6

Results in spaces of Hölder functions

6.1 Foreword

All the results in the previous chapters have been stated in the framework of the Sobolev spaces described in Chapter 1. The basic reason for using such spaces was explained in Section 1.1. However, one is mainly interested in statements claiming that the solution of a given boundary value problem has continuous derivatives up to a certain order. Such a property cannot be derived directly due to the bad behaviour of the kernels involved in the maximum norm (see Section 1.1 again). This is the reason why the classical property of continuity of some derivatives of the solution has been derived indirectly through the use of the Sobolev imbedding theorem of Subsection 1.4.4.

Another approach to the continuous differentiability of the solution of a boundary value problem consists in using spaces of functions with derivatives up to a certain order (say m a nonnegative integer) which are Hölder continuous (with exponent σ a real number between zero and one). Of course these spaces are closer to the classical spaces of continuously differentiable functions than the Sobolev spaces. However, they have few nice properties besides the ones which are obvious from the definition. Precise statements are given in Section 6.2. Fortunately a very nice multiplier theorem, similar to Mikhlin's theorem (Theorem 2.3.2.1) for the Lebesgue space $L_p(\mathbb{R}^n)$, holds for the Hölder spaces. As it turned out in Chapters 2 and 4, the multiplier theorem was the basic tool for proving the *a priori* inequalities. Accordingly similar *a priori* inequalities hold in the framework of Hölder spaces. They will be proved in this chapter and the corresponding regularity (or singularity) results will be derived.

To conclude this introductory section, let us mention some references about regularity in Hölder spaces. A wide set of results is derived in the classical book Miranda (1970) which deals with second-order problems in a domain with smooth boundary. Some of these results are extended to problems of higher order in Agmon *et al.* (1959). Second-order problems in domains with corners are studied in Volkoff (1965a,b), Azzam (1979, 1981) and Moussaoui (1971). The first two authors restrict their purpose to the Dirichlet problem. Their results are included in the present chapter.

6.2 A brief review of Hölder spaces

In this section, after defining precisely the spaces under consideration, we shall review their basic properties. In doing this, we shall follow the same plan as in Chapter 1 for the Sobolev spaces. Here Ω denotes any open subset of $\mathbb{R}^n$.

Definition 6.2.1 *Let m be a nonnegative integer and σ a real number such that $0<\sigma\leq 1$. We denote by $C^{m,\sigma}(\bar{\Omega})$ the space of all functions u defined in $\bar{\Omega}$ whose derivatives, up to the order m, are continuous and bounded in $\bar{\Omega}$ and whose derivatives of order m are uniformly Hölder continuous with exponent σ.*

We define a Banach norm on $C^{m,\sigma}(\bar{\Omega})$ by setting

$$\|u\|_{s,\infty,\bar{\Omega}}=\sum_{|\alpha|\leq m}\underset{x\in\bar{\Omega}}{\text{l.u.b.}}\,|D^\alpha u(x)|+\sum_{|\alpha|=m}\underset{x,y\in\bar{\Omega}}{\text{l.u.b.}}\,\frac{|D^\alpha u(x)-D^\alpha u(y)|}{|x-y|^\sigma},\qquad(6,2,1)$$

where $s=m+\sigma$. Observe that this definition includes the case when $\Omega=\mathbb{R}^n$.

In order to be able to describe the traces on the boundary Γ of Ω of such functions, we need a definition for similar spaces on Γ. This requires some smoothness assumption on Γ. Precisely we assume that Ω is bounded and that its boundary is of class $C^{k,1}$ with $k+1\geq s=m+\sigma$. We use the same notation as in Chapter 1; in particular Φ is defined by the identity (1,3,3,1), where φ is described in the Definition 1.2.1.1.

Definition 6.2.2 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a boundary of class $C^{k,1}$, where k is a nonnegative integer. Let Γ_0 be an open subset of Γ. A function u defined in Γ_0 belongs to $C^{m,\sigma}(\bar{\Gamma}_0)$, m a nonnegative integer, $\sigma\in\,]0,1]$, $s=m+\sigma\leq k+1$, iff $u\circ\Phi\in C^{m,\sigma}(V'\cap\Phi^{-1}(\Gamma_0\cap V))$ for all possible V and φ fulfilling the assumptions in Definition* 1.2.1.1.

It follows plainly from this definition that u belongs to $C^{0,\sigma}(\Gamma)$ iff u is Hölder continuous with exponent σ in the usual sense. In addition, it is clear that the trace $u|_\Gamma$ of a function $u\in C^{m,\sigma}(\bar{\Omega})$ is well defined and belongs to $C^{m,\sigma}(\Gamma)$. A converse statement will be derived later.

Some of the properties of the Sobolev spaces have analogues which are just obvious; among them are the following. First $C^{m,\sigma}(\bar{\Omega})$ and $C^{m,\sigma}(\Gamma)$ are algebras for the usual multiplication. Next the differentiation operator D_i, $1 \leqslant i \leqslant n$, is a continuous mapping from $C^{m,\sigma}(\bar{\Omega})$ into $C^{m-1,\sigma}(\bar{\Omega})$. Finally the natural imbedding of $C^{m,\sigma}(\bar{\Omega})$ into $C^{m',\sigma'}(\bar{\Omega})$ (and of $C^{m,\sigma}(\Gamma)$ into $C^{m',\sigma'}(\Gamma)$) is compact provided $m'+\sigma' < m+\sigma$; this is a simple consequence of the well-known Ascoli theorem.

On the other hand some of the most useful properties of the Sobolev spaces have no analogue at all for the Hölder spaces. For instance, there is no convenient density result. Indeed it is easy to check that any function u in the closure of $\mathcal{D}(\mathbb{R}^n)$ for the norm of $C^{m,\sigma}(\mathbb{R}^n)$, has the following extra properties:

(a) $D^\alpha u(x) \to 0$ when $|x| \to +\infty$, for $|\alpha| \leqslant m$

(b) $\dfrac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\sigma} \to 0$ when $|x-y| \to 0$, for $|\alpha| = m$.

The main reason for introducing these Hölder spaces is the following multiplier theorem.

Theorem 6.2.3 *Let $a \in C^n(\mathbb{R}^n)$ be such that there exists a constant C with*

$$|D^\alpha a(\xi)| \leqslant C(1+|\xi|)^{-|\alpha|} \tag{6,2,2}$$

for all $\xi \in \mathbb{R}^n$ and $|\alpha| \leqslant n$. Then the operator

$$g \mapsto F^{-1} a F g$$

is continuous in $C^{m,\sigma}(\mathbb{R}^n)$ for all m (a nonnegative integer) and $\sigma \in]0, 1[$.

A short outline of the proof may be found in Meyer (1978) while a detailed proof is given in Triebel (1978).

Let us now focus our attention on the continuation property. The case of a Lipschitz condition, i.e. the case when $\sigma = 1$, is very peculiar. For instance, if Ω is bounded, any function $u \in C^{m,1}(\bar{\Omega})$ is the restriction to $\bar{\Omega}$ of some function $U \in C^{m,1}(\mathbb{R}^n)$. This very strong result may be found in Schwarz (1965) for instance. Unfortunately, since Theorem 6.2.3 excludes the case when $\sigma = 1$, we shall mainly use the spaces with $0 < \sigma < 1$ in studying boundary value problems. There exists a Hölder version of Theorem 1.4.3.1: under the same assumptions, P_s maps $C^{m,\sigma}(\bar{\Omega})$ into $C^{m,\sigma}(\mathbb{R}^n)$, where $s = m + \sigma$. It is hard to give a precise reference for this result. However, the proof is just the same as the corresponding proof for the Sobolev spaces with Theorem 2.3.2.1 replaced by Theorem 6.2.3. The following statements, whose proofs are easy, are sufficient for our purpose.

Theorem 6.2.4 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a boundary Γ of class $C^{m,1}$. Then there exists a continuous linear operator P_{m+1} from $C^{m,\sigma}(\bar{\Omega})$ into $C^{m,\sigma}(\mathbb{R}^n)$ for every $\sigma \in]0, 1]$, such that*

$$P_{m+1}\, u|_{\bar{\Omega}} = u. \tag{6,2,3}$$

Outline of proof This is quite similar to the proof of Theorem 3.9, Section 3, Chapter 2, in Nečas (1967). The first step is a reduction to the case when Ω is replaced by a half space $\mathbb{R}^n_+$ defined by $x_n > 0$. This is achieved through the use of local coordinates near the boundary and a partition of unity. The second step consists in defining U by

$$U(x_1, \ldots, x_n) = \begin{cases} u(x_1, \ldots, x_n), & x_n \geq 0 \\ \sum_{i=1}^{m+1} \lambda_i u(x_1, \ldots, x_{n-1}, -ix_n), & x_n < 0, \end{cases}$$

assuming that

$$1 = \sum_{i=1}^{m+1} (-i)^\alpha \lambda_i, \qquad \alpha = 0, 1, \ldots, m. \tag{6,2,4}$$

It is very easy to check that $U \in C^{m,\sigma}(\mathbb{R}^n)$, when u is given in $C^{m,\sigma}(\bar{\mathbb{R}}^n_+)$, and that $U_{|\bar{\mathbb{R}}^n_+} = u$. ■

Theorem 6.2.5 *Let Ω be a bounded open subset of $\mathbb{R}^2$ whose boundary Γ is a curvilinear polygon of class $C^{m,1}$. There exists a continuous linear operator P_{m+1} from $C^{m,\sigma}(\bar{\Omega})$ into $C^{m,\sigma}(\mathbb{R}^2)$, for every $\sigma \in]0, 1]$, such that*

$$P_{m+1} u|_{\bar{\Omega}} = u$$

for every $u \in C^{m,\sigma}(\bar{\Omega})$.

Proof This is basically a repeat of the previous proof. Again we use a partition of unity and local coordinates. According to Definition 1.4.5.1, the problem is reduced to one of the following cases:

(a) Ω is a half space and we proceed as in the proof of Theorem 6.2.4;
(b) Ω is a quadrant, defined by $x_1 > 0, x_2 > 0$ and we define U as follows:

$$U(x_1, x_2) = \begin{cases} V(x_1, x_2), & x_2 \geq 0 \\ \sum_{i=1}^{m+1} \lambda_i V(x_1, -ix_2), & x_2 < 0 \end{cases}$$

where

$$V(x_1, x_2) = \begin{cases} u(x_1, x_2), & x_1 \geq 0, x_2 \geq 0 \\ \sum_{i=1}^{m+1} \lambda_i u(-ix_1, x_2), & x_1 < 0, x_2 \geq 0. \end{cases}$$

assuming again that (6,2,4) holds. Clearly $U \in C^{m,\sigma}(\mathbb{R}^2)$, when u is given in $C^{m,\sigma}(\bar{\Omega})$ and $U|_{\bar{\Omega}} = u$.

