An Undergraduate Course in Partial Differential Equations

Hung Vinh Tran

Department of Mathematics, University of Wisconsin Madison, Van Vleck hall, 480 Lincoln Drive, Madison, WI 53706, USA

Email address: hung@math.wisc.edu

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Contents

Preface

Partial Differential Equations (PDEs) are often used to describe various phenomena across the natural sciences (physics, chemistry), social sciences (economics), engineering (electrical and mechanical engineering), and other scientific domains. From a mathematical viewpoint, many important PDEs are grounded in rich physical, geometric, and probabilistic contexts.

My goal here is to provide an overview of core concepts related to some fundamental PDEs, tailored for an upper-level undergraduate PDE class. I envision that students taking this course will have prerequisites in:

- multivariable calculus,
- ordinary differential equations and linear algebra,
- real analysis (e.g., at the level of *baby Rudin*).

It is not required that students have taken measure theory or functional analysis prior to this class. Thus, the distinctive feature of this text is that it avoids requiring prior knowledge of measure theory, function spaces, and functional analysis.

Since PDEs form a large subject with many important and influential topics, it is not possible to study all fascinating equations in a single undergraduate course. Instead, we focus on selected foundational equations. The book should be thought of as a bridge to prepare students for graduate-level PDE courses. In particular, it will not contain contents related to measure theory and functional analysis.

The current lecture notes are based on the undergraduate PDE course (Math 619) that I taught at UW–Madison in Spring 2018, 2021, and 2024.

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This version is intended for educational purposes only. You are more than welcome to use the book for either teaching or studying or reference purpose. Please give me feedbacks about its content, and specifically, what else you think that can be included. Note that I aim at an upper-level undergraduate course without requirements on measure theory and functional analysis, and such a course seems suitable at a US institution.

Hung Vinh Tran
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Chapter 1

Introduction to Partial Differential Equations

1.1. Introduction

Firstly, let us distinguish between ordinary differential equations (ODEs) and partial differential equations (PDEs).

Definition 1.1. ODEs are equations involving derivatives of exactly one variable of some unknown functions.

For example, for $\phi = \phi(t) : \mathbb{R} \to \mathbb{R}$ is the unknown, some typical ODEs that we have seen are:

(1) $\phi'(t) + 10\phi(t) = 20 \sin t$;
(2) $a\phi''(t) + b\phi'(t) + c\phi(t) = 0$;
(3) $a\phi'''(t) + b\phi''(t) + c\phi'(t) + d\phi(t) = f(t)$.

Here, $a, b, c, d \in \mathbb{R}$ are given constants, and f is a given function.

Definition 1.2. PDEs are equations involving partial derivatives of some unknown functions.

Basically, if u is the unknown, then a PDE for u is an equation that has partial derivatives of u in it. We can view ODEs as a subset of PDEs, but surely not vice versa.

PDEs are often used to describe various phenomena coming from:

- **Natural Sciences**: physics, chemistry, ...
- **Social Sciences**: economics, ...

- **Engineering**: electrical engineering, mechanical engineering, ...

For example, denote by

$$u(x,t) = \text{temperature at location } x \text{ and time } t \geq 0,$$

then under some appropriate conditions, u solves a *heat equation*.

There are many important PDEs with rich backgrounds from physical, geometric, and probabilistic phenomena, and it is not possible to study them all in an undergraduate PDE course. We can only select some basic and important ones to focus on. We will give motivations to each PDE we study in each chapter.

Before we move on, let us standardize our set of notations in this book.

1.2. Notations

1.2.1. Functions with spatial variables. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. We write:

$$x = (x_1, \ldots, x_n) \mapsto u(x) \in \mathbb{R}.$$

Here, $x \in \mathbb{R}^n$ typically represent the spatial variable or the location.

Partial Derivatives. The first-order partial derivatives of u are written as:

$$u_{x_1} = \frac{\partial u}{\partial x_1}, \quad u_{x_2} = \frac{\partial u}{\partial x_2}, \quad \ldots, \quad u_{x_n} = \frac{\partial u}{\partial x_n}.$$

The second-order partial derivatives of u are

$$u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j} \quad \text{for } 1 \leq i, j \leq n.$$

We use similar notations for higher-order partial derivatives of u.

Gradient of u. The gradient of u, denoted by Du or ∇u, is:

$$Du(x) = \nabla u(x) = \left(u_{x_1}, u_{x_2}, \ldots, u_{x_n}\right)(x) \quad \text{for } x \in \mathbb{R}^n.$$

It is clear that $Du(x) = \nabla u(x)$ is a vector in $\mathbb{R}^n$.

Hessian of u. The Hessian matrix of u, which is an $n \times n$ matrix, is given by:

$$\mathrm{Hess}(u)(x) = D^2 u(x) = \begin{pmatrix} u_{x_1x_1} & u_{x_1x_2} & \cdots & u_{x_1x_n} \\ u_{x_2x_1} & u_{x_2x_2} & \cdots & u_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{x_nx_1} & u_{x_nx_2} & \cdots & u_{x_nx_n} \end{pmatrix}.$$

As u is smooth, we have $u_{x_ix_j} = u_{x_jx_i}$ for $1 \leq i, j \leq n$. In particular, $\mathrm{Hess}(u)(x) = D^2u(x)$ is a symmetric matrix.

Divergence of a vector field. For $\mathbf{F} = \left(f^1, f^2, \ldots, f^n\right) : \mathbb{R}^n \to \mathbb{R}^n$ be a given smooth vector field,

$$\operatorname{div} \mathbf{F} = f^1_{x_1} + f^2_{x_2} + \cdots + f^n_{x_n}.$$

Laplacian of u. The Laplacian of u is

$$\Delta u(x) = u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n} = \operatorname{trace}\left(D^2u(x)\right) = \operatorname{div}\left(Du(x)\right).$$

Example 1.3. Consider $u : \mathbb{R}^2 \to \mathbb{R}$, where

$$(x_1, x_2) \mapsto u(x_1, x_2) = e^{x_1} \sin x_2.$$

Then, the first-order partial derivatives are:

$$u_{x_1} = \frac{\partial u}{\partial x_1} = e^{x_1} \sin x_2, \quad u_{x_2} = \frac{\partial u}{\partial x_2} = e^{x_1} \cos x_2.$$

The gradient is:

$$Du(x) = \nabla u(x) = (u_{x_1}, u_{x_2}) = (e^{x_1} \sin x_2, e^{x_1} \cos x_2).$$

And

$$\Delta u(x) = u_{x_1x_1} + u_{x_2x_2} = e^{x_1} \sin x_2 + (e^{x_1}(-\sin x_2)) = 0.$$

Higher-order partial derivatives. We can similarly define $D^k u$ for any $k \in \mathbb{N}$ as the set of all partial derivatives of order k of u and regard $D^k u(x)$ as a vector in $\mathbb{R}^{n^k}$.

1.2.2. Functions with space and time variables. Let $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ be a given smooth function. We write:

$$(x, t) \mapsto u(x, t) \in \mathbb{R}.$$

Typically, $x \in \mathbb{R}^n$ represents location, and $t \geq 0$ represents time. We the same set of notations as above. In particular,

$$\begin{cases} u_t = \frac{\partial u}{\partial t}, u_{tt} = \frac{\partial^2 u}{\partial t^2}, \\ u_{x_i} = \frac{\partial u}{\partial x_i}, u_{x_ix_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \\ Du(x,t) = D_x u(x,t) = (u_{x_1}, \ldots, u_{x_n}), \\ \Delta u(x,t) = \Delta_x u(x,t) = u_{x_1x_1} + \cdots + u_{x_nx_n}. \end{cases}$$

1.2.3. Specific cases in 1D, 2D, or 3D. Sometimes, we are only in one dimension (1D), two dimensions (2D), or three dimensions (3D) in terms of the spatial variable. In such cases, our notations might be a bit different.

In 1D, for

$$\begin{aligned} u \, &: \mathbb{R} \times [0, \infty) \to \mathbb{R}, \\ &(x, t) \mapsto u(x, t) \in \mathbb{R}, \end{aligned}$$

denote by

$$\begin{cases} u_t = \frac{\partial u}{\partial t}, \\ u_x = \frac{\partial u}{\partial x}, \\ u_{xx} = \frac{\partial^2 u}{\partial x^2} = \Delta_x u(x,t). \end{cases}$$

In 2D, for

$$\begin{aligned} u \ :& \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}, \\ & (x,y,t) \mapsto u(x,y,t) \in \mathbb{R}, \end{aligned}$$

we write

$$\begin{cases} u_t = \frac{\partial u}{\partial t}, \\ u_x = \frac{\partial u}{\partial x}, u_y = \frac{\partial u}{\partial y}, \\ \Delta u = u_{xx} + u_{yy}. \end{cases}$$

And in 3D, for

$$\begin{aligned} u \ :& \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}, \\ & (x,y,z,t) \mapsto u(x,y,z,t) \in \mathbb{R}, \end{aligned}$$

we write

$$\begin{cases} u_t = \frac{\partial u}{\partial t}, \\ u_x = \frac{\partial u}{\partial x}, u_y = \frac{\partial u}{\partial y}, u_z = \frac{\partial u}{\partial z}, \\ \Delta u = u_{xx} + u_{yy} + u_{zz}. \end{cases}$$

1.3. An example of a PDE

Let $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ be an unknown.

$$\begin{aligned} u \ :& \mathbb{R} \times [0,\infty) \to \mathbb{R}, \\ & (x,t) \mapsto u(x,t) \in \mathbb{R}, \end{aligned}$$

Here, x represents for the location, and t represents for the time. A transport equation is typically of the form:

$$u_t + a u_x = 0 \quad \text{in } \mathbb{R} \times (0,\infty). \tag{1.1}$$

Here, $a \in \mathbb{R}$ is a given constant.

Clearly, the above equation is not an ODE. We will try to find some solutions to (1.1).

Definition 1.4 (Classical solutions to PDEs)**.** A solution to a PDE (such as (1.1)) is a real-valued (or sometimes complex-valued) function that satisfies the equation.

Solutions to (1.1)**.** By direct computations, we find some solutions as follows.

- $u(x,t) \equiv 0$. Then $u_t = u_x \equiv 0$, and $u_t + au_x = 0 + a \cdot 0 = 0$.
- $u(x,t) = \sin(x - at)$. We compute

  $$u_t = -a\cos(x-at), \quad u_x = \cos(x-at).$$

  Then,

  $$u_t + au_x = -a\cos(x-at) + a\cos(x-at) = 0.$$

- $u(x,t) = \cos(x - at)$.
- $u(x,t) = e^{x-at}$.
- $u(x,t) \equiv C$ for a given constant $C \in \mathbb{R}$.

From these guessworks, it seems that

$$u(x,t) = \psi(x - at)$$

for $\psi \in C^1(\mathbb{R})$ is a solution to (1.1). Indeed, by the chain rule,

$$u_t = \psi'(x-at)\cdot(-a), \quad u_x = \psi'(x-at),$$

and hence,

$$u_t + au_x = 0$$

It is natural to ask that why do we have many solutions to (1.1)? *What if we impose some conditions?*

Initial condition. Maybe, if we know the value of u at time $t = 0$, then we know the value of $u(x,t)$ for $t \geq 0$.

When $t = 0$, assume we have the *initial condition* that $u(x,0) = g(x)$ for a given function $g \in C^1(\mathbb{R})$. With this given initial data, we have a *Cauchy problem* for u

$$\begin{cases} u_t + au_x = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}. \end{cases} \tag{1.2}$$

This is also called an initial value problem. From the above point, we guess that $u(x,t) = \psi(x - at)$. At $t = 0$, we use the given initial condition to yield that $u(x,0) = g(x) = \psi(x)$. Thus, $g = \psi$, and (1.2) has a solution

$$u(x,t) = g(x - at). \tag{1.3}$$

Naturally, we wonder if this is the only solution to (1.2)? It is not that clear at the moment, and we will come back to this point to show rigorously that (1.2) admits a unique solution, which is given by (1.3).

1.4. Linear PDEs and four important equations

In general, a PDE can be written as:

$$L[u] = 0. \tag{1.4}$$

Here, u is the unknown, and $L[\cdot]$ is a certain differential operator.

Another form that we also see is

$$L[u] = f(x). \tag{1.5}$$

Here, u is the unknown, and $L[\cdot]$ is a certain differential operator. The right-hand side $f = f(x)$ is a given function, which typically represents a source term.

Definition 1.5. We say that $L[\cdot]$ is a linear differential operator if

$$L[su + rv] = sL[u] + rL[v]$$

for all $s, r \in \mathbb{R}$ and u, v smooth functions.

If $L[\cdot]$ is a linear differential operator, we say that (1.4) and (1.5) are linear PDEs.

Assume that (1.4) is a linear PDE. By the definition, we see that for u_1, u_2 are two solutions, then so is its linear combination $su_1 + rv_1$ for $s, t \in \mathbb{R}$. This simple but very important point is called the *superposition principle*. We will use this superposition principle a lot when we study linear PDEs.

There are many important linear PDEs, but we will focus on the **4 most basic and important ones**.

1.4.1. Linear transport equation. A linear transport equation is of the form

$$L[u] = u_t + cu_x = 0 \quad \text{in } \mathbb{R} \times (0, \infty)$$

Typically, it comes with an initial condition $u(x, 0) = g(x)$. This represents exactly the Cauchy problem (1.2) considered in the previous section with $c = a$.

Example 1.6. Is

$$L[u] = u_t + cu_x + 2 \sin x$$

linear?

Answer: L is not a linear differential operator. Indeed,

$$L[2u] = 2u_t + 2cu_x + 2 \sin x \quad \text{but} \quad 2L[u] = 2u_t + 2cu_x + 4 \sin x.$$

Therefore,

$$2L[u] \neq L[2u],$$

which yields that L is not a linear differential operator.

1.4.2. Laplace's equation. A Laplace PDE is of the form

$$L[u] = -\Delta u(x) = 0 \quad \text{in } \mathbb{R}^n.$$

Recall that $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and

$$\Delta u(x) = u_{x_1 x_1} + u_{x_2 x_2} + \cdots + u_{x_n x_n} = \sum_{i=1}^{n} u_{x_i x_i}(x).$$

1.4.3. Heat equation. A heat PDE is of the form

$$L[u] = u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

Typically, it comes with an initial condition $u(x, 0) = g(x)$. Here, $u = u(x, t)$ is the unknown.

In case $u_t = 0$ for all $t > 0$, we end up with $u(x, t) = g(x)$ for all $(x, t) \in \mathbb{R}^n \times (0, \infty)$ and the Laplace equation:

$$-\Delta u(x, t) = -\Delta g(x) = 0.$$

1.4.4. Wave equation. A wave PDE is of the form

$$L[u] = u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

Again, $u = u(x, t)$ is the unknown. Typically, it comes with an initial condition $u(x, 0) = g(x), u_t(x, 0) = h(x)$.

We note the following points.

(1) Laplace's equation represents an **elliptic PDE**.
(2) Heat equation represents a **parabolic PDE**.
(3) Wave equation represents a **hyperbolic PDE**.

Example 1.7. For what function f does

$$L[u] = \Delta u(x) - f(x)$$

remain a linear operator?

We compute

$$2L[u] = 2\left(\Delta u(x) - f(x)\right) = 2\Delta u(x) - 2f(x),$$

and

$$L[2u] = \Delta(2u(x)) - f(x) = 2\Delta u(x) - f(x).$$

Thus, to have $L[2u] = 2L[u]$, we need

$$f(x) = 0 \quad \text{for all } x.$$

By double-checking, L is indeed linear if and only if $f \equiv 0$.

1.5. Some nonlinear equations

Most of the important PDEs in practice are nonlinear, that is, $L[u]$ is not a linear operator. We will not be able to study all such important nonlinear PDEs, and we will only focus on some cases.

Here are some representative examples.

1.5.1. Burgers' equation.

Here,

$$u = u(x,t) : \mathbb{R} \times (0,\infty) \to \mathbb{R}$$

is the unknown. The Burgers' PDE is of the form

$$u_t + uu_x = 0 \quad \text{in } \mathbb{R} \times (0,\infty)$$

Typically, it comes with an initial condition $u(x,0) = g(x)$.

This is a very important equation in the literature, and we will use the *method of characteristics* to study it.

Example 1.8. Let us note that

$$L[u] = u_t + uu_x \quad \text{is not a linear operator.}$$

Indeed,

$$L[2u] = (2u)_t + (2u)(2u_x) = 2u_t + 4uu_x, \quad 2L[u] = 2u_t + 2uu_x.$$

Thus, for $u(x,t) = e^x$, we see that

$$L[2u] \neq 2L[u].$$

As noted, Burgers' equation is not linear. Still, if we naively assume that $u = a$, then we get (1.2), and

$$u(x,t) = g(x - u(x,t)t).$$

This is an implicit relation as u occurs on both sides. If this holds, then does u solve the Burgers' equation? This will be the content of a homework problem.

1.5.2. Hamilton–Jacobi equation.

Here,

$$u = u(x,t) : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$$

is the unknown. A Hamilton–Jacobi equation is of the form

$$u_t + H(Du) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

It often comes with an initial condition $u(x,0) = g(x)$.

Here, $H : \mathbb{R}^n \to \mathbb{R}$ is a given function. It is called the *Hamiltonian* in the literature. If H is not a linear function, then clearly, the PDE is not linear.

Example 1.9. Both of the following Hamiltonians are not linear

$$H(p) = |p|, \quad H(p) = |p|^2 \quad \text{for } p \in \mathbb{R}^n.$$

We note that the superposition principle is not true for nonlinear PDEs, and this is one of the reasons why it is rather hard to study nonlinear PDEs in general.

1.6. Some more characterizations of PDEs

We give a definition of an order of a PDE.

Definition 1.10 (Order of a PDE)**.** We say that a PDE is of order $k \in \mathbb{N}$ if k is the highest order of the partial derivatives of the unknown in the PDE. In this case, we can write in an abstract form that

$$L[u] = F(x, u(x), Du(x), \dots, D^k u(x)) = f(x).$$

We give some further definitions.

Definition 1.11. Consider a k^{th}-order PDE

$$L[u] = F(x, u(x), Du(x), \dots, D^k u(x)) = f(x).$$

(a) We say that the equation is linear if

$$L[u] = \sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u(x).$$

Here, for an index $\alpha = (\alpha_1, \dots, \alpha_n)$, we write

$$|\alpha| = \alpha_1 + \dots + \alpha_n, \quad D^\alpha u = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u.$$

(b) We say that the equation is semilinear if

$$L[u] = \sum_{|\alpha| = k} a_\alpha(x) D^\alpha u(x) + a_0(x, u(x), Du(x), \dots, D^{k-1} u(x)),$$

where a_0 is not linear in a slot concerning $u(x), Du(x), \dots, D^{k-1} u(x)$.

(c) We say that the equation is quasilinear if

$$L[u] = \sum_{|\alpha| = k} a_\alpha(x, u(x), Du(x), \dots, D^{k-1} u(x)) D^\alpha u(x) \\ + a_0(x, u(x), Du(x), \dots, D^{k-1} u(x)),$$

where a_α is not linear in a slot concerning $u(x), Du(x), \dots, D^{k-1} u(x)$ for one index $|\alpha| = k$.

(d) We say that the equation is fully nonlinear if $L[u]$ depends nonlinearly on $D^k u$.

1.7. Our goals in studying PDEs

Firstly, for each meaningful PDE, we will give motivation of why we study it. Then, of course, we aim at the following:

(1) **Solve the PDE explicitly if possible.** This is doable for some equations, but not so in general.

(2) When our PDEs are **not explicitly solvable**, then what do we do? We want to develop some theories to study PDE and their solutions.

Typically, in the second scenario, the first main goal is the *wellposedness theory*, which consists of three main questions as follows.

- **Existence**: Does the PDE have a solution?
- **Uniqueness**: Is there only one solution to the PDE?
- **Stability**: Do small changes in initial or boundary conditions create only small changes in solutions?

We say that a PDE is *wellposed* if all three points above hold. For problems arising from physical applications, it is rather important to have one and only one solution, which is consistent with the phenomena. Further, we would prefer that our problem is stable, that is, the unique solution would change just a little if the conditions specifying the problem change a little. This stability is also extremely important when we need to approximate the problem either theoretically or numerically.

After the wellposedness theory is established, we study properties of solutions such as:

- **Regularity theory**: How regular/smooth is our solution?
- **Asymptotic behavior**: What is the behavior of $u(x,t)$ with respect to the parameters (typically the small ones) in the problem?
- **Large time behavior**: What is the behavior of $\lim_{t\to\infty} u(x,t)$?

These are very important questions in applications.

1.8. Exercises

Exercise 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that $u(x,t) = f(x-4t)$ satisfies the linear transport equation

$$u_t + 4u_x = 0 \qquad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}.$$

Exercise 2. Find an equation relating the parameters k, m, n so that the function $u(x,t) = e^{mt}\sin(nx)$ solves the heat equation

$$u_t = ku_{xx} \qquad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}.$$

Exercise 3. Find an equation relating the parameters c, m, n so that the function $u(x,t) = \sin(mt)\cos(nx)$ solves the heat equation

$$u_{tt} = c^2 u_{xx} \qquad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}.$$

Exercise 4. Find all functions $a, b, c : \mathbb{R} \to \mathbb{R}$ such that $u(x,t) = a(t)e^{2x} + b(t)e^x + c(t)$ solves the heat equation

$$u_t = u_{xx} \qquad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}.$$

Exercise 5. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Assume that $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is differentiable and satisfies that

$$u(x,t) = f(x - u(x,t)t) \qquad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}.$$

Show that u is a solution to the following equation (Burgers' PDE)

$$\begin{cases} u_t + uu_x = 0 & \text{for } (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Exercise 6. Find all functions $a_k : \mathbb{R} \to \mathbb{R}$ such that $u(x,t) = \sum_{k\in\mathbb{Z}} a_k(t)e^{kx}$ solves

$$u_t = u_{xx} \qquad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}.$$

Exercise 7. Check whether each of the PDEs given bellow is linear or not.

(1) $u_t + u_{xxx} = 0$ in $\mathbb{R} \times (0,\infty)$.
(2) $u_{tt} + u_{xxxx} = 0$ in $\mathbb{R} \times (0,\infty)$.
(3) $u_{tt} + 2u_t - u_{xx} = 0$ in $\mathbb{R} \times (0,\infty)$.
(4) $|Du| = 1$ in $\mathbb{R}^n$.
(5) $\operatorname{div}(|Du|^3 Du) = 0$ in $\mathbb{R}^n$.
(6) $\det(D^2 u) = 1$ in $\mathbb{R}^n$.

1.9. Notes and references

(1) There are many excellent PDE textbooks already in the literature. We list here some representative ones: Evans [**Eva10**], John [**Joh91**], Strauss [**Str07**]. The current version of the text is strongly influenced by these books.

Transport equation

2.1. Transport equation with constant coefficients

2.1.1. Derivation of a transport equation.

$$\begin{aligned} u\ :\mathbb{R} \times (0, \infty) &\to \mathbb{R} \\ (x,t) &\mapsto u(x,t) \in \mathbb{R} \end{aligned}$$

is the unknown. Here, x represents the location, t represents the time, and $u(x,t)$ is the density of cars at x at time t.

We think of a given highway as an infinite line, which is represented by the real line $\mathbb{R}$. In practice, $u(x,t)$ is number of cars per unit length at x of the highway per unit time at t. We want to model such density of cars and write down an equation for it. This can be done via a *balance law*.

Fix two points $a < b$. Think of them as points of observation. See Figure 1.

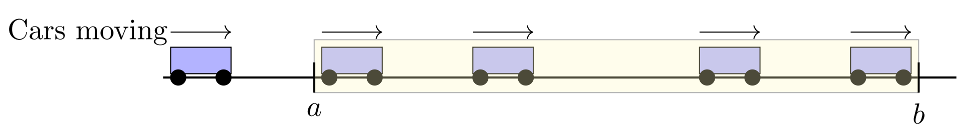

Figure 1. Cars moving on a highway with observation points a, b

We want to measure the change of the number of cars on the whole interval $[a,b]$. On the one hand, the number of cars on $[a,b]$ at time t is:

$$N(t) = \int_a^b u(x,t)\,dx.$$

So, the rate of change at time t is:

$$N'(t) = \int_a^b u_t(x,t)\,dx.$$

On the other hand, let $Q(x,t)$ be the number of cars past x per unit of time t. People also call $Q(x,t)$ the *flux* of cars past x at time t. Then, we can see that:

$$N'(t) = \int_a^b u_t(x,t)\,dx = Q(a,t) - Q(b,t)$$

where $Q(a,t)$ is the number of cars entering, and $Q(b,t)$ is the number of cars exiting $[a,b]$.

Using the fundamental theorem of calculus:

$$Q(b,t) - Q(a,t) = \int_a^b Q_x(x,t)\,dx.$$

Thus, we obtain the relation:

$$\int_a^b u_t(x,t)\,dx = \int_a^b -Q_x(x,t)\,dx.$$

As this identity holds true for all $a < b$, we imply that

$$u_t + Q_x = 0. \tag{2.1}$$

We will prove this claim in a homework problem.

What is the flux Q? Intuitively, we see that

$$Q(x,t) = \text{speed} \times \text{density} = V(x,t)u(x,t).$$

We consider first the simplest case where $V(x,t) = c$, which means that all cars move with a constant speed $c \in \mathbb{R}$.

This gives us $Q(x,t) = cu(x,t)$ and a transport equation:

$$\begin{cases} u_t(x,t) + cu_x(x,t) = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}. \end{cases} \tag{2.2}$$

Here, u is the unknown, $c \in \mathbb{R}$ is a given constant speed, and the initial data $g \in C^1(\mathbb{R})$ is given.

2.1.2. Solving the transport equation. Let us proceed to solve this transport PDE.

By the guessing from the previous chapter, we see that $u(x,t) = g(x-ct)$ is a solution to (2.2). However, we do not know if there are other solutions.

Theorem 2.1. *Equation* (2.2) *admits a unique solution*

$$u(x,t) = g(x-ct) \quad \textit{for } (x,t) \in \mathbb{R} \times [0,\infty).$$

Proof. The idea is simple. Consider a passenger who starts at x, hops in a car traveling with speed c, and observes the cars' environment on the highway. As every car travels with the same speed, it is intuitively clear that this passenger sees no changes in the cars' environment around. See Figure 2.

Here is a rigorous way to write it down. We fix $x \in \mathbb{R}$, and denote by $z(t) = u(x + ct, t)$ for $t \geq 0$.

Then, for $t > 0$, by the chain rule,

$$z'(t) = u_t(x + ct, t) + cu_x(x + ct, t) = 0.$$

Hence, $z(\cdot)$ is a constant function, and, for $t \geq 0$,

$$z(t) = z(0) = u(x, 0) = g(x),$$

which implies that $u(x + ct, t) = u(x, 0) = g(x)$ for $t \geq 0$.

In particular, $u(x,t) = g(x - ct)$. We already checked that this solves (2.2). Thus, we conclude that

$$\boxed{u(x,t) = g(x - ct)}$$

is the unique solution to (2.2).

□

2.2. Non-homogeneous transport equation

We consider

$$\begin{cases} u_t(x,t) + cu_x(x,t) = f(x,t) & \text{in } \mathbb{R} \times (0, \infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}. \end{cases} \tag{2.3}$$

Here, we have the presence of $f(x,t)$ on the right-hand side, which stands for a source term. Basically, we can think of f as the number of cars enter/exit the highway at (x,t). As usual, u is the unknown, $c \in \mathbb{R}$ is a given constant speed, and the initial data $g \in C^1(\mathbb{R})$ is given. Assume further that $f \in C(\mathbb{R} \times [0, \infty))$.

Theorem 2.2. *Equation* (2.3) *admits a unique solution*

$$u(x,t) = g(x - ct) + \int_0^t f(x - c(t - s), s)\, ds \qquad \text{for } (x,t) \in \mathbb{R} \times [0, \infty).$$

Proof. Fix $x \in \mathbb{R}$. Let $z(t) = u(x + ct, t)$ for $t \geq 0$. See Figure 2. Then

$$z'(t) = u_t(x + ct, t) + cu_x(x + ct, t) = f(x + ct, t).$$

Hence,

$$z(t) = z(0) + \int_0^t f(x + cs, s)\, ds = g(x) + \int_0^t f(x + cs, s)\, ds.$$

In particular,

$$\boxed{u(x,t) = g(x-ct) + \int_0^t f(x-c(t-s),s)\,ds.}$$

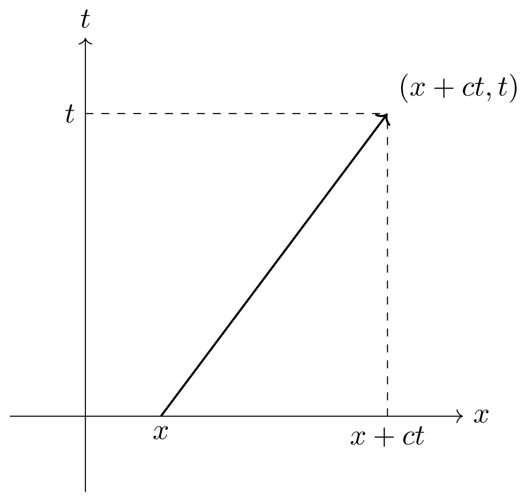

Figure 2. Position of a passenger starting from $(x, 0)$

□

We will revisit general first-order PDEs later in a systematic way. Let us now address some general transport equations.

2.3. General transport equations

We study a linear transport PDE of the form

$$\begin{cases} u_t(x,t) + c(x,t)u_x(x,t) = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}, \end{cases} \tag{2.4}$$

where we have that $c(x,t)$ is a given function in (x,t) and is not a constant. As usual, u is the unknown, and the initial data $g \in C^1(\mathbb{R})$ is given.

A natural idea to solve this problem is the go-with-the-flow viewpoint earlier. Fix $x \in \mathbb{R}$. Let $x(t)$ be such that

$$\begin{cases} x'(t) = c(x(t),t) & \text{for } t \geq 0, \\ x(0) = x. \end{cases}$$

Consider $z(t) = u(x(t), t)$ for $t \geq 0$. Then

$$z'(t) = u_t(x(t), t) + u_x(x(t), t) \cdot x'(t) = u_t(x(t), t) + u_x(x(t), t) \cdot c(x(t), t) = 0.$$

Thus, $z(t) = z(0) = g(x)$ for $t \geq 0$. In particular,

$$u(x(t), t) = u(x(0), 0) = g(x) \quad \text{for } t \geq 0.$$

This is not explicit, as we might not be able to find $x(t)$ for $t \geq 0$, but overall, this gives a good big picture. See Figure 3.

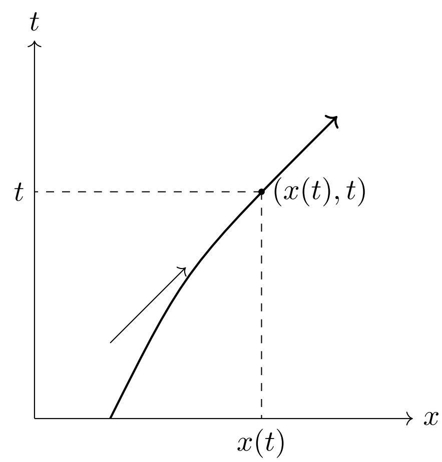

Figure 3. Position of $(x(t), t)$

Let us now consider some explicit examples.

Example 2.3.

$$\begin{cases} u_t + t u_x = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) = g(x) & \text{on } \mathbb{R}. \end{cases}$$

Here, $c(x, t) = t$.

As above, consider $x(t)$ such that

$$\begin{cases} x'(t) = c(x(t), t) = t, \\ x(0) = x. \end{cases}$$

Then

$$x(t) = x + \frac{t^2}{2}.$$

We showed already that

$$u(x(t), t) = u(x(0), 0) = g(x) \quad \text{for all } t \geq 0.$$

Hence,

$$u\left(x+\frac{t^2}{2},t\right)=g(x) \quad \text{for all } t\geq 0.$$

Tracing back, let $y=x+\frac{t^2}{2}$, so $x=y-\frac{t^2}{2}$. Then,

$$\boxed{u(y,t)=g\left(y-\frac{t^2}{2}\right).}$$

Example 2.4. Let us consider

$$\begin{cases} u_t+x^3u_x=0 & \text{in } \mathbb{R}\times(0,\infty),\\ u(x,0)=g(x) & \text{on } \mathbb{R}. \end{cases}$$

We solve

$$\begin{cases} x'(t)=x(t)^3 & \text{for } t>0,\\ x(0)=x>0. \end{cases}$$

Then

$$\frac{x'(t)}{x(t)^3}=1 \quad \Rightarrow \quad \frac{d}{dt}(x(t)^{-2})=(-2)\frac{x'}{x^3}=-2.$$

Therefore,

$$x(t)^{-2}=-2t+C$$

for some constant $C\in\mathbb{R}$. As $x(0)=x$, we yield $C=\frac{1}{x^2}$. We thus get

$$x(t)^{-2}=\frac{1}{x^2}-2t \quad \Rightarrow \quad x(t)=\frac{1}{\sqrt{\frac{1}{x^2}-2t}},$$

which is only defined for $t<\frac{1}{2x^2}$. In particular, as $x\to\infty$, the time that $x(t)$ is well-defined vanishes.

2.4. Transport equation in multi-dimensions

The results in previous sections hold also in multi-dimensions. Let us state the two relevant theorems here.

Theorem 2.5. *Consider*

$$\begin{cases} u_t(x,t)+c\cdot Du(x,t)=0 & \textit{in } \mathbb{R}^n\times(0,\infty),\\ u(x,0)=g(x) & \textit{on } \mathbb{R}^n. \end{cases} \tag{2.5}$$

Here, u is the unknown, $c\in\mathbb{R}^n$ is a given constant speed, and the initial data $g\in C^1(\mathbb{R}^n)$ is given. Then, (2.5) *admits a unique solution*

$$u(x,t)=g(x-ct) \quad \textit{for } (x,t)\in\mathbb{R}^n\times[0,\infty).$$

The proof of this theorem is exactly the same as that of Theorem 2.1.

Theorem 2.6. *We consider*

$$\begin{cases} u_t(x,t) + c\cdot Du(x,t) = f(x,t) & \text{in } \mathbb{R}^n\times(0,\infty),\\ u(x,0)=g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{2.6}$$

Here, u is the unknown, $c\in\mathbb{R}^n$ is a given constant speed, and the initial data $g\in C^1(\mathbb{R}^n)$ is given. Assume further that the source term $f\in C(\mathbb{R}^n\times[0,\infty))$. Equation (2.6) admits a unique solution

$$u(x,t) = g(x-ct) + \int_0^t f(x-c(t-s),s)\,ds \quad \text{for } (x,t)\in\mathbb{R}^n\times[0,\infty).$$

The proof of this theorem is exactly the same as that of Theorem 2.2.

2.5. Exercises

Exercise 8. Let $f\in C(\mathbb{R})$ be a given function such that, for all $a<b$,

$$\int_a^b f(x)\,dx = 0.$$

Show that $f\equiv 0$.

Exercise 9. Solve the initial value problem

$$\begin{cases} u_t + 4u_x = 0 & \text{for } (x,t)\in\mathbb{R}\times(0,\infty),\\ u(x,0) = \frac{1}{1+x^2} & \text{for } x\in\mathbb{R}. \end{cases}$$

Exercise 10. Solve the following transport equation

$$\begin{cases} u_t + u_x + 3u = e^{2x+t} & \text{in } \mathbb{R}\times(0,\infty),\\ u(x,0) = x & \text{on } \mathbb{R}. \end{cases}$$

Exercise 11. Solve the following transport equation

$$\begin{cases} (1+t^2)u_t + u_x = 0 & \text{in } \mathbb{R}\times(0,\infty),\\ u(x,0) = \cos x & \text{on } \mathbb{R}. \end{cases}$$

Exercise 12. Solve the following transport equation

$$\begin{cases} u_t(x,t) + c\cdot Du(x,t) + \lambda u = f(x,t) & \text{in } \mathbb{R}^n\times(0,\infty),\\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases}$$

Here, $c\in\mathbb{R}^n, \lambda\in\mathbb{R}$ are given, and $g\in C^1(\mathbb{R}^n)$, $f\in C(\mathbb{R}^n\times[0,\infty))$ are given functions.

Exercise 13. Solve the following transport equation

$$\begin{cases} u_t + 2u_x = 0 & \text{for } x>0, t>0,\\ u(x,0) = \sin x & \text{for } x\geq 0,\\ u(0,t) = te^{-t} & \text{for } t\geq 0. \end{cases}$$

Here $u = u(x,t) : [0,\infty) \times [0,\infty) \to \mathbb{R}$.

