7

A model fourth-order problem

7.1 Introductory results

In this chapter we shall study the properties of the solutions of the first boundary value problem for the biharmonic operator in a plane domain with a polygonal boundary. All the notation concerning the domain Ω will be the same as in Chapter 4. Our main goal is this: Given $f \in W_p^k(\Omega)$ with $1 < p < +\infty$, k integer $\geqslant -1$, we look for a solution $u \in W_p^{k+4}(\Omega)$ of

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ \gamma_j u = 0 & \text{on } \Gamma_j, \quad j = 1, 2, \ldots, N \\ \gamma_j \dfrac{\partial u}{\partial \nu_j} = 0 & \text{on } \Gamma_j, \quad j = 1, 2, \ldots, N. \end{cases} \tag{7,1,1}$$

The reason why it is useful to consider f given in $W_p^{-1}(\Omega)$ will appear clearly in Section 7.4, which is devoted to the related Stokes problem.

We shall start from a variational solution $u \in \mathring{H}^2(\Omega)$ to problem (7,1,1). Then localizing the problem near one corner, we shall apply the method introduced in Kondratiev (1967a), to study the behaviour of u near the corners. This is done in the framework of the weighted Sobolev spaces which we defined in Subsection 4.3.2. Then the trace theorems of Subsection 1.5.2 allow one to get rid of the weights in a very simple way. Finally we shall extend all the results to the case $p \neq 2$, by a technique using *a priori* estimates, very similar to those of Subsection 4.3.2.

First let us recall briefly the classical variational approach to the problem (7,1,1). We apply the Lax–Milgram lemma (see Lemma 2.2.1.1) with the following choice of V and a:

$$V = \mathring{H}^2(\Omega) = \left\{ u \in H^2(\Omega) \mid \gamma_j u = \gamma_j \frac{\partial u}{\partial \nu_j} = 0 \text{ on } \Gamma_j, j = 1, 2, \ldots, N \right\}$$

and

$$a(u; v) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial^2 u}{\partial x_i \, \partial x_j} \frac{\partial^2 v}{\partial x_i \, \partial x_j} \, dx.$$

It is obvious that

$$a(u;u)=\sum_{i,j=1}^{2}\left\|\frac{\partial^2 u}{\partial x_i\,\partial x_j}\right\|^2 \geq \alpha\,\|u\|^2_{2,2,\Omega}$$

for $\alpha>0$, due to Poincaré's inequality. Therefore a is a continuous and coercive bilinear form on V. Applying Lemma 2.2.1.1 we get the following result.

Lemma 7.1.1 *For any $f\in W_p^{-1}(\Omega)$ given, there exists a unique $u\in \mathring{H}^2(\Omega)$ solution of problem* (7,1,1).

Indeed we solve the variational problem:

$$a(u;v)=\langle f;v\rangle$$

for all $v\in \mathring{H}^2(\Omega)$. The Sobolev imbedding of $\mathring{H}^2(\Omega)$ into $\mathring{W}^1_q(\Omega)$ for any $q\in\,]1,\infty[$, implies that f is a continuous linear form on V.

In order to be able to study the behaviour of u only near the corners we need a smoothness result away from the corners.

Theorem 7.1.2 *Let $u\in \mathring{H}^2(\Omega)$ be the solution of the problem* (7,1,1) *with $f\in W_p^k(\Omega)$. Then $u\in W_p^{k+4}(\Omega\setminus V)$ for any neighbourhood V of the corners.*

The smoothness of u inside Ω is well known. Indeed we have $\varphi u\in W_p^{k+4}(\Omega)$ for every $\varphi\in\mathcal{D}(\Omega)$. The corresponding smoothness result near the sides Γ_j of Ω, deserves a proof. Here, we denote by $\mathbb{R}^2_+$ the half plane defined by $x_2>0$; γ is the trace operator on $\{x_2=0\}$.

Lemma 7.1.3 *Let $v\in W_p^l(\mathbb{R}^2_+)$, $l\geq 2$ be a solution with bounded support of*

$$\begin{cases}\Delta^2 v=g\in W_p^{l-3}(\mathbb{R}^2_+)\\[2mm] \gamma v=\gamma\dfrac{\partial v}{\partial x_2}=0.\end{cases}$$

Then $v\in W_p^{l+1}(\mathbb{R}^2_+)$

Proof We shall prove that $\gamma\Delta v\in W_p^{l-1-1/p}(\mathbb{R})$. The claim will follow from the known results for the Dirichlet problem for the Laplace equation (see Subsection 2.5.1) applied to $\psi=\Delta v$ and then to v.

First we shall prove a representation formula for v in terms of $(-\Delta+1)^2 v$, γv and $\gamma\,\partial v/\partial x_2$. For this purpose, we approximate v by a sequence of functions v_m, $m=1,2,\ldots$, such that

$$v_m\in W_p^l(\mathbb{R}^2_+)\cap H^4(\mathbb{R}^2_+)$$

and such that $v_m \to v$ in $W_p^l(\mathbb{R}_+^2)$. Then we write

$$v_m = w_m + E * E * Ph_m$$

where E is the elementary solution for $-\Delta + 1$ introduced in Subsection 2,

$$h_m = (-\Delta + 1)^2 v_m,$$

P is a continuation operator form $W_p^{l-3}(\mathbb{R}_+^2)$ into $W_p^{l-3}(\mathbb{R}^2)$ and from $L^2(\mathbb{R}_+^2)$ into $L^2(\mathbb{R}^2)$. Obviously we have

$$\begin{cases} (-\Delta + 1)^2 w_m = 0 & \text{in } \mathbb{R}_+^2 \\ \gamma w_m = \gamma v_m - \gamma E * E * Ph_m \in W_p^{l-1/p}(\mathbb{R}) \cap H^{7/2}(\mathbb{R}) \\ \gamma \dfrac{\partial w_m}{\partial x_2} = \gamma \dfrac{\partial v_m}{\partial x_2} - \gamma \dfrac{\partial}{\partial x_2} E * E * Ph_m \in W_p^{l-1-1/p}(\mathbb{R}) \cap H^{5/2}(\mathbb{R}). \end{cases}$$

Let us now perform a Fourier transform in x_1. We get

$$\begin{cases} (1 + \xi_1^2 - D_2^2)^2 \hat{w}_m = 0 & \text{a.e. in } \mathbb{R}_+^2 \\ \hat{w}_m(\xi_1, 0) = \hat{\varphi}_{0,m}(\xi_1) & \text{a.e. in } \mathbb{R} \\ D_2 \hat{w}_m(\xi_1, 0) = \hat{\varphi}_{1,m}(\xi_1) & \text{a.e. in } \mathbb{R}, \end{cases}$$

where $\varphi_{0,m} = \gamma w_m$ and $\varphi_{1,m} = \gamma D_2 w_m$. It follows that

$$\hat{w}_m(\xi_1, x_2) = \exp[-\surd(1 + \xi_1^2)x_2][\hat{\varphi}_{0,m}(\xi_1)\{1 + x_2\surd(1 + \xi_1^2)\} + \hat{\varphi}_{1,m}(\xi_1)x_2].$$

Accordingly we have

$$\Delta \hat{w}_m(\xi_1, 0) = -(1 + 2\xi_1^2)\hat{\varphi}_{0,m} - 2\surd(1 + \xi_1^2)\hat{\varphi}_{1,m}.$$

In other words, if we denote by T the operator defined as follows:

$$(T\hat{\varphi})(\xi_1) = \surd(1 + \xi_1^2)\hat{\varphi}(\xi_1),$$

we have

$$\gamma \Delta w_m = \varphi_{0,m} - 2T\varphi_{1,m} - 2T^2 \varphi_{0,m}.$$

Equivalently, we have

$$\begin{aligned} \gamma\, \Delta v_m &= \gamma\, \Delta w_m + \gamma\, \Delta E * E * Ph_m \\ &= (\gamma - 2T\gamma D_2 - 2T^2\gamma)(v_m - E * E * Ph_m) - \gamma\, \Delta E * E * Ph_m. \end{aligned}$$

Taking the limit in m, we obtain finally

$$\gamma\, \Delta v = (\gamma - 2T\gamma D_2 - 2T^2\gamma)(v - E * E * Ph) - \gamma\, \Delta E * E * Ph,$$

where $h = (-\Delta + 1)^2 v$. Due to the assumptions on v, we conclude that

$$\gamma\, \Delta v = -(\gamma - 2T\gamma D_2 - 2T^2\gamma + \gamma\Delta) E * E * P\{g - 2\, \Delta v + v\}.$$

The right-hand side of the identity belongs to $W_p^{l-1-1/p}(\mathbb{R})$ due to Theorem 2.3.2.1. ■

Proof of Theorem 7.1.2 It is a step by step proof using Lemma 7.1.3 at each step.

Assuming first that p is larger than two and that $f \in W_p^{-1}(\Omega)$ we show that $u \in W_p^2(\Omega \backslash V)$. Otherwise there is nothing to prove since $H^2(\Omega) \subset W_p^2(\Omega)$ when $p \leqslant 2$. Let us consider one of the sides Γ_j. After translation and rotation we can assume for convenience that Γ_j lies in $\{x_2 = 0\}$. Let us consider $\eta \in \mathcal{D}(\bar{\Omega})$ a cut-off function whose support is contained in $\{x_2 \geqslant 0\}$ and does not meet Γ_l for $l \neq j$. Finally let us define v by

$$v = \widetilde{\varphi u}.$$

Therefore $v \in H^2(\mathbb{R}_+^2)$, $\gamma v = \gamma\, \partial v/\partial x_2 = 0$ and

$$\Delta^2 v = \widetilde{\varphi f} + ([\Delta^2; \varphi]u)^{\sim} \in H^{-1}(\mathbb{R}_+^2).$$

Applying Lemma 7.1.3 shows that $v \in H^3(\mathbb{R}_+^2)$. Then varying φ and j shows that $u \in H^3(\Omega \backslash V) \subset W_p^2(\Omega \backslash V)$.

Now we assume that we know that $u \in W_p^l(\Omega \backslash V)$, $2 \leqslant l < k+3$ for every neighbourhood V of the corners and we show that this implies that $u \in W_p^{l+1}(\Omega \backslash V)$. Indeed we go through the same steps as before. Setting

$$v = \widetilde{\varphi u},$$

we have $v \in W_p^l(\mathbb{R}_+^2)$, $\gamma v = \gamma\, \partial v/\partial x_2 = 0$ and

$$\Delta^2 v = \widetilde{\varphi f} + ([\Delta^2; \varphi]u)^{\sim} \in W_p^{l-3}(\mathbb{R}_+^2).$$

Lemma 7.1.3 shows that $v \in W_p^{l+1}(\mathbb{R}_+^2)$ and therefore varying φ and j shows that $u \in W_p^{l+1}(\Omega \backslash V)$.

We conclude by induction. ■

In order to study the behaviour of u the solution of the problem (7,1,1), near one of the corners, say S_j, we use the related polar coordinates (r_j, θ_j) as in Subsection 4.3.2. We also use a cut-off function η_j which is equal to one near S_j and has a bounded support which does not intersect any of the $\bar{\Gamma}_l$ but Γ_j and Γ_{j+1}. Therefore $u_j = \widetilde{\eta_j u}$ is the solution of a boundary value problem in the infinite sector

$$G_j = \{r_j e^{i\theta_j} \mid r_j > 0, 0 < \theta_j < \omega_j\}.$$

Precisely we have $u_j \in \mathring{H}^2(G_j)$ and

$$\Delta^2 u_j = (\eta_j \Delta^2 u)^{\sim} + ([\eta_j; \Delta^2]u)^{\sim} = f_j \in W_p^k(G_j)$$

and in addition u_j has a bounded support. Dropping, for convenience, the subscript j, we are left with the problem of investigating the behaviour of

$u \in \mathring{H}^2(G)$ a solution with bounded support of the problem

$$\Delta^2 u = f \in W^k_p(G) \tag{7,1,2}$$

in an infinite plane sector G with angle ω.

7.2 Singular solutions, the L_2 case

7.2.1 Kondratiev's method in weighted spaces

In this section we shall study the problem (7,1,2) in the framework of the spaces $P^k_2(G)$ defined in Subsection 4.3.2. Briefly, we recall that a function u belongs to $P^k_p(G)$ iff

$$r^{-k+|\alpha|}D^\alpha u \in L_p(G), \qquad |\alpha| \leq k.$$

Theorem 4.3.2.2 has been useful in comparing the weighted space $P^k_p(G)$ with the usual Sobolev space $W^k_p(G)$; unfortunately it excluded the case when $p=2$. A corresponding weaker statement, when $p=2$, is the following:

Theorem 7.2.1.1 *Let $u \in \mathring{H}^k(G)$, then $u \in P^k_2(G)$.*

The proof of this result is quite similar to the corresponding part of the proof of Theorem 1.4.4.4.

For technical reasons which will become obvious later, it is also convenient to introduce a weighted space of order -1:

Definition 7.2.1.2 *We denote by $P^{-1}_p(G)$ the space of all the distributions*

$$T = \frac{g_0}{r} + D_1 g_1 + D_2 g_2 \tag{7,2,1,1}$$

where $g_j \in L_p(G)$, $0 \leq j \leq 2$.

A Banach norm on $P^{-1}_p(G)$ is the following.

$$T \mapsto \text{g.l.b.} \sum_{j=1}^{2} \|g_j\|_{0,p,G},$$

where the g.l.b. is taken with respect to all the functions g_j, $0 \leq j \leq 2$ belonging to $L_p(G)$ and such that (7,2,1,1) holds.

Now let us go back to the problem (7,1,2). Accordingly we consider

$$u \in P^2_2(G)$$

(remember Theorem 7.2.1.1) such that u has a bounded support and such that

$$\Delta^2 u = f \in P_2^k(G)$$

with $k \geq -1$ and such that $\gamma u = \gamma\, \partial u/\partial \nu = 0$. We shall denote by R a number such that

$$u(r, \theta) = 0 \qquad \text{for } r \geq R.$$

The Kondratiev method consists in performing the same change of variable $r = e^t$ as in Subsection 4.3.2 and then solving the problem by Fourier transform with respect to t. Thus we replace the equation $\Delta^2 u = f$ in the infinite sector G by a similar equation in an infinite strip

$$B = \mathbb{R} \times]0, \omega[.$$

This change of variable also replaces the weighted Sobolev spaces by ordinary Sobolev spaces. More precisely, the rule is the following.

Lemma 7.2.1.3 *Assume that $\varphi \in P_p^k(G)$ with $k \geq -1$ and define ψ by*

$$\psi(t, \theta) = \varphi(e^t \cos \theta, e^t \sin \theta) e^{(-k+2/p)t},$$

then $\psi \in W_p^k(B)$.

Proof This result is rather obvious when k is nonnegative. Consequently we leave its proof to the reader. However the case when k is -1 is less obvious and deserves a detailed proof.

From Definition 7.2.1.2 we know that

$$\varphi = \frac{g_0}{r} + D_1 g_1 + D_2 g_2$$

where $g_j \in L_p(G)$, $0 \leq j \leq 2$. Using polar coordinates this means that

$$\varphi = \frac{k_0}{r} + \frac{\partial k_1}{\partial r} + \frac{1}{r}\frac{\partial k_2}{\partial \theta},$$

where $k_j \in L_p(G)$, $j = 0, 1, 2$. Consequently we have

$$\begin{aligned}\psi(t; \theta) = e^{(2/p)t}\left\{k_0(e^t \cos \theta, e^t \sin \theta) - \frac{2}{p} k_1(e^t \cos \theta, e^t \sin \theta)\right\} \\ + D_t\{e^{(2/p)t} k_1(e^t \cos \theta, e^t \sin \theta)\} \\ + D_\theta\{e^{(2/p)t} k_2(e^t \cos \theta, e^t \sin \theta)\}.\end{aligned}$$

Here, each function

$$t, \theta \mapsto e^{(2/p)t} k_j(e^t \cos \theta, e^t \sin \theta), \qquad j = 0, 1, 2$$

belongs to $L_p(B)$. This clearly implies that ψ belongs to $W_p^{-1}(B)$. ∎

Since u belongs to $P_2^2(G)$ by assumption, we set

$$v(t,\theta)=e^{-t}u(e^t\cos\theta;\,e^t\sin\theta) \tag{7,2,1,2}$$

in accordance with Lemma 7.2.1.3. Hence

$$v\in \mathring{H}^2(B) \tag{7,2,1,3}$$

since, in addition, v and $\partial v/\partial\theta$ have zero traces on $F_0=\mathbb{R}\times\{0\}$ and $F_1=\mathbb{R}\times\{\omega\}$. The equation of v is

$$(D_t^4-2D_t^2+1)v+2(D_t^2+1)D_\theta^2 v+D_\theta^4 v=g \tag{7,2,1,4}$$

in B where

$$g(t,\theta)=e^{3t}f(e^t\cos\theta;\,e^t\sin\theta). \tag{7,2,1,5}$$

We observe that the assumption that f belongs to $P_2^k(G)$ implies that

$$e^{-(k+2)t}g\in H^k(B). \tag{7,2,1,6}$$

Finally, since u vanishes for $r\geq R$, it follows that v vanishes for $t\geq\log R$.