(c) Ω is the complement of a quadrant, defined by $x_1 \geq 0$ or $x_2 \geq 0$. Here we proceed by steps. We start from $u \in C^{m,\sigma}(\bar{\Omega})$ and we denote by u_1 the restriction of u to the half plane defined by $x_1 \geq 0$. Then we define V_1 by

$$V_1(x_1, x_2) = \begin{cases} u_1(x_1, x_2), & x_1 \geq 0 \\ \sum_{i=1}^{m+1} \lambda_i u_1(-ix_1, x_2), & x_1 < 0 \end{cases}$$

still assuming that (6,2,4) holds. Clearly V_1 belongs to $C^{m,\sigma}(\mathbb{R}^2)$ and $V_1 = u$ for $x_1 \geq 0$.

Next we set $w = u - V_1$ in the half plane defined by $x_2 \geq 0$. We have $w \in C^{m,\sigma}(\mathbb{R}^2_+)$ and $w = 0$ in the first quadrant ($x_1 \geq 0$ and $x_2 \geq 0$). We finally define a continuation W for w by

$$W(x_1, x_2) = \begin{cases} w(x_1, x_2), & x_2 \geq 0 \\ \sum_{i=1}^{m+1} \lambda_i w(x_1, -ix_2), & x_2 < 0 \end{cases}$$

again. The function W belongs to $C^{m,\sigma}(\mathbb{R}^2)$ and vanishes for $x_1 \geq 0$. Consequently the function

$$U = V_1 + W$$

is a suitable continuation for u. ∎

As a consequence of the above continuation property, it is easy to derive the following inequality. Assuming that $s' > s'' > s''' > 0$ and that Ω is a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary, there exists a constant K (depending on s', s'', s''' and Ω) such that

$$\|u\|_{s'',\infty,\bar{\Omega}} \leq \varepsilon \|u\|_{s',\infty,\bar{\Omega}} + \varepsilon^{-(s''-s''')/(s'-s'')} \|u\|_{s''',\infty,\bar{\Omega}} \tag{6,2,5}$$

for all $\varepsilon > 0$ and all $u \in C^{m,\sigma}(\bar{\Omega})$, $s' = m + \sigma$.

We shall conclude this section by a brief survey of trace results for the Hölder spaces. We shall obviously need them in the study of boundary value problems. Their proofs are much easier than the corresponding proofs for Sobolev spaces.

Theorem 6.2.6 *Let Ω be either a half space or a bounded open subset of $\mathbb{R}^n$ with a $C^{k,1}$ boundary Γ. Then the mapping*

$$u \mapsto \left\{\gamma u, \gamma\frac{\partial u}{\partial \nu}, \ldots, \gamma\frac{\partial^l u}{\partial \nu^l}\right\} \tag{6,2,6}$$

maps $C^{m,\sigma}(\bar{\Omega})$ onto $\prod_{j=0}^{l} C^{m-j,\sigma}(\Gamma)$, provided $l \leq m$ and $m + \sigma \leq k + 1$.

As in the previous chapters γ denotes the mapping

$$u \mapsto u|_{\Gamma}.$$

Here all the functions we consider the traces of are continuous and accordingly no extension is needed to define γ (compare with Subsection 1.5.1).

Theorem 6.2.6 is proved by reduction to the case when Ω is a half space using local coordinates and a partition of unity. Assume that the half-space is defined by $x_n > 0$, then the claim is that

$$u \mapsto \{\gamma_n u, \gamma_n D_n u, \ldots, \gamma_n D_n^l u\}$$

maps $C^{m,\sigma}(\bar{\mathbb{R}}^n_+)$ onto $\prod_{j=0}^{l} C^{m-j,\sigma}(\mathbb{R}^{n-1})$. Indeed starting from $u \in C^{m,\sigma}(\bar{\mathbb{R}}^n_+)$, it is clear that $D_n^j u|_{x_n=0}$ belongs to $C^{m-j,\sigma}(\mathbb{R}^{n-1})$. Therefore the mapping in (6,2,6) is well defined. To show that it is onto one can follow the method of proof of Theorem 5.8, Chapter 2 in Nečas (1967). (Observe that the same kind of proof works when Ω is an infinite strip $]a, b[\times\mathbb{R}$ (with $a, b \in \mathbb{R}$; $a < b$). This will be useful later.)

We also need trace results for domains with corners. The model result (corresponding to Theorem 1.5.2.4) is the following:

Theorem 6.2.7 *The mapping $u \mapsto (\{f_k\}_{k=0}^m; \{g_l\}_{l=0}^m)$ defined by*

$$f_k = D_y^k u|_{y=0}, \qquad g_l = D_x^l u|_{x=0} \tag{6,2,7}$$

is a continuous mapping from $C^{m,\sigma}(\overline{\mathbb{R}_+ \times \mathbb{R}_+})$ onto the subspace of

$$T = \prod_{k=0}^{m} C^{m-k,\sigma}(\bar{\mathbb{R}}_+) \times \prod_{l=0}^{m} C^{m-l,\sigma}(\bar{\mathbb{R}}_+)$$

defined by

$$D_x^l f_k(0) = D_y^k g_l(0), \qquad l + k \leq m. \tag{6,2,8}$$

Proof The direct part of the statement is easy. Indeed when u is given in $C^{m,\sigma}(\overline{\mathbb{R}_+ \times \mathbb{R}_+})$, it is obvious from the definitions that

$$(\{f_k\}_{k=0}^m; \{g_l\}_{l=0}^m)$$

belongs to T and satisfies the compatibility conditions (6,2,8). We just have to prove the converse, i.e. that the mapping is onto.

For this purpose we start from $f_k \in C^{m-k,\sigma}(\bar{\mathbb{R}}_+)$, $0 \leq k \leq m$ and $g_l \in C^{m-l,\sigma}(\bar{\mathbb{R}}_+)$, $0 \leq l \leq m$, given such that the conditions (6,2,8) are fulfilled. We have to find $u \in C^{m,\sigma}(\overline{\mathbb{R}_+ \times \mathbb{R}_+})$ such that (6,2,7) holds. As we did in the proof of Theorem 1.5.2.4, we first select functions $G_l \in C^{m-l,\sigma}(\mathbb{R})$ such that

$$G_{l|\mathbb{R}_+} = g_l, \qquad 0 \leq l \leq m.$$

Then, by the trace theorem for Hölder functions in a half space, there exists $U \in C^{m,\sigma}(\mathbb{R}^2_+)$ such that

$$D_x^l U|_{x=0} = G_l, \qquad 0 \leq l \leq m.$$

Next, instead of looking for u, we look for $v = u - U$, i.e. for a function $v \in C^{m,\sigma}(\overline{\mathbb{R}_+ \times \mathbb{R}_+})$ such that

$$\begin{cases} D_y^k v|_{y=0} = f_k - D_y^k U|_{y=0} = h_k, & 0 \leq k \leq m \\ D_x^l v|_{x=0} = 0, & 0 \leq l \leq m. \end{cases}$$

In other words, we have reduced our problem to the particular case when g_l is replaced by zero for all l. It is clear from (6,2,8) that

$$D_x^l h_k(0) = 0, \qquad l + k \leq m$$

and consequently $\tilde{h}_k \in C^{m,\sigma}(\mathbb{R})$. (Again, $\sim$ denotes the continuation by zero out of the domain of definition of the function.)

Applying again the trace results concerning a half space, we can find $w \in C^{m,\sigma}(\mathbb{R}^2_+)$ such that

$$D_y^k w|_{y=0} = \tilde{h}_k, \qquad 0 \leq k \leq m.$$

We obtain v as follows:

$$v(x, y) = w(x, y) - \sum_{j=1}^{m+1} \lambda_j w(-jx, y), \qquad x > 0, \quad y > 0,$$

assuming that

$$\sum_{j=1}^{m+1} (-j)^l \lambda_j = 1, \qquad 0 \leq l \leq m. \quad \blacksquare$$

This theorem implies a similar result on curvilinear polygons whose proof uses the same techniques as the proof of Theorem 1.5.2.8.

Corollary 6.2.8 *Let Ω be a bounded open subset of $\mathbb{R}^2$ whose boundary Γ is a curvilinear polygon of class C^∞. Then the mapping*

$$u \mapsto \left\{ \gamma_j \frac{\partial^l u}{\partial \nu_j^l} \right\}, \qquad 1 \leq j \leq N, \quad 0 \leq l \leq m,$$

is linear continuous from $C^{m,\sigma}(\bar{\Omega})$ onto the subspace of

$$T = \prod_{j=1}^{N} \prod_{l=0}^{m} C^{m-l,\sigma}(\bar{\Gamma}_j)$$

defined by the following conditions: Let L be any linear differential operator with coefficients of class C^∞ and order $d \leq m$. Denote by $P_{j,l}$ the differential *operators tangential to Γ_j such that*

$$L = \sum_{l \geq 0} P_{j,l} \frac{\partial^l}{\partial \nu_j^l};$$

then

$$\sum_{l \geq 0} (P_{j,l} f_{j,l})(S_j) = \sum_{l \geq 0} (P_{j+1,l} f_{j+1,l})(S_j). \tag{6,2,9}$$

The notation is the same as in the previous chapters. Namely Γ_j is the jth side of Γ, $\boldsymbol{\nu}_j$ the corresponding outward normal vector field and γ_j is the corresponding trace operator, $1 \leq j \leq N$. Following the direct orientation, $\bar{\Gamma}_j$ ends at S_j.

Another consequence of Theorem 6.2.7 through Corollary 6.2.8, is a statement similar to Theorem 1.6.1.4. Here we keep the same notation.

