CHAPMAN & HALL/CRC Research Notes in Mathematics 431

G Lemarié-Rieusset

Recent developments
n the Navier-Stokes
problem

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CRC Press UK
Chapman & Hall/CRC Statistics and Mathematics
23 Blades Court
Deodar Road
London SW15 2NU
Tel: 020 8875 4370

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CHAPMAN & HALL/CRC

A CRC Press Company
Boca Raton London New York Washington, D.C.

Library of Congress Cataloging-in-Publication Data

Lemarié, Pierre Gilles, 1960-
Recent developments in the Navier-Stokes problem / P.G. Lemarié-Rieusset.
p. cm. -- (Chapman & Hall/CRC research notes in mathematics series ; 431)
Includes bibliographical references and index.
ISBN 1-58488-220-4 (alk. paper)
1. Navier-Stokes equations. I. Title. II. Series.

QA374 .L39 2002
515′.353--dc21 2002018858

Direct all inquiries to CRC Press LLC, 2000 N.W. Corporate Blvd., Boca Raton, Florida 33431.

International Standard Book Number 1-58488-220-4
Library of Congress Card Number 2002018858
Printed in the United States of America 1 2 3 4 5 6 7 8 9 0
Printed on acid-free paper

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Preface

This book is a self–contained exposition of recent results on the Navier–Stokes equations, presented from the point of view of real harmonic analysis. A quarter of the book is an introduction to real harmonic analysis, where all the material we need in the book is introduced and proved (this part is based on a lecture given at Paris XI–Orsay in February-June 1998); the reader is assumed to have a basic knowledge of functional analysis, including the theory of the Fourier transform of tempered distributions. The other parts of the book are devoted to the Navier–Stokes equations on the whole space and include many recent results, such as the Koch and Tataru theorem on existence of mild solutions [KOCT 01], the results of Brandolese [BRA 01] and Miyakawa on the decay of solutions in space [MIY 00] or time (with Schonbek [MIYS 01]), many results on uniqueness (Chemin [CHE 99], Furioli, Lemarié–Rieusset and Terraneo [FURLT 00], Lions and Masmoudi [LIOM 98], May [MAY 02], Meyer [MEY 99] and Monniaux [MON 99]), results on Leray's self-similar solutions (Nečas, Ružička and Šverák [NECRS 96] and Tsai [TSA 98]), results on the decay of Lebesgue or Besov norms of solutions (Kato [KAT 90], Cannone and Planchon [CANP 00]), and the existence of solutions for a uniformly square integrable initial value [LEM 98b]. Older classical results are included, such as the existence of Leray weak solutions [LER 34], the uniqueness theorems of Serrin [SER 62] and Sohr and Von Wahl [WAH 85], Kato's theorems on the existence of mild solutions [FUJK 64], [KAT 84], [KAT 92], and the regularity criterion of Caffarelli, Kohn and Nirenberg [CAFKN 82].

Many proofs and statements are original. I tried to give general statements in the theorems and to remain in the setting of real harmonic analysis when proving the theorems. At some points, I have chosen not to give the shortest proofs, but to give proofs using only materials that are found in the limits of this book.

I am a newcomer in the vast realm of the theory of the Navier–Stokes equations, beginning to work seriously in this field in 1995, when I moved from the Université Paris XI-Orsay to the Université d'Évry and when G. Furioli and E. Terraneo began to prepare their theses with me.

At that time, I had no specific knowledge in the theory of PDEs, working rather in the field of real harmonic analysis. I was a specialist in wavelets, and before their invention by Meyer in 1985, I had worked on singular integrals, the Littlewood–Paley decomposition, and Besov spaces.

I took interest in the Navier–Stokes equations when Cannone finished his thesis [CAN 95], where the main tools were precisely wavelets, the Littlewood–Paley decomposition and Besov spaces. In February 1997, using Besov spaces, Furioli, Terraneo and I were able to prove uniqueness of solutions in the space $\mathcal{C}([0,T),(L^3(\mathbb{R}^3))^3)$ [FURLT 00]. Some monthes later, Meyer gave a simpler proof of this uniqueness result, using Lorentz spaces instead of Besov spaces [MEY 99].

I then taught at Université Paris XI-Orsay lecturing on the Navier–Stokes equations viewed from the point of view of real harmonic analysis, including introductory lessons on Besov and Lorentz spaces. Though I had heard about Lorentz spaces for fifteen years, this was still a brand new topic for me. The book is mainly based on my efforts to give a simple and efficient introduction to those technical spaces, in order to get efficient tools for proving inequalities. Thus, I have chosen to introduce Besov and Lorentz spaces through the discrete J-method of real interpolation, as the most simple and direct way to get sharp inequalities.

The efficiency of this approach may be seen in the chapter on Leray's self-similar solutions (Chapter 34), where we give a simplified proof of Tsai's results [TSA 98] and in the chapters on uniqueness of mild solutions (Chapters 27 and 28).

I owe the writing of this book to many people: H. Brezis, who asked me to write it; the members of the Department of Mathematics at Université d'Évry, who have created a very agreeable working environment; the Department of Mathematics at Université Paris-XI Orsay (especially, the Équipe d'Analyse Harmonique) who gave me the opportunity to lecture on the Navier–Stokes equations and thus to get a safer and more basic introductory point of view on this topic; my students and co-workers G. Furioli, R. May, E. Terraneo, E. Zahrouni and A. Zhioua who have helped me so much in the understanding of the Navier–Stokes equations; M. Cannone and F. Planchon who gave me their stimulating preprints on which so many chapters in this book are based; Y. Meyer who taught me so much and who took a constant interest in my work; and of course my wife and daughter who had to live in the same house with a monomaniac cyclothymic writer.

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Introduction

Chapter 1

What is this book about?

There is a huge literature on the mathematical theory of the Navier-Stokes equations, including the classical books by R. Temam [TEM 77], O.A. Ladyzhenskaya [LAD 69] or P. Constantin and C. Foias [CONF 88]; a more recent reference is the book by P.L. Lions [LIO 96]. Modern references on mild solutions and self-similar solutions in the setting of $\mathbb{R}^3$ are the books by M. Cannone [CAN 95] and Y. Meyer [MEY 99]. Another useful reference is the book by W. von Wahl [WAH 85].

In this book, we shall examine the Navier–Stokes equations in d dimensions (especially in the case $d = 3$) in a very restricted setting: we consider a viscous, homogeneous, incompressible fluid that fills the entire space and is not submitted to external force. The equations describing the evolution of the motion $\vec{u}(t,x)$ of the fluid element at time t and position x are given by:

$$(1.1) \qquad \begin{cases} \rho\ \partial_t \vec{u} = \mu\ \Delta\vec{u} - \rho\ (\vec{u}.\vec{\nabla})\ \vec{u} - \vec{\nabla} p \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

The *divergence free* condition $\vec{\nabla}.\vec{u} = 0$ expresses the incompressibility of the fluid. In equation (1.1), ρ is the (constant) *density* of the fluid, μ is the *viscosity* coefficient, and p is the (unknown) *pressure*, whose action is to maintain the divergence of $\vec{u}$ to be 0. We may assume with no loss of generality that $\rho = \mu = 1$ (changing the unknown $\vec{u}(x,t)$ and $p(x,t)$ into $\vec{u}(\frac{\mu x}{\rho}, \frac{\mu t}{\rho})$ and $\frac{1}{\rho} p(\frac{\mu x}{\rho}, \frac{\mu t}{\rho})$).

Since $\vec{\nabla}.\vec{u} = 0$, equation (1.1) can be rewritten (as far as $\vec{u}$ is a regular function):

$$(1.2) \qquad \begin{cases} \partial_t \vec{u} = \Delta\vec{u} - \vec{\nabla}.(\vec{u} \otimes \vec{u}) - \vec{\nabla} p \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

which is a condensed form of

$$(1.2') \qquad \begin{cases} \text{For } 1 \leq k \leq d,\ \partial_t u_k = \Delta u_k - \sum_{l=1}^d \partial_l (u_l u_k) - \partial_k p \\ \sum_{l=1}^d \partial_l u_l = 0 \end{cases}$$

Taking the divergence of (1.2), we obtain

$$\Delta p = -\vec{\nabla} \otimes \vec{\nabla}.(\vec{u}. \otimes \vec{u}) = -\sum_{k=1}^{d} \sum_{l=1}^{d} \partial_k \partial_l (u_k u_l) \tag{1.3}$$

Thus, we formally derive the equations

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{u}) \\ \vec{\nabla}.\vec{u} = 0 \end{cases} \tag{1.4}$$

where $\mathbb{P}$ is defined as

$$\mathbb{P}\vec{f} = \vec{f} - \vec{\nabla}\frac{1}{\Delta}(\vec{\nabla}.\vec{f}) \tag{1.5}$$

We study the Cauchy problem for equation (1.4) (looking for a solution on $(0,T) \times \mathbb{R}^d$ with initial value $\vec{u_0}$) and transform (1.4) into the integral equation

$$\begin{cases} \vec{u} = e^{t\Delta}\vec{u_0} - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{u})\, ds \\ \vec{\nabla}.\vec{u_0} = 0 \end{cases} \tag{1.6}$$

We consider weak solutions to equation (1.2), (1.4) or (1.6). In (1.2), we take the derivatives in the distribution sense; thus, (1.2) is meaningful as soon as $\vec{u}$ is locally square-integrable. We need extra information on $\vec{u}$ to give meaning to (1.4) or (1.6) (and, in some cases, to prove that the systems are equivalent to each other). Since we work on the whole space $\mathbb{R}^d$, the operators $e^{t\Delta}$ and $e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}$ that we use to write (1.6) are convolution operators. Therefore, we put special emphasis on shift-invariant estimates; this means that we are going to work in functional spaces invariant under spatial translations.

Part 1 is devoted to the recalling of some (presumably) well-known results of harmonic analysis on some special spaces of functions or distributions, and on some convolution operators (fractional integration, Calderón–Zygmund operators, Riesz transforms, etc.). In Parts 2 to 6, we apply those tools to the study of the Cauchy problem for the Navier–Stokes equations: Part 2 presents some general shift-invariant estimates for the Navier-Stokes equations; Part 3 reviews the classical existence results of Leray (weak solutions $\vec{u}$ such that $\vec{u} \in L^\infty((0,\infty),(L^2)^d)$, $\vec{\nabla} \otimes \vec{u} \in L^2((0,\infty),(L^2)^{d^2})$ [LER 34]) and Kato and Fujita (mild solutions in $\mathcal{C}([0,T],(H^s)^d), s \geq d/2-1$ [FUJK 64], or in $\mathcal{C}([0,T],(L^p)^d), p \geq d$ [KAT 84]); Part 4 and 5 describe some recent results on mild solutions (generalizations of Kato's results), including the theorem of Koch and Tataru on the existence of solutions for data in $BMO^{(-1)}$ [KOCT 01] and Cannone's theory of self-similar solutions [CAN 95]; Part 6 considers suitable solutions when $d = 3$, the main tool is the local energy inequality of Scheffer [SCH 77] and the regularity criterion of Caffarelli, Kohn and Nirenberg

[CAFKN 82], with applications to the study of weak solutions with infinite energy.

1. Uniform weak solutions for the Navier–Stokes equations

We will focus on the invariance of equation (1.2) under spatial translations and dilations, as we consider the problem on the whole space $\mathbb{R}^d$. We begin by defining what we call a weak solution for the Navier–Stokes equations.

Definition 1.1: (Weak solutions)

A weak solution of the Navier–Stokes equations on $(0,T)\times\mathbb{R}^d$ *is a distribution vector field* $\vec{u}(t,x)$ *in* $(\mathcal{D}'((0,T)\times\mathbb{R}^d))^d$ *where*
a) $\vec{u}$ *is locally square integrable on* $(0,T)\times\mathbb{R}^d$
b) $\vec{\nabla}.\vec{u}=0$
c) $\exists p\in\mathcal{D}'((0,T)\times\mathbb{R}^d)$ $\partial_t\vec{u}=\Delta\vec{u}-\vec{\nabla}.(\vec{u}\otimes\vec{u})-\vec{\nabla}p$

Notice that this is not the usual definition for weak solutions (as given in the books of Temam [TEM 77] or Von Wahl [WAH 85]).

Throughout the book, we use the following invariance of the set of solutions:
a) shift invariance: if $\vec{u}(t,x)$ is a weak solution of the Navier–Stokes equations on $(0,T)\times\mathbb{R}^d$, then $\vec{u}(t,x-x_0)$ is a weak solution on $(0,T)\times\mathbb{R}^d$;
b) dilation invariance: for $\lambda>0$, $\frac{1}{\lambda}\vec{u}(\frac{t}{\lambda^2},\frac{x}{\lambda})$ is a solution on $(0,\lambda^2T)\times\mathbb{R}^d$;
c) delay invariance: if $\vec{u}(t,x)$ is a weak solution of the Navier-Stokes equations on $(0,T)\times\mathbb{R}^d$ and if $t_0\in(0,T)$ then $\vec{u}(t+t_0,x)$ is a weak solution of the Navier-Stokes equations on $(0,T-t_0)\times\mathbb{R}^d$.

In order to use the space invariance, we introduce a more restrictive class of solutions:

Definition 1.2: (Uniformly locally square integrable weak solutions)

A weak solution of the Navier–Stokes equations on $(0,T)\times\mathbb{R}^d$ *is said to be uniformly locally square-integrable if for all* $\varphi\in\mathcal{D}((0,T)\times\mathbb{R}^d)$ *we have* $\sup_{x_0\in\mathbb{R}^d}\int\int|\varphi(x-x_0,t)\vec{u}(t,x)|^2\,dx\,dt<\infty$.

Equivalently, $\vec{u}$ *is uniformly locally square-integrable if and only if for all* $t_0<t_1\in(0,T)$*, the function* $U_{t_0,t_1}(x)=(\int_{t_0}^{t_1}|\vec{u}(t,x)|^2\,dt)^{1/2}$ *belongs to the Morrey space* L^2_{uloc}*. We then write* $\vec{u}\in\cap_{0<t_0<t_1<T}(L^2_{uloc,x}L^2_t((t_0,t_1)\times\mathbb{R}^d))^d$.

We now explain the utility of introducing uniform weak solutions. In order to understand equation (1.2), we only need to assume that $\vec{u}$ is locally square-integrable on $(0,T)\times\mathbb{R}^d$. We need a stronger assumption to make sense of

(1.4), since $\mathbb{P}$ is a non local operator. More precisely, we need to make sense of $\vec{\nabla}(\frac{1}{\Delta}\vec{\nabla} \otimes \vec{\nabla}.\vec{u} \otimes \vec{u})$. In Part 2, Chapter 11, we prove the following:

Theorem 1.1: (Elimination of the pressure)

i) If $\vec{u}$ is uniformly locally square-integrable on $(0,T) \times \mathbb{R}^d$ (in the sense of Definition 1.2), then $\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})$ is well defined in $(\mathcal{D}'((0,T)\times\mathbb{R}^d))^d$ and there exists a distribution $P \in \mathcal{D}'((0,T)\times\mathbb{R}^d)$ so that $\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u}) = \vec{\nabla}.(\vec{u}\otimes\vec{u}) + \vec{\nabla}P$. Thus, if $\vec{u}$ is a solution of (1.4), then it is a solution of (1.2).

ii) Conversely, if $\vec{u}$ is a uniformly locally square-integrable weak solution of (1.2), and if $\vec{u}$ vanishes at infinity in the sense that for all $t_0 < t_1 \in (0,T)$, we have

$$\lim_{R\to\infty} \sup_{x_0\in\mathbb{R}^d} \frac{1}{R^d} \int_{t_0}^{t_1} \int_{|x-x_0|<R} |\vec{u}|^2 \, dx \, dt = 0,$$

then $\vec{u}$ is a solution of (1.4).

The next step, when looking at such weak solutions, attempts to define the initial value problem and equation (1.6). It is easy to see that if we assume the solution $\vec{u}$ is uniformly locally square-integrable *up to the border* $t = 0$ (i.e., that for all $t_1 < T$, we have $(\int_0^{t_1} |u(t,x)|^2 \, dt)^{1/2} \in L^2_{uloc}(\mathbb{R}^d)$), then $\vec{u}_0$ is well defined. In Chapter 11, we prove more precisely the following theorem:

Theorem 1.2: (The equivalence theorem)

Let $\vec{u} \in \cap_{t_1<T}(L^2_{uloc,x}L^2_t((0,t_1)\times\mathbb{R}^d))^d$. Then, the following assertions are equivalent:

(A1) $\vec{u}$ is a solution of the differential Navier–Stokes equations

$$\begin{cases} \partial_t\vec{u} = \Delta\vec{u} - \mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u}) \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

(A2) $\vec{u}$ is a solution of the integral Navier–Stokes equations

$$\exists \vec{u}_0 \in (\mathcal{S}'(\mathbb{R}^d))^d \quad \begin{cases} \vec{u} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u}) \, ds \\ \vec{\nabla}.\vec{u}_0 = 0 \end{cases}$$

2. Mild solutions

Given $\vec{u}_0 \in \mathcal{S}'(\mathbb{R}^d)$, in order to find a solution to (1.6), a natural approach is to iterate the transform $\vec{v} \mapsto e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{v}\otimes\vec{v}) \, ds$ and to find a fixed point $\vec{u}$ for this transform. This is the so-called Picard contraction method, already in use by Oseen at the beginning of the 20th century to establish the (local) existence of a classical solution to the Navier–Stokes equations for a regular initial value [OSE 27].

A simple approach to this problem is trying to find a subspace $\mathcal{E}_T$ of $L^2_{uloc,x}L^2_t((0,T)\times \mathbb{R}^d)$ so that the bilinear transform

$$B(\vec{u},\vec{v}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\ ds$$

is bounded from $\mathcal{E}_T^d \times \mathcal{E}_T^d$ to $\mathcal{E}_T^d$. Then, we may consider the space $E_T \subset \mathcal{S}'$ defined by $f \in E_T$ iff $f \in \mathcal{S}'$ and $(e^{t\Delta}f)_{0<t<T} \in \mathcal{E}_T$. We reach the following easy existence result:

Theorem 1.3: (The Picard contraction principle)

Let $\mathcal{E}_T \subset L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d)$ be such that the bilinear transform B is bounded on $\mathcal{E}_T^d$. Then:
a) If $\vec{u} \in \mathcal{E}_T^d$ is a weak solution of the Navier–Stokes equations: $\partial\vec{u} = \Delta\vec{u} - \mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})$ and $\vec{\nabla}.\vec{u} = 0$, then the associated initial value $\vec{u}_0$ belongs to E_T^d.
b) Conversely, there exists a positive constant C such that for all $\vec{u}_0 \in E_T^d$ satisfying $\vec{\nabla}.\vec{u}_0 = 0$ and $\|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_T} < C$ there exists a weak solution $\vec{u} \in \mathcal{E}_T^d$ of the Navier–Stokes equations associated to the initial value $\vec{u}_0$: $\vec{u} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})\ ds$.

Part 4 (Chapters 16 to 22) is devoted to examples of such spaces $\mathcal{E}_T$. The solutions we obtain through the Picard contraction principle are called mild solutions. (This is a slight abuse; the notion of mild solutions was introduced by Browder [BRO 64] and Kato [KAT 65] in an abstract setting; we use a simplified definition because we work in a special simple setting).

We call a space $\mathcal{E}_T$ if we may apply the Picard contraction principle as an *admissible path space* for the Navier–Stokes equations, and the associated space E_T as an *adapted value space*. Thus, the "rule of the game" consists of identifying admissible path spaces and then characterizing the associated adapted value spaces. We find this rule more natural than the equivalent approach of Cannone [CAN 95] or Meyer [MEY 99] (which motivated ours): those authors start from the adapted value spaces before, specifying, when necessary, associated admissible path spaces.

Classical admissible spaces are provided by the L^p theory of Kato [KAT 84] and Weissler [WEI 81] (see Chapter 15):
– for $d < p < \infty$, $\mathcal{C}([0,T], L^p)$ is admissible with associated adapted space $L^p(\mathbb{R}^d)$;
– for $p = d$, the space

$$\{f \in \mathcal{C}([0,T], L^d)\ /\ \sup_{0<t<T}\sqrt{t}\|f\|_{L^\infty(dx)} < \infty \text{ and } \lim_{t\to 0}\sqrt{t}\|f\|_{L^\infty(dx)} = 0\}$$

is admissible with associated adapted space $L^d(\mathbb{R}^d)$;
– for $T = \infty$ (i.e., for global solutions in L^d), we use the admissible space

$$\{f \in \mathcal{C}(\mathbb{R}^+, L^d) / \sup_{0<t}\|f\|_{L^d(dx)} < \infty,\ \sup_{0<t}\sqrt{t}\|f\|_{L^\infty(dx)} < \infty, \lim_{t\to 0}\sqrt{t}\|f\|_{L^\infty(dx)} = 0\}$$

with associated adapted space $L^d(\mathbb{R}^d)$.

We may choose other admissible spaces for the same adapted space; for instance, for $0 < T \leq \infty$, we may consider the admissible space $\{f \in \mathcal{C}([0,T), L^d) / \sup_{0<t<T} |f(t,x)| \in L^d(\mathbb{R}^d)\}$ with associated adapted space $L^d(\mathbb{R}^d)$ (following Cannone[CAN 95] and C. Calderón [CAL 93]).

Other admissible spaces are based on Lebesgue spaces: for instance, we may consider the admissible space

$$\{f \in L^p([0,T], L^q(\mathbb{R}^d)) \ / \ \sup_{0<t<T} \sqrt{t}\|f\|_{L^\infty(dx)} < \infty\}$$

(for $q > d$ and $2/p + d/q \leq 1$), which is associated to the adapted space $B_q^{-2/p,p}(\mathbb{R}^d)$ (a Besov space). L^pL^q solutions have been considered by many authors in the 60's (Serrin [SER 62], Faber, Jones and Riviere [FABJR 72]), without reference to Besov spaces (see Chapter 14). They have been thoroughly investigated by Giga in the 80's [GIG 86] and recently by Cannone and Planchon [CAN 95] [PLA 96].

In the 90's, those results have been extended to spaces based on Morrey–Campanato spaces instead of Lebesgue spaces (see for instance Federbush [FED 93], Kato [KAT 92], Taylor [TAY 92], Cannone [CAN 95], Kozono and Yamazaki [KOZY 97], Furioli, Lemarié-Rieusset and Terraneo [FURLT 00]; see also Chapters 17 to 21).

All the adapted spaces we considered in the previous paragraphs are subspaces of the Besov space $B_\infty^{-1,\infty}$ (which may be defined through the heat kernel by $\sup_{0<t<T} \sqrt{t}\|e^{t\Delta}f\|_\infty < \infty$). As a matter of fact, they are included in a smaller space, the Triebel–Lizorkin space $F_\infty^{-1,2}$ of distributions f which may be decomposed into $f = f_0 + \sum_{1\leq k\leq d} \partial_k f_k$ with the functions f_k in the local space bmo for $0 \leq k \leq d$. This space is called bmo^{-1} by Koch and Tataru (and in this book). A useful characterization of this space is given by the following boundedness criterion in $L^2_{uloc,x}L^2_t$ under dilations : $f \in F_\infty^{-1,2}$ if and only if (for some $T > 0$), letting $F(t,x) = e^{t\Delta}f(x)$, we have $\sup_{0<\lambda<1} \|\lambda F(\lambda^2 t, \lambda x)\|_{L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d)} < \infty$ (or equivalently if and only if $\sup_{0<t<T, x_0\in\mathbb{R}^d} \frac{1}{t^{d/2}} \int\int_{0<s<t, |x-x_0|<\sqrt{t}} |F(s,x)|^2 \, ds \, dx < \infty$). Conversely, a recent theorem of Koch and Tataru [KOCT 01] proves (see Chapter 16) that $F_\infty^{-1,2}$ is an adapted value space for the Navier–Stokes equations (and the homogeneous space $\dot F_\infty^{-1,2}$ an adapted value space for global solutions of the Navier–Stokes equations).

Theorem 1.4: (The Koch and Tataru theorem)

a) [Admissible path space] If $\vec u$ is a solution of the Navier-Stokes equations on $(0,T) \times \mathbb{R}^d$,

$$\begin{cases} \partial_t \vec u = \Delta \vec u - \mathbb{P}\vec\nabla.\vec u \otimes \vec u \\ \vec\nabla.\vec u = 0 \end{cases}$$

so that

i) $\sqrt{t}\ \vec{u}(t,x) \in (L^\infty((0,T)\times \mathbb{R}^d)^d$

ii) $\sup_{t<T, x_0\in\mathbb{R}^d} \frac{1}{t^{d/2}} \int\int_{0<s<t, |x-x_0|<\sqrt{t}} |\vec{u}|^2\ ds\ dx < \infty$

then $\lim_{t\to 0} \vec{u} \in (F_\infty^{-1,2})^d$.

b) [Local existence] There exists a constant $\epsilon_0 > 0$ *such that for all* $T \in (0,\infty]$ *and for all* $\vec{u}_0 \in (F_\infty^{-1,2})^d$ *such that*

i) $\sup_{0<t<T} \sqrt{t}\|e^{t\Delta}\vec{u}_0\|_{L^\infty(dx)} < \epsilon_0$

ii) $\sup_{0<t<T, x_0\in\mathbb{R}^d} \frac{1}{t^{d/2}} \int\int_{0<s<t, |x-x_0|<\sqrt{t}} |e^{s\Delta}\vec{u}_0|^2\ ds\ dx < \epsilon_0^2$

iii) $\vec{\nabla}.\vec{u}_0 = 0$

there exists a weak solution $\vec{u}$ *of the Navier–Stokes equations on* $(0,T)\times\mathbb{R}^d$*:*

$$\begin{cases} \partial_t \vec{u} = \Delta\vec{u} - \mathbb{P}\vec{\nabla}.\vec{u}\otimes\vec{u} \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

so that

j) $\sqrt{t}\ \vec{u}(t,x) \in (L^\infty((0,T)\times \mathbb{R}^d)^d$

jj) $\sup_{t<T, x_0\in\mathbb{R}^d} \frac{1}{t^{d/2}} \int\int_{0<s<t, |x-x_0|<\sqrt{t}} |\vec{u}|^2\ ds\ dx < \infty$

jjj) $\lim_{t\to 0} \vec{u} = \vec{u}_0$.

In particular, for all $T > 0$*, there exists a constant* $\epsilon_T > 0$ *so that*

$$\|\vec{u}_0(x)\|_{F_\infty^{-1,2}} < \epsilon_T \Rightarrow i) \text{ and } ii),$$

hence, so that $\|\vec{u}_0(x)\|_{F_\infty^{-1,2}} < \epsilon_T$ *and* $\vec{\nabla}.\vec{u}_0 = 0$ *imply the existence of a solution on* $(0,T)\times\mathbb{R}^d$.

c) [Regular initial values] If $\vec{u}_0$ *belongs to the closure of the test functions* $(\mathcal{D}(\mathbb{R}^d))^d$ *in* $(F_\infty^{-1,2})^d$*, then* $\lim_{\lambda\to 0^+} \|\lambda\vec{u}_0(\lambda x)\|_{F_\infty^{-1,2}} = 0$*. If* $\vec{\nabla}.\vec{u}_0 = 0$ *and* $\|\lambda\vec{u}_0(\lambda x)\|_{F_\infty^{-1,2}} < \epsilon_T$*, then the Cauchy problem for the Navier–Stokes equations with* $\vec{u}_0$ *as initial value has a solution on* $(0,\lambda^2 T)\times\mathbb{R}^d$.

d) [Global solutions] There exists a positive constant ϵ_∞ *so that for all* $\vec{u}_0 \in (\dot{F}_\infty^{-1,2})^d$ *with* $\|\vec{u}_0\|_{\dot{F}_\infty^{-1,2}} < \epsilon_\infty$ *and* $\vec{\nabla}.\vec{u}_0 = 0$*, there exists a weak solution* $\vec{u}$ *of the Navier–Stokes equations on* $(0,\infty)\times\mathbb{R}^d$*:*

$$\begin{cases} \partial_t \vec{u} = \Delta\vec{u} - \mathbb{P}\vec{\nabla}.\vec{u}\otimes\vec{u} \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

so that

i) $\sqrt{t}\ \vec{u}(t,x) \in (L^\infty((0,\infty)\times \mathbb{R}^d)^d$

ii) $\sup_{0<t, x_0\in\mathbb{R}^d} \frac{1}{t^{d/2}} \int\int_{0<s<t, |x-x_0|<\sqrt{t}} |\vec{u}|^2\ ds\ dx < \infty$

iii) $\lim_{t\to 0} \vec{u} = \vec{u}_0$.

Besides existence theorems, we briefly discuss recent results on the uniqueness of mild solutions (following the theorem on uniqueness of solutions in $(\mathcal{C}([0,T),L^d))^d$ by Furioli, Lemarié-Rieusset and Terraneo [FURLT 00]; see also Meyer [MEY 99], Monniaux [MON 99], and May [MAY 02]).

Another interesting point will be regularity estimates for the mild solutions. A *persistency principle* (Furioli, Lemarié-Rieusset, Zahrouni and Zhioua [FURLZZ 00]) will be systematically developed : if $\vec{u}_0 \in B_\infty^{-1,\infty}$ is such that the iterated sequence defined by $\vec{u}_{(0)} = 0$ and $\vec{u}_{(n+1)} = e^{t\Delta}\vec{u_0} - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}_{(n)} \otimes \vec{u}_{(n)})\, ds$ converges exponentially in the sense that for all $n \in \mathbb{N}$ $\sqrt{t}\ \vec{u}_{(n)} \in (L^\infty((0,T)\times\mathbb{R}^d))^d$ and

$$\lim\sup_{n\to\infty} \|\sqrt{t}(\vec{u}_{(n+1)} - \vec{u}_{(n)})\|_{L^\infty((0,T)\times\mathbb{R}^d)}^{1/n} < 1 \tag{1.7}$$

then for a large class of functional spaces E (such as Besov spaces $B_p^{s,q}$ with $s > -1$, Lebesgue spaces L^p, ...) we have that whenever $\vec{u}_0 \in E^d$, $\vec{u}_{(n)}$ converges in $L^\infty((0,T), E^d)$. Thus, existence of solutions in E is reduced to the estimate (1.7), which may be proved by the Koch and Tataru theorem.

Other regularity estimates (such as analyticity norms or Lyapounov functionals for global solutions) will be discussed in Part 5.

3. Energy inequalities

We now consider the case $d = 3$ and we look at (locally square-integrable) weak solutions $\vec{u}$ so that $\vec{\nabla}\otimes\vec{u}$ is locally square-integrable on $(0,T)\times\mathbb{R}^3$. Then, we may write $\vec{\nabla}.(\vec{u}\otimes\vec{u}) = (\vec{u}.\vec{\nabla})\vec{u} + (\vec{\nabla}.\vec{u})\vec{u} = (\vec{u}.\vec{\nabla})\vec{u}$ (since $\vec{u}$ is divergence free; moreover, we may write $(\vec{u}.\vec{\nabla})\vec{u} = -\vec{u}\wedge(\vec{\nabla}\wedge\vec{u}) + \vec{\nabla}(\frac{1}{2}|\vec{u}|^2)$; thus, the Navier–Stokes equations may be rewritten as

$$\begin{cases} \partial_t\vec{u} - \vec{u}\wedge(\vec{\nabla}\wedge\vec{u}) = \Delta\vec{u} - \vec{\nabla}(\frac{1}{2}|\vec{u}|^2 + p) \\ \vec{\nabla}\vec{u} = 0 \end{cases} \tag{1.8}$$

If $\vec{u}$ is regular enough to allow (1.8) to be multiplied pointwise by $\vec{u}$, we write $\vec{u}.\partial_t\vec{u} = \partial_t(\frac{1}{2}|\vec{u}|^2)$, $\vec{u}.\{\vec{u}\wedge(\vec{\nabla}\wedge\vec{u})\} = 0$, $\vec{u}.\Delta\vec{u} = \Delta(\frac{1}{2}|\vec{u}|^2) - |\vec{\nabla}\otimes\vec{u}|^2$ and (since $\vec{\nabla}.\vec{u} = 0$) $\vec{u}.\vec{\nabla}(\frac{1}{2}|\vec{u}|^2 + p) = \vec{\nabla}.(\{\frac{1}{2}|\vec{u}|^2 + p\}\vec{u})$. Thus, (1.8) gives (for regular solutions) the energy equality

$$\partial_t|\vec{u}|^2 + 2|\vec{\nabla}\otimes\vec{u}|^2 = \Delta|\vec{u}|^2 - \vec{\nabla}.(\{|\vec{u}|^2 + 2p\}\vec{u}) \tag{1.9}$$

However, the regularity of weak solutions to the Navier–Stokes equations is still an open question and the energy equality (1.9) is not known to be satisfied by those irregular solutions. The best estimate we can get is the local energy inequality underlined by Scheffer [SCH 77] and Caffarelli, Kohn and Nirenberg [CAFKN 82]. We are able to construct solutions so that for some non-negative local measure μ, we have:

$$\partial_t|\vec{u}|^2 + 2|\vec{\nabla}\otimes\vec{u}|^2 = \Delta|\vec{u}|^2 - \vec{\nabla}.(\{|\vec{u}|^2 + 2p\}\vec{u}) - \mu \tag{1.10}$$

Following [CAFKN 82], we introduce the following:

Definition 1.3: (Suitable solutions)

A suitable weak solution for the Navier–Stokes equations on $(0,T) \times \mathbb{R}^3$ *is a vector field* $\vec{u}(t,x)$ *so that*

a) $\vec{u}$ *is locally square-integrable on* $(0,T) \times \mathbb{R}^3$

b) $\vec{\nabla}.\vec{u} = 0$

c) $\exists p \in \mathcal{D}'((0,T) \times \mathbb{R}^3)$ $\partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla}.(\vec{u} \otimes \vec{u}) - \vec{\nabla} p$

d) for all $0 < T_1 < T_2 < T$ *and all compact subset* $K \subset \mathbb{R}^3$*, we have* $\sup_{T_1<t<T_2} \int_K |\vec{u}|^2 \, dx < \infty$.

e) $\vec{\nabla} \otimes \vec{u}$ *is locally square integrable on* $(0,T) \times \mathbb{R}^3$

f) p *is locally* $L^{3/2}$ *on* $(0,T) \times \mathbb{R}^3$

g) for all $\phi \in \mathcal{D}((0,T) \times \mathbb{R}^3)$ *so that* $\phi \geq 0$ *we have*

$$2 \iint |\vec{\nabla} \otimes \vec{u}|^2 \phi \, dx \, dt \leq \iint |\vec{u}|^2 (\partial_t \phi + \Delta \phi) \, dx \, dt + \iint (|\vec{u}|^2 + 2p)(\vec{u}.\vec{\nabla})\phi \, dx \, dt \tag{1.11}$$

Suitable solutions are the most regular global solutions we are able to construct from a square-integrable initial value. We describe this regularity in Chapter 30.

Inequality (1.11) is the key for proving the existence of solutions when the initial data does not belong to $F_\infty^{-1,2}$. In this case, the Picard contraction algorithm does not work. The idea is then to modify the equation in order to make the algorithm work and to prove an energy estimate on the solution uniformly with respect to the perturbation; then we use a limiting process to go back to the initial equation, where the energy inequality allows weak limits and provides a weak solution. This method was initiated by Leray in 1934 [LER 34] in the case of $\vec{u}_0 \in L^2(\mathbb{R}^3)$. Part 5 is devoted to the extension of this theorem to the case of uniformly locally square-integrable vector fields:

Theorem 1.5: (Uniformly locally square integrable vector fields)

Let E_2 *be the closure of* $\mathcal{D}(\mathbb{R}^3)$ *in* L^2_{uloc}. *(Thus,* $f \in E_2$ *if and only if* f *is locally* L^2 *on* $\mathbb{R}^3$ *and* $\lim_{x\to\infty} \int_{|y-x|\leq 1} |f(y)|^2 \, dy = 0$*). For all* $\vec{u}_0 \in (E_2(\mathbb{R}^3))^3$ *so that* $\vec{\nabla}.\vec{u}_0 = 0$*, there exists a weak solution* $\vec{u}$ *on* $(0,\infty) \times \mathbb{R}^3$ *for the Navier–Stokes initial value problem associated to* $\vec{u}_0$ *so that:*

a) $\vec{u} \in \cap_{0<T} L^\infty((0,T),(E_2)^3)$*:*

$\sup_{x_0 \in \mathbb{R}^3, s<T} \int_{|x-x_0|\leq 1} |\vec{u}(s,x)|^2 \, dx < \infty$

b) $\vec{\nabla} \otimes \vec{u} \in \cap_{0<T} (L^2_{uloc,x} L^2_t((0,T) \times \mathbb{R}^3))^9$*:*

$\sup_{x_0 \in \mathbb{R}^3} \int\int_{0<s<T, |x-x_0|\leq 1} |\vec{\nabla} \otimes \vec{u}|^2 \, dx \, ds < \infty$

c) $\lim_{t\to 0^+} \|\vec{u} - \vec{u}_0\|_{E_2} = 0$

d) $\vec{u}$ *is suitable in the sense of Caffarelli, Kohn and Nirenberg.*

Part 1:

Some results of real harmonic analysis

Chapter 2

Real interpolation, Lorentz spaces, and Sobolev embeddings

Lorentz spaces $L^{p,q}$ were introduced by Lorentz in 1950 [LOR 50]. We shall consider $L^{p,q}$ only for $1 < p < \infty$ and $1 \leq q \leq \infty$ (they are defined for all $p > 0$ and $q > 0$, but in that case their topology is no longer locally convex). We do not detail the properties of Lorentz spaces and we shall only define them as real interpolation spaces of Lebesgue spaces $L^p(X, d\mu)$ (where (X, μ) is a σ-finite measured space). In this setting, useful theorems such as the Sobolev embeddings or size estimates on the pointwise multiplication of functions in Lorentz spaces are easily proved. For further information on Lorentz spaces, the reader may consult the books by Bergh and Löfström [BERL 76] or by Stein and Weiss [STEW 71] and the papers by Hunt [HUN 67] and O'Neil [ONE 63].

1. A primer to real interpolation theory

An interpolation functor F associates to a couple of Banach spaces (A_0, A_1), (which are assumed to be continuously embedded into a common topological vector space V so that $A_0 \cap A_1$ and $A_0 + A_1$ are well-defined Banach spaces), another Banach space $F(A_0, A_1) \subset A_0 + A_1$ defined in such a way that, whenever $F(A_0, A_1)$ is associated to A_0, A_1, $F(B_0, B_1)$ is associated to B_0, B_1, and T is a bounded linear operator from A_0 to B_0 and from A_1 to B_1, then T is bounded as well from $F(A_0, A_1)$ to $F(B_0, B_1)$.

The theory of interpolation spaces was introduced in the early 1960s (by Lions and Peetre [LIOP 64] for the real method and by Calderón [CAL 64] for the complex method). A useful reference on interpolation theory is the book by Bergh and Löfström [BERL 76].

In this section, we introduce the "real" method. There are many equivalent ways to define real interpolation spaces; we shall present the discrete J–method and the discrete K–method which are the simplest ones.

The J–method is named after the J–functional defined for $t > 0$ and $a \in A_0 \cap A_1$ by $J(t, a) = \max(\|a\|_{A_0}, t\|a\|_{A_1})$.

Definition 2.1: (J-method of interpolation)

For $0 < \theta < 1$ and $1 \leq q \leq \infty$, the interpolation space $[A_0, A_1]_{\theta,q,J}$ is defined by: $a \in [A_0, A_1]_{\theta,q,J}$ if and only if a can be written as a sum $a = \sum_{j\in\mathbb{Z}} u_j$, where the series converges in $A_0 + A_1$, where each u_j belongs to $A_0 \cap A_1$, and where $(2^{-j\theta}\ J(2^j, u_j))_{j\in\mathbb{Z}} \in l^q(\mathbb{Z})$.

It is equivalent to require that (when $q < \infty$) $\sum_j\ 2^{-j\theta q}\|u_j\|^q_{A_0} < \infty$ and $\sum_j\ 2^{+j(1-\theta)q}\|u_j\|^q_{A_1} < \infty$. In particular, we have $\sum_{j\leq 0}\|u_j\|_{A_0} < \infty$ and $\sum_{j>0}\|u_j\|_{A_1} < \infty$, so that the sum of the series is well-defined in $A_0 + A_1$.

The norm of $[A_0, A_1]_{\theta,q,J}$ is then defined by

$$\|a\|_{[A_0,A_1]_{\theta,q,J}} = \min_{a=\sum_j u_j} (\sum_j\ 2^{-j\theta q}\|u_j\|^q_{A_0})^{1/q} + (\sum_j\ 2^{+j(1-\theta)q}\|u_j\|^q_{A_1})^{1/q}$$

Proposition 2.1:

(A) For any $\rho > 1$, a belongs to $[A_0, A_1]_{\theta,q,J}$ if and only if a can be written as a sum $a = \sum_{j\in\mathbb{Z}} u_j$, where each u_j belongs to $A_0 \cap A_1$ and where $(\rho^{-j\theta}\ J(\rho^j, u_j))_{j\in\mathbb{Z}} \in l^q(\mathbb{Z})$.

(B) For $0 < \theta < 1$ and $1 \leq q \leq \infty$, $[A_0, A_1]_{\theta,q,J}$ is a Banach space.

(C) $[A_0, A_1]_{\theta,q,J} = [A_1, A_0]_{1-\theta,q,J}$

(D) For $q_1 \leq q_2$, $[A_0, A_1]_{\theta,q_1,J} \subset [A_0, A_1]_{\theta,q_2,J}$

(E) If $T : A_0 + A_1 \to B_0 + B_1$ is a linear operator which is bounded from A_0 to B_0 and from A_1 to B_1 ($\|T(a)\|_{B_0} \leq M_0\|a\|_{A_0}$ and $\|T(a)\|_{B_1} \leq M_1\|a\|_{A_1}$) then, for $0 < \theta < 1$ and $1 \leq q \leq \infty$, the operator T is bounded from $[A_0, A_1]_{\theta,q,J}$ to $[B_0, B_1]_{\theta,q,J}$ and the operator norm $M_{\theta,q,J}$ of T from $[A_0, A_1]_{\theta,q,J}$ to $[B_0, B_1]_{\theta,q,J}$ is controlled by $M_{\theta,q,J} \leq CM_0^{1-\theta}M_1^{\theta}$.

Proof: (A) is quite obvious: if $r > 1$ and $\rho > 1$ and if $a = \sum_{j\in\mathbb{Z}} u_j$ where $(\rho^{-j\theta}\ J(\rho^j, u_j))_{j\in\mathbb{Z}} \in l^q(\mathbb{Z})$, then we define $v_j = \sum_{r^j\leq\rho^k<r^{j+1}} u_k$, and we easily check that $(r^{-j\theta}\ J(r^j, v_j))_{j\in\mathbb{Z}} \in l^q(\mathbb{Z})$.

(B) is easy as well: it is enough to show that for any family $(a_k)_{k\in\mathbb{N}}$ so that $\sum_k \|a_k\|_{[A_0,A_1]_{\theta,q,J}} < \infty$, the series $\sum_k a_k$ converges in $[A_0, A_1]_{\theta,q,J}$; we just write $\sum_k(\sum_j u_{j,k}) = \sum_j(\sum_k u_{j,k})$.

(C) and (D) are obvious.

(E) is easy. It is enough to write $T(\sum u_j) = \sum_j T(u_j)$ with $J_B(2^j, T(u_j)) \leq \max(M_0, M_1)\ J_A(u_j)$ to get the boundedness of T on $[A_0, A_1]_{\theta,q,J}$. In order to estimate the operator norm of T, we choose k so that $M_1/M_0 \leq 2^k < 2M_1/M_0$, and we write $T(\sum_j u_j) = \sum_j v_j$ with $v_j = T(u_{j+k})$. $\square$

We now consider another method for real interpolation, namely the K–method. The K–method is named after the K–functional defined for $t > 0$ and $a \in A_0 + A_1$ by $K(t, a) = \min\{\|a_0\|_{A_0} + t\|a_1\|_{A_1}\ /\ a = a_0 + a_1\ \}$.

Definition 2.2: (K-method of interpolation)

For $0 < \theta < 1$ and $1 \le q \le \infty$, the space $[A_0, A_1]_{\theta,q,K}$ is defined by: $a \in [A_0, A_1]_{\theta,q,K}$ if and only if $a \in A_0 + A_1$ and $(2^{-j\theta}\ K(2^j, a))_{j\in\mathbb{Z}} \in l^q(\mathbb{Z})$.

The norm of $[A_0, A_1]_{\theta,q,K}$ is defined by

$$\|a\|_{[A_0,A_1]_{\theta,q,K}} = (\sum_j 2^{-j\theta q} K(2^j, a)^q)^{1/q}$$

Proposition 2.2: (Equivalence theorem)

For $0 < \theta < 1$ and $1 \le q \le \infty$, $[A_0, A_1]_{\theta,q,K} = [A_0, A_1]_{\theta,q,J}$. We shall designate this space as $[A_0, A_1]_{\theta,q}$.

Proof: If $a \in [A_0, A_1]_{\theta,q,J}$, we may write $a = \sum_j u_j$ with $\|u_j\|_{A_0} + 2^j\|u_j\|_{A_1} \le 2^{j\theta}\epsilon_j$ where $(\epsilon_j)_{j\in\mathbb{Z}} \in l^q$. In order to estimate $2^{-j\theta}K(2^j, a)$, we define $b_j = \sum_{l<j} u_l$ and $c_j = \sum_{l\ge j} u_l$. Then $2^{-j\theta}K(2^j, a) \le 2^{-j\theta}\|b_j\|_{A_0} + 2^{+j(1-\theta)}\|c_j\|_{A_1} \le \sum_{l<j} 2^{(l-j)\theta}\epsilon_l + \sum_{l<j} 2^{(j-l)(1-\theta)}\epsilon_l$, so that $a \in [A_0, A_1]_{\theta,q,K}$.

Conversely, if $a \in [A_0, A_1]_{\theta,q,K}$, then we have, for all j, $a = b_j + c_j$ with $\|b_j\|_{A_0} + 2^j\|c_j\|_{A_1} \le 2^{j\theta}\epsilon_j$, where $(\epsilon_j)_{j\in\mathbb{Z}} \in l^q$. We may write $a = \sum_j (b_{j+1} - b_j)$: indeed, $\sum_{j<0} b_{j+1} - b_j = b_0$ and $\sum_{j\ge 0}(c_j - c_{j+1}) = c_0$. This obviously gives $a \in [A_0, A_1]_{\theta,q,J}$. □

We shall use only two theorems of real interpolation theory: the duality theorem and the reiteration theorem.

Theorem 2.1: (Duality theorem for the real method)

If $A_0 \cap A_1$ is dense in A_0 and in A_1 and if $1 \le q < \infty$ and $0 < \theta < 1$, then $[A_0, A_1]'_{\theta,q} = [A'_0, A'_1]_{\theta,q'}$ where $\frac{1}{q} + \frac{1}{q'} = 1$ (with equivalence of norms).

Proof: If $A_0 \cap A_1$ is dense in A_0 and in A_1, then A'_0 and A'_1 are embedded in $(A_0 \cap A_1)'$ and we have $(A_0 \cap A_1)' = A'_0 + A'_1$. We shall check more precisely that the functional J on $A_0 \cap A_1$ and the functional K on $A'_0 + A'_1$ define dual norms: $K(t, a') = \sup_{a\in A_0\cap A_1} \frac{|\langle a'|a\rangle|}{J(1/t,a)}$. Equality $[A_0, A_1]'_{\theta,q} = [A'_0, A'_1]_{\theta,q'}$ is then straightforward.

Indeed, let us consider $a' \in (A_0 \cap A_1)'$. We define $D = \sup_{a\in A_0\cap A_1} \frac{|\langle a'|a\rangle|}{J(1/t,a)}$ and we associate to a' a linear form L on $E = \{(a_0, a_1)\ /\ a_0 = a_1\}$, defined by $L(a_0, a_1) = \langle a'|(a_0 + a_1)/2\rangle$. According to the Hahn-Banach theorem, we may extend L to $A_0 \times A_1$ in such a way that $|L(a_0, a_1)| \le D\ \max(\|a_0\|_{A_0}, \frac{\|a_1\|_{A_1}}{t})$. We then have $L(a_0, a_1) = a'_0(a_0) + a'_1(a_1)$ with $\|a'_0\|_{A'_0} + t\|a'_1\|_{A'_1} \le D$, which proves that $a' \in A'_0 + A'_1$ and that $K(t, a') = D$. □

Theorem 2.2: (Reiteration theorem)

If $0 \leq \theta_j \leq 1$, $\theta_0 \neq \theta_1$, and if X_0, X_1 are two Banach spaces so that $A_1 \cap A_2 \subset X_j \subset A_1 + A_2$ and $\forall x \in X_j$ $\forall t > 0$ $t^{-\theta_j} K(t,x) \leq C \|x\|_{X_j}$ and $\forall x \in A_1 \cap A_2$ $\forall t > 0$ $\|x\|_{X_j} \leq C\ t^{-\theta_j} J(t,x)$, then for $0 < \eta < 1$ and $1 \leq q \leq \infty$ we have $[X_0, X_1]_{\eta,q} = [A_0, A_1]_{\theta,q}$ where $\theta = (1-\eta)\theta_0 + \eta\theta_1$ (with equivalence of norms).

In particular, if for $j = 0, 1$ $X_j = [A_0, A_1]_{\theta_j, q_j}$ where $0 < \theta_j < 1$, $\theta_0 \neq \theta_1, 1 \leq q_j \leq \infty$, then for $0 < \eta < 1$ and $1 \leq q \leq \infty$ we have $[X_0, X_1]_{\eta,q} = [A_0, A_1]_{\theta,q}$ where $\theta = (1-\eta)\theta_0 + \eta\theta_1$.

Proof: Let us consider $x \in [X_0, X_1]_{\eta,q}$: we decompose x into $x = \sum_j x_j$ with $(2^{-j\eta} J_X(2^j, x_j))_{j \in \mathbb{Z}} \in l^q$. We know that x_j may be decomposed into $a_j + b_j$ with

$$r_0^{-k\theta}(\|a_j\|_{A_0} + r_0^k \|b_j\|_{A_1}) \leq C r_0^{k(\theta_i - \theta)} \|x_j\|_{X_i} \leq r_0^{(k-j)(\theta_i - \theta)} (r_0^{\theta_i - \theta} 2^{\eta - \delta_{i,1}})^j \epsilon_j$$

where $(\epsilon_j)_j \in l^q$. For $r_0^{\theta_1 - \theta_0} = 2$, we get $(r_0^{-j\theta} K_A(r_0^j, x))_{j \in \mathbb{Z}} \in l^q$.

Conversely, if $a \in [A_0, A_1]_{\theta,q}$, we decompose a into $a = \sum_j a_j$ with $(2^{-j\theta} J_A(2^j, a_j))_{j \in \mathbb{Z}} \in l^q$. We know that $\|a_j\|_{X_i} \leq C 2^{-j\theta_i} J_A(2^j, a_j)$; this gives $(r_1^{-j\eta} K_X(r_1^j, x))_{j \in \mathbb{Z}} \in l^q$ for $r_1 = 2^{\theta_1 - \theta_0}$. □

Remarks: a) We borrowed the text of Theorem 2.2 from the book by Bergh and Löfström [BERL 76]: the only reason why we have to consider other spaces, X_j, than the interpolation spaces $[A_0, A_1]_{\theta_j, q_j}$ is the fact that we have to interpolate between A_0 or A_1 and $[A_0, A_1]_{\theta_j, q_j}$.

b) There is a reiteration result when $\theta_0 = \theta_1$ but we do not need it. See [BERL 76] for further details.

2. Lorentz spaces

We may now introduce the Lorentz spaces:

Definition 2.3: (Lorentz spaces)

For $1 < p < \infty$ and $1 \leq q \leq \infty$, the Lorentz space $L^{p,q}$ is defined by $L^{p,q} = [L^1, L^\infty]_{1-\frac{1}{p}, q}$.

We need to characterize precisely the spaces $L^{p,p}$ and $L^{p,\infty}$:

Theorem 2.3: (Characterization of Lorentz spaces)

(A) For $1 < p < \infty$ and $1 \leq q \leq \infty$, $f \in L^{p,q}$ if and only if f may be decomposed as a sum $f = \sum_{j \in \mathbb{Z}} f_j$ with $(2^{-j(1-1/p)} \|f_j\|_1)_{j \in \mathbb{Z}} \in l^q(\mathbb{Z})$ and $(2^{j/p} \|f_j\|_\infty)_{j \in \mathbb{Z}} \in l^q(\mathbb{Z})$. Moreover, we may choose the f_j's with disjoint measurable supports.

(B) $L^p = L^{p,p}$.

(C) $L^{p,\infty} = L^{p,*}$, where $f \in L^{p,*} \Leftrightarrow \sup_{\lambda > 0} \lambda^p \mu(\{x / |f(x)| > \lambda\}) < \infty$.

Remark: As shown in Chapter 6, $L^{p,*}$ is the so-called weak L^p space.

Proof: (A) corresponds to the definition of the real interpolation spaces of L^1 and of L^∞ through the J-method. We must show that we may choose the functions f_j with disjoint supports. So, we consider a decomposition $f = \sum_{j\in\mathbb{Z}} f_j$ with $(2^{-j(1-1/p)}\|f_j\|_1)_{j\in\mathbb{Z}} \in l^q(\mathbb{Z})$ and $(2^{j/p}\|f_j\|_\infty)_{j\in\mathbb{Z}} \in l^q(\mathbb{Z})$. We define $g(x) = \sum_j \ |f_j(x)|$; g is finite almost everywhere since $g \in L^1 + L^\infty$. We then define $\epsilon_j = 2\sum_{k>j} \|f_k\|_\infty$, $E_j = \{x \ / \ \epsilon_{j+1} \leq g(x) < \epsilon_j\}$ and $v_j = f \ 1_{E_j}$. We have $f = \sum_j \ v_j$. We have $2^{j/p}\|v_j\|_\infty \leq 2^{j/p}\epsilon_j \leq 2\sum_{k>j} 2^{(j-k)/p} \ 2^{k/p}\|f_k\|_\infty$ and thus $(2^{j/p}\|v_j\|_\infty)_{j\in\mathbb{Z}} \in l^q$. Besides, we have $\sum_{k>j+1} \|f_k\|_\infty \leq \epsilon_{j+1}/2$; hence, we have on E_j that $|f| \leq 2\sum_{k\leq j+1} |f_k|$, that gives $2^{-j(1-1/p)}\|v_j\|_1 \leq 2\sum_{k\leq j+1} 2^{-(j-k)(1-1/p)} \ 2^{-k(1-1/p)}\|f_k\|_1$ and finally, $(2^{-j(1-1/p)}\|v_j\|_1)_{j\in\mathbb{Z}} \in l^q$.

Now, we prove (B). $[L^1, L^\infty]_{\theta,p} \subset L^p$ is a direct consequence of (A) : we decompose again f into $\sum_j v_j$ where the functions v_j have disjoint supports and we get (since $\|v_j\|_p^p \leq \|v_j\|_1 \|v_j\|_\infty^{p-1}$):

$$\|f\|_p^p = \sum_j \|v_j\|_p^p \leq (\sum_j 2^{-j(p-1)} \|v_j\|_1^p)^{1/p} \ (\sum_j 2^j \|v_j\|_\infty^p)^{1-1/p} < \infty.$$

Conversely, considering $q = \frac{p}{p-1}$ and L_0^∞ the closure of $L^1 \cap L^\infty$ in L^∞, we have $[L^1, L^\infty]_{1-\theta,q} = [L^1, L_0^\infty]_{1-\theta,q} \subset L^q$ (with dense injection), so that the duality theorem gives that $L^p \subset [L^\infty, (L_0^\infty)']_{1-\theta,p} = [L^1, L^\infty]_{\theta,p}$ (since $L^\infty \cap (L_0^\infty)' = L^\infty \cap L^1$).

Now, we prove (C). We first consider $f \in L^{p,\infty}$. For all $t > 0$, f can be decomposed into $f = g_t + h_t$, with $\|g_t\|_1 \leq Ct^{1-1/p}$ and $\|h_t\|_\infty \leq Ct^{-1/p}$. For $\lambda > 0$, we take t so that $2 \ Ct^{-1/p} = \lambda$; we then define $E = \{x \ / \ |f(x)| > \lambda\}$ and get that $x \in E \Rightarrow |g_t(x)| > \lambda/2$, so that $\mu(E) \leq \frac{2}{\lambda}\|g_t\|_1 \leq t = (\lambda/2C)^{-p}$.

Conversely, if $f \in L^{p,*}$, we have to estimate $\int_{|f(x)|>t^{-1/p}} |f(x)| \ d\mu = \sum_{j\geq 0} \int_{2^j < t^{1/p}|f(x)| \leq 2^{j+1}} |f(x)| \ d\mu$ and we obviously have:

$$\sum_{j\geq 0} \int_{2^j < t^{1/p}|f(x)|\leq 2^{j+1}} |f(x)| \ d\mu \leq C \ \sum_{j\geq 0} 2^{j+1} t^{-1/p} (2^j t^{-1/p})^{-p} = C' t^{1-1/p}$$

Thus, the theorem is proved. □

We now prove two easy and useful results on pointwise multiplication and convolution in Lorentz spaces.

Proposition 2.3: (Pointwise product in Lorentz spaces)

Let $1 < p < \infty$, $1 \leq q \leq \infty$, $1/p' + 1/p = 1$ and $1/q' + 1/q = 1$. Then pointwise multiplication is a bounded bilinear operator:

a) from $L^{p,q} \times L^\infty$ *to* $L^{p,q}$ *;*

b) from $L^{p,q} \times L^{p',q'}$ *to* L^1 *;*

c) from $L^{p,q} \times L^{p_1,q_1}$ *to* L^{p_2,q_2} *for* $p' < p_1 < \infty$, $q' \leq q \leq \infty$, $1/p_2 = 1/p + 1/p_1$ *and* $1/q_2 = 1/q + 1/q_1$.

Moreover, if $q < \infty$, *then the dual of* $L^{p,q}$ *is equal to* $L^{p',q'}$, *the duality bracket being given by the integral of the pointwise product.*

Proof: The duality theorem for real interpolation gives that $(L^{p,q})' = L^{p',q'}$ if $q < \infty$. Besides, points a) and b) are obvious. In order to prove c), we write, for $f \in L^{p,q}$ and $g \in L^{p_1,q_1}$, $f = \sum_j f_j$ and $g = \sum_j g_j$ and we decompose fg into $\sum_j v_j + w_j$ with $v_j = f_j \sum_{k>j} g_k$ and $w_j = g_j \sum_{k\geq j} f_k$. Since $\|g_j\|_\infty 2^{j/p_1} \in l^{q_1}$, we get that $2^{j/p_1} \sum_{k>j} \|g_k\|_\infty \in l^{q_1}$. Thus, we find that $2^{j(1/p+1/p_1)}\|v_j\|_\infty \in l^{q_2}$ and that $2^{j(1/p+1/p_1-1)}\|v_j\|_1 \in l^{q_2}$, which gives $\sum_j v_j \in L^{p_2,q_2}$. The same proof may be used for $\sum_j w_j$. □

We now consider $X = \mathbb{R}^d$ endowed with the Lebesgue measure and consider the properties of convolution between Lorentz spaces:

Proposition 2.4: (Convolution of Lorentz spaces)

Let $1 < p < \infty$, $1 \leq q \leq \infty$, $1/p' + 1/p = 1$ *and* $1/q' + 1/q = 1$. *Then convolution is a bounded bilinear operator :*

a) from $L^{p,q}(\mathbb{R}^d) \times L^1(\mathbb{R}^d)$ *to* $L^{p,q}(\mathbb{R}^d)$ *;*

b) from $L^{p,q}(\mathbb{R}^d) \times L^{p',q'}(\mathbb{R}^d)$ *to* $L^\infty(\mathbb{R}^d)$ *;*

c) from $L^{p,q}(\mathbb{R}^d) \times L^{p_1,q_1}(\mathbb{R}^d)$ *to* $L^{p_2,q_2}(\mathbb{R}^d)$ *for* $1 < p_1 < p'$, $q' \leq q \leq \infty$, $1/p_2 = 1/p + 1/p_1 - 1$ *and* $1/q_2 = 1/q + 1/q_1$.

Proof: Points a) and b) are obvious. In order to prove c), we write, for $f \in L^{p,q}$ and $g \in L^{p_1,q_1}$, $f = \sum_j f_j$ and $g = \sum_j g_j$, and we decompose fg into $\sum_j v_j + w_j$, with $v_j = f_j * \sum_{k<j} g_k$ and $w_j = g_j \sum_{k\leq j} f_k$. Since $\|g_j\|_1 2^{j(1/p_1-1)} \in l^{q_1}$, we get that $2^{j(1/p_1-1)} \sum_{k<j} \|g_k\|_1 \in l^{q_1}$. Hence, we get that $2^{j(1/p+1/p_1-2)}\|v_j\|_1 \in l^{q_2}$ and that $2^{j(1/p+1/p_1-1)}\|v_j\|_\infty \in l^{q_2}$, which gives $\sum_j v_j \in L^{p_2,q_2}$. We use the same proof for $\sum_j w_j$. □

3. Sobolev inequalities

Proposition 2.4 may be applied to easily get the Sobolev inequalities:

Theorem 2.4: (Sobolev inequalities)

Let $\alpha \in (0,d)$. *Then* $\frac{1}{(\sqrt{-\Delta})^\alpha}$ *is bounded:*

i) from $L^1(\mathbb{R}^d)$ *to* $L^{\frac{d}{d-\alpha},\infty}(\mathbb{R}^d)$*;*

ii) from $L^{\frac{d}{\alpha},1}(\mathbb{R}^d)$ *to* $L^\infty(\mathbb{R}^d)$*;*

iii) from $L^{p,q}(\mathbb{R}^d)$ *to* $L^{p_1,q}(\mathbb{R}^d)$ *for* $1 < p < \frac{d}{\alpha}$ *and* $1/p_1 = 1/p - \alpha/d$.

Proof: This is a convolution operator with kernel $\frac{c_{\alpha,d}}{|x|^{d-\alpha}} \in L^{\frac{d}{d-\alpha},\infty}(\mathbb{R}^d)$. $\square$

Now, we prove an estimate which will be useful in the study of the Navier–Stokes equations:

Proposition 2.5: (Heat kernel and Lorentz spaces.)
Let $\alpha \in \mathbb{R}$, $0 < \alpha < d$. *Then:*
(A) If $1 < p < \infty$ *and* $\alpha < d/p$ *and if* $\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{d}$, *there exists a constant* $C_{p,\alpha}$ *so that, for all* $f \in L^{p,1}(\mathbb{R}^d)$, *we have* $\int_0^\infty \|e^{t\Delta}\Delta\frac{1}{(\sqrt{-\Delta})^\alpha}\, f\|_{L^{r,1}}\, dt \le C_{p,\alpha}\, \|f\|_{L^{p,1}}$.
(B) If $\alpha = d/p$, *then there exists a constant* $C_{p,\alpha}$ *so that, for all* $f \in L^{p,1}(\mathbb{R}^d)$, *we have the estimate* $\int_0^\infty \|e^{t\Delta}\Delta\frac{1}{(\sqrt{-\Delta})^\alpha}\, f\|_\infty\, dt \le C_{p,\alpha}\, \|f\|_{L^{p,1}}$

Remark: This proposition means that $\frac{1}{(\sqrt{-\Delta})^\alpha}$ maps $L^{p,1}$ to the Besov space $\dot{B}^{0,1}_{L^{r,1}}$, a slightly more precise result than Theorem 2.4 (since $\dot{B}^{0,1}_{L^{r,1}} \subset L^{r,1}$). See Chapters 3, 4, and 5 for more information on Besov spaces.

Proof: The proposition is quite obvious; we write $f = \sum_{n\in\mathbb{N}} \lambda_n\, f_n$, with $\|f_n\|_1 \le A_n^p$, $\|f_n\|_\infty \le A_n^{-p/(p-1)}$ and $\sum_{n\in\mathbb{N}} |\lambda_n| \le C\|f\|_{L^{p,1}}$. By convexity, it is enough to show that $\int_0^\infty \|e^{t\Delta}\Delta\frac{1}{(\sqrt{-\Delta})^\alpha}\, f_n\|_{L^{r,1}}\, dt \le C_{p,\alpha}$ (uniformly with respect to n). Then, we define $1 < p_1 < p < p_2 < r$ by $1/p_1 = 1/2(1 + 1/p)$ and $1/p_2 = 1/2(1/r + 1/p)$. We have $\|f_n\|_{L^{p_j,1}} \le CA_n^{p/p_j} A_n^{-p(p_j-1)/p_j(p-1)}$. Now, $(\sqrt{-\Delta})^{-\beta_j}$ maps boundedly $L^{p_j,1}$ to $L^{r,1}$ ($r < \infty$) or L^∞ ($r = \infty$) for $\beta_j = d(1/p_j - 1/r) = d(1/p_j - 1/p) + \alpha$. Since, for all $\gamma \ge 0$, $(\sqrt{-\Delta})^\gamma e^\Delta$ is a convolution operator with a kernel $k_\gamma \in L^1$ (as proved in the lemma below), we find that $\|e^{t\Delta}\Delta(\sqrt{-\Delta})^{-\alpha}\, f_n\|_{L^{r,1}} = \|e^{t\Delta}(\sqrt{-\Delta})^{2-\alpha+\beta_j}\, (\sqrt{-\Delta})^{-\beta_j}\, f_n\|_{L^{r,1}} \le CA_n^{p/p_j} A_n^{-p(p_j-1)/p_j(p-1)} \frac{t^{1/2\ (1/p-1/p_j)}}{t}$. We integrate this estimate with $j = 2$ from $t = 0$ to $t = T_n$ and with $j = 1$ from $t = T_n$ to ∞ and we get that $\int_0^\infty \|e^{t\Delta}\Delta\frac{1}{(\sqrt{-\Delta})^\alpha}\, f_n\|_{L^{r,1}}\, dt \le C(T_n^{\frac{p_2-p}{2pp_2}} A_n^{\frac{p(p-p_2)}{p_2(p-1)}} + T_n^{\frac{p_1-p}{2pp_1}} A_n^{\frac{p(p-p_1)}{p_1(p-1)}})$; we may conclude by choosing $T_n = A_n^{\frac{2p^2}{p-1}}$. $\square$

To finish the proof, we need to prove the following lemma:

Lemma 2.1: *For all* $\gamma \ge 0$, $(\sqrt{-\Delta})^\gamma e^\Delta$ *is a convolution operator with a kernel* $k_\gamma \in L^1$.

Proof: We have $\hat{k}_\gamma(\xi) = |\xi|^\gamma e^{-|\xi|^2} \in L^1$; thus k_γ is a continuous bounded function. Moreover, if $\gamma > 0$, we introduce a function $\omega \in \mathcal{D}(\mathbb{R}^d)$ so that $0 \notin \text{Supp}\,\omega$ and $\sum_{j\in\mathbb{Z}} \omega(2^j\xi) = 1$. Then $|\xi|^\gamma\omega(\xi) \in \mathcal{D}$, and if we write $|\xi|^\gamma\omega(\xi) =$

$\hat{\Omega}_\gamma(\xi)$ and $\theta = 1 - \sum_{j\geq 0} \omega(2^j\xi)$, we have $\hat{k}_\gamma(\xi) = \sum_{j\geq 0} 2^{-j\gamma}\hat{\Omega}_\gamma(2^j\xi)e^{-|\xi|^2} + \theta(\xi)\ |\xi|^\gamma e^{-|\xi|^2}$, hence,

$$\|k_\gamma\|_1 \leq \sum_{j\geq 0} 2^{-j\gamma}\|\Omega_\gamma\|_1\|\mathcal{F}^{-1}(e^{-|\xi|^2})\|_1 + \|\mathcal{F}^{-1}(\theta(\xi)\ |\xi|^\gamma e^{-|\xi|^2})\|_1 < \infty.$$

□

Chapter 3

Besov spaces and Littlewood–Paley decomposition

In this chapter, we introduce a basic tool for this book: the Littlewood–Paley decomposition, $f = S_0 f + \sum_{j \geq 0} \Delta_j f$, and the Besov spaces $B_p^{s,q}$. Special attention throughout the book will be given to the spaces $B_\infty^{s,\infty}$. We will discuss as well the homogeneous decomposition for distributions modulo the polynomials. Our main references for Besov spaces are the books by Bergh and Löfström [BERL 76], by Meyer [MEY 92], and by Peetre [PEE 76], and the paper of Bourdaud [BOU 88a]. Another classical reference is the book by Triebel [TRI 83].

1. The Littlewood–Paley decomposition of tempered distributions

One of the main tools in this book is the Littlewood–Paley decomposition of distributions into dyadic blocks of frequencies:

Definition 3.1: (Dyadic blocks)

Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ be a non negative function so that $|\xi| \leq \frac{1}{2} \Rightarrow \varphi(\xi) = 1$ and $|\xi| \geq 1 \Rightarrow \varphi(\xi) = 0$. Let ψ be defined as $\psi(\xi) = \varphi(\xi/2) - \varphi(\xi)$. Let S_j and Δ_j be defined as the Fourier multipliers $\mathcal{F}(S_j f) = \varphi(\xi/2^j)\mathcal{F}f$ and $\mathcal{F}(\Delta_j f) = \psi(\xi/2^j)\mathcal{F}f$. The distribution $\Delta_j f$ is called the j–th dyadic block of the Littlewood–Paley decomposition of f.

Theorem 3.1: (Littlewood–Paley decomposition)

For all $N \in \mathbb{Z}$ and all $f \in \mathcal{S}'(\mathbb{R}^d)$, we have $f = S_N f + \sum_{j \geq N} \Delta_j f$ in $\mathcal{S}'(\mathbb{R}^d)$. This equality is called the Littlewood–Paley decomposition of the distribution f. If, moreover, $\lim_{N \to -\infty} S_N f = 0$ in $\mathcal{S}'$, then the equality $f = \sum_{j \in \mathbb{Z}} \Delta_j f$ is called the homogeneous Littlewood–Paley decomposition of f.

Proof: Clearly, we have: $\langle S_N f + \sum_{N \leq j < N+K} \Delta_j f | g \rangle_{\mathcal{S}',\mathcal{S}} = \langle S_{N+K} f | g \rangle_{\mathcal{S}',\mathcal{S}} = \langle f | S_{N+K} g \rangle_{\mathcal{S}',\mathcal{S}}$; thus, taking the Fourier transform $h = \hat{g}$ of g, it is enough to check that, for any $h \in \mathcal{S}(\mathbb{R}^d)$, we have $\lim_{N \to \infty} \varphi(\frac{\xi}{2^n}) h(\xi) = h(\xi)$ strongly in $\mathcal{S}$. □

Definition 3.2: (Distributions vanishing at infinity)

We define the space of tempered distributions vanishing at infinity as the space $\mathcal{S}'_0(\mathbb{R}^d)$ *of distributions so that* $\lim_{N\to-\infty} S_N f = 0$ *in* $\mathcal{S}'$.

For more general distributions, we cannot recover them from their homogeneous Littlewood–Paley decomposition but modulo polynomials:

Proposition 3.1: (Homogeneous decomposition)

For all $f \in (\mathcal{S})'$, *there is an integer* N *and a sequence of polynomials* $P_j, j \in \mathbb{Z}$ *of degree less or equal to* N *so that* $\sum_{j\in\mathbb{Z}}(\Delta_j f + P_j)$ *converges to* f *in* $(\mathcal{S})'$. *Thus the equality* $f = \sum_{j\in\mathbb{Z}} \Delta_j f$ *holds in* $\mathcal{S}'/\mathbb{C}[X_1, \dots, X_d]$.

Proof: We just have to recall that the Fourier transform $S_0 f$ is compactly supported, hence is of finite order K: $|\langle S_0 f|g\rangle| \leq C \sum_{|\alpha|\leq K} \|\frac{\partial^\alpha}{\partial\xi^\alpha}\hat{g}\|_\infty$. Now, we recall that S_0 is defined by a Fourier multiplier φ equal to 1 in a neighbourhood of 0; thus, we may write, for all $g \in \mathcal{S}$, $\hat{g} = (\sum_{|\alpha|\leq K} \frac{\xi^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial\xi^\alpha}\hat{g}(0))\varphi(\xi) + g_K(\xi)$. Then, clearly, $\lim_{j\to-\infty}\langle S_j f|g_K\rangle = \lim_{j\to-\infty}\langle S_0 f|S_j g_K\rangle = 0$. Thus, the series $\sum_{j\in\mathbb{Z}} T_j$, where the distributions T_j are defined by $\langle T_j|g\rangle = \langle\Delta_j f|g_K\rangle$, converges in $\mathcal{S}'$ to a limit T_∞. But $T_j - \Delta_j f$ are polynomials, and so is $T_\infty - f$ (since its Fourier transform is supported by $\{0\}$). $\square$

When dealing with the Littlewood–Paley decomposition, it is convenient to introduce the functions $\tilde{\varphi}(\xi) = \varphi(\xi/2)$ and $\tilde{\psi}(\xi) = \varphi(\xi/4) - \varphi(4\xi)$ as well as the operators $\tilde{S}_j$ and $\tilde{\Delta}_j$ defined by $\mathcal{F}(\tilde{S}_j f) = \tilde{\varphi}(\xi/2^j)\mathcal{F}f$ and $\mathcal{F}(\tilde{\Delta}_j f) = \tilde{\psi}(\xi/2^j)\mathcal{F}f$ (so that $S_j = \tilde{S}_j S_j$ and $\Delta_j = \tilde{\Delta}_j\Delta_j$). Using these operators, we obviously get the *Bernstein inequalities*:

Proposition 3.2: (Bernstein inequalities)

For all $\alpha \in \mathbb{N}^d$ *and* $\sigma \in \mathbb{R}$, *for all* $j \in \mathbb{Z}$, *for all* $1 \leq p \leq q \leq \infty$ *and for all* $f \in \mathcal{S}'(\mathbb{R}^d)$, *we have:*

(a) $\|\frac{\partial^\alpha}{\partial x^\alpha} S_j f\|_p \leq \|\frac{\partial^\alpha}{\partial x^\alpha}\mathcal{F}^{-1}\tilde{\varphi}\|_1\ \|S_j f\|_p\ 2^{j|\alpha|}$

(b) $\|\frac{\partial^\alpha}{\partial x^\alpha}\Delta_j f\|_p \leq \|\frac{\partial^\alpha}{\partial x^\alpha}\mathcal{F}^{-1}\tilde{\psi}\|_1\ \|\Delta_j f\|_p\ 2^{j|\alpha|}$

(c) $\|(\sqrt{-\Delta})^\sigma\Delta_j f\|_p \leq \|\mathcal{F}^{-1}(|\xi|^\sigma\ \tilde{\psi})\|_1\ \|\Delta_j f\|_p\ 2^{j\sigma}$

(d) $\|\Delta_j f\|_p \leq \sum_{l=1}^d \|\mathcal{F}^{-1}(\frac{\xi_l}{|\xi|^2}\tilde{\psi})\|_1\ \|\frac{\partial}{\partial x_l}\Delta_j f\|_p\ 2^{-j}$

(e) $\|S_j f\|_q \leq \|\mathcal{F}^{-1}\tilde{\varphi}\|_r\|S_j f\|_p 2^{j(d/p-d/q)}$ *with* $1/r = 1 - (1/p - 1/q)$.

(f) $\|\Delta_j f\|_q \leq \|\mathcal{F}^{-1}\tilde{\psi}\|_r\|\Delta_j f\|_p 2^{j(d/p-d/q)}$ *with* $1/r = 1 - (1/p - 1/q)$.

Another important tool in Littlewood–Paley analysis is the paraproduct operator introduced by J.M. Bony [BON 81]:

Proposition 3.3: (Paraproduct operator)

Let $f, g \in \mathcal{S}'(\mathbb{R}^d)$. *Then:*

(a) The series $\sum_{j\in\mathbb{N}} S_{j-2} f\Delta_j g$ *converges in* $\mathcal{S}'$. *Its sum,* $\pi(f,g)$, *is called the paraproduct of* f *and* g.

(b) The series $\sum_{j\in\mathbb{Z}} S_{j-2}f\Delta_j g$ *converges in* $\mathcal{S}'/\mathbb{C}[X_1,\dots,X_d]$*. Its sum,* $\dot\pi(f,g)$*, is called the homogeneous paraproduct of* f *and* g*.*

Proof: Since the Fourier transform of a product is the convolution of the Fourier transforms, we easily check that the spectrum of the product $S_{j-2}f\Delta_j g$ is supported on the set $\{\xi \ / \ \frac{1}{4}2^j|\xi| \le 4\ 2^j\}$. Moreover, f is tempered, so that we may write $\hat f = \sum_{|\alpha|\le N} \partial^\alpha f_\alpha$ with $|f_\alpha(\xi)| \le C(1+|\xi|)^M$ for some constants C and M and some measurable functions f_α, and similar estimates for $\hat g$, and we get, for $j \ge 0$, $\mathcal{F}(S_{j-2}f\Delta_j g) = \sum_{|\alpha|\le N'} \partial^\alpha f_{\alpha,j}$ with $|f_{\alpha,j}(\xi)| \le C'(1+|\xi|)^{M'}$ for some constants C', N' and M' which do not depend on j. Thus, $\pi(f,g)$ is well defined in $\mathcal{S}'$.

For $j<0$, we similarly get $\mathcal{F}(S_{j-2}f\Delta_j g) = \sum_{|\alpha|\le N'} \partial^\alpha f_{\alpha,j}$, with estimates $|f_{\alpha,j}(\xi)| \le C''|\xi|^{-M''}$ for some constants C'', N'' and M'' which do not depend on j. This gives that there is an integer K and a sequence of polynomials $P_j, j<0$, of degree less or equal to K so that $\sum_{j<0}(S_{j-2}f\Delta_j g + P_j)$ converges in $\mathcal{S}'$. □

2. Besov spaces as real interpolation spaces of potential spaces

In this section, we introduce the potential spaces H_p^σ and the Besov spaces $B_p^{\sigma,q}$:

Definition 3.3: (Potential spaces)
For $\sigma\in\mathbb{R}$ *and* $1\le p\le\infty$*, the potential space* H_p^σ *is defined as the space* $(Id-\Delta)^{-\sigma/2}L^p$*, equipped with the norm* $\|f\|_{H_p^\sigma} = \|(Id-\Delta)^{\sigma/2}f\|_p$*.*

We need a precise description of the convolution operator $(Id-\Delta)^{\sigma/2}$, which we obtain by using the Littlewood–Paley decomposition:

Proposition 3.4: (Bessel potentials)
For $\sigma\in\mathbb{R}$*, let* $k_\sigma\in\mathcal{S}'(\mathbb{R}^d)$ *be the distribution defined by* $\mathcal{F}k_\sigma(\xi) = (1+|\xi|^2)^{\sigma/2}$ *and* $(Id-\Delta)^{\sigma/2}$ *be the convolution operator with* k_σ*. Then:*

(a) $(Id-\Delta)^{\sigma/2}$ *operates boundedly on* $\mathcal{S}(\mathbb{R}^d)$ *and on* $\mathcal{S}'(\mathbb{R}^d)$*.*

(b) For all $j\in\mathbb{Z}$*,* $S_jk_\sigma\in\mathcal{S}(\mathbb{R}^d)$ *and* $\Delta_jk_\sigma\in\mathcal{S}(\mathbb{R}^d)$*.*

(c) For all $N\in\mathbb{N}$*, there exists a constant* $C_{\sigma,N}$ *so that for all* $j\ge0$*, we have for all* $x\in\mathbb{R}^d$

$$|\Delta_jk_\sigma(x)| \le C_{\sigma,N}\ 2^{j(d+\sigma)}\ (1+2^j|x|)^{-N}$$

(d) If $\sigma<0$*, then* $k_\sigma\in L^1(\mathbb{R}^d)$*. More precisely, for all* $N\in\mathbb{N}$*, there exists a constant* $D_{\sigma,N}$ *so that we have for all* $x\in\mathbb{R}^d$

$$|k_\sigma(x)| \le D_{\sigma,N}\ \omega_\sigma(x)(1+|x|)^{-N}$$

with $\omega_\sigma(x) = |x|^{-d-\sigma}$ *if* $d+\sigma > 0$, $= 1+\ln^+ \frac{1}{|x|}$ *if* $d+\sigma = 0$ *and* 1 *if* $d+\sigma < 0$.

Proof: (a) is obvious since $\mathcal{F}k_\sigma$ is a smooth function whose derivatives of any order are slowly increasing functions. (b) is obvious as well since the Fourier transforms of $S_j k_\sigma$ and of $\Delta_j k_\sigma$ are compactly supported smooth functions.

To prove (c), we may assume $j \geq 1$ so that we have

$$(2\pi)^d \|\Delta_j k_\sigma\|_\infty \leq \int (1+|\xi|^2)^{\sigma/2} |\psi(\xi/2^j)|\, d\xi \leq (1+2^{\sigma/2}) \int |\xi|^\sigma |\psi(\xi)|\, d\xi \; 2^{j(d+\sigma)}$$

We estimate in a similar way $\| |x|^N \Delta_j k_\sigma \|_\infty \leq \sqrt{d}^N \max_{1\leq l\leq d} \|x_l^N \; \Delta_j k_\sigma\|_\infty \leq \sqrt{d}^N \max_{1\leq l\leq d} (\frac{1}{2\pi})^d \int |\frac{\partial^N}{\partial x_l^N}((1+|\xi|^2)^{\sigma/2} \; \psi(\xi/2^j))|\, d\xi$ and get $\| |x|^N \Delta_j k_\sigma \|_\infty = O(2^{j(d+\sigma-N)})$.

We now prove (d). We write $k_\sigma = S_0 k_\sigma + \sum_{j\geq 0} \Delta_j k_\sigma$. Since $S_0 k_\sigma \in \mathcal{S}$, $S_0 k$ fulfills the desired size estimate. For $\sum_{j\geq 0} \Delta_j k_\sigma$, we consider two cases : if $|x| > 1$, we write for any $M > d+\sigma$ that $|\Delta_j k_\sigma(x)| \leq C_{M,\sigma} 2^{j(d+\sigma-M)} |x|^{-M}$ and just sum over $j \in \mathbb{N}$; if $|x| \leq 1$, we fix $M > d+\sigma$ and write $|\Delta_j k_\sigma(x)| \leq C_{M,\sigma} 2^{j(d+\sigma)}$ for $2^j|x| \leq 1$ and $|\Delta_j k_\sigma(x)| \leq C_{M,\sigma} 2^{j(d+\sigma-M)} |x|^{-M}$ for $2^j|x| > 1$; the summation over the j-s such that $2^j|x| > 1$ gives an estimate $O(|x|^{-d-\sigma})$; the summation over the j-s such that $2^j|x| \leq 1$ gives an estimate $O(|x|^{-d-\sigma})$ if $d+\sigma > 0$, $0(1-\ln|x|)$ if $d+\sigma = 0$ and $O(1)$ if $d+\sigma < 0$. $\square$

We may now define the Besov spaces:

Definition 3.4: (Besov spaces)

For $1 \leq p \leq \infty$, $\sigma \in \mathbb{R}$, $1 \leq q \leq \infty$, *the Besov space* $B_p^{\sigma,q}$ *is defined as* $B_p^{\sigma,q} = [H_p^{\sigma_0}, H_p^{\sigma_1}]_{\theta,q}$ *where* $\sigma_0 < \sigma < \sigma_1$ *and* $\sigma = (1-\theta)\sigma_0 + \theta\sigma_1$.

We easily prove that this definition does not depend on the choice of σ_0 and σ_1 by giving the Littlewood–Paley decomposition of distributions in $B_p^{\sigma,q}$:

Proposition 3.5: (Littlewood–Paley decomposition of Besov spaces)

For $1 \leq p \leq \infty$, $\sigma \in \mathbb{R}$, $1 \leq q \leq \infty$, $N \in \mathbb{Z}$ *and* $f \in \mathcal{S}'(\mathbb{R}^d)$, *the following assertions are equivalent :*

(A) $f \in B_p^{\sigma,q}$

(B) $S_N f \in L^p$, *for all* $j \geq N$ $\Delta_j f \in L^p$ *and* $(2^{j\sigma}\|\Delta_j f\|_p)_{j\geq N} \in l^q$.

Moreover, if $\sigma > 0$, *if we define for* $k \in \mathbb{N}$ *the Sobolev space* $W^{k,p}$ *as* $W^{k,p} = \{f \in L^p \;/\; \forall \alpha \in \mathbb{N}^d$ *with* $|\alpha| \leq k$, $\frac{\partial^\alpha}{\partial x^\alpha} f \in L^p\}$, *if* $k_0 < \sigma < k_1$, $\sigma = (1-\theta)k_0 + \theta k_1$, *then* $B_p^{\sigma,q} = [W^{k_0,p}, W^{k_1,p}]_{\theta,q}$.

Proof: $(B) \Rightarrow (A)$ is quite obvious. Indeed, if (B) holds for some N, it holds as well for $N = 0$. Now, we write $f = S_0 f + \sum_{j\geq 0} \Delta_j f$; then $S_0 f$ belongs to $H_p^{\sigma_1} \subset H_p^{\sigma_0}$ and so does $\Delta_j f$. Thus, it is enough to check that, for some positive $\rho \neq 1$, $\rho^{-j\theta} J(\rho^j, \Delta_j f) \in l^q$. We start from inequality $\|\Delta_j f\|_{H_p^{\sigma_i}} =$

$\|\tilde{\Delta}_j k_{\sigma_i} * \Delta_j f\|_p \leq C_{\sigma_i} 2^{j\sigma_i} \|\Delta_j f\|_p$ (according to Proposition 3.4) and write, for $\epsilon_j = \rho^{-j\theta} J(\rho^j, \Delta_j f) = \max(\rho^{-j\theta} \|\Delta_j f\|_{H_p^{\sigma_0}}, \rho^{j(1-\theta)} \|\Delta_j f\|_{H_p^{\sigma_1}})$ that $\epsilon_j \leq C \max(\rho^{-j\theta} 2^{-j\theta(\sigma_1-\sigma_0)}, \rho^{j(1-\theta)} 2^{j(1-\theta)(\sigma_1-\sigma_0)}) 2^{j\sigma} \|\Delta_j f\|_p$; we then conclude by taking $\rho = 2^{-(\sigma_1-\sigma_0)}$.

We now prove $(A) \Rightarrow (B)$. We write, for some positive $\rho \neq 1$, $f = \sum_{j\in\mathbb{Z}} f_j$ with $\rho^{-j\theta} J(\rho^j, f_j) \in l^q$. We then compute $\Delta_j f = \sum_{l\in\mathbb{Z}} \Delta_j f_l$ and $S_N f = \sum_{l\in\mathbb{Z}} S_N f_l$. We have $\|f_l\|_{H^{\sigma_0}} \leq \rho^{l\theta} \epsilon_l$ and $\|f_l\|_{H^{\sigma_1}} \leq \rho^{l(\theta-1)} \epsilon_l$, with $(\epsilon_l) \in l^q$. Since $f_l = k_{-\sigma_i} * (k_{\sigma_i} * f_l)$, we find (using Proposition 3.4) that, for $j \geq N$, $\|\Delta_j f_l\|_p \leq C_{\sigma_i,N} 2^{-j\sigma_i} \|f_l\|_{H^{\sigma_i}}$ and $\|S_N f_l\|_p \leq C_{\sigma_i,N} \|f_l\|_{H^{\sigma_i}}$. Since $\sum_l \min(\|f_l\|_{H^{\sigma_0}}, \|f_l\|_{H^{\sigma_1}}) < \infty$, we find that $S_N f \in L^p$. Moreover, we have $2^{j\sigma} \|\Delta_j f_l\|_p \leq C \min(\rho^{\theta l} 2^{j\theta(\sigma_1-\sigma_0)}, \rho^{(\theta-1)l} 2^{j(\theta-1)(\sigma_1-\sigma_0)})\ \epsilon_l$ and we conclude by taking $\rho = 2^{-(\sigma_1-\sigma_0)}$: we obtain, for $0 < \alpha = \rho^(1-\theta) < 1 < \beta = \rho^{-\theta}$, that $2^{j\sigma} \|\Delta_j f\|_p \leq C \sum_{l\in\mathbb{Z}} \min(\alpha^{j-l}, \beta^{j-l}) \epsilon_l$ with $\sum_{l\in\mathbb{Z}} \min(\alpha^l, \beta^l) < \infty$. Since $l^1 * l^q \subset l^q$, we have finished to prove (A).

We now prove that $B_p^{\sigma,q} = [W^{k_0,p}, W^{k_1,p}]_{\theta,q}$. We use the reiteration theorem for the real interpolation method: it is enough to check that $B_p^{k,1} \subset W^{k,p} \subset B_p^{k,\infty}$. But this is a direct consequence of the Bernstein inequalities. □

The duality theorem for the real interpolation method combined with the Riesz theorem gives that the Besov spaces are dual spaces:

Proposition 3.6: (Duality and Besov spaces)

For $1 \leq p \leq \infty$, $\sigma \in \mathbb{R}$, $1 \leq q \leq \infty$, let p', q' be the conjugate exponents of p and q and let $\tilde{B}_{p'}^{-\sigma,q'}$ be the closure of $\mathcal{S}$ in $B_{p'}^{-\sigma,q'}$. Then $B_p^{\sigma,q}$ is the dual space of $\tilde{B}_{p'}^{-\sigma,q'}$.

Proof: The case $p > 1$ is a straightforward application of the duality theorem. Clearly, $\mathcal{S}$ is dense in $H_{p'}^{\sigma_i}$ for $\sigma_0 < -\sigma < \sigma_1$, and this gives (for $-\sigma = \theta\sigma_1 + (1-\theta)\sigma_0$) that $\mathcal{S}$ is dense in $B_{p'}^{-\sigma,q'}$ if $q' < \infty$ while the dual of $B_{p'}^{-\sigma,q'} = [H_{p'}^{\sigma_0}, H_{p'}^{\sigma_1}]_{\theta,q'}$ is equal to $[H_p^{-\sigma_0}, H_p^{-\sigma_1}]_{\theta,q} = B_p^{\sigma,q}$. The case $q = 1$ follows the same lines, since $l^1(\mathbb{N})$ is the dual of c_0: it is then enough to check that f belongs to $\tilde{B}_{p'}^{-\sigma,\infty}$ $(p' < \infty)$ if and only if $S_0 f \in L^{p'}$ and $(2^{-j\sigma} \|\Delta_j f\|_{p'})_{j\geq 0} \in c_0$. The case $p = 1$ requires only a slight modification: for $q' < \infty$ we have $\tilde{B}_\infty^{-\sigma,q'} = [(Id-\Delta)^{-\sigma_0/2} \mathcal{C}_0, (Id-\Delta)^{-\sigma_1/2} \mathcal{C}_0]_{\theta,q'}$ (and a similar result for $q' = \infty$ changing l^∞ into c_0); the duality theorem gives that the dual space of $\tilde{B}_\infty^{-\sigma,q'}$ is equal to $[(Id-\Delta)^{\sigma_0/2} M, (Id-\Delta)^{\sigma_1/2} M]_{\theta,q}$ (where M is the space of finite complex Borel measures on $\mathbb{R}^d$). We easily find that $f \in [(Id-\Delta)^{\sigma_0/2} M, (Id-\Delta)^{\sigma_1/2} M]_{\theta,q}$ if and only if $S_0 f \in M$ and $(2^{j\sigma} \|\Delta_j f\|_M)_{j\geq 0} \in l^q$; but $S_0 f = \tilde{S}_0(S_0 f)$ and $\Delta_j f = \tilde{\Delta}_j(\Delta_j f)$ and $\tilde{S}_0$ and $\tilde{\Delta}_j$ map M to L^∞; since for $g \in M \cap L^\infty$, we have $g \in L^1$ and $\|g\|_M = \|g\|_1$, this gives that the dual space of $\tilde{B}_\infty^{-\sigma,q'}$ is $B_1^{\sigma,q}$. □

Chapter 7 demonstrates that for $1 < p < \infty$ the space H_p^σ is easily characterized by the Littlewood–Paley decomposition as well:

Proposition 3.7: (Littlewood–Paley decomposition of potential spaces)

For $1 < p < \infty$, $\sigma \in \mathbb{R}$, $N \in \mathbb{Z}$ and $f \in \mathcal{S}'(\mathbb{R}^d)$, the following assertions are equivalent:

(A) $f \in H_p^\sigma$

(B) $S_N f \in L^p$, for all $j \geq N$ $\Delta_j f \in L^p$ and $(\sum_{j\geq N} 4^{j\sigma}|\Delta_j f|^2)^{1/2} \in L^p(dx)$.

We now prove the following corollary of this result:

Corollary: *For $1 < p < \infty$ and $\sigma \in \mathbb{R}$, we have $B_p^{\sigma,\min(p,2)} \subset H_p^\sigma \subset B_p^{\sigma,\max(p,2)}$.*

Proof: For $2 \leq p < \infty$, we use the Fubini theorem and the inclusion $l^2 \subset l^p$ to get the inclusion $H_p^\sigma \subset B_p^{\sigma,p}$: we just have to write the inequalities

$$(\sum_{j\geq N} 2^{jp\sigma}\|\Delta_j f\|_p^p)^{1/p} = \|(\sum_{j\geq N} 2^{jp\sigma}|\Delta_j f|^p)^{1/p}\|_p \leq \|(\sum_{j\geq N} 4^{j\sigma}|\Delta_j f|^2)^{1/2}\|_p.$$

To get $B_p^{\sigma,2} \subset H_p^\sigma$, it is enough to use the Minkovski inequality in $L^{p/2}$:

$$\|(\sum_{j\geq N} 4^{j\sigma}|\Delta_j f|^2)^{1/2}\|_p = \|\sum_{j\geq N} 4^{j\sigma}|\Delta_j f|^2\|_{p/2}^{1/2} \leq (\sum_{j\geq N} 4^{j\sigma}\|(\Delta_j f)^2\|_{p/2})^{1/2}.$$

The result for $p \leq 2$ follows by duality. $\square$

3. Homogeneous Besov spaces

In this section, we are interested in distributions modulo polynomials. We start from this easy proposition:

Proposition 3.8: (Distributions modulo polynomials)
(A) The space $\mathcal{S}_\infty = \{f \in \mathcal{S}(\mathbb{R}^d)/\forall\alpha \in \mathbb{N}^d \ \int x^\alpha\ f(x)\ dx = 0\}$ is closed in $\mathcal{S}$.
(B) f belongs to $\mathcal{S}_\infty$ if and only if $f \in \mathcal{S}$ and $\sum_{j\in\mathbb{Z}} \Delta_j f$ converges to f in $\mathcal{S}$.
(C) The orthogonal $(\mathcal{S}_\infty)^\perp$ in $(\mathcal{S})'$ is the space of polynomials $\mathbb{C}[X_1, \ldots, X_d]$. Thus, $(\mathcal{S}_\infty)'$ is the space of tempered distributions modulo the polynomials $\mathcal{S}'/\mathbb{C}[X_1, \ldots, X_d]$.
(D) For all $f \in (\mathcal{S})'$, there is a sequence of polynomials $P_j, j \in \mathbb{Z}$ such that $\sum_{j\in\mathbb{Z}}(\Delta_j f + P_j)$ converges to f in $(\mathcal{S})'$. Thus the equality $f = \sum_{j\in\mathbb{Z}} \Delta_j f$ holds in $\mathcal{S}'/\mathbb{C}[X_1, \ldots, X_d]$.

Proof: (A) is obvious. For (B), we have only to check that $S_j f$ goes to 0 in $\mathcal{S}$ as j goes to $-\infty$ when f belongs to $\mathcal{S}_\infty$. Using the Fourier transform, we have to check that $\varphi(\xi/2^j)\hat{f}(\xi)$ goes to 0 in $\mathcal{C}^\infty$ whenever $\hat{f}$ vanishes at

0 with infinite order; this is easy to check since, for all $\alpha \in \mathbb{N}^d$, we have $|\frac{\partial^\alpha}{\partial\xi^\alpha}[\varphi(\frac{\xi}{2^j})]| \le C_\alpha|\xi|^{-|\alpha|}$ and $\lim_{\epsilon\to 0^+}\sup_{|\xi|\le\epsilon}|\xi|^{-|\alpha|}\hat f(\xi) = 0$. (C) is easy as well: it is enough to check that $f \in \mathcal{S}'$ is orthogonal to $\mathcal{S}_\infty$ if and only if $\hat f$ is supported by $\{0\}$. (D) was proved in Proposition 3.1. □

We may now define homogeneous Besov spaces in the following way:

Definition 3.5: (Homogeneous Besov spaces)
Let $1 \le p \le \infty$, $\sigma \in \mathbb{R}$ and $1 \le q \le \infty$. Then, the homogeneous Besov space $\dot{\mathcal{B}}_p^{\sigma,q}$ is defined as the Banach space of distributions $f \in \mathcal{S}'/\mathbb{C}[X_1,\dots,X_d]$ such that for all $j \in \mathbb{Z}$ $\Delta_j f \in L^p$ and $(2^{j\sigma}\|\Delta_j f\|_p)_{j\in\mathbb{Z}} \in l^q(\mathbb{Z})$.

The duality theorem has a homogeneous counterpart:

Proposition 3.9: (Duality and homogeneous Besov spaces)
For $1 \le p \le \infty$, $\sigma \in \mathbb{R}$, $1 \le q \le \infty$, let p', q' be the conjugate exponents of p and q and let $\tilde{\dot{\mathcal{B}}}_{p'}^{-\sigma,q'}$ be the closure of $\mathcal{S}_\infty$ in $\dot{\mathcal{B}}_{p'}^{-\sigma,q'}$. Then $\dot{\mathcal{B}}_p^{\sigma,q}$ is the dual space of $\tilde{\dot{\mathcal{B}}}_{p'}^{-\sigma,q'}$: a distribution $f \in \mathcal{S}'$ belongs to $\dot{\mathcal{B}}_p^{\sigma,q}$ if and only if there exists a constant C so that for all $\omega \in \mathcal{S}_\infty$ we have $|\langle f|\omega\rangle| \le C\|\omega\|_{\dot{\mathcal{B}}_{p'}^{-\sigma,q'}}$.

Proof: This is a direct consequence of the duality theorem for non homogeneous Besov spaces. □

This proposition should be modulated in the following manner: the duality bracket in Proposition 8 is the bracket between $\mathcal{S}_\infty$ and its dual; but for a given σ, $\dot{\mathcal{B}}_{p'}^{-\sigma,q'}$ contains a bigger subspace of $\mathcal{S}$ than $\mathcal{S}_\infty$, and the duality bracket should be the one between this subspace and its dual. We thus begin with the following lemma:

Lemma 3.1: *$f \in \mathcal{S}(\mathbb{R}^d)$ belongs to $\dot{\mathcal{B}}_{p'}^{-\sigma,\infty}$ if and only if $\int x^\alpha f(x)\,dx = 0$ for all $\alpha \in \mathbb{N}^d$ so that $|\alpha| < \sigma - d/p$. Similarly, f belongs to $\dot{\mathcal{B}}_{p'}^{-\sigma,q'}$ with $q' < \infty$ if and only if $\int x^\alpha f(x)\,dx = 0$ for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| \le \sigma - d/p$.*

Proof: Let N be an integer so that $\int x^\alpha f(x)\,dx = 0$ for all $\alpha \in \mathbb{N}^d$ so that $|\alpha| < N$. We write $\hat f(\xi) = \sum_{|\alpha|=N}\frac{\xi^\alpha}{\alpha!}\frac{\partial^\alpha f}{\partial x^\alpha}(0)\ \varphi(\xi) + \sum_{|\alpha|=N+1}\xi^\alpha \hat f_\alpha(\xi)$ where φ is the function associated to the Littlewood–Paley operator S_0 and where f_α belongs to $\mathcal{S}$. Then, for $j \le -2$, we have $\Delta_j f = 2^{j(d+N)}\Psi_f(2^j x) + \sum_{|\alpha|=N+1} f_\alpha * (2^{j(d+N+1)}\Psi_\alpha(2^j x))$ with $\mathcal{F}\Psi_f = \sum_{|\alpha|=N}\frac{\xi^\alpha}{\alpha!}\frac{\partial^\alpha f}{\partial x^\alpha}(0)\ \psi(\xi)$ and $\mathcal{F}\Psi_\alpha = \xi^\alpha\ \psi(\xi)$. This gives $\|\Delta_j f\|_{p'} = 2^{j(N+d/p)}\|\Psi_f\|_{p'} + O(2^{j(N+1+d/p)}$. □

This lemma leads us to the following alternative definition of homogeneous Besov spaces:

Definition 3.6: (Realization of homogeneous Besov spaces)

The realization $\dot{B}^{\sigma,q}_p$ *of the homogeneous Besov space* $\dot{\mathcal{B}}^{\sigma,q}_p$ *is defined in the following manner: we define* $N(\sigma,p,q)$ *as the smallest integer* N *so that* $N > \sigma - d/p$ *if* $q > 1$ *(or so that* $N \geq \sigma - d/p$ *if* $q = 1$*); then, a distribution* $f \in \mathcal{S}'$ *belongs to* $\dot{B}^{\sigma,q}_p$ *if and only if there exists a constant* C *so that for all* $\omega \in \mathcal{S}$ *with* $\int x^\alpha \omega \, dx = 0$ *for all* $\alpha \in \mathbb{N}^d$ *with* $|\alpha| < N(\sigma,p,q)$ *we have* $|\langle f|\omega\rangle| \leq C\|\omega\|_{\dot{\mathcal{B}}^{-\sigma,q'}_{p'}}$.

Example: If $\sigma < d/p$ (or if $\sigma = d/p$ and $q = 1$), then for all $f \in \dot{\mathcal{B}}^{\sigma,q}_p$, the decomposition $\sum_{j\in\mathbb{Z}} \Delta_j f$ converges in $\mathcal{S}'$; in that case, we can easily check that the realization of the Besov space $\dot{\mathcal{B}}^{\sigma,q}_p$ is equal to $\dot{B}^{\sigma,q}_p = \dot{\mathcal{B}}^{\sigma,q}_p \cap \mathcal{S}'_0$ and that the mapping $f \mapsto \sum_{j\in\mathbb{Z}} \Delta_j f$ is an isomorphism from $\dot{\mathcal{B}}^{\sigma,q}_p$ to $\dot{B}^{\sigma,q}_p$.

More generally, $\dot{B}^{\sigma,q}_p$, for $\sigma \in \mathbb{R}$, is a space of distributions modulo polynomials of degree less than $N(\sigma,p,q)$.

Chapter 4

Shift-invariant Banach spaces of distributions and related Besov spaces

We shall often use Banach spaces of distributions whose norm is invariant under translations $\|T(x-x_0)\| = \|T\|$ and on which dilations operate boundedly. This chapter is devoted to the basic definitions and results in this setting.

1. Shift-invariant Banach spaces of distributions

Definition 4.1: (Shift-invariant Banach spaces of distributions.)

A) A shift-invariant Banach space of test functions is a Banach space E such that we have the continuous embeddings $\mathcal{D}(\mathbb{R}^d) \subset E \subset \mathcal{D}'(\mathbb{R}^d)$ and so that:
(a) for all $x_0 \in \mathbb{R}^d$ and for all $f \in E$, $f(x-x_0) \in E$ and $\|f\|_E = \|f(x-x_0)\|_E$.
(b) for all $\lambda > 0$ there exists $C_\lambda > 0$ so that for all $f \in E$ $f(\lambda x) \in E$ and $\|f(\lambda x)\|_E \leq C_\lambda \|f\|_E$.
(c) $\mathcal{D}(\mathbb{R}^d)$ is dense in E

B) A shift-invariant Banach space of distributions is a Banach space E which is the topological dual of a shift-invariant Banach space of test functions $E^{(*)}$. The space $E^{(0)}$ of smooth elements of E is defined as the closure of $\mathcal{D}(\mathbb{R}^d)$ in E.

Remark: An easy consequence of hypothesis (a) is that a shift-invariant Banach space of test functions E satisfies $\mathcal{S}(\mathbb{R}^d) \subset E$ and a consequence of hypothesis (b) is that $E \subset \mathcal{S}'(\mathbb{R}^d)$. Similarly, we have for a shift-invariant Banach space of distributions E that $\mathcal{S}(\mathbb{R}^d) \subset E^{(0)} \subset E \subset \mathcal{S}'(\mathbb{R}^d)$. In particular, $E^{(0)}$ is a shift-invariant Banach space of test functions.

Shift-invariant Banach spaces of distributions are adapted to convolution with integrable kernels:

Proposition 4.1: (Convolution in shift-invariant spaces of distributions.)

If E is a shift-invariant Banach space of test functions or of distributions and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, then for all $f \in E$ we have $f * \varphi \in E$ and $\|f * \varphi\|_E \leq \|f\|_E \, \|\varphi\|_1$.

Moreover, convolution may be extended into a bounded bilinear operator from $E \times L^1$ to E and we have for all $f \in E$ and all $g \in L^1$ the inequality $\|f * g\|_E \leq \|f\|_E \|g\|_1$.

Proof: It is enough to check that for $f, g \in \mathcal{S}(\mathbb{R}^d)$ the Riemann sums $\frac{1}{N^d} \sum_{k \in \mathbb{Z}^d} g(\frac{k}{N}) f(x - \frac{k}{N})$ converge to $f * g$ in $\mathcal{S}(\mathbb{R}^d)$ as N goes to ∞. □

Due to this proposition, we very often use very often convolution operators with an integrable kernel. We thus introduce a convenient notation:

Definition 4.2: (Convolution operators)

If $k \in L^1(\mathbb{R}^d)$ and $m = \hat{k}$, we define the operator $m(D)$ as the convolution operator $m(D)f = f * k$ and we define the norm $\|\|m(D)\|\|_1$ as $\|\|m(D)\|\|_1 = \|k\|_1$.

A special case of convolution operator is the j–th dyadic block of the Littlewood–Paley decomposition $\Delta_j f = \psi(D/2^j) f$: we obtain the estimates $\|S_j f\|_E \leq \|\|S_0\|\|_1 \|f\|_E$ and $\|\Delta_j f\|_E \leq \|\|\Delta_0\|\|_1 \|f\|_E$. Due to the Littlewood–Paley decomposition, we may easily compare E to a Besov space:

Proposition 4.2:

Let E be a shift-invariant Banach space of distributions. Let $C_{1/2}$ be the operator norm in $\mathcal{L}(E, E)$ of the operator $f \mapsto f(x/2)$. Then, $E \subset B_\infty^{-\ln_2(C_{1/2}), \infty}$.

Proof: Since the norm of E is shift-invariant and since $E \subset \mathcal{S}'$, we find that Δ_j maps E to L^∞. Moreover, since dilations operate boundedly on E and on L^∞, we may write $D_j(f) = f(2^j x)$ and $\Delta_j f = D_j \Delta_0 D_{-j} f$ which gives that for $j \geq 0$ Δ_j maps E to L^∞ with an operator norm $O((C_{1/2})^j)$. □

As for the case of Lebesgue norms, we have the *Bernstein inequalities*:

Proposition 4.3: (Bernstein inequalities)

Let E be a shift-invariant Banach space of distributions. Then, for all $\alpha \in \mathbb{N}^d$ and $\sigma \in \mathbb{R}$, for all $j \in \mathbb{Z}$ and for all $f \in \mathcal{S}'(\mathbb{R}^d)$, we have (whenever the right-hand sides are well-defined):

(a) $\|\frac{\partial^\alpha}{\partial x^\alpha} S_j f\|_E \leq \|\frac{\partial^\alpha}{\partial x^\alpha} \mathcal{F}^{-1} \tilde{\varphi}\|_1 \; \|S_j f\|_E \; 2^{j|\alpha|}$

(b) $\|\frac{\partial^\alpha}{\partial x^\alpha} \Delta_j f\|_E \leq \|\frac{\partial^\alpha}{\partial x^\alpha} \mathcal{F}^{-1} \tilde{\psi}\|_1 \; \|\Delta_j f\|_E \; 2^{j|\alpha|}$

(c) $\|(\sqrt{-\Delta})^\sigma \Delta_j f\|_E \leq \|\mathcal{F}^{-1}(|\xi|^\sigma \; \tilde{\psi})\|_1 \; \|\Delta_j f\|_E \; 2^{j\sigma}$

(d) $\|\Delta_j f\|_E \leq \sum_{l=1}^d \|\mathcal{F}^{-1}(\frac{\xi_l}{|\xi|^2} \tilde{\psi})\|_1 \; \|\frac{\partial}{\partial x_l} \Delta_j f\|_E \; 2^{-j}$

Another interesting example of convolution operator that will be considered throughout this book is the convolution with the heat kernel. We have a semigroup of operators $e^{t\Delta}$ with the following regularity estimates:

Proposition 4.4: (Heat kernel)

The heat kernel $e^{t\Delta}$ *satisfies* $\|\|e^{t\Delta} - e^{\theta\Delta}\|\|_1 \leq C|\ln t - \ln\theta|$. *Hence:*

a) If E *is a shift-invariant Banach space of test functions and if* $f \in E$, *then* $e^{t\Delta}f \in \mathcal{C}_b([0,\infty), E)$, *i.e., the map* $t \mapsto e^{t\Delta}f$ *is continuous and bounded from* $[0,\infty)$ *to* E.

b) If E *is a shift-invariant Banach space of distributions and if* $f \in E$, *then* $e^{t\Delta}f \in \mathcal{C}_*([0,\infty), E)$, *i.e., the map* $t \mapsto e^{t\Delta}f$ *is continuous from* $(0,\infty)$ *to* E *and continuous at* $t = 0$ *for the *-weak topology* $\sigma(E, E^{(*)})$ *on* E.

Proof: We write $e^{t\Delta} - e^{\theta\Delta} = \int_\theta^t \Delta e^{s\Delta}\, ds$; hence, $\|\|e^{t\Delta} - e^{\theta\Delta}\|\|_1 \leq \int_\theta^t \|\|\Delta e^{\Delta}\|\|_1 \frac{ds}{s}$. For the continuity at $t = 0$, it is enough to check that for $f \in \mathcal{S}(\mathbb{R}^d)$ $t \mapsto e^{t\Delta}f$ is continuous from $[0,\infty)$ to $\mathcal{S}$. □

Sometimes, we consider a special case of shift-invariant spaces of distributions: spaces of local measures, where the spaces are requested to remain invariant through pointwise multiplication with bounded continuous functions:

Definition 4.3: (Shift-invariant spaces of local measures)

A shift-invariant Banach space of local measures is a shift-invariant Banach space of distributions E *so that for all* $f \in E$ *and all* $g \in \mathcal{S}(\mathbb{R}^d)$ *we have* $fg \in E$ *and* $\|fg\|_E \leq C_E \|f\|_E \|g\|_\infty$, *where* C_E *is a positive constant (which depends neither on* f *nor on* g*).*

As a matter of fact, we have a more precise result on pointwise multiplications:

Lemma 4.1:

Let E *be a shift-invariant Banach space of local measures. Then:*

a) the elements of E *are local measures (i.e. are distributions of order 0). More precisely, they belong to the Morrey space* M^1_{uloc} *of uniformly locally finite measures.*

b) E*, its predual* $E^{(*)}$ *and the space of smooth elements* $E^{(0)}$ *in* E *are stable under pointwise multiplication by bounded continuous functions.*

Proof: a) is obvious: if $f \in E$ and if K is a compact subset of $\mathbb{R}^d$, then we consider a function $\phi_K \in \mathcal{D}(\mathbb{R}^d)$ which is equal to 1 on K, and we write for all $\omega \in \mathcal{D}(\mathbb{R}^d)$ with support included in K:

$$|\langle f|\omega\rangle| = |\langle f\omega|\phi_K\rangle| \leq \|f\|_E \; \|\phi_K\|_{E^{(*)}} \; \|\omega\|_\infty$$

Thus, f is a local measure $d\mu$. Moreover, taking $K = x_0 + [-1,1]^d$ and $\phi_K = \phi_{[-1,1]^d}(x - x_0)$, we find that $\sup_{x_0 \in \mathbb{R}^d} |\mu|(x_0 + [-1,1]^d) < \|\phi_{[-1,1]^d}\|_{E^{(*)}} \|f\|_E$, and thus $f \in M^1_{uloc}$.

b) is easy. We notice that $E^{(*)}$ is the closure of $\mathcal{D}$ in E'. Now, if K is a compact subset of $\mathbb{R}^d$, if $\phi_K \in \mathcal{D}(\mathbb{R}^d)$ satisfies $1_K \leq \phi_K \leq 1$ everywhere, then we have for all $\omega \in \mathcal{D}(\mathbb{R}^d)$ with support included in K and all $g \in \mathcal{C}^\infty(\mathbb{R}^d)$

that $\|\omega g\|_{E'} = \|\omega g \phi_K\|_{E'} \leq \|\omega\|_{E'} \|g\|_\infty$. Thus, pointwise multiplication may be extended to a bounded bilinear operator from $E^{(*)} \times \mathcal{C}_b$ to $E^{(*)}$. $\square$

2. Besov spaces

In this section, we introduce the potential spaces over a shift-invariant Banach space of distributions:

Definition 4.4: (Potential spaces)

Let E be a shift-invariant Banach space of distributions. Then, for $\sigma \in \mathbb{R}$, the space H^σ_E is defined as the space $(Id - \Delta)^{-\sigma/2} E$, equipped with the norm $\|f\|_{\sigma,E} = \|(Id - \Delta)^{\sigma/2} f\|_E$.

We now define Besov spaces over a shift-invariant Banach space of distributions, in a very similar way as for the Besov spaces $B^{s,q}_p$ based on the Lebesgue spaces L^p:

Definition 4.5: (Besov spaces)

Let E be a shift-invariant Banach space of distributions. For $\sigma \in \mathbb{R}$, $1 \leq q \leq \infty$, the Besov space $B^{s,q}_E$ is defined as $B^{s,q}_E = [H^{\sigma_0}_E, H^{\sigma_1}_E]_{\theta,q}$ where $\sigma_0 < \sigma < \sigma_1$ and $\sigma = (1-\theta)\sigma_0 + \theta\sigma_1$.

Just like for the Besov spaces $B^{s,q}_p$, we may give an easy characterization of the Besov spaces $B^{s,q}_E$ through the Littlewood–Paley decomposition:

Proposition 4.5: (Littlewood–Paley decomposition of Besov spaces)

Let E be a shift-invariant Banach space of distributions. For $\sigma \in \mathbb{R}$, $1 \leq q \leq \infty$, $N \in \mathbb{Z}$ and $f \in \mathcal{S}'(\mathbb{R}^d)$, the following assertions are equivalent:

(A) $f \in B^{\sigma,q}_E$

(B) $S_N f \in E$, for all $j \geq N$ $\Delta_j f \in E$ and $(2^{j\sigma} \|\Delta_j f\|_E)_{j \geq N} \in l^q$.

Moreover, if $\sigma > 0$, if we define for $k \in \mathbb{N}$ the Sobolev space $W^{k,E} = \{f \in E \ / \ \forall \alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$ $\frac{\partial^\alpha}{\partial x^\alpha} f \in E\}$, if $k_0 < \sigma < k_1$, $\sigma = (1-\theta)k_0 + \theta k_1$, then $B^{s,q}_E = [W^{k_0,E}, W^{k_1,E}]_{\theta,q}$.

Proof: This is just the same proof as for the spaces $B^{s,q}_p$, since we only used a convolution estimate with the integrable kernel of $\Delta_j k_\sigma$. $\square$

We now give a simple example of application of the Littlewood–Paley decomposition: pointwise multiplication is a bounded bilinear mapping on regular enough Besov spaces.

Theorem 4.1: (Pointwise multiplication in Besov spaces)

Let E be a shift-invariant Banach space of local measures. Then:

(a) $E \cap L^\infty$ is an algebra for pointwise multiplication.

(b) For $1 \leq q \leq \infty$ and $\sigma > 0$, $B^{\sigma,q}_E \cap L^\infty$ is an algebra for pointwise multiplication.

(c) For $1 \leq q \leq \infty$, $\sigma > 0$ and $\tau \in (-\sigma, 0)$, the pointwise multiplication is a bounded bilinear operator from $(B_E^{\sigma,q} \cap B_\infty^{\tau,\infty}) \times (B_E^{\sigma,q} \cap B_\infty^{\tau,\infty})$ to $B_E^{\sigma+\tau,q}$.

The proof is based on a basic lemma:

Lemma 4.2:
Let E be a shift-invariant Banach space of distributions, $\sigma \in \mathbb{R}$, $1 \leq q \leq \infty$ and let $(f_j)_{j\in\mathbb{N}}$ a sequence of elements of E so that $(2^{j\sigma}\|f_j\|_E)_{j\in\mathbb{N}} \in l^q(\mathbb{N})$. Then:

a) If for some constants $0 < A < B < \infty$, we have for all $j \in \mathbb{N}$ that $\hat{f}_j$ is supported by $\{\xi \in \mathbb{R}^d \ / \ A2^j \leq |\xi| \leq B2^j\}$, then $\sum_{j\in\mathbb{N}} f_j$ converges in $\mathcal{S}'$ to a distribution $f \in B_E^{\sigma,q}$.

b) If for some constant $0 < B < \infty$, we have for all $j \in \mathbb{N}$ that $\hat{f}_j$ is supported by $\{\xi \in \mathbb{R}^d \ / \ |\xi| \leq B2^j\}$ and if $\sigma > 0$, then $\sum_{j\in\mathbb{N}} f_j$ converges in $\mathcal{S}'$ to a distribution $f \in B_E^{\sigma,q}$.

Proof of the lemma: The lemma is obvious. The convergence of the series $\sum_{j\in\mathbb{N}} f_j$ is obvious in case b) since $\sum_{j\in\mathbb{N}} \|f_j\|_E < \infty$; in case a), we may write $f_j = \Delta^N g_j$ with $2N + \sigma > 0$ and we find that $\sum_{j\in\mathbb{N}} \|g_j\|_E < \infty$. Moreover, if $k \in \mathbb{N}$ is such that $2^{-k+1} < A < B < 2^{k-1}$, we have:
– in a), $\Delta_j f = \sum_{j-k\leq l\leq j+k} \Delta_j f_l$, hence $2^{j\sigma}\|\Delta_j f\|_E \leq C \sum_{j-k\leq l\leq j+k} 2^{l\sigma}\|f_l\|_E$
– in b), $\Delta_j f = \sum_{0\leq l\leq j+k} \Delta_j f_l$, hence $2^{j\sigma}\|\Delta_j f\|_E \leq C \sum_{0\leq l\leq j+k} 2^{(j-l)\sigma} 2^{l\sigma}\|f_l\|_E$,
and in both cases we may conclude that $f \in B_E^{\sigma,q}$. □

Proof of Theorem 4.1: We write $fg = \pi(f,g) + \pi(g,f) + R(f,g)$ where π is the paraproduct operator:

$$\begin{cases} \pi(f,g) = \sum_{j\in\mathbb{N}} S_{j-2}f\Delta_j g \\ \pi(g,f) = \sum_{j\in\mathbb{N}} S_{j-2}g\Delta_j f \\ R(f,g) = S_0 f S_0 g + \Delta_{-2}f\Delta_0 g + \Delta_{-1}f\Delta_0 g + \Delta_{-1}f\Delta_1 g + \rho(f,g) \\ \rho(f,g) = \sum_{j\in\mathbb{N}} \Delta_j f(\sum_{j-2\leq k\leq j+2} \Delta_k g) \end{cases}$$

We may now easily conclude, applying Lemma 4.2 to $\pi(f,g)$, $\pi(g,f)$ and $\rho(f,g)$: $S_{j-2}f\Delta_j g$ has its spectrum contained in $\{\xi \in \mathbb{R}^d \ / \ |\xi| \leq 5 \ 2^{j-1}\}$ and $\|S_{j-2}f\Delta_j g\|_E \leq C_E\|S_{j-2}f\|_\infty\|\Delta_j g\|_E$; moreover, we have $f \in L^\infty \Leftrightarrow \sup_{j\geq 0}\|S_j f\|_\infty < \infty$ and, for $\tau < 0$, $f \in B_\infty^{\tau,\infty} \Leftrightarrow \sup_{j\geq 0} 2^{j\tau}\|S_j f\|_\infty < \infty$. □

As a corollary, if E is a shift-invariant Banach space of local measures and if $E \subset B_\infty^{\sigma,\infty}$ for some $\sigma \leq 0$, then $B_E^{\rho,q}$ is a Banach algebra for pointwise multiplication for $\rho + \sigma > 0$ and $1 \leq q \leq \infty$, or for $\rho = -\sigma > 0$ and $q = 1$.

3. Homogeneous spaces

We now define homogeneous shift-invariant spaces:

Definition 4.6: (Homogeneous spaces)

A) A homogeneous shift-invariant Banach space of test functions is a Banach space E so that we have the continuous embeddings $\mathcal{S}_\infty(\mathbb{R}^d) \subset E \subset \mathcal{S}'(\mathbb{R}^d)/\mathbb{C}[X_1,\dots,X_d]$ and so that:
(a) for all $x_0 \in \mathbb{R}^d$ and for all $f \in E$, $f(x-x_0) \in E$ and $\|f\|_E = \|f(x-x_0)\|_E$.
(b) for all $\lambda > 0$, there exists $C_\lambda > 0$ so that for all $f \in E$ $f(\lambda x) \in E$ and $\|f(\lambda x)\|_E \le C_\lambda \|f\|_E$.
(c) $\mathcal{S}_\infty(\mathbb{R}^d)$ is dense in E

B) A homogeneous shift-invariant Banach space of distributions is a Banach space E that is the topological dual of a homogeneous shift-invariant Banach space of test functions $E^{(*)}$. The space $E^{(0)}$ of smooth elements of E is defined as the closure of $\mathcal{S}_\infty(\mathbb{R}^d)$ in E.

We sometimes need homogeneous spaces modulo polynomials of a given degree. As with homogeneous Besov spaces, we define the realization of the homogeneous space of distributions through a duality setting:

Lemma 4.3: *Let E be a homogeneous shift-invariant Banach space of test functions. Then, there exists an integer $N \in \mathbb{N}$ such that the space $\mathcal{S}_N = \{f \in \mathcal{S}(\mathbb{R}^d) \ / \ \forall \alpha \in \mathbb{N}^d$ such that $|\alpha| < N \ \int x^\alpha \ f(x) \ dx = 0\}$ is continuously embedded in E.*

Proof: The continuous embedding of $\mathcal{S}_\infty$ into E gives the inequality, for some constants $C \ge 0$ and $M \in \mathbb{N}$, $\|f\|_E \le C \sum_{|\alpha|\le M} \sum_{|\beta|\le M} \|\xi^\alpha \frac{\partial^\beta}{\partial \xi^\beta} \hat{f}\|_\infty$. If $f \in \mathcal{S}_{M+1}$, we easily check that, using the operators S_j of the Littlewood–Paley decomposition, the approximation $f - S_j f$ of f satisfies $f - S_j f \in \mathcal{S}_\infty$ and $\lim_{j\to-\infty} \sum_{|\alpha|\le M} \sum_{|\beta|\le M} \|\xi^\alpha \frac{\partial^\beta}{\partial \xi^\beta} (\varphi(\frac{\xi}{2^j}) \hat{f})\|_\infty = 0$, hence that $f \in E$. □

Definition 4.7: (Realization of a homogeneous Banach space of distributions)

Let E be a homogeneous shift-invariant Banach space of distributions and let N be the greatest integer so that $\mathcal{S}_N \subset E^{(*)}$. Then, the realization E_r of E is the space of distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ so that there exists a constant C so that for all $\phi \in \mathcal{S}_N$ we have $|\langle f|\phi\rangle| \le C\|\phi\|_{E^{(*)}}$.

We may now extend the notion of homogeneous Besov spaces to Besov spaces based over shift-invariant Banach spaces of distributions:

Definition 4.8: (Homogeneous Besov spaces)

Let E be a shift-invariant Banach space of distributions or a homogeneous shift-invariant Banach space of distributions, $\sigma \in \mathbb{R}$ and $1 \le q \le \infty$. Then, the homogeneous Besov space $\dot{\mathcal{B}}_E^{\sigma,q}$ is defined as the Banach space of distributions $f \in \mathcal{S}'/\mathbb{C}[X_1,\dots,X_d]$ so that for all $j \in \mathbb{Z}$ $\Delta_j f \in E$ and $(2^{j\sigma}\|\Delta_j f\|_E)_{j\in\mathbb{Z}} \in l^q(\mathbb{Z})$.

Those homogeneous Besov spaces are indeed homogeneous shift-invariant Banach spaces of distributions:

Proposition 4.6: (Duality and homogeneous Besov spaces)

Let E be a homogeneous shift-invariant Banach space of distributions, $\sigma \in \mathbb{R}$, $1 \leq q \leq \infty$, and let q' be the conjugate exponent of q. We define $\tilde{\dot{\mathcal{B}}}_{E^{(*)}}^{-\sigma,q'}$ as the closure of $\mathcal{S}_\infty$ for the norm $\|2^{-j\sigma}\|\Delta_j f\|_{E^{(*)}}\|_{l^{q'}(\mathbb{Z})}$. Then, $\tilde{\dot{\mathcal{B}}}_{E^{(*)}}^{-\sigma,q'}$ is a homogeneous shift-invariant Banach space of test functions and $\dot{\mathcal{B}}_E^{\sigma,q}$ is the dual space of $\tilde{\dot{\mathcal{B}}}_{E^{(*)}}^{-\sigma,q'}$: a distribution $f \in \mathcal{S}'$ belongs to $\dot{\mathcal{B}}_E^{\sigma,q}$ if and only if there exists a constant C so that for all $\omega \in \mathcal{S}_\infty$ we have $|\langle f|\omega\rangle| \leq C\|\omega\|_{\tilde{\dot{\mathcal{B}}}_{E^{(*)}}^{-\sigma,q'}}$.

As for the usual Besov spaces over Lebesgue spaces, we shall write $\dot{B}_E^{\sigma,q}$ for the realization of the homogeneous Besov space $\dot{\mathcal{B}}_E^{\sigma,q}$.

Chapter 5

Vector-valued integrals

We often consider statements such as $f(t,x) \in L^p((0,T),E)$, where E is a Banach space of distributions on $\mathbb{R}^d$. We do not want to get into deep considerations on measurability of vector valued functions. Therefore, we introduce here some *ad hoc* definitions for such integrals in our setting, putting a special emphasis on the spaces $L^p(L^q)$ for which we shall need a duality theorem.

1. The case of Lebesgue spaces

We consider here two measured space (X,μ) and (Y,ν), where the measures μ and ν are positive σ–finite measures. In particular, we may apply the Fubini theorem to integrate over the product space $(X \times Y, \mu \otimes \nu)$.

Definition 5.1: (Spaces $L^p(L^q)$)

For $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$, we define the space $L^p(X, L^q(Y))$ as the space of measurable functions on $X \times Y$ so that $\| \, \|f\|_{L^q(d\nu(y))} \|_{L^p(d\mu(x))} < \infty$.

Remark: For $p = \infty$, this is not the usual definition.

We have the following easy result on $L^p(X, L^q(Y))$ spaces:

Proposition 5.1:

(A) If $1 \leq p, q \leq \infty$, if f is a measurable function on $X \times Y$, then $\| \|f\|_{L^q(d\nu(y))} \|_{L^p(d\mu(x))} = 0$ if and only if f is equal to 0 almost everywhere on $X \times Y$.

(B) If $1 \leq p, q \leq \infty$, the space of measurable functions on $X \times Y$ so that $\| \|f\|_{L^q(d\nu(y))} \|_{L^p(d\mu(x))} < \infty$, quotiented by the space of functions which are equal to 0 almost everywhere, is a Banach space.

Proof: This is very easy. Indeed, (A) is a direct consequence of the Fubini theorem. To get point (B), we consider a sequence of functions f_n so that $\sum_{n \in \mathbb{N}} \|f_n\|_{L^p(L^q)} < \infty$. Now, if $X' \subset X$ and $Y' \subset Y$ have finite measures, we get that $\sum_{n \in \mathbb{N}} \|f_n\|_{L^1(X' \times Y')} < \infty$. This proves that $\sum_{n \in \mathbb{N}} f_n(x,y)$ converges for almost every $(x,y) \in X \times Y$ to a measurable function f and the convergence of $\sum f_n$ to f in $L^p(L^q)$ is then easy to check. $\square$

The main result we shall use on $L^p(L^q)$ is the following duality theorem:

Theorem 5.1: (Duality for $L^p(L^q)$ spaces)
For $1 \leq p < \infty$ and $1 \leq q < \infty$, the dual space of $L^p(X, L^q(Y))$ is the space $L^{p'}(X, (L^{q'}(Y))$ with $1/p + 1/p' = 1/q + 1/q' = 1$. More precisely, if $f \in L^p(X, L^q(Y))$ and if $g \in L^{p'}(X, (L^{q'}(Y))$ then $fg \in L^1(X \times Y)$ and to every bounded linear functional L on $L^p(X, L^q(Y))$ we may associate a unique $g \in L^{p'}(X, (L^{q'}(Y))$ so that for all $f \in L^p(X, L^q(Y))$ we have $L(f) = \int\int_{X\times Y} f\ \bar{g}\ d\mu(x)\ d\nu(y)$.

Proof: We want to identify the bounded linear functionals L on $L^p(L^q)$. If $r = \max(p, q)$, we see that for all $X' \subset X$ and $Y' \subset Y$ with finite measures, $L^r(X' \times Y') \subset L^p(X', L^q(Y'))$ (with a dense embedding) so that L is given on $L^p(X', L^q(Y'))$ by the integration against a function $g' \in L^{r'}(X' \times Y')$, due to the Riesz representation theorem. Thus, we easily conclude that there is a measurable function g on $X \times Y$ so that $L(f) = \int\int_{X\times Y} f\ \bar{g}\ d\mu(x)\ d\nu(y)$ (at least when $f \in L^r(X \times Y)$ and $\operatorname{Supp} f$ is contained in a product $X' \times Y'$ with $\mu(X') < \infty$ and $\nu(Y') < \infty$). The point is now to prove that g belongs to $L^{p'}(L^{q'})$. We fix $X' \subset X$ and $Y' \subset Y$ with finite measures and $N \geq 0$. If $p, q > 1$, we define $E = \{(x, y)/x \in X', y \in Y', |g(x, y)| \leq N\}$, $\gamma = 1_E(x, y)|g|$ and $f = \operatorname{sgn}(g(x, y))\ \gamma(x, y)^{\frac{1}{q-1}}\ (\int_Y \gamma(x, z)^{\frac{q}{q-1}}\ d\nu(z))^{\frac{p'}{q'}-1}$. Then we have $|L(f)| = \int_X\ (\int_Y \gamma(x, z)^{q'}\ d\nu(z))^{\frac{p'}{q'}}\ d\mu(x) \leq \|L\|_{op}\|f\|_{L^p(L^q)} = \|L\|_{op}(\int_X\ (\int_Y \gamma(x, z)^{q'}\ d\nu(z))^{\frac{p'}{q'}}\ d\mu(x))^{1/p}$. Thus, we have proved the inequality $(\int_X\ (\int_Y 1_E(x, y)\ |g(x, y)|^{q'}\ d\nu(y))^{\frac{p'}{q'}}\ d\mu(x))^{1/p'} \leq \|L\|_{op}$. Letting N go to ∞, X' go to X and Y' go to Y gives the desired estimate on g.

The case $p = 1, q > 1$ may be dealt with in the same way. We define E and γ as we did in the previous case; then, we define for $A \geq 0$ $F = \{x\ /\ \int_Y \gamma(x, y)^{q'}\ d\nu(y) > A^{q'}\}$ and $f = \operatorname{sgn}(g(x, y))\ \gamma(x, y)^{\frac{1}{q-1}}\ 1_F(x)$. We find that

$$\begin{aligned} A^{q'}\mu(F) &\leq |L(f)| = \int_F\ (\int_Y \gamma(x, z)^{q'}\ d\nu(z))\ d\mu(x) \\ &\leq \|L\|_{op}\|f\|_{L^1(L^q)} = \|L\|_{op}(\int_F(\int_Y \gamma(x, z)^{q'}\ d\nu(z))^{\frac{1}{q}}\ d\mu(x)) \\ &\leq \|L\|_{op}\ (\mu(F))^{\frac{1}{q'}}(\int_F\ (\int_Y \gamma(x, z)^{q'}\ d\nu(z))\ d\mu(x))^{\frac{1}{q}} \end{aligned}$$

This gives $\mu(F) = 0$ for $A > \|L\|_{op}$.

The case of $q = 1$ is harder (except for $p = q = 1$ where we have the usual duality between $L^1(X \times Y)$ and $L^\infty(X \times Y)$). We are going to show that for almost every $x \in X$, the function $g(x, y)$ is essentially bounded on Y. For this, we write $X = \cup^{\uparrow}_{n\in\mathbb{N}} X_n$ with $\mu(X_n) < \infty$. We fix A a positive real number, and we define the set E (depending on n and A) as $E = \{(x, y)/x \in X_n, y \in Y, |g(x, y)| < A\}$ and the function γ as $\gamma = 1_E(x, y)|g|$. Moreover, we write $Y = \cup^{\uparrow}_{k\in\mathbb{N}} Y_k$, we fix $\epsilon > 0$ and define for $x \in X$ the set $E_x = \{y \in Y_k\ /\ \gamma(x, y) > (1 - \epsilon)\|\gamma(x, .)\|_{L^\infty(Y)}\}$. We then define $h =$

$\text{sgn} g \|\gamma(x,.)\|^{p'-1}_{L^\infty(Y_k)} \frac{1_{E_x}}{\mu(E_x)}$. We have $\|h\|_{L^p(L^1)} = \|\gamma\|^{p'/p}_{L^{p'}(L^\infty(Y_k))}$ whereas $L(h) = \int\int \gamma(x,y)\|\gamma(x,.)\|^{p'-1}_{L^\infty(Y_k)} \frac{1_{E_x}(y)}{\mu(E_x)}\ d\nu(y)\ d\mu(x)$ which gives that $|L(h)| \geq (1-\epsilon)\|\gamma\|^{p'}_{L^{p'}(L^\infty(Y_k))}$ Thus, we get $\|\gamma\|_{L^{p'}(L^\infty(Y_k))} \leq \frac{1}{1-\epsilon}\|L\|_{op}$. We let first ϵ go to 0, then we let Y_k go to Y (and check that for a measurable function f on Y we have $\|f\|_{L^\infty(Y)} = \sup_{k\in\mathbb{N}} \|f\|_{L^\infty(Y_k)})$; the monotone convergence theorem gives us that $\|\gamma\|_{L^{p'}(L^\infty(Y))} \leq \|L\|_{op}$. We may let X_n go to X and thus get that $\|g_A\|_{L^{p'}(L^\infty(Y))} \leq \|L\|_{op}$ where $g_A(x) = g(x,y)$ if $|g(x,y)| < A$ and $g_A = 0$ otherwise. Since g is finite almost everywhere on $X\times Y$, we know that $g(x,.)$ is finite almost everywhere on Y for almost every $x \in X$, so that almost everywhere on X we have $\lim_{A\to\infty} \|g_A\|_{L^\infty(Y)} = \|g\|_{L^\infty(Y)}$, hence we get by the monotone convergence theorem that $\lim_{A\to\infty} \|g_A\|_{L^{p'}(L^\infty)} = \|g\|_{L^{p'}(L^\infty)} \leq \|L\|_{op}$. □

We complete this section with a density result:

Proposition 5.2:

If X and Y are locally compact σ-compact metric spaces and if μ and ν are regular Borel measures on X and Y, then $\mathcal{C}_{comp}(X\times Y)$ is dense in $L^p(X;L^q(Y))$ for $1 \leq p < \infty$ and $1 \leq q < \infty$.

Proof: It is obvious (by the monotone convergence theorem) that, for $r = \max(p,q)$, $L^r_{comp}(X\times Y)$ is dense in $L^p(L^q)$. The proposition is then obvious, since $\mathcal{C}_{comp}(X\times Y)$ is dense in $L^r_{comp}(X\times Y)$. □

2. Spaces $L^p(E)$

A nice space of test functions on $(0,T)\times\mathbb{R}^d$ is the space $\mathcal{T}((0,T)\times\mathbb{R}^d)$ of the smooth functions, which are compactly supported in time and have rapid decay in space. We write $(0,T) = \cup^{\uparrow}_{n\in\mathbb{N}}[a_n,b_n]$; then $\mathcal{T}$ is the inductive limit of the Frechet spaces $\mathcal{T}_n = \{f \in \mathcal{T}/\text{Supp} f \subset [a_n,b_n]\times\mathbb{R}^d\}$, where the space $\mathcal{T}_n$ is equipped with the semi-norms

$$\sup_{a_n<t<b_n} \sup_{x\in\mathbb{R}^d} |x^\alpha \frac{\partial^\beta}{\partial x^\beta}\frac{\partial^p}{\partial t^p} f(t,x)|,\ \alpha\in\mathbb{N}^d,\ \beta\in\mathbb{N}^d,\ p\in\mathbb{N}.$$

We are thus able to take a partial Fourier transform with respect to the spatial variable:

$$\mathcal{F}f(t,\xi) = \int_{\mathbb{R}^d} f(t,x)e^{-ix.\xi}\ dx.$$

The space $\mathcal{T}'$ will then be the space of distributions ω on $(0,T)\times\mathbb{R}^d$ so that for all $[a,b]\subset(0,T)$ there exist $C\geq 0$ and $N\in\mathbb{N}$ such that for all $\varphi\in\mathcal{D}((0,T)\times\mathbb{R}^d)$, with $\text{Supp}\varphi\subset[a,b]\times\mathbb{R}^d$, we have the inequality $|\langle\omega|\varphi\rangle| \leq C\sum_{|\alpha|\leq N}\sum_{|\beta|\leq N}\sum_{p\leq N}\|x^\alpha\frac{\partial^\beta}{\partial x^\beta}\frac{\partial^p}{\partial t^p}f(t,x)\|_\infty$. The Fourier transformation may then be defined on $\mathcal{T}'$ as usual by duality, through the formula

$\langle \mathcal{F}\omega(t,x)|\varphi(t,x)\rangle = \langle \omega(t,x)|\mathcal{F}\varphi(t,-x)\rangle$. (We use the Hermitian duality bracket $\langle T|\varphi\rangle = \int T(t,x)\ \bar{\varphi}(t,x)\ dt\ dx$.)

Now, we write Δ for the spatial Laplacian operator $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_j^2}$. Then for all $\sigma \in \mathbb{C}$ the operator $(Id-\Delta)^\sigma$ operates on $\mathcal{S}(\mathbb{R}^d)$, hence on $\mathcal{T}((0,T)\times\mathbb{R}^d)$ and on $\mathcal{T}'((0,T)\times\mathbb{R}^d)$.

We may now define the spaces $L^p(E)$ in use throughout the book:

Definition 5.2: (Space $L^p(E)$)

Let E be a shift-invariant Banach space of distributions and let $E \subset B_\infty^{s,\infty}$ for some $s \in \mathbb{R}$. Let $M \in \mathbb{N}$, $M > s/2$. Then for $1 \le p \le \infty$ and $T \in (0,\infty]$, we define $L^p((0,T),E)$ as the subspace of $\mathcal{T}'((0,T)\times\mathbb{R}^d)$ of distributions f such that:
a) $(Id-\Delta)^{-M} f \in L^1_{loc}((0,T)\times\mathbb{R}^d)$;
b) for almost all $t \in (0,T)$, $f(t,.) \in E$;
c) $\| \|f(t,x)\|_E \|_{L^p((0,T))} < \infty$.

Let us make some comments on this definition. $(Id-\Delta)^{-M}$ operates on $\mathcal{T}'$, hence (a) is meaningful. Then, using the Fubini theorem, we get that for almost all $t \in (0,T)$ $(Id-\Delta)^{-M} f(t,.)$ belongs to $L^1_{loc}(\mathbb{R}^d)$ and that for all test function $\varphi \in \mathcal{D}((0,T)\times\mathbb{R}^d)$ we have the equality $\langle f|\varphi\rangle_{(0,T)\times\mathbb{R}^d} = \int_0^T \langle (Id-\Delta)^{-M} f(t,.)|(Id-\Delta)^M \varphi(t,.)\rangle_{\mathbb{R}^d}\, dt$ (where $\langle\ |\ \rangle_\Omega$ is the duality bracket between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$).

We thus have given sense to $f(t,.)$ in $\mathcal{D}'(\mathbb{R}^d)$ for almost all $t \in (0,T)$. Thus, assertion b) is meaningful. In order to give sense to assertion c), we prove that $t \mapsto \|f(t,.)\|_E$ is Lebesgue measurable:

Lemma 5.1:

Let E be a shift-invariant Banach space of distributions. Then, under the hypotheses a) and b) of Definition 5.2, we have:
i) For all $\phi \in \mathcal{D}(\mathbb{R}^d)$, the map $t \mapsto \langle f(t,.)|\phi\rangle_{\mathbb{R}^r}$ is Lebesgue measurable;
ii) For all $e \in E$, the map $t \mapsto \|f(t,.)-e\|_E$ is Lebesgue measurable.

Proof: We know that for some M, we have $f(t,x) = (Id-\Delta)^M g(t,x)$ where $g \in L^p((0,T),L^\infty)$. Thus, $f(t,.)$ is defined for almost every t as $f(t,.) = (Id-\Delta)^M g(t,.)$. If we want to compute $\|f(t)-e\|_E$, we write $\|f(t)-e\|_E = \sup_{\phi\in H} |\langle f(t)-e|\phi\rangle|$ where H is a countable set of test functions ($H \subset \mathcal{S}(\mathbb{R}^d)$) which is dense in the unit ball of the closure of $\mathcal{S}$ in $E^{(*)}$ (the existence of H is easy to check: for instance, the linear span of the Meyer wavelet basis $\big((\varphi(x-k))_{k\in\mathbb{Z}^d}, (2^{jd/2}\psi_\epsilon(2^j x-k))_{1\le\epsilon\le 2^d-1, j\in\mathbb{N}, k\in\mathbb{Z}^d}\big)$ [MEY 92] is dense in $E^{(*)}$). Since $\langle f(t)-e|\phi\rangle = \langle g(t)|(Id-\Delta)^M\phi\rangle - \langle e|\phi\rangle$, we see (using the Fubini theorem) that $t \mapsto \langle f(t)-e|\phi\rangle$ is measurable, hence $t \mapsto \|f(t)-e\|_E$ is measurable. $\square$

We now give a characterization of our spaces in terms of Bochner vector integration:

Theorem 5.2: (Bochner integrability)

Let E be a shift-invariant Banach space of distributions. Then:

(A) For $1 \leq p < \infty$, $\mathcal{D}((0,T) \times \mathbb{R}^d)$ is dense in $L^p((0,T), E^{(0)})$ and $L^p((0,T), E^{(0)})$ is the space of strongly measurable f with values $f(t,.)$ in $E^{(0)}$ such that $\int_0^T \|f\|_E^p \, dt < \infty$.

(B) If $1 \leq p \leq \infty$, if $1/p + 1/q = 1$, then for all $f \in L^p((0,T), E)$ and $g \in L^q(0,T)$, we have $\int_0^T g(t) f(t,x) \, dt \in E$. If f belongs more precisely to $L^p((0,T), E^{(0)})$, then $\int_0^T g(t) f(t,x) \, dt \in E^{(0)}$.

To prove (A), we use the following characterization of Bochner integrability:

Lemma 5.2:

A function f defined on $(0,T)$ with values in a separable Banach space E is strongly measurable if and only if for all $e \in E$ the function $\|f(t) - e\|_E$ is measurable.

Proof of the lemma: This lemma is easy to prove. If f is strongly measurable, i.e., if there is a sequence of simple functions $f_n = \sum_{1 \leq k \leq N_n} 1_{X_{k,n}}(t) \, e_{k,n}$ which converges almost everywhere to f, then it is obvious that the function $\|f(t) - e\|_E$ is measurable.

Conversely, since E is separable, for all $n \in \mathbb{N}^*$, there is a sequence $(x_{k,n})_{k \in \mathbb{N}}$ so that $E = \cup_{k \in \mathbb{N}} B(x_{k,n}, 1/2^n)$. Then, we define the set $Y_{k,n} = \{t \in (0,T) \ / \ f(t) \in B(x_{k,n}, 1/2^n) \text{ and } \forall p < k \ f(t) \notin B(x_{p,n}, 1/2^n)\}$. We have $(0,T) = \sum_{k \in \mathbb{N}} Y_{k,n}$. Then, we choose $K(n)$ so that $\sum_{k \leq K(n)} \ |Y_{k,n} \cap (0,n)| \geq \min(T,n) - \frac{1}{2^n}$. By construction, the sequence $\sum_{k \leq K(n)} 1_{Y_{k,n}}(t) \, x_{k,n}$ converges almost everywhere to f. Thus, Lemma 5.2 is proved. □

Proof of Theorem 5.2: We now prove the theorem. If $f \in L^p((0,T), E^{(0)})$ and $p < \infty$, it is easy to see that we may apply lemma 5.2 ; $E^{(0)}$ is separable since $\mathcal{S}$ is dense in $E^{(0)}$ and since the imbedding of $\mathcal{S}$ into $E^{(0)}$ is continuous. We now apply Lemmas 5.1 and 5.2 to conclude that f is strongly measurable. Now, we know that we may approximate f by a simple function g_n so that $Y_n = \text{Supp}(g_n) \subset (0, \min(T,n))$, $|Y_n| \geq \min(T,n) - \frac{1}{2^n}$ and $\sup_{t \in Y_n} \|f(t) - g_n(t)\|_E < \frac{1}{2^n}$. We find that $\|f - g_n\|_{L^p((0,T),E} \leq \frac{n}{2^n} + (\int_{t \notin Y_n} \|f(t)\|_E^p \, dt)^{1/p}$. Since p is finite, we find that g_n converges to f in $L^p((0,T), E^{(0)})$. Now, when g is a simple function, it is obvious that g can be approximated in $L^p((0,T), E^{(0)})$ by a sequence of test functions. Thus, (A) is proved.

(B) is quite obvious. If $f \in L^p((0,T), E)$ and $g \in L^q((0,T), E)$, then we write $f = (Id - \Delta)^M h$ and we see that $\int_0^T f(t,x) g(t) \, dt$ is defined in $\mathcal{S}'$ as $(Id - \Delta)^M \int_0^T g(t) h(t,x) \, dt$; moreover, we have $\langle \int_0^T f(t,x) g(t) \, dt | \phi(x) \rangle = \int_0^T \langle f(t,x) | \phi(x) \rangle \, g(t) \, dt$ (by the Fubini theorem). Thus, $f(t,x)g(t)$ belongs to $L^1((0,T), E)$ and $|\langle \int_0^T f(t,x) g(t) \, dt | \phi(x) \rangle| \leq \|f(t,x)g(t)\|_{L^1((0,T),E)} \ \|\phi\|_{E^{(*)}}$.

This gives that $\int_0^T f(t,x)g(t)\,dt$ belongs to E. If $f \in L^p((0,T),E^{(0)}$, we approximate $f(t,x)g(t)$ in $L^1((0,T),E^{(0)})$ by a sequence of test functions $\gamma_n \in \mathcal{D}((0,T)\times \mathbb{R}^d)$; we then find that $\int_0^T \gamma_n\,dt$ is a Cauchy sequence in $E^{(0)}$ which converges in $\mathcal{S}'$ to $\int_0^T f(t,x)g(t)\,dt$. $\square$

3. Heat kernel and Besov spaces

Since throughout the book we use the heat kernel operating on shift-invariant Banach spaces of distributions, it will be very useful to characterize the action of the heat kernel on the Besov spaces associated with such spaces.

Theorem 5.3: (Heat kernel and Besov spaces)

Let E be a shift-invariant Banach space of distributions. For $\sigma \in \mathbb{R}$, $1 \le q \le \infty$, $t_0 > 0$, $\alpha \ge 0$ so that $\alpha > \sigma$ and $f \in \mathcal{S}'(\mathbb{R}^d)$, the following assertions are equivalent :
(A) $f \in B_E^{\sigma,q}$
(B) for all $t > 0$, $e^{t\Delta} f \in E$ and $(t^{-\sigma/2}(\sqrt{-t\Delta})^\alpha e^{t\Delta} f)_{0<t<t_0} \in L^q((0,t_0),\frac{dt}{t},E)$.

Moreover, the norms $\|e^{t_0\Delta} f\|_E + \|t^{-\sigma/2}(\sqrt{-t\Delta})^\alpha e^{t\Delta} f)\|_{L^q((0,t_0),\frac{dt}{t},E)}$ and $\|f\|_{B_E^{s,q}}$ are equivalent.

Proof: We first notice that $(\sqrt{-t\Delta})^\alpha e^{t\Delta} = (\sqrt{-t\Delta})^\alpha e^{t/2\ \Delta} e^{t/2\ \Delta}$; the operator $(\sqrt{-t\Delta})^\alpha e^{t/2\ \Delta}$ is a convolution operator with a kernel $\frac{1}{t^{d/2}} k_\alpha(\frac{x}{\sqrt{t}})$ where $k_\alpha \in L^1$, thus $\|(\sqrt{-t\Delta})^\alpha e^{t\Delta} f\|_E \le C_\alpha \|e^{t/2\ \Delta} f\|_E$.

$(A) \Rightarrow (B)$: We may assume $t_0 = 1$. We write $f = S_0 f + \sum_{j\ge 0} \Delta_j f$ with $S_0 f \in E$ and $\|\Delta_j f\|_E = 2^{-j\sigma}\epsilon_j$ with $(\epsilon_j)_{j\in\mathbb{N}} \in l^q$. We estimate the norm $\|t^{-\sigma/2}(\sqrt{-t\Delta})^\alpha e^{t\Delta} f\|_E$ by

$$\|t^{-\sigma/2}(\sqrt{-t\Delta})^\alpha e^{t\Delta} S_0 f\|_E \le t^{(\alpha-\sigma)/2} \|(\sqrt{-\Delta})^\alpha \tilde{S}_0 S_0 f\|_E \le C_\alpha t^{(\alpha-\sigma)/2} \|S_0 f\|_E$$

since $(\sqrt{-\Delta})^\alpha \tilde{S}_0$ is a convolution operator with an integrable kernel. Similarly, writing $\Delta_j f = \tilde{\Delta}_j \Delta_j f$ and using the integrability of the kernel of the convolution operator $(\sqrt{-\Delta})^\alpha \tilde{\Delta}_0$, we get

$$\|t^{-\sigma/2}(\sqrt{-t\Delta})^\alpha e^{t\Delta} \Delta_j f\|_E \le C_\alpha t^{(\alpha-\sigma)/2} 2^{j\alpha} \|\Delta_j f\|_E.$$

Then, we write $(\sqrt{-t\Delta})^\alpha e^{t\Delta}\Delta_j f = t^{(\alpha-N)/2}(\sqrt{-t\Delta})^N e^{t\Delta}(\sqrt{-\Delta})^{\alpha-N}\tilde{\Delta}_j \Delta_j f$ and, owing to the integrability of the kernel of $(\sqrt{-\Delta})^{\alpha-N}\tilde{\Delta}_0$, we get that, for $N \ge 0$,

$$\|t^{-\sigma/2}(\sqrt{-t\Delta})^\alpha e^{t\Delta}\Delta_j f\|_E \le C_{\alpha,N} t^{(\alpha-\sigma-N)/2} 2^{j(\alpha-N)} \|\Delta_j f\|_E.$$

For $\alpha = 0$ and $N > 0$, we thus have proved that $e^{t\Delta} f \in E$ for all $t > 0$. Now, if $t \le 1$, we choose j_0 so that $1/4 < 4^{j_0} t \le 1$ and we choose $N > \alpha - \sigma$; we obtain that $\|t^{-\sigma/2}(\sqrt{-t\Delta})^\alpha e^{t\Delta} f\|_E$ is bounded by η_{j_0} where

$$\eta_{j_0} = C_\alpha(2^{-j_0(\alpha-\sigma)}\|S_0 f\|_E + \sum_{j\le j_0} 2^{(j-j_0)(\alpha-\sigma)}\epsilon_j) + C_{\alpha,N}\sum_{j>j_0} 2^{(j-j_0)(\alpha-\sigma-N)}\epsilon_j.$$

We have $\|t^{-\sigma/2}(\sqrt{-t\Delta})^\alpha e^{t\Delta} f)\|_{L^q((0,1),\frac{dt}{t},E)} \leq (\ln 4)^{1/q} \|(\eta_j)_{j\in\mathbb{N}}\|_{l^q(\mathbb{N})}$ and we may conclude since the sequence $(\lambda_j = 2^{j(\alpha-\sigma)}$ if $j \leq 0, = 2^{j(\alpha-\sigma-N)}$ if $j > 0)$ belongs to $l^1(\mathbb{Z})$, hence $(\epsilon_j) * (\lambda_j) \in l^q(\mathbb{N})$.

$(B) \Rightarrow (A)$: We may assume $t_0 = 1$. We get that $S_0 f \in E$ by writing $S_0 f = e^{-\Delta} S_0 e^{\Delta} f$ and using the integrability of the kernel of the convolution operator $e^{-\Delta} S_0$. Similarly, we write $\Delta_j f = e^{-t\Delta}(\sqrt{-t\Delta})^{-\alpha}\Delta_j(\sqrt{-t\Delta})^\alpha e^{t\Delta} f$. For $1/4 < 4^j t \leq 1$, the convolution operator $e^{-t\Delta}(\sqrt{-t\Delta})^{-\alpha}\Delta_j$ has an integrable kernel $k_{\alpha,j,t}$ with $\|k_{\alpha,j,t}\|_1 \leq C_\alpha$, so that we finally obtain $2^{j\sigma}\|\Delta_j f\|_E \leq \max(1, 2^{-\sigma}) C_\alpha t^{-\sigma/2} \|(\sqrt{-t\Delta})^\alpha e^{t\Delta} f\|_E$ for $1/4 < 4^j t \leq 1$. □

This theorem may be easily applied to the case of homogeneous Besov spaces. In order to avoid any problem of definition for $(\sqrt{-t\Delta})^\alpha e^{t\Delta} f$, we shall consider only the case $\alpha = 0$ and $\sigma < 0$:

Theorem 5.4: (Heat kernel and homogeneous Besov spaces)

Let E be a shift-invariant Banach space of distributions. Let $\sigma \in \mathbb{R}$, $\sigma < 0$, $1 \leq q \leq \infty$, and $f \in \mathcal{S}'(\mathbb{R}^d)$. Then:

(A) If, for all $t > 0$, $e^{t\Delta} f \in E$ and if $t^{-\sigma/2} e^{t\Delta} f \in L^q((0,\infty), \frac{dt}{t}, E)$, then f belongs to $\dot{\mathcal{B}}_E^{\sigma,q} \cap \mathcal{S}_0'(\mathbb{R}^d)$;

(B) Conversely, if f belongs to $\dot{\mathcal{B}}_E^{\sigma,q} \cap \mathcal{S}_0'(\mathbb{R}^d)$, then, for all $t > 0$, $e^{t\Delta} f \in E$ and $t^{-\sigma/2} e^{t\Delta} f \in L^q((0,\infty), \frac{dt}{t}, E)$.

Moreover, in that case, the norms $\|t^{-\sigma/2} e^{t\Delta} f\|_{L^q((0,\infty),\frac{dt}{t},E)}$ and $\|f\|_{B_E^{\sigma,q}}$ are equivalent.

Proof: Let us first remark that S_0 maps E to L^∞, hence maps $\dot{\mathcal{B}}_E^{\sigma,q}$ to $\dot{\mathcal{B}}_\infty^{\sigma,\infty}$. Thus, we may define the realization $\dot{B}_E^{\sigma,q} \subset \mathcal{S}'(\mathbb{R}^d)$ of $\dot{\mathcal{B}}_E^{\sigma,q}$ by the isomorphism $f \mapsto \sum_{j\in\mathbb{Z}} \Delta_j f$.

The proof of Theorem 4 is exactly the same as for non homogeneous spaces, since we have $e^{t\Delta} f = \sum_{j\in\mathbb{Z}} e^{t\Delta}\Delta_j f$ for $f \in \mathcal{S}_0'$. In particular, if $t^{-\sigma/2} e^{t\Delta} f \in L^q((0,\infty), \frac{dt}{t}, E)$, then f belongs to $\dot{\mathcal{B}}_E^{\sigma,q}$; thus, we may write $f = \sum_{j\in\mathbb{Z}} \Delta_j f + P = f_0 + P$ with $P \in \mathbb{C}[X_1, \ldots, X_d]$; since $f_0 \in \dot{\mathcal{B}}_E^{\sigma,q} \cap \mathcal{S}_0'(\mathbb{R}^d)$, we find that $t^{-\sigma/2} e^{t\Delta} f_0 \in L^q((0,\infty), \frac{dt}{t}, E)$, hence $t^{-\sigma/2} e^{t\Delta} P \in L^q((0,\infty), \frac{dt}{t}, E)$; but $P = e^{-t\Delta} S_0\, e^{t\Delta} P$, hence $P \in E$; now, this would give that, for any $\omega \in \mathcal{D}$, we should have $\sup_{x_0\in\mathbb{R}^d} |\int \omega(x - x_0) P(x)\, dx| < \infty$ and this implies that P is a constant $P = p \in \mathbb{C}$; but in that case we have $e^{t\Delta} P = P$ and we find that $t^{-\sigma/2} P \in L^q((0,\infty), \frac{dt}{t}, E)$, hence that $P = 0$ and f belongs to $\mathcal{S}_0'$. □

We give an example of application with the Sobolev embeddings:

Proposition 5.3: (Heat kernel and Sobolev embeddings)

Let $\alpha \in \mathbb{R}$, $0 < \alpha < d$. Then :

(A) If $1 < p < \infty$, $1 \leq q \leq \infty$ and $\alpha < d/p$ and if $\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{d}$, then the operator $\frac{1}{(\sqrt{-\Delta})^\alpha}$ is continuous from $L^{p,q}$ to $\dot{\mathcal{B}}_{L^{r,1}}^{0,q} \cap \mathcal{S}_0'$.

In particular, there exists a constant $C_{p,\alpha}$ *so that, for all* $f \in L^{p,1}(\mathbb{R}^d)$*, we have* $\int_0^\infty \|e^{t\Delta}\Delta \frac{1}{(\sqrt{-\Delta})^\alpha} f\|_{L^{r,1}}\, dt \leq C_{p,\alpha}\, \|f\|_{L^{p,1}}$.

(B) If $\alpha = d/p$*, then the operator* $\frac{1}{(\sqrt{-\Delta})^\alpha}$ *is continuous from* $L^{p,q}$ *to* $\dot{B}^{0,q}_\infty$.

In particular, there exists a constant $C_{p,\alpha}$ *so that, for all* $f \in L^{p,1}(\mathbb{R}^d)$*, we have the estimate* $\int_0^\infty \|e^{t\Delta}\Delta \frac{1}{(\sqrt{-\Delta})^\alpha} f\|_\infty\, dt \leq C_{p,\alpha}\, \|f\|_{L^{p,1}}$

Proof: We use the Sobolev embeddings for $1/p_1 = 1/p - \epsilon$ and $\alpha_1 = \alpha - d\epsilon$ and $1/p_2 = 1/p + \epsilon$ and $\alpha_2 = \alpha + d\epsilon$ and for $1/r = 1/p - \alpha/d = 1/p_i - \alpha_i/d$ to get that $\frac{1}{(\sqrt{-\Delta})^\alpha}$ maps $L^{p_1,1}$ to $\dot{H}^{d\epsilon}_{L^{r,1}}$ and $L^{p_2,1}$ to $\dot{H}^{-d\epsilon}_{L^{r,1}}$, hence by interpolation $L^{p,q}$ to $\dot{\mathcal{B}}^{0,q}_{L^{r,1}}$. The same proof works to get that $\frac{1}{(\sqrt{-\Delta})^{d/p}}$ maps $L^{p,q}$ to $\dot{\mathcal{B}}^{0,q}_\infty$.

Moreover, if $\alpha < d/p$, we know that $\frac{1}{(\sqrt{-\Delta})^\alpha}$ maps $L^{p,q}$ to $L^{r,q} \subset \mathcal{S}_0'$. If $\alpha = d/p$ and $q = 1$, we already know that $\frac{1}{(\sqrt{-\Delta})^{d/p}}$ maps $L^{p,1}$ to L^∞ ; we easily check that test functions are dense in $L^{p,1}$ and that for $f \in \mathcal{S}$ we have $\frac{1}{(\sqrt{-\Delta})^{d/p}} f \in \mathcal{C}_0$, $\frac{1}{(\sqrt{-\Delta})^{d/p}}$ maps more precisely $L^{p,q}$ to $\mathcal{C}_0 \subset \mathcal{S}_0'$. For $q > 1$, we shall see in Chapter 6 that $\frac{1}{(\sqrt{-\Delta})^{d/p}}$ maps the Hardy space $\mathcal{H}^1$ to $L^{p/(p-1),1}$, hence maps $L^{p,\infty}$ to $BMO \subset \dot{B}^{0,\infty}_\infty$; this is enough to get that $\frac{1}{(\sqrt{-\Delta})^{d/p}}$ maps $L^{p,q}$ to the realization $\dot{B}^{0,q}_\infty$ of $\dot{\mathcal{B}}^{0,q}_\infty$.

The final estimates on the heat kernel are then provided by the equality $\Delta e^{t\Delta} f = \sum_{j\in\mathbb{Z}} \Delta_j \Delta e^{t\Delta} f$ for $f \in \mathcal{S}_0'$. $\square$

Chapter 6

Complex interpolation, Hardy space, and Calderón–Zygmund operators

1. The Marcinkiewicz interpolation theorem and the Hardy–Littlewood maximal function

In this section, we begin by recalling a basic theorem of modern analysis, namely, the Marcinkiewicz interpolation theorem stated without proof by Marcinkiewicz in 1939 [MAR 39] and proved by Zygmund [ZYG 56] in 1956.

Definition 6.1: (Weak L^p space)

For $1 \leq p < \infty$, the space $L^{p,*}$ (which is called the weak L^p space) is defined by

$$f \in L^{p,*} \Leftrightarrow \exists C \geq 0 \ \forall \lambda > 0 \ \mu(\{x \ / \ |f(x)| > \lambda\}) \leq C\lambda^{-p}$$

A map T defined from $L^p(U, d\mu)$ to $L^{p,*}(V, d\nu)$ is a sublinear operator if it satisfies $|T(\lambda f + \mu g)| \leq |\lambda||T(f)| + |\mu||T(g)|$. A sublinear operator T is bounded from $L^p(U, d\mu)$ to $L^{p,*}(V, d\nu)$ if there exists a constant C so that for all $\lambda > 0$ and all $f \in L^p(U, d\mu)$ we have $\nu(\{x \ / \ |Tf(x)| > \lambda\}) \leq C\|f\|_p^p \lambda^{-p}$. (If $p = \infty$, we shall say that T is bounded from $L^p(U, d\mu)$ to $L^{p,*}(V, d\nu)$ if T is bounded from L^∞ to L^∞).

Theorem 6.1: (The Marcinkiewicz interpolation theorem)

If $1 \leq p_0 < p_1 \leq \infty$ and if T is a bounded sublinear operator from $L^{p_i}(U, d\mu)$ to $L^{p_i,*}(V, d\nu)$, then T is bounded from $L^p(U, d\mu)$ to $L^p(V, d\nu)$ for $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, 0 < \theta < 1$.

Proof: We just write $\|Tf\|_p^p \approx \sum_{j \in \mathbb{Z}} 2^{jp} \nu(\{x / |Tf| > 2^j\})$ and decompose f into $f_j + g_j$ where $f_j = f \ 1_{\{x/|f| > 2^j\}}$.

If $p_1 = \infty$, we choose $k \in \mathbb{Z}$ so that $2^k \|T\|_{op(L^\infty, L^\infty)} \leq 1/2$ and we write

$$\nu\{x/|Tf| > 2^j\} \leq \nu\{x/|Tf_{j+k}| > 2^{j-1}\} \leq \|T\|_{op(L^{p_0}, L^{p_0,*})}^{p_0} \|f_{j+k}\|_{p_0}^{p_0} 2^{(1-j)p_0}$$

and thus we get for $M_0 = \|T\|_{op(L^{p_0}, L^{p_0,*})}$ and $M_1 = \|T\|_{op(L^\infty, L^\infty)}$:

$$\|Tf\|_p^p \leq M_0^{p_0} \int_U |f(x)|^{p_0} \sum_{j \in \mathbb{Z}, 2^{j+k} < |f(x)|} 2^{jp} 2^{(1-j)p_0} \ d\mu(x) \leq C M_0^{p_0} M_1^{p-p_0} \|f\|_p^p$$

If $p_1 < \infty$, we write $C_i = \|T\|_{op(L^{p_i}, L^{p_i,*})}$ and we choose $k \in \mathbb{Z}$ so that $2^{k(p_0-p)} C_0^{p_0} \approx 2^{k(p_1-p)} C_1^{p_1} \approx C_0^{\frac{(p_1-p)p_0}{p_1-p_0}} C_1^{\frac{(p-p_0)p_1}{p_1-p_0}}$. We write : $\nu\{x / |Tf| > 2^j\} \leq \nu\{x/|Tf_{j+k}| > 2^{j-1}\} + \nu\{x/|Tg_{j+k}| > 2^{j-1}\}$ and get

$$\nu\{x/|Tf| > 2^j\} \leq C_0^{p_0} \|f_{j+k}\|_{p_0}^{p_0} 2^{(1-j)p_0} + C_1^{p_1} \|g_{j+k}\|_{p_1}^{p_1} 2^{(1-j)p_1}$$

which gives

$$\|Tf\|_p^p \leq \left\{ \begin{array}{ll} & C_0^{p_0} \int_U |f(x)|^{p_0} \sum_{j \in \mathbb{Z}, 2^{j+k} < |f(x)|} 2^{jp} 2^{(1-j)p_0} \ d\mu(x) \\ + & \\ & C_1^{p_1} \int_U |f(x)|^{p_1} \sum_{j \in \mathbb{Z}, 2^{j+k} \geq |f(x)|} 2^{jp} 2^{(1-j)p_1} \ d\mu(x) \end{array} \right.$$

and thus

$$\|Tf\|_p^p \leq C(2^{k(p_0-p)} C_0^{p_0} + 2^{k(p_1-p)} C_1^{p_1}) \|f\|_p^p$$

□

Remark: The Marcinkiewicz theorem remains valid for an operator on vector valued functions: for instance, if $X = L^{q_0}(E_0)$ and $Y = L^{q_1}(E_1)$, if T satisfies $\|T(\lambda f)\|_Y = |\lambda| \|T(f)\|_Y$, $\|T(f+g)\|_Y \leq C\ (\|T(f)\|_Y + \|T(g)\|_Y)$ and $\nu(\{v \in V\ /\ \|T(f)(v)\|_Y > \lambda\}) \leq C_i^{p_i} \|f\|_{L^{p_i}(U,\mu;X)}^{p_i}\ \lambda^{-p_i}$, then T is bounded from $L^p(U,\mu;X)$ to $L^p(V,\nu;Y)$ for $p_0 < p < p_1$.

Now, we prove two classical covering lemmas: the lemmas of Vitali and Whitney. We consider a separable quasimetric space (X,d), i.e., X is a separable metrizable space, the function d is continuous on $X \times X$ and satisfies:

i) $d(x,y) \leq 0$ and $d(x,y) = 0 \Leftrightarrow x = y$

ii) for all $x \in X$ the family $B(x,\epsilon) = \{y \in X\ /\ d(x,y) < \epsilon\}$ is a basis of neighbourhoods of x

iii) there exists a constant K such that, for all $x,y,z \in X$, $d(x,y) \leq K(d(x,z) + d(y,z))$.

Proposition 6.1: (The Vitali covering lemma)

Let $E \subset X$ be decomposed as a union of balls $E = \cup_{\alpha \in \mathcal{A}} B(x_\alpha, r_\alpha)$, where $(B(x_\alpha, r_\alpha))_{\alpha \in \mathcal{A}}$ is a family of balls so that $\sup_\alpha r_\alpha < \infty$. Then there exists a (countable) subfamily of balls $(B(x_\alpha, r_\alpha))_{\alpha \in \mathcal{B}}$ ($\mathcal{B} \subset \mathcal{A}$) so that $\alpha \neq \beta \Rightarrow B(x_\alpha, r_\alpha) \cap B(x_\beta, r_\beta) = \emptyset$ and so that $E \subset \cup_{\alpha \in \mathcal{B}} B(x_\alpha, 5K^2 r_\alpha)$.

Proof: We construct $\mathcal{B}$ by transfinite induction. Let Ω be the first uncountable cardinal number; then we define $(B_\nu)_{\nu \leq \Omega}$ in the following way: for choosing B_ν, we consider whether there exist balls $B(x_\alpha, r_\alpha)$ that do not meet $O_\nu = \cup_{\xi < \nu} B_\xi$. If so, we choose α_ν so that $B(x_{\alpha_\nu}, r_{\alpha_\nu}) \cap O_\nu = \emptyset$ and $r_{\alpha_\nu} > \frac{1}{2} \sup\{r_\alpha\ /\ B(x_\alpha, r_\alpha) \cap O_\nu = \emptyset\}$; in that case, we choose $B_\nu = B(x_{\alpha_\nu}, r_{\alpha_\nu})$; otherwise, we choose $B_\nu = \emptyset$. It is easy to check that $B_\Omega = \emptyset$: X is separable and the balls B_ν are disjoint open sets. For $\alpha \in \mathcal{A}$, we define $\nu =$

$\min\{\xi \leq \Omega \ / \ B(x_\alpha, r_\alpha) \cap O_\xi \neq \emptyset\}$; then, we have $B(x_\alpha, r_\alpha) \cap B(x_{\alpha_\nu}, r_{\alpha_\nu}) \neq \emptyset$ and $r_\alpha \leq 2r_{\alpha_\nu}$. If $y \in B(x_\alpha, r_\alpha) \cap B(x_{\alpha_\nu}, r_{\alpha_\nu})$ and $z \in B(x_\alpha, r_\alpha)$, we have $d(x_{\alpha_\nu}, z) \leq K d(x_{\alpha_\nu}, y) + K^2(d(x_\alpha, y) + d(x_\alpha, z)) \leq 5K^2 r_{\alpha_\nu}$. □

Remark: An equivalent proof may be done without introducing transfinite induction : let E be the set of of families of balls $(B(x_\alpha, r_\alpha))_{\alpha \in \mathcal{F}}$ so that the balls of $\mathcal{F}$ are disjoints and such that for all $\alpha \in \mathcal{A}$ either $B(x_\alpha, r_\alpha)$ does not meet $\cup_{\beta \in \mathcal{F}} B(x_\beta, r_\beta)$ or there exists $\beta \in \mathcal{F}$ with $B(x_\alpha, r_\alpha) \cap B(x_\beta, r_\beta) \neq \emptyset$ and $r_\alpha \leq 2r_\beta$; then E is inductive (with respect to usual set ordering) and a maximal element of E (whose existence is granted by Zorn's lemma) satisfies the conclusion of Proposition 1.

Proposition 6.2: (The Whitney covering lemma)

For all $M > 1$, if G is an open subset of X then we may write G as a disjoint union $G = \sum_{n \in \mathbb{N}} G_n$ where the sets G_n are Borelian and satisfy (when not empty) diam $G_n \leq \frac{1}{M}$ dist$(G_n, X \backslash G)$.

Proof: Let D be a countable dense subset of X. If $x \in G$, there exists $y \in D$ so that $d(x, y) < \frac{1}{M_1} \text{dist}(x, X \backslash G)$. Let $r \in \mathbb{Q} \cap]d(x, y), \frac{1}{M_1} \text{dist}(x, X \backslash G)[$. Then, $x \in B(y, r)$ with $\text{diam}(B(y, r)) \leq 2Kr$ while for $z \in B(y, r)$ and $w \in X \backslash G$, we have $d(z, w) \geq \frac{d(x,w)}{K} - 2Kr \geq (M_1/K - 2K)r$. Thus, it is enough to choose M_1 so that $\frac{2K^2}{M_1 - 2K^2} < \frac{1}{M}$. □

Finally, we introduce the main result of this section: the boundedness of the Hardy–Littlewood maximal function on L^p. We consider a separable quasi-metric space (X, d) and μ a Borelian measure on X. The Hardy–Littlewood maximal function M_f for a measurable function f is defined (μ almost everywhere) by $M_f(x) = \sup_{r>0,\ 0<\mu(B(x,r))<\infty} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)| \, d\mu(y)$. Similarly, the modified Hardy–Littlewood maximal function of f is defined by $\tilde{M}_f(x) = \sup_{r>0,\ 0<\mu(B(x,5K^2r))<\infty} \frac{1}{\mu(B(x,5K^2r))} \int_{B(x,r)} |f(y)| \, d\mu(y)$.

Theorem 6.2: (Integrability of the Hardy–Littlewood maximal function)

(a) If μ has the doubling property, then the map $f \to M_f$ is bounded from $L^1(X, \mu)$ to $L^{1,*}(X, \mu)$ and from $L^p(X, \mu)$ to $L^p(X, \mu)$ for $p > 1$.

(b) If μ does not have the doubling property, the same conclusion holds for the modified Hardy–Littlewood maximal function $\tilde{M}_f$.

Proof: The case $p = \infty$ is obvious. Therefore, it is enough to consider the case $p = 1$. We define $E_N = \{x \ / \ \exists r < N \text{ so that } 0 < \mu(B(x, 5K^2 r) < \infty \text{ and } \frac{1}{\mu(B(x,5K^2 r)} \int_{B(x,r)} |f(y)| \, d\mu(y) > \lambda\}$ and $E = \{x \ / \ M_f(x) > \lambda\}$; then E is the (monotone) union of the sets E_N. The Vitali covering lemma gives

$$\mu(E_N) \leq \sum_{\alpha \in \mathcal{B}} \mu(B(x_\alpha, 5K^2 r_\alpha)) \leq \frac{1}{\lambda} \sum_{\alpha \in \mathcal{B}} \int_{B(x_\alpha, r_\alpha)} |f(x)| \, d\mu(x) \leq \frac{1}{\lambda} \int_X |f| \, d\mu$$

and this is enough to get $\mu(E) \leq \frac{1}{\lambda} \int_X |f(x)|\, d\mu(x)$. □

2. The complex method in interpolation theory

The complex method in interpolation theory was introduced by Calderón [CAL 64], as a generalization of the well-known interpolation theorem of Riesz-Thorin. We make little use of complex interpolation. The reader may find a precise description of this theory in Bergh and Löfström [BERL 76].

Definition 6.2: *For $0 < \theta < 1$, we define the complex interpolation space $[A_0, A_1]_\theta$ of A_0 and A_1 as follows:*

Let $\mathcal{F}(A_0, A_1)$ be the space of bounded continuous functions f from $\{z \in \mathbb{C}/0 \leq \Re z \leq 1\}$ to $A_0 + A_1$ so that
a) f is analytic in $\{z \in \mathbb{C}/0 < \Re z < 1\}$;
b) for $j = 0, 1$ the function $t \to f(j + it)$ is bounded from $\mathbb{R}$ to A_j and goes to 0 at infinity.

Then, the space $[A_0, A_1]_\theta$ is the space of the elements $a \in A_0 + A_1$ so that there exists $f \in \mathcal{F}(A_0, A_1)$ with $f(\theta) = a$. It is endowed with the norm $\|a\|_{[A_0,A_1]_\theta} = \inf_{a=f(\theta)} \max(\sup_t \|f(it)\|_{A_0}, \sup_t \|f(1+it)\|_{A_1})$.

Theorem 6.3: *Let T be a linear operator that is bounded from A_0 to B_0 and from A_1 to B_1 (with operator norms M_0 and M_1). Then, T is bounded from $[A_0, A_1]_\theta$ to $[B_0, B_1]_\theta$ with operator norm $M_\theta \leq M_0^{1-\theta}\ M_1^\theta$.*

Proof: Let $a \in [A_0, A_1]_\theta$, $a = f(\theta)$ with $f \in \mathcal{F}(A_0, A_1)$. We then define $g(z) = M_0^{-(1-z)} M_1^{-z} T(f(z))$. Clearly, $g \in \mathcal{F}(B_0, B_1)$ and $T(a) = M_0^{1-\theta} M_1^\theta g(\theta) \in [B_0, B_1]_\theta$; this choice of g shows that $\|T(a)\|_{[B_0,B_1]_\theta} \leq M_0^{1-\theta} M_1^\theta \|a\|_{[A_0,A_1]_\theta}$. □

We shall mainly use two examples of complex interpolation :

Theorem 6.4: (L^p spaces)

$[L^{p_0}, L^{p_1}]_\theta = L^p$ *for $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ with equality of norms.*

Theorem 6.5: (Weighted Lebesgue spaces)

Let (X, μ) be a measured space and w_0, w_1 two positive weights on X. Then, for $1 \leq p < \infty$ we have $[L^p(w_0\ d\mu), L^p(w_1\ d\mu)]_\theta = L^p(w_0^{1-\theta} w_1^\theta\ d\mu)$ for $0 < \theta < 1$ with equality of norms.

Proof: $L^p \subset [L^{p_0}, L^{p_1}]_\theta$ and $L^p(w_0^{1-\theta} w_1^\theta\ d\mu) \subset [L^p(w_0\ d\mu), L^p(w_1\ d\mu)]_\theta$ is obvious. Indeed, if $f \in L^p$, with $\|f\|_p = 1$, we define for $\epsilon > 0$ $F_\epsilon(z) = e^{\epsilon(z^2-\theta^2)} \frac{f}{|f|} |f|^{(1-z)\frac{p}{p_0} + z\frac{p}{p_1}}$. For $z = \tau + i\eta$, we find that $F_\epsilon(z) \in L^q$ with $1/q = (1-\tau)/p_0 + \tau/p_1$ and $\|F_\epsilon(z)\|_q = e^{\epsilon(\tau^2-\eta^2-\theta^2)}$; this shows that $f \in$

$[L^{p_0}, L^{p_1}]_\theta$ and that $\|f\|_{[L^{p_0},L^{p_1}]_\theta} \leq e^{\epsilon(\tau^2-\theta^2)}$; then, letting ϵ go to 0, we find that $\|f\|_{[L^{p_0},L^{p_1}]_\theta} \leq \|f\|_p$. Similarly, if $f \in L^p(w_\theta\ d\mu)$ where $w_\theta = w_0^{1-\theta} w_1^\theta\ d\mu$, we define for $\epsilon > 0$ $F_\epsilon(z) = e^{\epsilon(z^2-\theta^2)} \left(\frac{w_\theta}{w_0^{1-z} w_1^z}\right)^{1/p} f$; then, for $z = \tau + i\eta$, we find that $F_\epsilon(z) \in L^p(w_0^{1-\tau} w_1^\tau\ d\mu)$ and $\|F_\epsilon(z)\|_{L^p(w_\tau\ d\mu)} = e^{\epsilon(\tau^2-\eta^2-\theta^2)}\|f\|_{L^p(w_\theta\ d\mu)}$; this shows that $f \in [L^p(w_0\ d\mu), L^p(w_1\ d\mu)]_\theta$ and that $\|f\|_{[L^p(w_0\ d\mu),L^p(w_1\ d\mu)]_\theta} \leq \|f\|_{L^p(w_\theta\ d\mu)}$.

The converse inequalities are quite as easy. We consider $f \in [L^{p_0}, L^{p_1}]_\theta$, $f = F(\theta)$ with $F \in \mathcal{F}(L^{p_0}, L^{p_1})$. We define q_0, q_1, q as the conjugate exponents of p_0, p_1, p. If g is a simple function with $\|g\|_q \leq 1$, we define $G(z) = \frac{g}{|g|}|g|^{(1-z)\frac{q}{q_0}+z\frac{q}{q_1}}$. For every z, G is a simple function,thus $I(z) = \int F(z)G(z)\,d\mu$ is well defined. This is a continuous bounded function on $\{z \in \mathbb{C}/0 \leq \Re z \leq 1\}$, analytical on $\{z \in \mathbb{C}/0 < \Re z < 1\}$; moreover, on the lines $\Re z = 0$ or $\Re z = 1$, $I(z)$ is bounded by $\max(\sup_t \|F(it)\|_{L^{p_0}}, \sup_t \|F(1+it)\|_{L^{p_1}}) = M_F$; the Phragmen-Lindelöf principle gives that $|I(z)| \leq M_F$; for $z = \theta$, we find that $f \in (L^q)' = L^p$ and $\|f\|_p \leq M_F$. Taking the infimum over all F gives $\|f\|_p \leq \|f\|_{[L^{p_0},L^{p_1}]_\theta}$.

Now, we consider the case of $f \in [L^p(w_0\ d\mu), L^p(w_1\ d\mu)]_\theta$, $f = F(\theta)$ with $F \in \mathcal{F}(L^p(w_0\,d\mu), L^p(w_1\,d\mu))$. For $\epsilon > 0$, we define $E_\epsilon = \{x \in X\ /\ \epsilon < w_0(x) < 1/\epsilon,\ \epsilon < w_1(x) < 1/\epsilon,\ \epsilon < |f(x)| < 1/\epsilon\ \}$. Then, we have $\int |f|^p\ w_\theta\ d\mu = \lim_{\epsilon\to 0}\int_{E_\epsilon} |f|^p\ w_\theta\ d\mu$. We then define $G_\epsilon(z) = 1_{E_\epsilon}\frac{\bar f}{|f|}|f|^{p-1}w_\theta\left(\frac{w_0^{1-z}w_1^z}{w_\theta}\right)^{1/p}$. For every z, $G(z)$belongs to $L^q(w_O^{-q/p}\ d\mu) \cap L^q(w_1^{-q/p}\ d\mu)$, so that $I(z) = \int F(z)G(z)\ d\mu$ is a well-defined continuous bounded function on $\{z \in \mathbb{C}/0 \leq \Re z \leq 1\}$, analytical on $\{z \in \mathbb{C}/0 < \Re z < 1\}$; moreover, $I(z)$ is controlled on the lines $\Re z = 0$ or $\Re z = 1$ by $I(z) \leq\ M_F\ \|f1_{E_\epsilon}\|^{p-1}_{L^p(w_\theta\ d\mu)}$ where $M_F = \max(\sup_t \|F(it)\|_{L^p(w_0\ d\mu)}, \sup_t \|F(1+it)\|_{L^p(w_1\ d\mu)})$; the Phragmen-Lindelöf principle gives that $|I(z)| \leq M_F\ \|f1_{E_\epsilon}\|^{p-1}_{L^p(w_\theta\ d\mu)}$; it is then enough to let ϵ go to 0 and then to take the infimum over all F to get $\|f\|_{L^p(w_\theta\ d\mu)} \leq \|f\|_{[L^p(w_0\ d\mu),L^p(w_1\ d\mu)]_\theta}$. □

We now mention the duality theorem for complex interpolation:

Theorem 6.3: (Duality)

If $A_0 \cap A_1$ is dense in A_0 and in A_1 and if A_0 or A_1 is reflexive, then $[A_0, A_1]'_\theta = [A'_0, A'_1]_\theta$ for $0 < \theta < 1$ with equality of norms.

Proof: See the original paper of Calderón [CAL 64] or the book of Bergh and Löfström [BERL 76]. □

3. Atomic Hardy space and Calderón–Zygmund operators

In this section, we study the interpolation between the Hardy space $\mathcal{H}^1$ and the Lebesgue spaces L^p.

Definition 6.3: (Atomic Hardy space)

The (atomic) Hardy space $\mathcal{H}^1(\mathbb{R}^d)$ is defined as: $f \in \mathcal{H}^1$ if and only if f may be decomposed into a series $f = \sum_{j \in \mathbb{N}} \lambda_j a_j$ where Supp a_j is contained in a ball $B(x_j, R_j)$, $\|a_j\|_\infty \leq (R_j)^{-d}$, $\int a_j \, dx = 0$ and $\sum |\lambda_j| < \infty$. The function a_j is called an **atom** *carried by the ball $B(x_j, R_j)$.*

It is normed by $\|f\|_{\mathcal{H}^1} = \inf\{\sum_j |\lambda_j| \ / \ f = \sum_j \lambda_j \, a_j\}$ where the minimum runs over all the atomic decompositions of f. Then, it is easy to see that $\mathcal{H}^1$ is a Banach space which is continuously embedded in L^1.

Theorem 6.7: $[\mathcal{H}^1, L^p]_\theta = L^q$ *where* $\frac{1}{q} = 1 - \theta + \frac{\theta}{p}$ *and* $1 < p < \infty$.

Proof: Since $\mathcal{H}^1 \subset L^1$, we have obviously $[\mathcal{H}^1, L^p]_\theta \subset [L^1, L^p]_\theta = L^q$. In order to prove the reverse inclusion, we begin by the construction of atomic decompositions in L^q. For $f \in L^q$, $1 < p < \infty$, we define f^* its Hardy-Littlewood maximal function and $\Omega_k = \{x / \ f^*(x) > 2^k\}$. Ω_k is an open set with finite measure (since $q < \infty$). Now, we call $\mathcal{Q}_k$ the family of dyadic cubes $Q = l/2^j + 1/2^j[0,1]^d \subset \Omega_k$ so that the double cube ${}^2Q = l/2^j + 2/2^j[0,1]^d$ is still included in Ω_k but not the quadruple cube ${}^4Q = l/2^j + 4/2^j[0,1]^d$; we then have $\Omega_k = \cup_{Q \in \mathcal{Q}_k} Q$ and $|\Omega_k| = \sum_{Q \in \mathcal{Q}_k} |Q|$.

Now, for $k \in \mathbb{Z}$ and $Q \in \mathcal{Q}_k$, we introduce the function

$$\beta_{Q,k} = 1_Q \ (f - m_Q f - \sum_{R \in \mathcal{Q}_{k+1}} 1_R \ (f - m_R f)) \text{ where } m_R f = \frac{1}{|R|} \int_R f(x) \, dx.$$

We first notice that for any two dyadic cubes Q, R, $1_Q 1_R \neq 0$ if and only if $Q \subset R$ or $R \subset Q$; for $R \in \mathcal{Q}_{k+1}$ and $Q \in \mathcal{Q}_k$, we notice that $R \subset \Omega_k$ and ${}^2R \subset \Omega_k$, so that $1_Q 1_R \neq 0$ if and only if $R \subset Q$; this gives that $Supp \beta_{k,Q} \subset Q$ and $\int \beta_{Q,k} \, dx = 0$. Clearly, we have that $\sum_{Q \in \mathcal{Q}_k} \beta_{Q,k} = \sum_{Q \in \mathcal{Q}_k} 1_Q \ (f - m_Q f) - \sum_{R \in \mathcal{Q}_{k+1}} 1_R \ (f - m_R f)$. Let b_k be the function $b_k = \sum_{Q \in \mathcal{Q}_k} 1_Q \ (f - m_Q f)$ and $g_k = f - b_k = f \ 1_{\mathbb{R}^d - \Omega_k} + \sum_{Q \in \mathcal{Q}_k} 1_Q \ m_Q f$. By construction, $|g_k| \leq C \ 2^k$ and $|b_k| \leq C \ f^* \ 1_{\Omega_k}$, hence:

$$f = \sum_{k \in \mathbb{Z}} g_{k+1} - g_k = \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_k} \beta_{Q,k}$$

Moreover, we have $|\beta_{k,Q}| \leq C \ 2^k$: since the family $\mathcal{Q}_k$ is essentially disjoint, we may write $\beta_{Q,k} = 1_Q \ f \ 1_{\mathbb{R}^d - \Omega_{k+1}} - 1_Q m_Q f + \sum_{R \in \mathcal{Q}_{k+1}} 1_Q 1_R \ m_R f$ and get the required estimate.

We thus got an atomic decomposition $f = \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_k} \beta_{Q,k}$. Now, we define for $\Re z \in [0,1]$, $F(z) = \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_k} 2^{kq(1 - z + \frac{z}{p})} 2^{-k} \ \beta_{Q,k}$; then, we have for $\Re z = 0$ that $F(z) \in H^1$ (Hardy) and $\|F\|_{H^1} \leq \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_k} 2^{kq} |Q| \leq C \|f^*\|_q^q$. On the other hand, for $\Re z = 1$, we have $F(z) \in L^p$, since $|F(z)(x)| \leq$

$\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_k} 2^{kq/p}1_Q(x) \leq C(f^*(x))^{q/p}$. This gives $L^q \subset [\mathcal{H}^1, L^p]_\theta$ for $\theta = \frac{1-1/q}{1-1/p}$. □

We have proved another interesting result concerning $\mathcal{H}^1$.

Proposition 6.3: (p-atoms)

Let $\mathcal{H}^1_p$ be the space of functions f, which may be decomposed into a series $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ where Supp a_j is contained in a ball $B(x_j, R_j)$, $\|a_j\|_p \leq (R_j)^{d/p-d}$, $\int a_j\, dx = 0$ and $\sum|\lambda_j| < \infty$. Then $\mathcal{H}^1_p = \mathcal{H}^1_\infty$ for all $p \in (1,\infty)$.

Proof: It is enough to prove that a p-atom supported by the ball $B(0,1)$ belongs to $\mathcal{H}^1$ with a bounded norm. We thus take $a \in L^p$ with *Supp* $a \subset B(0,1)$, $\|a\|_p \leq 1$ and $\int a\, dx = 0$. We use the atomic decomposition of a described above: $a = \sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_k}\beta_{Q,k} = \sum_k g_{k+1} - g_k$. But we better write $a = g_0 + \sum_{k\in\mathbb{N}} g_{k+1} - g_k$. Indeed, the function $h = a - g_0$ is supported by Ω_0 which is bounded (since $a^*(x) \leq \frac{C}{(\|x\|-1)^d}$ for $\|x\| \geq 2$). Moreover, we have $h = \sum_{k\in\mathbb{N}}\sum_{Q\in\mathcal{Q}_k}\beta_{Q,k}$ and $\|h\|_{\mathcal{H}^1} \leq \sum_{k\in\mathbb{N}}\sum_{Q\in\mathcal{Q}_k} 2^k|Q| \leq C\int_{\Omega_0} a^*\, dx \leq C|\Omega_0|^{(p-1)/p}\|a^*\|_p$. Thus, $\int h\, dx = 0$ and finally $g_0 = a - h$ is a bounded function with a bounded support and a zero integral ; thus, $g_0 \in \mathcal{H}^1$ and this concludes the proof. □

We now give an easy application of the preceding theorem to the analysis of some classical operators. Let us recall the definition of (generalized) Calderón–Zygmund operators given by Coifman and Meyer [COIM 78] [MEY 97]:

Definition 6.4: (Calderón–Zygmund operators.)

