k-ϵ Equation for Compressible Reciprocating Engine Flows

Sherif H. El Tahry*
General Motors Research Laboratories, Warren, Michigan

The k-ϵ model currently is being used by several investigators as a submodel for the calculation of turbulent flows in engines using multidimensional procedures. As this submodel mainly was developed and tested for incompressible flows, attempts were made and described in the literature to modify the model to suit engine applications. In these attempts, little effort was made to examine the basic k and ϵ equations closely. Because these equations contain information that could be quite helpful in developing the model, in the present study the basic k and ϵ equations are examined with an analysis tailored to identify important terms that appear during compression/expansion in engine cylinders. It transpires from the analysis that some terms that were neglected basically in earlier studies could be quite large. Modeling some of these terms in the ϵ equation was relatively straightforward and did not add any unknown constants. Thus, these terms were added to the ϵ equation in the new model. The remaining terms, which are significant only during combustion, are more complex, and it is doubtful that they could be modeled without the addition of model constants. Because of a lack of data necessary to estimate the model constants, these terms were neglected. A comparison between the new and an earlier version of the model revealed that the new version yielded more physically plausible results.

I. Introduction

RECENTLY there has been an increasing trend toward applying multidimensional models to compute fluid motion in internal combustion engines. One of the popular models of turbulence used in these procedures is the k-ϵ model (see Gosman et al.,[1] Syed and Bracco,[2] and Morel and Mansour[3]). While this model has several advantages over simpler models like the zero-equation model (e.g., the mixing-length model) or a one-equation model[4] it still has shortcomings. The most fundamental criticism of the model is its basic assumption that turbulence reacts to the mean strain rate in a similar manner to a Newtonian molecular response. The validity of such an assumption rests on a variety of factors.[5,6] The most relevant among them is the magnitude of the ratios of time (α) and length scales (β) between turbulent and mean motions. The model is most accurate when these ratios are small compared to unity.

In engine flows the length and time scales of mean and turbulent motion vary continuously in time and space, and the values of α and β will, in turn, depend on their location in the cylinder as well as on time. In general, however, it is reasonable to expect that on the average these ratios (i.e., α and β) will be somewhat less than unity (of the order of the ratio of turbulent velocity to the mean velocity). Whether this condition is adequate for the model to be valid will have to be decided by comparing model results to measurements.

So far, few detailed comparisons have been made with experiments. The main reason for this, besides the novelty of multidimensional engine modeling, is the lack of well-defined data that could be used for validation. One of the few attempts to validate the k-ϵ model in engine-type flows was made by Gosman et al.,[1] who compared their model predictions to the measurements of Witze (reported privately to Gosman et al.) and Morse et al.[7] The latter measurements are probably the most appropriate for validation of the two-dimensional axisymmetric computer codes currently available. The experiments, however, lacked measurements of the velocity profile within the annular jet flowing into the cylinder; these velocities are expected to have a significant effect on the predicted flowfield. Assumptions, therefore, were made by Gosman et al.,[1] about the shape and angle of the incoming jet. The agreement between model results and the data was shown to be satisfactory (at least qualitatively), which may suggest that the model assumptions for predicting such a flow might be acceptable. However, in view of the uncertainties existing in the experiments, it would be imprudent to consider these results as a validation of the model. More comparisons with experiments over a wider range of conditions are required before the model can be deemed sufficiently accurate in predicting engine flows. Having mentioned these precautions, for the purpose of the present work we will suppose that the basic assumption (i.e., turbulent stress and rate-of-strain relation on which the model is based) is valid and proceed to examine the behavior of the k-ϵ model when the flow is compressed.

Because the flow is in the same geometrical enclosure the length scales of the mean motion and turbulence are not expected to differ significantly between a compressing and a noncompressing (i.e., with open valve) engine; it is plausible to assume that the constitutive relation (relating turbulent stress to mean rate of strain) assumed valid in the noncompressing case is valid in the case of compression. Hence, if the k-ϵ model equations are adjusted to account for compressibility effects, the model presumably can be used in the compressing case. This brings us to the subject of this study.

As reported by Morel and Mansour,[3] the k-ϵ model originally was developed and tested (see Launder and Spalding[4] and Rodi[8]) in incompressible flows of a relatively simple nature (mainly of the boundary-layer type). It subsequently was used with various degrees of success in more complicated recirculating flows that included density variation caused by thermal stratification[9] and free shear flows influenced by Mach number effects.[10] The model version in the latter calculations was not much different from that used with the incompressible flow calculations. Watkins[11] was one of the first to use the k-ϵ model to predict engine cylinder flows, where large density changes are created by a moving piston. In his analysis, Watkins included the effect of compression in the constitutive equation and also accounted for variations of density with time, but otherwise neglected compression effects. Reynolds[12] objected that with the final form of the dissipation equation used by Watkins, the model is not able to predict the case of rapid spherical compression correctly. Consequently, Reynolds in effect adjusted a constant in the ϵ equation to remove this "shortcoming." Other values for the model "constants" were used by Morel and Mansour[3] and Ramos and Sirignano.[13]

Received Feb. 3, 1982; revision received June 14, 1982.

*Senior Research Engineer, Engine Research Department.

So far most of the workers that attempted to develop the ϵ equation to "suit" engine applications have started from the modeled equations rather than starting with the exact transport equations. It is this author's view that a lot of information can be gained by examining the exact transport equations, and from there one can proceed to account for the unknown terms. This will be the procedure adopted in the present study, where the exact transport equations for k and ϵ will be presented and an order of magnitude (OM) analysis will be applied to the terms appearing in the equations. Such an OM analysis was attempted previously only for incompressible flows. The OM analysis will permit us to neglect higher-order terms. It will transpire from the analysis that in the absence of combustion, terms in the ϵ equation that were neglected essentially by Watkins[11] cannot be neglected. These terms, as will be shown, can be modeled without the addition of unknown constants. In the case of combustion near the flame other terms become significant. However, no attempt is made to model all such terms, and because no combustion will be considered these terms will be neglected.

II. Analysis of the Dissipation Equation

To derive the equation of the dissipation rate ϵ, the equation for the component of the turbulent velocity in the x_i direction (Cartesian coordinates will be used with axes x_1,x_2,x_3) u_i is differentiated with respect to x_ℓ, and the result is multiplied by the instantaneous kinematic diffusivity ν and the velocity gradient $\partial u_i/\partial x_\ell$. The product is then averaged. By denoting spatial derivatives of a variable ϕ in the x_i direction by $\phi_{,i}$ (using a similar definition for temporal derivatives) and using the convention that repeated indices imply summation (e.g., $u_{i,i}=u_{1,1}+u_{2,2}+u_{3,3}$) and with overbars and primes indicating ensemble averages and turbulent quantities, respectively, for a compressible flow the equation leading to the dissipation equation reads

$$\overset{1.\ R^{-0.5},R^{-0.5}}{\overline{\rho}\;\overline{\nu u_{i,\ell}u_{i,t\ell}}} + \overset{2.\ M^2R^{-0.5},R^{-0.5}}{\overline{\rho}_{,t}\;\overline{\nu u_{i,\ell}u_{i,t}}} + \overset{3.\ M^2R^{-1},R^{-1}\,\overline{|t'|/T}}{\overline{\rho'_{,\ell}\nu u_{i,\ell}}\;\overline{U}_{i,t}} + \overset{4.\ M^2R^{-1.5},R^{-1.5}\,\overline{|t'|/T}}{\overline{\nu u_{i,\ell}\rho'}\;\overline{U}_{i,t\ell}}$$

$$+\overset{5.\ M^2R^{-0.5},R^{-0.5}\,\overline{|t'|/T}}{\overline{\nu u_{i,\ell}\rho'_{,\ell}u_{i,t}}} + \overset{6.\ M^2R^{-0.5},R^{-0.5}\,\overline{|t'|/T}}{\overline{\nu u_{i,\ell}\rho' u_{i,t\ell}}} - \overset{7.\ M^4R^{-1.5},R^{-1.5}(\overline{|t'|/T})^2}{\overline{\nu' u_{i,\ell}}\;(\overline{\rho' u_{i,t}})_{,\ell}}$$

$$+\overset{8.\ M^2R^{-1},R^{-1}\,\overline{|t'|/T}}{\overline{\nu u_{i,\ell}\rho'_{,\ell}}\;\overline{U}_j\;\overline{U}_{i,j}} + \overset{9.\ M^2R^{-1.5},R^{-1.5}\,\overline{|t'|/T}}{\overline{\nu u_{i,\ell}\rho'}\;\overline{U}_{j,\ell}\;\overline{U}_{i,j}} + \overset{10.\ M^2R^{-1.5},R^{-1.5}\,\overline{|t'|/T}}{\overline{\nu u_{i,\ell}\rho'}\;\overline{U}_j\;\overline{U}_{i,j\ell}}$$

