2

Regular second-order elliptic boundary value problems

2.1 Foreword

In the following chapters, we shall carry out the study of some elliptic boundary value problems in domains whose boundaries are not smooth: for example, domains with polygonal boundaries. Throughout this study, we shall make an extensive use of results concerning the same kind of boundary value problems in domains with regular boundaries. (We shall call these problems 'regular'.) The theory of such boundary value problems can be found in Hörmander (1963) and Lions and Magenes (1960–63), for instance. These authors consider problems of arbitrary order in domains with a C^∞ boundary. Less general boundary value problems are solved in domains with less smooth boundaries by Agmon (1965), Miranda (1970), Něcas (1967).

In spite of the great number of possible references on elliptic boundary value problems, we shall devote this chapter to a self-contained study of second-order strongly elliptic boundary value problems in regular domains. Apart from the objective of making this book as self-contained as possible, the purpose of this chapter is two-fold.

Three kinds of methods, at least, have proved to be quite successful in solving regular elliptic boundary value problems. Namely,

(a) *A priori* estimates as in Agmon (1965), Lions and Magenes (1960–63), Miranda (1970) and Něcas (1967);
(b) parametrices as in Hörmander (1963);
(c) pseudo-differential operators as in Seeley (1966).

These methods have long been known to allow one to solve elliptic boundary value problems involving operators with coefficients only a few times differentiable, in domains with boundaries also only a few times differentiable. However, most of the available references deal only with the C^∞ case. It is within the scope of this book to try to see to what extent the assumptions on the coefficients and on the boundary can be weakened when applying those methods. Actually we shall restrict ourselves to the *a priori* estimates method, which seems to be more flexible in this respect. It turns out that the most general domains that one is able to handle with such methods, have a boundary of class $C^{1,1}$. This assumption clearly excludes polygonal boundaries.

The second purpose of this chapter is to give a brief account of the L_p theory. The L_p theory of linear elliptic boundary value problems is of the utmost importance in the study of nonlinear problems. The reason is that, for a given m and a given domain Ω, the Sobolev space $W_p^m(\Omega)$ is more likely to be an algebra when p is large. The core of the L_p theory is the celebrated L_p *a priori* estimate proved by Agmon *et al.* (1959). These authors deal with problems of great generality. Their proofs can hardly be found, even in simpler particular cases, outside this original reference (but see Freeman and Schechter (1974)). We give here a simplified proof of the L_p estimate in the case of second-order strongly elliptic boundary value problems. This proof is closer to the L_2 proof, since it uses the partial Fourier transform, with the Plancherel theorem being replaced by the famous L_p multiplier theorem of Mih'lin (1956) (see also Hörmander (1960)). The proof makes use of a technical idea introduced for a different purpose by Boutet de Monvel (1971). The related existence and uniqueness results will be worked out in domains whose boundary is only of class $C^{1,1}$. This does not seem to be standard material and will be useful in the next chapters. (Here we attempt to work with the weakest assumptions on the domain but not on the coefficients of the operators. Indeed, in most practical cases one deals with simple operators—such as operators with constant coefficients—in bad domains.)

Let us now introduce the following framework for the remainder of this chapter. The domain Ω will be a bounded open subset of $\mathbb{R}^n$. The operator A is a second-order strongly elliptic real operator in Ω, and B is a real boundary operator of order d ($d=0$ or 1). In most of the forthcoming sections, we shall make the following assumptions:

(a) *the boundary Γ of Ω is of class $C^{1,1}$ (see Definition 1.2.1.1)*
(b) *the operator A is in divergence form:*

$$Au = \sum_{i,j=1}^{n} D_i(a_{ij}D_j u)$$

with $a_{i,j} = a_{j,i} \in C^{0,1}(\bar{\Omega})$ and there exists $\alpha > 0$ such that

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \leqslant -\alpha\, |\xi|^2 \tag{2,1,1}$$

for all $x \in \bar{\Omega}$ and $\xi \in \mathbb{R}^n$.

(c) *B is either the identity operator (thus $d = 0$) or*

$$Bu = \sum_{i=1}^{n} b_i D_i u \tag{2,1,2}$$

with $b_i \in C^{0,1}(\bar{\Omega})$, $i \leqslant i \leqslant n$ (then $d = 1$) and $\sum_{i=1}^{n} b_i \nu^i \neq 0$ everywhere on Γ. (In other words, Γ is nowhere characteristic for B.)

For a given function f defined in Ω and a given function g defined on Γ, we shall look for u defined in $\bar{\Omega}$ such that

$$\begin{cases} Au = f & \text{in } \Omega \\ Bu = g & \text{on } \Gamma. \end{cases} \tag{2,1,3}$$

For some technical reasons, it will often be convenient to consider the related problem with an extra real parameter λ as follows.

$$\begin{cases} Au + \lambda u = f & \text{in } \Omega \\ \qquad Bu = g & \text{on } \Gamma. \end{cases} \tag{2,1,4}$$

Later on, we shall add lower order terms to A and B and get rid of λ.

In the particular case where $B = I$, our problem is just a Dirichlet problem for the equation $Au + \lambda u = f$. Another particular case is when Bu is the 'co-normal derivative' of u corresponding to A, i.e.

$$b_i = \sum_{j=1}^{n} a_{i,j} \nu^i \qquad \text{on } \Gamma,$$

where ν^j, $1 \leqslant j \leqslant n$ are the components of the unit outer normal vector field on Γ. Then, our problem is a Neumann problem for the equation $Au + \lambda u = f$. In the general case when $d = 1$, we are solving the equation $Au + \lambda u = f$ with an 'oblique' boundary condition.

Actually, we shall pose the problem (2,1,4) in the framework of Sobolev spaces. Thus we shall look for conditions ensuring that

$$T_{p,\lambda} : u \mapsto \{Au + \lambda u, \gamma Bu\}$$

is an isomorphism from $W_p^2(\Omega)$ onto $L_p(\Omega) \times W_p^{2-d-1/p}(\Gamma)$, $1 < p < \infty$.

Let us conclude this introductory section with some examples of the results which we will look for in this chapter. These examples are related to the Laplace operator Δ. First, Theorem 2.4.2.5 implies that for every $f \in L_p(\Omega)$ and every $g \in W_p^{2-1/p}(\Gamma)$, there exists a unique $u \in W_p^2(\Omega)$ which is a solution of

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \gamma u = g & \text{on } \Gamma \end{cases}$$

provided $1 < p < \infty$ and Ω is a bounded open subset of $\mathbb{R}^n$ with a $C^{1,1}$

boundary. Then, Theorem 2.4.2.6 implies that for every $f \in L_p(\Omega)$ and every $g \in W_p^{1-1/p}(\Gamma)$ there exists a unique $u \in W_p^2(\Omega)$ which is a solution of

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \gamma\left(\dfrac{\partial u}{\partial \nu} + b_0 u\right) = g & \text{on } \Gamma \end{cases}$$

provided $b_0 \in C^{0,1}(\bar{\Omega})$ and $b_0 > 0$ everywhere on Γ, under the same assumptions as above on p and Ω. Similarly, Theorem 2.4.2.7 implies that for every $f \in L_p(\Omega)$ and every $g \in W_p^{1-1/p}(\Gamma)$ there exists a unique $u \in W_p^2(\Omega)$ which is a solution of

$$\begin{cases} -\Delta u + a_0 u = f & \text{in } \Omega \\ \gamma \dfrac{\partial u}{\partial \nu} = g & \text{on } \Gamma \end{cases}$$

provided $a_0 \in L^\infty(\Omega)$ and $a_0 \geqslant \beta > 0$ a.e. in $\bar{\Omega}$, under the same assumptions as before on p and Ω. Oblique boundary conditions are also considered in those theorems.

Unless otherwise indicated, we only consider real-valued functions in this chapter (with the exception of some proofs in Section 2.3.2 which require the use of the Fourier transform).

2.2 Variational solution of special problems

The roots of almost all the forthcoming results lie in a basic existence and uniqueness theorem for solutions in $H^1(\Omega)$. This result is proved by the variational method introduced first by Euler. A much more detailed description of the extent of this powerful method can be found in Magenes and Stampacchia (1958), Lions (1956), Něcas (1967) and Agmon (1959) for instance. We quote here the minimal material that we will need in the following chapters. In particular, we restrict ourselves to Dirichlet's and Neumann's problems although the variational approach allows us to solve problems with an oblique boundary condition.

2.2.1 Existence and uniqueness

According to what is said above we are looking for u which is a solution of

$$Au + \lambda u = \sum_{i,j=1}^{n} D_i(a_{ij} D_j u) + \lambda u = f \qquad \text{in } \Omega \tag{2,2,1,1}$$

with either a Dirichlet boundary condition

$$u = 0 \qquad \text{on } \Gamma \tag{2,2,1,2}$$

or a Neumann boundary condition

$$\frac{\partial u}{\partial \nu_A} = \sum_{i,j=1}^{n} a_{ij} \nu^i \frac{\partial u}{\partial x_i} = g \qquad \text{on } \Gamma. \tag{2,2,1,3}$$

Euler's variational approach to these problems consists of viewing them as the equation of critical points for some functional (see Section 1.1). However, we shall use a slightly different setting based on the famous Lax–Milgram Lemma. This will allow us also to deal with oblique boundary conditions later on.

Lemma 2.2.1.1 *Let V be a Hilbert space and let a be a continuous bilinear form on $V \times V$. (a does not need to be symmetric.) Assume that a is coercive, i.e. that there exists a constant $\alpha > 0$ such that*

$$a(u; u) \geq \alpha \|u\|_V^2$$

for all $u \in V$. Then for every continuous linear form l on V, there exists a unique $u \in V$ such that

$$a(u; v) = l(v) \tag{2,2,1,4}$$

for every $v \in V$.

Now the problem is to convert equation (2,2,1,1) and the boundary condition into a problem of the form (2,2,1,4). This is achieved by performing integration by parts, using Theorem 1.5.3.1. Let us assume, for instance, that $u \in H^2(\Omega)$ is a solution of (2,2,1,1), (2,2,1,2) and that $v \in \mathring{H}^1(\Omega)$. Then we have

$$\begin{aligned} \int_\Omega fv \, dx &= \sum_{i,j=1}^{n} \int_\Omega (D_i a_{ij} D_j u) v \, dx + \lambda \int_\Omega uv \, dx \\ &= -\sum_{i,j=1}^{n} \int_\Omega a_{ij} D_j u D_i v \, dx + \lambda \int_\Omega uv \, dx. \end{aligned} \tag{2,2,1,5}$$

It is therefore natural to define a and l on $V = \mathring{H}^1(\Omega)$ as follows:

$$a(u, v) = -\sum_{i,j=1}^{n} \int_\Omega a_{ij} D_j u D_i v \, dx + \lambda \int_\Omega uv \, dx$$

$$l(v) = \int_\Omega fv \, dx.$$

With this choice of V, a and l, our u is a solution of problem (2,2,1,4).

Conversely, it is easily seen that a is bilinear, continuous and coercive on V for $\lambda \geq 0$, while l is continuous for $f \in L_2(\Omega)$. Applying Lemma 2.2.1.1, we obtain the basic existence and uniqueness result for Dirichlet's problem.

Theorem 2.2.1.2 *For every $f \in L_2(\Omega)$ there exists a unique $u \in H^1(\Omega)$ solution of equation (2,2,1,1), with the boundary condition $\gamma u = 0$, provided $\lambda \geq 0$.*

Proof Identity (2,2,1,4) with all $v \in \mathscr{D}(\Omega)$, means that $Au + \lambda u = f$ in the sense of distributions. This is all the information that we can get from (2,2,1,4) since $\mathscr{D}(\Omega)$ is dense in $\mathring{H}^1(\Omega)$. The fulfillment of the boundary condition $\gamma u = 0$ follows from Corollary 1.5.1.6. ■

We turn now to the Neumann problem. Let us assume, as a starting point, that $u \in H^2(\Omega)$ is a solution of (2,2,1,1), (2,2,1,3) and that we have $v \in H^1(\Omega)$. Then we have

$$\int_\Omega fv \, dx = \sum_{i,j=1}^n \int_\Omega (D_i a_{ij} D_j u) v \, dx + \lambda \int_\Omega uv \, dx$$
$$= -\sum_{i,j=1}^n \int_\Omega a_{ij} D_j u D_i v \, dx + \int_\Gamma gv \, d\sigma + \lambda \int_\Omega uv \, dx. \qquad (2,2,1,6)$$

Accordingly, we define a as above and l as follows on $V = H^1(\Omega)$:

$$l(v) = \int_\Omega fv \, dx - \int_\Gamma gv \, d\sigma.$$

It follows again that u is a solution of problem (2,2,1,4).

Conversely, it is easily seen that a is bilinear, continuous and coercive on V for $\lambda > 0$, while l is continuous provided $f \in L_2(\Omega)$ and $g \in L_2(\Gamma)$. We again apply Lemma 2.2.1.1 for proving the basic existence and uniqueness result for Neumann's problem:

Theorem 2.2.1.3 *For every $f \in L_2(\Omega)$ and $g \in L_2(\Gamma)$ there exists a unique $u \in H^1(\Omega)$ such that*

$$-\sum_{i,j=1}^n \int_\Omega a_{ij} D_j u D_i v \, dx + \lambda \int_\Omega uv \, dx = \int_\Omega fv \, dx - \int_\Gamma gv \, d\sigma \qquad (2,2,1,7)$$

for all $v \in H^1(\Omega)$, provided $\lambda > 0$.

If we restrict identity (2,2,1,7) to $v \in \mathscr{D}(\Omega)$ only, we check that $Au + \lambda u = f$ in the sense of distributions. Consequently we have $u \in E(A, L_2(\Omega))$ (a space defined in 1.5) and $\gamma\, \partial u/\partial \nu_A$ is defined as an element of $H^{-1/2}(\Gamma)$. This allows one to prove that $\gamma\, \partial u/\partial \nu_A = g$ on Γ in the sense of $H^{-1/2}(\Gamma)$ (see details in Lions (1961a)), but we do not need this in the sequel.

2.2.2 Smoothness

In this short section, we shall prove that the solutions to the Dirichlet and Neumann problems that we obtained in 2.2.1 actually belong to $H^2(\Omega)$. The main tool for proving this is the well-known method of tangential differential quotients due to Nirenberg. We shall use this method only near a flat boundary, taking advantage of the invariance of our set of problems under cut offs and $C^{1,1}$ changes of coordinates.

Let us begin with the Dirichlet problem. Thus, let $u \in \overset{\circ}{H}{}^1(\Omega)$ be a solution of

$$-\sum_{i,j=1}^{n} \int_\Omega a_{ij} D_j u D_i v \, dx + \lambda \int_\Omega uv \, dx = \int_\Omega fv \, dx \qquad (2,2,2,1)$$

for all $v \in \overset{\circ}{H}{}^1(\Omega)$. Let θ be any function in $\mathscr{D}(\bar{\Omega})$ and set $u_1 = \theta u$. It is clear that $u_1 \in \overset{\circ}{H}{}^1(\Omega)$ and that

$$-\sum_{i,j=1}^{n} \int_\Omega a_{ij} D_j u_1 D_i v \, dx + \lambda \int_\Omega u_1 v \, dx = \int_\Omega f_1 v \, dx$$

for all $v \in \overset{\circ}{H}{}^1(\Omega)$ with

$$f_1 = \theta f + \sum_{i,j=1}^{n} \{a_{ij}(D_i\theta)(D_j u) + D_i[a_{ij}(D_j\theta)u]\}.$$

This function f_1 is again in $L_2(\Omega)$ since $a_{ij} \in C^{1,1}(\bar{\Omega})$ and $u \in H^1(\Omega)$.

Now let V be an open subset of $\mathbb{R}^n$ and let Φ be a $C^{1,1}$ diffeomorphism of $\bar{V}$ onto a neighbourhood of the support of θ. Assume that

$$\Phi^{-1}(\Omega \cap \Phi(V)) = U = \mathbb{R}^n_+ \cap V.$$

(We recall that we denote by $\mathbb{R}^n_+$ the half space defined by $x_n > 0$. In addition, possibly, V does not cut the hyperplane $\{x_n = 0\}$.) Then we consider $u_2 = u_1 \circ \Phi$. Again we have $u_2 \in \overset{\circ}{H}{}^1(U)$ and setting $\Psi = \Phi^{-1}$, we have

$$-\sum_{k,l=1}^{n} \int_U a^{\#}_{k,l} D_k u_2 D_l v \, dx + \lambda \int_U |D\Phi| \, u_2 v \, dx = \int_U f_2 v \, dx \qquad (2,2,2,2)$$

for all $v \in \overset{\circ}{H}{}^1(U)$, where

$$a^{\#}_{k,l} = \sum_{i,j=1}^{n} [a_{i,j}(D_i\Psi_k)(D_j\Psi_l)] \circ \Phi \, |D\Phi|$$

$$f_2 = |D\Phi| \, f \circ \Phi.$$

It is clear that $f_2 \in L_2(U)$, $a^{\#}_{k,l} \in C^{0,1}(\bar{U})$. In addition, it follows from (2,1,1) that

$$\sum_{k,l=1}^{n} a^{\#}_{k,l}(y)\xi_k\xi_l \leq -\alpha^{\#}\,|\xi|^2 \tag{2,2,2,3}$$

for all $\xi \in \mathbb{R}^n$ and $y \in \bar{U}$, with some $\alpha^{\#} > 0$.

The first step is the following.

Lemma 2.2.2.1 *Under the above hypotheses, we have* $u_2 \in H^2(U)$.

Proof We shall use identity (2,2,2,2) with a special test function v deduced from u_2. We observe that the support of u_2 is contained in the inverse image of the support of θ, by Φ. Consequently, the support of u_2 is compact in V and cuts the boundary of U only on the hyperplane $x_n = 0$. We extend u_2 to $\tilde{u}_2$, a function which is zero outside of V. It is clear that $\tilde{u}_2 \in \mathring{H}^1(\mathbb{R}^n_+)$.

