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Boundary layer analysis of the Navier–Stokes equations with generalized Navier boundary conditions

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ABSTRACT

We study the weak boundary layer phenomenon of the Navier–Stokes equations with generalized Navier friction boundary conditions, $u \cdot \boldsymbol{n} = 0$, $[\mathbf{S}(u)\boldsymbol{n}]_{\tan} + \mathcal{A}u = 0$, in a bounded domain in $\mathbb{R}^3$ when the viscosity, $\varepsilon > 0$, is small. Here, $\mathbf{S}(u)$ is the symmetric gradient of the velocity, u, and $\mathcal{A}$ is a type $(1,1)$ tensor on the boundary. When $\mathcal{A} = \alpha I$ we obtain Navier boundary conditions, and when $\mathcal{A}$ is the shape operator we obtain the conditions, $u \cdot \boldsymbol{n} = (\operatorname{curl} u) \times \boldsymbol{n} = 0$. By constructing an explicit corrector, we prove the convergence, as ε tends to zero, of the Navier–Stokes solutions to the Euler solution both in the natural energy norm and uniformly in time and space.

1. Introduction

The flow of an incompressible, constant-density, constant-viscosity Newtonian fluid is described by the Navier–Stokes equations,

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} - \varepsilon \Delta u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon + \nabla p^\varepsilon = f & \text{in } \Omega \times (0,T), \\ \operatorname{div} u^\varepsilon = 0 & \text{in } \Omega \times (0,T), \\ u^\varepsilon\big|_{t=0} = u_0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

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The fluid is contained in the bounded domain, $\Omega \subset \mathbb{R}^3$, with smooth boundary, Γ. The parameter, $\varepsilon > 0$, is the viscosity and $T > 0$ is fixed (see Theorem 2.2). The equations are to be solved for the velocity of the fluid, u^ε, and pressure, p^ε, given the forcing function, f, and initial velocity, u_0. The regularity of Γ, f, and u_0 that we assume is specified in (1.7), but our emphasis is not on optimal regularity requirements. We also impose Navier boundary conditions on u^ε, described below, which include the impermeable condition, $u^\varepsilon \cdot \boldsymbol{n} = 0$.

When $\varepsilon = 0$, we formally obtain the Euler equations,

$$\begin{cases} \dfrac{\partial u^0}{\partial t} + (u^0 \cdot \nabla) u^0 + \nabla p^0 = f & \text{in } \Omega \times (0, T), \\ \operatorname{div} u^0 = 0 & \text{in } \Omega \times (0, T), \\ u^0 \cdot \boldsymbol{n} = 0 & \text{on } \Gamma \times (0, T), \\ u^0 \big|_{t=0} = u_0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $\boldsymbol{n}$ is the outer unit normal vector on Γ.

The Euler equations, being first-order, need only the impermeable boundary condition, $u^0 \cdot \boldsymbol{n} = 0$, reflecting no entry or exit of fluid from the domain. No-slip boundary conditions, $u^\varepsilon = 0$ on Γ, are those most often prescribed for the Navier–Stokes equations. This, of necessity, leads to a discrepancy between u^ε and u^0 at the boundary, resulting in boundary layer effects. Prandtl [46] was the first to make real progress on analyzing these effects, and much of a pragmatic nature has been discovered, but to this day the mathematical understanding is woefully inadequate. (See [15,9,40,17] for reviews of the mathematical literature. See [24], which builds on linear results of [18,19,25], for ill-posedness of Prandtl's boundary layer equations; [19] gives a review of earlier ill-posedness results. See [31,57,60,10,33,34,49] for conditional results on convergence in the vanishing viscosity limit.)

In part because of these difficulties with no-slip boundary conditions, and in part because of very real physical applications, researchers have turned to other boundary conditions. Of particular interest are boundary conditions variously called Navier friction, Navier slip, or simply Navier boundary conditions (other names have been used as well). These boundary conditions can be written as

$$u^\varepsilon \cdot \boldsymbol{n} = 0, \qquad \left[\mathbf{S}(u^\varepsilon)\boldsymbol{n} + \alpha u^\varepsilon\right]_{\tan} = 0 \quad \text{on } \Gamma, \tag{1.3}$$

where

$$\mathbf{S}(u) := \frac{1}{2}\left(\nabla u + (\nabla u)^{\mathsf{T}}\right) = \left(\frac{1}{2}\frac{\partial u_j}{\partial x_i} + \frac{1}{2}\frac{\partial u_i}{\partial x_j}\right)_{1 \leqslant i,j \leqslant 3}, \quad \text{for } u = (u_1, u_2, u_3). \tag{1.4}$$

Here (x_1, x_2, x_3) (or (x, y, z) in Section 3), denotes the Cartesian coordinates of a point $\boldsymbol{x} \in \mathbb{R}^3$, α is the (positive or negative) friction coefficient, which is independent of ε. The notation $[\cdot]_{\tan}$ in (1.3) denotes the tangential components of a vector on Γ.

In this paper, we use the generalization of (1.3),

$$\begin{cases} u^\varepsilon \cdot \boldsymbol{n} = 0 & \text{on } \Gamma, \\ \left[\mathbf{S}(u^\varepsilon)\boldsymbol{n}\right]_{\tan} + \mathcal{A} u^\varepsilon = 0 & \text{on } \Gamma, \end{cases} \tag{1.5}$$

of the Navier boundary conditions. Here, $\mathcal{A}$ is a type $(1,1)$ tensor on the boundary having at least C^2-regularity. In coordinates on the boundary, $\mathcal{A}$ can be written in matrix form as $\mathcal{A} = (\alpha_{ij})_{1 \leqslant ij \leqslant 2}$. Note that u^ε lies in the tangent plane, as does $\mathcal{A}u^\varepsilon$.

It is easy to see that when $\mathcal{A} = \alpha I$, the product of a function α on Γ and the identity tensor, the generalized Navier boundary conditions, (1.5), reduce to the usual Navier friction boundary conditions, (1.3). In fact, the analysis using a general $\mathcal{A}$ in place of αI is changed only slightly from using αI with α a constant (we say a bit more on this in Remark 2.4).

The primary motivation for generalizing Navier boundary conditions in this manner is that when $\mathcal{A}$ is the shape operator (Weingarten map) on Γ, one obtains, as a special case, the boundary conditions,

$$u^\varepsilon \cdot \boldsymbol{n} = \left(\operatorname{curl} u^\varepsilon\right) \times \boldsymbol{n} = 0, \tag{1.6}$$

as we show in Appendix B. (This fact is implicit in [3].) Such boundary conditions have been studied (in 3D) by several authors, including [2,3,61] (and see the references therein), [7,6] for an inhomogeneous version of (1.6), and [4,5] for related boundary conditions. In this special case, stronger convergence can be obtained (at least in a channel), in large part because vorticity can be controlled near the boundary. Hence, somewhat different issues arise, and the bodies of literature studying boundary conditions (1.6) and (1.3) are somewhat disjoint.

We introduce the Hilbert space,

$$H = \left\{u \in L^2(\Omega)^3 \colon \operatorname{div} u = 0 \text{ and } u \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\right\},$$

equipped with the usual L^2 inner product. Then, letting $T > 0$ be an arbitrary time less than any T appearing in Theorems 2.2 and 2.3, we state our main result:

Theorem 1.1. *Let Γ_a be the interior tubular neighborhood of Ω with width $a > 0$. Assume that*

$$u_0 \in H \cap H^m(\Omega), \qquad f \in C^\infty_{loc}\big([0,\infty); C^\infty(\Omega)\big), \qquad \Gamma \text{ is } C^{m+2} \quad \text{for } m \geqslant 5. \tag{1.7}$$

Then u^ε, a solution of the Navier–Stokes equations, (1.1), with Generalized Navier boundary conditions, (1.5), converges to u^0, the solution of the Euler equations, (1.2), as the viscous parameter ε tends to zero, in the sense that

$$\left\|u^\varepsilon - u^0\right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant \kappa \varepsilon^{\frac{3}{4}}, \qquad \left\|u^\varepsilon - u^0\right\|_{L^2(0,T;H^1(\Omega))} \leqslant \kappa \varepsilon^{\frac{1}{4}}, \tag{1.8}$$

for some $T > 0$ and for a constant $\kappa = \kappa(T, \overline{\alpha}, u_0, f)$, $\overline{\alpha} = \|\mathcal{A}\|_{C^m(\Gamma)}$. If $m > 6$ and $f \equiv 0$ then

$$\left\|u^\varepsilon - u^0\right\|_{L^\infty([0,T]\times\Gamma_a)} \leqslant \kappa \varepsilon^{\frac{3}{8} - \frac{3}{8(m-1)}}, \qquad \left\|u^\varepsilon - u^0\right\|_{L^\infty([0,T]\times\Omega\setminus\Gamma_a)} \leqslant \kappa \varepsilon^{\frac{3}{4} - \frac{9}{8m}}, \tag{1.9}$$

where now $\kappa = \kappa(T, \overline{\alpha}, m, a, u_0, f)$.

Because we will only have existence of u^ε when $5 \leqslant m \leqslant 6$ (Theorem 2.1), by u^ε we mean an arbitrary choice of the possibly multiple solutions when we consider the limit as $\varepsilon \to 0$. When $m > 6$ the solutions are unique as shown by Masmoudi and Rousset (see Theorem 2.3), and this arbitrary choice becomes unnecessary.

Remark 1.2. Standard boundary layer analysis indicates that a linear corrector will be of order $\varepsilon^{1/2}$ in $L^\infty([0,T] \times \Omega)$, so an exponent of $\frac{1}{2}$ rather than $\frac{3}{8}$ in (1.9) should formally be considered optimal (for C^∞ initial data).

Navier boundary conditions go back to [44], in which Navier first proposed them, and to [42], in which Maxwell derived them from the kinetic theory of gases. There has been intermittent interest in them since, but revival of active interest in the mathematical community working on the vanishing viscosity limit started with the paper of Clopeau, Mikelić, and Robert [13], which gives a vanishing viscosity result in two dimensions. Also, the work of Coron in [14] on the controllability of the 2D Navier–Stokes equations with Navier boundary conditions, which precedes [13], initiated interest in these boundary conditions in the PDE control theory community. By now there is a fairly substantial mathematical literature on the subject, but the three papers, [29,30,41], are of particular concern to us here. Both [29] and [41] give existence theorems for solutions to (1.1), (1.3), with uniqueness holding for stronger initial data. We quote these results in Theorems 2.1 and 2.3.

Even with Navier boundary conditions there is a discrepancy between u^0 and u^ε on the boundary, so we expect boundary layer effects to occur. As first shown (in 3D) by Iftimie and Planas in [29], however, this boundary layer effect is mild enough to allow convergence of u^ε to u^0 in $L^\infty(0,T;L^2(\Omega))$ without using any artificial function correcting the difference $u^\varepsilon - u^0$ on the boundary. (This result of [29] was for $\alpha \geqslant 0$, but the argument is easily modified to allow α to be negative.) Thus, it makes sense to refer to the boundary layer as weak.

Specifically, Iftimie and Planas show in [29] that

$$\left\|u^\varepsilon - u^0\right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\varepsilon^{\frac{1}{2}}. \tag{1.10}$$

Iftimie and Sueur [30] use a corrector, $\widetilde{v}$, to improve the convergence rate in (1.10) to $\varepsilon^{3/4}$ in this energy norm. More precisely, they consider an asymptotic expansion of u^ε as the sum of u^0 and $\widetilde{v}$, where $\widetilde{v}$ is a corrector whose tangential components are defined as a solution of a linearized Prandtl-type system of coupled equations. Using the properties of $\widetilde{v}$, they show that

$$u^\varepsilon - \left(u^0 + \widetilde{v}\right) \quad \text{is order } \varepsilon \text{ in } L^\infty\big(0,T;L^2(\Omega)\big), \quad \text{order } \varepsilon^{\frac{1}{2}} \text{ in } L^2\big(0,T;H^1(\Omega)\big).$$

These bounds with estimates on the corrector $\widetilde{v}$ then give

$$\left\|u^\varepsilon - u^0\right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\varepsilon^{\frac{3}{4}}, \qquad \left\|u^\varepsilon - u^0\right\|_{L^2(0,T;H^1(\Omega))} \leqslant C\varepsilon^{\frac{1}{4}}. \tag{1.11}$$

We, on the other hand, use an asymptotic expansion of u^ε in the form $u^\varepsilon \cong u^0 + \theta^\varepsilon$, where the main part of the explicitly defined corrector, θ^ε, exponentially decays from the boundary; see (3.8), (3.9) and (4.24), (4.25).[1] Both correctors are linear and both can be used to obtain order $\varepsilon^{3/4}$ convergence in the vanishing viscosity limit, (1.8), but Iftimie and Sueur achieve an order of convergence of ε for the *corrected* difference, $u^\varepsilon - u^0 - \widetilde{v}$, while our corrected difference still gives order $\varepsilon^{3/4}$. The tradeoff is simplicity of the corrector versus rate of convergence of the corrected velocity.

We wish to emphasize that the techniques we employ in this paper differ considerably from those of [30]. While the approach in both papers originates in the work of Prandtl [46], our approach adheres much more closely to a by now well-established approach to boundary layer analysis, which we adapt to treat Navier boundary conditions. In this regard our arguments will be more familiar to many researchers, and hence, ultimately, we believe, easier to incorporate into the existing understanding of boundary layers as they appear in a variety of physical problems. (For a description of the general theory of boundary layer analysis, see, for example, [15,16,20,22,28,38,45,47,58]. Concerning boundary layer analysis related to the Navier–Stokes equations, we refer readers to [21,23,26,27,30,35,51,52,55–57].)

A key aspect of our corrector is that it is coordinate-independent. This not only gives it geometric meaning, it removes the need for a partition of unity to patch together the corrector defined in charts throughout the boundary layer. Nonetheless, the corrector has a particularly simple form in principal curvature coordinates, which we discuss in Section 4.2. (Such coordinates are used in much the same way, though for different purposes, in [3].)

We also, in (1.9), obtain convergence uniformly in time and space of order $\varepsilon^{3/8-\delta}$ near the boundary and $\varepsilon^{3/4-\delta'}$ in the interior, with δ, δ' decreasing as the regularity of the initial velocity is increased, by employing an anisotropic embedding inequality developed in Appendix A. We take great advantage of the regularity result of Masmoudi and Rousset [41] (Theorem 2.3) to obtain this convergence. The

[1] Here, and in all that follows, $A \cong B$ is used to indicate that $A = B + O(\varepsilon^r)$ for some $r > 0$. We write this only to motivate the asymptotic expansions we make, but always show that such expansions are, in fact, valid in specific norms.

authors of [41] themselves take a similar approach; however, the anisotropic inequality they use requires control on norms higher than those in (1.11), and this is only sufficient to obtain boundedness of the sequence of solutions to (1.1), (1.3). A compactness argument then gives convergence uniformly in time and space, though without a rate of convergence.

The body of this paper is organized as follows: The existence and uniqueness results for solutions to the Navier–Stokes equations and Euler equations that we will need are given in Section 2. We give the proof of (1.8), the first part of Theorem 1.1, in Sections 3 and 4. To avoid the geometrical difficulties of a curved boundary, which obscure the key ingredients of the argument, we first prove (1.8) for the case of a three-dimensional periodic channel domain. We do this in Section 3. Then, in Section 4, as a generalization of Section 3, we treat the case of a bounded domain in $\mathbb{R}^3$ with smooth (curved) boundary. In Section 5, we present the (very short) proof of (1.9), the second part of Theorem 1.1, which relies on the anisotropic Agmon's inequality, which we establish in Theorem A.2. In Appendix B, we prove that (1.5) reduces to (1.6) when $\mathcal{A}$ is the shape operator. Finally, Appendix C contains some standard lemmas which we state without proof.