Theorem 6.2.9 *Let Ω be a bounded open subset of $\mathbb{R}^2$ whose boundary is a curvilinear polygon of class C^∞. Let $\{B_{j,k}\}_{k=1}^{K_j}$ be for each j, a system of differential operators in Ω, with coefficients belonging to $C^\infty(\bar{\Omega})$, which is normal on Γ_j. Then the mapping*

$$u \mapsto \{f_{j,k} = \gamma_j B_{j,k} u\}, \qquad 1 \leq j \leq N, \quad 1 \leq k \leq K_j$$

maps $C^{m,\sigma}(\bar{\Omega})$ onto the subspace of

$$T = \prod_{j=1}^{N} \prod_{k=1}^{K_j} C^{m-d_{j,k},\sigma}(\bar{\Gamma}_j)$$

defined by the conditions (1, 6, 1, 2) *for $d \leq m$ and all possible systems of differential operators $\{P_{j,k}\}_{k=1}^{K_j}$ tangential to Γ_j and $\{Q_{j+1,k}\}_{k=1}^{K_{j+1}}$ tangential to Γ_{j+1}, such that identity* (1,6,1,1) *holds.*

The proof is quite similar to that of Theorem 1.6.1.4.

Now we conclude this section with one technical result which is useful in the remainder of this chapter. It is an extension of Theorem 1.4.5.3.

Theorem 6.2.10 *Let Ω be a bounded open subset of $\mathbb{R}^2$, whose boundary Γ is a curvilinear polygon. Assume that $0 \in \Gamma$. Let V be a neighbourhood of O such that*

$$V \cap \bar{\Omega} \subseteq \{(r\cos\theta, r\sin\theta);\ r \geq 0,\ a \leq \theta \leq b\}$$

with $b - a < 2\pi$. Finally let u be a function which is smooth in $\Omega \setminus \{0\}$ and which is equal to

$$r^\alpha \varphi(\theta)$$

in $V \cap \Omega$, where $\varphi \in C^\infty([a:b])$. Then

$$u \in C^{m,\sigma}(\bar{\Omega}) \tag{6,2,10}$$

for $\alpha \geqslant m+\sigma$, *while*

$$u \notin C^{m,\sigma}(\bar{\Omega}) \tag{6,2,11}$$

for $\alpha < m+\sigma$, *when* α *is not an integer.*

Observe that (6,2,10) follows from (1,4,5,1) with the help of the Sobolev imbedding theorem, when α is strictly larger than $m+\sigma$. Otherwise it is a matter of direct elementary proof.

6.3 Regular second-order elliptic boundary value problems revisited

6.3.1 The Schauder inequality

Here we shall derive a Hölder version of the *a priori* estimate proved in Section 2.3. Let us first briefly recall the notation (which we keep consistent with that in Chapter 2). The domain Ω is a bounded open subset of $\mathbb{R}^n$ with a $C^{2,1}$ boundary; $-A$ is a strongly elliptic real second order operator in Ω and B is a real boundary operator of order d ($d=0$ or 1) which is nowhere characteristic on Γ. The reader is referred to Section 2.1 for the detailed assumptions on the coefficients. Our aim is to prove that there exists a constant C such that

$$\|u\|_{\sigma+2,\infty,\bar{\Omega}} \leqslant C[\|Au\|_{\sigma,\infty,\bar{\Omega}} + \|\gamma Bu\|_{\sigma+2-d,\infty,\Gamma} + \|u\|_{\sigma+1,\infty,\bar{\Omega}}] \tag{6,3,1,1}$$

for all $u \in C^{2,\sigma}(\bar{\Omega})$, $0<\sigma<1$.

Exactly as in Section 2.3 the first step is the proof of an inequality in the half space. Here we make use of the notation of Subsection 2.3.2. Namely, L is a strongly elliptic, real, homogeneous second-order operator which has constant coefficients, while M is a real, homogeneous, first-order operator with constant coefficients, noncharacteristic on the hyperplane $x_n=0$. We now intend to prove that there exists a constant C such that

$$\|u\|_{2+\sigma,\infty,\bar{\mathbb{R}}^n_+} \leqslant c[\|Lu\|_{\sigma,\infty,\bar{\mathbb{R}}^n_+} + \|\gamma_n Mu\|_{1+\sigma,\infty,\mathbb{R}^{n-1}} + \|u\|_{1+\sigma,\infty,\mathbb{R}^n_+}] \tag{6,3,1,2}$$

for all $u \in C^{2,\sigma}(\bar{\mathbb{R}}^n_+)$ whose support is bounded.

Observing that such a function u also belongs to $H^2(\mathbb{R}^n_+)$, we can use the representation formula (2,3,2,12). Thus we have

$$u = E * \left(F + l_{n,n}k_0 \otimes \delta'_n + \left[l_{n,n}k_1 + 2\sum_{j=1}^{n-1} l_{n,j}D_jk_0\right] \otimes \delta_n\right). \tag{6,3,1,3}$$

Here F is a suitable continuation of $f=Lu+u$ such that $F \in C^{0,\sigma}(\mathbb{R}^n)$ and has compact support; E is the elementary solution for $L+1$ defined in

Subsection 2.3.2. In addition we have

$$\begin{aligned} \hat{k}_0 &= \left(m_n p_- + \sum_{j=1}^{n-1} i m_j \xi_j\right)^{-1} \hat{h} \\ \hat{k}_1 &= p_- \hat{k}_0 \\ h &= g - \gamma_n M E * F \end{aligned} \tag{6,3,1,4}$$

where $g = \gamma_n M u$.

Now it is very tempting to proceed with the same proofs as in Subsection 2.3.2, just substituting Theorem 6.2.3 in Mikhlin's multiplier theorem 2.3.2.1. Unfortunately this method of proof requires using spaces W_p^s corresponding to a negative order of differentiation s. This is in particular necessary for Lemma 2.3.2.2. A theory of Hölder spaces with negative order of differentiation is not yet well established. We shall not attempt to define such spaces and accordingly we shall not be able to derive any property of the operator γ_n^* in the framework of Hölder spaces. However, we shall be able to conclude by deriving directly the properties of the Poisson operator

$$P: \varphi \mapsto E * (\varphi \otimes \delta_n'). \tag{6,3,1,5}$$

Lemma 6.3.1.1 *When u belongs to $H^2(\mathbb{R}_+^n)$ then*

$$u = E * (F + \varphi \otimes \delta_n'), \qquad x_n > 0, \tag{6,3,1,6}$$

where

$$\hat{\varphi} = + l_{n,n} \hat{k}_0 + \frac{1}{p_-} \left[l_{n,n} \hat{k}_1 + 2 \sum_{j=1}^{n-1} l_{n,j} D_j \hat{k}_0 \right]. \tag{6,3,1,7}$$

In other words u is the solution of a Dirichlet problem

$$\begin{cases} Lu + u = f, & x_n > 0 \\ \gamma_n u = \varphi, & x_n = 0. \end{cases} \tag{6,3,1,8}$$

Proof This is a consequence of identity (6,3,1,3), observing that

$$\hat{E}(\xi, x_n) = -\frac{1}{l_{n,n}} \frac{e^{x_n p_-(\xi)}}{p_+(\xi) - p_-(\xi)}$$

for $x_n > 0$. ■

From identities (6,3,1,4) and (6,3,1,7) and from the multiplier theorem 6.2.3, it follows that there exists a constant C such that

$$\|\varphi\|_{2+\sigma,\infty,\mathbb{R}^{n-1}} \leq C\{\|f\|_{\sigma,\infty,\bar{\mathbb{R}}_+^n} + \|g\|_{1+\sigma,\infty,\mathbb{R}^{n-1}}\}. \tag{6,3,1,9}$$

Next we focus our attention on problem (6,3,1,8). A suitable linear transformation of coordinates reduces $L+1$ to $-\Delta+1$, and accordingly we can assume now that L is simply $-\Delta$. The basic estimate is the following.

Lemma 6.3.1.2 *For* $0<\sigma<1$, *there exists a constant* C *such that*

$$\|u\|_{2+\sigma,\infty,\bar{\mathbb{R}}_+^n} \leqslant C\{\|-\Delta u + u\|_{\sigma,\infty,\bar{\mathbb{R}}_+^n} + \|\gamma_n u\|_{2+\sigma,\infty,\mathbb{R}^{n-1}}\} \tag{6,3,1,10}$$

for all $u \in C^{2,\sigma}(\bar{\mathbb{R}}_+^n)$.

Proof We first reduce the general case to the particular one when both $\gamma_n u$ and $\gamma_n f$ vanish. Indeed, by the trace theorem 6.2.6, we know that

$$\gamma_n u \in C^{2,\sigma}(\mathbb{R}^{n-1}) \qquad \text{and} \qquad \gamma_n f \in C^{0,\sigma}(\mathbb{R}^{n-1}),$$

and consequently there exists $v \in C^{2,\sigma}(\bar{\mathbb{R}}_+^n)$ such that

$$\begin{cases} \gamma_n v = \gamma_n u \\ \gamma_n D_n^2 v = \left(-\sum\limits_{j=1}^{n-1} D_j^2 + I\right)\gamma_n u - \gamma_n f. \end{cases}$$

In addition, v depends continuously on $\gamma_n u$ and f, i.e. there exists C_1 such that

$$\|v\|_{2+\sigma,\infty,\bar{\mathbb{R}}_+^n} \leqslant C_1\{\|f\|_{\sigma,\infty,\bar{\mathbb{R}}_+^n} + \|\gamma_n u\|_{2+\sigma,\infty,\mathbb{R}^{n-1}}\}. \tag{6,3,1,11}$$

Then we look at $w = u - v$; this function is a solution of a homogenous Dirichlet problem:

$$\begin{cases} -\Delta w + w = g, & x_n > 0 \\ \gamma_n w = 0, & x_n = 0, \end{cases}$$

where $g = f - (-\Delta v + v)$. Therefore $g \in C^{0,\sigma}(\bar{\mathbb{R}}_+^n)$ and $\gamma_n g = 0$. We now perform an odd reflection through the hyperplane $x_n = 0$, i.e. we define W as follows:

$$W(x', x_n) = \begin{cases} w(x', x_n), & x_n \geqslant 0 \\ -w(x', -x_n), & x_n < 0. \end{cases}$$

