COMPUTER METHODS IN APPLIED MECHANICS AND ENGINEERING 9 (1976) 153–164

SKEW UPSTREAM DIFFERENCING SCHEMES FOR PROBLEMS INVOLVING FLUID FLOW

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Received 16 May 1975

When the convection terms in the conservation equations are approximated using upstream or upwind differences, a large error may result in the regions of the flow where the grid line and velocity directions are not closely aligned. Two new schemes are introduced which reduce this error but retain the advantages of upstream differencing. The schemes are particularly accurate in problems where diffusion and convection play dominant roles in establishing the distribution of the dependent variable. Solutions to two example problems illustrate substantial improvement in accuracy resulting from the new schemes.

1. Introduction

Engineers are often interested in obtaining approximate solutions to complex multidimensional problems involving fluid flow, heat and mass transfer. Finite difference methods are frequently chosen as the only viable means. When the Reynolds (or Peclet) number of the flow is large, upstream differencing of the convective terms in the conservation equations is often used to obtain these solutions simply and at reasonable cost.

A companion paper [1] illustrates that this leads also to accurate solutions when it is convection that is mainly responsible for establishing the streamwise distribution of the dependent variable. If high frequency transients occur, if the grid Reynolds or Peclet number (based on the grid dimension in the flow direction) is of order five or smaller, and if source terms take certain forms, then substantial errors may arise from the upstream difference approximation. For may problems of practical interest these conditions are not overly restrictive, that is the grid Peclet number *is* high, source terms are either nonexistent or *are* of a suitable form, and either the transients are slow enough or a steady-state solution is sought in which case inaccuracies in the transient solution are irrelevant. However, there is one additional condition, namely that, if there are large gradients in the dependent variable in the cross-flow direction, the grid line direction and flow direction should be closely aligned. For boundary layer problems, this is also not restrictive. However, in recirculating flow problems it is the violation of this condition which often leads to inaccuracies and valid criticism of the upstream difference approximation.

This paper presents two new upstream schemes which are designed to remove some of these restrictions. Because so many misunderstandings over the use of upstream differences have arisen in the past, it is perhaps appropriate to specify first what restrictions are *not* removed. First, the new schemes will be no more appropriate for transient problems than those presently in use. Thus the meteorologist interested in large-scale transient advection problems is warned that he will find little help here. However, quasi-steady problems can be handled (as with the present upstream scheme); these include some urban-scale atmospheric problems and a majority of the problems arising in industry. Second, the restrictions on the source terms remain. However, lest a potential user decides at this point to turn to another type of difference scheme to avoid this source-term restriction, it is fair to warn that other schemes are also subject to large errors in the presence of certain types of source terms; it is just that these restrictions are often not spelled out.

The first scheme, and simplest scheme, presented in this paper also does *not* remove the restriction that the grid Peclet or Reynolds number be large. However, it *does* lead to a large reduction in the error arising from the fact that the flow cuts across the grid at a large angle. This scheme is called a Skew Upstream Differencing Scheme (abbreviated as SUDS) because the representation of the convection terms depends on the angle which the flow makes to the grid line direction.

The second scheme is more complex, but the error arising from the flow cutting across the grid is more accurately removed and the restriction on grid Peclet number is removed. In the latter respect, it bears strong similarities to hybrid [2], [3] or upstream-weighted [4] schemes. It is called a Skew Upstream Weighted Differencing Scheme (abbreviated as SUWDS).

It is also convenient here to introduce three additional abbreviations. When the convection and diffusion terms in the finite difference representation of the conservation equation are approximated using central differences, the name Central Difference Scheme (or CDS) is applied. When the convection terms are represented by upstream differences and the diffusion terms by central differences, the name Upstream Difference Scheme (or UDS) is applied. Reference is also made to the Upstream Weighted Difference Scheme (or UWDS) described in [4].

The remainder of the paper is laid out as follows: 1) a general conservation equation, which is to be approximated by finite differences, is stated, 2) the simpler of the two new schemes is derived, 3) the second more complex scheme is derived, 4) both of these are applied to two simple two-dimensional problems in which the flow cuts across the grid at large angles, and 5) a preliminary report on the application of the simpler scheme to a complex problem is made.

