right)^2.$$

2.3.3 Area variations for the surface of the ball

In (2.3.24) we computed the second variation for the area for Hadamard perturbations. We now consider more general perturbations $v = \rho(x)\nu(x) + v^\tau(x)$ on ∂B_R which contain also tangential components. According to Theorem 2.1 and in particular (2.2.21), the perturbation

$$\Phi_t(x) = x + t\left[\rho(x)\nu(x) + v^\tau(x)\right] + \frac{t^2}{2}w(x)$$

can be replaced by the Hadamard perturbation

$$\hat{\Phi}(\hat{x}) = \hat{x} + t\rho(\hat{x}) + \frac{t^2}{2}\left\{\frac{|v^\tau(\hat{x})|^2}{R} - 2v^\tau(\hat{x})\cdot\nabla^\tau\rho(\hat{x}) + w(\hat{x})\right\}\nu(\hat{x}) + o(t^2).$$

We can apply (2.3.24) provided $(w\cdot\nu)$ is replaced by

$$\omega := (w\cdot\nu) + \frac{|v^\tau(\hat{x})|^2}{R} - 2\big(v^\tau(\hat{x})\cdot\nabla^\tau\rho(\hat{x})\big).$$

Thus, the second variation of the perimeter for general perturbations reads as

$$\ddot{\mathcal{S}}(0) = \oint_{\partial B_R}\left(g^{ij}\rho_{\xi_i}\rho_{\xi_j} + \frac{(n-1)(n-2)}{R^2}\rho^2 + \frac{n-1}{R}\omega\right)dS. \tag{2.3.25}$$

Recall that (see (2.3.14))

$$\ddot{\mathcal{V}}(0) = \oint_{\partial B_R}\left(\frac{n-1}{R}\rho^2 - 2(v^\tau\cdot\nabla^\tau\rho) + \frac{|v^\tau|^2}{R} + (w\cdot\nu)\right)dS = \oint_{\partial B_R}\left[\frac{n-1}{R}\rho^2 + \omega\right]dS.$$

Consequently, the second variation of the area is

$$\ddot{\mathcal{S}}(0) = \oint_{\partial B_R} g^{ij}\rho_{\xi_i}\rho_{\xi_j}\,dS - \frac{n-1}{R^2}\oint_{\partial B_R}\rho^2\,dS + \frac{n-1}{R}\ddot{\mathcal{V}}(0). \tag{2.3.26}$$

Recall that $g^{ij}\rho_{\xi_i}\rho_{\xi_j} = |\nabla^\tau(v\cdot\nu)|^2$. For later use we set

$$\ddot{\mathcal{S}}_0(0) := \int_{\partial B_R}|\nabla^\tau(v\cdot\nu)|^2\,dS - \frac{n-1}{R^2}\int_{\partial B_R}(v\cdot\nu)^2\,dS. \tag{2.3.27}$$

Thus,

$$\ddot{\mathcal{S}}(0) = \ddot{\mathcal{S}}_0(0) + \frac{n-1}{R}\ddot{\mathcal{V}}(0). \tag{2.3.28}$$

Consider now volume preservation in the sense of Definition 2.2. Then $\ddot{\mathcal{S}}(0) = \ddot{\mathcal{S}}_0(0)$. From $\dot{\mathcal{V}}(0) = 0$ it follows that $\oint_{B_R}\rho\,dS = 0$. Denote

$$\mathcal{K}_1 := \left\{\rho \in C^1(\partial B_R) : \oint_{\partial B_R}\rho\,dS = 0\right\}.$$

Then (cf. Appendix D.2.1)

$$\min_{\mathcal{K}_1}\frac{\oint_{\partial B_R} g^{ij}\rho_{\xi_i}\rho_{\xi_j}\,dS}{\oint_{\partial B_R}\rho^2\,dS} = \frac{n-1}{R^2}.$$

Equality holds if and only if ρ is a spherical harmonic of degree 1. It belongs to the n-dimensional eigenspace $\mathcal{H}_1$. This eigenspace is spanned by $\{\frac{x_i}{R}\}_{i=1}^n$. Consequently, $\ddot{\mathcal{S}}(0) = 0$ for $\rho \in \mathcal{H}_1$. In particular we proved the following lemma.

Lemma 2.3. *Let $\Phi_t : B_R \to \Omega_t$ be a perturbation which is volume preserving, i. e., $\dot{\mathcal{V}}(0) = \ddot{\mathcal{V}}(0) = 0$. Then $\dot{\mathcal{S}}(0) = 0$, $\ddot{\mathcal{S}}(0) \geq 0$, and equality holds if and only if $\rho = (\upsilon \cdot \nu) \in \mathcal{H}_1$.*

Definition 2.3. A perturbation is said to be in the kernel of $\ddot{\mathcal{S}}(0)$ if $\dot{\mathcal{S}}(0) = \ddot{\mathcal{S}}(0) = 0$.

The perturbation $\Phi_t = x + t\frac{x_i}{R}e_i + \frac{t^2}{2}w$, where w is such that

$$\ddot{\mathcal{V}}(0) = \oint\limits_{\partial B_R} \left(\frac{n-1}{R^3}(x_i)^2 + (w \cdot \nu) \right) dS = 0,$$

is in the kernel of $\ddot{\mathcal{S}}$ of the ball.

We now restrict to the following class of volume preserving perturbations in order to obtain a strictly positive $\ddot{\mathcal{S}}$. Let

$$\mathcal{K}_2 := \left\{ \rho \in C^1(\partial B_R) : \oint\limits_{\partial B_R} \rho \, dS = 0, \oint\limits_{\partial B_R} x\rho(x) \, dS = 0 \right\},$$

$$\min_{\mathcal{K}_2} \frac{\oint_{\partial B_R} g^{ij} \rho_{\xi_i} \rho_{\xi_j} \, dS}{\oint_{\partial B_R} \rho^2 \, dS} = \frac{2n}{R^2}.$$

The eigenspace of functions for this eigenvalue is denoted by $\mathcal{H}_2$ (cf. Appendix D.2.1). Thus, for any $\rho \in \mathcal{H}_2$ we have

$$\int\limits_{\partial B_R} |\nabla^\tau \rho|^2 \, dS \geq \frac{2n}{R^2} \oint\limits_{\partial B_R} \rho^2 \, dS. \tag{2.3.29}$$

Consequently,

$$\ddot{\mathcal{S}}(0) \geq \frac{n+1}{R^2} \oint\limits_{\partial B_R} \rho^2 \, dS.$$

Definition 2.4. For any domain $\Omega \subset \mathbb{R}^n$ the barycenter is defined as

$$b := \frac{1}{|\Omega|} \int\limits_{\Omega} x \, dx.$$

We consider volume preserving perturbations which leave the barycenter unchanged in first order:

$$b_t := \frac{1}{|\Omega_t|} \int\limits_{\Omega_t} y \, dy = \frac{1}{|\Omega|} \int\limits_{\Omega} x \, dx + o(t) = b + o(t).$$

This condition is equivalent to

$$\frac{1}{|\Omega|}\int_{\Omega}(x+tv(x))(1+t\,\mathrm{div}\,v)\,dx = b + o(t).$$

A straightforward computation yields $\oint_{\partial\Omega}(v\cdot\nu)x\,dS = 0$.

Definition 2.5. A perturbation Φ_t satisfies the barycenter condition if $\oint_{\partial\Omega}(v\cdot\nu)x\,dS = 0$.

If the barycenter condition is satisfied, a stronger estimate in Lemma 2.3 is available.

Lemma 2.4. *Let $\Phi_t : B_R \to \Omega_t$ be a perturbation which is volume preserving, i. e., $\dot{\mathcal{V}}(0) = \ddot{\mathcal{V}}(0) = 0$. Moreover, let Φ_t satisfy the barycenter condition. Then $\dot{\mathcal{S}}(0) = 0$ and*

$$\ddot{\mathcal{S}}(0) \geq \frac{n+1}{R^2}\oint_{\partial B_R}\rho^2\,dS.$$

2.3.4 Hadamard perturbations

In this subsection we evaluate the second variation of the area (2.3.23) for Hadamard perturbations.

For the computations we will use Weingarten's formula (see (2.2.4)):

$$\tilde{\nu}_{\xi_i} = g^{jk}L_{ji}\tilde{x}_{\xi_k}.$$

Thus,

$$\partial_i^*\tilde{v} = g^{ij}\frac{\partial}{\partial\xi_j}(\tilde{\rho}\tilde{\nu}) = g^{ij}(\tilde{\rho}_{\xi_j}\tilde{\nu} + \tilde{\rho}\tilde{\nu}_{\xi_j}) \quad \text{and} \quad \partial_i^* x = g^{ij}x_{\xi_j}.$$

Hence,

$$\begin{aligned}
\mathrm{div}_{\partial\Omega}\,\tilde{v} &= (\partial_i^*\tilde{v}\cdot\tilde{x}_{\xi_i}) = \tilde{\rho}g^{ij}(\tilde{\nu}_{\xi_j}\cdot\tilde{x}_{\xi_i}) = \tilde{\rho}g^{ij}L_{ij},\\
(\partial_i^*\tilde{v}\cdot\tilde{x}_{\xi_k}) &= \tilde{\rho}g^{ij}g^{sm}L_{js}(\tilde{x}_{\xi_m}\cdot\tilde{x}_{\xi_k}) = \tilde{\rho}g^{ij}L_{jk},\\
(\tilde{v}_{\xi_i}\cdot\partial_k^*\tilde{x}) &= \tilde{\rho}g^{sm}L_{is}g^{kl}g_{ml} = \tilde{\rho}g^{sk}L_{is}.
\end{aligned}$$

Let $\mathcal{L}$ denote the Weingarten operator (see (2.2.2)) and $T(\mathcal{L})$ its trace. Then

$$\begin{aligned}
\mathrm{div}_{\partial\Omega}\,\tilde{v} &= \tilde{\rho}T(\mathcal{L}),\\
(\partial_i^*\tilde{v}\cdot\tilde{x}_{\xi_k})(\partial_k^*\tilde{v}\cdot\tilde{x}_{\xi_i}) &= (\partial_i^*\tilde{v}\cdot\tilde{x}_{\xi_k})(\tilde{v}_{\xi_i}\cdot\partial_k^* x) = \tilde{\rho}^2T(\mathcal{L}^2).
\end{aligned}$$

Moreover,

$$(\partial_j^*\tilde{v}\cdot\tilde{v}_{\xi_j}) = g^{js}\tilde{\rho}_{\xi_s}\tilde{\rho}_{\xi_j} + \tilde{\rho}^2(g^{js}\nu_{\xi_s}\cdot\nu_{\xi_j}) = g^{js}\tilde{\rho}_{\xi_s}\tilde{\rho}_{\xi_j} + \tilde{\rho}^2T(\mathcal{L}^2).$$

Inserting these relations into (2.3.22), we get

$$\ddot{m}(0) = g^{js}\tilde{\rho}_{\xi_s}\tilde{\rho}_{\xi_j} + \tilde{\rho}^2\big[(T(\mathcal{L}))^2 - \mathcal{L} : \mathcal{L}\big] + \operatorname{div}_{\partial\Omega} \tilde{w}. \tag{2.3.30}$$

We are now able to write down the second variation of area:

$$\ddot{\mathcal{S}}(0) = \sum_{i=1}^{i_0} \int_{U_i} \{g^{ij}\tilde{\rho}_{\xi_i}\tilde{\rho}_{\xi_j} + \tilde{\rho}^2\big[(T(\mathcal{L}))^2 - \mathcal{L} : \mathcal{L}\big] + (n-1)H(\tilde{w}\cdot\tilde{\nu})\}\, dS. \tag{2.3.31}$$

The eigenvalues of $\mathcal{L}$ are the principal curvatures $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$ of $\partial\Omega$. The traces of the matrices $\mathcal{L}$ and $\mathcal{L}^2$ can be expresses as follows:

$$\begin{aligned} T(\mathcal{L}) &= \kappa_1 + \kappa_2 + \cdots + \cdots + \kappa_{n-1} = (n-1)H, \\ \mathcal{L} : \mathcal{L} &= T(\mathcal{L}^2) = \kappa_1^2 + \kappa_2^2 + \cdots + \cdots + \kappa_{n-1}^2. \end{aligned} \tag{2.3.32}$$

Consequently,

$$\ddot{m}(0) = g^{sj}\tilde{\rho}_{\xi_s}\tilde{\rho}_{\xi_j} + 2\tilde{\rho}^2 \sum_{i<j} \tilde{\kappa}_i\tilde{\kappa}_j + \operatorname{div}_{\partial\Omega} \tilde{w}. \tag{2.3.33}$$

From (2.3.16) it follows that

$$\ddot{\mathcal{S}}(0) = \sum_{i=1}^{i_0} \int_{U_i} g^{sj}\tilde{\rho}_{\xi_s}\tilde{\rho}_{\xi_j} + 2\tilde{\rho}^2 \sum_{i<j} \tilde{\kappa}_i\tilde{\kappa}_j + (n-1)H(\tilde{w}\cdot\nu)\, dS. \tag{2.3.34}$$

2.3.5 Tangential perturbations

Assume that on $\partial\Omega$ the first order perturbation is tangential, i. e., $v = v^\tau$. From (2.3.20) it follows that

$$\dot{\mathcal{S}}(0) = 0.$$

In order to compute the second variation we apply Theorem 2.1, which states that a tangential perturbation can be replaced by a Hadamard perturbation of the form

$$\Psi_t = x + \frac{t^2}{2}\big(v^\tau \mathcal{L} v^\tau \nu + w\big) =: x + \frac{t^2}{2}\hat{w}.$$

Then from (2.3.22), we have

$$\ddot{m}(0) = \operatorname{div}_{\partial\Omega} \hat{w} \quad \Longrightarrow \quad \ddot{\mathcal{S}}(0) = \oint_{\partial\Omega} \operatorname{div}_{\partial\Omega} \hat{w}\, dS. \tag{2.3.35}$$

If we set $\tilde{\rho} = 0$ in (2.3.31) and replace w by $v^\tau L v^\tau \nu + w$, we obtain

$$\ddot{\mathcal{S}}(0) = (n-1) \oint_{\partial\Omega} H\left[v^\tau \mathcal{L} v^\tau + (w \cdot \nu)\right] dS. \tag{2.3.36}$$

2.4 Intrinsic versus extrinsic

In the previous section we used local coordinates to express geometric quantities on $\partial\Omega$. Another way to capture these quantities is to write $\partial\Omega$ as the zero level set of the distance function. The geometric quantities on the boundary can be described in terms of this function.

2.4.1 The distance function

For $x \in \mathbb{R}^n$ we define the distance function of $\partial\Omega$:

$$\operatorname{dist}(x, \partial\Omega) := \inf\{|x - z| : z \in \partial\Omega\}.$$

Then

$$\delta(x, \partial\Omega) := \begin{cases} -\operatorname{dist}(x, \partial\Omega), & \text{if } x \in \mathbb{R}^n \setminus \Omega, \\ \operatorname{dist}(x, \partial\Omega), & \text{if } x \in \overline{\Omega}, \end{cases}$$

is the signed distance function of $\partial\Omega$. We will also write $\delta(x) = \delta(x, \partial\Omega)$ for short.

The next statement is Lemma 14.1 in [65].

Lemma 2.5. *Let Ω be a bounded domain and let $\partial\Omega \in C^k$ for some $k \geq 2$. Then there exists a constant ρ_0 depending on Ω such that $\delta \in C^k(\Omega_{\rho_0})$, where*

$$\Omega_{\rho_0} := \{x \in \mathbb{R}^n : -\rho_0 < \delta(x) < \rho_0\}$$

is a parallel set to $\partial\Omega$.

Consequently, any point $x \in \Omega_{\rho_0}$ can be uniquely written as

$$x = \pi(x) + \delta(x, \partial\Omega)\nu(\pi(x)),$$

where $\pi(x)$ is the nearest point of $\partial\Omega$ to x:

$$|\pi(x) - x| = \operatorname{dist}(x, \partial\Omega).$$

For $x \in \partial\Omega$ it is easy to show that

$$\nu(x) = -\nabla\delta(x).$$

We define $\hat{\nu}(x) := -\nabla\delta(x)$ for $x \in \Omega_{\rho_0}$ and thus obtain an extension of the normal field to $\partial\Omega$ into the parallel set to $\partial\Omega$. We continue to write $\nu(x)$ rather than $\hat{\nu}(x)$ for $x \in \Omega_{\rho_0}$.

In particular, $|\nabla\delta(x)| = 1$ for all $x \in \partial\Omega$ and

$$\nu \cdot \nu = 1, \quad \nu D_\nu = 0, \quad D_\nu \nu = 0. \tag{2.4.1}$$

This follows from the following computation, where $\partial_i = \frac{\partial}{\partial x_i}$:

$$\nu D_\nu = \nu_i \partial_i \nu_j = \partial_i \delta(x,0)\partial_i\partial_j\delta(x,0) = \frac{1}{2}\partial_j\left|\nabla\delta(x,0)\right|^2 = 0 \tag{2.4.2}$$

and

$$D_\nu \nu = \partial_i \nu_j \nu_j = \frac{1}{2}\partial_j\left|\nabla\delta(x,0)\right|^2 = 0. \tag{2.4.3}$$

Consequently,

$$(\nu \cdot D_\nu \nu) = 0.$$

The representation $\nu = -\nabla\delta$ also implies

$$\xi \cdot D_\nu \eta = -\xi_i \partial_i \partial_j \eta_j \delta = \eta \cdot D_\nu \xi, \quad \forall\, \xi, \eta \in \mathbb{R}^n. \tag{2.4.4}$$

In local coordinates D_ν reads as

$$D_\nu = \nabla^\tau \nu := g^{ij}\frac{\partial \nu}{\partial \xi_j}\tilde{x}_{\xi_i}. \tag{2.4.5}$$

In the sequel $(\tilde{x}_{\xi_l})_i$ denotes the i-th component of the vector $\tilde{x}_{\xi_l}$. If we take into account that $(\nu \cdot \tilde{x}_{\xi_i})_{\xi_j} = 0$, we arrive at

$$\begin{aligned}\partial_i \nu_j = \nabla_i^\tau \nu_j &= g^{kl}(\tilde{\nu}_j)_{\xi_k}(\tilde{x}_i)_{\xi_l} = g^{kl}(\tilde{\nu}_{\xi_k})_j(\tilde{x}_{\xi_l})_i = g^{kl}\big((\tilde{\nu}_{\xi_k}\cdot\tilde{x}_{\xi_m})\tilde{x}_{\xi_m}\big)_j(\tilde{x}_{\xi_l})_i\\ &= g^{kl}(\tilde{\nu}_{\xi_k}\cdot\tilde{x}_{\xi_m})(\tilde{x}_{\xi_m})_j(\tilde{x}_{\xi_l})_i = -g^{kl}(\tilde{\nu}\cdot\tilde{x}_{\xi_k\xi_m})(\tilde{x}_{\xi_m})_j(\tilde{x}_{\xi_l})_i.\end{aligned}$$

Hence,

$$\partial_i \nu_j = g^{kl}L_{km}(\tilde{x}_{\xi_m})_j(\tilde{x}_{\xi_l})_i.$$

For the special case of the sphere ∂B_R this yields

$$\partial_i \nu_j = \frac{1}{R}g^{kl}g_{km}(\tilde{x}_{\xi_m})_j(\tilde{x}_{\xi_l})_i = \frac{1}{R}(\tilde{x}_{\xi_m})_j(\tilde{x}_{\xi_m})_i. \tag{2.4.6}$$

From this we deduce $\mathcal{L} : \mathcal{L} = D_\nu : D_\nu$. As a further application we compute $\ddot{m}(0)$ from (2.3.30). For this purpose we apply (2.2.3). We have

$$\ddot{m}(0) = \left|\nabla^\tau \rho\right|^2 + (n-1)^2\rho^2 H^2 - \rho^2 D_\nu : D_\nu + \operatorname{div}_{\partial\Omega} w. \tag{2.4.7}$$

Then (2.3.31) becomes

$$\ddot{\mathcal{S}}(0) = \oint_{\partial\Omega} \left|\nabla^\tau \rho\right|^2 + (n-1)^2\rho^2 H^2 - \rho^2 D_\nu : D_\nu + (n-1)H(w\cdot\nu)\, dS. \tag{2.4.8}$$

2.4.2 Shape derivatives

We now consider a family of perturbed domains $\{\Omega_t\}_{|t|<t_0}$ in the sense of Section 2.1. Choosing $t_0 > 0$ sufficiently small we can assume $\partial\Omega_t \subset \Omega_{\rho_0}$ for all $|t| < t_0$. In this text we consider families of functions $u(y,t)_{|t|<t_0}$ for $y \in \overline{\Omega}_t$. As in (2.1.1) we set $y = \Phi_t(x)$ and write $\tilde{u}(t) = u(\Phi_t, t)$. Then

$$\frac{d}{dt}\tilde{u}(t) = \left(\nabla_y u(y,t)\cdot \partial_t \Phi_t\right) + \partial_t u(y,t).$$

Definition 2.6. The function $\dot{u}(0) := \frac{d}{dt}\tilde{u}(t)|_{t=0}$ is called the material derivative and $\partial_t u(y,t)|_{t=0} =: u'(x)$ the shape derivative.

For example, the material and shape derivatives of the distance function $\delta(y,t) := \delta(y, \partial\Omega_t)$ are computed. Clearly,

$$\delta(y,t) = \delta\left(\Phi_t(x), t\right) \quad \forall\, x \in \partial\Omega.$$

The shape derivative of $\delta(y,t)$ is defined as

$$\delta'(x) := \partial_t \delta\left(\Phi_t(x), t\right)\big|_{t=0}.$$

We define

$$\nu^t(y) := -\nabla_y \delta(y,t)$$

as the unit normal vector on $\partial\Omega_t$. The shape derivative ν' of ν is

$$\nu'(x) := \partial_t \nu^t(y)|_{t=0}. \tag{2.4.9}$$

Lemma 2.6. *The shape derivative ν' of ν^t satisfies the following identity:*

$$\nu' + \nabla^\tau \rho = 0 \quad \text{on } \partial\Omega, \tag{2.4.10}$$

where $\rho = (\upsilon \cdot \nu)$.

Proof. For $x \in \partial\Omega$ we have $\delta(\Phi_t(x), t) = 0$ for all $t \in (-t_0, t_0)$. Hence,

$$0 = \frac{d}{dt}\delta(\Phi_t(x), t) = \partial_t \Phi_t(x) \cdot \nabla_y \delta(\Phi_t(x), t) + \partial_t \delta(\Phi_t(x), t).$$

For $t = 0$ this yields

$$\partial_t \delta(\Phi_t(x), t)\big|_{t=0} = \delta'(x, 0) =: \delta'(x) = (v \cdot \nu). \tag{2.4.11}$$

Next we compute ν' in terms of the distance function δ. We have

$$\nu' = \partial_t \nu(y, t)|_{t=0} = -\partial_t \nabla_y \delta(y, t)|_{t=0} = -\nabla_y \partial_t \delta(y, t)|_{t=0} = -\nabla_x \delta'(x).$$

We decompose the gradient into its tangential and normal components. This implies

$$\nabla_x \delta'(x) = \nabla^\tau \delta'(x, 0) + (\nu \cdot \nabla_x \delta'(x, 0))\nu = \nabla^\tau \delta'(x, 0) - (\nu \cdot \nu')\nu = \nabla^\tau \delta'(x, 0).$$

From (2.4.11) we deduce

$$\nu' = -\nabla^\tau (v \cdot \nu) = -\nabla^\tau \rho,$$

which establishes the claim. □

A direct consequence is

$$\nu'(x) \cdot \nu(x) = 0 \quad \text{on } \partial\Omega. \tag{2.4.12}$$

Remark 2.4. If $u = \text{const.}$ on $\partial\Omega$, the decomposition $\nabla u|_{\partial\Omega} = (\nu \cdot \nabla u)\nu$ and (2.4.12) imply

$$(\nabla u \cdot \nu') = (\nabla u \cdot \nu)(\nu \cdot \nu') = 0 \quad \text{on } \partial\Omega. \tag{2.4.13}$$

We will write $H^t := H(y, t)$ for the mean curvature of $\partial\Omega_t$. With the help of Lemma 2.6 we will compute the material derivative of H^t for $y \in \partial\Omega_t$ in $t = 0$. For simplicity we restrict ourselves to Hadamard perturbations.

Lemma 2.7. *The normal derivative of the mean curvature is given by*

$$(n-1)\partial_\nu H = -D_\nu : D_\nu.$$

Proof. Let $x \in \Omega \cap U(\partial\Omega)$. Thus, $\delta(x) := \delta(x, 0)$ is smooth and $|\nabla\delta(x)|^2 = 1$. Hence,

$$\begin{aligned}0 = \Delta(|\nabla\delta|^2) &= 2\partial_i(\partial_j \delta \partial_j \partial_i \delta)\\ &= 2\partial_j \partial_i \delta \partial_j \partial_i \delta + 2\partial_j \delta \partial_j (\Delta\delta)\\ &= 2D_\nu : D_\nu + 2\nabla\delta \cdot \nabla(\Delta\delta).\end{aligned}$$

Hence, (2.4.3) and (2.2.10) imply

$$(n-1)\partial_\nu H = \partial_\nu(\operatorname{div}_{\partial\Omega} \nu) = \partial_\nu(\operatorname{div} \nu - \nu \cdot D_\nu \nu) = \partial_\nu(\operatorname{div} \nu) = \nabla\delta \cdot \nabla(\Delta\delta).$$

From this the claim follows easily. □

Lemma 2.8. *Let $\{H^t(y)\}_t$ be the family of mean curvatures in a point $y \in \partial\Omega_t$ and let $|t| < t_0$. The material derivative of H^t in $t = 0$ is equal to*

$$(n-1)\frac{d}{dt}H^t(y)\Big|_{t=0} = (n-1)\big(H' + v \cdot \nabla H\big) = -\Delta^* \rho - \rho D_\nu : D_\nu \quad \text{on } \partial\Omega,$$

where $H'(x) = \partial_t H^t(y)|_{t=0}$.

Proof. Lemma 2.6 implies

$$\partial_t H^t(y)|_{t=0} = \operatorname{div}_{\partial\Omega} \nu' = -\Delta^*(v \cdot \nu). \tag{2.4.14}$$

For Hadamard perturbations, we have

$$v \cdot \nabla H = (v \cdot \nu)\partial_\nu H = \rho \partial_\nu H. \tag{2.4.15}$$

We apply Lemma 2.7 and obtain

$$(n-1)\partial_\nu H = -D_\nu : D_\nu.$$

This yields

$$(n-1)v \cdot \nabla H = -\rho D_\nu : D_\nu.$$

Together with (2.4.14) this gives the claim. □

The second shape derivative of ν is defined as

$$\nu''(0,x) := \frac{\partial^2 \nu(t, \Phi(t,x))}{\partial t^2}\Big|_{t=0}.$$

We write ν'' rather than $\nu''(0,x)$ for short. Since $(\nu \cdot \nu) = 1$ we have

$$\nu \cdot \nu'' = -\big(\nu'\big)^2. \tag{2.4.16}$$

The following formula will be applied frequently.

Lemma 2.9. *Let $\partial\Omega$ be smooth and let ν and ν' be as above. Then*

$$(\nu \cdot D_{\nu'} \nu) = 0 \quad \text{on } \partial\Omega.$$

Proof. Since $\nu' = -\nabla^\tau(\upsilon \cdot \nu)$ (Lemma 2.6) and $\nabla_j^\tau = \partial_j - \nu_j \nu_k \partial_k$, we obtain

$$\nu_i \partial_i \nu_j' \nu_j = -\nu_i \partial_i [\partial_j(\upsilon \cdot \nu) - \nu_j \nu_k \partial_k (\upsilon \cdot \nu)] \nu_j.$$

We now differentiate the term inside square brackets:

$$\begin{aligned} \partial_i [\partial_j(\upsilon \cdot \nu) - \nu_j \nu_k \partial_k(\upsilon \cdot \nu)] &= \partial_i \partial_j(\upsilon \cdot \nu) - \partial_i \nu_j \nu_k \partial_k (\upsilon \cdot \nu) \\ &\quad - \nu_j \partial_i \nu_k \partial_k (\upsilon \cdot \nu) - \nu_j \nu_k \partial_i \partial_k (\upsilon \cdot \nu). \end{aligned}$$

We multiply all terms by ν_i and ν_j and sum over i, j, and k. Note that (2.4.1) implies

$$\sum_{i=1}^n \nu_i \partial_i \nu_k = \sum_{i=1}^n \nu_i \partial_i \nu_j = 0.$$

Hence,

$$\nu_i \partial_i \nu_j' \nu_j = -\nu_i \nu_j \partial_i \partial_j (\upsilon \cdot \nu) - \nu_i \nu_k \partial_i \partial_k (\upsilon \cdot \nu) = 0.$$

This is the claim. □

2.4.3 Notes

The basic concepts of differential geometry used in this chapter are found in classical texts on differential geometry (see for instance [40]). It seems difficult to trace back the formulas for the first and second variations of the volume and the surface area. They have also been derived in [66] and in [86]. Garabedian and Schiffer [63] called perturbations (variations) defined in $\overline{\Omega}$ *interior variations*.

In physics the material and the shape derivatives are related to the Lagrangian and the Eulerian description of a moving object. It has been introduced in [110]; for more information see also [111]. The shape derivatives of the normal and the mean curvature are well known. Lemma 2.6 can also be found in [75, Proposition 5.4.14] with a slightly different proof. Other proofs were given in [8] and [21, Lemma 39]. A proof of Lemma 2.7 is found in [43, Lemma 3.2].

3 Spherical harmonics and eigenvalue problems

In many cases the first variation vanishes for the ball. To check whether it maximizes or minimizes the functional we will expand the variations of the sphere in spherical harmonics. In this chapter we discuss properties of these functions and give some applications, such as the Funk–Hecke formula. We show that eigenfunctions of elliptic operators on balls and on other radial domains, such as the annulus, can be represented in terms of spherical harmonics. Among these elliptic operators are the Steklov operator and two fourth order operators.

3.1 Spherical harmonics

Let $B_1 \subset \mathbb{R}^n$ be the ball of radius 1 centered at the origin. In spherical coordinates the Laplacian is given by (see (B.1.2))

$$\Delta u = \partial_r^2 u + \frac{n-1}{r}\partial_r u + \frac{1}{r^2}\Delta^* u,$$

where

$$\Delta^* = \operatorname{div}_{\partial B_1} \nabla^\tau$$

is the Laplace–Beltrami operator on ∂B_1. We are interested in the eigenvalue problem

$$\Delta^* \phi + \Lambda\phi = 0 \text{ on } \partial B_1, \quad \phi \in C^\infty(\partial B_1). \tag{3.1.1}$$

The spectral theory for compact self-adjoint elliptic operators applies to this case. There exist infinitely many distinct nonnegative eigenvalues, each of finite multiplicity, with infinity as the only accumulation point. The lowest eigenvalue is $\Lambda_0 = 0$ with the corresponding eigenfunction $\phi_0 = \text{const}$. The Rayleigh quotient corresponding to (3.1.1) is

$$\mathcal{R}[u] = \frac{\oint_{\partial B_1} |\nabla^\tau u|^2 \, dS}{\oint_{\partial B_1} u^2 \, dS}.$$

The lowest positive eigenvalue has the variational representation

$$\Lambda_1 = \min_{\mathcal{K}} \mathcal{R}[u] \quad \text{where } \mathcal{K} = \left\{ u \in W^{1,2}(\partial B_1) : \oint_{\partial B_1} u \, dS = 0 \right\}.$$

The eigenfunctions are obtained as follows. A polynomial $P = P_k : \mathbb{R}^n \to \mathbb{R}$ is called *homogeneous of degree* k, $k \in \mathbb{N} \cup \{0\}$, if $P_k(\lambda x) = \lambda^k P_k(x)$. Obviously, $P_0 = \text{const}$. Among the homogeneous polynomials of degree k we consider only those which are harmonic, i. e., those for which $\Delta P_k(x) = 0$.

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Example 3.1. 1. In the plane $n = 2$ the harmonic homogeneous polynomials of degree k are linear combinations of

$$\mathrm{Re}(x_1 + ix_2)^k, \quad \mathrm{Im}(x_1 + ix_2)^k.$$

2. In $\mathbb{R}^3$ the harmonic, homogeneous polynomials of degree 2 are given by linear combinations of

$$\begin{gathered} x_1x_2, \quad x_2x_3, \quad x_1x_3, \\ 2x_3^2 - x_1^2 - x_2^2, \quad x_1^1 - x_2^2. \end{gathered}$$

Since P_k is homogeneous of degree k,

$$P_k(x) = P_k\left(|x|\frac{x}{|x|}\right) = |x|^k P_k\left(\frac{x}{|x|}\right).$$

Thus, $P_k(x)$ is uniquely determined by its values on ∂B_1. In the sequel we set $\xi = \frac{x}{|x|}$. From

$$0 = \Delta P_k(x) = k(n + k - 2)|x|^{k-2} P_k(\xi) + |x|^{k-2}\Delta^* P_k(\xi),$$

it follows that the restriction of a harmonic homogeneous polynomial $P_k(\xi)$ to ∂B_1 is an eigenfunction of the Laplace–Beltrami operator Δ^* on ∂B_1, corresponding to the eigenvalue

$$\Lambda_k = k(n + k - 2). \tag{3.1.2}$$

Definition 3.1. A *spherical harmonic of degree k* is the restriction of a homogeneous polynomial of degree k to ∂B_1, that is, $Y_k(\xi) = P_k(\xi)$, where $\xi = \frac{x}{|x|}$.

By Green's theorem and the fact that $\partial_\nu P_m(r\xi)|_{r=1} = mP_m(\xi)$ we obtain

$$0 = \int\limits_{B_1} [P_k \Delta P_q - P_q \Delta P_k]\, dx = \oint\limits_{\partial B_1} [P_k(\xi)P_q(\xi)(q - k)]\, dS.$$

Hence,

$$\int\limits_{\partial B_1} Y_k(\xi)Y_q(\xi)\, dS = 0 \quad \text{if } k \neq q. \tag{3.1.3}$$

The space of spherical harmonics of degree k will be denoted by $\mathcal{H}_k$. For $k \in \mathbb{N}$ its dimension is given by

$$d_k := \begin{cases} 2 & \text{if } n = 2, \\ \frac{(2k+n-2)(n+k-3)!}{(n-2)!k!} & \text{if } n \geq 3 \end{cases} \quad \text{and} \quad d_0 = 1. \tag{3.1.4}$$

For a given k we can find a system of d_k linearly independent spherical harmonics $Y_{k,i}(\xi)$ with $i = 1, \dots, d_k$ and $k = 0, 1, \dots$. We shall assume that in the Hilbert space $L^2(\partial B_1)$ we have the normalization

$$\oint_{\partial B_1} Y_{k,i}(\xi) Y_{k,j}(\xi)\, dS = \delta_{ij}. \tag{3.1.5}$$

From Example 3.1 we deduce that:
- in two dimensions, $Y_{0,1} = \frac{1}{\sqrt{2\pi}}$, $Y_{k,1}(\xi) = \frac{1}{\sqrt{\pi}} \cos(k\theta)$, and $Y_{k,2}(\xi) = \frac{1}{\sqrt{\pi}} \sin(k\theta)$, where $\xi = (\cos\theta, \sin\theta)$.
- in three dimensions, the spherical harmonics of degree $k = 2$ are

$$\begin{aligned} Y_{2,1}(\xi) &= \frac{1}{4}\sqrt{\frac{5}{\pi}}(2\xi_3^2 - \xi_1^2 - \xi_2^2), \quad Y_{2,2}(\xi) = \frac{1}{2}\sqrt{\frac{15}{\pi}}\xi_1\xi_3, \\ Y_{2,3}(\xi) &= \frac{1}{4}\sqrt{\frac{15}{\pi}}(\xi_1^1 - \xi_2^2), \quad Y_{2,4}(\xi) = \frac{1}{2}\sqrt{\frac{15}{\pi}}\xi_1\xi_2, \\ Y_{2,5}(\xi) &= \frac{1}{2}\sqrt{\frac{15}{\pi}}\xi_2\xi_3. \end{aligned}$$

For any dimension n, $\mathcal{H}_0$ is spanned by the constant function and $\mathcal{H}_1$ is spanned by $\{\xi_i : i = 1, \dots, n\}$. It is well known (see for instance [53]) that

$$L^2(\partial B_1) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k,$$

where the right-hand side is an orthonormal direct sum with respect to the scalar product on $L^2(\partial B_1)$. Together with the Stone–Weierstrass approximation theorem this leads to the following property of the spherical harmonics.

Any function $f(\xi)$ which is continuous on ∂B_1 can be expanded pointwise into a series of spherical harmonics

$$f(\xi) = \sum_{k=0}^{\infty} \sum_{i=1}^{d_k} c_{ki} Y_{k,i}.$$

The convergence is uniform. By (3.1.5),

$$c_{ki} = \oint_{\partial B_1} f(\xi) Y_{k,i}(\xi)\, dS.$$

It will often be convenient to write

$$f(\xi) = \sum_{k=0}^{\infty} c_k Y_k(\xi),$$

where $Y_k = \gamma_k \sum_{i=1}^{d_k} c_{ki} Y_{k,i}(\xi)$. The normalization factor γ_k is chosen such that $\oint_{\partial B_1} Y_k^2\, dS = 1$. Thus, by the orthonormality of $Y_{k,i}$,

$$\gamma_k = \frac{1}{\sqrt{\sum_{i=1}^{d_k} c_{ki}^2}}.$$

Hence,

$$Y_k(\xi) = \left\{ \sum_{k=1}^{d_k} c_{ki}^2 \right\}^{-1/2} \sum_{k=1}^{d_k} c_{ki} Y_{k,i}.$$

Clearly, by (3.1.3)

$$\oint_{\partial B_1} Y_i(\xi) Y_j(\xi)\, dS = \delta_{ij}$$

and

$$c_k = \oint_{\partial B_1} f(\xi) Y_k(\xi)\, dS.$$

We number the eigenvalues $\{\Lambda_k\}_{k=0}^{\infty}$ of (3.1.2) taking into account their multiplicity. For instance,

$$\begin{aligned} &\Lambda_0 = \tilde{\Lambda}_0 = 0, \\ &\Lambda_1 = \tilde{\Lambda}_1 = \tilde{\Lambda}_2 = \cdots = \tilde{\Lambda}_n = n - 1, \\ &\Lambda_2 = \tilde{\Lambda}_{n+1} = \tilde{\Lambda}_{n+2} = \cdots = \tilde{\Lambda}_{\frac{n^2+n-2}{2}} = 2n \\ &\ldots \end{aligned}$$

For functions $f \in W^{1,2}(\partial B_1)$ it follows that

$$\oint_{\partial B_1} (\nabla^\tau f \cdot \nabla^\tau f)\, dS = \sum_{k=1}^{\infty} \Lambda_k c_k^2.$$

A special case of homogeneous, harmonic polynomials are the *Legendre functions* $L_k(x)$ of degree k, which are defined as follows:

1. $L_k(Ox) = L_k(x)$ for all orthogonal transformations O which leave the vector $e_n := (0, 0, \ldots, 1)$ unchanged;
2. $L_k(e_n) = 1$.

Any $\xi \in \partial B_1$ can be written as $\xi = te_n + \sqrt{1 - t^2}\xi'$ with $\xi' \in \mathbb{R}^{n-1}$ and $|\xi'| = 1$. Clearly, $t = (\xi \cdot e_n)$. From this definition it follows that $L_k(x)$ depends only on t. Hence, $L_k(x) =:$

$P_k(t)$, $t \in (-1, +1)$, where $P_k(t)$ is called the *Legendre* polynomial. It can be computed by means of the *Rodrigues* formula

$$P_k(t) = \frac{1}{2^k k!} \frac{d^k}{dt^k} (t^2 - 1)^k.$$

From this formula we obtain

$$P_0(t) = 1, \quad P_1(t) = t, \quad P_2(t) = \frac{3t^2 - 1}{2}.$$

The Funk–Hecke formula

Suppose that $f : (-1, 1) \to \mathbb{R}$ satisfies the condition

$$\int_{-1}^{1} |f(t)| (1 - t^2)^{(n-3)/2} \, dt < \infty.$$

The Funk–Hecke formula states that the spherical harmonics of degree k are the eigenfunctions of the integral equation

$$\oint_{\partial B_1} f(\xi \cdot \eta) Y_{k,i}(\eta) \, dS_\eta = \lambda Y_{k,i}(\xi), \quad \text{where } \lambda = |\partial B_1| \int_{-1}^{1} f(t) P_k(t) (1 - t^2)^{(n-3)/2} \, dt. \tag{3.1.6}$$

Proof. We consider the special case $n = 2$. Then $(\xi \cdot \eta) = \cos(\theta_\xi - \theta_\eta)$ and $Y_{k,1}(\eta) = \frac{1}{\sqrt{\pi}} \cos(k\theta_\eta)$. Thus,

$$I := \oint_{\partial B_1} f(\xi \cdot \eta) Y_{k,1}(\eta) dS_\eta = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f\big(\cos(\theta_\xi - \theta_\eta)\big) \cos k\theta_\eta \, d\theta_\eta.$$

After the change of variables $\theta_\zeta = \theta_\eta - \theta_\xi$,

$$\begin{aligned} I &= \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(\cos\theta_\zeta) \cos k(\theta_\zeta + \theta_\xi) \, d\theta_\zeta \\ &= \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(\cos\theta_\zeta)(\cos k\theta_\zeta \cos k\theta_\xi - \sin k\theta_\zeta \sin k\theta_\xi) \, d\theta_\zeta. \end{aligned}$$

Observe that

$$\begin{aligned} f\big(\cos(\pi - \theta)\big) &= f\big(\cos(\pi + \theta)\big), \\ \sin(k\pi - k\theta) &= -\cos k\pi \sin k\theta = -\sin(k\pi + k\theta). \end{aligned}$$

This implies that

$$\int_0^{2\pi} f(\cos\theta_\zeta)\sin k\theta_\zeta\, d\theta_\zeta = 0.$$

Consequently,

$$I = \frac{1}{\sqrt{\pi}} \cos k\theta_\xi \int_0^{2\pi} f(\cos\theta_\zeta)\cos k\theta_\zeta\, d\theta_\zeta.$$

This completes the proof of the Funk–Hecke formula in two dimensions. For higher dimensions we refer to the Lecture Notes by C. Müller [91]. □

The values $\lambda = \lambda_k$ ($k \geq 0$) are the eigenvalues of the compact operator

$$K(g)(\xi) := \oint_{\partial B_1} f((\xi\cdot\eta))g(\eta)\, dS_\eta.$$

They are ordered as

$$|\lambda_0| \geq |\lambda_1| \geq |\lambda_2| \geq \cdots,$$

with zero as the only possible accumulation point. The eigenvalue λ_0 is simple, while the multiplicity of λ_1 is the space dimension n.

3.2 The Steklov eigenvalue problem

3.2.1 The ball

The classical Steklov eigenvalue problem in the ball is

$$\Delta\phi = 0 \text{ in } B_R, \quad \partial_\nu\phi = \mu\phi \text{ on } \partial B_R. \tag{3.2.1}$$

In spherical coordinates (r,ξ) the Laplacian Δ is of the form

$$\Delta = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta^*,$$

where Δ^* is the Laplace–Beltrami operator on ∂B_1. Separation of variables $\phi = a(r)b(\xi)$ leads to

$$0 = a_{rr}(r)b(\xi) + \frac{n-1}{r}a_r(r)b(\xi) + \frac{a(r)}{r^2}\Delta^* b(\xi).$$

If $b(\xi) = Y_{k,i}(\xi)$ is a spherical harmonic, the equation for $a(r)$ becomes

$$a_{rr}(r) + \frac{n-1}{r}a_r(r) - \frac{k(n+k-2)}{r^2}a(r) = 0.$$

The solutions are $a_1(r) = r^k$ and $a_2(r) = r^{-k-n+2}$.

Since for $k = 0, 1, \ldots, \infty$, $i = 1, \ldots, d_k$, the functions $Y_{k,i}(\xi)$ are a basis in $L^2(\partial B_1)$, every continuous harmonic function in B_R can be represented as a Fourier series

$$h(r,\xi) = h_{01}Y_{0,1}(\xi) + \sum_{k=1}^{\infty}\sum_{i=1}^{d_k} h_{ki}r^k Y_{k,i}(\xi),$$

where h_{01} and h_{ik} are real numbers.

The eigenfunctions of the classical Steklov problem in the ball are of the form

$$\phi(r,\xi) = \left(\frac{r}{R}\right)^k Y_{k,i}(\xi), \quad k = 0, 1\ldots, \; i = 1, 2, \ldots, d_k, \tag{3.2.2}$$

and the corresponding eigenvalues are

$$\mu_k = \frac{k}{R}. \tag{3.2.3}$$

Their multiplicity is the same as the dimension of $\mathcal{H}_k$. We number the eigenvalues $\{\tilde{\mu}_k\}_{k=0}^{\infty}$ of (3.2.1) taking into account their multiplicity. For example, we have

$$\begin{aligned} &\mu_0 = \tilde{\mu}_0 = 0, \quad \phi_0 = \text{const.}, \\ &\mu_1 = \tilde{\mu}_1 = \tilde{\mu}_2 = \cdots = \tilde{\mu}_n = \frac{1}{R}, \quad \phi_i = \frac{x_i}{R}, \quad i = 1, \ldots, n. \end{aligned} \tag{3.2.4}$$

The eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$ form an orthonormal basis for $L^2(\partial B_R)$ with respect to the inner product $\oint_{\partial B_R} uv\,dS$.

Remark 3.1. A slightly modified eigenvalue problem with a given parameter $\alpha \in \mathbb{R}$ is

$$\Delta\phi = 0 \text{ in } B_R, \quad \partial_\nu\phi + \alpha\phi = \mu\phi \text{ on } \partial B_R.$$

In complete analogy the eigenfunctions and eigenvalues are computed as

$$\phi(r,\xi) = \left(\frac{r}{R}\right)^k Y_{k,i}(\xi) \quad \text{and} \quad \mu_k = \frac{k}{R} + \alpha.$$

The multiplicity is the same as for (3.2.1).

3.2.2 The outer domain of a ball

Harmonic functions in $L^2(\mathbb{R}^n \setminus B_R)$ can be represented in the form

$$h(r,\xi) = \sum_{k=1}^{\infty}\sum_{i=1}^{d_k} h_{ki} r^{2-n-k} Y_{k,i}(\xi). \tag{3.2.5}$$

As in the previous section we will write $h(r,\xi) = \sum_{k=1}^{\infty} h_k Y_k$, thus suppressing the degeneracy.

The Steklov eigenvalue problem in outer balls reads as

$$\Delta\phi = 0 \text{ in } \mathbb{R}^n \setminus B_R, \quad \partial_\nu \phi = \mu\phi \text{ on } \partial B_R, \quad \phi \to 0 \text{ as } r \to \infty,$$

where ν points into the ball of radius R. The asymptotic $\phi \to 0$ as $r \to 0$ excludes the constant function and the logarithmic function (for $n = 2$) for the eigenvalue $\mu = 0$. From (3.2.5) and the boundary condition we deduce for $n \geq 3$

$$\phi(r,\xi) = \left(\frac{r}{R}\right)^{2-n-k} Y_{k,i}(\xi) \quad \text{and} \quad \mu_k = \frac{n-2+k}{R},$$

for $k = 1, \dots$ and $i = 1, \dots, d_k$. If $n \geq 2$, the multiplicity is the same as for Λ_k. Hence,

$$\begin{aligned} &\mu_0 = \tilde{\mu}_1 = \frac{n-2}{R}, \quad \phi_1 = \frac{c}{r^{n-2}}, \\ &\mu_1 = \tilde{\mu}_2 = \tilde{\mu}_3 = \tilde{\mu}_3 = \cdots \tilde{\mu}_{n-1} = \frac{n-1}{R}, \quad \phi_i = \frac{cx_i}{R}, \quad i = 1, 2, \dots, n. \end{aligned} \tag{3.2.6}$$

Following the arguments in Remark 3.1, the eigenfunctions and eigenvalues for the modified problem

$$\Delta\phi = 0 \text{ in } \mathbb{R}^n \setminus B_R, \quad \partial_\nu \phi + \alpha\phi = \mu\phi \text{ on } \partial B_R, \quad \phi \to 0 \text{ as } r \to \infty$$

are

$$\phi(r,\xi) = \left(\frac{r}{R}\right)^{2-n-k} Y_{k,i}(\xi) \quad \text{and} \quad \mu_k = \frac{n-2+k}{R} + \alpha.$$

The multiplicity is the same as in (3.2.6).

3.2.3 A Steklov–Dirichlet eigenvalue problem in annular domains

Consider the annular domain

$$\Omega = B_R \setminus B_{\kappa R}(0), \quad R > 0 \text{ and } 0 < \kappa < 1.$$

The Steklov–Dirichlet eigenvalue problem reads as

$$\Delta\phi = 0 \text{ in } B_R \setminus B_{\kappa R}, \quad \phi = 0 \text{ on } \partial B_R \quad \text{and} \quad \partial_\nu \phi = \mu\phi \text{ on } \partial B_{\kappa R}.$$

As in the previous sections, the separation of variables leads to the following eigenfunctions and eigenvalues:

$$\phi_{k,i} = (r^{-k-n+2} - R^{-2k-n+2} r^k) Y_{k,i}(\xi) \quad \text{and} \quad \mu_k := \frac{\kappa^{2k+n-2}k + k + n - 2}{\kappa R(1 - \kappa^{2k+n-2})}.$$

If $k = 0$, then

$$\phi_0(r) = \begin{cases} c(r^{-n+2} - R^{-n+2}) \text{ and } \mu_0 := \frac{n-2}{\kappa R(1-\kappa^{n-2})} & \text{if } n > 2, \\ c(\log r - \log R) \text{ and } \mu_0 = -\frac{1}{\kappa R \log \kappa} & \text{if } n = 2. \end{cases}$$

Thus, for $\mu_0 > 0$ the first eigenfunction is a nonconstant radial function.

If we set $\kappa := \frac{\tilde{R}}{R}$ for some fixed $\tilde{R} > 0$, then in the limit $R \to \infty$ we obtain the outer domain $\Omega = \mathbb{R}^n \setminus B_{\tilde{R}}$. For the eigenvalue μ_k, $k > 0$, we obtain the asymptotic formula $\lim_{R\to\infty} \mu_k = \frac{k+n-2}{\tilde{R}}$, which is the k-th eigenvalue of the Steklov problem in the exterior of the ball $B_{\tilde{R}}$.

A slight variant is the eigenvalue problem

$$\Delta\phi = 0 \text{ in } B_R \setminus B_{\kappa R}, \quad \phi = 0 \text{ on } \partial B_R, \quad \text{and} \quad \partial_\nu \phi + \alpha\phi = \mu\phi \text{ on } \partial B_{\kappa R}. \tag{3.2.7}$$

In this case

$$\phi_{k,i} = (r^{-k-n+2} - R^{-2k-n+2} r^k) Y_{k,i}(\xi) \tag{3.2.8}$$

and

$$\mu_k := \frac{\kappa\alpha R + \kappa^{2k+n-2}(k - \kappa\alpha R) + k + n - 2}{\kappa R(1 - \kappa^{2k+n-2})}. \tag{3.2.9}$$

3.2.4 Steklov eigenvalues in annular domains

In the annular domain

$$\Omega = B_R \setminus B_{\kappa R}(0), \quad R > 0 \text{ and } 0 < \kappa < 1,$$

the Steklov eigenvalue problem reads as

$$\Delta\phi = 0 \text{ in } B_R \setminus B_{\kappa R}, \quad \partial_\nu \phi + \alpha\phi = \mu\phi \text{ on } \partial B_R \cup \partial B_{\kappa R}. \tag{3.2.10}$$

Clearly, $\mu_0 = \alpha$ with $\phi = \text{const.}$ is the lowest eigenvalue. Separation of variables leads to

$$\phi_{k,i}^{\pm}(r,\xi) = (r^k + A^{\pm} r^{-k-n+2}) Y_{k,i}(\xi),$$

with

$$A^{\pm} := (\kappa R)^{n+2k-2} \frac{\kappa \alpha R - \mu_k^{\pm}\kappa R - k}{\kappa \alpha R - \mu_k^{\pm}\kappa R + k + n - 2}$$

and

$$\mu_k^{\pm} = \alpha + \frac{(k+n-2)(1+\kappa^{2k+n-1})}{2\kappa R(1-\kappa^{2k+n-2})} + \frac{\kappa k(1+\kappa^{2k+n-2})}{2\kappa R(1-\kappa^{2k+n-2})} \pm \frac{\sqrt{Z}}{2\kappa R(1-\kappa^{2k+n-2})}.$$

Here,

$$\begin{aligned} Z := {} & \big[(1-\kappa)k + n - 2\big]^2 + \big[(1-\kappa)k - \kappa(n-2)\big]^2 \kappa^{4k+2n-4} \\ & + 2\kappa^{2k+n-2}\big[(\kappa^2 + 6\kappa + 1)k(k+n-2) + \kappa(n-2)^2\big]. \end{aligned}$$

Figure 3.1 shows $\mu_k^{\pm}$ as a function of k for $\alpha = -4$, $n = 3$, and $R = 1$. In Figure 3.1a we choose $\kappa = 0.1$, while in Figure 3.1b $\kappa = 0.9$.

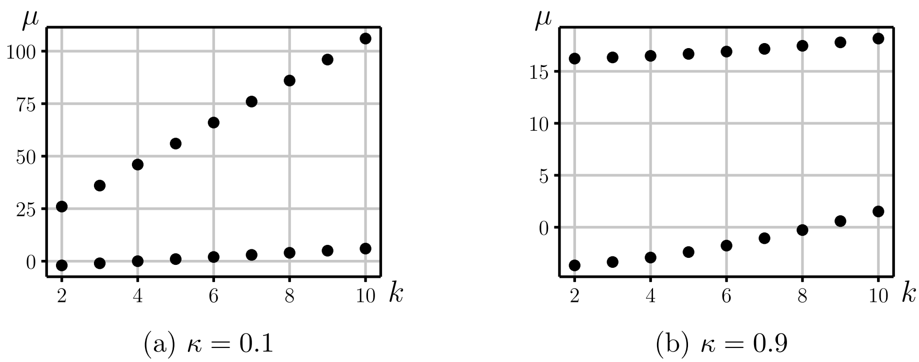

Figure 3.1: Steklov eigenvalues in annular domains.

Remark 3.2. The following asymptotic behavior can be shown.
1. For $\kappa \to 0$ we have $\lim_{\kappa\to 0}\mu_k^+(\kappa) = \infty$ and $\lim_{\kappa\to 0}\mu_k^-(\kappa) = \mu_k = \alpha + \frac{k}{R}$, which is the k-th Steklov eigenvalue of the ball (see Remark 3.1). On the other hand, $\lim_{\kappa\to 1}\mu_k^+(\kappa) = \infty$ and $\lim_{\kappa\to 1}\mu_k^-(\kappa) = 0$.
2. Let $\kappa = \frac{\tilde{R}}{R}$ for some fixed $\tilde{R} > 0$. Then

$$\lim_{R\to\infty}\mu_k^+ = \alpha + \frac{k+n-2}{\tilde{R}} \quad \text{and} \quad \lim_{R\to\infty}\mu_k^- = 0.$$

Thus, $\mu_k^+(R)$ converges for $R \to \infty$ to the k-th Steklov eigenvalue of the outer ball domain $\mathbb{R}^n \setminus B_{\tilde{R}}$ (see Section 3.2.2).

3.3 Membrane eigenvalue problems

In this section we consider the Helmholtz equation

$$\Delta u + \lambda u = 0 \quad \text{in } B_R, \tag{3.3.1}$$

subject to one of the following boundary conditions:
- *Dirichlet*, $u = 0$;
- *Neumann*, $\partial_\nu u = 0$;
- *Robin*, $\partial_\nu u + \alpha u = 0$, $\alpha \in \mathbb{R}$.

Separation of variables $u = r^\gamma J(r) Y_k(\theta)$, where $Y_k(\xi)$ is a spherical harmonic of degree k, leads to

$$J''(r) + J'(r)\left(\frac{2\gamma + n - 1}{r}\right) + J(r)\left(\frac{\gamma(\gamma + n - 2) - k(k + n - 2)}{r^2} + \lambda\right) = 0.$$

If we set $\gamma = -\frac{n-2}{2}$, then

$$r^2 J''(r) + rJ'(r) + (\lambda r^2 - \nu^2)J(r) = 0, \quad \nu = k + \frac{n-2}{2}.$$

The solutions which are regular at the origin are the *Bessel function* $J_\nu(\sqrt{\lambda} r)$ if $\lambda > 0$ and the *modified Bessel functions* $I_\nu(\sqrt{|\lambda|} r)$ if $\lambda < 0$. The boundary conditions give rise to a countable number of eigenfunctions and eigenvalues $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ tending to infinity. We shall write

$$r^{-(n-2)/2} \mathcal{J}_\nu(\sqrt{|\lambda|} r) Y_k(\xi), \quad \nu = k + (n-2)/2,$$

where $\mathcal{J}_\nu$ stands for either the Bessel function or the modified Bessel function.

The lowest eigenvalues λ^D, λ^N, and λ^R corresponding to Dirichlet, Neumann, and Robin boundary conditions, respectively, are

$$\lambda_1^D = \frac{j_0^2}{R^2}, \quad \text{where } j_0 \text{ is the first zero of } J_{(n-2)/2},$$

$$\lambda_1^N = 0 \text{ and } \lambda_2^N = \tilde{\lambda}_2^N = \cdots = \tilde{\lambda}_{n+1}^N = \frac{j_1^2}{R^2}, \quad \text{where } j_1 \text{ is the first zero of } J_{1+(n-2)/2},$$

$$\lambda_1^R = \text{lowest solution of}$$

$$\frac{2-n}{2}\mathcal{J}_{\frac{n-2}{2}}(\sqrt{|\lambda|}R) + \sqrt{|\lambda|}R\mathcal{J}'_{\frac{n-2}{2}}(\sqrt{|\lambda|}R) + \alpha R \mathcal{J}_{\frac{n-2}{2}}(\sqrt{|\lambda|}R) = 0,$$

where $\mathcal{J} = J$ or I depending on the sign of λ.

3.3.1 The generalized Steklov problem

We are looking for an eigenvalue μ such that for fixed λ the problem

$$\Delta\phi + \lambda\phi = 0 \text{ in } B_R, \quad \partial_\nu\phi = \mu\phi \text{ on } \partial B_R \tag{3.3.2}$$

has a nontrivial solution. In contrast to the classical Steklov problem, $\mu = 0$ is not the lowest eigenvalue anymore. If we separate variables

$$\phi_k(r,\xi) = c_k(r)Y_k(\xi),$$

we obtain

$$c_k(r) = r^{-(n-2)/2}\mathcal{J}_\nu(\sqrt{|\lambda|}r), \quad \text{where } \nu = k + \frac{n-2}{2}.$$

The eigenvalue μ_k is then determined by

$$\frac{2-n}{2}\mathcal{J}_{k+\frac{n-2}{2}}(\sqrt{|\lambda|}R) + \sqrt{|\lambda|}R\mathcal{J}'_{k+\frac{n-2}{2}}(\sqrt{|\lambda|}R) = \mu_k R\mathcal{J}_{k+\frac{n-2}{2}}(\sqrt{|\lambda|}R).$$

The lowest eigenvalue corresponds to the lowest root for $k = 0$. It is simple and the eigenfunction is radial. Note that

$$\oint_{\partial B_R} \phi_k\phi_i \, dS = 0 \quad \text{if } k \neq i.$$

3.4 The buckling plate

The eigenvalue problem of the buckling plate is

$$\Delta^2 u + \Lambda\Delta u = 0 \text{ in } B_R, \quad u = \partial_\nu u = 0 \text{ on } \partial B_R.$$

The eigenvalues are positive. This is easily seen by testing this equation with u. The differential equation can be written as $(\Delta + \Lambda)\Delta u = 0$, and it therefore breaks into two equations, namely $\Delta u = v$ and $\Delta v + \Lambda v = 0$. Consequently,

$$\Delta\left(u + \frac{v}{\Lambda}\right) = 0.$$

Hence, $u + \frac{v}{\Lambda}$ is a harmonic function. Every solution of $\Delta^2 u + \Lambda\Delta u = 0$ is therefore of the form $u = h + \phi$, where $\Delta h = 0$ and $\Delta\phi + \Lambda\phi = 0$. It can be written in the form

$$u(r,\xi) = \sum_{k=0}^{\infty}(h_k r^k + c_k r^{-(n-2)/2} J_{\nu_k}(\sqrt{\Lambda}r))Y_k(\xi),$$

where $\nu_k = k + (n-2)/2$. The boundary conditions together with $\oint_{\partial B_1} Y_m Y_k \, dS = \delta_{km}$ lead to a linear system for the unknowns h_k and c_k:

$$h_k R^k + c_k R^{-(n-2)/2} J_{\nu_k}(\sqrt{\Lambda} R) = 0,$$
$$k h_k R^{k-1} - c_k \frac{n-2}{2} R^{-n/2} J_{\nu_k}(\sqrt{\Lambda} R) + c_k R^{-(n-2)/2} \sqrt{\Lambda} J'_{\nu_k}(\sqrt{\Lambda} R) = 0.$$

It has a nontrivial solution provided the corresponding determinant vanishes. Thus,

$$R^{k-(n-2)/2} \sqrt{\Lambda} J'_{\nu_k}(\sqrt{\Lambda} R) - R^{k-n/2} \nu_k J_{\nu_k}(\sqrt{\Lambda} R) = 0.$$

Replacing $J'_\nu(z)$ by $\frac{\nu}{z} J_\nu(z) - J_{\nu+1}$, we get

$$-\sqrt{\Lambda} R^{k-(n-2)/2} J_{\nu_k+1}(\sqrt{\Lambda} R) = 0.$$

Obviously, $\Lambda = 0$ cannot be an eigenvalue. Therefore, $\sqrt{\Lambda} R = j_{\nu_k+1}(\ell)$, where $j_\nu(\ell)$ is the ℓ-th zero of J_ν. An argument based essentially on the completeness of the spherical harmonics on ∂B_1 reveals that we have found all eigenvalues. Hence,

$$\Lambda_1(B_R) = \left(\frac{j_{\frac{n}{2}}(1)}{R} \right)^2, \tag{3.4.1}$$
$$u_1(r) = r^{-\frac{n-2}{2}} R J_{\frac{n-2}{2}}(\sqrt{\Lambda} r) - R^{-\frac{n-2}{2}} r J_{\frac{n-2}{2}}(\sqrt{\Lambda} R).$$

Consequently, the first eigenvalue is simple and its eigenfunction is radial and monotone. Note that in the ball Λ_1 coincides with the second eigenvalue of the membrane with Dirichlet boundary conditions.

3.4.1 Fourth order Steklov eigenvalue problem

Consider the eigenvalue problem

$$\Delta^2 \varphi = 0 \text{ in } B_R, \quad \varphi = 0 \text{ on } \partial B_R, \quad \Delta\varphi = d\partial_\nu \varphi \text{ on } \partial B_R. \tag{3.4.2}$$

From $\Delta^2 = \Delta\Delta = 0$ it follows that $\Delta\varphi = h$, where h is harmonic. The sequence $\{h_k\}_0^\infty$ with $h_k = r^k Y_k(\xi)$ is a basis for the harmonic functions in B_R. The solutions of $\Delta u_k = a_k r^k Y_k(\xi)$ can be obtained by separation of variables, $u_k = A_k(r) Y_k(\xi)$. The function A_k is a regular solution of

$$A''_k + \frac{n-1}{r} A'_k - \frac{k(n-2+k)}{r^2} A_k = a_k r^k.$$

It is of the form $A_k(r) = \frac{a_k}{2(n+2k)} r^{k+2} + b_k r^k$. Hence,

$$\varphi(r,\xi) = \sum_{k=0}^{\infty}\left(b_k r^k + \frac{a_k}{2(n+2k)} r^{k+2}\right) Y_k(\xi).$$

The boundary conditions lead to

$$\varphi(R,\xi) = 0 \quad \Longrightarrow \quad b_k = -\frac{a_k R^2}{2(n+2k)}$$

and

$$\Delta\varphi = d\partial_\nu\varphi \quad \Longrightarrow \quad a_k R^k = d\left(k b_k R^{k-1} + a_k \frac{(k+2)R^{k+1}}{2(n+2k)}\right).$$

Consequently, the eigenvalues of (3.4.2) are

$$d_k = \frac{n+2k}{R}, \quad k = 0, 1, 2, \ldots,$$

and the corresponding eigenfunctions are

$$\varphi_k = a_k\left(r^k - \frac{r^{k+2}}{R^2}\right) Y_k(\xi).$$

The first eigenvalue is simple and its eigenfunction is radial and of constant sign.

3.5 Notes

References to spherical harmonics and to the Funk–Hecke formula are for instance the Lecture Notes of C. Müller [91].

4 Variational formulas

We present two methods to derive variational formulas for domain functionals. The change of variables method maps the regions via a diffeomorphism into a fixed domain and expresses the functionals over this fixed domain. In this case one perturbs the coefficients. Its implementation requires only the chain rule. The moving surface method captures the change in shape by means of a boundary displacement. This displacement is expressed by a one-parameter family of perturbations. It is used to understand the evolution of a given domain functional as a function of the parameter.

4.1 The change of variables method

4.1.1 Introduction

Let $\{\Omega_t\}_{|t|<t_0}$ be a family of domains introduced in Chapter 2 and let $\Omega = \Omega_0$. We shall consider two types of functionals:

1. Domain functionals,

$$\mathcal{G}(t) := \int_{\Omega_t} G(u(y,t), \nabla_y u(y,t), y, t)\, dy,$$

where $G : \mathbb{R} \times \mathbb{R}^n \times \Omega_t \times \mathbb{R} \to \mathbb{R}$ and $u : \Omega_t \times \mathbb{R} \to \mathbb{R}$ are smooth functions. Moreover, u may be subject to further conditions such as being the solution of a differential equation.

2. Boundary functionals,

$$\mathcal{B}(t) := \oint_{\partial\Omega_t} B(u(y,t), y, t)\, dS_{\partial\Omega_t},$$

where $B : \mathbb{R} \times \mathcal{U} \times \mathbb{R} \to \mathbb{R}$ and $u : \mathcal{U} \times \mathbb{R} \to \mathbb{R}$ are smooth functions. Here $\mathcal{U}$ is a tubular neighborhood of $\partial\Omega$.

Our goal is to compute the first and second variations of $\mathcal{G}(t)$ and $\mathcal{B}(t)$ at $t = 0$. With this aim in mind we shall assume sufficient regularity to carry out our computations.

We make the change of variable $y \to x$, where

$$y = \Phi_t(x) = x + tv(x) + \frac{t^2}{2} w(x) + o(t^2).$$

By the chain rule and (2.1.5) we have

$$\mathcal{G}(t) = \int_{\Omega} G\left(u(\Phi_t(x), t), \partial_j u \frac{\partial x_j}{\partial y_1}, \partial_j u \frac{\partial x_j}{\partial y_2}, \ldots, \partial_j u \frac{\partial x_j}{\partial y_n}, \Phi_t(x), t \right) J(t)\, dx. \tag{4.1.1}$$

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Similarly we get in view of (2.3.16)

$$\mathcal{B}(t) = \oint_{\partial\Omega} B\big(u(\Phi_t(x), t), \Phi_t(x), t\big) m(t)\, dS(x). \tag{4.1.2}$$

For $|t| < t_0$ the functionals are now defined on a fixed domain. They can be differentiated with respect to t. In the next subsections we shall carry out this differentiation in some special cases.

4.1.2 Volume integrals which depend on u only

The first example is a volume integral where $G = G(u(y, t))$. For simplicity we set

$$\tilde{u}(t) := u\big(\Phi_t(x), t\big).$$

Consequently,

$$\dot{\tilde{u}}(t) = \big((\nabla_y u(y, t) \cdot \partial_t \Phi_t) + \partial_t u(y, t)\big)\big|_{y=\Phi_t(x)}. \tag{4.1.3}$$

According to Definition 2.6 we get for $t = 0$ the following relation between the material derivative $\dot{\tilde{u}}(0)$ and the shape derivative $u'(x)$:

$$\dot{\tilde{u}}(0) = \partial_i u(x) v_i(x) + u'(x). \tag{4.1.4}$$

By (4.1.1),

$$\mathcal{G}(t) = \int_{\Omega_t} G\big(u(y, t)\big)\, dy = \int_{\Omega} G\big(\tilde{u}(t)\big) J(t)\, dx.$$

Differentiation with respect to t implies

$$\dot{\mathcal{G}}(t) = \int_{\Omega} \big[G'\big(\tilde{u}(t)\big)\dot{\tilde{u}}(t) J(t) + G\big(\tilde{u}(t)\big)\dot{J}(t)\big]\, dx, \tag{4.1.5}$$

where $G'(s) = \frac{d}{ds} G(s)$. Since $J(0) = 1$ and $\dot{J}(0) = \operatorname{div} v$, the divergence theorem leads to

$$\dot{\mathcal{G}}(0) = \oint_{\partial\Omega} G\big(u(x)\big)(v \cdot \nu)\, dS + \int_{\Omega} G'\big(u(x)\big) u'(x)\, dx. \tag{4.1.6}$$

The formula for $\dot{\mathcal{G}}(t)$ is obtained if we replace the x-coordinates by the y-coordinates. We then get

$$\int_{\Omega} G'(\tilde{u}(t))\dot{\tilde{u}}(t)J(t)\,dx = \int_{\Omega_t} G'(u(y,t))\partial_t u(y,t)\,dy + \int_{\Omega_t} G'(u(y,t))(\nabla u \cdot \partial_t \Phi_t(\Phi_t^1(y)))\,dy.$$

Moreover, by Jacobi's formula for determinants (A.1.1),

$$\dot{J}(t) = \frac{\partial}{\partial t}\det D_{\phi_t} = J(t)T\big((D_{\Phi_t})^{-1}\partial_t D_{\Phi_t}\big) = J(t)\frac{\partial_t(\Phi_t)_j}{\partial y_j}.$$

Hence,

$$\int_{\Omega} G(\tilde{u}(t))\dot{J}(t)]\,dx = \int_{\Omega_t} G(u(y,t))\operatorname{div}(\partial_t\Phi_t(\Phi_t^{-1}(y)))\,dy.$$

Then (4.1.5) can be expressed as

$$\dot{\mathcal{G}}(t) = \int_{\Omega_t} G'(u(y,t),t)\partial_t u(y,t)\,dy + \oint_{\partial\Omega_t} G(u(y,t))(\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot \nu^t)\,dS. \tag{4.1.7}$$

This formula is attributed to Reynolds. In Section 4.2 we will present a different proof of it.

For the second variation a straightforward differentiation of $\mathcal{G}(t)$ leads to

$$\ddot{\mathcal{G}}(0) = \int_{\Omega} (G''(u)\dot{\tilde{u}}^2(0) + G'(u)\ddot{\tilde{u}}(0) + 2G'(u)\dot{\tilde{u}}(0)\dot{J}(0) + G(u)\ddot{J}(0))\,dx.$$

If we differentiate $\dot{\tilde{u}}(t)$ in (4.1.3) with respect to t, we obtain

$$\ddot{\tilde{u}}(0) = u'' + 2(v\cdot\nabla u') + (v\cdot D^2 uv) + (w\cdot\nabla u). \tag{4.1.8}$$

Here $D^2u = \{\partial_i\partial_j u\}_{i,j=1}^n$ denotes the Hesse matrix. We replace $\dot{\tilde{u}}(0)$ by (4.1.4) and take into account that by (2.3.4), $\ddot{J}(0) = \operatorname{div}[v\operatorname{div} v - vD_v + w]$. Hence

$$\begin{aligned}\ddot{\mathcal{G}}(0) = {} & \int_{\Omega} G'(u)u''\,dx + \int_{\Omega} G''(u)u'^2\,dx + 2\int_{\Omega}\operatorname{div}[G'(u)u'v]\,dx \\ & + \int_{\Omega} G'(u)(v\cdot D^2uv)\,dx + \int_{\Omega} G''(u)(v\cdot\nabla u)^2\,dx \\ & + 2\int_{\Omega} G'(u)(v\cdot\nabla u)\operatorname{div} v\,dx + \int_{\Omega} G'(u)(w\cdot\nabla u)\,dx \\ & + \int_{\Omega} G(u)\operatorname{div}[v\operatorname{div} v - vD_v + w]\,dx.\end{aligned} \tag{4.1.9}$$

This expression can be simplified by means of the following two identities. The first identity is

$$G'(u)(v\cdot D^2uv) = \operatorname{div}[vG'(u)(v\cdot\nabla u)] - G''(u)(v\cdot\nabla u)^2 \\ - G'(u)[\operatorname{div} v(v\cdot\nabla u) + (v\cdot D_v\nabla u)],$$

which implies that

$$G'(u)(v\cdot D^2uv) + G''(u)(v\cdot\nabla u)^2 + G'(u)\operatorname{div} v(v\cdot\nabla u) \\ = \operatorname{div}[vG'(u)(v\cdot\nabla u)] - G'(u)(v\cdot D_v\nabla u).$$

The second identity is

$$\begin{aligned} &G'(u)\operatorname{div} v(v\cdot\nabla u) + G'(u)(w\cdot\nabla u) + G(u)\operatorname{div}[v\operatorname{div} v - vD_v + w] \\ &\quad = (v\cdot\nabla G(u))\operatorname{div} v + (w\cdot\nabla G(u)) + \operatorname{div}[G(u)(v\operatorname{div} v - vD_v + w)] \\ &\quad - (v\cdot\nabla G(u))\operatorname{div} v + (v\cdot D_v\nabla G(u)) - (w\cdot\nabla G(u)) \\ &\quad = \operatorname{div}[G(u)(v\operatorname{div} v - vD_v + w)] + G'(u)(v\cdot D_v\nabla u). \end{aligned}$$

Inserting these two identities into (4.1.9), we obtain

$$\begin{aligned} \ddot{\mathcal{G}}(0) = &\int_\Omega G'(u)u''\,dx + \int_\Omega G''(u)u'^2\,dx + 2\int_{\partial\Omega} G'(u)u'(v\cdot\nu)\,dS \\ &+ \int_{\partial\Omega} G'(u)(v\cdot\nabla u)(v\cdot\nu)\,dS + \int_{\partial\Omega} G(u)[(v\cdot\nu)\operatorname{div} v - (v\cdot D_v\nu) + (w\cdot\nu)]\,dS. \end{aligned} \tag{4.1.10}$$

Applications of this formula will be discussed in Chapter 5.

4.1.3 Boundary integrals which depend on *u* only

We consider boundary integrals of the type

$$\mathcal{B}(u) = \oint_{\partial\Omega_t} B(u(y,t))\,dS_t = \oint_{\partial\Omega} B(\tilde u(t))m(t)\,dS.$$

Compared to (4.1.2) the integrand does not depend explicitly on y and t. From (4.1.2), straightforward differentiation with respect to t yields

$$\begin{aligned} \dot{\mathcal{B}}(t) &= \oint_{\partial\Omega} B'(\tilde u(t))\dot{\tilde u}(t)m(t)\,dS(x) + \oint_{\partial\Omega} B(\tilde u(t))\dot m(t)\,dS(x) \\ &= \oint_{\partial\Omega} B'(\tilde u(t))[\partial_t\Phi_t(x)\cdot\nabla\tilde u(t) + \partial_t\tilde u(t)]m(t)\,dS \\ &\quad + \oint_{\partial\Omega} B(\tilde u(t))\dot m(t)\,dS. \end{aligned}$$

For $t = 0$, (4.1.4) and (2.3.19) imply

$$\dot{\mathcal{B}}(0) = \oint_{\partial\Omega} B'(u(x))[v \cdot \nabla u(x) + u'(x)]\, dS + \oint_{\partial\Omega} B(u(x))[\operatorname{div}_{\partial\Omega} v^\tau + (n-1)H(v \cdot \nu)]\, dS. \tag{4.1.11}$$

The expression for general t is obtained by replacing $\partial\Omega$ by $\partial\Omega_t$ and the geometrical quantities ν and H by the corresponding expressions ν^t and H^t on $\partial\Omega_t$. Gauss' theorem on surfaces (2.2.18) then leads to the second Reynolds formula

$$\dot{\mathcal{B}}(t) = \oint_{\partial\Omega_t} B'(u(y,t))\partial_t u(y,t)\, dS + \oint_{\partial\Omega_t} \partial_{\nu^t} B(u(y,t))(\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu^t)\, dS + (n-1) \oint_{\partial\Omega_t} B(u(y,t))H^t(y)(\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu^t)\, dS. \tag{4.1.12}$$

Later on this domain variation will be derived by a more direct geometrical argument.

The second derivative in $t = 0$ is given by

$$\ddot{\mathcal{B}}(0) = \oint_{\partial\Omega} B''(u)\dot{\tilde{u}}^2(0)m(0)\, dS + \oint_{\partial\Omega} B'(u)\ddot{\tilde{u}}(0)m(0)\, dS + 2\oint_{\partial\Omega} B'(u)\dot{\tilde{u}}(0)\dot{m}(0)\, dS + \oint_{\partial\Omega} B(u)\ddot{m}(0)\, dS.$$

If we replace $\dot{\tilde{u}}$ and $\ddot{\tilde{u}}$ by (4.1.4) and (4.1.8) and take into account that $m(0) = 1$, we obtain

$$\ddot{\mathcal{B}}(0) = \oint_{\partial\Omega} B'(u)u'' + B''(u)u'^2 + 2[B''(u)(v \cdot \nabla u) + B'(u)\dot{m}(0)]u'\, dS + 2\oint_{\partial\Omega} B'(u)(v \cdot \nabla u')\, dS + \oint_{\partial\Omega} B''(u)(v \cdot \nabla u)^2 + B(u)\ddot{m}(0)\, dS + \oint_{\partial\Omega} B'(u)[(v \cdot D^2 u v) + 2(v \cdot \nabla u)\dot{m}(0) + (w \cdot \nabla u)]\, dS, \tag{4.1.13}$$

where $\dot{m}(0)$ and $\ddot{m}(0)$ are defined in (2.3.19) and (2.3.22).

For simplicity we consider Hadamard perturbations $v = \rho\nu$ with $\rho = (v \cdot \nu)$ on $\partial\Omega$. By (2.3.19) we have $\dot{m}(0) = (n-1)H\rho$. Then (4.1.13) becomes

$$\begin{aligned}\ddot{\mathcal{B}}(0) = &\oint_{\partial\Omega} B'(u)u'' + B''(u)u'^2 + 2[B''(u)\partial_\nu u\rho + (n-1)B'(u)H\rho]u'\,dS\\
&+ 2\oint_{\partial\Omega} B'(u)\partial_\nu u'\rho\,dS + \oint_{\partial\Omega} B''(u)(\partial_\nu u)^2\rho^2\,dS\\
&+ \oint_{\partial\Omega} B'(u)[(\nu\cdot D^2u\nu)\rho^2 + 2(n-1)\partial_\nu uH\rho^2 + (w\cdot\nabla u)]\,dS\\
&+ \oint_{\partial\Omega} B(u)\ddot{m}(0)\,dS. \qquad (4.1.14)\end{aligned}$$

Note that for Hadamard perturbations $\ddot{m}(0)$ is given in (2.3.33).

Remark 4.1. In the above formula (4.1.14) the terms containing w in $\ddot{\mathcal{B}}(0)$ are

$$I := \oint_{\partial\Omega} B(u)\operatorname{div}_{\partial\Omega} w\,dS + \oint_{\partial\Omega} B'(u)(w\cdot\nabla u)\,dS.$$

We claim that I depends only on the normal component of w. This follows by applying (2.2.18) to the first integral of I. Then

$$\begin{aligned}I &= -\oint_{\partial\Omega} w\cdot\nabla^\tau B(u)\,dS + (n-1)\oint_{\partial\Omega} B(u)(w\cdot\nu)H\,dS + \oint_{\partial\Omega} B'(u)(w\cdot\nabla u)\,dS\\
&= (n-1)\oint_{\partial\Omega} B(u)(w\cdot\nu)H\,dS + \oint_{\partial\Omega} B'(u)\partial_\nu u(w\cdot\nu)\,dS,\end{aligned}$$

which proves the claim.