A Calderón–Zygmund operator is a bounded linear operator T on $L^2(\mathbb{R}^d)$ ($T \in \mathcal{L}(L^2, L^2)$) so that there exists a function $K(x,y)$ defined on $\mathbb{R}^d \times \mathbb{R}^d - \Delta$ (where Δ is the diagonal set $x = y$) which satisfies:

i) There exists a positive constant C_0 so that

$$\forall x\ \forall y \quad |K(x,y)| \leq \frac{C_0}{|x-y|^d}$$

ii) There exists two positive constants ϵ and C_1 so that

$$\forall x\ \forall y\ \forall z \in B(0, \frac{1}{2}|x-y|) \quad |K(x+z,y) - K(x,y)| \leq C_1\ \frac{|z|^\epsilon}{|x-y|^{d+\epsilon}}$$

iii) There exists two positive constants ϵ and C_1 so that

$$\forall x\ \forall y\ \forall z \in B(0, \frac{1}{2}|x-y|) \quad |K(x,y+z) - K(x,y)| \leq C_1\ \frac{|z|^\epsilon}{|x-y|^{d+\epsilon}}$$

iv) For all $f \in L^2_{comp}$ and all $g \in L^2_{comp}$,

$$Supp f \cap Supp g = \emptyset \Rightarrow \langle Tf|g\rangle = \int\int K(x,y)f(y)\bar{g}(x)\ dx\ dy$$

Theorem 6.8: *Let T be a Calderón–Zygmund operator. Then, T is bounded from $\mathcal{H}^1$ (Hardy) to L^1, from L^∞ to BMO and from L^p to L^p for $1 < p < \infty$.*

Proof: We notice that the estimates on T are invariant under transposition, under translation and under dilation. Then, the proof is easy. Just test T on an atom supported by $B(0,0)$. $T(a)$ is locally integrable, since it belongs to L^2. Since $\int a\ dx = 0$, we use the regularity assumption on K to get that $T(a)$ is $O(\|x\|^{-d-\epsilon})$ at infinity. Then, we use translations and dilations to deal with a general atom. This gives the boundedness from $\mathcal{H}^1$ to L^1. Interpolation between $\mathcal{H}^1$ and L^2 gives the result for $1 < p \leq 2$ and transposition gives the result for $p > 2$ (since $BMO = (\mathcal{H}^1)^*$). □

Example: A typical example of a Calderón–Zygmund operator is the Riesz transform $R_j = \frac{\partial_j}{\sqrt{-\Delta}}$. The boundedness of R_j on $L^2(\mathbb{R}^d)$ is obvious: using the Fourier transform, we see that R_j is a Fourier multiplier with a bounded function $\mathcal{F}(R_j f)(\xi) = \frac{i\xi_j}{|\xi|}\mathcal{F}f(\xi)$. The L^p boundedness (for $1 < p < \infty$) is then a consequence of the Calderón–Zygmund theory since $\frac{\partial_j}{\sqrt{-\Delta}}\delta$ is a homogeneous distribution with homogeneity degree $-d$ on $\mathbb{R}^d$.

We may also use atomic decomposition and complex interpolation to get an elementary proof of the Sobolev inequalities :

Theorem 6.9: (Sobolev inequalities.)

a) For $0 < \alpha < d$, the operator $(\frac{1}{\sqrt{-\Delta}})^\alpha$ is bounded from the Hardy space $\mathcal{H}^1$ to $L^{\frac{d}{d-\alpha}}$ and from $L^{d/\alpha}$ to $BMO = (\mathcal{H}^1)^*$.

b) For $1 < p < \infty$ and $0 < \alpha < n/p$ the operator $(\frac{1}{\sqrt{-\Delta}})^\alpha$ is bounded from L^p to L^q where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$.

Proof: It is enough to check that the transform $(\frac{1}{\sqrt{-\Delta}})^\alpha(a)$ of an atom a defined on $B(0,1)$ belongs to $L^{\frac{d}{d-\alpha}}$ with a bounded norm. Then, using the invariance through translations and the homogeneity of the kernel, we see that this gives the boundedness from $\mathcal{H}^1$ to $L^{\frac{d}{d-\alpha}}$, then self-adjointness gives the boundedness from $L^{\frac{d}{\alpha}} = (L^{\frac{d}{d-\alpha}})^*$ to $(\mathcal{H}^1)^*$. Finally, we conclude by interpolation (using Theorem 6.6 and the duality theorem).

To estimate the transform of an atom a, we note that the transform $(\frac{1}{\sqrt{-\Delta}})^\alpha$ is a convolution operator with a kernel $\frac{c_\alpha}{|x|^{d-\alpha}}$; this kernel is uniformly locally integrable, and thus $(\frac{1}{\sqrt{-\Delta}})^\alpha a$ is bounded. Moreover, if $|x| > 10$, we may write $(\frac{1}{\sqrt{-\Delta}})^\alpha a(x) = c_\alpha \int_{|y|\leq 1} a(y)\big(\frac{1}{|x-y|^{d-\alpha}} - \frac{1}{|x|^{d-\alpha}}\big)\, dy$, thus $(\frac{1}{\sqrt{-\Delta}})^\alpha a(x)$ is $O(\frac{1}{|x|^{d+1-\alpha}})$. We thus get that $(\frac{1}{\sqrt{-\Delta}})^\alpha a$ belongs to L^r for all $r > \frac{d}{d+1-\alpha}$. In particular, $(\frac{1}{\sqrt{-\Delta}})^\alpha a$ belongs to $L^{\frac{d}{d-\alpha}}$ (and even to $L^{\frac{d}{d-\alpha},1}$). $\square$

The proof gives a better statement for the space $\mathcal{H}^1$:

Proposition 6.4: *For $0 < \alpha < d$, the operator $(\frac{1}{\sqrt{-\Delta}})^\alpha$ is bounded from the Hardy space $\mathcal{H}^1$ to the Lorentz space $L^{\frac{d}{d-\alpha},1}$ and from $L^{d/\alpha,\infty}$ to $BMO = (\mathcal{H}^1)^*$.*

Chapter 7

Vector-valued singular integrals

In this chapter, we recall the basic result of the Calderón–Zygmund theory on vector-valued singular integrals. Then, we give two applications : the characterization of L^p by the the Littlewood–Paley decomposition and the maximal $L^p(L^q)$ regularity of the heat kernel.

1. Calderón–Zygmund operators

We consider three locally compact σ–compact metric spaces X, X_1, X_2, regular Borelian measures μ, μ_1, μ_2 on those spaces and three numbers $p, p_1, p_2 \in (1, \infty)$. We define $E = L^{p_1}(X_1, \mu_1)$ and $F = L^{p_2}(X_2, \mu_2)$. We consider a linear operator T bounded from $L^p(X, \mu; E)$ to $L^p(X, \mu; F)$. We assume that there exists a continuous function $\mathcal{L}(x, y; x_1, x_2)$ defined on $(X \times X - \Delta) \times X_1 \times X_2$ (where Δ is the diagonal set of $X \times X$) so that, whenever $x \notin \text{Supp} f$, i.e., when there exists $r > 0$ so that f is equal to 0 $\mu \otimes \mu_1$-almost everywhere on $B(x, r) \times X_1$, then $Tf(x)$ is given by $Tf(x, x_2) = \int_X \int_{X_1} \mathcal{L}(x, y; x_1, x_2) f(y, x_1)\, d\mu(y)\, d\mu_1(x_1)$. We define $L(x, y)$ the operator from $E = L^{p_1}(X_1, \mu_1)$ to $F = L^{p_2}(X_2, \mu_2)$ given by the integral $L(x, y) f(x_2) = \int_{X_1} \mathcal{L}(x, y; x_1, x_2) f(x_1)\, d\mu_1(x_1)$. The Calderón–Zygmund theory gives sufficient condition to extrapolate from the L^p boundedness to the L^q boundedness for other values of q:

Theorem 7.1: (Calderón–Zygmund operators)

We consider a quasi-distance d on X (defining the topology of X), satisfying the quasi-triangular inequality $d(x, y) \leq K(d(x, z) + d(z, y))$, such that there exists positive real numbers n and C_μ such that for all $x \in X$ and all $r > 0$ $\mu(B(x, r)) \leq C_\mu\, r^n$.

(A) Assume that T is bounded from $L^p(X, \mu; E)$ to $L^p(X, \mu; F)$:

$$\int \|T(f)(x)\|_F^p\, d\mu(x) \leq C_0^p \int \|f(x)\|_E^p\, d\mu(x)$$

Assume that whenever $x \notin$ Supp$f$ we have $T(f)(x) = \int L(x, y)\ f(y)\ d\mu(y)$ where the function L is continuous from $X \times X - \Delta$ to $\mathcal{L}(E, F)$ and satisfies for some positive ϵ:

$$(CZ) \quad \begin{cases} \|L(x, y)\|_{op(E,F)} \leq C_1\ \frac{1}{d(x,y)^n} \\ d(z, y) \leq \frac{1}{2K} d(x, y) \Rightarrow \|L(x, y) - L(x, z)\|_{op(E,F)} \leq C_2\ \frac{d(z,y)^\epsilon}{d(x,y)^{n+\epsilon}} \end{cases}$$

Then, T is bounded from $L^1(X;E)$ to $L^{1,*}(X;F)$ and, for $1 < q < p$, from $L^q(X;E)$ to $L^q(X;F)$.

(B) If, moreover, L satisfies

$$(CZ^*) \qquad d(x,z) \leq \frac{1}{2K} d(x,y) \Rightarrow \|L(x,y) - L(z,y)\|_{op(E,F)} \leq C_2 \ \frac{d(x,z)^\epsilon}{d(x,y)^{n+\epsilon}}$$

then T is bounded from $L^q(X;E)$ to $L^q(X;F)$ for $p < q < \infty$.

Remark: We give here a proof derived from the recent paper of Nazarov, Treil and Volberg [NAZTV 98]. In contrast to classical proofs (such as in Coifman and Meyer [COIM 78], Coifman and Weiss [COIW 77], Stein [STE 93],...), no doubling property is assumed on the measure μ.

Proof: We consider $\lambda > 0$ and $f \in \mathcal{C}_{comp}(X;E)$ and we need to estimate $\mu(\{x \ / \ \|Tf(x)\|_F > \lambda\})$.

We define $G_\lambda = \{x \ / \ \|f(x)\|_E > \lambda\}$, $f_\lambda = f \ 1_{G_\lambda}$ and $g_\lambda = f - f_\lambda$. We first notice that G_λ is an open set (since f is continuous) and has a finite measure: $\mu(G_\lambda) \leq \frac{\|f\|_{L^1(E)}}{\lambda}$. Now, we define $X_\lambda = (\{x \ / \ \|Tf_\lambda(x)\|_F > \lambda/2\})$ and $Y_\lambda = (\{x \ / \ \|Tg_\lambda(x)\|_F > \lambda/2\})$. We have clearly $\mu(\{x \ / \ \|Tf(x)\|_F > \lambda\}) \leq \mu(X_\lambda) + \mu(Y_\lambda)$.

Since $g_\lambda \in L^p(X;E)$, we may easily estimate the measure of Y_λ by $\mu(Y_\lambda) \leq (\frac{2\|Tg_\lambda\|_{L^p(X,F)}}{\lambda})^p \leq (2C_0)^p \frac{\int \|f\|_E \ d\mu}{\lambda}$. Besides, we have $\mu(X_\lambda) \leq \mu(G_\lambda) + \mu(X_\lambda - G_\lambda)$. We now use a Whitney decomposition of G_λ: $G_\lambda = \sum_i G_i$ with $\text{diam}(G_i) \leq \frac{1}{2K} d(G_i, X \backslash G_\lambda)$. We may write $f_\lambda = \sum_i f_i$ in $L^p(X;E)$ with $f_i = f_\lambda 1_{G_i}$, so that $T(f_\lambda) = \sum_i T(f_i)$ in $L^p(X;F)$; thus, we may find a sequence (N_k) so that $T(f_\lambda)(x) = \lim_{N_k \to \infty} \sum_{i \leq N_k} T(f_i)(x)$ almost everywhere. For each G_i, we choose a point $x_i \in G_i$ and then we write for $x \notin G_\lambda$

$$T(f_i)(x) = \int (L(x,y) - L(x,x_i)) f_i(y) \ d\mu(y) + L(x,x_i) \int f_i(y) \ d\mu(y).$$

Thus, we get $\|T(f_\lambda)(x)\|_F \leq A(x) + B(x)$ where

$$A(x) = \sum_i \| \int (L(x,y) - L(x,x_i)) f_i(y) \ d\mu(y) \|_F$$

and $B(x) = \| \sum_i L(x,x_i) \int f_i(y) \ d\mu(y) \|_F$. We then write $X_\lambda \subset G_\lambda \cup A_\lambda \cup B_\lambda$ with $A_\lambda = \{x \notin G_\lambda \ / \ A(x) > \lambda/4\}$ and $B_\lambda = \{x \notin G_\lambda \ / \ B(x) > \lambda/4\}$. We have $\mu(G_\lambda) < \frac{\int \|f\|_E \ d\mu}{\lambda}$. We easily estimate $\mu(A_\lambda)$ by estimating $I = \int_{X \backslash G_\lambda} A(x) \ d\mu(x)$. Indeed, we have:

$$I \leq C_2 \ \sum_i \int (\frac{d(x_i,y)}{d(x,x_i)})^\epsilon \ \frac{1}{d(x,x_i)^n} \|f_i(y)\|_F \ d\mu(y) \ d\mu(x)$$

For $y \in G_i$, $x \notin G_\lambda$ and $d_i = \mathrm{diam} G_i$, we have $d(x_i, y) \leq d_i \leq \frac{1}{2K} d(x, x_i)$ while for $l \in \mathbb{N}$ we have

$$\int_{2^l d_i \leq 2K d(x,x_i) < 2^{l+1} d_i} \left(\frac{d_i}{d(x,x_i)}\right)^\epsilon \frac{1}{d(x,x_i)^n} \, d\mu(x) \leq C_\mu 2^n (2K)^\epsilon \, 2^{-l\epsilon}$$

and finally, $I \leq C_2 \frac{C_\mu 2^n (2K)^\epsilon}{1-2^{-\epsilon}} \sum_i \int_{G_i} \|f\|_E \, d\mu$. This gives the estimate $\mu(A_\lambda) < \frac{4C_2 C_\mu 2^n (2K)^\epsilon}{1-2^{-\epsilon}} \frac{\int \|f\|_E \, d\mu}{\lambda}$.

We now estimate $\mu(B_\lambda)$. We begin by checking that, if $\mu(X) > \frac{\int \|f\|_E \, d\mu}{\lambda}$, we may find disjoint sets E_i and positive real numbers δ_i so that (writing $B_i = B(x_i, \delta_i)$ and $B^i = B_{cl}(x_i, \delta_i)$ for the open and closed balls with center x_i and radius δ_i) we have :

$$E_i \cap E_j = \emptyset \text{ (if } i \neq j), \; E_i \subset B^i, \; B_i \subset \cup_{j \leq i} E_j \text{ and } \mu(E_i) = \frac{\int_{G_i} \|f\| \, d\mu}{\lambda}.$$

We build E_i by induction. We have $\mu(X - \cup_{j<i} E_j) > \frac{\int_{G_i} \|f\| \, d\mu}{\lambda}$. Then, we define δ_i by $\delta_i = \min\{d \;/\; \mu(B(x_i, d) - \cup_{j<i} E_j) > \frac{\int_{G_i} \|f\| \, d\mu}{\lambda}\}$. We thus have $\mu(B_i - \cup_{j<i} E_j) \leq \frac{\int_{G_i} \|f\| \, d\mu}{\lambda} \leq \mu(B^i - \cup_{j<i} E_j)$. Then, we choose E_i such that $B_i - \cup_{j<i} E_j \subset E_i \subset B^i - \cup_{j<i} E_j$ and $\mu(E_i) = \frac{\int_{G_i} \|f\| \, d\mu}{\lambda}$. Such a set E_i exists, and we may check it by the following easy lemma:

Lemma 7.1:

If A, B are two measurable sets in X and $r > 0$, if moreover $A \subset B$ and $\mu(A) \leq r \leq \mu(B)$, then there exists C so that $A \subset C \subset B$ and $\mu(C) = r$.

This lemma is easy. We may assume that $\mu(A) < r < \mu(B)$. Let $D = \{d_0, d_1, \ldots\}$ be a countable subset of X. Let $\rho > 0$. Then $X = \cup_{i \in \mathbb{N}} B(d_i, \rho)$. We fix N the smallest N so that $\mu(A \cup (\cup_{i \leq N} (B(d_i, \rho) \cap B)))$ is bigger than r. Then, $A_N = (A \cup (\cup_{i<N} (B(d_i, \rho) \cap B))$ satisfies $A \subset A_N \subset B$ and $r - C_\mu \rho^n \leq \mu(A_N) \leq r$. Thus, by induction, we may find a sequence of sets A_i with $A \subset A_i \subset A_{i+1} \subset B$ and $r - 2^{-i} \leq \mu(A_i) \leq r$. We then define C as $C = \cup_{i \in \mathbb{N}} A_i$ and conclude by the monotone convergence theorem.

Now, we write $\int_{G_i} f \, d\mu = \lambda \int 1_{E_i} \, d\mu \; u_i$ with $\|u_i\|_E = 1$. Thus, we have

$$L(x, x_i) \int_{G_i} f \, d\mu = \lambda \; L(x, x_i) \int 1_{E_i} u_i \, d\mu$$

The next step is to compare in some way $\sum_i L(x, x_i) \int f_i(y) \, d\mu(y)$ to

$$\lambda \, T\left(\sum_i 1_{E_i} u_i\right) = \lambda \; \sum_i \int L(x, y) 1_{E_i}(y) u_i \, d\mu(y)$$

Indeed, we have

$$\mu(\{x/\lambda\,\|T(\sum_i 1_{E_i}u_i)\|_F > \frac{\lambda}{4}\}) \le (4C_0)^p \|\sum_i 1_{E_i}u_i\|^p_{L^p(X;E)} \le \frac{(4C_0)^p\int\|f\|_E d\mu}{\lambda}$$

Alas, the estimate we need is not so direct. We look only at the case $x \notin E_\infty = \cup_i E_i$ since $\mu(\cup_i E_i) \le \frac{\int \|f\|_E\ d\mu}{\lambda}$. Let $\beta(x) = \sum_i L(x,x_i)\int f_i(y)d\mu(y)$ and $\gamma(x) = \lambda\ \sum_i 1_{X\setminus B(x_i,2K\delta_i)}(x)\int L(x,y)1_{E_i}(y)\,u_i\ d\mu(y)$. In order to estimate $\beta-\gamma$, we write $\int_{X\setminus E_\infty}\|\beta-\gamma\|_F\ d\mu \le U+V$ where

$$U = \sum_i \lambda\ \int_{X\setminus B(x_i,2K\delta_i)}\int \|L(x,x_i)-L(x,y)\|_{op(E,F)}1_{E_i}(y)\ d\mu(y)\ d\mu(x)$$

and

$$V = \sum_i \int_{\delta_i\le d(x,x_i)\le 2K\delta_i} \|L(x,x_i)\|_{op(E,F)}\|\int_{G_i} f\ d\mu\|_E\ d\mu(x)$$

On E_i, $d(x_i,y)\le\delta_i$, thus $\int_{X\setminus B(x_i,2K\delta_i)}\|L(x,x_i)-L(x,y)\|_{op(E,F)}d\mu(x) \le C_\epsilon C_2$, which gives $U \le C_\epsilon\ C_2\ \lambda\ \mu(E_\infty) \le C_\epsilon C_2\int\|f\|_E\ d\mu$. We get a similar estimate for V: $V \le C_1 C_\mu(2K)^n\ \int\|f\|_E\ d\mu$ since for $\delta_i \le d(x,x_i)\le 2K\delta_i$ we have

$$\|L(x,x_i)\|_{op(E,F)} \le C_1\frac{1}{d(x,x_i)^n} \le C_1C_\mu(2K)^n\frac{1}{\mu(B(x_i,2K\delta_i))}$$

To estimate $\mu(B_\lambda)$, we may write $B_\lambda \subset E_\infty\cup C_\lambda\cup D_\lambda$ where we have $C_\lambda = \{x\notin E_\infty\ /\ \|\beta-\gamma\|_F > \lambda/8\}$ and $D_\lambda = \{x\notin E_\infty\ /\ \|\gamma\|_F>\lambda/8\}$. We know that we have a good control on $\mu(E_\infty)$ and on $\mu(C_\lambda)$, so that we now have to deal with $\mu(D_\lambda)$. We estimate $\mu(D_\lambda)$ by estimating $\int\|\gamma\|^p_F d\mu$. We know that

$$(\int\|\gamma\|^p_F d\mu)^{1/p} = \sup\{\ |\int\langle g|\gamma\rangle_{F',F}\ d\mu|\ /\ \int\|g\|^{\frac{p}{p-1}}_{F'}d\mu = 1\ \}$$

Thus, we consider $g\in L^{p/(p-1)}(X;F')$ and we write

$$\int\langle g|\gamma\rangle_{F',F}d\mu = \lambda\sum_i\int_{E_i}\langle\int_{X\setminus B(x_i,2K\delta_i)}L^*(x,y)g(x)d\mu(x)|u_i\rangle_{E',E}d\mu(y) = X+Y$$

with

$$\begin{cases} X = \lambda\sum_i\int_{E_i}\langle\int_{X\setminus B(y,3\delta_i)}L^*(x,y)g(x)\ d\mu(x)|u_i\rangle_{E',E}\ d\mu(y) \\ Y = \lambda\sum_i\int_{E_i}\langle\int_{B(y,3K^2\delta_i)\setminus B(x_i,2K\delta_i)}L^*(x,y)g(x)\ d\mu(x)|u_i\rangle_{E',E}\ d\mu(y)\end{cases}$$

Let $T^{(*)}$ be the operator $T^{(*)}f(y) = \sup_{r>0}\|\int_{X\setminus B(y,r)}L^*(x,y)f(x)\ d\mu(x)\|_{E'}$; we estimate $|X|$ by

$$|X| \le \lambda\int_{E_\infty}T^{(*)}g(y)\ d\mu(y) \le C_p\ \lambda\ \|T^{(*)}g\|_{\frac{p}{(p-1)},\infty}\mu(E_\infty)^{1/p}$$

(since $\|1_{E_\infty}\|_{L^{p,1}} \leq C_p \|1_{E_\infty}\|_1^{1/p}\|1_{E_\infty}\|_\infty^{1-1/p} = C_p\ \mu(E_\infty)^{1/p}$). The estimate on Y is easy: for $y \in E_i$ and $x \notin B(x_i, 2K\delta_i)$ we have

$$d(x,y) \geq (1/K)d(x,x_i) - d(x_i,y) \geq \delta_i$$

and thus $\|L^*(x,y)\|_{op(F',E')} \leq C_1 \frac{1}{d(x,y)^n} \leq C_1 C_\mu (15K^4)^n \frac{1}{\mu(B(y,15K^4\delta_i))}$, hence we get that $|Y| \leq C_1 C_\mu (15K^4)^n \lambda \int_{E_\infty} \tilde{M}_{\|g\|_{F'}}(y) d\mu(y)$ and finally the estimate $|Y| \leq C_1 C_\mu (15K^4)^n \lambda \|\tilde{M}_{\|g\|_{F'}}\|_{\frac{p}{p-1}} (\mu(E_\infty))^{1/p}$.

We easily control Y, since we already know that $g \mapsto \tilde{M}_{\|g\|_{F'}}$ is bounded from $L^{\frac{p}{p-1}}(X;F')$ to $L^{\frac{p}{p-1}}$. To control X, we shall prove that $T^{(*)}$ is bounded from $L^{\frac{p}{p-1}}(X;F')$ to $L^{\frac{p}{p-1},\infty}$. We will then conclude that we have $\mu(D_\lambda) \leq \left(\frac{8\|\gamma\|_{L^p(X;F)}}{\lambda}\right)^p \leq C\mu(E_\infty) \leq C\frac{\int \|f\|_E\ d\mu}{\lambda}$.

To get a control on $T^{(*)}g$,, we establish the following inequality: for μ-almost every y we have $T^{(*)}g(y) \leq C(\tilde{M}_{\|T^*g\|_{E'}}(y) + (\tilde{M}_{\|g\|_{F'}^{p/(p-1)}}(y))^{1-1/p})$. Indeed, let $y \in \mathrm{Supp}\mu$ and $r > 0$. We define $r_j = (5K^2)^j r$ and we consider the smallest integer $k \geq 1$ so that $\mu(B(y,r_{k+1})) < 4\ (25K^4)^n \mu(B(y,r_{k-1}))$. (Such a k exists since for $2l < k$ we have $4^l \leq \frac{\mu(B(y,(25K^4)^l r))}{(25K^4)^{ln}\mu(B(y,r))} \leq \frac{C_\mu r^n}{\mu(B(y,r))} < \infty$). We take $R = (5K^2)^{k-1} r$ and begin by comparing $\int_{X\setminus B(y,r)} L^*(x,y)g(x)\ d\mu(x)$ to $\int_{X\setminus B(y,5K^2R)} L^*(x,y)g(x)\ d\mu(x)$:

$$\begin{aligned}&\|\textstyle\int_{B(y,5K^2R)\setminus B(y,r)} L^*(x,y)g(x)\ d\mu(x)\|_{E'} \leq \\ &\leq \textstyle\sum_{j=1}^k \int_{B(y,r_j)\setminus B(y,r_{j-1})} C_1 \frac{1}{d(x,y)^n} \|g(x)\|_{F'} d\mu(x) \leq \\ &\leq M_g(y) \textstyle\sum_{j=1}^k C_1 \mu(B(y,r_{j+1}))\ r_{j-1}^{-n} = I\end{aligned}$$

We know the following estimates:
– $\mu(B(y,r_{k+1})) \leq 4\ (25K^4)^n \mu(B(y,r_{k-1})) \leq (4\ (25K^4)^n)^1 \mu(B(y,r_k))$;
– $\mu(B(y,r_k)) \leq (4\ (25K^4)^n)^{1/2} \mu(B(y,r_k))$;
– $\mu(B(y,r_{k-1})) \leq (4\ (25K^4)^n)^0 \mu(B(y,r_k))$;
– for $j \leq k-2$, $\mu(B(y,r_j)) \leq (4\ (25K^4)^n)^{-1} \mu(B(y,r_{j+2}))$.
Thus, we get $I \leq C_1 \tilde{M}_g(y) \sum_{j=1}^k 2^{j-k}\ 4(125K^3)^n C_\mu \leq 8\ (125K^3)^n C_\mu C_1 \tilde{M}_g(y)$.

The next step is to compare $\int_{X\setminus B(y,5K^2R)} L^*(x,y)g(x)\ d\mu(x)$ to $\epsilon_R(y) = \frac{1}{\mu(B(y,R))} \int_{B(y,R)} T^*g(z)\ d\mu(z)$ (whose norm in E' is bounded by $\|\epsilon_R\|_{E'} \leq 4\ (25K^2)^n \tilde{M}_{\|T^*g\|_{E'}}(y)$).We write $\int_{X\setminus B(y,5K^2R)} L^*(x,y)g(x)\ d\mu(x) - \epsilon_R(y) = H + K$ with

$$\begin{cases} H = & \frac{1}{\mu(B(y,R))} \int_{B(y,R)} (\int_{X\setminus B(y,5K^2R)} (L^*(x,y) - L^*(x,z))g(x)\ d\mu(x))\ d\mu(z) \\ K = & \frac{1}{\mu(B(y,R))} \int_{B(y,R)} T^*(g 1_{B(y,5K^2R)})(z)\ d\mu(z) \end{cases}$$

It is enough to write that $\|\int_{X\setminus B(y,5R)} (L^*(x,y) - L^*(x,z))g(x)\ d\mu(x)\|_{E'} \leq CC_2 M_g(y)$ for $d(y,z) < R$ to see that $\|H\|_{E'}$ is controlled by $\tilde{M}_{\|g\|_{F'}}(y)$. Finally, to estimate $\|K\|_{E'}$, we take $\omega \in E$ with $\|\omega\| = 1$ and we write

$$\langle K|\omega\rangle_{E',E} = \frac{1}{\mu(B(y,R))} \langle g 1_{B(y,5K^2R)} | T(1_{B(y,R)}\omega)\rangle_{L^{\frac{p}{p-1}}(X,F'),L^p(X,F)}$$

which finally gives

$$\|K\|_{E'} \leq (4\ (25K^4)^n)^{1-1/p} \|T\|_{op(Lp(X,E),L^p(X,F)}(\tilde{M}_{\|g\|_{F'}^{p/(p-1)}}(y))^{1-1/p}$$

and thus (A) is proved.

(B) is then easy : if T satisfies (CZ^*), we find that T^* is bounded from $L^{q'}(F')$ to $L^{q'}(E')$ for $1 < q' < \frac{p}{p-1}$ and by duality T is bounded from $L^q(E)$ to $L^q(F)$ for $p < q < \infty$.

Theorem 7.1 is proved. □

We have proved a usable estimate:

Corollary : *Under the same hypotheses (T is bounded from $L^p(X,E)$ to $L^p(X,F)$, T satisfies (CZ) and (CZ^*)), the maximal operator $\tilde{T}$ defined by $\tilde{T}f(x) = \sup_{r>0} \| \int_{d(x,y)\geq r} L(x,y)f(y)\ d\mu(y)\|_F$ is bounded from $L^q(X;E)$ to $L^q(X)$ for all $q \in (1,\infty)$.*

Proof: Indeed, (switching the role of T and T^*), we proved that when $1 < r < \infty$ and T^* is bounded from $L^{\frac{r}{r-1}}(X;F')$ to $L^{\frac{r}{r-1}}(X,E')$ then $\tilde{T}$ is bounded from $L^r(X,E)$ to $L^{r,\infty}$. Besides, Theorem 7.1 proved that T^* is bounded from $L^\rho(X,F')$ to $L^\rho(X,E')$ for all $1 < \rho < \infty$. . We then conclude using the Marcinkiewicz theorem by choosing $1 < r_0 < q < r_1 < \infty$. The boundedness $L^{r_i}(X,E)$ to $L^{r_i,\infty}$ then gives the boundedness $L^q(X,E)$ to L^q. □

2. Littlewood–Paley decomposition in L^p

Let us now recall the definition of the Littlewood–Paley decomposition :

Definition 7.1: (Littlewood–Paley decomposition)

Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ be such that $|\xi| \leq \frac{1}{2} \Rightarrow \varphi(\xi) = 1$ and $|\xi| \geq 1 \Rightarrow \varphi(\xi) = 0$. Let ψ be defined as $\psi(\xi) = \varphi(\xi/2) - \varphi(\xi)$. Let S_j and Δ_j be defined as the Fourier multipliers $\mathcal{F}(S_j f) = \varphi(\xi/2^j)\mathcal{F}f$ and $\mathcal{F}(\Delta_j f) = \psi(\xi/2^j)\mathcal{F}f$. Then for all $N \in \mathbb{Z}$ and all $f \in \mathcal{S}'(\mathbb{R}^d)$ we have $f = S_N f + \sum_{j\geq N} \Delta_j f$ in $\mathcal{S}'(\mathbb{R}^d)$. This equality is called the Littlewood–Paley decomposition of the distribution f.

The following result is then a direct consequence of Theorem 7.1:

Theorem 7.2: (Littlewood–Paley decomposition of $L^p(\mathbb{R}^d)$)

Let $f \in \mathcal{S}'(\mathbb{R}^d)$ and $1 < p < \infty$. Then the following assertions are equivalent:

(A) $f \in L^p(\mathbb{R}^d)$.

(B) $S_0 f \in L^p(\mathbb{R}^d)$ and $(\sum_{j\in\mathbb{N}} |\Delta_j f(x)|^2)^{1/2} \in L^p(\mathbb{R}^d)$.

(C) $f = \sum_{j\in\mathbb{Z}} \Delta_j f$ and $(\sum_{j\in\mathbb{Z}} |\Delta_j f(x)|^2)^{1/2} \in L^p(\mathbb{R}^d)$.

Moreover, the following norms are equivalent on L^p*:*
$\|f\|_p$, $\|S_0 f\|_p + \|(\sum_{j\in\mathbb{N}} |\Delta_j f(x)|^2)^{1/2}\|_p$ *and* $\|(\sum_{j\in\mathbb{Z}} |\Delta_j f(x)|^2)^{1/2}\|_p$.

Proof: We prove the equivalence of (A) and (C), the proof for (A) and (B) is similar. If $f \in L^2$, it is obvious (by the dominated convergence theorem applied to $\mathcal{F}(S_j f)$) that $\lim_{j\to-\infty} \|S_j f\|_2 = 0$. Thus, we have $f = \sum_{j\in\mathbb{Z}} \Delta_j f$. The Fubini and Plancherel theorems give

$$\|(\sum_{j\in\mathbb{Z}} |\Delta_j f(x)|^2)^{1/2}\|_2^2 = (\frac{1}{2\pi})^d \int |f(\xi)|^2 \, (\sum_{j\in\mathbb{Z}} |\psi(\xi/2^j)|^2) \, d\xi \approx \|f\|_2^2$$

since there exists two positive constants $0 < A_0 < A_1$ so that for $\xi \neq 0$ we have $A_0 \leq \sum_{j\in\mathbb{Z}} |\psi(\xi/2^j)|^2 \leq A_1$. Thus, the mapping $f \mapsto (\Delta_j f)_{j\in\mathbb{Z}}$ is bounded from $L^2(\mathbb{R}^d) = L^2(\mathbb{R}^d;\mathbb{C})$ to $L^2(\mathbb{R}^d, l^2(\mathbb{Z}))$.

We may then apply the theory of singular integrals to this mapping (interpreting $\mathbb{C}$ as $l^2(\{0\})$). Indeed, the kernel $L(x,y)$ is the operator $z \in \mathbb{C} \mapsto (2^{jd}\mathcal{F}^{-1}\psi(2^j(x-y))z)_{j\in\mathbb{Z}} \in l^2$. We have

$$\begin{aligned} \|L(x,y)\|_{op} &= (\textstyle\sum_{j\in\mathbb{Z}} 4^{jd}|\mathcal{F}^{-1}\psi(2^j(x-y))|^2)^{1/2} \\ \|\frac{\partial}{\partial x_j} L(x,y)\|_{op} = \|\frac{\partial}{\partial y_j} L(x,y)\|_{op} &= (\textstyle\sum_{j\in\mathbb{Z}} 4^{j(d+1)}|(\frac{\partial}{\partial x_j}\mathcal{F}^{-1}\psi)(2^j(x-y))|^2)^{1/2} \end{aligned}$$

and (writing $2^{j\alpha}|\omega(2^j x)| \leq \min(2^{j\alpha}\|\omega\|_\infty, 2^{-j\alpha}|x|^{-2\alpha}\|y^{2\alpha}\omega(y)\|_\infty)$ we check easily that for $\omega \in \mathcal{S}(\mathbb{R}^d)$, $\alpha > 0$ and $x \in \mathbb{R}^d$ we have $(\sum_{j\in\mathbb{Z}} 4^{j\alpha}|\omega(2^j x)|^2)^{1/2} \leq C_{\alpha,\omega}\,|x|^{-\alpha}$. Thus, Theorem 7.1 gives that $\|\Delta_j f\|_{L^p(l^2)} \leq C_p \|f\|_p$ for $1 < p < \infty$.

Conversely, let $\tilde{\psi} \in \mathcal{D}(\mathbb{R}^d)$ so that $0 \notin \operatorname{Supp} \tilde{\psi}$ and $\psi\tilde{\psi} = \psi$. Let $\tilde{\Delta}_j = \tilde{\psi}(D/2^j)$. Then, we have $f = \sum_{j\in\mathbb{Z}} \tilde{\Delta}_j \Delta_j f$. Next, look at the operator $(f_j)_{j\in\mathbb{Z}} \mapsto \sum_{j\in\mathbb{Z}} \tilde{\Delta}_j f_j$. This operator is bounded from $L^2(l^2)$ to L^2: since $0 \notin \operatorname{Supp} \tilde{\psi}$, there is a number M so that if $|j-l| > M$ then $\tilde{\Delta}_j f_j$ and $\tilde{\Delta}_l f_l$ have disjoint spectrums (i.e., the Fourier transforms have disjoint supports), so that $\|\sum_{j\in\mathbb{Z}} \tilde{\Delta}_j f_j\|_2^2 \leq (2M+1)(\sum_{j\in\mathbb{Z}} \|\tilde{\Delta}_j f_j\|_2^2) \leq (2M+1)\|\tilde{\psi}\|_\infty^2 \|(f_j)\|_{L^2(l^2)}^2$. We again apply the theory of singular integrals. Indeed, the kernel $\tilde{L}(x,y)$ of the operator is the operator

$$(\lambda_j)_{j\in\mathbb{Z}} \mapsto \sum_j \lambda_j 2^{jd}\mathcal{F}^{-1}\tilde{\psi}(2^j(x-y)).$$

We have $\|\tilde{L}(x,y)\|_{op} = (\sum_{j\in\mathbb{Z}} 4^{jd}|\mathcal{F}^{-1}\tilde{\psi}(2^j(x-y))|^2)^{1/2}$, $\|\frac{\partial}{\partial x_j}\tilde{L}(x,y)\|_{op} = \|\frac{\partial}{\partial y_j}\tilde{L}(x,y)\|_{op} = (\sum_{j\in\mathbb{Z}} 4^{j(d+1)}|\frac{\partial}{\partial x_j}\mathcal{F}^{-1}\tilde{\psi}(2^j(x-y))|^2)^{1/2}$ and the theory of Calderón–Zygmund operators gives that $\|\sum_{j\in\mathbb{Z}} \tilde{\Delta}_j f_j\|_p \leq C_p \|(f_j)\|_{L^p(l^2)}$. $\square$

We now extend this theorem to the potential spaces H_p^σ:

Proposition 7.1: (Potential spaces)

Let $f \in \mathcal{S}'(\mathbb{R}^d)$, $\sigma \in \mathbb{R}$ and $1 < p < \infty$. Then the following assertions are equivalent:

(A) $f \in H_p^\sigma(\mathbb{R}^d)$.

(B) $S_0 f \in L^p(\mathbb{R}^d)$ and $(\sum_{j\in\mathbb{N}} |4^{j\sigma}\ \Delta_j f(x)|^2)^{1/2} \in L^p(\mathbb{R}^d)$.

Moreover, the following norms are equivalent on $H_p^\sigma(\mathbb{R}^d)$:

$\|(Id-\Delta)^{\sigma/2} f\|_p$ and $\|S_0 f\|_p + \|(\sum_{j\in\mathbb{N}} |4^{j\sigma}\ \Delta_j f(x)|^2)^{1/2}\|_p$.

Proof: $(Id-\Delta)^{\sigma/2}$ is an isomorphism between H_p^σ and L^p. Thus, it is enough to check that we may apply the theory of vector-valued Calderón–Zygmund operators to the operators $f \mapsto (2^{j\sigma}\Delta_j((Id-\Delta)^{-\sigma/2}f))_{j\geq 0}$ and $(f_j)_{j\geq 0} \mapsto (Id-\Delta)^{\sigma/2}\sum_{j\geq 0}\tilde{\Delta}_j(2^{-j\sigma}f_j)$. This is easily done using the estimates on the Bessel potentials proved in Chapter 3: if k_σ is the kernel of the convolution operator $(Id-\Delta)^{\sigma/2}$, then, for all $N \in \mathbb{N}$, there exists a constant $C_{\sigma,N}$ so that for all $j \geq 0$ we have $|\Delta_j k_\sigma(x)| \leq C_{\sigma,N}\ 2^{j(d+\sigma)}\ (1+2^j|x|)^{-N}$ for all $x \in \mathbb{R}^d$. □

3. Maximal $L^p(L^q)$ regularity for the heat kernel

In this section, we prove the maximal $L^p(L^q)$ regularity theorem for the heat kernel:

Theorem 7.3: (Maximal $L^p(L^q)$ regularity for the heat kernel.)

The operator A defined by $f(t,x) \mapsto Af(t,x) = \int_0^t e^{(t-s)\Delta}\Delta f(s,.)\ ds$ is bounded from $L^p((0,T), L^q(\mathbb{R}^d))$ to $L^p((0,T), L^q(\mathbb{R}^d))$ for every $T \in (0,\infty]$, $1 < p < \infty$ and $1 < q < \infty$.

Proof: We may suppose $T = \infty$ (if $T < \infty$, we may extend f by $f = 0$ on (T,∞) ; this is harmless since $Af(t,x)$ depends only from the values of f on $(0,t)\times\mathbb{R}^d$). Moreover, we extend f and Af to negative values of t by $f = Af = 0$ on $(-\infty,0)$.Then, Theorem 7.3 is proved in three steps.

Step 1: A is bounded on $L^2(L^2)$.

Indeed, let W be the kernel of e^Δ: $W(x) = (4\pi)^{-d/2}e^{-\frac{\|x\|^2}{4}}$ and let Ω defined by $\Omega(t,x) = \frac{1}{t^{d/2}}(\Delta W)(\frac{x}{\sqrt{t}})$ for $t > 0$ and by $\Omega(t,x) = 0$ for $t < 0$. Then, we have $Af(t,x) = \int_{s\in\mathbb{R}}\int_{x\in\mathbb{R}^d}\frac{1}{t-s}\Omega(t-s,x-y)f(s,y)\ ds\ dy$. Thus, A is a convolution operator on $L^2(\mathbb{R}\times\mathbb{R}^d)$. We compute the Fourier transform (in t and x) of $\frac{1}{t}\Omega$: the Fourier transform in x gives for $t > 0$ $\int\Omega(t,x)e^{-ix.\xi}\ dx = -|\xi|^2e^{-t|\xi|^2}$ and then the Fourier transform in t gives $\mathcal{F}(\tau,\xi) = -\int_0^\infty|\xi|^2e^{-t|\xi|^2}e^{-it\tau}\ dt = -\frac{|\xi|^2}{|\xi|^2+i\tau}$. Since $|\mathcal{F}(\tau,\xi)| \leq 1$, we find that A is bounded on $L^2(\mathbb{R}\times\mathbb{R}^d)$.

Step 2: A is bounded on $L^p(L^p)$ for $1 < p < \infty$.

We interpret A as a Calderón–Zygmund operator on $\mathbb{R}\times\mathbb{R}^d$ endowed with the Lebesgue measure on $\mathbb{R}^{d+1}$ and with the quasi-distance $d((t,x),(s,y)) = (|x-y|^4+|t-s|^2)^{1/4}$. We have for all $(t,x)\in\mathbb{R}\times\mathbb{R}^d$ and all $r>0$ $|B((t,x),r)| = C\ r^{d+2}$. The kernel is given by

$$L((t,x),(s,y)) = 1_{(0,\infty)}(t-s)\frac{1}{(t-s)^{(d+2)/2}}(\Delta W)(\frac{x-y}{\sqrt{t-s}}).$$

If $|x-y|\leq\sqrt{t-s}$, we write $|L((t,x),(s,y))| \leq \frac{\|\Delta W\|_\infty}{|t-s|^{(d+2)/2}}$, while if $|x-y|\geq\sqrt{t-s}$, we write $|L((t,x),(s,y))| \leq \frac{\||z|^{d+2}\ \Delta W(z)\|_\infty}{|x-y|^{d+2}}$; thus, we obtain that $|L((t,x),(s,y))| \leq \frac{C}{d((t,x),(s,y))^{d+2}}$. In the same way, we obtain the estimates $|\frac{\partial}{\partial t}L((t,x),(s,y))| \leq \frac{C}{d((t,x),(s,y))^{d+4}}$ and $|\frac{\partial}{\partial x_j}L((t,x),(s,y))| \leq \frac{C}{d((t,x),(s,y))^{d+3}}$. This gives for (θ,h) small with respect to $d((t,x),(s,y))$ that

$$|L((t,x),(s,y))-L((t+\theta,x+h),(s,y))|\leq C(\frac{|\theta|}{d((t,x),(s,y))^{d+4}}+\frac{|h|}{d((t,x),(s,y))^{d+3}})$$

which gives

$$|L((t,x),(s,y)) - L((t+\theta,x+h),(s,y))| \leq \frac{C'd((t,x),(t+\theta,x+h))}{d((t,x),(s,y))^{d+3}}$$

We have similar estimates for the regularity of L with respect to (s,y) and thus we get the boundedness of A on $L^p(\mathbb{R}\times\mathbb{R}^d) = L^p(\mathbb{R},L^p(\mathbb{R}^d))$.

Step 3: A is bounded on $L^p(L^q)$ for $1<p<\infty$ and $1<q<\infty$.

We now interpret A as a Calderón–Zygmund operator on $\mathbb{R}$. The kernel $L(t,s)$ is now $\Delta e^{(t-s)\Delta}$ and we have $\|L\|_{op(L^q,L^q)} = \frac{C}{t-s}$ and $\|\frac{\partial}{\partial t}L\|_{op} = \|\frac{\partial}{\partial s}L\|_{op} = \|\Delta^2 e^{(t-s)\Delta}\|_{op} = \frac{C}{(t-s)^2}$. Thus, the $L^q(L^q)$ boundedness implies the $L^p(L^q)$ boundedness. □

Chapter 8

A primer to wavelets

Wavelet theory was introduced in the 1980's as an efficient tool for signal analysis. We do not give a detailed presentation of the theory or discuss its applications (see the excellent books of Daubechies [DAU 92] or Mallat [MAL 98]). We are interested in wavelets as tools for getting basic results of real harmonic analysis, in the spirit of the books of Coifman and Meyer [MEY 97] or Kahane and Lemarié-Rieusset [KAHL 95].

Definition 8.1: (Wavelet bases)

A wavelet basis of $L^2(\mathbb{R}^d)$ is a family of functions $(\psi_{\epsilon,j,k})_{1\leq\epsilon\leq 2^d-1, j\in\mathbb{Z}, k\in\mathbb{Z}^d}$ such that
i) they are derived through dyadic dilations and translations from a finite set of functions $(\psi_\epsilon)_{1\leq\epsilon\leq 2^d-1}$:

$$\psi_{\epsilon,j,k}(x) = 2^{jd/2}\psi_\epsilon(2^jx-k)$$

ii) the family is a Riesz basis of $L^2(\mathbb{R}^d)$, i.e., the mapping

$$(\lambda_{\epsilon,j,k})_{1\leq\epsilon\leq 2^d-1, j\in\mathbb{Z}, k\in\mathbb{Z}^d} \mapsto \sum_{1\leq\epsilon\leq 2^d-1, j\in\mathbb{Z}, k\in\mathbb{Z}^d} \lambda_{\epsilon,j,k}\ \psi_{\epsilon,j,k}$$

is an isomorphism between $l^2(\{1,\ldots,2^d-1\}\times\mathbb{Z}\times\mathbb{Z}^d)$ and $L^2(\mathbb{R}^d)$.
iii) The dual basis has the same structure : for all $f\in L^2$ we have $f = \sum_{1\leq\epsilon\leq 2^d-1, j\in\mathbb{Z}, k\in\mathbb{Z}^d}\langle f|\psi^*_{\epsilon,j,k}\rangle\ \psi_{\epsilon,j,k}$ with $\psi^*_{\epsilon,j,k}(x) = 2^{jd/2}\psi^*_\epsilon(2^jx-k)$.

The associated projection operators Q_j are defined by

$$Q_jf = \sum_{1\leq\epsilon\leq 2^d-1, k\in\mathbb{Z}^d} \langle f|\psi^*_{\epsilon,j,k}\rangle\ \psi_{\epsilon,j,k}.$$

The range W_j of Q_j is the closed linear span of the functions $\psi_{\epsilon,j,k}, 1\leq\epsilon\leq 2^d-1, k\in\mathbb{Z}^d$, and its kernel W_j^* is the closed linear span of the functions $\psi^*_{\epsilon,j,k}, 1\leq\epsilon\leq 2^d-1, k\in\mathbb{Z}^d$.

Scaling functions φ, φ^* associated with the wavelets are (if they exist) functions in L^2 so that the family $(\varphi(x-k))_{k\in\mathbb{Z}^d}$ is a Riesz basis of $V_0 =$

$\bigoplus_{j<0} W_j$, $(\varphi^*(x-k))_{k\in\mathbb{Z}^d}$ is a Riesz basis of $V_0^* = \bigoplus_{j<0} W_j^*$ and, defining the projection operator $P_j = \sum_{l<j} Q_l$ (with range V_j and kernel V_j^*), we have $P_0 f = \sum_{k\in\mathbb{Z}^d} \langle f|\varphi^*(x-k)\rangle\ \varphi(x-k)$.

When $\varphi = \varphi^*$ and $\psi_\epsilon = \psi_\epsilon^*$ for $1 \le \epsilon \le 2^d - 1$, we shall speak of orthogonal scaling functions and of orthogonal wavelets.

1. Multiresolution analysis

We now construct wavelet bases on $\mathbb{R}$, preceded by a loose introduction to the construction of wavelet bases.

Assume that φ, φ^* are scaling functions associated with some biorthogonal wavelet basis. The *multiresolution analysis* associated to the scaling function φ is the family of closed linear subspaces $V_j =$ Im P_j (i.e., V_j is the range of the projection operator P_j) ; V_j satisfies the following properties:

i) $V_j \subset V_{j+1}$

ii) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$ and $\cup_{j\in\mathbb{Z}} V_j$ is dense in L^2

iii) $f \in V_j$ if and only if $f(2x) \in V_{j+1}$

iv) V_0 has a shift–invariant Riesz basis $(\varphi(x-k)_{k\in\mathbb{Z}})$.

In particular, we have, since $\varphi \in V_0 \subset V_1$,

$$\varphi(x) = \sum_{k\in\mathbb{Z}} \lambda_k \varphi(2x-k) \tag{8.1}$$

for some sequence $(\lambda_k) \in l^2(\mathbb{Z})$. Thus, taking the Fourier transform of (8.1), we get that $\hat\varphi(\xi) = m_0(\xi/2)\hat\varphi(\xi/2)$ where $m_0 \in L^2(\mathbb{R}/2\pi\mathbb{Z})$ is the periodical function $m_0(\xi) = \frac{1}{2}\sum_{k\in\mathbb{Z}} \lambda_k e^{-ik\xi}$. Similarly, since φ and φ^* play symmetrical roles, we find that $\hat\varphi^*(\xi) = m_0^*(\xi/2)\hat\varphi^*(\xi/2)$ for some periodical function $m_0^*(\xi) \in L^2(\mathbb{R}/2\pi\mathbb{Z})$.

Now, we use the following Poisson summation formula: if $\omega, \Omega \in L^2(\mathbb{R})$, then the equality $\sum_{k\in\mathbb{Z}} \hat\omega(\xi+2k\pi)\bar{\hat\Omega}(\xi+2k\pi) = \sum_{k\in\mathbb{Z}} \langle\omega(x)|\Omega(x-k)\rangle e^{-ik.\xi}$ holds for almost every $\xi \in \mathbb{R}$. We write $C(\omega,\Omega)(\xi)$ for the periodical function $C(\omega,\Omega)(\xi) = \sum_{k\in\mathbb{Z}} \langle\omega(x)|\Omega(x-k)\rangle e^{-ik.\xi}$. Since $\hat\varphi(\xi) = m_0(\xi/2)\hat\varphi(\xi/2)$ and $\hat\varphi^*(\xi) = m_0^*(\xi/2)\hat\varphi^*(\xi/2)$, we find that

$$C(\varphi,\varphi^*)(\xi) = m_0(\frac{\xi}{2})\bar m_0^*(\frac{\xi}{2})C(\varphi,\varphi^*)(\frac{\xi}{2}) + m_0(\frac{\xi}{2}+\pi)\bar m_0^*(\frac{\xi}{2}+\pi)C(\varphi,\varphi^*)(\frac{\xi}{2}+\pi)$$

thus, the duality between the system $(\varphi(x-k)_{k\in\mathbb{Z}}$ and the system $(\varphi^*(x-k)_{k\in\mathbb{Z}}$ gives that $C(\varphi,\varphi^*) = 1$ and hence that $m_0(\frac{\xi}{2})\bar m_0^*(\frac{\xi}{2}) + m_0(\frac{\xi}{2}+\pi)\bar m_0^*(\frac{\xi}{2}+\pi) = 1$.

In order to construct φ,φ^*, we thus start with a pair of univariate trigonometric polynomials $m_0 = \sum_{k_0\le k\le k_1} a_k e^{-ik\xi}$ and $m_0^* = \sum_{l_0\le l\le l_1} \alpha_l e^{-il\xi}$ with real–valued coefficients such that:

j) $m_0(0) = m_0^*(0) = 1$ and $m_0(\xi)\bar{m}_0^*(\xi) + m_0(\xi+\pi)\bar{m}_0^*(\xi+\pi) = 1$.
jj) m_0 and m_0^* have no zero on $[-\pi/2, +\pi/2]$.
jjj) the functions $\hat{\varphi} = \prod_{j=1}^{\infty} m_0(\frac{\xi}{2^j})$ and $\hat{\varphi}^* = \prod_{j=1}^{\infty} m_0^*(\frac{\xi}{2^j})$ are square integrable on $\mathbb{R}$.

Lemma 8.1: *Let m_0 be a trigonometric polynomial $m_0 = \sum_{k_0 \leq k \leq k_1} a_k e^{-ik\xi}$ with $m_0(0) = 1$ and let $\hat{\varphi} = \prod_{j=1}^{\infty} m_0(\frac{\xi}{2^j})$. Then $\hat{\varphi}$ is the Fourier transform of a compactly supported distribution φ with support contained in $[k_0, k_1]$.*

Proof: Since $m_0(0) = 1$, we have $|m_0(\xi) - 1| \leq \|m_0'\|_\infty |\xi|$ and thus the infinite product is convergent for all $\xi \in \mathbb{R}$. Moreover, for $|\xi| \leq \pi$, we have $\sup_{N \geq 1} \prod_{j=1}^{N} |m_0(\xi/2^j)| \leq \prod_{j=1}^{\infty}(1 + \frac{\pi \|m_0'\|_\infty}{2^j}) = \Gamma < \infty$, while for $2^n\pi \leq |\xi| \leq 2^{n+1}\pi$ we have $\sup_{N\geq 1} \prod_{j=1}^{N} |m_0(\xi/2^j)| \leq \Gamma \|m_0\|_\infty^{n+1} \leq A|\xi|^B$ with $A = \Gamma\|m_0\|_\infty$ and $B = \ln \|m_0\|_\infty / \ln 2$. Thus, the finite products $\prod_{j=1}^{N} m_0(\xi/2^j)$ converge to $\hat{\varphi}$ in $\mathcal{S}'$; taking the inverse Fourier transforms gives that φ is the limit in $\mathcal{S}'$ of the convolution products $2\mu(2x) * 4\mu(4x) * \ldots * 2^N\mu(2^N x)$ with $\mu(x) = \sum_{k_0 \leq k \leq k_1} a_k \delta(x-k)$. The support of μ is $\{k_0, \ldots, k_1\} \subset [k_0, k_1]$ and thus the support of the convolution $2\mu(2x) * 4\mu(4x) * \ldots * 2^N\mu(2^N x)$ is contained in $[(1/2 + 1/4 + \ldots + 1/2^N)k_0, (1/2 + 1/4 + \ldots + 1/2^N)k_1] \subset (1 - 2^{-N})[k_0, k_1]$.
□

Lemma 8.2: *Let m_0 and m_0^* satisfy hypothesis j). Then, if θ_p and θ_p^* are defined as*

$$\hat{\theta}_p(\xi) = 1_{[-2^p\pi, 2^p\pi]}(\xi) \prod_{j=1}^{p} m_0(\frac{\xi}{2^j}) \text{ and } \hat{\theta}_p^*(\xi) = 1_{[-2^p\pi, 2^p\pi]}(\xi) \prod_{j=1}^{p} m_0^*(\frac{\xi}{2^j})$$

we have $\theta_p \in L^2(\mathbb{R})$, $\theta_p^* \in L^2(\mathbb{R})$ and $\langle \theta_p(x) | \theta_p^*(x-k) \rangle = \delta_{k,0}$.

If moreover m_0 and m_0^* satisfy hypotheses jj) and jjj), then θ_p converges to φ and θ_p^* converges to φ in $L^2(\mathbb{R})$ as p goes to ∞, and thus $\langle \varphi(x) | \varphi^*(x-k) \rangle = \delta_{k,0}$.

Proof: We use again the Poisson summation formula to estimate $C(\omega, \Omega)(\xi) = \sum_{k \in \mathbb{Z}} \langle \omega(x) | \Omega(x-k) \rangle e^{-ik.\xi}$. Since $\hat{\theta}_p(\xi) = m_0(\xi/2)\hat{\theta}_{p-1}(\xi/2)$ and $\hat{\theta}_p^*(\xi) = m_0^*(\xi/2)\hat{\theta}_{p-1}^*(\xi/2)$, we find that the function $C_p(\xi) = C(\theta_p, \theta_p^*)(\xi)$ satisfies the following recursion formula :

$$C_p(\xi) = m_0(\frac{\xi}{2})\bar{m}_0^*(\frac{\xi}{2})C_{p-1}(\frac{\xi}{2}) + m_0(\frac{\xi}{2}+\pi)\bar{m}_0^*(\frac{\xi}{2}+\pi)C_{p-1}(\frac{\xi}{2}+\pi)$$

thus, by induction, $C_p = 1$ and we have $\langle \theta_p(x) | \theta_p^*(x-k) \rangle = \delta_{k,0}$.

If we assume that hypotheses jj) and jjj) are satisfied, then , we know that on $[-2^p\pi, 2^p\pi]$ we have $\hat{\varphi}(\xi) = \hat{\theta}_p(\xi)\hat{\varphi}(\xi/2^p)$ while on $[-\pi, \pi]$ the continuous function $\hat{\varphi}(\xi)$ does not vanish (since none of the terms $m_0(\xi/2^j)$, $1 \leq j$, vanishes and since the infinite product is convergent in $\mathbb{C}^*$); thus, the function $\hat{\theta}_p(\xi)$ satisfies $|\hat{\theta}_p(\xi)| \leq A|\hat{\varphi}(\xi)|$ with $A = \max_{|\eta|\leq\pi} \frac{1}{|\hat{\varphi}(\eta)|}$ and Lebesgue's dominated convergence theorem gives that θ_p converges to φ in L^2 as p goes to ∞. Similarly, θ_p^* converges to φ^* in L^2 as p goes to ∞ and this gives that $\langle\varphi(x)|\varphi^*(x-k)\rangle = \delta_{k,0}$. □

Lemmas 8.1 and 8.2 allow us to construct scaling functions φ, φ^*. In order to get wavelets ψ, ψ^*, we need to have a simple criterion to check the unconditionality of wavelet bases. This is performed through the vaguelettes lemma of Y. Meyer [MEY 92] (see also Kahane and Lemarié-Rieusset [KAHL 95]):

Proposition 8.1: (The vaguelettes lemma)

A) Let ω be a compactly supported square integrable function on $\mathbb{R}$ ($\omega \in L^2_{comp}(\mathbb{R})$). Then, there is a constant C_ω such that for all $(\lambda_k)_{k\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$, $\|\sum_{k\in\mathbb{Z}} \lambda_k\omega(x-k)\|_2^2 \leq C_\omega \sum_{k\in\mathbb{Z}} |\lambda_k|^2$.

B) Let ω, ω^* be compactly supported square integrable functions on $\mathbb{R}$ such that $\langle\omega(x)|\omega^*(x-k)\rangle = \delta_{k,0}$ ($= 0$ if $k \in \mathbb{Z}\backslash\{0\}$ and $= 1$ if $k = 0$). Then, there are two constants C_ω and D_{ω^*} so that

$$D_{\omega^*} \sum_{k\in\mathbb{Z}} |\lambda_k|^2 \leq \|\sum_{k\in\mathbb{Z}} \lambda_k\omega(x-k)\|_2^2 \leq C_\omega \sum_{k\in\mathbb{Z}} |\lambda_k|^2$$

C) [The vaguelettes lemma] Let ω be a compactly supported square integrable function on $\mathbb{R}$ so that $\omega \in H^1$ (i.e. $\omega' \in L^2$) and $\int \omega\, dx = 0$. Then, there exists a constant E_ω so that for all $(\lambda_{j,k})_{(j,k)\in\mathbb{Z}^2} \in \mathbb{C}^{\mathbb{Z}^2}$, we have $\|\sum_{(j,k)\in\mathbb{Z}^2} \lambda_{j,k}2^{j/2}\omega(2^jx-k)\|_2^2 \leq E_\omega \sum_{(j,k)\in\mathbb{Z}^2} |\lambda_{j,k}|^2$.

D) Let ω, ω^* be compactly supported square integrable functions on $\mathbb{R}$ such that $\langle 2^{j/2}\omega(2^jx-k)|2^{j'/2}\omega^*(2^{j'}x-k')\rangle = \delta_{j,j'}\delta_{k,k'}$. Assume that ω and ω^* belong to H^1 and satisfy $\int \omega\, dx = \int \omega^*\, dx = 0$. Then, there are two constants E_ω and F_{ω^*} so that

$$F_{\omega^*} \sum_{(j,k)\in\mathbb{Z}^2} |\lambda_{j,k}|^2 \leq \|\sum_{(j,k)\in\mathbb{Z}^2} \lambda_{j,k}2^{j/2}\omega(2^jx-k)\|_2^2 \leq E_\omega \sum_{(j,k)\in\mathbb{Z}^2} |\lambda_{j,k}|^2$$

Proof: (A) is obvious; we just use the Cauchy-Schwarz inequality and write $|\sum_{k\in\mathbb{Z}} \lambda_k\omega(x-k)| \leq \left(\sum_{k\in\mathbb{Z}} |\lambda_k|^2|\omega(x-k)|^2\right)^{1/2}\left(\sum_{k\in\mathbb{Z}} 1_{\text{Supp }\omega}(x-k)\right)^{1/2}$; thus, we find the result with $C_\omega = \|\omega\|_2^2\, \|\sum_{k\in\mathbb{Z}} 1_{\text{Supp }\omega}(x-k)\|_\infty$.

To prove (B), we write $\sum_{|k|\leq K} |\lambda_k|^2 = \langle\sum_{k\in\mathbb{Z}^d} \lambda_k\omega(x-k)|\sum_{|k|\leq K} \lambda_k\omega^*(x-k)\rangle$ and using (A), we get $\sum_{|k|\leq K} |\lambda_k|^2 \leq \|\sum_{k\in\mathbb{Z}^d} \lambda_k\omega(x-k)\|_2\|\sum_{|k|\leq K} \lambda_k\omega^*(x-k)\|_2 \leq \|\sum_{k\in\mathbb{Z}^d} \lambda_k\omega(x-k)\|_2 C_{\omega^*}^{1/2}\left(\sum_{|k|\leq K} |\lambda_k|^2\right)^{1/2}$, which gives (B) with $D_{\omega^*} = 1/C_{\omega^*}$.

To prove (C), it is enough to prove it in the case where all the $\lambda_{j,k}$'s but finitely many are equal to 0. Then, when we have obtained the estimate with a constant E_ω independent from the number of non vanishing coefficients, we may conclude that the whole series converges in L^2 and satisfies the required estimate. For a finite numbers of coefficients $\lambda_{j,k}$, we may write (defining $\omega_{j,k} = 2^{j/2}\omega(2^j x - k)$) $\| \sum_{(j,k)\in A} \lambda_{j,k}\omega_{j,k}\|_2^2 = \sum_{(j,k)\in A}\sum_{(j',k')\in A} \lambda_{j,k}\bar{\lambda}_{j',k'}\langle \omega_{j,k}|\omega_{j',k'}\rangle$ and then we integrate by parts: since $\omega \in L^2_{comp}$ and $\int \omega \, dx = 0$, we may find a function $\gamma \in L^2_{comp}$ so that $\omega = \frac{d}{dx}\gamma$, so (defining $\gamma_{j,k} = 2^{j/2}\gamma(2^j x - k)$ and $\beta_{j,k} = 2^{j/2}\frac{d\omega}{dx}(2^j x - k)$) we have $\langle \omega_{j,k}|\omega_{j',k'}\rangle = -2^{j'-j}\langle \beta_{j,k}|\gamma_{j',k'}\rangle$. Then, applying (A), we get for $j, j' \in \mathbb{Z}$:

$$|\sum_{(j,k)\in A}\sum_{(j',k')\in A} \lambda_{j,k}\bar{\lambda}_{j',k'}\langle \omega_{j,k}|\omega_{j',k'}\rangle| \le C(\omega)2^{-|j-j'|}\sqrt{\sum_{k\in\mathbb{Z}}|\lambda_{j,k}|^2}\sqrt{\sum_{k'\in\mathbb{Z}}|\lambda_{j',k'}|^2}$$

This is enough to prove assertion (C), since for any non negative sequence ϵ_j we have the inequality $\sum_{j\in\mathbb{Z}}\sum_{j'\in\mathbb{Z}} 2^{-|j-j'|}\epsilon_j\epsilon_{j'} \le 3\sum_{j\in\mathbb{Z}}\epsilon_j^2$.

(D) is then a direct consequence of (C), just as (B) is a direct consequence of (A). □

Remark: The vaguelettes lemma may be easily extended to the multivariate case: let ω be a compactly supported square integrable function on $\mathbb{R}^d$ so that $\omega \in H^1$ (i.e. $\frac{\partial}{\partial x_i}\omega \in L^2$ for $i = 1, \dots, d$) and $\int \omega \, dx = 0$; then, we have $\| \sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}^d} \lambda_{j,k}2^{jd/2}\omega(2^j x - k)\|_2^2 \le E_\omega \sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}^d}|\lambda_{j,k}|^2$. The proof is the same as for $d = 1$, except that we have to now use d functions for the integration by parts: since $\omega \in L^2_{comp}$ and $\int \omega \, dx = 0$, we may find d functions $\omega_i \in L^2_{comp}$ such that $\omega = \sum_{i=1}^d \frac{\partial}{\partial x_i}\omega_i$.

Recollecting the above results, we obtain the following theorems on the construction of wavelets :

Theorem 8.1: (Orthogonal wavelets)

Let m_0 be a trigonometric polynomial $m_0 = \sum_{k_0\le k\le k_1} a_k e^{-ik\xi}$ with real-valued coefficients so that:
i) $m_0(0) = 1$ and $|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1$.
ii) m_0 has no zero on $[-\pi/2, +\pi/2]$.

Define φ by $\hat{\varphi} = \prod_{j=1}^{\infty} m_0(\frac{\xi}{2^j})$. Then the function ψ defined by $\hat{\psi}(\xi) = e^{-i\xi/2}\bar{m}_0^*(\xi/2+\pi)\hat{\varphi}(\xi/2)$ generate an orthonormal wavelet basis $(\psi_{j,k})_{j\in\mathbb{Z},k\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ with associated scaling function φ.

The wavelet ψ and the scaling function φ have compact supports: Supp $\varphi \subset [k_0, k_1]$ and Supp $\psi \subset [(k_0 - k_1 + 1)/2, (k_1 - k_0 + 1)/2]$.

Theorem 8.2: (Bi-orthogonal wavelets)

Let m_0, m_0^* be trigonometric polynomials $m_0 = \sum_{k_0\le k\le k_1} a_k e^{-ik\xi}$ and $m_0^* = \sum_{l_0\le l\le l_1} \alpha_l e^{-il\xi}$ with real-valued coefficients so that:

i) $m_0(0) = m_0^*(0) = 1$ *and* $m_0(\xi)\bar{m}_0^*(\xi) + m_0(\xi+\pi)\bar{m}_0^*(\xi+\pi) = 1$.
ii) m_0 *and* m_0^* *have no zero on* $[-\pi/2, +\pi/2]$.
iii) the functions φ *and* φ^* *defined by* $\hat{\varphi} = \prod_{j=1}^{\infty} m_0(\frac{\xi}{2^j})$ *and* $\hat{\varphi}^* = \prod_{j=1}^{\infty} m_0^*(\frac{\xi}{2^j})$ *belong to* H^1.

Then the functions ψ *and* ψ^* *defined by*

$$\hat{\psi}(\xi) = e^{-i\xi/2}\bar{m}_0^*(\xi/2+\pi)\hat{\varphi}(\xi/2) \text{ and } \hat{\psi}^*(\xi) = e^{-i\xi/2}\bar{m}_0(\xi/2+\pi)\hat{\varphi}^*(\xi/2)$$

generate a wavelet basis $(\psi_{j,k})_{j\in\mathbb{Z},k\in\mathbb{Z}}$ *of* $L^2(\mathbb{R})$ *and its dual wavelet basis* $(\psi^*_{j,k})_{j\in\mathbb{Z},k\in\mathbb{Z}}$*, with associated scaling functions* φ, φ^*.

Moreover, the wavelets ψ, ψ^* *and the scaling functions* φ, φ^* *have compact supports; more precisely,* $\mathit{Supp}\,\varphi \subset [k_0, k_1]$, $\mathit{Supp}\,\varphi^* \subset [l_0, l_1]$, $\mathit{Supp}\,\psi \subset [(k_0-l_1+1)/2, (k_1-l_0+1)/2]$ *and* $\mathit{Supp}\,\psi^* \subset [(l_0-k_1+1)/2, (l_1-k_0+1)/2]$.

Proof:
Step 1: We first check that, under the hypotheses of Theorem 8.1 (orthonormal scaling filters), φ belongs to L^2. We already know (from the proof of Lemma 8.2) that θ_p belongs to L^2 and that $\|\theta_p\|_2 = 1$. Since θ_p is weakly convergent to φ (the Fourier transform of θ_p is bounded by 1 and converges pointwise to $\hat{\varphi}$, hence θ_p converges to φ in $\mathcal{S}'$), this gives that $\varphi \in L^2$. Thus, we may apply Lemma 8.2 and get that the family $(\varphi(x-k))_{k\in\mathbb{Z}}$ is orthonormal.
Step 2: Let V_j the closed linear span of the family $(2^{j/2}\varphi(2^jx-k))_{k\in\mathbb{Z}}$. Then we are going to show that:
k) $V_j \subset V_{j+1}$
kk) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$ and $\cup_{j\in\mathbb{Z}} V_j$ is dense in L^2
kkk) $f \in V_j$ if and only if $f(2x) \in V_{j+1}$

Indeed, kkk) is obvious and k) is a direct consequence of the equality $\varphi(x) = \sum a_k 2\varphi(2x-k)$. If $V = \cup_{j\in\mathbb{Z}} V_j$, we find that its closure is invariant under any dyadic translation, hence under any translation. A classic result of harmonic analysis states that there exists a Borel set E so that $f \in \bar{V}$ if and only if Supp $\hat{f} \subset E$. Since $\hat{\varphi}$ is analytic, E must be equal to $\mathbb{R}$.

Now, point B) of Proposition 8.1 proves that there exists a positive constant A such that for all $f \in V_0$ we have $\|f\|_2^2 \le A\sum_{k\in\mathbb{Z}} |\langle f|\varphi^*(x-k)\rangle|^2$. Thus, if $f \in \cap_{j\in\mathbb{Z}} V_j$, $\|f\|_2^2 \le A \min_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\langle f|2^{j/2}\varphi^*(2^jx-k)\rangle|^2$. But the operators U_j defined by $U_j g = \sum_{k\in\mathbb{Z}} |\langle g|2^{j/2}\varphi^*(2^jx-k)\rangle|^2$ are equicontinuous on L^2 and it is easy to check that for $g \in L^2_{comp}$ we have $\lim_{j\to-\infty} U_j g = 0$ (indeed, for such a g, there exists a $K \in \mathbb{N}$ so that for all $j \le 0$ we have $U_j g = \sum_{|k|\le K} |\langle g|2^{j/2}\varphi^*(2^jx-k)\rangle|^2$, while it is obvious that for each fixed $k \in \mathbb{Z}$ we have $\lim_{j\to-\infty}\langle g|2^{j/2}\varphi^*(2^jx-k)\rangle = 0$).
Step 3: In this step, we are looking for a basis for $W_0 = V_1 \cap (V_0^*)^\perp$. f belongs to V_1 if and only if f may be decomposed as $f = \sum_{k\in\mathbb{Z}} \lambda_k \varphi_{1,k}$ for some sequence $(\lambda_k) \in l^2(\mathbb{Z})$. Equivalently, f belongs to V_1 if and only $\hat{f} = \mu(\xi/2)\hat{\varphi}(\xi/2)$ for some 2π–periodical locally square-integrable function μ. On the other hand, f

belongs to $(V_0^*)^\perp$ if and only $\langle f|\varphi^*(x-k)\rangle = 0$ for all $k \in \mathbb{Z}$, or equivalently (using the Poisson summation formula for $F(x) = \langle f(y)|\varphi^*(y-x)\rangle$) if and only if $\sum_{k\in\mathbb{Z}} \bar{\hat{f}}(\xi+2k\pi)\hat{\varphi}^*(\xi+2k\pi) = 0$ almost everywhere. Writing $\hat{f} = \mu(\xi/2)\hat{\varphi}(\xi/2)$ and $\hat{\varphi}^* = m_0^*(\xi/2)\hat{\varphi}^*(\xi/2)$, we get $\mu(\xi/2)\bar{m}_0^*(\xi/2)+\mu(\xi/2+\pi)\bar{m}_0^*(\xi/2+\pi) = 0$. We then write $\mu(\xi) = \mu(\xi)\big(m_0(\xi)\bar{m}_0^*(\xi) + m_0(\xi+\pi)\bar{m}_0^*(\xi+\pi)\big)$ and get $\mu(\xi) = \bar{m}_0^*(\xi+\pi)(-\mu(\xi+\pi)m_0(\xi) + \mu(\xi)m_0(\xi+\pi)) = e^{-i\xi}\bar{m}_0^*(\xi+\pi)A(2\xi)$ for some 2π-periodical function A. This gives that W_0 is the closed linear span of the functions $\psi(x-k)$, and a similar proof gives that $W_0^* = V_1^* \cap (V_0)^\perp$ is the closed linear span of the functions $\psi^*(x-k)$ and that $\langle\psi(x)|\psi^*(x-k)\rangle = \delta_{k,0}$.

Step 4: We may now easily conclude. We know that the family $(\psi_{j,k})$ is orthonormal (under the hypotheses of Theorem 8.1) or that the family $(\psi_{j,k})$ is the Riesz basis of some closed subspace of L^2 (from point D) of Proposition 8.1). We need to now show that the closed linear span of the family $(\psi_{j,k})$ is the whole of L^2. We write P_j for the projection operator with range V_j and kernel $(V_j^*)^\perp$ and Q_j for the projection operator with range W_j and kernel $(W_j^*)^\perp$, where W_j is the closed linear span of the family $(\psi_{j,k})_{k\in\mathbb{Z}}$; we have $P_{j_1} - P_{j_0} = \bigoplus_{j_0\le j\le j_1-1} Q_j$. For all $f \in L^2$, $\lim_{j\to+\infty} \|f - P_j f\|_2 = 0$ (by equicontinuity of the family P_j and density of $\cup_{j\in\mathbb{Z}} V_j$). Moreover, since $\|P_j f\|_2$ is equivalent to $\sqrt{U_j f}$ (where U_j is defined in Step 2), we get that, for all $f \in L^2$, $\lim_{j\to-\infty} \|P_j f\|_2 = 0$. $\square$

2. Daubechies wavelets

We now prove the existence of scaling functions and of wavelets with high regularity.

Theorem 8.3: (Daubechies wavelets [DAU 92])

For $N \in \mathbb{N}^*$, let $m_0(\xi)$ be the trigonometric polynomial defined by :
i) the degree of m_0 is equal to $2N-1$
ii) $m_0(\xi) = \sum_{0\le k\le 2N-1} a_k e^{-ik\xi}$ with $a_k \in \mathbb{R}$
iii) $m_0(0) = 1$
iv) For $0 \le k \le N-1$, $\frac{d^k}{d\xi^k} m_0(\pi) = 0$
v) $|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1$.

Then, the function $\hat{\varphi}(\xi) = \prod_{j=1}^\infty m_0(\frac{\xi}{2^j})$ is the Fourier transform of an orthonormal scaling function φ. Moreover, φ is compactly supported with support $[0, 2N-1]$ and, for $N \ge 2$, φ is continuous. More precisely, φ belongs to C^{α_N} with $\alpha_N = \frac{\ln(4/3)}{2\ln 2} N + o(N)$ as N goes to ∞.

Proof: Due to hypothesis iv), we may factorize $(1+e^{-i\xi})^N$ in m_0 and thus get $|m_0(\xi)|^2 = (\frac{1+\cos\xi}{2})^N P(\cos\xi)$ for some polynomial P. Hypothesis v) then gives that $(\frac{1+X}{2})^N P(X) + (\frac{1-X}{2})^N P(-X) = 1$. The Bezout identity grants that there exists one and only one solution P with a degree less or equal to $N-1$. We write $1 = (\frac{1+X}{2} + \frac{1-X}{2})^{2N-1}$ and find $P(X) = \sum_{k=0}^{N-1} \begin{pmatrix} N+k \\ 2N-1 \end{pmatrix} (\frac{1+X}{2})^k (\frac{1-X}{2})^{N-1-k}$.

Another way to find P is to define $Y = (\frac{1-X}{2})$ and to write $P(1-2Y) = \frac{1}{(1-Y)^N} - Y^N \frac{P(2Y-1)}{(1-Y)^N}$; then a Taylor expansion gives that $P(X) = P(1-2Y) = \sum_{k=0}^{N-1} \binom{N+k-1}{N-1} Y^k$.

The polynomial P is clearly non negative on $[-1,1]$. Then the Riesz lemma states that $P(\cos\xi)$ is the square modulus $|Q(\xi)|^2$ of a trigonometric polynomial $Q(\xi) = \sum_{k=0}^{degP} \alpha_k e^{-ik\xi}$ with real-valued coefficients α_k. (Q is not uniquely defined. If we require that $Q(0) = 1$ and that all the roots of $\sum_{k=0}^{degP} \alpha_k z^k = 0$ have a modulus less or equal to 1, then Q is uniquely defined.) We choose Q with $Q(0) = 1$ and then define $m_0(\xi) = (1+e^{-i\xi})^N Q(\xi)$.

Since m_0 does not vanish on $(-\pi,\pi)$, we may apply Theorem 8.1 and check that m_0 is the scaling filter associated to a Hilbertian basis of compactly supported wavelets. We shall now estimate the regularity of the scaling function φ. We write $m_0(\xi) = (\frac{1+e^{-i\xi}}{2})^N Q(\xi)$, and thus $\hat\varphi = (\frac{1-e^{-i\xi}}{i\xi})^N \prod_{j=1}^\infty Q(\xi/2^j)$. For $2^K \leq |\xi| < 2^{K+1}$, we write

$$|\hat\varphi(\xi)| \leq (\frac{2}{|\xi|})^N \|Q\|_\infty^K \sup_{|\eta|\leq 2\pi} |\prod_{j=1}^\infty Q(\frac{\eta}{2^j})|$$

and get, for $|\xi| \geq 1$, that $|\hat\varphi(\xi)| \leq C|\xi|^{-N+\frac{\log\|Q\|_\infty}{\log 2}}$. But $\|Q\|_\infty = \sqrt{P(-1)} = \sqrt{\binom{N}{2N-1}}$; for $N \geq 2$, we have $2\binom{N}{2N-1} = \binom{N}{2N-1} + \binom{N-1}{2N-1} < (1+1)^{2N-1}$, thus we find that $-N + \frac{\log\|Q\|_\infty}{\log 2} < -1$ so that $\hat\varphi$ is integrable.

In order to get better information on the regularity of φ, we notice that for a compactly supported function φ, the regularity exponent $\alpha = \max\{\beta \;/\; \varphi \in C^\beta\}$ and the decay exponent for its Fourier transform $\gamma = \max\{\beta \;/\; |\xi|^\beta\hat\varphi \in L^\infty\}$ are related by: $\gamma - 1 \leq \alpha \leq \gamma$. Thus, we shall try to estimate γ. Since m_0 does not vanish on $(-\pi,\pi)$, we find that $\hat\varphi(2\pi/3) \neq 0$ and that $|\hat\varphi(2^L\, 2\pi/3)| = \left(\frac{|Q(2\pi/3)|}{2^N}\right)^L |\hat\varphi(2\pi/3)|$, hence $\gamma \leq N - \frac{\log|Q(2\pi/3)|}{\log 2}$. Moreover, for $2^K \leq |\xi| < 2^{K+1}$, we write

$$|\hat\varphi(\xi)| \leq (\frac{2}{|\xi|})^N |\prod_{j=1}^K Q(\frac{\xi}{2^j})| \sup_{|\eta|\leq 2\pi} |\prod_{j=1}^\infty Q(\frac{\eta}{2^j})|$$

and get, writing $\mu = \limsup_{K\to\infty} \|\prod_{j=1}^K Q(\frac{\xi}{2^j})\|_\infty^{1/K}$, $\gamma \geq N - \frac{\log\mu}{\log 2}$. We shall now prove that $\mu \leq 3|Q(2\pi/3)|$; as a matter of fact, a more precise result of Cohen and Conze states that $\mu = |Q(2\pi/3)|$. We introduce $\nu = \max(\sup_{-1/2\leq X\leq 1} P(X), \sup_{-1\leq X\leq -1/2} \sqrt{P(X)P(2X^2-1)})$. We get, estimating $P(\cos(\frac{\xi}{2^j}))$ alone if $\cos(\frac{\xi}{2^j}) > -1/2$ or estimating $P(\cos(\frac{\xi}{2^j}))P(\cos(\frac{2\xi}{2^j}))$ in the other case, that $|\prod_{j=1}^K Q(\frac{\xi}{2^j})|^2 \leq P(-1)\nu^{K-1}$ and thus $\mu \leq \sqrt{\nu}$. We write $|Q(2\pi/3)|^2 = P(-1/2) \geq \binom{2N-2}{N-1} (\frac{3}{4})^{N-1}$ (which is the last term of

P expressed as the Taylor expansion of $\frac{1}{1-(\frac{1-X}{2})^N}$). On the other hand, P is decreasing on $[-1,1]$ and thus $\sup_{-1/2\leq X\leq 1} P(X) = P(-1/2)$. In order to estimate $\sqrt{P(X)P(2X^2-1)}$ for $-1 \leq X \leq -1/2$, we write for $X < 0$

$$P(X) = \sum_{k=0}^{N-1} \binom{N+k-1}{N-1} (\frac{1-X}{2})^k \leq \binom{2N-2}{N-1} \sum_{k=0}^{N-1} (\frac{1-X}{2})^k$$

and thus get $P(X) \leq \binom{2N-2}{N-1} (\frac{1-X}{2})^{N-1}(\frac{1-X}{-X})$. If $X \leq -1/8$, this gives more simply $P(X) \leq 9\binom{2N-2}{N-1}(\frac{1-X}{2})^{N-1}$. We then consider two domains for X:

- $-1 \leq X \leq -\frac{\sqrt{7}}{4}$: $2X^2-1 \geq -1/8$ so that

$$P(X)P(2X^2-1) \leq P(-1)P(-1/8) \leq 81 \binom{2N-2}{N-1}^2 1^{N-1}(\frac{9}{16})^{N-1}$$

- $-\frac{\sqrt{7}}{4} < X \leq -1/2$: in that case, X and $2X^2-1$ are less than $-1/8$, so that we have $P(X)P(2X^2-1) \leq 81\binom{2N-2}{N-1}^2 (\frac{1-X}{2})^{N-1}(1-X^2)^{N-1}$. Since $x \mapsto (1-x)(1-x^2)$ increases on $[-\frac{\sqrt{7}}{4}, -1/2]$, we obtain

$$P(X)P(2X^2-1) \leq 81 \binom{2N-2}{N-1}^2 (\frac{3}{4})^{N-1}(\frac{3}{4})^{N-1}$$

Since $\binom{2N-2}{N-1}(\frac{3}{4})^{N-1} \leq |Q(2\pi/3)|^2$, this gives $\mu \leq 3|Q(2\pi/3)|$. This finishes the proof, since we have proved that $\binom{2N-2}{N-1}(\frac{3}{4})^{N-1} \leq |Q(2\pi/3)|^2 \leq 9\binom{2N-2}{N-1}(\frac{3}{4})^{N-1}$. $\square$

Another useful way of constructing wavelets is integration and differentiation:

Proposition 8.2: (Differentiation of wavelets)

Let (V_j), (V_j^*) be a bi-orthogonal multi-resolution analysis of $L^2(\mathbb{R})$ with compactly supported dual scaling functions φ, φ^* and associated scaling filters m_0, m_0^*. Assume that φ belongs to the Sobolev space H^2. Then:

i) The derivative φ' of φ can be written as $\varphi'(x) = \tilde{\varphi}(x) - \tilde{\varphi}(x-1)$ where $\tilde{\varphi}$ is a compactly supported scaling function associated to the scaling filter $\tilde{m}_0(\xi) = \frac{2}{1+e^{-i\xi}} m_0(\xi)$.

ii) The primitive $\int_{-\infty}^x \varphi^*(t)\,dt$ *of* φ^* *satisfies* $\int_x^{x+1} \varphi^*(t)\,dt = \tilde\varphi^*(x)$ *where* $\tilde\varphi$ *is a compactly supported scaling function with associated scaling filter* $\tilde m_0^*(\xi) = \frac{1+e^{i\xi}}{2} m_0^*(\xi)$.

iii) $\tilde\varphi$ *and* $\tilde\varphi^*$ *are compactly supported dual scaling functions for a bi-orthogonal multiresolution analysis* $(\tilde V_j)$, $(\tilde V_j^*)$. *Moreover, the projection operators* P_j *onto* V_j *in the direction of* $(V_j^*)^\perp$ *and* $\tilde P_j$ *onto* $\tilde V_j$ *in the direction of* $(\tilde V_j^*)^\perp$ *satisfy:*

$$\frac{d}{dx} \circ P_j = \tilde P_j \circ \frac{d}{dx}. \tag{8.2}$$

iv) The bi–orthogonal wavelets ψ *and* ψ^* *associated with* φ, φ^* *and the wavelets* $\tilde\psi$ *and* $\tilde\psi^*$ *associated with* $\tilde\varphi$, $\tilde\varphi^*$ *satisfy:*

$$\tilde\psi = \frac{1}{4}\frac{d}{dx}\psi \text{ and } \tilde\psi^* = -4\int_{-\infty}^x \psi^*(t)\,dt. \tag{8.3}$$

Proof: We easily prove this proposition. First, we prove

$$m_0(\pi) = m_0^*(\pi) = 0.$$

If g belongs to $H^1(\mathbb{R})$ then $\sum_{k\in\mathbb{Z}} |g(\xi+2k\pi)|^2$ converges uniformly on $[-\pi,\pi]$: we write $H^1 \subset \mathcal{C}_0$, hence, using a smooth function ω equal to 1 on $[-\pi,\pi]$ and supported in $[-2\pi,2\pi]$, we find the inequality $\|g\|_{L^\infty([(2k-1)\pi,(2k+1)\pi])} \le C\|g(x)\omega(x-2k\pi)\|_{H^1} \le C'\|g\|_{L^2([(2k-2)\pi,(2k+2)\pi])} + \|g'\|_{L^2([(2k-2)\pi,(2k+2)\pi])})$. We apply this estimate to the Fourier transforms $\hat\varphi$, $\hat\varphi^*$ of the compactly supported square–integrable dual scaling function φ, φ^*. We find that the sum $\sum_{k\in\mathbb{Z}} \hat\varphi(\xi+2k\pi)\bar{\hat\varphi}^*(\xi+2k\pi)$ is equal to 1 for every $\xi \in \mathbb{R}$ (we already know by the Poisson sumation formula and the bi–orthogonality of the scaling functions that the sum is equal to 1 almost everywhere; the uniform convergence of the series gives us that the sum is a continuous function, hence that it must be identically equal to 1). Hence, we find that there exists a positive constant γ such that for every $\xi \in \mathbb{R}$ we have $\sum_{k\in\mathbb{Z}} |\hat\varphi(\xi+2k\pi)|^2 \ge \gamma$ and $\sum_{k\in\mathbb{Z}} |\hat\varphi^*(\xi+2k\pi)|^2 \ge \gamma$. We then write for $A = \sum_{k\in\mathbb{Z}} |\hat\varphi(4k\pi)|^2$, $B = \sum_{k\in\mathbb{Z}} |\hat\varphi(2\pi+4k\pi)|^2$ and $C = \sum_{k\in\mathbb{Z}} |\hat\varphi(\pi+2k\pi)|^2$ the equalities $A = |m_0(0)|^2(A+B) = A+B$, hence $0 = B = |m_0(\pi)|^2 C$, and (since $C \ne 0$) $m_0(\pi) = 0$. We obtain $m_0^*(\pi) = 0$ by the same way.

We now prove the proposition. Using $m_0(\pi) = 0$, we obtain that, for any $L \in \mathbb{Z}$ with $L \ne 0$, $\hat\varphi(2L\pi) = 0$: write $L = 2^N(2k+1)$ and $\hat\varphi(2^N(2k+1)\pi) = m_0(0)^{N-1} m_0(\pi)\hat\varphi((2k+1)\pi) = 0$. Then,using the Poisson summation formula, we get

$$\sum_{k\in\mathbb{Z}} \varphi(x-k) = \sum_{k\in\mathbb{Z}} \hat\varphi(2k\pi)e^{2ik\pi x} = 1.$$

Thus, $\sum_{k\in\mathbb{Z}}\varphi'(x-k)=0$. We then define $\tilde\varphi(x)=\sum_{k\geq 0}\varphi'(x-k)$; we have of course $\varphi'(x)=\tilde\varphi(x)-\tilde\varphi(x-1)$. Moreover, $\tilde\varphi$ is clearly equal to 0 on $x<\min\mathrm{Supp}\varphi$ and (since $\tilde\varphi(x)=-\sum_{k<0}\varphi'(x-k)$ as well) on $x>\max\mathrm{Supp}\varphi-1$. Since this locally finite sum of functions belonging to H^1 belongs locally to H^1 and is compactly supported, we find that $\tilde\varphi$ belongs to H^1. Moreover, we have $i\xi\hat\varphi(\xi)=(1-e^{-i\xi})\hat{\tilde\varphi}(\xi)$, which gives

$$\hat{\tilde\varphi}(2\xi)=\frac{2i\xi}{i\xi}\frac{1-e^{-i\xi}}{1-e^{-2i\xi}}m_0(\xi)\hat{\tilde\varphi}(\xi)=\frac{2}{1+e^{-i\xi}}m_0(\xi)\hat{\tilde\varphi}(\xi).$$

Similarly, $\tilde\varphi^*(x)=\int_x^{x+1}\varphi^*(t)\,dt$ is a compactly supported function (with support contained in $[\min\mathrm{Supp}\varphi^*-1,\max\mathrm{Supp}\varphi^*])$ which belongs to H^1 and we have $i\xi\hat{\tilde\varphi}^*(\xi)=(e^{i\xi}-1)\hat{\varphi}^*(\xi)$, giving $\hat{\tilde\varphi}^*(2\xi)=\frac{1+e^{i\xi}}{2}m_0^*(\xi)\hat{\tilde\varphi}^*(\xi)$. Moreover, we clearly have $\hat{\tilde\varphi}^*\bar{\hat{\tilde\varphi}}=\hat{\varphi}^*\bar{\hat\varphi}$, hence $\sum_{k\in\mathbb{Z}}\hat{\tilde\varphi}^*(\xi+2k\pi)\bar{\hat{\tilde\varphi}}(\xi+2k\pi)=\sum_{k\in\mathbb{Z}}\hat{\varphi}^*(\xi+2k\pi)\bar{\hat\varphi}(\xi+2k\pi)=1$. Thus, $\tilde\varphi$ and $\tilde\varphi^*$ are dual scaling functions.

We still have to prove the commutation formula $\frac{d}{dx}\circ P_j=\tilde P_j\circ\frac{d}{dx}$. It is enough to prove it for $j=0$ (by homogeneity of the differentiation operator). We compute $\frac{d}{dx}\circ P_0 f$ by an Abel transformation of the series

$$\sum_{k\in\mathbb{Z}}\langle f|\varphi^*(x-k)\rangle(\tilde\varphi(x-k)-\tilde\varphi(x-k-1))=\sum_{k\in\mathbb{Z}}\langle f|(\varphi^*(x-k)-\varphi^*(x-k+1)\rangle\tilde\varphi(x-k)$$

then we integrate by parts

$$\langle f|(\varphi^*(x-k)-\varphi^*(x-k+1)\rangle=-\langle f|\frac{d}{dx}\tilde\varphi^*(x-k)\rangle=\langle\frac{d}{dx}f|\tilde\varphi^*(x-k)\rangle.$$

□

3. Multivariate wavelets

The quickest way to construct a wavelet basis for $L^2(\mathbb{R}^d)$ is to start from d univariate multiresolution analysis and to construct by tensorization a d-variate multiresolution analysis :

Proposition 8.3: (Separable wavelets)

Let $(V_j^{(i)})_{j\in\mathbb{Z}}$, $(V_j^{*(i)})_{j\in\mathbb{Z}}$ $(1\leq i\leq d)$ be d bi-orthogonal multiresolution analyses of $L^2(\mathbb{R})$ with associated scaling functions $\varphi^{(i)}$, $\varphi^{*(i)}$ and associated wavelets $\psi^{(i)}$, $\psi^{*(i)}$. Then:

i) The space $V_j=V_j^{(1)}\hat\otimes\ldots\hat\otimes V_j^{(d)}$ (the closure of $V_j^{(1)}\otimes\ldots\otimes V_j^{(d)}$ in $L^2(\mathbb{R}^d)$) defines a multi-resolution analysis:

a) $V_j\subset V_{j+1}$

b) $\cap_{j\in\mathbb{Z}}V_j=\{0\}$ and $\cup_{j\in\mathbb{Z}}V_j$ is dense in L^2

c) $f\in V_j$ if and only if $f(2x)\in V_{j+1}$

d) V_0 has a shift-invariant Riesz basis $(\varphi(x-k)_{k\in\mathbb{Z}^d})$

with $\varphi(x) = \varphi^{(1)} \otimes \ldots \otimes \varphi^{(d)}(x) = \varphi^{(1)}(x_1) \ldots \varphi^{(d)}(x_d)$.
ii) Similarly, the space $V_j^* = V_j^{*(1)} \hat{\otimes} \ldots \hat{\otimes} V_j^{*(d)}$ *defines a multi-resolution analysis and* V_0^* *has a shift-invariant Riesz basis* $(\varphi^*(x-k)_{k \in \mathbb{Z}^d})$ *with* $\varphi^* = \varphi^{*(1)} \otimes \ldots \otimes \varphi^{*(d)}$.
iii) $\langle \varphi(x-k) | \varphi^*(x-k') \rangle = \delta_{k,k'}$.
iv) $W_j = V_{j+1} \cap (V_j^*)^\perp$ *has as Riesz basis* $(\psi_{\epsilon,j,k})_{1 \le \epsilon \le 2^d-1, k \in \mathbb{Z}^d}$, *where* $\psi_{\epsilon,j,k}$ *is defined by* $\psi_{\epsilon,j,k}(x) = 2^{jd/2}\psi_\epsilon(2^j x - k)$ *and* ψ_ϵ *by:* $\varphi^{(i),0} = \varphi^{(i)}$, $\varphi^{(i),1} = \psi^{(i)}$ *and, for* $\epsilon = \sum_{i=0}^d \epsilon_i 2^{i-1}$ *the dyadic decomposition of* ϵ, $\psi_\epsilon = \varphi^{(1),\epsilon_1} \otimes \ldots \otimes \varphi^{(d),\epsilon_d}$.
v) Similarly, $W_j^* = V_{j+1}^* \cap (V_j)^\perp$ *has as Riesz basis* $(\psi^*_{\epsilon,j,k})_{1 \le \epsilon \le 2^d-1, k \in \mathbb{Z}^d}$, *where* $\psi^*_{\epsilon,j,k}(x) = 2^{jd/2}\psi^*_\epsilon(2^j x - k)$ *and* $\psi^*_\epsilon = \varphi^{*(1),\epsilon_1} \otimes \ldots \otimes \varphi^{*(d),\epsilon_d}$
vi) The family $(\psi_{\epsilon,j,k})_{1 \le \epsilon \le 2^d-1, j \in \mathbb{Z}, k \in \mathbb{Z}^d}$ *is a wavelet basis of* $L^2(\mathbb{R}^d)$ *with dual basis* $(\psi^*_{\epsilon,j,k})_{1 \le \epsilon \le 2^d-1, j \in \mathbb{Z}, k \in \mathbb{Z}^d}$.

Proof: This is obvious. □

Chapter 9

Wavelets and functional spaces

We may now begin the description of functional spaces through wavelet coefficients. We do not attempt to give optimal statements (see Kahane and Lemarié-Rieusset [KAHL 95] for the case of univariate wavelets), since wavelets are only a tool to prove some useful estimates. We consider very optimistic hypotheses, wasting a lot of regularity in our assumptions, but this is harmless since we have shown that there exists compactly supported wavelets with arbitrarily high regularity. Thus, we shall only consider *N-regular wavelets*:

Definition 9.1: (N-regular wavelet bases)

A wavelet basis of $L^2(\mathbb{R}^d)$ $(\psi_{\epsilon,j,k})_{1\le\epsilon\le 2^d-1, j\in\mathbb{Z}, k\in\mathbb{Z}^d}$ is N-regular ($N \geq 1$) when it satisfies that:
i) this is a separable wavelet basis
ii) the wavelets ψ_ϵ and the dual wavelets ψ^*_ϵ are compactly supported functions ;
iii) they are associated to compactly supported scaling functions φ, φ^* ;
iv) the dual scaling function φ^* is continuously differentiable ;
v) the scaling function φ is N times continuously differentiable.

1. Lebesgue spaces

A well-known property of wavelet bases is that they provide Riesz bases to the Lebesgue spaces $L^p(\mathbb{R}^d)$:

Theorem 9.1: (Lebesgue spaces)

Let $(\psi_{\epsilon,j,k})_{1\le\epsilon\le 2^d-1, j\in\mathbb{Z}, k\in\mathbb{Z}^d}$ be a 1-regular wavelet basis (with associated dual wavelet ψ^* and associated scaling functions φ, φ^*). Then:
i) for $1 < p < \infty$, the family $(\psi_{\epsilon,j,k})_{1\le\epsilon\le 2^d-1, j\in\mathbb{Z}, k\in\mathbb{Z}^d}$ is a Riesz basis of $L^p(\mathbb{R}^d)$; more precisely, there exists two positive constants $0 < A_p \leq B_p < \infty$ so that for all $f \in L^p$ we have, writing $\chi(x) = 1_{[0,1]^d}(x)$,:

$$A_p\|f\|_p \leq \|\big(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{1\le\epsilon\le 2^d-1} 2^{jd}|\langle f|\psi^*_{\epsilon,j,k}\rangle|^2\chi(2^jx-k)\big)^{1/2}\|_p \leq B_p\|f\|_p$$

ii) for $1 < p < \infty$, the family $(\varphi(x-k))_{k\in\mathbb{Z}^d} \cup (\psi_{\epsilon,j,k})_{1\le\epsilon\le 2^d-1, j\in\mathbb{N}, k\in\mathbb{Z}^d}$ is a Riesz basis of $L^p(\mathbb{R}^d)$; more precisely, there exists two positive constants

$0 < A_p \leq B_p < \infty$ so that for all $f \in L^p$ we have, writing $\chi(x) = 1_{[0,1]^d}(x)$, $A_p\|f\|_p \leq N_p(f) \leq B_p\|f\|_p$ where

$$N_p(f) = \Big(\sum_{k\in\mathbb{Z}^d} |\langle f|\varphi^*_{0,k}\rangle|^p\Big)^{1/p} + \|\Big(\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}^d}\sum_{1\leq\epsilon\leq 2^d-1} 2^{jd}|\langle f|\psi^*_{\epsilon,j,k}\rangle|^2\chi(2^jx-k)\Big)^{1/2}\|_p$$

Proof: We choose a function $\omega \in \mathcal{D}(\mathbb{R}^d)$ equal to 1 on $[0,1]^d$, we write $\omega_{j,k}(x) = 2^{jd/2}\omega(2^jx-k)$, and we introduce the operator

$$f \mapsto A(f) = \Big(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{1\leq\epsilon\leq 2^d-1} |\langle f|\psi^*_{\epsilon,j,k}\rangle|^2\omega^2_{j,k}\Big)^{1/2}$$

which we associate to the vector-valued kernel

$$K(x,y) = (\bar{\psi}^*_{\epsilon,j,k}(y)\omega_{j,k}(x))_{j\in\mathbb{Z},k\in\mathbb{Z}^d,1\leq\epsilon\leq 2^d-1}$$

and the operator

$$f \mapsto Bf(x) = \int K(x,y)\ f(y)\ dy.$$

The vaguelettes lemma gives us that B is bounded from $L^2(\mathbb{R}^d)$ to $L^2_x l^2_{j,k,\epsilon}$, since $\|Bf\|^2_{L^2l^2} = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{1\leq\epsilon\leq 2^d-1}|\langle f|\psi^*_{\epsilon,j,k}\rangle|^2\|\omega\|_2^2$. Moreover, B is a vector-valued Calderón–Zygmund operator:

$$\|K(x,y)\|_{l^2} + \|x-y\|\ \|\vec{\nabla}_x K(x,y)\|_{l^2} + \|x-y\|\ \|\vec{\nabla}_y K(x,y)\|_{l^2} \leq C\|x-y\|^{-d}$$

and we may thus extrapolate its L^2, L^2l^2 boundedness to the L^p, L^pl^2 boundedness for $1 < p < \infty$.

Thus, we have $\|Af\|_p = \|Bf\|_{L^pl^2} \leq B_p\|f\|_p$ for $1 < p < \infty$. Since $\chi \leq \omega^2$, this gives that for $1 < p < \infty$

$$\|\Big(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{1\leq\epsilon\leq 2^d-1} 2^{jd}|\langle f|\psi^*_{\epsilon,j,k}\rangle|^2\chi(2^jx-k)\Big)^{1/2}\|_p \leq B_p\|f\|_p$$

We have, of course, a similar estimate with the dual system: for $1 < q < \infty$

$$\|\Big(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{1\leq\epsilon\leq 2^d-1} 2^{jd}|\langle f|\psi_{\epsilon,j,k}\rangle|^2\chi(2^jx-k)\Big)^{1/2}\|_q \leq B^*_q\|f\|_q$$

Now, for $f \in L^p \cap L^2$ and $g \in L^q \cap L^2$ $(1/p + 1/q = 1)$, we have

$$\begin{aligned}\int f\bar{g}\ dx &= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{1\leq\epsilon\leq 2^d-1}\langle f|\psi^*_{\epsilon,j,k}\rangle\ \langle\psi_{\epsilon,j,k}|g\rangle \\ &= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{1\leq\epsilon\leq 2^d-1}\langle f|\psi^*_{\epsilon,j,k}\rangle\ \langle\psi_{\epsilon,j,k}|g\rangle\int 2^{jd}\chi(2^jx-k)\ dx \\ &= \int\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{1\leq\epsilon\leq 2^d-1}\langle f|\psi^*_{\epsilon,j,k}\rangle\ \chi_{j,k}(x)\ \langle\psi_{\epsilon,j,k}|g\rangle\chi_{j,k}(x)\ dx\end{aligned}$$

with $\chi_{j,k}(x) = 2^{jd/2}\chi(2^j x - k)$. We then use the Cauchy–Schwarz inequality in $l^2_{j,k,\epsilon}$ and the Hölder inequality for the spaces $L^p(dx), L^q(dx)$ to get that $|\int f\bar{g}\ dx| \leq \|A(f)\|_p\|A(g)\|_q \leq \|A(f)\|_p B^*_q\|g\|_q$. We thus get that for $f \in L^p \cap L^2$ we have $\|f\|_p \leq B^*_q\|A(f)\|_p$. This is easily extended to the whole space L^p: if $f \in L^p$ we choose $f_\epsilon \in L^p \cap L^2$ with $\|f - f_\epsilon\|_p < \epsilon$ and we write: $\|f\|_p \leq \|f_\epsilon\|_p + \epsilon \leq B^*_q\|A(f_\epsilon)\|_p + \epsilon \leq B^*_q\|A(f)\|_p + B^*_q\|A(f - f_\epsilon)\|_p + \epsilon \leq B^*_q\|A(f)\|_p + (B_pB^*_q + 1)\epsilon$.

Thus, (A) is proved. The proof of (B) is exactly the same. □

2. Besov spaces

In order to analyze spaces of regular functions, we shall need a lemma on regular wavelets:

Lemma 9.1: (Oscillation of wavelets)

Let $(\psi_{j,k})_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$ be a N-regular wavelet basis of $L^2(\mathbb{R})$ (with associated dual wavelet ψ^* and associated scaling functions φ, φ^*). Then, for $0 \leq l \leq N$, we have $\int x^l\psi^*(x)\ dx = 0$.

In particular, there exists a compactly supported function Ψ^* so that $\psi^* = \frac{d^N}{dx^N}\Psi^*$.

Proof: Let L be the smallest integer so that $\int x^L\psi^*(x)\ dx \neq 0$ and let us assume that $L \leq N$. Let $x_0 \in \mathbb{R}$ so that $\varphi^{(L)}(x_0) \neq 0$, and for $j \geq 0$, let q the biggest integer smaller than $2^j x_0$; we have $0 = \int \psi^*_{j,q}\ \varphi\ dx = 2^{-j/2}\int \psi^*(y)\ \varphi(\frac{k+y}{2^j})\ dy$. But we easily check that $\lim_{j\to\infty} 2^{j(1/2-L)}\int \psi^*_{j,q}\ \varphi\ dx = (\int x^L\psi^*(x)\ dx)\ \frac{\varphi^{(L)}(x_0)}{L!}$; hence, we get a contradiction. Thus, $L > N$. □

Remark: Since the number L in the proof above is necessarily finite (otherwise, we would get $\psi^*_\epsilon = 0$), this proves that φ cannot be indefinitely smooth.

We begin by giving a stronger version of the vaguelettes lemma (extended to the case of multivariate functions) :

Lemma 9.2: (The vaguelettes lemma)

A) Let $0 < \delta \geq 1$ and let $(g_k)_{k\in\mathbb{Z}^d}$ be a sequence of functions such that $|g_k(x)| \leq \frac{1}{(1+|x|)^{d+\delta}}$ and $|g_k(x) - g_k(y)| \leq |x-y|^\delta(\frac{1}{(1+|x|)^{d+\delta}} + \frac{1}{(1+|y|)^{d+\delta}})$. Then, we have, for $\gamma \in \{\delta/8, -\delta/8\}$, $\|(-\Delta)^\gamma(\sum_{k\in\mathbb{Z}^d}\lambda_k g_k)\|_2 \leq C(\sum_{k\in\mathbb{Z}^d}|\lambda_k|^2)^{1/2}$.

B) Let $\delta > 0$ and let $(g_{j,k})_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$ be a sequence of functions so that $|g_{j,k}(x)| \leq 2^{jd/2}\frac{1}{(1+|2^jx-k|)^{d+\delta}}$, and

$$|g_{j,k}(x) - g_{j,k}(y)| \leq 2^{j(d/2+\delta)}|x-y|^\delta(\frac{1}{(1+|2^jx-k|)^{d+\delta}} + \frac{1}{(1+|2^jy-k|)^{d+\delta}}).$$

Then, we have $\|\sum_{,j\in\mathbb{Z},k\in\mathbb{Z}^d}\lambda_{j,k}g_{j,k}\|_2 \leq C(\sum_{j\in\mathbb{Z},k\in\mathbb{Z}^d}|\lambda_{j,k}|^2)^{1/2}$.

(C) With the same hypotheses, we have $\sum_{,j\in\mathbb{Z},k\in\mathbb{Z}^d} |\langle f|g_{j,k}\rangle|^2 \leq C\|f\|_2^2$.

Proof: For proving (A), we want to prove that for $\gamma \in \{\delta/8, -\delta/8\}$, the matrix $M = (m_{k,l} = \langle(-\Delta)^\gamma g_k|(-\Delta)^\gamma g_l\rangle)_{k,l\in\mathbb{Z}^d}$ is bounded on $l^2(\mathbb{Z}^d)$. We write $m_{k,l} = \langle(-\Delta)^{3\gamma} g_k|(-\Delta)^{-\gamma} g_l\rangle = \langle(-\Delta)^{-\gamma} g_k|(-\Delta)^{3\gamma} g_l\rangle$. Thus, if we can prove that for $0 < \gamma_1, \gamma_2 < \delta/2$ the matrix $\tilde{M} = (\langle(-\Delta)^{\gamma_1} g_k|(-\Delta)^{-\gamma_2} g_l\rangle)_{k,l\in\mathbb{Z}^d}$ is bounded on $l^1(\mathbb{Z}^d)$, we would obtain that M and ${}^tM$ are bounded on l^1, hence that M is bounded on l^1 and l^∞, hence on l^2.

We first prove that $\|(-\Delta)^{-\gamma_2} g_l\|_1 \leq C_{\gamma_2}$. Let K_{γ_2} be the kernel of the convolution operator $(-\Delta)^{-\gamma_2}$; $K_{\gamma_2}(x) = \frac{c_{\gamma_2}}{|x|^{d-2\gamma_2}}$. $K_{\gamma_2} \in L^{\frac{d}{d-2\gamma_2},\infty}$ and $\|g_l\|_{L^{\frac{d}{2\gamma_2},1}} \leq C$; hence, $\|(-\Delta)^{-\gamma_2} g_l\|_\infty \leq C$. Now, for $|x-l| \geq 1$, we write $(-\Delta)^{-\gamma_2} g_l(x) = I_1 + I_2 + I_3 = \int_{|y-l|\leq|x-l|/2} (K_{\gamma_2}(x-y) - K_{\gamma_2}(x-l)) g_l(y)\,dy + \int_{|y-l|\leq|x-l|/2} K_{\gamma_2}(x-l) g_l(y)\,dy + \int_{|y-l|\geq|x-l|/2} K_{\gamma_2}(x-y) g_l(y)\,dy$. We have $|I_1| \leq C\frac{1}{|x-l|^{d-2\gamma_2+\alpha}} \int \frac{|y-l|^\alpha}{(1+|y-l|)^{d+\delta}}\,dy$; this gives (choosing $2\gamma_2 < \alpha < \delta$) $\int |I_1|\,dx \leq C$. We have $I_2 = -K_{\gamma_2}(x-l)\int_{|y-l|>|x-l|/2} g_l(y)\,dy$; hence, we have $|I_2| \leq C\frac{1}{|x-l|^{d-2\gamma_2}}\frac{1}{|x-l|^\delta}$ and thus $\int |I_2|\,dx \leq C$. We have $|I_3| \leq C\int_{|x-y|\leq 3|y-l|} \frac{1}{|x-y|^{d-2\gamma_2}}\frac{1}{(1+|y-l|)^{d+\delta}}\,dy$ and we get easily by the Fubini theorem that $\int |I_3|\,dx \leq C\int \frac{|y-l|^{2\gamma_2}}{(1+|y-l|)^{d+\delta}}\,dy \leq C'$. Thus, $\|(-\Delta)^{-\gamma_2}(\sum_{l\in\mathbb{Z}^d} \mu_l g_l)\|_1 \leq C_{\gamma_2}\|(\mu_l)\|_1$.

Moreover, we have $\|\sum_{k\in\mathbb{Z}^d} \lambda_k g_k\|_{B_\infty^{\delta,\infty}} \leq C\|(\lambda_k)\|_\infty$: the estimation of $\sum_{k\in\mathbb{Z}^d} \lambda_k g_k(x)$ and of $\sum_{k\in\mathbb{Z}^d} \lambda_k \frac{g_k(x)-g_k(y)}{|x-y|^\delta}$ (for $|x-y| \leq 1$) is straightforward; hence, for $\gamma_1 < \delta/2$, $(-\Delta)^{\gamma_1}(\sum_{k\in\mathbb{Z}^d} \lambda_k g_k) \in B_\infty^{\delta-2\gamma_1,\infty} \subset L^\infty$ and $\|(-\Delta)^{\gamma_1}(\sum_{k\in\mathbb{Z}^d} \lambda_k g_k)\|_\infty \leq C_{\gamma_1}\|(\lambda_k)\|_\infty$. This proves that $|\langle\tilde{M}(\mu_l)|(\lambda_k)\rangle| \leq C\|(\mu_l)\|_1\|(\lambda_k)\|_\infty$, and (A) is proved.

(B) is then easy : we have by (A) that

$$\|(-\Delta)^{\delta/8}(\sum_{k\in\mathbb{Z}^d} \lambda_{j,k} g_{j,k})\|_2 \leq C 2^{j\delta/4}(\sum_{k\in\mathbb{Z}^d} |\lambda_{j,k}|^2)^{1/2}$$

and

$$\|(-\Delta)^{-\delta/8}(\sum_{k\in\mathbb{Z}^d} \lambda_{j,k} g_{j,k})\|_2 \leq C 2^{-j\delta/4}(\sum_{k\in\mathbb{Z}^d} |\lambda_{j,k}|^2)^{1/2}$$

hence, in conclusion, we find that

$$|\langle\sum_{k\in\mathbb{Z}^d} \lambda_{j,k} g_{j,k}|\sum_{k\in\mathbb{Z}^d} \lambda_{l,k} g_{l,k})\rangle| \leq C 2^{-|j-l|\delta/4}(\sum_{k\in\mathbb{Z}^d} |\lambda_{j,k}|^2)^{1/2}\sum_{k\in\mathbb{Z}^d} |\lambda_{l,k}|^2)^{1/2}$$

(C) is a direct consequence of (B). □

We now begin the analysis of Besov spaces with the description of the easy case of quadratic Sobolev spaces.

Theorem 9.2: (Sobolev spaces)

Let $(\psi_{\epsilon,j,k})_{1\leq\epsilon\leq 2^d-1, j\in\mathbb{Z}, k\in\mathbb{Z}^d}$ be a N-regular wavelet basis (with associated dual wavelet ψ^* and associated scaling functions φ, φ^*). Then for $0 < s < N$, the family $(\varphi(x-k))_{k\in\mathbb{Z}^d}\cup(\psi_{\epsilon,j,k})_{1\leq\epsilon\leq 2^d-1, j\in\mathbb{N}, k\in\mathbb{Z}^d}$ is a Riesz basis of $H^s(\mathbb{R}^d)$; more precisely, there exists two positive constants $0 < A_s \leq B_s < \infty$ so that for all $f \in H^s$ we have

$$A_s\|f\|_{H^s} \leq \Big(\sum_{k\in\mathbb{Z}^d}|\langle f|\varphi^*_{0,k}\rangle|^2 + \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}^d}\sum_{1\leq\epsilon\leq 2^d-1} 4^{js}|\langle f|\psi^*_{\epsilon,j,k}\rangle|^2\Big)^{1/2} \leq B_s\|f\|_{H^s}$$

Proof: We write $f_j = \sum_{k\in\mathbb{Z}^d}\sum_{1\leq\epsilon\leq 2^d-1}\langle f|\psi^*_{\epsilon,j,k}\rangle\ \psi_{j,k,\epsilon}$ and we define $\eta_j = \big(\sum_{k\in\mathbb{Z}^d}\sum_{1\leq\epsilon\leq 2^d-1}|\langle f|\psi^*_{\epsilon,j,k}\rangle|^2\big)^{1/2}$. We have to prove that for some constants C_s, D_s, we have : $C_s\|\sum_{j\geq 0} f_j\|_{H^s} \leq (\sum_j 4^{js}\eta_j^2)^{1/2} \leq D_s\|\sum_{j\geq 0} f_j\|_{H^s}$. Using the vaguelettes lemma, we obviously find that for $\sigma = 0$ or $\sigma = N$ we have $\|f_j\|_{H^\sigma} \leq C\ 2^{j\sigma}\eta_j$. This gives $2^{js}\max(\|f_j\|_2, 2^{-jN}\|f_j\|_{H^N}) \leq C2^{js}\eta_j$; hence, $\|f\|_{[L^2,H^N]_{s/N,2}} \leq C\|2^{js}\eta_j\|_{l^2}$. Since $[L^2, H^N]_{s/N,2} = H^s$, this gives $C_s\|\sum_{j\geq 0} f_j\|_{H^s} \leq (\sum_j 4^{js}\eta_j^2)^{1/2}$.

Conversely, we have (using Lemma 9.1) that for $1 \leq \epsilon \leq 2^d - 1$ the (separable) wavelet ψ^*_ϵ may be written as $\psi^*_\epsilon = \frac{\partial^N}{\partial x_l^N}\Psi^*_\epsilon$, where l depends on ϵ and $\Psi^*_\epsilon \in L^2_{comp}$. Using integration by parts and the vaguelettes lemma then gives that $\eta_j \leq C\min(\|f\|_2, 2^{-jN}\sum_{1\leq l\leq N}\|\frac{\partial^N}{\partial x_l^N}f\|_2)$. Now, if $f \in H^s$, we may write $f = \sum_{p\geq 0} F_p$ with $(2^{ps}\max(\|F_p\|_2, 2^{-pN}\|F_p\|_{H^N})) \in l^2(\mathbb{N})$. Then, we find that

$$2^{js}\eta_j \leq C\Big(\sum_{p<j} 2^{(j-p)(s-N)}2^{ps}2^{-pN}\|F_p\|_{H^N} + \sum_{p\geq j} 2^{(j-p)s}2^{ps}\|F_p\|_2\Big)$$

and this is enough to get that $(\sum_{j\in\mathbb{N}} 4^{js}\eta_j^2)^{1/2} \leq C\|f\|_{H^s}$. □

We use a simple application of this characterization to prove the compactness of the inclusion $H^s_0(\Omega) \subset H^\sigma_0(\Omega)$ for Ω bounded and $s > \sigma \geq 0$.

Definition 9.2: (Sobolev space on a domain) *For Ω a bounded open subset of $\mathbb{R}^n$ and $s \geq 0$ the Sobolev space $H^s_0(\Omega)$ is the space of functions $f \in H^s(\mathbb{R}^n)$ such that $f = 0$ almost everywhere outside from Ω. It is a closed subspace of $H^s(\mathbb{R}^n)$.*

Proposition 9.1: (Compact inclusion)

For Ω a bounded open subset of $\mathbb{R}^n$ and $s > \sigma \geq 0$ the inclusion operator $H^s_0(\Omega) \to H^\sigma_0(\Omega)$ is a compact operator.

Proof: Let $f_n \in H^s_0(\Omega)$, $n \in \mathbb{N}$, be a bounded sequence in $H^s(\mathbb{R}^d)$. Then a subsequence $g_p = f_{n_p}$, $p \in \mathbb{N}$, is weakly convergent to some $g \in H^s(\mathbb{R}^d)$:

for all $\varphi \in \mathcal{D}$, $\lim_{n\to\infty} \int g_p\varphi\, dx = \int g\varphi\, dx$. We must prove that the convergence is strong in $H^\sigma(\mathbb{R}^d)$ for $\sigma < s$. We use the wavelet decomposition $Id = P_j + \sum_{q\geq j} Q_q$ on a N-regular wavelet basis ($N > s$), where $P_j f = \sum_{k\in\mathbb{Z}^d}\langle f|\varphi^*_{j,k}\rangle\varphi_{j,k}$ and $Q_q f = \sum_{1\leq\epsilon\leq 2^d-1}\sum_{k\in\mathbb{Z}^d}\langle f|\psi^*_{\epsilon,q,k}\rangle\psi_{\epsilon,q,k}$. Theorem 9.2 gives that, for $j\geq 0$, $\|\sum_{q\geq j} Q_q(g_p-g)\|_{H^\sigma} \leq C2^{j(\sigma-s)}\sup_p \|g_p\|_{H^s}$. Moreover, if K_j is the (finite) set of indexes such that Supp $\varphi^*_{j,k}\cap\Omega\neq\emptyset$, then we have $\|P_j(g_p-g)\|_{H^\sigma}\leq C2^{j\sigma}\sum_{k\in K_j}|\langle g_p-g|\varphi^*_{j,k}\rangle|$, hence $\lim_{p\to\infty}\|P_j(g_p-g)\|_{H^\sigma}=0$.
□

Another easy case to deal with is the analysis of Hölder spaces:

Theorem 9.3: (Hölder spaces)

Let $(\psi_{\epsilon,j,k})_{1\leq\epsilon\leq 2^d-1, j\in\mathbb{Z}, k\in\mathbb{Z}^d}$ be a N-regular wavelet basis (with associated dual wavelet ψ^* and associated scaling functions φ, φ^*). Then for $0<\alpha<N$, there exists two positive constants $0 < A_\alpha \leq B_\alpha < \infty$ so that for all $f \in B^{\alpha,\infty}_\infty$ the series $\sum_{k\in\mathbb{Z}^d}\langle f|\varphi^*_{0,k}\rangle\varphi_{0,k} + \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}^d}\sum_{1\leq\epsilon\leq 2^d-1}\langle f|\psi^*_{\epsilon,j,k}\rangle\psi_{\epsilon,j,k}$ converges to f in $\mathcal{D}'$ and we have

$$A_\alpha\|f\|_{B^{\alpha,\infty}_\infty} \leq \sup_{k\in\mathbb{Z}^d}|\langle f|\varphi^*_{0,k}\rangle| + \sup_{j\in\mathbb{N}}\sup_{k\in\mathbb{Z}^d}\sup_{1\leq\epsilon\leq 2^d-1} 2^{j(\alpha+\frac{d}{2})}|\langle f|\psi^*_{\epsilon,j,k}\rangle| \leq B_\alpha\|f\|_{B^{\alpha,\infty}_\infty}$$

Proof: The convergence of the wavelet series is obvious. If M is the maximum of the diameters of the supports of the functions φ, φ^*, ψ_ϵ and ψ^*_ϵ, then the behavior of the wavelet series of f is determined in a neighborhood of any compact subset K of $\mathbb{R}^d$ by the values of f on the set $\{x\in\mathbb{R}^d \,/\, d(x,K)\leq 2M\}$. It is then sufficient to notice that the elements of $B^{\alpha,\infty}_\infty$ for $\alpha>0$ are locally square integrable and that the wavelet series of a square-integrable function converges to this function in L^2 norm.

We now give two proofs of Theorem 9.3. The first one works only for $\alpha\notin\mathbb{N}$. In that case, $B^{\alpha,\infty}_\infty$ is equal to the Hölder space C^α defined by: if $\alpha = m+\rho$, $m\in\mathbb{N}$ et $0<\rho<1$, then $f\in C^\alpha$ if and only the first derivatives of f, up to order m, are continuous and bounded and the mth derivatives of f are Hölderian of exponent ρ; C^α is normed by

$$\|f\|_{C^\alpha} = \sum_{|\beta|\leq m}\|\frac{\partial^\beta}{\partial x^\beta}f\|_\infty + \sum_{|\beta|=m}\sup_{x\neq y}\frac{|\frac{\partial^\beta}{\partial x^\beta}f(x) - \frac{\partial^\beta}{\partial x^\beta}f(y)|}{|x-y|^\rho}.$$

If $f\in C^\alpha$, we use the Taylor expansion of $f(x)$:

$$\sum_{|\beta|<m}(x-\frac{k}{2^j})^\beta\frac{\frac{\partial^\beta}{\partial x^\beta}f(\frac{k}{2^j})}{\beta!} + \int_0^1 d^m f(\frac{k}{2^j}+t(x-\frac{k}{2^j})).(x-\frac{k}{2^j},\dots,x-\frac{k}{2^j})\frac{(1-t)^{m-1}}{(m-1)!}dt$$

then we use the oscillation of the wavelets ψ^*_ϵ (Lemma 9.1) to get

$$\langle \sum_{|\beta|<m}(x-\frac{k}{2^j})^\beta \frac{\frac{\partial^\beta}{\partial x^\beta}f(\frac{k}{2^j})}{\beta!}|\psi^*_{\epsilon,j,k}\rangle = 0$$

and

$$\langle \int_0^1 d^m f(\frac{k}{2^j}).(x-\frac{k}{2^j},\ldots,x-\frac{k}{2^j})\frac{(1-t)^{m-1}}{(m-1)!}dt|\psi^*_{\epsilon,j,k}\rangle = 0$$

to get $|\langle f|\psi^*_{\epsilon,j,k}\rangle| \leq C\|f\|_{C^\alpha}\frac{1}{2^{j\alpha}}\ \frac{1}{2^{jd/2}}\|\psi^*_\epsilon\|_1$; we easily obtain $|\langle f|\varphi^*_{0,k}\rangle| \leq \|f\|_\infty\|\varphi^*\|_1$ and thus control the size of the coefficients of the wavelet expansions of f.

Conversely, if we control the size of the coefficients of the wavelet expansion of f, we define $f_{-1} = \sum_{k\in\mathbb{Z}^d}\langle f|\varphi^*_{0,k}\rangle\ \varphi_{0,k}$ and, for $j\geq 0$, $f_j = \sum_{k\in\mathbb{Z}^d}\sum_{1\leq\epsilon\leq 2^d-1}\langle f|\psi^*_{\epsilon,j,k}\rangle\ \psi_{j,k,\epsilon}$. f_{-1} is of class $\mathcal{C}^N$ and all its derivatives are bounded ; thus, f_{-1} belongs to C^α. Similarly, f_j is of class $\mathcal{C}^N$ and its derivatives of order k ($k\leq N$) are bounded by $C2^{j(k-\alpha)}$. This gives that $\sum_{j\geq 0} f_j$ is a function of class $\mathcal{C}^m$ and that its derivatives up to order m are bounded. If $|\beta| = m$, we control $|\frac{\partial^\beta}{\partial x^\beta}f_j(x)-\frac{\partial^\beta}{\partial x^\beta}f_j(y)|$ by $\min(2\|\frac{\partial^\beta}{\partial x^\beta}f_j\|_\infty, \|\vec{\nabla}(\frac{\partial^\beta}{\partial x^\beta}f_j)\|_\infty|x-y|) \leq C2^{-j\rho}\min(1,2^j|x-y|)$; summing over j, we get $|\frac{\partial^\beta}{\partial x^\beta}f_j(x)-\frac{\partial^\beta}{\partial x^\beta}f_j(y)| \leq C|x-y|^\rho$.

We thus have proved Theorem 9.3 for $\alpha\notin\mathbb{N}$. When $\alpha\in\mathbb{N}^*$, the space $B^{\alpha,\infty}_\infty$ is a Zygmund class C^α_*, and the proof is more delicate. We then prefer to use the characterization of the Besov space as a real interpolation space; we have shown in Chapter 3 that $B^{\alpha,\infty}_\infty = [L^\infty, W^{N,\infty}]_{\alpha/N,\infty}$. If we control the size of the coefficients in the wavelet expansion of f, we again introduce $f_{-1} = \sum_{k\in\mathbb{Z}^d}\langle f|\varphi^*_{0,k}\rangle\ \varphi_{0,k}$ and, for $j\geq 0$, $f_j = \sum_{k\in\mathbb{Z}^d}\sum_{1\leq\epsilon\leq 2^d-1}\langle f|\psi^*_{\epsilon,j,k}\rangle\ \psi_{j,k,\epsilon}$. Then, $f_{-1}\in W^{N,\infty}\subset B^{\alpha,\infty}_\infty$ while we have $\|f_j\|_\infty \leq C2^{-j\alpha}$ and $\|f_j\|_{W^{N,\infty}} \leq C2^{j(N-\alpha)}$ which gives $\sum_{j\geq 0} f_j \in [L^\infty, W^{N,\infty}]_{\alpha/N,\infty}$. Conversely, if we assume that $f\in[L^\infty, W^{N,\infty}]_{\alpha/N,\infty}$, then $f\in L^\infty$ so that we control the coefficients $\langle f|\varphi^*_{0,k}\rangle$; moreover, we may decompose for all $j\geq 0$ f into $f = g_j + h_j$ where $\|g_j\|_\infty \leq C2^{-j\alpha}$ and $\|h_j\|_{W^{N,\infty}} \leq C2^{j(N-\alpha)}$; using the Taylor expansion of h_j up to order N and the oscillation of the wavelet ψ^*_ϵ, we get $|\langle f|\psi^*_{\epsilon,j,k}\rangle| \leq C\|\psi^*_\epsilon\|_1(\|g_j\|_\infty + 2^{-jN}\|h_j\|_{W^{N,\infty}}) \leq C'2^{-jd/2}2^{-j\alpha}$. □

An example of application of the wavelet analysis of Hölder spaces, the characterization of (Morrey)-Campanato spaces follows:

Definition 9.3: (Campanato spaces)
a) For $1 < p \leq q \leq \infty$, the Morrey space $M^{p,q}(\mathbb{R}^d)$ is the space of locally L^p functions so that $\sup_{0<R\leq 1}\sup_{x_0\in\mathbb{R}^d} R^{d(p/q-1)}\int_{|x-x_0|<R}|f(x)|^p\ dx < \infty$, normed by $\|f\|_{M^{p,q}} = \sup_{0<R\leq 1}\sup_{x_0\in\mathbb{R}^d} R^{d(1/q-1/p)}(\int_{|x-x_0|<R}|f(x)|^p\ dx)^{1/p}$.

We write L^p_{uloc} for $M^{p,p}$; this is the space of uniformly locally L^p functions.

Similarly, for $1 \leq q \leq \infty$*, the Morrey space* $M^{1,q}(\mathbb{R}^d)$ *is the space of locally finite measures* $d\mu$ *so that* $\sup_{0<R\leq 1} \sup_{x_0\in\mathbb{R}^d} R^{d(1/q-1)} \int_{|x-x_0|<R} d|\mu| < \infty$*, normed by* $\|d\mu\|_{M^{1,q}} = \sup_{0<R\leq 1} \sup_{x_0\in\mathbb{R}^d} R^{d(1/q-1)} \int_{|x-x_0|<R} d|\mu|$*. We write* M^1_{uloc} *for* $M^{1,1}$*; this is the space of uniformly locally finite measures.*
b) For $1 < p \leq \infty$*,* $N \in \mathbb{N}$ *and* $\beta \geq 0$*, the Campanato space* $\Lambda_p^{\beta,N}(\mathbb{R}^d)$ *is the space of uniformly locally* L^p *functions so that*

$$\sup_{0<R\leq 1} \sup_{x_0\in\mathbb{R}^d} \inf_{P\in\mathbb{C}[X_1,\ldots,X_d],\deg P<N} R^{-\beta p} \int_{|x-x_0|<R} |f-P|^p \, dx < \infty$$

normed by

$$\|f\|_{\Lambda_p^{\beta,N}} = \|f\|_{L^p_{uloc}} + \sup_{0<R\leq 1} \sup_{x_0\in\mathbb{R}^d} \inf_{P\in\mathbb{C}[X_1,\ldots,X_d],\deg P<N} R^{-\beta} \left(\int_{|x-x_0|<R} |f-P|^p dx\right)^{1/p}$$

Similarly, the Campanato space $\Lambda_1^{\beta,N}(\mathbb{R}^d)$ *is the space of uniformly locally finite measures* $d\mu$ *so that*

$$\sup_{0<R\leq 1} \sup_{x_0\in\mathbb{R}^d} \inf_{d\nu=P\,dx,P\in\mathbb{C}[X_1,\ldots,X_d],\deg P<N} R^{-\beta} \int_{|x-x_0|<R} d|\mu-\nu| < \infty$$

normed by

$$\|d\mu\|_{\Lambda_1^{\beta,N}} = \|d\mu\|_{M^1_{uloc}} + \sup_{0<R\leq 1} \sup_{x_0\in\mathbb{R}^d} \inf_{d\nu=P\,dx,\deg P<N} R^{-\beta} \int_{|x-x_0|<R} d|\mu-\nu|$$

.

Remark: We are not interested in $\beta < 0$ since, in that case, the space $\Lambda_p^{\beta,N}$ would coincide with $\Lambda_p^{0,N} = L^p_{uloc}$ ($p > 1$) [uniformly locally L^p functions] or $\Lambda_1^{0,N} = M^1_{uloc}$ ($p = 1$).

Proposition 9.2: *Let* $p \in [1, +\infty]$*,* $\beta > 0$ *and* $N \in \mathbb{N}$*. If* $\beta > d/p$ *and* $\beta - d/p \notin \mathbb{N}$*, then, for* $N < \beta - d/p$*,* $\Lambda_p^{\beta,N} = \mathbb{C}$ *and, for* $N > \beta - d/p$*,* $\Lambda_p^{\beta,N} = B_\infty^{\alpha,\infty}$ *for* $\alpha = \beta - d/p$*.*

Proof: Let us choose $M > \max(N, \alpha)$ (where $\alpha = \beta - d/p$); we then use a basis of M-regular wavelets. Let $f \in \Lambda_p^{\beta,N}$. The coefficients $\langle f|\varphi^*_{0,k}\rangle$ are bounded since $f \in L^p_{uloc}$ or $f \in M^1_{uloc}$ and $\varphi^* \in L^q$ ($1/p + 1/q = 1$) or $\varphi^* \in \mathcal{C}_0$. Now, for any polynomial P with degree less than N, the oscillation of ψ^*_ϵ gives $\langle f|\psi^*_{\epsilon,j,k}\rangle = \langle f - P|\psi^*_{\epsilon,j,k}\rangle$; hence, for $j \geq 0$ we have $|\langle f|\psi^*_{\epsilon,j,k}\rangle| \leq C 2^{-j\beta} 2^{j(d/p-d/2)} \|\psi^*_\epsilon\|_q$. Thus, we have proved that $\Lambda_p^{\beta,N} \subset B_\infty^{\alpha,\infty}$. Conversely, if $f \in B_\infty^{\alpha,\infty}$ and $\alpha = m + \rho$ ($m \in \mathbb{N}$ and $0 < \rho < 1$), then if P_{x_0} is the Taylor expansion of f at x_0 up to order m, we have $|f(x) - P_{x_0}(x)| \leq C|x - x_0|^\alpha$; hence, $\left(\int_{|x-x_0|\leq R} |f(x) - P_{x_0}(x)|^p \, dx\right)^{1/p} \leq C R^{\alpha+d/p}$. Thus, $\Lambda_p^{\beta,N} = B_\infty^{\alpha,\infty}$ for

$N \geq m+1$. If $N \leq m$, we introduce the difference operators $\Delta_{i,\epsilon}$ defined by $\Delta_{i,\epsilon} g(x) = g(x+\epsilon e_i) - g(x)$ where $(e_1, \ldots, e_d)$ is the canonical basis of $\mathbb{R}^d$. Then, we have for $|x-x_0| \leq \epsilon$ $|\Delta_{i_1,\epsilon} \ldots \Delta_{i_N,\epsilon} f(x) - \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_N}} f(x_0)\epsilon^N| \leq C\epsilon^{\min(N+1,\alpha)}$ while for any P with degree less than N we have $\Delta_{i_1,\epsilon} \ldots \Delta_{i_N,\epsilon} P = 0$; this gives $(\int_{|x-x_0|\leq\epsilon} |\Delta_{i_1,\epsilon} \ldots \Delta_{i_N,\epsilon} f|^p \, dx)^{1/p} \leq C\epsilon^{\beta}$; hence, we find that $|\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_N}} f(x_0)| \leq C\epsilon^{\min(1,\alpha-N)}$; thus, f is a polynomial with degree less than N; its uniform integrability gives moreover that f is constant. □

Theorems 9.2 and 9.3 are easily extended to the case of Besov spaces:

Theorem 9.4: (Besov spaces)

Let $(\psi_{\epsilon,j,k})_{1\leq\epsilon\leq 2^d-1, j\in\mathbb{Z}, k\in\mathbb{Z}^d}$ be a N-regular wavelet basis (with associated dual wavelet ψ^* and associated scaling functions φ, φ^*). Then for $0 < s < N$, $1 \leq p < \infty$, $1 \leq q < \infty$, the family $(\varphi(x-k))_{k\in\mathbb{Z}^d} \cup (\psi_{\epsilon,j,k})_{1\leq\epsilon\leq 2^d-1, j\in\mathbb{N}, k\in\mathbb{Z}^d}$ is a Riesz basis of $B_p^{s,q}(\mathbb{R}^d)$; more precisely, there exists two positive constants $0 < A_{s,p,q} \leq B_{s,p,q} < \infty$ so that for all $f \in B_p^{s,q}$ we have

$$A_{s,p,q}\|f\|_{B_p^{s,q}} \leq N_{s,p,q}(f) \leq B_{s,p,q}\|f\|_{B_p^{s,q}}$$

with

$$N_{s,p,q}(f) = \Big(\sum_{k\in\mathbb{Z}^d} |\langle f|\varphi_{0,k}^*\rangle|^p\Big)^{1/p} + \Big(\sum_{j\in\mathbb{N}} 2^{j(s+\frac{d}{2}-\frac{d}{p})q} \Big(\sum_{k\in\mathbb{Z}^d} \sum_{1\leq\epsilon\leq 2^d-1} |\langle f|\psi_{\epsilon,j,k}^*\rangle|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}$$

Remark: The equivalence of norms is also true in the case $p = \infty$ or $q = \infty$, but in that case we have only weak convergence of the wavelet series.

Proof: We write $f_j = \sum_{k\in\mathbb{Z}^d} \sum_{1\leq\epsilon\leq 2^d-1} \langle f|\psi_{\epsilon,j,k}^*\rangle \; \psi_{j,k,\epsilon}$ and we define $\eta_j = \big(\sum_{k\in\mathbb{Z}^d} \sum_{1\leq\epsilon\leq 2^d-1} |\langle f|\psi_{\epsilon,j,k}^*\rangle|^p\big)^{1/p}$. We must prove that for some constants C, D, we have: $C\|\sum_{j\geq 0} f_j\|_{B_p^{s,q}} \leq (\sum_j 2^{jsq}\eta_j^q)^{1/q} \leq D\|\sum_{j\geq 0} f_j\|_{B_p^{s,q}}$. We check easily that for $\sigma = 0$ or $\sigma = N$ we have $\|f_j\|_{W^{\sigma,p}} \leq C\, 2^{j\sigma}\eta_j$. This gives, using the J-method of interpolation, that $\|f\|_{B_p^{s,q}} \leq C\|2^{js}\eta_j\|_{l^q}$.

Conversely, we write again, for $1 \leq \epsilon \leq 2^d - 1$, the (separable) wavelet ψ_ϵ^* as $\psi_\epsilon^* = \frac{\partial^N}{\partial x_l^N}\Psi_\epsilon^*$ where l depends on ϵ and $\Psi_\epsilon^* \in L_{comp}^{\frac{p}{p-1}}$. Using integration by parts and a one-scale vaguelettes lemma for L^p then gives that $\eta_j \leq C 2^{j(d/p-d/2)} \min(\|f\|_p, 2^{-jN}\sum_{1\leq l\leq N} \|\frac{\partial^N}{\partial x_l^N} f\|_p)$. Now, if $f \in B_p^{s,q}$, we may write $f = \sum_{l\geq 0} F_l$ with $(2^{ls}\max(\|F_l\|_p, 2^{-lN}\|F_l\|_{W^{N,p}}) \in l^q(\mathbb{N})$. Then, we conclude that

$$2^{j(s+d/2-d/p)}\eta_j \leq C\Big(\sum_{l<j} 2^{(j-l)(s-N)} 2^{ls} 2^{-lN}\|F_l\|_{W^{N,p}} + \sum_{l\geq j} 2^{(j-l)s} 2^{ls}\|F_l\|_p\Big) \square$$

3. Singular integrals

Wavelets are a useful tool for the study of singular integral operators.

Definition 9.4: (Singular integral operators)

A singular integral operator is a continuous linear operator T from $\mathcal{D}(\mathbb{R}^d)$ to $\mathcal{D}'(\mathbb{R}^d)$ so that there exists a continuous function K defined on $\mathbb{R}^d \times \mathbb{R}^d - \Delta$ (where Δ is the diagonal set $x = y$), which satisfies:
i) $\exists C_0 > 0\ \forall x\ \forall y \quad |K(x,y)| \leq \frac{C_0}{|x-y|^d}$
ii) $\exists \delta > 0\ \exists C_1 > 0\ \forall x, y, z, |z| < \frac{1}{2}|x-y| \Rightarrow |K(x+z,y) - K(x,y)| \leq C_1 \frac{|z|^\delta}{|x-y|^{d+\delta}}$
iii) $\exists \delta > 0\ \exists C_1 > 0\ \forall x, y, z, |z| < \frac{1}{2}|x-y| \Rightarrow |K(x,y+z) - K(x,y)| \leq C_1 \frac{|z|^\delta}{|x-y|^{d+\delta}}$
iv) $\forall f, g \in \mathcal{D}(\mathbb{R}^d),\ Supp f \cap Supp g = \emptyset \Rightarrow \langle Tf|g\rangle = \int\int K(x,y) f(y)\bar{g}(x)\, dx\, dy$

A *Calderón–Zygmund operator* is a singular integral operator T, which is bounded on $L^2(\mathbb{R}^d)$ ($T \in \mathcal{L}(L^2, L^2)$). When $\delta = 1$, we define the Calderón–Zygmund norm of T as the sum $\|T\|_{CZO} = \|T\|_{\mathcal{L}(L^2,L^2)} + \||x-y|^d\, K(x,y)\|_\infty + \||x-y|^{d+1}\ \vec{\nabla}_x K(x,y)\|_\infty + \||x-y|^{d+1}\ \vec{\nabla}_y K(x,y)\|_\infty \|T\|_{\mathcal{L}(L^2,L^2)}$.

We now introduce two useful notions associated to singular integral operators: the *weak boundedness property* and the distribution $T(1)$.

Definition 9.5: (Weak boundedness property)

A continuous linear operator T from $\mathcal{D}(\mathbb{R}^d)$ to $\mathcal{D}'(\mathbb{R}^d)$ satisfies the weak boundedness property (what we shall write $T \in WBP$) if there exist a constant C and a number N such that for all $\phi, \psi \in \mathcal{D}(\mathbb{R}^d)$ with support in $B(0,1)$, all $x_0 \in \mathbb{R}^d$ and all $R > 0$, writing $\phi_{x_0,R}(x) = \phi(\frac{x-x_0}{R})$ and $\psi_{x_0,R}(x) = \psi(\frac{x-x_0}{R})$, we have

$$|\langle T(\phi_{x_0,R})|\psi_{x_0,R}\rangle| \leq C(\sum_{|\alpha|\leq N} R^{|\alpha|}\|\partial^\alpha(\phi_{x_0,R})\|_2)\ (\sum_{|\alpha|\leq N} R^{|\alpha|}\|\partial^\alpha(\psi_{x_0,R})\|_2).$$

Definition 9.6: (The distribution $T(1)$)

If T is a singular integral operator, we may define $T(1) \in \mathcal{D}'(\mathbb{R}^d)/\mathbb{C}$ by choosing $\varphi \in \mathcal{D}(\mathbb{R}^d)$ equal to 1 in a neighborhood of 0 and computing for $\psi \in \mathcal{D}(\mathbb{R}^d)$ with $\int \psi\, dx = 0$ $\langle T(1)|\psi\rangle$ as $\langle T(1)|\psi\rangle = \lim_{R\to\infty} \langle T(\varphi(\frac{x}{R}))|\psi\rangle$. We may see easily that, if $\omega \in \mathcal{D}(\mathbb{R}^d)$ is equal to 1 in a neighborhood of Supp ψ and $x_0 \in$ Supp ψ, then

$$\langle T(1)|\psi\rangle = \langle T(\omega)|\psi\rangle + \int\int (K(x,y) - K(x_0,y))(1-\omega(y))\ \psi_{\epsilon,j,k}(x)\ dx\ dy.$$

We shall now prove the $T(1)$ theorem of David and Journé [DAVJ 84] in the case $T(1) = T^*(1) = 0$ (the general $T(1)$ theorem is proved in Chapter 10):

Theorem 9.4: (The $T(1)$ theorem)

Let T be a singular integral operator. If $T \in WBP$ and $T(1) = T^*(1) = 0$, then T is bounded from L^2 to L^2.

Proof: Since $T \in WBP$, the bracket $\langle Tf|g\rangle$ may be extended to $\mathcal{C}^N_{comp} \times \mathcal{C}^N_{comp}$ for some N. We thus introduce an orthonormal wavelet basis with $\varphi \in \mathcal{C}^N_{comp}$. Since P_j maps $\mathcal{C}^N_{comp}$ to $\mathcal{C}^N_{comp}$, the operator P_jTP_j is well defined and we have, for all $f, g \in \mathcal{C}^N_{comp}$, $\lim_{j\to+\infty}\langle TP_jf|P_jg\rangle = \langle Tf|g\rangle$. Moreover, we have $\lim_{j\to-\infty}\langle TP_jf|P_jg\rangle = 0$: let K be the (finite) set of indexes $k \in \mathbb{Z}^d$ so that $0 \in$ Supp $\varphi(x-k)$; then, for f, g with compact support, we have for j close enough to $-\infty$ that $P_jf = \sum_{k\in K}\langle f|\varphi_{j,k}\rangle\varphi_{j,k}$ and $P_jg = \sum_{k\in K}\langle g|\varphi_{j,k}\rangle\varphi_{j,k}$; thus, using the weak boundedness property to get that $\sup_{j\in\mathbb{Z}}\sup_{k,k'\in K}|\langle T\varphi_{j,k}|\varphi_{j,k'}\rangle| < \infty$, we find that $|\langle TP_jf|P_jg\rangle| \leq C\sup_{k\in K}|\langle f|\varphi_{j,k}\rangle| \ \sup_{k\in K}|\langle g|\varphi_{j,k}\rangle| \leq C'2^j\|f\|_1\|g\|_1$. Thus, T may be approximated by $P_JTP_J - P_{-J}TP_{-J}$ with J going to $+\infty$. Writing $P_{j+1}TP_{j+1} - P_jTP_j = P_jTQ_j + Q_jTP_j + Q_jTQ_j$, we find that, for f, g with compact support,

$$\langle Tf|g\rangle = \lim_{N\to\infty}\sum_{|j|\leq N}\langle TQ_jf|P_jg\rangle + \sum_{|j|\leq N}\langle TQ_jf|Q_jg\rangle + \sum_{|j|\leq N}\langle TP_jf|Q_jg\rangle.$$

Thus, the theorem will be proved when we have shown that the operators

$$\begin{array}{l} f \mapsto A_{\epsilon,N}f = \sum_{|j|\leq N}\sum_{k\in\mathbb{Z}^d}\langle f|\psi_{\epsilon,j,k}\rangle P_jT\psi_{\epsilon,j,k} \\ f \mapsto B_{\epsilon,N}f = \sum_{|j|\leq N}\sum_{k\in\mathbb{Z}^d}\langle f|\psi_{\epsilon,j,k}\rangle Q_jT\psi_{\epsilon,j,k} \\ f \mapsto C_{\epsilon,N}f = \sum_{|j|\leq N}\sum_{k\in\mathbb{Z}^d}\langle f|\psi_{\epsilon,j,k}\rangle P_jT^*\psi_{\epsilon,j,k} \end{array}$$

are bounded in L^2 norm with a bound that does not depend on N. This is easily accomplished. Indeed, let $\theta_{\epsilon,j,k} = P_jT\psi_{\epsilon,j,k}$. We have $\theta_{\epsilon,j,k} = \sum_{l\in\mathbb{Z}^d}\alpha_{\epsilon,j,k,l}\varphi_{j,l}$ with $|\alpha_{\epsilon,j,k,l}| \leq C\frac{1}{(1+|k-l|)^{d+\delta}}$: for k close to l, we use the weak boundedness property; for k and l far from each other, we write $\alpha_{\epsilon,j,k,l} = \langle T\psi_{\epsilon,j,k}|\varphi_{j,l}\rangle = \int\int(K(x,y) - K(x,k/2^j))\psi_{\epsilon,j,k}(y)\varphi_{j,l}(x)\ dx\ dy$. The estimates on $\alpha_{\epsilon,j,k,l}$ then give the estimates $|\theta_{\epsilon,j,k}(x)| \leq C2^{jd/2}\frac{1}{1+|2^jx-k|^{d+\delta}}$ and

$$|\theta_{\epsilon,j,k}(x) - \theta_{\epsilon,j,k}(y)| \leq C2^{j(d/2+\delta)}|x-y|^\delta\left(\frac{1}{(1+|2^jx-k|)^{d+\delta}} + \frac{1}{(1+|2^jy-k|)^{d+\delta}}\right)$$

Moreover, we have $\int\theta_{\epsilon,j,k}\ dx = 0$: indeed, we know that $\sum_{k\in\mathbb{Z}^d}\varphi(x-k) = 1$; we choose L large enough so that $\sum_{|l|\leq L}\varphi(2^jx - l) = 1$ is equal to 1 on a neighborhood of the support of $\psi_{\epsilon,j,k}$. Then we have $\langle T^*(1)|\psi_{\epsilon,j,k}\rangle = A + B$ with

$$\begin{array}{c} A = \langle T^*(\sum_{|l|\leq L}2^{-jd/2}\varphi_{j,l})|\psi_{\epsilon,j,k}\rangle \\ B = \int\int(K(x,y) - K(x,k/2^j))(\sum_{|l|>L}2^{-jd/2}\varphi_{j,l}(x))\ \psi(y)\ dy\ dx \end{array}.$$

We have $A = \sum_{|l| \leq L} 2^{-jd/2} \alpha_{\epsilon,j,k,l}$. Similarly, we have

$$B = \lim_{T \to \infty} \int \int (K(x,y) - K(x, k/2^j)) (\sum_{L < |l| < T} 2^{-jd/2} \varphi_{j,l}(x)) \; \psi(y) \; dy \; dx$$

which gives $B = \lim_{T \to \infty} \sum_{L < |l| < T} 2^{-jd/2} \alpha_{\epsilon,j,k,l}$. Thus, we get $\int \theta_{\epsilon,j,k} \; dx = \langle T^*(1) | \psi_{\epsilon,j,k} \rangle = 0$. We may now use the vaguelettes lemma (Lemma 9.2) to get that the operators $A_{\epsilon,N}$ are bounded in L^2 norm with a bound which does not depend on N. The same proof holds true for the operators $B_{\epsilon,N}$ and $C_{\epsilon,N}$. $\square$

Chapter 10

The space BMO

In this chapter, we present the space BMO and three basic results of harmonic analysis: the celebrated duality theorem of Fefferman and Stein [FEFS 72], the $T(1)$ theorem of David and Journé [DAVJ 84] (see also Meyer [MEY 97]) and Calderón's commutator theorem [MEY 97].

1. Carleson measures and the duality between $\mathcal{H}^1$ and BMO

We give several equivalent characterizations of BMO. For one characterization, we need the Littlewood–Paley decomposition, i.e., the operators S_j and Δ_j defined by $\mathcal{F}(S_j f) = \varphi(\xi/2^j)\mathcal{F}f$ and $\mathcal{F}(\Delta_j b) = \psi(\xi/2^j)\mathcal{F}b$, where $\varphi \in \mathcal{D}(\mathbb{R}^d)$ is such that $|\xi| \leq \frac{1}{2} \Rightarrow \varphi(\xi) = 1$ and $|\xi| \geq 1 \Rightarrow \varphi(\xi) = 0$ and where ψ is defined as $\psi(\xi) = \varphi(\xi/2) - \varphi(\xi)$.

Before stating the theorem, we prove a useful result on Carleson measures:

Proposition 10.1: (Carleson measures)

Let $\beta_j(x)$ be a sequence of measurable functions on $\mathbb{R}^d$ defining a Carleson measure on $\mathbb{Z} \times \mathbb{R}^d$:

$$\sup_{x_0 \in \mathbb{R}^d, R>0} \frac{1}{R^d} \sum_{2^j R > 1} \int_{|x-x_0|<R} |\beta_j(x)|^2 \, dx < \infty$$

Let $\varphi \in L^1(\mathbb{R}^d)$ so that $(1+|x|)^{d+1}\varphi \in L^\infty$ and let $\varphi_j(x) = 2^{jd}\varphi(2^j x)$. Then, for all $f \in L^2(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} \sum_{j \in \mathbb{Z}} |f * \varphi_j|^2 |\beta_j|^2 \, dx \leq C \|f\|_2^2 \sup_{x_0 \in \mathbb{R}^d, R>0} \frac{1}{R^d} \sum_{2^j R>1} \int_{|x-x_0|<R} |\beta_j(x)|^2 \, dx$$

where C does not depend neither on f, nor on φ nor on $(\beta_j)_{j \in \mathbb{Z}}$.

Proof: It is sufficient to prove the proposition for $\varphi = 1_{B(0,1)}$. Indeed, if $|\varphi(x)| \leq (1+|x|)^{-d}$, we may write $|\varphi(x)| \leq \sum_{k \geq 0} 2^{-(d+1)(k-1)} 1_{B(0,2^k)}(x)$; moreover, $\sum_{j \in \mathbb{Z}} |f * 2^{jd} 1_{B(0,2^k)}(2^j x)|^2 |\beta_j|^2 = 2^{-2kd} \sum_{j \in \mathbb{Z}} |f * 2^{jd} 1_{B(0,1)}(2^j x)|^2 |\beta_{j+k}|^2$ and

$$\sup_{x_0,R} \frac{1}{R^d} \sum_{2^j R>1} \int_{|x-x_0|<R} |\beta_{j+k}(x)|^2 dx \leq \sup_{x_0,R} \frac{1}{R^d} \sum_{2^j R>1} \int_{|x-x_0|<R} |\beta_j(x)|^2 dx$$

for $k \geq 0$. Thus, we may deduce the general case from the case $\varphi = 1_{B(0,1)}$.

We thus try to estimate $\int_{\mathbb{R}^d} \sum_{j\in\mathbb{Z}} (|m_{B(x,2^{-j})}|f|)^2 \,|\beta_j|^2 \,dx$. We introduce, for the Hardy–Littlewood maximal function M_f of f, $\Omega_k = \{x \,/\, M_f(x) > 2^k\}$,$\Gamma_k$ the set of maximal dyadic open cubes contained in Ω_k (so that $\Omega_k = \sum_{Q\in\Gamma_k} Q \cup N_k$ with $|N_k| = 0$) and $\Lambda_k = \{(j,x) \,/\, 2^k < |m_{B(x,2^{-j})}| \leq 2^{k+1}\}$. When $(j,x) \in \Lambda_k$, then we clearly have $x \in \Omega_k$; moreover, the ball $B(x, 2^{-j})$ is wholly contained in Ω_k; it contains a dyadic cube Q_0 of size $2^{-j}/\sqrt{d}$; if $Q = \frac{k_Q}{2^{l_Q}} + \frac{1}{2^{l_Q}}(0,1)^d$ is the maximal dyadic cube of Ω containing Q_0, then clearly $x \in B(\frac{k_Q}{2^{l_Q}}, 2\sqrt{d}2^{-l_Q})$ and $2^{j+l_Q} \geq \sqrt{d}$. Thus, we have

$$\int_{\mathbb{R}^d} \sum_{j\in\mathbb{Z}} (|m_{B(x,2^{-j})}|f|)^2 \,|\beta_j|^2 \,dx \leq \sum_{k\in\mathbb{Z}} 2^{2k+2} \sum_{Q\in\Gamma_k} \int_{B(\frac{k_Q}{2^{l_Q}}, 2\sqrt{d}2^{-l_Q})} \sum_{2^{j+l_Q}\geq\sqrt{d}} |\beta_j|^2 \,dx$$

$$\leq C \sum_{k\in\mathbb{Z}} 2^{2k+2} \sum_{Q\in\Gamma_k} |Q| = C \sum_{k\in\mathbb{Z}} 2^{2k+2} |\Omega_k| \leq C' \int |M_f(x)|^2 \,dx \leq C'' \|f\|_2^2$$

□

The classical characterizations of BMO follows:

Theorem 10.1: (Characterizations of BMO)

Let $b \in \mathcal{D}'(\mathbb{R}^d)$. Then, the following assertions are equivalent:

(A1) $b \in (\mathcal{H}^1)'$: more precisely, there exists a constant C so that for all $\omega \in \mathcal{D}(\mathbb{R}^d)$ with $\int \omega \,dx = 0$, we have $|\langle b|\omega\rangle| \leq C \,\|\omega\|_{\mathcal{H}^1}$.

(A2) b is locally integrable and we have: $\sup_{B\in\mathcal{B}} \frac{1}{|B|}\int_B \,|b - m_B b| \,dx < \infty$ where $\mathcal{B}$ is the collection of all balls $B(x,r)$, $x \in \mathbb{R}^d, r > 0$, and $m_B b = \frac{1}{|B|}\int_B b(x)\,dx$.

(A3) For $q \in [1,\infty)$, b is locally L^q and: $\sup_{B\in\mathcal{B}} \frac{1}{|B|}\int_B \,|b - m_B b|^q dx < \infty$.

(A4) $b \in \mathcal{S}'(\mathbb{R}^d)$ and for all $\psi \in \mathcal{S}(\mathbb{R}^d)$ so that $\int \psi \,dx = 0$, defining $\psi_t(x) = \frac{1}{t^d}\psi(\frac{x}{t})$, the measure $|b * \psi_t(x)|^2 \frac{dt}{t} dx$ is a Carleson measure on $(0,\infty) \times \mathbb{R}^d$:

$$\sup_{R>0, x_0\in\mathbb{R}^d} \frac{1}{R^d} \int\int_{0<t<R, |x-x_0|<R} |b * \psi_t(x)|^2 \frac{dt}{t} dx < \infty$$

(A5) $b \in \mathcal{S}'(\mathbb{R}^d)$ and

$$\text{for } j = 1, \ldots, d, \qquad \sup_{R>0, x_0\in\mathbb{R}^d} \frac{1}{R^d} \int\int_{0<t<R^2, |x-x_0|<R} |\frac{\partial}{\partial x_j} e^{t\Delta} b|^2 dt \,dx < \infty$$

(A6) $b \in \dot{B}^{0,\infty}_\infty$ [i.e. $b \in \mathcal{S}'(\mathbb{R}^d)$, $(\Delta_j b)_{j\in\mathbb{Z}} \in l^\infty(\mathbb{Z}, L^\infty)$ and there exist a sequence of constants $(\gamma_j) \in \mathbb{C}^{\mathbb{Z}}$ so that $b = \sum_{j\in\mathbb{Z}} \Delta_j b + \gamma_j$ in $\mathcal{S}'(\mathbb{R}^d)$] and the operator $f \mapsto \pi(b,f) = \sum_{j\in\mathbb{Z}} S_{j-1} f \;\Delta_j b$ is bounded on $L^2(\mathbb{R}^d)$.

Proof: We begin with some comments on the space $\dot{B}^{0,\infty}_\infty$. This is the realization of the homogeneous space $\dot{\mathcal{B}}^{0,\infty}_\infty$. Indeed, if we consider a sequence of bounded functions $(f_j)_{j\in\mathbb{Z}} \in l^\infty(\mathbb{Z}, L^\infty)$ so that $\hat{f}_j$ is supported in $\{\xi \in \mathbb{R}^d \,/\, \frac{1}{2}2^j \leq |\xi| \leq 2\,2^j\}$; then it is easy to check that $\sum_{j<0} f_j(x) - f_j(0) + \sum_{j\geq 0} f_j$ is convergent in $\mathcal{S}'(\mathbb{R}^d)$: for $j < 0$ we have $|f_j(x) - f_j(0)| \leq C|x|2^j$ while for $j \geq 0$, we have $f_j = \Delta F_j$ with $\|F_j\|_\infty \leq C4^{-j}$. Thus, we may define $\dot{B}^{0,\infty}_\infty$ as a Banach space of tempered distributions modulo the constants through the double condition, for b to belong to $\dot{B}^{0,\infty}_\infty$, that $(\Delta_j b)_{j\in\mathbb{Z}} \in l^\infty(\mathbb{Z}, L^\infty)$ and that there exist a sequence of constants $(\gamma_j) \in \mathbb{C}^{\mathbb{Z}}$ so that $b = \sum_{j\in\mathbb{Z}} \Delta_j b + \gamma_j$ in $\mathcal{S}'(\mathbb{R}^d)$. Equivalently, $b \in \mathcal{S}'(\mathbb{R}^d)$, $(\Delta_j b)_{j\in\mathbb{Z}} \in l^\infty(\mathbb{Z}, L^\infty)$ and for $l = 1, \ldots, d$, $\partial_l b = \sum_{j\in\mathbb{Z}} \partial_l \Delta_j b$ in $\mathcal{S}'(\mathbb{R}^d)$: indeed, if $(\Delta_j b)_{j\in\mathbb{Z}} \in l^\infty(\mathbb{Z}, L^\infty)$ and if $\beta = \sum_{j<0} \Delta_j b(x) - \Delta_j b(0) + \sum_{j\geq 0} \Delta_j b$, we find that $\hat{\beta} - \hat{b}$ is supported in $\{0\}$, hence is a polynomial P ; but a polynomial has a homogeneous Littlewood–Paley decomposition equal to 0, so that the assumption $\partial_l b = \sum_{j\in\mathbb{Z}} \partial_l \Delta_j b$ gives $\vec{\nabla} P = 0$; hence, $\beta - b$ is constant.

In particular, we see that BMO (defined as the dual of $\mathcal{H}^1$) is a subspace of $\dot{B}^{0,\infty}_\infty$: $\Delta_j b(x) = \langle b(y) | 2^{jd}\Psi(2^j(x-y))\rangle$ where $\Psi = \mathcal{F}^{-1}\psi$; hence, we find that $\|\Delta_j b\|_\infty \leq \|b\|_{BMO}\|\Psi\|_{\mathcal{H}^1}$; moreover, $\partial_l S_j b = \langle b(y)|2^{j(d+1)}\Psi_l(2^j(x-y))\rangle$, where $\Psi_l = \partial_l \mathcal{F}^{-1}\varphi$; this gives $\|\partial_l S_j b\|_\infty \leq 2^j \|b\|_{BMO}\|\Psi_l\|_{\mathcal{H}^1}$ and $\partial_l b = \sum_{j\in\mathbb{Z}} \partial_l \Delta_j b$.

We now prove the theorem :

$(A1) \Rightarrow (A3)$: for $q > 1$, use the atomic decomposition with L^p atoms, where $1/p + 1/q = 1$; for $q = 1$, use the result for $q > 1$ and the Hölder inequality.

$(A3) \Rightarrow (A2)$ is obvious (Hölder inequality).

$(A2) \Rightarrow (A1)$: use the atomic decomposition with bounded atoms.

$(A3) \Rightarrow (A4)$: We first assume that ψ is compactly supported, with support in $B(0,1)$. Then for $|x - x_0| \leq R$ and $0 < t < R$, the values of $b * \psi_t(x)$ depends only on the values of b on the ball $B(x_0, 2R)$ and, since $\int \psi\, dx = 0$, we may change b into $b - m_{B(x_0,2R}b$. Moreover, for $f \in L^2$, we have

$$\iint |f * \psi_t(x)|^2 \frac{dt}{t} dx = \frac{1}{(2\pi)^d} \iint |\hat{f}(\xi)|^2\ |\hat{\psi}(t\xi)|^2\ \frac{dt}{t} d\xi \leq C_\psi \|f\|_2^2$$

with $C_\psi = \max_{|\xi|=1} \int_0^{+\infty} |\hat{\psi}(t\xi)|^2\ \frac{dt}{t} \leq C\|\psi\|^2_{W^{1,1}}$. This gives

$$\iint_{0<t<R, |x-x_0|<R} |b * \psi_t(x)|^2 \frac{dt}{t} dx \leq C_\psi \int_{B(x_0,2R)} |b - m_{B(x_0,2R}b|^2\ dx \leq CR^d.$$

This proves $(A4)$ for ψ with support in $B(0,1)$. For $\psi \in \mathcal{S}'$ with $\int \psi\, dx = 0$, we write $\psi = \sum_{j=1}^d \partial_j \omega_j$ with $\omega_j \in \mathcal{S}$. ω_j may be decomposed as a sum of $\omega_j = \sum_{n\in\mathbb{N}} \omega_{j,n}$, with $\omega_{j,n}$ supported in $B(0, 2^n)$ and $\sum_{n\in\mathbb{N}} 2^{n(d+2)}\|\omega_{j,n}\|_{W^{2,1}} < \infty$, and this is enough to arrive at an estimate.

$(A4) \Rightarrow (A5)$: write $\frac{\partial}{\partial x_j} e^{t\Delta} b = \frac{1}{\sqrt{t}}\ b * \psi_{\sqrt{t}}$ for some $\psi \in \mathcal{S}(\mathbb{R}^d)$ and, for $\sqrt{t} = \tau$, $\frac{dt}{t} = \frac{2d\tau}{\tau}$.

$(A5) \Rightarrow (A6)$: We first prove that $\partial_i b \in \mathcal{S}'_0$ for $i = 1, \ldots, d$. Indeed, for $j \in \mathbb{Z}$ and $0 < t < 4^{-j}$, we may write $S_j \partial_i b = e^{-t\Delta} S_j e^{t\Delta} \partial_i b$. The operator

$e^{-t\Delta}S_j$ is a convolution operator with $2^{jd}F_{4^j t}(2^j x)$, where the functions $F_\theta = \mathcal{F}^{-1}(e^{\theta|\xi|^2}\varphi(\xi))$ $(0<\theta<1)$ are a bounded set in $\mathcal{S}(\mathbb{R}^d)$. Thus, $|2^{jd}F_{4^j t}(2^j x)| \leq C\frac{2^{jd}}{(1+|2^j x|)^{d+1}}$. Summing for $t\in(0,4^{-j})$, we get

$$|S_j\partial_i b(x)| \leq 4^j\int_0^{4^{-j}}\int_{\mathbb{R}^d}\frac{2^{jd}}{(1+|2^j(x-y)|)^{d+1}}|e^{t\Delta}\partial_i b(y)|\ dy\ dt$$

which gives

$$|S_j\partial_i b(x)| \leq C2^j\sum_{k\in\mathbb{Z}^d}\frac{1}{(1+|k|)^{d+1}}\big(2^{jd}\int_0^{4^{-j}}\int_{|y-x-\frac{k}{2^j}|\leq\frac{\sqrt{d}}{2^j}}|e^{t\Delta}\partial_i b(y)|^2\ dy\ dt\big)^{1/2}$$

and thus, $\|S_j\partial_i b\|_\infty \leq C2^j$. Thus, $\partial_i b\in\mathcal{S}_0'$ and $b\in\dot{B}^{-1,\infty}_\infty$.

We now prove that the family $(\Delta_j b)$ defines a Carleson measure on $\mathbb{Z}\times\mathbb{R}^d$. For $j\in\mathbb{Z}$ and $4^{-j-1}<t<4^{-j}$, we may write $\Delta_j b = \sum_{i=1}^d e^{-t\Delta}\frac{\partial_i}{\Delta}\Delta_j e^{t\Delta}\partial_i b$. The operator $e^{-t\Delta}\frac{1}{\Delta}\Delta_j$ is a convolution operator with $2^{j(d-2)}F_{4^j t}(2^j x)$, where the functions $F_\theta = \mathcal{F}^{-1}(e^{\theta|\xi|^2}\frac{\psi(\xi)}{|\xi|^2})$ $(1/4<\theta<1)$ are a bounded set in $\mathcal{S}(\mathbb{R}^d)$. We then decompose F_θ into $F_\theta\varphi(x)+\sum_{k\geq 0}F_\theta\psi(\xi/2^k) = g_{-1,\theta}+\sum_{k\geq 0}g_{k,\theta}$, with φ supported in $B(0,1)$ and ψ supported in $\{x\ /\ 1/2\leq|x|\leq 2\}$. We have, for $|\alpha|\leq 2$ and $L\in\mathbb{N}$, $|\partial^\alpha g_{k,\theta}|\leq C_L 2^{-kL}\frac{1}{(1+|x|)^{d+1}}$. This gives for every $L\in\mathbb{N}$:

$$\|2^{jd}(\partial_i g_{k,\theta})(2^j x)*f\|_2^2 \leq C_L 2^{-2kL}\|f\|_2^2$$

We thus write

$$\Delta_j b = 4^{j+1}\int_{4^{-(j+1)}}^{4^{-j}}\sum_{i=1}^d 2^{j(d-1)}\sum_{k\geq -1}(\partial_i g_{k,4^j t})(2^j x)*(e^{t\Delta}\partial_i b)\ dt$$

and get

$$\int_{|x-x_0|\leq R}|\Delta_j b(x)|^2\ dx \leq$$

$$\leq 4^{(j+2)}\sum_{k\geq -1}2^k\int_{4^{-(j+1)}}^{4^{-j}}\int_{|x-x_0|\leq R}|\sum_{i=1}^d 2^{j(d-1)}(\partial_i g_{k,4^j t})(2^j x)*(e^{t\Delta}\partial_i b)|^2\ dt\ dx$$

$$\leq C_L\sum_{k\geq -1}2^{k(1-2L)}\int_{4^{-(j+1)}}^{4^{-j}}\int_{|x-x_0|\leq R+2^{k+1-j}}|e^{t\Delta}\partial_i b|^2\ dt\ dx.$$

Finally:

$$\sum_{2^j R\geq 1}\int_{|x-x_0|\leq R}|\Delta_j b(x)|^2\ dx \leq$$

$$\leq C_L\sum_{k\geq -1}2^{k(1-2L)}\int_0^{R^2}\int_{|x-x_0|\leq R(1+2^{k+1})}|e^{t\Delta}\partial_i b|^2\ dt\ dx$$

$$\leq CC_L R^d \sum_{k\geq -1} 2^{k(d+1-2L)}$$

which gives the result for $2L > d+1$.

The Fourier spectrum of $S_{j-2}f\Delta_j b$ is contained in the annulus $\{\xi \;/\; 2^{j-2} \leq |\xi| \leq 2^{j+2}\}$, thus we have : $\|\pi(f,b)\|_2^2 \leq C\sum_{j\in\mathbb{Z}}\int |S_{j-2}f|^2\, |\Delta_j b|^2\, dx$. We may then apply Proposition 10.1 to get the boundedness of $\pi(.,b)$ on L^2.

$(A6) \Rightarrow (A1)$: We easily check that if $b \in \dot{B}^{0,\infty}_\infty$ and $f \in L^2$ then the paraproduct $\pi(b,f) = \sum_{j\in\mathbb{Z}} S_{j-1}f\ \Delta_j b$ is well defined in $\mathcal{S}'(\mathbb{R}^d)$ (it belongs to $B^{-d/2,\infty}_\infty$). The Schwartz kernel theorem for continuous linear operators from $\mathcal{D}$ to $\mathcal{D}'$ allows us to speak from the distribution kernel $K(x,y)$ of $\pi(b,.)$. We have $K(x,y) = \sum_{j\in\mathbb{Z}} k_j(x,y)$ with $k_j(x,y) = 2^{jd}\Phi(2^j(x-y))\Delta_j b(y)$ with $\Phi \in \mathcal{S}(\mathbb{R}^d)$. This gives $|k_j(x,y)| \leq C2^{jd}\min(1, \frac{1}{(2^j|x-y|)^{d+1}}$ and $|\vec{\nabla}_x k_j(x,y)| + |\vec{\nabla}_y k_j(x,y)| \leq C2^{j(d+1)}\min(1, \frac{1}{(2^j|x-y|)^{d+2}})$, and finally we get that, outside of the diagonal set Δ, the distribution K is a locally Lipschitzian function so that $|K(x,y)| \leq C\frac{1}{|x-y|^d}$ and $|\vec{\nabla}_x K(x,y)| + |\vec{\nabla}_y K(x,y)| \leq C\frac{1}{|x-y|^{d+1}}$.

If $\pi(b,.)$ is a bounded operator on L^2, then it is a Calderón–Zygmund operator and we have seen in Chapter 6 that by duality, it maps L^∞ to BMO. Thus, there exists $\beta \in BMO$ so that, for all $\psi \in \mathcal{D}(\mathbb{R}^d)$ with $\int \psi\, dx = 0$, $\int {}^tT\psi\, dx = \langle\beta|\psi\rangle$. But $\int {}^tT\psi\, dx$ may easily be computed: we have, for $\omega \in \mathcal{D}$ equal to 1 in a neighborhood of the support of ψ and for $y_0 \in$ Supp ψ, ${}^tT\psi = \omega\sum_{j\in\mathbb{Z}}\int k_j(y,x)\psi(y)\ dy + (1-\omega(x))\sum_{j\in\mathbb{Z}}\int(k_j(y,x) - k_j(y_0,x))\psi(y)\ dy$. Since $\sum_{j\in\mathbb{Z}}\int |(k_j(y,x)-k_j(y_0,x))\psi(y)|\ dy$ is bounded by $\frac{C}{|x-y_0|^{d+1}}$ far from the support of ψ, we obtain $\int\ {}^tT\psi\ dx = \sum_{j\in\mathbb{Z}}\int\int k_j(y,x)\psi(y)\ dy\ dx = \sum_{j\in\mathbb{Z}}\int \Delta_j b(y)\psi(y)\ dy$. This gives that $\bar{\beta} = \sum_{j\in\mathbb{Z}}\Delta_j b$ in $\mathcal{D}'/\mathbb{C}$. Thus, $b \in BMO$. ◻

2. The $T(1)$ theorem

David and Journé have given the following L^2 boundedness criterion for singular integral operators [DAVJ 84]:

Theorem 10.2: (The $T(1)$ theorem)

Let T be a singular integral operator. Then the following assertions are equivalent :

(A1) T is bounded on L^2: there exists a constant C so that for all $\phi \in \mathcal{D}(\mathbb{R}^d)$ $T(\phi) \in L^2$ and $\|T(\phi)\|_2 \leq C\|\phi\|_2$.

(A2) $T(1) \in BMO$, $T^*(1) \in BMO$ and $T \in WBP$.

Proof: $(A1) \Rightarrow (A2)$ is obvious: if T is a singular integral operator bounded on L^2, then T and T^* are Calderón–Zygmund operators and we know that a Calderón–Zygmund operator maps L^∞ to BMO.

$(A2) \Rightarrow (A1)$ is a direct consequence of Theorem 10.1, since we may modify T by $T - \pi(., T(1)) - \pi^*(., T^*(1))$ to get a singular integral operator $\tilde{T}$ so that $\tilde{T} \in WBP$, $\tilde{T}(1) = \tilde{T}^*(1) = 0$. We have proved in Chapter 9 that such a singular integral operator is bounded on L^2. □

We may now collect some results on singular integral operators:

Proposition 10.2:

a) Let $\omega_{j,k}$ and $\theta_{j,k}$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$, be two sequences of functions on $\mathbb{R}^d$ so that for some positive constants C and δ:

$$\begin{cases} |\omega_{j,k}(x)| \leq C 2^{jd/2} \frac{1}{1+|2^j x - k|^{d+\delta}} \\ |\omega_{j,k}(x) - \omega_{j,k}(y)| \leq C 2^{j(d/2+\delta)} |x-y|^\delta \left(\frac{1}{(1+|2^j x - k|)^{d+\delta}} + \frac{1}{(1+|2^j y - k|)^{d+\delta}}\right) \\ \int \omega_{j,k}\, dx = 0 \end{cases}$$

and

$$\begin{cases} |\theta_{j,k}(x)| \leq C 2^{jd/2} \frac{1}{1+|2^j x - k|^{d+\delta}} \\ |\theta_{j,k}(x) - \theta_{j,k}(y)| \leq C 2^{j(d/2+\delta)} |x-y|^\delta \left(\frac{1}{(1+|2^j x - k|)^{d+\delta}} + \frac{1}{(1+|2^j y - k|)^{d+\delta}}\right) \end{cases}$$

Then the operator $T(f) = \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^d} \langle f | \omega_{j,k} \rangle \theta_{j,k}$ is a singular integral operator, $T \in WBP$ and $T(1) = 0$.

b) When $b \in BMO$ and when $(\psi_{\epsilon,j,k})_{1 \leq \epsilon \leq 2^d-1, j \in \mathbb{Z}, k \in \mathbb{Z}^d}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ with compactly supported wavelets ψ_ϵ and compactly supported scaling function φ (with φ continuously differentiable), then the operator $\Pi(b,.)$ defined by $\Pi(b,f) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \epsilon \leq 2^d-1} 2^{jd/2} \langle f | \varphi_{j,k} \rangle \, \langle b | \psi_{\epsilon,j,k} \rangle \, \psi_{\epsilon,j,k}$ is a Calderón–Zygmund operator T so that $T(1) = b$ and $T^*(1) = 0$.

c) When T is a Calderón–Zygmund operator, then it may be decomposed into $T = T_0 + \Pi(T(1),.) + \Pi(T^*(1),.)^*$ with

$$T_0(f) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \epsilon \leq 2^d-1} \langle f | \psi_{\epsilon,j,k} \rangle \omega_{\epsilon,j,k}$$

where $\psi_{\epsilon,j,k}$ are the wavelets described in b) and where the functions $\omega_{\epsilon,j,k} = T_0(\psi_{\epsilon,j,k})$ satisfy for some positive constants C and δ:

$$\begin{cases} |\omega_{\epsilon,j,k}(x)| \leq C 2^{jd/2} \frac{1}{1+|2^j x - k|^{d+\delta}} \\ |\omega_{\epsilon,j,k}(x) - \omega_{\epsilon,j,k}(y)| \leq C 2^{j(d/2+\delta)} |x-y|^\delta \left(\frac{1}{(1+|2^j x - k|)^{d+\delta}} + \frac{1}{(1+|2^j y - k|)^{d+\delta}}\right) \\ \int \omega_{\epsilon,j,k}\, dx = 0 \end{cases}$$

d) When T and S are two Calderón–Zygmund operators so that $T(1) = 0$ and $S^*(1) = 0$, then $T \circ S$ is a Calderón–Zygmund operator.

e) When T is a Calderón–Zygmund operator so that $T^*(1) = 0$, then T is bounded from the Hardy space $\mathcal{H}^1$ to $\mathcal{H}^1$.

f) When T is a Calderón–Zygmund operator so that $T(1) = 0$, then T is bounded from BMO to BMO.

Remark: $\Pi(b,.)$ is a slight modification , proposed by Y. Meyer [MEY 92], of the paraproduct operator $\pi(b,.)$ of Bony [BON 81] (described in Section 1).

Proof: a) We first estimate the sum $\sum_{k\in\mathbb{Z}^d} \bar{\omega}_{j,k}(y)\theta_{j,k}(x) = k_j(x,y)$:
- $|k_j(x,y)| \le C2^{jd}\Omega(2^j(x-y))$ with $\Omega(x) = \frac{1}{1+|x|^{d+\delta}}$ with $\Omega(x) = \frac{1}{1+|x|^{d+\delta}}$;
- for $|h| \le \max(2^{-j}, |x-y|/2)$:
  - $|k_j(x,y) - k_j(x+h,y)| \le C2^{jd}(2^j|h|)^{\delta/2}\Omega(2^j(x-y))$
  - $|k_j(x,y) - k_j(x,y+h)| \le C2^{jd}(2^j|h|)^{\delta/2}\Omega(2^j(x-y))$.

This gives
– $\sum_{j\in\mathbb{Z}} |k_j(x,y)| \le C(\sum_{2^j|x-y|\le 1} 2^{jd} + \sum_{2^j|x-y|>1} 2^{-j\delta}|x-y|^{-d-\delta}) \le C'|x-y|^{-d}$
– for $|h| \le |x-y|/2$:
  – $\sum_{j\in\mathbb{Z}} |k_j(x,y) - k_j(x+h,y)| \le \sum_{j\in\mathbb{Z}} C2^{jd}(2^j|h|)^{\delta/2}\Omega(2^j(x-y)) \le C'|h|^{\delta/2}|x-y|^{-d-\delta/2}$
  – $\sum_{j\in\mathbb{Z}} |k_j(x,y) - k_j(x,y+h)| \le C'|h|^{\delta/2}|x-y|^{-d-\delta/2}$.

Moreover, if $x_0 \in \mathbb{R}^d$, $r_0 > 0$ and if f and g are supported in $B(x_0,r_0)$, we have:
- $|\int\int k_j(x,y)f(y)\bar{g}(x)\ dx\ dy| \le C2^{jd}\|f\|_1\|g\|_1$, hence we get the estimate $\sum_{2^j r_0\le 1} |\int\int k_j(x,y)f(y)\bar{g}(x)\ dx\ dy| \le Cr_0^{-d}\|f\|_1\|g\|_1 \le C'\|f\|_2\|g\|_2$;
- since $\int \omega_{j,k}dx = 0$, we may write $\omega_{j,k} = \sum_{l=1}^d \partial_l\omega_{j,k,l}$ where $\omega_{j,k,l}(x) = C\int \omega_{j,k}(x-y)(\frac{y_l}{|y|^d} - \frac{x_l}{|x|^d})\ dy$, we find $|\omega_{j,k,l}(x)| \le 2^{j(d-1)}(1+|2^jx|)^{-d+1-\delta}$ and $\sum_{k\in\mathbb{Z}^d} |\omega_{j,k,l}(y)\theta_{j,k}(x)| \le C2^{j(d-1)}(1+|2^j(x-y)|)^{-d+1-\delta}$; we then choose α so that $\frac{1}{d+\delta-1} < \alpha < \frac{d}{d-1}$ (which is always possible, since $d(d+\delta-1)-(d-1) = (d-1)^2 + d\delta > 0$) and we get

$$|\int\int k_j(x,y)f(y)\bar{g}(x)\ dx\ dy| \le C2^{j(d-1-d/\alpha)}\|\vec{\nabla}f\|_{\frac{\alpha}{\alpha-1}}\|g\|_1;$$

hence, $\sum_{2^j r_0>1} |\int\int k_j(x,y)f(y)\bar{g}(x)\ dx\ dy| \le Cr_0^{-d+1+d\alpha}\|\vec{\nabla}f\|_{\frac{\alpha}{\alpha-1}}\|g\|_1$ and this allows us to conclude that T is well defined and satisfies the weak boundedness property.

Now, we prove $T(1) = 0$. Indeed, let $g \in \mathcal{D}$ and let $\varphi \in \mathcal{D}$. We want to prove that, if $\int g\ dx = 0$, then $\lim_{R\to\infty}\langle T(\varphi(x/R))|g\rangle = 0$. We have proved that

$$|\int\int k_j(x,y)\varphi(y/R)\bar{g}(x)\ dx\ dy| \le C(2^jR)^{(d-1-d/\alpha)}\|\vec{\nabla}\varphi\|_{\frac{\alpha}{\alpha-1}}\|g\|_1,$$

with $d-1-d/\alpha < 0$, and thus we have no problem with the terms indexed by $j \ge 0$. For the sum over $j > 0$, we use the condition $\int g\ dx = 0$: we find that $\int\int |(k_j(x,y) - k_j(y,y))g(x)|\ dy\ dx \le C2^{j\delta}$ while $\int\int k_j(x,y)\bar{g}(x)\ dy\ dx = 0$, thus $\sum_{j<0}\int k_j(x,y)\bar{g}(x)\ dx$ converges in $L^1(dy)$ to a function whose integral is equal to 0.

b) is now obvious. We first notice that $2^{jd/2}|\langle b|\psi_{\epsilon,j,k}\rangle| \leq C\|b\|_{BMO}$. Hence, we may deduce from a) that $T = \Pi(b,.)$ is a singular integral operator, $T \in WBP$ and $T^*(1) = 0$. Moreover, $T(1) = b$: indeed, if $f \in \mathcal{D}$ with $\int f\ dx = 0$, we have $\langle f|T(1)\rangle = \int T^* f\ dx = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{1\leq\epsilon\leq 2^d-1}\langle f|\psi_{\epsilon,j,k}\rangle\ \langle\psi_{\epsilon,j,k}|b\rangle$. If f is supported by $B(x_0,r_0)$, the series limited to $j > j_0$ uses the value of b only on the ball $B(x_0, r_0 + C2^{-j_0})$; since b is locally square integrable, we may rewrite the truncated series as $\int f(b - m_{B(x_0,r_0)}b)\ dx - \sum_{k\in\mathbb{Z}^d}\langle f|\varphi_{j_0,k}\rangle\ \langle\varphi_{j_0,k}|b - m_{B(x_0,r_0)}b\rangle$. When j_0 goes to $-\infty$, there exist an index J and a fixed finite set of indices K so that, for $j_0 < J$ and $k \notin K$, $\langle f|\varphi_{j_0,k}\rangle = 0$, while, for $k \in K$, $|\langle f|\varphi_{j_0,k}\rangle| \leq C2^{j_0(d/2+1)}$ and $|\langle\varphi_{j_0,k}|b - m_{B(x_0,r_0)}b\rangle| \leq C|j_0|2^{-j_0 d/2}$ (see Stein [STE 93] for the last estimate).

In order to prove c), we just need to estimate the size of the functions $T_0(\psi_{\epsilon,j,k}) = \omega_{\epsilon,j,k}$. We easily estimate the size of $\omega_{\epsilon,j,k}(x)$ when $|x - k/2^j| \geq C2^{-j}$: we write, using the kernel K_0 of T_0, that $\omega_{\epsilon,j,k}(x) = \int(K_0(x,y) - K_0(x,k/2^j))\psi_{\epsilon,j,k}(y)\ dy$; hence, $|\omega_{\epsilon,j,k}(x)| \leq C2^{-j\delta}|x - k/2^j|^{-d-\delta}\|\psi_{\epsilon,j,k}\|_1 \leq C2^{jd/2}|2^j x - k|^{-d-\delta}$, while for $|2^j h| \leq 1$ we have $|\omega_{\epsilon,j,k}(x) - \omega_{\epsilon,j,k}(x+h)| \leq \int|K_0(x,y) - K_0(x+h,y)||\psi_{\epsilon,j,k}(y)|\ dy \leq C2^{jd/2}(2^j|h|)^{\delta}|2^j x - k|^{-d-\delta}$. We now consider the case $|x - k/2^j| \leq C2^{-j}$. We have already proved (through Theorem 9.4) that $T_0 f = \sum_{l\in\mathbb{Z}}\int K_l(x,y)f(y)\ dy$, where for some positive constants C and δ,

$$
\begin{gathered}
|K_l(x,y)| \leq C2^{ld}\frac{1}{1+|2^l(x-y)|^{d+\delta}} \\
|K_l(x,y) - K_l(x+z,y)| \leq C2^{l(d+\delta/2)}|z|^{\delta/2}\left(\frac{1}{(1+|2^l(x-y)|)^{d+\delta}} + \frac{1}{(1+|2^l(x+z-y)|)^{d+\delta}}\right) \\
|K_l(x,y) - K_l(x,y+z)| \leq C2^{l(d+\delta/2)}|z|^{\delta/2}\left(\frac{1}{(1+|2^l(x-y)|)^{d+\delta}} + \frac{1}{(1+|2^l(x-z-y)|)^{d+\delta}}\right) \\
\int K_l(x,y)\ dy = 0
\end{gathered}
$$

We then write $\int K_l(x,y)\psi_{\epsilon,j,k}(y)\ dy = \int(K_l(x,y) - K_l(x,k/2^j))\psi_{\epsilon,j,k}(y)\ dy = \int K_l(x,y)(\psi_{\epsilon,j,k}(y) - \psi_{\epsilon,j,k}(x))\ dy$ and we obtain $|\int K_l(x,y)\psi_{\epsilon,j,k}(y)\ dy| \leq 2^{jd/2}2^{-|j-l|\delta/2}$. Hence, $|\omega_{\epsilon,j,k}(x)| \leq C2^{jd/2}$. Similarly, we find $|\int(K_l(x,y) - K_l(x+h,y))\psi_{\epsilon,j,k}(y)\ dy| \leq 2^{jd/2}\min(2^{l\delta/2}|h|^{\delta/2}, 2^{(j-l)\delta/2})$ and thus $|\omega_{\epsilon,j,k}(x) - \omega_{\epsilon,j,k}(x+h)| \leq C(2^j|h|)^{\delta/4}$.

d) is a direct consequence of c). Indeed, write $T = \Pi(T^*(1),.)^* + T_0$ and $S = \Pi(S(1),.) + S_0$ and write the kernels of T and S as the series (converging in $\mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$): $\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{1\leq\epsilon\leq 2^d-1}T(\psi_{\epsilon,j,k})(x)\psi_{\epsilon,j,k}(y)$ and $\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{1\leq\epsilon\leq 2^d-1}\bar{S}^*(\psi_{\epsilon,j,k})(y)\psi_{\epsilon,j,k}(x)$; we obtain that the kernel of $T \circ S$ is given by $\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{1\leq\epsilon\leq 2^d-1}T(\psi_{\epsilon,j,k})(x)\bar{S}^*(\psi_{\epsilon,j,k})(y)$ and thus may be written as $\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{1\leq\epsilon\leq 2^d-1}\theta_{\epsilon,j,k}(x)\bar{\omega}_{\epsilon,j,k}(y)$, where

$$
\left\{
\begin{array}{c}
\theta_{\epsilon,j,k}(x) = 2^{jd/2}\langle\psi_{\epsilon,j,k}|T^*(1)\rangle\varphi_{j,k}(x) + T_0(\psi_{\epsilon,j,k})(x) \\
\text{and} \\
\omega_{\epsilon,j,k}(y) = 2^{jd/2}\langle\psi_{\epsilon,j,k}|S(1)\rangle\varphi_{j,k}(x) + S_0^*(\psi_{\epsilon,j,k})(y)
\end{array}
\right.
$$

Now, we apply the same computations as for point a) to conclude that we have good estimates for the kernel of $T \circ S$.

e) may be viewed as a consequence of d) since $\mathcal{H}^1$ is characterized through the Riesz transforms as $\mathcal{H}^1 = \{f \in L^1 \ / \ \text{For } j = 1, \dots, d, \ R_j f \in L^1\}$ (Fefferman and Stein [FEFS 72]). Another way is to check directly that the image of an atom is bounded in $\mathcal{H}^1$: we have seen in Chapter 6 that the image of an atom a supported by $B(0,0)$ through a Calderón-Zygmund operator T is locally square-integrable and is bounded near infinity with a size $0(|x|^{-d-\epsilon}$; if $T^*(1) = 0$, we have, moreover, $\int T(a)\, dx = 0$ and this is enough to prove that $T(a) \in \mathcal{H}^1$.

f) is a consequence of e) (through the duality between $\mathcal{H}^1$ and BMO). □

An important example of Calderón–Zygmund operator is the so-called Calderón commutator:

Theorem 10.3: (The Calderón commutator)

Let A be a Lipschitzian function on $\mathbb{R}^d$ (i.e., $\vec{\nabla} A \in (L^\infty)^d$) and σ be a smooth function on $\mathbb{R}^d \setminus \{0\}$ homogeneous with degree 1. Let M_A be the pointwise multiplication operator by A ($M_A f = Af$) and T_σ the Fourier multiplication operator of symbol σ ($\mathcal{F}(T_\sigma f) = \sigma \hat{f}$). Then the commutator between T_σ and M_A, $[T_\sigma, M_A] = T_\sigma M_A - M_A T_\sigma$, is a Calderón–Zygmund operator and satisfies $\|[T_\sigma, M_A]\|_{CZO} \le C \|\vec{\nabla} A\|_\infty \sum_{|\alpha| < d+3} \| |\xi|^{|\alpha|-1} \partial^\alpha \sigma \|_{L^\infty(\mathbb{R}^d \setminus \{0\})}$.

Proof: The operator T_σ is a convolution operator with a distribution k_σ. Since σ is smooth outside from 0 and homogeneous with degree 1, we find that, outside from 0, k_σ is smooth and homogeneous with degree $-d-1$: on $\mathbb{R}^d \setminus \{0\}$, $k_\sigma(x) = \omega(\frac{x}{|x|}) \frac{1}{|x|^{d+1}}$ where ω is smooth on S^{d-1}. This gives that the distribution kernel of $T = [T_\sigma, M_A]$ is given outside from the diagonal set Δ by $K(x,y) = \omega(\frac{x-y}{|x-y|}) \frac{A(y)-A(x)}{|x-y|^{d+1}}$. Thus, T is a singular integral operator. We shall now apply the $T(1)$ theorem to T.