1. The conservation equation

A two-dimensional form of the equation, describing the conservation of any quantity Φ, can be expressed (after certain assumptions about the form of the diffusion terms) as

$$\underbrace{\frac{\partial}{\partial t}(\rho\Phi)}_{\mathrm{I}} + \underbrace{\frac{\partial}{\partial x}(\rho u\Phi)}_{\mathrm{IIa}} + \underbrace{\frac{\partial}{\partial y}(\rho v\Phi)}_{\mathrm{IIb}} = \underbrace{\frac{\partial}{\partial x}\left(\Gamma\frac{\partial\Phi}{\partial x}\right)}_{\mathrm{IIIa}} + \underbrace{\frac{\partial}{\partial y}\left(\Gamma\frac{\partial\Phi}{\partial y}\right)}_{\mathrm{IIIb}} \underbrace{-\dot{S}\Phi}_{\mathrm{IV}} + \underbrace{\dot{P}}_{\mathrm{V}}. \tag{1}$$

This is the equation given by Gosman et al. [5] except for terms IV and V; the form of equation (1) has been recommended by Patankar [6]. The respective velocities in the x and y directions are u and v, $\dot{S}\Phi$ is a linear source term, $\dot{P}$ represents a production term for Φ, and Φ may be vorticity, specific enthalpy, mass fraction, specific kinetic energy etc. The corresponding diffusion coefficients Γ are generally different for each case.

The above equation applies to an infinitesimal control volume but is often the beginning point for writing finite difference equations. An alternative, and perhaps more instructive, beginning point is the integral form of equation (1) expressing conservation of Φ on the average over some finite time interval Δt and over some finite volume G in space. This equation can be obtained by integrating equation (1) from t to $t + \Delta t$ and over the volume G as in [1].

Using equation (1) as the starting point, the derivatives are replaced by finite differences; for example the convection terms could be replaced by one-sided (upstream) differences and the diffusion terms by central differences. Starting from the integral equations, the same final algebraic equations can be obtained, but the simplification requires (a) that the shape of the Φ profile be assumed in order that fluxes across surfaces of the control volume and (b) that the total generation and accumulation rate of Φ within the control volume can be estimated in terms of values of Φ at grid points. In what follows no explicit distinction according to starting points (or viewpoints) will be made; however, whenever profile assumptions are discussed, the latter is implied.

2. Skew upstream differencing scheme

It has been argued previously [1] that when convection dominates in establishing the spatial distribution of the conserved quantity Φ, the gradient in Φ in the streamwise direction vanishes, and the cross-flow gradient depends on upstream boundary conditions. These properties should be reflected in the profile shape assumed when estimating the transport of Φ across control-volume surfaces. For problems in which, over a length scale comparable to grid dimensions, convection dominates and the variation in Φ normal to the flow direction can be approximated by a linear equation, the local profile shape should be

$$\Phi \simeq \phi = C_1 + C_2 n = C_1 + C_2\left(y \frac{u}{V} - x \frac{v}{V}\right), \tag{2}$$

where ϕ is the approximation to the exact distribution Φ, n is the distance normal to the flow (see figure 1). To find Φ at a particular point, C_1 and C_2 are to be evaluated at locations where Φ is known *upstream* of the point of interest, and equation (2) is then used to obtain the required estimate.

Assumed Φ distributions which correspond, or lead to, the upstream approximation of the convection terms have the properties of equation (2) only when cross-flow gradients in Φ vanish (i.e. $C_2 = 0$ in (2)) or when the flow moves along one of the grid line directions. The error due to the upstream difference, in situations where large cross-flow gradients exist *and* the flow cuts across the grid at a large angle, can be attributed to a failure of the scheme to embody equation (2). The SUDS corrects this deficiency by using equation (2) directly in estimating fluxes by convection, as will now be seen.

Attention is focused on the control volume in figure 1 centered at point P (or equivalently at point (i, j), where i and j number the grid lines in the x and y directions, respectively). Moreover, N, NW, ... designate grid points north, north-west, ... of P, respectively. Equation (2) is now used to estimate the flux of Φ by convection across the western face of the control volume. First, this equation is rewritten as

$$\phi = C_1' + C_2'\left(y' \frac{u_{\mathrm{w}}}{V_{\mathrm{w}}} - x' \frac{v_{\mathrm{w}}}{V_{\mathrm{w}}}\right), \tag{3}$$

where x' and y' are measured from w, and it is assumed that u and v at w prevail in the vicinity of w. The constants C_1' and C_2' are to be determined from the values of Φ (or ϕ if only approximate values at the grid points are known) *upstream* of w. If a restriction is made that the two points be chosen only from the six surrounding w, the constraints are in general