We recall that we define the partial Fourier transform of v with respect to t by

$$\hat v(\tau,\theta)=\frac{1}{\sqrt{(2\pi)}}\int_{\infty}^{+\infty}e^{-it\tau}v(t,\theta)\,dt. \tag{7,2,1,7}$$

Here τ is the dual variable of t and is possibly a complex number. From (7,2,1,3) and from the fact that v vanishes for $t\geq\log R$, we derive that $\hat v$ is defined for $\operatorname{Im}\tau\geq 0$, analytic in $\operatorname{Im}\tau>0$ and that

$$\left\{\sum_{j=0}^{2}\int_{-\infty}^{+\infty}|\tau_1+i\tau_2|^{4-2j}\,\|\hat v(\tau_1+i\tau_2,\theta)\|^2_{j,2,]0,\omega[}\,d\tau_1\right\}^{1/2}\leq R^{\tau_2}\|v\|_{2,2,B} \tag{7,2,1,8}$$

for every $\tau_2\geq 0$. Here we have applied the Paley–Wiener and Plancherel theorems.

In addition the function $\theta\mapsto\hat v(\tau,\theta)$ is, for almost every $\tau_1(\tau=\tau_1+i\tau_2$, $\tau_2\geq 0)$, an element of the space $\mathring{H}^2(]0,\omega[)$. This implies four boundary conditions on v (valid for every τ with $\operatorname{Im}\tau>0$ since $\hat v$ is analytic).

$$\hat v(\tau,0)=\hat v(\tau,\omega)=D_\theta\hat v(\tau,0)=D_\theta\hat v(\tau,\omega)=0. \tag{7,2,1,9}$$

From (7,2,1,4), we derive the equation of $\hat v$:

$$(\tau^4+2\tau^2+1)\hat v+(2-2\tau^2)D_\theta^2\hat v+D_\theta^4\hat v=\hat g. \tag{7,2,1,10}$$

For each τ (7,2,1,10) and (7,2,1,9) define a two-point boundary value problem for a fourth-order differential equation in $]0,\omega[$. Solving such a problem is easy. It is uniquely solvable away from a discrete set of characteristic values for τ. This will make $\hat v$ an analytic function of τ in any subset E of the complex plane where $\hat{g}$ is an analytic function of τ, away from the characteristic values. Eventually we take advantage of (7,2,1,6), from which we derive that $\hat{g}$ is defined for $\operatorname{Im}\tau \geq -(k+2)$. This will provide us with an analytic continuation for $\hat{v}$ which implies more regularity for v.

Now we shall make all the previous outline more precise, step by step. First we investigate how well posed the problem (7,2,1,9), (7,2,1,10) is. The characteristic equation for the differential equation is

$$p^4 + 2(1-\tau^2)p^2 + (\tau^4 + 2\tau^2 + 1) = 0$$

and its roots are $p = \pm\tau \pm \mathrm{i}$.† Accordingly, a fundamental system of solutions for the fourth-order equation is

$$\sin\theta \sinh \tau\theta, \quad \sin\theta \cosh\tau\theta, \quad \cos\theta \sinh\tau\theta, \quad \cos\theta\cosh\tau\theta$$

when τ is different from 0 and $\pm\mathrm{i}$. In the particular case when $\tau = 0$, a fundamental system of solutions is

$$\sin\theta, \quad \cos\theta, \quad \theta\sin\theta, \quad \theta\cos\theta$$

while in the particular case when $\tau = \pm\mathrm{i}$, a fundamental system of solutions is

$$1, \quad \theta, \quad \sin 2\theta, \quad \cos 2\theta.$$

For each τ, a Fredholm alternative holds; namely, for a given $\hat{g}$ the problem (7,2,1,9) (7,2,1,10) has a unique solution iff the corresponding homogeneous problem has only the zero solution. In other words we are reduced to checking whether the following problem has only the zero solution:

$$\begin{cases} (\tau^4 + 2\tau^2 + 1)\psi + 2(1-\tau^2)\psi'' + \psi^{(\mathrm{IV})} = 0 & \text{in }]0, \omega[\\ \psi(0) = \psi(\omega) = \psi'(0) = \psi'(\omega) = 0. \end{cases} \tag{7,2,1,11}$$

Later we shall call regular the values of τ for which (7,2,1,11) has only the zero solution.

Lemma 7.2.1.4 *The problem* (7,2,1,11) *has only the zero solution in the following cases*

(a) *τ is not a root of the characteristic equation*

$$\sinh^2(\tau\omega) = \tau^2 \sin^2\omega \tag{7,2,1,12}$$

(b) $\tau = 0$,
(c) $\tau = \pm\mathrm{i}$ *if* $\omega \neq \tan\omega$ *and* $\omega \neq 2\pi$.

† When $\tau = 0$, $p = \pm\mathrm{i}$ are double roots and when $\tau = \pm\mathrm{i}$, $p = 0$ is double root.

Consequently the characteristic values are the roots of (7,2,1,12) except 0 and $\pm$i in the general case and except 0 when $\tan\omega=\omega$.

Proof When τ is neither zero nor $\pm$i, ψ solution of the equation in (7,2,1,11) is of the following form

$$\psi=\sin\theta[\alpha\sinh\tau\theta+\beta\cosh\tau\theta]+\cos\theta[\gamma\sinh\tau\theta+\delta\cosh\tau\theta]$$

where α, β, γ, δ are complex numbers. Substituting ψ in the boundary conditions, one finds an homogeneous system of four equations in the four unknowns α, β, γ, δ. The corresponding determinant is

$$\sinh^2(\tau\omega)-\tau^2\sin^2\omega.$$

When $\tau=0$, ψ is of the particular form

$$\psi=\sin\theta[\alpha+\beta\theta]+\cos\theta[\gamma+\delta\theta].$$

The determinant is now

$$\sin^2\omega-\omega^2,$$

which is not zero.

Finally when $\tau=\pm\text{i}$, ψ is of the particular form

$$\psi=\alpha+\beta\theta+\gamma\sin 2\theta+\delta\cos 2\theta.$$

The corresponding determinant is proportional to

$$\sin\omega\,[\sin\omega-\omega\cos\omega].$$

This is not zero unless $\omega=\tan\omega$ or 2π.

In each case, $\alpha,\beta,\gamma,\delta$ are all zero and thus ψ vanishes unless the determinant is zero. ■

Let us denote by D the set of all the regular values (i.e. the noncharacteristic values) for the problem (7,2,1,10) (7,2,1,9). D is the complement of a discrete set in the complex plane. When $\tau\in D$ the problem has a unique solution

$$\hat{v}\in\mathring{H}^2(]0,\omega[)$$

provided $\hat{g}$ is given in $H^{-2}(]0,\omega[)$. In addition, let E be any open subset of the complex plane such that

$$\tau\mapsto\hat{g}$$

is analytic from E into $H^{-2}(]0,\omega[)$, then

$$\tau\mapsto\hat{v}$$

is analytic from $D\cap E$ into $\mathring{H}^2(]0,\omega[)$. Let us now find a subset E.

We recall that g, like u, vanishes for $t>\log R$. Applying the Paley–Wiener and Plancherel theorems, we derive from (7,2,1,6) that, when k is nonnegative g is defined for $\operatorname{Im}\tau\geqslant-(k+2)$, analytic for $\operatorname{Im}\tau>-(k+2)$ and in addition that

$$\left\{\sum_{j=0}^{k}\int_{-\infty}^{+\infty}|\tau_1+i(\tau_2+k+2)|^{2(k-j)}\|\hat{g}(\tau_1+i\tau_2,\theta)\|^2_{j,2,]0,\omega[}\,d\tau_1\right\}^{1/2}$$
$$\leqslant R^{\tau_2+(k+2)}\|e^{-(k+2)t}g\|_{k,2,B}. \tag{7,2,1,13}$$

A similar result when $k=-1$ deserves a detailed proof.

Lemma 7.2.1.5 *Assume that* $e^{-t}g\in H^{-1}(B)$ *and that* g *vanishes for* $t>R$. *Then* $\hat{g}$ *is defined in* $\operatorname{Im}\tau\geqslant-1$, *analytic in* $\operatorname{Im}\tau>-1$ *with values in* $H^{-1}(]0,\omega[)$. *Furthermore for each* $R'>R$, *there exists* $\hat{g}_1$ *and* $\hat{g}_2$ *such that*

$$\hat{g}=\hat{g}_1+\hat{g}_2$$

where $\hat{g}_1$, *respectively* $\hat{g}_2$, *is defined in* $\operatorname{Im}\tau\geqslant-1$, *analytic in* $\operatorname{Im}\tau>-1$ *with values in* $L_2(]0,\omega[)$, *respectively* $H^{-1}(]0,\omega[)$. *In addition there exists a constant* C *such that*

$$\left\{\int_{-\infty}^{+\infty}[|\tau_1+i\tau_2+i|^{-2}\|\hat{g}_1(\tau_1+i\tau_2,\theta)\|^2_{0,2,]0,\omega[}\right.$$
$$\left.+\|\hat{g}_2(\tau_1+i\tau_2,\theta\|^2_{-1,2,]0,\omega[}]\,d\tau_1\right\}^{1/2}\leqslant CR'^{(\tau_2+1)} \tag{7,2,1,14}$$

for every $\tau_2\geqslant-1$.

Proof The assumption that $e^{-t}g\in H^{-1}(B)$ implies that

$$e^{-t}g=f_0+D_tf_1+D_\theta f_2 \tag{7,2,1,15}$$

where $f_j\in L_2(B)$, $0\leqslant j\leqslant 2$. Now we fix $R'>R$; then with the help of a cut-off function we can modify the f_j in order that they vanish for $t\geqslant R'$. Taking the Fourier transform in (7,2,1,15) we derive that

$$\hat{g}(\tau-i,\theta)=\hat{f}_0(\tau,\theta)+i\tau\hat{f}_1(\tau,\theta)+D_\theta\hat{f}_2(\tau,\theta),$$

where the f_j are defined in $\operatorname{Im}\tau\geqslant0$, analytic in $\operatorname{Im}\tau>0$ and there exists a constant C_1 such that:

$$\int_{-\infty}^{+\infty}\|\hat{f}_j(\tau_1+i\tau_2,\theta)\|^2_{0,2,]0,\omega[}\,d\tau_1\leqslant C_1R'^{2\tau_2},\qquad 0\leqslant j\leqslant 2$$

for every $\tau_2\geqslant0$. One obtains the desired result by setting

$$\hat{g}_1(\tau,\theta)=i(\tau+i)\hat{f}_1(\tau+i,\theta)$$
$$\hat{g}_2(\tau,\theta)=\hat{f}_0(\tau+i,\theta)+D_\theta f_2(\tau+i,\theta).$$

Observe that this method of proof lets $\hat{g}_1$ and $\hat{g}_2$ depend on R'. ■

Going back to the problem (7,2,1,9) (7,2,1,10), we know that $\hat{v}$ is analytic in $\operatorname{Im}\tau>0$, while $\hat{g}$ is analytic in $\operatorname{Im}\tau>-(k+2)$. Consequently $\hat{v}$ has an analytic continuation to the domain

$$\{-(k+2)<\operatorname{Im}\tau\}\cap D$$

where D is the set of the regular values defined above. We still denote this continuation by $\hat{v}$. Furthermore, from (7,2,1,13), (7,2,1,14) we shall derive some growth condition on $\hat{v}$.

Lemma 7.2.1.6 *Assume that* (7,2,1,3), (7,2,1,6), (7,2,1,9) *and* (7,2,1,10) *hold and that v vanishes for $t\geq\log R$. Then there exists K such that*

$$\underset{-(k+2)\leq\tau_2\leq 0}{\text{l.u.b.}}\left\{\int_{|\tau_1|\geq K}\sum_{j=0}^{k+4}|\tau_1|^{2(k+4-j)}\|\hat{v}(\tau_1+i\tau_2,\theta)\|^2_{j,2,]0,\omega[}\,d\tau_1\right\}<\infty. \tag{7,2,1,16}$$

Proof The main step is to find a bound for $\hat{v}$ in term of $\hat{g}$ at least for large values of $|\tau_1|$. This is straightforward for $k=-1$ and $k=0$. Indeed we calculate $\langle D_\theta^{2l}\hat{v};\hat{g}\rangle$ for $l=0,1,2$, where the brackets denote the pairing between distributions and functions in $]0,\omega[$. Integrating by parts, we find a constant C and a number K such that

$$\sum_{j=0}^{k+4}|\tau_1|^{k+4-j}\|\hat{v}\|_{j,2,]0,\omega[}\leq C\|\hat{g}\|_{k,2,]0,\omega[} \tag{7,2,1,17}$$

for $k=-2,-1,0$, $|\tau_1|\geq K$, $-(k+2)\leq\tau_2\leq 0$.

It is not possible to estimate further derivatives of $\hat{v}$ by mere integration by parts. We shall prove the corresponding inequality later.

Lemma 7.2.1.7 *For every nonnegative integer k, there exists a constant C and a number K such that the solution $\hat{v}$ of* (7,2,1,9) (7,2,1,10) *verifies*

$$\sum_{j=0}^{k+4}|\tau_1|^{k+4-j}\|\hat{v}\|_{j,2,]0,\omega[}\leq C\sum_{j=0}^{k}|\tau_1|^{k-j}\|\hat{g}\|_{j,2,]0,\omega[} \tag{7,2,1,18}$$

for $|\tau_1|\geq K$, $-(k+2)\leq\tau_2\leq 0$.

It is clear that (7,2,1,16) follows from (7,2,1,13) and (7,2,1,18) when k is nonnegative. Now let us look at the case when k is -1. The inequality (7,2,1,17) implies in particular that the problem (7,2,1,9) (7,2,1,10) is well posed for $|\tau_1|\geq K$, $-1\leq\tau_2\leq 0$. Thus we can write

$$\hat{v}=\hat{v}_1+\hat{v}_2$$

where

$$(\tau^4+2\tau^2+1)\hat{v}_j+(2-2\tau^2)D_\theta^2\hat{v}_j+D_\theta^4\hat{v}_j=\hat{g}_j$$

$j=1,2$, with $\hat{g}_1$ and $\hat{g}_2$ given by Lemma 7.2.1.5 and $\hat{v}_1$, $\hat{v}_2$ fulfilling the boundary conditions (7,2,1,9). The inequality (7,2,1,16) follows by applying (7,2,1,17) with $k=0$ to $\hat{v}_1$ and with $k=-1$ to $\hat{v}_2$. ■

Proof of Lemma 7.2.1.7 First we consider an auxiliary problem on the half-line $\mathbb{R}_+=]0,\infty[$: $w\in \mathring{H}^2(\mathbb{R}_+)$ is a solution of

$$\tau_1^4 w-2\tau_1^2 w''+w^{(\mathrm{iv})}=h \qquad \text{in } \mathbb{R}_+. \tag{7,2,1,19}$$

It is clear that $w\in H^{k+4}(\mathbb{R}_+)$ when g is given in $H^k(\mathbb{R}_+)$. In addition, for $\tau_1=1$, there exists a constant K_k such that

$$\|w\|_{k+4,2,\mathbb{R}_+}\le K_k\,\|h\|_{k,2,\mathbb{R}_+}. \tag{7,2,1,20}$$

Then we observe that replacing θ by $\theta/|\tau_1|$ reduces the equation (7,2,1,19) to a similar one where $\tau_1=1$. Performing this change of variable in (7,2,1,20) leads to

$$\sum_{j=0}^{k+4}|\tau_1|^{k+4-j}\,\|w\|_{j,2,\mathbb{R}_+}\le K_k'\sum_{j=0}^{k}|\tau_1|^{k-j}\,\|h\|_{j,2,\mathbb{R}_+} \tag{7,2,1,21}$$

for all $\tau_1\in\mathbb{R}$.

Next it is easy to check that the same inequality holds for $w\in \mathring{H}^2(]0,\omega[)$ solution of the same equation in the interval $]0,\omega[$. (Use a cut-off function and continuation by zero, then apply inequality (7,2,1,21)).