Remark 1.3 *(Notational conventions).* Let $I = [a, b]$ be a time interval and X a function space on Ω. By $L^p(a, b; X)$, $1 \leqslant p \leqslant \infty$, we mean the space of all functions, f, for which $f(t)$ lies in X for almost all t in I and for which its norm,

$$\left(\int_a^b \|f(t)\|_X^p \, dt\right)^{1/p} \quad \text{for } 1 \leqslant p < \infty \quad \text{or} \quad \operatorname*{ess\,sup}_{t\in(a,b)} \|f(t)\|_X \quad \text{for } p = \infty,$$

is finite. The space, $C^k(I; X)$, is the space of all functions, f, for which $t \mapsto \partial_t^j f(t)$ is continuous as a function from I to X for all derivatives of order $j \leqslant k$. Its norm is

$$\sum_{j=0}^{k} \operatorname*{ess\,sup}_{t\in I} \|\partial_t^j f(t)\|_X.$$

Similarly, $C_w^0(I; X)$ is the space of all functions weakly continuous from I to X.

The abbreviation, $e.s.t.$, stands for "exponentially small term". More precisely, $e.s.t.$ is a function (or a constant) whose norm in all Sobolev spaces, H^s, and thus Hölder spaces, C^s, is exponentially small with a bound of the form, $c_1 \exp(-c_2/\varepsilon^\gamma)$, $c_1, c_2, \gamma > 0$, for each s.

2. Existence and uniqueness theorems

Thanks to Lemma C.1, by applying the Galerkin method, one can construct solutions to (1.1) with (1.5) in the following sense, as shown in [30] (see Remark 2.4):

Theorem 2.1. *(See Iftimie, Sueur [30].) Assuming that u_0 lies in H and f lies in $L^2(0, T; L^2(\Omega)^3)$, there exists a weak solution $u^\varepsilon \in C_w^0([0, T]; H) \cap L^2(0, T; H^1(\Omega)^3)$ of the Navier–Stokes equations, (1.1), with the generalized Navier friction boundary conditions, (1.5), in the sense that, for any $v \in C^\infty([0, T]; C^\infty(\overline{\Omega}) \cap H)$ with $v(T) = 0$,*

$$-\int_0^T \int_\Omega u^\varepsilon \cdot \frac{\partial v}{\partial t} \, d\boldsymbol{x}\, dt + 2\varepsilon \int_0^T \int_\Omega \mathbf{S}(u^\varepsilon) \cdot \mathbf{S}(v) \, d\boldsymbol{x}\, dt + 2\varepsilon \int_0^T \int_\Gamma \mathcal{A}u^\varepsilon \cdot v \, dS\, dt$$
$$+ \int_0^T \int_\Omega (u^\varepsilon \cdot \nabla) u^\varepsilon \cdot v \, d\boldsymbol{x}\, dt = \int_\Omega u_0 \cdot v|_{t=0} \, d\boldsymbol{x}.$$

We have the following well-posedness result for solutions to the Euler equations:

Theorem 2.2. *Suppose that u^0 lies in $H \cap H^m(\Omega) \cap C^{1,\mu}(\Omega)$, μ in $(0,1]$, $m \geqslant 3$ is an integer, f lies in $C^\infty_{loc}([0,\infty); C^\infty(\overline{\Omega}))$, and Γ is of class C^{m+2}. Then for some time $T > 0$ there exists a unique solution, u^0, to (1.2) lying in $C^1_b([0,T] \times \Omega) \cap C([0,T]; H^m(\Omega))$. The corresponding pressure, p^0, lies in $L^\infty(0,T; H^{m+1}(\Omega))$ and is unique up to an additive function of time.*

Proof. Combining Theorem 1 and Theorem 2 part 3 of [37] gives the existence, uniqueness, and regularity of u when $f \equiv 0$ and the boundary is smooth. The proof is straightforward to adapt to smooth forcing, and the strongest restriction on the smoothness of the boundary comes through the use of the Leray projector (Lemma 2 of [37]), where, however, C^{m+2}-regularity suffices. The regularity of the pressure (as well as the well-posedness in Sobolev spaces) is proved in [54,50]. □

When (1.7) holds, by virtue of Theorem 2.2 and Sobolev embedding, for some $T > 0$,

$$u^0 \in C^1_b\big([0,T] \times \Omega\big) \cap C\big([0,T]; H^m(\Omega)\big), \qquad p^0 \in L^\infty\big(0,T; H^{m+1}(\Omega)\big) \quad \text{for } m \geqslant 5. \tag{2.1}$$

The regularity in (2.1) of the solution is what we require; we do not claim that the assumptions in (1.7) are the minimal ones that guarantee such regularity, however.

In [41], Masmoudi and Rousset obtain the well-posedness result that we state in Theorem 2.3 for solutions to (1.1), (1.5) in the conormal Sobolev spaces of Definition 1 (see Remark 2.4).

Definition 1. Let Ω be a d-dimensional manifold, $d \geqslant 1$, with C^k-boundary, $k \geqslant 1$. Viewing vector fields as derivations of $C^\infty(\Omega)$, we say that a vector field, X, is tangent to $\partial\Omega$ if $Xf = 0$ on $\partial\Omega$ whenever f is constant on Ω. Let $(Z_j)_{j=1}^N$ be a set of generators of vector fields tangent to $\partial\Omega$. (Locally, only d such vector fields are needed, but for a global basis, N will be greater than d.)

The following are examples of generators for the set of vector fields tangent to the boundary for two different domains:

(i) For a channel domain, $\Omega = \mathbb{R}^2 \times (0,h)$, periodic in x and y, let ζ be a nonnegative $C^\infty(\overline{\Omega})$-function positive on Ω with $\zeta(x,y,z) = z$ near the lower boundary and $\zeta(x,y,z) = h - z$ near the upper boundary. Then $(\partial_1, \partial_2, \zeta\partial_3)$ defined globally on the channel generate the set of vector fields tangent to the boundary.

(ii) For the unit disk, D, let ζ be a nonnegative $C^\infty(\overline{D})$-function positive on D with $\zeta(r,\theta) = 1 - r$ for all $r > \frac{1}{2}$. Let φ be a compactly supported cutoff function on Ω. Then $((1-\varphi)\partial_\theta, (1-\varphi)\zeta\partial_r, \varphi\partial_x, \varphi\partial_y)$ generate the set of vector fields tangent to the boundary.

At the beginning of the proof of Theorem A.2 we construct a set of generators for vector fields tangential to the boundary in charts supported in tubular neighborhoods of the boundary of an arbitrary bounded domain in $\mathbb{R}^3$.

For a multiindex, β, let $Z^\beta = Z_1^{\beta_1} \cdots Z_N^{\beta_N}$. Define

$$H^m_{co}(\Omega) = \big\{ f \in L^2(\Omega) \colon Z^\beta f \in L^2(\Omega) \text{ for all } |\beta| \leqslant m \big\}$$

with

$$\|f\|^2_{H^m_{co}(\Omega)} = \sum_{|\beta| \leqslant m} \big\| Z^\beta f \big\|^2_{L^2(\Omega)}.$$

We say that f is in the space, $W^{m,\infty}_{co}$, if

$$\|f\|_{W^{m,\infty}_{co}} := \sum_{|\beta| \leqslant m} \|Z^\beta u\|_{L^\infty(\Omega)} < \infty$$

and we define the space E^m by

$$E^m := \{u \in H^m_{co}(\Omega) \mid \nabla u \in H^{m-1}_{co}(\Omega)\}$$

with the obvious norm.

Theorem 2.3. *(See Masmoudi, Rousset [41].) Let m be an integer satisfying $m > 6$ and Ω be a C^{m+2} domain. Consider $u_0 \in E^m \cap H$ such that $\nabla u_0 \in W^{1,\infty}_{co}$. Then there exists $T > 0$ such that for all sufficiently small ε there exists a unique solution, $u^\varepsilon \in C([0,T], E^m)$, to (1.1), (1.5) with $f = 0$ and such that $\|\nabla u^\varepsilon\|_{W^{1,\infty}_{co}}$ bounded on $[0,T]$. Moreover, there exists $C = C(\overline{\alpha}) > 0$, where $\overline{\alpha} = \|\mathcal{A}\|_{C^m(\Gamma)}$, such that*

$$\sup_{[0,T]} \left(\|u^\varepsilon(t)\|_{H^m_{co}(\Omega)} + \|\nabla u^\varepsilon(t)\|_{H^{m-1}_{co}(\Omega)} + \|\nabla u^\varepsilon(t)\|_{W^{1,\infty}_{co}}\right) + \varepsilon \int_0^T \|\nabla^2 u(s)\|^2_{H^{m-1}_{co}(\Omega)}\, ds \leqslant C. \tag{2.2}$$

Remark 2.4. Theorems 2.1, 2.2, and 2.3 were proved for a bounded domain, but each of the proofs extends easily to a 3D channel. Theorems 2.1 and 2.3 were also proved assuming that $\mathcal{A} = \alpha I$, where α is a constant, but they easily extend to a general $\mathcal{A}$ by using $\overline{\alpha} = \|\mathcal{A}\|_{C^m(\Gamma)}$ in place of α in certain boundary terms, much as we do in Sections 3.2 and 4.3.

3. Channel domain

In this section, we prove (1.8) for a periodic channel domain in $\mathbb{R}^3$. We set $\Omega_\infty := \mathbb{R}^2 \times (0,h)$, and consider solutions to (1.1), (1.5) in a channel domain Ω_∞. That is,

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} - \varepsilon \Delta u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon + \nabla p^\varepsilon = f, \quad \text{in } \Omega_\infty \times (0,T), \\ \operatorname{div} u^\varepsilon = 0, \quad \text{in } \Omega_\infty \times (0,T), \\ u^\varepsilon \text{ and } p^\varepsilon \text{ are periodic in } x \text{ and } y \text{ directions with periods } L_1 \text{ and } L_2, \\ u^\varepsilon|_{t=0} = u_0, \quad \text{in } \Omega_\infty. \end{cases} \tag{3.1}$$

Here, f and u_0, satisfying (1.7), are assumed to be periodic in x and y directions with periods L_1 and L_2, respectively.

For the sake of convenience, we set

$$\Omega := (0, L_1) \times (0, L_2) \times (0, h),$$

and assume (to simplify the expressions in (3.6)) that

$$\varepsilon < (h/8)^2.$$

Since $\boldsymbol{n} = (0,0,-1)$ at $z = 0$ and $\boldsymbol{n} = (0,0,1)$ at $z = h$, we can write the Generalized Navier boundary condition, appearing in (1.5) with (1.4), in the form

$$\begin{cases} u_3^\varepsilon = 0, & \text{at } z = 0, h, \\ \dfrac{\partial u_i^\varepsilon}{\partial z} - 2\displaystyle\sum_{j=1}^{2} \alpha_{ij} u_j^\varepsilon = 0, & i = 1, 2, \ \text{at } z = 0, \\ \dfrac{\partial u_i^\varepsilon}{\partial z} + 2\displaystyle\sum_{j=1}^{2} \alpha_{ij} u_j^\varepsilon = 0, & i = 1, 2, \ \text{at } z = h. \end{cases} \tag{3.2}$$

The corresponding limit problem, (1.2), can be written as

$$\begin{cases} \dfrac{\partial u^0}{\partial t} + (u^0 \cdot \nabla) u^0 + \nabla p^0 = f, \quad \text{in } \Omega_\infty \times (0, T), \\ \operatorname{div} u^0 = 0, \quad \text{in } \Omega_\infty \times (0, T), \\ u^0 \text{ and } p^0 \text{ are periodic in } x \text{ and } y \text{ directions with periods } L_1 \text{ and } L_2, \\ u_3^0 = 0, \quad \text{at } z = 0, h, \\ u^0\big|_{t=0} = u_0, \quad \text{in } \Omega_\infty. \end{cases} \tag{3.3}$$

To study the boundary layer associated with the Navier–Stokes problem (3.1) with the Navier friction boundary conditions (3.2), we propose an asymptotic expansion of u^ε with respect to small viscosity ε,

$$u^\varepsilon \cong u^0 + \theta^\varepsilon, \tag{3.4}$$

where u^0 is the solution of (3.3) and θ^ε is a divergence-free corrector, which will be determined below. The main role of θ^ε is to correct the tangential error related to the normal derivative of $u^\varepsilon - u^0$ on the boundary; see (3.5) below.

3.1. The corrector

To define a corrector, $\theta^\varepsilon = (\theta_1^\varepsilon, \theta_2^\varepsilon, \theta_3^\varepsilon)$, using the ansatz $\theta_3^\varepsilon \cong \varepsilon^{1/2}\theta_i^\varepsilon$, $i = 1, 2$, with respect to the order of ε in any Sobolev space, we first devote ourselves to find a suitable boundary condition for θ_i^ε, $i = 1, 2$. By inserting the expansion (3.4) into $(3.2)_{2,3}$, we find that, for $i = 1, 2$,

$$\begin{cases} \dfrac{\partial u_i^0}{\partial z} - 2\displaystyle\sum_{j=1}^{2} \alpha_{ij} u_j^0 + \dfrac{\partial \theta_i^\varepsilon}{\partial z} - 2\displaystyle\sum_{j=1}^{2} \alpha_{ij} \theta_j^\varepsilon \cong 0, \quad \text{at } z = 0, \\ \dfrac{\partial u_i^0}{\partial z} + 2\displaystyle\sum_{j=1}^{2} \alpha_{ij} u_j^0 + \dfrac{\partial \theta_i^\varepsilon}{\partial z} + 2\displaystyle\sum_{j=1}^{2} \alpha_{ij} \theta_j^\varepsilon \cong 0, \quad \text{at } z = h. \end{cases}$$

For smooth α_{ij}, $1 \leqslant i, j \leqslant 2$ on Γ, independent of ε, we expect that $\partial\theta_i^\varepsilon/\partial z \gg 2\sum_{j=1}^{2} \alpha_{ij}\theta_j^\varepsilon$, $i = 1, 2$. Hence, we use the Neumann boundary condition for θ_i^ε,

$$\begin{cases} \dfrac{\partial \theta_i^\varepsilon}{\partial z} = \widetilde{u}_{i,L} := -\left(\dfrac{\partial u_i^0}{\partial z} - 2\displaystyle\sum_{j=1}^{2} \alpha_{ij} u_j^0 \right), \quad \text{at } z = 0 \text{ for } i = 1, 2, \\ \dfrac{\partial \theta_i^\varepsilon}{\partial z} = \widetilde{u}_{i,R} := -\left(\dfrac{\partial u_i^0}{\partial z} + 2\displaystyle\sum_{j=1}^{2} \alpha_{ij} u_j^0 \right), \quad \text{at } z = h \text{ for } i = 1, 2. \end{cases} \tag{3.5}$$

In the theory of boundary layer analysis, it is well known that the Neumann type boundary condition, (3.5), is useful when treating any weak boundary layer phenomenon. More precisely, to improve the convergence given in (1.10), it is sufficient to construct a corrector function that fixes the normal derivative of the difference $u^\varepsilon - u^0$ on the boundary, instead of the difference itself.