4.1.4 The Dirichlet integral

The Dirichlet integral of a function $u(y,t)$ in Ω_t is defined as

$$\mathcal{D}(t) := \int_{\Omega_t} |\nabla u(y,t)|^2\,dy.$$

Recall that $y = \Phi_t(x)$, $x_i = \Phi^{-1}_{t,i}(y)$, and $\tilde{u}(t) := u(\Phi_t(x),t)$. By the chain rule,

$$\partial_{y_k} u(y,t) = \partial_i\tilde{u}(t)\partial_{y_k}\Phi^{-1}_{t,i}(\Phi_t(x)) = \partial_i\tilde{u}(t)\frac{\partial x_i}{\partial y_k}.$$

Thus,

$$|\nabla u(y,t)|^2 = \partial_{y_k}\Phi^{-1}_{t,i}(\Phi_t(x))\partial_i\tilde{u}(t)\partial_{y_k}\Phi^{-1}_{t,j}(\Phi_t(x))\partial_j\tilde{u}(t).$$

Define

$$A_{ij}(t) := \partial_{y_k}\Phi_{t,i}^{-1}(\Phi_t(x))\partial_{y_k}\Phi_{t,j}^{-1}(\Phi_t(x))J(t). \tag{4.1.15}$$

After a change of variables, the Dirichlet integral becomes

$$\mathcal{D}(t) := \int_\Omega \partial_i\tilde{u}(t)\partial_j\tilde{u}(t)A_{ij}(t)\,dx.$$

Straightforward differentiation with respect to t leads to

$$\dot{\mathcal{D}}(t) = \int_\Omega \left[2\partial_i\dot{\tilde{u}}(t)\partial_j\tilde{u}(t)A_{ij}(t) + \partial_i\tilde{u}(t)\partial_j\tilde{u}(t)\dot{A}_{ij}(t)\right]dx.$$

Differentiating the integrand once more we obtain

$$\begin{aligned}\ddot{\mathcal{D}}(t) = &\int_\Omega \left[2\partial_i\ddot{\tilde{u}}(t)\partial_j\tilde{u}(t)A_{ij}(t) + 2\partial_i\dot{\tilde{u}}(t)\partial_j\dot{\tilde{u}}(t)A_{ij}(t)\right]dx\\ &+ \int_\Omega \left[4\partial_j\tilde{u}(t)\partial_i\dot{\tilde{u}}(t)\dot{A}_{ij}(t) + \partial_i\tilde{u}(t)\partial_j\tilde{u}(t)\ddot{A}_{ij}(t)\right]dx.\end{aligned}$$

Lemma 4.1. *We have*

$$\begin{aligned}A_{ij}(0) &= \delta_{ij},\\ \dot{A}_{ij}(0) &= \operatorname{div} v\delta_{ij} - \partial_j v_i - \partial_i v_j,\\ \ddot{A}_{ij}(0) &= \left((\operatorname{div} v)^2 - D_v : D_v + \operatorname{div} w\right)\delta_{ij} + 2(\partial_k v_i\partial_j v_k + \partial_k v_j\partial_i v_k)\\ &\quad + 2\partial_k v_i\partial_k v_j - 2\operatorname{div} v(\partial_j v_i + \partial_i v_j) - \partial_i w_j - \partial_j w_i.\end{aligned}$$

Proof. Set $A(t) =: B(t)J(t)$. From (4.1.15) and

$$\frac{\partial y_k}{\partial x_i} = \delta_{ik} + t\partial_i v_k + \frac{t^2}{2}\partial_i w_k = (D_{\Phi_t})_{ik},$$

it follows that

$$\big(B(t)\big)_{ij} = \big((D_{\Phi_t})^{-1}\big)_{ik}\big((D_{\Phi_t})^{-1}\big)_{jk}.$$

For small $|t|$ the Neumann series for $(D_{\Phi_t})^{-1}$ given in (2.1.3) implies that

$$\begin{aligned}\big(B(t)\big)_{ij} = \delta_{ij} - t(\partial_i v_j + \partial_j v_i) + t^2\Big[&\partial_i v_k\partial_j v_k + \partial_j v_k\partial_k v_i + \partial_i v_k\partial_k v_j\\ &- \frac{1}{2}(\partial_i w_j + \partial_j w_i)\Big] + o(t^2).\end{aligned}$$

Recall that by (2.1.5) $J(t) = 1 + t \operatorname{div} v + \frac{t^2}{2}((\operatorname{div} v)^2 - D_v : D_v + \operatorname{div} w) + o(t^2)$. This together with (4.1.15) proves the first assertion. The second assertion follows from

$$\dot{A}(0) = J(0)\dot{B}(0) + \dot{J}(0)B(0) = \dot{B}(0) + \dot{J}(0).$$

The last statement is a consequence of

$$\ddot{A}(0) = \ddot{B}(0) + 2\dot{B}(0)\dot{J}(0) + \ddot{J}(0)$$

and the particular expressions for $B(t)$ and $J(t)$. □

A useful identity

For two smooth functions we consider the expression $\nabla u \cdot \dot{A}(0)\nabla\phi$. Lemma 4.1 yields

$$\nabla u \cdot \dot{A}\nabla\phi = \operatorname{div} v \nabla u \cdot \nabla\phi - (\nabla u \cdot D_v\nabla\phi) - (\nabla\phi \cdot D_v\nabla u).$$

Then

$$\begin{aligned}\nabla u \cdot \dot{A}\nabla\phi = {} & \partial_i(v\nabla u \cdot \nabla\phi) - v \cdot D^2 u\nabla\phi - v \cdot D^2\phi\nabla u \\ & - \partial_i(\partial_i u v \cdot \nabla\phi) + \Delta u v \cdot \nabla\phi + v \cdot D^2\phi\nabla u \\ & - \partial_i(\partial_i \phi v \cdot \nabla u) + \Delta\phi v \cdot \nabla u + v \cdot D^2 u\nabla\phi,\end{aligned}$$

and therefore

$$\begin{aligned}\nabla u\dot{A}\nabla\phi = {} & \partial_i(v\nabla u \cdot \nabla\phi) - \partial_i(\partial_i u v \cdot \nabla\phi) - \partial_i(\partial_i\phi v \cdot \nabla u) \\ & + \Delta u(\nabla\phi \cdot v) + \Delta\phi(\nabla u \cdot v).\end{aligned}$$

Integration of this identity yields

$$\begin{aligned}\int_\Omega \nabla u\dot{A}\nabla\phi\, dx = {} & \oint_{\partial\Omega} \{(v \cdot \nu)(\nabla u \cdot \nabla\phi) - \partial_\nu\phi(\nabla u \cdot v) - \partial_\nu u(\nabla\phi \cdot v)\}\, dS \\ & + \int_\Omega [\Delta\phi(v \cdot \nabla u) + \Delta u(v \cdot \nabla\phi)]\, dx. \end{aligned} \tag{4.1.16}$$

4.2 The moving surface method

Instead of working on a fixed domain with variable coefficients, the moving surface method operates directly on the perturbed domains.

We repeat the main result, which goes back to Reynolds [102] and which has been derived in Sections 4.1.2 and 4.1.3 (see in particular (4.1.7) and (4.1.12)). Our goal is to present a direct geometrical proof.

Without loss of generality we may assume that

$$\mathcal{G}(t) = \int_{\Omega_t} G(y,t)\,dy \quad \text{and} \quad \mathcal{B}(t) = \oint_{\partial\Omega_t} B(y,t)\,dS.$$

We denote by ν^t the outer unit normal of $\partial\Omega_t$ and by H^t its mean curvature.

Theorem 4.1. *The first shape derivatives of $\mathcal{G}(t)$ and $\mathcal{B}(t)$ are*

$$\dot{\mathcal{G}}(t) := \int_{\Omega_t} \partial_t G(y,t)\,dy + \oint_{\partial\Omega_t} G(y,t)(\partial_t\Phi_t(\Phi_t^{-1}(y)) \cdot \nu^t)\,dS_t \tag{4.2.1}$$

and

$$\begin{aligned}\dot{\mathcal{B}}(t) := \oint_{\partial\Omega_t} \partial_t B(y,t)\,dS_t &+ \oint_{\partial\Omega_t} \partial_{\nu^t} B(y,t)(\partial_t\Phi_t(\Phi_t^{-1}(y)) \cdot \nu^t)\,dS_t \\ &+ (n-1)\oint_{\partial\Omega_t} B(y,t)H^t(y)(\partial_t\Phi_t(\Phi_t^{-1}(y)) \cdot \nu^t)\,dS_t.\end{aligned} \tag{4.2.2}$$

Proof. 1. We begin with the proof of (4.2.1). Assume that $G(y,t) : \Omega_t \times (-t_0, t_0) \to \mathbb{R}$ is a smooth function. We write (see Figure 4.1a)

$$\begin{aligned}&\int_{\Omega_{t+\epsilon}} G(y,t+\epsilon)\,dy - \int_{\Omega_t} G(y,t)\,dy \\ &= \int_{\Omega_{t+\epsilon}\cap\Omega_t} \{G(y,t+\epsilon) - G(y,t)\}\,dy + \int_{\Omega_{t+\epsilon}\setminus\Omega_t} G(y,t+\epsilon)\,dy - \int_{\Omega_t\setminus\Omega_{t+\epsilon}} G(y,t)\,dy.\end{aligned}$$

Then under the assumption that for small t_0 and fixed $y \in \bigcap_{|t|<t_0} \Omega_t$

$$G(y,t+\epsilon) = G(y,t) + \epsilon\partial_t G(y,t) + \frac{\epsilon^2}{2}\partial_t^2 G(y,t) + o(\epsilon^2), \tag{4.2.3}$$

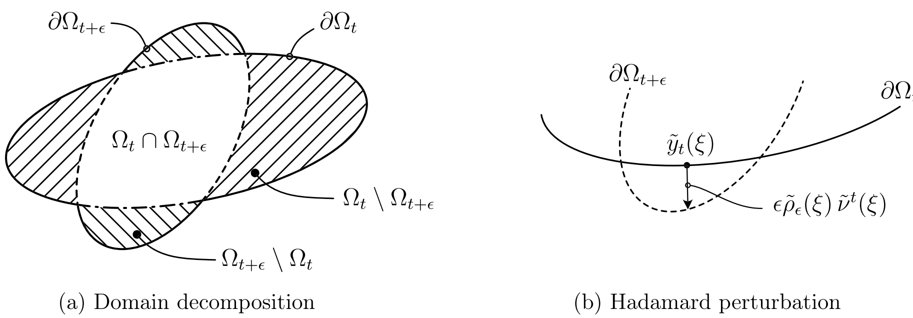

Figure 4.1: Moving surfaces.

we conclude that

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\Omega_{t+\epsilon}\cap\Omega_t}\{G(y,t+\epsilon)-G(y,t)\}\,dy=\int_{\Omega_t}\partial_t G(y,t)\,dy. \tag{4.2.4}$$

Next we compute the remaining integrals.

Suppose that

$$\partial\Omega_t\cap V=\{\tilde{y}_t(\xi)=\Phi_t(\tilde{x}(\xi)):\xi\in U,\tilde{x}(\xi)\in\partial\Omega\},$$

where $\tilde{x}:U\to V$ is a local parametrization. The boundary of $\partial\Omega_{t+\epsilon}$ is obtained by a displacement of $\partial\Omega_t$ of the form

$$\tilde{y}_{t+\epsilon}=\Phi_{t+\epsilon}(\tilde{x}(\xi))=\Phi_t(\tilde{x}(\xi))+\epsilon\tilde{v}(\xi)+\left(t\epsilon+\frac{\epsilon^2}{2}\right)\tilde{w}(\xi)+o(\epsilon^2).$$

In a neighborhood V of $\partial\Omega_t$ we introduce normal coordinates (ξ,r) (cf. Appendix B). Any point in the neighborhood of $\partial\Omega_t\cap V$ is given by

$$y=\tilde{y}_t(\xi)+r\tilde{\nu}^t(\xi),$$

where $r\in(-r_0,r_0)$.

According to Theorem 2.1 we can replace the perturbation $\Phi_{t+\epsilon}(\tilde{x}(\xi))$ by a Hadamard perturbation (see Figure 4.1b)

$$\tilde{y}_t(\xi)+\epsilon\tilde{\rho}_\epsilon(\xi)\tilde{\nu}^t(\xi)+o(\epsilon^2),\quad \text{where } \tilde{\rho}_\epsilon:=(\tilde{v}+t\tilde{w})\cdot\tilde{\nu}^t+o(\epsilon,t).$$

In normal coordinates the volume element is of the form (cf. (B.1.3))

$$(1+(n-1)H^t r+o(r))dr\,dS_t.$$

We are now in a position to compute the remaining integrals:

$$\Delta\mathcal{G}_\epsilon:=\int_{\Omega_{t+\epsilon}\setminus\Omega_t}G(y,t+\epsilon)\,dy-\int_{\Omega_t\setminus\Omega_{t+\epsilon}}G(y,t)\,dy.$$

We observe that $\tilde{\rho}_\epsilon<0$ on $\partial(\Omega_t\setminus\Omega_{t+\epsilon})$, while $\tilde{\rho}_\epsilon>0$ on $\partial(\Omega_{t+\epsilon}\setminus\Omega_t)$. Hence,

$$\begin{aligned}\Delta\mathcal{G}_\epsilon=&\oint_{\partial\Omega_t\cap\{\tilde{\rho}_\epsilon\geq 0\}}\left(\int_0^{\epsilon\tilde{\rho}_\epsilon}(1+(n-1)H^t r+o(r))G(\tilde{y}_t+r\tilde{\nu}^t,t+\epsilon)\,dr\right)dS_t\\&-\oint_{\partial\Omega_t\cap\{\tilde{\rho}_\epsilon<0\}}\left(\int_{\epsilon\tilde{\rho}_\epsilon}^0(1+(n-1)H^t r+o(\tilde{r}))G(\tilde{y}_t+r\tilde{\nu}^t,t)\,dr\right)dS_t.\end{aligned}$$

We expand $G(y, t+\epsilon)$ on the line $y = \tilde{y}_t + r\tilde{\nu}^t$, $|r| \leq \max|\epsilon\tilde{\rho}_\epsilon|$, with respect to r and ϵ and obtain

$$G(\tilde{y}_t + r\tilde{\nu}^t, t+\epsilon) = G(\tilde{y}_t, t) + \partial_{\tilde{\nu}^t} G(\tilde{y}_t, t) r + \partial_t G(\tilde{y}_t, t)\epsilon + o(r) + o(\epsilon).$$

This implies that

$$\int_0^{\epsilon\tilde{\rho}_\epsilon} (1 + (n-1)\tilde{H}^t r + o(r)) G(\tilde{y}_t + r\tilde{\nu}^t, t+\epsilon)\, dr = \epsilon\tilde{\rho}_\epsilon G(\tilde{y}_t, t) + o(\epsilon).$$

Consequently,

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \Delta\mathcal{G}_\epsilon = \oint_{\partial\Omega_t} G(\tilde{y}_t, t)\tilde{\rho}_0\, dS_t,$$

which together with (4.2.4) implies that

$$\dot{\mathcal{G}}(t) = \int_\Omega \partial_t G(y, t)\, dy + \oint_{\partial\Omega_t} G(\tilde{y}_t, t)(\tilde{v} + t\tilde{w}) \cdot \tilde{\nu}^t\, dS_t.$$

This completes the proof of the first assertion (i).

2. Next we will prove (4.2.1). Without loss of generality we can make a shift in time and compute $\dot{\mathcal{B}}(0)$. The points on $\partial\Omega_t$ are given by

$$y = \tilde{x}(\xi) + t\tilde{\rho}\tilde{\nu}(\xi) + o(|t\tilde{\rho}|), \quad \text{where } \tilde{\rho} = (\tilde{v} \cdot \tilde{\nu}).$$

As before we assume that Φ_t is a Hadamard perturbation. The Taylor expansion yields $B(y, t) = B(\tilde{x}, 0) + t\tilde{\rho}_t \partial_{\tilde{\nu}(\xi)} B(\tilde{x}, 0) + t\partial_t B(\tilde{x}, 0) + o(|t\tilde{\rho}_t|)$, and by (2.3.16), $dS_t = m(t)dS$. Hence,

$$\oint_{\partial\Omega_t} B(y, t)\, dS_t = \oint_{\partial\Omega} \{B(\tilde{x}, 0) + t\tilde{\rho}_t \partial_{\tilde{\nu}(\xi)} B(\tilde{x}, 0) + t\partial_t B(\tilde{x}, 0) + o(|t\rho_t|)\} m(t)\, dS.$$

We are now in a position to differentiate with respect to t. For Hadamard perturbations (2.3.19) implies that $\dot{m}(0) = (n-1)H\tilde{\rho}$. Consequently,

$$\dot{\mathcal{B}}(0) = \oint_{\partial\Omega} \{\partial_t B(x, 0) + \partial_\nu B(x, 0)\rho + (n-1)B(x, 0)H\rho\}\, dS.$$

The claim now follows by replacing $\partial\Omega$ by Ω_t and ρ by ρ_t as well as ν by ν^t and H by H^t. $\square$

4.2.1 Second variation for special cases

We consider the same functionals as in Section 4.1.2, and derive their second variations by means of the moving surface method.

Volume integrals which depend on *u* only
We apply the divergence theorem to the formula

$$\dot{\mathcal{G}}(t) = \int_{\Omega_t} \partial_t G(u(y,t))\,dy + \oint_{\partial\Omega_t} G(u(y,t))(\partial_t\Phi_t(\Phi_t^{-1}(y)) \cdot \nu^t)\,dS_t$$

derived in Theorem 4.1 and obtain

$$\dot{\mathcal{G}}(t) = \int_{\Omega_t} \partial_t G(u(y,t))\,dy + \int_{\Omega_t} \operatorname{div}_y[G(u(y,t))\partial_t\Phi_t(\Phi_t^{-1}(y))]\,dy.$$

Theorem 4.1 implies

$$\begin{aligned}\ddot{\mathcal{G}}(t) = {} & \int_{\Omega_t} \partial_t^2 G(u(y,t))\,dy + \oint_{\partial\Omega_t} \partial_t G(u(y,t))(\partial_t\Phi_t(\Phi_t^{-1}(y)) \cdot \nu^t)\,dS_t \\ & + \int_{\Omega_t} \partial_t(\operatorname{div}_y[G(u(y,t))\partial_t\Phi_t(\Phi_t^{-1}(y))])\,dy \\ & + \oint_{\partial\Omega_t} \operatorname{div}_y[G(u(y,t))\partial_t\Phi_t(\Phi_t^{-1}(y))](\partial_t\Phi_t(\Phi_t^{-1}(y)) \cdot \nu^t)\,dS_t.\end{aligned}$$

Interchanging the t- and the y- derivative, we obtain

$$\begin{aligned}\ddot{\mathcal{G}}(t) = {} & \int_{\Omega_t} \partial_t^2 G(u(y,t))\,dy + \oint_{\partial\Omega_t} \partial_t G(u(y,t))(\partial_t\Phi_t(\Phi_t^{-1}(y)) \cdot \nu^t)\,dS_t \\ & + \int_{\Omega_t} (\operatorname{div}_y[\partial_t G(u(y,t))\partial_t\Phi_t(\Phi_t^{-1}(y)) + G(u(y,t))\partial_t\{\partial_t\Phi_t(\Phi_t^{-1}(y))\}])\,dy \\ & + \oint_{\partial\Omega_t} \operatorname{div}_y[G(u(y,t))\partial_t\Phi_t(\Phi_t^{-1}(y))](\partial_t\Phi_t(\Phi_t^{-1}(y)) \cdot \nu^t)\,dS_t.\end{aligned}$$

For $t = 0$ we have

$$\partial_t G(u(y,t))|_{t=0} = G'(u)u' \quad \text{and} \quad \partial_t^2 G(u(y,t))|_{t=0} = G''(u)u'^2 + G'(u)u''.$$

Thus

$$\begin{aligned}&\operatorname{div}_y[\partial_t G(u(y,t))\partial_t\Phi_t(\Phi_t^{-1}(y)) + G(u(y,t))\partial_t\{\partial_t\Phi_t(\Phi_t^{-1}(y))\}]\big|_{t=0}\\ &\quad = \operatorname{div}[G'(u)u'v + G(u)w - G(u)vD_v].\end{aligned}$$

Furthermore

$$\begin{aligned}&\operatorname{div}_y[G(u(y,t))\partial_t\Phi_t(\Phi_t^{-1}(y))](\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t)\big|_{t=0}\\ &\quad = G'(u)v\cdot\nabla u(v\cdot\nu) + G(u)\operatorname{div} v(v\cdot\nu).\end{aligned}$$

Finally

$$\begin{aligned}\ddot{\mathcal{G}}(0) = {}& \int_\Omega G'(u)u''\,dx + \int_\Omega G''(u)u'^2\,dx + 2\oint_{\partial\Omega} G'(u)u'(v\cdot\nu)\,dS\\ &+ \oint_{\partial\Omega} G'(u)(v\cdot\nabla u)(v\cdot\nu)\,dS\\ &+ \oint_{\partial\Omega} G(u)[(v\cdot\nu)\operatorname{div} v - (v\cdot D_v\nu) + (w\cdot\nu)]\,dS.\end{aligned}$$

This is in accordance with (4.1.10).

Boundary integrals depending on $u(y,t)$ only

We now consider the functional

$$\mathcal{B}(t) = \oint_{\partial\Omega_t} B(u(y,t))\,dS_t.$$

From Theorem 4.1 we get

$$\begin{aligned}\dot{\mathcal{B}}(t) &= \frac{d}{dt}\oint_{\partial\Omega_t} B(u(y,t))\,dS_t = \oint_{\partial\Omega_t}\partial_t B(u(y,t))\,dS_t\\ &\quad + \oint_{\partial\Omega_t}[\partial_{\nu^t}B(u(y,t)) + (n-1)B(u(y,t))H^t(y)](\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t)\,dS_t\\ &= \oint_{\partial\Omega_t} B'(u(y,t))\partial_t u(y,t)\,dS_t + \oint_{\partial\Omega_t}\partial_{\nu^t}B(u(y,t))(\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t)\,dS_t\\ &\quad + (n-1)\oint_{\partial\Omega_t}[B(u(y,t))H^t(y)](\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t)\,dS_t\\ &=: \dot{\mathcal{B}}_1(t) + \dot{\mathcal{B}}_2(t) + \dot{\mathcal{B}}_3(t).\end{aligned}$$

For $t = 0$ this corresponds to (4.1.11).

We now compute the second derivative $\ddot{\mathcal{B}}(t)$ by applying Theorem 4.1 to each $\dot{\mathcal{B}}_i(t)$, $i = 1,2,3$:

$$\ddot{\mathcal{B}}_1(t) = \oint_{\partial\Omega_t} \partial_t [B'(u(y,t))\partial_t u(y,t)]\, dS_t$$
$$+ \oint_{\partial\Omega_t} \partial_{v^t} [B'(u(y,t))\partial_t u(y,t)](\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot v^t)\, dS_t$$
$$+ (n-1) \oint_{\partial\Omega_t} B'(u(y,t))\partial_t u(y,t) H^t(y)(\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot v^t)\, dS_t.$$

We then differentiate the square brackets and set $t = 0$. Hence,

$$\ddot{\mathcal{B}}_1(0) = \oint_{\partial\Omega} [B''(u)u'^2 + B'(u)u'']\, dS + \oint_{\partial\Omega} [B''(u)\partial_v u u' + B'(u)\partial_v u'](v \cdot v)\, dS$$
$$+ (n-1) \oint_{\partial\Omega} B'(u)u' H(v \cdot v)\, dS.$$

The computation for $\dot{\mathcal{B}}_2(t)$ is similar. We obtain

$$\ddot{\mathcal{B}}_2(t) = \oint_{\partial\Omega_t} \partial_t [B'(u(y,t))\partial_{v^t} u(y,t)(\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot v^t)]\, dS_t$$
$$+ \oint_{\partial\Omega_t} \partial_{v^t} [B'(u(y,t))\partial_{v^t} u(y,t)(\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot v^t)](\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot v^t)\, dS_t$$
$$+ (n-1) \oint_{\partial\Omega_t} B'(u(y,t))\partial_{v^t} u(y,t) H^t(y)(\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot v^t)^2\, dS_t.$$

The differentiation of the square brackets in the first two boundary integrals is done in several steps.

1. Differentiation of the integrand of the first boundary integral of $\ddot{\mathcal{B}}_2(t)$ yields

$$\begin{aligned} &\partial_t [B'(u(y,t))\partial_{v^t} u(y,t)(\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot v^t)] \\ &\quad = \partial_t [B'(u(y,t))\partial_{v^t} u(y,t)](\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot v^t) \\ &\quad\ + B'(u(y,t))\partial_{v^t} u(y,t)\partial_t [(\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot v^t)]. \end{aligned}$$

For $t = 0$ this expression simplifies to

$$\begin{aligned} &\partial_t [B'(u(y,t))\partial_{v^t} u(y,t)\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot v^t]\big|_{t=0} \\ &\quad = [B''(u)u'\partial_v u + B'(u)(v' \cdot \nabla u) + B'(u)\partial_v u'](v \cdot v) \\ &\quad\ + B'(u)\partial_v u \underbrace{\{\partial_t [\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot v^t]\}\big|_{t=0}}_{(*)}. \end{aligned}$$

2. Differentiation of the integrand of the second boundary integral of $\ddot{\mathcal{B}}_2(t)$ yields

$$\begin{aligned} &\partial_{v^t} [B'(u(y,t))\partial_{v^t} u(y,t)(\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot v^t)](\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot v^t) \\ &\quad = \partial_{v^t} [B'(u(y,t))\partial_{v^t} u(y,t)](\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot v^t)^2 \\ &\quad\ + B'(u(y,t))\partial_{v^t} u(y,t)\partial_{v^t} [\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot v^t](\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot v^t). \end{aligned}$$

For $t = 0$ we obtain

$$
\begin{aligned}
&\partial_{\nu^t}[B'(u(y,t))\partial_{\nu^t}u(y,t)\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t]\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t|_{t=0}\\
&\quad= [B''(u)(\partial_\nu u)^2 + B'(u)(\nu\cdot D^2u\nu)](\upsilon\cdot\nu)^2\\
&\quad\quad+ B'(u)\partial_\nu u\underbrace{\{\partial_{\nu^t}[\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t]\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t\}|_{t=0}}_{(**)}.
\end{aligned}
$$

3. The underbracketed terms $(*)$ and $(**)$ become

$$
\begin{aligned}
&\partial_t[\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t] + \partial_{\nu^t}[\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t](\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t)\\
&\quad= \partial_t^2\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t + [\partial_k\partial_t\Phi_t(\Phi_t^{-1}(y))\partial_t\Phi_{t,k}^{-1}(y)]\cdot\nu^t + \partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\partial_t\nu^t\\
&\quad\quad+ (\partial_{\nu^t}\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t)(\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t)\\
&\quad\quad+ (\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\partial_{\nu^t}\nu^t)(\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t).
\end{aligned}
$$

We recall that $\partial_t\Phi_t^{-1}(y)|_{t=0} = -\upsilon(x)$ and that ν' denotes the shape derivative of ν (see (2.4.9)). For Hadamard perturbations, (2.4.12) implies that $(\upsilon\cdot\nu') = 0$, $\upsilon^\tau = 0$, and (2.4.1) implies $D_{\nu^t}\nu^t = 0$ for all $|t| < t_0$. For $t = 0$ we have

$$
\begin{aligned}
&\partial_t[\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t] + \partial_{\nu^t}[\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t](\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t)\big|_{t=0}\\
&\quad= (w\cdot\nu) - \upsilon_k\partial_k\upsilon_j\nu_j + (\upsilon\cdot\nu') + \nu_k\partial_k\upsilon_j\nu_j(\upsilon\cdot\nu)\\
&\quad= (w\cdot\nu) - (\upsilon^\tau\cdot D_\upsilon\nu) + (\upsilon\cdot\nu') = (w\cdot\nu).
\end{aligned}
\tag{4.2.5}
$$

We then get

$$
\begin{aligned}
\ddot{\mathcal{B}}_2(0) &= \oint_{\partial\Omega} B''(u)u'\partial_\nu u(\upsilon\cdot\nu) + B'(u)(\nu'\cdot\nabla u)(\upsilon\cdot\nu) + B'(u)\partial_\nu u'(\upsilon\cdot\nu) + B'(u)\partial_\nu u(w\cdot\nu)\,dS\\
&\quad+ \oint_{\partial\Omega}[B''(u)(\partial_\nu u)^2 + B'(u)(\nu\cdot D^2u\nu)](\upsilon\cdot\nu)^2\,dS\\
&\quad+ (n-1)\oint_{\partial\Omega} B'(u)\partial_\nu uH(\upsilon\cdot\nu)^2\,dS.
\end{aligned}
$$

By Theorem 4.1 $\ddot{\mathcal{B}}_3(0)$ becomes

$$
\begin{aligned}
\ddot{\mathcal{B}}_3(t) &= (n-1)\oint_{\partial\Omega_t}\partial_t[B(u(y,t))H^t(y)(\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t)]\,dS_t\\
&\quad+ (n-1)\oint_{\partial\Omega_t}\partial_{\nu^t}[B(u(y,t))H^t(y)(\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t)](\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t)\,dS_t\\
&\quad+ (n-1)^2\oint_{\partial\Omega_t}[B(u(y,t))H^t(y)(\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t)]H^t(y)(\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t)\,dS_t.
\end{aligned}
$$

The same computations as for $\ddot{\mathcal{B}}_2(0)$ lead to

$$\begin{aligned}\ddot{\mathcal{B}}_3(0) = (n-1)\oint_{\partial\Omega} & B'(u)u'H(v\cdot\nu) + B(u)H'(v\cdot\nu) + B(u)H(w\cdot\nu) + B(u)H(v\cdot\nu')\,dS \\ &+ (n-1)\oint_{\partial\Omega}[B'(u)\partial_\nu uH(v\cdot\nu) + B(u)\partial_\nu H(v\cdot\nu)](v\cdot\nu)\,dS \\ &+ (n-1)^2\oint_{\partial\Omega} B(u)H^2(v\cdot\nu)^2\,dS.\end{aligned}$$

From (2.4.12) we have $(v\cdot\nu') = 0$. Hence,

$$\begin{aligned}\ddot{\mathcal{B}}_3(0) &= (n-1)\oint_{\partial\Omega} B'(u)u'H\rho + B'(u)\partial_\nu uH\rho^2\,dS \\ &\quad + (n-1)\oint_{\partial\Omega} B(u)[H'\rho + \partial_\nu H\rho^2 + (n-1)H^2\rho^2]\,dS \\ &\quad + (n-1)\oint_{\partial\Omega} B(u)H(w\cdot\nu)\,dS.\end{aligned}$$

Let us write for short

$$\ddot{\mathcal{B}}_3(0) = (n-1)\oint_{\partial\Omega} B'(u)u'H\rho + B'(u)\partial_\nu uH\rho^2\,dS + \oint_{\partial\Omega} z(0)\,dS,$$

where

$$z(0) := (n-1)B(u)[H'\rho + \partial_\nu H\rho^2 + (n-1)H^2\rho^2 + H(w\cdot\nu)]. \tag{4.2.6}$$

Adding all expressions in $\ddot{\mathcal{B}}_i(0)$, $i = 1,2,3$, we get the following expression for $\ddot{\mathcal{B}}(0)$

$$\begin{aligned}\ddot{\mathcal{B}}(0) = \oint_{\partial\Omega}&[B''(u)u'^2 + B'(u)u'']\,dS + 2\oint_{\partial\Omega}[B''(u)\partial_\nu uu' + B'(u)\partial_\nu u']\rho\,dS \\ &+ \oint_{\partial\Omega} 2(n-1)B'(u)u'H\rho\,dS + \oint_{\partial\Omega} B'(u)\partial_\nu u(w\cdot\nu)\,dS \\ &+ \oint_{\partial\Omega}[B''(u)(\partial_\nu u)^2 + B'(u)\nu\cdot(D^2u\nu) + 2(n-1)B'(u)\partial_\nu uH]\rho^2\,dS \\ &+ \oint_{\partial\Omega} B'(u)(\nu'\cdot\nabla u)\rho\,dS + \oint_{\partial\Omega} z(0)\,dS.\end{aligned} \tag{4.2.7}$$

Next we shall prove that (4.2.7) is identical with (4.1.14). The two formulas are identical if

$$\oint_{\partial\Omega} B(u)\ddot{m}(0)\,dS + \oint_{\partial\Omega} B'(u)w\cdot\nabla u\,dS$$
$$= \oint_{\partial\Omega} B'(u)\partial_\nu u(w\cdot\nu)\,dS + \oint_{\partial\Omega} B'(u)(\nu'\cdot\nabla u)\rho\,dS + \oint_{\partial\Omega} z(0)\,dS.$$

This follows from the next lemma.

Lemma 4.2. *The following relation holds between* $z(0)$ *in* (4.2.6) *and* $\ddot{m}(0)$*:*

$$\oint_{\partial\Omega} B'(u)(\nu'\cdot\nabla u)\rho\,dS + \oint_{\partial\Omega} z(0)\,dS$$
$$= \oint_{\partial\Omega} B(u)\ddot{m}(0)\,dS + \oint_{\partial\Omega} B'(u)(w\cdot\nabla^\tau u)\,dS, \tag{4.2.8}$$

where ν' *is the shape derivative of* ν *(see Lemma* 2.6*).*

Proof. From Lemma 2.6 we have

$$\oint_{\partial\Omega} B'(u)(\nu'\cdot\nabla u)\rho\,dS = -\oint_{\partial\Omega} B'(u)(\nabla^\tau\rho\cdot\nabla u)\rho\,dS = -\oint_{\partial\Omega} \nabla^\tau\rho\cdot\nabla^\tau B(u)\rho\,dS$$
$$= \oint_{\partial\Omega} B(u)\Delta^*\rho\rho\,dS + \oint_{\partial\Omega} B(u)|\nabla^\tau\rho|^2\,dS.$$

Next we recall Lemma 2.8. Hence,

$$z(0) = B(u)[\rho(-\Delta^*\rho - \rho D_\nu : D_\nu) + (n-1)^2H^2\rho^2 + (n-1)H(w\cdot\nu)].$$

Consequently, the left-hand side of (4.2.8) reads as

$$\oint_{\partial\Omega} B'(u)(\nu'\cdot\nabla u)\rho\,dS + \oint_{\partial\Omega} z(0)\,dS$$
$$= \oint_{\partial\Omega} B(u)[|\nabla^\tau\rho|^2 - \rho^2D_\nu : D_\nu + (n-1)^2\rho^2H^2]\,dS$$
$$+ (n-1)\oint_{\partial\Omega} B(u)(w\cdot\nu)H\,dS.$$

In order to analyze the right-hand side of (4.2.8) we recall (2.4.7). We have

$$\ddot{m}(0) = |\nabla^\tau\rho|^2 + (n-1)^2\rho^2H^2 - \rho^2D_\nu : D_\nu + \operatorname{div}_{\partial\Omega} w.$$

Hence, the claim (4.2.8) is equivalent to

$$(n-1)\oint_{\partial\Omega} B(u)H(w\cdot\nu)\,dS = \oint_{\partial\Omega} B(u)\operatorname{div}_{\partial\Omega} w\,dS + \oint_{\partial\Omega} B'(u)(w\cdot\nabla^\tau u)\,dS.$$

This equality follows from the Gauss Theorem on surfaces (2.2.18). □

In particular this lemma implies that (4.2.7) and (4.1.14) are the same.

Example 4.1. For $B(u) = 1$, Theorem 4.1 yields

$$\dot{\mathcal{S}}(t) = (n-1) \oint_{\partial\Omega_t} H^t(y)\partial_t\Phi_t(\Phi_t^{-1}(y)) \cdot \nu^t \, dS_t.$$

Evaluation at $t = 0$ implies

$$\dot{\mathcal{S}}(0) = (n-1) \oint_{\partial\Omega} H\rho \, dS. \tag{4.2.9}$$

This corresponds to (2.3.20).

In order to compute the second variation we apply the second assertion of Theorem 4.1 to $\dot{\mathcal{S}}(t)$ and obtain

$$\begin{aligned}\ddot{\mathcal{S}}(t) = (n-1) &\int_{\partial\Omega_t} \partial_t[H^t(y)\partial_t\Phi_t(\Phi_t^{-1}(y) \cdot \nu^t)] \, dS_t \\ &+ (n-1) \oint_{\partial\Omega_t} \partial_{\nu^t} [H^t(y)\partial_t\Phi_t(\Phi_t^{-1}(y)) \cdot \nu^t] \, dS_t \\ &+ (n-1)^2 \oint_{\partial\Omega_t} (\partial_t\Phi_t(\Phi_t^{-1}(y)) \cdot \nu^t)^2 H^t(y)^2 \, dS_t.\end{aligned}$$

By (2.1.3) we have

$$\partial_t\Phi_t(\Phi^{-1}(y))\big|_{t=0} = \rho(x)\nu(x) \quad \text{and} \quad (\partial_t + \partial_{\nu^t})\partial_t\Phi_t(\Phi^{-1}(y))\big|_{t=0} = w(x),$$

and by (2.4.12) we have $(\nu \cdot \nu') = 0$. Taking these identities into account we obtain

$$\begin{aligned}\ddot{\mathcal{S}}(0) = &\oint_{\partial\Omega} |\nabla^\tau \rho|^2 \, dS - \oint_{\partial\Omega} \rho^2 \mathcal{L} : \mathcal{L} \, dS + (n-1)^2 \oint_{\partial\Omega} \rho^2 H^2 \, dS \\ &+ (n-1) \oint_{\partial\Omega} (w \cdot \nu) H \, dS.\end{aligned} \tag{4.2.10}$$

This formula is the same as (2.3.31).

4.3 Notes

The modern theory of shape derivatives or domain variations goes back to J. Hadamard. In [69] he derived a variational formula for the Green's function. His proof is given in Section 1.3 and is based on a moving surface approach. In a seminal paper [63] Garabedian and Schiffer extended Hadamard's result to the Green's function of more general elliptic differential operators and derived the first and second variations of several domain-dependent physical quantities, like the electrostatic capacity, the virtual mass, and eigenvalues of the membrane and the plate. To give a rigorous proof of Hadamard's result they introduced the method of change of variables, which they called interior variation. Since it uses the divergence theorem it is only applicable to sufficiently regular domains.

Later on proofs of some variational formulas on Lipschitz domains were given by Elcrat and Miller [44]. Generalizations to the Green's function of general elliptic boundary value problems are found in [59]. The question of smoothness for the validity of Hadamard's formula has also been studied in [117].

A few decades after Garabedian and Schiffer's research the study of domain variations received new attention, especially in France and Japan. The main results of this chapter are also found in the books of Henrot and Pierre [75], Henry [76], and Sokolowski and Zolesio [111]. A short survey is given by Grinfeld [68].

5 Geometric inequalities, convolutions, cost functions

This chapter deals with applications of the variational formulas derived in Chapter 4. We start with a generalization of various moments of inertia. The results can then be used to derive local isoperimetric inequalities in spaces of constant curvature. A detailed discussion is devoted to convolutions and applied to a liquid model. We conclude this chapter with an optimal control problem for the cost function related to a boundary value problem.

5.1 Moment of inertia

Volume integrals

Consider volume integrals of the form

$$\mathcal{G}(t) = \int_{\Omega_t} G(|y|^2)\, dy.$$

We assume that G is a continuously differentiable function. In this case $\mathcal{G}(t)$ corresponds to (4.1.6) in Section 4.1.2 with $u(y,t) := |y|^2$. Since it is independent of t, both shape derivatives u' and u'' vanish. Then

$$\dot{\mathcal{G}}(0) = \oint_{\partial\Omega} G(|x|^2)(v \cdot \nu)\, dS. \tag{5.1.1}$$

From (4.1.10) and $\nabla u(x) = 2x$, taking into account that $u' = u'' = 0$, we obtain

$$\begin{aligned}\ddot{\mathcal{G}}(0) = 2\oint_{\partial\Omega} G'(|x|^2)(v \cdot x)(v \cdot \nu)\, dS \\ + \oint_{\partial\Omega} G(|x|^2)\big[(v \cdot \nu)\operatorname{div} v - (v \cdot D_v \nu) + (w \cdot \nu)\big]\, dS.\end{aligned} \tag{5.1.2}$$

The first variation vanishes for volume preserving perturbations $\dot{\mathcal{V}}(0) = \oint_{\partial\Omega}(v\cdot\nu)\, dS = 0$ if and only if $G(|x|^2) = \text{const.}$ on $\partial\Omega$. This implies that Ω must be a ball B_R.

Under the additional condition $\ddot{\mathcal{V}}(0) = 0$, the second variation assume the form

$$\ddot{\mathcal{G}}(0) = 2RG'(R^2) \oint_{\partial B_R} (v \cdot \nu)^2\, dS. \tag{5.1.3}$$

If $G'(R^2) > 0$ the ball is a local minimum and if $G'(R^2) < 0$ it is a local maximum of $\mathcal{G}(t)$.

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By means of the method of rearrangement (see for instance [87, Theorem 3.4]) it can be shown that for increasing $G(t)$ the ball is also a global minimum and for decreasing $G(t)$ it is a global maximum.

A simple example is the moment of inertia of a three-dimensional domain Ω where the barycenter is located at the origin. The moment of inertia with respect to the plane $x_k = 0$ is

$$J_k(\Omega) = \int_\Omega x_k^2\, dx,$$

and

$$J_0(\Omega) = \sum_{k=1}^n J_k(\Omega) = \int_\Omega |x|^2\, dx$$

is the polar moment of inertia. We consider volume preserving perturbations which leave the barycenter unchanged in the sense of Definition 2.5. In this case we have $G(s) = s$. For the ball, formula (5.1.3) implies

$$\ddot{J}_0(\Omega_t)|_{t=0} = 2R \oint_{\partial B_R} (v \cdot \nu)^2\, dS.$$

A more detailed analysis of the isoperimetric properties of moments of inertia is found in [73].

Boundary integrals

Secondly we consider the corresponding boundary integrals

$$\mathcal{B}(t) = \oint_{\partial\Omega_t} B(|y|^2)\, dS_t.$$

For $B(|y|^2) = |y|^2$ it corresponds to the moment of inertia with respect to a constant boundary measure.

We would like to compute $\dot{\mathcal{B}}(0)$ for Hadamard perturbations. To apply (4.1.11) one has to take into account that $u(y,t) = |y|^2$; thus, $u' = 0$. Then

$$\dot{\mathcal{B}}(0) = \oint_{\partial\Omega} [2B'(r^2)(v \cdot x) + (n-1)H(v \cdot \nu)B(r^2)]\, dS, \tag{5.1.4}$$

where $r = |x|$. For $\Omega = B_R$, (5.1.4) simplifies:

$$\dot{\mathcal{B}}(0) = 2B'(R^2) \oint_{\partial B_R} (v \cdot x)\, dS + \frac{n-1}{R} B(R^2) \oint_{\partial B_R} (v \cdot \nu)\, dS.$$

Since $(v \cdot x) = R(v \cdot \nu)$ on ∂B_R, the ball is a critical domain in the class of volume preserving perturbations.

For the computation of $\ddot{\mathcal{B}}(0)$ we also restrict ourselves to Hadamard perturbations, i. e., $v = \rho \nu$ on the boundary $\partial\Omega$. We apply (4.1.14) and use $u'' = 0$ and $u(y,t) = |y|^2$. Then

$$\ddot{\mathcal{B}}(0) = 4 \oint_{\partial\Omega} B''(r^2)(x \cdot \nu)^2 \rho^2 \, dS + 2 \oint_{\partial\Omega} B'(r^2)[\rho^2 + 2(n-1)(x \cdot \nu) H \rho^2 + (w \cdot x)] \, dS \\ + \oint_{\partial\Omega} B(r^2) \ddot{m}(0) \, dS.$$

A special case is $\Omega = B_R$. Then $(x \cdot \nu) = R$ and $(w \cdot x) = R(w \cdot \nu)$. Moreover, $\oint_{\partial\Omega} B(r^2)\ddot{m}(0)\, dS = B(R^2)\ddot{\mathcal{S}}(0)$. Thus, $\ddot{\mathcal{B}}(0)$ takes the following form:

$$\ddot{\mathcal{B}}(0) = 4R^2 B''(R^2) \oint_{\partial B_R} \rho^2 \, dS + 2B'(R^2) \oint_{\partial B_R} [\rho^2 + 2(n-1)\rho^2 + R(w \cdot \nu)] \, dS \\ + B(R^2)\ddot{\mathcal{S}}(0).$$

This gives

$$\ddot{\mathcal{B}}(0) = \big(4R^2 B''(R^2) + 2(2n-1)B'(R^2)\big) \oint_{\partial B_R} \rho^2 \, dS + 2B'(R^2) R \oint_{\partial B_R} (w \cdot \nu) \, dS \\ + B(R^2)\ddot{\mathcal{S}}(0). \tag{5.1.5}$$

For volume preserving perturbation, (2.3.12) implies

$$\oint_{\partial B_R} (w \cdot \nu) \, dS = -\frac{n-1}{R} \oint_{\partial B_R} \rho^2 \, dS.$$

Moreover, in view of (2.3.28), $\ddot{\mathcal{S}}(0) = \ddot{\mathcal{S}}_0(0) + \frac{n-1}{R}\ddot{\mathcal{V}}(0) = \ddot{\mathcal{S}}_0(0)$; thus,

$$\ddot{\mathcal{B}}(0) = \big(4R^2 B''(R^2) + 2nB'(R^2)\big) \oint_{\partial B_R} \rho^2 \, dS + B(R^2)\ddot{\mathcal{S}}_0(0).$$

If $B(R^2) \geq 0$, the second term is nonnegative by the isoperimetric inequality. This implies

$$\ddot{\mathcal{B}}(0) \geq \big(4R^2 B''(R^2) + 2nB'(R^2)\big) \oint_{\partial B_R} \rho^2 \, dS.$$

If the term in parentheses is positive, the ball is a local minimizer among all volume preserving perturbations.

Problem 5.1. Are there conditions on $B(r^2)$ such that the ball is the only critical domain for volume preserving perturbations?

Weighted isoperimetric inequalities

A further application is related to isoperimetric inequalities with density. We consider the weighted volume of Ω and the weighted surface area of $\partial\Omega$, defined as

$$\mathcal{V}_\omega(\Omega) := \int_\Omega \omega(|x|^2)\,dx, \quad \mathcal{S}_\omega(\partial\Omega) := \oint_{\partial\Omega} \omega(|x|^2)\,dS,$$

where $\omega : (0,\infty) \to (0,\infty)$ is a positive smooth function.

We are interested in minimizing $\mathcal{S}_\omega(\partial\Omega)$ among all bounded smooth domains with prescribed volume $\mathcal{V}_\omega(\Omega)$. We shall apply (5.1.1) and (5.1.2). In the sequel we restrict ourselves to perturbations such that

$$\dot{\mathcal{V}}_\omega(0) = \oint_{\partial\Omega} \omega(|x|^2)\rho\,dS = 0, \quad \rho = (v \cdot \nu),$$

and

$$\begin{aligned}\ddot{\mathcal{V}}_\omega(0) &= 2\oint_{\partial\Omega} \omega'(|x|^2)(v\cdot x)\rho\,dS + \oint_{\partial\Omega} \omega(|x|^2)[\rho \operatorname{div} v - (v \cdot D_v \nu) + (w\cdot\nu)]\,dS \\ &= 0.\end{aligned}$$

For the ball B_R this leads to

$$0 = \oint_{\partial B_R} \rho\,dS = 2R\omega'(R^2)\oint_{\partial B_R} \rho^2\,dS + \omega(R^2)\ddot{\mathcal{V}}(0). \tag{5.1.6}$$

We now use (5.1.4) and (5.1.6). If $\Omega = B_R$, we obtain

$$\dot{\mathcal{S}}_\omega(0) = 0 \quad \text{where } \mathcal{S}_\omega(t) = \mathcal{S}_\omega(\partial\Omega_t)$$

and

$$\begin{aligned}\ddot{\mathcal{S}}_\omega(0) &= \omega(R^2)\ddot{\mathcal{S}}(0) + \{2(2n-1)\omega'(R^2) + 4R^2\omega''(R^2)\}\oint_{\partial B_R} \rho^2\,dS \\ &\quad + 2\omega'(R^2)R\oint_{\partial B_R} (w\cdot\nu)\,dS.\end{aligned}$$

Suppose that the barycenter condition is satisfied. Then it follows from Section 2.3.3 that

$$\ddot{\mathcal{S}}(0) \geq \frac{n+1}{R^2}\oint_{\partial B_R} \rho^2\,dS + \frac{n-1}{R}\ddot{\mathcal{V}}(0).$$

For Hadamard perturbations, (2.3.12) implies

$$\oint_{\partial B_R} (w \cdot \nu)\, dS = \ddot{\mathcal{V}}(0) - \frac{n-1}{R} \oint_{\partial B_R} \rho^2\, dS.$$

Note that $\ddot{\mathcal{V}}(0)$ is given in (5.1.6). After rearranging terms, we get

$$\begin{aligned}\ddot{\mathcal{S}}_\omega(0) &\geq \left(\frac{n+1}{R^2}\omega(R^2) + 2\omega'(R^2) - \frac{4R^2\omega'^2(R^2)}{\omega(R^2)} + 4R^2\omega''(R^2)\right) \oint_{\partial B_R} \rho^2\, dS \\ &= \left(\frac{n+1}{R^2}\omega(R^2) + \omega(R^2)\frac{d^2}{dr^2}\log\omega(r^2)\Big|_{r=R}\right) \oint_{\partial B_R} \rho^2\, dS.\end{aligned}$$

Proposition 5.1. *Assume that* $\log\omega(r^2)$ *is a convex function of* r*. Then among all domains of given weighted volume* $\mathcal{V}_\omega$*, satisfying the barycenter condition the ball is a local minimum of weighted surface area* $\mathcal{S}_\omega$*. Furthermore, the estimate*

$$\ddot{\mathcal{S}}_\omega(0) \geq \frac{n+1}{R^2}\omega(R^2) \oint_{\partial B_R} \rho^2\, dS$$

holds.

Chambers [36] proved that among all domains of given volume $\mathcal{V}_\omega(\Omega)$, $\mathcal{S}_\omega(\partial\Omega)$ achieves its minimum if and only if Ω is a ball.

5.2 Isoperimetric inequalities in spaces of constant curvature

5.2.1 Spherical space $\mathbb{S}^n$

We consider the unit sphere $\mathbb{S}^n$ as an embedded hypersurface into $\mathbb{R}^{n+1}$, i. e.,

$$\mathbb{S}^n := \{X := (X_0, X_1, \dots, X_n) \in \mathbb{R}^{n+1} : (X \cdot X) = 1\},$$

where the point $N := (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$ denotes the north pole (see Figure 5.1).

For our purpose it will be convenient to map $\mathbb{S}^n$ by means of a stereographic projection into $\mathbb{R}^n$. It is defined as

$$P : \mathbb{S}^n \setminus \{N\} \subset \mathbb{R}^{n+1} \to \mathbb{R}^n, \quad X \to P(X) := \left(\frac{X_1}{1 - X_0}, \dots, \frac{X_n}{1 - X_0}\right).$$

Its inverse is given by

$$P^{-1} : \mathbb{R}^n \to \mathbb{S}^n \setminus \{N\} \subset \mathbb{R}^{n+1}, \quad x \to P^{-1}(x) := \left(\frac{|x|^2 - 1}{|x|^2 + 1}, \frac{2x_1}{|x|^2 + 1}, \dots, \frac{2x_n}{|x|^2 + 1}\right).$$

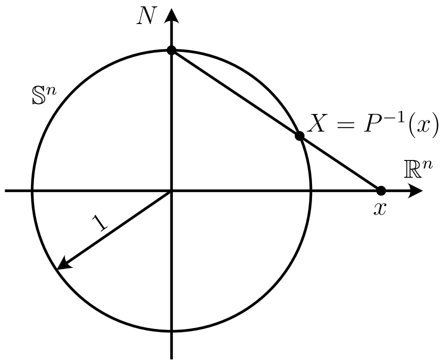

Figure 5.1: Stereographic projection.

The volume of a subdomain $D \subset S^n \setminus \{N\}$ is given by the area formula

$$\mathcal{V}_{\mathbb{S}}(D) := \int\limits_{P^{-1}(D)} \sqrt{\det[(D_{P^{-1}})^T D_{P^{-1}}]}\, dx = \int\limits_{P^{-1}(D)} \frac{2^n}{(|x|^2+1)^n}\, dx.$$

Similarly, for the perimeter we get

$$S_{\mathbb{S}}(\partial D) = \int\limits_{P^{-1}(\partial D)} \frac{2^{n-1}}{(|x|^2+1)^{n-1}}\, dS.$$

The volume and the surface area of a family of perturbations of $\{D_t\}_{|t|<t_0}$ are then expressed as

$$\mathcal{V}_{\mathbb{S}}(t) := \int\limits_{\Omega_t} \left(\frac{2}{1+|y|^2}\right)^n dy,$$

where $\Omega_t = P^{-1}(D_t)$, and

$$\mathcal{S}_{\mathbb{S}}(t) := \oint\limits_{\partial\Omega_t} \left(\frac{2}{1+|y|^2}\right)^{n-1} dS_t,$$

where $\partial\Omega_t := P^{-1}(\partial D_t)$.

The functionals $V_{\mathbb{S}}(t)$ (resp. $S_{\mathbb{S}}(t)$) correspond to $\mathcal{G}(t)$ (resp. $\mathcal{B}(t)$), where

$$G(u) = \left(\frac{2}{1+u}\right)^n, \quad B(u) = \left(\frac{2}{1+u}\right)^{n-1}, \quad \text{and} \quad u(y) = |y|^2.$$

Consider the functionals

$$\mathcal{G}(t) = \int\limits_{\Omega_t} \left(\frac{2}{1+|y|^2}\right)^n dy \quad \text{and} \quad \mathcal{B}(t) = \oint\limits_{\partial\Omega_t} \left(\frac{2}{1+|y|^2}\right)^{n-1} dS_t.$$

Here $\{\Omega_t\}_{|t|<t_0}$ is a family of nearly spherical domains of given volume, such that

$$\Omega_0 = B_R \quad \text{and} \quad \mathcal{V}_{\mathbb{S}}(t) = \mathcal{V}_{\mathbb{S}}(0) + o(t^2). \tag{5.2.1}$$

We want to apply the arguments of Section 5.1. Since the perturbations are volume preserving, (5.2.1) implies

$$\dot{\mathcal{G}}(0) = \dot{\mathcal{V}}_{\mathbb{S}}(0) = 0 \quad \text{and} \quad \ddot{\mathcal{G}}(0) = \ddot{\mathcal{V}}_{\mathbb{S}}(0) = 0.$$

From (5.1.1) we obtain the constraint

$$0 = \dot{\mathcal{V}}_{\mathbb{S}}(0) = G(R^2) \oint\limits_{\partial B_R} \rho \, dS = \left(\frac{2}{1+R^2}\right)^n \oint\limits_{\partial B_R} \rho \, dS \quad \Rightarrow \quad \oint\limits_{\partial B_R} \rho \, dS = 0.$$

From formula (5.1.2) it follows that for the special case $\Omega = B_R$,

$$0 = \ddot{\mathcal{V}}_{\mathbb{S}}(0) = 2G'(R^2) \oint\limits_{\partial B_R} (v \cdot x)(v \cdot \nu) \, dS + G(R^2)\ddot{\mathcal{V}}(0),$$

where $\mathcal{V}(t) = |\Omega_t|$ and $\mathcal{V}(0) = |B_R|$. This implies

$$\ddot{\mathcal{V}}(0) = -\frac{2RG'(R^2)}{G(R^2)} \oint\limits_{\partial B_R} \rho^2 \, dS = \frac{2nR}{1+R^2} \oint\limits_{\partial B_R} \rho^2 \, dS. \tag{5.2.2}$$

From now on we restrict ourselves to Hadamard perturbations. By (2.3.12),

$$\ddot{\mathcal{V}}(0) = \frac{n-1}{R} \oint\limits_{\partial B_R} (v \cdot \nu)^2 \, dS + \oint\limits_{\partial B_R} (w \cdot \nu) \, dS.$$

This is a constraint on $\oint_{\partial B_R} (w \cdot \nu) \, dS$. We have

$$\begin{aligned}\oint\limits_{\partial B_R} (w \cdot \nu) \, dS &= \ddot{\mathcal{V}}(0) - \frac{n-1}{R} \oint\limits_{\partial B_R} (v \cdot \nu)^2 \, dS \\ &= \left(\frac{2nR}{1+R^2} - \frac{n-1}{R}\right) \oint\limits_{\partial B_R} (v \cdot \nu)^2 \, dS.\end{aligned}$$

Hence,

$$\oint\limits_{\partial B_R} (w \cdot \nu) \, dS = -\frac{(n-1)-(n+1)R^2}{R(1+R^2)} \oint\limits_{\partial B_R} \rho^2 \, dS. \tag{5.2.3}$$

We want to apply (5.1.5). For this purpose we need

$$B'(u) = -\frac{n-1}{2}\left(\frac{2}{1+u}\right)^n \quad \text{and} \quad B''(u) = \frac{n(n-1)}{4}\left(\frac{2}{1+u}\right)^{n+1}.$$

Then

$$\begin{aligned}\ddot{\mathcal{B}}(0) = B(R^2)\ddot{\mathcal{S}}(0) + [4R^2 B''(R^2) + 2(2n-1)B'(R^2)] \oint_{\partial B_R} (v \cdot \nu)^2 \, dS \\ + 2B'(R^2)R \oint_{\partial B_R} (w \cdot \nu) \, dS.\end{aligned}$$

Recall that from (2.3.28) and (5.2.2),

$$\ddot{\mathcal{S}}(0) = \ddot{\mathcal{S}}_0(0) + \frac{n-1}{R}\ddot{\mathcal{V}}(0) = \ddot{\mathcal{S}}_0(0) + \frac{2n(n-1)}{1+R^2} \oint_{\partial B_R} \rho^2 \, dS.$$

With (5.2.3) we can rewrite $\ddot{\mathcal{B}}(0)$ as

$$\begin{aligned}\ddot{\mathcal{B}}(0) &= B(R^2)\ddot{\mathcal{S}}_0(0) + \frac{2n(n-1)}{1+R^2}B(R^2) \oint_{\partial B_R} \rho^2 \, dS \\ &\quad + [4R^2 B''(R^2) + 2(2n-1)B'(R^2)] \oint_{\partial B_R} \rho^2 \, dS \\ &\quad - 2B'(R^2)\frac{(n-1)-(n+1)R^2}{1+R^2} \oint_{\partial B_R} \rho^2 \, dS \\ &= B(R^2)\ddot{\mathcal{S}}_0(0).\end{aligned}$$

Thus,

$$\ddot{\mathcal{B}}(0) = \left(\frac{2}{1+R^2}\right)^n \ddot{\mathcal{S}}_0(0).$$

Since $\ddot{\mathcal{S}}_0(0) \geq 0$, we have $\ddot{\mathcal{B}}(0) \geq 0$.

Proposition 5.2. *Among all domains $D_t \subset \mathbb{S}^n$ of constant volume $\mathcal{V}_{\mathbb{S}}(t)$, the perimeter $\mathcal{S}_{\mathbb{S}}$ has a local minimum for the geodesic ball.*

It is well known that this isoperimetric inequality holds for any domain on the sphere.

5.2.2 Hyperbolic space $\mathbb{H}^n$

We consider a model of the hyperbolic space $\mathbb{H}^n$ as an embedded surface into $\mathbb{R}^{n+1}$,

$$\mathbb{H}^n := \{X \in \mathbb{R}^{n+1} : X \star X = -1\},$$

where

$$X \star X = \sum_{i=1}^{n} X_i^2 - X_{n+1}^2.$$

The stereographic projection (see Figure 5.2) leads to the Poincaré model in $\mathbb{R}^n$. Analogously to the spherical case, we define ($N := (0, \dots, 0, -1) \in \mathbb{R}^{n+1}$)

$$P : \mathbb{H}^n \setminus \{N\} \subset \mathbb{R}^{n+1} \to \mathbb{R}^n, \quad X \to P(X) := \frac{X_i}{1 + X_{n+1}}$$

and its inverse

$$P^{-1} : \mathbb{R}^n \to \mathbb{H}^n \setminus \{N\} \subset \mathbb{R}^{n+1}, \quad x \to P^{-1}(x) := \left(\frac{2x_1}{1 - |x|^2}, \dots, \frac{2x_n}{1 - |x|^2}, \frac{1 + |x|^2}{1 - |x|^2} \right).$$

As in the spherical case, the area formula gives expressions for the volume and the surface area of a family of perturbations $\{D_t\}_{|t|<t_0}$ of a domain D in $\mathbb{H}^n$. The volume of $\Omega_t := P^{-1}(D_t)$ is

$$\mathcal{V}_{\mathbb{H}}(t) = \int_{\Omega_t} \left(\frac{2}{1 - |y|^2} \right)^n dy,$$

and the surface area of $\partial\Omega_t := P^{-1}(\partial D_t)$ is

$$S_{\mathbb{H}}(t) = \oint_{\partial\Omega_t} \left(\frac{2}{1 - |y|^2} \right)^{n-1} dS_t.$$

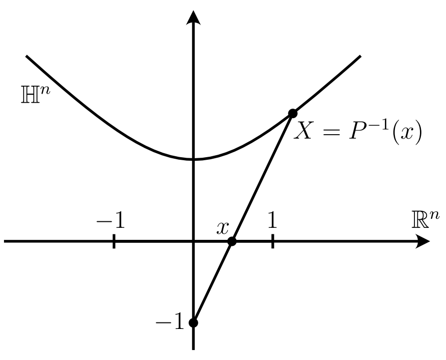

Figure 5.2: Poincaré model.

We are interested in computing the second derivatives $\ddot{\mathcal{B}}(0)$ and $\ddot{\mathcal{G}}(0)$, where

$$\mathcal{G}(t) = \int\limits_{\Omega_t} \left(\frac{2}{1-|y|^2}\right)^n dy, \quad \mathcal{B}(t) = \oint\limits_{\partial\Omega_t} \left(\frac{2}{1-|y|^2}\right)^{n-1} dS_{\partial\Omega_t},$$

and $\Omega_0 = B_R$. We want to apply (4.1.10). In this case,

$$G(u) = \left(\frac{2}{1-u}\right)^n, \quad B(u) = \left(\frac{2}{1-u}\right)^{n-1}, \quad \text{and} \quad u(y) = |y|^2 < 1.$$

Again volume preserving perturbations (i. e., $\dot{\mathcal{G}}(0) = \ddot{\mathcal{G}}(0) = 0$) are considered. The same computations as for the spherical space yield

$$\ddot{\mathcal{B}}(0) = \left(\frac{2}{1-R^2}\right)^{n-1} \ddot{\mathcal{S}}_0(0) - (n+1)(n-2)\left(\frac{2}{1-R^2}\right)^n \oint\limits_{\partial B_R} \rho^2\, dS. \tag{5.2.4}$$

The isoperimetric inequality in the Euclidean space, (2.3.28), and (5.2.2) imply

$$\begin{aligned} 0 \le \ddot{\mathcal{S}}(0) &= \ddot{\mathcal{S}}_0(0) + \frac{n-1}{R}\ddot{\mathcal{V}}(0) \\ &= \ddot{\mathcal{S}}_0(0) - \frac{2RG'(R^2)}{G(R^2)} \oint\limits_{\partial B_R} \rho^2\, dS. \end{aligned}$$

The explicit form of $G(u)$ gives the lower bound

$$\ddot{\mathcal{S}}_0(0) \ge \frac{2RG'(R^2)}{G(R^2)} \oint\limits_{\partial B_R} \rho^2\, dS = n(n-1)\left(\frac{2}{1-R^2}\right) \oint\limits_{\partial B_R} \rho^2\, dS.$$

Consequently, (5.2.4) yields

$$\begin{aligned} \ddot{\mathcal{B}}(0) &\ge (n(n-1) - (n+1)(n-2))\left(\frac{2}{1-R^2}\right)^n \oint\limits_{\partial B_R} \rho^2\, dS \\ &= 2\left(\frac{2}{1-R^2}\right)^n \oint\limits_{\partial B_R} \rho^2\, dS > 0. \end{aligned}$$

Proposition 5.3. *Among all nearly spherical domains on $\mathbb{H}^n$ of given volume, the ball has the smallest perimeter.*

This isoperimetric inequality is true for any domain in $\mathbb{H}^n$.

5.3 Convolutions

In this section we consider domain variations of functionals of the form

$$\mathcal{C}(t) = \int\limits_{\Omega_t \times \Omega_t} F(|\xi - \eta|^2)\, d\xi\, d\eta.$$

Let

$$\xi(x) = x + tv(x) + \frac{t^2}{2} w(x) \quad \text{and} \quad \eta(y) = y + tv(y) + \frac{t^2}{2} w(y), \quad x, y \in \Omega.$$

The change of variables $\xi \longrightarrow x$ and $\eta \longrightarrow y$ leads to

$$\mathcal{C}(t) = \int\limits_{\Omega \times \Omega} F\big(\big|\xi(x) - \eta(y)\big|^2\big) J(t, x) J(t, y)\, dx\, dy,$$

where $J(t, x)$ (resp. $J(t, y)$) is the Jacobi determinant (2.1.5) with respect to x (resp. y). We write for short

$$D(t) := \big|\xi(x) - \eta(y)\big|^2 = \left| x - y + t\big(v(x) - v(y)\big) + \frac{t^2}{2}\big(w(x) - w(y)\big) \right|^2.$$

With this notation,

$$\mathcal{C}(t) = \int\limits_{\Omega \times \Omega} F\big(D(t)\big) J(t, x) J(t, y)\, dx\, dy.$$

We also set

$$\delta v := v(x) - v(y), \quad \delta w := w(x) - w(y).$$

The first variation of $\mathcal{C}$ is

$$\dot{\mathcal{C}}(t) = \int\limits_{\Omega \times \Omega} F'\big(D(t)\big)\dot{D}(t) J(t, x) J(t, y) + F\big(D(t)\big)\big(\dot{J}(t, x) J(t, y) + J(t, x)\dot{J}(t, y)\big)\, dx\, dy. \tag{5.3.1}$$

Since

$$\dot{D}(t) = 2\delta v \cdot (x - y) + 2t\big(|\delta v|^2 + \delta w \cdot (x - y)\big) + O(t^2),$$

it follows that

$$\dot{\mathcal{C}}(0) = \int\limits_{\Omega \times \Omega} 2F'\big(|x - y|^2\big)\delta v \cdot (x - y) + F\big(|x - y|^2\big)\big(\operatorname{div} v(x) + \operatorname{div} v(y)\big)\, dx\, dy.$$

The two relations

$$\operatorname{div}_x(F(|x-y|^2)v(x)) = 2F'(|x-y|^2)v(x)\cdot(x-y) + F(|x-y|)\operatorname{div}_x v(x) \tag{5.3.2}$$

and

$$\operatorname{div}_y(F(|x-y|^2)v(y)) = -2F'(|x-y|^2)v(y)\cdot(x-y) + F(|x-y|)\operatorname{div}_y v(y) \tag{5.3.3}$$

will be used frequently. If we insert these identities into the formula for $\dot{\mathcal{C}}(0)$ and apply the divergence theorem, we get

$$\dot{\mathcal{C}}(0) = 4\oint_{\partial\Omega}\Big(\int_{\Omega} F(|x-y|^2)\,dy\Big)(v(x)\cdot\nu(x))\,dS. \tag{5.3.4}$$

Example 5.1. Let $\Omega = B_R$. The invariance with respect to rotations implies that for $x \in \partial B_R$,

$$\int_{B_R} F(|x-y|^2)\,dy = N(R). \tag{5.3.5}$$

This can easily be checked. In fact, consider the point $\mathcal{O}x \in \partial B_R$, where $\mathcal{O}$ is such that $\mathcal{O}\mathcal{O}^T = I$ is a rotation. Then

$$|\mathcal{O}x-y|^2 = \big|\mathcal{O}x - \mathcal{O}\mathcal{O}^T y\big|^2 = \big|x - \mathcal{O}^T y\big|^2,$$

and a change of variables shows that $\int_{B_R} F(|x-y|^2)\,dy$ only depends on R. Consequently, (5.3.5) implies that

$$\dot{\mathcal{C}}(0) = 4N(R)\oint_{\partial B_R}(v\cdot\nu)\,dS.$$

Clearly, for volume preserving perturbations the ball is a critical point of $\mathcal{C}$.

Remark 5.1. If a domain is a critical point of $\mathcal{C}$ among all domains of fixed volume, then by (5.3.4)

$$\int_{\Omega} F(|x-y|^2)\,dy = \text{const.} \quad \text{on } \partial\Omega.$$

What domains have this property? This question was addressed for the first time by Fraenkel [54] for the gravitational potential $c\int_{\Omega}|x-y|^{2-n}dy$. He showed that the ball is the only such domain. W. Reichel [101] showed that Fraenkel's result remains true if $n-2$ is replaced by $n-\alpha$, $\alpha > 2$, and if Ω is convex.

We now compute the second variation at $t = 0$. Straightforward differentiation of $\dot{\mathcal{C}}(t)$ (see (5.3.1)) yields

$$\begin{aligned}\ddot{\mathcal{C}}(0) = &\int\limits_{\Omega\times\Omega} F''(|x-y|^2)\dot{D}^2(0)\,dx\,dy + \int\limits_{\Omega\times\Omega} F'(|x-y|^2)\ddot{D}(0)\,dx\,dy\\ &+ 2\int\limits_{\Omega\times\Omega} F'(|x-y|^2)\dot{D}(0)[\dot{J}(0,x)+\dot{J}(0,y)]\,dx\,dy\\ &+ \int\limits_{\Omega\times\Omega} F(|x-y|^2)[\ddot{J}(0,x)+2\dot{J}(0,x)\dot{J}(0,y)+\ddot{J}(0,y)]\,dx\,dy. \end{aligned}\tag{5.3.6}$$

Our aim is to convert all domain integrals into boundary integrals. The calculations are based on the following formulas:

$$v(x)\cdot\nabla_x F(|x-y|^2) = 2F'(|x-y|^2)(v(x)\cdot(x-y)),\tag{5.3.7}$$

$$v(y)\cdot\nabla_y F(|x-y|^2) = -2F'(|x-y|^2)(v(y)\cdot(x-y)).\tag{5.3.8}$$

Formulas (5.3.7) and (5.3.8) are equivalent versions of (5.3.2) and (5.3.3). We also have

$$\begin{aligned}v(x)\cdot\nabla_x(v(x)\cdot\nabla_x F(|x-y|^2)) &= 4F''(|x-y|^2)(v(x)\cdot(x-y))^2\\ &+ 2F'(|x-y|^2)v(x)\cdot D_{v(x)}(x-y) + 2F'(|x-y|^2)|v(x)|^2,\end{aligned}\tag{5.3.9}$$

$$\begin{aligned}&v(x)\cdot\nabla_x(v(y)\cdot\nabla_y F(|x-y|^2))\\ &= -4F''(|x-y|^2)(v(x)\cdot(x-y))(v(y)\cdot(x-y))\\ &\quad - 2F'(|x-y|^2)(v(x)\cdot v(y)),\end{aligned}\tag{5.3.10}$$

$$\begin{aligned}v(y)\cdot\nabla_y(v(y)\cdot\nabla_y F(|x-y|^2)) &= 4F''(|x-y|^2)(v(y)\cdot(x-y))^2\\ &- 2F'(|x-y|^2)v(y)\cdot D_{v(y)}(x-y) + 2F'(|x-y|^2)|v(y)|^2.\end{aligned}\tag{5.3.11}$$

In (5.3.9) we have by definition $v(x)\cdot D_{v(x)}(x-y) = v_i(x)\partial_{x_i}v_j(x)(x_j-y_j)$. The same equality holds in (5.3.11) with x replaced by y. Finally we note that

$$v(y)\cdot\nabla_y(v(x)\cdot\nabla_x F(|x-y|^2)) = v(x)\cdot\nabla_x(v(y)\cdot\nabla_y F(|x-y|^2)).\tag{5.3.12}$$

Clearly,

$$\dot{D}(0) = 2(v(x)-v(y))\cdot(x-y) \quad\text{and}\quad \ddot{D}(0) = 2(v(x)-v(y))^2 + 2(w(x)-w(y))\cdot(x-y).$$

Step 1. We transform the integrand of the first integral in (5.3.6). From (5.3.9), (5.3.10), and (5.3.11) we deduce that

$$
\begin{aligned}
F''(|x-y|^2)\dot{D}(0)^2 &= 4F''(|x-y|^2)[(v(x)-v(y))\cdot(x-y)]^2 \\
&= \underbrace{v(x)\cdot\nabla_x(v(x)\cdot\nabla_x F(|x-y|^2))}_{(1)} + \underbrace{2v(x)\cdot\nabla_x(v(y)\cdot\nabla_y F(|x-y|^2))}_{(3)} \\
&\quad + \underbrace{v(y)\cdot\nabla_y(v(y)\cdot\nabla_y F(|x-y|^2))}_{(2)} - \underbrace{2F'(|x-y|^2)v(x)\cdot D_{v(x)}(x-y)}_{(1)} \\
&\quad + \underbrace{2F'(|x-y|^2)v(y)\cdot D_{v(y)}(x-y)}_{(2)} - \underbrace{2F'(|x-y|^2)(|v(x)|^2+|v(y)|^2)}_{(*)} \\
&\quad + \underbrace{4F'(|x-y|^2)(v(x)\cdot v(y))}_{(*)}.
\end{aligned}
$$

Step 2. The integrand of the second integral is

$$
\begin{aligned}
F'(|x-y|^2)\ddot{D}(0) = 2F'(|x-y|^2)(&\underbrace{|v(x)|^2+|v(y)|^2}_{(*)} - \underbrace{2(v(x)\cdot v(y))}_{(*)} \\
&+ \underbrace{(w(x)\cdot(x-y)) - (w(y)\cdot(x-y))}_{(4)}),
\end{aligned}
$$

and the integrand of the third integral can be written as

$$
\begin{aligned}
&2F'(|x-y|^2)\dot{D}(0)[\dot{J}(0,x)+\dot{J}(0,y)] \\
&\quad = 4F'(|x-y|^2)(\underbrace{(v(x)\cdot(x-y))\operatorname{div}_x v(x)}_{(1)} + \underbrace{(v(x)\cdot(x-y))\operatorname{div}_y v(y)}_{(3)} \\
&\qquad - \underbrace{(v(y)\cdot(x-y))\operatorname{div}_x v(x)}_{(3)} - \underbrace{(v(y)\cdot(x-y))\operatorname{div}_y v(y)}_{(2)}).
\end{aligned}
$$

Finally, we compute the integrand of the last integral in (5.3.6). We set

$$
A(x) := v(x)\operatorname{div}_x v(x) - v(x)\cdot D_{v(x)}, \quad \text{i. e.,} \quad A_i(x) = v_i(x)\operatorname{div}_x v(x) - v_j(x)\partial_{x_j} v_i(x),
$$

for $i,j \in \{1,\dots,n\}$. Then (2.3.4) gives $\ddot{J}(0,x) = \operatorname{div}_x A(x) + \operatorname{div}_x w(x)$. Hence,

$$
\begin{aligned}
&F(|x-y|^2)[\ddot{J}(0,x)+2\dot{J}(0,x)\dot{J}(0,y)+\ddot{J}(0,y)] \\
&\quad = F(|x-y|^2)(\underbrace{\operatorname{div}_x A(x)}_{(1)} + \underbrace{2\operatorname{div}_x v(x)\operatorname{div}_y v(y)}_{(3)} + \underbrace{\operatorname{div}_y A(y)}_{(2)} \\
&\qquad + \underbrace{\operatorname{div}_x w(x) + \operatorname{div}_y w(y)}_{(4)}).
\end{aligned}
$$

Step 3. We collect the four terms underbracketed with (1). The first and the fourth term can be written as a divergence, and the remaining terms are canceled out. Formulas (5.3.7) and (5.3.8) are applied in these computations. We have

$$
\underbrace{\dots\dots}_{(1)} = \operatorname{div}_x(v(x)v(x)\cdot\nabla_x F(|x-y|^2)) + \operatorname{div}_x(F(|x-y|^2)A(x)).
$$

A repetition of these computations for the four terms underbracketed with (2) yields

$$\underbrace{\ldots\ldots}_{(2)} = \operatorname{div}_y[v(y)(v(y)\cdot\nabla_y F(|x-y|^2))] + \operatorname{div}_y(F(|x-y|^2)A(y)).$$

There are four terms underbracketed with (3). We apply (5.3.7). They add up to

$$\underbrace{\ldots\ldots}_{(3)} = 2\operatorname{div}_x\operatorname{div}_y(v(x)v(y)F(|x-y|^2)).$$

Finally, we sum the terms underbracketed with (4):

$$\underbrace{\ldots\ldots}_{(4)} = \operatorname{div}_x(w(x)F(|x-y|^2)) + \operatorname{div}_y(w(y)F(|x-y|^2)).$$

Clearly, the terms which are underbracketed with (*) add up to zero.

Step 4. With the computations in Steps 1–3 and the divergence theorem, the second variation (5.3.6) becomes

$$\begin{aligned}\ddot{C}(0) = &\int_{\Omega_y}\oint_{\partial\Omega_x}(v(x)\cdot\nu(x))v(x)\cdot\nabla_x F(|x-y|^2) + (A(x)\cdot\nu(x))F(|x-y|^2)\,dy\,dS(x)\\ &+\oint_{\partial\Omega_y}\int_{\Omega_x}(v(y)\cdot\nu(y))v(y)\cdot\nabla_y F(|x-y|^2) + (A(y)\cdot\nu(y))F(|x-y|^2)\,dx\,dS(y)\\ &+2\oint_{\partial\Omega}\oint_{\partial\Omega}(v(x)\cdot\nu(x))(v(y)\cdot\nu(y))F(|x-y|^2)\,dS(y)\,dS(x)\\ &+\int_{\Omega_y}\oint_{\partial\Omega_x}(w(x)\cdot\nu(x))F(|x-y|^2)\,dS(x)\,dy\\ &+\oint_{\partial\Omega_y}\int_{\Omega_x}(w(y)\cdot\nu(y))F(|x-y|^2)\,dx\,dS(y).\end{aligned}$$

We now take into account that $\nabla_x F(|x-y|^2) = -\nabla_y F(|x-y|^2)$. The divergence then yields

$$\int_{\Omega_y}(v(x)\cdot\nu(x))v(x)\cdot\nabla_x F(|x-y|^2)\,dy = -\oint_{\partial\Omega_y}(v(x)\cdot\nu(x))(v(x)\cdot\nu(y))F(|x-y|^2)\,dS(y).$$

Analogously,

$$\int_{\Omega_x}(v(y)\cdot\nu(y))v(y)\cdot\nabla_y F(|x-y|^2)\,dy = -\oint_{\partial\Omega_x}(v(y)\cdot\nu(y))(v(y)\cdot\nu(x))F(|x-y|^2)\,dS(x)$$

holds. If

$$\hat{F}(x) = \int_{\Omega_y}F(|x-y|^2)\,dy,$$

then

$$\ddot{C}(0) = 2 \oint_{\partial\Omega_y} \oint_{\partial\Omega_x} (v(x) \cdot \nu(x))((v(y) - v(x)) \cdot \nu(y)) F(|x-y|^2)\, dS(x)\, dS(y) \\ + 2 \oint_{\partial\Omega_x} \big[(v(x) \cdot \nu(x)) \operatorname{div}_x v(x) - v(x) \cdot D_{v(x)} \nu(x) + (w(x) \cdot \nu(x))\big] \hat{F}(x)\, dS(x). \tag{5.3.13}$$

We summarize the results in the following proposition.

Proposition 5.4. *Let Ω be a bounded smooth domain and let $F : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function. For any smooth family $\{\Omega_t\}_{|t|<t_0}$ of perturbations of Ω let*

$$C(t) := \int_{\Omega_t} \int_{\Omega_t} F(|\xi - \eta|^2)\, d\xi\, d\eta.$$

Then $\dot{C}(0)$ is given by (5.3.4) *and $\ddot{C}(0)$ by* (5.3.13).

5.3.1 The sign of $\ddot{C}(0)$ for the ball

We consider the case where $\Omega = B_R(0)$ and the perturbations are restricted to the class of volume preserving Hadamard perturbations, i. e.,

$$v(x) = \rho(x)\nu(x), \quad v(y) = \rho(y)\nu(y), \quad \text{for } x, y \in \partial B_R, \quad \text{and} \quad \oint_{\partial B_R} \rho(x)\, dS_x = 0.$$

Here $\nu(x) = x/R$ and $\nu(y) = y/R$. If $y \in \partial B_R$, then $\hat{F}(x) = N(R)$ is independent of x (see (5.3.5)). Consequently, in view of (2.3.4), the first integral in (5.3.13) becomes

$$\oint_{\partial\Omega_x} [v(x) \cdot \nu(x) \operatorname{div}_x v(x) - v(x) \cdot D_v \nu(x) + w(x) \cdot \nu(x)] \hat{F}(x)\, dS_x = N(R) \ddot{V}(0) = 0.$$

Hence,

$$\ddot{C}(0) = \oint_{\partial B_R} \oint_{\partial B_R} \rho(x)[\rho(y) - \rho(x) R^{-2} (x \cdot y)] F(|x-y|^2)\, dS_x\, dS_y.$$

We want to study the sign of $\ddot{C}(0)$. For convenience we introduce the variables $x = R\xi$ and $y = R\eta$. Then

$$\ddot{C}(0) = R^{2n-2} \oint_{\partial B_1 \times \partial B_1} [\rho(R\xi)\rho(R\eta) - \rho^2(R\xi)(\xi \cdot \eta)] F(2R^2(1 - (\eta \cdot \xi)))\, dS_\xi\, dS_\eta.$$

The discussion of the expression above requires some results for integral operators. Suppose that $\oint_{\partial B_1} |F(R^2|\xi - \eta|^2)|\, dS_\eta < \infty$. Consider the operator

$$K : L^2(\partial B_1) \to L^2(\partial B_1), \quad (Kf)(\xi) := \oint\limits_{\partial B_1} F(2R^2|\xi - \eta|^2) f(\eta)\, dS_\eta.$$

By the formula of Funk–Hecke (see (3.1.6)) the eigenfunctions of K are the spherical harmonics $Y_{k,j}(\eta)$ of degree k and degeneration d_k. The eigenvalues of K form a decreasing sequence $|\lambda_0| \geq |\lambda_1| \geq \cdots$.