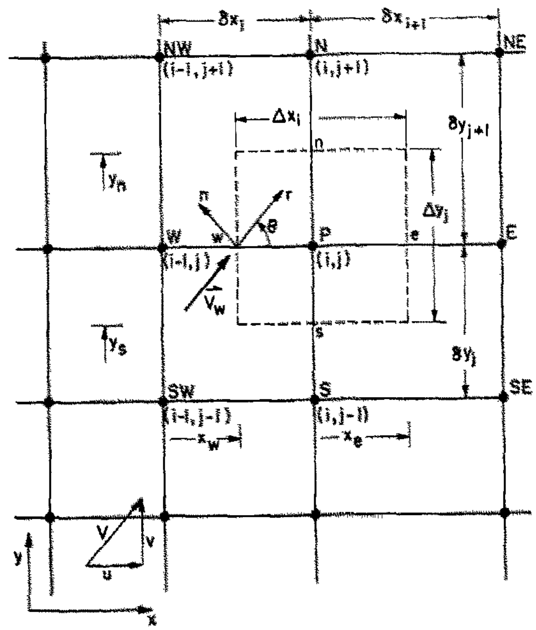

Fig. 1. A segment of the calculation region showing grid lines, a control volume (dotted lines) and notation.

$$\phi = \phi_{lw,j} \text{ at } x' = -\tfrac{1}{2}S_{uw}\,\delta x_i,\ y' = 0, \tag{4a}$$

$$\phi = \phi_{lw,mw} \text{ at } x' = -\tfrac{1}{2}S_{uw}\,\delta x_i,\ y' = -S_{vw}\,\delta y_{kw}, \tag{4b}$$

where $lw = i - \frac{1}{2}(1 + S_{uw})$, $mw = j - S_{vw}$, and $kw = j + \frac{1}{2}(1 - S_{vw})$. S_{uw} has a magnitude of unity and the sign of u_w; similarly S_{vw} is unity with the sign of v_w. Referring to figure 1, when

$$|v_w|/|u_w| > \delta y_{kw}/(\delta x_i/2),$$

the restriction that the two points be chosen from the six adjacent to w means that ϕ_w is obtained by extrapolation rather than interpolation, and the extrapolated value can be in considerable error if $|u|/|v| \to 0$. Therefore, when $|v_w|/|u_w| > \delta y_{kw}/(\delta x_i/2)$, ϕ is taken as $\phi_{lw,mw}$. The estimate of the convected flux through the w face therefore becomes

$$\begin{aligned} \rho u_w \Delta y_j \phi_w &= 2(L_w - K_w)\phi_{lw,j} + (2K_w)\phi_{lw,mw} \\ &= (L_w - K_w)(1 + S_{uw})\phi_W + (L_w - K_w)(1 - S_{uw})\phi_P + 2K_w\phi_{lw,mw}, \end{aligned} \tag{5}$$

where $L_w = \frac{1}{2}\rho_w u_w \Delta y_j$ is one-half the mass flow through the w face, and

$$K_w = S_{uw} \min\left[|L_w|,\ \rho_w \frac{\Delta y_j}{\delta y_{kw}} |v_w| \delta x_i/4\right] \tag{6}$$

It should also be noted that, if desired, one may write $\phi_{lw,mw}$ explicitly in terms of surrounding grid values as

$$\phi_{lw,mw} = (1+S_{uw})(1+S_{vw})\tfrac{1}{4}\phi_{SW} + (1+S_{uw})(1-S_{vw})\tfrac{1}{4}\phi_{NW} + (1-S_{uw})(1-S_{vw})\tfrac{1}{4}\phi_N + (1-S_{uw})(1+S_{vw})\tfrac{1}{4}\phi_S. \tag{7}$$

A linear profile is assumed in approximating the diffusive fluxes (although a weighting factor could be used to increase the accuracy [4]), so that the flux of Φ across the western boundary by diffusion is estimated by

$$-\left[\Gamma\frac{\partial\phi}{\partial x}\right]_w \Delta y_j = -\Lambda_w\phi_P + \Lambda_w\phi_W, \qquad \Lambda_w = \Gamma_w\Delta y_j/\delta x_i. \tag{8}$$

The total estimated flux by convection and diffusion J_w is

$$J_w = \rho_w u_w \Delta y_j \phi_w - \left[\Gamma_\phi\frac{\partial\phi}{\partial x}\right]_w \Delta y_j. \tag{9}$$

This is obtained by adding equations (5) and (8). The expressions for the total flux across the east J_e, north J_n and south boundaries J_s can similarly be found.

These flux expressions replace previous flux expressions in the finite difference representation of the conservation equation [4], [5]. Transient and source terms may also be present. The solution to the equation for each ϕ can now be obtained by the usual methods.