We clearly have $\|T_\sigma f\|_2 \le C \|\vec{\nabla} f\|_2$. Thus, if f and g are supported by the ball $B(0,1)$, if $x_0 \in \mathbb{R}^d$ and $R > 0$, we write $[T_\sigma, M_A] = [T_\sigma, M_{A-A(x_0)}]$ and $|\langle M_{A-A(x_0)} T_\sigma(u(\frac{x-x_0}{R})) | v(\frac{x-x_0}{R}) \rangle| \le \|\vec{\nabla} A\|_\infty R \|T_\sigma(u(\frac{x-x_0}{R}))\|_2 \|v(\frac{x-x_0}{R})\|_2$ $\le C R^d \|\vec{\nabla} A\|_\infty \|\vec{\nabla} u\|_2 \|v\|_2$. Thus, $T \in WBP$.

Now, we compute $T^*(1)$. We may assume $A(0) = 0$, hence $|A(x)| \le \|\vec{\nabla} A\|_\infty |x|$. We write T_σ as $T = \sum_{i=1}^d \partial_i \frac{\partial_i}{\Delta} T_\sigma = \sum_{i=1}^d \partial_i T_i$ where T_i are Calderón-Zygmund operators. Thus, for $\psi \in \mathcal{D}$ with $\int \psi \, dx = 0$ (so that $\psi = \sum_{k=1}^d \Psi_k$ with $\Psi_k \in \mathcal{D}$), $[T_\sigma, M_A]\psi = T_\sigma(A\psi) - A T_\sigma(\psi) = \sum_{i=1}^d \partial_i T_i(A\psi) - \sum_{i=1}^d \sum_{k=1}^d \partial_i \partial_k (A T_i(\Psi_k)) + \sum_{i=1}^d \sum_{k=1}^d \partial_i((\partial_k A) T_i(\Psi_k)) + \sum_{i=1}^d (\partial_i A) T_i(\psi)$. Now, for $\varphi \in \mathcal{D}$, we write for $p \in (1, \frac{d}{d-1})$ and $1/p + 1/q = 1$:

- $|\langle \varphi(\frac{x}{R}) | \partial_i T_i(A\psi) \rangle| \le C \|A\psi\|_p \|\partial_i \varphi\|_q R^{-1+d/q}$.
- $|\langle \varphi(\frac{x}{R}) | \partial_i((\partial_k A) T_i(\Psi_k)) \rangle| \le C \|\partial_k A\|_\infty \|\Psi_k\|_p \|\partial_i \varphi\|_q R^{-1+d/q}$.
- $|\langle \varphi(\frac{x}{R}) | \partial_i \partial_k (A T_i(\Psi_k)) \rangle| \le C \|\vec{\nabla} A\|_\infty \|\Psi_k\|_p \| \, |x| \partial_i \partial_k \varphi\|_q R^{-1+d/q}$.

Thus, we have

$$\lim_{R\to\infty}\langle\varphi(\frac{x}{R})|T_\sigma(A\psi)-AT_\sigma(\psi)\rangle=\lim_{R\to\infty}\langle\varphi(\frac{x}{R})|\sum_{i=1}^{d}(\partial_i A)T_i(\psi)\rangle$$

which gives $T^*(1)=\sum_{i=1}^d T_i^*(\partial_i\bar{A})\in BMO$. Similarly, we have $T(1)\in BMO$. □

3. The local Hardy space h^1 and the local space *bmo*

The Hardy space $\mathcal{H}^1$ and the space BMO are homogeneous spaces in the sense that their norms are homogeneous under dilations (for all positive λ, $\|f(\lambda x)\|_{\mathcal{H}^1}=\lambda^{-d}\ \|f\|_{\mathcal{H}^1}$ and $\|f(\lambda x)\|_{BMO}=\|f\|_{BMO}$); but BMO is not embedded into $\mathcal{D}'$, its elements being defined only modulo constants. These spaces admit a local counterpart, where dilations are not isometries but which are Banach spaces of distributions: the local Hardy space h^1 and the local space *bmo*:

Definition 10.4: (Local atomic Hardy space)

The local (atomic) Hardy space $h^1(\mathbb{R}^d)$ is defined as: $f\in\mathcal{H}^1$ if and only if f may be decomposed into a series $f=\sum_{j\in\mathbb{N}}\lambda_j a_j$ where Supp a_j is contained in a ball $B(x_j,R_j)$, $R_j\leq 1$, $\|a_j\|_\infty\leq(R_j)^{-d}$, $\int a_j\ dx=0$ if $R_j<1$ and $\sum|\lambda_j|<\infty$. The function a_j is called an atom carried by the ball $B(x_j,R_j)$.

It is normed by $\|f\|_{h^1}=\inf\{\sum_j\ |\lambda_j|\ /\ f=\sum_j\ \lambda_j\ a_j\}$ where the minimum runs over all the atomic decompositions of f.

The local space bmo is defined as $bmo=(h^1)^*$.

Theorem 10.4: (Characterizations of *bmo*)

Let $b\in\mathcal{D}'(\mathbb{R}^d)$. Then, the following assertions are equivalent:

(A1) $b\in(h^1)'$: there exists a constant C such that for all $\omega\in\mathcal{D}(\mathbb{R}^d)$ we have $|\langle b|\omega\rangle|\leq C\ \|\omega\|_{h^1}$.

(A2) b is a uniformly locally integrable function ($\sup_{x_0\in\mathbb{R}^d}\int_{|x-x_0|<1}\ |b|\ dx<\infty$) and satisfies: $\sup_{|B|<1}\frac{1}{|B|}\int_B\ |b-m_Bb|\ dx<\infty$.

(A3) For $q\in[1,\infty)$, we have that b is uniformly locally L^q and satisfies: $\sup_{|B|<1}\frac{1}{|B|}\int_B\ |b-m_Bb|^q dx<\infty$.

(A4) b is a uniformly locally integrable function and for all $\psi\in\mathcal{S}(\mathbb{R}^d)$ such that $\int\psi\ dx=0$, defining $\psi_t(x)=\frac{1}{t^d}\psi(\frac{x}{t})$, the measure $|b*\psi_t(x)|^2\frac{dt}{t}dx$ is a local Carleson measure on $(0,\infty)\times\mathbb{R}^d$:

$$\forall T>0\qquad\sup_{0<R<T,x_0\in\mathbb{R}^d}\frac{1}{R^d}\int\int_{0<t<R,|x-x_0|<R}\ |b*\psi_t(x)|^2\frac{dt}{t}dx<\infty$$

(A5) $b\in\mathcal{S}'(\mathbb{R}^d)$, $e^\Delta b\in L^\infty$ and

$$\text{for } j=1,\ldots,d,\qquad\sup_{0<R<1,x_0\in\mathbb{R}^d}\frac{1}{R^d}\int\int_{0<t<\sqrt{R},|x-x_0|<R}\ |\frac{\partial}{\partial x_j}e^{t\Delta}b|^2dt\ dx<\infty$$

(A6) $b \in B^{0,\infty}_{\infty}$ *and the operator* $f \mapsto \pi(b, f) = \sum_{j\in\mathbb{N}} S_{j-1}f\ \Delta_j b$ *is bounded on* $L^2(\mathbb{R}^d)$.

Proof: This is essentially the same proof as for Theorem 10.1. $\square$

Part 2:

A general framework for shift-invariant estimates for the Navier–Stokes equations

Chapter 11

Weak solutions for the Navier–Stokes equations

In this chapter, we shall present the integral formulation for the Navier–Stokes equations and discuss its equivalence with the differential formulation. The results presented in this chapter are a slight extension of results contained in the paper of Furioli, Lemarié-Rieusset and Terraneo [FURLT 00].

1. The Leray projection operator and the Oseen kernel

In this book, we consider the Navier–Stokes equations in d dimensions in the special setting of a viscous, homogeneous, incompressible fluid which fills the entire space and is not submitted to external forces. Thus, the equations we consider are the system:

$$(11.1)\qquad \begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla}.(\vec{u} \otimes \vec{u}) - \vec{\nabla} p \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

which is a condensed writing for

$$(11.1')\qquad \begin{cases} \text{For } 1 \leq k \leq d,\ \partial_t u_k = \Delta u_k - \sum_{l=1}^d \partial_l.(u_l u_k) - \partial_k p \\ \sum_{l=1}^d \partial_l u_l = 0 \end{cases}$$

The unknown quantities are the *velocity* $\vec{u}(x,t)$ of the fluid element at time t and position x and the *pressurep*, whose action is to maintain the divergence of $\vec{u}$ to be 0 (this *divergence free* condition expresses the incompressibility of the fluid).

Taking the divergence of (11.1), we obtain:

$$(11.2)\qquad \Delta p = -\vec{\nabla} \otimes \vec{\nabla}.(\vec{u}. \otimes \vec{u}) = -\sum_{k=1}^{d} \sum_{l=1}^{d} \partial_k \partial_l (u_k u_l)$$

Thus, we formally get the equations

$$(11.3)\qquad \begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \mathbb{P} \vec{\nabla}.(\vec{u} \otimes \vec{u}) \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

where $\mathbb{P}$ is defined as

$$\mathbb{P}\vec{f} = \vec{f} - \vec{\nabla}\frac{1}{\Delta}(\vec{\nabla}.\vec{f}) \tag{11.4}$$

We shall study the Cauchy problem for equation (11.3) (looking for a solution on $(0,T)\times\mathbb{R}^d$ with initial value $\vec{u}_0$), and transform (11.3) into the integral equation

$$\begin{cases} \vec{u} = e^{t\Delta}\vec{u_0} - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})\,ds \\ \vec{\nabla}.\vec{u_0} = 0 \end{cases} \tag{11.5}$$

Our first task is then to give a meaning to the operators $\mathbb{P}$ (the **Leray projection operator**) and $e^{t\Delta}\mathbb{P}$ (a convolution operator with the **Oseen kernel**).

On many occasions, we use a quite direct definition of $\mathbb{P}$. Let us recall that the Riesz transforms R_j are defined by $R_j = \frac{\partial_j}{\sqrt{-\Delta}}$, i.e. for $f \in L^2$ by $\mathcal{F}(R_j f) = \frac{i\xi_j}{|\xi|}\hat{f}(\xi)$. Then $\mathbb{P}$ is easily defined on $(L^2)^d$ as $\mathbb{P} = Id + \mathcal{R}\otimes\mathcal{R}$ where $\mathcal{R}$ is the vector of the Riesz transformations: $(\mathbb{P}\vec{f})_j = f_j + \sum_{1\leq k\leq d} R_j R_k f_k$. Since $R_j R_k$ is a Calderón–Zygmund operator, $\mathbb{P}$ may be defined on many Banach spaces.

However, we never quite use $\mathbb{P}$, but more often the operator $\mathbb{P}(\vec{\nabla}.)$. The use of one differentiation allows one to extend the definition of the operator to a larger class of Banach spaces. To give precise assertions, we use an auxiliary space:

Definition 11.1: (The space WL^∞)

The space WL^∞ is defined as the Banach space of Lebesgue measurable functions φ on $\mathbb{R}^d$ so that the norm $\|\varphi\|_{WL^\infty} = \sum_{k\in\mathbb{Z}^d}\sup_{x-k\in[0,1]^d}|\varphi(x)|$ is finite.

Lemma 11.1: (The Leray projection operator)

For $1 \leq k,l,m \leq d$, the operator $\frac{1}{\Delta}\partial_k\partial_l\partial_m$ is a convolution operator with a distribution $T_{k,l,m}$ which may be decomposed into $T_{k,l,m} = \alpha_{k,l,m} + \partial_k\partial_l\beta_m$ where $\alpha_{k,l,m} \in WL^\infty$ and $\beta_m \in L^1_{comp}$.

Remark: This lemma proves that $\mathbb{P}\vec{\nabla}.$ is defined on the space $(L^1_{uloc}(\mathbb{R}^d))^{d\times d}$ where L^1_{uloc} is the space of uniformly locally integrable functions on $\mathbb{R}^d$, since $L^1_{comp} * L^1_{uloc} \subset L^1_{uloc}$ and $WL^\infty * L^1_{uloc} \subset L^\infty$. In the next section, $\mathbb{P}\vec{\nabla}$ is defined in a slightly more general setting.

Proof: Let $\omega \in \mathcal{D}$ be equal to 1 in a neighborhood of 0. Let T_m be the distribution $\mathcal{F}^{(-1)}(\frac{i\xi_m}{\Delta})$. The distribution $T_{k,l,m}$ is equal to $\alpha_{k,l,m} + \partial_k\partial_l\beta_m$

where $\alpha_{k,l,m} = \partial_k\partial_l\big((1-\omega)T_m\big)$ and $\beta_m = \omega T_m$. We have $|\beta_m(x)| \leq C|x|^{-d+1}$, thus $\beta_m \in L^1_{comp}$. We have $|\alpha_{k,l,m}(x)| \leq C|x|^{-d-1}$, thus $\alpha_{k,l,m} \in WL^\infty$. $\square$

We now turn to the Oseen kernel. The main property we use throughout this book is that the operator $e^{t\Delta}\mathbb{P}\vec{\nabla}.$ is a matrix of convolution operators with bounded integrable kernels.

Proposition 11.1: (The Oseen kernel)

For $1 \leq j,k \leq d$ and $t > 0$, the operator $O_{j,k,t} = \frac{1}{\Delta}\partial_j\partial_k e^{t\Delta}$ is a convolution operator $O_{j,k,t}f = K_{j,k,t} * f$, where the kernel $K_{j,k,t}(x)$ satisfies $K_{j,k,t}(x) = \frac{1}{t^{d/2}}K_{j,k}(\frac{x}{\sqrt{t}})$ for a smooth function $K_{j,k}$ such that

$$\text{for all } \alpha \in \mathbb{N}^d \quad (1+|x|)^{d+|\alpha|}\partial^\alpha K_{j,k} \in L^\infty(\mathbb{R}^d)$$

Proof: We have $K_{j,k} = -\mathcal{F}^{(-1)}\big(\frac{\xi_j\xi_k}{|\xi|^2}e^{-|\xi|^2}\big)$, thus, for all $\alpha \in \mathbb{N}^d$, $\partial^\alpha K_{j,k} \in L^\infty(\mathbb{R}^d)$. For $|x| \geq 1$, we use the Littlewood–Paley decomposition and write $K_{j,k} = (Id - S_0)K_{j,k} + \sum_{l<0}\Delta_l K_{j,k}$. We have $(Id - S_0)K_{j,k} \in \mathcal{S}$, and we have $\Delta_l K_{j,k} = 2^{ld}\omega_{j,k,l}(2^l x)$ with $\hat{\omega}_{j,k,l}(\xi) = \psi(\xi)\frac{\xi_j\xi_k}{|\xi|^2}e^{-|2^l\xi|^2}$. The functions $\omega_{j,k,l}$, $l < 0$, are a bounded set in $\mathcal{S}$ and, thus, we may write for every $N \in \mathbb{N}$ and uniformly in l that $(1+2^l|x|)^N 2^{l(d+|\alpha|)}|\partial_\alpha\big(\Delta_l K_{j,k}\big)(x)| \leq C_N$. This gives (for $N > d + \alpha$)

$$|\partial_\alpha S_0 K_{j,k}(x)| \leq C \sum_{2^l|x|\leq 1} 2^{l(d+|\alpha|)} + \sum_{2^l|x|>1} 2^{l(d+|\alpha|-N}|x|^{-N} \leq C|x|^{-d-|\alpha|}.$$

$\square$

This kernel was introduced by Oseen [OSE 27] in $\mathbb{R}^3$ and used by Fabes, Riviere and Jones for describing strong solutions in $L^p(\mathbb{R}^d)$ [FABJR 72].

2. Elimination of the pressure

We focus on the invariance of equation (11.1) under spatial translations and dilations, in that we consider the problem on the whole space $\mathbb{R}^d$. We begin by defining what we call a "weak solution" for the Navier–Stokes equations:

Definition 11.2: (Weak solutions)

A weak solution of the Navier–Stokes equations on $(0,T) \times \mathbb{R}^d$ is a distribution vector field $\vec{u}(t,x)$ in $(\mathcal{D}'((0,T) \times \mathbb{R}^d)^d$ so that:

a) $\vec{u}$ is locally square integrable on $(0,T) \times \mathbb{R}^d$

b) $\vec{\nabla}.\vec{u} = 0$

c) $\exists p \in \mathcal{D}'((0,T) \times \mathbb{R}^d)$ $\partial_t\vec{u} = \Delta\vec{u} - \vec{\nabla}.(\vec{u} \otimes \vec{u}) - \vec{\nabla}p$

The following invariance of the set of solutions is used throughout the book:
i) shift-invariance: if $\vec{u}(t,x)$ is a weak solution of the Navier–Stokes equations on $(0,T)\times\mathbb{R}^d$, then $\vec{u}(t,x-x_0)$ is a weak solution on $(0,T)\times\mathbb{R}^d$
ii) dilation invariance: for $\lambda>0$, $\frac{1}{\lambda}\vec{u}(\frac{t}{\lambda^2},\frac{x}{\lambda})$ is a solution on $(0,\lambda^2T)\times\mathbb{R}^d$
iii) delay invariance: if $\vec{u}(t,x)$ is a weak solution of the Navier–Stokes equations on $(0,T)\times\mathbb{R}^d$ and if $t_0\in(0,T)$ then $\vec{u}(t+t_0,x)$ is a weak solution of the Navier–Stokes equations on $(0,T-t_0)\times\mathbb{R}^d$.

In order to use the space invariance, we introduce a more restrictive class of solutions:

Definition 11.3: (Uniformly locally square integrable weak solutions)

A weak solution of the Navier–Stokes equations on $(0,T)\times\mathbb{R}^d$ *is said to be uniformly locally square-integrable if for all* $\varphi\in\mathcal{D}((0,T)\times\mathbb{R}^d)$ *we have* $\sup_{x_0\in\mathbb{R}^d}\iint|\varphi(x-x_0,t)\vec{u}(t,x)|^2\,dx\,dt<\infty$.

Equivalently, $\vec{u}$ *is uniformly locally square-integrable if and only if for all* $t_0<t_1\in(0,T)$, *the function* $U_{t_0,t_1}(x)=(\int_{t_0}^{t_1}|\vec{u}(t,x)|^2\,dt)^{1/2}$ *belongs to the Morrey space* L^2_{uloc}.

We introduce some useful notations about uniform local integrability:

Definition 11.4: (Uniform local integrability)

For $1\leq p\leq\infty$, *the Morrey space of uniformly locally* L^p *functions on* $\mathbb{R}^d$ *is the Banach space* L^p_{uloc} *of Lebesgue measurable functions* f *on* $\mathbb{R}^d$ *so that the norm* $\|f\|_{p,uloc}=\sup_{x_0\in\mathbb{R}^d}\left(\int_{|x-x_0|<1}|f(x)|^p\,dx\right)^{1/p}$ *is finite.*

For $t_0<t_1$, $1\leq p,q\leq\infty$ *the space* $L^p_{uloc,x}L^q_t((t_0,t_1)\times\mathbb{R}^d)$ *is the Banach space of Lebesgue measurable functions* f *on* $(t_0,t_1)\times\mathbb{R}^d$ *such that the norm* $\sup_{x_0\in\mathbb{R}^d}\left(\int_{|x-x_0|<1}\{\int_{t_0}^{t_1}|f(t,x)|^q\,dt\}^{p/q}\,dx\right)^{1/p}$ *is finite.*

We now explain the utility of introducing uniform weak solutions. In order to make sense of equation (11.1), we only need to assume that $\vec{u}$ is locally square-integrable on $(0,T)\times\mathbb{R}^d$. But we need an extra assumption on $\vec{u}$ to assign meaning to (11.3) (namely, to $\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})$). We use Lemma 11.1 to give sense to $\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})$ with the help of the following obvious lemma:

Lemma 11.2: *If* φ *belongs to* $WL^\infty(\mathbb{R}^d)$, *then the (spatial) convolution operator with* φ: $f(t,x)\mapsto\int_{\mathbb{R}^d}f(t,y)\,\varphi(x-y)\,dy=f*\varphi(t,x)$ *is well defined and operates boundedly from* $L^1_{uloc,x}L^1_t((t_0,t_1)\times\mathbb{R}^d)$ *to* $L^1_{uloc,x}L^1_t((t_0,t_1)\times\mathbb{R}^d)$. *The same conclusion holds if* φ *belongs to* $L^1_{comp}(\mathbb{R}^d)$ *(i.e., if* φ *is compactly supported and belongs to* L^1*).*

We may now give a precise description of the relationships between equations (11.1) and (11.3):

Theorem 11.1: (Elimination of the pressure)

i) If $\vec{u}$ is uniformly locally square-integrable on $(0,T)\times\mathbb{R}^d$ (in the sense of Definition 3), then $\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})$ is well defined in $(\mathcal{D}'((0,T)\times\mathbb{R}^d))^d$, and there exists a distribution $P\in\mathcal{D}'((0,T)\times\mathbb{R}^d)$ so that $\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})=\vec{\nabla}.(\vec{u}\otimes\vec{u})+\vec{\nabla}P$. Thus, if $\vec{u}$ is a solution for (11.3), then it is also a solution for (11.1).

ii) Conversely, if $\vec{u}$ is a uniform weak solution for (11.1), and if $\vec{u}$ vanishes at infinity in the sense that for all $t_0<t_1\in(0,T)$ we have

$$\lim_{R\to\infty}\sup_{x_0\in\mathbb{R}^d}\frac{1}{R^d}\int_{t_0}^{t_1}\int_{|x-x_0|<R}|\vec{u}|^2\,dx\,dt=0,$$

then $\vec{u}$ is a solution for (11.3).

Proof:

We begin by proving that a solution $\vec{u}$ for (11.1) in $\mathcal{D}'$, which is uniformly locally square-integrable on $(0,T)\times\mathbb{R}^d$, is solution for (11.1) in a more precise space of distributions, namely in the space $\mathcal{T}'$ of spatially tempered distributions (defined in Chapter 5). Recall that the space $\mathcal{T}'$ is the space of distributions ω on $(0,T)\times\mathbb{R}^d$ so that for all $[a,b]\subset(0,T)$ there exist $C\geq 0$ and $N\in\mathbb{N}$ such that for all $\varphi\in\mathcal{D}((0,T)\times\mathbb{R}^d$ with $\mathrm{Supp}\varphi\subset[a,b]\times\mathbb{R}^d$, we have the inequality $|\langle\omega|\varphi\rangle|\leq C\sum_{|\alpha|\leq N}\sum_{|\beta|\leq N}\sum_{p\leq N}\|x^\alpha\frac{\partial^\beta}{\partial x^\beta}\frac{\partial^p}{\partial t^p}f(t,x)\|_\infty$. $\mathcal{T}'$ is the dual space of the space $\mathcal{T}((0,T)\times\mathbb{R}^d)$ of smooth functions compactly supported in time and rapidly decaying in space. $\mathcal{T}$ is the inductive limit of the Frechet spaces $\mathcal{T}_n=\{f\in\mathcal{T}\ /\ \mathrm{Supp}f\subset[1/n,T-1/n]\times\mathbb{R}^d\}$, where the space $\mathcal{T}_n$ is equipped with the semi-norms:

$$\sup_{1/n<t<T-1/n}\sup_{x\in\mathbb{R}^d}|x^\alpha\frac{\partial^\beta}{\partial x^\beta}\frac{\partial^p}{\partial t^p}f(t,x)|,\ \alpha\in\mathbb{N}^d,\ \beta\in\mathbb{N}^d,\ p\in\mathbb{N}.$$

Since the components of $\vec{u}$ are in $L^2_{uloc,x}L^2_t((t_0,t_1)\times\mathbb{R}^d)$ for all $0<t_0<t_1<T$, we easily find that $\vec{u}$ belongs to $(\mathcal{T}')^d$; hence, so do $\partial_t\vec{u}$ and $\Delta\vec{u}$. Moreover, the components of $\vec{u}\otimes\vec{u}$ are in $L^1_{uloc,x}L^1_t((t_0,t_1)\times\mathbb{R}^d)$ for all $0<t_0<t_1<T$; thus $\vec{u}\otimes\vec{u}$ belongs to $(\mathcal{T}')^{d\times d}$ and $\vec{\nabla}.\vec{u}\otimes\vec{u}$ belongs to $(\mathcal{T}')^d$.

Since all the other terms in (11.1) belong to $(\mathcal{T}')^d$, we find that $\vec{\nabla}p$ belongs to $(\mathcal{T}')^d$.

We shall now define the Leray projection operator $\mathbb{P}$. On $\mathcal{T}'$, we may use the Littlewood–Paley decomposition with respect to the space variable x. This wil be very useful in defining $\mathbb{P}$.

High frequencies do not present a problem, since $\mathbb{P}(Id-S_0)$ operates boundedly on $\mathcal{T}'$. In order to deal with low frequencies, we use the following properties:

- for $f\in L^1_{uloc}(\mathbb{R}^d)$ and $g\in WL^\infty$, $fg\in L^1$ and $\|fg\|_1\leq C\|f\|_{L^1_{uloc}}\|g\|_{WL^\infty}$;

• more precisely, WL^∞ is the dual space of E_1, the closure of $\mathcal{D}$ in L^1_{uloc} and, thus, WL^∞ is a shift-invariant Banach space of distributions;
• we thus may conclude that for $f \in L^1(\mathbb{R}^d)$ and $g \in WL^\infty$, $f * g \in WL^\infty$ and $\|f * g\|_{WL^\infty} \leq C\|f\|_1 \|g\|_{WL^\infty}$;
• If $\varphi(x)$ belongs to $WL^\infty(\mathbb{R}^d)$ and $f(t,x)$ belongs to $L^1_{uloc,x} L^1_t((t_0,t_1)\times \mathbb{R}^d)$, then the (spatial) convolution $f * \varphi(t,x) = \int_{\mathbb{R}^d} f(t,y)\,\varphi(x-y)\,dy$ is well defined and belongs to $L^\infty_x L^1_t((t_0,t_1)\times\mathbb{R}^d)$ and $\|f*\varphi\|_{L^\infty_x L^1_t} \leq C\|f\|_{L^1_{uloc,x}L^1_t}\|\varphi\|_{WL^\infty}$.

Now, let us consider a vector field $\vec{g}$ with $g_i = \sum_{k=1}^d \partial_k f_{i,k}$ with $f_{i,k} \in \cap_{0<t_0<t_1<T} L^1_{uloc,x} L^1_t((t_0,t_1)\times\mathbb{R}^d)$. We want to define $\mathbb{P} S_0 \vec{g}$. Since $S_j \vec{g}$ satisfies for $j<0$ (writing $S_j g = 2^{jd}\Phi(2^j x) * g$ and $S_j \partial_k g = 2^{j(d+1)}\partial_k \Phi(2^j x) * S_0 g$),

$$\|S_j \vec{g}\|_{L^\infty_x L^1_t((t_0,t_1)\times\mathbb{R}^d)} \leq C 2^j \sum_{i,k} \|f_{i,k}\|_{L^1_{uloc,x} L^1_t((t_0,t_1)\times\mathbb{R}^d)} \|\Phi\|_{WL^\infty} \sum_k \|\partial_k \Phi\|_1,$$

we find that we have $S_0\vec{g} = \sum_{j<0}\Delta_j \vec{g}$ in $\mathcal{T}'$. Since $\mathbb{P}\Delta_j$ operates on $(\mathcal{T}')^d$, we shall try to define $\mathbb{P} S_0 \vec{g}$ as the sum $\sum_{j<0}\Delta_j \mathbb{P}\vec{g}$. Since we have $(\Delta_j \mathbb{P}\vec{g})_i = \sum_k 2^{jd}\Psi_{i,k}(2^j x) * g_k$ with $\Psi_{i,k}\in\mathcal{S}$, we find that

$$\|\Delta_j \mathbb{P}\vec{g}\|_{L^\infty_x L^1_t((t_0,t_1)\times\mathbb{R}^d)} \leq C 2^j \sum_{i,k} \|f_{i,k}\|_{L^1_{uloc,x} L^1_t((t_0,t_1)\times\mathbb{R}^d)}$$

and thus the series $\sum_{j<0}\Delta_j \mathbb{P}\vec{g}$ is convergent in $\mathcal{T}'((0,T)\times\mathbb{R}^d)$.

In particular, when $\vec{u}$ belongs to $\cap_{0<t_0<t_1<T}(L^2_{uloc,x}L^2_t((t_0,t_1)\times\mathbb{R}^d))^d$, then $\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})$ is well defined.

We now prove point i) in Theorem 11.1. We consider a solution $\vec{u}$ of (11.3) with $\vec{u}\in\cap_{0<t_0<t_1<T}(L^2_{uloc,x}L^2_t((t_0,t_1)\times\mathbb{R}^d))^d$. Thus, $\partial_t\vec{u} = \Delta\vec{u} - \mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})$, with $\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u}) = \sum_{j\in\mathbb{Z}}\Delta_j\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})$. We define $p_j = -\sum_{i,l}\frac{\partial_i\partial_l}{\Delta}\Delta_j(u_i u_l)$, so that $\Delta_j\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u}) = \Delta_j\vec{\nabla}.(\vec{u}\otimes\vec{u}) + \vec{\nabla}p_j$. We modify p_j for $j<0$ by $q_j(t,x) = p_j(t,x) - p_j(t,0)$. Then, $\sum_{j\geq 0} p_j$ is convergent in $\mathcal{T}'$; moreover, $q_j(t,x) = \sum_{k=1}^d x_k \int_0^1 \partial_k p_j(t,\theta x)\,d\theta$; we easily find that $\|\partial_k p_j(t,x)\|_{L^\infty_x L^1_t((t_0,t_1)\times\mathbb{R}^d)} \leq C 2^j \|\vec{u}\|_{L^2_{uloc,x}L^2_t((t_0,t_1)\times\mathbb{R}^d)}$, which gives

$$\int_{t_0}^{t_1} |q_j(t,x)|\,dt \leq C 2^j |x| \|\vec{u}\|_{L^2_{uloc,x}L^2_t((t_0,t_1)\times\mathbb{R}^d)},$$

and this is enough to grant that $\sum_{j<0} q_j$ is convergent in $\mathcal{T}'$. Thus, $\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u}) = \vec{\nabla}.(\vec{u}\otimes\vec{u}) + \vec{\nabla}p$ with $p = \sum_{j\geq 0}p_j + \sum_{j<0} q_j$ and $\vec{u}$ is a solution of (11.1).

We now prove point ii) of Theorem 11.1. We start from the property

$$\lim_{R\to\infty}\sup_{x_0\in\mathbb{R}^d}\frac{1}{R^d}\int_{t_0}^{t_1}\int_{|x-x_0|<R}|\vec{u}|^2\,dx\,dt = 0,$$

and we estimate $S_j\vec{u}$ by $\|S_j\vec{u}\|_{L^\infty_x L^1_t} \leq C\big(\sup_{x_0\in\mathbb{R}^d}\frac{1}{R^d}\int_{t_0}^{t_1}\int_{|x-x_0|<R}|\vec{u}|^2\,dx\,dt\big)^{1/2}$, which gives that $\lim_{j\to-\infty} S_j\vec{u}$ is equal to 0 in $\mathcal{T}'$. Hence, $\lim_{j\to-\infty} S_j\partial_t\vec{u} = \lim_{j\to-\infty} S_j\Delta\vec{u} = 0$ in $\mathcal{T}'$. Thus, we have

$$\begin{array}{rcl}\partial_t\vec{u} & = & \lim_{j\to-\infty}(Id-S_j)\partial_t\vec{u}\\ & = & \lim_{j\to-\infty}\mathbb{P}(Id-S_j)\partial_t\vec{u}\\ & = & \lim_{j\to-\infty}\mathbb{P}(Id-S_j)(\Delta\vec{u}-\vec{\nabla}.(\vec{u}\otimes\vec{u})-\vec{\nabla}p)\\ & = & \Delta\vec{u}-\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})-\lim_{j\to-\infty}\mathbb{P}(Id-S_j)\vec{\nabla}p\end{array}$$

Thus, the proof of point ii) is reduced to proving that $\mathbb{P}(Id-S_j)\vec{\nabla}p = 0$. Of course, $\mathbb{P}(Id-S_j)\vec{\nabla}$ is equal to 0 on $\mathcal{T}'$, but we do not know whether p belongs to $\mathcal{T}'$. If $\vec{\omega}\in(\mathcal{T})^d$, we have $\langle\mathbb{P}(Id-S_j)\vec{\nabla}p|\vec{\omega}\rangle = \langle\vec{\nabla}p|\mathbb{P}(Id-S_j)\vec{\omega}\rangle$. We thus have to prove that, when $p\in\mathcal{D}'((0,T)\times\mathbb{R}^d)$ is such that $\vec{\nabla}p\in(\mathcal{T}')^d$ and $\vec{\psi}\in(\mathcal{T})^d$ is such that $\vec{\nabla}.\vec{\psi}=0$, then $\langle\vec{\nabla}p|\vec{\psi}\rangle = 0$. We are going to prove that there exists a matrix $\Phi = (\varphi_{k,l})_{1\leq k,l\leq d}\in\mathcal{T}^{d\times d}$ with $\varphi_{k,l} = -\varphi_{l,k}$ so that $\vec{\psi} = \vec{\nabla}.\Phi$. Then, it will be enough to approximate Φ by a sequence $\Phi_n\in\mathcal{D}^{d\times d}$ (with ${}^t\Phi_n = -\Phi_n$) to get:

$$\langle\vec{\nabla}p|\vec{\psi}\rangle = \lim_{n\to\infty}\langle\vec{\nabla}p|\vec{\nabla}.\Phi_n\rangle = -\lim_{n\to\infty}\langle p|\vec{\nabla}.(\vec{\nabla}.\Phi_n)\rangle = 0.$$

The construction of Φ is simple. We first notice that a function f in $\mathcal{T}$ may be written as $f = \partial_1 g$ with $g\in\mathcal{T}$ if and only if $\int f(t,x_1,\ldots,x_d)\,dx_1 = 0$ for all $t\in(0,T)$ and all $(x_2,\ldots,x_d)\in\mathbb{R}^{d-1}$: indeed, if θ is a univariate cut-off function equal to 1 on a neighbourhood of 0, we define g as $\mathcal{F}g(t,\xi) = -i\theta(\xi_1)\int_0^1\frac{\partial}{\partial\xi_1}\mathcal{F}f(t,\lambda\xi_1,\xi_2,\ldots,\xi_d)\,d\lambda + \frac{1-\theta(\xi_1)}{i\xi_1}\mathcal{F}f(t,\xi)$. We thus write $\vec{\psi} = \vec{\alpha}+\vec{\beta}$, where

$$\vec{\alpha} = \begin{pmatrix}\alpha_1\\ \alpha_2\\ \ldots\\ \alpha_d\end{pmatrix} = \begin{pmatrix}0\\ \big(\int\psi_2(t,y_1,x_2\ldots,x_d)\,dy_1\big)\omega(x_1)\\ \ldots\\ \big(\int\psi_d(t,y_1,x_2\ldots,x_d)\,dy_1\big)\omega(x_1)\end{pmatrix}$$

with $\int\omega(x_1)\,dx_1 = 1$, and $\vec{\beta} = \vec{\psi}-\vec{\alpha}$. We have $\vec{\nabla}.\vec{\alpha} = 0$; hence, $\vec{\nabla}.\vec{\beta} = 0$. We know that for $2\leq k\leq d$ we have $\int\beta_k(t,y_1,x_2,\ldots,x_d)\,dy_1 = 0$; hence, there exists $\varphi_{1,k}$ so that $\beta_k = \partial_1\varphi_{1,k}$. Now, we use $\vec{\nabla}.\vec{\beta} = 0$ to get $\partial_1\beta_1 = -\partial_1\sum_{k\geq2}\partial_k\varphi_{1,k}$, hence $\beta_1 = -\sum_{k\geq2}\partial_k\varphi_{1,k}$. Then, we deal with $\vec{\alpha}$: we have $\vec{\alpha} = \omega(x_1)\vec{\gamma}$ with $\gamma_1 = 0$, $\vec{\delta}(t,x_2,\ldots,x_d) = \begin{pmatrix}\gamma_2\\ \ldots\\ \gamma_d\end{pmatrix}\in(\mathcal{T}((0,T)\times\mathbb{R}^{d-1}))^{d-1}$ and $\vec{\nabla}.\vec{\delta} = 0$; we thus may iterate the construction and finally get the matrix Φ. $\square$

3. Differential formulation and integral formulation for the Navier–Stokes equations

We now turn to the equivalence between equations (11.3) and (11.5):

Theorem 11.2: (The equivalence theorem)
Let $\vec{u} \in \cap_{t_1<T}(L^2_{uloc,x}L^2_t((0,t_1) \times \mathbb{R}^d))^d$. Then, the following assertions are equivalent:
(A1) $\vec{u}$ is a solution of the differential Navier–Stokes equations

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{u}) \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

(A2) $\vec{u}$ is a solution of the integral Navier–Stokes equations

$$\exists \vec{u_0} \in (\mathcal{S}'(\mathbb{R}^d))^d \ \begin{cases} \vec{u} = e^{t\Delta}\vec{u_0} - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{u})\, ds \\ \vec{\nabla}.\vec{u_0} = 0 \end{cases}$$

We begin with an easy and useful lemma:

Lemma 11.3:
Let $T \in (0,\infty)$ and $w \in L^1_{uloc,x}L^1_t((0,T) \times \mathbb{R}^d)$. Then, $\int_0^t e^{(t-s)\Delta}w(s,.)ds$ defines a function $W(t,x) \in L^1_{uloc,x}L^1_t((0,T) \times \mathbb{R}^d)$. Moreover:
i) $\partial_t W = \Delta W + w$
ii) for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with Supp $\varphi \subset [-1,1]^d$, we have that $t \mapsto \varphi\, W(t,.)$ is continuous from $[0,T]$ to $L^1(\mathbb{R}^d)$ and satisfies $\sup_{0<t<T} \|\varphi(x)W(x,t)\|_{L^1(dx)} \leq C\|\varphi\|_\infty \|w(x,t)\|_{L^1_{uloc,x}L^1_t}$ and $\lim_{t\to 0}\|\varphi W(t,.)\|_1 = 0$.
iii) $t \mapsto W(t,.)$ is weakly continuous from $(0,T)$ to $\mathcal{S}'$ and $\lim_{t\to 0} W(t,.) = 0$.

Proof: We begin by proving that $W \in L^1_{uloc,x}L^1_t$. We have more precisely $\sup_{x_0\in\mathbb{R}^d, 0<t<T}\int_{x_0-x\in[-1,1]^d}|W(x,t)|\ dx \leq C\|w(x,t)\|_{L^1_{uloc,x}L^1_t}$. Indeed, let $x_0 \in \mathbb{R}^d$. We split w in two parts: $w = \alpha + \beta$ where $\alpha = w 1_{|x-x_0|\leq 100\sqrt{d}}\, w$. Since $\alpha \in L^1((0,T) \times \mathbb{R}^d)$, we have $|\int_{\mathbb{R}^d} e^{(t-s)\Delta}\alpha\ dx| \leq \int_{\mathbb{R}^d}|\alpha|\ dx$ and, thus, we get

$$\int_0^t \int_{\mathbb{R}^d} |e^{(t-s)\Delta}\alpha| ds\, dx \leq \int_0^t \int_{|x-x_0|\leq 100\sqrt{d}} |w(t,x)|\ ds\, dx \leq C\|w\|_{L^1_{uloc,x}L^1_t((0,T)\times\mathbb{R}^d)}.$$

Now, we estimate the contribution of β. When $|x - x_0| \leq \sqrt{d}$ and $|x-y| \geq 100\sqrt{d}$, we may easily check that the kernel $K(t-s, x-y)$ of $e^{(t-s)\Delta}$ satisfies

$0 \leq K(t-s, x-y) \leq C\frac{\sqrt{t-s}}{|x_0-y|^{d+1}}$. This gives $\int_0^t \int_{x_0-x\in[-1,1]^d} |e^{(t-s)\Delta}\beta| ds\ dx \leq C\sqrt{t}\int_0^t \int_{|x_0-y|\geq 99\sqrt{d}} \frac{|w(s,y)|}{|x_0-y|^{d+1}}\ ds\ dy \leq C\sqrt{t}\|w(x,t)\|_{L^1_{uloc,x}L^1_t((0,T)\times\mathbb{R}^d)}$.

We obviously see that $t \mapsto \int_0^t e^{(t-s)\Delta}\beta ds$ is continuous from $(0,T)$ to $L^1(B(x_0,1))$ and that $\lim_{t\to 0} \| \int_0^t e^{(t-s)\Delta}\beta ds\|_{L^1(B(x_0,1))} = 0$. The estimate $\lim_{t\to 0} \| \int_0^t e^{(t-s)\Delta}\alpha \ \ ds\|_{L^1(B(x_0,1))} = 0$ is easy as well, since we have $\lim_{T\to 0} \|w(x,t)\|_{L^1_{uloc,x}L^1_t((0,T)\times\mathbb{R}^d)} = 0$. For the continuity in L^1-norm of $t \mapsto A(t) = \int_0^t e^{(t-s)\Delta}\alpha ds$, it is enough to notice that $\|e^{\tau\Delta}(e^{\theta\Delta} - Id)\|_1 \leq C\frac{\theta}{\tau}$; thus, writing for $0 < t_1 < t_2 < T$ $\tau = (t_1+t_2)/2$ and $\theta = |t_1 - t_2|/2$, we have $\|A(t_1) - A(t_2)\|_{L^1(dx)} \leq C\big(\int_{\tau-\sqrt{\theta\tau}}^{\tau+\theta} \|\alpha(s)\|_{L^1(dx)}\ ds + \frac{\sqrt{\theta}}{\sqrt{\tau}-\sqrt{\theta}}\big(\int_0^{\tau-\sqrt{\theta\tau}} \|\alpha(s)\|_{L^1(dx)}\ ds\big)$, hence $\lim_{t_2\to t_1} \|A(t_1) - A(t_2)\|_{L^1(dx)} = 0$.

The simplest way to compute $\partial_t W$ is to notice that for $T \in L^1_{uloc,x}L^1_t$ we may easily check that $\partial(e^{t\Delta}T) = e^{t\Delta}(\partial_t T + \Delta T)$ (just use the partial Fourier transform) and that $\partial_t(\int_0^t T(s)\ ds) = T$; then, we write, for the Littlewood–Paley approximation of W, that $S_j W = e^{t\Delta}\int_0^t (S_j e^{-s\Delta})w\ ds$; hence, $S_j\partial_t W = \partial_t S_j W = \Delta S_j W + e^{t\Delta}(S_j e^{-t\Delta})w = S_j(\Delta W + w)$; letting j go to $+\infty$ gives then the required result. □

Proof of Theorem 11.2 : Using Lemma 11.3, we find that, for $\vec{u}_0 \in (\mathcal{S}')^d$ and $\vec{u} \in (L^1_{uloc,x}L^1_t)^d$, we find that $\vec{F}(\vec{u}) = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})\ ds$ is a solution for $\partial_t\vec{F}(\vec{u}) = \Delta\vec{F}(\vec{u}) - \mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})$. Thus, if $\vec{u} = \vec{F}(\vec{u})$, then $\vec{u}$ is a solution for the Navier–Stokes equations.

Conversely, if $\vec{u} \in (L^1_{uloc,x}L^1_t((0,T)\times\mathbb{R}^d))^d$ and $\partial_t\vec{u} = \Delta\vec{u} - \mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})$, then, choosing $\omega \in \mathcal{D}(\mathbb{R})$ equal to 1 on $[0, T/4]$ and to 0 on $[T/2, +\infty)$ and defining $\vec{v} = \omega(t)(Id - \Delta)^{-2}\vec{u}$, we find that $\partial_t\vec{v} \in (L^1_{uloc,x}L^1_t((0,T)\times\mathbb{R}^d))^d$; hence, $\vec{v}(t) = -\int_t^T \partial_t\vec{v}(s)\ ds \in (\mathcal{C}([0,T], L^1_{uloc}))^d$; thus we find that the mapping $t \mapsto (Id-\Delta)^{-2}\vec{u}$ is continuous from $[0,T)$ to $(L^1_{uloc})^d$ and that $t \mapsto \vec{u}$ is weakly continuous from $[0,T)$ to $(\mathcal{S}')^d$. Thus, $\vec{u}_0 = \lim_{t\to 0}\vec{u}$ is well defined.

We again introduce $\vec{F}(\vec{u}) = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})\ ds$; we have $\partial_t(\vec{u} - \vec{F}(\vec{u})) = \Delta(\vec{u} - \vec{F}(\vec{u}))$ and $\lim_{t\to 0}\vec{u} - \vec{F}(\vec{u}) = 0$. Using the Littlewood–Paley approximation S_j, we write: $\partial_t\big((e^{-t\Delta}S_j)(\vec{u} - \vec{F}(\vec{u}))\big) = 0$ and $\lim_{t\to 0}(e^{-t\Delta}S_j)(\vec{u} - \vec{F}(\vec{u})) = 0$; hence, $(e^{-t\Delta}S_j)(\vec{u} - \vec{F}(\vec{u})) = 0$; we get $S_j(\vec{u} - \vec{F}(\vec{u})) = 0$ and finally $\vec{u} = \vec{F}(\vec{u}))$. □

Remark: We proved that $t \mapsto \vec{u}$ is continuous from $[0,T)$ to $((Id-\Delta)^2 L^1_{uloc})^d$. It means that the distribution $\vec{u}$, which is a locally integrable vector function $(0,T)\times\mathbb{R}^d$ and hence may be modified arbitrarily on subsets of measure 0, may be defined by a function such that the mapping $t \mapsto \vec{u}$ is continuous. **In this book, we always assume that $\vec{u}$ denotes this representative.**

Chapter 12

Divergence-free vector wavelets

In this chapter, we consider divergence-free vector wavelets as a tool for approximating divergence free vector fields vanishing at infinity by compactly supported divergence-free vector fields.

1. A short survey in divergence-free vector wavelets

Wavelets have been advocated by many authors as a tool for investigating the structure of turbulent solutions of the Navier-Stokes equations (Farge, Kevlahan, Perrier and Schneider [FARKPS 97], Frick and Zimin [FRIZ 93]). According to Farge, applications of the wavelet transform to the theory of turbulence should be developed in three directions: analysis, filtering, and numerical modeling. The wavelet analysis relies on the localisation of wavelets both in space and frequency; thus, wavelets may be a useful tool for relating the dynamics of coherent structures in physical space to the redistribution of energy among the Fourier modes. Adaptive nonlinear filtering of the wavelet coefficients appears to help in isolating dynamically active structures supported by thin sets and exploring their interaction. Then a wavelet-based numerical modeling could be developed at a reasonable computational cost.

Special attention has been given to divergence-free vector wavelet bases, i.e., bases for $H = \{\vec{f} \in (L^2(\mathbb{R}^d))^d \ / \ \vec{\nabla}.\vec{f} = 0\}$ which are derived through dyadic dilations and translations from a finite number of basic functions: the basis is of the form $(2^{jd/2}\vec{\psi}_\alpha(2^j x - k))_{1\leq\alpha\leq(d-1)(2^d-1), j\in\mathbb{Z}, k\in\mathbb{Z}^d}$.

Battle and Lemarié-Rieusset noticed independently that in the case of $d = 2$, the construction of an Hilbertian basis for H is very easy: if R_1, R_2 are the Riesz transforms $\frac{\partial_1}{\sqrt{-\Delta}}$, $\frac{\partial_2}{\sqrt{-\Delta}}$ and if $(\psi_{\epsilon,j,k})_{1\leq\epsilon\leq 3, j\in\mathbb{Z}, k\in\mathbb{Z}^d}$ is a Hilbertian basis of $L^2(\mathbb{R}^2)$, then we have a basis $(\vec{\psi}_{\epsilon,j,k})_{1\leq\epsilon\leq 3, j\in\mathbb{Z}, k\in\mathbb{Z}^d}$ of H with $\vec{\psi}_\epsilon = (R_2\psi_\epsilon, -R_1\psi_\epsilon)$. We may choose $\vec{\psi}_\epsilon$ with good regularity and localization properties:

- If we start from a Meyer–Lemarié wavelet basis ($\psi_\epsilon \in \mathcal{S}(\mathbb{R}^2)$ and $\hat{\psi}_\epsilon(\xi) = 0$ on a neighbourhood of 0 [LEMM 86]), then $\vec{\psi}_\epsilon$ belongs to the Schwartz class $(\mathcal{S}(\mathbb{R}^2))^2$.
- If we start from a Battle basis $\psi_\epsilon = \sqrt{-\Delta}\Psi_\epsilon$ where the wavelets Ψ_ϵ and their first derivatives have exponential decay at infinity (Hibertian basis for

the homogeneous Sobolev scalar product $\int \sum_j \partial_j u \bar{\partial_j v}\, dx$ [BAT 87]), then the divergence-free vector wavelets $\vec{\psi}_\epsilon$ have exponential decay.

• However, there does not exist a Hilbertian wavelet basis for H with compactly supported wavelets $\vec{\psi}_\epsilon$ (Lemarié-Rieusset [LEM 94]).

• Moreover, the projection operator on the low-frequency vector wavelets is always ill-localized: if we define $\vec{Q}_j \vec{f}$ as $\vec{Q}_j \vec{f} = \sum_\epsilon \sum_{k\in\mathbb{Z}^d} \langle \vec{f} | \vec{\psi}_{\epsilon,j,k}\rangle \vec{\psi}_{\epsilon,j,k}$, we have $\mathbb{P}\vec{f} = \sum_{j\in\mathbb{Z}} \vec{Q}_j \vec{f} = \vec{P}_0 \vec{f} + \sum_{j\geq 0} \vec{Q}_j \vec{f}$; the kernel of $\mathbb{P} - \vec{P}_0$ is rapidly decaying at infinity (provided that the vector wavelets $\vec{\psi}_\epsilon$ have rapid decay) while the kernel of $\mathbb{P}$ is decaying with order $O(|x-y|^{-d})$.

Battle and Federbush have constructed Hilbertian bases of divergence-free vector wavelets in higher dimensions [BATF 95]. The construction is much more intricate than in dimension $d = 2$. Federbush [FED 93] used those bases to construct mild solutions for the Navier–Stokes equations on $\mathbb{R}^3$ through a Galerkin approach. (See also the book by Cannone [CAN 95]). This approach was criticized in the book by Meyer [MEY 99] who claimed that Cannone's approach by Littlewood–Paley estimates [CAN 95] and even a direct approach using spatial estimates as in the papers of Kato on L^d solutions [KAT 84] or on solutions in the Morrey–Campanato spaces [KAT 92] could give the same results (or even more precise results) at lesser expense. Meyer also proved that the interaction of two vector wavelets was not well localized in frequency and thus required many wavelet coefficients to be restored.

Lemarié-Rieusset proposed another construction [LEM 92] that is described in the next section. The idea was to get rid of orthogonality to allow a direct and local representation of the approximation on the low frequencies. We start from a bi-orthogonal multiresolution analysis $\vec{V}_j$ of the whole space $(L^2(\mathbb{R}^d)^d$ so that the projection operators $\vec{P}_j$ on V_j are local. Moreover, we construct $\vec{V}_j$ so that $\vec{P}_j(H) \subset H$. Then, we define a wavelet basis of $\vec{W}_j \cap H$, where $\vec{W}_j = (\vec{P}_{j+1} - \vec{P}_j)((L^2)^d)$. Thus, the approximation of $\vec{f} \in H$ by $\vec{P}_j \vec{f}$ may be described both as an external approximation (using a well-localized basis $(2^{j/2}\vec{\varphi}_l(2^j x - k))_{1\leq l\leq d, k\in\mathbb{Z}^d}$ of $\vec{V}_j$) or as an internal approximation (using the well-localized divergence-free vector wavelets $(\vec{\psi}_{\epsilon,l,k})_{1\leq\epsilon\leq(d-1)(2^d-1), l<j, k\in\mathbb{Z}^d})$. Such representations have been used by Urban [URB 95] to numerically approximate the Stokes problem.

2. Bi-orthogonal bases

We now introduce the divergence-free vector basis of compactly supported wavelets constructed in [LEM 92].

Let φ, φ^* be compactly supported univariate bi-orthogonal scaling functions so that, for instance, φ and φ^* are $\mathcal{C}^2$. Let $\tilde{\varphi}$ and $\tilde{\varphi}^*$ be the scaling functions defined by:

$$\frac{d}{dx}\varphi(x) = \tilde{\varphi}(x) - \tilde{\varphi}(x-1)$$

$$\tilde{\varphi}^*(x) = \int_x^{x+1} \varphi^*(t)\ dt$$

If $V_j, V_j^*, \tilde{V}_j$ and $\tilde{V}_j^*$ are the multiresolution analyses generated by $\varphi, \varphi^*, \tilde{\varphi}$ and $\tilde{\varphi}^*$, if P_j is the projection operator onto V_j in the direction of $(V_j^*)^\perp$ and if $\tilde{P}_j$ is the projection operator onto $\tilde{V}_j$ in direction of $(\tilde{V}_j^*)^\perp$, then we have:

$$\frac{d}{dx}(P_j f) = \tilde{P}_j(\frac{d}{dx} f)$$

where the formula is valid for all locally integrable f.

We then define the vector projection operator $\vec{P}_j = (P_j^{[1]}, P_j^{[2]}, \ldots, P_j^{[d]})$ with $P_j^{[k]} = P_j^{\{\delta_{k,1}\}} \otimes \tilde{P}_j^{\{\delta_{k,2}\}} \ldots \otimes \tilde{P}_j^{\{\delta_{k,d}\}}$ with $P_j^{\{0\}} = \tilde{P}_j$ and $P_j^{\{1\}} = P_j$ so that for all $f \in (L^1_{loc})^d$ we have:

$$\vec{\nabla}.\vec{P}_j(\vec{f}) = \tilde{P}_j \otimes \tilde{P}_j \otimes \ldots \otimes \tilde{P}_j(\vec{\nabla}.\vec{f}).$$

Thus, a divergence-free vector field is projected onto a vector field which is still divergence free. The main property of this approximation process for vector fields is that the term $(\vec{P}_{j+1} - \vec{P}_j)\vec{f}$ may be locally reorganized into divergence free "atoms" when $\vec{f}$ is itself a divergence free vector field. In the "classical" wavelet theory, we would decompose $(\vec{P}_{j+1} - \vec{P}_j)\vec{f}$ into

$$(\vec{P}_{j+1} - \vec{P}_j)\vec{f} = \sum_{1\leq l\leq d} \sum_{1\leq \alpha \leq 2^d-1} \sum_{k\in \mathbb{Z}^d} \langle f_l | \psi^*_{\alpha,l,j,k}\rangle\ \vec{\psi}_{\alpha,l,j,k}$$

where $\psi^*_{\alpha,l,j,k} = 2^{dj/2}\ \psi^*_{\alpha,l}(2^j x - k)$ and $\vec{\psi}_{\alpha,l,j,k} = 2^{dj/2}\ \vec{\psi}_{\alpha,l}(2^j x - k)$ and where the $d(2^d - 1)$ wavelets $\vec{\psi}_{\alpha,l}$ and their $d(2^d - 1)$ dual wavelets $\psi^*_{\alpha,l}$ are derived from the dual wavelets ψ, ψ^* (associated to φ, φ^*) and $\tilde{\psi}, \tilde{\psi}^*$ (associated to $\tilde{\varphi}, \tilde{\varphi}^*$) through the following formulas where we write $(\vec{e}_l)_{1\leq l \leq d}$ for the canonical basis of $\mathbb{R}^d$ (so that $\vec{f} = \sum_l\ f_l\ \vec{e}_l$):

$$\varphi^\beta_{[l,n]} = \begin{cases} = \varphi \text{ for } l = n \text{ and } \beta = 0,\ = \tilde{\varphi} \text{ for } l \neq n \text{ and } \beta = 0 \\ = \psi \text{ for } l = n \text{ and } \beta = 1,\ = \tilde{\psi} \text{ for } l \neq n \text{ and } \beta = 1 \end{cases}$$

$$\varphi^{*\beta}_{[l,n]} = \begin{cases} = \varphi^* \text{ for } l = n \text{ and } \beta = 0,\ = \tilde{\varphi}^* \text{ for } l \neq n \text{ and } \beta = 0 \\ = \psi^* \text{ for } l = n \text{ and } \beta = 1,\ = \tilde{\psi}^* \text{ for } l \neq n \text{ and } \beta = 1 \end{cases}$$

$$\vec{\psi}_{\alpha,l} = \varphi^{\beta_1}_{[l,1]} \otimes \varphi^{\beta_2}_{[l,2]} \otimes \ldots \otimes \varphi^{\beta_d}_{[l,d]} \quad \vec{e}_l \qquad \text{for } \alpha = \sum_{n=1}^{d}\ \beta_n\ 2^n$$

$$\psi^*_{\alpha,l} = \varphi^{*\beta_1}_{[l,1]} \otimes \varphi^{*\beta_2}_{[l,2]} \otimes \ldots \otimes \varphi^{*\beta_d}_{[l,d]} \qquad \text{for } \alpha = \sum_{n=1}^{d}\ \beta_n\ 2^n$$

In the theory of divergence-free vector wavelets, we decompose $(\vec{P}_{j+1}-\vec{P}_j)\vec{f}$ in 2^d-1 groups (following the values of α):

$$\vec{Q}_{\alpha,j}\vec{f}=\sum_{1\leq l\leq d}\ \sum_{k\in\mathbb{Z}^3}\langle f_l|\psi^*_{\alpha,l,j,k}\rangle\ \vec{\psi}_{\alpha,l,j,k}$$

which satisfy a formula of the kind $\vec{\nabla}.\vec{Q}_{\alpha,j}\vec{f}=\tilde{Q}_{\alpha,j}(\vec{\nabla}.\vec{f})$. Finally, we notice that, since $\alpha\neq 0$, at least one of the associated β_n's is equal to 1; we call n the smallest index so that $\beta_n=1$ and we use the fact that $\tilde{\psi}=\frac{d}{dx}\psi$ to get that for $l\neq n$ we have $\vec{\psi}_{\alpha,l}=\partial_n\Psi_{\alpha,l}\ \vec{e}_l$ where $\Psi_{\alpha,l}$ is defined as $\vec{\Psi}_{\alpha,l}=\phi^{\beta_1}_{[l,1]}\otimes\phi^{\beta_2}_{[l,2]}\otimes\ldots\otimes\phi^{\beta_d}_{[l,d]}$ with $\phi^{\beta_k}_{[l,k]}=\varphi^{\beta_k}_{[l,k]}$ for $k\neq n$ and $\phi^{\beta_n}_{[l,n]}=\psi$. We then define $\vec{\gamma}_{\alpha,l}=\partial_n\Psi_{\alpha,l}\ \vec{e}_l-\partial_l\Psi_{\alpha,l}\ \vec{e}_n$ and $\vec{\gamma}_{\alpha,l,j,k}=2^{dj/2}\ \vec{\gamma}_{\alpha,l}(2^jx-k)$. We may write:

$$\vec{Q}_{\alpha,j}\vec{f}=\sum_{l\neq n}\ \sum_{k\in\mathbb{Z}^d}\langle f_l|\psi^*_{\alpha,l,j,k}\rangle\ \vec{\gamma}_{\alpha,l,j,k}+R_n(\vec{f})\ \vec{e}_n$$

When $\vec{f}\in(E_1)^d$ and $\vec{\nabla}.\vec{f}=0$, we get that $\partial_n R_n(\vec{f})=0$ with $R_n(\vec{f})$ vanishing at infinity, and thus $R_n(\vec{f})=0$.

We may now define three series of operators on $(L^2(\mathbb{R}^d)^d$, defining $\vec{\psi}^*_{\alpha,l}=\psi^*_{\alpha,l}(x)\vec{e}_l$, $\vec{\gamma}^*_{\alpha,l}=\mathbb{P}\vec{\psi}^*_{\alpha,l}$ and $\vec{\psi}^*_{\alpha,l,j,k}(x)=2^{jd/2}\vec{\psi}^*_{\alpha,l}(2^jx-k)$, $\vec{\gamma}^*_{\alpha,l,j,k}(x)=2^{jd/2}\vec{\gamma}^*_{\alpha,l}(2^jx-k)$:

$$\begin{cases}\vec{Q}_j\vec{f} &= \sum_{1\leq l\leq d}\ \sum_{1\leq\alpha\leq 2^d-1}\ \sum_{k\in\mathbb{Z}^d}\langle\vec{f}|\vec{\psi}^*_{\alpha,l,j,k}\rangle\ \vec{\psi}_{\alpha,l,j,k}\\ \vec{R}_j\vec{f} &= \sum_{1\leq\alpha\leq 2^d-1}\ \sum_{1\leq l\leq d,l\neq n(\alpha)}\ \sum_{k\in\mathbb{Z}^d}\langle\vec{f}|\vec{\psi}^*_{\alpha,l,j,k}\rangle\ \vec{\gamma}_{\alpha,l,j,k}\\ \vec{S}_j\vec{f} &= \sum_{1\leq\alpha\leq 2^d-1}\ \sum_{1\leq l\leq d,l\neq n(\alpha)}\ \sum_{k\in\mathbb{Z}^d}\langle\vec{f}|\vec{\gamma}^*_{\alpha,l,j,k}\rangle\ \vec{\gamma}_{\alpha,l,j,k}\end{cases}$$

Then, we have

i) The family $(\vec{\psi}_{\alpha,l,j,k})_{1\leq\alpha\leq 2^d-1,1\leq l\leq d,j\in\mathbb{Z},k\in\mathbb{Z}^d}$ is a bi-orthogonal wavelet basis of $(L^2(\mathbb{R}^d))^d$ with dual basis the wavelets $(\vec{\psi}^*_{\alpha,l,j,k})_{1\leq\alpha\leq 2^d-1,1\leq l\leq d,j\in\mathbb{Z},k\in\mathbb{Z}^d}$: $\vec{Q}_j=\vec{P}_{j+1}-\vec{P}_j$ and $\sum_{j\in\mathbb{Z}}\vec{Q}_j=Id_{(L^2)^d}$. The wavelets $\vec{\psi}_{\alpha,l}$ and $\vec{\psi}^*_{\alpha,l}$ have compact supports and so do the associated scaling functions $\varphi_{\alpha,l}\vec{e}_l$ and $\varphi^*_{\alpha,l}\vec{e}_l$.

ii) $(\vec{\gamma}_{\alpha,l,j,k})_{1\leq\alpha\leq 2^d-1,1\leq l\leq d,l\neq n(\alpha),j\in\mathbb{Z},k\in\mathbb{Z}^d}$ is a bi-orthogonal wavelet basis of H with dual basis the wavelets $(\vec{\gamma}^*_{\alpha,l,j,k})_{1\leq\alpha\leq 2^d-1,1\leq l\leq d,l\neq n(\alpha),j\in\mathbb{Z},k\in\mathbb{Z}^d}$: $\vec{S}_j\vec{f}=\vec{Q}_j\mathbb{P}\vec{f}$ and $\sum_{j\in\mathbb{Z}}\vec{S}_j=\mathbb{P}$. The wavelets $\vec{\gamma}_{\alpha,l}$ have compact supports but the associated dual wavelets $\vec{\gamma}^*_{\alpha,l}$ have only polynomial decay at infinity: for some $N\in\mathbb{N}$, $|\vec{\gamma}^*_{\alpha,l}(x)|=O(|x|^{-N})$.

iii) The use of $\vec{R}_j$ is intermediate between i) and ii): on H, we have $\vec{R}_j=\vec{Q}_j=\vec{S}_j$, but on $H^\perp$ the three operators take different values. The operator $\sum_{j\in\mathbb{Z}}\vec{R}_j$ is a Calderón–Zygmund operator with range H and it is a bounded oblique projection operator from $(L^2)^d$ onto H. The main features of $\vec{R}_j$ is that the related wavelets $\vec{\gamma}_{\alpha,l}$ and $\vec{\psi}^*_{\alpha,l}$ have compact supports and that we may use the compactly supported scaling functions associated with the analysis $(\vec{Q}_j)_{j\in\mathbb{Z}}$ for expressing the low-frequency approximate $\sum_{j<0}\vec{R}_j\vec{f}$ of a divergence-free vector field $\vec{f}$.

We shall now prove a useful lemma on approximation of divergence-free vector fields by square-integrable divergence-free vector fields:

Proposition 12.1: (The approximation lemma)

Let $p\in[1,\infty)$ and let E_p be the space of locally L^p functions which vanish at infinity (defined by $\|f\|_{E_p}=\sup_{x\in\mathbb{R}^d}(\int_{|x-y|<1}|f(y)|^p\,dy)^{1/p}<\infty$ and $\lim_{x\to\infty}\int_{|x-y|<1}|f(y)|^p\,dy=0$). Let $\vec{u}\in(L^2+E_p)^d$ such that $\vec{\nabla}.\vec{u}=0$. Then for all $\epsilon>0$, there exists $\vec{v}_\epsilon\in(L^2)^d,\vec{w}_\epsilon\in(E_p)^d,\vec{\nabla}.\vec{v}_\epsilon=\vec{\nabla}.\vec{w}_\epsilon=0$ so that $\vec{u}=\vec{v}_\epsilon+\vec{w}_\epsilon$ et $\|\vec{w}_\epsilon\|_{E_p}\leq\epsilon$.

For $p=\infty$, the same result holds when defining E_∞ as $E_\infty=\mathcal{C}_0$.

Proof: Since $\mathcal{D}$ is dense in E_p, we may decompose each component u_i of $\vec{u}$ in V_i+W_i where the E_p norm of W_i is small. The difficulty lies in the requirement that $\vec{v}$ and $\vec{w}$ be divergence-free. It is not enough to project $\vec{V}$ and $\vec{W}$ on divergence-free vector fields with the help of the projection operator $\mathbb{P}$, because $\mathbb{P}$ (defined by $\mathbb{P}\vec{u}=\vec{u}-\vec{\nabla}(\frac{1}{\Delta}\vec{\nabla}.\vec{u})$) is not bounded on $(E_p)^d$. Instead of $\mathbb{P}$, we use our oblique projection operators $\vec{R}_j$ onto the divergence-free vector wavelet basis.

We first consider the projection operators $\vec{P}_j$ on $(E_p)^d$. We have, for any $\vec{v}\in(E_p)^d$, $\lim_{j\to+\infty}\|\vec{P}_j\vec{v}-\vec{v}\|_{E_p}=0$ and $\lim_{j\to-\infty}\|\vec{P}_j\vec{v}\|_{E_p}=0$: indeed, the operators $\vec{P}_j$ are equicontinuous on $(E_p)^d$, while for $\vec{v}\in(\mathcal{D})^d$ we obviously have that $\lim_{j\to+\infty}\|\vec{P}_j\vec{v}-\vec{v}\|_\infty=0$ and $\lim_{j\to-\infty}\|\vec{P}_j\vec{v}\|_\infty=0$. When $\vec{v}$ is divergence free, we find that for N large enough $\vec{z}_N=\vec{P}_{-N}\vec{v}+\vec{v}-\vec{P}_{N+1}\vec{v}$ is divergence free and has a norm in $(E_p)^d$ less than $\epsilon/2$. Thus, we just have to split

$$\vec{P}_{N+1}\vec{v}-\vec{P}_{-N}\vec{v}=\sum_{-N\leq j\leq N}\ \sum_{1\leq\alpha\leq 2^d-1}\ \sum_{l\neq n(\alpha)}\ \sum_{k\in\mathbb{Z}^d}\langle\vec{v}|\psi^*_{\alpha,l,j,k}\rangle\,\vec{\gamma}_{\alpha,l,j,k}.$$

We consider the operators $\vec{A}_{j,\alpha,l,K}(\vec{v})=\sum_{k\in\mathbb{Z}^d,|k|>K}\langle\vec{v}|\psi^*_{\alpha,l,j,k}\rangle\,\vec{\gamma}_{\alpha,l,j,k}$. When j is fixed, the operators $\vec{A}_{j,\alpha,l,K}$ are equicontinuous on $(E_p)^d$ and we have for every $\vec{v}\in(E_p)^d$ $\lim_{K\to+\infty}\|\vec{A}_{j,\alpha,l,K}\vec{v}\|_{E_p}=0$. Since we consider a finite number of j, we may find for every $\vec{v}$ a $K\in\mathbb{N}$ such that $\vec{Z}_{N,K}=\sum_{-N\leq j\leq N}\sum_{1\leq\alpha\leq 2^d-1}\sum_{l\neq n(\alpha)}\vec{A}_{j,\alpha,l,K}(\vec{v})$ has a norm in $(E_p)^d$ smaller than $\epsilon/2$. We then define $\vec{w}_\epsilon=\vec{z}_N+\vec{Z}_{N,K}$ and $\vec{v}_\epsilon=\vec{v}-\vec{w}_\epsilon$.

The proof gives a more precise result: we may choose $\vec{v}_\epsilon$ to have a compact support. □

3. The div-curl theorem

The following theorem proved by Coifman, Lions, Meyer and Semmes [COILMS 92] will be very useful :

Theorem 12.1: (The div-curl theorem)

Let $\vec{u} \in (L^2(\mathbb{R}^d))^d$ with $\vec{\nabla}.\vec{u} = 0$ and let $\vec{v} = \vec{\nabla} V$ with $V \in \dot{H}^1(\mathbb{R}^d)$. Then $\vec{u}.\vec{v}$ belongs to the Hardy space $\mathcal{H}^1$ and $\|\vec{u}.\vec{v}\|_{\mathcal{H}^1} \leq C\|\vec{u}\|_2\|\vec{v}\|_2$ where the constant C does depend only on d.

Proof: We adapt the proof given by Dobyinsky [DOB 92]. We use the multi-resolution analysis $\vec{P}_j$ described in Section 2 associated with the projection operators P_j and $\tilde{P}_j$, which satisfy $\frac{d}{dx} \circ P_j = \tilde{P}_j \circ \frac{d}{dx}$, and we use as well the projection operator $\mathcal{P}_j^* = \tilde{P}_j^* \otimes \ldots \otimes \tilde{P}_j^*$. We define $\vec{Q}_j = \vec{P}_{j+1} - \vec{P}_j$ and $\mathcal{Q}_j = \mathcal{P}_{j+1}^* - \mathcal{P}_j^*$. We then write $\vec{u}.\vec{v} = \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \vec{Q}_j \vec{u}.\vec{\nabla} \mathcal{Q}_l V = A(\vec{u}, \vec{v}) + B(\vec{u}, \vec{v}) + C(\vec{u}, \vec{v})$ with $A(\vec{u}, \vec{v}) = \sum_{j \in \mathbb{Z}} \vec{Q}_j \vec{u}.\vec{\nabla} \mathcal{Q}_j V$, $B(\vec{u}, \vec{v}) = \sum_{j \in \mathbb{Z}} \vec{Q}_j \vec{u}.\vec{\nabla} \mathcal{P}_j V$, $C(\vec{u}, \vec{v}) = \sum_{j \in \mathbb{Z}} \vec{P}_j \vec{u}.\vec{\nabla} \mathcal{Q}_j V$.

We replace $\vec{Q}_j$ by $\vec{R}_j$ (which is harmless on divergence-free vector fields). We write $\mathcal{Q}_j V = \sum_{1 \leq \epsilon \leq 2^d - 1} \sum_{k \in \mathbb{Z}^d} \langle V | 2^{jd/2} \Psi_\epsilon(2^j x - k) \rangle 2^{jd/2} \Psi_\epsilon^*(2^j x - k)$; for each wavelet Ψ_ϵ, we may write, for some index $l(\epsilon)$ and some compactly supported function Θ_ϵ, $\Psi_\epsilon = -\partial_{l(\epsilon)} \Theta_\epsilon$ so that we get $\vec{\nabla} \mathcal{Q}_j V = \vec{T}_j \vec{v}$ with $\vec{T}_j \vec{f} = \sum_{1 \leq \epsilon \leq 2^d - 1} \sum_{k \in \mathbb{Z}^d} \langle f_{l(\epsilon)} | 2^{jd/2} \Theta_\epsilon(2^j x - k) \rangle 2^{jd/2} (\vec{\nabla} \Psi_\epsilon^*)(2^j x - k)$. We thus replace A, B and C by $\tilde{A}(\vec{u}, \vec{v}) = \sum_{j \in \mathbb{Z}} \vec{R}_j \vec{u}.\vec{T}_j \vec{v}$, $\tilde{B}(\vec{u}, \vec{v}) = \sum_{j \in \mathbb{Z}} \vec{R}_j \vec{u}.\vec{P}_j^* \vec{v}$, $\tilde{C}(\vec{u}, \vec{v}) = \sum_{j \in \mathbb{Z}} \vec{P}_j \vec{u}.\vec{T}_j \vec{v}$.

We first check that $\tilde{B}$ and $\tilde{C}$ map $(L^2)^d \times (L^2)^d$ to $\mathcal{H}^1$: we use the duality of H^1 and CMO (the closure of $\mathcal{C}_0$ in BMO) (see Coifman and Weiss [COIW 77] and Bourdaud [BOU 01]) and try to prove that the operators $\mathcal{B}(\vec{f}, g) = \sum_{j \in \mathbb{Z}} \vec{R}_j^*(g \vec{P}_j^* \vec{f})$ and $\mathcal{C}(\vec{f}, g) = \sum_{j \in \mathbb{Z}} \vec{T}_j^*(g \vec{P}_j \vec{f})$ map $(L^2)^d \times BMO$ to $(L^2)^d$. First we notice that $\vec{R}_j^* \circ \vec{P}_j^* = 0$ and $\vec{T}_j^* \circ \vec{P}_j = 0$, so that we may add any constant to g and find the same result (recall that BMO is defined modulo the constants). Moreover, we find that $\mathcal{B}(., g)$ and $\mathcal{C}(., g)$ are matrices of singular integrals operators $(B_{i,j}(., g))_{1 \leq i,j \leq d}$ and $(C_{i,j}(., g))_{1 \leq i,j \leq d}$: expand $\vec{P}_j$, $\vec{P}_j^*$ using scaling functions and $\vec{R}_j^*$ and $\vec{T}_j^*$ with help of wavelets. There are only a few terms that interact, because of the localization of the supports. To estimate the size of the kernels of $\mathcal{B}$ and $\mathcal{C}$ and the size of their derivatives, it must be shown that

$$\sup_{\begin{cases} j \in \mathbb{Z}, k_0 \in \mathbb{Z}^d \\ k_1 \in \mathbb{Z}^d, 1 \leq l \leq d \\ 1 \leq \alpha \leq 2^d - 1, m \neq n(\alpha) \end{cases}} |\langle 2^{jd/2} \vec{\gamma}_{\alpha,m}(2^j x - k_1) | g 2^{jd/2} \psi_{0,l}^*(2^j x - k_2) \vec{e}_l \rangle| < \infty$$

and

$$\sup_{\begin{cases} j \in \mathbb{Z}, k_0 \in \mathbb{Z}^d \\ k_1 \in \mathbb{Z}^d, 1 \leq l \leq d \\ 1 \leq \epsilon \leq 2^d - 1 \end{cases}} |\langle 2^{jd/2} \vec{\nabla} \Psi_\epsilon^*(2^j x - k_1) | g 2^{jd/2} \psi_{0,l}(2^j x - k_2) \vec{e}_l \rangle| < \infty.$$

Proposition 10.2 allows us to conclude that $B_{i,j}(.,g)$ is a singular integral operator with $B_{i,j}(.,g) \in WBP$ and $B_{i,j}(.,g)^*(1) = 0$ and similarly, that $C_{i,j}(.,g)$ is a singular integral operator with $C_{i,j}(.,g) \in WBP$ and $C_{i,j}(.,g)^*(1) = 0$. Moreover, $\mathcal{B}(.,g)(1\vec{e}_l) = (\sum_{j\in\mathbb{Z}} \vec{R}_j^*) g \vec{e}_l$ and we may again apply Proposition 10.2 to conclude that the matrix of Calderón–Zygmund operators $\sum_{j\in\mathbb{Z}} \vec{R}_j^*$ maps BMO^d to BMO^d; this gives that for all i, j in $\{1, \ldots, d\}$ we have $B_{i,j}(.,g)(1) \in BMO$; hence, that $B_{i,j}(.,g)$ is bounded on L^2. The same proof works for $C_{i,j}$, using the matrix of Calderón–Zygmund operators $\sum_{j\in\mathbb{Z}} \vec{T}_j^*$.

Estimating $I_{j,k_0,k_1,l,\alpha,m} = \langle 2^{jd/2} \vec{\gamma}_{\alpha,m}(2^j x - k_1) | g 2^{jd/2} \psi_{0,l}^*(2^j x - k_2) \vec{e}_l \rangle$ and on $J_{j,k_0,k_1,l,\epsilon} = \langle 2^{jd/2} \vec{\nabla} \Psi_\epsilon^*(2^j x - k_1) | g 2^{jd/2} \psi_{0,l}(2^j x - k_2) \vec{e}_l \rangle$ is easy. Indeed, let M be large enough so that the supports of the scaling functions $\psi_{0,l}^*$, and $\psi_{0,l}$ are contained in $B(0, M)$ for every $l \in \{1, \ldots, d\}$; then, since we have $\langle \vec{\gamma}_{\alpha,m}(2^j x - k_1) | \psi_{0,l}^*(2^j x - k_2) \vec{e}_l \rangle = 0$ and $\langle \vec{\nabla} \Psi_\epsilon^*(2^j x - k_1) | \psi_{0,l}(2^j x - k_2) \vec{e}_l \rangle = 0$, we may write (writing B_{j,k_2} for the ball $B_{j,k_2} = B(\frac{k_2}{2^j}, \frac{M}{2^j})$):

$$I_{j,k_0,k_1,l,\alpha,m} = \int_{B_{j,k_2}} 2^{jd} (\bar{g}(x) - m_{B_{j,k_2}} \bar{g}) \psi_{0,l}^*(2^j x - k_2) \vec{\gamma}_{\alpha,m}(2^j x - k_1) . \vec{e}_l \, dx$$

and

$$J_{j,k_0,k_1,l,\epsilon} = \int_{B_{j,k_2}} 2^{jd} (\bar{g}(x) - m_{B_{j,k_2}} \bar{g}) \psi_{0,l}(2^j x - k_2) \partial_l \Psi_\epsilon^*(2^j x - k_1) \, dx$$

which gives the required estimates $|I_{j,k_0,k_1,l,\alpha,m}| \leq C \|g\|_{BMO} \|\psi_{0,l}^*\|_\infty \|\vec{\gamma}_{\alpha,m}\|_\infty$ and $|J_{j,k_0,k_1,l,\epsilon}| \leq C \|g\|_{BMO} \|\psi_{0,l}\|_\infty \|\partial_l \Psi_\epsilon^*\|_\infty$.

We still have to deal with $\tilde{A}(\vec{u}, \vec{v}) = \sum_{j\in\mathbb{Z}} \vec{R}_j \vec{u} . \vec{T}_j \vec{v}$. But this remainder term obviously belongs to $\mathcal{H}^1$, since expanding $\vec{R}_j \vec{u}$ and $\vec{T}_j \vec{v}$ in their wavelet bases yields directly an atomic decomposition: we have indeed

$$\vec{R}_j \vec{u} = \sum_{1\leq\alpha\leq 2^d-1} \sum_{1\leq m\leq d, m\neq n(\alpha)} \sum_{k\in\mathbb{Z}^d} \lambda_{j,k,\alpha,m} 2^{jd/2} (\vec{\gamma}_{\alpha,m})(2^j x - k)$$

with $\sum_{1\leq\alpha\leq 2^d-1} \sum_{1\leq m\leq d, m\neq n(\alpha)} \sum_{k\in\mathbb{Z}^d} |\lambda_{j,k,\alpha,m}|^2 \leq C \|\vec{u}\|_2^2$ and

$$\vec{T}_j \vec{v} = \sum_{1\leq\epsilon\leq 2^d-1} \sum_{k\in\mathbb{Z}^d} \mu_{j,k,\epsilon} 2^{jd/2} (\vec{\nabla} \Psi_\epsilon^*)(2^j x - k)$$

with $\sum_{1\leq\epsilon\leq 2^d-1}\sum_{k\in\mathbb{Z}^d}|\mu_{j,k,\epsilon}|^2 \leq C\|\vec{v}\|_2^2$; now, we recall that the pointwise products $\vec{\gamma}_{\alpha,m}(2^jx-k_1).\vec{\nabla}\Psi^*_\epsilon(2^jx-k_2)$ are identically equal to O for $|k_1-k_2|\geq K$, where K is a constant depending only on the supports of Ψ^*_ϵ and of $\vec{\gamma}_{\alpha,m}$ and we obtain that

$$\tilde{A}(\vec{u},\vec{v}) = \sum_{(j,k_1,k_2,\alpha,m,\epsilon)\in I} \lambda_{j,k_1,\alpha,m}\bar{\mu}_{j,k_2,\epsilon}2^{jd}\vec{\gamma}_{\alpha,m}(2^jx-k_1).(\vec{\nabla}\Psi^*_\epsilon)(2^jx-k_2)$$

with $I = \{(j,k_1,k_2,\alpha,m,\epsilon)\ /\ j\in\mathbb{Z}, 1\leq\alpha\leq 2^d-1, 1\leq m\leq d, m\neq n(\alpha), k_1\in\mathbb{Z}^d, 1\leq\epsilon\leq 2^d-1, k_2\in\mathbb{Z}^d, |k_2-k_1|<K\}$. We may now conclude, since $\sum_{(j,k_1,k_2,\alpha,m,\epsilon)\in I}|\lambda_{j,k_1,\alpha,m}\bar{\mu}_{j,k_2,\epsilon}|<\infty$ and $2^{jd}\vec{\gamma}_{\alpha,m}(2^jx-k_1).(\vec{\nabla}\Psi^*_\epsilon)(2^jx-k_2)$ is a bounded atom in $\mathcal{H}^1$. □

Chapter 13

The mollified Navier–Stokes equations

In this chapter, we present a classical tool of Leray [LER 34] to prove the existence of weak solutions: in order to be able to use the the Picard contraction algorithm, we first mollify the equations, replacing the nonlinearity $\vec{u} \otimes \vec{u}$ into the smoother term $(\vec{u} * \omega_\epsilon) \otimes \vec{u}$ with $\omega \in \mathcal{D}(\mathbb{R}^d)$, $\omega \geq 0$, $\int_{\mathbb{R}^d} \omega \, dx = 1$ and $\omega_\epsilon = \frac{1}{\epsilon^d} \omega(\frac{x}{\epsilon})$; then we try to get uniform estimates on the solutions with respect to the perturbation and to use a limiting process to go back to the initial equations.

1. The mollified equations

In order to introduce the mollified equations, we choose a function $\omega \in \mathcal{D}(\mathbb{R}^d)$ with $\omega \geq 0$ and $\int_{\mathbb{R}^d} \omega \, dx = 1$; then the mollified equations are given for $\epsilon > 0$ by

$$\left\{ \begin{array}{c} \partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla}.((\vec{u} * \omega_\epsilon) \otimes \vec{u}) - \vec{\nabla} p \\ \vec{\nabla}.\vec{u} = 0 \\ \vec{u}(0,.) = \vec{u}_0 \end{array} \right. \tag{13.1}$$

where $\omega_\epsilon = \frac{1}{\epsilon^d} \omega(\frac{x}{\epsilon})$.

For $\vec{u}_0 \in (B_\infty^{-N,\infty})^d$ for some $N > 0$, we seek a solution $\vec{u} \in (L_t^\infty B_\infty^{-N,\infty})^d$. We replace the term $\vec{\nabla}.((\vec{u} * \omega_\epsilon) \otimes \vec{u}) + \vec{\nabla} p$ by the term $\mathbb{P}\vec{\nabla}.((\vec{u} * \omega_\epsilon) \otimes \vec{u})$. We have no problem defining $\mathbb{P}$ in that case, since $(\vec{u} * \omega_\epsilon) \otimes \vec{u} \in (L_t^\infty B_\infty^{-N,\infty})^{d \times d}$ and since $\mathbb{P}\vec{\nabla}.$ maps $(B_\infty^{-N,\infty})^{d \times d}$ to $(B_\infty^{-N-1,\infty})^d$. Thus, we shall solve the equation

$$\left\{ \begin{array}{c} \partial_t \vec{u} = \Delta \vec{u} - \mathbb{P}\vec{\nabla}.((\vec{u} * \omega_\epsilon) \otimes \vec{u}) \\ \vec{\nabla}.\vec{u} = 0 \\ \vec{u}(0,.) = \vec{u}_0 \end{array} \right. \tag{13.2}$$

which is a special case of (13.1). Equation (13.2) is equivalent to the fixed-point problem $\vec{u} = e^{t\Delta} \vec{u}_0 - B_\epsilon(\vec{u}, \vec{u})$ where the bilinear operator B_ϵ is given by

$$B_\epsilon(\vec{u}, \vec{v}) = \int_0^t e^{(t-s)\Delta} \mathbb{P}\vec{\nabla}.((\vec{u} * \omega_\epsilon) \otimes \vec{v}) \, ds.$$

It is very easy to construct a local solution to (13.2):

Theorem 13.1: (Solution of the mollified equations)

i) Let $N > 0$ *and* $\vec{u}_0 \in (B_\infty^{-N,\infty})^d$ *with* $\vec{\nabla}.\vec{u}_0 = 0$. *Then there exists a maximal* $T^* > 0$ *so that the equations (13.2) have a solution* $\vec{u}$ *with* $\vec{u} \in \cap_{T<T^*}(L^\infty((0,T), B_\infty^{-N,\infty}(\mathbb{R}^d)))^d$. *If* $T^* < \infty$, *then* $\lim_{t\to T^*} \|\vec{u}\|_{B_\infty^{-N,\infty}} = +\infty$.

ii) The solution $\vec{u}$ *is unique and* $\mathcal{C}^\infty$ *on* $(0,T^*)\times\mathbb{R}^d$. *Moreover, the pressure* p *satisfying* $\vec{\nabla}p = (\mathbb{P} - Id)\vec{\nabla}.\big((\vec{u}*\omega_\epsilon)\otimes\vec{u}\big)$ *may be chosen* $\mathcal{C}^\infty$ *on* $(0,T^*)\times\mathbb{R}^d$.

iii) If we have more precisely $\vec{u}_0 \in (B_{\mathcal{C}_0}^{-N,\infty})^d$, *then* $T^* = +\infty$.

The construction of the solution is based on the Picard contraction principle:

Theorem 13.2: (The Picard contraction principle)

Let E *be a Banach space and let* B *be a bounded bilinear transform from* $E\times E$ *to* E:

$$\|B(e,f)\|_E \leq C_B\|e\|_E\|f\|_E.$$

Then, if $0 < \delta < \frac{1}{4C_B}$ *and if* $e_0 \in E$ *is such that* $\|e_0\|_E \leq \delta$, *the equation* $e = e_0 - B(e,e)$ *has a solution with* $\|e\|_E \leq 2\delta$. *This solution is unique in the ball* $\bar{B}(0,2\delta)$. *Moreover, the solution continuously depends on* e_0: *if* $\|f_0\|_E \leq \delta$, $f = f_0 - B(f,f)$ *and* $\|f\|_E \leq 2\delta$, *then* $\|e-f\|_E \leq \frac{1}{1-4C_B\delta}\|e_0 - f_0\|_E$.

Proof: We construct e by iteration. We start from e_0 and we define inductively a sequence e_n by $e_{n+1} = e_0 - B(e_n,e_n)$. By induction, we show that $\|e_n\|_E \leq 2\delta$: indeed, we have $\|e_{n+1}\|_E \leq \|e_0\|_E + C_B\|e_n\|_E^2 \leq \delta + 4C_B\delta^2 \leq 2\delta$. Moreover, we have

$$\|e_{n+1}-e_n\|_E = \|B(en-e_{n-1},e_n)+B(e_{n-1},e_n-e_{n-1}\|_E \leq (4C_B\delta)\|e_n-e_{n-1}\|_E;$$

hence, $\|e_{n+1} - e_n\|_E \leq (4C_B\delta)^n\|e_1 - e_0\|_E$. Since $4C_B\delta < 1$, we find that e_n converges to a limit e, which is the required solution. The uniqueness of e in $\bar{B}(0,2\delta)$ is then obvious. The continuous dependence on e_0 is easy as well: just write $e - f = e_0 - f_0 - B(e-f,e) - B(f,e-f)$, which gives $\|e-f\|_E \leq \|e_0-f_0\|_E + C_B\|e\|_E\|e-f\|_E + C_B\|f\|_E\|e-f\|_E \leq \|e_0-f_0\|_E + 4C_B\delta\|e-f\|_E$. $\square$

We may now prove Theorem 13.1:

Proof of Theorem 13.1: For $T>0$, let $E_T = (L^\infty((0,T), B_\infty^{-N,\infty}(\mathbb{R}^d)))^d$. If $\vec{u}_0$ belongs to $(B_\infty^{-N,\infty})^d$, then we have for all $T>0$ $\|e^{t\Delta}\vec{u}_0\|_{E_T} \leq \|\vec{u}_0\|_{B_\infty^{-N,\infty}}$. Moreover, B_ϵ is continuous from $E_T\times E_T$ to E_T: first, we have for $\alpha \in B_\infty^{-N,\infty}$ and $\epsilon > 0$ $\alpha*\omega_\epsilon \in B_\infty^{N+1,\infty}$ and $\|\alpha*\omega_\epsilon\|_{B_\infty^{N+1,\infty}} \leq C(1+\epsilon^{-2N-1})\|\alpha\|_{B_\infty^{-N,\infty}}$; moreover, for $\beta \in B_\infty^{N+1,\infty}$ and $\gamma \in B_\infty^{-N,\infty}$, we have $\beta\gamma \in B_\infty^{-N,\infty}$ and $\|\beta\gamma\|_{B_\infty^{-N,\infty}} \leq C\|\beta\|_{B_\infty^{N+1,\infty}}\|\gamma\|_{B_\infty^{-N,\infty}}$; finally, the operator $e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.$ is given by convolutions with functions whose L^1 norm is controlled in $(t-s)^{-1/2}$ (see Chapter 11); thus, we get

$$\|B_\epsilon(\vec{u},\vec{v})\|_{E_T} \leq C\sqrt{T}(1+\epsilon^{-2N-1})\|\vec{u}\|_{E_T}\|\vec{v}\|_{E_T}$$

where the constant C does not depend on T nor on ϵ. Thus, Theorem 13.2 gives that we have existence of a solution $\vec{u}$ in E_T of $\vec{u} = e^{t\Delta}\vec{u}_0 - B_\epsilon(\vec{u},\vec{u})$ as soon as $4C\sqrt{T}(1+\epsilon^{-2N-1})\|\vec{u}_0\|_{B_\infty^{-N,\infty}} < 1$, thus for T small enough.

A solution in E_T of (2) satisfies that $\partial_t\vec{u} \in (L^\infty((0,T), B_\infty^{-N-2,\infty}))^d$; hence, that $\vec{u} \in (\mathcal{C}([0,T], B_\infty^{-N-2,\infty}))^d$. Thus, uniqueness of solutions in E_T is obvious: the contraction principle proves that two solutions of (13.2) in E_T must coincide on $(0,T_0)$ where $C\sqrt{T_0}(1+\epsilon^{-2N-1})(\|\vec{u}\|_{E_T} + \|\vec{v}\|_{E_T}) < 1$. Now, if T_1 is the supremum of the times t so that $\vec{u} = \vec{v}$ on $[0,t)$, we have $\vec{u}(T_1) = \vec{v}(T_1)$ in $(B_\infty^{-N-2,\infty})^d$ by continuity; hence, in $(B_\infty^{-N,\infty})^d$ by *-weak continuity; if $T_1 < T$, we may notice that $\vec{u}(t+T_1)$ and $\vec{v}_(t+T_1)$ are solutions in E_{T-T_1} of a similar Cauchy problem (replacing the initial value $\vec{u}_0$ by $\vec{u}(T_1)$); thus, we would find that $\vec{u} = \vec{v}$ on $[T_1, T_1+T_2)$ for small enough T_2, and this is in contradiction with the definition of T_1 : thus $T_1 = T$ and we have uniqueness in E_T.

Because of this uniqueness, we find that there is a maximal interval $[0,T^*)$ on which $\vec{u}$ is defined. If $T^* < \infty$, we must have for $t < T^*$ that $\|\vec{u}(t)\|_{B_\infty^{-N,\infty}} \geq \frac{1}{4C\sqrt{T^*-t}(1+\epsilon^{-2N-1})}$, hence $\lim_{t\to T^*} \|\vec{u}(t)\|_{B_\infty^{-N,\infty}} = +\infty$.

We now prove the regularity of $\vec{u}$. More precisely, we prove by induction that $\vec{u} \in \cap_{0<T_1<T_2<T^*}(L^\infty((T_1,T_2), B_\infty^{-N+k/2,\infty}))^d$ for all $k \in \mathbb{N}$. Indeed, we have for $0 < T_0 < T_1 < t < T_2$,

$$\vec{u}(t) = e^{(t-T_0)\Delta}\vec{u}(T_0) - \int_{T_0}^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.((\vec{u}*\omega_\epsilon)\otimes\vec{u})\,ds;$$

we have $\|e^{(t-T_0)\Delta}\vec{u}(T_0)\|_{B_\infty^{-N+k/2,\infty}} \leq C\sup((t-T_0)^{-k/4},1)\|\vec{u}(T_0)\|_{B_\infty^{-N,\infty}}$; moreover, if $\vec{u} \in (B_\infty^{-N+k/2,\infty})^d$ for some $k \in \mathbb{N}$, then $(\vec{u}*\omega_\epsilon)\otimes\vec{u} \in (B_\infty^{-N+k/2,\infty})^{d\times d}$; thus, we find

$$\|\int_0^t \mathbb{P}\vec{\nabla}.((\vec{u}*\omega_\epsilon)\otimes\vec{u})\,ds\|_{B_\infty^{-N+(k+1)/2,\infty}} \leq$$

$$\leq C(1+\epsilon^{-N-1})\|\vec{u}\|^2_{L^\infty((T_0,T_2),B_\infty^{-N+k/2,\infty})}\int_{T_0}^t \sup(\frac{1}{(t-s)^{3/4}},1)\,ds.$$

Thus far, we have proved the spatial regularity of $\vec{u}$. In particular, we have $\vec{u} \in \cap_{0<T_1<T_2<T^*}(L^\infty((T_1,T_2), B_\infty^{k,\infty})^d$ for all $k \in \mathbb{N}$; a similar estimate then holds for $\partial_t\vec{u}$ since Δ maps $B_\infty^{k,\infty}$ to $B_\infty^{k-2,\infty}$ and $\mathbb{P}\vec{\nabla}.$ maps $(B_\infty^{k,\infty})^{d\times d}$ to $(B_\infty^{k-1,\infty})^d$. Then, we may conclude by induction that all derivatives $\frac{\partial^p}{\partial t^p}\vec{u}$ satisfy similar estimates, since we have $\frac{\partial^{p+1}}{\partial t^{p+1}}\vec{u} = \Delta\frac{\partial^p}{\partial t^p}\vec{u} - \mathbb{P}\vec{\nabla}.\big(\frac{\partial^p}{\partial t^p}((\vec{u}*\omega_\epsilon)\otimes\vec{u})\big)$. This proves that $\vec{u}$ is smooth on $(0,T^*)\times\mathbb{R}^d$.

We now consider the case of $\vec{u}_0 \in B_{\mathcal{C}_0}^{-N,\infty}$. First, we notice that $\vec{u}$ belongs to $(\mathcal{C}_0(\mathbb{R}^d))^d$ for all $t \in (0,T^*)$. Indeed, we use the Picard contraction principle in

$F_T = \{\vec{u} \in E_T \ / \ \vec{u} \in (\mathcal{C}((0,T), \mathcal{C}_0(\mathbb{R}^d)))^d$ and $\sup_{0<t<T} t^{N/2} \|\vec{u}\|_{L^\infty(\mathbb{R}^d)} < \infty\}$. We have for all $T \in (0,1)$ $\|t^{N/2} e^{t\Delta}\vec{u}_0\|_{L^\infty(\mathbb{R}^d)} \leq C\|\vec{u}_0\|_{B_\infty^{-N,\infty}}$. Moreover, B_ϵ is continuous from $F_T \times F_T$ to F_T: we write $B_\epsilon(\vec{u},\vec{v})(t) = e^{\frac{t}{2}\Delta} B_\epsilon(\vec{u},\vec{v})(t/2) + \int_{t/2}^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.((\vec{u} * \omega_\epsilon) \otimes \vec{v})\ ds$; we write (for $0 < t < T < 1$)

$$\begin{aligned} \|(\tfrac{t}{2})^{N/2} e^{\frac{t}{2}\Delta} \vec{B}_\epsilon(\vec{u},\vec{v})(\tfrac{t}{2})\|_\infty &\leq C\|\vec{B}_\epsilon(\vec{u},\vec{v})(\tfrac{t}{2})\|_{B_\infty^{-N,\infty}} \\ &\leq C'\sqrt{T}(1+\epsilon^{-2N-1})\|\vec{u}\|_{E_T}\|\vec{v}\|_{E_T}, \end{aligned}$$

while

$$\begin{aligned} \|\int_{t/2}^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.((\vec{u} * \omega_\epsilon) \otimes \vec{v}) ds\|_\infty &\leq C(1+\epsilon^{-N})\|\vec{u}\|_{E_T}\|\vec{v}\|_{F_T} \int_{t/2}^t \frac{1}{\sqrt{t-s}} \frac{ds}{s^{N/2}} \\ &\leq C' \frac{\sqrt{T}}{t^{N/2}}(1+\epsilon^{-N-1})\|\vec{u}\|_{E_T}\|\vec{v}\|_{F_T}. \end{aligned}$$

Moreover, $B_\epsilon(\vec{u},\vec{v})(t)$ belongs to $(\mathcal{C}_0)^d$: $B_\epsilon(\vec{u},\vec{v})(t) = e^{(t-\eta)\Delta}B_\epsilon(\vec{u},\vec{v})(\eta) + \int_\eta^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.((\vec{u} * \omega_\epsilon) \otimes \vec{v})\ ds$ where the first term has a L^∞-norm which is $O(\frac{\eta}{(t-\eta)^{N/2}})$, while the second term belongs to $(\mathcal{C}_0)^d$. Now, for the continuity in L^∞-norm of $t \mapsto B_\epsilon(\vec{u},\vec{v})(t)$, we write for $0 < t_1 < t_2 < T$ $\tau = (t_1+t_2)/2$ and $\theta = |t_1 - t_2|/2$ and easily check that $\|B_\epsilon(\vec{u},\vec{v})(t_1) - B_\epsilon(\vec{u},\vec{v})(t_2)\|_{L^\infty(dx)} \leq \|\int_{\tau-\sqrt{\theta\tau}}^{t_1} e^{(t_1-s)\Delta}\mathbb{P}\vec{\nabla}.((\vec{u}*\omega_\epsilon)\otimes\vec{v})ds\|_\infty + \|\int_{\tau-\sqrt{\theta\tau}}^{t_2} e^{(t_2-s)\Delta}\mathbb{P}\vec{\nabla}.((\vec{u}*\omega_\epsilon)\otimes\vec{v})ds\|_\infty$ $+\|\int_0^{\tau-\sqrt{\theta\tau}}(e^{(t_1-s)\Delta} - e^{(t_2-s)\Delta})\mathbb{P}\vec{\nabla}.((\vec{u} * \omega_\epsilon) \otimes \vec{v})\ ds\|_\infty$; the first two norms are controlled by $C(\tau\theta)^{1/4}\sup_{\tau-\sqrt{\theta\tau}\leq s\leq\tau+\theta}\|\vec{u}\|_\infty\vec{v}\|_\infty$; the last norm may be controlled, using a fixed small $\eta > 0$ and splitting the integral between $0 < s < \eta$ and $\eta < s < \tau - \sqrt{\tau\theta}$, by $C\big(\frac{\sqrt{\theta}}{\sqrt{\tau}-\sqrt{\theta}}\int_\eta^{\tau-\sqrt{\theta\tau}} \frac{1}{\sqrt{\tau-\theta-s}}\|\vec{u}(s)\|_\infty\|\vec{v}(s)\|_\infty\ ds + \frac{\theta}{\tau-\theta-\eta}\frac{1}{(\tau-\theta-\eta)^{N/2}}\|B_\epsilon(\vec{u},\vec{v})(\eta)\|_{B_\infty^{N,\infty}}\big)$; hence, we may easily conclude that we have $\lim_{t_2\to t_1}\|B_\epsilon(\vec{u},\vec{v})(t_1) - B_\epsilon(\vec{u},\vec{v})(t_2)\|_{L^\infty(dx)} = 0$.

Thus, there exists a positive constant C so that (13.2) has a solution in F_T as soon as $4C\sqrt{T}(1+\epsilon^{-2N-1})\|\vec{u}_0\|_{B_\infty^{-N,\infty}} < 1$. By uniqueness in E_T, we find that the solution $\vec{u}$ constructed on $(0,T^*)$ belongs to F_T where T depends only on $\|\vec{u}_0\|_{B_\infty^{-N,\infty}}$. Now, since $\vec{u}$ belongs to $(\mathcal{C}_0)^d$ for every $t \in (0,T)$, we find that $\vec{u}$, which is known to remain bounded in $B_\infty^{N,\infty}$ norm on any compact subset of $[0,T^*)$, will belong to the closed subspace $(B_{\mathcal{C}_0}^{N,\infty})^d$; thus, we may reiterate the construction and find finally that $\vec{u} \in (\mathcal{C}((0,T^*), \mathcal{C}_0(\mathbb{R}^d))^d$.

The problem is now to prove that $T^* = +\infty$. We are going to use the energy estimate of Leray [LER 34] for square-integrable solutions and a splitting argument introduced to deal with infinite-energy weak solutions (Calderón [CAL 90], Lemarié-Rieusset [LEM 98b]). Assume that $T^* < +\infty$. We fix $T \in (0,T^*)$ and $\rho > 0$ and we split $\vec{u}(T)$ in $\vec{V} + \vec{W}$ with $\vec{\nabla}.\vec{V} = \vec{\nabla}.\vec{W} = 0$, $\vec{W} \in (L^2)^d$, $\vec{V} \in (\mathcal{C}_0)^d$ and $\|\vec{V}\|_\infty < \rho$ (the existence of such an approximation was proved in Chapter 12). Then, we solve

$$\begin{cases} \partial_t\vec{v} = \Delta\vec{v} - \mathbb{P}\vec{\nabla}.((\vec{v} * \omega_\epsilon) \otimes \vec{v}) \\ \vec{\nabla}.\vec{v} = 0 \\ \vec{v}(T,.) = \vec{V} \end{cases}$$

by the Picard contraction principle in $(\mathcal{C}([T,T_1],\mathcal{C}_0(\mathbb{R}^d))^d$. We find that, for a constant C which does not depend on T nor ρ, the problem may be solved on $[T,T_1]$ (with $\sup_{T\leq t\leq T_1}\|\vec{v}\|_\infty \leq 2\rho$) as soon as $C\rho\sqrt{T_1-T}<1$. We shall then choose ρ in order to get $T_1>T^*$. Now, we write $\vec{u}=\vec{v}+\vec{w}$ where $\vec{w}$ is solution of

$$\begin{cases} \partial_t\vec{w} = \Delta\vec{w} - \mathbb{P}\vec{\nabla}.((\vec{v}*\omega_\epsilon)\otimes\vec{w} + (\vec{w}*\omega_\epsilon)\otimes\vec{v} + (\vec{w}*\omega_\epsilon)\otimes\vec{w}) \\ \vec{\nabla}.\vec{w}=0 \\ \vec{w}(T,.)=\vec{W} \end{cases}$$

This may be solved by the Picard contraction principle in $(\mathcal{C}([T,T_2],L^2(\mathbb{R}^d))^d$: we find that, for a constant C, which does not depend on T nor ρ, the problem may be solved on $[T,T_2]$ as soon as $C(\rho+\|\vec{W}\|_2)\sqrt{T_2-T}<1$. This is the same solution as $\vec{u}-\vec{w}$, because of uniqueness of solutions in $(L^\infty((T,T_2),B_\infty^{-N,\infty}))^d$ (we may assume $N>d/2$, hence $L^2\subset B_\infty^{-N,\infty}$). Now, we shall prove the energy estimate to get that this solution $\vec{w}$ never explodes in L^2 norm, hence may be continued up to T^*, and even beyond T^*; this will prove that $\vec{u}$ may be continued beyond T^*, in contradiction with the definition of T^*. Indeed, if $\vec{w}\in(\mathcal{C}([T,T_3],L^2(\mathbb{R}^d))^d$, we define $p\in\mathcal{C}([T,T_3],L^2(\mathbb{R}^d)$ by $p=-\frac{1}{\Delta}\sum_j\sum_k\partial_j\partial_k(w_k(w_j*\omega_\epsilon)+w_k(v_j*\omega_\epsilon)+v_k(w_j*\omega_\epsilon))$; since $\vec{w}=\vec{u}-\vec{v}$ is smooth on $(0,T_3)\times\mathbb{R}^d$, we have

$$\partial_t|\vec{w}|^2 + 2|\vec{\nabla}\otimes\vec{w}|^2 = \Delta|\vec{w}|^2 \ -\vec{\nabla}.\vec{Z} + 2\vec{\nabla}\otimes\vec{w}.(\vec{w}*\omega_\epsilon)\otimes\vec{v}$$

where $\vec{Z}=|\vec{w}|^2(\vec{w}*\omega_\epsilon+\vec{v}*\omega_\epsilon)+2(\vec{v}.\vec{w})\,\vec{w}*\omega_\epsilon+2p\,\vec{w}$ and where $\vec{\nabla}\otimes\vec{w}.(\vec{w}*\omega_\epsilon)\otimes\vec{v}=\sum_{1\leq j\leq d}\sum_{1\leq k\leq d}(w_j*\omega_\epsilon)\,v_k\,\partial_j w_k$; we now notice that $|\vec{w}|^2$, $|\vec{\nabla}\otimes\vec{w}|^2$ and $\vec{Z}$ are integrable on $[0,T_3]\times\mathbb{R}^d$: we just have to check it for $|\vec{\nabla}\otimes\vec{w}|^2$; we just write

$$\vec{\nabla}\otimes\vec{w} = \vec{\nabla}\otimes e^{(t-T)\Delta}\vec{W} + \int_T^t e^{(t-s)\Delta}\Delta\frac{\vec{\nabla}}{\sqrt{-\Delta}}\otimes\left(\mathbb{P}\frac{\vec{\nabla}}{\sqrt{-\Delta}}.M\right)\,ds$$

with $M=(\vec{v}*\omega_\epsilon)\otimes\vec{w}+(\vec{w}*\omega_\epsilon)\otimes\vec{v}+(\vec{w}*\omega_\epsilon)\otimes\vec{w}$; $\vec{\nabla}\otimes e^{(t-T)\Delta}\vec{W}$ is square-integrable: $\int\int_T^{+\infty}|\vec{\nabla}\otimes e^{(t-T)\Delta}\vec{W}|^2\,dt=\|\vec{W}\|_2^2$; moreover M belongs to $(\mathcal{C}([T,T_3],L^2))^{d\times d}$, hence is square-integrable on $(T,T_3)\times\mathbb{R}^d$; the maximal regularity theorem for the heat kernel (Chapter 7) then gives that $\int_T^t e^{(t-s)\Delta}\Delta\frac{\vec{\nabla}}{\sqrt{-\Delta}}\otimes\left(\mathbb{P}\frac{\vec{\nabla}}{\sqrt{-\Delta}}.M\right)\,ds$ is square-integrable. We now choose $\theta\in\mathcal{D}((\mathbb{R}^d)$ equal to 1 in a neighborhood of 0 and integrate this equality against $\alpha(t)\theta(x/R)$ for $\alpha\in\mathcal{D}((0,T_3))$ and $R>0$, and then let R go to infinity; we get

$$\|\vec{w}(t)\|_2^2 + 2\int\int_T^t|\vec{\nabla}\otimes\vec{w}|^2\,dx\,ds = \|\vec{W}\|_2^2 + 2\int\int_T^t\vec{\nabla}\otimes\vec{w}.(\vec{w}*\omega_\epsilon)\otimes\vec{v}\,ds$$

which gives (using the inequality $2ab\leq a^2+b^2$)

$$\|\vec{w}(t)\|_2^2 \leq \|\vec{W}\|_2^2 + \int\int_T^t|(\vec{w}*\omega_\epsilon)\otimes\vec{v}|^2ds \leq \|\vec{W}\|_2^2 + C\rho^2\int\int_T^t|(\vec{w}(s)|^2ds$$

which gives $\|\vec{w}(t)\|_2 \leq \|\vec{W}\|_2 e^{-C\rho^2(t-T)/2}$. Thus, we get $T^* = +\infty$. $\square$

2. The limiting process

In this section, we discuss how to recover a solution $\vec{u}$ for the Navier–Stokes equations:

$$(13.3) \qquad \begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \mathbb{P}\vec{\nabla}.((\vec{u} * \omega_\epsilon) \otimes \vec{u}) \\ \vec{\nabla}.\vec{u} = 0 \\ \vec{u}(0,.) = \vec{u}_0 \end{cases}$$

from the solutions $\vec{u}_\epsilon$ of the mollified equations,

$$(13.4) \qquad \begin{cases} \partial_t \vec{u_\epsilon} = \Delta \vec{u_\epsilon} - \mathbb{P}\vec{\nabla}.((\vec{u_\epsilon} * \omega_\epsilon) \otimes \vec{u_\epsilon}) \\ \vec{\nabla}.\vec{u_\epsilon} = 0 \\ \vec{u}_\epsilon(0,.) = \vec{u}_0 \end{cases}$$

Following the discussion in Chapter 11, we search for a solution $\vec{u}$ in the space $\cap_{0<T<T_0}(L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d))^d$ for some positive T_0. We then assume that, given $\vec{u}_0$ with $\vec{\nabla}.\vec{u}_0 = 0$, we are able to prove uniform estimates in $L^2_{uloc,x}L^2_t$ norm for the solutions $\vec{u}_\epsilon$ for $0 < \epsilon < \epsilon_0$ and for all times $T < T_0$. Since $L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d)$ is the dual space of the separable Banach space

$$\{f \in L^2_{loc}((0,T)\times\mathbb{R}^d) / \sum_{k\in\mathbb{Z}^d} \|f\|_{L^2((0,T)\times k+[0,1]^d)} < \infty\} \approx l^1(\mathbb{Z}, L^2((0,T)\times[0,1]^d))$$

we may extract from those solutions $\vec{u}_\epsilon$ (through a diagonal process) a subsequence $\vec{u}_{\epsilon_k}$ (with $\lim_{k\to\infty}\epsilon_k = 0$), which is *-weakly convergent to some function $\vec{u}$ in the space $\cap_{0<T<T_0}(L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d))^d$. But we need more information before concluding that $\vec{u}$ is a solution for (13.3), because weak convergence is not sufficient to control the nonlinear term $\vec{u}_\epsilon * \omega_\epsilon \otimes \vec{u}_\epsilon$. We have only $\partial_t\vec{u} = \Delta\vec{u} - \lim_{k\to\infty}\mathbb{P}\vec{\nabla}.((\vec{u_{\epsilon_k}} * \omega_{\epsilon_k}) \otimes \vec{u_{\epsilon_k}})$. Following the example of the Leray solutions (Leray [LER 34]), we prove:

Theorem 13.3: (Convergence to a weak solution)

Let $\vec{u}_0$ with $\vec{\nabla}.\vec{u}_0 = 0$, $\epsilon_0 > 0$, $T_0 > 0$ and $\eta > 0$.
i) Assume that the solutions $\vec{u}_\epsilon$ of (13.4) are defined on $(0,T_0)\times\mathbb{R}^d$ for all $\epsilon \in (0,\epsilon_0)$ and that we have the following uniform estimates on $\vec{u}_\epsilon$:

$$\text{for all } T \in (0,T_0), \quad \sup_{0<\epsilon<\epsilon_0} \|\vec{u}_\epsilon\|_{L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d)} < \infty;$$

*then there exists $\vec{u}\, \cap_{0<T<T_0}\ (L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d))^d$ and a subsequence $\vec{u}_{\epsilon_k}$ (with $\lim_{k\to\infty}\epsilon_k = 0$), which is *-weakly convergent to $\vec{u}$.*

ii) Assume that we have the following uniform estimation of spatial regularity:

$$\text{for all } \phi \in \mathcal{D}((0,T_0)\times \mathbb{R}^d), \sup_{0<\epsilon<\epsilon_0} \|(Id-\Delta)^{\eta/2}(\phi\vec{u}_\epsilon)\|_{L^2((0,T_0)\times\mathbb{R}^d)} < \infty;$$

then $\vec{u}$ is solution of the Navier–Stokes equations (13.3).

Proof: We have already proved the existence of a *-weak limit $\vec{u}$ for some subsequence $\vec{u}_{\epsilon_k}$. Thus, we have just to prove point ii). We first estimate the local regularity of $\vec{u}_\epsilon$ (uniformly with respect to ϵ). We have, for $\phi \in \mathcal{D}((0,T_0)\times\mathbb{R}^d)$, $\partial_t(\phi\vec{u}_\epsilon) = (\partial_t\phi)\vec{u}_\epsilon + \phi\Delta\vec{u}_\epsilon - \phi\mathbb{P}\vec{\nabla}.((\vec{u}_\epsilon * \omega_\epsilon)\otimes\vec{u}_\epsilon)$. Since we have uniform estimates in $L^2_{uloc,x}L^2_t$ norm on $\vec{u}_\epsilon$ on $(0,T)\times\mathbb{R}^d$ (where $\operatorname{Supp}\phi \subset (0,T)\times\mathbb{R}^d$) and since the kernel of $\mathbb{P}\vec{\nabla}.$ is a sum of a matrix of convolutions with kernels in WL^∞ and of a matrix of convolutions with kernels that are a sum of double derivatives of functions in L^1_{comp} (Lemma 11.1), we find that $\partial_t(\phi\vec{u}_\epsilon) = \sum_{j,k\leq d}\partial_j\partial_k\vec{v}_{j,k,\phi,\epsilon} + \sum_{j\leq d}\partial_j\vec{v}_{j,\phi,\epsilon} + \vec{v}_{\phi,\epsilon}$, where the functions $\vec{v}_{j,k,\phi,\epsilon}$, $\vec{v}_{j,\phi,\epsilon}$ and $\vec{v}_{\phi,\epsilon}$ are (uniformly with respect to ϵ) in $(L^1_{comp}((0,T_0)\times\mathbb{R}^d))^d$ and are supported in a compact neighborhood V of $\operatorname{Supp}\phi$ independent of ϵ. Thus, $(Id-\Delta)^{-2}\partial_t(\phi\vec{u}_\epsilon) \in (L^1(\mathbb{R}\times\mathbb{R}^d))^d$ uniformly with respect to ϵ. Let us call $\vec{F}_\epsilon(\tau,\xi)$ the Fourier transform (in t and x) of $\phi\vec{u}_\epsilon \in (L^2(\mathbb{R}\times\mathbb{R}^d))^d$. We proved that $\frac{|\tau|}{(1+|\xi|^2)^2}|\vec{F}_\epsilon(\tau,\xi)| \leq C(\phi)$ with $C(\phi)$ a constant independent of ϵ, hence $\int\int(1+\tau^2)^{1/4}(1+|\xi|^2)^{-(d+9)/2}|\vec{F}_\epsilon(\tau,\xi)|^2\, d\tau\, d\xi \leq D(\phi)$ with $D(\phi)$ a constant independent of ϵ.. Moreover, the hypothesis in point ii) states that $\int\int(1+|\xi|^2)^\eta|\vec{F}_\epsilon(\tau,\xi)|^2\, d\tau\, d\xi \leq E(\phi)$ with $E(\phi)$ a constant independent of ϵ. This gives for $\gamma > 0$ small enough (so that $4\gamma < 1$ and $\frac{(2d+19)\gamma}{(1-4\gamma)} \leq \eta$) that $\int\int(1+\tau^2+|\xi|^2)^\gamma|\vec{F}_\epsilon(\tau,\xi)|^2\, d\tau\, d\xi \leq F_\gamma(\phi)$ with $F_\gamma(\phi)$ a constant independent of ϵ. This proves that the family $(\phi\vec{u}_\epsilon)_{0<\epsilon<\epsilon_0}$ is bounded in $(H^\gamma(\mathbb{R}\times\mathbb{R}^d))^d$; hence, the subsequence $\phi\vec{u}_{\epsilon_k}$ is weakly convergent to $\phi\vec{u}$ in $(H^\gamma(\mathbb{R}\times\mathbb{R}^d))^d$ and thus strongly convergent to $\phi\vec{u}$ in $(L^2(\mathbb{R}\times\mathbb{R}^d))^d$ since all the functions are supported in the compact set $\operatorname{Supp}\phi$ (Proposition 9.1).

If we look more closely at the proof of Lemma 11.1, we easily see (replacing in the proof $\omega(x)$ by $\omega(x/R)$ and letting R go to $=\infty$) that the kernel of $\mathbb{P}\vec{\nabla}.$ is a sum of a matrix of convolutions with kernels in WL^∞ with arbitrarily small norms in WL^∞ and of a matrix of convolutions with kernels that are a sum of double derivatives of functions in L^1_{comp}. Now, if we look at $\phi\mathbb{P}\vec{\nabla}.((\vec{u}_{\epsilon_k} * \omega_{\epsilon_k})\otimes\vec{u}_{\epsilon_k})$, the convolution with the WL^∞ kernel gives a result in $(L^1(\mathbb{R}\times\mathbb{R}^d))^d$ with a small norm (uniformly in ϵ) and the convolution with the compactly supported kernel converges to the convolution with $\vec{u}\otimes\vec{u}$, since we have local strong convergence of $(\vec{u}_{\epsilon_k} * \omega_{\epsilon_k})\otimes\vec{u}_{\epsilon_k}$ to $\vec{u}\otimes\vec{u}$ in L^1 norm. This proves that $\mathbb{P}\vec{\nabla}.((\vec{u}_{\epsilon_k} * \omega_{\epsilon_k})\otimes\vec{u}_{\epsilon_k})$ converges to $\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})$ in $(\mathcal{D}'((0,T_0)\times\mathbb{R}^d))^d$. $\square$

3. Mild solutions

In this section, we consider the case of $\vec{u}_0$ so that the Picard contraction principle works directly for the Navier–Stokes equations and not only for the mollified equations.

Theorem 13.4: (Convergence to mild solutions)

Let $\mathcal{E}_T \subset L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d)$ be such that :

i) the bilinear transform B defined by $B(\vec{u},\vec{v}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\, ds$ is bounded on $\mathcal{E}_T^d$: $\|B(\vec{u},\vec{v}))\|_{\mathcal{E}_T} \leq C_0\|\vec{u}\|_{\mathcal{E}_T}\|\vec{v}\|_{\mathcal{E}_T}$;

ii) for all $f\in\mathcal{E}_T$ and all $\psi\in\mathcal{D}(\mathbb{R}^d)$, the function $f*\psi$ defined by $f*\psi(t,x) = \int f(t,y)\,\psi(x-y)\,dy$ belongs to $\mathcal{E}_T$ and $\|f*\psi\|_{\mathcal{E}_T}\leq\|f\|_{\mathcal{E}_T}\|\psi\|_1$;

iii) there exists a $\eta>0$ so that for all $\vec{u}$, $\vec{v}$ in $\mathcal{E}_T^d$ and all $\phi\in\mathcal{D}(0,T)\times\mathbb{R}^d)$, $\|(Id-\Delta)^{\eta/2}(\phi B(\vec{u},\vec{v}))\|_{L^2((0,T)\times\mathbb{R}^d)}\leq C(\phi)\|\vec{u}\|_{\mathcal{E}_T}\|\vec{v}\|_{\mathcal{E}_T}$;

iv) $\mathcal{E}_T$ is the dual space of a Banach space in which $\mathcal{D}((0,T)\times\mathbb{R}^d)$ is densely included and the norm $\|\ \|_{\mathcal{E}_T}$ is the dual norm.

Then:

a) For all $\vec{u}_0$ satisfying $\vec{\nabla}.\vec{u}_0 = 0$ and $\|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_T} < \frac{1}{4C_0}$, there exists a weak solution $\vec{u}\in\mathcal{E}_T^d$ of the Navier–Stokes equations associated with the initial value $\vec{u}_0$: $\vec{u} = e^{t\Delta}\vec{u_0} - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})\, ds$.

*b) Moreover, for every $\epsilon>0$, the mollified Navier–Stokes equations have a solution $\vec{u}_\epsilon$ defined on $(0,T)\times\mathbb{R}^d$ $\vec{u_\epsilon} = e^{t\Delta}\vec{u_0} - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.((\vec{u_\epsilon}*\omega_\epsilon\otimes\vec{u_\epsilon})\, ds$. The solutions $\vec{u}_\epsilon$ belong to $(\mathcal{E}_T)^d$ uniformly with respect to ϵ and converge *-weakly to $\vec{u}$.*

c) If, moreover, $\mathcal{D}((0,T)\times\mathbb{R}^d)$ is continuously and densely included in $\mathcal{E}_T$, then the solutions $\vec{u}_\epsilon$ converge strongly to $\vec{u}$ in $\mathcal{E}_T$.

Proof: Point a) is a direct application of the Picard contraction principle (Theorem 13.2). To prove point b) we first note that the Picard contraction principle works for finding a solution for the mollified equations in $\mathcal{E}_T$, since $\|\int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.((\vec{u}*\omega_\epsilon)\otimes\vec{v})\,ds\|_{\mathcal{E}_T}\leq C_0\|\vec{u}\|_{\mathcal{E}_T}\|\vec{v}\|_{\mathcal{E}_T}$ with the same constant C_0 as for B. Moreover, choosing $\theta\in\mathcal{D}((0,\infty))$ with $\theta(1)=1$, and writing $e^{t\Delta}\vec{u}_0 = \int_0^t\partial_s(\theta(s/t)e^{s\Delta}\vec{u}_0)\,ds = \int_0^t\frac{\theta'(s/t)}{t}e^{s\Delta}\vec{u}_0\,ds + \Delta\int_0^t\theta(s/t)e^{s\Delta}\vec{u}_0\,ds$, we find that $\sqrt{t}(Id-\Delta)^{-1}e^{t\Delta}\vec{u}_0\in(L^\infty_t L^2_{uloc,x}((0,T)\times\mathbb{R}^d))^d$; hence, $\vec{u}_0\in(B^{-3,\infty}_{L^2_{uloc}})^d\subset(B_\infty^{-3-d/2,\infty})^d$. Moreover, it is easy to see that the bilinear operator B maps $(L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d))^d\times(L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d))^d$ to $(L^\infty((0,T),B_\infty^{-d-1,\infty}))^d$. Thus, the solution $\vec{u}_\epsilon$ computed in $\mathcal{E}_T^d$ is the same one as the solution computed in $(L^\infty((0,T),B_\infty^{-\max(d+1,3+d/2),\infty}))^d$. We know (Theorem 13.3) that given any sequence $\vec{u}_{\epsilon_n}$ with $\epsilon_n\to 0$, we may find a subsequence $\vec{u}_{\epsilon_{n_k}}$, which is convergent in $(\mathcal{D}'((0,T)\times\mathbb{R}^d)^d$ to a solution $\vec{v}$ of the Navier–Stokes equations. But, since $\mathcal{E}_T$ is a dual space and since $\|\vec{u}_\epsilon\|_{\mathcal{E}_T}\leq 2\|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_T}$, we obtain $\|\vec{v}\|_{\mathcal{E}_T}\leq 2\|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_T}$; hence, $\vec{u}=\vec{v}$ by uniqueness of solutions in the ball $\{\vec{g}\ /\ \|\vec{g}\|_{\mathcal{E}_T}\leq 2\|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_T}$. Thus, point b) is proved.

To prove point c), we write

$$\vec{u}_\epsilon - \vec{u} = B(\vec{u}, \vec{u} - \vec{u}_\epsilon) + B((\vec{u} - \vec{u}_\epsilon) * \omega_\epsilon, \vec{u}_\epsilon) + B(\vec{u} - (\vec{u} * \omega_\epsilon), \vec{u}_\epsilon)$$

which gives

$$\|\vec{u} - \vec{u}_\epsilon\|_{\mathcal{E}_T} \leq \frac{2C_0 \|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_T}}{1 - 4C_0\|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_T}} \|\vec{u} - \vec{u} * \omega_\epsilon\|_{\mathcal{E}_T}.$$

The operators $f \mapsto f*\omega_\epsilon$ are equicontinuous on $\mathcal{E}_T$ and when $\theta \in \mathcal{D}((0,T)\times\mathbb{R}^d)$ the functions $\theta * \omega_\epsilon$ converge to θ in $\mathcal{D}$ (hence in $\mathcal{E}_T$) as ϵ goes to O. Thus, we have $\lim_{\epsilon\to 0} \|\vec{u} - \vec{u} * \omega_\epsilon\|_{\mathcal{E}_T} = 0$. □

Thus, we have proved that the limiting process in Theorem 13.3 and the direct application of the Picard contraction principle lead to the same solution.

Part 3:

Classical existence results for the Navier–Stokes equations

Chapter 14

The Leray solutions for the Navier–Stokes equations

In this chapter, we shall present the classical proof of the Leray theorem on existence of square integrable weak solutions:

Theorem 14.1: (Leray's theorem [LER 34])

For all $\vec{u}_0 \in (L^2(\mathbb{R}^d))^d$ so that $\vec{\nabla}.\vec{u}_0 = 0$, there exists a weak solution $\vec{u} \in L^\infty((0,\infty),(L^2)^d) \cap L^2((0,\infty),(\dot{H}^1)^d)$ for the Navier–Stokes equations on $(0,\infty) \times \mathbb{R}^d$ so that $\lim_{t\to 0^+} \|\vec{u} - \vec{u}_0\|_2 = 0$. Moreover, we may choose this solution $\vec{u}$ fullfilling the Leray energy inequality:

$$\forall t > 0 \quad \|\vec{u}(t,.)\|_2^2 + 2\int_0^t \int_{\mathbb{R}^d} |\vec{\nabla} \otimes \vec{u}|^2 \, dx \, ds \leq \|\vec{u}_0\|_2^2 \tag{14.1}$$

where

$$\|\vec{u}(t,.)\|_2^2 = \sum_{i=1}^d \int u_i(t,x) \, dx \ \text{and} \ |\vec{\nabla} \otimes \vec{u}(s,x)|^2 = \sum_{i=1}^d \sum_{j=1}^d |\partial_i u_j(s,x)|^2.$$

Then, we introduce the results of Serrin [SER 62] giving criteria to ensure uniqueness for the Leray solutions.

1. The energy inequality

We first recall that, when looking for a solution in $L^\infty((0,\infty),(L^2(\mathbb{R}^d)^d)$, the Navier–Stokes equations may be equivalently formulated (see Chapter 11) as

$$\left\{ \begin{array}{c} \vec{u} \in L^\infty((0,\infty),(L^2(\mathbb{R}^d)^d) \\ \exists p \in \mathcal{D}'((0,\infty) \times \mathbb{R}^d) \quad \partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla}.(\vec{u} \otimes \vec{u}) - \vec{\nabla} p \\ \vec{\nabla}.\vec{u} = 0 \\ \lim_{t\to 0} \vec{u} = \vec{u}_0 \end{array} \right. \tag{14.2}$$

or as

$$(14.3)\qquad \begin{cases} \vec{u} \in L^\infty((0,\infty),(L^2(\mathbb{R}^d)^d) \\ \partial_t \vec{u} = \Delta\vec{u} - \mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u}) \\ \vec{\nabla}.\vec{u} = 0 \\ \lim_{t\to 0}\vec{u} = \vec{u}_0 \end{cases}$$

or as

$$(14.4)\qquad \begin{cases} \vec{u} \in L^\infty((0,\infty),(L^2(\mathbb{R}^d)^d) \\ \vec{u} = e^{t\Delta}\vec{u_0} - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})\ ds \\ \vec{\nabla}.\vec{u_0} = 0 \end{cases}$$

We now turn to the proof of Theorem 14.1. As a matter of fact, we have already proved Theorem 14.1. Indeed, the method introduced by Leray in [LER 34] for solving the Navier–Stokes equations is the mollification discussed in Chapter 13. Theorem 13.1 proves that, when $\vec{u}_0 \in (L^2)^d$ with $\vec{\nabla}.\vec{u}_0 = 0$, the mollified equations

$$(14.5)\qquad \begin{cases} \partial_t\vec{u} = \Delta\vec{u} - \vec{\nabla}.((\vec{u}*\omega_\epsilon)\otimes\vec{u}) - \vec{\nabla}p \\ \vec{\nabla}.\vec{u} = 0 \\ \vec{u}(0,.) = \vec{u}_0 \end{cases}$$

have a solution on $\vec{u}_\epsilon$ on $(0,+\infty)\times\mathbb{R}^d$; this theorem proved that we have for all $t_0 \geq 0$ and all $t \geq t_0$ the energy equality $\|\vec{u}_\epsilon(t,.)\|_2^2 + 2\int_{t_0}^t\int_{\mathbb{R}^d}|\vec{\nabla}\otimes\vec{u}_\epsilon|^2 dx\ ds = \|\vec{u}(t_0,.)\|_2^2$. Thus, the functions $\vec{u}_\epsilon$ are (uniformly with respect to ϵ) in $L^\infty((0,\infty),(L^2)^d)\cap L^2((0,T),(\dot{H}^1)^d)$ (where $\dot{H}^1$ is the homogeneous Sobolev space $\dot{H}^1 = \{f \in \mathcal{S}_0' \ /\ \vec{\nabla}f \in (L^2)^d\}$ with norm $\|f\|_{\dot{H}^1} = \|\vec{\nabla}f\|_2$). Thus, we may apply Theorem 13.3 and conclude that there exists a subsequence $\vec{u}_{\epsilon_k}$, which converges in $(\mathcal{D}'((0,\infty)\times\mathbb{R}^d))^d$ to a solution $\vec{u}$.

If $\alpha(t)$ is a function in $\mathcal{D}((0,+\infty))$, we use the weak convergence of $\alpha\vec{u}_{\epsilon_k}$ in $(L^2((0,+\infty)\otimes\mathbb{R}^d)^d$ (since we have convergence in $(\mathcal{D}')^d$ and uniform control in L^2 norm on $(0,T)\times\mathbb{R}^d$ for every finite T) and the weak convergence of $\vec{\nabla}\otimes\vec{u}_{\epsilon_k}$ in $(L^2((0,+\infty)\otimes\mathbb{R}^d)^{d\times d}$ (convergence in $(\mathcal{D}')^{d\times d}$ and uniform control in L^2 norm) to get that

$$\begin{array}{rc} & \iint|\alpha(t)|^2|\vec{u}(t)|^2\ dt\ dx + 2\int|\alpha(t)|^2(\int_0^t\int|\vec{\nabla}\otimes\vec{u}(s)|^2 ds\ dx)dt \\ \leq & \liminf_{\epsilon_k\to 0}\iint|\alpha(t)|^2|\vec{u}_{\epsilon_k}(t)|^2\ dt\ dx + 2\int|\alpha(t)|^2(\int_0^t\int|\vec{\nabla}\otimes\vec{u}_{\epsilon_k}(s)|^2 ds\ dx)\ dt \\ \leq & \|\vec{u}_0\|_2^2\ \int|\alpha(t)|^2\ dt. \end{array}$$

We choose $\theta \in \mathcal{D}(\mathbb{R})$ with $\int|\theta|^2\ dt = 1$ and we take for $t_0 > 0$ and $\eta > 0$ $\alpha(t) = \frac{1}{\sqrt{\eta}}\theta(\frac{t-t_0}{\eta})$; we then obtain that

$$\limsup_{\eta\to 0}\int\frac{1}{\eta}|\theta(\frac{t-t_0}{\eta})|^2\|\vec{u}(t)\|_{L^2(dx)}^2\ dt + 2\int_0^{t_0}\int|\vec{\nabla}\otimes\vec{u}(s)|^2 ds\ dx \leq \|\vec{u}_0\|_2^2.$$

If t_0 is a Lebesgue point for the measurable function $t \mapsto \|\vec{u}\|_{L^2(dx)}$, then the limit in the left-hand side of the above inequality is equal to $\|\vec{u}(t_0)\|^2_{L^2(dx)}$. Thus, inequality (14.1) is valid for almost every $t > 0$. Moreover, since $t \mapsto \vec{u}$ is bounded in L^2-norm and continuous in some Besov norm (namely, in $B_\infty^{-\max(2+d/2,d+1),\infty}$ norm), we find that this is continuous from $[0,\infty)$ to $(L^2)^d$ endowed with the weak topology. If $t > 0$ and if t_n is a sequence of times so that (14.1) is valid for t_n and so that t_n converges to t, we find that $\|\vec{u}(t)\|_2^2 \leq \liminf_{t_n \to t} \|\vec{u}(t_n)\|_2^2$ and $\int_0^t \int |\vec{\nabla} \otimes \vec{u}(s)|^2 ds\, dx = \lim_{t_n \to t} \int_0^{t_n} \int |\vec{\nabla} \otimes \vec{u}(s)|^2 ds\, dx$. Thus, (14.1) is valid everywhere.

The continuity of $\vec{u}$ in $L^2(dx)$ norm at $t = 0$ is then obvious: since $\vec{u}(t)$ is weakly convergent to $\vec{u}_0$, we just have to check that $\lim_{t\to 0^+} \|\vec{u}(t)\|_2^2 = \|\vec{u}_0\|_2^2$. But the weak convergence gives that $\|\vec{u}_0\|_2^2 \leq \liminf_{t\to 0^+} \|\vec{u}(t)\|_2^2$ while inequality (14.1) gives that $\limsup_{t\to 0^+} \|\vec{u}(t)\|_2^2 \leq \|\vec{u}_0\|_2^2$. $\square$

We may now define the Leray solutions for the Navier–Stokes equations:

Definition 14.1: (Leray solution)

(A) Let $\vec{u}_0 \in (L^2(\mathbb{R}^d))^d$ so that $\vec{\nabla}.\vec{u}_0 = 0$. A Leray solution on $(0,T)$ for the Navier–Stokes equations with initial value $\vec{u}_0$ is a weak solution $\vec{u}$ for

$$\begin{cases} \exists p \in \mathcal{D}'((0,T)\times \mathbb{R}^d) & \partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla}.(\vec{u} \otimes \vec{u}) - \vec{\nabla} p \\ & \vec{\nabla}.\vec{u} = 0 \end{cases}$$

so that $\vec{u} \in L^\infty((0,T),(L^2)^d) \cap L^2((0,T),(\dot{H}^1)^d)$, $\lim_{t\to 0^+} \|\vec{u} - \vec{u}_0\|_2 = 0$ and, for all $t \in (0,T)$, $\|\vec{u}(t,.)\|_2^2 + 2\int_0^t \int_{\mathbb{R}^d} |\vec{\nabla} \otimes \vec{u}|^2 \, dx\, ds \leq \|\vec{u}_0\|_2^2$.

(B) A restricted Leray solution on $(0,T)$ is a solution provided by the limiting process applied to the solutions of the mollified equations.

The last two sections of this chapter are devoted to classical criteria for uniqueness of the Leray solutions. We shall say that we have uniqueness in the Leray class on $(0,T)$ if there is only one Leray solution on $(0,T)$ associated with $\vec{u}_0$; we have uniqueness in the restricted Leray class if there is only one restricted Leray solution on $(0,T)$ associated with $\vec{u}_0$. Proofs of uniqueness in the Leray class are based on the energy inequality ; proofs of uniqueness in the restricted Leray class are based on direct estimates for the mild solutions of the equations (Theorem 13.4).

Before turning our attention to uniqueness issues, we recall a regularity result for (restricted) Leray solutions when $d \leq 4$:

Proposition 14.1: (Strong energy inequality)

When $d \leq 4$, inequality (14.1) may be strengthened into:

$$(14.6)\quad \textit{for almost all } t_0 > 0, \forall t > t_0 \ \|\vec{u}(t,.)\|_2^2 + 2\int_{t_0}^t \int_{\mathbb{R}^d} |\vec{\nabla} \otimes \vec{u}|^2 dx\, ds \leq \|\vec{u}(t_0,.)\|_2^2$$

This strong energy inequality is fulfilled by all the restricted Leray solutions.

Proof: Let us consider a restricted Leray solution. We follow the proof of the Leray energy inequality in Theorem 14.1. For a fixed $t_0 > 0$ and for $t_1 > t_0$, we write $\alpha(t) = \frac{1}{\sqrt{\eta}}\theta(\frac{t-t_1}{\eta})$ with $\int |\theta|^2\, dt = 1$ and

$$\begin{array}{ll} & \int\int |\alpha(t)|^2|\vec{u}(t)|^2\, dt\, dx + 2\int |\alpha(t)|^2(\int_{t_0}^t \int |\vec{\nabla}\otimes\vec{u}(s)|^2 ds\, dx)\, dt \\ \leq & \liminf_{\epsilon_k\to 0} \int\int |\alpha(t)|^2|\vec{u}_{\epsilon_k}(t)|^2\, dt\, dx + 2\int |\alpha(t)|^2(\int_{t_0}^t\int|\vec{\nabla}\otimes\vec{u}_{\epsilon_k}(s)|^2 ds\, dx) dt \\ \leq & \liminf_{\epsilon_k\to 0} \|\vec{u}_{\epsilon_k}(t_0)\|_2^2. \end{array}$$

Thus, we obtain

$$\|\vec{u}(t_1)\|^2_{L^2(dx)} + 2\int_{t_0}^{t_1}\int |\vec{\nabla}\otimes\vec{u}(s)|^2 ds\, dx \leq \lim\inf_{\epsilon_k\to 0} \|\vec{u}_{\epsilon_k}(t_0)\|_2^2.$$

for any Lebesgue point $t_1 > t_0$ of the measurable function $t \mapsto \|\vec{u}\|_{L^2(dx)}$; hence, for every $t_1 > t_0$ (due to the continuity in the weak L^2 topology of $t \mapsto \vec{u}$). Thus, we shall prove Proposition 14.1 by establishing that there is a subsequence ϵ_l of the sequence ϵ_k such that $\vec{u}_{\epsilon_l}(t_0)$ converges to $\vec{u}(t_0)$ strongly in $(L^2(dx))^d$ for almost every $t_0 > 0$.

Recall that we proved (at least for a subsequence of $(\vec{u}_{\epsilon_k})$) that we have strong convergence in $(L^2_{loc}(\mathbb{R}\times\mathbb{R}^d))^d$. (See the proof of Theorem 13.3.) Now, we are going to prove that we have more precisely, for every $0 < T_0 < T_1 < \infty$, $\lim_{\epsilon_k\to 0}\int_{T_0}^{T_1} \|\vec{u}_{\epsilon_k} - \vec{u}\|_2^2\, dt = 0$. This is sufficient to reach a conclusion, since we shall then be able to extract a subsequence so that $\lim_{\epsilon_l\to 0}\|\vec{u}_{\epsilon_l} - \vec{u}\|_2^2 = 0$ almost everywhere.

In order to estimate $\int_{T_0}^{T_1} \|\vec{u}_{\epsilon_k} - \vec{u}\|_2^2\, dt$, we choose a cutoff function $\varphi \in \mathcal{D}(\mathbb{R}^d)$ so that $0 \leq \varphi \leq 1$ and $\varphi(x) = 1$ for $|x| \leq 1$ and define for $R > 0$ $\varphi_R(x) = \varphi(x/R)$. We know that $\lim_{\epsilon_l\to 0}\int_{T_0}^{T_1} \|\varphi_R(x)\, (\vec{u}_{\epsilon_k} - \vec{u})\|_2^2\, dt = 0$, so that

$$\begin{array}{ll} & \limsup_{\epsilon_l\to 0}\int_{T_0}^{T_1} \|\vec{u}_{\epsilon_k} - \vec{u}\|_2^2\, dt \\ \leq & (T_1 - T_0)\limsup_{R\to\infty}\limsup_{\epsilon_l\to 0}\sup_{T_0<t<T_1} \|(1-\varphi_R(x))\, (\vec{u}_{\epsilon_k} - \vec{u})\|_2^2 \\ \leq & 4(T_1 - T_0)\limsup_{R\to\infty}\limsup_{\epsilon_l\to 0}\sup_{T_0<t<T_1} \|(1-\varphi_R(x))\, \vec{u}_{\epsilon_k}\|_2^2. \end{array}$$

We go back to the mollified equations

$$\begin{cases} \partial_t\vec{u_\epsilon} = \Delta\vec{u_\epsilon} - \vec{\nabla}.((\vec{u_\epsilon} * \omega_\epsilon)\otimes\vec{u_\epsilon}) - \vec{\nabla}p_\epsilon \\ \vec{\nabla}.\vec{u_\epsilon} = 0 \\ \vec{u}_\epsilon(0,.) = \vec{u}_0 \end{cases}$$

Since $\vec{u}_\epsilon \in (\mathcal{C}([0,+\infty), L^2(\mathbb{R}^d))^d$, we may define $p_\epsilon \in \mathcal{C}([0,+\infty), L^2(\mathbb{R}^d)$ by $p_\epsilon = -\frac{1}{\Delta}\sum_j\sum_k \partial_j\partial_k(u_{\epsilon,k}(u_{\epsilon,j} * \omega_\epsilon)$; since u_ϵ is smooth on $(0,+\infty)\times\mathbb{R}^d$, we have, defining $\vec{u}_{R,\epsilon} = (1-\varphi_R)\vec{u}_\epsilon$, : $\partial_t|\vec{u}_{R,\epsilon}|^2 + 2|\vec{\nabla}\otimes\vec{u}_{R,\epsilon}|^2 = \Delta|\vec{u}_{R,\epsilon}|^2 - \vec{\nabla}.\vec{Z}_{R,\epsilon} + |\vec{u}_\epsilon|^2(\vec{u}_\epsilon * \omega_\epsilon).\vec{\nabla}(1-\varphi_R)^2 + 2p_\epsilon\vec{u}_\epsilon.\vec{\nabla}(1-\varphi_R)^2$, where $\vec{Z}_{R,\epsilon}$ is defined as $\vec{Z}_{R,\epsilon} = (1-\varphi_R)^2(|\vec{u}_\epsilon|^2\vec{u_\epsilon} * \omega_\epsilon + 2p_\epsilon\ \vec{u}_\epsilon)$. Since $\vec{Z}_{R,\epsilon}$ is integrable on every strip $(S_1, S_2) \times \mathbb{R}^d$ with $0 < S_1 < S_2 < +\infty$, we find the equality $\|\vec{u}_{R,\epsilon}(t)\|_2^2 + 2\int\int_0^t |\vec{\nabla} \otimes \vec{u}_{R,\epsilon}|^2\ dx\ ds = \|\vec{u}_{R,\epsilon}(0)\|_2^2 + \int\int_0^t |\vec{u}_\epsilon|^2(\vec{u}_\epsilon * \omega_\epsilon).\vec{\nabla}(1-\varphi_R)^2\ dx\ ds + 2\int\int_0^t p_\epsilon\vec{u}_\epsilon.\vec{\nabla}(1-\varphi_R)^2\ dx\ ds$. Now, we use the fact that $\vec{u}_\epsilon \in (L^2((0,+\infty), \dot{H}^1))^d$ and that (through the Sobolev embeddings) $\dot{H}^1(\mathbb{R}^d) \subset L^{\frac{2d}{d-2}}$ $(d \geq 3)$ or BMO $(d = 2)$. Since we have for all $T > 0$ $\vec{u} \in (L^\infty((0,T), L^2)^d \subset (L^2((0,T), L^2)^d$, we find that $\vec{u} \in (L^2((0,T), L^4)^d$ provided that $\frac{2d}{d-2} \geq 4$, i.e. $d \leq 4$. This gives $|\vec{u}_\epsilon|^2(\vec{u}_\epsilon * \omega_\epsilon) \in (L^1((0,T) \times \mathbb{R}^d)^d$ and $p_\epsilon\vec{u}_\epsilon \in (L^1((0,T) \times \mathbb{R}^d)^d$; moreover, their norms in L^1 are controlled uniformly with respect to ϵ, hence we find that for every $T > 0$ there exists a positive constant C_T (independent from ϵ) such that for $0 < t < T$ we have $\|\vec{u}_{R,\epsilon}(t)\|_2^2 \leq \|(1-\varphi_R)\vec{u}_0\|_2^2 + C_T\frac{1}{R}$. Thus, the proof is complete. □

2. Energy equality

We begin with a celebrated theorem of Serrin [SER 62] on cases where we have equality, not inequality as in (14.1):

Proposition 14.2: (Energy equality)

Let $T \in (0,+\infty]$ and let $\vec{u} \in L^\infty((0,T), (L^2(\mathbb{R}^d)^d)$, be a solution for the Navier–Stokes equations:

$$\begin{cases} \exists p \in \mathcal{D}'((0,\infty) \times \mathbb{R}^d) & \partial_t\vec{u} = \Delta\vec{u} - \vec{\nabla}.(\vec{u} \otimes \vec{u}) - \vec{\nabla}p \\ & \vec{\nabla}.\vec{u} = 0 \end{cases}$$

Assume that:
i) $\vec{u} \in L^2((0,T), (\dot{H}^1(\mathbb{R}^d)^d)$
ii) For some $q \in [d, +\infty]$, $\vec{u} \in L^p((0,T), (L^q(\mathbb{R}^d)^d)$, *with* $\frac{1}{p} = \frac{1}{2} - \frac{d}{2q}$.

Then, $\vec{u} \in \mathcal{C}([0,T), (L^2(\mathbb{R}^d)^d)$ *and the energy equality*

$$\|\vec{u}(t,.)\|_2^2 + 2\int_\tau^t \int_{\mathbb{R}^d} |\vec{\nabla} \otimes \vec{u}|^2\ dx\ ds = \|\vec{u}(\tau,.)\|_2^2$$

holds for all τ, t so that $0 \leq \tau < t < T$.

The same result holds when, for $q = +\infty$, we replace the assumption $\vec{u} \in L^2((0,T), (L^\infty(\mathbb{R}^d)^d)$ by the weaker assumption $\vec{u} \in L^2((0,T), (BMO(\mathbb{R}^d)^d)$.

We even prove a more general result:

Proposition 14.3:

Let $T \in (0,+\infty]$ and let $\vec{u}_1$ and $\vec{u}_2$ be two solutions for the Navier–Stokes equations:

$$for\ i = 1,2 \begin{cases} \exists p_i \in \mathcal{D}'((0,\infty) \times \mathbb{R}^d) & \partial_t\vec{u_i} = \Delta\vec{u_i} - \vec{\nabla}.(\vec{u_i} \otimes \vec{u_i}) - \vec{\nabla}p_i \\ & \vec{\nabla}.\vec{u_i} = 0 \end{cases}$$

Assume that:
i) for $i=1,2$, $\vec{u}_i \in L^\infty((0,T),(L^2(\mathbb{R}^d)^d)$,
ii) for $i=1,2$, $\vec{u}_i \in L^2((0,T),(\dot{H}^1(\mathbb{R}^d)^d)$
iii) For some $q\in[d,+\infty]$, $\vec{u}_1 \in L^p((0,T),(L^q(\mathbb{R}^d)^d)$, *with* $\frac{1}{p}=\frac{1}{2}-\frac{d}{2q}$.

Then, $t\mapsto \int \vec{u}_1(t,x).\vec{u}_2(t,x)\,dx$ *is continuous on* $[0,T)$ *and we have the equality*

$$\int \vec{u}_1(t,x).\vec{u}_2(t,x)\,dx + 2\int_\tau^t \int_{\mathbb{R}^d} \vec{\nabla}\otimes\vec{u}_1.\vec{\nabla}\otimes\vec{u}_2\,dx\,ds =$$

$$\int_\tau^t\!\!\int_{\mathbb{R}^d} \vec{u}_1.(\vec{u}_1.\vec{\nabla})\vec{u}_2\,dx\,ds - \int_\tau^t\!\!\int_{\mathbb{R}^d} \vec{u}_1.(\vec{u}_2.\vec{\nabla})\vec{u}_2\,dx\,ds + \int \vec{u}_1(\tau,x).\vec{u}_2(\tau,x)\,dx$$

for all τ, t *so that* $0\le\tau<t<T$.

The same result holds when, for $q=+\infty$, *we replace the assumption* $\vec{u}_1 \in L^2((0,T),(L^\infty(\mathbb{R}^d)^d)$ *by the weaker assumption* $\vec{u}_1\in L^2((0,T),(BMO(\mathbb{R}^d)^d)$.

We begin the proof with an useful lemma:

Lemma 14.1: *Let* $T\in(0,+\infty]$ *and* $j\in\{1,\dots,d\}$. *If* $f\in L^2((0,T)\times\mathbb{R}^d)$, *then* $\int_0^t e^{(t-s)\Delta}\partial_j f\,ds \in \mathcal{C}_b([0,T),L^2(\mathbb{R}^d))$.

Proof: Let $g(t)=\int_0^t e^{(t-s)\Delta}\partial_j f\,ds$ and let $\varphi\in L^2(\mathbb{R}^d)$. We write

$$\langle g(t)|\varphi\rangle_{L^2,L^2} = -\int_0^t \langle f(s)|e^{(t-s)\Delta}\partial_j\varphi\rangle_{L^2,L^2}\,ds$$

and thus,

$$\begin{aligned}|\langle g(t)|\varphi\rangle_{L^2,L^2}| &\le \left(\int_0^t\!\int |f|^2\,ds\,dx\right)^{1/2}\left(\int_0^t\!\int |e^{(t-s)\Delta}\partial_j\varphi|^2\,ds\,dx\right)^{1/2}\\ &\le \left(\int_0^T\!\int |f|^2\,ds\,dx\right)^{1/2}\left(\int_0^{+\infty}\!\int |e^{s\Delta}\partial_j\varphi|^2\,ds\,dx\right)^{1/2};\end{aligned}$$

we then use the Plancherel equality to get

$$\int_0^{+\infty}\int |e^{s\Delta}\partial_j\varphi|^2\,ds\,dx = \frac{1}{(2\pi)^d}\int_0^{+\infty}\int |e^{-s|\xi|^2}\xi_j\hat{\varphi}(\xi)|^2\,d\xi\,ds \le \frac{1}{2}\|\varphi\|_2^2.$$

Thus, the mapping $f\mapsto g$ is bounded from $L^2_tL^2_x$ to $L^\infty_tL^2_x$; when f belongs to $\mathcal{D}((0,T)\times\mathbb{R}^d)$, we check easily that $g\in\mathcal{C}([0,T],L^2)$; since $\mathcal{D}((0,T)\times\mathbb{R}^d)$ is dense in $L^2((0,T)\times\mathbb{R}^d$, we may conclude that the mapping $f\mapsto g$ is bounded from $L^2_tL^2_x$ to $\mathcal{C}_b([0,T],L^2)$. $\square$

Proof of Proposition 14.3: We use a smoothing function $\theta(t,x)=\alpha(t)\beta(x)\in\mathcal{D}(\mathbb{R}^{d+1})$, where α is supported in $[-1,1]$, with $\int\!\int\theta\,dx\,dt=1$, and define, for $\epsilon>0$, $\theta_\epsilon(t,x)=\frac{1}{\epsilon^{d+1}}\theta(\frac{t}{\epsilon},\frac{x}{\epsilon})$. Then, $\theta_\epsilon*\vec{u}_i$ is a smooth function on $(\epsilon,T-\epsilon)\times\mathbb{R}^d$

and we may write $\partial_t\big((\theta_\epsilon*\vec{u}_1).(\theta_\epsilon*\vec{u}_2)\big)=\partial_t\big((\theta_\epsilon*\vec{u}_1)\big).(\theta_\epsilon*\vec{u}_2)+(\theta_\epsilon*\vec{u}_1).\partial_t\big((\theta_\epsilon*\vec{u}_2)\big)=(\theta_\epsilon*\partial_t\vec{u}_1).(\theta_\epsilon*\vec{u}_2)+(\theta_\epsilon*\vec{u}_1).(\theta_\epsilon*\partial_t\vec{u}_2)$. We then write for $i,j\in\{1,2\}$:

$$\begin{gathered}(\theta_\epsilon*\partial_t\vec{u}_i).(\theta_\epsilon*\vec{u}_j)=\\ \vec{\nabla}.\big((\theta_\epsilon*[\vec{\nabla}\otimes\vec{u}_i]).(\theta_\epsilon*\vec{u}_j)\big)-(\theta_\epsilon*[\vec{\nabla}\otimes\vec{u}_i]).(\theta_\epsilon*[\vec{\nabla}\otimes\vec{u}_j])\\ -\vec{\nabla}.\big((\theta_\epsilon*[\vec{u}_i\otimes\vec{u}_i]).(\theta_\epsilon*\vec{u}_j)\big)+(\theta_\epsilon*[\vec{u}_i\otimes\vec{u}_i]).(\theta_\epsilon*[\vec{\nabla}\otimes\vec{u}_j])\\ -\vec{\nabla}.\big((\theta_\epsilon*p_i)(\theta_\epsilon*\vec{u}_j)\big)\end{gathered}$$

We then integrate this equality against a test function $\psi(t)\varphi(x/R)$ with φ equal to 1 on a neighbourhood of 0 and we let R go to ∞. The terms which are written $\vec{\nabla}.\vec{F}$ with $\vec{F}\in(L^1((0,T)\times\mathbb{R}^d))^d$ will have a null contribution. Since $\vec{u}_i\in(L^\infty L^2)^d\cap(L^2\dot{H}^1)^d\subset(L^4\dot{H}^{1/2})^d\subset(L^4L^q)^d$ with $1/q=1/2-1/(2d)$, we find that $p_i\in L^2L^{d/(d-1)}$, $\theta_\epsilon*p_i\in L^2L^{d/(d-1)}$ and $\theta_\epsilon*\vec{u}_j\in(L^2L^d)^d$, so that $(\theta_\epsilon*p_i)(\theta_\epsilon*\vec{u}_j)\in(L^1((0,T)\times\mathbb{R}^d))^d$. Hence, we get the equality in $\mathcal{D}'(\epsilon,T-\epsilon)$: $\partial_t\int(\theta_\epsilon*\vec{u}_1).(\theta_\epsilon*\vec{u}_2)\,dx=-2\int(\theta_\epsilon*[\vec{\nabla}\otimes\vec{u}_1]).(\theta_\epsilon*[\vec{\nabla}\otimes\vec{u}_2])\,dx+\int(\theta_\epsilon*[\vec{u}_1\otimes\vec{u}_1]).(\theta_\epsilon*[\vec{\nabla}\otimes\vec{u}_2])\,dx+\int(\theta_\epsilon*[\vec{u}_2\otimes\vec{u}_2]).(\theta_\epsilon*[\vec{\nabla}\otimes\vec{u}_1])\,dx$. We may rewrite the last summand in $\int(\theta_\epsilon*[\vec{u}_2\otimes\vec{u}_2]).(\theta_\epsilon*[\vec{\nabla}\otimes\vec{u}_1])\,dx=-\int(\theta_\epsilon*[\vec{\nabla}.(\vec{u}_2\otimes\vec{u}_2)]).(\theta_\epsilon*\vec{u}_1)\,dx$.

The next step is devoted to proving that $\vec{u}_1\otimes\vec{u}_1\in(L^2((0,T)\times\mathbb{R}^d))^{d\times d}$. We first deal with the case $d>2$ and $q<\infty$. (Notice that, when $d=2$, $\dot{H}^1(\mathbb{R}^2)\subset BMO(\mathbb{R}^2)$ so that we do not need the assumption iii) on $\vec{u}_1$). We write (for $d>2$) $L^2_t\dot{H}^1\cap L^\infty_tL^2_x\subset L^\rho_tL^r_x$ for $2\le r\le 2d/(d-2)$ and $2/\rho=d/2-d/r$; we then write $\vec{u}_1\in(L^p((0,T),L^q))^d$ for some (p,q) with $d\le q<\infty$ and $1/p=1/2-d/(2q)$ (assumption iii)), while $\vec{u}_1\in(L^\rho((0,T),L^r))^d$ where we choose $1/r=1/2-1/q$ so that $1/r+1/q=1/2$ and $1/\rho+1/p=d/4-d/2(1/2-1/q)+1/2-d/(2q)=1/2$; this gives $\vec{u}_1\otimes\vec{u}_1\in(L^2((0,T)\times\mathbb{R}^d))^{d\times d}$. We now discuss the case $\vec{u}_1\in(L^2((0,T),BMO))^d$. We know that the complex interpolation space $[L^2,BMO]_{1/2}$ is equal to L^4, hence that $\|f\|_4\le C\sqrt{\|f\|_2\|f\|_{BMO}}$. This gives the inclusion $L^\infty((0,T),L^2)\cap L^2((0,T),BMO)\subset L^4((0,T)\times\mathbb{R}^d)$. Hence, $\vec{u}_1\otimes\vec{u}_1\in(L^2((0,T)\times\mathbb{R}^d))^{d\times d}$.

We now let ϵ go to 0. When $\epsilon\to0$ and $f\in L^2((0,T)\times\mathbb{R}^d)$, $\theta_\epsilon*f$ (which may be defined on $(0,T)\times\mathbb{R}^d$ by first defining $\theta_\epsilon*f$ by the convolution on $(\epsilon,T-\epsilon)$ and by extending it by 0 on $(0,\epsilon)\times\mathbb{R}^d$ and on $(T-\epsilon,T)\times\mathbb{R}^d$) is strongly convergent to f in $L^2((0,T)\times\mathbb{R}^d)$. We therefore have the following convergences in $\mathcal{D}'(0,T)$: $\partial_t\int(\theta_\epsilon*\vec{u}_1).(\theta_\epsilon*\vec{u}_2)\,dx\to\partial_t\int\vec{u}_1.\vec{u}_2\,dx$, $\int(\theta_\epsilon*[\vec{\nabla}\otimes\vec{u}_1]).(\theta_\epsilon*[\vec{\nabla}\otimes\vec{u}_2])\,dx\to\int\vec{\nabla}\otimes\vec{u}_1.\vec{\nabla}\otimes\vec{u}_2\,dx$ and $\int(\theta_\epsilon*[\vec{u}_1\otimes\vec{u}_1]).(\theta_\epsilon*[\vec{\nabla}\otimes\vec{u}_2])\,dx\to\int\vec{u}_1\otimes\vec{u}_1.\vec{\nabla}\otimes\vec{u}_2\,dx=\int\vec{u}_1.(\vec{u}_1.\vec{\nabla})\vec{u}_2\,dx$. Thus, we just have to study the convergence of the last summand $\int(\theta_\epsilon*[\vec{\nabla}.(\vec{u}_2\otimes\vec{u}_2)]).(\theta_\epsilon*\vec{u}_1)\,dx$.

We begin by rewriting $\theta_\epsilon*[\vec{\nabla}.(\vec{u}_2\otimes\vec{u}_2)]$ as $\theta_\epsilon*[(\vec{u}_2.\vec{\nabla})\vec{u}_2]$. Indeed, we easily check that, for $\vec{v}\in(L^\infty((0,T),L^2)^d$ with $\vec{\nabla}.\vec{v}=0$ and $\vec{w}\in(L^2((0,T),H^1)^d$, we have $\vec{\nabla}.\{\vec{v}\otimes\vec{w}\}=(\vec{v}.\vec{\nabla}).\vec{w}$ in $(\mathcal{D}'((0,T)\times\mathbb{R}^d))^d$: just smoothen one more time $\vec{v}$ into $\theta_\epsilon*\vec{v}$ and $\vec{w}$ into $\theta_\epsilon*\vec{w}$ and let ϵ go to 0. Now, if we have $\vec{u}_1\in(L^p((0,T),L^q))^d$ with $1/p=1/2-d/(2q)$ and $q<\infty$, we shall write $\vec{u}_2\in(L^\rho((0,T),L^r))^d$ where we choose $1/r=1/2-1/q$ so that $1/r+$

$1/q = 1/2$ and $1/\rho + 1/p = 1/2$. Since $\vec{\nabla} \otimes \vec{u}_2 \in (L^2((0,T) \times \mathbb{R}^d))^{d \times d}$, this gives that $(\vec{u}_2.\vec{\nabla}).\vec{u}_2 \in (L^{p/(p-1)}((0,T), L^{q/(q-1)}))^d$. When $\epsilon \to 0$ and $f \in L^{p/(p-1)}((0,T), L^{q/(q-1)}(\mathbb{R}^d))$, then $\theta_\epsilon * f$ is strongly convergent to f in $L^{p/(p-1)}((0,T), L^{q/(q-1)})$ while when $g \in L^p((0,T), L^q(\mathbb{R}^d))$, then $\theta_\epsilon * g$ is strongly convergent to g in $L^p((0,T), L^q)$. This gives the convergence (in $\mathcal{D}'((0,T))$) $\int (\theta_\epsilon * [\vec{\nabla}.(\vec{u}_2 \otimes \vec{u}_2)]).(\theta_\epsilon * \vec{u}_1)\, dx \to \int \vec{u}_1.(\vec{u}_2.\vec{\nabla})\vec{u}_2\, dx$.

The proof when $\vec{u}_1 \in (L^2((0,T), BMO))^d$ is quite similar. We use the div-curl theorem in Chapter 12: we get that, for $\vec{u}_2 \in (L^\infty((0,T), L^2)^d \cap (L^2((0,T), H^1)^d$ with $\vec{\nabla}.\vec{u}_2 = 0$, we have $(\vec{u}_2.\vec{\nabla}).\vec{u}_2 \in (L^2((0,T), \mathcal{H}^1(\mathbb{R}^d)))^d$ where $\mathcal{H}^1$ is the Hardy space. When $\epsilon \to 0$ and $f \in L^2((0,T), \mathcal{H}^1(\mathbb{R}^d))$, $\theta_\epsilon * f$ is strongly convergent to f in $L^2((0,T), \mathcal{H}^1(\mathbb{R}^d))$, while when g belongs to $L^2((0,T), BMO(\mathbb{R}^d))$, $\theta_\epsilon * g$ is *-weakly convergent to g in $L^2((0,T), BMO)$. Thus, we obtain again the convergence in $\mathcal{D}'((0,T))$ of $\int (\theta_\epsilon * [\vec{\nabla}.(\vec{u}_2 \otimes \vec{u}_2)]).(\theta_\epsilon * \vec{u}_1)\, dx$ to $\int \vec{u}_1.(\vec{u}_2.\vec{\nabla})\vec{u}_2\, dx$.

We have thus obtained the following equality in $\mathcal{D}'(0,T)$: $\partial_t \int \vec{u}_1.\vec{u}_2\, dx = -2 \int \vec{\nabla} \otimes \vec{u}_1.\vec{\nabla} \otimes \vec{u}_2\, dx + \int \vec{u}_1.(\vec{u}_1.\vec{\nabla})\vec{u}_2\, dx - \int \vec{u}_1.(\vec{u}_2.\vec{\nabla})\vec{u}_2\, dx$.

Using Lemma 14.1, we get that the map $t \mapsto \vec{u}_1$ is continuous from $[0,T]$ to $(L^2(dx))^d$ (since $\vec{u}_1 \otimes \vec{u}_1 \in (L^2((0,T) \times \mathbb{R}^d))^{d \times d}$ and $\vec{u}_1 = e^{t\Delta}\vec{u}_1(0) - \mathbb{P} \int_0^t e^{(t-s)\Delta} \vec{\nabla}.(\vec{u}_1 \otimes \vec{u}_1)\, ds$). Since $t \mapsto \vec{u}_2$ is weakly continuous from $[0,T]$ to $(L^2(dx))^d$, we find that $t \mapsto \int \vec{u}_1.\vec{u}_2\, dx$ is continuous. Thus, we may integrate our equality and obtain the equality in Proposition 14.3. □

Remarks: i) The condition $\vec{u} \in (L^p_t L^q)^d$ is a limitation on the initial data $\vec{u}_0$. Indeed, if $\vec{u} \in (L^p_t L^q)^d$ with $d < q < \infty$ and $1/p = 1/2 - d/(2q)$ is a solution of the Navier–Stokes equations, then in Chapter 20, we see that we must have $\vec{u}_0 \in (B_q^{d/q-1,p})^d$. We see conversely that when $\vec{u}_0 \in (B_q^{d/q-1,p})^d$ with $d < q < \infty$ and $1/p = 1/2 - d/(2q)$ then there exist a positive T and a mild solution $\vec{u} \in (L^p((0,T), L^q(\mathbb{R}^r)))^d$ for the Navier–Stokes equations with initial value $\vec{u}_0$; when we have $\vec{u}_0 \in (B_q^{d/q-1,p} \cap L^2)^d$, this mild solution is a Leray solution, hence hypotheses i) and ii) for $\vec{u}_1$ are always fullfilled.

ii) Similarly, the condition $\vec{u} \in (L^\infty_t L^d)^d$ implies obviously (by weak continuity of $t \mapsto \vec{u}(t,.)$) that $\vec{u}_0 \in (L^d)^d$. Conversely, we see in Chapter 15 that when $\vec{u}_0 \in (L^d)^d$, then there exist a positive T and a mild solution $\vec{u} \in (\mathcal{C}([0,T], L^d(\mathbb{R}^r)))^d$ of the Navier–Stokes equations with initial value $\vec{u}_0$; when we have $\vec{u}_0 \in (L^d \cap L^2)^d$, this mild solution is a Leray solution.

iii) The replacement of hypothesis $\vec{u} \in (L^2_t L^\infty)^d$ by $\vec{u} \in (L^2_t BMO)^d$ was recently discussed in a similar context by Kozono and Taniuchi [KOZT 00].

3. Uniqueness theorems

Theorem 14.2: (Serrin's uniqueness theorem)

Let $\vec{u}_0 \in (L^2(\mathbb{R}^d))^d$ with $\vec{\nabla}.\vec{u}_0 = 0$. Assume that there exists a solution $\vec{u}$ for the Navier–Stokes equations on $(0,T) \times \mathbb{R}^d$ (for some $T \in (0, +\infty]$) with initial value $\vec{u}_0$ so that:

i) $\vec{u} \in L^\infty((0,T),(L^2(\mathbb{R}^d)^d)$;
ii) $\vec{u} \in L^2((0,T),(\dot{H}^1(\mathbb{R}^d)^d)$;
iii) For some $q \in (d,+\infty]$, $\vec{u} \in L^p((0,T),(L^q(\mathbb{R}^d)^d)$, *with* $\frac{1}{p} = \frac{1}{2} - \frac{d}{2q}$.

Then, $\vec{u}$ *is the unique Leray solution associated with* $\vec{u}_0$ *on* $(0,T)$.

The same result holds when the assumption $\vec{u} \in (L^2_t(L^\infty_x))^d$ *is replaced by* $\vec{u} \in (L^2_t(BMO))^d$

The same result is also true for $q = d$ *provided that the norm* $\|\vec{u}\|_{L^\infty(L^d)}$ *is smaller than a constant* δ_d.

Proof: The proof is easy, due to Proposition 14.3. If $\vec{v}$ is another Leray solution, we write that $\|\vec{u}(t,.) - \vec{v}(t,.)\|_2^2 = \|\vec{u}(t,.)\|_2^2 + \|\vec{v}(t,.)\|_2^2 - 2\int \vec{u}(t,.).\vec{v}(t,.)\, dx$ $\leq \|\vec{u}_0\|_2^2 - 2\int_0^t\int_{\mathbb{R}^d} |\vec{\nabla}\otimes\vec{u}|^2\, dx\, ds + \|\vec{u}_0\|_2^2 - 2\int_0^t\int_{\mathbb{R}^d} |\vec{\nabla}\otimes\vec{v}|^2\, dx\, ds - 2\|\vec{u}_0\|_2^2 +$ $4\int_0^t\int_{\mathbb{R}^d} \vec{\nabla}\otimes\vec{u}.\vec{\nabla}\otimes\vec{v}\, dx\, ds - 2\int_0^t\int_{\mathbb{R}^d}\vec{u}.(\vec{u}.\vec{\nabla})\vec{v}\, dx\, ds + 2\int_0^t\int_{\mathbb{R}^d}\vec{u}.(\vec{v}.\vec{\nabla})\vec{v}\, dx\, ds =$ $-2\int_0^t\int_{\mathbb{R}^d} |\vec{\nabla}\otimes(\vec{u}-\vec{v})|^2\, dx\, ds - 2\int_0^t\int_{\mathbb{R}^d}\vec{u}.\big((\vec{u}-\vec{v}).\vec{\nabla}\big)\vec{v}\, dx\, ds$. Moreover, we have the antisymmetry property $\int_0^t\int_{\mathbb{R}^d}\vec{u}.\big((\vec{u}-\vec{v}).\vec{\nabla}\big)\vec{u}\, dx\, ds = 0$. Now, when $d \leq q < \infty$, we define $r = d/q$. We have $\vec{v} - \vec{u} \in (L^\infty_t L^2_x \cap L^2_t\dot{H}^1)^d \subset (L^{2/r}_t\dot{H}^r)^d$ (since $\|f\|_{\dot{H}^r} = \frac{1}{(2\pi)^{d/2}}\||\xi|^r\hat{f}\|_2 \leq \|f\|_2^{1-r}\|f\|^r_{\dot{H}^1}$) ; hence $\vec{u}-\vec{v} \in (L^{2/r}_t L^\sigma_x)^d$ with $1/\sigma = 1/2 - r/d$, $\vec{\nabla}\otimes(\vec{u}-\vec{v}) \in (L^2L^2)^d$ and $\vec{u} \in (L^pL^q)^d$ with $1/\sigma + 1/2 + 1/q = 1$ and $r/2 + 1/2 + 1/p = 1$; this gives for every $0 \leq \tau < t < T$:

$$\begin{aligned}
&|\textstyle\int_\tau^t\int_{\mathbb{R}^d}\vec{u}.\big((\vec{u}-\vec{v}).\vec{\nabla}\big)(\vec{v}-\vec{u})\, dx\, ds| \\
&\leq C_r(\textstyle\int_\tau^t \|\vec{u}\|_q^p\, ds)^{1/p}(\int_\tau^t \|\vec{v}-\vec{u}\|^2_{\dot{H}^1}\, ds)^{1/2}(\int_\tau^t \|\vec{v}-\vec{u}\|^{\frac{2}{r}}_{\dot{H}^r}\, ds)^{\frac{r}{2}} \\
&\leq C'_r(\textstyle\int_\tau^t \|\vec{u}\|_q^p\, ds)^{1/p}(\int_0^t\int_{\mathbb{R}^d} |\vec{\nabla}\otimes(\vec{v}-\vec{u})|^2\, ds\, dx)^{\frac{1+r}{2}} \sup_{0<s<t}\|\vec{v}-\vec{u}\|_2^{1-r} \\
&\leq C'_r(\textstyle\int_\tau^t \|\vec{u}\|_q^p\, ds)^{1/p}(\frac{1+r}{2}\int_0^t\int_{\mathbb{R}^d} |\vec{\nabla}\otimes(\vec{v}-\vec{u})|^2 ds\, dx + \frac{1-r}{2}\sup_{0<s<t}\|\vec{v}-\vec{u}\|_2^2)
\end{aligned}$$

If $\vec{u} = \vec{v}$ on $[0,\tau]$ and if $t > \tau$ is such that $\frac{1+r}{2}C'_r(\int_\tau^t \|\vec{u}\|_q^p\, ds)^{1/p} < 1$, we obtain $\sup_{0<s<t}\|\vec{v}-\vec{u}\|_2^2 \leq \frac{1-r}{2}C'_r(\int_\tau^t \|\vec{u}\|_q^p\, ds)^{1/p}\sup_{0<s<t}\|\vec{v}-\vec{u}\|_2^2$, hence $\vec{u} = \vec{v}$ on $[0,t]$. When $p < \infty$, this gives uniqueness on $[0,T)$; when $p = \infty$, we must assume that $C'_1\|\vec{u}\|_{L^\infty L^d} < 1$ to grant uniqueness. Thus, the theorem is proved for $q < \infty$. The proof is similar in the case of $\vec{u} \in (L^2_t BMO)^d$. □

As a direct consequence, we obtain uniqueness for the 2D case:

Theorem 14.3: ($2D$ Navier–Stokes equations)

Let $\vec{u}_0 \in (L^2(\mathbb{R}^2))^2$ *with* $\vec{\nabla}.\vec{u}_0 = 0$. *Then, for the Navier–Stokes equations on* $(0,+\infty)\times\mathbb{R}^d$ *with initial value* $\vec{u}_0$, *there is existence and uniqueness of the solution in* $L^\infty((0,+\infty),(L^2(\mathbb{R}^2)^2) \cap L^2((0,+\infty),(\dot{H}^1(\mathbb{R}^2)^2)$.

Proof: As already stated, $\dot{H}^1 \subset BMO$; hence, if the solution $\vec{u}$ belongs to $(L^\infty(L^2))^2 \cap (L^2(\dot{H}^1))^2$, then $\vec{u}$ satisfies the energy equality and thus Theorem 14.2 gives uniqueness (since $L^\infty(L^2) \cap L^2(\dot{H}^1) \subset L^4(L^4)$). □

Since the case $d = 2$ is well understood, we shall mainly be interested in the case $d \geq 3$, discussed in the chapters that follow.

In Theorem 14.2, the case $q = d$ remains open. This is dealt with by the uniqueness theorem of Sohr and Von Wahl [WAH 85] :

Theorem 14.4: (Von Wahl's uniqueness theorem)

Let $\vec{u}_0 \in (L^2(\mathbb{R}^d))^d$ *with* $\vec{\nabla}.\vec{u}_0 = 0$. *Assume that there exist a solution* $\vec{u}$ *of the Navier–Stokes equations on* $(0,T) \times \mathbb{R}^d$ *(for some* $T \in (0, +\infty]$*) with initial value* $\vec{u}_0$ *so that :*

i) $\vec{u} \in L^\infty((0,T), (L^2(\mathbb{R}^d)^d)$;
ii) $\vec{u} \in L^2((0,T), (\dot{H}^1(\mathbb{R}^d)^d)$;
iii) $\vec{u} \in \mathcal{C}([0,T), (L^d(\mathbb{R}^d)^d)$.

Then, $\vec{u}$ *is the unique Leray solution associated to* $\vec{u}_0$ *on* $(0,T)$.

Proof: The proof is very easy. If $T_0 < T$, then for each $\epsilon > 0$ we may split $\vec{u}$ on $[0,T_0]$ in $\vec{u} = \vec{\alpha} + \vec{\beta}$ with $\vec{\beta} \in (L^\infty((0,T_0) \times \mathbb{R}^d)^d$ and $\|\vec{\alpha}\|_{L^\infty L^d} < \epsilon$: indeed, by uniform continuity of $t \in [0,T_0] \mapsto \vec{u}(t,.) \in L^d$, we may find N so that $\|\vec{u} - \sum_{0 \leq k < N} 1_{[k/N,(k+1)/N]}(t)\vec{u}(k/N,.)\|_{(L^\infty((0,T_0),L^d))^d} < \epsilon/2$; we may approximate each $\vec{u}(k/N,.)$ by a vector function $\vec{\beta}_{k,n} \in (L^\infty)^d$ with an error controlled in L^d norm by $\|\vec{u}(k/N,.) - \vec{\beta}_{k,n}\|_d < \epsilon/2$; thus, we define β as $\beta(t,x) = \sum_{0 \leq k < N} 1_{[k/N,(k+1)/N]}(t)\vec{\beta}_{k,N}(x)$.

Then we write

$$\begin{gathered}
|\int_0^t\int_{\mathbb{R}^d}\vec{u}.\big((\vec{u}-\vec{v}).\vec{\nabla}\big)(\vec{v}-\vec{u})\ dx\ ds| \\
\leq C\|\vec{\alpha}\|_{L^\infty L^d} \int_0^t\int_{\mathbb{R}^d}|\vec{\nabla} \otimes (\vec{v}-\vec{u})|^2\ dx\ ds \\
+\|\vec{\beta}\|_\infty (\int_0^t\int_{\mathbb{R}^d}|\vec{\nabla} \otimes (\vec{v}-\vec{u})|^2\ dx\ ds)^{1/2} (\int_0^t\int_{\mathbb{R}^d}|\vec{v}-\vec{u}|^2\ dx\ ds)^{1/2} \\
\leq 2C\epsilon \int_0^t\int_{\mathbb{R}^d}|\vec{\nabla} \otimes (\vec{v}-\vec{u})|^2\ dx\ ds + \tfrac{4}{C\epsilon}\|\vec{\beta}\|_\infty^2 \int_0^t\int_{\mathbb{R}^d}|\vec{v}-\vec{u}|^2\ dx\ ds.
\end{gathered}$$

Choosing ϵ so that $2C\epsilon < 1$, we get

$$\|\vec{v}(t,.) - \vec{u}(t,.)\|_2^2 \leq \frac{4}{C\epsilon}\|\vec{\beta}\|_\infty^2 \int_0^t \|\vec{v}(s,.) - \vec{u}(s,.)\|_2^2\ ds.$$

The Gronwall lemma gives then that $\vec{u} = \vec{v}$. □

Chapter 15

The Kato theory of mild solutions for the Navier–Stokes equations

In this chapter, we present the classical results of Kato and Fujita on the existence of mild solutions [FUJK 64] [KAT 84]. The idea is to construct the solution $\vec{u}$ for the Navier–Stokes equations as a solution for the integral equation

$$\vec{u} = e^{t\Delta}\vec{u_0} - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})\, ds \tag{15.1}$$

and, thus, to search for a fixed point of the transform

$$\vec{v} \mapsto e^{t\Delta}\vec{u_0} - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{v}\otimes\vec{v})\, ds = e^{t\Delta}\vec{u_0} - B(\vec{v},\vec{v}) \tag{15.2}$$

This is the so-called Picard method, which had been already used by Oseen in the beginning of the 20th century to establish the existence of a classical solution for a regular initial value [OSE 27].

1. Picard's contraction principle

A simple approach to problem (15.2) is trying to find a Banach space $\mathcal{E}_T$ of functions defined on $(0,T)\times\mathbb{R}^d$ so that the bilinear transform

$$B(\vec{u},\vec{v}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\, ds$$

is bounded from $\mathcal{E}_T^d \times \mathcal{E}_T^d$ to $\mathcal{E}_T^d$. Then, we may consider the space $E_T \subset \mathcal{S}'$ defined by $f\in E_T$ iff $f\in\mathcal{S}'$ and $(e^{t\Delta}f)_{0<t<T}\in\mathcal{E}_T$. We get the following easy existence result:

Theorem 15.1: (The Picard contraction principle)

Let $\mathcal{E}_T \subset L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d)$ be such that the bilinear transform B is bounded on $\mathcal{E}_T^d$. Then:

a) If $\vec{u}\in\mathcal{E}_T^d$ is a weak solution for the Navier–Stokes equations:

$$\partial_t\vec{u} = \Delta\vec{u} - \mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u}) \text{ and } \vec{\nabla}.\vec{u} = 0$$

then the associated initial value $\vec{u}_0$ belongs to E_T^d.
b) Conversely, there exists a positive constant C so that for all $\vec{u}_0 \in E_T^d$ satisfying $\vec{\nabla}.\vec{u}_0 = 0$ and $\|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_T} < C$ there exists a weak solution $\vec{u} \in \mathcal{E}_T^d$ of the Navier–Stokes equations associated with the initial value $\vec{u}_0$:

$$\vec{u} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})\; ds.$$

The requirement that $\mathcal{E}_T \subset L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d)$ grants that the operator B is well-defined: indeed, we saw in Chapter 11 that B is well-defined on $(L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d))^d \times (L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d))^d$ (with values in $(\mathcal{S}')^d$) and that the equation " $\partial_t\vec{u} = \Delta\vec{u} - \mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})$ and $\vec{\nabla}.\vec{u} = 0$ " for $\vec{u} \in (L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d))^d$ is equivalent to the existence of some $\vec{u}_0 \in (\mathcal{S}')^d$ with $\vec{\nabla}.\vec{u}_0 = 0$ so that $\vec{u} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})\; ds$.

Proof: If $\vec{u} \in (\mathcal{E}_T)^d$ and B maps $(\mathcal{E}_T)^d \times (\mathcal{E}_T)^d$ to $(\mathcal{E}_T)^d$ and if $\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u},\vec{u})$, then $e^{t\Delta}\vec{u}_0$ belongs to $(\mathcal{E}_T)^d$ and thus $\vec{u}_0$ belongs to $(E_T)^d$.

Conversely, if $\|B(\vec{v},\vec{w})\|_{\mathcal{E}_T} \leq C_0\|\vec{v}\|_{\mathcal{E}_T}\|\vec{w}\|_{\mathcal{E}_T}$, then if $\|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_T} \leq \delta < \frac{1}{4C_0}$ we find that the map $\vec{v} \mapsto e^{t\Delta}\vec{u}_0 - B(\vec{v},\vec{v})$ is contractive on the ball $\{\vec{v}\ /\ \|\vec{v}\|_{\mathcal{E}_T} \leq 2\delta\}$. Hence, the sequence defined by the recursion $\vec{u}^{(0)} = e^{t\Delta}\vec{u}_0$, $\vec{u}^{(n+1)} = e^{t\Delta}\vec{u}_0 - B(\vec{u}^{(n)},\vec{u}^{(n)})$ converges to a fixed point $\vec{u}$ of the map; hence, we have a solution $\vec{u}$ in $(\mathcal{E}_T)^d$. □

We then define *Kato's mild solutions* as solutions obtained through the iterative process $\vec{u}^{(0)} = e^{t\Delta}\vec{u}_0$, $\vec{u}^{(n+1)} = e^{t\Delta}\vec{u}_0 - B(\vec{u}^{(n)},\vec{u}^{(n)})$. We call a space $\mathcal{E}_T$, on which we may apply the Picard contraction principle, an *admissible path space* for the Navier–Stokes equations, and the associated space E_T an *adapted value space*.

At times, we consider a slight generalization of the pairing between admissible path spaces and adapted value spaces, because we may use a weaker norm on $e^{t\Delta}\vec{u}_0$ than the norm in $(\mathcal{E}_T)^d$ to grant existence of a solution in E_T:

Theorem 15.1 bis: (Variation on the Picard contraction principle)

Let $\mathcal{E}_T \subset \mathcal{F}_T \subset L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d)$ (with continuous embeddings) be such that the bilinear transform B is bounded from $\mathcal{F}_T^d \times \mathcal{F}_T^d$ to $\mathcal{E}_T^d$. Then:
a) Let $\vec{u} \in \mathcal{F}_T^d$ be a weak solution to the Navier–Stokes equations:

$$\partial_t\vec{u} = \Delta\vec{u} - \mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u}) \text{ and } \vec{\nabla}.\vec{u} = 0.$$

If, moreover, the initial value $\vec{u}_0$ belongs to E_T^d, then $\vec{u}$ belongs more precisely to $\mathcal{E}_T^d$.

b) Conversely, there exists a positive constant C so that for all $\vec{u}_0 \in E_T^d$ satisfying $\vec{\nabla}.\vec{u}_0 = 0$ and $\|e^{t\Delta}\vec{u}_0\|_{\mathcal{F}_T} < C$ there exists a weak solution $\vec{u} \in \mathcal{E}_T^d$ to the Navier–Stokes equations associated with the initial value $\vec{u}_0$:

$$\vec{u} = e^{t\Delta}\vec{u_0} - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})\, ds.$$

Proof: The theorem is a direct consequence of Theorem 15.1 b). □

We may often need a useful regularity criterion for the solutions to the Navier–Stokes equations:

Proposition 15.1: (Regularity criterion)
Let $\vec{u} \in (L^\infty((0,T)\times\mathbb{R}^d))^d$ be a solution to the Navier–Stokes equations

$$\left\{\begin{array}{c} \partial_t\vec{u} = \Delta\vec{u} - \mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u}) \\ \vec{\nabla}.\vec{u} = 0 \end{array}\right. \tag{15.3}$$

Then $\vec{u}$ is $\mathcal{C}^\infty$ on $(0,T)\times\mathbb{R}^d$.

Proof: We follow the proof of Theorem 13.1, first proving by induction that $\vec{u} \in \cap_{0<T_1<T_2<T}(L^\infty((T_1,T_2), B_\infty^{k/2,\infty}))^d$ for all $k \in \mathbb{N}$. Indeed, we have for $0 < T_0 < T_1 < t < T_2$,

$$\vec{u}(t) = e^{(t-T_0)\Delta}\vec{u}(T_0) - \int_{T_0}^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})\, ds;$$

we have $\|e^{(t-T_0)\Delta}\vec{u}(T_0)\|_{B_\infty^{k/2,\infty}} \le C\sup((t-T_0)^{-k/4},1)\|\vec{u}(T_0)\|_\infty$; moreover, if $\vec{u} \in (B_\infty^{k/2,\infty})^d$ for some $k \in \mathbb{N}^*$, then $\vec{u}\otimes\vec{u} \in (B_\infty^{k/2,\infty})^{d\times d}$ (while, for $k=0$, we write that $\vec{u}\otimes\vec{u} \in (L^\infty)^{d\times d} \subset (B_\infty^{0,\infty})^{d\times d}$; thus, we find

$$\|\int_0^t \mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})\, ds\|_{B_\infty^{(k+1)/2,\infty}} \le C\|\vec{u}\|^2_{L^\infty((T_0,T_2),B_\infty^{k/2,\infty})}\int_{T_0}^t \sup(\frac{1}{(t-s)^{3/4}},1)\, ds$$

(for $k=0$, replace $\|\vec{u}\|^2_{L^\infty((T_0,T_2),B_\infty^{k/2,\infty})}$ by $\|\vec{u}\|^2_{L^\infty((T_0,T_2)\times\mathbb{R}^d)}$), which yields the result. Thus far, we have proved the spatial regularity of $\vec{u}$. Moreover, similar estimates then hold for $\partial_t\vec{u}$ since Δ maps $B_\infty^{k,\infty}$ to $B_\infty^{k-2,\infty}$ and $\mathbb{P}\vec{\nabla}.$ maps $(B_\infty^{k,\infty})^{d\times d}$ to $(B_\infty^{k-1,\infty})^d$. Then, we may conclude by induction that all derivatives $\frac{\partial^p}{\partial t^p}\vec{u}$ satisfy similar estimates, since we have $\frac{\partial^{p+1}}{\partial t^{p+1}}\vec{u} = \Delta\frac{\partial^p}{\partial t^p}\vec{u} - \mathbb{P}\vec{\nabla}.\big(\frac{\partial^p}{\partial t^p}(\vec{u}\otimes\vec{u})\big)$. This proves that $\vec{u}$ is smooth on $(0,T)\times\mathbb{R}^d$. □

2. Kato's mild solutions in H^s, $s \geq d/2-1$

Mild solutions were first constructed by Kato and Fujita in the Sobolev spaces $H^s(\mathbb{R}^d)$, $s \geq d/2-1$ [FUJK 64]. The space E_T is then $H^s(\mathbb{R}^d)$ and the space $\mathcal{E}_T$ is the space $\mathcal{C}([0,T], H^{d/2-1})$ if $s > d/2-1$ and $\mathcal{C}([0,T], H^{d/2-1}) \cap L^2((0,T), H^{d/2})$ for $s = d/2-1$. A modern treatment for mild solutions in H^s was given by Chemin [CHE 92].

Theorem 15.2:

(A) Let $s > d/2-1$. Then for all $\vec{u}_0 \in (H^s(\mathbb{R}^d))^d$ so that $\vec{\nabla}.\vec{u}_0 = 0$, there exist a positive T^* and a (unique) weak solution $\vec{u} \in \mathcal{C}([0,T^*), (H^s(\mathbb{R}^d))^d)$ for the Navier–Stokes equations on $(0,T^*) \times \mathbb{R}^d$ so that $\vec{u}(0,.) = \vec{u}_0$. This solution is then smooth on $(0,T^*) \times \mathbb{R}^d$.

(B) For all $\vec{u}_0 \in (H^{d/2-1}(\mathbb{R}^d))^d$ so that $\vec{\nabla}.\vec{u}_0 = 0$, there exist a positive T^* and a weak solution $\vec{u} \in \mathcal{C}([0,T^*), (H^{d/2-1}(\mathbb{R}^d))^d)$ for the Navier–Stokes equations on $(0,T^*) \times \mathbb{R}^d$ so that $\vec{u}(0,.) = \vec{u}_0$. Moreover, this solution may be chosen so that for all $T \in (0,T^*)$, we have $\vec{u} \in L^2((0,T), (H^{d/2}(\mathbb{R}^d))^d)$. With this extra condition on the $H^{d/2}$ norm, such a solution is unique and, moreover, it is smooth on $(0,T^*) \times \mathbb{R}^d$.

(C) There exists $\epsilon_0 > 0$ such that if $(\int |\xi|^{d-2}\ |\hat{\vec{u}}_0(\xi)|^2\ d\xi)^{1/2} < \epsilon_0$ then the existence time T^* in point (A) or (B) for the smooth solution $\vec{u}$ is equal to $+\infty$.

(D) The smooth solutions $\vec{u}$ described in (A) and (B) satisfy $\vec{u}(t,.) \in (\mathcal{C}_0(\mathbb{R}^d))^d$ for all $t < T^*$. The maximal existence time T^* is finite if and only if $\lim_{t\to T^*} \|\vec{u}\|_\infty = +\infty$. We may replace this latter condition by: $T^* < \infty$ if and only if $\int_{T^*/2}^{T^*} \|\vec{u}(t,.)\|_{BMO}^2\ dt = +\infty$.