Finally we can consider $\hat{v}$ solution of (7,2,1,9) (7.2.1.10). We observe that $w=\hat{v}$ is solution of

$$\tau_1^4\hat{v}-2\tau_1^2\hat{v}''+\hat{v}^{(\mathrm{iv})}=h \qquad \text{in }]0,\omega[,$$

where

$$h=\hat{g}+(\tau_1^4-\tau^4-2\tau^2-1)\hat{v}+(2\tau^2-2-2\tau_1^2)D_\theta^2\hat{v}.$$

Consequently we can apply the previous inequality to $\hat{v}$. We get

$$\sum_{j=0}^{k+4}|\tau_1|^{k+4-j}\,\|\hat{v}\|_{j,2,]0,\omega[}\le K_k''$$

$$\sum_{j=0}^{m}|\tau_1|^{k-j}\{\|\hat{g}\|_{j,2,]0,\omega[}+|\tau_1|^3\,\|\hat{v}\|_{j,2,]0,\omega[}+|\tau_1|\,\|\hat{v}\|_{j+2,2,]0,\omega[}\}$$

since we assume that $-(k+2)\le\tau_2\le 0$. It is now clear that we can choose K large enough such that (7,2,1,18) holds. ■

Remark 7.2.1.8 The inequality (7,2,1,18) is also a particular case of some more general inequalities proved in Agranovitch and Višik (1964).

We shall now consider the continuation $\hat{v}$ of the solution $\hat{v}$ of problem (7,2,1,9) (7,2,1,10) on the horizontal line

$$\tau_2 = -(k+2).$$

The function $\hat{v}$ is well defined almost everywhere on this line provided there is no characteristic value (for the problem (7,2,1,11)). In addition, it follows from (7,2,1,16) that

$$\int_{-\infty}^{+\infty} \sum_{j=0}^{k+4} |\tau_1|^{2(k+4-j)} \|\hat{v}(\tau_1 - \mathrm{i}[k+2])\|^2_{j,2,]0,\omega[} \, d\tau_1 < +\infty.$$

This inequality implies that

$$\tau_1 \mapsto \hat{v}(\tau_1 - \mathrm{i}[k+2])$$

is the Fourier transform of a function

$$w(t,\theta) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{+\infty} e^{\mathrm{i}t\tau_1} \hat{v}(\tau_1 - \mathrm{i}[k+2], \theta) \, d\tau_1$$

which belongs to $H^{k+4}(B)$.

It is easy to compare v with w. Indeed from (7,2,1,7) it follows that

$$v(t,\theta) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{+\infty} e^{\mathrm{i}t\tau_1} \hat{v}(\tau_1, \theta) \, d\tau_1$$

and by Cauchy's formula that

$$v(t,\theta) = e^{t(k+2)} w(t,\theta) + \sum_{-(k+2) < \operatorname{Im} \tau_m < 0} s_m(t,\theta) \tag{7,2,1,22}$$

where τ_m, $m = 1, 2, \ldots$ denotes the sequence of the characteristic values and s_m is the residue of

$$\tau \mapsto \mathrm{i}\sqrt{(2\pi)} e^{\mathrm{i}t\tau} \hat{v}(\tau, \theta)$$

at $\tau = \tau_m$.

We shall now calculate these residues. Some additional understanding of the characteristic values is necessary for this purpose.

Lemma 7.2.1.9 *Let τ_m be any characteristic value of the problem (7,2,1,11); then for $\tau = \tau_m$ the solutions of the problem (7,2,1,11) span a one-dimensional space. In addition let $\hat{g}$ be any analytic function in a neighbourhood of τ_m, with values in $H^{-2}(]0,\omega[)$, then the corresponding solution of the problem (7,2,1,9) (7,2,1,10) has the following Laurent* *expansion near τ_m:*

(a) $$\hat{v}(\tau,\theta)=\frac{\psi_m(\theta)}{\tau-\tau_m}+\hat{w}_m(\tau,\theta) \qquad (7,2,1,23)$$

where ψ_m is a solution of (7,2,1,11) *with $\tau=\tau_m$, and $\hat{w}_m$ is an analytic function near τ_m, with values in $H^2(]0,\omega[)$, provided τ_m is a simple root of* (7,2,1,12).

(b) $$\hat{v}(\tau,\theta)=\frac{\psi_m(\theta)}{(\tau-\tau_m)^2}+\frac{\varphi_m(\theta)}{\tau-\tau_m}+\hat{w}_m(\tau,0) \qquad (7,2,1,24)$$

with similar properties for ψ_m and $\hat{w}_m$, while φ_m is a solution of

$$\begin{aligned}&(\tau_m^4+2\tau_m^2+1)\varphi_m+2(1-\tau_m^2)\varphi_m''+\varphi_m^{(\mathrm{iv})}\\&\quad=-4\tau_m(\tau_m^2+1)\psi_m+4\tau_m\psi_m''\end{aligned} \qquad (7,2,1,25)$$

in $]0,\omega[$, with the boundary conditions (7,2,1,9) *provided τ_m is a double root of* (7,2,1,12).

Finally, the equation (7,2,1,12) *has no root with a multiplicity larger than two.*

Proof It is readily seen that any solution of the equation is (7,2,1,11) which fulfils the boundary conditions at zero, is a linear combination of u_1 and u_2 defined below:

$$u_1(\tau,\theta)=\frac{\sin\theta\sinh\tau\theta}{\tau}, \qquad \tau\neq 0$$

$$u_1(0,\theta)=\theta\sin\theta$$

$$u_2(\tau,\theta)=\frac{1}{\tau^2+1}\left\{\frac{\cos\theta\sinh\tau\theta}{\tau}-\sin\theta\cosh\tau\theta\right\}, \qquad \tau\neq 0,\pm\mathrm{i}$$

$$u_2(0,\theta)=\theta\cos\theta-\sin\theta$$

$$u_2(\pm\mathrm{i},\theta)=\tfrac{1}{2}\left\{\frac{\sin 2\theta}{2}-\theta\right\}.$$

These functions are entire analytic functions of τ. Then it follows from the general results about the two-point boundary value problems that $\hat{v}$, the solution of (7,2,1,9) (7,2,1,10), is such that $d\hat{v}$ is analytic near τ_m, where d is the determinant

$$\begin{aligned}d(\tau)&=u_1(\tau,\omega)D_\theta u_2(\tau,\omega)-u_2(\tau,\omega)D_\theta u_1(\tau,\omega)\\&=\frac{1}{(\tau^2+1)}\left[\sin^2\omega-\frac{\sinh^2\tau\omega}{\tau^2}\right].\end{aligned}$$

The zeros of d are described in Lemma 7.2.1.4. They are either the solutions of the equation (7,2,1,12) with $\tau \neq 0, \pm i$ or $\pm i$ in the particular case when $\omega = \tan \omega$ or $\omega = 2\pi$. In addition the order of the zeros of d is the multiplicity of the solutions of (7,2,1,12) when $\tau \neq 0, \pm i$, while $\pm i$ is a simple zero of d when $\tan \omega = \omega$ or $\omega = 2\pi$.

Differentiating the identity (7,2,1,12) with respect to τ shows that the multiplicity of the solutions is at most two.

If we assume that d has a simple zero at τ_m, then $\hat{v}$ has a simple pole at τ_m and consequently (7,2,1,23) holds. Applying the differential operator

$$(\tau^4 + 2\tau^2 + 1) + (2 - 2\tau^2)D_\theta^2 + D_\theta^4 = L(\tau, D_\theta)$$

to both sides of this identity multiplied by $(\tau - \tau_m)$ yields

$$(\tau - \tau_m)\hat{g} = L(\tau, D_\theta)\psi_m + (\tau - \tau_m)L(\tau, D_\theta)\hat{w}_m.$$

It follows obviously that $L(\tau_m, D_\theta)\psi_m = 0$. The boundary conditions on ψ_m are obvious and thus ψ_m is a solution of the homogeneous problem (7,2,1,11) at $\tau = \tau_m$.

Let us now assume that d has a double zero at τ_m, then $\hat{v}$ has a double pole at τ_m and consequently (7,2,1,24) holds. Applying the differential operator $L(\tau, D_\theta)$ and multiplying by $(\tau - \tau_m)^2$, we obtain

$$(\tau - \tau_m)^2\hat{g} = L(\tau, D_\theta)\psi_m + (\tau - \tau_m)L(\tau, D_\theta)\varphi_m + (\tau - \tau_m)^2 L(\tau, D_\theta)\hat{w}_m.$$

Again it is obvious that $L(\tau_m, D_\theta)\psi_m = 0$ and that ψ_m fulfils the boundary conditions in (7,2,1,11). Next we have

$$(\tau - \tau_m)\hat{g} = \frac{L(\tau; D_\theta)\psi_m}{\tau - \tau_m} + L(\tau; D_\theta)\varphi_m + (\tau - \tau_m)L(\tau, D_\theta)\hat{w}_m,$$

and consequently

$$L(\tau; D_\theta)\varphi_m = -\frac{L(\tau; D_\theta) - L(\tau_m; D_\theta)}{\tau - \tau_m}\psi_m + (\tau - \tau_m)\{\hat{g} - L(\tau, D_\theta)\hat{w}_m\}.$$

Taking the limit when $\tau \to \tau_m$ implies the equation for ψ_m, namely

$$L(\tau_m, D_\theta)\varphi_m = -4\tau_m(\tau_m^2 + 1)\psi_m + 4\tau_m\psi_m''.$$

Again, the boundary conditions on φ_m are obvious. ■

Remark 7.2.1.10 The existence of φ_m solution of (7,2,1,25) with the boundary conditions (7,2,1,9) is not obvious since τ_m is a characteristic value. Accordingly we must check that the right-hand side

$$-4\tau_m(\tau_m^2 + 1)\psi_m + 4\tau_m\psi_m'' = -L_\tau'(\tau_m; D_\theta)\psi_m$$

is orthogonal to the kernel of the transposed problem. In other words we must check that this function is orthogonal to every function η which is a solution of

$$\begin{cases} L(\bar{\tau}_m; D_\theta)\eta = 0 & \text{in }]0, \omega[\\ \eta(0) = \eta(\omega) = \eta'(0) = \eta'(\omega) = 0. \end{cases}$$

An auxiliary result for this verification, is the following.

Lemma 7.2.1.11 *The double solutions of the equation* (7,2,1,12) *are all imaginary.*

Proof All the solutions of (7,2,1,12) are solutions of either

$$\sinh(\tau\omega) = \tau \sin \omega \tag{7,2,1,26}$$

or

$$\sinh(\tau\omega) = -\tau \sin \omega \tag{7,2,1,27}$$

Let us consider the first of these equations, for instance, and assume that τ_m is a double root. We have

$$\sinh(\tau_m\omega) = \tau_m \sin \omega,$$

together with the differentiated equation

$$\omega \cosh(\tau_m\omega) = \sin \omega.$$

If we denote by ξ_m and η_m the real part and imaginary part of τ_m, respectively, we derive

$$\begin{cases} \sinh(\xi_m\omega)\cos(\eta_m\omega) = \xi_m \sin \omega \\ \cosh(\xi_m\omega)\sin(\eta_m\omega) = \eta_m \sin \omega \\ \omega \cosh(\xi_m\omega)\cos(\eta_m\omega) = \sin \omega \\ \sinh(\xi_m\omega)\sin(\eta_m\omega) = 0. \end{cases}$$

From the last equation, it follows that we have either $\xi_m = 0$ or $\eta_m = k\pi/\omega$, where k is an integer. Assuming that $\xi_m \neq 0$ and accordingly that $\eta_m = k\pi/\omega$, it follows from the second equation that $k = 0$, i.e. $\eta_m = 0$. Then the first equation yields

$$\frac{\sinh(\xi_m\omega)}{\xi_m\omega} = \frac{\sin \omega}{\omega}.$$

This equation is impossible since we have $|(\sinh t)/t| \geq 1$ for every t, while we have $|(\sin \omega)/\omega| < 1$.

In conclusion, all the double roots are such that $\xi_m = 0$. ■

Since τ_m, a double root, is imaginary, we have $\bar{\tau}_m = -\tau_m$ and consequently $L(\tau_m; D_\theta) = L(\bar{\tau}_m; D_\theta)$. Going back to the existence condition on φ_m we must check that $L'_\tau(\tau_m, D_\theta)\psi_m$ is orthogonal to all the solutions of the problem (7,2,1,11) with $\tau = \tau_m$.

Let $u(\tau, \theta)$ be the solution of

$$\begin{cases} L(\tau; D_\theta)u = 0 & \text{in }]0, \omega[, \\ u(\tau, 0) = 0, \\ D_\theta u(\tau, 0) = 0, \\ u(\tau, \omega) = 0, \\ D_\theta^2 u(\tau, 0) = 1. \end{cases}$$

It is easily seen that u exists and is unique near each characteristic value τ_m. The function $\psi_m(\theta)$ is a scalar multiple of $u(\tau_m, \theta)$, since the space of the solutions of the problem (7,2,1,11) is one-dimensional. Let us differentiate with respect to τ the identity

$$\int_0^\omega L(\tau; D_\theta)u(\tau; \theta)\overline{u(\tau; \theta)}\, d\theta = 0.$$

This yields

$$\int_0^\omega L'_\tau(\tau; D_\theta)u(\tau; \theta)\overline{u(\tau; \theta)}\, d\theta + \int_0^\omega L(\tau; D_\theta)u'_\tau(\tau, \theta)\overline{u(\tau, \theta)}\, d\theta = 0.$$

At $\tau = \tau_m$, we obtain

$$\int_0^\omega L'_\tau(\tau_m, D_\theta)u(\tau_m, \theta)\overline{u(\tau_m, \theta)}\, d\theta + \int_0^\omega u'_\tau(\tau_m, \theta)\overline{L(\tau_m; D_\theta)u(\tau_m; \theta)}\, d\theta = 0.$$

Consequently $L'_\tau(\tau_m, D_\theta)u(\tau_m; \theta)$ is orthogonal to $u(\tau_m, \theta)$ and the same way $L'_\tau(\tau_m, D_\theta)\psi_m(\theta)$ is orthogonal to ψ_m in $L_2(]0, \omega[)$. This shows that the solution φ_m of equation of (7,2,1,25) with the boundary conditions of (7,2,1,9) actually exists.

Now going back to identity (7,2,1,22), we have

$$v(t, \theta) = e^{t(k+2)}w(t, \theta) + \sum_{-(k+2)<\operatorname{Im}\tau'_m<0} i\sqrt{(2\pi)}e^{it\tau'_m}\psi_m(\theta) + \sum_{-(k+2)<\operatorname{Im}\tau''_m<0} i\sqrt{(2\pi)}e^{it\tau''_m}\{\varphi_m(\theta) + it\psi_m(\theta)\},$$

where we denote by τ'_m the characteristic values which are simple and by τ''_m the double ones.

Summing up, we have proved the following statement where we have performed the change of variable $r = e^t$ and used Lemma 7.2.1.3.

Theorem 7.2.1.12 *We assume that $u \in P_2^2(G)$ is a solution with bounded support of*

$$\Delta^2 u = f \qquad \text{in } G$$

with the boundary conditions

$$\gamma u = \gamma \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial G$$

We assume in addition, that $f \in P_2^k(G)$ with $k \geqslant -1$ and that the problem (7,2,1,11) *has no characteristic value on the line*

$$\operatorname{Im} \tau = -(k+2).$$

Then $u = u_r + u_s$, where $u_r \in P_2^{k+4}(G)$ and

$$u_s(r;\theta) = \sum_{-(k+2)<\operatorname{Im} \tau'_m<0} \mathrm{i}\sqrt{(2\pi)}\, r^{1+\mathrm{i}\tau'_m} \psi_m(\theta) + \sum_{-(k+2)<\operatorname{Im} \tau''_m<0} \mathrm{i}\sqrt{(2\pi)}^{1+\mathrm{i}\tau''_m} \{\varphi_m(\theta) + \mathrm{i}(\ln r)\psi_m(\theta)\} \qquad (7,2,1,28)$$

where τ'_m, $m = 1, 2, \ldots$ denotes the sequence of the simple roots of the characteristic equation (7,2,1,12) *(augmented with $\pm\mathrm{i}$ when $\tan \omega = \omega$), τ''_m, $m = 1, 2, \ldots$ denotes the sequence of the double roots of equation* (7,2,1,12), *ψ_m is a solution of* (7,2,1,11) *with $\tau = \tau'_m$ or τ''_m and finally φ_m is a solution of* (7,2,1,25) *and* (7,2,1,9) *with $\tau = \tau''_m$.*

This result holds in particular when f is given in $\mathring{H}^k(G)$ by Theorem 7.2.1.1. It implies that u_r belongs to $H^{k+4}(G)$.