For simplicity here, the steady solution to the conservation equation is sought. For this case

$$J_w + J_s - J_e - J_n = -\dot{S}\phi_P\Delta x_i\Delta y_j + \dot{P}\Delta x_i\Delta y_j. \tag{10}$$

Substituting the flux expressions, one obtains

$$a_P\phi_P = a_E\phi_E + a_W\phi_W + a_N\phi_N + a_S\phi_S + 2K_w\phi_{lw,mw} + 2K_s\phi_{ls,ms} - 2K_e\phi_{le,me} - 2K_n\phi_{ln,mn} + \dot{S}\phi_P\Delta x_i\Delta y_j - \dot{P}\Delta x_i\Delta y_j, \tag{11}$$

where

$$a_E = \Lambda_e - (L_e - K_e)(1-S_{ue}), \quad \Lambda_e = \Gamma_e\Delta y_j/\delta x_{i+1}, \tag{11a}$$

$$a_W = \Lambda_w + (L_w - K_w)(1+S_{uw}), \quad \Lambda_w = \Gamma_w\Delta y_j/\delta x_i,$$

$$a_N = \Lambda_n - (L_n - K_n)(1-S_{vn}), \quad \Lambda_n = \Gamma_n\Delta x_i/\delta y_{j+1}, \tag{11c}$$

$$a_S = \Lambda_s + (L_s - K_s)(1+S_{vs}), \qquad \Lambda_s = \Gamma_s\Delta x_i/\delta y_j, \tag{11d}$$

$$a_P = a_E + a_W + a_N + a_S + 2K_w + 2K_s - 2K_e - 2K_n, \tag{11e}$$

$$K_n = S_{vn}\min\left[|L_n|, \rho_n\frac{\Delta x_i}{\delta x_{kn}}|u_n|\delta y_{j+1}/4\right], \quad kn = i + \tfrac{1}{2}(1-S_{un}), \tag{11f}$$

where K_w has been defined previously in equation (6). Similar expressions follow for K_e and K_s. The continuity equation has been used in obtaining the expression for a_P. The resulting set of equations (one for each control volume) can be solved by an iterative or direct method.

3. Skew upstream weigthed difference scheme (SUWDS)

The differencing scheme described above will be most accurate for problems where convection dominates in establishing the distribution of Φ in the flow direction. When the grid Peclet number is of order 1, the scheme may be considerably in error. If the problem is one dimensional in the x direction, for example, the flux of Φ across the western boundary of the control volume in figure 1 by convection will fall between $\rho_w u_w \phi_W \Delta y_j$ and $\frac{1}{2}\rho_w u_w [\phi_W + \phi_P] \Delta y_j$ for $u_w > 0$. Similarly, the diffusive flux will be overestimated by the assumed linear profile.

Upstream weighted or hybrid differencing schemes have been developed [2], [3], [4] which improve the flux estimates, independent of the relative importance of the convective and diffusion terms, when the flow moves nearly parallel to grid lines. A new upstream weighted scheme is now developed which is accurate even when the flow crosses the grid lines at large angles.

If it is assumed that the velocity components, density etc. are constant in the vicinity of the western face of the control volume in figure 1 and equal to the values at w, the conservation equation becomes *locally*

$$\rho_W u_w \frac{\partial \phi}{\partial x} + \rho_W v_W \frac{\partial \phi}{\partial y} = \Gamma_w \frac{\partial^2 \phi}{\partial x^2} + \Gamma_w \frac{\partial^2 \phi}{\partial y^2}. \tag{12}$$

A solution to this equation is

$$\phi = C_1 + C_2 \left(y' \frac{u_w}{V_w} - x' \frac{v_w}{V_w}\right) + C_3 \exp\left(\frac{\rho_w u_w x'}{\Gamma_w} + \frac{\rho_w v_w y'}{\Gamma_w}\right), \tag{13}$$

where x' and y' are measured from w and $V_w = \sqrt{u_w^2 + v_w^2}$

This is then the assumed shape of the ϕ profile in the vicinity of the w face of the control volume and is used as the interpolation formula for estimating the convective and diffusive fluxes across that face. The constants C_1, C_2 and C_3 are determined from three values of ϕ at adjacent grid points, two of which lie upstream of w, and one lying downstream. Using the notation of the previous section, these are

$$\phi = \phi_P \qquad \text{at } x' = \delta x_i/2, \qquad y' = 0 \tag{14a}$$

$$\phi = \phi_W \qquad \text{at } x' = -\delta x_i/2, \qquad y' = 0 \tag{14b}$$

$$\phi = \phi_{lw,mw} \qquad \text{at } x' = -S_{uw}\delta x_i/2, \qquad y' = -S_{vw}\delta y_{kw}. \tag{14c}$$