$$+\overset{11.\ M^2R^{-1.5},R^{-1.5}}{\overline{\nu u_{i,\ell}u_j}\;\overline{\rho}_{,\ell}\;\overline{U}_{i,j}} + \overset{12.\ R^{-1.5},R^{-1.5}}{\overline{\nu u_{i,\ell}u_j}\;\overline{\rho}\;\overline{U}_{i,j\ell}} + \overset{13.\ R^{-0.5},R^{-0.5}}{\overline{\nu u_{i,\ell}u_{j,\ell}}\;\overline{U}_{i,j}\;\overline{\rho}} + \overset{14.\ M^2R^{-1},R^{-1}\,\overline{|t'|/T}}{\overline{\nu u_{i,\ell}u_{j,\ell}\rho'}\;\overline{U}_{i,j}}$$

$$+\overset{15.\ M^2R^{-1.5},R^{-1.5}\,\overline{|t'|/T}}{\overline{\nu u_{i,\ell}u_j\rho'_{,\ell}}\;\overline{U}_{i,j}} + \overset{16.\ M^2R^{-1.5},R^{-1.5}\,\overline{|t'|/T}}{\overline{\nu u_{i,\ell}u_j\rho'}\;\overline{U}_{i,j\ell}} + \overset{17.\ R^{-0.5},R^{-0.5}}{\overline{\nu u_{i,\ell}u_{i,j\ell}}\;\overline{\rho}\;\overline{U}_j}$$

$$+\overset{18.\ M^2R^{-0.5},R^{-0.5}}{\overline{\nu u_{i,\ell}u_{i,j}}\;\overline{\rho}_{,\ell}\;\overline{U}_j} + \overset{19.\ R^{-0.5},R^{-0.5}}{\overline{\nu u_{i,\ell}u_{i,j}}\;\overline{\rho}\;\overline{U}_{j,\ell}} + \overset{20.\ M^2R^{-0.5},R^{-0.5}\,\overline{|t'|/T}}{\overline{\nu u_{i,\ell}\rho'_{,\ell}u_{i,j}}\;\overline{U}_j}$$

$$+\overset{21.\ M^2R^{-1},R^{-1}\,\overline{|t'|/T}}{\overline{\nu u_{i,\ell}\rho' u_{i,j}}\;\overline{U}_{j,\ell}} + \overset{22.\ M^2R^{-0.5},R^{-0.5}\,\overline{|t'|/T}}{\overline{\nu u_{i,}u_{i,j\ell}\rho'}\;\overline{U}_j} + \overset{23.\ M^2R^{-1},R^{-1}}{\overline{\nu u_{i,\ell}u_ju_{i,j}}\;\overline{\rho}_{,\ell}}$$

$$+\overset{24.\ 1,\ 1}{\overline{\nu u_{i,\ell}u_{j,\ell}u_{i,j}}\;\overline{\rho}} + \overset{25.\ R^{-0.5},R^{-0.5}}{\overline{\nu u_{i,\ell}u_ju_{i,j\ell}}\;\overline{\rho}} + \overset{26.\ M^2R^{-0.5},R^{-0.5}\,\overline{|t'|/T}}{\overline{\nu u_{i,\ell}u_{j,\ell}\rho' u_{i,j}}} + \overset{27.\ M^2R^{-1},R^{-1}\,\overline{|t'|/T}}{\overline{\nu u_{i,\ell}u_j\rho'_{,\ell}u_{i,j}}}$$

$$+\overset{28.\ M^2R^{-0.5},R^{-0.5}\,\overline{|t'|/T}}{\overline{\nu u_{i,\ell}u_j\rho' u_{i,j\ell}}} - \overset{29.\ M^4R^{-1.5},R^{-1.5}(\overline{|t'|/T})^2}{\overline{\nu' u_{i,\ell}}\;\overline{u_j\rho'}\;\overline{U}_{i,j\ell}} - \overset{30.\ M^4R^{-1.5},R^{-1.5}(\overline{|t'|/T})^2}{\overline{\nu' u_{i,\ell}}\;(\overline{u_j\rho'})_{,\ell}\;\overline{U}_{i,j}}$$

$$-\overset{31.\ M^4R^{-1.5},R^{-1.5}(\overline{|t'|/T})^2}{\overline{\nu' u_{i,\ell}}\;\overline{\rho' u_{i,j}}\;\overline{U}_{j,\ell}} - \overset{32.\ M^4R^{-1.5},R^{-1.5}(\overline{|t'|/T})^2}{\overline{\nu' u_{i,\ell}}\;(\overline{\rho' u_{i,j}})_{,\ell}\;\overline{U}_j} - \overset{33.\ M^4R^{-1.5},R^{-1.5}\,\overline{|t'|/T}}{\overline{\nu' u_{i,\ell}}\;(\overline{u_ju_{i,j}})\overline{\rho}_{,\ell}}$$

$$-\overset{34.\ M^2R^{-1.5},R^{-1.5}\,\overline{|t'|/T}}{\overline{\nu' u_{i,\ell}}\;(\overline{u_ju_{i,j}})_{,\ell}\;\overline{\rho}} - \overset{35.\ M^4R^{-1.5},R^{-1.5}(\overline{|t'|/T})^2}{(\overline{u_j\rho' u_{i,j}})_{,\ell}\;\overline{\nu' u_{i,\ell}}} = -\overset{36.\ R^{-1},R^{-1}}{\overline{\nu u_{i,\ell}p'_{,i\ell}}}$$

$$+\overset{37.\ M^2R^{-2},R^{-2}\,\overline{|t'|/T}}{\overline{\nu u_{i,\ell}\mu'_{,\ell j}}\;(\overline{U}_{i,j}+\overline{U}_{j,i})} + \overset{38.\ M^2R^{-2},R^{-2}\,\overline{|t'|/T}}{\overline{\nu u_{i,\ell}\mu'_{,j}}\;(\overline{U}_{i,j\ell}+\overline{U}_{j,i\ell})} + \overset{39.\ M^2R^{-2},R^{-2}\,\overline{|t'|/T}}{\overline{\nu u_{i,\ell}\mu'_{,\ell}}\;(\overline{U}_{i,jj}+\overline{U}_{j,ij})}$$

$$+\overset{40.\ M^2R^{-2.5},R^{-2.5}\,\overline{|t'|/T}}{\overline{\nu u_{i,\ell}\mu'}\;(\overline{U}_{i,j\ell j}+\overline{U}_{j,i\ell j})} + \overset{41.\ 1,1}{\overline{\nu u_{i,\ell}u_{i,j\ell j}}\;\overline{\mu}} + \overset{42.\ M^2,1}{\overline{\nu u_{i,\ell}u_{j,i\ell j}}\;\overline{\mu}}$$

$$+\overset{43.\ M^2R^{-1.5},R^{-1.5}}{\overline{\nu u_{i,\ell}u_{i,jj}}\;\overline{\mu}_{,\ell}} + \overset{44.\ M^2R^{-1.5},R^{-1.5}}{\overline{\nu u_{i,\ell}u_{j,ij}}\;\overline{\mu}_{,\ell}} + \overset{45.\ M^2R^{-1.5},R^{-1.5}}{\overline{\nu u_{i,\ell}u_{i,j\ell}}\;\overline{\mu}_{,j}}$$

