

Optimal control of Navier–Stokes equations by Oseen approximation[☆]

Miroslav Pošta^{a,*}, Tomáš Roubíček^{b,c}

^a Department of Mathematical Analysis, Math.-Phys. Faculty, Charles University, Sokolovská 83, CZ-186 75 Praha 8, Czech Republic

^b Mathematical Institute, Charles University, Sokolovská 83, CZ-186 75 Praha 8, Czech Republic

^c Institute of Information Theory and Automation, Academy of Sciences, Pod vodárenskou věží 4, CZ-182 08 Praha 8, Czech Republic

Received 13 September 2005; accepted 27 February 2006

Abstract

A non-standard sequential-quadratic-programming-type iterational process based on Oseen's approximation is proposed and analyzed to solve an optimal control problem for the steady-state Navier–Stokes equations. Further numerical approximation by a finite-element method and sample computational experiments are presented, too.

© 2007 Elsevier Ltd. All rights reserved.

Keywords: Incompressible flow; Steady state; Optimization; Quadratic-programming approximation; Banach contraction principle

1. Introduction

Flow of *incompressible viscous Newtonian fluids* is described by Navier–Stokes system. Optimization of such flow received significant attention both for its industrial applications and for its theoretical and computational difficulties. In this paper, we confine ourselves to steady-state problems. Optimal control problem of this sort was already studied in particular by Bilić [1], Bubák [2], Burkardt and Peterson [3], Casas [4], Desai and Ito [5], Gattas and Bark [6], Gunzburger, Hou and Svobodny [7–9], Hou and Ravindran [10,11], Heinkenschloss [12], Lions [13], Málek in [14], Tröltzsch in [15], Tröltzsch and Wachsmuth [16], and also [17,18], but the relevant literature is, of course, more extensive. Besides, optimization of transient regimes, i.e. governed by the evolution Navier–Stokes system, is more difficult because the uniqueness of the response is still the well-known open problem for 3-dimensional flows in general situations. Anyhow, even this evolution variant has been intensively scrutinized e.g. in [19–26,13,27,28].

As the governing Navier–Stokes equations are nonlinear, the resulting optimization problem is generally nonlinear and efficient numerical strategies are not simple. Often, numerical approaches are based on *sequential-quadratic programming* (SQP). This is an iterative algorithm whose philosophy is to apply the Newton method to the

[☆] Comments of Prof. M. Feistauer to Remark 3.3 and of Prof. M. Hinze to [M. Pošta, Aplikace sekvenciálně-kvadratického programování v optimálním řízení Navier–Stokesových rovnic, MS-Thesis, Math.-Phys. Faculty, Charles Univ., Praha, 2004 (in English)] are very acknowledged.

* Corresponding author. Tel.: +420 221 913 382; fax: +420 222 323 390.

E-mail address: postm9am@plk.mff.cuni.cz (M. Pošta).

1st-order optimality conditions which results in solving of the system of linear equations or equally in a linear-quadratic program. This conventional approach (or its modifications by using the quasi-Newton method) for Navier–Stokes (or similar) equations was scrutinized (often rather in the evolution variant or with state-space constraints) e.g. by Gattas and Bark [6], Heinkenschloss [12], Hintermüller and Hinze [29,30], Hinze [31], Hinze and Kunisch [25], Hou and Ravindran [11], Tröltzsch and Volkwein [32].

Another 2nd-order method used in the context of time-dependent fluid flow consists in replacing the reduced cost functional $\hat{J}(f) := J(u(f), f)$ (for the definition of J, u and f see below) by its second-order Taylor expansion with the derivatives of \hat{J} being expressed via the implicit function theorem, cf. Hinze [24], Hinze and Kunish [25,26]. This method, however, requires evaluating $u = u(f)$, i.e. solving the nonlinear Navier–Stokes equations, at each iteration in contrast to the SQP method which contains only linearized Navier–Stokes equations. This difference is not so significant in the case of the time-dependent problem because the nonlinear equations as well as the linearized equations are solved iteratively.

In this paper, we propose still another linearization strategy based on *Oseen's linearization* of the controlled Navier–Stokes equations. This linearization is known to have advantageous numerical properties as well as allowing for similar a-priori estimates as the original Navier–Stokes equations, and leads already to a linear-quadratic optimization problem provided the cost functional is quadratic but, on the other hand, the convergence is expectedly not of the 2nd-order. The general philosophy behind such strategy is that there is no need to solve the nonlinear Navier–Stokes equations exactly at each iteration of the optimization algorithm because this effort is partly lost in the next iteration, and it suffices to get the desired effect only in a limit. In Section 2, we will scrutinize this strategy on an *optimal control problem* for the *steady-state Navier–Stokes* system:

$$(\mathfrak{P}) \left\{ \begin{array}{ll} \text{Minimize} & J(f, u) := \int_{\Omega} \frac{\alpha}{2} |u - u_d|^2 + \frac{\beta}{2} |\operatorname{rot} u|^2 + \frac{1}{2} |f|^2 \, dx & \text{(cost functional)} \\ \text{subject to} & (u \cdot \nabla)u - \nu \Delta u + \nabla p = f \quad \text{on } \Omega, & \text{(state system)} \\ & \operatorname{div} u = 0 \quad \text{on } \Omega, & \text{(incompressibility)} \\ & u = 0 \quad \text{on } \Gamma, & \text{(boundary conditions)} \\ & f \in \mathcal{F}_{\text{ad}} & \text{(control constraints)} \\ & u \in W^{1,2}(\Omega; \mathbb{R}^n), \quad p \in L_0^2(\Omega), \quad f \in L^2(\Omega; \mathbb{R}^n), \end{array} \right.$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a Lipschitz boundary $\Gamma := \partial\Omega$, $n = 2$ or $n = 3$, $\nu > 0$ is the viscosity, f is a distributed control, and (u, p) a state response, i.e. the velocity and the pressure profiles, respectively, u_d is a given desired velocity profile, and $\alpha, \beta \geq 0$. As usual, $\operatorname{rot} u$ denotes the vorticity, namely the vector function

$$\operatorname{rot} u = \begin{cases} \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) & \text{if } n = 3, \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} & \text{if } n = 2. \end{cases} \quad (1.1)$$

We also define the vector function $\operatorname{rot} \operatorname{rot} u = \left(\frac{\partial \operatorname{rot} u}{\partial x_2}, -\frac{\partial \operatorname{rot} u}{\partial x_1} \right)$ in the case $n = 2$. Moreover, we use the usual notation $L^2(\Omega; \mathbb{R}^n)$ for the Banach space of (classes of) Lebesgue measurable square integrable functions $\Omega \rightarrow \mathbb{R}^n$, while $W^{1,2}(\Omega; \mathbb{R}^n)$ denotes the Sobolev space of functions $u \in L^2(\Omega; \mathbb{R}^n)$ whose distributional gradient ∇u belongs to $L^2(\Omega; \mathbb{R}^{n \times n})$. We utilize $W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n) := \{v \in W^{1,2}(\Omega; \mathbb{R}^n); v|_{\Gamma} = 0, \operatorname{div} v = 0\}$ where $v|_{\Gamma}$ is the trace of v on Γ and $\operatorname{div} v$ is understood in the sense of distributions, and finally we denote $L_0^2(\Omega) := \{p \in L^2(\Omega); \int_{\Omega} p \, dx = 0\}$. We will consider $W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n)$ endowed with the norm $\|u\|_{W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n)} := \|\nabla u\|_{L^2(\Omega; \mathbb{R}^{n \times n})} := (\int_{\Omega} |\nabla u(x)|^2 \, dx)^{1/2}$. Let us note that the distributed control is rather artificial and usually a control through boundary conditions occurs in engineering applications, but nevertheless even a distributed control can be realized through electromagnetic forcing in polarizable fluids, cf. [33]. The quadratic velocity-tracking term (i.e. the “ α -term”) in the cost functional J is a standard option in flow control, see Gunzburger [7] or also, e.g., [1,23,12,29,25,26,16]. The “ β -term” in J is another standard option, see again [7] or [12,25,10], to make the vorticity of the optimal flow small. The last term in J penalizes the control force. All terms are quadratic, which still has reasonable applicability and simultaneously simplifies the analysis considerably. Anyhow, (\mathfrak{P}) is obviously not a linear-quadratic problem due to the bilinear convective term $(u \cdot \nabla)u$ in the state equation.

The philosophy of cumulating the accuracy of solving nonlinear state problems only in the limit can be combined with numerical approximation of the controlled state equations by, e.g., finite-element method (=FEM), which is presented in Section 3. This makes the method ready to be implemented on computers and to perform computational experiments, which are reported in Section 4.