Example 5.2. A special case of (3.1.6) is $k = 1$ and $d_1 = n$. Up to a multiplicative constant we have $Y_{1,j}(\eta) = \eta_j$ for $j \in \{1, \ldots, n\}$. Scalar multiplication with ξ yields

$$\oint\limits_{\partial B_1} F(2R^2(1 - (\xi \cdot \eta)))(\xi \cdot \eta)\, dS_\eta = \lambda_1, \tag{5.3.14}$$

since $|\xi| = 1$.

The corresponding eigenvalues λ_k are given in (3.1.6). In our context they read as

$$\lambda_k = |\partial B_1| \int\limits_{-1}^{1} F(2R^2(1 - t)) P_k(t) (1 - t^2)^{(n-3)/2}\, dt. \tag{5.3.15}$$

Here $t = (\xi \cdot \eta)$ and $P_k(t)$ is the Legendre polynomial of degree k.

Example 5.3. We have $P_0 = 1$, $P_1 = t$, and $P_2 = \frac{3t^2-1}{2}$. Hence,

$$\begin{aligned}
\lambda_0 &= |\partial B_1| \int\limits_{1}^{1} F(2R^2(1 - t))(1 - t^2)^{(n-3)/2}\, dt, \\
\lambda_1 &= |\partial B_1| \int\limits_{-1}^{1} F(2R^2(1 - t)) t (1 - t^2)^{(n-3)/2}\, dt \\
&= -\frac{|\partial B_1|}{n-1} [F(2R^2(1 - t))(1 - t^2)^{(n-1)/2}]_{-1}^{1} \\
&\quad + \frac{|\partial B_1|}{n-1} \int\limits_{-1}^{1} \frac{dF}{dt}(2R^2(1 - t))(1 - t^2)^{(n-1)/2}\, dt, \\
\lambda_2 &= |\partial B_1| \int\limits_{-1}^{1} F(2R^2(1 - t)) \left[(1 - t^2)^{(n-3)/2} - \frac{3}{2}(1 - t^2)^{(n-1)/2}\right] dt.
\end{aligned} \tag{5.3.16}$$

The eigenvalue λ_1 has multiplicity n.

From Section 3.1 we know that the spherical harmonics $\{Y_{k,j}(\xi)\}_{k,j}$ provide an orthonormal basis in $L^2(\partial B_1)$. Thus, any integrable function $\rho : \partial B_R \to \mathbb{R}$ can be written as a Fourier series

$$\rho(R\xi) = c_{0,1}Y_{0,1} + \sum_{k=1}^{\infty}\sum_{j=1}^{d_k} c_{k,j}Y_{k,j}(\xi) \quad \text{with } c_{k,j} = \oint_{\partial B_1} \rho(R\xi)Y_{k,j}(\xi)\, dS_\xi.$$

By assumption, $\dot{\mathcal{V}}(0) = \oint_{\partial B_R} \rho\, dS = 0$. Keeping in mind that $Y_{0,1}$ is a constant, we conclude that $c_{0,1} = 0$ and we therefore get $\rho = \sum_{k=1}^{\infty}\sum_{j=1}^{d_k} c_{k,j}Y_{k,j}$. Next we apply the barycenter condition (see Definition 2.5). Since $Y_{1,j} = a_j x_j$ for $j \in \{1, \ldots, n\}$ for some positive constants a_j this implies

$$\rho = \sum_{k=2}^{\infty}\sum_{j=1}^{d_k} c_{k,j}Y_{k,j}.$$

Suppressing the degeneration we will write for short

$$\rho = \sum_{k=2}^{\infty} c_k Y_k.$$

Replacing ρ by its Fourier series we obtain

$$(K\rho)(\xi) = \oint_{\partial B_1} F\big(2R^2(1 - (\xi \cdot \eta))\big) \sum_{k=2}^{\infty} c_k Y_k(\eta)\, dS_\eta = \sum_{k=2}^{\infty} \lambda_k c_k Y_k(\xi).$$

This implies that

$$\oint_{\partial B_1} \rho(R\xi)(K\rho)(\xi)\, dS_\xi = \sum_{k=2}^{\infty} \lambda_k c_k^2.$$

Furthermore, from Example 5.2 we have $\oint_{\partial B_1} (\xi \cdot \eta)F(2R^2(1 - (\xi \cdot \eta)))\, dS_\eta = \lambda_1$. Thus,

$$\oint_{\partial B_1} \rho^2(R\xi) \oint_{\partial B_1} (\xi \cdot \eta)F\big(2R^2(1 - (\xi \cdot \eta))\big)\, dS_\eta = \lambda_1 \sum_{k=2}^{\infty} c_k^2.$$

This leads to the final result

$$\ddot{\mathcal{C}}(0) = R^{2n-2} \sum_{k=2}^{\infty} (\lambda_k - \lambda_1)c_k^2. \tag{5.3.17}$$

The sign depends on the monotonicity of F. This can be derived from the representation of λ_1 in (5.3.16). Let us assume $F(0) = 0$. Then

$$\lambda_1 = \frac{|\partial B_1|}{n-1} \int_{-1}^{1} \frac{dF}{dt}\big(2R^2(1-t)\big)(1-t^2)^{(n-1)/2}\, dt$$
$$= -2R^2 \frac{|\partial B_1|}{n-1} \int_{-1}^{1} F'\big(2R^2(1-t)\big)(1-t^2)^{(n-1)/2}\, dt.$$

If $F(s) > 0$ for $s > 0$ and $F'(\cdot) < 0$, then $0 < \lambda_1 < \lambda_0$. Since $|\lambda_k| < \lambda_1$ for all $k \geq 2$, we conclude that $\ddot{\mathcal{C}}(0) \leq 0$. Equality holds only if $c_k = 0$ for all $k \geq 2$, that is, $\rho = a_k x_k$.

In this case the ball is a local maximum of $\mathcal{C}(t)$. This is in accordance with the Riesz inequality [103], which states that the ball is also a global maximum.

On the other hand, if $F'(\cdot) > 0$, $\lambda_1 < 0$, then $|\lambda_1| > |\lambda_k|$, for $k = 2, \ldots$, implies that $\ddot{\mathcal{C}}(0) \geq 0$. In this case the ball is a local minimum.

5.3.2 Gamov's liquid model

In Gamov's liquid model the energy of a water drop is given by the surface energy and the Coulomb force. After suitable normalization it takes the form

$$\mathcal{E}(\Omega) = |\partial\Omega| + \frac{1}{8\pi} \int_\Omega \int_\Omega \frac{1}{|x-y|}\, dx\, dy,$$

where Ω is a bounded domain in $\mathbb{R}^3$. Let $\Omega = B_R$ and consider the family of volume preserving perturbations. Then $\dot{\mathcal{E}}(0) = 0$, and by (2.3.26) and the previous computations,

$$\ddot{\mathcal{E}}(0) = \ddot{\mathcal{S}}_0(0) + \frac{1}{8\pi}\ddot{\mathcal{C}}(0) = \oint_{\partial B_R} |\nabla^\tau \rho|^2\, dS - \frac{2}{R^2} \oint_{\partial B_R} \rho^2\, dS$$
$$- \frac{1}{8\pi} \oint_{\partial B_R} \oint_{\partial B_R} \rho(x)[\rho(y) - \rho(x) x \cdot y]|x-y|^{-1}\, dS_x\, dS_y.$$

The sign is not clear because the second variation of the surface area is positive and that of the convolution is negative. A more subtle analysis is needed to determine which one dominates. In [32] the author gave a detailed analysis of volume preserving perturbations of the first and the second order. We consider the special case $n = 3$ and

$$F(u) := \frac{1}{\sqrt{u}}. \tag{5.3.18}$$

Then $F > 0$ and $F' \leq 0$ and

$$
\begin{aligned}
\lambda_1 &= |\partial B_1| \int_{-1}^{1} F\big(2R^2(1-t)\big)t(1-t^2)^{\frac{n-3}{2}}\, dt \\
&= \frac{|\partial B_1|}{\sqrt{2}} \int_{-1}^{1} \frac{t}{\sqrt{R^2(1-t)}}\, dt = \frac{8\pi}{3R}.
\end{aligned} \tag{5.3.19}
$$

By (5.3.18) it follows $\oint_{\partial B_1} |F(R^2|\xi-\eta|^2)\, dS_\eta < \infty$. Hence, (5.3.17) applies and thus

$$
\ddot{\mathcal{E}}(0) = \ddot{\mathcal{S}}_0(0) + \frac{R^4}{8\pi} \sum_{k=2}^{\infty} (\lambda_k - \lambda_1) c_k^2.
$$

Since the perturbations is volume preserving and since the barycenter condition holds, we get the lower bound

$$
\ddot{\mathcal{E}}(0) \geq \frac{4}{R^2} \oint_{\partial B_R} \rho^2\, dS + \frac{R^4}{8\pi} \sum_{k=2}^{\infty} (\lambda_k - \lambda_1) c_k^2,
$$

where

$$
\rho = \rho(R\xi) = \sum_{k=2}^{\infty} c_k Y_k(\xi).
$$

Since $|\lambda_k| \leq \lambda_1$ for all $k \geq 2$,

$$
\ddot{\mathcal{E}}(0) \geq 4 \sum_{k=2}^{\infty} c_k^2 - 2\lambda_1 \frac{R^4}{8\pi} \sum_{k=2}^{\infty} c_k^2.
$$

We apply (5.3.19) and obtain

$$
\ddot{\mathcal{E}}(0) \geq 4 \sum_{k=2}^{\infty} c_k^2 - \frac{2}{3} R^3 \sum_{k=2}^{\infty} c_k^2.
$$

Hence, for $R < 6^{\frac{1}{3}} \sim 1.8$ the ball is a local minimizer for the Gamov energy. For large R such a statement cannot hold.

As an example we consider perturbations such that

$$
\rho = Y_k, \quad k \geq 2.
$$

Then ρ is volume preserving up to first order (since $k \geq 2$) and w can be chosen such that $(\Phi_t)_t$ is volume preserving up to second order. As a consequence we have $c_i = \delta_{ik}$. Hence,

$$
\ddot{\mathcal{C}}(0) = R^4(\lambda_k - \lambda_1).
$$

Since $n = 3$ the surface variation reads as

$$\ddot{\mathcal{S}}_0(0) = \oint_{\partial B_R} |\nabla^\tau \rho|^2 - \frac{4}{R^2}\rho^2 \, dS = k(k+1) - 4, \quad \text{for } k \geq 2.$$

Hence, the Gamov energy becomes

$$\ddot{\mathcal{E}}(0) = k(k+1) - 4 + \frac{R^4}{8\pi}(\lambda_k - \lambda_1), \quad k \geq 2.$$

From (5.3.16) and (5.3.19) we deduce

$$\lambda_1 = \frac{8\pi}{3R}.$$

Moreover (again applying (5.3.16)), a direct calculation yields

$$\lambda_2 = \frac{8\pi}{5R}.$$

Note that $|\lambda_k| - \lambda_2 \leq 0$ for $k \geq 2$. Then, for $k \geq 2$ we have

$$\begin{aligned}\ddot{\mathcal{E}}(0) &\leq k(k+1) - 4 + \frac{R^4}{8\pi}(\lambda_2 - \lambda_1) = k(k+1) - 4 - \frac{R^4}{8\pi}\left(\frac{16\pi}{15R}\right) \\ &= k(k+1) - 4 - \frac{2}{15}R^3.\end{aligned}$$

Clearly, $\ddot{\mathcal{E}}(0) < 0$ if

$$R > R^* := \left(\frac{15}{2}k(k+1) - 30\right)^{\frac{1}{3}}. \tag{5.3.20}$$

We summarize our results.

Proposition 5.5. *Let Ω be a bounded smooth domain in $\mathbb{R}^3$. The Gamov energy is given by*

$$\mathcal{E}(\Omega) = |\partial\Omega| + \frac{1}{8\pi}\int_\Omega\int_\Omega \frac{1}{|x-y|}\, dx\, dy.$$

Then among all domains of given volume the ball is a critical point of the energy. Moreover, the following statements hold true:
- *If $R < 6^{\frac{1}{3}} \sim 1.8$, the ball is a local minimizer.*
- *For large volumes (resp. R) the ball is not a local minimizer. Moreover,* (5.3.20) *gives a lower bound R^* such that $\ddot{\mathcal{E}}(0) < 0$ for $R > R^*$.*
- *No result is known for $6^{\frac{1}{3}} < R < R^*$.*

This result corresponds to an earlier work [79].

5.4 An optimal control problem

We compute the domain variations of the cost function

$$\mathcal{C}(t) = \int_{\Omega_t} C\big(u(y,t)\big)\, dy,$$

where u is a solution of the boundary value problem $\Delta u + g(u) = 0$ in Ω_t and $u = 0$ on $\partial\Omega_t$. We assume that Ω_t is obtained from B_R by means of the diffeomorphism Φ_t described in the previous chapters.

From formula (4.1.6) we obtain

$$\dot{\mathcal{C}}(0) = \int_{B_R} C'(u)u'\, dx + C(0) \oint_{\partial B_R} (v \cdot \nu)\, dS,$$

and from (4.1.10) it follows that

$$\begin{aligned}\ddot{\mathcal{C}}(0) = &\int_{B_R} C'(u)u''\, dx + \int_{B_R} C''(u)u'^2\, dx + 2C'(0) \oint_{\partial B_R} u'(v \cdot \nu)\, dS \\ &+ C'(0) \oint_{\partial B_R} (v \cdot \nabla u)(v \cdot \nu)\, dS + C(0)\ddot{\mathcal{V}}(0).\end{aligned}$$

Suppose that $u = u(r)$ is radial. Here $r = |x|$ and $u_r(r) = \frac{d}{dr}u(r)$. The first shape derivative satisfies

$$\Delta u' + g'(u)u' = 0 \text{ in } B_R, \quad u' = -(\nabla u \cdot v) = -u_r(R)(v \cdot \nu) \text{ on } \partial B_R. \tag{5.4.1}$$

The second shape derivative u'' solves the equation

$$\Delta u'' + g''(u)u'^2 + g'(u)u'' = 0 \quad \text{in } B_R.$$

Formula (4.1.8) implies that on the boundary,

$$\begin{aligned}u'' = &-(v \cdot D^2 u v) - 2(v \cdot \nabla u') - (w \cdot \nabla u) = \\ &- u_{rr}(R)(v \cdot \nu)^2 - u_r(R)\frac{|v^\tau|^2}{R} - 2v \cdot \nabla u' - u_r(R)(w \cdot \nu).\end{aligned}$$

The fact that the differential equation for u'' depends also on u' makes the discussion more difficult. This is not the case in the next example.

5.4.1 The torsion problem

We consider the special case $g(u) = 1$. Then

$$u(x) = \frac{1}{2n}(R^2 - |x|^2).$$

The first and second shape derivatives, u' and u'', are harmonic functions. For Hadamard perturbations they satisfy the boundary conditions

$$u' = \frac{R}{n}(v \cdot \nu) \quad \text{and} \quad u'' = -2(v \cdot \nu)\partial_\nu u' + \frac{1}{n}(v \cdot \nu)^2 + \frac{R}{n}(w \cdot \nu).$$

In the boundary condition for u'', we replace $(v \cdot \nu)$ by u':

$$u'' = -\frac{2n}{R} u' \partial_\nu u' + \frac{n}{R^2} u'^2 + \frac{R}{n}(w \cdot \nu) \quad \text{on } \partial B_R. \tag{5.4.2}$$

For the first variation we obtain

$$\dot{\mathcal{C}}(0) = \int_{B_R} C'(u(r))u' \, dx + C(0)\dot{\mathcal{V}}(0).$$

Since u' is harmonic, the mean value theorem implies

$$\frac{1}{r^{n-1}} \oint_{\partial B_r} u' \, dS = \frac{1}{R^{n-1}} \oint_{\partial B_R} u' \, dS \tag{5.4.3}$$

for all $0 \le r \le R$. Hence,

$$\begin{aligned}
\int_{B_R} C'(u(r))u' \, dx &= \int_0^R C'(u(r)) \oint_{\partial B_r} u' \, dS_r \, dr = \int_0^R C'(u(r)) \frac{r^{n-1}}{r^{n-1}} \oint_{\partial B_r} u' \, dS_r \, dr \\
&= \int_0^R C'(u(r)) \frac{r^{n-1}}{R^{n-1}} \oint_{\partial B_R} u' \, dS_R \, dr \\
&= \frac{1}{R^{n-1}} \oint_{\partial B_R} u' \, dS_R \int_0^R C'(u(r)) r^{n-1} \, dr.
\end{aligned}$$

Together with the boundary conditions of u' this leads to

$$\dot{\mathcal{C}}(0) = \left\{ \frac{1}{nR^{n-2}} \int_0^R C'(u) r^{n-1} \, dr + C(0) \right\} \dot{\mathcal{V}}(0).$$

If $\dot{\mathcal{V}}(0) = 0$, the first variation vanishes. Thus, the ball is a critical point of the cost functionals for all volume preserving perturbations.

The second variation is of the form

$$\ddot{\mathcal{C}}(0) = \int_{B_R} C'(u)u''\,dx + \int_{B_R} C''(u)u'^2\,dx + \frac{nC'(0)}{R}\oint_{\partial B_R} u'^2\,dS + C(0)\ddot{\mathcal{V}}(0). \tag{5.4.4}$$

We will determine the sign under the assumptions

$$\dot{\mathcal{V}}(0) = \ddot{\mathcal{V}}(0) = 0 \quad \text{and} \quad C'(u),\ C''(u) \geq 0.$$

The first two terms of $\ddot{\mathcal{C}}(0)$ will be discussed separately.

1. $\int_{B_R} C'(u)u''\,dx$

Since u'' is harmonic, the mean value theorem (see (5.4.3)) applies. Moreover, u'' satisfies the boundary conditions (5.4.2). Hence,

$$\begin{aligned}\int_{B_R} C'(u)u''\,dx &= R^{1-n}\oint_{\partial B_R} u''\,dS \int_0^R C'(u(r))r^{n-1}\,dr \\ &= R^{1-n}\int_0^R C'(u(r))r^{n-1}\,dr \oint_{\partial B_R}\left(-\frac{2n}{R}u'\partial_\nu u' + \frac{n}{R^2}u'^2 + \frac{R}{n}(w\cdot\nu)\right)dS.\end{aligned}$$

From (2.3.12) and the boundary condition for u' we obtain

$$\oint_{\partial B_R}(w\cdot\nu)\,dS = -\frac{n-1}{R}\oint_{\partial B_R}(v\cdot\nu)^2\,dS = -\frac{n^2(n-1)}{R^3}\oint_{\partial B_R} u'^2\,dS.$$

Consequently,

$$\begin{aligned}\int_{B_R} C'(u)u''\,dx &= R^{1-n}\oint_{\partial B_R} u''\,dS \int_0^R C'(u(r))r^{n-1}\,dr \\ &= R^{1-n}\int_0^R C'(u(r))r^{n-1}\,dr \oint_{\partial B_R}\left(-\frac{2n}{R}u'\partial_\nu u' + \frac{n(n-2)}{R^2}u'^2\right)dS. \end{aligned}\tag{5.4.5}$$

2. $\int_{B_R} C''(u)u'^2\,dx$

As before we introduce polar coordinates and write

$$\int_{B_R} C''(u(r))u'^2\,dx = \int_0^R C''(u(r))\oint_{\partial B_r} u'^2\,dS\,dr. \tag{5.4.6}$$

Since u'^2 is subharmonic, the mean value theorem does not apply. We therefore pursue a different scheme.

Let $\{\phi_k\}_{k=0}^{\infty}$ be the orthonormal basis of Steklov eigenfunctions (see Section 3.2.1)

$$\Delta\phi_k = 0 \text{ in } B_R, \quad \partial_\nu\phi_k = \mu_k\phi_k \text{ on } \partial B_R.$$

Expressed in polar coordinates, the eigenfunctions are of the form

$$\phi_0(r,\xi) = \frac{1}{|\partial B_R|^{1/2}}, \quad \phi_k(r,\xi) = \frac{r^k}{R^{(n-1+2k)/2}} Y_{k,i}(\xi), \tag{5.4.7}$$

where $Y_{k,i}$ is a normalized spherical harmonic such that

$$\oint_{\partial B_1} Y_{k,i}^2(\xi)\, dS = 1 \quad \text{and} \quad \oint_{\partial B_1} Y_{k,i}(\xi) Y_{s,j}(\xi)\, dS = 0 \quad \text{if } (k,i) \neq (s,j).$$

In order to estimate the integral $\oint_{\partial B_r} u'^2\, dS$ we expand u' into the Fourier series

$$u' = \sum_{k=0}^{\infty} c_k\phi_k.$$

Since $\dot{\mathcal{V}}(0) = \frac{n}{R}\oint_{\partial B_R} u'\, dS = 0$, it follows that $c_0 = 0$. Hence,

$$\oint_{\partial B_R} u'^2\, dS = \sum_{k=1}^{\infty} c_k^2. \tag{5.4.8}$$

For $r \le R$ we obtain

$$\oint_{\partial B_r} u'^2\, dS = r^{n-1} \oint_{\partial B_1} u'^2(r,\xi)\, dS.$$

Consequently,

$$\oint_{\partial B_r} u'^2(R,\xi)\, dS = r^{n-1} \oint_{\partial B_1} u'^2(r,\xi)\, dS = \sum_{k=1}^{\infty} \frac{r^{n-1+2k}}{R^{n-1+2k}} c_k^2.$$

Since $r \le R$ and $k \ge 1$,

$$\oint_{\partial B_r} u'^2(r,\xi)\, dS \le \sum_{k=2}^{\infty} \frac{r^{n+1}}{R^{n+1}} c_k^2 = \frac{r^{n+1}}{R^{n+1}} \oint_{\partial B_R} u'^2\, dS.$$

Then by (5.4.6)

$$\int\limits_{B_R} C''(u(r))u'^2\,dx \le \frac{1}{R^{n+1}} \oint\limits_{\partial B_R} u'^2\,dS \int\limits_0^R C''(u(r))r^{n+1}\,dr.$$

Furthermore, since $C''(u) = -\frac{n}{r}\frac{d}{dr}C'(u)$, we have

$$\int\limits_0^R C''(u)r^{n+1}\,dr = -nC'(0)R^n + n^2 \int\limits_0^R C'(u)r^{n-1}\,dr.$$

This leads to the final estimate

$$\int\limits_{B_R} C''(u(r))u'^2\,dx \le \frac{1}{R^{n+1}} \oint\limits_{\partial B_R} u'^2\,dS\left(-nC'(0)R^n + n^2 \int\limits_0^R C'(u)r^{n-1}\,dr\right). \tag{5.4.9}$$

These two steps enable us to derive an upper bound for $\ddot{C}(0)$.

If we insert (5.4.5) and (5.4.9) into (5.4.4) and use Green's theorem

$$\int\limits_{B_R} |\nabla u'|^2\,dx = \oint\limits_{\partial B_R} u'\partial_\nu u'\,dS,$$

we obtain the estimate

$$\begin{aligned}
\ddot{C}(0) \le{}& R^{1-n}\int\limits_0^R C'(u)r^{n-1}\,dr\left(-2\frac{n}{R}\int\limits_{B_R}|\nabla u'|^2\,dx - \frac{n(n-2)}{R^2}\oint\limits_{\partial B_R} u'^2\,dS\right)\\
&+ \frac{1}{R^{n+1}}\oint\limits_{\partial B_R} u'^2\,dS\left(n^2\int\limits_0^R C'(u)r^{n-1}\,dr - nC'(0)R^n\right)\\
&+ \frac{n}{R}C'(0)\oint\limits_{\partial B_R} u'^2\,dS.
\end{aligned}$$

Since $\mu_1 = \frac{1}{R}$ and $\dot{\mathcal{V}}(0) = 0$, we have

$$\int\limits_{B_R} |\nabla u'|^2\,dx \ge \frac{1}{R}\oint\limits_{\partial B_R} u'^2\,dS.$$

Finally we obtain

$$\ddot{C}(0) \le \frac{2}{R^n}\int\limits_0^R C'(u)r^{n-1}\,dr\left\{R^{-1}\oint\limits_{\partial B_R} u'^2\,dS - \int\limits_{B_R}|\nabla u'|^2\,dx\right\} \le 0. \tag{5.4.10}$$

The fact that the second variation is nonpositive is compatible with the isoperimetric inequality (see [9]), which states that among all domains of fixed volume, $\mathcal{C}$ is maximal for the ball.

A special case for $\mathcal{C}$
Let $C(u) = u^p$, $p > 1$, and consider the functional

$$\mathcal{U}(t) = \mathcal{C}(t)^{1/p}.$$

From

$$\mathcal{C}(t)^{1/p} = \exp\left\{\frac{1}{p}\log\left(u_{\max}^p \int_{\Omega_t} \left(\frac{u}{u_{\max}}\right)^p dy\right)\right\},$$

it follows that

$$\max_{y\in\Omega_t} u(y,t) = \lim_{p\to\infty} \mathcal{U}(t).$$

Consequently, for nearly spherical domains of given volume, $\max_{\Omega_t} u(x,t) \le \frac{R^2}{2n}$. By means of symmetrization it can be shown that this result holds globally for any domain with the same volume as B_R [9, 119].

Problem 5.2. Are the results of this section also true for torsion problems with Robin boundary conditions?

Is it possible to discuss Payne–Rayner type inequalities [94] for the first eigenfunctions of the membrane problem?

5.5 Notes

Isoperimetric inequalities for functionals of the type $\mathcal{G}(t)$ considered in Section (5.1) have been dealt with the techniques of rearrangement and symmetrization. They go back to Hardy, Littlewood, and Pólya [72] and Pólya and Szegö [98]. An excellent presentation of the essential ingredients is given in [87]. For further developments and an extensive bibliography we refer to [28]. Global inequalities for weighted isoperimetric inequalities are derived in [22]. By means of symmetrization it is shown that under certain conditions, among all domains of given weighted volume, the weighted surface area attains its minimum for the ball.

The classical isoperimetric inequalities for the spherical and hyperbolic spaces were first proved by E. Schmidt in 1940 and 1943. Proofs and further references are found in [34].

Convolutions play an important role in potential theory. The effect of symmetrization on convolutions is completely explained by Riesz' inequality [103]; see also [72] and [87]. In this case, among all domains of given area, $\mathcal{C}(t)$ achieves its maximum for the ball. A survey of the existence and nonexistence of global minimizers in the Gamov liquid problem is given in [37].

For an overview of control problems we refer to [120].

6 Domain variations for energies

This chapter deals with domain functionals (energies) depending on the solutions of semilinear elliptic boundary value problems. In contrast to optimal control problems these solutions are critical points of the energies in the sense of Fréchet. The Euler–Lagrange equations allow to eliminate the shape derivative in the first domain variation. Consequently, the second domain variation depends only on first order shape derivatives. In this chapter we use the change of variables method.

6.1 Energies and critical points

In this section we show that the first variation of a functional governed by a function satisfying the Euler–Lagrange equation does not contain the shape derivative of the first order. Let

$$\mathcal{E}(t) = \int_{\Omega_t} G\big(y, u(y,t), \nabla u(y,t)\big)\, dy,$$

where $G(y,u,p) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function in all $2n+1$ variables. Suppose that u satisfies the Euler–Lagrange equation

$$G_u(y,u,\nabla u) - \frac{\partial}{\partial y_i} G_{p_i}(y,u,\nabla u) = 0 \text{ in } \Omega_t, \quad u = 0 \text{ on } \partial\Omega_t. \tag{6.1.1}$$

If Reynolds' theorem (Theorem 4.1) applies, then

$$\dot{\mathcal{E}}(t) = \int_{\Omega_t} \partial_t G\, dy + \oint_{\partial\Omega_t} (\partial_t \Phi_t \cdot \nu^t) G\, dS.$$

In this case, $\partial_t G = G_u \partial_t u + G_{p_i} \frac{\partial}{\partial y_i} \partial_t u$. The divergence theorem implies that

$$\int_{\Omega_t} \partial_t G\, dx = \int_{\Omega_t} \left\{ G_u - \frac{\partial}{\partial y_i} G_{p_i} \right\} u\, dx + \oint_{\partial\Omega_t} \nu_i^t G_{p_i} \partial_t u\, dS.$$

The boundary conditions $u(y,t) = 0$ on $\partial\Omega_t$ yield (cf. Section 2.4.2)

$$0 = \partial_t \Phi_t \cdot \nabla_y u(y,t) + \partial_t u(y,t).$$

Finally we get

$$\dot{\mathcal{E}}(t) = -\oint_{\partial\Omega_t} \nu_i^t G_{p_i} (\nabla_y u \cdot \partial_t \Phi_t)\, dS_t + \oint_{\partial\Omega_t} G(\partial_t \Phi_t \cdot \nu^t)\, dS_t. \tag{6.1.2}$$

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In particular,

$$\dot{\mathcal{E}}(0) = -\oint_{\partial\Omega} \nu_i G_{p_i}(\nabla u \cdot v)\, dS + \oint_{\partial\Omega} G\,(v \cdot \nu)\, dS.$$

Consider now the more general case

$$\mathcal{E}(t) = \int_{\Omega_t} G(y, u(y,t), \nabla u(y,t))\, dy + \oint_{\partial\Omega_t} B(u(y,t))\, dS_t.$$

If $u(y,t)$ is a critical point of $\mathcal{E}(t)$, it satisfies the Euler–Lagrange equation (6.1.1) and the boundary conditions $\nu_i^t G_{p_i} + B_u(u) = 0$. Then by Reynolds' theorem (Theorem 4.1),

$$\dot{\mathcal{E}}(t) = \oint_{\partial\Omega_t} [G(y,u,\nabla u) + B_u(u)\partial_{\nu^t} u + (n-1)H^t B(u)](\partial_t \Phi_t \cdot \nu^t)\, dS_t.$$

Two remarks stand out:
1. If u satisfies the Euler–Lagrange equations corresponding to $\mathcal{E}(t)$, then its first variation contains only boundary integrals.
2. If in addition u satisfies the boundary conditions associated to the Euler–Lagrange equation, $\dot{\mathcal{E}}(t)$ is independent of the shape derivative u'.

6.2 Elliptic boundary value problems

6.2.1 General setting, change of variables

Let $\{\Omega_t\}_{|t|<t_0}$ be a family of smooth domains described in Section 2.1, and let $u(y,t) \in C^{2,\alpha}(\Omega_t) \cap C^1(\overline{\Omega_t})$ be a solution of

$$\Delta_y u + g(u) = 0 \quad \text{in } \Omega_t. \tag{6.2.1}$$

On the boundary u satisfies either *Dirichlet conditions*

$$u = 0 \quad \text{on } \partial\Omega_t \tag{6.2.2}$$

or *Robin conditions*

$$\partial_{\nu_t} u + \alpha u = 0 \quad \text{on } \partial\Omega_t, \quad \alpha \in \mathbb{R}, \tag{6.2.3}$$

where ν_t stands for the outer normal of $\partial\Omega_t$. If $\alpha = 0$, u satisfies *Neumann conditions.*

Problem (6.2.1) is the Euler–Lagrange equation corresponding to the energy

$$\mathcal{E}_D(t) := \int_{\Omega_t} (|\nabla_y u|^2 - 2G(u))\, dy \tag{6.2.4}$$

in the case of Dirichlet boundary conditions, or to

$$\mathcal{E}_R(t) := \int_{\Omega_t} \left(|\nabla_y u|^2 - 2G(u)\right) dy + \alpha \oint_{\partial\Omega_t} u^2 \, dS_t \tag{6.2.5}$$

in the case of Robin boundary conditions. We are interested in the question how the energies vary with t.

By a change of variables the energy will be expressed in terms of integrals over Ω and $\partial\Omega$. We first transform the energy back onto Ω and then compute its domain variations. Let $\tilde{u}(t) := u(\Phi_t(x), t)$. After the change of variables $x = \Phi_t^{-1}(y)$, we get in the case of Robin boundary conditions(see (4.1.15), (4.1.5), and (4.1.1))

$$\mathcal{E}_R(t) = \int_\Omega \partial_i \tilde{u}(t) \partial_j \tilde{u}(t) A_{ij}(t) \, dx - 2 \int_\Omega G(\tilde{u}(t)) J(t) \, dx + \alpha \oint_{\partial\Omega} \tilde{u}^2(t) m(t) \, dS, \tag{6.2.6}$$

and

$$\mathcal{E}_D(t) = \int_\Omega \partial_i \tilde{u}(t) \partial_j \tilde{u}(t) A_{ij}(t) \, dx - 2 \int_\Omega G(\tilde{u}(t)) J(t) \, dx \tag{6.2.7}$$

in the case of Dirichlet boundary conditions.

The function $\tilde{u}(t)$ is a critical point of $\mathcal{E}(t)$. It solves therefore the Euler–Lagrange equation

$$L_A \tilde{u}(t) + g(\tilde{u}(t)) J(t) = 0 \quad \text{in } \Omega, \tag{6.2.8}$$

together with the boundary conditions

$$\begin{aligned} \tilde{u}(t) &= 0 \qquad \text{on } \partial\Omega \text{ in the Dirichlet case,} \\ \partial_{\nu_A} \tilde{u}(t) + \alpha m(t) \tilde{u}(t) &= 0 \qquad \text{on } \partial\Omega \text{ in the Robin case.} \end{aligned} \tag{6.2.9}$$

Here

$$L_A = \partial_j (A_{ij}(t) \partial_i) \quad \text{and} \quad \partial_{\nu_A} = \nu_i A_{ij}(t) \partial_j. \tag{6.2.10}$$

6.2.2 Boundary value problem for the shape derivative

In this chapter we assume that the problem (6.2.1) with the corresponding boundary conditions has a solution which is smooth and differentiable with respect to t near $t = 0$.

Differentiation of (6.2.8) leads to

$$L_{A(0)} \dot{\tilde{u}}(0) + L_{\dot{A}(0)} \tilde{u}(0) + g'(\tilde{u}(0)) \dot{\tilde{u}}(0) J(0) + g(\tilde{u}(0)) \dot{J}(0) = 0 \quad \text{in } \Omega. \tag{6.2.11}$$

The material derivative $\dot{\tilde{u}}(t)$ is equal to $\nabla_y u(y,t)\cdot\partial_t\Phi_t(\Phi_t^{-1}(y))+\partial_t u(y,t)$ (see Section 2.4.2). From Lemma 4.1 and Definition 2.6 ($\partial_t u(y,t)|_{t=0} =: u'(x)$) we deduce

$$
\begin{aligned}
L_{A(0)}\dot{\tilde{u}}(0) &= \Delta(u' + (v\cdot\nabla u)),\\
L_{\dot{A}(0)}\tilde{u}(0) &= \partial_j(\operatorname{div} v\partial_j u) - \partial_j(\partial_j v_i\partial_i u) - \partial_j(\partial_i v_j\partial_i u).
\end{aligned}
$$

It follows immediately from (6.2.1) and $\dot{J}(0) = \operatorname{div} v$ that the shape derivative $u'(x)$ satisfies

$$\Delta u' + g'(u)u' = 0 \quad \text{in } \Omega. \tag{6.2.12}$$

If u satisfies Dirichlet boundary conditions, then $\tilde{u}(t) = 0$ on $\partial\Omega$ for all $|t| < t_0$. Hence, $\dot{\tilde{u}}(t) = 0$ on $\partial\Omega$ for all $|t| < t_0$. For $t = 0$ this implies

$$u'(x) = -v\cdot\nabla u \quad \text{on } \partial\Omega. \tag{6.2.13}$$

In the case of Robin boundary conditions we compute the boundary conditions as follows. From (6.2.10) and (2.3.19) we get

$$
\begin{aligned}
\partial_{\nu_{A(0)}}\dot{\tilde{u}}(0) &= \partial_\nu(v\cdot\nabla u) + \partial_\nu u',\\
\partial_{\nu_{\dot{A}(0)}}\tilde{u}(0) &= \operatorname{div} v\partial_\nu u - \nu_j\partial_j v_i\partial_i u - \partial_i u\partial_i v_j\nu_j\\
&= \operatorname{div} v\partial_\nu u - \nu\cdot D_v\nabla u - \nabla u\cdot D_v\nu,\\
am(0)\dot{\tilde{u}}(0) &= av\cdot\nabla u + au',\\
a\dot{m}(0)\tilde{u}(0) &= a(n-1)(v\cdot\nu)Hu + au\operatorname{div}_{\partial\Omega} v^\tau.
\end{aligned}
$$

Inserting these expressions into (6.2.9) and using the boundary condition $\partial_\nu u + au = 0$ on $\partial\Omega$, we obtain

$$
\begin{aligned}
\partial_\nu u' + au' = {}&-\partial_\nu(v\cdot\nabla u) + au\operatorname{div} v + \nu\cdot D_v\nabla u + \nabla u\cdot D_v\nu - av\cdot\nabla u\\
&- a(n-1)uH(v\cdot\nu) - au\operatorname{div}_{\partial\Omega} v^\tau.
\end{aligned}
$$

The right side can be simplified using the results of Chapter 2. From (2.2.14) and (2.2.10) it follows that

$$\operatorname{div} v = \operatorname{div}_{\partial\Omega} v^\tau + (n-1)(v\cdot\nu)H + \nu\cdot D_v\nu \quad \text{on } \partial\Omega.$$

Moreover,

$$-\partial_\nu(v\cdot\nabla u) + (\nu\cdot D_v\nabla u) = -(v\cdot D^2u\nu), \quad \text{where } D^2u = \partial_i\partial_j u.$$

This leads to

$$\partial_\nu u' + au' = -(v\cdot D^2u\nu) + au\nu\cdot D_v\nu + \nabla u\cdot D_v\nu - av\cdot\nabla u$$

on $\partial\Omega$. The boundary condition for u implies

$$\partial_\nu u' + \alpha u' = -(v \cdot D^2 u\nu) + (\nabla u^\tau \cdot D_v \nu) - \alpha(v \cdot \nabla u). \tag{6.2.14}$$

The right-hand side of (6.2.14) is independent of the extension of v into $\overline{\Omega}$. This follows immediately from

$$(\nabla^\tau u \cdot D_v \nu) = \nabla^\tau u \cdot (\nabla^\tau (v \cdot \nu) - D_\nu v).$$

If $\partial\Omega$ is sufficiently regular, say $C^{2,\alpha}$, then the differential equation $\Delta u + g(u) = 0$ holds up to the boundary. In this case no normal derivatives appear in (6.2.14). More precisely, we have the following lemma.

Lemma 6.1. *Let u' be the shape derivative of u in the case of Robin boundary data. Then*

$$\begin{aligned}\partial_\nu u' + \alpha u' = {} & (v \cdot \nu)\big[g(u) - \alpha(n-1)uH + \alpha^2 u + \Delta^* u\big] \\ & + \nabla^\tau u \cdot \nabla(v \cdot \nu)\end{aligned}$$

on $\partial\Omega$.

Proof. To obtain the claim we will use (6.2.14). The equation and the boundary condition for u will be used in the following computations.

Step 1. By (B.1.5) it follows that

$$\Delta = \frac{\partial^2}{\partial \nu^2} + (n-1)H\frac{\partial}{\partial \nu} + \Delta^* \quad \text{on } \partial\Omega,$$

where $\Delta^* u$ is the Laplace–Beltrami operator on $\partial\Omega$. Moreover, we observe that $\frac{\partial^2 u}{\partial \nu^2} = (\nu \cdot D^2 u\nu)$. This follows immediately by using a local orthonormal frame whose origin lies on a boundary and whose e_n-axis points in the normal direction. Another possibility is to write $(\nu \cdot D^2 u\nu) = \nu_i \partial_i(\partial_\nu u) - \nu_i \partial_i \nu_j \partial_j u$ and use the fact that $D_\nu = \nabla^\tau \nu$ (cf. (2.4.5)).

Consequently the first term on the right-hand side of (6.2.14) can be written as

$$\begin{aligned}-(v \cdot D^2 u\nu) &= -(v \cdot \nu)(\nu \cdot D^2 u\nu) - (v^\tau \cdot D^2 u\nu) \\ &= -(v \cdot \nu)\big[\Delta u - \alpha(n-1)uH - \Delta^* u\big] - (v^\tau \cdot D^2 u\nu).\end{aligned}$$

The last term is not yet in final form. A straightforward computation yields

$$-(v^\tau \cdot D^2 u\nu) = -v^\tau \cdot \nabla(\nu \cdot \nabla u) + (v^\tau \cdot D_\nu \nabla u).$$

The tangential derivative of the boundary condition for u implies

$$-(v^\tau \cdot D^2 u\nu) = -v^\tau \cdot \nabla(-\alpha u) + (v^\tau \cdot D_\nu \nabla^\tau u) - \alpha u(v^\tau \cdot D_\nu \nu).$$

From (2.4.1) we obtain $(v^\tau \cdot D_\nu \nu) = 0$. Thus

$$-(v^\tau \cdot D^2 u\nu) = \alpha v^\tau \cdot \nabla u + (v^\tau \cdot D_\nu \nabla^\tau u).$$

Hence, taking into account that $\Delta u + g(u) = 0$ up to the boundary and $\partial_\nu u + \alpha u = 0$ on $\partial\Omega$ we find

$$\begin{aligned} -(v \cdot D^2 u\nu) = (v \cdot \nu) g(u) + \alpha(n-1)(v \cdot \nu) uH + (v \cdot \nu)\Delta^* u \\ + \alpha(v^\tau \cdot \nabla u) + (v^\tau \cdot D_\nu \nabla^\tau u). \end{aligned}$$

Step 2. In this step we rewrite the second and the third term in (6.2.14) as

$$\begin{aligned} &(\nabla^\tau u \cdot D_v \nu) - \alpha(v \cdot \nabla u) \\ &\quad = \nabla^\tau u \cdot \nabla(v \cdot \nu) - (\nabla^\tau u \cdot D_\nu v) - \alpha(v^\tau \cdot \nabla u) + \alpha^2 u(v \cdot \nu). \end{aligned}$$

From (2.4.4) and (2.4.1) we deduce

$$(v^\tau \cdot D_\nu \nabla u) = (\nabla u \cdot D_\nu v^\tau) = (\nabla^\tau u \cdot D_\nu v^\tau).$$

Step 3. In summary, we obtain

$$\begin{aligned} \partial_\nu u' + \alpha u' = (v \cdot \nu) g(u) + \alpha(n-1)(v \cdot \nu) uH + (v \cdot \nu)\Delta^* u \\ + \nabla^\tau u \cdot \nabla(v \cdot \nu) + \alpha^2 u(v \cdot \nu). \end{aligned}$$

The assertion now follows. □

Note that on the boundary u' is independent of the normal derivatives of v.

Example 6.1. Of special interest will be the case where Ω is the ball B_R of radius R centered at the origin and where u is a radial solution of $\Delta u + g(u) = 0$ in B_R with $\partial_\nu u + \alpha u = 0$ on ∂B_R. Then on the boundary (6.2.14) becomes

$$\partial_\nu u' + \alpha u' = \left(g(u(R)) - \frac{\alpha(n-1)}{R} u(R) + \alpha^2 u(R) \right)(v \cdot \nu). \tag{6.2.15}$$

For the torsion problem $g(u) = 1$ we have

$$u(x) = \frac{R}{\alpha n} + \frac{1}{2n}(R^2 - |x|^2).$$

We insert $u(R) = \frac{R}{\alpha n}$ and $g'(u) = 0$ into (6.2.12) and apply Lemma 6.1. Hence

$$\begin{aligned} \Delta u' &= 0 \quad \text{in } B_R, \\ \partial_\nu u' + \alpha u' &= \left(\frac{1 + \alpha R}{n} \right)(v \cdot \nu) \quad \text{on } \partial B_R. \end{aligned} \tag{6.2.16}$$

6.3 Differentiation of the energy

6.3.1 First and second domain variation

In this section we compute the first and the second domain variation of the energy (6.2.6). A direct computation yields for the Robin energy

$$\begin{aligned}\dot{\mathcal{E}}(t) = &\int_\Omega (\dot{A}(t)\nabla\tilde{u}\cdot\nabla\tilde{u})\,dx - 2\int_\Omega G(\tilde{u})\dot{J}(t)\,dx + \alpha\oint_{\partial\Omega}\tilde{u}^2\dot{m}(t)\,dS \\ &+ 2\int_\Omega (A(t)\nabla\tilde{u}\cdot\nabla)\dot{\tilde{u}}\,dx - 2\int_\Omega g(\tilde{u})\dot{\tilde{u}}J(t)\,dx + 2\alpha\oint_{\partial\Omega}\dot{\tilde{u}}\tilde{u}m(t)\,dS. \end{aligned} \tag{6.3.1}$$

In the case of the Dirichlet energy the boundary integral is missing. As already observed at the end of Section 3.2, the material derivative $\dot{\tilde{u}}$ can be eliminated by means of the Euler–Lagrange equations (6.2.8)–(6.2.9). Testing (6.2.8) with $\dot{\tilde{u}}$ we find

$$\int_\Omega (A(t)\nabla\dot{\tilde{u}}\cdot\nabla\tilde{u})\,dx = \int_\Omega g(\tilde{u})\dot{\tilde{u}}J(t)\,dx - \alpha\oint_{\partial\Omega}\dot{\tilde{u}}\tilde{u}m(t)\,dS.$$

Inserting this expression in (6.3.1) we get

$$\dot{\mathcal{E}}(t) = \int_\Omega (\dot{A}(t)\nabla\tilde{u}\cdot\nabla\tilde{u})\,dx - 2\int_\Omega G(\tilde{u})\dot{J}(t)\,dx + \alpha\oint_{\partial\Omega}\tilde{u}^2\dot{m}(t)\,dS. \tag{6.3.2}$$

If we differentiate once more, we get the second variation:

$$\begin{aligned}\ddot{\mathcal{E}}(t) = &\int_\Omega [(\ddot{A}(t)\nabla\tilde{u}\cdot\nabla\tilde{u}) + 2(\dot{A}(t)\nabla\tilde{u}\cdot\nabla\dot{\tilde{u}}) - 2g(\tilde{u})\dot{\tilde{u}}\dot{J}(t) - 2G(\tilde{u})\ddot{J}(t)]\,dx \\ &+ \alpha\oint_{\partial\Omega}(2\tilde{u}\dot{\tilde{u}}\dot{m}(t) + \tilde{u}^2\ddot{m}(t))\,dS. \end{aligned} \tag{6.3.3}$$

This formula can be further simplified. In fact, differention of the Euler–Lagrange equation (6.2.8) with respect to t leads to

$$L_A\dot{\tilde{u}} + L_{\dot{A}}\tilde{u} + g'(\tilde{u})J(t)\dot{\tilde{u}} + g(\tilde{u})\dot{J}(t) = 0 \quad \text{in } \Omega, \tag{6.3.4}$$

and on the boundary $\partial\Omega$

$$\begin{aligned} \dot{\tilde{u}}(t) &= 0 \quad \text{for Dirichlet conditions,} \\ \nu_{\dot{A}}\tilde{u} + \nu_A\dot{\tilde{u}} + \alpha\dot{m}\tilde{u} + \alpha m\dot{\tilde{u}} &= 0 \quad \text{for Robin conditions.} \end{aligned}$$

Testing (6.3.4) with $\dot{\tilde{u}}$ we obtain

$$\int_\Omega (\dot{A}(t)\nabla\tilde{u}\cdot\nabla\dot{\tilde{u}})\,dx = -\int_\Omega A_{ij}(t)\partial_j\dot{\tilde{u}}\partial_i\dot{\tilde{u}}\,dx + \int_\Omega g'(\tilde{u})\dot{\tilde{u}}^2 J(t)\,dx + \int_\Omega g(\tilde{u})\dot{\tilde{u}}\dot{J}(t)\,dx$$
$$+ \oint_{\partial\Omega}\{\nu_i\dot{A}_{ij}(t)\partial_j\tilde{u}\dot{\tilde{u}} + \nu_i A_{ij}(t)\partial_j\dot{\tilde{u}}\dot{\tilde{u}}\}\,dS.$$

Hence,

$$\ddot{\mathcal{E}}_D(t) = \int_\Omega (\ddot{A}(t)\nabla\tilde{u}\cdot\nabla\tilde{u})\,dx - 2\int_\Omega (A(t)\nabla\dot{\tilde{u}}\cdot\nabla\dot{\tilde{u}})\,dx$$
$$- 2\int_\Omega G(\tilde{u})\ddot{J}(t)\,dx + 2\int_\Omega g'(\tilde{u})\dot{\tilde{u}}^2 J(t)\,dx. \tag{6.3.5}$$

In the case of Robin conditions we have

$$-\oint_{\partial\Omega}\{\nu_i\dot{A}_{ij}(t)\partial_j\tilde{u}\dot{\tilde{u}} + \nu_i A_{ij}(t)\partial_j\dot{\tilde{u}}\dot{\tilde{u}}\}\,dS = \alpha\oint_{\partial\Omega}[\dot{\tilde{u}}\dot{\tilde{u}}m(t) + \dot{\tilde{u}}\tilde{u}\dot{m}(t)]\,dS.$$

Inserting these expressions into (6.3.3) we obtain

$$\ddot{\mathcal{E}}_R(t) = \int_\Omega (\ddot{A}(t)\nabla\tilde{u}\cdot\nabla\tilde{u})\,dx - 2\int_\Omega (A(t)\nabla\dot{\tilde{u}}\cdot\nabla\dot{\tilde{u}})\,dx$$
$$- 2\int_\Omega G(\tilde{u})\ddot{J}(t)\,dx + 2\int_\Omega g'(\tilde{u})\dot{\tilde{u}}^2 J(t)\,dx$$
$$+ \alpha\oint_{\partial\Omega}\tilde{u}^2\ddot{m}(t)\,dS - 2\alpha\oint_{\partial\Omega}\dot{\tilde{u}}^2 m(t)\,dS. \tag{6.3.6}$$

In accordance with the first variation which does not depend on $\dot{\tilde{u}}$, the second derivative does not depend on $\ddot{\tilde{u}}$.

In view of $m(0) = 1$, $A(0) = I$, $J(0) = 1$, and $\tilde{u}(0) = u$, we have

$$\ddot{\mathcal{E}}_D(0) = \int_\Omega (\ddot{A}(0)\nabla u\cdot\nabla u)\,dx - 2\int_\Omega |\nabla\dot{\tilde{u}}|^2\,dx$$
$$- 2\int_\Omega G(u)\ddot{J}(0)\,dx + 2\int_\Omega g'(u)\dot{\tilde{u}}^2\,dx \tag{6.3.7}$$

and

$$\ddot{\mathcal{E}}_R(0) = \int_\Omega (\ddot{A}(0)\nabla u\cdot\nabla u)\,dx - 2\int_\Omega |\nabla\dot{\tilde{u}}|^2\,dx$$
$$- 2\int_\Omega G(u)\ddot{J}(0)\,dx + 2\int_\Omega g'(u)\dot{\tilde{u}}^2\,dx$$
$$+ \alpha\oint_{\partial\Omega}u^2\ddot{m}(0)\,dS - 2\alpha\oint_{\partial\Omega}\dot{\tilde{u}}^2\,dS. \tag{6.3.8}$$

6.4 The first domain variation

The goal of this section is to transform $\dot{\mathcal{E}}(0)$ into a boundary integral. To this aim we shall use the explicit expressions for $\dot{A}_{ij}(0)$ and $\dot{J}(0)$ given in Lemma 4.1 and in (2.1.5). The volume integrals in (6.3.2) are

$$\mathcal{E}_1 := \int_\Omega \dot{A}_{ij}(0)\partial_i u \partial_j u \, dx \quad \text{and} \quad \mathcal{E}_2 := \int_\Omega G(u)\dot{J}(0)\, dx. \tag{6.4.1}$$

Integration by parts (see (4.1.16)) yields

$$\mathcal{E}_1 = \oint_{\partial\Omega} [|\nabla u|^2 (v \cdot \nu) - 2(v \cdot \nabla u)\partial_\nu u] \, dS + 2 \int_\Omega \Delta u (v \cdot \nabla u)\, dx\, dS.$$

Since $\dot{J}(0) = \operatorname{div} v$, integration by parts leads to

$$\mathcal{E}_2 = \oint_{\partial\Omega} G(u)(v \cdot \nu)\, dS - \int_\Omega g(u)(v \cdot \nabla u)\, dx.$$

For the first variation of the Dirichlet energy $\mathcal{E}_D = \mathcal{E}_1 - 2\mathcal{E}_2$, keeping in mind that $\Delta u + g(u) = 0$ in Ω and $\nabla u = \partial_\nu u \nu$ on $\partial\Omega$, we find

$$\dot{\mathcal{E}}_D(0) = -\oint_{\partial\Omega} (v \cdot \nu)\{|\nabla u|^2 + 2G(u)\}\, dS. \tag{6.4.2}$$

Next we compute the first variation of the Robin energy. From (6.3.2) we have

$$\dot{\mathcal{E}}_R(0) = \mathcal{E}_1 - 2\mathcal{E}_2 + \alpha \oint_{\partial\Omega} \tilde{u}^2 \dot{m}(t)\, dS.$$

Replacing $\dot{m}(0)$ by (2.3.19) we get

$$\begin{aligned}\dot{\mathcal{E}}_R(0) = &\oint_{\partial\Omega} [|\nabla u|^2 - 2G(u)](v \cdot \nu) + 2\alpha (v \cdot \nabla u) u \, dS \\ &+ \alpha \oint_{\partial\Omega} u^2 [\operatorname{div}_{\partial\Omega} v^\tau + (n-1)(v \cdot \nu) H]\, dS.\end{aligned}$$

Decomposing v and ∇u into tangential and normal components we get

$$v \cdot \nabla u = (v^\tau + (v \cdot \nu)\nu) \cdot (\nabla^\tau u + (\partial_\nu u)\nu) = v^\tau \cdot \nabla^\tau u - \alpha (v \cdot \nu) u. \tag{6.4.3}$$

Thus,

$$\begin{aligned}\dot{\mathcal{E}}_R(0) = &\oint_{\partial\Omega} (v \cdot \nu)[|\nabla u|^2 - 2G(u) - 2\alpha^2 u^2 + \alpha(n-1)H u^2]\, dS \\ &+ \alpha \oint_{\partial\Omega} [2 v^\tau u \nabla^\tau u + u^2 \operatorname{div}_{\partial\Omega} v^\tau]\, dS.\end{aligned}$$

The last integral vanishes by the Gauss Theorem (2.2.18). Hence,

$$\dot{\mathcal{E}}_R(0) = \oint_{\partial\Omega} (v \cdot \nu)\big[|\nabla u|^2 - 2G(u) - 2\alpha^2 u^2 + \alpha(n-1)Hu^2\big]\, dS. \tag{6.4.4}$$

In particular we observe that $\dot{\mathcal{E}}_{D/R}(0) = 0$ for all purely tangential deformations.

6.4.1 The first domain variation for volume preserving perturbations

We now consider volume preserving perturbations Φ_t of the first order. According to Definition 2.2 this means that

$$\oint_{\partial\Omega} \rho\, dS = 0, \quad \text{where } \rho = (v \cdot \nu).$$

From (6.4.2) it then follows that

$$\dot{\mathcal{E}}_D(0) = -\oint_{\partial\Omega} \rho |\nabla u|^2\, dS.$$

Definition 6.1. If $\dot{\mathcal{E}}(0) = 0$, Ω is a critical domain of the energy $\mathcal{E}(t)$.

Theorem 6.1. *If Ω is a critical domain of the energy for all volume preserving perturbations, then u satisfies on $\partial\Omega$*

$$\begin{cases} |\nabla u|^2 = \textit{const.} & \textit{if } u = 0 \textit{ on } \partial\Omega, \\ |\nabla u|^2 - 2G(u) - 2\alpha^2 u^2 + \alpha(n-1)Hu^2 = \textit{const.} & \textit{if } \partial_\nu u + \alpha u = 0 \textit{ on } \partial\Omega. \end{cases}$$

Proof. We give a proof for the second statement. Let us write for short

$$z(x) := |\nabla u|^2 - 2G(u) - 2\alpha^2 u^2 + \alpha(n-1)H \quad \text{and} \quad \overline{z} := |\partial\Omega|^{-1} \oint_{\partial\Omega} z\, dS.$$

Then, since $\oint_{\partial\Omega} (v \cdot \nu)\, dS = 0$,

$$\dot{\mathcal{E}}(0) = \oint_{\partial\Omega} (v \cdot \nu) z\, dS = \oint_{\partial\Omega} (v \cdot \nu)(z - \overline{z})\, dS.$$

Put $Z^\pm = \max\{0, \pm(z - \overline{z})\}$. Hence,

$$\oint_{\partial\Omega} (v \cdot \nu) z\, dS = \oint_{\partial\Omega} (v \cdot \nu)(Z^+ - Z^-)\, dS.$$

Suppose that $z \neq$ const. Then $Z^{\pm} \neq 0$ and we can construct a volume preserving perturbation such that $(v \cdot \nu) > 0$ in $\operatorname{supp} Z^{+}$ and $(v \cdot \nu) < 0$ in $\operatorname{supp} Z^{-}$. In this case we get $\dot{\mathcal{E}}(0) > 0$, which is obviously a contradiction. The first assertion is proved in the same way. □

Example 6.2. If $\Omega = B_R$ and $u(x) = u(|x|)$, then $\dot{\mathcal{E}}_{D/R}(0) = 0$.

By means of the moving plane method – proposed by Serrin in [107] – the existence of positive solutions with $u = 0$ and $|\nabla u| =$ const. on $\partial\Omega$ implies that Ω is a ball. This method does not seem to apply to Robin boundary conditions.

Problem 6.1. Is the overdetermined problem $\Delta u + g(u) = 0$ in Ω, $\partial_\nu u + \alpha u = 0$ on $\partial\Omega$, and

$$|\nabla u|^2 - 2G(u) - 2\alpha^2 u^2 + \alpha(n-1)u^2 H = \text{const.} \quad \text{on } \partial\Omega$$

only solvable in a ball?

6.5 The second domain variation

The aim of this section is to find a suitable form of the second variation of the energy in order to determine its sign. By (6.3.7) we have

$$\ddot{\mathcal{E}}_D(0) = \mathcal{F}_1 + \mathcal{F}_2,$$

where

$$\mathcal{F}_1(0) := \int_\Omega \ddot{A}_{ij}(0)\partial_i u \partial_j u \, dx - 2\int_\Omega |\nabla \dot{\tilde{u}}(0)|^2 \, dx,$$
$$\mathcal{F}_2(0) := -2\int_\Omega G(u)\ddot{J}(0)\, dx + 2\int_\Omega g'(\tilde{u})\dot{\tilde{u}}^2(0)\, dx.$$

By (6.3.8), the Robin energy is

$$\ddot{\mathcal{E}}_R(0) = \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3,$$

where

$$\mathcal{F}_3(0) := \alpha \oint_{\partial\Omega} u^2 \ddot{m}(0)\, dS - 2\alpha \oint_{\partial\Omega} \dot{\tilde{u}}^2 \, dS.$$

6.5.1 The transformations of $\mathcal{F}_1(0)$, $\mathcal{F}_2(0)$, and $\mathcal{F}_3(0)$

Step (i). We start with $\mathcal{F}_1(0)$.

From Lemma 4.1 we have

$$\begin{aligned}\ddot{A}_{ij}(0)\partial_i u\partial_j u = {} & [(\operatorname{div} v)^2 - D_v : D_v]|\nabla u|^2 + 4(\nabla u D_v \cdot D_v \nabla u) \\ & + 2D_v \nabla u \cdot D_v \nabla u - 4 \operatorname{div} v(\nabla u \cdot D_v \nabla u) \\ & + \operatorname{div} w|\nabla u|^2 - 2(\nabla u \cdot D_w \nabla u).\end{aligned} \tag{6.5.1}$$

From (4.1.4) it follows that

$$\nabla\dot{\tilde{u}}(0) = \nabla(v \cdot \nabla u) + \nabla u' = D_v \nabla u + D^2 u v + \nabla u'.$$

Thus,

$$\begin{aligned}-2\left|\nabla\dot{\tilde{u}}(0)\right|^2 = {} & -2(D_v \nabla u \cdot D_v \nabla u) - 4(D_v \nabla u \cdot D^2 u v) - 2(v D^2 u \cdot v D^2 u) \\ & - 4(\nabla u' \cdot D_v \nabla u) - 4(v \cdot D^2 u \nabla u') - 2\left|\nabla u'\right|^2.\end{aligned}$$

Clearly,

$$\begin{aligned}-4(\nabla u' \cdot D_v \nabla u) - 4(v \cdot D^2 u \nabla u') &= -4 \operatorname{div}[\nabla u'(v \cdot \nabla u)] + 4\Delta u'(v \cdot \nabla u) \\ &\quad + 4(\nabla u' \cdot D^2 u v) - 4(v \cdot D^2 u \nabla u') \\ &= -4 \operatorname{div}[\nabla u'(v \cdot \nabla u)] + 4\Delta u'(v \cdot \nabla u).\end{aligned}$$

Thus,

$$\begin{aligned}-2\left|\nabla\dot{\tilde{u}}(0)\right|^2 = {} & -2(D_v \nabla u \cdot D_v \nabla u) - 4(D_v \nabla u \cdot D^2 u v) - 2(v D^2 u \cdot v D^2 u) \\ & - 4 \operatorname{div}[(v \cdot \nabla u)\nabla u'] + 4(v \cdot \nabla u)\Delta u' - 2\left|\nabla u'\right|^2.\end{aligned}$$

Inserting this expression and (6.5.1) in $\mathcal{F}_1(0)$ we find

$$\mathcal{F}_1(0) = \int_\Omega (E_0 + E_1 + E_2)\, dx,$$

where

$$\begin{aligned}E_0 := {} & \underbrace{((\operatorname{div} v)^2 - D_v : D_v)|\nabla u|^2}_{(1)} \underbrace{+4(\nabla u D_v \cdot D_v \nabla u)}_{(*)} \\ & \underbrace{-4 \operatorname{div} v(\nabla u \cdot D_v \nabla u) - 4(D_v \nabla u \cdot D^2 u v)}_{(2)} \underbrace{-2(v D^2 u \cdot v D^2 u)}_{(3)}.\end{aligned}$$

The terms containing w are collected in E_1 and partially expressed in divergence form:

$$\begin{aligned}E_1 &:= \operatorname{div} w|\nabla u|^2 - 2(\nabla u \cdot D_w \nabla u) \\ &= \operatorname{div}[w|\nabla u|^2 - 2(w \cdot D^2 u \nabla u) - 2 \operatorname{div}[\nabla u(w \cdot \nabla u)] + 2\Delta u(w \cdot \nabla u) \\ &\quad + 2(w \cdot D^2 u \nabla u) \\ &= \operatorname{div}[w|\nabla u|^2 - 2\nabla u(w \cdot \nabla u)] + 2\Delta u(w \cdot \nabla u).\end{aligned}$$

The term E_2 collects all terms containing u':

$$E_2 = -4\operatorname{div}[(v\cdot\nabla u)\nabla u'] + 4(v\cdot\nabla u)\Delta u' - 2|\nabla u'|^2.$$

The goal is to write as many terms of E_0 as possible in divergence form so that the domain integrals of the remaining terms are canceled out.

With the help of (2.3.4) the first term in the expression of E_0 can be rewritten as

$$\begin{aligned}((\operatorname{div} v)^2 - D_v : D_v)|\nabla u|^2 &= \operatorname{div}[(v\operatorname{div} v - vD_v)]|\nabla u|^2\\ &= \operatorname{div}[(v\operatorname{div} v - vD_v)|\nabla u|^2] - 2\operatorname{div} v(v\cdot D^2u\nabla u)\\ &\quad + 2(vD_v\cdot D^2u\nabla u).\end{aligned}$$

Moreover,

$$\begin{aligned}&-4\operatorname{div} v(\nabla u\cdot D_v\nabla u) - 4(D_v\nabla u\cdot D^2uv)\\ &\quad = 4\operatorname{div}[\nabla u(v\cdot D_v\nabla u) - v(\nabla u\cdot D_v\nabla u)] + 4(vD^2u\cdot\nabla uD_v)\\ &\qquad \underbrace{-4(\nabla uD_v\cdot D_v\nabla u)}_{(*)} - 4\Delta u(v\cdot D_v\nabla u) - 4(vD_v\cdot D^2u\nabla u).\end{aligned}$$

This is best seen by calculating the first term on the right-hand side. The terms underbracketed with a $(*)$ in the above formula and in E_0 are canceled out. Hence,

$$\begin{aligned}E_0 = {}&\underbrace{\operatorname{div}[(v\operatorname{div} v - vD_v)|\nabla u|^2]}_{(1)}\ \underbrace{-2\operatorname{div} v(v\cdot D^2u\nabla u)}_{(1)}\\ &\underbrace{-2(vD_v\cdot D^2u\nabla u)}_{(1),(2)}\ \underbrace{+4\operatorname{div}[\nabla u(v\cdot D_v\nabla u) - v(\nabla u\cdot D_v\nabla u)]}_{(2)}\\ &\underbrace{+4(vD^2u\cdot\nabla uD_v)}_{(2)}\ \underbrace{-4\Delta u(v\cdot D_v\nabla u)}_{(2)}\ \underbrace{-2(vD^2u\cdot vD^2u)}_{(3)}.\end{aligned}$$

The four terms in E_0 which are neither in divergence form nor contain the Δ operator will be considered separately separately. Let us sum up the first three terms:

$$\begin{aligned}&-2(vD_v\cdot D^2u\nabla u) - 2(vD^2u\cdot vD^2u) - 2\operatorname{div} v(vD^2u\cdot\nabla u)\\ &\quad = -2\operatorname{div}[v(vD^2u\cdot\nabla u)]\ \underbrace{+2v_iv_j\partial_i\partial_j\partial_k u\partial_k u}_{(*)}.\end{aligned}$$

Furthermore,

$$\begin{aligned}4(vD^2u\cdot\nabla uD_v) = {}&4\operatorname{div}[\nabla u(vD^2u\cdot v)] - 4\Delta u(v\cdot D^2uv)\\ &-4v_iv_j\partial_i\partial_j\partial_k u\partial_k u - 4(vD^2u\cdot\nabla uD_v).\end{aligned}$$

Note that the term on the left side also appears on the right side with reversed sign. Consequently,

$$4(\nabla u D_v \cdot D^2 u v) = 2\operatorname{div}[\nabla u(v D^2 u \cdot v)] - 2\Delta u(v \cdot D^2 u v) \underbrace{-2 v_i v_j \partial_i \partial_j \partial_k u \partial_k u}_{(*)}.$$

The four terms underbracketed with $(*)$ are canceled out when we sum them all. Inserting them into E_0 we obtain

$$\begin{aligned} E_0 = {} & \operatorname{div}[(v \operatorname{div} v - v \cdot D_v)|\nabla u|^2] + 4\operatorname{div}[\nabla u(v \cdot D_v \nabla u) - v(\nabla u \cdot D_v \nabla u)] \\ & - 4\Delta u(v \cdot D_v \nabla u) - 2\operatorname{div}[v(v \cdot D^2 u \nabla u) - \nabla u(v \cdot D^2 u v)] - 2\Delta u(v \cdot D^2 u v). \end{aligned} \tag{6.5.2}$$

The divergence theorem then implies the following proposition.

Proposition 6.1. *We have*

$$\begin{aligned} \mathcal{F}_1(0) = {} & \oint_{\partial\Omega} [v \cdot \nu \, div \, v - (v \cdot D_v \nu) + (w \cdot \nu)] |\nabla u|^2 \, dS \\ & + 4 \oint_{\partial\Omega} [\partial_\nu u(v \cdot D_v \nabla u) - (v \cdot \nu)(\nabla u \cdot D_v \nabla u)] \, dS \\ & - 2 \oint_{\partial\Omega} [(v \cdot \nu)(v \cdot D^2 u \nabla u) - \partial_\nu u(v \cdot D^2 u v) + \partial_\nu u(w \cdot \nabla u)] \, dS \\ & - 2 \int_\Omega \Delta u[2(v \cdot D_v \nabla u) + (v \cdot D^2 u v) - (w \cdot \nabla u)] \, dx \\ & - 2 \int_\Omega [|\nabla u'|^2 - 2\Delta u'(v \cdot \nabla u)] \, dx - 4 \oint_{\partial\Omega} (v \cdot \nabla u) \partial_\nu u' \, dS. \end{aligned}$$

Step (ii). Consider now $\mathcal{F}_2(0)$.

The integral $\mathcal{F}_2(0)$ is transformed similarly to $\mathcal{F}_1(0)$. From (2.1.5) and the definition of the material derivative $\dot{u}(0)$ we have

$$\begin{aligned} \mathcal{F}_2(0) &= -2 \int_\Omega G(\tilde{u}(0)) \ddot{J}(0) \, dx + 2 \int_\Omega g'(u) \dot{u}^2(0) \, dx \\ &= -2 \int_\Omega G(u(x))((\operatorname{div} v)^2 - D_v : D_v + \operatorname{div} w) \, dx \\ & \quad + 2 \int_\Omega g'(u)[u'^2 + 2u'(v \cdot \nabla u)] \, dx + \int_\Omega \underbrace{2g'(u)(v \cdot \nabla u)^2}_{(*)} \, dx. \end{aligned}$$

We apply (2.3.4) and obtain

$$\ddot{J}(0) = (\operatorname{div} v)^2 - D_v : D_v + \operatorname{div} w = \operatorname{div}(v \operatorname{div} v - v D_v + w).$$

Since $g(u) = G'(u)$,

$$-2G(u)\ddot{J}(0) = -2\operatorname{div}[G(u)(v \operatorname{div} v - v D_v + w)] + 2(\nabla G(u) \cdot [v \operatorname{div} v - v D_v + w])$$

$$
\begin{aligned}
&= -2\operatorname{div}\big[G(u)(v\operatorname{div}v - vD_v + w)\big]\\
&\quad + 2g(u)\big[(v\cdot\nabla u)\operatorname{div}v - (v\cdot D_v\nabla u) + (w\cdot\nabla u)\big].
\end{aligned}
$$

The term $2g(u)(v\cdot\nabla u)\operatorname{div}v$ can be written in divergence form with some additional terms:

$$
\begin{aligned}
2g(u)(v\cdot\nabla u)\operatorname{div}v &= 2\operatorname{div}\big[g(u)(v\cdot\nabla u)v\big] - 2v\cdot\nabla\big[g(u)(v\cdot\nabla u)\big]\\
&= 2\operatorname{div}\big[g(u)(v\cdot\nabla u)v\big] - \underbrace{2g'(u)(v\cdot\nabla u)^2}_{(*)} - 2g(u)(v\cdot D^2uv)\\
&\quad - 2g(u)(v\cdot D_v\nabla u).
\end{aligned}
$$

The underbracketed terms are canceled out; hence,

$$
\begin{aligned}
\mathcal{F}_2(0) = &-2\oint_{\partial\Omega} G(u)\big[(v\cdot\nu)\operatorname{div}v - (v\cdot D_v\nu) + (w\cdot\nu)\big]\,dS\\
&+ 2\oint_{\partial\Omega} g(u)(v\cdot\nu)(v\cdot\nabla u)\,dS\\
&- 2\int_{\Omega} g(u)\big[2(v\cdot D_v\nabla u) + (v\cdot D^2uv) - (w\cdot\nabla u)\big]\,dx\\
&+ 2\int_{\Omega} g'(u)\big[u'^2 + 2(v\cdot\nabla u)u'\big]\,dx.
\end{aligned}
$$

This together with Proposition 6.1 yields the following proposition.

Proposition 6.2. *We have*

$$
\begin{aligned}
\mathcal{F}_1(0) + \mathcal{F}_2(0) = &\oint_{\partial\Omega}\big[(v\cdot\nu)\,\mathit{div}\,v - v\cdot(D_v\nu) + (w\cdot\nu)\big]\big(|\nabla u|^2 - 2G(u)\big)\,dS\\
&+ 2\oint_{\partial\Omega} g(u)(v\cdot\nu)(v\cdot\nabla u)\,dS + 4\oint_{\partial\Omega}\big[\partial_\nu u(v\cdot D_v\nabla u) - (v\cdot\nu)(\nabla u\cdot D_v\nabla u)\big]\,dS\\
&- 2\oint_{\partial\Omega}\big[(v\cdot\nu)(v\cdot D^2u\nabla u) - \partial_\nu u(v\cdot D^2uv) + \partial_\nu u(w\cdot\nabla u)\big]\,dS\\
&- 2\int_{\Omega}(\Delta u + g(u))\big[2(v\cdot D_v\nabla u) + (v\cdot D^2uv) - (\nabla u\cdot w)\big]\,dx\\
&- 2\int_{\Omega}\big[|\nabla u'|^2 - g'(u)u'^2\big]\,dx\\
&+ 4\int_{\Omega}(\Delta u' + g'(u)u')(v\cdot\nabla u)\,dx - 4\oint_{\partial\Omega}(v\cdot\nabla u)\partial_\nu u'\,dS.
\end{aligned}
$$

Step (iii). We continue with the transformation of $\mathcal{F}_3(0)$.

In $\mathcal{F}_3(0) = \alpha \oint_{\partial\Omega} [u^2 \ddot{m} - 2\dot{\tilde{u}}^2(0)]\, dS$ we replace $\dot{\tilde{u}}(0)$ by $(v \cdot \nabla u) + u'$ and obtain

$$\mathcal{F}_3(0) = \alpha \oint_{\partial\Omega} [u^2(x)\ddot{m}(0) - 2(v \cdot \nabla u)^2 - 2u'^2 - 4(v \cdot \nabla u)u']\, dS. \tag{6.5.3}$$

6.5.2 Main results

We now collect the previous transformations to express the second variation of the energy.

Dirichlet energy

In this case, $\ddot{\mathcal{E}}_D(0) = \mathcal{F}_1(0) + \mathcal{F}_2(0)$. In view of equation (6.2.1), the boundary conditions (6.2.2), and $\nabla u = \partial_\nu u \nu$, Proposition 6.2 implies

$$\begin{aligned}
\ddot{\mathcal{E}}_D(0) = &\oint_{\partial\Omega} [\underbrace{(v \cdot \nu)\operatorname{div} v - v \cdot (D_v \nu)}_{=:I_1} + (w \cdot \nu)](|\nabla u|^2 - 2G(0))\, dS \\
&+ 2g(0) \oint_{\partial\Omega} \partial_\nu u (v \cdot \nu)^2\, dS + 4 \oint_{\partial\Omega} (\partial_\nu u)^2\, (\underbrace{v - (v \cdot \nu)\nu}_{=v^\tau}) \cdot D_v \nu)\, dS \\
&- 2 \oint_{\partial\Omega} [\underbrace{(v \cdot \nu)(v \cdot D^2 u \nabla u) - \partial_\nu u (v \cdot D^2 u v)}_{=:I_2} + (\partial_\nu u)^2 (w \cdot \nu)]\, dS \\
&+ R(u').
\end{aligned} \tag{6.5.4}$$

Here $R(u')$ stands for the two integrals in Proposition 6.2 which contain u'. According to (2.2.14) we have $\operatorname{div} v = \operatorname{div}_{\partial\Omega} v - (\nu \cdot D_v \nu)$ on $\partial\Omega$. Hence,

$$I_1 = (v \cdot \nu)\operatorname{div}_{\partial\Omega} v - (v^\tau \cdot D_v \nu). \tag{6.5.5}$$

Decomposing $v = (v \cdot \nu)\nu + v^\tau$ and $\nabla u = \partial_\nu u \nu + \nabla^\tau u = \partial_\nu \nu$, we obtain

$$I_2 = -\partial_\nu u (v \cdot D^2 u v^\tau) = -\partial_\nu u (v^\tau \cdot D^2 u v). \tag{6.5.6}$$

By Proposition 6.2 we have

$$\begin{aligned}
R(u') = &-2 \int_\Omega [|\nabla u'|^2 - g'(u)u'^2]\, dx + 4 \int_\Omega (\Delta u' + g'(u)u')(v \cdot \nabla u)\, dx \\
&- 4 \oint_{\partial\Omega} (v \cdot \nabla u)\partial_\nu u'\, dS.
\end{aligned}$$

Since $\Delta u' + g'(u)u' = 0$ in Ω and $u' = -(v \cdot \nabla u)$ on $\partial\Omega$, it follows that

$$R(u') = -2\int_\Omega [|\nabla u'|^2 - g'(u)u'^2]\,dx + 4\oint_{\partial\Omega} u'\partial_\nu u'\,dS.$$

For the last integral, the divergence theorem together with $\Delta u' + g'(u)u' = 0$ yields

$$4\oint_{\partial\Omega} u'\partial_\nu u'\,dS = 4\int_\Omega [|\nabla u'|^2 - g'(u)u'^2]\,dx.$$

Hence, $R(u') = 2Q_0(u')$, where

$$Q_0(u') := \int_\Omega \{|\nabla u'|^2 - g'(u)u'^2\}\,dx. \tag{6.5.7}$$

Inserting these identities into (6.5.4) we are led to the final form of $\ddot{\mathcal{E}}_D(0)$.

Theorem 6.2. *Suppose that $\Delta u + g(u) = 0$ in Ω and $u = 0$ on $\partial\Omega$. Then the second variation $\ddot{\mathcal{E}}_D(0)$ assumes the form*

$$\begin{aligned}\ddot{\mathcal{E}}_D(0) = &\oint_{\partial\Omega}[(v\cdot\nu)\,div_{\partial\Omega}\,v - (v^\tau\cdot D_v\nu) + (w\cdot\nu)](|\nabla u|^2 - 2G(0))\,dS\\ &+ 2g(0)\oint_{\partial\Omega}\partial_\nu u(v\cdot\nu)^2\,dS + 4\oint_{\partial\Omega}(\partial_\nu u)^2(v^\tau\cdot D_v\nu)\,dS\\ &+ 2\oint_{\partial\Omega}\partial_\nu u(v^\tau\cdot D^2uv)\,dS - 2\oint_{\partial\Omega}(\partial_\nu u)^2(w\cdot\nu)\,dS + 2Q_0(u').\end{aligned}$$

Discussion

We now transform various terms in $\ddot{\mathcal{E}}_D(0)$ to get a better insight in its structure. As before we write $v = v^\tau + \rho\nu$. By (2.3.9),

$$(v^\tau\cdot D_v\nu) = (v^\tau\cdot\nabla^\tau\rho) - v^\tau\mathcal{L}v^\tau.$$

Hence,

$$4\oint_{\partial\Omega}(\partial_\nu u)^2(v^\tau\cdot D_v\nu)\,dS = 4\oint_{\partial\Omega}(\partial_\nu u)^2[(v^\tau\cdot\nabla^\tau\rho) - v^\tau\mathcal{L}v^\tau]\,dS.$$

Let P be any fixed point on $\partial\Omega$ which we choose to be the origin of the Cartesian coordinate system such that $e_1,\dots,e_{n-1}$ are in the tangent space and e_n points in the direction of the outer normal. Since $u = 0$ on $\partial\Omega$, we have $\partial_i\partial_j u = 0$ for $i,j = 1,\dots,n-1$. Therefore,

$$
\begin{aligned}
(v^\tau \cdot D^2 u v) &= (v_1^\tau, v_2^\tau, \dots, v_{n-1}^\tau, 0) \begin{pmatrix} 0 & 0 & \dots & 0 & \partial_1\partial_n u \\ 0 & 0 & \dots & 0 & \partial_2\partial_n u \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \partial_n\partial_1 u & \partial_n\partial_2 & \dots & \partial_n\partial_{n-1}u & \partial_n\partial_n u \end{pmatrix} \begin{pmatrix} v_1^\tau \\ v_2^\tau \\ \vdots \\ \rho \end{pmatrix} \\
&= \rho v_i^\tau \partial_i \partial_\nu u.
\end{aligned}
$$

Since this expression is independent of the special Cartesian coordinate system, it follows that

$$
2 \oint_{\partial\Omega} \partial_\nu u(v^\tau \cdot D^2 u v)\, dS = \oint_{\partial\Omega} \rho(v^\tau \cdot \nabla^\tau (\partial_\nu u)^2)\, dS.
$$

These transformations lead to the following observation.