$$
\begin{aligned}
&+\overset{46.\ M^2R^{-1.5},R^{-1.5}}{\overline{\nu u_{i,\ell}u_{j,i\ell}}\ \ \overline{\mu_{,\ell}}}
+\overset{47.\ M^2R^{-1.5},R^{-1.5}}{\overline{\nu u_{i,\ell}u_{i,j}}\ \ \overline{\mu_{,\ell j}}}
+\overset{48.\ M^2R^{-1.5},R^{-1.5}}{\overline{\nu u_{i,\ell}u_{j,i}}\ \ \overline{\mu_{,\ell j}}} \\
&+\overset{49.\ M^2R^{-1.5},R^{-1.5}\ |t'|/\overline{T}}{\overline{\nu u_{i,\ell}\mu'_{,\ell j}(u_{i,j}+u_{j,i})}}
+\overset{50.\ M^2R^{-1.5},R^{-1.5}\ |t'|/\overline{T}}{\overline{\nu u_{i,\ell}\mu'_{,j}(u_{i,j\ell}+u_{j,i\ell})}} \\
&+\overset{51.\ M^2R^{-1.5},R^{-1.5}\ |t'|/\overline{T}}{\overline{\nu u_{i,\ell}\mu'_{,\ell}u_{i,jj}}}
+\overset{52.\ M^4R^{-1.5},R^{-1.5}\ |t'|/\overline{T}}{\overline{\nu u_{i,\ell}\mu_{,\ell}u_{j,ij}}}
+\overset{53.\ M^4R^{-0.5},R^{-0.5}\ |t'|/\overline{T}}{\overline{\nu u_{i,\ell}\mu' u_{i,j\ell j}}} \\
&+\overset{54.\ M^4R^{-0.5},R^{-0.5}\ |t'|/\overline{T}}{\overline{\nu u_{i,\ell}\mu' u_{j,i\ell j}}}
-2/3\ \overset{55.\ M^4R^{-1.5},R^{-1.5}}{\overline{\nu u_{i,\ell}u_{n,n}}\ \ \overline{\mu_{,\ell i}}}
-2/3\ \overset{56.\ M^4R^{-1.5},R^{-1.5}}{\overline{\nu u_{i,\ell}u_{n,n\ell}}\ \ \overline{\mu_{,i}}}
-2/3\ \overset{57.\ M^4R^{-1.5},R^{-1.5}}{\overline{\nu u_{i,\ell}u_{n,ni}}\ \ \overline{\mu_{,\ell}}} \\
&-2/3\ \overset{58.\ M^2,1}{\overline{\nu u_{i,\ell}u_{n,n\ell i}}\ \ \overline{\mu}}
-2/3\ \overset{59.\ M^2R^{-2},R^{-2}\ |t'|/\overline{T}}{\overline{\nu u_{i,\ell}\mu'_{,\ell i}}\ \ \overline{U_{j,j}}}
-2/3\ \overset{60.\ M^2R^{-2},R^{-2}\ |t'|/\overline{T}}{\overline{\nu u_{i,\ell}\mu'_{,i}}\ \ \overline{U_{j,j\ell}}}
-2/3\ \overset{61.\ M^2R^{-2},R^{-2}\ |t'|/\overline{T}}{\overline{\nu u_{i,\ell}\mu'_{,\ell}}\ \ \overline{U_{j,ji}}} \\
&-2/3\ \overset{62.\ M^2R^{-2.5},R^{-2.5}\ |t'|/\overline{T}}{\overline{\nu u_{i,\ell}\mu'}\ \ \overline{U_{j,j\ell i}}}
-2/3\ \overset{63.\ M^4R^{-1.5},R^{-1.5}\ |t'|/\overline{T}}{\overline{\nu u_{i,\ell}\mu'_{,\ell i}u_{n,n}}}
-2/3\ \overset{64.\ M^4R^{-1.5},R^{-1.5}\ |t'|/\overline{T}}{\overline{\nu u_{i,\ell}\mu'_{,i}u_{n,n\ell}}} \\
&-2/3\ \overset{65.\ M^4R^{-1.5},R^{-1.5}\ |t'|/\overline{T}}{\overline{\nu u_{i,\ell}\mu'_{,\ell}u_{n,ni}}}
-2/3\ \overset{66.\ M^4R^{-0.25},R^{-0.25}\ |t'|/\overline{T}}{\overline{\nu u_{i,\ell}\mu' u_{n,n\ell i}}}
\end{aligned}
\tag{1}
$$

where the numbers from 1 to 66 appearing on top of each term are used for identification purposes. The symbols adjacent to the numbers are the orders of magnitude of the term, normalized by $\mu(\overline{u_i^2})^{1/2}/\lambda$ (μ being the laminar viscosity and λ the Taylor microscale), in the absence and presence of combustion, respectively. Thus, for example, (23. M^2R^{-1}, $R^{-1}/\overline{\nu u_{i,\ell}u_j u_{i,j}}\ \overline{\rho_{,\ell}}$) indicates that this term will be referred to as term 23 and it has an order of magnitude of M^2R^{-1} ($M \equiv$ Mach number and $R \equiv$ Reynolds number) in the absence of combustion and R^{-1} when combustion is present. Other undefined symbols are ρ, which is the density, t the time, $\underline{U}_i$ the velocity component in the x_i direction, and t' and $\overline{T}$ the fluctuating and mean temperature, respectively. For simplicity the overbars henceforth will be omitted from the mean velocity component since at no occasion will the instantaneous value of U_i be required.

A. Order of Magnitude Analysis

At this stage we proceed to assess the orders of magnitude of the various terms appearing in Eq. (1). The procedure that is adopted in estimating the OM of terms involving turbulent components will be, first, to write down such components as a product of root mean squares of the variables involved in the correlation and a correlation coefficient. For example, to assess the OM of a correlation such as $\overline{u_{i,k}u_\ell}$ from the definition of the correlation coefficient r^- between $u_{i,k}$ and u_ℓ we write

$$\overline{u_{i,k}u_\ell} = |u_{i,k}|\ |u_\ell|\, r \tag{2}$$

where the $|\ |$ denote root mean square of the enclosed quantity. We then estimate the OM of the rms values of the variables appearing (i.e., of $|u_{i,k}|$, $|u_\ell|$) in Eq. (2) and the correlation coefficient. Thus, to assess the magnitudes of the terms in Eq. (1) we need the OM of the rms values of each of the turbulent components appearing in these terms (i.e., rms values of ρ', $u_{i,i}$, μ',..., etc.), the accompanying correlation coefficient, and also the OM of the mean quantities (e.g., $U_{i,j}$, $\overline{\rho_{,j}}$).

There are general (i.e., independent of type of flow) "rules" which apply in estimating the rms values and correlation coefficients. These are reviewed briefly in the Appendix (see also Tennekes and Lumley[5]). The rms values and mean quantities that depend on the flow situation will be discussed next.

OM of rms Value and Mean Quantities

With the contents of the Appendix, to complete the OM analysis we still require the values of $|\rho'|$, $\overline{\rho_{,i}}$, $\overline{\rho_{,t}}$, $U_{i,i}$, $|u_{i,i}|$, $\overline{\mu_{,\ell}}$, and $|\mu'|$ and $|\nu'|$. These will be examined sequentially.

The equation of state can be used to obtain the following equation for the pressure fluctuation p' in which variations in the gas constant R are neglected:

$$p' = R(\rho'\overline{T} + t'\overline{\rho} + \rho' t' - \overline{\rho' t'}) \tag{3}$$

If pressure fluctuations due to buoyancy effects are neglected, then p' is of the order (see Hinze[14]) of ρu^2 (where $u^2 = \frac{1}{3}\overline{u_i^2}$). If $|t'|$ is of OM $\overline{T}$, or smaller, then at most $|\rho'|$ is of the order (indicated by the symbol $\sim$)

$$|\rho'| \sim \overline{\rho}\ \max\left(\frac{|t'|}{\overline{T}}, M^2\right) \tag{4}$$

where max ( ) signifies the largest of the enclosed variables and M is the Mach number based on the velocity u.

To obtain the OM of $\overline{\rho_{,i}}$ we differentiate the averaged equation of state with respect to x_i. This yields

$$\frac{\overline{P_{,i}}}{R\overline{T}} = \overline{\rho_{,i}} + \frac{\overline{\rho}}{\overline{T}} T_{,i} + \frac{1}{\overline{T}} (\overline{\rho' t'})_{,i} \tag{5}$$

where $\overline{P_{,i}}$ is the mean pressure gradient which is of order ρU_i^2. Thus the OM of $\overline{\rho_{,i}}$, after making use of relation (4), can be at most

$$\overline{\rho_{,i}} \sim \frac{\overline{\rho}}{\ell} \max\left(M^2, \frac{\ell}{\ell_f}\frac{\Delta T}{\overline{T}}, \frac{\ell}{\ell_f}\frac{|t'|\,|t'|}{\overline{T}^2}\right) \tag{6}$$

where ℓ is the length scale of the turbulence and ΔT the mean temperature difference across the length scale ℓ_f. The latter scale may be different than ℓ. Relation (6) shows that $\overline{\rho_{,i}}$ is dependent on ℓ/ℓ_f, $\Delta T/\overline{T}$, and $|t'|/\overline{T}$ which we next discuss.