Toward this end, we first define cutoff functions, σ_L, σ_R, belonging to $C^\infty([0,h])$, by

$$\sigma_L(z) := \begin{cases} 1, & 0 \leqslant z \leqslant h/8, \\ 0, & h/4 \leqslant z \leqslant h, \end{cases} \qquad \sigma_R(z) := \sigma_L(h-z). \tag{3.6}$$

Then we define the tangential component θ_i^ε, $i = 1, 2$, of the corrector $\theta^\varepsilon = (\theta_1^\varepsilon, \theta_2^\varepsilon, \theta_3^\varepsilon)$ as

$$\theta_i^\varepsilon := \theta_{i,L}^\varepsilon + \theta_{i,R}^\varepsilon, \quad i = 1, 2, \tag{3.7}$$

where

$$\begin{aligned} \theta_{i,L}^\varepsilon &= -\sqrt{\varepsilon}\widetilde{u}_{i,L}(x,y;t)\sigma_L(z)e^{-\frac{z}{\sqrt{\varepsilon}}} - \varepsilon\widetilde{u}_{i,L}(x,y;t)\sigma_L'(z)\big(1 - e^{-\frac{z}{\sqrt{\varepsilon}}}\big) \\ &= -\varepsilon\widetilde{u}_{i,L}(x,y;t)\frac{\partial}{\partial z}\big\{\sigma_L(z)\big(1 - e^{-\frac{z}{\sqrt{\varepsilon}}}\big)\big\}, \\ \theta_{i,R}^\varepsilon &= \sqrt{\varepsilon}\widetilde{u}_{i,R}(x,y;t)\sigma_R(z)e^{-\frac{h-z}{\sqrt{\varepsilon}}} - \varepsilon\widetilde{u}_{i,R}(x,y;t)\sigma_R'(z)\big(1 - e^{-\frac{h-z}{\sqrt{\varepsilon}}}\big) \\ &= -\varepsilon\widetilde{u}_{i,R}(x,y;t)\frac{\partial}{\partial z}\big\{\sigma_R(z)\big(1 - e^{-\frac{h-z}{\sqrt{\varepsilon}}}\big)\big\}. \end{aligned} \tag{3.8}$$

To make θ^ε divergence-free, we must define the normal component θ_3^ε of θ^ε as

$$\theta_3^\varepsilon = \theta_{3,L}^\varepsilon + \theta_{3,R}^\varepsilon,$$

where

$$\begin{aligned} \theta_{3,L}^\varepsilon &= \varepsilon\left(\frac{\partial\widetilde{u}_{1,L}}{\partial x} + \frac{\partial\widetilde{u}_{2,L}}{\partial y}\right)(x,y;t)\sigma_L(z)\big(1 - e^{-\frac{z}{\sqrt{\varepsilon}}}\big), \\ \theta_{3,R}^\varepsilon &= \varepsilon\left(\frac{\partial\widetilde{u}_{1,R}}{\partial x} + \frac{\partial\widetilde{u}_{2,R}}{\partial y}\right)(x,y;t)\sigma_R(z)\big(1 - e^{-\frac{h-z}{\sqrt{\varepsilon}}}\big). \end{aligned} \tag{3.9}$$

(This form of the corrector is as in [35], adapted to Navier boundary conditions.)

Thanks to (3.6), (3.8), by differentiating (3.7) with respect to the normal variable z, one can easily verify that the tangential components, θ_1^ε, θ_2^ε, satisfy the desired boundary condition (3.5). Moreover, from (3.9), we infer that

$$\theta_3^\varepsilon = 0, \quad \text{at } z = 0, h. \tag{3.10}$$

3.2. Bounds on the corrector

We introduce the following convenient notation:

$$\frac{\partial^k}{\partial\tau^k} := \left(\begin{array}{l}\text{any differential operator of order } k \\ \text{with respect to the tangential variables } x \text{ and } y\end{array}\right), \quad k \geqslant 0.$$

We also use the convention that $\kappa_T = \kappa_T(T, u_0, f)$ is a constant that depends on T, u_0, and f, but is independent of ε and $\mathcal{A}$, and may vary from occurrence to occurrence.

Letting

$$\overline{\alpha} = \|\mathcal{A}\|_{C^m(\Gamma)}, \quad m > 6, \tag{3.11}$$

(3.5) through (3.9) give

$$\left\| \frac{\partial \theta_i^\varepsilon}{\partial \tau} \right\|_{L^\infty([0,T]\times\overline{\Omega})} \leqslant \kappa_T (1+\overline{\alpha})\varepsilon^{\frac{1}{2}}, \qquad \left\| \frac{\partial \theta_i^\varepsilon}{\partial z} \right\|_{L^\infty([0,T]\times\overline{\Omega})} \leqslant \kappa_T (1+\overline{\alpha}), \quad i = 1, 2, \tag{3.12}$$

$$\left\| \frac{\partial \theta_3^\varepsilon}{\partial \tau} \right\|_{L^\infty([0,T]\times\overline{\Omega})} \leqslant \kappa_T (1+\overline{\alpha})\varepsilon, \qquad \left\| \frac{\partial \theta_3^\varepsilon}{\partial z} \right\|_{L^\infty([0,T]\times\overline{\Omega})} \leqslant \kappa_T (1+\overline{\alpha})\varepsilon^{\frac{1}{2}}. \tag{3.13}$$

We have the following bounds on the corrector:

Lemma 3.1. *Assume (1.7) holds and that $k, l, n \geqslant 0$ are integers either $l = 1$, $k = 0$ or $l = 0$, $0 \leqslant k \leqslant 2$. Then the corrector, θ^ε, defined by (3.7) through (3.9), satisfies*

$$\begin{cases} \left\| \dfrac{\partial^{l+k+n}\theta_i^\varepsilon}{\partial t^l \partial \tau^k \partial z^n} \right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C(1+\overline{\alpha})\varepsilon^{\frac{3}{4}-\frac{n}{2}}, \quad i = 1, 2, \\ \left\| \dfrac{\partial^{l+k}\theta_3^\varepsilon}{\partial t^l \partial \tau^k} \right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C(1+\overline{\alpha})\varepsilon, \quad \left\| \dfrac{\partial^{l+k+n+1}\theta_3^\varepsilon}{\partial t^l \partial \tau^k \partial z^{n+1}} \right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C(1+\overline{\alpha})\varepsilon^{\frac{3}{4}-\frac{n}{2}} \end{cases} \tag{3.14}$$

for a constant, $C = C(T, l, k, n, u_0, f)$.

Proof. The assumptions (1.7) give the regularity of u^0 in (2.1), and since $m \geqslant 5$, this allows k to be at least as large as 2. To prove the lemma, using (3.6) through (3.9), we first notice that it is sufficient to verify (3.14) with θ_i^ε replaced by $\theta_{i,L}^\varepsilon$, $1 \leqslant i \leqslant 3$.

For $(3.14)_1$ with $\theta_{i,L}^\varepsilon$, using $(3.8)_1$, we write

$$\frac{\partial^{l+k}\theta_{i,L}^\varepsilon}{\partial t^l \partial \tau^k} = -\varepsilon^{\frac{1}{2}} \frac{\partial^{l+k}\widetilde{u}_{i,L}}{\partial t^l \partial \tau^k}\big(\sigma_L(z) - \varepsilon^{\frac{1}{2}}\sigma_L'(z)\big)e^{-\frac{z}{\sqrt{\varepsilon}}} - \varepsilon \frac{\partial^{l+k}\widetilde{u}_{i,L}}{\partial t^l \partial \tau^k}\sigma_L'(z). \tag{3.15}$$

Then, by differentiating (3.15) n times in the z variable, and using (3.5), (3.6), we find

$$\left| \frac{\partial^{l+k+n}\theta_{i,L}^\varepsilon}{\partial t^l \partial \tau^k \partial z^n} \right| \leqslant C(1+\overline{\alpha})\varepsilon^{\frac{1}{2}-\frac{n}{2}} e^{-\frac{z}{\sqrt{\varepsilon}}} + C(1+\overline{\alpha})\varepsilon + e.s.t. \tag{3.16}$$

(See Remark 1.3 for the meaning of the abbreviation, $e.s.t.$) Hence, we find

$$\begin{aligned} \left\| \frac{\partial^{l+k+n}\theta_{i,L}^\varepsilon}{\partial t^l \partial \tau^k \partial z^n} \right\|_{L^\infty(0,T;L^2(\Omega))} &\leqslant C(1+\overline{\alpha})\varepsilon^{\frac{1}{2}-\frac{n}{2}} \left(\int_0^h e^{-\frac{2z}{\sqrt{\varepsilon}}}\, dz \right)^{\frac{1}{2}} + C(1+\overline{\alpha})\varepsilon \\ &\qquad \big(\text{setting } z' = z/\sqrt{\varepsilon}\,\big) \\ &\leqslant C(1+\overline{\alpha})\varepsilon^{\frac{3}{4}-\frac{n}{2}} \left(\int_0^\infty e^{-2z'}\, dz' \right)^{\frac{1}{2}} + C(1+\overline{\alpha})\varepsilon \\ &\leqslant C(1+\overline{\alpha})\varepsilon^{\frac{3}{4}-\frac{n}{2}}, \quad \text{for } l, k, n \geqslant 0. \end{aligned} \tag{3.17}$$

To prove $(3.14)_2$ with $\theta^\varepsilon_{3,L}$, using $(3.9)_1$, we write

$$\frac{\partial^{l+k}\theta^\varepsilon_{3,L}}{\partial t^l\partial\tau^k} = \varepsilon\frac{\partial^{l+k}}{\partial t^l\partial\tau^k}\left(\frac{\partial\widetilde{u}_{1,L}}{\partial x}+\frac{\partial\widetilde{u}_{2,L}}{\partial y}\right)\sigma_L(z) - \varepsilon\frac{\partial^{l+k}}{\partial t^l\partial\tau^k}\left(\frac{\partial\widetilde{u}_{1,L}}{\partial x}+\frac{\partial\widetilde{u}_{2,L}}{\partial y}\right)\sigma_L(z)e^{-\frac{z}{\sqrt{\varepsilon}}}.$$

Hence, $(3.14)_2$ with $\theta^\varepsilon_{3,L}$ follows by applying exactly the same computations as (3.16), (3.17), and the proof of Lemma 3.1 is complete. $\square$

We define a continuous piecewise linear function, $\zeta(z)$, by

$$\zeta(z) := \begin{cases} z, & 0 \leqslant z \leqslant h/4, \\ h/4, & h/4 \leqslant z \leqslant 3h/4, \\ h-z, & 3h/4 \leqslant z \leqslant h. \end{cases} \tag{3.18}$$

Then, using the analog of the proof of Lemma 3.1, one can verify that, $i = 1, 2$,

$$\left\| \frac{\zeta(z)}{\sqrt{\varepsilon}}\frac{\partial\theta^\varepsilon_i}{\partial z}\right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C(1+\overline{\alpha})\varepsilon^{\frac14}, \qquad \left\| \frac{\zeta(z)}{\sqrt{\varepsilon}}\frac{\partial\theta^\varepsilon_3}{\partial z}\right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C(1+\overline{\alpha})\varepsilon^{\frac12}. \tag{3.19}$$

3.3. *Error analysis*

We set the remainder:

$$w^\varepsilon := u^\varepsilon - u^0 - \theta^\varepsilon. \tag{3.20}$$

Then, using (3.1) through (3.3) with (3.5), (3.10), (3.20), the equations for w^ε read

$$\begin{cases} \dfrac{\partial w^\varepsilon}{\partial t} - \varepsilon\Delta w^\varepsilon + \nabla\bigl(p^\varepsilon - p^0\bigr) = \varepsilon\Delta u^0 + R_\varepsilon\bigl(\theta^\varepsilon\bigr) - J_\varepsilon\bigl(u^\varepsilon, u^0\bigr), \quad \text{in } \Omega_\infty\times(0,T), \\ \operatorname{div} w^\varepsilon = 0, \quad \text{in } \Omega_\infty\times(0,T), \\ w^\varepsilon \text{ is periodic in } x \text{ and } y \text{ directions with periods } L_1 \text{ and } L_2, \\ w^\varepsilon_3 = 0, \quad \text{at } z = 0, h, \\ \dfrac{\partial w^\varepsilon_i}{\partial z} - 2\displaystyle\sum_{j=1}^2 \alpha_{ij}w^\varepsilon_j = 2\sum_{j=1}^2\alpha_{ij}\theta^\varepsilon_j, \quad i = 1,2, \text{ at } z = 0, \\ \dfrac{\partial w^\varepsilon_i}{\partial z} + 2\displaystyle\sum_{j=1}^2 \alpha_{ij}w^\varepsilon_j = -2\sum_{j=1}^2\alpha_{ij}\theta^\varepsilon_j, \quad i = 1,2, \text{ at } z = h, \\ w^\varepsilon\big|_{t=0} = -\theta^\varepsilon\big|_{t=0}, \quad \text{in } \Omega_\infty, \end{cases} \tag{3.21}$$

where

$$R_\varepsilon(v) := -\frac{\partial v}{\partial t} + \varepsilon\Delta v, \quad \text{for any smooth vector field } v, \tag{3.22}$$

$$J_\varepsilon\bigl(u^\varepsilon, u^0\bigr) := \bigl(u^\varepsilon\cdot\nabla\bigr)u^\varepsilon - \bigl(u^0\cdot\nabla\bigr)u^0. \tag{3.23}$$

We multiply $(3.21)_1$ by w^ε, integrate over Ω and then, integrate it by parts. As a result, after applying the Schwarz and Young inequalities as well, we find:

$$\frac{d}{dt}\|w^\varepsilon\|^2_{L^2(\Omega)} + 2\varepsilon\|\nabla w^\varepsilon\|^2_{L^2(\Omega)} \leqslant \varepsilon^2\|\Delta u^0\|^2_{L^2(\Omega)} + \|R_\varepsilon(\theta^\varepsilon)\|^2_{L^2(\Omega)} + 2\|w^\varepsilon\|^2_{L^2(\Omega)}$$
$$+ 2\varepsilon\int_{\{z=0,h\}} (\nabla w^\varepsilon \boldsymbol{n})\cdot w^\varepsilon\,dS - 2\int_\Omega J_\varepsilon(u^\varepsilon, u^0)\cdot w^\varepsilon\,d\boldsymbol{x}. \tag{3.24}$$

Thanks to Lemma 3.1 and (3.22) with v replaced by θ^ε, we find that

$$\|R_\varepsilon(\theta^\varepsilon)\|^2_{L^2(\Omega)} \leqslant \kappa_T(1+\overline{\alpha}^2)\varepsilon^{\frac{3}{2}}. \tag{3.25}$$

On the other hand, by remembering that $\boldsymbol{n} = (0,0,-1)$ at $z=0$ and $\boldsymbol{n} = (0,0,1)$ at $z=h$, and using $(3.21)_{5,6}$, we notice that

$$[\nabla w^\varepsilon \boldsymbol{n}]_{\tan} = \begin{cases} -(\frac{\partial w_1^\varepsilon}{\partial z}, \frac{\partial w_2^\varepsilon}{\partial z}), & \text{at } z=0, \\ (\frac{\partial w_1^\varepsilon}{\partial z}, \frac{\partial w_2^\varepsilon}{\partial z}), & \text{at } z=h, \end{cases}$$
$$= -2\left(\sum_{j=1}^2 \alpha_{1j}(w_j^\varepsilon + \theta_j^\varepsilon), \sum_{j=1}^2 \alpha_{2j}(w_j^\varepsilon + \theta_j^\varepsilon)\right), \quad \text{at } z=0,h. \tag{3.26}$$

Then, using (3.26), we find that

$$2\varepsilon\left|\int_{\{z=0,h\}} (\nabla w^\varepsilon \boldsymbol{n})\cdot w^\varepsilon\,dS\right| \leqslant \kappa_T\overline{\alpha}\varepsilon\|[w^\varepsilon+\theta^\varepsilon]_{\tan}\|_{L^2(\Gamma)}\|[w^\varepsilon]_{\tan}\|_{L^2(\Gamma)}$$
$$\leqslant \kappa_T\overline{\alpha}\varepsilon\|w^\varepsilon\|^2_{L^2(\Gamma)} + \kappa_T\overline{\alpha}\varepsilon\|[\theta^\varepsilon]_{\tan}\|_{L^2(\Gamma)}\|w^\varepsilon\|_{L^2(\Gamma)}$$
$$\big(\text{using Lemma C.2, (3.8), and the Poincaré inequality for } w^\varepsilon\big)$$
$$\leqslant \kappa_T\overline{\alpha}\varepsilon\|w^\varepsilon\|_{L^2(\Omega)}\|\nabla w^\varepsilon\|_{L^2(\Omega)} + \kappa_T(1+\overline{\alpha}^2)\varepsilon^{\frac{3}{2}}\|\nabla w^\varepsilon\|_{L^2(\Omega)}$$
$$\leqslant \varepsilon\|\nabla w^\varepsilon\|^2_{L^2(\Omega)} + \kappa_T\overline{\alpha}^2\varepsilon\|w^\varepsilon\|^2_{L^2(\Omega)} + \kappa_T(1+\overline{\alpha}^4)\varepsilon^2. \tag{3.27}$$