Once C_1, C_2 and C_3 have been obtained, the values of ϕ_w and $(\partial\phi/\partial x)_w$ can be obtained directly from the interpolation equation. Making the approximation that ϕ_w and $(\partial\phi/\partial x)_w$ prevail over the western face, the convected and diffusive fluxes become, respectively,

$$\rho_w u_w \Delta y_j \phi_w = 2L_w\{(\tfrac{1}{2} + A'_w)\phi_W + (\tfrac{1}{2} - A'_w - B'_w)\phi_P + B'_w \phi_{lw,mw}\}, \tag{15a}$$

$$-\left(\Gamma \frac{\partial \phi}{\partial y}\right)_w \Delta y_j = -\Lambda_w\{(1 - A''_w - B''_w)\phi_P - (1 - A''_w)\phi_W + B''_w \phi_{lw,mw}\}, \tag{15b}$$

where

$$A'_{\mathrm{w}} = (\cosh(P_{x\mathrm{w}}/2) - 1)A_{\mathrm{w}}, \qquad B'_{\mathrm{w}} = (\cosh(P_{x\mathrm{w}}/2) - 1)B_{\mathrm{w}},$$

$$A''_{\mathrm{w}} = S_{u\mathrm{w}}(\sinh(P_{x\mathrm{w}}/2) - P_{x\mathrm{w}})A_{\mathrm{w}}, \qquad B''_{\mathrm{w}} = S_{u\mathrm{w}}(\sinh(P_{x\mathrm{w}}/2) - P_{x\mathrm{w}})B_{\mathrm{w}},$$

$$A_{\mathrm{w}} = \frac{\beta_{\mathrm{w}} - |\alpha_{\mathrm{w}}/2|\,(S_{u\mathrm{w}} + 1)}{|\alpha_{\mathrm{w}}|\,\exp(-P_{x\mathrm{w}}/2)(1 - \exp(-P_{y\mathrm{w}})) + 2|\beta_{\mathrm{w}}|\,\sinh(P_{x\mathrm{w}}/2)},$$

$$B_{\mathrm{w}} = \frac{|\alpha_{\mathrm{w}}|}{|\alpha_{\mathrm{w}}|\,\exp(-P_{x\mathrm{w}}/2)(1 - \exp(-P_{y\mathrm{w}})) + 2|\beta_{\mathrm{w}}|\,\sinh(P_{x\mathrm{w}}/2)},$$

$$\alpha_{\mathrm{w}} = v_{\mathrm{w}}\,\delta x_i/V_{\mathrm{w}}, \quad \beta_{\mathrm{w}} = u_{\mathrm{w}}\,\delta y_{k\mathrm{w}}/V_{\mathrm{w}}, \quad P_{x\mathrm{w}} = |u_{\mathrm{w}}|\delta x_i/\Gamma_{\mathrm{w}}, \; P_{y\mathrm{w}} = |v_{\mathrm{w}}|\delta y_{k\mathrm{w}}/\Gamma_{\mathrm{w}}.$$

The $P_{x\mathrm{w}}$, and $P_{y\mathrm{w}}$ represent Peclet numbers based on local velocities and grid dimensions in the x and y directions, respectively. Similar expressions for the fluxes across the other faces can be derived or obtained directly from a rotation of the coordinate system; A_{w}, B_{w} etc. and corresponding values for the other faces can be interpreted as weighting factors. The solution of transient or steady-state problems then proceeds following the usual steps. It should be noted that, in the special case when the flow is aligned with one of the grid line directions, the above equations reduce to the hybrid scheme of [2].

There is some question about the practicality of the method due to the computational time required for the evaluation of exponential functions. Some success has been achieved by using very crude approximations of these functions which lead to the correct values of A'_{w}, B'_{w}, A''_{w}, B''_{w} (and corresponding parameters for the other faces) in the limiting cases of large and small values of P. However, in this paper attention is confined to results achieved only by accurately computing these functions. This will avoid confusion and also give the "best" available flux estimates.

Again it should be pointed out that the flux expressions have been calculated from an interpolation equation which is valid when ϕ is transported by diffusion and convection alone. When the transient or source terms in the transport equation become important and convection also remains important, the interpolation equation is subject to error. In these cases, the utilization of the above equations for the SUWDS, without reducing the computational cost by approximating the weighting factors, would be difficult to justify.

4. Application of results

Example 1. Transport of a step change in ϕ in a uniform velocity field

Calculations are carried out in the domain shown in figure 2. The velocity is uniform and parallel to the solid line (cutting across the grid) everywhere in the region, with boundary values $\Phi^* = 0$ below this line and $\Phi^* = 1$ above. Here Φ^* is a nondimensional value of Φ. If the solid line intersects a grid point on the boundary, a value of $\Phi^* = 0.5$ is assigned to this point. There are no source terms for Φ, and the steady state solution is sought. This problem has been discussed previously by Wolfstein [5].