2. The SQP-type conceptual algorithm

Let us first specify the basic assumptions we will need as to the parameters α, β , the desired velocity profile u_d , and the set of admissible controls \mathcal{F}_{ad} . We assume

$$\alpha \geq 0, \quad \beta \geq 0, \quad u_d \in L^2(\Omega; \mathbb{R}^n), \quad (2.1a)$$

$$\mathcal{F}_{\text{ad}} \text{ is closed, convex subset of } L^2(\Omega; \mathbb{R}^n), \quad (2.1b)$$

$$\forall f \in \mathcal{F}_{\text{ad}} : \|f\|_{L^2(\Omega; \mathbb{R}^n)} < \frac{v^2}{N_2 N_4^2}, \quad (2.1c)$$

with $N_p, p < 2n/(n-2)$, denoting the norm of the embedding $W_0^{1,2}(\Omega; \mathbb{R}^n) \subset L^p(\Omega)$. In particular, the condition (2.1c) guarantees by standard arguments (see e.g. [14]) uniqueness of the response u of the Navier–Stokes equations for a given control f and also uniqueness of the corresponding adjoint state w used below.

For convenience, we recall the frequently used notation. In $L^2(\Omega; \mathbb{R}^n)$ we introduce the scalar product $(u, v) := \int_{\Omega} \sum_{i=1}^n u_i v_i dx$ while $(U : V) := \int_{\Omega} \sum_{i,j=1}^n U_{ij} V_{ij} dx$ is the associated one in $L^2(\Omega, \mathbb{R}^{n \times n})$. Further, $(u \cdot \nabla)u$ denotes the vector valued function $\sum_{k=1}^n u_k \frac{\partial}{\partial x_k} u$ and $(\nabla u)^{\top}$ is the matrix having the column vectors $\nabla u_1, \dots, \nabla u_n$. In the context of Navier–Stokes equations, it is common to use the trilinear form $b : W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n)^3 \rightarrow \mathbb{R}$,

$$b(w, u, v) := ((w \cdot \nabla)u, v). \quad (2.2)$$

It is known that $b(w, u, v) = -b(w, v, u)$ if $\text{div } w = 0$ and the normal component of w on Γ vanishes. Here we will always have even $w|_{\Gamma} = 0$. In particular, these assumptions imply $b(w, u, u) = 0$.

We call $u \in W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n)$ a *weak solution* to the no-slip boundary-value problem for the steady-state Navier–Stokes system in (\mathfrak{P}) if the variational equation

$$((u \cdot \nabla)u, v) + \nu(\nabla u : \nabla v) = (f, v) \quad (2.3)$$

is satisfied for all $v \in W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n)$.

Let us remind ourselves of the 1st-order necessary optimality conditions for (\mathfrak{P}) , cf. [15,16] for more details. Considering a locally optimal pair $\{u^*, f^*\}$, they can formally be found by applying the well-known Lagrange principle, where the state-equations (2.3) are eliminated by the *Lagrange function*

$$L(u, f, w) = J(u, f) - (f - (u \cdot \nabla)u, w) + \nu(\nabla u : \nabla w). \quad (2.4)$$

Obviously, for a fixed multiplier $w \in W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n)$, the Lagrange function $L(\cdot, \cdot, w) : W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{R}^n) \rightarrow \mathbb{R}$ is quadratic and continuous, hence it is a C^2 -function. According to the Lagrange principle, $\{u^*, f^*\}$ should satisfy the necessary optimality conditions for minimizers of L with respect to $f \in \mathcal{F}_{\text{ad}}$, i.e. $L'_u(u^*, f^*, w)(u) = 0$ for all $u \in W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n)$ and $L'_f(u^*, f^*, w)(f - f^*) \geq 0$ for all $f \in \mathcal{F}_{\text{ad}}$. The first relation leads to the *adjoint system* to the Navier–Stokes equations linearized at $u = u^*$, i.e.

$$-\nu \Delta w + (\nabla u^*)^{\top} w - (u^* \cdot \nabla)w + \nabla \pi = \alpha(u_d - u^*) + \beta \text{rot rot } u^*, \quad (2.5a)$$

$$\text{div } w = 0, \quad (2.5b)$$

for the so-called *adjoint state* w and the adjoint pressure π , which vanishes in the weak formulation. Under a weak solution to the adjoint system (2.5) we understand any $w \in W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n)$ satisfying the integral identity

$$\begin{aligned} a(u^*; w, v) &:= \nu(\nabla w : \nabla v) - ((u^* \cdot \nabla)w, v) + ((v \cdot \nabla)u^*, w) \\ &= \alpha(u_d - u^*, v) - \beta(\text{rot } u^*, \text{rot } v) \end{aligned} \quad (2.6)$$

for all $v \in W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n)$. The condition (2.1c) provides the estimate (for some $\epsilon > 0$)

$$\begin{aligned} a(u^*; v, v) &\geq v \|\nabla v\|_{L^2(\Omega; \mathbb{R}^n)}^2 + ((v \cdot \nabla)u^*, v) \geq v \|\nabla v\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \|\nabla u^*\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \|v\|_{L^4(\Omega; \mathbb{R}^n)}^2 \\ &\geq \left(v - \frac{N_2}{v} \|f^*\|_{L^2(\Omega; \mathbb{R}^n)} N_4^2 \right) \|\nabla v\|_{L^2(\Omega; \mathbb{R}^{n \times n})}^2 \geq \epsilon \|\nabla v\|_{L^2(\Omega; \mathbb{R}^{n \times n})}^2. \end{aligned} \quad (2.7)$$

Thus, by the Lax–Milgram lemma, the adjoint equation (2.6) has a unique weak solution $w = w(u^*)$ for $\{u, f\}$ in question.

Now we formulate the standard *first-order necessary optimality conditions*. They were proven (mostly for the case without control constraints) in the reference mentioned in Section 1. This proof extends to control constraints by obvious modifications.

Proposition 2.1. *Let (2.1) hold, and let f^* be a locally optimal control for (\mathfrak{P}) with associated state $u^* = u(f^*)$. Then the variational inequality*

$$(f^* - w^*, f - f^*) \geq 0 \quad \forall f \in \mathcal{F}_{\text{ad}} \quad (2.8)$$

is satisfied for $w^ = w(u^*) \in W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n)$ being the unique weak solution to the adjoint Eq. (2.5).*

The point $\{u^*, f^*\}$ satisfying (2.3) with $u := u^*$ and $f := f^*$, (2.6) and (2.8) is called *critical* for (\mathfrak{P}) . The philosophy of our iterative procedure is to find a critical point for (\mathfrak{P}) as a limit of a sequence of solutions of suitable linear-quadratic problems. As already announced, we want to replace the original nonlinear Navier–Stokes equations by linear Oseen equations but then we must augment the cost functional by a suitable correction term, and we will see in the proof of Proposition 2.3 below that this term must be $-(u \cdot \nabla \bar{u}) \cdot \bar{w}$ to obtain the desired result. To be more specific, for (\bar{u}, \bar{w}) denoting the velocity profile and the adjoint state from the former iteration, our auxiliary linear-quadratic problem is:

$$(\mathfrak{P}_{\text{LQ}}) \quad \begin{cases} \text{Minimize} & \int_{\Omega} \frac{\alpha}{2} |u - u_d|^2 + \frac{\beta}{2} |\text{rot } u|^2 + \frac{1}{2} |f|^2 - (u \cdot \nabla \bar{u}) \cdot \bar{w} \, dx \\ \text{subject to} & (\bar{u} \cdot \nabla)u - v \Delta u + \nabla p = f, \quad \text{div } u = 0, \\ & u \in W_0^{1,2}(\Omega; \mathbb{R}^n), \quad p \in L_0^2(\Omega), \quad f \in \mathcal{F}_{\text{ad}}. \end{cases}$$

Obviously, (2.1a) and (2.1b) makes $(\mathfrak{P}_{\text{LQ}})$ a problem with strictly convex functional on a linear manifold, and thus it has a unique solution for \bar{u} and \bar{w} given. The strict convexity of $(\mathfrak{P}_{\text{LQ}})$ also implies that its first-order necessary optimality conditions are also the sufficient ones. Then the corresponding adjoint equation has a form:

$$-v \Delta w - (\bar{u} \cdot \nabla)w + \nabla \pi = \alpha(u_d - u) + \beta \text{rot rot } u - (\nabla \bar{u})^\top \bar{w}, \quad (2.9a)$$

$$\text{div } w = 0. \quad (2.9b)$$

For \bar{u} , \bar{w} , and u given, the proof of uniqueness of the adjoint state $w \in W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n)$ governed by (2.9) is similar to the proof of uniqueness of the adjoint state of the nonlinear problem (\mathfrak{P}) : We assume that (2.1c) and (2.3) hold with $f := \bar{f}$ and $u := \bar{u}$, where \bar{f} is the distributed control from the previous iteration, and we utilize the fact that the Oseen equations provide the same a-priori estimate for $\|\nabla \bar{u}\|_{L^2(\Omega; \mathbb{R}^{n \times n})}$ as the Navier–Stokes equations, cf. (2.7). Thus, for \bar{u} and \bar{w} given, the unique optimal solution $\{u, f\}$ to $(\mathfrak{P}_{\text{LQ}})$ determines uniquely by (2.9) the adjoint state w . Therefore, we can consider the mapping

$$M : (\bar{u}, \bar{w}) \mapsto (u, w).$$

Our next goal is to seek a fixed point of this mapping M by a Banach contraction-principle argument, which gives also an efficient numerical strategy after an additional discretization.