Corollary 6.1. *The second variation $\ddot{\mathcal{E}}_D(0)$ depends only on the boundary values of v, w, $\partial_\nu u$ and their tangential derivatives. It is independent of the extension of the perturbation inside Ω.*

Robin energy

In this case, $\ddot{\mathcal{E}}_r(0) = \mathcal{F}_1(0) + \mathcal{F}_2(0) + \mathcal{F}_3(0)$. Adding up the contributions in Proposition 6.1 and (6.5.3), we get

$$
\begin{aligned}
\ddot{\mathcal{E}}_R(0) = &\oint_{\partial\Omega} [\underbrace{(v\cdot\nu)\operatorname{div} v - (v \cdot D_v \nu)}_{I_1} + (w\cdot\nu)](|\nabla u|^2 - 2G(u))\, dS \\
&+ 4 \oint_{\partial\Omega} (\underbrace{\partial_\nu u(v \cdot D_v \nabla u) - (v\cdot\nu)(\nabla u \cdot D_v \nabla u)}_{I_2})\, dS \\
&+ 2 \oint_{\partial\Omega} (\underbrace{\partial_\nu u(v \cdot D^2 u v) - (v\cdot\nu)(v \cdot D^2 u \nabla u)}_{I_3})\, dS \\
&+ 2 \oint_{\partial\Omega} g(u)(v\cdot\nu)(v\cdot\nabla u)\, dS - 4 \oint_{\partial\Omega} (v\cdot\nabla u)(\partial_\nu u' + \alpha u')\, dS \\
&- 2\alpha \oint_{\partial\Omega} (v\cdot\nabla u)^2\, dS - 2 \oint_{\partial\Omega} (w\cdot\nabla u)\partial_\nu u\, dS + \alpha \oint_{\partial\Omega} u^2(x)\ddot{m}(0)\, dS \\
&- 4 \int_\Omega (\Delta u + g(u))(v \cdot D_v \nabla u)\, dx - 2 \int_\Omega (\Delta u + g(u))(v \cdot D^2 u v)\, dx \\
&+ 2 \int_\Omega (\Delta u + g(u))(w\cdot\nabla u)\, dx + 4 \int_\Omega (\Delta u' + g'(u)u')(v\cdot\nabla u)\, dx - 2Q_g(u'),
\end{aligned} \tag{6.5.8}
$$

where

$$
\begin{aligned}
Q_g(u') &:= \int_\Omega |\nabla u'|^2\, dx - \int_\Omega g'(u)u'^2\, dx + \alpha \oint_{\partial\Omega} u'^2\, dS \\
&= \oint_{\partial\Omega} u'(\partial_\nu u' + \alpha u')\, dS
\end{aligned} \tag{6.5.9}
$$

is a quadratic form in u'.

Formula (6.5.8) can further be simplified. The last four integrals vanish because $\Delta u + g(u) = 0$ and $\Delta u' + g'(u)u' = 0$ in Ω. We apply (6.5.5), and we obtain

$$I_1 = (v \cdot \nu) \operatorname{div}_{\partial\Omega} v - (v^\tau \cdot D_v \nu).$$

We decompose v and ∇u in their normal and tangential parts:

$$\begin{aligned} I_2 &= \partial_\nu u(v^\tau \cdot D_v \nabla u) + \partial_\nu u(v \cdot \nu)(\nu \cdot D_v \nabla u) \\ &\quad - (v \cdot \nu)(\nabla^\tau u \cdot D_v \nabla u) - (v \cdot \nu)\partial_\nu u(\nu \cdot D_v \nabla u) \\ &= \partial_\nu u(v^\tau \cdot D_v \nabla u) - (v \cdot \nu)(\nabla^\tau u \cdot D_v \nabla u). \end{aligned}$$

The Robin boundary condition then implies

$$I_2 = -\alpha u(v^\tau \cdot D_v \nabla u) - (v \cdot \nu)(\nabla^\tau u \cdot D_v \nabla u).$$

Similarly, we obtain

$$\begin{aligned} I_3 &= \partial_\nu u(v \cdot D^2 u v) - (v \cdot \nu)(v \cdot D^2 u \nabla u) \\ &= \partial_\nu u(v \cdot D^2 u v^\tau) - (v \cdot \nu)(v \cdot D^2 u \nabla^\tau u) \\ &= -\alpha u(v \cdot D^2 u v^\tau) - (v \cdot \nu)(v \cdot D^2 u \nabla^\tau u). \end{aligned}$$

Inserting all these expressions into (6.5.4), we obtain the final result.

Theorem 6.3. *Suppose that $\Delta u + g(u) = 0$ in Ω and $\partial_\nu u + \alpha u = 0$ on $\partial\Omega$. Then the second variation $\ddot{\mathcal{E}}_R(0)$ assumes the form*

$$\begin{aligned} \ddot{\mathcal{E}}_R(0) = &\oint_{\partial\Omega} [(v \cdot \nu)\, div_{\partial\Omega} v - (v^\tau \cdot D_v \nu) + (w \cdot \nu)](|\nabla u|^2 - 2G(u))\, dS \\ &- 4 \oint_{\partial\Omega} \{\alpha u(v^\tau \cdot D_v \nabla u) + (v \cdot \nu)(\nabla^\tau u \cdot D_v \nabla u)\}\, dS \\ &- 2 \oint_{\partial\Omega} \{\alpha u(v \cdot D^2 u v^\tau) + (v \cdot \nu)(v \cdot D^2 u \nabla^\tau u)\}\, dS \\ &+ 2 \oint_{\partial\Omega} g(u)(v \cdot \nu)(v \cdot \nabla u)\, dS - 4 \oint_{\partial\Omega} (v \cdot \nabla u)(\partial_\nu u' + \alpha u')\, dS \\ &- 2\alpha \oint_{\partial\Omega} (v \cdot \nabla u)^2\, dS + 2\alpha \oint_{\partial\Omega} (w \cdot \nabla u) u\, dS \\ &+ \alpha \oint_{\partial\Omega} u^2 \ddot{m}(0)\, dS - 2Q_g(u'), \end{aligned}$$

where $Q_g(u')$ is given by (6.5.9).

Discussion

We are mainly interested in the terms on $\partial\Omega$ which contain derivatives in the normal direction. Let $\{e_i\}_1^\infty$ be the same coordinate system as in the discussion of Theorem 6.3, i. e., e_n points in the direction of the outer normal ν. Then on $\partial\Omega$ we have $v = v_i^\tau e_i + \rho e_n$ and $\nabla u = \nabla^\tau u - \alpha e_n$, $i = 1, \ldots, n-1$. Consequently,

$$(v \cdot \nabla u) = \left(v^\tau \cdot \nabla^\tau u\right) - \alpha\rho.$$

Set $D_{v^\tau} = (\partial_i v_j^\tau)$.

A straightforward computation yields

$$\begin{aligned}
&-4 \oint_{\partial\Omega} \{\alpha u(v^\tau \cdot D_v \nabla u) + (v \cdot \nu)(\nabla^\tau u \cdot D_v \nabla u)\}\, dS \\
&\quad = -4 \oint_{\pm\Omega} (\alpha u v^\tau + \rho \nabla^\tau u) \cdot D_{v^\tau} \nabla^\tau u \, dS + 4 \oint_{\partial\Omega} (\alpha^2 u^2 + \alpha u \rho)(\nabla^\tau u \cdot \nabla^\tau \rho)\, dS.
\end{aligned}$$

On the boundary the matrix $D^2 u$ is of the form

$$D^2 u = \begin{pmatrix}
\partial_1\partial_1 u & \partial_1\partial_2 u \ldots & \partial_1\partial_{n-1} u & -\alpha\partial_1 u \\
\partial_2\partial_1 u & \partial_2\partial_2 u \ldots & \partial_2\partial_{n-1} u & -\alpha\partial_2 u \\
\vdots & \vdots \ldots & \vdots & \vdots \\
-\alpha\partial_1 u & -\alpha\partial_2 u \ldots & -\alpha\partial_{n-1} u & \alpha^2 u
\end{pmatrix}.$$

Thus, $D^2 u$ does not contain derivatives in the normal direction. By Lemma 6.1, $\partial_\nu u' + \alpha u'$ depends on u and v on $\partial\Omega$ only. The same is true for $Q(u') = \oint_{\partial\Omega} u'(\partial_\nu u' + \alpha u')\, dS$. Hence, we have proved the following property of the second variation.

Corollary 6.2. *The second variation $\ddot{\mathcal{E}}_R(0)$ depends only on the boundary values of v, w, u and their tangential derivatives. It is independent of the extension of the perturbation Φ_t into Ω.*

6.5.3 Notes

The computation of the second variation of the energy was first carried out in [13] and [14]. In [75] the second variation of $\mathcal{E}$ is presented in an abstract form.

7 Discussion of the main results

In the first section some simple variations are presented which lead to integral identities for boundary value problems. The rest of this chapter is devoted to the first and second domain variation of energies under Hadamard and tangential perturbations. Special emphasis is given to volume preserving perturbations. It turns out that tangential perturbations have an effect on the second domain variation only.

7.1 Translations and rotations

Translations

The perturbation $\Phi_t(x) = x + ta$ where a is a constant vector in $\mathbb{R}^n$ corresponds to a displacement of the domain Ω in the direction a. In this case obviously the volume and the surface area remain unchanged. Clearly, the energies $\mathcal{E}_D(t) = \text{const.}$ and $\mathcal{E}_R(t) = \text{const.}$ Therefore, all shape derivatives vanish. By (6.4.2),

$$\dot{\mathcal{E}}_D(0) = -\oint_{\partial\Omega} (a \cdot \nu)|\nabla u|^2 \, dS = 0.$$

This identity can also be checked directly. Indeed,

$$\begin{aligned} -\oint_{\partial\Omega} (a \cdot \nu)|\nabla u|^2 \, dS &= -\int_\Omega a_i \partial_i(\partial_j u \partial_j u)\, dx = -2\int_\Omega a_i \partial_i \partial_j u \partial_j u \, dx \\ &= -2\int_\Omega a_i u \partial_i \Delta u \, dx = 2\int_\Omega a_i u \partial_i g(u)\, dx = 2\int_\Omega a_i \partial_i \big(g(u)u - G(u)\big)\, dx \\ &= -2G(0)\oint_{\partial\Omega} (a \cdot \nu)\, dS = 0. \end{aligned}$$

Similarly, the invariance of $\mathcal{E}_R(t)$ with respect to translations and (6.4.4) imply that for any domain Ω,

$$\dot{\mathcal{E}}_R(0) = \oint_{\partial\Omega} (a \cdot \nu)\big[|\nabla u|^2 - 2G(u) - 2\alpha^2 u^2 + \alpha(n-1)Hu^2\big]\, dS = 0.$$

Obviously an additional second order term affects the second domain variation.

Rotations

Let Φ_t be a rotation of the (x_1, x_2)-plane considered in Remark 2.3. Expansion up to the second order yields

$$\Phi_t(x) = x + t\,\underbrace{(-x_2, x_1, 0, \dots, 0)}_{v} + \frac{t^2}{2}(-x_1, -x_2, 0, \dots, 0) + o(t^2).$$

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The first variation of the Dirichlet energy is then by (6.4.2)

$$\dot{\mathcal{E}}_D(0) = \oint_{\partial\Omega} (-x_2\nu_1 + x_1\nu_2)|\nabla u|^2\, dS = 0.$$

Similarly, by (6.4.4)

$$\dot{\mathcal{E}}_R(0) = \oint_{\partial\Omega} (-x_2\nu_1 + x_1\nu_2)[|\nabla u|^2 - 2G(u) - 2\alpha^2 u^2 + \alpha(n-1)Hu^2]\, dS = 0.$$

This identity holds for any Ω and for any solution of the Robin boundary value problem.

Pohozaev's identity

In 1965 Pohozaev [97] published an identity which he used to prove nonexistence of positive solutions of some Dirichlet boundary value problems in star-shaped domains. This identity is related to the first variation $\dot{\mathcal{E}}_D(0)$ of the Dirichlet energy.

From (6.3.2) and Lemma 4.1 it follows that

$$\dot{\mathcal{E}}_D(0) = \int_\Omega \operatorname{div} v|\nabla u|^2\, dx - 2\int_\Omega (\nabla u \cdot D_v \nabla u)\, dx - 2\int_\Omega G(u) \operatorname{div} v\, dx,$$

and from (6.4.2) and $G(0) = 0$ we have

$$\dot{\mathcal{E}}_D(0) = -\oint_{\partial\Omega} (v \cdot \nu)|\nabla u|^2\, dS.$$

If we equate these two terms, we obtain

$$\int_\Omega \operatorname{div} v|\nabla u|^2\, dx - 2\int_\Omega (\nabla u \cdot D_v \nabla u)\, dx - 2\int_\Omega G(u) \operatorname{div} v\, dx = -\oint_{\partial\Omega} (v \cdot \nu)|\nabla u|^2\, dS.$$

Choosing $v = x$ and keeping in mind that $\int_\Omega |\nabla u|^2\, dx = \int_\Omega f(u)u\, dx$, we are led to Pohozaev's identity

$$\oint_{\partial\Omega} (x \cdot \nu)|\nabla u|^2\, dS = (2-n)\int_\Omega f(u)u\, dx + 2n\int_\Omega G(u)\, dx. \tag{7.1.1}$$

In star-shaped domains the left-hand side is nonnegative, whereas for supercritical nonlinearities such as $f(u) = u^p$ with $p \geq \frac{n+2}{n-2}$ it is negative. This is impossible and therefore no positive solution exists.

7.2 Hadamard perturbations

This section focuses on Hadamard perturbations $v = (v \cdot \nu)\nu = \rho\nu$. The first variation is given in Section 6.4 and depends only on ρ.

For the computation of the second domain variations we use Theorems 6.2 and 6.3. We start with the Dirichlet energy (see Section 6.2.1). From Theorem 6.2 we obtain

$$\begin{aligned}\ddot{\mathcal{E}}_D(0) = &\oint_{\partial\Omega} [(v \cdot \nu)\operatorname{div}_{\partial\Omega} v + (w \cdot \nu)]\big(|\nabla u|^2 - 2G(0)\big)\, dS \\ &+ 2g(0)\oint_{\partial\Omega} \partial_\nu u\rho^2\, dS - 2\oint_{\partial\Omega}(w \cdot \nu)(\partial_\nu u)^2\, dS + 2Q_0(u').\end{aligned}$$

Since by (2.2.16)

$$(v \cdot \nu)\operatorname{div}_{\partial\Omega} v = \rho\operatorname{div}_{\partial\Omega}(\rho\nu) = (n-1)\rho H,$$

it follows that

$$\begin{aligned}\ddot{\mathcal{E}}_D(0) = &\oint_{\partial\Omega} [(n-1)\rho^2 H + (w \cdot \nu)]\big(|\nabla u|^2 - 2G(0)\big)\, dS \\ &+ 2g(0)\oint_{\partial\Omega} \partial_\nu u\rho^2\, dS - 2\oint_{\partial\Omega}(w \cdot \nu)(\partial_\nu u)^2\, dS + 2Q_0(u'). \qquad (7.2.1)\end{aligned}$$

For the second variation of the Robin energy we use the equations for u and u'. Theorem 6.3 then implies

$$\begin{aligned}\ddot{\mathcal{E}}_R(0) = &\oint_{\partial\Omega} [(v \cdot \nu)\operatorname{div}_{\partial\Omega} v + (w \cdot \nu)]\big(|\nabla u|^2 - 2G(u)\big)\, dS \\ &- 4\oint_{\partial\Omega}(v \cdot \nu)(\nabla^\tau u \cdot D_v \nabla u)\, dS - 2\oint_{\partial\Omega}(v \cdot \nu)(v \cdot D^2 u \nabla^\tau u)\, dS \\ &+ 2\oint_{\partial\Omega} g(u)(v \cdot \nu)(v \cdot \nabla u)\, dS - 4\oint_{\partial\Omega}(v \cdot \nabla u)(\partial_\nu u' + \alpha u')\, dS \\ &- 2\alpha\oint_{\partial\Omega}(v \cdot \nabla u)^2\, dS + 2\alpha\oint_{\partial\Omega}(w \cdot \nabla u)u\, dS \\ &+ \alpha\oint_{\partial\Omega} u^2\ddot{m}(0)\, dS - 2Q_g(u'),\end{aligned}$$

where $Q_g(u')$ is defined in (6.5.9). We consider the various integrals separately.

The first integral is the same as (7.2.1). In the second integral we set

$$(D_v)_{ij} = \partial_i v_j = \partial_i(\rho\nu_j) = \partial_i\rho\nu_j + \rho\partial_i\nu_j \quad \text{and} \quad \nabla u = (\partial_\nu u)\nu + \nabla^\tau u. \qquad (7.2.2)$$

Herewith we calculate $\nabla^\tau u \cdot D_v \nabla u$. Keeping in mind that by (2.4.3), $\nabla_i^\tau \partial_i \nu_j \nu_j = 0$, we get

$$\begin{aligned}
&-4 \oint_{\partial\Omega} (v \cdot \nu)(\nabla^\tau u \cdot D_v \nabla u)\, dS \\
&\quad = -4 \oint_{\partial\Omega} \rho \partial_\nu u (\nabla^\tau u \cdot \nabla^\tau \rho)\, dS + 4 \oint_{\partial\Omega} \rho^2 (\nabla^\tau u \cdot D_\nu \nabla^\tau u)\, dS.
\end{aligned}$$

Replacing v by $\rho\nu$ we obtain for the third integral

$$-2 \oint_{\partial\Omega} (v \cdot \nu)(v \cdot D^2 u \nabla^\tau u)\, dS = -2 \oint_{\partial\Omega} \rho^2 (\nabla^\tau u \cdot D^2 u \nu)\, dS.$$

In view of (6.2.12) the shape derivative u' solves $\Delta u' + g'(u)u' = 0$ in Ω. By (6.2.14) it satisfies for Hadamard perturbations the boundary condition

$$\begin{aligned}
\partial_\nu u' + \alpha u' &= -(v \cdot D^2 u \nu) + (\nabla^\tau u \cdot D_v \nu) - \alpha (v \cdot \nabla u) \\
&= -\rho \partial_\nu^2 u + \nabla_i^\tau u \partial_i v_j \nabla_j^\tau \rho - \alpha^2 \rho \partial_\nu u \quad \text{on } \partial\Omega.
\end{aligned}$$

For the second term in the last sum we apply (7.2.2). This implies

$$\begin{aligned}
\nabla_i^\tau u \partial_i v_j \nabla_j^\tau \rho &= \nabla_i^\tau u (\partial_i \rho \nu_j + \rho \partial_i \nu_j) \nabla_j^\tau \rho = \rho \nabla_i^\tau u \partial_i \nu_j \nabla_j^\tau \rho \\
&= \rho (\nabla^\tau u \cdot D_\nu \nabla^\tau \rho).
\end{aligned}$$

Hence,

$$\begin{aligned}
&-4 \oint_{\partial\Omega} (v \cdot \nabla u)(\partial_\nu u' + \alpha u')\, dS \\
&\quad = 4 \oint_{\partial\Omega} \rho^2 \partial_\nu^2 u \partial_\nu u\, dS - 4 \oint_{\partial\Omega} \rho^2 \partial_\nu u (\nabla^\tau u \cdot D_\nu \nabla^\tau \rho)\, dS - 4\alpha^2 \oint_{\partial\Omega} \rho^2 \partial_\nu u u\, dS.
\end{aligned}$$

In conclusion, we obtain the following theorem.

Theorem 7.1. *Assume $\Delta u + g(u) = 0$ in Ω. Let Φ_t be a Hadamard perturbation and let $\rho = (v \cdot \nu)$.*

1. *If $u = 0$ on $\partial\Omega$, then*

$$\begin{aligned}
\ddot{\mathcal{E}}_D(0) = &\oint_{\partial\Omega} [(n-1)\rho^2 H + (w \cdot \nu)](|\nabla u|^2 - 2G(0))\, dS \\
&+ 2g(0) \oint_{\partial\Omega} \partial_\nu u \rho^2\, dS - 2 \oint_{\partial\Omega} (w \cdot \nu)(\partial_\nu u)^2\, dS + 2Q_0(u'),
\end{aligned}$$

where $Q_0(u') = \int_\Omega [|\nabla u'|^2 - 2g'(u)u'^2]\, dx$.

2. *If $\partial_\nu u + \alpha u = 0$ on $\partial\Omega$, then*

$$\begin{aligned}
\ddot{\mathcal{E}}_R(0) = &\oint_{\partial\Omega} [(n-1)\rho^2 H + w\cdot\nu](|\nabla u|^2 - 2G(u))\,dS \\
&+ 4\alpha \oint_{\partial\Omega} \rho u(\nabla^\tau u \cdot \nabla^\tau \rho)\,dS + 4\oint_{\partial\Omega} \rho^2(\nabla^\tau u \cdot D_\nu \nabla^\tau u)\,dS \\
&- 2\oint_{\partial\Omega} \rho^2(\nabla^\tau u \cdot D^2 u\nu)\,dS - 2\alpha \oint_{\partial\Omega} g(u)\rho^2 u\,dS \\
&+ 4\oint_{\partial\Omega} \rho^2 \partial_\nu^2 u \partial_\nu u\,dS + 4\alpha \oint_{\partial\Omega} \rho^2 u(\nabla^\tau u \cdot D_\nu \nabla^\tau \rho)\,dS \\
&+ 2\alpha^3 \oint_{\partial\Omega} \rho^2 u^2\,dS + 2\alpha \oint_{\partial\Omega} (w\cdot\nabla u)u\,dS \\
&+ \alpha \oint_{\partial\Omega} u^2 \ddot{m}(0)\,dS - 2Q_g(u'),
\end{aligned}$$

where $Q_g(u') = \int_\Omega |\nabla u'|^2 - g'(u)u'^2\,dx + \alpha \oint_{\partial\Omega} u'^2\,dS$ *(see* (6.5.9)*) and* $\ddot{m}(0)$ *is given in* (2.3.30).

These formulas simplify if we assume volume or area preserving perturbations, or if we consider special geometries like the ball.

7.3 Tangential perturbations $v = v^\tau$

In the case of purely tangential perturbations $\Phi_t = x + tv^\tau + \frac{t^2}{2}w$, the first variations of the volume and the surface area vanish. This follows from (2.3.2) and (2.3.20). The tangential perturbations have an effect only in the second variations (see (2.3.13) and (2.3.36)). A similar phenomenon holds for the energies.

By (6.4.2) and (6.4.4) the first variation of the Dirichlet and the Robin energy vanishes for all tangential perturbations.

Theorem 6.2 and the subsequent discussion imply that

$$\begin{aligned}
\ddot{\mathcal{E}}_D(0) = &\oint_{\partial\Omega} [v^\tau \mathcal{L} v^\tau + (w\cdot\nu)](|\nabla u|^2 - 2G(0))\,dS - 2\oint_{\partial\Omega} |\nabla u|^2 (2v^\tau \mathcal{L} v^\tau + (w\cdot\nu))\,dS \\
&+ 2Q_0(u').
\end{aligned}$$

By (6.2.13) the shape derivative satisfies $u' = -(v\cdot\nu)\partial_\nu u = 0$ on $\partial\Omega$. Consequently, (6.5.7) implies that $u' = 0$ in Ω. Thus, $Q_0(u') = 0$ and

$$\ddot{\mathcal{E}}_D(0) = \oint_{\partial\Omega} [v^\tau \mathcal{L} v^\tau + (w \cdot \nu)][(\partial_\nu u)^2 - 2G(0)]\, dS$$
$$- 2 \oint_{\partial\Omega} (\partial_\nu u)^2 [2v^\tau \mathcal{L} v^\tau + (w \cdot \nu)]\, dS. \tag{7.3.1}$$

In convex domains $\mathcal{L}$ is positive. This leads to the following observation.

Corollary 7.1. *Let $\Phi_t = x + tv^\tau$ be a first order tangential perturbation, let $G(0) = 0$, and let Ω be convex. Then*

$$\dot{\mathcal{E}}_D(0) = 0 \quad \textit{and} \quad \ddot{\mathcal{E}}_D(0) = -\oint_{\partial\Omega} (\partial_\nu u)^2 v^\tau \mathcal{L} v^\tau\, dS \le 0.$$

In the case of Robin boundary conditions, Lemma 6.1 implies that $\partial_\nu u' + \alpha u' = 0$ on $\partial\Omega$, and thus by (6.5.9)

$$Q_g(u') = 0.$$

By Theorem 2.1 and Proposition 2.1 we can replace the tangential perturbation $x + tv^\tau + \frac{t^2}{2} w$ locally by the Hadamard perturbation $x + \frac{t^2}{2}\hat{w}$, where

$$\hat{w} = v^\tau \mathcal{L} v^\tau \nu + w.$$

If we introduce $\hat{w}$ into Theorem 7.1, we get

$$\ddot{\mathcal{E}}_R(0) = \oint_{\partial\Omega} (v^\tau \mathcal{L} v^\tau + w \cdot \nu)(|\nabla u|^2 - 2G(u) - 2\alpha^2 u^2) + \alpha u^2 \ddot{m}(0)\, dS.$$

By (2.3.35) it follows that $\ddot{m}(0) = \operatorname{div}_{\partial\Omega} \hat{w}$. Gauss' theorem (2.2.18) then implies that

$$\alpha \oint_{\partial\Omega} u^2 \ddot{m}(0)\, dS = -\alpha \oint_{\partial\Omega} (\hat{w} \cdot \nabla^\tau u^2)\, dS + \alpha(n-1) \oint_{\partial\Omega} u^2 (\hat{w} \cdot \nu) H\, dS.$$

Since $\hat{w} \cdot \nabla^\tau u^2 = w \cdot \nabla^\tau u^2$ we obtain

$$\ddot{\mathcal{E}}_R(0) = \oint_{\partial\Omega} (v^\tau \mathcal{L} v^\tau + w \cdot \nu)(|\nabla u|^2 - 2G(u) - 2\alpha^2 u^2)\, dS$$
$$- \alpha \oint_{\partial\Omega} (w \cdot \nabla^\tau u^2)\, dS + \alpha(n-1) \oint_{\partial\Omega} u^2 (\hat{w} \cdot \nu) H\, dS. \tag{7.3.2}$$

In conclusion, the tangential perturbations have an effect on the second variation of the energy.

7.4 Volume preserving perturbations

The effect of volume preserving perturbations in the sense of Definition 2.2 for the first variation of the energy has been discussed in Section 6.4.1. There only $\dot{\mathcal{V}}(0) = 0$ played a role.

In this section we therefore restrict ourselves to the second variation. In addition to $\dot{\mathcal{V}}(0) = 0$, also $\ddot{\mathcal{V}}(0) = 0$ will be assumed.

7.4.1 Dirichlet energy

Assume that Ω is a critical domain of the Dirichlet energy, i. e., $\dot{\mathcal{E}}_D(0) = 0$. By Theorem 6.1, $\partial_\nu u = \text{const.}$ on $\partial\Omega$. Under the assumption

$$\ddot{\mathcal{V}}(0) = \oint_{\partial\Omega} [(v \cdot \nu)\operatorname{div}_{\partial\Omega} v - (v^\tau \cdot D_v \nu) + (w \cdot \nu)]\, dS = 0,$$

if we take into account that $g(u) = g(0)$ and $G(u) = G(0)$ on $\partial\Omega$, Theorem 6.2 reads as

$$\begin{aligned}\ddot{\mathcal{E}}_D(0) = 2g(0) \oint_{\partial\Omega} \partial_\nu u (v \cdot \nu)^2\, dS + 4 \oint_{\partial\Omega} (\partial_\nu u)^2 (v^\tau \cdot D_v \nu)\, dS \\ + 2 \oint_{\partial\Omega} \partial_\nu u (v^\tau \cdot D^2 u v)\, dS - 2 \oint_{\partial\Omega} (\partial_\nu u)^2 (w \cdot \nu)\, dS + 2Q_0(u').\end{aligned}$$

This expression can be simplified. In fact, $\partial_\nu u = \text{const.}$ and $\ddot{\mathcal{V}}(0) = 0$ can be read as an integral equation for $\oint_{\partial\Omega} (w \cdot \nu)\, dS$. We obtain

$$\begin{aligned}&4 \oint_{\partial\Omega} (\partial_\nu u)^2 (v^\tau \cdot D_v \nu)\, dS - 2 \oint_{\partial\Omega} (\partial_\nu u)^2 (w \cdot \nu)\, dS \\ &\quad = 4(\partial_\nu u)^2 \oint_{\partial\Omega} (v^\tau \cdot D_v \nu)\, dS - 2(\partial_\nu u)^2 \oint_{\partial\Omega} (w \cdot \nu)\, dS \\ &\quad = 2(\partial_\nu u)^2 \oint_{\partial\Omega} [(v^\tau \cdot D_v \nu) + \rho \operatorname{div}_{\partial\Omega} v]\, dS.\end{aligned}$$

Moreover, on $\partial\Omega$ we have

$$(v^\tau \cdot D^2 u v) = 0.$$

This is easily seen by choosing an orthonormal frame such that the origin lies in an arbitrary, but fixed boundary point. The e_n-axis points in the direction of the outer normal and $\mathbb{R}^{n-1}$ is the tangent space. At the origin we have $\partial_i \partial_j u = 0$ for $i, j = 1, 2, \ldots, n-1$. Thus,

$$(v^\tau \cdot D^2uv) = (v_1^\tau, v_2^\tau, \dots, v_{n-1}^\tau, 0) \begin{pmatrix} 0 & \dots & \partial_1\partial_n u \\ 0 & \dots & \partial_2\partial_n u \\ \vdots & \vdots & \partial_{n-1}\partial_n u \\ 0 & \dots & \partial_n\partial_n u \end{pmatrix} \begin{pmatrix} v_1^\tau \\ \vdots \\ v_{n-1}^\tau \\ \rho \end{pmatrix} = \rho(v^\tau \cdot \nabla^\tau \partial_\nu u).$$

The claim now follows from the boundary condition $\partial_\nu u = \text{const.}$ Gauss' theorem (2.2.18) then leads to the following proposition.

Proposition 7.1. *Let u be a solution of $\Delta u + g(u) = 0$ in Ω and $u = 0$ on $\partial\Omega$ and let u' be the corresponding shape derivative solving* (6.2.12)–(6.2.13)*. We assume $\dot{\mathcal{E}}(0) = 0$. Then the second variation $\ddot{\mathcal{E}}(0)$ for volume preserving perturbations can be expressed in the form*

$$\ddot{\mathcal{E}}_D(0) = 2\int_\Omega [|\nabla u'|^2 - g'(u)u'^2]\,dx + 2\frac{g(0)}{\partial_\nu u}\oint_{\partial\Omega} u'^2\,dS + 2(n-1)\oint_{\partial\Omega} u'^2 H\,dS.$$

In particular, $\partial_\nu u = \text{const.} \neq 0$ on $\partial\Omega$.

If $v = v^\tau$ is tangential to $\partial\Omega$, then $u' = 0$ on $\partial\Omega$ and thus

$$\int_\Omega [|\nabla u'|^2 - g'(u)u'^2]\,dx = \oint_{\partial\Omega} u'\partial_\nu u'\,dS = 0.$$

Consequently, we have the following corollary.

Corollary 7.2. *The tangential perturbations $v = v^\tau$ are in the kernel of $\ddot{\mathcal{E}}_D(0)$, i. e., $\dot{\mathcal{E}}_D(0) = \ddot{\mathcal{E}}_D(0) = 0$.*

7.4.2 Robin energy

For the Robin boundary condition a domain Ω is critical for volume preserving perturbations if $\dot{\mathcal{E}}_R = 0$. This results in an overdetermined boundary value problem given in Theorem 6.1, where the additional boundary

$$|\nabla u|^2 - 2G(u) - 2\alpha^2u^2 + (n-1)u^2H = \text{const.} \quad \text{on } \partial\Omega$$

was derived. We denote this constant by C. The second domain derivative $\ddot{\mathcal{E}}_R(0)$ was computed in Theorem 7.1. With the additional boundary condition the first integral is

$$\begin{aligned} I &:= \oint_{\partial\Omega} [(n-1)\rho^2 H + w\cdot\nu](|\nabla u|^2 - 2G(u))\,dS \\ &= \oint_{\partial\Omega} [(n-1)\rho^2 H + w\cdot\nu](C + 2\alpha^2u^2 - (n-1)u^2H)\,dS. \end{aligned}$$

From (2.3.12) we deduce

$$I = \oint_{\partial\Omega} [(n-1)\rho^2 H + w \cdot \nu](2\alpha^2 u^2 - (n-1)u^2 H)\, dS.$$

The other integrals in $\ddot{\mathcal{E}}_R(0)$ in Theorem 7.1 remain unchanged.

Next we restrict ourselves to volume preserving tangential perturbations. We assume again that Ω is a critical domain for $\mathcal{E}_R$. Thus, the overdetermined boundary condition holds in this case as well. Moreover, by (2.3.35) we have $\ddot{m}(0) = \operatorname{div}_{\partial\Omega} \hat{w}$, where $\hat{w} = (v^\tau \mathcal{L} v^\tau)\nu + w$. Then (7.3.2) together with (2.2.18) implies that

$$\ddot{\mathcal{E}}_R(0) = C \oint_{\partial\Omega} (\hat{w} \cdot \nu)\, dS - (n-1)\alpha \oint_{\partial\Omega} u^2 H (\hat{w} \cdot \nu)\, dS + \alpha \oint_{\partial\Omega} u^2 \operatorname{div}_{\partial\Omega} \hat{w}\, dS.$$

Moreover, we have $(\hat{w} \cdot \nabla^\tau u^2) = (w \cdot \nabla^\tau u^2)$ and

$$\alpha \oint_{\partial\Omega} u^2 \operatorname{div}_{\partial\Omega} \hat{w}\, dS = \alpha \oint_{\partial\Omega} [-(\hat{w} \cdot \nabla^\tau u^2) + (n-1)u^2 (\hat{w} \cdot \nu) H]\, dS.$$

With these observations we get

$$\ddot{\mathcal{E}}_R(0) = C \oint_{\partial\Omega} (\hat{w} \cdot \nu)\, dS - \alpha \oint_{\partial\Omega} (w \cdot \nabla^\tau u^2)\, dS. \tag{7.4.1}$$

From (2.3.13) we get the additional condition

$$0 = \oint_{\partial\Omega} v^\tau \mathcal{L} v^\tau + (w \cdot \nu)\, dS = \oint_{\partial\Omega} \hat{w} \cdot \nu\, dS.$$

Hence,

$$\ddot{\mathcal{E}}_R(0) = -\alpha \oint_{\partial\Omega} (w \cdot \nabla^\tau u^2)\, dS. \tag{7.4.2}$$

In summary, we have the following proposition.

Proposition 7.2. *Let Ω be a critical domain for the energy $\mathcal{E}_R$ w. r. t. volume preserving perturbations. Then for tangential volume preserving perturbations $\Phi_t = x + tv^\tau + \frac{t^2}{2} w$, under the additional condition $w^\tau = 0$ we have*

$$\ddot{\mathcal{E}}_R(0) = 0.$$

In accordance with Definition 2.3 we now have the following corollary.

Corollary 7.3. *The second order volume preserving tangential perturbations with $w^\tau = 0$ are in the kernel of $\ddot{\mathcal{E}}_R(0)$.*

7.5 Neumann energy

Consider the Neumann problem

$$\begin{aligned} \Delta u + g(u) &= 0 \quad \text{in } \Omega, \\ \partial_\nu u &= 0 \quad \text{in } \partial\Omega. \end{aligned} \tag{7.5.1}$$

For the existence of a solution the compatibility condition

$$\oint_{\partial\Omega} g(u)\, dS = 0$$

has to be satisfied. The Neumann energy is given by

$$\mathcal{E}_N(\Omega, u) := \int_\Omega |\nabla u|^2\, dy - 2\int_\Omega G(u)\, dx. \tag{7.5.2}$$

From (6.4.4) we get the first variation

$$\dot{\mathcal{E}}_N(0) = \oint_{\partial\Omega} (v \cdot \nu)\{|\nabla u|^2 - 2G(u)\}\, dS.$$

According to Theorem 6.1, $\dot{\mathcal{E}}_N(0) = 0$ for all volume preserving perturbations if and only if u satisfies in addition to (7.5.1) the boundary condition

$$|\nabla u|^2 - 2G(u) = C = \text{const.} \quad \text{on } \partial\Omega. \tag{7.5.3}$$

For the second variation we use Theorem 6.3 with $\alpha = 0$. Then in consideration of (7.5.3) this yields

$$\begin{aligned} \ddot{\mathcal{E}}_N(0) = C\ddot{\mathcal{V}}(0) &- 4\oint_{\partial\Omega} \rho(\nabla^\tau u \cdot D_v \nabla u)\, dS - 2\oint_{\partial\Omega} \rho(v \cdot D^2 u \nabla^\tau u)\, dS \\ &+ 2\oint_{\partial\Omega} \rho(v \cdot \nabla u) g(u)\, dS - 4\oint_{\partial\Omega} (v \cdot \nabla u)\partial_\nu u'\, dS - 2Q_g^N(u'), \end{aligned}$$

where

$$Q_g^N(u') := \int_\Omega |\nabla u'|^2\, dx - \int_\Omega g'(u) u'^2\, dx. \tag{7.5.4}$$

The shape derivative satisfies (6.2.12) and (6.2.14) with $\alpha = 0$. Hence,

$$\Delta u' + g'(u)u' = 0 \quad \text{in } \Omega, \quad \partial_\nu u' = -(v \cdot D^2 u \nu) + (\nabla^\tau u \cdot D_v \nu).$$

By Lemma 6.1

$$\partial_\nu u' = \rho[g(u) + \Delta^* u] + (\nabla^\tau u \cdot \nabla\rho).$$

The following Lemma will allow us to simplify the expression for $\ddot{\mathcal{E}}_N(0)$.

Lemma 7.1. *We have*

$$\begin{aligned}-4\oint_{\partial\Omega}(\upsilon\cdot\nabla^\tau u)\partial_\nu u'\,dS &= -4\oint_{\partial\Omega}(\upsilon\cdot\nabla^\tau u)g(u)\rho\,dS\\ &\quad + 4\oint_{\partial\Omega}\rho(\upsilon\cdot\nabla^\tau u)(\nabla^\tau u\cdot\nabla^\tau \upsilon)\,dS + 4\oint_{\partial\Omega}\rho(\nabla^\tau u\cdot D_\upsilon\nabla^\tau u)\,dS\\ &\quad + 4\oint_{\partial\Omega}\rho(\nabla^\tau u\cdot D^2 u\upsilon)\,dS - 4(n-1)\oint_{\partial\Omega}\rho(\nabla u\cdot\upsilon)H\,dS.\end{aligned}$$

Proof. In view of the boundary condition for u' mentioned above we have

$$\begin{aligned}-4\oint_{\partial\Omega}(\upsilon\cdot\nabla^\tau u)\partial_\nu u'\,dS &= -4\oint_{\partial\Omega}g(u)\rho(\upsilon\cdot\nabla^\tau u)\,dS\\ &\quad - 4\oint_{\partial\Omega}(\upsilon\cdot\nabla^\tau u)\rho\Delta^* u\,dS - 4\oint_{\partial\Omega}(\upsilon\cdot\nabla^\tau u)(\nabla^\tau u\cdot\nabla\rho)\,dS.\end{aligned}$$

Since $\Delta^* u = \operatorname{div}_{\partial\Omega}\nabla^\tau u$ we can apply the Gauss theorem (2.2.18) and get

$$\begin{aligned}-4\oint_{\partial\Omega}(\upsilon\cdot\nabla^\tau u)\rho\Delta^* u\,dS &= 4\oint_{\partial\Omega}(\upsilon\cdot\nabla u)(\nabla^\tau u\cdot\nabla^\tau\rho)\,dS\\ &\quad + 4\oint_{\partial\Omega}\rho\nabla^\tau u\cdot\nabla^\tau(\upsilon\cdot\nabla^\tau u)\,dS - 4(n-1)\oint_{\partial\Omega}\rho(\upsilon\cdot\nabla u)H\,dS.\end{aligned}$$

Moreover,

$$4\oint_{\partial\Omega}\rho\nabla^\tau u\cdot\nabla^\tau(\upsilon\cdot\nabla^\tau u)\,dS = 4\oint_{\partial\Omega}\rho(\nabla^\tau u\cdot D_\upsilon\nabla^\tau u)\,dS + 4\oint_{\partial\Omega}\rho(\nabla^\tau u\cdot D^2 u\upsilon)\,dS.$$

The assertion now follows. □

This lemma then implies

$$\begin{aligned}\ddot{\mathcal{E}}_N(0) &= C\ddot{\mathcal{V}}(0) + 2\oint_{\partial\Omega}\rho\upsilon D^2 u\nabla^\tau u\,dS - 2\oint_{\partial\Omega}\rho(\upsilon\cdot\nabla u)g(u)\,dS\\ &\quad + 4\oint_{\partial\Omega}\rho(\upsilon\cdot\nabla^\tau u)(\nabla^\tau u\cdot\nabla^\tau\upsilon)\,dS - 4(n-1)\oint_{\partial\Omega}\rho(\nabla^\tau u\cdot\upsilon)H\,dS - 2Q_g^N(u'). \quad (7.5.5)\end{aligned}$$

This formula has been derived under the assumption that $\dot{\mathcal{V}}(0) = 0$ and therefore $|\nabla u|^2 - 2G(u) = \text{const.}$ on $\partial\Omega$. If we impose in addition that $\ddot{\mathcal{V}}(0) = 0$, then

$$\ddot{\mathcal{E}}_N(0) = 2\oint_{\partial\Omega} \rho(v \cdot D^2u\nabla^\tau u)\, dS - 2\oint_{\partial\Omega} \rho(v \cdot \nabla u)g(u)\, dS$$
$$+ 4\oint_{\partial\Omega} \rho(v \cdot \nabla^\tau u)(\nabla^\tau u \cdot \nabla^\tau v)\, dS - 4(n-1)\oint_{\partial\Omega} \rho(\nabla^\tau u \cdot v)H\, dS$$
$$- 2Q_g^N(u').$$

For Hadamard perturbations this leads to

$$\ddot{\mathcal{E}}_N(0) = 2\oint_{\partial\Omega} \rho(v \cdot D^2u\nabla^\tau u)\, dS - 2Q_g^N(u').$$

The first term can be written in a more accessible form. The Hadamard perturbation $v = \rho\nu$ implies $\rho(v \cdot D^2u\nabla u^\tau) = \rho^2(\nabla u^\tau \cdot D^2u\nu)$. Furthermore, by $\partial_\nu u = 0$

$$(\nabla^\tau u \cdot D^2u\nu) = (\nabla^\tau u \cdot \nabla\partial_\nu u) - (\nabla^\tau u \cdot D_\nu\nabla^\tau u) = -(\nabla^\tau u \cdot D_\nu\nabla^\tau u).$$

Consequently,

$$\ddot{\mathcal{E}}_N(0) = -2\oint_{\partial\Omega} \rho^2(\nabla^\tau u \cdot D_\nu\nabla^\tau u)\, dS - 2Q_g^N(u').$$

In summary, we obtain the following proposition.

Proposition 7.3. *Let u be a solution of $\Delta u + g(u) = 0$ in Ω and $\partial_\nu u = 0$ on $\partial\Omega$ and let $\mathcal{E}_N(t)$ be the corresponding Neumann energy (see (7.5.2)). Assume that Φ_t is a Hadamard, volume preserving perturbation. Under the assumption that $\dot{\mathcal{E}}_N(0)$ the second variation is*

$$\ddot{\mathcal{E}}_N(0) = -2\oint_{\partial\Omega} \rho^2(\nabla^\tau u \cdot D_\nu\nabla^\tau u)\, dS - 2Q_g^N(u'),$$

where $Q_g^N(u')$ is given by (7.5.4).

Note that by (2.3.8) the term D_ν contains the second fundamental form.

7.5.1 Notes

The relation between the first domain variation and Pohozaev's identity was discussed in [123]. This paper also contains a proof that the overdetermined torsion problem is solvable in balls only. A related interpretation of Pohozaev's identity is found in [100]. This text studies the influence of transformation groups to critical points for abstract functionals. Pohozaev's identity follows there as a special case.

8 General strategy and applications

In this chapter we present a strategy to determine the sign of the second variation. It is based on a generalized Steklov eigenvalue problem associated to the linearization of the elliptic equation. Our goal is to express the shape derivative as a Fourier series of the corresponding Steklov eigenfunctions. The lower eigenvalues turn out to be essential for the sign of the second variation of the energy. We start with a general discussion of this eigenvalue problem. The strategy of Fourier series expansions is then applied to domain functionals defined on nearly spherical domains. We obtain a lower bound for the second variation of the Robin energy for $\alpha > 0$. In the special case of the torsion energy this lower bound establishes the positivity of the second variation. Finally, we apply the same method to the Dirichlet energy and obtain a lower bound as well.

8.1 A generalized Steklov eigenvalue problem

It was shown in Section 6.2.2 that the shape derivatives of the boundary value problem $\Delta u + g(u) = 0$ in Ω with Robin boundary conditions satisfy $\Delta u' + g'(u)u' = 0$ in Ω and $\partial_\nu u' + \alpha u' = k_R(u, x)$ on $\partial\Omega$. The solutions of such problems will be computed by means of the eigenfunctions of the *Steklov type* eigenvalue problem

$$\Delta\phi + \sigma(x)\phi = 0 \text{ in } \Omega, \quad \partial_\nu\phi + \alpha\phi = \mu\phi \text{ on } \partial\Omega, \quad \text{where } \sigma(x) = g'(u). \tag{8.1.1}$$

In order to write it in a weak form we introduce the bilinear forms defined in the Hilbert spaces $W^{1,2}(\Omega)$ and $L^2(\partial\Omega)$, namely

$$a_\sigma(\phi, v) := \int_\Omega (\nabla\phi \cdot \nabla v)\, dx - \int_\Omega \sigma(x)\phi v\, dx$$

and

$$b(\phi, v) := \oint_{\partial\Omega} \phi v\, dS.$$

Since $\partial\Omega$ is Lipschitz, the embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\partial\Omega)$ is a compact linear operator. The space $W_0^{1,2}(\Omega)$ is the kernel of the trace operator. We write $\mathcal{H}$ for the quotient $W^{1,2}(\Omega)/W_0^{1,2}(\Omega)$. The weak formulation of (8.1.1) is: find $\phi \in \mathcal{H}$ such that

$$a_\sigma(\phi, v) = (\mu - \alpha)\, b(\phi, v) \quad \text{for all } v \in \mathcal{H}. \tag{8.1.2}$$

If $\sigma(x) \in C^0(\overline{\Omega})$,

$$a_\sigma(\phi, \phi) \geq \|\nabla\phi\|^2_{L^2(\Omega)} - \sigma_\infty\|\phi\|^2_{L^2(\Omega)}, \quad \text{where } \sigma_\infty := \max_{\overline{\Omega}}\{\sigma(x), 0\}.$$

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From (C.0.5) it follows that

$$\sigma_\infty \int_\Omega \phi^2\,dx \le \frac{1}{2}\int_\Omega |\nabla\phi|^2\,dx + c_{1/2}\oint_{\partial\Omega} \phi^2\,dS.$$

Hence, the corresponding Rayleigh quotient is bounded from below by

$$R_S(\phi) := \frac{a_\sigma(\phi,\phi)}{b(\phi,\phi)} \ge -c_{1/2}.$$

Consequently, $\inf_{\mathcal{H}} R_S(\phi) =: \mu_0 - \alpha > -\infty$.

Lemma 8.1. *The Steklov eigenvalue problem* (8.1.1), *resp.* (8.1.2), *possesses:*
- *countably many eigenvalues of finite multiplicity* $\mu_0 < \tilde{\mu}_1 \le \tilde{\mu}_2 \le \cdots$ *with* $\lim_{n\to\infty} \tilde{\mu}_n = \infty$,
- *a system of eigenvalues* $\{\phi_i\}_0^\infty$ *which is orthonormal with respect to the inner product* $b(\phi_i,\phi_j)$ *and complete in* $\mathcal{H}$.

Moreover:
- *The eigenvalues are characterized by the Rayleigh principle*

$$\mu_i - \alpha = \min_{v\in\mathcal{H}} R_S(v), \quad \textit{where } b(v,\phi_j) = 0 \textit{ for } j = 0,1,2,\dots,i-1.$$

- *The eigenspace corresponding to the lowest eigenvalue is one-dimensional and the eigenfunction is of constant sign.*

Proof. We prove the existence of μ_0 and show that the corresponding eigenspace is one-dimensional. For the proof of the existence of higher eigenvalues we refer to the classical literature on compact operators in Hilbert spaces.

Let $\{\varphi_n\}_0^\infty$ be a minimizing sequence in $\mathcal{H}$ such that $\lim_{n\to\infty} R_S(\varphi_n) = \mu_0 - \alpha$. It can be chosen such that $R_S(\varphi_n) \in (\mu_0 - \alpha - \epsilon_0, \mu_0 - \alpha + \epsilon_0)$ for all $n > 1$. Observe that $R_S(\varphi)$ does not change if φ is replaced by $\alpha\varphi$. Consequently, we can assume that $b(\varphi_n,\varphi_n) = 1$. The modified Friedrich inequality (C.0.5) implies that

$$\begin{aligned}\int_\Omega |\nabla\varphi_n|^2\,dx + \int_\Omega \sigma^-\varphi_n^2\,dx &\le \mu - \alpha + \epsilon_0 + \int_\Omega \sigma^+\varphi_n^2\,dx \\ &\le \mu - \alpha + \epsilon_0 + \sigma_\infty\Big[\epsilon\int_\Omega |\nabla\varphi_n|^2\,dx + c_\epsilon\Big],\end{aligned}$$

where $\sigma^\pm = \max\{o, \pm\phi\}$. If we choose $\epsilon = \frac{1}{2\sigma_\infty}$, then

$$\frac{1}{2}\int_\Omega |\nabla\varphi_n|^2\,dx \le C,$$

for some constant C which is independent of n. By Friedrich's inequality (C.0.4) and the normalization of φ_n it follows that also $\|\varphi_n\|_{L^2(\Omega)}$ is uniformly bounded. Thus, $\{\varphi_n\}_{n=1}^{\infty}$ is a bounded sequence in $W^{1,2}(\Omega)$. There exists a subsequence which converges weakly to a function $\phi_0 \in \mathcal{H}$. Consequently, the infimum of R_S is attained in $\mathcal{H}$. If $\sigma(x) = 0$, then $\mu - \alpha = 0$ and $\phi_0 =$ constant.

Next we prove that the minimizer is a solution of (8.1.2). Set for short $T_\sigma(\phi_0, v) := a_\sigma(\phi_0, v) - (\mu_0 - \alpha)b(\phi_0, v)$. Since ϕ_0 is a minimizer, we have

$$0 \le T_\sigma(\phi_0 + tv, \phi_0 + tv) = T_\sigma(\phi_0, \phi_0) + t^2 T_\sigma(v, v) + 2tT_\sigma(\phi_0, v).$$

If $T_\sigma(v, v) > 0$, the minimum is attained for $t = -\frac{T_\sigma(\phi_0,v)}{T_\sigma(v,v)}$. Consequently,

$$\left(T_\sigma(\phi_0, v)\right)^2 \le T_\sigma(v, v)T_\sigma(\phi_0, \phi_0).$$

Since $T_\sigma(\phi_0, \phi_0)$ vanishes, we have $T_\sigma(v, \phi_0) = 0$. Hence, ϕ_0 is the eigenfunction of the Steklov problem (8.1.1) and μ_0 is the corresponding eigenvalue.

If the eigenfunction ϕ_0 changes sign, then $|\phi_0|$ is also a minimizer. It is therefore a weak solution of (8.1.1). By the regularity theory – in particular Harnack's inequality – $|\phi_0|$ is a classical solution; hence, $|\phi_0| = \phi_0$.

Suppose that there exist two different eigenfunctions ϕ and ψ to the lowest eigenvalue μ. Then any linear combination $\zeta = c_1\phi + c_2\psi$ is again an eigenfunction to μ. If ϕ and ψ are different, we can find constants c_1 and c_2 such that ζ changes sign. This is a contradiction to the previous observation. □

Notation

According to Lemma 8.1, any function $u \in \mathcal{H}$ can be written as a Fourier series. We denote by $\phi_{k,1}, \dots, \phi_{k,d_k}$ the eigenfunctions corresponding to μ_k, and d_k denotes its multiplicity. Then

$$u = \sum_{k=0}^{\infty} \sum_{i=1}^{d_k} c_{k,i}\phi_{k,i}.$$

The eigenfunctions $\{\phi_{k,i}\}$, where $k = 0, 1, \dots$ and $i = 1, 2, \dots, d_k$, can always be chosen such that

$$\oint_{\partial\Omega} \phi_{k,i}\phi_{k,j}\, dS = \delta_{ij} \quad \text{for } i, j = 1, \dots, d_k$$

and

$$\oint_{\partial\Omega} \phi_{k,i}\phi_{\ell,j}\, dS = \delta_{k\ell} \quad \text{for } k, \ell = 0, 1, 2, \dots.$$

Then

$$c_{k,i} = \oint_{\partial\Omega} u\phi_{k,i}\, dS.$$

To simplify notation we will often set

$$\phi_k = \left\{\sum_{i=1}^{d_k} c_{k,i}^2\right\}^{-1/2} \sum_{i=1}^{d_k} c_{k,i}\phi_{k,i} \quad \text{and} \quad c_k = \left\{\sum_{i=1}^{d_k} c_{k,i}^2\right\}^{1/2}. \tag{8.1.3}$$

In this notation u can be written as

$$u = \sum_{k=0}^{\infty} c_k\phi_k. \tag{8.1.4}$$

Observe that

$$\oint_{\partial\Omega} \phi_k\phi_\ell\, dS = \delta_{k\ell}.$$

Representation of $Q_g(u')$

We apply the expansion described above to the quadratic form

$$Q_g(u') = \int_\Omega [|\nabla u'|^2 - g'(u)u'^2]\, dx + \alpha \oint_{\partial\Omega} u'^2\, dS,$$

which appears in the second variation of the Robin energy (cf. Theorem 6.3).

From the partial differential equation $\Delta u' + g'(u)u' = 0$ in Ω it follows that

$$Q_g(u') = \oint_{\partial\Omega} (u'\partial_\nu u' + \alpha u'^2)\, dS.$$

We now use (8.1.4) and insert $u' = \sum_{i=0}^{\infty} c_i\phi_i$ into $Q_g(u')$. Since ϕ_i is a Steklov eigenfunction corresponding to μ_i we get

$$Q_g(u') = \sum_{i,j} c_i c_j \oint_{\partial\Omega} (\phi_i\partial_\nu\phi_j + \alpha\phi_i\phi_j)\, dS = \sum_{i,j} \mu_j c_i c_j \oint_{\partial\Omega} \phi_i\phi_j\, dS.$$

From the orthonormality of $\{\phi_i\}_{i=0}^{\infty}$ it then follows that

$$Q_g(u') = \sum_{i=0}^{\infty} \mu_i c_i^2 = \sum_{i=0}^{\infty} \mu_i \sum_{j=1}^{d_i} c_{i,j}^2. \tag{8.1.5}$$

The Fourier coefficients are obtained from the boundary condition $\partial_\nu u' + au' =: K(u,\rho)$, derived in Lemma 6.1. Multiplication with ϕ_i and integration yields

$$\oint_{\partial\Omega} (\partial_\nu u'\phi_i + au'\phi_i)\, dS = \oint_{\partial\Omega} K(u,\rho)\phi_i\, dS.$$

Replacing u' by its expansion and using the orthonormality of $\{\phi_i\}_{i=0}^{\infty}$ we find that

$$c_i = \frac{1}{\mu_i} \oint_{\partial\Omega} K(u,\rho)\phi_i(x)\, dS. \tag{8.1.6}$$

8.2 Nearly spherical domains

We call a domain $\overline{\Omega}$ *nearly spherical* if there are a constant $c > 0$ and a $C^{1,1}$-diffeomorphism $\Phi : \overline{B_R} \to \overline{\Omega}$ such that $\|\Phi\|_{C^{1,1}} \leq c$. As a consequence there exists a radius $R_0 = R_0(c)$ such that $\overline{\Omega} \subset B_{R_0}$.

With regard to Section 2.1,

$$\Omega_t = \left\{ y = x + tv(x) + \frac{t^2}{2} w(x) + o(t^2) : x \in B_R, |t| < t_0 \right\}$$

describes a family of nearly spherical domains for t_0 sufficiently small (see Lemma 2.1).

8.2.1 Robin energy

Our goal is to discuss the first and second domain variation of the energy $\mathcal{E}_R(t)$ (cf. (6.2.5)) corresponding to the boundary value problem (6.2.1) with Robin boundary conditions.

We will assume that the solution $u = u(r)$ of problem (6.2.1), (6.2.3) in B_R is radial. Then according to (6.4.4) the first variation is of the form

$$\dot{\mathcal{E}}_R(0) = -\left(2G(u(R)) + \left[a^2 - a\frac{n-1}{R} \right] u^2(R) \right) \oint_{\partial B_R} (v \cdot \nu)\, dS. \tag{8.2.1}$$

This together with (2.3.2) and (2.3.20) leads to the following simple conclusions:
1. $\dot{\mathcal{E}}_R(0) = 0$ if $\dot{\mathcal{V}}(0) = 0$,
2. $\dot{\mathcal{E}}_R(0) = 0$ if $\dot{\mathcal{S}}(0) = 0$,
3. $\dot{\mathcal{E}}_R(0) = 0$ if $v = v^\tau$ is a purely tangential perturbation.

We now proceed to the calculation of $\ddot{\mathcal{E}}_R(0)$. To this end we need the shape derivative which by Lemma 6.1 solves

$$\Delta u' + g'(u)u' = 0 \quad \text{in } B_R, \tag{8.2.2}$$

$$\partial_\nu u' + au' = k_g(u(R))(\upsilon \cdot \nu) \quad \text{on } \partial B_R, \tag{8.2.3}$$

where

$$k_g(u(R)) = g(u(R)) - \frac{a(n-1)}{R}u(R) + a^2 u(R). \tag{8.2.4}$$

In order to determine $\ddot{\mathcal{E}}_R(0)$ for a general perturbation Φ_t we use Theorem 6.3. Since $\Omega = B_R$ and $u = u(|x|)$, the formula in Theorem 6.3 simplifies.

Step 1. Since u is radial, tangential derivatives of u in $\ddot{\mathcal{E}}_R(0)$ are zero:

$$\nabla u|_{\partial B_R} = u_r(R)\nu \quad \text{and} \quad \nabla^\tau u|_{\partial B_R} = 0.$$

Moreover,

$$\partial_i \partial_j u(x)|_{\partial B_R} = \partial_i \partial_j u(|R|) = u_{rr}(R)\nu_i \nu_j + \frac{u_r(R)}{R}(\delta_{ij} - \nu_i \nu_j), \quad \text{where } \nu_i = \frac{x_i}{R}.$$

The last identity implies

$$(\upsilon \cdot D^2 u \upsilon^\tau) = \upsilon_i \partial_i \partial_j u \upsilon_j^\tau = \frac{u_r(R)}{R}|\upsilon^\tau|^2.$$

We put this information into $\ddot{\mathcal{E}}_R(0)$ and write down the integrals in the same order as in Theorem 6.3. Taking innto account the boundary condition $u_r(R) + au(R) = 0$ we have

$$\begin{aligned}
\ddot{\mathcal{E}}_R(0) = {} & [u_r^2(R) - 2G(u(R))]\ddot{V}(0) + 4a^2 u^2(R) \oint_{\partial B_R} (\upsilon^\tau \cdot D_\upsilon \nu)\, dS \\
& \underbrace{-2\frac{u(R)u_r(R)}{R} \oint_{\partial B_R} |\upsilon^\tau|^2\, dS}_{(1)} - 2au(R)g(u(R)) \oint_{\partial B_R} (\upsilon \cdot \nu)^2\, dS \\
& + 4au(R) \oint_{\partial B_R} (\upsilon \cdot \nu)(\partial_\nu u' + au')\, dS - 2a^3 u^2(R) \oint_{\partial B_R} (\upsilon \cdot \nu)^2\, dS \\
& \underbrace{-2a^2 u^2(R) \oint_{\partial B_R} (w \cdot \nu)\, dS}_{(2)} + au^2(R)\ddot{S}(0) - 2Q_g(u').
\end{aligned}$$

Step 2. For the first integral on the right side we recall (2.3.11). We have

$$4u_r^2(R)\oint_{\partial B_R}(v^\tau\cdot D_v\nu)\,dS = -2u_r^2(R)\ddot{\mathcal{V}}(0)+\frac{2(n-1)}{R}u_r^2(R)\oint_{\partial B_R}(v\cdot\nu)^2\,dS$$
$$\underbrace{+\frac{2u_r^2(R)}{R}\oint_{\partial B_R}|v^\tau|^2\,dS}_{(1)}\underbrace{+2u_r^2(R)\oint_{\partial B_R}(w\cdot\nu)\,dS}_{(2)}.$$

Since $u_r(R) = -au(R)$, the underbracketed integrals are canceled out. This implies

$$\begin{aligned}\ddot{\mathcal{E}}_R(0) &= -[a^2u^2(R)+2G(u(R))]\ddot{\mathcal{V}}(0)\\ &\quad+\left(\frac{2(n-1)}{R}u_r^2(R)-2au(R)g(u(R))-2a^3u^2(R)\right)\oint_{\partial B_R}(v\cdot\nu)^2\,dS\\ &\quad+4au(R)\oint_{\partial B_R}(v\cdot\nu)(\partial_\nu u'+au')\,dS\\ &\quad+au^2(R)\ddot{\mathcal{S}}(0)-2Q_g(u').\end{aligned}$$

Step 3. We insert the boundary condition (8.2.3) for u'. With (8.2.4) we obtain

$$\begin{aligned}\ddot{\mathcal{E}}_R(0) &= -[a^2u^2(R)+2G(u(R))]\ddot{\mathcal{V}}(0)+2au(R)k_g(u(R))\oint_{\partial B_R}(v\cdot\nu)^2\,dS\\ &\quad+au^2(R)\ddot{\mathcal{S}}(0)-2Q_g(u').\end{aligned}$$

Theorem 8.1. *Let $\{\Omega_t\}_{|t|<t_0}$ be a family of nearly spherical domains and let $u(y,t)$ be solutions of $\Delta_y u + g(u) = 0$ in Ω_t and $\partial_\nu u + au = 0$ on $\partial\Omega_t$. Let $\mathcal{E}_R(t) = \int_{\Omega_t}\{|\nabla u|^2 - 2G(u)\}\,dy + a\oint_{\partial\Omega_t} u^2\,dS$ be the corresponding energy. Assume that the solution u in B_R is radial. Then*

$$\dot{\mathcal{E}}_R(0) = -\left(2G(u(R))+\left[1-\frac{n-1}{aR}\right]a^2u^2(R)\right)\oint_{\partial B_R}(v\cdot\nu)\,dS.$$

The second domain variation is

$$\begin{aligned}\ddot{\mathcal{E}}_R(0) &= -[a^2u^2(R)+2G(u(R))]\ddot{\mathcal{V}}(0)+2au(R)k_g(u(R))\oint_{\partial B_R}(v\cdot\nu)^2\,dS\\ &\quad+au^2(R)\ddot{\mathcal{S}}(0)-2Q_g(u').\end{aligned}$$

Discussion of $\ddot{\mathcal{E}}_R(0)$

In (2.3.28) it was shown that $\ddot{\mathcal{S}}(0) = \ddot{\mathcal{S}}_0(0) + \frac{n-1}{R}\ddot{\mathcal{V}}(0)$, where

$$\ddot{\mathcal{S}}_0(0) := \int\limits_{B_R} \left\{ \left|\nabla^\tau(v \cdot \nu)\right|^2 - \frac{n-1}{R^2}(v \cdot \nu)^2 \right\} dS. \tag{8.2.5}$$

Inserting this formula into $\ddot{\mathcal{E}}_R(0)$ given in Theorem 8.1 we get

$$\begin{aligned}\ddot{\mathcal{E}}_R(0) = &-\Big(\alpha^2 u^2(R) - \frac{n-1}{R}\alpha u^2(R) + 2G(u(R))\Big)\ddot{\mathcal{V}}(0) \\ &+ 2\alpha u(R) k_g(u(R)) \oint\limits_{\partial B_R} (v \cdot \nu)^2\, dS + \alpha u^2(R)\ddot{\mathcal{S}}_0(0) - 2Q_g(u').\end{aligned} \tag{8.2.6}$$

For volume preserving perturbations we have

$$\ddot{\mathcal{E}}_R(0) = 2\alpha u(R) k_g(u(R)) \oint\limits_{\partial B_R} (v \cdot \nu)^2\, dS + \alpha u^2(R)\ddot{\mathcal{S}}_0(0) - 2Q_g(u'). \tag{8.2.7}$$

For area preserving perturbations the analogous result

$$\begin{aligned}\ddot{\mathcal{E}}_R(0) = &\frac{R}{n-1}\Big(\alpha^2 u^2(R) - \frac{n-1}{R}\alpha u^2(R) + 2G(u(R))\Big)\ddot{\mathcal{S}}_0(0) \\ &+ 2\alpha u(R) k_g(u(R)) \oint\limits_{\partial B_R} (v \cdot \nu)^2\, dS - 2Q_g(u').\end{aligned} \tag{8.2.8}$$

The sign of $\ddot{\mathcal{E}}_R(0)$

This paragraph deals with the sign of $\ddot{\mathcal{E}}_R(0)$ for volume and area preserving perturbations. For this purpose the expansion into Fourier series developed in Section 8.1 comes into play.

For radial solutions $u(r)$ in balls the Steklov problem (8.1.1) is of the form

$$\Delta\phi + g'(u(r))\phi = 0 \text{ in } B_R, \quad \phi_r(R,\xi) + \alpha\phi(R,\xi) = \mu\phi(R,\xi). \tag{8.2.9}$$

Lemma 8.2. *Assume that $g'(u(r))$ is continuous in $\overline{B}_R$. Then problem* (8.2.9) *possesses a complete system of eigenfunctions of the form*

$$\phi_{k,i} = a_k(r) Y_{k,i}(\xi), \quad \textit{where } Y_{k,i} \textit{ is a spherical harmonic of degree } k.$$

The corresponding eigenvalues $\{\mu_k\}_{k=0}^\infty$ are ordered as the eigenvalues of the Laplace–Beltrami operator on ∂B_R:

$$\mu_k < \mu_{k+1} \quad \textit{for } k = 0, 1, 2, \ldots.$$

Furthermore,

$$\mu_k = \mu_{k,1} = \mu_{k,2} = \cdots = \mu_{k,d_k}, \quad \textit{where } d_k \textit{ is the multiplicity of } \Lambda_k.$$

Proof. *(i) Existence.* We write (8.2.9) in polar coordinates. Hence,

$$a_{rr}(r)b(\xi) + \frac{n-1}{r}a_r(r)b(\xi) + \frac{1}{r^2}a(r)\Delta^* b(\xi) + g'(u(r))a(r)b(\xi) = 0$$

and

$$a_r(R) + \alpha a(R) = \mu a(R).$$

We look for solutions of the form $\phi(r,\xi) = a(r)b(\xi)$. We set $b(\xi) = Y_{k,i}(\xi)$, where $Y_{k,i}$ is a spherical harmonic of degree k. For regular solutions we must have $a(0) < \infty$ and $a_r(0) = 0$. The function $a(r)$ is an eigenfunction of

$$\begin{aligned} a_{rr} + \frac{n-1}{r}a_r - \frac{k(n-2+k)}{r^2}a + g'(u(r))a = 0 \text{ in } (0,R), \quad a_r(0) = 0,\\ a_r(R) + \alpha a(R) = \mu a(R). \end{aligned} \tag{8.2.10}$$

Suppose that $g'(u(r))$ is real analytic in $\overline{B}_R$. Then $a(r)$ satisfies an ordinary differential equation of Fuchsian type. Its indicial equation is

$$\gamma(\gamma-1) + (n-1)\gamma - k(n-2+k) = 0.$$

Its roots are

$$\gamma_+ = k \quad \text{and} \quad \gamma_- = -k-n+2.$$

By the classical theory of ordinary differential equations, (8.2.10) has two solutions $a_\pm(r)$ for each k. In a neighborhood of zero $a_+(r) = O(r^{\gamma_+})$ and $a_-(r) = O(r^{\gamma_-})$ if $n \neq 2$ and $k \neq 0$; otherwise, $a_-(r) = O(-\log r)$. Clearly, $a_-(r)$ is singular at $r = 0$ and therefore it is not a solution of (8.2.10).

Consequently, $a_k(r) = a_+(r)$ is the desired solution and

$$\phi_{k,i} = a_k(r)Y_{k,i}(\xi), \quad k = 0,1,\dots, \; i = 1,\dots,d_k,$$

is an eigenfunction of (8.2.9).

(ii) Next we show that $\mu_k < \mu_{k+1}$. To see this we write equation (8.2.9) as a Sturm–Liouville problem:

$$(r^{n-1}a_r)_r + \underbrace{r^{n-1}\left[g'(u) - \frac{k(n-2+k)}{r^2}\right]}_{q_k(r)} a = 0.$$

Since $q_k(r) > q_{k+m}(r)$ for any $m \in \mathbb{R}^+$, Sturm's first comparison theorem [71, Theorem 3.1] applies and yields $\mu_k < \mu_{k+m}$.

The eigenvalues μ_k are computed from the boundary condition at $r = R$, i. e.,

$$\mu_k = \frac{\frac{d}{dr} a_k(R)}{a_k(R)} + \alpha.$$

Hence, the dimension of the eigenspace of μ_k is the dimension of the space spanned by the spherical harmonics of degree k. Thus, $\mu_{k,1} = \mu_{k,2} = \cdots = \mu_{k,d_k} = \mu_k$.

(iii) Completeness. To show that the system $\{\phi_{k,i}\}$, $k \geq 0, i = 1, \ldots, d_k$, is complete, we assume that there exists an eigenvalue $\mu \neq \mu_{i,k}$ for all i and k. If ϕ is the corresponding eigenfunction, then

$$(\mu_{k,i} - \alpha) \oint_{\partial B_R} \phi_{k,i}\phi \, dS = \int_{B_R} \left[(\nabla\phi_{k,i} \cdot \nabla\phi) - g'(u)\phi_{k,i}\phi + \frac{k(n-2+k)}{r^2}\phi_{k,i}\phi \right] dx,$$

$$(\mu - \alpha) \oint_{\partial B_R} \phi_{k,i}\phi \, dS = \int_{B_R} \left[(\nabla\phi_{k,i} \cdot \nabla\phi) - g'(u)\phi_{k,i}\phi + \frac{k(n-2+k)}{r^2}\phi_{k,i}\phi \right] dx.$$

Hence,

$$\oint_{\partial B_1} Y_{k,i}\phi \, dS = 0, \quad \text{for all } k = 0, \ldots \text{ and } i = 1, 2, 3, \ldots, d_k.$$

The completeness of the spherical harmonics implies that $\phi \equiv 0$. Hence, no other eigenvalues and eigenfunctions exist.

If $g'(u(r))$ is continuous in $\overline{B_R}$, we approximate it by analytic functions and use the continuous dependence of the differential equation on its data. This establishes the existence of a solution to (8.2). ☐

As in (8.1.4) we set

$$u' = \sum_{i=0}^{\infty} c_i\phi_i \quad \text{and} \quad (v \cdot \nu) = \sum_{i=0}^{\infty} b_i\phi_i,$$

where $\{\phi_i\}_{i\geq 0}$ is an orthonormal basis of eigenfunctions of the Steklov problem (8.2.9). By (8.1.6) and the boundary condition of u' (cf. (8.2.3)),

$$c_i = \frac{k_g(u(R))}{\mu_i} \oint_{\partial B_R} (v \cdot \nu)\phi_i \, dS = \frac{k_g(u(R))}{\mu_i} b_i. \tag{8.2.11}$$

In view of (8.1.5) we have

$$Q_g(u') = \sum_{i=0}^{\infty} \mu_i c_i^2.$$

Let us write for short

$$\mathcal{F} := -2Q_g(u') + 2au(R)k_g(u(R)) \oint_{\partial B_R} (v \cdot \nu)^2\, dS.$$

The boundary condition $\partial_\nu u' + au' = k_g(u(R))(v \cdot \nu)$ and $\phi_0(R) = \text{const.}$ imply that

$$\oint_{\partial B_R} (\partial_\nu u' + au')\phi_0\, dS = k_g(u(R))\phi_0(R) \oint_{\partial B_R} (v \cdot \nu)\, dS.$$

For volume and area preserving perturbations it follows from (2.3.2) and (2.3.20) that

$$\oint_{\partial B_R} (v \cdot \nu)\, dS = 0.$$

Hence,

$$0 = \oint_{\partial B_R} (\partial_\nu u' + au')\phi_0\, dS = \oint_{\partial B_R} \sum_{i=0}^{\infty} c_i(\partial_\nu \phi_i + a\phi_i)\phi_0\, dS = c_0\mu_0.$$

This implies $b_0 = 0$.

In $\mathcal{F}$ we replace $Q_g(u')$ by $\sum_{i=1}^{\infty} \mu_i c_i^2$ and $\oint_{\partial B_R} (v \cdot \nu)^2\, dS$ by $\sum_{i=1}^{\infty} b_i^2$, and we obtain

$$\begin{aligned} \mathcal{F} &= 2\sum_{i=1}^{\infty} c_i^2 \mu_i^2 \left[\frac{au(R)}{k_g(u(R))} - \frac{1}{\mu_i} \right] \\ &= 2\sum_{i=1}^{\infty} b_i^2 \left[au(R)k_g(u(R)) - \frac{k_g^2(u(R))}{\mu_i} \right]. \end{aligned} \tag{8.2.12}$$

From (8.2.7) we then obtain the following proposition.

Proposition 8.1. *Let Φ_t be a volume preserving perturbation. Then*

$$\ddot{\mathcal{E}}_R(0) = au^2(R)\ddot{\mathcal{S}}_0(0) + \mathcal{F}.$$

For tangential perturbations we have $(v \cdot \nu) = 0$ and therefore $b_i = 0$ for all i. This implies $\dot{\mathcal{E}}_R(0) = 0$ and by (2.3.27), $\ddot{\mathcal{E}}_R(0) = 0$. Consequently, the volume preserving tangential perturbations are in the kernel of $\mathcal{E}_R(0)$.

From (8.2.8) we get the following proposition.

Proposition 8.2. *Let Φ_t be an area preserving perturbation. Then*

$$\ddot{\mathcal{E}}_R(0) = \frac{R}{n-1}(k_g(u(R))u(R) + 2G(u(R)) - u(R)g(u(R)))\ddot{\mathcal{S}}_0(0) + \mathcal{F}.$$

With (8.2.4) *this reads as*

$$\ddot{\mathcal{E}}_R(0) = \frac{R}{n-1}\left(a^2u^2(R) - \frac{n-1}{R}au^2(R) + 2G(u(R))\right)\ddot{\mathcal{S}}_0(0) + \mathcal{F}.$$

Note that also in this case area preserving tangential perturbations belong to the kernel of $\ddot{\mathcal{E}}_R(0)$.

Lower bounds for $\ddot{\mathcal{E}}_R(0)$

The sign of $\ddot{\mathcal{E}}_R(0)$ depends on the solution and the data of the Robin boundary value problem. We consider volume or area preserving perturbations.

We now impose the barycenter condition

$$\oint_{\partial B_R} x(v(x)\cdot\nu(x))\,dS = 0.$$

Recall that $\phi_1 = a_1(r)Y_1(\xi)$, where $\xi = \frac{x}{|x|}$ and $x \in \mathcal{H}_1$. Then the orthonormality of $\{\phi_i\}_{i=1}^{\infty}$ implies

$$0 = \oint_{\partial B_R} \phi_1(R,\xi)\left(\sum_{i=1}^{\infty} b_i\phi_i(R,\xi)\right)dS = b_1.$$

Consequently,

$$\mathcal{F} = 2\sum_{i=2}^{\infty} b_i^2\left[au(R)k_g(u(R)) - \frac{k_g^2(u(R))}{\mu_i}\right]. \tag{8.2.13}$$

Let μ_p be the smallest positive eigenvalue of (8.2.9). Then

$$\mathcal{F} \geq \left[au(R)k_g(u(R)) - \frac{k_g^2(u(R))}{\mu_p}\right]\oint_{\partial B_R}\rho^2\,dS.$$

We recall that $\ddot{\mathcal{S}}_0(0)$ is nonnegative (cf. Section 2.3.3). By Lemma 2.4 the barycenter condition leads to the stronger estimate

$$\ddot{\mathcal{S}}(0) \geq \frac{n+1}{R^2}\oint_{\partial B_R}(v\cdot\nu)^2\,dS.$$

This together with the estimates for $\mathcal{F}$ leads to the following theorem.

Theorem 8.2. *Let $\{\Phi_t\}_{|t|<t_0}$ be a family of domain perturbations of B_R satisfying the barycenter condition and let $\mathcal{E}_R(t) = \mathcal{E}_R(\Omega_t)$.*

– *If the perturbations are volume preserving, then $\dot{\mathcal{E}}(0) = 0$ and*

$$\ddot{\mathcal{E}}_R(0) \geq \left\{ \frac{(n+1)\alpha u^2(R)}{R^2} + \alpha u(R) k_g\big(u(R)\big) - \frac{k_g^2(u(R))}{\mu_p} \right\} \oint_{\partial B_R} \rho^2 \, dS.$$

– *If the perturbations are area preserving, then $\dot{\mathcal{E}}(0) = 0$ and*

$$\begin{aligned} \ddot{\mathcal{E}}_R(0) \geq \bigg\{ & \frac{n+1}{(n-1)R} \big[k_g\big(u(R)\big)u(R) + 2G\big(u(R)\big) - u(R)g\big(u(R)\big)\big] \\ & + \alpha u(R) k_g\big(u(R)\big) - \frac{k_g^2(u(R))}{\mu_p} \bigg\} \oint_{\partial B_R} \rho^2 \, dS, \end{aligned}$$

where $k_g(u(R))$ is given in (8.2.4) *and $\mu_p = \min\{\mu_k : \mu_k \geq 0, k \geq 2\}$.*

8.2.2 The torsion problem with $\alpha > 0$

We apply the previous arguments to the torsion problem

$$\Delta u + 1 = 0 \text{ in } \Omega_t, \quad \partial_\nu u + \alpha u = 0 \text{ on } \partial\Omega_t, \quad \alpha > 0,$$

in nearly spherical domains. If $\alpha > 0$ the problem has a unique positive solution. In the ball it is therefore radial and of the form $u(r) = \frac{R}{\alpha n} + \frac{R^2}{2n} - \frac{r^2}{2n}$. In this case we have

$$u(R) = \frac{R}{\alpha n} \quad \text{and} \quad k_1\big(u(R)\big) = \frac{1+\alpha R}{n}.$$

The corresponding Steklov problem is

$$\Delta\phi = 0 \text{ in } B_R, \quad \partial_\nu \phi + \alpha\phi = \mu\phi \text{ on } \partial B_R,$$

where by Remark 3.1, $\mu_i = \frac{i}{R} + \alpha$.

From (8.2.1) the first variation is

$$\dot{\mathcal{E}}_R(0) = -\frac{R}{\alpha n^2}(\alpha R + n + 1) \oint_{\partial B_R} (v \cdot \nu) \, dS. \tag{8.2.14}$$

Clearly, it vanishes for volume or area preserving perturbations. It is negative, resp. positive, if $|\Omega_t| > |B_R]$, resp. $|\Omega_t| < |B_R]$.

For volume preserving perturbations Proposition 8.1 gives

$$\ddot{\mathcal{E}}_R(0) = \frac{R^2}{\alpha n^2}\ddot{\mathcal{S}}_0(0) + 2\sum_{i=1}^{\infty} b_i^2 \frac{R(1+\alpha R)(i-1)}{n^2(i+\alpha R)}. \tag{8.2.15}$$

Consequently, for volume preserving perturbations, $\ddot{\mathcal{E}}_R(0) \geq 0$. If $(v \cdot \nu) = b_1\phi_1$, then $\ddot{\mathcal{S}}_0(0) = 0$ and thus $\ddot{\mathcal{E}}_R(0) = 0$.

If we impose the barycenter condition, then $b_1 = 0$. Consequently,

$$\ddot{\mathcal{E}}_R(0) = \frac{R^2}{\alpha n^2}\ddot{\mathcal{S}}_0(0) + 2\sum_{i=2}^{\infty} b_i^2 \frac{R(1+\alpha R)(i-1)}{n^2(i+\alpha R)}.$$

In view of Theorem 8.2 we obtain

$$\ddot{\mathcal{E}}_R(0) \geq \left\{\frac{n+1}{\alpha n^2} + \frac{2(1+\alpha R)R}{n^2(2+\alpha R)}\right\} \oint_{\partial B_R} (v \cdot \nu)^2 \, dS > 0, \tag{8.2.16}$$

provided Φ_t is not a purely tangential perturbation.

In this case the ball is among all nearly spherical domains of prescribed volume a local minimizer of the Robin energy. This is in accordance with the result of Buçur and Giacomini [29], who showed that the ball has the minimal energy among all domains of given volume.