Chapter 3

Laplace and heat equations

3.1. Derivations of the Laplace and heat equations

3.1.1. Steepest Gradient Descent. Assume $u : \mathbb{R}^n \to \mathbb{R}$ is smooth, and $y \in \mathbb{R}^n$ is fixed.

Question. In what direction from y does u increase the fastest? In what direction does u decrease the fastest? Basically, we want to find a unit vector e such that $u(y+se)$ takes either the biggest value or the smallest value for $s>0$ extremely small.

Answer. We use the Taylor expansion of u around y, for $s>0$ small enough,

$$u(y+se) = u(y) + Du(y)\cdot(se) + |s|\omega(|s|),$$

where, ω is a modulus of continuity, that is, $\lim_{r\to 0}\omega(r)=0$. Hence,

$$u(y+se) - u(y) = sDu(y)\cdot e + s\omega(s).$$

Thus, the direction that u increases the fastest should be e so that $Du(y)\cdot e$ achieves its max value. Naturally, if $Du(y)\neq 0$, then

$$e = \frac{Du(y)}{|Du(y)|}.$$

Similarly, if $Du(y)\neq 0$, the direction that u decreases the fastest should be

$$\tilde{e} = -\frac{Du(y)}{|Du(y)|}.$$

Here, we are talking about unit directions.

Remark 3.1. If we do not need to normalize to unit directions, then we can say that the direction u increases the most is Du, and decreases the most is $-Du$.

3.1.2. Derivation of the heat equation. Let $u(x,t)$ be the temperature at $x \in \mathbb{R}^n$ at time $t \geq 0$.

Air flows from the region of hot temperature to regions of cold temperature. If the environment is homogeneous, then the flow can be written as

$$\mathbf{F}(x,t) = -cDu(x,t) \tag{3.1}$$

where $c > 0$ is a constant (depending on the environment). Basically, the nature is optimal in the sense that the air flows in the direction that the temperature decreases the quickest. We note that (3.1) has many names in the literature. It is called Fick's law of diffusion, or Fourier's law of heat conduction, or Ohm's law of electrical conduction. See Figure 1.

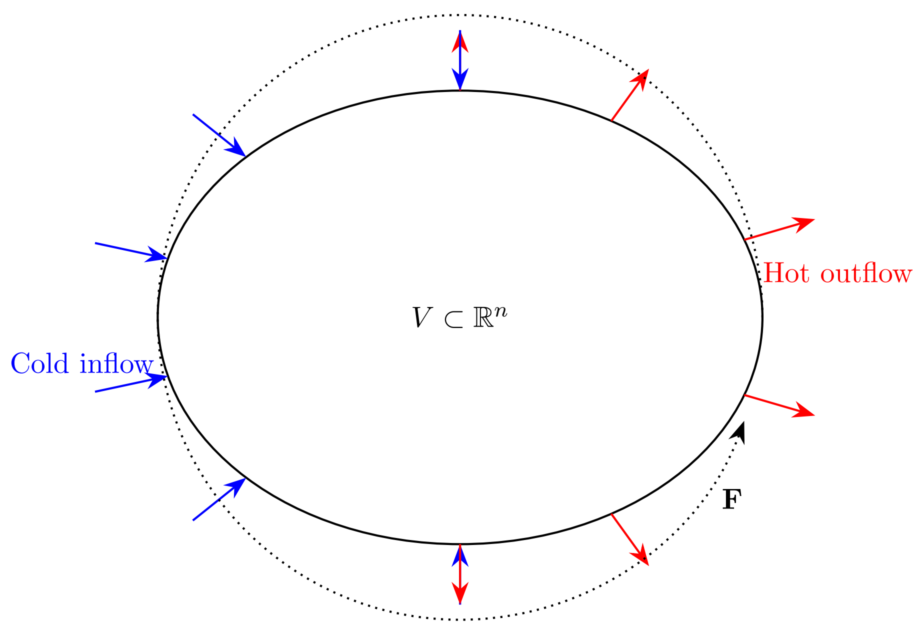

Figure 1. Air flows through a region V

Consider an arbitrary bounded region $V \subset \mathbb{R}^n$ with smooth boundary. Let $\mathbf{n} = \mathbf{n}(x)$ be the unit outward normal vector to ∂V at $x \in \partial V$. The total change of temperature in $V \subset \mathbb{R}^n$ can be measured in two ways.

First way to compute:

$$\frac{d}{dt}\int_V u(x,t)\,dx = \int_V u_t(x,t)\,dx.$$

Second way to compute: We use the divergence theorem to imply

$$-\int_{\partial V} \mathbf{F}(x,t)\cdot\mathbf{n}\,dS = -\int_V \operatorname{div}\mathbf{F}\,dx = c\int_V \Delta u\,dx.$$

By matching the two ways, we yield

$$\int_V (u_t(x,t) - c\Delta u)\,dx = 0$$

for all arbitrary bounded regions $V \subset \mathbb{R}^n$ with smooth boundaries. Thus, we deduce that

$$u_t - c\Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty). \tag{3.2}$$

This is a heat PDE.

Remark 3.2. When the environment is not homogeneous, we have $c = c(x)$, which is dependent on the position. Then, the above derivation gives us the following divergence-form parabolic PDE

$$u_t - \operatorname{div}(c(x)Du) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

3.1.3. Derivation of the Laplace equation. Consider (3.2). If we are at equilibrium, that is, the temperature u does not change with respect to time anymore. Then

$$u(x,t) = w(x) \quad \text{for all } t \geq 0,$$

which implies that $u_t = 0$. We use this in (3.2) to get

$$-\Delta w(x) = 0 \quad \text{in } \mathbb{R}^n.$$

This is our Laplace equation.

3.1.4. Examples of solutions to the Laplace equation. Let us consider

$$-\Delta u(x) = 0 \quad \text{in } \mathbb{R}^n. \tag{3.3}$$

We aim at finding some specific solutions to (3.3).

(1) **One dimension ($n = 1$).** We have

$$-u''(x) = 0 \qquad \Longrightarrow \qquad u(x) = ax + b$$

where $a, b \in \mathbb{R}$ are constants.

(2) **Two dimensions ($n = 2$).**

$$-\left(u_{x_1x_1} + u_{x_2x_2}\right) = 0 \quad \text{in } \mathbb{R}^2.$$

There are plenty of examples of such functions that we can think of as follows.

- $u(x_1,x_2) = ax_1 + bx_2 + c$, where $a, b, c \in \mathbb{R}$ are constants.
- $u(x_1,x_2) = x_1x_2$.

- $u(x_1, x_2) = x_1^2 - x_2^2$.
- $u(x_1, x_2) = e^{x_1} \sin x_2$.

3.2. Laplace's equation and Poisson's equation

We now focus on:

- Laplace's equation: $-\Delta u = 0$ (or $\Delta u = 0$),
- Poisson's equation: $-\Delta u = f$ in $\mathbb{R}^n$.

We note that Poisson's equation has a source term f, which is nonlinear unless $f \equiv 0$.

3.2.1. Fundamental solutions to the Laplace PDE. Consider Laplace's equation

$$-\Delta u = 0 \quad \text{ in } \mathbb{R}^n.$$

As noted, the Laplace equation is fully solved when $n = 1$. We only consider $n \geq 2$ here. We seek for special solutions. A natural choice is one with symmetry: radially symmetric u.

Consider $u(x) = v(|x|) = v(r)$ for $x \in \mathbb{R}^n$ and $r = |x| \in [0, \infty)$. First,

$$r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}.$$

For $r = |x| > 0$, we use the chain rule to compute

$$r_{x_i} = \frac{1}{2} \left(x_1^2 + \cdots + x_n^2 \right)^{-1/2} (2x_i) = \frac{x_i}{|x|},$$

and

$$r_{x_i x_i} = \left(\frac{x_i}{r} \right)_{x_i} = \frac{1}{r} + (-1) \frac{x_i}{r^2} \frac{x_i}{r} = \frac{1}{r} - \frac{x_i^2}{r^3}.$$

Then, by the chain rule again,

$$u_{x_i} = v'(r) \frac{x_i}{r}, \quad u_{x_i x_i} = v''(r) \frac{x_i^2}{r^2} + v'(r) \frac{r^2 - x_i^2}{r^3}.$$

Thus, by noting that $\sum_{i=1}^n x_i^2 = r^2$, we deduce that

$$\Delta u = v''(r) + \frac{n-1}{r} v'(r).$$

Therefore, the Laplace PDE becomes

$$-\Delta u = 0 \quad \Longrightarrow \quad v''(r) + \frac{n-1}{r} v'(r) = 0.$$

This is an ODE to solve for $v(r)$ for $r > 0$.

Solving the ODE. Multiply the ODE by r^{n-1} to imply

$$r^{n-1} v''(r) + (n-1) r^{n-2} v'(r) = 0.$$

Thus,

$$\frac{d}{dr}\left(r^{n-1}v'(r)\right) = 0 \qquad \Longrightarrow \qquad r^{n-1}v'(r) = C_1,$$

where $C_1 \in \mathbb{R}$ is a constant. Then,

$$v'(r) = \frac{C_1}{r^{n-1}}.$$

There are two cases to be considered.

- If $n = 2$: For some constants $C_1, C_2 \in \mathbb{R}$,

$$v(r) = C_1 \log r + C_2.$$

- If $n \geq 3$: For some constants $C_2, C_3 \in \mathbb{R}$,

$$v(r) = \frac{C_3}{r^{n-2}} + C_2.$$

The key point here is that, by using the symmetry that $u(x) = v(|x|)$, we go from a complicated PDE to a simple enough ODE, which is explicitly solvable.

Definition 3.3 (Fundamental solutions to the Laplace equation)**.** Define, for $x \neq 0$,

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log|x| & \text{if } n = 2, \\ \frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}} & \text{if } n \geq 3. \end{cases}$$

Here, ω_n is the volume of the unit ball in $\mathbb{R}^n$. We can write $\omega_n = |B_1|$.

The function Φ is called the **fundamental solution** to the Laplace PDE for $n \geq 2$.

The normalized constants are to be understood later. We often write:

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log|x| & \text{if } n = 2, \\ \frac{1}{n(n-2)\omega_n} |x|^{2-n} & \text{for } n \geq 3, \end{cases} \qquad (\text{for } x \neq 0).$$

Remark 3.4. By abuse of notations, we write, for $x \neq 0$,

$$\Phi = \Phi\,(x) = \Phi(|x|).$$

By direct computations, we see that, for $x \neq 0$,

$$|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}, \quad |\Delta\Phi(x)| \leq \frac{C}{|x|^{n}}.$$

Here, $C > 0$ is some constant dependent only on dimension n.

Remark 3.5. It is important noting that if $u(x) = v(|x|) = v(r)$, then:

$$\Delta u(x) = v''(r) + \frac{n-1}{r} v'(r),$$

which gives an important formula of the Laplacian in polar coordinates.

It is worth noting that $\Phi(x)$ is singular at 0, that is, $\Phi(x) \to +\infty$ as $|x| \to 0$, and we actually **do not** have that $\Delta\Phi(x) = 0$ in $\mathbb{R}^n$. We only have:

$$\Delta\Phi(x) = 0 \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}.$$

See Figure 2.

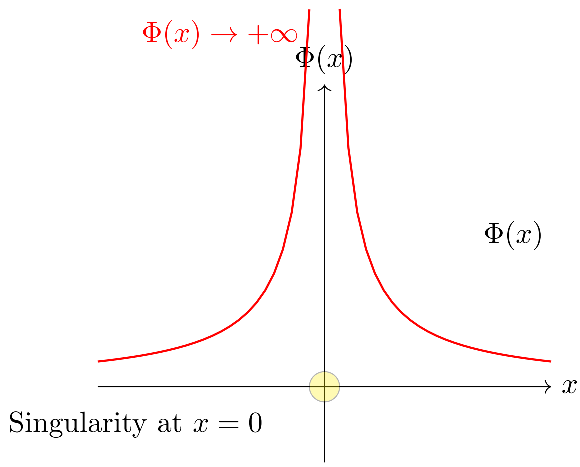

Figure 2. The fundamental solution Φ when $n \geq 3$

Therefore, to use $\Phi(x)$, we need to be very careful with the singularity at $x = 0$ later. A natural idea is to consider $\mathbb{R}^n \setminus B(0, \varepsilon)$, then let $\varepsilon \to 0$ to avoid this singularity.

Next, we use fundamental solutions to build solutions to Poisson's equation.

3.2.2. Poisson's problem. The Poisson equation is

$$-\Delta u = f \quad \text{in } \mathbb{R}^n.$$

Before stating the main result, we would like to discuss how to construct a solution u to the above. The key idea is to combine the fundamental solutions together.

Here is the first simple observation. If

$$-\Delta v_1 = g_1 \quad \text{and} \quad -\Delta v_2 = g_2,$$

then for $v = v_1 + v_2$ and $g = g_1 + g_2$, we have:

$$-\Delta v = g \quad \text{in } \mathbb{R}^n.$$

This is based on the linearity of $v \mapsto -\Delta v$, or the superposition principle.

Question. Assume for each $y \in \mathbb{R}^n$, and each function $g(x,y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, we can find a function $\varphi(x,y)$ such that

$$-\Delta_x \varphi(x,y) = -\sum_{i=1}^{n} \varphi_{x_i x_i}(x,y) = g(x,y).$$

(Here, y is a fixed parameter.) Let

$$g(x) = \int_{\mathbb{R}^n} g(x,y)\, dy.$$

Find $v(x)$ so that

$$-\Delta v(x) = -\sum_{i=1}^{n} v_{x_i x_i}(x) = g(x).$$

Answer. Remember that $\int$ is essentially a summation. Then, we need to "sum" in the y parameter. Define:

$$v(x) = \int_{\mathbb{R}^n} \varphi(x,y)\, dy.$$

Then,

$$v_{x_i}(x) = \int_{\mathbb{R}^n} \varphi_{x_i}(x,y)\, dy,$$

$$v_{x_i x_i}(x) = \int_{\mathbb{R}^n} \varphi_{x_i x_i}(x,y)\, dy.$$

Thus,

$$-\Delta v(x) = \int_{\mathbb{R}^n} -\Delta_x \varphi(x,y)\, dy = \int_{\mathbb{R}^n} g(x,y)\, dy.$$

By the given hypothesis, we conclude that

$$-\Delta v(x) = \int_{\mathbb{R}^n} g(x,y)\, dy = g(x).$$

We are now ready to state the main theorem in this section.

Theorem 3.6. *Assume that $f \in C_c^2(\mathbb{R}^n)$. Set, for $x \in \mathbb{R}^n$,*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy.$$

We have the following properties.

(i) $u \in C^2(\mathbb{R}^n)$.

(ii) *u solves the Poisson equation*

$$-\Delta u = f \quad \text{in } \mathbb{R}^n. \tag{3.4}$$

Proof. Note that $\Phi(\cdot)$ is singular at 0, and we need to be careful with the integral in the definition of u.

We have

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy = \int_{\mathbb{R}^n} \Phi(y) f(x-y)\, dy.$$

It is then fine to differentiate u and f in x inside the second integral as $f \in C_c^2(\mathbb{R}^n)$. We compute, for $1 \le i, j \le n$,

$$u_{x_i x_i}(x) = \int_{\mathbb{R}^n} \Phi(y) f_{x_i x_i}(x-y) dy,$$

$$u_{x_i x_j}(x) = \int_{\mathbb{R}^n} \Phi(y) f_{x_i x_j}(x-y) dy.$$

Hence, $u \in C^2(\mathbb{R}^n)$.

We now prove that u solves (3.4), that is, we need to show

$$-\Delta u(x) = -\int_{\mathbb{R}^n} \Phi(y) \Delta_x f(x-y)\, dy = f(x).$$

We write $\Delta_x f(x-y) = \sum_{i=1}^n f_{x_i x_i}(x-y)$. It is clear that

$$\Delta_x f(x-y) = \sum_{i=1}^n f_{x_i x_i}(x-y) = \sum_{i=1}^n f_{y_i y_i}(x-y) = \Delta_y f(x-y).$$

Therefore,

$$\begin{aligned} &\int_{\mathbb{R}^n} \Phi(y) \Delta_x f(x-y)\, dy \\ &= \int_{B(0,\varepsilon)} \Phi(y) \Delta_x f(x-y)\, dy + \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y) \Delta_x f(x-y)\, dy \\ &= A_\varepsilon + B_\varepsilon. \end{aligned}$$

The first integral

$$A_\varepsilon = \int_{B(0,\varepsilon)} \Phi(y) \Delta_x f(x-y)\, dy$$

contains the origin, the singular point of Φ. The second integral

$$B_\varepsilon = \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y) \Delta_x f(x-y)\, dy$$

does not contain any singularities.

For A_ε, we can bound it in a crude way:

$$|A_\varepsilon| \le \|D^2 f\|_{L^\infty} \int_{B(0,\varepsilon)} |\Phi(y)|\, dy \le \begin{cases} C\varepsilon^2 & \text{if } n \ge 3, \\ C\varepsilon^2 \log \varepsilon & \text{if } n = 2. \end{cases} \tag{3.5}$$

Here is a proof of this for $n \geq 3$:

$$\begin{aligned}\int_{B(0,\varepsilon)} \Phi(y)dy &= \frac{1}{n(n-2)\omega_n} \int_{B(0,\varepsilon)} \frac{1}{|x|^{n-2}}\, dx \\ &= \frac{1}{n(n-2)\omega_n} \int_0^\varepsilon \int_{\partial B(0,r)} \frac{1}{r^{n-2}}\, dS(x)dr \\ &= \frac{|\partial B_1|}{n(n-2)\omega_n} \int_0^\varepsilon r\, dr \leq C\varepsilon^2.\end{aligned}$$

In the above, we used $|\partial B(0,r)| = r^{n-1}|\partial B(0,1)| = r^{n-1}|\partial B_1|$. In fact, we will prove later that $|\partial B_1| = n\omega_n$. We leave the proof of (3.5) when $n = 2$ as an exercise. Thus,

$$\lim_{\varepsilon\to 0} |A_\varepsilon| = 0,$$

which means that A_ε does not play an role as $\varepsilon \to 0$.

Next, we handle B_ε. Using integration by parts,

$$\begin{aligned}B_\varepsilon &= \int_{\mathbb{R}^n\setminus B(0,\varepsilon)} \Phi(y)\Delta_y f(x-y)\, dy \\ &= -\int_{\mathbb{R}^n\setminus B(0,\varepsilon)} D\Phi(y)\cdot D_y f(x-y)\, dy + \int_{\partial B(0,\varepsilon)} \Phi(y)\frac{\partial f(x-y)}{\partial \nu}\, dS \\ &= D_\varepsilon + E_\varepsilon.\end{aligned}$$

Here, ν is the outward unit normal vector to $\mathbb{R}^n \setminus B(0,\varepsilon)$ on $\partial B(0,\varepsilon)$. For $y \in \partial B(0,\varepsilon)$, we see that

$$\nu(y) = -\frac{y}{|y|}.$$

We can bound E_ε as

$$|E_\varepsilon| \leq \|Df\|_{L^\infty} \int_{\partial B(0,\varepsilon)} |\Phi(y)|\, dS(y) \leq \begin{cases} C\varepsilon & \text{if } n \geq 3, \\ C\varepsilon\log\varepsilon & \text{if } n = 2.\end{cases}$$

The proof of this is simpler than that of (3.5). Indeed, for $n \geq 3$,

$$\int_{\partial B(0,\varepsilon)} |\Phi(y)|\, dS(y) = \frac{|\partial B(0,\varepsilon)|}{n(n-2)\omega_n\varepsilon^{n-2}} = \frac{\varepsilon}{n-2}.$$

Of course,

$$\lim_{\varepsilon\to 0} |E_\varepsilon| = 0,$$

which means that E_ε also does not play an role as $\varepsilon \to 0$.

For D_ε, we use integration by parts once more:

$$D_\varepsilon = \int_{\mathbb{R}^n\setminus B(0,\varepsilon)} \Delta_y\Phi(y)f(x-y)\, dy - \int_{\partial B(0,\varepsilon)} f(x-y)\frac{\partial\Phi(y)}{\partial\nu}\, dS.$$

Since $\Delta\Phi(y) = 0$ for $y \neq 0$, we get:

$$D_\varepsilon = -\int_{\partial B(0,\varepsilon)} f(x-y)\frac{\partial \Phi(y)}{\partial \nu}\, dS.$$

We will only do the computation for $n \geq 3$. The case $n = 2$ is left as an exercise. Recall that, for $y \neq 0$,

$$\Phi(y) = \frac{1}{n(n-2)\omega_n}\frac{1}{|y|^{n-2}}.$$

Hence, for $y \neq 0$,

$$D\Phi(y) = -\frac{1}{n\omega_n}\frac{y}{|y|^n}.$$

As $\nu = -\frac{y}{|y|}$,

$$\frac{\partial \Phi(y)}{\partial \nu} = D\Phi(y)\cdot\nu = \frac{1}{n\omega_n}\frac{1}{|y|^n}.$$

Thus,

$$D_\varepsilon = -\int_{\partial B(0,\varepsilon)} f(x-y)\frac{1}{n\omega_n \varepsilon^{n-1}}\, dS = -\frac{1}{|\partial B(0,\varepsilon)|}\int_{\partial B(0,\varepsilon)} f(x-y)\, dS.$$

By the intermediate value theorem,

$$\lim_{\varepsilon\to 0} D_\varepsilon = -f(x).$$

We combine everything together to conclude the proof. □

Lemma 3.7. *For $r > 0$, we have*

$$|\partial B(0,r)| = r^{n-1}|\partial B(0,1)| = n\omega_n r^{n-1}.$$

Proof. We compute

$$\begin{aligned}\omega_n &= \int_{B(0,1)} 1\, dx = \int_0^1 \int_{\partial B(0,r)} 1\, dS dr\\ &= \int_0^1 r^{n-1}|\partial B(0,1)|\, dr = |\partial B(0,1)|\int_0^1 r^{n-1}\, dr = \frac{1}{n}|\partial B(0,1)|.\end{aligned}$$

Hence, $|\partial B(0,1)| = n\omega_n$, and the proof is complete. □

We say that $|\partial B(0,1)| = n\omega_n$ is the surface area of the unit sphere in $\mathbb{R}^n$.

Remark 3.8. $\Phi(\cdot)$ is the fundamental solution to the Laplace PDE

$$\begin{cases} -\Delta\Phi(y) = 0 \text{ for } y \neq 0,\\ \Phi \text{ blows up at } 0.\end{cases}$$

Formally, we see that $-\Delta\Phi(y) = \delta_0$, a Dirac delta at 0. Let us give an explanation. For each $\varepsilon > 0$,

$$\int_{B(0,\varepsilon)} -\Delta\Phi(y)\,dy = \int_{\partial B(0,\varepsilon)} \frac{\partial\Phi(y)}{\partial\nu}\,ds = \int_{\partial B(0,\varepsilon)} \frac{1}{|\partial B(0,\varepsilon)|}\,ds = 1.$$

As $\varepsilon \to 0$, this gives us a Dirac delta mass.

Once we have that $-\Delta\Phi = \delta_0$, then it is very natural to see the idea that

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\,dy.$$

Indeed,

$$-\Delta u(x) = \int_{\mathbb{R}^n} -\Delta\Phi(x-y) f(y)\,dy = \int_{\mathbb{R}^n} \delta_0(x-y) f(y)\,dy = f(x).$$

This fits well with the discussions we had before Theorem 3.6. Of course, this is a heuristic calculation only as δ_0 is not a function.

3.3. Mean value property of harmonic functions

Definition 3.9. Let $U \subset \mathbb{R}^n$ be a given open set. Let $u \in C^2(U)$. If $-\Delta u = 0$ in U, then we say that u is a harmonic function in U.

Definition 3.10 (Average integral)**.** For a given set E, denote by $|E|$ the volume of E. We define

$$⨍_E f(x)\,dx = \frac{1}{|E|}\int_E f(x)\,dx.$$

Here, we do not talk about measurability issue as we do not yet touch measure theory.

Theorem 3.11. *Assume $u \in C^2(\mathbb{R}^n)$ is harmonic. Then,*

$$u(x) = ⨍_{B(x,r)} u(y)\,dy = ⨍_{\partial B(x,r)} u(y)\,dS(y) \tag{3.6}$$

for all $x \in \mathbb{R}^n$, $r > 0$.

If u satisfies (3.6), then we say that u has the mean value property.

Proof. Fix $x \in \mathbb{R}^n$. For $r > 0$, let

$$\phi(r) = ⨍_{\partial B(x,r)} u(y)\,dS(y).$$

Change of variables: let $y = x + rz$. Then, $y \in \partial B(x,r)$ if and only if $z \in \partial B(0,1)$. Besides, in terms of surface area,

$$dS(y) = r^{n-1} dS(z).$$

Thus,

$$\phi(r) = \fint_{\partial B(0,1)} u(x+rz)\,dS(z) = \frac{1}{n\omega_n}\int_{\partial B(0,1)} u(x+rz)\,dS(z).$$

The domain is fixed now, so we can differentiate things more easily. We compute

$$\phi'(r) = \frac{1}{n\omega_n}\int_{\partial B(0,1)} \frac{d}{dr}(u(x+rz))\,dS(z) = \frac{1}{n\omega_n}\int_{\partial B(0,1)} Du(x+rz)\cdot z\,dS(z).$$

Observe that $z=\nu$ is the unit outer normal to $B(0,1)$ on $\partial B(0,1)$. Thus,

$$\phi'(r) = \frac{1}{n\omega_n}\int_{\partial B(0,1)} \frac{\partial u(x+rz)}{\partial \nu}\,dS.$$

Using the divergence theorem:

$$\phi'(r) = \frac{1}{n\omega_n}\int_{B(0,1)} \mathrm{div}_z(Du(x+rz))\,dz = \frac{r}{n\omega_n}\int_{B(0,1)} \Delta u(x+rz)\,dz = 0.$$

Hence, $r \mapsto \phi(r)$ is constant for $r>0$.

Note that

$$\lim_{r\to 0}\phi(r) = \lim_{r\to 0}\fint_{\partial B(x,r)} u(y)\,dS(y) = u(x).$$

Thus,

$$u(x) = \phi(r) = \fint_{\partial B(x,r)} u(y)\,dS(y) \quad \text{for all } r>0. \tag{3.7}$$

Finally, we use (3.7) to compute

$$\begin{aligned}
\fint_{B(x,r)} u(y)\,dy &= \frac{1}{\omega_n r^n}\int_{B(x,r)} u(y)\,dy \\
&= \frac{1}{\omega_n r^n}\int_0^r \left(\int_{\partial B(x,s)} u(y)\,dS(y)\right) ds \\
&= \frac{1}{\omega_n r^n}\left(\int_0^r n\omega_n s^{n-1} u(x)\,ds\right) = u(x).
\end{aligned}$$

□

Definition 3.12. Let $U \subset \mathbb{R}^n$ be a given open set. Let $u \in C^2(U)$.

If $-\Delta u \le 0$ in U, we say that u is *subharmonic*.

If $-\Delta u \ge 0$ in U, we say that u is *superharmonic*.

Clearly, u is harmonic if it is both subharmonic and superharmonic.

By following step by step the proof of Theorem 3.11, we immediately have the corollary below.

Corollary 3.13. *Let ϕ be as in the proof of Theorem 3.11.*

If $u \in C^2(\mathbb{R}^n)$ is subharmonic, then for all $x \in \mathbb{R}^n$, $r > 0$, $\phi'(r) \geq 0$, and

$$u(x) \leq \fint_{\partial B(x,r)} u(y)\, dS(y),$$

$$u(x) \leq \fint_{B(x,r)} u(y)\, dy.$$

If $u \in C^2(\mathbb{R}^n)$ is superharmonic, then for all $x \in \mathbb{R}^n$, $r > 0$, $\phi'(r) \leq 0$, and

$$u(x) \geq \fint_{\partial B(x,r)} u(y)\, dS(y),$$

$$u(x) \geq \fint_{B(x,r)} u(y)\, dy.$$

Theorem 3.14. *If $u \in C^2(\mathbb{R}^n)$ and u satisfies the mean value property, that is, for all $x \in \mathbb{R}^n$, $r > 0$,*

$$u(x) = \fint_{\partial B(x,r)} u(y)\, dS(y),$$

then u is harmonic.

Proof. Assume by contradiction that u is not harmonic. Then, $-\Delta u \neq 0$. Without loss of generality, we can find $x \in \mathbb{R}^n$, $s > 0$ such that

$$-\Delta u(x) < 0 \quad \text{for all } y \in B(x,s).$$

By the proof of Theorem 3.11, for all $0 < r \leq s$,

$$\phi'(r) = \frac{r}{n\omega_n} \int_{B(0,1)} \Delta u(x + rz)\, dz > 0.$$

Therefore, $r \mapsto \phi(r)$ is strictly increasing on $(0, s]$. In particular,

$$u(x) < \fint_{\partial B(x,r)} u(y)\, dS(y) \quad \text{for } 0 < r \leq s,$$

which contradicts our assumption. $\square$

3.4. Strong maximum principle and a uniqueness result for harmonic functions

3.4.1. Maximum principles for harmonic functions.

Theorem 3.15 (Maximum principles for harmonic functions)**.** *Let $U \subset \mathbb{R}^n$ be a bounded domain. If $u \in C^2(U) \cap C(\overline{U})$ is harmonic in U, then the following claims hold.*

(i) $\max_{\overline{U}} u = \max_{\partial U} u$.

(ii) *If there exists $z_0 \in U$ such that $u(z_0) = \max_{\overline{U}} u$, then $u \equiv u(z_0)$.*

Property (i) in Theorem 3.15 is called the maximum principle for harmonic functions. Property (ii) in Theorem 3.15 is called the strong maximum principle for harmonic functions.

Proof. (i) Assume not, then there exists $y \in U$ such that

$$u(y) = \max_{\overline{U}} u > \max_{\partial U} u.$$

Let $r > 0$ be the maximum radius such that $B(y,r) \subset \overline{U}$. In particular, $B(y,r) \cap \partial\overline{U} \neq \emptyset$. In fact, we can see that $r = \mathrm{dist}(y, \partial U)$. See Figure 3.

Let $z \in B(y,r) \cap \partial\overline{U}$. As

$$\max_{\overline{U}} u = u(y) = \fint_{\partial B(y,r)} u(x)\, dS(x) \leq \fint_{\partial B(y,r)} \max_{\overline{U}} u \, dS(x) = \max_{\overline{U}} u,$$

we must have

$$u(x) = u(y) \quad \forall x \in \partial B(y,r).$$

In particular, $u(z) = u(y)$, which is absurd.

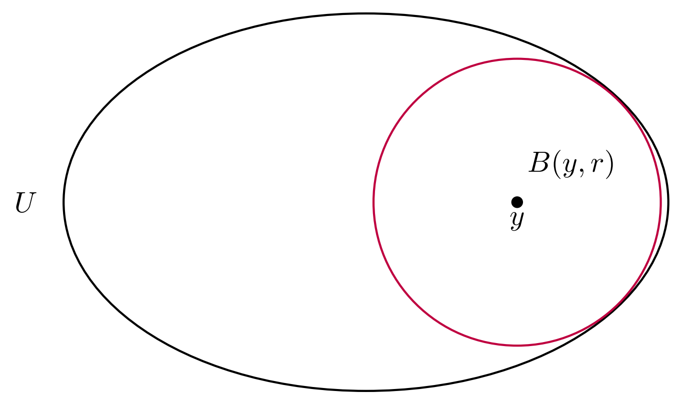

Figure 3. U and $B(y,r)$

(ii) Let

$$V = \{x \in U : u(x) = u(z_0) = \max u\} = u^{-1}(\{u(z_0)\}).$$

Clearly, $V \neq \emptyset$. As u is continuous, $V = u^{-1}(\{u(z_0)\})$ is relatively closed in U.

Besides, by the above proof, we see that V is open in U. Indeed, for each $x \in V$ and $r > 0$ such that $B(x,r) \subset U$, we imply that $u = u(z_0)$ in $B(x,r)$.

Thus, V is both relatively open and closed in U and $V \neq \emptyset$. As U is a domain, it is connected, which means that $V = U$. $\square$

We note that the same claims in the above theorem hold for minimum value. Let us state them as a theorem for completeness.

Theorem 3.16. *Let $U \subset \mathbb{R}^n$ be a bounded domain. If $u \in C^2(U) \cap C(\overline{U})$ is harmonic in U, then the following claims hold.*

(i) $\min_{\overline{U}} u = \min_{\partial U} u$.

(ii) *If there exists $z_0 \in U$ such that $u(z_0) = \min_{\overline{U}} u$, then $u \equiv u(z_0)$.*

Remark 3.17. By Corollary 3.13, we see that Theorem 3.15 holds if we only require that u is subharmonic. Similarly, we have Theorem 3.16 under a weaker assumption that u is superharmonic.

3.4.2. A uniqueness result.

Theorem 3.18 (Uniqueness via the maximum principle)**.** *Let $U \subset \mathbb{R}^n$ be a bounded domain. Let $f \in C(\overline{U}), g \in C(\partial U)$. If $u_1, u_2 \in C^2(U) \cap C(\overline{U})$ are both solutions to*

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u = g & \text{on } \partial U, \end{cases} \tag{3.8}$$

then $u_1 = u_2$.

Equation (3.8) is often called a Dirichlet problem. The boundary condition $u = g$ on ∂U is called the Dirichlet boundary condition.

Proof. Let $v = u_1 - u_2$. Then, $v \in C^2(U) \cap C(\overline{U})$ solves

$$\begin{cases} -\Delta v = 0 & \text{in } U, \\ v = 0 & \text{on } \partial U, \end{cases}$$

By the maximum principle,

$$\max_{\overline{U}} v = \max_{\partial U} v = 0,$$

and

$$\min_{\overline{U}} v = \min_{\partial U} v = 0.$$

Thus, $v \equiv 0$. $\square$

We note that the uniqueness result in the above theorem confirms that (3.8) has at most one solution. However, we have not shown yet if (3.8) has a solution. We will touch upon this point later.

So far, we have always had to assume that $u \in C^2$ in our analysis for harmonic functions. Is it necessary? Can we not assume it? This is a regularity question.

Theorem 3.19. *If $u \in C(\mathbb{R}^n)$ satisfies the mean value property, then immediately, $u \in C^\infty(\mathbb{R}^n)$.*

Proof. Let φ be a standard mollifier, that is,

$$\begin{cases} \varphi \in C_c^\infty(\mathbb{R}^n), \quad \varphi \geq 0, \quad \varphi(x) = \varphi(|x|), \\ \operatorname{supp}(\varphi) \subset B(0,1), \quad \int_{\mathbb{R}^n} \varphi(x)\,dx = 1. \end{cases}$$

Here, we assume that φ is radially symmetric and we write $\varphi(x) = \varphi(|x|)$ by abuse of notations. For $\varepsilon > 0$, let

$$\varphi^\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right).$$

Then,

$$\operatorname{supp}(\varphi^\varepsilon) \subset B(0,\varepsilon), \quad \int_{\mathbb{R}^n} \varphi^\varepsilon(x)\,dx = 1.$$

By abuse of notations, we write $\varphi^\varepsilon(x) = \varphi^\varepsilon(|x|)$ too. Let

$$u^\varepsilon(x) = \int_{\mathbb{R}^n} u(x-y)\varphi^\varepsilon(y)dy = \int_{B_\varepsilon(x)} u(y)\varphi^\varepsilon(x-y)dy.$$

It's clear that $u^\varepsilon \in C^\infty(\mathbb{R}^n)$. Let us use the mean value property of u to compute

$$\begin{aligned} u^\varepsilon(x) &= \int_{B_\varepsilon(x)} \varphi^\varepsilon(x-y)u(y)\,dy \\ &= \int_0^\varepsilon \left(\fint_{\partial B(x,s)} u(y)\,dS(y)\right)\varphi^\varepsilon(s)\,ds \\ &= \int_0^\varepsilon u(x)\left(\fint_{\partial B(x,1)} 1\,dS(y)\right)\varphi^\varepsilon(s)\,ds \\ &= u(x)\int_{B_\varepsilon(x)} \varphi^\varepsilon(x-y)\,dy = u(x). \end{aligned}$$

Thus, $u = u^\varepsilon$, and hence, $u \in C^\infty(\mathbb{R}^n)$. Of course, we also yield that u is harmonic.

□

We can sum all this up as the following.

Theorem 3.20. *Let $u \in C(\mathbb{R}^n)$. Then, the followings are equivalent.*

(i) *u satisfies the mean value property.*

(ii) *u is harmonic and $u \in C^\infty(\mathbb{R}^n)$.*

Remark 3.21. Theorem 3.20 still holds true for open set $U \subset \mathbb{R}^n$ and $u \in C(\overline{U})$.