We define v as follows:

$$v = \left(\frac{\tau_{i,h} + \tau_{i,-h} - 2}{h^2}\right)\tilde{u}_2$$

where $\tau_{i,h}$ is the operator defined by

$$\tau_{i,h}\varphi(x) = \varphi(x + he_i), \qquad 1 \leq i \leq n-1,\ h \in \mathbb{R},$$

e_i being the unit vector in the direction of x_i. We have $v \in \mathring{H}^1(U)$ for h small enough. Writing identity (2,2,2,2) with this particular v, we get

$$-\sum_{k,l=1}^{n}\int_U a^{\#}_{k,l}D_k\tilde{u}_2 D_l\left(\frac{\tau_{i,h}-1}{h}\right)\left(\frac{\tau_{i,-h}-1}{h}\right)\tilde{u}_2\,\mathrm{d}x$$
$$+\lambda\int_U |D\Phi|\,\tilde{u}_2\left(\frac{\tau_{i,h}-1}{h}\right)\left(\frac{\tau_{i,-h}-1}{h}\right)\tilde{u}_2\,\mathrm{d}x = \int_U f_2\frac{\tau_{i,h}-1}{h}\frac{\tau_{i,-h}-1}{h}\tilde{u}_2\,\mathrm{d}x.$$

This identity implies the following, which is obtained through a discrete integration by parts, the adjoint of the operator $\tau_{i,h}$ being $\tau_{i,-h}$.

$$-\sum_{k,l=1}^{n}\int_U \left[\left(\frac{\tau_{i,+h}-1}{h}\right)a^{\#}_{k,l}D_k\tilde{u}_2\right]D_l\left[\frac{\tau_{i,h}-1}{h}\tilde{u}_2\right]\mathrm{d}x$$
$$+\lambda\int_U \left[\left(\frac{\tau_{i,h}-1}{h}\right)|D\Phi|\,\tilde{u}_2\right]\left[\frac{\tau_{i,h}-1}{h}\tilde{u}_2\right]\mathrm{d}x = \int_U f_2\frac{\tau_{i,h}-1}{h}\frac{\tau_{i,-h}-1}{h}\tilde{u}_2\,\mathrm{d}x.$$

Then we observe that for any two functions a and φ we have

$$\frac{\tau_{i,h}-1}{h}(a\varphi) = \left[\frac{\tau_{i,h}-1}{h}a\right]\tau_{i,h}\varphi + a\left[\frac{\tau_{i,h}-1}{h}\right]\varphi.$$

Therefore we find

$$
\begin{aligned}
&-\sum_{k,l=1}^{n}\int_U a^{\#}_{k,l}D_k\left[\frac{\tau_{i,h}-1}{h}\tilde{u}_2\right]D_l\left[\frac{\tau_{i,h}-1}{h}\tilde{u}_2\right]\mathrm{d}x\\
&=\sum_{k,l=1}^{n}\int_U\left[\frac{\tau_{i,h}-1}{h}a^{\#}_{k,l}\right][\tau_{i,h}D_k\tilde{u}_2]D_l\left[\frac{\tau_{i,h}-1}{h}\tilde{u}_2\right]\mathrm{d}x\\
&\quad-\lambda\int_U|D\Phi|\left[\frac{\tau_{i,h}-1}{h}\tilde{u}_2\right]^2\mathrm{d}x-\lambda\int_U\left[\frac{\tau_{i,h}-1}{h}|D\Phi|\right]\\
&\quad\times[\tau_{i,h}\tilde{u}_2]\left[\frac{\tau_{i,h}-1}{h}\tilde{u}_2\right]\mathrm{d}x+\int_U f_2\frac{\tau_{i,h}-1}{h}\frac{\tau_{i,-h}-1}{h}\tilde{u}_2\,\mathrm{d}x.
\end{aligned}
$$

From this and inequality (2,2,2,3) we deduce the following:

$$
\begin{aligned}
\alpha^{\#}\sum_{k=1}^{n}\left\|D_k\frac{\tau_{i,h}-1}{h}\tilde{u}_2\right\|^2 &\leqslant \sum_{k,l=1}^{n} M\,\|D_k\tilde{u}_2\|\left\|D_l\frac{\tau_{i,h}-1}{h}\tilde{u}_2\right\|\\
&\quad+\lambda N\left\|\frac{\tau_{i,h}-1}{h}\tilde{u}_2\right\|^2+\lambda N'\,\|\tilde{u}_2\|\left\|\frac{\tau_{i,h}-1}{h}\tilde{u}_2\right\|^2\\
&\quad+\|f_2\|\left\|\frac{\tau_{i,h}-1}{h}\frac{\tau_{i,-h}-1}{h}\tilde{u}_2\right\|.
\end{aligned}
\tag{2,2,2,4}
$$

Here we use the norm of $L_2(U)$, while M is a bound for all the Lipschitz constants of the functions $a^{\#}_{k,l}$, $1\leqslant k,\ l\leqslant n$, N is the maximum of $|D\Phi|$ and finally N' is the Lipschitz constant of $|D\Phi|$. We already know that $u_2\in H^1(U)$, therefore, from (2,2,2,4) we deduce that there exist two constants C_1 and C_2 such that

$$
\sum_{k=1}^{n}\left\|D_k\frac{\tau_{i,h}-1}{h}\tilde{u}_2\right\|^2\leqslant C_1+C_2\left\|\frac{\tau_{i,h}-1}{h}\frac{\tau_{i,-h}-1}{h}\tilde{u}_2\right\| \tag{2,2,2,5}
$$

owing to the following lemma (the proof is easy and left to the reader):

Lemma 2.2.2.2 *For $\varphi\in H^1(\mathbb{R}^n_+)$ we have*

$$
\left\|\frac{\tau_{i,h}-1}{h}\varphi\right\|\leqslant\|D_i\varphi\|,\qquad 1\leqslant i\leqslant n-1.
$$

Next we again apply Lemma 2.2.2.2 to $\varphi=((\tau_{i,h}-1)/h)\tilde{u}_2$; we thus get

$$
\sum_{k=1}^{n}\left\|D_k\frac{\tau_{i,h}-1}{h}\tilde{u}_2\right\|^2\leqslant C_1+C_2\left\|D_i\frac{\tau_{i,h}-1}{h}\tilde{u}_2\right\|
$$

and consequently

$$\sum_{k=1}^{n} \left\| D_k \frac{\tau_{i,h}-1}{h} \tilde{u}_2 \right\|^2 \leqslant 2C_1 + C_2^2 \tag{2,2,2,6}$$

for $1 \leqslant i \leqslant n-1$.

To conclude, for each i, we consider any sequence $h_j \searrow 0$, such that

$$D_k \frac{\tau_{i,h_j}-1}{h_j} \tilde{u}_2, \qquad 1 \leqslant k \leqslant n$$

converges weakly to some limit $\varphi_{k,i}$ in $L_2(\mathbb{R}^n_+)$. This is clearly possible, due to the properties of bounded sequences in a Hilbert space. We have obviously

$$D_k \frac{\tau_{i,h_j}-1}{h_j} \tilde{u}_2 \to D_k D_i \tilde{u}_2$$

in the sense of distributions, and consequently

$$D_k D_i \tilde{u}_2 = \varphi_{k,i} \in L_2(\mathbb{R}^n_+) \tag{2,2,2,7}$$

for $1 \leqslant i \leqslant n-1$, $1 \leqslant k \leqslant n$. This shows that all second derivatives of u_2 except $D_n^2 u_2$ are square integrable in U. However, it follows from (2,2,2,2) that

$$+ \sum_{k,l=1}^{n} D_l a^{\#}_{k,l} D_k u_2 + \lambda \, |D\Phi| \, u_2 = f_2$$

in U. Furthermore, from (2,2,2,3), we have $a^{\#}_{n,n} \leqslant -\alpha^{\#}$, so that we can write

$$D_n^2 u_2 = \frac{1}{a^{\#}_{n,n}} \left\{ f_2 - \lambda \, |D\Phi| \, u_2 - \sum_{k+l \leqslant 2n-1} D_l a^{\#}_{k,l} D_k u_2 - (D_n a^{\#}_{n,n}) D_n u_2 \right\}$$

and this shows that $D_n^2 u_2 \in L_2(U)$. The proof of Lemma 2.2.2.1 is complete. ■

Now we prove the global result corresponding to Lemma 2.2.2.2.

Theorem 2.2.2.3 *For every $f \in L_2(\Omega)$ there exists a unique $u \in H^2(\Omega)$ solving equation (2,2,1,1) with the boundary condition $\gamma u = 0$, provided $\lambda \geqslant 0$.*

Proof We recall that from the beginning we assume (a), (b) in Section 2.1. Thus Ω is bounded and has a $C^{1,1}$ boundary, while $a_{i,j} \in C^{0,1}(\bar{\Omega})$ for $i, j = 1, \ldots, n$. It is therefore possible to find a finite number of open subsets V_k, $1 \leqslant k \leqslant \kappa$, of $\mathbb{R}^n$ together with $C^{1,1}$ diffeomorphisms Φ_k from

$\bar{V}_k$ onto $\Phi_k(\bar{V}_k)$, $1 \leq k \leq \kappa$ such that

(a) $\Phi_k(V_k)$, $1 \leq k \leq \kappa$, is a covering of $\bar{\Omega}$
(b) $\Phi_k^{-1}(\Omega \cap \Phi_k(V_k)) = U_k = \mathbb{R}^n_+ \cap V_k, \qquad 1 \leq k \leq \kappa.$

We observe that V_k need not meet the hyperplane $x_n = 0$, in order that the $\Phi_k(V_k)$ also cover Ω. We used here Theorem 1.2.1.5 which allows us to consider $\bar{\Omega}$ as a n-dimensional manifold with boundary, of class $C^{1,1}$ in $\mathbb{R}^n$.

With this covering of $\bar{\Omega}$ we associate a partition of the unity θ_k, $1 \leq k \leq \kappa$, such that

(c) $\theta_k \in \mathcal{D}(\bar{\Omega})$
(d) the support of θ_k is included in $\Phi_k(V_k)$
(e) $\sum_{k=1}^{\kappa} \theta_k = 1$ on $\bar{\Omega}$.

We apply Theorem 2.2.1.2 to prove the existence of a solution $u \in \mathring{H}^1(\Omega)$ to equation (2,2,1,1). Then Lemma 2.2.2.1 shows that for each k

$$(\theta_k u) \circ \Phi_k \in H^2(U_k).$$

We conclude by reconstructing u as follows

$$u = \sum_{k=1}^{\kappa} \theta_k u = \sum_{k=1}^{\kappa} (\theta_k u) \circ \Phi_k \circ \Phi_k^{-1} \in H^2(\Omega)$$

due to Lemma 1.3.3.1. ■

Corollary 2.2.2.4 *The mapping*

$$u \mapsto \{Au + \lambda u;\ \gamma u\}$$

is invertible from $H^2(\Omega)$ onto $L_2(\Omega) \times H^{3/2}(\Gamma)$, for $\lambda > 0$.

This is an obvious consequence of Theorem 2.2.2.3 using Theorem 1.5.1.2.

We shall now prove the same kind of results for the Neumann problem. We start from $u \in H^1(\Omega)$ fulfilling the same identity (2,2,2,1) for all $v \in H^1(\Omega)$ (instead of $\mathring{H}^1(\Omega)$). Such a solution u exists by Theorem 2.2.1.3 with $g = 0$. Then exactly the same proof as in Lemma 2.2.2.1 shows that

$$(\theta u) \circ \Phi \in H^2(U).$$

The corresponding global result is this:

Theorem 2.2.2.5 *For every $f \in L_2(\Omega)$ there exists a unique $u \in H^2(\Omega)$ solving equation (2,2,1,1) with the boundary condition $\gamma\, \partial u/\partial \nu_A = 0$, provided $\lambda > 0$.*

Proof The property that $u \in H^2(\Omega)$ is proved exactly as in Theorem 2.2.2.3. Then identity (2,2,2,1) shows that $Au + \lambda u = f$ in the sense of distributions (this uses $v \in \mathcal{D}(\Omega)$). This allows one to rewrite (2,2,2,1) as follows:

$$-\sum_{i,j=1}^{n} \int_{\Omega} a_{ij} D_j u D_i v \, dx = \sum_{i,j=1}^{n} \int_{\Omega} (D_i a_{i,j} D_j u) v \, dx$$

for all $v \in H^1(\Omega)$. Finally due to Theorem 1.5.3.1, this identity is equivalent to

$$\sum_{i,j=1}^{n} \int_{\Gamma} \gamma(a_{ij} \nu_i D_j u) \gamma v \, d\sigma = 0$$

for all $\gamma v \in H^{1/2}(\Gamma)$. This shows that

$$\gamma \frac{\partial u}{\partial \nu_A} = \sum_{i,j=1}^{n} \gamma(a_{ij} \nu_i D_j u) = 0$$

in the space $H^{1/2}(\Gamma)$ (since u belongs to $H^2(\Omega)$). ■

Corollary 2.2.2.6 *The mapping*

$$u \mapsto \left\{ Au + \lambda u; \gamma \frac{\partial u}{\partial \nu_A} \right\}$$

is invertible from $H^2(\Omega)$ onto $L_2(\Omega) \times H^{1/2}(\Gamma)$ for $\lambda > 0$.

This follows from Theorems 2.2.2.5 and 1.5.1.2.

2.3 *A priori* estimates

We now consider the general operators A and B introduced in Section 2.1. We no longer restrict ourselves to Dirichlet or Neumann problems. We shall prove the basic *a priori* estimate:

$$\|u\|_{2,p,\Omega} \leq C\{\|Au + \lambda u\|_{0,p,\Omega} + \|\gamma Bu\|_{2-d-1/p,p,\Gamma}\} \qquad (2,3,1,1)$$

for $u \in W_p^2(\Omega)$. This estimate holds only for λ large enough. This is essentially the inequality in Agmon *et al.* (1959); however, the proof given here is slightly different.

2.3.1 An inequality based on the duality mapping

The duality mapping from $L_p(\Omega)$ into its dual $L_q(\Omega)$ (with $1/p + 1/q = 1$) is the mapping $u \mapsto u^*$ defined by

$$u^*(x) = \begin{cases} |u(x)|^{p-1} \operatorname{sgn} u(x) & \text{if } u(x) \neq 0 \\ 0 & \text{if } u(x) = 0. \end{cases}$$

The reason for introducing u^* is that it is the unique function in $L_1(\Omega)$, such that

$$\begin{cases} \|u^*\|_{0,q,\Omega}^q = \|u\|_{0,p,\Omega}^p \\ \int_\Omega uu^*\,dx = \|u\|_{0,p,\Omega}\,\|u^*\|_{0,q,\Omega}. \end{cases}$$

The strong ellipticity of A allows us to prove some very useful estimates for $\|u\|_{0,p,\Omega}$, just by multiplying the equation $Au+\lambda u=f$, by u^* and integrating by parts. The boundary condition allows one to drop or to estimate the boundary integrals that appear in the integration by parts. This is the purpose of this subsection.

The differentiation of u^* will be difficult at points where u vanishes, since the sign of u will be undefined. So we shall approximate u^* by u^*_ε defined as follows for $\varepsilon>0$:

$$u^*_\varepsilon(x)=(u(x)^2+\varepsilon)^{(p-2)/2}u(x). \tag{2,3,1,2}$$

Assuming that $u\in C^1(\bar\Omega)$, we can differentiate u^*_ε as follows:

$$\begin{aligned} D_iu^*_\varepsilon(x) &= (u(x)^2+\varepsilon)^{(p-2)/2}D_iu(x)\\ &\quad +(p-2)(u(x)^2+\varepsilon)^{(p-4)/2}u(x)^2D_iu(x). \end{aligned}\tag{2,3,1,3}$$

Lemma 2.3.1.1 *For $u\in C^1(\bar\Omega)$, we have*

$$-\sum_{i,j=1}^n a_{ij}(x)D_iuD_ju^*_\varepsilon \geq \alpha'(u(x)^2+\varepsilon)^{(p-2)/2}|\nabla u(x)|^2 \tag{2,3,1,4}$$

for all $x\in\bar\Omega$, where $\alpha'=\alpha\inf\{1,p-1\}$.

Proof We have

$$\begin{aligned} -\sum_{i,j=1}^n a_{ij}D_iuD_ju^*_\varepsilon &= -(u^2+\varepsilon)^{(p-4)/2}\{u^2+\varepsilon+(p-2)u^2\}\sum_{i,j=1}^n a_{ij}D_iuD_ju\\ &\geq -\inf\{1,p-1\}(u^2+\varepsilon)^{(p-2)/2}\sum_{i,j=1}^n a_{ij}D_iuD_ju\\ &\geq \alpha\inf\{1,p-1\}(u^2+\varepsilon)^{(p-2)/2}|\nabla u|^2. \quad \blacksquare \end{aligned}$$

Lemma 2.3.1.2 *Let P be any first-order differential operator, with Lipschitz coefficients, tangential to Γ, everywhere on Γ. Then there exists β such that*

$$\begin{aligned} \int_\Omega (Au+\lambda u)u^*_\varepsilon\,dx &\geq \int_\Gamma\left(\frac{\partial u}{\partial\nu_A}+Pu\right)u^*_\varepsilon\,d\sigma-\beta\int_\Omega(u^2+\varepsilon)^{p/2}\,dx\\ &\quad+\lambda\int_\Omega(u^2+\varepsilon)^{(p-2)/2}|u|^2\,dx \end{aligned}\tag{2,3,1,5}$$

for all $u\in C^2(\bar\Omega)$.

Proof We have

$$\int_{\Omega} (Au+\lambda u)u_\varepsilon^* \,dx = -\sum_{i,j=1}^{n} \int_{\Omega} a_{ij}D_juD_iu_\varepsilon^* \,dx + \int_{\Gamma} \frac{\partial u}{\partial \nu_A} u_\varepsilon^* \,d\sigma + \int_{\Omega} \lambda u u_\varepsilon^* \,dx. \tag{2,3,1,6}$$

Using Lemma 2.3.1.1 we deduce the following inequality:

$$\int_{\Omega} (Au+\lambda u)u_\varepsilon^* \,dx \geqslant \alpha' \int_{\Omega} (u^2+\varepsilon)^{(p-2)/2} |\nabla u|^2 \,dx + \int_{\Gamma} \frac{\partial u}{\partial \nu_A} u_\varepsilon^* \,d\sigma + \lambda \int_{\Omega} (u^2+\varepsilon)^{(p-2)/2} |u|^2 \,dx. \tag{2,3,1,7}$$

We then transform the boundary integral. We have

$$\int_{\Gamma} \frac{\partial u}{\partial \nu_A} u_\varepsilon^* \,d\sigma = \int_{\Gamma} \left(\frac{\partial u}{\partial \nu_A} + Pu\right) u_\varepsilon^* \,d\sigma - \int_{\Gamma} Puu_\varepsilon^* \,d\sigma$$

and

$$\begin{aligned}\int_{\Gamma} Puu_\varepsilon^* \,d\sigma &= \int_{\Gamma} (u^2+\varepsilon)^{(p-2)/2}(Pu)u \,d\sigma \\ &= \int_{\Gamma} (u^2+\varepsilon)^{(p-2)/2} \tfrac{1}{2} Pu^2 \,d\sigma \\ &= \frac{1}{p} \int_{\Gamma} P[(u^2+\varepsilon)^{p/2}] \,d\sigma.\end{aligned}$$

We use the following auxiliary lemma, which we shall prove later.

