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REVIEW ARTICLE

Topology optimization approaches

A comparative review

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Abstract Topology optimization has undergone a tremendous development since its introduction in the seminal paper by Bendsøe and Kikuchi in 1988. By now, the concept is developing in many different directions, including "density", "level set", "topological derivative", "phase field", "evolutionary" and several others. The paper gives an overview, comparison and critical review of the different approaches, their strengths, weaknesses, similarities and dissimilarities and suggests guidelines for future research.

Keywords Structural optimization · Topology optimization · Density methods · Level set methods · Phase field methods · Topological derivatives

1 Introduction

Topology optimization gives answers to the fundamental engineering question: how to place material within a prescribed design domain in order to obtain the best structural performance? The concept was initiated for mechanical design problems but has spread to a wide range of other physical disciplines, including fluids, acoustics, electromagnetics, optics and combinations thereof. The method builds on repeated analysis and design update steps, mostly guided by gradient computation.

A number of prior developments within homogenization theory and numerical optimization methods provided the foundation for the seminal paper on numerical topology optimization by Bendsøe and Kikuchi (1988). Since the original "homogenization approach to topology optimization" the concept has developed in a number of different directions, including: a) density (Bendsøe 1989; Zhou and Rozvany 1991; Mlejnek 1992); b) level set (Allaire et al. 2002, 2004; Wang et al. 2003); c) topological derivative (Sokołowski and Zochowski 1999); d) phase field (Bourdin and Chambolle 2003); e) evolutionary approaches (Xie and Steven 1993) and several others. The density and evolutionary approaches use simple element or nodal-based design variables whereas the level set approach, which in 2D must be combined with topological derivatives, uses shape derivatives for the development of the optimal topology. Lately, hybrid approaches have appeared, such as level set approaches that use shape derivatives for design updates but, somewhat against the original definition of the level set concept, do allow for introduction of holes without the use of topological derivatives (Yamada et al. 2010). Also, filtered density fields used in recent projection techniques (Guest et al. 2004; Sigmund 2007; Xu et al. 2010; Wang et al. 2011) are becoming more and more similar to the level set function used in the level set approaches.

Based on above observations, it seems timely to perform a comparative and critical review of topology optimization approaches. How should different approaches by

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categorized? What are the strengths and weaknesses of various approaches? How efficient are the various approaches? Which approaches show the biggest potential and widest applicability? May there be possibilities for further merging of methods for the benefit of speed of convergence? Are there directions of developments which are missing?

The paper will try to give answers to the questions above and hopefully form a basis for future nomenclature, discussions and developments in the exciting and rewarding area of topology optimization.

The developments discussed in this paper have mainly taken part after the comprehensive review of topology optimization by Bendsøe and Sigmund (2004). Since then the method's popularity, as well as the number of researchers pursuing different directions, has exploded and hence it is an almost impossible task to perform a comprehensive review that covers all aspects and branches. Therefore, the aim of the present paper is not to perform a comprehensive review but rather to observe and point out trends and directions in the field and to try to tie many of these developments together since they turn out not to be too different after all. The papers included in this review are the ones that we consider of major importance for the development of topology optimization methods and do not include the many papers that extend the methodology to exciting new application areas. For another review with a somewhat narrower perspective (approaches that have reached the stage of applications in industrial software) the readers are referred to Rozvany (2009). A recent comprehensive review of level set methods for structural topology optimization is provided by van Dijk et al. (2013) and a review of the broader field of topology optimization also including applications can be found in Deaton and Grandhi (2013).

The paper is composed as follows: In Section 2 we review the general topology optimization problem and briefly introduce different ways of solving it. In Section 3 we review the element-based solution approaches in more detail, including density (Section 3.1), topological derivatives (Section 3.2), level sets (Section 3.3) and phase field (Section 3.4) approaches. Discrete solution strategies are discussed in Section 4 and shape parameterized Lagrangian approaches in Section 5. In Section 6 the various approaches are compared and we provide recommendations for future research. Section 7 concludes on the review.