By applying (3.25), (3.27) to (3.24), we obtain

$$\frac{d}{dt}\|w^\varepsilon\|^2_{L^2(\Omega)} + \varepsilon\|\nabla w^\varepsilon\|^2_{L^2(\Omega)} \leqslant \kappa_T(1+\overline{\alpha}^4)\varepsilon^{\frac{3}{2}}$$
$$+ \kappa_T(1+\overline{\alpha}^2)\|w^\varepsilon\|^2_{L^2(\Omega)} - 2\int_\Omega J_\varepsilon(u^\varepsilon,u^0)\cdot w^\varepsilon\,d\boldsymbol{x}. \tag{3.28}$$

To estimate the last term on the right-hand side of (3.28), using (3.20), (3.23), we first notice that

$$J_\varepsilon(u^\varepsilon,u^0) = (u^\varepsilon\cdot\nabla)w^\varepsilon + (w^\varepsilon\cdot\nabla)(u^\varepsilon - w^\varepsilon) + (u^0\cdot\nabla)\theta^\varepsilon + (\theta^\varepsilon\cdot\nabla)u^0 + (\theta^\varepsilon\cdot\nabla)\theta^\varepsilon. \tag{3.29}$$

Then, we write:

$$\int_\Omega J_\varepsilon(u^\varepsilon,u^0)\cdot w^\varepsilon\,d\boldsymbol{x} := \sum_{j=1}^5 \mathcal{J}_\varepsilon^j, \tag{3.30}$$

where

$$\begin{cases} \mathcal{J}_\varepsilon^1 = \int\limits_\Omega (u^\varepsilon \cdot \nabla) w^\varepsilon \cdot w^\varepsilon \, d\boldsymbol{x} = 0, \quad \mathcal{J}_\varepsilon^2 = \int\limits_\Omega (w^\varepsilon \cdot \nabla)(u^\varepsilon - w^\varepsilon) \cdot w^\varepsilon \, d\boldsymbol{x}, \\ \mathcal{J}_\varepsilon^3 = \int\limits_\Omega (\theta^\varepsilon \cdot \nabla) u^0 \cdot w^\varepsilon \, d\boldsymbol{x}, \quad \mathcal{J}_\varepsilon^4 = \int\limits_\Omega (u^0 \cdot \nabla) \theta^\varepsilon \cdot w^\varepsilon \, d\boldsymbol{x}, \\ \mathcal{J}_\varepsilon^5 = \int\limits_\Omega (\theta^\varepsilon \cdot \nabla) \theta^\varepsilon \cdot w^\varepsilon \, d\boldsymbol{x}. \end{cases} \tag{3.31}$$

To bound $\mathcal{J}_\varepsilon^2$, using (3.20), we first write

$$\mathcal{J}_\varepsilon^2 = \int\limits_\Omega (w^\varepsilon \cdot \nabla) u^0 \cdot w^\varepsilon \, d\boldsymbol{x} + \int\limits_\Omega (w^\varepsilon \cdot \nabla) \theta^\varepsilon \cdot w^\varepsilon \, d\boldsymbol{x}.$$

Then

$$\left| \int\limits_\Omega (w^\varepsilon \cdot \nabla) u^0 \cdot w^\varepsilon \, d\boldsymbol{x} \right| \leqslant \kappa_T \|\nabla u^0\|_{L^\infty(\Omega)} \|w^\varepsilon\|_{L^2(\Omega)}^2 \leqslant \kappa_T \|w^\varepsilon\|_{L^2(\Omega)}^2$$

and, thanks to (3.12), (3.13),

$$\left| \int\limits_\Omega (w^\varepsilon \cdot \nabla) \theta^\varepsilon \cdot w^\varepsilon \, d\boldsymbol{x} \right| \leqslant \kappa_T \|\nabla \theta^\varepsilon\|_{L^\infty(\Omega)} \|w^\varepsilon\|_{L^2(\Omega)}^2 \leqslant \kappa_T (1 + \overline{\alpha}) \|w^\varepsilon\|_{L^2(\Omega)}^2.$$

Thus,

$$\|\mathcal{J}_\varepsilon^2\| \leqslant \kappa_T (1 + \overline{\alpha}) \|w^\varepsilon\|_{L^2(\Omega)}^2. \tag{3.32}$$

Using Lemma 3.1, we bound $\mathcal{J}_\varepsilon^3$ by

$$\begin{aligned} |\mathcal{J}_\varepsilon^3| &\leqslant \|\nabla u^0\|_{L^\infty(\Omega)} \|\theta^\varepsilon\|_{L^2(\Omega)} \|w^\varepsilon\|_{L^2(\Omega)} \leqslant \kappa_T \|\theta^\varepsilon\|_{L^2(\Omega)} \|w^\varepsilon\|_{L^2(\Omega)} \\ &\leqslant \kappa_T (1 + \overline{\alpha}) \varepsilon^{\frac{3}{4}} \|w^\varepsilon\|_{L^2(\Omega)} \leqslant \kappa_T (1 + \overline{\alpha}^2) \varepsilon^{\frac{3}{2}} + \|w^\varepsilon\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.33}$$

Since u_3^0 vanishes at $z = 0$ or h, using the regularity of u^0, we bound $\mathcal{J}_\varepsilon^4$ by

$$\begin{aligned} |\mathcal{J}_\varepsilon^4| &\leqslant \sum_{j=1}^3 \left| \int\limits_\Omega \left(u_1^0 \frac{\partial \theta_j^\varepsilon}{\partial x} + u_2^0 \frac{\partial \theta_j^\varepsilon}{\partial y} \right) w_j^\varepsilon \, d\boldsymbol{x} \right| + \sum_{j=1}^3 \left| \int\limits_\Omega u_3^0 \frac{\partial \theta_j^\varepsilon}{\partial z} w_j^\varepsilon \, d\boldsymbol{x} \right| \\ &\leqslant \|u^0\|_{L^\infty(\Omega)} \sum_{j=1}^3 \left\| \frac{\partial \theta_j^\varepsilon}{\partial x} + \frac{\partial \theta_j^\varepsilon}{\partial y} \right\|_{L^2(\Omega)} \|w_j^\varepsilon\|_{L^2(\Omega)} \\ &\quad + \varepsilon^{\frac{1}{2}} \sum_{j=1}^3 \left\| \frac{u_3^0}{\zeta(z)} \right\|_{L^\infty(\Omega)} \left\| \frac{\zeta(z)}{\sqrt{\varepsilon}} \frac{\partial \theta_j^\varepsilon}{\partial z} \right\|_{L^2(\Omega)} \|w_j^\varepsilon\|_{L^2(\Omega)} \end{aligned}$$

$$\big(\text{using (3.18), (3.19) and Lemma 3.1}\big)$$
$$\leqslant \kappa_T(1+\bar{\alpha})\varepsilon^{\frac{3}{4}}\|w^\varepsilon\|_{L^2(\Omega)} \leqslant \kappa_T\big(1+\bar{\alpha}^2\big)\varepsilon^{\frac{3}{2}} + \|w^\varepsilon\|^2_{L^2(\Omega)}. \tag{3.34}$$

Thanks to (3.12), (3.13) and Lemma 3.1, was can bound $\mathcal{J}^5_\varepsilon$ by

$$\begin{aligned}
\big|\mathcal{J}^5_\varepsilon\big| &\leqslant \sum_{j=1}^{3}\left|\int_\Omega \left(\theta^\varepsilon_1\frac{\partial\theta^\varepsilon_j}{\partial x} + \theta^\varepsilon_2\frac{\partial\theta^\varepsilon_j}{\partial y}\right) w^\varepsilon_j\,d\boldsymbol{x}\right| + \sum_{j=1}^{3}\left|\int_\Omega \theta^\varepsilon_3\frac{\partial\theta^\varepsilon_j}{\partial z} w^\varepsilon_j\,d\boldsymbol{x}\right| \\
&\leqslant \sum_{j=1}^{3}\left\{\|\theta^\varepsilon_1\|_{L^\infty(\Omega)}\left\|\frac{\partial\theta^\varepsilon_j}{\partial x}\right\|_{L^2(\Omega)}\|w^\varepsilon_j\|_{L^2(\Omega)} + \|\theta^\varepsilon_2\|_{L^\infty(\Omega)}\left\|\frac{\partial\theta^\varepsilon_j}{\partial y}\right\|_{L^2(\Omega)}\|w^\varepsilon_j\|_{L^2(\Omega)}\right\} \\
&\quad + \sum_{j=1}^{3}\|\theta^\varepsilon_3\|_{L^\infty(\Omega)}\left\|\frac{\partial\theta^\varepsilon_j}{\partial z}\right\|_{L^2(\Omega)}\|w^\varepsilon_j\|_{L^2(\Omega)} \\
&\leqslant \kappa_T\big(1+\bar{\alpha}^2\big)\varepsilon^{\frac{5}{4}}\|w^\varepsilon\|_{L^2(\Omega)} \leqslant \kappa_T\big(1+\bar{\alpha}^4\big)\varepsilon^{\frac{5}{2}} + \|w^\varepsilon\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.35}$$

Then, using (3.32) through (3.35), (3.30) gives

$$\left|\int_\Omega J_\varepsilon\big(u^\varepsilon, u^0\big)\cdot w^\varepsilon\,d\boldsymbol{x}\right| \leqslant \kappa_T\big(1+\bar{\alpha}^4\big)\varepsilon^{\frac{3}{2}} + \kappa_T(1+\bar{\alpha})\|w^\varepsilon\|^2_{L^2(\Omega)}. \tag{3.36}$$

Applying (3.36) to (3.28), we obtain

$$\frac{d}{dt}\|w^\varepsilon\|^2_{L^2(\Omega)} + \varepsilon\|\nabla w^\varepsilon\|^2_{L^2(\Omega)} \leqslant \kappa_T\big(1+\bar{\alpha}^4\big)\varepsilon^{\frac{3}{2}} + \kappa_T\big(1+\bar{\alpha}^2\big)\|w^\varepsilon\|^2_{L^2(\Omega)}.$$

Moreover, using (3.7) through (3.9) and $(3.21)_7$, we observe that

$$\|w^\varepsilon|_{t=0}\|_{L^2(\Omega)} = \|\theta^\varepsilon|_{t=0}\|_{L^2(\Omega)} \leqslant \kappa_T(1+\bar{\alpha})\varepsilon^{\frac{1}{2}}\big\|e^{-\frac{z}{\sqrt{\varepsilon}}}\big\|_{L^2(\Omega)} + l.o.t. \leqslant \kappa_T(1+\bar{\alpha})\varepsilon^{\frac{3}{4}}.$$

Thanks to the Gronwall inequality, we finally have the bounds on the remainder, w^ε,

$$\|w^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leqslant \kappa(T,\bar{\alpha},u_0,f)\varepsilon^{\frac{3}{4}}, \qquad \|w^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \leqslant \kappa(T,\bar{\alpha},u_0,f)\varepsilon^{\frac{1}{4}}. \tag{3.37}$$

3.4. *Proof of convergence*

Using (3.20), we first notice that

$$\big|u^\varepsilon - u^0\big| \leqslant \big|w^\varepsilon\big| + \big|\theta^\varepsilon\big| \quad \text{pointwise in } \Omega_\infty\times(0,T). \tag{3.38}$$

Then, using (3.37) and Lemma 3.1, (1.8) follows from (3.38). □

4. Bounded domain

In this section we consider the Navier–Stokes equations, (1.1), (1.5), and the Euler equations, (1.2), in a bounded domain Ω in $\mathbb{R}^3$ with boundary Γ having regularity as in (1.7). To handle the geometric difficulties of a curved boundary, we must treat Ω as a manifold with boundary, first constructing charts on $\Gamma = \partial\Omega$ in a special way, as we describe below.

We consider the boundary, Γ as a submanifold of $\mathbb{R}^3$. Then, since Γ is a compact and smooth surface in $\mathbb{R}^3$, we construct a system of finitely many charts where each chart is a C^m-map, $\widetilde{\psi}$, from a domain, $\widetilde{U}$, in $\mathbb{R}^2$ to a domain, $\widetilde{V}$, in Γ. More precisely, we choose an orthogonal curvilinear system $(\xi') = (\xi_1, \xi_2)$ in $\widetilde{U}$ so that, for any point $\widetilde{\boldsymbol{x}}$ on $\widetilde{V} \subset \Gamma$, we write

$$\widetilde{\boldsymbol{x}} = \widetilde{\psi}(\xi'), \quad \xi' = (\xi_1, \xi_2) \in \widetilde{U}. \tag{4.1}$$

Differentiating (4.1) with respect to ξ_i, $i = 1, 2$, variables, we obtain the covariant basis on $\widetilde{U}$ and the metric tensor:

$$\widetilde{\boldsymbol{g}}_i(\xi') := \frac{\partial \widetilde{\boldsymbol{x}}}{\partial \xi_i}, \quad i = 1, 2, \tag{4.2}$$

and

$$\big(\widetilde{g}_{ij}(\xi')\big)_{1\leqslant i,j\leqslant 2} := (\widetilde{\boldsymbol{g}}_i \cdot \widetilde{\boldsymbol{g}}_j)_{1\leqslant i,j\leqslant 2} = \operatorname{diag}(\widetilde{\boldsymbol{g}}_1 \cdot \widetilde{\boldsymbol{g}}_1, \widetilde{\boldsymbol{g}}_2 \cdot \widetilde{\boldsymbol{g}}_2). \tag{4.3}$$

Moreover, we see that the determinant of the metric tensor is strictly positive;

$$\widetilde{g}(\xi') := \det(\widetilde{g}_{ij}) > 0, \quad \text{for all } \xi' \text{ in the closure of } \widetilde{U}. \tag{4.4}$$

For any smooth 2d compact manifold Γ in $\mathbb{R}^3$, one can construct a system of finitely many charts, which satisfy (4.3), (4.4). Hence, the class of domains under consideration in this paper covers all bounded domains in $\mathbb{R}^3$ having boundary regularity as in (1.7). Moreover, as we will see below in Section 4.2, the construction of the corrector is independent of our choice of charts. Hence, it is sufficient to restrict our attention to a single chart only, since any estimates we develop will apply equally to all of Ω.

We define Γ_c to be the interior tubular neighborhood of Ω with width c for any sufficiently small $c > 0$, and let $a > 0$ be small enough that Γ_{3a} is such a tubular neighborhood. We can globally define the coordinate ξ_3 on Γ_{3a} to be distance from the boundary, with positive distances directed inward.