Lemma 2.3.1.3 *For all $\varphi \in C^1(\bar{\Omega})$, we have*

$$\left|\int_{\Gamma} P\varphi \,d\sigma\right| \leqslant C \int_{\Gamma} |\varphi| \,d\sigma. \tag{2,3,1,8}$$

Setting $\varphi = (u^2+\varepsilon)^{p/2}$, we finally obtain

$$\left|\int_{\Gamma} Puu_\varepsilon^* \,d\sigma\right| \leqslant \frac{C}{p} \int_{\Gamma} (u^2+\varepsilon)^{p/2} \,d\sigma.$$

We now take advantage of inequality (1,5,1,2). This leads to

$$\left|\int_{\Gamma} Puu_\varepsilon^* \,d\sigma\right| \leqslant \frac{CK}{p} \left[\sqrt{\delta} \int_{\Omega} |\nabla (u^2+\varepsilon)^{p/4}|^2 \,dx + \frac{1}{\sqrt{\delta}} \int_{\Omega} (u^2+\varepsilon)^{p/2} \,dx\right]$$

for all $\delta > 0$. In other words, we have

$$\left|\int_\Gamma Puu_\varepsilon^* \,d\sigma\right| \leqslant \frac{CK}{p}\left[\sqrt{\delta}\frac{p^2}{4}\int_\Omega (u^2+\varepsilon)^{(p-2)/2}|\nabla u|^2\,dx + \frac{1}{\sqrt{\delta}}\int_\Omega (u^2+\varepsilon)^{p/2}\,dx\right]. \tag{2,3,1,9}$$

We can choose δ such that $(CKp/4)\sqrt{\delta} = \alpha'$ and summing up from (2,3,1,7) we obtain

$$\int_\Omega (Au+\lambda u)u_\varepsilon^*\,dx \geqslant \int_\Gamma \left(\frac{\partial u}{\partial \nu_A} + Pu\right)u_\varepsilon^*\,d\sigma + \lambda\int_\Omega (u^2+\varepsilon)^{(p-2)/2}|u|^2\,dx - \frac{CK}{p\sqrt{\delta}}\int_\Omega (u^2+\varepsilon)^{p/2}\,dx.$$

If we choose β large enough, this implies (2,3,1,5). ■

Proof of Lemma 2.3.1.3 Using a partition of unity and local $(C^{1,1})$ coordinates, it is enough to prove (2,3,1,8) when Γ is replaced by $\mathbb{R}^{n-1}$, P is a first-order operator with Lipschitz coefficients on $\mathbb{R}^{n-1}$ and φ has compact support. Thus we have

$$\int_{\mathbb{R}^{n-1}} P\varphi\,dx = \sum_{k=1}^{n-1}\int_{\mathbb{R}^{n-1}} a_k D_k\varphi\,dx = -\int_{\mathbb{R}^{n-1}} \varphi\left\{\sum_{k=1}^{n-1} D_k a_k\right\}dx$$

and (2,3,1,8) follows. We observe that the constant C depends only on bounds for the coefficients of P and their first derivatives. ■

Lemma 2.3.1.4 *Under the assumptions of Lemma 2.3.1.2, we have*

$$\int_\Omega |u|^p\,dx \leqslant \frac{1}{\lambda-\beta}\left[\left(\int_\Omega |Au+\lambda u|^p\,dx\right)^{1/p}\left(\int_\Omega |u|^p\,dx\right)^{1/q} + \left(\int_\Gamma \left|\gamma\left(\frac{\partial u}{\partial \nu_A}+Pu\right)\right|^p d\sigma\right)^{1/p}\left(\int_\Gamma |\gamma u|^p\,d\sigma\right)^{1/q}\right] \tag{2,3,1,10}$$

for all $u \in W_p^2(\Omega)$.

Proof We begin with $u \in C^2(\bar{\Omega})$ and let $\varepsilon \to 0$ in (2,3,1,5). It is obvious that $u_\varepsilon^* \to u^*$ pointwise everywhere and that u_ε^* remains uniformly bounded in $\bar{\Omega}$ when $\varepsilon \to 0$, because u is continuous in $\bar{\Omega}$. Consequently, by Lebesgue theorem we know that

$$u_\varepsilon^* \to u^*$$

strongly in $L_q(\Omega)$ and in $L_q(\Gamma)$. Thus from (2,3,1,5) we deduce that

$$\int_\Omega (Au+\lambda u)u^*\,dx \geqslant \int_\Gamma \left(\frac{\partial u}{\partial \nu_A}+Pu\right)u^*\,d\sigma+(\lambda-\beta)\int_\Omega uu^*\,dx.$$

Now applying Hölder's inequality, we obtain

$$\int_\Omega |u|^p\,dx \leqslant \frac{1}{\lambda-\beta}\left[\left(\int_\Omega |Au+\lambda u|^p\,dx\right)^{1/p}\left(\int_\Omega |u|^p\,dx\right)^{1/q}\right.$$
$$\left.+\left(\int_\Gamma \left|\frac{\partial u}{\partial \nu_A}+Pu\right|^p d\sigma\right)^{1/p}\left(\int_\Gamma |u|^p\,d\sigma\right)^{1/q}\right].$$

This is exactly (2,3,1,10) when $u\in C^2(\bar\Omega)$. However, this inequality does not involve u^*, so that it is easily extended to all $u\in W_p^2(\Omega)$ by density. ■

We are now able to prove

Theorem 2.3.1.5 *Let A and B satisfy the assumptions in Section 2.1; then there exists λ_0 such that, for $\lambda>\lambda_0$:*

$$\|u\|_{0,p,\Omega} \leqslant \frac{1}{\lambda-\lambda_0}\|Au+\lambda u\|_{0,p,\Omega} \tag{2,3,1,11}$$

for all $u\in W_p^2(\Omega)$, such that $\gamma Bu=0$.

Proof We consider first the case when the order of B is one, i.e. $d=1$. We observe that the boundary condition $\gamma Bu=0$ may be rewritten as $\gamma(\partial u/\partial\nu_A+Pu)=0$, for some tangential operator P with Lipschitz coefficients. Indeed we have:

$$Bu=\sum_{i=1}^n b_iD_iu=b_\nu\frac{\partial u}{\partial\nu}+\mathbf{b}_T\cdot\nabla_T u$$

where b_ν is the component of the vector $\mathbf{b}=(b_1,\ldots,b_n)$ in the direction of $\boldsymbol{\nu}$ and $\mathbf{b}_T$ is the projection of $\mathbf{b}$ on the tangent hyperplane to Γ. We denote by ∇_T the tangential gradient on Γ. In the same way, we have

$$\frac{\partial u}{\partial\nu_A}=\sum_{i=1}^n c_iD_iu=c_\nu\frac{\partial u}{\partial\nu}+\mathbf{c}_T\cdot\nabla_T u$$

where $c_i=\sum_{j=1}^n a_{ij}\nu^j$.

We assumed that Γ is not characteristic for B and this means that b_ν does not vanish on Γ. Consequently we have

$$\frac{c_\nu}{b_\nu}Bu=\frac{\partial u}{\partial\nu_A}-\mathbf{c}_T\cdot\nabla_T u+\frac{c_\nu}{b_\nu}\mathbf{b}_T\cdot\nabla_T u.$$

We define P by

$$Pu = \left(\frac{c_\nu}{b_\nu}\mathbf{b}_T - \mathbf{c}_T\right) \cdot \nabla_T u$$

and the boundary condition $\gamma Bu = 0$ implies

$$\gamma\left(\frac{\partial u}{\partial \nu_A} + Pu\right) = 0.$$

We observe the c_ν, b_ν, $\mathbf{b}_T$, $\mathbf{c}_T$ are all Lipschitz functions since $a_{i,j}$ and b_i are so and Γ is of class $C^{1,1}$ by assumption.

We now make use of inequality (2,3,1,10) and get the following:

$$\|u\|^p_{0,p,\Omega} \leqslant \frac{1}{\lambda - \beta} \|Au + \lambda u\|_{0,p,\Omega} \|u\|^{p/q}_{0,p,\Omega}.$$

Inequality (2,3,1,11) follows easily.

So far we have left out the case of a Dirichlet problem (i.e. $B = 1$ and $d = 0$). In that case (2,3,1,11) follows obviously from (2,3,1,10), with $P = 0$, say, because $\gamma u = 0$ on Γ. Consequently (2,3,1,11) holds. ■

2.3.2 An inequality in the half space

Here we consider a second-order, homogeneous, strongly elliptic operator L, with constant real coefficients

$$L = \sum_{j,k=1}^{n} l_{j,k} D_j D_k$$

together with a first-order, homogeneous differential operator M, with constant real coefficients

$$M = \sum_{j=1}^{n} m_j D_j.$$

We assume that the hyperplane $x_n = 0$ is not characteristic for M. This means that $m_n \neq 0$. The strong ellipticity of $-L$ means that there exists $\mu > 0$ such that

$$\sum_{j,k=1}^{n} l_{j,k}\xi_j\xi_k \leqslant -\mu\, |\xi|^2 \tag{2,3,2,1}$$

for all $\xi \in \mathbb{R}^n$. The corresponding boundary value problem in $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$ is

$$\begin{cases} Lu = f & \text{in } \mathbb{R}^n_+ \\ \gamma_n Mu = g & \text{on } \mathbb{R}^{n-1} \end{cases} \tag{2,3,2,2}$$

where γ_n is the trace operator on $x_n = 0$.

The purpose of this subsection is to prove an estimate for a $u \in W_p^2(\mathbb{R}_+^n)$ which is a solution of (2,3,2,2). Namely, we shall prove that there exists some constant C such that

$$\|u\|_{2,p,\mathbb{R}_+^n} \leq C[\|Lu\|_{0,p,\mathbb{R}_+^n} + \|\gamma_n Mu\|_{1-1/p,p,\mathbb{R}^{n-1}} + \|u\|_{1,p,\mathbb{R}_+^n}]. \qquad (2,3,2,3)$$

We shall also prove the corresponding estimate for the Dirichlet problem (i.e. when M is replaced by I). This is the first step of the proof of Agmon *et al.*'s inequality. The proof presented here is different.

We shall use two auxiliary results, one of which is the powerful L_p-multiplier theorem of Mih'lin (1956) (see also Hörmander (1960)). Here, we denote by $\mathbb{R}_*^n$ the set $\mathbb{R}^n \setminus \{0\}$. (In the rest of this section, we allow our functions to be complex valued.)

Theorem 2.3.2.1 *Let $a \in C^n(\mathbb{R}_*^n)$ be such that there exists a constant C with*

$$|\xi|^{|\alpha|} |D^\alpha a(\xi)| \leq C \qquad (2,3,2,4)$$

for all $\xi \in \mathbb{R}_*^n$ and $|\alpha| \leq n$. Then the operator

$$g \mapsto F^{-1} aFg$$

is continuous in $L_p(\mathbb{R}^n)$ and there exists a function $n, p \mapsto K(n, p)$ such that

$$\|F^{-1} aFg\|_{0,p,\mathbb{R}^n} \leq K(n, p) C \|g\|_{0,p,\mathbb{R}^n} \qquad (2.3.2.5)$$

for all $g \in L_p(\mathbb{R}^n)$ and $1 < p < \infty$.

We emphasize the fact that $K(n, p)$ blows up when $p \to 1$ and when $p \to \infty$. The case $p = 2$ is useless since (2,3,2,5) holds with $K(n, 2) = 1$ and $C = \max_{\mathbb{R}^n} |a|$, by Plancherel's theorem.

Lemma 2.3.2.2 *The mapping*

$$\gamma_n^* : g \mapsto g \otimes \delta_n$$

where δ_n denotes the Dirac measure in the variable x_n, is continuous from $W_p^s(\mathbb{R}^{n-1})$ into $W_p^{s+1/p-1}(\mathbb{R}^n)$ provided $s < 0$.

Proof This is a very simple consequence of Theorem 1.5.1.1 since γ_n^* is obviously the transposed operator of the trace operator γ_n. ■

We shall also use an elementary solution E for $L + 1$, defined by

$$FE(\xi) = \left[-\sum_{j,k=1}^{n} l_{j,k} \xi_j \xi_k + 1 \right]^{-1}, \qquad \xi \in \mathbb{R}^n.$$

The assumption (2,3,2,1) implies the existence of a constant C such that

$a = D^\beta FE$ fulfils (2,3,2,4) when $|\beta| \leq 2$. Consequently the convolution operator by E maps $L_p(\mathbb{R}^n)$ into $W_p^2(\mathbb{R}^n)$.

We now use the elementary solution for reducing the boundary value problem

$$\begin{cases} Lu + u = f & \text{in } \mathbb{R}^n_+ \\ \gamma_n Mu = g & \text{on } \mathbb{R}^{n-1} \end{cases} \tag{2,3,2,6}$$

to an equation on $\mathbb{R}^{n-1}$. For that purpose we set

$$v = u - E * \tilde{f}. \tag{2,3,2,7}$$

We then have

$$\begin{cases} Lv + v = 0 \\ \gamma_n Mv = h \end{cases} \tag{2,3,2,8}$$

where $h = g - \gamma_n ME * \tilde{f}$. We now denote by $\hat{\varphi}$ the partial Fourier transform of φ, in $x_1, \ldots, x_{n-1}$ (or Fourier transform on $\mathbb{R}^{n-1}$), i.e.,

$$\hat{\varphi}(\xi, x_n) = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi} \varphi(x', x_n)\, dx'$$

where $x' = (x_1, \ldots, x_{n-1})$. We use the same notation for the Fourier transform of a function defined on $\mathbb{R}^{n-1}$

$$\hat{\psi}(\xi) = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\mathbb{R}^{n-1}} e^{-ix\cdot\xi} \psi(x)\, dx.$$

It follows from (2,3,2,8) that

$$\begin{cases} l_{n,n} D_n^2 \hat{v} + 2i \sum_{j=1}^{n-1} l_{n,j} \xi_j D_n \hat{v} + \left(1 - \sum_{j,k=1}^{n-1} l_{j,k} \xi_j \xi_k\right) \hat{v} = 0, & x_n > 0 \\ m_n D_n \hat{v} + i \sum_{j=1}^{n-1} m_j \xi_j \hat{v} = \hat{h}, & x_n = 0. \end{cases} \tag{2,3,2,9}$$

We now solve the differential equation in (2,3,2,9). The corresponding characteristic equation is

$$l_{n,n} \tau^2 + 2i \sum_{j=1}^{n-1} l_{n,j} \tau \xi_j + \left(1 - \sum_{j,k=1}^{n-1} l_{j,k} \xi_j \xi_k\right) = 0. \tag{2,3,2,10}$$

If we set $\zeta = (\xi_1, \ldots, \xi_{n-1}, \tau/i)$, this is equivalent to

$$1 - \sum_{j,k=1}^{n} l_{j,k} \zeta_j \zeta_k = 0.$$

This equation has no real solution, due to the strong ellipticity of $-L$ (see (2,3,2,1)). Furthermore, since the $l_{j,k} \cdot s$ are real, the solutions are conjugates. It follows that (2,3,2,10) has two symmetric, nonimaginary solutions in τ, which are functions of ξ. We denote them by

$$p_{\pm}(\xi)$$

with Re $p_+(\xi)>0$ and Re $p_-(\xi)<0$. It follows that $\hat{v}$ is, for almost every ξ, a linear combination of the functions

$$\exp x_n p_+(\xi), \qquad \exp x_n p_-(\xi).$$

However, due to the assumption that $\hat{v}$ is a Fourier transform, the fast increasing component has to be excluded.

Lemma 2.3.2.3 *Let $v \in H^2(\mathbb{R}^n_+)$*† *be a solution of*

$$Lv + v = 0 \qquad \text{in } \mathbb{R}^n_+$$

then

$$\hat{v}(\xi, x_n) = \widehat{\gamma_n v}(\xi) \exp x_n p_-(\xi), \qquad x_n > 0$$

for almost every $\xi \in \mathbb{R}^{n-1}$.

Proof At first glance we have

$$\hat{v}(\xi, x_n) = \alpha(\xi) \exp x_n p_-(\xi) + \beta(\xi) \exp x_n p_+(\xi), \qquad x_n > 0$$

for almost every ξ, where α and β are some functions. From this, it follows that

$$\begin{cases} \alpha + \beta = \widehat{\gamma_n v} & \text{a.e.} \\ p_-\alpha + p_+\beta = \widehat{\gamma_n D_n v} & \text{a.e.} \end{cases}$$

and this shows that α and β are measurable functions since $\widehat{\gamma_n v}$ and $\widehat{\gamma_n D_n v}$ are square integrable measurable functions.

Then, from the fact that v belongs to $L_2(\mathbb{R}^n_+)$ we deduce that $\hat{v}$ also belongs to $L_2(\mathbb{R}^n_+)$ (in the variables (ξ, x_n)) and consequently $\beta = 0$ a.e. It follows that $\alpha = \widehat{\gamma_n v}$. ■

An immediate consequence of (2,3,2,10) is that

$$\widehat{\gamma_n D_n v} = p_- \widehat{\gamma_n v} \qquad \text{a.e.}$$

and thus the boundary condition in (2,3,2,8) may be rewritten as follows, where $k_0 = \gamma_n v$:

$$\left(m_n p_- + \sum_{j=1}^{n-1} i m_j \xi_j\right)\hat{k}_0 = \hat{h}. \tag{2,3,2,11}$$

† It is enough to consider here the case $p = 2$, since later on we shall take advantage of the density of $W^2_p(\mathbb{R}^n_+) \cap H^2(\mathbb{R}^n_+)$ in $W^2_p(\mathbb{R}^n_+)$.

This is the equation on the boundary, which is equivalent to problem (2,3,2,9).

The equation (2,3,2,11) is obviously uniquely solvable since the function $\xi \mapsto (m_n p_- + \sum_{j=1}^{n-1} im_j\xi_j)$ does not vanish on $\mathbb{R}^n$. Indeed the m_j are all real and $\operatorname{Re} p_-(\xi) < 0$ everywhere. This leads to the representation formula in Lemma 2.3.2.4.