3.4.3. Some discussions. We would like to give some discussions about the main results in this section.

3.4.3.1. *Discussion 1.* We require connectedness of U in (ii) of Theorem 3.15, which is the strong maximum principle. Is it really needed?

It turns out this is really needed, and claim (ii) would not hold if we do not have connectedness as shown in the example below.

Example 3.22. Take

$$y = 2e_1 = (2, 0, \ldots, 0), \quad z = -2e_1 = (-2, 0, \ldots, 0).$$

Let

$$U = B(y, 1) \cup B(z, 1).$$

See Figure 4. Denote by

$$u(x) = \begin{cases} 1 & \text{for } x \in B(y, 1), \\ 2 & \text{for } x \in B(z, 1). \end{cases}$$

Then, u is harmonic in U and (ii) fails.

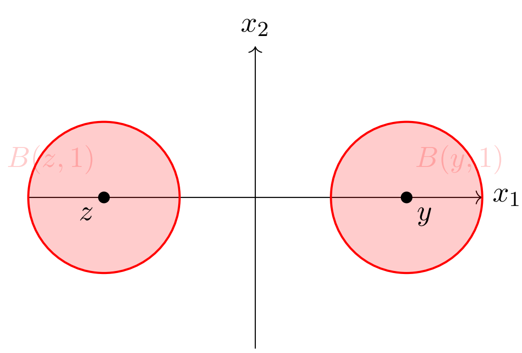

Figure 4. Strong maximum principle fails when U is not connected

3.4.3.2. *Discussion 2.* The uniqueness result for the Dirichlet problem in Theorem 3.18 is really important as it helps us to see that the physical solution is unique. Therefore, if we can find one solution to (3.8), then it is automatically the only one.

Example 3.23. Let $n = 2$, $U = (0, 1)^2 \subset \mathbb{R}^2$. Consider the following Dirichlet problem

$$\begin{cases} -\Delta u(x) = \sin(2\pi x_1) & \text{in } U, \\ u(x) = g(x) & \text{on } \partial U, \end{cases}$$

where

$$g(x) = \begin{cases} 0 & x_1 = 0 \text{ or } 1, \quad 0 \le x_2 \le 1, \\ \frac{1}{4\pi^2}\sin(2\pi x_1) & x_2 = 0 \text{ or } 1, \quad 0 \le x_1 \le 1. \end{cases}$$

This is an easy case as we can guess right away the form of solutions. Consider

$$u(x) = \frac{1}{4\pi^2}\sin(2\pi x_1) \quad \text{for } x \in \overline{U}.$$

Then, u satisfies the boundary condition, and

$$u_{x_1x_1} = -\sin(2\pi x_1), \quad u_{x_2x_2} = 0.$$

Hence, $-\Delta u(x) = \sin(2\pi x_1)$ in U. We thus get that u is the unique solution to the above Dirichlet problem.

3.5. Liouville's theorem

We show that all bounded harmonic functions on the whole $\mathbb{R}^n$ are constant.

Theorem 3.24 (Liouville's theorem). *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a bounded harmonic function. Then u is constant.*

Proof. As u is bounded, there exists $C > 0$ such that

$$|u(x)| \le C \quad \text{for all } x \in \mathbb{R}^n.$$

Note that u_{x_i} is also harmonic for $1 \le i \le n$. Fix $y \in \mathbb{R}^n$. For $r > 0$, we denote by $\nu = (\nu_1, \ldots, \nu_n)$ the unit outward normal to $B(y,r)$ on $\partial B(y,r)$. By the mean value property and the divergence theorem,

$$\begin{aligned} |u_{x_i}(y)| &= \left| \fint_{B(y,r)} u_{x_i}(x)\,dx \right| \\ &= \frac{1}{\omega_n r^n} \left| \int_{B(y,r)} u_{x_i}(x)\,dx \right| \\ &= \frac{1}{\omega_n r^n} \left| \int_{\partial B(y,r)} u(x)\nu_i(x)\,dS(x) \right| \\ &\le \frac{1}{\omega_n r^n} C n \omega_n r^{n-1} = \frac{Cn}{r} \qquad (\text{as } |u(x)| \le C). \end{aligned}$$

By letting $r \to \infty$, we imply that $u_{x_i}(y) = 0$. Hence, $Du = 0$ in $\mathbb{R}^n$, which yields that u is constant in $\mathbb{R}^n$. $\square$

Remark 3.25. From the above proof, we also obtain an important inequality

$$|u_{x_i}(y)| \le \frac{n\|u\|_{L^\infty(B(y,r))}}{r}.$$

We give an application of the Liouville theorem.

Theorem 3.26. *Assume that $n \geq 3$ and $f \in C_c^2(\mathbb{R}^n)$. Then any bounded solution to*

$$-\Delta u = f \quad \text{in } \mathbb{R}^n \tag{3.9}$$

is of the form

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy + C \quad \text{for } x \in \mathbb{R}^n.$$

Here, $C \in \mathbb{R}$ is a constant.

Proof. Thanks to Theorem 3.6,

$$\overline{u}(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$$

is a solution to (3.9). As f has compact support and Φ decays to 0 at infinity, we see that $\overline{u}$ is a bounded.

Let u be another bounded solution to (3.9). Set $v = u - \overline{u}$. Then, v is bounded and harmonic in $\mathbb{R}^n$. By Liouville's theorem, v is constant. The proof is complete. □

We hence obtain the uniqueness of bounded solutions (up to additive constants) of Poisson's equation (3.9).

3.6. Energy method for the Laplace equation

3.6.1. Uniqueness via the energy method. Another way to prove uniqueness of solutions to the Dirichlet problem (3.8) is the energy method.

Theorem 3.27 (Uniqueness via the energy method)**.** *Let $U \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary. Let $f \in C(\overline{U}), g \in C(\partial U)$. If $u_1, u_2 \in C^2(U) \cap C(\overline{U})$ are both solutions to*

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u = g & \text{on } \partial U, \end{cases} \tag{3.10}$$

then $u_1 = u_2$.

Proof. Let u_1, u_2 be two solutions. Let $v = u_1 - u_2$. Then,

$$\begin{cases} -\Delta v = 0 & \text{in } U, \\ v = 0 & \text{on } \partial U. \end{cases}$$

Instead of using the maximum principle here, we use integration by parts. Multiply the PDE by v and integrate:

$$0 = \int_U -\Delta v \cdot v\, dx = \int_U |Dv|^2\, dx - \int_{\partial U} v\, Dv \cdot \mathbf{n}\, dS.$$

We used either integration by parts or the divergence theorem in the above noting that $\operatorname{div}(vDv) = |Dv|^2 + v\Delta v$. Here, $\mathbf{n}$ is the unit outward normal vector to U on ∂U. The boundary integral above vanishes as $v = 0$ on ∂U. Therefore,

$$\int_U |Dv|^2\,dx = 0 \quad \Longrightarrow \quad Dv \equiv 0.$$

Thus, v is constant. As $v = 0$ on ∂U, we conclude $v \equiv 0$. □

3.6.2. Other types of boundary conditions. Let us consider other boundary conditions and give some discussions.

3.6.2.1. *Neumann boundary condition.*

$$\begin{cases} -\Delta u = f & \text{in } U, \\ \frac{\partial u}{\partial \mathbf{n}} = h & \text{on } \partial U. \end{cases} \tag{3.11}$$

Here, $\mathbf{n}$ is the unit outward normal vector to U on ∂U, and $\frac{\partial u}{\partial \mathbf{n}}$ represents the partial derivative of u in the direction normal to the boundary. We have $f \in C(U)$ and $h \in C(\partial U)$ are given functions. The boundary condition $\frac{\partial u}{\partial \mathbf{n}} = h$ on ∂U is called the Neumann boundary condition.

Let u_1, u_2 be two smooth solutions of (3.11). Let $v = u_1 - u_2$. Then, v solves

$$\begin{cases} -\Delta v = 0 & \text{in } U, \\ \frac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial U. \end{cases}$$

Multiply the PDE by v and integrate:

$$0 = \int_U -\Delta v \cdot v\,dx = \int_U |Dv|^2\,dx - \int_{\partial U} vDv \cdot \mathbf{n}\,dS.$$

Since $\frac{\partial v}{\partial \mathbf{n}} = Dv \cdot \mathbf{n} = 0$ on ∂U, the boundary integral vanishes. We get

$$\int_U |Dv|^2\,dx = 0 \quad \Longrightarrow \quad Dv \equiv 0.$$

Thus, v is constant. But we cannot conclude that $v \equiv 0$. In fact, any constant function v is a solution to the above PDE.

Thus, we can only conclude that the solution to (3.11) is unique up to an additive constant. In particular, there exists $C \in \mathbb{R}$ such that

$$u_1 = u_2 + C.$$

3.6.2.2. *Robin boundary condition.*

$$\begin{cases} -\Delta u = f & \text{in } U, \\ \frac{\partial u}{\partial \mathbf{n}} + \alpha u = g & \text{on } \partial U, \end{cases} \tag{3.12}$$

where $\alpha > 0$ is a given constant, $f \in C(U)$ and $h \in C(\partial U)$ are given functions. The boundary condition $\frac{\partial u}{\partial \mathbf{n}} + \alpha u = g$ on ∂U is called the Robin boundary condition.

We claim that if u_1, u_2 are smooth two solutions, then $u_1 = u_2$ (i.e., uniqueness holds). Let $v = u_1 - u_2$. Then,

$$\begin{cases} -\Delta v = 0 & \text{in } U, \\ \frac{\partial v}{\partial \mathbf{n}} + \alpha v = 0 & \text{on } \partial U. \end{cases}$$

Multiplying by v and integrating:

$$0 = \int_U -\Delta v \cdot v \, dx = \int_U |Dv|^2 \, dx - \int_{\partial U} v Dv \cdot \mathbf{n} \, dS.$$

From the Robin boundary condition, we get

$$\int_{\partial U} v \left(\frac{\partial v}{\partial \mathbf{n}} \right) dS = -\alpha \int_{\partial U} v^2 \, dS.$$

Combine the two lines of equalities above to yield

$$0 = \int_U |Dv|^2 \, dx + \alpha \int_{\partial U} v^2 \, dS.$$

Both terms are non-negative, which implies:

$$\begin{cases} Dv \equiv 0 & \text{in } U, \\ v \equiv 0 & \text{on } \partial U. \end{cases}$$

Therefore, $v \equiv 0$ in U.

Hence, $u_1 = u_2$, which proves uniqueness. This uniqueness result however relies on the strong assumption that $\alpha > 0$. We have seen that this uniqueness fails if $\alpha = 0$. It also fails if $\alpha < 0$.

3.7. Heat equation

We normalize $c = 1$ in (3.2) and consider

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty).$$

Here, x represents the position, t represents the time, and $u(x,t)$ is the temperature at $(x,t) \in \mathbb{R}^n \times [0, \infty)$. We have already discussed the physical interpretation of the heat equation. Let us therefore proceed.

3.7.1. Fundamental solution to the heat equation. There are many ways to guess a form of a fundamental solution to the heat equation. We here just provide one such way.

Consider

$$u(x,t) = \frac{1}{t^\alpha} v \left(\frac{x}{t^\beta} \right) = \frac{1}{t^\alpha} v(y)$$

for $y = \frac{x}{t^\beta}$. This is just an educated guess that u has some scaling property with α, β to be chosen.

Then,

$$\begin{aligned} u_t(x,t) &= -\alpha t^{-(\alpha+1)} v(y) - \beta t^{-(\alpha+\beta+1)} Dv\left(\frac{x}{t^\beta}\right) \cdot x \\ &= -t^{-(\alpha+1)}\left[\alpha v(y) + \beta Dv(y) \cdot y\right], \end{aligned}$$

and

$$\Delta u(x,t) = t^{-(\alpha+2\beta)} \Delta v(y).$$

Thus, we get

$$t^{-(\alpha+1)}\left[\alpha v(y) + \beta Dv(y) \cdot y\right] + t^{-(\alpha+2\beta)} \Delta v(y) = 0.$$

In order to balance things out, we want

$$\alpha + 1 = \alpha + 2\beta \quad \Longrightarrow \quad \beta = \frac{1}{2}.$$

We end up with a new PDE for v:

$$\alpha v(y) + \frac{1}{2} Dv(y) \cdot y + \Delta v(y) = 0.$$

Next, we want v to be radially symmetric, that is,

$$v(y) = w(r) = w(|y|) \quad \text{for } r = |y|.$$

By the same computations that we did for the Laplace equation, we get, for $r > 0$,

$$\alpha w(r) + \frac{1}{2} r w'(r) + w''(r) + \frac{n-1}{r} w'(r) = 0.$$

It is now an ODE, and we need to use some tricks to group things together. Multiply the ODE by r^{n-1} to get:

$$\left[\alpha r^{n-1} w + \frac{1}{2} r^n w'\right] + \left[r^{n-1} w'' + (n-1) r^{n-2} w'\right] = 0.$$

As suggested, we choose

$$\alpha = \frac{n}{2}$$

to have

$$\frac{n}{2} r^{n-1} w + \frac{1}{2} r^n w' = \frac{1}{2}\left(r^n w\right)'.$$

Therefore,

$$\frac{1}{2}\left(r^n w'\right)' + \left(r^{n-1} w'\right)' = 0.$$

This implies

$$\frac{1}{2} r^n w + r^{n-1} w' = C,$$

for some constant $C \in \mathbb{R}$. By physical interpretation of temperatures, w, w' decay quickly to 0 as $r \to \infty$, so one can choose $C = 0$. Hence,

$$\frac{1}{2}r^n w + r^{n-1}w' = 0.$$

Rearrange this to yield

$$\frac{w'}{w} = -\frac{1}{2}r \qquad \Longrightarrow \qquad (\log w)' = -\frac{1}{2}r.$$

Hence,

$$\log w = -\frac{r^2}{4} + C_1 \qquad \Longrightarrow \qquad w(r) = C_2 e^{-\frac{r^2}{4}}.$$

Putting things together, we see that

$$u(x,t) = \frac{1}{t^{n/2}} v\left(\frac{x}{t^{1/2}}\right) = \frac{C_2}{t^{n/2}} e^{-\frac{|x|^2}{4t}}.$$

Definition 3.28 (Fundamental solution of the heat equation)**.** The function

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & x \in \mathbb{R}^n, t > 0 \\ 0 & x \in \mathbb{R}^n, t < 0 \end{cases}$$

is called the fundamental solution of the heat equation.

Below, we will see why we choose the constant for the fundamental solution $C_2 = \frac{1}{(4\pi)^{n/2}}$.

Lemma 3.29. *For all $t > 0$,*

$$\int_{\mathbb{R}^n} \Phi(x,t)\, dx = 1.$$

This is often called a conservation of heat/energy.

Proof. We perform a change of variable $z = \frac{x}{(4t)^{1/2}}$. Then,

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(x,t)\, dx &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}}\, dx = \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2}\, dz \\ &= \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|z_1|^2} e^{-|z_2|^2} \ldots e^{-|z_n|^2}\, dz_1 dz_2 \ldots dz_n \\ &= \left(\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-s^2} ds\right)^n \quad \text{(by Fubini's theorem).} \end{aligned}$$

We now show that

$$\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-s^2}\, ds = 1.$$

This problem often appears in multivariable calculus. We have, by Fubini's theorem again,

$$
\begin{aligned}
\left(\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-s^2}\,ds\right)^2 &= \frac{1}{\pi}\int_{\mathbb{R}^2} e^{-|z|^2}\,dz \\
&= \frac{1}{\pi}\int_0^\infty \left(\int_{\partial B(0,r)} e^{-|z|^2}\,dS(z)\right)\,dr \\
&= \frac{1}{\pi}\int_0^\infty \left(e^{-r^2}\cdot 2\pi r\right)\,dr \\
&= \int_0^\infty e^{-r^2}\,d(r^2) = -e^{-r^2}\Big]_{r=0}^{r=\infty} = 1.
\end{aligned}
$$

The proof is complete. □

Remark 3.30. We note that

$$s \mapsto \frac{1}{\sqrt{\pi}}e^{-s^2} \quad \text{is a Gaussian.}$$

And for each fixed $t > 0$,

$$x \mapsto \Phi(x,t) = \frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}} \quad \text{is a Gaussian.}$$

See Figure 5. Besides, $\max_x \Phi(x,t) = \Phi(0,t) = \frac{1}{(4\pi t)^{n/2}}$, and we see that

$$\lim_{t\to\infty}\Phi(0,t) = \lim_{t\to\infty}\frac{1}{(4\pi t)^{n/2}} = 0.$$

On the other hand,

$$\lim_{t\to 0^+}\Phi(0,t) = \lim_{t\to 0^+}\frac{1}{(4\pi t)^{n/2}} = +\infty,$$

and for $x \neq 0$,

$$\lim_{t\to 0^+}\Phi(x,t) = \lim_{t\to 0^+}\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}} = 0.$$

These points strongly suggest that as $t \to 0^+$, $\Phi(x,t)$ tends to the Dirac delta δ_0. We often write

$$\begin{cases} \Phi_t - \Delta\Phi = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ \Phi(\cdot,0) = \delta_0 & \text{on } \mathbb{R}^n. \end{cases}$$

Theorem 3.31. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a given continuous and bounded function. Denote by*

$$
\begin{aligned}
u(x,t) &= \int_{\mathbb{R}^n}\Phi(x-y,t)g(y)\,dy \\
&= \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}g(y)\,dy \qquad \textit{for } x\in\mathbb{R}^n, t>0.
\end{aligned}
$$

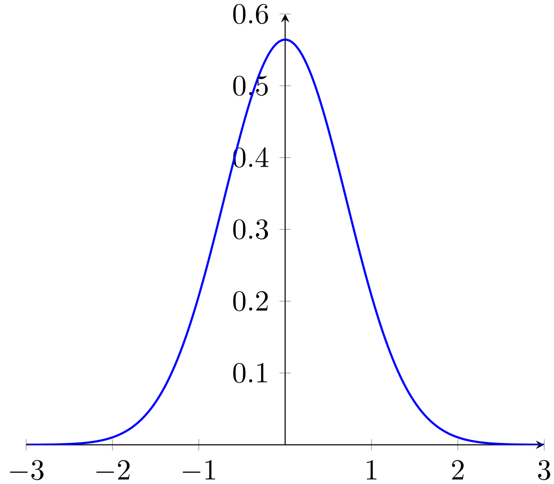

Figure 5. Graph of $\Phi(\cdot, t)$ for a fixed $t > 0$

Then, the following properties hold.

(i) $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$*;*

(ii) $u_t - \Delta u = 0$ *in* $\mathbb{R}^n \times (0,\infty)$*;*

(iii) $\lim_{(x,t)\to(x^0,0)} u(x,t) = g(x^0)$ *for each* $x^0 \in \mathbb{R}^n$*.*

Proof. We prove (i). Note that for $t > 0$, the mapping $x \mapsto \Phi(x-y,t)$ is smooth, and

$$u_{x_i}(x,t) = \int_{\mathbb{R}^n} \Phi_{x_i}(x-y,t)g(y)\,dy, u_{x_ix_j}(x,t) \quad = \int_{\mathbb{R}^n} \Phi_{x_ix_j}(x-y,t)g(y)\,dy,$$

and so on. Besides, for $t > 0$, the mapping $t \mapsto \Phi(x-y,t)$ is smooth, and

$$u_t(x,t) = \int_{\mathbb{R}^n} \Phi_t(x-y,t)g(y)\,dy,$$

and so on. Thus, $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$.

We next prove (ii). For $(x,t) \in \mathbb{R}^n \times (0,\infty)$,

$$u_t - \Delta u(x,t) = \int_{\mathbb{R}^n} [\Phi_t - \Delta\Phi]\,(x-y,t)g(y)\,dy = 0.$$

This is easy and clear as everything is nice and smooth for $t > 0$.

Finally, we need to prove (iii), which says that the initial condition holds. This is the most complicated point in this proof. Fix $x^0 \in \mathbb{R}^n$ and $\varepsilon > 0$. As g is continuous, we can find $\delta > 0$ such that

$$|g(y) - g(x^0)| < \varepsilon \quad \text{for } |y - x^0| < \delta. \tag{3.13}$$

See Figure 6.

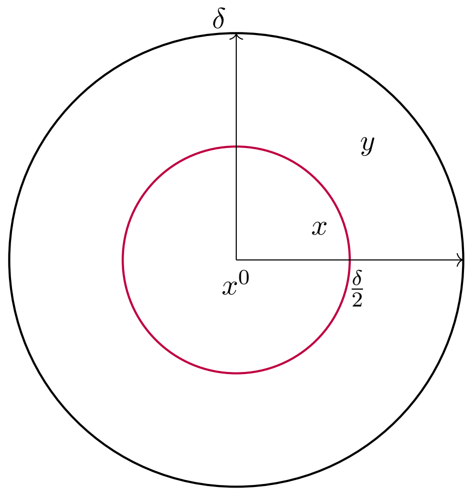

Figure 6. $x \in B(x^0, \frac{\delta}{2})$

For $x \in B(x^0, \frac{\delta}{2})$ and $t > 0$,

$$
\begin{aligned}
|u(x,t) - g(x^0)| &= \left| \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy - g(x^0) \right| \\
&= \left| \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy - \int_{\mathbb{R}^n} \Phi(x-y,t) g(x^0)\, dy \right| \\
&\leq \int_{\mathbb{R}^n} \Phi(x-y,t) \left| g(y) - g(x^0) \right| dy \\
&= \int_{B(x^0,\delta)} \Phi(x-y,t) |g(y) - g(x^0)|\, dy \\
&\qquad + \int_{\mathbb{R}^n \setminus B(x^0,\delta)} \Phi(x-y,t) |g(y) - g(x^0)|\, dy \\
&= I + J.
\end{aligned}
$$

It is very easy to handle I thanks to (3.13)

$$
\begin{aligned}
I &= \int_{B(x^0,\delta)} \Phi(x-y,t) |g(y) - g(x^0)|\, dy \\
&\leq \varepsilon \int_{B(x^0,\delta)} \Phi(x-y,t) dy \leq \varepsilon.
\end{aligned}
$$

Thus, $I \leq \varepsilon$. We next bound J. For $x \in B(x^0, \frac{\delta}{2})$ and $y \notin B(x^0, \delta)$,

$$
|y - x| \geq |y - x^0| - |x - x^0| \geq \frac{1}{2} |y - x^0|.
$$

As g is bounded by C by the assumptions, $|g(y) - g(x^0)| \leq 2C$. Then,

$$\begin{aligned}
J &\leq 2C \int_{\mathbb{R}^n \setminus B(x^0,\delta)} \Phi(x-y,t)\,dy \\
&= \frac{2C}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n \setminus B(x^0,\delta)} e^{-\frac{|x-y|^2}{4t}}\,dy \\
&\leq \frac{2C}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n \setminus B(x^0,\delta)} e^{-\frac{|y-x^0|^2}{16t}}\,dy \\
&= \frac{2C}{(4\pi t)^{n/2}} \int_{\delta}^{\infty} \left(\int_{\partial B(x^0,r)} e^{-\frac{r^2}{16t}}\,dS \right) dr \\
&= \frac{2C}{(4\pi t)^{n/2}} \int_{\delta}^{\infty} e^{-\frac{r^2}{16t}} n\omega_n r^{n-1}\,dr \\
&\leq C \int_{\delta}^{\infty} \frac{e^{-\frac{r^2}{16t}}}{t^{n/2}} r^{n-1}\,dr.
\end{aligned}$$

As $t \to 0^+$, the term $\frac{e^{-\frac{r^2}{16t}}}{t^{n/2}} r^{n-1}$ goes to 0 exponentially fast for $r > \delta$. Thus, we get that

$$\lim_{t \to 0} J = 0.$$

Let $t \to 0^+$ and $\varepsilon \to 0$ in this order to yield that

$$u(x,t) \to g(x^0) \quad \text{as} \quad (x,t) \to (x^0, 0).$$

$\square$

Remark 3.32. Consider the heat equation with initial data $u(x,0) = g(x)$. Heuristically, we have

$$u(x,0) = g(x) = \int_{\mathbb{R}^n} \delta_0(x-y) g(y)\,dy.$$

Therefore, in light of the superposition principle, it is natural to expect that

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\,dy.$$

Assume now that $g \geq 0$ and $g \neq 0$. Then, by the formula above, we have that

$$u(x,t) > 0 \quad \text{for all } (x,t) \in \mathbb{R}^n \times (0,\infty).$$

Thus, the temperature $u(x,t)$ becomes positive right away even if we only start with the initial data with a compact support. This phenomenon is called the *infinite speed of propagation*. Of course, $u(x,t)$ decays exponential fast to 0 as $|x| \to \infty$.

3.8. Heat equation on the first quadrant

We now apply Theorem 3.31 to solve some heat equations on the first quadrant when $n = 1$. Denote by

$$V = \{(x,t) \in \mathbb{R}^2 \, : \, x > 0, t > 0\}.$$

We consider a Dirichlet boundary problem and a Neumann boundary problem.

3.8.1. Dirichlet boundary problem.

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } V, \\ u(0,t) = 0 & \text{for } t \geq 0, \\ u(x,0) = g(x) & \text{for } x \geq 0. \end{cases} \tag{3.14}$$

Here, $g \in C([0,\infty)$ is a given bounded function. For compatibility conditions, we assume $g(0) = 0$. The Dirichlet boundary condition $u(0,t) = 0$ for $t \geq 0$ means that the temperature at the boundary $x = 0$ is kept to be always zero.

Idea. Extend this problem to all $\mathbb{R}$. We need to do it in such a way that $u(0,t) = 0$. It is thus natural to extend the initial data $g(x)$ in the odd fashion to keep this property of $u(0,t) = 0$. Let $\widetilde{g}$ be the odd extension of g, that is,

$$\widetilde{g}(x) = \begin{cases} g(x) & \text{for } x \geq 0, \\ -g(-x) & \text{for } x < 0. \end{cases}$$

Consider the heat PDE

$$\begin{cases} \widetilde{u}_t - \widetilde{u}_{xx} = 0 & \text{for } x \in \mathbb{R}, t > 0, \\ \widetilde{u}(x,0) = \widetilde{g}(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Thanks to Theorem 3.31, we have the formula

$$\widetilde{u}(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \widetilde{g}(y) \, dy.$$

Lemma 3.33. *Let $\widetilde{u}$ be defined as above. Then, $x \mapsto \widetilde{u}(x,t)$ is odd. In particular, $\widetilde{u}(0,t) = 0$.*

Proof. As $\widetilde{g}$ is odd,

$$\begin{aligned} \widetilde{u}(-x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(-x-y)^2}{4t}} \widetilde{g}(y) \, dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-z)^2}{4t}} (-\widetilde{g}(z)) \, dz \quad \text{(change of variable } z = -y) \\ &= -\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-z)^2}{4t}} \widetilde{g}(z) \, dz = -\widetilde{u}(x,t). \end{aligned}$$

So, $x \mapsto \widetilde{u}(x,t)$ is odd, and in particular, $\widetilde{u}(0,t) = 0$. $\square$

Hence, for $x > 0$, $t > 0$,

$$\begin{aligned} u(x,t) = \widetilde{u}(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \widetilde{g}(y)\, dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \left[e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right] g(y)\, dy. \end{aligned}$$

We record this as a theorem.

Theorem 3.34. *For $x > 0$, $t > 0$, denote by*

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \left[e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right] g(y)\, dy.$$

Then, u solves (3.14).

3.8.2. Neumann boundary problem.

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } V, \\ u_x(0,t) = 0 & \text{for } t \geq 0, \\ u(x,0) = g(x) & \text{for } x \geq 0. \end{cases} \tag{3.15}$$

Here, $g \in C([0,\infty)$ is a given bounded function. For compatibility conditions, we assume $g'(0) = 0$. The Neumann boundary condition $u_x(0,t) = 0$ for $t \geq 0$ means that no heat flux goes out at the boundary $x = 0$. It is also called the insulated boundary condition.

Idea. Extend this problem to all $\mathbb{R}$. We need to do it in such a way that $u_x(0,t) = 0$.
It turns out that in this case it is best to extend the initial data in the even fashion.

Lemma 3.35. *Let $f : \mathbb{R} \to \mathbb{R}$ be C^1. Assume further that f is even, that is,*

$$f(y) = f(-y) \quad \text{for all } y \in \mathbb{R}.$$

Then, $f'(0) = 0$.

Proof. Since $f(y) = f(-y)$, we differentiate in y to yield

$$f'(y) = -f'(-y).$$

At $y = 0$, $f'(0) = -f'(0)$. Thus, $f'(0) = 0$. $\square$

Hence, we let $\widetilde{g}$ be the even extension of g, that is,

$$\widetilde{g}(x) = \begin{cases} g(x) & x > 0, \\ g(-x) & x < 0. \end{cases}$$

Consider the heat PDE

$$\begin{cases} \widetilde{u}_t - \widetilde{u}_{xx} = 0 & \text{for } x \in \mathbb{R}, t > 0, \\ \widetilde{u}(x,0) = \widetilde{g}(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Thanks to Theorem 3.31, we have the formula

$$\widetilde{u}(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \widetilde{g}(y)\, dy.$$

Lemma 3.36. *Let $\widetilde{u}$ be defined as above. Then, $x \mapsto \widetilde{u}(x,t)$ is even. In particular, $\widetilde{u}_x(0,t) = 0$.*

Proof. As $\widetilde{g}$ is even,

$$\begin{aligned} \widetilde{u}(-x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(-x-y)^2}{4t}} \widetilde{g}(y)\, dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-z)^2}{4t}} \widetilde{g}(z)\, dz \quad \text{(change of variable } z = -y) \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-z)^2}{4t}} \widetilde{g}(z)\, dz = \widetilde{u}(x,t). \end{aligned}$$

Hence, $x \mapsto \widetilde{u}(x,t)$ is even, and in particular, $\widetilde{u}_x(0,t) = 0$. □

Therefore, for $x > 0$, $t > 0$,

$$\begin{aligned} u(x,t) &= \widetilde{u}(x,t) \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \widetilde{g}(y)\, dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{0}^{\infty} \left[e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}} \right] g(y)\, dy. \end{aligned}$$

We record this as a theorem.

Theorem 3.37. *For $x > 0$, $t > 0$, denote by*

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{0}^{\infty} \left[e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}} \right] g(y)\, dy.$$

Then, u solves (3.15).

3.9. Maximum principle for the heat equation

3.9.1. Maximum principle. We now study the maximum principle for the heat equation. We note that there is still a mean value formula for the heat equation, which is more complicated and harder to understand. Specifically, the heat ball is not as intuitive as the usual ball/sphere. We will proceed to prove the maximum principle in a different way.

Here is our setting. Let $U \subset \mathbb{R}^n$ be a given bounded domain. Let $T > 0$ be fixed. Let $U_T = U \times (0, T]$. We say that U_T is a parabolic cylinder. Denote by $\overline{U}_T$ the closure of U_T, that is, $\overline{U}_T = \overline{U} \times [0, T]$. Consider the heat equation in U_T

$$\begin{cases} u_t = \Delta u & \text{in } U_T, \\ u \in C_1^2(U_T) \cap C\left(\overline{U}_T\right). \end{cases} \tag{3.16}$$

Here, $u \in C_1^2(U_T)$ means that u is C^2 in x and C^1 in t in U_T.

Definition 3.38 (Parabolic boundary)**.** Note that U_T includes $t = T$. Denote the parabolic boundary by

$$\Gamma_T = \overline{U}_T \setminus U_T = (\partial V \times (0, T]) \cup (V \times \{t = 0\}).$$

We emphasize that Γ_T does not include $t = T$. See Figure 7.

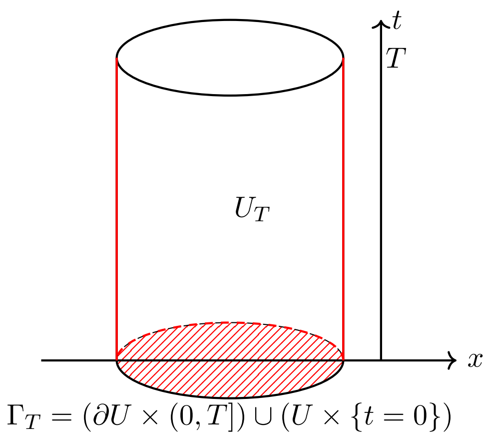

Figure 7. Parabolic cylinder and parabolic boundary

Theorem 3.39. *Let u be a solution to the heat equation* (3.16)*. Then:*

$$\max_{\overline{U}_T} u = \max_{\Gamma_T} u.$$

Proof. We divide the proof into two steps.

Step 1. We first prove the claim under the condition that

$$u_t - \Delta u < 0 \quad \text{in} \quad U_T. \tag{3.17}$$

We will explain why we can put this condition later.

Assume that there exists $(x_0, t_0) \in U_T$ such that

$$u(x_0, t_0) = \max_{\overline{U}_T} u.$$

Then $x_0 \in U$, and $0 < t_0 \leq T$.

We note that $t \mapsto u(x_0, t)$ has a maximum at $t = t_0$. There are two cases.

- If $t_0 \in (0, T)$, then we see that $u_t(x_0, t_0) = 0$.
- If $t_0 = T$, we can conclude that $u_t(x_0, T) \geq 0$.

In both situations, we have

$$u_t(x_0, t_0) \geq 0.$$

Besides, we have that $x \mapsto u(x, t_0)$ has a maximum at $x_0 \in U$. Hence, for $1 \leq i \leq n$,

$$\begin{cases} u_{x_i}(x_0, t_0) = 0 \\ u_{x_i x_i}(x_0, t_0) \leq 0 \end{cases} \qquad \Longrightarrow \qquad \Delta u(x_0, t_0) \leq 0$$

Combining these two points,

$$u_t(x_0, t_0) - \Delta u(x_0, t_0) \geq 0,$$

which contradicts (3.17).

Step 2. Now, we need to justify why we can put the strict inequality as in (3.17). Recall that u solves (3.16), that is,

$$u_t - \Delta u = 0 \quad \text{ in } U_T.$$

For $\varepsilon > 0$, set

$$u^\varepsilon(x, t) = u(x, t) + \varepsilon |x|^2 \quad \text{ for } (x, t) \in \overline{U}_T.$$

Then

$$u^\varepsilon_t = u_t, \quad u^\varepsilon_{x_i} = u_{x_i} + 2\varepsilon x_i, \quad u^\varepsilon_{x_i x_i} = u_{x_i x_i} + 2\varepsilon.$$

Thus,

$$u^\varepsilon_t - \Delta u^\varepsilon = u_t - \Delta u - 2n\varepsilon = -2n\varepsilon < 0.$$

We can then apply Step 1 for u^ε to have

$$\max_{\overline{U}_T} u^\varepsilon = \max_{\Gamma_T} u^\varepsilon.$$

Let $\varepsilon \to 0^+$ to deduce the desired result. We emphasize that this kind of approximation/perturbation is an important and simple idea in analysis.

$\square$

Remark 3.40. Let us discuss in depth the maximum principle here.

(1) For fixed $t_0 > 0$,

$$x \mapsto u(x, t_0) \text{ has a maximum at } x_0.$$

For any fixed vector $\xi \in \mathbb{R}^n$, the single variable function $s \mapsto \varphi(s) = u(x_0 + s\xi, t_0)$ has a maximum at $s = 0$. Hence,

$$\varphi'(0) = 0, \quad \varphi''(0) \leq 0.$$

By computations, we then get

$$\begin{cases} Du(x_0, t_0) \cdot \xi = 0, \\ \sum_{i,j} u_{x_i x_i}(x_0, t_0)\xi_i \xi_j \le 0. \end{cases}$$

In particular, we say $D^2u(x_0, t_0) \le 0$, that is, $D^2u(x_0, t_0)$ is a nonpositive definite matrix.

(2) As t_0 can be the fixed terminal time T, we can only have the one-sided control on $u_t(x_0, t_0)$ as done in the proof.

The same result holds for minimum value. Let us state it here.

Theorem 3.41. *Let u be a solution to the heat equation* (3.16)*. Then:*

$$\min_{\overline{U}_T} u = \min_{\Gamma_T} u.$$

3.9.2. Maximum principle for the Cauchy problem. We prove the maximum principle for the Cauchy problem on $\mathbb{R}^n \times [0, T]$ for any fixed $T > 0$.

Theorem 3.42 (Maximum principle for the Cauchy problem)**.** *Let $u \in C^2_1(\mathbb{R}^n \times (0, T]) \cap C(\mathbb{R}^n \times [0, T])$ solve*

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, T), \\ u(x, 0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{3.18}$$

Here, $g \in C(\mathbb{R}^n)$ is a given bounded initial data. Assume that there exist $c, C > 0$ such that

$$u(x, t) \le Ce^{c|x|^2} \quad \text{on } \mathbb{R}^n \times [0, T].$$

Then,

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g.$$

Proof. We just need to prove the result for $T = \frac{1}{16c}$, and then iterate it on time intervals $[0, T]$, $[T, 2T]$, and so forth to conclude for any given positive time.