We define G in a similar fashion. Since $\gamma_n g = 0$, $\gamma_n w = 0$ and $\gamma_n D_n^2 w = 0$, it follows that $G \in C^{0,\sigma}(\mathbb{R}^n)$ and $W \in C^{2,\sigma}(\mathbb{R}^n)$. In addition we have

$$(-\Delta + 1)W = G \qquad \text{in } \mathbb{R}^n.$$

Consequently the multiplier Theorem 6.2.3 shows that there exists a constant C_2 such that

$$\|W\|_{2+\sigma,\infty,\mathbb{R}^n} \leqslant C_2\|G\|_{\sigma,\infty,\mathbb{R}^n}.$$

By restriction to $\bar{\mathbb{R}}^n_+$ we derive the existence of another constant C_3 such that

$$\|w\|_{2+\sigma,\infty,\bar{\mathbb{R}}^n_+} \leq C_3 \|g\|_{\sigma,\infty,\bar{\mathbb{R}}^n_+}. \qquad (6,3,1,12)$$

Finally inequality (6,3,1,10) follows from (6,3,1,11) and (6,3,1,12). ■

With the help of inequality (6,3,1,9) we conclude:

Theorem 6.3.1.3 *Let $-L$ be a homogeneous strongly elliptic second-order operator with real constant coefficients and let M be either the identity operator or a homogeneous first-order operator with constant coefficients for which the hyperplane $x_n = 0$ is not characteristic. Then for $0 < \sigma < 1$ there exists a constant C such that*

$$\|u\|_{2+\sigma,\infty,\bar{\mathbb{R}}^n_+} \leq C[\|Lu\|_{\sigma,\infty,\mathbb{R}_+} + \|\gamma_n Mu\|_{2+\sigma-d,\infty,\mathbb{R}^{n-1}} + \|u\|_{1+\sigma,\infty,\bar{\mathbb{R}}^n_+}] \qquad (6,3,1,13)$$

for all $u \in C^{2,\sigma}(\bar{\mathbb{R}}^n_+)$ with bounded support $(0 < \sigma < 1)$.

This statement is quite similar to Theorem 2.3.2.7 in the framework of Hölder spaces. Then by applying exactly the same technique as in the proof of Theorem 2.3.3.2, we derive a statement concerning operators with variable coefficients:

Theorem 6.3.1.4 *Let A, B and Ω fulfil the assumptions in Section* 2.1 *Assume in addition that*

(a) *the boundary Γ of Ω is of class $C^{2,1}$*
(b) $a_{i,j} \in C^{1,\sigma}(\bar{\Omega})$, $1 \leq i, j \leq n$
(c) $b_i \in C^{1,\sigma}(\bar{\Omega})$.

Then for $0 < \sigma < 1$ there exists a constant C such that (6,3,1,1) *holds for all $u \in C^{2,\sigma}(\bar{\Omega})$.*

We first prove that it is enough that each $x \in \bar{\Omega}$ has a neighbourhood V_x so that (6,3,1,1) holds for all $u \in C^{2,\sigma}(\bar{\Omega})$ whose support is contained in V_x. This is a statement similar to Lemma 2.3.3.1. The proof is exactly the same in the Hölder norms due to the extra assumptions that are made on the coefficients of A and B.

Then the existence of V_x is checked in the two particular cases (a) and (b) similar to those in the proof of Theorem 2.3.3.2. Accordingly we have two lemmas:

Lemma 6.3.1.5 *For every $y \in \Omega$, there exists a neighbourhood V of y in Ω such that* (6,3,1,1) *holds for all $u \in C^{2,\sigma}(\bar{\Omega})$ whose support is contained in V.*

Some minor modifications of the proof of Lemma 2.3.3.3 are necessary. This is why we shall detail the proof of Lemma 6.3.1.5.

Proof We freeze the coefficients of A at y and thus we obtain an operator L with constant coefficients such that $-L$ is strongly elliptic:

$$L = \sum_{i,j=1}^{n} l_{i,j} D_i D_j$$

where $l_{i,j} = a_{i,j}(y)$.

Then if the support of u is contained in V and $\bar{V} \subset \Omega$ we have

$$L\tilde{u} + \tilde{u} = \widetilde{Au} - \left[\sum_{i,j=1}^{n} (\alpha_{i,j} - l_{i,j}) \widetilde{D_i D_j u} + \sum_{i,j=1}^{n} (D_i \alpha_{i,j}) \widetilde{D_j u} - \tilde{u} \right]$$

where $\alpha_{i,j}$ are functions belonging to $C^{1,\sigma}(\mathbb{R}^n)$ such that $\alpha_{i,j} = a_{i,j}$ in $\bar{\Omega}$. Let E be the elementary solution for $L+1$ introduced in Section 2.3.2. It follows that

$$\tilde{u} = E * \widetilde{Au} - E * \left[\sum_{i,j=1}^{n} (\alpha_{i,j} - l_{i,j}) \widetilde{D_i D_j u} + \sum_{i,j=1}^{n} (D_i \alpha_{i,j}) \widetilde{D_j u} - \tilde{u} \right].$$

By Theorem 6.2.3, $E*$ is a linear continuous mapping from $C^{0,\sigma}(\mathbb{R}^n)$ into $C^{2,\sigma}(\mathbb{R}^n)$ since the Fourier transform of any derivative of E up to order 2 fulfils the assumptions of that theorem. Therefore there exists a constant C_1 such that

$$\|u\|_{2+\sigma,\infty,\bar{\Omega}} \leq C_1 \left[\|Au\|_{\sigma,\infty,\bar{\Omega}} + \|u\|_{1+\sigma,\infty,\bar{\Omega}} + \sum_{i,j=1}^{n} \|(a_{i,j} - l_{i,j}) D_i D_j u\|_{\sigma,\infty,\bar{\Omega}} \right]. \tag{6,3,1,14}$$

Handling the last term is slightly more tricky than in the case of the Sobolev norms. Actually we have

$$\|(a_{i,j} - l_{i,j}) D_i D_j u\|_{\sigma,\infty,\bar{\Omega}}$$
$$\leq \max_{x \in V} |a_{i,j} - l_{i,j}| \, \|D_i D_j u\|_{\sigma,\infty,\bar{\Omega}} + \|a_{i,j} - l_{i,j}\|_{\sigma,\infty,\bar{\Omega}} \max_{x \in V} |D_i D_j u|,$$

provided u has its support in V. If we assume that the diameter of V is $\leq \delta$, then we have

$$\max_{x \in V} |a_{i,j}(x) - l_{i,j}| \leq K\delta^\sigma$$

since $a_{i,j} \in C^{0,\sigma}(\bar{\Omega})$ and we also have

$$\max_{x \in V} |D_i D_j u(x)| \leq \delta^\sigma \, \|u\|_{2+\sigma,\infty,\bar{\Omega}}.$$

Accordingly there exists a constant C_2 such that

$$\|u\|_{2+\sigma,\infty,\bar{\Omega}} \leq C_2 [\|Au\|_{\sigma,\infty,\bar{\Omega}} + \|u\|_{1+\sigma,\infty,\bar{\Omega}} + \delta^\sigma \, \|u\|_{2+\sigma,\infty,\bar{\Omega}}]. \tag{6,3,1,15}$$

We conclude by choosing δ small enough to ensure that $C_2\delta^\sigma$ is less than 1. Then (6,3,1,13) holds for all $u \in C^{2,\sigma}(\bar{\Omega})$ with support in V, provided $\bar{V}$ is contained in Ω and the diameter of V is less than δ.

The technical lemma corresponding to Lemma 2.3.3.4 is the following:

Lemma 6.3.1.6 *Let $y \in \Gamma$ have a neighbourhood W in Γ, contained in the hyperplane $\{x_n = 0\}$. Then there exists a neighbourhood U of y in $\bar{\Omega}$ such that* (6,3,1,13) *holds for all $u \in C^{2,\sigma}(\bar{\Omega})$, whose support is contained in U.*

Translating the proof of Lemma 2.3.3.4 into the framework of Hölder spaces requires the same kind of modifications as for Lemma 6.3.1.5. It is not worth detailing a theorem.

6.3.2 Smoothness

We shall now derive some regularity results similar to those in Section 2.5.1. We assume again that A and B fulfil the assumptions in Section 2.1. In addition we assume that

(d) *the boundary Γ of Ω is of class $C^{2,1}$*
(e) $a_{ij} \in C^{1,1}(\bar{\Omega})$, $b_i \in C^{1,1}(\bar{\Omega})$.

These requirements are slightly more restrictive than those in Theorem 6.3.1.4 (about the *a priori* inequalities). These extra assumptions could be avoided, but they will save many boring technicalities.

Theorem 6.3.2.1 *Let $u \in W_p^2(\Omega)$, with $p > n$, be such that*

$$\begin{cases} Au = f \in C^{0,\sigma}(\bar{\Omega}) \\ \gamma Bu = g \in C^{2-d,\sigma}(\Gamma) \end{cases}$$

with $0 < \sigma < 1$; then $u \in C^{2,\sigma}(\bar{\Omega})$.

Basically we shall approximate the data f and g by better ones to which the regularity result of Theorem 2.5.1.1 may be applied. Then we shall be able to take limits with the help of the *a priori* inequalities of Subsections 2.3.3 and 6.3.1. However, the lack of convenient density results in the Hölder spaces introduces an additional complication. The following statement is a possible substitute for a density result.

Lemma 6.3.2.2 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary Γ. Then each $u \in C^{m,\sigma}(\bar{\Omega})$ can be approximated by a sequence u_ν, $\nu = 1, \ldots$ such that*

(a) $u_\nu \in \mathscr{D}(\bar{\Omega})$,

(b) $\|u_\nu\|_{m+\sigma,\infty,\bar\Omega}$ *is bounded when* $\nu \to +\infty$,
(c) $\|u_\nu - u\|_{m+\sigma',\infty,\bar\Omega} \to 0$ *when* $\nu \to +\infty$ *for every* $\sigma' < \sigma$.

Proof This is straightforward. One can define u_ν as follows:

$$u_\nu = (\rho_\nu * U)|_{\bar\Omega},$$

where $U \in C^{m,\sigma}(\mathbb{R}^n)$ is a continuation of u (i.e. $U|_{\bar\Omega} = u$) and ρ_ν, $\nu = 1, \ldots$ is an approximation of identity; in other words

$$\rho_\nu \in \mathcal{D}(\mathbb{R}^n), \qquad \int_{\mathbb{R}^n} \rho_\nu \, dx = 1$$

and the support of ρ_ν converges to $\{0\}$ when $\nu \to \infty$. ■

Proof of Theorem 6.3.2.1 We first choose λ large enough for inequality (2,3,3,7) to hold. Then we set $h = f + \lambda u$; it is clear that $h \in C^{0,\sigma}(\bar\Omega)$.