Lemma 2.3.2.4 *Let $u \in H^2(\mathbb{R}^n_+)$ be the solution of problem (2,3,2,6); then we have*

$$u = E * \left(\tilde{f} + l_{n,n}k_0 \otimes \delta'_n + \left[l_{n,n}k_1 + 2\sum_{j=1}^{n-1} l_{n,j}D_jk_0\right] \otimes \delta_n\right), \qquad (2,3,2,12)$$

where $\hat{k}_0 = (m_n p_- + \sum_{j=1}^{n-1} im_j\xi_j)^{-1}\hat{h}$, $\hat{k}_1 = p_- \hat{k}_0$ *and* $h = g - \gamma_n ME * \tilde{f}$.

Proof We first observe that we can apply Lemma 2.3.2.3 to $v = u - E * \tilde{f}$, since $f \in L_2(\mathbb{R}^n_+)$ and consequently $E * \tilde{f} \in H^2(\mathbb{R}^n)$.

Let us consider the equation of $\tilde{v}$ (again $\tilde{u}$ is the extension of u defined by $\tilde{u} = 0$ for $x_n < 0$):

$$\begin{aligned} L\tilde{v} + \tilde{v} &= \sum_{j,k=1}^{n} l_{j,k}D_jD_k\tilde{v} + \tilde{v} \\ &= l_{n,n}D_n^2\tilde{v} + 2\sum_{j=1}^{n-1} l_{j,n}D_jD_n\tilde{v} + \sum_{j,k=1}^{n-1} l_{j,k}D_jD_k\tilde{v} + \tilde{v} \\ &= l_{n,n}(\gamma_n v \otimes \delta'_n + \gamma_n D_n v \otimes \delta_n) + 2\sum_{j=1}^{n-1} l_{j,n}D_j\gamma_n v \otimes \delta_n. \end{aligned}$$

Since $\tilde{v}$ is a tempered distribution we check by Fourier transform that

$$\tilde{v} = E * \left(l_{n,n}[\gamma_n v \otimes \delta'_n + \gamma_n D_n v \otimes \delta_n] + 2\sum_{j=1}^{n-1} l_{j,n}D_j\gamma_n v \otimes \delta_n\right).$$

Then we derive $k_0 = \gamma_n v$ from (2,3,2,11) and substitute $\gamma_n D_n v$ by k_1, where $\hat{k}_1 = p_- \hat{k}_0$ by Lemma 2.3.2.3. ■

The representation formula (2,3,2,12) is the key tool for proving the estimate (2,3,2,3). We need two more auxiliary lemmas.

Lemma 2.3.2.5 *Let a fulfil the assumptions in Theorem 2.3.2.1. Then the operator*

$$M_a : g \mapsto F^{-1}aFg$$

is continuous in $W_p^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.

This result can be found in Triebel (1978), however, we can also obtain it as a consequence of Theorem 2.3.2.1.

Proof of Lemma 2.3.2.5 Applying Theorem 2.3.2.1 with a replaced by

$$\xi \mapsto (1+|\xi|^2)^{m/2} a(\xi)(1+|\xi|^2)^{-m/2}$$

and remembering Definition 1.3.1.3, we check that M_a is continuous in $H_p^m(\mathbb{R}^n)$ for all $m \in \mathbb{Z}$. Now, $H_p^m(\mathbb{R}^n)$ is the same space as $W_p^m(\mathbb{R}^n)$. The result stated in Lemma 2.3.2.5 for a non-integer s, follows by the interpolation Theorem 1.4.3.5. ■

Lemma 2.3.2.6 *The functions*

$$\xi \mapsto |\xi|^{|\alpha|-1} D^\alpha p_\pm(\xi)$$

are bounded on $\mathbb{R}^{n-1}$ for all α.

Proof This is easily checked on the explicit formula for the roots of (2,3,2,10)

$$p_\pm(\xi) = l_{n,n}^{-1}\left\{-\mathrm{i}\sum_{j=1}^{n-1} l_{n,j}\xi_j \mp \left[-\left(\sum_{j=1}^{n-1} l_{n,j}\xi_j\right)^2 - l_{n,n}\left(1-\sum_{j,k=1}^{n-1} l_{j,k}\xi_j\xi_k\right)\right]^{1/2}\right\}.$$

It follows from (2,3,2,1) that $l_{n,n} < 0$ and that the polynomial in the bracket is always strictly positive. ■

We are now able to prove the basic estimate.

Theorem 2.3.2.7 *Let $-L$ be a homogeneous strongly elliptic second-order operator with constant coefficients and let M be a homogeneous first-order operator with constant coefficients. Assume that $x_n = 0$ is not characteristic for M. Then there exists a constant C such that (2,3,2,3) holds for all $u \in W_p^2(\mathbb{R}_+^n)$.*

Proof It is enough to prove inequality (2,3,2,3) for $u \in H^2(\mathbb{R}_+^n) \cap W_p^2(\mathbb{R}_+^n)$, since this is a dense subspace of $W_p^2(\mathbb{R}_+^n)$. This allows one to use the representation formula (2,3,2,12) for u. We shall consider each term separately.

We start from $f \in L_p(\mathbb{R}_+^n)$ and $h \in W_p^{1-1/p}(\mathbb{R}^{n-1})$. Since E maps $L_p(\mathbb{R}^n)$ into $W_p^2(\mathbb{R}^n)$, we have

$$\|E * \tilde{f}\|_{2,p,\mathbb{R}_+^n} \leq C_1 \|f\|_{0,p,\mathbb{R}_+^n}. \tag{2,3,2,13}$$

Then let us set

$$b(\xi) = \left(m_n p_-(\xi) + \mathrm{i}\sum_{j=1}^{n-1} m_j\xi_j\right)^{-1}.$$

It follows from Lemma 2.3.2.6 that b, $\xi_j b$, $1 \leqslant j \leqslant n-1$ and $p_- b$, all fulfil (2,3,2,4). Consequently the mappings

$$h \mapsto k_0 = F^{-1} bFh$$
$$h \mapsto D_j k_0 = F^{-1} \mathrm{i}\xi_j bFh, \qquad 1 \leqslant j \leqslant n-1$$
$$h \mapsto k_1 = F^{-1} p_- bFh$$

are continuous operators in $W_p^{1-1/p}(\mathbb{R}^{n-1})$ owing to Lemma 2.3.2.5. Thus we have

$$\|k_0\|_{2-1/p,p,\mathbb{R}^{n-1}} + \|k_1\|_{1-1/p,p,\mathbb{R}^{n-1}} \leqslant C_2 \|h\|_{1-1/p,p,\mathbb{R}^{n-1}}. \tag{2,3,2,14}$$

From this we deduce that

$$\zeta = l_{n,n} k_1 + 2 \sum_{j=1}^{n-1} l_{n,j} D_j k_0 \in W_p^{1-1/p}(\mathbb{R}^{n-1}).$$

Then $\zeta \in W_p^{1-1/p}(\mathbb{R}^{n-1})$ and $D_j \zeta \in W_p^{-1/p}(\mathbb{R}^{n-1})$, $1 \leqslant j \leqslant n-1$. Lemma 2.3.2.2 implies that

$$\zeta \otimes \delta_n \in W_p^{-1}(\mathbb{R}^n) \quad \text{and} \quad D_j \zeta \otimes \delta_n \in W_p^{-1}(\mathbb{R}^n), \qquad 1 \leqslant j \leqslant n-1.$$

Consequently

$$E * (\zeta \otimes \delta_n) \in W_p^1(\mathbb{R}^n)$$
$$D_j E * (\zeta \otimes \delta_n) = E * (D_j \zeta \otimes \delta_n) \in W_p^1(\mathbb{R}^n), \qquad 1 \leqslant j \leqslant n-1$$

by Lemma 2.3.2.5. In other words, we have shown that

$$E * (\zeta \otimes \delta_n) \in L_p(\mathbb{R}^n)$$
$$D_j E * (\zeta \otimes \delta_n) \in L_p(\mathbb{R}^n), \qquad 1 \leqslant j \leqslant n$$
$$D_j D_k E * (\zeta \otimes \delta_n) \in L_p(\mathbb{R}^n), \qquad 2 \leqslant j+k \leqslant 2n-1.$$

We just need to check that $D_n^2 E * (\zeta \otimes \delta_n) \in L_p(\mathbb{R}_+^n)$ in order to prove that $E * (\zeta \otimes \delta_n) \in W_p^2(\mathbb{R}_+^n)$. This is achieved by using the fact that E is an elementary solution for $L+1$. This implies that in $\mathbb{R}_+^n$ we have

$$D_n^2 E * (\zeta \otimes \delta_n) = -\frac{1}{l_{n,n}} \left[E * (\zeta \otimes \delta_n) + \sum_{j+k \leqslant 2n-1} l_{j,k} D_j D_k E * (\zeta \otimes \delta_n) \right].$$

Summing up, we have shown that

$$E * \left[l_{n,n} k_1 + 2 \sum_{j=1}^{n-1} l_{n,j} D_j k_0 \right] \otimes \delta_n \in W_p^2(\mathbb{R}_+^n)$$

and in addition, we have

$$\left\| E * \left[l_{n,n} k_1 + 2 \sum_{j=1}^{n-1} l_{n,j} D_j k_0 \right] \otimes \delta_n \right\|_{2,p,\mathbb{R}_+^n}$$
$$\leqslant C_3 [\|k_0\|_{2-1/p,p,\mathbb{R}^{n-1}} + \|k_1\|_{1-1/p,p,\mathbb{R}^{n-1}}] \tag{2.3.2.15}$$

owing to the continuity of all the involved operators.

Finally, let us consider $E * (k_0 \otimes \delta_n')$. We start from $k_0 \in W_p^{2-1/p}(\mathbb{R}^{n-1})$, so that $k_0 \in W_p^{-1/p}(\mathbb{R}^{n-1})$, $D_j k_0 \in W_p^{-1/p}(\mathbb{R}^{n-1})$, $1 \leq j \leq n-1$ and $D_j D_k k_0 \in W_p^{-1/p}(\mathbb{R}^{n-1})$, $1 \leq j, k \leq n-1$. From Lemmas 2.3.2.2 and 2.3.2.5 it follows that

$$\begin{aligned}
&E * (k_0 \otimes \delta_n') \in L_p(\mathbb{R}^n) \\
&D_j E * (k_0 \otimes \delta_n') \in L_p(\mathbb{R}^n), && 1 \leq j \leq n-1 \\
&D_j D_k E * (k_0 \otimes \delta_n') \in L_p(\mathbb{R}^n), && 1 \leq j, k \leq n-1.
\end{aligned}$$

Then we write that

$$\begin{aligned}
D_n E * (k_0 \otimes \delta_n') &= D_n^2 E * (k_0 \otimes \delta_n) \\
&= -\frac{1}{l_{n,n}} \Big[E * (k_0 \otimes \delta_n) + \sum_{j+k \leq 2n-1} l_{j,k} D_j D_k E * (k_0 \otimes \delta_n) \Big]
\end{aligned}$$

since E is an elementary solution for $L+1$. It follows that in $\mathbb{R}_+^n$, we have

$$\begin{aligned}
D_n E * (k_0 \otimes \delta_n') = -\frac{1}{l_{n,n}} \Big[& E * (k_0 \otimes \delta_n) + \sum_{j,k=1}^{n-1} l_{j,k} E * (D_j D_k k_0) \otimes \delta_n \\
& + 2 \sum_{j=1}^{n-1} l_{j,n} E * (D_j k_0 \otimes \delta_n') \Big]
\end{aligned}$$

and that

$$\begin{aligned}
D_i D_n E * (k_0 \otimes \delta_n') = -\frac{1}{l_{n,n}} \Big[& E * (D_i k_0 \otimes \delta_n) \\
& + \sum_{j,k=1}^{n-1} l_{j,k} D_i E * (D_j D_k k_0) \otimes \delta_n \\
& + 2 \sum_{j=1}^{n-1} l_{j,n} E * (D_i D_j k_0 \otimes \delta_n') \Big], \qquad 1 \leq i \leq n-1.
\end{aligned}$$

Again applying Lemmas 2.3.2.2 and 2.3.2.5, we show that

$$\begin{aligned}
&D_n E * (k_0 \otimes \delta_n') \in L_p(\mathbb{R}_+^n) \\
&D_i D_n E * (k_0 \otimes \delta_n') \in L_p(\mathbb{R}_+^n), && 1 \leq i \leq n-1.
\end{aligned}$$

The only derivative missing for proving that $E * (k_0 \otimes \delta_n') \in W_p^2(\mathbb{R}_+^n)$ is $D_n^2 E * (k_0 \otimes \delta_n')$. Using the property of the elementary solution E, we derive this last fact as we did for $D_n^2 E * (\zeta \otimes \delta_n)$. Summing up, we have proved that

$$E * (k_0 \otimes \delta_n') \in W_p^2(\mathbb{R}_+^n)$$

and in addition,

$$\|E * (k_0 \otimes \delta_n')\|_{2,p,\mathbb{R}_+^n} \leq C_4 \|k_0\|_{2-1/p,p,\mathbb{R}^{n-1}}. \tag{2,3,2,16}$$

Putting together identity (2,3,2,12) with inequalities (2,3,2,13) to (2,3,2,16), we obtain the existence of a constant C such that

$$\|u\|_{2,p,\mathbb{R}^n_+} \leq C[\|Lu+u\|_{0,p,\mathbb{R}^n_+} + \|\gamma_n Mu\|_{1-1/p,p,\mathbb{R}^{n-1}}]$$

for all $u \in W^2_p(\mathbb{R}^n_+) \cap H^2(\mathbb{R}^n_+)$. By density, the same is true for all $u \in W^2_p(\mathbb{R}^n_+)$ and inequality (2,3,2,3) follows obviously. This completes the proof of Theorem 2.3.2.7. ■

Remark 2.3.2.8 Inspection shows that the constant C (deduced from C_1 to C_4) is bounded by a continuous function of the $l_{j,k}$ and the m_j.

The similar statement concerning Dirichlet's problem is this.

Theorem 2.3.2.9 *Let $-L$ be a homogeneous strongly elliptic second-order operator with constant coefficients. Then there exists a constant C such that*

$$\|u\|_{2,p,\mathbb{R}^n_+} \leq C[\|Lu\|_{0,p,\mathbb{R}^n_+} + \|\gamma_n u\|_{2-1/p,p,\mathbb{R}^{n-1}} + \|u\|_{1,p,\mathbb{R}^n_+}] \qquad (2,3,2,17)$$

for all $u \in W^2_p(\mathbb{R}^n_+)$.

Proof We use here the same representation formula (2,3,2,12) but with $k_0 = h = g - \gamma_n E * \bar{f}$ and $\hat{k}_1 = p_- \hat{k}_0$. The rest of the proof is exactly similar to that of Theorem 2.3.2.7. ■

2.3.3 A general *a priori* estimate

We consider again the general operators A and B of Section 2.1, in a general bounded domain Ω with a $C^{1,1}$ boundary. We shall now extend inequality (2,3,2,3) to this general case. Namely, we shall prove that there exists a constant C such that

$$\|u\|_{2,p,\Omega} \leq C[\|Au\|_{0,p,\Omega} + \|\gamma Bu\|_{2-d-1/p,p,\Gamma} + \|u\|_{1,p,\Omega}]. \qquad (2,3,3,1)$$

Then we shall combine inequalities (2,3,1,1) and (2,3,3,1) to obtain the basic inequality for the remaining sections of this chapter.

Inequality (2,3,3,1) is very flexible because of the norm of u in $W^1_p(\Omega)$ that appears on the right-hand side. Indeed, it allows us to 'localize' the inequality. This property is rigorously stated as follows.

Lemma 2.3.3.1 *Assume that each point $x \in \bar{\Omega}$ has a neighbourhood V_x such that (2,3,3,1) holds for all the functions u in $W^2_p(\Omega)$ which have their support in V_x. Then (2,3,3,1) holds for all $u \in W^2_p(\Omega)$ (with possibly another constant).*

Proof The compactness of $\bar{\Omega}$ allows us to find a finite number of points $x_1, \ldots, x_N$ in $\bar{\Omega}$, such that $\bar{\Omega}$ is covered by the interiors of V_{x_j}, $1 \leq j \leq N$.

Then we choose a partition of unity corresponding to this covering. Namely we assume that

$$1=\sum_{j=1}^{N}\theta_j$$

on $\bar{\Omega}$, where $\theta_j\in\mathscr{D}(\bar{\Omega})$ and the support of θ_j is contained in the interior of V_{x_j}, $1\leqslant j\leqslant N$.

The assumption of Lemma 2.3.3.1 is that (2,3,3,1) holds in particular for all the $\theta_j u$. It follows that

$$\begin{aligned}\|u\|_{2,p,\Omega}&\leqslant\sum_{j=1}^{N}\|\theta_j u\|_{2,p,\Omega}\\&\leqslant C\sum_{j=1}^{N}[\|A(\theta_j u)\|_{0,p,\Omega}+\|\gamma B(\theta_j u)\|_{2-d-1/p,p,\Gamma}+\|\theta_j u\|_{1,p,\Omega}]\\&\leqslant C_1[\|Au\|_{0,p,\Omega}+\|\gamma Bu\|_{2-d-1/p,p,\Gamma}+\|u\|_{1,p,\Omega}]\\&\quad+C\sum_{j=1}^{N}[\|[A;\theta_j]u\|_{0,p,\Omega}+\|\gamma[B;\theta_j]u\|_{2-d-1/p,p,\Gamma}].\end{aligned}\tag{2,3,3,2}$$

Here $[A;\theta_j]$ is a first-order operator with continuous coefficients so that there exists C_2 such that

$$\|[A;\theta_j]u\|_{0,p,\Omega}\leqslant C_2\|u\|_{1,p,\Omega},\qquad 1\leqslant j\leqslant N\tag{2,3,3,3}$$

for all $u\in W_p^2(\Omega)$. The same way, $[B;\theta_j]$ is either 0 when $d=0$ (i.e. $B=I$) or the multiplication by a Lipschitz-continuous function when $d=1$. In both cases there exists C_3 such that

$$\|\gamma[B;\theta_j]u\|_{2-d-1/p,p,\Gamma}\leqslant C_3\|u\|_{1,p,\Omega},\qquad 1\leqslant j\leqslant N\tag{2,3,3,4}$$

for all $u\in W_p^2(\Omega)$. Inequality (2,3,3,1) follows from (2,3,3,2), (2,3,3,3) and (2,3,3,4) by addition. ■

We shall now prove inequality (2,3,3,1).