Fix $y \in \mathbb{R}^n$ and $\mu > 0$. For $(x, t) \in \mathbb{R}^n \times [0, T]$, set

$$v(x, t) = u(x, t) - \frac{\mu}{(4\pi(2T - t))^{n/2}} e^{\frac{|x-y|^2}{4(2T-t)}} = u(x, t) - \mu\Phi(i(x - y), 2T - t).$$

Then, in $\mathbb{R}^n \times (0, T)$,

$$v_t - \Delta v = (u_t + \mu\Phi_t) - (\Delta u + \mu\Delta\Phi) = 0.$$

For $r > 0$, denote by $U = B(y, r)$. Then, by the maximum principle in Theorem 3.39,

$$\max_{\overline{U}_T} v = \max_{\Gamma_T} v.$$

By the definition of v, it is clear that

$$v(x, 0) \le u(x, 0) = g(x).$$

For $(x, t) \in \partial B(y, r) \times (0, T)$, we use $T = \frac{1}{16c}$ to compute

$$\begin{aligned} v(x,t) &= u(x,t) - \frac{\mu}{(4\pi(2T-t))^{n/2}} e^{\frac{|x-y|^2}{4(2T-t)}} \\ &\le Ce^{c|x|^2} - \frac{\mu}{(4\pi T)^{n/2}} e^{\frac{r^2}{4T}} \\ &\le Ce^{c(|y|+r)^2} - \frac{\mu}{(4\pi T)^{n/2}} e^{4cr^2} \\ &\le Ce^{2c|y|^2} e^{2cr^2} - \frac{\mu}{(4\pi T)^{n/2}} e^{4cr^2} < \sup_{\mathbb{R}^n} g \end{aligned}$$

for $r > 0$ sufficiently large. Therefore, we get

$$\max_{\overline{U}_T} v \le \sup_{\mathbb{R}^n} g.$$

Let $r \to +\infty$ and $\mu \to 0^+$ in this order to conclude. □

3.9.3. Uniqueness via the maximum principle.

Theorem 3.43 (Uniqueness for Dirichlet boundary problem)**.** *Consider the heat equation with the Dirichlet boundary condition*

$$\begin{cases} u_t - \Delta u = f(x,t) & \text{in } U_T = U \times (0, T), \\ u(x,t) = g(x,t) & \text{on } \Gamma_T. \end{cases} \tag{3.19}$$

Assume $f \in C(U_T)$ and $g \in C(\Gamma_T)$. If $u_1, u_2 \in C_1^2(U_T) \cap C(\overline{U}_T)$ are both solutions to (3.19)*, then $u_1 \equiv u_2$.*

Proof. Let $v = u_1 - u_2$. Then v solves:

$$\begin{cases} v_t - \Delta v = 0 & \text{in } U_T, \\ v = 0 & \text{on } \Gamma_T. \end{cases}$$

Besides, $v \in C_1^2(U_T) \cap C(\overline{U}_T)$.

By the maximum principle, we have that

$$\begin{cases} \max_{\overline{U}_T} v = \max_{\Gamma_T} v = 0, \\ \min_{\overline{U}_T} v = \min_{\Gamma_T} v = 0, \end{cases} \implies \max_{\overline{U}_T} v = \min_{\overline{U}_T} v = 0 \implies v \equiv 0.$$

The proof is complete. □

Theorem 3.44 (Uniqueness for the Cauchy problem)**.** *Let $g \in C(\mathbb{R}^n)$ and $f \in C(\mathbb{R}^n \times [0,T])$. Then, there exists at most one solution $u \in C^2_1(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ to the following Cauchy problem*

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n \end{cases} \tag{3.20}$$

satisfying

$$u(x,t) \le Ce^{c|x|^2} \quad \text{on } \mathbb{R}^n \times [0,T]$$

for some $c, C > 0$.

Proof. Let $u_1, u_2 \in C^2_1(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ be two solutions to (3.20). Let $v = u_1 - u_2$. Then, by Theorem 3.42, we imply that $v \le 0$. Apply Theorem 3.42 to $-v$, we also get $-v \le 0$. Thus, $v \equiv 0$. $\square$

By combining Theorem 3.31 and 3.44, we conclude with the following theorem.

Theorem 3.45 (Wellposedness for the Cauchy problem)**.** *Let $g \in C(\mathbb{R}^n)$ be a given bounded initial data. Fix $T > 0$. Then, there exists a unique solution $u \in C^2_1(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ to the following Cauchy problem*

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n \end{cases}$$

satisfying

$$u(x,t) \le Ce^{c|x|^2} \quad \text{on } \mathbb{R}^n \times [0,T]$$

for some $c, C > 0$. The unique solution is given by

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy \\ &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy \qquad \text{for } x \in \mathbb{R}^n, t \in (0,T]. \end{aligned}$$

3.10. Maximum principle for the Laplace equation – revisited

We proved the maximum principle for the Laplace equation by using the mean value property. It turns out that we can prove this in a similar way to the proof of Theorem 3.39, that is, we can bypass the usage of the mean value property.

Theorem 3.46 (Maximum principles for harmonic functions – revisited)**.** *Let $U \subset \mathbb{R}^n$ be a bounded domain. If $u \in C^2(U) \cap C(\overline{U})$ is harmonic in U, then*

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

Proof. We divide the proof into two steps.

Step 1. We first prove the claim under the condition that

$$-\Delta u < 0 \quad \text{in} \quad U. \tag{3.21}$$

Assume that there exists $x_0 \in U$ such that

$$u(x_0) = \max_{\overline{U}} u.$$

Then, for $1 \leq i \leq n$,

$$\begin{cases} u_{x_i}(x_0) = 0 \\ u_{x_i x_i}(x_0) \leq 0 \end{cases} \quad \Longrightarrow \quad -\Delta u(x_0) \geq 0,$$

which contradicts (3.21).

Step 2. Now, we need to justify why we can put the strict inequality as in (3.21). Recall that u solves

$$-\Delta u = 0 \quad \text{in } U.$$

For $\varepsilon > 0$, set

$$u^\varepsilon(x) = u(x) + \varepsilon |x|^2 \quad \text{for } x \in \overline{U}.$$

Then

$$u^\varepsilon_{x_i} = u_{x_i} + 2\varepsilon x_i, \quad u^\varepsilon_{x_i x_i} = u_{x_i x_i} + 2\varepsilon.$$

Thus,

$$-\Delta u^\varepsilon = -\Delta u - 2n\varepsilon = -2n\varepsilon < 0.$$

We can then apply Step 1 for u^ε to have

$$\max_{\overline{U}} u^\varepsilon = \max_{\partial U} u^\varepsilon.$$

Let $\varepsilon \to 0^+$ to deduce the desired result.

$\square$

Remark 3.47. It is important noting that the proofs of Theorem 3.39 and Theorem 3.46 work for general elliptic and parabolic PDEs.

3.11. Energy method for the heat equation

Similar to the Laplace equation, we are able to give another proof of uniqueness via the energy method. In particular, we do not need to use the maximum principle here.

Theorem 3.48. *Assume ∂U is C^1. Assume $f \in C(U_T)$ and $g \in C(\Gamma_T)$. If $u_1, u_2 \in C^2_1(U_T) \cap C(\overline{U}_T)$ are both solutions to* (3.19)*, then $u_1 \equiv u_2$.*

Proof. Let $v = u_1 - u_2$. Then v solves:

$$\begin{cases} v_t - \Delta v = 0 & \text{in } U_T, \\ v = 0 & \text{on } \Gamma_T. \end{cases}$$

Besides, $v \in C_1^2(U_T) \cap C(\overline{U}_T)$.

The idea now is to multiply the PDE by v and integrate over U. For each $t \in (0, T)$:

$$\int_U v_t(x,t)v(x,t)\,dx = \int_U \Delta v(x,t)v(x,t)\,dx.$$

Integrate by parts to yield

$$\frac{1}{2}\frac{d}{dt}\left(\int_U v(x,t)^2\,dx\right) = \int_U -|Dv|^2\,dx + \int_{\partial U} vDv \cdot \mathbf{n}\,dS$$

The boundary integral vanishes as $v = 0$ on ∂U. Therefore,

$$\frac{1}{2}\frac{d}{dt}\int_U v(x,t)^2\,dx = \int_U -|Dv(x,t)|^2\,dx \leq 0.$$

Let

$$\phi(t) = \int_U v(x,t)^2\,dx.$$

Then $\phi(t) \geq 0$ and $\phi(0) = 0$. By the above:

$$\phi'(t) = 2\int_U -|Dv(x,t)|^2\,dx \leq 0.$$

Hence, we must have $\phi(t) = 0$ for all t, which yields further that

$$v(x,t) = 0 \quad \text{for all } (x,t) \in U_T.$$

□

What about Neumann boundary conditions? Let us consider

$$\begin{cases} u_t - \Delta u = f(x,t) & \text{in } U_T = U \times (0,T), \\ \frac{\partial u}{\partial \mathbf{n}}(x,t) = h(x,t) & \text{on } \partial U \times (0,T), \\ u(x,0) = g(x) & \text{on } \overline{U}. \end{cases} \tag{3.22}$$

Here, we only impose the Neumann boundary condition on $\partial U \times (0,T)$, which is different from the Neumann problem for Laplace equation in (3.11). We impose the usual initial condition on $\overline{U} \times \{0\}$. In this case, we do have the uniqueness result thanks to the given initial condition.

Theorem 3.49. *Assume ∂U is C^1. Assume $f \in C(U_T)$, $g \in C(\overline{U})$, and $h \in C(\partial U \times (0,T))$. If $u_1, u_2 \in C_1^2(U_T) \cap C^1(\overline{U}_T)$ are both solutions to (3.22), then $u_1 \equiv u_2$.*

Proof. Let $v = u_1 - u_2$. Then v solves:

$$\begin{cases} v_t - \Delta v = 0 & \text{in } U_T = U \times (0,T), \\ \frac{\partial v}{\partial \mathbf{n}}(x,t) = 0 & \text{on } \partial U \times (0,T), \\ v(x,0) = 0 & \text{on } \overline{U}. \end{cases}$$

Besides, $v \in C_1^2(U_T) \cap C^1(\overline{U}_T)$.

The idea now is to multiply the PDE by v and integrate over U. For each $t \in (0,T)$:

$$\int_U v_t(x,t)v(x,t)\,dx = \int_U \Delta v(x,t)v(x,t)\,dx.$$

Integrate by parts to yield

$$\frac{1}{2}\frac{d}{dt}\left(\int_U v(x,t)^2\,dx\right) = \int_U -|Dv|^2\,dx + \int_{\partial U} vDv\cdot\mathbf{n}\,dS$$

The boundary integral vanishes as $Dv\cdot\mathbf{n} = \frac{\partial v}{\partial \mathbf{n}} = 0$ on $\partial U \times (0,T)$. Therefore,

$$\frac{1}{2}\frac{d}{dt}\int_U v(x,t)^2\,dx = \int_U -|Dv(x,t)|^2\,dx \le 0.$$

Let

$$\phi(t) = \int_U v(x,t)^2\,dx.$$

Then $\phi(t) \ge 0$ and $\phi(0) = 0$. By the above:

$$\phi'(t) = 2\int_U -|Dv(x,t)|^2\,dx \le 0.$$

Hence, we must have $\phi(t) = 0$ for all t, which yields further that

$$v(x,t) = 0 \quad \text{for all } (x,t) \in U_T.$$

□

3.12. Non-homogeneous heat equation

We first study a non-homogeneous heat PDE of the form

$$\begin{cases} u_t - \Delta u = f(x,t) & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = 0 & \text{on } \mathbb{R}^n. \end{cases} \tag{3.23}$$

Here, $f = f(x,t) \in C(\mathbb{R}^n \times [0,\infty))$ is the given source term. Assume further that f is bounded. For the initial data, we assume $u(x,0) \equiv 0$, which means that the system has no initial heat.

Interpretation. We can think of $f(x,t)$ as the source that pumps heat into the system instantly. Basically, think of a heater $f(x,t)$. Each time that we have a new heat source pumped in, we can think of this as an isolated effect, which does not related to other events. As such, we can think about

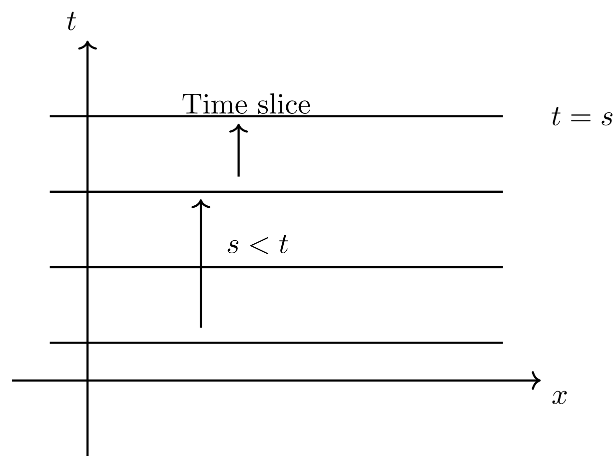

Figure 8. Time slices in a non-homogeneous heat PDE

this new heat source as a new "initial data" that will be evolved under the heat equation as time goes. And since the differential operator $u \mapsto u_t - \Delta u$ is linear, we can sum up all such effects to get the solution to (3.23). This is essentially based upon the superposition principle.

Here is a precise formulation of the above interpretation. For each time slice $s \geq 0$, we can think of the problem starting at s with initial data $f(x, s)$. See Figure 8. Consider $u^s(x, t)$ such that:

$$\begin{cases} u_t^s(x,t) - \Delta u^s(x,t) = 0 & \text{in } \mathbb{R}^n \times (s, \infty), \\ u(x,s) = f(x,s) & \text{on } \mathbb{R}^n. \end{cases}$$

We have:

$$u^s(x,t) = \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dy \quad \text{for} \quad x \in \mathbb{R}^n,\ t \geq s.$$

Now, to get the solution to (3.23), we can sum up all the heat sources $f(x, s)$ for $0 \leq s \leq t$, that is,

$$\begin{aligned} u(x,t) &= \int_0^t u^s(x,t)\, ds \\ &= \int_0^t \left(\int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dy \right) ds. \end{aligned}$$

The principle explained here is called *Duhamel's principle*. We formulate this into the following theorem.

Theorem 3.50. *For* $(x, t) \in \mathbb{R}^n \times (0, \infty)$*, define*

$$u(x,t) = \int_0^t \left(\int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dy \right) ds.$$

Then, u solves (3.23).

Proof. We have

$$u(x,t) = \int_0^t u^s(x,t)\,ds.$$

Differentiate with respect to t to get

$$u_t(x,t) = u^t(x,t) + \int_0^t u_t^s(x,t)\,ds.$$

At $s=t$, the initial data of u^s at $s=t$ is $f(x,t)$, hence:

$$u_t(x,t) = f(x,t) + \int_0^t u_t^s(x,t)\,ds.$$

Also,

$$\Delta u(x,t) = \int_0^t \Delta u^s(x,t)\,ds.$$

As $u_t^s = \Delta u^s$, we see that:

$$u_t(x,t) = f(x,t) + \Delta u.$$

Finally, by the definition of u, we see immediately that $u(x,0)=0$. $\square$

By the superposition principle again, we can consider the case with general initial data too.

Theorem 3.51. *Let $f = f(x,t) \in C(\mathbb{R}^n \times [0,\infty))$ and $g = g(x) \in C(\mathbb{R}^n)$ be given bounded functions. For $(x,t) \in \mathbb{R}^n \times (0,\infty)$, define*

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)\,dy + \int_0^t \left(\int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)\,dy \right) ds.$$

Then, u solves

$$\begin{cases} u_t - \Delta u = f(x,t) & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases}$$

3.13. Exercises

Exercise 14. Let $f \in C(\mathbb{R}^n)$ be a given function such that, for all open bounded set U,

$$\int_U f(x)\,dx = 0.$$

Show that $f \equiv 0$.

Exercise 15. Give three examples of harmonic functions that are different from the ones given in the book. Check your examples rigorously. Here, u is harmonic if $-\Delta u = 0$.

Exercise 16. We are in two dimensions ($\mathbb{R}^2$) here. Let $r = \sqrt{x^2+y^2}$, which is the length of vector $(x,y) \in \mathbb{R}^2$. Assume that
$$u(x,y) = v\left(\sqrt{x^2+y^2}\right) = v(r).$$
Compute $\Delta u(x,y)$ in term of v and its derivatives.

Exercise 17. Let $f \in C_c^2(\mathbb{R}^2)$, and for $\varepsilon > 0$, write
$$A_\varepsilon = \int_{B(0,\varepsilon)} \Phi(y) \Delta_y f(x-y)\, dy.$$
Here, Φ is the fundamental solution to the Laplace equation in two dimensions. Show that
$$\lim_{\varepsilon \to 0} A_\varepsilon = 0.$$

Exercise 18. Let $n = 2$ and Φ be the fundamental solution to the Laplace equation in two dimensions. Show that
$$\lim_{\varepsilon \to 0} \int_{\partial B(0,\varepsilon)} |\Phi(y)|\, dS(y) = 0.$$

Exercise 19. We are in two dimensions ($\mathbb{R}^2$) here. For $x \in \mathbb{R}^2$, we write $x = (x_1, x_2)$. Show that, for each $r > 0$,
$$\int_{B(0,r)} (x_1^2 - x_2^2)\, dx = 0.$$

Exercise 20. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a smooth function.

(a) Let $v(x) = u(x)^2$ for all $x \in \mathbb{R}^n$. Compute Δv in terms of u.

(b) Let $w(x) = e^{u(x)}$ for all $x \in \mathbb{R}^n$. Compute Δw in terms of u.

Exercise 21. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a given smooth function. Define $v, w : \mathbb{R}^n \to \mathbb{R}$ such that $v(x) = u(x)^2$ and $w(x) = e^{u(x)}$ for $x \in \mathbb{R}^n$. Assume that u is harmonic in $\mathbb{R}^n$.

(a) Show that v satisfies
$$-\Delta v(x) \le 0 \quad \text{for all } x \in \mathbb{R}^n.$$

(b) Show that w satisfies
$$-\Delta w(x) \le 0 \quad \text{for all } x \in \mathbb{R}^n.$$

Exercise 22. Assume that $u \in C^2(\mathbb{R}^n)$ is a harmonic function. Let $v : \mathbb{R}^n \to \mathbb{R}$ be such that
$$v(x) = \sqrt{9 + (u(x))^2} \qquad \text{for } x \in \mathbb{R}^n.$$
Show that

(a) v is subharmonic, that is, $-\Delta v \le 0$ in $\mathbb{R}^n$.

(b) Find u so that v is harmonic in $\mathbb{R}^n$.

Exercise 23. Assume $U \subset \mathbb{R}^n$ is an open, bounded, and connected set. Let $u \in C^2(U) \cap C(\overline{U})$ be a harmonic function. Assume further that $u(x) \geq 0$ for all $x \in \partial U$, and there exists $y \in \partial U$ such that $u(y) > 0$. Show that

$$u(x) > 0 \qquad \text{for all } x \in U.$$

Exercise 24. Let $v : \mathbb{R}^n \to \mathbb{R}$ be a smooth and subharmonic function. Show that, for all $x \in \mathbb{R}^n$ and $r > 0$,

$$v(x) \leq ⨍_{B(x,r)} v(y)\, dy.$$

Exercise 25. Show that

$$\lim_{R\to\infty} \int_R^\infty e^{-y^2}\, dy = 0.$$

Exercise 26. Let $c \in \mathbb{R}$ be a given constant. Find the formula of the solution u to the following equation

$$\begin{cases} u_t + cu = \Delta u & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases}$$

Exercise 27. Let $d \in \mathbb{R}$ be a given constant. Find the formula of the solution u to the following equation

$$\begin{cases} u_t + du_x = u_{xx} & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}. \end{cases}$$

Exercise 28. Let u be the solution to

$$\begin{cases} u_t = u_{xx} & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = x^2 & \text{on } \mathbb{R}. \end{cases}$$

(a) Show that $v(x,t) = u_{xxx}(x,t)$ also satisfies the heat equation with initial data $v(x,0) = 0$. Find v.

(b) Use v to find an explicit formula for u.

Exercise 29. Assume $n = 1$.

(a) Let $g : [0,\infty) \to \mathbb{R}$ be a bounded and continuous function. Show that

$$u(x,t) = \int_0^\infty (\Phi(x-y,t) - \Phi(x+y,t))g(y)\, dy$$

is an odd function of x for each fixed $t > 0$.

(b) Let $h : \mathbb{R} \to \mathbb{R}$ be an odd, bounded, and continuous function. Show that

$$u(x,t) = \int_{-\infty}^\infty \Phi(x-y,t)h(y)\, dy$$

is an odd function of x for each fixed $t > 0$. This implies that the odd symmetry property is preserved under the heat equation.

Exercise 30. Let $c(t)$ be a continuous function. Find the formula of the solution u to the following equation

$$\begin{cases} u_t + c(t)u = \Delta u & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases}$$

Exercise 31. Let U be an open, connected and bounded set in $\mathbb{R}^n$ and $T > 0$. Let U_T and Γ_T be the corresponding parabolic cylinder and its boundary, respectively. Let $u = u(x,t) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ be a smooth function.

(a) Assume that $u_t - \Delta u > 0$ in U_T. Show that

$$\min_{\overline{U}_T} u = \min_{\Gamma_T} u.$$

(b) Use part (a) to show that the same conclusion still holds if we only have $u_t - \Delta u \geq 0$ in U_T.

Exercise 32. Let $k > 0$ be a given number. Use the representation formula of the heat PDE derived in the book (for the PDE $w_t = w_{xx}$) to write down the formula of the solution to

$$\begin{cases} u_t = k u_{xx} & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}. \end{cases}$$

Here, the initial data g is a given continuous and bounded function in $\mathbb{R}$.

Exercise 33. Solve the following heat equation in one dimension

$$\begin{cases} u_t = u_{xx} & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = e^{px} & \text{on } \mathbb{R}. \end{cases}$$

Here, $p > 0$ is a given constant.

Exercise 34. Suppose u is smooth and solves $u_t = \Delta u$ in $\mathbb{R}^n \times (0,\infty)$. For each $\lambda \in \mathbb{R}$ fixed, define

$$u^\lambda(x,t) := u(\lambda x, \lambda^2 t) \qquad \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty).$$

Show that u^λ also solves the heat PDE.

Exercise 35. Assume $n = 1$ and $u(x,t) = v\left(\frac{x}{\sqrt{t}}\right)$ for $(x,t) \in \mathbb{R} \times (0,\infty)$.

(a) Show that $u_t = u_{xx}$ if and only if

$$v''(z) + \frac{z}{2} v'(z) = 0.$$

(b) Solve this ODE for v and show that the general solution to this ODE is

$$v(z) = c \int_0^z e^{-s^2/4}\, ds + d$$

for $c, d \in \mathbb{R}$ are constants.

(c) Differentiate $u(x,t) = v\left(\frac{x}{\sqrt{t}}\right)$ with respect to x and select the constant c properly to obtain the fundamental solution Φ to the heat equation in one dimension.

Exercise 36. Write down an explicit formula for solution u of

$$\begin{cases} u_t - \Delta u + cu = f(x,t) & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases}$$

Here, $c \in \mathbb{R}$ is a given constant. The source term f and the initial data g are given continuous and bounded functions.

3.14. Notes and references

(1) A large part of this chapter is based on the content in the book of Evans [**Eva10**]. See also John [**Joh91**].

(2) For the Cauchy problem for the heat equation, we have the uniqueness and the wellposedness (Theorem 3.44 and Theorem 3.45) if we impose some growth conditions at infinity. Without some appropriate conditions at infinity, this Cauchy problem has infinity many solutions. See Tychonov's solution, which is non-physical, in Section 5.7.

Chapter 4

Wave equations

4.1. Derivation of the wave equation in one dimension

Let us give a derivation of the wave equation in $\mathbb{R} \times (0, \infty)$. We consider a thin elastic rod undergoing only longitudinal deformation (extension or compression) with no bending. The displacement at time $t \geq 0$ of a point $x \in \mathbb{R}$ on the rod is denoted by $u(x, t)$.

$$\begin{aligned} u \ &: \mathbb{R} \times [0, \infty) \to \mathbb{R} \\ &(x, t) \mapsto u(x, t) : \text{displacement of the rod at } x \text{ at time } t. \end{aligned}$$

Let $V = (a, b) \subset \mathbb{R}$ be an arbitrary interval. The acceleration of the displacement u within V is given by:

$$\frac{d^2}{dt^2} \int_a^b u(x, t)\, dx = \int_a^b u_{tt}(x, t)\, dx.$$

Let the mass density of the rod be unity, that is, $m = 1$. By Newton's law, the net force within V is:

$$F(a, t) - F(b, t) = ma = a = \int_a^b u_{tt}(x, t)\, dx.$$

Here, $F(x, t)$ represents the force at position x at time t. See Figure 1.

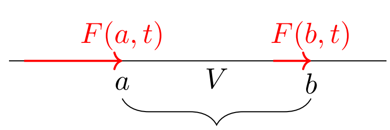

Figure 1. Acceleration of the displacement of the rod within V

Thus, the force equation becomes:

$$-\int_a^b F_x(x,t)\,dx = \int_a^b u_{tt}(x,t)\,dx$$

for all interval $(a,b) \subset \mathbb{R}$. This implies

$$u_{tt}(x,t) = -F_x(x,t).$$

What is the force F? For elastic bodies, the force is typically a function of u_x (strain). Therefore, we can expect that

$$F = F(u_x).$$

Assuming that the displacement u is small, which intuitively suggests that$u_x \approx 0$). In this regime, we might consider that F is linear in u_x. Using the steepest gradient descent principle, we assume:

$$F(u_x) = -cu_x,$$

where $c > 0$ is a constant.

Thus, the wave equation becomes:

$$u_{tt} = (cu_x)_x = cu_{xx}.$$

Normalize $c = 1$, so the wave equation becomes:

$$u_{tt} = u_{xx} \quad \text{in } \mathbb{R} \times (0,\infty).$$

4.2. Wave equation in one dimension

The wave equation in one dimension is given by:

$$u_{tt} = u_{xx} \quad \text{in } \mathbb{R} \times (0,\infty).$$

4.2.1. Special solutions to the wave equation in one dimension. We note the following key identity

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right).$$

To find special solutions to $u_{tt} - u_{xx} = 0$, we find functions $v(x,t)$, $w(x,t)$ as follows.

- We look for v such that

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) v = v_t - v_x = 0$$

  which gives

$$v(x,t) = F(x+t) \quad \text{for some } F : \mathbb{R} \to \mathbb{R}.$$

  Of course, $v_{tt} - v_{xx} = 0$ thanks to the above key identity.

- Similarly, for $w(x,t)$, we require

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)w = w_t + w_x = 0$$

which implies

$$w(x,t) = G(x-t) \quad \text{for some } G:\mathbb{R}\to\mathbb{R}.$$

Again, $w_{tt} - w_{xx} = 0$ thanks to the above key identity.

Since the wave equation is linear, we use the superposition principle to yield special solutions of the form

$$u(x,t) = v(x,t) + w(x,t) = F(x+t) + G(x-t) \tag{4.1}$$

for some $F, G : \mathbb{R}\to\mathbb{R}$, with $F, G \in C^2$.

Interpretation. A special solution $u(x,t)$ to the wave equation of the form (4.1) can be interpreted as a sum of two parts that travel in opposite directions with unit velocity. Specifically, $F(x+t)$ represents the part that travels to the left with speed 1. And $G(x-t)$ represents the part that travels to the right with speed 1. See Figure 2.

However, at this point, we are not sure if the special solutions (4.1) represent all possible general solutions to the wave equation in one dimension.

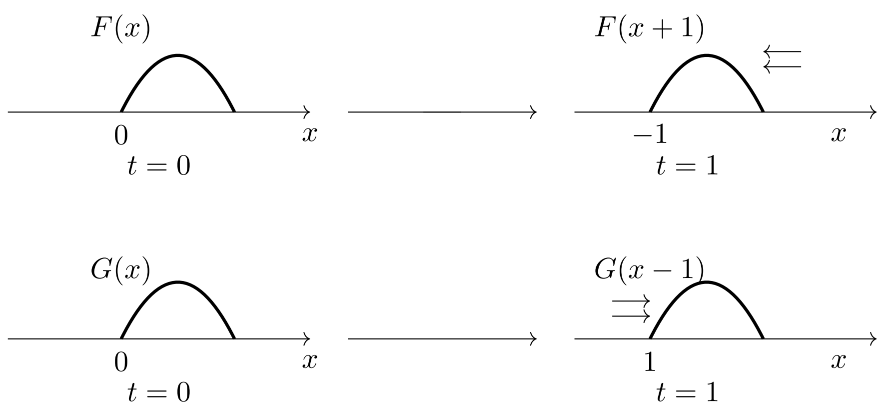

Figure 2. Waves traveling to the left and the right with unit velocity

4.2.2. D'Alembert's formula in one dimension. We now consider the following initial value problem

$$\begin{cases} u_{tt} = u_{xx} & \text{in } \mathbb{R}\times(0,\infty),\\ u(x,0) = g(x) & \text{on } \mathbb{R},\\ u_t(x,0) = h(x) & \text{on } \mathbb{R}. \end{cases} \tag{4.2}$$

Since we take two derivatives in time, we need two initial conditions on $u(x,0)$ and $u_t(x,0)$.

Let us now aim at solving this PDE. Recall again that

$$u_{tt} - u_{xx} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u.$$

Let:

$$v = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = u_t - u_x.$$

Then:

$$0 = u_{tt} - u_{xx} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) v = v_t + v_x.$$

As $v_t + v_x = 0$, we yield that

$$v(x,t) = a(x-t),$$

where

$$v(x,0) = a(x) = u_t(x,0) - u_x(x,0) = h(x) - g'(x).$$

Thus,

$$u_x - u_t = a(x-t).$$

We use Theorem 2.2 to get that

$$u(x,t) = \int_0^t a(x+t-2s)\,ds + g(x+t).$$

Change of variables to imply

$$\begin{aligned} u(x,t) &= \frac{1}{2}\int_{x-t}^{x+t} a(z)\,dz + g(x+t) \\ &= \frac{1}{2}\int_{x-t}^{x+t} (h(z) - g'(z))\,dz + g(x+t) \\ &= \frac{1}{2}\int_{x-t}^{x+t} h(z)\,dz + \frac{1}{2}(g(x+t) + g(x-t)). \end{aligned}$$

Thus, we conclude that

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} h(z)\,dz + \frac{1}{2}(g(x+t) + g(x-t)).$$

We formulate this as a theorem.

Theorem 4.1. *Consider* (4.2) *with $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$. Then,* (4.2) *has a unique solution u, where*

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} h(z)\,dz + \frac{1}{2}(g(x+t) + g(x-t)). \tag{4.3}$$

We note that (4.3) is called the D'Alembert formula.

Remark 4.2. The following comments are in order.

(1) u is clearly of the form:

$$u(x,t) = F(x+t) + G(x-t).$$

Hence, the special solutions (4.1) represent all possible general solutions to the wave equation in one dimension.

(2) The regularity of u is not better than g. If $g \in C^k, h \in C^{k-1}$ for some $k \in \mathbb{N}$, then $u \in C^k$. This regularity property is like that of the transport equation, and is very different from those of the Laplace and heat equations.

4.3. Wave equation in the first quadrant

We now apply Theorem 4.1 to solve a wave equation on the first quadrant when $n = 1$. Denote by

$$V = \{(x,t) \in \mathbb{R}^2 \,:\, x > 0, t > 0\}.$$

We consider

$$\begin{cases} u_{tt} = u_{xx} & \text{in } V, \\ u(x,0) = g(x) & \text{on } [0,\infty), \\ u_t(x,0) = h(x) & \text{on } [0,\infty), \\ u(0,t) = 0 & \text{on } [0,\infty). \end{cases} \tag{4.4}$$

For compatibility conditions, we also assume $g(0) = h(0) = 0$.

Similar to the situation of the heat equation on the first quadrant, as we have $u(0,t) = 0$, we perform an odd extension. Denote by

$$\widetilde{g}(x) = \begin{cases} g(x) & x \geq 0, \\ -g(-x) & x < 0, \end{cases}$$

and

$$\widetilde{h}(x) = \begin{cases} h(x) & x \geq 0, \\ -h(-x) & x < 0. \end{cases}$$

Consider the following problem:

$$\begin{cases} \widetilde{u}_{tt} = \widetilde{u}_{xx} & \text{in } \mathbb{R} \times (0,\infty), \\ \widetilde{u}(x,0) = \widetilde{g}(x) & \text{on } \mathbb{R}, \\ \widetilde{u}_t(x,0) = \widetilde{h}(x) & \text{on } \mathbb{R}. \end{cases}$$

By the D'Alembert formula,

$$\widetilde{u}(x,t) = \frac{1}{2}\left(\widetilde{g}(x+t) + \widetilde{g}(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} \widetilde{h}(z)\, dz.$$

It is clear that $\widetilde{u}(x,t)$ is an odd function in x. We record this in the following theorem.

Theorem 4.3. *Equation* (4.4) *has a unique solution u, which has the representation formula: for $(x,t) \in V$,*

$$u(x,t) = \frac{1}{2}\left(\widetilde{g}(x+t) + \widetilde{g}(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} \widetilde{h}(z)\,dz.$$

This can be written in terms of g and h as follows.

- *If $x \geq t \geq 0$, then*
$$u(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(z)\,dz.$$
- *If $t \geq x \geq 0$, then*
$$u(x,t) = \frac{1}{2}\left(g(x+t) + g(t-x)\right) + \frac{1}{2}\int_{t-x}^{x+t} h(z)\,dz.$$

4.4. Geometric interpretation of solutions in one dimension

Recall the wave equation (4.2) with $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$. Then, by D'Alembert's formula, for $(x,t) \in \mathbb{R} \times [0,\infty)$,

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} h(z)\,dz + \frac{1}{2}(g(x+t) + g(x-t)).$$

4.4.1. Domain of dependence. From the formula, the solution $u(x,t)$ is determined by the information of initial data g, h on the interval

$$I_{x,t} = \{z \in \mathbb{R} \,:\, x-t \leq z \leq x+t\}.$$

Let

$$D(x,t) = \{(z,s) \in \mathbb{R} \times [0,t] \,:\, x-t+s \leq z \leq x+t-s\}.$$

See Figure 3.

We also have, for $0 < s < t$, by D'Alembert's formula again,

$$u(x,t) = \frac{1}{2}\left(u(x+t-s,s) + u(x-t+s,s)\right) + \frac{1}{2}\int_{x-t+s}^{x+t-s} u_t(z,s)\,dz.$$

Basically, we think of $u(\cdot,s)$ as the initial data, and run the wave equation for time $t-s$ and compute $u(x,t)$. Thus, $u(x,t)$ is determined by everything in $D(x,t)$. In other words, if we change the information outside of $D(x,t)$, then $u(x,t)$ is not affected.

Definition 4.4. We say that $D(x,t)$ is the *domain of dependence* of the wave PDE at (x,t).

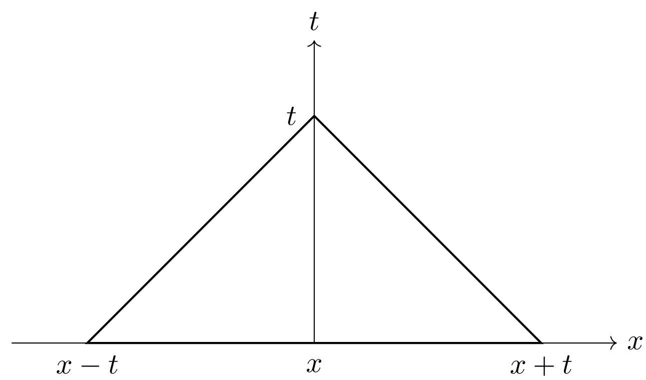

Figure 3. Domain of dependence

4.4.2. Region of influence. We can also ask a converse question. Fix initial data g, h on $\mathbb{R}$ and an interval $[a, b] \subset \mathbb{R}$. Then, the data of g, h on $[a, b] \subset \mathbb{R}$ can influence the value of $u(x, t)$ in what region?

Denote by

$$\mathcal{I}([a, b]) = \{(x, t) \,:\, a - t \le x \le b + t, t \ge 0\}\,.$$

See Figure 4.