Proof: (A) is easy. We want to prove that $H^s(\mathbb{R}^d)$ is an adapted value space when $s > d/2-1$, with adapted path space $\mathcal{E}_T = L^\infty([0,T], H^s(\mathbb{R}^d))$. We first estimate the pointwise product of two functions in H^s. We have, due to the Bernstein inequalities, $H^s = B_2^{s,2} \subset B_\infty^{s-d/2,\infty}$; thus (applying Theorem 4.1), we have that, whenever $f \in H^s$ and $g \in H^s$, $fg \in H^s$ $(s > d/2)$ or $fg \in H^{2s-d/2}$ $(d/4 < s < d/2)$; for $s = d/2$, we shall write that for any positive ϵ we have $fg \in H^{d/2-\epsilon}$ when $f, g \in H^{d/2}$. Now, let $\vec{f}$ and $\vec{g}$ belong to $\mathcal{E}_T^d$, with $\mathcal{E}_T = L^\infty([0,T], H^s(\mathbb{R}^d))$; let us estimate $\vec{h} = B(\vec{f},\vec{g}) = \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f}\otimes\vec{g})\ d\tau$. We write $s = \sigma + \rho$ with $\sigma = s$ and $\rho = 0$ for $s > d/2$, $\sigma = 2s - d/2$ and $\rho = d/2 - s$ for $d/2-1 < s < d/2$ and $\sigma = d/2 - 1/2$ and $\rho = 1/2$ for $s = d/2$; we then have $\|\vec{h}(t,.)\|_{H^s} \leq \int_0^t \|e^{(t-\tau)\Delta}(Id-\Delta)^{\rho/2}\mathbb{P}\vec{\nabla}.(Id-\Delta)^{\sigma/2}(\vec{f}\otimes\vec{g})\|_2 d\tau \leq C_s\ \sup_{0<\tau<t}\|\vec{f}(\tau,.)\|_{H^s}\ \sup_{0<\tau<t}\|\vec{g}(\tau,.)\|_{H^s}\ \int_0^t \max(\frac{1}{(t-\tau)^{1/2}}, \frac{1}{(t-\tau)^{(1+\rho)/2}})\ d\tau$. This gives $\|B(\vec{f},\vec{g})\|_{\mathcal{E}_T} \leq C's\|\vec{f}\|_{\mathcal{E}_T}\|\vec{g}\|_{\mathcal{E}_T} T^{(1-\rho)/2}(1+T^{\rho/2})$. Since $\|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_T} \leq \|\vec{u}_0\|_{H^s}$, we find that we have a mild solution in $\mathcal{E}_T^d$ for T small enough. Moreover, we have (locally in time) uniqueness in $(L_t^\infty H_x^s)^d$; since we have *-weak continuity of $t \mapsto \vec{u}(t,.) \in (H^s(\mathbb{R}^d))^d$, we have (globally) uniqueness in

$\cap_{0<T<T^*}(L^\infty((0,T),H^s))^d$ and we may consider the maximal interval on which $\vec{u}$ is defined as a (locally bounded) function with values in $(H^d)^d$. We now check easily that $t \mapsto \vec{u}$ is continuous from $[0,T^*)$ to $(H^s)^d$. First of all, the Kato algorithm associated to an initial value $\vec{u}_0 \in (H^s)^d$ converges to a solution $\vec{u} \in \mathcal{C}([0,T],(H^s)^d)$ where T may be estimated by $T \geq \min(1, \gamma_s \|\vec{u}_0\|_{H^s}^{-\frac{4}{1-\rho}})$ where γ_s is a positive constant (which depends only on d and s) and ρ was defined in the proof of existence of a solution in $\mathcal{E}_T^d$: since $\mathcal{C}([0,T],(H^s)^d)$ is closed in $\mathcal{E}_T^d$, it is enough to check that each iterate $\vec{u}^{(k)}$ defined by

$$\vec{u}^{(0)} = e^{it\Delta}\vec{u}_0 \text{ and } \vec{u}^{(k)} = \vec{u}^{(0)} - B(\vec{u}^{(k)}, \vec{u}^{(k)})$$

belongs to $\mathcal{C}([0,T],(H^s)^d)$. The fact that $\vec{u}^{(0)} \in \mathcal{C}([0,T],(H^s)^d)$ is obvious (since test functions are dense in the shift-invariant space H^s) ; the fact that B maps $\mathcal{C}([0,T],(H^s)^d) \times \mathcal{C}([0,T],(H^s)^d)$ to $\mathcal{C}([0,T],(H^s)^d)$ is easily checked: it is enough to prove it on a dense subspace, and, thus, we just have to check that B maps $\mathcal{C}([0,T],(H^{s+2})^d) \times \mathcal{C}([0,T],(H^{s+2})^d)$ to $\mathcal{C}([0,T],(H^s)^d)$: indeed, for $\vec{f}$ and $\vec{g}$ in $(L^\infty((0,T),H^{s+2}))^d$, we have $B(\vec{f},\vec{g}) \in (L^\infty((0,T),H^{s+2}))^d$ and $\mathbb{P}\vec{\nabla}.(\vec{f}\otimes\vec{g}) \in (L^\infty(0,T),H^{s+1}))^d$, hence $\partial_t B(\vec{f},\vec{g}) \in (L^\infty(0,T),H^s))^d$.

Moreover, we easily check that the solution $\vec{u}$ is $\mathcal{C}^\infty$ on $(0,T^*)\times\mathbb{R}^d$. Indeed, since $H^s \subset B_\infty^{s-d/2,\infty}$, we may replace the space $\mathcal{E}_T = L^\infty([0,T],H^s(\mathbb{R}^d))$ by $\mathcal{E}_T = \{f \in L^\infty([0,T],H^s(\mathbb{R}^d)) \ / \ t^\theta f \in L^\infty((0,T)\times\mathbb{R}^d)\}$ where we define θ as $\theta = 0$ if $s > d/2$, $\theta = d/4 - s/2$ if $d/2-1 < s < d/2$ and $\theta = 1/4$ if $s = d/2$. Since $\|t^\theta B(\vec{f},\vec{g})(t)\|_{L^\infty((0,T)\times\mathbb{R}^d)} \leq C_\theta \, T^{1/2-\theta} \, \|t^\theta \vec{f}\|_{L^\infty((0,T)\times\mathbb{R}^d)} \, \|t^\theta \vec{g}\|_{L^\infty((0,T)\times\mathbb{R}^d)}$, where the constant C_θ does not depend on T, we find that the Picard contraction principle works in our new $\mathcal{E}_T$ for some T which can be estimated by $T \geq \min(1, \gamma_s\|\vec{u}_0\|_{H^s}^{-\frac{4}{1-\rho}}, \gamma_s'\|\vec{u}_0\|_{H^s}^{-\frac{4}{1-2\theta}})$. Hence, the solution $\vec{u}$ we constructed in $\cap_{0<T<T^*}(L^\infty([0,T],H^s(\mathbb{R}^d)))^d$ belongs to $\cap_{0<T_0<T_1<T^*}(L^\infty([T_0,T_1]\times\mathbb{R}^d))^d$. Then, Proposition 15.1 gives us that $\vec{u}$ is smooth.

To prove (B), we shall prove that $H^{d/2-1}(\mathbb{R}^d)$ is an adapted value space, with adapted path space $\mathcal{E}_T = L^\infty([0,T],H^{d/2-1}(\mathbb{R}^d)) \cap L^2([0,T],H^{d/2}(\mathbb{R}^d))$ $(T < \infty)$. Indeed, we have $\mathcal{E}_T \subset L^4([0,T],H^{(d-1)/2})$. Hence, for $\vec{f}$ and $\vec{g}$ in $(\mathcal{E}_T)^d$, we have $\vec{f}\otimes\vec{g} \in (L^2([0,T],H^{d/2-1})^{d\times d}$ and this gives that $B(\vec{f},\vec{g}) \in (L^2([0,T],H^{d/2}(\mathbb{R}^d))^d$ (due to the maximal regularity theorem) and $B(\vec{f},\vec{g}) \in (\mathcal{C}_b([0,T],H^{d/2-1}(\mathbb{R}^d))^d$ (see Lemma 14.1). We thus find that we may apply Theorem 1 bis with $\mathcal{F}_T = L^4([0,T],H^{(d-1)/2})$. We have the inequality $\|B(\vec{f},\vec{g})\|_{\mathcal{F}_T} \leq C_T\|\vec{f}\|_{\mathcal{F}_T}\|\vec{g}\|_{\mathcal{F}_T}$ where C_T is a non-decreasing function of T. Thus, in order to prove (locally in time) existence of a solution, we just have to prove that for $\vec{u}_0 \in (H^{d/2-1})^d$, we have $\lim_{T\to 0}\int_0^T \|e^{t\Delta}\vec{u}_0\|_{H^{(d-1)/2}}^4 \, dt = 0$, which is obvious.

We could as well use the adapted path space $\mathcal{G}_T = L^4([0,T],H^{(d-1)/2}) \cap \{f \ / \ \sqrt{t}f \in L^\infty((0,T)\times\mathbb{R}^d)\}$. The bilinear operator B operates boundedly on $\mathcal{G}_T$, since it is bounded on $\mathcal{F}_T$ and since we may write $B(\vec{f},\vec{g})(t) =$

$e^{t\Delta/2}B(\vec{f},\vec{g})(t/2) + \int_{t/2}^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{f}\otimes\vec{g}\,ds$; hence,

$$\|B(\vec{f},\vec{g})(t)\|_\infty \leq C\min(1,t)^{-1/2}\|B(\vec{f},\vec{g})(t/2)\|_{H^{\frac{d}{2}-1}} + C\int_{t/2}^t \|\vec{f}\|_\infty\|\vec{g}\|_\infty \frac{ds}{\sqrt{t-s}}$$

where C does not depend on T. Moreover, we have $\lim_{t\to 0}\sqrt{t}\|e^{t\Delta}\vec{u}_0\|_\infty = 0$ and we find that the Picard iteration scheme will provide a solution in $\mathcal{G}_T$ for some small enough positive T. Our estimation on T depends only on the size of $\|e^{t\Delta}\vec{u}_0\|_{L^4_t H^{(d-1)/2}_x}$ and $\sup_{0<t<T}\sqrt{t}\|e^{t\Delta}\vec{u}_0\|_\infty$; thus, given a solution $\vec{u}$ in $\cap_{t<T^*}(L^4((0,t),H^{(d-1)/2}))^d \cap (\mathcal{C}([0,T^*),H^{d/2-1})^d$, we find that $\vec{u}(.+t_0)$ belongs to $\mathcal{G}_{T(\vec{u}(t_0))}$ where $T(\vec{u}(t_0))$ is uniformly bounded by below on any compact subset of $[0,T^*)$. This gives that $\vec{u}\in(L^\infty_{loc,t}L^\infty_x)^d$; hence, that $\vec{u}$ is smooth on $(0,T^*)\times\mathbb{R}^d$.

To prove (C), we follow the proof of (B), but with homogeneous Sobolev spaces: we just check that $\dot{H}^{d/2-1}(\mathbb{R}^d)$ is an adapted value space, with adapted path space

$$\mathcal{E} = L^\infty((0,\infty),\dot{H}^{d/2-1}) \cap L^2((0,\infty),\dot{H}^{d/2}) \cap \{f \;/\; \sqrt{t}f \in L^\infty((0,\infty)\times\mathbb{R}^d)\}$$

by proving the boundedness of B on

$$\mathcal{G} = L^4((0,\infty),\dot{H}^{(d-1)/2}) \cap \{f \;/\; \sqrt{t}f \in L^\infty((0,\infty)\times\mathbb{R}^d)\}.$$

Thus, we get that there exists a positive ϵ_0 so that, if $\|\vec{u}_0\|_{\dot{H}^{d/2-1}} < \epsilon_0$, then the Navier–Stokes equations with initial value $\vec{u}_0$ has a solution $\vec{u}$ in $\mathcal{E}^d$. Moreover, $\vec{u}\in\mathcal{C}_b((0,\infty),(\dot{H}^{d/2-1})^d)$ and for any $t_0\geq 0$ $\vec{u}(t+t_0)$ may be computed through the Picard algorithm for the Navier–Stokes equations with initial value $\vec{u}(t_0)$. Now, if $\vec{u}_0$ belongs moreover to a space $(H^s)^d$ with $s\geq d/2-1$ and if we look at the unique solution $\vec{v}\in\mathcal{C}((0,T^*),(H^s)^d\}$ $(s>d/2-1)$ or $\vec{v}\in\mathcal{C}((0,T^*),(H^{d/2-1})^d\}\cap_{T<T^*}L^4((0,T),(H^{(d-1)/2})^d)$, we first see by using the Picard process on $[t_0,t_0+\epsilon(t_0))$ for every $t_0<T^*$ that $\vec{v}=\vec{u}$ on $[0,T^*)$. Moreover, the L^2 norm of $\vec{u}$ is bounded by the L^2 norm of $\vec{u}_0$ (since we know that the limiting process for the mollified equation [which provides a Leray solution] and the Picard algorithm give us the same solution). Thus, $T^*=+\infty$ if $s=d/2-1$. If $s>d/2-1$, we write (using the Littlewood–Paley decomposition) $fg=\sum_{j\in\mathbb{Z}}S_jf\Delta_jg+S_{j+1}g\Delta_jf$, which gives the estimate $\|fg\|_{H^s}\leq C(\|f\|_\infty\|g\|_{H^s}+\|g\|_\infty\|f\|_{H^s}$. This gives the inequality for $t<T^*$

$$\|\vec{u}(t)\|_{H^s} \leq \|\vec{u}_0\|_{H^s} + C_0\int_0^t \|\vec{u}(\tau)\|_\infty\|\vec{u}(\tau)\|_{H^s}\frac{d\tau}{\sqrt{t-\tau}}$$

Now, we use the estimate $\|\vec{u}(\tau)\|_\infty \leq C_1\tau^{-1/2}$ and we get

$$\|\vec{u}(t)\|_{H^s} \leq \|\vec{u}_0\|_{H^s} + C_0C_1\int_0^t \|\vec{u}(\tau)\|_{H^s}\frac{d\tau}{\sqrt{t-\tau}\sqrt{\tau}}.$$

If $\omega(t) = \sup_{\tau<t} \|\vec{u}(\tau)\|_{H^s}$, we find that, for $\epsilon \in (0,1)$,

$$\omega(t) \leq (1 + C_0 C_1 \int_0^{1-\epsilon} \frac{d\tau}{\sqrt{1-\tau}\sqrt{\tau}})\omega((1-\epsilon)t) + C_0 C_1 \int_{1-\epsilon}^1 \frac{d\tau}{\sqrt{1-\tau}\sqrt{\tau}}\omega(t)$$

which gives, for a small enough ϵ, $\omega(t) \leq C(\epsilon)\omega((1-\epsilon)t)$. Thus, the norm of $\vec{u}$ in H^s remains bounded on every compact subset of $[0,\infty)$ and we know from (A) that the maximal existence time of the solution in H^s is then $T^* = +\infty$. (As a matter of fact, we may even prove that the H^s norm remains globally bounded on $[0,\infty)$: see the persistency theorem in Chapter 19).

We now prove (D). We know from Proposition 15.1 that $\vec{u}(t)$ is Lipschitz for positive t (since $\vec{\nabla} \otimes \vec{u} \in (L^\infty(\mathbb{R}^d))^{d\times d}$); since $\vec{u}$ is square-integrable, we find that $\vec{u}(t) \in (\mathcal{C}_0(\mathbb{R}^d))^d$ for all positive $t < T^*$. Moreover, we check easily that B is bounded on $(L^\infty([0,T) \times \mathbb{R}^d))^d$: $\sup_{0<t<T} \|B(\vec{f},\vec{g})\|_\infty \leq C\sqrt{T}\ \sup_{0<t<T} \|\vec{f}\|_\infty\ \sup_{0<t<T} \|\vec{g}\|_\infty$. This proves that, for every $t < T^*$, $\vec{u}(t+s)$ remains bounded for $s \in [0, T(t)]$ where $T(t) = O(1/\|\vec{u}(t)\|_\infty^2)$. Thus, if $\limsup_{t\to T^*} \|\vec{u}(t)\|_\infty = +\infty$, we must have $\|\vec{u}\|_\infty \geq \gamma(T^*-t)^{-1/2}$ for some positive constant γ and for all $t < T^*$, so that $\lim_{t\to T^*} \|\vec{u}\|_\infty = \infty$. Thus, if $\liminf_{t\to T^*} \|\vec{u}(t)\|_\infty < +\infty$, we have that $\sup_{T^*/2<t<T^*} \|\vec{u}(t)\|_\infty < \infty$. We then get the estimate for $T^*/2 < t < T^*$

$$\|\vec{u}(t)\|_{H^s} \leq \|\vec{u}(T^*/2)\|_{H^s} + C_0 \int_{T^*/2}^t \|\vec{u}(\tau)\|_\infty \|\vec{u}(\tau)\|_{H^s}\ \frac{d\tau}{\sqrt{t-\tau}}$$

Now, we use the estimate $\|\vec{u}(\tau)\|_\infty \leq C_1$ and we get

$$\begin{aligned} \|\vec{u}(t)\|_{H^s} &\leq \|\vec{u}(T^*/2)\|_{H^s} + C_0 C_1 \int_{T^*/2}^t \|\vec{u}(\tau)\|_{H^s}\ \tfrac{d\tau}{\sqrt{t-\tau}} \\ &\leq C_2 \|\vec{u}(T^*/2)\|_{H^s} + C_3 \int_{T^*/2}^t \|\vec{u}(\tau)\|_{H^s}\ d\tau \end{aligned}$$

with $C_2 = 1 + 2C_0C_1\sqrt{T^*/2}$ and $C_3 = (C_0C_1)^2 \int_0^1 \frac{d\sigma}{\sqrt{1-\sigma}\sqrt{\sigma}}$. The Gronwall lemma shows then that the H^s norm of $\vec{u}$ remains bounded. This is enough to get that $\vec{u}$ exists in H^s beyond T^* if $s > d/2-1$, since we know that the existence time is related to the size of the norm. If $s = d/2-1$, we notice that $\vec{u} \in (H^{d/2})^d$ for $0 < t < T^*$; thus, we find that the solution $\vec{u}$ exists in $H^{d/2}$ beyond T^*, hence exists as well in $H^{d/2-1}$ beyond T^*.

Finally, we prove in the next section that, if T^* is finite, if $\vec{u}_0 \in (L^p)^d$ for some $p \in [d,+\infty)$ and if $\int_{T^*/2}^{T^*} \|\vec{u}(t,.)\|_{BMO}^2\ dt < +\infty$, then we have $\limsup_{t\to T^*} \|\vec{u}\| < \infty$. Since $H^s \subset L^d$ for $s \geq d/2-1$, this concludes the proof. □

3. Kato's mild solutions in L^p, $p \geq d$

In 1984, following the ideas of Weissler [WEI 81], Kato proved the existence of mild solutions in L^p. Existence of solutions for initial values in L^p had also been studied by Fabes, Jones, and Rivière [FABJR 72] and by Giga [GIG 86] (see Chapter 20).

Theorem 15.3: (Kato's theorem [KAT 84])

(A) Let $p \in (d, \infty)$. Then, for all $\vec{u}_0 \in (L^p(\mathbb{R}^d))^d$ so that $\vec{\nabla}.\vec{u}_0 = 0$, there exist a positive T^* and a (unique) weak solution $\vec{u} \in \mathcal{C}([0, T^*), (L^p)^d)$ for the Navier–Stokes equations on $(0, T^*) \times \mathbb{R}^d$ so that $\vec{u}(0, .) = \vec{u}_0$. This solution is then smooth on $(0, T^*) \times \mathbb{R}^d$.

(B) For all $\vec{u}_0 \in (L^d(\mathbb{R}^d))^d$ so that $\vec{\nabla}.\vec{u}_0 = 0$, there exist a positive T^* and a weak solution $\vec{u} \in \mathcal{C}([0, T^*), (L^d)^d)$ for the Navier–Stokes equations on $(0, T^*) \times \mathbb{R}^d$ so that $\vec{u}(0, .) = \vec{u}_0$. This solution may be chosen so that for all $T \in (0, T^*)$ we have $\sup_{0<t<T} \sqrt{t}\, \|\vec{u}(t, .)\|_\infty < \infty$ and $\lim_{t \to 0} \sqrt{t}\, \|\vec{u}(t, .)\|_\infty = 0$. With this extra condition on the L^∞ norm, such a solution is unique and, moreover, it is smooth on $(0, T^*) \times \mathbb{R}^d$.

(C) There exists $\epsilon_0 > 0$ so that if $\vec{u}_0 \in (L^p \cap L^d)^d$ and if $\|\vec{u}_0\|_d < \epsilon_0$, then the existence time T^* in point (A) or (B) for the smooth solution $\vec{u}$ is equal to $+\infty$.

(D) The smooth solutions $\vec{u}$ described in (A) and (B) satisfy $\vec{u}(t, .) \in (\mathcal{C}_0(\mathbb{R}^d))^d$ for all $t < T^*$. The maximal existence time T^* is finite if and only if $\lim_{t \to T^*} \|\vec{u}\|_\infty = +\infty$. We may replace this latter condition by : $T^* < \infty$ if and only if $\int_{T^*/2}^{T^*} \|\vec{u}(t, .)\|_{BMO}^2 \, dt = +\infty$.

Proof: The proof of Theorem 15.3 is the same as the proof of Theorem 15.2. (As a matter of fact, we shall adapt this proof to the setting of many Banach spaces in Part 4.)

(A) is easy: we want to prove that $L^p(\mathbb{R}^d)$ is an adapted value space when $d < p < +\infty$, with adapted path space $\mathcal{E}_T = L^\infty([0, T], L^p(\mathbb{R}^d))$.Let $\vec{f}$ and $\vec{g}$ belong to $\mathcal{E}_T^d$, with $\mathcal{E}_T = L^\infty([0, T], L^p(\mathbb{R}^d))$; let us estimate $\vec{h} = B(\vec{f}, \vec{g}) = \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \vec{\nabla}.(\vec{f} \otimes \vec{g})\, d\tau$. We know that the kernel of the convolution operator $e^{(t-\tau)\Delta} \mathbb{P} \vec{\nabla}.$ has a $L^1(dx)$ norm which is $O((t-\tau)^{-1/2})$ and a $L^\infty(dx)$ norm which is $O((t - \tau)^{-(d+1)/2})$, hence has a $L^{p/(p-1)}(dx)$ norm which is $O((t - \tau)^{-(d+p)/2p})$.

Thus, we easily get the estimate $\|\vec{h}(t, .)\|_p \le \int_0^t \|e^{(t-\tau)\Delta} \mathbb{P} \vec{\nabla}.(\vec{f} \otimes \vec{g})\|_p d\tau \le C_p \sup_{0<\tau<t} \|\vec{f}(\tau, .)\|_p \sup_{0<\tau<t} \|\vec{g}(\tau, .)\|_p \int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}+\frac{d}{2p}}} \le C' s \|\vec{f}\|_{\mathcal{E}_T} \|\vec{g}\|_{\mathcal{E}_T} T^{\frac{1}{2}-\frac{d}{2p}}$. Since $\|e^{t\Delta} \vec{u}_0\|_{\mathcal{E}_T} \le \|\vec{u}_0\|_p$, we find that we have a mild solution in $\mathcal{E}_T^d$ for T small enough. Moreover, we have (locally in time) uniqueness in $(L_t^\infty L_x^p)^d$; since we have *-weak continuity of $t \mapsto \vec{u}(t, .) \in (L^p(\mathbb{R}^d))^d$, we have (globally) uniqueness in $\cap_{0<T<T^*} (L^\infty((0, T), L^p))^d$ and we may consider the maximal interval on which $\vec{u}$ is defined as a (locally bounded) function with values in $(L^p)^d$. We now check easily that $t \mapsto \vec{u}$ is continuous from $[0, T^*)$ to $(L^p)^d$. First of all, the Kato algorithm associated to an initial value $\vec{u}_0 \in (L^p)^d$ converges to a solution $\vec{u} \in \mathcal{C}([0, T], (L^p)^d)$ where T may be estimated by $T \ge \gamma_p \|\vec{u}_0\|_p^{-\frac{2p}{p-d}}$ where γ_p is a positive constant (which depends only on d

and p): since $\mathcal{C}([0,T],(L^p)^d)$ is closed in $\mathcal{E}_T^d$, it is enough to check that each iterate $\vec{u}^{(k)}$ defined by $\vec{u}^{(0)} = e^{it\Delta}\vec{u}_0$ and $\vec{u}^{(k)} = \vec{u}^{(0)} - B(\vec{u}^{(k)},\vec{u}^{(k)})$ belongs to $\mathcal{C}([0,T],(L^p)^d)$. The fact that $\vec{u}^{(0)} \in \mathcal{C}([0,T],(L^p)^d)$ is obvious (since test functions are dense in the shift-invariant space L^p); the fact that B maps $\mathcal{C}([0,T],(L^p)^d)\times\mathcal{C}([0,T],(L^p)^d)$ to $\mathcal{C}([0,T],(L^p)^d)$ is easily checked: it is enough to prove it on a dense subspace, and, thus, we just have to check that B maps $\mathcal{C}([0,T],(H_p^{2+d/p})^d) \times \mathcal{C}([0,T],(H_p^{2+d/p})^d)$ to $\mathcal{C}([0,T],(L^p)^d)$: indeed, for $\vec{f}$ and $\vec{g}$ in $(L^\infty((0,T),H_p^{2+d/p}))^d$, we have $B(\vec{f},\vec{g}) \in (L^\infty((0,T),H_p^{2+d/p}))^d$ and $\mathbb{P}\vec{\nabla}.(\vec{f}\otimes\vec{g}) \in (L^\infty(0,T),H_p^{1+d/p}))^d$; hence, $\partial_t B(\vec{f},\vec{g}) \in (L^\infty(0,T),L^p))^d$.

Moreover, we easily check that the solution $\vec{u}$ is $\mathcal{C}^\infty$ on $(0,T^*)\times\mathbb{R}^d$. Indeed, since $L^p \subset B_\infty^{-d/p,\infty}$, we may replace the space $\mathcal{E}_T = L^\infty([0,T],L^p(\mathbb{R}^d))$ by $\mathcal{E}_T = \{f \in L^\infty([0,T],L^p(\mathbb{R}^d)) \ / \ t^{\frac{d}{2p}} f \in L^\infty((0,T)\times\mathbb{R}^d)\}$. We find that the Picard contraction principle works in our new $\mathcal{E}_T$ for some T which can be estimated by $T \geq \gamma_p \|\vec{u}_0\|_p^{-\frac{4}{1-\rho}}$. Hence, the solution $\vec{u}$ we have constructed in $\cap_{0<T<T^*}(L^\infty([0,T],L^p(\mathbb{R}^d)))^d$ belongs to $\cap_{0<T_0<T_1<T^*}(L^\infty([T_0,T_1]\times\mathbb{R}^d))^d$. Then, Proposition 15.1 gives us that $\vec{u}$ is smooth.

To prove (B), we prove that $L^d(\mathbb{R}^d)$ is an adapted value space, with adapted path space $\mathcal{E}_T = \{f \in L^\infty([0,T],L^d(\mathbb{R}^d)) \ / \ \sqrt{t}\, f(t,.) \in L^\infty([0,T],L^\infty)$ and $\lim_{t\to 0}\sqrt{t}\|f(t,.)\|_\infty = 0\}$. Indeed, we have

$$\mathcal{E}_T \subset \mathcal{F}_T = \{f \ / \ t^{1/4}f(t,.) \in L^\infty([0,T],L^{2d}) \text{ and } \lim_{t\to 0} t^{1/4}\|f(t,.)\|_{2d} = 0\}$$

For $\vec{f}$ and $\vec{g}$ in $(\mathcal{F}_T)^d$, we have

$$\|B(\vec{f},\vec{g})\|_{2d} \leq C\int_0^t \frac{1}{(t-s)^{3/4}} \frac{s^{1/4}\|\vec{f}\|_{2d}\ s^{1/4}\|\vec{g}\|_{2d}}{s^{1/2}}\, ds$$

hence, $B(\vec{f},\vec{g}) \in (\mathcal{F}_T)^{d\times d}$, while for $\vec{f}$ and $\vec{g}$ in $(\mathcal{E}_T)^d$ we have

$$\|B(\vec{f},\vec{g})\|_d + \|B(\vec{g},\vec{f})\|_d \leq C\int_0^t \frac{1}{(t-s)^{3/4}} \frac{s^{1/4}\|\vec{f}\|_{2d}\ \|\vec{g}\|_d}{s^{1/4}}\, ds$$

and

$$\|B(\vec{f},\vec{g})\|_\infty + \|B(\vec{g},\vec{f})\|_\infty \leq C\int_0^t \frac{1}{(t-s)^{3/4}} \frac{s^{1/4}\|\vec{f}\|_{2d}\ s^{1/2}\|\vec{g}\|_\infty}{s^{3/4}}\, ds$$

hence, $B(\vec{f},\vec{g}) \in (\mathcal{E}_T)^{d\times d}$ and $\|B(\vec{f},\vec{g})\|_{\mathcal{E}_T} + \|B(\vec{g},\vec{f})\|_{\mathcal{E}_T} \leq C\ \|\vec{f}\|_{\mathcal{F}_T}\ \|\vec{g}\|_{\mathcal{E}_T}$. The estimate in $\mathcal{F}_T$ proves that the Picard iteration scheme will provide a solution to the Navier–Stokes equations $\vec{u} \in (\mathcal{F}_T)^d$ provided that the norm of $(e^{t\Delta}\vec{u}_0)_{0<t<T}$ in $(\mathcal{F}_T)^d$ is small enough; moreover, if $\vec{u}_0 \in L^d$, we get a control on the nth iterate $\vec{u}^{(n)}$ in the Picard iteration scheme for the $\mathcal{E}_T$ norm: $\|\vec{u}^{(n+1)} - \vec{u}^{(n)}\|_{\mathcal{E}_T} \leq C\|\vec{u}^{(n)} - \vec{u}^{(n-1)}\|_{\mathcal{F}_T}\ (\|\vec{u}^{(n)}_{\mathcal{E}_T} + \|\vec{u}^{(n)}\|_{\mathcal{E}_T})$ and this gives that

$\vec{u}^{(n)}$ converges to $\vec{u}$ in the $\mathcal{E}_T$ norm. Moreover, for $\vec{f}$ and $\vec{g}$ in $(\mathcal{E}_T)^d$, we find easily that $B(\vec{f},\vec{g}) \in (\mathcal{C}([0,T], L^d(\mathbb{R}^d))^d$: clearly, we have from the above estimates that $\lim_{t\to 0} \|B(\vec{f},\vec{g})\|_d = 0$; for $0 < t \le T$ and for $t < \theta \le \min(T, 3t/2)$, we write $\theta = (1+\gamma)t$ and $\tau = (1-\gamma)t$ (with $0 < \gamma \le 1/2$) and

$$\begin{array}{rcl} B(\vec{f},\vec{g})(\theta) - B(\vec{f},\vec{g})(t) & = & \int_0^{t/2} (e^{(\theta-t)\Delta} - Id) e^{(t-s)\Delta} \mathbb{P}\vec{\nabla}.\vec{f}\otimes\vec{g}\, ds \\ & + & \int_{t/2}^{\tau} (e^{(\theta-t)\Delta} - Id) e^{(t-s)\Delta} \mathbb{P}\vec{\nabla}.\vec{f}\otimes\vec{g}\, ds \\ & + & \int_{\tau}^{t} (e^{(\theta-t)\Delta} - Id) e^{(t-s)\Delta} \mathbb{P}\vec{\nabla}.\vec{f}\otimes\vec{g}\, ds \\ & - & \int_t^{\theta} e^{(\theta-s)\Delta} \mathbb{P}\vec{\nabla}.\vec{f}\otimes\vec{g}\, ds \\ & = & I_1 + I_2 + I_3 + I_4 \end{array}$$

For I_1 and I_2, we write $\|(e^{(\theta-t)\Delta} - Id)f\|_d \le C(\theta - t)\|\Delta f\|_d$ and get:
– $\|I_1\|_d \le C\big(\frac{\theta-t}{t}\big)\ \sup_{0<s<T} \|\vec{f}\|_d\ \sup_{0<s<T} \sqrt{s}\ \|\vec{g}\|_\infty$
– $\|I_2\|_d \le C\big(\frac{\theta-t}{\sqrt{t}\ \sqrt{t-\tau}}\big)\ \sup_{0<s<T} \|\vec{f}\|_d\ \sup_{0<s<T} \sqrt{s}\ \|\vec{g}\|_\infty$

For I_3, we write $\|(e^{(\theta-t)\Delta} - Id)f\|_d \le 2\|\ f\|_d$ and get:
– $\|I_3\|_d \le C\big(\frac{\sqrt{t-\tau}}{\sqrt{t}}\big)\ \sup_{0<s<T} \|\vec{f}\|_d\ \sup_{0<s<T} \sqrt{s}\ \|\vec{g}\|_\infty$

For I_4, we easily get:
– $\|I_4\|_d \le C\big(\frac{\sqrt{\theta-t}}{\sqrt{t}}\big)\ \sup_{0<s<T} \|\vec{f}\|_d\ \sup_{0<s<T} \sqrt{s}\ \|\vec{g}\|_\infty$

Adding the four estimates, we get (since $\theta - t = t - \tau$) that for all t and θ in $[0,T]$ we have

$$\|B(\vec{f},\vec{g})(\theta) - B(\vec{f},\vec{g})(t)\|_d \le C\sqrt{\frac{|\theta - t|}{t+\theta}}\ \sup_{0<s<T} \|\vec{f}\|_d\ \sup_{0<s<T} \sqrt{s}\ \|\vec{g}\|_\infty$$

Finally, we conclude that (B) is valid. So far, we have proved the inequality $\|B(\vec{f},\vec{g})\|_{\mathcal{F}_T} \le C\|\vec{f}\|_{\mathcal{F}_T}\|\vec{g}\|_{\mathcal{F}_T}$ where C is a positive constant that does not depend on T. Thus, in order to prove (locally in time) existence of a solution, we just prove that for $\vec{u}_0 \in (L^d)^d$, we have $\lim_{T\to 0} \sup_{0<t<T} t^{1/4}\|e^{t\Delta}\vec{u}_0\|_{2d} = 0$: we use the inequalities $\sup_{0<t} \|e^{t\Delta}\vec{u}_0\|_d = \|\vec{u}_0\|_d$, $\sup_{0<t} \sqrt{t}\|e^{t\Delta}\vec{u}_0\|_\infty \le C\|\vec{u}_0\|_{\dot{B}_\infty^{-1,\infty}} \le C'\|\vec{u}_0\|_d$ and $\lim_{t\to 0} \sqrt{t}\|e^{t\Delta}\vec{u}_0\|_\infty = 0$ (which is obvious for $\vec{u}_0 \in (L^d \cap L^\infty)^d$ and is straightforwardly extended to $(L^d)^d$ due to the equicontinuity of the family of operators $(\sqrt{t}e^{t\Delta})_{\mathcal{L}(L^d, L^\infty)}$ and to the density of $L^d \cap L^\infty$ in L^d).

To prove (C), we follow the proof of (B), but with $T = +\infty$. This does not present a problem, since all the involved spaces are homogeneous: we just check that $L^d(\mathbb{R}^d)$ is an adapted value space, with adapted path space $\mathcal{E} = \{f \in L^\infty((0,\infty), L^d(\mathbb{R}^d))/\ \sqrt{t}\ f(t,.) \in L^\infty((0,\infty), L^\infty)$ and $\lim_{t\to 0} \sqrt{t}\|f(t,.)\|_\infty = 0\}$ by proving the boundedness of B on $\mathcal{F} = \{f\ /\ t^{1/4} f(t,.) \in L^\infty((0,\infty), L^{2d})$ and $\lim_{t\to 0} t^{1/4}\|f(t,.)\|_{2d} = 0\}$. Thus, we get that there exists a positive ϵ_0 so that, if $\|\vec{u}_0\|_d < \epsilon_0$, then the Navier–Stokes equations with initial value $\vec{u}_0$ has a solution $\vec{u}$ in $\mathcal{E}^d$. Moreover, $\vec{u} \in \mathcal{C}_b((0,\infty), (L^d)^d)$ and for any $t_0 \ge 0$ $\vec{u}(t + t_0)$ may be computed through the Picard algorithm for the Navier–Stokes equations with initial value $\vec{u}(t_0)$. The end of (C) is contained in assertion (D), which we prove now.

We know from Proposition 15.1 that $\vec{u}(t)$ is Lipschitz for positive t (since $\vec{\nabla} \otimes \vec{u} \in (L^\infty(\mathbb{R}^d))^{d\times d}$); since $\vec{u}$ is in L^p (with $p < \infty$), we find that $\vec{u}(t) \in (\mathcal{C}_0(\mathbb{R}^d))^d$ for all positive $t < T^*$. Moreover, we have seen in the proof of Theorem 15.1 that B is bounded on $(L^\infty([0,T)\times\mathbb{R}^d))^d$ and we thus get that, for every $t < T^*$, $\vec{u}(t+s)$ remains bounded for $s \in [0, T(t)]$ where $T(t) = O(1/\|\vec{u}(t)\|_\infty^2)$. Thus, we know that either $\limsup_{t\to T^*} \|\vec{u}(t)\|_\infty < +\infty$ or $\lim_{t\to T^*} \|\vec{u}\|_\infty = \infty$. In the first case, we get that $\sup_{T^*/2<t<T^*} \|\vec{u}(t)\|_\infty < \infty$. We then get the estimate for $T^*/2 < t < T^*$

$$\|\vec{u}(t)\|_p \leq \|\vec{u}(T^*/2)\|_p + C_0 \int_{T^*/2}^t \|\vec{u}(\tau)\|_\infty \|\vec{u}(\tau)\|_p \, \frac{d\tau}{\sqrt{t-\tau}}$$

As in the case of the H^s norm, we get the inequality

$$\|\vec{u}(t)\|_p \leq C_1 \|\vec{u}(T^*/2)\|_p + C_2 \int_{T^*/2}^t \|\vec{u}(\tau)\|_p \, d\tau$$

and we may use the Gronwall lemma to prove that the L^p norm of $\vec{u}$ remains bounded. This is enough to get that $\vec{u}$ exists in L^p beyond T^* if $p > d$, since we know that the existence time is related to the size of the norm. If $p = d$, we notice that $\vec{u} \in (L^{2d})^d$ for $0 < t < T^*$, thus we find that the solution $\vec{u}$ exists in L^{2d} beyond T^*, hence exists as well in L^d beyond T^*. Indeed, we write $\vec{u}(t + T^*/2) = e^{t\Delta}\vec{u}(T^*/2) - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}/\vec{u}(s+T^*/2) \otimes \vec{u}(s+T^*/2)\, ds = e^{t\Delta}\vec{u}(T^*/2) - \vec{w}(t)$. Since $\vec{u}(T^*/2) \in L^d$, the term $e^{t\Delta}\vec{u}(T^*/2)$ belongs to L^d for every positive t, while, since $\vec{u} \in \mathcal{C}([T^*/2, T^*+\epsilon], (L^{2d})^d]$, we find that $\vec{w} \in \mathcal{C}([T^*/2, T^*+\epsilon], (L^d)^d]$.

Finally, we consider the case where $\int_{T^*/2}^{T^*} \|\vec{u}(t,.)\|_{BMO}^2 \, dt < +\infty$. We may consider only the case $d < p < \infty$ (for $p = d$, we use the result for $p = 2d$ just as in the preceding paragraph).In this case, we have only to prove that the L^p norm remains bounded on the whole interval $[0, T^*)$. Thus, we consider, for $0 < t < T^*/2$, $\vec{w}(t) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{u}(s+T^*/2) \otimes \vec{u}(s+T^*/2)\, ds$. Let $\vec{\varphi} \in (L^{\frac{p}{p-1}})^d$. We have

$$\int_{\mathbb{R}^d} \vec{\varphi}(x).\vec{w}(t,x)\, dx = \int_0^t \int_{\mathbb{R}^d} (e^{(t-s)\Delta}\mathbb{P}\vec{\nabla})^*\vec{\varphi}.\vec{u}(s+T^*/2) \otimes \vec{u}(s+T^*/2)\, dx\, ds$$

hence

$$\begin{aligned} |\textstyle\int_{\mathbb{R}^d} \vec{\varphi}(x).\vec{w}(t,x)\, dx| &\leq C \textstyle\int_0^t \|e^{(t-s)\Delta}\sqrt{-\Delta}\vec{\varphi}\|_{\frac{p}{p-1}} \|\vec{u}(s+T^*/2)\|_{2p}^2 \, ds \\ &\leq C\big(\textstyle\int_0^t \|e^{(t-s)\Delta}\sqrt{-\Delta}\vec{\varphi}\|_{\frac{p}{p-1}}^2 ds\big)^{1/2} \big(\textstyle\int_0^t \|\vec{u}(s+\frac{T^*}{2})\|_{2p}^4 ds\big)^{1/2} \end{aligned}$$

Since $p > 2$, we have $L^{\frac{p}{p-1}} \subset \dot{B}^{0,2}_{\frac{p}{p-1}}$; hence, we get

$$\|\vec{w}(t,x)\|_p \leq C\big(\int_0^t \|\vec{u}(s+\frac{T^*}{2})\|_{2p}^4 ds\big)^{1/2}$$

But we know that $L^{2p} = [L^p, BMO]_{1/2}$ (Chapter 6). Thus, we find at last

$$\|\vec{u}(t+T^*/2)\|_p^2 \leq C(\|\vec{u}(T^*/2)\|_p^2 + \int_0^t \|\vec{u}(s+\frac{T^*}{2})\|_p^2 \|\vec{u}(s+\frac{T^*}{2})\|_{BMO}^2 ds)$$

and we conclude with the Gronwall lemma.

To conclude the proof, we notice that since $\vec{u}$ is extended beyond T^*, we have that $\vec{u}$ remains bounded in L^∞ norm on $[T^*/2, T^*]$ as stated in the proof of Theorem 15.1. □

There is another proof of existence of global solutions given by Calderón [CAL 93] and by Cannone [CAN 95]:

Proposition 15.2: *There exists $\epsilon_0 > 0$ (which depends only on the dimension d) such that if $\vec{u}_0 \in (L^d(\mathbb{R}^d))^d$ with $\vec{\nabla}.\vec{u}_0 = 0$ and $\|\vec{u}_0\|_d < \epsilon_0$, there exists a weak solution $\vec{u}$ for the Navier–Stokes equations on $(0,+\infty) \times \mathbb{R}^d$ with initial value $\vec{u}(0,.) = \vec{u}_0$ so that $\sup_{0<t<+\infty} |\vec{u}(t,x)| \in L^d(\mathbb{R}^d)$.*

Proof: To prove Proposition 15.2, we change the adapted path space associated with the adapted value space $L^d(\mathbb{R}^d)$ into $\mathcal{E} = \{f \in L^\infty([0,+\infty), L^d(\mathbb{R}^d)) \ / \ \sup_{0<t<+\infty} |f(t,x)| \in L^d(\mathbb{R}^d)\}$. We easily check that B is bounded on $\mathcal{E}^d$, writing that if $|\vec{u}(t,x)| \leq U(x)$ and $|\vec{v}(t,x)| \leq V(x)$ for all $t > 0$, then

$$\begin{array}{rcl} |B(\vec{u},\vec{v})| & \leq & C \int\int_{0<s<t} \frac{1}{(\sqrt{t-s}+|x-y|)^{d+1}} U(y)V(y) \, dy \, ds \\ & \leq & C' \int \frac{1}{|x-y|^{d-1}} U(y)V(y) \, dy \end{array}$$

Now, $UV \in L^{d/2}$ while $\frac{1}{|x|^{d-1}} \in L^{\frac{d}{d-1},\infty}$, hence $W = \frac{1}{|x|^{d-1}} * UV \in L^d$. □

Remark: The extra conditions on the solutions $\vec{u} \in (\mathcal{C}([0,T^*), L^d(\mathbb{R}^d)))^d$ are used to introduce good adapted path spaces for the proofs of existence of solutions in Theorem 15.2 and Proposition 15.2. We do not need them to prove uniqueness (Furioli, Lemarié-Rieusset and Terraneo [FURLT 00], Meyer [MEY 99], Monniaux [MON 99], Lions and Masmoudi [LIOM 98]); this point will be discussed in Chapter 27.

Part 4:

New approaches to mild solutions

Chapter 16

The mild solutions of Koch and Tataru : the space BMO^{-1}

1. The space BMO^{-1}

The space BMO^{-1} is defined as the space of derivatives of functions in BMO. It plays a natural role in the search of global mild solutions for the Navier–Stokes equations. We have seen that the initial value problem for these equations was meaningful for solutions belonging to the space $L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d)$. When looking for solutions through the iterative scheme of Picard, we usually require the first term of the iteration, $e^{t\Delta}\vec{u}_0$, to belong to the space where solutions are searched for. Moreover, the search for global solutions is often derived from the proof of local existence provided that the space we work in satisfies that its norm is invariant through the transforms $\vec{u}\mapsto\lambda\vec{u}(\lambda^2 t,\lambda x)$, $\lambda>0$, (which operate on the set of solutions to the Navier–Stokes equations). $BMO^{(-1)}$ may be characterized precisely as the space of tempered distributions f so that the distribution $F=e^{t\Delta}f$ satisfies that the family $\lambda F(\lambda^2 t,\lambda x)$, $\lambda>0$, is bounded in $L^2_{uloc,x}L^2_t((0,1)\times\mathbb{R}^d)$.

Definition 16.1: ($bmo^{(-1)}$ and $BMO^{(-1)}$)

We define bmo^{-1} as the space of tempered distributions f on $\mathbb{R}^d$ so that for all $T\in(0,\infty)$ we have $\sup_{0<t<T}\sup_{x_0\in\mathbb{R}^d}\frac{1}{t^{d/2}}\int_0^t\int_{|x-x_0|\leq\sqrt{t}}|e^{s\Delta}f(x)|^2\,ds\,dx<\infty$.

Similarly, $BMO^{(-1)}$ is the space of the distributions $f\in bmo^{(-1)}$ such that $\sup_{0<t}\sup_{x_0\in\mathbb{R}^d}\frac{1}{t^{d/2}}\int_0^t\int_{|x-x_0|\leq\sqrt{t}}|e^{s\Delta}f(x)|^2\,ds\,dx<\infty$.

We define the norms of $bmo^{(-1)}$ and $BMO^{(-1)}$ by

$$\|f\|_{bmo^{(-1)}}=\Big(\sup_{0<t<1}\sup_{x_0\in\mathbb{R}^d}\frac{1}{t^{d/2}}\int_0^t\int_{|x-x_0|\leq\sqrt{t}}|e^{s\Delta}f(x)|^2\,ds\,dx\Big)^{1/2}$$

and

$$\|f\|_{BMO^{(-1)}}=\Big(\sup_{0<t}\sup_{x_0\in\mathbb{R}^d}\frac{1}{t^{d/2}}\int_0^t\int_{|x-x_0|\leq\sqrt{t}}|e^{s\Delta}f(x)|^2\,ds\,dx\Big)^{1/2}$$

We are going to check that those spaces are shift-invariant Banach spaces of distributions.

Lemma 16.1: ($bmo^{(-1)}$ and Besov spaces)
There is a constant C so that for any $f \in \mathcal{S}'(\mathbb{R}^d)$ and any $t > 0$, we have

$$\|e^{t\Delta} f\|_\infty \le C \frac{1}{\sqrt{t}} \sup_{x_0 \in \mathbb{R}^d} \Big(\frac{1}{t^{d/2}} \int_0^{t/2} \int_{|x-x_0| \le \sqrt{t/2}} |e^{s\Delta} f(x)|^2 \, ds \, dx \Big)^{1/2}.$$

In particular, $bmo^{(-1)} \subset B_\infty^{-1,\infty}$ and $BMO^{-1} \subset \dot{B}_\infty^{-1,\infty}$

Proof: This is very easy. It is enough to write that $e^{t\Delta} f = e^{(t-s)\Delta} e^{s\Delta} f$ and thus $e^{t\Delta} f = \frac{2}{t} \int_0^{t/2} e^{(t-s)\Delta} e^{s\Delta} f \, ds$. Now, $e^{\theta\Delta}$ is a convolution operator with a positive kernel $W_\theta(x) = \frac{1}{\theta^{d/2}} W(\frac{x}{\sqrt{\theta}})$, where $W(x) = \frac{1}{(4\pi)^{d/2}} e^{-\frac{|x|^2}{4}}$. Thus, we use the Cauchy–Schwarz inequality to get: $|e^{\theta\Delta} f(x_0)| = |\int W_\theta(x - x_0) f(x) \, dx| \le \big(\int W_\theta(x-x_0) |f(x)|^2 \, dx \big)^{1/2}$, and using again the Cauchy–Schwarz inequality, we obtain: $|e^{t\Delta} f(x_0)| \le \big(\frac{2}{t} \int_0^{t/2} \int_{\mathbb{R}^d} W_{t-s}(x - x_0) |e^{s\Delta} f(x)|^2 \, ds \, dx \big)^{1/2}$. We then conclude by writing, for $0 < s < t/2$, $k \in \mathbb{Z}^d$ and $\frac{x-x_0}{\sqrt{t}} \in k + [0,1]^d$ that $W_{t-s}(x - x_0) \le C \frac{1}{t^{d/2}} \frac{1}{1+|k|^{d+1}}$. □

We may give the following useful characterization of $BMO^{(-1)}$:

Proposition 16.1: ($BMO^{(-1)}$ as derivatives of BMO)
i) A distribution f belongs to $bmo^{(-1)}$ if and only if there exists $d+1$ distributions $f_0, \dots, f_d \in bmo$ with $f = f_0 + \sum_{1 \le i \le d} \partial_i f_i$. Moreover, the norm of f in $bmo^{(-1)}$ is equivalent to the minimum of the sums $\sum_{i=0}^d \|f_i\|_{bmo}$ for all decompositions $f = f_0 + \sum_{1 \le i \le d} \partial_i f_i$.
ii) Similarly, a distribution f belongs to $BMO^{(-1)}$ if and only if there exists d distributions $f_1, \dots, f_d \in BMO$ with $f = \sum_{1 \le i \le d} \partial_i f_i$. Moreover, the norm of f in $BMO^{(-1)}$ is equivalent to the minimum of the sums $\sum_{i=1}^d \|f_i\|_{BMO}$ for all decompositions $f = \sum_{1 \le i \le d} \partial_i f_i$.

Proof: We know that a distribution b belongs to BMO if and only if

$$\text{for } j = 1, \dots, d, \qquad \sup_{R>0, x_0 \in \mathbb{R}^d} \frac{1}{R^d} \int\int_{0<t<\sqrt{R}, |x-x_0|<R} |\frac{\partial}{\partial x_j} e^{t\Delta} b|^2 dt \, dx < \infty,$$

thus, if $f = \sum_{1 \le i \le d} \partial_i f_i$ with $f_i \in BMO$, we find that $f \in BMO^{(-1)}$ and that $\|f\|_{BMO^{(-1)}} \le \sum_{1 \le i \le d} \|f_i\|_{BMO}$. A similar result holds for $bmo^{(-1)}$ and a sum $\sum_{1 \le i \le d} \partial_i f_i$ with $f_i \in bmo$. We easily check that $bmo \subset bmo^{(-1)}$: indeed, we write for $f \in bmo$ $f = S_0 f + \sum_{j \ge 0} \Delta_j f$. We have $S_0 f \in L^\infty$; hence, $\|e^{t\Delta} S_0 f\|_\infty \le \|S_0 f\|_\infty$ and this implies that $S_0 f \in bmo^{(-1)}$. On the other hand, we write $\Delta_j f = \sum_{i=1}^d \partial_i \frac{\partial_i}{\Delta} \Delta_j f$ and get $\|\Delta_j f\|_{bmo^{(-1)}} \le C 2^{-j} \|\Delta_j f\|_{bmo} \le C' 2^{-j} \|f\|_{bmo}$.

We now prove the converse implications. Since $BMO^{(-1)} \subset \dot{B}_\infty^{-1,\infty}$, we may write $f = \sum_{1\le i\le d} \partial_i f_i$ with $f_i = \sum_{j<0} f_{i,j} - f_{i,j}(0) + \sum_{j\ge 0} f_{i,j}$ where $f_{i,j} = \Delta_j \frac{\partial_i}{\Delta} f$. We have $f_i \in \dot{B}_\infty^{0,\infty}$. We must check that $f_i \in BMO$. It means (from Theorem 10.1) that we must check for $f \in BMO^{(-1)}$ that

$$\text{for } i,j = 1,\ldots,d, \quad \sup_{R>0, x_0 \in \mathbb{R}^d} \frac{1}{R^d} \int\int_{0<t<R^2, |x-x_0|<R} \left|\frac{\partial_i\partial_j}{\Delta} e^{t\Delta} f\right|^2 dt\ dx < \infty$$

Let ω be a fixed function in $\mathcal{D}(\mathbb{R}^d)$ with $\operatorname{Supp} \omega \subset B(0,1)$ and $\int \omega\ dx = 1$; we define $\omega_R(x) = \frac{1}{R^d}\omega(\frac{x}{R})$. We then write $\frac{\partial_i\partial_j}{\Delta} e^{t\Delta} f = f_R + g_R$ with $g_R = \omega_R * \frac{\partial_i\partial_j}{\Delta} e^{t\Delta} f$ (the convolution is with respect with the x variable in $\mathbb{R}^d$). We have, for fixed t, $\|g_R\|_\infty \le \|\omega_R\|_{\dot{B}_1^{1,1}} \|\frac{\partial_i\partial_j}{\Delta} e^{t\Delta} f\|_{\dot{B}_\infty^{-1,\infty}} \le C\frac{1}{R}\|f\|_{\dot{B}_\infty^{-1,\infty}}$ and this gives obviously that $\int\int_{0<t<R^2, |x-x_0|<R} |g_R(t,x)|^2 dt\ dx \le C R^d \|f\|^2_{\dot{B}_\infty^{-1,\infty}}$. We now estimate f_R. We choose γ a fixed function in $\mathcal{D}(\mathbb{R}^d)$ with $\gamma = 1$ identically on $B(0,10)$ and define $\gamma_{R,x_0}(x) = \gamma((x-x_0)/R)$. We write $f_R = \alpha_{R,x_0} + \beta_{R,x_0}$ where $\beta_{R,x_0} = \frac{\partial_i\partial_j}{\Delta}(\gamma_{R,x_0} e^{t\Delta} f) - \omega_R * \frac{\partial_i\partial_j}{\Delta}(\gamma_{R,x_0} e^{t\Delta} f)$. We have $\int\int_{0<t<R^2, x\in\mathbb{R}^d} |\beta_{R,x_0}(t,x)|^2 dt\ dx \le C \int\int_{0<t<R^2, x\in\mathbb{R}^d} |\gamma_{R,x_0}(t,x) e^{t\Delta} f|^2 dt\ dx \le C' R^d \|f\|^2_{BMO^{(-1)}}$. Finally, we write

$$\alpha_{R,x_0}(t,x) = \int\int \omega_R(x-z)(K_{i,j}(x-y) - K_{i,j}(z,y))(1-\gamma_{R,x_0}(y))(e^{t\Delta} f)(t,y)\, dy\, dz$$

where $K_{i,j}$ is the kernel of $\frac{\partial_i\partial_j}{\Delta}$. We get, for $|x-x_0| < R$, that

$$|\alpha_{R,x_0}(t,x)| \le C \int_{|x_0-y|\ge 10R} \frac{R}{|x_0-y|^{d+1}} |(e^{t\Delta} f)(t,y)|\ dy$$

hence

$$\int_{|x-x_0|<R} |\alpha_{R,x_0}(t,x)|^2\ dx \le C' R^d \int_{|x_0-y|\ge 10R} \frac{R}{|x_0-y|^{d+1}} |(e^{t\Delta} f)(t,y)|^2\ dy$$

and $\int\int_{0<t<R^2, x\in\mathbb{R}^d} |\alpha_{R,x_0}(t,x)|^2 dt\ dx \le C'' R^d \|f\|^2_{BMO^{(-1)}}$. This completes the proof for the case $f \in BMO^{(-1)}$.

For $f \in bmo^{(-1)}$, we may give a similar proof by writing $f = f_0 + \sum_{1\le i\le d} \partial_i f_i$ with $f_0 = (Id-\Delta)^{-1} f$ and $f_i = -\partial_i (Id-\Delta)^{-1} f$. Another way is to write $f = S_0 f + (Id - S_0) f$. We have $S_0 f \in L^\infty \subset bmo$. Moreover, we check that $(Id - S_0) f \in BMO^{(-1)}$, so that we may use the result for $BMO^{(-1)}$ and write $(Id - S_0) f = \sum_{1\le i\le d} \partial_i f_i$ with $f_i \in BMO^{(-1)}$. We have $f_i = (Id - S_0)\frac{\partial_i}{\Delta} f$ and we obviously get that $e^{\Delta} f_i \in L^\infty$, this gives that we have more precisely $f_i \in bmo$ (Theorem 10.4). Thus, we just have to prove that $(Id - S_0) f \in BMO^{(-1)}$, i.e., to prove $\sup_{1<t}\sup_{x_0\in\mathbb{R}^d}\frac{1}{t^{d/2}}\int_0^t\int_{|x-x_0|\leq\sqrt{t}}|e^{s\Delta}(Id-S_0)f(x)|^2\,ds\,dx<\infty$. For $s>1$, we write that $\|\Delta e^{s\Delta}f\|_\infty\leq Cs^{-1}\|f\|_{B_\infty^{-1,\infty}}$, and, since $\|\frac{(Id-S_0)}{\Delta}\|_1<\infty$, we get $\int_1^t\int_{|x-x_0|\leq\sqrt{t}}|e^{s\Delta}(Id-S_0)f(x)|^2\,ds\,dx<Ct^{d/2}\|f\|^2_{B_\infty^{-1,\infty}}\int_1^{\sqrt{t}}\frac{ds}{s^2}<Ct^{d/2}\|f\|^2_{B_\infty^{-1,\infty}}$. For $s<1$, we write

$$\int_0^1\int_{|x-x_0|\leq\sqrt{t}}|e^{s\Delta}(Id-S_0)f|^2ds\,dx<Ct^{d/2}\sup_{z\in\mathbb{R}^d}\int_0^1\int_{|x-z|\leq 1}|e^{s\Delta}(Id-S_0)f|^2ds\,dx$$

and we easily check that $\int_0^1\int_{|x-z|\leq 1}|e^{s\Delta}S_0f|^2ds\,dx\leq C\|f\|^2_{bmo^{(-1)}}$ $\square$

2. Local and global existence of solutions

We may now state the theorem of Koch and Tataru:

Theorem 16.1: (Koch and Tataru [KOCT 01])

a) [Admissible path space] If $\vec{u}$ is a solution of the Navier–Stokes equations on $(0,T)\times\mathbb{R}^d$

$$\begin{cases}\partial_t\vec{u}=\Delta\vec{u}-\mathbb{P}\vec{\nabla}.\vec{u}\otimes\vec{u}\\ \vec{\nabla}.\vec{u}=0\end{cases}$$

so that
i) $\sqrt{t}\,\vec{u}(t,x)\in(L^\infty((0,T)\times\mathbb{R}^d)^d$
ii) $\sup_{t<T,x_0\in\mathbb{R}^d}\frac{1}{t^{d/2}}\int\int_{0<s<t,|x-x_0|<\sqrt{t}}|\vec{u}|^2\,ds\,dx<\infty$
then $\lim_{t\to 0}\vec{u}\in(bmo^{(-1)})^d$.

b) [Local existence] For $T>0$ there exists a $\epsilon_T>0$ such that for all $\vec{u}_0\in(bmo^{(-1)})^d$ with $\|\vec{u}_0\|_{bmo^{(-1)}}<\epsilon_T$ and $\vec{\nabla}.\vec{u}_0=0$ there exists a weak solution $\vec{u}$ of the Navier–Stokes equations on $(0,T)\times\mathbb{R}^d$:

$$\begin{cases}\partial_t\vec{u}=\Delta\vec{u}-\mathbb{P}\vec{\nabla}.\vec{u}\otimes\vec{u}\\ \vec{\nabla}.\vec{u}=0\end{cases}$$

such that
i) $\sqrt{t}\,\vec{u}(t,x)\in(L^\infty((0,T)\times\mathbb{R}^d)^d$
ii) $\sup_{t<T,x_0\in\mathbb{R}^d}\frac{1}{t^{d/2}}\int\int_{0<s<t,|x-x_0|<\sqrt{t}}|\vec{u}|^2\,ds\,dx<\infty$
iii) $\lim_{t\to 0}\vec{u}=\vec{u}_0$.

c) [Regular initial values] If $\vec{u}_0$ belongs to the closure of the test functions $(\mathcal{D}(\mathbb{R}^d))^d$ in $(bmo^{(-1)})^d$, then $\lim_{\lambda\to 0^+}\|\lambda\vec{u}_0(\lambda x)\|_{bmo^{(-1)}}=0$. If $\vec{\nabla}.\vec{u}_0=0$ and $\|\lambda\vec{u}_0(\lambda x)\|_{bmo^{(-1)}}<\epsilon_T$, then the Cauchy problem for the Navier–Stokes equations with $\vec{u}_0$ as initial value has a solution on $(0,\lambda^2T)\times\mathbb{R}^d$.

The proof relies on the following estimates:

Lemma 16.2:

For $\alpha(t,x)$ defined on $(0,1)\times\mathbb{R}^d$, let $A(\alpha)$ be the quantity

$$A(\alpha)=\sup_{x_0\in\mathbb{R}^d}\sup_{0<t<1} t^{-d/2}\int_0^t\int_{|x_0-x|<\sqrt{t}}|\alpha(s,x)|\ ds\ dx$$

Then, the following inequality holds for $\beta(t,x)=\sqrt{-\Delta}e^{t\Delta}\int_0^t\alpha\ ds$:

$$\int_0^1\int_{\mathbb{R}^d}|\beta(s,x)|^2\ ds\ dx\le CA(\alpha)\ \int_0^1\int_{\mathbb{R}^d}|\alpha|\ ds\ dx$$

Proof: We write

$$\begin{gathered}\int_0^1\int_{\mathbb{R}^d}|\beta(t,x)|^2\ dt\ dx=\\ \int_0^1\langle\int_0^t\sqrt{-\Delta}e^{t\Delta}\alpha(s,.)\ ds|\int_0^t\sqrt{-\Delta}e^{t\Delta}\alpha(\sigma,.)\ d\sigma\rangle_{L^2(dx)}\ dt=\\ 2Re\int\int\int_{0<\sigma<s<t<1}\langle\sqrt{-\Delta}e^{t\Delta}\alpha(s,.)|\sqrt{-\Delta}e^{t\Delta}\alpha(\sigma,.)\rangle_{L^2(dx)}\ ds\ d\sigma\ dt=\\ 2Re\int\int_{0<\sigma<s<1}\langle\alpha(s,.)|\int_s^1(-\Delta)e^{2t\Delta}\alpha(\sigma,.)\ dt\rangle_{L^2(dx)}\ ds\ d\sigma=\\ Re\int\int_{0<\sigma<s<1}\langle\alpha(s,.)|(e^{2\Delta}-e^{2s\Delta})\alpha(\sigma,.)\rangle_{L^2(dx)}\ ds\ d\sigma=\\ Re\int_{0<s<1}\langle\alpha(s,.)|\int_{0<\sigma<s}(e^{2\Delta}-e^{2s\Delta})\alpha(\sigma,.)\ d\sigma\rangle_{L^2(dx)}\ ds.\end{gathered}$$

We then write $\gamma=\sup_{0<s\le1}\|\int_0^s e^{2s\Delta}|\alpha(\sigma,.)|\ d\sigma\|_{L^\infty(dx)}$ and we get

$$\int_0^1\int_{\mathbb{R}^d}|\beta(s,x)|^2\ ds\ dx\le 2\gamma\int_0^1\int_{\mathbb{R}^d}|\alpha(t,x)|\ dt\ dx$$

and we conclude by using the kernel W of e^Δ ($W(x)=(4\pi)^{-d/2}e^{-\frac{\|x\|^2}{4}}$):

$$\begin{gathered}\int_0^s e^{2s\Delta}|\alpha(\sigma,.)|\ d\sigma=\int_0^s\int_{\mathbb{R}^d}\frac{1}{(2s)^{d/2}}W(\frac{x-y}{\sqrt{2s}})|\alpha(\sigma,y)|\ dy\ d\sigma=\\ \sum_{k\in\mathbb{Z}^d}\int_0^s\int_{x-\sqrt{s}(k+[0,1]^d)}\frac{1}{(2s)^{d/2}}W(\frac{x-y}{\sqrt{2s}})|\alpha(\sigma,y)|\ dy\ d\sigma\le\\ \sum_{k\in\mathbb{Z}^d}\sup_{z\in k+[0,1]^d}W(\frac{z}{\sqrt{2}})\frac{1}{(2s)^{d/2}}\int_0^s\int_{x-\sqrt{s}(k+[0,1]^d)}|\alpha(\sigma,y)|\ dy\ d\sigma\le CA(\alpha)\end{gathered}$$

□

Lemma 16.3: *For every $T\in(0,+\infty]$, the bilinear operator B defined by*

$$B(\vec{u},\vec{v})=\int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\ ds$$

is continuous from $\mathcal{E}_T^d\times\mathcal{E}_T^d$ to $\mathcal{E}_T^d$, where $\mathcal{E}_T\subset L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d$ is defined by

$$f\in\mathcal{E}_T\Leftrightarrow\left\{\begin{array}{c}\sup_{0<t<T}\sqrt{t}\|f\|_{L^\infty(dx)}<\infty\\ \text{and}\\ \sup_{x_0\in\mathbb{R}^d}\sup_{0<t<T}\left(t^{-d/2}\int_0^t\int_{|x-x_0|<\sqrt{t}}|f(s,x)|^2\ ds\ dx\right)^{1/2}<\infty\end{array}\right.$$

with norm

$$\|f\|_{\mathcal{E}_T} = \sup_{0<t<T} \sqrt{t}\|f\|_\infty + \sup_{x_0\in\mathbb{R}^d}\sup_{0<t<T} \big(t^{-d/2}\int_0^t\int_{|x-x_0|<\sqrt{t}} |f(s,x)|^2\ ds\ dx\big)^{1/2}.$$

Proof: The estimate on the L^∞ norm is easy. We just split the integral defining B in two domains $0<s<t/2$ and $t/2\le s<t$. For $t/2\le s<t$, we just write $\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\| \le C\frac{1}{\sqrt{t-s}}\|\vec{u}\|_\infty\vec{v}\|_\infty \le C\frac{1}{s\sqrt{t-s}}\|\vec{u}\|_{\mathcal{E}_T}\|\vec{v}\|_{\mathcal{E}_T}$. For $0<s<t/2$, we write (using the estimate on the Oseen kernel from Proposition 11.1) that

$$|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})| \le C\int_{\mathbb{R}^d}\frac{1}{(\sqrt{t}+|x-y|)^{d+1}}|\vec{u}(s,y)||\vec{v}(s,y)|\ dy.$$

This gives

$$|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})| \le C\sum_{k\in\mathbb{Z}^d}\frac{\int_{x-y\in\sqrt{t}(k+[0,1]^d)}|\vec{u}(s,y)||\vec{v}(s,y)|\ dy}{t^{(d+1)/2}(1+|k|)^{d+1}}.$$

Since we have

$$\int_0^t\int_{x-y\in\sqrt{t}(k+[0,1]^d)}|\vec{u}(s,y)||\vec{v}(s,y)|\ dy\ ds \le Ct^{d/2}\|\vec{u}\|_{\mathcal{E}_T}\|\vec{v}\|_{\mathcal{E}_T},$$

we obtain the desired estimate.

We now turn to the estimate on the local L^2 norm: we try to estimate for $0<t<T$ and $x_0\in\mathbb{R}^d$ the quantity

$$I = \int_0^t\int_{|x-x_0|\le\sqrt{t}}|B(\vec{u},\vec{v})|^2\ ds\ dx = \int_0^t\int_{|x-x_0|\le\sqrt{t}}\Big|\int_0^s e^{(s-\sigma)\Delta}\mathbb{P}\vec{\nabla}.\vec{u}\otimes\vec{v}\ d\sigma|^2\ ds\ dx.$$

We use the function $\chi_{t,x_0}(y) = 1_{B(x_0,10\sqrt{t})}(y)$ and write $B(\vec{u},\vec{v}) = B_1 - B_2 - B_3$ with

$$\begin{cases} B_1 = \int_0^s e^{(s-\sigma)\Delta}\mathbb{P}\vec{\nabla}.((1-\chi_{t,x_0})\vec{u}\otimes\vec{v})\ d\sigma \\ B_2 = \frac{1}{\sqrt{-\Delta}}\mathbb{P}\vec{\nabla}.\int_0^s e^{(s-\sigma)\Delta}\Delta\big(\frac{1}{\sqrt{-\Delta}}(Id-e^{\sigma\Delta})(\chi_{t,x_0}\vec{u}\otimes\vec{v})\big)\ d\sigma \\ B_3 = \frac{1}{\sqrt{-\Delta}}\mathbb{P}\vec{\nabla}.\sqrt{-\Delta}e^{s\Delta}(\int_0^s\chi_{t,x_0}\vec{u}\otimes\vec{v}\ d\sigma) \end{cases}$$

B_1 is easily estimated for $s<t$ and $|x-x_0|<\sqrt{t}$:

$$\begin{aligned} |B_1| &\le C\int_0^s\int_{|y-x_0|\ge10\sqrt{t}}\frac{1}{(\sqrt{s-\sigma}+|x-y|)^{d+1}}|\vec{u}(\sigma,y)||\vec{v}(\sigma,y)|\ d\sigma \\ &\le C'\int_0^t\int_{|y-x_0|\ge10\sqrt{t}}\frac{1}{|x_0-y|^{d+1}}|\vec{u}(\sigma,y)||\vec{v}(\sigma,y)|\ d\sigma \\ &\le C''\frac{1}{\sqrt{t}}\|\vec{u}\|_{\mathcal{E}_T}\|\vec{v}\|_{\mathcal{E}_T} \end{aligned}$$

which gives $\int_0^t \int_{|x-x_0|\leq\sqrt{t}} |B_1|^2\, ds\, dx \leq C t^{d/2} \|\vec{u}\|_{\mathcal{E}_T}^2 \|\vec{v}\|_{\mathcal{E}_T}^2$.

To estimate B_2 and B_3, we write $M(\sigma,y) = \chi_{t,x_0}(y)\vec{u}(\sigma,y)\otimes\vec{v}(\sigma,y)$. We use the boundedness of the Riesz transforms on $L^2(\mathbb{R}^d)$ (hence, on $L^2_t L^2_x$) to get rid of $\frac{1}{\sqrt{-\Delta}}\mathbb{P}\vec{\nabla}$ and get

$$\int_0^t \int_{\mathbb{R}^d} |B_2|^2\, ds\, dx \leq C \int_0^t \int_{\mathbb{R}^d} \Big| \int_0^s e^{(s-\sigma)\Delta}\Delta\big(\frac{1}{\sqrt{-\Delta}}(Id - e^{\sigma\Delta})M\big)\, d\sigma\Big|^2\, ds\, dx$$

Now, we use the maximal L^2L^2 regularity of the heat kernel (Theorem 7.3) and get

$$\int_0^t \int_{\mathbb{R}^d} |B_2|^2\, ds\, dx \leq C \int_0^t \int_{\mathbb{R}^d} \Big|\frac{1}{\sqrt{-\Delta}}(Id - e^{\sigma\Delta})M\Big|^2\, d\sigma\, dy.$$

Since the function $\lambda \mapsto \frac{1-e^{-\lambda^2}}{\lambda}$ is bounded, we find that $\frac{1}{\sqrt{-\Delta}}(Id - e^{\sigma\Delta})$ operates on $L^2(\mathbb{R}^d)$ with operator norm $O(\sqrt{\sigma})$, which gives

$$\int_0^t\int_{\mathbb{R}^d} |B_2|^2 ds\, dx \leq C \int_0^t\int_{\mathbb{R}^d} \sigma|M|^2 d\sigma\, dy \leq C\|\sigma M\|_{L^\infty((0,t)\times\mathbb{R}^d)}\|\sigma M\|_{L^1((0,t)\times\mathbb{R}^d)}$$

and we easily conclude, since we have $\|\sigma M\|_{L^\infty((0,t)\times\mathbb{R}^d)} \leq C\|\vec{u}\|_{\mathcal{E}_T}\|\vec{v}\|_{\mathcal{E}_T}$ and $\|\sigma M\|_{L^1((0,t)\times\mathbb{R}^d)} \leq C t^{d/2}\|\vec{u}\|_{\mathcal{E}_T}\|\vec{v}\|_{\mathcal{E}_T}$.

Similarly,

$$\int_0^t \int_{\mathbb{R}^d} |B_3|^2\, ds\, dx \leq C \int_0^t \int_{\mathbb{R}^d} \Big|\sqrt{-\Delta} e^{s\Delta}\big(\int_0^s M\, d\sigma\big)\Big|^2\, ds\, dx$$

We make the change of variables $s = t\tau$, $\sigma = t\theta$, $x = \sqrt{t}\, z$, $y = \sqrt{t}\, w$ and we find

$$\int_0^t \int_{\mathbb{R}^d} |B_3|^2\, ds\, dx \leq C t^{2+d/2} \int_0^1 \int_{\mathbb{R}^d} \Big|\sqrt{-\Delta} e^{\tau\Delta}\big(\int_0^\tau N\, d\theta\big)\Big|^2\, d\tau\, dz$$

with $N(\theta,w) = M(t\theta, \sqrt{t}\, w)$. We then use Lemma 16.2 and get

$$\int_0^t \int_{\mathbb{R}^d} |B_3|^2\, ds\, dx \leq C t^{2+d/2} A(N) \|N\|_{L^1((0,1)\times\mathbb{R}^d}.$$

Since $A(N) \leq C t^{-1}\|\vec{u}\|_{\mathcal{E}_T}\|\vec{v}\|_{\mathcal{E}_T}$ and $\|N\|_{L^1((0,1)\times\mathbb{R}^d} \leq C t^{-1}\|\vec{u}\|_{\mathcal{E}_T}\|\vec{v}\|_{\mathcal{E}_T}$, we get the required estimate. □

Proof of Theorem 16.1: Point a) is easy. By assumption, $\vec{u}$ belongs to $\mathcal{E}_T^d$. Through Lemma 16.3, we find that $B(\vec{u},\vec{u})$ belongs to $\mathcal{E}_T^d$. Since $\vec{u}$ is a solution of the Navier–Stokes equations which belongs to $(L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d))^d$, we find that $\vec{u}_0 = \lim_{t\to 0}\vec{u}$ is well defined (Theorem 11.2) and satisfies $e^{t\Delta}\vec{u}_0 = \vec{u} + B(\vec{u},\vec{u})$. Hence, $e^{t\Delta}\vec{u}_0$ belongs to $\mathcal{E}_T^d$ and $\vec{u}_0 \in (bmo^{(-1)})^d$.

Point b) is a direct consequence of Lemma 16.3, due to the Picard contraction principle.

Point c) is quite obvious: we have $e^{t\Delta}(\lambda f(\lambda .))(x) = \lambda(e^{\lambda^2 t\Delta} f)(\lambda x)$; hence,

$$\|\lambda f(\lambda x)\|^2_{bmo^{(-1)}} = \sup_{0<t<\lambda^2} \sup_{x_0 \in \mathbb{R}^d} \frac{1}{t^{d/2}} \int_0^t \int_{|x-x_0|\leq\sqrt{t}} |e^{s\Delta} f(x)|^2 \, ds \, dx$$

Thus, the maps $f \mapsto \lambda f(\lambda .)$ are equicontinuous on $bmo^{(-1)}$ for $\lambda \in (0,1)$; since we have for $f \in L^\infty$ $\|\lambda f(\lambda .)\|_{bmo^{(-1)}} \leq C\lambda$, we see that whenever f belongs to the closure of $\mathcal{D}(\mathbb{R}^d)$ in $bmo^{(-1)}$, then we have $\lim_{\lambda\to 0} \|\lambda f(\lambda .)\|_{bmo^{(-1)}} = 0$. Now, if $T \in (0,\infty)$ and if $\vec{u}_0 \in (bmo^{(-1)})^d$ is such that $\vec{\nabla}.\vec{u}_0 = 0$ and $\lim_{\lambda\to 0} \|\lambda \vec{u}_0(\lambda .)\|_{bmo^{(-1)}} = 0$, we find that for some small enough $\lambda > 0$, we have $\|\lambda \vec{u}_0(\lambda .)\|_{bmo^{(-1)}} < \epsilon_T$; point ii) gives that we have a solution $\vec{v}$ on $(0,T) \times \mathbb{R}^d$ of the Navier–Stokes problem with initial value $\lambda \vec{u}_0(\lambda .)$; then, $\vec{u} = \frac{1}{\lambda}\vec{v}(\frac{t}{\lambda^2}, \frac{x}{\lambda})$ is a solution on $(0, \lambda^2 T) \times \mathbb{R}^d$ of the Navier–Stokes problem with initial value $\vec{u}_0$. □

Existence of global solutions for the Navier–Stokes equations may be proved in the same way, just replacing the "local" space $bmo^{(-1)}$ by the "global space" $BMO^{(-1)}$.

Theorem 16.2: (Global mild solutions)

There exists a positive constant ϵ_∞ so that for all $\vec{u}_0 \in (BMO^{(-1)})^d$ with $\|\vec{u}_0\|_{BMO^{(-1)}} < \epsilon_\infty$ and $\vec{\nabla}.\vec{u}_0 = 0$ there exists a weak solution $\vec{u}$ of the Navier–Stokes equations on $(0,\infty) \times \mathbb{R}^d$:

$$\begin{cases} \partial_t \vec{u} = \Delta\vec{u} - \mathbb{P}\vec{\nabla}.\vec{u} \otimes \vec{u} \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

so that
i) $\sqrt{t}\ \vec{u}(t,x) \in (L^\infty((0,\infty) \times \mathbb{R}^d)^d$
ii) $\sup_{0<t, x_0 \in \mathbb{R}^d} \frac{1}{t^{d/2}} \iint_{0<s<t, |x-x_0|<\sqrt{t}} |\vec{u}|^2 \, ds \, dx < \infty$
iii) $\lim_{t\to 0} \vec{u} = \vec{u}_0$.

Conversely, if $\vec{u}$ is a weak solution of the Navier–Stokes equations on $(0,\infty) \times \mathbb{R}^d$:

$$\begin{cases} \partial_t \vec{u} = \Delta\vec{u} - \mathbb{P}\vec{\nabla}.\vec{u} \otimes \vec{u} \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

so that
i) $\sqrt{t}\ \vec{u}(t,x) \in (L^\infty((0,\infty) \times \mathbb{R}^d)^d$
ii) $\sup_{0<t, x_0 \in \mathbb{R}^d} \frac{1}{t^{d/2}} \iint_{0<s<t, |x-x_0|<\sqrt{t}} |\vec{u}|^2 \, ds \, dx < \infty$
then the limit $\vec{u}_0$ of $\vec{u}$ for $t \to 0$ belongs to $BMO^{(-1)}$.

Proof: From Lemma 16.3, we know that B is bounded from $\mathcal{E}^d_\infty \times \mathcal{E}^d_\infty$ to $\mathcal{E}^d_\infty$ and this will produce the desired result. □

We conclude by describing some properties of the iterates $\vec{u}_{(n)}$ which appear in the Picard algorithm when solving the Navier–Stokes equations for $\vec{u}_0 \in (BMO^{(-1)})^d$:

Corollary: *Let* $\vec{u}_0 \in (BMO^{(-1)})^d$ *with* $\|\vec{u}_0\|_{BMO^{(-1)}} < \epsilon_\infty$ *and* $\vec{\nabla}.\vec{u}_0 = 0$. *Let* $\vec{u}_{(n)}$ *be defined by* $\vec{u}_{(0)} = 0$ *and* $\vec{u}_{(n+1)} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}_{(n)} \otimes \vec{u}_{(n)})\, ds$. *Then the sequence* $\vec{w}_{(n)} = \vec{u}_{(n+1)} - \vec{u}_{(n)}$ *staisfies for some positive constants* C *and* ρ *with* $\rho < 1$ *(which depend on* $\vec{u}_0$ *but not on* n*):*
i) for all $n \in \mathbb{N}$, $\|\vec{w}_{(n)}(t,x)\|_{\mathcal{E}_\infty} \leq C\rho^n$
ii) for all $n \in \mathbb{N}$, $\|\sqrt{t}\ \vec{w}_{(n)}(t,x)\|_{L^\infty((0,\infty)\times\mathbb{R}^d)} \leq C\rho^n$
iii) for all $n \in \mathbb{N}$, $\sup_{t>0} \|\vec{w}_{(n)}(t,x)\|_{BMO^{(-1)}(\mathbb{R}^d)} \leq C\rho^n$.

Proof: i) is of course a direct consequence of the contraction principle. ii) is contained in i). iii) is a consequence of i): indeed, we have that $\vec{w}_{n+1} = -\mathbb{P}\vec{\nabla}.\int_0^t e^{(t-s)\Delta}(\vec{w}_n \otimes \vec{u}_{n+1} + \vec{u}_n \otimes \vec{w}_n)\, ds$. A proof similar to the proof of Lemma 16.3 gives that $(f,g) \mapsto \int_0^t e^{(t-s)\Delta} fg\, ds$ is bounded from $\mathcal{E}_\infty \times \mathcal{E}_\infty$ to $L^\infty((0,\infty)\times\mathbb{R}^d)$. Thus, we find that

$$\sup_{0<t} \|\vec{w}_{(n+1)}\|_{BMO^{(-1)}(\mathbb{R}^d)} \leq C\|\vec{w}_{(n)}\|_{\mathcal{E}_\infty}(\|\vec{u}_{(n)}\|_{\mathcal{E}_\infty} + \|\vec{u}_{(n+1)}\|_{\mathcal{E}_\infty}) \leq C'\rho^{n+1}$$

$\square$

We shall use mainly point ii) in the persistency theorems in the chapters that follow; at times we shall use point iii) as well. We thus introduce the following definition :

Definition 16.2: *Let* $0 < T \leq \infty$. *If* $\vec{u}_0 \in (BMO^{(-1)})^d$, *we say that the iterated sequence defined by* $\vec{u}_{(0)} = 0$ *and* $\vec{u}_{(n+1)} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}_{(n)} \otimes \vec{u}_{(n)})\, ds$ *converges exponentially on* $(0,T)$ *(in* $t^{-1/2}L^\infty$*) if*

$$\limsup_{n\to\infty} \|\sqrt{t}(\vec{u}_{(n+1)} - \vec{u}_{(n)})\|_{L^\infty((0,T)\times\mathbb{R}^d)}^{1/n} < 1.$$

We have moreover exponential convergence in $\dot{B}_\infty^{-1,\infty}$ *if*

$$\limsup_{n\to\infty} \|\vec{u}_{(n+1)} - \vec{u}_{(n)}\|_{L^\infty((0,T),\dot{B}_\infty^{-1,\infty}(\mathbb{R}^d))}^{1/n} < 1.$$

3. Fourier transform, Navier–Stokes, and $BMO^{(-1)}$

For the study of space analyticity of solutions to the Navier–Stokes equations, it may be more convenient to study the Fourier transform of solutions. (See Chapter 24.) We thus consider a version of the Koch and Tataru theorem adapted to the Fourier transform.

Lemma 16.4:

i) Let $T \in (0,\infty]$. Let $\mathcal{E}_T \subset L^2_{uloc,x}L^2_t((0,T)\times \mathbb{R}^d$ be defined by

$$f \in \mathcal{E}_T \Leftrightarrow \begin{cases} \sup_{0<t<T} \sqrt{t}\|f\|_{L^\infty(dx)} < \infty \\ \text{and} \\ \sup_{x_0\in\mathbb{R}^d} \sup_{0<t<T} \left(t^{-d/2}\int_0^t \int_{|x-x_0|<\sqrt{t}} |f(s,x)|^2 \, ds \, dx\right)^{1/2} < \infty \end{cases}$$

with norm

$$\|f\|_{\mathcal{E}_T} = \sup_{0<t<T} \sqrt{t}\|f\|_\infty + \sup_{x_0\in\mathbb{R}^d} \sup_{0<t<T} \left(t^{-d/2}\int_0^t \int_{|x-x_0|<\sqrt{t}} |f(s,x)|^2 \, ds \, dx\right)^{1/2}.$$

Then the norm $\|.\|_{\mathcal{E}_T}$ is equivalent to the norm $\|.\|^*_{\mathcal{E}_T}$ defined by

$$\|f\|^*_{\mathcal{E}_T} = \sup_{0<t<T} \sqrt{t}\|f\|_\infty + \sup_{x_0\in\mathbb{R}^d} \sup_{0<t<T} \left(\int_0^t e^{t\Delta}(|f(s,.)|^2)(x_0)\, ds\right)^{1/2}.$$

ii) Let $f \in \mathcal{E}_T$ be a real-valued function so that the spatial Fourier transform $\hat f(t,\xi)$ is a nonnegative locally integrable function on $(0,T)\times\mathbb{R}^d$. Then

$$\begin{cases} \|f\|^*_{\mathcal{E}_T} = \frac{1}{(2\pi)^d} \sup_{0<t<T} \sqrt{t}\|\hat f\|_{L^1(d\xi)} + N_T(f) \\ \text{with} \\ N_T(f) = \frac{1}{(2\pi)^{2d}} \sup_{0<t<T} \left(\int_0^t \int\int e^{-t|\xi|^2} \hat f(s,\xi-\eta)\hat f(s,\eta)(x_0)\, ds\, d\xi\, d\eta\right)^{1/2} \end{cases}$$

iii) Let $\mathcal{F}_T$ be the space of real-valued $f\in\mathcal{E}_T$ so that the spatial Fourier transform $\hat f(t,\xi)$ is a locally integrable function on $(0,T)\times\mathbb{R}^d$ and so that $\mathcal{F}^{-1}|\hat f(t,\xi)| \in \mathcal{E}_T$. Then $\|f\|^*_{\mathcal{E}_T} \le \|\mathcal{F}^{-1}|\hat f(t,\xi)|\|^*_{\mathcal{E}_T}$. Moreover, the map $f \mapsto \|\mathcal{F}^{-1}|\hat f(t,\xi)|\|^*_{\mathcal{E}_T}$ is a norm on $\mathcal{F}_T$, which is then a Banach space.

Proof: i) is easy. ii) is a simple consequence of the fact that when $\hat f(\xi)$ is a nonnegative locally integrable function, then f is bounded if and only if $\hat f$ is integrable (and, then, $\|f\|_\infty = \frac{1}{(2\pi)^d}\|\hat f\|_1$).

iii) is easy as well. The only point to check is the triangle inequality for the norm $\|f\|_{\mathcal{F}_T} = \|\mathcal{F}^{-1}|\hat f(t,\xi)|\|^*_{\mathcal{E}_T}$. But we have $|\hat f(t,\xi)+g(t,\xi)| \le |\hat f(t,\xi)| + |\hat g(t,\xi)|$ and point ii) then gives that

$$\|\mathcal{F}^{-1}(|\hat f(t,\xi)+\hat g(t,\xi)|)\|^*_{\mathcal{E}_T} \le \|\mathcal{F}^{-1}(|\hat f(t,\xi)|+|\hat g(t,\xi)|)\|^*_{\mathcal{E}_T}$$

and we conclude by using the triangle inequality on $\|.\|^*_{\mathcal{E}_T}$. □

Lemma 16.5: *i) The bilinear operator $\mathcal{A}(f,g) = \int_0^t \sqrt{-\Delta}e^{(t-s)\Delta}(fg)\, ds$ is bounded from $\mathcal{E}_T\times\mathcal{E}_T$ to $\mathcal{E}_T$ and from $\mathcal{F}_T\times\mathcal{F}_T$ to $\mathcal{F}_T$.*

ii) The bilinear operator $B(\vec u,\vec v) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec\nabla.\vec u\otimes\vec v\, ds$ is bounded from $\mathcal{F}_T^d\times\mathcal{F}_T^d$ to $\mathcal{F}_T^d$.

Proof: The proof of the boundedness of $\mathcal{A}$ on $\mathcal{E}_T$ follows the same line as the proof of Lemma 16.3. (The same estimates are valid on the kernel $\sqrt{-\Delta}e^{(t-s)\Delta}$ of $\mathcal{A}$ and on the kernel $e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.$ of B.) Since $\mathcal{A}$ has a nonnegative symbol:

$$\mathcal{F}\mathcal{A}(f,g)(t,\xi) = \frac{|\xi|}{(2\pi)^d}\int_0^t \int e^{-(t-s)|\xi|^2}\hat{f}(s,\xi-\eta)\hat{g}(s,\eta)\ ds\ d\eta$$

(where $\mathcal{F}$ is the spatial Fourier transform), the boundedness on $\mathcal{F}_T$ is a direct consequence of the boundedness on $\mathcal{E}_T$:

$$|\mathcal{F}\mathcal{A}(f,g)| \leq \mathcal{F}\mathcal{A}(\mathcal{F}^{-1}(|\mathcal{F}f|), \mathcal{F}^{-1}(|\mathcal{F}g|))$$

and thus

$$\begin{array}{rcl} \|\mathcal{A}(f,g)\|_{\mathcal{F}_T} & \leq & \|\mathcal{A}(\mathcal{F}^{-1}(|\mathcal{F}f|), \mathcal{F}^{-1}(|\mathcal{F}g|))\|_{\mathcal{F}_T} \\ & = & \|\mathcal{A}(\mathcal{F}^{-1}(|\mathcal{F}f|), \mathcal{F}^{-1}(|\mathcal{F}g|))\|^*_{\mathcal{E}_T} \\ & \leq & C\|\mathcal{F}^{-1}(|\mathcal{F}f|)\|^*_{\mathcal{E}_T}\|\mathcal{F}^{-1}(|\mathcal{F}g|))\|^*_{\mathcal{E}_T} \\ & = & C\|f\|_{\mathcal{F}_T}\|g\|_{\mathcal{F}_T} \end{array}$$

.

The Riesz transforms (in space variable x) operate boundedly on $\mathcal{F}_T$ (since $|\mathcal{F}(R_j f)(t,\xi)| \leq |\hat{f}(t,\xi)|$). Since B is composed from the scalar operator $\mathcal{A}$ and of Riesz transforms, it is obvious that B is bounded on $\mathcal{F}_T^d$. □

Theorem 16.3:

a) [Local existence] For $0 < T < \infty$ *there exists a* $\eta_T > 0$ *so that for all* $\vec{u}_0 \in (bmo^{(-1)})^d$ *with* $\mathcal{F}^{-1}(|\mathcal{F}\vec{u}_0|) \in bmo^{(-1)}$ *and with* $\|\mathcal{F}^{-1}(|\mathcal{F}\vec{u}_0|)\|_{bmo^{(-1)}} < \eta_T$ *and* $\vec{\nabla}.\vec{u}_0 = 0$, *there exists a weak solution* $\vec{u}$ *of the Navier–Stokes equations on* $(0,T)\times\mathbb{R}^d$:

$$\begin{cases} \partial_t \vec{u} = \Delta\vec{u} - \mathbb{P}\vec{\nabla}.\vec{u}\otimes\vec{u} \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

so that
i) $\sqrt{t}\ \mathcal{F}\vec{u}(t,\xi) \in (L^\infty_t L^1_\xi((0,T)\times\mathbb{R}^d)^d$
ii) $\sup_{t<T}\int_0^t\int\int e^{-t|\xi|^2}|\mathcal{F}\vec{u}(s,\xi-\eta)||\mathcal{F}\vec{u}(s,\eta)|\ ds\ d\xi\ d\eta < \infty$
iii) $\lim_{t\to 0}\vec{u} = \vec{u}_0$.

B) [Global existence] A similar result holds for global solutions : there exists a $\eta_\infty > 0$ *so that for all* $\vec{u}_0 \in (BMO^{(-1)})^d$ *with* $\mathcal{F}^{-1}(|\mathcal{F}\vec{u}_0|) \in BMO^{(-1)}$ *and with* $\|\mathcal{F}^{-1}(|\mathcal{F}\vec{u}_0|)\|_{BMO^{(-1)}} < \eta_\infty$ *and* $\vec{\nabla}.\vec{u}_0 = 0$, *there exists a weak solution* $\vec{u}$ *of the Navier–Stokes equations on* $(0,\infty)\times\mathbb{R}^d$:

$$\begin{cases} \partial_t \vec{u} = \Delta\vec{u} - \mathbb{P}\vec{\nabla}.\vec{u}\otimes\vec{u} \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

so that
j) $\sqrt{t}\ \mathcal{F}\vec{u}(t,\xi) \in (L^\infty_t L^1_\xi((0,\infty)\times\mathbb{R}^d)^d$
jj) $\sup_{t>0}\int_0^t\int\int e^{-t|\xi|^2}|\mathcal{F}\vec{u}(s,\xi-\eta)||\mathcal{F}\vec{u}(s,\eta)|\ ds\ d\xi\ d\eta < \infty$
jjj) $\lim_{t\to 0}\vec{u} = \vec{u}_0$.

Proof: This is direct application of the Picard contraction principle according to Lemma 16.5. □

Chapter 17

Generalization of the L^p theory: Navier–Stokes and local measures

The results of Chapter 15 may be generalized in a direct way to the setting of spaces of local measures, such as Morrey–Campanato spaces or Lorentz spaces.

1. Shift-invariant spaces of local measures

We consider in this chapter a special class of shift-invariant spaces of distributions: the spaces which are invariant under pointwise multiplication by $\mathcal{C}_0$ functions. We have introduced those spaces in Chapter 4 as the shift-invariant spaces of local measures. Let us recall first the definition of shift-invariant spaces:

Definition 17.1: (shift-invariant Banach spaces of distributions and of local measures)

A) A shift-invariant Banach space of test functions is a Banach space E so that we have the continuous embeddings $\mathcal{D}(\mathbb{R}^d) \subset E \subset \mathcal{D}'(\mathbb{R}^d)$ and so that:
(a) for all $x_0 \in \mathbb{R}^d$ and for all $f \in E$, $f(x-x_0) \in E$ and $\|f\|_E = \|f(x-x_0)\|_E$.
(b) for all $\lambda > 0$ there exists $C_\lambda > 0$ so that for all $f \in E$ $f(\lambda x) \in E$ and $\|f(\lambda x)\|_E \leq C_\lambda \|f\|_E$.
(c) $\mathcal{D}(\mathbb{R}^d)$ is dense in E

B) A shift-invariant Banach space of distributions is a Banach space E, which is the topological dual of a shift-invariant Banach space of test functions $E^{(*)}$. The space $E^{(0)}$ of smooth elements of E is defined as the closure of $\mathcal{D}(\mathbb{R}^d)$ in E.

C) A shift-invariant Banach space of local measures is a shift-invariant Banach space of distributions E so that for all $f \in E$ and all $g \in \mathcal{S}(\mathbb{R}^d)$ we have $fg \in E$ and $\|fg\|_E \leq C_E \|f\|_E \|g\|_\infty$, where C_E is a positive constant (which depends neither on f nor on g).

We recall that we proved the following useful lemma on shift-invariant spaces of local measures:

Lemma 17.1: *Let E be a shift-invariant Banach space of local measures. Then:*

a) the elements of E are local measures (i.e., are distributions of order 0). More precisely, they belong to the Morrey space M^1_{uloc} of uniformly locally finite measures.

b) E, its predual $E^{(*)}$ and the space of smooth elements $E^{(0)}$ in E are stable under pointwise multiplication by bounded continuous functions.

We now prove an easy result on real interpolation on local measures:

Proposition 17.1: [Real interpolation on local measures]

Let E be a shift-invariant space of local measures. Let us define $E^{(1/2)} = [E, L^\infty(\mathbb{R}^d)]_{1/2,1}$ and $E^{(1/2,*)} = [E, \mathcal{C}_b(\mathbb{R}^d)]_{1/2,1}$. Then
i) $E^{(1/2)}$ is a shift-invariant space of local measures and is continuously embedded into $L^2_{loc}(\mathbb{R}^d)$. Moreover, if $\mathcal{D}(\mathbb{R}^d) \subset E$ with continuous embedding and dense range, then $E^{(1/2)} = E^{(1/2,*)}$.
ii) If $f \in E^{(1/2,*)}$ and $g \in E^{(1/2,*)}$, then we have $fg \in E$ and $\|fg\|_E \leq C\|f\|_{E^{(1/2,*)}}\|g\|_{E^{(1/2,*)}}$. Moreover, we may replace $E^{(1/2,*)}$ by $E^{(1/2)}$ if $\mathcal{D}(\mathbb{R}^d) \subset E$ with continuous embedding and dense range or if pointwise multiplication by L^∞ functions operate boundedly on E.

Proof:

To prove point i), let $E^{(*)}$ be the pre-dual of E. We have $E^{(1/2)} = ([E^{(*)}, L^1]_{1/2,c_0})'$ (where $[A,B]_{\theta,c_0}$ is the space of elements of $A+B$, which may be decomposed into $f = \sum_{k\in\mathbb{Z}} f_k$ with $\sup_{k\in\mathbb{Z}} \max(2^{-k\theta}\|f_k\|_A, 2^{k(1-\theta)}\|f_k\|_B) < \infty$ and $\lim_{|k|\to\infty} \sup_{k\in\mathbb{Z}} \max(2^{-k\theta}\|f_k\|_A, 2^{k(1-\theta)}\|f_k\|_B) = 0$). Thus, $E^{(1/2)}$ is a shift-invariant Banach space of local measures.

ii) is very easy. If $f \in E^{(1/2,*)}$ and $g \in E^{(1/2,*)}$, we may write $f = \sum_{k\in\mathbb{N}} \lambda_k f_k$ with $f_k \in E \cap \mathcal{C}_b$, $\|f_k\|_E \, \|f_k\|_\infty \leq 1$ and $\sum_{k\in\mathbb{N}} |\lambda_k| \leq C\|f\|_{E^{(1/2,*)}}$, and the same for $g = \sum_{k\in\mathbb{N}} \mu_k g_k$. Then, we write $\|f_k g_p\|_E \leq C\|f_k\|_E\|g_p\|_\infty$ and $\|f_k g_p\|_E \leq C\|f_k\|_\infty\|g_p\|_E$, thus $\|f_k g_p\|_E \leq C\sqrt{\|f_k\|_E\|g_p\|_\infty\|f_k\|_E\|g_p\|_\infty} \leq C$ and $\|fg\|_E \leq C(\sum_{k\in\mathbb{N}} |\lambda_k|)(\sum_{p\in\mathbb{N}} |\mu_p|)$. □

We now list examples of shift-invariant Banach spaces of local measures:
i) Lebesgue spaces: $E = L^p(\mathbb{R}^d)$ for $1 < p \leq \infty$; we then have $E^{(*)} = L^q$ with $1/p + 1/q = 1$; moreover, $E^{(0)} = L^p$ when $p < \infty$ and $E^{(0)} = \mathcal{C}_0(\mathbb{R}^d)$ for $p = \infty$.
ii) bounded Borel measures: $E = M(\mathbb{R}^d)$; we have $E^{(*)} = \mathcal{C}_0(\mathbb{R}^d)$ and $E^{(0)} = L^1$.
iii) Morrey spaces of uniformly locally L^p functions: $E = L^p_{uloc}(\mathbb{R}^d)$ is defined for $1 < p < \infty$ by $\sup_{x_0\in\mathbb{R}^d} \int_{|x-x_0|<1} |f(x)|^p \, dx < \infty$; the predual of E is then the Wiener space $E^{(*)} = WL^q$ with $1/p + 1/q = 1$, defined by $\sum_{k\in\mathbb{Z}^d} \left(\int_{[0,1]^d} |f(x+k)|^q \, dx \right)^{1/q} < \infty$; moreover, $E^{(0)} = E^p$ is the subspace of L^p_{uloc} characterized by $\lim_{x_0\to\infty} \int_{|x-x_0|<1} |f(x)|^p \, dx = 0$.
iv) Morrey–Campanato spaces $M^{p,q}$ $(1 < p \leq q < \infty)$ of locally L^p functions f so that

$$\sup_{x_0\in\mathbb{R}^d} \sup_{0<R\leq 1} R^{d(1/q-1/p)} \left(\int_{|x-x_0|<R} |f(x)|^p \, dx\right)^{1/p} < \infty;$$

the predual of E is then the space of functions f, which may be decomposed as a series $\sum_{n\in\mathbb{N}} \lambda_n f_n$ with f_n supported by a ball $B(x_n, R_n)$, $R_n \leq 1$, $f_n \in L^{p/(p-1)}$, $\|f_n\|_{p/(p-1)} \leq R_n^{d(1/q-1/p)}$, and $\sum_{n\in\mathbb{N}} |\lambda_n| < \infty$; moreover, $E^{(0)}$ is the subspace of $M^{p,q}$ characterized by $\lim_{x_0\to\infty} \int_{|x-x_0|<1} |f(x)|^p \, dx = 0$ and $\lim_{R\to 0} \sup_{x_0\in\mathbb{R}^d} R^{d(1/q-1/p)} (\int_{|x-x_0|<R} |f(x)|^p \, dx)^{1/p} = 0$.
v) We consider as well Morrey spaces based on Lorentz spaces: $E = ML^{p,\infty}(\mathbb{R}^d)$ is defined for $1 < p < \infty$ by $\sup_{x_0\in\mathbb{R}^d} \|1_{[0,1]^d}(x - x_0) \, f(x)\|_{L^{p,\infty}} < \infty$; the predual of E is then the Wiener space $E^{(*)} = WL^{q,1}$ with $1/p + 1/q = 1$, defined by $\sum_{k\in\mathbb{Z}^d} \|1_{[0,1]^d}(x - k) \, f(x)\|_{L^{q,1}} < \infty$.
v) In Chapter 21, we shall consider the multiplier space X_r defined for $0 \leq r < d/2$ as the space of functions that are locally square-integrable on $\mathbb{R}^d$ and such that pointwise multiplication with these functions maps boundedly $H^r(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. The norm of X_r is given by the operator norm of pointwise multiplication:

$$\|f\|_{X_r} = \sup\{ \, \|fg\|_2 \; / \; \|g\|_{H^r} \leq 1\}.$$

The predual of X_r is then the space of functions f, which may be decomposed as a series $\sum_{n\in\mathbb{N}} \lambda_n f_n g_n$, with $\|f_n\|_2 \leq 1$, $\|g_n\|_{H^1} \leq 1$ and $\sum_{n\in\mathbb{N}} |\lambda_n| < \infty$.

2. Kato's theorem for local measures: the direct approach

Now, it is quite obvious how Kato's theorem may be generalized to shift-invariant Banach spaces of local measures. We assign a useful criterion for the boundedness of the bilinear operator B on $(L^\infty((0,T), E))^d$:

Theorem 17.1: [Local measures and Navier–Stokes]
Let E be a shift-invariant Banach space of local measures. Assume that E is continuously embedded into $L^2_{loc}(\mathbb{R}^d)$ and that there exists a a shift-invariant Banach space of local measures F so that the pointwise product maps boundedly $E \times E$ to F: for $f \in E$ and $g \in E$, $\|fg\|_F \leq C\|f\|_E\|g\|_E$. Then:
A) If, moreover,
i) F is boundedly embedded into $B_\infty^{-\sigma,\infty}(\mathbb{R}^d)$ for some $\sigma < 2$
ii) $[F, \mathcal{C}_b(\mathbb{R}^d)]_{1/2,1} \subset E$: for $f \in F \cap \mathcal{C}_b$, $\|f\|_E \leq C\sqrt{\|f\|_F\|f\|_\infty}$
then, defining $\mathcal{E}_T = L^\infty((0,T), E^d)$, the bilinear operator B defined by $B(\vec{u}, \vec{v}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v}) \, ds$ is bounded from $\mathcal{E}_T \times \mathcal{E}_T$ to $\mathcal{E}_T$:

$$\|B(\vec{u}, \vec{v})\|_{\mathcal{E}_T} \leq CT^{\frac{1}{2}-\frac{\sigma}{4}}(1+T)^{\frac{\sigma}{4}}\|\vec{u}\|_{\mathcal{E}_T}\|\vec{v}\|_{\mathcal{E}_T}$$

*for a positive constant C that does not depend on T. Moreover, $B(\vec{u}, \vec{v}) \in \mathcal{C}((0,T], E^d)$ and converges *-weakly to 0 as t goes to 0.*

Thus, for all $\vec{u}_0 \in (E^d)$ (with $\vec{\nabla}.\vec{u}_0 = 0$), the initial value problem for the Navier–Stokes equations with initial data $\vec{u}_0$ has a solution $\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u}, \vec{u})$ with $\vec{u} \in \mathcal{E}_T$ with $T \geq \min(1, \gamma_E\|\vec{u}_0\|^{\frac{4}{2-\sigma}})$.

B) Similarly, if
i) F is boundedly embedded into $B_\infty^{-2,\infty}(\mathbb{R}^d)$
ii) $[F, \mathcal{C}_b(\mathbb{R}^d)]_{1/2,\infty} \subset E$
then, defining $\mathcal{E}_T = L^\infty((0,T), E^d)$*, the bilinear operator* B *defined by* $B(\vec{u},\vec{v}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\, ds$ *is bounded from* $\mathcal{E}_T \times \mathcal{E}_T$ *to* $\mathcal{E}_T$*:*

$$\|B(\vec{u},\vec{v})\|_{\mathcal{E}_T} \leq C(1+T)\|\vec{u}\|_{\mathcal{E}_T}\|\vec{v}\|_{\mathcal{E}_T}$$

for a positive constant C *that does not depend on* T*. Moreover,* $B(\vec{u},\vec{v}) \in \mathcal{C}((0,T], E^d)$ *and converges *-weakly to* 0 *as* t *goes to* 0*.*

Thus, there exists a positive constant γ_E *so that for all* $\vec{u}_0 \in (E^d)$ *(with* $\vec{\nabla}.\vec{u}_0 = 0$*) with* $\|\vec{u}_0\|_E < \gamma_E$*, the initial value problem for the Navier–Stokes equations with initial data* $\vec{u}_0$ *has a solution* $\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u},\vec{u})$ *with* $\vec{u} \in \mathcal{E}_T$ *with* $T = 1$*.*

C) Similarly, if
i) F is boundedly imbedded into $\dot{B}_\infty^{-2,\infty}(\mathbb{R}^d)$
ii) $[F, \mathcal{C}_b(\mathbb{R}^d)]_{1/2,\infty} \subset E$
then, then, defining $\mathcal{E}_\infty = L^\infty((0,\infty), E^d)$*, the bilinear operator* B *defined by* $B(\vec{u},\vec{v}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\, ds$ *is bounded from* $\mathcal{E}_\infty \times \mathcal{E}_\infty$ *to* $\mathcal{E}_\infty$*:*

$$\|B(\vec{u},\vec{v})\|_{\mathcal{E}_\infty} \leq C\|\vec{u}\|_{\mathcal{E}_\infty}\|\vec{v}\|_{\mathcal{E}_\infty}$$

for a positive constant C*.* $B(\vec{u},\vec{v}) \in \mathcal{C}_b((0,\infty], E^d)$ *and converges *-weakly to* 0 *as* t *goes to* 0*.*

Thus, there exists a positive constant γ_E *so that for all* $\vec{u}_0 \in (E^d)$ *(with* $\vec{\nabla}.\vec{u}_0 = 0$*) with* $\|\vec{u}_0\|_E < \gamma_E$*, the initial value problem for the Navier–Stokes equations with initial data* $\vec{u}_0$ *has a global solution* $\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u},\vec{u})$ *with* $\vec{u} \in \mathcal{E}_\infty$*.*

Proof: A) We just write $\|e^{\frac{(t-s)}{2}\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\|_F \leq C\frac{1}{\sqrt{t-s}}\|u\|_E\|v\|_E$, which gives

$$\begin{cases} \|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\|_F \leq C\frac{1}{\sqrt{t-s}}\|u\|_E\|v\|_E \\ \|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\|_\infty \leq C\frac{1}{\sqrt{t-s}}\left(\frac{|t-s|}{1+|t-s|}\right)^{-\sigma/2}\|u\|_E\|v\|_E \end{cases}$$

hence,

$$\begin{aligned} \|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\|_E &\leq C\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\|_{[F,\mathcal{C}_b(\mathbb{R}^d)]_{1/2,1}} \\ &\leq C'\frac{1}{\sqrt{t-s}}\left(\frac{|t-s|}{1+|t-s|}\right)^{-\sigma/4}\|\vec{u}\|_E\|\vec{v}\|_E \end{aligned}$$

and thus, for $0 < t < T$,

$$\|\int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\, ds\|_E \leq CT^{\frac{1}{2}-\frac{\sigma}{4}}(1+T)^{\frac{\sigma}{4}}\sup_{0<s<T}\|\vec{u}\|_E\sup_{0<s<T}\|\vec{v}\|_E.$$

This is enough to grant that we have a solution in $L^\infty((0,T),E^d)$ as soon as T is small enough: $T \leq \min(1, \gamma \|\vec{u}_0\|^{\frac{4}{2-\sigma}})$ where γ depends only on E.

B) We split the integral $I = \int_0^t \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\, ds$ into $G_A + H_A$ with

$$\begin{cases} G_A = \int_0^A e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\, ds \\ H_A = \int_A^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\, ds \end{cases}$$

We then use the inequalities

$$\begin{cases} \|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\|_F \leq C\frac{1}{\sqrt{t-s}}\|u\|_E\|v\|_E \\ \|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\|_\infty \leq C\frac{1}{\sqrt{t-s}}\left(\frac{|t-s|}{1+|t-s|}\right)^{-1}\|u\|_E\|v\|_E \end{cases}$$

We obtain, for $0 < t < T$, $\|G_A\|_F \leq CA^{1/2}\sup_{0<s<T}\|\vec{u}\|_E \sup_{0<s<T}\|\vec{v}\|_E$ and $\|H_A\|_\infty \leq C(1+T)A^{-1/2}\sup_{0<s<T}\|\vec{u}\|_E\sup_{0<s<T}\|\vec{v}\|_E$. Since such splitting may be done for any positive A, we get that $I \in ([F, \mathcal{C}_b]_{1/2,\infty})^d \subset E^d$.

C) is proved in the same way, using the inequalities

$$\begin{cases} \|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\|_F \leq C\frac{1}{\sqrt{t-s}}\|u\|_E\|v\|_E \\ \|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\|_\infty \leq C\frac{1}{|t-s|^{3/2}}\|u\|_E\|v\|_E \end{cases}$$

□

Examples: (A) may be applied to the Lebesgue spaces L^p, $p > d$, to the Lorentz spaces $L^{p,q}$, $d < p < \infty$, $1 \leq q \leq \infty$, to the Morrey–Campanato spaces $M^{p,q}$, $2 \leq p \leq q$ and $d < q$ and to the multiplier spaces X_r, $0 < r < 1$. (B) may be applied to the spaces $L^{d,\infty}$ and $ML^{d,\infty}$ and (C) to the space $L^{d,\infty}$. □

3. Kato's theorem for local measures: the role of $B_\infty^{-1,\infty}$

We generalize Kato's theorem to shift-invariant Banach spaces of local measures by using the regularizing effect of the heat kernel:

Theorem 17.2: [The second theorem on local measures and Navier–Stokes]

Let E be a shift-invariant Banach space of local measures.

*a) Let $\mathcal{E}_E = L^\infty((0,1),E^d)$ and $\mathcal{B}_\infty = \{\vec{u} \,/\, \sqrt{t}\,\vec{u} \in (L^\infty((0,T),\mathcal{C}_b(\mathbb{R}^d))^d\}$. The bilinear operator B defined by $B(\vec{u},\vec{v}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\, ds$ is bounded from $\mathcal{E}_E \times \mathcal{B}_\infty$ to $\mathcal{E}_E$ and from $\mathcal{B}_\infty \times \mathcal{E}_E$ to $\mathcal{E}_E$. Moreover, $B(\vec{u},\vec{v}) \in \mathcal{C}((0,1],E^d)$ and converges *-weakly to 0 as t goes to 0. If $\lim_{t\to 0}\sqrt{t}\|\vec{u}\|_\infty = 0$, if $\vec{u} \in \mathcal{B}_\infty$ and $\vec{v} \in \mathcal{E}_E$, then the convergence is strong.*

b) If E is continuously embedded in the Besov space $B_\infty^{-1,\infty}$, then the bilinear operator B is bounded as well from $(\mathcal{E}_E \cap \mathcal{B}_\infty) \times \mathcal{B}_\infty$ to $\mathcal{B}_\infty$ and from $\mathcal{B}_\infty \times (\mathcal{E}_E \cap \mathcal{B}_\infty)$ to $\mathcal{B}_\infty$. Hence, there exists a constant $\gamma_E > 0$ so that for all $\vec{u}_0 \in (E^d)$ (with $\vec{\nabla}.\vec{u}_0 = 0$) with $\|\vec{u}_0\|_E \leq \gamma_E$ then the initial value problem for the Navier–Stokes equations with initial data $\vec{u}_0$ has a solution $\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u}, \vec{u})$ with $\vec{u} \in \mathcal{E}_E \cap \mathcal{B}_\infty$.

c) If E is continuously embedded in the Besov space $B_\infty^{-1,\infty}$, then there exists a positive constant C_E so that, defining for $\vec{u}_0 \in E^d$ and $T > 0$ the semi-norm $N_\infty^T(\vec{u}_0) = \sup_{0<t<T} \sqrt{t}\ \|e^{t\Delta}\vec{u}_0\|_\infty$, the initial value problem for the Navier–Stokes equations with initial data $\vec{u}_0$ has a solution $\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u}, \vec{u})$ in $L^\infty((0,T), E^3)$ with $\sup_{0<t<T} \sqrt{t}\ \|\vec{u}\|_\infty < \infty$ for all $T \in (0,1]$ and all $\vec{u}_0 \in E^3$ with

$$C_E\ N_\infty^T(\vec{u}_0)\ \left(1 + \ln^+\left(\frac{C_E\|\vec{u}_0\|_E}{N_\infty^T(\vec{u}_0)}\right)\right) < 1.$$

Thus, if E is continuously embedded in the Besov space $B_\infty^{-\alpha,\infty}$ for some $\alpha < 1$ or if $\vec{u}_0$ is a smooth element of E^d (i.e. $\vec{u}_0 \in (E^{(0)})^d$), then the initial value problem for the Navier–Stokes equations with initial data $\vec{u}_0$ has a local solution.

*This solution $\vec{u}$ belongs to $\mathcal{C}((0,1], E^d)$ and converges *-weakly to $\vec{u}_0$ as t goes to 0; moreover, this convergence is strong if $\vec{u}_0$ is a smooth element of $(E^{(0)})^d$.*

d) If E is continuously embedded in the Besov space $\dot{B}_\infty^{-1,\infty}$, then there exists a constant $\delta_E > 0$ so that for all $\vec{u}_0 \in (E^d)$ (with $\vec{\nabla}.\vec{u}_0 = 0$) with $\|\vec{u}_0\|_E \leq \gamma_E$ then the initial value problem for the Navier–Stokes equations with initial data $\vec{u}_0$ has a global solution $\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u}, \vec{u})$ in $L^\infty((0,\infty), E^d)$ with $\sup_{0<t} \sqrt{t}\ \|\vec{u}\|_\infty < \infty$. Defining for $\vec{u}_0 \in E^d$ the seminorm $N_\infty(\vec{u}_0) = \sup_{0<t} \sqrt{t}\ \|e^{t\Delta}\vec{u}_0\|_\infty$, the same result holds for all $\vec{u}_0 \in E^d$ so that

$$D_E\ N_\infty(\vec{u}_0)\ \left(1 + \ln^+\left(\frac{C_E\|\vec{u}_0\|_E}{N_\infty(\vec{u}_0)}\right)\right) < 1$$

where D_E is a positive constant.

Remark: A sufficient condition for E to be continuously embedded in the Besov space $B_\infty^{-\alpha,\infty}$ (for some $\alpha \geq 0$) is the existence of a constant $C \geq 0$ so that for all $f \in E$ we have $\sup_{\lambda \leq 1} \lambda^\alpha \|f(\lambda x)\|_E \leq C\ \|f\|_E$.

Similarly, a sufficient condition for E to be continuously embedded in the Besov space $\dot{B}_\infty^{-1,\infty}$ is the existence of a constant $C \geq 0$ such that for all $f \in E$ we have $\sup_{\lambda > 0} \lambda \|f(\lambda x)\|_E \leq C\ \|f\|_E$.

Proof: The boundedness of B from $\mathcal{E}_E \times \mathcal{B}_\infty$ to $\mathcal{E}_E$ and from $\mathcal{B}_\infty \times \mathcal{E}_E$ to $\mathcal{E}_E$ is obvious, since $\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\|_{\mathcal{L}(E^{d\times d}, E^d)} \leq C\frac{1}{\sqrt{t-s}}$ (according to the estimate on the Oseen kernel (see Chapter 11) and the invariance of E under convolution with L^1 kernels). Thus, we get

$$\|B(\vec{f}, \vec{g})\|_E \leq C \int_0^t \frac{1}{\sqrt{t-s}\sqrt{s}}\ ds\ \sup_{0<s<t} \|\vec{f}\|_E\ \sup_{0<s<t} \sqrt{s}\|\vec{g}\|_\infty$$

which gives the result, since

$$\int_0^t \frac{1}{\sqrt{t-s}\sqrt{s}}\,ds = \int_0^1 \frac{1}{\sqrt{1-s}\sqrt{s}}\,ds = \pi.$$

We now prove more precisely that $B(\vec{f},\vec{g}) \in (\mathcal{C}((0,1),E))^d$ and that we have, for $0 < t_1 < t_2 < 1$,

$$\|B(\vec{f},\vec{g})(t_1) - B(\vec{f},\vec{g})(t_2)\|_E \le C\sqrt{\frac{t_2-t_1}{t_1}} \sup_{0<s<t_2} \|\vec{f}\|_E \sup_{0<s<t_2} \sqrt{s}\|\vec{g}\|_\infty.$$

We write, for $s < t_1$, $e^{(t_2-s)\Delta} - e^{(t_1-s)\Delta} = \int_0^{t_2-t_1} e^{\theta\Delta}\,d\theta\ \Delta e^{(t_1-s)\Delta}$, which gives

$$\|(e^{(t_2-s)\Delta} - e^{(t_1-s)\Delta})\mathbb{P}\vec{\nabla}.\|_{\mathcal{L}(E^{d\times d},E^d)} \le C\min(\frac{1}{\sqrt{t_1-s}}, \frac{t_2-t_1}{|t_1-s|^{3/2}})$$

Thus, we write $B(\vec{f},\vec{g})(t_2) - B(\vec{f},\vec{g})(t_1) = A_1 + A_2$ with

$$\begin{cases} A_1 = \int_0^{t_1}(e^{(t_2-s)\Delta} - e^{(t_1-s)\Delta})\mathbb{P}\vec{\nabla}.(\vec{f}\otimes\vec{g})\,ds \\ \text{and} \\ A_2 = \int_{t_1}^{t_2} e^{(t_2-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f}\otimes\vec{g})\,ds \end{cases}$$

We have $\|A_1\|_E \le C\int_0^{t_1}\min(\frac{1}{\sqrt{t_1-s}}, \frac{t_2-t_1}{|t_1-s|^{3/2}})\frac{ds}{\sqrt{s}}\|\vec{f}\|_{\mathcal{E}_E}\|\vec{g}\|_{\mathcal{B}_\infty}$; for $s < t_1/2$, we write $\min(\frac{1}{\sqrt{t_1-s}}, \frac{t_2-t_1}{|t_1-s|^{3/2}})\frac{1}{\sqrt{s}} \le \sqrt{\frac{2(t_2-t_1)}{t_1}}\frac{1}{\sqrt{t_1-s}\sqrt{s}}$, while for $s \ge t_1/2$ we write $\min(\frac{1}{\sqrt{t_1-s}}, \frac{t_2-t_1}{|t_1-s|^{3/2}})\frac{1}{\sqrt{s}} \le \sqrt{\frac{2(t_2-t_1)}{t_1}}\min(\sqrt{\frac{t_2-t_1}{t_1-s}}, \sqrt{\frac{t_1-s}{t_2-t_1}})\frac{1}{t_1-s}$, which gives $\|A_1\|_E \le C'\sqrt{\frac{2(t_2-t_1)}{t_1}}\|\vec{f}\|_{\mathcal{E}_E}\|\vec{g}\|_{\mathcal{B}_\infty}$. On the other hand, we have $\|A_2\|_E \le C\int_{t_1}^{t_2}\frac{1}{\sqrt{t_2-s}}\frac{ds}{\sqrt{s}}\|\vec{f}\|_{\mathcal{E}_E}\|\vec{g}\|_{\mathcal{B}_\infty}$, which gives (by writing $\frac{1}{\sqrt{s}} \le \frac{1}{\sqrt{t_1}}$) that $\|A_2\|_E \le 2C\sqrt{\frac{t_2-t_1}{t_1}}\|\vec{f}\|_{\mathcal{E}_E}\|\vec{g}\|_{\mathcal{B}_\infty}$.

We may easily check that $B(\vec{f},\vec{g})$ goes weakly to 0 when t decreases to 0: since we know that $B(\vec{f},\vec{g})$ is bounded in E^d for $0 < t < 1$, hence is bounded in $(\mathcal{S}')^d$, and since, for every $\omega \in \mathcal{S}$, $S_j\omega$ goes to 0 strongly in $\mathcal{S}$ as j goes to $+\infty$ (where S_j is the jth partial sum operator in the Littlewood–Paley decomposition), it is enough to chek that, for all $j \in \mathbb{N}$, $\lim_{t\to 0} S_jB(\vec{f},\vec{g}) = 0$. But this is obvious: since $\|S_j\mathbb{P}\vec{\nabla}.\|_1 \le C2^j$, we find that $\|S_jB(\vec{f},\vec{g})\|_E \le C2^j\sqrt{t}\|\vec{f}\|_{\mathcal{E}_E}\|\vec{g}\|_{\mathcal{B}_\infty}$. Thus, point a) is proved.

Point b) is obvious. We write, for $s > t/2$, $\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f}\otimes\vec{g})\|_\infty \le C\frac{1}{\sqrt{t-s}}\frac{1}{s}\|\vec{f}\|_{\mathcal{B}_\infty}\|\vec{g}\|_{\mathcal{B}_\infty}$ and, for $s < t/2$ and $t < 1$, $\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f}\otimes\vec{g})\|_\infty \le C\frac{1}{t-s}\frac{1}{\sqrt{s}}\|\vec{f}\|_{\mathcal{B}_\infty}\|\vec{g}\|_{\mathcal{E}_E}$.

In order to prove point c) with $T = 1$, we introduce the set $K = \{\vec{f} \in \mathcal{E}_E \cap \mathcal{B}_\infty\ /\ \|\vec{f}\|_{\mathcal{E}_E} \le 2\|\vec{u}_0\|_E$ and $\|\vec{f}\|_{\mathcal{B}_\infty} \le 2\|e^{t\Delta}\vec{u}_0\|_{\mathcal{B}_\infty}\}$. For $\vec{f}$ and $\vec{g} \in K$, we write $\|B(\vec{f},\vec{f}) - B(\vec{g},\vec{g})\|_E \leq C\int_0^t \frac{ds}{\sqrt{t-s}\sqrt{s}}\|\vec{f}-\vec{g}\|_{\mathcal{E}_E}(\|\vec{f}\|_{\mathcal{B}_\infty} + \|\vec{g}\|_{\mathcal{B}_\infty}) \leq 4\pi C\|\vec{u}_0\|_\infty\|\vec{f}-\vec{g}\|_{\mathcal{E}_E}$. Now, to estimate the L^∞ norm, we write

$$\omega(t,s) = \min(\frac{1}{\sqrt{t-s}}\frac{1}{s}(\|\vec{f}\|_{\mathcal{B}_\infty} + \|\vec{g}\|_{\mathcal{B}_\infty}), \frac{1}{t-s}\frac{1}{\sqrt{s}}(\|\vec{f}\|_{\mathcal{E}_E} + \|\vec{g}\|_{\mathcal{E}_E}))$$

and

$$\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f}\otimes\vec{f} - \vec{g}\otimes\vec{g})\|_\infty \leq C\omega(t,s)\|\vec{f}-\vec{g}\|_{\mathcal{B}_\infty}.$$

We then write $\alpha = \|e^{t\Delta}\vec{u}_0\|_{\mathcal{B}_\infty}$ and $\beta = \|\vec{u}_0\|_E$; we have

$$\int_0^t \omega(t,s)\ ds \leq \frac{2}{\sqrt{t}}\int_0^1 \min(\frac{\alpha}{s\sqrt{1-s}}, \frac{\beta}{(1-s)\sqrt{s}})\ ds \leq \frac{8\alpha}{\sqrt{t}}\int_0^{\pi/2}\frac{\beta\ d\theta}{\alpha\cos\theta + \beta\sin\theta}$$

Since $E \subset B_\infty^{-1,\infty}$, we have $\|e^{t\Delta}\vec{u}_0\|_{\mathcal{B}_\infty} \leq C(E)\|\vec{u}_0\|_E$; hence, we write $\alpha/\beta = \tan\theta_0$ with $\theta_0 \in [0, \arctan C(E) = \theta_E]$ and we find

$$\begin{aligned}\int_0^t \omega(t,s)\ ds \leq \frac{4\alpha}{\sqrt{t}}\ln\left(\frac{(1+\cos\theta_0)(1+\sin\theta_0)}{(1-\cos\theta_0)(1-\sin\theta_0)}\right) \quad &\leq C_E\frac{\alpha(1+|\ln\theta_0|)}{\sqrt{t}} \\ &\leq C'_E\frac{\alpha(1+\ln^+(\frac{\beta}{\alpha}))}{\sqrt{t}}.\end{aligned}$$

Thus, if $\alpha(1+\ln^+(\frac{\beta}{\alpha}))$ is small enough, $\vec{f} \mapsto e^{t\Delta}\vec{u}_0 - B(\vec{f},\vec{f})$ is a contraction on K. This proves point c) for $T = 1$.

The same estimates are valid for $0 < T \leq 1$, since $N_\infty^T(\vec{u}_0) \leq N_\infty^1(\vec{u}_0) \leq C(E)\|\vec{u}_0\|_E$. Thus, if $N_\infty^T(\vec{u}_0)(1+\ln^+(\frac{\|\vec{u}_0\|_E}{N_\infty^T(\vec{u}_0)}))$ is small enough ($N_\infty^T(\vec{u}_0)(1+\ln^+(\frac{\|\vec{u}_0\|_E}{N_\infty^T(\vec{u}_0)})) < \gamma_E$, where the positive constant γ_E does not depend on T), then $\vec{f} \mapsto e^{t\Delta}\vec{u}_0 - B(\vec{f},\vec{f})$ is a contraction on the set $\{\vec{f} \in (L^\infty((0,T),E))^d/ \sup_{0<t<T}\|\vec{f}\|_E \leq 2\|\vec{u}_0\|_E$ and $\sup_{0<t<T}\sqrt{t}\|\vec{f}\|_\infty \leq 2N_\infty^T(\vec{u}_0))\}$.

Point d) is proved in the same way, since we now have the inequality $N_\infty(\vec{u}_0) \leq C(E)\|\vec{u}_0\|_E$. $\square$

Chapter 18

Further results on local measures

In this chapter, we restate the results of Chapter 17 by describing more precisely the role of the space $bmo^{(-1)}$ in existence theorems for mild solutions in spaces of local measures (Sections 1 and 2). We also give some alternate proofs for the existence of global solutions (Section 3).

1. The role of the Morrey–Campanato space $M^{1,d}$ and of $bmo^{(-1)}$

In Chapter 17 (Theorem 17.2), we studied the special case of shift-invariant Banach spaces of local measures that are continuously embedded in $B_\infty^{-1,\infty}$. As a matter of fact, we may easily characterize the maximal space of this type as being the Morrey–Campanato space $M^{1,d}$:

Theorem 18.1: ($M^{1,d}$ and local measures)

Let E be a shift-invariant Banach space of local measures on $\mathbb{R}^d$. Then E is continuously embedded in $B_\infty^{-1,\infty}$ if and only if E is continuously embedded in the Morrey–Campanato space $M^{1,d}$.

Proof: We easily check that $M^{1,d}$ is continuously embedded in $B_\infty^{-1,\infty}$. Indeed, we write for $0 < t < 1$ that

$$e^{t\Delta} f(x) = \int \frac{1}{t^{d/2}} W(\frac{y}{\sqrt{t}}) f(x-y)\, dy = \int W(y) f(x-\sqrt{t}y)\, dy$$

hence,

$$|e^{t\Delta} f(x)| \leq \sum_{k\in\mathbb{Z}^d} \sup_{y\in k+[0,1]^d} |W(y)| \int_{k+[0,1]^d} |f(x-\sqrt{t}y)|\, dy.$$

We then write

$$\int_{k+[0,1]^d} |f(x-\sqrt{t}y)|\, dy = t^{-d/2} \int_{x-k\sqrt{t}+\sqrt{t}[0,1]^d} |f(z)|\, dz \leq C t^{-d/2} \|f\|_{M^{1,d}} (\sqrt{t})^{d-1}$$

and we get the inclusion $M^{1,d} \subset B_\infty^{-1,\infty}$.

Conversely, if $E \subset B_\infty^{-1,\infty}$, we obtain $\sup_{\omega\in\mathcal{C}_0, \|\omega\|_\infty\leq 1} \|\omega f\|_{B_\infty^{-1,\infty}} \leq C\|f\|_E$. Let $\varphi \in \mathcal{D}$ be equal to 1 on $B(0,1)$ and let $\omega \in \mathcal{D}$ be supported in $B(x_0, 1)$. We have $|\langle f|\omega\rangle = |\langle\bar{\omega} f|\varphi(x-x_0)\rangle \leq C\|\omega f\|_{B_\infty^{-1,\infty}}\|\varphi\|_{B_1^{1,1}}$. This gives $\|f\|_{M^1_{uloc}} \leq C\sup_{\omega\in\mathcal{C}_0, \|\omega\|_\infty\leq 1} \|\omega f\|_{B_\infty^{-1,\infty}}$. Now, we may define the norm of $B_\infty^{-1,\infty}$ with help of the heat kernel by $\|f\|_{B_\infty^{-1,\infty}} = \sup_{0<t<1} \sqrt{t}\|e^{t\Delta} f\|_{L^\infty(dx)}$; we then have for $0 < \lambda < 1$ $\|\lambda f(\lambda x)\|_{B_\infty^{-1,\infty}} \leq \|f\|_{B_\infty^{-1,\infty}}$; since $\|\omega(\lambda x)\|_\infty = \|\omega\|_\infty$ and $\|f\|_{M^{1,d}} = \sup_{0<\lambda<1} \|\lambda f(\lambda x)\|_{M^1_{uloc}}$, we get the inclusion $E \subset M^{1,d}$. $\square$

Remark: The Morrey–Campanato space $M^{1,d}$ may be precisely defined as the spaces of locally finite measures $d\mu$ so that the positive measure $d|\mu|$ is a tempered distribution and so that $\sup_{0<t<1} \sqrt{t}\|e^{t\Delta}(d|\mu|)\|_\infty < \infty$, i.e., so that $d|\mu| \in B_\infty^{-1,\infty}$.

Now, we discuss Theorem 17.2, which may be viewed as a special case of the Koch and Tataru theorem: the smallness condition in Theorem 17.2 implies the smallness of the norm in $bmo^{(-1)}$.

Theorem 18.2:

Let E be a shift-invariant Banach space of local measures that is continuously embedded in $B_\infty^{-1,\infty}$. Then E is continuously embedded in bmo^{-1}. Moreover, there exists a constant C_E so that

$$\|f\|_{bmo^{-1}} \leq C_E\|f\|_{B_\infty^{-1,\infty}}\sqrt{1+\ln^+(\frac{\|f\|_E}{\|f\|_{B_\infty^{-1,\infty}}})}$$

Proof: According to Theorem 18.1, we may assume that $E = M^{1,d}$. We want to estimate

$$\|f\|_{bmo^{-1}} = (\sup_{0<t<1}\sup_{x_0\in\mathbb{R}^d} \frac{1}{t^{d/2}}\int_0^t\int_{|x-x_0|\leq\sqrt{t}} |e^{s\Delta}f(x)|^2\ ds\ dx)^{1/2}.$$

But for $f \in B_\infty^{-1,\infty}$ and $0 < s < 1$, we have that $\|e^{s\Delta}f\|_\infty \leq C\frac{1}{\sqrt{s}}\|f\|_{B_\infty^{-1,\infty}}$; $e^{s\Delta}f$ is smooth on $(0,1)\times\mathbb{R}^d$. Thus, we have

$$\|\ |e^{s\Delta}f|^2\ \|_{M^{1,d}} \leq C\frac{\|f\|_{M^{1,d}}\|f\|_{B_\infty^{-1,\infty}}}{\sqrt{s}};$$

hence, $\|\int_0^t |e^{s\Delta}f|^2\ ds\|_{M^{1,d}} \leq C\|f\|_{M^{1,d}}\|f\|_{B_\infty^{-1,\infty}}\sqrt{t}$. This gives $\|f\|_{bmo^{-1}} \leq C\sqrt{\|f\|_{M^{1,d}}\|f\|_{B_\infty^{-1,\infty}}} \leq C'\|f\|_{M^{1,d}}$.

To prove the more precise estimate given in Theorem 18.2, we may assume that $f \neq 0$ and that we have $\|f\|_{B_\infty^{-1,\infty}} < \|f\|_{M^{1,d}}/2$ (in the other case, we would have $\|f\|_{M^{1,d}} \leq 2\|f\|_{B_\infty^{-1,\infty}}\big(1+\ln^+(\frac{\|f\|_{M^{1,d}}}{\|f\|_{B_\infty^{-1,\infty}}})\big)^{1/2}\big)$. We define $\tau = \frac{\|f\|^2_{B_\infty^{-1,\infty}}}{\|f\|^2_{M^{1,d}}} \in (0,1/4)$ and we write, for $t \in (0,1)$ and $x_0 \in \mathbb{R}^d$,

$$\begin{array}{lccc} & \int_0^t\int_{|x-x_0|\leq\sqrt{t}} |e^{s\Delta}f(x)|^2\,ds\,dx & & \\ = & \int_0^{\tau t}\int_{|x-x_0|\leq\sqrt{t}} |e^{s\Delta}f(x)|^2\,ds\,dx & + & \int_{\tau t}^t\int_{|x-x_0|\leq\sqrt{t}} |e^{s\Delta}f(x)|^2\,ds\,dx \\ = & I_1(t,x_0) & + & I_2(t,x_0). \end{array}$$

We estimate I_1 with the inequality $\|\,|e^{s\Delta}f|^2\,\|_{M^{1,d}} \leq C\frac{\|f\|_{M^{1,d}}\|f\|_{B_\infty^{-1,\infty}}}{\sqrt{s}}$, and we estimate I_2 with the inequality $\|\,|e^{s\Delta}f|^2\,\|_{M^{1,d}} \leq C\frac{\|f\|^2_{B_\infty^{-1,\infty}}}{s}$. Thus, we get $I_1(t,x_0) \leq C\sqrt{\tau t}(\sqrt{t})^{d-1}\|f\|_{M^{1,d}}\|f\|_{B_\infty^{-1,\infty}} = Ct^{d/2}\|f\|^2_{B_\infty^{-1,\infty}}$ and $I_2(t,x_0) \leq C\ln\frac{1}{\tau}\,(\sqrt{t})^d\|f\|^2_{B_\infty^{-1,\infty}} = 2Ct^{d/2}\|f\|^2_{B_\infty^{-1,\infty}}\ln(\frac{\|f\|_{M^{1,d}}}{\|f\|_{B_\infty^{-1,\infty}}})$. □

2. A persistency theorem

A consequence of Theorem 18.2 is that we must study the size of the norm of $\vec{u}_0$ in $(bmo^{(-1)})^d$ to prove the existence of a mild solution for the Navier–Stokes equations associated with $\vec{u}_0$. When $\vec{u}_0$ belongs more precisely to E^d, where E is a shift-invariant Banach space of local measures, the solution $\vec{u}$ provided by the Koch and Tataru theorem is then expected to remain in E^d. This is no longer a theorem on the existence of solutions; this is rather a regularity theorem for the solutions associated with regular data. What is important is that this point may be drastically separated from the problem of existence: if E is any shift-invariant Banach space of local measures and if $\vec{u}_0 \in (E \cap bmo^{(-1)})^d$, then if the Picard contraction principle gives a solution in the path space associated with $bmo^{(-1)}$, this solution will remain in E^d, however big the norm of $\vec{u}_0$ is in E^d. We shall call such a result a persistency theorem (following Furioli, Lemarié-Rieusset, Zahrouni and Zhioua [FURLZZ 00]).

Definition 18.1: (Exponential convergence of the Navier–Stokes Picard sequence)

Let $\vec{u}_0 \in (bmo^{(-1)})^d$ with $\vec{\nabla}.\vec{u}_0 = 0$. The Navier–Stokes Picard sequence associated with $\vec{u}_0$ is defined by induction as $\vec{u}_{(0)} = e^{t\Delta}\vec{u}_0$ and $\vec{u}_{(n+1)} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}_{(n)} \otimes \vec{u}_{(n)})\,ds$.

We say that the Navier–Stokes Picard sequence associated with $\vec{u}_0$ is exponentially convergent on $(0,T)$ if we have, for all $T_0 \in (0,T)$,

$$\limsup_{n\to\infty} \|\sqrt{t}(\vec{u}_{(n+1)} - \vec{u}_{(n)})\|^{1/n}_{L^\infty((0,T_0)\times\mathbb{R}^d)} < 1.$$

Lemma 18.1: *If $\vec{u}_0 \in (bmo^{(-1)})^d$ with $\vec{\nabla}.\vec{u}_0 = 0$ is such that the Navier–Stokes Picard sequence associated with $\vec{u}_0$ is exponentially convergent on $(0,T)\times$*

$\mathbb{R}^d$, *then the sequence converges to a solution* $\vec{u}$ *of the Navier–Stokes equations.*

Proof: Of course, $\vec{u}_{(n)}$ is a Cauchy sequence in $(L^\infty((t_0, T_0) \times \mathbb{R}^d))^d$ for every $0 < t_0 < T_0 < T$; hence, we may consider the limit $\vec{u}$ of $\vec{u}_{(n)}$ as n goes to ∞. We thus obtain, from $\partial_t \vec{u}_{(n+1)} = \Delta \vec{u}_{(n+1)} - \mathbb{P}\vec{\nabla}.\vec{u}_{(n)} \otimes \vec{u}_{(n)}$, that $\partial_t \vec{u} = \Delta \vec{u} - \mathbb{P}\vec{\nabla}.\vec{u} \otimes \vec{u}$. □

Remark: The solution provided by the Koch and Tataru theorem (for $\vec{u}_0 \in (bmo^{(-1)})^d$) is given as the limit of an exponentially convergent Navier–Stokes Picard sequence.

Theorem 18.3: (Persistency theorem for local measures)

Let E be a shift-invariant Banach space of local measures and let $\vec{u}_0 \in E^d$ with $\vec{\nabla}.\vec{u}_0 = 0$. If $\vec{u}_0 \in (bmo^{(-1)})^d$ is such that the Navier–Stokes Picard sequence associated to $\vec{u}_0$ is well defined and exponentially convergent on $(0,T) \times \mathbb{R}^d$, then the sequence converges to a solution $\vec{u}$ of the Navier–Stokes equations, which satisfies the regularity property $\vec{u} \in \cap_{0<T_0<T}(L^\infty((0,T_0), E))^d$.

*More precisely, in that case, $\vec{u} \in \mathcal{C}((0,T), E^d)$ and is *-weakly convergent to $\vec{u}_0$ in E as $t \to 0$. If $\lim_{t\to 0} \sqrt{t}\|\vec{u}(t,.)\|_\infty = 0$, then $\lim_{t\to 0} \|\vec{u} - e^{t\Delta}\vec{u}_0\|_E = 0$.*

In particular, a sufficient condition to have a global solution in E is that $\vec{u}_0 \in (BMO^{(-1)})^d$ and that $\|\vec{u}_0\|_{BMO^{(-1)}}$ be small enough to grant that the Picard iteration scheme described by the Koch and Tataru theorem works on $(0,\infty) \times \mathbb{R}^d$. This smallness condition is independent from E and from the size of $\|\vec{u}_0\|_E$.

Remark: A sufficient condition to grant that $\lim_{t\to\infty} \sqrt{t}\|\vec{u}(t,.)\|_\infty = 0$ is that $\vec{u}_0 \in (B_\infty^{-\alpha,\infty})^d$ for some $\alpha < 1$ (see Chapter 20) or that $\vec{u}_0 \in (cmo^{(-1)})^d$, where $cmo^{(-1)}$ is the space of smooth elements in $bmo^{(-1)}$ (see Chapter 16).

Proof: The proof is based on the identity relating $\vec{w}_{(n)} = \vec{u}_{(n+1)} - \vec{u}_{(n)}$ to $\vec{u}_{(n)}$: $\vec{w}_{(n)} = -B(\vec{w}_{(n-1)}, \vec{u}_{(n)}) - B(\vec{u}_{(n-1)}, \vec{w}_{(n-1)})$, where B is the bilinear operator $B(\vec{f},\vec{g})(t,x) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{f} \otimes \vec{g}\, ds$.

We define $M_n = \max_{0\le j\le n} \sup_{0<t<T_0} \|\vec{u}_{(j)}\|_E$ and $\alpha_n = \sup_{0<t<T_0} \|\vec{w}_{(n)}\|_E$. From the inequalities $\|fg\|_E \le C_E\|f\|_E\ \|g\|_\infty$ and $\|f * g\|_E \le \|f\|_E\ \|g\|_1$, we easily obtain by induction that $\vec{u}_{(n)} \in L^\infty((0,T_0), E^d)$. (See the proof of Theorem 17.2.) We then use the identity on $\vec{w}_{(n)}$ to get that, for a positive constant C_E (depending only on E),

$$\|\vec{w}_{(n)}(t)\|_E \le C_E \int_0^t \frac{1}{\sqrt{t-s}}\frac{1}{\sqrt{s}}\sqrt{s}\|\vec{w}_{(n-1)}(s)\|_\infty\ (\|\vec{u}_{(n)}\|_E + \|\vec{u}_{(n-1)}\|_E)\ ds.$$

Due to the exponential convergence of the Navier–Stokes Picard sequence, we know there exists a positive constant C_0 and a positive $\rho \in (0,1)$ so that, for every $n \in \mathbb{N}$, $\sup_{0<t<T_0} \sqrt{t}\|\vec{w}_{(n)}(t)\|_\infty \leq C_0\rho^n$; hence, $\alpha_n \leq C\ C_0\rho^{n-1}M_n$ and thus, $M_{n+1} \leq M_n\ (1 + C\ C_0\rho^{n-1})$. We find

$$M_n \leq M_\infty = M_1 \prod_{j=0}^{\infty}(1 + C\ C_0\rho^j) < \infty$$

and thus, $\sum_{n=0}^{\infty} \alpha_n < \infty$.

Now, we check the continuity of $\vec{u}$. The continuity at $t = 0$ of $\vec{u} - e^{t\Delta}\vec{u}_0$ is obvious, since we have for a positive constant C_E

$$\|\vec{u} - e^{t\Delta}\vec{u}_0\|_E = \|B(\vec{u}, \vec{u})\|_E \leq C_E \sup_{0<s<t} \|\vec{u}\|_E \sup_{0<s<t} \sqrt{s}\|\vec{u}(s,.)\|_\infty.$$

The continuity for $t > 0$ of $\vec{u}$ is easy as well: we already know that $t \mapsto e^{t\Delta}\vec{u}_0$ is continuous from $(0, +\infty)$ to E (Proposition 4.4); for $\vec{w} = -B(\vec{u}, \vec{u})$, we write, for $0 < t < T_0$ and $t/2 < \tau < \min(3t/2, T_0)$, defining $\tau_1 = t - 2|t-\tau|$, $\vec{w}(\tau) - \vec{w}(t) = (e^{(\tau-\tau_1)\Delta} - e^{(t-\tau_1)\Delta})\vec{w}(\tau_1) - \int_{\tau_1}^{\tau} e^{(\tau-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{u} \otimes \vec{u}\ ds + \int_{\tau_1}^{t} e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{u} \otimes \vec{u}\ ds$, which gives

$$\|\vec{w}(\tau) - \vec{w}(t)\|_E \leq C|\ln\frac{\tau}{t}| \sup_{0<s<T_0} \|\vec{w}\|_E + C\sqrt{|1 - \frac{\tau}{t}|} \sup_{0<s<T_0} \|\vec{u}\|_E \sup_{0<s<T_0} \sqrt{s}\|\vec{u}\|_\infty.$$

Finally, we note that the *-weak continuity at $t = 0$ is provided by the weak continuity in $\mathcal{S}'$ and the boundedness in $E = (E^{(*)})'$. $\square$

3. Some alternate proofs for the existence of global solutions

We first check that the proof based on maximal functions given for L^d by C. Calderón [CAL 93] and by Cannone [CAN 95] may easily be adapted to the case of Lorentz spaces $L^{d,q}$ $(1 \leq q \leq \infty)$. Moreover, the persistency theorem states that the Koch and Tataru solutions satisfies similar maximal estimates when $\vec{u}_0$ belongs to a homogeneous Morrey–Campanato spaces $\dot{M}^{p,q}$ $(1 < p \leq q)$.

Theorem 18.4: (Maximal functions and Navier–Stokes equations)

Let E be a Lorentz space $L^{p,q}(\mathbb{R}^d)$ $(1 < p < \infty,\ 1 \leq q \leq \infty)$ or a homogeneous Morrey–Campanato space $\dot{M}^{p,q}$ $(1 < p \leq q)$. Let $\mathcal{E}_E$ be the space of measurable functions $F(t,x)$ defined on $(0,\infty) \times \mathbb{R}^d$ so that there exists a function $F^* \in E$ with $|F(t,x)| \leq F^*(x)$ almost everywhere on $(0,\infty) \times \mathbb{R}^d$, normed with $\|F\|_{\mathcal{E}_E} = \min_{|F| \leq F^*} \|F^*\|_E$.

A) $f \in E \Leftrightarrow e^{t\Delta} f \in \mathcal{E}_E$.

B) Let E be a Lorentz space $L^{d,q}(\mathbb{R}^d)$. Then $\mathcal{E}_E$ is an admissible path space for the Navier–Stokes equations: the bilinear operator B defined by $B(\vec{f}, \vec{g})(t,x) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{f} \otimes \vec{g}\ ds$ is continuous from $\mathcal{E}^d \times \mathcal{E}^d$ to $\mathcal{E}^d$. In particular, there *exists a positive constant $\gamma_{d,q}$ so that, if $\vec{u}_0 \in E^d$ with $\vec{\nabla}.\vec{u}_0 = 0$ and $\|\vec{u}_0\|_E < \gamma_{d,q}$, then the Navier–Stokes problem with initial value $\vec{u}_0$ has a solution $\vec{u} \in (\mathcal{E}_E)^d$.*

C) Let E be a Lorentz space $L^{p,q}(\mathbb{R}^d)$ ($1 < p < \infty$, $1 \leq q \leq \infty$) or a homogeneous Morrey–Campanato space $\dot{M}^{p,q}$ ($1 < p \leq q$). Let $\vec{u}_0 \in E^d$ with $\vec{\nabla}.\vec{u}_0 = 0$. If $\vec{u}_0 \in (BMO_\infty^{-1,\infty})^d$ is small enough to satisfy the conclusion of the Koch and Tataru theorem for global solutions, then the Koch and Tataru solution $\vec{u}$ of the Navier–Stokes equations satisfies the regularity property $\vec{u} \in (\mathcal{E}_E)^d$.

Proof: The proof of point A) relies on the observation that there exists a constant C_0 so that for any f in the Morrey space L^1_{uloc} of uniformly locally integrable functions and for any positive t, we have $|e^{t\Delta}f| \leq C_0 M_f(x)$, where M_f is the Hardy–Littlewood maximal function of f. Thus, it is enough to show that $M_f \in E$ whenever $f \in E$. We know (Theorem 6.2) that $f \in L^p$ and $p > 1$ imply $M_f \in L^p$; hence, by interpolation of sublinear operators we find that $f \in L^{p,q}$ and $1 < p < \infty$ imply $M_f \in L^{p,q}$. Thus, we must only study the case of homogeneous Morrey-Campanato spaces $E = \dot{M}^{p,q}$, $1 < p \leq q$. We simply write $M_r f(x) = \sup_{0<\rho<r} \frac{1}{\rho^d} \int_{|x-y|\leq \rho} |f(y)|\, dy$; then we have $M_f(x) \leq C(M_r f(x) + r^{-d/q}\|f\|_{\dot{M}^{p,q}})$; this gives that $\int_{|x-y|\leq r} |M_f(y)|^p\, dy \leq C(\int_{|x-y|\leq r} |M_r f(y)|^p\, dy + r^{d(1-p/q)}\|f\|^p_{\dot{M}^{p,q}}) \leq C'(\int_{|x-y|\leq 2r} |f(y)|^p\, dy + r^{d(1-p/q)}\|f\|^p_{\dot{M}^{p,q}}) \leq C'' r^{d(1-p/q)}\|f\|^p_{\dot{M}^{p,q}}$. Thus, $\|M_f\|_{\dot{M}^{p,q}} \leq C\|f\|_{\dot{M}^{p,q}}$.

Conversely, since f is the limit of $e^{t\Delta}f$ in $\mathcal{S}'$ as $t \to 0$ and since $|g| \leq |h| \Rightarrow \|g\|_E \leq \|h\|_E$, we find that $\|f\|_E \leq \liminf_{t\to 0} \|e^{t\Delta}f\|_E \leq \|\sup_{t>0} |e^{t\Delta}f|\,\|_E$.

We now prove point B). We just have to prove that B is bounded from $(\mathcal{E}_E)^d \times (\mathcal{E}_E)^d$ to $(\mathcal{E}_E)^d$. Let $\vec{f}$ and $\vec{g} \in \mathcal{E}_E^d$, $sup_{t>0}|\vec{f}(t,x)| \leq F(x)$ and $sup_{t>0}|\vec{g}(t,x)| \leq G(x)$ and let $\vec{h} = B(\vec{f}, \vec{g})$; we have

$$|\vec{h}(t,x)| \leq C \int_0^t \int_{\mathbb{R}^d} \frac{1}{|t-s|^{(d+1)/2}} \frac{1}{\left(1 + \frac{|x-z|}{\sqrt{t-s}}\right)^{1+d}} F(z)G(z)\, dz\, ds$$

Since F and G belong to $L^{d,q}$, the product FG belongs to $L^{d/2,q}$. If we look at the integral with respect to s, we have

$$\int_0^t \frac{1}{\left(\sqrt{t-s} + |x-z|\right)^{1+d}}\, ds = \frac{1}{|x-z|^{d-1}} \int_0^{\frac{t}{|x-z|^2}} \frac{d\sigma}{\left(1+\sqrt{\sigma}\right)^{1+d}} \leq \frac{C_d}{|x-z|^{d-1}}$$

with $C_d = \int_0^{+\infty} \frac{d\sigma}{\left(1+\sqrt{\sigma}\right)^{1+d}} < \infty$. Thus, $|\vec{h}(t,x)| \leq C_d \frac{1}{|x|^{d-1}} * (FG)$. Since $\frac{1}{|x|^{d-1}} \in L^{\frac{d}{d-1},\infty}$ and since $L^{\frac{d}{d-1},\infty} * L^{d/2,q} \subset L^{d,q}$, we get the required estimate.

We now prove point C). The Picard sequence $\vec{u}_{(n)}$ satisfies, for some $\rho \in (0,1)$, $\sup_{n\in\mathbb{N}} \rho^{-n}\sqrt{t}\|\vec{u}_{(n+1)} - \vec{u}_{(n)}\|_\infty < \infty$. By induction on n, we are able to prove that $\vec{u}_{(n)} \in \mathcal{E}^d$: let $\vec{f} \in \mathcal{E}_E^d$, $sup_{t>0}|\vec{f}(t,x)| \le F(x)$ and let $\vec{g}$ be such that $\sqrt{t}\ \vec{g} \in (L^\infty((0,\infty)\times \mathbb{R}^d))^d$ and let $\vec{h} = B(\vec{f},\vec{g})$; we have

$$|\vec{h}(t,x)| \le C \int_0^t\int_{\mathbb{R}^d} \frac{1}{|t-s|^{d/2}} \frac{1}{\left(1+\frac{|x-z|}{\sqrt{t-s}}\right)^{1+d}} F(z)\, dz\ \frac{ds}{\sqrt{t-s}\sqrt{s}} \sup_{t>0} \sqrt{t}\|g(t,.)\|_\infty$$

thus, writing M_F for the Hardy–Littlewood maximal function of F, we find that $|\vec{h}(t,x)| \le C' \int_0^t M_F(x)\ \frac{ds}{\sqrt{t-s}\sqrt{s}} \sup_{t>0}\sqrt{t}\|g(t,.)\|_\infty$ with $\int_0^t \frac{ds}{\sqrt{t-s}\sqrt{s}} = \pi$. But we already know that $F \in E$ and $p > 1$ imply $M_F \in E$ (see the proof of point A). Then, the proof follows the same line as for Theorem 18.3: we may use the identity on $\vec{w}_{(n)} = \vec{u}_{n+1} - \vec{u}_n$ to get for some positive constant C (depending only on E and $\vec{u}_0$) $\|\vec{u}_{n+1} - \vec{u}_n\|_{\mathcal{E}} \le C\rho^n(\|\vec{u}_{(n)}\|_{\mathcal{E}} + \|\vec{u}_{(n-1)}\|_{\mathcal{E}})$. □

We now describe a curious result of Gallagher [GAL 99], which deals with $\vec{u}_0 \in (L^2_{loc}(\mathbb{R}^d))^d$, depending only on the two first variables x_1, x_2 and such that $\int\int |\vec{u}_0(x_1,x_2,x')|^2\ dx_1\ dx_2 < \infty$ (where $x' = (x_3,\dots,x_d)$).

Theorem 18.5:

Let E be the space $E = \{f \in L^2_{loc}(\mathbb{R}^d)/$ for $j = 3,\dots,d$, $\frac{\partial}{\partial x_j} f = 0$ and $\int\int |f(x_1,x_2,x')|^2\ dx_1\ dx_2 < \infty\}$ (where $x' = (x_3,\dots,x_d)$) normed with $\|f\|_E = \left(\int\int |f(x_1,x_2,x')|^2\ dx_1\ dx_2\right)^{1/2}$. Let $\mathcal{C}_{x_1,x_2}$ be the space $\mathcal{C}_{x_1,x_2} = \{f \in \mathcal{C}_b(\mathbb{R}^d)\ /\ $ for $j = 3,\dots,d$, $\frac{\partial}{\partial x_j} f = 0\}$, normed with the norm $\|.\|_\infty$ of uniform convergence. Then:
i) $E \subset M^{1,d}(\mathbb{R}^d)$;
ii) pointwise multiplication is a bounded operator from $E \times \mathcal{C}_{x_1,x_2}$ to E;
iii) the convolution is a bounded operator from $E \times L^1(\mathbb{R}^d)$ to E and from $\mathcal{C}_{x_1,x_2} \times L^1$ to $\mathcal{C}_{x_1,x_2}$;
iv) e^Δ maps boundedly E to $\mathcal{C}_{x_1,x_2}$;
v) for all $f \in E$, $\lim_{\lambda\to 0^+} \|\lambda f(\lambda x)\|_{bmo^{(-1)}} = 0$.

Given $\vec{u}_0 \in E^d$ with $\vec{\nabla}.\vec{u}_0 = 0$, the solution to the Navier–Stokes equations with initial value $\vec{u}_0$ may be solved through the following methods:
j) the Picard contraction principle converges on some interval $[0,T]$ to a solution $\vec{u} \in \mathcal{C}([0,T],E^d)$; this solution may be uniquely continued to a solution $\vec{u} \in \mathcal{C}([0,+\infty),E^d)$;
jj) the solution $\vec{u}_\epsilon$ of the mollified equations is defined on $(0,+\infty)\times\mathbb{R}^d$ and converges (strongly in $\mathcal{C}([0,T],E^d)$ for every $T > 0$) to the unique solution $\vec{u}$ of the Navier–Stokes equations in $\mathcal{C}_b([0,\infty),E^d)$.

Proof: Points i) to iv) are obvious. In order to prove point v), we use Theorem 18.2: it is then enough (since $\|\lambda f(\lambda x)\|_E = \|f\|_E$) to check that $\lim_{\lambda\to 0}\|\lambda f(\lambda x)\|_{B_\infty^{-1,\infty}} = 0$. This is equivalent to $\lim_{t\to 0}\sqrt{t}\|e^{t\Delta}f\|_\infty = 0$: since f may be considered as a function defined on $\mathbb{R}^2$: $f(x_1,x_2,x') = g(x_1,x_2)$ with $g \in L^2(\mathbb{R}^2)$ and since $e^{t\Delta}f(x_1,x_2,x') = e^{t(\partial_1^2+\partial_2^2)}g(x_1,x_2)$ and the test functions are dense in $L^2(\mathbb{R}^2)$, this limit is easily computed.

We now turn to point j). Since $\lim_{\lambda\to 0^+}\|\lambda\vec{u}_0(\lambda x)\|_{bmo^{(-1)}} = 0$, the Koch and Tataru theorem grants that the Picard iteration sequence will converge on $(0,T)\times\mathbb{R}^d$ for some positive T. Due to points i) to v), we find that the iterates $\vec{u}_{(n)}$ belong (for all $t\in(0,T)$) to the space $\mathcal{C}_{x_1,x_2}$ and that we may apply the persistency theorem (Theorem 18.3) to get that $\vec{u}_{(n)}$ converge to a solution in $(\mathcal{C}([0,T],E))^d$.

In order to get a global solution, we could conclude directly with the above analysis in the case that $\|\vec{u}_0\|_{BMO^{(-1)}}$ is small enough to apply the Koch and Tataru theorem. But we want to prove global existence for all $\vec{u}_0$, not only for the small ones. Global existence is then proved through energy estimates. For $\vec{u}\in(\mathcal{C}([0,T],E))^d$, $\vec{u}=((u_1,u_2,\vec{u}'))$, we write $\vec{u}''=(u_1,u_2)$ and $\vec{\nabla}''=(\partial_1,\partial_2)$; then the Navier–Stokes equations are split in two systems:

$$\partial_t\vec{u}''=\Delta\vec{u}''-(\vec{u}''.\vec{\nabla}'')\vec{u}''-\vec{\nabla}''p \text{ and } \vec{\nabla}''.\vec{u}''=0$$

and

$$\partial_t\vec{u}'=\Delta\vec{u}'-(\vec{u}''.\vec{\nabla}'')\vec{u}'.$$

The system on $\vec{u}''$ are the two-dimensional Navier–Stokes equations and we know that we may find a solution $\vec{u}''$ in $(\mathcal{C}_b([0,\infty),L^2))^2\cap(L^2_t\dot{H}^1)^2$ and we easily check that the linear equations on $\vec{u}'$ have a solution $\vec{u}'$ in $(\mathcal{C}_b([0,\infty),L^2))^{d-2}\cap(L^2_t\dot{H}^1)^{d-2}$.

Point jj) is quite obvious. We have uniform energy estimates on $\vec{u}_\epsilon$:

$$\sup_{\epsilon>0}\sup_{t>0}\iint|\vec{u}_\epsilon|^2\,dx_1\,dx_2<\infty \text{ and } \sup_{\epsilon>0}\int_0^\infty\iint(|\partial_1\vec{u}_\epsilon|^2+|\partial_2\vec{u}_\epsilon|^2)\,dx_1\,dx_2<\infty$$

The limiting process then gives the existence of a sequence that converges weakly to a solution; this solution belongs to $(L^\infty_t E)^d$ and its partial derivatives belong to $(L^2_t E)^d$: this limit then belongs to $\mathcal{C}_b([0,\infty),E^d)$ and thus is equal to $\vec{u}$. We now estimate the size of $\vec{u}-\vec{u}_\epsilon$. We use the energy equality:

$$\begin{gathered}\|\vec{u}(t)-\vec{u}_\epsilon(t)\|_E^2+2\int_0^t\|\vec{\nabla}\otimes(\vec{u}(s)-\vec{u}_\epsilon(s))\|_E^2\,ds=\\ \int_0^t\iint\vec{u}.[(\vec{u}-\omega_\epsilon*\vec{u}).\vec{\nabla}](\vec{u}-\vec{u}_\epsilon)\,dx_1\,dx_2\,ds\\ +\int_0^t\iint\vec{u}.\big([(\vec{u}-\vec{u}_\epsilon)*\omega_\epsilon].\vec{\nabla}\big)(\vec{u}-\vec{u}_\epsilon)\,dx_1\,dx_2\,ds\end{gathered}$$

On $[0,T]$, we split, for a given $\gamma>0$, $\vec{u}$ in $\vec{u}=\vec{\alpha}+\vec{\beta}$ where $\|\vec{\alpha}\|_{L^\infty E}<\gamma$ and where $\beta\in(L^\infty((0,T)\times\mathbb{R}^d))^d$. This gives

$$\begin{gathered}\|\vec{u}(t)-\vec{u}_\epsilon(t)\|_E^2+2\int_0^t\|\vec{\nabla}\otimes(\vec{u}(s)-\vec{u}_\epsilon(s))\|_E^2\,ds\le\\ C\|\vec{u}_0\|_E(\|\vec{u}-\omega_\epsilon*\vec{u}\|_{L^\infty E}+\|\vec{\nabla}\otimes(\vec{u}-\omega_\epsilon*\vec{u})\|_{L^2E})\big(\int_0^t\|\vec{\nabla}\otimes(\vec{u}-\vec{u}_\epsilon)\|_E^2\,ds\big)^{\frac12}\\ +C\|\vec{u}_0\|_E^4\gamma+C\gamma\int_0^t\|\vec{\nabla}\otimes(\vec{u}-\vec{u}_\epsilon)\|_E^2\,ds\\ +C\|\vec{\beta}\|_\infty^2\int_0^t\|\vec{u}-\vec{u}_\epsilon\|_E^2\,ds+\int_0^t\|\vec{\nabla}\otimes(\vec{u}-\vec{u}_\epsilon)\|_E^2\,ds\end{gathered}$$

Now, if γ is small enough and if $\epsilon<\epsilon_0(\gamma,T,\vec{u})$ this gives

$$\|\vec{u}(t)-\vec{u}_\epsilon(t)\|_E^2\le C_0\|\vec{u}_0\|_E^4\gamma+C_1\|\vec{\beta}\|_\infty^2\int_0^t\|\vec{u}-\vec{u}_\epsilon\|_E^2\,ds$$

for two constants C_0 and C_1 which do not depend on γ. Thus, we find that for every $\gamma > 0$, there exists $\epsilon_0 = \epsilon_0(\frac{\gamma}{C_0\|\vec{u}_0\|_E^4}, T, \vec{u}) > 0$ so that for $\epsilon < \epsilon_0$, we have $\|\vec{u}(t) - \vec{u}_\epsilon(t)\|_E^2 \leq \gamma\ e^{C_1 t\|\vec{\beta}\|_\infty^2}$. ◻

The space E considered by Gallagher is not a space of local measures, since it is not invariant through pointwise multiplication by bounded continuous functions, but the scheme of the proof is the same as for local measures. We may consider spaces of functions that are invariant through pointwise multiplication by functions whose Fourier transforms are integrable (a proper subclass of bounded continuous functions):

Theorem 18.6: (Wiener algebra and Navier–Stokes equations)

Let E be a shift-invariant Banach space of distributions. We assume that there exists a positive constant C_E so that, for all $T \in E$ and all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\|\varphi T\|_E \leq C_E\|\hat{\varphi}\|_1\|T\|_E$ (so that pointwise multiplication of distributions in E by functions in the Wiener algebra $\mathcal{F}L^1$ is well defined). Let $\vec{u}_0 \in E^d$ with $\vec{\nabla}.\vec{u}_0 = 0$. If $\vec{u}_0 \in (bmo^{(-1)})^d$ is such that $\mathcal{F}\vec{u}_0$ is locally integrable and $\mathcal{F}^{(-1)}(|\mathcal{F}\vec{u}_0|) \in (bmo^{(-1)})^d$ and if the Navier–Stokes Picard sequence associated with $\vec{u}_0$ is exponentially convergent on $(0,T) \times \mathbb{R}^d$ in the Wiener norm, i.e., there exists two positive constants C and ρ with $\rho < 1$ so that, for all $n \in \mathbb{N}$, $\|\mathcal{F}(\vec{u}_{(n+1)} - \vec{u}_{(n)})\|_1 \leq C\rho^n$, then the sequence converges to a solution $\vec{u}$ of the Navier–Stokes equations which satisfies the regularity property $\vec{u} \in \cap_{0<T_0<T}(L^\infty((0,T_0),E))^d$.

*More precisely, in that case, $\vec{u} \in \mathcal{C}((0,T),E^d)$ and is *-weakly convergent to $\vec{u}_0$ in E as $t \to 0$. If $\lim_{t\to 0}\sqrt{t}\|\mathcal{F}\vec{u}(t,.)\|_1 = 0$, then $\lim_{t\to 0}\|\vec{u} - e^{t\Delta}\vec{u}_0\|_E = 0$.*

In particular, a sufficient condition to have a global solution in E is that $\mathcal{F}^{(-1)}(|\mathcal{F}\vec{u}_0|) \in (BMO^{(-1)})^d$ and that $\|\mathcal{F}^{(-1)}(|\mathcal{F}\vec{u}_0|)\|_{BMO^{(-1)}}$ is small enough to grant that the Picard iteration scheme described by the modified Koch and Tataru theorem (Theorem 16.3) works on $(0,\infty)\times\mathbb{R}^d$. This smallness condition is independent of E and of the size of $\|\vec{u}_0\|_E$.

Proof: The same proof as for the persistency theorem (Theorem 18.3) may be used. ◻

Chapter 19

Regular initial values

1. Cannone's adapted spaces

In his book [CAN 95], Cannone studied Banach spaces X so that the bilinear operator B defined by

$$B(\vec{f}, \vec{g})(t) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \vec{\nabla}.\vec{f} \otimes \vec{g}\, ds$$

is bounded from $L^\infty((0,T), X^d) \times L^\infty((0,T), X^d)$ to $L^\infty((0,T), X^d)$. (In our presentation, X is an adapted value space with $L^\infty((0,T), X^d)$ as admissible path space.)

Definition 19.1: (Cannone's adapted spaces)

According to Cannone, a Banach space X is adapted to the Navier–Stokes equations if the following assertions are satisfied:
a) X is a shift-invariant Banach space of distributions.
b) the pointwise product between two elements of X is still well defined as a tempered distribution
c) there is a sequence of real numbers $\eta_j > 0,\ j \in \mathbb{Z}$, so that

$$\sum_{j\in\mathbb{Z}} 2^{-|j|}\ \eta_j < \infty$$

and so that

$$\forall j \in \mathbb{Z}, \forall f \in X, \forall g \in X\ \ \|\Delta_j(fg)\|_X \leq \eta_j\ \|f\|_X\ \|g\|_X$$

Theorem 19.1: (Cannone's theorem for adapted spaces)

Let X be a Banach space adapted (according to Cannone) to the Navier–Stokes equations. Then, for all $\vec{u_0} \in X^d$ so that $\vec{\nabla}.\vec{u_0} = 0$, there is a $T^* = T^*(\|\vec{u_0}\|_X)$ and a unique solution $\vec{u}$ of the Navier–Stokes equations on $(0, T^*)$

with initial value $\vec{u}_0$ *so that* $\vec{u} - e^{t\Delta}\vec{u}_0 \in \mathcal{C}([0,T^*), X^d)$. *More precisely, we have* $\vec{u} - e^{t\Delta}\vec{u}_0 \in \mathcal{C}([0,T^*), (\dot{B}_X^{0,1})^d)$ *and we have the inequality:*

$$\forall t \in (0,T^*) \ \|\vec{u} - e^{t\Delta}\vec{u}_0\|_{\dot{B}_X^{0,1}} \leq C \ (t \sum_{4^j t \leq 1} 2^j \eta_j + \sum_{4^j t > 1} 2^{-j}\eta_j) \ \|\vec{u}_0\|_X^2$$

Proof: The proof is based on the Picard iteration scheme. The main point is to prove that the bilinear transform B is continuous on $L^\infty(X^d)$. It is enough to notice the following properties of this operator:
$\alpha)$ $e^{(t-s)\Delta}\mathbb{P}(\vec{\nabla}.\ )$ is a matrix of convolution operators;
$\beta)$ since X is a shift-invariant Banach space of distributions, we have for $f \in X$ and $g \in L^1$ $\|f * g\|_X \leq \|f\|_X \ \|g\|_1$;
$\gamma)$ for all j we have $\Delta_j = (\sum_{k=j-2}^{k=j+2} \Delta_k) \ \Delta_j$;
$\delta)$ we may conclude that

$$\|e^{(t-s)\Delta}\mathbb{P}(\vec{\nabla}.\Delta_j(\vec{u}\otimes\vec{v}))\|_X \leq C \min(2^j, 4^{-j}(t-s)^{-3/2}) \ \|\Delta_j(\vec{u}\otimes\vec{v})\|_X$$

The continuity of the bilinear transform is then easily established, writing

$$\|B(\vec{u},\vec{v})\|_X \leq \|B(\vec{u},\vec{v})\|_{\dot{B}_X^{0,1}} \leq \int_0^t \sum_{j\in\mathbb{Z}} \|e^{(t-s)\Delta}\mathbb{P}(\vec{\nabla}.\Delta_j(\vec{u}\otimes\vec{v}))\|_X \ ds$$

and estimating the integral $\int_0^t \sum_{j\in\mathbb{Z}} \min(2^j, 4^{-j}(t-s)^{-3/2}) \ \eta_j \ ds$. ◻

If we look accurately at the definition of an adapted space, we may see that the theorem is not really a result of *existence* of solutions but a result of *regularity* at $t = 0$ of mild solutions. Indeed, the Navier–Stokes equations may be solved in a greater space: all Cannone's adapted spaces are included in the Morrey–Campanato space $M^{2,d}$, and we know the existence of a local solution for the Navier–Stokes equations with initial value $\vec{u}_0 \in (M^{2,d})^d$ with $\vec{\nabla}.\vec{u}_0 = 0$ provided that $\lim_{t\to 0}\sqrt{t}\|e^{t\Delta}\vec{u}_0\|_\infty = 0$ (see Chapter 17). We easily check the following inclusion result:

Proposition 19.1:
Let X *be an adapted space (according to Cannone) for the Navier–Stokes equations and assume that*
i) $\mathcal{S}$ *is *-weakly dense in* X*;*
ii) there is a Banach space F *continuously embedded into* $\mathcal{S}'(\mathbb{R}^d)$ *so that* $(f,g) \mapsto fg$ *is bounded from* $X \times X$ *to* F.
Then X *is continuously embedded in* $M^{2,d}$ *and, for all* $f \in X$, $\lim_{t\to 0}\sqrt{t}\|e^{t\Delta}f\|_\infty = 0$.

Proof: The first point is to check that X is embedded into L^2_{loc}. Let K be a compact subset of $\mathbb{R}^d$ and ω_K a compactly supported nonnegative $\mathcal{C}^\infty$ function on $\mathbb{R}^d$ so that ω_K is identically equal to 1 in a neighborhood of K. There exists a constant D_K so that for all $f \in \mathcal{S}$, we have $\int_{\mathbb{R}^n} |f(x)|^2 \, \omega_K(x) \, dx \leq D_K \, \|f^2\|_F \leq C D_K \|f\|_X^2$ (we consider real-valued functions). We may conclude (according to Cauchy–Schwarz) that for all $f \in \mathcal{S}$ and all $\varphi \in \mathcal{D}_K$ we have

$$| \int_{\mathbb{R}^d} f(x) \, \varphi(x) \, dx | \leq \sqrt{C D_K} \|f\|_X \|\varphi\|_2$$

Now, if $T \in X$, we approximate T by a sequence of functions in the Schwartz class and get that for all $\varphi \in \mathcal{D}_K$ we have

$$| \, \langle T | \varphi \rangle \, | \leq \sqrt{C D_K} \|T\|_X \|\varphi\|_2$$

Thus, $X \subset L^2_{loc}(\mathbb{R}^d, dx)$.

Since the norm of X is shift invariant, the embedding $X \subset L^2_{loc}$ is equivalent to $X \subset L^2_{uloc}$. We may define the following numbers:

$$\alpha(r) = \sup_{\|f\|_X \leq 1, \ x \in \mathbb{R}^d} \int_{|x-y| \leq r} |f(y)|^2 dy$$

We estimate $\alpha(r)$ for $r \leq 1$. We select a smooth nonnegative function θ, which is equal to 1 on $|x| \leq 1$ and to 0 for $|x| \geq 2$, and we write $f^2 = S_0(f^2) + (\sum_{j \geq 0} \Delta_j(f^2)) = g + h$. It is easy to see that g is bounded (since $f^2 \in L^1_{uloc}$ and that S_0 maps L^1_{uloc} into L^∞) so that $| \int g(y) \ \theta(\frac{y-x}{r}) \ dy \ | \leq C r^d \|f\|_X^2$. We may write h as $\sum_{1 \leq j \leq d} \partial_j h_j$, where $h_j \in X$ and $\|h_j\|_X \leq C \|f\|_X^2$; we then have $| \int h(y) \ \theta(\frac{y-x}{r}) \ dy \ | \leq \sum_{1 \leq j \leq d} 1/r \ | \int h_j(y) \ \partial_j \theta(\frac{y-x}{r}) \ dy \ | \leq C \ r^{d/2-1} \ \sqrt{\alpha(2r)} \|f\|_X^2 \leq C' \ r^{d/2-1} \ \sqrt{\alpha(r)} \|f\|_X^2$; since $\int_{|x-y| \leq r} |f(y)|^2 dy \leq \int (g(y) + h(y)) \, \theta(\frac{y-x}{r}) \, dy$, we obtain that for $r \leq 1$, $\alpha(r) \leq C(r^d + \sqrt{\alpha(r)} r^{d/2-1})$, which gives $\alpha(r) \leq C r^{d-2}$; thus, X is embedded in the Morrey–Campanato space $M^{2,d}$.

This implies that $X \subset B_\infty^{-1,\infty}$ (and also that $\sup_{0 \leq t \leq 1} \sqrt{t} \ \|e^{t\Delta} f\|_\infty \leq C \|f\|_X$). We are now going to show that we have $\lim_{t \to 0} \sqrt{t} \|e^{t\Delta} f\|_\infty = 0$. Let us define

$$C_j = \sup_{\|f\|_X \leq 1} \|\Delta_j f\|_\infty.$$

Since X is invariant through dyadic dilations, we see that C_j is slowly varying: $\exists \gamma > 1 \ \forall j \in \mathbb{Z} \ \frac{1}{\gamma} \leq \frac{C_j}{C_{j+1}} \leq \gamma$. If f_j satisfies $\|f_j\|_X = 1$ and $|\Delta_j f_j(0)| \geq 1/2 \ C_j$, then $f_j \ f_{j+3} = \sum_{j+1 \leq k \leq j+5} \Delta_k(f_j \ f_{j+3})$; hence,

$$1/4 \ C_j \ C_{j+3} \leq (\sum_{j+1 \leq k \leq j+5} C_k) \ (\sum_{j \leq l \leq j+6} \eta_l)$$

Thus, we get $\lim_{j \to +\infty} 2^{-j} C_j = 0$. Then, write for $f \in X$ $f = S_N f + \sum_{j \geq N} \Delta_j f = f_N + g_N$; for $t \in (0,1]$ and $N \geq 1$ we have: $\sqrt{t} \ \|e^{t\Delta} f\|_\infty \leq$

$\sqrt{t}\ \|e^{t\Delta} f_N\|_\infty + \sqrt{t}\ \|e^{t\Delta} g_N\|_\infty \ \leq\ \sqrt{t}\ \|f_N\|_\infty + C\ \|g_N\|_{B_\infty^{-1,\infty}} \ \leq\ C(2^N\sqrt{t} + sup_{j\geq N} 2^{-j} C_j)\|f\|_X$. □

We have shown that the spaces studied by Cannone are subspaces of one space, the space $M^{2,d}$. Moreover, for all the examples quoted in [CAN 95] [Sobolev spaces $H_p^s (p < d, s > \frac{d}{p} - 1)$, Morrey–Campanato spaces $M^{2,p}(p > d)$, Lebesgue spaces $L^p, (p > d)$], we have the property that

$$(19.1) \qquad \forall f, g \in X \cap L^\infty\ \|fg\|_X \leq C\ (\|f\|_X \|g\|_\infty + \|g\|_X \|f\|_\infty);$$

we know that the bilinear transform B is bounded on $\mathcal{E}_T^d$, where $\mathcal{E}_T$ is the admissible path space $L^\infty((0,T), M^{2,d}) \cap \{f\ /\ \sqrt{t} f \in L^\infty((0,T) \times \mathbb{R}^d)\}$, and we obviously find (using inequality (19.1)) that it is bounded on $\mathcal{F}_T^d$, where $\mathcal{F}_T$ is the admissible path space $\mathcal{E}_T \cap L^\infty((0,T), X)$. This gives another proof of Theorem 19.1 (under the assumption that (19.1) is valid).

2. Sobolev spaces and Besov spaces of positive order

While Cannone developped an abstract theory of adapted spaces, all the examples which have been discussed [CAN 95] were either shift-invariant Banach spaces of local measures (as, for example, Morrey–Campanato spaces - Kato [KAT 92], Taylor [TAY 92]) or Besov spaces or potential spaces with positive regularity over local measures, such as $\dot{H}_p^s$ (Kato and Ponce [KATP 88]). See also for Besov spaces Planchon [PLA 96] and Terraneo [TER 99]. All the latter examples may be treated in a much easier way using the regularity property given by inequality (19.1):

Definition 19.2: (Pointwise product regularity)

Let E be a shift-invariant Banach space of distributions. Then E has the pointwise product regularity property if

$$(19.2) \qquad \forall f, g \in E \cap L^\infty\ \|fg\|_E \leq C\ (\|f\|_E \|g\|_\infty + \|g\|_E \|f\|_\infty);$$

Lemma 19.1: *Let F be a shift-invariant Banach space of local measures. Then, for any positive s, the Besov spaces $B_F^{s,\infty}$ and $\dot{B}_F^{s,\infty}$ have the pointwise product regularity property.*

Proof: Just apply Lemma 4.2 (i.e., use paraproduct operators). □

Theorem 19.2: [Besov regularity and Navier–Stokes]

(A) Let E be a shift-invariant Banach space of distributions with the pointwise product regularity property. Then:

a) Let $\mathcal{E}_E = L^\infty((0,1), E^d)$, $\mathcal{B}_\infty = \{\vec{u} \,/\, \sqrt{t}\,\vec{u} \in (L^\infty((0,1), \mathcal{C}_b(\mathbb{R}^d))^d\}$ and let $\mathcal{F}_E = \mathcal{E}_E \cap \mathcal{B}_\infty$. The bilinear operator B defined by

$$B(\vec{u}, \vec{v}) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \vec{\nabla}.(\vec{u} \otimes \vec{v})\; ds$$

*is bounded from $\mathcal{F}_E \times \mathcal{F}_E$ to $\mathcal{E}_E$. $B(\vec{u}, \vec{v}) \in \mathcal{C}((0,1], E^d)$ and converges *-weakly to 0 as t goes to 0. If $\lim_{t\to 0} \sqrt{t}\|\vec{u}\|_\infty = \lim_{t\to 0} \sqrt{t}\|\vec{v}\|_\infty = 0$, then the convergence is strong.*

b) If E is continuously embedded in the Besov space $B^{-1,\infty}_\infty$, then the bilinear operator B is bounded as well from $\mathcal{F}_E \times \mathcal{F}_E$ to $\mathcal{B}_\infty$. Hence, there exists a constant $\gamma_E > 0$ so that, for all $\vec{u}_0 \in E^d$ (with $\vec{\nabla}.\vec{u}_0 = 0$) with $\|\vec{u}_0\|_E \le \gamma_E$, the initial value problem for the Navier–Stokes equations with initial data $\vec{u}_0$ has a solution $\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u}, \vec{u})$ with $\vec{u} \in \mathcal{E}_E \cap \mathcal{B}_\infty$.

c) If moreover E is continuously imbedded in the Besov space $\dot{B}^{-1,\infty}_\infty$, then there exists a constant $\delta_E > 0$ such that, for all $\vec{u}_0 \in E^d$ (with $\vec{\nabla}.\vec{u}_0 = 0$) with $\|\vec{u}_0\|_E \le \gamma_E$, the initial value problem for the Navier–Stokes equations with initial data $\vec{u}_0$ has a global solution $\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u}, \vec{u})$ in $L^\infty((0,\infty), E^d)$ with $\sup_{0<t} \sqrt{t}\;\|\vec{u}\|_\infty < \infty$.

(B) Let F be a shift-invariant Banach space of local measures and let E be a space of regular distributions over F: for some positive s and for some $q \in [1, +\infty]$, $E = H^s_F = (Id - \Delta)^{-s/2}F$, $E = \dot{H}^s_F$, $E = B^{s,q}_F$ or $E = \dot{B}^{s,q}_F$. Then, the same conclusions a), b), and c) hold as in point A).

Proof: The proof is similar to the proof in the case of local measures (Theorem 17.2). The boundedness of B from $\mathcal{F}_E \times \mathcal{F}_E$ to $\mathcal{E}_E$ is obvious, since $\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\|_{\mathcal{L}(E^{d\times d}, E^d)} \le C\frac{1}{\sqrt{t-s}}$:

$$\|B(\vec{f}, \vec{g})\|_E \le C \int_0^t \frac{1}{\sqrt{t-s}\sqrt{s}}\; ds \sup_{0<s<t} (\|\vec{f}\|_E\; \sqrt{s}\|\vec{g}\|_\infty + \|\vec{g}\|_E\; \sqrt{s}\|\vec{f}\|_\infty).$$

We may similarly prove that we have, for $0 < t_1 < t_2 < 1$, $\|B(\vec{f}, \vec{g})(t_1) - B(\vec{f}, \vec{g})(t_2)\|_E \le C\sqrt{\frac{t_2 - t_1}{t_1}} \sup_{0<s<t_2}(\|\vec{f}\|_E\; \sqrt{s}\|\vec{g}\|_\infty + \|\vec{g}\|_E\; \sqrt{s}\|\vec{f}\|_\infty)$ and easily check that $B(\vec{f}, \vec{g})$ goes weakly to 0 when t decreases to 0 (since we have, for all $j \in \mathbb{N}$, $\|S_j B(\vec{f}, \vec{g})\|_E \le C 2^j \sqrt{t} \sup_{0<s<t}(\|\vec{f}\|_E\; \sqrt{s}\|\vec{g}\|_\infty + \|\vec{g}\|_E\; \sqrt{s}\|\vec{f}\|_\infty))$. Thus, point a) is proved.

Point b) is obvious. Indeed we write, for $s > t/2$, $\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f} \otimes \vec{g})\|_\infty \le C\frac{1}{\sqrt{t-s}}\frac{1}{s}\|\vec{f}\|_{\mathcal{B}_\infty}\|\vec{g}\|_{\mathcal{B}_\infty}$ and, for $s < t/2$ and $t < 1$, $\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f} \otimes \vec{g})\|_\infty \le C\frac{1}{t-s}\frac{1}{\sqrt{s}}(\|\vec{f}\|_{\mathcal{B}_\infty}\|\vec{g}\|_{\mathcal{E}_E} + \|\vec{g}\|_{\mathcal{B}_\infty}\|\vec{f}\|_{\mathcal{E}_E})$. Point c) is treated in the same way.

Now, we prove point B). We just notice that, when B is any shift-invariant Banach space of distributions, we have, for $\alpha \in (-1, 1)$ and $b \in B^{d\times d}$, the estimate $\|(\sqrt{-\Delta})^\alpha e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.b\|_B \le C_{\alpha,B}\|b\|_B(t-s)^{-(1+\alpha)/2}$. We interpolate, since $[\dot{H}^\alpha_B, \dot{H}^{-\alpha}_B]_{1/2,1} = \dot{B}^{0,1}_B$ (see Chapter 4), and get $\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.b\|_B \le$

$C_{\alpha,B}\|b\|_B(t-s)^{-1/2}$. We conclude the proof as follows: for $E = H^s_F$ or $E = B^{s,q}_F$, we have $E \subset B^{s,\infty}_F$; $B^{s,\infty}_F$ has the pointwise product regularity property, so that we may apply the results obtained in point A); moreover, we have a more precise result, and find that the bilinear operator B maps $\mathcal{F} \times \mathcal{F}$ to $\mathcal{E}$ with $\mathcal{F} = L^\infty((0,1),(B^{s,\infty}_F)^d) \cap \mathcal{B}_\infty$ and $\mathcal{E} = L^\infty((0,1),(\dot{B}^{0,1}_{B^{s,\infty}_F})^d)$; we then easily conclude, since $\dot{B}^{0,1}_{B^{s,\infty}_F} \subset B^{0,1}_{B^{s,\infty}_F} = B^{s,1}_F \subset E$. The case of $E = \dot{H}^s_F$ or $E = \dot{B}^{s,q}_F$ is similar, using the embeddings $\dot{B}^{0,1}_{\dot{B}^{s,\infty}_F} = \dot{B}^{s,1}_F \subset E \subset \dot{B}^{s,\infty}_F$. □

We now present an example of Iftimie [IFT 99], which may be seen as a generalization of Gallagher's construction.

Proposition 19.2: *We define the partial Littlewood-Paley decomposition operators $S^{[d-2]}_j$ and $\Delta^{[d-2]}_j$ as the tensor products $S^{[d-2]}_j = Id_{\mathcal{S}'(\mathbb{R}^2)} \otimes S^{(d-2)}_j$ and $\Delta^{[d-2]}_j = Id_{\mathcal{S}'(\mathbb{R}^2)} \otimes \Delta^{(d-2)}_j$, where $S^{(d-2)}_j$ and $\Delta^{(d-2)}_j$ are the Littlewood-Paley operators on $\mathcal{S}'(\mathbb{R}^{d-2})$. Let B be the Banach space of distributions defined by $f \in B \Leftrightarrow \sum_{j\in\mathbb{Z}} 2^{j\frac{d-2}{2}} \|\Delta^{[d-2]}_j f\|_2 < \infty$. Then B is contained in $\dot{B}^{-1,\infty}_\infty$ and has the pointwise product regularity property.*

Proof: The pointwise regularity property is easily proved by using a $d-2$-dimensional paraproduct decomposition of the pointwise product. □

3. Persistency results

When we look for global solutions in Besov spaces $B^{s,q}_E$, we apply the same method as for global solutions in shift-invariant Banach spaces of local measures: we have a persistency theorem that states that the Koch and Tataru solutions retain their initial regularity (see Furioli [FUR 99] and Furioli, Lemarié–Rieusset, Zahrouni and Zhioua [FURLZZ 00]).

Theorem 19.3: (Persistency theorem for Besov spaces or potential spaces over local measures)

Let F be a shift-invariant Banach space of local measures and let E be a space of regular distributions over F: for some positive s and for some $q \in [1,+\infty]$, $E = H^s_F = (Id-\Delta)^{-s/2}F$, $E = B^{s,q}_F$ or $E = \dot{B}^{s,q}_F$. Let $\vec{u}_0 \in E^d$ with $\vec{\nabla}.\vec{u}_0 = 0$. If $\vec{u}_0 \in (BMO^{-1})^d$ is small enough to satisfy the conclusion of the Koch and Tataru theorem for global solutions, then the Koch and Tataru solution $\vec{u}$ of the Navier–Stokes equations satisfies the regularity property $\vec{u} \in (L^\infty((0,\infty),E))^d$.

*More precisely, in that case, $\vec{u} \in \mathcal{C}((0,T),E^d)$ and is *-weakly convergent to $\vec{u}_0$ in E as $t \to 0$. If $\lim_{t\to 0} \sqrt{t}\|\vec{u}(t,.)\|_\infty = 0$, then $\lim_{t\to 0} \|\vec{u} - e^{t\Delta}\vec{u}_0\|_E = 0$.*

In particular, a sufficient condition to have a global solution in E is that $\vec{u}_0 \in (BMO^{-1})^d$ and that $\|\vec{u}_0\|_{BMO^{-1}}$ is small enough to grant that the Picard *iteration scheme described by the Koch and Tataru theorem works on* $(0,\infty)\times\mathbb{R}^d$*. This smallness condition is independent of* E *and of the size of* $\|\vec u_0\|_E$.

The proof relies on the following lemma:

Lemma 19.2: *Let* F *be a shift-invariant Banach space of local measures, let* $s>0$*, and let* $\lambda>0$ *satisfy* $s/(s+1)<\lambda<1$*. Let* $\vec f$ *and* $\vec g$ *belong to the space* $L^\infty((0,\infty),(\dot B_F^{s,\infty}\cap\dot B_\infty^{-1,\infty})^d)$ *and be such that* $\sup_{t>0}\sqrt t\|\vec f\|_\infty<\infty$ *and* $\sup_{t>0}\sqrt t\|\vec g\|_\infty<\infty$*. Then* $B(\vec f,\vec g)\in L^\infty((\dot B_F^{s,1})^d)$*. More precisely, if we define the norms* $N_s(f)=\|f\|_{L^\infty_t\dot B_F^{s,1}}$*,* $N_\infty(f)=\sup_{t>0}\sqrt t\|f\|_\infty$*, and* $\tilde N_\infty=\|f\|_{L^\infty_t\dot B_\infty^{-1,\infty}}$*, and if we write* $\alpha_\infty=N_\infty(\vec f-\vec g)$*,* $\gamma_s=N_s(\vec f-\vec g)$*,* $\Gamma_s=N_s(\vec f)+N_s(\vec g)$ *and* $\tilde A_\infty=\tilde N_\infty(\vec f)+\tilde N_\infty(\vec g)$*, we have:*

$$\|B(\vec f,\vec f)-B(\vec g,\vec g)\|_{L^\infty_t\dot B_F^{s,1}}\le C(\lambda,s,E)(\alpha_\infty\Gamma_s+(\alpha_\infty\Gamma_s)^{1-\lambda}(\gamma_s\tilde A_\infty)^\lambda)$$

Proof: We begin by proving that if $f\in\dot B_F^{s,\infty}\cap\dot B_\infty^{-1,\infty}$ and $g\in\dot B_F^{s,\infty}\cap L^\infty$, then we may write $fg=u+v$ where $\|u\|_{\dot B_F^{s,\infty}}\le C(\lambda,s,F)\ \|f\|_{\dot B_F^{s,\infty}}\ \|g\|_\infty$ and $\|v\|_{\dot B_F^{s-\lambda,\infty}}\le C_{\lambda,s,F}\ (\|f\|_{\dot B_F^{s,\infty}}\ \|g\|_\infty)^{1-\lambda}\ (\|f\|_{\dot B_\infty^{-1,\infty}}\ \|g\|_{\dot B_F^{s,\infty}})^\lambda$. In order to prove it, we use the paraproduct operators of Bony. We use the Littlewood–Paley decomposition of f and of g and write

$$fg=\sum_{j\in\mathbb{Z}}S_{j-2}f\Delta_jg+\sum_{j\in\mathbb{Z}}S_{j+3}g\Delta_jf=\dot\varpi(f,g)+\dot\rho(f,g).$$

We easily obtain the classical estimate $\|\dot\rho(f,g)\|_{\dot B_F^{s,\infty}}\le C(s,F)\ \|f\|_{\dot B_F^{s,\infty}}\ \|g\|_\infty$ (see Lemma 4.2). To prove the estimate on $\dot\varpi(f,g)$, we must estimate, for all $j\in\mathbb{Z}$, $2^{j(s-\lambda)}\|S_{j-2}f\Delta_jg\|_F$. Writing $S_{j-2}f=\sum_{k\le j-3}\Delta_kf$, we use the estimates:

$$\|\Delta_kf\Delta_jg\|_F\le C_F\|\Delta_kf\|_F\|\Delta_jg\|_\infty\le C2^{-ks}\|f\|_{\dot B_F^{s,\infty}}\|g\|_\infty$$

and

$$\|\Delta_kf\Delta_jg\|_F\le C_F\|\Delta_kf\|_\infty\|\Delta_jg\|_F\le C2^k\|f\|_{\dot B_\infty^{-1,\infty}}2^{-js}\|g\|_{\dot B_F^{s,\infty}}$$

which give (writing $\|\Delta_kf\Delta_jg\|_F=(\|\Delta_kf\Delta_jg\|_F)^{1-\lambda}(\|\Delta_kf\Delta_jg\|_F)^\lambda$)

$$\|\Delta_kf\Delta_jg\|_F\le C2^{k(\lambda(s+1)-s)}2^{-j\lambda s}(\|f\|_{\dot B_F^{s,\infty}}\ \|g\|_\infty)^{1-\lambda}\ (\|f\|_{\dot B_\infty^{-1,\infty}}\ \|g\|_{\dot B_F^{s,\infty}})^\lambda$$

and summing over $k\le j-3$ we get (since $\lambda(s+1)-s>0$)

$$\|S_{j-2}f\Delta_jg\|_F\le C2^{j(\lambda-s)}(\|f\|_{\dot B_F^{s,\infty}}\ \|g\|_\infty)^{1-\lambda}\ (\|f\|_{\dot B_\infty^{-1,\infty}}\ \|g\|_{\dot B_F^{s,\infty}})^\lambda.$$

Now, the proof is easy. We write $B(\vec f,\vec f)-B(\vec g,\vec g)=B(\vec f,\vec f-\vec g)+B(\vec f-\vec g,\vec g)$. Then we notice that the operator $e^{(t-\tau)\Delta}\mathbb{P}\vec\nabla.$ maps the Besov space $(\dot B_F^{\sigma,\infty})^{d\times d}$ to the Besov space $(\dot B_F^{\theta,\infty})^d$ for $\sigma-1\leq\theta<\infty$ with an operator norm $O((t-\tau)^{-1/2-(\theta-\sigma)/2})$; hence, (by interpolation) maps $(\dot B_F^{s,\infty})^{d\times d}$ to $(\dot B_F^{s,1})^d$ with an operator norm $O((t-\tau)^{-1/2})$ and $(\dot B_F^{\sigma-\lambda,\infty})^{d\times d}$ to $(\dot B_F^{s,1})^d$ with an operator norm $O((t-\tau)^{-1/2-\lambda/2})$. Thus, we find $\|B(\vec f,\vec f-\vec g)(t)\|_{\dot B_F^{s,1}}\leq C(s,F)A+C(s,F,\lambda)B$ with $A=\int_0^t\frac{1}{\sqrt{t-\tau}}\frac{1}{\sqrt\tau}\sqrt\tau\|\vec f(\tau)-\vec g(\tau)\|_\infty\ \|\vec f(\tau)\|_{\dot B_F^{s,\infty}}\,d\tau$ and $B=\int_0^t(t-\tau)^{-1/2-\lambda/2}\tau^{-1/2+\lambda/2}\ \omega(\tau)\,d\tau$ with

$$\omega(\tau)=\|\vec f\|_{\dot B_F^{s,\infty}}^{1-\lambda}\ \|\vec f\|_{\dot B_\infty^{-1,\infty}}^{\lambda}\ \|\vec f-\vec g\|_{\dot B_F^{s,\infty}}^{\lambda}\ (\sqrt\tau\|\vec f(\tau)-\vec g(\tau)\|_\infty)^{1-\lambda}.$$

The lemma is proved. $\square$

Proof of Theorem 19.3: We may assume that E is a homogeneous space $\dot H_F^s$ or $\dot B_F^{s,q}$, since we know that we control the norm of $\vec u$ in F^d (persistency theorem for local measures). Thus, we may assume that $\dot B_F^{s,1}\subset E\subset\dot B_F^{s,\infty}$.