In the case of area preserving transformations, Proposition 8.2 yields

$$\begin{aligned}\ddot{\mathcal{E}}_R(0) &= \frac{1}{\alpha(n-1)n^2}(\alpha R^3 + R^2(n+1))\ddot{\mathcal{S}}_0(0) + 2\sum_{i=1}^{\infty} b_i^2 \frac{R(1+\alpha R)(i-1)}{n^2(i+\alpha R)} \\ &\geq \left\{\frac{(\alpha R+n+1)(n+1)}{\alpha(n-1)n^2} + \frac{2(1+\alpha R)}{(2+\alpha R)n^2}\right\} \oint_{\partial B_R} \rho^2 \, dS.\end{aligned}$$

Clearly, $\ddot{\mathcal{E}}_R(0)$ is nonnegative. Imposing the barycenter condition we obtain similar lower estimates as for volume preserving perturbations. Hence, the ball is a local minimizer for area preserving perturbations.

Problem 8.1. If there exists an optimal domain, must it be a ball?

8.2.3 Dirichlet energy

Let $\mathcal{E}_D(t)$ be the Dirichlet energy (cf. Section 6.2.1) defined in the nearly spherical domain Ω_t. Assume that the corresponding Dirichlet problem has a solution $u(y,t)$ which is differentiable in t. In addition, we require that in the ball u is radially symmetric. From (6.4.2) we get

$$\dot{\mathcal{E}}_D(0) = -(u_r^2(R) + 2G(0)) \oint_{\partial B_R} (v \cdot \nu)\, dS.$$

It vanishes whenever $\dot{\mathcal{V}}(0) = 0$, $\dot{\mathcal{S}}(0) = 0$, or $v = v^\tau$.

From Theorem 6.2 it follows that

$$\begin{aligned}\ddot{\mathcal{E}}_D(0) = (u_r^2(R) - G(0))\ddot{\mathcal{V}}(0) + 2g(0)u_r(R) \oint_{\partial B_R} (v \cdot \nu)^2\, dS \\ + u_r^2(R) \oint_{\partial B_R} \{4(v^\tau \cdot D_v \nu) - 2(w \cdot \nu)\}\, dS \\ + 2u_r(0) \oint_{\partial B_R} (v^\tau \cdot D^2 u v)\, dS + 2Q_0(u').\end{aligned}$$

By (2.3.7),

$$-2\oint_{\partial B_R} (w \cdot \nu)\, dS = -2\ddot{\mathcal{V}}(0) + 2\oint_{\partial B_R} (v \cdot \nu)\operatorname{div}_{\partial B_R} v\, dS - 2\oint_{\partial B_R} (v^\tau \cdot D_v \nu)\, dS.$$

For radial solutions u we have from Subsection 8.2.1 $(v^\tau \cdot D^2 u v) = \frac{u_r(R)}{R}|v^\tau|^2$. Hence,

$$\begin{aligned}\ddot{\mathcal{E}}_D(0) = -(u_r^2(R) + G(0))\ddot{\mathcal{V}}(0) + 2g(0)u_r(R) \oint_{\partial B_R} (v \cdot \nu)^2\, dS \\ + 2u_r^2(R) \oint_{\partial B_R} \left\{(v^\tau \cdot D_v \nu) + (v \cdot \nu)\operatorname{div}_{\partial B_R} v + \frac{1}{R}|v^\tau|^2\right\} dS \\ + 2Q_0(u').\end{aligned}$$

On the sphere we have $(v^\tau \cdot D_v \nu) = v^\tau \cdot \nabla^\tau (v \cdot \nu) - |v^\tau|^2/R$. By Gauss' theorem (2.2.18),

$$\begin{aligned}\ddot{\mathcal{E}}_D(0) = -\,(u_r^2(R) + G(0))\ddot{\mathcal{V}}(0) + 2g(0)u_r(R) \oint_{\partial B_R} (v \cdot \nu)^2\, dS \\ + 2u_r^2(R)\frac{(n-1)}{R} \oint_{\partial B_R} (v \cdot \nu)^2\, dS + 2Q(u').\end{aligned} \tag{8.2.17}$$

For volume preserving perturbations, taking into account the boundary condition $u' = -u_r(R)(v \cdot \nu)$ on ∂B_R, we get

$$\ddot{\mathcal{E}}_D(0) = 2\frac{g(0)}{u_r(R)} \oint_{\partial B_R} u'^2\, dS + 2\frac{(n-1)}{R} \oint_{\partial B_R} u'^2\, dS + 2Q(u'). \tag{8.2.18}$$

This is the formula given in Proposition 7.1.

The strategy used for the Robin energy requires eigenfunctions of the Steklov problem

$$\Delta\phi + g'(u)\phi = 0 \text{ in } B_R, \quad \partial_\nu\phi = \mu\phi \text{ in } \partial B_R.$$

Set $u' = \sum_{i=0}^{\infty} c_i\phi_i$ (cf. (8.1.4)). Then

$$\partial_\nu u' = \sum_{i=0}^{\infty} \mu_i c_i \phi_i = -(\nabla u \cdot v) = -u_r(R)(v \cdot \nu).$$

Consequently,

$$(v \cdot \nu) = -\sum_{i=0}^{\infty} \frac{\mu_i c_i}{u_r(R)}\phi_i =: \sum_{i=0}^{\infty} b_i\phi_i \quad \text{and} \quad \oint_{\partial B_R} (v \cdot \nu)^2\, dS = \sum_{i=0}^{\infty} b_i^2.$$

By Lemma 8.1, ϕ_0 is of constant sign. The condition $\dot{\mathcal{V}}(0) = 0$ implies that

$$0 = \oint_{\partial B_R} (v \cdot \nu)\, dS = b_0.$$

Furthermore,

$$Q_0(u') = \sum_{i=1}^{\infty} \mu_i c_i^2 = u_r^2(R) \sum_{i=1}^{\infty} \frac{b_i^2}{\mu_i}.$$

Observe that $\mu_i c_i = 0$ implies $b_i = 0$. Thus volume preserving perturbations yield

$$\ddot{\mathcal{E}}_D(0) = 2\left\{g(0)u_r(R) + \frac{n-1}{R}u_r^2(R)\right\} \oint_{\partial B_R} (v \cdot \nu)^2\, dS + 2u_r^2(R) \sum_{i=1}^{\infty} \frac{b_i^2}{\mu_i}. \tag{8.2.19}$$

We set $\mu_p := \min\{\mu_k : \mu_k \geq 0, k \geq 1\}$ and take into account the boundary condition $u' = -u_r(R)(v \cdot \nu)$ on ∂B_R. Then

$$\ddot{\mathcal{E}}_D(0) \geq 2\left\{\frac{g(0)}{u_r(R)} + \frac{n-1}{R} - \frac{2}{\mu_p}\right\} \oint_{\partial B_R} u'^2\, dS. \tag{8.2.20}$$

Remark 8.1. 1. If $u(y, t)$ is a positive minimizer of $\mathcal{E}_D(t)$, then Schwarz symmetrization applies. It shows that among all domains of given volume the ball yields the smallest energy. Moreover, the ball is the unique domain with such a property (see [119] and [9]).

2. Formula (8.2.19) is valid for sign changing solutions and for any critical point of $\mathcal{E}_D(0)$.

Problem 8.2. Does $\dot{\mathcal{E}}_D(0) = 0$ and $\ddot{\mathcal{E}}_D(0) > 0$ imply that the ball is the only optimal domain?

8.3 Notes

A detailed discussion of related Steklov eigenvalue problems is found in the work of Auchmuty; see for instance [6], [7], and the references cited therein. A discussion of the sign of the second variation in nearly spherical domains was first given in [13] and [14].

9 Eigenvalue problems

This chapter is devoted to domain variations of membrane eigenvalue problems. The computations rely strongly on the calculations in Chapter 6. It is shown that among all nearly spherical domains of given volume the ball has the smallest first eigenvalue.

The second part of this chapter deals with nonlinear eigenvalue problems. The peculiarity of these problems is that there exists a finite interval of eigenvalues for which solutions exist. The endpoint (turning point) of this interval where the solution branch bends is of particular interest. It can be expressed by means of a Rayleigh quotient depending on two functions. An important example is the Gelfand problem. We discuss the one-dimensional and the radial two-dimensional case analytically and compute the solution branches numerically for Dirichlet and Robin boundary conditions. At the end the first variation of the turning point of general nonlinear eigenvalue problems is derived for volume preserving perturbations.

9.1 Robin eigenvalue problem

Let $\{\Omega_t\}_{|t|<t_0}$ be the family of domains obtained from Ω by a perturbation $\Phi_t(\Omega)$. In Ω_t we consider the eigenvalue problem

$$\Delta_y u + \lambda(\Omega_t)u = 0 \quad \text{in } \Omega_t,$$

subject to the following boundary condition:

$$\partial_{\nu^t} u + \alpha u = 0 \quad \text{on } \partial\Omega_t \text{ and } \alpha > 0.$$

If we transform this problem back onto Ω, the transformed solution $\tilde{u}(t) := u(\Phi_t(x), t)$ solves (see Section 6.2.1)

$$L_{A(t)}\tilde{u}(t) + \lambda(t)\tilde{u}(t)J(t) = 0 \quad \text{in } \Omega. \tag{9.1.1}$$

Here we set $\lambda(t) := \lambda(\Omega_t)$ and $\lambda(0) = \lambda(\Omega)$. On $\partial\Omega$ we obtain

$$\partial_{\nu_A}\tilde{u}(t) + \alpha m(t)\tilde{u}(t) = 0. \tag{9.1.2}$$

We assume that $\tilde{u}(t) \in W^{1,2}(\Omega)$. The solutions of (9.1.1) satisfy

$$\int_\Omega \partial_i\tilde{u}A_{ij}(t)\partial_j\psi\,dx + \alpha\oint_{\partial\Omega}\tilde{u}\psi m(t)\,dS = \lambda(t)\int_\Omega \tilde{u}\psi J(t)\,dx, \quad \forall\,\psi \in W^{1,2}(\Omega).$$

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For $\psi = \tilde{u}$ we obtain

$$\lambda(t) = \frac{\int_\Omega A_{ij}(t)\partial_i \tilde{u}(t)\partial_j \tilde{u}(t)\,dx + \alpha \oint_{\partial\Omega} \tilde{u}^2(t)m(t)\,dS}{\int_\Omega \tilde{u}^2(t)J(t)\,dx}. \tag{9.1.3}$$

Observe that

$$\mathcal{G}_{\tilde{u}}(t)[\psi] := \int_\Omega \partial_i \tilde{u} A_{ij}(t)\partial_j \psi\,dx + \alpha \oint_{\partial\Omega} \tilde{u}\psi m(t)\,dS : W^{1,2}(\Omega) \to \mathbb{R}$$

is a self-adjoint linear operator, analytic in t for small $|t|$. The same is true for the linear operator

$$\mathcal{B}_{\tilde{u}}(t)[\psi] := \int_\Omega \tilde{u}\psi J(t)\,dx : L^2(\Omega) \to \mathbb{R}.$$

The following result holds true for the eigenvalue problem $\mathcal{G}_{\tilde{u}} = \lambda(t)\mathcal{B}_{\tilde{u}}$.

Theorem 9.1. *Let $\lambda(0)$ be an eigenvalue of multiplicity d. Then there exists $\delta > 0$ such that for $|t| < \delta$ there exist exactly d eigenvalues $\lambda_1(t), \lambda_2(t), \dots, \lambda_d(t)$. After a possible renaming these eigenvalues and the corresponding eigenfunctions are analytic in t.*

A proof of this result is given for example in the book of Chow and Hale [38, Theorem 5.5]. Figure 9.1a illustrates the branches and Figure 9.1b shows that the order of the eigenvalues can change at the bifurcation point.

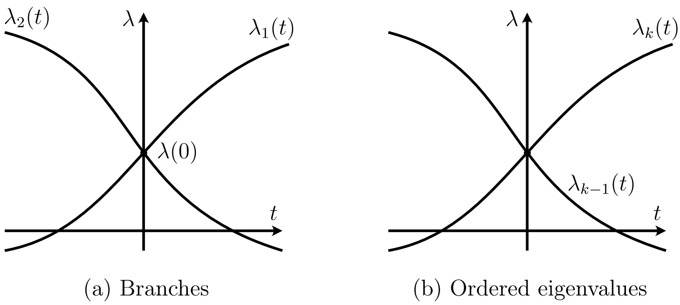


Figure 9.1: Bifurcations.

9.1.1 Domain variations for Robin eigenvalues with $\alpha > 0$

Let $\lambda(t)$ be a differentiable branch of eigenvalues in Ω_t with $\alpha > 0$. We use the same arguments as for the energy in Section 6.3 to compute the derivatives of $\lambda(t)$.

Differentiation of (9.1.3) and the normalization

$$\int_\Omega \tilde{u}^2(t)J(t)\,dx = 1 \tag{9.1.4}$$

lead to

$$\begin{aligned}\dot{\lambda}(t) = 2\int_\Omega \big(\nabla\dot{\tilde{u}}(t)\cdot A(t)\nabla\tilde{u}(t)\big)\,dx + \int_\Omega \big(\nabla\tilde{u}(t)\cdot \dot{A}(t)\nabla\tilde{u}(t)\big)\,dx\\ + 2\alpha\oint_{\partial\Omega} \dot{\tilde{u}}(t)\tilde{u}(t)m(t)\,dS + \alpha\oint_{\partial\Omega}\tilde{u}^2(t)\dot{m}(t)\,dS.\end{aligned}$$

The expressions containing $\dot{\tilde{u}}$ can be eliminated. If we test (9.1.1) with $\dot{\tilde{u}}$, we get

$$\lambda(t)\int_\Omega \dot{\tilde{u}}(t)u(t)J(t)\,dx = \int_\Omega \big(\nabla\dot{\tilde{u}}(t)\cdot A(t)\nabla\tilde{u}(t)\big)\,dx + \alpha\oint_{\partial\Omega}\dot{\tilde{u}}(t)\tilde{u}(t)m(t)\,dS.$$

The normalization (9.1.4) implies

$$\frac{d}{dt}\int_\Omega \tilde{u}^2(t)J(t)\,dx = \int_\Omega \big[2\tilde{u}(t)\dot{\tilde{u}}(t)J(t) + \tilde{u}(t)^2\dot{J}(t)\big]\,dx = 0 \tag{9.1.5}$$

for all $t \in (-t_0, t_0)$. Hence,

$$\begin{aligned}\dot{\lambda}(t) = \int_\Omega \big(\nabla\tilde{u}(t)\cdot\dot{A}(t)\nabla\tilde{u}(t)\big)\,dx - \lambda(t)\int_\Omega \tilde{u}^2(t)\dot{J}(t)\,dx\\ + \alpha\oint_{\partial\Omega}\tilde{u}^2(t)\dot{m}(t)\,dS.\end{aligned} \tag{9.1.6}$$

Next we compute the second derivative of $\lambda(t)$:

$$\begin{aligned}\ddot{\lambda}(t) = \int_\Omega \big(\nabla\tilde{u}(t)\cdot\ddot{A}(t)\nabla\tilde{u}(t)\big)\,dx + 2\int_\Omega \big(\nabla\dot{\tilde{u}}(t)\cdot\dot{A}(t)\nabla\tilde{u}(t)\big)\,dx - \dot{\lambda}(t)\int_\Omega \tilde{u}^2(t)\dot{J}(t)\,dx\\ - 2\lambda(t)\int_\Omega \dot{\tilde{u}}(t)\tilde{u}(t)\dot{J}(t)\,dx - \lambda(t)\int_\Omega \tilde{u}^2(t)\ddot{J}(t)\,dx\\ + 2\alpha\oint_{\partial\Omega}\tilde{u}(t)\dot{\tilde{u}}(t)\dot{m}(t)\,dS + \alpha\oint_{\partial\Omega}\tilde{u}(t)^2\ddot{m}(t)\,dS.\end{aligned}$$

As for the energy in Section 6.3 we can eliminate the domain integrals which contain both $\tilde{u}$ and $\dot{\tilde{u}}$. For this purpose we differentiate (9.1.1) with respect to t, multiply it by $2\dot{\tilde{u}}$, and integrate over Ω. We obtain

$$
\begin{aligned}
0 = &-2\int_\Omega \big(\nabla\tilde u(t)\cdot \dot A(t)\nabla\dot{\tilde u}(t)\big)\,dx - 2\int_\Omega \big(\nabla\dot{\tilde u}(t)\cdot A(t)\nabla\dot{\tilde u}(t)\big)\,dx\\
&+2\oint_{\partial\Omega}\big(\nu\cdot \dot A(t)\nabla\tilde u(t)\big)\dot{\tilde u}(t)\,dS + 2\oint_{\partial\Omega}\big(\nu\cdot A(t)\nabla\dot{\tilde u}(t)\big)\dot{\tilde u}(t)\,dS\\
&+2\dot\lambda(t)\int_\Omega \tilde u(t)\dot{\tilde u}(t)J(t)\,dx + 2\lambda(t)\int_\Omega \tilde u(t)\dot{\tilde u}(t)\dot J(t)\,dx\\
&+2\lambda(t)\int_\Omega \dot{\tilde u}^2(t)J(t)\,dx.
\end{aligned}
$$

Adding this expression to $\ddot\lambda(t)$ we obtain

$$
\begin{aligned}
\ddot\lambda(t) = &\int_\Omega \big(\nabla\tilde u(t)\cdot \ddot A(t)\nabla\tilde u(t)\big)\,dx - 2\int_\Omega \big(\nabla\dot{\tilde u}(t)\cdot A(t)\nabla\dot{\tilde u}(t)\big)\,dx\\
&+2\oint_{\partial\Omega}\dot{\tilde u}(t)\big(\big(\nu\cdot \dot A(t)\nabla\tilde u(t)\big) + \big(\nu\cdot A(t)\nabla\dot{\tilde u}(t)\big) + \alpha\tilde u(t)\dot m(t)\big)\,dS\\
&+\alpha\oint_{\partial\Omega}\tilde u^2(t)\ddot m(t)\,dS + \dot\lambda(t)\int_\Omega 2\tilde u(t)\dot{\tilde u}(t)J(t) - \tilde u^2(t)\dot J(t)\,dx\\
&+\lambda(t)\int_\Omega 2\dot{\tilde u}^2(t)J(t) - \tilde u^2(t)\ddot J(t)\,dx.
\end{aligned}
$$

From the boundary condition (9.1.2) it follows that

$$
\nu_i\dot A_{ij}\partial_j\tilde u + \nu_i A_{ij}\partial_j\dot{\tilde u} + \alpha\dot m\tilde u + \alpha m\dot{\tilde u} = 0 \quad \text{on } \partial\Omega.
$$

Hence,

$$
\begin{aligned}
\ddot\lambda(t) = &\int_\Omega \big(\nabla\tilde u(t)\cdot \ddot A(t)\nabla\tilde u(t)\big)\,dx - 2\int_\Omega \big(\nabla\dot{\tilde u}(t)\cdot A(t)\nabla\dot{\tilde u}(t)\big)\,dx\\
&-2\alpha\oint_{\partial\Omega}\dot{\tilde u}^2(t)m(t)\,dS + \alpha\oint_{\partial\Omega}\tilde u^2(t)\ddot m(t)\,dS\\
&+\lambda(t)\int_\Omega 2\dot{\tilde u}^2(t)J(t) - \tilde u^2(t)\ddot J(t)\,dx - 2\dot\lambda(t)\int_\Omega \tilde u^2(t)\dot J(t)\,dx.
\end{aligned} \tag{9.1.7}
$$

9.1.2 The discussion of $\dot{\lambda}(0)$

The same arguments as in Section 6.4 with $G(u)$ replaced by $\lambda(0)\frac{u^2}{2}$ imply that (cf. (6.4.4))

$$
\dot\lambda(0) = \oint_{\partial\Omega}(v\cdot\nu)\big[|\nabla u|^2 - \lambda(0)u^2 - 2\alpha^2u^2 + \alpha(n-1)Hu^2\big]\,dS. \tag{9.1.8}
$$

This expression characterizes the critical domains for which $\dot{\lambda}(0) = 0$. We will write $\lambda := \lambda(0)$ for short. In analogy to Theorem 6.1 we have the following.

Theorem 9.2. *Let $\Omega_t = \Phi_t(\Omega)$ be a family of volume preserving perturbations of Ω. Then Ω is a critical domain for the eigenvalue $\lambda(t)$, i. e., $\dot{\lambda}(0) = 0$, if and only if*

$$|\nabla u|^2 - \lambda u^2 - 2\alpha^2 u^2 + \alpha(n-1)u^2 H = const. \quad on\ \partial\Omega. \tag{9.1.9}$$

This additional condition leads to an overdetermined boundary value problem for u. The radial solutions in the ball clearly satisfy this condition.

Problem 9.1. Is the ball the unique domain for which this overdetermined problem has a solution?

Example 9.1. Let $\Omega = B_R$ and let λ be the lowest eigenvalue. The corresponding eigenfunction is radial and of constant sign. Then

$$\dot{\lambda}(0) = -u^2(R)\left(\alpha^2 - \alpha\frac{n-1}{R} + \lambda\right) \oint_{\partial B_R} (v \cdot \nu)\, dS. \tag{9.1.10}$$

The sign of

$$K := \alpha^2 - \alpha\frac{n-1}{R} + \lambda$$

can be determined as follows. Since $u(r)$ is of constant sign, the function $z = \frac{u_r}{u}$ is well-defined in (r_0, R) and satisfies the Riccati equation

$$\frac{dz}{dr} + z^2 + \frac{n-1}{r}z + \lambda = 0 \quad \text{in } (0, R).$$

At the endpoint $z(R) = -\alpha$, $z(0) = 0$, and $z_r(R) = -K$. Assume $\alpha > 0$ and let $u(r) > 0$ in $(0, R)$. If $z_r(R) > 0$, then there exists a number $\rho \in (0, R)$ such that $z_r(\rho) = 0$, $z(\rho) < 0$, and $z_{rr}(\rho) \geq 0$. From the equation we get $z_{rr}(\rho) = \frac{n-1}{\rho^2}z < 0$, which leads to a contradiction. Hence,

$$K > 0 \quad \text{if } \alpha > 0. \tag{9.1.11}$$

By the same argument,

$$K < 0 \quad \text{if } \alpha < 0. \tag{9.1.12}$$

From the representation

$$\dot{\lambda}(0) = -Ku^2(R) \oint_{\partial B_R} (v \cdot \nu)\, dS$$

we deduce that for $\alpha > 0$,

$$\dot{\lambda}(0) < 0$$

for all volume increasing perturbations $\oint_{\partial B_R} v \cdot \nu \, dS > 0$.

9.1.3 Discussion of $\ddot{\lambda}(0)$ for the ball

From (9.1.7) we obtain for $t = 0$ and $\lambda = \lambda(0)$

$$\ddot{\lambda}(0) = \int_\Omega (\ddot{A}(0)\nabla u \cdot \nabla u)\, dx - 2\int_\Omega |\nabla \dot{\tilde{u}}(0)|^2 \, dx + \lambda \int_\Omega [2\dot{\tilde{u}}^2(0) - u^2 \ddot{J}(0)]\, dx$$
$$- 2\dot{\lambda}(0) \int_\Omega u^2 \dot{J}(0)\, dx + \alpha \oint_{\partial\Omega} u^2 \ddot{m}(0)\, dS - 2\alpha \oint_{\partial\Omega} \dot{\tilde{u}}^2(0)\, dS.$$

Next we discuss the case where Ω is a ball B_R and the eigenfunction u corresponding to $\lambda(0)$ is radial. Because of the similarity between $\ddot{\lambda}(0)$ and $\ddot{\mathcal{E}}(0)$ we can proceed exactly as in Section 8.2. We consider volume preserving perturbations from now on; hence, $\dot{\lambda}(0) = 0$ for the ball.

By (6.2.12) and Lemma 6.1 the shape derivative u' solves

$$\begin{aligned} \Delta u' + \lambda u' &= 0 \quad \text{in } B_R, \\ \partial_\nu u' + \alpha u' &= k_{\lambda u}(u(R))(v \cdot \nu) \quad \text{on } \partial B_R, \end{aligned} \tag{9.1.13}$$

where

$$k_{\lambda u}(u(R)) = \left(\lambda - \frac{\alpha(n-1)}{R} + \alpha^2\right) u(R).$$

Then by (8.2.7)

$$\begin{aligned} \ddot{\lambda}(0) = &-2Q_\lambda(u') - u^2(R)(\lambda + \alpha^2)\ddot{\mathcal{V}}(0) \\ &+ 2\alpha u(R) k_{\lambda u} \oint_{\partial B_R} (v \cdot \nu)^2 \, dS + \alpha u^2(R) \ddot{\mathcal{S}}(0). \end{aligned} \tag{9.1.14}$$

Here

$$Q_\lambda(u') = \int_{B_R} |\nabla u'|^2 \, dx + \alpha \oint_{\partial B_R} u'^2 \, dS - \lambda \int_{B_R} u'^2 \, dx.$$

By the observation in Example 9.1 we have $\alpha u(R) k_{\lambda u} = \alpha u^2(R) K \geq 0$ independently of the sign of α.

The sign of $\ddot{\lambda}(0)$ for volume preserving perturbations

In this section we shall assume that $\lambda(t)$ is the lowest eigenvalue.

By the Bossel–Daners inequality the principal eigenvalue $\lambda(\Omega)$ is minimal for all domains Ω such that $|\Omega| = |B|$. We will show that the local result follows directly from the positivity $\ddot{\lambda}(0)$.

We assume $\dot{\mathcal{V}}(0) = \ddot{\mathcal{V}}(0) = 0$. In particular, $\dot{\mathcal{V}}(0) = 0$ implies $\dot{\lambda}(0) = 0$. We shall show that the strategy developed in Section 8 leads to $\ddot{\lambda}(0) > 0$. Under the assumptions stated above we have by (9.1.14)

$$\ddot{\lambda}(0) = -2Q_\lambda(u') + 2au(R)k_{\lambda u} \oint\limits_{\partial B_R} (v \cdot \nu)^2 \, dS + au^2(R)\ddot{\mathcal{S}}_0(0).$$

Following the strategy of Chapter 8 we shall express the quadratic form $Q_{\lambda u}$ in terms of the Steklov eigenfunctions

$$\begin{aligned} \Delta\phi + \lambda\phi = 0 \quad & \text{in } B_R, \\ \partial_\nu \phi + a\phi = \mu\phi \quad & \text{in } \partial B_R. \end{aligned} \tag{9.1.15}$$

Let $\{\phi_i\}_{i=0}^\infty$ be normalized such that

$$\oint\limits_{\partial B_R} \phi_i \phi_j \, dS = \delta_{ij} \Longrightarrow \int\limits_{B_R} [(\nabla\phi_i \cdot \nabla\phi_j) - \lambda\phi_i\phi_j] \, dx = (\mu_i - a)\delta_{ij}.$$

The first eigenfunction of the Robin eigenvalue problem in the ball is radial and is of the form

$$u(r) = C_n J_{\frac{n-2}{2}}(\sqrt{\lambda_1} r) r^{-\frac{n-2}{2}}, \tag{9.1.16}$$

where C_n is determined by the normalization $\int_{B_R} u^2 \, dx = 1$ (see Section 3.3.1). The function $u(r)$ is also a solution of the Steklov problem (9.1.15) with $\mu = 0$. Since it is of constant sign, $\mu = 0$ is the lowest eigenvalue.

Consequently, $0 = \mu_0 < \tilde{\mu}_1 \le \tilde{\mu}_2 \le \cdots$. Let $u' = \sum_{i=0}^\infty c_i \phi_i$ and $(v \cdot \nu) = \sum_{i=0}^\infty b_i \phi_i$ (see Section 8.1 for the notation). By the orthonormality of the Steklov eigenfunctions we obtain

$$c_j = \oint\limits_{\partial B_R} u' \phi_j \, dS \quad \text{and} \quad b_j = \oint\limits_{\partial B_R} (v \cdot \nu)\phi_j \, dS.$$

Since ϕ_0 is radial, $b_0 = 0$ for volume preserving perturbations. The relations between b_j and c_j for $j \ge 1$ are obtained from (9.1.13) and (9.1.15). In fact,

$$0 = \int_{B_R} (\Delta u' + \lambda u')\phi_j \, dx = \oint_{\partial B_R} (\phi_j \partial_\nu u' - u' \partial_\nu \phi_j) \, dS$$
$$= \oint_{\partial B_R} (\phi_j k_{\lambda u} (v \cdot \nu) - \mu_j u' \phi_j) \, dS.$$

Thus,

$$k_{\lambda u} b_j = \mu_j c_j, \quad j \geq 1.$$

Recall that $Q_\lambda(u') = \sum_{i=1}^\infty \mu_i c_i^2$. If we insert this expression and

$$\oint_{\partial B_R} (v \cdot \nu)^2 \, dS = \sum_{i=1}^\infty b_i^2 = \sum_{i=1}^\infty \left(\frac{\mu_i c_i}{k_{\lambda u}} \right)^2$$

into $\ddot{\lambda}(0)$, we obtain

$$\ddot{\lambda}(0) = 2 \sum_{i=1}^\infty c_i^2 \mu_i^2 \left[\frac{\alpha u(R)}{k_{\lambda u}(u(R))} - \frac{1}{\mu_i} \right] + \alpha u^2(R) \ddot{\mathcal{S}}_0(0).$$

Replacing $\mu_i c_i$ by $k_{\lambda u} b_i$ and $k_{\lambda u}$ by $K u(R)$, we get

$$\ddot{\lambda}(0) = 2u^2(R) K^2 \sum_{i=1}^\infty b_i^2 \left[\frac{\alpha}{K} - \frac{1}{\mu_i} \right] + \alpha u^2(R) \ddot{\mathcal{S}}_0(0). \tag{9.1.17}$$

Next we shall show that the term inside brackets is nonnegative. Since $\mu_i > \mu_1$ for $i > 1$,

$$\frac{\alpha}{K} - \frac{1}{\mu_i} > \frac{\alpha}{K} - \frac{1}{\mu_1}.$$

In Example 9.1 it was shown that K and α are positive. The inequality $\frac{\alpha}{K} - \frac{1}{\mu_1} > 0$ is equivalent to

$$L := \mu_1 - \alpha + \frac{n-1}{R} - \frac{\lambda}{\alpha} \geq 0.$$

To establish this inequality we have to compute μ_1. By Lemma 8.2 the eigenfunctions of μ_1 are of the form

$$\phi(x) = \sum_i c_i a_1(r) Y_{1,i}(\xi), \quad \xi \in S^{n-1},$$

where $Y_{1,i}(\xi)$, $i = 1, \ldots, n$, denotes the i-th spherical harmonics of the first order and

$$a_1(r) = r^{\frac{2-n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} r).$$

The boundary condition in (9.1.15) implies that

$$a_1'(R) = (\mu_1 - \alpha)a_1(R).$$

From $u_r(R) + \alpha u(R) = 0$, (9.1.16), and the well-known Bessel identity

$$(z^{-\nu}J_\nu(z))_z = -r^{-\nu}J_{\nu+1}(z),$$

it follows that

$$\alpha = \sqrt{\lambda}\frac{J_{n/2}(\sqrt{\lambda}R)}{J_{(n-2)/2}(\sqrt{\lambda}R)}. \tag{9.1.18}$$

The same argument and the boundary condition for a_1 implies

$$\frac{1}{R} + \alpha - \sqrt{\lambda}\frac{J_{n/2+1}(\sqrt{\lambda}R)}{J_{n/2}(\sqrt{\lambda}R)} = \mu_1.$$

Introducing these two expressions in L, we obtain

$$L = \frac{n}{R} - \frac{\sqrt{\lambda}}{J_{\frac{n}{2}}(\sqrt{\lambda}R)}(J_{\frac{n}{2}+1}(\sqrt{\lambda}R) + J_{\frac{n}{2}-1}(\sqrt{\lambda}R)).$$

The recurrence relation for Bessel functions

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z}J_\nu \tag{9.1.19}$$

implies that $L = 0$ and $K = \alpha\mu_1$. Consequently,

$$\frac{\alpha}{K} - \frac{1}{\mu_1} = \frac{\alpha L}{K\mu_1} = 0.$$

Then (9.1.17) becomes

$$\ddot{\lambda}(0) = 2u^2(R)K^2\sum_{i=2}^{\infty} b_i^2\left[\frac{\alpha}{K} - \frac{1}{\mu_i}\right] + \alpha u^2(R)\ddot{\mathcal{S}}_0(0).$$

Since $0 < \mu_i < \mu_{i+1}$ for $i \geq 1$, $\frac{\alpha}{K} - \frac{1}{\mu_i} \geq \frac{\alpha}{K} - \frac{1}{\mu_{i+1}}$ and we get

$$\ddot{\lambda}(0) \geq 2K^2u^2(R)\left[\frac{\alpha}{K} - \frac{1}{\mu_2}\right]\sum_{i=2}^{\infty} b_i^2 + \alpha u^2(R)\ddot{\mathcal{S}}_0(0) \geq 0. \tag{9.1.20}$$

By imposing the barycenter condition (cf. Definition 2.5), Lemma 2.4 yields

$$\ddot{\lambda}(0) \geq \left(2K^2\left[\frac{\alpha}{K} - \frac{1}{\mu_2}\right] + \alpha\frac{n+1}{R^2}\right)u^2(R)\oint_{\partial B_R}(\upsilon\cdot\nu)^2\,dS > 0.$$

Theorem 9.3. *Let λ be the principal eigenvalue of the Robin eigenvalue problem with $\alpha > 0$. Then among all nearly spherical domains of equal volume the ball yields the lowest eigenvalue.*

9.2 Nonlinear eigenvalue problems

In this chapter we discuss nonlinear eigenvalue problems of the form

$$\Delta u + \lambda f(u) = 0 \text{ in } \Omega, \quad \partial_\nu u + \alpha u = 0 \text{ on } \partial\Omega, \ \alpha > 0, \tag{9.2.1}$$

where $\lambda > 0$ and f is a positive twice differentiable function such that $f(0) > 0$ and $f(u)/u \to \infty$ as $u \to \infty$.

We assume that there exists a *turning point* $0 < \lambda^* < \infty$ such that:
- for $0 < \lambda < \lambda^*$ there exists a classical minimal solution,
- for $\lambda = \lambda^*$ there is a unique classical solution,
- for $\lambda > \lambda^*$ there are no solutions.

For convex and monotone functions f the existence of such a turning point has been proved for instance in [39, 16]. Problems of this type arise in a variety of models such as combustion, thermal explosions, and gravitational equilibrium of polytropic stars (cf. Crandall and Rabinowitz [39]).

The turning point is characterized by two different boundary value problems, namely

$$\text{original problem:} \quad \Delta u + \lambda^* f(u) = 0 \text{ in } \Omega, \quad \partial_\nu u + \alpha u = 0 \text{ on } \partial\Omega, \tag{9.2.2}$$

$$\text{linearized problem:} \quad \Delta \phi + \lambda^* f'(u)\phi = 0 \text{ in } \Omega, \quad \partial_\nu \phi + \alpha \phi = 0 \text{ on } \partial\Omega, \tag{9.2.3}$$

where λ^* is a simple eigenvalue of (9.2.3). It can be expressed in terms of the Rayleigh quotient

$$\lambda^* = \frac{\int_\Omega (\nabla u \cdot \nabla \phi)\, dx + \alpha \oint_{\partial\Omega} u\phi\, dS}{\int_\Omega f(u)\phi\, dx} = \frac{\int_\Omega (\nabla u \cdot \nabla \phi)\, dx + \alpha \oint_{\partial\Omega} u\phi\, dS}{\int_\Omega f'(u)u\phi\, dx},$$

where u and ϕ are solutions of (9.2.2) and (9.2.3).

Let $\{\Omega_t\}$, $|t| \le t_0$, be a family of small perturbations of Ω as considered throughout this text. Next we will show that problem (9.2.1) has a turning point $\lambda(t)$ in Ω_t. The next lemma and its proof are taken from Henry's book [76].

Lemma 9.1. *Let u^* and ϕ^* be solutions of* (9.2.2) *and* (9.2.3)*. Assume that λ^* is simple. Furthermore, let $f'(u) > 0$ and*

$$\int_\Omega f''(u)(\phi^*)^2\, dx \ne 0. \tag{9.2.4}$$

Then in each Ω_t there exists a turning point $\lambda^*(t)$ such that $\lambda^*(0) = \lambda^*$. Moreover, it is twice continuously differentiable for small $|t|$.

Proof. We shall use the notation of Section 9.1 and denote $\mathcal{B}_A := \partial_{\nu_A} + am$ and $\mathcal{B}_0 := \partial_\nu + \alpha$ for short. Choose $p > n/2$ and set

$$G: \quad (\lambda, v, \phi, t) \to \Big(L_A v + \lambda f(v)J, \mathcal{B}_A v, L_A\phi + \lambda f'(v)J\phi, \mathcal{B}_A\phi, \int_\Omega \phi^2 J\,dx\Big),$$

where

$$\begin{aligned} G: \quad & \mathbb{R} \times W^{2,p}(\Omega) \times W^{2,2}(\Omega) \times (-t_0, t_0) \\ & \to C^0(\Omega) \times C^0(\partial\Omega) \times L^2(\Omega) \times L^2(\partial\Omega) \times \mathbb{R}. \end{aligned}$$

If ϕ^* is normalized such that $\int_\Omega (\phi^*)^2\,dx = 1$, then clearly $G(\lambda^*, u^*, \phi^*, 0) = (0,0,0,0,1)$. We are looking for a solution $G(\lambda, u, \phi, t) = (0,0,0,0,1)$ in a neighborhood of $(\lambda^*, u^*, \phi^*, 0)$. We will apply the implicit function theorem. The derivative of $G(\lambda, v, \phi, t)$ with respect to t evaluated at $t = 0$ is the quintuple $(T_1, T_2, T_3, T_4, T_5)$, where

$$\begin{aligned} & T_1 := \Delta\dot v + \dot\lambda f(u^*) + \lambda^* f'(u^*)\dot v, \quad T_2 := \mathcal{B}_0\dot v, \\ & T_3 := \Delta\dot\phi + \dot\lambda f'(u^*)\phi^* + \lambda^* f'(u^*)\dot\phi + \lambda^* f''(u^*)\dot v\phi^*, \\ & T_4 := \mathcal{B}_0\dot\phi, \quad \text{and} \quad T_5 := 2\int_\Omega \phi^*\dot\phi\,dx. \end{aligned}$$

We have to show that it vanishes if and only if $(\dot v, \dot\phi, \dot\lambda) = (0,0,0)$.

From (9.2.3) and

$$\Delta\dot v + \dot\lambda f(u^*) + \lambda^* f'(u^*)\dot v = 0 \text{ in } \Omega, \quad \partial_\nu\dot v + \alpha\dot v = 0 \text{ on } \partial\Omega,$$

it follows that $\dot\lambda \int_\Omega f(u^*)\phi^*\,dx = 0$. Our assumption $f'(u) > 0$ implies that ϕ^* does not change sign and thus

$$\int_\Omega f(u^*)\phi^*\,dx \neq 0.$$

Hence, $\dot\lambda = 0$. Since λ^* is simple it follows that $\dot v = c\phi^*$. The equation

$$\Delta\dot\phi + \lambda^* f'(u^*)\dot\phi + c\lambda^* f''(u^*)(\phi^*)^2 = 0$$

together with (9.2.3) and (9.2.4) implies that $c = 0$. It follows now immediately that $\dot\phi = b\phi^*$ and the normalization yields $b = 0$. Hence, by the implicit function theorem $G = (0,0,0,0,1)$ has for $|t| < t_0$ a unique solution $(\lambda(t), u(t), \phi(t), t)$ such that $(\lambda(0), u(0), \phi(0)) = (\lambda^*, u^*, \phi^*)$.

In addition to the existence of a turning point the implicit function theorem implies that for small $|t|$, $u(t), \phi(t), \lambda(t)$ are twice continuously differentiable with respect to t. □

9.2.1 Examples

A classical example is the Gelfand problem $\Delta u + \lambda e^u = 0$ [64] for $\lambda > 0$.

The $1 - d$ Gelfand equation

Let Ω be the interval $(-a, a)$. By scaling, the differential equation $u'' + \lambda e^u = 0$ transforms to the equation $u'' + a^2\lambda e^u = 0$ on $(-1, 1)$. We set $\lambda_a := a^2\lambda$. Without specifying the boundary conditions the solution is

$$u(x) := \ln\left(\frac{1 - \tanh^2(\frac{x+d}{2c})}{2c^2\lambda_a}\right). \tag{9.2.5}$$

We consider only symmetric solutions and set therefore $d = 0$. In particular, $u'(0) = 0$. For Dirichlet data $u(\pm 1) = 0$, (9.2.5) yields

$$1 = \frac{1 - \tanh^2(\frac{1}{2c})}{2c^2\lambda_a}.$$

For a given λ_a this equation gives a condition on $c \geq 0$. It is equivalent to

$$Z_D(c, \lambda_a) := h_D(c) - \lambda_a = 0, \quad \text{where } h_D(c) := \frac{1}{2c^2}\left(1 - \tanh^2\left(\frac{1}{2c}\right)\right),$$

for all $c \geq 0$. Clearly, $h \geq 0$ for $c \geq 0$ and $h(0) = \lim_{c\to\infty} h(c) = 0$ (see Figure 9.2, where we set $a = 1$). There is a critical c_* such that the function Z_D has exactly two zeros c_1 and

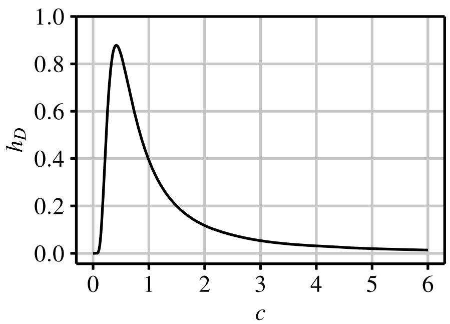


Figure 9.2: The function h_D.

c_2 for $0 < \lambda_a < h_D(c_*)$. Consequently,

$$u_i(x) = \ln\left(\frac{1 - \tanh^2(\frac{x}{2c_i})}{2c_i^2\lambda_a}\right) \quad \text{for } i = 1, 2$$

both solve the Dirichlet boundary value problem. Note that for $i = 1, 2$

$$\|u_i\|_{L^\infty(-1,1)} = u_i(0) = \left(\frac{1}{2c_i^2\lambda_a}\right).$$

Figure 9.3a shows the values of $u(0)$ in dependence of λ_a for $a = 1$.

For the Robin boundary condition $\pm u'(\pm 1) + \alpha a u(\pm 1) = 0$ with $\alpha > 0$, the constant c must satisfy

$$Z_R(c, \lambda_a, \alpha) := h_R(c, \alpha) - \lambda_a = 0.$$

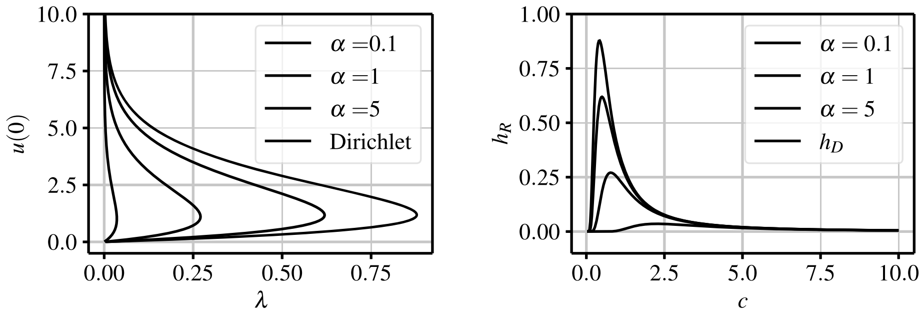
Here

$$h_R(c, \alpha) := \frac{2}{c^2}\left(1 + e^{\frac{1}{c}}\right)^{-2} e^{\frac{A(c,\alpha)}{B(c,\alpha)}}$$

with

$$A(c, \alpha) = 1 + \alpha + (\alpha - 1)e^{\frac{1}{c}} \quad \text{and} \quad B(c, \alpha) = \alpha c\left(1 + e^{\frac{1}{c}}\right).$$

Note that $\lim_{\alpha\to\infty} h_R(c, \alpha) = h_D(c)$.



(a) Branches (b) Positive α

Figure 9.3: The function h_R.

Figure 9.3b shows h_R as a function of c for different values of α and $a = 1$. As before, there is a critical value c_* such that there are two zeros c_1 and c_2 of Z_R for $0 < \lambda_a < h_R(c_*)$. The two values c_1 and c_2 give two different solutions u_1 and u_2.

The turning point λ^* for the Dirichlet (resp. Robin) boundary value problem is the largest value such that Z_D (resp. Z_R) has two solutions for $0 < \lambda_a < \lambda^*$. Thus, $\lambda^* = h_D(c_*)$ (resp. $\lambda^* = h_R(c_*)$).

Remark 9.1. The solution of the one-dimensional Gelfand equation such that $u'(0) = 0$ without taking into account any boundary conditions can be calculated by multiplying $u'' + \lambda_a e^u = 0$ by u' and then integrating it from 0 to $x \in (-a, a)$. This leads to

$$u'^2(x) + 2\lambda_a\big(e^{u(x)} - e^{u(0)}\big) = 0. \tag{9.2.6}$$

Because of the equation $u'' = -\lambda_a e^u < 0$ for $\lambda_a > 0$, u is concave on $(-a, a)$. Hence, (9.2.6) implies

$$u'(x) = -\sqrt{2\lambda_a[e^{u(0)} - e^{u(x)}]}.$$

Separation of variables and integration yield

$$\begin{aligned} u(x) &= u(0) - 2\log\cosh\bigg(\sqrt{\frac{\lambda_a e^{u(0)}}{2}}x\bigg), \\ u'(x) &= -2\sqrt{\frac{\lambda_a e^{u(0)}}{2}}\tanh\bigg(\sqrt{\frac{\lambda_a e^{u(0)}}{2}}x\bigg). \end{aligned}$$

The Robin boundary condition at $x = \pm 1$ leads to the following relations between λ and $u(0) = \|u\|_{L^\infty}$:

$$2\sqrt{\frac{\lambda_a e^{u(0)}}{2}}\tanh\bigg(\sqrt{\frac{\lambda_a e^{u(0)}}{2}}\bigg) = \alpha\bigg[u(0) - 2\ln\cosh\bigg(\sqrt{\frac{\lambda_a e^{u(0)}}{2}}\bigg)\bigg].$$

We write this equation as

$$2\sqrt{z}\tanh(\sqrt{z}) = \alpha\bigg[\ln\bigg(\frac{2z}{\lambda_a}\bigg) - 2\ln\big(\cosh(\sqrt{z})\big)\bigg], \quad z := \frac{\lambda_a e^{u(0)}}{2},$$

and solve it for λ_a:

$$\lambda_a = \frac{8z}{(1+e^{2\sqrt{z}})^2}e^{A(z,\alpha)} := \tilde{h}_R(z,\alpha), \quad A(z,\alpha) = \frac{2\sqrt{z}[e^{2\sqrt{z}}\alpha - e^{2\sqrt{z}} + \alpha + 1]}{\alpha(1+e^{2\sqrt{z}})}.$$

There is a maximal range of values for λ_a for which there exist two values of $u(0)$, i. e., two solutions of the $1-d$ Gelfand equation.

The radial Gelfand equation in the disc

The Gelfand problem $\Delta u+\lambda e^u = 0$ in the disc B_R can be transformed after scaling $r \to r/R$ into the problem $\Delta u + \lambda_R e^u = 0$ in B_1, where $\lambda_R = \lambda R^2$. The radial solutions are of the form

$$u(r) = \ln \frac{8\beta}{\lambda_R(r^2+\beta)^2}, \quad \lambda_R := R^2\lambda.$$

In the case of Dirichlet data, $u(1) = 0$. By the maximum principle the solution is positive.

There are two values of β for which the boundary condition is satisfied, namely

$$\beta_1 = \frac{1}{\lambda}(4 - \lambda_R + 2\sqrt{4-2\lambda_R}\,) \quad \text{and} \quad \beta_2 = \frac{1}{\lambda}(4 - \lambda_R - 2\sqrt{4-2\lambda_R}\,).$$

The corresponding solutions $u_i(r)$ (for $\beta = \beta_i$) have a unique maximum in $r = 0$; thus, $\|u_i\|_{L^\infty(B_1)} = u_i(0)$. Obviously, there is no solution if $\lambda_R > 2$. Consequently, the turning point for B_R is $\lambda^* = \frac{2}{R^2}$.

After scaling, the Robin boundary conditions $\partial_\nu u + \alpha u = 0$ on ∂B_R become $u_r(1) + \alpha R u(1) = 0$ with $\alpha > 0$. The parameter β is then determined by the zeros of

$$Z_{\text{rad}} := \frac{8\beta}{(1+\beta)^2} e^{-\frac{4}{(1+\beta)\alpha R}} - \lambda_R.$$

Set

$$h_{\text{rad}}(\beta) := \frac{8\beta}{(1+\beta)^2} e^{-\frac{4}{(1+\beta)\alpha R}}.$$

Note that $\lim_{\alpha\to\infty} h_{\text{rad}} = \frac{8\beta}{(1+\beta)^2}$ corresponds to the case of Dirichlet data.

Clearly, $h_{\text{rad}} \geq 0$ for $\beta \geq 0$ and $\lim_{\beta\to 0} h_{\text{rad}}(\beta) = \lim_{\beta\to\infty} h_{\text{rad}}(\beta) = 0$. This is true for all $\alpha > 0$.

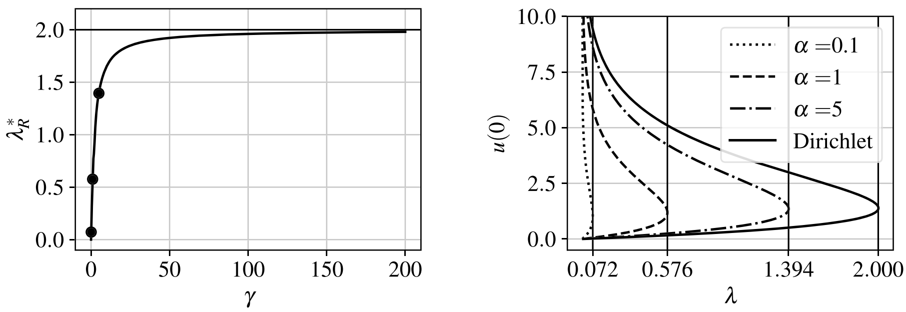
The λ_R corresponding to the turning point is given by

$$\lambda_R^* = h_{\text{rad}}(\beta^*) = \max_{\beta>0}\left\{\frac{8\beta}{(1+\beta)^2} e^{-\frac{4}{\alpha R(1+\beta)}}\right\}.$$

The maximum is attained for $\beta^* = \frac{2}{\gamma} + \sqrt{\frac{4}{\gamma^2}+1}$, where $\gamma = \alpha R$. This leads to

$$\lambda_R^* = \lambda^* R^2 = \frac{8\gamma(2+\sqrt{\gamma^2+4}) \exp(\frac{-4}{\gamma+2+\sqrt{\gamma^2+4}})}{(\gamma - 2 + \sqrt{\gamma^2+4})^2}.$$

Figure 9.4 shows that λ^* tends to 2 as γ tends to ∞. Hence, for fixed $R > 0$ the turning point for the Robin problem converges to the turning point of the Dirichlet problem as $\alpha \to \infty$. The three dots in Figure 9.4 (a) show the different values of α for which the graph of $\lambda \to u(0)$ is plotted in Figure 9.4 (b) for $R = 1$.



(a) Turning point λ_R^* as a function of αR (b) Solution branches for B_1

Figure 9.4: Results for the ball.

9.2.2 First variation of the turning point

In this section we compute the first variation of the turning point of the nonlinear problem (9.2.1). For simplicity we shall write $\lambda(t)$ instead of $\lambda^*(t)$. We will use the same notation as in Chapter 6, that is, $\tilde{u}$ is the solution of (9.2.2) and $\tilde{\phi}$ is the solution of (9.2.3) in Ω_t after pullback to Ω.

It follows from the general discussion in Section 9.2 that $\lambda(t)$ can be represented with a Rayleigh quotient. After the change of variables it reads as

$$\lambda(t) = \frac{\int_\Omega (\nabla\tilde{u}(t)\cdot A(t)\nabla\tilde{\phi}(t))\,dx + \alpha\oint_{\partial\Omega}\tilde{u}(t)\tilde{\phi}(t)m(t)\,dS}{\int_\Omega f(\tilde{u}(t))\tilde{\phi}(t)J(t)\,dx}.$$

Differentiation in $t = 0$ together with the fact that $J(0) = m(0) = 1$ and $A_{ij}(0) = \delta_{ij}$ yields

$$\begin{aligned}\dot{\lambda}(0)\int_\Omega f(u)\phi\,dx + \lambda\int_\Omega [f'(u)\dot{u}\phi + f(u)\dot{\phi} + f(u)\phi\dot{J}(0)]\,dx \\ = \int_\Omega [(\nabla\dot{u}\cdot\nabla\phi) + (\nabla u\cdot\dot{A}\nabla\phi) + (\nabla u\cdot\nabla\dot{\phi})]\,dx \\ + \alpha\oint_{\partial\Omega}[\dot{u}\phi + u\dot{\phi} + u\phi\dot{m}(0)]\,dS.\end{aligned} \tag{9.2.7}$$

Testing $\Delta u + \lambda f(u) = 0$ in Ω and $\partial_\nu u + \alpha u = 0$ on $\partial\Omega$ with $\dot{\phi}(0)$ we get

$$\lambda\int_\Omega f(u)\dot{\phi}\,dx = \int_\Omega (\nabla\dot{\phi}\cdot\nabla u)\,dx + \alpha\oint_{\partial\Omega}\dot{\phi}u\,dS.$$

Testing $\Delta\phi + \lambda f'(u)\phi = 0$ in Ω and $\partial_\nu\phi + \alpha\phi = 0$ on $\partial\Omega$ with $\dot{u}(0)$ we get

$$\lambda \int_\Omega f'(u)\phi\dot{u}\,dx = \int_\Omega (\nabla\phi \cdot \nabla\dot{u})\,dx + \alpha \oint_{\partial\Omega} \phi\dot{u}\,dS.$$

Introducing these expressions in (9.2.7) we obtain

$$\dot{\lambda} \int_\Omega f(u)\phi\,dx + \lambda \int_\Omega f(u)\phi\dot{J}\,dx = \int_\Omega (\nabla u \cdot \dot{A}\nabla\phi)\,dx + \alpha \oint_{\partial\Omega} u\phi\dot{m}\,dS. \tag{9.2.8}$$

In the sequel we shall write u^* and ϕ^* for the functions u and ϕ corresponding to λ^*. In $t = 0$ we apply (2.1.5), (4.1.16), (2.3.19), and (2.3.21). Hence,

$$\int_\Omega f(u)\phi\,dx = \int_\Omega f(u^*)\phi^*\,dx,$$

$$\int_\Omega f(u)\phi\dot{J}\,dx = \oint_{\partial\Omega} f(u^*)\phi^*(v\cdot\nu)\,dS - \int_\Omega \Big[f'(u^*)\phi^*(\nabla u^* \cdot v)\,dx - \int_\Omega f(u^*)(\nabla\phi^* \cdot v)\Big]\,dx,$$

$$\int_\Omega (\nabla u \cdot \dot{A}\nabla\phi)\,dx = \oint_{\partial\Omega} [(v\cdot\nu)(\nabla u^* \cdot \nabla\phi^*) + \alpha\phi^*(\nabla u^* \cdot v) + \alpha u^*(\nabla\phi^* \cdot v)]\,dS - \lambda^* \int_\Omega [f'(u^*)\phi^*(\nabla u \cdot v) + f(u^*)(\nabla\phi^* \cdot v)]\,dx,$$

and

$$\oint_{\partial\Omega} u\phi\dot{m}(0)\,dS = \oint_{\partial\Omega} [(n-1)H(v\cdot\nu)u^*\phi^* - (v^\tau \cdot \nabla^\tau(u^*\phi^*))]\,dS.$$

Consequently,

$$\begin{aligned}\dot{\lambda}(0) \int_\Omega f(u^*)\phi^*\,dx = \alpha &\oint_{\partial\Omega} [(v\cdot\nabla u^*)\phi^* + (v\cdot\nabla\phi^*)u^*]\,dS \\ &+ \oint_{\partial\Omega} (v\cdot\nu)[(\nabla u^* \cdot \nabla\phi^*) - \lambda^* f(u^*)\phi^*]\,dS \\ &+ \alpha \oint_{\partial\Omega} [(n-1)H(v\cdot\nu)u^*\phi^* - (v^\tau \cdot \nabla^\tau(u^*\phi^*))]\,dS.\end{aligned} \tag{9.2.9}$$

For $\rho = (v\cdot\nu)$ this becomes

$$\begin{aligned}\dot{\lambda}(0) \int_\Omega f(u^*)\phi^*\,dx = -2\alpha^2 \oint_{\partial\Omega} \rho u^*\phi^*\,dS + \alpha(n-1) \oint_{\partial\Omega} H\rho u^*\phi^*\,dS \\ + \oint_{\partial\Omega} \rho[(\nabla u^* \cdot \nabla\phi^*) - \lambda^* f(u^*)\phi^*]\,dS.\end{aligned}$$

On the boundary ∇u^* and $\nabla\phi^*$ can be decomposed in their normal and tangential components. Thus, in view of the Robin boundary conditions

$$(\nabla u^* \cdot \nabla\phi^*) = \alpha^2 u^* \phi^* + (\nabla^\tau u^* \cdot \nabla^\tau \phi^*).$$

Finally we get for Hadamard perturbations

$$\begin{aligned}\dot{\lambda}(0) &\int_\Omega f(u^*)\phi^*\, dx \\ &= \oint_{\partial\Omega} \rho[(\nabla^\tau u^* \cdot \nabla^\tau \phi^*) - \alpha^2 u^*\phi^* - \lambda^* f(u^*)\phi^* + \alpha(n-1)Hu^*\phi^*]\, dS. \qquad (9.2.10)\end{aligned}$$

For volume preserving perturbations we get an overdetermined boundary value problem for the bilinear form in u^* and ϕ^*.

Example 9.2. Let Ω be the ball B_R. In this case $u := u^*$ and $\phi := \phi^*$ are radial. The boundary conditions $\nabla u(R) = -\alpha u(R)\nu$ and $\nabla\phi(R) = -\alpha\phi(R)\nu$ and (9.2.10) implies

$$\begin{aligned}\dot{\lambda}(0)\int_{B_R} f(u)\phi\, dx = \alpha\phi(R)u(R) \oint_{\partial B_R} \dot{m}(0)\, dS + \alpha^2 u(R)\phi(R) \oint_{\partial B_R} (v\cdot\nu)\, dS \\ - \lambda f(u(R))\phi(R) \oint_{\partial B_R} (v \cdot \nu)\, dS.\end{aligned}$$

Since by (2.3.19), $\dot{m}(0) = \operatorname{div}_{\partial\Omega} v^\tau + \frac{n-1}{R}(v\cdot\nu)$, we obtain

$$\dot{\lambda}(0)\int_{B_R} f(u)\phi\, dx = \left[\alpha\frac{n-1}{R} - \alpha^2 - \lambda\frac{f(u(R))}{u(R)}\right]\phi(R)u(R) \oint_{\partial B_R} (v\cdot\nu)\, dS.$$

This formula reveals that $\dot{\lambda}(0)$ vanishes for volume preserving perturbations.

In the case of Dirichlet boundary conditions, (9.2.8) together with (4.1.16) implies that

$$\dot{\lambda}^*(0)\int_\Omega f(u^*)\phi^*\, dx = -\oint_{\partial\Omega} \rho(\nabla u^* \cdot \nabla\phi^*)\, dS. \qquad (9.2.11)$$

This result was first obtained in [88].

Problem 9.2. Does the overdetermined boundary condition (9.2.11) for volume preserving perturbations imply that Ω is a ball? A first step in this direction would be to show that for a critical domain $\partial_\nu u = \text{const.}$ and $\partial_\nu \phi = \text{const.}$ on the boundary.

Problem 9.3. The second variation is rather involved because it depends on two functions u^* and ϕ^* and their shape derivatives. The determination of its sign remains open.

9.3 Notes

The first and the second domain variation of the first Dirichlet eigenvalue of the Laplace operator were derived in [63]. In this paper the second domain variation is expressed in terms of the material derivatives, which makes it difficult to discuss their sign. The Rayleigh–Faber–Krahn inequality implies that the ball is not only a local but also a global minimizer of the first eigenvalue among all domains of given volume [98]. This is also true for the first Robin eigenvalue (see [25, 41]). For further studies see [74].

Daners and Kennedy proved [42] that the ball is the only minimizer. A different approach which requires less regularity has been taken by Buçur and Giacomini [29]. Problems with large α are studied in [85].

The nonlinear eigenvalue problem was studied in [16]. In [17] and for more general problems in [18] it was shown that in the case of Dirichlet boundary conditions the ball has the smallest turning point among all domains of given volume. The Gelfand problem has applications in several fields, like astrophysics and the theory of combustion. It is the stationary solution of the Frank-Kamenetskii model in combustion. In 1973 Joseph and Lundgren [78] completely characterized the structure of the radial solutions of the Gelfand equation with Dirichlet data for all dimensions. A detailed discussion of this equation is found in [118]. Mignot, Murat, and Puel [88] studied the variation of the turning point of the Gelfand problem in the case of Dirichlet data. Henry [76] extended their result to more general nonlinearities.

10 Quantitative estimates

The stability of an optimal domain plays an important role in applications. By stability we mean the existence of a neighborhood of the optimal domain in which there is no other optimal domain. A quantitative inequality gives an estimate for the size of this neighborhood. We focus mainly on nearly spherical domains. A rather general criterion based on the second variation is derived and then applied to various examples.

10.1 Preliminary remark

In the calculus of variations it is well known that a critical point with a positive second variation may not be a strict (isolated) minimum. This is shown by an example taken from [35]. Let $J : L^2(0,1) \to \mathbb{R}$ be given by

$$J(u) := \int_0^1 \sin(u(s))\, ds.$$

Clearly, $u^*(s) := -\frac{\pi}{2}$ is a global minimizer. It is easily checked that:
(1) $J'(u^*)[\varphi] = 0$ and
(2) $J''(u^*)[\varphi, \varphi] = \|\varphi\|_{L^2(0,1)}$.

Property (2) is often referred to as “strict coercivity” with respect to L^2. The family of functions

$$u_\epsilon(s) := \begin{cases} -\frac{\pi}{2} & \text{if } s \in [0, 1-\epsilon], \\ \frac{3\pi}{2} & \text{if } s \in [1-\epsilon, 1] \end{cases}$$

also satisfies $J(u_\epsilon) = J(u^*) = 0$ for all $0 < \epsilon < 1$. Moreover, $\|u^* - u_\epsilon\|_{L^2} = 2\pi\sqrt{\epsilon}$. Thus, in any L^2-neighborhood of u^* there are infinitely many different global minimizers.

The reason for this phenomenon is a lack of differentiability of the functional J. While it is twice differentiable on $L^\infty(0,1)$ it is not on the larger space $L^2(0,1)$. One might be tempted to restrict the functional J to the smaller space $L^\infty(0,1)$. However, then property (2) is not satisfied: There does not exist any $\delta > 0$ such that $J''(u^*)[\varphi, \varphi] \geq \|\varphi\|_{L^\infty(0,1)}$. This is called the “norm discrepancy phenomenon.”

For the domain functionals the situation is similar. They are differentiable in the class of smooth domains; however, the second domain variation typically provides an inequality of the type $\ddot{\mathcal{E}}(0) \geq c\|v\|_{L^2}$. We will show that the norm discrepancy phenomenon cannot occur because of the smoothness of the domain perturbations. Thus, any optimal domain is isolated in a sense that will be specified in the next section.

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10.2 The Fraenkel asymmetry function

Let $\{\Omega_t\}_{|t|<t_0}$ be a family of nearly spherical domains with $|\Omega_t| = |B_R|$ and $\Omega_0 = B_R$. We consider functionals $\mathcal{F}(\Omega_t) := \mathcal{F}(t)$. For small $|t|$ and sufficiently smooth $\mathcal{F}(\cdot)$ the expansion

$$\mathcal{F}(t) = \mathcal{F}(0) + t\dot{\mathcal{F}}(0) + \frac{t^2}{2}\ddot{\mathcal{F}}(0) + o(t^2)$$

holds. Suppose that B_R is optimal, i. e., $\dot{\mathcal{F}}(0) = 0$ and $\ddot{\mathcal{F}}(0) \geq 0$. Our aim is to study the stability of $\mathcal{F}(0)$, in the sense that a small deficiency

$$\delta\mathcal{F}(t) := \frac{\mathcal{F}(t)}{\mathcal{F}(0)} - 1$$

implies that Ω_t deviates only slightly from the ball. A classical tool to measure this deviation is the *Fraenkel asymmetry function*:

$$\mathcal{A}(\Omega) = \inf\left\{\frac{|\Omega \triangle B|}{|\Omega|} : B \text{ is a ball such that } |\Omega| = |B|\right\}, \tag{10.2.1}$$

where $\Omega \triangle B = (\Omega \setminus B) \cup (B \setminus \Omega)$ denotes the symmetric difference of the sets Ω and B. The volume $|\Omega \triangle B|$ can be computed explicitly for nearly spherical domains. For Hadamard perturbations we have $\Phi_t(x) = x + t\rho\nu + \frac{t^2}{2}w + o(t^2)$ on ∂B_R.

If we introduce normal coordinates in a neighborhood of ∂B_R (see (B.1.3)) and $\omega = (w \cdot \nu)$ we find

$$\begin{aligned} |\Omega_t \triangle B_R| &= \oint_{\partial B_R} dS\left(\int_0^{|t\rho+\frac{t^2}{2}\omega|}\left(1 + \frac{n-1}{R}r + o(r)\right)dr\right) \\ &= \oint_{\partial B_R} \left|t\rho + \frac{t^2}{2}\omega\right| + \frac{n-1}{2R}\left|t\rho + \frac{t^2}{2}\omega\right|^2 dS. \end{aligned}$$

Near $t = 0$ we get

$$|\Omega_t \triangle B_R| = |t| \oint_{\partial B_R} |\rho|\, dS + t^2 R(\rho,\omega) + o(t^2),$$

where $|R(\rho,\omega)| \leq c_0$.

Consider the deficiency of the surface area $\mathcal{S}(t) := |\partial\Omega_t|$. Then by Taylor's formula

$$\delta\mathcal{S}(t) = \frac{t^2}{2}\frac{\ddot{\mathcal{S}}(0)}{|\partial B_R|} + o(t^2).$$

If Ω_t satisfies the barycenter condition, Lemma 2.4 implies

$$\ddot{\mathcal{S}}(0) \geq \frac{n+1}{R^2} \oint_{\partial B_R} \rho^2 \, dS \geq \frac{n+1}{R^2} \frac{(\oint_{\partial B_R} \rho \, dS)^2}{|\partial B_R|}.$$

Hence,

$$\delta \mathcal{S}(t) \geq \frac{n+1}{2R^2 |\partial B_R|^2} |\Omega_t \triangle B_R|^2 + o(t^2) \geq \frac{n+1}{2n^2} \mathcal{A}^2(t) + o(t^2).$$

Fusco, Maggi, and Pratelli [60] showed that such an inequality holds for arbitrary domains $\Omega \subset \mathbb{R}^n$. Consequently the ball is an isolated minimizer among all smooth domains of equal volume and the quantitative estimate

$$\mathcal{S}(t) - \mathcal{S}(0) \geq \frac{n+1}{2n^2} \mathcal{A}^2(t) \mathcal{S}(0) + o(t^2)$$

holds.

10.2.1 Stability criterion

In this section we follow an idea in [93]. Let $\{\mathcal{I}(t)\}_{|t|<t_0}$ be a family of smooth functions , such that $\mathcal{I}(0) \neq 0, \dot{\mathcal{I}}(0) = 0$, and $\ddot{\mathcal{I}}(0) > 0$. The aim is to derive a quantitative inequality of the form

$$\mathcal{F}(t) - \mathcal{F}(0) \geq C(\mathcal{I}(t) - \mathcal{I}(0))$$

for a given family of domain functionals $\{\mathcal{F}(t)\}_{|t|<t_0}$. As a consequence a quantitative inequality for $\mathcal{I}$ implies a quantitative inequality for $\mathcal{F}$. To this aim we distinguish between two cases.

(i) $\mathcal{I}(0)$ and $\mathcal{F}(0)$ are of equal sign. Set

$$\delta(t) := \frac{\delta \mathcal{F}(t)}{\delta \mathcal{I}(t)}.$$

We expand $\delta(t)$ with respect to t and take into account that $\dot{\mathcal{F}}(0) = 0$.

$$\delta(t) = \frac{\frac{\ddot{\mathcal{F}}(0)}{\mathcal{F}(0)} + \frac{o(t^2)}{t^2}}{\frac{\ddot{\mathcal{I}}(0)}{\mathcal{I}(0)} + \frac{o(t^2)}{t^2}} = \frac{\mathcal{I}(0)}{\mathcal{F}(0)} \frac{\ddot{\mathcal{F}}(0)}{\ddot{\mathcal{I}}(0)} + o(1).$$

If there exists a positive number $\gamma > 0$ such that

$$\frac{\mathcal{I}(0)}{\mathcal{F}(0)} \frac{\ddot{\mathcal{F}}(0)}{\ddot{\mathcal{I}}(0)} \geq \gamma, \tag{10.2.2}$$

then for small $|t|$

$$\gamma \le \delta(t) = \frac{\frac{\mathcal{F}(t)}{\mathcal{F}(0)} - 1}{\frac{\mathcal{I}(t)}{\mathcal{I}(0)} - 1} = \frac{\mathcal{I}(0)}{\mathcal{F}(0)} \frac{\mathcal{F}(t) - \mathcal{F}(0)}{\mathcal{I}(t) - \mathcal{I}(0)}.$$

This implies

$$\mathcal{F}(t) - \mathcal{F}(0) \ge \gamma \frac{\mathcal{F}(0)}{\mathcal{I}(0)} (\mathcal{I}(t) - \mathcal{I}(0)). \tag{10.2.3}$$

Hence, stability of $\mathcal{I}(0)$ implies stability of $\mathcal{F}(0)$.

(ii) $\mathcal{I}(0)$ and $\mathcal{F}(0)$ are of opposite sign. We restrict ourselves to the case $\mathcal{I}(0) > 0$ and $\mathcal{F}(0) < 0$. We define

$$\delta(t) := -\frac{\delta \mathcal{F}(t)}{\delta \mathcal{I}(t)}.$$

Expansion with respect to t yields

$$\delta(t) = -\frac{\mathcal{I}(0)}{\mathcal{F}(0)} \frac{\ddot{\mathcal{F}}(0)}{\ddot{\mathcal{I}}(0)} + o(1).$$

Assume there is a $\gamma^- > 0$ such that

$$\gamma^- \le \delta(t) = -\frac{\mathcal{I}(0)}{\mathcal{F}(0)} \frac{\mathcal{F}(t) - \mathcal{F}(0)}{\mathcal{I}(t) - \mathcal{I}(0)}. \tag{10.2.4}$$

Then we deduce

$$\mathcal{F}(t) - \mathcal{F}(0) \ge \gamma^- \left| \frac{\mathcal{F}(0)}{\mathcal{I}(0)} \right| (\mathcal{I}(t) - \mathcal{I}(0)). \tag{10.2.5}$$

Clearly, the same result holds if $\mathcal{I}(0) < 0$ and $\mathcal{F}(0) > 0$.

Next we will apply (10.2.3) and (10.2.5) for different domain functions.

10.3 Example

10.3.1 The Robin energy for $\alpha > 0$

Let $\Omega = B_R$, $\alpha > 0$ and consider the Robin energy

$$\mathcal{E}(u, \Omega) = \int_\Omega |\nabla u|^2 - 2G(u)\, dx + \alpha \oint_{\partial\Omega} u^2\, dS,$$

where u solves

$$\begin{aligned} \Delta u + g(u) &= 0 \quad \text{in } \Omega, \\ \partial_\nu u + \alpha u &= 0 \quad \text{in } \partial\Omega. \end{aligned}$$

We assume

$$\int_{\Omega} ug(u) - 2G(u)\,dx < 0.$$

This ensures $\mathcal{E}(0) < 0$.

Let $\{\Omega_t\}_{|t|<t_0}$ be a family of smooth nearly spherical domains such that

$$|\Omega_t| = |B_R| + o(t^2).$$

By Proposition 8.1,

$$\ddot{\mathcal{E}}(0) = au^2(R)\ddot{\mathcal{S}}(0) + \mathcal{F}.$$

Let $\mathcal{S}(t)$ be the surface area of $\partial\Omega_t$. Then a natural choice is $\mathcal{I}(t) = \mathcal{S}(t)$. Since $\mathcal{S}(0) > 0$ and $\mathcal{E}(0) < 0$ we are in the case of Section 10.2.1. We like to verify (10.2.4). Note that

$$-\frac{\mathcal{I}(0)}{\mathcal{E}(0)}\frac{\ddot{\mathcal{E}}(0)}{\ddot{\mathcal{I}}(0)} = -\frac{\mathcal{S}(0)}{\mathcal{E}(0)}\left(au^2(R) + \frac{\mathcal{F}}{\ddot{\mathcal{S}}(0)}\right).$$

For the torsion problem $g = 1$ and $a > 0$ it follows that $\mathcal{F} \geq 0$. In this case,

$$\liminf_{t\to 0} \delta(t) = \liminf_{t\to 0} \frac{\delta(\mathcal{E}(t))}{\delta(\mathcal{S}(t))} = \frac{S(0)}{\mathcal{E}(0)}au^2(R) > 0.$$

Thus, there exists a $t_0 > 0$ such that for all $|t| < t_0$ we have $\delta(t) > \frac{S(0)}{2\mathcal{E}(0)}au^2(R)$. Thus the following proposition holds.

Proposition 10.1. *Assume there exists a $t_0 > 0$ such that the isoperimetric deficit $\delta S(t) > c_0$ for all $|t| < t_0$. Assume $\mathcal{F}$ is nonnegative. Then we have*

$$\mathcal{E}(t) - \mathcal{E}(0) \geq \frac{1}{2}au^2(R)(S(t) - S(0)).$$

Hence, the ball is a strict local minimizer for the Robin energy.

Robin eigenvalue problem

Next we discuss the stability of the lowest eigenvalue $\lambda(t)$ for nearly spherical domains. For this purpose we assume that $\lambda(t)$ is three times continuously differentiable or equivalently v and w are in $C^{3,1}$. We set

$$\delta\lambda(t) := \frac{\lambda(t)}{\lambda(0)} - 1 \quad \text{and} \quad \delta(t) := \frac{\delta\lambda(t)}{\delta\mathcal{S}(t)}.$$

Then expansion of $\lambda(t)$ around $t = 0$ together with the lower bound (9.1.20) for $\ddot{\lambda}(0)$ leads to

$$\begin{aligned}\delta(t) &= \frac{\mathcal{S}(0)\ddot{\lambda}(0)}{\lambda(0)\ddot{\mathcal{S}}(0)} + o(1) \geq \frac{\mathcal{S}(0)u^2(R)}{\lambda(0)} \left\{ \frac{2K^2}{\ddot{\mathcal{S}}(0)} \left[\frac{\alpha}{K} - \frac{1}{\mu_2} \right] \sum_{i=2}^{\infty} b_i^2 + \alpha \right\} \\ &\geq \alpha \frac{\mathcal{S}(0)u^2(R)}{\lambda(0)}.\end{aligned}$$

Recall that $\frac{\alpha}{K} - \frac{1}{\mu_2} > 0$. Hence,

$$\lambda(t) - \lambda(0) \geq \alpha \mathcal{S}(0)u^2(R).$$

Another example of a quantitative estimate is the first Steklov eigenvalue for the bi-Laplace operator (see Chapter 16).

10.3.2 Notes

Quantitative estimates for optimal domains attracted recently a lot of attention. The first results date back to the first half of the nineteenth century, when Bernstein and Bonnesen proved quantitative estimates for the isoperimetric inequality. Fuglede [58] extended their results to higher dimensions. In the last decades there has been growing interest in sharpening the classical isoperimetric inequality in terms of an asymmetric functional. For this purpose Fusco et al. [60] used the Fraenkel asymmetry measure. We refer to [61] and to [62] for a review on this topic. Subsequently, Brasco et al. [26] studied improvements of isoperimetric inequalities related to eigenvalues. More recent results can be found in [27].

11 The Robin eigenvalues for $\alpha < 0$

Eigenvalue problems with negative α are of interest for bounded smooth domains Ω and for its complement $\mathbb{R}^n \setminus \overline{\Omega}$. Domain variations lead to different results than the problems with positive α. It will be shown that the ball and the outer ball are local maximizers for the first eigenvalue $\lambda(\Omega)$ among all nearly spherical and exterior spherical domains of equal volume.

11.1 The interior eigenvalue problem

The equation

$$\begin{aligned} \Delta u + \lambda(\Omega) u &= 0 \quad \text{in } \Omega, \\ \partial_\nu u + \alpha u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{11.1.1}$$

with $\alpha < 0$ was studied by Bareket [20]. The trace inequality (see Section D.1) implies $\lambda_1(\Omega) > -\infty$. By the direct method of the calculus of variations there exists a minimizer $u \in W^{1,2}(\Omega)$ which is a solution to (11.1.1). As for $\alpha > 0$ this problem also has a countable number of eigenvalues tending to infinity. The minimizer is of constant sign and the eigenspace of the smallest eigenvalue is one-dimensional (see Theorem D.1 and Section D.2.2). Hence,

$$\lambda(\Omega) < \lambda_2(\Omega) \leq \cdots \leq \lambda_n(\Omega) \leq \cdots .$$

The variational characterization of the first eigenvalue $\lambda(\Omega)$ is given by

$$\lambda(\Omega) = \inf\left\{ \int_\Omega |\nabla v|^2 \, dx + \alpha \oint_{\partial\Omega} v^2 \, dS : v \in W^{1,2}(\Omega), \int_\Omega v^2 \, dx = 1 \right\}. \tag{11.1.2}$$

Let $v = \frac{1}{\sqrt{|\Omega|}}$. Then

$$\lambda(\Omega) \leq \alpha \frac{|\partial\Omega|}{|\Omega|}.$$

Thus, $\lambda(\Omega)$ is negative and $\lim_{\alpha\to-\infty} \lambda(\Omega) = -\infty$.

By Poincaré's principle (see Theorem D.1), any higher eigenfunction u_i satisfies

$$\int_\Omega u u_i \, dx = 0.$$

Since the first eigenfunction u is of constant sign, u_i for $i > 1$ has to change sign.

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Under the assumption that the eigenfunction of the perturbed domain Ω_t, $u(y,t)$, is normalized by $\int_{\Omega_t} u(y,t)^2\,dy = 1$, the first variation of a simple eigenvalue $\lambda(\Omega)$ is the same as for positive α, namely (see (9.1.8))

$$\dot{\lambda}(0) = \oint_{\partial\Omega} (v \cdot \nu)\big[|\nabla u|^2 - u^2(\lambda(0) + 2\alpha^2 - \alpha(n-1)H)\big]\,dS.$$

Domain variations for nearly spherical domains

Let $\{\Omega_t\}_{|t|<t_0}$ be a family of nearly spherical domains such that $\Omega_0 = B_R$.

We consider the eigenvalue problem

$$\begin{aligned} \Delta_y u(y,t) + \lambda(\Omega_t)u(y,t) &= 0 \quad \text{in } \Omega_t, \\ \partial_{\nu_t} u(y,t) + \alpha u(y,t) &= 0 \quad \text{on } \partial\Omega_t. \end{aligned} \tag{11.1.3}$$

The first eigenfunction in the ball is radial and it is a solution of

$$u''(r) + \frac{n-1}{r}u'(r) + \lambda(B_R)u(r) = 0, \quad u'(R) + \alpha u(R) = 0. \tag{11.1.4}$$

The solution is (see (9.1.16))

$$u(r) = C_n r^{\frac{2-n}{2}} I_{\frac{n-2}{2}}\big(\sqrt{|\lambda(B_R)|}\,r\big), \tag{11.1.5}$$

where I_ν is the modified Bessel function of order $\nu = \frac{n-2}{2}$ and the constants C_n are given by the normalization $\int_{B_R} u^2\,dx = 1$. The eigenvalue $\lambda(B_R)$ is determined implicitly by the boundary condition.

We will use the notation $\lambda := \lambda(0) = \lambda(B_R)$ for the lowest eigenvalue. Its first variation is

$$\dot{\lambda}(0) = -Ku^2(R) \oint_{\partial B_R} (v \cdot \nu)\,dS, \quad \text{where } K = \alpha^2 - \frac{n-1}{R}\alpha + \lambda.$$

In this case, $K < 0$ by (9.1.12). Suppose that $|\Omega_t| > |B_R|$ for small $|t|$. Then

$$\dot{\lambda}(0) > 0.$$

This improves the local monotonicity property derived in [19]. Note that this observation extends partly the result of Giorgi and Smits [67], who proved that $\lambda(\Omega) > \lambda(B_R)$ for any $B_R \subset \Omega$.