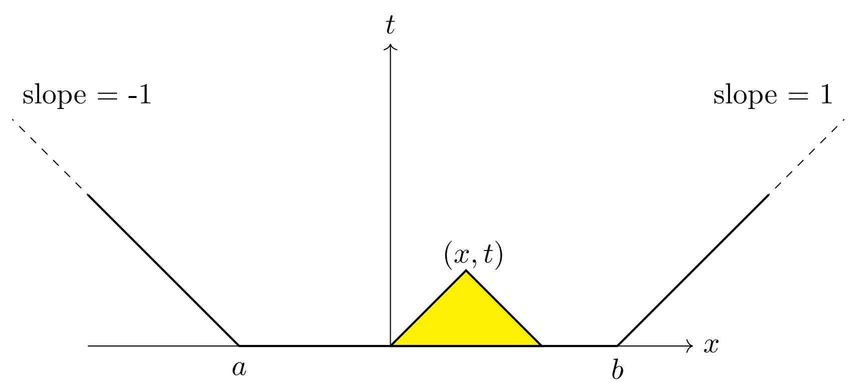

Figure 4. Region of influence

Definition 4.5. We say that $\mathcal{I}([a, b])$ is the *region of influence* of the interval $[a, b]$.

Basically, if $x \in \mathcal{I}([a, b])$, then $u(x, t)$ is being determined or influenced at least partially by initial data on $[a, b]$ as

$$[x - t, x + t] \cap [a, b] \neq \emptyset.$$

Example 4.6. Consider the wave equation:

$$u_{tt} - u_{xx} = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty).$$

If we know that $u(z, 0) = 0$, $u_t(z, 0) = 0$ for $x - t \leq z \leq x + t$, then $u = 0$ on $D(x, t)$.

4.5. Energy method and uniqueness of solutions

Let $U = (a, b) \subset \mathbb{R}$ and $T > 0$. Denote by

$$U_T = U \times (0, T] = (a, b) \times (0, T]$$

the parabolic cylinder. The parabolic boundary is

$$\Gamma_T = (\{a, b\} \times (0, t)) \cup ([a, b] \times \{t = 0\}).$$

Theorem 4.7. *There exists at most one solution $u \in C^2(\overline{U}_T)$ to*

$$\begin{cases} u_{xx} - u_{tt} = f & \text{in } U_T, \\ u = g & \text{on } \Gamma_T, \\ u_t = h & \text{on } [a, b] \times \{t = 0\}. \end{cases}$$

We remark that this result holds in multi-dimensions as well. The proof for the multi-dimensional case is the same as the following.

Proof. Let $u_1, u_2 \in C^2(\overline{U}_T)$ be two solutions to the wave equation above. Set $v = u_1 - u_2$. Then,

$$\begin{cases} v_{xx} - v_{tt} = 0 & \text{in } U_T, \\ v = 0 & \text{on } \Gamma_T, \\ v_t = 0 & \text{on } [a, b] \times \{t = 0\}. \end{cases}$$

Denote the energy $E(t)$ by: for $t \in [0, t]$,

$$E(t) = \frac{1}{2} \int_U \left(v_x^2 + v_t^2 \right) \, dx.$$

Clearly, $E \geq 0$. At $t = 0$, $v_x(x,0) = 0$ and $v_t(x,0) = 0$ for $x \in (a,b)$ imply that $E(0) = 0$. We compute

$$\begin{aligned}
\frac{d}{dt}E(t) &= \frac{d}{dt}\left(\frac{1}{2}\int_U \left(v_x^2 + v_t^2\right)\, dx\right) \\
&= \int_U \left(v_x v_{xt} + v_t v_{tt}\right)\, dx \\
&= \int_U v_x v_{xt} dx + \int_U v_t v_{tt}\, dx \\
&= \int_U v_t \left(v_{tt} - v_{xx}\right)\, dx + v_x v_t\Big]_{x=b}^{x=a} \quad \text{(by integration by parts)} \\
&= v_x v_t\Big]_{x=b}^{x=a}.
\end{aligned}$$

As $v(a,t) = v(b,t) = 0$ for all $t \in (0,t]$, this gives

$$v_t(a,t) = v_t(b,t) = 0 \quad \text{for all } t \in (0,t].$$

Thus, $E'(t) = 0$ for all $t \in (0,t]$, which together with $E(0) = 0$, implies that

$$E(t) = 0 \qquad \text{for all } t \geq 0.$$

Thus v is constant. As $v = 0$ on Γ_T, we imply that $v \equiv 0$. The proof is complete. □

4.6. Duhamel's principle for the non-homogeneous wave equation

We now consider the wave equation in one dimension with a source term

$$\begin{cases} u_{tt} = u_{xx} + f(x,t) & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = 0 & \text{on } \mathbb{R}, \\ u_t(x,0) = 0 & \text{on } \mathbb{R}. \end{cases} \tag{4.5}$$

Here, $f \in C(\mathbb{R} \times [0,\infty))$ is the given source term.

We will use the Duhamel principle similar to the way that we use it for the heat equation. For the wave equation, it is however a bit more complex. It is necessary to understand the intuition behind a bit more.

4.6.1. Intuition. Assume $f(x,t) = f(t)$, that is, there is no x-dependence. In this situation, we can search for solutions to (4.5) independent of x, that is, $u(x,t) = V(t)$.

Then, our PDE (4.5) becomes an ODE:

$$\begin{cases} V''(t) = f(t) & t > 0, \\ V(0) = 0,\ V'(0) = 0. \end{cases}$$

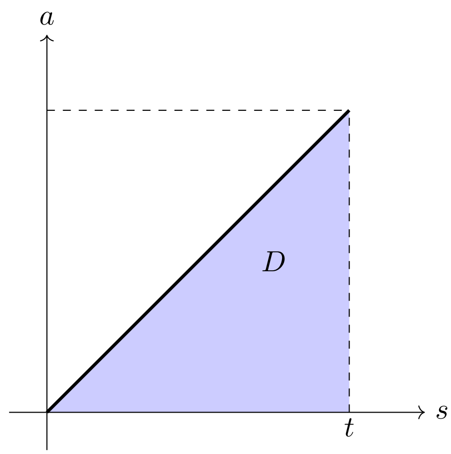

Figure 5. Domain of integration D

We can then integrate to find the solution V:

$$V(t) = V(0) + \int_0^t V'(s)\,ds = \int_0^t \left(\int_0^s f(a)\,da \right) ds = \iint_D f(a)\,dads.$$

By Fubini's theorem,

$$V(t) = \int_0^t \left(\int_a^t f(a)\,ds \right) da = \int_0^t (t-a) f(a)\,da.$$

Thus,

$$u(x,t) = \int_0^t (t-a) f(a)\,da.$$

We have another way to look at the problem as follows. Recall that $V''(t) = f(t)$ and

$$V'(t) = \int_0^t f(s)\,ds.$$

For each fixed $s > 0$, study

$$\begin{cases} (V^s)''(s) = 0, \\ V^s(s) = 0,\ (V^s)'(s) = f(s), \end{cases} \qquad \text{for } t \in \mathbb{R}.$$

Then, it is precisely the case that

$$(V^s)'(t) = f(s) \quad \text{ for all } t \in \mathbb{R}.$$

Integrate in t to yield

$$V^s(t) = (t-s) f(s) \quad \text{ for all } t \in \mathbb{R}.$$

Thus, we obtain the important identity:

$$V(t) = \int_0^t (t-s) f(s)\, ds = \int_0^t V^s(t)\, ds.$$

4.6.2. Duhamel's principle. The above intuition leads us to the Duhamel principle. For each $s \le t$, consider $u^s(x,t)$ solving

$$\begin{cases} u^s_{tt} = u^s_{xx} & \text{for } x \in \mathbb{R},\ t > s, \\ u^s(x,s) = 0 & \text{for all } x \in \mathbb{R}, \\ u^s_t(x,s) = f(x,s) & \text{for all } x \in \mathbb{R}. \end{cases}$$

Then, we have the formula: for $t > s$,

$$u^s(x,t) = \frac{1}{2} \int_{x-(t-s)}^{x+(t-s)} f(y,s)\, dy.$$

Theorem 4.8 (Duhamel's Principle for wave equations)**.** *For* $(x,t) \in \mathbb{R}^n \times [0,\infty)$*, set*

$$u(x,t) = \int_0^t u^s(x,t)\, ds = \frac{1}{2} \int_0^t \left(\int_{x-(t-s)}^{x+(t-s)} f(y,s)\, dy \right) ds.$$

Then, u *solves* (4.5)*.*

Proof. We differentiate u with respect to x twice to yield

$$u_x(x,t) = \int_0^t u^s_x(x,t)\, ds,$$

and

$$u_{xx}(x,t) = \int_0^t u^s_{xx}(x,t)\, ds.$$

Next, differentiate u with respect to t to get

$$u_{tt}(x,t) = \int_0^t u^s_t(x,t)\, ds + u^t(x,t) = \int_0^t u^s_t(x,t)\, ds$$

as $u^t(x,t) = 0$. Differentiate the above equality once more with respect to t to imply

$$u_{tt}(x,t) = \int_0^t u^s_{tt}(x,t)\, ds + u^t_t(x,t).$$

Thus,

$$u_{tt}(x,t) - u_{xx}(x,t) = \int_0^t (u^s_{tt} - u^s_{xx})\, ds + u^t_t(x,t) = f(x,t)$$

as $u^t_t(x,t) = f(x,t)$. The proof is complete. $\square$

Remark 4.9. The formula of u from Theorem 4.8 reads

$$\begin{aligned} u(x,t) &= \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)\,dyds \\ &= \frac{1}{2}\iint_{D(x,t)} f(y,s)\,dyds. \end{aligned}$$

See Figure 6.

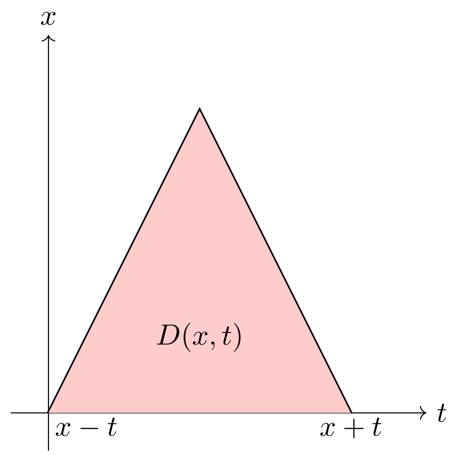

Figure 6. $D(x,t)$ from the Duhamel principle

4.7. Wave equations in two dimensions and three dimensions

This is more complicated, and we aim at using **spherical means** to solve the wave PDE.

4.7.1. General Setting & Spherical Means. Consider the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g & \text{on } \mathbb{R}^n, \\ u_t(x,0) = h & \text{on } \mathbb{R}^n. \end{cases} \tag{4.6}$$

Definition 4.10 (Spherical means)**.** For $x \in \mathbb{R}^n$, $t > 0$, and $r > 0$, define:

$$\begin{aligned} U(x,r,t) &= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y,t)\,dS(y) = ⨍_{\partial B(x,r)} u(y,t)\,dS(y), \\ G(x,r) &= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} g(y)\,dS(y) = ⨍_{\partial B(x,r)} g(y)\,dS(y), \\ H(x,r) &= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} h(y)\,dS(y) = ⨍_{\partial B(x,r)} h(y)\,dS(y). \end{aligned}$$

Lemma 4.11. *Fix x. Then $U : (0,\infty) \times [0,\infty) \to \mathbb{R}$ solves:*

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 & \text{in } (0,\infty) \times (0,\infty), \\ U = G & \text{on } (0,\infty) \times \{t = 0\}, \\ U_t = H & \text{on } (0,\infty) \times \{t = 0\}. \end{cases}$$

We note that this is not surprising, as Δu in polar coordinates is

$$u_{rr} + \frac{n-1}{r} u_r.$$

Sketch of the proof. We start with:

$$U(x,r,t) = \fint_{\partial B(x,r)} u(y,t)\, dS(y) = \fint_{\partial B(0,1)} u(x+rz,t)\, dS(z).$$

Now, differentiate U with respect to r:

$$\begin{aligned} U_r &= \fint_{\partial B(0,1)} Du(x+rz,t) \cdot z\, dS(z) \\ &= \frac{1}{n\omega_n} \int_{\partial B(0,1)} Du(x+rz,t) \cdot \mathbf{n}\, dS(z) \\ &= \frac{r}{n\omega_n} \int_{B(0,1)} \Delta u(x+rz,t)\, dz \\ &= \frac{r}{n} \fint_{B(x,r)} \Delta u(y,t)\, dy. \end{aligned}$$

Similarly,

$$U_{rr} = \left(\frac{1}{n} - 1\right) \fint_{B(x,r)} \Delta u(y,t)\, dy + \fint_{\partial B(x,r)} \Delta u(y,t)\, dS(y).$$

Combine the two terms to get

$$U_{rr} + \frac{n-1}{r} U_r = \fint_{\partial B(x,r)} \Delta u\, dS(y) = \fint_{\partial B(x,r)} u_{tt}\, dS(y) = U_{tt},$$

which gives us the desired result. □

4.7.2. Wave equation in three dimensions – Kirchhoff's formula. When $n = 3$, by the spherical means, we have:

$$U_{tt} - U_{rr} - \frac{2}{r} U_r = 0.$$

We aim to employ some tricks to combine the terms U_{rr} and $\frac{2}{r} U_r$.

Multiplying the ODE by r, we get:

$$rU_{tt} - rU_{rr} - 2U_r = 0.$$

Let $\widetilde{U} = rU$. Then:

$$\widetilde{U}_{tt} = rU_{tt}.$$

Interestingly, we also have:

$$\widetilde{U}_r = rU_r + U,$$

and

$$\widetilde{U}_{rr} = rU_{rr} + U_r + U_r = rU_{rr} + 2U_r.$$

Thus, $\widetilde{U}$ solves the equation:

$$\begin{cases} \widetilde{U}_{tt} - \widetilde{U}_{rr} = 0 & \text{in } (0,\infty) \times (0,\infty), \\ \widetilde{U} = \widetilde{G} & \text{on } (0,\infty) \times \{t = 0\}, \\ \widetilde{U}_t = \widetilde{H} & \text{on } (0,\infty) \times \{t = 0\}, \end{cases}$$

where

$$\widetilde{G} = rG, \quad \widetilde{H} = rH.$$

Further, as $\widetilde{U} = rU$,

$$\widetilde{U}(0,t) = 0 \quad \text{for } t \in [0,\infty).$$

By the odd reflection method in Theorem 4.3, we get, for $0 < r < t$,

$$\widetilde{U}(x,r,t) = \frac{1}{2}\left[\widetilde{G}(r+t) - \widetilde{G}(t-r)\right] + \frac{1}{2}\int_{t-r}^{r+t} \widetilde{H}(y)\,dy.$$

By the mean value theorem:

$$u(x,t) = \lim_{r\to 0^+} \frac{\widetilde{U}(x,r,t)}{r} = \lim_{r\to 0^+} U(x,r,t),$$

which simplifies to:

$$u(x,t) = \widetilde{G}'(t) + \widetilde{H}(t).$$

Thus, we have:

$$u(x,t) = \frac{\partial}{\partial t}\left[t \fint_{\partial B(x,t)} g\,dS\right] + t\fint_{\partial B(x,t)} h\,dS.$$

A careful computation leads to

$$u(x,t) = \fint_{\partial B(x,t)} \left[th(y) + g(y) + Dg(y)\cdot(y-x)\right]\,dS(y).$$

We formulate this into a theorem.

Theorem 4.12. *Assume $n = 3$, $g \in C^2(\mathbb{R}^3)$ and $h \in C^1(\mathbb{R}^3)$. Then,* (4.6) *has a unique solution given by*

$$u(x,t) = \fint_{\partial B(x,t)} \left[th(y) + g(y) + Dg(y)\cdot(y-x)\right]\,dS(y). \tag{4.7}$$

The formula (4.7) is called Kirchhoff's formula.

4.7.3. Wave Equation in two dimensions – Poisson's Formula. We first note that by spherical means, we have:

$$U_{tt} - U_{rr} - \frac{1}{r}U_r = 0.$$

Interestingly, in this setting, it is not possible to combine U_{rr} and $\frac{1}{r}U_r$ using the previous trick. People have tried various approaches, but it seems that it is not possible to solve the problem directly in 2D.

The approach we take now is to introduce a fake variable to get to the three dimensional case and then use the formula in 3D (Kirchhoff's formula). Once solved, we go back to the 2D case.

Our wave PDE in two dimensions reads

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^2 \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^2, \\ u_t(x,0) = h(x) & \text{on } \mathbb{R}^2. \end{cases} \tag{4.8}$$

Consider a new variable x_3 and set:

$$\begin{cases} \widetilde{u}(x_1,x_2,x_3,t) = u(x_1,x_2,t), \\ \widetilde{g}(x_1,x_2,x_3) = g(x_1,x_2), \\ \widetilde{h}(x_1,x_2,x_3) = h(x_1,x_2). \end{cases}$$

Then all partial derivatives in x_3 of $\widetilde{u}, \widetilde{g}, \widetilde{h}$ are zero.

Thus, $\widetilde{u}$ solves:

$$\begin{cases} \widetilde{u}_{tt} - \Delta \widetilde{u} = 0 & \text{in } \mathbb{R}^3 \times (0,\infty), \\ \widetilde{u}(x,0) = \widetilde{g} & \text{on } \mathbb{R}^3, \\ \widetilde{u}_t(x,0) = \widetilde{h} & \text{on } \mathbb{R}^3. \end{cases}$$

By Kirchhoff's formula in three dimensions, we have:

$$\widetilde{u}(\bar{x},t) = ⨍_{\partial B(\bar{x},t)} \left[t\widetilde{h}(y) + \widetilde{g}(y) + D\widetilde{g}(y)\cdot(y-\bar{x}) \right] dS(y).$$

Note, however, that $\bar{x}, y \in \mathbb{R}^3$, and $\partial B(\bar{x},t)$ is the sphere in three dimensions. To move back to two dimensions, one needs to be careful.

At $\bar{x} = (x,0)$, after an appropriate change of variable, we deduce

$$\widetilde{u}(x,0,t) = u(x,t) = \frac{1}{2}\int_{B(x,t)} \frac{tg(y) + t^2 h(y) + Dg(y)\cdot(y-x)}{(t^2 - |y-x|^2)^{1/2}}\, dy.$$

We skip the proof of this formula.

Theorem 4.13. *Assume $n = 2$, $g \in C^2(\mathbb{R}^2)$ and $h \in C^1(\mathbb{R}^2)$. Then,* (4.8) *has a unique solution given by*

$$u(x,t) = \frac{1}{2}\int_{B(x,t)} \frac{tg(y) + t^2 h(y) + Dg(y)\cdot(y-x)}{(t^2 - |y-x|^2)^{1/2}}\, dy. \tag{4.9}$$

The formula (4.9) is called Poisson's formula.

4.7.4. Wave equations in higher dimensions. The approaches go in the same way as those for three dimensions and two dimensions, but the calculations are a lot more involved. The case when n is odd and $n \geq 5$ is similar to that of $n = 3$. The case when n is even is similar to that of $n = 2$. We skip the cases in higher dimensions here, and we refer the interested reader to Evans [**Eva10**].

4.7.5. Huygen's principle. From Kirchhoff's formula and Poisson's formula, we have the following observations, which is called the Huygen principle.

(1) In three dimensions, $u(x,t)$ is totally determined by only data of g, h on the sphere $\partial B(x,t)$. This is also true for odd dimensions $n \geq 3$. As a matter of fact, we see sharp wave fronts.

(2) In two dimensions, $u(x,t)$ is determined by all data of g, h on the whole ball $B(x,t)$. This is also true for even dimensions $n \geq 2$. Hence, we do not see sharp wave fronts here.

4.8. Exercises

Exercise 37. We study the wave equation in one dimension.

(a) Show that the general solution of the PDE $u_{xy} = 0$ is

$$u(x,y) = F(x) + G(y)$$

for arbitrary differentiable functions F, G.

(b) Use change of variable $\xi = x + t, \eta = x - t$ to show that

$$u_{tt} - u_{xx} = 0 \iff u_{\xi\eta} = 0.$$

(c) Use (a) and (b) to rederive d'Alembert's formula.

Exercise 38. Consider the wave equation in 1D

$$u_{tt} = u_{xx}.$$

Let $e(t) = \frac{1}{2}(u_t(x,t)^2 + u_x(x,t)^2)$ (energy density), and $p(t) = u_t(x,t)u_x(x,t)$ (momentum density).

(a) Show that $e_t = p_x$ and $e_x = p_t$.

(b) Show that both e and p satisfy the wave equation in 1D.

Exercise 39. Consider the wave equation in 1D

$$u_{tt} = u_{xx}.$$

(a) For each fixed $y \in \mathbb{R}$, show that $v(x,t) = u(x-y,t)$ also solves the wave equation.

(b) Show that u_t, u_x, u_{tt}, u_{xx} all solve the wave equation.

(c) For $r > 0$, let $w(x,t) = u(rx, rt)$. Show that w is a solution to the wave equation.

Exercise 40. Consider the wave equation in 1D

$$u_{tt} = u_{xx}.$$

Show that for all $x, t, h, k \in \mathbb{R}$,

$$u(x+h, t+k) + u(x-h, t-k) = u(x+k, t+h) + u(x-k, t-h).$$

Exercise 41. Another wave equation in the first quadrant is

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{for } x > 0, t > 0, \\ u(x,0) = g(x) & \text{for } x > 0, \\ u_t(x,0) = h(x) & \text{for } x > 0, \\ u_x(0,t) = 0 & \text{for } t > 0. \end{cases}$$

Note that we have here $u_x(0,t) = 0$ (instead of $u(0,t) = 0$ as done in the book). Solve this equation.

Exercise 42. Consider the one dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{for } x \in \mathbb{R}, t > 0, \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}, \\ u_t(x,0) = h(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Assume that g, h are smooth functions with compact supports. Use the energy method to prove that there exists a unique smooth solution u such that u has compact support on $\mathbb{R} \times [0,T]$ for each given time $T > 0$.
Here, u has compact support on $\mathbb{R} \times [0,T]$ for each given time $T > 0$ means that there exists $R_T > 0$ such that

$$u(x,t) = 0 \qquad \text{for } |x| > R_T, 0 \le t \le T.$$

(Note that as you are not in bounded domain, you need to verify carefully that your integrals and integration by parts make sense. Follow the proof of uniqueness for the wave equation in bounded domains in the book with some modifications.)

Exercise 43. Consider the one dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{for } x \in \mathbb{R}, t > 0, \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}, \\ u_t(x,0) = h(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Assume that g, h are smooth functions with compact supports. Define the kinetic energy to be $k(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t(x,t)^2\,dx$ and potential energy to be $p(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x(x,t)^2\,dx$. Show that

(a) $k(t) + p(t)$ is always constant in t.

(b) $k(t) = p(t)$ for all large enough times t.

Exercise 44. Consider the one dimensional wave equation with frictional damping

$$u_{tt} + \delta u_t - u_{xx} = 0 \quad \text{for } x \in \mathbb{R}, t > 0.$$

Here, $\delta > 0$ is a given damping constant. Assume that $u \in C^2$ is a solution and u_x decays fast to 0 as $|x| \to \infty$ so that

$$E(t) = \int_{-\infty}^{\infty} \frac{1}{2}(u_t(x,t)^2 + u_x(x,t)^2)\,dx$$

is well-defined. Prove that $t \mapsto E(t)$ is nonincreasing.

Exercise 45. Let $u, v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be C^2 functions such that, for all $(x,t) \in \mathbb{R} \times \mathbb{R}$,

$$\begin{cases} u_t(x,t) + u_x(x,t) + 10(u(x,t) - v(x,t)) = 0, \\ v_t(x,t) - v_x(x,t) + 10(v(x,t) - u(x,t)) = 0. \end{cases}$$

Show that both u and v solve the following equation

$$w_{tt} + 20w_t - w_{xx} = 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}.$$

(That is, $w := u$ and $w := v$ solve the above PDE.)

4.9. Notes and references

(1) A large part of this chapter is based on the content in the book of Evans [**Eva10**, Chapter 2]. We skip the formulas of the solutions to the wave equation in higher dimensions ($n \geq 4$) as the calculations are more involved.

Chapter 5

Separation of variables and Fourier series

So far, we have developed a systematic way to study basic, prototypical PDEs: transport, Laplace, heat, and wave equations. In particular, we constructed solutions using various methods and proved uniqueness results for each.

Here, in this chapter, we aim at developing some methods to construct solutions for heat, wave, and Laplace PDEs. As we already have uniqueness results in the given contexts, the solutions we get here are automatically the unique ones. We will use:

- separation of variables;
- Fourier series.

5.1. Quick overview of Fourier series

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, which is periodic of period $2L$, that is,

$$f(x) = f(x + 2L) \quad \text{for } x \in \mathbb{R}.$$

Then, f can be represented by a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

Here, $\cos\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{n\pi x}{L}\right)$ are periodic with period $2L$. Besides, we have the orthogonal property as follows.

- $\int_{-L}^{L} \cos\left(\frac{j\pi x}{L}\right) \sin\left(\frac{k\pi x}{L}\right)\, dx = 0$ for all $j, k \geq 0$.

- $\int_{-L}^{L} \cos\left(\frac{j\pi x}{L}\right) \cos\left(\frac{k\pi x}{L}\right)\, dx = L\delta_{jk}$.
- $\int_{-L}^{L} \sin\left(\frac{j\pi x}{L}\right) \sin\left(\frac{k\pi x}{L}\right)\, dx = L\delta_{jk}$ as well.

Here, δ_{jk} is the Dirac delta satisfying

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

We note that a_n is called the Fourier cosine coefficient, and

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\, dx$$

and b_n is called the Fourier sine coefficient, and

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx.$$

5.1.1. Two Special Cases.

5.1.1.1. *Odd functions.* If f is odd, that is, $f(x) = -f(-x)$, then $a_n = 0$ for all $n \geq 0$, and

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx.$$

5.1.1.2. *Even functions.* If f is even, that is, $f(x) = f(-x)$, then $b_n = 0$ for all $n \geq 1$, and

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\, dx = \frac{2}{L} \int_{0}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\, dx.$$

Let us now proceed to study the aforementioned PDEs using Fourier series and separation of variables.

5.2. Homogeneous heat equations

5.2.1. Homogeneous heat equations with Dirichlet boundary conditions. Let $k, L > 0$ be given. We consider the homogeneous heat equation

$$\begin{cases} u_t = k u_{xx} & 0 < x < L,\ t > 0, \\ u(0,t) = u(L,t) = 0 & t > 0, \\ u(x,0) = g(x) & 0 < x < L. \end{cases} \tag{5.1}$$

Here, $g \in C([0, L])$ is given with $g(0) = g(L) = 0$ for compatibility conditions. See Figure 1.

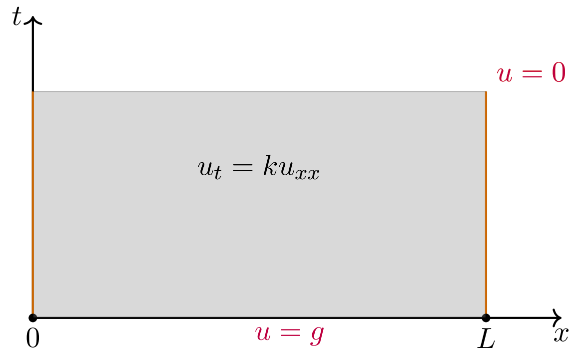

Figure 1. Homogeneous heat equations with Dirichlet boundary conditions

The overall strategy is this. We search for solutions in the special separable form

$$u(x,t) = v(x)w(t),$$

which typically leads to a family of functions $\{u_n(x,t)\}_{n\in\mathbb{N}}$. Because of the superposition principle, we can then write a general solution $u(x,t)$ as an infinite series in terms of $\{u_n\}$:

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t), \quad (c_n \in \mathbb{R}).$$

Let us now give a rigorous derivation of each step.

Step 1. Write $u(x,t) = v(x)w(t)$. Then

$$u_t = v(x)w'(t), \quad u_{xx} = v''(x)w(t).$$

So

$$u_t = ku_{xx} \quad \Longleftrightarrow \quad v(x)w'(t) = kv''(x)w(t).$$

This leads to the separation of variables

$$\frac{w'(t)}{kw(t)} = \frac{v''(x)}{v(x)}.$$

Since the left-hand side depends only on t and the right-hand side depends only on x, both must equal a constant independent of x and t. Thus, we set:

$$\frac{w'(t)}{kw(t)} = \frac{v''(x)}{v(x)} = -\lambda \quad \text{for some } \lambda \in \mathbb{R}.$$

For $w(t)$:

$$w'(t) = -\lambda k w(t) \quad \Longrightarrow \quad w(t) = Ce^{-\lambda k t}.$$

As it is typically the case that the solution to the heat equation decays as $t \to \infty$, it is intuitive to require that $\lambda > 0$.

For $v(x)$:

$$v''(x) = -\lambda v(x) \quad \Longrightarrow \quad v'' + \lambda v = 0.$$

The characteristic equation is:

$$\alpha^2 + \lambda = 0 \quad \Longrightarrow \quad \alpha = \pm i\sqrt{\lambda}.$$

Thus, the general solution is:

$$v(x) = C_1 \cos\left(\sqrt{\lambda} x\right) + C_2 \sin\left(\sqrt{\lambda} x\right).$$

We need to impose the boundary conditions $v(0) = v(L) = 0$. From $v(0) = 0$, we have $C_1 = 0$. From $v(L) = 0$, we get:

$$\sin\left(\sqrt{\lambda} L\right) = 0.$$

Thus, $\sqrt{\lambda} L = n\pi$ for $n \in \mathbb{N}$, which gives:

$$\lambda = \frac{n^2\pi^2}{L^2} \quad \text{for } n \in \mathbb{N}.$$

Thus, we get a family of special solutions:

$$u_n(x,t) = e^{-k\frac{n^2\pi^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right).$$

Step 2. A general solution is of the form:

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) \quad (c_n \in \mathbb{R} \text{ to be determined}).$$

Thus, we have:

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\frac{n^2\pi^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right).$$

As $g \in C([0,L])$ is given with $g(0) = g(L) = 0$, we can extend g to be an odd function in $[-L, L]$. We then extend g to be a periodic function of period $2L$ in $\mathbb{R}$. By abuse of notations, we still call this extension g. Then,

$$g(x) = u(x,0) = \sum_{n=0}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right).$$

To compute c_n, we use the formula that

$$c_n = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right)\, dx, \quad n \geq 0.$$

Example 5.1. Solve

$$\begin{cases} u_t = 7u_{xx} & 0 < x < 1, t > 0, \\ u(0,t) = u(1,t) = 0 & t > 0, \\ u(x,0) = 3\sin(\pi x) - 7\sin(5\pi x) & 0 < x < 1. \end{cases}$$

Solution. By our method above, a general solution to this heat equation is, for $k = 7$, $L = 1$,

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-7n^2\pi^2 t} \sin(n\pi x).$$

Since $u(x,0) = 3\sin(\pi x) - 7\sin(5\pi x)$, we see that $c_1 = 3$, $c_5 = -7$, and $c_n = 0$ otherwise.

This leads to our unique solution:

$$u(x,t) = 3e^{-7\pi^2 t} \sin(\pi x) - 7e^{-175\pi^2 t} \sin(5\pi x).$$

5.2.2. Homogeneous Heat equation with Neumann boundary condition. Here, we have $k, L > 0$ are given. The PDE is

$$\begin{cases} u_t = ku_{xx}, & 0 < x < L,\ t > 0, \\ u_x(0,t) = u_x(L,t) = 0 & t > 0, \\ u(x,0) = g(x) & 0 < x < L. \end{cases} \tag{5.2}$$

By compatibility conditions, we assume that $g \in C^1[0,L])$ and $g'(0) = g'(L) = 0$. We proceed as in the previous section.

Step 1: Search for a separable solution. We consider u of the form $u(x,t) = V(x)W(t)$. Then

$$\frac{W'}{kW} = \frac{V''}{V} = -\lambda \quad \text{for } \lambda > 0 \text{ a constant.}$$

As $W'(t) = -\lambda k W(t)$, we get $W(t) = Ce^{-\lambda kt}$ as above. Further, by the same manner,

$$V'' + \lambda V = 0 \quad \Longrightarrow \quad V(x) = C_1 \cos\left(\sqrt{\lambda}x\right) + C_2 \sin\left(\sqrt{\lambda}x\right).$$

What is different here is the Neumann boundary condition. Differentiate V to get

$$V'(x) = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + C_2\sqrt{\lambda}\cos(\sqrt{\lambda}x).$$

We need to require $V'(0) = V'(L) = 0$. Firstly,

$$V'(0) = 0 \quad \Longrightarrow \quad C_2 = 0 \quad \Longrightarrow \quad V'(x) = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}x).$$

To have $V'(L) = 0$, we need $\sqrt{\lambda}L = n\pi$ for $n \in \mathbb{N}$, which means

$$\lambda = \frac{n^2\pi^2}{L^2} \quad \text{for } n \in \mathbb{N}.$$

We hence get a family of special solutions, for $n \in \mathbb{N}$,

$$u_n(x,t) = e^{-k\frac{n^2\pi^2}{L^2}t} \cos\left(\frac{n\pi x}{L}\right).$$

Step 2: General solution. A general solution is of the form

$$u(x,t) = \sum_{n=0}^{\infty} c_n u_n(x,t) = \sum_{n=0}^{\infty} c_n e^{-k\frac{n^2\pi^2}{L^2}t} \cos\left(\frac{n\pi x}{L}\right),$$

where $c_n \in \mathbb{R}$ is to be determined. Then,

$$g(x) = u(x,0) = \sum_{n=0}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right).$$

As $g \in C^1[0,L])$ and $g'(0) = g'(L) = 0$, we can extend g to be an even function in $[-L,L]$. We then extend g to be a periodic function of period $2L$ in $\mathbb{R}$. By abuse of notations, we still call this extension g. To compute c_n, we use the formula that

$$c_n = \frac{2}{L}\int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \quad n \geq 0.$$

5.2.3. Quick discussion on Robin boundary condition. We will only provide a brief discussion here.

$$\begin{cases} u_t = k u_{xx} & 0 < x < L,\ t > 0,\\ u_x(0,t) - a_0 u(0,t) = 0 & t > 0,\\ u_x(L,t) + a_L u(L,t) = 0 & t > 0,\\ u(x,0) = g(x) & 0 < x < L. \end{cases}$$

Here we have

$$\begin{cases} a_0, a_L > 0 \text{ correspond to radiating boundary conditions,}\\ a_0, a_L < 0 \text{ correspond to absorbing boundary conditions.} \end{cases}$$

When we do separation of variables $u(x,t) = V(x)W(t)$, then we still have

$$\frac{W''}{kW} = \frac{V''}{V} = \lambda \quad \text{for some } \lambda > 0.$$

Then

$$\begin{cases} W(t) = Ce^{-\lambda k t},\\ V(x) = C_1 \cos\left(\sqrt{\lambda}x\right) + C_2 \sin\left(\sqrt{\lambda}x\right). \end{cases}$$

What is different (and a bit more difficult) here is that we need to consider the boundary condition:

$$V'(0) - a_0 V(0) = 0 = V'(L) + a_L V(L).$$

By plugging in this boundary conditions, we can find the relations between λ, C_1, C_2 and proceed.

5.3. Non-homogeneous heat equations

$$(5.3) \qquad \begin{cases} u_t = ku_{xx} + f(x,t), & 0 < x < \pi,\ t > 0, \\ u(0,t) = u(\pi,t) = 0, & t > 0, \\ u(x,0) = g(x) & 0 < x < \pi. \end{cases}$$

Here, the source term $f = f(x,t) \in C([0,\pi] \times [0,\infty))$ and the initial data $g \in C([0,\pi])$ are given. For compatibility conditions, we assume $f(0,t) = f(\pi,t) = 0$ for $t \geq 0$ and $g(0) = g(\pi) = 0$.

Without the source term, that is, when $f \equiv 0$, then the general form of solutions that we derived is

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2 t} \sin(nx).$$

With the appearance of f, we guess that u is of the form:

$$u(x,t) = \sum_{n=1}^{\infty} C_n(t) \sin(nx),$$

where $C_n(t)$ is to be found.