Let $\vec u_{(n)}$ be defined by $\vec u_{(0)}=0$ and $\vec u_{(n+1)}=e^{t\Delta}\vec u_0-\int_0^t e^{(t-s)\Delta}\mathbb{P}\vec\nabla.(\vec u_{(n)}\otimes\vec u_{(n)})\,ds$, and let $(\vec w_n)$ be the sequence defined by $\vec w_{(n)}=\vec u_{(n+1)}-\vec u_{(n)}$ The proofs relies on the identity relating $\vec w_{(n)}$ to $\vec u_{(n)}$:

$$\vec w_{(n)}=-B(\vec w_{(n-1)},\vec u_{(n)})-B(\vec u_{(n-1)},\vec w_{(n-1)}).$$

We define $M_n=\max_{0\leq j\leq n}\sup_{t>0}\|\vec u_{(j)}\|_E$ and $\alpha_n=\sup_{t>0}\|\vec w_{(n)}\|_E$. We apply Lemma 19.2 and get the estimate $\sup_{t>0}\|\vec w^{(n)}\|_E\leq C'_EA+C'_{E,\lambda}B$ with

$$A=\sup_{0<t}\sqrt t\|\vec w_{(n-1)}\|_\infty\ (\sup_{t>0}\|\vec u_{(n)}\|_E+\sup_{t>0}\|\vec u_{(n-1)}\|_E)$$

and

$$\begin{aligned}B\ =\ &(\|\vec u_{(n-1)}\|_{L_t^\infty E}^{1-\lambda}\ \|\vec u_{(n-1)}\|_{L_t^\infty\dot B_\infty^{-1,\infty}}^{\lambda}+\|\vec u_{(n)}\|_{L_t^\infty E}^{1-\lambda}\ \|\vec u_{(n)}\|_{L_t^\infty\dot B_\infty^{-1,\infty}}^{\lambda})\\ \times\ &\|\vec w_{(n-1)}\|_{L_t^\infty E}^{\lambda}\ (\sup_{t>0}\sqrt t\|\vec w_{(n-1)}\|_\infty)^{1-\lambda}\end{aligned}$$

But we know that for a global Koch and Tataru solution given by a Picard iterative scheme we have

$$\sup_{n\in\mathbb{N}}\sup_{t>0}\|\vec u_{(n)}\|_{\dot B_\infty^{-1,\infty}}<\infty\text{ and }\lim\sup_{n\to\infty}\big(\sup_{t>0}\|\vec w_{(n)}\|_\infty\big)^{1/n}<1.$$

We find that for some constants C_0 and ρ, which depend on E, λ and $\vec u_0$, with $0\leq C_0$ and $0\leq\rho<1$, we have

$$\alpha_n\leq C_0\rho^{n-1}M_n+C_0M_n^{1-\lambda}\alpha_{n-1}^{\lambda}\rho^{(n-1)(1-\lambda)}.$$

The Young inequality gives us that for any positive ϵ we have a constant C_ϵ (depending only on ϵ and λ) so that

$$\alpha_n\leq C_0\epsilon\alpha_{n-1}+C_\epsilon C_0\rho^{n-1}M_n.$$

We then fix σ so that $\rho<\sigma<1$ and we choose $\epsilon=(\sigma-\rho)/C_0$; we also select $\Gamma>0$ so that $C_\epsilon C_0\leq\Gamma$ and $\alpha_1\leq\Gamma M_1$. We then prove by induction that for all $n\geq1$ we have $\alpha_n\leq\Gamma\sigma^{n-1}M_n$: we get that $\alpha_{n+1}\leq C_0\epsilon\alpha_n+C_\epsilon C_0\rho^nM_{n+1}\leq\Gamma C_0\epsilon\sigma^{n-1}M_n+\Gamma\sigma^{n-1}\rho M_{n+1}\leq\Gamma\sigma^nM_{n+1}$, finally yielding that $M_n\leq M_\infty=M_1\prod_{j=0}^\infty(1+\Gamma\sigma^j)<\infty$ and thus $\sum_{n=0}^\infty\alpha_n<\infty$. $\square$

Chapter 20

Besov spaces of negative order

1. $L^p(L^q)$ solutions

In Chapter 14, we saw the uniqueness criterion of Serrin expressed in terms of L^pL^q-norms with $\frac{1}{p} = \frac{1}{2} - \frac{d}{2q}$. Existence of solutions in L^pL^q has been discussed by Fabes, Jones, and Riviere [FABJR 72] and more extensively by Giga [GIG 86].

Theorem 20.1:

(A) Let $q \in (d, \infty)$ and p_q be defined by $\frac{1}{p_q} = \frac{1}{2} - \frac{d}{2q}$ Let $p \geq p_q$. Then, for all $\vec{u}_0 \in (\mathcal{S}'(\mathbb{R}^d))^d$ with $e^{t\Delta}\vec{u}_0 \in (L^p_t L^q_x((0,1)\times\mathbb{R}^d))^d$ and $\vec{\nabla}.\vec{u}_0 = 0$, there exist a positive T^* and a (unique) weak solution $\vec{u} \in \cap_{T<T^*}(L^p_t L^q_x((0,T)\times\mathbb{R}^d))^d$ for the Navier–Stokes equations on $(0,T^*)\times\mathbb{R}^d$ so that $\vec{u}(0,.) = \vec{u}_0$. This solution is then smooth on $(0,T^*)\times\mathbb{R}^d$.

(B) Let $d < q < \infty$ and $p = p_q$; then there exists a positive ϵ_q so that $T^* = \infty$ whenever $\|e^{t\Delta}\vec{u}_0\|_{L^pL^q((0,\infty)\times\mathbb{R}^d)} < \epsilon_q$.

Proof: To prove (A), we first consider the boundedness of B on $(L^p_t L^q_x)^d$. The pointwise product maps $L^q \times L^q$ to $L^{q/2}$. Now, $e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.$ maps $(L^{q/2})^{d\times d}$ to L^q with a norm $O((t-s)^{-\frac{1}{2}-\frac{d}{2q}})$. We find that

$$\|B(\vec{f},\vec{g})\|_{L^q(dx)} \leq C\int_0^t \frac{1}{(t-s)^{\frac{1}{2}+\frac{d}{2q}}}\|\vec{f}\|_q\|\vec{g}\|_q\, ds.$$

Since $t \mapsto 1_{t>0}\, t^{-\frac{q+d}{2q}}$ belongs to $L^{\frac{2q}{q+d},\infty}(\mathbb{R})$, we find that $B(\vec{f},\vec{g})$ belongs to L^rL^q with $\frac{1}{r} = \frac{2}{p} + (\frac{1}{2}+\frac{d}{2q}) - 1 \leq \frac{1}{p}$ since $\frac{1}{p} \leq \frac{1}{p_q} = \frac{1}{2} - \frac{d}{2q}$. Finally,

$$\|B(\vec{f},\vec{g})\|_{L^pL^q((0,T)\times\mathbb{R}^d)} \leq C\, T^{1/p-1/r}\|\vec{f}\|_{L^pL^q((0,T)\times\mathbb{R}^d)}\|\vec{g}\|_{L^pL^q((0,T)\times\mathbb{R}^d)}$$

and this gives the existence of a solution for the Navier–Stokes equations on the interval $(0,T)$ for $4CT^{1/p-1/r}\|e^{t\Delta}\vec{u}_0\|_{L^pL^q((0,T)\times\mathbb{R}^d)} < 1$. This condition is fullfilled for $T = T(\vec{u}_0)$ small enough: this is obvious for $p > p_d$ since $\lim_{T\to 0} T^{1/p-1/r} = 0$; for $p = p_d$, the condition is fullfilled since we have $\lim_{T\to 0}\|e^{t\Delta}\vec{u}_0\|_{L^pL^q((0,T)\times\mathbb{R}^d)} = 0$.

We now consider the regularity of $\vec{u}$. We know, according to the regularity criterion proved in Chapter 15 (Proposition 15.1), that it is enough to prove that $\vec{u}$ is bounded on every strip $(t_1, t_2) \times \mathbb{R}^d$ with $0 < t_1 < t_2 < T^*$. We first prove that this is true in the strips close to 0 (i.e., with $t_2 < \epsilon(\vec{u}_0)$). The condition $e^{t\Delta}\vec{u}_0 \in (L^p_t L^q_x((0,1) \times \mathbb{R}^d))^d$ is equivalent to $\vec{u}_0 \in (B_q^{-2/p,p}(\mathbb{R}^d))^d$. For $p \geq p_q$, we have, according to the Bernstein inequalities, $B_q^{-2/p,p} \subset B_q^{-2/p_q,p_q} \subset B_\infty^{-1,\infty}$; we may thus assume that $p = p_q$ and we have $\sup_{0<t<1} \sqrt{t}\|e^{t\Delta}\vec{u}_0\|_{L^\infty(dx)} < \infty$ and $\lim_{t\to 0} \sqrt{t}\|e^{t\Delta}\vec{u}_0\|_\infty = 0$ (since the test functions are dense in $B_q^{-2/p_q,p_q}$). We introduce the path space

$$\mathcal{E}_T = \{f \in L^{p_q}L^q((0,T) \times \mathbb{R}^d) \ / \ \sup_{t<T} \sqrt{t}\|f\|_\infty < \infty \text{ and } \lim_{t\to 0} \sqrt{t}\|f\|_\infty\}$$

and we easily check that the bilinear operator B is bounded from $\mathcal{E}_T^d \times \mathcal{E}_T^d$ to $\mathcal{E}_T^d$. Indeed, we estimate $\|B(\vec{f},\vec{g})\|_\infty$ by splitting the integral on $s \in (0,t)$ in two parts: $I_1 = \int_0^{t/2} e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f} \otimes \vec{g})\, ds$ and $I_2 = \int_{t/2}^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f} \otimes \vec{g})\, ds$. We then get:

i) $|I_1| \leq \int_0^{t/2} \frac{C}{(t-s)^{1/2+d/q}}\|\vec{f}\|_q\|\vec{g}\|_q\, ds \leq \frac{C'}{t^{1/2+d/q}} t^{1-2/p_q}\|\vec{f}\|_{L^{p_q}L^q}\|\vec{g}\|_{L^{p_q}L^q}$

ii) $|I_2| \leq \int_{t/2}^t \frac{C}{(t-s)^{1/2}}\sqrt{s}\|\vec{f}\|_\infty\sqrt{s}\|\vec{g}\|_\infty \frac{ds}{s} \leq \frac{C'}{\sqrt{t}} \sup_{s<t}\sqrt{s}\|\vec{f}\|_\infty \sup_{s<t}\sqrt{s}\|\vec{g}\|_\infty$

Thus, the Picard iteration scheme converges to a solution in $\mathcal{E}_T^d$ for T small enough (to ensure that $\|e^{t\Delta}\vec{u}_0\|_{L^{p_q}L^q((0,T)\times\mathbb{R}^d)} + \sup_{0<t<T}\sqrt{t}\|e^{t\Delta}\vec{u}_0\|_\infty < C_q$ where C_q depends only on q). We note that this condition is stable under a small perturbation of $\vec{u}_0$: for $\vec{w}_0 \in B_q^{-2/p,p}$ and $0 < T < 1$, we have

$$\begin{aligned} &\|e^{t\Delta}\vec{w}_0\|_{L^{p_q}L^q((0,T)\times\mathbb{R}^d)} + \sup_{0<t<T}\sqrt{t}\|e^{t\Delta}\vec{w}_0\|_\infty \\ &\leq \|e^{t\Delta}\vec{u}_0\|_{L^{p_q}L^q((0,T)\times\mathbb{R}^d)} + \sup_{0<t<T}\sqrt{t}\|e^{t\Delta}\vec{u}_0\|_\infty + C\|\vec{w}_0 - \vec{u}_0\|_{B_q^{-2/p,p}} \end{aligned}$$

We propagate this estimate along the interval $(0,T^*)$, following May [MAY 02] who uses a compactness argument of Brezis [BRE 94]. We first check that $\vec{u}$ belongs to $(B_q^{-2/p,p})^d$; if $\theta < T < T^*$, we write, for $0 < t < T - \theta$,

$$\vec{u}(t+\theta) = e^{t\Delta}\vec{u}(\theta) - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}(s+\theta) \otimes \vec{u}(s+\theta))\, ds;$$

since $\vec{u} \in (L^pL^q((0,T) \times \mathbb{R}^d))^d$, we have $\vec{u}(t+\theta) \in (L^pL^q((0,T-\theta) \times \mathbb{R}^d))^d$, and so does $\int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}(s+\theta) \otimes \vec{u}(s+\theta))\, ds$ (by the boundedness of the bilinear operator B on $(L^pL^q)^d$); hence, $e^{t\Delta}\vec{u}(\theta) \in (L^pL^q((0,T-\theta) \times \mathbb{R}^d))^d$, which shows that $\vec{u}(\theta) \in (B_q^{-2/p,p})^d$. In particular, we get uniqueness on $(0,T^*)$ of the solution which belongs to $\cap_{0<T<T^*}(L^pL^q((0,T) \times \mathbb{R}^d))^*$. We now prove that $t \mapsto \vec{u}(t)$ is continuous from $[0,T^*)$ to $(B_q^{-2/p,p})^d$: for $\epsilon \in (0,T^*/2)$, we write $Q_\epsilon = (0,\epsilon) \times \mathbb{R}^d$ and we estimate, for $0 \leq t_1 < t_2 \leq T^* - 2\epsilon$, the size of $\|\vec{u}(t_1) - \vec{u}(t_2)\|_{B_q^{-2/p,p}}$ by the equivalent norm $\|e^{t\Delta}(\vec{u}(t_1) - \vec{u}(t_2))\|_{L^pL^q(Q_\epsilon)}$, which in turn is controlled (according to the Navier–Stokes equations and the boundedness of B) by

$$\|\vec{u}(t+t_1)-\vec{u}(t+t_2)\|_{L^pL^q(Q_\epsilon)}(1+\|\vec{u}(t+t_1)\|_{L^pL^q(Q_\epsilon)}+\|\vec{u}(t+t_2)\|_{L^pL^q(Q_\epsilon)})$$

Since the test functions are dense in L^pL^q, we see that $\theta\mapsto\vec{u}(t+\theta)$ is uniformly continuous from $[0,T^*-2\epsilon]$ to $(L^pL^q(Q_\epsilon))^d$ and this gives the continuity of the map $t\mapsto\vec{u}(t)$ from $[0,T^*)$ to $(B_q^{-2/p,p})^d$. We now apply May's proof: if $T<T^*$, the range of $t\mapsto\vec{u}(t)$, $0\le t\le T$, is a compact subset of $(B_q^{-2/p,p})^d$; thus, there are two fixed positive numbers δ and D so that for each $\theta\in[0,T]$, we have for $t\in(0,2\delta]$ $\sqrt{t}\|\vec{u}(t+\theta)\|_\infty\le D$, finding that $\vec{u}$ is controlled by $Dt^{-1/2}$ on $(0,\delta]$ and by $D\delta^{-1/2}$ on $[\delta,T]$. Thus, $\vec{u}$ is smooth.

(B) is a direct consequence of the proof given for (A). The norm of $e^{t\Delta}\vec{u}_0$ in $(L^{p_d}L^q((0,\infty)\times\mathbb{R}^d))^d$ is equivalent to the norm of $\vec{u}_0$ in the homogeneous Besov space $(\dot{B}_q^{-2/p_q,p_q})^d$. □

We note that, for $d=3$ or 4, this yields a new proof of the Kato theorem:

Proposition 20.1: (Kato's theorem)

Let $d=3$ or $d=4$. For all $\vec{u}_0\in(L^d(\mathbb{R}^d))^d$ such that $\vec{\nabla}.\vec{u}_0=0$, there exist a positive T^* and a weak solution $\vec{u}\in\mathcal{C}([0,T^*),(L^d)^d)$ for the Navier–Stokes equations on $(0,T^*)\times\mathbb{R}^d$ so that $\vec{u}(0,.)=\vec{u}_0$. This solution may be chosen so that for all $T\in(0,T^*)$, we have $\vec{u}\in(L^4_tL^{2d}_x((0,T)\times\mathbb{R}^d))^d$. With this extra condition on the L^4L^{2d} norm, such a solution is unique and, moreover, it is smooth on $(0,T^*)\times\mathbb{R}^d$. This solution is equal to the Kato solution provided by Theorem 15.3.

Proof: We borrow the proof from Planchon's thesis [PLA 96]. If we look for a solution in $(L^4_tL^{2d}_x((0,T)\times\mathbb{R}^d))^d$, we see that $\vec{u}_0$ must belong to $(B_{2d}^{-1/2,4})^d$. We know that $L^d(\mathbb{R}^d)\subset B_d^{0,d}\subset B_{2d}^{-1/2,d}$ (for $d\ge2$), and then $L^d(\mathbb{R}^d)\subset B_d^{-1/2,4}$ when $d=3$ or $d=4$.

Thus, when $d=3$ or $d=4$, when $\vec{u}_0\in(L^d)^d$, we may construct a solution so that for some positive T^* and for all $T\in(0,T^*)$ we have $\vec{u}\in(L^4_tL^{2d}_x((0,T)\times\mathbb{R}^d))^d$. We prove now that $\vec{u}$ belongs to $(L^d)^d$: if $\vec{\varphi}(x)\in(L^{\frac{d}{d-1}}(\mathbb{R}^d))^d$, we have

$$|\int\vec{u}(t,x).\vec{\varphi}(x)\,dx|\le\|\vec{u}_0\|_d\|\vec{\varphi}\|_{\frac{d}{d-1}}+C(\int_0^t\|e^{(t-s)\Delta}\vec{\nabla}\otimes\vec{\varphi}\|^2_{\frac{d}{d-1}}ds)^{1/2}(\int_0^t\|\vec{u}\|^4_{2d}ds)^{1/2}$$

and we may control $(\int_0^t\|e^{(t-s)\Delta}\vec{\nabla}\otimes\vec{\varphi}\|^2_{\frac{d}{d-1}}\,ds)^{1/2}$ by $(\int_0^\infty\|e^{t\Delta}\vec{\nabla}\otimes\vec{\varphi}\|^2_{\frac{d}{d-1}}\,ds)^{1/2}$, or equivalently by the norm of $\vec{\nabla}\otimes\vec{\varphi}$ in $(\dot{B}^{-1,2}_{\frac{d}{d-1}})^{d\times d}$; we use the embedding $L^{\frac{d}{d-1}}\subset\dot{B}^{0,2}_{\frac{d}{d-1}}$ to get the inequality

$$|\int\vec{u}(t,x).\vec{\varphi}(x)\,dx|\le\|\vec{\varphi}\|_{\frac{d}{d-1}}\big(\|\vec{u}_0\|_d+C(\int_0^t\|\vec{u}\|^4_{2d}\,ds)^{1/2}\big).$$

Thus, we have $\vec{u} \in ((L^{\frac{d}{d-1}})')^d = (L^d)^d$. In particular, we have proved that B maps $(L^4_t L^{2d}_x)^d \times (L^4_t L^{2d}_x)^d$ to $(L^\infty L^d)^d$; since the test functions are dense in $L^4_t L^{2d}_x$, we may improve this result and conclude that B maps $(L^4_t L^{2d}_x)^d \times (L^4_t L^{2d}_x)^d$ to $(\mathcal{C}_b([0,T], L^d)^d$. This proves the proposition. Since we know that the solutions in $(L^4_t L^{2d}_x)^d$ are bounded by $Ct^{-1/2}$, we find that they are the same as the solutions provided by Kato's theorem. □

2. Potential spaces and Besov spaces

The results on $L^p L^q$ solutions may be easily generalized to the setting of initial data in Besov spaces with negative exponent over shift-invariant spaces of local measures. The case of Besov spaces on L^p has been investigated by Cannone [CAN 95] and the case of Banach spaces on Morrey–Campanato spaces has been investigated by Kozono and Yamazaki [KOZY 97]. A useful tool will be again the regularizing properties of the heat kernel:

Theorem 20.2: *Let F be a shift-invariant Banach space of local measures and let $\sigma \in (-1, 0)$ and $q \in [1, \infty]$.*
(A) Let $E = B^{\sigma,q}_F$, $\mathcal{E} = \{\vec{f} \in L^1((0,1), F^d)\ /\ t^{-\sigma/2}\|\vec{f}\|_F \in (L^q \cap L^\infty)((0,1), \frac{dt}{t})\}$, $\mathcal{B}_\infty = \{\vec{u}\ /\ \sqrt{t}\ \vec{u} \in (L^\infty((0,1), \mathcal{C}_b(\mathbb{R}^d))^d\}$ and $\mathcal{F} = \mathcal{E} \cap \mathcal{B}_\infty$. Then:

a) The bilinear operator B defined by

$$B(\vec{u}, \vec{v}) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \vec{\nabla}.(\vec{u} \otimes \vec{v})\ ds$$

is bounded from $\mathcal{F} \times \mathcal{F}$ to $\mathcal{E}$.

b) If F is continuously embedded in the Besov space $B^{-1-\sigma,\infty}_\infty$ (or, equivalently, if E is continuously embedded in the Besov space $B^{-1,\infty}_\infty$), then the bilinear operator B is bounded as well from $\mathcal{F} \times \mathcal{F}$ to $\mathcal{B}_\infty$. Hence, there exists a constant $\gamma_E > 0$ so that, for all $\vec{u}_0 \in E^d$ (with $\vec{\nabla}.\vec{u}_0 = 0$) with $\|\vec{u}_0\|_E \le \gamma_E$, the initial value problem for the Navier–Stokes equations with initial data $\vec{u}_0$ has a solution $\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u}, \vec{u})$ with $\vec{u} \in \mathcal{E} \cap \mathcal{B}_\infty$.

(B) Similarly, if F is continuously embedded in the homogeneous Besov space $\dot{B}^{-1-\sigma,\infty}_\infty$ and if $E = \dot{B}^{\sigma,q}_F$ (so that E is continuously embedded in the Besov space $\dot{B}^{-1,\infty}_\infty$), then there exists a constant $\delta_E > 0$ so that, for all $\vec{u}_0 \in E^d$ (with $\vec{\nabla}.\vec{u}_0 = 0$) with $\|\vec{u}_0\|_E \le \delta_E$, the initial value problem for the Navier–Stokes equations with initial data $\vec{u}_0$ has a global solution $\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u}, \vec{u})$ with $t^{-\sigma/2}\|\vec{u}\|_F \in (L^q \cap L^\infty)((0,\infty), \frac{dt}{t})$ and $\sup_{0<t} \sqrt{t}\ \|\vec{u}\|_\infty < \infty$.

We begin the proof with an easy lemma:

Lemma 20.1: *For $\alpha \in]0, 1/2[$ and $q \in [1, \infty]$, the operator $f \mapsto F$ where $F(t) = \int_0^t \frac{1}{\sqrt{t-s}\sqrt{s}} \frac{t^\alpha}{s^\alpha} f(s)\ ds$ is bounded on $L^q((0,\infty), \frac{dt}{t})$.*

Proof: For $q = \infty$, this is obvious since $\int_0^t \frac{1}{\sqrt{t-s}\sqrt{s}} \frac{t^\alpha}{s^\alpha}\, ds = \int_0^1 \frac{1}{\sqrt{1-s}\sqrt{s}} \frac{ds}{s^\alpha} < \infty$; similarly, for $q = 1$, we just write

$$s \int_s^{+\infty} \frac{1}{\sqrt{t-s}\sqrt{s}} \frac{t^\alpha}{s^\alpha}\ \frac{dt}{t} = \int_1^{+\infty} \frac{t^\alpha}{\sqrt{t-1}} \frac{dt}{t} < \infty.$$

For $1 < q < \infty$, the result follows by interpolation. □

Proof of Theorem 20.2: We prove only point (A), since point (B) is proved in exactly the same way. To prove (a), we note that the boundedness of B from $\mathcal{F} \times \mathcal{F}$ to $\mathcal{E}$ is a direct consequence of Lemma 20.1; more precisely, B is bounded from $\mathcal{E} \times \mathcal{B}_\infty$ to $\mathcal{E}$ and from $\mathcal{B}_\infty \times \mathcal{E}$ to $\mathcal{E}$.

To prove (b), we estimate $\|B(\vec{f}, \vec{g})\|_\infty$ by using the inequality, for $0 < s < t < 1$, $\|e^{\frac{t-s}{2}\Delta} f\|_\infty \leq C(t-s)^{-(1+\sigma)/2} \|f\|_{B_\infty^{-1-\sigma,\infty}}$):

$$\begin{aligned} |B(\vec{f}, \vec{g})(t)| \quad &\leq \int_0^t \frac{C}{(t-s)^{1+\frac{\sigma}{2}}} \|\vec{f}\|_F \|\vec{g}\|_\infty\, ds \\ &\leq \int_0^t \frac{C}{(t-s)^{1+\frac{\sigma}{2}}}\ \frac{ds}{s^{\frac{1-\sigma}{2}}} \|\vec{f}\|_{\mathcal{E}} \|\vec{g}\|_{B_\infty} \\ &= C_\sigma t^{-1/2} \|\vec{f}\|_{\mathcal{E}} \|\vec{g}\|_{B_\infty} \end{aligned}$$

with $C_\sigma < \infty$ since $\sigma \in (-1, 0)$. □

Finally, we discuss the approach of global solutions by the use of maximal functions (as developped for L^d by Calderón [CAL 93] and Cannone [CAN 95]):

Proposition 20.2: *For $d < p < \infty$ and $\alpha = 1 - d/p$, there exists $\epsilon_0(d,p) > 0$ (which depends only on p and on the dimension d) so that if $\vec{u}_0 \in (\dot{H}_p^{-\alpha}(\mathbb{R}^d))^d$ with $\vec{\nabla}.\vec{u}_0 = 0$ and $\|\vec{u}_0\|_{\dot{H}_p^{-\alpha}} < \epsilon_0(d,p)$, there exists a weak solution $\vec{u}$ for the Navier–Stokes equations on $(0, +\infty) \times \mathbb{R}^d$ with initial value $\vec{u}(0,.) = \vec{u}_0$ so that* $\sup_{0<t<+\infty} t^{\alpha/2} |\vec{u}(t,x)| \in L^p(\mathbb{R}^d)$.

Proof: We first notice that, if $u_0 \in \dot{H}_p^{-\alpha}$, then $u_0 = (-\Delta)^{\alpha/2} v_0$ with $v_0 \in L^p$; hence, $t^{\alpha/2} |e^{t\Delta} u_0| \leq C_\alpha M_{v_0}$, where M_{v_0} is the Hardy–Littlewood maximal function of v_0 (hence $M_{v_0} \in L^p$). To prove Proposition 20.2, we introduce a new adapted path space associated with the adapted value space $\dot{H}_p^{-\alpha}(\mathbb{R}^d)$ as the space $\mathcal{E} = \{f \in L^\infty([0, +\infty), \dot{H}_p^{-\alpha}(\mathbb{R}^d))\ /\ \sup_{0<t<+\infty} t^{\alpha/2} |f(t,x)| \in L^p(\mathbb{R}^d)\}$. We easily check that B is bounded on $\mathcal{E}^d$: we just write that if $t^{\alpha/2} |\vec{u}(t,x)| \leq U(x)$ and $t^{\alpha/2} |\vec{v}(t,x)| \leq V(x)$ for all $t > 0$, then

$$\begin{aligned} t^{\alpha/2} |B(\vec{u}, \vec{v})| \quad &\leq \quad C \iint_{0<s<t} \frac{|t-s|^{\alpha/2} + s^{\alpha/2}}{(\sqrt{t-s} + |x-y|)^{d+1}} U(y) V(y)\ dy\ \frac{ds}{s^\alpha} \\ &\leq \qquad C' \int \frac{1}{|x-y|^{d-1+\alpha}} U(y) V(y)\ dy \end{aligned}$$

Now, $UV \in L^{p/2}$ while $\frac{1}{|x|^{d-1+\alpha}} \in L^{\frac{d}{d-1+\alpha}, \infty}$; hence, $W = \frac{1}{|x|^{d-1}} * UV \in L^p$, since $1/p = 2/p + \alpha/d - 1/d$. Similarly, we have

$$\begin{aligned} |(-\Delta)^{-\alpha/2} B(\vec{u}, \vec{v})| \quad &\leq \quad C \iint_{0<s<t} \frac{|t-s|^{\alpha/2}}{(\sqrt{t-s} + |x-y|)^{d+1}} U(y) V(y)\ dy\ \frac{ds}{s^\alpha} \\ &\leq \qquad C' \int \frac{1}{|x-y|^{d-1+\alpha}} U(y) V(y)\ dy \end{aligned}$$

and the proposition is proved. □

3. Persistency results

One more time, Theorem 20.2 is not a new existence result, since we could apply the Koch and Tataru theorem:

Proposition 20.3:
Let $\sigma \in (-1,0)$ and let F be a shift-invariant Banach space of local measures. Then:
(A) If F is continuously embedded in the Besov space $B_\infty^{-1-\sigma,\infty}$, then F is more precisely continuously embedded in the Morrey–Campanato space $M_\sigma = M^{1,\frac{d}{1+\sigma}}$ (defined by $\sup_{x_0\in\mathbb{R}^d,0<t<1} t^{1+\sigma-d}\int_{|x-x_0|<t} d|\mu(x)| < \infty$) and we then have, for $1 \le q \le \infty$, the embeddings $B_F^{\sigma,q} \subset B_{M_\sigma}^{\sigma,\infty} \subset bmo^{-1}$.
(B) If F is continuously embedded in the homogeneous Besov space $\dot{B}_\infty^{-1-\sigma,\infty}$, then F is more precisely continuously embedded in the Morrey–Campanato space $\dot{M}_\sigma = \dot{M}^{1,\frac{d}{1+\sigma}}$ (defined by $\sup_{x_0\in\mathbb{R}^d,0<t} t^{1+\sigma-d}\int_{|x-x_0|<t} d|\mu(x)| < \infty$) and we then have, for $1 \le q \le \infty$, the embeddings $\dot{B}_F^{\sigma,q} \subset \dot{B}_{\dot{M}_\sigma}^{\sigma,\infty} \subset BMO^{-1}$.

Proof: We prove only (A), (B) can be proved in the same way. We mimic the proof of Theorem 18.1, starting from the inequality

$$\|f\|_{M^1_{uloc}} \le C \sup_{\omega\in\mathcal{C}_0,\|\omega\|_\infty\le 1} \|\omega f\|_{B_\infty^{-1-\sigma,\infty}},$$

and we write

$$\begin{aligned}\|f\|_{M_\sigma} &= \sup_{0<\lambda<1} \|\lambda^{1+\sigma} f(\lambda x)\|_{M^1_{uloc}} \\ &\le C \sup_{0<\lambda<1} \sup_{\omega\in\mathcal{C}_0,\|\omega\|_\infty\le 1} \|\lambda^{1+\sigma}\omega(x) f(\lambda x)\|_{B_\infty^{-1-\sigma,\infty}} \\ &= C \sup_{0<\lambda<1} \sup_{\omega\in\mathcal{C}_0,\|\omega\|_\infty\le 1} \|\lambda^{1+\sigma}\omega(\lambda x) f(\lambda x)\|_{B_\infty^{-1-\sigma,\infty}} \\ &\le C' \sup_{\omega\in\mathcal{C}_0,\|\omega\|_\infty\le 1} \|\omega f\|_{B_\infty^{-1-\sigma,\infty}} \\ &\le C'' \sup_{\omega\in\mathcal{C}_0,\|\omega\|_\infty\le 1} \|\omega f\|_F \le C'''\|f\|_F.\end{aligned}$$

This gives $F \subset M_\sigma$. Now, for $f \in E_\sigma = B_{M_\sigma}^{\sigma,\infty} \subset B_\infty^{-1,\infty}$, we write for $x_0 \in \mathbb{R}^d$ and $0 < t < 1$

$$\begin{aligned}\int_{|x-x_0|<\sqrt{t}} \int_0^t |e^{s\Delta} f(x)|^2\, dx\, ds &\le C(\sqrt{t})^{d-1-\sigma} \int_0^t \|e^{s\Delta} f\|_{M_\sigma} \|e^{s\Delta} f\|_\infty\, ds \\ &\le C(\sqrt{t})^{d-1-\sigma} \|f\|^2_{E_\sigma} \int_0^t s^{\sigma/2} s^{-1/2}\, ds \\ &\le C' t^{d/2} \|f\|^2_{E_\sigma};\end{aligned}$$

this proves that $B_{M_\sigma}^{\sigma,\infty} \subset bmo^{-1}$. □

Theorem 20.2 is primarily a regularity result for the Koch and Tataru solutions when the initial value belongs more precisely to a Besov space $B_F^{\sigma,q}$. Again, this regularity can be handled with a persistency theorem (Furioli, Lemarié–Rieusset, Zahrouni and Zhioua [FURLZZ 00]):

Theorem 20.3: (Persistency theorem for Besov spaces with negative exponents)

Let F be a shift-invariant Banach space of local measures and let E be a homogeneous Besov space of singular distributions over F: for some negative s and for some $q \in [1, +\infty]$, $E = \dot{B}_F^{s,q}$. Let $\vec{u}_0 \in E^d$ with $\vec{\nabla}.\vec{u}_0 = 0$. If $\vec{u}_0 \in (BMO^{-1})^d$ is small enough to satisfy the conclusion of the Koch and Tataru theorem for global solutions, then the Koch and Tataru solution $\vec{u}$ of the Navier–Stokes equations satisfies the regularity property $t^{-s/2}\|\vec{u}\|_F \in L^q_t((0,\infty), \frac{dt}{t})$.

Proof: Let the sequence $\vec{u}_{(n)}$ be defined by $\vec{u}_{(0)} = 0$ and $\vec{u}_{(n+1)} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}_{(n)} \otimes \vec{u}_{(n)})\, ds$, and let $(\vec{w}_n)$ be the sequence defined by $\vec{w}_{(n)} = \vec{u}_{(n+1)} - \vec{u}_{(n)}$ The proof relies, as usual, on the identity relating $\vec{w}_{(n)}$ to $\vec{u}_{(n)}$:

$$\vec{w}_{(n)} = -B(\vec{w}_{(n-1)}, \vec{u}_{(n)}) - B(\vec{u}_{(n-1)}, \vec{w}_{(n-1)}).$$

We then define the norm $\|f(t,x)\|_{F,s} = \|t^{-s/2}\|f\|_F\|_{L^q(\frac{dt}{t})}$ and the quantities $M_n = \max_{0\le j\le n}\|\vec{u}_{(j)}\|_{F,s}$ and $\alpha_n = \|\vec{w}_{(n)}\|_{F,s}$. Starting from the estimate

$$\|\vec{w}_{(n)}\|_F \le C\int_0^t \frac{1}{\sqrt{t-s}}\,\|\vec{w}_{(n-1)}(s)\|_\infty(\|\vec{u}_{(n)}(s)\|_F + \|\vec{u}_{(n-1)}(s)\|_F)\, ds,$$

we apply Lemma 20.1 to get the estimate

$$\|\vec{w}_{(n)}\|_{F,s} \le C_E \sup_{0<t}\sqrt{t}\|\vec{w}_{(n-1)}\|_\infty\ (\|\vec{u}_{(n)}\|_{F,s} + \|\vec{u}_{(n-1)}\|_{F,s})$$

But we know that for a global Koch and Tataru solution given by a Picard iterative scheme we have

$$\limsup_{n\to\infty}\left(\sup_{t>0}\|\vec{w}_{(n)}\|_\infty\right)^{1/n} < 1.$$

We find that for some constants C_0 and ρ, which depend on E and $\vec{u}_0$, with $0 \le C_0$ and $0 \le \rho < 1$, we have

$$\alpha_n \le C_0\rho^{n-1}M_n.$$

We get that $M_n \le M_\infty = M_1\prod_{j=0}^\infty(1 + C_0\rho^j) < \infty$ and thus $\sum_{n=0}^\infty \alpha_n < \infty$. $\square$

Chapter 21

Pointwise multipliers of negative order

In the classical L^pL^q uniqueness criterion of Serrin for Leray solutions, one estimates integrals $\int_0^t \int_{\mathbb{R}^d} u(s,x)v(s,x)\partial_j w(s,x)\,dx$ where $w \in L^2_t \dot H^1_x$ (so that $\partial_j w \in L^2_t L^2_x$) and $v \in L^2_t \dot H^1_x \cap L^\infty_t L^2_x$; one then uses the embeddings $L^2_t \dot H^1_x \cap L^\infty_t L^2_x \subset L^\sigma_t \dot H^r_x$ with $1/\sigma = r/2$ and $\dot H^r(\mathbb{R}^d) \subset L^t$ with $1/t = 1/2 - r/d$; thus, the integral is well defined when $u \in L^p_t L^q_x$ with $1 = 1/2 + 1/t + 1/q = 1 - r/d + 1/q$ (hence $q = d/r$) and $1 = 1/2 + 1/\sigma + 1/p = r + 1/p$ (hence $2/p = 1 - d/q = 1 - r$). One direct way to generalize this result is to study functions $u(s,x) \in L^p \dot X_r$ with $2/p = 1 - d/q = 1 - r$ and with the space $\dot X_r$ defined by the condition that the pointwise product between a function in $\dot X_r$ and a function in $\dot H^r$ belongs to L^2. An equivalent requirement is that the pointwise product between a function in $\dot X_r$ and a function in L^2 belongs to $\dot H^{-r}$. Since we have a loss of regularity, we shall say that elements in $\dot X_r$ are pointwise multipliers of negative order.

1. Multipliers and Morrey–Campanato spaces

We first recall the definition of Morrey–Campanato spaces:

Definition 21.1: (Morrey–Campanato spaces)

a) For $1 < p \le q \le \infty$ the Morrey–Campanato space $M^{p,q}$ is defined by: $f \in M^{p,q}$ if and only if f is locally L^p on $\mathbb{R}^d$ and $\|f\|_{M^{p,q}} < \infty$ where

$$\|f\|_{M^{p,q}} = \sup_{x\in\mathbb{R}^d} \sup_{0<R<1} R^{d/q-d/p}\ \|f(y)1_{B(x,R)}(y)\|_{L^p(dy)}$$

For $p = 1$, we require that f is a locally finite measure so that

$$\sup_{x\in\mathbb{R}^d} \sup_{0<R<1} R^{d(1/q-1)} \int_{B(x,R)} d|f|(y) < \infty$$

b) The closure of $\mathcal{S}$ in $M^{p,p}$ is called E_p.

It is easy to check that (for $p < \infty$) f belongs to E_p if and only if f is locally L^p and f vanishes at infinity in the sense that $\lim_{x_0\to\infty} \int_{|x-x_0|<1} |f(x)|^p\,dx = 0$.

The space $X_r(\mathbb{R}^d)$ of pointwise multipliers which map L^2 into H^{-r} is defined in the following way:

Definition 21.2: (Pointwise multipliers of negative order.)

For $0 \leq r < d/2$, we define the space $X_r(\mathbb{R}^d)$ as the space of functions, which are locally square-integrable on $\mathbb{R}^d$ and such that pointwise multiplication with these functions maps boundedly $H^r(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. The norm of X_r is given by the operator norm of pointwise multiplication:

$$\|f\|_{X_r} = \sup\{ \ \|fg\|_2 \ / \ \|g\|_{H^r} \leq 1\}$$

We have the following comparison between X_r and Morrey–Campanato spaces:

Theorem 21.1: (Comparison theorem)

For $2 < p \leq d/r$ and $0 < r$ we have $M^{p,d/r} \subset X_r \subset M^{2,d/r}$.

Proof: The injection $X_r \subset M^{2,d/r}$ is easy: indeed, since $H^r.X_r \subset L^2$, we have that $X_r \subset L^2_{loc}$. Choosing ω equal to 1 on the unit ball of $\mathbb{R}^d$, we have for $R < 1$:

$$\|f(y)1_{B(x,R)}(y)\|_{L^2(dy)} \leq \|f\|_{X_r}\|\omega((x-y)/R)\|_{H^r(dy)} \leq C \ \|f\|_{X_r}R^{d/2-r}$$

For the injection $M^{p,d/r} \subset X_r$, we consider a dual formulation of the problem, introducing the predual of $M^{p,d/r}$.

Definition 21.3: (The predual $N^{p',q'}$)

For $1 \leq q' \leq p'$, we define $N^{p',q'}$ as the subspace of $L^{q'}(\mathbb{R}^d)$ of functions f, which can be decomposed into an atomic series $f = \sum_{k\in\mathbb{N}} g_k$ where the functions g_k are in $L^{p'}$ with compact support and satisfy the following inequalities for the diameter d_k of the support of g_k: $d_k \leq 1$ and $\sum_{k\in\mathbb{N}} d_k^{d/q'-d/p'} \ \|g_k\|_{p'} < \infty$. $N^{p',q'}$ is normed by $\|f\|_{N^{p',q'}} = \inf_{f=\sum_{k\in\mathbb{N}} g_k} \ \sum_{k\in\mathbb{N}} d_k^{d/q'-d/p'} \ \|g_k\|_{p'}$.

Remark: It is obvious that for $q' < p' < p''$, we have $N^{p'',q'} \subset N^{p',q'} \subset L^{q',1} \subset L^{q'}$ since $\|g_k\|_{p'} \leq (C \ d_k^d)^{1/p'-1/p''} \ \|g_k\|_{p''}$.

Lemma 21.1: *The dual space of $N^{p',q'}$ for $1 \leq q' \leq p'$ is $M^{p,q}$ where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$.*

Proof: It is obvious since $L^{p'}_{comp} \subset N^{p',q'}$ and thus $(N^{p',q'})^* \subset L^p_{loc}$. □

According to Lemma 21.1, we may reformulate Theorem 21.1 in the following proposition:

Proposition 21.1: *Pointwise multiplication is a bounded bilinear operator from $L^2(\mathbb{R}^d) \times H^r$ to $N^{p',q'}$ for $1 \leq q' \leq p' < 2$ and $1/q' = 1 - r/d$.*

Pointwise multiplication by L^2 maps obviously H^r to $L^{q'}$ because of the Sobolev embedding theorem ($H^r \subset L^s$ with $1/s = 1/2 - r/d$); it is well known that it maps more precisely H^r into the Lorentz space $L^{q',1}$ since the sharp Sobolev embedding theorem states that $H^r \subset L^{s,2}$.

Now, we must prove a still sharper Sobolev embedding estimate. We are going to introduce an atomic decomposition for functions in the Sobolev space H^r, which will be transformed, by multiplication with a function in L^2, into an atomic decomposition in $N^{p',q'}$.

The key idea for decomposing $v \in H^r$ is to start from an atomic decompostion of $(\sqrt{-\Delta})^r v \in L^2$, where L^2 is to be considered as the real Hardy space $\mathcal{H}^2$ (Stein [STE 93]), then to apply fractional integration $(\sqrt{-\Delta})^{-r}$ to each atom of the decomposition of $(\sqrt{-\Delta})^r v$ to arrive at an atom of the decomposition of v. However, in order to preserve the compactness of the supports of the atoms, it will be more convenient to work with the decomposition of v on compactly supported orthonormal wavelets (Kahane and Lemarié-Rieusset [KAHL 95], Meyer [MEY 92]), and to define the atomic decomposition through the wavelet coefficients instead of using $(\sqrt{-\Delta})^r v$.

Let us recall that, if $\phi \in L^2(\mathbb{R})$ is a compactly supported orthonormal scaling function so that $\phi, \phi', \ldots, \phi^{(m)}$ are bounded functions for some $m > r$, if $\varphi = \phi \otimes \ldots \otimes \phi$ is the d-dimensional scaling function associated with ϕ and if $\psi_\epsilon, \epsilon = 1, \ldots, 2^d - 1$ are the associated separable wavelets, then we have, decomposing f into

$$f = \sum_{k \in \mathbb{Z}^d} c_k \ \varphi(x-k) + \sum_{j \geq 0} \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \epsilon \leq 2^d - 1} c_{j,k,\epsilon} \ 2^{dj/2} \ \psi_\epsilon(2^j x - k)$$

and letting $Q_{j,k}$ be the dyadic cube $Q_{j,k} = \{x / \ 2^j x - k \in [0,1]^d\}$, the following equivalence of norms:

$$\|f\|_{H^r} \approx \|(c_k)\|_{l^2} + (\sum_{j \geq 0} \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \epsilon \leq 2^d - 1} 4^{jr} |c_{j,k,\epsilon}|^2)^{1/2}$$

$$\approx \|(c_k)\|_{l^2} + \sum_{1 \leq \epsilon \leq 2^d - 1} \|\Big(\sum_{j \geq 0} \sum_{k \in \mathbb{Z}^d} 4^{jr} \ |c_{j,k,\epsilon}|^2 \ 2^{dj} 1_{Q_{j,k}}(x)\Big)^{1/2}\|_2$$

and

$$\forall p \in (1, \infty) \quad \|f\|_p \approx \|(c_k)\|_{l^p} + \sum_{1 \leq \epsilon \leq 2^d - 1} \|\Big(\sum_{j \geq 0} \sum_{k \in \mathbb{Z}^d} |c_{j,k,\epsilon}|^2 \ 2^{dj} 1_{Q_{j,k}}(x)\Big)^{1/2}\|_p$$

In order to prove Proposition 21.1, we take $u \in L^2$ and $v \in H^r$ and write

$$v = w_0 + \sum_{1\le\epsilon\le 2^d-1} w_\epsilon = \sum_{k\in\mathbb{Z}^d} c_k\, \varphi(x-k) + \sum_{1\le\epsilon\le 2^d-1} \Big(\sum_{j\ge 0}\sum_{k\in\mathbb{Z}^d} c_{j,k,\epsilon}\, 2^{dj/2}\, \psi_\epsilon(2^j x - k)\Big)$$

Lemma 21.2: $u\, w_0 \in N^{p',q'}$.

Proof: This is obvious. We have $\varphi \in L^s$ where $1/s + 1/2 = 1/p'$, and thus, writing $K = \mathrm{Supp}\ \varphi$:

$$\sum_{k\in\mathbb{Z}^d} |c_k|\, \|u\, \varphi(x-k)\|_{p'} \le \|\varphi\|_s \|(c_k)\|_{l^2} \Big(\sum_{k\in\mathbb{Z}^d}\int |u|^2 1_K(x-k)\, dx\Big)^{1/2} \le C\|u\|_2 \|v\|_{H^r}$$

□

Lemma 21.3: *For* $\epsilon = 1, \ldots, 2^d - 1$, $u\, w_\epsilon \in N^{p',q'}$.

Proof: To avoid notational burden, we drop the index ϵ.

We define $W(x) = \big(\sum_{j\ge 0}\sum_{k\in\mathbb{Z}^d} 4^{jr}\ |c_{j,k}|^2\ 2^{dj} 1_{Q_{j,k}}(x)\big)^{1/2}$. For $N \in \mathbb{Z}$, we consider $E_N = \{x/\ W(x) > 2^N\}$. Then, $2^{j(r+d/2)}\ |c_{j,k}| > 2^N$ implies $Q_{j,k} \subset E_N$. We write $F_N = \{(j,k) \in \mathbb{N}\times\mathbb{Z}^d/\ Q_{j,k} \subset E_N\}$ and we get $c_{j,k} \neq 0 \Rightarrow \exists N\ Q_{j,k} \in F_N$ and $Q_{j,k} \notin F_{N+1}$.Thus, we may write:

$$w = \sum_{N\in\mathbb{Z}} \Big(\sum_{Q_{j,k}\in F_N - F_{N+1}} c_{j,k}\ 2^{dj/2}\ \psi(2^j x - k)\Big) = \sum_{N\in\mathbb{Z}} w_N$$

Last, we consider for a given N the collection H_N of maximal dyadic cubes in $F_N - F_{N+1}$, and we decompose w_N into

$$w_N = \sum_{Q\in H_N} \Big(\sum_{Q_{j,k}\in F_N - F_{N+1}, Q_{j,k}\subset Q} c_{j,k}\ 2^{dj/2}\ \psi(2^j x - k)\Big) = \sum_{\in H_N} w_{N,Q}$$

This is the atomic decomposition we are looking for. We first notice that:

$$\Big(\sum_{Q_{j,k}\in F_N - F_{N+1}, Q_{j,k}\subset Q} 4^{jr}|c_{j,k}|^2\ 2^{dj}\ 1_{Q_{j,k}}(x)\Big)^{1/2} \le C\ 2^N 1_{\tilde{Q}}(x)$$

where $\tilde{Q}_{j,k} = \{x/\ 2^j x - k \in [-M, M]^d\}$ with a constant M, depending only on the support of ψ. In particular, we get that *Supp* $w_{N,Q} \subset \tilde{Q}$ and that for $1 < s < \infty$ we have $\|w_{N,Q}\|_s \le C_s\ 2^N\ d_Q^{r+d/s}$ where d_Q is the diameter of Q.

We then fix $s, t < \infty$ so that $1/2 + 1/s + 1/t = 1/p'$ and we write (taking σ as $1/\sigma = 1/2 + 1/s$)

$$d_Q^{d/q'-d/p'} \|u\, w_{N,Q}\|_{p'} \le C\|u\, 1_{\tilde{Q}}\|_\sigma\ 2^N d_Q^{d/q'-d/p'}\ d_Q^{r+d/t} = C\|u\, 1_{\tilde{Q}}\|_\sigma\ 2^N d_Q^{d(1-1/\sigma)}$$

and thus

$$\sum_N \sum_{Q\in H_N} d_Q^{\frac{d}{q'}-\frac{d}{p'}} \|u\, w_{N,Q}\|_{p'} \le C\Big(\sum_N \sum_{Q\in H_N} \|u\, 1_{\tilde Q}\|_\sigma^\sigma\, 2^{N(2-\sigma)}\Big)^{\frac{1}{\sigma}} \Big(\sum_N \sum_{Q\in H_N} d_Q^d 2^{2N}\Big)^{1-\frac{1}{\sigma}}$$

This latter inequality may be rewritten as

$$\sum_N \sum_{Q\in H_N} d_Q^{\frac{d}{q'}-\frac{d}{p'}} \|u\, w_{N,Q}\|_{p'} \le C\|u\, \Big(\sum_N \sum_{Q\in H_N} 2^{N(2-\sigma)}\, 1_{\tilde Q}\Big)^{\frac{1}{\sigma}}\|_\sigma \Big(\sum_N \sum_{Q\in H_N} d_Q^d 2^{2N}\Big)^{1-\frac{1}{\sigma}}$$

and thus we get

$$\|u\, w\|_{N^{p',q'}} \le C\, \|u\|_2\, \|\Big(\sum_N \sum_{Q\in H_N} 2^{\frac{4N}{(s+2)}}\, 1_{\tilde Q}\Big)^{1/2+1/s}\|_s\, \Big(\sum_N \sum_{Q\in H_N} d_Q^d\, 2^{2N}\Big)^{1/2-1/s}$$

$$= C\, \|u\|_2\, \|\sum_N \sum_{Q\in H_N} 2^{4N/(s+2)}\, 1_{\tilde Q}\|_{1+s/2}^{1/2+1/s}\, \Big(\sum_N \sum_{Q\in H_N} d_Q^d\, 2^{2N}\Big)^{1/2-1/s}$$

We then use the classical result that for any sequence of positive real numbers α_j and any sequence of cubes Q_j we have $\|\sum_j \alpha_j\, 1_{\tilde Q_j}\|_{1+s/2} \le C\, \|\sum_j \alpha_j\, 1_{Q_j}\|_{1+s/2}$: indeed, let L^τ be the dual space of $L^{1+s/2}$, then we have for $g \in L^\tau$ and $x \in Q_j$: $\frac{1}{|Q_j|}\int_{\tilde Q_j} |g(y)|\, dy \le Cg^*(x)$ (where g^* is the Hardy–Littlewood maximal function of g), hence:

$$\int \sum_j \alpha_j 1_{\tilde Q_j}(y)\, |g(y)|\, dy \le C \int \sum_j \alpha_j 1_{Q_j}\, g^*(y)\, dy \le C'\|\sum_j \alpha_j\, 1_{Q_j}\|_{1+s/2}\, \|g\|_\tau$$

Now, we use the inequalities $\sum_N \sum_{Q\in H_N} 2^{4N/(s+2)}\, 1_Q \le C\, W(x)^{2/(1+s/2)}$ and $\sum_N \sum_{Q\in H_N} d_Q^d\, 2^{2N} \le C \int W(x)^2\, dx$ to conclude:

$$\|u\, w\|_{N^{p',q'}} \le C\, \|u\|_2\, \|W\|_2 \le C'\|u\|_2\, \|w\|_{H^r}$$

□

Thus, Theorem 21.1 is proved. As a direct corollary, we deal with homogeneous spaces:

Proposition 21.2:

Let us define the homogeneous spaces $\dot M^{p,q}$ for $p \le q \le \infty$ and $\dot X_r$ for $0 \le r < d/2$) by

i) $\|f\|_{\dot M^{p,q}} = \sup_{x\in\mathbb{R}^d} \sup_{0<R} \big(R^{d(p/q-1)} \int_{B(x,R)} |f(y)|^p\, dy\big)^{1/p}$

ii) $\|f\|_{\dot X_r} = \sup\{\, \|fg\|_2\ /\ \|g\|_{\dot H^r} \le 1\}$.

Then, for $2 < p \le d/r$ and $0 < r < d/2$, we have the embeddings $\dot M^{p,d/r} \subset \dot X_r \subset \dot M^{2,d/r}$.

Proof: We have the homogeneity properties $\|\lambda^{d/2} f(\lambda x)\|_2 = \|f(x)\|_2$ and $\|\lambda^{d/2-r} f(\lambda x)\|_{\dot{H}^r} = \|f(x)\|_{\dot{H}^r}$. This gives a homogeneity property for the norm of $\dot{X}_r$: $\|\lambda^r f(\lambda x)\|_{\dot{X}_r} = \|f(x)\|_{\dot{X}_r}$.

We will achieve a more precise result. Since $H^r = L^2 \cap \dot{X}_r$ and $\|f\|_{\dot{H}^r} \leq \|f\|_{H^r}$, we see that $\dot{X}_r \subset X_r$ and $\|f\|_{X_r} \leq \|f\|_{\dot{X}_r}$; by homogeneity, we have $\sup_{\lambda>0} \|\lambda^r f(\lambda x)\|_{X_r} \leq \|f\|_{\dot{X}_r}$. Conversely, we have, for every $g \in H^r$, the inequality $\|fg\|_2 \leq \|\lambda^r f(\lambda x)\|_{X_r} \|\lambda^{d/2-r} g(\lambda x)\|_{H^r}$; since we have $\lim_{\lambda\to+\infty} \|\lambda^{d/2-r} g(\lambda x)\|_{H^r} = \|g\|_{\dot{H}^r}$ and since H^r is dense in $\dot{H}^r$, we get the equality $\sup_{\lambda>0} \|\lambda^r f(\lambda x)\|_{X_r} = \|f\|_{\dot{X}_r}$ Proposition 21.2 is clearly a direct consequence of Theorem 21.1 and of the property $\sup_{\lambda>0} \|\lambda^{d/q} f(\lambda x)\|_{M^{p,q}} = \|f\|_{\dot{M}^{p,q}}$. □

The embedding $\dot{X}_r \subset \dot{M}^{2,d/r}$ is not an isomorphism, at least when $2/r \in \mathbb{N}$ (May [MAY 02], following an idea of Meyer [MEY 99]):

Proposition 21.3: *Let $r \in (0, d/2)$ and assume that $2/r = k \in \mathbb{N}$. Let $\varphi \in L^2(\mathbb{R}^k)$ and $\varphi \neq 0$. Then the function ϕ defined by $\phi(x_1, \ldots, x_d) = \varphi(x_1, \ldots, x_k)$ (i.e. $\phi = \varphi \otimes 1$) belongs to $\dot{M}^{2,d/r}(\mathbb{R}^d)$ and not to $\dot{X}_r$.*

Proof: Obviously, writing $x = (x_1, \ldots, x_d)$, $x' = (x_1, \ldots, x_k)$ and $x'' = (x_{k+1}, \ldots, x_d)$, we have

$$\int_{B(x,R)} |\phi(y)|^2 \, dy \leq \int_{B(x',R)} |\varphi(y')|^2 \, dy' \int_{B(x'',R)} dy'' \leq C R^{d-k} \|\varphi\|_2^2$$

so that $\phi \in \dot{M}^{2,\frac{2d}{k}}(\mathbb{R}^d)$. We now check that $\phi \notin \dot{X}_r$ with $r = k/2$ by looking the operation of pointwise multiplication with ϕ on functions $f(x) = \bar{\varphi}(x')g(x'') = \bar{\varphi} \otimes g$, where $g \in L^2(\mathbb{R}^{d-k})$. If $\phi \in \dot{X}_r$, we should have the existence of a constant C_r so that for all $g \in L^2(\mathbb{R}^{d-k)}$ the inequality $\|\phi \, (\bar{\varphi} \otimes g)\|_{\dot{H}^r} = \||\varphi|^2 \otimes g\|_{\dot{H}^r} \leq C_0 \|g\|_2$ holds. Using Parseval equalities, we would have

$$\iint \frac{|\int \hat{\varphi}(\eta') \bar{\hat{\varphi}}(\eta' - \xi') \, d\eta'|^2}{(|\xi'|^2 + |\xi''|^2)^r} |\hat{g}(\xi'')|^2 \, d\xi' \, d\xi'' \leq C_0^2 \, (2\pi)^k \int |\hat{g}(\xi'')|^2 \, d\xi''$$

which is equivalent to

$$\int \frac{|\int \hat{\varphi}(\eta') \bar{\hat{\varphi}}(\eta' - \xi') \, d\eta'|^2}{|\xi'|^{2r}} \, d\xi' \leq C_0^2 \, (2\pi)^k$$

In the neighborhood of $\xi' = 0$, we have the equivalence

$$\frac{|\int \hat{\varphi}(\eta') \bar{\hat{\varphi}}(\eta' - \xi') \, d\eta'|^2}{|\xi'|^{2r}} \sim \frac{\|\varphi\|_2^2}{|\xi'|^{2r}} = \frac{\|\varphi\|_2^2}{|\xi'|^k}$$

which gives $\int \frac{|\int \hat{\varphi}(\eta') \bar{\hat{\varphi}}(\eta' - \xi') \, d\eta'|^2}{|\xi'|^{2r}} \, d\xi' = \infty$. Thus, $\phi \notin \dot{X}_r$. □

2. Solutions in X_r

If we want to study the existence of solutions for the Navier–Stokes equations in $(\mathcal{C}([0,T), X_r))^d$ or $(L^p_t X_r)^d$, we may directly apply the results of the preceding chapters. Indeed, X_r is a Banach space of shift–invariant local measures:

Proposition 21.4: *Let $r \in (0, d/2)$.*

(A) i) Let us define the space Y_r in the following way: $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to Y_r if and only if f can be decomposed as a series $f = \sum_{n\in\mathbb{N}} f_n g_n$ with $f_n \in L^2$, $g_n \in H^r$ and $\sum_{n\in\mathbb{N}} \|f_n\|_2 \|g_n\|_{H^r} < \infty$. We define the norm of Y_r as $\|f\|_{Y_r} = \inf\{\sum_{n\in\mathbb{N}} \|f_n\|_2 \|g_n\|_{H^r} \ / \ f = \sum_{n\in\mathbb{N}} f_n g_n\}$. Then Y_r is a shift-invariant Banach space of test functions.

ii) X_r is a shift-invariant Banach space of local measures. More precisely, X_r is the dual of the space Y_r.

(B) Similar results hold for $\dot{Y}_r$ and $\dot{X}_r$, where the Banach space $\dot{Y}_r$ is defined by $\|f\|_{\dot{Y}_r} = \inf\{\sum_{n\in\mathbb{N}} \|f_n\|_2 \|g_n\|_{\dot{H}^r} \ / \ f = \sum_{n\in\mathbb{N}} f_n g_n\}$.

Proof: We first prove that $\|.\|_{Y_r}$ and $\|.\|_{\dot{Y}_r}$ are norms. With the inequalities $\|g\|_q \leq C_r \|g\|_{\dot{H}^r}$ for $1/q = 1/2 - d/r$ and $\|g\|_{\dot{H}^r} \leq \|g\|_{H^r}$, we get that, for $1/p = 1 - d/r$, $\|f\|_p \leq C_r \|f\|_{\dot{Y}_r} \leq C_r \|f\|_{Y_r}$. Therefore, if the distribution f belongs to Y_r and has a norm equal to 0, it must be the null distribution $f = 0$ (and the same for $\dot{Y}_r$).

Since the test functions are dense in L^2, H^r and $\dot{H}^r$, we find that they are dense in Y_r and $\dot{Y}_r$. So our spaces are shift-invariant Banach spaces of test functions.

X_r operates on Y_r: if $f \in Y_r$ is decomposed as $f = \sum_{n\in\mathbb{N}} f_n g_n$, we saw that $f \in L^p$ $(1/p = 1 - d/r)$, hence is a measurable function; if $h \in X_r$, h is a measurable function too, since $X_r \subset M^{2,d/r}$; moreover, we have $|hf| \leq \sum_{n\in\mathbb{N}} |h(x) f_n(x) g_n(x)|$; since $\|h f_n g_n\|_1 \leq \|f_n\|_2 \|h g_n\|_2 \leq \|f_n\|_2 \|g_n\|_{H^r} \|h\|_{X_r}$, we find that $h\bar{f}$ is integrable over $\mathbb{R}^d$, and we may define $\langle h|f\rangle = \int h\bar{f}\, dx$. We have $|\langle h|f\rangle| \leq \sum_{n\in\mathbb{N}} \|h f_n g_n\|_1 \leq \|h\|_{X_r} \sum_{n\in\mathbb{N}} \|f_n\|_2 \|g_n\|_{H^r}$. This gives $|\langle h|f\rangle| \leq \|h\|_{X_r} \|f\|_{Y_r}$. Thus, $X_r \subset (Y_r)'$ and $\|f\|_{Y_r'} \leq \|f\|_{X_r}$.

Conversely, let $T \in (Y_r')$. Let K be a compact subset of $\mathbb{R}^d$ and ω_K be a compactly supported smooth function equal to 1 in a neighborhood of K; then, if $\varphi \in \mathcal{D}_K$, we have

$$|\langle T|\varphi\rangle| = |\langle T|\varphi\omega_K\rangle| \leq \|T\|_{Y_r'} \|\varphi\|_2 \|\omega_K\|_{H^r}$$

Thus, T is defined by a locally square-integrable function h: $\langle T|\varphi\rangle = \int h\bar{\varphi}\, dx$. If φ and ψ are two test functions, we get $|\int h\bar{\varphi}\psi\, dx| \leq \|T\|_{Y_r'} \|\varphi\|_2 \|\psi\|_{H^r}$. This gives that the function $h\psi$ belongs to L^2 with a norm less or equal to $\|T\|_{Y_r'} \|\psi\|_{H^r}$. Since $\mathcal{D}$ is dense in H^r, this inequality extends to the whole of

H^r, and this proves that $h \in X_r$ and $\|h\|_{X_r} \leq \|T\|_{Y'_r}$. Thus, $X_r = Y'_r$ with equality of norms. □

Thus, we know how to solve the Navier–Stokes equations for initial values in X_r with $r \leq 1$ (Chapter 17) or in $B^{s,q}_{X_r}$ with $r \leq s+1$ and with $s > 0$ (Chapter 19) or $-1 < s < 0$ (Chapter 20).

Similarly, we may complete the uniqueness criterion of Serrin (Theorem 14.2):

Definition 21.3: *A vector field $\vec{u}$ on $(0,T) \times \mathbb{R}^d$ will be labeled r-regular for some $r \in [0,1]$ if $\vec{u}$ belongs to $L^\sigma((0,T),(X_r)^d)$ with $2/\sigma = 1 - r$ $(0 < r < 1)$ or to $\mathcal{C}([0,T],(X_1^{(0)})^d)$ $(r = 1)$, where $X_1^{(0)}$ is the space of smooth elements in X_1 (i.e. the closure in X_1 of the space of the test functions).*

A smooth r-regular vector field is an r-regular vector field so that, for almost every t, $\vec{u}$ belongs to $(X_r^{(0)})^d)$ where $X_r^{(0)}$ is the space of smooth elements in X_r.

r-regular solutions of the Navier–Stokes equations are provided by initial values in $(X_1^{(0)})^d$ or in $(B^{-2/\sigma,\sigma}_{X_r})^d$ $(0 < r < 1)$.

Theorem 21.2: (Serrin's uniqueness theorem)

Let $\vec{u}_0 \in (L^2(\mathbb{R}^d))^d$ with $\vec{\nabla}.\vec{u}_0 = 0$. Assume that there exists a solution $\vec{u}$ of the Navier–Stokes equations on $(0,T) \times \mathbb{R}^d$ (for some $T \in (0,+\infty]$) with initial value $\vec{u}_0$ so that:
i) $\vec{u} \in L^\infty((0,T),(L^2(\mathbb{R}^d)^d)$;
ii) $\vec{u} \in L^2((0,T),(H^1(\mathbb{R}^d)^d)$;
iii) For some $r \in (0,1]$, $\vec{u}$ is r-regular.

Then, $\vec{u}$ is the unique Leray solution associated with $\vec{u}_0$ on $(0,T)$.

Proof: Let $\vec{v}$ be another solution associated with $\vec{u}_0$ on $(0,T)$ (with associated pressure q) so that $\vec{v} \in L^\infty((0,T),(L^2(\mathbb{R}^d)^d) \cap L^2((0,T),(H^1(\mathbb{R}^d)^d)$. As in Chapter 14, we use a smoothing function $\theta(t,x) = \alpha(t)\beta(x) \in \mathcal{D}(\mathbb{R}^{d+1})$ where α is supported in $[-1,1]$, with $\iint \theta\, dx\, dt = 1$, and define for $\epsilon > 0$, $\theta_\epsilon(t,x) = \frac{1}{\epsilon^{d+1}}\theta(\frac{t}{\epsilon},\frac{x}{\epsilon})$. Then, $\theta_\epsilon * \vec{u}$ and $\theta_\epsilon * \vec{v}$ are smooth functions on $(\epsilon, T-\epsilon) \times \mathbb{R}^d$ and we may write $\partial_t\big((\theta_\epsilon * \vec{u}).(\theta_\epsilon * \vec{v})\big) = (\theta_\epsilon * \partial_t\vec{u}).(\theta_\epsilon * \vec{v}) + (\theta_\epsilon * \vec{u}).(\theta_\epsilon * \partial_t\vec{v})$. We then obtain:

$$\begin{gathered}
\partial_t\big((\theta_\epsilon * \vec{u}).(\theta_\epsilon * \vec{v})\big) = \\
\vec{\nabla}.\big((\theta_\epsilon * [\vec{\nabla} \otimes \vec{u}]).(\theta_\epsilon * \vec{v})\big) - (\theta_\epsilon * [\vec{\nabla} \otimes \vec{u}]).(\theta_\epsilon * [\vec{\nabla} \otimes \vec{v}]) \\
-\vec{\nabla}.\big((\theta_\epsilon * [\vec{u} \otimes \vec{u}]).(\theta_\epsilon * \vec{v})\big) + (\theta_\epsilon * [\vec{u} \otimes \vec{u}]).(\theta_\epsilon * [\vec{\nabla} \otimes \vec{v}]) \\
-\vec{\nabla}.\big((\theta_\epsilon * p)(\theta_\epsilon * \vec{v})\big) \\
+\vec{\nabla}.\big((\theta_\epsilon * \vec{u}).(\theta_\epsilon * [\vec{\nabla} \otimes \vec{v}])\big) - (\theta_\epsilon * [\vec{\nabla} \otimes \vec{u}]).(\theta_\epsilon * [\vec{\nabla} \otimes \vec{v}]) \\
-\vec{\nabla}.\big((\theta_\epsilon * \vec{u}).(\theta_\epsilon * [\vec{v} \otimes \vec{v}])\big) + (\theta_\epsilon * [\vec{\nabla} \otimes \vec{u}]).(\theta_\epsilon * [\vec{v} \otimes \vec{v}]) \\
-\vec{\nabla}.\big((\theta_\epsilon * q)(\theta_\epsilon * \vec{u})\big)
\end{gathered}$$

We integrate this equality against a test function $\psi(t)\varphi(x/R)$ with φ equal to 1 in a neighborhood of 0, and we let R go to ∞. The terms, which are written $\vec{\nabla}.\vec{F}$ with $\vec{F} \in (L^1((0,T)\times \mathbb{R}^d))^d$, will have a null contribution. Since $\vec{u}$ and $\vec{v}$ belong to $(L^\infty L^2)^d \cap (L^2 H^1)^d \subset (L^4 H^{1/2})^d \subset (L^4 L^q)^d$ with $1/q = 1/2 - 1/(2d)$, we find that the pressures p and q belong to $L^2 L^{d/(d-1)}$, and so do $\theta_\epsilon * p$ and $\theta_\epsilon * q$; since $\theta_\epsilon * \vec{u}$ and $\theta_\epsilon * \vec{v}$ belong to $(L^\infty_t L^2_x)^d \cap L^\infty((0,T)\times\mathbb{R}^d))^d$, we find that they belong to $(L^2 L^d)^d$, so that $(\theta_\epsilon * p)(\theta_\epsilon * \vec{v})$ and $(\theta_\epsilon * q)(\theta_\epsilon * \vec{u})$ belong to $(L^1((0,T)\times\mathbb{R}^d))^d$. Hence, we get the equality in $\mathcal{D}'(\epsilon, T-\epsilon)$:

$$\begin{aligned}\partial_t \int (\theta_\epsilon * \vec{u}).(\theta_\epsilon * \vec{v})\,dx = \quad & -2\int (\theta_\epsilon * [\vec{\nabla}\otimes\vec{u}]).(\theta_\epsilon * [\vec{\nabla}\otimes\vec{v}])\,dx \\ & +\int (\theta_\epsilon * [\vec{u}\otimes\vec{u}]).(\theta_\epsilon * [\vec{\nabla}\otimes\vec{v}])\,dx \\ & +\int (\theta_\epsilon * [\vec{v}\otimes\vec{v}]).(\theta_\epsilon * [\vec{\nabla}\otimes\vec{u}])\,dx\end{aligned}$$

We may rewrite the last summand as

$$\int (\theta_\epsilon * [\vec{v}\otimes\vec{v}]).(\theta_\epsilon * [\vec{\nabla}\otimes\vec{u}])\,dx = -\int (\theta_\epsilon * [\vec{\nabla}.(\vec{v}\otimes\vec{v})]).(\theta_\epsilon * \vec{u})\,dx$$

The next step is to check that $\vec{u}\otimes\vec{u} \in (L^2((0,T)\times\mathbb{R}^d))^{d\times d}$. We write $L^2_t H^1 \cap L^\infty_t L^2_x \subset L^{2/r}_t H^r_x$ for $0 \le r \le 1$; we then write that the pointwise product maps $L^{2/r}H^r \times L^{2/(1-r)}X_r$ to L^2L^2.

We now let ϵ go to 0. When $\epsilon \to 0$ and $f \in L^2((0,T)\times\mathbb{R}^d)$, $\theta_\epsilon * f$ is strongly convergent to f in $L^2((0,T)\times\mathbb{R}^d)$. Thus, we have the following convergences in $\mathcal{D}'(0,T)$: $\partial_t \int (\theta_\epsilon * \vec{u}).(\theta_\epsilon * \vec{v})\,dx \to \partial_t \int \vec{u}.\vec{v}\,dx$, $\int (\theta_\epsilon * [\vec{\nabla}\otimes\vec{u}]).(\theta_\epsilon * [\vec{\nabla}\otimes\vec{v}])\,dx \to \int \vec{\nabla}\otimes\vec{u}.\vec{\nabla}\otimes\vec{v}\,dx$ and $\int (\theta_\epsilon * [\vec{u}\otimes\vec{u}]).(\theta_\epsilon * [\vec{\nabla}\otimes\vec{v}])\,dx \to \int \vec{u}\otimes\vec{u}.\vec{\nabla}\otimes\vec{v}\,dx = \int \vec{u}.(\vec{u}.\vec{\nabla})\vec{v}\,dx$. Thus, we must study the convergence of the last summand $\int (\theta_\epsilon * [\vec{\nabla}.(\vec{v}\otimes\vec{v})]).(\theta_\epsilon * \vec{u})\,dx$.

We begin by rewriting $\theta_\epsilon * [\vec{\nabla}.(\vec{v}\otimes\vec{v})]$ as $\theta_\epsilon * [(\vec{v}.\vec{\nabla})\vec{v}]$. We know (from Proposition 21.4) that pointwise product maps $H^r \times L^2$ in the predual Y_r of X_r and that smooth functions are dense in Y_r; thus, $\theta_\epsilon * [(\vec{v}.\vec{\nabla})\vec{v}]$ converges strongly to $(\vec{v}.\vec{\nabla})\vec{v}$ in $(L^{\frac{2}{1+r}}Y_r)^d$ while $\theta_\epsilon * \vec{u}$ converges weakly to $\vec{u}$ in $(L^{\frac{2}{1-r}}X_r)^d$. This gives the convergence (in $\mathcal{D}'((0,T))$) $\int (\theta_\epsilon * [\vec{\nabla}.(\vec{v}\otimes\vec{v})]).(\theta_\epsilon * \vec{u})\,dx \to \int \vec{u}.(\vec{v}.\vec{\nabla})\vec{v}\,dx$. We have thus obtained the following equality in $\mathcal{D}'(0,T)$:

$$\partial_t \int \vec{u}.\vec{v}\,dx = -2\int \vec{\nabla}\otimes\vec{u}.\vec{\nabla}\otimes\vec{v}\,dx + \int \vec{u}.(\vec{u}.\vec{\nabla})\vec{v}\,dx - \int \vec{u}.(\vec{v}.\vec{\nabla})\vec{v}\,dx$$

Since $\vec{u}\otimes\vec{u} \in (L^2((0,T)\times\mathbb{R}^d))^{d\times d}$ and $\vec{u} = e^{t\Delta}\vec{u}(0) - \mathbb{P}\int_0^t e^{(t-s)\Delta}\vec{\nabla}.(\vec{u}\otimes\vec{u})\,ds$, $t \mapsto \vec{u}$ is continuous from $[0,T]$ to $(L^2(dx))^d$ and $t \mapsto \int \vec{u}.\vec{v}\,dx$ is continuous. Thus, we may integrate our equality and obtain the equality

$$\int \vec{u}(t,x).\vec{v}(t,x)\,dx + 2\int_0^t \int_{\mathbb{R}^d} \vec{\nabla}\otimes\vec{u}.\vec{\nabla}\otimes\vec{v}\,dx\,ds =$$

$$\|\vec{u}_0\|_2^2 + \int_0^t \int_{\mathbb{R}^d} \vec{u}.(\vec{u}.\vec{\nabla})\vec{v}\ dx\ ds - \int_0^t \int_{\mathbb{R}^d} \vec{u}.(\vec{v}.\vec{\nabla})\vec{v}\ dx\ ds$$

Of course, this equality holds as well for $\vec{v} = \vec{u}$.

Now, if we assume that $\vec{v}$ satisfies the Leray inequality

$$\|\vec{v}(t)\|_2^2 + 2\int_0^t \|\vec{\nabla} \otimes \vec{v}\|_2^2\ ds \le \|\vec{u}_0\|_2^2,$$

we get the following inequality for $\vec{u} - \vec{v}$:

$$\|\vec{u}(t,.)-\vec{v}(t,.)\|_2^2 \le -2\int_0^t\int_{\mathbb{R}^d} |\vec{\nabla} \otimes (\vec{u}-\vec{v})|^2\ dx\ ds - 2\int_0^t \int_{\mathbb{R}^d} \vec{u}.\big((\vec{u}-\vec{v}).\vec{\nabla}\big)\vec{v}\ dx\ ds$$

Moreover, we have the antisymmetry property $\int_0^t\int_{\mathbb{R}^d}\vec{u}.\big((\vec{u}-\vec{v}).\vec{\nabla}\big)\vec{u}\ dx\ ds = 0$.

For $r < 1$, we quickly conclude as in Serrin's theorem (Theorem 14.3) since we have

$$\begin{array}{c} |\int_\tau^t\int_{\mathbb{R}^d}\vec{u}.\big((\vec{u}-\vec{v}).\vec{\nabla}\big)(\vec{v}-\vec{u})\ dx\ ds| \\ \le C_r(\int_\tau^t \|\vec{u}\|_{X_r}^{\frac{2}{1-r}}\ ds)^{(1-r)/2}(\int_0^t \|\vec{v}-\vec{u}\|_{H^1}^2\ ds)^{1/2}(\int_\tau^t \|\vec{v}-\vec{u}\|_{H^r}^{\frac{2}{r}}\ ds)^{\frac{r}{2}} \end{array}$$

If $\vec{u} = \vec{v}$ on $[0,\tau]$, we find that

$$\sup_{0<s\le t} \|\vec{u}-\vec{v}\|_2^2 \le C_r''(\int_\tau^t \|\vec{u}\|_{X_r}^{\frac{2}{1-r}}\ ds)^{1/2} \sup_{0<s\le t} \|\vec{u}-\vec{v}\|_2^2$$

and uniqueness is valid on a bigger interval. By weak continuity of $t \mapsto \vec{v}$, we find $\vec{u} = \vec{v}$.

For $r = 1$, we easily conclude as in Von Wahl's theorem (Theorem 14.4). If $T_0 < T$, then for each $\epsilon > 0$ we may split $\vec{u}$ on $[0,T_0]$ in $\vec{u} = \vec{\alpha} + \vec{\beta}$ with $\vec{\beta} \in (L^\infty((0,T_0)\times\mathbb{R}^d)^d$ and $\|\vec{\alpha}\|_{L^\infty X_1} < \epsilon$. Then we write

$$\begin{array}{c} |\int_0^t\int_{\mathbb{R}^d}\vec{u}.\big((\vec{u}-\vec{v}).\vec{\nabla}\big)(\vec{v}-\vec{u})\ dx\ ds| \\ \le C\|\vec{\alpha}\|_{L^\infty X_1}\int_0^t \|\vec{v}-\vec{u}\|_{H^1}^2\ ds \\ +\|\vec{\beta}\|_\infty(\int_0^t\int_{\mathbb{R}^d}|\vec{\nabla}\otimes(\vec{v}-\vec{u})|^2\ dx\ ds)^{1/2}(\int_0^t\int_{\mathbb{R}^d}|\vec{v}-\vec{u}|^2\ dx\ ds)^{1/2} \\ \le 2C\epsilon\int_0^t\int_{\mathbb{R}^d}|\vec{\nabla}\otimes(\vec{v}-\vec{u})|^2\ dx\ ds + (\frac{4}{C\epsilon}\|\vec{\beta}\|_\infty^2 + C\epsilon)\int_0^t\int_{\mathbb{R}^d}|\vec{v}-\vec{u}|^2\ dx\ ds. \end{array}$$

Choosing ϵ so that $2C\epsilon < 1$, we get

$$\|\vec{v}(t,.)-\vec{u}(t,.)\|_2^2 \le (\frac{4}{C\epsilon}\|\vec{\beta}\|_\infty^2 + C\epsilon)\int_0^t \|\vec{v}(s,.)-\vec{u}(s,.)\|_2^2\ ds.$$

The Gronwall lemma gives then that $\vec{u} = \vec{v}$. $\square$

3. Perturbated Navier–Stokes equations

We now consider perturbated Stokes equations or Navier–Stokes equations, where the perturbations are defined with the help of r-regular vector fields:

Theorem 21.3:

Let $T < \infty$, $r_1, r_2, r_3 \in [0,1]$, let $\vec{V}$ be a smooth r_1-regular vector field on $(0,T)\times \mathbb{R}^d$ and let $\vec{W}$ be a smooth r_2-regular vector field on $(0,T)\times\mathbb{R}^d$. Let P_η be the perturbated Stokes equations ($\eta = 0$) or Navier–Stokes equations ($\eta = 1$):

$$(P_\eta) \qquad \begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \eta \vec{\nabla}.(\vec{u}\otimes\vec{u}) - \vec{\nabla}.(\vec{V}\otimes\vec{u}) - \vec{\nabla}.(\vec{u}\otimes\vec{W}) - \vec{\nabla} p \\ \vec{\nabla}.\vec{u} = 0 \\ \vec{u}(0,.) = \vec{u}_0 \end{cases}$$

with

$$\begin{cases} \eta \in \{0,1\} \\ \vec{\nabla}.\vec{V} = \vec{\nabla}.\vec{W} = 0 \\ \vec{\nabla}.\vec{u}_0 = 0 \qquad \vec{u}_0 \in (L^2(\mathbb{R}^d))^d \end{cases}$$

Then the problem (P_η) has a solution $\vec{u} \in L^\infty((0,T),(L^2)^d) \cap L^2((0,T),(H^1)^d)$ and $\lim_{t\to 0} \|\vec{u}-\vec{u}_0\|_2 = 0$.

If $\eta = 0$ (linear case), then the solution $\vec{u}$ is unique and we have the energy equality:

$$(21.1) \qquad \|\vec{u}(t)\|_2^2 + 2\int_0^t \|\vec{\nabla}\otimes\vec{u}\|_2^2 \, ds = \|\vec{u}_0\|_2^2 + 2\int\int_0^t ((\vec{u}.\vec{\nabla})\ \vec{u}).\vec{W}\ dx\ ds$$

If $\eta = 1$ (non-linear case), then one may choose $\vec{u}$ such that we have the energy inequality:

$$(21.2) \qquad \|\vec{u}(t)\|_2^2 + 2\int_0^t \|\vec{\nabla}\otimes\vec{u}\|_2^2 \, ds \le \|\vec{u}_0\|_2^2 + 2\int\int_0^t ((\vec{u}.\vec{\nabla})\ \vec{u}).\vec{W}\ dx\ ds$$

Moreover, if $\vec{u}$ and $\vec{v}$ are two solutions of (P_η) with the same initial value, if $\vec{u}$ and $\vec{v}$ belong to $L^\infty((0,T),(L^2)^d) \cap L^2((0,T),(H^1)^d)$, if $\vec{u}$ satisfies inequality (21.2) and $\vec{v}$ is r_3-regular, then $\vec{u} = \vec{v}$ (Serrin's uniqueness criterion).

Proof:

Step 1: Elimination of the pressure

If we take the divergence of equation (P_η), we obtain

$$\Delta p = -\eta(\vec{\nabla}\otimes\vec{\nabla}).(\vec{u}\otimes\vec{u}) - (\vec{\nabla}\otimes\vec{\nabla}).(\vec{V}\otimes\vec{u}) - (\vec{\nabla}\otimes\vec{\nabla}).(\vec{u}\otimes\vec{W})$$

For general classes of solutions, this allows elimination of the pressure by the formula $\vec{\nabla} p = \vec{\nabla}\frac{1}{\Delta}\Delta p$. We just adapt the proofs of Theorems 11.1 and 11.2 to deal with equation (P_η). In order to define $\vec{u}\otimes\vec{W}$, we will not consider the class $(L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d))^d$, but the smaller class $(L^2_tL^2_{uloc,x}((0,T)\times\mathbb{R}^d))^d$ (as in Furioli, Lemarié-Rieusset and Terraneo [FURLT 00]). Indeed, the product of a function in L^2_{uloc} by a function in X_r is uniformly locally in H^{-r} and since a function f in the Schwartz class will satisfy, for any $\phi\in\mathcal{D}(\mathbb{R}^d)$ with $\sum_{k\in\mathbb{Z}^d}\phi(x-k)=1$, the inequality $\sum_{k\in\mathbb{Z}^d}\|\phi(x-k)f\|_{H^r}<\infty$, we get for $j<0$ the inequality $\|\mathbb{P}\Delta_j\vec{\nabla}.(\vec{u}\otimes\vec{W})\|_\infty\leq C2^j\|S_0(\vec{u}\otimes\vec{W})\|_\infty\leq C'2^j\|\vec{u}\|_{L^2_{uloc}}\|\vec{W}\|_{X_r}$ so that $\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{W})$ is well defined. Thus, we get from (P_η) the equations:

$$\lim_{j\to-\infty}(Id-S_j)(\partial_t\vec{u}-\Delta\vec{u})=-\eta\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})-\mathbb{P}\vec{\nabla}.(\vec{V}\otimes\vec{u})-\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{W})$$

To get rid of S_j, we must assume that $\vec{u}$ vanishes at infinity, hence we replace L^2_{uloc} by the subspace E_2. We then obtain the following result:

Lemma 21.4: (Equivalent formulations for the Navier–Stokes equations)

Let $\vec{u}\in\cap_{T<T^*}L^2((0,T),(E_2)^d)$. Then the following three assertions are equivalent:

(A) $\vec{u}$ is a weak solution of the generalized Stokes or Navier–Stokes equations on $(0,T^*)\times\mathbb{R}^d$

$$\begin{cases}\partial_t\vec{u}=\Delta\vec{u}-\eta\vec{\nabla}.(\vec{u}\otimes\vec{u})-\vec{\nabla}.(\vec{V}\otimes\vec{u})-\vec{\nabla}.(\vec{u}\otimes\vec{W})-\vec{\nabla}p\\ \vec{\nabla}.\vec{u}=0\\ \vec{u}(0,.)=\vec{u}_0\end{cases}$$

where $\eta,\vec{V}$ and $\vec{W}$ satisfy the hypotheses of Theorem 21.2.

(B) $\vec{u}$ is a solution of $\vec{\nabla}.\vec{u}=0,\partial_t\vec{u}=\Delta\vec{u}-\mathbb{P}\vec{\nabla}.(\eta\,\vec{u}\otimes\vec{u}+\vec{V}\otimes\vec{u}+\vec{u}\otimes\vec{W})$

(C) there exists $\vec{u}_0\in(\mathcal{S}'(\mathbb{R}^3))^3$, $\vec{\nabla}.\vec{u}_0=0$, so that $\vec{u}=e^{t\Delta}\vec{u}_0-\int_0^te^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\eta\,\vec{u}\otimes\vec{u}+\vec{V}\otimes\vec{u}+\vec{u}\otimes\vec{W})\,ds$.

Step 2: The mollified equations

We follow the lines of Chapter 13 and introduce a modified equation, choosing a function $\omega\in\mathcal{D}(\mathbb{R}^d)$, with $\int_{\mathbb{R}^d}\omega\,dx=1$ and a function $\zeta\in\mathcal{D}(\mathbb{R}^{d+1})$ with $\int_{\mathbb{R}^{d+1}}\zeta\,dt\,dx=1$ and solving for $\epsilon>0$ the equations $(P_{\eta,\epsilon})$ given by

$$\begin{cases}\partial_t\vec{u}=\Delta\vec{u}-\eta\vec{\nabla}.((\vec{u}*\omega_\epsilon)\otimes\vec{u})-\vec{\nabla}.((\vec{V}*\zeta_\epsilon)\otimes\vec{u})-\vec{\nabla}.(\vec{u}\otimes(\vec{W}*\zeta_\epsilon))-\vec{\nabla}p\\ \vec{\nabla}.\vec{u}=0\\ \vec{u}(0,.)=\vec{u}_0\end{cases}$$

where $\omega_\epsilon=\frac{1}{\epsilon^d}\omega(\frac{x}{\epsilon})$ and $\zeta_\epsilon=\frac{1}{\epsilon^{d+1}}\zeta(\frac{t}{\epsilon},\frac{x}{\epsilon})$ (we extend $\vec{V}$ to the whole $\mathbb{R}\times\mathbb{R}^d$ by taking $\vec{V}=0$ for $t<0$ or $t>T$ $(0\leq r<1)$ or, for $r=1$ and for a $\gamma\in\mathcal{D}(\mathbb{R})$ so that $\gamma(0)=1$, $\vec{V}=\gamma(t)\vec{V}(0)$ for $t<0$ and $\vec{V}=\gamma(t-T)\vec{V}(T)$ for $t>T$). Following Lemma 21.4, we are led to solve the fixed-point problem $\vec{u}=e^{t\Delta}\vec{u}_0-B_\epsilon(\vec{u})-A_\epsilon(\vec{u})$, where the bilinear operator B_ϵ is given by

$B_\epsilon(\vec{u},\vec{v}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.((\vec{u}*\omega_\epsilon)\otimes\vec{v})\ ds$ and the linear operator A_ϵ is given by $A_\epsilon(\vec{u}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.((\vec{V}*\zeta_\epsilon)\otimes\vec{u}+\vec{u}\otimes(\vec{W}*\zeta_\epsilon))\ ds$. Now, we notice that for $\vec{u}_0\in(L^2)^d$, we have $e^{t\Delta}\vec{u}_0\in\mathcal{C}([0,T],(L^2)^d)\cap L^2([0,T],(H^1)^d)$. Moreover $\|B_\epsilon(\vec{u},\vec{v})(t)\|_2 \le C_\epsilon\sqrt{t}\ \sup_{0<s<t}\|\vec{u}\|_2\ \sup_{0<s<t}\|\vec{v}\|_2$ and $\|A_\epsilon(\vec{u})(t)\|_2 \le C_\epsilon\sqrt{t}\ \sup_{0<s<t}\|\vec{u}\|_2$ where C_ϵ does not depend on t. This proves that one can find a solution of $(P_{\epsilon,\eta})$ in $\mathcal{C}([0,T_\epsilon],(L^2)^d)$ with a positive T_ϵ which is $O(\|\vec{u}_0\|_2^{-2})$. Moreover, this solution belongs to $L^2([0,T_\epsilon],(H^1)^d)$ since we have $\|B_\epsilon(\vec{u},\vec{u})\|_{L^2([0,T_\epsilon],(\dot{H}^1)^d)} \le C'_\epsilon\sup_{0<s<t}\|\vec{u}\|_2^2$ and $\|A_\epsilon(\vec{u})(t)\|_{L^2([0,T_\epsilon],(\dot{H}^1)^d)} \le C'_\epsilon\sup_{0<s<t}\|\vec{u}\|_2$.

This solution can be extended to $[0,T]\times\mathbb{R}^d$: we compute $\partial_t|\vec{u}|^2$ as $2\langle\partial_t\vec{u}|\vec{u}\rangle$ (since $\vec{u}\in L^2([0,T_\epsilon],(H^1)^d)$ while $\partial_t\vec{u}\in L^2([0,T_\epsilon],(H^{-1})^d)$), and writing $\vec{V}_\epsilon$ for $\vec{V}*\zeta_\epsilon$ and $\vec{W}_\epsilon$ for $\vec{W}*\zeta_\epsilon$, we obtain

$$\begin{aligned}\partial_t|\vec{u}|^2+2|\vec{\nabla}\otimes\vec{u}|^2 = &\ \Delta|\vec{u}|^2-\eta\ \vec{\nabla}.(|\vec{u}|^2(\vec{u}*\omega_\epsilon))-\vec{\nabla}.(|\vec{u}|^2\vec{V}_\epsilon)-2\vec{\nabla}.((\vec{u}.\vec{W}_\epsilon)\ \vec{u})\\ &+2((\vec{u}.\vec{\nabla})\ \vec{u}).\vec{W}_\epsilon-2\vec{\nabla}.(p\ \vec{u})\end{aligned}$$

and finally (integrating in x)

$$\|\vec{u}(t)\|_2^2+2\int\int_0^t|\vec{\nabla}\otimes\vec{u}|^2\ dx\ ds = \|\vec{u}_0\|_2^2+2\int\int_0^t((\vec{u}.\vec{\nabla})\ \vec{u}).\vec{W}_\epsilon\ dx\ ds$$

Since $\vec{W}_\epsilon$ is bounded, we find that

$$2\int\int_0^t((\vec{u}.\vec{\nabla})\ \vec{u}).\vec{W}_\epsilon\ dx\ ds \le \int\int_0^t|\vec{\nabla}\otimes\vec{u}|^2\ dx\ ds + C_\epsilon\int_0^t\|\vec{u}\|_2^2\ ds$$

where C_ϵ depends only on $\|\vec{W}_\epsilon\|_\infty$ and not on t nor on T_ϵ. The Gronwall lemma then gives the existence of $\vec{u}$ on the whole $[0,T]\times\mathbb{R}^d$.

Step 3: The weak limit

We call $\vec{u}_\epsilon$ the solution of $(P_{\eta,\epsilon})$. We have seen that

$$\|\vec{u}_\epsilon(t)\|_2^2+2\int\int_0^t|\vec{\nabla}\otimes\vec{u}_\epsilon|^2\ dx\ ds = \|\vec{u}_0\|_2^2+2\int\int_0^t((\vec{u}_\epsilon.\vec{\nabla})\ \vec{u}_\epsilon).\vec{W}_\epsilon\ dx\ ds.$$

Since the norm of X_r is invariant under translation, we have that $\|\vec{W}_\epsilon\|_{L^\sigma((0,T),(X_r)^d)} \le \|\vec{W}\|_{L^\sigma((0,T),(X_r)^d)}$ and thus

$$2\int\int_0^t((\vec{u}_\epsilon.\vec{\nabla})\ \vec{u}_\epsilon).\vec{W}_\epsilon\ dx\ ds \le C\|\vec{u}_\epsilon\|_{L^2(I,(\dot{H}^1)^d)}\|\vec{u}_\epsilon\|_{L^\rho(I,(H^r)^d)}\|\vec{W}\|_{L^\sigma(I,(X_r)^d)}$$

where $I=(0,T)$, $2/\sigma = 1-r$ and $1/\rho = r/2$. If $r<1$, we write for every positive α:

$$\begin{aligned}\|\vec{u}_\epsilon\|_{L^2((H^1)^d)}\|\vec{u}_\epsilon\|_{L^\rho((H^r)^d)} &\le \|\vec{u}_\epsilon\|_{L^2((H^1)^d)}^{1+r}\|\vec{u}_\epsilon\|_{L^\infty(,(L^2)^d)}^{1-r}\\ &\le \tfrac{(1+r)\alpha}{2}\|\vec{u}_\epsilon\|_{L^2((H^1)^d)}^2+\tfrac{(1-r)}{2\alpha^{(1+r)/(1-r)}}\|\vec{u}_\epsilon\|_{L^\infty((L^2)^d)}^2\end{aligned}$$

For α small enough, this gives (by the Gronwall lemma) that we have $\sup_{\epsilon>0}\sup_{0<t<T}\|\vec{u}_\epsilon\|_2<\infty$ and $\sup_{\epsilon>0}\int_0^T\|\vec{u}_\epsilon\|_{H^1}\,dt<\infty$. For the case $r=1$, we follow Von Wahl [WAH 85] and split $\vec{W}$ in $\vec{X}+\vec{Y}$, where $\sup_{0<t<T}\|\vec{X}\|_{X_1}\leq\alpha$ and $\vec{Y}\in(L^\infty((0,T)\times\mathbb{R}^d)^d$ (we use the density of tests functions in $X_1^{(0)}$ and the compactness of $[0,T]$). We then obtain

$$2\int\int_0^t((\vec{u}_\epsilon.\vec{\nabla})\ \vec{u}_\epsilon).\vec{W}_\epsilon\ dx\ ds\leq$$
$$\leq C\|\vec{u}_\epsilon\|_{L^2((\dot{H}^1)^d)}\big(\|\vec{u}_\epsilon\|_{L^2((H^1)^d)}\|\vec{X}\|_{L^\infty((X_1)^d)}+\sqrt{t}\ \|\vec{u}_\epsilon\|_{L^\infty((H^1)^d)}\|\vec{Y}\|_\infty\big)$$

which gives again (for α small enough) the control of $\vec{u}_\epsilon$ in norms $L^\infty((L^2)^d)$ and $L^2((H^1)^d)$.

The control on $\vec{u}$ gives a control on $\partial_t\vec{u}_\epsilon$; we have

$$\sup_\epsilon\int_0^T\|\vec{V}_\epsilon\otimes\vec{u}_\epsilon+\vec{u}_\epsilon\otimes\vec{W}_\epsilon\|_2^2\ ds<\infty\text{ and }\sup_\epsilon\int_0^T\|\vec{u}_\epsilon\otimes\vec{u}_\epsilon\|^2_{\frac{d}{d-1}}\ ds<\infty,$$

hence, we get $\sup_\epsilon\int_0^T\|\partial_t\vec{u}_\epsilon\|^2_{H^{-\tau}}\ ds<\infty$ with $\tau=1$ if $\eta=0$ and $\tau=d/2-1$ if $\eta=1$. Now, we use the following lemma (see Theorem 13.3).

Lemma 21.5: *a) Let $\tau>0$. If $(f_\epsilon)_{0<\epsilon<1}$ is a family of functions on $(0,T)\times\mathbb{R}^d$ so that for all $\phi\in\mathcal{D}((0,T)\times\mathbb{R}^d)$ we have: $(\phi f_\epsilon)_{0<\epsilon<1}$ is bounded in $L^2((0,T),H^1)$ and $(\phi\ \partial_t f_\epsilon)_{0<\epsilon<1}$ is bounded in $L^2((0,T),H^{-\tau})$, then there exists $f_\infty\in L^2_{loc}((0,T)\times\mathbb{R}^d)$ and a sequence ϵ_n converging to 0 so that for all $\phi\in\mathcal{D}((0,T)\times\mathbb{R}^d)$ the sequence ϕf_{ϵ_n} converges to ϕf_∞ strongly in $L^2((0,T)\times\mathbb{R}^d)$ and weakly in $L^2((0,T),H^1)$.*

b) If, moreover, for all $\phi\in\mathcal{D}(]0,T[\times\mathbb{R}^3)$ we have that $(\phi f_\epsilon)_{0<\epsilon<1}$ is bounded in $L^\infty((0,T),L^2)$, then the sequence ϕf_{ϵ_n} converges to ϕf_∞ strongly in $L^p((0,T),L^2)$ for all $p\in[1,\infty)$ and strongly in $L^3((0,T),L^{\frac{2d}{d-1}})$.

We apply the lemma to $\vec{u}_\epsilon$. We find a sequence ϵ_n and a vector field $\vec{u}$ so that for all $\phi\in\mathcal{D}((0,T)\times\mathbb{R}^3)$ we have convergence of $\phi\ \vec{u}_{\epsilon_n}$ to $\phi\ \vec{u}$, strongly in $L^2((0,T),(L^2)^d)$ and weakly in $L^2((0,T),(H^1)^d)$. Moreover, p_ϵ may be written as $p_\epsilon=q_\epsilon+\eta r_\epsilon$ with $q_\epsilon=-\frac{1}{\Delta}\vec{\nabla}\otimes\vec{\nabla}.(\vec{V}_\epsilon\otimes\vec{u}_\epsilon+\vec{u}_\epsilon\otimes\vec{W}_\epsilon)$ and $r_\epsilon=-\frac{1}{\Delta}\vec{\nabla}\otimes\vec{\nabla}.\vec{u}_\epsilon\otimes\vec{u}_\epsilon$: we have $\sup_{\epsilon>0}\|q_\epsilon\|_{L^2((0,T)\times\mathbb{R}^d)}<\infty$ and $\sup_{\epsilon>0}\|r_\epsilon\|_{L^2((0,T),L^{\frac{d}{d-1}})}<\infty$; hence, we may impose in the same way weak convergence of ϕp_{ϵ_n} to ϕp. Then, we have that each term of equations $(P_{\eta,\epsilon})$ applied to $\vec{u}_\epsilon$ converges in $\mathcal{D}'((0,T)\times\mathbb{R}^d)$ (we write ϵ instead of ϵ_n): $\vec{u}_\epsilon$ converges to $\vec{u}$; hence, $\partial_t\vec{u}_\epsilon$ converges to $\partial_t\vec{u}$ and $\Delta\vec{u}_\epsilon$ converges to $\Delta\vec{u}$; p_ϵ converges to p; hence, $\vec{\nabla}p_\epsilon$ converges to $\vec{\nabla}p$; moreover $\vec{V}_\epsilon$ converges strongly to $\vec{V}$ in $L^\sigma((0,T),(X_r)^d)$ (with $2/\sigma=1-r$) while $\vec{u}_\epsilon$ converges strongly in $(L^2_{loc}((0,T)\times\mathbb{R}^d)^d)$, and this gives local strong convergence of $\vec{V}_\epsilon\otimes\vec{u}_\epsilon$ in $L^{2/(2-r)}((0,T),(H^{-r})^{d\times d})$; thus, $\vec{\nabla}.(\vec{V}_\epsilon\otimes\vec{u}_\epsilon+\vec{u}_\epsilon\otimes\vec{W}_\epsilon)$ converges to $\vec{\nabla}.(\vec{V}\otimes\vec{u}+\vec{u}\otimes\vec{W})$; finally, $\vec{u}_\epsilon\otimes\vec{u}_\epsilon$ converges strongly in $(L^{3/2}_{loc}((0,T),L^{\frac{d}{d-1}}(\mathbb{R}^d)))^d$ to $\vec{u}\otimes\vec{u}$; hence, $\vec{\nabla}.((\vec{u}_\epsilon*\omega_\epsilon)\otimes\vec{u}_\epsilon)$ converges to $\vec{\nabla}.(\vec{u}\otimes\vec{u})$.

Moreover, for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $\varphi\, \partial_t \vec{u}_\epsilon$ is bounded in $L^2((0,T),(H^{-\tau})^d)$ and thus $\|\phi(\vec{u}_\epsilon(t) - \vec{u}_0)\|_{H^{-\tau}} \leq C(\varphi)\sqrt{t}$ where $C(\varphi)$ does not depend on t nor on ϵ; this proves that $\vec{u}(t)$ converges to $\vec{u}_0$ in $H^{-\tau}$. Thus, $\vec{u}$ is a solution of (P_η).

Step 4: Energy estimates

If $\eta = 0$, we find that $\vec{u} \in L^2((0,T),(H^1)^d)$ and $\partial_t \vec{u} \in L^2((0,T),(H^{-1})^d)$, so that $\partial_t |\vec{u}|^2 = 2\langle \partial_t \vec{u} | \vec{u}\rangle$ and equality (21.1) is obvious. If $\eta = 1$, we start from the equality obtained in Step 2:

$$\|\vec{u}_\epsilon(t)\|_2^2 + 2\int\int_0^t |\vec{\nabla} \otimes \vec{u}_\epsilon|^2 \; dx \; ds = \|\vec{u}_0\|_2^2 + 2\int\int_0^t ((\vec{u}_\epsilon.\vec{\nabla})\; \vec{u}_\epsilon).\vec{W}_\epsilon \; dx \; ds$$

The strong convergence of $\vec{u}_{\epsilon_n}$ in $(L^2_{loc}((0,T)\times\mathbb{R}^d))^d$ shows (through a diagonal extraction process) that we may find a subsequence (still written (ϵ_n)) so that we have pointwise convergence almost everywhere on $(0,T) \times \mathbb{R}^d$; hence, we have, for almost every t, pointwise convergence almost everywhere on $\mathbb{R}^d$. Since we have weak convergence of $\vec{\nabla}\otimes\vec{u}_{\epsilon_n}$ in $(L^2((0,T)\times\mathbb{R}^d))^{d\times d}$, we get that almost everywhere on $(0,T)$ we have

$$\|\vec{u}(t)\|_2^2 + 2\int\int_0^t |\vec{\nabla}\otimes\vec{u}_\epsilon|^2 \; dx \; ds \leq \|\vec{u}_0\|_2^2 + 2\limsup_{\epsilon_n \to 0} \int\int_0^t ((\vec{u}_{\epsilon_n}.\vec{\nabla})\; \vec{u}_{\epsilon_n}).\vec{W}_\epsilon \; dx \; ds$$

We now study the convergence of $((\vec{u}_{\epsilon_n}.\vec{\nabla})\; \vec{u}_{\epsilon_n}).\vec{W}_{\epsilon_n}$ to $((\vec{u}.\vec{\nabla})\; \vec{u}).\vec{W}$. We should use the convergence of $\vec{\nabla} \otimes \vec{u}_{\epsilon_n}$ to $\vec{\nabla} \otimes \vec{u}$ in $L^2((0,T),(L^2)^{d\times d})$ and of $\vec{u}_{\epsilon_n}$ to $\vec{u}$ in $L^{2/r_2}((0,T),(H^{r_2})^d)$, but both are weak convergences, and, thus,we cannot deal directly with the bilinear term. However, the solution is easy. Let us assume $r_2 < 1$. For any positive α, we may split $\vec{W}$ in $\vec{X} + \vec{Y}$, where $\|\vec{X}\|_{L^{\sigma_2}((0,T),(X_{r_2})^d} < \alpha$ and $\vec{Y} \in (\mathcal{D}((0,T) \times \mathbb{R}^d))^d$ (since we work in the subspace $X_{r_2}^{(0)}$); we have that $\int\int_0^t ((\vec{u}_{\epsilon_n}.\vec{\nabla})\; \vec{u}_{\epsilon_n}).\vec{Y}_{\epsilon_n} \; dx \; ds$ converges to $\int\int_0^t ((\vec{u}.\vec{\nabla})\; \vec{u}).\vec{Y} \; dx \; ds$ while $\sup_\epsilon \int_0^T |((\vec{u}_\epsilon.\vec{\nabla})\; \vec{u}_\epsilon).\vec{X}_\epsilon| \; dx \; ds < C\alpha$, and letting α go to 0, we arrive at the desired convergence result. If $r_2 = 1$, we split $\vec{W}$ in $\vec{X} + \vec{Y}$ where $\|\vec{X}\|_{\mathcal{C}([0,T],(X_1)^d} < \alpha$ and $\vec{Y} \in (\mathcal{D}([0,T) \times \mathbb{R}^d))^d$; then, we split $\vec{Y}$ in $\vec{X}' + \vec{Y}'$ where $\|\vec{X}'\|_{L^4((0,T),(X_{1/2})^d} < \alpha$ and $\vec{Y}' \in (\mathcal{D}((0,T) \times \mathbb{R}^d))^d$ and the end of the proof is similar to the case $r_2 < 1$.

We attained inequality (21.2) for almost every t; we then conclude that it is valid for every t since the continuity of $\vec{u}$ in $(H^{-\tau}_{loc}(\mathbb{R}^d))^d$ and the boundedness of $\vec{u}$ in $(L^2)^d$ imply the weak convergence of $\vec{u}(t')$ to $\vec{u}(t)$ in $(L^2)^d$ when t' goes to t.

Step 5: Uniqueness

The proof is exactly the same as the proof of the classical theorem of Serrin (see Theorem 21.3). Since $\vec{v}$ is r_3-regular, we obtain the energy inequality

$$\|\vec{u}(t) - \vec{v}(t)\|_2^2 + 2\int_0^t \|\vec{\nabla} \otimes (\vec{u} - \vec{v})\|_2^2 \; ds \leq$$
$$-2\eta \int_0^t\int_{\mathbb{R}^d} \vec{v}.\big((\vec{v} - \vec{u}).\vec{\nabla}\big)(\vec{u} - \vec{v}) \; dx \; ds + 2\int\int_0^t \big(((\vec{v} - \vec{u}).\vec{\nabla})\; (\vec{v} - \vec{u})\big).\vec{W} \; dx \; ds$$

and we conclude as for Theorem 21.3. □

Chapter 22

Further adapted spaces for the Navier–Stokes equations

1. The analysis of Meyer and Muschietti

In his book, Y. Meyer presents a generalization of Cannone's adapted spaces to deal with the case of some Besov spaces with null or negative regularity exponent [MEY 99]. In order to find global solutions, they restrict their analysis to the case of homogeneous spaces ($\|\lambda f(\lambda x)\|_E = \|f\|_E$).

Definition 22.1: (Meyer and Muschietti's adapted spaces)

According to Meyer and Muschietti, a Banach space X is adapted to the Navier–Stokes equations if the following assertions are satisfied:
a) X is a shift-invariant Banach space of distributions.
b) the norm of X is homogeneous of exponent -1: $\|\lambda f(\lambda x)\|_X = \|f\|_X$
c) if $f \in L^\infty$ and $\vec{\nabla} f \in X^d$, then f is a pointwise multiplier of X:

$$\|fg\|_X \leq C(\|f\|_\infty + \sum_{i=1}^{d} \|\partial_i f\|_X)\|g\|_X. \tag{22.1}$$

The theorem of Meyer and Muschietti then states the following result:

Theorem 22.1: (Meyer and Muschietti's theorem for adapted spaces)

Let X be a Banach space adapted (according to Meyer and Muschietti) to the Navier–Stokes equations. Then, there exists a positive ϵ_X so that for all $\vec{u_0} \in X^d$ such that $\vec{\nabla}.\vec{u_0} = 0$ and $\|\vec{u}_0\|_X < \epsilon_X$, there is a solution $\vec{u}$ of the Navier–Stokes equations on $(0,\infty)$ with initial value $\vec{u_0}$ so that $\vec{u} \in \mathcal{C}_b((0,\infty), X^d)$.

Proof: Let $M(X)$ be the Banach space of pointwise multipliers of X, normed by $\|f\|_{M(X)} = \sup_{g\in X, \|f\|_X \leq 1} \|fg\|_X$. Of course, $M(X)$ is a Banach algebra: $\|fg\|_{M(X)} \leq \|f\|_{M(X)}\|g\|_{M(X)}$. The proof then follows the line of Theorem 17.2. We introduce as admissible path space $\mathcal{X}$ associated to X the space $\mathcal{E}_X \cap \mathcal{B}_X$ where $\mathcal{E}_X = L^\infty((0,\infty), X^d)$ and $\mathcal{B}_X = \{\vec{u} \ / \ \sqrt{t}\ \vec{u} \in (L^\infty((0,\infty), (M(X))^d\}$.

Clearly, $\vec{u}_0 \mapsto (e^{t\Delta}\vec{u}_0)_{t>0}$ maps boundedly X^d to $\mathcal{X}$ (according to inequality (22.1)). The boundedness of B from $\mathcal{E}_X \times \mathcal{B}_X$ to $\mathcal{E}_X$ is obvious, since we have $\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\|_{\mathcal{L}(X^{d\times d},X^d)} \leq C\frac{1}{\sqrt{t-s}}$. We get a more precise estimate, for $0 < t_1 < t_2$,

$$\|B(\vec{f},\vec{g})(t_1) - B(\vec{f},\vec{g})(t_2)\|_X \leq C\sqrt{\frac{t_2 - t_1}{t_1}} \sup_{0<s<t_2} \|\vec{f}\|_X \sup_{0<s<t_2} \sqrt{s}\|\vec{g}\|_{M(X)}$$

which shows the continuity of $t \mapsto B(\vec{f},\vec{g})$ from $(0,\infty)$ to X^d.

Now, we prove that B is bounded from $\mathcal{B}_X \times (\mathcal{B}_X \cap \mathcal{E}_X)$ to $\mathcal{B}_X$: we write, for $s > t/2$, $\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f}\otimes\vec{g})\|_{M(X)} \leq C\frac{1}{\sqrt{t-s}}\frac{1}{s}\|\vec{f}\|_{\mathcal{B}_X}\|\vec{g}\|_{\mathcal{B}_X}$ and, for $s < t/2$ and $t < 1$, $\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f}\otimes\vec{g})\|_{M(X)} \leq C\frac{1}{t-s}\frac{1}{\sqrt{s}}\|\vec{f}\|_{\mathcal{B}_X}\|\vec{g}\|_{\mathcal{E}_X}$.

Thus, if $\|\vec{u}_0\|_X$ is small enough, we get a global solution $\vec{u} \in \mathcal{B}_X \cap \mathcal{E}_X$. $\square$

Meyer and Muschietti have given examples of spaces adapted to the Navier–Stokes equations. A trivial example is given by the shift-invariant Banach spaces of local measures with homogeneous norms (with homogeneity exponent -1): L^d, Lorentz spaces $L^{d,q}$, homogeneous Morrey–Campanato spaces $\dot{M}^{p,d}$. Another example is given by the Besov spaces $\dot{B}_q^{-1+d/q,\infty}$ for $1 \leq q < 2d$. We give a little more precise statement:

Proposition 22.1:

(A) Let X be a Banach space adapted (according to Meyer and Muschietti) to the Navier–Stokes equations. Then X is continuously embedded into $\dot{B}^{-1/2,\infty}_{\dot{M}^{2,2d}}$.

(B) Let E be a shift-invariant Banach space of local measures. Assume that E is continuously embedded into $L^2_{loc}(\mathbb{R}^d)$ and that there exists a shift-invariant Banach space of local measures F so that the pointwise product maps boundedly $E \times E$ to F: for $f \in E$ and $g \in E$, $\|fg\|_F \leq C\|f\|_E\|g\|_E$. Assume, moreover, that:

i) the norm of E is homogeneous with homogeneity exponent $-\alpha$ for some $\alpha \in (1/2, 1]$ (i. e. $\|\lambda^\alpha f(\lambda x)\|_E = \|f\|_E$)

ii) $[F, \mathcal{C}_b(\mathbb{R}^d)]_{1/2,1} \subset E$: for $f \in F \cap \mathcal{C}_b$, $\|f\|_E \leq C\sqrt{\|f\|_F\|f\|_\infty}$

then, for $1 \leq q \leq \infty$, the space $X = \dot{B}_E^{-1+\alpha,q}$ is a Banach space adapted (according to Meyer and Muschietti) to the Navier–Stokes equations.

Proof: (A) is clear. We start from the embedding $X \subset \dot{B}_X^{0,\infty} = \dot{B}^{-1/2,\infty}_{\dot{B}_X^{1/2,1}}$. We want to prove $\dot{B}_X^{1/2,1} \subset \dot{M}^{2,2d}$. We have $\|f\|_{\dot{M}^{2,2d}} = \sup_{k\in\mathbb{Z}} 2^{k/2}\|f(2^k x)\|_{L^2_{uloc}}$ and $\|f\|_{\dot{B}_X^{1/2,1}} = 2^{k/2}\|f(2^k x)\|_{\dot{B}_X^{1/2,1}}$; thus it is enough to prove the continuous embedding $\dot{B}_X^{1/2,1} \subset L^2_{uloc}$, and even $\dot{B}_X^{1/2,1} \subset L^2_{loc}$ since the norm of $\dot{B}_X^{1/2,1}$ is shift-invariant. We have seen that it is sufficient to check that the pointwise product of two elements in $\dot{B}_X^{1/2,1}$ is defined as a distribution (see the proof of Proposition 19.1). Since $\dot{B}^{0,1}_X \subset X \subset \dot{B}^{0,\infty}_X$, we have $\dot{B}^{1/2,1}_X = [X, \dot{B}^{1,1}_X]_{1/2,1}$, thus we may write $u \in \dot{B}^{1/2,1}_X$ as a sum $u = \sum_{n\in\mathbb{N}} \lambda_n u_n$ with $\sup_{n\in\mathbb{N}} \|u_n\|_X \ \|u_n\|_{\dot{B}^{1,1}_X} \leq 1$ and $\sum_{n\in\mathbb{N}} \ |\lambda_n| < \infty$. If u and v satisfy $\|u\|_X \|u\|_{\dot{B}^{1,1}_X} \leq 1$ and $\|v\|_X \|v\|_{\dot{B}^{1,1}_X} \leq 1$, we use inequality (22.1), which gives $\|fg\|_X \leq C\|f\|_{\dot{B}^{1,1}_X} \|g\|_X$ to get

$$\|uv\|_X = \sqrt{\|uv\|_X}\sqrt{\|uv\|_X} \leq C\sqrt{\|u\|_{\dot{B}^{1,1}_X}\|v\|_X}\sqrt{\|v\|_{\dot{B}^{1,1}_X}\|u\|_X} \leq C.$$

Thus, pointwise product maps boundedly $\dot{B}^{1/2,1}_X \times \dot{B}^{1/2,1}_X$ to X.

To prove (B), we use the paradifferential calculus of Bony and write

$$\begin{aligned} fg &= \textstyle\sum_{j\in\mathbb{Z}} S_{j-2} f \Delta_j g + \sum_{j\in\mathbb{Z}} S_{j-2} g \Delta_j f + \sum_{j\in\mathbb{Z}} \sum_{|j-k|\leq 2} \Delta_j f \Delta_k g \\ &= \dot{\pi}(f,g) + \dot{\pi}(g,f) + \dot{\rho}(f,g). \end{aligned}$$

We control the paraproduct $\dot{\pi}(f,g)$ by

$$\begin{aligned} \|\dot{\pi}(f,g)\|_X &\leq C\|2^{j(\alpha-1)}\|S_{j-2} f \Delta_j g\|_E\|q \ \leq C'\|2^{j(\alpha-1)}\|S_{j-2} f\|_\infty \|\Delta_j g\|_E\|q \\ &\leq C''\|f\|_\infty \|2^{j(\alpha-1)}\|\Delta_j g\|_E\|q \qquad = C''\|f\|_\infty \|g\|_X, \end{aligned}$$

and we control the paraproduct $\dot{\pi}(g,f)$ by

$$\begin{aligned} \|\dot{\pi}(g,f)\|_X &\leq C\|2^{j(\alpha-1)}\|S_{j-2} g \Delta_j f\|_E\|q \\ &\leq C'\|2^{j(\alpha-1)}\|S_{j-2} g\|_\infty \|\Delta_j f\|_E\|q \\ &\leq C''\|(\textstyle\sum_{k\leq j-3} 2^{k\alpha}\|\Delta_k g\|_E)\|\Delta_j f\|_X\|q \\ &\leq C'''\|(\textstyle\sum_{k\leq j-3} 2^{k\alpha}\|\Delta_k g\|_E) 2^{-j}\|\vec{\nabla}\Delta_j f\|_X\|q \\ &\leq C''''\|\vec{\nabla} f\|_X \|\textstyle\sum_{k\leq j-3} 2^{k-j} 2^{k(\alpha-1)}\|\Delta_k g\|_E\|q \\ &\leq C'''''\|\vec{\nabla} f\|_X \|g\|_X. \end{aligned}$$

Then, we control the remainder by writing the Bernstein inequality $\|\Delta_j f\|_E \leq C\sqrt{\|\Delta_j f\|_F \|\Delta_j f\|_\infty} \leq C'\sqrt{\|\Delta_j f\|_F 2^{2j\alpha}\|\Delta_j f\|_F} = C' 2^{j\alpha}\|\Delta_j f\|_F$:

$$\begin{aligned} \|\dot{\rho}(f,g)\|_X &\leq C\|2^{j(\alpha-1)}\Delta_j\|\dot{\rho}(f,g)\|_E\|q \\ &\leq C'\|2^{j(2\alpha-1)}\Delta_j\|\dot{\rho}(f,g)\|_F\|q \\ &\leq C''\|2^{j(2\alpha-1)}\textstyle\sum_{k\geq j-4}\|\Delta_k g\|_E\|\sum_{|l-k|\leq 2}\Delta_l f\|_E\|q \\ &\leq C'''\|2^{j(2\alpha-1)}\textstyle\sum_{k\geq j-4}\|\Delta_k g\|_E 2^{-k(\alpha-1)}\|\sum_{|l-k|\leq 2}\Delta_l f\|_X\|q \\ &\leq C''''\|\vec{\nabla} f\|_X \|2^{j(2\alpha-1)}\textstyle\sum_{k\geq j-4}\|\Delta_k g\|_E 2^{-k\alpha}\|_q \\ &\leq C'''''\|\vec{\nabla} f\|_X \|\textstyle\sum_{k\geq j-4} 2^{(k-j)(1-2\alpha)} 2^{k(\alpha-1)}\|\Delta_k g\|_E\|q \\ &\leq C''''''\|\vec{\nabla} f\|_X \|g\|_X \end{aligned}$$

since $2\alpha - 1 > 0$. □

Whenever X is adapted in the sense of Meyer and Muschietti, we have the inequality $\|fg\|_X \leq C\|f\|_{\dot{B}^{1,1}_X}\|g\|_X$. This provides an easy way to generalize the theorem of Meyer and Muschietti:

Definition 22.2: *A Banach space X is adapted in the generalized sense of Meyer and Muschietti to the Navier–Stokes equations if the following assertions are satisfied:*
a) X is a shift-invariant Banach space of distributions.
b) the norm of X is homogeneous with exponent -1: $\|\lambda f(\lambda x)\|_X = \|f\|_X$
c) for a $\sigma \in (0,1)$ X satisfies the regularity property (P_σ): if f and g belong to $\dot{B}_X^{1,1} \cap \dot{B}_X^{\sigma,1}$, then

$$(P_\sigma) \qquad \|fg\|_{\dot{B}_X^{\sigma,\infty}} \leq C(\|f\|_{\dot{B}_X^{1,1}}\|g\|_{\dot{B}_X^{\sigma,1}} + \|g\|_{\dot{B}_X^{1,1}}\|f\|_{\dot{B}_X^{\sigma,1}})$$

Theorem 22.2:

(A) If X is adapted in the sense of Meyer and Muschietti, then X satisfies (P_σ) for all $\sigma \in (0,1)$.

(B) Let E be a shift-invariant Banach space of local measures. Assume that E is continuously embedded into $L^2_{loc}(\mathbb{R}^d)$ and that there exists a shift-invariant Banach space of local measures F so that the pointwise product maps boundedly $E \times E$ to F: for $f \in E$ and $g \in E$, $\|fg\|_F \leq C\|f\|_E\|g\|_E$. Assume that:
i) the norm of E is homogeneous with homogeneity exponent $-\alpha$ for some $\alpha \in (0,1]$ (i. e. $\|\lambda^\alpha f(\lambda x)\|_E = \|f\|_E$)
ii) $[F, \mathcal{C}_b(\mathbb{R}^d)]_{1/2,1} \subset E$: for $f \in F \cap \mathcal{C}_b$, $\|f\|_E \leq C\sqrt{\|f\|_F\|f\|_\infty}$
then, for $1 \leq q \leq \infty$, the space $X = \dot{B}_E^{-1+\alpha,q}$ is a Banach space adapted (in the generalized sense of Meyer and Muschietti) to the Navier–Stokes equations. More precisely, X satisfies (P_σ) for all $\sigma \in (0,1)$ so that $\sigma + 2\alpha \geq 1$.

(C) Let X be a Banach space adapted (in the generalized sense of Meyer and Muschietti) to the Navier–Stokes equations. Then, there exists a positive ϵ_X so that for all $\vec{u_0} \in X^d$ so that $\vec{\nabla}.\vec{u_0} = 0$ and $\|\vec{u}_0\|_X < \epsilon_X$, there is a solution $\vec{u}$ of the Navier–Stokes equations on $(0,\infty)$ with initial value $\vec{u_0}$ so that $\vec{u} \in \mathcal{C}_b((0,\infty), X^d)$.

Proof: We first prove (A). Proving inequality (P_σ) is equivalent to proving the inequality on dyadic blocks. On dyadic blocks, (P_σ) reads as

$$2^{l\sigma}\|\Delta_l(\Delta_j f \Delta_k g)\|_X \leq C(2^j\|\Delta_j f\|_X 2^{k\sigma}\|\Delta_k g\|_X + 2^k\|\Delta_k g\|_X 2^{j\sigma}\|\Delta_j f\|_X)$$

But we know that $\Delta_l(\Delta_j f \Delta_k g) = 0$ if $l \geq \max(j,k)+3$. On the other hand, if $l \leq k+2$, we have

$$2^{l\sigma}\|\Delta_l(\Delta_j f \Delta_k g)\|_X \leq C 2^{k\sigma}\|(\Delta_j f \Delta_k g)\|_X \leq C' 2^{k\sigma} 2^j \|\Delta_j f\|_X \|\Delta_k g\|_X.$$

To prove (B), we want to estimate

$$\begin{aligned} 2^{l\sigma}\|\Delta_l(\Delta_j f \Delta_k g)\|_X &\sim 2^{l(\sigma+\alpha-1)}\|\Delta_l(\Delta_j f \Delta_k g)\|_E \\ &\leq C 2^{l(\sigma+\alpha-1)} \min\left(2^{l\alpha}\|\Delta_l(\Delta_j f \Delta_k g)\|_F, \|\Delta_j f \Delta_k g\|_E\right) \\ &\leq C' 2^{l(\sigma+\alpha-1)} \min\left(2^{l\alpha}\|\Delta_j f\|_E\|\Delta_k g\|_E, \|\Delta_j f\|_\infty\|\Delta_k g\|_E, \|\Delta_j f\|_E\|\Delta_k g\|_\infty\right) \\ &\leq C'' 2^{l(\sigma+\alpha-1)} \min\left(2^{l\alpha}2^{(j+k)(1-\alpha)}, 2^j 2^{k(1-\alpha)}, 2^{j(1-\alpha)}2^k\right)\|\Delta_j f\|_X\|\Delta_k g\|_X \end{aligned}$$

Recall that $\Delta_l(\Delta_j f \Delta_k g) = 0$ unless $|j-k| \leq 2$ and $l \leq \max(j,k)+3$ or $\min(j,k) \leq \max(j,k)-3$ and $|l-\max(j,k)| \leq 3$. In the case $|j-k| \leq 2$, we write

$$2^{l(\sigma+2\alpha-1)}2^{(j+k)(1-\alpha)}\|\Delta_j f\|_X\|\Delta_k g\|_X \leq C2^j\|\Delta_j\|_X 2^{k\sigma}\|\Delta_k g\|_X 2^{(l-j)(\sigma+2\alpha-1)}$$

while we write in the case $k \leq j-3$

$$2^{l(\sigma+\alpha-1)}2^j 2^{k(1-\alpha)}\|\Delta_j f\|_X\|\Delta_k g\|_X \leq C2^j\|\Delta_j f\|_X 2^{k\sigma}\|\Delta_k g\|_X$$

and a similar estimate when $j \leq k-3$. This gives the inequality (P_σ) provided that $\sigma+2\alpha-1 \geq 0$.

The proof for (C) follows the line of Theorem 22.1. We introduce as admissible path space $\mathcal{X}$ associated to X the space $\mathcal{E}_X \cap \mathcal{B}_X$, where $\mathcal{E}_X = L^\infty((0,\infty), X^d)$ and $\mathcal{B}_X = \{\vec{u} \ / \ \sqrt{t}\ \vec{u} \in (L^\infty((0,\infty),(\dot{B}^{1,1}_X)^d\}$. Clearly, we have that $\vec{u}_0 \mapsto (e^{t\Delta}\vec{u}_0)_{t>0}$ maps boundedly X^d to $\mathcal{X}$.

The boundedness of B from $\mathcal{X} \times \mathcal{X}$ to $\mathcal{X}$ is obvious: if $\vec{u} \in \mathcal{X}$, we have $\|\vec{u}\|_{\dot{B}^{\sigma,1}_X} \leq C\|\vec{u}\|^{\sigma}_{\dot{B}^{1,1}_X}\|\vec{u}\|^{1-\sigma}_{\dot{B}^{0,1}_X} \leq Ct^{-\sigma/2}\|\vec{u}\|_{\mathcal{X}}$. Thus, if $\vec{u}$ and $\vec{v}$ belong to $\mathcal{X}$, we find that $\vec{u}(t) \otimes \vec{v}(t)$ belongs to $(\dot{B}^{\sigma,\infty}_X)^{d\times d}$ with a norm bounded by $Ct^{-(1+\sigma)/2}\|\vec{u}\|_{\mathcal{X}}\|\vec{v}\|_{\mathcal{X}}$. We have, for $\epsilon \in -1, 1$,

$$\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\|_{\mathcal{L}((\dot{B}^{\sigma,\infty}_X)^{d\times d},(\dot{B}^{\sigma+\epsilon,\infty}_X)^d)} \leq C\frac{1}{(t-s)^{(1+\epsilon)/2}}$$

This gives by interpolation:

$$\|B(\vec{u},\vec{v})\|_{\dot{B}^{0,1}_X} \leq C\int_0^t \frac{1}{(t-s)^{(1-\sigma)/2}}\frac{1}{s^{(1+\sigma)/2}}\ ds\|\vec{u}\|_{\mathcal{X}}\|\vec{v}\|_{\mathcal{X}}$$

and

$$\|B(\vec{u},\vec{v})\|_{\dot{B}^{1,1}_X} \leq C\int_0^t \frac{1}{(t-s)^{(2-\sigma)/2}}\frac{1}{s^{(1+\sigma)/2}}\ ds\|\vec{u}\|_{\mathcal{X}}\|\vec{v}\|_{\mathcal{X}}$$

and finally $\|B(\vec{u},\vec{v})\|_{\mathcal{X}} \leq C\|\vec{u}\|_{\mathcal{X}}\|\vec{v}\|_{\mathcal{X}}$.

We now prove more precisely that $B(\vec{u},\vec{v}) \in (\mathcal{C}((0,T),X))^d$. For $0 < t_1 < t_2$, we write, for $s < t_1$, $e^{(t_2-s)\Delta} - e^{(t_1-s)\Delta} = \int_0^{t_2-t_1} e^{\theta\Delta}\ d\theta\ \Delta e^{(t_1-s)\Delta}$, which gives

$$\|(e^{(t_2-s)\Delta}-e^{(t_1-s)\Delta})\mathbb{P}\vec{\nabla}.\|_{\mathcal{L}((\dot{B}^{\sigma,\infty}_X)^{d\times d},(\dot{B}^{0,1}_X)^d)} \leq C\min\Big(\frac{1}{(t_1-s)^{\frac{1-\sigma}{2}}}, \frac{t_2-t_1}{|t_1-s|^{\frac{3-\sigma}{2}}}\Big)$$

which then gives

$$\|B(\vec{u},\vec{v})(t_1) - B(\vec{u},\vec{v})(t_2)\|_X \leq C\Big(\frac{t_2-t_1}{t_1}\Big)^{\frac{1+\sigma}{2}}\|\vec{u}\|_{\mathcal{X}}\|\vec{v}\|_{\mathcal{X}}.$$

Thus, Theorem 22.2 is proved. □

2. The case of Besov spaces of null regularity

We previously discussed a persistency theorem in the case of initial data given in a Besov space over local measures when the Besov regularity exponent was positive (Chapter 19) or negative (Chapter 20). We have as well a persistency result for the Koch and Tataru solutions in the case of a null Besov exponent (Furioli, Lemarié-Rieusset, Zahrouni and Zhioua [FURLZZ 00]):

Theorem 22.3: (Persistency theorem for Besov spaces with null regularity exponents)

Let E be a shift-invariant Banach space of local measures and assume that $\mathcal{D}$ is dense in E. Let $\vec{u}_0 \in (\dot{B}^{0,\infty}_E \cap BMO^{-1})^d$ with $\vec{\nabla}.\vec{u}_0 = 0$. If $\vec{u}_0 \in (BMO^{-1})^d$ is small enough to satisfy the conclusion of the Koch and Tataru theorem for global solutions, then the Koch and Tataru solution $\vec{u}$ of the Navier–Stokes equations satisfies the regularity property $\vec{u} - e^{t\Delta}\vec{u}_0 \in (L^\infty((0,\infty), \dot{B}^{0,1}_E))^d$.

Proof: In Chapter 17, we introduced the space $E^{(1/2)} = [E, \mathcal{C}_b(\mathbb{R}^d)]_{1/2,1}$. We obviously have $\dot{B}^{0,\infty}_E \cap \dot{B}^{-1,\infty}_\infty \subset \dot{B}^{-1/2,\infty}_{E^{(1/2)}}$. We know that $E^{(1/2)}$ is a shift-invariant Banach space of local measures. We may apply the persistency theorem Theorem 20.3 and conclude that $\vec{u}$ satisfies the estimate $\sup_{t>0} t^{1/4}\|\vec{u}\|_{E^{(1/2)}} < \infty$. Moreover, we know that the pointwise product maps $E^{(1/2)} \times E^{(1/2)}$ to E. We thus get that $\|e^{\frac{(t-s)}{2}\Delta}\frac{1}{(-\Delta)^{1/4}}\mathbb{P}\vec{\nabla}.\vec{u}\otimes\vec{u}\|_E \le \frac{C}{(t-s)^{1/4}\sqrt{s}}$. This gives $\|(-\Delta)^{-1/4}B(\vec{u},\vec{u})\|_E \le C^- t^{1/4}$ and $\|(-\Delta)^{1/4}B(\vec{u},\vec{u})\|_E \le C^+ t^{-1/4}$, and then $\|B(\vec{u},\vec{u})\|_{\dot{B}^{0,1}_E} \le C\sqrt{\|(-\Delta)^{-1/4}B(\vec{u},\vec{u})\|_E \|(-\Delta)^{1/4}B(\vec{u},\vec{u})\|_E} \le \sqrt{C^-C^+}$. □

3. The analysis of Auscher and Tchamitchian

In their paper, Auscher and Tchamitchian gave another generalization of Cannone's adapted spaces [AUST 99]. We present in detail two results from this paper:

i) a remark on the range of convergence for the Picard method in the case of bilinear equations; (see Theorem 22.4 bis below)

ii) an existence result for the Navier–Stokes equations with a criterion based on some estimates for the pointwise product of dyadic blocks in the Littlewood-Paley decomposition (see Theorem 22.5 below).

We first recall the Picard contraction principle (Theorems 13.2 and 15.1):

Theorem 22.4: (The Picard contraction principle)

Let E be a Banach space and let B be a bounded bilinear transform from $E \times E$ to E:

$$\|B(e,f)\|_E \le C_B \|e\|_E \|f\|_E.$$

Then, if $0 < \delta < \frac{1}{4C_B}$ *and if* $e_0 \in E$ *is such that* $\|e_0\|_E \leq \delta$*, the equation* $e = e_0 - B(e,e)$ *has a solution so that* $\|e\|_E \leq 2\delta$*. This solution is unique in the ball* $\bar{B}(0,2\delta)$ *and depends continuously on* e_0*: if* $\|f_0\|_E \leq \delta$*,* $f = f_0 - B(f,f)$ *and* $\|f\|_E \leq 2\delta$*, then* $\|e-f\|_E \leq \frac{1}{1-4C_B\delta}\|e_0 - f_0\|_E$*.*

The proof is based on the fact that whenever $\|e_0\|_E \leq \delta < \frac{1}{4C_B}$ the map $e \mapsto e_0 - B(e,e)$ is a contraction on the ball $\bar{B}(0,2\delta)$, so that the solution may be constructed recursively by choosing $f_0 \in \bar{B}(0,2\delta)$ and defining $f_{n+1} = e_0 - B(f_n, f_n)$: the unique solution $e = e_0 - B(e,e)$ is the limit of the sequence f_n. Auscher and Tchamitchian extended this result to the ball $\bar{B}(0, \frac{1}{2C_B})$ and $\|e_0\| \leq \frac{1}{4C_B}$, though the map is no longer a contraction:

Theorem 22.4 bis: (The full Picard contraction principle)

Let E *be a Banach space and let* B *be a bounded bilinear transform from* $E \times E$ *to* E*:*

$$\|B(e,f)\|_E \leq C_B \|e\|_E \|f\|_E.$$

Then, if $e_0 \in E$ *is so that* $\|e_0\|_E \leq \frac{1}{4C_B}$*, the equation* $e = e_0 - B(e,e)$ *has a solution such that* $\|e\|_E \leq \frac{1}{2C_B}$*. This solution is unique in the ball* $\bar{B}(0, \frac{1}{2C_B})$*.*

Moreover, the solution may be constructed recursively by choosing $f_0 \in \bar{B}(0, \frac{1}{2C_B})$ *and defining* $f_{n+1} = e_0 - B(f_n, f_n)$*: the unique solution* $e = e_0 - B(e,e)$ *is the limit of the sequence* f_n*.*

Proof: Let F be the map $F(e) = e_0 - B(e,e)$. If $\|e_0\|_E \leq \frac{1}{4C_B}$, then F maps the ball $\bar{B}(0, \frac{1}{2C_B})$ to itself: $\|F(e)\|_E \leq \|e_0\|_E + C_B\|e\|_E^2 \leq \frac{1}{4C_B} + C_B(\frac{1}{2C_B})^2 = \frac{1}{2C_B}$. We want to show that F has a fixed point e and that the sequence f_n is convergent to this fixed point e, whatever the choice of f_0. In particular, this fixed point is unique (if e' is another fixed point, look at the sequence f_n associated to $f_0 = e'$).

We first look at the case $E = \mathbb{C}$ and $B(z,w) = zw/4$. We want to solve $z = z_0 + z^2/4$ when $|z_0| \leq 1$. The discriminant of the equation is $\Delta = 1 - z_0$; we define the square root $\sqrt{Z}$ for $\mathcal{R}Z \geq 0$ by writing $Z = \rho e^{i\theta}$ with $\theta \in [-\pi/2, \pi/2]$ and $\sqrt{Z} = \sqrt{\rho}e^{i\theta/2}$. The solutions are then $z^+ = 2(1+\sqrt{1-z_0})$ and $z^- = 2(1-\sqrt{1-z_0})$. We have $z^+z^- = 4z_0$ and $|z^+| \geq 2(1+\mathcal{R}\sqrt{1-z_0}) \geq 2$ (with equality if and only if $z_0 = 1$ and $z^+ = z^- = 2$), and thus $|z^-| \leq 2$ Thus, the unique solution in $\bar{D}(0,2)$ is $z^- = 2(1-\sqrt{1-z_0})$. For $|z_0| < 1$, we can develop z^- in a Taylor series

$$z^- = z + \sum_{k=2}^{\infty} \frac{\Gamma(k-1/2)}{\Gamma(1/2)\Gamma(k+1)} z_0^k = \sum_{k=1}^{\infty} c_k z_0^k.$$

Since the Taylor coefficients are positive, we have $\sum_{k=1}^{\infty} c_k = \lim_{t\to 1} \sum_{k=1}^{\infty} c_k t^k$ $= 2(1-\sqrt{1-1}) = 2$. Thus, the Taylor expansion is still valid for $|z_0| = 1$.

This expansion will be generalized to the Banach space setting. We are going to expand in a Taylor series the solution of $e = ze_0 - B(e,e)$ for $|z| < 1$

and then show that the expansion is valid for $z = 1$. We first look at the analytic functions $f_n(z)$ defined by $f_0 = 0$ and $f_{n+1}(z) = ze_0 - B(f_n(z), f_n(z))$. The Picard contraction principle (Theorem 22.4) shows that the sequence $f_n(z)$ is uniformly convergent on any closed disk $\bar{D}(0, \rho)$ with $\rho < 1$. Thus, the limit $f_\infty(z)$ is analytic. We write $f_\infty(z) = \sum_{k=1}^\infty z^k \epsilon_k$ and, substituting the series in the identity $f_\infty(z) = ze_0 - B(f_\infty(z), f_\infty(z))$, we get $\epsilon_1 = e_0$ and, for $2 \leq n$, $\epsilon_n = -\sum_{k=1}^{n-1} B(\epsilon_k, \epsilon_{n-k})$. Moreover, we may easily compare $\|\epsilon_k\|_E$ to the kth coefficient c_k in the Taylor expansion of z^-. We can prove by induction that $\|\epsilon_k\|_E \leq \frac{c_k}{4C_B}$: we have $\|\epsilon_1\|_E = \|e_0\|_E \leq \frac{1}{4C_B} = \frac{c_1}{4C_B}$; now, we write $\|\epsilon_{n+1}\|_E \leq \sum_{k=1}^n \|B(\epsilon_k, \epsilon_{n+1-k})\|_E \leq C_B \sum_{k=1}^n \frac{c_k}{4C_B}\frac{c_{n+1-k}}{4C_B} = \frac{1}{4C_B}\sum_{k=1}^n \frac{c_k c_{n+1-k}}{4} = \frac{c_{n+1}}{4C_B}$. Since $\sum_{k=1}^\infty c_k < \infty$, we find that $\lim_{t\to 1} f_\infty(z) = \sum_{k=1}^\infty \epsilon_k$. If $e = \sum_{k=1}^\infty \epsilon_k$, we have $e = \lim_{t\to 1} te_0 - B(f_\infty(t), f_\infty(t)) = e_0 - B(e, e)$. Hence, the map F has at least one fixed point in $\bar{B}(0, \frac{1}{2C_B})$.

We now select $f_0 \in \bar{B}(0, \frac{1}{2C_B})$ and define $f_{n+1} = e_0 - B(f_n, f_n)$: we must prove the convergence of f_n to $\sum_{k=1}^\infty \epsilon_k$. By induction, we attempt to prove that $\|f_n - \sum_{k=1}^n \epsilon_k\|_E \leq \frac{1}{4C_B}\sum_{k=n+1}^\infty c_k$. Let $g_n = \sum_{k=1}^n \epsilon_k$. We have $\|f_0 - g_0\|_E = \|f_0\|_E \leq \frac{1}{2C_B} = \frac{\sum_{k=1}^\infty c_k}{4C_B}$; thus, the inequality is true for $n = 0$. We split $f_{n+1} - g_{n+1}$ into $f_{n+1} - g_{n+1} = h_n - B(f_n - g_n, g_n) - B(g_n, f_n - g_n) - B(f_n - g_n, f_n - g_n)$ with $h_n = e_0 - B(g_n, g_n) - g_{n+1}$. We have $e_0 - B(g_n, g_n) - g_{n+1} = -\sum_{k\leq n, q\leq n, k+q\geq n+2} T(\epsilon_k, \epsilon_q)$; hence, $\|h_n\|_E \leq C_B \sum_{k\leq n, q\leq n, k+q\geq n+2} \frac{c_k}{4C_B}\frac{c_q}{4C_B}$. On the other hand, the induction hypothesis gives

$$\begin{cases} \|B(f_n - g_n, g_n)\|_E \leq C_B\left(\frac{1}{4C_B}\sum_{k=n+1}^\infty c_k\right)\sum_{k=1}^n \frac{c_k}{4C_B} \\ \|B(g_n, f_n - g_n)\|_E \leq C_B\left(\frac{1}{4C_B}\sum_{k=n+1}^\infty c_k\right)\sum_{k=1}^n \frac{c_k}{4C_B} \\ \|B(f_n - g_n, f_n - g_n)\|_E \leq C_B\left(\frac{1}{4C_B}\sum_{k=n+1}^\infty c_k\right)^2 \end{cases}$$

and this gives

$$\|f_{n+1} - g_{n+1}\|_E \leq C_B\left([\frac{1}{4C_B}\sum_{k=1}^\infty c_k]^2 - \sum\sum_{k+q\leq n+1}\frac{c_k}{4C_B}\frac{c_q}{4C_B}\right) = \frac{1}{4C_B}\sum_{k=n+2}^\infty c_k.$$

This yields the convergence of f_n to the (unique) fixed point $\sum_{k=1}^\infty g_k$. $\square$

We now consider the generalization of Theorem 22.2 to spaces closer to $\dot{B}_\infty^{-1,1}$. We replace (P_σ) (controlling the pointwise product in $\dot{B}_X^{1,1} \cap \dot{B}_X^{\sigma,1}$ by a condition on the pointwise product in the whole $\dot{B}_X^{1,1}$. We first consider the inequality (P_1):

Lemma 22.1: *Let X be a shift-invariant Banach space of distributions. Then X satisfies the regularity property (P_1):*

$$(P_1) \qquad \|fg\|_{\dot{B}_X^{1,\infty}} \leq C\|f\|_{\dot{B}_X^{1,1}}\|g\|_{\dot{B}_X^{1,1}}$$

if and only if we have for all $j, k, l \in \mathbb{Z}$ and all $f, g \in X$:

$$\|\Delta_j(\Delta_k f \Delta_l g)\|_X \leq C2^{k+l-j}\|\Delta_k f\|_X\|\Delta_l g\|_X$$

Proof: This is obvious. ◻

This property may be precised in the case of Besov spaces of local measures:

Proposition 22.2: *Let E be a shift-invariant Banach space of local measures. Assume that E is continuously embedded into $L^2_{loc}(\mathbb{R}^d)$ and that there exists a shift-invariant Banach space of local measures F so that the pointwise product maps boundedly $E \times E$ to F: for $f \in E$ and $g \in E$, $\|fg\|_F \le C\|f\|_E\|g\|_E$. Assume that:*
i) the norm of E is homogeneous with homogeneity exponent $-\alpha$ for some $\alpha \in (0,1]$ (i. e. $\|\lambda^\alpha f(\lambda x)\|_E = \|f\|_E$)
ii) $[F, \mathcal{C}_b(\mathbb{R}^d)]_{1/2,1} \subset E$: for $f \in F \cap \mathcal{C}_b$, $\|f\|_E \le C\sqrt{\|f\|_F\|f\|_\infty}$
then, for $1 \le q \le \infty$, the space $X = \dot{B}_E^{-1+\alpha,q}$ satisfies (P_1) and satisfies more precisely the inequalities:

$$\|\Delta_j(\Delta_k f \Delta_l g)\|_X \le C 2^{-2\alpha|l-\max(k,j)|} 2^{k+l-j} \|\Delta_k f\|_X \|\Delta_l g\|_X.$$

Proof: We have $\|\Delta_j f\|_X \sim 2^{j(\alpha-1)}\|f\|_E$. If $k \le j-3$, we just write that $\Delta_j(\Delta_k f \Delta_l g) = 0$ if $|j-l| \ge 4$ while $\|\Delta_k f \Delta_l g\|_E \le C\| \Delta_k f\|_\infty \|\Delta_l g\|_E \le C' 2^k \|\Delta_k f\|_X 2^{l(1-\alpha)}\|\Delta_l g\|_X$. If $|k-l| \le 2$, we write that $\Delta_j(\Delta_k f \Delta_l g) = 0$ if $j \ge l+6$ while $\|\Delta_j(\Delta_k f \Delta_l g\|_E \le C 2^{j\alpha}\|\Delta_k f \Delta_l g\|_F \le C' 2^{j\alpha}\|\Delta_k f\|_E \|\Delta_l g\|_E \le C'' 2^{(j-k-l)\alpha} 2^k \|\Delta_k f\|_X 2^l \|\Delta_l g\|_X$. ◻

The theorem of Auscher and Tchamitchian is discussed below:

Theorem 22.5: (Auscher and Tchamitchian's adapted spaces)
Let X be a shift-invariant Banach space of distributions.such that:
i) the norm of X is homogeneous of exponent -1: $\|\lambda f(\lambda x)\|_X = \|f\|_X$
ii) there exists a sequence of positive numbers $(\eta_n)_{n\in\mathbb{N}}$ so that we have the inequalities for all $j,k,l \in \mathbb{Z}$ and all $f, g \in X$:

$$\|\Delta_j(\Delta_k f \Delta_l g)\|_X \le \eta_{|l-\max(k,j)|} 2^{k+l-j} \|\Delta_k f\|_X \|\Delta_l g\|_X.$$

Then:

(A) If $\sum_{n\in\mathbb{N}} \eta_n < \infty$, the bilinear operator B is bounded on $\mathcal{E}_X \cap \mathcal{B}_X$ where $\mathcal{E}_X = L^\infty((0,\infty), (\dot{B}_X^{0,\infty})^d)$ and $\mathcal{B}_X = \{\vec{u} \ / \ t^{3/4}\ \vec{u} \in (L^\infty((0,\infty), (\dot{B}_X^{3/2,\infty})^d\}$. Thus, there exists a positive ϵ_X so that for all $\vec{u_0} \in X^d$ so that $\vec{\nabla}.\vec{u_0} = 0$ and $\|\vec{u}_0\|_X < \epsilon_X$, there is a solution $\vec{u}$ of the Navier–Stokes equations on $(0,\infty)$ with initial value $\vec{u_0}$ so that $\vec{u} \in \mathcal{E}_X \cap \mathcal{B}_X$.

(B) If $\sum_{n\in\mathbb{N}} n\eta_n < \infty$, the bilinear operator B is bounded on $\mathcal{F}_X \cap \mathcal{B}_X$ where $\mathcal{F}_X = L^\infty((0,\infty), X^d)$ and $\mathcal{B}_X = \{\vec{u} \ / \ t^{3/4}\ \vec{u} \in (L^\infty((0,\infty), (\dot{B}_X^{3/2,\infty})^d\}$. Thus, there exists a positive η_X so that for all $\vec{u_0} \in X^d$ so that $\vec{\nabla}.\vec{u_0} = 0$ and $\|\vec{u}_0\|_X < \eta_X$, there is a solution $\vec{u}$ of the Navier–Stokes equations on $(0,\infty)$ with initial value $\vec{u_0}$ so that $\vec{u} \in L^\infty((0,\infty), X^d)$.

(C) If $\sum_{n\in\mathbb{N}} n\eta_n < \infty$ and if Δ_0 maps X to $\mathcal{C}_0$, X is continuously embedded into $BMO^{(-1)}$.

Proof: The proof of (A) is quite easy. If $\vec{f} \in \mathcal{E}_X \cap \mathcal{B}_X$, we have $\|\vec{f}(t)\|_{\dot{B}_X^{0,\infty}} \le \|\vec{f}\|_{\mathcal{E}_X}$ and $\|\vec{f}(t)\|_{\dot{B}_X^{3/2,\infty}} \le \|\vec{f}\|_{\mathcal{B}_X} t^{-3/4}$; hence,

$$\|\Delta_j \vec{f}(t)\|_X \le \min(\|\vec{f}\|_{\mathcal{E}_X}, \|\vec{f}\|_{\mathcal{B}_X} \frac{1}{(2^j\sqrt{t})^{3/2}}).$$

In order to prove that for $\vec{f}$ and $\vec{g}$ in $\mathcal{E}_X \cap \mathcal{B}_X$, $B(\vec{f},\vec{g})$ is still in $\mathcal{E}_X \cap \mathcal{B}_X$, we must estimate the size of $\|\Delta_j(B(\vec{f},\vec{g})(t))\|_X$.

We define the norm of $\mathcal{X} = \mathcal{E}_X \cap \mathcal{B}_X$ as

$$\|\vec{f}\|_{\mathcal{X}} = \sup_{t>0, j\in\mathbb{Z}} \max(1, 2^{3j/2} t^{3/4}) \|\Delta_j \vec{f}(t)\|_X.$$

We then write $B(\vec{f},\vec{g}) = B_1(\vec{f},\vec{g}) + B_2(\vec{f},\vec{g}) + B_3(\vec{f},\vec{g})$ with

$$\begin{cases} B_1(\vec{f},\vec{g}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\sum_{j\in\mathbb{Z}} S_{j-2}\vec{f}\otimes\Delta_j\vec{g}\, ds \\ B_2(\vec{f},\vec{g}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\sum_{j\in\mathbb{Z}} \Delta_j\vec{f}\otimes S_{j-2}\vec{g}\, ds \\ B_3(\vec{f},\vec{g}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\sum_{j\in\mathbb{Z}}\sum_{|j-k|\le 2} \Delta_j\vec{f}\otimes\Delta_k\vec{g}\, ds. \end{cases}$$

We have $\Delta_j(B_1(\vec{f},\vec{g})(t)) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\Delta_j(\sum_{|l-j|\le 3} S_{l-2}\vec{f}\otimes\Delta_l\vec{g})\, ds$ so that (writing $\vec{\nabla}\Delta_j = \Delta\vec{\nabla}\frac{1}{\Delta}\Delta_j$)

$$\begin{gathered}\|\Delta_j(B_1(\vec{f},\vec{g})(t))\|_X \\ \le \int_0^t \frac{C\|\vec{f}\|_{\mathcal{X}}\|\vec{g}\|_{\mathcal{X}}}{\sqrt{t-s}} \min(1, \frac{1}{4^j(t-s)}) \sum_{k\le j} 2^k \min(1, \frac{1}{2^{\frac{3k}{2}} s^{\frac{3}{4}}}) \min(1, \frac{1}{2^{\frac{3j}{2}} s^{\frac{3}{4}}})\, ds\end{gathered}$$

Writing $\sum_{k\le j} 2^k \min(1, \frac{1}{2^{3k/2}s^{3/4}}) \min(1, \frac{1}{2^{3j/2}s^{3/4}}) \le \sum_{k\le j} 2^k \frac{1}{2^j\sqrt{s}} = \frac{2}{\sqrt{s}}$ and $\min(1, \frac{1}{4^j(t-s)}) \le 1$ gives that $\|\Delta_j(B_1(\vec{f},\vec{g})(t))\|_X \le C'\|\vec{f}\|_{\mathcal{X}}\|\vec{g}\|_{\mathcal{X}}$. On the other hand, we have that $\sum_{k\in\mathbb{Z}} 2^k \min(1, \frac{1}{2^{3k/2}s^{3/4}}) \le C\frac{1}{\sqrt{s}}$ and we write

$$\begin{gathered}\int_0^t \frac{1}{\sqrt{t-s}\sqrt{s}} \min(1, \frac{1}{4^j(t-s)}) \min(1, \frac{1}{2^{3j/2}s^{3/4}})\, ds \\ \le \int_0^\infty \frac{1}{4^j}\left(\frac{2}{t}\right)^{3/4} \frac{1}{\sqrt{s}} \min(1, \frac{1}{2^{3j/2}s^{3/4}})\, ds + \int_{t/2}^t \frac{1}{\sqrt{t-s}\sqrt{s}}\left(\frac{2}{4^j t}\right)^{3/4} ds \\ \le C\frac{1}{2^{3j/2}t^{3/4}}.\end{gathered}$$

Thus, we get that $2^{3j/2}t^{3/4}\|\Delta_j(B_1(\vec{f},\vec{g})(t))\|_X \le C\|\vec{f}\|_{\mathcal{X}}\|\vec{g}\|_{\mathcal{X}}$.

We have a similar estimate for $\Delta_j(B_2(\vec f,\vec g)(t))$. To estimate B_3, we write $\Delta_j(B_3(\vec f,\vec g)(t)) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec\nabla.\Delta_j(\sum_{l\geq j-3}\sum_{|l-k|\leq 2}\Delta_k\vec f\otimes\Delta_l\vec g)\,ds$ and $\mathbb{P}\vec\nabla\Delta_j = (-\Delta)^{1/4}\mathbb{P}\vec\nabla\frac{1}{(-\Delta)^{1/4}}\Delta_j = Id\circ(\mathbb{P}\vec\nabla\Delta_j)$, so that

$$\begin{aligned}&\|\Delta_j(B_3(\vec f,\vec g)(t))\|_X\\ \leq \int_0^t C\|\vec f\|_{\mathcal{X}}\|\vec g\|_{\mathcal{X}}\min(2^j,\tfrac{1}{2^{j/2}(t-s)^{3/4}})&\sum_{k\geq j-5}4^k2^{-j}\eta_{|k-j|}\min(1,\tfrac{1}{2^{3k}s^{3/2}})\,ds\end{aligned}$$

Since $\min(2^j,\frac{1}{2^{j/2}(t-s)^{3/4}})\leq 2^j$, $\eta_{|k-j|}\leq\|(\eta_n)\|_\infty$ and $\int_0^\infty 4^k\min(1,\frac{1}{2^{3k}s^{3/2}})\,ds = \int_0^\infty\min(1,\frac{1}{s^{3/2}})\,ds<\infty$, we get that $\|\Delta_j(B_3(\vec f,\vec g)(t))\|_X\leq C\|\vec f\|_{\mathcal{X}}\|\vec g\|_{\mathcal{X}}$. On the other hand, we write $\min(2^j,\frac{1}{2^{j/2}(t-s)^{3/4}})\leq\frac{1}{2^{j/2}(t-s)^{3/4}}$ and then

$$\begin{aligned}&\int_0^t\tfrac{1}{(t-s)^{3/4}}4^k\min(1,\tfrac{1}{2^{3k}s^{3/2}})\,ds\\ \leq&\int_0^\infty\left(\tfrac{2}{t}\right)^{3/4}4^k\min(1,\tfrac{1}{2^{3k}s^{3/2}})\,ds+\int_{t/2}^t\tfrac{1}{(t-s)^{3/4}}\tfrac{2}{t}\,ds\\ \leq&\, Ct^{-3/4}.\end{aligned}$$

Since $\sum_{k\geq j-5}\eta_{|k-j|}=\sum_{n\geq -5}\eta_{|n|}<\infty$, we get that $2^{3j/2}t^{3/4}\|\Delta_j(B_3(\vec f,\vec g)(t))\|_X\leq C\|\vec f\|_{\mathcal{X}}\|\vec g\|_{\mathcal{X}}$ and (A) is proved.

We now prove (B). Since $\mathcal{F}_X$ is continuously embedded into $\mathcal{E}_X$, it is sufficient (according to (A)) to prove that B maps $(\mathcal{E}_X\cap\mathcal{B}_X)\times(\mathcal{E}_X\cap\mathcal{B}_X)$ to $\mathcal{F}_X$. In the proof of (A), the estimates for B_1 showed that $\|\sum_{l\in\mathbb{Z}}S_{l-2}\vec f(s)\otimes\Delta_l g(s)\|_{\dot B_X^{0,\infty}}\leq C\frac{\|\vec f\|_{\mathcal{X}}\|\vec g\|_{\mathcal{X}}}{\sqrt s}$; we have seen in several instances that such an estimate gives that $B_1(\vec f,\vec g)\in L^\infty((0,\infty),(\dot B_X^{0,1})^d)$. We use the same estimates for B_2. The difficulty lies in the third term B_3. We want to show that we have the inequality $\sup_{t>0}\sum_{j\in\mathbb{Z}}\|\Delta_jB_3(\vec f,\vec g)\|_X\leq C\|\vec f\|_{\mathcal{X}}\|\vec g\|_{\mathcal{X}}$. Since $\|\Delta_jB_3(\vec f,\vec g)\|_X\leq C\|\vec f\|_{\mathcal{X}}\|\vec g\|_{\mathcal{X}}(2^j\sqrt t)^{-3/2}$, we may limit the sum to the j's such that $2^j\sqrt t\leq 1$. We want to estimate

$$\begin{aligned}I(t)&=\sum_{2^j\sqrt t\leq 1}\int_0^t\min(2^j,\tfrac{1}{2^{j/2}(t-s)^{3/4}})\sum_{k\geq j-5}4^k2^{-j}\eta_{|k-j|}\min(1,\tfrac{1}{2^{3k}s^{3/2}})\,ds\\ &=\sum_{2^j\sqrt t\leq 1}\int_0^t\sum_{k\geq j-5}4^k\eta_{|k-j|}\min(1,\tfrac{1}{2^{3k}s^{3/2}})\,ds\end{aligned}$$

We write that $\int_0^t 4^k\min(1,\frac{1}{2^{3k}s^{3/2}})\,ds\leq C\min(1,4^kt)$ and get, writing $k=n+j$, that $I(t)\leq C\sum_{n\geq 5}\eta_{|n|}\sum_{2^j\sqrt t\leq 1}\min(4^{j+n}t,1)$. The sum on the j's such that $4^{n+j}t\leq 1$ may be estimated through $\sum_{4^{j+n}t\leq 1}4^{n+j}t\leq 4/3$, while the sum on the j's such that $2^j\sqrt t\leq 1$ and $4^{j+n}t>1$ may be estimated through $\sum_{1/(2^n\sqrt t)<2^j\leq 1/\sqrt t}1\leq 1+n-\ln t/ln2\leq 3n+1$. Thus, $I(t)\leq C\sum_{n\in\mathbb{N}}(1+n)\eta_n$.

We now prove (C). Due to the characterizations of BMO (Proposition 10.1 and Theorem 10.1) and of $BMO^{(-1)}$ (Proposition 16.1), it is sufficient to prove that for $f\in X$ the sequence $(2^{-j}\Delta_jf)_{j\in\mathbb{Z}}$ is a Carleson measure on $\mathbb{Z}\times\mathbb{R}^d$, i.e., we must show that

$$\sup_{x_0\in\mathbb{R}^d,R>0}R^{-d}\int_{B(x_0,R)}\sum_{2^jR\geq 1}4^{-j}|\Delta_jf|^2\,dx<\infty.$$

We write $I(x_0,R)=\int_{B(x_0,R)}\sum_{2^jR\geq 1}4^{-j}|\Delta_j f|^2\,dx$. We choose $\varphi\in\mathcal{D}(\mathbb{R}^d)$ so that $\varphi\geq 0$ and $\varphi=1$ on $B(0,1)$ and we define $\varphi_{x_0,R}=\varphi(\frac{x-x_0}{R})$. Thus $I(x_0,R)\leq\sum_{2^jR\geq 1}4^{-j}\int\varphi_{x_0,R}|\Delta_j f|^2\,dx$. Since $\lim_{x\to\infty}\Delta_j f(x)=0$, we have that $|\Delta_j f|^2=\sum_{l\in\mathbb{Z}}\Delta_l|\Delta_j f|^2$ in $\mathcal{S}'$. Thus, we may write $\int\varphi_{x_0,R}|\Delta_j f|^2\,dx=\sum_{l\in\mathbb{Z}}\int\tilde{\Delta}_l\varphi_{x_0,R}\Delta_l|\Delta_j f|^2\,dx$ with $\tilde{\Delta}_l=\sum_{|k-l|\leq 2}\Delta_k$. We now write

$$\begin{gathered}\|\Delta_l|\Delta_j f|^2\|_X\leq C4^j2^{-l}\eta_{|j-l|}\|f\|_X^2 \text{ for } l\leq j+3\\ \Delta_l|\Delta_j f|^2=0 \text{ for } l\geq j+4.\end{gathered}$$

This gives

$$I(x_0,R)\leq C\|f\|_X^2\sum_{2^jR\geq 1}\sum_{l\leq j+3}2^{-l}\eta_{|j-l|}\|\tilde{\Delta}_l\varphi_{x_0,R}\|_{X^{(*)}}$$

where $X^{(*)}$ is the predual of X. The norm of $X^{(*)}$ (which is isometrically embedded into the dual X^* of X) is shift-invariant and has homogeneity exponent $1-d$: $\|g(\lambda x)\|_{X^{((*)}}=\lambda^{1-d}\|g\|_{X^{(*)}}$. The norm of $\dot{B}^{-1,\infty}_{X^{(*)}}$ has then homogeneity exponent $-d$. We then write $2^{-l}\|\tilde{\Delta}_l\varphi_{x_0,R}\|_{X^{(*)}}\leq CR^{d-1}\|\varphi\|_{X^{(*)}}$ and $2^{-l}\|\tilde{\Delta}_l\varphi_{x_0,R}\|_{X^{(*)}}\leq CR^d\|\varphi\|_{\dot{B}^{-1,\infty}_{X^{(*)}}}$ (we have $\varphi\in\dot{B}^{0,\infty}_1\subset\dot{B}^{-1,\infty}_{X^{(*)}}$, since $X\subset\dot{B}^{-1,\infty}_\infty$). We get

$$I(x_0,R)\leq CR^d\|f\|_X^2\sum_{2^jR\geq 1}\sum_{l\leq j+3}\eta_{|j-l|}\min(1,\frac{1}{2^lR}).$$

It is then enough to write $\sum_{\{j\in\mathbb{Z}/\ 2^{j-n}R\geq 1\}}\frac{1}{2^{j-n}R}\leq 2$ and $\sum_{\{j\in\mathbb{Z}/\ 1\leq 2^jR<2^n\}}1$ $\leq(1+3n)$ to get that $I(x_0,R)\leq CR^d\|f\|_X^2\sum_{n\in\mathbb{N}}(1+n)\eta_n$. □

Chapter 23

Cannone's approach of self-similarity

In his book [CAN 95], Cannone gave a very clear strategy for exhibiting selfsimilar solutions to the Navier–Stokes equations. We take a shift-invariant Banach space of distributions X whose norm is homogeneous with the good scaling for the Navier–Stokes equations ($\|f(\lambda x)\|_X = \frac{1}{\lambda}\|f\|_X$). This implies that $X \subset \dot{B}_\infty^{-1,\infty}$, or equivalently that for all $f \in X$ we have $\sup_{0<t} \sqrt{t}\ \|e^{t\Delta} f\|_\infty \leq C\|f\|_X$. We assume that X contains nontrivial homogeneous distributions, and we try to arrive at a theorem of global existence and uniqueness for solutions of the Navier–Stokes equations. Such a theorem will provide us with self-similar solutions when the initial data is homogeneous (with a small norm).

We write $B(\vec{f}, \vec{g}) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \vec{\nabla}.(\vec{f} \otimes \vec{g})\, ds$, and we check on which Banach space based on X and containing the tendencies $e^{t\Delta}\vec{u}_0$ for $\vec{u}_0 \in X^d$ the bilinear transform B is continuous. The first space we can try is $\mathcal{E} = L^\infty((0,\infty), X^d)$. Of course, the bilinear product uv should be defined for elements of X; hence, we should assume that X is embedded in L^2_{loc}; we then find that $X \subset \dot{M}^{2,d}$, and this case has already been discussed in Chapter 17. There are many instances of spaces X which can be treated this way and provide selfsimilar solutions: homogeneous Besov spaces $\dot{B}_p^{s,\infty}$ where $p < d$ and $s = d/p - 1$ (Cannone [CAN 95], Chemin [CHE 99], Furioli, Lemarié-Rieusset and Terraneo [FURLT 00]), the Lorentz space $L^{d,\infty}$ (Meyer [MEY 97]), and the space $\dot{B}_{PM}^{d-1,\infty} = \{f / |\xi|^{d-1}\ \hat{f} \in L^\infty\}$ used by Le Jan and Sznitman [LEJS 97].

We may use the smoothing effect of $e^{t\Delta}$ and start from a more singular initial value. Cannone [CAN 95] and Planchon [PLA 96] have shown that it is possible to take $\vec{u}_0 \in \dot{B}_p^{s,\infty}$ where $p \in (d,\infty)$ and $s = d/p - 1$; then $\mathcal{E} = \{\vec{f} / \sup_{0<t} t^{1/2-d/2p}\|\vec{f}(t,.)\|_p < \infty\}$ is a good choice.

This latter example can even be generalized by replacing L^p by a Morrey–Campanato space (Kozono and Yamazaki [KOZY 97]). As a matter of fact, the first instance of self-similar solutions was constructed with help of Morrey–Campanato spaces (Giga and Miyakawa [GIGM 89]).

1. Besov spaces

In his book, Cannone [CAN 95] describes, how Besov spaces are used to provide self-similar solutions for the Navier–Stokes equations. The general strategy may be described by the following theorem:

Theorem 23.1:

Let E be a shift-invariant Banach space of local measures so that the norm of E is homogeneous: for some $\alpha \in (0,1]$, we have, for all $f \in E$ and all $\lambda > 0$, $\lambda^\alpha \|f(\lambda x)\|_E = \|f\|_E$. Then:
(A) the bilinear operator B is bounded on $\mathcal{E}_\alpha^d$ where $\mathcal{E}_\alpha$ is defined by

$$f \in \mathcal{E}_\alpha \Leftrightarrow \begin{cases} t^{(1-\alpha)/2} f \in L^\infty((0,+\infty), E) \\ \quad\quad \text{and} \\ \sqrt{t}\ f \in L^\infty((0,\infty) \times \mathbb{R}^d) \end{cases}$$

with norm $\|f\|_{\mathcal{E}_\alpha} = \sup_{t>0} t^{(1-\alpha)/2} \|f\|_E + \sup_{t>0} \sqrt{t} \|f\|_\infty$.
(B) Let E_α be defined as $E_\alpha = E$ if $\alpha = 1$ and $E_\alpha = \dot{B}_E^{-1+\alpha,\infty}$ if $0 < \alpha < 1$. Then, for $f_0 \in \mathcal{S}'(\mathbb{R}^d)$, $f_0 \in E_\alpha \Leftrightarrow e^{t\Delta} f_0 \in \mathcal{E}_\alpha$.
(C) Let C_0 be the norm of the bilinear operator B on $\mathcal{E}_\alpha^d$:

$$C_0 = \sup_{\|\vec{f}\|_{\mathcal{E}_\alpha} \le 1, \|\vec{g}\|_{\mathcal{E}_\alpha} \le 1} \|B(\vec{f}, \vec{g})\|_{\mathcal{E}_\alpha}$$

and let C_1 be the norm of the linear operator $\vec{f}_0 \mapsto e^{t\Delta} \vec{f}_0$ from E_α^d to $\mathcal{E}_\alpha^d$. Then if $\|\vec{u}_0\|_{E_\alpha} \le \frac{1}{4C_0C_1}$ and $\vec{\nabla}.\vec{u}_0 = 0$, there exists a solution $\vec{u} \in \mathcal{E}_\alpha^d$ for the Navier–Stokes equations on $(0,\infty) \times \mathbb{R}^d$ with initial value $\vec{u}_0$, with $\|\vec{u}\|_{\mathcal{E}_\alpha} \le \frac{1}{2C_0}$. This solution is unique in the closed ball $\{\vec{f} \ / \ \|\vec{f}\|_{\mathcal{E}_\alpha} \le \frac{1}{2C_0}\}$ and is smooth on $(0,\infty) \times \mathbb{R}^d$.

In particular, if $\|\vec{u}_0\|_{E_\alpha} \le \frac{1}{4C_0C_1}$ and $\vec{\nabla}.\vec{u}_0 = 0$, if $\lambda > 0$ and if $\vec{v}_0(x) = \lambda \vec{u}_0(\lambda x)$, then $\|\vec{v}_0\|_{E_\alpha} = \|\vec{v}_0\|_{E_\alpha}$ and $\vec{\nabla}.\vec{v}_0 = 0$. Moreover, the solutions $\vec{u}$ and $\vec{v}$ of the Navier–Stokes equations on $(0,\infty) \times \mathbb{R}^d$ with initial value $\vec{u}_0$ and $\vec{v}_0$ described in point (C) satisfy $\vec{v}(t,x) = \lambda \vec{u}(\lambda^2 t, \lambda x)$.
(D) If $\|\vec{u}_0\|_{E_\alpha} \le \frac{1}{4C_0C_1}$ and $\vec{\nabla}.\vec{u}_0 = 0$, if $\lambda > 0$ and if $\vec{u}_0$ is homogeneous: for all $\lambda > 0$, $\lambda \vec{u}_0(\lambda x) = \vec{u}_0(x)$, then the solution $\vec{u}$ of the Navier–Stokes equations on $(0,\infty) \times \mathbb{R}^d$ with initial value $\vec{u}_0$ described in point (C) satisfies $\vec{u}(t,x) = \frac{1}{\sqrt{t}} \vec{u}(1, \frac{x}{\sqrt{t}})$.

Proof: Point (A) was proved in Chapters 17 and 20. Point (B) was proved in Chapter 5. Point (C) is based on the Picard contraction principle when $\|\vec{u}_0\|_{E_\alpha} < \frac{1}{4C_0C_1}$; the case $\|\vec{u}_0\|_{E_\alpha} = \frac{1}{4C_0C_1}$ was proved in Chapter 28. The identity $\vec{v}(t,x) = \lambda \vec{u}(\lambda^2 t, \lambda x)$ is a consequence of the homogeneity of the Navier–Stokes equations, of the identity $(e^{t\Delta} \vec{v}_0)(x) = \lambda (e^{\lambda^2 t \Delta} \vec{u}_0)(\lambda x)$, of the homogeneity of the norms in E_α and $\mathcal{E}_\alpha$ and of the uniqueness of solutions in the ball $\bar{B}(0, \frac{1}{2C_0})$. Point (D) is a direct consequence of Point (C). $\square$

Cannone studied the case of the Besov spaces $\dot{B}_p^{d/p-1,\infty}$ based on Lebesgue spaces. Such spaces contain nontrivial homogeneous distributions. For instance, $\frac{1}{|x|}$ belongs to all the spaces $\dot{B}_p^{d/p-1,\infty}(\mathbb{R}^d)$, $1 \le p \le \infty$. (Notice that those spaces form a monotonous scale: $\dot{B}_p^{d/p-1,\infty} \subset \dot{B}_q^{d/q-1,\infty}$ for $1 \leq p \leq q \leq \infty$). Cannone [CAN 95] gave a precise description of homogeneous distributions in $\dot{B}_p^{d/p-1,\infty}$. First, we need to define Besov spaces on the sphere S^{d-1}. We begin with an easy lemma:

Lemma 23.1:

a) Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Then the pointwise multiplication operator M_φ defined by $M_\varphi f = \varphi f$ is bounded on every Besov space $B_q^{s,p}$, $s \in \mathbb{R}$, $p \in [1,\infty]$ and $q \in [1,\infty]$.

b) Let Ω_1 and Ω_2 be two diffeomorphic open subsets of $\mathbb{R}^d$ and h be a smooth diffeomorphism from Ω_1 onto Ω_2. For every $\varphi \in \mathcal{D}(\Omega_1)$, the map $f \mapsto \varphi\ f \circ h$ defined from $\mathcal{D}'(\Omega_2)$ to $\mathcal{D}'(\Omega_1)$ is continuous in the $B_q^{s,p}$ norm for every $s \in \mathbb{R}$, $p \in [1,\infty]$ and $q \in [1,\infty]$: if K is a compact subset of Ω_2 so that $h($ Supp $\varphi)$ is contained in the interior set of K and if $f \in B_q^{s,p}(\mathbb{R}^d)$ has its support contained in K, then $\|\varphi\ f \circ h\|_{B_q^{s,p}(\mathbb{R}^d)} \leq C(K,s,p,q)\|f\|_{B_q^{s,p}(\mathbb{R}^d)}$.

Proof: a) is very easy. Since pointwise multiplication by a bounded function is a bounded operator on L^p for every $p \in [1,\infty]$, the Leibnitz rule for computing the derivatives of φf shows that M_φ is a bounded operator on the Sobolev spaces $W^{k,p}$ for every $k \in \mathbb{N}$ and every $p \in [1,\infty]$. By interpolation, we get that M_φ is a bounded operator on the Besov spaces $B_q^{s,p}$ for every $s > 0$, every $p \in [1,\infty]$ and every $q \in [1,\infty]$. When $s \leq 0$, we may write $s = \sigma - 2N$ with $\sigma > 0$ and $N \in \mathbb{N}$. Then, f belongs to $B_q^{s,p}$ if and only if $(Id-\Delta)^{-N} f$ belongs tp $B_q^{\sigma,p}$. Thus, we write $M_\varphi f = M_\varphi(Id-\Delta)^N(Id-\Delta)^{-N} f$, and we again use the Leibnitz rule to find smooth, compactly supported functions $\varphi_{\alpha,N}$ so that, for $g \in \mathcal{S}'$, $M_\varphi(Id-\Delta)^N g = \sum_{|\alpha|\leq 2N} \partial^\alpha(M_{\varphi_{\alpha,N}} g)$.

We now prove b). For $f \in \mathcal{D}'(\Omega_2)$, the distribution $f \circ h \in \mathcal{D}'(\Omega_1)$ is defined by $\langle f \circ h|\omega\rangle_{\mathcal{D}'(\Omega_1),\mathcal{D}(\Omega_1)} = \langle f|\ |\det\ (dh^{-1})|\ \omega \circ h^{-1}\rangle_{\mathcal{D}'(\Omega_2),\mathcal{D}(\Omega_2)}$ where dh^{-1} is the Jacobian matrix of h^{-1}. In particular, we easily check that $\frac{\partial}{\partial x_k}(f \circ h) = \sum_{j=1}^d \frac{\partial}{\partial x_k} h_j\ (\frac{\partial}{\partial x_j} f) \circ h$. Thus, if $\varphi \in \mathcal{D}(\Omega_1)$ and if $\psi \in \mathcal{D}(\Omega_2)$ is identically equal to 1 in a neighborhood of $h($ Supp $\varphi)$, the mapping $f \mapsto \varphi\ (\psi f) \circ h$ is a bounded operator on every Sobolev space $W^{k,p}$ with $k \in \mathbb{N}$ and $p \in [1,\infty]$; hence, by interpolation, on every Besov space $B_q^{s,p}$ with $s > 0$, $p \in [1,\infty]$ and $q \in [1,\infty]$. Similarly, we find that the mapping $\omega \mapsto \alpha\ (\beta\omega) \circ h^{-1}$ is bounded on the closure of the test functions in every Besov space $B_q^{s,p}$ with $s > 0$, $p \in [1,\infty]$ and $q \in [1,\infty]$, where $\alpha(x) = |\det\ (dh^{-1})|\ \varphi \circ h^{-1}$ and $\beta(x) = \psi \circ h$; thus, by duality, we get that $f \mapsto \varphi\ (\psi f) \circ h$ is bounded on every Besov space $B_q^{s,p}$ with $s < 0$, $p \in [1,\infty]$ and $q \in [1,\infty]$. The case $s = 0$ then follows by interpolation between the case $s > 0$ and the case $s < 0$. □

We then may define Besov spaces on a compact smooth manifold:

Definition 23.1: (Besov spaces on a compact manifold)

Let M be a compact smooth manifold of dimension δ. The Besov space $B_q^{s,p}(M)$ is then defined by: $T \in \mathcal{D}'(M)$ belongs to $B_q^{s,p}(M)$ if and only if for *every* (Ω, h, φ) *such that* Ω *is an open subset of* $\mathbb{R}^\delta$, $h : \Omega \mapsto M$ *is a smooth diffeomorphism from* Ω *onto* $h(\Omega)$ *and* $\varphi \in \mathcal{D}(\Omega)$ *the distribution* $T_{\Omega,h,\varphi}$ *defined by* $T_{\Omega,h,\varphi}(f) = T((\varphi f) \circ h^{-1})$ *belongs to* $B_q^{s,p}(\mathbb{R}^\delta)$.

Let $(\Omega_\alpha, h_\alpha)_{\alpha \in A}$ be a finite atlas of M: Ω_α is an open subset of $\mathbb{R}^\delta$, h_α is a diffeomorphism from Ω_α to $h(\Omega_\alpha)$, $M = \cup_{\alpha \in A} h_\alpha(\Omega_\alpha)$ and A is finite. Let $(\varphi_\alpha)_{\alpha \in A}$ be a partition of unity subordinated to the $h_\alpha(\Omega_\alpha)$. We may then define the norm of $B_q^{s,p}(M)$ as $\|T\|_{B_q^{s,p}} = \sum_{\alpha \in A} \|T_{\Omega_\alpha, h_\alpha, \varphi_\alpha \circ h_\alpha}\|_{B_q^{s,p}(\mathbb{R}^\delta)}$. According to Lemma 23.1, we easily see that the Banach space $B_q^{s,p}(M)$ does not depend on the choice of the atlas, since all those norms are equivalent.

Definition 23.2:

Let $F \in \mathcal{D}'(S^{d-1})$ *and let* $f \in \cap_{T>0} L^1((0,T), r^{d-1}\, dr)$. *Then the formula*

$$\langle F(\sigma) f(r) | \varphi(x) \rangle_{\mathcal{D}'(\mathbb{R}^d), \mathcal{D}(\mathbb{R}^d)} = \langle F(\sigma) | \int_0^\infty \varphi(r\sigma) \bar{f}(r) r^{d-1}\, dr \rangle_{\mathcal{D}'(S^{d-1}), \mathcal{D}(S^{d-1})}$$

is well defined and defines a distribution $F(\sigma) f(r) \in \mathcal{D}'(\mathbb{R}^d)$.

This definition can be useful when characterizing the homogeneous distributions:

Lemma 23.2:

Let T *be a distribution on* $\mathbb{R}^d$ *and* $\alpha > -d$. *Then the following assertions are equivalent:*
(A) T *is homogeneous with degree* α:

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^d) \quad \forall \lambda > 0 \quad \langle T(x) | \lambda^d \varphi(\lambda x) \rangle = \lambda^{-\alpha} \langle T(x) | \varphi(x) \rangle$$

(B) There exists $\omega \in \mathcal{D}'(S^{d-1})$ *such that* $T(x) = \omega(\sigma) r^\alpha$.
The distribution ω *is then unique. We shall write* $\omega = T|_{S^{d-1}}$.

Proof: (A) is obviously a consequence of (B). Conversely, let us define a distribution $\omega \in \mathcal{D}'(S^{d-1})$ by choosing a real-valued function $\beta \in \mathcal{D}((0,\infty))$ so that $\int_0^\infty \beta(r) r^{\alpha+d-1}\, dr = 1$: we then define ω by $\langle \omega | \varphi(\sigma) \rangle_{\mathcal{D}'(S^{d-1}), \mathcal{D}(S^{d-1})} = \langle T | \varphi(\frac{x}{|x|}) \beta(|x|) \rangle_{\mathcal{D}'(\mathbb{R}^d), \mathcal{D}(\mathbb{R}^d)}$. We first prove that we have $T(x) = \omega(\sigma) r^\alpha$ in $\mathcal{D}'(\mathbb{R}^d - \{0\})$, i.e., that for all $\varphi \in \mathcal{D}(\mathbb{R}^d - \{0\})$, defining $\psi \in \mathcal{D}(\mathbb{R}^d - \{0\})$ by

$$\psi(x) = \varphi(x) - \beta(|x|) \int_0^\infty \varphi(r \frac{x}{|x|}) r^{\alpha+d-1}\, dr,$$

we have $T(\psi) = 0$. We write $x = r\sigma$ in polar coordinates; hence, $\psi(x) = F(r, \sigma)$ with $F \in \mathcal{D}((0,\infty) \times S^{d-1})$. We have $\int_0^\infty F(r \frac{x}{|x|}) r^{\alpha+d-1}\, dr = 0$, so that $F(r,\sigma) = r^{1-d-\alpha} \frac{\partial}{\partial r} G(r,\sigma)$ with $G \in \mathcal{D}((0,\infty) \times S^{d-1})$. Let $r^{-d-\alpha} G(r,\sigma) = \theta(x)$; we get $\psi(x) = r \frac{\partial}{\partial r}\theta + (d+\alpha)\theta = \sum_{i=1}^d x_i \frac{\partial}{\partial x_i}\theta + (d+\alpha)\theta$. Since T

is homogeneous, it satisfies the Euler equation $\sum_{i=1}^d x_i \frac{\partial}{\partial x_i} T = \alpha T$, and thus $\langle T|\psi\rangle =< -\sum_{i=1}^d x_i \frac{\partial}{\partial x_i} T + \alpha T|\theta\rangle = 0$. We find that the distribution $T - \omega(\sigma) r^\alpha$ is supported in $\{0\}$, thus its Fourier transform is a polynomial and must be a homogeneous distribution with homogeneity exponent $-d-\alpha$; since $-d-\alpha < 0$, we find that this must be the null polynomial. $\square$

A final lemma will be useful:

Lemma 23.3:

a) Let $\alpha \in \mathcal{D}(\mathbb{R})$ with $\alpha(0) \neq 0$. Then for a distribution $T \in \mathcal{D}'(\mathbb{R}^{d-1})$ and for $s \in \mathbb{R}$, $p \in [1, \infty]$ and $q \in [1, \infty]$, the following assertions are equivalent:
i) $T \in B_q^{s,p}(\mathbb{R}^{d-1})$
ii) $T \otimes \alpha \in B_q^{s,p}(\mathbb{R}^d)$.

b) Let $\alpha \in \mathcal{D}((0, \infty))$ with $\alpha(1) \neq 0$. Then for a distribution $T \in \mathcal{D}'(S^{d-1})$ and for $s \in \mathbb{R}$, $p \in [1, \infty]$ and $q \in [1, \infty]$, the following assetions are equivalent:
i) $T \in B_q^{s,p}(S^{d-1})$
ii) $T(\sigma)\alpha(r) \in B_q^{s,p}(\mathbb{R}^d)$.

Proof: We choose $\beta \in \mathcal{D}(\mathbb{R})$ so that $\int \alpha\beta \, dx = 1$ and we define the operators A_α and B_β by:

$$T \in \mathcal{D}'(\mathbb{R}^{d-1}) \mapsto A_\alpha(T) = T \otimes \alpha \in \mathcal{D}'(\mathbb{R}^d)$$

and

$$T \in \mathcal{D}'(\mathbb{R}^d) \mapsto B_\beta(T) = \int T \, \beta(x_d) \, dx_d \in \mathcal{D}'(\mathbb{R}^{d-1})$$

(i.e., $\langle B_\beta(T)|\omega\rangle = \langle T|\omega \otimes \bar{\beta}\rangle$). We have $B_\beta \circ A_\alpha = Id$; thus, we are able to prove the lemma by showing that A_α maps boundedly $B_q^{s,p}(\mathbb{R}^{d-1})$ to $B_q^{s,p}(\mathbb{R}^d)$ and that B_β maps boundedly $B_q^{s,p}(\mathbb{R}^d)$ to $B_q^{s,p}(\mathbb{R}^{d-1})$. Obviously, A_α maps boundedly the Sobolev space $W^{k,q}(\mathbb{R}^{d-1})$ to $W^{k,q}(\mathbb{R}^d)$ and B_β maps boundedly the Sobolev space $W^{k,q}(\mathbb{R}^d)$ to $W^{k,q}(\mathbb{R}^{d-1})$ for every $k \in \mathbb{N}$ and every $q \in [1, \infty]$; by interpolation, this gives the required estimates for Besov spaces $B_q^{s,p}$ with positive exponent s. We deal easily with the remaining exponents by noting that, writing Δ_d for the d-dimensional Laplacian and Δ_{d-1} for the $(d-1)$-dimensional Laplacian, we have, for $T \in \mathcal{D}'(\mathbb{R}^{d-1})$, $A_\alpha(\Delta_{d-1} T) = \Delta_{d-1} A_\alpha(T)$ in $\mathcal{D}'(\mathbb{R}^d)$ and, for $T \in \mathcal{D}'(\mathbb{R}^d)$, $B_\beta(\Delta_d T) = \Delta_{d-1} B_\beta(T) + B_{\frac{d^2\beta}{dx^2}}(T)$ in $\mathcal{D}'(\mathbb{R}^{d-1})$.

Point b) is a direct consequence of Point a) and Lemma 23.1, by using an atlas on S^{d-1}. $\square$

Cannone's result follows:

Proposition 23.1:

Let $p \in [1, +\infty]$ and T be a distribution on $\mathbb{R}^d$ which is homogeneous with degree -1. Then the following assertions are equivalent:
(A) $T \in \dot{B}_p^{d/p-1,\infty}(\mathbb{R}^d)$
(B) $T|_{S^{d-1}} \in B_p^{d/p-1,p}(S^{d-1})$.

Proof: The proof relies on the characterization of Besov spaces by means of wavelets (see Chapter 9). According to Lemma 23.3, we may replace assertion (B) by $\alpha(r)T(x)(= r\alpha(r)T|_{S^{d-1}}(\sigma)) \in B_p^{d/p-1,p}(\mathbb{R}^d)$, where $\alpha \in \mathcal{D}((0,\infty))$ with $\alpha(1) \neq 0$. We use d-variate Daubechies compactly supported wavelets. Since T is homogeneous, it is a distribution with finite order K and thus both T and $\alpha(r)T$ may be decomposed on a wavelet basis $(\varphi(x-k))_{k\in\mathbb{Z}^d} \cup (\psi_{\epsilon,j,k})_{1\leq\epsilon\leq 2^d-1, j\in\mathbb{N}, k\in\mathbb{Z}^d}$, where $(\psi_{\epsilon,j,k})_{1\leq\epsilon\leq 2^d-1, j\in\mathbb{Z}, k\in\mathbb{Z}^d}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ formed by d-variate Daubechies compactly supported wavelets and where the associated compactly supported scaling function φ is of class $\mathcal{C}^N$ with $N > K$. Moreover, we assume $N > d-1$.

Let Q_j be the projection operator $Q_j f = \sum_{1\leq\epsilon\leq 2^d-1}\sum_{k\in\mathbb{Z}^d}\langle f|\psi_{\epsilon,j,k}\rangle\psi_{\epsilon,j,k}$ and let P_j be the projection operator $P_j f = \sum_{k\in\mathbb{Z}^d}\langle f|\varphi_{j,k}\rangle\varphi_{j,k}$. Then we have the following equivalences for a distribution T of order $K < N$:

$$T \in \dot{B}_p^{d/p-1,\infty} \Leftrightarrow \sup_{j\in\mathbb{Z}} 2^{j(d/p-1)}\|Q_jT\|_p < \infty \text{ and } \lim_{j\to-\infty}\|P_jT\|_\infty = 0$$

and

$$\alpha(r)T \in B_p^{d/p-1,p} \Leftrightarrow (2^{j(d/p-1)}\|Q_j(\alpha T)\|_p)_{j\in\mathbb{N}} \in l^p(\mathbb{N}) \text{ and } \|P_0(\alpha T)\|_p < \infty.$$

In the case of a distribution T, which is homogeneous with homogeneity exponent -1, we have $2^{j(d/p-1)}\|Q_jT\|_p = \|Q_0T\|_p$ and $2^{-j}\|P_jT\|_\infty = \|P_0T\|_\infty$, so that we have:

$$T \in \dot{B}_p^{d/p-1,\infty} \Leftrightarrow Q_0T \in L^p \text{ and } P_0T \in L^\infty.$$

If $T|_{s^{d-1}}$ belongs to $B_p^{s,p}$ and if we choose $\alpha \in D((0,\infty))$ with $\alpha = 1$ on $[1/2, 4]$, then if δ is the maximum of the diameters of the supports of the wavelets ψ_ϵ $(1 \leq \epsilon < 2^d)$ and of the scaling function φ, we have Supp $\psi_\epsilon(x-k) \subset \{x \in \mathbb{R}^d \ / \ |k|-\delta \leq |x| \leq |k|+\delta\}$ (and the same for the support of $\varphi(x-k)$), so that if $|k| \geq 2\delta$ and if we write $|k| \in [2^{j_0}, 22^{j_0})$, we find that the support of $\psi_\epsilon(x-k)$ and the support of $\varphi(x-k)$ are contained in the annulus $\{x \in \mathbb{R}^d \ / \ \frac{1}{2} \leq |\frac{x}{2^{j_0}}| \leq 4\}$, and we find

$$\|Q_0T\|_p \leq C \sum_{1\leq\epsilon<2^d} \| \ \langle T|\psi_\epsilon(x-k)\rangle \ \|_{l^p(\mathbb{Z})} \leq C \sum_{1\leq\epsilon<2^d}(A_\epsilon + B_\epsilon)$$

with

$$\begin{cases} A_\epsilon = \| \ \langle T|\psi_\epsilon(x-k)\rangle \ 1_{[0,4\delta]}(|k|)\|_{l^p(\mathbb{Z})} \\ B_\epsilon = \|1_{[2\delta,\infty)}(2^j)\| \ \langle T|\psi_\epsilon(x-k)\rangle \ 1_{[2^j,22^j)}(|k|)\|_{l^p(\mathbb{Z})}\|_{l^p(\mathbb{Z})}. \end{cases}$$

There is only a finite number of indices k so that $|k| \leq 4\delta$; moreover, $T = \frac{1}{r}T|_{S^{(d-1)}}(\sigma)$; hence, (since $\int_0^1 \frac{1}{r}r^{d-1}\, dr < \infty$), the distribution T is controlled by the distribution $T|_{S^{(d-1)}}$ and we may write $|\ \langle T|\psi_\epsilon(x-k)\rangle\ | \leq C(k)\|T|_{S^{(d-1)}}\|_{B_p^{d/p-1,p}}$. Thus, A_ϵ is well controlled.