Theorem 2.3.3.2 *Let A and B fulfil the assumptions in Section 2.1; then there exists a constant C such that (2,3,3,1) holds for all $u\in W_p^2(\Omega)$.*

Taking advantage of Lemma 2.3.3.1 we shall restrict ourselves to proving inequality (2,3,3,1) in those two particular cases.

Case (a) the support of u is compact in Ω.
Case (b) the support of u is contained in $\Phi(V)$ where V is an open neighbourhood of O and Φ is a $C^{1,1}$ diffeomorphism of V onto $\Phi(V)$ such that

$$\Phi^{-1}(\Omega\cap\Phi(V))=U=\mathbb{R}_+^n\cap V.$$

We observe that the norms involved in (2,3,3,1) are invariant under $C^{1,1}$ changes of coordinates. Furthermore the properties of A and B are also invariant under $C^{1,1}$ changes of coordinates. That is why we shall consider $u \circ \Phi$ instead of u. This reduces the proof to the particular case where the intersection of the support of u with Γ is contained in $\{x_n = 0\}$.

The case (a) is solved with the help of this lemma.

Lemma 2.3.3.3 *For all $y \in \Omega$, there exists a neighbourhood V of y in Ω, such that (2,3,3,1) holds for all $u \in W_p^2(\Omega)$, whose support is contained in V.*

Proof We use the famous perturbation argument known as Korn's procedure. Freezing the coefficients of A at y, we obtain an operator with constant coefficients

$$L = \sum_{i,j=1}^{n} l_{i,j} D_i D_j,$$

where $l_{i,j} = a_{i,j}(y)$, which satisfies the assumptions of Section 2.3.2.

We then observe that

$$Lu + u = Au - \left[\sum_{i,j=1}^{n} D_i (a_{i,j} - l_{i,j}) D_j u - u \right]$$

in Ω. If we denote by $\alpha_{i,j}$ any Lipschitz functions defined everywhere, such that $\alpha_{i,j} = a_{i,j}$ on $\bar{\Omega}$, we have

$$L\tilde{u} + \tilde{u} = \widetilde{Au} - \left[\sum_{i,j=1}^{n} (\alpha_{i,j} - l_{i,j}) \widetilde{D_i D_j u} + \sum_{i,j=1}^{n} (D_i \alpha_{i,j}) \widetilde{D_j u} - \tilde{u} \right].$$

Now we assume that the support of u is contained in V such that $\bar{V} \subset \Omega$. Using the elementary solution E introduced in Section 2.3.2, we obtain

$$\tilde{u} = E * A\tilde{u} - E * \left[\sum_{i,j=1}^{n} (a_{i,j} - l_{i,j}) \widetilde{D_i D_j u} + \sum_{i,j=1}^{n} (D_i \alpha_{i,j}) \widetilde{D_j u} - \tilde{u} \right].$$

Since $E *$ is a linear continuous map from $L_p(\mathbb{R}^n)$ into $W_p^2(\mathbb{R}^n)$, it follows that

$$\|u\|_{2,p,\Omega} \leq C \left[\|Au\|_{0,p,\Omega} + \sum_{i,j=1}^{n} \|(a_{i,j} - l_{i,j}) D_i D_j u\|_{0,p,\Omega} \right] + C_1 \|u\|_{1,p,\Omega}.$$

Let us now call δ the diameter of V and K a bound for the Lipschitz constants of all the $a_{i,j}$ in $\bar{\Omega}$. We then have

$$\|u\|_{2,p,\Omega} \leq C[\|Au\|_{0,p,\Omega} + n^2 K \delta \|u\|_{2,p,\Omega}] + C_1 \|u\|_{1,p,\Omega}.$$

We conclude by choosing δ small enough; indeed, if we assume that δ is less than or equal to $1/2Cn^2K$, we have

$$\|u\|_{2,p,\Omega} \leq 2C\,\|Au\|_{0,p,\Omega} + 2C_1\,\|u\|_{1,p,\Omega}.$$

This is exactly inequality (2,3,3,1) for u, since the support of u is contained in $V \subset \Omega$ and consequently $\gamma Bu = 0$. The proof of Lemma 2.3.3.3 is complete. ■

Let us now consider the case (b). It is solved with the help of this last lemma.

Lemma 2.3.3.4 *Let $y \in \Gamma$ have a neighbourhood W in Γ, contained in the hyperplane $\{x_n = 0\}$. Then there exists a neighbourhood U of y in $\bar{\Omega}$ such that (2,3,3,1) holds for all $u \in W_p^2(\Omega)$, whose support is contained in U.*

Proof We shall use the same perturbation argument as in the proof of the previous lemma. We freeze the coefficients of A and B at y and obtain operators with constant coefficients:

$$L = \sum_{i,j=1}^{n} l_{i,j} D_i D_j$$

$$M = \text{either } I \text{ or } \sum_{j=1}^{n} m_j D_j$$

where $l_{i,j} = a_{i,j}(y)$ and $m_j = b_j(y)$ (when $d = 1$). These operators satisfy the assumptions of Section 2.3.2.

We start with an open neighbourhood U of y in $\bar{\Omega}$ such that $U \cap \Gamma \subset W$, and we assume that the support of u is contained in U. We then have

$$Lu = Au - \left[\sum_{i,j=1}^{n} D_i(a_{i,j} - l_{i,j}) D_j u\right]$$

in $U \cap \Omega$ and

$$\gamma_n Mu = \gamma_n Bu - \gamma_n\left[\sum_{j=1}^{n} (b_j - m_j) D_j u\right]$$

in W if $d = 1$, while $\gamma_n u = \gamma_n Bu$ if $d = 0$. We again denote by $\alpha_{i,j}$ (respectively β_j) any Lipshitz functions defined everywhere, such that $\alpha_{i,j} = a_{i,j}$ in $\bar{\Omega}$ (respectively $\beta_j = b_j$ in $\bar{\Omega}$). We have

$$L\tilde{u} = \widetilde{Au} - \left[\sum_{i,j=1}^{n} (\alpha_{i,j} - l_{i,j}) \widetilde{D_i D_j u} + \sum_{i,j=1}^{n} (D_i \alpha_{i,j}) \widetilde{D_j u}\right]$$

in $\mathbb{R}_+^n$ and

$$\gamma_n M\tilde{u} = \gamma_n Bu - \gamma_n\left[\sum_{j=1}^{n} (\beta_j - m_j) \widetilde{D_j u}\right]$$

on $\{x_n = 0\}$ if $d = 1$, while $\gamma_n \tilde{u} = \gamma_n \widetilde{Bu}$ if $d = 0$. Since $\tilde{u} \in W_p^2(\mathbb{R}_+^n)$ we can use estimate (2,3,2,3) proved in Theorems 2.3.2.7 for $d = 1$ and 2.3.2.9 for $d = 0$. It follows that

$$\|u\|_{2,p,\Omega} = C\Big\{\|Au\|_{0,p,\Omega} + \sum_{i,j=1}^{n} \|(a_{i,j} - l_{i,j})D_iD_ju\|_{0,p,\Omega} + \|\gamma Bu\|_{2-d-1/p,p,\Gamma}$$
$$+ \sum_{j=1}^{n} \|(b_j - m_j)D_ju\|_{2-d-1/p,p,\Gamma}\Big\} + C_1\|u\|_{1,p,\Omega}$$

when $d = 1$ (the additional boundary term $\sum_{j=1}^{n} \|(b_j - m_j)D_ju\|_{2-d-1/p,p,\Gamma}$ does not appear when $d = 0$).

Let us again denote by δ a bound for the diameter of U and by K a bound for the Lipschitz constants of all the $a_{i,j}$ and b_j. It follows that

$$\|u\|_{2,p,\Omega} \leqslant C\{\|Au\|_{0,p,\Omega} + \|\gamma Bu\|_{2-d-1/p,p,\Gamma}\} + C_1\|u\|_{1,p,\Omega}$$
$$+ n^2KC\delta\|u\|_{2,p,\Omega} + \sum_{j=1}^{n} C_2\|(b_j - m_j)D_ju\|_{1,p,\Omega} \tag{2,3,3,5}$$

by the trace theorem (Section 1.5.1). Let us consider separately the last term of this inequality. We have

$$\|(b_j - m_j)D_ju\|_{1,p,\Omega}^p = \sum_{i=1}^{n} \|D_i(b_j - m_j)D_ju\|_{0,p,\Omega}^p + \|(b_j - m_j)D_ju\|_{0,p,\Omega}^p$$
$$\leqslant (nK\delta)^p\|u\|_{2,p,\Omega}^p + C_3\|u\|_{1,p,\Omega}^p. \tag{2,3,3,6}$$

From (2,3,3,5) and (2,3,3,6) we deduce that

$$\|u\|_{2,p,\Omega} \leqslant C\{\|Au\|_{0,p,\Omega} + \|\gamma Bu\|_{2-d-1/p,p,\Gamma}\} + C_4\|u\|_{1,p,\Omega}$$
$$+ \{n^2C + nC_5\}K\delta\|u\|_{2,p,\Omega}.$$

Choosing δ small enough so that $\{n^2C + nC_5\}K\delta \leqslant \frac{1}{2}$, we obtain finally

$$\|u\|_{2,p,\Omega} \leqslant 2C\{\|Au\|_{0,p,\Omega} + \|\gamma Bu\|_{2-d-1/p,p,\Gamma}\} + 2C_4\|u\|_{1,p,\Omega}.$$

This is inequality (2,3,3,1). ■

Proof of Theorem 2.3.3.2 We apply Lemma 2.3.3.1. The existence of V_x follows from Lemma 2.3.3.3 when $x \in \Omega$ while it follows from Lemma 2.3.3.4 after change of coordinates, when $x \in \Gamma$. In this last case we assume at once that $V_x \subset \Phi(V)$, where (V, Φ) is a map of the manifold $\bar{\Omega}$, near x (see notation above). This allows us to 'flatten' the boundary Γ, near x, by replacing u by $u \circ \Phi$. ■

Remark 2.3.3.5 Actually we have proved a little more than Theorem 2.3.3.2 and this will be useful in the next subsection. In the proof of Lemma 2.3.3.1, it is enough to cover the support of u by the interiors of

V_{x_j}, $1 \leq j \leq N$. Consequently we can release the assumptions on Ω. Let us assume that Ω is a (possibly unbounded) open subset of $\mathbb{R}^n$ with a $C^{1,1}$ boundary. Then for each compact subset K of $\bar{\Omega}$, there exists a constant C (depending on K) such that inequality (2,3,3,1) holds for all $u \in W_p^2(\Omega)$, with support in K.

We are now able to perform the final step of our search for *a priori* estimates.

Theorem 2.3.3.6 *Let A and B fulfil the assumptions in Section 2.1, then there exist C and λ_0 such that*

$$\|u\|_{2,p,\Omega} \leq C[\|Au+\lambda u\|_{0,p,\Omega} + \|\gamma Bu\|_{2-d-1/p,p,\Gamma}] \tag{2,3,3,7}$$

for all $u \in W_p^2(\Omega)$ and $\lambda > \lambda_0$.

Proof We first improve inequality (2,3,3,1), using Theorem 1.4.3.3. We have

$$\|u\|_{1,p,\Omega} \leq \varepsilon \|u\|_{2,p,\Omega} + \frac{K}{\varepsilon}\|u\|_{0,p,\Omega} \tag{2,3,3,8}$$

for all $u \in W_p^2(\Omega)$ and $\varepsilon > 0$. Choosing $\varepsilon > 0$ small enough and substituting (2,3,3,8) in (2,3,3,1) we obtain:

$$\begin{aligned} \|u\|_{2,p,\Omega} &\leq C_1[\|Au\|_{0,p,\Omega} + \|\gamma Bu\|_{2-d-1/p,p,\Gamma} + \|u\|_{0,p,\Omega}] \\ &\leq C_1[\|Au+\lambda u\|_{0,p,\Omega} + \|\gamma Bu\|_{2-d-1/p,p,\Gamma} + (1+\lambda)\|u\|_{0,p,\Omega}]. \end{aligned} \tag{2,3,3,9}$$

We now take advantage of (2,3,1,11). We have

$$\|u\|_{0,p,\Omega} \leq \frac{1}{\lambda-\lambda_0}\|Au+\lambda u\|_{0,p,\Omega}$$

for $u \in W_p^2(\Omega)$ such that $\gamma Bu = 0$. From (2,3,3,9) it follows that

$$\|u\|_{2,p,\Omega} \leq C_1\left(1+\frac{1+\lambda}{\lambda-\lambda_0}\right)\|Au+\lambda u\|_{0,p,\Omega} = C_2\|Au+\lambda u\|_{0,p,\Omega}.$$

This is exactly (2,3,3,7) in the particular case where $\gamma Bu = 0$.

Let us now consider the general case. From Theorem 1.6.1.3 we know that there exists a linear continuous operator R from $W_p^{2-d-1/p}(\Gamma)$ into $W_p^2(\Omega)$ such that

$$\gamma BRg = g$$

for all $g \in W_p^{2-d-1/p}(\Gamma)$. We set $v = u - R\gamma Bu$. It is clear that $v \in W_p^2(\Omega)$

and that $\gamma Bv=0$ so that (2,3,3,7) holds for v. It follows that

$$\begin{aligned}\|u\|_{2,p,\Omega}&\leq\|v\|_{2,p,\Omega}+\|R\gamma Bu\|_{2,p,\Omega}\\&\leq C_2\,\|Av+\lambda v\|_{0,p,\Omega}+C_3\,\|\gamma Bu\|_{2-d-1/p,p,\Gamma}\\&\leq C_2\{\|Au+\lambda u\|_{0,p,\Omega}+\|(A+\lambda)R\gamma Bu\|_{0,p,\Omega}\}+C_2\,\|\gamma Bu\|_{2-d-1/p,p,\Gamma}\\&\leq C_4\{\|Au+\lambda u\|_{0,p,\Omega}+\|\gamma Bu\|_{2-d-1/p,p,\Gamma}\}.\end{aligned}$$

This is (2,3,3,7). ■

2.4 Existence and uniqueness, the general case

In this section we derive a general existence and uniqueness result for problem (2,1,4) as a consequence of the *a priori* estimate of Section 2.3. Then we remove the parameter λ and attempt to solve problem (2,1,3).

2.4.1 The basic result

We shall show that under the assumptions of Section 2.1, the mapping

$$T_{p,\lambda}: u\mapsto\{Au+\lambda u;\ \gamma Bu\}$$

is an isomorphism from $W^2_p(\Omega)$ onto $L_p(\Omega)\times W_p^{2-d-1/p}(\Gamma)$, for λ large enough. For that purpose, we shall consider successively the three cases (a) $p=2$, (b) $p<2$, (c) $p>2$.

The starting point of the proof consists in observing that $T_{2,\lambda}$ is a semi-Fredholm operator for λ large enough. Indeed, from (2,3,3,7), it follows that $T_{2,\lambda}$ is one to one and that the image of $T_{2,\lambda}$ is closed in $L_2(\Omega)\times H^{3/2-d}(\Gamma)$. This allows us to consider the index of $T_{2,\lambda}$ which is (in this particular case):

$$\chi(T_{2,\lambda})=-\operatorname{def} R(T_{2,\lambda})$$

where $R(T_{2,\lambda})$ is the image (range) of $T_{2,\lambda}$ and $\operatorname{def} R(T_{2,\lambda})$ is the codimension (possibly infinite) of $R(T_{2,\lambda})$ i.e., the dimension of

$$\{L_2(\Omega)\times H^{3/2-d}(\Gamma)\}/R(T_{2,\lambda}).$$

It is well known that the index remains constant when one performs a homotopy from an operator to another, remaining in the set of all semi-Fredholm operators (see Kato (1966), Chapter IV, §5). We shall only use the following very simple form of this general principle:

Lemma 2.4.1.1 *Let X, Y be a pair of Banach spaces and let $t\to T_t$ be a continuous mapping from $[a,b]$ (a and b are any real numbers) into the space $L(X;Y)$ of all continuous linear operators from X into Y. Assume* *that for each t, there exists C_t such that*

$$\|x\|_X \leqslant C_t \, \|T_t x\|_Y, \qquad x \in X \tag{2,4,1,1}$$

Assume T_a is an isomorphism; then T_b is also an isomorphism.

A direct elementary proof, avoiding the general theory of semi-Fredholm operators, can be built from the fact that isomorphisms define an open subset of $L(X; Y)$.

Many of the estimates that we have derived involve a parameter λ whose lower bound depends on the particular problem which is under consideration. In performing homotopies from one problem to another, this will cause problems. This is why we shall use the following technical lemma.

Lemma 2.4.1.2 *Let X, Y be a pair of Banach spaces. Let $t \mapsto T_t$ be a continuous mapping from $[a, b]$ into $L(X, Y)$. Let also S be a fixed element of $L(X; Y)$. Assume that for each t there exists C_t and λ_t such that*

$$\|x\|_X \leqslant C_t \, \|T_t x + \lambda S x\|_Y, \qquad x \in X \tag{2,4,1,2}$$

for all $\lambda \geqslant \lambda_t$. Then there exists $\bar{C}$ and $\bar{\lambda}$ such that

$$\|x\|_X \leqslant \bar{C} \, \|T_t x + \lambda S x\|_Y, \qquad x \in X \tag{2,4,1,3}$$

for all $\lambda \geqslant \bar{\lambda}$ and $t \in [a, b]$.

Proof Consider any pair of numbers t and t' in $[a, b]$. Then we have, for $\lambda \geqslant \lambda_t$,

$$\|x\|_X \leqslant C_t \, \|T_t x + \lambda S x\|_Y \leqslant C_t \, \|T_t - T_{t'}\|_{X \to Y} \, \|x\|_X + C_t \, \|T_{t'} x + \lambda S x\|_Y.$$

If we assume that t and t' are close enough to one another, we have

$$\|T_t - T_{t'}\|_{X \to Y} \leqslant \tfrac{1}{2} C_t$$

since the mapping $t \mapsto T_t$ is continuous. It follows that

$$\|x\|_X \leqslant 2C_t \, \|T_{t'} x + \lambda S x\|_Y$$

for $\lambda \geqslant \lambda_t$. This shows the existence of $\bar{C}$ and $\bar{\lambda}$ locally. The desired result follows, since $[a, b]$ is compact. ■

We now prove our basic result

Theorem 2.4.1.3 *Let A, B and Ω fulfil the assumptions in Section 2.1. Then for $1 < p < \infty$, there exists λ_p such that*

$$T_{p,\lambda} : u \mapsto \{Au + \lambda u; \gamma B u\}$$

is an isomorphism from $W_p^2(\Omega)$ onto $L_p(\Omega) \times W_p^{2-d-1/p}(\Gamma)$ for all $\lambda \geqslant \lambda_p$.