Before this, we will still mention a 2nd-order analysis of the original problem (\mathfrak{P}) , cf. e.g. [29,15,16]. The second-order differential of $L(\cdot, \cdot, w)$ at a point $\{u, f\}$, denoted as $L''(u, f, w) : [W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{R}^n)]^2 \rightarrow \mathbb{R}$, is given by

$$\begin{aligned} L''(u, f, w)[(u_1, f_1), (u_2, f_2)] &= \alpha(u_1, u_2) + \beta(\text{rot } u_1, \text{rot } u_2) \\ &\quad + (f_1, f_2) + ((u_1 \cdot \nabla)u_2, w) + ((u_2 \cdot \nabla)u_1, w). \end{aligned} \quad (2.10)$$

This quadratic form is obviously symmetric and independent of $\{u, f\}$, and even bounded due to the estimate

$$\begin{aligned} |L''(u, f, w)[(u_1, f_1), (u_2, f_2)]| &\leq \left(\alpha N_2^2 + 2\beta + 2N_4^2 \|\nabla w\|_{L^2(\Omega; \mathbb{R}^n)} \right) \\ &\quad \times \|u_1\|_{W^{1,2}(\Omega; \mathbb{R}^n)} \|u_2\|_{W^{1,2}(\Omega; \mathbb{R}^n)} + \|f_1\|_{L^2(\Omega; \mathbb{R}^n)} \|f_2\|_{L^2(\Omega; \mathbb{R}^n)} \end{aligned} \quad (2.11)$$

where we used the estimate $\|\operatorname{rot} u\|_{L^2(\Omega; \mathbb{R}^n)} \leq \sqrt{2} \|\nabla u\|_{L^2(\Omega; \mathbb{R}^{n \times n})}$.

The boundedness of the quadratic form $L''(u, f, w)$ is even uniform with respect to all w under consideration. We need only the restriction of $L''(u, f, w)$ to the diagonal of $[W_{0,\operatorname{DIV}}^{1,2}(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{R}^n)]^2$, and then we simply write $L''(u, f, w)(\tilde{u}, \tilde{f})^2 := L''(u, f, w)[(\tilde{u}, \tilde{f}), (\tilde{u}, \tilde{f})]$. Due to $((\tilde{u} \cdot \nabla) \tilde{u}, w) = -((\tilde{u} \cdot \nabla) w, \tilde{u})$, this restricted second-order differential takes the form

$$L''(u, f, w)(\tilde{u}, \tilde{f})^2 = \alpha \|\tilde{u}\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \beta \|\operatorname{rot} \tilde{u}\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \|\tilde{f}\|_{L^2(\Omega; \mathbb{R}^n)}^2 - 2((\tilde{u} \cdot \nabla) w, \tilde{u}). \quad (2.12)$$

The standard *second-order sufficient optimality condition*, often abbreviated as (SSC), at (u^*, f^*, w^*) requires existence of a positive δ such that the coercivity condition

$$L''(u^*, f^*, w^*)(u, f)^2 \geq \delta \|f\|_{L^2(\Omega; \mathbb{R}^n)}^2 \quad (2.13)$$

holds for all $\{u, f\}$ solving the Navier–Stokes system linearized at $\{u^*, f^*\}$, i.e. in the weak formulation

$$((u \cdot \nabla) u^*, v) + ((u^* \cdot \nabla) u, v) + v(\nabla u : \nabla v) = (f, v) \quad (2.14)$$

for all $v \in W_{0,\operatorname{DIV}}^{1,2}(\Omega; \mathbb{R}^n)$.

Proposition 2.2. *Let (2.1c) hold, and let $\{u^*, f^*, w^*\}$ satisfy the first-order necessary conditions (2.3) (with $u := u^*$ and $f := f^*$) and (2.6) hold together with the second-order sufficient condition (SSC). Then $\{u^*, f^*\}$ is the locally optimal pair for (\mathfrak{P}) with respect to the topology of $W_{0,\operatorname{DIV}}^{1,2}(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{R}^n)$.*

The proof of the above assertion is essentially due to Casas and Tröltzsch [34], cf. also [15, Prop.2.6]. We can apply it directly to our iteration strategy:

Proposition 2.3. *Again, let (2.1c) hold and let $\{u^*, w^*\}$ be a fixed point of M with f^* being the corresponding control, i.e. $\{u^*, f^*\}$ is a solution of $(\mathfrak{P}_{\operatorname{LQ}})$ with $\bar{u} = u^*, \bar{w} = w^*$ while w^* is a weak solution of Eq. (2.9) with $\bar{u} = u^*, \bar{w} = w^*$. Then $\{u^*, f^*\}$ is a critical point for the nonlinear problem (\mathfrak{P}) . If, moreover, (SSC) are satisfied at this point $\{u^*, f^*\}$, then it is a local minimizer for (\mathfrak{P}) .*

Proof. As already mentioned, the linear-quadratic problem $(\mathfrak{P}_{\operatorname{LQ}})$ is strictly convex, and thus it has a unique minimizer $\{u, f\}$. This minimizer satisfies the first-order necessary (and now also sufficient) optimality conditions, i.e. the Oseen state problem in $(\mathfrak{P}_{\operatorname{LQ}})$ in the weak formulation governed by the identity

$$((\bar{u} \cdot \nabla) u, v) + v(\nabla u : \nabla v) = (f, v) \quad (2.15)$$

holding for all $v \in W_{0,\operatorname{DIV}}^{1,2}(\Omega; \mathbb{R}^n)$, the adjoint equation (2.9) in the weak formulation governed by the identity

$$v(\nabla v : \nabla w) + ((\bar{u} \cdot \nabla) v, w) + ((v \cdot \nabla) \bar{u}, \bar{w}) = \alpha(u_d - u, v) - \beta(\operatorname{rot} u, \operatorname{rot} v) \quad (2.16)$$

holding for all $v \in W_{0,\operatorname{DIV}}^{1,2}(\Omega; \mathbb{R}^n)$, and also the inequality

$$(f - w, \tilde{f} - f) \geq 0 \quad \forall \tilde{f} \in \mathcal{F}_{\operatorname{ad}}. \quad (2.17)$$

Now, if $\{u^*, w^*\}$ is a fixed point of the mapping M with $f^* \in \mathcal{F}_{\operatorname{ad}}$ the corresponding control, it holds $\bar{u} = u = u^*, \bar{w} = w = w^*$ and the first-order optimality conditions (2.15)–(2.17) coincide with (2.3) (with $u = u^*$ and $f = f^*$), (2.6) and (2.8), and therefore $\{u^*, f^*\}$ is a critical point for (\mathfrak{P}) .

If it happens that also (SSC) holds, then Proposition 2.2 says that $\{u^*, f^*\}$ is a local minimizer for (\mathfrak{P}) . \square

Now, an important question is whether there is a set, say D , which is mapped by M into itself and a norm with respect to which M is a contraction on D . The following assertion answers it affirmatively on the condition that the fluid (i.e. $\nu > 0$) as well as the domain Ω are given and thus assumed not subjected to any choice.

Proposition 2.4. Let (2.1) hold with $\alpha \geq 0$ and $\beta \geq 0$ sufficiently small, u_d be sufficiently small in L^2 -norm, and let the set of admissible controls \mathcal{F}_{ad} be bounded in L^2 -norm by a (sufficiently small) constant $R_1 > 0$; in view of (2.1c), always $R_1 < \nu/(N_2 N_4^2)$. Then, the mapping $M : (\bar{u}, \bar{w}) \mapsto (u, w)$ is contractive on the set $(=$ a complete metric space endowed with the norm $W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n)^2$)

$$D := \left\{ (u, w) \in W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n)^2; \|u\|_{W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n)} \leq \frac{N_2 R_1}{\nu}, \|w\|_{W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n)} \leq R_2 \right\} \quad (2.18)$$

for a suitable $R_2 > 0$.

Proof. Let (u_1, w_1, f_1) and (u_2, w_2, f_2) be the solution of the optimality conditions (2.15)–(2.17) corresponding to the quantities (\bar{u}_1, \bar{w}_1) and (\bar{u}_2, \bar{w}_2) , respectively. We will abbreviate also $u_{12} := u_1 - u_2$, $f_{12} := f_1 - f_2$, $\bar{u}_{12} := \bar{u}_1 - \bar{u}_2$, $\bar{w}_{12} := \bar{w}_1 - \bar{w}_2$, etc.