We fix the orientation of ξ' variables on $\widetilde{V}$ so that

$$\boldsymbol{n}(\xi') := -\frac{\widetilde{\boldsymbol{g}}_1 \times \widetilde{\boldsymbol{g}}_2}{|\widetilde{\boldsymbol{g}}_1 \times \widetilde{\boldsymbol{g}}_2|}\big(\widetilde{\psi}(\xi')\big), \tag{4.5}$$

where $\boldsymbol{n}(\xi')$ is the unit outer normal vector on $\widetilde{V}$. Then, letting $U = \widetilde{U} \times (0, 3a)$, we define a chart $\psi: U \to V$ (giving what are sometimes called *boundary normal coordinates*):

$$\boldsymbol{x} = \psi(\boldsymbol{\xi}) = \widetilde{\psi}(\xi') - \xi_3 \boldsymbol{n}(\xi'), \quad \boldsymbol{x} = (x_1, x_2, x_3) \in \Gamma_{3a}. \tag{4.6}$$

By differentiating ψ in $\boldsymbol{\xi}$ variables and using (4.2), we define the covariant basis of the curvilinear system $\boldsymbol{\xi}$:

$$\boldsymbol{g}_i(\boldsymbol{\xi}) = \widetilde{\boldsymbol{g}}_i(\xi') - \xi_3 \frac{\partial \boldsymbol{n}}{\partial \xi_i}(\xi'), \quad i = 1, 2, \qquad \boldsymbol{g}_3(\boldsymbol{\xi}) = -\boldsymbol{n}(\xi'); \tag{4.7}$$

hence, from (4.5), (4.7), we see that the covariant basis satisfies the right-hand rule.

One important observation here is that the orthogonality, on $\widetilde{V}$, of $\widetilde{\boldsymbol{g}}_i$, $i = 1, 2$, does not imply the orthogonality, in V, of $\boldsymbol{g}_i$, $i = 1, 2, 3$. To see this, we first notice that

$$\frac{\partial \boldsymbol{n}}{\partial \xi_i} = (\text{linear combination of } \widetilde{\boldsymbol{g}}_1 \text{ and } \widetilde{\boldsymbol{g}}_2), \quad i = 1, 2. \tag{4.8}$$

Thus, $\boldsymbol{g}_i \cdot \boldsymbol{g}_3 = 0$ for $i = 1, 2$, but $\boldsymbol{g}_1 \cdot \boldsymbol{g}_2 \neq 0$ in general. Consequently, the metric tensor $(g_{ij}(\boldsymbol{\xi}))_{1 \leqslant i,j \leqslant 3} := (\boldsymbol{g}_i \cdot \boldsymbol{g}_j)_{1 \leqslant i,j \leqslant 3}$ satisfies:

$$\begin{cases} g_{ij} = 0, \quad i = 3 \text{ and } j = 1, 2, \text{ or } i = 1, 2 \text{ and } j = 3, \\ g_{33} = 1. \end{cases} \tag{4.9}$$

Moreover, thanks to (4.4), by choosing the thickness $3a > 0$ of the tubular neighborhood Γ_{3a} small enough, we see that

$$g(\boldsymbol{\xi}) := \det(g_{ij})_{1 \leqslant i,j \leqslant 3} > 0 \quad \text{for all } \boldsymbol{\xi} \text{ in the closure of } U = \widetilde{U} \times (0, 3a); \tag{4.10}$$

The function, $\sqrt{g} := g^{1/2}$, is the magnitude of the Jacobian determinant of the chart, ψ.

The matrix of the contravariant metric components are defined in the closure of U as well:

$$(g^{ij})_{1 \leqslant i,j \leqslant 3} = (g_{ij})^{-1}_{1 \leqslant i,j \leqslant 3} = \frac{1}{g} \begin{pmatrix} g_{22} & -g_{12} & 0 \\ -g_{12} & g_{11} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.11}$$

We introduce the normalized covariant vectors:

$$\boldsymbol{e}_i = \frac{\boldsymbol{g}_i}{|\boldsymbol{g}_i|}, \quad 1 \leqslant i \leqslant 3. \tag{4.12}$$

Then, for a vector valued function F, defined on U, in the form

$$F = \sum_{i=1}^{3} F_i \boldsymbol{e}_i,$$

one can classically express the divergence operator acting on F in the $\boldsymbol{\xi}$ variable (see [11] or [36]) as

$$\operatorname{div} F = \frac{1}{\sqrt{g}} \sum_{i=1}^{2} \frac{\partial}{\partial \xi_i} \left(\frac{\sqrt{g}}{\sqrt{g_{ii}}} F_i \right) + \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g} F_3)}{\partial \xi_3}. \tag{4.13}$$

We write the Laplacian of F as

$$\Delta F = \sum_{i=1}^{3} \left(\mathcal{S}^i F + \mathcal{L}^i F_i + \frac{\partial^2 F_i}{\partial \xi_3^2} \right) \boldsymbol{e}_i, \tag{4.14}$$

where

$$\begin{cases} \mathcal{S}^i F = \begin{pmatrix} \text{linear combination of tangential derivatives} \\ \text{of } F_j, 1 \leqslant j \leqslant 3, \text{ in } \xi', \text{ up to order 2} \end{pmatrix}, \\ \mathcal{L}^i F_i = \left(\text{proportional to } \dfrac{\partial F_i}{\partial \xi_3} \right). \end{cases} \tag{4.15}$$

Remark 4.1. Note that the coefficients of $\mathcal{S}^i$ and $\mathcal{L}^i$, $1 \leqslant i \leqslant 3$ in (4.15), are multiples of $\sqrt{g}$, $1/\sqrt{g}$, $\sqrt{g_{ii}}$, $1/\sqrt{g_{ii}}$, $i = 1, 2$, g_{12}, g_{21}, and their derivatives. Thanks to (4.10), all these quantities are well-defined because of the regularity assumed for Γ in (1.7).

Remark 4.2. Thanks to (4.9), we notice that the tangential directions are perpendicular to the normal direction in the tubular neighborhood Γ_{3a}. Indeed this property enables us to obtain the expression of Laplacian as (4.14), (4.15), which is essentially the same as for the case of orthogonal curvilinear system. The explicit expression of Laplacian in orthogonal system appears in, e.g., [21].

For smooth vector fields $F, G : U \to \mathbb{R}^3$, we consider $\nabla_F G$, the covariant derivative of G in the direction F, which gives $F \cdot \nabla G$ in the Cartesian coordinate system. More precisely, we consider the smooth functions F and G in the form

$$F = \sum_{i=1}^{3} F_i \boldsymbol{e}_i, \qquad G = \sum_{i=1}^{3} G_i \boldsymbol{e}_i.$$

Then, one can write $\nabla_F G$ in the $\boldsymbol{\xi}$ variable,

$$\nabla_F G = \sum_{i=1}^{3} \left\{ \mathcal{P}_i \left(F_1, F_2 : \frac{\partial G_i}{\partial \xi_1}, \frac{\partial G_i}{\partial \xi_2} \right) + F_3 \frac{\partial G_i}{\partial \xi_3} + \mathcal{Q}_i(F : G) \right\} \boldsymbol{e}_i, \tag{4.16}$$

where

$$\begin{cases} \mathcal{P}_i \left(F_1, F_2 : \dfrac{\partial G_i}{\partial \xi_1}, \dfrac{\partial G_i}{\partial \xi_2} \right) = \begin{pmatrix} \text{product of a linear combination of} \\ \text{the tangential components } F_1, F_2 \text{ of } F, \\ \text{and sum of tangential derivatives of } G_i \end{pmatrix}, \\ \mathcal{Q}_i(F : G) = (\text{linear combination of the products } F_j G_k,\ 1 \leqslant j, k \leqslant 3). \end{cases} \tag{4.17}$$

Remark 4.3. $\mathcal{Q}_i(F : G)$, $1 \leqslant i \leqslant 3$, are related to the Christoffel symbols of the second kind, which comes from the twisting effects of the curvilinear system $\boldsymbol{\xi}$. For the case of an orthogonal system, the explicit expression of (4.16) is given in Appendix 2 of [1].

Using the expression of contravariant components of the strain rate tensor, and by remembering, from (4.3), (4.7), (4.9), that the covariant basis (and hence the normalized covariant basis) is triply orthogonal on Γ, we write the generalized Navier boundary conditions, (1.5), for $F = \sum_{i=1}^{3} F_i \boldsymbol{e}_i$ as

$$\begin{cases} F_3 = 0, & \text{at } \xi_3 = 0, \\ -\dfrac{1}{2} \dfrac{\partial F_i}{\partial \xi_3} + \mathcal{M}_i \left(F_i, \dfrac{\partial F_3}{\partial \xi_i} \right) + \displaystyle\sum_{j=1}^{2} \alpha_{ij} F_j = 0, & \text{in } \boldsymbol{e}_i \text{ direction at } \xi_3 = 0,\ i = 1, 2, \end{cases} \tag{4.18}$$

where

$$\mathcal{M}_i \left(F_i, \frac{\partial F_3}{\partial \xi_i} \right) = \begin{pmatrix} \text{linear combination of the tangential component } F_i \text{ and} \\ \text{the derivative in } \xi_i \text{ of the normal component } F_3 \end{pmatrix}. \tag{4.19}$$

Remark 4.4. Thanks to (1.7), (4.10), the coefficients of $\mathcal{M}_i(F)$, $i = 1, 2$, are well-defined. Concerning an orthogonal system, the explicit expression of $\mathcal{M}_i(F)$, $i = 1, 2$, appears on p. 115 of [43].

4.1. The corrector

In defining the corrector, θ^ε, we parallel the strategy we used in Section 3 for a periodic channel domain as closely as possible, employing an asymptotic expansion, $u^\varepsilon \cong u^0 + \theta^\varepsilon$, as in (3.4), but adapting the corrector to the curved boundary.

With the unit vectors, $\boldsymbol{e}_i$, defined as in (4.12), on U, θ^ε can be written

$$\theta^\varepsilon := \sum_{i=1}^{3} \theta_i^\varepsilon \boldsymbol{e}_i. \tag{4.20}$$

The tangential components θ_i^ε, $i = 1, 2$, will be constructed to correct the tangential discrepancy in the boundary conditions related to the normal derivative of $u^\varepsilon - u^0$ on the boundary. Then the normal component θ_3^ε will be deduced from the divergence-free condition on θ^ε.

To define the corrector θ^ε appearing in (4.20), since $u^0 \cdot \boldsymbol{n} = 0$ on Γ, we first set

$$\widetilde{u} = 2\big(\big[\mathbf{S}\big(u^0\big)\boldsymbol{n}\big]_{\tan} + \mathcal{A}u^0\big),$$

defined on all of Γ, and write, in coordinates,

$$\widetilde{u}\big(\xi'; t\big) = \sum_{i=1}^{2} \widetilde{u}_i\big(\xi'; t\big)\boldsymbol{e}_i\big|_{\xi_3=0}, \qquad \widetilde{u}_i\big(\xi'; t\big) := \widetilde{u}\big(\xi'; t\big) \cdot \boldsymbol{e}_i\big|_{\xi_3=0}. \tag{4.21}$$

Then, we insert the expansion $u^\varepsilon \cong u^0 + \theta^\varepsilon$ into the generalized Navier boundary conditions, (1.5), and, thanks to $(4.18)_2$, we find that, for $i = 1, 2$,

$$\frac{1}{2}\widetilde{u}_i\big(\xi'; t\big) - \frac{1}{2}\frac{\partial \theta_i^\varepsilon}{\partial \xi_3} + \mathcal{M}_i\left(\theta_i^\varepsilon, \frac{\partial \theta_3^\varepsilon}{\partial \xi_i}\right) + \sum_{j=1}^{2} \alpha_{ij}\theta_j^\varepsilon \cong 0, \quad \text{at } \xi_3 = 0.$$

Using (4.19), we expect that $\partial\theta_i^\varepsilon/\partial\xi_3 \gg \mathcal{M}_i(\theta_i^\varepsilon, \partial\theta_3^\varepsilon/\partial\xi_i)$ or $\sum_{j=1}^{2} \alpha_{ij}\theta_j^\varepsilon$, $i = 1, 2$, for smooth α_{ij}, $1 \leqslant i, j \leqslant 2$, independent of ε. Hence, we use the Neumann boundary condition for θ_i^ε,

$$\left.\frac{\partial \theta_i^\varepsilon}{\partial \xi_3}\right|_{\xi_3=0} = \widetilde{u}_i\big(\xi'; t\big), \quad i = 1, 2. \tag{4.22}$$

We can now model the corrector after the flat-space version in (3.8). We define a smooth cutoff function, $\sigma(\xi_3)$, with

$$\sigma(\xi_3) := \begin{cases} 1, & 0 \leqslant \xi_3 \leqslant a, \\ 0, & \xi_3 \geqslant 2a. \end{cases} \tag{4.23}$$

Letting

$$\gamma_i := \gamma_i\big(\xi'\big) = \left.\frac{\sqrt{g}}{\sqrt{g_{ii}}}\right|_{\xi_3=0},$$

we define the tangential components of the corrector by

$$\theta_i^\varepsilon(\boldsymbol{\xi};t) := -\varepsilon \frac{\sqrt{g_{ii}}(\boldsymbol{\xi})}{\sqrt{g}(\boldsymbol{\xi})}\left[(\gamma_i \widetilde{u}_i)(\xi';t)\frac{\partial}{\partial \xi_3}\left(\sigma(\xi_3)\left(1-e^{-\frac{\xi_3}{\sqrt{\varepsilon}}}\right)\right)\right], \quad i=1,2. \tag{4.24}$$

It follows from (4.13) that

$$\begin{aligned}\sqrt{g}\operatorname{div}\theta^\varepsilon &= -\varepsilon\sum_{i=1}^{2}\frac{\partial}{\partial \xi_i}(\gamma_i\widetilde{u}_i)(\xi';t)\frac{\partial}{\partial \xi_3}\left(\sigma(\xi_3)\left(1-e^{-\frac{\xi_3}{\sqrt{\varepsilon}}}\right)\right)+\frac{\partial(\sqrt{g}\theta_3^\varepsilon)}{\partial \xi_3}\\ &= -\varepsilon\frac{\partial}{\partial \xi_3}\left\{\sum_{i=1}^{2}\frac{\partial}{\partial \xi_i}(\gamma_i\widetilde{u}_i)(\xi';t)\sigma(\xi_3)\left(1-e^{-\frac{\xi_3}{\sqrt{\varepsilon}}}\right)\right\}+\frac{\partial(\sqrt{g}\theta_3^\varepsilon)}{\partial \xi_3}.\end{aligned}$$

Thus, we can easily ensure that θ^ε is divergence-free by letting

$$\theta_3^\varepsilon(\boldsymbol{\xi};t) := \varepsilon\frac{1}{\sqrt{g}(\boldsymbol{\xi})}\left\{\sum_{i=1}^{2}\frac{\partial}{\partial \xi_i}(\gamma_i\widetilde{u}_i)(\xi';t)\right\}\sigma(\xi_3)\left(1-e^{-\frac{\xi_3}{\sqrt{\varepsilon}}}\right). \tag{4.25}$$

It is easy to see that θ_3^ε vanishes at $\xi_3=0$, and by differentiating (4.24) and using (4.23), we see that each tangential component θ_i^ε, $i=1,2$, satisfies the boundary condition (4.22) to within order $\sqrt{\varepsilon}$:

$$\left.\frac{\partial \theta_i^\varepsilon}{\partial \xi_3}\right|_{\xi_3=0} = \widetilde{u}_i(\xi';t)-\sqrt{\varepsilon}E(\xi';t), \tag{4.26}$$

where

$$E(\xi';t) = \gamma_i(\xi')\frac{\partial}{\partial \xi_3}\left(\frac{\sqrt{g_{ii}}}{\sqrt{g}}\right)\bigg|_{\xi_3=0}\widetilde{u}_i(\xi';t). \tag{4.27}$$

Due to the presence of σ in (4.24), (4.25), we also have

$$\left.\frac{\partial^k \theta_i^\varepsilon}{\partial \xi_j^k}\right|_{\xi_3\geqslant 2a} = 0, \quad 1\leqslant i,j\leqslant 3,\ k\geqslant 0. \tag{4.28}$$

Remark 4.5. For the case of a flat boundary Γ, one can choose curvilinear coordinates with the metric tensor $(g_{ij})_{1\leqslant i,j\leqslant 3}$, defined in (4.9), as the identity matrix $\boldsymbol{I}_{3\times 3}$, and hence $\sqrt{g}$, $\sqrt{g_{ii}}$ and γ_i, $i=1,2$, appearing in (4.24), (4.25), (4.27), are equal to 1. This implies that the expression of the corrector defined by (4.24), (4.25) are identical to (3.8), (3.9) in a channel domain where the error E in (4.27) is now equal to 0.