2 General design problem

We consider the general topology optimization problem: find the material distribution that minimizes an objective function F, subject to a volume constraint $G_0 \leq 0$ and possibly M other constraints $G_i \leq 0,\ i = 1 \ldots M$. The material distribution is described by the density variable $\rho(\mathbf{x})$ that can take either the value 0 (void) or 1 (solid material) at any point in the design domain Ω.[1] This optimization problem can be written in mathematical form as

$$\left.\begin{array}{ll} \min\limits_{\rho}: & F = F(\mathbf{u}(\rho),\rho) = \int_\Omega f(\mathbf{u}(\rho),\rho)dV \\ s.t.: & G_0(\rho) = \int_\Omega \rho(\mathbf{x})dV - V_0 \leq 0 \\ & : G_i(\mathbf{u}(\rho),\rho) \leq 0,\ \ j = 1,\ldots,M \\ & : \rho(\mathbf{x}) = 0 \text{ or } 1,\ \forall\, \mathbf{x} \in \Omega \end{array}\right\}, \tag{1}$$

where the state field $\mathbf{u}$ satisfies a linear or non-linear state equation. For simplicity of later notations we here assume that the objective function can be calculated as the integral over a local function $f(\mathbf{u}(\rho),\rho)$, as e.g. the strain energy density when considering compliance optimization.

Note that we include M extra constraints in our general formulation. This is because most real life applications require satisfaction of several design criteria which mostly come in the form of additional constraints. Hence, the popular design formulations that only allow for one simple linear volume constraint must be seen as mainly of academic interest. Also, there exist design problems where the volume constraint is not important, however, we include it here as a basic constraint since even for these kinds of problems a volume constraint may improve general convergence. Furthermore, for some problems that do not naturally include a volume constraint it may still be an advantage to solve an initial problem with volume constraint in order to find a feasible starting guess for the real optimization problem.

The basic topology optimization problem (1) can be attacked in two ways: either as a shape optimization problem or as a density approach (nodal or element-wise densities). These two approaches may also be coined as Lagrangian (boundary following mesh) and Eulerian (fixed mesh) approaches, respectively. The former approach requires a possibility for introducing new holes in order to be categorized as a topology optimization approach. Such approaches include the bubble-method by Eschenauer and co-workers (1994) and a number of other Lagrangian shape-based approaches with conformal meshes that will be reviewed in Section 5. The concept behind level set approaches is also to operate with boundaries instead of local density variables. However, in practise, most level set approaches operate with ersatz materials and fixed Eulerian meshes and hence we here categorize them as being part of the (modified) density approaches discussed in the following.

Considering the density approach, the topology optimization problem above is typically solved by discretizing the domain Ω into a large number of finite elements and letting the density distribution be described by N element or nodal

[1] The one material formulation can easily be extended to multiple material phases—see e.g. Sigmund and Torquato (1997); Bendsøe and Sigmund (1999); Sigmund (2001b).

design variables. In this case, the design problem may be rewritten as

$$\left.\begin{array}{ll} \min\limits_{\rho\rho} : & F(\mathbf{u}(\boldsymbol{\rho}), \boldsymbol{\rho}) = \sum_i \int_{\Omega_i} f(\mathbf{u}(\rho_i), \rho_i) dV \\ s.t. : & G_0(\boldsymbol{\rho}) = \sum_i v_i \rho_i - V_0 \leq 0 \\ & : G_j(\mathbf{u}(\boldsymbol{\rho}), \boldsymbol{\rho}) \leq 0, \ j = 1, \ldots, M \\ & : \rho_i = 0 \text{ or } 1, \ i = 1, \ldots, N \end{array}\right\}, \tag{2}$$

where $\boldsymbol{\rho}$ denotes the design variable vector of length N. This discretized formulation[2] constitutes the basis for the major part of topology optimization approaches.

It is by now well-known that the topology optimization problems (1) and (2) lack solutions in general (Sigmund and Petersson 1998). For many problems, generating ever more holes will decrease the objective function, i.e. there is a lack of closedness of the design space. Similarly, more holes can appear for the discretized problem (2) for increasing N, which is referred to as mesh-dependence. To circumvent this problem one may resort to relaxation or restriction. The former refers to the homogenization approach to topology optimization as introduced by Bendsøe and Kikuchi (1988) but includes also the so-called free material approaches (Bendsøe et al. 1994). Restriction refers to approaches that prevent rapid oscillations of the density distribution. Although the free-material approach seems to have gained some momentum (Kocvara et al. 2008), restriction methods are far more popular and will provide the basis for this review.