In what follows it will be convenient to restate the expansion (7,2,1,28) in a slightly different way. We define the function s_m by

$$\begin{cases} (\tau_m^4 + 2\tau_m^2 + 1)s_m + 2(1-\tau_m^2)s''_m + s_m^{(\mathrm{iv})} = 0 & \text{in }]0, \omega[\\ s_m(0) = s_m(\omega) = s'_m(0) = s'_m(\omega) = 0 \end{cases} \qquad (7,2,1,29)$$

with the normalization condition

$$\int_0^\omega |s_m(\theta)|^2 \, d\theta = 1 \qquad (7,2,1,30)$$

and we define the function σ_m by

$$\begin{cases} (\tau_m''^4 + 2\tau_m''^2 + 1)\sigma_m + 2(1-\tau_m''^2)\sigma''_m + \sigma_m^{(\mathrm{iv})} \\ \quad = -4\tau''_m(\tau_m''^2 + 1)s_m + 4\tau''_m s''_m, & \text{in }]0, \omega[\\ \sigma_m(0) = \sigma_m(\omega) = \sigma'_m(0) = \sigma'_m(\omega) = 0 \end{cases} \qquad (7,2,1,31)$$

with the orthogonality condition

$$\int_0^\omega s_m(\theta)\overline{\sigma_m(\theta)}\,d\theta = 0. \tag{7,2,1,32}$$

The functions s_m and σ_m are uniquely determined and ψ_m is a multiple of s_m, say,

$$i\sqrt{(2\pi)}\psi_m = \lambda_m s_m$$

for some complex number λ_m; then obviously $i\sqrt{(2\pi)}\varphi_m - \lambda_m\sigma_m$ is a multiple of s_m, say,

$$i\sqrt{(2\pi)}\varphi_m = \lambda_m\sigma_m + \mu_m s_m$$

for another complex number μ_m. Accordingly we have

$$\begin{aligned} u_s(r,\theta) = & \sum_{-(k+2)<\operatorname{Im}\tau'_m<0} \lambda_m r^{1+i\tau'_m} s_m(\theta) \\ & + \sum_{-(k+2)<\operatorname{Im}\tau''_m<0} \lambda_m r^{1+i\tau''_m}\{\sigma_m(\theta) + i(\ln r)s_m(\theta)\} + \mu_m r^{1+i\tau''_m} s_m(\theta). \end{aligned} \tag{7,2,1,33}$$

Remark 7.2.1.13 When the assumption on the characteristic values of the problem (7,2,1,11) is not satisfied, one can prove some partial results. Indeed if there is some characteristic value on the line $\operatorname{Im}\lambda = -(k+2)$, there exists $\varepsilon > 0$ arbitrarily small such that there is no characteristic value on the line $\operatorname{Im}\lambda = -(k+2)+\varepsilon$. Again

$$\tau_1 \mapsto \hat{v}(\tau_1 - i[k+2] + i\varepsilon)$$

is the Fourier transform of a function belonging to $H^{k+3}(B)$. Consequently we can replace (7,2,1,22) by

$$v(t,\theta) = e^{t(k+2-\varepsilon)}\rho(t,\theta) + \sum_{-(k+2)+\varepsilon<\operatorname{Im}\tau_m<0} S_m(t,\theta)$$

where $\rho \in H^{k+3}(B)$. Then the corresponding expansion in Theorem 7.2.1.12 implies that $u = u_r + u_s$, where $u_r \in P_2^{k+3}(G)$ and u_s is given again by (7,2,1,28). This is not the best possible result but this will be technically convenient in Subsection 7.3.2.

Remark 7.2.1.14 It is easily checked that s_m is proportional to the function

$$-\sinh\tau_m\omega\{\sinh\tau_m\theta\sin(\theta-\omega)\} + \tau_m\sin\omega\{\sinh\tau_m(\theta-\omega)\sin\theta\}$$

and that σ_m is a linear combination of the function above and its derivative (with respect to τ_m), i.e. the function

$$\begin{aligned}&-\omega \cosh \tau_m''\omega\{\sinh \tau_m''\theta \sin(\theta-\omega)\}+\sin\omega\{\sinh \tau_m''(\theta-\omega)\sin\theta\}\\&-\sinh \tau_m''\omega\{\theta\cosh \tau_m''\theta\sin(\theta-\omega)\}\\&\quad+\tau_m''\sin\omega\{(\theta-\omega)\cosh\tau_m''(\theta-\omega)\sin\theta\}.\end{aligned}$$

Remark 7.2.1.15 One can derive similar results for the boundary value problem (7,2,1,3) (7,2,1,4) in a strip B whose width is $\omega=2\pi$. This takes care of fracture problems. The characteristic values are the numbers $\tau_m=-im/2$ with m an integer ($m\neq 0$). The multiplicities are 2 unless $|m|=2$. It is readily seen that the solutions of the homogeneous problem (7,2,1,11) span the two-dimensional space generated by the functions

$$g_m(\theta)=\sin(1+m/2)\theta-\frac{m+2}{m-2}\sin(-1+m/2)\theta$$

and

$$h_m(\theta)=\cos(1+m/2)\theta-\cos(-1+m/2)\theta.$$

This implies that the function ψ_m defined by (7,2,1,23) is a linear combination of g_m and h_m. On the other hand, when τ_m is a double root, ψ_m defined by (7,2,1,24) vanishes while φ_m is again a linear combination of g_m and h_m (this follows from the solvability condition for the problem (7,2,1,25)). Consequently the expansion (7,2,1,28) is simply:

$$u_s=\sum_{0<m<2(k+2)} r^{1+m/2}\{\alpha_m g_m(\theta)+\beta_m h_m(\theta)\},$$

where α_m and β_m are constants. Actually only the terms corresponding to odd values of m are relevant in this expansion (since the other terms are simply polynomial). Thus it will be convenient to relabel everything by replacing m by $2m-1$. The expansion (7,2,1,33) now has the following form:

$$u_s=\sum_{1\leq m<k+5/2}\lambda_m r^{m+1/2}s_m^{(1)}(\theta)+\nu_m r^{m+1/2}s_m^{(2)}(\theta),$$

where

$$s_m^{(1)}(\theta)=\sin(m+\tfrac{1}{2})\theta-\frac{2m+1}{2m-3}\sin(m-\tfrac{3}{2})\theta$$

and

$$s_m^{(2)}(\theta)=\cos(m+\tfrac{1}{2})\theta-\cos(m-\tfrac{3}{2})\theta.$$

Again we have $u-u_S\in P_2^{k+3}(G)$ according to Remark 7.2.1.13.

7.2.2 Getting rid of the weights

First we go back to the original problem (7,1,1) in a polygon Ω. This will be merely a matter of notation. Again we denote by Γ_j, $1 \leq j \leq N$, the sides of Ω, S_j being the final point of Γ_j. We denote by ω_j the measure of the angle at S_j and finally we use again the polar coordinates with origin at S_j.

Now we define the singular functions corresponding to each corner. The sequence $\lambda_{j,m}$, $m = 1, 2, \ldots$ denotes the set of all the roots of the characteristic equation

$$\sinh^2(\lambda\omega_j) = \lambda^2 \sin^2 \omega_j \tag{7,2,2,1}$$

excluding 0 and $+\mathrm{i}$ when $\tan \omega_j \neq \omega_j$ and excluding only zero when $\tan \omega_j = \omega_j$. Then the sequence $\lambda'_{j,m}$, $m = 1, 2, \ldots$ denotes the set of the simple roots of (7,2,2,1) including $+\mathrm{i}$ when $\tan \omega_j = \omega_j$. Finally $\lambda''_{j,m}$, $m = 1, 2, \ldots$ denotes the set of the double roots of (7,2,2,1). Accordingly we have

$$\{\lambda_{j,m}\}_{m=1,2,\ldots} = \{\lambda'_{j,m}\}_{m=1,2,\ldots} \cup \{\lambda''_{j,m}\}_{m=1,2,\ldots}$$

and the numbers $\lambda''_{j,m}$ are all imaginary. Next we set

$$\mathcal{S}_{j,m}(r_j, \theta_j) = r_j^{1+\mathrm{i}\lambda'_{j,m}} s_{j,m}(\theta_j)\eta_j(r_j \mathrm{e}^{\mathrm{i}\theta_j}), \tag{7,2,2,2}$$

where η_j is a cut-off function which is equal to one near S_j and vanishes near all the other corners and near all the sides but Γ_j and Γ_{j+1} and where $s_{j,m}$ is a solution of the equation.

$$(\lambda'^4_{j,m} + 2\lambda'^2_{j,m} + 1)s_{j,m} + 2(1 - \lambda'^2_{j,m})s''_{j,m} + s^{(\mathrm{iv})}_{j,m} = 0 \tag{7,2,2,3}$$

in $]0, \omega_j[$ with the boundary conditions

$$s_{j,m}(0) = s_{j,m}(\omega_j) = s'_{j,m}(0) = s'_{j,m}(\omega_j) = 0 \tag{7,2,2,4}$$

and the normalization condition

$$\int_0^{\omega_j} |s_{j,m}(\theta)|^2 \, \mathrm{d}\theta = 1.$$

We also set

$$\mathcal{T}_{j,m}(r_j, \theta_j) = r_j^{1+\mathrm{i}\lambda''_{j,m}} t_{j,m}(\theta_j)\eta_j(r_j \mathrm{e}^{\mathrm{i}\theta_j}), \tag{7,2,2,5}$$

where $t_{j,m}$ is a solution of the same problem as $s_{j,m}$ with $\lambda'_{j,m}$ replaced by $\lambda'_{j,m}$ and

$$\mathcal{U}_{j,m}(r_j, \theta_j) = r_j^{1+\mathrm{i}\lambda''_{j,m}}\{u_{j,m}(\theta_j) + \mathrm{i}(\ln r_j)t_{j,m}(\theta_j)\}\eta_j(r_j \mathrm{e}^{\mathrm{i}\theta_j}), \tag{7,2,2,6}$$

where $u_{j,m}$ is a solution of the equation

$$\begin{aligned}(\lambda''^4_{j,m} + 2\lambda''^2_{j,m} + 1)u_{j,m} &+ 2(1 - \lambda''^2_{j,m})u''_{j,m} + u^{(\mathrm{iv})}_{j,m} \\ &= -4\lambda''_{j,m}(\lambda''^2_{j,m} + 1)t_{j,m} + 4\lambda''_{j,m}t''_{j,m}\end{aligned} \tag{7,2,2,7}$$

in $]0, \omega_j[$ with the boundary conditions

$$u_{j,m}(0) = u_{j,m}(\omega_j) = u'_{j,m}(0) = u'_{j,m}(\omega_j) = 0 \tag{7,2,2,8}$$

and the orthogonality condition

$$\int_0^{\omega_j} u_{j,m}(\theta)\overline{t_{j,m}(\theta)}\, d\theta = 0.$$

Our starting point in this section is the following statement.

Theorem 7.2.2.1 *We assume that $u \in \mathring{H}^2(\Omega)$ is a solution of*

$$\Delta^2 u = f \qquad \text{in } \Omega$$

with $f \in \mathring{H}^k(\Omega)$, $k \geq -1$. (Let us agree for convenience that $\mathring{H}^{-1}(\Omega) = H^{-1}(\Omega)$.) We assume in addition that the equations (7,2,2,1) for $j = 1, 2, \ldots, N$, have no roots (other than $-i$) on the line

$$\operatorname{Im} \lambda = -(k+2)$$

and we exclude the case $k = -1$ when $\tan \omega_j = \omega_j$ for at least one value of j, $1 \leq j \leq N$. Then u belongs to the space spanned by $H^{k+4}(\Omega)$, the functions $\mathcal{S}_{j,m}$ which correspond to

$$-(k+2) < \operatorname{Im} \lambda'_{j,m} < 0$$

and the functions $\mathcal{T}_{j,m}$ and $\mathcal{U}_{j,m}$ which correspond to

$$-(k+2) < \operatorname{Im} \lambda''_{j,m_0} < 0.$$

Proof It follows from Theorem 7.1.2 that $u \in H^{k+4}(\Omega \setminus V)$ for any neighbourhood V of the corners. Then we proceed as we did at the end of the Subsection 7.1, considering $u_j = \eta_j u$ in the infinite sector G_j corresponding to the corner S_j (after a rotation and a translation, possibly). The function u_j belongs to $\mathring{H}^2(G_j)$, has a bounded support and

$$\Delta^2 u_j = f_j,$$

where $f_j \in \mathring{H}^k(G_j)$ has a bounded support and coincides with f near S_j. By Theorem 7.2.1.1, this implies that

$$f_j \in P_2^k(G_j).$$

We can conclude by applying Theorem 7.2.1.12 to u_j, $1 \leq j \leq N$.

The purpose of the remainder of this subsection is to eliminate the very unnatural assumption that f belongs to $\mathring{H}^k(\Omega)$ instead of $H^k(\Omega)$ (this is an actual assumption only when $k \geq 1$). This will be achieved with the help of a new trace theorem. Later in this chapter we shall also need a similar result in the framework of the Sobolev spaces related to L_p with

$p \neq 2$. This is the reason why we state and prove this trace theorem in the general case, at once.

Theorem 7.2.2.2 *Assume that* $k \geq -1$ *and* $1 < p < \infty$ *and let* $f \in W_p^k(\Omega)$ *and* $\varphi_j \in W_p^{k+4-1/p}(\Gamma_j)$, $\psi_j \in W_p^{k+3-1/p}(\Gamma_j)$, $1 \leq j \leq N$; *then there exists*

$$v \in W_p^{k+4}(\Omega)$$

such that

$$\begin{cases} \gamma_j v = \varphi_j, & 1 \leq j \leq N \qquad (7,2,2,9) \\ \gamma_j \dfrac{\partial v}{\partial \nu_j} = \psi_j, & 1 \leq j \leq N \qquad (7,2,2,10) \\ \Delta^2 v - f \in \mathring{W}_p^k(\Omega) & \qquad (7,2,2,11) \end{cases}$$

iff

$$\begin{cases} \varphi_j(S_j) = \varphi_{j+1}(S_j) \\ \dfrac{\partial \varphi_j}{\partial \tau_j}(S_j) = -\cos \omega_j \dfrac{\partial \varphi_{j+1}}{\partial \tau_{j+1}}(S_j) + \sin \omega_j\, \psi_{j+1}(S_j) \\ \psi_j(S_j) = -\sin \omega_j \dfrac{\partial \varphi_{j+1}}{\partial \tau_{j+1}}(S_j) - \cos \omega_j\, \psi_{j+1}(S_j) \end{cases} \qquad (7,2,2,12)$$

for $j = 1, 2, \ldots, N$, *and iff in addition*

$$-\cos \omega_j \frac{\partial^2 \varphi_j}{\partial \tau_j^2}(S_j) - \sin \omega_j \frac{\partial \psi_j}{\partial \tau_j}(S_j) = -\cos \omega_j \frac{\partial^2 \varphi_{j+1}}{\partial \tau_{j+1}^2}(S_j) + \sin \omega_j \frac{\partial \psi_{j+1}}{\partial \tau_{j+1}}(S_j) \qquad (7,2,2,13)$$

when $p > 2$ *or* $k \geq 0$ *and*

$$\int_0^{\delta_j} \left| \cos \omega_j \left\{ -\frac{\partial^2 \varphi_j}{\partial \tau_j^2}(x_j(-\sigma)) + \frac{\partial^2 \varphi_{j+1}}{\partial \tau_{j+1}^2}(x_j(\sigma)) \right\} + \sin \omega_j \left\{ -\frac{\partial \psi_j}{\partial \tau_j}\left(x_j(-\sigma) - \frac{\partial \psi_{j+1}}{\partial \tau_{j+1}}(x_j(\sigma))\right) \right\} \right|^2 \frac{d\sigma}{\sigma} < +\infty \qquad (7,2,2,14)$$

for $j = 1, 2, \ldots, N$ *when* $p = 2$ *and* $k = -1$.

(We again identify $N+1$ with 1; the notation δ_j, x_j has been introduced in Subsection 1.5.2.)

Proof First, we must view the property (7,2,2,11) as a trace property. Indeed $\Delta^2 v - f$ belongs to $\mathring{W}_p^k(\Omega)$ iff

$$\gamma_j \frac{\partial^l}{\partial \nu_j^l} \Delta^2 v = \gamma_j \frac{\partial^l f}{\partial \nu_j^l} \qquad \text{on } \Gamma_j \qquad (7,2,2,15)$$

for $0 \leqslant l \leqslant k-1$, $1 \leqslant j \leqslant N$. Consequently we look for a function v which satisfies (7,2,2,9), (7,2,2,10) and (7,2,2,15).