Write:

$$\begin{aligned} g(x) &= \sum_{n=1}^{\infty} \alpha_n \sin(nx), \\ f(x,t) &= \sum_{n=1}^{\infty} f_n(t) \sin(nx). \end{aligned}$$

We can compute:

$$\begin{aligned} \alpha_n &= \frac{2}{\pi} \int_0^{\pi} g(x) \sin(nx)\, dx, \\ f_n(t) &= \frac{2}{\pi} \int_0^{\pi} f(x,t) \sin(nx)\, dx. \end{aligned}$$

Thus, as

$$u(x,t) = \sum_{n=1}^{\infty} C_n(t) \sin(nx),$$

we see that:

$$u_t = \sum_{n=1}^{\infty} C_n'(t) \sin(nx),$$
$$u_{xx} = \sum_{n=1}^{\infty} -n^2 C_n(t) \sin(nx).$$

Plug everything into our PDE to imply:

$$\sum_{n=1}^{\infty} \big(C_n'(t) + n^2 C_n(t) - f_n(t)\big) \sin(nx) = 0.$$

Then, for $n \in \mathbb{N}$,

$$\begin{cases} C_n'(t) + n^2 C_n(t) = f_n(t), & t > 0, \\ C_n(0) = \alpha_n. \end{cases}$$

Thus, we can solve this ODE to find $C_n(t)$. We have

$$\left(e^{n^2 t} C_n(t)\right)' = e^{n^2 t} f_n(t),$$

which implies

$$e^{n^2 t} C_n(t) = e^{n^2 t} \alpha_n + \int_0^t e^{n^2 s} f_n(s)\, ds.$$

Hence,

$$C_n(t) = \alpha_n e^{-n^2 t} + e^{-n^2 t} \int_0^t e^{n^2 s} f_n(s)\, ds.$$

We conclude that

$$u(x,t) = \sum_{n=1}^{\infty} \left(\alpha_n e^{-n^2 t} + e^{-n^2 t} \int_0^t e^{n^2 s} f_n(s)\, ds\right) \sin(nx).$$

5.4. Homogeneous wave equations

We now use the same approach to study the wave PDE. Consider

$$\begin{cases} u_{tt} = c^2 u_{xx}, & 0 < x < L,\ t > 0, \\ u(0,t) = u(L,t) = 0, & t > 0, \\ u(x,0) = g(x) & 0 < x < L, \\ u_t(x,0) = h(x), & 0 < x < L. \end{cases} \tag{5.4}$$

Here, $c > 0$ is a given constant. The initial data $g, h \in C([0,L])$ are given. We assume $g(0) = g(L) = 0$ and $h(0) = h(L) = 0$ for compatibility.

We follow the same approach like earlier and use the earlier intuition to do it efficiently.

Step 1: Find special solutions of separable form. Assume

$$u(x,t) = V(x)W(t).$$

Then,

$$u_{tt} = V(x)W''(t), \quad u_{xx} = V''(x)W(t).$$

Thus,

$$\frac{W''(t)}{c^2W(t)} = \frac{V''(x)}{V(x)} = \lambda,$$

where λ must be a constant independent of x and t.

By the earlier intuition:

$$\frac{W''(t)}{c^2W(t)} = \frac{V''(x)}{V(x)} = -\lambda \quad \text{for some } \lambda > 0.$$

Thus,

$$V''(x) + \lambda V(x) = 0,$$
$$W''(t) + \lambda c^2 W(t) = 0,$$

so

$$V(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x),$$
$$W(t) = C_3 \cos(\sqrt{\lambda}ct) + C_4 \sin(\sqrt{\lambda}ct).$$

As $V(0) = V(L) = 0$, we get $C_1 = 0$, and

$$\sqrt{\lambda}L = n\pi \implies \lambda = \frac{n^2\pi^2}{L^2}.$$

Thus, we get special solutions:

$$u_n(x,t) = \left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right),$$

where $a_n, b_n \in \mathbb{R}$ are to be chosen.

Step 2: General solution. Because of the superposition principle, we can then write a general solution $u(x,t)$ as an infinite series in terms of $\{u_n\}$:

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right).$$

To find a_n, b_n, we need to use the initial data:

$$u(x,0) = g(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right),$$
$$u_t(x,0) = h(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin\left(\frac{n\pi x}{L}\right).$$

Recall that

$$g_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx, \quad h_n = \frac{2}{L}\int_0^L h(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Therefore, for $n \in \mathbb{N}$,

$$\begin{cases} a_n = g_n, \\ b_n = \frac{L}{n\pi c} h_n. \end{cases}$$

Remark 5.2. Each $u_n(x,t)$ has the form:

$$u_n(x,t) = \left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right).$$

This is a function periodic in x with period $2L$ and periodic in t with period $\frac{2L}{nc}$. We can see that this looks like a periodic wave in space and time.

$$\sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2}\left[\cos\left(\frac{n\pi}{L}(x-ct)\right) - \cos\left(\frac{n\pi}{L}(x+ct)\right)\right]$$

$$\cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2}\left[\sin\left(\frac{n\pi}{L}(x+ct)\right) + \sin\left(\frac{n\pi}{L}(x-ct)\right)\right]$$

Thus, we deduce that $u_n(x,t)$ is of the form:

$$u_n(x,t) = G_n(x+ct) + H_n(x-ct),$$

where G_n, H_n are some functions. This is exactly the form we care about earlier.

5.5. Separation of variables for a porous medium equation

Typically, separation of variables works well for linear equations, but it does not work well for nonlinear PDEs.

Recall the superposition principle we already discussed. Assume that

$$L[u_1] = f_1, \quad L[u_2] = f_2.$$

If L is linear, then $L[u_1+u_2] = f_1+f_2$. But if L is nonlinear, then in general

$$L(u_1+u_2) \neq f_1+f_2.$$

Still, sometimes, separation of variables works for some nonlinear equations. We consider one such in this section.

Let $m > 0$ be a fixed constant. We consider a parabolic PDE:

$$u_t = \Delta(u^m) \quad \text{in } \mathbb{R}^n \times (0,\infty). \tag{5.5}$$

This equation is called a porous medium equation in the literature. We note that u^m is nonlinear in u for all $m \neq 1$. If $m \neq 1$, (5.5) is a nonlinear heat equation. If $m = 1$, we get back to the usual heat equation.

We now fix $m > 1$. Let us now find solutions to (5.5) of separable form $u(x,t) = v(x)w(t)$. Then,

$$\begin{cases} u_t = v(x)w'(t), \\ \Delta(u^m) = \Delta\left(v(x)^m\right) w(t)^m. \end{cases}$$

Thus,

$$v(x)w'(t) = \Delta(v(x)^m)w(t)^m.$$

Rearrange to yield

$$\frac{w'(t)}{w(t)^m} = \frac{\Delta(v(x)^m)}{v(x)} = \mu.$$

Here, $\mu \in \mathbb{R}$ is a constant independent of x, t.

To solve for $w(t)$, we have that

$$\frac{w'(t)}{w(t)^m} = \mu.$$

Integrate this relation to yield

$$w(t)^{1-m} = (1-m)\mu t + C,$$

where C is a constant. Hence,

$$w(t) = \left[C - (m-1)\mu t\right]^{\frac{1}{1-m}}. \tag{5.6}$$

We see right away that we need $C > 0$ for the above to be defined. Further, if $\mu \leq 0$, then w is well-defined for all $t \geq 0$. Otherwise, if $\mu > 0$, then w is well-defined only up to

$$T = \frac{C}{(m-1)\mu} > 0.$$

Besides, in the case that $\mu > 0$,

$$\lim_{t \to T^-} w(t) = +\infty.$$

Next, we need to find $v(x)$

$$\Delta(v(x)^m) = \mu v(x) \quad \text{ in } \mathbb{R}^n.$$

Consider the radially symmetric setting as usual:

$$v(x) = |x|^\alpha \quad \Longrightarrow \quad \phi(x) = v(x)^m = |x|^{m\alpha}.$$

Write $r = |x|$. By abuse of notations, we also write $\phi(x) = \phi(|x|) = \phi(r)$. Then,

$$\begin{aligned}
\Delta(u^m) = \Delta\phi &= \phi''(r) + \left(\frac{n-1}{r}\right)\phi' \\
&= m\alpha(m\alpha-1)r^{m\alpha-2} + \frac{n-1}{r}m\alpha r^{m\alpha-1} \\
&= m\alpha\left(m\alpha - 1 + n - 1\right)r^{m\alpha-2} \\
&= m\alpha(m\alpha+n-2)r^{m\alpha-2}.
\end{aligned}$$

Thus, we yield:

$$\mu r^{\alpha} = m\alpha(m\alpha+n-2)r^{m\alpha-2}.$$

By matching the terms,

$$\begin{cases} \alpha = \frac{2}{m-1} \\ \mu = m\alpha(m\alpha+n-2) \end{cases} \implies \mu = \frac{2m(n(m-1)+2)}{(m-1)^2}.$$

We see that $\mu > 0$ and

$$v(x) = |x|^{\frac{2}{m-1}}, \quad \mu = m\alpha(n+\alpha) = \frac{2m(n(m-1)+2)}{(m-1)^2}.$$

Combining everything together, we conclude

$$\begin{aligned}
u(x,t) = v(x)w(t) &= |x|^{\frac{2}{m-1}}\left[C - (m-1)\mu t\right]^{\frac{1}{1-m}} \\
&= |x|^{\frac{2}{m-1}}\left[C - 2m\frac{(n(m-1)+2)}{(m-1)}t\right]^{\frac{1}{1-m}}.
\end{aligned}$$

This special solution is typically called the Barenblatt–Kompaneets–Zeldovich solution in the literature.

Remark 5.3. This is again a very beautiful special solution to our nonlinear heat equation – a porous media one. What is remarkable is that the solution we found only exists for

$$0 < t < T = \frac{C}{(m-1)\mu} = \frac{C(m-1)}{2m(n(m-1)+2)}.$$

We do not discuss about the possible singularity nature when $x = 0$ here. This shows that nonlinear equations are really interesting and also harder in general.

5.6. Separation of variables for Laplace equation

We conclude the usage of the separation of variables method and Fourier series by demonstrating its application to the Laplace equation with mixed boundary conditions – both Dirichlet and Neumann boundary conditions.

5.6.1. An example.

Example 5.4. We are in $\mathbb{R}^2$ and a point is written as (x, y). Let $U = (0, a) \times (0, b)$ where $a, b > 0$ are given.

Consider the PDE

$$
\begin{cases}
-\Delta u = 0 & \text{in } U = (0, a) \times (0, b), \\
u = 0 & \text{on } \left(\{0\} \times [0, b]\right) \cup \left(\{a\} \times [0, b]\right), \\
\frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } [0, a] \times \{0\}, \\
u = f & \text{on } [0, a] \times \{b\}.
\end{cases} \tag{5.7}
$$

For compatibility, we assume that $f(0) = f(a) = 0$.

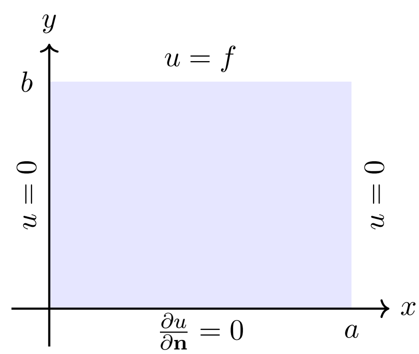

Figure 2. Domain and boundary conditions for u

This is a rather complicated mixed boundary problem, but it is the one that occurs a lot in practice. See Figure 2. We do have uniqueness of solutions to this problem by using the energy method (to be recalled and proved later).

Solution. Let us now attempt to solve the PDE first.

As $u(x, y) = 0$ for $x = 0$ and $x = a$, we guess that

$$u(x, y) = V(x)W(y)$$

with $V(0) = V(a) = 0$. And because of this boundary condition on V, we guess next that

$$V(x) = \sin\left(\frac{n\pi x}{a}\right) \quad \text{for } n \in \mathbb{N}.$$

For one such fixed $n \in \mathbb{N}$, we have:

$$u(x, y) = V(x)W(y) = \sin\left(\frac{n\pi x}{a}\right) W(y).$$

Then,

$$\begin{cases} u_{xx} = -\frac{n^2\pi^2}{a^2} \sin\left(\frac{n\pi x}{a}\right) W(y), \\ u_{yy} = \sin\left(\frac{n\pi x}{a}\right) W''(y). \end{cases}$$

Thus, in U, $-\Delta u = \Delta u = 0$ implies

$$W''(y) - \frac{n^2\pi^2}{a^2} W(y) = 0.$$

A general solution W has the form

$$W(y) = C_1 e^{\frac{n\pi}{a}y} + C_2 e^{-\frac{n\pi}{a}y}.$$

As $\frac{\partial u(x,y)}{\partial \mathbf{n}} = 0$ on $y = 0, 0 < x < a$, we get

$$W'(0) = 0.$$

Therefore,

$$0 = W'(0) = \frac{n\pi}{a} C_1 - \frac{n\pi}{a} C_2 \qquad \Longrightarrow \qquad C_1 = C_2.$$

We imply that

$$W(y) = C_1 \left(e^{\frac{n\pi}{a}y} + e^{-\frac{n\pi}{a}y}\right).$$

We now vary $n \in \mathbb{N}$ and use the superposition principle to see that the general solution to our Laplace PDE is

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) \left[e^{\frac{n\pi}{a}y} + e^{-\frac{n\pi}{a}y}\right].$$

Finally, we need to ensure that $u(x,b) = f(x)$ for $0 \le x \le a$, that is,

$$\sum_{n=1}^{\infty} C_n \left(e^{\frac{n\pi b}{a}} + e^{-\frac{n\pi b}{a}}\right) \sin\left(\frac{n\pi x}{a}\right) = f(x).$$

Hence, we get

$$C_n \left(e^{\frac{n\pi b}{a}} + e^{-\frac{n\pi b}{a}}\right) = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx.$$

This gives the formula for C_n for all $n \in \mathbb{N}$.

5.6.2. A uniqueness result.

Theorem 5.5. *Let $U = (0,a) \times (0,b)$ where $a, b > 0$ are given. If $u_1, u_2 \in C\left(\overline{U}\right) \cap C^2(U)$ are solutions to (5.7), then $u_1 \equiv u_2$.*

Proof. Assume that $u_1, u_2 \in C\left(\overline{U}\right) \cap C^2(U)$ are solutions to (5.7). Let $\varphi = u_1 - u_2$. Then φ solves

$$\begin{cases} -\Delta\varphi = 0 & \text{in } U, \\ \varphi = 0 & \text{on } \left(\{0\} \times [0,b]\right) \cup \left(\{a\} \times [0,b]\right) \cup \left[(0,a] \times \{b\}\right), \\ \frac{\partial\varphi}{\partial \mathbf{n}} = 0 & \text{on } [0,a] \times \{0\}. \end{cases}$$

See Figure 3.

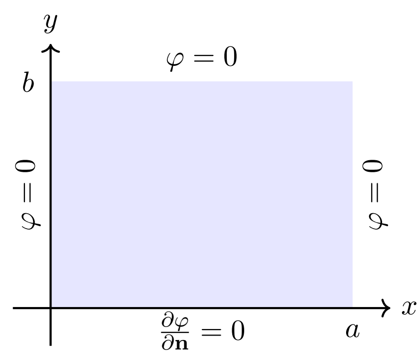

Figure 3. Domain and boundary conditions for φ

Multiply the equation by φ and integrate on U:

$$0 = \int_U -\Delta\varphi \cdot \varphi \, dx\, dy = \int_U |D\varphi|^2 \, dx\, dy + \int_{\partial U} \varphi \, D\varphi \cdot \mathbf{n} \, dS$$

The boundary term is zero as either $\varphi = 0$ or $\frac{\partial \varphi}{\partial \mathbf{n}} = 0$ on ∂U.

Thus,

$$\int_U |D\varphi|^2 \, dx\, dy = 0 \qquad \Longrightarrow \qquad D\varphi \equiv 0 \text{ in } U.$$

This implies φ is constant on $\overline{U}$. Since $\varphi = 0$ on 3 sides, we conclude $\varphi \equiv 0$. $\square$

5.6.3. General problems. We can use the above method together with others to solve more complicated problems, such as the following

$$\begin{cases} -\Delta u = 0 & \text{in } U = (0,a) \times (0,b), \\ \frac{\partial u}{\partial \mathbf{n}} = f_1 & \text{on } [0,a] \times \{0\}, \\ u = f_2 & \text{on } \{a\} \times [0,b], \\ u = f_3 & \text{on } [0,a] \times \{b\}, \\ u = f_4 & \text{on } \{0\} \times [0,b]. \end{cases} \tag{5.8}$$

See Figure 4. The key point here is that such a problem can be analyzed via four different subproblems. Each f_i can be handled separately for $1 \le i \le 4$. Let u_i be the solution corresponding to the subproblem with f_i for $1 \le i \le 4$, respectively. Then, by the superposition principle, $u = u_1 + u_2 + u_3 + u_4$ solves our original equation. We have the uniqueness of u via Theorem 5.5.

The four subproblems are pictured as follows. See Figures 5–6.

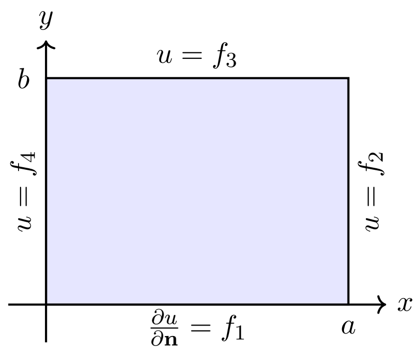

Figure 4. Domain with mixed boundary conditions

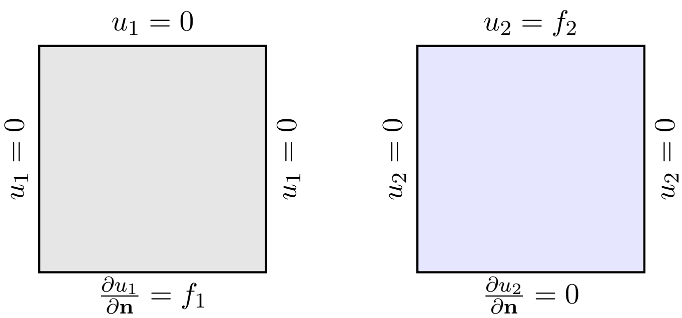

Figure 5. Decomposition of the problem into subproblems

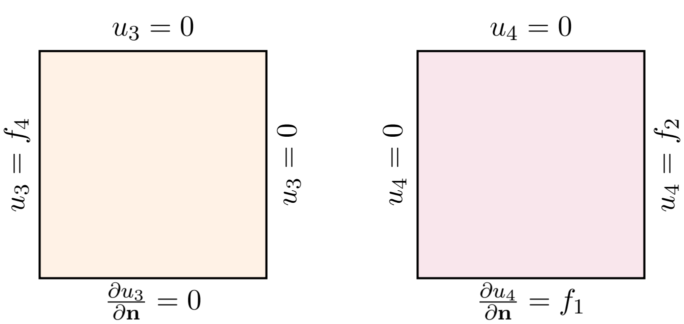

Figure 6. Decomposition of the problem into subproblems

5.7. Tychonov's solution to the heat equation in one dimension

We now construct a non-physical solution to the heat equation in one dimension. Instead of imposing the initial condition as we always did, we here require that

$$u(0,t) = \phi(t) \qquad \text{for } t \geq 0, \tag{5.9}$$

for a given function ϕ.

By using the same philosophy of using the separation of variables method to find special solutions and the superposition principle to find general solutions in this chapter, we look for solution to the heat equation of power series form

$$u(x,t) = \sum_{k=0}^{\infty} \phi_k(t) x^k.$$

Then $\phi_0 = \phi$. Assume that this power series is convergent and nice enough to differentiate in x, t, we find

$$u_t = \sum_{k=0}^{\infty} \phi_k'(t) x^k,$$

and

$$u_{xx} = \sum_{k=2}^{\infty} \phi_k(t) k(k-1) x^{k-2} = \sum_{k=0}^{\infty} \phi_{k+2}(t)(k+2)(k+1) x^k.$$

By matching the terms in the equality $u_t = u_{xx}$, we get

$$\phi_{2k} = \frac{\phi^{(k)}}{(2k)!}, \quad \phi_{2k+1} = \frac{\phi_1^{(k)}}{(2k+1)!}.$$

Choose $\phi_1 = 0$. We then get, heuristically,

$$u(x,t) = \sum_{k=0}^{\infty} \phi^{(k)}(t) \frac{x^{2k}}{(2k)!}.$$

Let us show that this heuristic idea can be made rigorous as follows. We first give a definition of the Gevrey class.

Definition 5.6 (Gevrey class)**.** Let $J \subset \mathbb{R}$ be a given interval and $\theta > 0$ be a given number. We say that $\phi \in C^\infty(J)$ is in the Gevrey class θ if

$$|\phi^{(k)}(t)| \leq A^{k+1} k^{\theta k} \quad \text{for all } t \in J,\ k \in \mathbb{N},$$

for some $A > 0$.

Example 5.7. The function

$$\phi(t) = \begin{cases} e^{-1/t^2} & \text{for } t > 0, \\ 0 & \text{for } t \leq 0 \end{cases}$$

is in the Gevrey class $3/2$.

Theorem 5.8 (Tychonov's solution)**.** *Assume that $\phi \in C^\infty(\mathbb{R})$ is in the Gevrey class θ for some given $\theta \in (1,2)$. Define, for $(x,t) \in \mathbb{R} \times [0,\infty)$,*

$$u(x,t) = \sum_{k=0}^{\infty} \phi^{(k)}(t) \frac{x^{2k}}{(2k)!}.$$

Then, u solves the heat equation and u satisfies (5.9)*.*

Proof. We will prove the convergence of the series and its derivatives later. We compute

$$u_t = \sum_{k=0}^{\infty} \phi^{(k+1)}(t) \frac{x^{2k}}{(2k)!},$$

and

$$u_{xx} = \sum_{k=1}^{\infty} \phi^{(k)}(t) \frac{x^{2(k-1)}}{(2(k-1))!}.$$

It follows that $u_t = u_{xx}$, that is, u is a solution of the heat equation.

We now prove the convergence of our power series in the definition of u. Fix $L > 0$. For $|x| \le L$, we use the definition of the Gevrey class to yield

$$\left| \frac{\phi^{(k)}(t) x^{2k}}{(2k)!} \right| \le A(AL^2)^k \frac{k^{\theta k}}{(2k)!} = M_k.$$

By the ratio test for series,

$$\begin{aligned} \lim_{k\to\infty} \frac{M_{k+1}}{M_k} &= \lim_{k\to\infty} (AL^2) \frac{(k+1)^\theta}{(2k+2)(2k+1)} \left(1 + \frac{1}{k}\right)^{\theta k} \\ &= (AL^2 e^\theta) \lim_{k\to\infty} \frac{(k+1)^\theta}{(2k+2)(2k+1)} = 0, \end{aligned}$$

where we used $\theta < 2$ in the last equality.

Thus, the power series for u converges uniformly for $t \in [0,\infty)$, $|x| \le L$, for any $L > 0$. The same arguments apply to the series for u_t and u_{xx}, and all higher partial derivatives of u. Therefore, we can differentiate the series for u term by term, and u is indeed a solution to the heat equation in $\mathbb{R} \times (0,\infty)$. □

We thus have the following puzzling corollary.

Corollary 5.9. *Let*

$$\phi(t) = \begin{cases} e^{-1/t^2} & \textit{for } t > 0, \\ 0 & \textit{for } t \le 0. \end{cases}$$

Define, for $(x,t) \in \mathbb{R} \times [0,\infty)$,

$$u(x,t) = \sum_{k=0}^{\infty} \phi^{(k)}(t) \frac{x^{2k}}{(2k)!}.$$

Then, $u \in C^\infty(\mathbb{R} \times [0,\infty))$ *solves*

$$\begin{cases} u_t = u_{xx} & \textit{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = 0 & \textit{on } \mathbb{R}. \end{cases} \tag{5.10}$$

Remark 5.10. We note that the physical solution to (5.10) is $u \equiv 0$. However, Corollary 5.9 implies that (5.10) has infinity many smooth solutions if we do not impose conditions on u at infinity. We say that the Tychonov's solution Corollary 5.9 is non-physical.

Once we impose appropriate conditions on the solutions at infinity, then we have the desired uniqueness result. We refer the reader back to Theorem 3.44 and Theorem 3.45.

5.8. Exercises

Exercise 46. Find all possible special solutions $u(x,t) = v(x)w(t)$ to the heat equation

$$\begin{cases} u_t - 20u_{xx} = 0 & \text{for } (x,t) \in (0,5) \times (0,\infty), \\ u(0,t) = u(5,t) = 0 & \text{for } t > 0. \end{cases}$$

Exercise 47. Find all possible special solutions $u(x,t) = v(x)w(t)$ to the heat equation

$$\begin{cases} u_t - 5u_{xx} = 0 & \text{for } (x,t) \in (0,\pi) \times (0,\infty), \\ u_x(0,t) = u_x(\pi,t) = 0 & \text{for } t > 0. \end{cases}$$

Exercise 48. Find all possible special solutions $u(x,t) = v(x)w(t)$ to the heat equation

$$\begin{cases} u_t + u - u_{xx} = 0 & \text{for } (x,t) \in (0,1) \times (0,\infty), \\ u(0,t) = u(1,t) = 0 & \text{for } t > 0. \end{cases}$$

Exercise 49. Solve

$$\begin{cases} u_t = 9u_{xx} & \text{for } 0 < x < \pi, t > 0, \\ u(0,t) = u(\pi,t) = 0 & \text{for } t > 0, \\ u(x,0) = \sum_{n=1}^{8} \frac{1}{n^4} \sin(nx) & \text{for } x \in [0,\pi]. \end{cases}$$

Exercise 50. Solve

$$\begin{cases} u_t = 4u_{xx} & \text{for } 0 < x < \pi, t > 0, \\ u_x(0,t) = u_x(\pi,t) = 0 & \text{for } t > 0, \\ u(x,0) = \sum_{n\in\mathbb{N}} \frac{1}{n^3}\cos(nx) & \text{for } x \in [0,\pi]. \end{cases}$$

Exercise 51. Solve the following heat equation

$$\begin{cases} u_t + u = 2u_{xx} & \text{for } 0 < x < \pi, t > 0, \\ u_x(0,t) = u_x(\pi,t) = 0 & \text{for } t > 0, \\ u(x,0) = \sum_{n\in\mathbb{N}} \frac{1}{n^4}\cos(nx) & \text{for } x \in [0,\pi]. \end{cases}$$

Exercise 52. Solve

$$\begin{cases} u_{tt} = 9u_{xx} & \text{for } 0 < x < 1, t > 0, \\ u(0,t) = u(1,t) = 0 & \text{for } t > 0, \\ u(x,0) = 2\sin(\pi x) + 7\sin(3\pi x) & \text{for } x \in [0,1], \\ u_t(x,0) = 2\sin(\pi x) & \text{for } x \in [0,1]. \end{cases}$$

Exercise 53. Let $U = (0,\pi) \times (0,1) \subset \mathbb{R}^2$. A point in $\mathbb{R}^2$ is written as (x,y) with $x, y \in \mathbb{R}$. Solve the PDE with the unknown $u : U \to \mathbb{R}$

$$\begin{cases} -\Delta u = 0 & \text{in } U, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } (\{0\} \times [0,1]) \cap (\{\pi\} \times [0,1]), \\ u = 0 & \text{on } [0,\pi] \times \{0\}, \\ u = 7\cos x + 9\cos(2x) + 11\cos(10x) & \text{on } [0,\pi] \times \{1\}. \end{cases}$$

Here, $\mathbf{n}$ is the outward unit normal vector to the boundary of U.

Exercise 54. Consider the following eigenvalue problem with Robin boundary condition

$$\begin{cases} v''(x) + \lambda v(x) = 0 & \text{for } 0 < x < L, \\ v'(0) - a_0 v(0) = v'(L) + a_L v(L) = 0. \end{cases}$$

Here, $L > 0$ and $a_0, a_L \in \mathbb{R}$ are given constants.

Suppose $\lambda = 0$ is an eigenvalue.

(a) Find the corresponding eigenfunctions.

(b) Find a necessary condition on the coefficients a_0, a_L.

(c) Prove that this condition is also sufficient to guarantee that $\lambda = 0$ is an eigenvalue.

Exercise 55. Consider the following eigenvalue problem with Robin boundary condition

$$\begin{cases} v''(x) + \lambda v(x) = 0 & \text{for } 0 < x < L, \\ v'(0) - a_0 v(0) = v'(L) + a_L v(L) = 0. \end{cases}$$

Here, $L > 0$ and $a_0, a_L \in \mathbb{R}$ are given constants. Consider the absorbing case with $a_0, a_L < 0$. Show that this problem has at least one negative eigenvalue.

Exercise 56. This is a problem concerning Fourier series only.

(a) Calculate the Fourier series of the function f, which is odd and (2π)-periodic, with $f(x) = 1$ for $0 < x < \pi$.

(b) Assume that the Fourier series converges to f at $x = \pi/4$. Use this to calculate the sum

$$1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots$$

(Note: $L = \pi$ here.)

5.9. Notes and references

(1) We only present the usage of the separation of variables and Fourier series for some prototypical equations in this section. We do not discuss in deep about the convergence of Fourier series and related topics.

(2) The material on Tychonov's solution in Section 5.7 is based on the content in the book of John [**Joh91**, Chapter 7]. Without some appropriate conditions at infinity, the heat equation might have infinity many solutions. Once some growth conditions are imposed, we do have the desired uniqueness result. We refer the reader back to Theorem 3.44 and Theorem 3.45.

Chapter 6

First-order equations

We focus on first-order equations in this chapter. We first study general first-order equations using the method of characteristics. We will then study Burgers' equation and Hamilton–Jacobi equations in further details.

Definition 6.1. A first-order PDE is a PDE that involves the unknown and its first-order partial derivatives only. In particular, it does not involve second-order or higher-order partial derivatives.

6.1. Cauchy problems for first-order PDE

We focus mostly on Cauchy problems for first-order PDEs. The most general form is

$$\begin{cases} u_t(x,t) + F(x,t,u,Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases}$$

Here,

$$\begin{aligned} u\ &: \mathbb{R}^n \times [0,\infty) \implies \mathbb{R} \text{ is the unknown} \\ &(x,t) \mapsto u(x,t) \in \mathbb{R}, \end{aligned}$$

and

$$\begin{cases} u_t(x,t) = \frac{\partial u}{\partial t}(x,t), \\ Du(x,t) = \nabla_x u(x,t) = (u_{x_1}, \ldots, u_{x_n}). \end{cases}$$

The function

$$\begin{aligned} F\ &: \mathbb{R}^n \times [0,\infty) \times \mathbb{R} \times \mathbb{R}^n \implies \mathbb{R} \\ &(x,t,z,p) \mapsto F(x,t,z,p) \in \mathbb{R} \end{aligned}$$

is given. And $g : \mathbb{R}^n \implies \mathbb{R}$ is the given initial data.

6.1.1. Examples of first-order PDEs. We already encountered two examples of first-order PDEs:

6.1.1.1. *Transport equation.* Let $c \in \mathbb{R}$ be a fixed constant. Consider

$$\begin{cases} u_t + cu_x = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}. \end{cases}$$

6.1.1.2. *Generalized transport equation with variable speed.* For $c(x,t)$ given, consider

$$\begin{cases} u_t + c(x,t)u_x = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}. \end{cases}$$

Let us recall quickly the idea to study this equation, the go-with-the-flow viewpoint. Consider the ODE

$$\begin{cases} x'(t) = c(x(t),t) & \text{for } t > 0, \\ x(0) = x_0 \in \mathbb{R}. \end{cases}$$

Here, in this ODE, $x(t)$ is the unknown and $x_0 \in \mathbb{R}$ is given. See Figure 1.

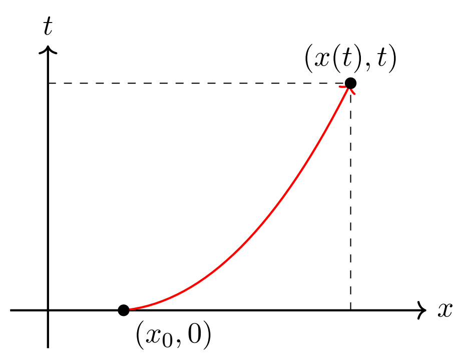

Figure 1. Illustration of a characteristic curve in (x,t)-plane

Let $\varphi(t) = u(x(t),t)$ for $t \geq 0$. Then

$$\begin{aligned} \varphi'(t) = \frac{d}{dt}(u(x(t),t)) &= u_t(x(t),t) + u_x(x(t),t) \cdot x'(t) \\ &= u_t(x(t),t) + c(x(t),t)u_x(x(t),t) = 0. \end{aligned}$$

This implies that $t \mapsto \varphi(t)$ is constant, and hence,

$$u(x(t),t) = u(x(0),0) = g(x_0).$$

6.1.2. The general case and the method of characteristics.

$$
\begin{cases} u_t(x,t) + F(x,t,u,Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{6.1}
$$

We will develop the characteristic method to study the problem, which shares the same philosophy with the go-with-the-flow viewpoint above.

Our goal is to follow a curve $(x(t),t)$ for $t \geq 0$ with $x(0) = x_0 \in \mathbb{R}^n$ fixed. We will write down some ODE for $x(t)$ and keep track of the PDE along $(x(t),t)$. See Figure 2.

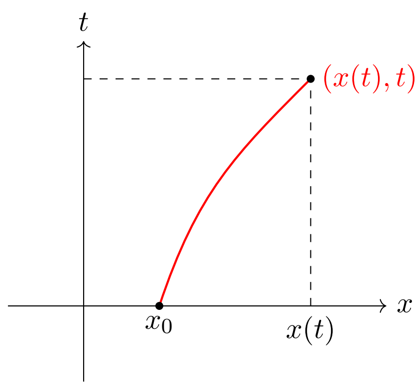

Figure 2. The curve $(x(t),t)$ from the characteristic method

For $t \geq 0$, denote by

$$
\begin{cases} z(t) = u(x(t),t), \\ p(t) = Du(x(t),t). \end{cases}
$$

Then we have

$$
\begin{cases} x(t): & \text{position at time } t, \\ z(t): & \text{value of } u \text{ at } (x(t),t), \\ p(t): & \text{spatial gradient of } u \text{ at } (x(t),t). \end{cases}
$$

We need to choose the ODE for $x(t)$ smartly so that we can keep track of $(x(t),z(t),p(t))$.

As $p_i(t) = u_{x_i}(x(t),t)$ for $1 \leq i \leq n$,

$$
p_i'(t) = \sum_{j=1}^{n} u_{x_i x_j}(x(t),t) x_j'(t) + u_{x_i t}(x(t),t).
$$

In other words,

$$
p_i'(t) = Du_{x_i}(x(t),t) \cdot x'(t) + u_{x_i t}(x(t),t).
$$

Recall the PDE

$$u_t + F(x, t, u, Du) = 0.$$

Differentiate the equation with respect to x_i to yield

$$u_{x_i t} + F_{x_i} + F_z u_{x_i} + D_p F \cdot D u_{x_i} = 0.$$

As $u_{x_i t}$ occurs in both equalities above, we want to match the Du_{x_i} term to omit it. We set

$$x'(t) = D_p F(x(t), t, z(t), p(t)).$$

Then,

$$p_i'(t) = u_{x_i t} + D u_{x_i} \cdot x'(t) = u_{x_i t} + D_p F \cdot D u_{x_i} = -F_{x_i} - F_z u_{x_i}.$$

Thus,

$$p'(t) = -D_x F(x(t), t, z(t), p(t)) - F_z(x(t), t, z(t), p(t)) p(t).$$

Finally, as $z(t) = u(x(t), t)$, we yield

$$\begin{aligned} z'(t) &= u_t(x(t), t) + Du(x(t), t) \cdot x'(t), \\ &= -F(x(t), t, z(t), p(t)) + p(t) \cdot D_p F(x(t), t, z(t), p(t)). \end{aligned}$$

Combining everything together, we get a system of ODE for $(x(t), z(t), p(t))$:

$$\begin{cases} x'(t) = D_p F(x(t), t, z(t), p(t)), \\ p'(t) = -D_x F(x(t), t, z(t), p(t)) - F_z(x(t), t, z(t), p(t)) p(t), \\ z'(t) = -F(x(t), t, z(t), p(t)) + p(t) \cdot D_p F(x(t), t, z(t), p(t)). \end{cases}$$

In a shortcut way, we can write that $(x(t), z(t), p(t))$ solves

$$\begin{cases} x'(t) = D_p F, \\ p'(t) = -D_x F - F_z p(t), \\ z'(t) = -F + p(t) \cdot D_p F. \end{cases} \tag{6.2}$$

At $t = 0$,

$$\begin{cases} x(0) = x_0, \\ z(0) = g(x_0), \\ p(0) = Dg(x_0). \end{cases}$$

Definition 6.2. We say that (6.2) are the characteristic equations for the first-order PDE (6.1). Each curve $(x(t), z(t), p(t))$ is called a characteristic in $\mathbb{R}^{2n+1}$. We also say that $(x(t), t)$ is a projected characteristic in $\mathbb{R}^{n+1}$.

The ODE system (6.2) consists of $2n + 1$ equations of $2n + 1$ unknowns (as $x(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}$, $p(t) \in \mathbb{R}^n$). The hope is that once we are able to solve the ODEs, we can track the value of u along the characteristics and hence solve the PDE. This process is called the method of characteristics.