The differential equation for Φ, which could be temperature or concentration for example, is nondimensionalized using the grid dimension $\Delta x = \Delta y$ as the reference length and the velocity $V =$

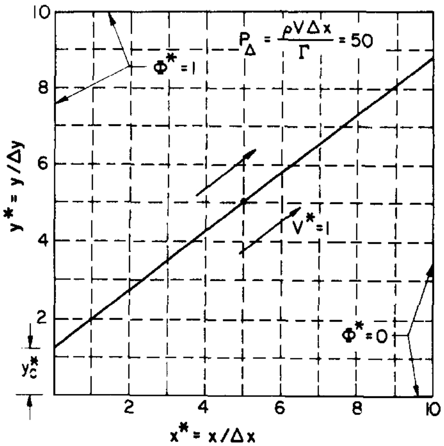

Fig. 2. Grid lines for Example 1 involving the convection and diffusion of a step change in ϕ in a constant-velocity region. $P_\Delta = 50$.

$\sqrt{u^2+v^2}$ as the reference velocity. The nondimensional quantities are designated by stars (x^*, y^*, u^*, v^*). The parameter which arises in the resulting equation is $1/P_\Delta = \Gamma/\rho V \Delta x$, the inverse of a Peclet number based on V and Δx. Solutions were carried out for $P_\Delta = 50$. The solutions were obtained using a Gauss-Seidel type of iteration, terminating the process when the maximum value of the change in ϕ^* (where ϕ^* is the finite-difference approximation of the exact solution, Φ^*) between two iteration levels was less than 10^{-6}. All calculations were carried out on an 11×11 grid for various angles of the velocity from the grid lines. The solid line in figure 2 always passed through the center of the domain.

The same type of iterative solutions were not possible when central differences were used to approximate the convective terms. If the solution were found by other means for this value of P_Δ, it is expected that the results would not be in general satisfactory.

Calculations were also carried out using an upstream approximation of the convective terms and central differences for the diffusion on terms (UDS). Similarly, calculations using the SUDS and the SUWDS as developed in this paper are reported. The results are plotted in figure 3 showing the variation in ϕ^* along the line of constant x falling through the center of the region.

The solid lines were obtained by assuming that diffusion *in the flow direction* is much less important than convection. The equation for Φ is therefore

$$\rho V \frac{\partial \Phi}{\partial r} = \Gamma \frac{\partial^2 \Phi}{\partial n^2}. \tag{16}$$

This is equivalent to a transient conduction problem in which time is replaced by r/V. The solution to this problem, subject to the step change initial condition [7] and transformed back to the dimensionless (x^*, y^*) coordinate system, is

$$\Phi^* = 0.5 \left[1 + \operatorname{erf}\left\{\frac{(\sqrt{P_\Delta}\,(y^* - y_c^*)u^* - x^* v)}{2\sqrt{(y^* - y_c^*)v^* + x^* u^*}}\right\}\right], \tag{17}$$

where erf is the error function and y_c^* is defined in figure 2. This solution should be valid far enough removed from the downstream boundaries and for $P_\Delta \gg 1$.

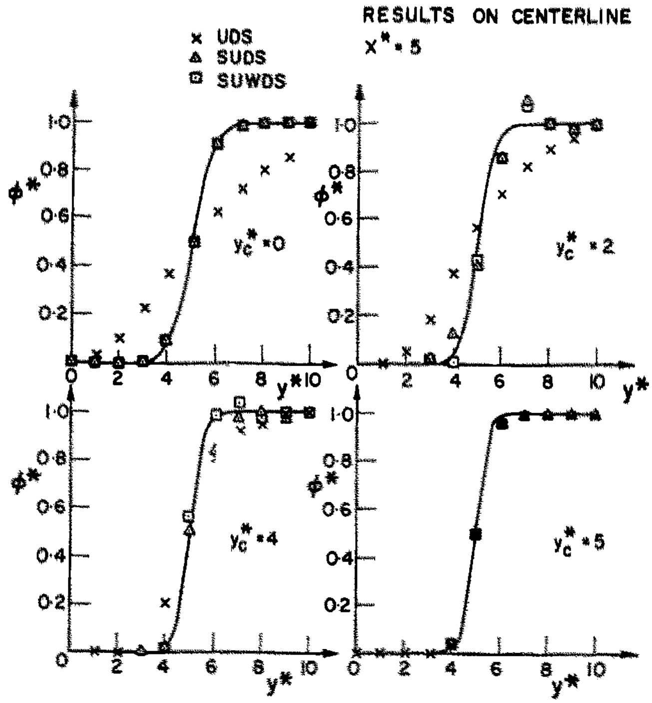

Fig. 3. Solutions to Example 1 using an upstream differencing scheme (UDS), a skew upstream differencing scheme (SUDS) and a skew upstream weighted differencing scheme (SUWDS).