Under motoring (i.e., noncombustion) conditions, temperature stratification can arise from kinetic heating or from differences in temperature between the combustion chamber walls and the in-cylinder gas. The normalized temperature variation (i.e., $\Delta T/\overline{T}$) arising from the former effect can be shown to be of the order of the Eckart number $u^2/c_p\overline{T}$ (where

c_p is the specific heat at constant pressure), which is a small quantity under engine running conditions. Stratification of temperature from the latter effect is more significant. In the absence of strong recirculating regions, such stratification is expected to be restricted to regions in the vicinity of the boundaries in a region of the order of the thickness of the thermal boundary layer. We, therefore, can assume that the major portion of the flowfield, under motoring conditions, will be approximately thermally homogeneous.

From the previous assumption of negligible thermal stratification in regions away from solid boundaries, it can be shown (by examining the equation of the mean of the square of the temperature fluctuations) that temperature fluctuations are negligibly small. Thus, relation (5) will, under motoring conditions, lead to the following:

$$\overline{\rho}_{,i} \sim (\rho/\ell) M^2 \tag{7}$$

In the presence of combustion the order of $|t'|$ compared to $\overline{T}$ will depend on how the former is generated. If it is initiated by turbulent motion in a temperature field having a gradient $\overline{T}_{,i}$, then (see, for example, Hinze[14])

$$|t'| \sim \ell \overline{T}_{,i} \sim \ell(\Delta T/\ell_f)$$

In the case of a flame traveling through the location under consideration, it can be argued that ℓ_f is of the order of ℓ and hence, $|t'| \sim \overline{T}$ when $\Delta \overline{T}$ is taken to be of the order of $\overline{T}$ (note that the maximum of $\Delta \overline{T}$ is the temperature difference between burnt and unburnt gasses). If, moreover, t' is caused additionally by random variation in the combustion process, then t' would attain values depending also on the fluctuating heat release. The values of $|t'|$ from the latter effect could be dependent on factors other than the temperature gradient. In general, the OM of $|t'|$ is expected to vary significantly depending on location in the flowfield and crank angle. It will here be assumed that $|t'|$ is an order of magnitude smaller than $\overline{T}$, acknowledging that at some instances and locations in the combustion chamber during combustion $|t'|$ could be larger. When $|t'|$ is equal to or less than $\overline{T}$, then relation (5) gives

$$\overline{\rho}_{,i} \sim \overline{\rho}/\ell \tag{8}$$

We next assess the OM of $\overline{\rho}_{,t}$. The density varies in time because of the piston motion and because of combustion. Under motoring conditions, a piston moving with velocity V_p produces the following temporal variations of spatial mean density ρ_m (assuming a flat piston and cylinder head and neglecting intake and exhaust cycles)

$$\rho_{m,t} = \rho_m (V_p/\ell_p) \tag{9}$$

where ℓ_p is the distance from the piston to the cylinder head. Since the piston motion is the source of motion, then V_p is of the order u (since $u \sim U$) and from geometric consideration for locations not too close to the boundaries $\ell \sim \ell_p$ and, thus, with Eq. (9) and Eq. (7) we have,

$$\overline{\rho}_{,t} \sim \rho(u/\ell) \tag{10}$$

With combustion occurring via a traveling flame with velocity U_f relative to a stationary coordinate, then the local variation of $\overline{\rho}$ due to combustion is

$$\overline{\rho}_{,t} \sim U_f \overline{\rho}_{,i} \tag{11}$$

The velocity U_f is the sum of flame velocity relative to the fluid plus the fluid velocity. From the experimental observation of Groff and Matekunas[15] the former is expected to be of the order of u and, hence, with Eq. (8) we find that the temporal variation of density due to combustion is of the order of

$$\overline{\rho}_{,t} \sim \overline{\rho(u/\ell)} \tag{12}$$

i.e., it is of the same order as in the case with no combustion.

Under general conditions (excluding mass sources) the averaged continuity equation can be written as

$$\overline{\rho}_{,t} + (\overline{\rho} U_i)_{,i} + (\overline{\rho' u_i})_{,i} = 0 \tag{13}$$

From this equation and relation (12) and (8) [or Eq. (7)] we find the order of $U_{i,i}$ can be as large as

$$U_{i,i} \sim u/\ell \tag{14}$$

This is so with or without combustion.

To estimate $|u_{i,i}|$ we will have to consider an equation for $\overline{\rho'^2}$. Such an equation can be derived from the continuity equation. This reads (again numerals on terms are for identification purposes)

$$\overset{\text{I}}{(\overline{\rho'^2})_{,t}} + \overset{\text{II}}{U_j(\overline{\rho'^2})_{,j}} \quad \overset{\text{III}}{2\overline{\rho'^2} U_{j,j}} + \overset{\text{IV}}{2\overline{\rho' u_j}\overline{\rho}_{,j}} + \overset{\text{V}}{2\overline{\rho} \quad \overline{\rho' u_{j,j}}}$$
$$+ \overset{\text{VI}}{\overline{\rho'^2 u_{j,j}}} + \overset{\text{VII}}{(\overline{u_j \rho'^2})_{,j}} \tag{15}$$

From Eqs. (5), (6), and (A6) we can deduce that terms I, II, III, IV, and VII have the following orders:

a) with combustion

$$\text{I,II,III,VII} \sim \frac{u}{\ell} \overline{\rho}^2 \left(\frac{|t'|}{\overline{T}} \right)^2$$

$$\text{IV} \sim \frac{u}{\ell} \overline{\rho}^2 \frac{|t'|}{\overline{T}}$$

b) with no combustion

$$\text{I,II,III,VII} \sim \frac{u}{\ell} \rho^2 M^4$$

At present the OM of terms V and VI are unknown, however, term V is expected to be larger than term VI (by the ratio $\overline{\rho}/|\rho'|$). We can deduce, therefore, that term V can at most be of the order $\rho^2(u/\ell)(|t'|/\overline{T})$ in the presence of combustion and of order $(u/\ell)\rho^2 M^4$ under motoring conditions. Thus (see Appendix),

a) with combustion

$$|u_{j,j}| \sim u/\lambda \tag{16}$$

b) with no combustion

$$|u_{j,j}| \sim (u/\lambda) M^2 \tag{17}$$

That is, in the absence of combustion, since M^2 is exceedingly small, the turbulence is incompressible.

Finally, to find the OM of $\overline{\mu}_{,\ell}$, $|\mu'|$, and $|\nu'|$, in accordance with the kinetic theory of gasses (cf. Bird et al.[16]), it is assumed that the laminar viscosity varies with temperature T according to

$$\mu = AT^{0.5} \tag{18}$$

where A for a specific gas is a weak function of temperature. By decomposing the temperature and viscosity in Eq. (18) into mean and fluctuating quantities, we can deduce that (when the

Table 1 Estimated OM of different variables

	With combustion	With no combustion
$\lvert\rho'\rvert$	$\sim\bar{\rho}\,\dfrac{\overline{\lvert t'\rvert}}{\bar{T}}$	$\sim\bar{\rho}\,M^2$
$\overline{\rho}_{,i}$	$\sim\dfrac{\bar{\rho}}{\ell}$	$\sim\dfrac{\bar{\rho}}{\ell}M^2$
$\overline{\rho}_{,t}$	$\sim\bar{\rho}\,\dfrac{u}{\ell}$	$\sim\bar{\rho}\,\dfrac{u}{\ell}$
$U_{i,i}$	$\sim\dfrac{u}{\ell}$	$\sim\dfrac{u}{\ell}$
$\lvert u_{i,i}\rvert$	$\sim\dfrac{u}{\lambda}$	$\sim\dfrac{u}{\lambda}M^2$
$\overline{\mu}_{,i}$	$\sim\dfrac{\bar{\mu}}{\ell}$	$\sim\dfrac{\bar{\mu}}{\ell}M^2$
$\lvert\mu'\rvert$	$\sim\bar{\mu}\,\dfrac{\overline{\lvert t'\rvert}}{\bar{T}}$	$\sim\bar{\mu}\,M^2$

temperature field has a length scale ℓ)

$$\overline{\mu}_{,i}\sim\mu/\ell,\qquad \lvert\mu'\rvert\sim\bar{\mu}(\overline{\lvert t'\rvert}/\bar{T}) \tag{19}$$

in the presence of combustion and

$$\overline{\mu}_{,i}\sim(\mu/\ell)M^2,\qquad \lvert\mu'\rvert\sim\bar{\mu}M^2 \tag{20}$$

under motoring conditions. Table 1 summarizes the maximum OM of the different variables. From the preceding analysis and the analysis given in the Appendix we are now in a position to estimate the orders of magnitude of the terms in Eq. (1). These are shown, for the general case with combustion and for the case with no combustion, above their corresponding terms in Eq. (1).