We shall solve this problem by applying Theorems 1.6.1.4 and 1.6.1.5. Accordingly, we have to define the operators $B_{j,l}$ which are involved in these statements. We set

$$\begin{cases} B_{j,1} = I \\ B_{j,2} = \dfrac{\partial}{\partial \nu_j} \\ B_{j,l} = \dfrac{\partial^{l-3}}{\partial \nu_j^{l-3}} \Delta^2, \qquad 3 \leqslant l \leqslant k+2. \end{cases} \tag{7,2,2,16}$$

The degree of $B_{j,l}$ is clearly $d_{j,l} = l-1$, when $l = 1, 2$ and $d_{j,l} = l+1$ when $l \geqslant 3$. The corresponding functions $f_{j,l}$ are

$$\begin{cases} f_{j,1} = \varphi_j \\ f_{j,2} = \psi_j \\ f_{j,l} = \gamma_j \dfrac{\partial^{l-3} f}{\partial \nu_j^{l-3}}, \qquad 3 \leqslant l \leqslant k+2. \end{cases} \tag{7,2,2,17}$$

Next we have to find the operators $P_{j,l}$ and $Q_{j+1,l}$ such that (1,6,1,1) holds. Due to Remark 1.6.1.8, we look for operators which are homogeneous and have constant coefficients. Consequently we have

$$P_{j,l} = a_l \left(\frac{\partial}{\partial \tau_j}\right)^{d-d_{j,l}}, \qquad Q_{j,l} = b_l \left(\frac{\partial}{\partial \tau_{j+1}}\right)^{d-d_{j+1,l}},$$

where for simplicity we do not make explicit the dependence of a_l and b_l also on j and d. The corresponding identity (1,6,1,1) reads as follows (when $d \geqslant 4$).

$$\begin{aligned} &a_1 \left(\frac{\partial}{\partial \tau_j}\right)^d + a_2 \left(\frac{\partial}{\partial \tau_j}\right)^{d-1} \frac{\partial}{\partial \nu_j} + \sum_{l=3}^{k+2} a_l \left(\frac{\partial}{\partial \tau_j}\right)^{d-l-1} \frac{\partial^{l-3}}{\partial \nu_j^{l-3}} \Delta^2 \\ &= b_1 \left(\frac{\partial}{\partial \tau_{j+1}}\right)^d + b_2 \left(\frac{\partial}{\partial \tau_{j+1}}\right)^{d-1} \frac{\partial}{\partial \nu_{j+1}} + \sum_{l=3}^{k+2} b_l \left(\frac{\partial}{\partial \tau_{j+1}}\right)^{d-l-1} \frac{\partial^{l-3}}{\partial \nu_{j+1}^{l-3}} \Delta^2. \end{aligned}$$

This implies the identity

$$a_1 \left(\frac{\partial}{\partial \tau_j}\right)^d + a_2 \left(\frac{\partial}{\partial \tau_j}\right)^{d-1} \frac{\partial}{\partial \nu_j} - b_1 \left(\frac{\partial}{\partial \tau_{j+1}}\right)^d - b_2 \left(\frac{\partial}{\partial \tau_{j+1}}\right)^{d-1} \frac{\partial}{\partial \nu_{j+1}} = R\Delta^2, \tag{7,2,2,18}$$

where R is a homogeneous differential operator of order $d-4$. Considering the corresponding symbols, this identity implies that the polynomial

$$a_1(-x\cos\omega_j - y\sin\omega_j)^d + a_2(-x\cos\omega_j - y\sin\omega_j)^{d-1} \times (y\cos\omega_j - x\sin\omega_j) - b_1x^d + b_2x^{d-1}y = S(x, y)$$

can be divided by $(x^2+y^2)^2$.

Equivalently, this means that $x = \pm iy$ are double roots of S. This yields the following system of equations for a_1, a_2, b_1, b_2:

$$a_1(-1)^d(\pm i\cos\omega_j + \sin\omega_j)^d + a_2(-1)^{d-1}(\pm i\cos\omega_j + \sin\omega_j)^{d-1} \times (\cos\omega_j \mp i\sin\omega_j) - b_1(\pm i)^d + b_2(\pm i)^{d-1} = 0$$

and

$$a_1d(-1)^d\cos\omega_j(\pm i\cos\omega_j + \sin\omega_j)^{d-1} + a_2\{(d-1)(-1)^{d-1}\cos\omega_j(\pm i\cos\omega_j + \sin\omega_j)^{d-2}(\cos\omega_j \mp i\sin\omega_j) + (-1)^d(\pm i\cos\omega_j + \sin\omega_j)^{d-1}\sin\omega_j\} - b_1d(\pm i)^{d-1} + b_2(d-1)(\pm i)^{d-2} = 0.$$

An easy but lengthy calculation shows that the corresponding determinant is proportional to

$$d(d-2)\sin^2\omega_j - \sin(d-2)\omega_j \sin d\omega_j.$$

Consequently the determinant does not vanish and the only solution of the system is the null solution.

Summing up, in the particular case under consideration here, there exists no nonzero operators $P_{j,l}$ and $Q_{j+1,l}$ such that (1,6,1,1) holds with $d \geqslant 4$. Let us now consider the cases when $0 \leqslant d < 4$.

First, when $d = 0$, we look for numbers a_1 and b_1 such that $a_1 = b_1$; the corresponding relation (1,6,1,2) is

$$\varphi_j(S_j) = \varphi_{j+1}(S_j). \tag{7,2,2,19}$$

Then, when $d = 1$ we look for numbers a_1, a_2, b_1, b_2 such that

$$a_1\frac{\partial}{\partial\tau_j} + a_2\frac{\partial}{\partial\nu_j} = b_1\frac{\partial}{\partial\tau_{j+1}} + b_2\frac{\partial}{\partial\nu_{j+1}}.$$

This can be any homogeneous first-order operator; the corresponding relations (1,6,1,2) are

$$\begin{cases} \dfrac{\partial\varphi_j}{\partial\tau_j}(S_j) = -\cos\omega_j\dfrac{\partial\varphi_{j+1}}{\partial\tau_{j+1}}(S_j) + \sin\omega_j\,\psi_{j+1}(S_j) \\ \psi_j(S_j) = -\sin\omega_j\dfrac{\partial\varphi_{j+1}}{\partial\tau_{j+1}}(S_j) - \cos\omega_j\,\psi_{j+1}(S_j), \end{cases} \tag{7,2,2,20}$$

since $\partial/\partial\tau_j$ and $\partial/\partial\nu_j$ generate all the first-order operators.

Next, when $d=2$ we look for numbers a_1, a_2, b_1, b_2 such that

$$a_1\left(\frac{\partial}{\partial\tau_j}\right)^2+a_2\frac{\partial^2}{\partial\tau_j\,\partial\nu_j}=b_1\left(\frac{\partial}{\partial\tau_{j+1}}\right)^2+b_2\frac{\partial^2}{\partial\tau_{j+1}\,\partial\nu_{j+1}}.$$

Equivalently, we look for numbers α_1, α_2, β_1, β_2 such that

$$\alpha_1\left(\frac{\partial}{\partial\tau_j}\right)^2+\alpha_2\frac{\partial^2}{\partial\tau_j\,\partial\tau_{j+1}}=\beta_1\left(\frac{\partial}{\partial\tau_{j+1}}\right)^2+\beta_2\frac{\partial^2}{\partial\tau_j\,\partial\tau_{j+1}}.$$

Obviously, we have $\alpha_1=\beta_1=0$ and $\alpha_2=\beta_2$. The corresponding relation (1,6,1,2) is

$$-\cos\omega_j\frac{\partial^2\varphi_{j+1}}{\partial\tau_{j+1}^2}(S_j)+\sin\omega_j\frac{\partial\psi_{j+1}}{\partial\tau_{j+1}}(S_j)=-\cos\omega_j\frac{\partial^2\varphi_j}{\partial\tau_j^2}(S_j)-\sin\omega_j\frac{\partial\psi_j}{\partial\tau_j}(S_j).\tag{7,2,2,21}$$

Finally, when $d=3$ we look for numbers a_1, a_2, b_1, b_2 such that

$$a_1\left(\frac{\partial}{\partial\tau_j}\right)^3+a_2\left(\frac{\partial}{\partial\tau_j}\right)^2\frac{\partial}{\partial\nu_j}=b_1\left(\frac{\partial}{\partial\tau_{j+1}}\right)^3+b_2\left(\frac{\partial}{\partial\tau_{j+1}}\right)^2\frac{\partial}{\partial\nu_{j+1}}.$$

Obviously the only solution is $a_1=a_2=b_1=b_2=0$, and there is no corresponding relation (1,6,1,2).

In conclusion the image of $W_p^{k+4}(\Omega)$ by the mapping

$$\{\gamma_jB_{j,l}\}_{1\le j\le N,\ 1\le l\le k+2}$$

is the subspace of

$$\prod_{j=1}^{N}\left(\prod_{l=1}^{k+2}W_p^{l+4-d_{j,l}-1/p}(\Gamma_j)\right)$$

defined by the conditions (7,2,2,19) (7,2,2,20) and (7,2,2,21), when $k\ge 0$ or when $k=-1$ and $p>2$. When $k=-1$ and $p<2$, only the conditions (7,2,2,19) and (7,2,2,20) occur, while, in the limit case $k=-1$ and $p=2$ (7,2,2,21) is replaced by the corresponding integral condition (the pattern being that (1,6,1,3) replaces (1,6,1,2)). This implies the claim of Theorem 7.2.2.2. ■

Going back to the particular case when $p=2$, we get the following consequence of Theorems 7.2.2.1 and 7.2.2.2:

Theorem 7.2.2.3 *Assume that $u\in H^2(\Omega)$ is a solution of*

$$\begin{cases}\Delta^2u=f & \text{in } \Omega\\ \gamma_ju=\varphi_j & \text{on } \Gamma_j,\quad 1\le j\le N\\ \gamma_j\dfrac{\partial u}{\partial\gamma_j}=\psi_j & \text{on } \Gamma_j,\quad 1\le j\le N\end{cases}\tag{7,2,2,22}$$

with $f \in H^k(\Omega)$, $\varphi_j \in H^{k+7/2}(\Gamma_j)$, $\psi_j \in H^{k+5/2}(\Gamma_j)$, $k \geqslant -1$ *that* (7,2,2,12) *holds and that* (7,2,2,13) *holds when* $k \geqslant 0$. *In addition we assume that* (7,2,2,14) *holds when* $k = -1$. *Finally we assume that the equations* (7,2,2,1) *for* $j = 1, 2, \ldots, N$ *have no root* (*other than* $-\mathrm{i}$) *on the line* $\operatorname{Im} \lambda = -(k+2)$ *and we exclude the case* $k = -1$, *when* $\tan \omega_j = \omega_j$ *for some* j. *Then* u *belongs to the space spanned by* $H^{k+4}(\Omega)$, *the functions* $\mathcal{S}_{j,m}$ *which correspond to* $\operatorname{Im} \lambda'_{j,m} \in]-(k+2), 0[$ *and the functions* $\mathcal{T}_{j,m}$ *and* $\mathcal{U}_{j,m}$ *which correspond to* $\operatorname{Im} \lambda''_{j,m} \in]-(k+2), 0[$.

Proof We merely apply Theorem 7.2.2.1 to

$$w = u - v$$

where $v \in H^{k+4}(\Omega)$ is a solution of (7,2,2,9) to (7,2,2,11) given by Theorem 7.2.2.2.

Remark 7.2.2.4 When the condition that no root of the equations (7,2,2,1) lies on the line $\operatorname{Im} = -(k+2)$ is not fulfilled, then from Remark 7.2.1.13 we conclude that u belongs to span of $H^{k+3}(\Omega)$ and the functions $\mathcal{S}_{j,m}$, $\mathcal{T}_{j,m}$ and $\mathcal{U}_{j,m}$ corresponding respectively to $\operatorname{Im} \lambda'_{j,m} \in [-(k+1), 0[$ and to $\operatorname{Im} \lambda''_{j,m} \in [-(k+1), 0[$ (see also Theorem 1.4.5.3).

Remark 7.2.2.5 If we allow cuts i.e. $\omega_j = 2\pi$ for some j, then the assumptions (7,2,2,12) to (7,2,2,14) must be replaced by the following:

$$\begin{cases} \dfrac{\partial^l \varphi_j}{\partial \tau_j^l}(S_j) = (-1)^l \dfrac{\partial^l \varphi_{j+1}}{\partial \tau_{j+1}^l}(S_j), & l < k+4-\dfrac{2}{p} \\[2ex] \dfrac{\partial^l \psi_j}{\partial \tau_j^l}(S_j) = -(-1)^l \dfrac{\partial^l \psi_{j+1}}{\partial \tau_{j+1}^l}(S_j), & l < k+3-\dfrac{2}{p} \end{cases} \tag{7,2,2,22}$$

for every p and by

$$\begin{cases} \displaystyle\int_0^{\delta_j} \left| \frac{\partial^l \varphi_j}{\partial \tau_j^l}(x_j(-\sigma)) - (-1)^l \frac{\partial^l \varphi_{j+1}}{\partial \tau_{j+1}^l}(x_j(\sigma)) \right|^2 \frac{\mathrm{d}\sigma}{\sigma} < +\infty, & l = k+3 \\[2ex] \displaystyle\int_0^{\delta_j} \left| \frac{\partial^l \psi_j}{\partial \tau_j^l}(x_j(-\sigma)) + (-1)^l \frac{\partial^l \psi_{j+1}}{\partial \tau_{j+1}^l}(x_j(\sigma)) \right|^2 \frac{\mathrm{d}\sigma}{\sigma} < +\infty, & l = k+2 \end{cases} \tag{7,2,2,23}$$

for $p = 2$. This result follows from Section 1.7.

Then we set $\lambda_{j,m} = -\mathrm{i}(m - \frac{1}{2})$ (m an integer) when $\omega_j = 2\pi$ and

$$\mathcal{S}^{(i)}_{j,m}(r_j, \theta_j) = r_j^{m+1/2} s^{(i)}_m(\theta_j) \eta_j(r_j e^{\mathrm{i}\theta_j}), \qquad i = 1, 2$$

where $s^{(i)}_m$, $i = 1, 2$ have been defined in the Remark 7.2.1.15.

The statement corresponding to the Theorem 7.2.2.3 is now that u, the solution of the problem (7,2,2,22), belongs to the span of $H^{k+3}(\Omega)$ and the functions

$$\mathcal{S}_{j,m}, \mathcal{T}_{j,m} \text{ and } \mathcal{U}_{j,m} \text{ corresponding to } \operatorname{Im} \lambda'_{j,m} \text{ (or } \lambda''_{j,m}) \in]-(k+2), 0[$$

when $\omega_j < 2\pi$ and the functions

$$\mathcal{S}^{(1)}_{j,m}, \mathcal{S}^{(2)}_{j,m} \text{ corresponding to } m < k + \tfrac{5}{2}$$

when $\omega_j = 2\pi$. This holds provided $f \in H^k(\Omega)$, $\varphi_j \in H^{k+7/2}(\Gamma_j)$, $\psi_j \in H^{k+5/2}(\Gamma_j)$, (7,2,2,12) and (7,2,2,13) hold when $\omega_j < 2\pi$, and (7,2,2,22) and (7,2,2,23) hold when $\omega_j = 2\pi$.

7.3 Singular solutions, the L_p case

7.3.1 *A priori* inequalities

Assuming that $1 < p < +\infty$ and $p \neq 2$, we shall prove the existence of a constant C such that

$$\|u\|_{k+4,p,\Omega} \leq C\{\|\Delta^2 u\|_{k,p,\Omega} + \|u\|_{k+3,p,\Omega}\} \tag{7,3,1,1}$$

for all $u \in W^{k+4}_p(\Omega)$ with $\gamma_j u = 0$ and $\gamma_j\, \partial u/\partial \nu_j = 0$ on Γ_j, $k \geq -1$, provided some conditions are satisfied by the angles. This inequality is similar to inequality (4,3,2,12) and we shall follow the same method of proof. Consequently the first step is the proof of the corresponding inequality for u belonging to the weighted space $P^{k+4}_p(\Omega)$ (see Definition 4.3.2.1). This is done locally by considering first the equation $\Delta^2 u = f$ in an infinite sector G.