Let $h_\nu \in C^1(\bar\Omega)$ and $g_\nu \in C^3(\Gamma)$ be such that $\|h_\nu\|_{\sigma,\infty,\bar\Omega}$ and $\|g_\nu\|_{2-d+\sigma,\infty,\Gamma}$ remain bounded and

$$\|h_\nu - h\|_{\sigma',\infty,\bar\Omega} \to 0, \qquad \nu \to \infty$$

$$\|g_\nu - g\|_{2-d+\sigma',\infty,\Gamma} \to 0, \qquad \nu \to \infty$$

for all $\sigma' < \sigma$. Such sequences can be found by Lemma 6.3.2.2. (In order to approximate g, we can consider g as the trace of a function G belonging to $C^{2-d,\sigma}(\bar\Omega)$ and apply Lemma 6.3.2.2 to G.) Finally let u_ν be solution of

$$\begin{cases} Au_\nu + \lambda u_\nu = h_\nu & \text{in } \Omega \\ \gamma B u_\nu = g_\nu & \text{on } \Gamma. \end{cases}$$

We know that u_ν exists by Theorem 2.4.1.3. Then Theorem 2.5.1.1 shows that $u_\nu \in W_p^3(\Omega)$ for every $p < +\infty$. Consequently $u_\nu \in C^{2,\sigma}(\bar\Omega)$.

In order to take the limit in ν, we make use of the two *a priori* inequalities (2,3,3,7) and (6,3,1,1). We first have

$$\|u_\nu - u\|_{2,p,\Omega} \le C[\|h_\nu - h\|_{0,p,\Omega} + \|g_\nu - g\|_{2-d-1/p,p,\Gamma}]$$

and consequently $u_\nu \to u$ in $W_p^2(\Omega)$, when $\nu \to +\infty$.

On the other hand, we have

$$\begin{aligned}\|u_\nu - u_{\nu'}\|_{\sigma'+2,\infty,\bar\Omega} \le C[&\|h_\nu - h_{\nu'}\|_{\sigma',\infty,\bar\Omega} + \|g_\nu - g_{\nu'}\|_{\sigma'+2-d,\infty,\Gamma} \\ &+ \|u_\nu - u_{\nu'}\|_{\sigma'+1,\infty,\bar\Omega}]\end{aligned}$$

and since $W_p^2(\Omega)$ is continuously imbedded in $C^{1,\sigma'}(\bar\Omega)$ for p large enough, this shows that u_ν, $\nu = 1, 2, \ldots$ is also a Cauchy sequence in $C^{2,\sigma'}(\bar\Omega)$ for all $\sigma' < \sigma$. It follows that $u \in C^{2,\sigma'}(\bar\Omega)$ and that $D_iD_ju_\nu(x) \to D_iD_ju(x)$ for all $x \in \bar\Omega$, $i, j = 1, 2, \ldots, n$.

Finally, applying again inequality (6,3,1,1), we see that $\|u_\nu\|_{2,\sigma,\bar\Omega}$ remains bounded as $\nu \to +\infty$. Accordingly, there exists a constant K such that

$$|D_iD_ju_\nu(x) - D_iD_ju_\nu(y)| \leqslant K\,|x-y|^\sigma$$

for all $i, j = 1, 2, \ldots, n$, $\nu = 1, 2 \ldots$, and $x, y \in \bar\Omega$. Taking the limit shows that all the second derivatives of u are Hölder continuous with exponent σ. ■

Remark 6.3.2.3 A proof of the same result assuming that a_{ij} and b_i belong to $C^{1,\sigma}(\bar\Omega)$ requires application of the same technique as in the proof of Lemma 2.4.1.4 (i.e. locally flattening the boundary and mollifying tangentially).

Remark 6.3.2.4 As a consequence of Theorem 6.3.2.1, we can restate each of the Theorems 2.4.2.5–2.4.2.7 in the framework of the Hölder spaces. For instance the result corresponding to Theorem 2.4.2.5 reads as follows. Assume that the hypotheses of Theorem 2.4.2.5 are fulfilled and that in addition the boundary Γ of Ω is of class $C^{2,1}$ and that $a_{i,j} \in C^{1,\sigma}(\bar\Omega)$, $1 \leqslant i, j \leqslant n$; $a_i \in C^{0,\sigma}(\bar\Omega)$, $0 \leqslant i \leqslant n$. Then, for every $f \in C^{0,\sigma}(\bar\Omega)$ and every $g \in C^{2,\sigma}(\Gamma)$, there exists a unique solution $u \in C^{2,\sigma}(\bar\Omega)$ of

$$\begin{cases} \sum_{i,j=1}^{n} D_i(a_{ij}D_ju) + \sum_{i=1}^{n} a_iD_iu + a_0u = f & \text{in } \Omega \\ \gamma u = g & \text{on } \Gamma. \end{cases}$$

6.4 Second-order elliptic problems in polygons revisited

6.4.1 The Schauder inequality in an infinite strip

We now look again at the boundary value problems of Section 4.2. Keeping the same notation, we are going to find sufficient conditions on the coefficients a, b, α_j, β_j, λ_j, $j = 0, 1$ for the existence of a constant C such that

$$\|u\|_{2+\sigma,\infty,\bar B} \leqslant C\,\|Lu\|_{\sigma,\infty,\bar B} \tag{6,4,1,1}$$

for all $u \in C^{2,\sigma}(\bar B)$ such that $\gamma_jM_ju = 0$ on F_j, $j = 0, 1$.

A first step will be the proof of the weaker inequality.

$$\max_{\bar B} |u| \leqslant C\,\|Lu\|_{\sigma,\infty,\bar B}. \tag{6,4,1,2}$$

The technique is quite similar to the one we used in Subsection 4.2.2, just replacing the multiplier theorem 2.3.2.1 by Theorem 6.2.3. Then inequality (6,4,1,2) makes it possible to replace L by $\Delta - 1$ to estimate the second derivatives of u in $C^{0,\sigma}(\bar{B})$.

Theorem 6.4.1.1 *Assume that $b > 0$, $a \neq 0$ and that for each $j = 0$ or 1, we have either $\alpha_j = 1$ or $\alpha_j = \beta_j = 0$ and $\lambda_j = 1$. Assume in addition that the characteristic equation* (4,2,1,2) *has no real root. Then for $0 < \sigma < 1$ there exists a constant C such that inequality* (6,4,1,2) *holds for $u \in C^{2,\sigma}(\bar{B})$ such that $\gamma_j M_j u = 0$ on F_j, $j = 0, 1$.*

Proof We first consider the particular case when $u \in C^{2,\sigma}(\bar{B})$ and has a bounded support. Thus u also belongs to $H^2(B)$ and we can apply Theorem 4.2.1.2. In other words identity (4,2,1,3) holds.

Then a Hölder version of Lemma 4.2.1.3 is this one:

Lemma 6.4.1.2 *Let $\xi, y, z \mapsto K(\xi, y, z)$ be a smooth function such that*

$$\max_{y \in]0,h[} \int_0^h \max_{\xi \in \mathbb{R}} \{|K(\xi, y, z)| + (1 + |\xi|)\, |D_\xi K(\xi, y, z)|\}\, dz < +\infty;$$

then the mapping $u \mapsto f$ defined by

$$\hat{u}(\xi, y) = \int_0^h K(\xi, y, z)\hat{f}(\xi, z)\, dz$$

is continuous from $C^{0,\sigma}(\bar{B})$ into the space of continuous bounded functions in $\bar{B}$.

Applying this lemma we obtain inequality (6,4,1,2) but only for $u \in C^{2,\sigma}(\bar{B}) \cap H^2(B)$ such that $\gamma_j M_j u = 0$ on F_j, $j = 0, 1$.

We shall reduce the general case to the previous particular one with the help of cut-off functions. Accordingly let $\theta \in \mathcal{D}(\mathbb{R})$ be such that

$$\begin{cases} \theta(x) = 1, & |x| \leq 1 \\ \theta(x) = 0, & |x| \geq 2 \end{cases}$$

and let for $\varepsilon > 0$, θ_ε be defined by

$$\theta_\varepsilon(x) = \theta(\varepsilon x).$$

Clearly $\theta_\varepsilon(x) \to 1$ when $\varepsilon \to 0$. Thus let us start from $u \in C^{2,\sigma}(\bar{B})$ such that $\gamma_j M_j u = 0$ on F_j, $j = 0, 1$. We have

$$\gamma_j M_j \theta_\varepsilon u = \beta_j \theta'_\varepsilon \gamma_j u \qquad \text{on } F_j, \qquad j = 0, 1.$$

By the trace theorem (Section 6.2) we know that there exists $v_\varepsilon \in C^{2,\sigma}(\bar{B})$

such that

$$\begin{cases} \gamma_j v_\varepsilon = 0 & \text{on } F_j, \quad j = 0, 1 \\ \gamma_j M_j v_\varepsilon = \beta_j \theta'_\varepsilon \gamma_j u & \text{on } F_j, \quad j = 0, 1 \end{cases}$$

and there exists a constant C_1 such that

$$\|v_\varepsilon\|_{2+\sigma,\infty,\bar{B}} \leqslant C_1 \sum_{j=0}^{1} \|\beta_j \theta'_\varepsilon \gamma_j u\|_{1+\sigma,\infty,F_j} \tag{6,4,1,3}$$

If we replace v_ε by $\theta_{2\varepsilon} v_\varepsilon$ we can assume in addition that v_ε has a bounded support. Consequently we have

$$\theta_\varepsilon u - v_\varepsilon \in C^{2,\sigma}(\bar{B}) \cap H^2(B)$$

and

$$\gamma_j M_j(\theta_\varepsilon u - v_\varepsilon) = 0 \qquad \text{on } F_j, \quad j = 0, 1.$$

Applying inequality (6,4,1,2) to $\theta_\varepsilon u - v_\varepsilon$, we get

$$\max_{\bar{B}} |\theta_\varepsilon u - v_\varepsilon| \leqslant C \, \|L(\theta_\varepsilon u) - L v_\varepsilon\|_{\sigma,\infty,\bar{B}}.$$