Example 6.3. Back to the generalized transport PDE

$$\begin{cases} u_t + c(x,t)u_x = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}. \end{cases}$$

Here, $F(x,t,z,p) = c(x,t)p$ for $x,t,z,p \in \mathbb{R}$. Then, $F_z = 0$ as F does not depend on z. We see that

$$\begin{cases} F_p = D_pF = c(x,t), \\ F_x = D_xF = D_xc(x,t)p = c_x(x,t)p. \end{cases}$$

The characteristic equations are

$$\begin{cases} x'(t) = D_pF = c(x(t),t), \\ z'(t) = -F + p(t) \cdot D_pF = -c(x(t),t)p(t) + c(x(t),t)p(t) = 0, \\ p'(t) = -F_x - F_zp(t) = -c_x(x(t),t)p(t). \end{cases}$$

In particular, we get

$$\begin{cases} x'(t) = c(x(t),t) \\ z'(t) = 0 \end{cases} \implies u(x(t),t) = g(x_0),$$

which is consistent with the go-with-the-flow approach we did earlier. Actually, through the characteristic equations, we get more that

$$p'(t) = -c_x(x(t),t)p(t).$$

Example 6.4 (Burgers' equation)**.** Consider

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}. \end{cases}$$

We have discussed a bit about Burgers' equation earlier, and we will study much deeper about it.

Here, $F(x,t,z,p) = zp$ for $x,t,z,p \in \mathbb{R}$. Then,

$$\begin{cases} F_x \equiv 0, \\ F_z = p, \\ F_p = z. \end{cases}$$

The characteristic equations: $(x(t), z(t), p(t))$ solves

$$\begin{cases} x'(t) = D_pF = z(t), \\ p'(t) = -D_xF - F_zp(t) = -p(t)^2, \\ z'(t) = -F + p(t) \cdot D_pF = -z(t)p(t) + p(t)z(t) = 0. \end{cases}$$

Thus, $z(t) = z(0) = g(x_0)$ for all $t \geq 0$. The ODE system becomes

$$\begin{cases} x'(t) = z(0) = g(x_0), \\ z'(t) = z(0) = g(x_0), \\ p'(t) = -p(t)^2. \end{cases}$$

In particular, for $t \geq 0$,

$$\begin{cases} x(t) = x_0 + g(x_0)t, \\ u(x_0 + g(x_0)t, t) = g(x_0). \end{cases}$$

See Figure 3.

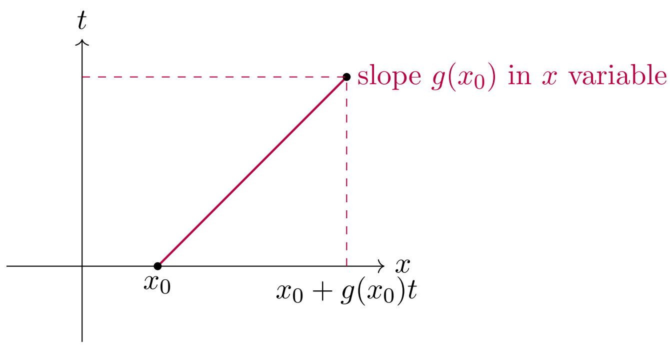

Figure 3. Projected characteristics are straight lines

6.2. Burgers' equation

Consider

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}. \end{cases} \tag{6.3}$$

Here, $g : \mathbb{R} \to \mathbb{R}$ is the given initial data. And $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ is the unknown. We will consider different cases.

6.2.1. Non-decreasing initial data. We assume in this section that g is C^1, bounded, and non-decreasing. This can also be rewritten as

$$\begin{cases} g \in C^1(\mathbb{R}),\ g'(x) \geq 0 \text{ for all } x \in \mathbb{R}, \\ -C \leq g(x) \leq C \text{ for some } C > 0,\ \text{for all } x \in \mathbb{R}. \end{cases} \tag{6.4}$$

As g is increasing, we can see, as in Figure 4, that the projected characteristics $(x(t), t)$ are well-ordered. More precisely, for $x_1 < x_2$, it is clear that $g(x_1) \leq g(x_2)$, and so

$$x_1 + g(x_1)t < x_2 + g(x_2)t \quad \text{for } t \geq 0.$$

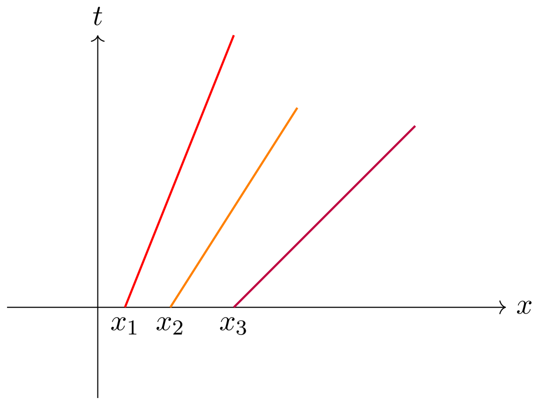

Figure 4. Ordered projected characteristics of Burgers' equation

As the projected characteristics are well-ordered and u is constant along each projected characteristic ray, everything is well-set, and the problem should have a unique solution. This is indeed the case.

Theorem 6.5. *Assume that g satisfies* (6.4). *Then, the Burgers equation* (6.3) *has a unique solution $u(x,t)$, which can be found by the method of characteristics. For every $x_0 \in \mathbb{R}$ and $t \geq 0$,*

$$u(x_0 + g(x_0)t, t) = u(x_0, 0) = g(x_0).$$

Proof. We have already seen that the projected characteristics are all well-ordered, and that, for $x_0 \in \mathbb{R}$,

$$u(x_0 + g(x_0)t, t) = u(x_0, 0) = g(x_0) \quad \text{for all } t \geq 0.$$

What is left to show here is only that: For any $(x,t) \in \mathbb{R} \times (0, \infty)$, it is reachable by one and only one projected characteristic.

Let us now use the fact that $-C \leq g(x) \leq C$ for all x. Pick

$$x_1 = x - Ct, \quad x_2 = x + Ct.$$

Clearly,

$$\begin{cases} x_1 + tg(x_1) = x - Ct + tg(x_1) \leq x - Ct + Ct = x, \\ x_2 + tg(x_2) = x + Ct + tg(x_2) \geq x + Ct - Ct = x. \end{cases}$$

Thus,

$$x_1 + tg(x_1) \leq x \leq x_2 + tg(x_2).$$

See Figure 5. By continuity of the map $x \mapsto x + tg(x)$, we can find $\bar{x} \in [x_1, x_2]$ such that

$$x = \bar{x} + tg(\bar{x}).$$

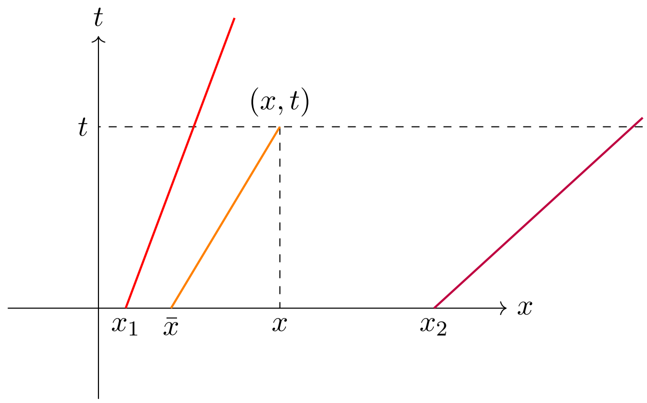

Figure 5. Finding a projected characteristic going through (x, t)

Thus, (x, t) belongs to the projected characteristic starting from $(\bar{x}, 0)$. As discussed, the projected characteristics are well-ordered, and hence, we get the desired claim. In particular,

$$u(x, t) = g(\bar{x}).$$

$\square$

6.2.2. What happens when g is not non-decreasing? Assume that $g \in C^1(\mathbb{R})$. We start with an example.

Example 6.6. We consider the case that $g(0) > g(1)$.

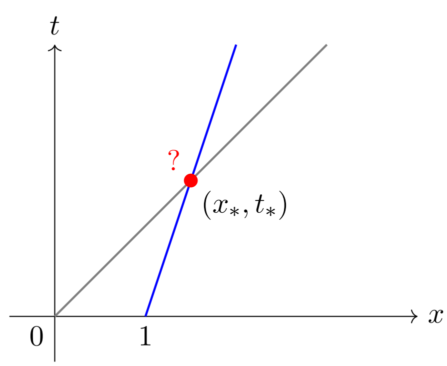

Figure 6. Projected characteristics cross each other

Then, we can see explicitly that the two projected characteristics starting from $(0, 0)$ and $(1, 0)$ cross each other. We can even find the crossing point

(x_*, t_*) explicitly. Indeed,

$$1 + g(1)t_* = g(0)t_* \qquad \Longrightarrow \qquad t_* = \frac{1}{g(0) - g(1)} > 0.$$

And

$$x_* = g(0)t_* = \frac{g(0)}{g(0) - g(1)}.$$

At (x_*, t_*), how do we assign $u(x_*, t_*)$? See Figure 6. If we follow each projected characteristic, we see that

$$u(x_*, t_*) = g(0) > g(1) = u(x_*, t_*).$$

We thus face real issues when projected characteristics cross each other as we do not know how to assign values to u at the crossing points.

From the above example, we can see that problems occur once projected characteristics cross each other. For $x_1 < x_2$, if $g(x_1) > g(x_2)$, then characteristics starting from $(x_1, 0)$ and $(x_2, 0)$ will meet at $(\bar{x}, \bar{t})$ where

$$\bar{x} + g(x_1)\bar{t} = x_2 + g(x_2)\bar{t},$$

which implies that

$$\bar{t} = \frac{x_2 - x_1}{g(x_1) - g(x_2)} = -\frac{x_2 - x_1}{g(x_2) - g(x_1)}.$$

By the mean value theorem, there exists $y \in (x_1, x_2)$ such that

$$g'(y) = \frac{g(x_2) - g(x_1)}{x_2 - x_1}.$$

Therefore,

$$\bar{t} = -\frac{1}{g'(y)} > 0 \quad \text{for some } y \in (x_1, x_2).$$

Besides, we have not looked into the behavior of

$$p(t) = u_x(x(t), t)$$

along the characteristics. Recall that p satisfies

$$\begin{cases} p'(t) = -p(t)^2, \\ p(0) = g'(y_0). \end{cases}$$

This ODE is known as a Riccati equation. Let us assume that $g'(y_0) < 0$. We now solve the Riccati equation

$$\frac{p'(t)}{p(t)^2} = -1 \qquad \Longrightarrow \qquad \left(\frac{1}{p(t)}\right)' = -\frac{p'(t)}{p(t)^2} = 1.$$

Thus,

$$\frac{1}{p(t)} = t + C \qquad \Longrightarrow \qquad p(t) = \frac{1}{t + C}.$$

At $t = 0$:

$$p(0) = \frac{1}{C} = g'(y_0) \implies C = \frac{1}{g'(y_0)}.$$

We arrive at

$$p(t) = u_x(x(t), t) = \frac{1}{\frac{1}{g'(y_0)} + t} = \frac{g'(y_0)}{1 + g'(y_0)t}.$$

This is well-defined for

$$t < \bar{t} = -\frac{1}{g'(y_0)},$$

and

$$\lim_{t \to \bar{t}^-} p(t) = \lim_{t \to \bar{t}^-} u_x(x(t), t) = -\infty.$$

In both analyses, it is all consistent that the solution is fine before the crossing of the projected characteristics, which can happen at time

$$\bar{t} = -\frac{1}{g'(y_0)}$$

for some $g'(y_0) < 0$.

This leads to the following theorem.

Theorem 6.7. *Assume that*

$$\begin{cases} g \in C^1(\mathbb{R}),\ g \text{ is bounded in } \mathbb{R}, \\ \inf_{x \in \mathbb{R}} g'(x) = \theta \le 0. \end{cases}$$

Let $T = -\frac{1}{\theta}$ (here, $T = \infty$ if $\theta = 0$). Then, the Burgers equation (6.3) *has a unique solution via the characteristic method $u(x, t)$ for $x \in \mathbb{R}$, $t \in (0, T)$.*

Moreover, if $T < \infty$, then projected characteristics cross at time T.

The proof of this theorem is already given in the discussions above.

Let us now consider the case that $T < \infty$. How can we define the solution for $t > T$? When the projected characteristics cross, how can we select one and only one value for u?

6.3. Rankine-Hugoniot condition for Burgers' equation

Recall the Burgers equation

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) = g(x) & \text{on } \mathbb{R}. \end{cases} \tag{6.5}$$

Here, $g : \mathbb{R} \to \mathbb{R}$ is the given initial data. And $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is the unknown. As discussed, we need to deal with the case where projected characteristics cross, and solutions might be discontinuous there.

6.3.1. Weak solutions. Let $v : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ be a smooth test function such that

$$\operatorname{supp}(v) = \{(x, t) : v(x, t) = 0\} \subset \mathbb{R} \times (0, \infty).$$

We say that $v \in C_c^\infty(\mathbb{R} \times (0, \infty))$. See Figure 7.

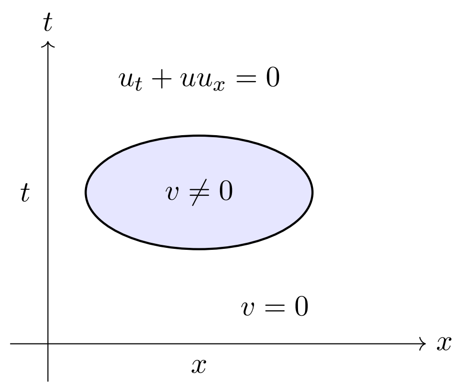

Figure 7. Test function v

Multiply (6.5) by v and integrate to yield:

$$\iint_{\mathbb{R}\times(0,\infty)} (u_t + \frac{1}{2}(u^2)_x)\, v\, dx\, dt = 0.$$

Integrate by parts and note that v has compact support, and $v = 0$ either at $t = 0$ or at infinity,

$$0 = \iint_{\mathbb{R}\times(0,\infty)} \left(u_t v + \frac{1}{2}(u^2)_x v \right) dx\, dt = - \iint_{\mathbb{R}\times(0,\infty)} \left(u\, v_t + \frac{1}{2} u^2 v_x \right) dx\, dt.$$

Thus, for any $v \in C_c^\infty(\mathbb{R} \times (0, \infty))$,

$$\iint_{\mathbb{R}\times(0,\infty)} \left(uv_t + \frac{1}{2} u^2 v_x \right) dx\, dt = 0.$$

Definition 6.8 (Weak solution)**.** We say that u is a weak solution to (6.5) if $u(x, 0) = g(x)$ on $\mathbb{R}$, and for any test function $v \in C_c^\infty(\mathbb{R} \times (0, \infty))$,

$$\iint_{\mathbb{R}\times(0,\infty)} \left(uv_t + \frac{1}{2} u^2 v_x \right) dx\, dt = 0.$$

6.3.2. Rankine-Hugoniot condition (R-H). Let

$$C = \{(s(t), t) : t \in [a, b]\} \subset \mathbb{R} \times [0, \infty)$$

be a shock curve. Consider an open region $V \subset \mathbb{R} \times (0, \infty)$. Let V_L be the part of V on the left side of C and V_R be the part of V on the right side

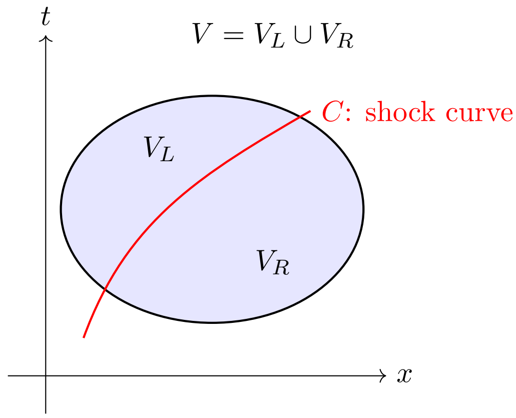

Figure 8. Shock curve C

of C. Assume that u, a weak solution to Burgers' equation, is smooth and nice in V_L and V_R, but discontinuous on C. See Figure 8.

Question. How do we find a compatibility condition on C, the shock curve? This is the main point that leads to the Rankine-Hugoniot condition.

Pick $v \in C_c^\infty(\mathbb{R} \times (0,\infty))$ so that $\operatorname{supp}(v) \subset V$. By the definition of weak solutions,

$$\begin{aligned} 0 &= \iint_{\mathbb{R}\times(0,\infty)} \left[uv_t + \frac{1}{2}u^2 v_x\right] dxdt = \iint_V \left[uv_t + \frac{1}{2}u^2 v_x\right] dxdt \\ &= \iint_{V_L} \left[uv_t + \frac{1}{2}u^2 v_x\right] dxdt + \iint_{V_R} \left[uv_t + \frac{1}{2}u^2 v_x\right] dxdt. \end{aligned} \tag{6.6}$$

Now, take v such that $\operatorname{supp}(v) \subset V_L$ to yield

$$\iint_{V_L} \left[uv_t + \frac{1}{2}u^2 v_x\right] dxdt = 0.$$

As u is smooth in V_L, integrate by parts again to imply

$$\iint_{V_L} \left(u_t v + \frac{1}{2}(u^2)_x v\right) dx\, dt = 0,$$

which gives us that

$$u_t + \frac{1}{2}(u^2)_x = u_t + uu_x = 0 \quad \text{ in } V_L.$$

Similarly,

$$u_t + \frac{1}{2}(u^2)_x = u_t + uu_x = 0 \quad \text{ in } V_R.$$

Again, the main trouble is that u is discontinuous across C.

Let $\mathbf{n}$ be the unit normal vector outward to V_L on C, and let

$$\mathbf{n} = (n_1, n_2).$$

By integration by parts:

$$\iint_{V_L} \left(uv_t + \frac{1}{2}u^2 v_x \right) \, dx\, dt$$
$$= \iint_{V_L} \left[-u_t v - \frac{1}{2}(u^2)_x v \right] \, dx\, dt + \int_C \left(\frac{1}{2}u_L^2 n_1 + u_L n_2 \right) v\, dl.$$

Here u_L denotes the value of u on $V_L \cap C$.

Thus,

$$\iint_{V_L} \left(uv_t + \frac{1}{2}u^2 v_x \right) \, dx\, dt = \int_C \left[\frac{1}{2}u_L^2 n_1 + u_L n_2 \right] v\, dl.$$

Similarly,

$$\iint_{V_R} \left(uv_t + \frac{1}{2}u^2 v_x \right) \, dx\, dt = -\int_C \left[\frac{1}{2}u_R^2 n_1 + u_R n_2 \right] v\, dl.$$

We note that the unit normal vector outward to V_R on C is $-\mathbf{n}$.

We combine the above two equalities and (6.6) to imply

$$\int_C \left[\frac{1}{2}(u_L^2 - u_R^2) n_1 + (u_L - u_R) n_2 \right] v\, dl = 0$$

for all test functions v.

Thus, we obtain the Rankine-Hugoniot (R-H) condition:

$$\frac{1}{2}(u_L^2 - u_R^2) n_1 + (u_L - u_R) n_2 = 0.$$

See Figure 9.

As $C = \{(s(t), t) : t \in (a, b)\}$, the tangent vector to C is $(s'(t), 1)$. So,

$$\mathbf{n} = \frac{(1, -s'(t))}{\sqrt{1 + (s'(t))^2}}.$$

We deduce that

$$\frac{1}{2}(u_L^2 - u_R^2) = s'(t)(u_L - u_R).$$

If $u_L \neq u_R$, then

$$s'(t) = \frac{1}{2}(u_L + u_R).$$

Definition 6.9 (Rankine-Hugoniot (R-H) condition)**.** Let

$$C = \{(s(t), t) : t \in [a, b]\} \subset \mathbb{R} \times [0, \infty)$$

be a shock curve. Consider an open region $V \subset \mathbb{R} \times (0, \infty)$. Let V_L be the part of V on the left side of C and V_R be the part of V on the right side

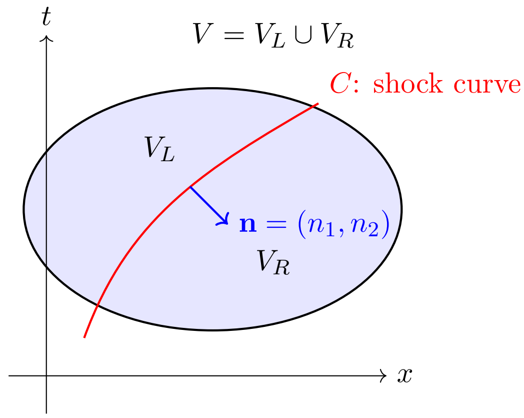

Figure 9. Rankine-Hugoniot (R-H) condition

of C. Assume that u, a weak solution to Burgers' equation, is smooth and nice in V_L and V_R, but discontinuous on C. Let u_L denote the value of u on $V_L \cap C$ and u_R denote the value of u on $V_R \cap C$. Then, along the curve C, $u_L \neq u_R$ and

$$s'(t) = \frac{1}{2}(u_L + u_R). \tag{6.7}$$

Equation (6.7) is called the Rankine-Hugoniot condition or R-H condition.

6.4. Shock waves and rarefaction waves for Burgers' equation

6.4.1. Shock waves. To start the discussion, we consider an example.

Example 6.10. Consider the initial data for the Burgers' PDE:

$$g(x) = \begin{cases} 1, & \text{if } x \leq 0, \\ 1 - x, & \text{if } 0 < x \leq 1, \\ 0, & \text{if } x > 1. \end{cases}$$

See Figure 10.

Clearly, g is non-increasing, so projected characteristics must cross after some finite time. Projected characteristics:

$$(x_0 + g(x_0)t, t) \quad \text{for } t \geq 0.$$

See Figure 11.

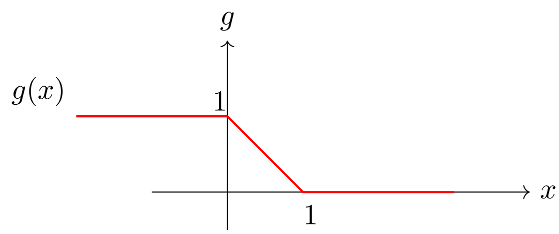

Figure 10. Graph of initial data g

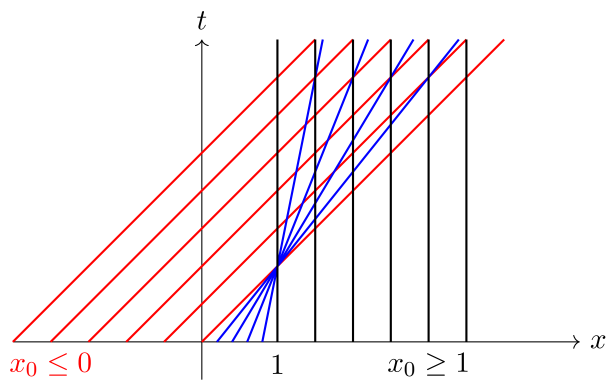

Figure 11. Projected characteristics

Thus, projected characteristics do not cross for $0 < t < 1$, and we can define the solution via the characteristic method here. We have

$$u(x,t) = \begin{cases} 1, & \text{if } x < t < 1, \\ 0, & \text{if } x > 1, \\ \frac{1-x}{1-t}, & \text{if } t \leq x \leq 1,\ 0 \leq t < 1. \end{cases}$$

At $t = 1$, we see that

$$u(x,1) = \begin{cases} 1, & \text{if } x < 1, \\ 0, & \text{if } x > 1. \end{cases}$$

This $u(x,1)$ is a shock. See Figure 12.

To continue defining the solution, we use the Rankine-Hugoniot condition. Shock curve:

$$C = \{(s(t),t) : t \geq 1\}.$$

As $u_L = 1$, $u_R = 0$, by the Rankine-Hugoniot condition, we see that:

$$s'(t) = \frac{1}{2}(u_L + u_R) = \frac{1}{2}$$

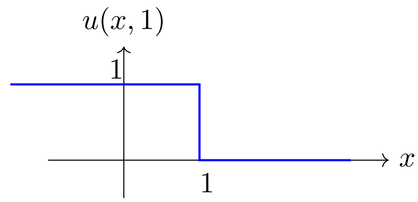

Figure 12. Graph of $u(x,1)$

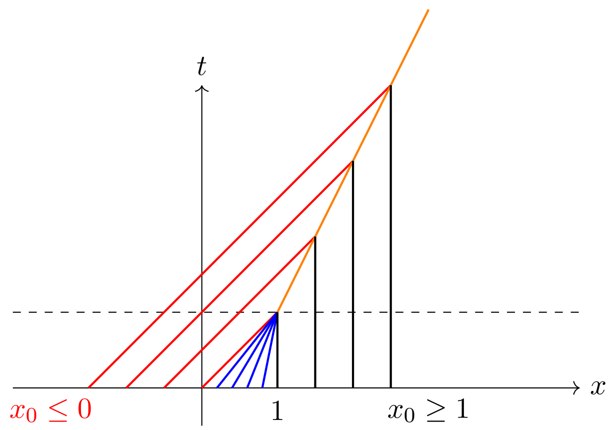

Figure 13. Projected characteristics and shock curve

and

$$s(1) = 1.$$

Thus,

$$s(t) = 1 + \frac{1}{2}(t-1) = \frac{1}{2}(t+1) \quad \text{for } t \geq 1.$$

Therefore, for $t \geq 1$,

$$u(x,t) = \begin{cases} 1 & \text{if } x < \frac{1}{2}(t+1),\ t \geq 1, \\ 0 & \text{if } x > \frac{1}{2}(t+1),\ t \geq 1. \end{cases}$$

See Figure 13.

The above example suggests that we should look into initial data of the form

$$u(x,0) = g(x) = \begin{cases} u_L & \text{for } x < 0, \\ u_R & \text{for } x > 0. \end{cases}$$

We assume $u_L \neq u_R$. The problem with initial data like this is called the Riemann problem. Naturally, there are two cases to be considered: either $u_L > u_R$ or $u_L < u_R$.

We consider first the case $u_L > u_R$. See Figure 14. Then g is decreasing, and the initial data g corresponds to a shock wave, which is similar to the example above when $t \geq 1$.

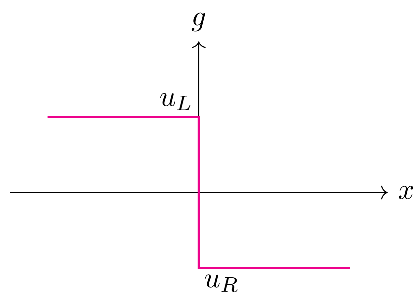

Figure 14. Graph of initial data g

Here, the shock curve is

$$C = \{(s(t), t) : t \geq 0\}.$$

By the Rankine-Hugoniot condition,

$$s'(t) = \frac{1}{2}(u_L + u_R) \quad \Longrightarrow \quad s(t) = \frac{1}{2}(u_L + u_R)t.$$

Therefore, we can conclude that:

$$u(x,t) = \begin{cases} u_L & \text{for } x < s(t) = \frac{1}{2}(u_L + u_R)t, \\ u_R & \text{for } x > s(t) = \frac{1}{2}(u_L + u_R)t. \end{cases}$$

See Figure 15.

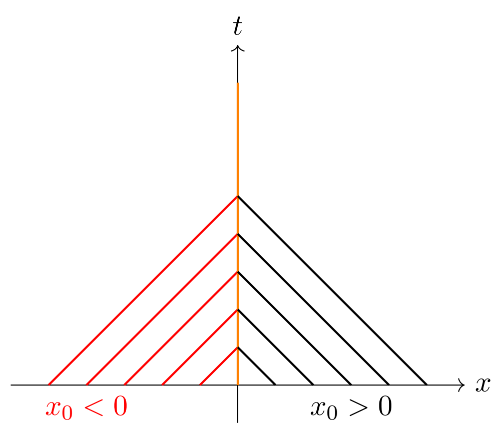

Figure 15. Projected characteristics and shock curve with $u_L = -u_R$

6.4.2. Rarefaction waves. Next, we consider the case $u_L < u_R$, then g is increasing. See Figure 16. As we discussed, we expect that there should not be any shocks (projected characteristics crossings) in this case. We consider an explicit example to study this case first.

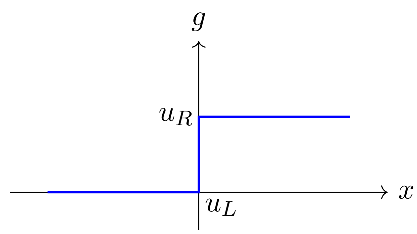

Figure 16. Graph of initial data g

Example 6.11. We consider the case where

$$u_L = 0 < u_R = 1.$$

Then, by the method of characteristics, we see the following.

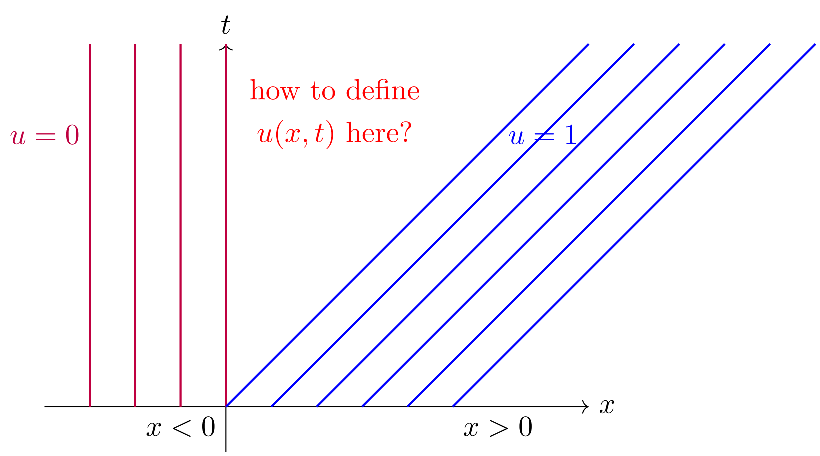

Figure 17. Region untouched by the projected characteristics

There is a region $\{(x,t) : 0 < x < t\}$ that is not touched by projected characteristics (see Figure 17), and we need to find a way to define $u(x,t)$ there.

Naive guess (WRONG one – non-physical solution). We artificially create a shock in this region following the Rankine-Hugoniot condition:

$$\begin{cases} s'(t) = \frac{1}{2}(u_L + u_R) = \frac{1}{2}, \\ s(0) = 0. \end{cases}$$

Then $s(t) = t/2$. This gives us the shock line:

$$C = \left\{ \left(\frac{t}{2}, t \right) : t > 0 \right\}.$$

See Figure 18.

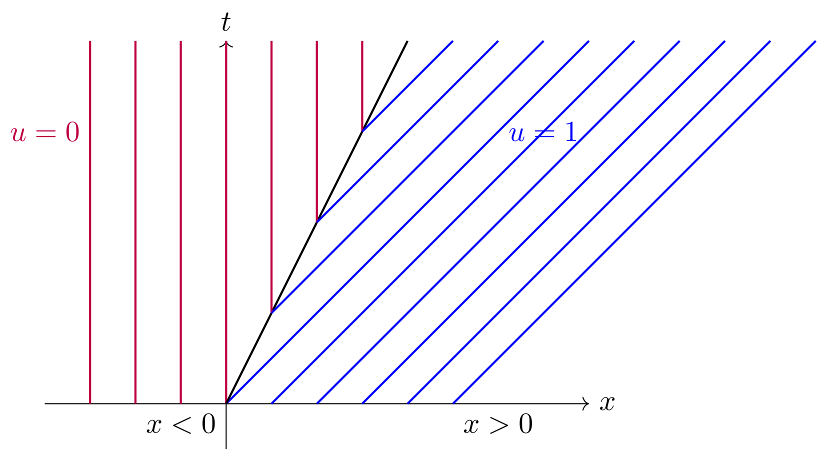

Figure 18. Non-physical shock

In this way, we have created a solution that satisfies the Rankine-Hugoniot condition. But there are two issues:

- There is still a shock, and this should not happen.
- Characteristics should not come out of a shock. It should be only physical if characteristics run into each other to form shocks, not the other way around.

Second guess – Rarefaction Waves. Somehow, we want to connect $u = 0$ to $u = 1$ nicely.

We search for special solutions that are constant along each line $x = \alpha t$ for $\alpha \in [0, 1]$. See Figure 19. This means that we want to find solutions of the form:

$$u(x,t) = v\left(\frac{x}{t}\right) \quad \text{for } y = \frac{x}{t} \in [0,1].$$

Then, we can compute:

$$u_t = v'\left(\frac{x}{t}\right)\left(\frac{x}{t}\right)_t = v'\left(\frac{x}{t}\right) \cdot \frac{-x}{t^2},$$
$$u_x = v'\left(\frac{x}{t}\right)\left(\frac{x}{t}\right)_x = v'\left(\frac{x}{t}\right) \cdot \frac{1}{t}.$$

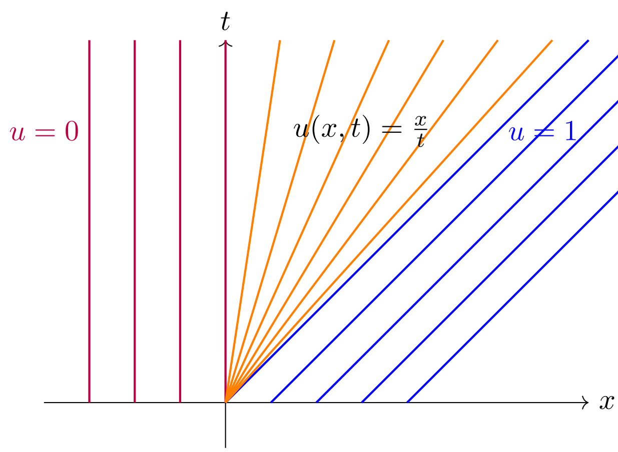

Figure 19. Rarefaction waves

Thus,

$$0 = u_t + uu_x = v'\left(\frac{x}{t}\right) \cdot \frac{-x}{t^2} + v'\left(\frac{x}{t}\right) \cdot v \cdot \frac{1}{t},$$
$$= v'\left(\frac{x}{t}\right) \cdot \frac{1}{t}\left(-\frac{x}{t} + v\right).$$

We conclude that, in the region $\{(x,t) : 0 < x < t\}$,

$$u(x,t) = v\left(\frac{x}{t}\right) = \frac{x}{t}.$$

We see that u is now continuous in $\mathbb{R} \times (0,\infty)$, there is no shock, and u satisfies the Burgers' PDE. We say that the part $u(x,t) = \frac{x}{t}$ is the *rarefaction wave*.

We now consider the general case $u_L < u_R$. Recall that

$$g(x) = \begin{cases} u_L & \text{for } x < 0, \\ u_R & \text{for } x > 0. \end{cases}$$

By following the same arguments as these of the example above, we see that the correct solution to our Burgers' PDE is

$$u(x,t) = \begin{cases} u_L & \text{for } x \le u_L t,\ t > 0, \\ u_R & \text{for } x > u_R t,\ t > 0, \\ \frac{x}{t} & \text{for } u_L t \le x \le u_R t,\ t > 0. \end{cases}$$

See Figure 20.

6.4.3. Entropy solution to Burgers' equation. Finally, we give a rigorous definition of what we mean by a physically correct solution.

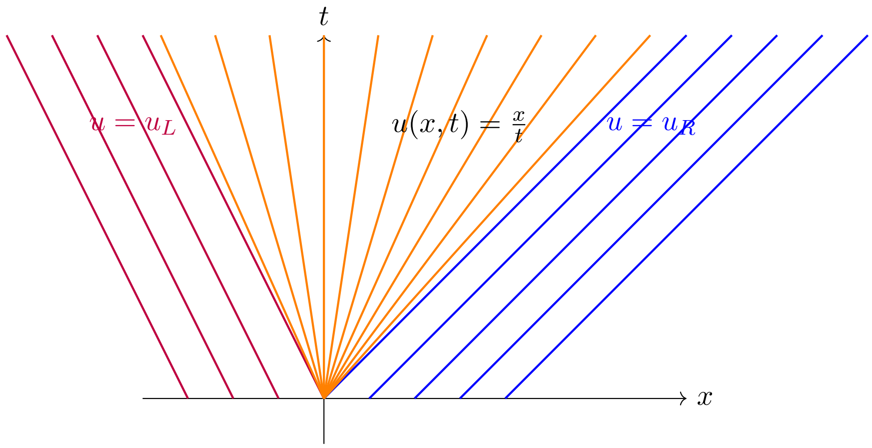

Figure 20. Rarefaction waves with general $u_L < u_R$

Definition 6.12 (Entropy solution to Burgers' equation)**.** We say that u is an entropy solution to the Burgers equation if, along any shock, $u_L > u_R$.