At first we test the inequality (2.17) for $f := f_1$ and $w := w_1$ by $\tilde{f} := f_2$

$$(f_1 - w_1, -f_{12}) \geq 0. \quad (2.19)$$

Similarly, for $f = f_2$, $w = w_2$ and $\tilde{f} := f_1$, we get

$$(f_2 - w_2, f_{12}) \geq 0. \quad (2.20)$$

Summing (2.19) with (2.20), we obtain the estimate

$$\|f_{12}\|_{L^2(\Omega; \mathbb{R}^n)} \leq \|w_{12}\|_{L^2(\Omega; \mathbb{R}^n)}. \quad (2.21)$$

Now we test the Oseen problem (2.15) for $u = u_1$ and $\bar{u} = \bar{u}_1$ (resp. $u = u_2$ and $\bar{u} = \bar{u}_2$) by $v = u_{12}$ and subtract the associated identities. We obtain

$$\nu(\nabla u_{12} : \nabla u_{12}) + ((\bar{u}_1 \cdot \nabla)u_1 - (\bar{u}_2 \cdot \nabla)u_2, u_{12}) = (f_{12}, u_{12}). \quad (2.22)$$

Using $(\bar{u}_1 \cdot \nabla)u_1 - (\bar{u}_2 \cdot \nabla)u_2 = (\bar{u}_1 \cdot \nabla)u_{12} + (\bar{u}_{12} \cdot \nabla)u_2$ and $b(w, v, v) = 0$, cf. (2.2), this equation implies the estimate:

$$\nu \|\nabla u_{12}\|_{L^2(\Omega; \mathbb{R}^{n \times n})}^2 \leq \|\nabla u_2\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \|u_{12}\|_{L^4(\Omega; \mathbb{R}^n)} \|\bar{u}_{12}\|_{L^4(\Omega; \mathbb{R}^n)} + \|f_{12}\|_{L^2(\Omega; \mathbb{R}^n)} \|u_{12}\|_{L^2(\Omega; \mathbb{R}^n)}. \quad (2.23)$$

Using (2.21), the Friedrichs inequality and the Sobolev embedding theorem, we can replace $\|f_{12}\|_{L^2(\Omega; \mathbb{R}^n)}$ by $\|w_{12}\|_{L^2(\Omega; \mathbb{R}^n)}$ and cancelate the term $\|\nabla u_{12}\|_{L^2(\Omega; \mathbb{R}^{n \times n})}$:

$$\nu \|\nabla u_{12}\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \leq N_4^2 \|\nabla u_2\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \|\bar{u}_{12}\|_{L^2(\Omega; \mathbb{R}^{n \times n})} + N_2^2 \|\nabla w_{12}\|_{L^2(\Omega; \mathbb{R}^{n \times n})}. \quad (2.24)$$

We proceed similarly with the adjoint equation (2.16). We substitute $(u_1, \bar{u}_1, w_1, \bar{w}_1)$ and $(u_2, \bar{u}_2, w_2, \bar{w}_2)$ for (u, \bar{u}, w, \bar{w}) in it, test both obtained equations by $v := w_{12}$ and subtract them. As a result, we get:

$$\begin{aligned} & \nu(\nabla w_{12} : \nabla w_{12}) + ((w_{12} \cdot \nabla)\bar{u}_1, \bar{w}_1) - ((w_{12} \cdot \nabla)\bar{u}_2, \bar{w}_2) + ((\bar{u}_1 \cdot \nabla)w_{12}, w_1) \\ & - ((\bar{u}_2 \cdot \nabla)w_{12}, w_2) + \alpha(u_{12}, w_{12}) + \beta(\text{rot } u_{12}, \text{rot } w_{12}) = 0. \end{aligned} \quad (2.25)$$

Using the properties of the trilinear form b generated by the convective term (2.2), we can rearrange the four terms above as follows:

$$((w_{12} \cdot \nabla)\bar{u}_1, \bar{w}_1) - ((w_{12} \cdot \nabla)\bar{u}_2, \bar{w}_2) = -((\bar{u}_{12} \cdot \nabla)w_1, w_{12}), \quad \text{and} \quad (2.26a)$$

$$((\bar{u}_1 \cdot \nabla)w_{12}, w_1) - ((\bar{u}_2 \cdot \nabla)w_{12}, w_2) = ((w_{12} \cdot \nabla)\bar{u}_1, \bar{w}_{12}) - ((w_{12} \cdot \nabla)\bar{w}_2, \bar{u}_{12}). \quad (2.26b)$$

Furthermore, we get the estimate

$$\begin{aligned} \nu \|\nabla w_{12}\|_{L^2(\Omega; \mathbb{R}^{n \times n})}^2 & \leq \alpha \|u_{12}\|_{L^2(\Omega; \mathbb{R}^n)} \|w_{12}\|_{L^2(\Omega; \mathbb{R}^n)} + \beta \|\text{rot } u_{12}\|_{L^2(\Omega; \mathbb{R}^n)} \|\text{rot } w_{12}\|_{L^2(\Omega; \mathbb{R}^n)} \\ & + \|\nabla w_1\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \|w_{12}\|_{L^4(\Omega; \mathbb{R}^n)} \|\bar{u}_{12}\|_{L^4(\Omega; \mathbb{R}^n)} \\ & + \|w_{12}\|_{L^4(\Omega; \mathbb{R}^n)} \|\nabla \bar{u}_1\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \|\bar{w}_{12}\|_{L^4(\Omega; \mathbb{R}^n)} \\ & + \|w_{12}\|_{L^4(\Omega; \mathbb{R}^n)} \|\nabla \bar{w}_2\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \|\bar{u}_{12}\|_{L^4(\Omega; \mathbb{R}^n)}. \end{aligned} \quad (2.27)$$

Using the Sobolev embedding theorem and the Friedrichs inequality, we can cancelate the terms $\|\nabla w_{12}\|_{L^2(\Omega; \mathbb{R}^{n \times n})}$ to obtain

$$\begin{aligned} v \|\nabla w_{12}\|_{L^2(\Omega; \mathbb{R}^{n \times n})} &\leq \alpha N_2^2 \|\nabla u_{12}\|_{L^2(\Omega; \mathbb{R}^{n \times n})} + 2\beta \|\nabla u_{12}\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \\ &\quad + N_4^2 \|\nabla w_1\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \|\nabla \bar{u}_{12}\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \\ &\quad + N_4^2 \|\nabla \bar{u}_1\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \|\nabla \bar{w}_{12}\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \\ &\quad + N_4^2 \|\nabla \bar{w}_2\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \|\nabla \bar{u}_{12}\|_{L^2(\Omega; \mathbb{R}^{n \times n})}. \end{aligned} \quad (2.28)$$

At the end, we add the estimates (2.24) and (2.28) multiplied by a suitable $\kappa > 0$. This yields

$$\begin{aligned} &(\kappa v - \alpha N_2^2 - 2\beta) \|\nabla u_{12}\|_{L^2(\Omega; \mathbb{R}^{n \times n})} + (v - \kappa N_2^2) \|\nabla w_{12}\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \\ &\leq \left(\kappa N_4^2 \|\nabla u_2\|_{L^2(\Omega; \mathbb{R}^{n \times n})} + N_4^2 \|\nabla w_1\|_{L^2(\Omega; \mathbb{R}^{n \times n})} + N_4^2 \|\nabla \bar{w}_2\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \right) \|\nabla \bar{u}_{12}\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \\ &\quad + N_4^2 \|\nabla \bar{u}_1\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \|\nabla \bar{w}_{12}\|_{L^2(\Omega; \mathbb{R}^{n \times n})}. \end{aligned} \quad (2.29)$$

We choose $\kappa = v/(2N_2^2)$. The estimate (2.29) then implies that M is contractive on D for sufficiently small α, β, R_1 and R_2 provided we prove that M maps D into itself. We need therefore to obtain bounds on $\|\nabla u\|_{L^2(\Omega; \mathbb{R}^{n \times n})}$ and $\|\nabla w\|_{L^2(\Omega; \mathbb{R}^{n \times n})}$ provided $(\bar{u}, \bar{w}) \in D$.

The condition $f \in \mathcal{F}_{\text{ad}}$ implies that $\|\nabla u\|_{L^2(\Omega; \mathbb{R}^{n \times n})} < N_2 R_1 / v$. This could be seen by testing the Oseen equation (2.15) by its solution u itself.