Thus we consider

$$u \in P^{k+4}_p(G)$$

such that

$$\Delta^2 u = f \in P^k_p(G)$$

with the boundary conditions $\gamma u = \gamma\, \partial u/\partial \nu = 0$ on the boundary of G, which is the infinite sector defined, in polar coordinates, by

$$G = \{(r\cos\theta, r\sin\theta);\ r > 0,\ 0 < \theta < \omega\}.$$

We use the change of variable $r = e^t$ in order to obtain an equation in the strip $B = \mathbb{R} \times]0, \omega[$. Setting

$$\begin{cases} u(r\cos\theta, r\sin\theta) = v(t, \theta) \\ f(r\cos\theta, r\sin\theta) = g(t, \theta) \end{cases}$$

we obtain the equation

$$(D_t^4 - 4D_t^3 + 4D_t^2 + 2D_t^2D_\theta^2 - 4D_tD_\theta^2 + D_\theta^4 + 4D_\theta^2)v = e^{4t}g.$$

However, due to Lemma 7.2.1.3, it is more natural to consider

$$\begin{cases} w = e^{(-k-4+2/p)t}v \in W_p^{k+4}(B) \cap \mathring{W}_p^2(B) \\ h = e^{(-k+2/p)t}g \in W_p^k(B) \end{cases}$$

and the corresponding equation is

$$\begin{aligned}&[(D_t-\rho)^4 - 4(D_t-\rho)^3 + (4+2D_\theta^2)(D_t-\rho)^2 \\ &\quad - 4D_\theta^2(D_t-\rho) + D_\theta^4 + 4D_\theta^2]w = h\end{aligned} \tag{7,3,1,2}$$

in B, where $\rho = -k-4+2/p$ (i.e. $w = e^{\rho t}v$).

Now we use again the method of Section 4.2 We study the well posedness of the problem by performing a partial Fourier transform in t. Thus $\hat{w}(\tau, \theta)$ is a solution of

$$\begin{cases} [(i\tau-\rho)^4 - 4(i\tau-\rho)^3 + 4(i\tau-\rho)^2]\hat{w} \\ \quad +[2(i\tau-\rho)^2 - 4(i\tau-\rho)+4]\hat{w}'' + \hat{w}^{(iv)} = \hat{h} \qquad \text{in }]0, \omega[\\ \hat{w}(\tau, 0) = \hat{w}(\tau, \omega) = \hat{w}'(\tau, 0) = \hat{w}'(\tau, \omega) = 0 \end{cases} \tag{7,3,1,3}$$

for every $\tau \in \mathbb{R}$, where the superscript $'$ denotes the differentiation in θ. Here we assume in addition that w has a bounded support (in t) in order to give a meaning to its Fourier transform everywhere in τ.

In the particular case when $p=2$ and $k=-1$, the problem (7,3,1,3) coincides with the problem (7,2,1,9) (7,2,1,10). In the general case let us set

$$\mathscr{E} = +\tau + i(\rho+1),$$

then the equation in (7,3,1,3) is just

$$(\mathscr{E}^4 + 2\mathscr{E}^2 + 1)\hat{w} + (2 - 2\mathscr{E}^2)\hat{w}'' + \hat{w}^{(iv)} = \hat{h}.$$

Consequently the well posedness of the problem (7,3,1,3) has been investigated in Lemma 7.2.1.4.

Lemma 7.3.1.1 *The problem* (7,3,1,3) *has a unique solution for every real* τ *iff the characteristic equation*

$$\sinh^2(\lambda\omega) = \lambda^2 \sin^2\omega \tag{7,3,1,4}$$

has no solution on the line $\operatorname{Im}\lambda = -(k+1+2/q)$ (*assuming* $p \neq 2$ *and* $k \geq -1$).

From now on, we assume that this condition is fulfilled. The solution of the problem (7,3,1,3) can be written down explicitly through the use of a

Green function:

$$\hat{w}(\tau;\theta)=\int_0^\omega K(\tau;\theta,\theta')\hat{h}(\tau,\theta')\,d\theta',$$

where the kernel K is smooth in τ and three times continuously differentiable in θ and θ'.

More precisely, let us set

$$\begin{cases}\alpha(\theta)=\sin\theta\sinh\mathscr{E}\theta\\ \beta(\theta)=\sinh\mathscr{E}\theta\cos\theta-\mathscr{E}\cosh\mathscr{E}\theta\sin\theta\\ \gamma(\theta)=\sinh\mathscr{E}\theta\cos\theta+\mathscr{E}\cosh\mathscr{E}\theta\sin\theta\end{cases}$$

and

$$\delta=\mathscr{E}^2\sin^2\omega-\sinh^2\mathscr{E}\omega;$$

then we have

$$\begin{aligned}&K(\mathscr{E}-\mathrm{i}[\rho+1];\theta,\theta')\\ &=\delta^{-1}\left\{\frac{\beta(\omega)}{2\mathscr{E}}\alpha(\theta)\alpha(\omega-\theta')-\frac{\alpha(\omega)}{2\mathscr{E}}\alpha(\theta)\beta(\omega-\theta')\right.\\ &\qquad\left.-\frac{\gamma(\omega)}{2\mathscr{E}(1+\mathscr{E}^2)}\beta(\theta)\beta(\omega-\theta')-\frac{\alpha(\omega)}{2\mathscr{E}}\beta(\theta)\alpha(\omega-\theta')\right\}\\ &\qquad\qquad\text{for } 0\leqslant\theta\leqslant\theta'\leqslant\omega\end{aligned}$$

and

$$\begin{aligned}&\delta^{-1}\left\{\frac{\beta(\omega)}{2\mathscr{E}}\alpha(\omega-\theta)\alpha(\theta')-\frac{\alpha(\omega)}{2\mathscr{E}}\beta(\omega-\theta)\alpha(\theta')\right.\\ &\qquad\left.-\frac{\gamma(\omega)}{2\mathscr{E}(1+\mathscr{E}^2)}\beta(\omega-\theta)\beta(\theta')-\frac{\alpha(\omega)}{2\mathscr{E}}\alpha(\omega-\theta)\beta(\theta')\right\}\\ &\qquad\qquad\text{for } 0\leqslant\theta'\leqslant\theta\leqslant\omega.\end{aligned}$$

It is not hard to check that the kernels K, τK and $D_{\theta'}K$ fulfil the conditions of Lemma 4.2.1.3. This yields the following inequality.

Theorem 7.3.1.2 *Assume that* $p\neq 2$, $k\geqslant -1$ *and that the characteristic equation* (7,3,1,4) *has no solution on the line* $\operatorname{Im}\lambda=-(k+1+2/q)$. *Then there exists a constant* C *such that*

$$\|w\|_{0,p,B}\leqslant C\,\|h\|_{k,p,B} \tag{7,3,1,5}$$

for every $w\in W_p^{k+4}(B)\cap \mathring{W}_p^2(B)$ *such that* (7,3,1,2) *holds.*

Proof When $k\geqslant 0$, we apply directly Lemma 4.2.1.3 and actually show that

$$\|w\|_{0,p,B}\leqslant\|h\|_{0,p,B}.$$

In the particular case when $k=-1$, we write

$$h = g_0 + D_t g_1 + D_\theta g_2$$

with $g_j \in L_p(B)$, $0 \leq j \leq 2$. Consequently we have

$$\hat{w}(\tau;\theta) = \int_0^\omega K(\tau;\theta,\theta')\hat{g}_0(\tau,\theta')\,d\theta' + \int_0^\omega i\tau K(\tau;\theta,\theta')\hat{g}_1(\tau,\theta)\,d\theta' - \int_0^\omega D_{\theta'}K(\tau;\theta,\theta')\hat{g}_2(\tau,\theta')\,d\theta'$$

since $K(\tau;\theta,0) = K(\tau;\theta,\omega) = 0$. Then applying Lemma 4.2.1.3 three times, we obtain

$$\|w\|_{0,p,B} \leq C\left\{\sum_{j=0}^{2} \|g_j\|_{0,p,B}\right\}$$

and consequently

$$\|w\|_{0,p,B} \leq C' \|h\|_{-1,p,B}$$

for some other constant C'.

Actually these estimates have been derived only for a w which has a bounded support in t, but it is easy to deduce the general case by taking limits. Indeed the functions with bounded support are dense in $W_p^{k+4}(B) \cap \mathring{W}_p^2(B)$ (use cut-off functions). ∎

Inequality (7,3,1,5) is just the analogue of inequality (4,2,1,4). Now we must find bounds for the derivatives of order $k+4$ of w. We proceed as in Subsection 4.2.2, namely we will neglect some non leading terms in the equation (7,3,1,2). For this purpose we rewrite the equation as follows

$$(D_t^4 + 2D_\theta^2 D_t^2 + D_\theta^4)w = h_1, \tag{7,3,1,6}$$

where clearly there exists some constant such that

$$\|h_1\|_{k,p,B} \leq \|h\|_{k,p,B} + C\,\|w\|_{k+3,p,B}. \tag{7,3,1,7}$$

Lemma 7.3.1.3 *There exists a constant C such that*

$$\|w\|_{k+4,p,B} \leq C\{\|\Delta^2 w\|_{k,p,B} + \|w\|_{k+3,p,B}\}$$

for every $w \in W_p^{k+4}(B) \cap \mathring{W}_p^2(B)$.

Before proving this lemma, let us take some preliminary steps. We shall start from the inequality

$$\|u\|_{2,p,B} \leq C_0\{\|\Delta u\|_{0,p,B} + \|u\|_{1,p,B}\} \tag{7,3,1,8}$$

which follows from (4,2,2) (see Theorem 4.2.2.4) for every $u \in W_p^2(B) \cap \mathring{W}_p^1(B)$. It implies the following.

Lemma 7.3.1.4 *For each integer $k \geqslant 0$ there exists a constant C_k such that*

$$\|u\|_{k+2,p,B} \leqslant C_k\{\|\Delta u\|_{k,p,B} + \|u\|_{k+1,p,B}\} \tag{7,3,1,9}$$

for every $u \in W_p^{k+2}(B) \cap \mathring{W}_p^1(B)$.

Proof Since we already know this result for $k=0$, we can proceed by induction. Thus we consider $u \in W_p^{k+2}(B) \cap \mathring{W}_p^1(B)$, assuming that C_{k-1} exists. We have to estimate the L_p norm of the derivatives of order $k+2$ of u. Applying (7,3,1,9) to

$$D_t u \in W_p^{k+1}(B) \cap \mathring{W}_p^1(B),$$

we get the desired bound for all these derivatives but $D_\theta^{k+2}u$. We conclude by writing

$$D_\theta^{k+2}u = D_\theta^k \Delta u - D_\theta^k D_t^2 u.$$

It follows that

$$\|D_\theta^{k+2}u\|_{0,p,B} \leqslant \|\Delta u\|_{k,p,B} + \|D_t u\|_{k+1,p,B}. \quad \blacksquare$$

Now using the trace theorem it is easy to deduce the following result:

Corollary 7.3.1.5 *For each integer $k \geqslant 0$ there exists a constant L_k such that*

$$\|u\|_{k+2,p,B} \leqslant L_k\left\{\|\Delta u\|_{k,p,B} + \sum_{j=0}^{1} \|\gamma_j u\|_{k+2-1/p,p,F_j} + \|u\|_{k+1,p,B}\right\} \tag{7,3,1,10}$$

for every $u \in W_p^{k+2}(B)$.

We recall that F_0 and F_1 denote the two components of the boundary of the strip B, i.e.

$$F_j = \mathbb{R} \times \{j\omega\}, \qquad j = 0, 1$$

We are now going to apply the inequality (7,3,1,10) to Δw and then to w. Clearly we have

$$\|w\|_{k+4,p,B} \leqslant L_{k+2}\{\|\Delta w\|_{k+2,p,B} + \|w\|_{k+3,p,B}\}. \tag{7,3,1,11}$$

Then we have to estimate $\|\Delta w\|_{k+2,p,B}$. We will apply (7,3,1,10) to w now. We get

$$\|\Delta w\|_{k+2,p,B} \leqslant L_k\left\{\|\Delta^2 w\|_{k,p,B} + \sum_{j=0}^{1} \|\gamma_j \Delta w\|_{k+2-1/p,p,F_j} + \|\Delta w\|_{k+1,p,B}\right\}. \tag{7,3,1,12}$$

The last step is the proof of an estimate for $\gamma_j \Delta w = \gamma_j D_\theta^2 w$ (since $\gamma_j D_t^2 w = D_t^2 \gamma_j w = 0$) for $j = 0, 1$.

Lemma 7.3.1.6 *Let $\varphi \in H^3(B)$ be such that*

$$(-\Delta+1)^2\varphi=0 \qquad \text{in } B$$

then we have

$$\begin{aligned}(\gamma_0 D_\theta^2\varphi)^\wedge(\tau) = \rho^2(\gamma_0\varphi)^\wedge(\tau) + 2\rho\{\sinh^2\rho\omega - \rho^2\omega^2\}^{-1} \\ \times\{\rho^2\omega \sinh \rho\omega(\gamma_1\varphi)^\wedge(\tau) \\ +[\sinh\rho\omega - \rho\omega\cosh\rho\omega](\gamma_1 D_\theta\varphi)^\wedge(\tau) \\ -\rho\sinh\rho\omega(\gamma_0\varphi)^\wedge(\tau) \\ +[\rho\omega - \sinh\rho\omega\cosh\rho\omega](\gamma_0 D_\theta\varphi)^\wedge(\tau)\}\end{aligned}$$

where $\rho=\sqrt{(1+\tau^2)}$.

Proof It is a simple calculation: we write that $\hat\varphi$, the Fourier transform of φ in t, is solution of the differential equation

$$(-D_\theta^2+1+\tau^2)^2\hat\varphi=0 \qquad \text{in }]0,\omega[.$$

Consequently we have

$$\hat\varphi(\tau,\theta)=(a(\tau)+b(\tau)\theta)\sinh\rho\theta+(c(\tau)+d(\tau)\theta)\cosh\rho\theta.$$

The explicit value of the functions a, b, c and d is obtained by substituting the above expression for φ in $\gamma_0\varphi$, $\gamma_1\varphi$, $\gamma_0 D_\theta\varphi$ and $\gamma_1 D_\theta\varphi$. ■

Obviously we have a similar formula (*mutatis mutandis*) for $(\gamma_1 D_\theta^2\varphi)^\wedge(\tau)$.

Actually we need a consequence of Lemma 7.3.1.6:

Corollary 7.3.1.7 *For every $k \geq -1$, there exists a constant K_k such that*

$$\|\gamma_j \Delta w\|_{k+2-1/p,p,F_j} \leq K_k \|(-\Delta+1)^2 w\|_{k,p,B} \tag{7,3,1,13}$$

$j=0, 1$, for every $w \in W_p^{k+4}(B)\cap \mathring{W}_p^2(B)$, $p\geq 2$.

Proof Let us set $\psi=(-\Delta+1)^2 w$ and assume in addition that w has a bounded support. Thus $\psi\in H^{-1}(B)\cap W_p^k(B)$. Let $l\in H^{-1}(\mathbb{R}^2)\cap W_p^k(\mathbb{R}^2)$ be a continuation of ψ out of B such that

$$\|l\|_{k,p,\mathbb{R}^2} \leq C_1\|\psi\|_{k,p,B} \tag{7,3,1,14}$$

for some constant C_1.

Then consider the elementary solution E for $-\Delta+1$ defined by

$$FE(\xi)=(|\xi|^2+1)^{-1}, \qquad \xi\in\mathbb{R}^2.$$

We write $w=E*E*l|_B+\varphi$. By the multiplier theorem 2.3.2.1, we know that

$$\|E*E*l\|_{k+4,p,\mathbb{R}^2} \leq C_2 \|l\|_{k,p,\mathbb{R}^2}. \tag{7,3,1,15}$$

Then we have $\varphi \in H^3(B) \cap W_p^{k+4}(B)$ and in addition

$$(-\Delta+1)^2 = 0 \qquad \text{in } B.$$

Therefore we can use the explicit formula for $\gamma_j \Delta\varphi$ given by Lemma 7.3.1.6. Applying again the multiplier theorem (or rather its corollary, Lemma 2.3.2.5) we obtain

$$\|\gamma_j \Delta\varphi\|_{k+2-1/p,p,F_j} \leq C_3 \sum_{j=0}^{1} \left\{ \|\gamma_j\varphi\|_{k+4-1/p,p,F_j} + \left\|\gamma_j \frac{\partial\varphi}{\partial\theta}\right\|_{k+3-1/p,p,F_j} \right\}. \tag{7,3,1,16}$$

On the other hand we have

$$\gamma_j\varphi = -\gamma_j(E*E*l)$$

$$\gamma_j \frac{\partial\varphi}{\partial\theta} = -\gamma_j \frac{\partial}{\partial\theta}(E*E*l)$$

and accordingly

$$\|\gamma_j\varphi\|_{k+4-1/p,pF_j} + \left\|\gamma_j \frac{\partial\varphi}{\partial\theta}\right\|_{k+3-1/p,p,F_j} \leq C_4 \|E*E*l\|_{k+4,p,B}. \tag{7,3,1,17}$$

Summing up the inequalities (7,3,1,14) to (7,3,1,17) imply that

$$\begin{aligned}\|\gamma_j \Delta w\|_{k+2-1/p,p,F_j} &\leq \|\gamma_j \Delta E*E*l\|_{k+2-1/p,p,F_j} + \|\gamma_j \Delta\varphi\|_{k+2-1/p,p,F_j} \\ &\leq C_5 \|E*E*l\|_{k+4,p,B} \leq C_2C_5 \|l\|_{k,p,\mathbb{R}^2} \\ &\leq C_1C_2C_5 \|\psi\|_{k,p,B} = C_1C_2C_5 \|(-\Delta+1)^2 w\|_{k,p,B}.\end{aligned}$$

This is the desired result when w has a bounded support. The general case follows by a density argument. ■

We are now able to derive the Lemma 7.3.1.3.