Next we want to discuss the second variation of the lowest eigenvalue in balls. The computations of Section 9.1.3 apply. The only difference is that λ is negative and therefore the Bessel functions in u and ϕ have to be replaced by modified Bessel functions.

We recall the Steklov problem (9.1.15) associated to the Robin eigenvalue problem with negative λ and α:

$$\Delta\phi + \lambda\phi = 0 \quad \text{in } B_R, \quad \partial_\nu\phi + \alpha\phi = \mu\phi \quad \text{on } \partial B_R. \tag{11.1.6}$$

It was discussed in Section 8.2.1, where $g'(u(r)) = \lambda < 0$ in this section. Let $\mu_0 < \mu_1 < \cdots$ denote eigenvalues of the eigenvalue equation (11.1.6) without counting their multiplicity (see Section 9.1.3). Since the first eigenfunction of the Robin eigenvalue problem u_1 is an eigenfunction of the Steklov problem corresponding to $\mu = 0$ and it does not change sign, we obtain $\mu = \mu_0 = 0$.

In Lemma 8.2 the eigenfunctions $\phi_{k,i}$ were expanded with respect to the spherical harmonics. The radial coefficients of this expansion with negative λ are given by

$$a_s(r) = a_n r^{\frac{2-n}{2}} I_{s+\frac{n}{2}-1}(\sqrt{|\lambda|}r).$$

Hence for volume preserving perturbations and by (9.1.17) we obtain

$$\ddot{\lambda}(0) = 2u^2(R)K^2 \sum_{i=1}^{\infty} b_i^2 \left[\frac{\alpha}{K} - \frac{1}{\mu_i}\right] + \alpha u^2(R)\ddot{\mathcal{S}}_0(0), \tag{11.1.7}$$

where

$$K = \alpha^2 - \frac{n-1}{R}\alpha + \lambda < 0. \tag{11.1.8}$$

The sum starts with $i = 1$ because the perturbations are volume preserving. Since (9.1.19) holds also for the modified Bessel functions

$$\left[\frac{\alpha}{K} - \frac{1}{\mu_1}\right] = 0.$$

From (11.1.7) and $\mu_i > 0$ we obtain the following upper bound:

$$\ddot{\lambda}(0) \le 2u^2(R)K\alpha \sum_{i=2}^{\infty} b_i^2 + \alpha u^2(R)\ddot{\mathcal{S}}_0(0). \tag{11.1.9}$$

In Lemma 2.1 it was shown that the barycenter condition implies

$$\ddot{\mathcal{S}}_0(0) \ge \frac{n+1}{R^2} \oint_{\partial B_R} (v\cdot\nu)^2 \, dS.$$

Thus,

$$\ddot{\lambda}(0) \le \alpha u^2(R)\left\{2K + \frac{n+1}{R^2}\right\} \oint_{\partial B_R} (v\cdot\nu)^2 \, dS. \tag{11.1.10}$$

Proposition 11.1. *Assume $\alpha < 0$. Then*

$$\ddot{\lambda}(0) \le \alpha u^2(R)\left\{2K + \frac{n+1}{R^2}\right\} \oint_{\partial B_R} (v \cdot \nu)^2 \, dS < 0$$

for all volume preserving perturbations satisfying the barycenter condition.

Proof. We will show the positivity of the bracket $\{2K + \frac{n+1}{R^2}\}$. From (11.1.8) and (11.1.10) this is the case if

$$\alpha^2 + \frac{n-1}{R}|\alpha| + \frac{n+1}{2R^2} > -\lambda. \tag{11.1.11}$$

Step 1. The value $\lambda(B_R)$ is implicitly given as a solution of

$$u'(R) + \alpha u(R) = 0,$$

where $u(r)$ is given in (11.1.5). Thus,

$$\left(-\frac{n-2}{2} + \alpha R\right) I_{\frac{n-2}{2}}(\sqrt{-\lambda}R) + \sqrt{-\lambda}R I'_{\frac{n-2}{2}}(\sqrt{-\lambda}R) = 0.$$

Next we use the rule for derivatives

$$I'_\nu(z) = \frac{\nu}{z} I_\nu(z) + I_{\nu+1}(z),$$

where $I'_\nu(z) = \partial_z I_\nu(z)$. This implies

$$\alpha I_{\frac{n-2}{2}}(\sqrt{-\lambda}R) + \sqrt{-\lambda} I_{\frac{n}{2}}(\sqrt{-\lambda}R) = 0.$$

We apply the recurrence identity

$$I_\nu(z) = \frac{2(\nu+1)}{z} I_{\nu+1}(z) + I_{\nu+2}(z)$$

for $\nu = \frac{n-2}{2}$. This gives

$$\alpha R\sqrt{-\lambda} I_{\frac{n}{2}+1}(\sqrt{-\lambda}R) - (\lambda R - n\alpha) I_{\frac{n}{2}}(\sqrt{-\lambda}R) = 0. \tag{11.1.12}$$

This equation needs to be satisfied by some λ for which also (11.1.11) holds.

Step 2. For $s \in [0, 1]$ we set

$$\ell(s) := s\left(\alpha^2 + |\alpha|\frac{n-1}{R} + \frac{n+1}{2R^2}\right). \tag{11.1.13}$$

We want to show that there exists an $s^* \in [0, 1]$ such that $-\ell(s^*) = \lambda$.

Step 3. We reformulate the problem. We set $y := |\alpha|R > 0$ and

$$z(s) := \sqrt{s\left(y^2 + (n-1)y + \frac{n+1}{2}\right)}.$$

Then $-\ell(s^*)$ is a solution of (11.1.12) if

$$F(s^*) := -I_{\frac{n}{2}+1}(z(s^*))yz(s^*) + I_{\frac{n}{2}}(z(s^*))(z^2(s^*) - ny) = 0. \tag{11.1.14}$$

We want to show that $F(s) < 0$ for s sufficiently small and $F(1) > 0$. The intermediate value theorem implies the existence of an s^* such that (11.1.14) holds.

Step 4. For s small enough such that $0 < z(s) < \sqrt{\frac{n}{2} + 1}$ we have the asymptotic expansion

$$F(s) \sim -nI_{\frac{n}{2}}(z(s))y < 0.$$

This implies $F(s) < 0$ for small s.

Step 5. For $s = 1$ we set $z := z(1)$ and show that

$$F(1) = -I_{\frac{n}{2}+1}(z)yz + I_{\frac{n}{2}}(z)(z^2 - ny) > 0, \tag{11.1.15}$$

where

$$z = \sqrt{y^2 + (n-1)y + \frac{n+1}{2}}.$$

We write (11.1.15) in the equivalent form, using the fact that $z^2 - ny > 0$ for all n and all $y > 0$, namely

$$F_1 := \frac{zI_{\frac{n}{2}}(z)}{I_{\frac{n}{2}+1}(z)} - \frac{yz^2}{z^2 - ny} > 0.$$

Step 6. The Simpson–Spector inequality states (see, e. g., [109, Theorem 2] or [128, formula (1.9)]) that for all $z > 0$ and all $n \geq 2$ there holds

$$\frac{zI_{\frac{n}{2}}(z)}{I_{\frac{n}{2}+1}(z)} \geq \frac{n+1}{2} + \sqrt{z^2 + \frac{n+1}{2}\frac{n+3}{2}}.$$

This lower bound reduces the proof of $F_1 > 0$ to

$$\sqrt{z^2 + \frac{n+1}{2}\frac{n+3}{2}} > \frac{yz^2}{z^2 - ny} - \frac{n+1}{2}.$$

Note that for small y the right-hand side becomes negative. We will show that for all $y \geq 0$,

$$z^2 + \frac{n+1}{2}\frac{n+3}{2} > \left(\frac{yz^2}{z^2 - ny} - \frac{n+1}{2}\right)^2.$$

This is equivalent to

$$y^4 - 2y^3 + \frac{1}{4}(2n^3 + 6n + 8)y^2 + \frac{1}{4}(2n^3 - 6n - 4)y + \frac{1}{4}(n+1)^3 > 0.$$

Note that the sum of the last two terms in parentheses is always positive. For the other terms we note that

$$y^4 - 2y^3 + \frac{1}{4}(2n^3 + 6n + 8)y^2 = y^2\left(y^2 - 2y + \frac{1}{4}(2n^3 + 6n + 8)\right) > 0.$$

This proves $F(1) > 0$ and hence the existence of an s^* such that (11.1.14) holds. □

Thus, among all domains of equal volume the ball is a local maximizer.

Problem 11.1. Is the statement in Proposition 11.1 true for all volume preserving perturbations even without requiring the barycenter condition?

Problem 11.2. Is it true that the ball has the largest first eigenvalue in the class of simply connected domains of prescribed volume?

11.2 The exterior eigenvalue problem

Let $\overline{\Omega}$ be a nearly spherical domain in the sense of Section 8.2. We will write $D := \mathbb{R}^n \setminus \overline{\Omega}$ and call D a *spherical exterior domain* (see Figure 11.1). For $\alpha < 0$ we consider the eigenvalue problem

$$\lambda(D) := \inf\left\{\int_D |\nabla u|^2\,dx + \alpha \oint_{\partial\Omega} u^2\,dS : u \in W^{1,2}(D), \int_D u^2\,dx = 1\right\}. \tag{11.2.1}$$

Lemma 11.1. *Let D be a spherical exterior domain. Then there exists a positive constant $c = c(\alpha)$ such that*

$$\lambda(D) \geq -c(\alpha).$$

Proof. It was shown in [108] that there exists a linear and continuous trace operator $T : W^{1,2}(D) \to W^{\frac{1}{2},2}(\partial\Omega)$ such that $T(u) = u|_{\partial\Omega}$ for $u \in C^1(\overline{D})$. For any Lipschitz boundary $\partial\Omega$ the imbedding $W^{\frac{1}{2},2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ is compact. Hence, the same argument as for bounded domains applies and proves the claim. □

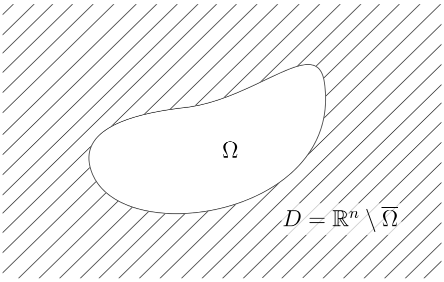

Figure 11.1: Exterior domain.

In [81] the following result for exterior domains D was shown.

Proposition 11.2. *There exists a number $\alpha_\star(D) \le 0$ such that $\lambda(D)$ is a negative discrete eigenvalue if and only if $\alpha < \alpha_\star(D)$. Moreover, $\alpha_\star(D) = 0$ for $n = 2$ and $\alpha_\star(D) < 0$ for $n \ge 3$. In particular, for the exterior of a ball of radius R there holds $\alpha_\star(\mathbb{R}^n \setminus \overline{B_R}) = -\frac{n-2}{R}$.*

It is well known that under the assumptions of Proposition 11.2 there exists an infinite sequence of eigenvalues

$$\lambda_1(D) < \lambda_2(D) \le \cdots$$

with ∞ as the only accumulation point.

For the rest of this section we assume $\alpha < \alpha_\star(D)$ and we set $D_R := \mathbb{R}^n \setminus \overline{B_R}$. From [65, Theorem 8.38] we know that $\lambda(D) := \lambda_1(D)$ is simple and the corresponding eigenfunction is of constant sign. The first eigenfunction in D_R is radial. It solves

$$u''(r) + \frac{n-1}{r}u'(r) + \lambda(D_R)u(r) = 0 \quad \text{in } (R, \infty) \tag{11.2.2}$$

with the boundary conditions

$$u'(R) - \alpha u(R) = 0 \quad \text{and} \quad \lim_{r\to\infty} u(r) = 0. \tag{11.2.3}$$

The solution is

$$u(r) = C_n r^{\frac{2-n}{2}} K_{\frac{n-2}{2}}\left(\sqrt{|\lambda(D_R)|}\,r\right), \tag{11.2.4}$$

where $K_{\frac{n-2}{2}}$ denotes the modified Bessel functions of second kind of order $\frac{n-2}{2}$. The constant C_n is determined by the normalization, while the eigenvalue is implicitly given by the boundary condition.

For given t_0 and $|t| < t_0$ let $\{\Omega_t\}_{|t|<t_0}$ be a family of nearly spherical domains. We assume $\Omega_t = \Phi_t(B_R)$ for some $\Phi_t \in C_0^{1,1}(B_{R_0})$ for all $|t| < t_0$. Here $C_0^{1,1}(B_{R_0}) \subset C_0^{1,1}(\overline{B_{R_0}})$ denotes the subspace of all diffeomorphisms with compact support in B_{R_0}. We set $D_t := \mathbb{R}^n \setminus \Omega_t$; hence, $\{D_t\}_{|t|<t_0}$ is a family of spherical exterior domains.

In this setting we compute the first and second domain variations of $\lambda(D_R)$ for volume preserving perturbations. The first variation is given by (9.1.8), where $\partial\Omega = \partial B_R$. Note that since ∂B_R is the boundary of the outer domain D, ν has to be replaced by $-\nu$ and $H = -\frac{1}{R}$. This implies

$$\dot{\lambda}(0) = u^2(R)\left(\alpha^2 + \alpha\frac{n-1}{R} + \lambda\right)\oint_{\partial B_R}(v\cdot\nu)\,dS.$$

We set

$$K := \alpha^2 + \alpha\frac{n-1}{R} + \lambda. \tag{11.2.5}$$

The next lemma determines the sign of K. The proof uses similar arguments as in Example 9.1.

Lemma 11.2. *We have*

$$K < 0.$$

Proof. **Step 1.** The function $z = \frac{u_r}{u}$ satisfies

$$\frac{dz}{dr} + z^2 + \frac{n-1}{r}z + \lambda = 0 \quad \text{in } (R,\infty). \tag{11.2.6}$$

From (11.2.3) we get the first boundary condition

$$z(R) = \alpha < 0.$$

We claim that the second boundary condition is given by

$$\lim_{r\to\infty} z(r) = -\sqrt{|\lambda(D_R)|}.$$

To prove this asymptotic behavior we write

$$z(r) = \frac{u'(r)}{u(r)} = \frac{\sqrt{|\lambda(D_R)|}K_{\frac{n}{2}}(\sqrt{|\lambda(D_R)|}r)\sqrt{|\lambda(D_R)|}r}{-K_{\frac{n+2}{2}}(\sqrt{|\lambda(D_R)|}r)\sqrt{|\lambda(D_R)|}r + nK_{\frac{n}{2}}(\sqrt{|\lambda(D_R)|}r)}.$$

The recurrence relation

$$K_{\mu-1}(s) = K_{\mu+1}(s) - \frac{2\mu}{s}K_\mu(s)$$

leads to

$$z(r) = -\sqrt{|\lambda(D_R)|}\frac{K_{\frac{n}{2}}(\sqrt{|\lambda(D_R)|}r)}{K_{\frac{n-2}{2}}(\sqrt{|\lambda(D_R)|}r)}.$$

The asymptotic behavior of the Bessel functions K_μ yields

$$\lim_{s\to\infty} \frac{K_{\frac{n}{2}}(s)}{K_{\frac{n-2}{2}}(s)} = 1.$$

This shows

$$\lim_{r\to\infty} z(r) = -\sqrt{|\lambda(D_R)|}.$$

Moreover, since the ratio $\frac{K_{\frac{n}{2}}(s)}{K_{\frac{n-2}{2}}(s)}$ is an increasing function in s we get $\alpha < -\sqrt{|\lambda(D_R)|}$.

Step 2. We consider (11.2.6) in the point $r = R$. Taking into account the boundary condition for z implies

$$\frac{dz}{dr}(R) = -\alpha^2 - \frac{n-1}{R}\alpha - \lambda = -K.$$

Assume $\frac{dz}{dr}(R) < 0$, i. e., $K > 0$. Since

$$z(R) = \alpha < -\sqrt{|\lambda(D_R)|} = \lim_{r\to\infty} z(r),$$

there exists a radius $R < \rho < \infty$ such that

$$z(\rho) < 0, \quad \frac{dz}{dr}(\rho) = 0, \quad \text{and} \quad \frac{d^2z}{dr^2}(\rho) \geq 0.$$

However if we differentiate (11.2.6) with respect to r we get for $r = \rho$

$$\begin{aligned} 0 &= \frac{d^2z}{dr^2}(\rho) + 2z(\rho)\frac{dz}{dr}(\rho) - \frac{n-1}{\rho^2}z(\rho) + \frac{n-1}{\rho}\frac{dz}{dr}(\rho) \\ &= \frac{d^2z}{dr^2}(\rho) - \frac{n-1}{\rho^2}z(\rho) > 0, \end{aligned}$$

which is a contradiction. Therefore it follows $K \leq 0$. The case $K = 0$ can be excluded in a similar way. □

For the exterior domain we also obtain the same upper bound as for the interior problem.

$$\ddot{\lambda}(0) \leq \alpha u^2(R)\left\{2K + \frac{n+1}{R^2}\right\} \oint_{\partial B_R} (v\cdot\nu)^2\, dS. \tag{11.2.7}$$

Note that in this inequality the solution $u(R)$ is expressed in terms of the Bessel functions K_μ rather than I_μ. The following result is analogous to Proposition 11.1.

Proposition 11.3. *Assume $\alpha < \alpha_\star(B_R)$ for $2 \le n \le 5$. Then*

$$\ddot{\lambda}(0) \le \alpha u^2(R)\left\{2K + \frac{n+1}{R^2}\right\} \oint_{\partial B_R} (v \cdot \nu)^2 \, dS < 0$$

for all volume preserving perturbations which also satisfy the barycenter condition.

Proof. From (11.2.7) we deduce that the proposition is proven if we can show that $\{2K + \frac{n+1}{R^2}\}$ is negative. Here K is given in (11.2.5). This is equivalent to

$$2\alpha^2 + 2\frac{n-1}{R}\alpha + \frac{n+1}{R^2} + 2\lambda < 0, \tag{11.2.8}$$

where λ solves $u'(R) - \alpha u(R) = 0$.

Step 1. We argue as in the proof of Proposition 11.1. Thus, the boundary condition for $u(r)$ becomes

$$\left(\frac{2-n}{2} - \alpha R\right)K_{\frac{n-2}{2}}(\sqrt{-\lambda}R) + \sqrt{-\lambda}RK'_{\frac{n-2}{2}}(\sqrt{-\lambda}R) = 0.$$

The differentiation rule

$$K'_\nu(z) = \frac{\nu}{z}K_\nu(z) - K_{\nu+1}(z),$$

where $K'_\nu(z) = \partial_z K_\nu(z)$, implies

$$\sqrt{-\lambda}RK_{\frac{n}{2}}(\sqrt{-\lambda}R) + \alpha RK_{\frac{n-2}{2}}(\sqrt{-\lambda}R) = 0.$$

Finally we apply the recurrence identity

$$K_\nu(z) = -\frac{2(\nu+1)}{z}K_{\nu+1}(z) + K_{\nu+2}(z)$$

for $\nu = \frac{n-2}{2}$. This gives

$$\alpha R^2\sqrt{-\lambda}K_{\frac{n}{2}+1}(\sqrt{-\lambda}R) - K_{\frac{n}{2}}(\sqrt{-\lambda}R)(R^2\lambda + n\alpha R) = 0. \tag{11.2.9}$$

Step 2. For $s \in [0, 1]$ we set

$$\ell(s) := s\left(\alpha^2 + \frac{n-1}{R}\alpha + \frac{n+1}{2R^2}\right) > 0. \tag{11.2.10}$$

Note that the assumptions on α ensure the positivity of $\alpha^2 + \frac{n-1}{R}\alpha + \frac{n+1}{2R^2}$. As in the proof of Proposition 11.1 it remains to show that there exists an $s^* \in [0, 1]$ such that the corresponding $-\ell(s^*)$ is a solution to (11.2.9).

Step 3. In this step we reformulate the problem. Set $y := |a|R$ and

$$z(s) := \sqrt{s\Big(y^2 - (n-1)y + \frac{n+1}{2}\Big)}.$$

Since $a < a_*$ for $2 \le n \le 5$ it follows from (11.2.10) that $y^2 - (n-1)y + \frac{n+1}{2} > 0$. Then (11.2.9) takes the form

$$F(s) := K_{\frac{n}{2}+1}(z(s))yz(s) - K_{\frac{n}{2}}(z(s))\big(z^2(s) + ny\big) = 0. \tag{11.2.11}$$

We claim that $F(s) > 0$ for s sufficiently small and $F(1) < 0$. Hence, the existence of an s^* such that (11.2.11) holds is established.

Step 4. In the sequel the Turan type inequality (see [106, Theorems 1 and 5 and Corollary 3])

$$\frac{\frac{\nu}{2} + \sqrt{\frac{\nu^2}{4} + z^2}}{z} < \frac{K_{\frac{\nu}{2}+1}(z)}{K_{\frac{\nu}{2}}(z)} < \frac{\frac{\nu+1}{2} + \sqrt{\frac{(\nu+1)^2}{4} + z^2}}{z} \tag{11.2.12}$$

for $\nu \ge 0$ and $z \ge 0$ will play a crucial role. To establish the lower bound for $F(s)$ we apply the lower bound in (11.2.12). Thus,

$$\begin{aligned} F(s) &> K_{\frac{n}{2}}(z(s))\Bigg(\frac{\frac{n}{2} + \sqrt{(\frac{n}{2})^2 + z^2(s)}}{z(s)} - \big(z^2(s) + ny\big)\Bigg) \\ &= K_{\frac{n}{2}}(z(s))\Big(\frac{n}{z(s)} - ny + O(s)\Big), \end{aligned}$$

where $O(s) \to 0$ as $s \to 0$. This implies $F(s) > 0$ for sufficiently small s.

Step 5. For $s = 1$ we set $z := z(1)$ and want to show that for $2 \le n \le 5$ and $y > n-2$

$$F(1) = K_{\frac{n}{2}+1}(z)yz - K_{\frac{n}{2}}(z)\big(z^2 + ny\big) < 0, \tag{11.2.13}$$

where

$$z = \sqrt{y^2 - (n-1)y + \frac{n+1}{2}} \quad \text{and} \quad y \ge n-2.$$

The cases $n = 2, 3, 4, 5$ will be discussed separately.

Step 6. If $n = 2$ and $y \geq 0$, then (11.2.13) reads as

$$F(y) := F(1) = K_2(z)yz - K_1(z)\left(y + y^2 + \frac{3}{2}\right) < 0 \tag{11.2.14}$$

where

$$z = \sqrt{y^2 - y + \frac{3}{2}}.$$

We will show that F is an increasing function in y and then we will use asymptotic behavior of K_1 and K_2. We first compute $F'(y)$ using the differentiation rule and the recurrence identity for the modified Bessel functions of second kind:

$$F'(y) = \frac{(-8y^4 + 4y^3 - 4y^2 - 10y + 6)K_1(z) + 4K_0(z)(y^3 + \frac{1}{2}y^2 + \frac{3}{4} + y)z}{8y^2 - 8y + 12}.$$

Note that the denominator of $F'(y)$ is positive. We apply (11.2.12) for $\nu = 0$. Hence,

$$K_0(z) \geq \frac{K_1(z)}{\frac{1}{2} + \sqrt{\frac{1}{4} + z^2}}.$$

We thus obtain the lower bound

$$F'(y) \geq K_1(z)\left(\frac{(-2y^2 - y + 1)\sqrt{4y^2 - 4y + 7} + 4y^3 + y + 4}{2 + 2\sqrt{4y^2 - 4y + 7}}\right).$$

It is easy to verify that the right-hand side is positive. Thus, $F(y)$ is strictly increasing in y. To show that $F(y) < 0$ for all y we apply the asymptotic formula

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}\left(1 + \frac{4\nu^2 - 1}{8z} + o\left(\frac{1}{z}\right)\right)$$

as $z \to \infty$. From (11.2.14) we thus get

$$\begin{aligned} F(y) \sim{} & \sqrt{\frac{\pi}{2z}}e^{-z}\left(1 + \frac{15}{8z} + o\left(\frac{1}{z}\right)\right)yz \\ & - \sqrt{\frac{\pi}{2z}}e^{-z}\left(1 + \frac{3}{8z} + o\left(\frac{1}{z}\right)\right)\left(y^2 + y + \frac{3}{2}\right). \end{aligned}$$

Thus,

$$F(y) \sim -\sqrt{\pi}e^{-z}\frac{2z(8y^2 - 7y + 12) - 16y^3 + 22y^2 - 18y + 9 + o(\frac{1}{z})(yz - y^2 - y - \frac{3}{2})}{8(4y^2 - 4y + 6)^{\frac{3}{4}}}$$

as $y \to \infty$. From this formula it also follows that $\lim_{y\to\infty} F(y) = 0$.

Step 7. Let $n = 3$. In this case we use the explicit representation,

$$K_{\frac{3}{2}}(z) = \frac{\sqrt{\pi}e^{-z}(1 + \frac{1}{z})}{\sqrt{2z}},$$
$$K_{\frac{3}{2}+1}(z) = \frac{\sqrt{\pi}e^{-z}(1 + \frac{3}{z} + \frac{3}{z^2})}{\sqrt{2z}}.$$

For $y \geq 1$, $F(1)$ reads as

$$F(y) := F(1) = \frac{\sqrt{\pi}}{\sqrt{2}}(y^2 - 2y + 2)^{\frac{1}{4}}(y - 1 - \sqrt{y^2 - 2y + 2})e^{-\sqrt{y^2-2y+2}}.$$

Clearly, $F(y) < 0$ and consequently, (11.2.13) holds.

Step 8. We consider the case $n = 4$. To prove (11.2.13) we set

$$z = \frac{\sqrt{4y^2 - 12y + 10}}{2}.$$

For $y \geq 2$ we get

$$F(y) := F(1) = 2\frac{y^2 - 3y + \frac{5}{2}}{\sqrt{4y^2 - 12y + 10}}((y - 2)K_1(z) - zK_0(z)),$$

where again the differentiation rule for the modified Bessel functions of second kind has been used. From (11.2.12) we deduce

$$K_1(z) < \frac{\frac{1}{2} + \sqrt{\frac{1}{4} + z^2}}{z}K_0(z).$$

Then

$$F(y) < 2\frac{y^2 - 3y + \frac{5}{2}}{\sqrt{4y^2 - 12y + 10}}\left((y - 2)\frac{\frac{1}{2} + \sqrt{\frac{1}{4} + z^2}}{z} - z\right)K_0(z).$$

The large parentheses determines the sign of the right-hand side. It is negative if

$$4(y - 2)\left(\frac{1}{2} + \frac{1}{2}\sqrt{4y^2 - 12y + 11}\right) - 4y^2 + 12y - 10 < 0.$$

This follows easily for $y \geq 2$.

Step 9. We consider the case $n = 5$. In this case the following representation hold:

$$K_{\frac{5}{2}}(z) = \frac{((t^2 + 2t + 3)\sqrt{t(t + 2)} + 3t^2 + 6t)\sqrt{\pi}\sqrt{2}e^{-\sqrt{t(t+2)}}}{2(t(t + 2))^{\frac{3}{4}}t(t + 2)},$$

$$K_{\frac{5}{2}+1}(z) = \sqrt{\pi}\sqrt{2}e^{-\sqrt{t(t+2)}}\frac{(t^2+2t+15)\sqrt{t(t+2)}+6t^2+12t+15}{2(t(t+2))^{\frac{3}{4}}t(t+2)}.$$

For $y \geq 3$, $F(1)$ reads as

$$F(y) := F(1) = -\frac{(\sqrt{t(t+2)}t - t^2 + \sqrt{t(t+2)} - 2t)t\sqrt{\pi}\sqrt{2}e^{-\sqrt{t(t+2)}}}{2(t(t+2))^{\frac{3}{4}}}$$

for $y > 3$. Clearly, $F(y) < 0$ for $y > 3$; thus, (11.2.13) holds. □

Remark 11.1. The situation is more delicate if $n > 5$. Inequality (11.2.10) holds if and only if

$$\alpha < \alpha^* := -\frac{1}{2R}(n - 1 + \sqrt{n^2 - 4n - 1}). \tag{11.2.15}$$

It is easy to check that $\alpha^* \leq \alpha_*$ (see Proposition 11.2), with equality if and only if $n = 5$.

The condition $\alpha < \alpha^*$ seems to be too weak to guarantee (11.2.13). Figure 11.2 shows $F(1)$ as a function of $y := |\alpha|R$, where α satisfies (11.2.15). Thus, for $n > 5$ our result is weaker than in [81].

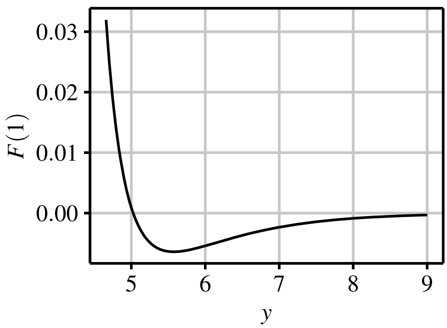

Figure 11.2: The graph of $F(1)$ as a function of y for $n = 6$.

Remark 11.2. L. Bundrock [33] proved Proposition 11.3 for all $n \geq 2$.

Krejcirik [80] has shown that for all $\alpha < 0$ and in the plane the ball is a maximizer among all smooth convex domains of equal volume. The same result holds true if we compare the ball with smooth convex domains of the same perimeter. The situation changes considerably if $n \geq 3$: We have $\lambda(\mathbb{R}^n \setminus \Omega) = 0$ for $\alpha = \alpha_*(\mathbb{R}^n \setminus \Omega)$. For the ball $\alpha_*(\mathbb{R}^n \setminus B_R) = -\frac{n-2}{R}$.

11.3 Notes

Eigenvalue problems with Robin boundary conditions with negative α appear in acoustic analysis. Bareket [20] showed that for nearly circular domains of given area in the plane the circle has the largest first eigenvalue λ for $\alpha < 0$. Her proof was based on the construction of suitable trial functions in the variational characterization. This result was extended to higher dimensions for nearly spherical domains by Ferone, Nitsch, and Trombetti [48]. The question whether or not the ball is optimal for all domains of the same volume remained open until Freitas and Krejcirik in [57] showed that annuli have a larger eigenvalue than the ball with the same volume for sufficiently small $\alpha < 0$ (see also [4]). Further properties of this problem are found in [19]. The exterior problem is discussed in a series of papers [80, 81, 33].

12 Problems with infinitely many positive and negative eigenvalues

This chapter deals with an eigenvalue problem which possesses two sequences of eigenvalues, one positive and one negative. The peculiarity of the problem is that the eigenvalues appear both in the equation in the domain and in the boundary condition. The smallest positive and the largest negative eigenvalue share some properties with the lowest Robin eigenvalue. Under certain conditions the ball is a global minimizer for the smallest positive eigenvalue and a local minimizer for the largest negative eigenvalue.

12.1 Eigenvalue problem with dynamical boundary conditions

A reaction–diffusion problem with dynamical boundary conditions gives rise to the eigenvalue problem

$$\Delta u + \lambda u = 0 \text{ in } \Omega, \quad \partial_\nu u = \lambda\sigma u \text{ in } \partial\Omega, \tag{12.1.1}$$

where $\sigma, \lambda \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. The special feature of this problem is that the eigenvalue appears in the domain and on the boundary. Except for $\lambda_0 = 0$ they are critical values of the functional

$$J_\sigma(v) := \frac{\int_\Omega v^2\,dx + \sigma\int_{\partial\Omega} v^2\,dS}{\int_\Omega |\nabla v|^2\,dx}, \quad v \in W^{1,2}(\Omega).$$

For positive σ, $J_\sigma(v)$ is positive and the classical theory for symmetric operators applies. François [55] showed that in this case the spectrum consists of countably many eigenvalues, which are bounded from below and tend to infinity. The lowest eigenvalue is zero and the corresponding eigenfunction is the constant.

In this section we are interested in the problem with negative σ. It has been studied in [12]. A more general approach is found in [45]. It is known that in addition to $\lambda = 0$ there exist two sequences of eigenvalues, one tending to $+\infty$ and the other tending to $-\infty$. The eigenfunctions are complete in $W^{1,2}(\Omega)$ except in the *resonance case* $|\Omega| + \sigma|\partial\Omega| = 0$.

The motivation to study this problem comes from the parabolic equations with dynamical boundary conditions

$$\begin{aligned} \partial_t u - \Delta u &= 0 \quad \text{in } (0,\infty)\times\Omega, \\ \sigma\partial_t u + \partial_\nu u &= 0 \quad \text{in } (0,\infty)\times\partial\Omega, \\ u(x,0) &= u_0(x) \quad \text{in } \Omega. \end{aligned}$$

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It is known that they are well posed for positive σ in the space $C([0,T], W^{1,2}(\Omega))$ in the sense of Hadamard. Moreover, there exists a smooth solution globally in time, whereas for $\sigma < 0$ this is not the case in dimensions $n \geq 2$.

12.2 Known results

Some results for the case $\sigma < 0$ are summarized below. The results presented here are taken from [12]. For $u, v \in W^{1,2}(\Omega)$ let

$$a(u,v) := \int_{\Omega} uv\,dx + \sigma \int_{\partial\Omega} uv\,dS$$

be an inner product on $L^2(\Omega) \oplus L^2(\partial\Omega)$. We define

$$\mathcal{K} := \left\{ u \in W^{1,2}(\Omega) : \int_{\Omega} |\nabla u|^2\,dx = 1, a(u,1) = 0 \right\}.$$

Moreover, let $(u,v) := \int_{\Omega}(\nabla u \cdot \nabla v)\,dx$. We consider critical points of the functional $\mathcal{J}_\sigma$ on $\mathcal{K}$. Any critical point solves (12.1.1) in the weak sense:

$$(u,v) = \lambda a(u,v) \quad \text{for all } v \in W^{1,2}(\Omega).$$

In [12] the authors showed the existence of two infinite sequences of eigenvalues. One sequence consists of negative eigenvalues $\{\lambda_{-k}\}_k$ and the other of positive eigenvalues $\{\lambda_k\}_k$. The corresponding eigenfunctions $\{u_{\pm k}\}_k \in \mathcal{K}$ solve (12.1.1). The eigenvalues are ordered like

$$\cdots \leq \lambda_{-k} \leq \cdots \leq \lambda_{-1} < \lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots.$$

Note that they depend on σ. For the case $n > 1$ one calculates the following asymptotes:

$$\lim_{k\to\infty} \lambda_{-k} = -\infty \quad \text{and} \quad \lim_{k\to\infty} \lambda_k = \infty.$$

In case $n = 1$ there are infinitely many positive eigenvalues but only finitely many negative ones.

We are interested in the bottoms of the positive and the negative spectra, λ_1 and λ_{-1}, for which the following theorem holds.

Theorem 12.1. 1. *If $\sigma < \sigma_0 = -\frac{|\Omega|}{|\partial\Omega|}$, then λ_1 is simple and*

$$\frac{1}{\lambda_1(\sigma)} = \sup_{v\in\mathcal{K}} \left\{ \int_{\Omega} v^2\,dx + \sigma \oint_{\partial\Omega} v^2\,dS \right\}$$

2. *If $\sigma_0 < \sigma < 0$, then λ_{-1} is simple and*

$$\frac{1}{\lambda_{-1}(\sigma)} = \inf_{v\in\mathcal{K}} \left\{ \int_\Omega v^2\,dx + \sigma \oint_{\partial\Omega} v^2\,dS \right\}.$$

Lemma 12.1. *The eigenvalue $\lambda_1(\sigma)$ is for $\sigma < \sigma_0$ and $\lambda_{-1}(\sigma)$ is for $0 > \sigma > \sigma_0$ monotone decreasing in σ.*

Proof. We assume $\sigma_1 > \sigma_2$. We distinguish two cases.

Case 1. From the characterization of λ_1 we get

$$\frac{1}{\lambda_1(\sigma_1)} \geq \int_\Omega u^2\,dx + \sigma_1 \int_{\partial\Omega} u^2\,dS \geq \int_\Omega u^2\,dx + \sigma_2 \int_{\partial\Omega} u^2\,dS.$$

For u we choose the eigenfunction of $\lambda_1(\sigma_2)$ and we obtain

$$\frac{1}{\lambda_1(\sigma_1)} \geq \frac{1}{\lambda_1(\sigma_2)}.$$

This gives $\lambda_1(\sigma_1) \leq \lambda_1(\sigma_2)$.

Case 2. From the characterization of λ_{-1} we get

$$\frac{1}{\lambda_{-1}(\sigma_2)} \leq \int_\Omega u^2\,dx + \sigma_2 \int_{\partial\Omega} u^2\,dS \leq \int_\Omega u^2\,dx + \sigma_1 \int_{\partial\Omega} u^2\,dS.$$

In this case we choose u as the eigenfunction of $\lambda_{-1}(\sigma_1)$ and we obtain

$$\frac{1}{\lambda_{-1}(\sigma_2)} \leq \frac{1}{\lambda_{-1}(\sigma_1)}.$$

Since $\lambda_{-1}(\sigma) < 0$, we have $\lambda_{-1}(\sigma_1) \leq \lambda_{-1}(\sigma_2)$. □

Remark 12.1. 1. In [12] the authors considered the map $\sigma \to \lambda(\sigma)$ given by

$$\lambda(\sigma) = \begin{cases} \lambda_1(\sigma) & \text{if } \sigma < \sigma_0, \\ 0 & \text{if } \sigma = \sigma_0, \\ \lambda_{-1}(\sigma) & \text{if } 0 > \sigma > \sigma_0. \end{cases} \tag{12.2.1}$$

They proved that it is a smooth curve satisfying

$$\lim_{\sigma\to-\infty} \lambda(\sigma) = \lambda_1^D \quad \text{and} \quad \lim_{\sigma\to 0} \lambda(\sigma) = -\infty,$$

where λ_1^D is the first Dirichlet eigenvalue for the Laplacian.

2. For $0 > \sigma > \sigma_0$, $\lambda_1(\sigma)$ is strictly decreasing and $\lim_{\sigma\to 0} \lambda_1(\sigma) = \lambda_1^N(\Omega)$, where $\lambda_1^N(\Omega)$ is the first nontrivial Neumann eigenvalue of the Laplacian.

   The different eigenvalues as functions of σ are shown in Figure 12.1.

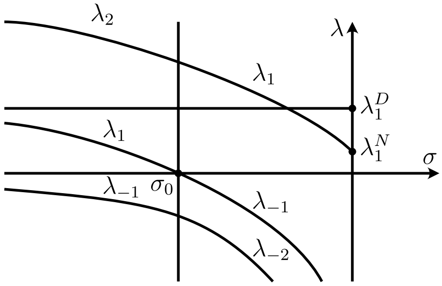

Figure 12.1: Eigenvalues as functions of σ.

We are interested in the domain dependence of $\lambda_{\pm 1}(\sigma, \Omega)$. Note that $\sigma_0 = \sigma_0(\Omega)$ depends on Ω as well. In particular for domains of given volume and for a ball B_R with the same volume the isoperimetric inequality gives

$$\sigma_0(\Omega) = -\frac{|\Omega|}{|\partial\Omega|} = -\frac{|B_R|}{|\partial\Omega|} \geq -\frac{|B_R|}{|\partial B_R|} = \sigma_0(B_R). \tag{12.2.2}$$

12.3 Comparison with balls of the same volume

By exploiting previous results on the Robin eigenvalue problem we can prove the following isoperimetric inequality.

Theorem 12.2. *Let $|\Omega| = |B_R|$ and assume $\sigma < \sigma_0(B_R) < 0$. Then*

$$\lambda_1(\sigma, \Omega) \geq \lambda_1(\sigma, B_R).$$

Equality holds if and only if $\Omega = B_R$.

Proof. Let $\lambda^R(\alpha, \Omega)$, $\alpha > 0$, be the lowest Robin eigenvalue defined in (11.1.1). The variational characterization (11.1.2) implies that $\lambda^R(\alpha, \Omega)$ is concave and increasing as a function of α and $\lambda^R(\alpha, \Omega) \leq \lambda_1^D(\Omega)$. We claim that

$$\lim_{\alpha\to\infty} \lambda^R(\alpha, \Omega) = \lambda_1^D(\Omega), \tag{12.3.1}$$

where $\lambda_1^D(\Omega)$ denotes the first Dirichlet eigenvalue of the Laplacian. This can be seen as follows. For any sequence $\{\alpha_i\}_{i\in\mathbb{N}}$ with $\lim_{i\to\infty} \alpha_i = \infty$ there exists a subsequence $\{\alpha_{i_k}\}_{k\in\mathbb{N}}$ such that the sequence $\{\lambda^R(\alpha_{i_k}, \Omega)\}_{k\in\mathbb{N}}$ converges to some $0 < \hat{\zeta} \leq \lambda_1^D(\Omega)$. By scaling we may assume that the corresponding eigenfunctions are normalized, i. e., $\|\varphi_k\|_{L^2(\Omega)} = 1$. Since φ_k solves

$$\int_\Omega (\nabla\varphi_k \cdot \nabla v)\,dx + \alpha_{i_k} \int_{\partial\Omega} \varphi_k v\,dS = \lambda^R(\alpha_{i_k}, \Omega) \int_\Omega \varphi_k v\,dx \quad \forall\, v \in W^{1,2}(\Omega), \tag{12.3.2}$$

we easily get the following convergence results:

(i) The sequence $\{\varphi_k\}_k$ is bounded in $W^{1,2}(\Omega)$; hence, there exists $\hat{\varphi} \in W^{1,2}(\Omega)$ such that $\varphi_k \to \hat{\varphi}$ as $k \to \infty$ weakly in $W^{1,2}(\Omega)$ and thus strongly in $L^2(\Omega)$. In particular, $\|\hat{\varphi}\|_{L^2(\Omega)} = 1$.

(ii) Since

$$\int_{\partial\Omega} |\varphi_k|^2\,dS \le \frac{\lambda^R(\alpha_{i_k}, \Omega)}{\alpha_{i_k}} \to 0 \quad \text{as } k \to \infty,$$

the compactness of the trace operator gives $\hat{\varphi} = 0$ a. e. on $\partial\Omega$. Hence, $\hat{v} \in W_0^{1,2}(\Omega)$.

(iii) We can pass to the limit $k \to \infty$ in (12.3.2) and get

$$\int_\Omega \nabla\hat{\varphi} \cdot \nabla v\,dx = \hat{\zeta} \int_\Omega \hat{\varphi} v\,dx \quad \forall\, v \in W_0^{1,2}(\Omega). \tag{12.3.3}$$

From $\hat{\varphi} \in W_0^{1,2}(\Omega)$ and the fact that $\hat{\varphi}$ solves (12.3.3) we deduce that $\hat{\varphi}$ is an eigenfunction of the Dirichlet Laplace operator and $\hat{\zeta}$ is a Dirichlet eigenvalue. Since $0 < \hat{\zeta} \le \lambda_1^D(\Omega)$ we necessarily get $\hat{\zeta} = \lambda_1^D(\Omega)$. This establishes (12.3.1).

The eigenvalue $\lambda_1(\sigma, \Omega)$ coincides with $\lambda^R(\alpha, \Omega)$ for $\alpha = -\sigma\lambda_1(\sigma, \Omega)$. Hence, it solves the equation

$$\lambda^R(|\sigma|\lambda, \Omega) = \lambda \quad \text{or equivalently} \quad \lambda^R(\alpha, \Omega) = \frac{\alpha}{|\sigma|}.$$

This equation has a solution if and only if (see Figure 12.2)

$$\frac{1}{|\sigma|} < \frac{d}{d\alpha}\lambda^R(\alpha, \Omega)\bigg|_{\alpha=0} = \frac{|\Omega|}{|\partial\Omega|}.$$

Consequently, we must have $\sigma < \sigma_0(\Omega)$. From the Bossel–Daners inequality it follows that $\lambda^R(\alpha, \Omega) \ge \lambda^R(\alpha, B_R)$. Since by assumption $\sigma < \sigma_0(B_R)$, the equation $\frac{\alpha}{|\sigma|} = \lambda^R(\alpha, B_R)$ has a solution α_1 which is smaller than the solution α_2 of $\frac{\alpha}{|\sigma|} = \lambda^R(\alpha, \Omega)$. The assertion now follows from

$$\alpha_1 = -\sigma\lambda_1(\sigma, B_R) \le \alpha_2 = -\sigma\lambda_1(\sigma, \Omega). \qquad \square$$

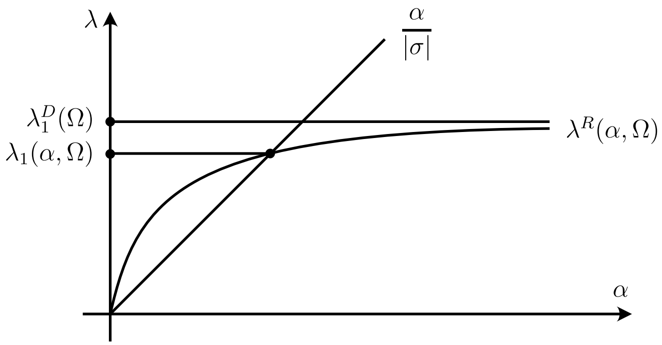

Figure 12.2: Construction of $\lambda_1(\alpha,\Omega)$.

The claim of Theorem 12.2 cannot be true for all $0 < \sigma < \sigma_0$ because by Remark 12.1 $\lambda_1(\sigma,\Omega)$ is close to $\lambda_1^N(\Omega)$. The Szegö–Weinberger inequality [124] states that $\lambda_1^N(\Omega) \le \lambda_1^N(B_R)$, which suggests that for small $|\sigma|$, $\lambda_1(\sigma,\Omega)$ is also smaller than $\lambda_1(\sigma,B_R)$.

Next we consider the negative eigenvalue $\lambda_{-1}(\sigma,\Omega_t)$ in $\sigma_0 < \sigma < 0$, where Ω_t is a nearly spherical domain. As mentioned before, $\lambda_{-1}(\sigma,\Omega_t)$ is simple. A similar argument as for the previous theorem leads to the following theorem.

Theorem 12.3. *Assume $\sigma_0(\Omega_t) < \sigma < 0$ and let Ω_t be a nearly spherical domain of the same volume as B_R. Then*

$$0 > \lambda_{-1}(\sigma,\Omega_t) \ge \lambda_{-1}(\sigma,B_R).$$

Proof. Since $\sigma\lambda_{-1}(\sigma,\Omega_t)$ is positive, $\lambda_{-1}(\sigma,\Omega_t)$ coincides with $\lambda^R(-\alpha,\Omega_t)$, where $\alpha = \sigma\lambda_{-1}(\sigma,\Omega_t)$ (see Figure 12.3). Hence, $\lambda_{-1}(\sigma,\Omega_t)$ satisfies

$$\lambda^R\big(-\sigma\lambda_{-1}(\sigma,\Omega_t),\Omega_t\big) = \lambda_{-1}(\sigma,\Omega_t).$$

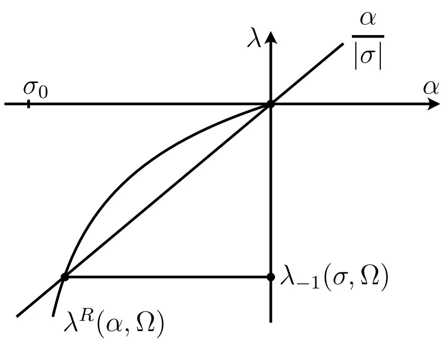

Figure 12.3: Construction of $\lambda_{-1}(\sigma,\Omega)$.

For negative α we have by Proposition 11.1, $\lambda^R(\alpha,\Omega_t) \le \lambda^R(\alpha,B_R)$. If $\sigma > \sigma_0(\Omega_t) > \sigma(B_R)$, then the equations

$$\lambda^R(-\sigma Z,\Omega_t) = Z \quad \text{and} \quad \lambda^R(-\sigma z,B_R) = z$$

have respectively a unique solution $Z \ge z$. This establishes the claim. □

In [85] it is shown that for $\alpha \to -\infty$, $\lambda^R(\alpha,\Omega) = -\alpha^2 + o(\alpha^{-2})$. This implies that

$$\lim_{\sigma\to 0} \sigma^2 \lambda_{-1}(\sigma,\Omega) = -1.$$

12.4 Notes

The parabolic problem has been studied by various authors (cf. for instance [46, 77], and [122]). A discussion of eigenvalue problems with eigenvalues in the interior and on the boundary is found in [45] (cf. also [121]). Generalizations to elliptic operators are treated in [11]. Monotonicity properties and isoperimetric inequalities for λ_1 and λ_{-1} are discussed in [11].

13 The torsion problem for $\alpha < 0$

In contrast to the problem with positive α, a solution of the torsion problem for negative α exists only under an additional compatibility condition. Special attention is paid to the energy of nearly spherical domains of equal volume. The stability of the ball is affected. The energy does not have a local minimum for the ball anymore. The sign of the second variation depends not only on the size of α but also on the dimension.

In the last part we split $u = h + z$, where h is a harmonic function satisfying a nonhomogeneous Robin boundary condition and z is the torsion function with homogeneous Dirichlet boundary conditions. This splitting leads to an alternative discussion of the stability analysis of the energy. At the end we discuss a related functional for which domain variations lead to a global result.

13.1 General remarks

We study the torsion problem

$$\Delta u + 1 = 0 \text{ in } \Omega, \quad \partial_\nu u + \alpha u = 0 \text{ on } \partial\Omega, \tag{13.1.1}$$

where $\alpha < 0$. Integration of this equation leads to

$$|\Omega| = \alpha \oint_{\partial\Omega} u \, dS.$$

In contrast to the case where $\alpha > 0$, the solution must therefore take on negative values.

The associated Steklov eigenvalue problem (8.1.1) is

$$\Delta\phi = 0 \text{ in } \Omega, \quad \partial_\nu \phi + \alpha\phi = \mu\phi \text{ in } \partial\Omega,$$

with the eigenvalues

$$\alpha = \mu_0 < \mu_1 < \cdots .$$

Testing problem (13.1.1) with the corresponding eigenfunctions ϕ_i and $i \geq 1$, we obtain

$$\begin{aligned} -\int_\Omega \phi_i \, dx &= \int_\Omega [\phi_i \Delta u - u\Delta\phi_i] \, dx = \oint_{\partial\Omega} [\phi_i \partial_\nu u - u \partial_\nu \phi_i] \, dS \\ &= -\mu_i \oint_{\partial\Omega} u\phi_i \, dS. \end{aligned} \tag{13.1.2}$$

This is a compatibility condition for u. If

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$$\int_\Omega \phi_i \, dx = 0 \quad \text{for some } i \in \mathbb{N}, \tag{13.1.3}$$

then either $\mu_i = 0$ or $\oint_{\partial\Omega} u\phi_i \, dS = 0$. If it is satisfied, then for all $\beta \in \mathbb{R}$, $u + \beta\phi_i$ is also a solution of (13.1.1). The solution is therefore not unique.

Any solution of (13.1.1) is a critical point of the energy

$$\mathcal{E}_R(u,\Omega) := \int_\Omega |\nabla u|^2 \, dx - 2\int_\Omega u \, dx + \alpha \oint_{\partial\Omega} u^2 \, ds \tag{13.1.4}$$

for $u \in W^{1,2}(\Omega)$. If $\mu_i = 0$ and if there exists a solution of the form $u + \beta\phi_i$, then

$$\mathcal{E}_R(u + \beta\phi_i, \Omega) = \mathcal{E}_R(u, \Omega).$$

Example 13.1. Let $\Omega = B_R$. The Steklov eigenfunctions for the ball are of the form $c_k r^k Y_k(\xi)$, where $\xi \in \partial B_1$, $k \in \mathbb{N} \cup \{0\}$, and $Y_k(\xi)$ are the spherical harmonics of degree k. The corresponding eigenvalues are $\mu_k = \frac{k}{R} + \alpha$, $\forall k$, without counting multiplicity. The compatibility condition (13.1.3) is satisfied for all ϕ_k, $k \geq 1$. Consequently (13.1.1) has a solution for all $\alpha \neq 0$ in the ball . It is of the form

$$u = \begin{cases} \frac{R^2}{2n} + \frac{R}{\alpha n} - \frac{r^2}{2n} & \text{if } 0 \neq \mu_j,\ j = 0, 1, 2, \dots, \\ \frac{R^2}{2n} + \frac{R}{\alpha n} - \frac{r^2}{2n} + \psi & \text{if } 0 = \mu_j, \end{cases}$$

where ψ is any function in the eigenspace of μ_j. In both cases we get

$$\mathcal{E}_R = -\int_{B_R} u \, dx = -|B_R| \left(\frac{R^2}{n(n+2)} + \frac{R}{\alpha n} \right). \tag{13.1.5}$$

In Figure 13.1 radial solutions are shown for $R = 1$ and different values of α.

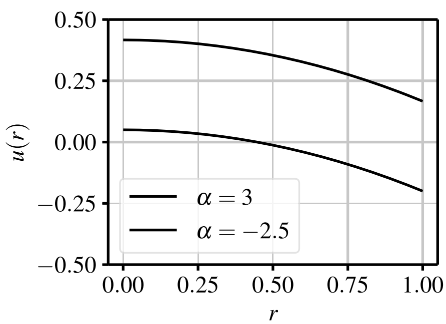

Figure 13.1: Radial solutions for $n = 2$.

13.2 Domain variations for nearly spherical domains

We now consider the torsion problem (13.1.1) in nearly spherical domains $\{\Omega_t\}_{|t|<t_0}$. From the smoothness of Φ_t we deduce $\mu_i(\Omega_t) = \mu_i(B_R) + O(|t|) \neq 0$ for all $i = 0, 1, 2, \ldots$ and for all $|t| < t_0$ (see, e. g., [38, Corollary 4.2]). If $\mu_i(\Omega_t) \neq 0$ then the torsion problem has a unique solution for each t. Moreover these solutions are differentiable with respect to t.

The formal computation of the domain variations of the energy $\mathcal{E}(u, \Omega_t) =: \mathcal{E}(t)$ in (13.1.4) is independent of the sign of α and has been carried out in Section 8.2.2 for positive α. The first variation is given in (8.2.14) and has the form

$$\dot{\mathcal{E}}_R(0) = -\frac{R}{an^2}(\alpha R + n + 1) \oint_{\partial B_R} (v \cdot \nu)\, dS.$$

This leads to the following result.

Proposition 13.1. *Let Ω_t be a family of nearly spherical domains such that $|\Omega_t| = |B_R| + o(t^2)$ and $\Omega_0 = B_R$. Assume that $\mu_i(\Omega_t) \neq 0$ for $i \geq 0$. Then $\dot{\mathcal{E}}(0) = 0$.*

Furthermore, the following local monotonicity property holds.

Corollary 13.1. *If $0 < \alpha R + n + 1$ and $|\Omega_t| > |B_R|$, then $\dot{\mathcal{E}}_R(0) > 0$; otherwise, if $\alpha R + n + 1 < 0$, then $\dot{\mathcal{E}}_R(0) < 0$.*

Note that $-\alpha R = n + 1$ is equivalent to $\mu_{n+1}(B_R) = 0$.

13.2.1 Second variation

The second variation for volume preserving perturbations was computed in (8.2.7) for problems with a nonlinearity $g(u)$. The formal computations are independent of the sign of α. The torsion problem for $\alpha > 0$ was treated in Section 8.2.2. Recall that u' stands for the shape derivative satisfying

$$\begin{aligned} \Delta u' &= 0 \quad \text{in } B_R, \\ \partial_\nu u' + au' &= \left(\frac{1 + \alpha R}{n}\right)(v \cdot \nu) \quad \text{on } \partial B_R. \end{aligned}$$

By (8.2.7) we have

$$\ddot{\mathcal{E}}_R(0) = -2Q(u') + \frac{2R}{n^2}(1 + \alpha R) \int_{\partial B_R} (v \cdot \nu)^2\, dS + \frac{R^2}{an^2}\ddot{\mathcal{S}}(0). \tag{13.2.1}$$

In this case $Q(u') := \int_{B_R} |\nabla u'|^2\, dx + \alpha \int_{\partial B_R} u'^2\, dS$. Testing the equation for u' with ϕ_k we get

$$\left(\frac{1+\alpha R}{n}\right)\oint_{\partial B_R} \phi_k (v\cdot\nu)\,dS = \mu_k \oint_{\partial B_R} \phi_k u'\,dS = \left(\frac{k}{R}+\alpha\right)\oint_{\partial B_R} \phi_k u'\,dS.$$

If $\mu_k(B_R) \neq 0$, the shape derivative exists, whereas if $\mu_k = 0$, it exists only under the assumption

$$\oint_{\partial B_R} (v\cdot\nu)\phi_k\,dS = 0.$$

As in Section 8.2 we set

$$u'(x) = \sum_{k=0}^{\infty} c_k\phi_k \quad \text{and} \quad (v\cdot\nu) = \sum_{k=0}^{\infty} b_k\phi_k,$$

where by (8.2.11)

$$b_k = \frac{nc_k\mu_k}{1+\alpha R} \quad \text{for } k = 0,1,2,\ldots.$$

Since the perturbation is volume preserving, we deduce that $b_0 = c_0 = 0$. By (8.2.15),

$$\ddot{\mathcal{E}}_R(0) = \frac{R^2}{\alpha n^2}\ddot{\mathcal{S}}_0(0) + 2\sum_{k=1}^{\infty} b_k^2 \frac{R(1+\alpha R)(k-1)}{n^2(k+\alpha R)},$$

where $\ddot{\mathcal{S}}_0(0)$ is given in (2.3.27).

Lemma 13.1. *The expansion of $\ddot{\mathcal{S}}_0(0)$ into a series of Steklov eigenvalues leads to*

$$\ddot{\mathcal{S}}_0(0) := \oint_{\partial B_R} \left[|\nabla^\tau\rho|^2 - \frac{n-1}{R^2}\rho^2\right] dS = R^{-2}\sum_{k=0}^{\infty}(k(n-2+k)-n+1)b_k^2.$$

Proof. The normalized Steklov eigenfunctions are of the form

$$\phi_k = \frac{1}{R^{\frac{n-1}{2}+k}} r^k Y_k(\xi).$$

Then by the orthonormality of the spherical harmonics,

$$\oint_{\partial B_R} |\nabla^\tau\rho|^2\,dS = \sum_{k,i=0}^{\infty} R^{-2}\oint_{\partial B_1}(\nabla^\tau Y_k\cdot\nabla^\tau Y_i)\,dS = \sum_{k=0}^{\infty} R^{-2}\Lambda_k b_k^2,$$

where by (3.1.2), $\Lambda_k = k(n-2+k)$. Moreover,

$$\oint \rho^2\,dS = \sum_{k=0}^{\infty} b_k^2.$$

This establishes the assertion. □

We insert the expansion for $\ddot{\mathcal{S}}_0(0)$ into $\ddot{\mathcal{E}}_R(0)$ and obtain

$$\ddot{\mathcal{E}}_R(0) = \sum_{k=1}^{\infty} \frac{b_k^2}{\alpha n^2} \underbrace{\left\{2(1+\alpha R)^2\left(\frac{\alpha R}{1+\alpha R} - \frac{\alpha R}{k+\alpha R}\right) + k(k+n-2) - n + 1\right\}}_{\ell_k}.$$

For the analysis of $\ell_k(\xi)$, $k = 1, 2, 3, \ldots$, we set $\xi := -\alpha R > 0$. Hence,

$$\begin{aligned} \ell_k = \ell_k(\xi) &= \frac{2\xi(\xi-1)(k-1)}{k-\xi} + k(k+n-2) - n + 1 \\ &= (k-1)\left[\frac{2\xi(\xi-1)}{k-\xi} + k + n - 1\right]. \end{aligned} \tag{13.2.2}$$

Figure 13.2 shows ℓ_k as a function of ξ for $k = n = 3$.

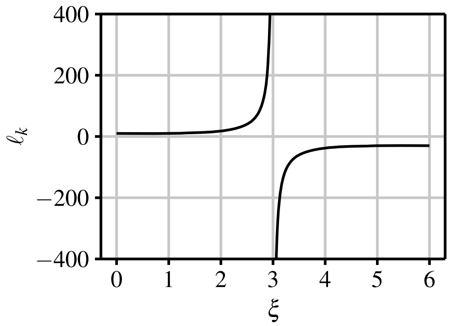

Figure 13.2: The function $\ell_k(\xi)$ with $k, n = 3$.

Clearly, $\ell_1(\xi) = 0$ for all ξ. For $k > 1$ note that $\ell_k(\xi) > 0$ if $0 < \xi < k$. If $\xi > k$, the sign of $\ell_k(\xi)$ is more involved. We have

$$\lim_{\xi \searrow k} \ell_k(\xi) = -\infty \quad \text{and} \quad \lim_{\xi \to \infty} \ell_k(\xi) = -\infty.$$

The maximum of $\ell_k(\xi)$ for $\xi > k$ is achieved at $\xi_m = k + \sqrt{k^2 - k}$ and takes the value

$$\ell(\xi_m) = n + 1 - 3k - 4\sqrt{k^2 - k}.$$

Since

$$\min_{k \geq 2}\left(3k + 4\sqrt{k^2 - k} - 1\right) = 10.65685425,$$

we have for all $k \geq 2$ and $\xi > k$

$$\ell_k(\xi, n) < 0 \quad \text{provided } n \leq 9.$$

On the other hand, if $n > 10$ we can find for any given $k_0 \geq 2$ an integer n_0 such that $\ell_{k_0}(\xi, n_0) > 0$ for ξ in a neighborhood of $\xi_m(k_0, n_0)$.

These observations lead to the following result concerning the sign of $\ddot{\mathcal{E}}(0)$.

Theorem 13.1. *Let $\mathcal{E}$ be the torsion energy given in* (13.1.4) *and let $\alpha < 0$. Then for volume preserving perturbations the following statements hold:*
(i) *Let $0 < -\alpha R < 2$, $-\alpha \neq \frac{1}{R}$, and $n \leq 10$. Then $\ddot{\mathcal{E}}(0) \leq 0$.*
(ii) *Assume $k < -\alpha R < k+1$, $k \geq 2$, and $n \leq 9$. If $b_j = 0$ for $j = 2, .., k-1$, then $\ddot{\mathcal{E}}(0) \geq 0$.*
(iii) *If $n > 10$, there exists a value $\hat{\xi}$ such that $\ell_k(\hat{\xi}) > 0$ for some $k \geq 2$.*

The last assertion implies that $\mathcal{E}(0)$ has a saddle point.

Example 13.2. Let $\Omega_t \subset \mathbb{R}^2$ be the ellipse whose boundary $\partial\Omega$ is given by

$$\left\{\frac{R\cos(\theta)}{1+t}, (1+t)R\sin(\theta)\right\},$$

where (r, θ) are the polar coordinates in the plane. This ellipse has the same area as the circle B_R. For small $|t|$ we have $y = x + t(-x_1, x_2) + \frac{t^2}{2}(x_1, 0) + o(t^2)$. The eigenvalues and eigenfunctions of the Steklov eigenvalue problem (8.1.1) in B_R are

$$\mu_k = \frac{k}{R} + \alpha \quad \text{and} \quad \phi_{k,i} = r^k \begin{cases} c_k \cos(k\theta), \\ c_k \sin(k\theta), \end{cases}$$

where $c_k = \frac{1}{\sqrt{\pi R^{2k+1}}}$ is the normalization constant. We have

$$(v \cdot \nu) = -R\cos(2\theta) = b_2\phi_2$$

and

$$\ddot{\mathcal{S}}(0) = \oint_{\partial B_R} \left(\left|\nabla^\tau (v\cdot\nu)\right|^2 - \frac{(v\cdot\nu)^2}{R^2} \right) dS = 3\pi R.$$

A straightforward computation yields

$$\ddot{\mathcal{E}}(0) = \left[\frac{3}{4\alpha} + \frac{R(1+\alpha R)}{2(2+\alpha R)}\right] \oint_{\partial B_R} (v\cdot\nu)^2 \, dS,$$

with $\oint_{\partial B_R} (v\cdot\nu)^2 \, dS = \pi R^3$. From this expression it follows immediately that

$$\ddot{\mathcal{E}}(0) \begin{cases} > 0 & \text{if } -\alpha R > 2, \\ < 0 & \text{if } -\alpha R < 2. \end{cases}$$

This result is in accordance with Theorem 13.1(i). In this example $b_i = 0$ for all $i \neq 2$.

13.3 An alternative approach: Robin versus Dirichlet torsion

In this section the Robin energy $\mathcal{E}_R(\Omega)$ is compared with the Dirichlet energy $\mathcal{E}_D(\Omega)$.

The following arguments are independent of the sign of α. Let $u = h + z$, where

$$\Delta z + 1 = 0 \text{ in } \Omega, \quad z = 0 \text{ on } \partial\Omega \tag{13.3.1}$$

and

$$\Delta h = 0 \text{ in } \Omega, \quad \partial_\nu h + \alpha h = -\partial_\nu z \text{ on } \partial\Omega. \tag{13.3.2}$$

We follow the general strategy presented in Section 8.1. Let ϕ_k be the eigenfunction of $\Delta\phi = 0$ in Ω, $\partial_\nu\phi + \alpha\phi = \mu\phi$ on $\partial\Omega$, corresponding to the eigenvalue μ_k.

(i) Assume $\mu_k \neq 0$ for all k. From Lemma 8.1 it follows that $h = \sum_{k=0}^{\infty} h_k\phi_k$, where

$$h_k = -\frac{\oint_{\partial\Omega} \phi_k \partial_\nu z \, dS}{\mu_k}.$$

This series converges in $W^{1,2}(\Omega) \cap L^2(\partial\Omega)$.

(ii) Assume $\mu_k = 0$. Let $\mathcal{H}_k$ be the linear space generated by the Steklov eigenfunctions belonging to the eigenvalue μ_k. Then (13.3.2) has a solution if and only if the compatibility condition

$$\oint_{\partial\Omega} \partial_\nu z\phi \, dS = 0$$

is satisfied for any $\phi \in \mathcal{H}_k$. In this case (13.3.2) has infinitely many solutions, which are expressed as

$$h = \sum_{\substack{i=0 \\ i\neq k}} h_i\phi_i + \mathcal{H}_k.$$

For the next considerations we assume $\mu_i \neq 0$ for any $i = 0, 1, 2, \ldots$. This is always true for positive α. Since $u = h + z$, a straightforward computation shows that

$$\mathcal{E}_R(\Omega) = -\int_\Omega u \, dx = -\int_\Omega (h + z) \, dx = -\sum_{i=0}^{\infty} h_i \int_\Omega \phi_i \, dx - \int_\Omega z \, dx.$$

Observe that

$$-\int_\Omega z \, dx = \min_{V \in W_0^{1,2}(\Omega)} \int_\Omega (|\nabla V|^2 - 2V) \, dx = \mathcal{E}_D(\Omega),$$

where $\mathcal{E}_D(\Omega)$ is the Dirichlet energy of the torsion problem (13.3.1). Moreover,

$$-\sum_{i=0}^{\infty} h_i \int_\Omega \phi_i\,dx = \sum_{i=0}^{\infty} \frac{\oint_{\partial\Omega} \phi_i \partial_\nu z\,dS}{\mu_i} \int_\Omega \phi_i\,dx$$

and

$$\int_\Omega \phi_i\,dx = -\int_\Omega \phi_i \Delta z\,dx = -\oint_{\partial\Omega} \phi_i \partial_\nu z\,dS.$$

Hence,

$$\mathcal{E}_R(\Omega) = \mathcal{E}_D(\Omega) - \sum_{i=0}^{\infty} \frac{(\oint_{\partial\Omega} \phi_i \partial_\nu z\,dS)^2}{\mu_i}. \tag{13.3.3}$$

Theorem 13.2. *Assume that $\mu_p < 0 < \mu_{p+1}$ for some $p \in \mathbb{N}$. Set*

$$\mathcal{E}^+ := -\sum_{i=0}^{p} \frac{(\oint_{\partial\Omega} \phi_i \partial_\nu z\,dS)^2}{\mu_i} \geq 0 \quad \textit{and} \quad \mathcal{E}^- := -\sum_{i=p+1}^{\infty} \frac{(\oint_{\partial\Omega} \phi_i \partial_\nu z\,dS)^2}{\mu_i} \leq 0.$$

Then the following statements hold true:
1. *We have $\mathcal{E}_R(\Omega) = \mathcal{E}_D(\Omega) + \mathcal{E}^+ + \mathcal{E}^-$.*
2. *Let $\mathcal{H}_p$ be the linear space spanned by the eigenfunctions corresponding to μ_p and let $\mathcal{H}_p^\infty$ be the orthogonal complement spanned by $\{\phi_j\}_{j=p+1}^\infty$. Set*

$$H(\psi) := \int_\Omega |\nabla\psi|^2\,dx + \alpha \oint_{\partial\Omega} \psi^2\,dS + 2\oint_{\partial\Omega} \psi \partial_\nu z\,dS.$$

   Then

$$\mathcal{E}^+ = \max_{\psi\in\mathcal{H}_p} H(\psi) \quad \textit{and} \quad \mathcal{E}^- = \min_{\psi\in\mathcal{H}_p^\infty} H(\psi).$$

3. *If $\mu_i = 0$, then* (13.1.1) *has a solution if and only if $\int_\Omega \phi_i\,dx = -\oint_{\partial\Omega} \phi_i \partial_\nu z\,dS = 0$ for all eigenfunctions in the eigenspace of μ_i.*

Proof. The first assertion follows from (13.3.3). Replacing $\psi \in \mathcal{H}_p$ by its series $\sum_{i=0}^p q_i\phi_i$ and applying the orthogonality, we find

$$H(\psi) = \sum_{i=1}^{p} q_i^2 \mu_i + 2\sum_{i=1}^{p} q_i \oint_{\partial\Omega} \phi_i \partial_\nu z\,dS.$$

By assumption $\mu_i < 0$ for $i = 0, 1, \ldots, p$. The maximum of $H(\psi)$ is therefore attained for $q_i = -\oint_{\partial\Omega} \phi_i \partial_\nu z\,dS/\mu_i = h_i$. Inserting this expression into $H(\psi)$ we obtain $\mathcal{E}^+$.

The same argument yields the result for $\mathcal{E}^-$. This establishes the second statement and the last assertion is the compatibility condition stated in (13.1.3). □

Remark 13.1. If α is positive, $\mu_i > 0$ and therefore $\mathcal{E}^+ = 0$. Consequently, $\mathcal{E}_R(\Omega) = \mathcal{E}_D(\Omega) + \mathcal{E}^-$.

13.3.1 The functional $\mathcal{J}(\Omega)$

Assume that $-\frac{1}{R} < \alpha < 0$. With $\mu_0 = \alpha$ and $\phi_0 = \frac{1}{\sqrt{|\partial\Omega|}}$, we get

$$\mathcal{E}^+ = -\frac{(\oint_{\partial\Omega} \phi_0 \partial_\nu z\, dS)^2}{\alpha} = -\frac{|\Omega|^2}{\alpha|\partial\Omega|}.$$

From Theorem 13.2 it follows that

$$\mathcal{E}_R(\Omega) \le \mathcal{E}_D(\Omega) - \frac{|\Omega|^2}{\alpha|\partial\Omega|}.$$

Define

$$\mathcal{J}(\Omega) := \mathcal{E}_D(\Omega) - \frac{|\Omega|^2}{\alpha|\partial\Omega|}, \quad \alpha < 0.$$

If $|\Omega| = |B_R|$, then by Schwarz symmetrization

$$\mathcal{E}_D(\Omega) \ge \mathcal{E}_D(B_R) \quad \text{and} \quad \frac{|\Omega|^2}{|\partial\Omega|} \le \frac{|B_R|^2}{|\partial B_R|}.$$

In $\mathcal{J}(\Omega)$ the functionals $\mathcal{E}_D(\Omega)$ and $-\frac{|\Omega|^2}{\alpha|\partial\Omega|}$ compete. It is not clear from the start which one prevails. In order to tackle this problem we compute its first and second domain variation.

13.3.2 Domain variations of $\mathcal{J}(t)$

Let $\Phi_t : \Omega \to \Omega_t$ be a volume preserving perturbation. Set $\mathcal{J}(t) := \mathcal{J}(\Omega_t)$ and $\mathcal{S}(t) = |\partial\Omega_t|$. Then the first variation of $\mathcal{J}(t)$ is

$$\dot{\mathcal{J}}(0) = \dot{\mathcal{E}}_D(0) + \frac{|\Omega|^2}{\alpha|\partial\Omega|^2}\dot{\mathcal{S}}_0(0),$$

where by (6.4.2) and (2.3.20)

$$\dot{\mathcal{E}}_D(0) = -\int_{\partial\Omega} |\nabla z|^2 (v\cdot\nu)\, dS, \quad \dot{\mathcal{S}}_0(0) = (n-1)\int_{\partial\Omega} (v\cdot\nu) H\, dS.$$

Consequently,

$$\dot{\mathcal{J}}(0) = -\oint_{\partial\Omega} \left(|\nabla z|^2 - \frac{|\Omega|^2}{\alpha|\partial\Omega|^2}(n-1)H \right)(v \cdot \nu)\, dS. \tag{13.3.4}$$

If Ω is a critical domain, that is $\dot{J}(0) = 0$ for all volume preserving perturbations, then in addition to $z = 0$ on $\partial\Omega$ the condition

$$|\nabla z|^2 - \frac{|\Omega|^2}{\alpha|\partial\Omega|^2}(n-1)H = \text{const.} \quad \text{on } \partial\Omega \tag{13.3.5}$$

must hold. By [107, Theorem 3] concerning overdetermined boundary value problems, the ball is the only domain for which z is constant and $|\nabla z| = c(H)$ for a nonincreasing function c on $\partial\Omega$.

Consequently, we have the following lemma.

Lemma 13.2. *Among all domains of equal volume, the ball is the only critical domain for the functional $\mathcal{J}(\Omega)$.*

13.3.3 The second variation of $\mathcal{J}(\Omega_t)$ in nearly spherical domains

The second variation for volume preserving perturbations for nearly spherical domains is

$$\ddot{\mathcal{J}}(0) = \ddot{\mathcal{E}}_D(0) - 2\frac{|B_R|^2}{\alpha|\partial B_R|^3}\dot{\mathcal{S}}_0^2(0) + \frac{|B_R|^2}{\alpha|\partial B_R|^2}\ddot{\mathcal{S}}_0(0).$$

In the ball we have $z(r) = \frac{R^2}{2n} - \frac{r^2}{2n}$ and its shape derivative satisfies

$$\Delta z' = 0 \text{ in } B_R, \quad z' = -(v \cdot \nabla z) = (v \cdot \nu)|\nabla z| \text{ on } \partial B_R.$$

From (8.2.18) it follows that

$$\ddot{\mathcal{E}}_D(0) = -\frac{2}{R}\oint_{\partial B_R} z'^2\, dS + 2\int_{B_R} |\nabla z'|^2\, dx.$$

For the ball B_R we have

$$\dot{\mathcal{S}}_0(0) = 0 \quad \text{and} \quad \ddot{\mathcal{S}}_0(0) = \oint_{\partial B_R} \left(\left|\nabla^\tau (v \cdot \nu)\right|^2 - \frac{n-1}{R^2}(v \cdot \nu)^2 \right) dS \geq 0.$$

Then

$$\ddot{\mathcal{J}}(0) = -\frac{2}{R}\int_{\partial B_R} z'^2\,dS + 2\int_{B_R} \left|\nabla z'\right|^2 dx - \frac{R^2}{an^2}\ddot{\mathcal{S}}(0).$$

If on ∂B_R we replace $(v \cdot \nu)$ by $\frac{n}{R}z'$, then $\ddot{\mathcal{J}}(0)$ depends only on z'. It is of the form

$$\ddot{\mathcal{J}}(0) = 2\int_{B_R} \left|\nabla z'\right|^2 dx - \frac{2}{R}\int_{\partial B_R} z'^2\,dS - \frac{1}{a}\int_{\partial B_R} \left(\left|\nabla^\tau z'\right|^2 - \frac{n-1}{R^2}z'^2\right) dS.$$

Sign of $\ddot{\mathcal{J}}(0)$

The volume constraint $\oint_{\partial B_R} z'\,dS = 0$ implies that

$$\int_{B_R} \left|\nabla z'\right|^2 dx \geq \frac{1}{R}\oint_{\partial B_R} z'^2\,dS.$$

We get the lower estimate

$$\ddot{\mathcal{J}}(0) \geq -\frac{1}{a}\int_{\partial B_R} \left(\left|\nabla^\tau z'\right|^2 - \frac{n-1}{R^2}z'^2\right) dS \geq 0. \tag{13.3.6}$$

Since $\mathcal{J}(\Omega)$ is bounded from below by $\mathcal{E}_D(B_R)$ for any domain of fixed volume $|B_R|$ and the ball is the only optimal domain, we have the following proposition.

Proposition 13.2. *Among all domains of prescribed volume, the ball minimizes $\mathcal{J}(\Omega)$.*

Some results in this chapter are published in [15].

14 Problems in annular domains

In this chapter we will discuss some examples of the previous chapters for domains with holes. To be as explicit as possible we will concentrate on small perturbations of annular domains. In general the spherical ring is a critical domain of the energy; however, in contrast to the ball, the sign of the second variation depends on the perturbation.

14.1 An eigenvalue problem related to trace inequalities

Let Ω and $K \subset \overline{\Omega}$ be two bounded smooth domains, and let

$$W_K^{1,2}(\Omega) := \{u \in W^{1,2}(\Omega) : u \equiv 0 \text{ on } K\}.$$

In [24] the authors considered the following eigenvalue problem:

$$\lambda := \lambda(\Omega \setminus K) = \inf\left\{ \frac{\int_\Omega (|\nabla u|^2 + u^2)\, dx}{\oint_{\partial\Omega} u^2\, dS} : u \in W_K^{1,2}(\Omega) \right\}.$$

Any minimizer satisfies

$$\Delta u = u \text{ in } \Omega \setminus \overline{K}, \quad \partial_\nu u = \lambda u \text{ on } \partial\Omega, \quad u = 0 \text{ in } \partial K. \tag{14.1.1}$$

A typical situation is shown in Figure 14.1.

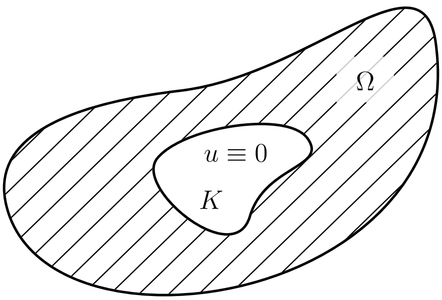

Figure 14.1: The domain $\Omega \setminus K$.

We are interested in the change of λ when K is perturbed and Ω remains unchanged. In this case the family of diffeomorphisms $\{\Phi_t\}_{|t|<t_0}$ is chosen such that $\Phi_t(\partial\Omega) = \partial\Omega$. In particular, this implies $v = w = 0$ on $\partial\Omega$. Thus,

$$\Phi_t(\Omega \setminus K) = \Omega \setminus K_t, \quad \text{where } K_t := \Phi_t(K).$$

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In the sequel we shall assume that Φ_t are Hadamard perturbations and that u is normalized such that

$$\oint_{\partial\Omega} u^2\,dS = 1.$$

In order to compute the domain variations of $\lambda(t) := \lambda(\Omega_t)$ we use the change of variables method and proceed as in Section 6.2. Then $\lambda(t)$ assumes the form

$$\lambda(t) = \int_{\Omega\setminus K} A_{ij}\partial_i\tilde{u}\partial_j\tilde{u}\,dx + \int_{\Omega\setminus K} \tilde{u}^2 J(t)\,dx.$$

Then differentiation with respect to t yields

$$\dot{\lambda}(t) = \int_{\Omega\setminus K} \dot{A}_{ij}\partial_i\tilde{u}\partial_j\tilde{u}\,dx + 2\int_{\Omega\setminus K} A_{ij}\partial_i\dot{\tilde{u}}\partial_j\tilde{u}\,dx + \int_{\Omega\setminus K} \tilde{u}^2\dot{J}(t)\,dx + 2\int_{\Omega\setminus K} \dot{\tilde{u}}\tilde{u}J(t)\,dx.$$

Testing (14.1.1) with $\dot{\tilde{u}}(t)$ we get

$$\int_{\Omega\setminus K} A_{ij}\partial\tilde{u}\partial\dot{\tilde{u}}\,dx = \lambda\oint_{\partial\Omega} \tilde{u}\dot{\tilde{u}}m\,dS - \int_{\Omega\setminus K} \tilde{u}\dot{\tilde{u}}J\,dx.$$

The normalization implies

$$2\oint_{\partial\Omega} \tilde{u}\dot{\tilde{u}}m\,dS + \oint_{\partial\Omega} \tilde{u}^2\dot{m}\,dS = 0.$$

Hence, since $\dot{m} = 0$ on $\partial\Omega$

$$\dot{\lambda}(t) = \int_{\Omega\setminus K} \dot{A}_{ij}\partial_i\tilde{u}\partial_j\tilde{u}\,dx + \int_{\Omega\setminus K} \tilde{u}^2\dot{J}\,dx.$$

This corresponds to the expression for the first variation of the Dirichlet energy. From the computation in Section 6.3.1 it follows that

$$\dot{\lambda}(0) = -\oint_{\partial K} (\partial_\nu u)^2(v\cdot\nu)\,dS.$$

Thus, if $\dot{\lambda}(0) = 0$ for all volume preserving perturbations, then necessarily $(\partial_\nu u)^2 =$ const. on ∂K.