We now test the adjoint equation (2.16) by its solution w itself:

$$\begin{aligned} v \|\nabla w\|_{L^2(\Omega; \mathbb{R}^{n \times n})}^2 &= -((w \cdot \nabla) \bar{u}, \bar{w}) + \alpha(u_d - u, w) - \beta(\text{rot } u, \text{rot } w) \\ &\leq N_4^2 \|\nabla w\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \|\nabla \bar{u}\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \|\nabla \bar{w}\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \\ &\quad + \alpha N_2^2 \|\nabla u\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \|\nabla w\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \\ &\quad + \alpha N_2 \|u_d\|_{L^2(\Omega; \mathbb{R}^n)} \|\nabla w\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \\ &\quad + 2\beta \|\nabla u\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \|\nabla w\|_{L^2(\Omega; \mathbb{R}^{n \times n})}. \end{aligned} \quad (2.30)$$

Assuming $\|\nabla \bar{w}\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \leq R_2$, this estimate implies

$$\|\nabla w\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \leq \frac{1}{v} \left(\frac{N_4^2 N_2 R_1 R_2}{v} + \frac{\alpha N_2^3 R_1}{v} + \alpha N_2 \|u_d\|_{L^2(\Omega; \mathbb{R}^n)} + \frac{2\beta N_2 R_1}{v} \right). \quad (2.31)$$

It is now possible to choose R_1 and u_d so small that $N_4^2 N_2 R_1 R_2 + \alpha N_2^3 R_1 + v \alpha N_2 \|u_d\|_{L^2(\Omega; \mathbb{R}^n)} + 2\beta N_2 R_1 \leq v^2 R_2$ and so M maps D into itself. \square

Remark 2.5. The smallness conditions on data can be formulated so that the functional $f \mapsto J(f, u(f))$ is convex on \mathcal{F}_{ad} ; if $\beta = 0$ see [14] or also [15] while for $\beta > 0$ a similar term has been analyzed in [18] under assumptions of a higher integrability of u_d and smoothness of Γ . Then the obtained fixed point, being a critical point, is also the global minimizer because the necessary optimality conditions are then also sufficient. The condition (SSC) is then satisfied automatically but it loses its importance in this context. In [2] it has been shown, however, that conditions guaranteeing such a convexity are quite severe.

Remark 2.6. An example for \mathcal{F}_{ad} involving point-wise constraints and satisfying (2.1b) and (2.1c) is

$$\mathcal{F}_{\text{ad}} = \{f : \Omega \rightarrow \mathbb{R}^n \text{ measurable} : |f(x)| \leq r \text{ a.e. on } \Omega\} \quad \text{with } r < \frac{v^2 |\Omega|^{-1/2}}{N_2 N_4^2}. \quad (2.32)$$

Remark 2.7. The case with no control constraints, i.e. $\mathcal{F}_{\text{ad}} = L^2(\Omega; \mathbb{R}^2)$, is obviously not consistent with (2.1c). Yet, we can adopt a philosophy that the problem is globally coercive due to the obvious estimate

$$\frac{1}{2} \|f^*\|_{L^2(\Omega; \mathbb{R}^n)} \leq J(u^*, f^*) \leq J(0, 0) = \frac{\alpha}{2} \|u_d\|_{L^2(\Omega; \mathbb{R}^n)} \quad (2.33)$$

because the pair $\{u, f\} \equiv \{0, 0\}$ obviously solves the Navier–Stokes equations with homogeneous Dirichlet boundary conditions, and therefore those f for which (2.1c) possibly would not hold, i.e. $\|f\|_{L^2(\Omega; \mathbb{R}^2)} \geq v^2/(N_2 N_4^2)$, cannot occur as minimizers. Then (2.8) turns simply into $w^* = f^*$ and $(\mathfrak{P}_{\text{LQ}})$ obviously admits a term $(u \cdot \nabla \bar{u}) \cdot \bar{f}$ instead of $(u \cdot \nabla \bar{u}) \cdot \bar{w}$ with obviously equivalent effects. This approach has been used in [17].

3. The SQP-type algorithm with a discretization

In numerical solution on computers, we need a further discretization of $(\mathfrak{P}_{\text{LQ}})$. In this section, we use an abstract discretization of $W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n)$ and \mathcal{F}_{ad} by some finite-dimensional subspaces V_h and $\mathcal{F}_{\text{ad},h}$, respectively. Here, $h > 0$ is an abstract discretization parameter, $V_{h_1} \subset V_{h_2} \subset W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n)$ and $\mathcal{F}_{\text{ad},h_1} \subset \mathcal{F}_{\text{ad},h_2} \subset \mathcal{F}_{\text{ad}}$ for $h_1 \geq h_2 > 0$. Then, instead of $(\mathfrak{P}_{\text{LQ}})$, we are to solve

$$(\mathfrak{P}_{\text{LQ},h}) \quad \begin{cases} \text{Minimize} & \int_{\Omega} \frac{\alpha}{2} |u - u_d|^2 + \frac{\beta}{2} |\text{rot } u|^2 + \frac{1}{2} |f|^2 - (u \cdot \nabla \bar{u}) \cdot \bar{w} \, dx \\ \text{subject to} & (\bar{u} \cdot \nabla) u - \nu \Delta u + \nabla p = f, \quad \text{div } u = 0, \\ & u \in V_h, \quad f \in \mathcal{F}_{\text{ad},h}, \quad p \in L_0^2(\Omega), \end{cases}$$

which determines the mapping $M_h : V_h^2 \rightarrow V_h^2 : (\bar{u}, \bar{w}) \mapsto (u, w)$ if the state equation in $(\mathfrak{P}_{\text{LQ},h})$ is assumed in the weak sense, i.e. (2.15) for all $v \in V_h$. Then the state Eq. (2.16) is a mapping $V_h \times \mathcal{F}_{\text{ad},h} \rightarrow V_h^*$ so that the adjoint state w indeed belongs to $V_h^{**} \cong V_h$.

The corresponding discretization of the original problem (\mathfrak{P}) results in the problem

$$(\mathfrak{P}_h) \quad \begin{cases} \text{Minimize} & \int_{\Omega} \frac{\alpha}{2} |u - u_d|^2 + \frac{\beta}{2} |\text{rot } u|^2 + \frac{1}{2} |f|^2 \, dx \\ \text{subject to} & (u \cdot \nabla) u - \nu \Delta u + \nabla p = f, \quad \text{div } u = 0, \\ & u \in V_h, \quad f \in \mathcal{F}_{\text{ad},h}, \quad p \in L_0^2(\Omega). \end{cases}$$

We naturally call $\{u^*, f^*\} \in V_h \times \mathcal{F}_{\text{ad},h}$ a *critical point* for (\mathfrak{P}_h) if (2.3) and (2.6) with some $w \in V_h$ hold for all $v \in V_h$, and if $(f^* - w, f - f^*) \geq 0$ for any $f \in \mathcal{F}_{\text{ad},h}$. A critical point just satisfies 1st-order optimality conditions for (\mathfrak{P}_h) .

Proposition 3.1. *Let the assumptions of Proposition 2.4 hold. Then M_d is contractive on $D \cap V_h^2$, and the fixed point $\{u_h, w_h\} \in V_h^2$ with the corresponding control $f_h \in \mathcal{F}_{\text{ad},h}$ form a critical point $\{u_h, f_h\}$ for (\mathfrak{P}_h) .*

Proof. This just modifies the proof of Propositions 2.3 and 2.4 by restriction on the finite-dimensional subspace in question. \square

Proposition 3.2. *Let us assume, in addition to the assumption of Proposition 2.4, that $\bigcup_{h>0} V_h$ is dense in $W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n)$ in the weak- $W^{1,2}$ -topology and $\bigcup_{h>0} \mathcal{F}_{\text{ad},h}$ is dense in \mathcal{F}_{ad} in the weak- L^2 -topology, and $n \leq 3$. Then the sequence $\{u_h, w_h, f_h\}_{h>0}$ obtained in Proposition 3.1 is bounded in $W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n)^2 \times L^2(\Omega; \mathbb{R}^n)$, and hence it contains a weakly convergent subsequence with a limit, say (u^*, w^*, f^*) , for $h \searrow 0$. Moreover, any $\{u^*, f^*\}$ thus obtained is a critical point for (\mathfrak{P}) .*

Proof. The boundedness of the sequence $\{u_h, w_h, f_h\}_{h>0}$ follows simply from the boundedness of D and of \mathcal{F}_{ad} . Let us consider a subsequence, denoted by the same indices for simplicity, such that

$$u_h \rightharpoonup u^* \quad \text{weakly in } W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n), \quad (3.1a)$$

$$w_h \rightharpoonup w^* \quad \text{weakly in } W_{0,\text{DIV}}^{1,2}(\Omega; \mathbb{R}^n), \quad (3.1b)$$

$$f_h \rightharpoonup f^* \quad \text{weakly in } L^2(\Omega; \mathbb{R}^n). \quad (3.1c)$$