Therefore it follows that

$$\max_{\bar{B}} |\theta_\varepsilon u - v_\varepsilon| \leqslant C\{\|\theta_\varepsilon L u\|_{\sigma,\infty,\bar{B}} + \|\theta''_\varepsilon u\|_{\sigma,\infty,\bar{B}} + 2\, \|\theta'_\varepsilon D_x u\|_{\sigma,\infty,\bar{B}} + |a| \, \|\theta'_\varepsilon u\|_{\sigma,\infty,\bar{B}} + C_2 \|v_\varepsilon\|_{2+\sigma,\infty,\bar{B}}\}$$

Finally with the help of (6,4,1,3) we get

$$\max_{\bar{B}} |\theta_\varepsilon u| \leqslant C\{\|\theta_\varepsilon L u\|_{\sigma,\infty,\bar{B}} + \|\theta''_\varepsilon u\|_{\sigma,\infty,\bar{B}} + 2\, \|\theta'_\varepsilon D_x u\|_{\sigma,\infty,\bar{B}} + |a| \, \|\theta'_\varepsilon u\|_{\sigma,\infty,\bar{B}} + C_3 \, \|\theta'_\varepsilon u\|_{1+\sigma,\infty,\bar{B}}\}$$

and taking the limit in $\varepsilon \to 0$ yields plainly

$$\max_{\bar{B}} |u| \leqslant \|Lu\|_{\sigma,\infty,\bar{B}};$$

this is the desired result. ■

Proof of Lemma 6.4.1.2 Again, as in the proof of Lemma 4.2.1.3, we denote by M the function

$$M(y; z) = \operatorname*{l.u.b.}_{\xi \in \mathbb{R}} \{|K(\xi, y, z)| + (1 + |\xi|) \, |D_\xi K(\xi, y, z)|\}.$$

It follows from Theorem 6.2.3 that there exists C such that

$$\operatorname*{l.u.b.}_{x\in\mathbb{R}} |u(x, y)| + \operatorname*{l.u.b.}_{\substack{x,x'\in\mathbb{R}\\ x\neq x'}} \frac{|u(x, y) - u(x', y)|}{|x - x'|^\sigma}$$

$$\leqslant C \int_0^h M(y, z) \left\{ \operatorname*{l.u.b.}_{x\in\mathbb{R}} |f(x, z)| + \operatorname*{l.u.b.}_{\substack{x,x'\in\mathbb{R}\\ x\neq x'}} \frac{|f(x, z) - f(x', z)|}{|x - x'|^\sigma} \right\} dz.$$

It follows that

$$\max_{\bar{B}} |u| \leqslant C \int_0^h M(y, z)\, dz\, \|f\|_{\sigma,\infty,\bar{B}}. \quad \blacksquare$$

We are now able to prove the stronger inequality (6,4,1,1).

Theorem 6.4.1.3 *Under the assumptions of Theorem* 6.4.1.1, *there exists a constant C such that inequality* (6,4,1,1) *holds for $u \in C^{2,\sigma}(\bar{B})$ such that $\gamma_j M_j u = 0$ on F_j, $j = 0, 1$.*

Proof Exactly as in the proof of Theorem 6.4.1.1, we begin with the particular case when u has a bounded support and therefore belongs to $H^2(B)$. We choose μ_j, $j = 0, 1$ as in Lemma 4.2.2.5 and set

$$\begin{cases} g = \Delta u - u \\ \Psi_j = \gamma_j(\alpha_j D_y u + \beta_j D_x u + \mu_j u), \qquad j = 0, 1. \end{cases}$$

Clearly we have

$$\begin{cases} g = Lu - aD_x u + (b - 1)u \\ \Psi_j = (\mu_j - \lambda_j)\gamma_j u, \qquad j = 0, 1. \end{cases} \tag{6,4,1,4}$$

Then we set $v = u - E * G$ where G is a continuation of g from $\bar{B}$ (i.e. $G|_{\bar{B}} = g$) such that $G \in C^{0,\sigma}(\mathbb{R}^2)$ has bounded support and

$$\|G\|_{\sigma,\infty,\mathbb{R}^2} \leqslant C_1 \|g\|_{\sigma,\infty,\bar{B}},$$

and where E is the elementary solution of $(\Delta - 1)$ defined by

$$E = -F^{-1}(1 + |\xi|^2)^{-1}.$$

By Theorem 6.2.3 the convolution by E maps continuously $C^{0,\sigma}(\mathbb{R}^2)$ into $C^{2,\sigma}(\mathbb{R}^2)$. Consequently there exists a constant C_2 such that

$$\|u\|_{2+\sigma,\infty,\bar{B}} \leqslant C_2\{\|v\|_{2+\sigma,\infty,\bar{B}} + \|g\|_{\sigma,\infty,\bar{B}}\}. \tag{6,4,1,5}$$

Now, we have to estimate v which is a solution of

$$\begin{cases} -\Delta v + v = 0 & \text{in } B \\ \gamma_j(\alpha_j D_y v + \beta_j D_x v + \mu_j v) = h_j, & j = 0, 1, \end{cases} \tag{6,4,1,6}$$

where h_j is defined by

$$h_j = \Psi_j - \gamma_j\{(\alpha_j D_y + \beta_j D_x + \mu_j)E * G\}, \qquad j = 0, 1. \tag{6,4,1,7}$$

Since G has a bounded support, it belongs to $L_2(\mathbb{R}^2)$ and $E * G$ belongs to $H^2(\mathbb{R}^2)$. It follows that v also belongs to $H^2(B)$. This is why we can use the calculations in Subsection 4.2.2.

This means that, setting $k_j = \gamma_j v$ and $l_j = \gamma_j D_y v$, $j = 0, 1$, we have:

$$\hat{k}_0 = \alpha,\ \hat{l}_0 = r\beta,$$

$$\hat{k}_1 = \alpha \cosh(rh) + \beta \sinh(rh)$$

$$\hat{l}_1 = \alpha r \sinh(rh) + \beta r \cosh(rh),$$

where

$$\alpha = \frac{1}{d}\{\alpha_0 r\hat{h}_1 - [\alpha_1 r \cosh(rh) + (\mathrm{i}\beta_1\xi + \mu_1)\sinh(rh)]\hat{h}_0\}$$

$$\beta = \frac{1}{d}\{-(\mathrm{i}\beta_0\xi + \mu_0)\hat{h}_1 + [\alpha_1 r \sinh(rh) + (\mathrm{i}\beta_1\xi + \mu_1)\cosh(rh)]\hat{h}_0\}$$

$$\begin{aligned} d &= r\cosh(rh)[\alpha_0(\mathrm{i}\beta_1\xi + \mu_1) - \alpha_1(\mathrm{i}\beta_0\xi + \mu_0)] \\ &\quad + \sinh(rh)[r^2\alpha_0\alpha_1 - (\mathrm{i}\beta_0\xi + \mu_0)(\mathrm{i}\beta_1\xi + \mu_1)] \\ r &= \sqrt{(1+\xi^2)}. \end{aligned}$$

It is worth recalling here that due to Lemma 4.2.2.5 d does not vanish for any real ξ. Applying Theorem 6.2.3 we get the following inequality:

$$\sum_{j=0}^{1} \{\|k_j\|_{2+\sigma,\infty,F_j} + \|l_j\|_{1+\sigma,\infty,F_j}\} \le C_3 \sum_{j=0}^{1} \|h_j\|_{2-d_j+\sigma,\infty,F_j}.$$

Then (6,4,1,4) and (6,4,1,7) imply that

$$\sum_{j=0}^{1} \{\|k_j\|_{2+\sigma,\infty,F_j} + \|l_j\|_{1+\sigma,\infty,F_j}\} \le C_4\{\|Lu\|_{\sigma,\infty,\bar{B}} + \|u\|_{1+\sigma,\infty,\bar{B}}\}. \tag{6,4,1,8}$$

To conclude we write the equation of $\tilde{v}$:

$$(-\Delta + 1)\tilde{v} = -k_0 \otimes \delta_0' + k_1 \otimes \delta_h' - l_0 \otimes \delta_0 + l_1 \otimes \delta_h.$$

Accordingly we have

$$\tilde{v} = -E * \{-k_0 \otimes \delta_0' + k_1 \otimes \delta_h' - l_0 \otimes \delta_0 + l_1 \otimes \delta_h\}.$$

And consequently (as in Lemma 6.3.1.1†)

$$v = -E * \{\varphi_0 \otimes \delta_0' + \varphi_1 \otimes \delta_h'\} \tag{6,4,1,9}$$

† Here

$$\hat{E}(\xi, x_2) = -\frac{\exp(-|x_2|\sqrt{(1+\xi^2)})}{2\sqrt{(1+\xi^2)}}.$$

where

$$\hat{\varphi}_0 = -\hat{k}_0 + \frac{\hat{l}_0}{\sqrt{(1+\xi^2)}}, \qquad \hat{\varphi}_1 = \hat{k}_1 + \frac{\hat{l}_1}{\sqrt{(1+\xi^2)}}. \tag{6,4,1,10}$$

Thus, in B,v is the sum of two functions v_0 and v_1 which are solutions of a Dirichlet problem in the half planes $x_2>0$ and $x_2<h$ respectively

$$v_0 = -E * \varphi_0 \otimes \delta_0', \qquad v_1 = -E * \varphi_1 \otimes \delta_h'$$

$$\begin{cases} -\Delta v_0 + v_0 = 0, & x_2>0 \\ \gamma_0 v_0 = -\frac{1}{2}\varphi_0 \end{cases}$$

$$\begin{cases} -\Delta v_1 + v_1 = 0, & x_2<h \\ \gamma_h v_1 = \frac{1}{2}\varphi_1. \end{cases}$$

Applying Lemma 6.3.1.2 twice, we get the inequality

$$\|v\|_{2+\sigma,\infty,\bar{B}} \leq C_5 \sum_{j=0}^{1} \|\varphi_j\|_{2+\sigma,\infty,F_j}.$$

This implies, by (6,4,1,10) and Theorem 6.2.3, that

$$\|v\|_{2+\sigma,\infty,\bar{B}} \leq C_6 \sum_{j=0}^{1} \{\|k_j\|_{2+\sigma,\infty,F_j} + \|l_j\|_{1+\sigma,\infty,F_j}\}. \tag{6,4,1,11}$$

Summing up, from the inequalities (6,4,1,5), (6,4,1,8) and (6,4,1,11) it follows that

$$\|u\|_{2+\sigma,\infty,\bar{B}} \leq C_7\{\|Lu\|_{\sigma,\infty,\bar{B}} + \|u\|_{1+\sigma,\infty,\bar{B}}\}. \tag{6,4,1,12}$$

Now we take advantage of the classical inequality (6,2,5) in the case of an infinite strip:

Lemma 6.4.1.4 *There exists a constant K such that*

$$\|u\|_{1+\sigma,\infty,\bar{B}} \leq \varepsilon \|u\|_{2+\sigma,\infty,\bar{B}} + K\varepsilon^{-(1+\sigma)} \max_{\bar{B}} |u|$$

for all $\varepsilon \in]0, 1]$ and all $u \in C^{2,\sigma}(\bar{B})$.