Remark 4.6. The form of our corrector (4.24), (4.25) is similar to the background flow in Lemma 1 of [59].

4.2. The corrector in principal curvature coordinates

In this section, we express the corrector in particularly convenient and geometrically meaningful coordinates called *principal curvature coordinates*.

We define an umbilical point of Γ to be a point at which the principal curvatures, κ_1 and κ_2, are equal (this also includes what some authors refer to as a planar point, where both curvatures vanish). By Lemma 3.6.6 of [36], in some neighborhood of any non-umbilical point there exists a chart in which the metric tensor of (4.2) is diagonal (as is the second fundamental form) and the coordinate lines are parallel to the principal directions at each point. Such a chart is also called a *principal curvature coordinate system*.

For now, we assume that we are working in such a chart, $\widetilde{\psi}_p: \widetilde{U}_p \to \widetilde{V} \subseteq \Gamma$,

$$\widetilde{\boldsymbol{x}} = \widetilde{\psi}_p(\eta'), \quad \eta' = (\eta_1, \eta_2) \in \widetilde{U}_p.$$

The corresponding covariant basis and metric tensor are

$$\begin{aligned} &\widetilde{\boldsymbol{q}}_i(\eta') = \frac{\partial \widetilde{\psi}_p}{\partial \eta_i}, \quad i = 1, 2, \\ &\big(\widetilde{q}_{ij}(\eta')\big)_{1 \leqslant i, j \leqslant 2} = (\widetilde{\boldsymbol{q}}_i \cdot \widetilde{\boldsymbol{q}}_j)_{1 \leqslant i, j \leqslant 2} = \mathrm{diag}(\widetilde{\boldsymbol{q}}_1 \cdot \widetilde{\boldsymbol{q}}_1, \quad \widetilde{\boldsymbol{q}}_2 \cdot \widetilde{\boldsymbol{q}}_2). \end{aligned} \tag{4.29}$$

Using (4.6) with ξ' and $\widetilde{\psi}$ replaced by η' and $\widetilde{\psi}_p$, we define a chart ψ_p from $U_p = \widetilde{U}_p \times (0, 3a)$ into Γ_{3a} by

$$\boldsymbol{x} = \psi_p(\boldsymbol{\eta}) = \widetilde{\psi}_p(\eta') - \xi_3 \boldsymbol{n}(\eta'), \quad \boldsymbol{\eta} = (\eta', \xi_3) \in U_p.$$

As before, ξ_3 is the distance from the boundary, which we note does not depend upon the choice of the boundary chart.

In the principal curvature coordinate system on $\widetilde{U}_p$, the unit outer normal vector $\boldsymbol{n}$ satisfies

$$\frac{\partial \boldsymbol{n}}{\partial \eta_i} = \kappa_i(\eta') \widetilde{\boldsymbol{q}}_i, \quad i = 1, 2.$$

Hence, differentiating ψ_p in the $\boldsymbol{\eta}$ variables gives the covariant basis of the coordinate system $\boldsymbol{\eta}$,

$$\boldsymbol{q}_i(\boldsymbol{\eta}) = \big(1 - \kappa_i(\eta')\xi_3\big)\widetilde{\boldsymbol{q}}_i(\eta'), \quad i = 1, 2, \qquad \boldsymbol{q}_3(\boldsymbol{\eta}) = -\boldsymbol{n}(\eta'). \tag{4.30}$$

Using (4.29), (4.30), the metric tensor $(q_{ij})_{1 \leqslant i, j \leqslant 3}$ is written in the form

$$(q_{ij})_{1 \leqslant i, j \leqslant 3} = \begin{pmatrix} (1 - \kappa_1(\eta')\xi_3)^2 \widetilde{q}_{11} & 0 & 0 \\ 0 & (1 - \kappa_2(\eta')\xi_3)^2 \widetilde{q}_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.31}$$

with its determinant, $q(\boldsymbol{\eta})$, bounded away from zero. This is guaranteed by simply choosing the thickness, $3a > 0$, of the tubular neighborhood small enough.

It is easy to see that the coordinate system, $\boldsymbol{\eta}$, derived from the principal curvature coordinate system, satisfies (4.4), (4.9), (4.10). Hence we use the expression of the corrector θ^ε, (4.20), (4.24), (4.25), in $\boldsymbol{\eta}$ coordinates and write

$$\theta^\varepsilon = \theta^\varepsilon_{\boldsymbol{\tau}} + \theta^\varepsilon_3 \boldsymbol{e}_3,$$

where

$$\begin{cases} \theta^\varepsilon_{\boldsymbol{\tau}} = -\varepsilon \dfrac{\partial}{\partial \xi_3}\big(\sigma(\xi_3)\big(1 - e^{-\frac{\xi_3}{\sqrt{\varepsilon}}}\big)\big) \displaystyle\sum_{i=1}^{2} \left\{ \frac{\sqrt{q_{ii}}}{\sqrt{q}} \left[\frac{\sqrt{q}}{\sqrt{q_{ii}}} \right]_{\xi_3 = 0} \widetilde{u}(\xi'; t) \cdot \widehat{\boldsymbol{e}}_i|_{\xi_3 = 0} \right\} \widehat{\boldsymbol{e}}_i, \\ \theta^\varepsilon_3 = \varepsilon \sigma(\xi_3)\big(1 - e^{-\frac{\xi_3}{\sqrt{\varepsilon}}}\big) \dfrac{1}{\sqrt{q}} \left\{ \displaystyle\sum_{i=1}^{2} \frac{\partial}{\partial \xi_i} \left(\frac{\sqrt{q}}{\sqrt{q_{ii}}} \bigg|_{\xi_3 = 0} \widetilde{u}(\xi'; t) \cdot \widehat{\boldsymbol{e}}_i|_{\xi_3 = 0} \right) \right\}, \end{cases} \tag{4.32}$$

for $\widehat{\boldsymbol{e}}_i = \boldsymbol{q}_i / |\boldsymbol{q}_i|$, $i = 1, 2$, and $\boldsymbol{e}_3 = -\boldsymbol{n}$.

Using (4.31), it is easy to see that

$$\frac{\sqrt{q_{ii}}}{\sqrt{q}}\left[\frac{\sqrt{q}}{\sqrt{q_{ii}}}\right]_{\xi_3=0} = \frac{1}{1-\kappa_j(\eta')\xi_3} \quad \text{for } i = 1, 2 \text{ and } j = 3 - i. \tag{4.33}$$

Then, combining (4.32), (4.33), we find

$$\theta^\varepsilon_{\boldsymbol{\tau}}(\boldsymbol{\eta}) = -\varepsilon \frac{\partial}{\partial \xi_3}\big(\sigma(\xi_3)\big(1 - e^{-\frac{\xi_3}{\sqrt{\varepsilon}}}\big)\big)\mathbf{M}(\boldsymbol{\eta})\widetilde{u}\big(\eta'; t\big), \tag{4.34}$$

where $\mathbf{M}$ is a smooth type $(1, 1)$ tensor defined, in our coordinates, by

$$\mathbf{M}(\boldsymbol{\eta})\widetilde{F} := \sum_{i=1}^{2} \frac{1}{1-\kappa_{3-i}(\eta')\xi_3}\widetilde{F}_i\widehat{\boldsymbol{e}}_i, \quad \widetilde{F} = \sum_{i=1}^{2}\widetilde{F}_i\widehat{\boldsymbol{e}}_i. \tag{4.35}$$

On the boundary, the divergence operator is

$$\operatorname{div}_{\boldsymbol{\tau}} = \frac{1}{\sqrt{\widetilde{q}}}\sum_{i=1}^{2}\frac{\partial}{\partial \eta_i}\left(\sqrt{\frac{\widetilde{q}}{\widetilde{q}_{ii}}}\widetilde{F}_i\right).$$

Then by (4.31),

$$\sqrt{q} = \big(1 - \kappa_1\big(\eta'\big)\xi_3\big)\big(1 - \kappa_2\big(\eta'\big)\xi_3\big)\sqrt{\widetilde{q}},$$

so we can write θ^ε_3 in (4.32) as

$$\theta^\varepsilon_3 = \varepsilon\sigma(\xi_3)\frac{1 - e^{-\frac{\xi_3}{\sqrt{\varepsilon}}}}{(1-\kappa_1(\eta')\xi_3)(1-\kappa_2(\eta')\xi_3)}\operatorname{div}_{\boldsymbol{\tau}}\widetilde{u}. \tag{4.36}$$

Although we assumed in its derivation that we were near a non-umbilical point so that we could construct a principal curvature coordinate system, our expression for θ^ε is perfectly valid at an umbilical point, where we simply have $\kappa_1 = \kappa_2$ thanks to the smoothness of the curvatures in the tangential variables.

Finally, a straightforward but lengthy calculation, which we omit, shows that (4.24), (4.25) transforms to (4.34), (4.36) under the change of variables from ψ to ψ_p, showing that our corrector in the form (4.24), (4.25) is covariant with respect to the changes of charts. (This is perhaps not immediately obvious, because (4.24), (4.25) involve the metrics both on the boundary and in the tubular neighborhood.)

Remark 4.7. For most smooth, bounded domains in $\mathbb{R}^3$, principal curvature coordinate systems can be constructed in the neighborhood of all but at most isolated points; in fact, having only isolated umbilical points is (in some sense) generic. And, for instance, a sphere, while it consists only of umbilical points, can be covered by two charts, both of which use principal curvature coordinates (essentially, spherical coordinates). When such coordinates suit the boundary of a domain, the expression for the tangential corrector in (4.34), (4.35) is both simpler to calculate and simpler to interpret than the expression in (4.24). In such coordinates, the expressions for the differential operators, such as div, curl, Δ, can also be written more simply. Though we used principal curvature coordinates in this section to prove that our corrector is independent of the choice of charts, we cannot restrict ourselves to such coordinates in the rest of our analysis, as that would put constraints on the geometry of the domains that we would be able to treat.

Remark 4.8. As a special example of a smooth bounded domain in $\mathbb{R}^3$, consider a solid torus, which has no umbilical points on its boundary. Hence, we need only one principal curvature coordinate system, if we allow it to be periodic in the tangential variables. In this sense, the solid torus is the simplest smooth bounded domain to work with in $\mathbb{R}^3$.

4.3. Bounds on the corrector

We follow the convention described at the beginning of Section 3.2, though now the tangential variables are ξ_1 and ξ_2, and as in (3.11), we let $\bar{\alpha} = \|\mathcal{A}\|_{C^m(\Gamma)}$, $m > 6$. Then, from (4.24), (4.25), we infer that

$$\left\| \frac{\partial \theta_i^\varepsilon}{\partial \tau} \right\|_{L^\infty([0,T]\times\bar{U})} \leqslant \kappa_T(1+\bar{\alpha})\varepsilon^{\frac{1}{2}}, \qquad \left\| \frac{\partial \theta_i^\varepsilon}{\partial \xi_3} \right\|_{L^\infty([0,T]\times\bar{U})} \leqslant \kappa_T(1+\bar{\alpha}), \quad i = 1, 2, \tag{4.37}$$

and

$$\left\| \frac{\partial \theta_3^\varepsilon}{\partial \tau} \right\|_{L^\infty([0,T]\times\bar{U})} \leqslant \kappa_T(1+\bar{\alpha})\varepsilon, \qquad \left\| \frac{\partial \theta_3^\varepsilon}{\partial \xi_3} \right\|_{L^\infty([0,T]\times\bar{U})} \leqslant \kappa_T(1+\bar{\alpha})\varepsilon^{\frac{1}{2}}. \tag{4.38}$$

We now state the estimates on the corrector in the lemma below, which we omit the proof as it is essentially the same as that of Lemma 3.1 because of (4.10).

Lemma 4.9. *Assume (1.7) holds and that $k, l, n \geqslant 0$ are integers either $l = 1$, $k = 0$ or $l = 0$, $0 \leqslant k \leqslant 2$. Then the corrector, θ^ε, defined by (4.24), (4.25), satisfies*

$$\begin{cases} \left\| \dfrac{\partial^{l+k+n}\theta_i^\varepsilon}{\partial t^l \partial \tau^k \partial \xi_3^n} \right\|_{L^\infty(0,T;L^2(U))} \leqslant C|V|^{\frac{1}{2}}(1+\bar{\alpha})\varepsilon^{\frac{3}{4}-\frac{n}{2}}, \quad i = 1, 2, \\ \left\| \dfrac{\partial^{l+k}\theta_3^\varepsilon}{\partial t^l \partial \tau^k} \right\|_{L^\infty(0,T;L^2(U))} \leqslant C|V|^{\frac{1}{2}}(1+\bar{\alpha})\varepsilon, \\ \left\| \dfrac{\partial^{l+k+n+1}\theta_3^\varepsilon}{\partial t^l \partial \tau^k \partial \xi_3^{n+1}} \right\|_{L^\infty(0,T;L^2(U))} \leqslant C|V|^{\frac{1}{2}}(1+\bar{\alpha})\varepsilon^{\frac{3}{4}-\frac{n}{2}}, \end{cases}$$

for $C = C(T, l, k, n, u_0, f) > 0$, independent of ε, $\mathcal{A}$ and the measure $|V|$ of $V = \psi(U)$.

In addition to the estimates in Lemma 4.9, as for the case of the channel domain, one can easily verify that the corrector θ^ε satisfies that, $i = 1, 2$,

$$\begin{aligned} \left\| \frac{\xi_3}{\sqrt{\varepsilon}} \frac{\partial \theta_i^\varepsilon}{\partial \xi_3} \right\|_{L^\infty(0,T;L^2(U))} &\leqslant C|V|^{\frac{1}{2}}(1+\bar{\alpha})\varepsilon^{\frac{1}{4}}, \\ \left\| \frac{\xi_3}{\sqrt{\varepsilon}} \frac{\partial \theta_3^\varepsilon}{\partial \xi_3} \right\|_{L^\infty(0,T;L^2(U))} &\leqslant C|V|^{\frac{1}{2}}(1+\bar{\alpha})\varepsilon^{\frac{1}{2}}. \end{aligned} \tag{4.39}$$

4.4. Error analysis

We set the remainder:

$$w^\varepsilon := u^\varepsilon - u^0 - \theta^\varepsilon. \tag{4.40}$$

Then, using (1.1), (1.2), (1.5), (4.40) and the fact that $\theta_3^\varepsilon = 0$ on Γ, the equations for w^ε read

$$\begin{cases} \dfrac{\partial w^\varepsilon}{\partial t} - \varepsilon \Delta w^\varepsilon + \nabla\big(p^\varepsilon - p^0\big) = \varepsilon \Delta u^0 + R_\varepsilon\big(\theta^\varepsilon\big) - J_\varepsilon\big(u^\varepsilon, u^0\big), & \text{in } \Omega \times (0,T), \\ \operatorname{div} w^\varepsilon = 0, & \text{in } \Omega \times (0,T), \\ w^\varepsilon \cdot \boldsymbol{n} = 0, & \text{on } \Gamma, \\ \big[\mathbf{S}\big(w^\varepsilon\big)\boldsymbol{n}\big]_{\tan} + \mathcal{A} w^\varepsilon = -\big[\mathbf{S}\big(u^0 + \theta^\varepsilon\big)\boldsymbol{n}\big]_{\tan} - \mathcal{A}\big(u^0 + \theta^\varepsilon\big), & \text{on } \Gamma, \\ w^\varepsilon\big|_{t=0} = -\theta^\varepsilon\big|_{t=0}, & \text{in } \Omega, \end{cases} \tag{4.41}$$

where $R_\varepsilon(\cdot)$ and $J_\varepsilon(\cdot,\cdot)$ are defined by (3.22) and (3.23).