By Rellich–Kondrachov’s theorem, $u_h \rightarrow u^*$ and $w_h \rightarrow w^*$ strongly in $L^4(\Omega; \mathbb{R}^n)$ if $n \leq 3$. Then we can pass to the limit in the optimality conditions for (\mathfrak{P}_h) , i.e.

$$((u_h \cdot \nabla) u_h, v) + \nu (\nabla u_h : \nabla v) = (f_h, v) \quad \forall v \in V_{h_0}, \quad (3.2a)$$

$$\begin{aligned} v(\nabla w_h : \nabla v) - ((u_h \cdot \nabla)w_h, v) + ((v \cdot \nabla)u_h, w_h) \\ = \alpha(u_d - u_h, v) - \beta(\operatorname{rot} u_h, \operatorname{rot} v) \quad \forall v \in V_{h_0}, \end{aligned} \quad (3.2b)$$

$$(f_h - w_h, f - f_h) \geq 0 \quad \forall f \in \mathcal{F}_{\text{ad}, h_0}. \quad (3.2c)$$

In fact, (3.2) holds for $h \geq h_0$. First, we fix h_0 and let $h \rightarrow 0$. Then all the terms in (3.2a) and (3.2b) allow for the limit passage. The passage in (3.2c) is only by weak-lower-semicontinuity of the functional $f \mapsto \|f\|_{L^2(\Omega; \mathbb{R}^n)}^2$:

$$\begin{aligned} 0 &\leq \limsup_{h \rightarrow 0} (f_h - w_h, f - f_h) = \lim_{h \rightarrow 0} (f_h - w_h, f) - \lim_{h \rightarrow 0} (w_h, f_h) - \liminf_{h \rightarrow 0} \|f_h\|_{L^2(\Omega; \mathbb{R}^n)}^2 \\ &\leq (f^* - w^*, f) - (w^*, f^*) - \|f^*\|_{L^2(\Omega; \mathbb{R}^n)}^2 = (f^* - w^*, f - f^*). \end{aligned} \quad (3.3)$$

In this way, we obtain

$$((u^* \cdot \nabla)u^*, v_{h_0}) + v(\nabla u^* : \nabla v_{h_0}) = (f^*, v_{h_0}) \quad \forall v_{h_0} \in V_{h_0}, \quad (3.4a)$$

$$\begin{aligned} v(\nabla w^* : \nabla v_{h_0}) - ((u^* \cdot \nabla)w^*, v_{h_0}) + ((v_{h_0} \cdot \nabla)u^*, w^*) \\ = \alpha(u_d - u^*, v_{h_0}) - \beta(\operatorname{rot} u^*, \operatorname{rot} v_{h_0}) \quad \forall v_{h_0} \in V_{h_0}, \end{aligned} \quad (3.4b)$$

$$(f^* - w^*, f_{h_0} - f^*) \geq 0 \quad \forall f_{h_0} \in \mathcal{F}_{\text{ad}, h_0}. \quad (3.4c)$$

Eventually, for any $f \in \mathcal{F}_{\text{ad}}$ we take a sequence $f_{h_0} \rightarrow f$ weakly in $L^2(\Omega; \mathbb{R}^n)$ with $f_{h_0} \in \mathcal{F}_{\text{ad}, h_0}$, and for any $v \in W_{0, \text{DIV}}^{1,2}(\Omega; \mathbb{R}^n)$ we take a sequence $v_{h_0} \rightarrow v$ weakly in $W^{1,2}(\Omega; \mathbb{R}^n)$ with $v_{h_0} \in V_{h_0}$. Then we pass to the limit in (3.4) for $h_0 \rightarrow 0$. \square

Remark 3.3. It is well known that the strong convergence for $h \rightarrow 0$ of the finite-element Oseen scheme would require a regularity of the solution, which would then require additional qualification of the domain Ω . In view of this, it seems that the weak mode of convergence (3.1) is optimal for FEM on general domains.

Remark 3.4. There are still interesting questions. e.g., can every cluster point obtained in Proposition 3.2 be identified with the fixed point obtained in Proposition 2.4? Can one make a limit passage directly in $(\mathfrak{P}_{\text{LQ}, h})$, i.e. make Banach fixed-point iterations simultaneously with refining the discretization?

4. Computational tests

We have carried out some simple computational tests of the algorithm analyzed theoretically in the preceding sections. These computations show the feasibility of this approach at least in specially qualified cases, cf. Proposition 2.4, essentially in the cases of small Reynold's numbers.

All computations were made only in two dimensions, $n = 2$. For the sake of simplicity, we considered no control constraints, i.e. $\mathcal{F}_{\text{ad}} = L^2(\Omega; \mathbb{R}^2)$, cf. Remark 2.7. As the assumption $\mathcal{F}_{\text{ad}} = L^2(\Omega; \mathbb{R}^2)$ implies that (2.8) as well as (2.17) reduces to $w^* = f^*$, it is sufficient to solve only the linear system containing the optimality conditions (2.15) and (2.16) with w^* replaced by f^* at each iteration. It is therefore obvious that it is not necessary to store both variables f and w together in the memory of a computer. Theoretical analysis of this case with $\beta = 0$ as well as simple numerical tests and comparison with other optimization algorithms, namely the SQP method and the steepest descent method, can be found in [17], cf. also Remark 4.1 below.

For numerical computations, we have used a slightly modified program written originally by Hron [35] based on quadrilateral finite elements. In view of this code, we did not approximate the space $W_{0, \text{DIV}}^{1,2}(\Omega; \mathbb{R}^2)$ by some finite element subspace V_h satisfying $V_h \subset W_{0, \text{DIV}}^{1,2}(\Omega; \mathbb{R}^2)$. Instead, we have utilized a modified weak formulation of the Oseen equations $(\mathfrak{P}_{\text{LQ}})$ and the adjoint equation (2.9) (with $w := f$) which includes the pressure p and π , respectively, namely

$$((\bar{u} \cdot \nabla)u, v) + v(\nabla u : \nabla v) + (p, \operatorname{div} v) = (f, v) \quad \forall v \in W_0^{1,2}(\Omega, \mathbb{R}^n), \quad (4.1a)$$

$$(\operatorname{div} u, q) = 0 \quad \forall q \in L^2(\Omega; \mathbb{R}^n), \quad (4.1b)$$

$$\begin{aligned} v(\nabla v : \nabla f) + ((\bar{u} \cdot \nabla)v, f) + ((v \cdot \nabla)\bar{u}, \bar{f}) + (\pi, \operatorname{div} v) \\ = \alpha(u_d - u, v) - \beta(\operatorname{rot} u, \operatorname{rot} v) \quad \forall v \in W_0^{1,2}(\Omega, \mathbb{R}^n) \end{aligned} \quad (4.1c)$$

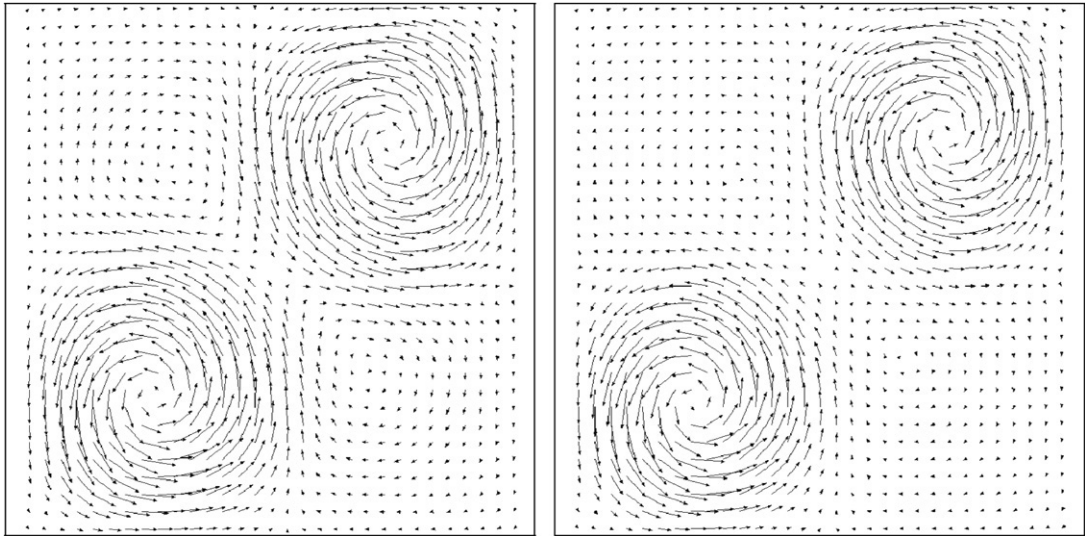


Fig. 1. $\nu = 0.1, \alpha = 0.1, \beta = 0$. Left: velocity u_h (the scale $\|u_h\|_{L^\infty(\Omega, \mathbb{R}^2)} = 0.0033$), Right: force f_h (the scale $\|f_h\|_{L^\infty(\Omega, \mathbb{R}^2)} = 0.0134$).