It turns out that the Burgers equation (6.5) admits a unique entropy solution, that is, we have the wellposedness theory for entropy solutions to (6.5). The proof of this claim however is outside of the scope of this book and is omitted.

6.5. Hamilton–Jacobi equations

We focus on the following PDE:

$$\begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = g(x) & \text{on } \mathbb{R}^n. \end{cases}$$

Here, $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the given Hamiltonian:

$$(x, p) \mapsto H(x, p) \in \mathbb{R}.$$

And $g : \mathbb{R}^n \to \mathbb{R}$ is given initial data. Surely, $u = u(x,t) : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ is the unknown. Recall that

$$u_t = \frac{\partial u}{\partial t}, \quad Du = D_x u = \nabla_x u = (u_{x_1}, \ldots, u_{x_n}).$$

This is a special case of the general first-order PDE that we considered earlier with

$$F(x, t, z, p) = H(x, p).$$

6.5.1. Motivation – A front propagation problem. Let Γ_0 be a given smooth closed hypersurface in $\mathbb{R}^n$. We want to study the evolution of Γ_0 with respect to time under a law of motion. And the law of motion is that, for $t \geq 0$, at each point $x \in \Gamma_t$, it is moving inward with normal velocity $V(x) = a(x)\mathbf{n}$. Here, $\mathbf{n}$ is the unit inward normal vector to the region enclosed by Γ_t at x. See Figure 21.

Question. Describe the motion of $\{\Gamma_t\}_{t\geq 0}$?

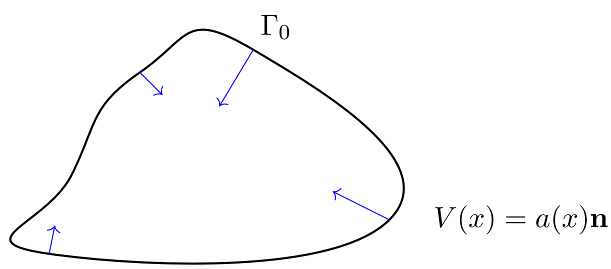

Figure 21. Γ_0 and the inward normal velocity

Example 6.13. Here is a specific example where

$$\begin{cases} a(x) = 1 \quad \text{for all } x \in \mathbb{R}, \\ \Gamma_0 = \partial B(0,1). \end{cases}$$

It's clear that $\Gamma_t = \partial B(0, 1-t)$ for $0 \leq t < 1$, and at $t = 1$, it shrinks to a point at the origin, and for $t > 1$, there is nothing left. See Figure 22.

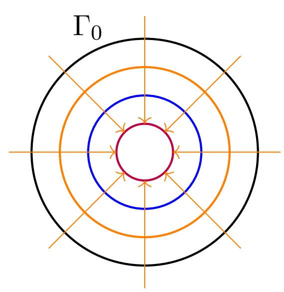

Figure 22. Motion of concentric spheres

For the general setting, it is much more complicated. We introduce a new method, *the level set method*. Let us assume that Γ_t is the 0-level set of a function $u(x,t)$ for $t \geq 0$, that is,

$$\Gamma_t = \{x \in \mathbb{R}^n : u(x,t) = 0\}.$$

Assume that $u(x,t) > 0$ in the region enclosed by Γ_t and $u(x,t) < 0$ elsewhere. Assume everything is nice. See Figure 23.

At $x \in \Gamma_t$, as Γ_t is the 0-level set of $u(\cdot,t)$, we can see that

$$\mathbf{n}(x) = \frac{Du(x,t)}{|Du(x,t)|}.$$

Note that $Du(x,t)$ points inward to the region enclosed by Γ_t in light of the steepest gradient descent.

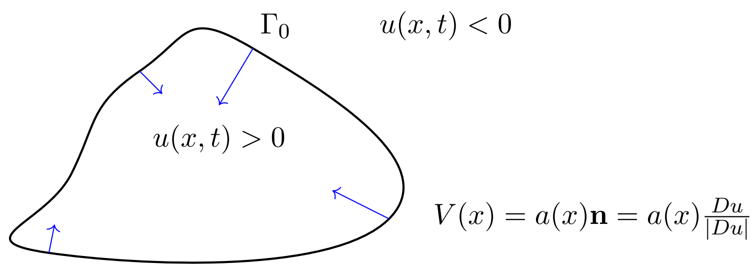

Figure 23. 0-level set of u

Now, for a given point $x(0) \in \Gamma_0$, we keep track with its position through the evolution. Let $x(t) \in \Gamma_t$ be its position at time $t \geq 0$. Firstly,

$$x'(t) = a(x(t))\mathbf{n}(x(t)) = a(x(t))\frac{Du(x(t),t)}{|Du(x(t),t)|}.$$

Besides, as $x(t) \in \Gamma_t$, $u(x(t),t) = 0$. Differentiate this with respect to t to yield

$$\frac{d}{dt}\left(u(x(t),t)\right) = u_t(x(t),t) + Du(x(t),t) \cdot x'(t) = 0.$$

We use the formula of $x'(t)$ to imply that

$$u_t(x(t),t) + a(x(t))|Du(x(t),t)| = 0.$$

Thus, we end up with a PDE

$$u_t + a(x(t))|Du| = 0 \quad \text{ in } \mathbb{R}^n \times (0,\infty).$$

This is a Hamilton–Jacobi equation with $H(x,p) = a(x)|p|$. This particular equation comes from front propagation as described above. It is also called an eikonal equation.

6.5.2. Method of characteristics for Hamilton–Jacobi equations. Recall that

$$F(x,t,z,p) = H(x,p).$$

We use the method of characteristics to study this case with

$$\begin{cases} x(t): & \text{position at time } t, \\ z(t): & \text{value of } u \text{ at } (x(t),t), \\ p(t): & \text{spatial gradient of } u \text{ at } (x(t),t). \end{cases}$$

See Figure 24.

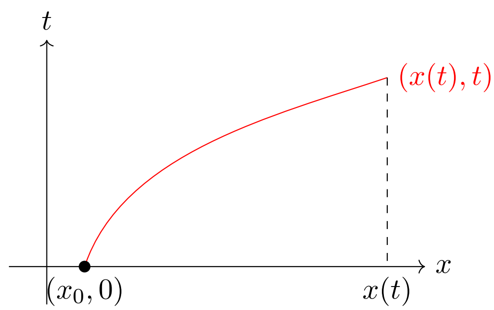

Figure 24. Projected characteristics for Hamilton–Jacobi equation

By the earlier computations, we have the following system of ODE:

$$\begin{cases} x'(t) = D_pH(x(t),p(t)), \\ p'(t) = -D_xH(x(t),p(t)), \\ z'(t) = p(t)\cdot D_pH(x(t),p(t)) - H(x(t),p(t)). \end{cases}$$

The initial data is

$$\begin{cases} x(0) = x_0, \\ p(0) = Dg(x_0), \\ z(0) = g(x_0). \end{cases}$$

Once we can solve this system of ODE, then we can solve the Hamilton–Jacobi PDE at least locally. Let us revisit the front propagation problem discussed earlier.

Example 6.14. For the front propagation problem, $H(x,p) = a(x)|p|$. Hence,

$$\begin{cases} D_xH(x,p) = |p|Da(x), \\ D_pH(x,p) = a(x)\frac{p}{|p|} \quad \text{for } p \neq 0. \end{cases}$$

We therefore have

$$\begin{cases} x'(t) = a(x(t))\frac{p(t)}{|p(t)|}, \\ p'(t) = -|p(t)|Da(x(t)), \\ z'(t) = p(t) \cdot \left(a(x(t))\frac{p(t)}{|p(t)|}\right) - a(x(t))|p(t)| = 0. \end{cases}$$

Hence,

$$\begin{cases} x'(t) = a(x(t))\frac{p(t)}{|p(t)|}, \\ p'(t) = -|p(t)|Da(x(t)), \\ z(t) = z(0). \end{cases}$$

As $p(t) = Du(x(t), t)$, we recover exactly that

$$x'(t) = a(x(t))\frac{Du(x(t), t)}{|Du(x(t), t)|}.$$

Further, $z(t) = z(0)$ means that $x(t)$ stays at the same 0-level set of u if we normalize it so that $z(0) = 0$.

6.6. Hamiltonian ODE

A system of ODE for $(x(t), p(t))$ from the method of characteristics is called a Hamiltonian ODE or a Hamiltonian system. This system plays an important role in classical mechanics, Hamiltonian dynamics, and symplectic geometry. We have

$$\begin{cases} x'(t) = D_pH(x(t), p(t)), \\ p'(t) = -D_xH(x(t), p(t)), \end{cases} \tag{6.8}$$

where $x(t) \in \mathbb{R}^n$ and $p(t) \in \mathbb{R}^n$.

In the context of classical mechanics, H is called the Hamiltonian and is the total energy.

Lemma 6.15 (Conservation of energy). *Let $(x(t), p(t))$ be a solution to (6.8). We have that $t \mapsto H(x(t), p(t))$ is constant.*

Proof. This is quite easy to show:

$$\begin{aligned} \frac{d}{dt}H(x(t), p(t)) &= D_xH(x(t), p(t)) \cdot x'(t) + D_pH(x(t), p(t)) \cdot p'(t) \\ &= -p' \cdot x' + x' \cdot p' = 0, \end{aligned}$$

where we used (6.8) in the last equality. Therefore, $t \mapsto H(x(t), p(t))$ is constant.

In other words, the path $(x(t), p(t))$ stays on a fixed energy level of H for all time. □

Example 6.16. Let $n = 1$ and
$$H(x,p) = \frac{p^2}{2} + \frac{x^2}{2} \quad \text{for } (x,p) \in \mathbb{R} \times \mathbb{R}.$$
Then
$$D_x H(x,p) = x, \quad D_p H(x,p) = p.$$
Our Hamiltonian system is
$$\begin{cases} x'(t) = 2p, \\ p'(t) = -2x. \end{cases}$$
Pick $(x(0), p(0)) = (-1, 0)$, then by conservation of energy,
$$x^2(t) + p(t)^2 = x(0)^2 + p(0)^2 = 1 \quad \text{for all } t \geq 0.$$
See Figure 25.

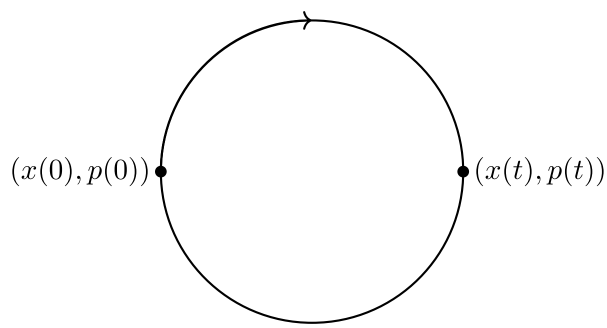

Figure 25. Orbit of $(x(t), p(t))$

Example 6.17 (Classical mechanics)**.** Consider a particle with unit mass, that is, $m = 1$. Let v be the particle's velocity, then
$$p = mv = v.$$
Consider the Hamiltonian
$$\begin{aligned} H(x,p) &= \text{kinetic energy} + \text{potential energy} \\ &= \frac{1}{2} m |v|^2 + V(x). \end{aligned}$$
Since $p = v$, we have
$$H(x,p) = \frac{1}{2}|p|^2 + V(x).$$
Then $D_x H(x,p) = DV(x)$ and $D_p H(x,p) = p$. The Hamiltonian ODE system is
$$\begin{cases} x'(t) = p(t), \\ p'(t) = -DV(x(t)). \end{cases}$$

In particular, we yield, along each trajectory $(x(t), p(t))$,

(1) conservation of total energy
$$\frac{1}{2}x'(t)^2 + V(x(t)) = \text{constant};$$
(2) $x''(t) = p'(t) = -DV(x(t))$.

As the potential energy is V, by the steepest gradient descent method, we see that the force field is
$$\mathbf{F}(x(t)) = -DV(x(t)).$$
Therefore,
$$\mathbf{F}(x(t)) = x''(t) = mx''(t) = ma(t),$$
which is exactly Newton's second law of motion.

When $V \equiv 0$, that is, the potential energy is zero, then $H(x,p) = \frac{1}{2}|p|^2$. We see that
$$\begin{cases} x' = p, \\ p' = 0. \end{cases}$$
We will discuss this in detail in the next section.

6.7. Spatially homogeneous Hamiltonians

6.7.1. Method of characteristics. If H is independent of x, we say that H is spatially homogeneous. In this section, we always assume that H is spatially homogeneous, and we write $H = H(p)$. Our Hamiltonian system is
$$\begin{cases} x'(t) = D_pH(p(t)) = DH(p(t)), \\ p'(t) = 0 \quad (\text{as } H \text{ is independent of } x). \end{cases}$$
The initial data is
$$\begin{cases} x(0) = x_0, \\ p(0) = p_0 = Dg(x_0), \\ z(0) = g(x_0). \end{cases}$$
We have that
$$\begin{cases} x(t) = x_0 + DH(p_0)t, \\ p(t) = p_0 \quad \text{for all } t \geq 0. \end{cases}$$
Thus, we achieve that
$$\begin{cases} p(t) = p_0 = Dg(x_0) \quad \text{for all } t \geq 0, \\ x(t) = x_0 + tDH(p_0), \\ z(t) = g(x_0) + t\left(p_0 \cdot DH(p_0) - H(p_0)\right). \end{cases}$$

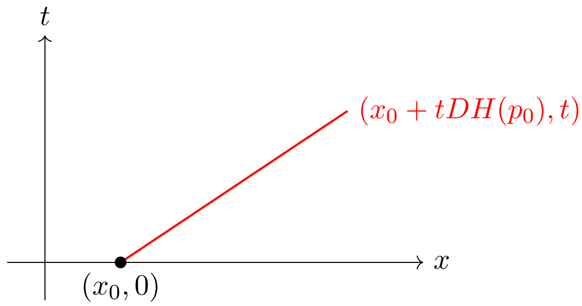

Figure 26. Projected characteristics when $H = H(p)$

It is intersecting that the projected characteristics are again straight lines (see Figure 26), and before they cross each other, we have

$$Du(x(t), t) = Dg(x_0),$$

which means that the spatial gradient of u is constant along characteristics.

Remark 6.18. As usual, in light of the projected characteristics, solutions can be found in short time before the projected characteristics cross each other.

6.7.2. Local wellposedness theory. Let us now discuss a local wellposedness result. We need some results on the inverse mapping theorem and the implicit mapping theorem.

Theorem 6.19 (Inverse mapping theorem)**.** *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map. Assume F is proper, that is, $F^{-1}(K)$ is compact if K is compact. Assume further that $\det DF(x) \neq 0$ for all $x \in \mathbb{R}^n$. Then F is a global C^1 diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$.*

Applying the inverse mapping theorem, we obtain the following implicit mapping theorem.

Theorem 6.20 (Implicit mapping theorem)**.** *Let $\Phi : \mathbb{R}^n \times [a, b] \to \mathbb{R}^n$ be a C^1 map. Suppose that there exists $M > 0$ so that*

$$|\Phi(z, t) - z| \leq M \quad \text{for all } (z, t) \in \mathbb{R}^n \times [a, b]. \tag{6.9}$$

Suppose also that $\det D_z\Phi(z, t) \neq 0$ for all $(z, t) \in \mathbb{R}^n \times [a, b]$. Then, there exists an image map $\Psi : \mathbb{R}^n \times [a, b] \to \mathbb{R}^n$ of class C^1 such that

$$\Phi(\Psi(x, t), t) = x \quad \text{for all } (x, t) \in \mathbb{R}^n \times [a, b].$$

Proof. For $t \in [a, b]$ fixed, let $F(z) = \Phi(z, t)$. Then, $F \in C^1(\mathbb{R}^n)$, and $\det DF(z) = \det D_zF(z, t) \neq 0$ for all $z \in \mathbb{R}^n$. We now show that F

is proper. For an arbitrary set K compact, $F^{-1}(K)$ is closed. By (6.9), $F^{-1}(K)$ is bounded. Thus, $F^{-1}(K)$ is compact, and F is proper.

By Theorem 6.19, F has its C^1 global inverse, that is, there exists a unique map $\Psi : \mathbb{R}^n \times [a,b] \to \mathbb{R}^n$ such that $\Psi(\cdot,t) \in C^1(\mathbb{R}^n)$ for each fixed $t \in [a,b]$, and

$$\Phi(\Psi(x,t),t) = x \quad \text{for all } (x,t) \in \mathbb{R}^n \times [a,b].$$

We still need to show Ψ is C^1 in both variables. Define

$$\widetilde{\Phi}(x,t) = (\Phi(x,t),t), \quad \widetilde{\Psi}(x,t) = (\Psi(x,t),t) \quad \text{for all } (x,t) \in \mathbb{R}^n \times [a,b].$$

Then $\widetilde{\Phi} \circ \widetilde{\Psi} = \mathrm{Id}$, the identity map, and

$$D\widetilde{\Phi}(x,t) = \begin{pmatrix} D_x\Phi(x,t) & \Phi_t(x,t) \\ 0 & 1 \end{pmatrix}.$$

As $\det D\widetilde{\Phi}(x,t) = \det D_x\Phi(x,t) \neq 0$, we use the inverse mapping theorem once more to imply that $\widetilde{\Psi} \in C^1$. Hence, $\Psi \in C^1$. $\square$

We are ready to prove a local existence result for the following Hamilton-Jacobi equation

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{6.10}$$

Theorem 6.21. *Let $H, g \in C^2(\mathbb{R}^n$ be given functions. Suppose that Dg, D^2g are bounded. Let*

$$T = \sup\left\{t \geq 0 \,:\, I_n + tD^2H(Dg(z))D^2g(z) \text{ is invertible for all } z \in \mathbb{R}^n\right\}.$$

Here, I_n is the identity matrix of size n. Then, (6.10) *has a unique solution $u \in C^1(\mathbb{R}^n \times [0,T))$.*

Sketch of the proof. By method of characteristics, we can define

$$X(z,t) = z + tDH(Dg(z)).$$

See Figure 27.

We have

$$D_zX(z,t) = I_n + tD^2H(Dg(z))D^2g(z).$$

By the definition of T, we see that

$$\det D_zX(z,t) \neq 0 \quad \text{for all } (z,t) \in \mathbb{R}^n \times [0,T).$$

By Theorem 6.20, we can find a unique map $Z \in C^1$ such that

$$X(Z(z,t),t) = z \quad \text{for all } (z,t) \in \mathbb{R}^n \times [0,T).$$

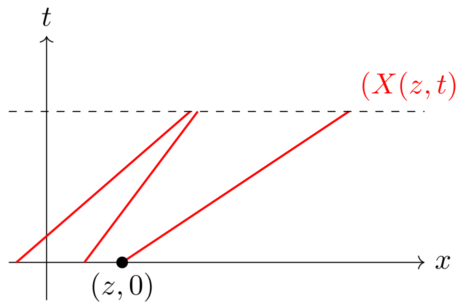

Figure 27. $X(z,t)$ when $H = H(p)$

Thus, projected characteristics do not cross for $t \in [0,T)$, and we have further that

$$u(x,t) = g(Z(x,t)) + t\left[DH(Dg(Z(x,t))) \cdot Dg(Z(x,t)) - H(Dg(Z(x,t)))\right].$$

We conclude that (6.10) has a unique solution $u \in C^1(\mathbb{R}^n \times [0,T))$. $\square$

Remark 6.22. Note that $\det D_z X(z,t) \neq 0$ gives a geometric picture that

$$X(z,t) \neq X(y,t) \quad \text{for } z \neq y.$$

Finally, we discuss a special case where H, g are convex.

Corollary 6.23. *Let $H, g \in C^2(\mathbb{R}^n$ be given functions. Suppose that Dg, D^2g are bounded. If H, g are convex, then the unique solution u to* (6.10) *is globally defined.*

Proof. Indeed, if H, g are convex, then $D^2H, D^2g \geq 0$, that is, D^2H, D^2g are nonnegative definite symmetric matrices. Therefore, for all $(z,t) \in \mathbb{R}^n \times [0,\infty)$,

$$tD^2H(Dg(z))D^2g(z) \geq 0,$$

and

$$I_n + tD^2H(Dg(z))D^2g(z) > 0.$$

Hence, $T = +\infty$ in Theorem 6.21. The proof is finished.

$\square$

6.8. Hopf–Lax formula for a quadratic Hamilton–Jacobi equation

Let us state right away the main theorem of this section.

Theorem 6.24. *Consider the Hamilton–Jacobi equation*

$$\begin{cases} u_t + \frac{|Du|^2}{2} = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{6.11}$$

Assume that $g \in C(\mathbb{R}^n)$ with $-C \le g \le C$ for some $C > 0$. Denote by

$$u(x,t) = \min_y \left\{ g(y) + \frac{|x-y|^2}{2t} \right\} \qquad \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty) \tag{6.12}$$

Then u is differentiable almost everywhere and u solves (6.11) *almost everywhere.*

We say that u in (6.12) is the Hopf–Lax solution to our Hamilton–Jacobi equation. Here, the Hamiltonian is $H(x,p) = \frac{|p|^2}{2}$. We say that H is a quadratic Hamiltonian.

We note that we have not touched measure theory in this book and we have not defined the meaning of the"almost everywhere sense". Therefore, we will only give a sketch of the proof of this important theorem.

6.8.1. Discussions. For each fixed y, consider

$$\varphi^y(x,t) := g(y) + \frac{|x-y|^2}{2t}, \quad x \in \mathbb{R}^n.$$

Then

$$D\varphi^y(x,t) = \frac{x-y}{t}, \quad \varphi^y_t(x,t) = -\frac{|x-y|^2}{2t^2}.$$

We see that

$$\varphi^y_t + \frac{|D\varphi^y|^2}{2} = 0,$$

which means that φ^y is a particular solution to (6.11).

We now assume that

$$\min_y \varphi^y(x,t) = \min_y \left\{ g(y) + \frac{|x-y|^2}{2t} \right\} = g(y_{x,t}) + \frac{|x-y_{x,t}|^2}{2t}$$

for a point $y_{x,t} \in \mathbb{R}^n$. Then we have, at the minimum point $y_{x,t}$,

$$Dg(y_{x,t}) = \frac{x-y_{x,t}}{t},$$

which is the slope of the line segment connecting $(y_{x,t},0)$ to (x,t). By plugging this into the formula of u, we see that

$$\begin{aligned} u(x,t) &= g(y_{x,t}) + \frac{|x-y_{x,t}|^2}{2t} \\ &= g(y_{x,t}) + t\left(\frac{x-y_{x,t}}{t} \cdot \frac{x-y_{x,t}}{t} - \frac{|x-y_{x,t}|^2}{2t^2} \right) \\ &= g(y_{x,t}) + t\left(Dg(y_{x,t}) \cdot DH(Dg(y_{x,t})) - H(Dg(y_{x,t})) \right). \end{aligned}$$

This is essentially the formula arising from the characteristic method.

6.8.2. Sketch of the proof of Theorem 6.24. We have the following lemma.

Lemma 6.25 (Dynamic Programming Principle)**.** *Let u be defined as in Theorem 6.24. Then, for any $0 < s < t$ and $x \in \mathbb{R}^n$,*

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left(u(y,s) + \frac{|x-y|^2}{2(t-s)} \right).$$

Proof. For $y \in \mathbb{R}^n$, assume that

$$u(y,s) = g(z) + \frac{|y-z|^2}{2s}$$

for some $z \in \mathbb{R}^n$. Then,

$$\begin{aligned} u(y,s) + \frac{|x-y|^2}{2(t-s)} &= g(z) + \frac{|y-z|^2}{2s} + \frac{|x-y|^2}{2(t-s)} \\ &= g(z) + t\left(\frac{s}{t}\frac{|y-z|^2}{2s^2} + \frac{t-s}{t}\frac{|x-y|^2}{2(t-s)^2} \right) \\ &\geq g(z) + t\frac{|x-z|^2}{2t^2} = g(z) + \frac{|x-z|^2}{2t}. \end{aligned}$$

We use Jensen's inequality in the last inequality above. We hence get

$$u(x,t) \leq \min_{y \in \mathbb{R}^n} \left(u(y,s) + \frac{|x-y|^2}{2(t-s)} \right).$$

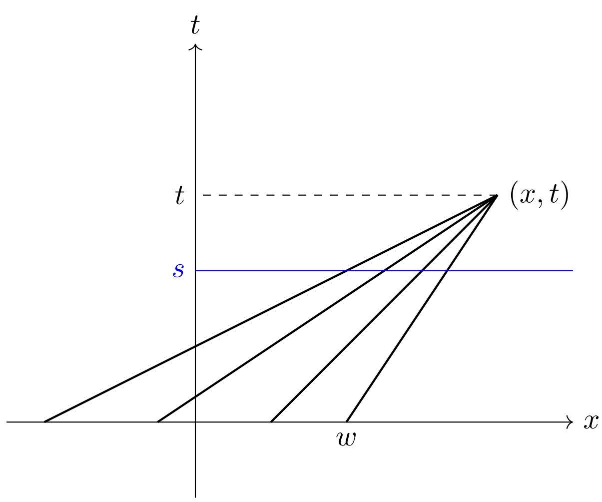

Figure 28. Dynamic Programming Principle

Next, pick $w \in \mathbb{R}^n$ such that

$$u(x,t) = g(w) + \frac{|x-w|^2}{2t}.$$

See Figure 28. Pick

$$y = \frac{s}{t}x + \frac{t-s}{t}w.$$

Then we get

$$\begin{aligned} u(x,t) &= g(w) + \frac{|x-w|^2}{2t} \\ &= g(w) + \frac{|y-w|^2}{2s} + \frac{|x-y|^2}{2(t-s)} \\ &\geq u(y,s) + \frac{|x-y|^2}{2(t-s)}. \end{aligned}$$

Thus, we obtain the Dynamic Programming Principle. Note that, by the choice of y, the inequality above is actually an equality.

□

Sketch of the proof of Theorem 6.24. It is not hard to see that u is Lipschitz through its given formula. Hence, u differentiable almost everywhere thanks to the Rademacher theorem.

Fix $(x,t) \in \mathbb{R}^n \times (0,\infty)$ such that u is differentiable at this point. For $h > 0$ small and any $v \in \mathbb{R}^n$,

$$u(x,t) \leq u(x-hv, t-h) + \frac{h^2|v|^2}{2h} = u(x-hv,t-h) + \frac{h|v|^2}{2}.$$

Therefore,

$$\lim_{h\to 0} \frac{u(x,t) - u(x-hv,t-h)}{h} \leq \frac{|v|^2}{2},$$

which gives us that

$$u_t(x,t) + v \cdot Du(x,t) - \frac{|v|^2}{2} \leq 0.$$

As

$$\max_{v\in\mathbb{R}^n} \left(v \cdot Du(x,t) - \frac{|v|^2}{2} \right) = \frac{|Du(x,t)|^2}{2},$$

we imply that

$$u_t(x,t) + \frac{|Du(x,t)|^2}{2} \leq 0.$$

We next prove the converse inequality. Pick w, y as in the last part of the proof of Lemma 6.25. Let

$$\overline{v} = \frac{x-w}{t}.$$

As the inequality in the last part of the proof of Lemma 6.25 becomes an equality, we have that

$$u(x,t) = u(x - h\overline{v}, t-h) + \frac{h^2|\overline{v}|^2}{2h} = u(x - h\overline{v}, t-h) + \frac{h|\overline{v}|^2}{2}.$$

By repeating the same calculations as above, we yield

$$u_t(x,t) + \overline{v} \cdot Du(x,t) - \frac{|\overline{v}|^2}{2} = 0.$$

We conclude that

$$u_t(x,t) + \frac{|Du(x,t)|^2}{2} = 0.$$

□

6.9. Connections between a quadratic Hamilton–Jacobi equation and Burgers' equation in one dimension

Consider the quadratic Hamilton–Jacobi equation in one dimension

$$\begin{cases} u_t + \frac{|u_x|^2}{2} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}. \end{cases} \tag{6.13}$$

As already discussed in the previous section, under some appropriate conditions, (6.13) has a C^1 solution at least locally on $\mathbb{R} \times [0,T)$ for some $T > 0$. We assume that u is sufficiently smooth on $\mathbb{R} \times [0,T)$. Differentiate this equation with respect to x to yield

$$u_{xt} + u_x u_{xx} = 0.$$

Set $w = u_x$. Then we see that w solves exactly the Burgers equation

$$\begin{cases} w_t + w w_x = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ w(x,0) = g'(x) & \text{on } \mathbb{R}. \end{cases} \tag{6.14}$$

This is an extremely beautiful and natural connection between the quadratic Hamilton–Jacobi equation and Burgers' equation in one dimension.

Recall the projected characteristics for (6.14): For $x(0) = x_0$ given and $t \geq 0$,

$$\begin{cases} x(t) = x_0 + g'(x_0)t, \\ w(x_0 + g'(x_0)t, t) = g'(x_0), \end{cases}$$

which agrees with the projected characteristics for (6.13) noting the fact that $w = u_x$.

Let us now connect the theorems done earlier for Hamilton–Jacobi equations and Burgers' equation in this setting. As $H(p) = \frac{p^2}{2}$, H is convex. We have for $g \in C^2(\mathbb{R})$, g is convex if and only if $g'' = (g')' \geq 0$. Thus, we have that Corollary 6.23 is equivalent to Theorem 6.5.

Next, $DH(p) = H'(p) = p$, $D^2H(p) = H''(p) = 1$, and for $g''(z) < 0$,

$$1 + tD^2H(Dg(z))D^2g(z) = 1 + tg''(z) > 0 \qquad \Longleftrightarrow \qquad t < -\frac{1}{g''(z)}.$$

Recall that, for $\inf_{\mathbb{R}} g'' = \theta \leq 0$, we defined

$$T = -\frac{1}{\theta}.$$

We hence deduce Theorem 6.21 is equivalent to Theorem 6.7.

6.10. Exercises

Exercise 57. Use the substitution $v = u_x$ to solve the PDE

$$\begin{cases} u_{xt} = 5u_x & \text{in } \mathbb{R} \times \mathbb{R}, \\ u(t,t) = 0, u_x(t,t) = 2 & \text{for } t \in \mathbb{R}. \end{cases}$$

Exercise 58. (a) Use the method of characteristics to solve the following problem

$$\begin{cases} u_t + tu_x = u^2 & \text{in } \mathbb{R} \times (0,1), \\ u(x,0) = \frac{1}{1+x^2} & \text{on } \mathbb{R}. \end{cases}$$

(b) Show that the solution blows up as $t \to 1^-$, that is,

$$\lim_{t \to 1^-} \max_{x \in \mathbb{R}} u(x,t) = \infty.$$

Exercise 59. Let $F = F(p) : \mathbb{R}^n \to \mathbb{R}$ be a given C^1 function. Consider the following PDE with the unknown $u = u(x,t) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$

$$\begin{cases} u_t + F(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}^n. \end{cases}$$

Here, the initial data $u_0 \in C^1(\mathbb{R}^n)$. Write down the ODE system of the characteristic method, and solve the system.

Exercise 60. Solve the following PDE with the unknown $u = u(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$\begin{cases} u_t + uu_x = 1 & \text{in } \mathbb{R} \times \mathbb{R}, \\ u(x,x) = \frac{x}{2} & \text{for } x \in \mathbb{R}. \end{cases}$$

Exercise 61. Consider the following PDE with the unknown $u = u(x,t) : \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}$

$$\begin{cases} u_t + x_2 u_{x_1} + (u(u-1))_{x_2} = 0 & \text{in } \mathbb{R}^2 \times (0,\infty), \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}^2. \end{cases}$$

Here, the initial data $u_0 \in C^1(\mathbb{R}^2)$. Write down the ODE system of the characteristic method, and solve the system.

Exercise 62. Consider the Burgers equation with the unknown $u = u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}. \end{cases}$$

Here,

$$g(x) = \begin{cases} 0 & \text{for } x < 0, \\ x & \text{for } 0 \le x \le 1, \\ 1 & \text{for } x > 1. \end{cases}$$

Use the method of characteristics to solve the equation and find u.

Exercise 63. Consider the Burgers equation with the unknown $u = u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}. \end{cases}$$

Here,

$$g(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x > 0. \end{cases}$$

Use the method of characteristics to solve the equation. Can you find all the values of $u(x,t)$ for all $(x,t) \in \mathbb{R} \times (0,\infty)$ this way? Explain why?

Exercise 64. Consider the Burgers equation with the unknown $u = u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}. \end{cases}$$

Here,

$$g(x) = \begin{cases} 2 & \text{for } x < 0, \\ 2 - \frac{x}{2} & \text{for } 0 \le x \le 4, \\ 0 & \text{for } x > 4. \end{cases}$$

Use the method of characteristics to find t^*, the earliest time that the characteristics cross with each other. Write down the value of $u(x,t^*)$ for $x \in \mathbb{R}$.

Exercise 65. Consider the Burgers equation with the unknown $u = u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}. \end{cases}$$

Give an example of an initial data g such that $t^* = 0$, where t^* is the earliest time that the characteristics cross with each other.

Exercise 66. Consider the Burgers equation with the unknown $u = u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}. \end{cases}$$

Here, g is C^1, bounded, and nondecreasing, that is, $g'(x) \geq 0$ for all $x \in \mathbb{R}$. Consider the method of characteristics with $x(0) = x_0$ for some given $x_0 \in \mathbb{R}$. Recall that $p(t) = u_x(x(t),t)$. Show that $\lim_{t\to\infty} p(t) = 0$.

Exercise 67. Let $V \subset \mathbb{R} \times (0,\infty)$ be a smooth bounded domain. Let $C = \{(s(t),t) : t \in [a,b]\}$ be a C^1 curve in V. Assume that f is continuous on C and

$$\int_C f v \, dl = 0$$

for all $v \in C_c^\infty(\mathbb{R} \times (0,\infty))$ with $\operatorname{supp}(v) \subset V$. Show that $f = 0$ on C.

Exercise 68. Consider the Burgers equation with the unknown $u = u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}. \end{cases}$$

Here,

$$g(x) = \begin{cases} 2 & \text{for } x < 0, \\ 1 & \text{for } 0 \leq x \leq 4, \\ 0 & \text{for } x > 4. \end{cases}$$

Solve the equation.

Exercise 69. Consider the Burgers equation with the unknown $u = u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}. \end{cases}$$

Here,

$$g(x) = \begin{cases} -1 & \text{for } x < 0, \\ 3 & \text{for } 0 \leq x \leq 2, \\ 5 & \text{for } x > 2. \end{cases}$$

Solve the equation.

Exercise 70. Consider the Burgers equation with the unknown $u = u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$

$$u_t + uu_x = 0 \quad \text{in } \mathbb{R} \times (0,\infty).$$

Assume that, for some given constants $b, c, d > 0$,

$$u(x,t) = \begin{cases} 0 & \text{for } x < -t^2, \\ -bt - \sqrt{cx + dt^2} & \text{for } x > -t^2. \end{cases}$$

Find b, c, d such that this function u is a weak solution to the Burgers equation in $\mathbb{R} \times (0, \infty)$.

(That is, besides the fact that it solves the equation classically when it is smooth, it has to satisfy the Rankine-Hugoniot condition along shock curves.)

Exercise 71. Consider a scalar conservation laws in one dimension

$$u_t + (F(u))_x = 0 \quad \text{in } \mathbb{R} \times [0, \infty).$$

Here, $F : \mathbb{R} \to \mathbb{R}$ is a given continuous function, and u is a weak solution. Derive the Rankine-Hugoniot condition for u along a shock curve C.

Exercise 72. Consider the Hamilton-Jacobi equation with the unknown $u = u(x,t) : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = g(x) & \text{on } \mathbb{R}^n. \end{cases}$$

Here, $H : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are given smooth functions.

(a) Write down the ODE system from the characteristic method. Solve the system.

(b) Assume further that $H(p) = |p|^2/2$ and $g(x) = |x|^2/2$. Prove that the characteristics do not cross each other in this case.

Exercise 73. Consider the Hamilton-Jacobi equation with the unknown $u = u(x) : [-1, 1] \to \mathbb{R}$

$$\begin{cases} |u'(x)| = 1 & \text{in } (-1, 1), \\ u(-1) = u(1) = 0. \end{cases}$$

(a) Show that this equation has no C^1 solutions.

(b) Show that this equation has infinitely many solutions in the sense that they solve the equation everywhere except a finite number of points in $(-1, 1)$.

6.11. Notes and references

(1) We refer the reader to [**Eva10**, Chapter 3] for related content. We do not cover the characteristic method for static first-order PDEs here, but the method to study them follows in the same manner as what is done here.

(2) For various topics in the theory of Hamilton–Jacobi equations with particular emphasis on modern approaches and viewpoints, we refer the reader to the book of Tran [**Tra21**].

(3) In this chapter, we put a lot of emphasis on studying basic properties of Burgers' equation and some prototypical Hamilton–Jacobi equations.

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