In order to control B_ϵ, we write for $2^j \geq 2\delta$ and $|k| \in [2^j, 22^j)$ that $\langle T|\psi_\epsilon(x-k)\rangle = \langle \alpha(2^{-j}r)T|\psi_\epsilon(x-k)\rangle = 2^{j(d-1)}\langle \alpha(r)T|\psi_\epsilon(2^j x-k)\rangle$; this gives for $2^j \geq 2\delta$,

$$\begin{aligned}\|\ \langle T|\psi_\epsilon(x-k)\rangle\ 1_{[2^j,22^j)}(|k|)\|_{l^p(\mathbb{Z})} &\leq 2^{j(d-1)}\|\ \langle \alpha(r)T|\psi_\epsilon(2^j x-k)\rangle\|_{l^p(\mathbb{Z})}\\ &\leq C2^{j(d/p-1)}\|Q_j(\alpha(r)T)\|_p.\end{aligned}$$

We find that $B_\epsilon \leq C\|T|_{S^{(d-1)}}\|_{B_p^{d/p-1,p}}$.

The same proof (replacing ψ_ϵ by φ and p by $+\infty$) gives that $\|P_0 T\|_\infty \leq C\|T|_{S^{(d-1)}}\|_{B_\infty^{-1,\infty}} \leq C'\|T|_{S^{(d-1)}}\|_{B_p^{d/p-1,p}}$.

Conversely, let us assume that $T \in \dot{B}_p^{d/p-1,\infty}$. Let $\alpha \in \mathcal{D}((0,\infty))$. We want to prove that $\alpha(r)T \in B_p^{d/p-1,p}$. We define for $j \geq 0$ $K_j = \{k \in \mathbb{Z}^d\ /\ \exists \epsilon \in \{1,\ldots,2^d-1\}\ \text{Supp}\ \psi_{\epsilon,j,k} \cap\ \text{Supp}\ \alpha(r) \neq \emptyset\}$ and we similarly define $K_{-1} = \{k \in \mathbb{Z}^d\ /\ \text{Supp}\ \varphi(x-k) \cap\ \text{Supp}\ \alpha(r) \neq \emptyset\}$. Then, we define the distribution

$$L = \sum_{k\in K_{-1}} \langle T|\varphi(x-k)\rangle\ \varphi(x-k) + \sum_{j\geq 0}\sum_{k\in K_j}\sum_{1\leq\epsilon\leq 2^d-1} \langle T|\psi_{\epsilon,j,k}\rangle\ \psi_{\epsilon,j,k}.$$

We have $\alpha(r)T = \alpha(r)L$; since the space $B_p^{d/p-1,p}$ is preserved by pointwise multiplication with smooth functions, hence we shall prove $T|_{S^{d-1}} \in B_p^{d/p-1,p}(S^{d-1})$ by proving that $L \in B_p^{d/p-1,p}(\mathbb{R}^d)$. This is obvious, since we have by homogeneity of T that

$$\sum_{j\geq 0} 2^{j(d-p)}\|Q_j L\|_p^p \leq C\sum_{j\geq 0}\sum_{1\leq\epsilon\leq 2^d-1}\sum_{k\in K_j} |\ \langle T|\psi_\epsilon(x-k)\rangle\ |^p \leq C'\|Q_0 T\|_p^p$$

since the family K_j satisfies $\sup_{k\in\mathbb{Z}^d}\sum_{j\in\mathbb{N}} 1_{K_j}(k) < \infty$. □

2. The Lorentz space $L^{d,\infty}$

Barraza [BAR 96] studied the existence of self-similar solutions for an initial value in the Lorentz space $(L^{d,\infty})^d$. Theorem 23.1 may be applied to $E = L^{d,\infty}$ (with $\alpha = 0$).

Moreover, Barraza was able to easily characterize the homogeneous distributions in $L^{d,\infty}$:

Proposition 23.2:

Let T be a distribution on $\mathbb{R}^d$ which is homogeneous with degree -1. Then the following assertions are equivalent:

(A) $T \in L^{d,\infty}(\mathbb{R}^d)$

(B) $T|_{S^{d-1}} \in L^d(S^{d-1})$.

Proof: Let T be a distribution on $\mathbb{R}^d$ which is homogeneous of degree -1 and locally integrable for the Lebesgue measure. Then $f = T|_{S^{d-1}}$ belongs to $L^1(S^{d-1})$. Moreover, T belongs to $L^{d,\infty}$ if and only if $\sup_{k\in\mathbb{Z}} 2^{kd} m(\{x \in \mathbb{R}^d \ / \ |T(x)| \in [2^k, 2^{k+1}]\}) < \infty$, while f belongs to $L^d(S^{d-1})$ if and only if $\sum_{k\in\mathbb{Z}} 2^{kd}\sigma(\{x \in S^{d-1} \ / \ |T(x)| \in [2^k, 2^{k+1}]\}) < \infty$. It is then enough to write that, for $r \in [2^j, 2^{j+1}]$, we have $|T(r\sigma)| \in [2^k, 2^{k+1}] \Rightarrow |f(\sigma)| \in [2^{k+j}, 2^{k+j+2}]$. Thus, if $\epsilon_k = m(\{x \in \mathbb{R}^d \ / \ |T(x)| \in [2^k, 2^{k+1}]\})$ and $\eta_k = \sigma(\{x \in S^{d-1} \ / \ |T(x)| \in [2^k, 2^{k+1}]\})$, then

$$2^{kd}\epsilon_k \leq C 2^{kd} \sum_{j\in\mathbb{Z}} 2^{jd}\eta_{k+j} \leq C'\|f\|_d^d.$$

On the other hand, we have

$$\sum_{k\in\mathbb{Z}} 2^{kd}\eta_k \leq C \sum_{k\in\mathbb{Z}} m(\{x \in \mathbb{R}^d \ /|x| \in [2^k, 2^{k+1}] \text{ and } |T(x)| \in [1/2, 2]\})$$

so that $\|f\|_d^d \leq Cm(\{x \in \mathbb{R}^d \ /|T(x)| \in [1/2, 2]\})$. □

As was noted by Cannone [CAN 95], following Taylor [TAY 92] and Federbush [FED 93], homogeneous Morrey–Campanato spaces may provide a good frame for the existence of global solutions and hence for self-similar solutions. Indeed, Theorem 23.1 may be applied to $E = \dot{M}^{p,d}$ (with $\alpha = 0$) for $1 \leq p < d$. We may easily extend the results of Proposition 23.2 to the setting of the Morrey–Campanato spaces.

Proposition 23.3:

Let T be a distribution on $\mathbb{R}^d$, which is homogeneous with degree -1. Let $p \in [1, d)$. Then the following assertions are equivalent:
(A) $T \in \dot{M}^{p,d}(\mathbb{R}^d)$
(B) $T|_{S^{d-1}} \in L^p(S^{d-1})$ *(if $d-1 \leq p < d$) or* $T|_{S^{d-1}} \in M^{p,d-1}(S^{d-1})$ *(if $1 \leq p < d-1$).*

Proof: Let us assume that $T(x) = \frac{f(\sigma)}{r}$ belongs to $\dot{M}^{p,d}$ with $p < d$. We see easily that $f \in L^p$: we have $\|f\|_p^p \leq \frac{1}{d-p} \int\int_0^1 |f(\sigma)|^p r^{d-p} \frac{dr}{r} d\sigma \leq C_p \int_{|x|\leq 1} |T(x)|^p \, dx$. If we assume that $p < d-1$, we must prove that we have moreover $\sup_{\sigma_0 \in S^{d-1}, \rho\in(0,1/2)} \rho^{p-d+1} \int_{|\sigma-\sigma_0|\leq\rho} |f(\sigma)|^p d\sigma < \infty$:

$$\begin{aligned}\int_{|\sigma-\sigma_0|\leq\rho} |f(\sigma)|^p d\sigma \quad & \leq \frac{1}{2\rho(1-\rho)^{d+1-p}} \int_{|\sigma-\sigma_0|\leq\rho} \int_{1-\rho}^{1+\rho} |f(\sigma)|^p r^{d-p} \frac{dr}{r} \\ & \leq C\rho^{-1} \int_{|x-\sigma|_0\leq 2\rho} |T(x)|^p \, dx \leq C\|T\|_{\dot{M}^{p,d}}^d \rho^{d-p-1}.\end{aligned}$$

Conversely, let us assume that f belongs to $L^p(S^{d-1})$ and, if $p < d-1$, more precisely to $M^{p,d-1}(S^{d-1})$. We want to estimate, for $r_0 > 0$ and $x_0 \in \mathbb{R}^d$,

$I(x_0,r_0) = r_0^{p-d}\int_{|x-x_0|\leq r_0}|T(x)|^p\ dx = r_0^{p-d}\int\int_{|r\sigma-x_0|\leq r_0}|f(\sigma)|^p\ r^{d-p}\ \frac{dr}{r}d\sigma$. We write $x_0 = \rho_0\sigma_0$ and $r = \rho_0\rho$. We have $I(x_0,r_0) = I(\sigma_0,\frac{r_0}{\rho_0})$. We may assume that $x_0 = \sigma_0$ and try to prove that $\sup_{\sigma_0\in S^{d-1},r_0>0} I(\sigma_0,r_0) < \infty$. We have $I(\sigma_0,r_0) \leq \|f\|_p^p\, r_0^{d-p}\int_0^{1+r_0} r^{d-p}\ \frac{dr}{r}$; we get the required inequality for $r_0 > 1/2$. If $r_0 < 1/2$, we write for $|r\sigma - x_0| < r_0$ that we have $r \in [1-r_0, 1+r_0]$ and $|\sigma-\sigma_0| < 2r_0$; this gives

$$\begin{aligned} I(\sigma_0,r_0) \quad &\leq r_0^{p-d}\int_{|\sigma-\sigma_0|<2r_0}|f(\sigma)|^p\ d\sigma\int_{1-r_0}^{1+r_0} r^{d-p}\ \frac{dr}{r} \\ &\leq r_0^{p-d}(2r_0)^{\max(d-p-1,0)}\|f\|_{X^p}^p r_0\max((1-r_0)^{d-p-1},(1+r_0)^{d-p-1}) \end{aligned}$$

with $X^p = L^p$ if $p\in[d-1,d)$ and $X^p = M^{p,d-1}$ if $p\in[1,d-1]$. □

3. Asymptotic self–similarity

Planchon has shown that the asymptotic behavior of solutions for the Navier–Stokes equations was controlled by the behaviour of the initial value at low frequencies [PLA 96]. We consider initial values with small norms in $(\dot{B}_p^{d/p-1,\infty}(\mathbb{R}^d))^d$. Recall that B is bounded on the space $\mathcal{E}_p^d$ $(d < p < \infty)$ where $\mathcal{E}_p = \{f\in L^\infty_{loc,t}L^p_x\ /\ \sup_{t>0} t^{\frac{1}{2}-\frac{d}{2p}}\|f\|_p < \infty\}$, since

$$\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{u}\otimes\vec{v}\|_p \leq C\frac{1}{(t-s)^{\frac{1}{2}+\frac{d}{2p}}}\frac{1}{s^{1-\frac{d}{p}}}\|\vec{u}\|_{\mathcal{E}_p}\|\vec{v}\|_{\mathcal{E}_p}.$$

In order to describe the asymptotic behavior of solutions as t goes to $+\infty$, we introduce the following definition:

Definition 23.3:
Let $\mathcal{E}_p = \{f\in L^\infty_{loc,t}L^p_x\ /\ \sup_{t>0} t^{\frac{1}{2}-\frac{d}{2p}}\|f\|_p < \infty\}$. Two functions $f,g\in\mathcal{E}_p$ are asymptotic in $\mathcal{E}_p$ if $\lim_{t\to+\infty} t^{\frac{1}{2}-\frac{d}{2p}}\|f(t,.)-g(t,.)\|_p = 0$.

We then have the following easy lemma:

Lemma 23.4:
If $\vec{f}$ and $\vec{g}$ belong to $\mathcal{E}_p^d$ and are asymptotic in $\mathcal{E}_p$, then $B(\vec{f},\vec{f})$ and $B(\vec{g},\vec{g})$ are asymptotic in $\mathcal{E}_p$.

Proof: Let $\|f\|_{p,t} = t^{\frac{1}{2}-\frac{d}{2p}}\|f(t,.)\|_p$. We have the identity $B(\vec{f},\vec{f})-B(\vec{g},\vec{g}) = B(\vec{f}-\vec{g},\vec{f}) + B(\vec{g},\vec{f}-\vec{g})$; hence,

$$\|B(\vec{f},\vec{f})-B(\vec{g},\vec{g})\|_{p,t} \leq C\int_0^1\frac{1}{(1-\tau)^{\frac{1}{2}+\frac{d}{2p}}}\frac{1}{\tau^{1-\frac{d}{p}}}(\|\vec{f}\|_{p,t\tau}+\|\vec{g}\|_{p,t\tau})\|\vec{f}-\vec{g}\|_{p,t\tau}\ d\tau$$

and we conclude by dominated convergence. $\square$

We have a "converse" result:

Lemma 23.5:
Let C_p be the norm of the bilinear operator B on $(\mathcal{E}_p)^d$. Let $\vec{f}$ and $\vec{g}$ belong to $\mathcal{E}_p^d$ with $\|\vec{f}\|_{\mathcal{E}_p} < \frac{1}{2C_p}$ and $\|\vec{g}\|_{\mathcal{E}_p} < \frac{1}{2C_p}$. If $\vec{f} + B(\vec{f},\vec{f})$ and $\vec{g} + B(\vec{g},\vec{g})$ are asymptotic in $\mathcal{E}_p$, then $\vec{f}$ and $\vec{g}$ are asymptotic in $\mathcal{E}_p$.

Proof: Let $\|f\|_{p,t} = t^{\frac{1}{2}-\frac{d}{2p}}\|f(t,.)\|_p$. Define $\alpha(t) = \|\vec{f}-\vec{g}\|_{p,t}$ and $\epsilon(t) = \|\vec{f}+B(\vec{f},\vec{f})-\vec{g}-B(\vec{g},\vec{g})\|_{p,t}$. Writing again $B(\vec{f},\vec{f})-B(\vec{g},\vec{g}) = B(\vec{f}-\vec{g},\vec{f}) + B(\vec{g},\vec{f}-\vec{g})$, we get for $R>1$ $\alpha(t) \le \epsilon(t) + I(R) + J(R)$ with

$$\begin{array}{ll} I(R) & = \| \int_0^{t/R} e^{(t-s)\delta}\mathbb{P}\vec{\nabla}.[\vec{f}\otimes(\vec{f}-\vec{g}) + (\vec{f}-\vec{g})\otimes\vec{g}]\; ds\|_{p,t} \\ & \le C\int_0^{1/R} \frac{1}{(1-\tau)^{\frac{1}{2}+\frac{d}{2p}}}\frac{1}{\tau^{1-\frac{d}{p}}}[\|f\|_{p,t\tau} + \|g\|_{p,t\tau}]\alpha(t\tau)\; d\tau \le \gamma(R), \end{array}$$

where $\gamma(R) = \frac{C}{4C_p^2}\int_0^{1/R} \frac{1}{(1-\tau)^{\frac{1}{2}+\frac{d}{2p}}}\frac{1}{\tau^{1-\frac{d}{p}}}\; d\tau$, and

$$\begin{array}{ll} J(R) & = \| \int_{t/R}^{t} e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.[\vec{f}\otimes(\vec{f}-\vec{g}) + (\vec{f}-\vec{g})\otimes\vec{g}]\; ds\|_{p,t} \\ & \le \delta \sup_{t/R<s<t} \|\vec{f}-\vec{g}\|_{p,s} \end{array}$$

with $\delta = C_p(\|\vec{f}\|_{\mathcal{E}_p} + \|\vec{g}\|_{\mathcal{E}_p}) < 1$. The main point is that δ does not depend on R and is less than 1 while $\lim_{R\to\infty}\gamma(R) = 0$. Finally, we define $\beta(t) = \sup_{s>t}\alpha(s)$ and $\eta(t) = \sup_{s>t}\epsilon(s)$. We have for $N\in\mathbb{N}$

$$\beta(R^{N+1}) \le \eta(R^{N+1}) + \gamma(R) + \delta\beta(R^N)$$

which gives

$$\beta(R^N) \le \sum_{k=1}^{N}\eta(R^k)\delta^{N-k} + \frac{1-\delta^N}{1-\delta}\gamma(R) + \delta^N\beta(1);$$

we then let N go to $+\infty$ and get $\limsup_{t\to+\infty}\alpha(t) \le \gamma(R)$ and then we let R go to $+\infty$ to get $\lim_{t\to+\infty}\alpha(t) = 0$. $\square$

Thus, we may describe the asymptotics of the Navier–Stokes solutions in $\mathcal{E}_p$:

Theorem 23.2: (Equivalence in $\mathcal{E}_p$)
Let $p\in(d,\infty)$. Let $\mathcal{E}_p = \{f\in L^\infty_{loc,t}L^p_x \;/\; \sup_{t>0} t^{\frac{1}{2}-\frac{d}{2p}}\|f\|_p < \infty\}$ and let C_p be the norm of the bilinear operator B on $(\mathcal{E}_p)^d$. Let $\vec{u}_0$ and $\vec{v}_0$ in $(\dot{B}_p^{d/p-1,\infty}(\mathbb{R}^d))^d$ be such that $\vec{\nabla}.\vec{u}_0 = \vec{\nabla}.\vec{v}_0 = 0$ and such that $\|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_p} < \frac{1}{4C_p}$

and $\|e^{t\Delta}\vec{v}_0\|_{\mathcal{E}_p} < \frac{1}{4C_p}$*). Let* $\vec{u}$ *and* $\vec{v}$ *be the mild solutions of the Navier–Stokes equations with initial value* $\vec{u}_0$ *and* $\vec{v}_0$ *(with* $\|\vec{u}\|_{\mathcal{E}_p} < \frac{1}{2C_p}$ *and* $\|\vec{v}\|_{\mathcal{E}_p} < \frac{1}{2C_p}$*). Then the following assertions are equivalent:*
(A) $\vec{u}$ *and* $\vec{v}$ *are asymptotic in* $\mathcal{E}_p$
(B) $e^{t\Delta}\vec{u}_0$ *and* $e^{t\Delta}\vec{v}_0$ *are asymptotic in* $\mathcal{E}_p$
(C) $\lim_{j\to-\infty} \|S_j(\vec{u}_0 - \vec{v}_0)\|_{\dot{B}_p^{d/p-1,\infty}} = 0$
(D) $\lim_{j\to-\infty} 2^{j(d/p-1)}\|\Delta_j(\vec{u}_0 - \vec{v}_0)\|_p = 0$.

Proof: (A) implies (B) by Lemma 23.4, since we have $e^{t\Delta}\vec{u}_0 - e^{t\Delta}\vec{v}_0 = \vec{u} - \vec{v} + B(\vec{u},\vec{u}) - B(\vec{v},\vec{v})$.
(B) implies (A) by Lemma 23.5.
(B) $\Rightarrow$ (D): We notice that $e^{-4^{-j}\Delta}\Delta_j$ is continuous on L^p with an operator norm which does not depend on j; thus, for $t = 4^{-j}$, $2^{j(d/p-1)}\|\Delta_j(\vec{u}_0 - \vec{v}_0)\|_p \leq Ct^{\frac{1}{2}-\frac{d}{2p}}\|e^{t\Delta}(\vec{u}_0 - \vec{v}_0)\|_p$.
(C) $\Leftrightarrow$ (D) is obvious.
(C) $\Rightarrow$ (B): For $f \in \dot{B}_p^{d/p-1,\infty}$ and $j \in \mathbb{Z}$, we have $(Id - S_j)f \in \dot{B}_p^{d/p-2,\infty}$; hence, $t^{\frac{1}{2}-\frac{d}{2p}}\|e^{t\Delta}(Id - S_j)f\|_p$ is $O(1/t)$ and goes to 0 as t goes to $+\infty$. Thus, we get that $\limsup_{t\to+\infty} t^{\frac{1}{2}-\frac{d}{2p}}\|e^{t\Delta}f\|_p \leq \|S_jf\|_{\dot{B}_p^{d/p-1,\infty}}$. □

From Theorem 23.3, Planchon's result is more precise:

Theorem 23.3:
Let $p \in (d,\infty)$*. Let* $\mathcal{E}_p = \{f \in L^\infty_{loc,t}L^p_x \ / \ \sup_{t>0} t^{\frac{1}{2}-\frac{d}{2p}}\|f\|_p < \infty\}$ *and let* C_p *be the norm of the bilinear operator* B *on* $(\mathcal{E}_p)^d$*. Let* $\vec{u}_0$ $(\dot{B}_p^{d/p-1,\infty}(\mathbb{R}^d))^d$ *be such that* $\vec{\nabla}.\vec{u}_0 = 0$ *and such that* $\|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_p} < \frac{1}{4C_p}$*. Let* $\vec{u}$ *be the mild solution of the Navier–Stokes equations with initial value* $\vec{u}_0$ *(with* $\|\vec{u}\|_{\mathcal{E}_p} < \frac{1}{2C_p}$*). Then the following assertions are equivalent:*
(A) $\vec{u}$ *is asymptotically self-similar in the sense that there exists a function* $\vec{V} \in (L^p(\mathbb{R}^d))^d$ *so that* $\lim_{t\to\infty} t^{\frac{1}{2}-\frac{d}{2p}}\|\vec{u} - \frac{1}{\sqrt{t}}\vec{V}(\frac{x}{\sqrt{t}})\|_p = 0$
(B) $e^{t\Delta}\vec{u}_0$ *is asymptotically self-similar in the sense that there exists a function* $\vec{U} \in (L^p(\mathbb{R}^d))^d$ *so that* $\lim_{t\to\infty} t^{\frac{1}{2}-\frac{d}{2p}}\|e^{t\Delta}\vec{u}_0 - \frac{1}{\sqrt{t}}\vec{U}(\frac{x}{\sqrt{t}})\|_p = 0$
(C) There exists a distribution $\vec{v}_0$ *in* $(\dot{B}_p^{d/p-1,\infty}(\mathbb{R}^d))^d$ *homogeneous with degree* -1 *so that* $\lim_{j\to-\infty} 2^{j(d/p-1)}\|\Delta_j(\vec{u}_0 - \vec{v}_0)\|_p = 0$.

In this case, we have $\vec{U} = e^{\Delta}\vec{v}_0$ *and the function* $\frac{1}{\sqrt{t}}\vec{V}(\frac{x}{\sqrt{t}})$ *is the mild solution of the Navier–Stokes equations with initial value* $\vec{v}_0$*.*

Proof: (A) implies (B) by Lemma 23.4: if $\vec{f}(t,x) = \frac{1}{\sqrt{t}}\vec{V}(\frac{x}{\sqrt{t}})$, we have that $e^{t\Delta}\vec{u}_0 = \vec{u} + B(\vec{u},\vec{u})$ is asymptotic in $\mathcal{E}_p$ to $\vec{f} + B(\vec{f},\vec{f}) = \vec{g}(t,x)$; we then define $\vec{U}$ as $\vec{g}(1,x)$.

(B) implies (C). We write $\vec{z}_{0,t}(x) = \sqrt{t}\vec{u}_0(\sqrt{t}\ x)$. (B) is equivalent to $\lim_{t\to+\infty} \|e^{\Delta}\vec{z}_{0,t} - \vec{U}\|_p = 0$. This gives for all $j \in \mathbb{Z}$ that $\lim_{t\to+\infty} \|S_j\vec{z}_{0,t} -$

$e^{-\Delta} S_j \vec{U}\|_p = 0$. Since the norms $\|\vec{z}_{0,t}\|_{\dot{B}_p^{d/p-1,\infty}}$ remain bounded, we find that $\vec{z}_{0,t}$ is weakly convergent to a distribution $\vec{v}_0$. This distribution is then homogeneous with degree -1. Moreover, we have $\vec{U} = e^{t\Delta}\vec{v}_0$. Then, Theorem 23.2 gives that $\lim_{j\to-\infty} 2^{j(d/p-1)}\|\Delta_j(\vec{u}_0 - \vec{v}_0)\|_p = 0$.

Finally, (C) implies (A): if (C) is satisfied, then $\vec{v}_0$ is the weak limit of $2^j\vec{u}_0(2^j x)$ as j goes to $-\infty$; hence, $\|e^{t\Delta}\vec{v}_0\|_p \leq \liminf_{j\to-\infty} 2^j\|e^{t\Delta}(2^j\vec{u}_0(2^j.))\|_p \leq t^{\frac{d}{2p}-\frac{1}{2}}\|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_p} < \frac{1}{2C_p}$. We then apply Theorem 23.2 and get that $\vec{u}$ is asymptotically self-similar, since the solution $\vec{v}(t,x)$ associated with $\vec{v}_0$ is self-similar. $\square$

Part 5:

Decay and regularity results for weak and mild solutions

Chapter 24

Solutions of the Navier–Stokes equations are space-analytical

In this chapter, we prove that an elementary modification of the proof of existence of mild solutions gives a proof of the spatial analyticity of those solutions. Spatial analyticity was first discussed by Masuda [MAS 67] and time analyticity was considered by Foias and Temam [FOIT 89].

We consider only spatial analyticity. In order to avoid technical notations, we only consider global solutions in three simple cases: global solutions in $\dot{B}^{d-1,\infty}_{PM}$ (Le Jan and Sznitman's solutions [LEJS 97]), global solutions in $\dot{H}^{d/2-1}$ and global solutions in L^d. For each case, we briefly recall the construction of global solutions based on the Picard contraction principle (see Chapter 15), then we prove spatial analyticity.

First, we state the Picard contraction principle for global solutions:

Proposition 24.1: (The Picard contraction principle)

Let $\mathcal{E}$ be a Banach space of functions defined on $(0,+\infty)\times\mathbb{R}^d$ so that the bilinear operator B defined by $B(\vec{u},\vec{v}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\,ds$ is bounded from $\mathcal{E}^d\times\mathcal{E}^d$ to $\mathcal{E}^d$. Let E be the space defined by $f\in E$ if and only if $f\in\mathcal{S}'(\mathbb{R}^d)$ and $(e^{t\Delta}f)_{t>0}\in\mathcal{E}$. Then, there exists a positive constant $C_{\mathcal{E}}$ so that for all $\vec{u}_0\in E^d$ with $\vec{\nabla}.\vec{u}_0=0$ and $\|(e^{t\Delta}\vec{u}_0)_{t>0}\|_{\mathcal{E}}<C_{\mathcal{E}}$, there exists a solution $\vec{u}\in\mathcal{E}^d$ for the Navier–Stokes equations with initial value $\vec{u}_0$: $\vec{u}=e^{t\Delta}\vec{u}_0-\int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})\,ds$.

1. The Le Jan and Sznitman solutions

Le Jan and Sznitman [LEJS 97] considered, as a very simple space convenient to the study of Navier–Stokes equations, the space E of tempered distributions $f\in\mathcal{S}'(\mathbb{R}^d)$ so that $\hat{f}(\xi)$ is a locally integrable function on $\mathbb{R}^d$ and $\sup_{\xi\in\mathbb{R}^d}|\xi|^{d-1}\,|\hat{f}(\xi)|<\infty$. This space may be defined as a Besov space based on the spaces PM of pseudomeasures (i.e. PM is the space of the Fourier transforms of essentially bounded functions: $PM=\mathcal{F}L^\infty$). More precisely, $E=\dot{B}^{d-1,\infty}_{PM}$.

Proposition 24.2: (Existence of global solutions)

i) $f \in \dot{B}^{d-1,\infty}_{PM}$ *if and only if* $e^{t\Delta} f \in \mathcal{E}$, *where* $\mathcal{E}$ *is defined by* $g \in \mathcal{E}$ *if and only if the spatial Fourier transform* $\hat{g}(t,\xi)$ *is a locally integrable function on* $(0,\infty) \times \mathbb{R}^d$ *and* $\sup_{t>0} \sup_{\xi \in \mathbb{R}^d} |\xi|^{d-1} \, |\hat{g}(t,\xi)| < \infty$.

ii) The bilinear operator B *is bounded from* $\mathcal{E}^d \times \mathcal{E}^d$ *to* $\mathcal{E}^d$.

iii) There exists a positive constant C_0 *so that for all* $\vec{u}_0 \in (\dot{B}^{d-1,\infty}_{PM})^d$ *with* $\vec{\nabla}.\vec{u}_0 = 0$ *and* $\|(e^{t\Delta}\vec{u}_0)_{t>0}\|_{\dot{B}^{d-1,\infty}_{PM}} < C_0$, *there exists a solution* $\vec{u} \in \mathcal{E}^d$ *for the Navier–Stokes equations with initial value* $\vec{u}_0$.

Proof: We have only to prove point ii). The proof is based on the following elementary lemmas:

Lemma 24.1: *If* $\vec{w} = B(\vec{u},\vec{v})$, *then we have*

$$|\hat{\vec{w}}(t,\xi)| \leq C_1 \, |\xi| \, \int_0^t e^{-(t-s)|\xi|^2} \, |\hat{\vec{u}}(s,\xi)| * |\hat{\vec{v}}(s,\xi)| \, ds$$

Lemma 24.2: *For all* $\xi \in \mathbb{R}^d$, $\int_{\mathbb{R}^d} \frac{1}{|\xi-\eta|^{d-1}} \, \frac{1}{|\eta|^{d-1}} \, d\eta \leq C_2 \, \frac{1}{|\xi|^{d-2}}$.

Lemma 24.3: *For all* $\xi \in \mathbb{R}^d$, $\int_0^t e^{-(t-s)|\xi|^2} ds \leq \frac{1}{|\xi|^2}$.

This obviously gives the boundedness of B. □

We now slightly modify the proof to get analyticity.

Theorem 24.1: (Analyticity of global solutions)

i) $f \in \dot{B}^{d-1,\infty}_{PM}$ *if and only if* $e^{t\Delta} f \in \mathcal{F}$, *where* $\mathcal{F}$ *is defined by* $g \in \mathcal{F}$ *if and only if the spatial Fourier transform* $\hat{g}(t,\xi)$ *is a locally integrable function on* $(0,\infty) \times \mathbb{R}^d$ *and* $\sup_{t>0} \sup_{\xi \in \mathbb{R}^d} e^{\sqrt{t}\,|\xi|} |\xi|^{d-1} \, |\hat{g}(t,\xi)| < \infty$.

ii) The bilinear operator B *is bounded from* $\mathcal{F}^d \times \mathcal{F}^d$ *to* $\mathcal{F}^d$.

iii) There exists a positive constant C_3 *such that for all* $\vec{u}_0 \in (\dot{B}^{d-1,\infty}_{PM})^d$ *with* $\vec{\nabla}.\vec{u}_0 = 0$ *and* $\|(e^{t\Delta}\vec{u}_0)_{t>0}\|_{\dot{B}^{d-1,\infty}_{PM}} < C_3$, *there exists a solution* $\vec{u} \in \mathcal{F}^d$ *for the Navier–Stokes equations with initial value* $\vec{u}_0$.

This solution is space-analytic on the domain $\{(x+iy) \in \mathbb{C}^d \ / \ |y| < \sqrt{t}\}$.

We just have to prove ii) by transforming Lemma 24.1 into the following lemma:

Lemma 24.4: *If* $\vec{w} = B(\vec{u},\vec{v})$, *and if* $\vec{u}(t,.) = e^{-\sqrt{-t\Delta}}\vec{U}$ *and* $\vec{v}(t,.) = e^{-\sqrt{-t\Delta}}\vec{V}$, *then* $\vec{w}(t,.) = e^{-\sqrt{-t\Delta}}\vec{W}$ *with*

$$|\hat{\vec{W}}(t,\xi)| \leq C_4 \, |\xi| \, \int_0^t e^{-(t-s)/2 \ |\xi|^2} \, |\hat{\vec{U}}(s,\xi)| * |\hat{\vec{V}}(s,\xi)| \, ds.$$

Proof: To prove Lemma 24.4, we prove, for $0 < s < t$ and $\xi, \eta \in \mathbb{R}^d$, the following inequality:

$$(24.1) \qquad e^{-(t-s)|\xi|^2} \, e^{-\sqrt{s}\,|\xi-\eta|} e^{-\sqrt{s}\,|\eta|} \leq e^2 \, e^{-\sqrt{t}\,|\xi|} e^{-\frac{1}{2}(t-s)|\xi|^2}$$

Such an inequality was already considered by Foias and Temam [FOIT 89]. We first notice that, due to the triangle inequality, we have

$$-\sqrt{s}\,|\xi-\eta| - \sqrt{s}\,|\eta| \leq -\sqrt{s}\,|\xi|,$$

hence $e^{-(t-s)|\xi|^2} \, e^{-\sqrt{s}\,|\xi-\eta|} e^{-\sqrt{s}\,|\eta|} \leq e^{-\frac{1}{2}(t-s)|\xi|^2 - \sqrt{s}|\xi|} e^{-\frac{1}{2}(t-s)|\xi|^2}$. Thus, we want to prove that $I = -\frac{1}{2}(t-s)\,|\xi|^2 + (\sqrt{t}-\sqrt{s})|\xi| \leq 2$. We write $I = (\sqrt{t}-\sqrt{s})|\xi|\big(1-\frac{1}{2}(\sqrt{t}+\sqrt{s})|\xi|\big)$. If $\sqrt{t}|\xi| \geq 2$, then we have $I \leq 0 < 2$, whereas if $\sqrt{t}|\xi| < 2$, we have $I \leq \sqrt{t}|\xi| < 2$. □

Theorem 24.1 is then easily proved by using Lemmas 24.4, 24.2, and 24.3.

2. Analyticity of solutions in $\dot{H}^{d/2-1}$

Foias and Temam [FOIT 89] proved spatial analyticity for solutions in Sobolev spaces of periodical functions in an elementary way. We adapt this proof to the case of the Sobolev space $\dot{H}^{d/2-1}$ defined on the whole space and follow the same line as for the Le Jan and Sznitman solutions. We start with an existence theorem of global solutions in $\dot{H}^{d/2-1}$ (with a different proof than in Chapters 15 and 19):

Proposition 24.3: (Existence of global solutions)

i) $f \in \dot{H}^{d/2-1}$ if and only if $e^{t\Delta} f \in \mathcal{E}$, where $\mathcal{E}$ is defined by $g \in \mathcal{E}$ if and only if the spatial Fourier transform $\hat{g}(t,\xi)$ is a locally integrable function on $(0,\infty) \times \mathbb{R}^d$ and $\int_{\mathbb{R}^d} |\xi|^{d-2} \left(\sup_{t>0} |\hat{g}(t,\xi)|\right)^2 \, d\xi < \infty$.

ii) The bilinear operator B is bounded from $\mathcal{E}^d \times \mathcal{E}^d$ to $\mathcal{E}^d$.

iii) There exists a positive constant C_0 such that for all $\vec{u}_0 \in (\dot{H}^{d/2-1})^d$ with $\vec{\nabla}.\vec{u}_0 = 0$ and $\|(e^{t\Delta}\vec{u}_0)_{t>0}\|_{\dot{H}^{d/2-1}} < C_0$, there exists a solution $\vec{u} \in \mathcal{E}^d$ for the Navier–Stokes equations with initial value $\vec{u}_0$.

We have only to prove point ii). Again using Lemmas 24.1 and 24.3, we replace Lemma 24.2 with the following elementary lemma:

Lemma 24.5: *Let $d \geq 3$. There exists a constant C_5 so that for all measurable functions U and V on $\mathbb{R}^d$ we have*

$$\int |\xi|^{d-4} \, |U * V|^2 \, d\xi \leq C_5 \Big(\int |\xi|^{d-2} |U(\xi)|^2 \, d\xi\Big)^{1/2} \Big(\int |\xi|^{d-2} |V(\xi)|^2 \, d\xi\Big)^{1/2}$$

Proof: Lemma 24.5 is equivalent to the fact that the pointwise product is a bounded operator from $\dot H^{d/2-1}\times\dot H^{d/2-1}$ to $\dot H^{d/2-2}$. Using the decomposition of the product into paraproducts, $fg=\dot\pi(f,g)+\dot\pi(g,f)+\dot\rho(f,g)$ with $\dot\pi(f,g)=\sum_{j\in\mathbb{Z}}S_{j-2}f\Delta_j g$, we find that

$$\begin{cases} \|\dot\pi(f,g)\|_{\dot H^{d/2-2}}\le C\|f\|_{\dot B_\infty^{-1,\infty}}\|g\|_{\dot H^{d/2-1}}\\ \|\dot\pi(g,f)\|_{\dot H^{d/2-2}}\le C\|g\|_{\dot B_\infty^{-1,\infty}}\|f\|_{\dot H^{d/2-1}}\\ \|\dot\rho(f,g)\|_{\dot H^{d/2-2}}\le C\|\dot\rho(f,g)\|_{\dot B_1^{d-2,1}}\le C'\|f\|_{\dot H^{d/2-1}}\|g\|_{\dot H^{d/2-1}} \end{cases}\quad\square$$

Lemma 24.5 readily implies Proposition 24.2 when $d\ge 3$. It remains to consider the case $d=2$, in which case $\dot H^{d/2-1}=L^d$. We prove analyticity in L^d in the next section. $\square$

We now slightly modify the proof to achieve analyticity.

Theorem 24.2: (Analyticity of global solutions)

i) $f\in\dot H^{d/2-1}$ if and only if $e^{t\Delta}f\in\mathcal{F}$, where $\mathcal{F}$ is defined by $g\in\mathcal{F}$ if and only if the spatial Fourier transform $\hat g(t,\xi)$ is a locally integrable function on $(0,\infty)\times\mathbb{R}^d$ and $|\xi|^{d/2-1}(\sup_{t>0}e^{\sqrt{t}\,|\xi|}\,|\hat g(t,\xi)|)\in L^2(\mathbb{R}^d)$.

ii) The bilinear operator B is bounded from $\mathcal{F}^d\times\mathcal{F}^d$ to $\mathcal{F}^d$.

iii) There exists a positive constant C_6 so that for all $\vec u_0\in(\dot H^{d/2-1})^d$ with $\vec\nabla.\vec u_0=0$ and $\|(e^{t\Delta}\vec u_0)_{t>0}\|_{\dot H^{d/2-1}}<C_6$, there exists a solution $\vec u\in\mathcal{F}^d$ for the Navier–Stokes equations with initial value $\vec u_0$.

This solution is space-analytic on the domain $\{(x+iy)\in\mathbb{C}^d\ /\ |y|<\sqrt{t}\}$.

Proof: We check that B is bounded from $\mathcal{F}^d\times\mathcal{F}^d$ to $\mathcal{F}^d$. We write $|\hat{\vec u}(t,\xi)|\le e^{-\sqrt{t}|\xi|}U(\xi)$ with $|\xi|^{d/2-1}U\in L^2$ and $|\hat{\vec v}(t,\xi)|\le e^{-\sqrt{t}|\xi|}V(\xi)$ with $|\xi|^{d/2-1}V\in L^2$. We apply Lemmas 24.4 and 24.3 and get that $\vec w=B(\vec u,\vec v)$ satisfies $\sup_{t>0}e^{\sqrt{t}\,|\xi|}\,|\hat{\vec w}(t,\xi)|\le 2C_4|\xi|^{-1}U*V(\xi)$. To conclude the proof, we must prove that $(\int|\xi|^{d-2}\frac{|U*V|^2}{|\xi|^2}\,d\xi)^{1/2}\le C_5(\int|\xi|^{d-2}|U(\xi)|^2\,d\xi)^{1/2}(\int|\xi|^{d-2}|V(\xi)|^2\,d\xi)^{1/2}$, which is exactly the content of Lemma 24.5. $\square$

3. Analyticity of solutions in L^d

Spatial analyticity for solutions in Lebesgue space was considered by Grujič and Kukavica [GRUK 98]. Their proof was based on the study of the equation satisfied by $\vec u(t,(1+i\alpha\sqrt{t})x)$ for small α's. We give here a different proof (Lemarié-Rieusset [LEM 00]) based on multilinear singular integrals.

Proposition 24.4: (Existence of global solutions)

i) $f \in L^d$ if and only if $e^{t\Delta} f \in \mathcal{E}_1 \cap \mathcal{E}_2$, where $\mathcal{E}_1 = \mathcal{C}_b([0,\infty), L^d(\mathbb{R}^d))$ and $\mathcal{E}_2$ is defined by:

$$g \in \mathcal{E}_2 \Leftrightarrow t^{1/8} g(t,x) \in L^\infty((0,\infty), L^{4d/3}(\mathbb{R}^d)) \text{ and } \lim_{t\to 0} t^{1/8} \|g(t,x)\|_{4d/3} = 0.$$

ii) The bilinear operator B is bounded from $\mathcal{E}_2^d \times \mathcal{E}_2^d$ to $\mathcal{E}_1^d \cap \mathcal{E}_2^d$.

iii) There exists a positive constant C_0 such that for all $\vec{u}_0 \in (L^d)^d$ with $\vec{\nabla}.\vec{u}_0 = 0$ and $\|(e^{t\Delta}\vec{u}_0)_{t>0}\|_{\mathcal{E}_2} < C_0$, there exists a solution $\vec{u} \in \mathcal{E}_1^d \cap \mathcal{E}_2^d$ for the Navier–Stokes equations with initial value $\vec{u}_0$.

Proof: Point i) is a simple consequence of the Bernstein inequalities (which give $L^d \subset \dot{B}_d^{0,\infty} \subset B_{4d/3}^{-1/4,\infty}$) and of the characterization of Besov spaces through the heat kernel. Point ii) is obvious: we just write for $\vec{u}$ and $\vec{v}$ in $\mathcal{E}_2^d$ that

$$\|B(\vec{u},\vec{v})\|_{4d/3} \leq C \int_0^t \frac{1}{|t-s|^{7/8}}\ s^{1/8}\|\vec{u}\|_{4d/3}\ s^{1/8}\|\vec{v}\|_{4d/3}\ \frac{ds}{s^{1/4}}$$

and

$$\|B(\vec{u},\vec{v})\|_{d} \leq C \int_0^t \frac{1}{|t-s|^{3/4}}\ s^{1/8}\|\vec{u}\|_{4d/3}\ s^{1/8}\|\vec{v}\|_{4d/3}\ \frac{ds}{s^{1/4}}.$$

This proves Proposition 24.4. □

Once again, we slightly alter the proof and get analyticity of the solutions:

Theorem 24.3: (Analyticity of global solutions)

Let Λ_1 be the operator defined by the Fourier multiplier $\|\xi\|_1 = \sum_{j=1}^d |\xi_j|$: $\Lambda_1 f(x) = \frac{1}{(2\pi)^d}\int e^{ix.\xi}\|\xi\|_1 \hat{f}(\xi)\ d\xi$. Then:

i) $f \in L^d$ if and only if $e^{t\Delta} f \in \mathcal{F}_1 \cap \mathcal{F}_2$, where

$$\mathcal{F}_1 = \{f \in \mathcal{C}_b([0,\infty), L^d(\mathbb{R}^d)) / \sup_{t>0} \|e^{\sqrt{t}\Lambda_1} f\|_d < \infty\}$$

and

$$\mathcal{F}_2 = \{g \in L^1_{loc}((0,\infty), L^{\frac{4d}{3}}) / \sup_{t>0} t^{\frac{1}{8}} \|e^{\sqrt{t}\Lambda_1} g\|_{\frac{4d}{3}} < \infty \text{ and } \lim_{t\to 0} t^{\frac{1}{8}}\|g\|_{\frac{4d}{3}} = 0\}.$$

ii) The bilinear operator B is bounded from $\mathcal{F}_2^d \times \mathcal{F}_2^d$ to $(\mathcal{F}_1 \cap \mathcal{F}_2)^d$.

iii) There exists a positive constant C_7 so that for all $\vec{u}_0 \in (L^d)^d$ with $\vec{\nabla}.\vec{u}_0 = 0$ and $\|(e^{t\Delta}\vec{u}_0)_{t>0}\|_d < C_7$, there exists a solution $\vec{u} \in (\mathcal{F}_1 \cap \mathcal{F}_2)^d$ for the Navier–Stokes equations with initial value $\vec{u}_0$.

This solution is space-analytic on the domain $\{(x+iy) \in \mathbb{C}^d\ /\ |y| < \sqrt{t}\}$.

The proof relies on several basic lemmas.

Lemma 24.6: *Let $1 \leq p < \infty$. If $T_j \in \mathcal{L}(L^p(\mathbb{R}), L^p(\mathbb{R}))$ for $1 \leq j \leq d$, then $T_1 \otimes \ldots \otimes T_d$ defined by $T_1 \otimes \ldots \otimes T_d(\varphi_1 \otimes \ldots \varphi_d) = T_1(\varphi_1) \otimes \ldots \otimes T_d(\varphi_d)$ belongs to $\mathcal{L}(L^p(\mathbb{R}^d), L^p(\mathbb{R}^d))$.*

Proof: We may assume that all the T_j but one are identity operators (then write $T_1 \otimes \ldots \otimes T_d = (T_1 \otimes Id \otimes \ldots \otimes Id) \circ (Id \otimes T_2 \otimes \ldots \otimes Id) \circ \ldots \circ (Id \otimes \ldots \otimes Id \otimes T_d))$. But we obviously see that $Id \otimes \ldots \otimes Id \otimes T_j \otimes Id \otimes \ldots \otimes Id$ operates on $L^p(\mathbb{R}^d) = L^p(dx_1 \ldots dx_{j-1}\ dx_{j+1} \ldots dx_d, L^p(dx_j))$. □

Lemma 24.7: *Let $1 < p < \infty$. The linear operators $A_{t,s}$ defined by $A_{t,s} = e^{(t-s)\Delta/2} e^{(\sqrt{t}-\sqrt{s})\ \Lambda_1}$ for $0 \leq s < t < \infty$ are equicontinuous on $L^p(\mathbb{R}^d)$.*

Proof: We may write $A_{t,s} = T_{t,s} \otimes \ldots \otimes T_{t,s}$ where $T_{t,s}$ is an operator on $L^p(\mathbb{R})$:

$$T_{t,s} f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) e^{-\frac{t-s}{2}\xi^2 + (\sqrt{t}-\sqrt{s})|\xi|}\ d\xi$$

We may then use the Littlewood–Paley theory for functions in L^p and use more precisely the Marcinkiewicz multiplier theorem: a Fourier multiplier $m(D)$ with $m \in \mathcal{C}^{\infty\infty}(\mathbb{R}^*)$ operates on $L^p(\mathbb{R})$ $(1 < p < \infty)$ if we have $m \in L^\infty(\mathbb{R}^*)$ and $\xi \frac{d}{d\xi} m(\xi) \in L^\infty(\mathbb{R}^*)$ and we have $\|m(D)\|_{\mathcal{L}(L^p,L^p)} \leq C_p(\|m\|_\infty + \|\xi \frac{d}{d\xi^k} m\|_\infty)$ (Stein [STE 70]). We have just to estimate $\alpha_{t,s} = \|e^{-\frac{t-s}{2}\xi^2 + (\sqrt{t}-\sqrt{s})|\xi|}\|_\infty$ and $\beta_{t,s} = \|((t-s)\xi^2 - (\sqrt{t}-\sqrt{s})|\xi|) e^{-\frac{t-s}{2}\xi^2 + (\sqrt{t}-\sqrt{s})|\xi|}\|_\infty$. We know, from the proof of Lemma 24.4, that $-\frac{1}{4}(t-s)\ |\xi|^2 + (\sqrt{t}-\sqrt{s})|\xi| \leq 4$. This gives that $\alpha_{t,s} \leq e^4$ and that $\beta_{t,s} \leq 5 \|\frac{1}{4}(t-s)\ |\xi|^2 e^{-\frac{1}{4}(t-s)|\xi|^2}\|_\infty \|e^{-\frac{1}{4}(t-s)|\xi|^2 + (\sqrt{t}-\sqrt{s})|\xi|}\|_\infty$ $+ \|e^{-\frac{1}{4}(t-s)\ |\xi|^2}\|_\infty \|\big(\frac{1}{4}(t-s)\ |\xi|^2 + (\sqrt{t}-\sqrt{s})|\xi|\big) e^{-\frac{1}{4}(t-s)\ |\xi|^2 + (\sqrt{t}-\sqrt{s})|\xi|}\|_\infty$ and finally $\beta_{t,s} \leq 5\ \frac{e^{-1/2}}{\sqrt{2}}\ e^4 + 4e^4$. □

Lemma 24.8: *Let $2 < p < \infty$. The bilinear operators B_t defined by $B_t(f,g) = e^{\sqrt{t}\ \Lambda_1}(e^{-\sqrt{t}\ \Lambda_1} f . e^{-\sqrt{t}\ \Lambda_1} g)$ for $t > 0$ are equicontinuous from $L^p \times L^p$ to $L^{p/2}$.*

Proof: We have

$$B_t(f,g) = \frac{1}{(2\pi)^d} \int\int e^{ix.(\xi+\eta)} e^{\sqrt{t}(\|\xi+\eta\|_1 - \|\xi\|_1 - \|\eta\|_1)} \hat{f}(\xi) \hat{g}(\eta)\ d\xi\ d\eta$$

We then split the domain of integration on subdomains, depending on the sign of ξ_j, of η_j and of $\xi_j + \eta_j$. For this, we introduce the monodimensional operators $K_1 f = \frac{1}{2\pi} \int_0^{+\infty} e^{ix\xi} \hat{f}(\xi)\ d\xi$, $K_{-1} f = \frac{1}{2\pi} \int_{-\infty}^0 e^{ix\xi} \hat{f}(\xi)\ d\xi$, $L_{t,\epsilon_1,\epsilon_2} f = f$ if $\epsilon_1\epsilon_2 = 1$ and $L_{t,\epsilon_1,\epsilon_2} f = \frac{1}{2\pi} \int e^{ix\xi} e^{-2\sqrt{t}|\xi|} \hat{f}(\xi) d\xi$ if $\epsilon_1\epsilon_2 = -1$. We define for

$\vec{\alpha} = (\alpha_1, \ldots, \alpha_d)$ and $\vec{\beta} = (\beta_1, \ldots, \beta_d)$ in $\{-1, 1\}^d$ and for $t > 0$ the operators $Z_{t,\vec{\alpha},\vec{\beta}} = K_{\beta_1} L_{t,\alpha_1,\beta_1} \otimes \ldots \otimes K_{\beta_d} L_{t,\alpha_d,\beta_d}$. Then,

$$B_t(f, g) = \sum_{(\vec{\alpha},\vec{\beta},\vec{\gamma}) \in \{-1,1\}^{d\times 3}} K_{\alpha_1} \otimes \ldots K_{\alpha_d} \big(Z_{t,\vec{\alpha},\vec{\beta}} f \; Z_{t,\vec{\alpha},\vec{\gamma}} g \big)$$

The operators $Z_{t,\vec{\alpha},\vec{\beta}}$ are equicontinuous on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$ (because the kernel of $e^{-|D|}$ in dimension 1 is given by $\frac{1}{\pi} \frac{1}{1+x^2} \in L^1(\mathbb{R})$ and since the operators K_1 and K^{-1} are combinations of the identity operator and of the Hilbert transform), and the operators $K_{\alpha_1} \otimes \ldots K_{\alpha_d}$ are bounded on $L^{p/2}$ for $2 < p < \infty$. This finishes the proof. □

Proof of Theorem 3: Point i) is a direct consequence of Lemma 24.7: we just write $e^{t\Delta} f = e^{-t\Lambda_1} A_{t,0} e^{t\Delta/2} f$. Since $e^{t\Delta/2} f$ belongs to $\mathcal{E}_1 \cap \mathcal{E}_2$, $e^{t\Delta} f$ belongs to $\mathcal{F}_1 \cap \mathcal{F}_2$.

In order to prove point ii), we write for $\vec{f}$ and $\vec{g}$ in $(\mathcal{F}_2)^d$: $\vec{f} = e^{-\sqrt{t}\Lambda_1} \vec{F}$ and $\vec{g} = e^{-\sqrt{t}\Lambda_1} \vec{G}$ with $t^{1/8} \vec{F}$ and $t^{1/8} \vec{G}$ in $(L^\infty_t L^{4d/3}_x)^d$. We define $\mathcal{B}_t(\vec{F}, \vec{G}) = (B_t(F_j, G_k))_{1 \le j,k \le d}$ and write

$$B(\vec{f}, \vec{g}) = e^{-\sqrt{t}\Lambda_1} \int_0^t A_{t,s} e^{(t-s)\Delta/2} \mathbb{P} \vec{\nabla}.\mathcal{B}_s(\vec{F}, \vec{G}) \, ds.$$

We then use Lemma 24.8 to estimate the norm of $\mathcal{B}_s(\vec{F}, \vec{G})$ in $(L^{2d/3}(\mathbb{R}^d))^{d\times d}$. □

Of course, this proof works as well for local solutions, hence for Lebesgue spaces L^p with $p > d$. An easy result follows:

Theorem 24.4:

For all $p \in [d, \infty)$ et $\vec{u}_0 \in (L^p)^d$ such that $\vec{\nabla}.\vec{u}_0 = 0$, there exists $T > 0$ and a solution $\vec{u} \in \mathcal{C}([0, T], (L^p)^d)$ of the Navier–Stokes equations with initial value $\vec{u}_0$: $\vec{u} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{u}) \, ds$ so that $\vec{u} = e^{-\sqrt{t}\,\Lambda} \vec{v}$ with $\vec{v} \in \mathcal{L}^\infty([0, T], (L^p)^d)$.

Chapter 25

Space localization and Navier–Stokes equations

In this chapter, we describe recent results of Brandolese [BRA 01] on spatial localization of solutions to the Navier–Stokes equations. Well-localized solutions were described by Furioli and Terraneo [FURT 01] and by Miyakawa [MIY 00]. Limitations on the localization were first described by Dobrokhotov and Shafarevich [DOBS 94] and extended by Brandolese, who gives some additional interesting constructions of well-localized solutions.

1. The molecules of Furioli and Terraneo

We begin with the molecules described by Furioli and Terraneo [FURT 01] and by Brandolese [BRA 01]. Those molecules were introduced with a double prospect. On one hand, Furioli and Terraneo began their thesis by studying mild solutions in $(\mathcal{C}([0,T), L^3(\mathbb{R}^3)))^3$ (a study which led to the theorem on uniqueness described in Chapter 27); a special case was the subspace of solutions which belong $(\mathcal{C}([0,T), \dot{F}_1^{2,2}))^3$; the Triebel-Lizorkin space $\dot{F}_1^{2,2}$ is the space of distributions vanishing at infinity and such that their Laplacian belongs to the Hardy space $\mathcal{H}^1$; this space appeared likely to provide a better insight into the solutions to the Navier–Stokes equations: if $\Delta\vec{u} \in (\mathcal{H}^1(\mathbb{R}^3))^3$, then $\vec{u} \in (L^3)^3$ and $\vec{\nabla} \otimes \vec{u} \in (L^{3/2})^{3\times 3}$, so that, by the div-curl theorem (Theorem 12.1), $(\vec{u}.\vec{\nabla})\vec{u} \in (\mathcal{H}^1(\mathbb{R}^3))^3$ and (since the Riesz transforms operate on $\mathcal{H}^1$), $\vec{\nabla}p \in (\mathcal{H}^1(\mathbb{R}^3))^3$; thus, the three terms which contribute to $\partial_t\vec{u}$ have the same regularity. On the other hand, the Hardy space has a very simple structure, according to the theory of atomic decompositions; an atomic decomposition could then be seen as a model for decomposing a Navier–Stokes flow into simple structures. In terms of the wavelet theory, instead of trying to decompose $\vec{u}$ on a wavelet basis (as discussed in Chapter 12, and as studied by Federbush [FED 93], Cannone [CAN 95], and Meyer [MEY 99]), we could try to decompose it as a vaguelettes series and see how the vaguelettes evolved. The first step looks at the evolution of a single vaguelette, before looking at the interaction of vaguelettes due to the nonlinearity. Because of the nonlocal character of the equation (because of the pressure), Furioli and Terraneo did not consider atoms (compactly supported elementary functions in $\mathcal{H}^1$), but molecules in the sense of Coifman and Weiss [COIW 77]. They considered an initial value $\vec{u}_0$ so that $\vec{u}_0 \in (\mathcal{S}_0')^3$, $\vec{\nabla}.\vec{u}_0 = 0$ and $\int_{\mathbb{R}^3}(1+|x|)^{2\delta}|\Delta\vec{u}_0|^2\, dx < \infty$. They showed that this localization and regularity were preserved on an interval $[0, T(\vec{u}_0)]$, provided that $\delta \in (3/2, 9/2)$ and $\delta - 1/2 \notin \mathbb{N}$. Brandolese [BRA 01] simplified this proof by using the localization of homogeneous Sobolev spaces introduced by Bourdaud [BOU 88a] [BOU 88b].

Theorem 25.1:

Let $\alpha \in \mathbb{N}$ so that $\alpha > d/2$. Let $\delta \in \mathbb{R}$ so that $\delta > d/2$ and $\delta \in (\alpha - d/2, \alpha + d/2 + 1)$. Let $E_{\alpha,\delta}$ be the space

$$E_{\alpha,\delta} = \{f \in L^1(\mathbb{R}^d) /\ |x|^\alpha f(x) \in H^\delta \text{and } \partial^\beta(|x|^\alpha f(x))|_{x=0} = 0 \text{ for } |\beta| < \delta - \frac{d}{2}\}$$

normed by $\|f\|_{E_{\alpha,\delta}} = \|\ |x|^\alpha f\|_{H^\delta}$ and let $F_{\alpha,\delta} = \{f \in \mathcal{C}_0(\mathbb{R}^d)\ /\ \hat{f}(\xi) \in E_{\alpha,\delta}\}$. Then:

a) $F_{\alpha,\delta} \subset \{f \in \mathcal{C}_0(\mathbb{R}^d)\ /\ (1+|x|)^{\delta+d/2-\alpha} f \in L^\infty\}$

b) For every $T > 0$, the bilinear operator B is continuous on $(\mathcal{C}([0,T], F_{\alpha,\delta}))^d$, with an operator norm $O\big(\sqrt{T}\ (1+\sqrt{T})\big)$

c) If $\vec{u}_0 \in (F_{\alpha,\delta})^d$ and $\vec{\nabla}.\vec{u}_0 = 0$, then there exists a positive T and a solution $\vec{u} \in (\mathcal{C}([0,T], F_{\alpha,\delta}))^d$ to the Navier–Stokes equations with initial value $\vec{u}_0$.

The theorem can be proved in the Fourier variable ξ by a series of lemmas on the space $E_{\alpha,\delta}$ and on the realization of the homogeneous Sobolev space $\dot{H}^\delta$. If we look at the definition of the homogeneous Besov space $\dot{\mathcal{B}}_2^{\delta,2}(\mathbb{R}^d)$ (Chapter 3), we have already seen two definitions. More precisely, define for $N \in \mathbb{N}$ and $f \in \mathcal{C}^\infty$ the polynomial $P_N(\partial)f$ as $P_N(\partial)f = \sum_{|\alpha|<N} \frac{1}{\alpha!}\frac{\partial^\alpha}{dx^\alpha} f(0) x^\alpha$ (so that $P_0 f = 0$ and $P_1 f(x) = f(0)$). We then have:

i) $\dot{\mathcal{B}}_2^{\delta,2}(\mathbb{R}^d)$ is defined as a subspace of $\mathcal{S}'/\mathbb{C}[X_1, \ldots, X_d]$; this is the space of tempered distributions f so that $\sum_{j\in\mathbb{Z}} 4^{j\delta}\|\Delta_j f\|_2^2 < \infty$ (Definition 3.5); moreover, f may be written as $f = \sum_{j<0} \Delta_j f - P_N(\partial)\Delta_j f + \sum_{j\geq 0}\Delta_j f + P(x)$, where $N > \delta - d/2$ and $P \in \mathbb{C}[X_1, \ldots, X_d]$.

ii) let $N(\delta)$ be the smallest integer N so that $N > \delta - d/2$; then, a distribution $f \in \mathcal{S}'$ belongs to $\dot{B}_2^{\delta,2}$ if and only if there exists a constant C so that for all $\omega \in \mathcal{S}$ with $\int x^\alpha \omega\, dx = 0$ for all $\alpha \in \mathbb{N}^d$ with $|\alpha| < N(\delta)$ we have $|\langle f|\omega\rangle| \leq C\|\omega\|_{\dot{\mathcal{B}}_2^{-\delta,2}}$ (Definition 3.6). If $\delta < d/2$, then the realization $\dot{B}_2^{\delta,2}$ of the Besov space $\dot{\mathcal{B}}_2^{\delta,2}$ is equal to $\dot{B}_2^{\delta,2} = \dot{\mathcal{B}}_2^{\delta,2} \cap \mathcal{S}_0'$ and thus is defined as a space of distributions. More generally, $\dot{B}_2^{\delta,2}$, for $\delta \in \mathbb{R}$, is a space of distributions modulo polynomials of degree less than $N(\delta)$: f may be written as $f = \sum_{j<0}\Delta_j f - P_N(\partial)\Delta_j f + \sum_{j\geq 0}\Delta_j f + P(x)$, where $N = N(\delta)$ and $P \in \mathbb{C}[X_1, \ldots, X_d]$ with degree less than $N(\delta)$.

We may now introduce the third definition:

Definition 25.1: (Realization of the homogeneous Sobolev spaces)

Let $\delta \in \mathbb{R}$ so that $\delta - d/2 \notin \mathbb{N}$. Let $N(\delta)$ be the smallest integer N so that $N > \delta - d/2$. Then the realization $\dot{H}^\delta$ of the homogeneous Sobolev space $\dot{\mathcal{B}}_2^{\delta,2}(\mathbb{R}^d)$ is defined as the space of distributions $f \in \dot{\mathcal{B}}_2^{\delta,2}(\mathbb{R}^d)$ so that $f = \sum_{j\in\mathbb{Z}} \Delta_j f - P_{N(\delta)}(\partial)\Delta_j f$.

For $\delta < d/2$, we have $\dot{H}^\delta = \dot{B}_2^{\delta,2}$. For $\delta > d/2$, we have $\dot{H}^\delta = \{f \in \dot{B}_2^{\delta,2} / \frac{\partial^\beta}{dx^\beta} f(0) = 0$ for $|\beta| < \delta - \frac{d}{2}\}$. $\dot{H}^\delta$ is thus a space of distributions: if $f = 0$ in $\dot{H}^\delta$, $f = 0$ in $\mathcal{S}'$.

We recall the fundamental result of Bourdaud [BOU 88b]:

Lemma 25.1: (Pointwise multipliers of $\dot{H}^\delta$)

Let $\delta \geq 0$ so that $\delta - d/2 \notin \mathbb{N}$. Let m be a function of class $\mathcal{C}^N$ on $\mathbb{R}^d\setminus\{0\}$ with $N \geq d + 1 + \max(0, \delta - d/2)$ so that, for all $\beta \in \mathbb{N}^d$ with $|\beta| \leq N$, $|x|^{|\beta|} \frac{\partial^\beta}{dx^\beta} m \in L^\infty$. Then the pointwise multiplier operator M_m defined by $M_m f(x) = m(x)f(x)$ is bounded on $\dot{H}^\delta$.

Proof: When $\delta < d/2$, we notice that the Fourier multiplier $m(D)$ (defined by $\mathcal{F}(m(D)f)(\xi) = m(\xi)\hat{f}(\xi)$) is a Calderón–Zygmund operator and that the weight $|x|^{2\delta}$ belongs to the Muckenhoupt class $\mathcal{A}_2$; thus, $m(D)$ is bounded on $L^2(|x|^{2\delta}\, dx)$ (Stein [STE 93]). This proves that M_m is bounded on $\dot{H}^\delta$ for $\delta < d/2$. We now prove the result for $\delta \in (K + d/2, K + 1 + d/2)$, by induction on $K \in \mathbb{N}$. We first notice that $f \in \dot{H}^\delta$ for $\delta > d/2$ if and only if $f(0) = 0$ and $\vec{\nabla} f \in (\dot{H}^{\delta-1})^d$. Thus, we try to check that $\partial_j(mf)$ belongs to $\dot{H}^{\delta-1}$; the case of $m\partial_j f$ is dealt with by the induction hypothesis. We write $(\partial_j m)f = (|x|\partial_j m)\frac{f}{|x|}$ and we conclude by showing that $\frac{f}{|x|}$ belongs to $\dot{H}^{\delta-1}$; indeed, since $f(0) = 0$, we write $\frac{f}{|x|} = \sum_{k=1}^d \frac{x_k}{|x|} \int_0^1 \partial_k f(tx)\, dt$; by induction hypothesis, $\frac{x_k}{|x|}$ is a pointwise multiplier of $\dot{H}^{\delta-1}$; moreover, by homogeneity, $\partial_k f(tx)$ belongs to $\dot{H}^{\delta-1}$ with norm $t^{\delta-d/2-1}\|\partial_k f\|_{\dot{H}^{\delta-1}}$; since $\delta > d/2$, we find that $\int_0^1 \|\partial_k f(tx)\|_{\dot{H}^{\delta-1}} dt < \infty$.
□

We now are able to begin the proof for Theorem 25.1. We first comment on the requirement "$f \in L^1$" in the definition of $E_{\alpha,\delta}$. This requirement expresses only the fact that f is defined by its restriction on $\mathbb{R}^d\setminus\{0\}$:

Lemma 25.2: *Let $\delta \in \mathbb{R}$ so that $\delta > d/2$ and $\delta - d/2 \notin \mathbb{N}$. Let $h \in H^\delta(\mathbb{R}^d)$ be such that $\partial^\beta h(0) = 0$ for $|\beta| < \delta - \frac{d}{2}$. Let $\alpha \in \mathbb{R}$ so that $d/2 < \alpha < \delta + d/2$. Then the function $H(x) = \frac{h(x)}{|x|^\alpha}$ belongs to $L^1(\mathbb{R}^d)$.*

Proof: The hypothesis on h may be written as $h \in L^2 \cap \dot{H}^\delta$. For $|x| > 1$, h is square-integrable and $|x|^{-\alpha}$ is square-integrable (since $\alpha > d/2$), so that their product H is integrable. For $|x| < 1$, we write that $h \in \dot{H}^\delta$; hence,

$h = \sum_{j\in\mathbb{Z}} \Delta_j h - P_{N(\delta)}(\partial)\Delta_j h$. To estimate $\epsilon_j(x) = \Delta_j h - P_{N(\delta)}(\partial)\Delta_j h$, we may estimate the size of each term in the sum and get (according to the Bernstein inequality)

$$|\epsilon_j(x)| \leq \|\Delta_j h\|_\infty + \sum_{|\beta|<N(\delta)} \frac{1}{\beta!}\|\partial^\beta \Delta_j h\|_\infty |x^\beta| \leq C2^{jd/2}(1+2^j|x|)^{N(\delta)-1}\|\Delta_j h\|_2$$

or we may estimate the size of the remainder in the Taylor integral expansion and get

$$|\epsilon_j(x)| \leq C \sum_{|\beta|=N(\delta)} \|\partial^\beta \Delta_j h\|_\infty |x^\beta| \leq C2^{jd/2}(2^j|x|)^{N(\delta)}\|\Delta_j h\|_2;$$

we now use the fact that $\|\Delta_j h\|_2 \leq C\|h\|_{\dot{H}^\delta} 2^{-j\delta}$ and write

$$|h(x)| \leq C\|h\|_{\dot{H}^\delta}\big(|x|^{N(\delta)} \sum_{2^j|x|<1} 2^{j(d/2-\delta+N(\delta)} + |x|^{N(\delta)-1} \sum_{2^j|x|\geq 1} 2^{j(d/2-\delta+N(\delta)-1)}\big);$$

since we have $N(\delta) - 1 < \delta - d/2 < N(\delta)$ this gives $|h(x)| \leq C\|h\|_{\dot{H}^\delta}|x|^{\delta-d/2}$. Since $\delta - d/2 - \alpha > -d$, this gives that H is integrable in the neighborhood of 0. □

Lemma 25.3: *Under the assumptions of Theorem 25.1 on α and δ, if f and g belong to $E_{\alpha,\delta}$ and if $\beta \in \mathbb{N}^d$ with $|\beta| = \alpha$, then $x^\beta(f * g)$ belongs to H^δ.*

Proof: Choose $\theta \in \mathcal{D}(\mathbb{R})$ so that $\operatorname{Supp}\theta \subset [-3/4, 3/4]$, $\theta(x) = \theta(-x)$ and $\sum_{k\in\mathbb{Z}} \theta(x-k) = 1$. Define $\omega(x,y) = \theta(\frac{|x|^2}{|x|^2+|y|^2})$ so that $\omega(x-y,y) + \omega(y,x-y) = 1$. We then write $f = \frac{F}{|x|^\alpha}$ and $g = \frac{G}{|x|^\alpha}$; thus, $f * g(x) = \int \frac{F(x-y)}{|x-y|^\alpha}\frac{G(y)}{|y|^\alpha}\, dy = \int \frac{F(x-y)}{|x-y|^\alpha}\frac{G(y)}{|y|^\alpha}\omega(y,x-y)\, dy + \int \frac{F(x-y)}{|x-y|^\alpha}\frac{G(y)}{|y|^\alpha}\omega(x-y,y)\, dy = \int \omega(y,x-y)\frac{F(x-y)}{|x-y|^\alpha}\frac{G(y)}{|y|^\alpha}\, dy + \int \omega(y,x-y)\frac{G(x-y)}{|x-y|^\alpha}\frac{F(y)}{|y|^\alpha}\, dy$. We want to prove that $H(x) = x^\beta \int \omega(y,x-y)\frac{F(x-y)}{|x-y|^\alpha}\frac{G(y)}{|y|^\alpha}\, dy$ belongs to H^δ. We have $\|H\|_{H^\delta} \leq \int \frac{|G(y)|}{|y|^\alpha}\, dy\ \sup_{y\in\mathbb{R}^d} \|\, (x-y+y)^\beta \omega(y,x-y)\frac{F(x-y)}{|x-y|^\alpha}\|_{H^\delta(dx)}$; since H^δ is a shift-invariant Banach space of distributions, we may rewrite this inequality into $\|H\|_{H^\delta} \leq \int \frac{|G(y)|}{|y|^\alpha}\, dy\ \sup_{y\in\mathbb{R}^d} \|\, (x+y)^\beta\omega(y,x)\frac{F(x)}{|x|^\alpha}\|_{H^\delta(dx)}$ and try to estimate $\mu(x,y)F(x)$ in $L^2(dx)$ and in $\dot{H}^\delta(dx)$, where $\mu(x,y) = \omega(y,x)\frac{(x+y)^\beta}{|x|^\alpha}$. The function $\omega(y,x)$ is smooth and homogeneous with degree 0 on $\mathbb{R}^{d\times d}\setminus\{(0,0)\}$ and is supported by $\{(x,y)\ /\ |y| \leq \sqrt{3}|x|\}$. Thus, μ is itself smooth and homogeneous with degree 0 on $\mathbb{R}^{d\times d}\setminus\{(0,0)\}$. Moreover, its derivatives with respect to x will satisfy, for all $\gamma \in \mathbb{N}^d$, $|\frac{\partial^\gamma}{\partial x^\gamma}\mu(x,y)| \leq C_\gamma|(x,y)|^{-|\gamma|} \leq C_\gamma|x|^{-|\gamma|}$. Thus, Lemma 25.1 gives us control of the norm of $\mu(x,y)F(x)$ in $\dot{H}^\delta$ uniformly with respect to y. Since we know that $\frac{G}{|x|^\alpha}$ is integrable, this proves that $H \in H^\delta$. □

Lemma 25.4:

Under the assumptions of Theorem 25.1 on α and δ, if f and g belong to $E_{\alpha,\delta}$, if j, k and l belong to $\{1,\ldots,d\}$ and if $t>0$, then the function $h(x)=\frac{x_j}{|x|}\frac{x_k}{|x|}e^{-t|x|^2}x_l\ f*g$ belongs to $E_{\alpha,\delta}$ with norm

$$\|h\|_{E_{\alpha,\delta}}\leq C\frac{1}{\sqrt{t}}(1+\sqrt{t})\|f\|_{E_{\alpha,\delta}}\|g\|_{E_{\alpha,\delta}}.$$

Proof: We have to prove that $|x|^\alpha h(x)\in L^2\cap\dot{H}^\delta$. We write

$$|x|^\alpha h(x)=\frac{x_j}{|x|}\frac{x_k}{|x|}\sum_{|\beta|=\alpha}\frac{|x|^\alpha x^\beta}{\sum_{|\gamma|=\alpha}|x^\gamma|^2}e^{-t|x|^2}x_l\ x^\beta\ f*g.$$

Using Lemma 25.1, we must prove that $h_\beta(x)=e^{-t|x|^2}x_l\ x^\beta\ f*g$ belongs to $L^2\cap\dot{H}^\delta$. From Lemma 25.3, we get that $\|h_\beta\|_2\leq C\|e^{-t|x|^2}x_l\|_\infty\|x^\beta\ f*g\|_2\leq C't^{-1/2}\|f\|_{E_{\alpha,\delta}}\|g\|_{E_{\alpha,\delta}}$. On the other hand, we know that $x^\beta\ f*g$ belongs to H^δ. We introduce a smooth function $\theta\in\mathcal{D}(\mathbb{R}^d)$, which is equal to 1 in the neighborhood of 0. We know that $(1-\theta(x))x^\beta\ f*g$ belongs to H^δ, hence to $\dot{H}^\delta$ since $1-\theta$ is equal identically to 0 in the neighborhood of 0. Similarly, we know that $x_l\theta(x)x^\beta\ f*g$ belongs to H^δ and we are going to check that its derivatives of order less than $\delta-d/2$ are equal to 0: if $\delta-d/2\geq\alpha$, we have $f\in L^\infty$ and $g\in L^1$, thus $f*g\in L^\infty$ and $x_l\theta(x)x^\beta\ f*g=O(|x|^{\alpha+1})$ with $\alpha+1>\delta-d/2$; if $\delta-d/2<\alpha$, we write that, for $y\in\mathbb{R}^d$, we have $|y|\geq|x|/2$ or $|x-y|\geq|x|/2$, hence $|f(y)|\leq C\|f\|_{E_{\alpha,\delta}}|x|^{\delta-d/2-\alpha}$ or $|g(x-y)|\leq C\|g\|_{E_{\alpha,\delta}}|x|^{\delta-d/2-\alpha}$, this gives that $f*g(x)=O(|x|^{\delta-d/2-\alpha})$ and thus that $x_l\theta(x)x^\beta\ f*g=O(|x|^{\delta-d/2+1})$. Now, we write

$$h_\beta=(e^{-t|x|^2}x_l)\ \big((1-\theta(x))\ x^\beta\ f*g\big)+(e^{-t|x|^2})\ \big(\theta(x)\ x_l\ x^\beta\ f*g\big);$$

Lemma 25.1 then gives that $\|h_\beta\|_{\dot{H}^\delta}\leq Ct^{-1/2}(1+\sqrt{t})\ \|f\|_{E_{\alpha,\delta}}\ \|g\|_{E_{\alpha,\delta}}$. □

Proof of Theorem 25.1: Point b) is a direct consequence of Lemma 25.4, since we may decompose the Fourier transform of $e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{f}\otimes\vec{g}$ with $\vec{f}(s)\in(F_{\alpha,\delta})^d$ and $\vec{g}(s)\in(F_{\alpha,\delta})^d$ as a sum of terms $\frac{\xi_j}{|\xi|}\frac{\xi_k}{|\xi|}e^{-t|\xi|^2}\xi_l\ \hat{f}_m*\hat{g}_n$ (with j, k, l, m and n in $\{1,\ldots,d\}$). Point c) is a consequence of point b), according to the Picard contraction principle.

We now prove point a). We already know that $F_{\alpha,\delta}\subset\mathcal{C}_0$. We want to estimate the decay of $f\in F_{\alpha,\delta}$ at infinity. We write for $f\in F_{\alpha,\delta}$ $h=(-\Delta)^{\alpha/2}f$ and $k_\alpha=|x|^{-d+\alpha}$ that $f=c_\alpha k_\alpha*h$ for some positive constant c_α. We choose a smooth function $\theta\in\mathcal{D}(\mathbb{R}^d)$ equal to 1 in a neighborhood of 0 and supported by the ball $B(0,1/2)$, and we define $K_\alpha(x,y)=c_\alpha\omega(\frac{x-y}{|x|})k_\alpha(x-y)$ and $H_\alpha(x,y)=c_\alpha(1-\omega(\frac{x-y}{|x|}))k_\alpha(x-y)$; then we have:

$$f(x)=\int h(y)\ K_\alpha(x-y)\ dy+\int h(y)\ (H_\alpha(x,y)-\sum_{|\beta|\leq\delta-d/2}\frac{\partial^\beta}{\partial y^\beta}H_\alpha(x,0)\frac{y^\beta}{\beta!})\ dy.$$

We then write that $(1+|y|)^{-\delta}|K_\alpha(x,y)| \le C(1+|x|)^{-\delta}|x-y|^{-d+\alpha}$ and thus

$$|\int h(y)\ K_\alpha(x-y)\ dy| \le C\|\hat{h}\|_{H^\delta}(1+|x|)^{-\delta}\Big(\int_{|x-y|\le|x|/2}\frac{dy}{|x-y|^{d+d-2\alpha}}\Big)^{1/2}$$

which is $O(|x|^{\alpha-d/2-\delta})$ since $\alpha > d/2$. Similarly, we have the estimate

$$|H_\alpha(x,y) - \sum_{|\beta|\le\delta-d/2}\frac{\partial^\beta}{\partial y^\beta}H_\alpha(x,0)\frac{y^\beta}{\beta!}| \le C\frac{|y|^{N(\delta)-1}}{|x|^{N(\delta)+d-\alpha-1}}\min(1,\frac{2|y|}{|x|}) :$$

this is obvious by the Taylor integral formula when $|y| < |x|/2$; on the other hand, when $|y| \ge |x|/2$, we control $H_\alpha(x,y)$ by $|x-y|^{-d+\alpha} \le C|x|^{-d+\alpha}$ and $\frac{\partial^\beta}{\partial y^\beta}H_\alpha(x,0)$ by $|x|^{-d+\alpha+|\beta|}$; we then multiply the estimate $O(\frac{|y|^{|\beta|}}{|x|^{d-\alpha+|\beta|}})$ by $(\frac{2|y|}{|x|})^{N(\delta)-|\beta|-1}$, which is greater than 1. This gives

$$\begin{gathered}|\int h(y)\ \big(H_\alpha(x,y) - \textstyle\sum_{|\beta|\le\delta-d/2}\frac{\partial^\beta}{\partial y^\beta}H_\alpha(x,0)\frac{y^\beta}{\beta!}\big)\ dy| \\ \le C\|\hat{h}\|_{H^\delta}|x|^{-N(\delta)-d+\alpha+1}\big(\int\frac{|y|^{2N(\delta)-2}}{(1+|y|)^{2\delta}}\min(1,\frac{4|y|^2}{|x|^2})\ dy\big)^{1/2}\end{gathered}$$

which is $O(|x|^{\alpha-d/2-\delta})$ since $d/2-1 < \delta - N(\delta) < d/2$. □

2. Spatial decay of velocities

Theorem 25.1 proves the existence of solutions that are decaying at infinity like $|x|^{-d-1+\epsilon}$ for any $\epsilon > 0$ (since we have the decay $0(|x|^{-\delta-d/2+\alpha})$ with the restriction $\delta < \alpha+1+d/2$). The decay $|x|^{-d-1}$ is easily obtained (Miyakawa [MIY 00], Brandolese [BRA 01]) as we shall see (in Proposition 25.1) and is optimal (Dobrokhotov and Shafarevich [DOBS 94], Brandolese [BRA 01], Theorem 25.2).

Proposition 25.1:

For $\delta \in \mathbb{R}$, let $Y_\delta = \{f \in L^1_{loc}\ /\ (1+|x|)^\delta f \in L^\infty\}$. Then:
a) For $\delta \in [0, d+1]$ and for every $T > 0$, the bilinear operator B is continuous on $(\mathcal{C}_b((0,T), Y_\delta))^d$, with an operator norm $O\big(\sqrt{T}\ (1+\sqrt{T})\big)$
b) If $\vec{u}_0 \in (Y_\delta)^d$ with $0 \le \delta \le d+1$ and $\vec{\nabla}.\vec{u}_0 = 0$, then there exists a positive T and a solution $\vec{u} \in (\mathcal{C}_b((0,T), Y_\delta))^d$ to the Navier–Stokes equations with initial value $\vec{u}_0$. Moreover, we may choose $T \ge \gamma\min(1, \|\vec{u}\|_{Y_\delta}^{-2})$ for a positive constant γ, which depends only on d and δ.
c) Let $\vec{u}_0 \in (L^\infty)^d$ and $\vec{\nabla}.\vec{u}_0 = 0$ and let, for some positive T, a solution $\vec{u} \in (\mathcal{C}_b((0,T), L^\infty)^d$ to the Navier–Stokes equations with initial value $\vec{u}_0$. If $\vec{u}_0$ belongs more precisely to $(Y_\delta)^d$ with $0 \le \delta \le d+1$, then $\vec{u}$ belongs to $(\mathcal{C}_b((0,T), Y_\delta))^d$.

Proof: a) is easily seen just by looking at the size of the Oseen kernel; from Proposition 11.1, we know that we have

$$|B(\vec{f},\vec{g})(t,x)| \leq C \int_0^t \int_{\mathbb{R}^d} \frac{1}{\left(|x-y|+\sqrt{t-s}\right)^{d+1}} |\vec{f}(s,y)||\vec{g}(s,y)| \, ds \, dy.$$

For $\delta = 0$, we write $\int \frac{dy}{\left(|x-y|+\sqrt{t-s}\right)^{d+1}} \leq \frac{C}{\sqrt{t-s}}$ and get $\|B(\vec{f},\vec{g})(t,.)\|_\infty \leq C\sqrt{t} \sup_{0<s<t} \|f\|_\infty \sup_{0<s<t} \|g\|_\infty$. This gives as well the control of the size of $|B(\vec{f},\vec{g})(t,x)|$ for $\delta > 0$ and $|x| \leq 1$. For $\delta > 0$ and $|x| > 1$, we write

$$|B(\vec{f},\vec{g})(t,x) \leq C(I(t,x)+J(t,x)) \sup_{0<s<t} \|f(s,.)\|_{Y_\delta} sup_{0<s<t} \|g(s,.)\|_\infty,$$

where we define I and J as

$$\begin{cases} I(t,x) = \int_0^t \int_{|y|\geq|x|/2} \frac{1}{\left(|x-y|+\sqrt{t-s}\right)^{d+1}} \frac{1}{(1+|y|)^\delta} \, ds \, dy \\ J(t,x) = \int_0^t \int_{|y|<|x|/2} \frac{1}{\left(|x-y|+\sqrt{t-s}\right)^{d+1}} \frac{1}{(1+|y|)^\delta} \, ds \, dy. \end{cases}$$

We have $I(t,x) \leq \frac{2^\delta}{|x|^\delta} \int_0^t \int_{\mathbb{R}^d} \frac{1}{\left(|x-y|+\sqrt{t-s}\right)^{d+1}} \, ds \, dy \leq \frac{C\sqrt{t}}{|x|^\delta}$. For $|y| < |x|/2$, we have $|x-y| \geq |x|/2$; for $\delta < d$, we obtain

$$J(t,x) \leq \int_0^t \int_{|y|<|x|/2} \frac{2^\delta}{|x-y|^{d-\delta} \sqrt{t-s} \, |x|^\delta} \frac{1}{|y|^\delta} ds \, dy \leq C\sqrt{t} \, |x|^{-\delta}.$$

For $d \leq \delta \leq d+1$, we write that $\int_{|y|<|x|/2} \frac{dy}{(1+|y|)^\delta} \leq C$ $(\delta > d)$ or $\leq C(1+\ln|x|)$ $(\delta = d)$ and that (for $|y| < |x|/2$) $\frac{1}{\left(|x-y|+\sqrt{t-s}\right)^{d+1}} \leq \frac{2^{d+1}}{|x|^{d+1}}$; hence, we get $J(t,x) \leq Ct|x|^{-d-1} \leq Ct|x|^{-\delta}$ for $\delta > d$ and $J(t,x) \leq Ct|x|^{-d-1}(1+\ln|x|) \leq Ct|x|^{-\delta}$ for $\delta = d$. This gives the boundedness of B from $(L^\infty((0,T),Y_\delta)^d \times (L^\infty((0,T),L^\infty)^d$ to $(L^\infty((0,T),Y_\delta)^d$ with an operator norm $O\big(\sqrt{T}\,(1+\sqrt{T})\big)$.

b) is a direct consequence of a), through the Picard contraction principle. We must check that $f_0 \mapsto (e^{t\Delta} f_0)_{0<t<T}$ maps Y_δ to $\mathcal{C}_b((0,T),Y_\delta)$. The proof is similar to the proof of point a). Indeed, the kernel $\frac{1}{t^{d/2}} W(\frac{x-y}{\sqrt{t}})$ of $e^{t\Delta}$ may be controlled by $\frac{1}{t^{d/2}}|W(\frac{x-y}{\sqrt{t}})| \leq C \frac{\sqrt{t}}{(|x-y|+\sqrt{t})^{d+1}}$. This gives $\|e^{t\Delta} f_0\|_{Y_\delta} \leq C(1+\sqrt{t})\|f_0\|_{Y_\delta}$.

c) is easily checked. We know that we have local (and global) uniqueness in $L^\infty((0,T),L^\infty))^d$. Thus, due to point b), if $\vec{u}(t_0) \in Y_\delta^d$, then $\vec{u}$ will remain in Y_δ^d for $t \leq t_0 + \theta$ with $\theta \geq \gamma \min(1, \|\vec{u}(t_0)\|_{Y_\delta}^{-2})$. Hence, we have only to prove that the norm of $\vec{u}$ in Y_δ does not blow up as long as the norm of $\vec{u}$ in L^∞ remains bounded. If $\vec{u}$ is a solution to the Navier–Stokes equations in

$\cap_{T_1<T}(\mathcal{C}_b((0,T_1),Y_\delta))^d$ so that $\|\vec{u}\|_\infty$ is bounded by a constant M on $(0,T)$, we find, from the proof of point a), that for all $t<T$ we have

$$\begin{aligned}\|\vec{u}(t,.)\|_{Y_\delta} &\leq \|e^{t\Delta}\vec{u}_0\|_{Y_\delta} + \|B(\vec{u},\vec{u})\|_{Y_\delta} \\ &\leq C(1+\sqrt{T})(\|\vec{u}_0\|_{Y_\delta} + M\int_0^t \frac{1}{\sqrt{t-s}}\|\vec{u}(s,.)\|_{Y_\delta}\, ds \\ &\leq C(1+\sqrt{T})\|\vec{u}_0\|_{Y_\delta} + MC(1+\sqrt{T})\int_0^t \frac{1}{\sqrt{t-s}}\, ds\|\vec{u}_0\|_{Y_\delta} \\ &+C^2(1+\sqrt{T})^2M^2\int_0^t\int_0^s \frac{1}{\sqrt{t-s}}\frac{1}{\sqrt{s-\sigma}}\|\vec{u}(\sigma,.)\|_{Y_\delta}\, d\sigma\, ds \\ &\leq A + B\int_0^t \|\vec{u}(\sigma,.)\|_{Y_\delta}\, d\sigma\end{aligned}$$

with $A = C(1+\sqrt{T})(1+2M\sqrt{T})\|\vec{u}_0\|_{Y_\delta}$ and $B = \pi C^2(1+\sqrt{T})^2M^2$. The Gronwall lemma then gives that $\|\vec{u}\|_{Y_\delta} \leq Ae^{Bt}$. $\square$

We now prove that this result cannot be true for $\delta > d+1$, by using the inclusion, for $\delta > d+1$, $Y_\delta \subset L^2(\mathbb{R}^d, (1+|x|)^{d+2}\, dx)$:

Theorem 25.2:

Let $\vec{u}_0$ be a divergence-free vector field in $(L^2(\mathbb{R}^d,(1+|x|)^{d+2}\, dx))^d$ and let, for some positive T, a solution $\vec{u} \in (\mathcal{C}([0,T], L^2(\mathbb{R}^d,(1+|x|)^{d+2}\, dx))^d$ to the Navier–Stokes equations with initial value $\vec{u}_0$. Then $\vec{u}$ satisfies that, for every $t\in[0,T]$, the matrix $(\int_{\mathbb{R}^d} u_j(x)\, u_k(x)\, dx)_{1\leq j,k\leq d}$ is a multiple of the identity: $\int_{\mathbb{R}^d} u_j(t,x)\, u_k(t,x)\, dx = \lambda(t)\delta_{j,k}$. In particular, the localization of $\vec{u}_0$ will not persist unless $\vec{u}_0$ satisfies the orthogonality properties $\int_{\mathbb{R}^d} u_{0,j}(x)\, u_{0,k}(x)\, dx = \lambda(0)\delta_{j,k}$.

Before proving Theorem 25.2, we prove some useful lemmas:

Lemma 25.5:

a) (Restriction theorem) For $s>1/2$, the mapping $f(x,y)\in\mathcal{S}(\mathbb{R}\times\mathbb{R}^{d-1})\mapsto f(0,y)\in\mathcal{S}(\mathbb{R}^{d-1})$ has a continuous extension from $H^s(\mathbb{R}^d)$ to $H^{s-1/2}(\mathbb{R}^{d-1})$.
b) For $s\geq 1$, the mapping $f(x,y)\in H^s(\mathbb{R}\times\mathbb{R}^{d-1})\mapsto \frac{f(x,y)-f(0,y)}{x}$ maps boundedly $H^s(\mathbb{R}^d)$ to $H^{s-1}(\mathbb{R}^d)$.

Proof: a) is quite obvious; looking at the Fourier transforms, we have to prove an inequality such as

$$\int_{\mathbb{R}^{d-1}}(1+|\eta|^2)^{s-1/2}|\int_{\mathbb{R}} f(\xi,\eta)\, d\xi|^2\, d\eta \leq \int_{\mathbb{R}\times\mathbb{R}^{d-1}}(1+|\eta|^2+|\xi|^2)^s|f(\xi,\eta)|^2 d\eta\, d\xi.$$

This is proved through the Cauchy–Schwarz inequality

$$|\int_{\mathbb{R}} f(\xi,\eta)\, d\xi|^2 \leq \int_{\mathbb{R}}(1+|\eta|^2+|\xi|^2)^s|f(\xi,\eta)|^2\, d\xi\int_{\mathbb{R}}(1+|\eta|^2+|\xi|^2)^{-s}\, d\xi$$

since $\int_{\mathbb{R}}(1+|\eta|^2+|\xi|^2)^{-s}\, d\xi = (1+|\eta|^2)^{1/2-s}\int_{\mathbb{R}}(1+X^2)^{-s}\, dX$.

To prove b), we write (at least for $f \in \mathcal{S}(\mathbb{R}^d)$) the identity $\frac{f(x,y)-f(0,y)}{x} = \int_0^1 \frac{\partial}{\partial x} f(\theta x, y)\, d\theta$. Of course, $\frac{\partial}{\partial x} f \in H^{s-1}$. For $\sigma \geq 0$ and $\theta \in]0,1]$ we have for $g \in H^\sigma$ $\|g(\theta x, y)\|_{H^\sigma} \leq C(\sigma)\|g\|_{H^\sigma}\theta^{-1/2}$. (This is obvious for $\sigma \in \mathbb{N}$ by computing the L^2 norms of the derivatives up to order σ; then extend the result to $\sigma \notin \mathbb{N}$ by interpolation.) Thus, we get $\|\frac{f(x,y)-f(0,y)}{x}\|_{H^{s-1}} \leq \int_0^1 \|\frac{\partial}{\partial x} f(\theta x, y)\|_{H^{s-1}}\, d\theta < \infty$. $\square$

Lemma 25.6: *Let $d \geq 2$. Let ω be a smooth function on $\mathbb{R}^d - \{0\}$ which is homogeneous with degree 0. If ω belongs locally to $H^{d/2}$ (i.e. if $\varphi\omega \in H^{d/2}(\mathbb{R}^d)$ for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$), then ω is constant.*

Proof: We prove the result by demonstrating that ω is constant on every subspace of dimension 2. Since the hypotheses are invariant under rotations, we only have to check that ω is constant on the plane $x_3 = \ldots = x_d = 0$. We use the restriction theorem on $H^{d/2}(\mathbb{R}^d)$ and find that $\omega(x_1, x_2, 0, \ldots, 0)$ belongs locally to $H^1(\mathbb{R}^2)$, and thus that $\partial_1\omega(x_1, x_2, 0, \ldots, 0)$ and $\partial_2\omega(x_1, x_2, 0, \ldots, 0)$ belong locally to $L^2(\mathbb{R}^2)$; since they are homogeneous with degree -1, we find that they must be equal to 0 (just integrate on the ball $|(x_1, x_2)| \leq 1$). $\square$

Proof of Theorem 25.2: We start from the identity

$$\vec{u}(t) = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\vec{\nabla}.\vec{u} \otimes \vec{u}\, ds - \int_0^t e^{(t-s)\Delta}\vec{\nabla}p\, ds$$

and we define $q_j = \int_0^t e^{(t-s)\Delta}\partial_j p\, ds$. We have $\partial_k q_j = \partial_j q_k$.

Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and let $\gamma_j(\xi) = \varphi(\xi)\hat{q}_j(\xi)$. Since $q_j = e^{t\Delta}u_{0,j} - u_j - \int_0^t e^{(t-s)\Delta}\vec{\nabla}.(u_j\vec{u})\, ds$ and $\vec{u}(t)$ is bounded in $L^2((1+|x|)^{d+2}\, dx)$, we find that $\hat{q}_j$ is the sum of a function in $H^{(d+2)/2}$ and a function in $B_\infty^{d+2,1}$. Thus, γ_j belongs to $H^{d/2+1}$, hence is continuous. Since $\xi_k\gamma_j = \xi_j\gamma_k$, γ_j is equal to 0 on the hyperplane $\xi_j = 0$. We then apply point b) in Lemma 25.5 and get that $\gamma_j = \xi_j g_j$ with $g_j \in H^{d/2}$. But g_j does not depend on j since we have $\xi_j\xi_k g_j = \xi_j\xi_k g_k$; hence, $g_j = g_k$ a.e.. We thus write $\gamma_j = \xi_j g$.

We use the fact that $\vec{u}$ is divergence free; hence, $\sum_{j\in\{1,\ldots,d\}} \partial_j q_j = -\sum_{j,k\in\{1,\ldots d\}} \int_0^t e^{(t-s)\Delta}\partial_j\partial_k(u_j u_k)\, ds$. Taking the Fourier transforms and multiplying by $\varphi(\xi)$, we obtain

$$i|\xi|^2 g(\xi) = \varphi(\xi) \sum_{j,k\in\{1,\ldots d\}} \int_0^t e^{(t-s)\Delta}\xi_j\xi_k\mathcal{F}(u_j(s)u_k(s))(\xi)\, ds.$$

Let $\int_0^t e^{(t-s)\Delta}\mathcal{F}(u_j(s)u_k(s))(\xi)\, ds = U_{j,k}(t,\xi)$. The function $U_{j,k}$ belongs to $L^\infty((0,T), B_\infty^{d+2,1})$. In particular, $\varphi(\xi)(U_{j,k}(t,\xi) - U_{j,k}(t,0))$ belongs to $L^2 \cap \dot{H}^{(d+1)/2}$ for every t, and so does $\frac{\xi_j\xi_k}{|\xi|^2}\varphi(\xi)(U_{j,k}(t,\xi) - U_{j,k}(t,0))$ (according to Lemma 25.1). Since $g \in H^{d/2}$ and since $L^2 \cap \dot{H}^{(d+1)/2} \subset H^{d/2}$, we get that $\left(\sum_{1\leq j,k\leq d}\frac{\xi_j\xi_k}{|\xi|^2}U_{j,k}(t,0)\right)\varphi(\xi)$ belongs to $H^{d/2}$ for every t. Lemma 25.5 gives that $\sum_{1\leq j,k\leq d}\frac{\xi_j\xi_k}{|\xi|^2}U_{j,k}(t,0) = C(t)$, thus $U_{j,k}(t,0) = C(t)\delta_{j,k}$. Since $U_{j,k}(t,0) = \int_0^t\int u_j(s,y)u_k(s,y)\ ds\ dy$, Theorem 25.2 is proved. □

Brandolese has shown how to construct solutions to the Navier–Stokes equations with a better localization than allowed by Theorem 25.2. Those localised solutions are symmetric in the following sense: let σ be the cycle in $\mathbb{R}^d$ $(x_1,x_2,\dots,x_d)\mapsto(x_2,x_3,\dots x_1)$; then a vector field $\vec{a}$ is symmetric in the sense of Brandolese if $\sigma\circ\vec{a}=\vec{a}\circ\sigma$ and if $\vec{a}_j$ is an even function with respect to the jth variable x_j and an odd function with respect to the kth variable x_k for $k\in\{1,\dots,d\}$, $k\neq j$. Then Brandolese shows that when $\vec{u}_0$ is a divergence-free symmetric vector field, which belongs to $(Y_\delta)^d$ with $d+1<\delta<d+2$, the mild solution associated with $\vec{u}_0$ remains in $(Y_\delta)^d$ [BRA 01].

3. Vorticities are well localized

Brandolese has shown that, in contrast to the case of velocities (Theorem 25.2), the vorticity may keep a very rapid decay at infinity [BRA 01].

We begin by introducing the vorticity matrix Ω associated to the velocity $\vec{u}$:

Lemma 25.7:

Let ROT and $\vec{K}$ be the operators $\vec{u}\mapsto ROT(\vec{u}) = (\partial_j u_k - \partial_k u_j)_{1\leq j,k\leq d}$ and $M=(m_{j,k})_{1\leq j,k\leq d}\mapsto\vec{K}(M) = (-\sum_{1\leq k\leq d}\frac{\partial_k}{\Delta}m_{j,k})_{1\leq j\leq d}$.

Let $E=\{u\in\mathcal{C}_0(\mathbb{R}^d)\ /\ \vec{\nabla}u\in(L^{d,1})^d\}$. Then ROT boundedly maps E^d to $(L^{d,1})^{d\times d}$ and $\vec{K}$ boundedly maps $(L^{d,1})^{d\times d}$ to E^d. Moreover, $\vec{K}\circ ROT = Id$ on E^d.

Proof: This is obvious since $\frac{\partial_k}{\Delta}$ is a convolution operator with a kernel that is homogeneous with degree $-d+1$ and smooth outside $\{0\}$ and thus belongs to $L^{\frac{d}{d-1},\infty}$. □

Definition 25.2: (Vorticity matrix)

If $\vec{u}$ is a solution to the Navier–Stokes equations, its vorticity matrix Ω is defined by $\Omega = ROT(\vec{u})$.

The Navier–Stokes equations may be expressed in terms of the vorticity matrix:

Proposition 25.2: (Navier–Stokes equations and vorticity matrix)

Let $E=\{u\in\mathcal{C}_0(\mathbb{R}^d)\ /\ \vec{\nabla}u\in(L^{d,1})^d\}$. Let $\vec{u}_0\in E^d$ with $\vec{\nabla}.\vec{u}_0=0$ and let $\Omega_0 = ROT(\vec{u}_0)$. Then:

(A) There exists a positive T and a solution $\vec{u}\in(\mathcal{C}([0,T),E)^d$ to the Navier–Stokes equations with initial value $\vec{u}_0$.

(B) The following assertions are equivalent for $\vec{u} \in (\mathcal{C}([0,T), E)^d$:

(B1) $\vec{u}$ is a solution to the Navier–Stokes equations with initial value $\vec{u}_0$: there exists $p \in \mathcal{D}'((0,T) \times \mathbb{R}^d)^d$ so that

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla}.\vec{u} \otimes \vec{u} - \vec{\nabla} p \\ \vec{\nabla}.\vec{u} = 0 \\ \vec{u}(0,.) = \vec{u}_0 \end{cases}$$

(B2) there exists $p \in (\mathcal{D}'(0,T) \times \mathbb{R}^d)^d$ so that

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} + ROT(\vec{u})\vec{u} - \vec{\nabla}(p + \frac{1}{2}|\vec{u}|^2) \\ \vec{\nabla}.\vec{u} = 0 \\ \vec{u}(0,.) = \vec{u}_0 \end{cases}$$

(B3) There exists $\Omega \in (\mathcal{C}([0,T), L^{d,1})^{d \times d}$ so that $\vec{u} = \vec{K}(\Omega)$ and

$$\begin{cases} \partial_t \Omega = \Delta \Omega + ROT(\Omega \vec{K}(\Omega)) \\ \Omega(0,.) = \Omega_0 \end{cases}$$

Proof: E is a Banach algebra for the pointwise product: if $\|u\|_E = \|u\|_\infty + \sum_{1 \le j \le d} \|\partial_j f\|_{L^{d,1}}$, then $\|uv\|_E \le \|u\|_E \|v\|_E$. We thus obtain the inequality $\|e^{(t-s)\Delta} \mathbb{P} \vec{\nabla}.\vec{u} \otimes \vec{v}\|_E \le C \frac{1}{\sqrt{t-s}} \|\vec{u}\|_E \|\vec{v}\|_E$. This gives (A).

The equivalence between (B1) and (B2) is a direct consequence of the identities $\vec{\nabla}.\vec{u} \otimes \vec{u} = (\vec{u}.\vec{\nabla})\vec{u} + (\vec{\nabla}.\vec{u})\vec{u}$ and $(\vec{u}.\vec{\nabla})\vec{u} = \vec{\nabla}(\frac{1}{2}|\vec{u}|^2) - ROT(\vec{u})\vec{u}$ (i.e. $\sum_{1 \le k \le d} u_k \partial_k u_j = \sum_{1 \le k \le d} u_k \partial_j u_k - \sum_{1 \le k \le d} u_k (\partial_j u_k - \partial_k u_j)$).

We obviously have (B2)$\Rightarrow$(B3). The converse implication is proved by the following remark. We have local uniqueness for the solutions of the problems (B2) and (B3); thus, if Ω is a solution of (B3) in $(\mathcal{C}([0,T_\Omega), L^{1,d}))^{d \times d}$ and $\vec{u}$ is a solution of (B2) in $(\mathcal{C}([0,T_{\vec{u}}), E))^d$, then we may assume $T_\Omega \ge T_{\vec{u}}$ and $\Omega = ROT(\vec{u})$ on $(0, T_{\vec{u}})$. Moreover, if $T_\Omega > T_{\vec{u}}$, we find that $\|\vec{u}\|_E$ remains bounded as t goes to $T_{\vec{u}}$; since the existence of a solution of (B2) on $[t_0, t_0 + \theta]$ is granted for $\theta = O(\|\vec{u}(t_0)\|_E^{-2})$, we see that $\vec{u}$ may be extended up to T_Ω. $\square$

Theorem 25.3:

Let $X = \{f \in L^1(\mathbb{R}^d) \ / \ \int e^{|x|} |f(x)| \, dx < \infty$ and $\int e^{|\xi|} |\hat{f}(\xi)| \, dx < \infty\}$, with norm $\|f\|_X = \|e^{|x|} f\|_1 + \|e^{|\xi|} \hat{f}\|_1$. Then:

A) i) $X \subset \mathcal{S}$.

ii) For $u \in X$, $v \in X$ and $1 \le k \le d$, $\|u \ \frac{\partial_k}{\Delta} v\|_X \le C \|u\|_X \|v\|_X$.

iii) For $u \in X$, $t \in (0,1]$ and $1 \le k \le d$, $\|e^{t\Delta} \partial_k u\|_X \le C \frac{1}{\sqrt{t}} \|u\|_X$.

B) Let $E = \{u \in \mathcal{C}_0(\mathbb{R}^d) \ / \ \vec{\nabla} u \in (L^{d,1})^d\}$. Let $\vec{u}_0 \in E^d$ with $\vec{\nabla}.\vec{u}_0 = 0$ and let $\Omega_0 = ROT(\vec{u}_0)$. If $\Omega_0 \in X^{d \times d}$, then there exists a positive T and a solution $\vec{u} \in (\mathcal{C}([0,T), E)^d$ to the Navier–Stokes equations with initial value $\vec{u}_0$ so that $ROT(\vec{u}) \in (\mathcal{C}([0,T), X)^{d \times d}$.

Proof: The Schwartz class $\mathcal{S}$ may be defined by the conditions $f \in \mathcal{S}$ if and only if, for all α and β in $\mathbb{N}^d$, $x^\alpha \partial^\beta f \in L^2$. Integration by parts and Leibnitz rule for differentiation then gives us that $f \in \mathcal{S}$ if and only if, for all α and β in $\mathbb{N}^d$, $x^\alpha f \in L^1$ and $\partial^\beta f \in \mathcal{C}_0$. Thus, $X \subset \mathcal{S}$.

We have $f \in X \Rightarrow \hat{f} \in L^1 \cap L^\infty \Rightarrow \frac{1}{|\xi|}\hat{f} \in L^1$. Thus, we get $\|e^{|x|} u \ \frac{\partial_k}{\Delta} v\|_1 \leq \frac{1}{(2\pi)^d}\|e^{|x|}u\|_1 \|\frac{1}{|\xi|}\hat{v}\|_1 \leq C\|u\|_X \|v\|_X$. Moreover, we may estimate the Fourier transform of $u \ \frac{\partial_k}{\Delta} v$ by $|\mathcal{F}(u \ \frac{\partial_k}{\Delta} v)| \leq \frac{1}{(2\pi)^d} \int |\hat{u}(\eta)| \ \frac{1}{|\xi-\eta|}|\hat{v}(\xi-\eta)| \ d\eta$; we write $e^{|\xi|} \leq e^{|\eta|}e^{|\xi-\eta|}$ to obtain

$$\int e^{|\xi|}|\mathcal{F}(u \ \frac{\partial_k}{\Delta} v)| \ d\xi \leq \frac{1}{(2\pi)^d} \int e^{|\xi|}|\hat{u}(\xi)| \ d\xi \ \int e^{|\xi|}\frac{1}{|\xi|}|\hat{v}(\xi)| \ d\xi;$$

since $e^{|\xi|}\hat{v} \in L^1 \cap L^\infty_{loc}$, we have $\frac{1}{|\xi|}e^{|\xi|}\hat{v} \in L^1$ and thus $\|e^{|\xi|}\mathcal{F}(u \ \frac{\partial_k}{\Delta} v)\|_1 \leq C\|u\|_X\|v\|_X$.

To control $e^{t\Delta}u$ for $t \leq 1$, we write

$$|\partial_k e^{t\Delta} u| \leq \int |u(y)| \frac{1}{(4\pi t)^{d/2}} \frac{|x-y|}{2t} e^{-\frac{|x-y|^2}{4t}} \ dy$$

and $e^{|x|} \leq e^{|y|}e^{\frac{|x-y|}{\sqrt{t}}}$ and, thus, we get

$$\int e^{|x|}|\partial_k e^{t\Delta}u| \ dx \leq \frac{C}{\sqrt{t}} \int e^{|x|}|u| \ dx \text{ with } C = \frac{\int e^{|x|}|x|e^{-\frac{|x|^2}{4}} \ dx}{2(4\pi)^{d/2}};$$

the control of the Fourier transform is simpler: $\mathcal{F}(\partial_k e^{t\Delta}u) = i\xi_k e^{-t|\xi|^2}\hat{u}(\xi)$, and thus, since $\|\xi_k e^{-t|\xi|^2}\|_\infty = \frac{1}{\sqrt{2e}}$,

$$\int e^{|\xi|}|\mathcal{F}(\partial_k e^{t\Delta}u)| \ d\xi \leq \frac{1}{\sqrt{2e}} \int e^{|\xi|}|\hat{u}| \ d\xi.$$

This constitutes proof for (A).

If Ω_0 belongs to $X^{d\times d}$, then $e^{t\Delta}\Omega_0 \in \mathcal{C}([0,1]), X^{d\times d})$. The bilinear operator β defined by $\beta(\Omega,\omega) = \int_0^t e^{(t-s)\Delta} ROT(\Omega\vec{K}(\omega)) \ ds$ is bounded on $\mathcal{C}([0,T], X^{d\times d})$ for every $T \leq 1$ with an operator norm $O(\sqrt{T})$. Thus, if T is small enough, we may apply the Picard contraction principle to the equation (B3) $\Omega = e^{t\Delta}\Omega_0 + \beta(\Omega,\Omega)$. Hence, (B) is proved. $\square$

Chapter 26

Time decay for the solutions to the Navier–Stokes equations

In this chapter, we describe some classical and some recent results on time decay of solutions to the Navier–Stokes equations. The problem of time decay of the solutions was described by Wiegner [WIE 87] and Schonbek [SCHO 85]. A discussion of limitations on the decay was recently given by Mikayawa and Schonbek [MIYS 01] and Brandolese [BRA 01] who gives some additional interesting constructions of rapidly decaying solutions.

1. Wiegner's fundamental lemma and Schonbek's Fourier splitting device

The Fourier splitting device introduced by Schonbek [SCHO 85] is based on the following idea: if we look at the heat kernel $e^{t\Delta}$, it may be viewed as a low-frequency filter that selects the frequencies ξ so that $t|\xi|^2 \leq 1$; thus, the frequency $1/\sqrt{t}$ is the limit over which high frequencies are to be dumped by the kernel. Then, we estimate the L^2 norm of the solution $\vec{u}_\epsilon$ of the mollified Navier–Stokes equations

$$\vec{u}_\epsilon = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.[(\vec{u}_\epsilon * \omega_\epsilon) \otimes \vec{u}_\epsilon]\ ds$$

(with $\omega_\epsilon(x) = \epsilon^{-d}\omega(x/\epsilon)$, $\omega \in \mathcal{D}(\mathbb{R}^d)$, $\omega \geq 0$ and $\int \omega\ dx = 1$) by splitting the norm into two pieces $(2\pi)^d\|\vec{u}_\epsilon(t,x)\|^2_{L^2(dx)} = \int_{|\xi|\leq f(t)} |\hat{u}_\epsilon(t,\xi)|^2\ d\xi + \int_{|\xi|>f(t)} |\hat{u}_\epsilon(t,\xi)|^2\ d\xi$ with $f(t) = O(t^{-1/2})$.

We prove more precisely the following lemma of Wiegner [WIE 87]:

Lemma 26.1:

Let $\vec{u}_0 \in (L^2)^d$ with $\vec{\nabla}.\vec{u}_0 = 0$ and let, for $\epsilon > 0$, $\vec{u}_\epsilon$ be the solution of the mollified Navier–Stokes equations $\vec{u}_\epsilon = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.[(\vec{u}_\epsilon * \omega_\epsilon) \otimes \vec{u}_\epsilon]\ ds$. Then there exists a constant γ_d, which does not depend on $\vec{u}_0$ nor on ϵ so that for any positive continuous function φ on $[0,\infty)$ and for $\Phi(t) = \int_0^t \varphi(s)\ ds$, we have the inequality, for all $t \in [0,\infty)$:

$$e^{\Phi(t)}\|\vec{u}_\epsilon(t,.)\|_2^2 \leq \|\vec{u}_0\|_2^2 + \gamma_d \int_0^t e^{\Phi(s)}\varphi(s)\big(\|e^{s\Delta}\vec{u}_0\|_2^2 + \varphi(s)^{\frac{d+2}{2}} R_\epsilon(s)\big)\ ds$$

with $R_\epsilon(s) = (\int_0^s \|\vec{u}_\epsilon(\sigma,.)\|_2^2 \, d\sigma)^2$.

Proof: We have

$$\frac{d}{dt}(e^\Phi \|\vec{u}_\epsilon\|_2^2) = e^\Phi(\varphi\|\vec{u}_\epsilon\|_2^2 + 2\langle \vec{u}_\epsilon | \partial_t \vec{u}_\epsilon\rangle) = e^\Phi(\varphi\|\vec{u}_\epsilon\|_2^2 - 2\|\vec{\nabla}\otimes\vec{u}_\epsilon\|_2^2),$$

hence $\frac{d}{dt}(e^\Phi\|\vec{u}_\epsilon\|_2^2) = \frac{e^{\Phi(t)}}{(2\pi)^d}\int \big(\varphi(t) - 2|\xi|^2\big)|\mathcal{F}\vec{u}_\epsilon|^2 \, d\xi$. Let $\mu \in \mathcal{D}(\mathbb{R}^d)$ with $\mu \geq 0$ and $\mu(\xi) = 1$ for $|\xi| \leq 1$ and $\mu(\xi) = 0$ for $|\xi| \geq 2$. We obviously have $\varphi(t) - 2|\xi|^2 \leq \varphi(t)\mu(\sqrt{\frac{2}{\varphi(t)}}\ \xi)$, and we thus get

$$\frac{d}{dt}(e^\Phi\|\vec{u}_\epsilon\|_2^2) \leq 2e^{\Phi(t)}\varphi(t)\Big(\|e^{t\Delta}\vec{u}_0\|_2^2 + (\int_0^t \|\mu(\sqrt{\frac{2}{\varphi(t)}}\ D)\vec{\nabla}.[(\vec{u}_\epsilon*\omega_\epsilon)\otimes\vec{u}_\epsilon]\|_2 ds)^2\Big)$$

We now write, for $j = 1,\ldots,d$ and for $\lambda > 0$, $\|\mu(\lambda D)\partial_j f\|_2 \leq C\lambda^{-(d+2)/2}\|f\|_1$ and we arrive at

$$\frac{d}{dt}(e^\Phi\|\vec{u}_\epsilon\|_2^2) \leq \varphi e^\Phi\big(2\|e^{t\Delta}\vec{u}_0\|_2^2 + C\varphi^{(d+2)/2}\ (\int_0^t \|\vec{u}_\epsilon\|_2^2 ds)^2\big).$$

$\square$

2. Decay rates for the L^2 norm

We may now introduce Wiegner's theorem [WIE 87] on the decay rate of restricted Leray weak solutions of the Navier–Stokes equations:

Theorem 26.1: (Wiegner's theorem)

Let $\vec{u}_0 \in (L^2(\mathbb{R}^d))^d$ *with* $\vec{\nabla}.\vec{u}_0 = 0$ *and let* $\vec{u}$ *be a retricted Leray weak solution for the Navier–Stokes equations with initial value* $\vec{u}_0$*. Then:*

(A) $\lim_{t\to\infty}\|\vec{u}(t,.)\|_2 = 0$.

(B) If $\vec{u}_0$ *belongs more precisely to* $(L^2\cap\dot{B}_2^{-\alpha,\infty})^d$ *with* $0 < \alpha \leq (d+2)/2$ *(so that* $\|e^{t\Delta}\vec{u}_0\|_2 \leq C(1+t)^{-\alpha/2}$*) then* $\sup_{t>0}(1+t)^{\alpha/2}\|\vec{u}(t,.)\|_2 < \infty$*. Moreover, if* $\alpha < (d+2)/2$, $\lim_{t\to 0} t^{\alpha/2}\|\vec{u} - e^{t\Delta}\vec{u}_0\|_2 = 0$.

Proof:

Step 1: Decay of the L^2 norm

(A) is a direct consequence of Lemma 26.1, with $\varphi(t) = \frac{1}{(t+e)\ln(t+e)}$, $\Phi(t) = \ln\ln(t+e)$ and $e^\Phi = \ln(t+e)$. Since $\|\vec{u}_\epsilon\|_2 \leq \|\vec{u}_0\|_2$, we have $R_\epsilon(s) \leq s^2\|\vec{u}_0\|_2^4$; thus, Lemma 26.1 then states that

$$\|\vec{u}_\epsilon(t,.)\|_2^2 \leq \frac{\|\vec{u}_0\|_2^2}{\ln(t+e)} + \gamma_d\frac{\int_0^t e^{\Phi(s)}\varphi(s)\ \big(\|e^{s\Delta}\vec{u}_0\|_2^2 + \varphi(s)^{(d+2)/2}s^2\|\vec{u}_0\|_2^4\big)\ ds}{\int_0^t e^{\Phi(s)}\varphi(s)\ ds}.$$

The same inequality holds with $\vec{u}$ instead of $\vec{u}_\epsilon$; we know from the study of the limiting process for the mollified equations (Chapter 13) that $\vec{u}$ is the strong limit of a sequence $\vec{u}_\epsilon$ in $(L^2_{loc}((0,\infty)\times \mathbb{R}^d))^d$; as a consequence, we get that there is a subsequence $\vec{u}_\epsilon$ such that for almost every t $\vec{u}_\epsilon(t,.)$ converges to $\vec{u}$ in $(L^2_{loc}(\mathbb{R}^d))^d$. Since the L^2 norm of $\vec{u}_\epsilon$ is bounded uniformly with respect to ϵ, this subsequence converges weakly to $\vec{u}$ in L^2, so that for almost every t we have $\|\vec{u}(t,.)\|_2 \leq \limsup_{\epsilon\to 0}\|\vec{u}_\epsilon(t,.)\|_2$. Thus, for almost every t, we have:

$$\|\vec{u}(t,.)\|_2^2 \leq \frac{\|\vec{u}_0\|_2^2}{\ln(t+e)} + \gamma_d \frac{\int_0^t e^{\Phi(s)}\varphi(s)\ \left(\|e^{s\Delta}\vec{u}_0\|_2^2 + \varphi(s)^{(d+2)/2}s^2\|\vec{u}_0\|_2^4\right)\ ds}{\int_0^t e^{\Phi(s)}\varphi(s)\ ds}.$$

Moreover, $t \mapsto \vec{u}(t,.)$ is weakly continuous from $[0,\infty)$ to $(L^2)^d$; since the majorant of $\|\vec{u}(t,.)\|_2^2$ is a continuous function, we get that the inequality is valid for all $t>0$. Since $\int_0^\infty e^{\Phi}\varphi\ ds = +\infty$, we conclude by checking that $\lim_{s\to+\infty}\|e^{s\Delta}\vec{u}_0\|_2 = 0$ and $\lim_{s\to+\infty}\varphi(s)^{\frac{d+2}{2}}s^2 = \lim_{s\to+\infty} s^{\frac{2-d}{2}}(\ln s)^{-\frac{d+2}{2}} = 0$ (for $d\geq 3$, the choice of $\varphi(t) = 1/(t+1)$ would have been sufficient; for $d=2$, we have to deal with a function φ decaying to 0 at ∞ more rapidly than $1/t$).

Step 2: Control of $t^{\alpha/2}\|\vec{u}(t,.)\|_2$

To prove (B), we choose $\varphi(t) = \frac{\gamma}{1+t}$; hence, $e^{\Phi} = (t+1)^\gamma$ with $\gamma > (d+2)/2$. Thus, Lemma 26.1 gives (for $\|e^{t\Delta}\vec{u}_0\|_2 \leq C(\vec{u}_0)(1+t)^{-\alpha/2}$)

$$(t+1)^\gamma\|\vec{u}_\epsilon(t,.)\|_2^2 \leq \|\vec{u}_0\|_2^2 + CC(\vec{u}_0)^2(t+1)^{\gamma-\alpha} + C\int_0^t (s+1)^{\gamma-1-(d+2)/2}R_\epsilon(s)\ ds.$$

where C depends only on d, γ and α. If we assume that we have the estimate $\|\vec{u}_\epsilon\|_2 \leq D(\beta,\vec{u}_0)(1+t)^{-\beta}$ for some $\beta\geq 0$, we get for $\beta\neq 1$

$$\int_0^t (s+1)^{\gamma-1-(d+2)/2}R_\epsilon(s)\ ds \leq C(d,\beta,\gamma)D(\beta,\vec{u}_0)^2(1+t)^{\gamma-(d+2)/2+\max(0,2-4\beta)}$$

and thus

$$\|\vec{u}_\epsilon(t,.)\|_2^2 \leq \frac{\|\vec{u}_0\|_2^2}{(t+1)^\gamma} + C\frac{C(\vec{u}_0)^2}{(t+1)^\alpha} + C\frac{D(\beta,\vec{u}_0)^2}{(t+1)^{(d+2)/2-\max(0,2-4\beta)}}$$

and finally $\|\vec{u}_\epsilon(t,.)\|_2 \leq D(\delta,\vec{u}_0)(1+t)^{-\delta}$ with $\delta = \min(\alpha/2, \frac{d-2}{4}+\min(1,2\beta))$. For $d\geq 3$, we start with $\beta=0$ and we conclude after finitely many iterations that $\|\vec{u}_\epsilon(t,.)\|_2 \leq D(\alpha/2,\vec{u}_0)(1+t)^{-\alpha/2}$; since the constant $D(\alpha/2,\vec{u}_0)$ does not depend on ϵ, we conclude that the same estimate is valid for $\|\vec{u}(t,.)\|_2$.

Step 3: The case $d=2$

The same proof is valid for $d=2$ provided that we can start with $\beta>0$, since we find in that case $\delta = \min(\alpha/2, \min(1,2\beta))$; if $\beta=0$, $\delta=0$ and iterating the estimate does not give anything. Up to now, we only have an estimate that was proved in (A):

$$\|\vec{u}_\epsilon(t,.)\|_2^2 \le \frac{\|\vec{u}_0\|_2^2}{\ln(t+e)} + \gamma_2 \frac{\int_0^t \frac{1}{s+e}\ \left(\|e^{s\Delta}\vec{u}_0\|_2^2 + \left\{\frac{\int_0^s \|\vec{u}_\epsilon\|_2^2\ d\sigma}{(s+e)\ln(s+e)}\right\}^2\right)\ ds}{\ln(t+e)}.$$

We then use the estimates $\|e^{t\Delta}\vec{u}_0\|_2 \le C(\vec{u}_0)(t+e)^{-\alpha/2}$ with $\alpha > 0$, $\|\vec{u}_\epsilon(t,.)\|_2 \le \|\vec{u}_0\|_2$ and $\int_0^\infty \frac{dt}{(t+e)(\ln(t+e))^2} = 1$ and we obtain

$$\|\vec{u}_\epsilon(t,.)\|_2^2 \le \frac{\|\vec{u}_0\|_2^2}{\ln(t+e)} + \frac{\gamma_2 C(\vec{u}_0)^2}{\alpha(t+e)^\alpha} + \frac{\gamma_2\|\vec{u}_0\|_2^4}{\ln(t+e)}.$$

Thus, we have $\|\vec{u}_\epsilon(t,.)\|_2^2 \le \frac{D(\vec{u}_0)}{\ln(t+e)}$ where the constant $D(\vec{u}_0)$ does not depend on ϵ. We again use Lemma 26.1, with $\varphi = \frac{2}{(t+e)\ln(t+e)}$ and $e^\Phi = (\ln(t+e))^2$. We obtain

$$\|\vec{u}_\epsilon(t,.)\|_2^2 \le \frac{\|\vec{u}_0\|_2^2}{(\ln(t+e))^2} + \gamma_d \frac{\int_0^t \frac{2\ln(s+e)}{s+e}\ \left(\|e^{s\Delta}\vec{u}_0\|_2^2 + \left\{2\frac{\int_0^s \|\vec{u}_\epsilon\|_2^2\ d\sigma}{(s+e)\ln(s+e)}\right\}^2\right)\ ds}{(\ln(t+e))^2}.$$

We then estimate $\int_0^s \|\vec{u}_\epsilon\|_2^2\ d\sigma$ by $\int_0^s \frac{d\sigma}{\ln(\sigma+e)} = [\frac{e+\sigma}{\ln(\sigma+e)}]_0^s + \int_0^s \frac{d\sigma}{(\ln(\sigma+e))^2} = \frac{e+s}{\ln(s+e)} + o(\int_0^s \frac{d\sigma}{\ln(\sigma+e)}) \le C\frac{e+s}{\ln(s+e)}$; hence,

$$\|\vec{u}_\epsilon(t,.)\|_2^2 \le C\Big(\frac{\|\vec{u}_0\|_2^2}{(\ln(t+e))^2} + \frac{C(\vec{u}_0)^2}{t^\alpha\ln(t+e)} + \frac{D(\vec{u}_0)^2}{(\ln(t+e))^2}\Big) \le \frac{E(\vec{u}_0)}{(\ln(t+e))^2}$$

where the constant $E(\vec{u}_0)$ does not depend on ϵ. As a consequence, we have $\int_0^s \|\vec{u}_\epsilon\|_2^2\ d\sigma \le C\ E(\vec{u}_0)^2(s+e)(\ln(s+e))^{-2}$. Thus, using Lemma 26.1 with $\varphi = \gamma(t+1)^{-1}$ and $e^\Phi = (t+1)^\gamma$ with $\gamma > \alpha$, we get

$$\begin{aligned} \|\vec{u}_\epsilon(t,.)\|_2^2 \quad & \le \frac{\|\vec{u}_0\|_2^2}{(t+1)^\gamma} + \gamma_d \frac{\int_0^t \gamma(1+s)^{\gamma-1}\ \left(\|e^{s\Delta}\vec{u}_0\|_2^2 + \left\{\gamma\frac{\int_0^s \|\vec{u}_\epsilon\|_2^2\ d\sigma}{(s+1)}\right\}^2\right)\ ds}{(t+1)^\gamma} \\ & \le \frac{\|\vec{u}_0\|_2^2}{(t+1)^\gamma} + \gamma_d\frac{\gamma}{\gamma-\alpha}\frac{C(\vec{u}_0)^2}{(1+t)^\alpha} + CE(\vec{u}_0)^2\frac{\int_0^t \frac{(1+s)^{\gamma-2}}{(\ln(s+e))^2}\ [\int_0^s \|\vec{u}_\epsilon\|_2^2\ d\sigma]\ ds}{(t+1)^\gamma}. \end{aligned}$$

Now, we define $\alpha_{N,\epsilon} = \max_{N\le t\le N+1}\|\vec{u}_\epsilon(t,.)\|_2^2$. We have $\alpha_{0,\epsilon} \le \|\vec{u}_0\|_2^2$ and, for $1 \le N$,

$$\alpha_{N,\epsilon} \le F(\vec{u}_0)N^{-\alpha} + G(\vec{u}_0)N^{-\gamma}\sum_{0\le k\le N}(1+k)^{\gamma-2}(\ln(k+e))^{-2}\sum_{0\le p\le k}\alpha_{p,\epsilon}.$$

For $\gamma > 1$, we have $\sum_{0\le k\le N}(1+k)^{\gamma-2}(\ln(k+e))^{-2} \le C(\gamma)N^{\gamma-1}(\ln N)^{-2}$ and thus,

$$\alpha_{N,\epsilon} \le F(\vec{u}_0)N^{-\alpha} + C(\gamma)\ G(\vec{u}_0)\frac{1}{N(\ln N)^2}\sum_{0\le k\le N}\alpha_{k,\epsilon}.$$

If N_0 is chosen so that $2C(\gamma)G(\vec{u}_0)\sum_{N\geq N_0}\frac{1}{N(\ln N)^2}\leq 1$, we obtain for $N\geq N_0$

$$\sum_{k=0}^{N}\alpha_{k,\epsilon}\leq N_0\|\vec{u}_0\|_2^2+F(\vec{u}_0)\sum_{k=N_0}^{N}k^{-\alpha}+\frac{1}{2}\sum_{k=0}^{N}\alpha_{k,\epsilon}$$

and we get that for some exponent $\delta\in[0,1)$ and some constant $H(\vec{u}_0)$ we have $\sum_{k=0}^{N}\alpha_{k,\epsilon}\leq H(\vec{u}_0)N^\delta$. We get that $\int_0^t\|\vec{u}_\epsilon(t,.)\|_2^2\,ds\leq CH(\vec{u}_0)(1+t)^\delta$, giving us

$$\frac{\int_0^t(1+s)^{\gamma-3}\left\{\int_0^s\|\vec{u}_\epsilon\|_2^2\,d\sigma\right\}^2ds}{(t+1)^\gamma}\leq\frac{1}{2-2\delta}H(\vec{u}_0)^2(1+t)^{-2+2\delta}$$

and we get that $\|\vec{u}_(\epsilon,.)\|_2^2\leq I(\vec{u}_0)(1+t)^{-\beta}$ with $\beta=\min(\alpha,2-2\delta)>0$.

Step 4: Control of the fluctuation

We now prove that, when $\alpha<(d+2)/2$, the tendency $e^{t\Delta}\vec{u}_0$ prevails over the fluctuation $\vec{w}=-B(\vec{u},\vec{u})$ when t goes to $+\infty$: when $\|e^{t\Delta}\vec{u}_0\|_2=O(t^{-\alpha/2})$, $\|\vec{u}_\epsilon-e^{t\Delta}\vec{u}_0\|_2=o(t^{-\alpha/2})$. Let us define $\vec{w}_\epsilon=\vec{u}_\epsilon-e^{t\Delta}\vec{u}_0$. We have

$$\partial_t\vec{w}_\epsilon=\Delta\vec{w}_\epsilon-\mathbb{P}\vec{\nabla}.[(\vec{u}_\epsilon*\omega_\epsilon)\otimes\vec{u}_\epsilon];$$

hence,

$$\begin{aligned}\tfrac{d}{dt}\|\vec{w}_\epsilon(t,.)\|_2^2 &= -2\|\vec{\nabla}\otimes w_\epsilon(t,.)\|_2^2+2\langle\vec{\nabla}\otimes\vec{w}_\epsilon|(\vec{u}_\epsilon*\omega_\epsilon)\otimes\vec{u}_\epsilon\rangle\\ &= -2\|\vec{\nabla}\otimes w_\epsilon(t,.)\|_2^2-2\langle\vec{\nabla}\otimes e^{t\Delta}\vec{u}_0|(\vec{u}_\epsilon*\omega_\epsilon)\otimes\vec{u}_\epsilon\rangle\end{aligned}$$

We now work for $t\geq 1$. We again consider a positive continuous function φ on $[1,\infty)$ and we define $\Phi(t)=\int_1^t\varphi(s)\,ds$. Then,

$$\begin{aligned}\tfrac{d}{dt}(e^\Phi\|\vec{w}_\epsilon\|_2^2) &= e^\Phi(\varphi\|\vec{w}_\epsilon\|_2^2-2\|\vec{\nabla}\otimes\vec{w}_\epsilon\|_2^2-2\langle\vec{\nabla}\otimes e^{t\Delta}\vec{u}_0|(\vec{u}_\epsilon*\omega_\epsilon)\otimes\vec{u}_\epsilon\rangle)\\ &\leq e^\Phi(\varphi\|\mu(\sqrt{\tfrac{2}{\varphi(t)}}\,D)\vec{w}_\epsilon\|_2^2-2\langle\vec{\nabla}\otimes e^{t\Delta}\vec{u}_0|(\vec{u}_\epsilon*\omega_\epsilon)\otimes\vec{u}_\epsilon\rangle).\end{aligned}$$

Since $\vec{w}_\epsilon=-\int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.[(\vec{u}_\epsilon*\omega_\epsilon)\otimes\vec{u}_\epsilon]\,ds$, we again find that

$$\|\mu(\sqrt{\frac{2}{\varphi(t)}}\,D)\vec{w}_\epsilon\|_2^2\leq C\varphi^{(d+2)/2}\,(\int_0^t\|\vec{u}_\epsilon\|_2^2ds)^2.$$

On the other hand, we have $\|\vec{\nabla}\otimes e^{t\Delta}\vec{u}_0\|_\infty\leq Ct^{-\frac{\alpha+1+d/2}{2}}\|\vec{u}_0\|_{\dot{B}_2^{-\alpha,\infty}}$ since $\dot{B}_2^{-\alpha,\infty}\subset\dot{B}_\infty^{-\alpha-d/2,\infty}$. Hence,

$$|\langle\vec{\nabla}\otimes e^{t\Delta}\vec{u}_0|(\vec{u}_\epsilon*\omega_\epsilon)\otimes\vec{u}_\epsilon\rangle|\leq Ct^{-\frac{\alpha+1+d/2}{2}}\|\vec{u}_0\|_{\dot{B}_2^{-\alpha,\infty}}\|\vec{u}_\epsilon\|_2^2.$$

Thus, we have the inequality

$$\frac{d}{dt}(e^\Phi\|\vec{w}_\epsilon\|_2^2)\leq C\varphi e^\Phi\big(\varphi^{(d+2)/2}\,(\int_0^t\|\vec{u}_\epsilon\|_2^2ds)^2+\varphi^{-1}t^{-\frac{\alpha+1+d/2}{2}}\|\vec{u}_0\|_{\dot{B}_2^{-\alpha,\infty}}\|\vec{u}_\epsilon\|_2^2\big).$$

We know that $\|\vec{u}_\epsilon(t,.)\|_2 \leq C(\vec{u}_0)(1+t)^{-\alpha/2}$, so that we have for $t \geq 1$ and $\alpha \neq 1$ $\int_0^t \|\vec{u}_\epsilon\|_2^2\, ds \leq D(\vec{u}_0) t^{\max(1-\alpha,0)}$, where the constant $D(\vec{u}_0)$ does not depend on ϵ. We then take $\varphi = \gamma t^{-1}$ with $\gamma > \max(\frac{d+2}{2}, \frac{3\alpha}{2} + \frac{d-2}{4})$. Thus, $e^\Phi = t^\gamma$ and we have

$$t^\gamma \|\vec{w}_\epsilon(t,.)\|_2^2 \leq \|\vec{w}_\epsilon(1,.)\|_2^2 + CD(\vec{u}_0)^2 t^{\gamma-\beta_0} + CC(\vec{u}_0)^2 \|\vec{u}_0\|_{\dot{B}_2^{-\alpha,\infty}} t^{\gamma-\beta_1}$$

with

$$\begin{cases} \beta_0 = \frac{d+2}{2} - \max(2-2\alpha, 0) \\ \beta_1 = \frac{3\alpha}{2} + \frac{d-2}{4} \end{cases}$$

If $\alpha \in (0, (d+2)/2)$ or if $\alpha = 0$ and $d \geq 3$, γ, β_0 and β_1 are greater than α, so we get the result (notice that $\|\vec{w}_\epsilon(1,.)\|_2 \leq \|\vec{u}_\epsilon(1,.)\|_2 + \|e^\Delta \vec{u}_0\|_2 \leq 2\|\vec{u}_0\|_2$ and that there exist a subsequence $\vec{w}_{\epsilon_n}$ so that, for almost every t, $\vec{w}_{\epsilon_n}(t,.)$ in weakly convergent to $\vec{w}(t,.)$ in $(L^2)^d$). If $\alpha = 0$ and $d = 2$, we must use the estimates $\lim_{t\to+\infty} \sup_{\epsilon>0} \frac{1}{t} \int_0^t \|\vec{u}_\epsilon\|_2^2\, ds = 0$ and $\lim_{t\to+\infty} t\|\vec{\nabla} \otimes e^{t\Delta}\vec{u}_0\|_\infty = 0$.
□

3. Optimal decay rate for the L^2 norm

We now prove that the results in Theorem 26.1 are optimal, in the sense that, generically, we cannot achieve a better decay.

We begin with the description of the asymptotic behavior of the solutions as t goes to $+\infty$ (a result of Carpio [CAR 96] and of Fujigaki and Miyakawa [FUJM 01]).

Theorem 26.2:
Let $\vec{u}_0 \in (L^2(\mathbb{R}^d) \cap \dot{B}_2^{-(d+2)/2,\infty})^d$ with $\vec{\nabla}.\vec{u}_0 = 0$. Let $\vec{u}$ be a retricted Leray weak solution for the Navier–Stokes equations with initial value $\vec{u}_0$. Then:
(A) $\vec{u}$ satisfies the inequality $\sup_{t>0}(1+t)^{\frac{d+2}{4}}\|\vec{u}(t,.)\|_2 < \infty$.
(B) Let $\vec{U}_0 = \vec{u}_0 - \mathbb{P}\vec{\nabla}.\int_0^\infty \vec{u}\otimes\vec{u}\, ds$. Then, $\lim_{t\to+\infty} t^{\frac{d+2}{4}}\|\vec{u} - e^{t\Delta}\vec{U}_0\|_2 = 0$.

Proof: (A) is a consequence of Theorem 26.1.

To prove (B), we introduce the vector $\vec{z}_\epsilon$ defined by

$$\begin{aligned} \vec{z}_\epsilon &= \int_0^t (e^{t\Delta} - e^{(t-s)\Delta})\mathbb{P}\vec{\nabla}.[(\vec{u}_\epsilon * \omega_\epsilon)\otimes\vec{u}_\epsilon]\, ds \\ &= e^{t\Delta}\mathbb{P}\vec{\nabla}.\int_0^t [(\vec{u}_\epsilon * \omega_\epsilon)\otimes\vec{u}_\epsilon]\, ds + \vec{u}_\epsilon - e^{t\Delta}\vec{u}_0. \end{aligned}$$

We have

$$\partial_t \vec{z}_\epsilon = \Delta\vec{z}_\epsilon - (Id - e^{t\Delta})\mathbb{P}\vec{\nabla}.[(\vec{u}_\epsilon * \omega_\epsilon)\otimes\vec{u}_\epsilon];$$

hence,

$$\begin{aligned} \frac{d}{dt}\|\vec{z}_\epsilon(t,.)\|_2^2 &= -2\|\vec{\nabla}\otimes z_\epsilon(t,.)\|_2^2 + 2\langle\vec{\nabla}\otimes(Id - e^{t\Delta})\vec{z}_\epsilon|(\vec{u}_\epsilon * \omega_\epsilon)\otimes\vec{u}_\epsilon\rangle \\ &= -2\|\vec{\nabla}\otimes w_\epsilon(t,.)\|_2^2 - 2\langle\vec{\nabla}\otimes e^{t\Delta}\vec{Z}_\epsilon|(\vec{u}_\epsilon * \omega_\epsilon)\otimes\vec{u}_\epsilon\rangle \end{aligned}$$

with $\vec{Z}_\epsilon = \vec{z}_\epsilon + \vec{u}_0 - \mathbb{P}\vec{\nabla}.\int_0^t [(\vec{u}_\epsilon * \omega_\epsilon) \otimes \vec{u}_\epsilon]\ ds$. We now work for $t \geq 1$. We again consider a positive continuous function φ on $[1, \infty)$ and we define $\Phi(t) = \int_1^t \varphi(s)\ ds$, arriving at

$$\begin{aligned} \tfrac{d}{dt}(e^\Phi \|\vec{z}_\epsilon\|_2^2) &= e^\Phi(\varphi\|\vec{z}_\epsilon\|_2^2 - 2\|\vec{\nabla}\otimes\vec{z}_\epsilon\|_2^2 - 2\langle\vec{\nabla}\otimes e^{t\Delta}\vec{Z}_\epsilon|(\vec{u}_\epsilon * \omega_\epsilon)\otimes\vec{u}_\epsilon\rangle) \\ &\leq e^\Phi(\varphi\|\mu(\sqrt{\tfrac{2}{\varphi(t)}}\ D)\vec{z}_\epsilon\|_2^2 - 2\langle\vec{\nabla}\otimes e^{t\Delta}\vec{Z}_\epsilon|(\vec{u}_\epsilon * \omega_\epsilon)\otimes\vec{u}_\epsilon\rangle). \end{aligned}$$

Since $(e^{t\Delta} - e^{(t-s)\Delta})\mathbb{P}\vec{\nabla}.[(\vec{u}_\epsilon * \omega_\epsilon)\otimes\vec{u}_\epsilon] = \mathbb{P}\vec{\nabla}.(e^{t\Delta} - e^{(t-s)\Delta})[(\vec{u}_\epsilon * \omega_\epsilon)\otimes\vec{u}_\epsilon]\ ds$, we get that

$$\begin{aligned} \|\mu(\sqrt{\tfrac{2}{\varphi(t)}}\ D)\vec{z}_\epsilon\|_2^2 &\leq C\varphi^{(d+2)/2}\ (\int_0^t \|(e^{t\Delta} - e^{(t-s)\Delta})[(\vec{u}_\epsilon * \omega_\epsilon)\otimes\vec{u}_\epsilon]\|_1 ds)^2 \\ &\leq C\varphi^{(d+2)/2}\ (\int_0^t \tfrac{s}{t}\|\vec{u}_\epsilon\|_2^2 ds)^2. \end{aligned}$$

On the other hand, we have $\|\vec{\nabla}\otimes e^{t\Delta}\vec{u}_0\|_\infty \leq Ct^{-(d+2)/2}\|\vec{u}_0\|_{\dot{B}_2^{-(d+2)/2,\infty}}$ since $\dot{B}_2^{-(d+2)/2,\infty} \subset \dot{B}_\infty^{-d-1,\infty}$. We know that

$$\|\vec{u}_\epsilon(t,.)\|_2 \leq C(\vec{u}_0)(1+t)^{-(d+2)/4}$$

and this gives

$$\begin{aligned} \|\vec{\nabla}\otimes e^{t\Delta}\mathbb{P}\vec{\nabla}.\int_0^t [(\vec{u}_\epsilon * \omega_\epsilon)\otimes\vec{u}_\epsilon]\ ds\|_\infty &\leq Ct^{-(d+1)/2}\|\int_0^t [(\vec{u}_\epsilon * \omega_\epsilon)\otimes\vec{u}_\epsilon]\ ds\|_1 \\ &\leq C'\ C(\vec{u}_0)^2 t^{-(d+1)/2}. \end{aligned}$$

Finally, we have $\|\vec{\nabla}\otimes e^{t\Delta}\vec{z}_\epsilon\|_\infty \leq \|\vec{\nabla}\otimes e^{t\Delta}\vec{u}_0\|_\infty + \|\vec{\nabla}\otimes e^{t\Delta}\vec{u}_\epsilon\|_\infty + \|\vec{\nabla}\otimes e^{t\Delta}\mathbb{P}\vec{\nabla}.\int_0^t [(\vec{u}_\epsilon * \omega_\epsilon)\otimes\vec{u}_\epsilon]\ ds\|_\infty$ and we have $\|\vec{\nabla}\otimes e^{t\Delta}\vec{u}_\epsilon\|_\infty \leq Ct^{-d/4-1/2}\|\vec{u}_\epsilon\|_2 \leq CC(\vec{u}_0)t^{-(d+2)/2}$. Hence, for $t \geq 1$,

$$|\langle\vec{\nabla}\otimes e^{t\Delta}\vec{Z}_\epsilon|(\vec{u}_\epsilon * \omega_\epsilon)\otimes\vec{u}_\epsilon\rangle| \leq D(\vec{u}_0)t^{-\frac{d+1}{2}}\|\vec{u}_\epsilon\|_2^2$$

where the constant $D(\vec{u}_0)$ does not depend on ϵ. Thus, we have the inequality

$$\frac{d}{dt}(e^\Phi\|\vec{z}_\epsilon\|_2^2) \leq C\varphi e^\Phi\left(\varphi^{(d+2)/2}\ (\int_0^t \frac{s}{t}\|\vec{u}_\epsilon\|_2^2 ds)^2 + D(\vec{u}_0)\varphi^{-1}t^{-\frac{d+1}{2}}\|\vec{u}_\epsilon\|_2^2\right).$$

According to the case $d = 2$, it is desirable to replace $\frac{s}{t}$ by $\sqrt{\frac{s}{t}}$. We then again write $\|\vec{u}_\epsilon(t,.)\|_2 \leq C(\vec{u}_0)(1+t)^{-(d+2)/4}$ and thus $\int_0^t \sqrt{\frac{s}{t}}\|\vec{u}_\epsilon\|_2^2 ds \leq CC(\vec{u}_0)^2 t^{-1/2}$. We then take $\varphi = \gamma t^{-1}$ with $\gamma > \max(\frac{d+4}{2}, \frac{2d+1}{2})$ and we get

$$t^\gamma\|\vec{z}_\epsilon(t,.)\|_2^2 \leq \|\vec{z}_\epsilon(1,.)\|_2^2 + CC(\vec{u}_0)^4 t^{\gamma-\beta_0} + CD(\vec{u}_0)C(\vec{u}_0)^2 t^{\gamma-\beta_1}$$

with

$$\begin{cases} \beta_0 = \frac{d+2}{2} + 1 \\ \beta_1 = \frac{d-1}{2} + \frac{d+2}{2} \end{cases}$$

Since $\|\vec{z}_\epsilon(1,.)\|_2 \leq C\int_0^1 \|\vec{u}_\epsilon\|_2^2 \, ds + \|\vec{u}_\epsilon(1,.)\|_2 + \|e^{\Delta}\vec{u}_0\|_2 \leq C\|\vec{u}_0\|_2^2 + 2\|\vec{u}_0\|_1$, we find that for some constant $E(\vec{u}_0)$ that does not depend on ϵ we have, for $t \geq 1$, $\|\vec{z}_\epsilon\|_2 \leq E(\vec{u}_0)t^{-(d+2)/4}t^{-1/4}$.

Moreover, we have

$$\begin{aligned} \|e^{t\Delta}\mathbb{P}\vec{\nabla}.\int_t^\infty [(\vec{u}_\epsilon * \omega_\epsilon)\otimes\vec{u}_\epsilon] \, ds\|_2 &\leq Ct^{-(d+2)/4}\int_t^\infty \|u_\epsilon\|_2^2 ds \\ &\leq C'C(\vec{u}_0)^2 t^{-(d+2)/4}t^{-d/2}. \end{aligned}$$

This gives for a constant $F(\vec{u}_0)$ which does not depend on ϵ and for $t \geq 1$

$$t^{(d+2)/4}\|\vec{u}_\epsilon - e^{t\Delta}\vec{u}_0 + e^{t\Delta}\mathbb{P}\vec{\nabla}.\int_0^\infty [(\vec{u}_\epsilon * \omega_\epsilon)\otimes\vec{u}_\epsilon] \, ds\|_2 \leq F(\vec{u}_0)t^{-1/4}.$$

If we take the sequence $\vec{u}_\epsilon$ which strongly converges to $\vec{u}$ in $(L^2_{loc}((0,\infty)\times\mathbb{R}^d))^d$, then $e^{t\Delta}\mathbb{P}\vec{\nabla}.\int_0^\infty[(\vec{u}_\epsilon * \omega_\epsilon)\otimes\vec{u}_\epsilon] \, ds$ weakly converges to $e^{t\Delta}\mathbb{P}\vec{\nabla}.\int_0^\infty[\vec{u}\otimes\vec{u}] \, ds$ in $(L^2(\mathbb{R}^d))^d$. This gives

$$t^{(d+2)/4}\|\vec{u} - e^{t\Delta}\vec{u}_0 + e^{t\Delta}\mathbb{P}\vec{\nabla}.\int_0^\infty [\vec{u}\otimes\vec{u}] \, ds\|_2 \leq F(\vec{u}_0)t^{-1/4}.$$

(B) has been proved. □

In order to describe the consequences of the asymptotic formula given in Theorem 26.2, we prove two useful lemmas:

Lemma 26.2:

Let $\vec{u}_0 \in (L^2(\mathbb{R}^d))^d$ with $\vec{\nabla}.\vec{u}_0 = 0$ and $\int(1+|x|)|\vec{u}_0(x)| \, dx < \infty$. Then:

(A) $\int \vec{u}_0 \, dx = \vec{0}$.

(B) $\vec{u}_0 \in (\dot{B}_2^{-(d+2)/2,\infty})^d$.

(C) If $\vec{u}_0 \in (\dot{B}_2^{-\alpha,\infty})^d$ for some $\alpha > \frac{d+2}{2}$, then $\int x_j\vec{u}_0(x) \, dx = \vec{0}$ for every $j \in \{1,\dots,d\}$.

Proof: (A) is easy: the Fourier transforms $\hat{u}_j$ are continuous and satisfy $\sum_{j=1}^d \xi_j\hat{u}_j(\xi) = 0$. Thus, on $\hat{u}_j(0,\dots,0,\xi_j,0,\dots,0) = 0$ for $\xi_j \neq 0$; by continuity, $\hat{u}_j(0) = 0$ and $\int u_j(x) \, dx = 0$.

(B) is equivalent to $\sup_{j\in\mathbb{Z}}\int_{1/2\leq 2^j|\xi|\leq 1} |\mathcal{F}(\vec{u}_0)|^2 \frac{d\xi}{|\xi|^{d+2}} < \infty$. Since we have $\int(1+|x|)|\vec{u}_0(x)| \, dx < \infty$, the Fourier transform of $\vec{u}_0$ has bounded derivatives, hence is Lipschitzian. Since $\mathcal{F}(\vec{u}_0)(0) = 0$, we have $|\mathcal{F}(\vec{u}_0)(\xi)| \leq C|\xi|$.

We now prove (C). If $\int x_k u_j \neq 0$, then we define the cone

$$\Gamma = \{\xi \ / \ \sum_{l\neq k} \|\partial_l\hat{u}_j\|_\infty|\xi_l| \leq \frac{1}{2}|\partial_k\hat{u}_j(0)||\xi_k|\}$$

We then have

$$\liminf_{j\to+\infty} 2^{j(d+2)}\int_{1/2\leq 2^j|\xi|\leq 1} |\mathcal{F}(\vec{u}_0)|^2 \, d\xi \geq \frac{|\partial_k u_j(0)|^2}{4}\int_{\xi\in\Gamma,1/2\leq|\xi|\leq 1} |\xi_k|^2 \, d\xi > 0$$

and thus $\vec{u}_0$ cannot belong to $(\dot{B}_2^{-\alpha,\infty})^d$ for $\alpha > (d+2)/2$. □

Lemma 26.3:

Let $g \in H^1(\mathbb{R}^d)$ and let $g_t(x) = t^{-d/2} g(x/\sqrt{t})$. Then:

(A) For $f \in L^1(\mathbb{R}^d)$, $\lim_{t\to+\infty} t^{d/4} \|f * g_t - (\int f\ dx) g_t\|_2 = 0$.

(B) For $f \in L^1(\mathbb{R}^d)$ with $\int (1+|x|)|f(x)|\ dx < \infty$ and $\int f\ dx = 0$, $\lim_{t\to+\infty} t^{(d+2)/4} \|f * g_t + \sum_{j=1}^d (\int x_j f\ dx)\partial_j g_t\|_2 = 0$.

Proof: (A) is an easy estimate on rescaling. Let $f_{[t]}(x) = t^{d/2} f(\sqrt{t}\, x)$. Then $f * g_t(x) = t^{-d/2}\int f(y) g((x-y)/\sqrt{t})\ dy = t^{-d/2}(f_{[t]} * g)(x/\sqrt{t})$ and $t^{d/4}\|f * g_t - (\int f\ dx) g_t\|_2 = \|f_{[t]} * g - (\int f\ dx) g\|_2 = \|\int t^{d/2} f(\sqrt{t}\, y)(g(x-y) - g(x))\ dy\|_2 \le \int |f(y)|\ \|g(x - \frac{y}{\sqrt{t}}) - g(x)\|_{L^2(dx)}\ dy$.

We now prove (B). Let $\omega \in \mathcal{D}(\mathbb{R})$ with $\int_{\mathbb{R}} \omega\ dx = 1$. For $f \in L^1(\mathbb{R}^d)$ with $\int (1+|x|)|f(x)|\ dx < \infty$ and for $0 \le k \le d$, we define f_k by $f_0 = f$ and, for $k \ge 1$, $f_k(x) = \int_{\mathbb{R}^k} f(y_1, \dots, y_k, x_{k+1}, \dots, x_d)\ dy_1 \dots dy_k\ \omega(x_1)\dots\omega(x_k)$. Then $f_{k-1} - f_k$ satisfies $\int_{\mathbb{R}} (f_{k-1} - f_k)(x_1, \dots, x_{k-1}, y, x_{k+1}, \dots, x_d)\ dy = 0$ and $\int_{\mathbb{R}} |(f_{k-1} - f_k)(x_1, \dots, x_{k-1}, y, x_{k+1}, \dots, x_d)|\ |y|\ dy \in L^1(\mathbb{R}^{d-1})$; thus, the function F_k, defined by

$$\begin{aligned} F_k(x) &= \int_{-\infty}^{x_k} (f_{k-1} - f_k)(x_1, \dots, x_{k-1}, y, x_{k+1}, \dots, x_d)\ dy \\ &= -\int_{-x_k}^{+\infty} (f_{k-1} - f_k)(x_1, \dots, x_{k-1}, y, x_{k+1}, \dots, x_d)\ dy, \end{aligned}$$

satisfies $F_k \in L^1(\mathbb{R}^d)$ and $\partial_k F_k = f_{k-1} - f_k$; thus, we get $f = f_n + \sum_{k=1}^d \partial_k F_k$ (with $f_n = 0$ if $\int f\ dx = 0$).

If $\int f\ dx = 0$, we get $f * g_t = \sum_{1\le k\le d} (\partial_k F_k) * g_t = t^{-1/2} \sum_{1\le k\le d} F_k * (\partial_k g)_t$. (A) gives then that $\lim_{t\to\infty} t^{(d+2)/4} \|f * g_t - \sum_{1\le k\le d} (\int F_k\ dx)\partial_k g_t\|_2 = 0$. Now, we have $\hat{f} = \sum_{1\le k\le d} i\xi_k \hat{F}_k$; hence, $\frac{\partial}{\partial \xi_k}\hat{f}(0) = i\int F_k\ dx$; since $\frac{\partial}{\partial \xi_k}\hat{f}(0) = -i\int x_k\ f(x)\ dx$, (B) is proved. □

We may now state the result of Miyakawa and Schonbek [MIYS 01]:

Theorem 26.2:

Let $\vec{u}_0 \in (L^2(\mathbb{R}^d))^d$ with $\vec{\nabla}.\vec{u}_0 = 0$ and $\int (1+|x|)|\vec{u}_0(x)|\ dx < \infty$. Let $\vec{u}$ be a retricted Leray weak solution for the Navier–Stokes equations with initial value $\vec{u}_0$. Define, for $1 \le j,k \le d$, $\alpha_{j,k} = \int x_k u_{0,j}(x)\ dx$ and $\beta_{j,k} = \int_0^\infty \int u_j(x,s) u_k(x,s)\ dx\ ds$ and let $W = \mathcal{F}^{-1}(e^{-|\xi|^2})$. We then have

$$\lim_{t\to+\infty} t^{\frac{d+2}{4}} \|\vec{u}\|_2 = \|\vec{V}\|_2$$

where, for $1 \le j \le d$,

$$V_j = \sum_{1\le l\le d} (\alpha_{j,l} + \beta_{j,l})\partial_l W - \sum_{1\le l\le d}\sum_{1\le k\le d} \beta_{k,l} \frac{\partial_l \partial_k \partial_j}{\Delta} W.$$

In particular, $\lim_{t\to+\infty} t^{\frac{d+2}{4}} \|\vec{u}\|_2 = 0$ *if and only if the two following conditions are fullfilled:*

(A) $\int x_j \vec{u}_0(x)\, dx = \vec{0}$ *for every* $j \in \{1, \dots, d\}$*.*

(B) The matrix $(\int\int u_j(t,x) u_k(t,x)\, dt\, dx)_{1\leq j,k\leq d}$ *is a scalar multiple of the identity.*

Proof: The formula $\lim_{t\to+\infty} t^{\frac{d+2}{4}} \|\vec{u}\|_2 = \|\vec{V}\|_2$ is a direct consequence of Theorem 26.2 and Lemma 26.3.

In particular, $\lim_{t\to+\infty} t^{\frac{d+2}{4}} \|\vec{u}\|_2 = 0$ if and only if $\vec{V} = O$. We have $\hat{V}_j(\xi) = i\frac{e^{-|\xi|^2}}{|\xi|^2} P_j(\xi)$ with $P_j(\xi) = \sum_{1\leq l\leq d}\sum_{1\leq k\leq d} (\alpha_{j,l} + \beta_{j,l})\xi_l \xi_k^2 - \beta_{k,l}\xi_j\xi_k\xi_l$. $\vec{V} = 0$ if and only if $P_j = 0$ for $1 \leq j \leq d$. For $j \neq k$, $j \neq l$ and $k \neq l$, we have $\partial_j \partial_k \partial_l P_j = -\beta_{k,l} - \beta_{l,k} = -2\beta_{k,l}$. Thus, if $\vec{V} = 0$, we get $\beta_{k,l} = 0$ for $k \neq l$ and $0 = P_j = \beta_{j,j}\xi_j|\xi|^2 + \sum_{1\leq l\leq d} \alpha_{j,l}\xi_l|\xi|^2 - \beta_{l,l}\xi_j\xi_l^2$. For $l \neq j$, we get $\partial_l^3 P_j = \alpha_{j,l}$; thus, $\alpha_{j,l} = 0$ for $j \neq l$ and $P_j = \xi_j \sum_{1\leq l\leq d} (\alpha_{j,j} + \beta_{j,j} - \beta_{l,l})\xi_l^2$. $P_j = 0$ then gives for $l = j$ $\alpha_{j,j} = 0$ and for $l \neq j$ $\beta_{l,l} = \beta_{j,j} + \alpha_{j,j} = \beta_{j,j}$. □

The condition (B) in Theorem 26.3 is not easily checked, since it depends on the solution $\vec{u}$. However, Brandolese could give examples of solutions with a decay of the L^2 norm better than $t^{-(d+2)/4}$, by using symmetric divergence-free vector fields as initial data [BRA 01].

Chapter 27

Uniqueness of L^d solutions

In this chapter, we briefly address the uniqueness problems for mild solutions in $\mathcal{C}([0,T^*),(L^d)^d)$. Given $\vec{u}_0 \in (L^d)^d$, we may not directly construct a solution in $\mathcal{C}([0,T^*),(L^d)^d)$, since B is not continuous on $\mathcal{C}([0,T^*),(L^d)^d)$ (Oru [ORU 98]; see Chapter 28). Solutions are always constructed in a smaller space (see Kato [KAT 84], Giga [GIG 86], Cannone [CAN 95], or Planchon [PLA 96]) and, thus, uniqueness was first granted only in the subspaces of $\mathcal{C}([0,T^*),(L^d)^d)$ where the iteration algorithm was convergent. In 1997, Furioli, Lemarié-Rieusset and Terraneo [FURLT 00] proved uniqueness in $\mathcal{C}([0,T^*),(L^d)^d)$:

Theorem 27.1: (Uniqueness)
If $\vec{u}$ and $\vec{v}$ are two weak solutions of the Navier–Stokes equations on $(0,T^*)\times \mathbb{R}^d$ so that $\vec{u}$ and $\vec{v}$ belong to $\mathcal{C}([0,T^*),(L^d(\mathbb{R}^d))^d)$ and have the same initial value, then $\vec{u}=\vec{v}$.

This theorem was later reproved by many authors through various methods. In this chapter, we detail the proofs of Furioli, Lemarié-Rieusset and Terraneo [FURLT 00], Meyer [MEY 99], and Monniaux [MON 99]. The proof of Lions and Masmoudi [LIOM 98] will be described in the next chapter. This theorem was extended to the case of Morrey-Campanato spaces by Furioli, Lemarié-Rieusset and Terraneo [FURLT 00] and May [MAY 02], as we shall see in the final section.

1. The uniqueness problem

We take a more general approach to the problem of uniqueness.

Definition 27.1: (Regular space)
A regular space is a Banach space X such that we have the continuous embeddings $\mathcal{D}(\mathbb{R}^d)\subset X\subset L^2_{loc}(\mathbb{R}^d)$ and such that moreover:
(a) for all $x_0\in\mathbb{R}^d$ and for all $f\in X$, $f(x-x_0)\in X$ and $\|f\|_X=\|f(x-x_0)\|_X$.
(b) for all $\lambda\in(0,1)$ and for all $f\in X$, $f(\lambda x)\in X$ and $\lambda\|f(\lambda x)\|_X\le C\|f\|_X$ for a constant C which depends neither on λ nor on f.
(c) $\mathcal{D}(\mathbb{R}^d)$ is dense in X.

We have the obvious result:

Lemma 27.1: (Regular spaces and Morrey–Campanato spaces). *Let X be a regular space so that X is continuously embedded into $L^p_{loc}(\mathbb{R}^d)$ for some $1 \leq p \leq \infty$. Then X is continuously embedded in $m^{p,d}$, the closure of the smooth compactly supported functions in the Morrey-Campanato space $M^{p,d}$.*

In particular, every regular space is continuously embedded in $m^{2,d}$.

We then consider the problem of uniqueness in $\mathcal{C}([0,T^*),X^d)$:

Uniqueness problem: *Let X be a regular space. If $\vec{u}$ and $\vec{v}$ are two weak solutions of the Navier–Stokes equations on $(0,T^*)\times\mathbb{R}^d$ so that $\vec{u}$ and $\vec{v}$ belong to $\mathcal{C}([0,T^*),X^d)$ and have the same initial value, then do we have $\vec{u}=\vec{v}$?*

Of course, we do not know whether the uniqueness problem has a positive answer in the maximal regular limit space $m^{2,d}$.

We first easily check that local uniqueness implies global uniqueness:

Lemma 27.2: *Let X be a regular space. Assume that we have local uniqueness:*

if $T^* > 0$ and if $\vec{u}$ and $\vec{v}$ are two weak solutions of the Navier–Stokes equations on $(0,T^*)\times\mathbb{R}^d$ so that $\vec{u}$ and $\vec{v}$ belong to $\mathcal{C}([0,T^*),X^d)$ and have the same initial value, then there exists a positive ϵ so that we have $\vec{u}=\vec{v}$ on $[0,\epsilon]\times\mathbb{R}^d$.

Then we have global uniqueness: $\vec{u}=\vec{v}$ on $[0,T^*)$.

Proof: Let $\tau=\max\{T>0\ \vec{u}=\vec{v}\text{ on }[0,T]\}$. If $\tau<T^*$, we have by continuity that $\vec{u}(\tau)=\vec{v}(\tau)$. Moreover, $\vec{u}(t+\tau)$ and $\vec{v}(t+\tau)$ are solutions of the Navier–Stokes equations on $(0,T^*-\tau)\times\mathbb{R}^d$; hence, by local uniqueness $\vec{u}(\tau+t)=\vec{v}(\tau+t)$ for $0\leq t\leq\epsilon$, which constitutes a contradiction. □

As a consequence, we may obviously check the following result:

Proposition 27.1: *For every $\epsilon>0$, we have uniqueness in $m^{2,d+\epsilon}$. In particular, we have uniqueness in every regular space X so that for some $\alpha\in[0,1)$ and some constant C, we have, for all $f\in X$, $\sup_{0<\lambda<1}\lambda^\alpha\|f(\lambda x)\|_X\leq C\|f\|_X$.*

Proof: We even have uniqueness for solutions in the larger space $(L^\infty((0,T^*),M^{2,d+\epsilon}))^d$. It is enough to check that we have local uniqueness, since the mapping $t\mapsto\vec{u}(t)$ is continuous from $[0,T^*)$ to $(M^{2,d+\epsilon})^d$, endowed with the $*$-weak topology. Hence, Lemma 27.2 may be modified easily to fit to the case of $(L^\infty((0,T^*),M^{2,d+\epsilon}))^d$.

Local uniqueness is then obvious, since we have, for two solutions $\vec{u}$ and $\vec{w}$ with the same initial value, $\vec{w}=\vec{u}-\vec{v}=-B(\vec{w},\vec{v})-B(\vec{u},\vec{w})$. Theorem 17.1 gives that

$$\|\vec{w}\|_{M^{2,d+\epsilon}}\leq\|B(\vec{w},\vec{v})\|_{M^{2,d+\epsilon}}+\|B(,\vec{u},\vec{w})\|_{M^{2,d+\epsilon}}$$

$$\leq CT^{\frac{\epsilon}{2(d+\epsilon)}} \sup_{0<s<t} \|\vec{w}(s)\|_{M^{2,d+\epsilon}} (\sup_{0<s<t} \|\vec{u}(s)\|_{M^{2,d+\epsilon}} + \sup_{0<s<t} \|\vec{v}(s)\|_{M^{2,d+\epsilon}}).$$

This then gives $\vec{u} = \vec{v}$ on $[0, \min(T_0, T^*))$ with $CT_0^{\frac{\epsilon}{2(d+\epsilon)}} (\sup_{0<s<t} \|\vec{u}(s)\|_{M^{2,d+\epsilon}} + \sup_{0<s<t} \|\vec{v}(s)\|_{M^{2,d+\epsilon}}) = 1$. □

Thus, the problem of uniqueness is open only for limit spaces (for which $\lambda \|f(\lambda x)\| \approx \|f\|_X$). According to Lemma 27.2, we focus on local uniqueness. The basic idea in Furioli, Lemarié-Rieusset and Terraneo [FURLT 00] is to split the solutions in tendency and fluctuation, and to use different estimates on each term. More precisely, we consider two mild solutions $\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u}, \vec{u}) = e^{t\Delta}\vec{u}_0 - \vec{w}_1$ and $\vec{v} = e^{t\Delta}\vec{u}_0 - B(\vec{v}, \vec{v}) = e^{t\Delta}\vec{u}_0 - \vec{w}_2$ in $\mathcal{C}([0, T^*), X^d)$ and write $\vec{w} = \vec{u} - \vec{v} = \vec{w}_2 - \vec{w}_1 = -B(\vec{w}, \vec{v}) - B(\vec{u}, \vec{w})$; finally,

$$\vec{w} = B(\vec{w}_1, \vec{w}) + B(\vec{w}, \vec{w}_2) - B(e^{t\Delta}\vec{u}_0, \vec{w}) - B(\vec{w}, e^{t\Delta}\vec{u}_0).$$

Thus the role of the fluctuations is clearly seen: they control the behavior of $\vec{w}$. We use the regularization properties of the heat kernel for the term $e^{t\Delta}\vec{u}_0$ (mainly, that $\lim_{t\to 0} \sqrt{t}\|e^{t\Delta}\vec{u}_0\|_\infty = 0$), while we use the fact that, for $i = 1$ or 2, we have $\lim_{\epsilon\to 0} \sup_{0\leq t\leq \epsilon} \|\vec{w}_i(t)\|_X = 0$; thus we may assume that the norm of $\vec{w}_i$ is very small.

2. Uniqueness in L^d

In this section, we give three proofs of Theorem 27.1. According to Lemma 27.2, we consider only the problem of local uniqueness.

Proof No. 1: The Besov space approach

The construction of solutions in $\mathcal{C}([0, T^*), (L^d(\mathbb{R}^d))^d)$ by Cannone and Planchon aimed at proving that the fluctuation $\vec{u} - e^{t\Delta}\vec{u}_0$ was in a smaller subspace (such as $\mathcal{C}([0, T^*), (Y)^d)$ with Y a Besov space ($Y = \dot{B}_d^{0,1}$ in Cannone [CAN 95] and Furioli, Lemarié-Rieusset and Terraneo [FURLT 00], or $Y = \dot{B}_d^{0,2}$ in Planchon [PLA 96]) or, as one may easily check for a mild solution $\vec{u} \in (L^\infty((0,T); L^d))^d$ so that $\sqrt{t}\vec{u} \in (L^\infty((0,T) \times \mathbb{R}^d))^d$, the Lorentz space $Y = L^{d,1}$). The idea in [FURLT 00] was that the problem of uniqueness could be treated in a *greater* space, since one did not bother to get an estimate especially in the L^d norm, the solutions already having been given.

The example of Le Jan and Sznitman [LEJS 97] is particularly interesting. They consider the limit space $X = \dot{B}_{PM}^{d-1,\infty}(\mathbb{R}^d)$ defined by: $f \in \dot{B}_{PM}^{d-1,\infty}(\mathbb{R}^d)$ iff $\hat{f} \in L^1_{loc}(\mathbb{R}^d)$ and $|\xi|^{d-1}\hat{f} \in L^\infty$, checking in a very simple way that the operator B is well behaved in this space. As a matter of fact, since the Riesz transforms are obviously bounded on this space, we may just look at the scalar bilinear operator $A(u, v) = \int_0^t e^{(t-s)\Delta}\sqrt{-\Delta}(uv)ds$ operating on $L^\infty(X)$. We do not try to prove that $\int_0^t \|e^{(t-s)\Delta}\sqrt{-\Delta}(uv)\|_X ds$ can be controlled; this is too coarse an estimate. The estimate we use is more direct: $\frac{1}{\sqrt{-\Delta}}(uv) \in X$ since (taking the Fourier transform) it is enough to notice that $\frac{1}{|\xi|^{d-1}} * \frac{1}{|\xi|^{d-1}} \leq \frac{C}{|\xi|^{d-2}}$ (an obvious inequality due to radial invariance and homogeneity); then:

$$\begin{aligned} |\mathcal{F}\{A(uv)(t,.)\}(\xi)| &\leq \frac{C}{|\xi|^{d-1}} \sup_{0<s<t} \|u\|_X \sup_{0<s<t} \|v\|_X \int_0^t e^{-(t-s)|\xi|^2} |\xi|^2 ds \\ &\leq \frac{C}{|\xi|^{d-1}} \sup_{0<s<t} \|u\|_X \sup_{0<s<t} \|v\|_X \int_0^\infty e^{-\tau} d\tau \end{aligned}$$

These simple estimates suggested that certain limit spaces could be handled, as far as their norm can be computed *frequency by frequency* or at least by frequency packets. This is the case of the Besov l^∞ spaces:

Lemma 27.3: (Besov spaces and heat kernel)
Let X be a shift-invariant Banach space of distributions so that $X \subset \dot{B}_\infty^{\alpha,\infty}$ for some $\alpha \in \mathbb{R}$. Let the Besov space $\dot{B}_X^{s,\infty}$ be defined for $s+\alpha < 0$ as the space of tempered distributions f so that $f = \sum_{j\in\mathbb{Z}} \Delta_j f$ in $\mathcal{S}'$ and so that $2^{js}\|\Delta_j f\|_X \in l^\infty(\mathbb{Z})$. Then

$$\forall f \in L^\infty((0,\infty), \dot{B}_X^{s,\infty}) \quad \int_0^\infty \Delta e^{\tau\Delta} f(\tau,.)\, d\tau \in \dot{B}_X^{s,\infty}.$$

Proof: According to the Bernstein inequalities, $(-\Delta)^\gamma$ maps $\dot{B}_X^{s,\infty}$ to $\dot{B}_X^{s-2\gamma,\infty}$ for $s+\alpha-2\gamma < 0$, while $\|(-\Delta)^{1-\gamma} e^{t\Delta} g\|_{\dot{B}_X^{s-2\gamma,\infty}} \leq C_\gamma t^{-(1-\gamma)} \|g\|_{\dot{B}_X^{s-2\gamma,\infty}}$ if $\gamma \leq 1$. Thus, we split the integral $F = \int_0^\infty \Delta e^{\tau\Delta} f(\tau,.)\, d\tau$ into $G_A + H_A$, with

$$\left\{ \begin{array}{l} G_A = -\int_0^A (-\Delta)^{1-\gamma} e^{\tau\Delta}\ (-\Delta)^\gamma f(\tau,.)\, d\tau \\ H_A = -\int_A^{+\infty} (-\Delta)^{1+\gamma} e^{\tau\Delta}\ (-\Delta)^{-\gamma} f(\tau,.)\, d\tau \end{array} \right.$$

with $0 < \gamma < \min(1, -s-\alpha)$. We obtain $\|G_A\|_{\dot{B}_X^{s-2\gamma,\infty}} \leq C_\gamma A^\gamma \|f\|_{L_t^\infty \dot{B}_X^{s,\infty}}$ and $\|H_A\|_{\dot{B}_X^{s+2\gamma,\infty}} \leq C_\gamma A^{-\gamma} \|f\|_{L_t^\infty \dot{B}_X^{s,\infty}}$. Since such splitting may be done for any positive A, we get that $F \in [\dot{B}_X^{s-2\gamma,\infty}, \dot{B}_X^{s+2\gamma,\infty}]_{1/2,\infty} = \dot{B}_X^{s,\infty}$. □

In order to use Lemma 27.3, we must apply some conditions for a pointwise product to belong to a Besov space:

Lemma 27.4: (Pointwise product in Besov spaces)
Let $2 \leq p < d$ and let $q \in [p, \frac{dp}{d-p})$. Then the pointwise product boundedly maps $\dot{B}_p^{d/p-1,\infty} \times \dot{B}_q^{0,\infty}$ to $\dot{B}_p^{d/p-1-d/q,\infty}$ and boundedly maps $\dot{B}_p^{d/p-1,\infty} \times \dot{B}_p^{d/p-1,\infty}$ to $\dot{B}_p^{d/p-2,\infty}$.

Proof: We use the paradifferential calculus (see Chapter 3), writing $uv = \dot{\pi}(u,v) + \dot{\pi}(u,v) + \dot{\rho}(u,v)$ with $\dot{\pi}(u,v) = \sum_{j\in\mathbb{Z}}(\sum_{k\leq j-3} \Delta_k u)\Delta_j v$, $\dot{\pi}(u,v) = \sum_{j\in\mathbb{Z}}(\sum_{k<\leq j-3} \Delta_k v \Delta_j u$ and $\dot{\rho} = \sum_{j\in\mathbb{Z}} \sum_{|k-j|\leq 2} \Delta_k u \Delta_j v$.

We assume that $u \in \dot B_p^{d/p-1,\infty}$ and that $v \in L^q$. The term $\dot\pi(v,u)$ is easy to estimate:

$$\|(\sum_{k\le j-3} \Delta_k v)\Delta_j u\|_p \le C 2^{jd/q}\|v\|_{\dot B_\infty^{-d/q,\infty}}\|\Delta_j u\|_p$$

and this gives $\|\dot\pi(v,u)\|_{\dot B_p^{d/p-1-d/q,\infty}} \le C\|u\|_{\dot B_p^{d/p-1,\infty}}\|v\|_{\dot B_\infty^{-d/q,\infty}}$.

On the other hand, we may estimate $\dot\pi(u,v)$ by

$$\|\dot\pi(u,v)\|_{\dot B_p^{d/p-1-d/q,\infty}} \le C\sup_{j\in\mathbb{Z}} 2^{j(d/p-1-d/q)}\|(\sum_{k\le j-3}\Delta_k u)\Delta_j v\|_p$$

and by writing $1/p = 1/q + 1/r$ with $r > d$. According to the Bernstein inequalities, we have that $\|\Delta_k u\|_r \le C2^{k(d/p-d/r)}\|\Delta_k u\|_p \le C2^{k(1-d/r)}\|u\|_{\dot B_p^{d/p-1,\infty}}$; thus,

$$\|(\sum_{k\le j-3}\Delta_k u)\Delta_j v\|_p \le (\sum_{k\le j-3}\|\Delta_k u\|_r\|\Delta_j v\|_q \le C2^{j(1-d/r)}\|u\|_{\dot B_p^{d/p-1,\infty}}\|v\|_{\dot B_q^{0,\infty}};$$

this gives $\|\dot\pi(u,v)\|_{\dot B_p^{d/p-1-d/q,\infty}} \le C\|u\|_{\dot B_p^{d/p-1,\infty}}\|v\|_{\dot B_q^{0,\infty}}$.

Moreover, the Bernstein inequalities give that

$$\|\Delta_l\dot\rho(u,v)\|_p \le C2^{ld/q}\|\Delta_l\dot\rho(u,v)\|_{\frac{pq}{p+q}} \le C'2^{ld/q}\sum_{j\ge l-4}\sum_{|k-j|\le 2}\|\Delta_k u\|_p\|\Delta_j v\|_q$$

and this gives $\|\dot\rho(u,v)\|_{\dot B_p^{d/p-1-d/q,\infty}} \le C\|u\|_{\dot B_p^{d/p-1,\infty}}\|v\|_{\dot B_q^{0,\infty}}$ since $d/p-1>0$.

Thus, the pointwise product is bounded from $\dot B_p^{d/p-1,\infty}\times \dot B_q^{0,\infty}$ to $\dot B_p^{d/p-1-d/q,\infty}$; just by using the embedding $\dot B_q^{0,\infty}\subset \dot B_\infty^{-d/q,\infty}$. Similarly, the pointwise product is bounded from $\dot B_p^{d/p-1,\infty}\times\dot B_p^{d/p-1,\infty}$ to $\dot B_p^{d/p-2,\infty}$: embedding $\dot B_p^{d/p-1,\infty}\subset\dot B_d^{-1,\infty}$ allows the control of $\pi(v,u)$ (and thus of $\dot\pi(u,v)$), while embedding $\dot B_p^{d/p-1,\infty}\subset\dot B_d^{0,\infty}$ gives control of $\dot\rho(u,v)$. $\square$

We now prove uniqueness in L^d. We begin by proving uniqueness in the smaller space $\dot H^{d/2-1}$ ($d\ge 3$). This result is a direct consequence of Lemmas 27.3 and 27.4. We just write that $\dot H^{d/2-1}\subset\dot B_2^{d/2-1,\infty}\subset\dot B_q^{d/q-1,\infty}$ with $d<q<2d/(d-2)$. The proof of uniqueness in $\dot H^{d/2-1}$ is then obvious. We write

$$\vec w = B(\vec w_1,\vec w) + B(\vec w,\vec w_2) - B(e^{t\Delta}\vec u_0,\vec w) - B(\vec w, e^{t\Delta}\vec u_0)$$

where we have $\vec w$, $\vec w_1$, $\vec w_2$ in $L^\infty((0,T^*),(\dot B_2^{d/2-1,\infty})^d)$ and $t^{(1-d/q)/2}e^{t\Delta}\vec u_0$ in $L^\infty((0,T^*),(L^q)^d)$. In order to estimate $B(\vec w_1,\vec w)$, we write $B(\vec w_1,\vec w) = \int_0^t e^{(t-s)\Delta}\Delta\vec z(s)\,ds$ with $\vec z = \frac{1}{\Delta}\mathbb{P}\vec\nabla.\vec w_1\otimes\vec w \in (L^\infty(0,T^*),(\dot B_2^{d/2-1,\infty})^d)$ [according to Lemma 27.4] and we apply Lemma 27.3. We proceed in the same way for $B(\vec w,\vec w_2)$. To estimate $B(e^{t\Delta}\vec u_0,\vec w)$, we use Lemma 27.4 and we write

$$\begin{aligned}&\|B(e^{t\Delta}\vec u_0,\vec w)\|_{\dot B_2^{d/2-1,\infty}}\\ &\le \int_0^t \frac{C}{(t-s)^{(1+d/q)/2}}\|\vec w(s)\|_{\dot B_2^{d/2-1,\infty}}\left(s^{(1-d/q)/2}\|e^{s\Delta}\vec u_0\|_q\right)\frac{ds}{s^{(1-d/q)/2}}.\end{aligned}$$

Thus, we find

$$\|\vec{w}(t)\|_{\dot{B}_2^{d/2-1,\infty}} \leq C \sup_{0<s<t} \|\vec{w}(s)\|_{\dot{B}_2^{d/2-1,\infty}} \sup_{0<s<t} A(s)$$

with

$$A(s) = \|\vec{w}_1(s)\|_{\dot{B}_2^{d/2-1,\infty}} + \|\vec{w}_2(s)\|_{\dot{B}_2^{d/2-1,\infty}} + s^{(1-d/q)/2}\|e^{s\Delta}\vec{u}_0\|_q.$$

Since $\lim_{t\to 0} A(t) = 0$, we get local uniqueness in $\dot{H}^{d/2-1}$.

This proof does not work directly for the space L^d; we should be operating with the Besov space $\dot{B}_d^{0,\infty}$, which is not embedded in L^2_{loc}. We then use the fact that the fluctuation has a better regularity than the tendency, in that it has a bounded norm in a Besov space with a positive regularity exponent:

Lemma 27.5:

The operator B is bounded from $L^\infty((0,T^*),(L^d)^d) \times L^\infty((0,T^*),(L^d)^d)$ to $L^\infty((0,T^*),(\dot{B}_p^{d/p-1,\infty})^d)$ for every $p \in [d/2,\infty]$.

Proof: This is a direct consequence of Lemma 27.3. We just write $B(\vec{u},\vec{v}) = \int_0^t e^{(t-s)\Delta}\Delta\vec{z}(s)\, ds$ with $\vec{z} = \frac{1}{\Delta}\mathbb{P}\vec{\nabla}.\vec{u}\otimes\vec{v}$. We have, for f and g in L^d, $fg \in L^{d/2} \subset \dot{B}_{d/2}^{0,\infty} \subset \dot{B}_p^{d/p-2,\infty}$ for all $p \geq d/2$ according to the Bernstein inequalities. We find that $\vec{z} \in L^\infty((0,T^*),(\dot{B}_p^{d/p-1,\infty})^d)$ and so does $B(\vec{u},\vec{v})$, according to Lemma 27.3. $\square$

The proof of uniqueness in L^d is now obvious. We write

$$\vec{w} = B(\vec{w}_1,\vec{w}) + B(\vec{w},\vec{w}_2) - B(e^{t\Delta}\vec{u}_0,\vec{w}) - B(\vec{w},e^{t\Delta}\vec{u}_0)$$

where we have $\vec{w}$, $\vec{w}_1$, $\vec{w}_2$ in $L^\infty((0,T^*),(\dot{B}_\infty^{d/p-1,\infty})^d)$ for every $p \geq d/2$ (for $\vec{w}_1$ and $\vec{w}_2$, apply Lemma 27.5; for $\vec{w}$, write that $\vec{w} = \vec{u} - \vec{v} = \vec{w}_2 - \vec{w}_1$) and $t^{(1-d/q)/2}e^{t\Delta}\vec{u}_0$ in $L^\infty((0,T^*),(L^q)^d)$ for every $q \geq d$. We then choose $p \in (\max(2,d/2),d)$ and $q \in (d, \frac{pd}{d-p})$. Lemmas 27.3 and 27.4 give the control

$$\|\vec{w}(t)\|_{\dot{B}_p^{d/p-1,\infty}} \leq C \sup_{0<s<t} \|\vec{w}(s)\|_{\dot{B}_p^{d/p-1,\infty}} \sup_{0<s<t} A(s)$$

with

$$A(s) = \|\vec{w}_1(s)\|_{\dot{B}_p^{d/p-1,\infty}} + \|\vec{w}_2(s)\|_{\dot{B}_p^{d/p-1,\infty}} + s^{(1-d/q)/2}\|e^{s\Delta}\vec{u}_0\|_q.$$

Since $\lim_{t\to 0} A(t) = 0$, we get local uniqueness in L^d.

Proof No. 2: The Lorentz space approach

For L^d, there is now a simpler way to achieve uniqueness: while the space $\dot{B}_d^{0,\infty}$ is not good for proving uniqueness, it can be replaced by another l^∞ space: $L^{d,\infty}$ (Meyer [MEY 99]).

Lemma 27.6: (Lorentz spaces and heat kernel)
Let $p \in (1,d)$ and let $1/q = 1/p - 1/d$. Then:

$$\forall f \in L^\infty((0,\infty), L^{p,\infty}) \quad \int_0^\infty \sqrt{-\Delta} e^{\tau\Delta} f(\tau,.)\, d\tau \in L^{q,\infty}.$$

Proof: According to the Sobolev inequalities, $(-\Delta)^{-\gamma}$ maps $L^{p,\infty}$ to $L^{r,\infty}$ for $0 < 2\gamma < d/p$ and $\frac{1}{r} = \frac{1}{p} - \frac{2\gamma}{d}$, while $\|(-\Delta)^{\frac{1}{2}+\gamma} e^{t\Delta} g\|_{L^{r,\infty}} \leq C_\gamma t^{-(\frac{1}{2}-\gamma)} \|g\|_{L^{r,\infty}}$. Thus, we split the integral $F = \int_0^\infty \sqrt{-\Delta} e^{\tau\Delta} f(\tau,.)\, d\tau$ into $G_A + H_A$, with

$$\begin{cases} G_A = \int_0^A (-\Delta)^{1-\gamma} e^{\tau\Delta}\ (-\Delta)^{\gamma-1/2} f(\tau,.)\, d\tau \\ H_A = \int_A^{+\infty} (-\Delta)^{1+\gamma} e^{\tau\Delta}\ (-\Delta)^{-\gamma-1/2} f(\tau,.)\, d\tau \end{cases}$$

with $0 < \gamma < \frac{1}{2}(\frac{1}{p} - \frac{1}{d})$. We obtain $\|G_A\|_{L^{\frac{dq}{d-2\gamma q},\infty}} \leq C_\gamma A^\gamma \|f\|_{L_t^\infty L^{p,\infty}}$ and $\|H_A\|_{L^{\frac{dq}{d+2\gamma q},\infty}} \leq C_\gamma A^{-\gamma} \|f\|_{L_t^\infty L^{p,\infty}}$. Since such splitting may be accomplished for any positive A, we get that $F \in [L^{\frac{dq}{d-2\gamma q},\infty}, L^{\frac{dq}{d+2\gamma q},\infty}]_{1/2,\infty} = L^{q,\infty}$. $\square$

The proof of uniqueness in L^d is now obvious. We write

$$\vec{w} = B(\vec{w}_1, \vec{w}) + B(\vec{w}, \vec{w}_2) - B(e^{t\Delta}\vec{u}_0, \vec{w}) - B(\vec{w}, e^{t\Delta}\vec{u}_0)$$

where we have $\vec{w}$, $\vec{w}_1$, $\vec{w}_2$ in $L^\infty((0,T^*), (L^{d,\infty})^d)$ (for $\vec{w}_1$, write $\vec{w}_1 = e^{t\Delta}\vec{u}_0 - \vec{u} \in L^\infty((0,T^*), (L^d)^d)$ and $L^d \subset L^{d,\infty}$; for $\vec{w}$, write that $\vec{w} = \vec{u} - \vec{v}$) and $t^{(1-d/q)/2} e^{t\Delta}\vec{u}_0$ in $L^\infty((0,T^*), (L^q)^d)$ for every $q \geq d$. Since the pointwise product maps $L^{d,\infty} \times L^{d,\infty}$ to $L^{d/2,\infty}$, Lemma 27.6 gives control of $B(\vec{w}_1, \vec{w})$ and of $B(\vec{w}, \vec{w}_2)$. Moreover, the operator $(f,g) \mapsto (-\Delta)^{-\frac{d}{2q}}(uv)$ boundedly maps $L^{d,\infty} \times L^{q,\infty}$ to $L^{d,\infty}$, according to the Sobolev inequalities, giving

$$\|B(e^{t\Delta}\vec{u}_0, \vec{w})\|_{L^{d,\infty}} \\ \leq \int_0^t \frac{C}{(t-s)^{(1+d/q)/2}} \|\vec{w}(s)\|_{L^{d,\infty}} \left(s^{(1-d/q)/2} \|e^{s\Delta}\vec{u}_0\|_q\right) \frac{ds}{s^{(1-d/q)/2}}.$$

Thus, we find

$$\|\vec{w}(t)\|_{L^{d,\infty}} \leq C \sup_{0<s<t} \|\vec{w}(s)\|_{L^{d,\infty}} \sup_{0<s<t} A(s)$$

with
$$A(s) = \|\vec{w}_1(s)\|_{L^{d,\infty}} + \|\vec{w}_2(s)\|_{L^{d,\infty}} + s^{(1-d/q)/2}\|e^{s\Delta}\vec{u}_0\|_q.$$
Since $\lim_{t\to 0} A(t) = 0$, we get local uniqueness in L^d.

Proof No. 3: The maximal regularity approach

Monniaux [MON 99] has given simpler proof based on the maximal $L^p(L^q)$ regularity property for the heat kernel. We recall this theorem (see Chapter 7, Theorem 7.3):

Theorem 27.3: (Maximal $L^p(L^q)$ regularity for the heat kernel.)

The operator A defined by $f(t,x) \mapsto Af(t,x) = \int_0^t e^{(t-s)\Delta}\Delta f(s,.)\, ds$ is bounded from $L^p((0,T), L^q(\mathbb{R}^d))$ to $L^p((0,T), L^q(\mathbb{R}^d))$ for every $T \in (0,\infty]$, $1 < p < \infty$ and $1 < q < \infty$.

The proof of uniqueness in L^d is now obvious. We select $p \in (2,\infty)$ and write
$$\vec{w} = B(\vec{w}_1, \vec{w}) + B(\vec{w}, \vec{w}_2) - B(e^{t\Delta}\vec{u}_0, \vec{w}) - B(\vec{w}, e^{t\Delta}\vec{u}_0)$$
where, for $T < T^*$, $\vec{w}_1$, $\vec{w}_2$ in $L^\infty((0,T),(L^d)^d)$, $\vec{w}$ in $L^p((0,T),(L^d)^d)$ and $\sqrt{t}e^{t\Delta}\vec{u}_0$ in $(L^\infty((0,T)\times\mathbb{R}^d))^d$. In order to estimate $B(\vec{w}_1,\vec{w})$, we write that $B(\vec{w}_1,\vec{w}) = \int_0^t e^{(t-s)\Delta}\Delta\vec{z}(s)\, ds$ with $\vec{z} = \frac{1}{\Delta}\mathbb{P}\vec{\nabla}.\vec{w}_1 \otimes \vec{w} \in (L^p(0,T),(L^d)^d)$ [according to the Sobolev embedding $\dot{H}^{-1}_{d/2} \subset L^d$] and we apply Theorem 27.3. To accomplish the estimate of $B(e^{t\Delta}\vec{u}_0, \vec{w})$, we write
$$\|B(e^{t\Delta}\vec{u}_0,\vec{w})\|_{L^p_t L^d_x} \leq \| \int_0^t \frac{C}{\sqrt{t-s}} \|\vec{w}(s)\|_d \left(\sqrt{s}\|e^{s\Delta}\vec{u}_0\|_\infty\right) \frac{ds}{\sqrt{s}}\|_p.$$
We write $t \mapsto \|\vec{w}(t)\|_d \in L^p(0,T)$ and $t \mapsto t^{-1/2} \in L^{2,\infty}((0,T))$. Since $p > 2$, we may use the estimates on pointwise product and on convolution in Lorentz spaces $L^{p,q}(\mathbb{R})$ (Chapter 2) and get that pointwise product maps $L^p \times L^{2,\infty}$ to $L^{2p/(p+2),p}$ and that convolution maps $L^{2p/(p+2),p} \times L^{2,\infty}$ to L^p.

Thus, we find for all $T \in (0,T^*)$:
$$\|\vec{w}(t)\|_{L^p_t L^d_x((0,T)\times\mathbb{R}^d)} \leq C\|\vec{w}\|_{L^p_t L^d_x((0,T)\times\mathbb{R}^d)} \sup_{0<s<T} A(s)$$
with
$$A(s) = \|\vec{w}_1(s)\|_{L^d} + \|\vec{w}_2(s)\|_{L^d} + \sqrt{s}\|e^{s\Delta}\vec{u}_0\|_\infty.$$
Since $\lim_{t\to 0} A(t) = 0$, we get local uniqueness in L^d.