We can see from Eq. (1) that at high turbulent Reynolds number R (based on velocity u and length scale ℓ) and/or low Mach numbers, a lot of the terms appearing then become small and could be neglected. In this study we will neglect terms of the order 10^{-2} and smaller. Under engine running conditions, when we exclude the intake- and exhaust-valve flows and divided chamber type geometries, the Mach number rarely exceeds 0.05, i.e., $M^2\sim10^{-3}$. The Reynolds number varies significantly with speed, swirl, squish, crank angle,..., etc. However, values greater than 10^3 are not uncommon. Thus, we will neglect terms having orders smaller than $R^{-1/2}$.

By neglecting terms of order smaller than $R^{-1/2}$ and with some manipulation Eq. (1) can be recast in the following form in terms of ϵ (where the terms have been renumbered).

$$\underset{1}{\bar{\rho}\epsilon_{,t}}+\underset{2}{\bar{\rho}U_j\epsilon_{,j}}+\underset{3}{(\bar{\rho}\,\overline{u_j\epsilon'})_{,j}}+\underset{4}{2\bar{\rho}\,\overline{\nu u_{i,\ell}u_{j,\ell}u_{i,j}}}+\underset{5}{2\bar{\mu}\,\overline{\nu u^2_{i,j\ell}}}+\underset{6}{2\,\overline{\nu u_{i,\ell}u_{j,\ell}}\,U_{i,j}\,\bar{\rho}}+\underset{7}{2\,\overline{\nu u_{i,\ell}u_{i,j}}\,\bar{\rho}\,U_{j,\ell}}+\underset{8}{2\,\overline{\nu u_{i,\ell}u_{i,j}}\,\bar{\rho}_{,\ell}U_j}$$

$$\underset{9}{-\frac{\epsilon}{\nu}\,\bar{\mu}_{,t}}+\underset{10}{\epsilon\,\bar{\rho}_{,t}}-\boxed{\underset{11}{\bar{\rho}\,U_j\frac{\epsilon}{\bar{\nu}}\,\bar{\nu}_{,j}}-\underset{12}{\bar{\rho}\,\frac{\overline{u_j\epsilon'}}{\bar{\nu}}\,\bar{\nu}_{,j}}-\underset{13}{\bar{\rho}\,\overline{u_{j,j}\epsilon'}}+\underset{14}{2/3\,\bar{\mu}\,\overline{\nu u_{i,\ell}u_{n,n\ell i}}}}-\boxed{\underset{15}{\overline{u_j\epsilon'}\,\bar{\rho}_{,j}}+\underset{16}{\rho_{,\ell}\overline{\nu u_{i,\ell}u_{i,t}}}}=0 \tag{21}$$

where ϵ' is the nonaveraged dissipation rate and the "boxed" terms can be neglected when the temperature field is not strongly stratified.

B. Modeling of the Dissipation Equation

Apart from the terms containing ϵ, all terms in Eq. (21) require modeling. Terms 3-5 have been modeled previously in the case of incompressible flows by Launder et al.[17] as

$$\underset{3}{(\bar{\rho}\,\overline{u_j\epsilon'})_{,j}}=-\left(\frac{\mu_t}{\sigma_\epsilon}\epsilon_{,j}\right)_{,j} \tag{22}$$

$$(\underset{4}{2\bar{\rho}\,\overline{\nu u_{i,\ell}u_{j,\ell}u_{i,j}}}+\underset{5}{2\bar{\mu}\,\overline{\nu u^2_{i,j\ell}}})=-c_1\frac{\mathcal{P}\epsilon}{k}+c_2\bar{\rho}\frac{\epsilon^2}{k} \tag{23}$$

where μ_t is the turbulent viscosity, σ_ϵ an effective "Prandtl" number, c_1 and c_2 model constants, and $\mathcal{P}$ the production of turbulence kinetic energy

$$\mathcal{P}=-\rho\,\overline{u_iu_j}U_{i,j}$$

where $\overline{u_iu_j}$ is written in the framework of the k-ϵ model as

$$\overline{u_iu_j}=\mu_t[U_{i,j}+U_{j,i}-2/3U_{m,m}\delta_{ij}]-2/3\rho k\delta_{ij} \tag{24}$$

To extend the application of the dissipation equation to compressible flows, Watkins[11] retained only term 10 from all the terms that result from compressibility effects and assumed that terms 6 and 7 (which are small in incompressible flows) are accounted for implicitly in the modeled term $c_1\mathcal{P}\epsilon/k$. Watkins' work yielded the following equation for ϵ:

$$(\bar{\rho}\epsilon)_{,t}+(\rho U_j\epsilon)_{,j}=\left(\frac{\mu_t}{\sigma_\epsilon}\epsilon_{,j}\right)_{,j}+c_3\bar{\rho}\epsilon U_{i,i}$$

$$+c_1\frac{\mathcal{P}\epsilon}{k}-c_2\bar{\rho}\frac{\epsilon^2}{k} \tag{25}$$

where c_3 is a constant equal to 1. Several workers employed this equation (e.g., Gosman et al.[1] and Syed and Bracco[2]). Ramos and Sirignano[13] neglected terms 9 and 10 in Eq. (21), which in effect means a zero value for c_3. Reynolds[12] "deduced" the value of c_3 from the constraint that the turbulence conserves its angular momentum when rapidly distorted. While using such a constraint to set model constants is appropriate, it is doubtful that the k-ϵ model can predict a rapid distorted turbulence. This is because in the limit of rapid distortion the turbulence has an elastic response to the distortion (see, for example, Crow[18]) rather than the viscous response implied in the k-ϵ model. This, however, does not necessarily disqualify the k-ϵ model to be used in engines since at no time is the turbulence expected to be rapidly distorted. Morel and Mansour[3] extended Reynolds' work to encompass more "modes" of compression than the spherical compression case employed by Reynolds. This led to a variable value of c_3 which depended on a dimensionless parameter characterizing the local strain-rate tensor.

Because of the lack of rigor in previous work in the determination of c_3, Ahmadi et al.[19] made computation in a combusting flow with different values of c_3. They found their results to be dependent on c_3 but when comparing their results with experimental observation they could not choose a value of c_3 that gave good agreement with experiment without that value being dependent on the constants of their accompanying combustion model. This emphasizes the need for a rigorous approach for the determination of c_3 like that described herein.

In the present work both the terms 9 and 10, which require no modeling, will be retained since these are equal to

$$\frac{\epsilon}{\nu}\overline{\mu}_{,t} \simeq -\overline{\rho}\,\frac{\epsilon}{2}\,(s-1)\,U_{i,i} \tag{26}$$

$$\epsilon\overline{\rho}_{,t} \simeq -\rho\epsilon U_{i,i} \tag{27}$$

with the assumption of a polytropic compression with exponent s and assuming small values of $|t'|/\overline{T}$. This shows that these two terms are of the same order as other terms retained in Eq. (21) and, therefore, should be retained.

We next turn attention to term 6. As mentioned in the Appendix, the main contribution to the correlation $\overline{u_{i,\ell}u_{i,j}}$ arises from motions in the high-frequency range of the energy spectrum. At sufficiently high Reynolds number these motions are close to isotropic. Therefore, we can use the isotropy condition to relate different correlations to each other. From the isotropic constraint it is possible to derive the following relation.

$$\overline{u_{i,k}u_{j,\ell}} = 1/2f[\delta_{j\ell}\delta_{ik} + \delta_{i\ell}\delta_{jk}] + g\delta_{\ell k}\delta_{ij} - \frac{\delta_{j\ell}\delta_{ik}}{2} - \frac{\delta_{i\ell}\delta_{jk}}{2} \tag{28}$$

where f and g are some scalar functions of u and the δ_{ij}'s are components of the unit tensor. It is straightforward to show that terms 6, 7, and 8 become, respectively,

$$2\nu\overline{u_{i,\ell}u_{j,\ell}}U_{i,j}\overline{\rho} = 2/3\epsilon U_{i,i}\overline{\rho} \tag{29}$$

$$2\nu\overline{u_{i,\ell}u_{i,j}}U_{j,\ell}\overline{\rho} = 2/3\epsilon U_{i,i}\overline{\rho} \tag{30}$$

$$2\nu\overline{u_{i,\ell}u_{i,j}}U_j\overline{\rho}_{,\ell} = 2/3\epsilon U_i\overline{\rho}_{,i} \tag{31}$$

The correlations between ϵ' and u_j appearing in terms 12 and 15 are modeled with a gradient-transport hypothesis consistent with what is implied in Eq. (22). Thus, the two terms become

$$\overline{\rho}\,\frac{\overline{u_j\epsilon'}}{\nu}\,\overline{\nu}_{,j} = \frac{\mu_t}{\sigma_\epsilon\,\nu}\,\overline{\nu}_{,j}\epsilon_{,j} \tag{32}$$

$$\overline{u_j\epsilon'}\;\overline{\rho}_{,j} = \frac{\mu_t}{\rho\sigma_\epsilon}\,\overline{\rho}_{,j}\epsilon_{,j} \tag{33}$$

We note here that no extra constants appear since the constants used are the same as those in Eq. (22).