We multiply Eq. $(4.41)_1$ by w^ε, integrate it over Ω and then use Lemma C.1. After applying the Schwarz and Young inequalities to the right-hand side of the resulting equation, we find:

$$\begin{aligned} &\frac{d}{dt}\big\|w^\varepsilon\big\|^2_{L^2(\Omega)} + 4\varepsilon\big\|\mathbf{S}\big(w^\varepsilon\big)\big\|^2_{L^2(\Omega)} \\ &\quad \leqslant \varepsilon^2\big\|\Delta u^0\big\|^2_{L^2(\Omega)} + \big\|R_\varepsilon\big(\theta^\varepsilon\big)\big\|^2_{L^2(\Omega)} + 2\big\|w^\varepsilon\big\|^2_{L^2(\Omega)} \\ &\qquad - 4\varepsilon\int_\Gamma \big(\mathcal{A} w^\varepsilon + \big[\mathbf{S}\big(u^0 + \theta^\varepsilon\big)\boldsymbol{n}\big]_{\tan} + \mathcal{A}\big(u^0 + \theta^\varepsilon\big)\big)\cdot w^\varepsilon\, dS \\ &\qquad - 2\int_\Omega J_\varepsilon\big(u^\varepsilon, u^0\big)\cdot w^\varepsilon\, d\boldsymbol{x}. \end{aligned} \tag{4.42}$$

To go further, using what is sometime called Korn's second inequality, which requires no boundary conditions (see [12] for proofs of this inequality and some comments on its history), we have

$$\kappa_{\mathbf{S}}\big\|\nabla w^\varepsilon\big\|^2_{L^2(\Omega)} \leqslant \big\|\mathbf{S}\big(w^\varepsilon\big)\big\|^2_{L^2(\Omega)} + \big\|w^\varepsilon\big\|^2_{L^2(\Omega)} \tag{4.43}$$

for a positive constant, $\kappa_{\mathbf{S}}$, depending on the domain, but independent of ε and α.

Restricted to the range, V, of any chart, ψ, we find, using (3.22) with v replaced by θ^ε and (4.14), (4.15) for each θ^ε, that

$$\big\|R_\varepsilon\big(\theta^\varepsilon\big)\big\|^2_{L^2(V)} \leqslant \left\|\frac{\partial \theta^\varepsilon}{\partial t}\right\|^2_{L^2(U)} + \varepsilon\sum_{i=1}^3\left\|\mathcal{S}^i\theta^\varepsilon + \mathcal{L}^i\theta_i^\varepsilon + \frac{\partial^2\theta_i^\varepsilon}{\partial \xi_3^2}\right\|^2_{L^2(U)} \leqslant \kappa_T\big(1+\overline{\alpha}^2\big)\varepsilon^{\frac{3}{2}},$$

where we also used Remark 4.1 and Lemma 4.9. Since we have a finite number of charts on Γ_{3a}, and since θ^ε is supported in Γ_{3a} by (4.28), the same estimate holds on Ω; namely,

$$\big\|R_\varepsilon\big(\theta^\varepsilon\big)\big\|^2_{L^2(\Omega)} \leqslant \kappa_T\big(1+\overline{\alpha}^2\big)\varepsilon^{\frac{3}{2}}. \tag{4.44}$$

To estimate the fourth term in the right-hand side of (4.42), we write

$$\begin{aligned} &\left|4\varepsilon\int_\Gamma \big(\mathcal{A} w^\varepsilon + \big[\mathbf{S}\big(u^0 + \theta^\varepsilon\big)\boldsymbol{n}\big]_{\tan} + \mathcal{A}\big(u^0 + \theta^\varepsilon\big)\big)\cdot w^\varepsilon\, dS\right| \\ &\quad \leqslant \kappa_T\varepsilon\overline{\alpha}\|w\|^2_{L^2(\Gamma)} + \kappa_T\varepsilon\big\|\big[\mathbf{S}\big(u^0 + \theta^\varepsilon\big)\boldsymbol{n}\big]_{\tan} + \mathcal{A}\big(u^0 + \theta^\varepsilon\big)\big\|_{L^2(\Gamma)}\|w\|_{L^2(\Gamma)}. \end{aligned} \tag{4.45}$$

On each $\widetilde{V} \subset \Gamma$, the range of the boundary chart, $\widetilde{\psi}$, using (4.18), (4.19), (4.21), (4.26), we have

$$\begin{aligned}
&\big(\big[\mathbf{S}\big(u^0+\theta^\varepsilon\big)\boldsymbol{n}\big]_{\tan}+\mathcal{A}\big(u^0+\theta^\varepsilon\big)\big)\big|_{\widetilde{V}}\\
&\quad=\sum_{i=1}^{2}\left[\frac{1}{2}\widetilde{u}_i-\frac{1}{2}\frac{\partial\theta_i^\varepsilon}{\partial\xi_3}+\mathcal{M}_i\left(\theta_i^\varepsilon,\frac{\partial\theta_3^\varepsilon}{\partial\xi_i}\right)+\sum_{j=1}^{2}\alpha_{ij}\theta_j^\varepsilon\right]_{\xi_3=0}\boldsymbol{e}_i|_{\xi_3=0}\\
&\quad=\sum_{i=1}^{2}\left[\mathcal{M}_i\left(\theta_i^\varepsilon,\frac{\partial\theta_3^\varepsilon}{\partial\xi_i}\right)+\sum_{j=1}^{2}\alpha_{ij}\theta_j^\varepsilon+\frac{1}{2}\sqrt{\varepsilon}E(\xi_1,\xi_2;t)\right]_{\xi_3=0}\boldsymbol{e}_i|_{\xi_3=0},
\end{aligned}$$

where $\widetilde{u}_i$ and E are defined by (4.21), (4.27). Since this bound holds for all charts, using Remark 4.4 and (4.19), (4.37), (4.38), we find that

$$\varepsilon\big\|\big[\mathbf{S}\big(u^0+\theta^\varepsilon\big)\boldsymbol{n}\big]_{\tan}+\mathcal{A}\big(u^0+\theta^\varepsilon\big)\big\|_{L^2(\Gamma)}\leqslant\kappa_T\big(1+\overline{\alpha}^2\big)\varepsilon^{\frac{3}{2}}. \tag{4.46}$$

Thanks to (4.43), (4.44), (4.45), (4.46), applying Lemma C.2 and the Poincaré inequality, (4.42) yields that

$$\begin{aligned}
&\frac{d}{dt}\big\|w^\varepsilon\big\|_{L^2(\Omega)}^2+2\kappa_{\mathbf{S}}\varepsilon\big\|\nabla w^\varepsilon\big\|_{L^2(\Omega)}^2\\
&\quad\leqslant\kappa_T\big(1+\overline{\alpha}^4\big)\varepsilon^{\frac{3}{2}}+\kappa_T\big(1+\overline{\alpha}^2\big)\big\|w^\varepsilon\big\|_{L^2(\Omega)}^2-2\int_\Omega J_\varepsilon\big(u^\varepsilon,u^0\big)\cdot w^\varepsilon\,d\boldsymbol{x}.
\end{aligned}\tag{4.47}$$

To estimate the last term of (4.47), using (3.29), we write

$$\int_\Omega J_\varepsilon\big(u^\varepsilon,u^0\big)\cdot w^\varepsilon\,d\boldsymbol{x}:=\sum_{j=1}^{5}\mathcal{J}_\varepsilon^j,$$

where $\mathcal{J}_\varepsilon^j$, $1\leqslant j\leqslant 5$, are given by (3.31). Due to (4.28), (4.37), (4.38) and Lemma 4.9, one can easily verifies that $\mathcal{J}_\varepsilon^j$, $j=2,3$, satisfies the same estimate, appearing in (3.32), (3.33), as for the case of a channel domain. That is,

$$\left|\sum_{j=1}^{3}\mathcal{J}_\varepsilon^j\right|\leqslant\kappa_T\big(1+\overline{\alpha}^2\big)\varepsilon^{\frac{3}{2}}+\kappa_T(1+\overline{\alpha})\big\|w^\varepsilon\big\|_{L^2(\Omega)}^2.$$

To bound $\mathcal{J}_\varepsilon^4$ and $\mathcal{J}_\varepsilon^5$, it is sufficient to work in a single chart, $\psi: U\to V$, as there a finite number, N, of them, which just introduces the constant, N.

For $\mathcal{J}_\varepsilon^4$, using $(3.31)_4$, (4.28), we write

$$\begin{aligned}
\big|\mathcal{J}_\varepsilon^4\big|&\leqslant\big\|\big(u^0\cdot\nabla\big)\theta^\varepsilon\big\|_{L^2(V)}\big\|w^\varepsilon\big\|_{L^2(\Omega)}\\
&\qquad\big(\text{using (4.16), (4.17) with } F, G \text{ replaced by } u^0,\theta^\varepsilon,\text{ respectively}\big)\\
&\leqslant\kappa_T\big\|u^0\big\|_{L^\infty(\Omega)}\sum_{i=1}^{3}\left\{\big\|\theta_i^\varepsilon\big\|_{L^2(U)}+\sum_{j=1}^{2}\left\|\frac{\partial\theta_i^\varepsilon}{\partial\xi_j}\right\|_{L^2(U)}\right\}\big\|w^\varepsilon\big\|_{L^2(\Omega)}\\
&\quad+\kappa_T\sqrt{\varepsilon}\left\|\frac{u^0\cdot\boldsymbol{e}_3}{\xi_3}\right\|_{L^\infty(\Gamma_{2a})}\sum_{i=1}^{3}\left\|\frac{\xi_3}{\sqrt{\varepsilon}}\frac{\partial\theta_i^\varepsilon}{\partial\xi_3}\right\|_{L^2(U)}\big\|w^\varepsilon\big\|_{L^2(\Omega)}
\end{aligned}$$

$$\big(\text{using (4.39), Lemma 4.9 and the regularity of } u^0 \text{ with } \big(u^0 \cdot \boldsymbol{e}_3\big)\big|_{\xi_3=0} = 0\big)$$
$$\leqslant \kappa_T (1+\bar{\alpha})\varepsilon^{\frac{3}{4}} \|w^\varepsilon\|_{L^2(\Omega)} \leqslant \kappa_T \big(1+\bar{\alpha}^2\big)\varepsilon^{\frac{3}{2}} + \|w^\varepsilon\|^2_{L^2(\Omega)}.$$

For $\mathcal{J}^5_\varepsilon$, using (4.16), (4.17) with G and F replaced by θ^ε, and using (4.28), we find

$$\begin{aligned}
\big|\mathcal{J}^5_\varepsilon\big| &\leqslant \big\|\big(\theta^\varepsilon \cdot \nabla\big)\theta^\varepsilon\big\|_{L^2(V)} \|w^\varepsilon\|_{L^2(\Omega)} \\
&\leqslant \kappa_T \|\theta^\varepsilon\|_{L^\infty(U)} \sum_{i=1}^{3} \left\{ \|\theta^\varepsilon_i\|_{L^2(U)} + \sum_{j=1}^{2} \left\| \frac{\partial \theta^\varepsilon_i}{\partial \xi_j} \right\|_{L^2(U)} \right\} \|w^\varepsilon\|_{L^2(\Omega)} \\
&\quad + \kappa_T \|\theta^\varepsilon_3\|_{L^\infty(U)} \sum_{i=1}^{3} \left\| \frac{\partial \theta^\varepsilon_i}{\partial \xi_3} \right\|_{L^2(U)} \|w^\varepsilon\|_{L^2(\Omega)} \\
&\quad \big(\text{using (4.37), (4.38) and Lemma 4.9}\big) \\
&\leqslant \kappa_T \big(1+\bar{\alpha}^2\big)\varepsilon^{\frac{5}{4}} \|w^\varepsilon\|_{L^2(\Omega)} \leqslant \kappa_T \big(1+\bar{\alpha}^4\big)\varepsilon^{\frac{5}{2}} + \|w^\varepsilon\|^2_{L^2(\Omega)}.
\end{aligned}$$

Using these bounds on $\mathcal{J}^i_\varepsilon$, $1 \leqslant i \leqslant 5$, (4.47) becomes

$$\frac{d}{dt}\|w^\varepsilon\|^2_{L^2(\Omega)} + 2\kappa_{\mathbf{S}}\varepsilon\big|\nabla w^\varepsilon\big|^2_{L^2(\Omega)} \leqslant \kappa_T \big(1+\bar{\alpha}^4\big)\varepsilon^{\frac{3}{2}} + \kappa_T\big(1+\bar{\alpha}^2\big)\|w^\varepsilon\|^2_{L^2(\Omega)}.$$

Moreover, using (4.24), (4.25), $(4.41)_5$, we see that

$$\big\|w^\varepsilon|_{t=0}\big\|_{L^2(\Omega)} = \big\|\theta^\varepsilon|_{t=0}\big\|_{L^2(\Gamma_{2a})} \leqslant \kappa_T(1+\bar{\alpha})\varepsilon^{\frac{1}{2}} \big\|e^{-\frac{\xi_3}{\sqrt{\varepsilon}}}\big\|_{L^2(\Gamma_{2a})} + l.o.t. \leqslant \kappa_T(1+\bar{\alpha})\varepsilon^{\frac{3}{4}}.$$

Thanks to the Gronwall inequality, we finally have the estimates,

$$\|w^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leqslant \kappa(T,\bar{\alpha},u_0,f)\varepsilon^{\frac{3}{4}}, \qquad \|w^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \leqslant \kappa(T,\bar{\alpha},u_0,f)\varepsilon^{\frac{1}{4}}. \tag{4.48}$$

4.5. Proof of convergence

Using (4.40), we first notice that

$$\big|u^\varepsilon - u^0\big| \leqslant \big|w^\varepsilon\big| + \big|\theta^\varepsilon\big| \quad \big(\text{pointwise in } \Omega \times (0,T)\big).$$

Then, using (4.48) and Lemma 4.9, (1.8) follows. □

5. Uniform convergence

With the estimates we now have, the proof of (1.9) is quite simple.

Because we assume that $m > 6$ in (1.7), $u_0 \in E^m \cap H$ and (by Sobolev embedding) $\nabla u_0 \in W^{1,\infty}_{co}$. Hence, both (2.1) and the hypotheses for Theorem 2.3 hold,[2] so we can use (2.1), (2.2) to conclude that

$$\big\|u^\varepsilon - u^0\big\|_{L^\infty(0,T;H^m_{co}(\Omega))}, \qquad \big\|\nabla\big(u^\varepsilon - u^0\big)\big\|_{L^\infty(0,T;H^{m-1}_{co}(\Omega))} \leqslant C.$$

Then using (1.8), Theorem A.2, and Remark A.3, (1.9) follows.

[2] The hypotheses in Theorem 1.1 are not the minimal ones insuring this.

Remark 5.1. Since also $u^\varepsilon - u^0$ lies in $L^\infty([0,T]; W^{1,\infty})$ by Theorem 2.3, we could use the Gagliardo–Nirenberg interpolation inequality (see, for instance, p. 314 of [8]),

$$\|u^\varepsilon - u^0\|_{L^\infty} \leqslant C\|u^\varepsilon - u^0\|_{L^2}^{\frac{2}{5}}\|u^\varepsilon - u^0\|_{W^{1,\infty}}^{\frac{3}{5}},$$

to give $\|u^\varepsilon - u^0\|_{L^\infty} \leqslant C\varepsilon^{3/10}$. This is the same rate that is obtained for $m = 6$ in (1.9); since, however, we require that $m > 6$, (1.9) always gives a better rate than this.

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Appendix A. An anisotropic Agmon's inequality

In this appendix we develop a version of Agmon's inequality in d dimensions, $d = 2$ or 3, that is suitable for applying to anisotropic problems in which there is more control over tangential (horizontal) derivatives than over normal (vertical) derivatives.