3. The case of Morrey–Campanato spaces

The proofs given in Section 2 work in a more general pattern. Furioli, Lemarié-Rieusset and Terraneo [FURLT 00] extended the proof through Besov spaces to the case of Morrey–Campanato spaces by using the Besov spaces over Morrey–Campanato spaces described by Kozono and Yamazaki [KOZY 97]:

Theorem 27.4: *Let $m^{p,d}$ be the closure of the smooth, compactly supported functions in the Morrey–Campanato space $M^{p,d}$. If $p > 2$ and $\vec{u}$ and $\vec{v}$ are two weak solutions of the Navier–Stokes equations on $(0, T^*) \times \mathbb{R}^d$ so that $\vec{u}$ and $\vec{v}$ belong to $\mathcal{C}([0, T^*), (m^{p,d})^d)$ and have the same initial value, then $\vec{u} = \vec{v}$.*

Hence, we have uniqueness in $\mathcal{C}([0, T^*), (Y)^d)$ for all regular space Y in which the test functions are dense and which is embedded into L^p_{loc} for some $p > 2$. We do not prove Theorem 27.4 here, since May [MAY 02] proved a more general result by extending the approach of Monniaux (i.e., by using the maximal $L^p L^q$ property of the heat kernel):

Theorem 27.5: *Let $X_1^{(0)}$ be the closure of the smooth compactly supported functions in the multiplier space X_1. If $\vec{u}$ and $\vec{v}$ are two weak solutions of the Navier–Stokes equations on $(0, T^*) \times \mathbb{R}^d$ so that $\vec{u}$ and $\vec{v}$ belong to $\mathcal{C}([0, T^*), (X_1^{(0)})^d)$ and have the same initial value, then $\vec{u} = \vec{v}$.*

Remark: Let us recall that we have, for all $p > 2$, $M^{p,d} \subset X^1 \subset M^{2,d}$ (Chapter 21).

Proof: We choose $p \in (2, \infty)$. We consider for $T < T^*$ the space $E_T = \{f \in L^2_{loc}((0,T) \times \mathbb{R}^d)) /\ \sup_{x_0 \in \mathbb{R}^d} \|\big(\int_{|x-x_0|<1} |f(x,t)|^2\, dx\big)^{1/2}\|_{L^p((0,T))} < \infty\}$. We then write

$$\vec{w} = B(\vec{w}_1, \vec{w}) + B(\vec{w}, \vec{w}_2) - B(e^{t\Delta}\vec{u}_0, \vec{w}) - B(\vec{w}, e^{t\Delta}\vec{u}_0)$$

where $\vec{w}_1$, $\vec{w}_2$ in $L^\infty((0,T), (X_1)^d)$, $\vec{w}$ in E_T^d and $\sqrt{t} e^{t\Delta}\vec{u}_0$ in $(L^\infty((0,T) \times \mathbb{R}^d))^d$.

We compute the norm in E_T by using a partition of unity: we choose $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with $\operatorname{Supp} \varphi \subset [-1,1]^d$ and $\sum_{k \in \mathbb{Z}^d} \varphi(x-k) = 1$, and we define $\varphi_k(x) = \varphi(x-k)$; then, the norm in E_T is equivalent to the norm $\sup_{k \in \mathbb{Z}^d} \|\varphi_k f\|_{L^p_t L^2_x((0,T) \times \mathbb{R}^d)}$. We define $K = \{k \in \mathbb{Z} \,/\, d(\operatorname{Supp} \varphi, \operatorname{Supp} \varphi_k) \leq 1\}$ and $\psi = \sum_{k \in K} \varphi_k$.

We first estimate $\varphi_k B(\vec{w}_1, \vec{w})$. We write $\vec{w} = \vec{\alpha} + \vec{\beta}$ where we have $\vec{\alpha} = \psi(x-k)\vec{w}$ and $\vec{\beta} = \sum_{l \notin K} \varphi_l(x-k)\vec{w}$. Then:

$$B(\vec{w}_1, \vec{\alpha}) = \int_0^t e^{(t-s)\Delta}(Id - \Delta)\vec{z}(s)\, ds \text{ with } \vec{z} = \frac{1}{Id - \Delta}\mathbb{P}\vec{\nabla}.\vec{w}_1 \otimes \vec{\alpha}$$

We have $\vec{z} \in (L^p(0,T), L^2(\mathbb{R}^d)))^d$ [since we have $\alpha \in (L^p(0,T), L^2(\mathbb{R}^d)))^d$; hence, $\vec{w}_1 \otimes \vec{\alpha} \in (L^p(0,T)L^2(\mathbb{R}^d)))^{d\times d}$]. According to Theorem 27.3,

$$\int_0^t e^{(t-s)\Delta}\Delta\vec{z}(s)\ ds \in (L^p(0,T), L^2(\mathbb{R}^d)))^d.$$

On the other hand, we have

$$\|\int_0^t e^{(t-s)\Delta}\Delta\vec{z}(s)\ ds\|_{L^2(dx)} \leq Ct^{1-1/p}\|\vec{z}\|_{L^pL^2((0,T)\times\mathbb{R}^d)}.$$

In order to estimate $\varphi_k B(\vec{w}_1, \vec{\beta})$, we write

$$\|\varphi_k B(\vec{w}_1, \vec{\beta})\|_{L^p_t L^2} \leq \sum_{l\notin K} \|\varphi_k B(\vec{w}_1, \varphi_{k+l}\vec{w})\|_{L^p_t L^2}$$

We know (Proposition 11.1) that the operators $\frac{1}{\Delta}\partial_i\partial_j\partial_k e^{(t-s)\Delta}$ have integrable kernels $\frac{1}{(t-s)^{(d+1)/2}}K_{i,j,k}(\frac{x}{\sqrt{t-s}})$ $(1 \leq i,j,k \leq d)$, which can be controlled by $\frac{1}{(t-s)^{(d+1)/2}}|K_{i,j,k}(\frac{x}{\sqrt{t-s}})| \leq \frac{C}{(\sqrt{t-s}+|x|)^{d+1}} \leq \frac{C}{|x|^{d+1}}$. Thus, we may control, for $l \notin K$, $\varphi_k B(\vec{w}_1, \varphi_{k+l}\vec{w})$ by

$$\begin{array}{c} |\varphi_k(x) B(\vec{w}_1, \varphi_{k+l}\vec{w})(t,x)| \\ \leq C|l|^{-(d+1)} \int_0^t \|\psi(x-k-l)\vec{w}_1(s,x)\|_{L^2(dx)}\|\varphi(x-k-l)\vec{w}(s,x)\|_{L^2(dx)}\ ds \\ \leq C'|l|^{-(d+1)} \sup_{0<s<t}\|\vec{w}_1(s,x)\|_{L^2_{uloc,x}} \int_0^t \|\varphi(x-k-l)\vec{w}(s,x)\|_{L^2(dx)}\ ds \\ \leq C''t^{1-1/p}|l|^{-(d+1)} \sup_{0<s<t}\|\vec{w}_1(s,x)\|_{L^2_{uloc,x}}\|\varphi(x-k-l)\vec{w}(s,x)\|_{L^p_t L^2_x} \end{array}$$

We then conclude, since $X_1 \subset L^2_{uloc}$ and $\sum_{l\neq 0}|l|^{-d-1} < \infty$, that

$$\|\varphi_k B(\vec{w}_1, \vec{\beta})\|_{L^pL^2} \leq CT^{2-1/p} \sup_{0<s<t}\|\vec{w}_1(s,x)\|_{X_1}\|\vec{w}\|_{E_T}.$$

We now estimate $\varphi_k B(e^{t\Delta}\vec{u}_0, \vec{w})$. We split again $\vec{w}$ in $\vec{w} = \vec{\alpha} + \vec{\beta}$. We first write (since $p > 2$):

$$\begin{aligned} \|B(e^{t\Delta}\vec{u}_0, \vec{\alpha})\|_{L^p_t L^2} &\leq \|\int_0^t \frac{C}{\sqrt{t-s}}\|\vec{\alpha}(s)\|_{L^2}\left(\sqrt{s}\|e^{s\Delta}\vec{u}_0\|_\infty\right)\frac{ds}{\sqrt{s}}\|_p \\ &\leq C'\|\vec{\alpha}\|_{L^pL^2}\sup_{0<s<T}\sqrt{s}\|e^{s\Delta}\vec{u}_0\|_\infty. \end{aligned}$$

In order to estimate $\varphi_k B(\vec{w}_1, \vec{\beta})$, we write

$$\|\varphi_k B(e^{t\Delta}\vec{u}_0, \vec{\beta})\|_{L^p_t L^2} \leq \sum_{l\notin K}\|\varphi_k B(e^{t\Delta}\vec{u}_0, \varphi_{k+l}\vec{w})\|_{L^p_t L^2}$$

which gives

$$\begin{array}{c} |\varphi_k(x) B(e^{t\Delta}\vec{u}_0, \varphi_{k+l}\vec{w})(t,x)| \\ \leq C|l|^{-(d+1)} \int_0^t \|\psi(x-k-l)e^{s\Delta}\vec{u}_0\|_{L^2(dx)}\|\varphi(x-k-l)\vec{w}(s,x)\|_{L^2(dx)}\ ds \\ \leq C'|l|^{-(d+1)} \sup_{0<s<t}\sqrt{s}\|e^{s\Delta}\vec{u}_0\|_\infty \int_0^t \|\varphi(x-k-l)\vec{w}(s,x)\|_{L^2(dx)}\ \frac{ds}{\sqrt{s}} \\ \leq C''t^{1/2-1/p}|l|^{-(d+1)} \sup_{0<s<t}\sqrt{s}\|e^{s\Delta}\vec{u}_0\|_\infty\|\varphi(x-k-l)\vec{w}(s,x)\|_{L^p_t L^2_x} \end{array}$$

and finally that

$$\|\varphi_k B(e^{t\Delta}\vec{u}_0, \vec{\beta})\|_{L^pL^2} \leq CT^{1+1/2-1/p} \sup_{0<s<t} \sqrt{s}\|e^{s\Delta}\vec{u}_0\|_\infty \|\vec{w}\|_{E_T}.$$

Thus, we find for all $T \in (0, T^*)$

$$\|\vec{w}(t)\|_{E_T} \leq C\|\vec{w}\|_{E_T} \sup_{0<s<T} A(s)$$

with

$$A(s) = \|\vec{w}_1(s)\|_{X_1} + \|\vec{w}_2(s)\|_{X_1} + \sqrt{s}\|e^{s\Delta}\vec{u}_0\|_\infty.$$

Since $\lim_{t\to 0} A(t) = 0$, we achieve local uniqueness in $X_1^{(0)}$. □

Chapter 28

Further results on uniqueness of mild solutions

In this chapter, we prove some further results on uniqueness. We first prove that uniqueness in $(\mathcal{C}([0,T],L^d(\mathbb{R}^d))^d$ could not be directly provided by the boundedness of the bilinear operator B: Oru [ORU 98] proved that B is not continuous on $(\mathcal{C}_b([0,T),L^d)^d$ (nor in the subspace of divergence-free vector fields). We then prove two additional uniqueness results: uniqueness in $(L^\infty([0,T],L^d(\mathbb{R}^d))^d$ when $d \geq 4$ (Lions and Masmoudi [LIOM 98]) and uniqueness in $(\mathcal{C}([0,T],B^\infty)^d \cap (L^\infty([0,T],L^2))^d \cap (L^2([0,T],H^1))^d$, when $\vec{u}_0 \in (B^p \cap L^2)^d$ ($p < \infty$) (Chemin [CHE 99], May [MAY 02]), where B^p is the closure of the test functions in $B_p^{d/p-1,\infty}$.

1. Nonboundedness of the bilinear operator B on $\mathcal{C}_b([0,T),(L^d)^d)$

Oru [ORU 98] proved the following result on the bilinear operator B:

Theorem 28.1: (Nonboundedness on $\mathcal{C}([0,T],(L^d)^d)$)
Let $d \geq 3$. Let B be the bilinear operator

$$B(\vec{u},\vec{v})(t,.) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{u}\otimes\vec{v}\,ds$$

and, for $T \in (0,+\infty]$, let $\mathcal{E}_T$ be the space

$$\mathcal{E}_T = \{\vec{f} \in (\mathcal{C}([0,T),\mathbb{R}^d))^d \;/\; \sup_{0<t<T} \|\vec{f}\|_d < \infty \text{ and } \vec{\nabla}.\vec{f} = 0\}.$$

Then B is not bounded from $\mathcal{E}_T \times \mathcal{E}_T$ to $\mathcal{E}_T$.

Proof: If B is continuous on $\mathcal{E}_\infty$, it is continuous on $\mathcal{E}_T$ for all T: if $\vec{u} \in \mathcal{E}_T$ and $\vec{v} \in \mathcal{E}_T$, we may define $\vec{u}_\epsilon \in \mathcal{E}_\infty$ by $\vec{u}_\epsilon(t,x) = \vec{u}(\min(t,T-\epsilon),x)$ and the same for $\vec{v}_\epsilon$; then, on $[0,T-\epsilon]$, we have $B(\vec{u},\vec{v}) = B(\vec{u}_\epsilon,\vec{v}_\epsilon)$. Letting ϵ go to 0 gives control of B on $\mathcal{E}_T$. Moreover, if B is continuous on $\mathcal{E}_{T_0}$ for one given T_0, then it is continuous on $\mathcal{E}_T$ for all T: define for $\vec{u} \in \mathcal{E}_T$ and $\lambda > 0$, $X_\lambda\vec{u}(t,x) =$

$\sqrt{\lambda}\vec{u}(\lambda t, \sqrt{\lambda}x)$; then we have $X_{\frac{T}{T_0}}\vec{u} \in \mathcal{E}_{T_0}$ and $B(\vec{u},\vec{v}) = X_{\frac{T_0}{T}} B(X_{\frac{T}{T_0}}\vec{u}, X_{\frac{T}{T_0}}\vec{v})$. Thus, we may assume $T = 1$.

We then consider $\theta \in \mathcal{S}(\mathbb{R}^d)$ so that θ is radial and its Fourier transorm $\hat{\theta}$ is supported by the ball $\{\xi \ / \ |\xi| \leq 1/2\}$. We define $\vec{f}(t,x) = (f_1, f_2, 0, \ldots, 0)$ with $f_1(t,x) = -\frac{x_2}{1-t}\theta(\frac{x}{\sqrt{1-t}})$ and $f_2(t,x) = \frac{x_1}{1-t}\theta(\frac{x}{\sqrt{1-t}})$. We have $\vec{f} \in \mathcal{E}_1$ and we are going to prove that $B(\vec{f},\vec{f}) \notin \mathcal{E}_1$. Let us write $\vec{g} = B(\vec{f},\vec{f})$. We have $\partial_t \vec{g} = \Delta\vec{g} + \mathbb{P}\vec{\nabla}.\vec{f}\otimes\vec{f}$; if $\vec{g} \in \mathcal{E}_1$, we have $\partial_t\vec{g} \in (L^\infty((0,1), \dot{B}_\infty^{-3,\infty}))^d$, thus $\vec{g}(t,.)$ converges weakly to a limit $\vec{h}(x)$ in $(\mathcal{S}'(\mathbb{R}^d))^d$ as $t \to 1$. Since $\vec{g}(t,.)$ is bounded in $(L^d)^d$ and since L^d is a dual space, we find that $\vec{h} \in (L^d)^d$. But we have $\vec{h} = \int_0^1 e^{(1-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{f}\otimes\vec{f}\, ds$. Let us compute h_d, the dth component of $\vec{h}$:

$$h_d(x) = -\int_0^1 e^{(1-s)\Delta}\frac{\partial_d}{\Delta}(\partial_1^2 f_1^2 + 2\partial_1\partial_2(f_1 f_2) + \partial_2^2 f_2^2)\, ds.$$

The Fourier transform of h_d is given by

$$\hat{h}_d(\xi) = -\frac{i}{(2\pi)^d}\int_0^1 e^{-(1-s)|\xi|^2}\frac{\xi_d}{|\xi|^2}(\xi_1^2\ \hat{f}_1 * \hat{f}_1 + 2\xi_1\xi_2\ \hat{f}_1 * \hat{f}_2 + \xi_2^2\ \hat{f}_2 * \hat{f}_2)\, ds.$$

We call $\omega_{j,k} = \frac{\partial}{\partial\xi_j}\hat{\theta} * \frac{\partial}{\partial\xi_k}\hat{\theta}$ and we define

$$\Omega(\xi) = \frac{\xi_d}{|\xi|^2}(\xi_1^2\ \omega_{2,2}(\xi) - 2\xi_1\xi_2\ \omega_{1,2}(\xi) + \xi_2^2\ \omega_{1,1}(\xi)).$$

We then may write

$$\hat{h}_d(\xi) = -\frac{i}{(2\pi)^d}\int_0^1 e^{-(1-s)|\xi|^2}(1-s)^{(d-3)/2}\Omega(\sqrt{1-s}\ \xi)\, ds$$

and, finally, writing $\sigma = (1-s)|\xi|^2$,

$$\hat{h}_d(\xi) = -\frac{i}{(2\pi)^d}\ \frac{1}{|\xi|^{d-1}}\ \int_0^{|\xi|^2} e^{-\sigma}\sigma^{(d-2)/2}\Omega(\sqrt{\sigma}\ \frac{\xi}{|\xi|})\, ds.$$

The support of Ω is contained in the ball $\bar{B}(0,1)$; thus, $\hat{h}_d$ is equal for $|\xi| \geq 1$ to $m(\xi)$ where

$$m(\xi) = -\frac{i}{(2\pi)^d}\ \frac{1}{|\xi|^{d-1}}\ \int_0^1 e^{-\sigma}\sigma^{(d-2)/2}\Omega(\sqrt{\sigma}\ \frac{\xi}{|\xi|})\, ds.$$

If M is the inverse Fourier transform of m, we find that $h_d = M + \rho$ where ρ is a smooth function (since its Fourier transform is compactly supported). But M is a homogeneous distribution with degree -1 ($M(\lambda x) = \frac{1}{\lambda}M(x)$ for $\lambda > 0$); hence, it cannot belong to L^d in the neighborhood of 0 unless it is identically equal to 0. Thus, we must prove that m is not a null function. This is easily proved, when we choose $\hat{\theta}(\xi) = \alpha(|\xi|^2)$ with α nonincreasing on $[0, \infty)$. Indeed, we compute $m(0, \xi_2, \ldots, \xi_d) = m(0, \xi')$ as

$$m(0, \xi') = -\frac{i}{(2\pi)^d} \frac{\xi_d \xi_2^2}{|\xi'|^{d+2}} \int_0^1 e^{-\sigma} \sigma^{(d-2)/2} \omega_{1,1}(\sqrt{\sigma}\ (0, \frac{\xi'}{|\xi'|}))\ ds;$$

hence, as

$$m(\xi) = \frac{4i}{(2\pi)^d} \frac{\xi_d \xi_2^2}{|\xi'|^{d+2}} \int_0^1 e^{-\sigma} \sigma^{\frac{d-2}{2}} [\int_{\mathbb{R}^d} \eta_1^2 \alpha'(|\eta|^2) \alpha'(\eta_1^2 + |\sqrt{\sigma} \frac{\xi'}{|\xi|} - \eta'|^2)\ d\eta]\ ds.$$

The integral does not vanish; hence, m is not identically equal to 0, and thus $h_d \notin L^d$. $\square$

2. Uniqueness in $L^\infty(L^d)$ $(d \geq 4)$

The uniqueness theorem in $(\mathcal{C}([0,T], L^d(\mathbb{R}^d))^d$(Furioli, Lemarié-Rieusset and Terraneo [FURLT 00]) described in Chapter 27 was extended in dimension $d \geq 4$ by Lions and Masmoudi [LIOM 98]:

Theorem 28.2: (uniqueness in $(L^\infty([0,T], L^d(\mathbb{R}^d))^d$)
If $\vec{u}$ and $\vec{v}$ are two weak solutions of the Navier–Stokes equations on $(0, T^*) \times \mathbb{R}^d$ so that $\vec{u}$ and $\vec{v}$ belong to $L^\infty([0, T^*), (L^d(\mathbb{R}^d))^d)$ and have the same initial value, then $\vec{u} = \vec{v}$.

Proof: Once again, we just have to prove local uniqueness. If $\vec{u}$ and $\vec{v}$ are equal on $[0, T)$, then we use the fact that they are weakly continuous in $(\mathcal{S}'(\mathbb{R}^d))^d$ to get that the equality is still valid for $t = T$; moreover, since L^d is a dual space and since $\vec{u}$ and $\vec{v}$ are bounded in $(L^d)^d$, we find that we have $\vec{u}(T) = \vec{v}(T)$ in $(L^d)^d$. If we can prove local uniqueness, we are able to get global uniqueness.

The proof then follows a theorem of Von Wahl [WAH 85] for solutions in $(L_t^\infty L_x^d \cap L_t^\infty L_x^2 \cap L_t^2 H_x^1)^d$. In order to prove that $\vec{u} = \vec{v}$ on some interval $[0, \epsilon)$, we may suppose that $\vec{u}$ is the Kato solution in $\mathcal{C}[0, \epsilon], (L^d)^d$; if we consider two bounded solutions in $(L^d)^d$ and if we prove that each bounded solution is equal to the continuous solution in $(L^d)^d$ for some interval, then the bounded solutions are equal on the smallest interval.

The proof of Von Wahl relies on an energy equality on $\vec{w} = \vec{u} - \vec{v}$. First, we write that $\vec{w} \in (L^2((0, T^*), H^1)^d$ and $\vec{u}$ and $\vec{v}$ are in $(L^\infty((0, T^*), L^d))^d$; then, we write $\partial_t w = \Delta \vec{w} - \mathbb{P}\vec{\nabla}(\vec{u} \otimes \vec{u} - \vec{v} \otimes \vec{v}) = \Delta \vec{w} - \mathbb{P}\vec{\nabla}(\vec{w} \otimes \vec{u} + \vec{v} \otimes \vec{w})$; the Sobolev embeddings (when $d \geq 3$) give that $H^1 \subset \dot{H}^1 \subset L^{2d/(d-2)}$ and thus the pointwise product maps $L^\infty L^d \times L^2 H^1$ to $L^2 L^2$; thus we obtain that $\partial_t \vec{w} \in (L^2((0, T^*), H^{-1}))^d$. Thus, we may write that $\partial_t |\vec{w}|^2 = 2\vec{w}.\partial_t \vec{w}$ and integration on x gives:

$$(28.1) \qquad \partial_t \|\vec{w}\|_2^2 + 2 \int_{\mathbb{R}^d} |\vec{\nabla} \otimes \vec{w}|^2\ dx = -2 \int_{\mathbb{R}^d} \vec{w}.([\vec{w}.\vec{\nabla}]\vec{u} + [\vec{v}.\vec{\nabla}]\vec{w})\ dx.$$

We use the equalities $\int_{\mathbb{R}^d} \vec{w}.[\vec{w}.\vec{\nabla}]\vec{u}\,dx = -\int_{\mathbb{R}^d} \vec{u}.[\vec{w}.\vec{\nabla}]\vec{w}\,dx$ and $\int_{\mathbb{R}^d} \vec{w}.[\vec{v}.\vec{\nabla}]\vec{w}\,dx = 0$. Then, we use the continuity of $\vec{u}$ and write $\vec{u} = e^{t\Delta}\vec{u}_0 - \vec{w}_1$ where $\lim_{t\to 0}\|\vec{w}_1\|_d = 0$, yielding

$$(28.2)\quad \frac{1}{2}\partial_t\|\vec{w}\|_2^2 + \int_{\mathbb{R}^d} |\vec{\nabla}\otimes\vec{w}|^2 dx \le |\int_{\mathbb{R}^d} \vec{w}_1.[\vec{w}.\vec{\nabla}]\vec{w}\,dx| + |\int_{\mathbb{R}^d} e^{t\Delta}\vec{u}_0.[\vec{w}.\vec{\nabla}]\vec{w})\,dx|.$$

Pointwise product maps $L^d \times \dot{H}^1 \times L^2$ to L^1; hence,

$$|\int_{\mathbb{R}^d} \vec{w}_1.[\vec{w}.\vec{\nabla}]\vec{w}\,dx| \le C_1\|\vec{w}_1\|_d\|\vec{\nabla}\otimes\vec{w}\|_2^2.$$

On the other hand we have $\vec{w} \in (L^\infty L^2)^d$. We then write $L^\infty L^2 \cap L^2\dot{H}^1 \subset L^\infty L^2 \cap L^2 L^{2d/(d-2)} \subset L^{d(1-2/p)}L^p$ for $2 < p < 2d/(d-2)$. We have $L^d \subset \dot{B}_d^{0,d} \subset B_q^{-d(1/d-1/q),d}$ for $d \ge q$. We select $q = d^2/(d-2)$ and $p = 2q/(q-2)$. The pointwise product maps $L^q \times L^p \times L^2$ to L^1, giving

$$\begin{aligned} |\textstyle\int_{\mathbb{R}^d} e^{t\Delta}\vec{u}_0.[\vec{w}.\vec{\nabla}]\vec{w}\,dx| &\le C\|e^{t\Delta}\vec{u}_0\|_q\|\vec{w}\|_p\|\vec{\nabla}\otimes\vec{w}\|_2 \\ &\le C_2\|e^{t\Delta}\vec{u}_0\|_q\|\vec{w}\|_2^{1-d(1/2-1/p)}\|\vec{\nabla}\otimes\vec{w}\|_2^{1+d(1/2-1/p)}. \end{aligned}$$

We have $d(1/2-1/p) = d/q$, $1-d/q = 2/d$ and $1+d/q = 2-2/d$, which then gives

$$\begin{aligned} |\textstyle\int_{\mathbb{R}^d} e^{t\Delta}\vec{u}_0.[\vec{w}.\vec{\nabla}]\vec{w}\,dx| &\le C\|e^{t\Delta}\vec{u}_0\|_q\|\vec{w}\|_2^{2/d}\|\vec{\nabla}\otimes\vec{w}\|_2^{2(1-1/d)} \\ &\le \tfrac{C_2^d}{d}\|e^{t\Delta}\vec{u}_0\|_q^d\|\vec{w}\|_2^2 + \tfrac{d-1}{d}\|\vec{\nabla}\otimes\vec{w}\|_2^2. \end{aligned}$$

On $[0,\epsilon)$ with ϵ small enough, we have $C_1\|\vec{w}_1\|_d \le 1/d$; thus, (28.2) finally yields

$$(28.3)\qquad \partial_t\|\vec{w}\|_2^2 \le 2\frac{C_2^d}{d}\|e^{t\Delta}\vec{u}_0\|_q^d\|\vec{w}\|_2^2.$$

Since $\vec{u}_0 \in (\dot{B}_q^{-2/d,d})^d$, we have $e^{t\Delta}\vec{u}_0 \in (L^d((0,\infty),L^q))^d$. The Gronwall lemma gives us that $\|\vec{w}(t)\|_2^2 \le \|\vec{w}(0)\|_2^2 e^{\frac{2C_2^d}{d}\int_0^t \|e^{s\Delta}\vec{u}_0\|_q^d\,ds} = 0$.

Now, if we want to prove the theorem of Lions and Masmoudi, we can prove that, if $\vec{u}$ and $\vec{v}$ are two solutions of the Navier–Stokes equations in $(L^\infty((0,T^*),L^d))^d$ with the same initial value $\vec{u}_0$, then $\vec{w} = \vec{u} - \vec{v}$ satisfies $\vec{w} \in (L^\infty L^2 \cap L^2 H^1)^d$ (at least, when $T^* < \infty$, which we may assume, since we are interested in local uniqueness). But it is enough to write that $\vec{w} = -B(\vec{w},\vec{u}) - B(\vec{v},\vec{w})$: if we know that $\vec{w} \in (L^\infty((0,T^*),L^q))^d$ for some $q \ge 1$, we may write that $\vec{w}\otimes\vec{u} \in (L^\infty((0,T^*),L^r))^{d\times d}$ with $r = dq/(d+q)$ if $q \ge d/(d-1)$ and $r = 1$ if $q \le d/(d-1)$ (since we then would have by interpolation between L^d and L^q that $\vec{w} \in (L^\infty((0,T^*),L^{d/(d-1)}))^d$), which gives

$$\|\vec{w}(t)\|_r \le C\sqrt{t}\sup_{0<s<t}\|\vec{w}(s)\|_{\max(q,d/(d-1))}\sup_{0<s<t}\|\vec{v}(s)\|_d + \|\vec{u}(s)\|_d.$$

Therefore, $\vec{w} \in (L^\infty L^q)^d$ for all $q \in [1, d]$. Moreover, we may write

$$(-\Delta)^{1/2}\vec{w} = -\int_0^t e^{(t-s)\Delta}\Delta\big(\frac{1}{(-\Delta)^{1/2}}\mathbb{P}\vec{\nabla}.[\vec{w}\otimes\vec{u}+\vec{v}\otimes\vec{w}]\big)\ ds.$$

For $d/(d-1) < q \leq d$, we have $\vec{w} \in (L^\infty L^q)^d \subset (L^2 L^q)^d$ (since we assume $T^* < \infty$), and thus $\frac{1}{(-\Delta)^{1/2}}\mathbb{P}\vec{\nabla}.[\vec{w}\otimes\vec{u}+\vec{v}\otimes\vec{w}] \in (L^2 L^{dq/(d+q)})^d$. The maximal $L^p L^q$ regularity theorem for the heat kernel (Theorem 7.3) gives then that $\vec{w} \in (L^2 L^r)^d$ with $r = dq/(d+q) \in (1, d/2]$. Provided that $r = 2 \leq d/2$, i.e., $d \geq 4$, the proof is complete. $\square$

3. A uniqueness result in $\dot{B}_\infty^{-1,\infty}$

We end the chapter with a result described in Chemin [CHE 99]:

Theorem 28.3: (Uniqueness in $(\mathcal{C}([0,T], B^{(-1)})^d)$

Let B^p be the closure of the test functions in $B_p^{d/p-1,\infty}$, and let $\vec{u}_0 \in (B^p \cap L^2)^d$ for some $p < \infty$, with $\vec{\nabla}.\vec{u}_0 = 0$.

If $\vec{u}$ and $\vec{v}$ are two weak solutions of the Navier–Stokes equations on $(0,T^*)\times \mathbb{R}^d$ such that $\vec{u}$ and $\vec{v}$ belong to the space $(\mathcal{C}([0,T^*), B^\infty)^d \cap (L^\infty([0,T^*), L^2))^d$ $\cap (L^2([0,T^*), H^1))^d$ and have the same initial value $\vec{u}_0$, then $\vec{u} = \vec{v}$.

Remark: Chemin proved the theorem for $d = 3$ using energy estimates proved by Chemin and Lerner [CHEL 95]. May [MAY 02] demonstrated that the proof was easily extended to greater values of d by using the maximal $L^p L^q$ regularity of the heat kernel instead of energy estimates.

We always assume $p > d$ and $T^* < \infty$, with no loss of generality since $B^p \subset B^q$ for $q > p$.

The proof is based on the paradifferential calculus of Bony [BON 81]. More precisely, let π and ρ be the operators $\pi(f,g) = \sum_{j\in\mathbb{N}} S_{j-2}f\Delta_j g$ and $\rho(f,g) = S_0 f S_0 g + \Delta_{-2}f\Delta_0 g + \Delta_{-1}f\Delta_0 g + \Delta_{-1}f\Delta_1 g + \sum_{j\in\mathbb{N}}\Delta_j f(\sum_{j-2\leq k\leq j+2}\Delta_k g)$, so that $fg = \pi(f,g) + \pi(g,f) + \rho(f,g)$. We define the paradifferential decompositions of the bilinear B, B_π, B_{π^*} and B_ρ, by the formulas

$$\begin{cases} B_\pi(\vec{f},\vec{g}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.M_\pi(\vec{f},\vec{g})\ ds \quad \text{with} \quad M_\pi(\vec{f},\vec{g}) = (\pi(f_j,g_k))_{1\leq j,k\leq d} \\ B_{\pi^*}(\vec{f},\vec{g}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.M_{\pi^*}(\vec{f},\vec{g})\ ds \quad \text{with} \quad M_{\pi^*}(\vec{f},\vec{g}) = (\pi(g_k,f_j))_{1\leq j,k\leq d} \\ B_\rho(\vec{f},\vec{g}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.M_\rho(\vec{f},\vec{g})\ ds \quad \text{with} \quad M_\pi(\vec{f},\vec{g}) = (\rho(f_j,g_k))_{1\leq j,k\leq d} \end{cases}$$

so that we have $B(\vec{f},\vec{g}) = B_\pi(\vec{f},\vec{g}) + B_{\pi^*}(\vec{f},\vec{g}) + B_\rho(\vec{f},\vec{g})$.

The proof begins by describing more precisely the regularity of s solution $\vec{u}$ of the Navier–Stokes equations in $(L^\infty([0,T^*), L^2))^d \cap (L^2([0,T^*), H^1))^d$:

Lemma 28.1:

i) If $\vec{f}$ *and* $\vec{g}$ *belong to* $(L^\infty([0,T^*),L^2))^d \cap (L^2([0,T^*),H^1))^d$*, then* $\vec{h} = B(\vec{f},\vec{g})$ *belongs to* $(L^{2/r}([0,T^*),H^{1+r-\alpha}_{d/(d-\alpha)}))^d$ *for* $0<\alpha<r\leq 1$.

ii) For $f\in B^{-1,\infty}_\infty$*, the operator* $g\mapsto \pi(f,g)$ *is bounded from* H^s_p *to* H^{s-1}_p *for every* $p\in(1,\infty)$ *and* $s\in\mathbb{R}$ *and* $g\mapsto\rho(f,g)$ *is bounded from* H^s_p *to* H^{s-1}_p *for every* $p\in(1,\infty)$ *and* $s>1$.

iii) For $\vec{f}\in L^\infty((0,T^*),(B^{-1,\infty}_\infty)^d)$*, the operator* $\vec{g}\mapsto B_\pi(\vec{f},\vec{g})$ *is bounded on* $L^q((0,T^*),(H^s_p)^d)$ *for every* p *and* q *in* $(1,\infty)$ *and every* $s\in\mathbb{R}$ *and the operator* $\vec{g}\mapsto B_\rho(\vec{f},\vec{g})$ *is bounded on* $L^q((0,T^*),(H^s_p)^d)$ *for every* p *and* q *in* $(1,\infty)$ *and every* $s>1$.

iv) Let $\vec{u}_0\in(B^{-1,\infty}_\infty\cap L^2)^d$*, with* $\vec{\nabla}.\vec{u}_0=0$*. If* $\vec{u}$ *is a weak solution of the Navier–Stokes equations on* $(0,T^*)\times\mathbb{R}^d$ *so that* $\vec{u}$ *belongs to the space* $(L^\infty([0,T^*),B^{-1,\infty}_\infty))^d\cap(L^\infty([0,T^*),L^2))^d\cap(L^2([0,T^*),H^1))^d$ *and has initial value* $\vec{u}_0$*, then* $\vec{w}=e^{t\Delta}\vec{u}_0-\vec{u}$ *is a solution of the fixed point-problem in* $L^q((0,T^*),(H^\sigma_p)^d)$ *(with* $q\in(2,\infty)$*,* $\sigma=1+1/q$ *and* $p=d/(d-1/q)$*)*

$$\vec{w}=-2B_\pi(\vec{u},\vec{w})-B_\rho(\vec{u},\vec{w})+\vec{w}_0$$

with

$$\vec{w}_0=2B_\pi(\vec{u},e^{t\Delta}\vec{u}_0)+B_\rho(\vec{u},e^{t\Delta}\vec{u}_0)\in L^q((0,T^*),(H^\sigma_p)^d).$$

Proof: We first study the pointwise product between two functions f and g in H^r with $0<r\leq 1$. We write $fg=\pi(f,g)+\pi(g,f)+\rho(f,g)$. Now, for $\alpha\in(0,r)$, we define $p_\alpha>2$ by $1/p_\alpha=1/2-\alpha/d$ and we write that $L^2\subset B^{-\alpha,2}_{p_\alpha}$ (due to the Bernstein inequalities) and this gives that $(2^{-j\alpha}\|\Delta_j f\|_{p_\alpha})_{j\in\mathbb{N}}\in l^2$ and $(2^{-j\alpha}\|S_j f\|_{p_\alpha})_{j\in\mathbb{N}}\in l^2$. Since $(2^{jr}\|\Delta_j g\|_2)_{j\in\mathbb{N}}\in l^2$ and since we have $r-\alpha>0$, we find that $\|\pi(f,g)\|_{B^{r-\alpha,1}_{d/(d-\alpha)}}\leq C\|f\|_2\|g\|_{H^r}$ and $\|\rho(f,g)\|_{B^{r-\alpha,1}_{d/(d-\alpha)}}\leq C\|f\|_2\|g\|_{H^r}$. Thus, we find that $fg\in H^{r-\alpha}_{d/(d-\alpha)}$, or more accurately $\|fg\|_{H^{r-\alpha}_{\frac{d}{(d-\alpha)}}}\leq C(\|f\|_2\|g\|_{H^r}+\|f\|_{H^r}\|g\|_2)$.

Since $L^\infty_t L^2_x\cap L^2_t H^1_x\subset L^{2/r}_t H^r_x$ for $0<r\leq 1$, we write

$$(\sqrt{-\Delta})^{1+r-\alpha}B(\vec{f},\vec{g})=-\int_0^t e^{(t-s)\Delta}\Delta\vec{z}\,ds$$

with $\vec{z}=(\sqrt{-\Delta})^{r-\alpha-1}\mathbb{P}\vec{\nabla}.\vec{f}\otimes\vec{g}$. We have $\vec{z}\in(L^{2/r}L^{d/(d-\alpha)})^d$. The maximal L^pL^q regularity property of the heat kernel gives then that $B(\vec{f},\vec{g})\in(L^{2/r}((0,T^*),\dot{H}^{1+r-\alpha}_{d/(d-\alpha)}))^d$. Moreover, $B(\vec{f},\vec{g})=-\int_0^t e^{(t-s)\Delta}(-\Delta)^{(1+\alpha-r)/2}\vec{z}\,ds$; hence, $\|B(\vec{f},\vec{g})\|_{d/(d-\alpha)}\leq\int_0^t\frac{C}{(t-s)^{(1+\alpha-r)/2}}\|\vec{z}\|_{d/(d-\alpha)}\,ds$. Since $T^*<\infty$ and thus $s\mapsto\frac{1}{s^{(1+\alpha-r)/2}}\in L^1((0,T^*))$, we find that $B(\vec{f},\vec{g})\in(L^{2/r}L^{d/(d-\alpha)})^d$. Thus, we proved that $B(\vec{f},\vec{g})\in(L^{2/r}((0,T^*),H^{1+r-\alpha}_{d/(d-\alpha)}))^d$.

Point ii) is a classical consequence of the following characterization of the space H^σ_p: $f \in H^\sigma_p$ ($1 < p < \infty$ and $\sigma \in \mathbb{R}$) if and only if $f \in \mathcal{S}'$, $S_0 f \in L^p$ and $(-\Delta)^{\sigma/2}(Id - S_0)f \in L^p$; moreover, for $1 < q < \infty$, $\|f\|_q \sim \|(\sum_{j\in\mathbb{Z}} 4^{-j\sigma}|\Delta_j(-\Delta)^{\sigma/2} f|^2)^{1/2}\|_q$. Now, if $T = \sum_{j\in\mathbb{N}} T_j$ with Supp $\hat{T}_j \subset \{\xi \ / \ 2^{j-3} \leq |\xi| \leq 2^{j+3}\}$ and if $f \in \mathcal{S}$, we write

$$|\langle(-\Delta)^{\sigma/2}T|f\rangle| \leq \sum_{j\in\mathbb{N}} \sum_{|j-k|\leq 5} |\langle T_j|\Delta_k(-\Delta)^{\sigma/2} f\rangle|;$$

hence, $\|(-\Delta)^{\sigma/2}T\|_p \leq C\|(\sum_{j\in\mathbb{N}} 4^{j\sigma}|T_j(x)|^2)^{1/2}\|_p$. This gives the control of $\pi(f,g)$ in H^{s-1}_p when $f \in B^{-1,\infty}_\infty$ (hence $\|S_j f\|_\infty \leq C2^j\|f\|_{B^{-1,\infty}_\infty}$ for $j \in \mathbb{N}$) and $g \in H^s_p$.

Similarly, when $\sigma > 0$ and $T = \sum_{j\in\mathbb{N}} T_j$ with Supp $\hat{T}_j \subset \{\xi \ / \ |\xi| \leq 2^{j+3}\}$, we get $\|T\|_{H^\sigma_p} \leq C\|(\sum_{j\geq 5} 4^{j\sigma}\big(\sum_{k\geq j-5}|T_k(x)|\big)^2)^{1/2}\|_p$. Since the convolution with $(2^{-j\sigma}1_{j\geq -5})_{j\in\mathbb{Z}}$ is a bounded mapping on $l^2(\mathbb{Z})$, we find that we have $\|T\|_{H^\sigma_p} \leq C'\|(\sum_{j\in\mathbb{N}} 4^{j\sigma}|T_j(x)|^2)^{1/2}\|_p$. This gives the control of $\rho(f,g)$ in H^{s-1}_p when $f \in B^{-1,\infty}_\infty$, $g \in H^s_p$ and $s > 1$.

Point iii) is a direct consequence of point ii) and of the maximal L^pL^q regularity property of the heat kernel. We just have to write $\mathbb{P}\vec{\nabla}.M_\pi(\vec{f},\vec{g}) = (Id - \Delta)\vec{Z}$ with $\vec{Z} = (Id-\Delta)^{-1}\mathbb{P}\vec{\nabla}.M_\pi(\vec{f},\vec{g}) \in (L^q((0,T^*), H^s_p))^d$.

Now, if $\vec{u}$ is a weak solution of the Navier–Stokes equations on $(0,T^*)\times\mathbb{R}^d$ so that $\vec{u}$ belongs to the space $(L^\infty([0,T^*), B^{-1,\infty}_\infty))^d \cap (L^\infty([0,T^*), L^2))^d \cap (L^2([0,T^*), H^1))^d$ and has initial value $\vec{u}_0$, and if $q \in (2,\infty)$, we apply point i) with $r = 2/q$ and $\alpha = 1/q$, to get that $\vec{w} = B(\vec{u},\vec{u}) \in (L^q((0,T^*), H^\sigma_p)^d$ with $\sigma = 1 + 1/q$ and $p = d/(d-1/q)$. Moreover, we have $\vec{w} = B(\vec{u},\vec{u}) = 2B_\pi(\vec{u},\vec{u}) + B_\rho(\vec{u},\vec{u}) = 2B_\pi(\vec{u}, e^{t\Delta}\vec{u}_0 - \vec{w}) + B_\rho(\vec{u}, e^{t\Delta}\vec{u}_0 - \vec{w})$. Point iii) gives then that $2B_\pi(\vec{u},\vec{w}) + B_\rho(\vec{u},\vec{w}) \in (L^q((0,T^*), H^\sigma_p)^d$ (since $1 < \sigma$), and so does $\vec{w}_0 = \vec{w} + 2B_\pi(\vec{u},\vec{w}) + B_\rho(\vec{u},\vec{w})$. □

The next step is to study the fixed point equation in another space:

Lemma 28.2:

i) For $f \in B^{-1,\infty}_\infty$, the operator $g \mapsto \pi(f,g)$ is bounded from $B^{s,\infty}_p$ to $B^{s-1,\infty}_p$ for every $p \in [1,\infty]$ and $s \in \mathbb{R}$ and $g \mapsto \rho(f,g)$ is bounded from $B^{s,\infty}_p$ to $B^{s-1,\infty}_p$ for every $p \in [1,+\infty]$ and $s > 1$.

ii) For $p \in (d,\infty)$, $\alpha \in (1, 1+d/p)$ and $T > 0$, we define $\mathcal{E}^{\alpha,p}_T$ as the space

$$\mathcal{E}^{\alpha,p}_T = \{f \in \mathcal{C}_b([0,T), B^p) \ / \ \begin{array}{c} t^{(1+\alpha-d/p)/2} f \in L^\infty((0,T), B^{\alpha,\infty}_p) \\ \textit{and} \\ \lim_{t\to 0} t^{(1+\alpha-d/p)/2}\|f\|_{B^{\alpha,\infty}_p} = 0 \end{array} \}.$$

Then, for $\vec{f} \in L^\infty((0,T), (B^{-1,\infty}_\infty)^d)$, the operators $\vec{g} \mapsto B_\pi(\vec{f},\vec{g})$ and $\vec{g} \mapsto B_\rho(\vec{f},\vec{g})$ are bounded on $(\mathcal{E}^{\alpha,p}_T)^d$ for every $p \in (d,\infty)$ and every $\alpha \in (1, 1+d/p)$.

Proof: Point i) is easily checked. Before proving point ii), we give an exact description of B^p for $p < \infty$: a distribution f belongs to B^p if and only if $f \in \mathcal{S}'_0$, $S_0 f \in L^p$, for all $j \in \mathbb{N}$ $\Delta_j f \in L^p$ and $\lim_{j\to+\infty} 2^{j(1-d/p)}\|\Delta_j f\|_p = 0$. Thus, we have, for $\epsilon > 0$, $B_p^{d/p-1+\epsilon,\infty} \subset B^p$.

If $\vec{f} \in L^\infty((0,T),(B_\infty^{-1,\infty})^d)$ and $t^{(1+\alpha-d/p)/2}\vec{g} \in (L^\infty((0,T),B_p^{\alpha,\infty}))^d$, with $\alpha > 1$, we find that $t^{\frac{(1+\alpha-d/p)}{2}}\mathbb{P}\vec{\nabla}.M_\pi(\vec{f},\vec{g})$ and $t^{\frac{(1+\alpha-d/p)}{2}}\mathbb{P}\vec{\nabla}.M_\rho(\vec{f},\vec{g})$ belong to $(L^\infty((0,T),B_p^{\alpha-2,\infty}))^d$. The norm of $B_\pi(\vec{f},\vec{g})$ in $(B_p^{\gamma,\infty})^d$ for $\gamma \geq \alpha-2$ may be estimated as $\|B_\pi(\vec{f},\vec{g})\|_{B_p^{\alpha-2,\infty}} + \|(-\Delta)^{(\gamma-\alpha+2)/2}B_\pi(\vec{f},\vec{g})\|_{B_p^{\alpha-2,\infty}}$. We have

$$\|(-\Delta)^{\frac{(\gamma-\alpha+2)}{2}}B_\pi(\vec{f},\vec{g})\|_{B_p^{\alpha-2,\infty}} \leq C\int_0^t \frac{ds}{(t-s)^{\frac{(\gamma-\alpha+2)}{2}}s^{\frac{(1+\alpha-d/p)}{2}}}N_1(\vec{f},t)N_2(\vec{g},t)$$

with

$$\begin{cases} N_1(\vec{f},t) = \sup_{0<s<t}\|\vec{f}(s)\|_{B_\infty^{-1,\infty}} \\ N_2(\vec{g},t) = \sup_{0<s<t} s^{\frac{(1+\alpha-d/p)}{2}}\|\vec{g}\|_{B_p^{\alpha,\infty}}. \end{cases}$$

For $\gamma \in [\alpha-2,\alpha)$, we get that

$$t^{\frac{(d/p-\gamma-1)}{2}}\|(-\Delta)^{\frac{(\gamma-\alpha+2)}{2}}B_\pi(\vec{f},\vec{g})\|_{B_p^{\alpha-2,\infty}} \leq C_\gamma N_1(\vec{f},t)N_2(\vec{g},t).$$

With the values $\gamma = \alpha-2$ and $\gamma = d/p-1+\epsilon$ ($\epsilon \in (0,\alpha+1-d/p)$), we find that $B_\pi(\vec{f},\vec{g})$ belongs to $(B^p)^d$, then with the value $\gamma = d/p-1$, we find that $B_\pi(\vec{f},\vec{g})$ belongs to $L^\infty((0,T),(B^p)^d)$ and that $\|B_\pi(\vec{f},\vec{g})\|_{B^p} \leq C(1+t)N_1(\vec{f},t)N_2(\vec{g},t)$. Since $\lim_{t\to0}N_2(\vec{g},t) = 0$, we find that $\lim_{t\to0}\|B_\pi(\vec{f},\vec{g})\|_{B^p} = 0$. Thus, $t \mapsto B_\pi(\vec{f},\vec{g}) \in (B^p)^d$ is continuous at O^+. We now prove the continuity of the map on $(0,T)$. For $0 < t_1 < t_2$, we write, as we have done before, the identity for $0 < s < t_1$: $e^{(t_2-s)\Delta} - e^{(t_1-s)\Delta} = \int_0^{t_2-t_1} e^{\theta\Delta}\,d\theta\ \Delta e^{(t_1-s)\Delta}$, which gives

$$\|(e^{(t_2-s)\Delta} - e^{(t_1-s)\Delta})\|_{\mathcal{L}((B_p^{\alpha-2,\infty})^d,(B_p^{\alpha-2,\infty})^d)} \leq C\min(1,\frac{t_2-t_1}{t_1-s})$$

and

$$\begin{gathered}\|(-\Delta)^{\frac{(d/p+1-\alpha)}{2}}(e^{(t_2-s)\Delta} - e^{(t_1-s)\Delta})\|_{\mathcal{L}((B_p^{\alpha-2,\infty})^d,(B_p^{\alpha-2,\infty})^d)} \\ \leq C\min(\frac{1}{(t_1-s)^{\frac{(d/p+1-\alpha)}{2}}},\frac{t_2-t_1}{|t_1-s|^{\frac{(d/p+3-\alpha)}{2}}})\end{gathered}$$

and we obtain

$$\|B_\pi(\vec{f},\vec{g})(t_1) - B_\pi(\vec{f},\vec{g})(t_2)\|_{B^p} \leq C_T\big(\frac{t_2-t_1}{t_2}\big)^{\frac{(1+\alpha-d/p)}{2}}N_1(\vec{f},t_2)N_2(\vec{g},t_2).$$

Now we deal with the value $\gamma = \alpha$, i.e., to estimate $\|B_\pi(\vec{f},\vec{g})\|_{B_p^{\alpha,\infty}}$. We want more accurately to estimate $t^{(1+\alpha-d/p)/2}\|\Delta\int_0^t e^{(t-s)\Delta}\vec{z}(s)\,ds\|_{B_p^{\alpha-2,\infty}}$

when $\sup_{0<s<t} s^{(1+\alpha-d/p)/2}\|\vec{z}(s)\|_{B_p^{\alpha-2,\infty}} < \infty$. We split the integral in two halves: for $0 < s < t$, we write directly

$$\int_0^{t/2} \|\Delta e^{(t-s)\Delta}\vec{z}(s)\|_{B_p^{\alpha-2,\infty}}\ ds \le C\int_0^{t/2} \frac{1}{t-s}\|\vec{z}(s)\|_{B_p^{\alpha-2,\infty}}\ ds;$$

for $t/2 < s < t$, we write that $1_{[t/2,t]}(s)\vec{z} \in (L^\infty((O,t), B_p^{\alpha-2,\infty})^d$ with, more precisely, $\sup_{t/2<s<t}\|\vec{z}(s)\|_{B_p^{\alpha-2,\infty}} \le Ct^{-\frac{(1+\alpha-d/p)}{2}}$. Thus, the proof will be complete when we prove that the map $f \mapsto \int_0^t e^{(t-s)\Delta}\Delta f\ ds$ is bounded on $L^\infty((0,T), B_p^{\alpha-2,\infty})$, or equivalently that

$$\|\int_0^t e^{(t-s)\Delta}\Delta f\ ds\|_{B_p^{\alpha-2,\infty}} \le C_T \sup_{0<s<t}\|f(s)\|_{B_p^{\alpha-2,\infty}}$$

for every $t < T$ and every $f \in L^\infty((0,T), B_p^{\alpha-2,\infty})$. We proved a similar result in Chapter 27 (Lemma 27.3) for homogeneous Besov spaces. This gives the control for $(Id-S_0)\int_0^t e^{(t-s)\Delta}\Delta f\ ds = \int_0^t e^{(t-s)\Delta}\Delta(Id-S_0)f\ ds$; the remainder is easily controlled by $\|\int_0^t e^{(t-s)\Delta}\Delta S_0 f\ ds\|_{B_p^{\alpha-2,\infty}} \le C\int_0^t \|f\|_{B_p^{\alpha-2,\infty}}\ ds$. □

The following point is to replace the estimate $\vec{f} \in (L^\infty(B_\infty^{-1,\infty}))^d$ by $\vec{f} \in (L^\infty((0,T^*)\times\mathbb{R}^d))^d$:

Lemma 28.3:

A) i) For $f \in L^\infty$, the operator $g \mapsto \pi(f,g)$ is bounded from H_p^s to H_p^s for every $p \in (1,\infty)$ and $s \in \mathbb{R}$ and $g \mapsto \rho(f,g)$ is bounded from H_p^s to H_p^s for every $p \in (1,\infty)$ and $s > 0$.

ii) For $T^* > 0$, if $\vec{f} \in (L^\infty((0,T^*)\times\mathbb{R}^d))^d)$, the operator $\vec{g} \mapsto B_\pi(\vec{f},\vec{g})$ is bounded on $L^q((0,T^*),(H_p^s)^d)$ for every p and q in $(1,\infty)$ and every $s \in \mathbb{R}$ with an operator norm bounded by $C\sqrt{T^*}\sup_{0<t<T^*}\|\vec{f}\|_\infty$ and the same result holds for the operator $\vec{g} \mapsto B_\rho(\vec{f},\vec{g})$ for every p in $(1,\infty)$, every q in $(1,\infty)$ and every $s > 0$.

B) j) For $f \in L^\infty$, the operator $g \mapsto \pi(f,g)$ is bounded from $B_p^{s,\infty}$ to $B_p^{s,\infty}$ for every $p \in [1,\infty]$ and $s \in \mathbb{R}$ and $g \mapsto \rho(f,g)$ is bounded from $B_p^{s,\infty}$ to $B_p^{s,\infty}$ for every $p \in [1,+\infty]$ and $s > 0$.

ii) For $T > 0$, if $\vec{f} \in (L^\infty((0,T)\times\mathbb{R}^d))^d)$, the operators $\vec{g} \mapsto B_\pi(\vec{f},\vec{g})$ and $\vec{g} \mapsto B_\rho(\vec{f},\vec{g})$ are bounded on $(\mathcal{E}_T^{\alpha,p})^d$ for every $p \in (d,\infty)$, every $\alpha \in (1, 1+d/p)$, with operator norms bounded by $C\sqrt{T}\sup_{0<t<T}\|\vec{f}\|_\infty$.

Proof: The boundedness of $\pi(f,.)$ and $\rho(g,.)$ on H_p^s and $B_p^{s,\infty}$ is proved in the same way as in Lemmas 28.1 and 28.2 (replacing for $j \in \mathbb{N}$ the estimate $\|S_j f\|_\infty \le 2^j\|f\|_{B_\infty^{-1,\infty}}$ by $\|S_j f\|_\infty \le C\|f\|_\infty$).

Now, we write $\|B_\pi(\vec{f},\vec{g})(t)\|_{H^s_p} \leq C\int_0^t \frac{1}{\sqrt{t-\tau}}\|M_\pi(\vec{f},\vec{g})\|_{H^s_p}\, d\tau$. Since we have $\|\frac{1}{\sqrt{t}}\|_{L^1(0,T^*)} = 2\sqrt{T}^*$, we find that

$$\|B_\pi(\vec{f},\vec{g})\|_{L^q((0,T^*),(H^s_p)^d)} \leq C\sqrt{T^*}\sup_{0<s<T^*}\|\vec{f}\|_\infty\|\vec{g}\|_{L^q((0,T^*),(H^s_p)^d)}.$$

Similarly, we write

$$\|B_\pi(\vec{f},\vec{g})(t)\|_{B^{\alpha,\infty}_p} \leq C\int_0^t \tfrac{1}{\sqrt{t-\tau}}\|M_\pi(\vec{f},\vec{g})\|_{B^{\alpha,\infty}_p}$$
$$\leq C' t^{1/2}\, t^{(1-\alpha-d/p)/2}\sup_{0<s<t}\|\vec{f}\|_\infty \sup_{0<s<t} s^{(\alpha+1-d/p)/2}\|\vec{g}\|_{B^{\alpha,\infty}_p}$$

Similarly, we write that $\vec{g} \in (B^{\gamma,\infty}_p)^d$ with $\gamma \in (0,d/p)$ and we write that $B_\pi(\vec{f},\vec{g}) = \int_0^t e^{(t-s)\Delta}\frac{1}{\sqrt{-\Delta}^{\gamma+1-d/p}}\mathbb{P}\vec{\nabla}.\sqrt{-\Delta}^{\gamma+1-d/p}M_\pi(\vec{f},\vec{g})\, ds$, which gives

$$\|B_\pi(\vec{f},\vec{g})(t)\|_{B^{d/p-1,\infty}_p} \leq C\int_0^t \frac{1}{|t-\tau|^{\frac{(d/p-\gamma)}{2}}}\|M_\pi(\vec{f},\vec{g})\|_{B^{\gamma,\infty}_p}$$

$$\leq C' t^{1/2}\sup_{0<s<t}\|\vec{f}\|_\infty \sup_{0<s<t}\|\vec{g}\|^{\frac{\alpha-\gamma}{\alpha+d/p-1}}_{B^{d/p-1,\infty}_p}\left(s^{(\alpha+1-d/p)/2}\|\vec{g}\|_{B^{\alpha,\infty}_p}\right)^{\frac{\gamma+1-d/p}{\alpha+d/p-1}}.$$

We do not have to check that $B_\pi(\vec{f},\vec{g})$ belongs more precisely to $(B^p)^d$ and that the map $t \mapsto B_\pi(\vec{f},\vec{g})$ is continuous, since $L^\infty \subset B^{-1,\infty}_\infty$, so that we may use the results of Lemma 28.2. □

Lemmas 28.1 to 28.3 then give the following fundamental regularity result:

Lemma 28.4:

Let $\vec{u}_0 \in (B^p \cap L^2)^d$ for some $p \in (d,\infty)$, with $\vec{\nabla}.\vec{u}_0 = 0$. If $\vec{u}$ is a weak solution of the Navier–Stokes equations on $(0,T^*)\times\mathbb{R}^d$ so that $\vec{u}$ belongs to the space $(\mathcal{C}([0,T^*),B^\infty)^d \cap (L^\infty([0,T^*),L^2))^d \cap (L^2([0,T^*),H^1))^d$ and has the initial value $\vec{u}_0$, then $\vec{u}$ belongs to the space $(\mathcal{E}^{\alpha,p}_T)^d$ for every $T<T^*$ and every $\alpha \in (1,1+d/p)$.

Proof: We know that $\vec{u}$ belongs to $L^q((0,T^*),(H^\sigma_r)^d)$ (with $q \in (2,\infty)$, $\sigma = 1+1/q$ and $r = d/(d-1/q)$ and that $\vec{w}$ is a fixed point of the mapping $\vec{g} \mapsto -2B_\pi(\vec{u},\vec{g}) - B_\rho(\vec{u},\vec{g}) + \vec{w}_0$ with $\vec{w}_0 = 2B_\pi(\vec{u}, 2^{t\Delta}\vec{u}_0) + B_\rho(\vec{u}, e^{t\Delta}\vec{u}_0)$.

We know that the operator $(\vec{f},\vec{g}) \mapsto 2B_\pi(\vec{f},\vec{g}) + B_\rho(\vec{f},\vec{g}) = A_{\vec{f}}(\vec{g})$ is continuous from $L^\infty((0,\delta),(B^{-1,\infty}_\infty)^d)) \times L^q((0,\delta),(H^\sigma_r)^d)$ to $L^q((0,\delta),(H^\sigma_r)^d)$ and that we have for a constant C_0, which does not depend on $\delta \in (0,T^*)$:

$$\|A_{\vec{f}}(\vec{g})\|_{L^q((0,\delta),(H^\sigma_r)^d)} \leq C_0\sup_{0<s<\delta}\|\vec{f}\|_{B^{-1,\infty}_\infty}\|\vec{g}\|_{L^q((0,\delta),(H^\sigma_r)^d)}$$

Moreover, the operator $(\vec{f}, \vec{g}) \mapsto A_{\vec{f}}(\vec{g})$ is continuous from $L^\infty((0,\delta),(L^\infty)^d)) \times L^q((0,\delta),(H^\sigma_r)^d)$ to $L^q((0,\delta),(H^\sigma_r)^d)$ and we have for a constant C_1 which does not depend on $\delta \in (0, T^*)$:

$$\|A_{\vec{f}}(\vec{g})\|_{L^q((0,\delta),(H^\sigma_r)^d)} \leq C_1\sqrt{\delta} \sup_{0<s<\delta} \|\vec{f}\|_\infty \|\vec{g}\|_{L^q((0,\delta),(H^\sigma_r)^d)}$$

Similarly, we know that the operator $(\vec{f}, \vec{g}) \mapsto A_{\vec{f}}(\vec{g})$ is continuous from $L^\infty((0,\delta),(B^{-1,\infty}_\infty)^d) \times (\mathcal{E}^{\alpha,p}_\delta)^d$ to $(\mathcal{E}^{\alpha,p}_\delta)^d$ and we have for a constant C_2 which does not depend on $\delta \in (0, T^*)$:

$$\|A_{\vec{f}}(\vec{g})\|_{(\mathcal{E}^{\alpha,p}_\delta)^d} \leq C_2 \sup_{0<s<\delta} \|\vec{f}\|_{B^{-1,\infty}_\infty} \|\vec{g}\|_{(\mathcal{E}^{\alpha,p}_\delta)^d}$$

Finally, the operator $(\vec{f}, \vec{g}) \mapsto A_{\vec{f}}(\vec{g})$ is continuous from $L^\infty((0,\delta),(L^\infty)^d)) \times (\mathcal{E}^{\alpha,p}_\delta)^d$ to $(\mathcal{E}^{\alpha,p}_\delta)^d$ and we have for a constant C_3 which does not depend on $\delta \in (0, T^*)$:

$$\|A_{\vec{f}}(\vec{g})\|_{(\mathcal{E}^{\alpha,p}_\delta)^d} \leq C_3\sqrt{\delta} \sup_{0<s<\delta} \|\vec{f}\|_\infty \|\vec{g}\|_{(\mathcal{E}^{\alpha,p}_\delta)^d}$$

We then fix $T < T^*$. Since the test functions are dense in B^∞ and since $[0,T]$ is compact, $L^\infty((0,T), L^\infty)$ is dense in $\mathcal{C}([0,T], B^\infty$. Hence, we may split $\vec{u}$ in $\vec{u} = \vec{X} + \vec{Y}$ with $\sup_{0<s<T} \|\vec{X}\|_{B^\infty} \leq \min(\frac{1}{4C_0}, \frac{1}{4C_2})$ and $\sup_{0<s<T} \|\vec{Y}\|_\infty < \infty$. We write then $M = \sup_{0<s<T} \|\vec{Y}\|_\infty < \infty$. We choose $\delta \in (0,T]$ so that $M\delta \leq \min(\frac{1}{4C_1}, \frac{1}{4C_3})$.

For $t_0 \in [0, T-\delta]$, we define $\vec{v}_{t_0}$ on $(0,\delta)$ by $\vec{v}_{t_0}(t) = \vec{v}(t+t_0)$. Of course, we have that $\vec{v}_{t_0} \in L^q((0,\delta),(H^\sigma_r)^d)$; moreover, $\vec{v}_{t_0}$ is a solution of the Navier–Stokes equations with initial value $\vec{u}(t_0)$. Thus, $\vec{v}_{t_0} - e^{t\Delta}\vec{u}(t_0) = \vec{w}_{t_0}$ is solution of the fixed point problem:

$$\vec{w}_{t_0} = -A_{\vec{X}(t+t_0)}(\vec{w}_{t_0}) - A_{\vec{Y}(t+t_0)}(\vec{w}_{t_0}) + A_{\vec{X}(t+t_0)}(e^{t\Delta}\vec{u}(t_0)) + A_{\vec{X}(t+t_0)}(e^{t\Delta}\vec{u}(t_0))$$

Since $\vec{z} \mapsto A_{\vec{u}(t+t_0)}(e^{t\Delta}\vec{u}(t_0) - \vec{z})$ is a contraction on $L^q((0,\delta),(H^\sigma_r)^d)$, $\vec{w}_{t_0}$ may be computed as the limit of the sequence $\vec{w}_{t_0,n}$ recursively defined by $\vec{w}_{t_0,0} = A_{\vec{u}(t+t_0)}(e^{t\Delta}\vec{u}(t_0))$ and $\vec{w}_{t_0,n+1} = A_{\vec{u}(t+t_0)}(e^{t\Delta}\vec{u}(t_0) - \vec{w}_{t_0,n})$. If $\vec{u}(t_0) \in (B^p)^d$, then we would have that $e^{t\Delta}\vec{u}(t_0) \in (\mathcal{E}^{\alpha,p}_\delta)^d$ and the same result would hold for $\vec{w}_{t_0,0}$; then, $\vec{z} \mapsto A_{\vec{u}(t+t_0)}(e^{t\Delta}\vec{u}(t_0) - \vec{z})$ would be a contraction on $(\mathcal{E}^{\alpha,p}_\delta)^d$ and $\vec{w}_{t_0}$ would be the limit of the sequence $\vec{w}_{t_0,n}$ in $(\mathcal{E}^{\alpha,p}_\delta)^d$ as well as in $L^q((0,\delta),(H^\sigma_r)^d)$. Thus, we see that $\vec{u}(t_0) \in (B^p)^d$ implies that $\vec{u} \in \mathcal{C}([t_0, t_0+\delta),(B^p)^d$. Since $\vec{u}_0 \in (B^p)^d$ and δ does not depend on t_0, we see that $\vec{u}(t+t_0) \in (\mathcal{E}^{\alpha,p}_\delta)^d$ for all $t_0 \in [0, T-\delta]$, and this gives $\vec{u} \in (\mathcal{E}^{\alpha,p}_T)^d$. □

Thus, Theorem 28.3 is a direct consequence of the following uniqueness theorem:

Proposition 28.1:

Let $\vec{u}_0 \in (B^p)^d$ for some $p \in (d, \infty)$, with $\vec{\nabla}.\vec{u}_0 = 0$. If $\vec{u}$ and $\vec{v}$ are two weak solutions of the Navier–Stokes equations on $(0,T) \times \mathbb{R}^d$ so that $\vec{u}$ and $\vec{v}$ belong to the space $(\mathcal{E}_T^{\alpha,p})^d$ with $\alpha \in (1, 1+d/p)$ and have the same initial value $\vec{u}_0$, then $\vec{u} = \vec{v}$.

Proof: We see easily that, one more time, it is enough to prove that we have local uniqueness: if $\vec{u}$ is a solution of the Navier–Stokes equations on $(0,T) \times \mathbb{R}^d$ and if $\vec{u} \in (\mathcal{E}_T^{\alpha,p})^d$, then for $t_0 \in (0,T)$ $\vec{u}(t+t_0)$ is a solution of the Navier–Stokes equations on $(0, T-t_0) \times \mathbb{R}^d$ with initial value $\vec{u}(t_0) \in (B^p)^d$ and $\vec{u}(t+t_0) \in (\mathcal{E}_{T-t_0}^{\alpha,p})^d$. Thus, local uniqueness would imply global uniqueness.

If $\vec{w} = \vec{u} - \vec{v}$, $\vec{w}_1 = B(\vec{u}, \vec{u})$ and $\vec{w}_2 = B(\vec{v}, \vec{v})$, we have

$$\vec{w} = -B(\vec{w}, \vec{u}) - B(\vec{v}, \vec{w}) = B(\vec{w}, \vec{w}_1) - B(\vec{w}_2, \vec{w}) - B(\vec{w}, e^{t\Delta}\vec{u}_0) - B(e^{t\Delta}\vec{u}_0, \vec{w}).$$

We are going to estimate

$$A(t, \vec{w}) = \|\vec{w}(t)\|_{B_p^{d/p-1,\infty}} + t^{(1+\alpha-d/p)/2} \|\vec{w}(t)\|_{B_p^{\alpha,\infty}}$$

with the help of the norms $\sup_{0<s<t} A(s, \vec{w})$, $\sup_{0<s<t} A(s, \vec{w}_1)$, $\sup_{0<s<t} A(s, \vec{w}_2)$ and $\sup_{0<s<t} \sqrt{s} \|e^{s\Delta}\vec{u}_0\|_\infty$. Indeed, we have $B = B\pi + B_{\pi^*} + B_\rho$, then use the embedding $B^p \subset B_\infty^{-1,\infty}$ and Lemma 28.2 to get

$$A(t, B_\pi(\vec{w}, \vec{w}_1)) \leq C \sup_{0<s<t} \|\vec{w}(s)\|_{B_\infty^{-1,\infty}} \sup_{0<s<t} s^{(1+\alpha-d/p)/2} \|\vec{w}_1(s)\|_{B_p^{\alpha,\infty}}$$

$$A(t, B_{\pi^*}(\vec{w}, \vec{w}_1)) \leq C \sup_{0<s<t} s^{(1+\alpha-d/p)/2} \|\vec{w}(s)\|_{B_p^{\alpha,\infty}} \sup_{0<s<t} \|\vec{w}_1(s)\|_{B_\infty^{-1,\infty}}$$

$$A(t, B_\rho(\vec{w}, \vec{w}_1)) \leq C \sup_{0<s<t} s^{(1+\alpha-d/p)/2} \|\vec{w}(s)\|_{B_p^{\alpha,\infty}} \sup_{0<s<t} \|\vec{w}_1(s)\|_{B_\infty^{-1,\infty}}$$

and

$$A(t, B_\pi(\vec{w}_2, \vec{w})) \leq C \sup_{0<s<t} s^{(1+\alpha-d/p)/2} \|\vec{w}(s)\|_{B_p^{\alpha,\infty}} \sup_{0<s<t} \|\vec{w}_2(s)\|_{B_\infty^{-1,\infty}}$$

$$A(t, B_{\pi^*}(\vec{w}_2, \vec{w})) \leq C \sup_{0<s<t} \|\vec{w}(s)\|_{B_\infty^{-1,\infty}} \sup_{0<s<t} s^{(1+\alpha-d/p)/2} \|\vec{w}_2(s)\|_{B_p^{\alpha,\infty}}$$

$$A(t, B_\rho(\vec{w}_2, \vec{w})) \leq C \sup_{0<s<t} s^{(1+\alpha-d/p)/2} \|\vec{w}(s)\|_{B_p^{\alpha,\infty}} \sup_{0<s<t} \|\vec{w}_1(s)\|_{B_\infty^{-1,\infty}}$$

Similarly, Lemma 28.3 gives

$$A(t, B_{\pi^*}(\vec{w}, e^{t\Delta}\vec{u}_0)) \leq C \sup_{0<s<t} s^{(1+\alpha-d/p)/2} \|\vec{w}(s)\|_{B_p^{\alpha,\infty}} \sup_{0<s<t} \sqrt{s} \|e^{s\Delta}\vec{u}_0\|_\infty$$

$$A(t, B_\rho(\vec{w}, e^{t\Delta}\vec{u}_0)) \leq C \sup_{0<s<t} s^{(1+\alpha-d/p)/2} \|\vec{w}(s)\|_{B_p^{\alpha,\infty}} \sup_{0<s<t} \sqrt{s} \|e^{s\Delta}\vec{u}_0\|_\infty$$

and

$$A(t, B_\pi(e^{t\Delta}\vec{u}_0, \vec{w})) \le C \sup_{0<s<t} s^{(1+\alpha-d/p)/2} \|\vec{w}(s)\|_{B_p^{\alpha,\infty}} \sup_{0<s<t} \sqrt{s}\|e^{s\Delta}\vec{u}_0\|_\infty$$

$$A(t, B_\rho(e^{t\Delta}\vec{u}_0, \vec{w})) \le C \sup_{0<s<t} s^{(1+\alpha-d/p)/2} \|\vec{w}(s)\|_{B_p^{\alpha,\infty}} \sup_{0<s<t} \sqrt{s}\|e^{s\Delta}\vec{u}_0\|_\infty$$

To control the remaining terms, we use the boundedness of the paraproduct π^* from $L^\infty \times B_p^{s,\infty}$ to $B_p^{s,\infty}$ for $p \in [1, \infty]$ and $s < 0$. This gives

$$A(t, B_\pi(\vec{w}, e^{t\Delta}\vec{u}_0)) \le C \sup_{0<s<t} \|\vec{w}(s)\|_{B_p^{d/p-1,\infty}} \sup_{0<s<t} \sqrt{s}\|e^{s\Delta}\vec{u}_0\|_\infty$$

and

$$A(t, B_{\pi^*}(e^{t\Delta}\vec{u}_0, \vec{w})) \le C \sup_{0<s<t} \|\vec{w}(s)\|_{B_p^{d/p-1,\infty}} \sup_{0<s<t} \sqrt{s}\|e^{s\Delta}\vec{u}_0\|_\infty$$

Thus, we get that for some constant C_0 we have

$$A(t, \vec{w}) \le C \sup_{0<s<t} A(s, \vec{w}) \sup_{0<s<t} \left(A(s, \vec{w}_1) + A(s, \vec{w}_2) + \sup_{0<s<t} \sqrt{s}\|e^{s\Delta}\vec{u}_0\|_\infty\right)$$

Since $\vec{u}_0 \in B^p$ (i.e., since it may be approximated through smooth functions), we have $\lim_{t\to 0} \sqrt{t}\|e^{t\Delta}\vec{u}_0\|_\infty = 0$; moreover, since $\alpha > d/p - 1$, we have $\lim_{t\to 0} t^{(1+\alpha-d/p)/2}\|e^{t\Delta}\vec{u}_0\|_{B_p^{\alpha,\infty}} = 0$. Since $\vec{u} \in (\mathcal{E}_T^{\alpha,p})^d$, we find that $\vec{w}_1 = e^{t\Delta}\vec{u}_0 - \vec{u}$ satisfies $\lim_{t\to 0} t^{(1+\alpha-d/p)/2}\|\vec{w}_1\|_{B_p^{\alpha,\infty}} = 0$. Finally, since $\vec{u}$ is continuous in B^p norm and $\lim_{t\to 0} \|e^{t\Delta}\vec{u}_0 - \vec{u}_0\|_{B_p^{d/p-1,\infty}} = 0$, we find that $\lim_{t\to 0} \|\vec{w}_1\|_{B_p^{d/p-1,\infty}} = 0$. Of course, $\vec{w}_2$ satisfies similar estimates and, for some positive ϵ, we have

$$\sup_{0<s<\epsilon} \left(A(s, \vec{w}_1) + A(s, \vec{w}_2) + \sup_{0<s<t} \sqrt{s}\|e^{s\Delta}\vec{u}_0\|_\infty\right) \le \frac{1}{2C_0},$$

which gives $\vec{w} = 0$ on $[0, \epsilon]$. Thus, Proposition 28.1 and Theorem 28.3 are proved. □

Chapter 29

Stability and Lyapunov functionals

A Lyapunov functional for the Navier–Stokes equations is a norm $\|\vec{u}(t,.)\|_E$ which is decreasing for $t \in [0,\infty)$. For instance, the energy norm $\|\vec{u}(t,.)\|_2$ is a Lyapunov functional for the set of Leray solutions (at least, for the Leray solutions that satisfy Serrin's criterion of uniqueness, for which we have the energy equality $\partial_t \|\vec{u}\|_2^2 + 2\|\vec{\nabla} \otimes \vec{u}\|_2^2 = 0$). Under the assumption that $\vec{u}_0 \in (L^p \cap L^d(\mathbb{R}^d))^d$ with a small norm in L^d, Kato [KAT 90] has proved that the L^p norm was a Lyapunov functional for the solutions $\vec{u} \in (\mathcal{C}([0,\infty), L^p \cap L^d))^d$ $(1 < p < \infty)$. Kato [KAT 90] has also studied the norm in the potential spaces H_p^s for $p \geq 2$ and $s > 0$ when the initial data has a small norm in $(L^d)^d$. More recently, Cannone and Planchon [CANP 00] discussed the case of Besov norms as Lyapunov functionals, under a condition of size of $\vec{u}$ in $(L^\infty([0,\infty), \dot{B}_\infty^{-1,\infty}))^d$. We adapt their proof, considering the data in $(\dot{B}_p^{s,q} \cap BMO^{-1})^d$ with no scaling condition on s instead of considering only data in $(\dot{B}_p^{s,q})^d$ with the scaling condition $s = d/p - 1$ (which ensures that $B_p^{s,q} \subset BMO^{-1}$); this adaptation has been made in a joint work with E. Zahrouni [ZAH 02].

1. Stability in Lebesgue norms

We first prove the stability theorem of Kato:

Theorem 29.1: (Stability in Lebesgue norms)

For $1 < p < \infty$ there exists a positive ϵ_p depending only on d and p so that:

(A) When $\vec{u}_0 \in (L^p(\mathbb{R}^d) \cap L^d(\mathbb{R}^d))^d$ with $\vec{\nabla}.\vec{u}_0 = 0$ and $\|\vec{u}_0\|_d < \epsilon_p$, then the solution $\vec{u} \in (\mathcal{C}([0,\infty), L^p \cap L^d))^d$ for the Navier–Stokes equations with initial data $\vec{u}_0$ has a norm in $(L^p(\mathbb{R}^d))^d$, which decreases on $[0,\infty)$ and $\lim_{t\to\infty} \|\vec{u}\|_p = 0$.

(B) When $p \geq 2$ we have a more general decay result: ϵ_p may be chosen so that, when $\vec{u}_0 \in (L^p(\mathbb{R}^d) \cap L^d(\mathbb{R}^d))^d$ with $\vec{\nabla}.\vec{u}_0 = 0$ and $\|\vec{u}_0\|_d < \epsilon_p$ and when $\vec{v}_0 \in (L^p(\mathbb{R}^d) \cap L^d(\mathbb{R}^d))^d$ with $\vec{\nabla}.\vec{v}_0 = 0$ and $\|\vec{v}_0\|_d < \epsilon_p$, then the solutions $\vec{u}$ and $\vec{v}$ in $(\mathcal{C}([0,\infty), L^p \cap L^d))^d$ for the Navier–Stokes equations with initial data $\vec{u}_0$ and $\vec{v}_0$ are getting closer in L^p norm as t increases: $t \mapsto \|\vec{u} - \vec{v}\|_p$ is decreasing on $[0,\infty)$.

Proof:

Step 1: Regularity of the solutions

We know (Theorem 15.3) that there exists $\epsilon(d) > 0$ so that, when $\vec{u}_0 \in (L^d(\mathbb{R}^d))^d$ with $\vec{\nabla}.\vec{u}_0 = 0$ and $\|\vec{u}_0\|_d < \epsilon(d)$, then the Navier–Stokes equations with initial data $\vec{u}_0$ have a global solution $\vec{u} \in (\mathcal{C}([0,\infty), L^d))^d$ with $\sup_{t>0} \|\vec{u}(t,.)\|_d \leq 2\|\vec{u}_0\|_d$. If moreover $\vec{u}_0 \in (L^p)^d$ for some $p \in [1,\infty)$, then the persistency theorem (Theorem 18.3) gives that $\vec{u} \in (\mathcal{C}([0,\infty), L^p))^d$.

This mild solution is smooth. We know (Proposition 15.1) that for all $0 < T_1 < T_2$ and all $\alpha \in \mathbb{N}^d$ we have $\sup_{T_1<t<T_2} \|\partial^\alpha \vec{u}(t,.)\|_\infty < \infty$. We easily check that for all $0 < T_1 < T_2$ and all $\alpha \in \mathbb{N}^d$ we have $\sup_{T_1<t<T_2} \|\partial^\alpha \vec{u}(t,.)\|_p < \infty$. In particular, we find that $\Delta\vec{u}$ and $\mathbb{P}\vec{\nabla}.\vec{u} \otimes \vec{u}$ (hence, $\partial_t \vec{u}$ as well) belong to $(\mathcal{C}((0,\infty), L^p))^d$. Moreover, at fixed t, $\vec{u}$ and $\vec{\nabla} \otimes \vec{u}$ go to 0 as x goes to ∞.

We now assume that $\vec{u}_0 \in (L^p(\mathbb{R}^d) \cap L^d(\mathbb{R}^d))^d$ with $\vec{\nabla}.\vec{u}_0 = 0$ and $\|\vec{u}_0\|_d < \epsilon(d)$ and that $\vec{v}_0 \in (L^p(\mathbb{R}^d) \cap L^d(\mathbb{R}^d))^d$ with $\vec{\nabla}.\vec{v}_0 = 0$ and $\|\vec{v}_0\|_d < \epsilon(d)$, and we consider the associated solutions $\vec{u}$ and $\vec{v}$. We want to show that, if we have $\|\vec{u}_0\|_d < \epsilon_p$ and $\|\vec{v}_0\|_d < \epsilon_p$, then $\|\vec{u}(t,.) - \vec{v}(t,.)\|_p$ is nonincreasing, where $\|\vec{f}\|_p = \left(\int_{\mathbb{R}^d} \left(\sum_{k=1}^d |f_k(x)|^2 \right)^{p/2} dx \right)^{1/p} = (\int |f|^p dx)^{1/p}$. (This has to be proved only with $\vec{v} = 0$ in the case $p < 2$).

Step 2: Derivative of the L^p norm

We first check that, on $(0,\infty)$,

$$\frac{d}{dt}\|\vec{u} - \vec{v}\|_p^p = p \int_{\mathbb{R}^d} (\partial_t \vec{u} - \partial_t \vec{v}).|\vec{u} - \vec{v}|^{p-2}(\vec{u} - \vec{v})\, dx. \tag{29.1}$$

Indeed, let $\Phi_\epsilon = \left(\epsilon e^{-|x|^2} + \sum_{k=1}^d |u_k(x) - v_k(x)|^2 \right)^{1/2}$. Then

$$\frac{d}{dt}\|\Phi_\epsilon\|_p^p = p \int_{\mathbb{R}^d} (\partial_t \vec{u} - \partial_t \vec{v}).\Phi_\epsilon^{p-2}(\vec{u} - \vec{v})\, dx.$$

Since, for $0 < \epsilon < 1$, we have $|\vec{u} - \vec{v}| \leq \Phi_\epsilon \leq e^{-|x|^2/2} + |\vec{u} - \vec{v}|$, we find by the dominated convergence theorem that

$$\lim_{\epsilon \to 0} \frac{d}{dt}\|\Phi_\epsilon\|_p^p = p \int_{\mathbb{R}^d} (\partial_t \vec{u} - \partial_t \vec{v}).|\vec{u} - \vec{v}|^{p-2}(\vec{u} - \vec{v})\, dx.$$

$\|\Phi_\epsilon\|_p^p \leq \|e^{-|x|^2/2}\|_p + \|\vec{u} - \vec{v}\|_p$, and the majorant is a continuous function of t; hence, we find that inequality (29.1) is valid at least in the sense of distributions on $(0,\infty)$; since the right-hand term in equality (29.1) is a continuous function of t, we find that $\|\vec{u} - \vec{v}\|_p^p$ is $\mathcal{C}^1$ and that (29.1) is valid as an equality between continuous functions.

Equation (29.1) may be rewritten as

$$\begin{aligned} \tfrac{d}{dt}\|\vec{u} - \vec{v}\|_p^p = & \quad p \textstyle\int_{\mathbb{R}^d} \Delta(\vec{u} - \vec{v}).|\vec{u} - \vec{v}|^{p-2}(\vec{u} - \vec{v})\, dx \\ & -p \textstyle\int_{\mathbb{R}^d} \left(\mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{u} - \vec{v} \otimes \vec{v}) \right).|\vec{u} - \vec{v}|^{p-2}(\vec{u} - \vec{v})\, dx \end{aligned} \tag{29.2}$$

Step 3: The term $\int_{\mathbb{R}^d} \Delta(\vec{u}-\vec{v}).|\vec{u}-\vec{v}|^{p-2}(\vec{u}-\vec{v})\ dx$

To deal with equality (29.2), we define $\vec{w}=\vec{u}-\vec{v}$ and look at the properties of $\int_{\mathbb{R}^d} \Delta\vec{w}.|\vec{w}|^{p-2}\vec{w}\ dx$.

Recall that we have defined $\Phi_\epsilon = (|\vec{w}|^2+\epsilon e^{-|x|^2})^{1/2}$. Φ_ϵ is a $\mathcal{C}^2$ positive function, and so is Φ_ϵ^α for every $\alpha\in\mathbb{R}$. We have, for $j\in\{1,\dots,d\}$, $\partial_j(\Phi_\epsilon^\alpha) = \alpha\partial_j\Phi_\epsilon\ \Phi_\epsilon^{\alpha-1}$. We then write

$$\begin{aligned}\int_{|x_j|\le R}\partial_j^2\vec{w}.\Phi_\epsilon^{p-2}\vec{w}\ dx_j = \quad & \left[\partial_j\vec{w}.\Phi_\epsilon^{p-2}\vec{w}\right]_{x_j=-R}^{x_j=+R}\\ & -\int_{|x_j|\le R}|\partial_j\vec{w}|^2\ \Phi_\epsilon^{p-2}\ dx_j\\ & -(p-2)\int_{|x_j|\le R}(\partial_j\vec{w}.\vec{w})\ \partial_j\Phi_\epsilon\ \Phi_\epsilon^{p-3}\ dx_j,\end{aligned}$$

followed by $\partial_j\Phi_\epsilon = (\partial_j\vec{w}.\vec{w} - x_j\epsilon e^{-|x|^2})\Phi_\epsilon^{-1}$. We get

$$\begin{aligned}&\int_{|x_j|\le R}|\partial_j\vec{w}|^2\ \Phi_\epsilon^{p-2}\ dx_j + (p-2)\int_{|x_j|\le R}|\partial_j\vec{w}.\vec{w}|^2\ \Phi_\epsilon^{p-4}\ dx_j = \\ &\left[\partial_j\vec{w}.\Phi_\epsilon^{p-2}\vec{w}\right]_{x_j=-R}^{x_j=+R}\\ &+(p-2)\int_{|x_j|\le R}(\partial_j\vec{w}.\vec{w})\ x_j\epsilon e^{-|x|^2}\Phi_\epsilon^{p-4}dx_j - \int_{|x_j|\le R}\partial_j^2\vec{w}.\Phi_\epsilon^{p-2}\vec{w}\ dx_j.\end{aligned} \tag{29.3}$$

We have $|\partial_j^2\vec{w}.\Phi_\epsilon^{p-2}\vec{w}| \le |\partial_j^2\vec{w}|\Phi_\epsilon^{p-1} \in L^1(\mathbb{R}^d)$ (note that $\partial_j^2\vec{w}\in(L^p)^d$ since $\Delta\vec{w}\in(L^p)^d$ and since the Riesz transform operates boundedly on L^p) and $|(\partial_j\vec{w}.\vec{w})\ x_j\epsilon e^{-|x|^2}\ \Phi_\epsilon^{p-4}| \le |\partial_j\vec{w}|\Phi_\epsilon^{p-1-\alpha}\ |x_j|\epsilon^{\alpha/2}e^{-\alpha|x|^2/2} \in L^1(\mathbb{R}^d)$ for $0<\alpha<p-1$ (we have $\partial_j\vec{w}\in(L^p)^d$ since $(Id-\Delta)\vec{w}\in(L^p)^d$ and since the kernel of the convolution operator $\frac{\partial_j}{Id-\Delta}$ belongs to $L^1(\mathbb{R}^d)$). We may then take the limit of (29.3) as R goes to $+\infty$, and then integrate with respect to the other variables x_k's, obtaining

$$\begin{aligned}&\int|\partial_j\vec{w}|^2\ \Phi_\epsilon^{p-2}\ dx + (p-2)\int|\partial_j\vec{w}.\vec{w}|^2\ \Phi_\epsilon^{p-4}\ dx = \\ &(p-2)\int(\partial_j\vec{w}.\vec{w})\ x_j\epsilon e^{-|x|^2}\ \Phi_\epsilon^{p-4}\ dx - \int\partial_j^2\vec{w}.\Phi_\epsilon^{p-2}\vec{w}\ dx.\end{aligned} \tag{29.4}$$

We now take the limit as ϵ goes to 0. For $p\ge 2$ we write $|\partial_j\vec{w}.\vec{w}|^2\ \Phi_\epsilon^{p-4} \le |\partial_j\vec{w}|^2\ \Phi_\epsilon^{p-2} \le |\partial_j\vec{w}|^2\ (|\vec{w}|+\sqrt{\epsilon}e^{-|x|^2/2})^{p-2}$ and we apply the dominated convergence theorem; for $p<2$, we use the monotone convergence theorem since Φ_ϵ^{-1} increases to $|\vec{w}|^{-1}$. Thus, we have $\lim_{\epsilon\to 0}\int|\partial_j\vec{w}|^2\ \Phi_\epsilon^{p-2}\ dx = \int|\partial_j\vec{w}|^2\ |\vec{w}|^{p-2}\ dx$ and $\lim_{\epsilon\to 0}\int|\partial_j\vec{w}.\vec{w}|^2\ \Phi_\epsilon^{p-4}\ dx = \int|\partial_j\vec{w}.\vec{w}|^2\ |\vec{w}|^{p-4}\ dx$. On the other hand, we have seen that $\int(\partial_j\vec{w}.\vec{w})\ x_j\epsilon e^{-|x|^2}\ \Phi_\epsilon^{p-4}\ dx = O(\epsilon^{\alpha/2})$ for $0<\alpha<p-1$ and that $|\partial_j^2\vec{w}.\Phi_\epsilon^{p-2}\vec{w}| \le |\partial_j^2\vec{w}|\Phi_\epsilon^{p-1}$, so that we may apply the dominated convergence and get $\lim_{\epsilon\to 0}\int\partial_j^2\vec{w}.\Phi_\epsilon^{p-2}\vec{w}\ dx = \int\partial_j^2\vec{w}.|\vec{w}|^{p-2}\vec{w}\ dx$.

Thus

$$\begin{aligned}&\int\Delta(\vec{u}-\vec{v}).|\vec{u}-\vec{v}|^{p-2}(\vec{u}-\vec{v})\ dx = \\ &-\sum_{j=1}^d\int|\partial_j(\vec{u}-\vec{v})|^2\ |\vec{u}-\vec{v}|^{p-2}\ dx\\ &-(p-2)\sum_{j=1}^d\int|\partial_j(\vec{u}-\vec{v}).(\vec{u}-\vec{v})|^2\ |\vec{u}-\vec{v}|^{p-4}\ dx.\end{aligned} \tag{29.5}$$

Step 4: The function $|\vec{u}-\vec{v}|^{p/2}$

If we approximate again $|\vec{u}-\vec{v}|$ by Φ_ϵ, we have

$$\partial_j\big(\Phi_\epsilon^{p/2}\big) = (p/2)(\partial_j(\vec{u}-\vec{v}).(\vec{u}-\vec{v}) - x_j\epsilon e^{-|x|^2})\Phi_\epsilon^{p/2-2}$$

and we have for $0<\alpha<p/2$ and $\vec{w}=\vec{u}-\vec{v}$

$$\frac{2}{p}|\partial_j\big(\Phi_\epsilon^{p/2}\big)| \leq \begin{cases} |\partial_j\vec{w}||\vec{w}|^{(p-2)/2} + \epsilon^{\alpha/2}|x_j|e^{-\alpha|x|^2/2}\Phi_\epsilon^{p/2-\alpha} & \text{if } p<2 \\ |\partial_j\vec{w}|\Phi_\epsilon^{(p-2)/2} + \epsilon^{\alpha/2}|x_j|e^{-\alpha|x|^2/2}\Phi_\epsilon^{p/2-\alpha} & \text{if } p\geq 2 \end{cases}$$

Thus $\partial_j\big(\Phi_\epsilon^{p/2}\big)$ remains bounded in L^2 as ϵ goes to 0. Taking the limit in $\mathcal{D}'$, we find that $\partial_j(|\vec{u}-\vec{v}|^{p/2})\in L^2$ and that

$$\|\vec{\nabla}(|\vec{u}-\vec{v}|^{p/2})\|_2 \leq C\sum_{j=1}^d \||\vec{u}-\vec{v}|^{p/2-1}\partial_j(\vec{u}-\vec{v})\|_2 \leq C'\|\Delta(\vec{u}-\vec{v})\|_p^{1/2}\|\,\vec{u}-\vec{v}\|_p^{p-1/2}.$$

Let us define $Q_p(\vec{f}) = \sum_{j=1}^d \|\,|\vec{f}|^{p/2-1}\partial_j\vec{f}\|_2$. Up to now we have proved the following results:

i) $|\vec{u}-\vec{v}|^{p/2}\in H^1\subset L^{\frac{2d}{d-2}}$ and

$$\||\vec{u}-\vec{v}|^{p/2}\|_{\frac{2d}{d-2}} \leq C\|\vec{\nabla}(|\vec{u}-\vec{v}|^{p/2})\|_2 \leq C'Q_p(\vec{u}-\vec{v})$$

ii) for some positive constant A_p

$$\frac{d}{dt}\|\vec{u}-\vec{v}\|_p^p \leq -A_pQ_p(\vec{u}-\vec{v})^2 - p\int \mathbb{P}\vec{\nabla}(\vec{u}\otimes\vec{u}-\vec{v}\otimes\vec{v}).|\vec{u}-\vec{v}|^{p-2}(\vec{u}-\vec{v})\,dx$$

Step 5: Control of $\int\big(\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u}-\vec{v}\otimes\vec{v})\big).|\vec{u}-\vec{v}|^{p-2}(\vec{u}-\vec{v})\,dx$ for $p\geq 2$

Let $\alpha = 2/p\in(0,1]$. We first write that

$$|\vec{u}\otimes\vec{u}-\vec{v}\otimes\vec{v}| = |\vec{u}\otimes(\vec{u}-\vec{v}) + (\vec{u}-\vec{v})\otimes\vec{v}| \leq (|\vec{u}-\vec{v}|^{p/2})^\alpha(|\vec{u}|+|\vec{v}|).$$

On the other hand, we have

$$|\partial_j\big(|\vec{u}-\vec{v}|^{p-2}(\vec{u}-\vec{v})\big)| \leq C|\partial_j(\vec{u}-\vec{v})|\,|\vec{u}-\vec{v}|^{p-2} = C|\partial_j(\vec{u}-\vec{v})|\,|\vec{u}-\vec{v}|^{\frac{p}{2}-1}(|\vec{u}-\vec{v}|^{\frac{p}{2}})^{1-\alpha}$$

Since $|\vec{u}|+|\vec{v}|\in L^d$, $(|\vec{u}-\vec{v}|^{p/2})^\alpha\in L^{\frac{2d}{(d-2)\alpha}}$ and $0<\frac{1}{d}+\alpha(\frac{1}{2}-\frac{1}{d})<1$, we may forget the operator $\mathbb{P}$ (the Riesz transforms operate boundedly on L^r for $1<r<\infty$); since $|\partial_j(\vec{u}-\vec{v})|\,|\vec{u}-\vec{v}|^{\frac{p}{2}-1}\in L^2$ and (for $p>2$) $(|\vec{u}-\vec{v}|^{\frac{p}{2}})^{1-\alpha}\in L^{\frac{2d}{(d-2)(1-\alpha)}}$, we use the Hölder inequality (since $\frac{1}{d}+\alpha(\frac{1}{2}-\frac{1}{d})+\frac{1}{2}+(1-\alpha)(\frac{1}{2}-\frac{1}{d})=1$), we find that

$$\begin{gathered}|\int\mathbb{P}\vec{\nabla}(\vec{u}\otimes\vec{u}-\vec{v}\otimes\vec{v}).|\vec{u}-\vec{v}|^{p-2}(\vec{u}-\vec{v})\,dx| \\ \leq C(\|\vec{u}\|_d+\|\vec{v}\|_d)\|\vec{u}-\vec{v}\|_{\frac{2d}{(d-2)\alpha}}\times \\ (\textstyle\sum_{j=1}^d\||\vec{u}-\vec{v}|^{p/2-1}\partial_j(\vec{u}-\vec{v})\|_2)\,\|\,|\vec{u}-\vec{v}|^{p/2-1}\|_{\frac{2d}{(d-2)(1-\alpha)}} \\ \leq C'(\|\vec{u}\|_d+\|\vec{v}\|_d)Q_p(\vec{u}-\vec{v})^\alpha Q_p(\vec{u}-\vec{v})Q_p(\vec{u}-\vec{v})^{1-\alpha}.\end{gathered}$$

We obtain for two positive constants A_p and B_p:

$$\frac{d}{dt}\|\vec{u}-\vec{v}\|_p^p \le -A_p\big(1 - B_p(\|\vec{u}\|_d + \|\vec{v}\|_d)\big)\ Q_p(\vec{u}-\vec{v})^2$$

and $\frac{d}{dt}\|\vec{u}-\vec{v}\|_p^p \le 0$ if $\|\vec{u}\|_d + \|\vec{v}\|_d \le 1/B_p$.

Step 6: Control of $\int(\mathbb{P}\vec{\nabla}.\vec{u}\otimes\vec{u}).|\vec{u}|^{p-2}\vec{u}\ dx$ for $p<2$

Let $\beta = \frac{2}{p} - 1 \in (0,1)$. Since $\vec{\nabla}.\vec{u} = 0$, we write $\vec{\nabla}.\vec{u}\otimes\vec{u} = (\vec{u}.\vec{\nabla})\vec{u}$ and thus

$$|\vec{\nabla}.\vec{u}\otimes\vec{u}| \le C|\vec{u}||\vec{u}|^{1-p/2}\sum_{j=1}^d |\partial_j\vec{u}||\vec{u}|^{p/2-1} = C|\vec{u}|(|\vec{u}|^{p/2})^\beta \sum_{j=1}^d |\partial_j\vec{u}||\vec{u}|^{p/2-1}.$$

We then write $|\vec{u}| \in L^d$, $(|\vec{u}|^{p/2})^\beta \in L^{\frac{2d}{\beta(d-2)}}$ and $\sum_{j=1}^d |\partial_j\vec{u}||\vec{u}|^{p/2-1} \in L^2$; since $0 < \frac{1}{d} + \frac{1}{2} + \beta(\frac{1}{2} - \frac{1}{d}) < 1$, we may again forget the operator $\mathbb{P}$. Moreover, $|\vec{u}|^{p-2}\vec{u} \in (L^{\frac{2d}{(1-\beta)(d-2)}})^d$ since $p-1 = \frac{p}{2}(1-\beta)$. Now, we obtain for two positive constants A_p and B_p:

$$\frac{d}{dt}\|\vec{u}\|_p^p \le -A_p\big(1 - B_p\|\vec{u}\|_d\big)Q_p(\vec{u})^2$$

and $\frac{d}{dt}\|\vec{u}\|_p^p \le 0$ if $\|\vec{u}\|_d \le 1/B_p$. $\square$

Remark: The theorem remains valid when we replace the L^d norm with the weaker $L^{d,\infty}$ norm. Starting from the statement that $|\vec{u}-\vec{v}|^{p/2} \in H^1$, we use the sharp Sobolev inequality $H^1 \subset L^{2d/(d-2),2}$. Thus, we have only to check that for $\alpha \in (0,1)$ and $q \in (1,\infty)$ and $r \in [1,\infty]$, we have for $f \in L^{q,r}$ $|f|^\alpha \in L^{q/\alpha, r/\alpha}$. But this is very easy: Theorem 2.3 states that f may be decomposed as $f = \sum_{j\in\mathbb{Z}} f_j$ with $(2^{-j(1-1/q)}\|f_j\|_1)_{j\in\mathbb{Z}} \in l^r(\mathbb{Z})$ and $(2^{j/q}\|f_j\|_\infty)_{j\in\mathbb{Z}} \in l^r(\mathbb{Z})$ and where the f_j's have disjoint measurable supports. Then, $|f|^\alpha = \sum_{j\in\mathbb{Z}} |f_j|^\alpha$ with $(2^{-j\alpha(1-1/q)}\||f_j|^\alpha\|_{\frac{1}{\alpha}})_{j\in\mathbb{Z}} \in l^{r/\alpha}(\mathbb{Z})$ and $(2^{j\alpha/q}\||f_j|^\alpha\|_\infty)_{j\in\mathbb{Z}} \in l^{r/\alpha}(\mathbb{Z})$, which gives $|f|^\alpha \in [L^{1/\alpha}, L^\infty]_{[1-1/q, r/\alpha]} = L^{q/\alpha, r/\alpha}$. This gives

$$\frac{d}{dt}\|\vec{u}-\vec{v}\|_p^p \le -A_p\big(1 - B'_p(\|\vec{u}\|_{L^{d,\infty}} + \|\vec{v}\|_{L^{d,\infty}})\big)\ Q_p(\vec{u}-\vec{v})^2 \text{ for } 2 \le p < \infty$$

and

$$\frac{d}{dt}\|\vec{u}\|_p^p \le -A_p\big(1 - B'_p\|\vec{u}\|_{L^{d,\infty}}\big)Q_p(\vec{u})^2 \text{ for } 1 < p < 2.$$

2. A new Bernstein inequality

Before turning our attention to the case of Besov norms, we consider a new kind of Bernstein inequality proved recently by Planchon [PLA 00] (generalizing a former result of Danchin [DAN 97]):

Proposition 29.1:

Let $p \in (2, \infty)$. Then there exists two positive constants c_p and C_p so that for every $f \in \mathcal{S}'(\mathbb{R}^d)$ and every $j \in \mathbb{Z}$ we have

$$c_p 2^{2j/p} \|\Delta_j f\|_p \leq \|\vec{\nabla}(|\Delta_j f|^{p/2})\|_2^{2/p} \leq C_p 2^{2j/p} \|\Delta_j f\|_p$$

where $\Delta_j f$ is the j-th dyadic block in the Littlewood-Paley decomposition of f.

Proof: By homogeneity, it is enough to prove the inequality for $j = 0$. Since $p/2 > 1$ and since $\Delta_0 f$ is smooth, we find that

$$\frac{\partial}{\partial x_k}(|\Delta_0 f|^{p/2}) = p/2 \ \mathrm{sgn}(\Delta_0 f)|\Delta_0 f|^{p/2-1} \frac{\partial}{\partial x_k} \Delta_0 f;$$

hence, we may write

$$\|\vec{\nabla}(|\Delta_0 f|^{\frac{p}{2}})\|_2^{\frac{2}{p}} = \frac{p}{2} \||\Delta_0 f|^{\frac{p}{2}-1} \vec{\nabla} \Delta_0 f\|_2^{\frac{2}{p}} \leq \frac{p}{2} \|\Delta_0 f\|_p^{1-\frac{2}{p}} \|\vec{\nabla}\Delta_0 f\|_p^{\frac{2}{p}} \leq C_p \|\Delta_0 f\|_p$$

according to the Bernstein inequality. For the converse part, we just define $g_k = \frac{\partial_k}{\Delta} \Delta_0 f$, so that $\Delta_0 f = \sum_{1 \leq k \leq d} \frac{\partial}{\partial x_k} g_k$. We then write

$$\begin{array}{rcl}
\int |\Delta_0 f|^p \, dx & = & \int (\sum_{1 \leq k \leq d} \frac{\partial}{\partial x_k} g_k) \Delta_0 \bar{f} |\Delta_0 f|^{p-2} \, dx \\
& = & -\sum_{1 \leq k \leq d} \int g_k \frac{\partial}{\partial x_k} (\Delta_0 \bar{f} |\Delta_0 f|^{p-2}) \, dx \\
& \leq & \sum_{1 \leq k \leq d} (p-1) \|g_k\|_p \ \| \ |\Delta_0 f|^{p-2} \ \frac{\partial}{\partial x_k} \Delta_0 f\|_{p/(p-1)} \\
& \leq & \sum_{1 \leq k \leq d} (p-1) \|g_k\|_p \ \| \ |\Delta_0 f|^{\frac{p}{2}-1} \frac{\partial}{\partial x_k} \Delta_0 f\|_2 \|\Delta_0 f\|_p^{p/2-1} \\
& \leq & C \|\Delta_0 f\|_p^{p/2} \| \ |\Delta_0 f|^{\frac{p}{2}-1} \frac{\partial}{\partial x_k} \Delta_0 f\|_2
\end{array}$$

where the last inequality follows from the Bernstein inequality $\|\frac{\partial_k}{\Delta} \Delta_0 f\|_p \leq C \|\Delta_0 f\|_p$. □

If we look at a vector-valued function, we have a similar statement:

Proposition 29.2:

Let $p \in (2, \infty)$. Then there exists two positive constants c_p and C_p so that for every (real-valued) $\vec{f} \in (\mathcal{S}'(\mathbb{R}^d))^d$ and every $j \in \mathbb{Z}$ we have

$$c_p \leq \frac{\sum_{1 \leq k \leq d} \| \ |\Delta_j \vec{f}|^{\frac{p}{2}-1} \frac{\partial}{\partial x_k} \Delta_j \vec{f}\|_2 + \sum_{1 \leq k \leq d} \| \ |\Delta_j \vec{f}|^{\frac{p}{2}-2} \frac{\partial}{\partial x_k} \Delta_j \vec{f} . \Delta_j \vec{f}\|_2}{2^j \|\Delta_j \vec{f}\|_p^{p/2}} \leq C_p$$

where $\Delta_j \vec{f}$ is the j-th dyadic block in the Littlewood-Paley decomposition of $\vec{f}$.

Proof: By homogeneity, it is enough to prove the inequality for $j = 0$. The majoration by C_p is a direct consequence of the Bernstein inequalities. For the converse, we define $\vec{g}_k = \frac{\partial_k}{\Delta}\Delta_0 \vec{f}$, so that $|\Delta_0 \vec{f}|^2 = \sum_{1\leq k\leq d} \frac{\partial}{\partial x_k}\vec{g}_k.\Delta_0\vec{f}$, and then write

$$\begin{aligned}
\int |\Delta_0 \vec{f}|^p \, dx &= \int (\textstyle\sum_{1\leq k\leq d} \frac{\partial}{\partial x_k}\vec{g}_k).\Delta_0 \vec{f}|\Delta_0\vec{f}|^{p-2} \, dx \\
&= -\textstyle\sum_{1\leq k\leq d} \int \vec{g}_k . \frac{\partial}{\partial x_k}(\Delta_0 \vec{f}|\Delta_0 \vec{f}|^{p-2}) \, dx \\
&\leq \textstyle\sum_{1\leq k\leq d}(p-1)\|\vec{g}_k\|_p \, \| \, |\Delta_0 \vec{f}|^{p-2} \, \frac{\partial}{\partial x_k}\Delta_0\vec{f}\|_{p/(p-1)} \\
&\quad +(p-2)\textstyle\sum_{1\leq k\leq d}\|\vec{g}_k\|_p \, \| \, |\Delta_0\vec{f}|^{p-3} \, \frac{\partial}{\partial x_k}\Delta_0\vec{f}.\Delta_0\vec{f}\|_{p/(p-1)} \\
&\leq \textstyle\sum_{1\leq k\leq d}\|\vec{g}_k\|_p \, \| \, |\Delta_0\vec{f}|^{\frac{p}{2}-1}\frac{\partial}{\partial x_k}\Delta_0\vec{f}\|_2 \|\Delta_0\vec{f}\|_p^{p/2-1} \\
&\quad +(p-2)\textstyle\sum_{1\leq k\leq d}\|\vec{g}_k\|_p\| \, |\Delta_0\vec{f}|^{\frac{p}{2}-2}\frac{\partial}{\partial x_k}\Delta_0\vec{f}.\Delta_0\vec{f}\|_2\|\Delta_0\vec{f}\|_p^{p/2-1} \\
&\leq C\|\Delta_0\vec{f}\|_p^{p/2}\textstyle\sum_{1\leq k\leq d}\| \, |\Delta_0\vec{f}|^{\frac{p}{2}-1}\frac{\partial}{\partial x_k}\Delta_0\vec{f}\|_2 \\
&\quad +C\|\Delta_0\vec{f}\|_p^{p/2}\textstyle\sum_{1\leq k\leq d}\| \, |\Delta_0\vec{f}|^{\frac{p}{2}-2}\frac{\partial}{\partial x_k}\Delta_0\vec{f}.\Delta_0\vec{f}\|_2.
\end{aligned}$$

$\square$

3. Stability and Besov norms

We may now state the theorem on the Besov norms as Lyapunov functionals. We define the norm of $\vec{u}$ in $(\dot{B}_p^{s,q})^d$ as

$$\|\vec{u}\|_{\dot{B}_p^{s,q}} = \Big(\sum_{j\in\mathbb{Z}} 2^{jsq}\Big(\int\Big(\sum_{k=1}^d |\Delta_j u_k|^2\Big)^{p/2} \, dx\Big)^{q/p}\Big)^{1/q}.$$

Theorem 29.2:

Let $p \in [2,\infty)$, $q \in [1,\infty)$, $s > -1$ so that $s + \frac{2}{q} > 0$. Then there exists a positive constant $\epsilon = \epsilon(p,q,s)$ so that whenever $\vec{u}_0$ belongs to $(CMO^{-1})^d$ (where CMO^{-1} is the closure of the test functions in BMO^{-1}) and to $(\dot{B}_p^{s,q})^d$ with $\vec{\nabla}.\vec{u}_0 = 0$ and $\|\vec{u}_0\|_{BMO^{-1}} < \epsilon$, then the Koch-Tataru solution $\vec{u} \in (\mathcal{C}([0,\infty), CMO^{-1}))^d$ for the Navier–Stokes equations with initial data $\vec{u}_0$ belongs to $(\mathcal{C}([0,\infty), \dot{B}_p^{s,q}))^d$; moreover, its norm in $(\dot{B}_p^{s,q}(\mathbb{R}^d))^d$ decreases on $[0,\infty)$ and $\lim_{t\to\infty}\|\vec{u}\|_{\dot{B}_p^{s,q}} = 0$.

Proof:

Step 1: Regularity of the solutions

According to the Koch and Tataru theorem, we know that there exists $\epsilon(d) > 0$ so that, when $\vec{u}_0 \in (BMO^{-1}(\mathbb{R}^d))^d$ with $\vec{\nabla}.\vec{u}_0 = 0$ and $\|\vec{u}_0\|_d < \epsilon(d)$, then the Navier–Stokes equations with initial data $\vec{u}_0$ have a global solution;

this solution satisfies $\sup_{t>0} \sqrt{t}\|\vec{u}\|_\infty \leq C\|\vec{u}_0\|_{BMO^{-1}}$ and we easily check that $\sup_{t>0} \|\vec{u}\|_{BMO^{-1}} \leq C\|\vec{u}_0\|_{BMO^{-1}}$ (see Theorem 16.2 and its corollary).

If moreover $\vec{u}_0 \in (\dot{B}_p^{s,q})^d$ for some $p \in [1,\infty)$, $q \in [1,\infty)$ and $s > -1$, then we have some information on the behavior of $\vec{u}$ due to the various persistency theorems already proved (see Theorem 19.3 for $s > 0$, Theorem 22.3 for $s = 0$ and Theorem 20.3 for $s < 0$). More precisely, we proved:
i) for $s > 0$: $\vec{u} \in (\mathcal{C}_b([0,\infty), \dot{B}_p^{s,q}))^d$.
ii) for $s = 0$: $\sup_{t>0} t^{1/4}\|\vec{u}\|_{2p} < \infty$.
iii) for $s < 0$: $t^{-s/2}\|\vec{u}\|_p \in (L^q((0,\infty), \frac{dt}{t}))^d$.

In all cases, we find that $\vec{u} \in (\mathcal{C}_b([0,\infty), \dot{B}_p^{s,q}))^d$: we easily see that $\vec{u} - e^{t\Delta}\vec{u}_0 \in (L^\infty([0,\infty), \dot{B}_p^{s,1}))^d$. Indeed, it is simple to estimate the size of $\|B(\vec{u},\vec{u})\|_{B_\infty^{s+\epsilon,\infty}}$ for $\epsilon \in (\max(-1,-1-s),1)$:
i) for $s > 0$, we write that $\|\vec{u}\otimes\vec{u}\|_{\dot{B}_p^{s,q}} \leq C\|\vec{u}\|_{\dot{B}_p^{s,q}}\|\vec{u}\|_\infty \leq C(\vec{u}_0)t^{-1/2}$ and thus

$$\begin{aligned} \|B(\vec{u},\vec{u})\|_{B_\infty^{s+\epsilon,\infty}} &\leq \int_0^t \|\|(-\Delta)^{\epsilon/2}e^{(t-\sigma)\Delta}\mathbb{P}\vec{\nabla}\|\|_1 C(\vec{u}_0)\frac{d\sigma}{\sqrt{\sigma}} \\ &\leq C_\epsilon \int_0^t (t-\sigma)^{-(1+\epsilon)/2}C(\vec{u}_0)\frac{d\sigma}{\sqrt{\sigma}} = C'_\epsilon t^{-\epsilon/2}C(\vec{u}_0). \end{aligned}$$

ii) for $s = 0$, we write that $\|\vec{u}\otimes\vec{u}\|_p \leq C\|\vec{u}\|_{2p}^2 \leq C(\vec{u}_0)t^{-1/2}$ and thus $\|B(\vec{u},\vec{u})\|_{B_\infty^{s+\epsilon,\infty}} \leq C_\epsilon t^{-\epsilon/2}C(\vec{u}_0)$.
iii) for $s \in (-1,0)$, we write that $\|\vec{u}\otimes\vec{u}\|_p \leq C\|\vec{u}\|_p\|\vec{u}\|_\infty \leq C(\vec{u}_0)t^{(s-1)/2}$ and thus

$$\begin{aligned} \|B(\vec{u},\vec{u})\|_{B_\infty^{s+\epsilon,\infty}} &\leq \int_0^t \|\|(-\Delta)^{(s+\epsilon)/2}e^{(t-\sigma)\Delta}\mathbb{P}\vec{\nabla}\|\|_1 C(\vec{u}_0)\frac{d\sigma}{\sigma^{(1-s)/2}} \\ &\leq C_\epsilon \int_0^t (t-\sigma)^{-(1+s+\epsilon)/2}C(\vec{u}_0)\frac{d\sigma}{\sigma^{(1-s)/2}} = C'_\epsilon t^{-\epsilon/2}C(\vec{u}_0). \end{aligned}$$

Since $\|B(\vec{u},\vec{u})\|_{\dot{B}_p^{s,1}} \leq C\sqrt{\|B(\vec{u},\vec{u})\|_{\dot{B}_p^{s+\epsilon,\infty}}\|B(\vec{u},\vec{u})\|_{\dot{B}_p^{s-\epsilon,\infty}}}$, we proved that $B(\vec{u},\vec{u}) \in (L^\infty((0,\infty), \dot{B}_p^{s,1}))^d$. Thus, $\vec{u} \in (L^\infty((0,\infty), \dot{B}_p^{s,q}))^d$. This gives that, for all $[T_1,T_2] \subset (0,\infty)$ and all $\sigma > s$ $\vec{u} \in (L^\infty((T_1,T_2), \dot{B}_p^{\sigma,q}))^d$; indeed, if $T_0 \in (0,T_1)$, $\vec{u} \in (L^\infty((T_0,T_2), \dot{B}_p^{\sigma,q}))^d$ and $\epsilon \in (0,1)$, then we use the fact that $\vec{u}$ and all its derivatives are bounded on $[T_0,T_2]$ (according to the regularity criterion Proposition 15.1) and we write that $\|\vec{u}\otimes\vec{u}\|_{\dot{B}_p^{\sigma,q}} \leq C\|\vec{u}\|_{\dot{B}_p^{\sigma,q}}\|\vec{u}\|_{B_\infty^{1+|\sigma|,\infty}} \leq C(\vec{u}_0,T_0,T_2)$ and we find that on $[T_0,T_2]$ we have $\|\vec{u} - e^{(t-T_0)\Delta}\vec{u}(T_0)\|_{\dot{B}_p^{\sigma+\epsilon,q}} \leq C_\epsilon(T_2-T_0)^{(1-\epsilon)/2}C(\vec{u}_0,T_0,T_2)$, while on $[T_1,T_2]$ we have $\|e^{(t-T_0)\Delta}\vec{u}(T_0)\|_{\dot{B}_p^{\sigma+\epsilon,q}} \leq C_\epsilon(T_1-T_0)^{-\epsilon/2}C(\vec{u}_0,T_0,T_2)$.

In particular, we find that $\Delta\vec{u}$ and $\mathbb{P}\vec{\nabla}.\vec{u}\otimes\vec{u}$ (hence, $\partial_t\vec{u}$) belong to $(L^\infty((T_1,T_2), \dot{B}_p^{\sigma,q}))^d$ for all $\sigma \geq s$ and all $[T_1,T_2] \subset (0,\infty)$; hence, we get $\vec{u} \in (\mathcal{C}((0,\infty), \dot{B}_p^{\sigma,q}))^d$. For $\sigma = s$, the continuity is still valid for $t = 0$: we have $\lim_{t\to 0} \|e^{t\Delta}\vec{u}_0 - \vec{u}_0\|_{\dot{B}_p^{s,q}} = 0$ by density of the smooth functions in $\dot{B}_p^{s,q}$, while $\lim_{t\to 0} \|B(\vec{u},\vec{u})\|_{\dot{B}_p^{s,q}} = 0$ is a consequence of $\lim_{t\to 0} t^{1/2}\|\vec{u}\|_\infty = 0$.

We thus have proved the following estimates, that will be useful in the proof of Theorem 29.2:

j) $\vec{u} \in (\mathcal{C}([0,\infty), \dot{B}_p^{s,q}))^d$.
jj) $\vec{u} \in (\mathcal{C}((0,\infty), \dot{B}_p^{s+2/q,q}))^d$.
jjj)For some $\sigma < s$, $\vec{u} - e^{t\Delta}\vec{u}_0 \in (L^\infty((T_1,T_2), \dot{B}_p^{\sigma,q}))^d$ for all $[T_1,T_2] \subset (0,\infty)$.

Step 2: Derivative of the $\dot{B}_p^{s,q}$ norm

Recall that we have

$$\|\vec{u}\|^q_{\dot{B}_p^{s,q}} = \sum_{j\in\mathbb{Z}} 2^{jsq}(\int(\sum_{k=1}^d |\Delta_j u_k|^2)^{p/2}\ dx)^{q/p}.$$

The series converges uniformly on all $[T_1,T_2] \subset (0,\infty)$, since $\|\Delta_j\vec{u}\|_p \le \|\Delta_j\vec{u}_0\|_p + \|\Delta_j B(\vec{u},\vec{u})\|_p$ and since $B(\vec{u},\vec{u})$ is bounded in $(\dot{B}_p^{s+2/q,q})^d \cap (\dot{B}_p^{\sigma,q})^d$ on $[T_1,T_2]$ with $\sigma < s$. Thus, we may write (at least in the sense of distributions):

$$\frac{d}{dt}\|\vec{u}\|^q_{\dot{B}_p^{s,q}} = \sum_{j\in\mathbb{Z}} 2^{jsq}\frac{d}{dt}(\int(\sum_{k=1}^d |\Delta_j u_k|^2)^{p/2}\ dx)^{q/p}.$$

Let $\Phi_{j,\epsilon} = \left(\epsilon e^{-|x|^2} + \sum_{k=1}^d |\Delta_j u_k(x)|^2\right)^{1/2}$. Then

$$\frac{d}{dt}\|\Phi_{j,\epsilon}\|_p^q = q\|\Phi_{j,\epsilon}\|_p^{q-p}\int_{\mathbb{R}^d}(\partial_t\Delta_j\vec{u}).\Phi_{j,\epsilon}^{p-2}\Delta_j\vec{u}\ dx.$$

We find that $|\frac{d}{dt}\|\Phi_{j,\epsilon}\|_p^q| \le \|\Phi_{j,\epsilon}\|_p^{q-1}\|\Delta_j\partial_t\vec{u}\|_p$. Since we know that $\partial_t\vec{u}$ is, locally in time, bounded in $(\dot{B}_p^{s,q})^d$, we find by the dominated convergence theorem that

$$\frac{d}{dt}\|\Delta_j\vec{u}\|_p^q = q\|\Delta_j\vec{u}\|_p^{q-p}\int_{\mathbb{R}^d}\partial_t\Delta_j\vec{u}.|\Delta_j\vec{u}|^{p-2}\Delta_j\vec{u}\ dx.$$

This equality may be rewritten as

$$\begin{aligned}\tfrac{d}{dt}\|\Delta_j\vec{u}\|_p^q = \quad & q\|\Delta_j\vec{u}\|_p^{q-p}\textstyle\int_{\mathbb{R}^d}\Delta(\Delta_j\vec{u}).|\Delta_j\vec{u}|^{p-2}\Delta_j\vec{u}\ dx \\ & -q\|\Delta_j\vec{u}\|_p^{q-p}\textstyle\int_{\mathbb{R}^d}(\Delta_j\mathbb{P}\vec{\nabla}.\vec{u}\otimes\vec{u}).|\Delta_j\vec{u}|^{p-2}\Delta_j\vec{u}\ dx.\end{aligned}$$

Step 3: The term $\int_{\mathbb{R}^d}\Delta(\Delta_j\vec{u}).|\Delta_j\vec{u}|^{p-2}\Delta_j\vec{u}\ dx$

Recall that we proved in Section 1 the identity

$$\begin{gathered}\textstyle\int\Delta(\Delta_j\vec{u}).|\Delta_j\vec{u}|^{p-2}\Delta_j\vec{u}\ dx = \\ -\textstyle\sum_{k=1}^d\int|\partial_k\Delta_j\vec{u}|^2\ |\Delta_j\vec{u}|^{p-2}\ dx \\ -(p-2)\textstyle\sum_{k=1}^d\int|\partial_k\Delta_j\vec{u}.\Delta_j\vec{u}|^2\ |\Delta_j\vec{u}|^{p-4}\ dx.\end{gathered}$$

We now use Proposition 29.2 and get the estimate

$$\int\Delta(\Delta_j\vec{u}).|\Delta_j\vec{u}|^{p-2}\Delta_j\vec{u}\ dx \le -A_p 2^{2j}\|\Delta_j\vec{u}\|_p^p.$$

for some positive constant A_p.

Step 4: Control of $\int (\Delta_j \mathbb{P}\vec{\nabla}.\vec{u}\otimes\vec{u}).|\Delta_j\vec{u}|^{p-2}\Delta_j\vec{u}\ dx$ when $q \geq 2$

We use the paraproduct decomposition of the products and write

$$\|\Delta_j \dot{\pi}(u_k, u_l)\|_p \leq C\|S_{j+2}\vec{u}\|_\infty \sum_{m=-2}^{2} \|\Delta_{j+m}\vec{u}\|_p$$

and

$$\|\Delta_j \dot{\rho}(u_k, u_l)\|_{p/2} \leq C \sum_{m \geq j-2} \|\Delta_m \vec{u}\|_p^2$$

and thus we get

$$\begin{array}{rl} & |\int (\Delta_j \mathbb{P}\vec{\nabla}.\vec{u}\otimes\vec{u}).|\Delta_j\vec{u}|^{p-2}\Delta_j\vec{u}\ dx| \\ \leq C2^j & \big(\|S_{j+2}\vec{u}\|_\infty \|\Delta_j\vec{u}\|_p^{p-1} \sum_{m=-2}^{2} \|\Delta_{j+m}\vec{u}\|_p \\ & +\|\Delta_j\vec{u}\|_\infty \|\Delta_j\vec{u}\|_p^{p-2} \sum_{m\geq j-2} \|\Delta_m\vec{u}\|_p^2\big) \\ \leq C' 2^{2j} \|\vec{u}\|_{\dot{B}^{-1,\infty}_\infty} & \big(\|\Delta_j\vec{u}\|_p^{p-1} \sum_{m=-2}^{2} \|\Delta_{j+m}\vec{u}\|_p \\ & +\|\Delta_j\vec{u}\|_p^{p-2} \sum_{m\geq j-2} \|\Delta_m\vec{u}\|_p^2\big). \end{array}$$

We now sum over j to get:

$$\begin{array}{c} \sum_{j\in\mathbb{Z}} 2^{jsq} q \|\Delta_j\vec{u}\|_p^{q-p} \int_{\mathbb{R}^d} \Delta(\Delta_j\vec{u}).|\Delta_j\vec{u}|^{p-2}\Delta_j\vec{u}\ dx \\ \leq -qA_p \sum_{j\in\mathbb{Z}} 2^{jsq} \|\Delta_j\vec{u}\|_p^{q-p} 2^{2j} q \|\Delta_j\vec{u}\|_p^p \\ = -qA_p \|\vec{u}\|^q_{\dot{B}^{s+2/q,q}_\infty} \end{array}$$

and

$$\begin{array}{c} \sum_{j\in\mathbb{Z}} 2^{jsq} q \|\Delta_j\vec{u}\|_p^{q-p} |\int (\Delta_j \mathbb{P}\vec{\nabla}.\vec{u}\otimes\vec{u}).|\Delta_j\vec{u}|^{p-2}\Delta_j\vec{u}\ dx| \\ \leq C \sum_{j\in\mathbb{Z}} \|\vec{u}\|_{\dot{B}^{-1,\infty}_\infty} 2^{j(sq+2)} \|\Delta_j\vec{u}\|_p^{q-2} \times \\ \big(\|\Delta_j\vec{u}\|_p \sum_{m=-2}^{2} \|\Delta_{j+m}\vec{u}\|_p + \sum_{m\geq j-2} \|\Delta_m\vec{u}\|_p^2\big) \\ \leq C' \|\vec{u}\|_{\dot{B}^{-1,\infty}_\infty} \|\vec{u}\|^q_{\dot{B}^{s+2/q,q}_\infty} \end{array}$$

(since $s + 2/q > 0$); hence,

$$\frac{d}{dt}\|\vec{u}\|^q_{\dot{B}^{s,q}_\infty} \leq -qA_p(1 - B_p\|\vec{u}\|_{\dot{B}^{-1,\infty}_\infty})\|\vec{u}\|^q_{\dot{B}^{s+2/q,q}_\infty}$$

for two positive constants A_p and B_p, so that $\frac{d}{dt}\|\vec{u}\|^q_{\dot{B}^{s,q}_\infty} \leq 0$ when $\|\vec{u}\|_{\dot{B}^{-1,\infty}_\infty} \leq 1/B_p$.

Step 5: Control of $\int (\Delta_j \mathbb{P}\vec{\nabla}.\vec{u}\otimes\vec{u}).|\Delta_j\vec{u}|^{p-2}\Delta_j\vec{u}\ dx$ when $q < 2$

We use the paraproduct decomposition of the products and write

$$\|\Delta_j \dot{\pi}(u_k, u_l)\|_p \leq C\|S_{j+2}\vec{u}\|_\infty \sum_{m=-2}^{2} \|\Delta_{j+m}\vec{u}\|_p$$

and

$$\|\Delta_j \dot\rho(u_k, u_l)\|_{p/q} \le C \sum_{m\ge j-2} \|\Delta_m \vec u\|_p^q \|\Delta_m \vec u\|_\infty^{2-q}$$

and thus we get

$$\begin{array}{ll} & |\int (\Delta_j \mathbb{P}\vec\nabla.\vec u \otimes \vec u).|\Delta_j \vec u|^{p-2}\Delta_j \vec u \; dx| \\ \le C2^j & \big(\|S_{j+2}\vec u\|_\infty \|\Delta_j \vec u\|_p^{p-1} \sum_{m=-2}^{2} \|\Delta_{j+m}\vec u\|_p \\ & +\|\Delta_j \vec u\|_\infty^{q-1}\|\Delta_j \vec u\|_p^{p-q} \sum_{m\ge j-2} \|\Delta_m \vec u\|_p^q \|\Delta_m \vec u\|_\infty^{2-q}\big) \\ \le C' 2^j \|\vec u\|_{\dot B_\infty^{-1,\infty}} & \big(2^j \|\Delta_j \vec u\|_p^{p-1} \sum_{m=-2}^{2} \|\Delta_{j+m}\vec u\|_p \\ & +2^{j(q-1)} \|\Delta_j \vec u\|_p^{p-q} \sum_{m\ge j-2} 2^{k(2-q)} \|\Delta_m \vec u\|_p^q\big). \end{array}$$

We now sum over j to get:

$$\begin{array}{c} \sum_{j\in\mathbb{Z}} 2^{jsq} q \|\Delta_j \vec u\|_p^{q-p} \int_{\mathbb{R}^d} \Delta(\Delta_j \vec u).|\Delta_j \vec u|^{p-2}\Delta_j \vec u \; dx \\ \le -qA_p \sum_{j\in\mathbb{Z}} 2^{jsq} \|\Delta_j \vec u\|_p^{q-p} 2^{2j} q \|\Delta_j \vec u\|_p^p \\ = -qA_p \|\vec u\|^q_{\dot B_\infty^{s+2/q,q}} \end{array}$$

and

$$\begin{array}{c} \sum_{j\in\mathbb{Z}} 2^{jsq} q \|\Delta_j \vec u\|_p^{q-p} |\int (\Delta_j \mathbb{P}\vec\nabla.\vec u \otimes \vec u).|\Delta_j \vec u|^{p-2}\Delta_j \vec u \; dx| \\ \le C\|\vec u\|_{\dot B_\infty^{-1,\infty}} \sum_{j\in\mathbb{Z}} 2^{j(sq+2)} \|\Delta_j \vec u\|_p^{q-1} \sum_{m=-2}^{2} \|\Delta_{j+m}\vec u\|_p \\ +C\|\vec u\|_{\dot B_\infty^{-1,\infty}} \sum_{j\in\mathbb{Z}} 2^{j(s+1)q} \sum_{m\ge j-2} 2^{k(2-q)} \|\Delta_m \vec u\|_p^q\big) \\ \le C' \|\vec u\|_{\dot B_\infty^{-1,\infty}} \|\vec u\|^q_{\dot B_\infty^{s+2/q,q}} \end{array}$$

(since $s+1>0$); hence,

$$\frac{d}{dt}\|\vec u\|^q_{\dot B_\infty^{s,q}} \le -qA_p(1 - B_p \|\vec u\|_{\dot B_\infty^{-1,\infty}}) \|\vec u\|^q_{\dot B_\infty^{s+2/q,q}}$$

for two positive constants A_p and B_p, so that $\frac{d}{dt}\|\vec u\|^q_{\dot B_\infty^{s,q}} \le 0$ when $\|\vec u\|_{\dot B_\infty^{-1,\infty}} \le 1/B_p$. □

Remark: Danchin [DAN 01] recently extended the Bernstein-like inequality described in Proposition 29.1 to the range $1 < p < \infty$. This allows us to extend Theorem 29.2 to Besov spaces $\dot B_p^{s,q}$ with $1 < p < 2$ (replacing the condition $s + \frac{q}{2} > 0$ by $s + \frac{q}{2} > \frac{2-p}{p}$).

Part 6:

Local energy inequalities for the Navier–Stokes equations on $\mathbb{R}^3$

Chapter 30

The Caffarelli, Kohn, and Nirenberg regularity criterion

1. Suitable solutions

Leray's theorem is based on an energy inequality. To give an accurate description of the local regularity of a Leray solution, Scheffer introduced a local version of this energy inequality [SCH 77]. Weak solutions satisfying the local energy inequality are termed *suitable* by Caffarelli, Kohn, and Nirenberg [CAFKN 82]. We begin by defining a loose version of weak solutions:

Definition 30.1: (Weak solutions)

Let $\vec{u}(t,x)$ be a weak solution for the Navier–Stokes equations on $(0,T) \times \mathbb{R}^3$:
i) $\vec{u}$ is locally square integrable on $(0,T) \times \mathbb{R}^3$
ii) $\vec{\nabla}.\vec{u} = 0$
iii) $\exists p \in \mathcal{D}'((0,T) \times \mathbb{R}^3)$ $\partial_t \vec{u} = \Delta\vec{u} - \vec{\nabla}.(\vec{u} \otimes \vec{u}) - \vec{\nabla} p$

Then we define suitable solutions:

Definition 30.1: (Suitable solutions)

Let $\vec{u}(t,x)$ be a weak solution for the Navier–Stokes equations on $(0,T) \times \mathbb{R}^3$. $\vec{u}$ is suitable *on the cylinder $Q = (a,b) \times B(x_0, r_0)$ (where $x_0 \in \mathbb{R}^3$, $r_0 > 0$ and $0 < a < b < T$) if $\vec{u}$ satisfies moreover*
j) $\sup_{a<t<b} \int_{B(x_0,r_0)} |\vec{u}|^2 \, dx < \infty$.
jj) $\int\int_Q |\vec{\nabla} \otimes \vec{u}|^2 \, dx \, dt < \infty$.
jjj) $\int\int_Q |p|^{3/2} \, dx \, dt < \infty$.
jv) for all $\phi \in \mathcal{D}(Q)$ such that $\phi \geq 0$ we have

$$(30.1) \quad 2 \iint |\vec{\nabla} \otimes \vec{u}|^2 \phi \, dx \, dt \leq \iint |\vec{u}|^2 (\partial_t \phi + \Delta\phi) \, dx \, dt + \iint (|\vec{u}|^2 + 2p)(\vec{u}.\vec{\nabla})\phi \, dx \, dt$$

$\vec{u}$ is suitable *on $(0,T) \times \mathbb{R}^3$ if $\vec{u}$ is suitable on all cylinders $Q \subset (0,T) \times \mathbb{R}^3$.*

Let us first comment on hypothesis jjj). In the original definition of Caffarelli, Kohn, and Nirenberg, the hypothesis for suitable solutions was $p \in$

$L^{5/4}(Q)$. The reason for using this exponent is that it was at that time the best exponent one could prove when exhibiting suitable solutions for the Navier–Stokes equations on a *bounded* domain associated with a square–integrable initial value $\vec{u}_0$ (for the whole space, it is easy to check that the restricted Leray solutions are suitable on $(0,\infty) \times \mathbb{R}^3$ and that the associated pressure may be chosen in $L^{5/3}_{loc}((0,\infty) \times \mathbb{R}^3)$; see Proposition 30.1 below). See the recent paper of Ladyzhenskaya and Seregin [LADS 99]. But Lin [LIN 98] proved the existence of suitable solutions with $p \in L^{3/2}_{loc}((0,T) \times \Omega)$ for bounded domains Ω, by using regularity estimates for the pressure obtained by Sohr and Von Wahl [SOHW 86]. The computations are easier with the hypothesis $p \in L^{3/2}(Q)$ and we always assume this hypothesis in our definitions of suitability.

Inequality (30.1) may be restated in the following form: there exists a local nonnegative measure μ on $(0,T) \times \mathbb{R}^3$ so that the following equality holds in $\mathcal{D}'((0,T) \times \mathbb{R}^3)$:

$$\partial_t |\vec{u}|^2 + 2|\vec{\nabla} \otimes \vec{u}|^2 = \Delta|\vec{u}|^2 - \vec{\nabla}.\{(|\vec{u}|^2 + 2p)\vec{u}\} - \mu \tag{30.2}$$

Existence of suitable solutions on $(0,\infty) \times \mathbb{R}^3$ is provided by the following remarks:

Proposition 30.1: (Leray solution)
(A) A restricted Leray solution on $(0,T) \times \mathbb{R}^3$ *is suitable.*
(B) A suitable Leray solution satisfies the strong energy inequality.

Proof: We first prove (A). Recall that a restricted Leray solution $\vec{u}$ associated with an initial data $\vec{u}_0 \in (L^2(\mathbb{R}^3))^3$ with $\vec{\nabla}.\vec{u}_0 = 0$ is given as a limit of solutions $\vec{u}_{\epsilon_n}$ for the mollified equations (with $\lim_{n\to\infty} \epsilon_n = 0$). We have $\vec{u} \in (L^\infty((0,T), L^2))^3 \cap (L^2((0,T), \dot{H}^1))^3$. Obviously,
i) $\sup_{0<t<T} \int |\vec{u}|^2 \, dx < \infty$.
ii) $\int_{\mathbb{R}^3} \int_0^T |\vec{\nabla} \otimes \vec{u}|^2 \, dx \, dt < \infty$.
iii) $\vec{u} \in (L^\infty_t L^2_x)^3 \cap (L^2_t L^6_x)^3 \subset (L^4_t L^3_x)^3$. Hence for all finite $T' \leq T$ we have $\int_0^{T'} \int_{\mathbb{R}^3} |\vec{u}|^3 \, dx \, dt < \infty$ and, thus, $\int_0^{T'} \int_{\mathbb{R}^3} |p|^{3/2} \, dx \, dt < \infty$, since p may be computed as $p = \sum_{j=1}^3 \sum_{k=1}^3 R_j R_k (u_j u_k)$.

We have only to prove the local energy inequality (30.1). We start from the equality for the smooth function $\vec{u}_\epsilon$:

$$\partial_t |\vec{u}_\epsilon|^2 + 2|\vec{\nabla} \otimes \vec{u}_\epsilon|^2 = \Delta|\vec{u}_\epsilon|^2 \ - \vec{\nabla}.(|\vec{u}_\epsilon|^2 (\vec{u_\epsilon} * \omega_\epsilon) + 2p_\epsilon \ \vec{u}_\epsilon)$$

and we let ϵ_n go to 0. We know that $\vec{u}_{\epsilon_n}$ converges strongly to $\vec{u}$ in $(L^2_{loc})^3$: for all $\phi \in \mathcal{D}((0,T) \times \mathbb{R}^3)$ we have $\lim_{\epsilon_n} \int\int |\phi(t,x)(\vec{u}_{\epsilon_n}(t,x) - \vec{u}(t,x))|^2 \, dt \, dx = 0$. This gives strong convergence in $((L^6_t L^2_x)_{loc})^3$ (since the functions $\vec{u}_\epsilon$ are uniformly bounded in $(L^\infty_t L^2_x)^3$) and strong convergence in $((L^3_{loc})^3$ (since the functions $\vec{u}_\epsilon$ are uniformly bounded in $(L^2_t L^6_x)^3$). Moreover, the functions p_ϵ are uniformly bounded in the space $L^2_t L^{3/2}_x$; hence, we may assume that the sequence p_{ϵ_n} converges weakly to p_ϵ in $L^2_t L^{3/2}_x$. Thus, we have the following convergences in $\mathcal{D}'$: $|\vec{u}_{\epsilon_n}|^2 \to |\vec{u}|^2$, $|\vec{u}_{\epsilon_n}|^2(\vec{u_{\epsilon_n}} * \omega_{\epsilon_n}) \to |\vec{u}|^2\vec{u}$ and $p_{\epsilon_n}\ \vec{u}_{\epsilon_n} \to p\ \vec{u}$. Hence, $2|\vec{\nabla} \otimes \vec{u}_{\epsilon_n}|^2$ converges in $\mathcal{D}'$; since $\vec{\nabla} \otimes \vec{u}_{\epsilon_n}$ is weakly convergent in $(L^2_{loc})^{3\times 3}$, we find that $2|\vec{\nabla} \otimes \vec{u}_{\epsilon_n}|^2 \to 2|\vec{\nabla} \otimes \vec{u}|^2 + \mu$ where μ is a nonnegative locally finite measure.

The proof of (B) follows the proof of the strong energy inequality for the restricted Leray solutions (Proposition 14.1). We choose $\theta \in \mathcal{D}(\mathbb{R})$ with $\theta \geq 0$ and $\int \theta\ dt = 1$ and we take for $t_1 > t_0 > 0$ and $\eta > 0$ $\alpha_\eta(t) = \frac{1}{\eta}\int_{-\infty}^t (\theta(\frac{s-t_0}{\eta}) - \theta(\frac{s-t_1}{\eta}))\ ds$; moreover, we take $\beta \in \mathcal{D}(\mathbb{R}^3)$ with $\beta = 1$ in a neighborhood of 0, $\beta \geq 0$, and we define $\beta_R(x) = \beta(x/R)$. We apply the local energy inequality to $\varphi = \alpha_\eta(t)\beta_R(x)$. We let R go to $+\infty$ and obtain

$$2\iint |\vec{\nabla} \otimes \vec{u}|^2\ \alpha_\eta(t)\ dx\ dt \leq \iint |\vec{u}|^2 \frac{1}{\eta}(\theta(\frac{t-t_0}{\eta}) - \theta(\frac{t-t_1}{\eta}))\ dx\ dt$$

If t_0 and t_1 are Lebesgue points for the measurable function $t \mapsto \|\vec{u}\|_{L^2(dx)}$, we get when η goes to 0 that

$$2\int_{t_0}^{t_1}\int |\vec{\nabla} \otimes \vec{u}|^2\ \alpha_\eta(t)\ dx\ dt \leq \int |\vec{u}(t_0,x)|^2\ dx - \int |\vec{u}(t_1,x)|^2\ dx.$$

For a fixed (Lebesgue point) t_0, this inequality is then extended to all $t_1 > t_0$ by weak continuity. □

There are many other ways of finding suitable solutions on the whole space; for instance, Beirão da Vega [BEI 85] uses the approximation to the Navier–Stokes equations by adding a bi-Laplacian in the equations ($\partial_t \vec{u}_\epsilon = -\epsilon\Delta^2\vec{u}_\epsilon + \Delta\vec{u}_\epsilon - \vec{\nabla}.\vec{u}_\epsilon \otimes \vec{u}_\epsilon - \vec{\nabla}p_\epsilon$).

The measure μ expresses the lack of regularity of the solution $\vec{u}$. This may be viewed in several ways. For instance, if we look at the proof of Serrin's theorem on energy equality, we find the following basic result:

Proposition 30.2: (Energy equality)

Let $\vec{u}$ be a weak solution for the Navier–Stokes equations on $(0,T)\times\mathbb{R}^3$ and assume that $\vec{u}$ satisfies hypotheses j), jj) and jjj) of Definition 30.2 on a cylinder $Q \subset (0,T)\times\mathbb{R}^3$. If $\vec{u} \in (L^p_t L^q_x(Q))^d$ for some $d \in [d,+\infty]$ and $\frac{1}{p} = \frac{1}{2} - \frac{d}{2q}$, then $\vec{u}$ satisfies (30.2) on Q with $\mu = 0$.

We begin the proof by recalling an elementary inequality:

Lemma 30.1:

Let $B = B(0,1) \subset \mathbb{R}^3$, $f \in L^2(B)$, *and* $\vec{\nabla} f \in (L^2(B))^d$. *Then, for* $0 < \rho < 1$, $f \in L^3(B(0,\rho))$ *and* $\int_{B(0,\rho)} |f|^3\, dx \le C_\rho(\|f\|^3_{L^2(B)} + \|f\|^{3/2}_{L^2(B)} \|\vec{\nabla} f\|^{3/2}_{L^2(B)})$.

Proof: Let $\varphi \in \mathcal{D}(\mathbb{R}^3)$ with $\varphi = 1$ for $|x| \le (1+2\rho)/3$ and $\varphi = 0$ for $|x| \ge (2+\rho)/3$. Then $\vec{\nabla}(\varphi f) \in L^2(\mathbb{R}^3)$; thus, the Sobolev inequality gives us that $\varphi f \in L^6(\mathbb{R}^3)$ and $\|f\|_{L^3(B(0,\rho)} \le \|f\varphi\|_2^{1/2} \|f\varphi\|_6^{1/2} \le C_\rho \|f\|_2^{1/2}(\|f\|^{1/2}_{L^2(B)} + \|\vec{\nabla} f\|^{1/2}_{L^2(B)})$. □

Proof of Proposition 2: We use a smoothing function $\theta \in \mathcal{D}(\mathbb{R}^d)$ which is supported in $B(0,1)$, and satisfies $\int\int \theta\, dx\, dt = 1$, and define, for $0 < \epsilon < r_0/2$, $\theta_\epsilon(x) = \frac{1}{\epsilon^d}\theta(\frac{x}{\epsilon})$. According to Lemma 30.1, $\theta_\epsilon * \vec{u}$ belongs to $(L^3((a,b) \times B(x_0, r_0 - 2\epsilon)))^3$ and $\theta_\epsilon * \partial_t \vec{u}$ belongs to $(L^{3/2}((a,b) \times B(x_0, r_0 - 2\epsilon)))^3$; hence, we may write $\partial_t |\theta_\epsilon * \vec{u}|^2 = 2\partial_t(\theta_\epsilon * \vec{u})\big).\theta_\epsilon * \vec{u} = (\theta_\epsilon * \partial_t \vec{u}).\theta_\epsilon * \vec{u}$. Then:

$$\begin{gathered}(\theta_\epsilon * \partial_t \vec{u}).\theta_\epsilon * \vec{u} = \\ \vec{\nabla}.\big((\theta_\epsilon * [\vec{\nabla} \otimes \vec{u}]).(\theta_\epsilon * \vec{u})\big) - |\theta_\epsilon * [\vec{\nabla} \otimes \vec{u}]|^2 \\ -\vec{\nabla}.\big((\theta_\epsilon * [\vec{u} \otimes \vec{u}]).(\theta_\epsilon * \vec{u})\big) + (\theta_\epsilon * [\vec{u} \otimes \vec{u}]).(\theta_\epsilon * [\vec{\nabla} \otimes \vec{u}]) \\ -\vec{\nabla}.\big((\theta_\epsilon * p)(\theta_\epsilon * \vec{u})\big)\end{gathered}$$

Since $\vec{u} \in (L^\infty_t L^2_x(Q))^3 \subset (L^2(Q))^3$, we have in $\mathcal{D}'((a,b) \times B(x_0, r_0 - 2\epsilon))$ that $\lim_{\epsilon \to 0} \partial_t |\theta_\epsilon * \vec{u}|^2 = \partial_t |\vec{u}|^2$ and $\lim_{\epsilon \to 0} \vec{\nabla}.\big((\theta_\epsilon * [\vec{\nabla} \otimes \vec{u}]).(\theta_\epsilon * \vec{u})\big) = \lim_{\epsilon \to 0} \frac{1}{2}\Delta.|\theta_\epsilon * \vec{u}|^2 = \frac{1}{2}\Delta.|\vec{u}|^2$. Similarly, since $\vec{\nabla} \otimes \vec{u} \in (L^2(Q))^{d\times d}$, we have in $\mathcal{D}'((a,b) \times B(x_0, r_0 - 2\epsilon))$ the equality $\lim_{\epsilon \to 0} |\theta_\epsilon * [\vec{\nabla} \otimes \vec{u}]|^2 = |\vec{\nabla} \otimes \vec{u}|^2$. We know from Lemma 30.1 and from assumptions j) and jj) that $\vec{u} \in (L^3((a,b) \times B(x_0, r_0 - \epsilon)))^3$ and from assumption jjj) that $p \in L^{3/2}((a,b) \times B(x_0, r_0)))$. Hence, we get the equalities in $\mathcal{D}'((a,b) \times B(x_0, r_0 - 2\epsilon))$: $\lim_{\epsilon \to 0} \vec{\nabla}.\big((\theta_\epsilon * [\vec{u} \otimes \vec{u}]).(\theta_\epsilon * \vec{u})\big) = \vec{\nabla}.\big(|\vec{u}|^2 \vec{u}\big)$ and $\lim_{\epsilon \to 0} \vec{\nabla}.\big((\theta_\epsilon * p)(\theta_\epsilon * \vec{u})\big) = \vec{\nabla}.\big(p\vec{u})\big)$.

Thus, under the sole assumptions j), jj) and jjj), we get that

$$\begin{aligned}\partial_t |\vec{u}|^2 = \quad & \Delta |\vec{u}|^2 - 2|\vec{\nabla} \otimes \vec{u}|^2 - 2\vec{\nabla}.\big(|\vec{u}|^2 \vec{u}\big) - 2\vec{\nabla}.\big(p\vec{u})\big) \\ & +2 \lim_{\epsilon \to 0} (\theta_\epsilon * [\vec{u} \otimes \vec{u}]).(\theta_\epsilon * [\vec{\nabla} \otimes \vec{u}])\end{aligned}$$

Now, if we assume that $\vec{u} \in (L^p_t L^q_x(Q))^3$, we find (from the proof of Lemma 30.1) that $\vec{u} \in (L^r_t L^s_x((a,b) \times B(x_0, r_0 - \epsilon)))^3$ with $1/p + 1/r = 1/q + 1/s = 2$, and thus in $\mathcal{D}'((a,b) \times B(x_0, r_0 - 2\epsilon))$ we have $\lim_{\epsilon \to 0} 2(\theta_\epsilon * [\vec{u} \otimes \vec{u}]).(\theta_\epsilon * [\vec{\nabla} \otimes \vec{u}]) = 2[\vec{u} \otimes \vec{u}].[\vec{\nabla} \otimes \vec{u}] = \lim_{\epsilon \to 0} [(\theta_\epsilon * \vec{u}) \otimes (\theta_\epsilon * \vec{u})].(\theta_\epsilon * [\vec{\nabla} \otimes \vec{u}]) = \lim_{\epsilon \to 0} \vec{\nabla}\big(|\theta_\epsilon * \vec{u}|^2 (\theta_\epsilon * \vec{u})\big) = \vec{\nabla}\big(|\vec{u}|^2 \vec{u}\big)$. □

Duchon and Robert [DUCR 99] proposed an impressive visualization of how the energy inequality expresses the lack of regularity of the solution:

Proposition 30.3:

Let $\vec{u}$ be a weak solution for the Navier–Stokes equations on $(0,T)\times \mathbb{R}^3$ and assume that $\vec{u}$ satisfies hypotheses j), jj) and jjj) of Definition 30.2 on a cylinder $Q \subset (0,T)\times\mathbb{R}^3$. If moreover $\vec{u} \in (L^p_t L^q_x(Q))^d$ for some $d \in [d,+\infty]$ and $\frac{1}{p} = \frac{1}{2} - \frac{d}{2q}$. Then

(A) The distribution $\mu = -\partial_t|\vec{u}|^2 - 2|\vec{\nabla}\otimes\vec{u}|^2 + \Delta|\vec{u}|^2 - \vec{\nabla}.\{(|\vec{u}|^2+2p)\vec{u}\}$ may be computed as the limit in $\mathcal{D}'(Q)$ of $\mu_\epsilon - 2\nu_\epsilon$, where

$$\mu_\epsilon = -\int \vec{\nabla}\varphi_\epsilon(\xi).\big(|\vec{u}(t,x-\xi)-\vec{u}(t,x)|^2(\vec{u}(t,x-\xi)-\vec{u}(t,x))\big)\ d\xi$$

and

$$\nu_\epsilon = (\vec{u}-\varphi_\epsilon*\vec{u}).\sum_{k=1}^d \int \partial_k\varphi_\epsilon(\xi)(u_k(t,x-\xi)-u_k(t,x))\ (\vec{u}(t,x-\xi)-\vec{u}(t,x))\ d\xi$$

where $\varphi_\epsilon = \frac{1}{\epsilon^3}\varphi(\frac{x}{\epsilon})$, φ is a smooth function supported in $B(0,1)$ with $\int\varphi\,dx = 1$ and where μ_ϵ is defined in $\mathcal{D}'((a,b)\times B(x_0,r_0-\epsilon))$.

(B) Let $b_3^{1/3,\infty}$ is the closure of $\mathcal{D}(\mathbb{R}^3)$ in $B_3^{1/3,\infty}$. If for all $\omega\in\mathcal{D}(Q)$ the vector distribution $\omega\vec{u}$ belongs to $(L^3((a,b),b_3^{1/3,\infty}))^3)$, then $\mu = 0$ on Q.

Proof: From the proof of Proposition 30.2, we already know that

$$\begin{aligned}\partial_t|\vec{u}|^2 = \ & \Delta|\vec{u}|^2 - 2|\vec{\nabla}\otimes\vec{u}|^2 - 2\vec{\nabla}.\big(|\vec{u}|^2\vec{u}\big) - 2\vec{\nabla}.\big(p\vec{u}\big)\big) \\ & +2\lim_{\epsilon\to0}(\varphi_\epsilon*[\vec{u}\otimes\vec{u}]).(\varphi_\epsilon*[\vec{\nabla}\otimes\vec{u}]).\end{aligned}$$

Hence $\mu = \vec{\nabla}.\big(|\vec{u}|^2\vec{u}\big) - 2\lim_{\epsilon\to0}(\varphi_\epsilon*[\vec{u}\otimes\vec{u}]).(\varphi_\epsilon*[\vec{\nabla}\otimes\vec{u}])$.

On the other hand, we develop $|\vec{u}(t,x-\xi)-\vec{u}(t,x)|^2(\vec{u}(t,x-\xi)-\vec{u}(t,x))$ and we get:

$$\begin{aligned}\mu_\epsilon \ = \ & -\varphi_\epsilon*\vec{\nabla}.(|\vec{u}|^2\vec{u}) + \vec{u}.\vec{\nabla}(\varphi_\epsilon*|\vec{u}|^2) - |\vec{u}|^2(\varphi_\epsilon*\vec{\nabla}.\vec{u}) + (\textstyle\int\vec{\nabla}\varphi_\epsilon\ dx).|\vec{u}|^2\vec{u} \\ & +2\vec{u}.\big(\varphi_\epsilon*(\vec{u}.\vec{\nabla})\vec{u}\big) - 2\vec{u}.(\vec{u}.\vec{\nabla})(\varphi_\epsilon*\vec{u}) \\ = & -\varphi_\epsilon*\vec{\nabla}.(|\vec{u}|^2\vec{u}) + \vec{u}.\vec{\nabla}(\varphi_\epsilon*|\vec{u}|^2) + 2\vec{u}.\big(\varphi_\epsilon*(\vec{u}.\vec{\nabla})\vec{u}\big) - 2\vec{u}.(\vec{u}.\vec{\nabla})(\varphi_\epsilon*\vec{u})\end{aligned}$$

We have

$$\lim_{\epsilon\to0}\varphi_\epsilon*\vec{\nabla}.(|\vec{u}|^2\vec{u}) - \vec{u}.\vec{\nabla}(\varphi_\epsilon*|\vec{u}|^2) = \lim_{\epsilon\to0}\vec{\nabla}.\big(\varphi_\epsilon*(|\vec{u}|^2\vec{u}) - \vec{u}(\varphi_\epsilon*|\vec{u}|^2)\big) = 0$$

in $\mathcal{D}'(Q)$. Thus, the only significant term in μ_ϵ when ϵ goes to 0 will be $2\vec{u}.\big(\varphi_\epsilon*(\vec{u}.\vec{\nabla})\vec{u}\big) - 2\vec{u}.(\vec{u}.\vec{\nabla})(\varphi_\epsilon*\vec{u})$.

Similarly, we develop $(\vec{u}_k(t,x-\xi)-u_k(t,x))(\vec{u}(t,x-\xi)-\vec{u}(t,x))$ and we get:

$$\begin{aligned}\nu_\epsilon = & (\vec{u}-\varphi_\epsilon*\vec{u}).\big[\varphi_\epsilon*(\vec{u}.\vec{\nabla})\vec{u} + (\textstyle\int\vec{\nabla}\varphi_\epsilon dx).\vec{u}\otimes\vec{u} - (\vec{u}.\vec{\nabla})\varphi_\epsilon*\vec{u} - (\varphi_\epsilon*(\vec{\nabla}.\vec{u})\vec{u}\big] \\ & = (\vec{u}-\varphi_\epsilon*\vec{u}).\big[\varphi_\epsilon*(\vec{u}.\vec{\nabla})\vec{u} - (\vec{u}.\vec{\nabla})\varphi_\epsilon*\vec{u}\big]\end{aligned}$$

Thus, we have

$$\lim_{\epsilon\to 0}\mu_\epsilon - 2\nu_\epsilon = 2\lim_{\epsilon\to 0}\varphi_\epsilon * \vec{u}.\big[\varphi_\epsilon * (\vec{u}.\vec{\nabla})\vec{u} - (\vec{u}.\vec{\nabla})\varphi_\epsilon * \vec{u}\big]$$
$$= \lim_{\epsilon\to 0} 2\vec{\nabla}.\big[\varphi_\epsilon * \vec{u}.\varphi_\epsilon * (\vec{u}\otimes\vec{u})\big] - 2(\varphi_\epsilon * [\vec{u}\otimes\vec{u}]).(\varphi_\epsilon * [\vec{\nabla}\otimes\vec{u}]) - \vec{\nabla}.(|\varphi_\epsilon * \vec{u}|^2\vec{u})$$
$$= \vec{\nabla}.(|\vec{u}|^2\vec{u}) - 2\lim_{\epsilon\to 0}(\varphi_\epsilon * [\vec{u}\otimes\vec{u}]).(\varphi_\epsilon * [\vec{\nabla}\otimes\vec{u}]) = \mu.$$

To prove (B), we fix $\epsilon_0 > 0$ and we consider $\epsilon < \epsilon_0$. We consider $\omega \in \mathcal{D}(Q)$ equal to 1 on $(a+\epsilon_0, b-\epsilon_0)\times B(x_0, r_0-\epsilon_0)$ and define $\vec{v} = \omega\vec{u}$. On $Q_0 = (a+\epsilon_0, b-\epsilon_0)\times B(x_0, r_0 - 2\epsilon_0)$, we have

$$\int_{Q_0} |\mu_\epsilon(t,x)|\, dx \le \frac{C}{\epsilon}\int_{a_0+\epsilon_0}^{b-\epsilon_0} \sup_{|\xi|<\epsilon} \|\vec{v}(t,x-\xi) - \vec{v}(t,x)\|_3^3\, dt$$

and we have the same estimate for $\int_{Q_0} |\nu_\epsilon(t,x)|\, dx$. We have the estimate $\sup_{|\xi|<\epsilon}\|\vec{v}(t,x-\xi)-\vec{v}(t,x)\|_3^3 \le C\epsilon\|\vec{v}(t,.)\|^3_{B_3^{1/3,\infty}} \in L^1((a_0+\epsilon_0, b_0-\epsilon_0))$ and we know that for almost every t $\lim_{\epsilon\to 0}\frac{1}{\epsilon}\sup_{|\xi|<\epsilon}\|\vec{v}(t,x-\xi)-\vec{v}(t,x)\|_3^3 = 0$. We then may conclude by dominated convergence. □

2. A fundamental inequality

Scheffer [SCH 77] described how to choose special test functions in order to use the local energy inequality:

Lemma 30.2: (Scheffer's test functions)

Let $\vec{u}$ be a weak solution for the Navier–Stokes equations on $(0,T)\times\mathbb{R}^3$, so that $\vec{u}$ is a suitable solution on the cylinder $(0,1)\times B(0,1)$. Let $\varphi(x)\in\mathcal{D}(\mathbb{R}^3)$ be a nonnegative function such that $\varphi\equiv 1$ on $B(0,1/4)$ and Supp $\varphi\subset B(0,1/2)$ and we define $\varphi_x(y) = \varphi(y-x)$.

Let $W(t,y)$ be the fundamental solution of the heat equation:

$$W(t,y) = \Big(\frac{1}{4\pi t}\Big)^{3/2} e^{-\frac{|y|^2}{4t}}$$

which is the solution of $\partial_t W - \Delta W = \delta$. The function $\psi_{x,s,k}$ is then defined as

$$\psi_{x,s,k}(\sigma,y) = W(4^{-k} + s - \sigma, y - x).$$

We then have the inequality for $x\in B(0,1/2)$, $k\ge 2$ and $3/4 - 4^{-k} < s < 1$

$$2^{3k}\int_{|x-y|<\frac{1}{2^k}} |\vec{u}(s,y)|^2\, dy \; + 2^{3k}\iint_{s-\frac{1}{4^k}<\sigma<s,\ |x-y|<\frac{1}{2^k}} |\vec{\nabla}\otimes\vec{u}(\sigma,y)|^2\, dy\, d\sigma \le$$
$$D_0\Big(\iint_{0<\sigma<1, |y|<1} |\vec{u}(\sigma,y)|^3\, d\sigma\, dy\Big)^{2/3} +$$
$$e^{1/4}(8\pi)^{3/2}\Big\{\iint_0^s (|\vec{u}(\sigma,y)|^2 + 2p(\sigma,y))\{\vec{u}(\sigma,y).\vec{\nabla}\}(\varphi_x(y)\psi_{x,s,k}(\sigma,y))\, dy\, d\sigma\Big\}$$

where D_0 depends neither on $\vec{u}$, nor on x, s, k.

Proof: Since $\vec{u}$ is locally L^3 and p is locally $L^{3/2}$, we find that $\partial_t \vec{u}$ is locally $L^1_t(H_x^{-3/2})$, which gives that $\vec{u}$ is weakly continuous from $(0,1)$ into $\mathcal{D}'(B(0,1))$; since $\vec{u} \in (L^\infty_t L^2_x((0,1) \times B(0,1)))^3$, we get that $\vec{u}$ is weakly continuous from $(0,1)$ into $(L^2(B(0,1)))^3$. Now, we fix $\theta \in \mathcal{D}(\mathbb{R})$, $\operatorname{Supp} \theta \subset [0,1]$, $\theta \geq 0$ and $\int \theta \, dt = 1$ and we introduce for $\epsilon > 0$ the function $\omega_{s,\epsilon}(\sigma) = \int_{-\infty}^{\sigma} 2\theta(2\tau) - \frac{1}{\epsilon}\theta(\frac{s-\tau}{\epsilon}) \, d\tau$. If $s > \frac{1}{2}$ and if ϵ is small enough ($\epsilon < s - \frac{1}{2}$), $\omega_{s,\epsilon}$ is nonnegative and bounded by 1. If for some $X \in (s,1)$, we have $\psi(\sigma,y) \in \mathcal{C}^\infty((0,X) \times B(0,1))$ and if ψ is nonnegative, the local energy inequality gives for $|x| < 1/2$ and $0 < s < 1$:

$$\begin{gathered}\iint |\vec{\nabla} \otimes \vec{u}(\sigma,y)|^2 \; \omega_{s,\epsilon}(\sigma)\varphi_x(y)\psi(\sigma,y) \; dy \; d\sigma \leq \\ \iint |\vec{u}(\sigma,y)|^2 \; \{\partial_\sigma + \Delta_y\}(\omega_{s,\epsilon}(\sigma)\varphi_x(y)\psi(\sigma,y)) \; dy \; d\sigma \\ + \iint (|\vec{u}(\sigma,y)|^2 + 2p(\sigma,y))\{\vec{u}(\sigma,y).\vec{\nabla}\}(\omega_{s,\epsilon}(\sigma)\varphi_x(y)\psi(\sigma,y)) \; dy \; d\sigma\end{gathered}$$

We let ϵ go to 0; then, for all Lebesgue point s of $s \mapsto \int_{|y|<1} |\vec{u}(s,y)|^2 \, dy$, we get that

$$\begin{gathered}\int |\vec{u}(s,y)|^2 \; \varphi_x(y)\psi(s,y) \; dy \; + \int\int_{\frac{1}{2}}^{s} |\vec{\nabla} \otimes \vec{u}(\sigma,y)|^2 \; \varphi_x(y)\psi(\sigma,y) \; dy \; d\sigma \leq \\ 2\|\theta\|_\infty \int\int_0^{\frac{1}{2}} |\vec{u}(\sigma,y)|^2 \; \varphi_x(y)\psi(\sigma,y) \; dy \; d\sigma + \\ \int\int_0^s |\vec{u}(\sigma,y)|^2 \; \{\partial_\sigma + \Delta_y\}(\varphi_x(y)\psi(\sigma,y)) \; dy \; d\sigma \\ + \int\int_0^s (|\vec{u}(\sigma,y)|^2 + 2p(\sigma,y))\{\vec{u}(\sigma,y).\vec{\nabla}\}(\varphi_x(y)\psi(\sigma,y)) \; dy \; d\sigma\end{gathered}$$

Since $\vec{u}$ is weakly continuous from $(0,1)$ into $(L^2((B(0,1))^3$, we find that this equality is valid for all $s \in (0,1)$.

We now let $\psi = \psi_{x,s,k}$. For $\sigma < s$, we have $\{\partial_\sigma + \Delta_y\}\psi_{x,s,k} = 0$. Moreover, if $s - 4^{-k} < \sigma < s$ and $|x-y| < 2^{-k}$, then $\psi_{x,s,k}(\sigma,y) \geq (8\pi)^{-3/2} 2^{3k} e^{-1/4}$. Thus, we get:

$$\begin{gathered}2^{3k} \int_{|x-y|<\frac{1}{2^k}} |\vec{u}(s,y)|^2 \; dy \; + 2^{3k} \iint_{s-\frac{1}{4^k}<\sigma<s, \; |x-y|<\frac{1}{2^k}} |\vec{\nabla} \otimes \vec{u}(\sigma,y)|^2 \; dy \; d\sigma \leq \\ e^{1/4}(8\pi)^{3/2} \quad \{2\|\theta\|_\infty \int\int_0^{\frac{1}{2}} |\vec{u}(\sigma,y)|^2 \; \varphi_x(y)\psi_{x,s,k}(\sigma,y) \; dy \; d\sigma + \\ \int\int_0^s |\vec{u}(\sigma,y)|^2 \; (\psi_{x,s,k}(\sigma,y) \; \Delta\varphi_x(y) + 2\vec{\nabla}\psi_{x,s,k}(\sigma,y).\vec{\nabla}\varphi_x(y)) \; dy \; d\sigma \\ + \int\int_0^s (|\vec{u}(\sigma,y)|^2 + 2p(\sigma,y))\{\vec{u}(\sigma,y).\vec{\nabla}\}(\varphi_x(y)\psi_{x,s,k}(\sigma,y)) \; dy \; d\sigma\}\end{gathered}$$

If $0 < \sigma < 1/2$ and $s > 3/4 - 4^{-k}$, or if $|x-y| > 1/4$, then $\psi_{x,s,k}(\sigma,y) \leq \left(\frac{3}{2e\pi}\right)^{3/2}$ (check that $W(t,y) \leq (4\pi)^{-3/2} \max(t^{-3/2}, (6/e)^{3/2} \; |y|^{-3}))$ and, for $1 \leq j \leq d$, $|\partial_j \psi_{x,s,k}(\sigma,y)| \leq \frac{2^{25}}{(2\pi)^{3/2} e^2}$ (check that $\frac{|y|}{2t} W(t,y) \leq \frac{1}{(4\pi)^{3/2}} \max(\frac{1}{\sqrt{2}t^2}, \frac{2^{17/2}}{e^2 \; |y|^4}))$. This gives us that:

$$\begin{gathered}e^{1/4}(8\pi)^{3/2}\{2\|\theta\|_\infty \int\int_0^{\frac{1}{2}} |\vec{u}(\sigma,y)|^2 \; \varphi_x(y)\psi_{x,s,k}(\sigma,y) \; dy \; d\sigma + \\ \int\int_0^s |\vec{u}(\sigma,y)|^2 \; (\psi_{x,s,k}(\sigma,y) \; \Delta\varphi_x(y) + 2\vec{\nabla}\psi_{x,s,k}(\sigma,y).\vec{\nabla}\varphi_x(y)) \; dy \; d\sigma\} \leq \\ C\{\iint_0^{\frac{1}{2}} |\vec{u}(\sigma,y)|^2 |\varphi_x(y)| dy \; d\sigma + \iint_0^s |\vec{u}(\sigma,y)|^2 (|\Delta\varphi_x(y)| + |\vec{\nabla}\varphi_x(y)|) dy \; d\sigma\} \leq \\ D_0\big(\int\int_{0<\sigma<1, |y|<1} |\vec{u}(\sigma,y)|^3 \; d\sigma \; dy\big)^{2/3}. \qquad \square\end{gathered}$$

3. The regularity criterion

We may now introduce the celebrated regularity criterion of Caffarelli, Kohn, and Nirenberg [CAFKN 82]:

Theorem 30.1: (Caffarelli, Kohn, and Nirenberg's regularity criterion)
There exist two positive constants ϵ_{CKN} and C_1 so that if $\vec{u}$ is a weak solution for the Navier–Stokes equations on $(0,T)\times\mathbb{R}^3$, if $x_0\in\mathbb{R}^3$ and $0<t_0$, if for some r in $(0,\sqrt{t_0})$ we have

$$(30.3)\qquad \iint_{|x-x_0|<r,t_0-r^2<s<t_0} |\vec{u}|^3+|p|^{3/2}\ dx\ ds<\epsilon_1\ r^2$$

where $\epsilon_1<\epsilon_{CKN}$ and if $\vec{u}$ is a suitable solution on the cylinder $(t_0-r^2,t_0)\times B(x_0,r)$, then

$$(30.4)\qquad \sup_{|x-x_0|<r/2,t_0-r^2/4<s<t_0} |\vec{u}|<C_1\epsilon_1^{1/3}r^{-1}.$$

Proof: We first notice that $\vec{v}(t,x)=r\vec{u}(r^2t,rx)$ is a weak solution for the Navier–Stokes equations on $(0,T/r^2)\times\mathbb{R}^3$ (with pressure $q(t,x)=r^2p(r^2t,rx)$), suitable on the cylinder $(t_0/r^2-1,t_0/r^2)\times B(x_0/r,1)$, and that we have the inequality $\iint_{|x-x_0/r|<1,t_0/r^2-1<s<t_0/r^2}|\vec{v}|^3+|q|^{3/2}\ dx\ ds<\epsilon_1$. Thus, we see that we may assume with no loss of generality that $r=1$.

Similarly, $\vec{w}(t,x)=\vec{v}(t+t_0/r^2-1,x+x_0/r)$ is a weak solution for the Navier–Stokes equations on $(0,1+(T-t_0)/r^2)\times\mathbb{R}^3$ (with pressure $r(t,x)=q(t+t_0/r^2-1,x+x_0/r)$), suitable on the cylinder $(0,1)\times B(0,1)$, and we have the inequality $\iint_{|x|<1,0<s<1}|\vec{w}|^3+|r|^{3/2}\ dx\ ds<\epsilon_1$. We see that we may assume with no loss of generality that $x_0=0$ and $t_0=1$.

Thus, we shall prove the theorem for $(\vec{u},p)$ satisfying inequality (30.3) with $x_0=0$, $t_0=1$, $r=1$.

Since $\vec{u}$ is square-integrable on $(0,1)\times B(0,1)$, inequality (30.4) will be proved by proving that:

$$(30.5)\qquad \sup_{|x|<\frac{1}{2}}\Big\{\sup_{k\geq 2}\ \sup_{\frac{3}{4}-\frac{1}{2^{2k}}<s<1} 2^{3k}\int_{|x-y|<\frac{1}{2^k}}|\vec{u}(s,y)|^2\ dy\Big\}\leq C_1^2\epsilon_1^{2/3}|B_1|$$

with $B_1=B(0,1)$.

Step 1: The induction

We introduce the function $p_{x,k}(\sigma)=\frac{1}{|B(x,2^{-k}\sqrt{2})|}\int_{|x-y|<2^{-k}\sqrt{2}}p(\sigma,y)\ dy$. We also define for $k\geq 2$, $x\in B(0,1/2)$ and $s\in(3/4-4^{-k},1)$ the sets $B_k(x)=$

$B(x, 2^{-k})$ and $Q_k(x,s) = (s - 4^{-k}, s) \times B(x, 2^{-k})$. We are going to prove by induction on k that (if ϵ_1 is small enough) there exists two constants D_1 and D_2 such that for all $x \in B(0, 1/2)$, $s \in (3/4 - 4^{-k}, 1)$ and all $k \geq 2$:

$$(A_k) \qquad \begin{cases} I_k = 2^{3k} \int_{B_k(x)} |\vec{u}(s,y)|^2 \, dy \leq D_1 \epsilon_1^{2/3} \\ J_k = 2^{3k} \iint_{Q_k(x,s)} |\vec{\nabla} \otimes \vec{u}(\sigma, y)|^2 \, dy \, d\sigma \leq D_1 \epsilon_1^{2/3} \end{cases}$$

and for all $k \geq 3$

$$(B_k) \qquad \begin{cases} K_k = 2^{5k} \iint_{Q_k(x,s)} |\vec{u}(\sigma,y)|^3 \, dy \, d\sigma \leq D_2 \epsilon_1^{2/3} \\ L_k = 2^{5k} \iint_{Q_k(x,s)} |p(\sigma,y) - p_{x,k}(\sigma)|^{3/2} dy \, d\sigma \leq D_2 \epsilon_1^{2/3} 2^{k/2} \end{cases}$$

In the proof of those inequalities, we find constraints on D_1, D_2 and ϵ_1. We first assume that $\epsilon_1 < 1$. When the constraint concerns only D_2, we assume that D_2 is big enough to fulfill the constraint. When the constraint concerns only D_1 and D_2, we assume that D_1 is big enough to fulfill the constraint (the size of D_1 will depend on the choice of D_2). When the constraint concerns the three parameters, we assume that ϵ_1 is small enough to fulfill the constraint (the size of ϵ_1 will depend on the choice of D_1 and D_2). The constants C, C', ..., which will appear in the computations, will not depend on D_1, D_2, ϵ_1 (nor on $\vec{u}$, x, s, k).

First, we take $k = 2$ and use the fundamental inequality, the fact that we control $|\vec{\nabla}(\varphi_x \psi_{x,s,2})|$ by D_2 [*Constraint 1*] and $\iint_{0<\sigma<1, |y|<1} |\vec{u}(\sigma,y)|^3 + |p(\sigma,y)|^{3/2} \, d\sigma \, dy$ by ϵ_1 to get that $I_2 + J_2 \leq D_0 \epsilon_1^{2/3} + 3e^{1/4}(8\pi)^{3/2} D_2 \epsilon_1$. This gives (A_2) if $\epsilon_1 < 1$ and $D_1 \geq D_0 + 3e^{1/4}(8\pi)^{3/2} D_2$ [*Constraint 2*].

Step 2: Localisation of the pressure

Before proceeding further, we need to have an expression for the pressure that uses only the knowledge of $\vec{u}$ on $(0,1) \times B(0,1)$, since we control $\vec{u}$ only on this domain. The Navier–Stokes equations gives us local information on the Laplacian of p:

$$\Delta p = - \sum_{j=1}^{d} \sum_{l=1}^{d} \partial_j \partial_l (u_j u_l).$$

We fix a function $\zeta \in \mathcal{D}(\mathbb{R}^3)$ with Supp $\zeta \subset B(0,1)$ and $\zeta = 1$ on $B(0, 3/4)$. We then write for $y \in B(0, 3/4)$:

$$p(t,y) = \frac{1}{\Delta} \Delta(\zeta(y) p(t,y)).$$

The kernel of $\frac{1}{\Delta}$ is the function $K(x) = -\frac{3}{4\pi |x|}$. We write $\Delta(\zeta(z)p(t,z)) = (\Delta\zeta)(z)p(t,z) + \zeta(z)\Delta p(t,z)) + 2\vec{\nabla}\zeta(z).\vec{\nabla}p(t,z)$ and we get $p = p_1 + p_2$ with $p_1(t,y) = -\sum_{j=1}^{d} \sum_{l=1}^{d} \partial_j \partial_l K * (\zeta u_j u_l)$ and

$$\begin{aligned} p_2 = \quad & -2 \sum_{j=1}^{d} \sum_{l=1}^{d} \partial_j K * (u_j u_l \partial_l \zeta) - \sum_{j=1}^{d} \sum_{l=1}^{d} K * (u_j u_l \partial_j \partial_l \zeta) \\ & -K * (p \Delta \zeta) + 2 \sum_{j=1}^{d} \partial_j K * (p \partial_j \zeta) \end{aligned}$$

If $y \in B(0,5/8)$, we find that, for all $\alpha \in \mathbb{N}^3$ and $\beta \in \mathbb{N}^3$, $\beta \neq (0,0,0)$, $\partial^\alpha K(y-z)$ is bounded on the set $\{z \in B(0,1) \ / \ \partial^\beta \zeta(z) \neq 0\}$, uniformly with respect to y and z, and thus, for all $\alpha \in \mathbb{N}^3$, $\sup_{0<t<1, y\in B(0,5/8)} |\partial^\alpha p_2(t,y)| \leq C_\alpha \int_{B(0,1)} |\vec{u}(t,z)|^2 + |p(t,z)| \, dz$. Thus, $\int_0^1 \|\partial^\alpha p_2(t,.)\|_{L^\infty(B(0,5/8))}^{3/2} \, dt \leq C_\alpha \epsilon_1$.

Step 3: (A_k) **true for** $2 \leq k \leq n$ **implies** (B_{n+1})

The control of $K_{n+1} = 2^{5(n+1)} \iint_{Q_{n+1}(x,s)} |\vec{u}(\sigma,y)|^3 \, dy \, d\sigma$ is plain. We choose $\omega \in \mathcal{D}$ with $\omega = 1$ on $B(0,1)$ and with $\operatorname{Supp} \omega \subset B(0,2)$ and we write that

$$\begin{aligned} \|\vec{u}(\sigma,.)\|_{L^3(B_{n+1}(x))} &\leq C\|\omega(2^{n+1}(x-y))\vec{u}(\sigma,.)\|_{H^{1/2}} \\ &\leq C'\|\omega(2^{n+1}(x-y))\vec{u}(\sigma,.)\|_2^{1/2}\|\omega(2^{n+1}(x-y))\vec{u}(\sigma,.)\|_{H^1}^{1/2} \\ &\leq C''\|\vec{u}(\sigma,.)\|_{L^2(B_n(x))}^{1/2}(2^n\|\vec{u}(\sigma,.)\|_{L^2(B_n(x))} + \|\vec{\nabla}\otimes\vec{u}(\sigma,.)\|_{L^2(B_n(x))})^{1/2}. \end{aligned}$$

We then use Young's inequality ($ab \leq \frac{2^{2n}a^4}{4} + \frac{3 2^{-2n/3} b^{4/3}}{4}$) and get

$$\|\vec{u}\|_{L^3(B_{n+1}(x))} \leq C\big(2^{\frac{n}{2}}\|\vec{u}\|_{L^2(B_n(x))} + 2^{2n}\|\vec{u}\|_{L^2(B_n(x))}^2 + 2^{-\frac{2n}{3}}\|\vec{\nabla}\otimes\vec{u}\|_{L^2(B_n(x))}^{2/3}\big).$$

Due to the induction hypothesis (A_n), we have $2^{\frac{n}{2}}\|\vec{u}\|_{L^2(B_n(x))} \leq \sqrt{D_1}\epsilon_1^{1/3}2^{-n}$, $2^{2n}\|\vec{u}\|_{L^2(B_n(x))}^2 \leq D_1\epsilon_1^{2/3}2^{-n}$, and $\int_{s-4^{-n}}^s \|\vec{\nabla}\otimes\vec{u}\|_{L^2(B_n)(x)}^2 \, d\sigma \leq D_1\epsilon_1^{2/3}2^{-3n}$. Thus, we find the estimate $K_{n+1} \leq C\epsilon_1^{2/3}(D_1^{3/2}\epsilon_1^{1/3} + D_1^3\epsilon_1^{4/3})$, and we have $K_{n+1} \leq D_2\epsilon_1^{2/3}$ if ϵ_1 is small enough [*Constraint 3*: $C(D_1^{3/2}\epsilon_1^{1/3} + D_1^3\epsilon_1^{4/3}) \leq D_2$].

We now study $L_{n+1} = 2^{5(n+1)} \iint_{Q_{n+1}(x,s)} |p(\sigma,y) - p_{x,n+1}(\sigma)|^{3/2} dy \, d\sigma$. We write $p = p_1 + p_2$ (see Step 3). We decompose the average pressure $p_{x,n+1}$ as $p_{x,n+1} = p_{1,x,n+1} + p_{2,x,n+1}$, with $p_{j,x,n+1}(\sigma) = \frac{1}{|B_{n+1}(x)|}\int_{B_{n+1}(x)} p_j(\sigma,y) \, dy$ for $j \in \{1,2\}$, and we similarly decompose L_{n+1} as $L_{n+1} \leq \sqrt{2}(L_{1,n+1} + L_{2,n+1})$ with $L_{j,n+1} = 2^{5(n+1)} \iint_{Q_{n+1}(x,s)} |p_j(\sigma,y) - p_{j,x,n+1}(\sigma)|^{3/2} dy \, d\sigma$.

$L_{2,n+1}$ is easily controlled. For $x \in B(0,1/2)$ and $n \geq 2$, $B_{n+1}(x) \subset B(0,5/8)$; moreover, we have on $B_{n+1}(x)$

$$|p_2 - p_{2,x,n+1}| \leq C 2^{-(n+1)} \|\vec{\nabla} p_2\|_{L^\infty(B(0,5/8))};$$

hence, we have

$$\int_{Q_{n+1}(x,s)} |p_2 - p_{2,x,n+1}|^{\frac{3}{2}} \, dy \, d\sigma \leq C 2^{-\frac{9n}{2}} \int_0^1 \|\vec{\nabla} p_2\|_{L^\infty(B(0,5/8))}^{3/2} \, d\sigma \leq C' 2^{-\frac{9n}{2}} \epsilon_1$$

Thus, we get $L_{2,n+1} \leq C 2^{(n+1)/2} \epsilon_1$.

We now study $L_{1,n+1}$. Recall that $p_1(t,y) = -\sum_{j=1}^d \sum_{l=1}^d \partial_j \partial_l K * (\zeta u_j u_l)$. We split $u_j u_l$ into $u_j u_l 1_{Bn} + u_j u_l (1 - 1_{B_n})$ and get $p_1 = q_{1,n} + r_{1,n}$ with $q_{1,n} = -\sum_{j=1}^d \sum_{l=1}^d \partial_j \partial_l K * (\zeta u_j u_l 1_{B_n})$. The average pressure $p_{1,x,n+1}$ may be decomposed as $p_{1,x,n+1} = q_{1,x,n+1} + r_{1,x,n+1}$ where $q_{1,x,n+1}$ and $r_{1,x,n+1}$

are defined as $q_{1,x,n+1}(\sigma) = \frac{1}{|B_{n+1}(x)|}\int_{B_{n+1}(x)} q_{1,n}(\sigma,y)\ dy$ and $r_{1,x,n+1}(\sigma) = \frac{1}{|B_{n+1}(x)|}\int_{B_{n+1}(x)} r_{1,n}(\sigma,y)\ dy$. We write also $L_{1,n+1} \le \sqrt{2}(L_{1,q,n+1} + L_{1,r,n+1})$ with $L_{1,q,n+1} = 2^{5(n+1)} \iint_{Q_{n+1}(x,s)} |q_{1,n}(\sigma,y) - q_{1,x,n+1}(\sigma)|^{3/2} dy\ d\sigma$ and $L_{1,r,n+1} = 2^{5(n+1)} \iint_{Q_{n+1}(x,s)} |r_{1,n}(\sigma,y) - r_{1,x,n+1}(\sigma)|^{3/2} dy\ d\sigma$. We have $|q_{1,x,n+1}(\sigma)| \le \big(\frac{1}{|B_{n+1}(x)|}\int_{B_{n+1}(x)} |q_{1,n}(\sigma,y)|^{3/2}\ dy\big)^{2/3}$; hence,

$$\int_{B_{n+1}(x)} |q_{1,n}(\sigma,y) - q_{1,x,n+1}(\sigma)|^{3/2} dy \le C\|q_{1,n}(\sigma,.)\|_{3/2}^{3/2} \le C\|\ |\vec{u}|^2 1_{B_n}\|_{3/2}^{3/2}$$

since the convolution with $-\partial_j\partial_l K$ is the Calderón–Zygmund operator R_jR_l and thus operates boundedly on $L^{3/2}$. This gives $L_{1,q,n+1} \le CK_n$. But we know that $K_n \le C\epsilon_1 D_1^{3/2}(1 + D_1^{3/2}\epsilon_1)$: for $n \ge 3$, this is the same proof as for K_{n+1}; for $n = 2$, we write $K_2 \le 2^6\epsilon_1 \le CD_1\epsilon_1$ [*Constraint 4*: $CD_1 > 2^6$].

To estimate $L_{1,r,n+1}$, we write on B_{n+1} that

$$|r_{1,n}(\sigma,y) - r_{1,x,n+1}(\sigma)| \le C2^{-(n+1)}\|\vec{\nabla} r_{1,n}(\sigma,.)\|_{L^\infty(B_{n+1})}.$$

We then write for $y \in B_{n+1}$

$$|\partial_m r_{1,n}(\sigma,y)| \le \sum_{j=1}^{d}\sum_{l=1}^{d} \int_{z\notin B_n} |\partial_j\partial_l\partial_m K(y-z)\zeta(z)u_j(z)u_l(z)|\ dz.$$

For $|x-y| \le 2^{-(n+1)}$ and $|x-z| \ge 2^{-n}$, we have $|\partial_j\partial_l\partial_m K(y-z)| \le C|y-z|^{-4} \le 2^4C|x-z|^{-4}$. Thus, $\|\vec{\nabla} r_{1,n}(\sigma,.)\|_{L^\infty(B_{n+1})} \le C\int_{|x-z|\ge 2^{-n}} \frac{|\vec{u}(\sigma,z)|^2}{|x-z|^4}|\zeta(z)|\ dz$.

We may choose ζ to be nonnegative. As in the proof of Lemma 30.2, we use the local energy inequality for the nonnegative test function $\phi(\sigma,y) = \zeta(y)\omega_{s,\epsilon}(\sigma)$, and then we let ϵ go to 0; we get for $1/2 < s < 1$:

$$\begin{gathered}
\int |\vec{u}(s,y)|^2\ \zeta(y)\ dy\ \ + \int\int_{\frac{1}{2}}^{s} |\vec{\nabla}\otimes\vec{u}(\sigma,y)|^2\ \zeta(y)\ dy\ d\sigma \le \\
2\|\theta\|_\infty \int\int_0^{\frac{1}{2}} |\vec{u}(\sigma,y)|^2\ \zeta(y)\ dy\ d\sigma + \\
\int\int_0^s |\vec{u}(\sigma,y)|^2\ \Delta\zeta(y)\ dy\ d\sigma \\
+ \int\int_0^s (|\vec{u}(\sigma,y)|^2 + 2p(\sigma,y))\{\vec{u}(\sigma,y).\vec{\nabla}\}\zeta(y)\ dy\ d\sigma \\
\le C_1\epsilon_1^{2/3} + C_2\epsilon_1.
\end{gathered}$$

Since $\epsilon_1 < 1$, we may write $\int |\vec{u}(s,y)|^2\ \zeta(y)\ dy \le C\epsilon_1^{2/3}$. By the induction hypothesis, we know that if $2 \le k \le n$ we have $\int_{|x-z|\in[2^{-k-1},2^{-k}]} |\vec{u}(\sigma,z)|^2\ dz \le D_1\epsilon_1^{2/3}2^{-3k}$ for $\sigma \in (s-4^{-k}, s)$; thus, we have

$$\begin{aligned}
L_{1,r,n+1} &\le C2^{(n+1)/2}\int_{s-4^{-(n+1)}}^{s}\Big(\int_{|x-z|\ge 2^{-n}} \frac{|\vec{u}(\sigma,z)|^2}{|x-z|^4}|\zeta(z)|\ dz\Big)^{3/2}\ d\sigma \\
&\le C'2^{(n+1)/2}\int_{s-4^{-(n+1)}}^{s}\Big(\sum_{k=0}^{n-1}\int_{2^{-k-1}\le|x-z|\le 2^{-k}} \frac{|\vec{u}(\sigma,z)|^2}{|x-z|^4}|\zeta(z)|dz\Big)^{3/2} d\sigma \\
&\le C''\epsilon_1 2^{(n+1)/2}\int_{s-4^{-(n+1)}}^{s}\Big(1 + \sum_{k=2}^{n-1} D_1 2^k\Big)^{3/2} d\sigma \\
&\le C'''D_1^{3/2}\epsilon_1 2^{2(n+1)}\int_{s-4^{-(n+1)}}^{s} d\sigma = C'''D_1^{3/2}\epsilon_1.
\end{aligned}$$

We find that $L_{n+1} \leq C\epsilon_1(D_1^{3/2} + D_1^3\epsilon_1 + 2^{(n+1)/2}) \leq D_2\epsilon_1^{2/3}2^{(n+1)/2}$ for ϵ_1 small enough [*Constraint 5*: $C(D_1^{3/2}\epsilon_1^{1/3} + D_1^3\epsilon_1^{4/3} + \epsilon_1^{1/3}) \leq D_2$]

Remark: We see that the discrepancy between the local scaling of $|\vec{u}|^2$ and the local scaling of $|p - p_{x,k}|$ is due to the non-local character of the pressure: the knowledge of p is not given by the local knowledge of $\vec{u}$, and the ill-controlled part in p is the term p_2.

Step 4: (B_k) true for $3 \leq k \leq n$ implies (A_n)

We use the fundamental inequality given in Lemma 2 and find that $I_n + J_n \leq U_n + V_n + W_n$ with

$$\begin{cases} U_n = D_0\big(\iint_{0<\sigma<1,|y|<1} |\vec{u}(\sigma,y)|^3\, d\sigma\, dy\big)^{2/3} \\ V_n = e^{1/4}(8\pi)^{3/2} \iint_0^s |\vec{u}(\sigma,y)|^2\{\vec{u}(\sigma,y).\vec{\nabla}\}(\varphi_x(y)\psi_{x,s,n}(\sigma,y))\, dy\, d\sigma \\ W_n = 2e^{1/4}(8\pi)^{3/2} \iint_0^s p(\sigma,y)\{\vec{u}(\sigma,y).\vec{\nabla}\}(\varphi_x(y)\psi_{x,s,n}(\sigma,y))\, dy\, d\sigma \end{cases}$$

We prove that $\max(U_n, V_n, W_n) \leq D_1\epsilon^{2/3}/3$. Indeed, we have $U_n \leq D_0\epsilon^{2/3} \leq D_1\epsilon^{2/3}/3$ if D_1 is big enough [*Constraint 6*: $3D_0 \leq D_1$]. In order to estimate V_n and W_n, we need to control the size of $\vec{\nabla}\ (\varphi_x(y)\psi_{x,s,n}(\sigma,y))$. We have $|\partial_j\varphi_x(y)|\ |\psi_{x,s,n}(\sigma,y)| \leq C 1_{|x-y|\geq 1/4}$. On the other hand,

$$|\varphi_x(y)|\ |\partial_j\psi_{x,s,n}(\sigma,y)| \leq C\frac{1}{\big(4^{-n} + s - \sigma + |x-y|^2\big)^2}.$$

This gives control on V_n:

$$\begin{aligned} V_n \quad & \leq C\iint_{(0,1)\times B(0,1)} |\vec{u}(\sigma,y)|^3 \frac{1}{\big(4^{-n}+s-\sigma+|x-y|^2\big)^2}\, dy\, d\sigma \\ & \leq C'\big(\iint_{(0,1)\times B(0,1)-Q_3} |\vec{u}(\sigma,y)|^3\, dy\, d\sigma \\ & + \sum_{k=3}^{n-1} 2^{4k}\iint_{Q_k-Q_{k+1}} |\vec{u}(\sigma,y)|^3 dy\, d\sigma + 2^{4n}\iint_{Q_n} |\vec{u}(\sigma,y)|^3\, dy\, d\sigma\big) \\ & \leq C''\big(\iint_{(0,1)\times B(0,1)} |\vec{u}(\sigma,y)|^3\, dy\, d\sigma + \sum_{k=3}^{n} 2^{-k}K_k\big) \end{aligned}$$

Thus, $V_n \leq C(\epsilon_1 + D_2\epsilon_1^{2/3}) \leq D_1\epsilon_1^{2/3}/3$ if D_1 is big enough [*Constraint 7*: $3C(1+D_2) \leq D_1$].