Terms 13, 14, and 16 which are expected to be of importance only in the vicinity of flames are not considered in the present work. Their neglect, however, will not influence the results of the example computations shown later since no combustion will be considered. Collecting the modeled terms and neglecting gradients of $\overline{u'\rho'}$ compared to gradients of $\overline{U\rho}$, the proposed dissipation equation is as follows (renumbering the terms again).

$$\overset{1}{(\rho\epsilon)_{,t}} + \overset{2}{(\rho U_j\epsilon)_{,j}} = \overset{3}{c_1\frac{\epsilon}{k}} \overset{4}{- c_2\frac{\epsilon^2}{k}} + \overset{5}{\left(\frac{\mu_t\epsilon_{,j}}{\sigma_\epsilon}\right)_{,j}} \overset{6}{- 1/3\,\overline{\rho}\,\epsilon\,U_{i,i}} + \overset{7}{\frac{\epsilon}{\nu}\,\overline{\mu}_{,t}} + \overset{8}{\overline{\rho}U_j\frac{\epsilon}{\nu}\,\overline{\nu}_{,j}} \overset{9}{- \frac{\mu_t}{\sigma_\epsilon\,\nu}\,\overline{\nu}_{,j}\epsilon_{,j}} \overset{10}{- \frac{\mu_t}{\rho\sigma_\epsilon}\,\overline{\rho}_{,j}\epsilon_{,j}} \tag{34}$$

We note that Eq. (34) contains no undefined constants. The value of $-1/3$ multiplying term 6 was obtained rigorously (by assuming local isotropy of the turbulence) while the constants in terms 9 and 10 are equal to constants appearing in counterpart terms modeled in incompressible flows. The remaining new terms added (i.e., 7 and 8) are exact and contain no new constants.

By comparing Eq. (34) to Eq. (25) used by Watkins[11] we find the following differences: 1) the value of c_3 used (term 6) is $-1/3$ as opposed to the value of 1.0 employed by Watkins[11] and 2) Eq. (34) contains terms 7-10 which were neglected in Eq. (25). The first of these terms is important in cases with and without combustion while the latter terms are expected to be significant only in the presence of combustion (or in other cases involving strong thermal stratification).

III. Analysis of the Turbulence Kinetic Energy Equation

The equation for k can be derived from the Navier-Stokes equation. This can be written as (cf. Favre[20])

$$\overset{1.\ 1,1}{\overline{(\rho k)}_{,t}} + \overset{2.\ 1,1}{U_j(\overline{\rho k})_{,j}} + \overset{3.\ M^2,\ |t'|/T}{\overline{(\rho' k')}_{,t}} + \overset{4.\ M^2,\ |t'|/\overline{T}}{U_j\ \overline{(\rho' k')}_{,j}} + \overset{5.\ M^2,\ |t'|/T}{\overline{\rho' u_i}U_{i,t}} + \overset{6.\ M^2,\ |t'|/T}{\overline{\rho' u_i}\ U_j\ U_{i,j}} = -\overset{7.\ 1,1}{\overline{\rho}\ \overline{u_iu_j}\ U_{i,j}}$$

$$- \overset{8.\ M^2,\ |t'|/\overline{T}}{\overline{\rho' u_iu_j}\ U_{i,j}} - \overset{9.\ 1,1}{k\ \overline{\rho}\ U_{j,j}} - \overset{10.\ M^2,\ |t'|/\overline{T}}{\overline{\rho' k'}\ U_{j,j}} - \overset{11.\ 1,1}{(\overline{u_jp'}} + \overset{12.\ M^2,\ |t'|/\overline{T}}{\overline{\rho' k' u_j}} + \overset{13.\ 1,1}{\overline{\rho}\ \overline{k'u_j})_{,j}} + \overset{14.\ M^2,1}{\overline{p'u_{i,i}}}$$

$$
\begin{aligned}
&+\overset{15.\; R^{-1},R^{-1}}{\overline{\mu}\;k_{,jj}}
-\overset{16.\; 1,1}{\overline{\rho\epsilon}}
+\overset{17.\; M^2,1}{\overline{\mu}\;\overline{u_iu_{j,ji}}}
+\overset{18.\; M^2R^{-1},R^{-1}}{\overline{\mu_{,j}}k_{,j}}
+\overset{19.\; M^2R^{-1},R^{-1}}{\overline{u_iu_{j,i}}\;\overline{\mu_{,j}}}
+\overset{20.\; M^2R^{-1},R^{-1}\,|t'|/\overline{T}}{\overline{u_i\mu'_{,j}}(U_{i,j}+U_{j,i})}\\
&+\overset{21.\; M^2R^{-1},R^{-1}\,|t'|/\overline{T}}{\overline{u_i\mu'}\;(U_{i,jj}+U_{j,ij})}
+\overset{22.\; M^2,\,|t'|/T}{\overline{\mu'_{,j}k'_{,j}}}
+\overset{23.\; M^2R^{-1},R^{-1}\,|t'|/\overline{T}}{\overline{u_i\mu'_{,j}u_{j,i}}}
+\overset{24.\; M^2,\,|t'|/\overline{T}}{\overline{\mu' k_{,jj}}}
+\overset{25.\; M^2,\,|t'|/\overline{T}}{\overline{\nu}\;\overline{\rho' u_{i,j}u_{i,j}}}\\
&-\overset{26.\; M^4,(|t'|/\overline{T})^2}{\overline{\rho'\nu'}\;\overline{u_{i,j}u_{i,j}}}
+\overset{27.\; M^4R^{-0.5},R^{-0.5}\,|t'|/\overline{T}}{\overline{\mu' u_iu_{j,ij}}}
-\overset{28.\; M^2,1}{\overline{2/3\;\mu}\;\;\overline{u_iu_{j,ij}}}
+\overset{29.\; M^4R^{-1},R^{-1}}{\overline{\mu_{,i}}\;\;\overline{u_iu_{j,j}}}\\
&+\overset{30.\; M^2R^{-1},R^{-1}\,|t'|/\overline{T}}{\overline{\mu' u_i}\;U_{j,ij}}
-\overset{31.\; M^2R^{-1},R^{-1}\,|t'|/\overline{T}}{\overline{u_i\mu'_{,i}}\;U_{j,j}}
+\overset{32.\; M^4R^{-0.5},R^{-0.5}\,|t'|/\overline{T}}{\overline{\mu' u_iu_{j,ij}}}
+\overset{33.\; M^4R^{-1},R^{-1}\,|t'|/T}{\overline{\mu'_{,i}u_{j,j}u_i}}
\end{aligned}
\tag{35}
$$

where k' is the nonaveraged value of k (i.e., $k' = 1/2\, u_i^2$).

By carrying out an OM analysis on Eq. (35) similar to that made with the dissipation equation, we can arrive at the estimation of the OM shown above each term. The OM's are normalized by $\mu u^2/\lambda^2$.