A comparison of this equation with the numerical results for $y_c^* = 0, 2, 4, 5$ (or the velocity making angles to the horizontal of between $\pi/4$ and 0) shows how the accuracy of the schemes are affected by the angle which the flow makes to the grid lines. For flow along the x^* direction ($y_c^* = 5$) all three schemes convect and diffuse the initial step function accurately. As y_c^* decreases to 0 and the angle of approach becomes $\pi/4$, the UDS results in a smearing of the step, as if the actual transport coefficient Γ were much larger. At $y_c^* = 0$ both the SUDS and the SUWDS are in almost perfect agreement with the solid curves. For intermediate angles the results are everywhere better than for the UDS. At $y_c^* = 2$ a small bump develops in the profiles causing the calculated value to exceed the maximum boundary value. No stability problems were encountered in obtaining iterative solutions, although the number of iterations required to obtain convergence was larger for the SUDS and SUWDS than for the UDS.

Example 2. Conduction in a rotating shaft

The problem of conduction in a rotating shaft has been used previously for testing differencing schemes [8], [4]. A square section of the northeast quadrant of the shaft is chosen as shown in figure 4 for the computational experiment. This is divided into equal spaces using 11 grid lines in each direction, and boundary conditions of the exact solution were applied. Because of symmetry there is a one-dimensional net transport of ϕ by diffusion in the radial direction. The selection of the grid in the (inappropriate) (x, y) coordinate system makes the problem appear fully two dimensional.

Problems involving constant and variable diffusion coefficients Γ have been previously solved [8]. Attention is focused here on the case in which $\Gamma \propto (x^2 + y^2)^{-2}$ so that the exact solu-

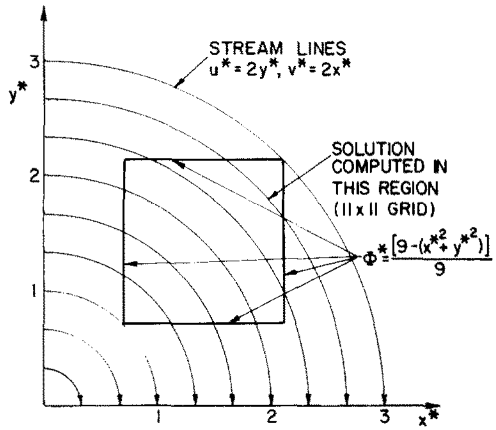

Fig. 4. Solution region for radial diffusion in a rotating geometry, Example 2.

tion for a nondimensional Φ^* using boundary conditions of unity at $(x^*)^2 + (y^*)^2 = 1$ and zero at $(x^{*2}) + (y^{*2}) = 3$ is $\Phi^* = [9 - (x^{*2}) + (y^{*2})]/8$. In the numerical experiment the parameter in the problem is $P_L = \rho V_r L_r / \Gamma_r$, where the subscript r denotes the reference values (of velocity, length, and Γ) used in nondimensionalizing the equation for Φ^*. Numerical solutions for the steady-state distribution of Φ^* were obtained as in example 1 with the same convergence criterion. Boundary conditions from the exact solution were applied.

Figure 5 shows the percentage error in the calculated value of ϕ^* at the center of the domain. Because of the form of the exact solution, the central difference scheme (CDS) leads to very precise results up to $P_L \simeq 2$. Beyond this value, where convection is more important in the transport of ϕ than diffusion, this scheme fails to converge.

Since the flow crosses the grid lines at large angles over most of the domain, the UDS produces much larger errors, the error at the center asymptotically approaching about 3 percent as $P_L \to \infty$. However, no problems with convergence are encountered. The best features of the UDS and CDS are combined in the upstream weighted differencing scheme (UWDS). These results correspond to

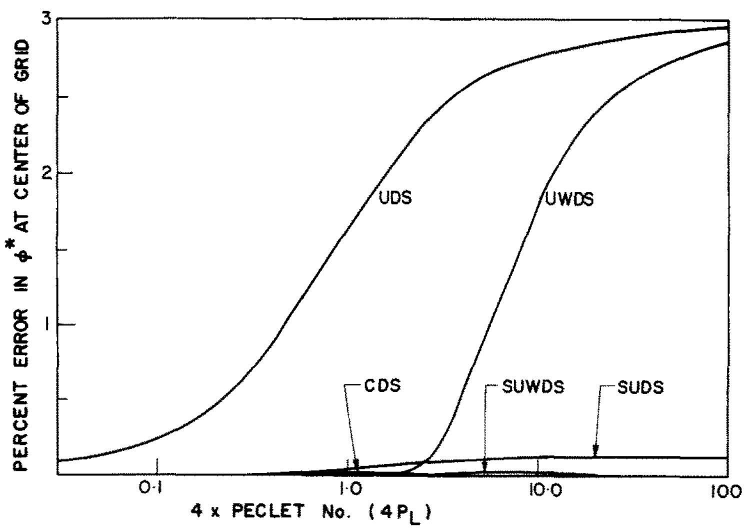

Fig. 5. The percentage error (between exact and numerical solution) at the mid-point of the solution region.