The control on W_n is obtained in a similar way: we let χ be a smooth function on $\mathbb{R} \times \mathbb{R}^3$ equal to 1 on $(0,1/4) \times B(0,1/2)$ and equal to 0 outside from $(0,1) \times B(0,1)$ and we define $\chi_{x,s,k}(\sigma,y) = \chi(4^k(s-\sigma), 2^k(x-y))$. We write $1 = 1 - \chi_{x,s,3}(\sigma,y) + \sum_{k=3}^{n-1}\chi_{x,s,k}(\sigma,y) - \chi_{x,s,k+1}(\sigma,y) + \chi_{x,s,n}(\sigma,y)$ and

$$\begin{aligned} & \iint_0^s p(\sigma,y)\{\vec{u}(\sigma,y).\vec{\nabla}\}\big((\chi_{x,s,k}(\sigma,y) - \chi_{x,s,k+1}(\sigma,y))\varphi_x(y)\psi_{x,s,n}(\sigma,y)\big)\, dy\, d\sigma \\ & = \iint_{Q_k} (p(\sigma,y) - p_{x,k}(\sigma))\{\vec{u}(\sigma,y).\vec{\nabla}\} \\ & \big((\chi_{x,s,k}(\sigma,y) - \chi_{x,s,k+1}(\sigma,y))\varphi_x(y)\psi_{x,s,n}(\sigma,y)\big)\, dy\, d\sigma \\ & \leq C2^{4k}\iint_{Q_k} |p(\sigma,y) - p_{x,k}(\sigma)|\ |\vec{u}(\sigma,y)|\, dy\, d\sigma. \end{aligned}$$

This gives

$$\begin{array}{rcl} W_n \leq & & C \iint_{(0,1)\times B(0,1)} |p(\sigma,y)||\vec{u}(\sigma,y)| \\ & & + \sum_{k=3}^n C2^{4k} \iint_{Q_k} |p(\sigma,y) - p_{x,k}(\sigma)|\ |\vec{u}(\sigma,y)|\ dy\ d\sigma \\ \leq & & C\|p\|_{L^{3/2}((0,1)\times B(0,1))}\|\vec{u}\|_{L^3((0,1)\times B(0,1))} + \sum_{k=3}^n C2^{-k} K_k^{1/3} L_k^{2/3} \\ \leq & & C\epsilon_1 + \sum_{k=3}^n CD_2\epsilon_1^{2/3} 2^{-2k/3}. \end{array}$$

Thus, $W_n \leq C(\epsilon_1 + D_2\epsilon_1^{2/3}) \leq D_1\epsilon_1^{2/3}/3$ if D_1 is big enough [*Constraint 8*: $3C(1+D_2) \leq D_1$].

Thus, Theorem 30.1 is proved. □

Chapter 31

On the dimension of the set of singular points

The main result of Caffarelli, Kohn, and Nirenberg [CAFKN 82] is their estimate on the Hausdorff dimension of the set of singular points of a suitable solution (generalizing results of Scheffer [SCH 77]).

1. Singular times

We begin by recalling the definition of the Hausdorff measure on $\mathbb{R}^d$:

Definition 31.1: (Hausdorff measure.)

i) For a sequence of open balls $\mathcal{B} = (B(x_i, r_i))_{i\in\mathbb{N}}$ of $\mathbb{R}^d$ and for $\alpha > 0$, we define $\sigma_\alpha(\mathcal{B}) = \sum_{i\in\mathbb{N}} r_i^\alpha$. The Hausdorff measure $\mathcal{H}^\alpha$ on $\mathbb{R}^d$ is defined for a Borel subset $B \subset \mathbb{R}^d$ by

$$\mathcal{H}^\alpha(B) = \lim_{\delta\to 0} \min\{\sigma_\alpha(\mathcal{B}) \ / \ \mathcal{B} = (B(x_i, r_i))_{i\in\mathbb{N}}, B \subset \cup_{i\in\mathbb{N}} B(x_i, r_i), \sup_{i\in\mathbb{N}} r_i < \delta\}$$

ii) The Hausdorff dimension $d_\mathcal{H}(B)$ of a Borel subset of $\mathbb{R}^d$ is defined as

$$d_\mathcal{H}(B) = \inf\{\alpha > 0 \ / \ \mathcal{H}^\alpha(B) = 0\} = \sup\{\alpha > 0 \ / \ \mathcal{H}^\alpha(B) = \infty\}.$$

A classical result (which goes back to the description of the structure of turbulent solutions by Leray [LER 34]) states that the set of singular times for a restricted Leray solution is small:

Proposition 31.1: (Singular times)

Let $\vec{u}$ be a weak Leray solution of the Navier–Stokes equations on $(0,\infty) \times \mathbb{R}^3$ which satisfies the strong energy inequality. Then there is compact set $\Sigma_t \subset [0,\infty)$ so that:

i) $\vec{u}$ is smooth outside from $\Sigma_t \times \mathbb{R}^3$

ii) $\mathcal{H}^{1/2}(\Sigma_t) = 0$.

Proof: We proved in Chapter 15 that, for an initial data in $(L^p(\mathbb{R}^3))^3$ with $p > 3$, we are able to find a solution of the Navier–Stokes equations in $(\mathcal{C}([0,T], L^p))^3$

with $T \geq C(p)\|\vec{u}_0\|_p^{-\frac{2p}{p-3}}$. If $p = 3$ and if $\|\vec{u}_0\|_3$ is small enough ($\|\vec{u}_0\|_3 < \epsilon_0$), we may find a global solution in $(\mathcal{C}([0,T], L^3))^3$.

If $\vec{u}$ is a weak Leray solution that fullfills the strong energy equality, we have that $\int_0^\infty \|\vec{u}(t,.)\|_3^4 \, dt < \infty$; thus, there is a subset of positive measure of $(0,\infty)$ on which $\|\vec{u}(t,.)\|_3$ is smaller than ϵ_0. If t_0 is a Lebesgue point of the map $t \mapsto \|\vec{u}(t,.)\|_2^2$ so that $\|\vec{u}(t_0,.)\|_3$ is smaller than ϵ_0, the theorems of Kato (existence of a global solution) and Serrin (uniqueness in the class of Leray weak solutions) show that $\vec{u}$ on $[t_0,\infty)$ is given by the Kato solution associated with the initial value $\vec{u}(t_0,.)$. Since this mild solution is smooth, we find that $\Sigma_t \subset [0,t_0]$; thus, Σ_t is compact.

Now, let us consider $\tau \in \Sigma_t$ and let $s < \tau$ be a Lebesgue point of the map $t \mapsto \|\vec{u}(t,.)\|_2^2$. Let $\gamma = \|\vec{u}(s,.)\|_6$. If $\gamma < \infty$, we know that there exists a mild solution $\vec{v}$ of the Navier–Stokes equations in $(\mathcal{C}([s,s+T], L^6))^3$ with $T \geq C\gamma^{-4}$, with initial value $\vec{u}(s)$. Since $\vec{u}$ is a Leray solution on $(s,s+T)$, we find that $\vec{u} = \vec{v}$ on $(s,s+T)$ (due to the Serrin uniqueness theorem). Since $\vec{v}$ is smooth on $(s,s+T)\times \mathbb{R}^3$, we find that $s+T < \tau$. Thus, $\|\vec{u}(s,.)\|_6 \geq C(\tau-s)^{-1/4}$.

If $I = (\tau-r,\tau+r)$ is an open interval centered on $\tau \in \Sigma_t$ and if $\tau > r$, we have for almost every $s \in [\tau-r,\tau]$ $\|\vec{u}(s,.)\|_6 \geq C(\tau-s)^{-1/4}$; hence, $\int_I \|\vec{u}\|_6^2 \, ds \geq Cr^{1/2}$. Now, let $\delta > 0$ and let us write $\Sigma_t \cap [\delta,\infty) \subset \cup_{\tau\in\Sigma_t,\tau\geq\delta}(\tau-\delta/5,\tau+\delta/5)$. The Vitali covering lemma (Proposition 6.1) gives us that we may find $N \in \mathbb{N}$ and $\tau_1,\ldots,\tau_N \in \Sigma_t \cap [\delta,\infty)$ so that $\Sigma_t \cap [\delta,\infty) \subset \cup_{i=1}^N(\tau_i-\delta,\tau_i+\delta)$ while $\min_{1\leq i<j\leq N} |\tau_i-\tau_j| \geq 2\delta/5$. We thus find that for $\mathcal{B} = ((\tau_i-\delta,\tau_i+\delta))_{0\leq i\leq N}$ with $\tau_0 = 0$, we have $\sigma_{1/2}(\mathcal{B}) = (N+1)\delta^{1/2} \leq \delta^{1/2} + C\sum_{i=1}^N \int_{\tau_i-2\delta/5}^{\tau_i} \|\vec{u}\|_6^2 \, ds \leq \delta^{1/2} + C\int_{d(s,\Sigma_t)\leq 2\delta/5} \|\vec{u}\|_6^2 \, ds$. Since we know that $\vec{u} \in (L^2L^6)^3$, we find that $\mathcal{H}^{1/2}(\Sigma_t) \leq C\int_{\Sigma_t} \|\vec{u}\|_6^2 \, ds$. In particular, $\mathcal{H}^{1/2}(\Sigma_t) < \infty$; hence, the Lebesgue measure of Σ_t is equal to 0; this gives $\int_{\Sigma_t} \|\vec{u}\|_6^2 \, ds = 0$ and finally $\mathcal{H}^{1/2}(\Sigma_t) = 0$.
$\square$

2. Hausdorff dimension of the set of singularities for a suitable solution

To more accurately describe the singularities in $\mathbb{R}\times\mathbb{R}^3$, Caffarelli, Kohn, and Nirenberg used another Hausdorff dimension, adapted to the scaling properties of the Navier–Stokes equations:

Definition 31.2: (Parabolic Hausdorff measure)

i) For a sequence of open parabolic cylinders $\mathcal{Q} = (Q((t_i,x_i),r_i))_{i\in\mathbb{N}}$ of $\mathbb{R}\times\mathbb{R}^d$ (where $Q((t_i,x_i),r_i) = \{(t,x) \ / \ |t-t_i| \leq r_i^2$ and $|x-x_i| \leq r_i\}$) and for $\alpha > 0$, we define $\tilde{\sigma}_\alpha(\mathcal{Q}) = \sum_{i\in\mathbb{N}} r_i^\alpha$. The parabolic Hausdorff measure $\mathcal{P}^\alpha$ on $\mathbb{R}\times\mathbb{R}^d$ is defined for a Borel subset $B \subset \mathbb{R}\times\mathbb{R}^d$ by

$$\mathcal{P}^\alpha(B) = \lim_{\delta\to 0} \min\{\tilde{\sigma}_\alpha(\mathcal{Q}) \ / B \subset \cup_{i\in\mathbb{N}} Q((t_i,x_i),r_i), \sup_{i\in\mathbb{N}} r_i < \delta\}$$

ii) The parabolic Hausdorff dimension $d_{\mathcal{H}}(B)$ of a Borel subset of $\mathbb{R} \times \mathbb{R}^d$ *is defined as*

$$d_{\mathcal{P}}(B) = \inf\{\alpha \ / \ \mathcal{P}^\alpha(B) = 0\} = \sup\{\alpha \ / \ \mathcal{P}^\alpha(B) = \infty\}.$$

We may now state the result of Caffarelli, Kohn, and Nirenberg:

Theorem 31.1: (Dimension of the singular set)

Let $\vec{u}$ be a weak solution for the Navier–Stokes equations on $(0,T) \times \mathbb{R}^3$, which is a suitable solution on the cylinder $Q_0 = (a,b) \times B(x_0, r_0)$. Let Σ be the smallest closed set in Q so that $\vec{u}$ is locally bounded on $Q_0 - \Sigma$. Then $\mathcal{P}^1(\Sigma) = 0$.

The proof is based on the following proposition (which we shall prove in the next section):

Proposition 31.2: (The second regularity criterion of Caffarelli, Kohn, and Nirenberg)

There exists a positive constant ϵ_0 so that if $\vec{u}$ is a weak solution for the Navier–Stokes equations on $(0,T) \times \mathbb{R}^3$, if $x_0 \in \mathbb{R}^3$ and $0 < t_0$, if $\vec{u}$ is a suitable solution on a neighbourhood of (t_0, x_0) and if

$$\text{(31.1)} \qquad \limsup_{r \to 0} r^{-1} \iint_{|x-x_0|<r, t_0 - \frac{7r^2}{8} < s < t_0 + \frac{r^2}{8}} |\vec{\nabla} \otimes \vec{u}|^2 \ dx \ ds < \epsilon_0$$

then we have

$$\text{(31.2)} \qquad \limsup_{r \to 0} r^{-2} \iint_{|x-x_0|<r, t_0 - \frac{7r^2}{8} < s < t_0 + \frac{r^2}{8}} |\vec{u}|^3 + |p|^{3/2} \ dx \ ds < \epsilon_{CKN}$$

where ϵ_{CKN} is the constant in the regularity criterion of Caffarelli, Kohn, and Nirenberg (Theorem 30.1). In particular, $\vec{u}$ is bounded on a cylinder $Q((t_0 + r^2/8, x_0), r/2) = (t_0 - r^2/8, t_0 + r^2/8) \times B(x_0, r/2)$.

Before proving Proposition 31.2 in the next section, we prove that Theorem 31.1 is a consequence of Proposition 31.2.

Proof of Theorem 1:

Let $\delta > 0$. Let $(t,x) \in \Sigma$. According to Proposition 31.2, we know that for r small enough, we have $\iint_{|y-x|<r, t-\frac{7r^2}{8} < s < t_+ \frac{r^2}{8}} |\vec{\nabla} \otimes \vec{u}|^2 \ dy \ ds \geq \epsilon_0 r$. Thus, we have $\Sigma \subset \cup_{Q \in \mathcal{Q}_\delta} Q$ where $\mathcal{Q}_\delta$ is the collection of open cylinders $Q((t,x),r) = (t - r^2, t) \times B(x,r)$ so that $Q \subset Q_0$, $r < \delta$ and $\iint_Q |\vec{\nabla} \otimes \vec{u}|^2 \ dy \ ds \geq \epsilon_0 r$. We use the parabolic distance $d_{\mathcal{P}}((t,x),(s,y)) = \max(|y-x|, \sqrt{2|t-s|})$; for this distance, an open cylinder $Q((t,x),r)$ is a ball $B((t + r^2/2, x), r)$, hence we may apply the Vitali covering lemma and find a countable subcollection $\mathcal{Q}_{[\delta]} = (Q((t_i, x_i), r_i))_{i\in\mathbb{N}}$ of $\mathcal{Q}_\delta$ so that $\Sigma \subset \cup_{i\in\mathbb{N}} Q((t_i, x_i), r_i)$ and $i \neq j \Rightarrow Q_i \cap Q_j = \emptyset$. We then have:

$$\tilde{\sigma}_1(\mathcal{Q}_{[\delta]}) = 5 \sum_{i\in\mathbb{N}} r_i \leq \frac{5}{\epsilon_0} \int\int_{(t,x)\in Q_0, d_{\mathcal{P}}((t,x),\Sigma)\leq\delta} |\vec{\nabla} \otimes \vec{u}|^2 \, dx \, dt.$$

This gives $\mathcal{P}^1(\Sigma) \leq \frac{5}{\epsilon_0} \int\int_\Sigma |\vec{\nabla} \otimes \vec{u}|^2 \, dx \, dt$. In particuler $\mathcal{P}^1(\Sigma) < \infty$; hence, the Lebesgue measure of Σ is equal to 0 and finally $\mathcal{P}^1(\Sigma) = 0$. □

3. The second regularity criterion of Caffarelli, Kohn, and Nirenberg

The proof of Proposition 31.2 runs along the same lines as for the the regularity criterion of Caffarelli, Kohn, and Nirenberg (Theorem 30.1). We introduce the same normalized quantities on a cylinder $Q((t,x),r) = (t-r^2, t) \times B(x,r)$:

$$\begin{cases} I_Q = \sup_{s-r^2<\sigma<s} r^{-3} \int_{B(x,r)} |\vec{u}(s,y)|^2 \, dy \\ J_Q = r^{-3} \iint_Q |\vec{\nabla} \otimes \vec{u}(\sigma,y)|^2 \, dy \, d\sigma \\ K_Q = r^{-5} \iint_Q |\vec{u}(\sigma,y)|^3 \, dy \, d\sigma \\ L_Q = r^{-5} \iint_Q |p(\sigma,y)|^{3/2} dy \, d\sigma \end{cases}$$

The hypothesis is that

$$\limsup_{r\to 0} r^{-1} \int\int_{|x-x_0|<r, t_0-\frac{7r^2}{8}<s<t_0+\frac{r^2}{8}} |\vec{\nabla} \otimes \vec{u}|^2 \, dx \, ds < \epsilon_0$$

which we may rewrite as

$$\limsup_{r\to 0} r^2 J_{Q((t_0+\frac{r^2}{8}, x_0), r)} < \epsilon_0,$$

and we want to prove that

$$\limsup_{r\to 0} r^3 (K_{Q((t_0+\frac{r^2}{8}, x_0), r)} + L_{Q((t_0+\frac{r^2}{8}, x_0), r)}) < \epsilon_{CKN}.$$

We write more concisely: $B_r = B(x_0, r)$, $Q_r = Q((t_0 + \frac{r^2}{8}, x_0), r)$, $I(r) = r^2 I_{Q_r}$, $J(r) = r^2 J_{Q_r}$, $K(r) = r^3 K_{Q_r}$ and $L(r) = r^3 L_{Q_r}$. Thus, we attempt to prove that for a weak solution suitable in the neighborhood of (t_0, x_0), we have

$$\limsup_{r\to 0} J(r) < \epsilon_0 \Rightarrow \limsup_{r\to 0} K(r) + L(r) < \epsilon_{CKN}.$$

As for Theorem 30.1, we shall try to estimate $K(r)$ and $L(r)$ from $I(\rho)$ and $J(\rho)$ with $\rho \geq 2r$ and to estimate $I(r)$ from $K(\rho)$ and $L(\rho)$ with $\rho \geq 2r$, but we have no need to estimate $J(r)$, since its size is controlled by inequality (31.1).

Step 1: Sobolev inequalities in the ball

Lemma 31.1:
Let $M_\rho f = \frac{1}{|B_\rho|}\int f(y)\,dy$. *Then:*
i) For $1 \leq p \leq \infty$ *and* $f \in L^p(B_\rho)$, $\|f - M_\rho f\|_{L^p(B_\rho)} \leq 2\|f\|_{L^p(B_\rho)}$.
ii) For $1 \leq p \leq \infty$ *and* $f \in L^p(B_\rho)$, $\|f - M_\rho f\|_{L^p(B_\rho)} \leq C_p \rho \|\vec{\nabla} f\|_{L^p(B_\rho)}$.
iii) For $1 < p < 3$, $f \in W^{1,p}(B_\rho)$ *and* $r = \frac{3p}{3-p}$, $\|f - M_\rho f\|_{L^r(B_{7\rho/8})} \leq C_p \|\vec{\nabla} f\|_{L^p(B_\rho)}$.

Proof: For i) and ii), we write

$$\|f - M_\rho f\|^p_{L^p(B_\rho)} \leq \frac{1}{|B_\rho|}\int\int_{B_\rho \times B_\rho} |f(y) - f(z)|^p \,dy\,dz.$$

We have $|f(y) - f(z)|^p \leq 2^p(|f(y)|^p + |f(z)|^p)$; thus i) is obvious. In order to arrive at ii), we write $|f(y) - f(z)|^p \leq \int_0^t |\vec{\nabla} f(ty + (1-t)z)|^p |y - z|^p \,dt$; thus,

$$\begin{aligned} \|f - M_\rho f\|^p_{L^p(B_\rho)} &\leq \tfrac{2\rho^p}{|B_\rho|}\int\int_{B_\rho \times B_\rho}\int_0^{1/2} |\vec{\nabla} f(ty + (1-t)z)|^p \,dy\,dz\,dt \\ &\leq \tfrac{2\rho^p}{|B_\rho|}\int\int_{B_\rho \times B_\rho}\int_0^{1/2} (1-t)^{-3}|\vec{\nabla} f(w)|^p \,dy\,dw\,dt \\ &= \tfrac{\rho^p}{4}\|\vec{\nabla} f\|^p_{L^p(B_\rho)}. \end{aligned}$$

To prove iii), we select $\omega \in \mathcal{D}$ with $\omega = 1$ on $B(0, 7/8)$ and with Supp $\omega \subset B(0,1)$ and we define $\omega_\rho(y) = \omega(y/\rho)$; then, conforming to the Sobolev inequalities,

$$\begin{aligned} \|f - M_\rho f\|_{L^r(B_{7\rho/8})} &\leq \|\omega_\rho(f - M_\rho f)\|_{W^{1,p}} \\ &\leq C(\|\vec{\nabla} f\|_{L^p(B_\rho)} + \rho^{-1}\|f - M_\rho f\|_{L^p(B_\rho)}) \\ &\leq C'\|\vec{\nabla} f\|_{L^p(B_\rho)}. \end{aligned}$$

□

Step 2: Estimating $K(r)$
The control of $K(r) = r^{-2}\iint_{Q_r} |\vec{u}(\sigma, y)|^3 \,dy\,d\sigma$ is not difficult. We choose $\omega \in \mathcal{D}$ with $\omega = 1$ on $B(0, 5/4)$ and with Supp $\omega \subset B(0, 7/4)$ and we write that

$$\begin{aligned} \|\vec{u}(\sigma,.)\|_{L^3(B_r)} &\leq C\|\omega(\tfrac{x-y}{r})\vec{u}(\sigma,.)\|_{\dot{H}^{1/2}} \\ &\leq C'\|\omega(\tfrac{x-y}{r})\vec{u}(\sigma,.)\|_2^{1/2}\|\omega(\tfrac{x-y}{r})\vec{u}(\sigma,.)\|_{\dot{H}^1}^{1/2} \\ &\leq C''\|\vec{u}(\sigma,.)\|^{1/2}_{L^2(B_{2r})}(r^{-1}\|\vec{u}(\sigma,.)\|_{L^2(B_{2r})} + \|\vec{\nabla} \otimes \vec{u}(\sigma,.)\|_{L^2(B_{2r})})^{1/2} \\ &\leq C'''\big(\rho^{1/4} I(\rho)^{1/4}\|\vec{\nabla} \otimes \vec{u}(\sigma,.)\|^{1/2}_{L^2(B_\rho)} + r^{-1/2}\|\vec{u}(\sigma,.)\|_{L^2(B_{2r})}\big). \end{aligned}$$

We want to introduce the estimate on $\vec{\nabla} \otimes \vec{u}$ in the last term of the estimate:

$$\begin{aligned} &\int_{B_{2r}} |\vec{u}(\sigma,y)|^2 \,dy = \tfrac{1}{|B_\rho|}\int_{y \in B_{2r}}\int_{z \in B_\rho} |\vec{u}(\sigma,y)|^2 - |\vec{u}(\sigma,z)|^2 + |\vec{u}(\sigma,z)|^2 \,dy\,dz \\ &\leq \tfrac{1}{|B_\rho|}(\textstyle\iint_{B_\rho \times B_\rho}|\vec{u}(\sigma,y) + \vec{u}(\sigma,z)|^2 dy\,dz)^{\frac{1}{2}}(\iint_{B_\rho \times B_\rho}|\vec{u}(\sigma,y) - \vec{u}(\sigma,z)|^2 dy\,dz)^{\frac{1}{2}} \\ &\quad + (\tfrac{r}{\rho})^3\int_{B_\rho} |\vec{u}(\sigma,z)|^2 \,dz \end{aligned}$$

We have $|\vec{u}(\sigma,y)-\vec{u}(\sigma,z)|^2 \le \int_0^1 |\vec{\nabla}\otimes\vec{u}(\sigma,y+t(z-y))|^2|y-z|^2\,dt$, hence $\iint_{B_\rho\times B_\rho}|\vec{u}(\sigma,y)-\vec{u}(\sigma,z)|^2dy\,dz \le 8\rho^2\iint_{B_\rho\times B_\rho}\int_0^{\frac12}|\vec{\nabla}\otimes\vec{u}(\sigma,y+t(z-y))|^2dy\,dz\,dt$ $\le 8\rho^2\iint_{B_\rho\times B_\rho}\int_0^{\frac12}|\vec{\nabla}\otimes\vec{u}(\sigma,w)|^2(1-t)^{-3}dw\,dz\,dt = 12\rho^2|B_\rho|\int_{B_\rho}|\vec{\nabla}\otimes\vec{u}(\sigma,w)|^2dw$. We thus obtain the estimate

$$\|\vec{u}(\sigma,.)\|_{L^3(B_r)} \le C\big(\sqrt{\frac{\rho}{r}}\rho^{1/4}I(\rho)^{1/4}\|\vec{\nabla}\otimes\vec{u}(\sigma,.)\|^{1/2}_{L^2(B_\rho)}+\frac{r}{\rho}I(\rho)^{1/2}\big)$$

and finally

$$(31.3)\ \ K(r)=r^{-2}\int_{t_0-\frac{7r^2}{8}}^{t_0+\frac{r^2}{8}}\|\vec{u}(\sigma,.)\|^3_{L^3(B_r)}d\sigma\le C\big((\frac{\rho}{r})^3I(\rho)^{\frac34}J(\rho)^{\frac34}+(\frac{r}{\rho})^3I(\rho)^{\frac32}\big)$$

Step 3: Estimating $L(r)$

We repeat

$$\Delta p = -\sum_{j=1}^d\sum_{l=1}^d\partial_j\partial_l(u_ju_l).$$

Since $\vec{u}$ is divergence free, we may write as well

$$\Delta p = -\sum_{j=1}^d\sum_{l=1}^d\partial_j\partial_l\big(u_j(u_l-M_\rho u_l)\big).$$

We then introduce $\zeta_\rho(z)=\omega(2(z-x_0)/\rho)$, where $\omega\in\mathcal{D}$ with $\omega=1$ on $B(0,5/4)$ and with Supp $\omega\subset B(0,7/4)$ and we write $\Delta(\zeta_\rho(z)p(\sigma,z))=\zeta_\rho(z)\Delta p(\sigma,z))+2\vec{\nabla}.(p(\sigma,z)\vec{\nabla}\zeta_\rho(z))-p(\sigma,z)\Delta\zeta_\rho(z)$. We again call K the kernel of $\frac{1}{\Delta}$ ($K(x)=-\frac{3}{4\pi|x|}$), obtaining $p=p_1+p_2$ with

$$p_1(\sigma,y)=-\sum_{j=1}^d\sum_{l=1}^d K*\Big(\zeta_\rho\partial_j\partial_l\big(u_j(u_l-M_\rho u_l)\big)\Big)$$

and

$$p_2(\sigma,y)=2\sum_{j=1}^d\partial_jK*(p\partial_j\zeta_\rho)-K*(p\Delta\zeta_\rho).$$

For $y\in B_r$ and $r\le\rho/2$ and for z in the support of $\partial^\alpha\zeta_\rho$ (with $\alpha\ne0$), we have $|z-y|\in[\rho/8,11\rho/8]$. This gives $|p_2(\sigma,y)|\le C\rho^{-3}\int_{B_\rho}|p(\sigma,z)|\,dz\le C'\big(\rho^{-3}\int_{B_\rho}|p(\sigma,z)|^{3/2}\,dz\big)^{2/3}$ and thus

$$r^{-2}\int\int_{Q_r}|p_2(\sigma,y)|^{3/2}\,dy\,d\sigma\le C\frac{r}{\rho}L(\rho).$$

In order to estimate p_1, we write $p_1 = p_3 + p_4$ with

$$p_3(\sigma, y) = -\sum_{j=1}^{d}\sum_{l=1}^{d} \partial_j\partial_l K * \big(\zeta_\rho u_j(u_l - M_\rho u_l)\big)$$

and

$$\begin{aligned} p_4(\sigma, y) = \quad & -\textstyle\sum_{j=1}^{d}\sum_{l=1}^{d} K * \big((\partial_j\partial_l\zeta_\rho)u_j(u_l - M_\rho u_l)\big) \\ & +\textstyle\sum_{j=1}^{d}\sum_{l=1}^{d} \partial_j K * \big((\partial_l\zeta_\rho)u_j(u_l - M_\rho u_l)\big) \\ & +\textstyle\sum_{j=1}^{d}\sum_{l=1}^{d} \partial_l K * \big((\partial_j\zeta_\rho)u_j(u_l - M_\rho u_l)\big). \end{aligned}$$

We have

$$\|p_4(\sigma,.)\|_{L^\infty(B_r)} \le C\rho^{-3}\|\vec{u}\|_{L^2(B_\rho)}\|\vec{u}-M_\rho\vec{u}\|_{L^2(B_\rho)} \le C'\rho^{-\frac{3}{2}}I(\rho)^{\frac{1}{2}}\|\vec{\nabla}\otimes\vec{u}\|_{L^2(B_\rho)};$$

hence, $\|p_4(\sigma,.)\|_{L^{3/2}(B_r)} \le C\frac{r^2}{\rho^{3/2}}I(\rho)^{1/2}\|\vec{\nabla}\otimes\vec{u}\|_{L^2(B_\rho)} \le C\rho^{1/2}I(\rho)^{1/2}\|\vec{\nabla}\otimes\vec{u}\|_{L^2(B_\rho)}$. On the other hand, $\frac{\partial_j\partial_l}{\Delta} = -R_jR_k$, and we have

$$\|p_3(\sigma,.)\|_{L^{\frac{3}{2}}(B_r)} \le C\|\,|\vec{u}|\,|\vec{u}-M_\rho\vec{u}|\,\|_{L^{\frac{3}{2}}(B_{\frac{7\rho}{8}})} \le C\|\vec{u}\|_{L^2(B_{\frac{7\rho}{8}})}\|\vec{u}-M_\rho\vec{u}\|_{L^6(B_{\frac{7\rho}{8}})}$$

hence, $\|p_3(\sigma,.)\|_{L^{3/2}(B_r)} \le C\rho^{1/2}I(\rho)^{1/2}\|\vec{\nabla}\otimes\vec{u}\|_{L^2(B_\rho)}$. Thus, we get

$$r^{-2}\int\!\!\int_{Q_r} |p_1(\sigma,y)|^{3/2}\, dy\, d\sigma \le C(\frac{\rho}{r})^{3/2}I(\rho)^{3/4}J(\rho)^{3/4}.$$

We obtain the estimate:

$$L(r) \le C\ (\frac{r}{\rho}L(\rho) + (\frac{r}{\rho})^{3/2}I(\rho)^{3/4}J(\rho)^{3/4}) \tag{31.4}$$

Step 4: Estimating $I(r)$

To estimate $I(r)$, we use the local energy inequality. We recall that $\vec{u}$ is suitable on a cylinder $[t_0 - 7R^2/8, t_0 + R^2/8] \times B(x_0, R)$. We fix R independent from ρ and r. We apply the local energy inequality to a test function $\omega_{s,\epsilon}\phi_\rho$ defined as follows. We define $\phi_\rho(\sigma, y) = \alpha((\sigma - t_0)/\rho^2)\beta((y - x_0)/\rho)$ with $\alpha \in \mathcal{D}(\mathbb{R})$ with $\alpha = 1$ on $[-7/30, 1/30]$ and Supp $\alpha \subset [-7/8, 1/8]$ and $\beta \in \mathcal{D}(\mathbb{R}^3)$ with $\beta = 1$ on $B(0, 5/8)$ and Supp $\beta \subset B(0, 7/8)$. Thus, $\phi_\rho = 1$ on a neighborhood of Q_r when $r \le \rho/2$.

Now, we fix $\theta \in \mathcal{D}(\mathbb{R})$, Supp $\theta \subset [0, 1]$, $\theta \ge 0$ and $\int \theta\, dt = 1$. For $s \in [t_0 - 7r^2/8, t_0 + r^2/8]$, $r \le \rho/2$ ($\rho < R$) we introduce for $\epsilon > 0$ the function $\omega_{s,\epsilon}(\sigma) = \int_{-\infty}^{\sigma} \frac{8}{R^2}\theta(\frac{8}{R^2}(\tau - t_0 + 3R^2/4)) - \frac{1}{\epsilon}\theta(\frac{s-\tau}{\epsilon})\, d\tau$. If ϵ is small enough ($\epsilon < \frac{13}{32}\rho^2$), $\omega_{s,\epsilon}$ is nonnegative and bounded by 1. The local energy inequality gives then for $s \in [t_0 - 7r^2/8, t_0 + r^2/8]$:

$$\begin{gathered} \textstyle\int\!\int |\vec{\nabla}\otimes\vec{u}(\sigma,y)|^2\ \omega_{s,\epsilon}(\sigma)\phi_\rho(\sigma,y)\ dy\ d\sigma \le \\ \textstyle\int\!\int |\vec{u}(\sigma,y)|^2\ \{\partial_\sigma + \Delta_y\}(\omega_{s,\epsilon}(\sigma)\phi_\rho(\sigma,y))\ dy\ d\sigma \\ +\textstyle\int\!\int (|\vec{u}(\sigma,y)|^2 + 2p(\sigma,y))\{\vec{u}(\sigma,y).\vec{\nabla}\}(\omega_{s,\epsilon}(\sigma)\phi_\rho(\sigma,y))\ dy\ d\sigma \end{gathered}$$

We let ϵ go to 0; then, for all Lebesgue point s of $s \mapsto \int_{B_\rho} |\vec{u}(s,y)|^2\, dy$, we get that

$$\begin{gathered}\int |\vec{u}(s,y)|^2\ \phi_\rho(s,y)\ dy\ + \iint_{t_0-7\rho^2/8}^{s} |\vec{\nabla}\otimes\vec{u}(\sigma,y)|^2\ \phi_\rho(\sigma,y)\ dy\ d\sigma \leq \\ C\rho^{-4}\iint_{t_0-7\rho^2/8}^{t_0-5\rho^2/8} |\vec{u}(\sigma,y)|^2\ \phi_\rho(\sigma,y)\ dy\ d\sigma + \\ \iint_{t_0-7\rho^2/8}^{s} |\vec{u}(\sigma,y)|^2\ \{\partial_\sigma+\Delta_y\}(\phi_\rho(\sigma,y))\ dy\ d\sigma \\ + \iint_{t_0-7\rho^2/8}^{s} (|\vec{u}(\sigma,y)|^2+2p(\sigma,y))\{\vec{u}(\sigma,y).\vec{\nabla}\}(\phi_\rho(\sigma,y))\ dy\ d\sigma\end{gathered}$$

Since $\vec{u}$ is weakly continuous from $[t_0-7r^2/8, t_0+r^2/8]$ into $(L^2((B_\rho)^3$, we find that this equality is valid for all $s \in [t_0-7r^2/8, t_0+r^2/8]$. Moreover, we have

$$\int |\vec{u}(\sigma,y)|^2\{\vec{u}(\sigma,y).\vec{\nabla}\}(\phi_\rho(\sigma,y))\ dy = \int (|\vec{u}(\sigma,y)|^2 - |M_\rho\vec{u}|^2))\{\vec{u}(\sigma,y).\vec{\nabla}\}\phi_\rho\ dy$$

since $\vec{u}$ is divergence free. Thus, we find $rI(r) \leq C(A(\rho)+B(\rho)+C(\rho)$ with

$$\begin{cases} A(\rho) = C\rho^{-2}\iint_{[t_0-7\rho^2/8, t_0+\rho^2/8]\times B_{7\rho/8}} |\vec{u}(\sigma,y)|^2\ dy\ d\sigma \\ B(\rho) = \rho^{-1}\iint_{[t_0-7\rho^2/8, t_0+\rho^2/8]\times B_{7\rho/8}} |\ |\vec{u}(\sigma,y)|^2 - |M_\rho\vec{u}|^2\ |\ |\vec{u}(\sigma,y)|\ dy\ d\sigma \\ C(\rho) = \rho^{-1}\iint_{[t_0-7\rho^2/8, t_0+\rho^2/8]\times B_{7\rho/8}} |p(\sigma,y)|\ |\vec{u}(\sigma,y)|\ dy\ d\sigma. \end{cases}$$

We have $A(\rho) \leq C\rho K(\rho)^{2/3}$. We estimate $B(\rho)$ by writing $|\vec{u}|^2 - |M_\rho\vec{u}|^2 = (\vec{u}+M_\rho\vec{u}).(\vec{u}+M_\rho\vec{u})$; hence,

$$\begin{aligned}\int_{B_{7\rho/8}} |\ |\vec{u}(\sigma,y)|^2 - |M_\rho\vec{u}|^2\ |\ |\vec{u}(\sigma,y)|\ dy \\ &\leq \|\vec{u}+M_\rho\vec{u}\|_{L^2(B_\rho)}\|\vec{u}-M_\rho\vec{u}\|_{L^5(B_{7\rho/8})}\|\vec{u}\|_{L^3(B_\rho)} \\ &\leq C\|\vec{u}\|_{L^2(B_\rho)}\|\vec{\nabla}\otimes\vec{u}\|_{L^2(B_\rho)}\|\vec{u}\|_{L^3(B_\rho)}\end{aligned}$$

and thus $B(\rho) \leq C\rho I(\rho)^{1/2}J(\rho)^{1/2}K(\rho)^{1/3}$. Finally, $C(\rho) \leq \rho K(\rho)^{1/3}L(\rho)^{2/3}$.

We obtain the estimate

$$I(r) \leq C\frac{\rho}{r}(K(\rho)^{2/3} + I(\rho)^{1/2}J(\rho)^{1/2}K(\rho)^{1/3} + K(\rho)^{1/3}L(\rho)^{2/3}) \tag{31.5}$$

Step 5: The recursion formula

Putting together estimates (31.3), (31.4), and (31.5), we obtain for $r \leq \rho/2 \leq \tau/4$ (and writing $I(\rho) \leq \frac{\tau}{\rho}I(\tau)$, $J(\rho) \leq \frac{\tau}{\rho}J(\tau)$, $K(\rho) \leq \frac{\tau^2}{\rho^2}K(\tau)$ and $L(\rho) \leq \frac{\tau^2}{\rho^2}L(\tau)$)

$$\begin{cases} I(r) \leq C\frac{\rho}{r}(K(\rho)^{2/3} + I(\rho)^{1/2}J(\rho)^{1/2}K(\rho)^{1/3} + K(\rho)^{1/3}L(\rho)^{2/3}) \\ \quad\leq C'(\frac{\tau^2}{r\rho}I(\tau)^{1/2}J(\tau)^{1/2} + \frac{\rho^3}{r\tau^2}I(\tau) + \frac{\tau^{11/6}}{r\rho^{5/6}}I(\tau)^{1/2}J(\tau)^{1/2}K(\tau)^{1/3} \\ \qquad + \frac{\tau^{5/3}}{r\rho^{2/3}}I(\tau)^{1/4}J(\tau)^{1/4}L(\tau)^{2/3} + \frac{\rho^{4/3}}{r\tau^{1/3}}I(\tau)^{1/2}L(\tau)^{2/3}) \\ K(r) \leq C\big((\frac{\tau}{r})^3 I(\tau)^{\frac{3}{4}}J(\tau)^{\frac{3}{4}} + (\frac{r}{\tau})^3 I(\tau)^{\frac{3}{2}}\big) \\ L(r) \leq C\ (\frac{r}{\tau}L(\tau) + (\frac{\tau}{r})^{3/2}I(\tau)^{3/4}J(\tau)^{3/4}). \end{cases}$$

Let $M(r) = I(r) + K(r)^{2/3} + L(r)^{4/3}$. We assume that τ is small enough to grant that $J(\tau) < \epsilon_0$. We find that $M(r) \leq \alpha(r,\rho,\tau)M(\tau) + \beta(r,\rho,\tau)M(\tau)^{1/2} + \gamma(r,\rho,\tau)M(\tau)^{3/4}$ with

$$\begin{cases} \alpha(r,\rho,\tau) = & C_1\ (\frac{\rho^3}{r\tau^2} + \frac{\tau^{11/6}}{r\rho^{5/6}}\epsilon_0^{1/2} + \frac{\rho^{4/3}}{r\tau^{1/3}} + \frac{r^2}{\tau^2} + \frac{r^{4/3}}{\tau^{4/3}} + \frac{\tau^2}{r^2}\epsilon_0) \\ \beta(r,\rho,\tau) = & C_2\ (\frac{\tau^2}{r\rho}\epsilon_0^{1/2} + \frac{\tau^2}{r^2}\epsilon_0^{1/2}) \\ \gamma(r,\rho,\tau) = & C_3\ \frac{\tau^{5/3}}{r\rho^{2/3}}\epsilon_0^{1/4}. \end{cases}$$

We want to prove that, for r small enough, $K(r) + L(r) < \epsilon_{CKN}$. We may assume that $\epsilon_{CKN} < 1$. It is then enough to show that $M(r) < 2^{-1/3}\epsilon_{CKN}^{4/3}$ (this will give $K(r) < 1$, hence $K(r) + L(r) \leq 2^{1/4}M(r)^{3/4} < \epsilon_{CKN}$). We thus assume that $M(\tau) \geq 2^{-1/3}\epsilon_{CKN}^{4/3}$ and we write

$$M(r) \leq (\alpha(r,\rho,\tau) + \beta(r,\rho,\tau)2^{1/6}\epsilon_{CKN}^{-2/3} + \gamma(r,\rho,\tau)2^{1/12}\epsilon_{CKN}^{-1/3})M(\tau)$$

We then choose to fix the ratio $\rho/r = 1/2$ and $\tau/\rho = \eta$ with η small enough to grant that

$$C_1\ (2\eta^2 + 2\eta^{1/3} + \frac{\eta^2}{4} + \frac{\eta^{4/3}}{2^{4/3}}) < 1/4$$

and then we require $\epsilon_0 < 1$ to be small enough to grant that

$$\left(C_1\ (\frac{2}{\eta^{-11/6}} + \frac{1}{4\eta^2}) + C_2\ (\frac{2}{\eta} + \frac{1}{4\eta^2})2^{1/6}\epsilon_{CKN}^{-2/3} + C_3\ \frac{2}{\eta^{5/3}}2^{1/12}\epsilon_{CKN}^{-1/3}\right)\ \epsilon_0^{1/4} < 1/4.$$

Thus, starting from an initial r_0 so that $\vec{u}$ is suitable on Q_{r_0} and so that $J(r) < \epsilon_0$ for $r \leq r_0$ and defining $r_n = (\eta/2)^n r_0$, we find that $M(r_n) < 2^{-1/3}\epsilon_{CKN}^{4/3}$ or $M(r_{n+1}) < \frac{1}{2}M(r_n)$. Thus, we may find some $n \in \mathbb{N}$ with $M(r_n) < 2^{-1/3}\epsilon_{CKN}^{4/3}$ and Proposition 31.2 is proved. $\square$

Chapter 32

Local existence (in time) of suitable local square-integrable weak solutions

In this chapter, we attempt to prove how the local energy inequality for suitable solutions may be turned into a tool for proving the existence of solutions for the Navier–Stokes initial value problem for a locally square-integrable initial data, just as the Leray energy inequality was a tool for proving existence of weak solutions for a square-integrable initial data.

We begin by defining the *local Leray solutions*:

Definition 32.1: (local Leray solutions)

Let $\vec{u}_0 \in (L^2_{uloc}(\mathbb{R}^3))^3$ *so that* $\vec{\nabla}.\vec{u}_0 = 0$. *A local Leray solution* $\vec{u}$ *on* $(0,T)\times\mathbb{R}^3$ *for the Navier–Stokes initial value problem associated with* $\vec{u}_0$ *is a weak solution for the Navier–Stokes equations on* $(0,T)\times\mathbb{R}^3$ *so that:*

α) $\vec{u} \in \cap_{t<T} L^\infty((0,t),(L^2_{uloc})^3)$: $\sup_{x_0\in\mathbb{R}^3, s<t} \int_{|x-x_0|\leq 1} |\vec{u}(s,x)|^2\, dx < \infty$

β) *For all* $t<T$ *we have* $\sup_{x_0\in\mathbb{R}^3} \int\int_{0<s<t, |x-x_0|\leq 1} |\vec{\nabla}\otimes\vec{u}|^2\, dx\, ds < \infty$

γ) *for all compact subset* K *of* $\mathbb{R}^3$, $\lim_{t\to 0^+} \int_K |\vec{u}-\vec{u}_0|^2\, dx = 0$

δ) $\vec{u}$ *is suitable in the sense of Caffarelli, Kohn, and Nirenberg.*

We are going to prove in this chapter an existence theorem for local Leray solutions:

Theorem 32.1: (locally square-integrable initial value)

a) For all $\vec{u}_0 \in (L^2_{uloc}(\mathbb{R}^3))^3$ *so that* $\vec{\nabla}.\vec{u}_0 = 0$, *there exists a positive real number* T *and a local Leray solution* $\vec{u}$ *on* $(0,T)\times\mathbb{R}^3$ *for the Navier–Stokes initial value problem associated with* $\vec{u}_0$.

b) If $\vec{u}_0$ *belongs more precisely to* E_2^3, *where* E_2 *is the closure of* $\mathcal{D}(\mathbb{R}^3)$ *in* L^2_{uloc}, *then* $\vec{u} \in \cap_{t<T} L^\infty((0,t),(E_2)^3)$ *and* $\lim_{t\to 0^+} \|\vec{u}-\vec{u}_0\|_{L^2_{uloc}} = 0$.

The proof of Theorem 32.1 is based on some local energy estimates for the solution of the mollified equations:

$$(32.1)\qquad \begin{cases} \partial_t \vec{u}_\epsilon = \Delta\vec{u}_\epsilon - \mathbb{P}\vec{\nabla}.((\vec{u}_\epsilon * \omega_\epsilon)\otimes\vec{u}_\epsilon) \\ \vec{\nabla}.\vec{u}_\epsilon = 0 \\ \vec{u}_\epsilon(0,.) = \vec{u}_0 \end{cases}$$

which we have introduced in Chapter 13. (We give here a proof slightly different from the proof we set forth in [LEM 98b]).

1. Size estimates for $\vec{u}_\epsilon$

In this section, we prove that, at least for small times, we may find an estimate for the L^2_{uloc} norm of the solution $\vec{u}_\epsilon$ of (32.1), which depends only on the L^2_{uloc} norm of the initial value (and is independent from ϵ):

Proposition 32.1:

Let $\vec{u}_0 \in (L^2_{uloc}(\mathbb{R}^3))^3$ be such that $\vec{\nabla}.\vec{u}_0 = 0$. Define $\alpha_0 = \|\vec{u}_0\|_{L^2_{uloc}}$ and $\alpha_1 = \min(1, \alpha_0)$. Then, there exists a positive constant C_0 (which does not depend on $\vec{u}$ nor on ϵ) so that the equations (32.1) have a solution $\vec{u}_\epsilon$ on $(0, T_0) \times \mathbb{R}^3$ with $T_0 = \min(1, \frac{\alpha_1^2}{\alpha_0^2 C_0^4})$ and so that for all $0 < t < T_0$ we have

$$\|\vec{u}_\epsilon(t,.)\|_{L^2_{uloc}} \le \sqrt{C_0}\|\vec{u}_0\|_{L^2_{uloc}}\big(1 - \frac{\alpha_0^2 C_0^4}{\alpha_1^2} t\big)^{-1/4} \tag{32.2}$$

Proof: We saw in Chapter 13 that equation (32.1) has a solution $\vec{u}_\epsilon$ defined on $(0, T^*_\epsilon) \times \mathbb{R}^3$ for some maximal $T^*_\epsilon > 0$ so that:

i) $\vec{u}_\epsilon \in \cap_{T<T^*_\epsilon} (L^\infty((0,T), B^{-3/2,\infty}_\infty(\mathbb{R}^3)))^3$.

ii) If $T^*_\epsilon < \infty$, then $\lim_{t\to T^*_\epsilon} \|\vec{u}_\epsilon\|_{B^{-3/2,\infty}_\infty} = +\infty$.

iii) The solution $\vec{u}_\epsilon$ is $\mathcal{C}^\infty$ on $(0, T^*_\epsilon) \times \mathbb{R}^d$.

iv) The pressure p_ϵ satisfying $\vec{\nabla} p_\epsilon = (\mathbb{P} - Id)\vec{\nabla}.\big((\vec{u}_\epsilon * \omega_\epsilon) \otimes \vec{u}_\epsilon\big)$ is $\mathcal{C}^\infty$ on $(0, T^*_\epsilon) \times \mathbb{R}^d$.

We now prove Proposition 32.1 by proving that inequality (32.2) is valid for all $t < \min(T_0, T^*_\epsilon)$, and thus that $T^*_\epsilon \ge T_0$.

A convenient tool to estimate L^2_{uloc} norms will be the use of a family of test functions $(\varphi_0(. - k))_{k\in\mathbb{Z}^3}$, generated through translations from one given smooth function $\varphi_0 \in \mathcal{D}(\mathbb{R}^3)$, to localize L^2 norms:

Definition 32.2:

Let $\varphi_0 \in \mathcal{D}(\mathbb{R}^3)$ so that $\varphi_0 \ge 0$ and $\sum_{k\in\mathbb{Z}^3} \varphi_0(x-k) = 1$.

(A) We define $\mathcal{B} = \{\varphi_0(x - x_0) \ / \ x_0 \in \mathbb{R}^3\}$; the norm $\|f\|_{L^2_{uloc}}$ is then equivalent to $\sup_{\varphi\in\mathcal{B}} \|f\varphi\|_2$.

(B) We fix ω_0 and $\psi_0 \in \mathcal{D}(\mathbb{R}^3)$ so that ω_0 is identically equal to 1 in the neighborhood of the support of φ_0 and similarly, ψ_0 is identically equal to 1 in the neighborhood of the support of ω_0. Then, for $\varphi \in \mathcal{B}$, $\varphi = \varphi_0(x - x_\varphi)$, we define $\psi = \psi_0(x - x_\varphi)$.

Now, we introduce the functions $\alpha_\epsilon(t) = \sup_{\varphi\in\mathcal{B}} \|\vec{u}_\epsilon(t,.)\varphi(x)\|_2^2$, $\beta_\epsilon(t) = \sup_{\varphi\in\mathcal{B}}\int_0^t \|(\vec{\nabla}\otimes\vec{u}_\epsilon(s,.))\varphi(x)\|_2^2 ds$ and $\beta_{\epsilon,\eta}(t) = \sup_{\varphi\in\mathcal{B}}\int_\eta^t \|(\vec{\nabla}\otimes\vec{u}_\epsilon(s,.))\varphi(x)\|_2^2 ds$. We estimate α_ϵ and β_ϵ with the help of a series of technical lemmas.

Lemma 32.1: *For $0 < \eta < T < T_\epsilon^*$, we have $\sup_{\eta<t<T} \alpha_\epsilon(t) < \infty$ and $\sup_{\eta<t<T} \beta_{\epsilon,\eta}(t) < \infty$.*

Proof: We know that on $[\eta, T] \times \mathbb{R}^3$ we have $\vec{u}_\epsilon \in (L^\infty([\eta,T]\times\mathbb{R}^3))^3$ and $\vec{\nabla}\otimes\vec{u}_\epsilon \in (L^\infty([\eta,T]\times\mathbb{R}^3))^{3\times 3}$. □

In the next lemma, we estimate the size of the pressure p_ϵ associated with $\vec{u}_\epsilon$:

Lemma 32.2: *For all $\varphi \in \mathcal{B}$ ($\varphi = \varphi_0(x - x_0)$ for some $x_0 \in \mathbb{R}^3$), there exists a function $p_{\epsilon,\varphi}(t)$ so that for all interval $I = (T_0, T_1)$ with $0 < T_0 < T_1 < T_\epsilon^*$:*

$$\begin{aligned}&\Big(\iint_{I\times\mathbb{R}^3} |p_\epsilon(t,x) - p_{\epsilon,\varphi}(t)|^{3/2}\varphi(x)\ dx\ dt\Big)^{2/3} \\ &\leq C\big(\|\vec{u}_\epsilon\|^2_{L^3(I,L^2_{uloc})} + \|\psi\vec{u}_\epsilon\|_{L^6(I,L^2)}\|\psi\vec{u}_\epsilon\|_{L^2(I,H^1)}\big)\end{aligned}$$

where C does not depend neither on φ nor on T_0, T_1 nor on ϵ.

Proof: In order to prove this estimate, we recall that p_ϵ is computed, at least formally, as $p_\epsilon = \sum_{1\leq j\leq 3}\sum_{1\leq k\leq 3} R_j R_k\big((u_{\epsilon,j} * \omega_\epsilon)u_{\epsilon,k}\big)$, where $R_j = \frac{\partial_j}{\sqrt{-\Delta}}$ is the jth Riesz transform and $u_{\epsilon,j}$ is the jth coordinate of the vector $\vec{u}_\epsilon$. Writing more compactly $p_\epsilon = \mathcal{G}.(\vec{u}_\epsilon * \omega_\epsilon)\otimes\vec{u}_\epsilon = \int G(x,y).(\vec{u}_\epsilon*\omega_\epsilon)(t,y)\otimes\vec{u}_\epsilon(y)\,dy$, we thus compute $p_\epsilon - p_{\epsilon,\varphi}(t)$ through the formula: $p_\epsilon(t,x) - p_{\epsilon,\varphi}(t) = p_1(t,x) + p_2(t,x)$ with

$$\begin{cases} p_1(t,x) = \mathcal{G}\big(\psi_0^2(y-x_0)(\vec{u}_\epsilon * \omega_\epsilon)(t,y)\otimes\vec{u}_\epsilon(t,y)\big)(x) \\ p_2(t,x) = \int (G(x,y) - G(x_0,y)).(1-\psi_0^2(y-x_0))(\vec{u}_\epsilon * \omega_\epsilon)(t,y)\otimes\vec{u}_\epsilon(t,y)\ dy \end{cases}$$

It is easy to check that, on the support of $\varphi(x) = \varphi_0(x - x_0)$, $p_2(t,x)$ is bounded by $C\|\vec{u}_\epsilon(t,.)\|^2_{L^2_{uloc}}$. On the other hand, $\|p_1(x,t)\|_{L^{3/2}([T_0,T_1]\times\mathbb{R}^3)} \leq C\|\psi^2\ ((\vec{u}_\epsilon * \omega_\epsilon)\otimes\vec{u}_\epsilon)\|_{L^{3/2}}$ ($\mathcal{G}$ being a matrix of Calderón–Zygmund operators). But $H^1(\mathbb{R}^3) \subset L^6$ and thus we have $L^6(L^2)\cap L^2(H^1) \subset L^3(L^3)$. This finally gives the following control on p_1:

$$\|p_1(x,t)\|_{L^{3/2}([T_0,T_1]\times\mathbb{R}^3)} \leq C\|\psi\vec{u}\|_{L^6([T_0,T_1],L^2)}\ \|\psi\vec{u}\|_{L^2([T_0,T_1],H^1)}. \quad \square$$

We now prove the key local energy estimate for $\vec{u}_\epsilon$.

Lemma 32.3: *For all $t \in (\eta, T_\epsilon^*)$ and all $\varphi \in \mathcal{B}$, we have the following equality:*

$$(32.3)\quad \begin{aligned} \|\vec{u}_\epsilon(t,.)\varphi(x)\|_2^2 + 2\int_\eta^t \|(\vec{\nabla}\otimes\vec{u}_\epsilon(s,.))\varphi(x)\|_2^2\ ds = \\ \|\vec{u}_\epsilon(\eta,.)\varphi(x)\|_2^2 + \iint_\eta^t |\vec{u}_\epsilon|^2\Delta(\varphi^2(x))\ dx\ ds \\ +\iint_\eta^t |\vec{u}_\epsilon|^2(\vec{u}_\epsilon * \omega_\epsilon).\vec{\nabla}\varphi^2(x)\ dx\ ds + 2\iint_\eta^t (p_\epsilon - p_{\epsilon,\varphi})(\vec{u}_\epsilon.\vec{\nabla})\varphi^2(x)\ dx\ ds \end{aligned}$$

Proof: Since $\vec{u}_\epsilon$ is smooth on $(0, T_\epsilon^*)\times \mathbb{R}^3$, we have

$$\partial_t|\vec{u}_\epsilon|^2 + 2|\vec{\nabla}\otimes\vec{u}_\epsilon|^2 = \Delta|\vec{u}_\epsilon|^2\ - \vec{\nabla}.(|\vec{u}_\epsilon|^2(\vec{u_\epsilon}*\omega_\epsilon) + 2p_\epsilon\ \vec{u}_\epsilon)$$

and equality (32.3) follows by integration, since the divergence free condition on $\vec{u}_\epsilon$ gives that $\vec{\nabla}.(p_\epsilon\ \vec{u}_\epsilon) = \vec{u}_\epsilon.\vec{\nabla}p_\epsilon = \vec{u}_\epsilon.\vec{\nabla}(p_\epsilon - p\epsilon,\varphi)$. □

Lemma 32.4: *For all $t \in (\eta, T_\epsilon^*)$ and all $\varphi \in \mathcal{B}$, we have the following inequality:*

$$\int\int_\eta^t |\vec{u}_\epsilon|^2\ \Delta(\varphi^2(x))\ dx\ ds \le C_1\int_\eta^t \alpha_\epsilon(s)\ ds$$

Proof: This is obvious. □

Lemma 32.5: *For all $t \in (\eta, T_\epsilon^*)$ and all $\varphi \in \mathcal{B}$, we have the following inequality:*

$$\iint_\eta^t |\vec{u}_\epsilon|^2(\vec{u}_\epsilon*\omega_\epsilon).\vec{\nabla}\varphi^2(x)\ dx\ ds \le C_2(\int_\eta^t \alpha_\epsilon(s)^3\ ds)^{1/4}\ \left(\beta_{\epsilon,\eta}(t) + \int_\eta^t \alpha_\epsilon(s)\ ds\right)^{3/4}$$

Proof: Let ψ_0 be equal to 1 on the support of φ_0 and let us write, for $\varphi(x) = \varphi_0(x - x_0)$ and $\psi(x) = \psi_0(x - x_0)$,

$$\iint_\eta^t |\vec{u}_\epsilon|^2(\vec{u}_\epsilon * \omega_\epsilon).\vec{\nabla}\varphi^2(x)\ dx\ ds \le C\int\int_\eta^t |\psi\ \vec{u}_\epsilon|^2\ |\psi\ \vec{u}_\epsilon * \omega_\epsilon|\ dx\ ds$$

We use the inclusions $L^2(L^6)\cap L^6(L^2) \subset L^3(L^3)$ and $H^1(\mathbb{R}^3) \subset L^6$ to get

$$\iint_\eta^t |\psi\vec{v}|^3\ dx\ ds \le C\big(\int_0^t \|\psi\vec{v}\|_2^6 ds\big)^{1/4}\big(\int_0^t \|\psi\vec{v}\|_{H^1}^2 ds\big)^{3/4}$$

and we must check the following four inequalities:

$\int(|\psi_0(x - x_0)|^2 + |\vec{\nabla}\psi_0(x - x_0)|^2)\ |\vec{u}_\epsilon(t,x)|^2\ dx \le C\alpha_\epsilon(t)$

$\int\int_\eta^t |\psi_0(x - x_0)|^2\ |\vec{\nabla}\otimes\vec{u}_\epsilon(s,x)|^2\ dx\ ds \le C\beta_{\epsilon,\eta}(t)$

$\int (|\psi_0(x-x_0)|^2 + |\vec{\nabla}\psi_0(x-x_0)|^2)\ |\vec{u}_\epsilon * \omega_\epsilon(t,x)|^2\ dx \leq C\alpha_\epsilon(t)$
$\int\int_\eta^t |\psi_0(x-x_0)|^2\ |\vec{\nabla}\otimes(\vec{u}_\epsilon * \omega_\epsilon)(s,x)|^2\ dx\ ds \leq C\beta_{\epsilon,\eta}(t).$

The first two inequalities are obvious, and the third inequality is easy, since L^2_{uloc} is a shift-invariant space of distributions, hence is stable through convolution with functions in L^1. For the last inequality, we notice that $\beta_{\epsilon,\eta}(t)$ is equivalent to the L^1_{uloc} norm of $\int_\eta^t |\vec{\nabla}\otimes\vec{u}|^2\ ds$; now, recall that $\omega \geq 0$ and $\int \omega\ dx = 1$, this gives that $\int_\eta^t |\vec{\nabla}\otimes(\vec{u}_\epsilon * \omega_\epsilon)|^2\ ds \leq \int_\eta^t |\vec{\nabla}\otimes\vec{u}_\epsilon|^2\ ds * \omega_\epsilon$, and we may conclude since L^1_{uloc} is stable through convolution with functions in L^1. □

Lemma 32.6: For all $t \in (\eta, T^*_\epsilon)$ and all $\varphi \in \mathcal{B}$, we have the following inequality:

$$\iint_\eta^t (p_\epsilon - p_{\epsilon,\varphi})(\vec{u}_\epsilon.\vec{\nabla})\varphi^2(x)\ dx\ ds \leq C_3\Big(\int_\eta^t \alpha_\epsilon(s)^3\ ds\Big)^{1/4}\Big(\beta_{\epsilon,\eta}(t) + \int_\eta^t \alpha_\epsilon(s)\ ds\Big)^{3/4}$$

Proof: We use the results of Lemma 32.2 to get that

$$\int\int_\eta^t (p_\epsilon - p_{\epsilon,\varphi})(\vec{u}_\epsilon.\vec{\nabla})\varphi_0^2(x-x_0)\ \ dx\ ds \leq$$

$$\leq C\Big(\iint_\eta^t |p_\epsilon(t,x) - p_{\epsilon,\varphi}(s)|^{3/2}\omega_0(x-x_0)\ dx\ dt\Big)^{2/3}\Big(\iint_\eta^t |\vec{u}_\epsilon|^3\omega_0(x-x_0)\ dx\ dt\Big)^{1/3}$$

$$\leq C'\big(\|\vec{u}_\epsilon\|^2_{L^3([\eta,t],L^2_{uloc})} + \|\psi_0(x-x_0)\vec{u}_\epsilon\|^2_{L^3([\eta,t]\times\mathbb{R}^3)}\big)\ \|\psi_0(x-x_0)\vec{u}_\epsilon\|_{L^3([\eta,t]\times\mathbb{R}^3)}$$

and we conclude since we know (as already used in the proof of Lemma 32.5) that we have

$$\|\psi_0(x-x_0)\vec{u}_\epsilon\|_{L^3([\eta,t]\times\mathbb{R}^3)} \leq C''\Big(\int_\eta^t \alpha_\epsilon(s)^3\ ds\Big)^{1/12}\Big(\beta_{\epsilon,\eta}(t) + \int_\eta^t \alpha_\epsilon(s)\ ds\Big)^{1/4}$$

and

$$\|\vec{u}_\epsilon\|_{L^3([\eta,t],L^2_{uloc})} \leq \|\vec{u}_\epsilon\|^{1/2}_{L^2([\eta,t],L^2_{uloc})}\|\vec{u}_\epsilon\|^{1/2}_{L^6([\eta,t],L^2_{uloc})} \leq C'''\Big(\int_\eta^t \alpha_\epsilon ds\Big)^{\frac{1}{4}}\Big(\int_\eta^t \alpha_\epsilon^3 ds\Big)^{\frac{1}{12}}$$

□

End of the proof: Now, clearly, these lemmas give us the following inequalities for $t \in (\eta, T^*_\epsilon)$ (just by looking at the suprema at time t of $\|\vec{u}_\epsilon(t,.)\varphi(x)\|_2^2$ and of $2\int_\eta^t \|(\vec{\nabla}\otimes\vec{u}_\epsilon(s,.))\varphi(x)\|_2^2\ ds$):

$$\alpha_\epsilon(t) \leq \alpha_\epsilon(\eta) + C_1\int_\eta^t \alpha_\epsilon(s)\ ds + (C_2 + 2C_3)\Big(\int_\eta^t \alpha_\epsilon(s)^3\ ds\Big)^{\frac{1}{4}}\Big(\beta_{\epsilon,\eta}(t) + \int_\eta^t \alpha_\epsilon(s)\ ds\Big)^{\frac{3}{4}}$$

$$2\beta_{\epsilon,\eta}(t)\leq\alpha_\epsilon(\eta)+C_1\int_\eta^t\alpha_\epsilon(s)\,ds+(C_2+2C_3)(\int_\eta^t\alpha_\epsilon(s)^3ds)^{\frac14}(\beta_{\epsilon,\eta}(t)+\int_\eta^t\alpha_\epsilon(s)\,ds)^{\frac34}$$

We then use the Young inequality $(\int_\eta^t\alpha_\epsilon(s)^3\;ds)^{\frac14}(\beta_{\epsilon,\eta}(t)+\int_\eta^t\alpha_\epsilon(s)\;ds)^{\frac34}\leq\frac{1}{4\delta^4}\int_\eta^t\alpha_\epsilon(s)^3\;ds\;+\frac{3\delta^{4/3}}{4}(\beta_{\epsilon,\eta}(t)+\int_\eta^t\alpha_\epsilon(s)\;ds)$ with $3/4\;(C_2+2C_3)\;\delta^{4/3}<1$ to eliminate $\beta_{\epsilon,\eta}(t)$ and we obtain

$$\alpha_\epsilon(t)\leq C_4\;(\alpha_\epsilon(\eta)+\int_\eta^t\alpha_\epsilon(s)\;ds+\int_\eta^t\alpha_\epsilon(s)^3\;ds)$$

We then let η go to 0. First, we notice that $\|\vec v*\omega_\epsilon(t,.)\|_\infty\leq C_\epsilon\|\vec v(t,.)\|_{B_\infty^{-3/2,\infty}}$, hence that

$$\|\int_0^t\mathbb{P}e^{(t-s)\Delta}\vec\nabla.(\vec v*\omega_\epsilon)\otimes\vec v\;ds\|_{L^2_{uloc}}\leq C_\epsilon\sqrt t\sup_{0<\tau<t}\|\vec v(\tau,.)\|_{B_\infty^{-\frac32,\infty}}\sup_{0<\tau<t}\|\vec v(\tau,.)\|_{L^2_{uloc}}$$

Thus, the Picard iteration scheme provides a solution not only in the space $(L^\infty((0,T_0(\epsilon)),B_\infty^{-3/2,\infty})^3)$ for some positive $T_0(\epsilon)$, but more precisely in the space $(L^\infty((0,T_1(\epsilon)),B_\infty^{-3/2,\infty})^3)\cap(L^\infty((0,T_1(\epsilon)),L^2_{uloc})^3)$ for some positive $T_1(\epsilon)$. We then write:

$$\alpha_\epsilon(t)\leq C\|\vec u(t,.)\|_{L^2_{uloc}}\leq C\|e^{t\Delta}\vec u_0\|_{L^2_{uloc}}+C(\vec u_\epsilon)\sqrt t\leq C\alpha_0+C(\vec u_\epsilon)\sqrt t$$

Thus, we get that $\alpha_\epsilon(t)$ is bounded in the neighborhood of $t=0$ (with a bound depending on ϵ) and that

$$\alpha_\epsilon(t)\leq C_5\;(\alpha_0+\int_0^t\alpha_\epsilon(s)\;ds+\int_0^t\alpha_\epsilon(s)^3\;ds).$$

We then write $\alpha_\epsilon(s)\leq\alpha_0+\frac{\alpha_\epsilon(s)^3}{\alpha_0^2}$ and get for $t<1$:

$$\alpha_\epsilon(t)\leq C_0\;(\alpha_0+\int_0^t\frac{\alpha_\epsilon(s)^3}{\alpha_1^2}\;ds)$$

Thus, the function $\gamma_\epsilon(t)=\alpha_0+\int_0^t\frac{\alpha_\epsilon(s)^3}{\alpha_1^2}\;ds$ is a solution of the differential inequality $\gamma_\epsilon'\leq C_0^3\gamma_\epsilon^3/\alpha_1^2$. Thus, we obtain $1/\alpha_0^2-1/\gamma_\epsilon(t)^2\leq C_0^3t/\alpha_1^2$, so that $\alpha_\epsilon(t)\leq C_0(1/\alpha_0^2-C_0^3t/\alpha_1^2)^{-1/2}$ and this completes the proof of Proposition 32.1. □

2. Local existence of solutions

We may now begin the proof of Theorem 32.1. In this section, we prove the existence of a suitable weak solution in $(0,T)\times\mathbb{R}^3$ for some positive T.

We consider an initial value $\vec{u}_0 \in (L^2_{uloc})^3$ with $\vec{\nabla}.\vec{u}_0 = 0$. We proved the following important estimates on the solutions $\vec{u}_\epsilon$ of the mollified equations:

$$\sup_{\varphi\in\mathcal{B}} \|\vec{u}_\epsilon(t,.)\varphi(x)\|_2^2 \leq C_0(1/\alpha_0^2 - C_0^3 t/\alpha_1^2)^{-1/2}$$

$$\sup_{\varphi\in\mathcal{B}} \int_0^t \|(\vec{\nabla}\otimes\vec{u}_\epsilon(s,.))\varphi(x)\|_2^2 ds \leq C_6(1/\alpha_0^2 - C_0^3 t/\alpha_1^2)^{-1/2}$$

which are valid on $(0, T_0)$. Then, we may use the limiting process described in Theorem 13.2 and get that a subsequence $\vec{u}_{\epsilon_n}$ is convergent to a solution $\vec{u}$ of the Navier–Stokes equations associated to $\vec{u}_0$. More precisely, when ϵ_n converges to 0, we have for all $\phi \in \mathcal{D}((0,T_0)\times \mathbb{R}^3)$ strong convergence of $\phi\vec{u}_\epsilon$ in $L^p((0,T_0),(L^2)^3)$ for all $p<\infty$ and weak convergence in $L^2((0,T_0),(H^1)^3))$. Moreover, if we use the formula:

$$p = \sum_{j<0} p_j(x) - p_j(0) + \sum_{j\geq 0} p_j(x)$$

we see that for all $t < T$ the pression p_ϵ is bounded in $L^\infty((0,t), B_\infty^{-3,\infty})$, which is a dual space of a separable Banach space; hence, we may have weak convergence of p_{ϵ_n}. The weak limits $(\vec{u}, p)$ are then a solution of the Navier–Stokes equations in $(0,T_0)\times\mathbb{R}^3$.

We demonstrate that this solution is suitable. Since the space L^2_{uloc} of uniformly locally square-integrable functions is a dual space, the control on the L^2_{uloc} norm of $\vec{u}_\epsilon$ gives $\vec{u} \in \cap_{t<T_0} L^\infty((0,t),(L^2_{uloc})^3)$. Similarly, we have for all $t < T_0$, $\sup_{\varphi\in\mathcal{B}}\int_0^t \|(\vec{\nabla}\otimes\vec{u}_\epsilon(s,.))\varphi(x)\|_2^2 ds < \infty$. Thus, we have only to check that p is locally in $L^{3/2}$ and that we have the local energy inequality. The fact that p is locally in $L^{3/2}$ is proved in the same way as in Lemma 32.2. For the energy inequality, this is the same proof as for the case of the restricted Leray solutions (Proposition 30.2). We start from the equality for the smooth function $\vec{u}_\epsilon$:

$$\partial_t|\vec{u}_\epsilon|^2 + 2|\vec{\nabla}\otimes\vec{u}_\epsilon|^2 = \Delta|\vec{u}_\epsilon|^2 \ - \vec{\nabla}.(|\vec{u}_\epsilon|^2(\vec{u_\epsilon} * \omega_\epsilon) + 2p_\epsilon \ \vec{u}_\epsilon)$$

We then use the following convergences in $\mathcal{D}'$: $|\vec{u}_{\epsilon_n}|^2 \to |\vec{u}|^2$, $|\vec{u}_{\epsilon_n}|^2(\vec{u_{\epsilon_n}} * \omega_{\epsilon_n}) \to |\vec{u}|^2\vec{u}$ and $\vec{\nabla}.(p_{\epsilon_n}\ \vec{u}_{\epsilon_n}) \to \vec{\nabla}.(p\ \vec{u})$. Hence, $2|\vec{\nabla}\otimes\vec{u}_{\epsilon_n}|^2$ converges in $\mathcal{D}'$ and we have that $2|\vec{\nabla}\otimes\vec{u}_{\epsilon_n}|^2 \to 2|\vec{\nabla}\otimes\vec{u}|^2 + \mu$ where μ is a nonnegative locally finite measure.

Now, we must prove that we have strong convergence of $\vec{u}(t)$ to $\vec{u}_0$ in $(L^2_{loc}(\mathbb{R}^3))^3$ when t goes to 0. We already know that we have convergence in $(\mathcal{S}'(\mathbb{R}^3)^3$ and thus, since the norm of $\vec{u}(t,.)$ in L^2_{uloc} is controlled and since L^2_{uloc} is a shift-invariant Banach space of distributions, we get that we have weak L^2 convergence on any compact subset of $\mathbb{R}^3$. To prove that we have strong local L^2 convergence, we just have to show that for any nonnegative $\varphi \in \mathcal{D}(\mathbb{R}^3)$ we have $\limsup_{t\to 0}\int |\vec{u}(t,x)|^2\varphi(x)\ dx \leq \int |\vec{u}_0(x)|^2\varphi(x)\ dx$. We know that we have strong convergence in $(L^2_{loc}((0,T_0)\times \mathbb{R}^3))^3$ of $\vec{u}_{\epsilon_n}$ toward $\vec{u}$; hence, (at least for a subsequence of the sequence ϵ_n) we have, for almost every $t\in(0,T_0)$, $\lim_{\epsilon_n\to 0}\int |\vec{u}_{\epsilon_n}(t,x)-\vec{u}(t,x)|^2\varphi(x)\,dx = 0$. Moreover, since $\|\vec{u}_\epsilon - e^{t\Delta}\vec{u}_0\|_{L^2_{uloc}} \le C_\epsilon\sqrt{t}$ and $\vec{u}_\epsilon$ is smooth on $(0,T_0)\times\mathbb{R}^3$, we find that $t\mapsto \int |\vec{u}_\epsilon(t,x)|^2\varphi(x)\,dx$ is continuous on $[0,T_0)$ and smooth on $(0,T_0)$; thus, $\int |\vec{u}_\epsilon(t,x)|^2\varphi(x)\,dx = \int |\vec{u}(0,x)|^2\varphi(x)\,dx + \int\int_0^t \partial_t|\vec{u}_\epsilon(s,x)|^2\varphi(x)\,dx\,ds$. We write $I_\epsilon(\varphi,t)=\int\int_0^t \partial_t|\vec{u}_\epsilon(s,x)|^2\varphi(x)\,dx\,ds$, and we have

$$\begin{aligned} I_\epsilon(\varphi,t)\le \int\int_0^t |\vec{u}_\epsilon|^2\,\Delta\varphi(x)\,dx\,ds \quad &+\int\int_0^t |\vec{u}_\epsilon|^2(\vec{u_\epsilon}*\omega_\epsilon).\vec{\nabla}\varphi(x)\,dx\,ds\\ &+2\int\int_0^t (p_\epsilon - p_{\epsilon,\varphi}(s))\vec{u}_\epsilon.\vec{\nabla}\varphi(x)\,dx\,ds\end{aligned}$$

Now, we write $\alpha(t)=\sup_{\epsilon>0}\alpha_\epsilon(t)$ and $\beta(t)=(t)=\sup_{\epsilon>0}\beta_\epsilon(t)$; we know that those two functions are bounded on every $[0,T]$ with $t<T_0$. The computations in the proof of Proposition 32.1 give us that $I_\epsilon(\varphi,t)\le C(\varphi)(\int_0^t\alpha(s)\,ds + (\int_0^t \alpha(s)^3\,ds)^{1/4}(\beta(t)+\int_0^t\alpha(s)\,ds)^{3/4}) = A(\varphi,t)$. This gives that we have for almost every $t\in(0,T_0)$

$$\int |\vec{u}(t,x)|^2\varphi(x)\,dx \le \int |\vec{u}(0,x)|^2\varphi(x)\,dx + A(\varphi,t).$$

Since $A(\varphi,.)$ is a continuous function on $[0,T_0)$ and since $\int|\vec{u}(.,x)|^2\varphi(x)\,dx$ is a lower semicontinuous function on $[0,T_0)$ (by *-weak continuity of $t\mapsto\vec{u}(t,.)\in(L^2_{uloc})^3$), we find that this equality holds for every t. Since $\lim_{t\to 0}A(\varphi,t)=0$, this gives the required estimate. □

3. Decay estimates for suitable solutions

In this section, we consider a local Leray solution associated with $\vec{u}_0\in(E_2)^3$ and we prove that we may find an estimate for the decay at infinity of the solution that depends only on the decay rate of the initial value and on its L^2_{uloc} norm:

Proposition 32.2: *Let $\vec{u}_0\in(E_2(\mathbb{R}^3))^3$ be such that $\vec{\nabla}.\vec{u}_0=0$ and let $\vec{u}$ be be a local Leray solution on $(0,T^*)\times\mathbb{R}^3$ for the Navier–Stokes initial value problem associated with $\vec{u}_0$. Let $\gamma\in\mathcal{D}(\mathbb{R}^3)$ be equal to 1 in a neighborhood of 0 and define $\chi_R(x)=1-\gamma(x/R)$. Then, for all $T<T^*$, there exists a positive constant C_T so that for all $0<t<T$ and all $R>1$ we have*

$$\|\vec{u}(t,.)\chi_R\|_{E_2}\le C_T\,(\|\vec{u}_0\chi_R\|_{E_2}+\sqrt{\frac{1+\ln R}{R}}) \tag{32.4}$$

The constant C_T depends only on T, $\sup_{\varphi\in\mathcal{B}}\int_0^T\|(\vec{\nabla}\otimes\vec{u}(s,.))\varphi(x)\|_2^2\,ds$ and $\sup_{t<T,\varphi\in\mathcal{B}}\|\vec{u}(t,.)\varphi(x)\|_2^2$.

Proof: We mimic the proof of Proposition 32.1. Thus, we define $\alpha(t) = \sup_{\varphi\in\mathcal{B}} \|\vec{u}(t,.)\varphi(x)\|_2^2$ and $\beta(t) = \sup_{\varphi\in\mathcal{B}} \int_0^t \|(\vec{\nabla}\otimes\vec{u}(s,.))\varphi(x)\|_2^2\, ds$. We define moreover the truncated estimates $\alpha_R(t) = \sup_{\varphi\in\mathcal{B}} \|\vec{u}(t,.)\varphi(x)\chi_R(x)\|_2^2$ and $\beta_R(t) = \sup_{\varphi\in\mathcal{B}} \int_0^t \|(\vec{\nabla}\otimes\vec{u}(s,.))\varphi(x)\chi_R(x)\|_2^2\, ds$. Lemmas 32.2 to 32.6 then are changed in those lemmas that follow:

Lemma 32.7: *We have the following estimate valid for all $R>1$ and all $\varphi\in\mathcal{B}$ ($\varphi(x)=\varphi(x-x_0)$ for some $x_0\in\mathbb{R}^3$): $(p-p_\varphi)\chi_R^2$ can be decomposed into $(p(t,x)-p_\varphi(t))\ \chi_R^2(x) = p_{\varphi,R}(t,x)+q_{\varphi,R}(t,x)$ so that*

$$\Big(\iint_{I\times\mathbb{R}^3} |p_{\varphi,R}(t,x)|^{3/2}\varphi(x)\,dx\,dt\Big)^{2/3}+\Big(\iint_{I\times\mathbb{R}^3} |\vec{\nabla}q_{\varphi,R}(t,x)|^{3/2}\varphi(x)\,dx\,dt\Big)^{2/3} \le$$

$$\le C\quad \Big(\|\chi_R\vec{u}\|^2_{L^3(I,E_2)} + \|\psi\chi_R\vec{u}\|_{L^6(I,L^2)}\ \|\psi\chi_R\vec{u}\|_{L^2(I,H^1)} + \frac{\ln R}{R}\|\vec{u}\|^2_{L^3(I,E_2)} + \frac{1}{R}\|\psi\vec{u}\|_{L^6(I,L^2)}\ \|\psi\vec{u}\|_{L^2([T_0,T_1],H^1)}\Big)$$

where we write $I=[T_0,T_1]$ and where C is a constant which depends neither on x_0, nor on R, nor on T_0,T_1.

Proof: As with the proof of Lemma 32.2, we split $p-p_\varphi$ into p_1+p_2 with

$$\begin{cases} p_1(t,x) = \mathcal{G}\big(\psi_0^2(y-x_0)(\vec{u}(t,y)\otimes\vec{u}(t,y))\big)(x) \\ p_2(t,x) = \int (G(x,y)-G(x_0,y)).(1-\psi_0^2(y-x_0))\ \vec{u}(t,y)\otimes\vec{u}(t,y)\ dy \end{cases}$$

Now, we just write $p_1(x,t)\chi_R^2(x) = q_1+q_2$ where

$$\begin{cases} q_1 = \mathcal{G}\big(\psi_0^2(y-x_0)\chi_R^2(y)\vec{u}(t,y)\otimes\vec{u}(t,y)\big)(x) \\ q_2 = -[\mathcal{G},\chi_R^2]\big(\psi_0^2(y-x_0)\vec{u}(t,y)\otimes\vec{u}(t,y)\big)(x) \end{cases}$$

and similarly $p_2 = q_3+q_4$ where

$$\begin{cases} q_3(t,x) = \int (G(x,y)-G(x_0,y))(1-\psi_0^2(y-x_0))\chi_R^2(y)\vec{u}(t,y)\otimes\vec{u}(t,y)\ dy \\ q_4(t,x) = \int (G(x,y)-G(x_0,y))(1-\psi_0^2(y-x_0))(\chi_R^2(x)-\chi_R^2(y))\ \vec{u}(t,y)\otimes\vec{u}(t,y)\ dy \end{cases}$$

On the support of $\varphi_0(x-x_0)$, $q_3(t,x)$ is bounded by $C\|\vec{u}(t,.)\chi_R\|^2_{E_2}$ and q_4 by $C\frac{\ln R}{R}\|\vec{u}(t,.)\|^2_{E_2}$. On the other hand, we control q_1 by $\|q_1(x,t)\|_{L^{3/2}([T_0,T_1]\times\mathbb{R}^3} \le C\|\psi\ \chi_R\ \vec{u}\|_{L^6([T_0,T_1],L^2)}\ \|\psi\ \chi_R\ \vec{u}\|_{L^2([T_0,T_1],H^1)}$. The term q_2 is the most difficult one. We compute $\vec{\nabla}q_2$ as $\vec{\nabla}q_2 = q_5+q_6$ where

$$\begin{cases} q_5(x,t) = (\vec{\nabla}\chi_R^2)(x)\mathcal{G}\big(\psi_0^2(y-x_0)\vec{u}(t,y)\otimes\vec{u}(t,y)\big)(x) \\ q_6(x,t) = -[\vec{\nabla}\mathcal{G},\chi_R^2]\big(\psi_0^2(y-x_0)\vec{u}(t,y)\otimes\vec{u}(t,y)\big)(x) \end{cases}$$

q_5 is controlled by $\|q_5\|_{L^{3/2}([T_0,T_1]\times\mathbb{R}^3} \le C\frac{1}{R}\|\psi\vec{u}\|_{L^6([T_0,T_1],L^2)}\ \|\psi\vec{u}\|_{L^2([T_0,T_1],H^1)}$. For q_6, we notice that the commutator $[\vec{\nabla}\mathcal{G},\chi_R^2]$ is a Calderón commutator between an operator of order 1 and a Lipschitz function; hence, it is a Calderón-Zygmund operator (see Chapter 9) whose operator norm is controlled by the L^∞

norm of $\vec{\nabla}\chi_R^2$; we thus obtain a control on q_6, namely, that $\|q_6(x,t)\|_{L^{3/2}([T_0,T_1]\times\mathbb{R}^3)} \le C\frac{1}{R}\|\psi\vec{u}\|_{L^6([T_0,T_1],L^2)}\ \|\psi\vec{u}\|_{L^2([T_0,T_1],H^1)}$. This proves the lemma. □

Lemma 32.8: *For all $t > 0$ and all $\varphi \in \mathcal{B}$, we have the following inequality:*

$$\|\vec{u}(t,.)\chi_R(x)\varphi(x)\|_2^2 + 2\int_0^t \|(\vec{\nabla}\otimes\vec{u}(s,.))\chi_R(x)\varphi(x)\|_2^2\ ds \le$$

$$\|\vec{u}_0\chi_R\varphi\|_2^2 + \iint_0^t |\vec{u}|^2\ \Delta(\chi_R^2\varphi^2)\ dx\ ds + \iint_0^t (|\vec{u}|^2 + 2(p-p_\phi))(\vec{u}.\vec{\nabla})(\chi_R^2\varphi^2)\ dx\ ds$$

Proof: We start from the local energy inequality

$$2\iint |\vec{\nabla}\otimes\vec{u}|^2\ \phi(t,x)\ dx\ dt \le \iint |\vec{u}|^2\ (\partial_t\phi + \Delta\phi)\ dx\ dt + \iint (|\vec{u}|^2 + 2p)(\vec{u}.\vec{\nabla})\phi\ \ dx\ dt$$

applied to $\phi(t,x) = \chi_R^2(x)\varphi^2(x)g_{\tau,\epsilon}(t)$ with $g_{\tau,\epsilon}(t) = G(t/\epsilon) - G((t-\tau)/\epsilon)$, where G is a fixed smooth function equal to 0 on $(-\infty,1]$ and to 1 on $[2,+\infty)$. We may replace p by $p - p_\varphi$ since we have $\int \vec{u}(t,x).\vec{\nabla}\varphi(x)\ dx = \int \vec{\nabla}.(\varphi(x)\ \vec{u}(t,x))\ dx = 0$. We fix $\tau \in (0,T^*)$ and let ϵ go to 0. We obtain that

$$\begin{gathered}\limsup_{\epsilon\to 0}\int_0^{T^*} \|\vec{u}(s,.)\chi_R(x)\varphi(x)\|_2^2\tfrac{1}{\epsilon}\big(G'(\tfrac{s-\tau}{\epsilon}) - G'(\tfrac{s}{\epsilon})\big)ds \\ +2\int_0^\tau \|(\vec{\nabla}\otimes\vec{u}(s,.))\ \chi_R(x)\varphi(x)\|_2^2 ds \\ \le \int\int_0^\tau |\vec{u}|^2\ \Delta(\chi_R^2\varphi^2)\ dx\ ds + \int\int_0^\tau (|\vec{u}|^2 + 2(p-p_\varphi))(\vec{u}.\vec{\nabla})(\chi_R^2\varphi^2)\ \ dx\ ds\end{gathered}$$

If τ is a Lebesgue point of $t \to \|\vec{u}(t,x)\chi_R(x)\varphi(x)\|_2^2$, we obtain that

$$\begin{gathered}\lim_{\epsilon\to 0}\int_0^{T^*} \|\vec{u}(s,.)\chi_R(x)\varphi(x)\|_2^2\tfrac{1}{\epsilon}\big(G'((s-\tau)/\epsilon) - G'(s/\epsilon)\big)\ ds \\ = \|\vec{u}(\tau,x)\chi_R(x)\varphi(x)\|_2^2 - \|\vec{u}_0\chi_R(x)\varphi(x)\|_2^2\end{gathered}$$

Thus, when φ and R are fixed, the inequality stated in Lemma 32.8 is true for almost every t in $(0,T^*)$. But $t \to \vec{u}(t,.)$ is weakly continuous and thus if τ_n is a sequence of times converging to t, we have $\|\vec{u}(t,x)\chi_R(x)\varphi(x)\|_2^2 \le \liminf \|\vec{u}(\tau_n,x)\chi_R(x)\varphi(x)\|_2^2$. Finally, we may conclude that the inequality is valid for all t. □

Lemma 32.9: *For all $t > 0$, all $\varphi \in \mathcal{B}$ and all $R > 1$, we have the following inequality:*

$$\int\int_0^t |\vec{u}|^2\ \Delta(\chi_R^2(x)\varphi^2(x))\ dx\ ds \le C_1\ (\int_0^t \alpha_R(s)\ ds + \frac{1}{R}\int_0^t \alpha(s)\ ds)$$

Proof: This is obvious. We just have to consider whether we differentiate χ_R (getting the $O(1/R)$ term) or not (getting the term involving α_R). $\square$

Lemma 32.10: *For all $t > 0$ and all $\varphi \in \mathcal{B}$, we have the following inequality:*

$$\iint_0^t |\vec{u}|^2(\vec{u}.\vec{\nabla})(\chi_R^2\varphi^2) \;\; dx \; ds \leq C_2 M(t)\{M_R(t)^2 + \frac{1}{R}M^2(t)\}$$

where M is the function $M(t) = (\int_0^t \alpha(s)^3 \; ds)^{1/12} \; \left(\beta(t) + \int_0^t \alpha(s) \; ds\right)^{1/4}$ and M_R is the function $M_R(t) = (\int_0^t \alpha_R(s)^3 \; ds)^{1/12} \; \left(\beta_R(t) + \int_0^t \alpha_R(s) \; ds\right)^{1/4}$.

Proof: We again consider whether or not we differentiate χ_R (getting the $O(1/R)$ term). When χ_R is not differentiated, we have to estimate the integral $\int\int_0^t |\vec{u}\chi_R(x)|^2(\vec{u}.\vec{\nabla})\varphi^2(x) \;\; dx \; ds$, and we proceed as in Lemma 32.5 to control the L^3 norm of $\psi(x) \; \vec{u}$ and of $\psi(x)\chi_R(x)\vec{u}$. $\square$

Lemma 32.11: *For all $t > 0$, all $\varphi \in \mathcal{B}$ and all $R > 1$, we have the following inequality:*

$$\iint_0^t (p - p_\varphi)(\vec{u}.\vec{\nabla})(\chi_R^2(x)\varphi^2(x))dx \; ds \leq C_3 M(t)\{M_R(t)^2 + \frac{1+\ln R}{R}M(t)^2\}$$

Proof: We consider one more time whether or not we differentiate χ_R (getting an $O(1/R)$ term). When χ_R is not differentiated, we have to estimate $\int\int_0^t (p - p_\varphi)\chi_R^2(x)(\vec{u}.\vec{\nabla})\varphi^2(x) \;\; dx \; ds$. We then decompose $(p - p_\varphi)\chi_R^2(x)$ into $p_{x_0,R}(t,x) + q_{x_0,R}(t,x)$ according to Lemma 32.7 and we write

$$\int\int_0^t (p - p_\varphi)\chi_R^2(x)(\vec{u}.\vec{\nabla})\varphi^2 \;\; dx \; ds =$$

$$= \int\int_0^t p_{x_0,R}(s,x)(\vec{u}.\vec{\nabla})\varphi^2(x) \;\; dx \; ds - \int\int_0^t \varphi^2(x) \; (\vec{u}.\vec{\nabla})q_{x_0,R}(s,x) \; dx \; ds$$

Putting together estimates of Lemma 32.7 and Lemma 32.6 gives the required estimate. $\square$

End of the proof: Now, clearly, these lemmas give us the following inequalities (just by looking at the extrema of the functions $\|\vec{u}(t,.)\chi_R(x)\varphi(x)\|_2^2$ and of $2\int_0^t \|(\vec{\nabla}\otimes\vec{u}(s,.))\chi_R(x)\varphi(x)\|_2^2 \; ds$ and by controlling the size of $M(t)$ on $(0,T)$): $\alpha_R(t) \leq \Gamma_R(t)$ and $2\beta_R(t) \leq \Gamma_R(t)$ where $\Gamma_R(t) = \alpha_R(0) + C_1\int_0^t \alpha_R(s) \; ds + A_T(\int_0^t \alpha_R(s)^3 \; ds)^{1/6}(\beta_R(t) + \int_0^t \alpha_R(s) \; ds)^{1/2} + B_T\frac{1+\ln R}{R}$. We use the Young inequality to control $(\int_0^t \alpha_R(s)^3 \;\; ds)^{1/6}(\beta_R(t) + \int_0^t \alpha_R(s) \;\; ds)^{1/2}$ by

$\frac{1}{2\epsilon_T^2}(\int_0^t \alpha_R(s)^3\ ds)^{1/3} + \frac{\epsilon_T^2}{2}(\beta_R(t) + \int_0^t \alpha_R(s)\ ds)$ with $A_T\epsilon_T^2/2 < 1$ to remove $\beta_R(t)$ and we obtain

$$\alpha_R(t) \leq D_T\ \left(\alpha_R(0) + \int_0^t \alpha_R(s)\ ds + (\int_0^t \alpha_R(s)^3\ ds)^{1/3} + \frac{1+\ln R}{R}\right)$$

Using the Hölder inequality, $\int_0^t \alpha_R(s)\ ds \leq T^{2/3}(\int_0^t \alpha_R(s)^3)^{1/3}$, we finally obtain:

$$\alpha_R^3(t) \leq E_T\ \left(\alpha_R^3(0) + \int_0^t \alpha_R(s)^3\ ds + (\frac{1+\ln R}{R})^3\right)$$

Thus, the function $\gamma(t) = \alpha_R^3(0) + \int_0^t \alpha_R(s)^3\ ds$ is a solution of the differential inequality $\gamma' \leq E_T\ (\gamma + (\frac{1+\ln R}{R})^3)$. This gives Proposition 32.2. $\square$

Final proof: We may now end the proof of Theorem 32.1 by proving that, when $\vec{u}_0 \in (E_2)^3$, $\vec{u} \in \cap_{0<T<T^*} L^\infty((0,T),(E_2)^3$ and $\lim_{t\to 0} \|\vec{u}(t,.)-\vec{u}_0\|_{L^2_{uloc}} = 0$.

The first point is easy. Indeed, $f \in E_2$ if and only if $f \in L^2_{uloc}$ and $\lim_{R\to\infty} \sup_{|x_0|>R} \int_{|x-x_0|\leq 1} |f(x)|^2\ dx = 0$. Thus, Proposition 32.2 gives that $\vec{u}(t,.) \in (E_2)^3$.

We have already proved, under the condition that $\vec{u}_0 \in (L^2_{uloc})^3$, that we have for all $R > 0$ $\lim_{t\to 0} \int_{|x|<R} |\vec{u}(t,x) - \vec{u}_0(x)|^2\ dx = 0$. On the other hand, Proposition 32.2 gives that, when $\vec{u}_0 \in (E_2)^3$, we have for any $T < T^*$

$$\lim_{R\to\infty} \sup_{0<t<T, x_0\in\mathbb{R}^3} \int_{|x-x_0|\leq 1, |x|>R} |\vec{u}(t,x)|^2\ dx = 0. \qquad \square$$

Chapter 33

Global existence of suitable local square-integrable weak solutions

In this chapter, we complete the result of Chapter 32 (where we got locally in time the existence of local Leray solutions for uniformly locally square-integrable initial data) by proving the existence of **global** weak solutions for locally square-integrable initial values vanishing at infinity:

Theorem 33.1: (locally square-integrable initial value)
For all $\vec{u}_0 \in (E_2(\mathbb{R}^3))^3$ so that $\vec{\nabla}.\vec{u}_0 = 0$, there exists a local Leray solution $\vec{u}$ on $(0,\infty) \times \mathbb{R}^3$ for the Navier–Stokes initial value problem associated with $\vec{u}_0$.

Existence of global weak solutions, generalizing the result of Leray for $\vec{u}_0 \in (L^2)^3$ was first established for $\vec{u}_0 \in (L^p)^3$ $(2 \leq p < \infty)$ by C. Calderón [CAL 90] and later by Lemarié-Rieusset [LEM 98a]. Theorem 1 was proved in [LEM 98b].

1. Regularity of uniformly locally L^2 suitable solutions

In the preceding chapter, we proved a local existence result of weak solutions for a locally square-integrable initial value:

Local existence: *For all $\vec{u}_0 \in (E_2(\mathbb{R}^3))^3$ so that $\vec{\nabla}.\vec{u}_0 = 0$, there exists a positive T and a local Leray solution $\vec{u}$ on $(0,T) \times \mathbb{R}^3$ for the Navier–Stokes initial value problem associated with $\vec{u}_0$. Moreover, $\vec{u} \in \cap_{t<T} L^\infty((0,t),(E_2)^3)$ and $\lim_{t\to 0^+} \|\vec{u}-\vec{u}_0\|_{L^2_{uloc}} = 0$.*

To achieve a solution beyond T, we need a better understanding of the local Leray solutions. We show that for almost every time t the solution $\vec{u}$ belongs to $(E_3)^3$, where E_3 is the closure of the test functions in L^3_{uloc}:

Proposition 33.1: *If $\vec{u} \in \cap_{t<T} L^\infty((0,t),(E_2)^3)$ is a local Leray solution on $(0,T) \times \mathbb{R}^3$ for the Navier–Stokes initial value problem associated with $\vec{u}_0$ (for some $\vec{u}_0 \in (E_2)^3$), then we have the following regularity results:*

(A) for almost all $t \in (0,T)$, $\lim_{s>t, s\to t} \|\vec{u}(s)-\vec{u}(t)\|_{E_2} = 0$.

(B) for all $[\delta, t] \subset (0, T)$, $\vec{u} \in L^4((\delta, t), (E_3)^3)$.

Proof: Point (A) is easy. We use the energy equality on suitable solutions to get that for all $\varphi \in \mathcal{B}$ and for all Lebesgue points t, τ of $\|\vec{u}\varphi\|_2$ with $t < \tau$ we have the inequality

$$\|\vec{u}(\tau,.)\varphi(x)\|_2^2 + 2\int_t^\tau \|(\vec{\nabla} \otimes \vec{u}(s,.))\varphi(x)\|_2^2 \, ds \leq$$

$$\|\vec{u}(t,.)\varphi(x)\|_2^2 + \int\int_t^\tau |\vec{u}|^2 \, \Delta(\varphi^2(x)) \, dx \, ds + \int\int_t^\tau (|\vec{u}|^2 + 2(p - p_\varphi))(\vec{u}.\vec{\nabla})\varphi \, ds \, dx$$

Then, we use the weak continuity of $t \to \vec{u}(t,.)$ to conclude that this equality is valid for almost all t and for all $\tau > t$. Thus, for all $\varphi \in \mathcal{B}$ and for all Lebesgue points t of $\|\vec{u}\varphi\|_2$ we have $\lim_{\tau > t, \tau \to t} \|(\vec{u}(\tau) - \vec{u}(t))\varphi\|_2 = 0$. Moreover, Proposition 32.2 implies a good uniform control of the decay of $\|\vec{u}(\tau)\varphi_0(x-k)\|_2$ when k goes to ∞, whereas we have a good control of $\|(\vec{u}(\tau) - \vec{u}(t))\varphi_0(x-k)\|_2$ on the points t which are Lebesgue points for all $\|\vec{u}\varphi_0(x-k)\|_2, k \in \mathbb{Z}^3$; hence, for almost all t. Since $\sum_k \varphi_0(x-k) = 1$, this gives (A). In order to prove point (B), we use again Proposition 32.2 and we see from the estimates on $\vec{u}$ and on p that for all $t < T$ we have

$$\lim_{R \to \infty} \sup_{|x_0| > R} \int\int_{|x-x_0| < \min(\sqrt{t}, 1), \sup(0, t-1) < s < t} |\vec{u}|^3 + |p(s,x) - p_{x_0}(s)|^{3/2} \, dx \, ds = 0$$

and thus (due to the Caffarelli, Kohn, and Nirenberg regularity criterion)

$$\sup_{|x_0| > R(t), |x-x_0| < \min(1,\sqrt{t})/2, \sup(3t/4, t-1/4) < s < t} |\vec{u}| < C_1 \frac{1}{\min(1, \sqrt{t})}.$$

Since we know that for every $R > 0$ and for every $[\delta, t] \subset (0, T)$ we have $\vec{u} \in L^4((\delta, t), (L^3(B(0,R))^3)$, we get the desired estimate, thus proving Proposition 33.1. □

2. A generalized Von Wahl uniqueness theorem

In this section, we generalize the Von Wahl theorem, replacing the Leray L^2 solutions by local Leray solutions (as defined in the preceding chapter) and the L^3 condition by a uniformly locally L^3 condition:

Theorem 33.2: (locally L^3 initial value)
If $\vec{u} \in \mathcal{C}([0, T), (E_3)^3)$ *is a solution for the Navier–Stokes equations, then the velocity* $\vec{u}$ *and the pressure* p *are smooth on* $(0, T) \times \mathbb{R}^3$ *and for all* $t < T$ *we have* $\sup_{x_0 \in \mathbb{R}^3} \int\int_{0<s<t, |x-x_0| \leq 1} |\vec{\nabla} \otimes \vec{u}|^2 \, dx \, ds < \infty$. *If* $\vec{v}$ *is a local Leray solution* $\vec{v}$ *on* $(0, T) \times \mathbb{R}^3$ *for the Navier–Stokes initial value problem associated with* $\vec{u}(0)$, *then* $\vec{v} = \vec{u}$.

Proof:

Step 1: $\sqrt{t}\vec{u}$ is bounded

There is a direct proof by May [MAY 02] that a solution in $\mathcal{C}([0,T],(E_3)^3)$ is bounded: $\sup_{0<t<T}\sqrt{t}\|\vec{u}\|_\infty < \infty$. Since L^3_{uloc} is a shift-invariant Banach space of local measures embedded in $B^{-1,\infty}_\infty$, we know (Theorem 17.2) that we may find for every $\theta \in [0,T]$ a solution $\vec{u}_\theta$ of the Navier–Stokes equations on $[0,T(\theta)]$ with initial value $\vec{u}(\theta)$ so that $\sup_{0<t<T(\theta)}\sqrt{t}\|\vec{u}_\theta(t)\|_\infty < \infty$ and that $T(\theta)$ depends continuously on $\vec{u}(\theta)$; $T(\theta)$ is determined by the conditions $T(\theta) < 1$ and

$$C\sup_{0<s<T(\theta)}\sqrt{s}\|e^{s\Delta}\vec{u}(\theta)\|_\infty(1+\ln^+(\frac{\|\vec{u}(\theta)\|_{L^3_{uloc}}}{\sup_{0<s<T(\theta)}\sqrt{s}\|e^{s\Delta}\vec{u}(\theta)\|_\infty})) < 1.$$

By compactness, we find that there exists some positive δ so that $T(\theta) > \delta$ for all $\theta \in [0,T]$. Moreover, uniqueness of the solutions of the Navier–Stokes equations in E_3 (Chapter 27) gives that $\vec{u}_\theta(t) = \vec{u}(t+\theta)$. It is then enough to write $[0,T] = \cup_{k=0}^{N-1}[kT/N,(k+1)T/N]$ with $T/N \leq \delta/2$ to get that $\sqrt{t}\|\vec{u}(t)\|_\infty \leq C$ on $[0,2T/N]$ and $\|\vec{u}(t)\|_\infty \leq 2C\delta^{-1}$ on $[2T/N,T]$ with $C = \sup_{0\leq k\leq N-1}\sup_{0<t<\delta}\sqrt{t}\|\vec{u}_{kT/N}(t)$.

We give here another (indirect) proof, in order to get a more precise statement. We first check that a solution in $\mathcal{C}([0,T],(E_3)^3)$ is suitable and then use the Caffarelli, Kohn, and Nirenberg regularity criterion.

In this subsection, we check that a *suitable* solution, which belongs to $\mathcal{C}([0,T],(E_3)^3)$, is indeed bounded in $1/\sqrt{t}$. The set $\{\vec{u}(t)\ /\ 0 \leq t \leq T\}$ is compact in $(E_3)^3$ and thus $\lim_{\delta\to 0}\sup_{x_0\in\mathbb{R}^3, t\leq T}\int_{|x-x_0|<\delta}|\vec{u}|^3\,dx = 0$. Moreover, we may easily control the pressure; we choose $\omega \in \mathcal{D}(\mathbb{R}^3)$ equal to 1 in the neighborhood of 0 and we compute $p(t,x)$ in the neighbourhood of x_0 up to a constant p_{t,x_0} by $p(t,x) - p_{t,x_0} = \mathcal{R}(\omega(y-x_0)(\vec{u}\otimes\vec{u})(t,y)) + \int(\mathcal{R}(x,y)-\mathcal{R}(x_0,y))(1-\omega(y-x_0))(\vec{u}\otimes\vec{u})(t,y)\,dy$. We find that $q_{t,x_0} = \mathcal{R}(\omega(y-x_0)(\vec{u}\otimes\vec{u})(t,y))$ belongs to a compact set of $L^{3/2}$ when x_0 varies in $\mathbb{R}^3$ and t in $(0,T]$ while $r_{t,x_0} = \int(\mathcal{R}(x,y)-\mathcal{R}(x_0,y))(1-\omega(y-x_0))(\vec{u}\otimes\vec{u})(t,y)\,dy$ remains bounded on $|x-x_0| \leq \gamma_0$ (where γ_0 is a positive constant which does not depend on t nor on x_0) uniformly in t and x_0. This gives that $\lim_{\delta\to 0}\sup_{x_0\in\mathbb{R}^3, t\leq T}\int_{|x-x_0|<\delta}|p-p_{x_0,t}|^{3/2}\,dx = 0$

We then fix δ_0 so that $\omega(\delta,t) = \sup_{x_0\in\mathbb{R}^3}\int_{|x-x_0|<\delta}|\vec{u}|^3 + |p-p_{x_0,t}|^{3/2}\,dx$ is smaller than ϵ_{CKN} for $\delta \leq \delta_0$ and $t \leq T$. For $t_0 \leq T$, we choose $\delta(t_0) = \min(\sqrt{t_0},\delta_0)$ and the inequality $\int_{t_0-\delta(t_0)^2}^{t_0}\omega(\delta(t_0),t)\,dt \leq \epsilon_1\ \delta(t_0)^2$ gives us (according to the Caffarelli, Kohn, and Nirenberg regularity criterion) that: $\|\vec{u}(t_0)\|_\infty \leq C_1\frac{1}{\delta(t_0)}$. This proves that $\|\vec{u}(t)\|_\infty$ is $O(1/\sqrt{t})$. This proves even that it is $o(1/\sqrt{t})$ (the Caffarelli, Kohn, and Nirenberg criterion [CAFKN 82] gives that the local L^∞ norm of $\vec{u}$ goes to 0 if the local L^3 norm of $\vec{u}$ and the local $L^{3/2}$ of p go to 0), but this is also a consequence of the uniqueness of weak solutions in $\mathcal{C}([0,T),(E_3)^3)$ and the fact that we may find for a small time T_0 a Kato solution in in $\mathcal{C}([0,T_0),(E_3)^3)$ whose L^∞ norm is $o(1/\sqrt{t})$. □

Step 2: Suitability of the E_3 solutions

Now, we prove that a solution $\vec{u} \in \mathcal{C}([0,T),(E_3)^3)$ is suitable. We shall first prove that the problem

$$(33.1)\qquad \begin{cases} \partial_t \vec{z} = \Delta\vec{z} - \vec{\nabla}.(\vec{u}\otimes\vec{z}) - \vec{\nabla}q \\ \vec{\nabla}.\vec{z} = 0 \\ \vec{z}(0,.) = \vec{u}_0 \end{cases}$$

(which has $(\vec{u},p)$ as solution) has a solution $(\vec{z},q)$ so that, for all $t<T$,

$$(33.2)\qquad \sup_{x_0\in\mathbb{R}^3, s<t} \int_{|x-x_0|\le 1} |\vec{z}(s,x)|^2 dx + \sup_{x_0\in\mathbb{R}^3} \iint_{0<s<t,|x-x_0|\le 1} |\vec{\nabla}\otimes\vec{z}|^2 dx\, ds < \infty$$

Indeed, let us approximate $\vec{u}_0$ by $\vec{u}_0 = \vec{v}_\epsilon + \vec{w}_\epsilon$ with divergence-free vector fields $\vec{v}_\epsilon$ and $\vec{w}_\epsilon$ so that $\vec{v}_\epsilon \in (L^2)^3$ and $\|\vec{w}_\epsilon\|_{E_3} < \epsilon$. Equation (33.1) with $\vec{v}_\epsilon$ as initial value instead of $\vec{u}_0$ may be solved by the Leray method, which provides a solution $\vec{z}_\epsilon$. We know that this solution is in $\cap_{t<T} L^\infty((0,t),(L^2)^3) \cap L^2((0,t),(H^1)^3)$. We define, as we have done many times now, the functions

$$\alpha_\epsilon(t) = \sup_{x_0\in\mathbb{R}^3} \int_{|x-x_0|\le 1} |\vec{z}_\epsilon(t,x)|^2\ dx$$

and

$$\beta_\epsilon(t) = \sup_{x_0\in\mathbb{R}^3} \int\int_{0<s<t,|x-x_0|\le 1} |\vec{\nabla}\otimes\vec{z}_\epsilon|^2\ dx\ ds.$$

We show that for all $t<T$ we have $\sup_\epsilon(\sup_{s<t}\alpha_\epsilon(s) + \beta_\epsilon(t)) < \infty$. Indeed, we start from the identities

$$\begin{cases} \partial_t|\vec{z}_\epsilon|^2 + 2|\vec{\nabla}\otimes\vec{z}_\epsilon|^2 = \Delta|\vec{z}_\epsilon|^2 - \vec{\nabla}.(|\vec{z}_\epsilon|^2\vec{u}) - 2\vec{\nabla}.(q_\epsilon\ \vec{z}_\epsilon) \\ q_\epsilon = \mathcal{R}(\vec{u}\otimes\vec{z}_\epsilon) \end{cases}$$

Now, we notice that

$$\int_{|x-x_0|<2} |q_\epsilon(s,x) - C_{s,x_0,\epsilon}|^2\ dx \le C(\{\int_{|x-x_0|<3} |\vec{z}(s,x)|^6\ dx\}^{1/3} + \alpha_\epsilon(s))\|\vec{u}(s)\|_{E_3}^2$$

where C does not depend on x_0 nor on s nor on ϵ and where the constant $C_{s,x_0,\epsilon}$ is formally given by $C_{s,x_0,\epsilon} = \int_{|y-x_0|>3} \mathcal{R}(x_0,y)\vec{u}\otimes\vec{z}_\epsilon(s,y)\ dy$. Thus, if we integrate the energy equality against $1_{(0,t)}(s)\varphi(x-x_0)$ where $\varphi\in\mathcal{D}(\mathbb{R}^3)$ is equal to 1 on the unit ball $B(0,1)$ and to 0 outside from the ball $B(0,2)$, we get

$$\int |\vec{z}_\epsilon(s,x)|^2\varphi(x-x_0)\ dx + 2\int\int_0^t |\vec{\nabla}\otimes\vec{z}_\epsilon(s,x)|^2\varphi(x-x_0)\ dx \le$$

$$\leq \int |\vec{z}_\epsilon(0,x)|^2 \varphi(x-x_0)\ dx + C_1 \int_0^t \alpha_\epsilon(s)\ ds +$$

$$+C_2 \sup_{0<s<t} \|\vec{u}(s)\|_{E_3} (\int_0^t \alpha_\epsilon(s)\ ds)^{1/2} (\int_0^t \alpha_\epsilon(s)\ ds + \beta_\epsilon(t))^{1/2}$$

$$\leq C_3 \alpha_\epsilon(0) + C_4(1 + \sup_{0<s<t} \|\vec{u}(s)\|_{E_3}^2) \int_0^t \alpha_\epsilon(s)\ ds + \beta_\epsilon(t)$$

For one choice of x_0 (depending on t and ϵ), $\alpha_\epsilon(t) \leq \int |\vec{z}_\epsilon(s,x)|^2 \varphi(x-x_0)\ dx$ while, for another good choice x_0', $\beta_\epsilon(t) \leq \int\int_0^t |\vec{\nabla} \otimes \vec{z}_\epsilon(s,x)|^2 \varphi(x - x_0')\ dx$. Thus, we have

$$\alpha_\epsilon(t) \leq 2\ (C_3\alpha_\epsilon(0) + C_4(1 + \sup_{0<s<t} \|\vec{u}(s)\|_{E_3}^2) \int_0^t \alpha_\epsilon(s)\ ds)$$

and thus $\alpha_\epsilon(t) \leq 2\ C_3 \alpha_\epsilon(0) e^{2\ C_4(1+\sup_{0<s<t} \|\vec{u}(s)\|_{E_3}^2)t}$. This gives the desired inequalities on α_ϵ and β_ϵ.

Now, we again use the Leray weak convergence lemma (Theorem 13.3) to get that for a sequence ϵ_n going to 0 $\vec{z}_{\epsilon_n}$ converges strongly in $(L^2_{loc}((0,T) \times \mathbb{R}^3)^3$ to a solution $\vec{z}$ of problem (33.1) with energy estimate (33.2). For such a solution, we may compute $\partial_t |\vec{z}|^2$ as $2\ \langle \partial_t \vec{z} | \vec{z} \rangle$ and get that

$$\partial_t |\vec{z}|^2 + 2|\vec{\nabla} \otimes \vec{z}|^2 = \Delta |\vec{z}|^2 - \vec{\nabla}.(|\vec{z}|^2 \vec{u}) - 2\vec{\nabla}.(q\ \vec{z}).$$

In order to prove the suitability of $\vec{u}$, we only need to show that $\vec{z} = \vec{u}$. Thus, we will show that Problem (33.1) has a unique solution in $\cap_{t<T} L^\infty((0,t),(E_2)^3)$. (It is not difficult to see that $\vec{z}$, which we proved to belong to the space $\cap_{t<T} L^\infty((0,t),(M^{2,2})^3)$, is indeed in $\cap_{t<T} L^\infty((0,t),(E_2)^3)$; we proved a similar statement in the preceding chapter in a more difficult case). The proof runs along the same lines as the proof of uniqueness of E_3 solutions: if $\vec{z}_1$ and $\vec{z}_2$ are two such solutions, the difference $\vec{w} = \vec{z}_1 - \vec{z}_2$ is a solution of the integral equation:

$$\vec{w} = -\int_0^t e^{(t-s)\Delta} \mathbb{P} \vec{\nabla}.(\vec{u} \otimes \vec{w})\ ds$$

For $T_2 < T$, we split $\vec{u}$ in $\vec{X} + \vec{Y}$ where $\sup_{0<s<T_2} \|\vec{X}\|_3 < \alpha$ with α small enough and $\sup_{0<s<T_2} \|\vec{Y}\|_\infty < \infty$. Since $\vec{w} \in L^\infty((0,T_2),(E_2)^3)$, we have that $\vec{w} \in L^\infty((0,T_2),(ML^{2,\infty})^3)$, where $ML^{2,\infty}$ is the space of functions which are uniformly locally in $L^{2,\infty}$. We shall prove that, if $\vec{w}$ is equal to 0 on $(0,t_0)$ for some $t_0 \in [0,T_2]$, then for all $t \geq t_0$ we have

$$\|\vec{w}(t)\|_{ML^{2,\infty}} \leq C \sup_{0<s<t} \|\vec{w}(s)\|_{ML^{2,\infty}} (\sup_{0<s<t} \|\vec{X}\|_3 + \sqrt{t-t_0} \sup_{0<s<t} \|\vec{Y}\|_\infty)$$

which is enough (if $C\alpha < 1$) to grant that $\vec{w} = 0$. The estimate on $\vec{Y}$ is obvious: since pointwise multiplication by a L^∞ function operates on $ML^{2,\infty}$ and since the norm of $ML^{2,\infty}$ is shift-invariant, we have $\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{Y}\otimes\vec{w})\|_{ML^{2,\infty}} \leq C\frac{1}{\sqrt{t-s}}\|\vec{Y}(s)\|_\infty \, \|\vec{w}(s)\|_{ML^{2,\infty}}$. For the estimate on $\int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{X}\otimes\vec{w})\, ds$, we write that $\vec{X}\otimes\vec{w} \in L^\infty((0,T_2),(ML^{6/5,\infty})^9)$, so that it is enough to check that $\vec{f}\to\int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{f}\, ds$ is bounded from $L^\infty((0,T_2),(ML^{6/5,\infty})^9)$ to $L^\infty((0,T_2),(ML^{2,\infty})^3)$. We are going to prove in an equivalent way that $\vec{f}\to A(\vec{f}) = \int_0^{T_2} e^{s\Delta}\mathbb{P}\vec{\nabla}.\vec{f}\, ds$ is bounded from $L^\infty((0,T_2),(ML^{6/5,\infty})^9)$ to $(ML^{2,\infty})^3$. We write

$$A(\vec{f}) = \ A(\vec{f}1_{[-2,2]^3}(y-x_0)) + \ A(\vec{f}(1-1_{[-2,2]^3}(y-x_0))) = \vec{f_1}+\vec{f_2}$$

Now, we estimate $\vec{f_1}$ by checking that A is a bounded linear operator from $L^\infty((0,T_2),(L^{6/5,\infty})^9)$ to $(L^{2,\infty})^3$ and we estimate $\vec{f_2}$ by checking that $f\to A(\vec{f}(1-1_{[-2,2]^3}(y-x_0)))$ operates boundedly from $L^\infty((0,T_2),(M^{1,1})^9)$ to $(L^\infty(x_0+[0,1]^3)^3)$. More precisely, we have estimates that, for $1<q<6$, $\|e^{s\Delta}\mathbb{P}\vec{\nabla}.\vec{g}\|_{L^{6,1}} \leq C_q\|\vec{g}\|_q \; s^{-(3/q+1/2)/2}$ so that $\int_0^\infty \|e^{s\Delta}\mathbb{P}\vec{\nabla}.\vec{g}\|_{L^{6,1}}\, ds \leq C\|\vec{g}(x)\|_{L^{2,1}}$ (because each $g\in L^{2,1}$ can be decomposed into $g=\sum_{n\in\mathbb{N}} g_n$ with $\sum_{n\in\mathbb{N}}\|g_n\|_1^{1/2}\|g_n\|_\infty^{1/2}\leq C\|g\|_{L^{2,1}}$) and by duality we see that A operates boundedly from $L^\infty((0,T_2),(L^{6/5,\infty})^9)$ to $(L^{2,\infty})^3$. On the other hand we have that $|e^{s\Delta}\mathbb{P}\vec{\nabla}.\vec{g}|\leq C\int_{\mathbb{R}^3}\frac{\sqrt{s}}{|x-y|^4}|\vec{g}(y)|\, dy$ and this gives the control on $\vec{f_2}$. □

Step 3: Local Leray solutions

We consider $(\vec{u},p)$ and $(\vec{v},q)$ two local Leray solutions on $(0,T)\times\mathbb{R}^3$ for the Navier–Stokes initial value problem associated with some $\vec{u}_0\in(E_3)^3$ and we assume that $\vec{u}\in\mathcal{C}([0,T),(E_3)^3)$. First, we notice that $\partial_t\langle\vec{u}|\vec{v}\rangle = \langle\partial_t\vec{u}|\vec{v}\rangle + \langle\vec{u}|\partial_t\vec{v}\rangle$ since for all $\phi\in\mathcal{D}((0,T)\times\mathbb{R}^3$ we have

$$\begin{gathered}\phi\vec{v}\in L^2((0,T),(H^1)^3)\\ \phi\partial_t\vec{v}\in L^2((0,T),(H^{-1})^3)+L^1((0,T),(L^{3/2})^3)\end{gathered}$$

and

$$\begin{gathered}\phi\vec{u}\in L^2((0,T),(H^1)^3)\cap\mathcal{C}([0,T],(L^3)^3)\\ \phi\partial_t\vec{u}\in L^2((0,T),(H^{-1})^3).\end{gathered}$$

This gives for some nonnegative local measure μ:

$$\begin{gathered}\partial_t|\vec{v}|^2+2|\vec{\nabla}\otimes\vec{v}|^2 = \Delta|\vec{v}|^2-\vec{\nabla}.(|\vec{v}|^2\vec{v})-2\vec{\nabla}.(q\;\vec{v})-\mu\\ \partial_t|\vec{u}|^2+2|\vec{\nabla}\otimes\vec{u}|^2 = \Delta|\vec{u}|^2-\vec{\nabla}.(|\vec{u}|^2\vec{u})-2\vec{\nabla}.(p\;\vec{u})\end{gathered}$$

and

$$\begin{gathered}\partial_t\langle\vec{u}|\vec{v}\rangle+2\langle\vec{\nabla}\otimes\vec{u}|\vec{\nabla}\otimes\vec{v}\rangle = \Delta\langle\vec{u}|\vec{v}\rangle-\vec{\nabla}.(\langle\vec{u}|\vec{v}\rangle\vec{u})+\langle((\vec{u}-\vec{v}).\vec{\nabla})(\vec{v}-\vec{u})|\vec{u}\rangle+\\ +\tfrac{1}{2}\vec{\nabla}.(|\vec{u}|^2(\vec{u}-\vec{v}))-\vec{\nabla}.(p\;\vec{v})-\vec{\nabla}.(q\;\vec{u})\end{gathered}$$

Hence, we get

$$\partial_t|\vec{v}-\vec{u}|^2+2|\vec{\nabla}\otimes(\vec{v}-\vec{u})|^2 =$$

$= \Delta|\vec{v}-\vec{u}|^2 - \vec{\nabla}.(|\vec{v}-\vec{u}|^2\vec{u}) - \vec{\nabla}.(\langle\vec{v}-\vec{u}|\vec{v}+\vec{u}\rangle(\vec{v}-\vec{u})) - 2\vec{\nabla}.((q-p)\ (\vec{v}-\vec{u})) - \mu$

with $q-p = \mathcal{R}(\vec{v}\otimes(\vec{v}-\vec{u}) + (\vec{v}-\vec{u})\otimes\vec{u})$. Now, we write $\vec{u}-\vec{v}=\vec{w}$, and we do the same estimations as in the preceding chapter for $\alpha(t,\vec{w})$ and $\beta(t,\vec{w})$, and we get

$$\alpha(t,\vec{w}) \leq \alpha(0,\vec{w}) + C_1\int_0^t \alpha(s,\vec{w})\ ds + C_2(\int_0^t \alpha(s,\vec{w})^3\ ds)^{1/6}(\beta(t,\vec{w}) + \int_0^t \alpha(s,\vec{w})\ ds)^{1/2}$$

$$2\beta(t,\vec{w}) \leq \alpha(0,\vec{w}) + C_1\int_0^t \alpha(s,\vec{w})\ ds + C_2(\int_0^t \alpha(s,\vec{w})^3\ ds)^{1/6}(\beta(t,\vec{w}) + \int_0^t \alpha(s,\vec{w})\ ds)^{1/2}$$

where C_2 depends on $\vec{u}$ and $\vec{v}$ and satisfies

$$C_2 \leq C_3(1+T)^{3/2}\{(\int_0^T \alpha(s,\vec{u})^3 ds)^{1/3} + \beta(T,\vec{u}) + (\int_0^T \alpha(s,\vec{v})^3 ds)^{1/3} + \beta(T,\vec{v})\}$$

Then, we can get rid of $\beta(t,\vec{w})$ and, since $\alpha(0,\vec{w}) = 0$, get

$$\alpha(t,\vec{w}) \leq C_4(T)(\int_0^t \alpha(s,\vec{w})^3\ ds)^{1/3}$$

which gives $\alpha(t,\vec{w}) = 0$ and, thus, $\vec{u} = \vec{v}$. □

Thus, Theorem 33.2 is proved. Besides, Theorem 33.2 provides an useful insight into the structure of global solutions in L^3:

Theorem 33.3: (Global solutions in L^3.)
If $\vec{u} \in \mathcal{C}([0,\infty),(L^3)^3)$ is a solution for the Navier–Stokes equations, then $\sup_{0<s}\sqrt{s}\ \|\vec{u}(s,.)\|_\infty < \infty$ *and* $\lim_{t\to\infty}\sqrt{t}\ \|\vec{u}(t,.)\|_\infty = 0$.

Proof: We now consider the case of a global L^3 solution $\vec{u} \in \mathcal{C}([0,\infty),(L^3)^3)$. We split $\vec{u}_0$ in $\vec{v}_0 + \vec{w}_0$ where $\vec{\nabla}.\vec{v}_0 = \vec{\nabla}.\vec{w}_0 = 0$, $\vec{v}_0 \in (L^2)^3$ and $\|\vec{w}_0\|_3 < \epsilon$ where $\epsilon > 0$ is small enough. Then, we write $\vec{W}$ the global Kato solution in $\mathcal{C}([0,\infty),(L^3)^3)$ of the Navier–Stokes equations with $\vec{w}_0$ as initial value; $\vec{v} = \vec{u} - \vec{W}$ is solution of the equation

$$\begin{cases} \partial_t\vec{v} = \Delta\vec{v} - \vec{\nabla}.(\vec{v}\otimes\vec{v}) - \vec{\nabla}.(\vec{W}\otimes\vec{v}) - \vec{\nabla}.(\vec{v}\otimes\vec{W}) - \vec{\nabla}p \\ \vec{\nabla}.\vec{v} = 0 \\ \vec{v}(0,.) = \vec{v}_0 \end{cases}$$

Since $\vec{v}_0 \in (L^2)^3$ and $\sup_{t>0}\|\vec{W}\|_3 < 2\epsilon$, we know that this problem has a Leray solution $(\vec{z},q)$ so that:

$$\sup_{t>0}\|\vec{z}\|_2 < \infty \text{ and } \int_0^\infty \|\vec{\nabla}\otimes\vec{z}\|_2^2\ ds < \infty$$

and (for some nonnegative local measure μ)

$$\partial_t|\vec{z}|^2 + 2|\vec{\nabla}\otimes\vec{z}|^2 =$$
$$\Delta|\vec{z}|^2 - \vec{\nabla}.(|\vec{z}|^2\vec{z}) - \vec{\nabla}.(|\vec{z}|^2\vec{W}) - 2\vec{\nabla}.((\vec{z}.\vec{W})\ \vec{z}) + 2((\vec{z}.\vec{\nabla})\ \vec{z}).\vec{W} - 2\vec{\nabla}.(q\ \vec{z}) - \mu$$

Moreover, we know that $\vec{u}$ and $\vec{W}$ are suitable and, therefore, that $\vec{v}$ is locally $L^2(H^1)$ and is $\mathcal{C}(L^3)$. We may then apply the Von Wahl uniqueness theorem [WAH 85]. Indeed, we have for $\vec{y} = \vec{z} - \vec{v}$:

$$\partial_t|\vec{y}|^2 + 2|\vec{\nabla}\otimes\vec{y}|^2 = \Delta|\vec{y}|^2 - \vec{\nabla}.(|\vec{z}|^2\vec{z}) - \vec{\nabla}.(|\vec{v}|^2\vec{v}) + 2((\vec{z}.\vec{\nabla})\ \vec{z}).\vec{v} +$$

$$+2((\vec{v}.\vec{\nabla})\ \vec{v}).\vec{z} - \vec{\nabla}.(|\vec{y}|^2\vec{W}) - 2\vec{\nabla}.((\vec{y}.\vec{W})\ \vec{y}) + 2((\vec{y}.\vec{\nabla})\ \vec{y}).\vec{W} - 2\vec{\nabla}.((q-p)\ \vec{y}) - \mu$$

and we write $((\vec{z}.\vec{\nabla})\ \vec{z}).\vec{v} + ((\vec{v}.\vec{\nabla})\ \vec{v}).\vec{z} = ((\vec{y}.\vec{\nabla})\ \vec{z}).\vec{v} + \vec{\nabla}((\vec{v}.\vec{z})\ \vec{v})$; now, all the quantities can be integrated (against $1_{[0,t]}\ \omega(x/R)$ and letting R go to ∞) to give

$$\|\vec{y}(t)\|^2 + 2\iint_0^t |\vec{\nabla}\otimes\vec{y}|^2\, ds\, dx \leq 2\iint_0^t ((\vec{y}.\vec{\nabla})\ \vec{y}).\vec{W}\, ds\, dx + 2\iint_0^t ((\vec{y}.\vec{\nabla})\ \vec{z}).\vec{v}\, ds\, dx$$

For finite T, we split $\vec{v}$ on $[0,T]$ in $\vec{X}+\vec{Y}$ where $\sup_t \|\vec{X}\|_3 < \epsilon$ and $\vec{Y} \in (L^\infty)^3$. Then, we get a Gronwall inequality on $\|\vec{y}\|_2^2$, which allows one to conclude that $\vec{y} = 0$. This implies that $\int_0^\infty \|\vec{u} - \vec{W}\|_3^4\, dt < \infty$ while $\sup_t \|\vec{W}\|_3 < 2\epsilon$; thus $\|\vec{u}\|_3$ take small values and the Kato theorem for small datas in L^3 gives us the behavior of $\vec{u}$ as t goes to ∞. □

3. Global existence of uniformly locally L^2 suitable solutions

We may now finish the proof of Theorem 33.1 about the existence of global Leray solutions.

Step 1: Local resolution in E_2

We consider the initial value $\vec{u}_0$ in $(E_2)^3$ and we solve the Navier–Stokes equations as explained in Chapter 32. We then have a local Leray solution $\vec{u}^{(0)}$ on $(0, T_0) \times \mathbb{R}^3$.

Step 2: Local resolution in E_3

Assume we have a local Leray solution $\vec{u}^{(N)}$ on some $(S_N, T_N) \times \mathbb{R}^3$. Then, according to Proposition 1, we may choose $S_{N+1} \in (\sup(S_N, T_N - 1), T_N)$ so that

$$\vec{u}^{(N)}(S_{N+1}) \in (E_3)^3 \text{ and } \lim_{s > S_{N+1}, s \to S_{N+1}} \|\vec{u}^{(N)}(s) - \vec{u}^{(N)}(S_{N+1})\|_{E_2} = 0$$

Then, due to the Kato theory of mild solutions (Chapter 17), we may find a time $U_{N+1} > S_{N+1}$ and a solution $\vec{v}^{(N+1)} \in \mathcal{C}([S_{N+1}, U_{N+1}], (E_3)^3)$ with $\vec{v}^{(N+1)}(S_{N+1}) = \vec{u}^{(N)}(S_{N+1})$.

Step 3: Resolution in $E_3 + L^2$

Using the approximation lemma (Chapter 12), we know that we may split $\vec{u}^{(N)}(S_{N+1})$ into $\vec{X}_{N+1} + \vec{Y}_{N+1}$ with $\vec{\nabla}.\vec{X}_{N+1} = \vec{\nabla}.\vec{Y}_{N+1} = 0$, $\vec{Y}_{N+1} \in (L^2)^3$ and $\|\vec{X}_{N+1}\|_{E_3} < \epsilon$, where we choose ϵ small enough to ensure that we may find a Kato solution $\vec{X}^{(N+1)} \in \mathcal{C}([S_{N+1}, S_{N+1}+2], (E_3)^3)$ with $\vec{X}^{(N+1)}(S_{N+1}) = \vec{X}_{N+1}$. Then, using the Leray theory developed in Chapters 14 and 21, we know that we may find a Leray solution $\vec{Y}^{(N+1)}$ of

$$\begin{cases} \partial_t \vec{Y} = \Delta \vec{Y} - \vec{\nabla}.(\vec{Y} \otimes \vec{Y}) - \vec{\nabla}.(X^{(\vec{N}+1)} \otimes \vec{Y}) - \vec{\nabla}.(\vec{Y} \otimes X^{(\vec{N}+1)}) - \vec{\nabla} p \\ \vec{\nabla}.\vec{Y} = 0 \\ \vec{Y}(S_{N+1}, .) = \vec{Y}_{N+1} \end{cases}$$

Then, $\vec{u}^{(N+1)} = \vec{X}^{(N+1)} + \vec{Y}^{(N+1)}$ is a local Leray solution on $(S_{N+1}, T_{N+1}) \times \mathbb{R}^3$ (with $T_{N+1} = S_{N+1} + 2$) of the Navier–Stokes initial value problem associated with $\vec{u}^{(N)}(S_{N+1})$.

Step 4: Identifications of solutions

If we use the generalized Von Wahl uniqueness theorem (theorem 33.2), we find that on $[S_{N+1}, U_{N+1}]$, we have $\vec{u}^{(N)} = \vec{v}^{(N+1)} = \vec{u}^{(N+1)}$.

Step 5: Construction of $\vec{u}$

By induction on N, we have a sequence of local Leray solutions $\vec{u}^{(N)}$ on $(S_N, U_{N+1}) \times \mathbb{R}^3$ so that:

i) $S_0 = 0$ and $\vec{u}^{(0)}(0) = \vec{u}_0$

ii) $\vec{u}^{(N)} = \vec{u}^{(N+1)}$ on $(S_{N+1}, U_{N+1}) \times \mathbb{R}^3$

iii) $S_N + 1 < S_{N+1} < U_{N+1} < S_N + 2$ so that $(0, \infty) = \cup_{N \in \mathbb{N}} (S_N, U_{N+1})$

This defines a unique distribution $\vec{u}$ on $(0, \infty) \times \mathbb{R}^3$. It is obvious that we have the desired estimates on $\vec{u}$, as it can be seen by using a nonnegative partition of unity adapted to $[0, U_1) \cup \cup_{N \geq 1} (S_N, U_{N+1})$, proving Theorem 33.1. $\square$

Chapter 34

Leray's conjecture on self-similar singularities

In 1934, Leray [LER 34] proposed to study self-similar solutions to the Navier–Stokes equations in order to try to exhibit solutions that blow up in a finite time. Leray's self-similar solutions on $(0, T^*) \times \mathbb{R}^d$ are given by $\vec{u}(t,x) = \lambda(t)\vec{U}(\lambda(t)x)$ with $\lambda(t) = \frac{1}{\sqrt{2a(T^*-t)}}$ for some positive a.

Recently, Nečas, Ružička, and Šverák [NECRS 96] proved that the only solution $\vec{U} \in (L^3(\mathbb{R}^3))^3$ was $\vec{U} = 0$. Their result was extended by Tsai [TSA 98] to the case $\vec{U} \in (\mathcal{C}_0(\mathbb{R}^3))^3$ or to the case of solutions $\vec{u}$ with local energy estimates near the blow-up point: $\sup_{T_0<t<T^*} \int_{|x|<1} |\vec{u}(t,x)|^2 \, dx < \infty$ and $\int_{T_0}^{T^*} \int_{|x|<1} |\vec{\nabla} \otimes \vec{u}(t,x)|^2 \, dx \, dt < \infty$. Their proof was based on Hopf's strong maximum principle applied to $\Pi = \frac{1}{2}|\vec{U}|^2 + P + a\vec{X}.\vec{U}$, where P is associated to the pressure $p(t,x)$ by $\vec{\nabla}p(t,x) = \lambda(t)^3(\vec{\nabla}P)(\lambda(t)x)$, and where $\vec{X}$ be the identical vector field on $\mathbb{R}^3$: $\vec{X}(x) = (x_1, x_2, x_3)$; another key ingredient was the regularity criterion of Caffarelli, Kohn, and Nirenberg.

We give here a simpler proof (still based on the strong maximum principle [section 1], but with no use of the regularity criterion of Caffarelli, Kohn, and Nirenberg): we prove in Section 2 that there is no non-trivial solution $\vec{U} \in (L^d(\mathbb{R}^d))^d$ or in $(\mathcal{C}_{\prime}(\mathbb{R}^d))^d$, and we give in Section 3 a simple proof of Tsai's result on solutions with local energy estimates near the blow-up point.

1. Hopf's strong maximum principle

In this section, we prove a lemma of Tsai, which is a variation on the maximum principle of Hopf:

Proposition 34.1: (Tsai's lemma)

Let Π, $b_1,\ldots,$ b_d be real-valued functions so that $\Pi \in \mathcal{C}^2(\mathbb{R}^d)$ and, for $j = 1\ldots,d$, $b_j \in \mathcal{C}_{\mathbb{R}}(\mathbb{R}^d)$ and $\lim_{x\to\infty} \frac{|b_j(x)|}{|x|} = 0$. Let $a > 0$, $\vec{b} = (b_1,\ldots,b_d)$ and let $\vec{X}$ be the identical vector field on $\mathbb{R}^d$: $\vec{X}(x) = (x_1,\ldots,x_d)$. If

$$\Delta\Pi - (\vec{b} + a\vec{X}).\vec{\nabla}\Pi \geq 0 \text{ and } \lim_{x\to\infty} \frac{|\Pi(x)|}{|x|^2} = 0,$$

then Π *is constant.*

Proof: Let us define $\mathcal{L}$ as the differential operator $\mathcal{L}f = \Delta f - (\vec{b} + a\vec{X}).\vec{\nabla} f$. The hypothesis is then $\mathcal{L}\Pi \geq 0$.

Let us assume that Π is not constant. We select two points X_0 and X_1 so that $\Pi(X_0) < \Pi(X_1)$. We select two real numbers R_0 and R_1 so that $0 < R_0 < |X_0 - X_1| < R_1$ and so that, for $|x - X_0| \geq R_1$, $|\vec{b}(x) + a\vec{X}_0| \leq a|x - X_0|/2$ and, for $|x - X_0| \leq R_0$, $\Pi(x) \leq (\Pi(X_0) + \Pi(X_1))/2$. Moreover, we require that $aR_1^2 > 2d$.

We then consider the function $V(x) = \Pi(x) + \alpha(e^{-\beta|x-X_0|^2} - \gamma|x - X_0|^2)$, where the positive numbers α, β and γ are chosen in the following way:
i) first, we choose β big enough to ensure that the function $\phi_\beta(x) = e^{-\beta|x-X_0|^2}$ satisfies $\mathcal{L}\phi_\beta > 0$ on $|x - X_0| \geq R_0$. This is always possible, since we have $\mathcal{L}\phi_\beta = e^{-\beta|x-X_0|^2}\beta((4\beta + 2a)|x - X_0|^2 - 2d + 2(\vec{b} + a\vec{X}_0).(\vec{X} - \vec{X}_0))$; since $|\vec{b}(x)|$ may be bounded by $C(R_0)|x - X_0|$ for $|x - X_0| \geq R_0$, it is enough to take $\beta > 0$ with $4\beta + 2a > C(R_0) + 2dR_0^{-2} + a|X_0|R_0^{-1}$.
ii) next, we choose γ small enough to ensure that the function $\psi(x) = |x - X_0|^2$ satisfies $\gamma \sup_{R_0 \leq |x-X_0| \leq R_1} |\mathcal{L}\psi(x)| < \min_{R_0 \leq |x-X_0| \leq R_1} \mathcal{L}\phi_\beta(x)$.
iii) finally, we choose α small enough to ensure that

$$\alpha \sup_{|x-X_0| \leq R_1} |\phi_\beta(x) - \gamma\psi(x)| < (\Pi(X_1) - \Pi(X_0))/4.$$

The function V is $\mathcal{C}^2$; we have $V(x) \sim -\alpha\gamma|x|^2$ when $x \to \infty$, thus V has a maximum at some point X_2; at this point, we must have $\vec{\nabla}V(X_2) = 0$ and $\Delta V(X_2) \leq 0$ (since $V(X_2) \geq \frac{1}{\sigma(S^{d-1})} \int_{S^{d-1}} V(X_2 + \epsilon\theta)\, d\sigma(\theta) = V(X_2) + \frac{\epsilon^2}{2d}\Delta V(X_2) + o(\epsilon^2)$). Thus, we have $\mathcal{L}V(X_2) \leq 0$.

For $|x - X_0| \leq R_0$, we have $V(x) \leq \Pi(X_0) + \frac{3}{4}(\Pi(X_1) - \Pi(X_0)) < V(X_1)$. Thus, we cannot have $|X_2 - X_0| \leq R_0$.

For $R_0 \leq |x - X_0| \leq R_1$, we have $\gamma|\mathcal{L}\psi(x)| < \mathcal{L}\phi_\beta(x)$ and $\mathcal{L}\Pi(x) \geq 0$, hence $\mathcal{L}V(x) > 0$. Thus, we cannot have $R_0 \leq |X_2 - X_0| \leq R_1$.

For $|x - X_0| \geq R_1$, we have $\mathcal{L}\phi_\beta(x) > 0$, $\mathcal{L}\Pi(x) \geq 0$, and $\mathcal{L}\psi(x) = 2d - 2(\vec{b} + a\vec{X}_0).(\vec{X} - \vec{X}_0) - 2a|x - X_0|^2 \leq 2d - a|x - X_0|^2 < 0$, hence $\mathcal{L}V(x) > 0$. Thus, we cannot have $|X_2 - X_0| \geq R_1$.

Therefore, Π must be constant. $\square$

2. The C_0 self-similar Leray solutions are equal to 0

We may now study self-similar solutions to the Navier–Stokes equations. Recall that a weak solution of the Navier–Stokes equations is a locally square-integrable vector field $\vec{u}(t,x)$ in $(L^2_{loc}((0,T^*) \times \mathbb{R}^d)^d$, which satisfies, for some distribution $p \in \mathcal{D}'((0,T^*) \times \mathbb{R}^d)$, the equations

$$\left\{ \begin{array}{c} \partial_t \vec{u} = \Delta\vec{u} - \vec{\nabla}.(\vec{u} \otimes \vec{u}) - \vec{\nabla}p \\ \vec{\nabla}.\vec{u} = 0. \end{array} \right. \tag{34.1}$$

A self-similar solution is a solution that can be written as

$$\vec{u}(t,x) = \lambda(t)\vec{U}(\lambda(t)x) \text{ with } \lambda(t) = \frac{1}{\sqrt{2a(T^* - t)}} \tag{34.2}$$

for some positive a and some locally square-integrable $\vec{U} \in (L^2_{loc}(\mathbb{R}^d))^d$. We then find the equation: $\vec{\nabla}p(t,x) = \lambda(t)^3\vec{Q}(\lambda(t)x)$ with $\vec{Q} = \Delta\vec{U} - \vec{\nabla}.\vec{U}\otimes\vec{U} - a\vec{U} - a(\vec{X}.\vec{\nabla})\vec{U}$. Thus, we shall have $p(t,x) = \lambda(t)^2 P(\lambda(t)x) + q(t)$ for some distributions $P \in \mathcal{D}'(\mathbb{R}^d)$ and $q \in \mathcal{D}'((0,T^*))$. We may assume with no loss of generality that $q(t) = 0$. Thus, self-similar solutions $(\vec{u}, p)$ are given by the equations

$$\begin{cases} \vec{u}(t,x) = \lambda(t)\vec{U}(\lambda(t)x) \\ p(t,x) = \lambda(t)^2 P(\lambda(t)x) \\ \\ \Delta\vec{U} = a\vec{U} + a(\vec{X}.\vec{\nabla})\vec{U} + \vec{\nabla}.\vec{U}\otimes\vec{U} + \vec{\nabla}P \\ \vec{\nabla}.\vec{U} = 0 \end{cases} \tag{34.3}$$

A direct consequence of Tsai's lemma is then the following theorem:

Theorem 34.1:
Let $\vec{U} \in (\mathcal{C}^2(\mathbb{R}^d))^d$ and $P \in \mathcal{C}^2(\mathbb{R}^d)$ be solutions of Leray's equations:

$$\begin{cases} \Delta\vec{U} = a\vec{U} + a(\vec{X}.\vec{\nabla})\vec{U} + \vec{\nabla}.\vec{U}\otimes\vec{U} + \vec{\nabla}P \\ \vec{\nabla}.\vec{U} = 0 \end{cases}$$

with $a > 0$. If moreover $\lim_{x\to\infty} \frac{|\vec{U}(x)|}{|x|} = 0$ and $\lim_{x\to\infty} \frac{|P(x)|}{|x|^2} = 0$ then $\vec{U}$ is constant.

Proof: We define $\Pi = \frac{1}{2}|\vec{U}|^2 + P + a\vec{X}.\vec{U}$ and $\mathcal{L}$ the differential operator $\mathcal{L}f = \Delta f - (\vec{U} + a\vec{X}).\vec{\nabla}f$. We apply Proposition 34.1 by proving that $\mathcal{L}\Pi \geq 0$.

Let us first notice that $(\vec{X}.\vec{\nabla})\vec{U}$ is divergence free since $\vec{\nabla}.\big((\vec{X}.\vec{\nabla})\vec{U}\big) = \vec{\nabla}.\vec{U} + (\vec{X}.\vec{\nabla})(\vec{\nabla}.\vec{U})$. Thus, we have

$$\Delta P = -\sum_{j=1}^{d}\sum_{k=1}^{d} \partial_j\partial_k(U_jU_k) = -\sum_{j=1}^{d}\sum_{k=1}^{d} \partial_jU_k\partial_kU_j$$

(by taking the divergence of Leray's equations). We have $\Delta(\vec{X}.\vec{U}) = \vec{X}.\Delta\vec{U} + \vec{\nabla}.\vec{U} = \vec{X}.\Delta\vec{U}$ and $\Delta(\frac{1}{2}|\vec{U}|)^2 = \vec{U}.\Delta\vec{U} + |\vec{\nabla}\otimes\vec{U}|^2$. On the other hand, we have

$$(\vec{U} + a\vec{X}).\vec{\nabla}\Pi = (\vec{U} + a\vec{X}).\big(\vec{\nabla}P + a\vec{U} + ((\vec{U} + a\vec{X}).\vec{\nabla})\vec{U}\big) = (\vec{U} + a\vec{X}).\Delta\vec{U}.$$

We find that

$$\mathcal{L}\Pi = |\vec{\nabla}\otimes\vec{U}|^2 - \sum_{j=1}^{d}\sum_{k=1}^{d} \partial_jU_k\partial_kU_j = \frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{d} |\partial_jU_k - \partial_kU_j|^2 \geq 0.$$

We may then apply Proposition 34.1. We have that Π is a constant function, hence $\mathcal{L}\Pi = 0$. This gives $\partial_j U_k = \partial_k U_j$ for $j, k \in \{1, \ldots, d\}$; since $\vec{U}$ is divergence free, we find that $\vec{U} = \vec{\nabla} F$ for some harmonic function F. Thus, for $1 \leq j \leq d$, U_j is a harmonic function on $\mathbb{R}^d$ that is $o(|x|)$ at infinity; thus U_j is a harmonic tempered distribution, hence a harmonic polynomial, and since it has a very slow growth at infinity, it is constant. □

Corollary 34.1: (Tsai's theorem)

If $\vec{u} = \lambda(t)\vec{U}(\lambda(t)x)$ is a self-similar Leray solution of the Navier–Stokes equations with $\vec{U} \in (\mathcal{C}_0)^d$, then $\vec{u} = 0$.

Proof: Since $\vec{u}$ vanishes at infinity, we may apply Theorem 11.1 to get that $\vec{u}$ is solution of the Navier–Stokes equations:

$$(34.4) \qquad \begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{u}) \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

It means that we may choose $p = (Id - \mathbb{P})(\vec{\nabla}.\vec{u} \otimes \vec{u})$. Moreover, we know that a solution of (34.4) with $\vec{u} \in (L^\infty((0,T) \times \mathbb{R}^d))^d$ is smooth: for all $T_0 < T$ and all $k \in \mathbb{N}$ we have $\vec{u} \in (L^\infty((T_0, T_1), B_\infty^{k,\infty}(\mathbb{R}^d)))^d$. Thus, $\vec{U}$ is smooth. We have to check that P is smooth and is $o(|x|^2)$ at infinity. It is enough to check that, for a fixed t_0, $\vec{\nabla} p(t_0, x)$ is smooth and is $o(|x|)$ at infinity. Let $Id = S_0 + \sum_{j \in \mathbb{Z}} \Delta_j$ be the Littlewood–Paley decomposition; $(Id - \mathbb{P})(Id - S_0)$ maps $B_\infty^{k,\infty}$ to $B_\infty^{k,\infty}$, so that $(Id - S_0)\vec{\nabla} p$ belongs to all the Besov spaces $B_\infty^{k,\infty}$, hence is smooth and bounded. Since $S_0 \vec{\nabla} P$ is smooth (because it has a compactly supported Fourier transform), $\vec{\nabla} P$ is smooth. Moreover, we have $\|(Id - \mathbb{P})\vec{\nabla} S_0(\vec{u} \otimes \vec{u})\|_\infty \leq C\|\vec{u}\|_\infty^2$, and thus, $\vec{\nabla} P$ is bounded. We may apply Theorem 34.1 and get that $\vec{U}$ is constant. Since it goes to 0 at infinity, we have $\vec{U} = 0$. □

Corollary 34.2:

If $\vec{u} = \lambda(t)\vec{U}(\lambda(t)x)$ is a self-similar Leray solution of the Navier–Stokes equations on $(0,T)^* \times \mathbb{R}^d$ with $\vec{U} \in (L^d)^d$, then $\vec{u} = 0$.

Proof: The result of Nečas, Ružička, and Šverák is now easy to prove. Since $\vec{u}$ vanishes at infinity, we may apply Theorem 11.1 to get that $\vec{u}$ is solution of the Navier–Stokes equations

$$(34.4) \qquad \begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{u}) \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

Moreover, we have uniqueness of the solutions of (34.4) in $(\mathcal{C}([0,T], L^d))^d$ (Chapter 27); Kato's algorithm provides a solution that belongs to $(\mathcal{C}([0,T], L^d))^d$

$\cap(\mathcal{C}((0,T],\mathcal{C}_0))^d$; thus, we get that $\vec{U} \in (\mathcal{C}_0)^d$. Tsai's theorem then gives $\vec{U} = 0$. □

We conclude this section with other results of Tsai:

Corollary 34.3:

Let $q \in (d,\infty)$. If $\vec{u} = \lambda(t)\vec{U}(\lambda(t)x)$ is a self-similar Leray solution of the Navier–Stokes equations on $(0,T)^* \times \mathbb{R}^d$ with $\vec{U} \in (L^q)^d$, then $\vec{u} = 0$.

The same conclusion holds for the weaker hypothesis $\vec{U} \in (E_q)^d$, where E_q is the closure of the test functions in L^q_{uloc}, i.e. if $\vec{U}$ is locally L^q and $\lim_{x\to\infty} \int_{|y-x|<1} |\vec{U}(y)|^q\ dy = 0$.

Proof: We use the same proof as for Corollary 34.2, since we have uniqueness of the solutions of (34.4) in $(\mathcal{C}([0,T],E_q))^d$. □

3. The case of local control

In this section, we deal with $d = 3$. We study self-similar Leray solutions under some energy estimates.

If we assume that we have a global control as for the square-integrable Leray solutions, i.e. $\vec{u} \in (L^\infty((0,T^*),L^2(\mathbb{R}^3)))^3 \cap (L^2((0,T^*),\dot{H}^1(\mathbb{R}^3)))^3$, then we have that $\vec{u} \in (L^4((0,T^*),L^3(L^3)))^3$, thus $\vec{U} \in (L^3)^3$, and according to Corollary 34.2 we may conclude that $\vec{u} = 0$.

Tsai [TSA 98] proved that the same results hold under the weaker assumption that we have control of the energy in the neighborhood of the blowing up point $(T^*,0)$:

Theorem 34.2:

Let $\vec{u}$ be a self-similar solution of the Navier–Stokes equations on $(0,T^*) \times \mathbb{R}^3$:

$$\begin{cases} \vec{u}(t,x) = \lambda(t)\vec{U}(\lambda(t)x) \\ p(t,x) = \lambda(t)^2 P(\lambda(t)x) \\ \\ \Delta\vec{U} = a\vec{U} + a(\vec{X}.\vec{\nabla})\vec{U} + \vec{\nabla}.\vec{U}\otimes\vec{U} + \vec{\nabla}P \\ \vec{\nabla}.\vec{U} = 0 \end{cases}$$

where $\lambda(t) = \frac{1}{\sqrt{2a(T^*-t)}}$ for some positive a. If we assume that for some $T_0 < T^*$ and some $r_0 > 0$, we have the energy estimates

$$\sup_{T_0<t<T^*} \int_{|x|<r_0} |\vec{u}(t,x)|^2\ dx < \infty \text{ and } \int_{T_0}^{T^*} \int_{|x|<r_0} |\vec{\nabla}\otimes\vec{u}(t,x)|^2\ dx\ dt < \infty$$

then $\vec{u} = 0$.

Proof: If we define $\Pi = \frac{1}{2}|\vec{U}|^2 + P + a\vec{X}.\vec{U}$ and $\mathcal{L}$ as the differential operator $\mathcal{L}f = \Delta f - (\vec{U} + a\vec{X}).\vec{\nabla} f$, we find again that

$$\mathcal{L}\Pi = \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3} |\partial_j U_k - \partial_k U_j|^2 \geq 0.$$

Thus, our strategy is clear: we check that $\vec{U}$ and $\vec{P}$ are smooth and that they can grow only at a slow rate at infinity, then we apply Theorem 34.1.

Step 1: Regularity of $\vec{U}$ and of P

We start from the following classical estimate: if $f \in \mathcal{D}'(\mathbb{R}^3)$ and if $\Delta f \in H^\sigma_{loc}(\mathbb{R}^3)$ for some $\sigma \in \mathbb{R}$, then $f \in H^{\sigma+2}_{loc}(\mathbb{R}^3)$. Indeed, if Ω is an open bounded subset of $\mathbb{R}^d$, we have $f \in H^s_{loc}(\Omega)$ for some $s \in \mathbb{R}$; now, just write for $\omega \in \mathcal{D}(\Omega)$ $\omega f = (Id - \Delta)^{-1}\big(f(Id - \Delta)\omega - \omega\Delta f - \vec{\nabla}\omega.\vec{\nabla} f\big)$, which gives

$$f \in H^s_{loc}(\Omega) \text{ and } \Delta f \in H^\sigma_{loc}(\Omega) \Rightarrow f \in H^{\min(s+1,\sigma+2)}_{loc}(\Omega)$$

and finally the required regularity on f. Now, the hypothesis on $\vec{u}$ gives that $\vec{U} \in (H^1_{loc}(\mathbb{R}^3))^3$: in order to prove that $\vec{\nabla}\otimes\vec{U}$ is square-integrable on $B(0,R)$, just consider $\vec{u}(t,.)$ with $T^* - t \leq \frac{1}{2a}\left(\frac{r_0}{R}\right)^2$. Now we write

$$\begin{cases} \Delta P = \sum_{j=1}^3 \sum_{k=1}^3 \partial_j\partial_k(U_jU_k) \\ \Delta\vec{U} = a\vec{U} + a(\vec{X}.\vec{\nabla})\vec{U} + \vec{\nabla}.\vec{U}\otimes\vec{U} + \vec{\nabla}P \end{cases}$$

and we check that for f and g in $H^s_{loc}(\mathbb{R}^3)$ with $s \geq 1$, we have $fg \in H^{s-\frac{1}{2}}_{loc}(\mathbb{R}^3)$; thus, $\vec{U} \in (H^s_{loc}(\mathbb{R}^3))^3$ with $s \geq 1$ gives that $P \in H^{s-\frac{1}{2}}_{loc}(\mathbb{R}^3)$ and then $\vec{U} \in (H^{s+\frac{1}{2}}_{loc}(\mathbb{R}^3))^3$. Thus, $\vec{U}$ and P are $\mathcal{C}^\infty$.

Step 2: Global estimates on $\vec{U}$

We use the energy inequalities. Let $A = \sup_{T_0<t<T^*}\int_{B(0,r_0)} |\vec{u}(t,x)|^2\, dx$ and $B(t) = \int_t^{T^*}\int_{B(0,r_0)} |\vec{\nabla}\otimes\vec{u}(s,x)|^2\, dx\, ds$. For $t > T_0$, we have

$$\int_{B(0,r_0)} |\vec{u}(t,x)|^2\, dx = \lambda(t)^{-1}\int_{B(0,\lambda(t)r_0)} |\vec{U}(x)|^2\, dx;$$

thus, $\int_{B(0,R)} |\vec{U}(x)|^2\, dx \leq \frac{AR}{r_0}$ for $R > \lambda(T_0)r_0$. Similarly, we have

$$\begin{aligned} B(t) &= \int_t^{T^*} \lambda(s)\int_{B(0,\lambda(s)r_0)} |\vec{\nabla}\otimes\vec{U}(x)|^2\, dx\, ds \\ &\geq (T^* - t)\lambda(t)\int_{B(0,\lambda(t)r_0)} |\vec{\nabla}\otimes\vec{U}(x)|^2\, dx \end{aligned}$$

thus, $\int_{B(0,R)} |\vec{\nabla}\otimes\vec{U}(x)|^2\, dx \leq \frac{2aB(t)R}{r_0}$ for $R > \lambda(T_0)r_0$ and $T^* - t = \frac{R^2}{2ar_0^2}$. This gives that $\|\vec{U}\|_{L^{6,2}(B(0,R))}$ is $o(\sqrt{R})$ as R goes to $+\infty$.

Step 3: Global estimates for P

We first check that we may define a pressure by the formula

$$P_0 = -\frac{1}{\Delta}\sum_{k=1}^{3}\sum_{l=1}^{3}\partial_k\partial_l(U_kU_l),$$

then we shall compare P_0 to P. Let us see how P_0 behaves on $B(0,R)$: we fix $\omega \in \mathcal{D}(\mathbb{R}^3)$ equal to 1 on $B(0,1)$ and supported by $B(0,2)$. We define, for $R \geq 1$, $\omega_R(y) = \omega(\frac{y}{4R})$. Then, for $x \in B(0,R)$, we write $P_0(x) = P_1(x) + P_2(x)$ with $P_1 = \sum_{k=1}^3\sum_{l=1}^3 R_kR_l(\omega_R U_kU_l)$. The distribution kernel of $\frac{1}{\Delta}\partial_k\partial_l$ is bounded by $C|x-y|^{-3}$ outside from the diagonal $x = y$, hence $|P_2(x)| \leq C\int_{|y|\geq 4R}\frac{1}{|x-y|^3}|\vec{U}(y)|^2\ dx \leq C'\sum_{n\in\mathbb{N}}\int_{4^nR\leq|y|\leq 4^{n+1}R}\frac{|\vec{U}(y)|^2}{4^{3n}R^3}\ dx \leq C''R^{-2}$. Thus, $\|P_2\|_{L^{3,1}(B(0,R))} \leq CR^{-1}$. On the other hand, the Riesz transforms operate on $L^{3,1}$, hence $\|P_1\|_{L^{3,1}} \leq C\|\omega_R|\vec{U}|^2\|_{L^{3,1}} \leq C'\|\vec{U}\|^2_{L^{6,2}(B(0,8R))} \leq C''\alpha(R)R$ with $\lim_{R\to\infty}\alpha(R) = 0$. This gives that P_0 is well defined and that $\|P_0\|_{L^{3,1}(B(0,R))} = o(R)$.

Moreover, $\partial_jP_0 = Q_1 + Q_2$ with $Q_1 = \sum_{k=1}^3\sum_{l=1}^3 R_kR_l(\omega_R\partial_j(U_kU_l))$. On one hand, we have $\|Q_1\|_{L^{3/2,1}} \leq C\|\omega_R|\vec{U}|\ |\vec{\nabla}\otimes\vec{U}|\ \|_{L^{3/2,1}} \leq C'\|\vec{U}\|_{L^{6,2}(B(0,8R))}$ $\|\vec{\nabla}\otimes\vec{U}\|^2_{L^2(B(0,8R))} \leq C''\beta(R)R$ with $\lim_{R\to\infty}\beta(R) = 0$. On the other hand, for $x \in B(0,R)$, we have $|Q_2(x)| \leq C\int_{|y|\geq 4R}\frac{1}{|x-y|^3}|\vec{U}(y)|\ |\vec{\nabla}\otimes\vec{U}(y)|\ dx \leq C'\sum_{n\in\mathbb{N}}\int_{4^nR\leq|y|\leq 4^{n+1}R}\frac{|\vec{U}(y)|\ |\vec{\nabla}\otimes\vec{U}(y)|}{4^{3n}R^3}\ dx \leq C''R^{-2}$. Thus, $\|Q_2\|_{L^{3/2,1}(B(0,R))}$ $\leq C$. This gives that $\|\vec{\nabla}P_0\|_{L^{3/2,1}(B(0,R))} = o(R)$.

We now compare P and P_0. We have $\Delta P = \Delta P_0$; moreover, P and P_0 are tempered distributions (because $\vec{U}$ has a small rate of growth at infinity). Thus, $P - P_0$ is a harmonic polynomial. Let $\partial_j(P-P_0) = \sum_{|\alpha|\leq N} c_{\alpha,j}x^\alpha$. For $|\alpha| \leq N$, let $\varphi_\alpha \in \mathcal{D}$ so that $\int x^\alpha\varphi_\alpha\ dx = 1$ and, for $|\beta| \leq N$ and $\beta \neq \alpha$, $\int x^\beta\varphi_\alpha\ dx = 0$ and let $\psi_\alpha \in \mathcal{D}$ so that $\psi_\alpha = 1$ on a neighborhood of the support of φ_α. Then, we have

$$\int \varphi_\alpha(\frac{x}{R})(\partial_jP(x) - \partial_jP_0(x))\ dx = c_{\alpha,j}R^{3+|\alpha|}.$$

On the other hand, we have

$$|\int \varphi_\alpha(\frac{x}{R})\partial_jP_0(x)\ dx| \leq \|\varphi_\alpha(\frac{x}{R})\|_{L^{3,\infty}}\|\psi_\alpha(\frac{x}{R})\vec{\nabla}P_0\|_{L^{3/2,1}} = o(R^2)$$

and

$$\begin{array}{rcl} |\int\varphi_\alpha(\frac{x}{R})\partial_jP(x)\ dx| & \leq & ((a+d)\|\varphi_\alpha(\frac{x}{R})\|_2 + \|\frac{1}{R^2}(\Delta\varphi_\alpha)(\frac{x}{R})\|_2)\|\psi_\alpha(\frac{x}{R})\vec{U}\|_2 \\ & & +\frac{1}{R}\|\vec{\nabla}\varphi_\alpha\|_\infty(\|\psi_\alpha(\frac{x}{R})\ |x|\ |\vec{U}|\|_1 + \|\psi_\alpha(\frac{x}{R})|\vec{U}|^2\|_1) \\ & = & O(R^2) \end{array}$$

and thus we get $c_{\alpha,j}=0$. Thus, $P-P_0$ is constant, and we may assume with no loss of generality that $P=P_0$.

Step 4: Local estimates for $\vec{U}$

We now estimate the size of $\vec{U}$. We cannot use the Leray equations for $\vec{U}$, since the estimate of the size of $\vec{X}\otimes\vec{U}$ cannot be $o(|x|)$. We thus go back to the Navier–Stokes equations.

Let $T=T^*/2$. We fix $\omega\in\mathcal{D}(\mathbb{R}^3)$, which is equal to 1 on the ball $B(0,1)$ and which is supported in the ball $B(0,2)$, and we define, for $|x|\geq 1$, ω_x as $\omega_x(y)=\omega(\frac{y}{4|x|})$. We have for $|x|\geq 1$

$$\begin{cases} \sup_{0\leq t\leq T}\|\vec{u}(t,y)\|_{L^{6,2}(B(0,8|x|))}\leq \alpha(x)|x|^{1/2} \\ \sup_{0\leq t\leq T}\|p(t,y)\|_{L^{3,1}(B(0,8|x|))}\leq \alpha(x)|x| \end{cases}$$

with $\lim_{x\to\infty}\alpha(x)=0$. We then write the equations

$$\begin{aligned}\partial_t(\omega_x\vec{u}) = & \quad \Delta(\omega_x\vec{u})-2\sum_{j=1}^3\partial_j((\partial_j\omega_x)\vec{u})+(\Delta\omega_x)\vec{u} \\ & -\vec{\nabla}.(\omega_x(\vec{u}\otimes\vec{u}))+(\vec{u}.\vec{\nabla}\omega_x)\vec{u}-\vec{\nabla}(\omega_xP)+P\vec{\nabla}\omega_x\end{aligned}$$

This gives for $t\in(0,T]$

$$\omega_x(y)\vec{u}(t,y)=e^{t\Delta}(\omega_x\vec{u}(0,.))+\int_0^t e^{(t-\tau)\Delta}(\vec{\beta}_x(\tau,.)-\sum_{j=1}^3\partial_j\vec{\gamma}_{j,x}(\tau,.))\,d\tau$$

with $\vec{\beta}_x=(\Delta\omega_x)\vec{u}+(\vec{u}.\vec{\nabla}\omega_x)\vec{u}+P\vec{\nabla}\omega_x$ and $\vec{\gamma}_{j,x}=2(\partial_j\omega_x)\vec{u}+\omega_xu_j\vec{u}+\omega_xPe_j$. Let us write $B_x=B(0,8|x|)$. We have

$$\begin{cases} \|\vec{\beta}_x\|_{L^{3/2,1}}\leq C\big(\|\vec{u}\|_{L^{6,2}(B_x)}|x|^{-2}|x|^{3/2}+(\|\vec{u}\|^2_{L^{6,2}(B_x)}+\|P\|_{L^{3,1}(B_x)})|x|^{-1}|x|\big) \\ \|\vec{\gamma}_{j,x}\|_{L^{3,1}}\leq C\big(\|\vec{u}\|_{L^{6,2}(B_x)}|x|^{-1}|x|^{1/2}+\|\vec{u}\|^2_{L^{6,2}(B_x)}+\|P\|_{L^{3,1}(B_x)}\big) \end{cases}$$

Thus, $\sup_{0\leq t\leq T}\|\vec{\beta}_x(t,.)\|_{L^{3/2,1}}=o(|x|)$ and $\sup_{0\leq t\leq T}\|\vec{\gamma}_{j,x}(t,.)\|_{L^{3,1}}=o(|x|)$.

We now use the estimates $\int_0^\infty\|e^{t\Delta}\sqrt{-\Delta}f\|_\infty\,dt\leq C\|f\|_{L^{3,1}(\mathbb{R}^3)}$ (Proposition 5.3) and $\int_0^T\|\int_0^t e^{(t-s)\Delta}f(s,.)\,ds\|_\infty\,dt\leq C\int_0^T\|f(s,.)\|_{L^{3,1}}\,ds$: indeed, if $f\in L^1((0,T),L^{3,1})$, it can be written as $f(s,y)=\sum_{n\in\mathbb{N}}\chi_n(s)f_n(y)$ with $\sum_{n\in\mathbb{N}}\|\chi_n\|_1\|f_n\|_{L^{3,1}}\leq C\int_0^t\|f(s,.)\|_{L^{3,1}}\,ds$; we have

$$\int_0^T\|\int_0^t e^{(t-s)\Delta}f(s,.)\,ds\|_\infty\,dt\leq\sum_{n\in\mathbb{N}}\int_0^T|\chi_n(s)|\int_0^{T-s}\|e^{\tau\Delta}f_n\|_\infty\,d\tau\,ds.$$

Since $\frac{1}{\sqrt{-\Delta}}$ maps $L^{3/2,1}$ to $L^{3,1}$, we find that the local tendency $\vec{w}_x(t,y)=\omega_x(y)\vec{u}(t,y)-e^{t\Delta}(\omega_x\vec{u}(0,.))$ satisfies

$$\begin{aligned}\int_0^T\|\vec{w}_x(t,y)\|_\infty\,dt &\leq C\int_0^T\|\vec{\beta}_x(t,.)\|_{L^{3/2,1}}+\sum_{j=1}^3\|\vec{\gamma}_{j,x}(t,.)\|_{L^{3,1}}\,dt \\ &\leq CT\sup_{0<t<T}\|\vec{\beta}_x(t,.)\|_{L^{3/2,1}}+\sum_{j=1}^3\|\vec{\gamma}_{j,x}(t,.)\|_{L^{3,1}} \\ &\leq \beta_T(x)|x|\end{aligned}$$

where $\lim_{x\to\infty} \beta_T(x) = 0$. Similarly,

$$\int_0^T \|e^{t\Delta}(\omega_x \vec{u}(0,.))\|_\infty \, dt \le CT^{1/4}\|\omega_x\vec{u}(0,.)\|_2 = o(|x|).$$

Thus, we find that $\int_0^T \|\omega_x\vec{u}(t,.)\|_\infty \, dt = o(|x|)$. Since we have, on $[0,T]$, $\lambda(0) \le \lambda(t) \le \lambda(T) = \sqrt{2}\lambda(0)$, we have

$$|\vec{U}(\lambda(T)x)| = \frac{1}{\lambda(t)}|\vec{u}(t, \frac{\lambda(T)}{\lambda(t)}x)| \le \frac{1}{\lambda(0)}|\omega_x(\frac{\lambda(T)}{\lambda(t)}x)\vec{u}(t, \frac{\lambda(T)}{\lambda(t)}x)|,$$

hence, $|\vec{U}(\lambda(T)x)| \le \frac{1}{T\lambda(0)}\int_0^T \|\omega_x\vec{u}(t,.)\|_\infty \, dt = o(|x|)$. Since $\lambda(T)$ is a constant, we find that $\lim_{x\to\infty} \frac{|\vec{U}(x)|}{|x|} = 0$.

Step 5: Local estimates for $\vec{\nabla} \otimes \vec{U}$

We may proceed in the same way in order to estimate the size of the derivatives of $\vec{U}$.

We keep the notations $T = T^*/2$ and $\omega_x(y) = \omega(\frac{y}{4|x|})$. Differentiating the formula for $\omega_x\vec{u}$ gives for $t \in (0,T]$

$$\partial_l\big(\omega_x(y)\vec{u}(t,y)\big) = \partial_l e^{t\Delta}(\omega_x\vec{u}(0,.)) + \int_0^t e^{(t-\tau)\Delta}\partial_l\vec{\delta}_x(\tau,.)\, d\tau$$

with $\vec{\delta}_x = \vec{\beta}_x - \sum_{j=1}^3 \partial_j\vec{\gamma}_{j,x} = -(\Delta\omega_x)\vec{u} + (\vec{u}.\vec{\nabla}\omega_x)\vec{u} - (\vec{\nabla}\omega_x.\vec{u})\vec{u} - \omega_x(\vec{u}.\vec{\nabla})\vec{u}$ $-\omega_x\vec{\nabla}P - 2\sum_{j=1}^3 \partial_j\omega_x\partial_j\vec{u}$. Let us write $B_x = B(0,8|x|)$. We have

$$\begin{aligned}\|\vec{\delta}_x\|_{L^{3/2,1}} \le \quad & C\big(\|\vec{u}\|_{L^{6,2}(B_x)}|x|^{-2}|x|^{3/2} + \|\vec{u}\|^2_{L^{6,2}(B_x)}|x|^{-1}|x| \\ & +\|\vec{u}\|_{L^{6,2}(B_x)}\|\vec{\nabla}\otimes\vec{u}\|_{L^2(B_x)} + \|\vec{\nabla}P\|_{L^{3/2,1}(B_x)} \\ & +\|\vec{\nabla}\otimes\vec{u}\|_{L^2(B_x)}|x|^{-1}|x|^{1/2}\big).\end{aligned}$$

Thus, $\sup_{0\le t\le T}\|\vec{\delta}_x(t,.)\|_{L^{3/2,1}} = o(|x|)$.

We again use Proposition 5.3 to get $\int_0^\infty \|e^{t\Delta}\sqrt{-\Delta}f\|_{L^{3,1}}dt \le C\|f\|_{L^{3/2,1}(\mathbb{R}^3)}$ and $\int_0^T \|\int_0^t e^{(t-s)\Delta}f(s,.)\, ds\|_{L^{3,1}} \, dt \le C\int_0^T \|f(s,.)\|_{L^{3/2,1}} \, ds$. We find that the local tendency $\vec{w}_x(t,y) = \omega_x(y)\vec{u}(t,y) - e^{t\Delta}(\omega_x\vec{u}(0,.))$ satisfies

$$\begin{aligned}\int_0^T \|\partial_p\vec{w}_x(t,y)\|_{L^{3,1}} \, dt \quad & \le \quad C\int_0^T \|\vec{\delta}_x(t,.)\|_{L^{3/2,1}} \, dt \\ & \le \quad CT\sup_{0<t<T}\|\vec{\delta}_x(t,.)\|_{L^{3/2,1}} \le \delta_T(x)|x|\end{aligned}$$

where $\lim_{x\to\infty}\delta_T(x) = 0$. Similarly, $e^{t\Delta}$ maps $\dot{H}^{-1}$ to $L^{3,1}$ with an operator norm $0(t^{-3/4})$; hence,

$$\int_0^T \|\partial_p e^{t\Delta}(\omega_x\vec{u}(0,.))\|_{L^{3,1}} \, dt \le CT^{1/4}\|\omega_x\vec{u}(0,.)\|_2 = o(|x|).$$

Thus, we find that $\int_0^T \|\partial_p\big(\omega_x\vec{u}(t,.)\big)\|_{L^{3,1}}\,dt = o(|x|)$. On $[0,T]\times B(0,2|x|)$, we have

$$|\partial_p\vec{U}(\lambda(T)y)| = \frac{1}{\lambda(t)^2}|\partial_p\vec{u}(t,\frac{\lambda(T)}{\lambda(t)}y)| \leq \frac{1}{\lambda(0)^2}|\partial_p\big(\omega_x\vec{u}(t,.)\big)(\frac{\lambda(T)}{\lambda(t)}y)|,$$

hence, $\|\partial_p\vec{U}\|_{L^{3,1}(B(0,2\lambda(T)|x|))} \leq \frac{1}{T\lambda(0)^2}\int_0^T \lambda(t)\|\partial_p(\omega_x\vec{u}(t,.))\|_{L^{3,1}}\ dt = o(|x|)$. This gives that $\lim_{R\to\infty}\frac{\|\vec{\nabla}\otimes\vec{U}\|_{L^{3,1}(B(0,R))}}{R} = 0$.

Step 6: Local estimates for P

We have $\omega_x P = \frac{1}{\Delta}(\Delta(\omega_x P)) = \frac{1}{\Delta}(-P\Delta\omega_x + \omega_x\Delta P + 2\vec{\nabla}.(P\vec{\nabla}\omega_x))$, and thus $\|\omega_x P\|_\infty \leq C\|\Delta(\omega_x P)\|_{L^{3/2,1}} \leq C'(\|P\|_{L^{3,1}(B_x)}|x|^{-1} + \|\Delta P\|_{L^{3/2,1}(B_x)} + \|\vec{\nabla}P\|_{L^{3/2,1}(B_x)}|x|^{-1})$; since $\Delta P = -\sum_{j=1}^3\sum_{k=1}^3 \partial_j U_k\partial_k U_j$, we have

$$\|\Delta P\|_{L^{3/2,1}(B_x)} \leq C\|\vec{\nabla}\otimes\vec{U}\|^2_{L^{3,1}(B_x)} = o(|x|^2).$$

Thus, we find that $\lim_{x\to\infty}\frac{P(x)}{|x|^2} = 0$.

Step 7: End of the proof

We may now apply Theorem 34.1 and we get that $\vec{U}$ is constant. Since $\|\vec{U}\|_{L^2(B(0,R)}$ is $O(\sqrt{R})$ as R goes to $+\infty$, this gives $\vec{U} = 0$. $\square$

Chapter 35

Singular initial values

1. Allowed initial values

The Navier–Stokes equations have a long history. Classical solutions were studied by Oseen at the beginning of the 20th century [OSE 27]. Then Leray introduced the notion of weak solutions [LER 34] and proved by using weak compactness the existence of global solutions associated with an initial value in $(L^2(\mathbb{R}^3))^3$ (see Chapter 14). Uniqueness in the class of Leray solutions remains an open question. Thirty years later, Kato and Fujita introduced mild solutions associated with more regular initial values [FUJK 64] (i.e., initial values in $(H^s(\mathbb{R}^3))^3$ with $s \geq 1/2$) (see Chapter 15). Then several authors tried to eliminate regularity in the search for mild solutions: Fabes, Jones, and Riviere [FABJR 72] and Giga [GIG 86] worked in $L^p_t L^q_x$ spaces, then Weissler [WEI 81] and Kato [KAT 84] worked in Lebesgue spaces (see Chapter 15), the next step was Morrey–Campanato spaces (Giga and Miyakawa [GIGM 89], Taylor [TAY 92], Kato [KAT 92], Federbush [FED 93], Cannone [CAN 95]) (see Chapters 17 and 18), then Besov spaces with negative regularity (Cannone [CAN 95], Kozono and Yamazaki [KOZY 97]) (see Chapter 20).

All these spaces in which mild solutions are constructed are contained in the Besov space $B_\infty^{-1,\infty}$ (or, when searching for global solutions, in the smaller space $\dot{B}_\infty^{-1,\infty}$). This is related to the scaling of the Navier–Stokes equations: if $\vec{u}(t,x)$ is a solution on $(0,T) \times \mathbb{R}^d$ with initial value $\vec{u}_0$, then $\lambda\vec{u}(\lambda^2 t, \lambda x)$ is a solution on $(0, T/\lambda^2) \times \mathbb{R}^d$ with initial value $\lambda\vec{u}_0(\lambda.)$. Thus, a theorem of existence up to a time T^* modulo a size condition $\|\vec{u}_0\|_E \leq C(T^*)$ would be consistent with an estimate for dilations of the type $\sup_{0<\lambda\leq 1} \|\lambda\vec{u}_0(\lambda x)\|_E \leq \|\vec{u}_0\|_E$. If, moreover, e^Δ is a bounded operator from E to L^∞, we find that $\sup_{t\in(0,1)} \sqrt{t}\|e^{t\Delta} f\|_\infty \leq \|e^\Delta\|_{\mathcal{L}(E,L^\infty)}\|f\|_E$, hence $E \subset B_\infty^{-1,\infty}$. There is a strong suspicion that the problem of finding mild solutions in the space $(B_\infty^{-1,\infty})^d$ is ill-posed (see the recent remarks of Montgomery-Smith [MOS 01] on a simplified model).

Auscher and Tchamitchian [AUST 99] proved that the model of adapted spaces developped by Cannone [CAN 95], Meyer and Muschietti [MEY 99] and Furioli, Lemarié-Rieusset, and Terraneo [FURLT 00], which was essentially based on a Littlewood–Paley decomposition of the initial value, could not be extended to the limit case of the Besov space $B_\infty^{-1,\infty}$, nor to the smaller space

bmo^{-1}. Auscher and Tchamitchian's result did not close the question of searching weaker conditions to grant the existence of mild solutions, since at the very same time Koch and Tataru [KOCT 01] were able to prove the existence of mild solutions associated with initial values in $(bmo^{-1})^d$. While Auscher and Tchamitchian focused on *a priori* estimates obtained by looking first for estimates in the space variable, then integrating in time, Koch and Tataru obtained a more precise estimate by integrating first in time, then estimating the size of the result in the space variable. Such an approach is customary for other nonlinear equations (as the Korteweg de Vries equation, for instance). In a similar way, one may first integrate in time then estimate the size of the result in the frequency variable, as Chemin and co-workers do in some limit cases (see for instance Chemin and Lerner [CHEL 95] or Chemin [CHE 99]).

Thus, bmo^{-1} has replaced $B_\infty^{-1,\infty}$ as the maximal space for mild solutions. The Besov space $B_\infty^{-1,\infty}$ is still used in some results as a tool to prove uniqueness or stability in smaller subspaces, as in the uniqueness theorem of Chemin [CHE 99] (see Chapter 28) or in the stability theorem of Cannone and Planchon [CANP 00] (see Chapter 29).

Similarly, the estimate $\lim_{t\to 0}\sqrt{t}\|\vec{u}\|_\infty = 0$, which is satisfied by mild solutions associated with smooth initial values, may often be used to prove uniqueness of solutions in subcritical classes (for example, if $X = (L^p((0,T),E_1))^d$, with $p > 2$ and E_1 the closure of the test functions in the Morrey space of uniformly locally finite measures $M^1_{uloc}(\mathbb{R}^d)$, and if $Y_\infty = \{\vec{u} \in (\mathcal{C}((0,T) \times \mathbb{R}^d))^d \ / \ \sup_{0<t<T}\sqrt{t}\|\vec{u}\|_\infty$ and $\lim_{t\to 0}\sqrt{t}\|\vec{u}\|_\infty = 0\}$, then if $\vec{u}$ and $\vec{v}$ belong to $X \cap Y_\infty$ and are two solutions of the Navier–Stokes equations with the same initial value, then $\vec{u} = \vec{v}$).

2. Maximal regularity and critical spaces

In most cases, when studying the existence of mild solutions associated with a Banach space E^d of initial values, we used regularizing properties of the heat kernel and obtained estimates not in $L^\infty((0,T),E^d)$ but in the smaller space $L^\infty((0,T),E^d) \cap Y_\infty$. We discussed the properties of the solutions in the whole class $L^\infty((0,T),E^d)$ at only two points: when discussing the generalization of Kato's theorem to shift-invariant Banach spaces of local measures (Theorem 17.1) and when discussing uniqueness of solutions in $\mathcal{C}([0,T),E^d)$ (Chapter 27).

When considering the uniqueness problem in $\mathcal{C}([0,T),E^d)$, we had to assume that E was continuously embedded into L^2_{loc} (in order to give sense to the nonlinear term $\vec{\nabla}.\vec{u}\otimes\vec{u}$ in the differential equation); then, the scaling condition $\sup_{0<\lambda\leq 1}\|\lambda\vec{u}_0(\lambda x)\|_E \leq \|\vec{u}_0\|_E$ and the assumption that E is a shift-invariant Banach space of test functions give $E \subset m^{2,d}$, the closure of the test functions in the Morrey–Campanato space $M^{2,d}$. When E is supercritical ($E \subset M^{2,q}$ with $q > d$ or $\sup_{0<\lambda\leq 1}\|\lambda^\alpha\vec{u}_0(\lambda x)\|_E \leq \|\vec{u}_0\|_E$ with $\alpha \in [0,1)$), the uniqueness is proved by a direct computation based on the estimate $\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{u}\otimes\vec{v}\|_{M^{2,q}} \leq C_q|t-s|^{-1/2}(1+|t-s|^{-d/(2q)})\|\vec{u}\|_{M^{2,q}}\|\vec{v}\|_{M^{2,q}}$. When E is critical, we find an estimate $\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{u}\otimes\vec{v}\|_E = O(|t-s|^{-1})\|\vec{u}\|_E\|\vec{v}\|_E$, which we cannot integrate. This difficulty may be circumvened in three ways:

i) by using the maximal L^pL^q regularity of the heat kernel: the integral we deal with is considered as a singular integral and studied through the theory of Calderón–Zygmund operators (Chapter 7). The control on the size of the integral is provided by the oscillations of the kernel, not only by its decay. This approach of Monniaux [MON 99], extended by May [MAY 02], allowed one to prove uniqueness in spaces close to the limit case $m^{2,d}$ – in fact, in the closure of the test functions in the multiplier space X^1, hence, in $m^{p,d}$ for every $p > 2$ (see Chapters 27 and 21).

ii) by using Besov spaces $B_p^{s,q}$ with $q = \infty$: the integral we deal with is considered as a sum of bandpass filters focusing on separate bands; since we do not sum the contributions of different frequencies ($q = \infty$), we suppress the problem of the divergence of the integral $\int_0^t \frac{ds}{t-s}$. This was the approach of Furioli, Lemarié-Rieusset, and Terraneo [FURLT 00].

iii) by considering the integral as a representation in a sum of Banach spaces $A_0 + A_1$. Then, the use of the J-method of real interpolation of Lions and Peetre (Chapter 2) will allow an estimation in $[A_0, A_1]_{\theta,q}$ with $q = \infty$. The choice of $q = \infty$ suppresses the summation over the whole line when estimating the A_0 norm or the A_1 norm. This approach may be used with Besov spaces (see Lemma 27.3) or with Lorentz spaces (see Lemma 27.6). The use of Lorentz spaces in this setting was proposed by Meyer [MEY 99].

The role of real interpolation spaces with $q = \infty$ has been underlined as well in the setting of Kato's theorem for critical spaces of local measures (Theorem 17.1). Moreover, the role of Besov spaces or Lorents spaces may be viewed as borderline cases of the maximal L^pL^q regularity theorem. Indeed, this maximal regularity theorem states that $\|\int_0^t e^{(t-s)\Delta}\Delta f\ ds\|_{L^p_tL^q_x} \leq C(p,q)\|f\|_{L^p_tL^q_x}$ for $1 < p < \infty$ and $1 < q < \infty$. For the limit cases $p = 1$ or $p = \infty$, we may use Besov spaces and Lorentz spaces: $\|\int_0^t e^{(t-s)\Delta}\Delta f\ ds\|_{L^\infty_t\dot{B}^{s,\infty}_q} \leq C(q)\|f\|_{L^\infty_t\dot{B}^{s,\infty}_q}$ and $\|\int_0^t e^{(t-s)\Delta}\Delta f\ ds\|_{L^1_t\dot{B}^{s,1}_q} \leq C(q)\|f\|_{L^1_t\dot{B}^{s,1}_q}$, or, for $d/(d-1) < q < \infty$ and $1/q = 1/r - 1/d$, $\|\int_0^t e^{(t-s)\Delta}\sqrt{-\Delta}f\ ds\|_{L^\infty_tL^{q,\infty}} \leq C(q)\|f\|_{L^\infty_tL^{r,\infty}}$ and $\|\int_0^t e^{(t-s)\Delta}\sqrt{-\Delta}f\ ds\|_{L^1_tL^{q,1}} \leq C(q)\|f\|_{L^1_tL^{r,1}}$.

Notice that the last estimate was useful in proving Tsai's theorem [TSA 98] in Chapter 34.

3. Mixed initial values

Leray's weak solutions are provided by energy inequalities and weak compactness. Kato's mild solutions are provided by the Picard iteration scheme. There is a possibility of combining the two methods to prove the existence of weak solutions in new classes. For instance, Calderón [CAL 90] and Lemarié-Rieusset [LEM 98a] discussed the case of an initial value in $(L^p)^d$ with $2 < p < d$. Their idea was to split the initial value $\vec{u}_0$ into the sum of two divergence-free vector fields $\vec{v}_0$ and $\vec{w}_0$ with $\vec{v}_0$ in $(L^d)^d$ with a small norm and $\vec{w}_0 \in (L^2)^d$.

One may then compute by Kato's algorithm a solution $\vec{v} \in (\mathcal{C}_b([0,\infty), L^d))^d$ to the Navier–Stokes equations with initial value $\vec{v}_0$:

$$\begin{cases} \partial_t \vec{v} = \Delta \vec{v} - \mathbb{P}\vec{\nabla}.\vec{v} \otimes \vec{v} \\ \vec{\nabla}.\vec{v} = 0 \\ \lim_{t\to 0} \vec{v} = \vec{v}_0 \end{cases}$$

and, thereafter, prove by Leray's method the existence of a solution $\vec{w} \in \cap_{T>0}(L^\infty((0,T), L^2) \cap L^2((0,T), H^1))^d$ to the perturbated equations

$$\begin{cases} \partial_t \vec{w} = \Delta \vec{w} - \mathbb{P}\vec{\nabla}.(\vec{w} \otimes \vec{w} + \vec{v} \otimes \vec{w} + \vec{w} \otimes \vec{v}) \\ \vec{\nabla}.\vec{w} = 0 \\ \lim_{t\to 0} \vec{w} = \vec{w}_0 \end{cases}$$

This result is easily extended to a large class of singular initial values:

Theorem 35.1: (Singular initial values)
Let $\tilde{X}_r$ be the the closure of the test functions in the multiplier space X_r and let E_2 be the the closure of the test functions in the Morrey space L^2_{uloc}.
A) Let $\vec{u}_0 = \vec{v}_0 + \vec{w}_0$ with $\vec{\nabla}.\vec{v}_0 = \vec{\nabla}.\vec{w}_0 = 0$, $\vec{w}_0 \in (L_2)^d$ and $\vec{v}_0 \in (B^{-1+r,2/(1-r)}_{\tilde{X}_r})^d$ and $0 < r < 1$. Then, there exists a weak solution $\vec{u} \in \cap_{T>0} L^2((0,T),(E_2)^d)$ for the Navier–Stokes equations on $(0,\infty) \times \mathbb{R}^d$ so that $\lim_{t\to 0^+} \vec{u} = \vec{u}_0$.
B) If $d = 3$, the same conclusion holds when $\vec{u}_0 = \vec{v}_0 + \vec{w}_0$ with $\vec{\nabla}.\vec{v}_0 = \vec{\nabla}.\vec{w}_0 = 0$, $\vec{w}_0 \in (E_2)^3$ and $\vec{v}_0 \in (B^{-1+r,2/(1-r)}_{\tilde{X}_r})^3$ and $0 < r < 1$. Moreover, $\vec{u}$ may be chosen to be suitable, in the sense of Caffarelli, Kohn, and Nirenberg.

Proof: We first prove (A). The theory of mild solutions in Besov spaces of negative regularity (Chapter 20) allows us to compute by Kato's algorithm a solution $\vec{v}$ to the Navier–Stokes equations with initial value $\vec{v}_0$; this solution will be defined on a strip $(0,T^*) \times \mathbb{R}^d$ and satisfy the following regularities:
i) $\vec{v}$ is smooth on $(0,T^*) \times \mathbb{R}^d$
ii) for all $t \in (0,T^*)$, $\vec{v}(t,.)$ belongs to $(\mathcal{C}_0(\mathbb{R}^d))^d$
iii) $\sup_{0<t<T^*} \sqrt{t}\|\vec{v}(t,.)\|_\infty < \infty$
iv) $\lim_{t\to 0} \sqrt{t}\|\vec{v}(t,.)\|_\infty = 0$
v) $\vec{v}$ belongs to $(L^p((0,T^*), \tilde{X}_r))^d$ with $p = \frac{2}{1-r}$. (Notice that $L^p((0,T^*), \tilde{X}_r) \subset L^2((0,T^*), E_2)$).

We may now use Leray's method to prove the existence of a solution $\vec{w} \in (L^\infty((0,T^*), L^2) \cap L^2((0,T^*), H^1))^d$ to the perturbated equations

$$\begin{cases} \partial_t \vec{w} = \Delta \vec{w} - \mathbb{P}\vec{\nabla}.(\vec{w} \otimes \vec{w} + \vec{v} \otimes \vec{w} + \vec{w} \otimes \vec{v}) \\ \vec{\nabla}.\vec{w} = 0 \\ \lim_{t\to 0} \vec{w} = \vec{w}_0 \end{cases}$$

This has been proved in Chapter 21 (Theorem 21.3). Thus, we proved the existence of a solution at least for small times. To get a global solution is an easy task: when we have a solution $\vec{u}_N$ on $(0, T_N) \times \mathbb{R}^d$, we choose $S_N < T_N$ and we write $\vec{u}(S_N) = \vec{v}(S_N) + \vec{w}(S_N)$ where $\vec{v}(S_N)$ and $\vec{w}(S_N)$ are divergence free, $\vec{v}(S_N) \in (\mathcal{C}_0)^d$ and $\vec{w}(S_N) \in (L^2)^d$. Then, the approximation lemma (Chapter 12) allows us to split $\vec{v}(S_N)$ into $\vec{\alpha}_N + \vec{\beta}_N$ where $\vec{\alpha}_N$ and $\vec{\beta}_N$ are divergence free, $\vec{\alpha}_N \in (\mathcal{C}_0)^d$ with an arbitrarily small norm and $\vec{\beta}_N \in (L^2)^d$. We may compute by Kato's algorithm a solution $\vec{v}_{N+1}$ to the Navier–Stokes equations on $(S_N, T_N + 2)$ with initial value $\vec{\alpha}_N$. We then use Leray's method to prove the existence of a solution $\vec{w}_{N+1} \in (L^\infty((S_N, T_N + 2), L^2) \cap L^2((S_N, T_N + 2), H^1))^d$ to the perturbated equations (with perturbation $\mathbb{P}\vec{\nabla}.(\vec{w}_{N+1} \otimes \vec{v}_{N+1} + \vec{v}_{N+1} \otimes \vec{w}_{N+1}))$. If we define $\vec{u}_{N+1}$ by $\vec{u}_N$ on $(0, S_N]$ and by $\vec{v}_N + \vec{w}_N$ on $(S_N, T_N + 2)$, we still have a solution of the Navier–Stokes equations: by weak continuity of $t \mapsto \vec{u}_N(t, .)$, we may differentiate by pieces the distribution $\vec{u}_{N+1}$.

We now prove (B). This is the very same proof as for (A), following the lines of Chapter 32 and 33. We may use a smoother patching in the definition of the global solution, using the suitability of the solution and Serrin's uniqueness theorem to enforce $\vec{u}_N$ and $\vec{v}_{N+1} + \vec{w}_{N+1}$ to coincide not only at $t = S_N$ but on a strip $[S_N, S_N + \epsilon)$ (see Chapter 33). □

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