Neglecting terms of order smaller than $R^{-1/2}$ and rearranging we get the following equation for k (with the terms being renumbered)

$$
\begin{aligned}
&\underset{}{\overset{1}{(\overline{\rho}k)_{,t}}}+\overset{2}{(\rho U_jk)_{,j}}+\boxed{\overset{3}{(\overline{\rho'k'})_{,t}}+\overset{4}{U_j(\overline{\rho'k'})_{,j}}-\overset{5}{\overline{\rho'u_i}\;U_{i,t}}}\\
&+\boxed{\overset{6}{\overline{\rho'u_i}\;U_jU_{i,j}}}=\overset{7}{-\overline{\rho}\;\overline{u_iu_j}U_{i,j}}-\boxed{\overset{8}{\overline{\rho'u_iu_j}\;U_{i,j}}-\overset{9}{\overline{\rho'k'}\;U_{j,j}}}\\
&-\Big(\overset{10}{\overline{u_jp'}}+\boxed{\overset{11}{\overline{\rho'k'u_j}}}+\overset{12}{\overline{\rho}\;\overline{k'u_j}}\Big)_{j,j}+\boxed{\overset{13}{\overline{p'u_{i,i}}}}\overset{14}{-\rho\epsilon}\\
&+\boxed{\overset{15}{\overline{\mu}\;\;\overline{u_iu_{j,ji}}}}+\boxed{\overset{16}{\overline{\mu'_{,j}k'_{,j}}}+\overset{17}{\overline{\mu'k_{,jj}}}-\overset{18}{2/3\;\overline{\mu}\;\;\overline{u_iu_{j,ij}}}}
\end{aligned}
\tag{36}
$$

where the boxed terms can be neglected in locations removed from the vicinity of flames. The terms 1, 2, 7, 10, 12, and 14 are the same terms that appear in the incompressible flow situations, and, hence, in the absence of combustion the modeled equation for k will retain the same form that was proposed for cases when the flow is incompressible. In the case of combustion all the boxed terms should be modeled. Their modeling, however, is beyond the scope of the present work. Thus, for the purpose of the present work (which will not include combustion) the boxed terms in the k equation will be dropped. This will give us the following modeled k equation version used by several authors (e.g., Watkins[11] and Gosman et al.[1]):

$$
(\rho k)_{,j}+(\rho U_jk)_{,j}=-\rho\;\overline{u_iu_j}U_{i,j}+\left(\frac{\mu_\epsilon}{\sigma_k}\;k_{,j}\right)_{,j}-\rho\epsilon \tag{37}
$$

IV. Results and Discussion

Due to the absence of adequate experimental data that include compression effects with which to compare the model results, comparison will be made between results obtained by employing the ϵ equation proposed in the present study and those obtained with the version used by Watkins.[11] This comparison will give us an idea of the effect of the proposed changes on the turbulence calculation.

The test calculations were made with the CONCHAS computer code[21] in which the k-ϵ model has been incorporated. In these calculations the engine is assumed to have a flat piston and a flat cylinder head and is motored (i.e., no combustion is present) at 1600 rev/min. Other relevant specifications are given in Table 2.

The computation started at 140 deg before top dead center (B.T.D.C.), where the intake value is assumed to close. Starting values of k were assumed spatially uniform and equal to 12% of the kinetic energy based on the maximum piston velocity. The initial values of ϵ were computed from

$$
\epsilon=c_\mu^{-3/4}k^{3/2}/y
$$

where c_μ is a constant (see Table 3 for value of constants used) and y is the distance to the nearest solid boundary. All the mean velocities were initially taken equal to zero, except for the axial velocity component U_2 which was calculated from

$$
U_2=V_p\,(1-x_2/\ell_p)
$$

where x_2 is the vertical distance from the location where U_2 is being calculated to the piston crown.

Results for ℓ ($\equiv k^{3/2}/\epsilon$) and the turbulent velocity u are shown in Figs. 1 and 2, respectively. These are given for two stationary locations in space situated in the vicinity of the centerline at 0.01 mm and 0.02 m from the cylinder head. It is seen from Fig. 1 that the effect of the new model is to decrease the length scale in comparison to that from Watkin's model. This was to be expected since during compression $U_{i,i}$ is negative, and with a negative value for c_3 in Eq. (25) (as is the case with the present model), $c_3U_{i,i}\rho\epsilon$ will give a positive contribution to ϵ. This in turn will reduce the value of ℓ. We also note from Fig. 1 that the trends in the variation of ℓ with crank angle given by the two models are different, the present model initially giving a region where ℓ is approximately constant followed by a decline in ℓ later in the compression stroke. With the Watkins model we see a sharp increase in ℓ at both locations as compression proceeds, with a decline in ℓ starting at 50° B.T.D.C. While the initial rise in the length scale is not a physical impossibility, it is unlikely for it to increase at such a rate, especially for the relatively simple piston/cylinder configuration used. Another alarming feature of the results obtained with the Watkins model is the levels of the length scales observed. We can see, for example, that at point 1, which is 0.01 m away from the cylinder head, ℓ (which

Table 2 Engine geometry

Bore	0.099 m
Stroke	0.0876 m
Connecting rod length	0.163 m
Compression ratio	10.5

Table 3 Model constants used in k-ϵ model

Constant	c_1	c_2	σ_ϵ	c_μ	σ_k
Value	1.44	1.9	1.3	0.09	1

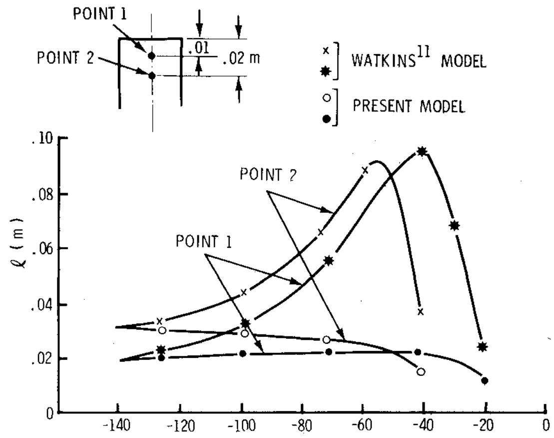

Fig. 1 Variation of computed length scale ℓ with crank angle.

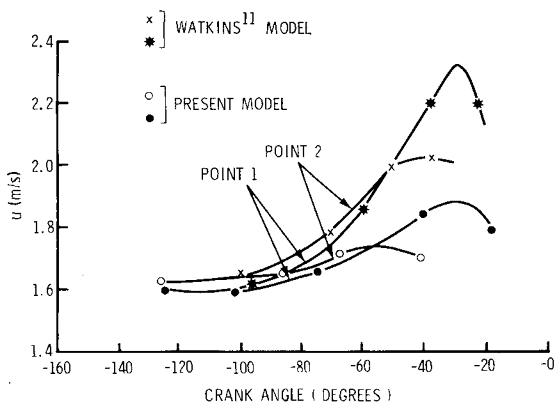

Fig. 2 Variation of turbulent velocity with crank angle.

is roughly the size of the energy containing eddies) reaches a value of about 0.094 m, i.e., 9.4 times its distance from the wall. This does not seem plausible since an "eddy" centered at some distance from the wall cannot have a "diameter" in excess of twice that distance.

The variation in k with crank angle shown in Fig. 2 also reveals lower values of k for the present model calculation. This again can be explained by the occurrence of a higher dissipation rate with the present model. We note here that the difference in values attained by the two models is expected to have a significant effect on combustion (if it were to occur) for, as was shown by Groff and Matekunas,[15] flame speeds are proportional to the value of u near top dead center. Thus, by using the maximum values of u predicted by the two models, a 25% slower combustion rate will be predicted by the present model.

From this study we can deduce that the results obtained with the new ϵ equation [Eq. (34)] are more plausible than those produced by the earlier version reported by Watkins.[11] This, of course, does not answer the question: is the present version of the k-ϵ model adequate to be used confidently in engine flow computations? Unfortunately, the answer to this question will have to await comparisons with experimental results. At this stage, however, it can be argued that if we accept the model version used by Watkins[11] as sufficiently accurate for predicting the flow in a noncompressing reciprocating engine, then the model equation proposed here would be capable of predicting the flow in the same engine motored with compression. The basis of this argument is the following. The main differences between the ϵ equations in the compressing and noncompressing situations are the terms 6, 7, 9, and 10 in Eq. (21). Among these terms, as was shown, there are some that do not require modeling and, hence, are exact, and the remaining have been modeled rigorously (assuming local isotropy of the turbulence). Thus any inaccuracies that will appear are expected to evolve from terms that are significant, both with and without compression, namely, the terms 3, 4, and 5 [in Eq. (21)] that were modeled in incompressible flows.

As was noted earlier there are several terms that are of significance (only) during combustion that were omitted from Eqs. (34) and (37). These terms were invariably neglected in all previous works involving combustion in which the k-ϵ model has been used (e.g., Khalil et al.[9] and Ahmadi-Befrui et al.[19]). Thus the present work makes no more severe assumptions than have been made previously. However, to take proper account of combustion it is quite likely that these terms should be considered. It should be stated, though, that modeling of these terms is not an easy task since any development of the model will require detailed measurements in a combusting environment, which, to the author's knowledge do not exist. Furthermore, the constants appearing in these terms will have to be closely associated with the constants selected in the necessarily accompanying combustion model.

V. Conclusions

1) An order-of-magnitude analysis has been applied to the basic equations for k and ϵ under engine running conditions. The analysis highlighted important terms in the ϵ equation that were in effect neglected in previous studies.

2) Under the assumption of high turbulence Reynolds number some of the terms mentioned were "rigorously" modeled. The modeled versions in this case did not contain any undefined constants. The remaining terms, which are mainly significant in the direct vicinity of a flame, were neglected as has been done by other workers.