the (α_i, β_j) scheme in [4]. At intermediate P_L the results are improved, but the error as $P_L \to \infty$ must be the same as for the UDS.

The two new schemes introduced above decrease the error greatly. For large P_L, the error at the center approaches 0.13 percent for the SUDS. The SUWDS has a maximum error of about 1/10th this amount at $P_L \simeq 2$, but the error decreases rapidly to zero for larger and smaller P_L. No stability problems were encountered with the new schemes; a Gauss-Seidel iteration technique was used to obtain all solutions.

Example 3. Recirculation flow problem

The examples already considered are very simple (and therefore "clean") problems. The schemes proposed are intended for much more complex problems involving particularly recirculating flow. The simpler (SUDS) scheme proposed above has now been implemented at this University by J. Militzer, and it is appropriate to discuss briefly his findings. Details of his experiments and analysis will appear in his Ph.D. dissertation and a subsequent publication.

The problem was to predict the interaction of two parallel two-dimensional (slot) jets issuing from a plane wall into a semi-infinite, quiescent fluid region. The fluids in the jets and receiving region have the same properties and are at the same temperature. There is a mutual attraction of the jets because both compete to entrain the fluid between them, so that they merge at some distance normal to the wall, forming a single jet. There is a recirculation zone between the jets and a sharp curvature of their centerlines. The resulting velocity distributions were determined experimentally.

A two-dimensional finite difference scheme, using upstream differencing of the convective terms, was used to predict the velocities. The (k, ϵ)-turbulence model was used [9]. The major features of the solution were correct, but the magnitudes of the velocities were rather poorly predicted; the jets behaved as if the viscosity of the fluid in certain regions of the flow was very high. Since (1) the flow did not move along the grid lines, and (2) were large gradients in the dependent variables in the cross-flow direction, this behaviour is not surprising [1]. Applying the SUDS to the equations of motion (but retaining upstream differences in the turbulence equations) resulted in greatly improved predictions with slightly greater computer-storage requirements, but with smaller processing times.

5. Discussion

The upstream schemes derived and tested here are successful in the sense that solutions were readily obtainable (no stability problems) and the accuracy was greatly improved. However, it is also instructive to describe one of the several *previous* attempts which "failed".

Referring to figure 1, and focusing attention on the determination of the flux by convection through the w-surface, one wishes to determine the value of Φ_w. Recognizing that the upstream value will prevail when the grid Peclet number is high, the velocity vector through w is extended upstream until it intersects one surface of the box SW - W - NW - N - P - S - SW. In figure 1 this intersection would occur on the SW - W segment of the box. A linear interpolation between W and SW could be used to estimate Φ at the intersection point, and this could be used as the estimated value of Φ convected across the w-surface. Such a procedure leads to a finite difference equation which is unstable at high grid Peclet numbers except when very slow and expensive solution procedures are used. However, solutions so obtained have been found to be accurate at high Peclet numbers. It will be noted that this interpolation procedure differs from that proposed in equation (2) in the sense that a component of the linear interpolation falls in the flow direction. This and other experience has led to the conclusion that, at high grid Peclet numbers, linear interpolations should be confined to the cross-flow direction to obtain stable schemes. This is also physically reasonable [1].

6. Summary

Upstream differencing schemes are particularly useful when convection and diffusion are primarily responsible for establishing the spatial distribution of the dependent variable. Former upstream schemes suffered a loss in accuracy, however, when the flow cut across the grid at large angles. Two new schemes are developed in this paper which largely remove this source of error. This is illustrated quantitatively in two example problems, and the results of other numerical experiments are described.

Acknowledgements

The author is grateful to Mr. W. Hallett who assisted in developing an earlier "skew" scheme which provided background and experience for developing the schemes in this paper. Discussions with Dr. S.V. Patankar and Dr. K.E. Torrance were very helpful. The financial support of the National Research Council of Canada is gratefully acknowledged.

Note added in proof

The authors' attention has been drawn to a scheme [10] which resembles the SUWDS described herein, and which was developed simultaneously and independently. This scheme appears to be simpler to apply, but has the disadvantage of being non-conservative [11].

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