Proof for $p=2$ This is nothing but Corollary 2.2.2.4 when $B=I$, i.e. when we are solving Dirichlet's problem. The same way, this is nothing but Corollary 2.2.2.6 when $B=\partial/\partial\nu_A$ on Γ, i.e. when we are solving Neumann's problem.

Let us consider now an oblique boundary condition (i.e. $d=1$). We shall perform a homotopy from Neumann's problem to our problem. For that purpose, we introduce the operators

$$B_t u=(1-t)\frac{\partial u}{\partial\nu_A}+tBu, \qquad t\in[0,1].$$

We observe that Γ is not characteristic for B_t, for all $t\in[0,1]$, provided $b_\nu=\sum_{i=1}^n b_i\nu_i<0$† (if this is not the case we replace B by $-B$). Accordingly, we can apply Theorem 2.3.3.6 to the mapping

$$T_t: u\rightarrow\{Au;\ \gamma B_t u\}$$

and there exists C_t and λ_t such that

$$\|u\|_{2,2,\Omega}\leq C_t[\|Au+\lambda u\|_{0,2,\Omega}+\|\gamma B_t u\|_{1/2,2,\Gamma}]$$

for all $u\in H^2(\Omega)$ and $\lambda\geq\lambda_t$.

Let us now set $X=H^2(\Omega)$, $Y=L_2(\Omega)\times H^{1/2}(\Gamma)$ and

$$S: u\mapsto\{u,0\}.$$

It is obvious that $t\mapsto T_t$ is continuous from $[0,1]$ into $L(X;Y)$. Applying Lemma 2.4.1.2, we find $\bar\lambda$ and $\bar C$ such that

$$\|u\|_{2,2,\Omega}\leq\bar C[\|Au+\lambda u\|_{0,2,\Omega}+\|\gamma B_t u\|_{1/2,2,\Gamma}] \tag{2,4,1,4}$$

for all $u\in H^2(\Omega)$, $\lambda\geq\bar\lambda$ and $t\in[0,1]$.

A first application of Lemma 2.4.1.1, using homotopy in λ instead of t, shows that $T_0+\lambda S$ is an isomorphism for all $\lambda\geq\bar\lambda$, since by Corollary 2.2.2.6 we already know that $T_0+\lambda S$ is an isomorphism for λ large enough.

A second application of Lemma 2.4.1.1, using homotopy in t with a fixed $\lambda\geq\bar\lambda$, shows that $T_1+\lambda S$ is an isomorphism, since $T_0+\lambda S$ is so. This proves Theorem 2.4.1.3 when $p=2$.

Proof of Theorem 2.4.1.3 for all $p<2$ Inequality (2,3,3,7) shows that $T_{p,\lambda}$ is one to one and has a closed range for λ large enough. On the other hand, the result already proved for $p=2$ shows that the range of $T_{p,\lambda}$ contains $L_2(\Omega)\times H^{3/2}(\Gamma)$, since $H^2(\Omega)\subseteq W_p^2(\Omega)$. Consequently, the range of $T_{p,\lambda}$ is also dense; this proves that $T_{p,\lambda}$ is onto.

† We recall that b_ν does not vanish on Γ, since Γ is not characteristic for B.

Proof of Theorem 2.4.1.3 for all $p>2$ We shall make use of this auxiliary smoothness result which will be proved later on.

Lemma 2.4.1.4 *Let A, B and Ω fulfil the assumptions in Section 2.1. Let $u \in W_r^2(\Omega)$ be a solution of*

$$\begin{cases} Au = f & \text{in } \Omega \\ \gamma Bu = g & \text{on } \Gamma \end{cases}$$

where $f \in L_p(\Omega)$, $g \in W_p^{2-d-1/p}(\Gamma)$. Assume that $p \leq rn/(n-r)$ if $r<n$. Then $u \in W_p^2(\Omega)$.

Exactly as in all the preceding cases, it follows from inequality (2,3,3,7) that $T_{p,\lambda}$ is one to one and that its range is closed. To prove that $T_{p,\lambda}$ is onto, we start from $f \in L_p(\Omega)$ and $g \in W_p^{2-d-1/p}(\Gamma)$. We have consequently $f \in L_2(\Omega)$ and $g \in H^{3/2-d}(\Gamma)$. Applying again the result already proved for $p=2$, we know that there exists $u \in H^2(\Omega)$ such that

$$\begin{cases} Au + \lambda u = f & \text{in } \Omega \\ \gamma Bu = g & \text{on } \Gamma \end{cases}$$

for λ large enough. A (possibly iterated) application of Lemma 2.4.1.4 shows that $u \in W_p^2(\Omega)$. Consequently, $T_{p,\lambda}$ is onto. ■

Proof of Lemma 2.4.1.4 It uses methods very similar to those in Section 2.2.2. Accordingly, we shall first localize our problem with the aid of cut-off functions. Then we shall use a $C^{1,1}$ change of coordinates to flatten the boundary. Finally we shall use Friedrichs' mollifiers method instead of Nirenberg's tangential differential quotients. This is to prove smoothness in the case where the boundary is flat.

Thus, let θ be any function in $\mathscr{D}(\bar{\Omega})$ and set $u_1 = \theta u$. It follows that $u_1 \in W_r^2(\Omega)$ and that

$$f_1 = Au_1 = \theta f + [A\,;\theta]u \in L_p(\Omega).$$

Indeed the assumption on p ensures that $W_r^1(\Omega)$ is contained in $L_p(\Omega)$ by Sobolev's imbedding theorem (see Section 1.4.4). Then we have

$$g_1 = \gamma Bu_1 = \theta g \in W_p^{2-1/p}(\Gamma)$$

when $d=0$ and

$$g_1 = \gamma Bu_1 = \theta g + \gamma[B\,;\theta]u \in W_p^{1-1/p}(\Gamma)$$

when $d=1$.

Now let V be an open subset of $\mathbb{R}^n$ and let Φ be a $C^{1,1}$ diffeomorphism of $\bar{V}$ onto a neighbourhood of the support of θ. Assume that

$$\Phi^{-1}(\Omega \cap \Phi(V)) = U = \mathbb{R}_+^n \cap V.$$

(We do not exclude here the possibility that $\Gamma \cap \Phi(V) = \emptyset$.) Then we consider $u_2 = u_1 \circ \Phi$. Again we have $u_2 \in W_r^2(U)$ and setting $\Psi = \Phi^{-1}$, we have

$$A^{\#}u_2 = \sum_{k,l=1}^{n} D_l a_{k,l}^{\#} D_k u_2 = f_2 \qquad \text{in } U \tag{2,4,1,5}$$

where

$$a_{k,l}^{\#} = \sum_{i,j=1}^{n} [a_{i,j}(D_i\Psi_k)(D_j\Psi_l)] \circ \Phi\, |D\Phi|,$$

$$f_2 = |D\Phi|\, f_1 \circ \Phi.$$

Clearly we have $f_2 \in L_p(U)$, $a_{k,l}^{\#} \in C^{0,1}(\bar{U})$ and in addition

$$\sum_{k,l=1}^{n} a_{k,l}^{\#}(y)\xi_k\xi_l \leq -\alpha^{\#}\, |\xi|^2$$

for all $\xi \in \mathbb{R}^n$ and all $y \in \bar{U}$, for some $\alpha^{\#} > 0$. We also have

$$\gamma u_2 = g_2 = g_1 \circ \Phi \qquad \text{in } V \cap \{x_n = 0\} \tag{2,4,1,6}$$

when $d = 0$ and

$$\gamma B^{\#}u_2 = \gamma \sum_{j=1}^{n} b_j^{\#} D_j u_2 = g_2 = g_1 \circ \Phi \qquad \text{in } V \cap \{x_n = 0\}, \tag{2,4,1,7}$$

where

$$b_j^{\#} = \sum_{i=1}^{n} [b_i(D_i\Psi_j)] \circ \Phi$$

if $d = 1$. Clearly again, we have $g_2 \in W_p^{2-d-1/p}(V \cap \{x_n = 0\})$, $b_j^{\#} \in C^{0,1}(\bar{U})$ and

$$b_n^{\#} = \sum_{i=1}^{n} [b_i(D_i\Psi_n)] \circ \Phi$$

does not vanish on $\{x_n = 0\}$ since $\boldsymbol{\nu}$ and the gradient of Ψ_n are parallel.
A first technical step is:

Lemma 2.4.1.5 *Under the above assumptions, we have $u_2 \in W_p^2(U)$.*

Then, since Ω is bounded and has a $C^{1,1}$ boundary, it is possible to find a finite number of open subsets V_k, $1 \leq k \leq N$ of $\mathbb{R}^n$, together with $C^{1,1}$ diffeomorphisms from $\bar{V}_k$ onto $\Phi_k(\bar{V}_k)$, $1 \leq k \leq N$, such that

(a) $\{\Phi_k(V_k)\}_{k=1}^{N}$ is a covering of $\bar{\Omega}$
(b) $\Phi_k^{-1}(\Omega \cap \Phi_k(V_k)) = U_k = \mathbb{R}_+^n \cap V_k, \qquad 1 \leq k \leq N.$

With this covering of $\bar{\Omega}$, we associate a partition of unity θ_k, $1 \leq k \leq N$, such that

(c) $\theta_k \in \mathcal{D}(\bar{\Omega})$
(d) the support of θ_k is included in $\Phi_k(V_k)$
(e) $\sum_{k=1}^{N} \theta_k = 1$ on $\bar{\Omega}$.

Now Lemma 2.4.1.5 shows that for each k, we have

$$(\theta_k u) \circ \Phi_k \in W_p^2(U_k).$$

Consequently

$$u = \sum_{k=1}^{N} \theta_k u = \sum_{k=1}^{N} [(\theta_k u) \circ \Phi_k \circ \Phi_k^{-1}]^{\sim} \in W_p^2(\Omega).$$

Here the symbol $\sim$ means that the function has been continued by zero in $\Omega \setminus \Phi_k(V_k)$. Lemma 2.4.1.4 is proved. ■

Before proving Lemma 2.4.1.5, let us quote and prove one particular form of the famous mollifiers lemma due to Friedrichs. Here we denote by ρ_m, $m = 1, 2, \ldots$ a sequence of functions belonging to $\mathcal{D}(\mathbb{R}^{n-1})$ such that

$$\rho_m(x') = m^{n-1}\rho(mx')$$

where $\rho \in \mathcal{D}(\mathbb{R}^{n-1})$ is such that $\int_{\mathbb{R}^{n-1}} \rho(x')\, dx' = 1$. Therefore, the convolution by ρ_m is an approximation of the identity operator when $m \to +\infty$.

Lemma 2.4.1.6 *Let a be a uniformly Lipschitz function on $\mathbb{R}_+^n$; then there exists a constant C such that*

$$\|a(\rho_m * D_i v) - \rho_m * (aD_i v)\|_{0,p,\mathbb{R}_+^n} \leq C\,\|v\|_{0,p,\mathbb{R}_+^n} \tag{2,4,1,8}$$

for all m and $1 \leq i \leq n-1$.

Proof Explicitly we have for $v \in \mathcal{D}(\mathbb{R}_+^n)$:

$$\begin{aligned}(a\rho_m * D_i v - \rho_m * aD_i v)(x) &= \int_{\mathbb{R}^{n-1}} \rho_m(x'-y')[a(x', x_n) - a(y', x_n)](D_i v)(y', x_n)\, dy' \\ &= \int_{\mathbb{R}^{n-1}} D_i\rho_m(x'-y')[a(x', x_n) - a(y', x_n)]v(y', x_n)\, dy' \\ &\quad + \int_{\mathbb{R}^{n-1}} \rho_m(x'-y')D_i a(y', x_n)v(y', x_n)\, dy'.\end{aligned}$$

Consequently we have the following estimate where K denotes the

Lipschitz constant of a:

$$|(a\rho_m * D_i v - \rho_m * aD_i v)(x)|$$
$$\leq K\int_{\mathbb{R}^{n-1}} \{|\rho_m(x'-y')| + |x'-y'|\,|D_i\rho_m(x'-y')|\}\,|v(y',x_n)|\,dy'.$$

Applying Young's inequality, we obtain

$$\|a\rho_m * D_i v - \rho_m * (aD_i v)\|_{0,p,\mathbb{R}^n_+}$$
$$\leq K\,\|v\|_{0,p,\mathbb{R}^n_+}\int_{\mathbb{R}^{n-1}} [|\rho_m(x')| + |x'|\,|D_i\rho_m(x')|]\,dx'$$
$$= K\,\|v\|_{0,p,\mathbb{R}^n_+}\int_{\mathbb{R}^{n-1}} [|\rho(x')| + |x'|\,|D_i\rho(x')|]\,dx'.$$

This is exactly (2,4,1,8) when $v\in\mathcal{D}(\mathbb{R}^n_+)$. The general case follows by density. ■

Proof of Lemma 2.4.1.5 The main idea is to apply inequality (2,3,3,1) to a sequence of smooth functions u_2^m, $m = 1, 2, \ldots$ which approximates u_2.

It is convenient to extend u_2 in $\tilde{u}_2$ defined, as usual by

$$\tilde{u}_2 = \begin{cases} u_2 & \text{in } U \\ 0 & \text{in } \mathbb{R}^n_+ \setminus U. \end{cases}$$

Since u_2 has compact support in V, it is clear that $\tilde{u}_2 \in W^2_r(\mathbb{R}^n_+)$. We extend f_2 and g_2 in a similar fashion. Then $\tilde{f}_2 \in L_p(\mathbb{R}^n_+)$ and $\tilde{g}_2 \in W_p^{2-d-1/p}(\mathbb{R}^{n-1})$. We extend also the functions $a^\#_{k,l}$ and $b^\#_j$ to the whole of $\mathbb{R}^n_+$ in any way that preserves all the properties of $A^\#$ and $B^\#$. Accordingly, we have

$$\begin{cases} A^\#\tilde{u}_2 = \tilde{f}_2 & \text{in } \mathbb{R}^n_+ \\ \gamma_n B^\#\tilde{u}_2 = \tilde{g}_2 & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

We now set

$$u_2^m = \rho_m * \tilde{u}_2.$$

We first show that $u_2^m \in W^2_p(\mathbb{R}^n_+)$. Indeed, we know that $\tilde{u}_2 \in W^1_p(\mathbb{R}^n_+)$ by the Sobolev imbedding. It follows that

$$D_iD_ju_2^m \in L_p(\mathbb{R}^n_+), \qquad i+j\leq 2n-1$$

since the effect of $\rho_m *$ is to smooth up the functions in the directions of x_i, $1\leq i\leq n-1$. It is a little more tricky to show that $D_n^2u_2^m \in L_p(\mathbb{R}^n_+)$. Indeed, we observe that

$$D_n^2\tilde{u}_2 = \frac{1}{a^\#_{n,n}}\Big\{A^\#\tilde{u}_2 - \sum_{i+j\leq 2n-1} D_i(a^\#_{i,j}D_j\tilde{u}_2) - (D_na^\#_{n,n})D_n\tilde{u}_2\Big\}.$$

Consequently, we have

$$D_n^2\tilde{u}_2 \in L_p(\mathbb{R}_+;\ W_p^{-1}(\mathbb{R}^{n-1}))$$

if we agree to consider $\tilde{u}_2$ as a vector-valued function of x_n. Smoothing with $\rho_m *$, we obtain

$$D_n^2 u_2^m \in L_p(\mathbb{R}_+^n).$$

We observe in addition that u_2^m remains bounded in $W_p^1(\mathbb{R}_+^n)$:

$$\|u_2^m\|_{1,p,\mathbb{R}_+^n} \leqslant \|\tilde{u}_2\|_{1,p,\mathbb{R}_+^n}. \tag{2,4,1,9}$$

We now show that $A^{\#}u_2^m = f_2^m$ remains bounded in $L_p(\mathbb{R}_+^n)$. We use Lemma 2.4.1.6 to compare $A^{\#}u_2^m$ with $\rho_m * \tilde{f}_2$. First, since $\tilde{u}_2 \in W_r^2(\mathbb{R}_+^n) \subseteq W_p^1(\mathbb{R}_+^n)$, we know that

$$a_{k,l}^{\#} D_k\rho_m * D_l\tilde{u}_2 - \rho_m * a_{k,l}^{\#} D_k D_l \tilde{u}_2$$

remains bounded in $L_p(\mathbb{R}_+^n)$ for $1 \leqslant k \leqslant n-1$, $1 \leqslant l \leqslant n$. Then we write

$$D_n^2\tilde{u}_2 = \frac{\tilde{f}_2}{a_{n,n}^{\#}} - \sum_{k=1}^{n-1}\sum_{l=1}^{n}\left\{D_k\left(\frac{a_{k,l}^{\#}}{a_{n,n}^{\#}} D_l\tilde{u}_2\right) + a_{k,l}^{\#}\frac{D_k a_{n,n}^{\#}}{a_{n,n}^{\#2}} D_l\tilde{u}_2\right\} - \frac{D_n a_{n,n}^{\#}}{a_{n,n}^{\#}} D_n\tilde{u}_2.$$

This shows that

$$D_n^2\tilde{u}_2 = F - \sum_{k=1}^{n-1} D_k G_k \tag{2,4,1,10}$$

where F and G_k, $1 \leqslant k \leqslant n-1$, belong to $L_p(\mathbb{R}_+^n)$. Thus we have

$$\begin{aligned} &a_{n,n}^{\#}\rho_m * D_n^2\tilde{u}_2 - \rho_m * a_{n,n}^{\#} D_n^2 \tilde{u}_2 \\ &\quad = [a_{n,n}^{\#}\rho_m * F - \rho_m * (a_{n,n}^{\#}F)] - \sum_{k=1}^{n-1}[a_{n,n}^{\#}\rho_m * D_k G_k - \rho_m * a_{n,n}^{\#} D_k G_k] \end{aligned}$$

and this is bounded in $L_p(\mathbb{R}_+^n)$, owing again to Lemma 2.4.1.6. Adding, we obtain the boundedness in $L_p(\mathbb{R}_+^n)$ of

$$\sum_{k,l=1}^{n} a_{k,l}^{\#} D_k D_l u_2^m - \rho_m * \sum_{k,l=1}^{n} a_{k,l}^{\#} D_k D_l \tilde{u}_2.$$

Clearly, it follows that there exists C_1 such that

$$\|A^{\#}u_2^m - \rho_m * \tilde{f}_2\|_{0,p,\mathbb{R}_+^n} \leqslant C_1. \tag{2,4,1,11}$$