Choosing ε small enough, inequality (6,4,1,1) follows from (6,4,1,12) and (6,4,1,2).

So far we have always assumed that u had a compact support. In order to remove this extra assumption, we approximate a general u by $\theta_\varepsilon u$ exactly as in the proof of Theorem 6.4.1.1. ■

Proof of Lemma 6.4.1.4 One first extends the function u into a function

$U \in C^{2,\sigma}(\mathbb{R}^2)$. Then the corresponding inequality on the whole plane, is an easy consequence of the Taylor formula. ■

6.4.2 The Schauder inequality in a polygon and its consequences

The notation is now that of Section 4.4. Thus Ω is a bounded open subset of $\mathbb{R}^2$ whose boundary Γ is a polygon. To each side Γ_j corresponds a boundary condition which is either the Dirichlet condition or an oblique condition (in the direction of $\boldsymbol{\mu}_j = \boldsymbol{\nu}_j + \beta_j \boldsymbol{\tau}_j$). Given $f \in C^{0,\sigma}(\bar{\Omega})$, $0 < \sigma < 1$, we look for a solution $u \in C^{2,\sigma}(\bar{\Omega})$ of

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \gamma_j u = 0 & \text{in } \Gamma_j, \quad j \in \mathcal{D} \\ \gamma_j \dfrac{\partial u}{\partial \nu_j} + \beta_j \dfrac{\partial}{\partial \tau_j} \gamma_j u = 0 & \text{on } \Gamma_j, \quad j \in \mathcal{N} \end{cases} \tag{6,4,2,1}$$

We begin with the proof of an estimate of the Schauder type namely

$$\|u\|_{\sigma+2,\infty,\Omega} \leqslant C\{\|\Delta u\|_{\sigma,\infty,\Omega} + \|u\|_{1+\sigma,\infty,\Omega}\}$$

for all $u \in C^{2,\sigma}(\bar{\Omega})$ which fulfil the boundary conditions in (6,4,2,1). Here we shall use the same local method of proof as in Subsection 4.3.2, using suitable weighted spaces (depending on ρ, the distance to the corners):

Definition 6.4.2.1 *We denote by $P_\infty^{m,\sigma}(\Omega)$ the space of all functions u defined in $\bar{\Omega}$ such that*

(a) *$\rho^{|\alpha|-m-\sigma} D^\alpha u$ is continuous and bounded in Ω for all α with $|\alpha| \leqslant m$,*
(b) *$D^\alpha u \in C^{0,\sigma}(\bar{\Omega})$ for $|\alpha| = m$.*

A Banach norm for this space

$$\|u\|_{P_\infty^{m,\sigma}(\bar{\Omega})} = \sum_{|\alpha| \leqslant m} \underset{\Omega}{\text{l.u.b.}}\, \rho^{|\alpha|-m-\sigma} |D^\alpha u| + \sum_{|\alpha|=m} \underset{x \in \bar{\Omega}, y \in \bar{\Omega}}{\text{l.u.b.}} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\sigma}.$$

The space $P_\infty^{m,\sigma}(\bar{\Omega})$ is a subspace of $C^{m,\sigma}(\bar{\Omega})$ while a converse inclusion is given by the following result.

Theorem 6.4.2.2 *Let $u \in C^{m,\sigma}(\bar{\Omega})$ be such that*

$$D^\alpha u(S_j) = 0 \quad \text{for} \quad |\alpha| \leqslant m$$

$j = 1, 2, \ldots, N$; then $u \in P_\infty^{m,\sigma}(\bar{\Omega})$.

This is an elementary application of Taylor's formula using polar coordinates near each corner (see the proof of Theorem 4.3.2.2).

The basic estimate here is the following.

***Theorem* 6.4.2.3** *For $0<\sigma<1$ there exists a constant C such that*

$$\|u\|_{P^{2,\sigma}_\infty(\bar\Omega)} \leq C\{\|f\|_{\sigma,\infty,\bar\Omega} + \|u\|_{1+\sigma,\infty,\bar\Omega}\} \tag{6,4,2,2}$$

for all solutions $u \in P^{2,\sigma}_\infty(\bar\Omega)$ of problem (6,4,2,1), provided

$$\frac{1}{\pi}(\phi_{j+1} - \phi_j - [2+\sigma]\omega_j)$$

is not an integer for any j, where $\phi_j = \arctan\beta_j$, $j \in \mathcal{N}$ and $\phi_j = \pi/2$, $j \in \mathcal{D}$.

Sketch of the proof We just outline this proof since it is very similar to the proof of Theorem 4.3.2.3. We consider the problem locally with the help of a partition of unity $\{\eta_j\}_{j=0,\dots N}$ on $\bar\Omega$ such that $\eta_j \in \mathcal{D}(\mathbb{R}^2)$ for each j and such that

(a) the support of η_0 does not meet any vertex of $\bar\Omega$.
(b) the support of η_j contains S_j and none of the S_k with $k \neq j$, $j = 1, 2, \dots, N$.
(c) we have

$$\frac{\partial\eta_j}{\partial\nu_k} + \beta_k\frac{\partial\eta_j}{\partial\tau_k} = 0 \qquad \text{on } \Gamma_k$$

for $k=j$ if $j \in \mathcal{N}$ and for $k = j+1$ if $j+1 \in \mathcal{N}$.

Thus we have

$$\|\Delta(\eta_j u) - \eta_j f\|_{\sigma,\infty,\bar\Omega} \leq K_j\,\|u\|_{1+\sigma,\infty,\bar\Omega} \tag{6,4,2,3}$$

and

$$\begin{cases} \gamma_k(\eta_j u) = 0 & \text{on } \Gamma_k,\quad k \in \mathcal{D} \\ \gamma_k\dfrac{\partial}{\partial\nu_k}(\eta_j u) + \beta_k\dfrac{\partial}{\partial\tau_k}\gamma_k(\eta_j u) = 0 & \text{on } \Gamma_k,\quad k \in \mathcal{N}, \end{cases}$$

$j = 1, 2, \dots, N$.

We can apply inequality (6,3,1,1) to $\eta_0 u$ and this yields the following inequality:

$$\|\eta_0 u\|_{P^{2,\sigma}_\infty(\bar\Omega)} \leq C_0\{\|f\|_{\sigma,\infty,\bar\Omega} + \|u\|_{1+\sigma,\infty,\bar\Omega}\}. \tag{6,4,2,4}$$

We now consider the functions $\eta_j u$, $1 \leq j \leq N$.

Using the polar coordinates with origin at S_j and setting (for a given j)

$$w(t,\theta) = e^{-(\sigma+2)t}(\widetilde{\eta_j u})(e^{t+i\theta}),$$

we obtain a function belonging to $C^{2,\sigma}(\bar{B})$ where $B=\mathbb{R}\times]0,\omega_j[$. In addition w is solution of a boundary value problem in B to which inequality (6,4,1,1) may be applied. This implies the following

$$\|\eta_j u\|_{P^{2,\sigma}_\infty(\bar{\Omega})}\leq C_j\,\|\Delta(\eta_j u)\|_{\sigma,\infty,\bar{\Omega}}. \tag{6,4,2,5}$$

Finally inequality (6,4,2,2) is a plain consequence of (6,4,2,3), (6,4,2,4) and (6,4,2,5). ■

Now Theorem 6.4.2.2 shows that $P^{2,\sigma}_\infty(\bar{\Omega})$ contains a subspace of $C^{2,\sigma}(\bar{\Omega})$ whose codimension is finite in $C^{2,\sigma}(\bar{\Omega})$. Therefore a proof similar to that of Theorem 4.3.2.4 yields the following:

Theorem 6.4.2.4 *Assume that $0<\sigma<1$ and that $1/\pi(\phi_{j+1}-\phi_j-[2+\sigma]\omega_j)$ is not an integer for any j. Then there exists a constant C such that*

$$\|u\|_{2+\sigma,\infty,\bar{\Omega}}\leq C\{\|\Delta u\|_{\sigma,\infty,\bar{\Omega}}+\|u\|_{1+\sigma,\infty,\bar{\Omega}}\} \tag{6,4,2,6}$$

for all $u\in C^{2,\sigma}(\bar{\Omega})$, a solution of the problem (6,4,2,1).

We are now going to draw the consequences of Theorem 6.4.2.4. The following existence result is closely related to the content of Subsection 5.1.3. Let us recall briefly some notation (taken from Definition 5.1.3.4):

$$\lambda_{j,m}=\frac{\phi_j-\phi_{j+1}+m\pi}{\omega_j},\qquad j=1,2,\ldots,N,\quad m\in\mathbb{Z},$$

$$\mathfrak{S}_{j,m}(r_je^{i\theta_j})=r_j^{-\lambda_{j,m}}\cos(\lambda_{j,m}\theta_j+\phi_{j+1})\eta_j(r_je^{i\theta_j})$$

where $\lambda_{j,m}$ is not an integer and

$$\mathfrak{S}_{j,m}(r_je^{i\theta_j})=r_j^{-\lambda_{j,m}}\{\ln r_j\cos(\lambda_{j,m}\theta_j+\phi_{j+1})$$
$$+\theta_j\sin(\lambda_{j,m}\theta_j+\phi_{j+1})\}\eta_j(r_je^{i\theta_j})$$

when $\lambda_{j,m}$ is an integer. Here again r_j and θ_j are the polar coordinates with origin at S_j.