3) Calculations performed using the modified ϵ equation showed more physically plausible results than with an earlier version used. The k equation used in the present computation basically was unchanged from the version used in the literature.

4) More experimental data are required for validation purposes.

Appendix

In this Appendix we will set up some relations and definitions that are necessary for estimating the OM's in Eq. (1). From the definition of the Taylor microscale λ we can generally write

$$|u_{i,j}| \sim u/\lambda \tag{A1}$$

Similarly, for temperature fluctuation t' and Taylor microscale of temperature λ_t, we have

$$|t'_{,j}| \sim t'/\lambda_t \tag{A2}$$

For a Prandtl number close to unity and a steady homogeneous shear flow (see, for example, Ref. 5)

$$\lambda_t \sim \lambda \tag{A3}$$

In the present work we will assume Eq. (A3) to be valid, and, in general, for any variable ϕ we will estimate $\phi_{,i}$ from

$$|\phi_{,i}| \sim \phi/\lambda \tag{A4}$$

The microscale λ is related to the length scale of the energy-containing eddies ℓ in an isotropic turbulence by (see Ref. 15)

$$\frac{\lambda}{\ell} = \left(\frac{15}{B}\right)^{1/2} R^{-1/2} \tag{A5}$$

where B is a constant of order unity.

A length scale L for an average quantity $\bar{\Phi}$ is defined from

$$\bar{\Phi}_{,i} \sim \frac{\Delta\Phi}{L} \tag{A6}$$

where $\Delta\Phi$ is the variation in $\bar{\Phi}$ over the length scale L. In shear-generated turbulence a relation between ℓ and L (when $\bar{\Phi} \simeq U$) is usually taken as (see Ref. 5)

$$\frac{u}{\ell} \sim \frac{U}{L} \tag{A7}$$

Because the turbulent kinetic energy in an engine is of the same order as that of the mean motion, i.e., $u \sim U$, then with relation Eq. (A7) we will take ℓ or L to be of the same order.

From the analysis given in Ref. 5 the root mean square of the second derivative of a fluctuating-turbulence component is related to λ and the Kolmogoroff length scale η by

$$|\phi_{,ij}| \sim |\phi|/\lambda\eta \tag{A8}$$

With relation Eq. (A8) and the preceding definitions it can be shown that the correlation coefficients between fluctuations and first derivatives of fluctuations are at best of the order λ/ℓ. Thus, for example,

$$|\phi\theta_{,i}| \sim |\phi| \frac{|\theta|}{\lambda} \cdot \frac{\lambda}{\ell} \tag{A9}$$

where θ and ϕ are any fluctuating components. It can also be shown that the correlation coefficient between fluctuations and second derivatives of fluctuations is of the order η/λ.

In an isotropic flowfield, correlations of the following form are zero.

$$\overline{u_{i,\ell}\phi_{,\ell}} = 0 \tag{A10}$$

$$\overline{u_{i,\ell}u_{j,m}} = 0 \quad \text{when } \ell = m \text{ and } i \neq j$$
$$\text{or } i = j \text{ and } \ell \neq m \tag{A11}$$

where ϕ in Eq. (A10) denotes a fluctuating scalar. In nonisotropic fields, relation Eqs. (A10) and (A11) are not necessarily satisfied. At high Reynolds numbers, however, the part of the motion (mainly that in the dissipation range of the energy spectrum) which contributes most[5] to correlations such as in Eqs. (A10) and (A11) is close to isotropic; the anisotropy there, as argued in Ref. 5, is proportional to the ratio of the strain rate of all the larger motions to the strain rate of the motions in the dissipation range. Thus, the value of the correlation Eqs. (A10) and (A11) in a nonisotropic field will be different from zero only in response to the degree of anisotropy in the dissipation range. Since the ratio of the strain rates between the large-scale motions and those in the dissipation range is of the order λ/ℓ, the orders of magnitude of the correlation Eqs. (A10) and (A11) become

$$\overline{u_{i,\ell}\phi_{,\ell}} \sim \frac{u|\phi|}{\lambda\ell}$$

$$\overline{u_{i,\ell}u_{j,m}} \sim \frac{u^2}{\lambda\ell} \quad \text{when } \ell = m \text{ and } i \neq j$$
$$\text{or } i = j \text{ and } \ell \neq m$$

Acknowledgments

The author is grateful for the interest and useful comments made by Drs. R. Diwakar and R. Krieger on this work.

References

[1] Gosman, A.D., Johns, R.J.R., and Watkins, A.P., "Development of Prediction Methods for In-Cylinder Processes in Reciprocating Engines," *Combustion Modeling in Reciprocating Engines,* edited by J.N. Mattavi and C.A. Amann, Plenum Press, New York, 1980, pp. 69-124.

[2] Syed, S.A. and Bracco, F.V., "Further Comparison of Computed and Measured Divided-Chamber Engine Combustion," SAE Paper 790247, 1979.

[3] Morel, T. and Mansour, N., "Modeling of Turbulence in Rapidly Compressed Flows," SAE Paper 820040, 1982.

[4] Launder, B.E. and Spalding, D.B., *Mathematical Models of Turbulence,* Academic Press, London and New York, 1972.

[5] Tennekes, H. and Lumley, J.L., *A First Course in Turbulence,* MIT Press, 1974.

[6] Lumley, J.L., "Rational Approach to Relations between Motions of Differing Scales in Turbulent Flows," *The Physics of Fluids,* Vol. 10, No. 7, 1967, p. 1405.

[7] Morse, A.P., Whitelaw, J.H., and Yianneskis, M., "The Influence of Swirl on the Flow Characteristics of a Reciprocating Piston-Cylinder Assembly," Imperial College Mechanical Engineering Dept. Rept. FS/78/24, 1978.

[8] Rodi, W., Ph.D. Thesis, Imperial College, University of London, 1972.

[9] Khalil, E.E., Spalding, D.B., and Whitelaw, J.H., "The Calculation of Local Flow Properties in Two-Dimensional Furnaces," *International Journal of Heat and Mass Transfer,* Vol. 18, 1975, p. 775.

[10] Launder, B.E., Morse, A., Rodi, W., and Spalding, D.B., "Prediction of Free Shear Flows—A Comparison of the Performance of Six Turbulence Models," NASA SP 321, Vol. 1, 1973.

[11] Watkins, A.P., Ph.D. Thesis, Imperial College, University of London, 1977.

[12] Reynolds, W.C., "Modeling of Fluid Motions in Engines—An Introductory Overview," *Combustion Modeling in Reciprocating Engines,* edited by J.N. Mattavi and C.A. Amann, Plenum Press, New York, 1980, pp. 41-68.

[13] Ramos, J.I. and Sirignano, W.A., "Axisymmetric Flow Model with and without Swirl in a Piston Cylinder Arrangement with Idealized Value Operation," SAE Paper 800284, 1980.

[14] Hinze, J.O., *Turbulence,* McGraw Hill Inc., New York, 1975.

[15] Groff, E.G. and Matekunas, F.A., "The Nature of Turbulent Flame Propagation in a Homogeneous Spark-Ignited Engine," General Motors Research Lab. Pub. GMR-3134R, Feb. 25, 1980.

[16] Bird, R.B., Stewart, W.E., and Lightfoot, E.N., *Transport Phenomena,* John Wiley and Sons, New York, 1960.

[17] Launder, B.E., Reece, G.J., and Rodi, W., "Progress in the Development of a Reynolds-Stress Turbulence Closure," *Journal of Fluid Mechanics,* Vol. 68, No. 3, p. 537.

[18] Crow, S.C., "Viscoelastic Properties of Fine-Grained Incompressible Turbulence," *Journal of Fluid Mechanics,* Vol. 33, No. 1, 1968, p. 1.

[19] Ahmadi-Befrui, B., Gosman, A.D., Lockwood, F.C., and Watkins, A.P., "Multidimensional Calculation of Combustion in an Idealised Homogeneous Charge Engine: A Progress Report," SAE Paper 810151, 1981.

[20] Favre, A., "Statistical Equations of Turbulent Gases," *Problems of Hydrodynamics and Continuum Mechanics,* SIAM, Philadelphia, 1969.

[21] Butler, T.D., Cloutman, L.D., Dukowicz, J.K., and Ramshaw, J.D., "CONCHAS: An Arbitrary Lagrangian-Eulerian Computer Code for Multicomponent Chemically Reactive Fluid Flow at all Speeds," Los Alamos Scientific Lab. Rept. LA-8129-MS, Nov. 1979.