We use the notation $A \ll B$ to mean that $A \leqslant CB$ for some constant, C, which may depend upon the geometry of an underlying domain but not upon anything else. If C depends on some parameter, m, then we write $A \ll_m B$.

Our starting point is the following simple lemma:

Lemma A.1. *Let U be a bounded domain in $\mathbb{R}^d$, $d = 1, 2, 3$. For any f in $H^k(U)$, $k \geqslant d$,*

$$\|f\|_{L^\infty(U)} \ll_k \|f\|_{L^2(U)}^{1-\frac{1}{2k}}\|f\|_{H^k(U)}^{\frac{1}{2k}} \quad \text{if } d = 1,$$

$$\|f\|_{L^\infty(U)} \ll_k \|f\|_{L^2(U)}^{1-\frac{1}{k}}\|f\|_{H^k(U)}^{\frac{1}{k}} \quad \text{if } d = 2,$$

$$\|f\|_{L^\infty(U)} \ll_k \|f\|_{L^2(U)}^{1-\frac{3}{2k}}\|f\|_{H^k(U)}^{\frac{3}{2k}} \quad \text{if } d = 3.$$

Proof. Combine the 1D Agmon's inequality, $\|f\|_{L^\infty(U)} \ll \|f\|_{L^2(U)}^{1/2}\|f\|_{H^1(U)}^{1/2}$, 2D Agmon's inequality, $\|f\|_{L^\infty(U)} \ll \|f\|_{L^2(U)}^{1/2}\|f\|_{H^2(U)}^{1/2}$, or 3D Agmon's inequality, $\|f\|_{L^\infty(U)} \ll \|f\|_{L^2(U)}^{1/4}\|f\|_{H^2(U)}^{3/4}$ with the Sobolev interpolation inequality, $\|f\|_{H^j(U)} \ll_k \|f\|_{L^2(U)}^{1-j/k}\|f\|_{H^k(U)}^{j/k}$, $0 \leqslant j \leqslant k$. $\square$

Theorem A.2. *Let Ω be a bounded domain in $\mathbb{R}^3$ with C^{m+1}-boundary, $m \geqslant 3$, and let Γ_a be the tubular neighborhood of fixed width $a > 0$ interior to Ω. Suppose that f and ∇f lie in the space $H^m_{co}(\Omega)$ of Definition 1. Then*

$$\|f\|_{L^\infty(\Gamma_a)} \ll_{m,a} \|f\|_{L^2(\Omega)}^{\frac{1}{2}-\frac{1}{2m}}\|f\|_{H^m_{co}(\Omega)}^{\frac{1}{2m}}\left[\|f\|_{L^2(\Omega)} + \|\nabla f\|_{H^m_{co}(\Omega)}\right]^{\frac{1}{2}},$$

$$\|f\|_{L^\infty(\Omega\setminus\Gamma_a)} \ll_{m,a} \|f\|_{L^2(\Omega)}^{1-\frac{3}{2m}}\|f\|_{H^m_{co}(\Omega)}^{\frac{3}{2m}}.$$

Proof. We define the chart, ψ, as in the beginning of Section 4. In this chart, we can define a local conormal basis, (X_1, X_2, X_3), by $X_i f(\boldsymbol{x}) = \partial_i(f \circ \psi)(\psi^{-1}(\boldsymbol{x}))$, $i = 1, 2$ and $X_3 f(\boldsymbol{x}) = \frac{\psi_3^{-1}}{1+\psi_3^{-1}}\partial_3(f \circ \psi)(\psi^{-1}(\boldsymbol{x}))$. We will have need, however, only for X_1 and X_2.

It suffices to assume that f lies in $C^\infty(\overline{\Omega})$. Restricting ourselves to the one chart, ψ, by Lemma A.1, we have, for any $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ in U,

$$f \circ \psi(\boldsymbol{\xi}) \ll_m \left\| f \circ \psi(\cdot, \cdot, \xi_3) \right\|_{L^2(U_0)}^{1-\frac{1}{m}} \left\| f \circ \psi(\cdot, \cdot, \xi_3) \right\|_{H^m(U_0)}^{\frac{1}{m}}.$$

Applying Lemma A.1 again, this time in 1D with $k = 1$, gives

$$\begin{aligned}
&\left\| f \circ \psi(\cdot, \cdot, \xi_3) \right\|_{L^2(U_0)}^2 \\
&= \int_{U_0} f \circ \psi\left(\xi_1', \xi_2', \xi_3\right)^2 d\xi_1'\, d\xi_2' \\
&\ll \int_{U_0} \left\| f \circ \psi\left(\xi_1', \xi_2', \cdot\right) \right\|_{L^2(0,a)} \left\| f \circ \psi\left(\xi_1', \xi_2', \cdot\right) \right\|_{H^1(0,a)} d\xi_1'\, d\xi_2' \\
&\leqslant \left(\int_{U_0} \left\| f \circ \psi\left(\xi_1', \xi_2', \cdot\right) \right\|_{L^2(0,a)}^2 d\xi_1'\, d\xi_2' \right)^{\frac{1}{2}} \left(\int_{U_0} \left\| f \circ \psi\left(\xi_1', \xi_2', \cdot\right) \right\|_{H^1(0,a)}^2 d\xi_1'\, d\xi_2' \right)^{\frac{1}{2}} \\
&= \left(\int_U \left| f \circ \psi(\boldsymbol{\xi}) \right|^2 d\boldsymbol{\xi} \right)^{\frac{1}{2}} \left(\int_U \left[\left| f \circ \psi(\boldsymbol{\xi}) \right|^2 + \left| \nabla f\left(\psi(\boldsymbol{\xi})\right) \cdot \partial_3 \psi(\boldsymbol{\xi}) \right|^2 \right] d\boldsymbol{\xi} \right)^{\frac{1}{2}} \\
&\ll_a \left(\int_U \left| f \circ \psi(\boldsymbol{\xi}) \right|^2 \left| J(\boldsymbol{\xi}) \right| d\boldsymbol{\xi} \right)^{\frac{1}{2}} \left(\int_U \left[\left| f \circ \psi(\boldsymbol{\xi}) \right|^2 + \left| \nabla f\left(\psi(\boldsymbol{\xi})\right) \right|^2 \right] \left| J(\boldsymbol{\xi}) \right| d\boldsymbol{\xi} \right)^{\frac{1}{2}} \\
&= \| f \|_{L^2(V)} \| f \|_{H^1(V)}.
\end{aligned}$$

The second $\ll$ followed because the magnitude of the Jacobian determinant, J, is bounded away from zero and $\partial_3 \psi$ is bounded above. Because there are a finite number of charts on Γ_a, the bounds are uniform over Γ_a.

Similarly, for all multiindices, $\alpha = (\alpha_1, \alpha_2, 0)$ with $|\alpha| \leqslant m$, applying Lemma A.1 with $k = 1$ gives

$$\left\| D^\alpha (f \circ \psi)(\cdot, \cdot, \xi_3) \right\|_{L^2(U_0)}^2 \ll_{m,a} \left\| X^\alpha f \right\|_{L^2(V)} \left\| X^\alpha f \right\|_{H^1(V)}.$$

But this is true for all $|\alpha| \leqslant m$, so

$$\left\| f \circ \psi(\cdot, \cdot, \xi_3) \right\|_{H^m(U_0)} \leqslant \| f \|_{H^m_{co}(V)}^{1/2} \left[\| f \|_{L^2(\Omega)} + \| \nabla f \|_{H^m_{co}(V)} \right]^{1/2},$$

where we can use the conormal Sobolev space since the only derivative in the normal direction occurs in ∇f itself. Also, because Γ is C^{m+1}, ψ can be chosen to be $C^{m+1}(\Gamma_a)$ and hence $f \circ \psi$ has sufficient smoothness.

Combining these bounds we have,

$$\begin{aligned}
\| f \|_{L^\infty(V)} &\ll_{m,a} \| f \|_{L^2(V)}^{\frac{1}{2} - \frac{1}{2m}} \| f \|_{H^1(V)}^{\frac{1}{2} - \frac{1}{2m}} \| f \|_{H^m_{co}(V)}^{\frac{1}{2m}} \left[\| f \|_{L^2(\Omega)} + \| \nabla f \|_{H^m_{co}(V)} \right]^{\frac{1}{2m}} \\
&\leqslant \quad \| f \|_{L^2(V)}^{\frac{1}{2} - \frac{1}{2m}} \| f \|_{H^m_{co}(V)}^{\frac{1}{2m}} \left[\| f \|_{L^2(\Omega)} + \| \nabla f \|_{H^m_{co}(V)} \right]^{\frac{1}{2}}.
\end{aligned}$$

Summing over all the V gives

$$\|f\|_{L^\infty(\Gamma_a)} \ll_{m,a} \|f\|_{L^2(\Omega)}^{\frac{1}{2}-\frac{1}{2m}} \|f\|_{H^m_{co}(\Omega)}^{\frac{1}{2m}} \left[\|f\|_{L^2(\Omega)} + \|\nabla f\|_{H^m_{co}(V)}\right]^{\frac{1}{2}}.$$

With $W = \Omega \setminus \Gamma_a$, Lemma A.1 gives

$$\|f\|_{L^\infty(W)} \ll_m \|f\|_{L^2(W)}^{1-\frac{3}{2m}} \|f\|_{H^m(W)}^{\frac{3}{2m}} \ll_a \|f\|_{L^2(\Omega)}^{1-\frac{3}{2m}} \|f\|_{H^m_{co}(\Omega)}^{\frac{3}{2m}}.$$

Combining these last two inequalities completes the proof. □

Remark A.3. It is easy to see that Theorem A.2 holds as well for a channel domain.

It is worth comparing the inequality in Theorem A.2 with the 3D Agmon's anisotropic inequality in Proposition 2.2 of [53], which can be written, for any f in $H^2(\Omega)$, as

$$\begin{aligned} \|f\|_{L^\infty(\Omega)} &\ll \|f\|_{L^2(\Omega)}^{\frac{1}{4}} \prod_{j=1}^{3} \left(\|f\|_{L^2(\Omega)} + \|\partial_j f\|_{L^2(\Omega)} + \|\partial_j \partial_j f\|_{L^2(\Omega)}\right)^{\frac{1}{4}} \\ &\ll \|f\|_{L^2(\Omega)}^{\frac{1}{4}} \left(\|f\|_{L^2(\Omega)} + \|\nabla f\|_{L^2(\Omega)} + \|\Delta f\|\right)_{L^2(\Omega)}^{\frac{3}{4}}. \end{aligned} \tag{A.1}$$

In [53], the authors have some control of the Laplacian but not (directly) of the full H^2 norm. This inequality would not work for us, however, as it includes $\partial_3^2 f$.

Both Theorem A.2 and the inequality in (A.1) are descendants in spirit of Solonnikov's Theorem 4 of [48]. The proof in (A.1) uses, in part, Solonnikov's approach. The approach we have taken is, however, more elementary and direct than that of [48].

Another type of anisotropic inequality that is not a descendant of Solonnikov's theorem (and is not of Agmon type) is the anisotropic embedding inequality of [35, Corollary 7.3], which originated in Remark 4.2 of [51], which states that for all f in $H^1_0(\Omega)$,

$$\|f\|_{L^\infty(\Omega)} \ll \|f\|_{L^2(\Omega)}^{\frac{1}{2}} \|\partial_3 f\|_{L^2(\Omega)}^{\frac{1}{2}} + \|\partial_1 f\|_{L^2(\Omega)}^{\frac{1}{2}} \|\partial_3 f\|_{L^2(\Omega)}^{\frac{1}{2}} + \|f\|_{L^2(\Omega)}^{\frac{1}{2}} \|\partial_1 \partial_3 f\|_{L^2(\Omega)}^{\frac{1}{2}}.$$

Its proof, however, is entirely different from that of Theorem A.2 or the inequalities in [48,53] described above. (A 3D version of it can, however, be obtained using an argument somewhat along the lines of the proof of Theorem A.2.)

Appendix B. Special boundary conditions

For the Navier–Stokes equations in 2D, when $\alpha = \kappa$, the Navier boundary conditions reduce to the conditions, $u \cdot \boldsymbol{n} = \omega(u) = 0$ [13,39,32].[3] Here, κ is the curvature of the boundary of a planar bounded domain and $\omega(u)$ is the scalar curl of u. The natural extension of this observation to 3D is Lemma B.1, which involves the shape operator,[4]

$$\mathcal{A}v := \frac{\partial \boldsymbol{n}}{\partial v} = \nabla_v \boldsymbol{n},$$

the directional derivative of $\boldsymbol{n}$ in the direction, v, for any vector, v, in the tangent plane.

Lemma B.1. *The boundary conditions in* (1.5) *reduce to those in* (1.6) *when $\mathcal{A}$ is the shape operator.*

[3] In these references, the relation is written $\alpha = 2\kappa$, since $2\mathcal{S}(u)$ rather than $\mathcal{S}(u)$ is used in the (2D version of) (1.3).

[4] When an inward unit normal convention is used, the expression for $\mathcal{A}$ contains a negative sign.

Proof. Let $\mathcal{A}$ be the shape operator and let $\boldsymbol{\tau}$ be any unit tangent vector. Then since the shape operator is symmetric, we can write (1.5) as $\mathbf{S}(u^\varepsilon)\boldsymbol{n} \cdot \boldsymbol{\tau} + \mathcal{A}\boldsymbol{\tau} \cdot u^\varepsilon = 0$. But as in [2], $2\mathbf{S}(u^\varepsilon)\boldsymbol{n} \cdot \boldsymbol{\tau} = (\operatorname{curl} u^\varepsilon \times \boldsymbol{n}) \cdot \boldsymbol{\tau} - 2u^\varepsilon \cdot \frac{\partial \boldsymbol{n}}{\partial \boldsymbol{\tau}}$, so (1.5) becomes

$$\left(\operatorname{curl} u^\varepsilon \times \boldsymbol{n}\right) \cdot \boldsymbol{\tau} = 2\left[\frac{\partial \boldsymbol{n}}{\partial \boldsymbol{\tau}} - \mathcal{A}\boldsymbol{\tau}\right] \cdot u^\varepsilon = 0. \qquad \square$$

Appendix C. Some lemmas

We assume that Ω is a bounded domain in $\mathbb{R}^3$ having a Lipschitz boundary, Γ.

Lemma C.1. *Let f be a divergence-free vector field in $H^2(\Omega)$ that satisfies*

$$\mathbf{S}(f)\boldsymbol{n} = \Phi \quad \text{on } \Gamma$$

in the sense of a trace, where Φ lies in $H^{3/2}(\Gamma)$. Then, for any vector field, g, in H^2, we have

$$-\int_\Omega \Delta f \cdot g \, d\boldsymbol{x} = 2\int_\Omega \mathbf{S}(f) \cdot \mathbf{S}(g) \, d\boldsymbol{x} - 2\int_\Gamma \Phi \cdot g \, dS,$$

where $A \cdot B = \sum_{1 \leqslant i,j \leqslant 3} a_{ij} b_{ij}$ for matrices $A = (a_{ij})_{1 \leqslant i,j \leqslant 3}$ and $B = (b_{ij})_{1 \leqslant i,j \leqslant 3}$.

We recall the following classical lemma:

Lemma C.2. *Let u be a divergence-free vector field, of class $H^1(\Omega)^3$, in a bounded domain, $\Omega \subset \mathbb{R}^3$, with a C^2-boundary, Γ. Then, if the normal component of u vanishes on Γ, we have*

$$|u|_{L^2(\Gamma)} \leqslant \kappa_\Omega |u|_{L^2(\Omega)}^{\frac{1}{2}} |\nabla u|_{L^2(\Omega)}^{\frac{1}{2}},$$

for a constant κ_Ω depending on the domain.

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