Finally we show that $\gamma_n B^{\#}u_2^m = g_2^m$ remains bounded in $W_p^{2-d-1/p}(\mathbb{R}^{n-1})$. In the case where $d=0$, we simply have $\gamma_n B^{\#}u_2^m = \gamma_n u_2^m = \rho_m * \tilde{g}_2$ and the claim is obvious. In the case where $d=1$, we again compare $\gamma_n B^{\#}u_2^m$ with $\rho_m * \tilde{g}_2$. First, since $\tilde{u}_2 \in W_p^1(\mathbb{R}_+^n)$, it is clear that

$$b_j^{\#}\rho_m * D_j\tilde{u}_2 - \rho_m * b_j^{\#} D_j \tilde{u}_2, \qquad 1 \leqslant j \leqslant n$$

remains bounded in $L_p(\mathbb{R}^n_+)$ and that

$$D_k(b_j^{\#}\rho_m * D_j\tilde{u}_2 - \rho_m * b_j^{\#}D_j\tilde{u}_2) = (D_k b_j^{\#})\rho_m * D_j\tilde{u}_2 - \rho_m * (D_k b_j^{\#})D_j\tilde{u}_2 + [b_j^{\#}\rho_m * D_jD_k\tilde{u}_2 - \rho_m * b_j^{\#}D_jD_k\tilde{u}_2]$$

also remains bounded in $L_p(\mathbb{R}^n_+)$ provided $1 \leqslant j \leqslant n-1$, $1 \leqslant k \leqslant n$, because of Lemma 2.4.1.6. Then we write

$$\begin{aligned} D_k(b_n^{\#}\rho_m * D_n\tilde{u}_2 - \rho_m * b_n^{\#}D_n\tilde{u}_2) &= (D_k b_n^{\#})(\rho_m * D_n\tilde{u}_2) - \rho_m * (D_k b_n^{\#})D_n\tilde{u}_2 \\ &\quad + [b_n^{\#}\rho_m * D_n^2\tilde{u}_2 - \rho_m * b_n^{\#}D_n^2\tilde{u}_2] \\ &= (D_k b_n^{\#})(\rho_m * D_n\tilde{u}_2) - \rho_m * (D_k b_n^{\#})D_n\tilde{u}_2 + b_n^{\#}\rho_m * F - \rho_m * b_n^{\#}F \\ &\quad - \sum_{k=1}^{n-1} [b_n^{\#}\rho_m * D_kG_k - \rho_m * b_n^{\#}D_kG_k] \end{aligned}$$

owing to (2,4,1,10). Using again Lemma 2.4.1.6 we show that there exists C_2 such that

$$\|B^{\#}u_2^m - \rho_m * B^{\#}\tilde{u}_2\|_{1,p,\mathbb{R}^n_+} \leqslant C_2.$$

Taking the traces, we deduce that

$$\|\gamma_n B^{\#}u_2^m - \rho_m * \tilde{g}_2\|_{1-1/p,p,\mathbb{R}^{n-1}} \leqslant C_3. \tag{2,4,1,12}$$

The conclusion of the proof is now straightforward. The functions u_2^m have their support in a fixed compact set. This allows us to use inequality (2,3,3,1) (see Remark 2.3.3.5). Accordingly there exists C_4 such that

$$\|u_2^m\|_{2,p,\mathbb{R}^n_+} \leqslant C_4\{\|A^{\#}u_2^m\|_{0,p,\mathbb{R}^n_+} + \|\gamma_n B^{\#}u_2^m\|_{2-d-1/p,p,\mathbb{R}^{n-1}} + \|u_2^m\|_{1,p,\mathbb{R}^n_+}\}.$$

The estimates (2,4,1,9), (2,4,1,11) and (2,4,1,12) imply then that u_2^m, $m = 1, 2, \ldots$ is a bounded sequence in $W_p^2(\mathbb{R}^n_+)$. On the other hand, we have

$$u_2^m \to \tilde{u}_2, \qquad m \to +\infty$$

in $W_p^1(\mathbb{R}^n_+)$. This implies that $\tilde{u}_2 \in W_p^2(\mathbb{R}^n_+)$. The proof of Lemma 2.4.1.5 is now complete. ■

2.4.2 Applications of the Fredholm theory and the maximum principle

So far, we have dealt with operators A and B respectively fulfilling the assumptions (b) and (c) introduced in Section 2.1. We are now able to widen our class of operators by adding lower-order terms. Thus we now assume that

$$Au = \sum_{i,j=1}^{n} D_i(a_{i,j}D_ju) + \sum_{i=1}^{n} a_iD_iu + a_0u$$

where $a_{i,j} = a_{j,i} \in C^{0,1}(\bar{\Omega})$ fulfil (2,1,1) again and where $a_i \in L^\infty(\Omega)$, $0 \leq i \leq n$. In addition B is either the identity operator ($d = 0$) or

$$Bu = \sum_{i=1}^{n} b_iD_iu + b_0u$$

where $b_i \in C^{0,1}(\bar{\Omega})$, $0 \leq i \leq n$ and $b_\nu = \sum_{i=1}^{n} b_i\nu^i$ does not vanish on Γ ($d = 1$). It will be convenient to assume that $b_\nu < 0$ on Γ (by possibly changing B to $-B$).

Adding lower-order terms to A and B means adding a compact operator to $T_{p,\lambda}$. Indeed, it follows from Theorem 1.4.3.2 that

$$u \mapsto \left\{ \sum_{i=1}^{n} a_iD_iu + a_0u - \lambda u;\ \gamma(b_0u) \right\}$$

is a compact mapping from $W_p^2(\Omega)$ into $L_p(\Omega) \times W_p^{1-1/p}(\Gamma)$. Adding this to $T_{p,\lambda}$, which is an isomorphism for λ large, implies the following lemma:

Lemma 2.4.2.1 *The mapping*

$$\mathsf{T}_p : u \mapsto \{Au, \gamma Bu\}$$

is a Fredholm operator of index zero from $W_p^2(\Omega)$ *into* $L_p(\Omega) \times W_p^{2-d-1/p}(\Gamma)$.

In other words, this means that the operator under consideration has a finite dimensional kernel and a range of finite codimension. In addition, the codimension μ of its range is equal to the dimension of its kernel (see, for instance, Theorem 5.26, §5, Chapter IV in Kato (1966)).

The problem of showing that the mapping T_p is actually an isomorphism is now reduced to showing that $\mu = 0$, i.e. that T_p is one to one. This is a much simpler question since we have some strong smoothness results for functions in the kernel of T_p.

Lemma 2.4.2.2 *Let* $u \in W_p^2(\Omega)$ *be a solution of*

$$\begin{cases} Au = 0 & \text{in } \Omega \\ \gamma Bu = 0 & \text{on } \Gamma; \end{cases} \tag{2,4,2,1}$$

then $u \in \bigcap_{1<q<\infty} W_q^2(\Omega) \subseteq C^1(\bar{\Omega})$.

By the way, this shows that $\ker \mathsf{T}_p$ does not depend on p.

Proof of Lemma 2.4.2.2 The differentiability of u up to the boundary follows from Lemma 2.4.1.4. This lemma does not apply directly to A and B since we have weakened our assumptions on these operators. However, it applies to A_0 and B_0 defined as follows:

$$A_0u = \sum_{i,j=1}^{n} D_i(a_{ij}D_ju)$$

and B_0u is either u $(d=0)$ or

$$B_0u = \sum_{i=1}^{n} b_iD_iu.$$

Indeed it follows from (2,4,2,1) that

$$D_iu \in W_p^1(\Omega)$$

and that $\gamma u = 0$ when $d=0$ and $\gamma B_0u \in W_p^{2-1/p}(\Gamma)$ when $d=1$. In all cases Sobolev's imbedding theorem implies that

$$\begin{cases} A_0u \in L_q(\Omega) \\ \gamma B_0u \in W_q^{2-d-1/q}(\Gamma) \end{cases}$$

where $q \leq pn/(n-p)$ for $p<n$ and $q<\infty$ for $p \geq n$. Lemma 2.4.1.4 shows that $u \in W_q^2(\Omega)$.

Iterating the previous procedure eventually shows that

$$u \in \bigcap_{1<q<\infty} W_q^2(\Omega).$$

We conclude by using again Sobolev's imbedding theorem which implies that $u \in C^1(\bar{\Omega})$. ■

Of course, the result of Lemma 2.4.2.2 is an invitation to use the maximum principle for showing uniqueness. The proof of uniqueness would be quite simple if we also knew that $u \in C^2(\Omega)$. However, this may not be true under our assumptions on the coefficients a_i, $1 \leq i \leq n$. Thus we shall make use of the generalized form of the maximum principle for weak solutions, due to Stampacchia (1965). This author considers general weak solutions in $H^1(\Omega)$. Here we shall take advantage of the smoothness proved in Lemma 2.4.2.2 to give a simpler proof.

Theorem 2.4.2.3 *Let $u \in \bigcap_{1<p<\infty} W_p^2(\Omega)$ be a solution of $Au=0$ in Ω. Assume that either $a_i=0$, $1 \leq i \leq n$ and $a_0 \geq 0$ or $a_0 \geq \beta > 0$, then*

$$\max_{x\in\bar{\Omega}} u(x) \leq \max\left(0, \max_{x\in\Gamma} u(x)\right). \tag{2,4,2,2}$$

The proof of this result will follow after some preliminaries. We set $k = \max(0, \max_{x\in\Gamma} u(x))$ and

$$u_k(x) = \max(u-k; 0) \tag{2,4,2,3}$$

Lemma 2.4.2.4 *u_k belongs to $\mathring{W}^1_p(\Omega)$ for all p.*

Proof We can redefine u_k as being $\varphi_k \circ u$, where

$$\varphi_k(t) = \max(t-k; 0)$$

This is a uniformly Lipschitz function with Lipschitz constant equal to one. Since $u \in C^1(\bar{\Omega})$, it follows that u_k is also a Lipschitz function and furthermore, applying a theorem of Rademacher (1919), we see that

$$D_i u_k = (\varphi'_k \circ u) D_i u$$

almost everywhere. It follows that $|D_i u_k| \leq |D_i u|$ and consequently $u_k \in W^1_p(\Omega)$ for all p. In addition, we have shown that

$$D_i u_k = \begin{cases} 0 \text{ a.e. in } \{x \mid u(x) \leq k\} \\ D_i u \text{ a.e. in } \{x \mid u(x) > k\}. \end{cases} \tag{2.4.2.4}$$

Finally, to show that $\gamma u_k = 0$, we approximate φ_k by means of a sequence of functions $\varphi_{k,m}$, $m = 1, 2, \ldots$ such that

(a) $\varphi_{k,m} \in C^1(\mathbb{R})$
(b) $\varphi_{k,m}$ is uniformly Lipschitz continuous, with Lipschitz constant equal to one
(c) $0 \leq \varphi_{k,m} \leq \varphi_k$
(d) $\varphi_{k,m} \to \varphi_k$ uniformly when $m \to +\infty$.

Then we approximate u_k by $\varphi_{k,m} \circ u$. It is obvious that $\varphi_{k,m} \circ u \in C^1(\bar{\Omega})$ and that $\varphi_{k,m} \circ u$ vanishes on Γ. Then we show that there exists an increasing sequence m_j, $j = 1, 2, \ldots$ such that $\varphi_{k,m_j} \circ u \to u_k$ in $W^1_p(\Omega)$. This implies that $\gamma u_k = \lim_{j\to\infty} \gamma(\varphi_{k,m_j} \circ u) = 0$, i.e., $u_k \in \mathring{W}^1_p(\Omega)$.

Actually we have

$$\varphi_{k,m} \circ u(x) \to u_k(x)$$

for all x and in addition

$$|(\varphi_{k,m} \circ u)(x)| \leq |u(x)|$$
$$|(D_i \varphi_{k,m} \circ u)(x)| \leq |D_i u(x)|, \qquad 1 \leq i \leq n$$

for all x. Applying Lebesgue's dominated convergence theorem we find an increasing sequence m_j and functions v_j, $0 \leq j \leq n$ in $L_p(\Omega)$ such that

$$\varphi_{k,m} \circ u \to v_0$$
$$D_i \varphi_{k,m} \circ u \to v_i, \qquad 1 \leq i \leq n$$

in $L_p(\Omega)$. Applying also Lebesgue's subsequence theorem we can achieve the choice of the sequence m_j in order that

$$\varphi_{k,m} \circ u \to v_0$$

almost everywhere. We conclude by observing that $v_0 = u_k$ almost everywhere on the one hand and that $v_i = D_i v_0$, $1 \leq i \leq n$ in the sense of distributions on the other hand. Thus $u_k = v_0 \in W_p^1(\Omega)$ and $\varphi_{k,m} \circ u \to u_k$ in $W_p^1(\Omega)$. This completes the proof of Lemma 2.4.2.4. ■

Now, as in Section 2.3.1, we shall consider the corresponding function u_k^* through the duality mapping from $L_p(\Omega)$ into $L_q(\Omega)$, i.e.,

$$u_k^*(x) = |u_k(x)|^{p-2}\, u_k(x). \tag{2,4,2,5}$$

Since we shall only use large values of p, we can view u_k^* as $\psi_p \circ u_k$, where

$$\psi_p(t) = |t|^{p-2}\, t \tag{2,4,2,6}$$

is a continuously differentiable function. Since function u_k is uniformly Lipschitz continuous, we can again apply Rademacher's theorem to differentiate u_k^*. This leads to the following identity

$$D_i u_k^*(x) = (p-1)\, |u_k(x)|^{p-2}\, D_i u_k(x) \tag{2,4,2,7}$$

almost everywhere, $1 \leq i \leq n$. This together with identity (2,4,2,4), implies the following:

$$D_i u_k^*(x) = \begin{cases} 0 & \text{a.e. in } \{x \mid u(x) \leq k\} \\ (p-1)\, |u_k(x)|^{p-2}\, D_i u(x) & \text{in } \{x \mid u(x) > k\} \end{cases} \tag{2,4,2,8}$$

Proof of Theorem 2.4.2.3 We start from the identity

$$\int_\Omega A u u_k^* \, dx = 0$$

which is obvious, and then we integrate by parts. Since $u_k \in \mathring{H}^1(\Omega)$, we obtain

$$-\sum_{i,j=1}^{n} \int_\Omega a_{ij} D_j u D_i u_k^* \, dx + \sum_{i=1}^{n} \int_\Omega a_i D_i u u_k^* \, dx + \int_\Omega a_0 u u_k^* \, dx = 0.$$

This is equivalent to the following, where A_k denotes the set $\{x \in \Omega \mid u(x) > k\}$:

$$-\sum_{i,j=1}^{n} (p-1) \int_{A_k} a_{ij}\, |u_k|^{p-2}\, D_j u_k D_i u_k \, dx + \sum_{i=1}^{n} \int_{A_k} a_i D_i u_k\, |u_k|^{p-2}\, u_k \, dx$$
$$+ \int_{A_k} a_0 (u_k + k)\, |u_k|^{p-2}\, u_k \, dx = 0.$$

We now denote by M an upper bound for $|a_i|$, $1 \leq i \leq k$ in $\bar{\Omega}$. It follows from (2,1,1) that we have:

$$\alpha(p-1)\int_{A_k} |u_k|^{p-2} |\nabla u_k|^2 \, dx - Mn \int_{A_k} |\nabla u_k| \, |u_k|^{p-1} \, dx + \int_{A_k} a_0 \, |u_k|^p \, dx$$
$$\leq -k \int_{A_k} a_0 \, |u_k|^{p-2} \, u_k \, dx \leq 0.$$

Then using Cauchy–Schwarz inequality, we get the following inequality for all $\varepsilon > 0$

$$\left[\alpha(p-1) - \frac{Mn}{2\varepsilon}\right] \int_{A_k} |u_k|^{p-2} |\nabla u_k|^2 \, dx + \left[\beta - \frac{Mn}{2}\varepsilon\right] \int_{A_k} |u_k|^p \, dx \leq 0, \tag{2,4,2,9}$$

where $\beta \leq a_0(x)$ a.e. We finally chose ε small enough so that $\beta - (Mn/2)\varepsilon \geq 0$. This is possible under the assumptions of Theorem 2.4.2.3, which mean that either $\beta > 0$ or $M = 0$ if $\beta = 0$. Once we have chosen ε we can find p large enough such that $\alpha(p-1) > Mn/2\varepsilon$. From (2,4,2,9) we conclude that

$$|u_k|^{p-2} \, |\nabla u_k|^2 = 0$$

a.e. in A_k. Equivalently, we have

$$(u-k) \, \nabla u = 0$$

everywhere in A_k (since $u \in C^1(\bar{\Omega})$). In other words

$$\nabla (u-k)^2 = 0$$

in A_k and consequently $u - k$ is constant in A_k. On the other hand, we have $u = k$ on the boundary of A_k; thus $u = k$ everywhere in A_k. This means that $u \leq k$. ■

It is now easy to deduce several uniqueness theorems corresponding to various kinds of boundary conditions, from Theorem 2.4.2.3. First let us consider the Dirichlet boundary condition.

Theorem 2.4.2.5 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a $C^{1,1}$ boundary. Let $a_{i,j}$ be uniformly Lipschitz functions and a_i be bounded measurable functions such that $a_{j,i} = a_{i,j}$, $1 \leq i, j \leq n$ and that there exists $\alpha > 0$ with*

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \leq -\alpha \, |\xi|^2$$

for all $\xi \in \mathbb{R}^n$ and for almost every $x \in \bar{\Omega}$. Assume in addition, that either

$a_i = 0$, $1 \leq i \leq n$ and $a_0 \geq 0$ a.e. or $a_0 \geq \beta > 0$ a.e. Then for every $f \in L_p(\Omega)$ and every $g \in W_p^{2-1/p}(\Gamma)$, there exists a unique $u \in W_p^2(\Omega)$ solution of

$$\begin{cases} \sum_{i,j=1}^{n} D_i(a_{i,j}D_ju) + \sum_{i=1}^{n} a_iD_iu + a_0u = f & \text{in } \Omega \\ \gamma u = g & \text{on } \Gamma. \end{cases}$$

Proof According to Lemma 2.4.2.1, we just have to prove that T_p is one to one. Thus let $u \in \ker T_p$. From Lemma 2.4.2.2, we know that actually $u \in \ker T_p$ for all p. Then applying Theorem 2.4.2.3, we have

$$\max_{x \in \bar{\Omega}} u(x) = 0$$

since $\max_{x \in \Gamma} u(x) = 0$. The same holds for $-u$, so that $u = 0$. ∎

Let us now consider the so-called third boundary value problem and more generally an operator B of order $d = 1$, with a nonzero coefficient b_0.