Table 1

Decrease of the cost functional J for various values of α and ν

Fig. no	Viscosity ν	α	β	Cost functional at initial guess	Cost functional at critical point	Number of iterations
Fig. 1.	0.1	0.1	0	3.27e-04	3.110e-04	3
Fig. 2.	0.03	100	0	3.27e-01	0.112e-01	3
Fig. 3.	0.055	1000	0	3.27e-00	0.047e-00	3

$$\begin{aligned}
 (\operatorname{div} f, q) &= 0 \quad \forall q \in L^2(\Omega; \mathbb{R}^n), \\
 u &\in W_0^{1,2}(\Omega, \mathbb{R}^n), \quad f \in W_0^{1,2}(\Omega, \mathbb{R}^n), \quad p \in L_0^2(\Omega), \quad \pi \in L_0^2(\Omega),
 \end{aligned} \tag{4.1d}$$

and chosen conforming biquadratic Q_2 -elements to approximate the space $W_0^{1,2}(\Omega; \mathbb{R}^2)$ and discontinuous affine P_1 -elements for $L^2(\Omega; \mathbb{R}^2)$. This pair is known to be stable for the problems with incompressibility constraint. This approach, however, does not allow for a direct usage of Proposition 3.1 to prove the contractiveness of the mapping $\{\bar{u}_h, \bar{f}_h\} \mapsto \{u_h, f_h\}$ but we rely on the fact that the difference between discretization of (4.1) by Q_2/P_1 -elements or by conformal elements assumed in Section 3 is not essential if $h > 0$ is small.

As a numerical example, we have considered a square domain $\Omega := [-1, 1] \times [-1, 1]$. The desired velocity profile u_d is formed by two vortices whose midpoints are $s := [\frac{1}{2}, \frac{1}{2}]$ and $-s$, see Fig. 4. To be more specific, we have taken

$$u_d(x) = \begin{cases} \left(\frac{1}{2} - |x - s|\right) \left(x_2 - \frac{1}{2}, -x_1 + \frac{1}{2}\right) & \text{if } |x - s| \leq \frac{1}{2}, \\ \left(\frac{1}{2} - |x + s|\right) \left(x_2 + \frac{1}{2}, -x_1 - \frac{1}{2}\right) & \text{if } |x + s| \leq \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

The distributions of the critical velocity u_h (=the response) and the corresponding distributed force f_h (=the control) for several values of viscosity ν and parameter α are shown on Figs. 1–3. The magnitudes (L^∞ -norms) of depicted vector fields are provided as information on the scales of the arrows. Note that for increasing values of α the response u becomes more and more similar to the desired velocity profile u_d . The Table 1 shows the decrease of the cost functional J after performing the optimization algorithm for various combinations of viscosity ν and parameter α while $\beta = 0$.

A reasonable choice for the stopping criterion is the requirement that the difference of the last two iterations should be small. We have used the criterion $\|\bar{u}_h - u_h\|_{L^2(\Omega, \mathbb{R}^2)} + \|\bar{f}_h - f_h\|_{L^2(\Omega, \mathbb{R}^2)} < 10^{-8}$. As an initial guess, we

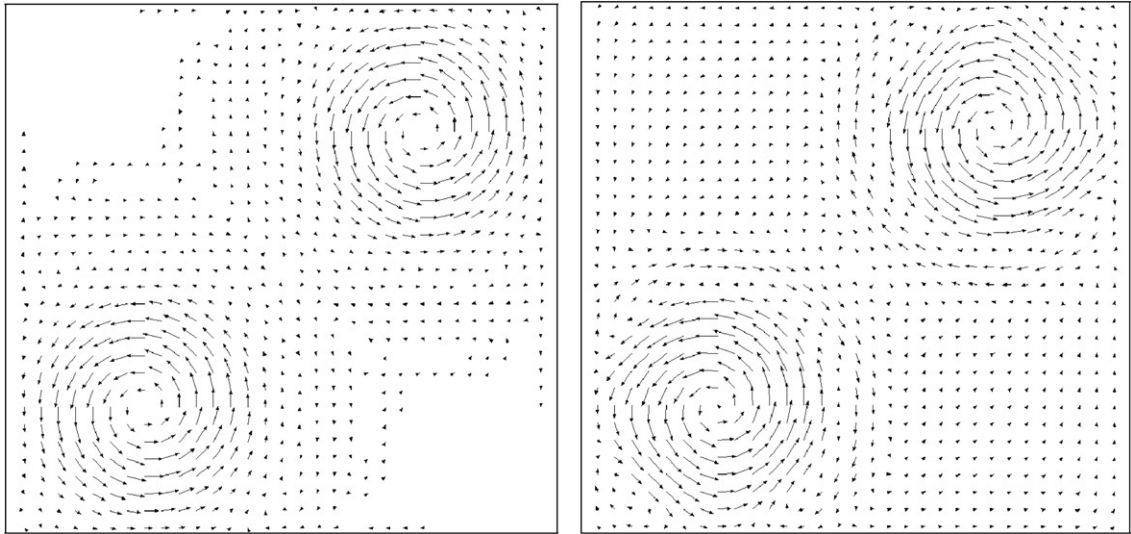


Fig. 2. $\nu = 0.03, \alpha = 100, \beta = 0$. Left: velocity u_h (the scale $\|u_h\|_{L^\infty(\Omega, \mathbb{R}^2)} = 0.0619$), Right: force f_h (the scale $\|f_h\|_{L^\infty(\Omega, \mathbb{R}^2)} = 0.106$).

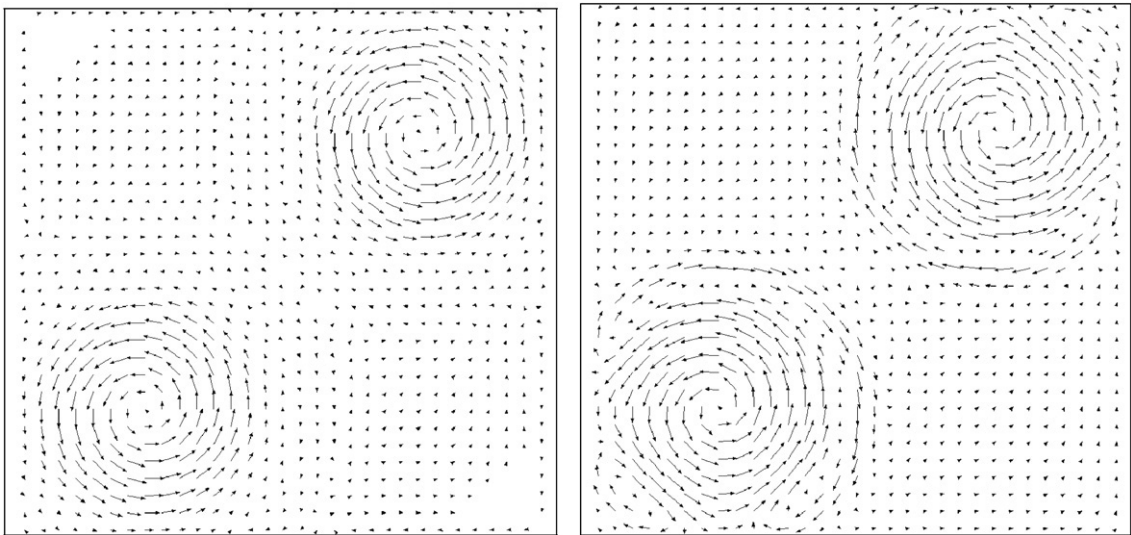


Fig. 3. $\nu = 0.055, \alpha = 1000, \beta = 0$. Left: velocity u_h (the scale $\|u_h\|_{L^\infty(\Omega, \mathbb{R}^2)} = 0.626$), Right: force f_h (the scale $\|f_h\|_{L^\infty(\Omega, \mathbb{R}^2)} = 0.201$).

have always chosen zero vector for all variables. All computations were performed on the same mesh containing 256 elements. Each one took about three minutes on a 64-bit Alpha processor EV5, 700 MHz.

The Reynold's numbers of the critical flow in all presented cases were rather small as the viscosities ν were relatively big and α, β , and $\|u_d\|_{L^2(\Omega; \mathbb{R}^n)}$ were small. In this case, the algorithm converges after a few iterations. Conversely, the algorithm fails if the Reynold's number is big, say bigger than 6. This case would require adding additional stabilization terms into the discretization of the system (4.1).