The basic result is the following where, for the sake of simplicity, we assume the uniqueness of the solution in $H^2(\Omega)$. Sufficient conditions for this uniqueness have been found in Theorem 4.4.1.3.

Theorem 6.4.2.5 *Assume that $\mathcal{D}$ is not empty and that at least two of the vectors μ_j are linearly independent. Assume in addition that $(1/\pi)(\phi_{j+1}+\phi_j-[2+\sigma]\omega_j)$ is not an integer for any j. Then for each $f\in C^{0,\sigma}(\bar{\Omega})$, with $0<\sigma<1$, there exists a function u and numbers $c_{j,m}$ such that*

$$u-\sum_{-(\sigma+2)<\lambda_{j,m}<0}c_{j,m}\mathfrak{S}_{j,m}\in C^{2,\sigma}(\bar{\Omega})$$

and u is solution of the problem (6,4,2,1).

Proof We choose $p \geq 2$ such that $(\phi_j - \phi_{j+1} + 2\omega_j/q)/\pi$ is not an integer for any j and apply Theorem 4.4.4.11. Since $f \in L_p(\Omega)$, there exists a function u and numbers $c_{j,m}$ such that

$$w = u - \sum_{\substack{1 \leq j \leq N \\ -2/q < \lambda_{j,m} < 0 \\ \lambda_{j,m} \neq -1}} c_{j,m} \mathfrak{S}_{j,m} \in W_p^2(\Omega)$$

and u is a solution of the problem (6,4,2,1). Now we study w. Clearly we have

$$\begin{cases} \Delta w = f - \displaystyle\sum_{\substack{1 \leq j \leq N \\ -2/q < \lambda_{j,m} < 0 \\ \lambda_{j,m} \neq -1}} c_{j,m} \Delta \mathfrak{S}_{j,m} = g & \text{in } \Omega \\ \gamma_j w = 0 & \text{on } \Gamma_j, \quad j \in \mathcal{D} \\ \gamma_j \dfrac{\partial w}{\partial \gamma_j} + \beta_j \dfrac{\partial}{\partial \tau_j} \gamma_j w = 0 & \text{on } \Gamma_j, \quad j \in \mathcal{N} \end{cases}$$

because the functions $\mathfrak{S}_{j,m}$ fulfil all the homogeneous boundary conditions in (6,4,2,1). In addition we have $g \in C^{0,\sigma}(\bar{\Omega})$ since $\mathfrak{S}_{j,m}$ is harmonic near S_j and smooth far from S_j.

On the other hand Δ is one to one (Theorem 4.4.1.3) and has finite index (Theorem 4.4.4.11) from

$$E = \left\{ w \in W_p^2(\Omega);\ \gamma_j w = 0 \text{ on } \Gamma_j, j \in \mathcal{D},\ \gamma_j \frac{\partial w}{\partial \nu_j} + \beta_j \frac{\partial}{\partial \tau_j} \gamma_j w = 0 \text{ on } \Gamma_j, j \in \mathcal{N} \right\}$$

into $L_p(\Omega)$. Let us denote by $v_1, \ldots, v_M$ a basis of the annihilator (formerly denoted by N_q in Chapter 4) of ΔE. Necessarily we have

$$(g; v_k) = 0, \qquad k = 1, 2, \ldots, M. \tag{6,4,2,7}$$

We shall now approximate w by smoother solutions. For this purpose we shall approximate g by smoother functions, trying to keep (6,4,2,7).

We apply Lemma 6.3.2.2. This provides us with a sequence g_ν, $\nu = 1, 2, \ldots$ such that

(a) $g_\nu \in \mathcal{D}(\bar{\Omega})$,
(b) $\|g_\nu\|_{\sigma,\infty,\bar{\Omega}}$ is bounded when $\nu \to +\infty$,
(c) $\|g_\nu - g\|_{\sigma',\infty,\bar{\Omega}} \to 0$ when $\nu \to \infty$ for every $\sigma' < \sigma$.

Unfortunately there is no reason why g_ν should be orthogonal to v_k, $k = 1, 2, \ldots, M$. Thus we introduce φ_k, $k = 1, 2, \ldots, M$ belonging to $\mathcal{D}(\Omega)$ and such that

$$(v_k, \varphi_l) = \delta_{k,l}, \qquad k, l = 1, 2, \ldots, M.$$

If we replace g_ν by $g_\nu - \sum_{k=1}^{M} (g_\nu; v_k)\varphi_k$, we get a new sequence that we still denote by g_ν, $\nu = 1, 2, \ldots$, such that the properties (a)–(c) above still hold and that in addition

$$(g_\nu; v_k) = 0, \qquad k = 1, 2, \ldots, M, \quad \nu = 1, 2 \ldots.$$

Accordingly there exists for each ν a unique $w_\nu \in E$ such that $\Delta w_\nu = g_\nu$. In addition $w_\nu \to w$ in $W^2_p(\Omega)$ when $\nu \to +\infty$.

Next Theorem 5.1.3.5 shows there exist real sequences $C_{j,m,\nu}$ such that

$$w_\nu - \sum_{-1-2/q_1<\lambda_{j,m}<-2/q} C_{j,m,\nu}\mathfrak{S}_{j,m} \in W^3_{p_1}(\Omega)$$

where p_1 has been chosen such that none of the $[\phi_{j+1} - \phi_j - \omega_j - 2\omega_j/q_1]\pi$ is an integer, $1/p_1 + 1/q_1 = 1$. If p_1 is large enough the Sobolev imbedding theorem implies that

$$W^3_{p_1}(\Omega) \subseteq C^{2,\sigma}(\bar{\Omega}).$$

Consequently we have

$$w_\nu - \sum_{-(2+\sigma)<\lambda_{j,m}<-2/q} C_{j,m,\nu}\mathfrak{S}_{j,m} \in C^{2,\sigma}(\bar{\Omega}).$$

In other words w_ν belongs to S, the space

$$\Sigma = \left\{ w \in C^{2,\sigma}(\bar{\Omega});\ \gamma_j w = 0 \text{ on } \Gamma_j,\ j \in \mathcal{D},\ \gamma_i \frac{\partial w}{\partial \nu_i} + \beta_j \frac{\partial}{\partial \tau_j}\gamma_j w = 0 \text{ on } \Gamma_j,\ j \in \mathcal{N} \right\}$$

augmented with the span of the functions $\mathfrak{S}_{j,m}$ corresponding to $-(2+\sigma) < \lambda_{j,m} < -2/q$, $j = 1, 2, \ldots, N$. If p is also large enough, the Sobolev imbedding theorem implies that

$$W^2_p(\Omega) \subset C^{1,\sigma}(\bar{\Omega})$$

and consequently we have $S \subset C^{1,\sigma}(\bar{\Omega})$. A natural Banach norm on S is

$$\|\omega\|_S = \inf \left\{ \|\varphi\|_{2+\sigma,\infty,\bar{\Omega}} + \sum_{-(2+\sigma)<\lambda_{j,m}<-2/q} |C_{j,m}| \right\},$$

where the infimum is taken over all the possible functions $\varphi \in \Sigma$ and numbers $C_{j,m}$ such that

$$w = \varphi + \sum_{-(2+\sigma)<\lambda_{j,m}<-2/q} C_{j,m}\mathfrak{S}_{j,m}.$$

In addition, Σ has a finite codimension in S. It follows from Theorem 6.4.2.4 that there exists a new constant C such that

$$\|w\|_S \leq C\{\|\Delta w\|_{\sigma,\infty,\bar{\Omega}} + \|w\|_{1+\sigma,\infty,\bar{\Omega}}\}.$$

We apply this last inequality to w_ν. It shows that w_ν, $\nu = 1, 2, \ldots$, is a bounded sequence in S. Remembering that $w_\nu \to w$ in $W^2_p(\Omega)$ when $\nu \to +\infty$, it is now easy to conclude that $w \in S$. Going back to u we see that it belongs to $C^{2,\sigma}(\bar{\Omega})$ augmented with the span of all the functions $\mathfrak{S}_{j,m}$ corresponding (with the exception of $\lambda_{j,m}=-1$) to $-(2+\sigma)<\lambda_{j,m}<0$, $j=1,2,\ldots,N$. ■

Theorem 6.4.2.5 is a Hölder version of Theorem 4.4.4.11. In Section 5.1, we proved a wide extension of Theorem 4.4.4.11, namely Theorem 5.1.3.5, where the behaviour of higher-order derivatives of the solution is investigated. This was achieved by differentiating the original problem (4,1,1) and taking advantage of the very general trace theorem 1.6.1.4. Replacing Theorem 1.6.1.4 by its Hölder version, Theorem 6.2.9 yields the following extension of Theorem 6.4.2.5.

Theorem 6.4.2.6 *Assume that $\mathcal{D}$ is not empty and that at least two of the vectors $\boldsymbol{\mu}_j$ are linearly independent. Assume in addition that*

$$(\phi_{j+1}-\phi_j-[k+2+\sigma]\omega_j)$$

is not an integer for any j. Then for each $f\in C^{k,\sigma}(\bar{\Omega})$ and $g_j\in C^{k+2,\sigma}(\bar{\Gamma}_j)$, $j\in\mathcal{D}$, $g_j\in C^{k+1,\sigma}(\bar{\Gamma}_j)$ $j\in\mathcal{N}$ with $0<\sigma<1$ such that $g_j(S_j)=g_{j+1}(S_j)$ when j and $j+1\in\mathcal{D}$, there exists a solution u of

$$\begin{cases} \Delta u=f & \text{in } \Omega \\ \gamma_j u=g_j & \text{on } \Gamma_j, \quad j\in\mathcal{D} \\ \gamma_j\dfrac{\partial u}{\partial\nu_j}+\beta_j\dfrac{\partial}{\partial\tau_j}\gamma_j u=g_j & \text{on } \Gamma_j, \quad j\in\mathcal{N} \end{cases}$$

and there exist real numbers $C_{j,m}$ such that

$$u-\sum_{-(k+2+\sigma)<\lambda_{j,m}<0} C_{j,m}\mathfrak{S}_{j,m}\in C^{k+2,\sigma}(\bar{\Omega}).$$

Remark 6.4.2.7 This statement is also valid when one allows cuts in Ω (i.e. $\omega_j=2\pi$ for some j).