Theorem 2.4.2.6 *Let Ω be a bounded open subset of $\mathbb{R}^n$, with a $C^{1,1}$ boundary. Let $a_{i,j}$ and b_i be uniformly Lipschitz functions and let a_i be bounded measurable functions in $\bar{\Omega}$. Assume that $a_{i,j} = a_{j,i}$, $1 \leq i, j \leq n$ and that there exists $\alpha > 0$ with*

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \leq -\alpha\ |\xi|^2$$

for all $\xi \in \mathbb{R}^n$ and almost every $x \in \bar{\Omega}$. Assume in addition, that either $a_i = 0$, $1 \leq i \leq n$ and $a_0 \geq 0$ a.e. or $a_0 \geq \beta > 0$ a.e. in $\bar{\Omega}$. Assume finally that

$$b_0 b_\nu = b_0 \sum_{j=1}^{n} b_j \nu^j > 0$$

on Γ. Then for every $f \in L_p(\Omega)$ and every $g \in W_p^{1-1/p}(\Gamma)$, there exists a unique $u \in W_p^2(\Omega)$ which is a solution of

$$\begin{cases} \sum_{i,j=1}^{n} D_i(a_{i,j}D_ju) + \sum_{i=1}^{n} a_iD_iu + a_0u = f & \text{in } \Omega \\ \gamma\left(\sum_{j=1}^{n} b_jD_ju + b_0u\right) = g & \text{on } \Gamma. \end{cases}$$

Proof Again, owing to Lemma 2.4.2.1, we just have to prove that T_p is one to one. Thus let $u \in \operatorname{Ker} T_p$. We know from Lemma 2.4.2.2 that $u \in \bigcap_{1<p<\infty} W_p^2(\Omega)$. This allows us to apply Theorem 2.4.2.3. We want to prove that

$$\max_{x\in\bar{\Omega}} u(x)\leqslant 0. \tag{2,4,2,10}$$

Assume the contrary; then necessarily, the maximum of u is attained on the boundary Γ. Since the first derivatives of u are continuous up to the boundary, the boundary condition is fulfilled in the classical sense. In other words, we have

$$\sum_{j=1}^{n} b_j(x)D_ju(x)+b_0(x)u(x)=0,$$

for all $x\in\Gamma$.

At the particular point x_0 where u reaches its maximum, the tangential derivatives of u vanish and the derivative of u in the direction of $\boldsymbol{\nu}$ is nonnegative. We can rewrite the boundary condition at x_0 as follows:

$$b_\nu(x_0)\frac{\partial u}{\partial \nu}(x_0)+b_0(x_0)u(x_0)=0.$$

This is contradictory since we assumed that $u(x_0)>0$ and that $b_\nu(x_0)$ and $b_0(x_0)$ are both nonzero numbers and have the same sign. This shows that (2,4,2,10) holds. The same holds for $-u$, so that $u=0$. ■

In the next statement we shall allow b_0 to vanish so as to be able to consider a Neumann boundary condition for instance. As a counterpart, we have to assume that $a_0\geqslant\beta>0$ a.e.

Theorem 2.4.2.7 *Let Ω be a bounded open subset of $\mathbb{R}^n$, with a $C^{1,1}$ boundary. Let $a_{i,j}$ and b_i be uniformly Lipschitz functions in $\bar{\Omega}$ and let a_i be bounded measurable functions in $\bar{\Omega}$. Assume that $a_{i,j}=a_{j,i}$, $1\leqslant i,j\leqslant n$ and that there exists $\alpha>0$ with*

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j\leqslant-\alpha\,|\xi|^2$$

for all $\xi\in\mathbb{R}^n$ and almost every $x\in\bar{\Omega}$. Assume in addition that $a_0\geqslant\beta>0$ a.e. in $\bar{\Omega}$ and that

$$b_0b_\nu=b_0\sum_{j=1}^{n} b_j\nu^j\geqslant 0,\qquad b_\nu\neq 0$$

on Γ. Then for every $f\in L_p(\Omega)$ and every $g\in W_p^{1-1/p}(\Gamma)$, there exists a unique $u\in W_p^2(\Omega)$, which is a solution of

$$\begin{cases}\displaystyle\sum_{i,j=1}^{n} D_i(a_{i,j}D_ju)+\sum_{i=1}^{n} a_iD_iu+a_0u=f & \text{in } \Omega\\ \displaystyle\gamma\Big(\sum_{j=1}^{n} b_jD_ju+b_0u\Big)=g & \text{on } \Gamma.\end{cases} \tag{2,4,2,11}$$

Proof We introduce a function $\rho \in C^{1,1}(\bar{\Omega})$ such that $\rho > 0$ in Ω, $\rho = 0$ on Γ and $\partial\rho/\partial\nu < 0$ on Γ.† Then we define v by setting

$$u = \exp(-\varepsilon\rho)v$$

and we show that u is a solution of problem (2,4,2,11) if and only if v is a solution of a problem which fulfils the assumptions of Theorem 2.4.2.6, at least for $\varepsilon > 0$ small enough. The result will follow by applying Theorem 2.4.2.6 to v.

Indeed, we have

$$D_j u = \exp(-\varepsilon\rho)(D_j v - \varepsilon[D_j\rho]v), \qquad 1 \leq j \leq n$$

and consequently

$$\begin{aligned} Au = &\sum_{i,j=1}^{n} D_i\{a_{i,j} \exp(-\varepsilon\rho)(D_j v - \varepsilon[D_j\rho]v)\} \\ &+ \sum_{j=1}^{n} \exp(-\varepsilon\rho)a_j(D_j v - \varepsilon[D_j\rho]v) + \exp(-\varepsilon\rho)a_0 v. \end{aligned}$$

It follows that

$$\begin{aligned} \exp(\varepsilon\rho)Au = &\sum_{i,j=1}^{n} \{D_i a_{i,j} D_j v - \varepsilon D_i(a_{i,j} D_j\rho)v - \varepsilon a_{i,j} D_j\rho D_i v \\ &- \varepsilon a_{i,j} D_i\rho(D_j v - \varepsilon[D_j\rho]v)\} + \sum_{j=1}^{n} a_j(D_j v - \varepsilon[D_j\rho]v) + a_0 v \end{aligned}$$

and finally that

$$\begin{aligned} &\sum_{i,j=1}^{n} D_i a_{i,j} D_j v + \sum_{j=1}^{n} \left[a_j - 2\varepsilon \sum_{i=1}^{n} a_{i,j} D_i\rho\right] D_j v \\ &+ \left[a_0 - \varepsilon \sum_{j=1}^{n} a_j D_j\rho - \varepsilon \sum_{i,j=1}^{n} D_i(a_{i,j} D_j\rho)\right. \\ &\left. + \varepsilon^2 \sum_{i,j=1}^{n} a_{i,j} D_i\rho D_j\rho\right] v = \exp(\varepsilon\rho)f. \end{aligned}$$

On the other hand, we have

$$Bu = \sum_{j=1}^{n} b_j \exp(-\varepsilon\rho)\,(D_j v - \varepsilon[D_j\rho]v) + b_0 \exp(-\varepsilon\rho)v$$

† It is easy to define ρ locally near the boundary. In the notations of Definition 1.2.1.1, we can define ρ_V as being $(y', y_n) \mapsto \varphi(y') - y_n$. Then covering Γ by a finite number of hypercubes such as V, we build up a function ρ from the $\rho_V \cdot$ s with the help of a partition of unity.

and consequently

$$\gamma\left(\sum_{j=1}^{n} b_j D_j v + \left[b_0 - \varepsilon \sum_{j=1}^{n} b_j D_j \rho\right] v\right) = g$$

on Γ.

We now check that the problem of which v is a solution fulfils the assumptions of Theorem 2.4.2.6. Indeed we have

$$a_0 - \varepsilon \sum_{j=1}^{n} a_j D_j \rho - \varepsilon \sum_{i,j=1}^{n} D_i(a_{i,j} D_j \rho) + \varepsilon^2 \sum_{i,j=1}^{n} a_{ij} D_i \rho D_j \rho \geq \beta/2 > 0$$

a.e. in $\bar{\Omega}$ for ε small enough, if $a_0 \geq \beta > 0$ a.e. in $\bar{\Omega}$. Then we have

$$\left[b_0 - \varepsilon \sum_{j=1}^{n} b_j D_j \rho\right] \sum_{j=1}^{n} b_j \nu^j = \left[b_0 - \varepsilon b_\nu \frac{\partial \rho}{\partial \nu}\right] b_\nu \geq -\varepsilon b_\nu^2 \frac{\partial \rho}{\partial \nu} > 0$$

since b_ν does not vanish and $\partial\rho/\partial\nu < 0$ everywhere on Γ. ■

2.5 Other kinds of solutions

2.5.1 More on smoothness

If we add the same amount of smoothness—so to speak—to the boundary of Ω, to the coefficients of the involved operators and to the data of our problem, we obtain eventually the same amount of extra smoothness for the solution. More precisely, let k be any positive integer and consider the operators A and B introduced in Section 2.1. Assume that

(d) *the boundary Γ of Ω is of class $C^{k+1,1}$*
(e) $a_{i,j} = a_{j,i} \in C^{k,1}(\bar{\Omega})$, $b_i \in C^{k,1}(\bar{\Omega})$.

Then we have the following smoothness result.

Theorem 2.5.1.1 *Let $u \in W_p^2(\Omega)$ be such that*

$$\begin{cases} Au = f \in W_p^k(\Omega) \\ \gamma Bu = g \in W_p^{2+k-d-1/p}(\Gamma) \end{cases}$$

then $u \in W_p^{k+2}(\Omega)$.

Proof This follows very closely the proof of Lemma 2.4.1.4 to begin with. That is why we use the same notation. In addition, it is clear that we can prove by induction on m that $u \in W_p^{2+m}(\Omega)$ implies that $u \in W_p^{2+m+1}(\Omega)$ provided $m \leq k-1$. Thus we have to prove that $u_2 \in W_p^{2+m+1}(U)$, knowing that $u_2 \in W_p^{2+m}(U)$.

Our additional smoothness hypotheses, imply that $a_{k,l}^{\#} \in C^{k,1}(\bar{U})$, $b_l^{\#} \in$

$C^{k,1}(\bar{U})$, $f_2 \in W_p^{m+1}(U)$ and $g_2 \in W_p^{m+1-d-1/p}(U)$. At this step, instead of using Friedrichs' mollifier technique, it is possible to go back to Nirenberg's tangential differential quotients as in the proof of Lemma 2.2.2.1. However, for the sake of using the notation of Lemma 2.4.1.4, we proceed with Friedrichs' technique.

With the help of Lemma 2.4.1.6, it is easy to check that

$$u_2^m = \rho_m * \tilde{u}_2 \in W_p^{2+m+1}(\mathbb{R}_+^n),$$

that u_2^m remains bounded in $W_p^{2+m}(\mathbb{R}_+^n)$ and that $A^{\#}u_2^m$ is bounded in $W_p^{m+1}(\mathbb{R}_+^n)$, while $\gamma B^{\#}u_2^m$ is bounded in $W_p^{2+m+1-d-1/p}(\mathbb{R}^{n-1})$. Then the inequality (2,3,3,1) shows that u_2^m is actually bounded in $W_p^{2+m+1}(\mathbb{R}_+^n)$. Letting $m \to \infty$, this shows that $\tilde{u}_2 \in W_p^{2+m+1}(\mathbb{R}_+^n)$ and consequently $u_2 \in W_p^{2+m}(U)$.

The remaining steps of the proof are exactly similar to the corresponding steps in the proof of Lemma 2.4.1.4. ■

Remark 2.5.1.2 To each of the existence and uniqueness results of Section 2.4.2, corresponds a result with additional smoothness, proved with the aid of Theorem 2.5.1.1. Briefly, those results are the following.

Under the assumptions of Theorem 2.4.2.5, plus the hypotheses (d) and (e) above and if $a_i \in C^{k-1,1}(\bar{\Omega})$, $0 \leqslant i \leqslant n$, the mapping (notation as in Section 2.4.2)

$$u \mapsto \{Au; \gamma u\}$$

is an isomorphism from $W_p^{k+2}(\Omega)$ onto $W_p^k(\Omega) \times W_p^{k+2-1/p}(\Gamma)$.

Under the assumptions of either Theorem 2.4.2.6 or Theorem 2.4.2.7, plus the hypotheses (d) and (e) above, and if $a_i \in C^{k-1,1}(\bar{\Omega})$, $b_i \in C^{k,1}(\bar{\Omega})$, the mapping (notations of Section 2.4.2)

$$u \mapsto \{Au; \gamma Bu\}$$

is an isomorphism from $W_p^{k+2}(\Omega)$ onto $W_p^k(\Omega) \times W_p^{k+1-1/p}(\Gamma)$.

Remark 2.5.1.3 Under the hypotheses of this section we have

$$\ker T_p \subseteq C^2(\bar{\Omega}).$$

This makes the proof of Theorem 2.4.2.3 much simpler (see Hopf (1927) for instance).

2.5.2 Very weak solution

Here, for the sake of later reference, we prove a simple basic result which is obtained from the results in Section 2.4.2, by applying the transposition procedure of Lions and Magenes (1960–63).

Let us begin with Dirichlet's problem.

Theorem 2.5.2.1 *Let the assumptions of Theorem 2.4.2.5 be fulfilled. Assume in addition that $a_i = 0$, $1 \leqslant i \leqslant n$. Then the mapping*

$$u \mapsto \{Au, \gamma u\}$$

is an isomorphism from $D(A; L_p(\Omega))$ onto $L_p(\Omega) \times W_p^{-1/p}(\Gamma)$.

Proof Let us consider the mapping

$$v \mapsto \{Av, \gamma v\}$$

which is an isomorphism from $W_q^2(\Omega)$ onto $L_q(\Omega) \times W_q^{2-1/q}(\Gamma)$. The transposed operator T_q^* is also an isomorphism. Assume that $p^{-1} + q^{-1} = 1$ and consider $f \in L_p(\Omega)$ and $g \in W_p^{-1/p}(\Gamma)$. Define a continuous linear form on $W_q^2(\Omega)$ by

$$v \mapsto l(v) = \int_\Omega fv \,dx + \left\langle g; \gamma \frac{\partial v}{\partial \nu_A} \right\rangle.$$

Here the brackets denote the duality pairing between $W_q^{1-1/q}(\Gamma)$ and $W_p^{-1/p}(\Gamma)$.

Since T_q^* is an isomorphism, there exists a unique $u \in L_p(\Omega)$ and a unique $\varphi \in W_p^{-1-1/p}(\Gamma)$ such that

$$l(v) = \int_\Omega uAv \,dx + \langle \varphi; \gamma v \rangle$$

for all $v \in W_q^2(\Omega)$. In other words, we have

$$\int_\Omega fv \,dx - \int_\Omega uAv \,dx = \langle \varphi; \gamma v \rangle - \left\langle g; \gamma \frac{\partial v}{\partial \nu_A} \right\rangle \tag{2,5,2,1}$$

for all $v \in W_q^2(\Omega)$.

If we use this identity with $v \in \mathcal{D}(\Omega)$ only, we check that $Au = f$. Consequently u belongs to $D(A, L_p(\Omega))$. This allows us to use Green's formula (see identity (1,5,3,5)): we have

$$\int_\Omega fv \,dx - \int_\Omega uAv \,dx = \left\langle \gamma \frac{\partial u}{\partial \nu_A}; \gamma v \right\rangle - \left\langle \gamma u; \gamma \frac{\partial v}{\partial \nu_A} \right\rangle \tag{2,5,2,2}$$

for all $v \in W_q^2(\Omega)$. It follows from (2,5,2,1) and (2,5,2,2) that

$$\left\langle \gamma \frac{\partial u}{\partial \nu_A} - \varphi; \gamma v \right\rangle = \left\langle \gamma u - g; \gamma \frac{\partial v}{\partial \nu_A} \right\rangle$$

for all $v \in W_q^2(\Omega)$. By the trace theorem 1.5.1.2 this implies that

$$\left\langle \gamma \frac{\partial u}{\partial \nu_A} - \varphi; \psi_0 \right\rangle = \langle \gamma u - g; \psi_1 \rangle$$

for all $\psi_0 \in W_q^{2-1/q}(\Omega)$ and all $\psi_1 \in W_q^{1-1/q}(\Gamma)$. Consequently we have

$$\gamma u = g$$

and

$$\varphi = \gamma \frac{\partial u}{\partial \nu_A}.$$

This proves the desired result. ■

Actually we shall only use this consequence of Theorem 2.5.2.1.

Corollary 2.5.2.2 *Let the assumptions of Theorem 2.4.2.5 be fulfilled. Assume in addition that $a_i = 0$, $1 \leq i \leq n$. Let $u \in D(A, L_p(\Omega))$* be a solution of

$$\begin{cases} Au = f \in L_p(\Omega) \\ \gamma u = g \in W_p^{2-1/p}(\Gamma) \end{cases}$$

then $u \in W_p^2(\Omega)$.

This is a straightforward consequence of Theorems 2.4.2.5 and 2.5.2.1. The corresponding result for Neumann's problem is this

Proposition 2.5.2.3 *Let the assumptions of Theorem 2.4.2.5 be fulfilled. Assume in addition that $a_i = 0$, $1 \leq i \leq n$ and that $a_0 \geq \beta > 0$ a.e. in $\bar{\Omega}$. Let $u \in D(A; L_p(\Omega))$ be a solution of*

$$\begin{cases} Au = f \in L_p(\Omega) \\ \gamma \dfrac{\partial u}{\partial \nu_A} = g \in W_p^{1-1/p}(\Gamma); \end{cases}$$

then $u \in W_p^2(\Omega)$.

The proof of Proposition 2.5.2.3 is similar to the proof of Corollary 2.5.2.2. The corresponding statement for an oblique boundary condition requires a little more smoothness on the coefficients.

Proposition 2.5.2.4 *Let the assumptions of Theorem 2.4.2.7 be fulfilled. Assume in addition that $a_i = 0$, $1 \leq i \leq n$ and that $b_j \in C^{1,1}(\bar{\Omega})$, $1 \leq j \leq n$. Let $u \in D(A; L_p(\Omega))$ be a solution of*

$$\begin{cases} Au = f \in L_p(\Omega) \\ \gamma Bu = g \in W_p^{1-1/p}(\Gamma) \end{cases}$$

then $u \in W_p^2(\Omega)$.