Proof of Lemma 7.3.1.3 From (7,3,1,11) and (7,3,1,12) we easily find a constant C such that

$$\|w\|_{k+4,p,B} \leq C\left\{ \|\Delta^2 w\|_{k,p,B} + \|w\|_{k+3,p,B} + \sum_{j=0}^{1} \|\gamma_j \Delta w\|_{k+2-1/p,p,F_j} \right\}.$$

Then applying inequality (7,3,1,13) to estimate $\gamma_j \Delta w$, we obtain the desired result. ■

We conclude this subsection with a result which summarizes Theorem 7.3.1.2 and Lemma 7.3.1.3:

Theorem 7.3.1.8 *Assume that* $1<p<+\infty$ *and* $p\neq 2$, $k\geq -1$ *and that the characteristic equation* (7,3,1,4) *has no solution on the line* $\operatorname{Im}\lambda = -(k+1+2/q)$. *Then there exists a constant* C *such that*

$$\|w\|_{k+4,p,B} \leq C\,\|h\|_{k,p,B} \tag{7,3,1,18}$$

for every $w\in W_p^{k+4}(B)\cap \mathring{W}_p^2(B)$ *such that* (7,3,1,2) *holds.*

Proof This is a direct consequence of the inequalities (7,3,1,5) (7,3,1,7) (1,4,3,2) and Lemma 7.3.1.3. ∎

Going back to the original coordinates ($r=\ln t$) we have proved that

$$\|u\|_{P_p^{k+4}(G)} \leq C\,\|\Delta^2 u\|_{P_p^k(G)}$$

provided $u\in P_p^{k+4}(G)$ and $\gamma u=\gamma\,\partial u/\partial\nu=0$ on ∂G. Finally with the help of Theorem 7.1.2 and a partition of unity we deduce the following:

Corollary 7.3.1.9 *Assume that* $1<p<\infty$, $p\neq 2$, $k\geq -1$ *and that the equations*

$$\sinh^2(\lambda\omega_j)=\lambda^2\sin^2\omega_j$$

have no solution on the line $\operatorname{Im}\lambda=-(k+1+2/q)$ *for any* $j=1,2,\ldots,N$. *Then there exists a constant* C *such that the inequality* (7,3,1,1) *holds for every*

$$u\in W_p^{k+4}(\Omega)\cap \mathring{W}_p^2(\Omega).$$

Proof First we obtain directly the inequality

$$\|u\|_{P_p^{k+4}(\Omega)} \leq C'\{\|\Delta^2 u\|_{P_p^k(\Omega)}+\|u\|_{k+3,p,\Omega}\}.$$

Then by Theorem 4.3.2.2 we know that $P_p^{k+4}(\Omega)$ has just a finite codimension in $W_p^{k+4}(\Omega)$. This implies the inequality (7,3,1,1) with, possibly, another constant (see the method of proof of Theorem 4.3.2.4). ∎

7.3.2 Smoothness

We extend now the results in Theorem 7.2.2.3 to the general case $1<p<+\infty$.

Theorem 7.3.2.1 *Assume that* $u\in H^2(\Omega)$ *is a solution of*

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega\\ \gamma_j u=\varphi_j & \text{on } \Gamma_j,\quad 1\leq j\leq N\\ \gamma_j\dfrac{\partial u}{\partial\nu_j}=\psi_j & \text{on } \Gamma_j,\quad 1\leq j\leq N \end{cases} \tag{7,3,2,1}$$

with $f \in W_p^k(\Omega)$, $\varphi_j \in W_p^{k+4-1/p}(\Gamma_j)$, $\psi_j \in W_p^{k+3-1/p}(\Gamma_j)$, $k \geq -1$ *such that* (7,2,2,12) *holds in any case and such that* (7,2,2,13) *holds when either* $k \geq 0$ *or* $p > 2$ *and* (7,2,2,14) *holds when* $k = -1$ *and* $p = 2$. *Assume in addition that the equations* (7,2,2,1) *for* $j = 1, 2, \ldots, N$ *have no root* (*other than* $-i$) *on the line*

$$\operatorname{Im} \lambda = -(k + 1 + 2/q)$$

and exclude finally the case $k = -1$ *when* $\tan \omega_j = \omega_j$ *for some* j. *Then* u *belongs to the space spanned by* $W_p^{k+4}(\Omega)$, *the functions* $\mathcal{S}_{j,m}$ *which correspond to*

$$\operatorname{Im} \lambda'_{j,m} \in \left]-\left(k+1+\frac{2}{q}\right), 0\right[\tag{7,3,2,2}$$

and the functions $\mathcal{T}_{j,m}$ *and* $\mathcal{U}_{j,m}$ *which correspond to*

$$\operatorname{Im} \lambda''_{j,m} \in \left]-\left(k+1+\frac{2}{q}\right), 0\right[\tag{7,3,2,3}$$

Proof First we approximate the data of the problem (7,3,2,1) by better ones. Indeed Theorem 7.2.2.2 shows that there exists $v \in W_p^{k+4}(\Omega)$ such that

$$\begin{cases} \gamma_j v = \varphi_j, & 1 \leq j \leq N \\ \gamma_j \dfrac{\partial v}{\partial \nu_j} = \psi_j, & 1 \leq j \leq N. \end{cases}$$

Then Theorem 1.4.2.1 implies that there exists a sequence v_l, $l = 1, 2, \ldots$ such that $v_l \in C^\infty(\bar{\Omega})$ and $v_l \to v$ in $W_p^{k+4}(\Omega)$; the corresponding traces (which are smooth)

$$\begin{cases} \varphi_{j,l} = \gamma_j v_l, & 1 \leq j \leq N, \quad l \geq 1 \\ \psi_{j,l} = \gamma_j \dfrac{\partial v_l}{\partial \nu_j}, & 1 \leq j \leq N, \quad l \geq 1 \end{cases}$$

converge respectively to φ_j and ψ_j in $W_p^{k+4-1/p}(\Gamma_j)$ and $W_p^{k+3-1/p}(\Gamma_j)$. In addition Theorem 7.2.2.2 implies that they fulfil the conditions (7,2,2,12) and (7,2,2,13) (7,2,2,14) when suitable. We also approximate f in $W_p^k(\Omega)$ by a sequence $f_l \in C^\infty(\bar{\Omega})$, $l = 1, 2, \ldots$.

Clearly we have $f_l \in H^{k+2}(\Omega)$, $\varphi_{j,l} \in H^{k+11/2}(\Gamma_j)$, $\psi_{j,l} \in H^{k+9/2}(\Gamma_j)$ and all the conditions for applying Theorem 7.2.2.3 with k replaced by $k+2$ are satisfied provided the equations (7,2,2,1) have no root on the line

$\operatorname{Im} \lambda = -(k+4)$. Let $u_l \in H^2(\Omega)$ be the solution of

$$\begin{cases} \Delta^2 u_l = f_l & \text{in } \Omega \\ \gamma_j u_l = \varphi_{j,l} & \text{on } \Gamma_j, \quad 1 \leqslant j \leqslant N \\ \gamma_j \dfrac{\partial u_l}{\partial \nu_j} = \psi_{j,l} & \text{on } \Gamma_j, \quad 1 \leqslant j \leqslant N. \end{cases}$$

Then u_l belongs to the span of $H^{k+6}(\Omega)$ and the functions $\mathcal{S}_{j,m}$, $\mathcal{T}_{j,m}$, $\mathcal{U}_{j,m}$ corresponding respectively to $-(k+4) < \operatorname{Im} \lambda'_{j,m} < 0$ and $-(k+4) < \operatorname{Im} \lambda''_{j,m} < 0$. When the conditions on the roots are not satisfied, one can only claim that u_l belongs to the span of $H^{k+5}(\Omega)$ and the same functions $\mathcal{S}_{j,m}$, $\mathcal{T}_{j,m}$ and $\mathcal{U}_{j,m}$; this follows from Remark 7.2.2.4.

The Sobolev theorem and Theorem 1.4.5.2 imply in both cases that

$$u_l \in E$$

where E is the span of $W_p^{k+4}(\Omega)$ and of the singular functions $\mathcal{S}_{j,m}$, $\mathcal{T}_{j,m}$ and $\mathcal{U}_{j,m}$ corresponding to the conditions (7,3,2,2) and (7,3,2,3).

To conclude we shall take advantage of the inequality in the following lemma, of which we postpone the proof.

Lemma 7.3.2.2 *There is a constant C such that*

$$\|u\|_E \leqslant C\left\{\|\Delta^2 u\|_{k,p,\Omega} + \sum_{j=1}^{N} \|\gamma_j u\|_{k+4-1/p,p,\Gamma_j} + \sum_{j=1}^{N} \left\|\gamma_j \frac{\partial u}{\partial \nu_j}\right\|_{k+3-1/p,p,\Gamma_j}\right\} \tag{7,3,2,4}$$

for all $u \in E$, where E is equipped with the natural norm

$$\text{g.l.b.} \left\{\|\varphi\|_{k+4,p,\Omega} + \sum_{-(k+1+2/q) < \lambda'_{j,m} < 0} |a_{j,m}| + \sum_{-(k+1-2/q) < \lambda''_{j,m} < 0} [|b_{j,m}| + |C_{j,m}|]\right\},$$

where

$$u = \varphi + \sum_{-(k+1+2/q) < \lambda'_{j,m} < 0} a_{j,m} \mathcal{S}_{j,m} + \sum_{-(k+1-2/q) < \lambda_{j,m} < 0} [b_{j,m} \mathcal{T}_{j,m} + C_{j,m} \mathcal{U}_{j,m}].$$

Let us apply this inequality to $u_{l'} - u_l$; this yields

$$\|u_{l'} - u_l\|_E \leqslant C\left\{\|f_{l'} - f_l\|_{k,p,\Omega} + \sum_{j=1}^{N} \|\varphi_{j,l'} - \varphi_{j,l}\|_{k+4-1/p,p,\Gamma_j} + \sum_{j=1}^{N} \|\psi_{j,l'} - \psi_{j,l}\|_{k+3-1/p,p,\Gamma_j}\right\}.$$

Consequently u_l, $l = 1, 2, \ldots$, is a Cauchy sequence in E; its limit u is the solution of problem (7,3,2,1). This shows that $u \in E$.

Proof of Lemma 7.3.2.2 First we observe that $\Delta^2\mathcal{S}_{j,m}$, $\gamma_s\mathcal{S}_{j,m}$ and $\gamma_s\, \partial\mathcal{S}_{j,m}/\partial\nu_s$ are smooth functions. The functions $\mathcal{T}_{j,m}$ and $\mathcal{U}_{j,m}$ have the same property.

Consequently

$$\left\{\Delta^2;\ \gamma_j,\ \gamma_j\frac{\partial}{\partial\nu_j},\ 1\leq j\leq N\right\}$$

actually maps E into

$$W_p^k(\Omega)\times\prod_{j=1}^N\{W_p^{k+4-1/p}(\Gamma_j)\times W_p^{k+3-1/p}(\Gamma_j)\}.$$

Consequently the inequality (7,3,2,4) is meaningful.

Now we proceed by steps starting from inequality (7,3,1,1). Combined with the obvious estimate

$$\|u\|_{2,2,\Omega}\leq C_1\,\|\Delta^2 u\|_{-2,2,\Omega}$$

and Theorem 1.4.3.3, it yields the existence of C_2 such that

$$\|u\|_{k+4,p,\Omega}\leq C_2\,\|\Delta^2 u\|_{k,p,\Omega}$$

for every $u\in W_p^{k+4}(\Omega)\cap \mathring{W}_p^2(\Omega)$, under the assumptions of Corollary 7.3.1.9.

Then the trace theorems imply the existence of C_3 such that

$$\|u\|_{k+4,p,\Omega}\leq C_3\left\{\|\Delta^2 u\|_{k,p,\Omega}+\sum_{j=1}^N\|\gamma_j u\|_{k+4-1/p,p,\Gamma_j}+\sum_{j=1}^N\left\|\gamma_j\frac{\partial u}{\partial\nu_j}\right\|_{k+3-1/p,p,\Gamma_j}\right\}$$

for every $u\in W_p^{k+4}(\Omega)$. Finally the inequality (7,3,2,4) follows by augmenting $W_p^{k+4}(\Omega)$ with the finite-dimensional space spanned by the functions $\mathcal{S}_{j,m}$, $\mathcal{T}_{j,m}$ and $\mathcal{U}_{j,m}$ corresponding to the conditions (7,3,2,2) and (7,3,2,3) (see the method of proof of Theorem 4.3.2.4).

Remark 7.3.2.3 Theorem 7.3.2.1 does not express a regularity result in general, since the solution does not belong to $W_p^{k+4}(\Omega)$. However, this is a regularity result when the equations (7,2,2,1) have no roots except $-i$ in the strip $-(k+1+2/q)\leq \operatorname{Im}\lambda<0$. Then the solution u of problem (7,3,2,1) belongs to $W_p^{k+4}(\Omega)$. Therefore the behaviour of the solution of the biharmonic equation is reduced just to the behaviour of the roots of equation (7,2,2,1). We now mention a very useful result in this direction.

Lemma 7.3.2.4 *Assume that $\lambda=\xi+i\eta$ is solution of*

$$\sinh^2(\lambda\omega)=\lambda^2\sin^2\omega$$

and assume that $0<\omega<\pi$; then $|\eta|$ is strictly larger than 1, unless $\xi=0$ and $|\eta|=1$.

Proof The equation is equivalent to

$$\sinh(\lambda\omega) = \pm\lambda \sin\omega.$$

Taking the imaginary part of this equation yields

$$\cosh(\xi\omega)\sin(\eta\omega) = \pm\eta \sin\omega. \tag{7,3,2,5}$$

Since the function

$$t \to \frac{\sin t}{t}$$

is decreasing in $[0, \pi]$, it follows that

$$\left|\frac{\sin \eta\omega}{\eta\omega}\right| \geqslant \frac{\sin\omega}{\omega}$$

for $\eta \in [-1, +1]$. Consequently, we have

$$\left|\frac{\sin \eta\omega}{\eta \sin\omega}\right| \geqslant 1$$

and identity (7,3,2,5) is impossible unless $\xi = 0$, since $\cosh(\xi\omega) > 1$ for $\xi \neq 0$.

This lemma together with Theorem 7.2.2.1 imply the following general principle.

Corollary 7.3.2.5 *Assume that Ω is a convex plane polygon; then Δ^2 is an isomorphism from $H^3(\Omega) \cap \mathring{H}^2(\Omega)$ onto $H^{-1}(\Omega)$.*

Remark 7.3.2.6 The whole Subsection 7.3.1 is valid for a strip with width $\omega = 2\pi$. Taking advantage of the remarks 7.2.2.5 and 7.2.1.15, and applying the techniques of the proof of Theorem 7.3.2.1, we derive the following statement where, for simplicity, we assume that $k \geqslant 0$ and $p \neq 2$.