The second variation $\ddot{\lambda}(0)$ will be computed for the special case $\Omega := B_R$ and $K := B_{\kappa R}$. Clearly, $\dot{\lambda}(0) = 0$ in this case. The shape derivative u' solves the boundary value problem

$$\Delta u' = u' \text{ in } \Omega\setminus\overline{K}, \quad \partial_\nu u' = \lambda u' \text{ on } \partial\Omega, \quad u' = -\partial_\nu u(v\cdot\nu) \text{ on } \partial K$$

for the shape derivative u'. Following the computations in Section 9.1.3 we obtain for volume preserving perturbations

$$\ddot{\lambda}(0) = 2 \int_{B_R \setminus B_{\kappa R}} (|\nabla u'|^2 + u'^2)\, dx - 2\lambda \oint_{\partial B_R} u'^2\, dS - \frac{2(n-1)}{\kappa R} \oint_{\partial B_{\kappa R}} u'^2\, dS. \tag{14.1.2}$$

The radial solution of $\Delta u = u$ in $B_R \setminus \overline{B_{\kappa R}}$ and $u_r(R) = \lambda u(R)$ is

$$\begin{aligned} u(r) := {} & R^{\frac{n-2}{2}} \frac{(K_{\frac{n-2}{2}}(R)\lambda + K_{\frac{n}{2}}(R)) r^{-\frac{n-2}{2}} I_{\frac{n-2}{2}}(r)}{I_{\frac{n}{2}}(R) K_{\frac{n-2}{2}}(R) + K_{\frac{n}{2}}(R) I_{\frac{n-2}{2}}(R)} \\ & + R^{\frac{n-2}{2}} \frac{(-I_{\frac{n-2}{2}}(R)\lambda + I_{\frac{n}{2}}(R)) r^{-\frac{n-2}{2}} K_{\frac{n-2}{2}}(r)}{I_{\frac{n}{2}}(R) K_{\frac{n-2}{2}}(R) + K_{\frac{n}{2}}(R) I_{\frac{n-2}{2}}(R)}. \end{aligned}$$

Here I_ν and K_ν denote the modified Bessel functions of first and second kind and of order ν (see Chapter 3).

The boundary condition $u(\kappa R) = 0$ yields

$$\lambda := -\frac{K_{\frac{n}{2}}(R) I_{\frac{n-2}{2}}(\kappa R) + I_{\frac{n}{2}}(R) K_{\frac{n-2}{2}}(\kappa R)}{K_{\frac{n-2}{2}}(R) I_{\frac{n-2}{2}}(\kappa R) - I_{\frac{n-2}{2}}(R) K_{\frac{n-2}{2}}(\kappa R)}.$$

We expand the shape derivative u' with respect to spherical harmonics (see Section 8.1):

$$u' = \sum_{k=1}^{\infty} a_k(r) Y_k(\xi), \quad \text{for } \kappa R < r < R,\ \xi \in \partial B_1,$$

where we use the notation from Section 8.1. Then

$$a_k''(r) + \frac{n-1}{r} a_k'(r) - \frac{k(k+n-2)}{r^2} a_k(r) = a_k(r) \quad \forall\, \kappa R < r < R,$$

with the boundary condition

$$a_k'(R) = \lambda a_k(R).$$

Let $N := n + 2(k-1)$. We get the solution

$$a_k(r) = r^{-\frac{n}{2}+1} (c_1 K_{\frac{N}{2}}(r) + c_2 I_{\frac{N}{2}}(r)),$$

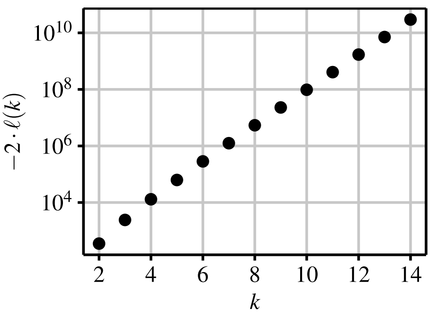
where

$$\begin{aligned} c_1 &:= R(k - \lambda R) I_{\frac{N}{2}+2}(R) + I_{\frac{N}{2}+1}(R) \left(R^2 - 2\lambda R \left(\frac{N}{2} + 1 \right) + 2k^2 + nk \right), \\ c_2 &:= -R(k - \lambda R) K_{\frac{N}{2}+2}(R) + K_{\frac{N}{2}+1}(R) \left(R^2 - 2\lambda R \left(\frac{N}{2} + 1 \right) + 2k^2 + nk \right). \end{aligned}$$

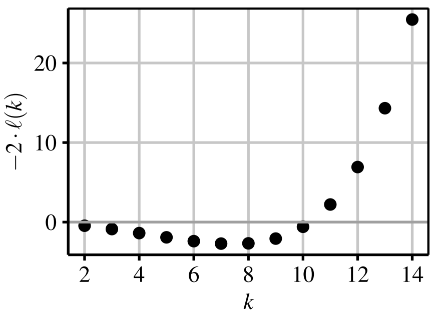
We integrate by parts in (14.1.2), apply the boundary condition for u', and express u' by its expansion. The orthogonality relations for the spherical harmonics then imply for volume preserving Hadamard perturbations

$$\ddot{\lambda}(0) = -2\sum_{k=1}^{\infty}\underbrace{\left(a_k'(\kappa R)a_k(\kappa R) + \frac{n-1}{\kappa R}a_k^2(\kappa R)\right)}_{=:\ell(k)}.$$

Figure 14.2 shows the values of $-2\ell(k)$ as a function of k with $n = 3$ and $R = 9$ for two different values of κ. Thus, for large κ the sign of the second variation depends on the perturbation.



(a) $\kappa = 0.5$



(b) $\kappa = 0.9$

Figure 14.2: Values of $-2\ell(k)$ as a function of k with $n = 3$ and $R = 9$.

14.2 Bernoulli's problem

Daniel Bernoulli (1700–1782) studied stream lines of a fluid around an obstacle. For bounded domains this leads to the following mathematical formulation.

Let Ω be a bounded domain and let $K \subset \Omega$ be the obstacle. The domain occupied by the fluid is $\Omega \setminus K$. The stream function u is a minimizer of the capacity

$$\operatorname{cap}(\Omega, K) := \inf\left\{\int_{\Omega\setminus K} |\nabla u|^2\,dx : u \in W_0^{1,2}(\Omega),\ u = 1 \text{ on } \partial K\right\}.$$

It solves

$$\Delta u = 0 \text{ in } \Omega \setminus K, \quad u = 0 \text{ on } \partial\Omega, \quad u = 1 \text{ on } \partial K.$$

Let $\mathcal{S}(\partial K)$ be the surface area of K. In [51] the authors studied this problem in the plane and asked the following question. Let Ω^* (resp. K^*) be a ball of the same volume as Ω (resp. K). Does the isoperimetric inequality

$$\frac{\operatorname{cap}(\Omega^*, K^*)}{\mathcal{S}(\partial K^*)} \leq \frac{\operatorname{cap}(\Omega, K)}{\mathcal{S}(\partial K)} \tag{14.2.1}$$

hold?

It turns out that in general the answer is no. We shall prove this by means of the method of domain variations leaving Ω^* fixed and perturbing only ∂K^*.

Let $\{\Phi_t\}_{|t|<t_0}$ be a family of volume preserving perturbations such that $\Phi_t(\partial\Omega^*) = \partial\Omega^*$. Set

$$\mathcal{E}(t) := \frac{\operatorname{cap}(\Omega, K_t)}{\mathcal{S}(\partial K_t)} \quad \text{and} \quad \mathcal{C}(t) := \operatorname{cap}(\Omega, K_t).$$

Let $\Omega^* = B_R$ for some $R > 0$ and $K^* = K_0 = B_{\kappa R}$ for some $0 < \kappa < 1$. The stream function is

$$u(r) = \begin{cases} \frac{1}{\ln \kappa} \ln \frac{r}{R} & \text{if } n = 2, \\ \frac{r^{2-n} - R^{2-n}}{(\kappa R)^{2-n} - R^{2-n}} & \text{if } n \geq 3. \end{cases}$$

A direct computation yields

$$\operatorname{cap}(B_R, B_{\kappa R}) = \begin{cases} -\frac{2\pi}{\ln \kappa} & \text{if } n = 2, \\ \frac{(n-2)\omega_n}{1-\kappa^{n-2}} (\kappa R)^{n-2} & \text{if } n \geq 3. \end{cases}$$

By means of the results in Section 6.4 it is easy to check that $\dot{\mathcal{E}}(0) = 0$. Consequently,

$$\mathcal{S}^2(0)\ddot{\mathcal{E}}(0) = \mathcal{S}(0)\ddot{\mathcal{C}}(0) - \ddot{\mathcal{S}}(0)\mathcal{C}(0).$$

If we restrict to volume preserving Hadamard perturbations, Proposition 7.1 yields

$$\ddot{\mathcal{C}}(0) = 2 \int\limits_{B_R \setminus B_{\kappa R}} |\nabla u'|^2 \, dx - \frac{2(n-1)}{\kappa R} \oint\limits_{\partial B_{\kappa R}} u'^2 \, dS,$$

while (2.3.27) and (2.3.28) imply

$$\ddot{\mathcal{S}}(0) = \ddot{\mathcal{S}}_0(0) = \oint\limits_{\partial B_{\kappa R}} |\nabla^\tau (v \cdot \nu)|^2 - \frac{n-1}{\kappa^2 R^2} (v \cdot \nu)^2 \, dS.$$

In this setting the shape derivative u' satisfies

$$\Delta u' = 0 \text{ in } B_R \setminus B_{\kappa R}, \quad u' = 0 \text{ on } \partial B_R,$$

and on the boundary $\partial B_{\kappa R}$

$$u' = \begin{cases} \frac{1}{\kappa R \ln \kappa}(v \cdot \nu) & \text{if } n = 2, \\ \frac{n-2}{\kappa R} \frac{1}{1-\kappa^{n-2}}(v \cdot \nu) & \text{if } n \geq 3. \end{cases}$$

Consider first the case $n = 2$. Replacing $(v \cdot \nu) = \rho$ by $\kappa R \ln \kappa = u'$ we get

$$\mathcal{S}_0^2(0)\ddot{\mathcal{E}}(0) = 4\pi\kappa R\Big[\int\limits_{B_R \setminus B_{\kappa R}} |\nabla u'|^2\, dx - \frac{1}{\kappa R} \oint\limits_{\partial B_{\kappa R}} u'^2\, dS\Big] + 2\pi \ln \kappa (\kappa R)^2 \oint\limits_{\partial B_{\kappa R}} |\nabla^\tau u'|^2 - \frac{1}{(\kappa R)^2} u'^2\, dS. \tag{14.2.2}$$

For $n \geq 2$ we consider the eigenvalue problem

$$\Delta\phi = 0 \text{ in } B_R \setminus B_{\kappa R}, \quad \partial_r \phi = 0 \text{ on } \partial B_R, \quad -\partial_r \phi = \mu\phi \text{ on } \partial B_{\kappa R}.$$

The solutions are known explicitly. The eigenvalues $\{\mu_k\}_k$ each with multiplicity d_k are

$$\mu_k = \frac{1}{\kappa R} \frac{k(k+n-2)(1-\kappa^{2k+n-2})}{k + (k+n-2)\kappa^{2k+n-2}}.$$

The lowest eigenvalue is $\mu_0 = 0$ and $\phi_0 = \text{const}$. For $i \leq d_k$ the corresponding eigenfunctions are

$$\phi_{i,k}(r,\xi) = \left(\frac{k+n-2}{k} R^{-2k-n+2} r^k + r^{-k-n+2}\right) Y_{i,k}(\xi).$$

The integrals in (14.2.2) containing u' can be expressed as boundary integrals over $\partial B_{\kappa R}$. On $\partial B_{\kappa R}$ the shape derivative u' can be expanded as $u' = \sum_k c_k \phi_k$. The following identities hold.

$$\int\limits_{B_R \setminus B_{\kappa R}} |\nabla u'|^2\, dx = -\oint\limits_{\partial B_{\kappa R}} \partial_\nu u'\, u'\, dS = \sum_{k=1}^{\infty} c_k^2 \mu_k,$$

$$\oint\limits_{\partial B_{\kappa R}} |\nabla^\tau u'|^2\, dS = \sum_{k=1}^{\infty} c_k^2 \frac{k(k+n-2)}{(\kappa R)^2}, \qquad \oint\limits_{\partial B_{\kappa R}} u'^2\, dS = \sum_{k=1}^{\infty} c_k^2.$$

For $n = 2$ we have $\mu_k = \frac{k(1-\kappa^{2k})}{(\kappa R)(1+\kappa^{2k})}$. Inserting these expansions into (14.2.2) and taking into account the volume preservation we get for $n = 2$

$$\mathcal{S}_0^2(0)\ddot{\mathcal{E}}(0) = 2\pi \sum_{k=1}^{\infty} c_k^2 \Big[\frac{2k(1-\kappa)^{2k}}{1+\kappa^{2k}} - 2 + (k^2 - 1)\ln \kappa\Big].$$

From the inequality $\ln \kappa \leq \kappa - 1$ it follows that

$$\left[\frac{2k(1-\kappa)^{2k}}{1+\kappa^{2k}} - 2 + (k^2-1)\ln\kappa\right] < 2k(1-\kappa) - 2 - (k^2-1)(1-\kappa) < 0.$$

Consequently, $\mathcal{S}^2(0)\ddot{\mathcal{E}}(0) < 0$. Therefore, $\ddot{\mathcal{E}}(0)$ has a local maximum among this particular family of nearly spherical domains.

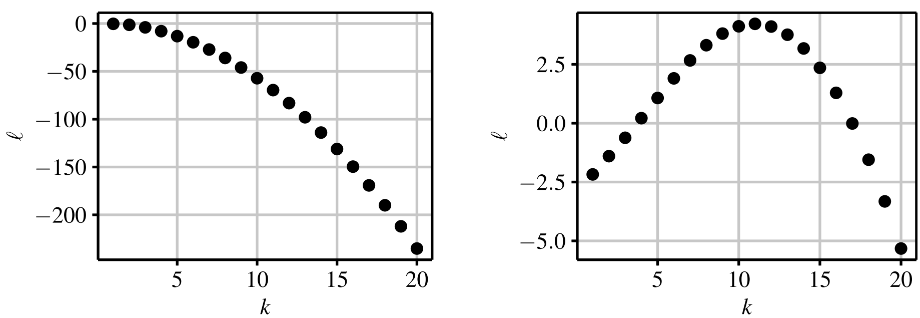
The situation is more involved for higher dimensions ($n \geq 3$). In this case we obtain

$$\begin{aligned}\mathcal{J}(u') := \frac{1}{|\partial B_1|}(\kappa R)^{2-n}\mathcal{S}^2(0)\ddot{\mathcal{E}}(0) &= 2(\kappa R)\int\limits_{B_R\setminus B_{\kappa R}} |\nabla u'|^2\,dx \\ &+ (n-1)\left\{\frac{1-\kappa^{n-2}}{n-2} - 2\right\}\oint\limits_{\partial B_{\kappa R}} u'^2\,dS - \frac{1-\kappa^{n-2}}{n-2}(\kappa R)^2\oint\limits_{\partial B_{\kappa R}} |\nabla^\tau u'|^2\,dS.\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{J}(u') = \sum_{k=1}^{\infty} c_k^2\Bigg(&\underbrace{\frac{(k+n-2)[(k-1)\kappa^{n-2}-k-2n+5](n-1+k)\kappa^{2k+n}}{[(k+n-2)\kappa^{2k+n}+\kappa^2 k](n-2)}}_{\ell_1(n,k,\kappa)} \\ &+ \underbrace{\frac{\kappa^2 k(k+1)[(n-1+k)\kappa^{n-2}-k+n-3]}{[(k+n-2)\kappa^{2k+n}+\kappa^2 k](n-2)}}_{\ell_2(n,k,\kappa)}\Bigg).\end{aligned}$$

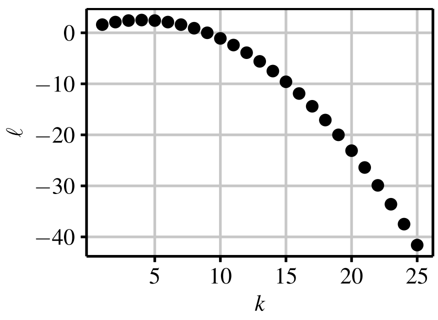
Set $\ell = \ell_1 + \ell_2$. Figure 14.3 shows $\ell(3,k,\kappa)$ for two different values of κ as a function of k. The second variation is negative for small κ. Therefore the annulus is a local maximizer. For large κ there exist values of k such that $\ell(3,k,\kappa)$ is positive. Consequently, the sign of the second variation depends on the perturbation.



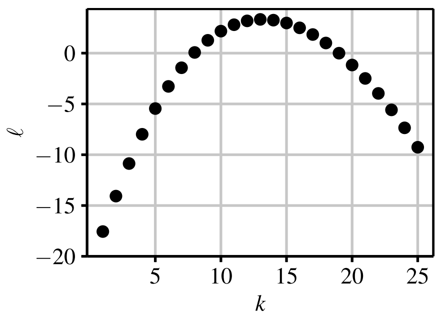
(a) $\kappa = 0.4$ (b) $\kappa = 0.9$

Figure 14.3: $\ell(3,k,\kappa)$.

Figure 14.4 shows the case where $n = 12$. The second variation has no constant sign in general: The function $\ell(12,k,\kappa)$ is positive for κ and k small, whereas for κ large there are values of k for which it is positive while it is negative for all other values of k.



(a) $\kappa = 0.1$



(b) $\kappa = 0.9$

Figure 14.4: $\ell(12, k, \kappa)$.

14.3 The Robin torsion problem for $\alpha > 0$

For $\alpha > 0$ we consider the torsion problem

$$\begin{aligned} \Delta u + 1 = 0 &\quad \text{in } B_R \setminus B_{\kappa R}, \\ \partial_\nu u + \alpha u = 0 &\quad \text{on } \partial(B_R \setminus B_{\kappa R}). \end{aligned} \tag{14.3.1}$$

There exists a unique solution which must be radial. It is of the form

$$u(r) = \begin{cases} -\frac{r^2}{2n} + cr^{2-n} + d & \text{if } n \geq 3, \\ -\frac{r^2}{4} + c \ln r + d & \text{if } n = 2. \end{cases} \tag{14.3.2}$$

The constants c and d are determined by the boundary conditions.

In particular, for $n = 3$ and $R = 1$ we obtain

$$u(\kappa) = \frac{2 + \alpha(1 - 3\kappa^2 + 2\kappa^3) - 2\kappa^3}{6\alpha[1 + \kappa^2 + \alpha\kappa(1 - \kappa)]}. \tag{14.3.3}$$

Let Ω_t be a volume preserving perturbation of $B_R \setminus B_{\kappa R}$ and let the outer boundary ∂B_R be unperturbed. Consider the torsion energy

$$\mathcal{E}_R(t) = \int_{\Omega_t} (|\nabla u(y)|^2 - 2u(y))\, dy + \alpha \oint_{\partial\Omega_t} u^2(y)\, dS(y) = -\int_{\Omega_t} u(y)\, dy.$$

Since u is radial we can apply the results in Section 6.4.1, and get

$$\dot{\mathcal{E}}(0) = 0.$$

For the computation of the second derivative $\ddot{\mathcal{E}}(0)$ we need the shape derivative u'. From (6.2.12) and Lemma 6.1 it follows that

$$\Delta u' = 0 \quad \text{in } B_R \setminus B_{\kappa R}$$

with the boundary conditions

$$u'_r + \alpha\, u' = 0 \text{ on } \partial B_R \quad \text{and} \quad -u'_r + \alpha\, u' = k_1(u(\kappa R))\rho \text{ on } \partial B_{\kappa R},$$

where u'_r denotes the derivative with respect to r and $\rho = (v \cdot \nu)$ on $\partial B_{\kappa R}$. Note that $\partial_\nu u' = -u'_r$ on $\partial B_{\kappa R}$ and

$$k_1(u(\kappa R)) = 1 + \alpha \frac{n-1}{\kappa R} u(\kappa R) + \alpha^2 u(\kappa R). \tag{14.3.4}$$

From (6.5.9) it follows that

$$Q(u') = \int_{B_R \setminus B_{\kappa R}} |\nabla u'|^2 + \alpha \oint_{\partial(B_R \setminus B_{\kappa R})} u'^2\, dS = k_1(u(\kappa R)) \oint_{\partial B_{\kappa R}} \rho u'\, dS.$$

For volume preserving Hadamard perturbations we can apply (8.2.7) and obtain

$$\begin{aligned}\ddot{\mathcal{E}}_R(0) = {} & 2\alpha u(\kappa R)\left(1 + \alpha \frac{n-1}{\kappa R} u(\kappa R) + \alpha^2 u(\kappa R)\right) \oint_{\partial B_{\kappa R}} \rho^2\, dS \\ & + \alpha u^2(\kappa R) \ddot{\mathcal{S}}_0(0) - 2Q(u').\end{aligned} \tag{14.3.5}$$

In summary, we get the following statement.

Lemma 14.1. *For $0 < \kappa < 1$ let $B_R \setminus B_{\kappa R}$ be an annulus in $\mathbb{R}^n$, $n \geq 2$. Then the first domain variation for volume preserving Hadamard perturbations, leaving the outer boundary ∂B_R unperturbed, is zero. The corresponding second domain variation is given in* (14.3.5).

Based on the strategy developed in Section 8.1, we discuss the sign of $\ddot{\mathcal{E}}_R(0)$ for the special case $n = 3$ and $R = 1$.

For this purpose we will expand u' into a series of eigenfunctions of the Steklov problem $\Delta\phi = 0$ in $B_1 \setminus B_\kappa$, with $\phi_r + \alpha\phi = 0$ on ∂B_1 and $-\phi_r + \alpha\phi = \mu\phi$ on ∂B_κ.

For $\xi \in \partial B_1$,

$$\phi_k(r, \xi) = \left(r^k + \frac{k+\alpha}{k+1-\alpha} r^{-k-1}\right) Y_k(\xi)$$

and

$$\mu_k := \frac{\alpha(k+1-\alpha)\kappa^{2k+2} - k(k+1-\alpha)\kappa^{2k+1} + (k+\alpha)(k+1+\alpha\kappa)}{\kappa[(k-\alpha+1)\kappa^{2k+1} + \alpha + k]}.$$

For $k = 0$ we obtain

$$\mu_0 = \frac{\alpha(1-\alpha)\kappa^2 + \alpha(\alpha\kappa + 1)}{\kappa[(1-\alpha)\kappa + \alpha]}$$

and $0 < \mu_0 < \mu_1 < \cdots < \mu_k < \mu_{k+1} < \cdots$.

Let the eigenfunctions $\{\phi_k\}_{k\geq 1}$ be normalized such that $\|\phi_k\|_{L^2(\partial B_1)} = 1$. An expansion on ∂B_κ with respect to $\{\phi_k\}_{k\geq 1}$ which takes into account the volume preserving property and the barycenter condition of the perturbation yields

$$u'(x) = \sum_{k=2}^{\infty} c_k \phi_k \quad \text{and} \quad \rho = \sum_{k=2}^{\infty} b_k \phi_k.$$

This implies the following identities:

$$1) \qquad \oint_{\partial B_\kappa} \rho^2 \, dS = \frac{1}{k_1^2(u(\kappa))} \sum_{k=2}^{\infty} c_k^2 \mu_k^2,$$

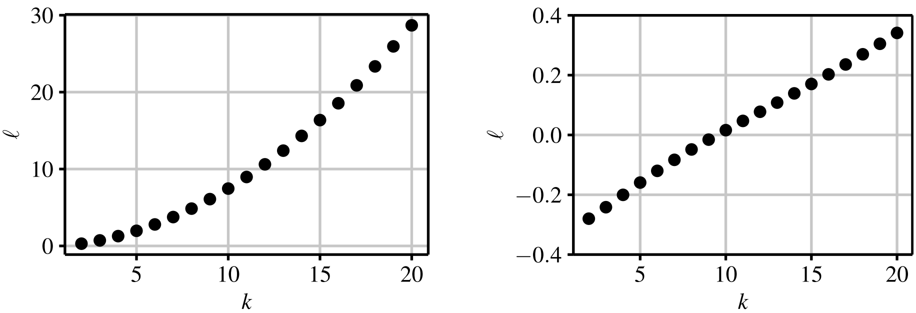
$$2) \qquad \ddot{\mathcal{S}}_0(0) = \frac{1}{\kappa^2 k_1^2(u(\kappa))} \sum_{k=2}^{\infty} (k(k+1) - 2) c_k^2 \mu_k^2 \quad \text{(see (2.3.27))},$$

$$3) \qquad -2Q(u') = -2k_1(u(\kappa)) \oint_{\partial B_\kappa} \rho u' \, dS = -2 \sum_{k=2}^{\infty} c_k^2 \mu_k,$$

where $u(\kappa)$ and k_1 are given in (14.3.3) and (14.3.4). The second domain variation then becomes

$$\ddot{\mathcal{E}}(0) = \sum_{k=2}^{\infty} \underbrace{\left(2 \frac{\alpha u(\kappa)}{k_1(u(\kappa))} + \frac{\alpha u^2(\kappa)}{\kappa^2 k_1^2(u(\kappa))} (k(k+1) - 2) - \frac{2}{\mu_k} \right)}_{\ell(k)} c_k^2 \mu_k^2.$$

Let $\alpha = 2.5$. Then Figure 14.5a (resp. Figure 14.5b) shows $\ell(k)$ for $\kappa = 0.1$ (resp. $\kappa = 0.9$) as a function of $k \geq 2$.



(a) $\kappa = 0.1$ (b) $\kappa = 0.9$

Figure 14.5: $\ell(k)$ for $n = 3$, $R = 1$, and $\alpha = 2.5$.

As in the first example (see Section 14.1), we see that for κ close to 1 the coefficient $\ell(k)$ becomes negative for small k.

Remark 14.1. We have discussed the case where only the inner sphere is perturbed. Alternatively, we could also perturb the outer sphere. In that case, the second variation is positive.

Problem 14.1. The case $\alpha < 0$ is more difficult. Numerical experiments show that there are values of α for which there is a unique solution. The solution of (14.3.1) is not unique if α coincides with an eigenvalue of the Steklov eigenvalue problem $\Delta\phi = 0$ in $B_R \setminus B_{\kappa R}$ and $\partial_\nu \phi = \mu\phi$ on $\partial(B_R \setminus B_{\kappa R})$. A complete analysis is not yet available.

14.4 Notes

For further reading about the mathematical background of problem (14.1.1) and its relation to trace inequalities we refer to [24, Section 1]. Information about applications and the solutions of the Bernoulli problem are found in [52]. Some important questions of optimal shapes related to the Bernoulli problem are discussed in [105].

15 The first buckling eigenvalue of a clamped plate

The first and second domain variation of the first buckling eigenvalue are computed by means of the moving surface method. It turns out that for optimal domains the second variation is a quadratic form in the shape derivative only. The positivity of this form together with Payne's inequality implies that among all domains of given volume the ball is the only optimal domain. This establishes Pólya and Szegö's conjecture for smooth domains.

15.1 The first eigenvalue

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected domain and let

$$\mathcal{R}(u,\Omega) := \frac{\int_\Omega |\Delta u|^2 \, dx}{\int_\Omega |\nabla u|^2 \, dx}$$

for $u \in W_0^{2,2}(\Omega)$. We set $\mathcal{R}(u,\Omega) = \infty$ if the denominator vanishes. Then the lowest *buckling eigenvalue of* Ω is given by

$$\Lambda(\Omega) := \inf\{\mathcal{R}(u,\Omega) : u \in W_0^{2,2}(\Omega),\ \|\nabla u\|_{L^2(\Omega)} = 1\}.$$

The infimum is attained by the first eigenfunction u which solves the Euler–Lagrange equation

$$\Delta^2 u + \Lambda(\Omega)\Delta u = 0 \quad \text{in } \Omega, \tag{15.1.1}$$

$$u = \partial_\nu u = 0 \quad \text{on } \partial\Omega. \tag{15.1.2}$$

The corresponding weak form is

$$\int_\Omega \Delta u \Delta \psi \, dx = \Lambda(\Omega) \int_\Omega (\nabla u \cdot \nabla \psi) \, dx, \quad \forall\, \psi \in C_0^\infty(\Omega).$$

Problem (15.1.1)–(15.1.2) possesses countable many eigenvalues tending to infinity. The corresponding eigenfunctions form a basis in $W_0^{2,2}(\Omega)$ which is orthonormal with respect to the inner product $\int_\Omega \Delta u \Delta v \, dx$.

In contrast to the membrane, the sign of the first eigenfunction may change depending on Ω. For the ball the first eigenvalue is simple and the first eigenfunction is radial; see Section 3.4 where the eigenfunctions have been computed explicitly.

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15.2 Shape derivatives

Let Ω be a bounded smooth domain in $\mathbb{R}^n$ (at least $C^{2,\alpha}$). For given $t_0 > 0$ and $t \in (-t_0, t_0)$ let $\{\Omega_t := \Phi_t(\Omega)\}_{|t|<t_0}$ be a family of perturbations of the domain $\Omega \subset \mathbb{R}^n$ given in Section 2.1. As before we will use the notation $y := \Phi_t(x)$. The eigenvalue $\Lambda(\Omega_t)$ is characterized by the variational principal

$$\Lambda(\Omega_t) := \inf\{\mathcal{R}(u, \Omega_t) : u \in W_0^{2,2}(\Omega_t)\}.$$

Throughout this chapter we assume that the first eigenvalue is simple.

Denote by $u_t \in W_0^{2,2}(\Omega_t)$ its minimizer. For each $t \in (-t_0, t_0)$ it solves

$$\Delta^2 u_t + \Lambda(\Omega_t)\Delta u_t = 0 \quad \text{in } \Omega_t, \tag{15.2.1}$$

$$u_t = |\nabla u_t| = 0 \quad \text{on } \partial\Omega_t. \tag{15.2.2}$$

For the minimzer u_t we set

$$\Lambda(t) := \Lambda(\Omega_t) = \mathcal{R}(u_t, \Omega_t). \tag{15.2.3}$$

The smoothness of Ω and Φ_t implies that the eigenfunction $u_t = u(y, t)$ is also smooth in t and y. This has several consequences which will be listed below.

The boundary condition $u_t = 0$ on $\partial\Omega_t$ for all $|t| < t_0$ implies

$$0 = \frac{d}{dt}u(\Phi_t(x), t) = \partial_t\Phi_t(x) \cdot \nabla u(\Phi_t(x), t) + \partial_t u(\Phi_t(x), t),$$

and $\nabla u_t = 0$ on $\partial\Omega_t$ implies that

$$\partial_t u_t = \partial_t u(\Phi_t(x), t) = 0 \quad \text{on } \partial\Omega_t,\ \forall\, t \in (-t_0, t_0). \tag{15.2.4}$$

Since $u_t = 0$ on $\partial\Omega_t$ we get

$$\Delta u_t = \partial^2_{\nu^t} u_t + (n-1)\partial_{\nu^t} u_t H^t \quad \text{on } \partial\Omega_t, \tag{15.2.5}$$

where ν^t is the outer normal and H^t is the mean curvature of $\partial\Omega_t$. Since $|\nabla u_t| = 0$ on $\partial\Omega_t$, then necessarily

$$\Delta u_t = \partial^2_{\nu^t} u_t \quad \text{on } \partial\Omega_t. \tag{15.2.6}$$

This holds for all $|t| < t_0$, in particular for $t = 0$.

We will need the first and second shape derivatives of u_t and ν^t. Recall that $\tilde{u}(t) = u_t(\Phi_t, t)$. By (4.1.4) and (4.1.8),

$$\begin{aligned}
&\dot{\tilde{u}}(0) = u'(x) + (v \cdot \nabla u(x)) \quad \text{and} \\
&\ddot{\tilde{u}}(0) = u''(x) + 2v(x) \cdot \nabla u'(x) + w(x) \cdot \nabla u(x) + v(x) \cdot D^2 u(x) v(x).
\end{aligned}$$

On $\partial\Omega$ it follows from the boundary conditions (15.2.2) that

$$0 = \dot{\ddot{u}}(0) = u'(x) \quad \forall x \in \partial\Omega, \tag{15.2.7}$$

and from $(\nabla u \cdot w) = \partial_\nu u(w \cdot \nu) = 0$ we conclude that

$$0 = \ddot{u}(0) = u''(x) + 2v(x) \cdot \nabla u'(x) + \big(v(x) \cdot D^2 u(x) v(x)\big) \quad \forall x \in \partial\Omega. \tag{15.2.8}$$

Lemma 15.1. *For all $t \in (-t_0, t_0)$ the following relation holds:*

$$\nu^t \cdot \nabla \partial_t u_t(y) = -\Delta u_t(y)(\nu^t \cdot \partial_t \Phi_t(\Phi_t^{-1}(y))) \quad \textit{for } y \textit{ in } \partial\Omega_t$$

Proof. Recall that $u_t(y) = u(\Phi_t(x), t)$. Since $\nabla u(\Phi_t(x), t) = 0$ for all $|t| < t_0$ and all $x \in \partial\Omega$, we have

$$0 = \frac{d}{dt} \nabla u\big(\Phi_t(x), t\big) = \nabla \partial_t u\big(\Phi_t(x), t\big) + D^2 u(\Phi_t(x), t) \partial_t \Phi_t(x). \tag{15.2.9}$$

We insert (15.2.4) in the first term on the right side:

$$\nabla \partial_t u\big(\Phi_t(x), t\big) = \nu^t (\nu^t \cdot \nabla \partial_t u_t(y)).$$

For the second term we use the fact that $\nabla u_t = 0$ in $\partial\Omega_t$. Hence, $D^2 u_t \tau_t = 0$ in $\partial\Omega_t$ for all tangent vectors τ_t. Multiplying (15.2.9) by ν^t we obtain

$$0 = \nu^t \cdot \nabla \partial_t u_t(y) + \nu^t \cdot D^2 u_t(y) \partial_t \Phi_t(\Phi_t^{-1}(y)) \quad \forall y \in \partial\Omega_t$$

for all $t \in (-t_0, t_0)$. Hence

$$\nu^t \cdot D^2 u_t(y) \partial_t \Phi_t(\Phi_t^{-1}(y)) = (\nu^t \cdot D^2 u_t(y) \nu^t)(\nu^t \cdot \partial_t \Phi_t(\Phi_t^{-1}(y))).$$

Thus

$$(\nu^t \cdot \nabla \partial_t u_t(y)) = -(\nu^t \cdot D^2 u_t(y) \nu^t)(\nu^t \cdot \partial_t \Phi_t(\Phi_t^{-1}(y))) \quad \forall y \in \partial\Omega_t.$$

By formula (15.2.6) we get

$$\nu^t \cdot \nabla \partial_t u_t(y) = -\Delta u_t(y)(\nu^t \cdot \partial_t \Phi_t(\Phi_t^{-1}(y))) \quad \forall y \in \partial\Omega_t$$

which completes the proof of the lemma. □

Hence, for $t = 0$ we have

$$\partial_\nu u'(x) = -\big(v(x) \cdot \nu(x)\big) \Delta u \quad \forall x \in \partial\Omega.$$

Lemma 15.2. *Let Ω be a critical domain for Λ, i. e., $\dot{\Lambda}(0) = 0$. Then the shape derivative u' solves the following boundary value problem:*

$$\Delta^2 u' + \Lambda(\Omega)\Delta u' = 0 \quad \text{in } \Omega,$$
$$u' = 0, \quad \text{and} \quad \partial_\nu u' = -(v\cdot\nu)\Delta u \quad \text{on } \partial\Omega.$$

Moreover,

$$u'' = (v\cdot\nu)^2\Delta u \quad \text{on } \partial\Omega.$$

Proof. The first statement is obvious therefore only the last statement needs a proof.

Since $u' = 0$ and $u = |\nabla u| = 0$ on $\partial\Omega$, (15.2.8) has the form

$$0 = u''(x) + 2(v\cdot\nu)\partial_\nu u' + (v\cdot\nu)^2\partial_\nu^2 u = u''(x) + 2(v\cdot\nu)\partial_\nu u' + (v\cdot\nu)^2\Delta u.$$

The boundary condition for $\partial_\nu u'$ then establishes the claim. □

15.3 The first domain variation

We will use Reynolds' transport theorem (Theorem 4.1) to compute the first domain variation of $\Lambda(t) = \frac{D(t)}{N(t)}$, where

$$D(t) := \int_{\Omega_t} |\Delta u_t|^2\,dy \quad \text{and} \quad N(t) := \int_{\Omega_t} |\nabla u_t|^2\,dy.$$

Without loss of generality we can assume that

$$N(t) = \int_{\Omega_t} |\nabla u_t|^2\,dy = 1 \quad \forall\, t \in (-t_0, t_0). \tag{15.3.1}$$

With this normalization (4.2.1) leads to

$$\dot{\Lambda}(t) = 2\int_{\Omega_t} \Delta u_t \Delta\partial_t u_t\,dy - 2\Lambda(t)\int_{\Omega_t} \nabla u_t\cdot\nabla\partial_t u_t\,dy$$
$$+ \oint_{\partial\Omega_t} |\Delta u_t|^2 \partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t(y)\,dS_t.$$

Integration by parts and (15.2.2) yield

$$\dot{\Lambda}(t) = 2\int_{\Omega_t} [\Delta^2 u_t + \Lambda(t)\Delta u_t]\partial_t u_t\,dy + 2\oint_{\partial\Omega_t} \Delta u_t \partial_{\nu^t}\partial_t u_t\,dS_t$$
$$- 2\oint_{\partial\Omega_t} \partial_{\nu^t}\Delta u_t \partial_t u_t\,dS_t + \oint_{\partial\Omega_t} |\Delta u_t|^2(\partial_t\Phi_t(\Phi_t^{-1}(y))\cdot\nu^t)\,dS_t.$$

Since u_t solves (15.2.1), the first integral vanishes. The third integral vanishes in view of (15.2.4). Finally, we use Lemma 15.1 and obtain the following theorem.

Theorem 15.1. *Let u_t be an eigenfunction of* (15.2.1)–(15.2.2), *normalized according to* (15.3.1). *Then*

$$\dot{\Lambda}(t) = -\oint_{\partial\Omega_t} |\Delta u_t|^2 (\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu^t(y))\, dS_t. \tag{15.3.2}$$

Remark 15.1. If $\big(\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu^t(y)\big) > 0$ for $y \in \partial\Omega_t$, then $\Omega_0 \subset \Omega_t$ for small t. Thus, $\dot{\Lambda}(t)$ is negative, which implies that the first buckling eigenvalue is decreasing under set inclusion. This is in accordance with the inequality $\Lambda(A) \geq \Lambda(B)$ for any eigenvalue of the buckling plate whenever $A \subset B$.

For $t = 0$ Theorem 15.1 implies that

$$\dot{\Lambda}(0) = -\oint_{\partial\Omega} |\Delta u|^2 (v \cdot \nu)\, dS.$$

Assume that Ω is a critical point of $\Lambda(t)$ for all volume preserving perturbations. From (2.3.2) it follows that $|\Delta u| = c_0$ for some constant c_0. This constant can be determined if we use the following representation of $\Lambda(\Omega)$.

If we multiply (15.1.1) by $(x \cdot \nabla u)$ and integrate by parts, we obtain

$$\Lambda(\Omega) = \frac{1}{2} \int_{\partial\Omega} |\Delta u|^2 (x \cdot \nu)\, dS, \qquad \int_{\Omega} |\nabla u|^2\, dx = 1. \tag{15.3.3}$$

Consequently,

$$c_0 := \sqrt{\frac{2\Lambda(0)}{|\Omega|}}.$$

From now on we assume that $\partial\Omega$ consists only of one connected component.

Corollary 15.1. *Let Ω_t be a family of volume preserving perturbations of Ω. Then Ω is a critical point of the energy $\Lambda(t)$, i. e., $\dot{\Lambda}(0) = 0$, if and only if*

$$|\Delta u| = \sqrt{\frac{2\Lambda(0)}{|\Omega|}} \quad \textit{on } \partial\Omega. \tag{15.3.4}$$

Proof. In this case u is a solution of the overdetermined boundary value problem (15.1.1)–(15.1.2) with $\Delta u = \text{const.}$ on $\partial\Omega$. This implies that

$$\Delta u + \Lambda(\Omega) u = \sqrt{\frac{2\Lambda(0)}{|\Omega|}} \quad \text{in } \overline{\Omega}. \tag{15.3.5}$$

In fact, if we set $U := \Delta u + \Lambda(\Omega)u$, (15.1.1)–(15.1.2) imply

$$\Delta U = 0 \text{ in } \Omega \quad \text{and} \quad U = \sqrt{\frac{2\Lambda(0)}{|\Omega|}} \text{ on } \partial\Omega.$$

Hence, $U = \text{const.}$ in $\overline{\Omega}$. From (15.3.5) we get in addition

$$\partial_\nu \Delta u = 0 \quad \text{in } \partial\Omega. \tag{15.3.6}$$

□

Remark 15.2. There exists a $k \in \mathbb{N}$ such that $\Lambda(0) = \mu_k$, where μ_k is the k-th eigenvalue of the membrane with Neumann boundary conditions. This follows immediately from the fact that $u - \sqrt{\frac{2}{|\Omega|\lambda(0)}}$ is an eigenfunction corresponding to μ_k. The value of k is not known.

In [127] it was shown that the overdetermined boundary value problem for $n = 2$ has a solution u if and only if Ω is a ball.

15.4 The second domain variation

We now compute $\ddot{\Lambda}(0)$ for volume preserving domain perturbations Φ_t and under the assumption that $\dot{\Lambda}(0) = 0$, i. e., that Ω is a critical domain.

It was observed previously that the last assumption implies that u satisfies the overdetermined boundary value problem. Consequently, by Lemma 15.2, the shape derivative is a solution of

$$\Delta^2 u' + \Lambda(\Omega)\Delta u' = 0 \quad \text{in } \Omega, \tag{15.4.1}$$

$$u' = 0 \quad \text{on } \partial\Omega, \tag{15.4.2}$$

$$\partial_\nu u' = -c_0(v \cdot \nu) \quad \text{on } \partial\Omega, \quad c_0 := \sqrt{\frac{2\Lambda(0)}{|\Omega|}}. \tag{15.4.3}$$

The normalization (15.3.1) and the fact that we restrict ourselves to volume preserving perturbations imply

$$\int_\Omega (\nabla u \cdot \nabla u')\, dx = 0 \quad \text{and} \quad \oint_{\partial\Omega} \partial_\nu u'\, dS = 0. \tag{15.4.4}$$

In view of Lemma 15.1, (15.3.2) assumes the form

$$\dot{\Lambda}(t) = \oint_{\partial\Omega_t} \Delta u_t (v^t \cdot \nabla(\partial_t u_t(y)))\, dS_t.$$

Next we apply Reynolds' theorem (Theorem 4.1). Let

$$B(u_t) := \Delta u_t(\nu^t \cdot \nabla(\partial_t u_t(y))).$$

The following quantities appear in Reynolds formula:

$$\begin{aligned}\partial_t B(u_t)|_{t=0} &= \Delta u' \partial_\nu u' + \Delta u(\nu' \cdot \nabla u') + \Delta u \partial_\nu u'',\\ \partial_{\nu^t} B(u_t)|_{t=0} &= (v \cdot \nu)\partial_\nu(\Delta u \partial_\nu u').\end{aligned}$$

Then (4.2.2) implies

$$\begin{aligned}\ddot{\Lambda}(0) = &\oint_{\partial\Omega} \Delta u' \partial_\nu u' \, dS + \oint_{\partial\Omega} \Delta u(\nu' \cdot \nabla u') \, dS + \oint_{\partial\Omega} \Delta u \partial_\nu u'' \, dS\\ &+ \oint_{\partial\Omega} (v \cdot \nu)\partial_\nu(\Delta u \partial_\nu u') \, dS + (n-1) \oint_{\partial\Omega} (v \cdot \nu)\Delta u \partial_\nu u' H \, dS.\end{aligned} \tag{15.4.5}$$

Recall that ν' is the shape derivative of ν defined in (2.4.9). By (15.4.2) we have $\nabla u' = \partial_\nu u' \nu$ and by (2.4.12) we obtain

$$\oint_{\partial\Omega} \Delta u(\nu' \cdot \nabla u') \, dS = 0.$$

For the fourth integral we apply (15.3.6) and (15.3.4). Then

$$\begin{aligned}\oint_{\partial\Omega} (v \cdot \nu)\partial_\nu(\Delta u \partial_\nu u') \, dS &= \oint_{\partial\Omega} (v \cdot \nu)\partial_\nu \Delta u \partial_\nu u' \, dS + \oint_{\partial\Omega} (v \cdot \nu)\Delta u \partial_\nu^2 u' \, dS\\ &= 0 + c_0 \oint_{\partial\Omega} (v \cdot \nu)\partial_\nu^2 u' \, dS.\end{aligned}$$

With the help of (15.4.2) (15.2.5) we obtain

$$\partial_\nu^2 u' = \Delta u' - (n-1)\partial_\nu u' H. \tag{15.4.6}$$

Hence, (15.4.3) yields

$$\oint_{\partial\Omega} (v \cdot \nu)\partial_\nu(\Delta u \partial_\nu u') \, dS = c_0 \oint_{\partial\Omega} (v \cdot \nu)\Delta u' \, dS - c_0(n-1) \oint_{\partial\Omega} (v \cdot \nu)\partial_\nu u' H \, dS.$$

With these computations, (15.4.5) simplifies to

$$\ddot{\Lambda}(0) = \oint_{\partial\Omega} \Delta u' \partial_\nu u' \, dS + \oint_{\partial\Omega} \Delta u \partial_\nu u'' \, dS + c_0 \oint_{\partial\Omega} (v \cdot \nu)\Delta u' \, dS.$$

In the first integral on the right-hand side we use (15.4.3) and apply Corollary 15.1 to the second integral.

$$\ddot{\Lambda}(0) = c_0 \oint_{\partial\Omega} \partial_\nu u'' \, dS. \tag{15.4.7}$$

Next it will be shown that $\ddot{\Lambda}(0)$ can be expressed in terms of the first shape derivative u'.

Differentiation of the equation in Lemma 15.1 with respect to t and evaluation at $t = 0$ implies by (15.3.6) and $\Delta u = c_0$ that

$$\begin{aligned}(\nu' \cdot \nabla u') + (\upsilon \cdot D_\nu \nabla u') + \partial_\nu u'' + (\nu \cdot D^2 u' \upsilon)\\ = -\Delta u'(\upsilon \cdot \nu) - c_0(\upsilon \cdot \nu') - c_0(\upsilon \cdot D_\nu \upsilon) - c_0(w \cdot \nu).\end{aligned}$$

Recall that $(\nu' \cdot \nabla u') = 0$ on $\partial\Omega$. Moreover, by (15.4.3)

$$(\upsilon \cdot D_\nu \nabla u') = -c_0(\upsilon \cdot D_\nu \nu)(\upsilon \cdot \nu) = 0.$$

The last equality follows from (2.4.3). Thus,

$$\begin{aligned}\ddot{\Lambda}(0) = &-c_0 \oint_{\partial\Omega}(\upsilon \cdot \nu)\Delta u' \, dS - c_0 \oint_{\partial\Omega}(\nu \cdot D^2 u' \upsilon) \, dS\\ &- c_0^2 \oint_{\partial\Omega}(\upsilon \cdot \nu') \, dS - c_0^2 \oint_{\partial\Omega}(\upsilon \cdot D_\nu \upsilon) \, dS - c_0^2 \oint_{\partial\Omega}(w \cdot \nu) \, dS.\end{aligned} \tag{15.4.8}$$

For the first integral we use (15.4.3) and then (15.4.2). We obtain

$$-c_0 \oint_{\partial\Omega}(\upsilon \cdot \nu)\Delta u' \, dS = \oint_{\partial\Omega} \Delta u' \partial_\nu u' \, dS - \oint_{\partial\Omega} u' \partial_\nu \Delta u' \, dS.$$

Gauss' theorem, partial integration, and equation (15.4.1) for u' lead to

$$-c_0 \oint_{\partial\Omega}(\upsilon \cdot \nu)\Delta u' \, dS = \int_\Omega |\Delta u'|^2 \, dx - \Lambda(\Omega) \int_\Omega |\nabla u'|^2 \, dx.$$

For the second integral we set $\upsilon^\tau = \upsilon - (\upsilon \cdot \nu)\nu$. Then (15.4.6) yields

$$-c_0 \oint_{\partial\Omega}(\upsilon \cdot D^2 u' \nu) \, dS = -c_0 \oint_{\partial\Omega}(\upsilon^\tau \cdot D^2 u' \nu) - (\upsilon \cdot \nu)(\Delta u' - (n-1)\partial_\nu u' H) \, dS.$$

The first integral on the right-hand side can further be simplified:

$$(D^2 u' \nu)_i = \partial_i \partial_j u' \nu_j = \partial_i(\partial_j u' \nu_j) - \partial_j u' \partial_i \nu_j.$$

Because $u' = 0$ on $\partial\Omega$, $\nabla u' = \partial_\nu u' \nu$ must necessarily hold. Hence,

$$(D^2 u' \nu)_i = \partial_i \partial_j u' \nu_j = \partial_i(\partial_j u' \nu_j) - \partial_k u' \nu_k \nu_j \partial_i \nu_j.$$

The last term is zero by (2.4.1). The boundary condition (15.4.3) then implies

$$-c_0 \oint_{\partial\Omega} v^\tau \cdot D(\partial_\nu u' \nu) \cdot \nu \, dS = c_0^2 \oint_{\partial\Omega} v^\tau \cdot \nabla^\tau (v \cdot \nu) \, dS.$$

For the third integral in (15.4.8) we recall Lemma 2.6, where it was shown that

$$\nu' := \partial_t \nu(t,y)|_{t=0} = -\nabla^\tau (v \cdot \nu) \quad \text{and} \quad (\nu \cdot \nu') = 0 \quad \text{on } \partial\Omega.$$

Consequently,

$$-c_0^2 \oint_{\partial\Omega} (v \cdot \nu') \, dS = c_0^2 \oint_{\partial\Omega} v \cdot \nabla^\tau (v \cdot \nu) \, dS = c_0^2 \oint_{\partial\Omega} v^\tau \cdot \nabla^\tau (v \cdot \nu) \, dS.$$

We define

$$\mathcal{E}(u') := \int_\Omega |\Delta u'|^2 \, dx - \Lambda(\Omega) \int_\Omega |\nabla u'|^2 \, dx. \tag{15.4.9}$$

With the simplifications derived above, (15.4.8) assumes the form

$$\begin{aligned} \ddot{\Lambda}(0) = 2\mathcal{E}(u') &+ 2c_0^2 \oint_{\partial\Omega} v^\tau \cdot \nabla^\tau (v \cdot \nu) \, dS \\ &- c_0^2 (n-1) \oint_{\partial\Omega} (v \cdot \nu)^2 H \, dS - c_0^2 \oint_{\partial\Omega} (v \cdot D_\nu v) \, dS \\ &- c_0^2 \oint_{\partial\Omega} (w \cdot \nu) \, dS. \end{aligned} \tag{15.4.10}$$

Next we use the volume constraint (2.3.7) and obtain

$$-c_0^2 \oint_{\partial\Omega} (w \cdot \nu) \, dS = c_0^2 \oint_{\partial\Omega} (v \cdot \nu) \operatorname{div}_{\partial\Omega} v \, dS - c_0^2 \oint_{\partial\Omega} (v^\tau \cdot D_v^\tau \nu) \, dS.$$

By Gauss' theorem (2.2.18)

$$\begin{aligned} -c_0^2 \oint_{\partial\Omega} (w \cdot \nu) \, dS = -c_0^2 \oint_{\partial\Omega} v^\tau \cdot \nabla^\tau (v \cdot \nu) \, dS &+ c_0^2 (n-1) \oint_{\partial\Omega} (v \cdot \nu)^2 H \, dS \\ &- c_0^2 \oint_{\partial\Omega} (v^\tau \cdot D_v^\tau \nu) \, dS. \end{aligned}$$

Thus, (15.4.10) becomes

$$\begin{aligned} \ddot{\Lambda}(0) = 2 \oint_{\partial\Omega} \partial_\nu u' \Delta u' \, dS &+ c_0^2 \oint_{\partial\Omega} v^\tau \cdot \nabla^\tau (v \cdot \nu) \, dS - c_0^2 \oint_{\partial\Omega} (v^\tau \cdot D_v^\tau \nu) \, dS \\ &- c_0^2 \oint_{\partial\Omega} (v \cdot D_\nu v) \, dS. \end{aligned}$$

The integrals which contain the vector field v cancel. To see this recall that

$$v^\tau \cdot D^\tau_v \nu = v^\tau \cdot \nabla^\tau (v \cdot \nu) - v^\tau \cdot D^\tau_\nu v$$

and

$$v^\tau \cdot D^\tau_\nu v = v^\tau \cdot D^\tau_\nu v^\tau + \underbrace{(v \cdot \nu) v^\tau \cdot D^\tau_\nu \nu}_{=0}.$$

Moreover (2.4.1) implies

$$v \cdot D_\nu v = v^\tau \cdot D_\nu v^\tau.$$

Inserting these terms in the formula for $\ddot{\Lambda}(0)$ we obtain the following result.

Theorem 15.2. *Let Φ_t be a volume preserving perturbation. Assume $\dot{\Lambda}(0) = 0$. Then*

$$\ddot{\Lambda}(0) = 2\mathcal{E}(u'),$$

where $\mathcal{E}(u')$ is defined in (15.4.9) *and the shape derivative u' is a solution of the boundary value problem* (15.4.1)–(15.4.3).

15.5 Minimization of the second domain variation

We consider the quadratic functional

$$\mathcal{E}(\varphi) := \int_\Omega |\Delta\varphi|^2 \, dx - \Lambda(\Omega) \int_\Omega |\nabla\varphi|^2 \, dx \tag{15.5.1}$$

for $\varphi \in W^{1,2}_0 \cap W^{2,2}(\Omega)$.

By the definition of Λ we have $\min_{W^{2,2}_0(\Omega)} \mathcal{E}(\varphi) = 0$. In this case the minimizer is a buckling eigenfunction. If the set of admissible functions φ is enlarged, $\mathcal{E}(\varphi)$ need not be bounded from below. A suitable subset of $W^{1,2}_0 \cap W^{2,2}(\Omega)$, containing the shape derivatives defined in (15.4.1)–(15.4.3), will be constructed.

It will be convenient to work with an alternative representation of $\mathcal{E}$. For $\varphi \in W^{1,2}_0 \cap W^{2,2}(\Omega)$ partial integration of the first integral leads to

$$\mathcal{E}(\varphi) = \int_\Omega \left|D^2\varphi\right|^2 - \Lambda(\Omega)|\nabla\varphi|^2 \, dx + \oint_{\partial\Omega} \Delta\varphi \partial_\nu \varphi - \varphi \cdot \left(D^2\varphi\nu\right) dS.$$

Taking into account (15.2.5), we obtain

$$\begin{aligned}\Delta\varphi\partial_\nu\varphi - \nabla\varphi \cdot \left(D^2\varphi\nu\right) &= \partial^2_\nu\varphi\partial_\nu\varphi + (n-1)(\partial_\nu\varphi)^2 H - \nabla\varphi \cdot \left(D^2\varphi\nu\right) \\ &= \nu \cdot D^2\varphi \cdot \nu(\nu \cdot \nabla\varphi) + (n-1)(\partial_\nu\varphi)^2 H - \nabla\varphi \cdot \left(D^2\varphi\nu\right) \\ &= (n-1)(\partial_\nu\varphi)^2 H.\end{aligned}$$

Consequently, we get

$$\mathcal{E}(\varphi) = \int_\Omega \left|D^2\varphi\right|^2 dx - \Lambda(\Omega) \int_\Omega |\nabla\varphi|^2 \, dx + (n-1) \oint_{\partial\Omega} (\partial_\nu \varphi)^2 H \, dS. \tag{15.5.2}$$

Remark 15.3. The functional $\mathcal{E}$ is lower semicontinuous with respect to weak convergence in $W_0^{1,2} \cap W^{2,2}(\Omega)$.

In the sequel we assume $\ddot{\Lambda}(0) \geq 0$. We know from Theorem 15.2 that this is equivalent to $\mathcal{E}(\varphi) \geq 0$ for all φ which are shape derivatives of u.

We set

$$\mathcal{Z} := \left\{\varphi \in W_0^{1,2} \cap W^{2,2}(\Omega) : \oint_{\partial\Omega} \partial_\nu \varphi \, dS = 0, \oint_{\partial\Omega} (\partial_\nu \varphi)^2 \, dS > 0, \int_\Omega \nabla u \cdot \nabla \varphi \, dx = 0\right\}.$$

All shape derivatives are in $\mathcal{Z}$ since they satisfy (15.4.4). However, $\mathcal{Z}$ also contains elements which are not shape derivatives.

The next lemma ensures that $\mathcal{Z}$ is not empty.

Lemma 15.3. *For each* $1 \leq k \leq n$ *the directional derivative* $\partial_k u$ *belongs to* $\mathcal{Z}$. *Furthermore,* $\mathcal{E}(\partial_k u) = 0$.

Proof. First we show that $\partial_k u$ is a shape derivative. Let $1 \leq k \leq n$. Due to (15.1.1) and (15.1.2), $\partial_k u$ satisfies

$$\begin{aligned} \Delta^2 \partial_k u + \Lambda(\Omega) \Delta \partial_k u &= 0 \quad \text{in } \Omega, \\ \partial_k u &= 0 \quad \text{in } \partial\Omega. \end{aligned} \tag{15.5.3}$$

According to (15.2.6) we have $\partial_\nu \partial_k u = c_0 \nu_k$ on $\partial\Omega$. Hence,

$$\oint_{\partial\Omega} \partial_\nu \partial_k u \, dS = c_0 \oint_{\partial\Omega} \nu_k \, dS = 0.$$

In addition, we find that

$$\int_\Omega \nabla u \cdot \nabla \partial_k u \, dx = \frac{1}{2} \oint_{\partial\Omega} |\nabla u|^2 \nu_k \, dS = 0.$$

We obtain that $\partial_\nu \partial_k u$ does not vanish identically on $\partial\Omega$. Thus, $\partial_k u \in \mathcal{Z}$. Moreover, (15.3.6) and (15.5.3) imply

$$\mathcal{E}(\partial_k u) = \int_\Omega (\Delta^2 \partial_k u + \Lambda(\Omega) \Delta \partial_k u) \partial_k u \, dx + \oint_{\partial\Omega} \partial_k \Delta u \partial_\nu \partial_k u \, dS = 0.$$

This proves the lemma. □

It will be shown that

$$\mathcal{E}|_{\mathcal{Z}} \geq 0.$$

For this purpose consider

$$\tilde{\mathcal{E}}(\varphi) := \frac{\mathcal{E}(\varphi)}{\oint_{\partial\Omega} (\partial_\nu \varphi)^2 \, dS},$$

where $\varphi \in \mathcal{Z}$ and we set $\tilde{\mathcal{E}} = \infty$ if $\oint_{\partial\Omega} (\partial_\nu \varphi)^2 \, dS = 0$. By the scaling invariance we may assume

$$\oint_{\partial\Omega} (\partial_\nu \varphi)^2 \, dS = 1.$$

The following statement holds.

Theorem 15.3. *The infimum of the functional $\tilde{\mathcal{E}}$ in $\mathcal{Z}$ is finite.*

Proof. We argue by contradiction. Let us assume that $\inf_{\mathcal{Z}} \tilde{\mathcal{E}} = -\infty$ and consider a sequence $\{\hat{w}_k\}_k \subset \mathcal{Z}$ such that

$$\oint_{\partial\Omega} (\partial_\nu \hat{w}_k)^2 \, dS = 1$$

and

$$\lim_{k\to\infty} \tilde{\mathcal{E}}(\hat{w}_k) = \lim_{k\to\infty} \mathcal{E}(\hat{w}_k) = -\infty.$$

Let $H_\infty := \max_{\partial\Omega} |H|$. Then

$$\left| \oint_{\partial\Omega} H (\partial_\nu \hat{w}_k)^2 \, dS \right| \leq H_\infty < \infty.$$

From (15.5.2) it follows that

$$\mathcal{E}(\hat{w}_k) = \tilde{\mathcal{E}}(\hat{w}_k) \geq -\Lambda(\Omega) \int_\Omega |\nabla \hat{w}_k| \, dx - (n-1) H_\infty. \tag{15.5.4}$$

The assumption $\lim_{k\to\infty} \tilde{\mathcal{E}}(\hat{w}_k) = -\infty$ implies

$$\int_\Omega |\nabla \hat{w}_k|^2 \, dx \overset{k\to\infty}{\longrightarrow} \infty.$$

We define

$$w_k := \frac{1}{\|\nabla \hat{w}_k\|_{L^2(\Omega)}} \hat{w}_k.$$

Then we have

$$\|\nabla w_k\|_{L^2(\Omega)} = 1 \quad \text{and} \quad \oint_{\partial\Omega} (\partial_\nu w_k)^2 \, dS \overset{k\to\infty}{\longrightarrow} 0. \tag{15.5.5}$$

Since $\mathcal{E}(\hat{w}_k) = \tilde{\mathcal{E}}(\hat{w}_k)$ for each $k \in \mathbb{N}$ the estimate (15.5.4) implies

$$\mathcal{E}(w_k) = \frac{\mathcal{E}(\hat{w}_k)}{\|\nabla \hat{w}_k\|^2_{L^2(\Omega)}} = \frac{\tilde{\mathcal{E}}(\hat{w}_k)}{\|\nabla \hat{w}_k\|^2_{L^2(\Omega)}} \geq -\Lambda(0) - C :$$

The infimum of $\mathcal{E}$ taken over $M := \{w_k : k \in \mathbb{N}\}$ is finite. Therefore, there exists a subsequence of $\{w_k\}_k$ – again denoted by $\{w_k\}_k$ – such that

$$\lim_{k\to\infty} \mathcal{E}(w_k) = \inf_M \mathcal{E}.$$

Poincaré's inequality and the previous estimates imply

$$\begin{aligned}
\|w_k\|^2_{W^{2,2}(\Omega)} &= \int_\Omega |D^2 w_k|^2 + |\nabla w_k|^2 + w_k^2 \, dx \\
&\leq \mathcal{E}(w_k) + C \int_\Omega |\nabla w_k|^2 \, dx + (n-1) \oint_{\partial\Omega} |H| (\partial_\nu w_k)^2 \, dS \\
&\leq C.
\end{aligned} \tag{15.5.6}$$

Thus, the sequence $(w_k)_k$ is uniformly bounded in $W^{2,2}(\Omega)$. There exists a $w \in W^{2,2}(\Omega)$ such that $(w_k)_k$ converges weakly to w. In view of (15.5.5), the limit function w satisfies $\|\nabla w\|_{L^2(\Omega)} = 1$ and $w = \partial_\nu w = 0$ on $\partial\Omega$. Hence, $w \in W_0^{2,2}(\Omega)$.

Since by our assumption $\tilde{\mathcal{E}}(\hat{w}_k)$ converges to $-\infty$, there exists a $k_0 \in \mathbb{N}$ such that

$$\mathcal{E}(w_k) = \frac{\tilde{\mathcal{E}}(\hat{w}_k)}{\|\nabla \hat{w}_k\|^2_{L^2(\Omega)}} < 0$$

for all $k \geq k_0$. The functional $\mathcal{E}$ is lower semicontinuous with respect to the weak convergence in $W^{2,2}(\Omega)$, hence $\mathcal{E}(w) < 0$. According to the definition of $\mathcal{E}$ in (15.5.1), this leads immediately to

$$\frac{\int_\Omega |\Delta w|^2 \, dx}{\int_\Omega |\nabla w|^2 \, dx} < \Lambda(\Omega).$$

Since $w \in W_0^{2,2}(\Omega)$, this contradicts the minimum property of $\Lambda(\Omega)$. □

We now consider a minimizing sequence $(\varphi_k)_k \subset \mathcal{Z}$ which satisfies

$$\oint_{\partial\Omega} (\partial_\nu \varphi_k)^2 \, dS = 1$$

for all $k \in \mathbb{N}$. The same arguments as in (15.5.6) imply

$$\|\varphi_k\|^2_{W^{2,2}(\Omega)} \le \tilde{\mathcal{E}}(\varphi_k) + C \int_\Omega |\nabla \varphi_k|^2 \, dx.$$

We apply the modified trace inequality (C.0.3) to $u = \partial_i \varphi_k$. Thus,

$$\int_\Omega |\nabla \varphi_k|^2 \, dx \le \epsilon \int_\Omega \left|D^2 \varphi_k\right|^2 dx + C(\epsilon) \oint_{\partial\Omega} (x \cdot \nu)^2 (\partial_\nu \varphi_k)^2 \, dS.$$

Hence, for sufficiently small $\epsilon > 0$ the sequence $(\varphi_k)_k$ is uniformly bounded in $W^{2,2}(\Omega)$ and φ_k converges weakly to a $\varphi^* \in W^{2,2}(\Omega)$. We deduce that $\varphi^* \in \mathcal{Z}$ and that it satisfies $\tilde{\mathcal{E}}(\varphi^*) = \inf_{\mathcal{Z}} \tilde{\mathcal{E}}$. In addition, we have

$$\oint_{\partial\Omega} (\partial_\nu \varphi^*)^2 \, dS = 1.$$

Hence, φ^* minimizes $\tilde{\mathcal{E}}$ in $\mathcal{Z}$.

Next we compute the first variation under the assumption $v \in W_0^{1,2} \cap W^{2,2}(\Omega)$. Since φ^* is a minimizer it satisfies the constraints in $\mathcal{Z}$.

$$\begin{aligned} &\frac{d}{dt} \frac{\mathcal{E}(\varphi^* + tv)}{\oint_{\partial\Omega} (\partial_\nu(\varphi^* + tv))^2 \, dS}\bigg|_{t=0} \\ &\quad + \mu_1 \frac{d}{dt} \oint_{\partial\Omega} \partial_\nu (\varphi^* + tv) \, dS \bigg|_{t=0} + \mu_2 \frac{d}{dt} \int_\Omega \nabla u \cdot \nabla(\varphi^* + tv) \, dx\big|_{t=0} = 0, \end{aligned}$$

where μ_1 and μ_2 are the Lagrange parameters for the two constraints

$$\oint_{\partial\Omega} \partial_\nu \varphi \, dS = 0, \qquad \int_\Omega \nabla u \cdot \nabla \varphi \, dx = 0.$$

Since $v \in \mathcal{Z}$ was chosen arbitrarily, φ^* satisfies the Euler–Lagrange equality

$$\begin{aligned} \Delta^2 \varphi^* + \Lambda(\Omega) \Delta \varphi^* &= \mu_2 \Delta u \quad \text{in} \quad \Omega, \\ \Delta \varphi^* - \ell \partial_\nu \varphi^* &= \text{const.} \quad \text{in} \quad \partial\Omega, \end{aligned}$$

where $\ell := \min_{\mathcal{Z}} \tilde{\mathcal{E}}$. It is now easy to see that $\mu_2 = 0$. In fact, if we multiply the first equation by u, integrate over Ω, and take into account $u \in W_0^{2,2}(\Omega)$ and (15.3.6), we get

$$\int_\Omega \varphi(\Delta u^2 + \Lambda(\Omega)\Delta u)\,dx = -\mu_2 \int_\Omega |\nabla u|^2\,dx.$$

The fact that u is the eigenfunction associated to $\Lambda(\Omega)$ and $\|\nabla u\|_{L^2(\Omega)} = 1$ implies $\mu_2 = 0$. The following theorem collects the previous results.

Theorem 15.4. *There exists a function $\varphi^* \in \mathcal{Z}$ such that $\tilde{\mathcal{E}}(\varphi^*) = \min_{\mathcal{Z}} \tilde{\mathcal{E}}$. Furthermore, any minimizer $\varphi^* \in \mathcal{Z}$ satisfies*

$$\Delta^2\varphi^* + \Lambda(\Omega)\Delta\varphi^* = 0 \quad \text{in } \Omega, \tag{15.5.7}$$

$$\Delta\varphi^* - \ell\partial_\nu\varphi^* = \text{const.} \quad \text{and} \quad \varphi^* = 0 \quad \text{in } \partial\Omega, \tag{15.5.8}$$

where $\ell = \min_{\mathcal{Z}} \tilde{\mathcal{E}}$.

The next theorem shows $\ell = 0$.

Theorem 15.5. *Suppose $\varphi^* \in \mathcal{Z}$ is a minimizer of $\tilde{\mathcal{E}}$. Hence, φ^* satisfies equations* (15.5.7)–(15.5.8)*. Then we have $\tilde{\mathcal{E}}(\varphi^*) = 0$. In particular, $\mathcal{E} \geq 0$ in $\mathcal{Z}$.*

Proof. Since φ^* satisfies equations (15.5.7)–(15.5.8) and $\partial\Omega$ is smooth, φ^* is a smooth function on $\overline{\Omega}$. Hence, we define a volume preserving perturbation Φ_t of Ω such that

$$\partial_\nu u'(x) = \partial_\nu\varphi^*(x) \quad \text{for } x \in \partial\Omega.$$

Note that this can be achieved by setting $v = -c_0^{-1}\nabla\varphi^*$ on $\partial\Omega$. In this way, each minimizer φ^* implies the existence of vector fields v and w in the sense of Section 2.1. We define $\psi := u' - \varphi^*$. Then $\psi \in W_0^{2,2}(\Omega)$ and

$$\Delta^2\psi + \Lambda(\Omega)\Delta\psi = 0 \quad \text{in } \Omega.$$

The uniqueness of u implies $\psi = \alpha u$ for an $\alpha \in \mathbb{R}$. Since $\varphi^* \in \mathcal{Z}$, equation (15.4.4) yields

$$0 = \int_\Omega \nabla u \cdot \nabla u'\,dx - \int_\Omega \nabla u \cdot \nabla\varphi^*\,dx = \int_\Omega \nabla u \cdot \nabla\psi\,dx = \alpha.$$

Consequently $u' \equiv \varphi^*$. Thus, φ^* is a shape derivative. Since Ω is optimal, $\tilde{\mathcal{E}}(\varphi^*) \geq 0$. Finally we apply Lemma 15.3 and find

$$0 \leq \tilde{\mathcal{E}}(\varphi^*) = \min_{\mathcal{Z}} \tilde{\mathcal{E}} \leq \tilde{\mathcal{E}}(\partial_k u) = 0.$$

This proves the claim. □

15.6 The optimal domain is a ball

We will apply an inequality due to L. E. Payne to show that the optimal domain Ω is a ball. Let λ_2 denote the second Dirichlet eigenfunction of the Laplacian. Payne's inequality (see [95] and [56]) states that for any open bounded domain G

$$\lambda_2(G) \leq \Lambda(G).$$

Equality holds if and only if G is a ball. Note that we do not require any smoothness of G.

We denote by u_1 and u_2 the first and the second Dirichlet eigenfunction for the Laplacian in Ω. Thus, for $k = 1, 2$ we have

$$\begin{aligned} \Delta u_k + \lambda_k(\Omega) u_k &= 0 \quad \text{in } \Omega, \\ u_k &= 0 \quad \text{in } \partial\Omega, \end{aligned}$$

where $\lambda_k(\Omega)$ is the k-th Dirichlet eigenvalue for the Laplacian in Ω. Note that $0 < \lambda_1(\Omega) < \lambda_2(\Omega)$. For short we will write λ_k instead of $\lambda_k(\Omega)$ and Λ instead of $\Lambda(\Omega)$. In addition we assume $\|u_k\|_{L^2(\Omega)} = 1$ and use the fact that

$$\int\limits_\Omega u_1 u_2 \, dx = 0.$$

Without loss of generality we may assume that

$$\int\limits_\Omega u_1 \, dx > 0 \quad \text{and} \quad \int\limits_\Omega u_2 \, dx \leq 0.$$

Consequently, there exists a $t \in (0, 1]$ such that

$$\int\limits_\Omega (1-t)\lambda_1 u_1 + t\lambda_2 u_2 \, dx = 0. \tag{15.6.1}$$

Lemma 15.4. *Set*

$$\psi(x) := (1-t)u_1(x) + tu_2(x) + cu(x) \quad \textit{for } x \in \overline{\Omega},$$

where u is the first buckling eigenfunction in Ω and c is defined as

$$c := \frac{1}{\Lambda} \int\limits_\Omega [(1-t)\lambda_1 u_1 + t\lambda_2 u_2] \Delta u \, dx.$$

Then $\psi \in \mathcal{Z}$.

Proof. 1. Note that $\psi \in W_0^{1,2} \cap W^{2,2}(\Omega)$. The boundary condition $\partial_\nu u = 0$, the equations for u_1 and u_2, and (15.6.1) imply

$$\oint_{\partial\Omega} \partial_\nu \psi \, dS = \int_\Omega (1-t)\Delta u_1 + t\Delta u_2 \, dx$$
$$= -\int_\Omega (1-t)\lambda_1 u_1 + t\lambda_2 u_2 \, dx = 0. \tag{15.6.2}$$

By classical arguments and the unique continuation principle, $\partial_\nu \psi$ does not vanish identically in $\partial\Omega$. Thus, to show that $\psi \in \mathcal{Z}$ it remains to prove that

$$\int_\Omega \nabla u \cdot \nabla \psi \, dx = 0. \tag{15.6.3}$$

We recall that $\Delta u = c_0$ in $\partial\Omega$. Since $\psi \in W^{2,2} \cap W_0^{1,2}(\Omega)$,

$$0 = \int_\Omega (\Delta^2 u + \Lambda \Delta u)\psi \, dx$$
$$= \int_\Omega \Delta u \Delta \psi \, dx - \Lambda \int_\Omega (\nabla u \cdot \nabla \psi) \, dx + c_0 \oint_{\partial\Omega} \partial_\nu \psi \, dS.$$

The last integral vanishes because of (15.6.2). We compute $\Delta\psi$ in the first integral and obtain

$$0 = -\int_\Omega [(1-t)\lambda_1 u_1 + t\lambda_2 u_2]\Delta u \, dx + c \int_\Omega |\Delta u|^2 \, dx - \Lambda \int_\Omega (\nabla u \cdot \nabla \psi) \, dx.$$

Since $\|\nabla u\|_{L^2(\Omega)} = 1$, the second integral is equal to Λ. Thus, the definition of c implies (15.6.3). $\square$

Unless $t = 1$ and Ω is a ball ψ doesn't satisfy (15.4.1). Therefore, it is not a shape derivative.

Since $\psi \in \mathcal{Z}$ Theorem 15.5 implies $\tilde{\mathcal{E}}(\psi) \geq 0$,

$$\mathcal{E}(\psi) = \int_\Omega |\Delta \psi|^2 - \Lambda |\nabla \psi|^2 \, dx \geq 0.$$

We write $\psi = \tilde{\psi} + cu$. After rearranging terms and partial integration we obtain

$$0 \leq \mathcal{E}(\psi) = \int_\Omega |\Delta \tilde{\psi}|^2 - \Lambda |\nabla \tilde{\psi}|^2 \, dx + 2c \int_\Omega \Delta \tilde{\psi}(\Delta u + \Lambda u) \, dx$$
$$+ c^2 \int_\Omega |\Delta u|^2 - \Lambda |\nabla u|^2 \, dx.$$

Since u is the first eigenfunction, the last integral vanishes, The overdetermination (15.3.5) then yields

$$0 \leq \mathcal{E}(\psi) = \int_\Omega |\Delta\tilde{\psi}|^2 - \Lambda|\nabla\tilde{\psi}|^2 \, dx + 2cc_0 \int_\Omega \Delta\tilde{\psi} \, dx.$$

Since $\psi \in \mathcal{Z}$ and $\partial_\nu u = 0$ on $\partial\Omega$,

$$0 = \oint_{\partial\Omega} \partial_\nu \psi \, dS = \oint_{\partial\Omega} \partial_\nu \tilde{\psi} \, dS = \int_\Omega \Delta\tilde{\psi} \, dx.$$

This leads to the final form

$$0 \leq \mathcal{E}(\psi) = \int_\Omega |\Delta\tilde{\psi}|^2 - \Lambda|\nabla\tilde{\psi}|^2 \, dx = \int_\Omega \Delta\tilde{\psi}(\Delta\tilde{\psi} + \Lambda\tilde{\psi}) \, dx.$$

Since $\tilde{\psi} = (1-t)u_1 + tu_2$ and u_1 and u_2 are eigenfunctions,

$$\Delta\tilde{\psi} = -(1-t)\lambda_1 u_1 - t\lambda_2 u_2.$$

Hence,

$$0 \leq \mathcal{E}(\psi) = (1-t)^2\lambda_1(\lambda_1 - \Lambda) + t^2\lambda_2(\lambda_2 - \Lambda).$$

Since $\lambda_1 - \Lambda < 0$ and $\lambda_2 - \Lambda \leq 0$, both summands in $\mathcal{E}(\psi)$ have to vanish. Consequently, $t = 1$ and $\lambda_2(\Omega) = \Lambda(\Omega)$. The case of equality in Payne's inequality implies that Ω is a ball. This proves the main theorem of the section.

Theorem 15.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, smooth domain. Assume the boundary $\partial\Omega$ is connected and Ω minimizes the first buckling eigenvalue $\Lambda(\Omega)$ among all domains of equal volume. Moreover, we assume the simplicity of this eigenvalue. Then Ω is a ball.*

15.7 Notes

One of the earliest papers on this plate model was published by Weinstock in 1937. In his doctoral thesis [125] he showed that there is a discrete spectrum of positive eigenvalues of finite multiplicity and their only accumulation point is ∞. Pólya and Szegö's conjecture is not yet proved in its full generality. However, partial results are known. In [116] Szegö proved the conjecture for all smooth plane domains under the additional assumption that $u > 0$ in Ω.

In [127] H. Weinberger and B. Willms proved the following uniqueness result for $n = 2$. If an optimal simply connected bounded plane domain Ω exists and $\partial\Omega$ is smooth (at least $C^{2,\alpha}$), then Ω is a disc. The result of H. Weinberger and B. Willms was generalized to arbitrary dimensions by Stollenwerk and Wagner in [115]. This paper was motivated by the work of E. Mohr [89], and [90] for a related problem.

M. S. Ashbaugh and D. Buçur [5] showed that among simply connected plane domains of prescribed volume there exists an optimal domain. In [112–114] K. Stollenwerk applied techniques from the theory of free boundary value problems to prove the existence of an optimal domain with prescribed volume. She also derived geometric properties of the optimal domain and regularity properties of the first eigenfunction.

16 A fourth order Steklov problem

In this chapter we investige the domain dependence of the first Steklov eigenvalue of the Bilaplace operator. However several partial results are known. The stability of the ball has not yet been investigated. For volume and area preserving perturbations we compute the first and second variation for the ball. In the plane the first eigenvalue has a local maximum for the disk. In general depending on the perturbation the second variation can be positive or negative.

16.1 The Steklov eigenvalue

We consider open bounded domains in $\mathbb{R}^n$ which are at least of class $C^{4,\alpha}$ for $0 < \alpha \leq 1$. Let

$$\mathcal{R}(\varphi, \Omega) := \frac{\int_\Omega |\Delta\varphi|^2\, dx}{\oint_{\partial\Omega} |\partial_\nu \varphi|^2\, dS} \tag{16.1.1}$$

on the space $W^{2,2} \cap W_0^{1,2}(\Omega)$. Set $\mathcal{R} = \infty$ for $\varphi \in W_0^{2,2}(\Omega)$ and normalize $\oint_{\partial\Omega} |\partial_\nu \varphi|^2\, dS = 1$. We denote

$$q_1(\Omega) = \inf\{\mathcal{R}(\varphi, \Omega) : \varphi \in W^{2,2} \cap W_0^{1,2}(\Omega)\}.$$

Any minimizer u solves the fourth order Steklov eigenvalue problem

$$\Delta^2 u = 0 \quad \text{in } \Omega, \tag{16.1.2}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{16.1.3}$$

$$\Delta u - q_1(\Omega)\partial_\nu u = 0 \quad \text{on } \partial\Omega. \tag{16.1.4}$$

We list some known properties from the literature.

- In [83] (see also [10]) it was shown that there exists a minimizer $u \in W^{2,2} \cap W_0^{1,2}(\Omega)$ which is of constant sign in Ω and which is unique up to a multiplicative constant. Thus, the first eigenvalue q_1 is simple.
- The eigenvalue problem (16.1.2)–(16.1.4) admits infinitely many eigenvalues. All eigenfunctions change sign except the one corresponding to the first eigenvalue $q_1(\Omega)$.
- The lowest eigenvalue of the ball is differentiable with respect to perturbations (see [30]).
- All eigenfunctions are as smooth as the boundary permits (i. e., $u \in C^{4,\alpha}(\overline{\Omega})$) (see [10, Theorem 1] and [50, Theorem 1]).

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16.2 Shape derivatives

For the reader's convenience we prove the following result.