Remark 4.1. In spite of the fact that the algorithm studied in this article is required to converge only linearly in contrast to the standard SQP method which provides locally quadratic convergence, our numerical experiments suggest that in case of small Reynold's numbers the rate of convergence for both methods is approximately the same. On the other hand, the steepest-descent method requires much more time to decrease the cost functional comparably with the other methods. However, the steepest descent method requires about four times less memory (on the same mesh) for

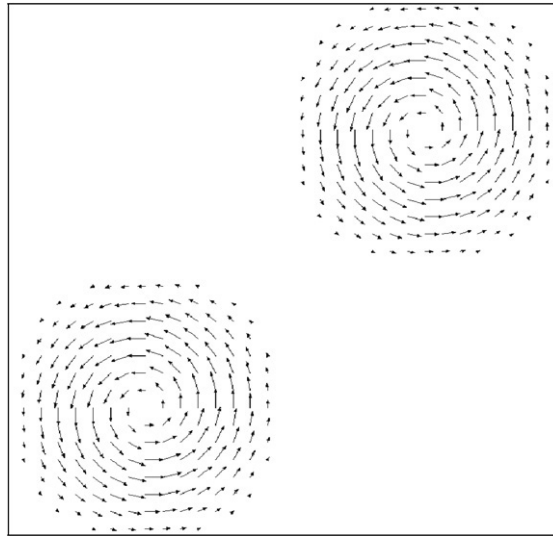


Fig. 4. The desired velocity profile u_d .

the storage of the stiffness matrix, because it solves the Navier–Stokes equations and the adjoint equation separately, cf. [25] for instance.

Acknowledgements

This work has been partly created as a research activity of “Nečas center for mathematical modeling” LC 06052 (MŠMT ČR).

This work has been supported partly through the grants A 1075402 (GA AV ČR), MSM 0021620839 (MŠMT ČR).

References

- [1] N. Bilić, Approximation of optimal distributed control problem for Navier–Stokes equations, in: Numerical Methods and Approx. Th., Univ. Novi Sad, Novi Sad, 1985, pp. 177–185.
- [2] P. Bubák, Optimal control of flow driven by the thermal field, MS-diploma Thesis, Math. -Phys. Faculty, Charles University, Praha, 2002.
- [3] J. Burkardt, J. Peterson, Control of steady incompressible 2D channel flow, in: M.D. Gunzburger (Ed.), Flow Control, in: IMA Vol. Math. Appl., vol. 68, Springer, New York, 1995, pp. 111–126.
- [4] E. Casas, Optimality conditions for some control problems of turbulent flow, in: M.D. Gunzburger (Ed.), Flow Control, in: IMA Vol. Math. Appl., vol. 68, Springer, New York, 1995, pp. 127–147.
- [5] M.C. Desai, K. Ito, Optimal control of Navier–Stokes equations, SIAM J. Control Optim. 32 (1994) 1428–1446.
- [6] O. Gattas, J. Bark, Optimal control of two- and three-dimensional incompressible Navier–Stokes flow, J. Comput. Phys. 136 (1997) 231–244.
- [7] M.D. Gunzburger, A prehistory of flow control and optimization, in: M.D. Gunzburger (Ed.), Flow Control, in: IMA Vol. Math. Appl., vol. 68, Springer, New York, 1995, pp. 185–195.
- [8] M.D. Gunzburger, L. Hou, T.P. Svobodny, Analysis and finite element approximation of optimal control problems for stationary Navier–Stokes equations with distributed and Neumann controls, Math. Comp. 57 (1991) 123–151.
- [9] M.D. Gunzburger, L. Hou, T.P. Svobodny, Boundary velocity control of incompressible flow with an application to viscous drag reduction, SIAM J. Control Optim. 30 (1992) 167–181.
- [10] L.S. Hou, S.S. Ravindran, A penalized Neumann control approach for solving an optimal Dirichlet control problem for the Navier–Stokes equations, SIAM J. Control Optim. 36 (1998) 1795–1814.
- [11] L.S. Hou, S.S. Ravindran, Numerical approximation of optimal flow control problems by a penalty method: Error estimates and numerical results, SIAM J. Sci. Comput. 20 (1999) 1753–1777.
- [12] M. Heinkenschloss, Formulation and analysis of a sequential quadratic programming method for the optimal Dirichlet boundary control of Navier–Stokes flow, in: W.H. Hager, P. Pardalos (Eds.), Optimal control: Theory, Algorithms, and Appl., Kluwer, Dordrecht, 1998, pp. 178–203.
- [13] J.L. Lions, Contrôle des systèmes distribués singuliers (Control of Distributed Singular Systems), Bordas, Paris, 1983. Gauthier-Villars, 1985 (Engl. transl.).
- [14] J. Málek, T. Roubíček, Optimization of steady flows for incompressible viscous fluids, in: A. Sequiera, H. Beirão da Vega, J.H. Videman (Eds.), Nonlinear Applied Analysis, Plenum Press, New York, 1999, pp. 355–372.

- [15] T. Roubíček, F. Tröltzsch, Lipschitz stability of optimal controls for the steady-state Navier–Stokes equations, *Control Cybern.* 32 (2003) 683–705.
- [16] F. Tröltzsch, D. Wachsmuth, Second-order sufficient optimality conditions for the optimal control of Navier–Stokes equations, *ESAIM Control Optim. Calc. Var.* 12 (2006) 93–119.
- [17] M. Pošta, Aplikace sekvenciálně-kvadratického programování v optimálním řízení Navier–Stokesových rovnic, MS-Thesis, Math. -Phys. Faculty, Charles Univ., Praha, 2004 (in English).
- [18] T. Roubíček, Optimization of steady-state flow of incompressible fluids, in: V. Barbu, I. Lasiecka, D. Tiba, C. Varsan (Eds.), *Proc. IFIP Conf. Analysis and Optimization of Differential Systems*, Kluwer Academic Publ., Boston, 2003, pp. 357–368.
- [19] H.O. Fattorini, Optimal chattering control for viscous flow, *Nonlinear Anal. TMA* 25 (1995) 763–797.
- [20] H.O. Fattorini, *Infinite Dimensional Optimization and Control Theory*, Cambridge Univ. Press, Cambridge, 1999.
- [21] H.O. Fattorini, S.S. Sritharan, Necessary and sufficient conditions for optimal control in viscous flow problems, *Proc. Roy. Soc. Edinburgh Sect. A* 124 (1994) 211–251.
- [22] A.V. Fursikov, *Optimal Control of Distributed Systems. Theory and Applications*, AMS, Providence, 2000.
- [23] M.D. Gunzburger, S. Manservigi, The velocity tracking problem for Navier–Stokes flow with bounded distributed controls, *SIAM J. Control Optim.* 37 (1999) 1913–1945.
- [24] M. Hinze, Optimal and instantaneous control of the instationary Navier–Stokes equations, *Habilitation Thesis*, Fachbereich Math., TU Berlin, 1999.
- [25] M. Hinze, K. Kunisch, Second order methods for optimal control of time-dependent fluid flow, *SIAM J. Control Optim.* 40 (2001) 925–946.
- [26] M. Hinze, K. Kunisch, Second order methods for boundary control of the instationary Navier–Stokes system, *Z. Angew. Math. Mech.* 84 (2004) 171–187.
- [27] S.S. Sritharan, Deterministic and stochastic control of Navier–Stokes equation with linear, monotone and hyperviscosities, *Appl. Math. Optim.* 41 (2000) 255–308.
- [28] R. Temam, Remarks on the control of turbulent flows, in: M.D. Gunzburger (Ed.), *Flow Control*, in: *IMA Vol. Math. Appl.*, vol. 68, Springer, New York, 1995, pp. 357–381.
- [29] M. Hintermüller, M. Hinze, Globalization of SQP-methods in control of the instationary Navier–Stokes equations, *Math. Model. Numer. Anal.* 36 (2002) 725–746.
- [30] M. Hintermüller, M. Hinze, A SQP-Semi-Smooth Newton-type algorithm applied to control of the instationary Navier–Stokes system subject to control constraints, *SIAM J. Optim.* 16 (4) (2006) 1177–1200.
- [31] M. Hinze, A remark on second order methods in control of fluid flow, *Z. Angew. Math. Mech.* 81 (Suppl. 3) (2001).
- [32] F. Tröltzsch, S. Volkwein, The SQP method for control constrained optimal control of the Burgers equation, *ESAIM: Control Optim. Calc. Var.* 6 (2001) 649–674.
- [33] T. Weier, G. Gerbeth, Control of separated flows by time periodic Lorentz forces, *European J. Mech. B Fluids* 23 (2004) 835–849.
- [34] E. Casas, F. Tröltzsch, Second order necessary and sufficient optimality conditions for optimization problems and applications to control theory, *SIAM J. Optim.* 13 (2002) 406–431.
- [35] J. Hron, Fluid structure interaction with applications in biomechanics, Ph.D. Thesis, Math. -Phys. Faculty, Charles Univ., Praha, 2001.