The solution of the problem (7,3,2,1) belongs to the span of $W_p^{k+4}(\Omega)$ and the functions $\mathcal{S}_{j,m}$, $\mathcal{T}_{j,m}$ and $\mathcal{U}_{j,m}$ corresponding to $\operatorname{Im}\lambda'_{j,m}$ (or $\lambda''_{j,m}) \in]-(k+1+2/q), 0[$, $\omega_j < 2\pi$ and $\mathcal{S}^{(1)}_{j,m}$, $\mathcal{S}^{(2)}_{j,m}$ corresponding to $m < 2(k+1+2/q)$ and $\omega_j = 2\pi$. This holds provided $f \in W_p^k(\Omega)$, $\varphi_j \in W_p^{k+4-1/p}(\Gamma_j)$, $\psi_j \in W_p^{k+3-1/p}(\Gamma_j)$ and the conditions

(7,2,2,12) and (7,2,2,13) hold when $\omega_j < 2\pi$,

(7,2,2,22) hold when $\omega_j = 2\pi$,

and the equations (7,2,2,1) for $\omega_j < 2\pi$ have no root (other than $-i$) on the line

$$\operatorname{Im}\lambda = -(k+1+2/q),$$

and $p \neq 4, 2, \frac{4}{3}$.

When the conditions (7,2,2,22) are not fulfilled the following additional singular solutions must be introduced:

$$r_j^l\eta_j(r_je^{i\theta_j})\Big\{(\ln r_j \sin l\theta_j+\theta_j\cos l\theta_j) \\ -\frac{l}{l-2}(\ln r_j \sin(l-2)\theta_j+\theta_j\cos(l-2)\theta_j)-\frac{2}{(l-2)^2}\sin(l-2)\theta_j\Big\}$$

together with

$$r_j^l\eta_j(r_je^{i\theta_j})\{(\ln r_j\cos l\theta_j-\theta_j\sin l\theta_j)-(\ln r_j\cos(l-2)\theta_j-\theta_j\sin(l-2)\theta_j)\}$$

for $l<k+4-2/p$.

7.3.3 The related Stokes problem

We consider here a given vector function

$$\mathbf{f}=(f_1,f_2)$$

in $H^{-1}(\Omega)^2$ and

$$\mathbf{v}=(v_1,v_2)$$

the solution in $\mathring{H}^1(\Omega)^2$ of the system of equations

$$\begin{cases}-\Delta\mathbf{v}+\nabla p=\mathbf{f}\\ \nabla\cdot\mathbf{v}=0\end{cases} \tag{7,3,3,1}$$

in Ω, where p is a scalar function in Ω.

The existence and uniqueness of $\mathbf{v}$ is well known. One can apply the variational method, i.e. Lemma 2.2.1.1, choosing H, V and a as follows:

V is the subspace of $\mathring{H}^1(\Omega)^2$ spanned by the divergence-free vector functions; H is the closure of V in $L_2(\Omega)^2$ and finally

$$a(\mathbf{u},\mathbf{v})=\sum_{j=1}^{2}\int_\Omega \nabla u_j\cdot\nabla v_j\,dx.$$

the space H defined above is characterized in Teman (1977). We are not going to detail this proof of existence and uniqueness here but we rather focus our attention on the regularity of $\mathbf{u}$. Thus, assuming that f is given in $W_p^k(\Omega)^2$, we ask whether $\mathbf{v}$ belongs to $W_p^{k+2}(\Omega)^2$ or not.

We reduce this problem to the corresponding one for the biharmonic equation by considering as usual the stream function $u\in\mathring{H}^2(\Omega)$ defined by

$$v_1=-\frac{\partial u}{\partial y},\qquad v_2=\frac{\partial u}{\partial x}. \tag{7,3,3,2}$$

This function u is well defined since $\mathbf{v}$ is divergence-free and Ω is simply connected (an assumption). It follows that u is a solution of

$$\Delta^2 u = g = \frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial x} \quad \text{in } \Omega. \tag{7,3,3,3}$$

This function g is given in $W_p^{k-1}(\Omega)$ and we can apply Theorem 7.3.2.1. provided k is nonnegative. This shows why we have always included the case $k=-1$ in previous subsections.

It follows that there exist a function u_r and numbers $a_{j,m}$, $b_{j,m}$ and $c_{j,m}$ such that

$$u = u_r + \sum_{-(k+2/q)<\operatorname{Im}\lambda'_{j,m}<0} a_{j,m}\mathcal{S}_{j,m} + \sum_{-(k+2/q)<\operatorname{Im}\lambda''_{j,m}<0} (b_{j,m}\mathcal{T}_{j,m} + c_{j,m}\mathcal{U}_{j,m})$$

where $u_r \in W_p^{k+3}(\Omega)$, provided the equations (7,2,2,1) have no root on the line

$$\operatorname{Im}\lambda = -\left(k+\frac{2}{q}\right).$$

Consequently we have the following expansions for v_1 and v_2:

$$v_l = v_{l,r} + (-1)^l \frac{\partial}{\partial x_{3-l}} \left\{ \sum_{-s<\operatorname{Im}\lambda'_{j,m}<0} a_{j,m}\mathcal{S}_{j,m} + \sum_{-s<\operatorname{Im}\lambda''_{j,m}<0} (b_{j,m}\mathcal{T}_{j,m} + c_{j,m}\mathcal{U}_{j,m}) \right\}$$

$l=1,2$, where $v_{l,r}$ belongs to $W_p^{k+2}(\Omega)$ and where we set

$$s = k + \frac{2}{q}.$$

If we go back to the identities defining $\mathcal{S}_{j,m}$, $\mathcal{T}_{j,m}$ and $\mathcal{U}_{j,m}$, i.e. (7,2,2,2), (7,2,2,5) and (7,2,2,6), we obtain directly the singular solutions corresponding to the Stokes problem. We shall use the following notation:

$$\begin{aligned}
S_{2,j,m}(r_j,\theta_j) &= r_j^{i\lambda'_{j,m}}\{(i\lambda'_{j,m}+1)\cos\theta_j\, S_{j,m}(\theta_j) - \sin\theta_j\, S'_{j,m}(\theta_j)\}\eta_j(r_j e^{i\theta_j})\\
S_{1,j,m}(r_j,\theta_j) &= -r_j^{i\lambda'_{j,m}}\{(i\lambda'_{j,m}+1)\sin\theta_j\, S_{j,m}(\theta_j) + \cos\theta_j\, S'_{j,m}(\theta_j)\}\eta_j(r_j e^{i\theta_j})\\
T_{2,j,m}(r_j,\theta_j) &= r_j^{i\lambda''_{j,m}}\{(i\lambda''_{j,m}+1)\cos\theta_j t_{j,m}(\theta_j) - \sin\theta_j t'_{j,m}(\theta_j)\}\eta_j(r_j e^{i\theta_j})\\
T_{1,j,m}(r_j,\theta_j) &= -r_j^{i\lambda''_{j,m}}\{(i\lambda''_{j,m}+1)\sin\theta_j t_{j,m}(\theta_j) + \cos\theta_j t'_{j,m}(\theta_j)\}\eta_j(r_j e^{i\theta_j})\\
U_{2,j,m}(r_j,\theta_j) &= r_j^{i\lambda''_{j,m}}\{[(1+i\lambda''_{j,m})(u_{j,m}(\theta_j) + i\ln r_j t_{j,m}(\theta_j)) + i t_{j,m}(\theta_j)]\\
&\qquad \times\cos\theta_j - [u'_{j,m}(\theta_j) + i\ln r_j\, t'_{j,m}(\theta_j)]\sin\theta_j\}\eta_j(r_j^{i\theta_j})\\
U_{1,j,m}(r_j,\theta_j) &= -r_j^{i\lambda''_{j,m}}\{[(1+i\lambda''_{j,m})(u_{j,m}(\theta_i) + i\ln r_j\, t_{j,m}(\theta_j)) + i t_{j,m}(\theta_j)]\\
&\qquad \times\sin\theta_j + [u'_{j,m}(\theta_i) + i\ln r_j\, t'_{j,m}(\theta_j)]\cos\theta_j\}\eta_j(r_j e^{i\theta_j}).
\end{aligned} \tag{7,3,3,4}$$

It is clear that $\{S_{1,j,m}, -S_{2,j,m}\}$, $\{T_{1,j,m}, -T_{2,j,m}\}$ and $\{U_{1,j,m}, U_{2,j,m}\}$ coincide respectively with $\nabla\mathcal{S}_{j,m}$, $\nabla\mathcal{T}_{j,m}$ and $\nabla\mathcal{U}_{j,m}$.

Theorem 7.3.3.1 *Let* $\mathbf{v}\in \mathring{H}^1(\Omega)^2$ *be the solution of the problem* (7,3,3,1) *with* $\mathbf{f}$ *given in* $W_p^k(\Omega)^2$. *Assume that the equations* (7,2,2,1) $j=1,2,\ldots,N$ *have no root* (*other than* $-\mathrm{i}$) *on the line*

$$\operatorname{Im}\lambda=-\left(k+\frac{2}{q}\right)$$

and exclude the case $k=0$ *when* $\tan\omega_j=\omega_j$ *for some* j. *Then* v_l *belongs to the span of* $W_p^{k+2}(\Omega)$, *the functions* $S_{l,j,m}$ *which correspond to*

$$\operatorname{Im}\lambda'_{j,m}\in\left]-\left(k+\frac{2}{q}\right),0\right[,\tag{7,3,3,5}$$

and the functions $T_{l,j,m}$ *and* $U_{l,j,m}$ *which correspond to*

$$\operatorname{Im}\lambda''_{j,m}\in\left]-\left(k+\frac{2}{q}\right),0\right[,\tag{7,3,3,6}$$

$l=1,2$.

Remark 7.3.3.2 This theorem implies that $\mathbf{v}\in W_p^{k+2}(\Omega)^2$ in the particular case when the equations (7,2,2,1) have no root (except $-\mathrm{i}$) in the strip

$$-\left(k+\frac{2}{q}\right)<\operatorname{Im}\lambda<0.\tag{7,3,3,7}$$

In particular, due to Lemma 7.3.2.4 (or the Corollary 7.3.2.5), if $\mathbf{f}$ is given in $L^2(\Omega)^2$ and Ω is a convex plane polygon, then the solution $\mathbf{v}\in\mathring{H}^1(\Omega)^2$ of (7,3,3,1) actually belongs to $H^2(\Omega)^2$. This is the result proved by Kellogg and Osborn (1976). In Chapter 3 we proved a similar result for many boundary value problems for a single Laplace equation in any convex domain and in any dimension. It would be very tempting to try to extend the previous result on the Stokes problem to a general plane convex domain. The technique in Chapter 3 was to take limits with respect to Ω. However, here, we are unable to achieve such an extension because we have no method of proof for an inequality similar to (3,1,2,1) providing good control of the constant $C(\Omega)$ (as a function of Ω).

Remark 7.3.3.3 Writing

$$\nabla p=\mathbf{f}-\Delta\mathbf{v}$$

and applying Theorem 7.3.3.1, yields that ∇p belongs to the span of $W_p^k(\Omega)^2$ and the functions $(\Delta S_{1,j,m}, \Delta S_{2,j,m})$, $(\Delta T_{1,j,m}, \Delta T_{2,j,m})$ and

$(\Delta U_{1,j,m}, \Delta U_{2,j,m})$ corresponding to (7,3,3,5) and (7,3,3,6). By integrating, one finds easily the singular part of p which does not belong to $W_p^{k+1}(\Omega)$. In particular $p \in W_p^{k+1}(\Omega)$ when there are no roots (except $-i$) of the equations (7,2,2,1) in the strip (7,3,3,7). When $\mathbf{f} \in L^2(\Omega)^2$ and Ω is convex, then $p \in H^1(\Omega)$.

We shall conclude this subsection with a few remarks concerning the Navier–Stokes equations. Let us first recall a now classical result (see Temam (1977) for instance). Let $\mathbf{f}$ (a force) be given in $L_2(\Omega)^2$; then there exists a unique solution $\mathbf{v}$ (a velocity in $\mathring{H}^1(\Omega)^2$) of

$$\begin{cases} -\Delta \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f} - \nabla p \\ \nabla \cdot \mathbf{v} = 0 \end{cases} \tag{7,3,3,8}$$

in Ω, where p (a pressure) belongs to $L_2(\Omega)$ and is unique up to the addition of some constant.

We shall derive some smoothness results for $\mathbf{v}$, just by rewriting this problem as a linear Stokes system

$$\begin{cases} -\Delta \mathbf{v} + \nabla p = \mathbf{f} - (\mathbf{v} \cdot \nabla)\mathbf{v} \\ \nabla \cdot \mathbf{v} = 0 \end{cases} \tag{7,3,3,9}$$

considering the components of $\mathbf{f} - (\mathbf{v} \cdot \nabla)\mathbf{v}$ as the data of our problem.

Theorem 7.3.3.4 *Let $\mathbf{v} \in H^1(\Omega)^2$ be the solution of the problem* (7,3,3,8) *with $\mathbf{f}$ given in $L_2(\Omega)^2$. Assume that the equations* (7,2,2,1), $j = 1, 2, \ldots, N$ *have no root (other than $-$i) on the line* $\operatorname{Im} \lambda = -1$, *and assume that $\omega_j \neq \tan \omega_j$ for every j. Then v_l belongs to the span of $H^2(\Omega)$, the functions $S_{l,j,m}$ $T_{l,j,m}$ and $U_{l,j,m}$ which correspond to* $-1 < \operatorname{Im} \lambda'_{j,m} < 0$ *and the functions $T_{l,j,m}$ and $U_{l,j,m}$ which correspond to* $-1 < \operatorname{Im} \lambda''_{j,m} < 0$, $l = 1, 2$.

Proof Knowing that $\mathbf{v}$ belongs to $H^1(\Omega)^2$ implies that $\mathbf{v} \in L_p(\Omega)^2$ for every p by Sobolev's imbedding theorem. Consequently

$$(\mathbf{v} \cdot \nabla)\mathbf{v} \in L_r(\Omega)^2$$

for every $r < 2$, by Hölder's inequality. We choose $r > 1$, such that the equations (7,2,2,1) have no root on the line

$$\operatorname{Im} \lambda = -\frac{2}{s}, \qquad \frac{1}{r} + \frac{1}{s} = 1$$

and we apply Theorem 7.3.3.1 with $p = r$. Thus v_l belongs to the span of $W_r^2(\Omega)$ and the functions $S_{l,j,m}$, $T_{l,j,m}$ and $U_{l,j,m}$ corresponding to

$$-\frac{2}{s} < \operatorname{Im} \lambda'_{j,m} < 0$$

and

$$-\frac{2}{s} < \operatorname{Im} \lambda''_{j,m} < 0.$$

Now we observe that the functions in $W_r^2(\Omega)$ and $S_{l,j,m}$, $T_{l,j,m}$ and $U_{l,j,m}$ are all bounded functions. Thus $\mathbf{v} \in L_\infty(\Omega)^2$ and consequently

$$(\mathbf{v} \cdot \nabla)\mathbf{v} \in L_2(\Omega)^2.$$

We apply again Theorem 7.3.3.1 with $p = 2$ and get the desired conclusion. ■

Corollary 7.3.3.5 *Let* $\mathbf{v} \in \mathring{H}^1(\Omega)^2$ *be the solution of problem* (7,3,3,8) *with* $\mathbf{f}$ *given in* $L_2(\Omega)^2$ *and assume that* Ω *is a convex plane polygon. Then* $\mathbf{v} \in H^2(\Omega)^2$ *(and consequently* $p \in H^1(\Omega)$*).*

Now applying the same procedure as before, one can obtain further results when Ω is convex.

Theorem 7.3.3.6 *Let* $\mathbf{v} \in \mathring{H}^1(\Omega)^2$ *be the solution of the problem* (7,3,3,8) *with* $\mathbf{f}$ *given in* $L_p(\Omega)^2$, $2 < p < \infty$. *Assume that* Ω *is convex and that the equations* (7,2,2,1), $j = 1, 2, \ldots, N$ *have no root on the line*

$$\operatorname{Im} \lambda = -\frac{2}{q}, \qquad \frac{1}{p} + \frac{1}{q} = 1$$

Then v_l *belongs to the span of* $W_p^2(\Omega)$, *the functions* $S_{l,j,m}$ *which correspond to* $-(2/q) < \operatorname{Im} \lambda'_{j,m} < -1$ *and the functions* $T_{l,j,m}$ *and* $U_{l,j,m}$ *which correspond to* $-(2/q) < \operatorname{Im} \lambda''_{j,m} < -1$.

Proof We already know that $\mathbf{v} \in H^2(\Omega)^2$ and consequently $\mathbf{v} \in L_\infty(\Omega)^2$ and $\nabla v_l \in L_p(\Omega)^2$, $l = 1, 2$, by the imbedding theorem. Consequently we have

$$(\mathbf{v} \cdot \nabla)\mathbf{v} \in L_p(\Omega)^2$$

and we can apply Theorem 7.3.3.1. ■