Lemma 16.1. *Let $u \in W^{2,2} \cap W_0^{1,2}(\Omega)$ be a minimizer of $\mathcal{R}(\varphi, \Omega)$. Then u is unique up to a multiplicative constant. Moreover it is of constant sign and q_1 is therefore simple.*

Proof. 1. Suppose that $u \in W^{2,2} \cap W_0^{1,2}(\Omega)$ is a minimizer which changes sign. Consider the solution $w \in W^{2,2} \cap W_0^{1,2}(\Omega)$ of

$$\Delta w = -|\Delta u| \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega.$$

By the strong maximum principle, $w > 0$ in Ω. Moreover,

$$\Delta(w \pm u) = -|\Delta u| \pm \Delta u \leq 0 \text{ in } \Omega, \quad w \pm u = 0 \text{ on } \partial\Omega.$$

We may assume that $\Delta(w \pm u)$ does not vanish identically in Ω. Otherwise by the strong maximum principle $w \pm u$ would vanish identically in Ω. Since u changes sign and $w > 0$ this is a contradiction.

As a consequence, $w \pm u > 0$ in Ω. Hopf's lemma and the smoothness of $w \pm u$ then imply

$$\partial_\nu(w \pm u) \leq 0 \text{ on } \partial\Omega \quad \text{and} \quad \partial_\nu(w \pm u) < 0 \text{ on } \Gamma \subset \partial\Omega$$

for some Γ with positive surface measure. This implies $|\partial_\nu w| \geq |\partial_\nu u|$ on $\partial\Omega$ and $|\partial_\nu w| > |\partial_\nu u|$ on Γ. Consequently

$$q_1 > \frac{\int_\Omega (\Delta w)^2 \, dx}{\oint_{\partial\Omega} (\partial_\nu w)^2 \, dS}.$$

Since w is admissible, this is contradictory. Hence, the minimizer is of constant sign.

2. Next we show that the minizer is unique up to a multiplicative constant. If there exist two linearly independent minimizers $u_1, u_2 \in W^{2,2} \cap W_0^{1,2}(\Omega)$, then any linear combination $c_1 u_1 + c_2 u_2 =: u$ is again a minimizer. It is always possible to find c_1 and c_2 such that u changes sign. This contradicts the previous observation in Step 1. □

Let $\{\Omega_t\}_{|t|<t_0}$ be a family of perturbations of the domain $\Omega \subset \mathbb{R}^n$ as described in Chapter 2 (see (2.1.1)). Recall that for $t \in (-t_0, t_0)$ and $t_0 > 0$ sufficiently small, $\Phi(t, \cdot) : \Omega \to \Omega_t$ is a diffeomorphism. As in Section 2.4.2, $\delta(\cdot, t)$ is the signed distance function with respect to $\partial\Omega_t$. Then $\nu^t(y) := -\nabla_y \delta(y, t)$ for the outer unit normal vector in $y \in \partial\Omega_t$.

We denote by $u_t(y) := u(y, t)$ the first Steklov eigenfunction on Ω_t, and we write for short $q_1(t) := q_1(\Omega_t)$ and $q_1 := q_1(0) = q_1(\Omega)$. Then u_t solves

$$\Delta^2 u_t = 0 \quad \text{in } \Omega_t, \tag{16.2.1}$$

$$u_t = 0 \quad \text{on } \partial\Omega_t, \tag{16.2.2}$$

$$\Delta u_t - q_1(t)\partial_{\nu^t} u_t = 0 \quad \text{on } \partial\Omega_t. \tag{16.2.3}$$

The shape derivative $u'(x) = \partial_t u(\Phi_t(x), t)|_{t=0}$ solves the following boundary value problem.

Lemma 16.2. *The shape derivative u' satisfies*

$$\Delta^2 u' = 0 \quad \textit{in } \Omega, \tag{16.2.4}$$

$$u' + v \cdot \nabla u = 0 \quad \textit{in } \partial\Omega, \tag{16.2.5}$$

$$\Delta u' - q_1 \partial_\nu u' - q_1(\nu \cdot D^2 u v) = -v \cdot \nabla \Delta u + \dot{q}_1(0)\partial_\nu u \quad \textit{in } \partial\Omega. \tag{16.2.6}$$

Proof. The assertions (16.2.4) and (16.2.5) are straightforward. We therefore prove only (16.2.6).

Let $y = \Phi_t(x)$ for $x \in \partial\Omega$. Then $u_t(y) = u(\Phi_t(x), t)$. Thus for all $|t| < t_0$, (16.2.3) may be rewritten as an equation on $\partial\Omega$:

$$\Delta_y u(\Phi_t(x), t) - q_1(t)\nu(\Phi_t(x), t) \cdot \nabla_y u(\Phi_t(x), t) = 0 \quad \forall\, x \in \partial\Omega,$$

where $\nu(\Phi_t(x), t) = \nu^t(y)$.

We now differentiate this equation with respect to t. Hence for all $x \in \partial\Omega$ and $|t| < t_0$,

$$0 = \frac{d}{dt}\{\Delta_y u(\Phi_t(x), t) - q_1(t)\nu(\Phi_t(x), t) \cdot \nabla_y u(\Phi_t(x), t)\}.$$

In order to evaluate the first term we observe that the partial derivatives with respect to t and y commute. Thus,

$$\begin{aligned} \frac{d}{dt}\Delta_y u(\Phi_t(x), t) &= \partial_t \Delta_y u(\Phi_t(x), t) + \partial_t \Phi_t \cdot \nabla_y \Delta_y u(\Phi_t(x), t) \\ &= \Delta_y \partial_t u(\Phi_t(x), t) + \partial_t \Phi_t \cdot \nabla_y \Delta_y u(\Phi_t(x), t). \end{aligned}$$

For $t = 0$ this reads as

$$\frac{d}{dt}\Delta_y u(\Phi_t(x), t)\Big|_{t=0} = \Delta u' + v \cdot \nabla \Delta u.$$

In addition we have

$$\frac{d}{dt}\nu(\Phi_t(x), t)\Big|_{t=0} = \nu' + (v \cdot D_\nu)$$

and

$$\frac{d}{dt}\nabla_y u(\Phi_t(x),t)\Big|_{t=0} = \nabla u' + vD^2 u.$$

By (2.4.12) we have $(\nu' \cdot \nabla u) = 0$. Moreover, since $u = 0$ on $\partial\Omega$ it follows from (2.4.1) that $v \cdot D_\nu \nabla u = \partial_\nu u(v \cdot D_\nu \nu) = 0$. Summation of all the contributions leads to (16.2.6) and establishes the proof of the lemma. □

16.3 First domain variation

In the sequel we will consider perturbations which are volume or perimeter preserving of second order. We shall apply the moving surface technique to establish the next result.

Theorem 16.1. 1. *Let Ω_t be a family of volume preserving perturbations of Ω such that $\dot{\mathcal{V}}(0) = 0$. Then Ω is a critical point of the eigenvalue $q_1(\Omega_t)$, i. e., $\dot{q}_1(0) = 0$, if and only if*

$$(\Delta u)^2 - (n-1)q_1(\Omega)(\partial_\nu u)^2 H - 2(\partial_\nu u)\partial_\nu \Delta u = const. \quad \text{on } \partial\Omega. \tag{16.3.1}$$

2. *Let Ω_t be a family of perimeter preserving perturbations of Ω such that $\dot{\mathcal{S}}(0) = 0$. Then Ω is a critical point of the eigenvalue $q_1(\Omega_t)$ if and only if*

$$\frac{1}{n-1}(\Delta u)^2 - q_1(\Omega)(\partial_\nu u)^2 - \frac{2}{n-1}(\partial_\nu u)\partial_\nu \Delta u = c\,H \quad \text{on } \partial\Omega. \tag{16.3.2}$$

for some constant c.

Proof. We write $q_1(t)$ as the Rayleigh quotient

$$q_1(\Omega_t) = \frac{\int_{\Omega_t} |\Delta u_t|^2\,dy}{\oint_{\partial\Omega_t} |\partial_{\nu^t} u_t|^2\,dS_t}, \tag{16.3.3}$$

where u_t is the first Steklov eigenfunction of problem (16.2.1)–(16.2.3).

The proof will be done in several steps.

Step 1 For the differentiation of the numerator we apply Reynolds' theorem (see (4.2.1)) and find

$$\frac{d}{dt}\int_{\Omega_t} |\Delta u_t|^2\,dy\Big|_{t=0} = 2\int_\Omega \Delta u \Delta u'\,dx + \oint_{\partial\Omega} |\Delta u|^2 (v\cdot\nu)\,dS.$$

If we integrate the first integral on the right side twice by parts and use the fact that u is biharmonic and $u = 0$ on $\partial\Omega$, we obtain

$$2\int_\Omega \Delta u \Delta u'\,dx = 2\oint_{\partial\Omega} \partial_\nu u \Delta u'\,dS.$$

Hence,

$$\frac{d}{dt}\int_{\Omega_t}[\Delta u_t]^2\,dy\bigg|_{t=0} = 2\oint_{\partial\Omega}\partial_\nu u\Delta u'\,dS + \oint_{\partial\Omega}|\Delta u|^2(v\cdot\nu)\,dS.$$

Step 2 For the differentiation of the denominator we apply (4.2.2) and get

$$\begin{aligned}\frac{d}{dt}\oint_{\partial\Omega_t}|\partial_{\nu_t}u_t|^2\,dS_t &= \oint_{\partial\Omega_t}\partial_t(|\partial_{\nu^t}u_t|^2)\,dS_t + \oint_{\partial\Omega_t}\partial_{\nu^t}(|\partial_{\nu_t}u_t|^2)(\partial_t\Phi_t\cdot\nu^t)\,dS_t\\ &\quad + (n-1)\oint_{\partial\Omega_t}|\partial_{\nu_t}u_t|^2 H_t(y)(\partial_t\Phi_t\cdot\nu^t)\,dS_t.\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{d}{dt}\oint_{\partial\Omega_t}|\partial_{\nu_t}u_t|^2\,dS_t\bigg|_{t=0}\\ &\quad = \oint_{\partial\Omega}2\partial_\nu u(\partial_\nu u' + \nabla u\cdot\nu') + (v\cdot\nu)[2\partial_\nu u\partial_\nu^2 u + (n-1)(\partial_\nu u)^2 H]\,dS.\end{aligned}$$

By (2.4.13), $\nabla u\cdot\nu' = 0$; hence,

$$\begin{aligned}\frac{d}{dt}\oint_{\partial\Omega_t}|\partial_{\nu_t}u_t|^2\,dS_t\bigg|_{t=0} &= 2\oint_{\partial\Omega}\partial_\nu u\partial_\nu u'\,dS + 2\oint_{\partial\Omega}(v\cdot\nu)\partial_\nu u\partial_\nu^2 u\,dS\\ &\quad + (n-1)\oint_{\partial\Omega}(v\cdot\nu)(\partial_\nu u)^2 H\,dS.\end{aligned}$$

Step 3 We now compute the first variation $\dot{q}_1(0)$. The normalization

$$\oint_{\partial\Omega_t}|\partial_{\nu_t}u_t|^2\,dS_t = 1 \tag{16.3.4}$$

together with (16.3.3) implies

$$\dot{q}_1(t) = \frac{d}{dt}\int_{\Omega_t}|\Delta u_t|^2\,dy - q_1(t)\frac{d}{dt}\oint_{\partial\Omega_t}|\partial_{\nu_t}u_t|^2\,dS_t.$$

For $t = 0$ the computations in Steps 1 and 2 yield

$$\begin{aligned}\dot{q}_1(0) &= 2\oint_{\partial\Omega}\partial_\nu u\Delta u'\,dS + \oint_{\partial\Omega}|\Delta u|^2 v\cdot\nu\,dS - 2q_1\oint_{\partial\Omega}\partial_\nu u\partial_\nu u'\,dS\\ &\quad - 2q_1\oint_{\partial\Omega}(v\cdot\nu)\partial_\nu u\partial_\nu^2 u\,dS - (n-1)q_1\oint_{\partial\Omega}(v\cdot\nu)(\partial_\nu u)^2 H\,dS.\end{aligned} \tag{16.3.5}$$

Some of the integrals in this expression can be simplified. This is done in the next two steps.

Step 4 First we multiply equation (16.2.6) by $\partial_\nu u$, integrate over $\partial\Omega$, and use the normalization (16.3.4) for $t = 0$. After rearranging the terms we get

$$2\oint_{\partial\Omega} \partial_\nu u \Delta u' \, dS - 2q_1 \oint_{\partial\Omega} \partial_\nu u \partial_\nu u' \, dS = 2\dot{q}_1(0) + 2q_1 \oint_{\partial\Omega} \partial_\nu u(\nu \cdot D^2 u v) \, dS - 2\oint_{\partial\Omega} \partial_\nu u(v \cdot \nabla\Delta u) \, dS.$$

The two terms on the left side correspond to the first and third integrals in (16.3.5). As a consequence all integrals in the expression for (16.3.5) which depend on u' can be replaced by integrals depending only on u and v. Consequently,

$$\begin{aligned}\dot{q}_1(0) = &-\oint_{\partial\Omega} |\Delta u|^2 (v \cdot \nu) \, dS + (n-1)q_1 \oint_{\partial\Omega} (v \cdot \nu)(\partial_\nu u)^2 H \, dS \\ &- 2\oint_{\partial\Omega} \partial_\nu u[q_1(\nu \cdot D^2 u v) - v \cdot \nabla\Delta u - q_1(v \cdot \nu)\partial_\nu^2 u] \, dS.\end{aligned} \tag{16.3.6}$$

Step 5 In this step we replace the term containing $\partial_\nu^2 u$. Since $\Delta u = q_1 \partial_\nu u$ on $\partial\Omega$, this equality remains after tangential differentiation:

$$v^\tau \cdot \nabla\Delta u = q_1 v^\tau \cdot \nabla\partial_\nu u.$$

Replacing v^τ on the right side of this equation by $v - (v \cdot \nu)\nu$ we get

$$v^\tau \cdot \nabla\Delta u = q_1(v \cdot D^2 u \nu) - q_1(v \cdot \nu)\partial_\nu^2 u.$$

Consequently,

$$\begin{aligned}-v \cdot \nabla\Delta u &= -(v \cdot \nu)\partial_\nu \Delta u - (v^\tau \cdot \nabla\Delta u) \\ &= -(v \cdot \nu)\partial_\nu \Delta u - q_1(v \cdot D^2 u \nu) + q_1(v \cdot \nu)\partial_\nu^2 u.\end{aligned}$$

We insert this into (16.3.6):

$$\begin{aligned}\dot{q}_1(0) = &-\oint_{\partial\Omega} |\Delta u|^2 (v \cdot \nu) \, dS + (n-1)q_1 \oint_{\partial\Omega} (v \cdot \nu)(\partial_\nu u)^2 H \, dS \\ &+ 2\oint_{\partial\Omega} \partial_\nu u(v \cdot \nu)\partial_\nu \Delta u \, dS.\end{aligned}$$

If the vector fields are volume preserving in the sense $\dot{\mathcal{V}}(0) = 0$, then $\dot{q}_1(0) = 0$ if and only if (16.3.1) holds.

An analogous result holds if $\dot{\mathcal{S}}(0) = 0$. This proves (16.3.2). □

16.4 First variation for the ball

From now on we consider a family $\{\Omega_t\}_{|t|<t_0}$ of nearly spherical domains such that $\Omega_0 = B_1$. As before we consider volume or area preserving perturbations only.

In Section 3.4.1 it was shown that the first eigenfunction u is

$$u(x) = 1 - |x|^2,$$

which leads to

$$q_1(B_1) = n, \quad \Delta u = -2n, \quad \partial_\nu u|_{\partial B_1} = -2, \quad \partial_i \partial_j u = -2\delta_{ij}. \tag{16.4.1}$$

By Lemma 16.2 the shape derivative solves

$$\Delta^2 u' = 0 \quad \text{in } B_1, \tag{16.4.2}$$

$$u' = 2(v \cdot \nu) \quad \text{and} \quad \Delta u' - n\partial_\nu u' + 2n(v \cdot \nu) = -2\dot{q}_1(B) \quad \text{in } \partial B_1. \tag{16.4.3}$$

We are now in position to prove the main result of this section.

Theorem 16.2. *Let $\{\Phi_t\}_{|t|<t_0}$ be a family of perturbations of the unit ball of given volume or given perimeter. Then the unit ball B_1 is a critical point, i. e., $\dot{q}_1(B_1) = \dot{q}_1(0) = 0$.*

Proof. It remains only to prove the first assertion. We integrate the second equation in (16.4.3) and get

$$2|\partial B_1|\dot{q}_1(0) = -\oint_{\partial B_1} \Delta u' \, dS + n \oint_{\partial B_1} \partial_\nu u' \, dS - 2n \oint_{\partial B_1} (v \cdot \nu) \, dS.$$

Since the perturbations are volume or perimeter preserving, the last integral vanishes. By (16.4.2), $\Delta u'$ is a harmonic function. Therefore, the mean value theorem applies and yields

$$n \oint_{\partial B_1} \partial_\nu u' \, dS = n \int_{B_1} \Delta u' \, dx = \oint_{\partial B_1} \Delta u' \, dS.$$

The first variation then becomes

$$\begin{aligned} 2|\partial B_1|\dot{q}_1(0) &= -\oint_{\partial B_1} \Delta u' \, dS + n \oint_{\partial B_1} \partial_\nu u' \, dS \\ &= -\oint_{\partial B_1} \Delta u' \, dS + n \int_{B_1} \Delta u' \, dS \\ &= 0. \end{aligned}$$

This proves the claim. ☐

16.5 Second variation for the ball

In this section we will compute $\ddot{q}_1(0)$ for volume and area preserving perturbations.

Instead of differentiating the Rayleigh quotient twice we prefer to differentiate the boundary condition (16.2.3). We have

$$\Delta u_t - q_1(t)\partial_{\nu^t} u_t = 0 \quad \text{on } \partial\Omega_t.$$

This approach is less involved, but requires more regularity of the solutions u_t.

We recall that

$$\frac{d}{dt} = \partial_t + \partial_t\Phi_t \cdot \nabla_y. \tag{16.5.1}$$

Hence, if we differentiate (16.2.3) and the interchange of ∂_t and ∂_{y_i} for all $i \in \{1,\dots,n\}$, we obtain

$$\begin{aligned}\dot{q}_1(t)\partial_{\nu^t} u_t = \Delta\partial_t u_t + (\partial_t\Phi_t \cdot \nabla_y \Delta_y u_t) - q_1(t)(\partial_t \nu^t \cdot \nabla_y u_t) - q_1(t)(\partial_t\Phi_t \cdot D_{\nu^t}\nabla u_t)\\ - q_1(t)(\nu^t \cdot \nabla_y \partial_t u) - q_1(t)(\partial_t\Phi_t \cdot D^2 u_t \nu^t).\end{aligned}$$

Next we differentiate this expression with respect to t once again and use $\dot{q}_1(0) = 0$. We consider each term at the right-hand side separately.

Step 1. We have

$$\frac{d}{dt}[\Delta\partial_t u_t]_{t=0} = \Delta u'' + (v \cdot \nabla\Delta u').$$

Step 2. For the calculation of the next expression we keep in mind that $u(x) = 1 - |x|^2$. Thus, all derivatives of order three and higher vanish. Hence,

$$\frac{d}{dt}[\partial_t\Phi_t \cdot \nabla_y \Delta_y u_t]_{t=0} = (v \cdot \nabla\Delta u').$$

Step 3. In order to compute the third term we note that $D^2 u = -2I$ and $\nabla u = -2\nu$, which implies

$$\begin{aligned}-\left[q_1(t)\frac{d}{dt}(\partial_t \nu^t \cdot \nabla_y u_t)\right]_{t=0} &= -q_1(\nu'' + v \cdot D_{\nu'}) \cdot \nabla u - q_1 \nu' \cdot (\nabla u' + v \cdot D^2 u)\\ &= 2q_1(\nu'' \cdot \nu) + 2q_1(v \cdot D_{\nu'}\nu) - q_1(\nu' \cdot \nabla u') + 2q_1(v \cdot \nu').\end{aligned}$$

From (2.4.12) it follows that $(\partial_t \nu_t \cdot \nabla u_t)|_{t=0} = 0$. Consequently

$$\begin{aligned}-\left[q_1(t)\frac{d}{dt}(\partial_t \nu^t \cdot \nabla_y u_t)\right]_{t=0} = q_1[2(\nu'' \cdot \nu) + 2(v \cdot D_{\nu'}\nu)\\ - (\nu' \cdot \nabla u') + 2(v \cdot \nu')].\end{aligned}$$

Step 4. For the computation of the fourth term we recall that $D_{\nu_t}\nabla u_t|_{t=0} = -2(\delta_{ij} - \nu_i\nu_j)\nu_j = 0$. We have

$$-\left[q_1(t)\frac{d}{dt}(\partial_t\Phi_t \cdot D_{\nu^t}\nabla u_t)\right]_{t=0} = -\left[q_1(t)\partial_t\Phi_t \cdot \left(\frac{d}{dt}(D_{\nu^t}\nabla u_t)\right)\right]_{t=0}.$$

Then direct calculation results in

$$\begin{aligned}-\left[q_1(t)\partial_t\Phi_t \cdot \left(\frac{d}{dt}(D_{\nu^t}\nabla u_t)\right)\right]_{t=0} &= 2q_1 \upsilon \cdot (D_{\nu'}\nu) + 2q_1 \sum_{i,j,k=1}^{n} \upsilon_j\upsilon_k\partial_k\partial_j\nu_i\partial_i u \\ &\quad - q_1\upsilon \cdot (D_\nu\nabla u') + 2q_1 \sum_{i,j,k=1}^{n} \upsilon_j\partial_j\nu_i\upsilon_k\partial_k\partial_i u.\end{aligned}$$

For the evaluation of the first sum we recall that $\partial_i u = -2\nu_i$. Hence,

$$\partial_k\partial_j\nu_i\partial_i u = -2[-\delta_{ij}\nu_k + 3\nu_i\nu_j\nu_k - \delta_{ik}\nu_j - \delta_{jk}\nu_i]\nu_i.$$

After adding up over i we get

$$\sum_{i=1}^{n}\partial_k\partial_j\nu_i\partial_i u = 2(\delta_{jk} - \nu_j\nu_k).$$

Since $\partial_k\partial_i u = -2\delta_{ik}$ the second sum is easier. Hence,

$$\begin{aligned}-\left[q_1(t)\frac{d}{dt}(\partial_t\Phi_t \cdot D_{\nu^t}\nabla u_t)\right]_{t=0} &= 2q_1\upsilon \cdot (D_{\nu'}\nu) + 4q_1|\upsilon^\tau|^2 - q_1\upsilon \cdot (D_\nu\nabla u') - 4q_1|\upsilon^\tau|^2 \\ &= 2q_1[(\upsilon \cdot D_{\nu'}\nu) - (\upsilon^\tau \cdot \nabla^\tau u')].\end{aligned}$$

Step 5. Straightforward computation yields

$$-\left[q_1(t)\frac{d}{dt}(\nu^t \cdot \nabla_y\partial_t u_t)\right]_{t=0} = -q_1[(\nu' \cdot \nabla u') + (\upsilon \cdot D_\nu\nabla u') + \partial_\nu u'' + (\upsilon \cdot D^2u'\nu)].$$

In view of (2.4.2) we obtain $\upsilon \cdot D_\nu\nabla u' = \upsilon^\tau \cdot \nabla u'$.

Step 6. We compute the last term. Taking into account that $D^2u = -2I$ and $D_\nu = \delta_{ij} - \nu_i\nu_j$ on ∂B, we obtain

$$-\left[q_1(t)\frac{d}{dt}(\partial_t\Phi_t \cdot D^2u_t\nu^t)\right]_{t=0} = -q_1[-2(w \cdot \nu) + (\upsilon \cdot D^2u'\nu) - 2(\upsilon \cdot \nu') - 2|\upsilon^\tau|^2].$$

In summary we get ($q_1 = n$)

$$\begin{aligned}-2\ddot{q}_1(0) = \Delta u'' &+ 2(\upsilon \cdot \nabla\Delta u') + 2n(\nu'' \cdot \nu) + 4n(\upsilon \cdot D_{\nu'}\nu) \\ &- 2n(\nu' \cdot \nabla u') + 4n(\upsilon \cdot \nu') - 2n(\upsilon^\tau \cdot \nabla^\tau u') - n\partial_\nu u'' \\ &- 2n(\upsilon \cdot D^2u'\nu) + 2n(w \cdot \nu) + 2n|\upsilon^\tau|^2.\end{aligned}$$

This formula holds in all points on ∂B_1.

Lemma 16.3. *Assume $\dot q_1(0) = 0$. Then*

$$
\begin{aligned}
-2\ddot q_1(0) = \Delta u'' - n\partial_\nu u'' + 2(v\cdot\nabla\Delta u') + \frac{n}{2}\left|\nabla^\tau u'\right|^2 \\
- 2n(v^\tau\cdot\nabla^\tau u') - 2n(v\cdot D^2 u'\nu) + 2n(w\cdot\nu) + 2n\left|v^\tau\right|^2,
\end{aligned}
$$

on ∂B_1.

Proof. By Lemma 2.6 we have

$$\nu' = -\nabla^\tau(v\cdot\nu) \quad \text{and} \quad (\nu\cdot\nu') = 0.$$

Since $u = 0$ on the boundary we get $(\nabla u\cdot\nu') = 0$ on ∂B_1. A direct calculation together with $u' = 2(v\cdot\nu)$ yields

$$(\nu\cdot\nu'') = -(\nu')^2 = -\frac{1}{4}\left|\nabla^\tau u'\right|^2.$$

From Lemma 2.9 we recall

$$(\nu\cdot D_{\nu'}\nu) = 0 \quad \text{on } \partial B_1.$$

We apply this result to

$$
\begin{aligned}
A &:= 4q_1(v\cdot D_{\nu'}\nu) + 4q_1(v\cdot\nu') \\
&= 4q_1(v^\tau\cdot D_{\nu'}\nu) + 4q_1(v\cdot\nu)(\nu\cdot D_{\nu'}\nu) + 4q_1(v^\tau\cdot\nu) + 4q_1(v\cdot\nu)(\nu\cdot\nu'),
\end{aligned}
$$

and obtain

$$A = 4q_1(v^\tau\cdot D_{\nu'}\nu) + 4q_1(v^\tau\cdot\nu).$$

Since $D_{\nu'}\nu = \partial_i\nu_j'\nu_j = \partial_i(\nu_j'\nu_j) - \nu_j'\partial_i\nu_j$ and $(D_\nu)_{ij} = \delta_{ij} - \nu_i\nu_j$ it follows that

$$
\begin{aligned}
A &= 4q_1 v^\tau\cdot\nabla(\nu\cdot\nu') - 4q_1(v^\tau\cdot D_\nu\nu') + 4q_1(v^\tau\cdot\nu') \\
&= 4q_1 v^\tau\cdot\nabla(\nu\cdot\nu') + 4q_1(v^\tau\cdot\nu)(\nu\cdot\nu') = 0.
\end{aligned}
$$

This completes the proof. □

In order to get rid of the expressions in u'' we integrate $-2\ddot q_1(0)$ over ∂B_1 and use the fact that $\Delta u''$ is harmonic and $q_1(0) = n$. The mean value theorem then yields

$$\oint_{\partial B_1}[\Delta u'' - n\partial_\nu u'']\,dS = \oint_{\partial B_1}\Delta u''\,dS - n\int_{B_1}\Delta u''\,dS = 0. \tag{16.5.2}$$

For further discussion of $\ddot q_1(0)$ we need the following identity.

Lemma 16.4. *Assume $\dot q_1(0) = 0$. Then the shape derivative u' satisfies the following identity:*

$$\begin{aligned}2(v \cdot \nabla \Delta u') &= 2n(v \cdot D^2 u' \nu) + 2n^2 (v \cdot \nu) u' - 2n(v^\tau \cdot \nabla^\tau u') \\ &\quad - 2n(v \cdot \nu)\partial_\nu u' + 2n(v \cdot \nu)\Delta^* u' + 2(v \cdot \nu)\partial_\nu \Delta u'\end{aligned}$$

on ∂B_1.

Proof. If we decompose v in its tangential and normal components, $v = v^\tau + (v \cdot \nu)\nu$, we get

$$2(v \cdot \nabla \Delta u') = 2(v^\tau \cdot \nabla^\tau \Delta u') + 2(v \cdot \nu)\partial_\nu \Delta u'.$$

From (16.4.3) and $\dot q_1(0) = 0$ it follows

$$\begin{aligned}2(v^\tau \cdot \nabla^\tau \Delta u') &= 2q_1(v^\tau \cdot \nabla^\tau \partial_\nu u') - 2q_1(v^\tau \cdot \nabla^\tau u') \\ &= 2q_1(v \cdot \nabla \partial_\nu u') - 2q_1(v \cdot \nu)\partial_\nu^2 u' - 2q_1(v^\tau \cdot \nabla^\tau u') \\ &= 2q_1(v \cdot D^2 u' \nu) - 2q_1(v \cdot \nu)\partial_\nu^2 u' - 2q_1(v^\tau \cdot \nabla^\tau u').\end{aligned} \tag{16.5.3}$$

By (B.1.5), $\Delta u'$ can be written on ∂B_1 as

$$\Delta u' = \partial_\nu^2 u' + (n-1)\partial_\nu u' + \Delta^* u'.$$

In (16.5.3) we replace $\partial_\nu^2 u'$ by $\Delta u' - (n-1)\partial_\nu u' - \Delta^* u'$, and obtain

$$\begin{aligned}2(v^\tau \cdot \nabla^\tau \Delta u') &= 2q_1(v \cdot D^2 u' \nu) - 2q_1(v \cdot \nu)\Delta u' \\ &\quad + 2(n-1)q_1(v \cdot \nu)\partial_\nu u' + 2q_1(v \cdot \nu)\Delta^* u' - 2q_1(v^\tau \cdot \nabla^\tau u').\end{aligned}$$

Finally we apply the boundary condition (16.4.3) to the second term $2q_1(v \cdot \nu)\Delta u'$ on the right-hand side. Hence,

$$\begin{aligned}2(v^\tau \cdot \nabla^\tau \Delta u') &= 2q_1(v \cdot D^2 u' \nu) - 2q_1(v \cdot \nu)(q_1 \partial_\nu u' - q_1 u') \\ &\quad + 2(n-1)q_1(v \cdot \nu)\partial_\nu u' + 2q_1(v \cdot \nu)\Delta^* u' - 2q_1 v^\tau \cdot \nabla^\tau u'.\end{aligned}$$

The assertion now follows. □

If we insert this identity into Lemma 16.3 and keep in mind that $u' = 2(v \cdot \nu)$, we are led to

$$\begin{aligned}-2\ddot q_1(0) &= \Delta u'' - n\partial_\nu^2 u'' + \frac{n}{2}|\nabla^\tau u'|^2 - 4n(v^\tau \cdot \nabla^\tau u') + 2n(w \cdot \nu) \\ &\quad + 2n|v^\tau|^2 + n^2 u'^2 - nu'\partial_\nu u' + nu'\Delta^* u' + u'\partial_\nu \Delta u'.\end{aligned}$$

Integration over ∂B_1 implies in view of (16.5.2) and Green's theorem ($\oint_{\partial B_1} u'\Delta^* u'\, dS = -\oint_{\partial B_1} |\nabla^\tau u'|^2\, dS$) yields

$$-2\ddot{q}_1(0)|\partial B_1| = -\frac{n}{2}\oint_{\partial B_1} |\nabla^\tau u'|^2\, dS - 4n\oint_{\partial B_1} (v^\tau \cdot \nabla^\tau u')\, dS + 2n\oint_{\partial B_1} (w \cdot \nu)\, dS$$
$$+ \oint_{\partial B_1} [2n|v^\tau|^2 + n^2 u'^2 - nu'\partial_\nu u' + u'\partial_\nu \Delta u']\, dS. \tag{16.5.4}$$

Since u' is biharmonic, integration by parts yields

$$\oint_{\partial B_1} u'\partial_\nu \Delta u'\, dS = -\int_{B_1} |\Delta u'|^2\, dx + \oint_{\partial B_1} \partial_\nu u' \Delta u'\, dS.$$

Moreover, by (16.4.3)

$$\oint_{\partial B_1} u'\partial_\nu \Delta u'\, dS = n\oint_{\partial B_1} (\partial_\nu u')^2\, dS - n\oint_{\partial B_1} \partial_\nu u' u'\, dS - \int_{B_1} (\Delta u')^2\, dx.$$

This together with (16.5.2) leads to

$$-2\ddot{q}_1(0)|B_1| = -\frac{n}{2}\oint_{\partial B_1} |\nabla^\tau u'|^2\, dS + n^2\oint_{\partial B_1} u'^2\, dS$$
$$+ n\oint_{\partial B_1} (\partial_\nu u')^2\, dS - 2n\oint_{\partial B_1} u'\partial_\nu u'\, dS - \int_{B_1} (\Delta u')^2\, dS$$
$$+ 2n\oint_{\partial B_1} (w \cdot \nu)\, dS - 4n\oint_{\partial B_1} (v^\tau \cdot \nabla^\tau u')\, dS + 2n\oint_{\partial B_1} |v^\tau|^2\, dS.$$

By (2.3.14) the last line corresponds to

$$2n\ddot{\mathcal{V}}(0) - 2n(n-1)\oint_{\partial B_1} (v \cdot \nu)^2\, dS = 2n\ddot{\mathcal{V}}(0) - \frac{n(n-1)}{2}\oint_{\partial B_1} u'^2\, dS.$$

In conclusion we have the following theorem.

Theorem 16.3. *Assume $\Omega = B_1$. For volume preserving perturbations there holds*

$$2\ddot{q}_1(0)|\partial B_1| = \frac{n}{2}\oint_{\partial B_1} |\nabla^\tau u'|^2\, dS - \frac{n(n-1)}{2}\oint_{\partial B_1} u'^2\, dS - 2n\ddot{\mathcal{V}}(0)$$
$$- n\oint_{\partial B_1} [u' - \partial_\nu u']^2\, dS + \int_{B_1} (\Delta u')^2\, dS.$$

Remark 16.1. According to (16.4.3), $n(\partial_\nu u' - u') = \Delta u'$. By Fichera's principle (see, e. g., [82]) q_1 can be characterized as

$$q_1 = \min_{\Delta w=0 \text{ on } \partial\Omega} \frac{\oint_{\partial\Omega} w^2\,dS}{\int_\Omega w^2\,dx}.$$

If we set $w = \Delta u'$, then $\oint_{\partial B_1} (\Delta u')^2\,dS \geq n\int_{B_1} (\Delta u')^2\,dx$. Hence, for volume preserving transformations

$$4\ddot{q}_1(0)|B_1| \leq \oint_{\partial B_1} |\nabla^\tau u'|^2\,dS - (n-1)\oint_{\partial B_1} u'^2\,dS = \ddot{\mathcal{S}}_0(0).$$

Discussion of the sign of $\ddot{q}_1(0)$ in the ball

It follows from Section 3.4.1 that the shape derivative can be written as a Fourier series of Steklov eigenvalues defined in (3.4.2). Since $\dot{\mathcal{V}}(0) = 0$ and therefore $\oint_{\partial B_1} (v\cdot\nu)\,dS = 0$, it follows that the term of Y_0 vanishes. Hence,

$$u'(x) = \sum_{k=1}^{\infty}\sum_{i=1}^{d_i}\left(b_{k,i}r^k + \frac{a_{k,i}}{4k+2n}r^{2+k}\right)Y_{k,i}(\xi), \quad \xi \in \partial B_1. \tag{16.5.5}$$

This implies

$$\Delta u'|_{\partial B_1} = \sum_{k=1}^{\infty}\sum_{i=1}^{d_i} a_{ki}Y_{k,i}(\xi), \quad \xi \in \partial B_1. \tag{16.5.6}$$

Analogously,

$$(v\cdot\nu)|_{\partial B_1} = \sum_{k=1}^{\infty}\sum_{i=1}^{d_i} c_{k,i}Y_{k,i}(\xi), \quad \xi \in \partial B_1. \tag{16.5.7}$$

In view of the boundary conditions, the coefficients $a_{k,i}$ and $b_{k,i}$ can be expressed in terms of $c_{k,i}$. In fact, $u' = 2(v\cdot\nu)$ on ∂B_1 implies

$$2c_{k,i} = b_{k,i} + \frac{a_{k,i}}{2(n+2k)}.$$

Under the assumption that $\dot{q}_1(0) = 0$, the second boundary condition is

$$\Delta u' = n(\partial_\nu u' - u').$$

By (16.5.6) this implies

$$a_{k,i} = n\left[(k-1)b_{k,i} + \frac{k+1}{4k+2n}a_{k,i}\right].$$

Consequently,

$$a_{k,i} = \frac{(n+2k)(k-1)n}{k} c_{k,i},$$
$$b_{k,i} = \frac{n-k(n-4)}{2k} c_{k,i}. \tag{16.5.8}$$

Straightforward computations lead to

$$\int_{B_1} (\Delta u')^2\,dx = \sum_{k=1}^{\infty}\sum_{i=1}^{d_i} \frac{a_{k,i}^2}{2k+n},$$

$$n \oint_{\partial B_1} [u' - \partial_\nu u']^2\,dS = \frac{1}{n}\oint_{\partial B_1} (\Delta u')^2\,dS = \frac{1}{n}\sum_{k=1}^{\infty}\sum_{i=1}^{d_i} a_{i,k}^2,$$

$$\frac{n}{2}\oint_{\partial B_1} |\nabla^\tau u'|^2\,dS - \frac{n(n-1)}{2}\oint_{\partial B_1} u'^2\,dS = 2n\sum_{k=1}^{\infty}\sum_{i=1}^{d_i} [k(k+n-2)-(n-1)]c_{k,i}^2.$$

In addition, we have

$$\begin{aligned}\int_{B_1} (\Delta u')^2 - \frac{1}{n}\oint_{\partial B_1} (\Delta u')^2\,dS &= -\sum_{k=1}^{\infty}\sum_{i=1}^{d_i} \frac{2k}{n(2k+n)} a_{k,i}^2\\ &= -2n\sum_{k=1}^{\infty}\sum_{i=1}^{d_i} \frac{(n+2k)(k-1)^2}{k} c_{k,i}^2.\end{aligned}$$

Introducing these expressions into the formula in Theorem 16.3, we find

$$\begin{aligned}2\ddot{q}_1(0)|\partial B_1| &= -2n\ddot{\mathcal{V}}(0) + 2n\sum_{k=1}^{\infty}\sum_{i=1}^{d_i} [k(k+n-2)-(n-1)]c_{k,i}^2\\ &\quad - \sum_{k=1}^{\infty}\sum_{i=1}^{d_i} \frac{2n(n+2k)(k-1)^2}{k} c_{k,i}^2\\ &= -2n\ddot{\mathcal{V}}(0) - 2n\sum_{k=1}^{\infty}\sum_{i=1}^{d_i} \frac{(k-1)(k^2-k-n)}{k} c_{k,i}^2.\end{aligned} \tag{16.5.9}$$

Volume preserving perturbations such that $(v\cdot\nu) = \sum_{i=1}^{n} c_{1,i} Y_{1,i}(\xi)$ are obviously in the kernel of $\ddot{q}_1(0)$. The kernel can be eliminated by imposing the barycenter condition.

The main results of this chapter are summarized as follows.

Theorem 16.4. *Assume $\Omega = B_1$ and let $\{\Omega_t\}_{t<|t_0|}$ be a family of nearly spherical domains.*
(i) If $\dot{\mathcal{V}}(0) = 0$, then $\dot{q}_1(0) = 0$.

(ii) *If $\dot{\mathcal{V}}(0) = \ddot{\mathcal{V}}(0) = 0$ and if the barycenter condition holds then*

$$2\ddot{q}_1(0)|\partial B_1| = -2n \sum_{k=2}^{\infty} \sum_{i=1}^{d_i} \frac{(k-1)(k^2-k-n)}{k} c_{k,i}^2.$$

This theorem allows us to deduce the following stability result.

Theorem 16.5. *Let $k^*(n) := [\frac{1+\sqrt{1+4n}}{2}]$.*
(i) *In the plane $\ddot{q}_1(0) \leq 0$ and $\ddot{q}_1(0) = 0$ iff $k = 2$.*
(ii) *If $n > 2$ then $\ddot{q}_1(0) \geq 0$ for all perturbations of the form*

$$\Phi_t(x) = x + t\rho\,\nu + \frac{t^2}{2} w + o(t^2), \quad \textit{where } \rho = \sum_{k=2}^{k^*(n)} c_k\, Y_k(\xi).$$

Moreover $\ddot{q}_1(0) \leq 0$ for all perturbations for which

$$\rho = \sum_{k=k^*(n)+1}^{\infty} c_k\, Y_k(\xi)$$

The ball is therfore neither stable nor unstable.

From (16.5.9) we can also discuss stability with respect to perimeter preserving perturbations, e. g., $\dot{\mathcal{S}}(0) = \ddot{\mathcal{S}}(0) = 0$. By (2.3.28) we have

$$\ddot{\mathcal{V}}(0) = \frac{1}{n-1}\ddot{\mathcal{S}}(0) - \frac{1}{n-1}\oint_{\partial B_1} |\nabla^\tau(v\cdot\nu)|^2\, dS + \oint_{\partial B_1} (v\cdot\nu)^2\, dS.$$

Then

$$2\ddot{q}_1(0)|\partial B_1| = -\frac{2n}{n-1}\ddot{\mathcal{S}}(0) - 2n \sum_{k=2}^{\infty} \sum_{i=1}^{d_i} \frac{(k-1)}{k(n-1)} [(n-2)k^2 - 2(n-1)k - n(n-1)] c_{k,i}^2.$$

Theorem 16.6. *Assume $\Omega = B_1$ and let $\{\Omega_t\}_{t<|t_0|}$ be a family of nearly spherical domains.*
(i) *If $\dot{\mathcal{S}}(0) = 0$, then $\dot{q}_1(0) = 0$.*
(ii) *If $\dot{\mathcal{S}}(0) = \ddot{\mathcal{S}}(0) = 0$ and if the barycenter condition holds then*

$$\ddot{q}_1(0)|\partial B_1| = -n \sum_{k=2}^{\infty} \sum_{i=1}^{d_i} \frac{k-1}{k(n-1)} [(n-2)k^2 - 2(n-1)k - n(n-1)] c_{k,i}^2.$$

This theorem allows us to deduce the following stability result.

Theorem 16.7. *Let $k^{**}(n) := [\frac{n-1+\sqrt{n^3-2n+1}}{n-2}]$ for $n > 2$.*

(i) In the plane

$$\ddot{q}_1(0)|\partial B_1| = 4\sum_{k=2}^{\infty}\sum_{i=1}^{d_i}\frac{k^2-1}{k}c_{k,i}^2 > 0.$$

Hence the ball is a local minimizer.

(ii) If $n > 2$ then $\ddot{q}_1(0) \geq 0$ for all perturbations of the form

$$\Phi_t(x) = x + t\rho\, \nu + \frac{t^2}{2}\, w + o(t^2), \quad \textit{where } \rho = \sum_{k=2}^{k^{**}(n)} c_k\, Y_k(\xi).$$

Moreover $\ddot{q}_1(0) \leq 0$ for all perturbations for which

$$\rho = \sum_{k=k^{**}(n)+1}^{\infty} c_k\, Y_k(\xi)$$

The ball is therfore neither stable nor unstable.

16.6 Notes

It is known that among all convex domains in $\mathbb{R}^n$ of given volume or perimeter there exists an optimal one which minimizes q_1 (see [31, Theorem 4.6] and [3, Theorem 5]). Kuttler [83] showed – in two dimensions – that a square has a first eigenvalue q_1, which is strictly smaller than the one of a disc. Ferrero, Gazzola, and Weth (see [50, formulas (1.14) and (1.15)]) improved this result. More recently Antunes and Gazzola [3] gave numerical evidence that the optimal planar shape is the regular pentagon. Buçur, Ferrero, and Gazzola [30] showed that the ball is a critical domain of q_1 among all volume and area preserving smooth perturbations. By means of the second domain variations it was shown in [84] that the ball is unstable.

A General remarks

A.1 Jacobi's formula for determinants

Let $A(t)$ be an $n \times n$-matrix depending on the parameter t. Then by Jacobi's formula

$$\frac{d}{dt}\det A(t) = \det A(t) \cdot T\left(A^{-1}(t)\frac{d}{dt}A(t)\right), \tag{A.1.1}$$

where $T(C)$ denotes the trace of the matrix C. Set $A(t) = I + tB + \frac{t^2}{2}C$. Then for small $|t|$,

$$\begin{aligned} A^{-1}(t) &= I - tB - \frac{t^2}{2}C + t^2B^2 + o(t^2), \\ \frac{d}{dt}A(t) &= B + tC, \\ A^{-1}(t)\frac{d}{dt}A(t) &= B + t(C - B^2) + t^2\left(B^3 - \frac{3CB}{2}\right) + o(t^2). \end{aligned}$$

We apply these computations to the Taylor expansion of

$$\det\left(I + tB + \frac{t^2}{2}C\right) = 1 + a_0 t + a_1\frac{t^2}{2} + O(|t|^3), \quad |t| \ll 1.$$

Then

$$\begin{aligned} a_0 &= \det(A(t))\big|_{t=0} = T(B), \\ a_1 &= \frac{d}{dt}\left(\det A(t)T\left(A^{-1}(t)\frac{d}{dt}A(t)\right)\right)\bigg|_{t=0} = T^2(B) + \frac{d}{dt}T(B + t(C - B^2) + o(t))\big|_{t=0} \\ &= T^2(B) - T(B^2) + T(C). \end{aligned}$$

A.2 Hölder continuity

Let $f(x)$ be a function defined in $\Omega \subset \mathbb{R}^n$ and let β_i, $i = 1, 2, \ldots, n$, be nonnegative integers. Set

$$D^\beta f = \frac{\partial f^{|\beta|}}{\partial x_1^{\beta_1}\partial x_2^{\beta_2}\ldots\partial x_n^{\beta_n}}, \quad \text{where } \beta_i \geq 0,\ \beta = (\beta_1, \beta_2, \ldots, \beta_n), \text{ and } |\beta| = \sum_{i=1}^n \beta_i.$$

- The function f is in the class C^k if $\|f\|_{C^k} := \max_{|\beta|\leq k}\sup_{x\in\Omega}|D^\beta f| < \infty$.
- The function f is in the class $C^{0,\alpha}$, $0 < \alpha < 1$, if the Hölder coefficient $|f|_{C^{0,\alpha}} = \sup_{x,y\in\Omega}\frac{|f(x)-f(y)|}{|x-y|^\alpha}$ is bounded.
- The function f is in the Hölder space $C^{k,\alpha}$ if $\|f\|_{C^{k,\alpha}} = \|f\|_{C^k} + \max_{|\beta|=k}|D^\beta f|_{C^{0,\alpha}} < \infty$.

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The following definition is taken from [65].

Definition A.1. A bounded domain Ω or equivalently $\partial\Omega$ is in C^k or $C^{m,\alpha}$ if at any point $x_0 \in \partial\Omega$ there is a ball B_{r_0} centered at x_0 and a one-to-one map $\Psi : B_{r_0} \to D \in \mathbb{R}^n$ such that:
1. $\Psi(x) > 0$ if $x \in B_{r_0} \cap \Omega$,
2. $\Psi(x) = 0$ if $x \in B_{r_0} \cap \partial\Omega$,
3. Ψ and Ψ^{-1} in C^k or $C^{m,\alpha}$.

Denote by

$$\delta(x, \partial\Omega) := \begin{cases} -\inf\{|x - z| : z \in \partial\Omega, x \in \mathbb{R}^n \setminus \Omega\}, \\ \inf\{|x - z| : z \in \partial\Omega, x \in \overline{\Omega}\} \end{cases}$$

the signed distance function. In [65, Lemma 14.16] it is shown that if $\partial\Omega \in C^k$ or $C^{k,\alpha}$, $k \geq 2$, and $\alpha \geq 0$, then there is a neighborhood of $\partial\Omega$ such that $\delta(\cdot, \partial\Omega)$ is in C^k or in $C^{k,\alpha}$, respectively.

A.2.1 Whitney's extension theorem

The deformation of domains can be described by the perturbation of the boundary. Especially for the change of variables method we need their extensions to the whole domain. This is possible by Whitney's theorem.

Theorem A.1. *Let $\partial\Omega \subset \mathbb{R}^n$ be locally parametrized by $\tilde{x}(\xi)$, where $\xi \in \mathbb{R}^{n-1}$ are local coordinates. On $\partial\Omega$ consider a vector field $\tilde{v}(\xi) \in C^m$. Then by Whitney's theorem* [126], *$\tilde{v}$ can be extended into Ω such that its extension $v : \overline{\Omega} \to \mathbb{R}^n$ with $v = \tilde{v}$ on $\partial\Omega$ and $v \in C^m(\overline{\Omega})$.*

A particular extension of a Hadamard diffeomorphism $\Phi_t : \partial\Omega \to \partial\Omega_t$ is constructed in the next theorem.

Theorem A.2. *Let $\partial\Omega \subset \mathbb{R}^n$ be locally parametrized by $\tilde{x}(\xi)$, where $\xi \in \mathbb{R}^{n-1}$ are local coordinates. Let $\Phi_t : \partial\Omega \to \partial\Omega_t$ be a diffeomorphism such that*

$$\tilde{\Phi}_t(\xi) = \tilde{x}(\xi) + t\tilde{p}(\xi)\tilde{\nu}(\xi) + \frac{t^2}{2}\tilde{w}(\xi).$$

Then there exists a diffeomorphism $\Phi_t : \overline{\Omega} \to \overline{\Omega}_t$ such that

$$\Phi_t|_{\partial\Omega} = \tilde{\Phi}_t|_{\partial\Omega}.$$

Proof. We follow the arguments in [117, Section 3.4]. We set

$$\Omega_\eta^+ := \{x \in \mathbb{R}^n \setminus \Omega : \operatorname{dist}(x, \partial\Omega) < \eta\}, \quad \Omega_\eta^- := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \eta\}.$$

It is well known (cf. the remark in the previous section) that if $\partial\Omega \in C^{2,\alpha}$, there exists a sufficiently small $\eta_0 > 0$ such that $\partial\Omega_{\eta_0}^- \in C^{2,\alpha}$ as well. As a consequence, there exists a local coordinate system such that any $x \in \Omega_{\eta_0}^-$ can be written in the form $x = \tilde{x}(\xi) - \eta\tilde{\nu}(\xi)$ where $0 < \eta < \eta_0$.

Next we assume that t_0 is chosen so small that for $|t| < t_0$, $\partial\Omega_t \subset \Omega_{\eta_0}^+ \cup \Omega_{\eta_0}^-$. This implies that for $0 < \eta < \frac{\eta_0}{2}$,

$$\tilde{x}(\xi) - \eta\tilde{\nu}(\xi) + \big(\tilde{\Phi}_t(\xi) - \tilde{x}(\xi)\big) \in \Omega_{\eta_0}^+ \cup \Omega_{\eta_0}^-.$$

Finally, we choose a cutoff function $\vartheta \in C^\infty(\Omega)$ with $0 \le \vartheta \le 1$ such that:
- $0 < \vartheta < 1$ in $\Omega_{\eta_0}^- \setminus \Omega_{\frac{\eta_0}{2}}^-$
- $\vartheta = 0$ in $\Omega \setminus \Omega_{\eta_0}^-$,
- $\vartheta = 1$ in $\Omega_{\frac{\eta_0}{2}}^-$.

We then define the following extension:

$$\Phi_t(x) := x + \vartheta(x)\big(\tilde{\Phi}_t(\xi) - \tilde{x}(\xi)\big). \qquad \square$$

B Geometry

B.1 Some concepts from differential geometry

Denote by $\{x_i\}_{i=1}^n$ the Cartesian coordinates and let $(a \cdot b)$ be the Euclidean scalar product of the vectors $a, b \in \mathbb{R}^n$. Let $\partial\Omega \subset \mathbb{R}^n$ be an $(n-1)$-dimensional surface given in local coordinates by $\tilde{x}(\xi)$, $\xi \in \mathbb{R}^{n-1}$. The vectors $\tilde{x}_{\xi_i}$, $i = 1, \dots, n-1$, form a basis in the tangent space. Let ν be the outer unit normal of $\partial\Omega$. Two tensors play an important role in geometry, namely:

– the metric tensor $g_{ij} = (\tilde{x}_{\xi_i} \cdot \tilde{x}_{\xi_j})$, g^{ij} is the inverse of g_{ij},
– the second fundamental form $L_{ij} = -(\tilde{x}_{\xi_i\xi_j} \cdot \nu) = \frac{1}{2}((x_{\tilde{\xi}_i} \cdot \nu_{\xi_j}) + (\tilde{x}_{\xi_j} \cdot \nu_{\xi_i}))$.

The surface element of $\partial\Omega$ is given by

$$dS = \sqrt{\det g_{ij}}\, d\xi, \quad d\xi := d\xi_1\, d\xi_2 \cdots d\xi_{n-1}.$$

The Laplace–Beltrami operator on $\partial\Omega$ is expressed in terms of the metric tensor as follows:

$$\Delta^* = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi_i}\left(\sqrt{g} g^{ij} \frac{\partial}{\partial \xi_j} \right), \quad g := \det g_{ij}.$$

The formulas of Gauss and Weingarten are

$$\begin{aligned} \tilde{x}_{\xi_i\xi_j} &= \Gamma^s_{ij} \tilde{x}_{\xi_s} - L_{ij}\nu, \quad \text{where } \Gamma^s_{ij} = \frac{1}{2} g^{sq}\left(\frac{\partial g_{iq}}{\partial \xi_j} - \frac{\partial g_{ij}}{\partial \xi_q} + \frac{\partial g_{jq}}{\partial \xi_i} \right), \\ \tilde{\nu}_{\xi_j} &= g^{ki} L_{ji} \tilde{x}_{\xi_k}. \end{aligned} \tag{B.1.1}$$

The operator $g^{ik}L_{kj}$ is called Weingarten operator . The eigenvalues κ_i of $g^{ik}L_{kj}$ are called *principal curvatures* and the eigenvectors are tangent to the lines of principal curvatures. An important notion in our study is the *mean curvature* H. It is defined as

$$H(\xi) = \frac{1}{n-1} g^{ik} L_{ik}(\xi).$$

The mean curvature is invariant with respect to changes of the coordinates. Therefore,

$$H = \frac{1}{n-1} \sum_{i=1}^{n-1} \kappa_i.$$

If the local coordinates are chosen such that x_{ξ_s} is an eigenvector, then

$$g^{\ell i} L_{ik} \frac{\partial \tilde{x}_k(\xi)}{\partial \xi_s} = \kappa_s \delta_{\ell k} \frac{\partial \tilde{x}_k(\xi)}{\partial \xi_s} \quad \Longrightarrow \quad L_{ik} \frac{\partial \tilde{x}_k(\xi)}{\partial \xi_s} = \kappa_s g_{ik} \frac{\partial \tilde{x}_k(\xi)}{\partial \xi_s}.$$

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Moreover, for $s \neq r$ we have $g_{ik} \frac{\partial x_k}{\partial \xi_s} \frac{\partial x_i}{\partial \xi_r} = 0$.

Example B.1. Let $\mathcal{S}$ be a surface represented by a graph $(x', f(x'))$, where $x' \in \mathbb{R}^{n-1}$. In this case

$$g_{ij} = \delta_{ij} + \partial_i f \partial_j f, \quad \tilde{\nu} = \frac{(-\nabla f, 1)}{\sqrt{1 + |\nabla f|^2}},$$
$$L_{ij} = -\frac{\partial_i \partial_j f}{\sqrt{1 + |\nabla f|^2}}.$$

Here $\tilde{\nu}$ is the outer normal of the domain $\{x : x \in \mathbb{R}^n,\ x_n < f(x')\}$. The principal curvatures κ_i are the eigenvalues of

$$-\frac{\partial_i \partial_j f}{\sqrt{1 + |\nabla f|^2}}.$$

We can always choose a coordinate system such that the origin lies on $P \in \partial\Omega$, $\mathbb{R}^{n-1}$ coincides with the tangent space at P, and the e_n-axis is the outer normal. In this case $\nabla f(0) = 0$, $g_{ij}(0) = \delta_{ij}$, and $L_{ij}(0) = -\partial_i \partial_j f(0)$.

B.1.1 Curvilinear coordinates in $\mathbb{R}^n$

Spherical coordinates

The spherical coordinates $(r, \theta_1, \theta_2, \dots, \theta_{n-1})$, $r > 0$, $\theta_j \in (0, \pi)$, if $j = 2, \dots, n-1$ and $\theta_1 \in (0, 2\pi)$, are defined as follows (see Figure B.1):

$$\begin{aligned}
x_1 &= r\left(\prod_{j=2}^{n-1} \sin\theta_j\right) \cos\theta_1,\\
x_2 &= r\left(\prod_{j=2}^{n-1} \sin\theta_j\right) \sin\theta_1,\\
x_3 &= r\left(\prod_{j=3}^{n-1} \sin\theta_j\right) \cos\theta_2,\\
&\ \vdots\\
x_{n-1} &= r \sin\theta_{n-1} \cos\theta_{n-2},\\
x_n &= r\cos\theta_{n-1}.
\end{aligned}$$

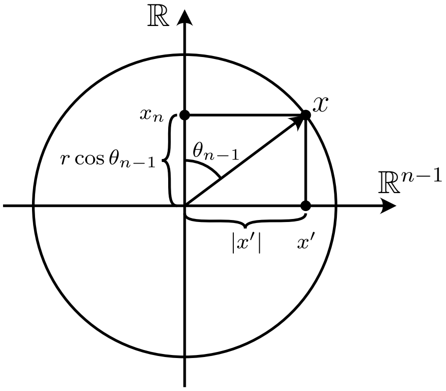

Figure B.1: Spherical coordinates.

The volume element is

$$dx = r^{n-1}\sin(\theta_1)^{n-2}\sin(\theta_2)^{n-3}\dots\sin(\theta_{n-2})\,dr\,d\theta_1\,d\theta_2\dots\,d\theta_{n-1}.$$

Set $\theta := (\theta_1, \theta_2, \dots, \theta_{n-1})$. A generic point in $\mathbb{R}^n$ is given by $x(r,\theta) = r\xi(\theta)$, where $\xi(\theta)$ is a point on $S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$. Clearly, $(\xi(\theta)\cdot\xi_{\theta_j}(\theta)) = 0$.

Hence, the square of the line element is $|dx|^2 = dr^2 + r^2(\xi_{\theta_i}\cdot\xi_{\theta_j})\,d\theta_i\,d\theta_j$. This implies that

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 G(S^{n-1}) \end{pmatrix} \quad \text{and} \quad g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2}G^{-1}(S^{n-1}) \end{pmatrix},$$

where $G(S^{n-1})$ is the metric tensor on S^{n-1}. In this case $g = r^{2(n-1)}\det G(S^{n-1})$ and thus

$$\Delta_x = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta^*, \quad \text{where } \Delta^* = \text{Laplace–Beltrami operator on } S^{n-1}. \tag{B.1.2}$$

B.1.2 Normal coordinates

Let $\partial\Omega \in C^k$ for $k > 2$. In a neighborhood of $\tilde{x}(\xi) \in \partial\Omega$ we introduce the coordinates (ξ, r) as follows:

$$y(\xi, r) = \tilde{x}(\xi) + r\tilde{\nu}(\xi), \quad \text{where } \tilde{x}(\xi) \in \partial\Omega,\ \xi = (\xi_1, \xi_2, \dots, \xi_{n-1}).$$

The square of the line element is

$$\begin{aligned} |dy|^2 &= \left|(\tilde{x}_{\xi_i} + r\tilde{\nu}_{\xi_i})\,d\xi_i + \tilde{\nu}(\xi)\,dr\right|^2 \\ &= \big[\tilde{x}_{\xi_i}\cdot\tilde{x}_{\xi_j} + r\underbrace{(\tilde{\nu}_{\xi_i}\cdot\tilde{x}_{\xi_j} + \tilde{\nu}_{\xi_j}\cdot\tilde{x}_{\xi_i})}_{2L_{ij}} + r^2\tilde{\nu}_{\xi_i}\cdot\tilde{\nu}_{\xi_j}\big]\,d\xi_i\,d\xi_j + dr^2. \end{aligned}$$

The volume element is then given by

$$dy = \sqrt{\det(g_{ij} + 2rL_{ij} + r^2 \nu_{\xi_i} \cdot \nu_{\xi_j})} \prod_{i=1}^{n-1} d\xi_i \, dr.$$

From Jacobi's formula (cf. Section A.1) and the definition of the mean curvature (2.2.3) we have for small $|r|$

$$\begin{aligned}\det[g_{ij} + 2rL_{ij} + o(r)] &= \det[g_{ij}(I + 2rg^{ik}L_{ik}) + o(r)]\\ &= \det[g_{ij}] \det[I + 2rg^{ik}L_{jk} + o(r)]\\ &= \det[g_{ij}][1 + 2r(n-1)H + o(r)].\end{aligned}$$

Since

$$\sqrt{1 + 2r(n-1)H + o(r)} = \big(1 + (n-1)rH + o(r)\big),$$

the volume element assumes now the form

$$dy = \big(1 + (n-1)rH + o(r)\big)\, dr \, dS_{\partial\Omega}. \tag{B.1.3}$$

Suppose that Ω is represented locally as a graph as in Example B.1. We also assume that $\mathbb{R}^{n-1}$ is in the tangent space at $P \in \partial\Omega$ and that the coordinate axes $\{e_j\}_{j=1}^{n-1}$ point in the direction of the principal curvatures.

Then, evaluating $|dy|^2$ at P we obtain

$$|dy|^2 = [\delta_{ij} + 2r\kappa_j\delta_{ij} + r^2\kappa_i\kappa_j\delta_{ij}]\, dx_i\, dx_j + dr^2 = \sum_{i=1}^{n-1}(1 + r\kappa_i)^2\, dx_i^2 + dr^2.$$

In order to compute the Laplacian, we need $\sqrt{g} = \prod_{i=1}^{n-1}(1 + \kappa_i r)$ and

$$g^{ij} = \begin{pmatrix} (1+r\kappa_1)^{-2} & 0 & \dots & \dots & 0 \\ 0 & (1+r\kappa_2)^{-2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & \vdots & \vdots & (1+r\kappa_{n-1})^{-2} & 0 \\ 0 & 0 & \vdots & 0 & 1 \end{pmatrix}.$$

Consequently,

$$\Delta = \frac{1}{\sqrt{g}}\partial_i\left\{\frac{\sqrt{g}}{(1+r\kappa_i)^2}\partial_i\right\} + \frac{\partial^2}{\partial r^2} + \frac{(n-1)H}{\sqrt{g}}\frac{\partial}{\partial r}. \tag{B.1.4}$$

For small r and for $\kappa_i \in C^1$ it follows that

$$\Delta = \Delta^* + \frac{\partial^2}{\partial r^2} + (n-1)H\frac{\partial}{\partial r} + O(r), \quad \Delta^* = \text{Laplace–Beltrami operator on } \partial\Omega. \tag{B.1.5}$$

B.2 Implicit function theorem

The fact that geometric quantities in perturbed domains $\{\Omega_t\}_t$ are differentiable with respect to the parameter t relies on the following version of the implicit function theorem (see, e. g., [23, Sections 3.1.10–3.1.13]). For two Banach spaces X and Z the space of bounded linear mappings from X to Z will be denoted by $\mathcal{L}(X, Z)$.

Theorem B.1. *Let X, Y, and Z be Banach spaces and let U and V be open subsets of X and Y, respectively. Let $F \in C^r(U \times V, Z)$, $r \geq 1$, and fix $(x_0, y_0) \in U \times V$. Suppose $\partial_x F(x_0, y_0) \in \mathcal{L}(X, Z)$ is an isomorphism. Then there exists an open neighborhood $U_1 \times V_1 \subset U \times V$ of (x_0, y_0) such that for each $y \in V_1$ there exists a unique point $(\xi(y), y) \in U_1 \times V_1$ satisfying $F(\xi(y), y) = F(x_0, y_0)$. Moreover, the map ξ is in $C^r(V_1, Z)$ and satisfies*

$$\nabla\xi(y) = -\big(\partial_x F(\xi(y), y)\big)^{-1} \circ \partial_y F(\xi(y), y).$$

C Sobolev spaces and inequalities

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. The Sobolev spaces $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$, where $1 < p < \infty$ and p is a nonnegative integer, are given by

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega),\ 0 \le |\alpha| \le n\}, \quad D^\alpha u \text{ weak derivative},$$
$$W_0^{m,p}(\Omega) := \text{closure of } C_0^\infty(\Omega) \text{ in } W^{p,m}(\Omega).$$

For $u \in W^{m,p}(\Omega)$ we set

$$\|u\|_{W^{m,p}}(\Omega) := \left\{ \sum_{0 \le |\alpha| \le m} \left|D^\alpha u\right|^p_{L^p(\Omega)} \right\}^{1/p}.$$

Equipped with this norm, $W^{m,p}(\Omega)$ is a reflexive Banach space.

In particular, the space $W^{1,2}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{1,2}(\Omega)} := \left(\int_\Omega |\nabla u|^2 + u^2\, dx \right)^{\frac{1}{2}}.$$

The expression

$$\left\||\nabla u|\right\|_{L^2(\Omega)} = \left(\int_\Omega |\nabla u|^2\, dx \right)^{\frac{1}{2}}$$

is a half-norm on $W^{1,2}(\Omega)$ which vanishes on the space $\mathcal{P}$ consisting of the constants. If $|u|_p$ is any norm on $\mathcal{P}$, then

$$\|u\| = \left(|u|_P^2 + \int_\Omega |\nabla u|^2\, dx \right)^{\frac{1}{2}}$$

is an equivalent norm on $W^{1,2}(\Omega)$.

Poincaré's inequality states that there exists a positive constant c_p such that

$$\int_\Omega u^2\, dx \le c_p \int_\Omega |\nabla u|^2\, dx \quad \forall\, u \in W_0^{1,2}(\Omega). \tag{C.0.1}$$

Hence, $\||\nabla u|\|_{L^2(\Omega)}$ is a norm in $W_0^{1,2}(\Omega)$.

A function $u \in W^{1,2}(\Omega)$ possesses a trace satisfying the *trace inequality*

$$\oint_{\partial\Omega} u^2\, dS \le c_T \|u\|^2_{W^{1,2}(\Omega)}, \tag{C.0.2}$$

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for a suitable positive constant c_T which depends only on Ω. A closer look at the proof (see, e. g., [47]) shows that Young's inequality gives the modified trace inequality: For any positive ϵ there exists a number $c_\epsilon > 0$ such that

$$\oint_{\partial\Omega} u^2\, dS \leq \epsilon \int_\Omega |\nabla u|^2\, dx + c_\epsilon \int_\Omega u^2\, dx \quad \forall\, u \in W^{1,2}(\Omega). \tag{C.0.3}$$

Friedrich's inequality states that there exists a $c_F > 0$ such that for all $u \in W^{1,2}(\Omega)$,

$$\int_\Omega u^2\, dx \leq c_F\Big(\int_\Omega |\nabla u|^2\, dx + \oint_{\partial\Omega} u^2\, dS\Big). \tag{C.0.4}$$

In our studies we shall often need the modified Friedrich inequality: For any $\epsilon > 0$ there exists c_ϵ such that

$$\int_\Omega u^2\, dx \leq \epsilon \int_\Omega |\nabla u|^2\, dx + c_\epsilon \oint_{\partial\Omega} u^2\, dS \quad \forall\, u \in W^{1,2}(\Omega). \tag{C.0.5}$$

In Lipschitz domains the following embedding theorems hold:

$$\begin{aligned} &W^{1,2}(\Omega) \hookrightarrow L^2(\Omega), \text{ compact},\\ &W^{1,2}(\Omega) \hookrightarrow L^2(\partial\Omega), \text{ compact}. \end{aligned} \tag{C.0.6}$$

For the discussion of fourth order problems we need the spaces $W^{2,2}(\Omega)$ and $W_0^{2,2}(\Omega)$, equipped with the norms

$$\|u\|_{W^{2,2}(\Omega)} = \Big(\int_\Omega [(\Delta u)^2 + |\nabla u|^2 + u^2]\, dx\Big)^{\frac{1}{2}} \quad \text{and} \quad \|u\|_{W_0^{2,2}(\Omega)} = \Big(\int_\Omega (\Delta u)^2\, dx\Big)^{\frac{1}{2}}.$$

If $\Omega \subset \mathbb{R}^n$ is bounded, then

$$W_0^{2,2}(\Omega) \hookrightarrow W_0^{1,2}(\Omega), \text{ compact} \tag{C.0.7}$$

see, e. g., [2, Theorem A8.1].

D Bilinear forms

D.1 Abstract setting

Let $\mathcal{H}$ and $\mathcal{L}$ be two real Hilbert spaces such that the embedding $\mathcal{H} \subset \mathcal{L}$ is compact. We consider a symmetric bilinear form

$$a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}, \quad (u, v) \to a(u, v)$$

which satisfies

$$a(u, v) \leq c\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}$$

for some positive constant c.

The inner product on $\mathcal{H}$ is denoted by $(u, v)_{\mathcal{H}}$ and induces the norm $\|u\|_{\mathcal{H}}$. Similarly, we denote by $(u, v)_{\mathcal{L}}$ the inner product on $\mathcal{L}$ and the induced norm by $\|u\|_{\mathcal{L}}$.

Then the variational formulation of the eigenvalue problem is as follows.

Find a pair $(u, \lambda) \in \mathcal{H} \times \mathbb{R}$ such that

$$a(u, v) = \lambda(u, v)_{\mathcal{L}} \quad \forall\, v \in \mathcal{H}. \tag{D.1.1}$$

The number λ is an *eigenvalue of a* if there exists an element $u \in \mathcal{H}$, $u \neq 0$, such that (D.1.1) holds. The element u is called *eigenvector corresponding to* λ. The number of independent eigenvectors corresponding to λ is called the *multiplicity of* λ. If the multiplicity is one, then the eigenvalue is *simple*. The smallest (first) eigenvalue λ_1 is characterized by the Rayleigh principle

$$\lambda_1 = \inf\left\{\frac{a(u, u)}{(u, u)_{\mathcal{L}}} : u \in \mathcal{H}, u \neq 0\right\}.$$

The *kernel* of the bilinear form a is the set of all elements $u \in \mathcal{H}$ such that $a(u, v) = 0$ for all $v \in \mathcal{H}$.

A special situation occurs if a is *coercive*, that is, if there exists a positive number γ such that

$$a(u, u) \geq \gamma\|u\|^2_{\mathcal{H}}.$$

In that case $a(\cdot, \cdot)$ defines an inner product on $\mathcal{H}$ which is equivalent to $(\cdot, \cdot)_{\mathcal{H}}$. As a consequence, (D.1.1) shows that any eigenvalue λ is positive. The following theorem can be found in many textbooks.

Theorem D.1. *Let $a(\cdot, \cdot)$ be a coercive symmetric bounded bilinear form on $\mathcal{H} \times \mathcal{H}$ and let $\mathcal{H} \subset \mathcal{L}$ be compactly embedded. Then the following holds true.*

(i) *There are at most countably many eigenvalues $(\lambda_k)_k$ with finite multiplicity each. There is no finite accumulation point.*

(ii) *The eigenvalues are ordered $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$.*

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(iii) *There are countably infinitely many eigenvectors corresponding to nonzero eigenvalues* $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots$. *They are orthonormal with respect to the inner product* $(\cdot,\cdot)_{\mathcal{L}}$.

(iv) *Let* E_{λ_k} *be the eigenspace corresponding to the eigenvalue* λ_k. *Then* $\dim E_{\lambda_k} < \infty$ *for all* $k \in \mathbb{N}$ *and* $\mathcal{H} = \bigoplus_{k=1}^{\infty} E_{\lambda_k}$. *Moreover, the eigenfunctions* $\{u_j\}_{j=1}^{\infty}$ *form an orthonormal basis for* $\mathcal{H}$.

(v) *The higher eigenvalues are characterized by Poincaré's principle*

$$\lambda_i = \min\left\{\frac{a(v,v)}{(v,v)_{\mathcal{L}}} \;:\; v \perp \bigoplus_{k=1}^{i-1} E_{\lambda_k}\right\}.$$

An equivalent version of this principle is as follows. Let $L_k \subset \mathcal{H}$ *be a k-dimensional linear subspace. Then*

$$\lambda_i = \min_{L_{i-1}\subset\mathcal{H}}\left(\max_{v\in L_{i-1}} \frac{a(v,v)}{(v,v)_{\mathcal{L}}}\right).$$

D.2 Examples

D.2.1 Eigenvalues on the sphere

Consider the eigenvalue problem

$$\Delta^*\phi + \Lambda\phi = 0 \quad \text{on } \partial B_1 \subset \mathbb{R}^n.$$

In this case

$$a(u,v) = \oint_{\partial B_1} (\nabla^\tau u \cdot \nabla^\tau v)\, dS \quad \text{and} \quad (u,v) = \oint_{\partial B_1} uv\, dS.$$

Here $\mathcal{H} = \mathcal{L} = W^{1,2}(\partial B_1)$ and $\mathcal{L} = L^2(\partial B_1)$. The eigenvalues are $\Lambda_k = \frac{k(n+k-2)}{R^2}$, $k = 0,1,\dots$, and the corresponding eigenfunctions are the spherical harmonics $Y_k\left(\frac{x}{|x|}\right)$. By Rayleigh's principle,

$$\Lambda_0 = 0 = \inf_v \frac{\oint_{\partial B_R} |\nabla^\tau v|^2\, dS}{\oint_{\partial B_R} v^2\, dS}, \quad v \in W^{1,2}(\partial B_R),$$

$$\Lambda_1 = \frac{n-1}{R^2} = \inf_v \frac{\oint_{\partial B_R} |\nabla^\tau v|^2\, dS}{\oint_{\partial B_R} v^2\, dS}, \quad v \in W^{1,2}(\partial B_R), \quad \oint_{B_R} v\, dS = 0,$$

$$\Lambda_2 = \frac{2n}{R^2} = \inf_v \frac{\oint_{\partial B_R} |\nabla^\tau v|^2\, dS}{\oint_{\partial B_R} v^2\, dS}, \quad v \in W^{1,2}(\partial B_R), \quad \oint_{B_R} v\, dS = 0 \text{ and } \oint_{\partial B_R} xv\, dS = 0.$$

D.2.2 Robin eigenvalues

Let Ω be a bounded domain. For $\mathcal{H} = W^{1,2}(\Omega)$ and $\mathcal{L} = L^2(\Omega)$ we consider the following bilinear form:

$$a(u,v) := \int_\Omega (\nabla u \cdot \nabla v)\,dx + \alpha \oint_{\partial\Omega} uv\,dS, \quad \alpha \in \mathbb{R}.$$

By Friedrich's inequality (C.0.4), we have for $\alpha > 0$

$$a(u,u) \geq c_F^{-1} \min\{\alpha, 1\} \int_\Omega u^2\,dx \quad \text{for all } u \in W^{1,2}(\Omega).$$

If $\alpha < 0$, the modified trace inequality (C.0.3) gives

$$a(u,u) \geq \int_\Omega |\nabla u|^2\,dx - |\alpha| \left[\epsilon \int_\Omega |\nabla u|^2\,dx + c_\epsilon \int_\Omega u^2\,dx\right] \geq -|\alpha| c_\epsilon \int_\Omega u^2\,dx.$$

Hence, for $\alpha > 0$ the bilinear form $a(u,v)$ is coercive. If $\alpha < 0$, there exists a first negative eigenvalue. Theorem D.1 applies in both cases.

The eigenspace of λ_1 is one-dimensional. In fact, if u is an eigenfunction of λ_1, then clearly $|u|$ is a minimizer of $\frac{a(u,u)}{\|u\|^2_{L^2(\Omega)}}$ as well. Both functions solve the Euler–Lagrange equation $\Delta w + \lambda_1 w = 0$ in Ω. Classical regularity theory then implies the analyticity of any solution. Hence, u cannot change sign. Any other eigenfunction is orthogonal to u by Theorem D.1(v). Thus, it must change sign. This gives the uniqueness of the first eigenfunction up to a multiplicative constant. Hence, λ_1 is simple.

D.2.3 Buckling eigenvalue

Let Ω be a bounded domain. For $\mathcal{H} = W_0^{2,2}(\Omega)$ and $\mathcal{L} = W_0^{1,2}(\Omega)$ we consider the following bilinear form:

$$a(u,v) := \int_\Omega \Delta u \Delta v\,dx, \quad (u,v)_{\mathcal{L}} := \int_\Omega (\nabla u \cdot \nabla v)\ dx.$$

Poincaré's inequality yields $a(u,u) \geq \gamma \|u\|^2_{W^{2,2}(\Omega)}$. Due to the compact embedding (C.0.7), the statements of Theorem D.1 apply.

D.2.4 Steklov eigenvalue of fourth order

Let Ω be a bounded Lipschitz domain. We equip the space $\mathcal{H} = W^{2,2} \cap W_0^{1,2}(\Omega)$ with the scalar product $(u, v)_{\mathcal{H}} = \int_\Omega \Delta u \Delta v \, dx$ (see, e. g., [49]). Let $\mathcal{L} = L^2(\partial\Omega)$. The bilinear form is defined as

$$a(u, v) := \int_\Omega \Delta u \Delta v \, dx, \quad (u, v)_{\mathcal{L}} := \int_{\partial\Omega} uv \, dS.$$

The embedding from $W^{2,2} \cap W_0^{1,2}(\Omega)$ to $L^2(\partial\Omega)$ is compact due to some results in [92, Theorem 6.2 in Chapter 2].

Notation

$\Phi_t : \Omega \to \Omega_t$	diffeomorphism
$\Omega_t = \Phi_t(\Omega) = \{x + tv(x) + \frac{t^2}{2} w(x) + o(t^2) : x \in \Omega\}$	domain perturbation
$\{\tilde{x}(\xi) : \xi \in \mathbb{R}^{n-1}\}$	parametric representation of $\partial\Omega$
$\rho = (v \cdot \nu)$	normal component of a vector field on $\partial\Omega$
$\delta(x) := \begin{cases} -\mathrm{dist}(x, \partial\Omega) & \text{if } x \in \mathbb{R}^n \setminus \Omega, \\ \mathrm{dist}(x, \partial\Omega) & \text{if } x \in \Omega \end{cases}$	signed distance function
$\tilde{f}(\xi) := f(x(\xi))$	function defined on $\partial\Omega$
$\nabla^\tau \tilde{f} := g^{ij} \tilde{f}_{\xi_j} \tilde{x}_{\xi_i}$	tangential gradient
$G = g_{ij} := (x_{\xi_i} \cdot x_{\xi_j}),\ G^{-1} := g^{ij}$	metric tensor on $\partial\Omega$
$\partial\Omega_t = \{\tilde{x}(\xi) + t\tilde{v}(\xi) + \frac{t^2}{2} \tilde{w}(\xi) + o(t^2) : \xi \in \mathbb{R}^{n-1}\}$	parametric representation of Ω_t
$\partial_i := \frac{\partial}{\partial x_i}$	derivative
$(D_v)_{ij} := \partial_i v_j$	Jacobi matrix
$(D_{\Phi_t})_{ij} = \partial_i (x_j + tv_j(x) + \frac{t^2}{2} w_j(x))$	
$\partial_i^* \tilde{v} := g^{ij} \tilde{v}_{\xi_j}$	
$\mathrm{div}_{\partial\Omega}\, \tilde{v} := (\partial_i^* \tilde{v} \cdot \tilde{v}_{\xi_i})$	tangential divergence
Δ^*	Laplace–Beltrami operator on $\partial\Omega$
$T(A) := a_{ii}$	trace of the matrix $(A)_{ij} = a_{ij}$
L_{ij}	second fundamental form
$\mathcal{L} = g^{ik} L_{kj}$	Weingarten's operator
$(A^t)_{ij} := a_{ji}$	transpose of A
$D_v : D_v := \partial_i v_j \partial_j v_i = T(D_v^2)$	trace of the square of D_v^2
$D^2 u(y) := \frac{\partial^2 u}{\partial y_i \partial y_j}$	Hessian matrix
$\tilde{u}(t) := u\left(x + tv(x) + \frac{t^2}{2} w(x) + o(t^2)\right)$	pullback of $u(y)$ to Ω
$b_t := \frac{1}{\lvert\Omega_t} \int_{\Omega_t} y\, dy$	barycenter of Ω_t
$\oint_{\partial\Omega} (v \cdot \nu) x\, dS = 0$	Φ_t satisfies the barycenter condition
$Q_g(u')$	quadratic form
$f^\pm(x) = \max\{\pm f(x), 0\}$	
$\mathbb{S}^n$ and $\mathbb{H}^n$	spherical and hyperbolic space
$\dot{\mathcal{V}}(0)$ and $\ddot{\mathcal{V}}(0)$	first and second variation of the volume
$\dot{\mathcal{S}}(0)$ and $\ddot{\mathcal{S}}(0)$	first and second variation of the surface area

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