Another issue with the discretized optimization problem (2) is that the design variables only can take discrete values: 0 or 1. The discrete nature of the problem makes it difficult to solve. There exist a large number of both mathematically well-founded as well as heuristic approaches that can solve discrete combinatorial problems. However, none of them is well-suited for the problems with thousands and up to millions of variables that are encountered in topology optimization approaches. Arguments against using so-called non-gradient discrete approaches in topology optimization can be found in a recent Forum article by the first author (Sigmund 2011). However, as will be discussed in Section 4, there exist ways for efficiently solving simple discrete problems; those approaches are inspired by continuous approaches and make use of gradient information.

Continuous density design variables allow for the use of efficient gradient-based optimization algorithms, in turn ensuring convergence within a reasonable number of iterations (from 10 s to 1000 s). The continuous topology optimization problem can be written as

$$\left.\begin{array}{ll} \min\limits_{\rho\rho} : & F(\mathbf{u}(\boldsymbol{\rho}), \boldsymbol{\rho}) = \sum_i \int_{\Omega_i} f(\mathbf{u}(\rho_i), \rho_i) dV \\ s.t. : & G_0(\boldsymbol{\rho}) = \sum_i v_i \rho_i - V_0 \leq 0 \\ & : G_j(\mathbf{u}(\boldsymbol{\rho}), \boldsymbol{\rho}) \leq 0, \ j = 1, \ldots, M \\ & : 0 \leq \rho_i \leq 1, \ i = 1, \ldots, N \end{array}\right\}. \tag{3}$$

This formulation constitutes the basis for a large part of the recent topology optimization literature. The interpolation between density and material property depends on the approach but can typically be written as $f(\mathbf{u}(\rho), \rho) = g(\rho) f_0(\mathbf{u})$, where $g(\rho)$ is the density interpolation function and $f_0(\mathbf{u})$ is a function of the field for solid material (i.e. the strain energy density in case of a compliance objective).

Most level set approaches using an ersatz material approach on fixed meshes can also be written in the form of (3). The difference between density and level set approaches lies in the way gradients are computed and how designs are updated.

In Section 3, we will review a number of topology optimization approaches that all make use of (3). In Section 4 we will review approaches that solve the discrete problem (2), and in Section 5 we will review approaches that attempt to solve the original topology optimization problem (1) by alternative parameterizations. The optimization formulations described above represent so-called nested approaches, i.e. the equilibrium equations are assumed to be satisfied for each optimization step. Alternatively, one may consider so-called simultaneous analysis and design approaches but they shall not be treated in detail here. These methods may improve the computational efficiency but the formulation of the optimization problem and its parameterizations are similar to those for the nested approach.

A problem often encountered in the early days of topology optimization was the checkerboard problem, referring to patches of alternating black and white elements which represent bad FE modeling but are favored by the unrestricted optimization process. As is by now well-known, checkerboards can be avoided by use of higher order elements or they are taken care of by all the restriction methods that ensure mesh-independence, and hence they will not be discussed further here. Interested readers are referred to the basic papers that explain the checkerboard issue. (Díaz and Sigmund 1995; Jog and Haber 1996) or a review paper (Sigmund and Petersson 1998). However, it should be briefly noted that both pure (low-order) element and nodal-based design variables create instabilities. Sometimes researchers remark that their schemes based on nodal variables with element-wise constant densities do not create checkerboard-type instabilities (e.g. Amstutz 2011). However, this is not surprising since the intrinsic mapping

[2] Although the discretized optimization problem (2) is a solid-void optimization problem it is for computational reasons common to treat it as a "solid-almost void" problem, meaning that void is mimicked by a very soft material, hence avoiding to have to remesh or renumber the finite element mesh in between iterations. Hence throughout the paper, unless otherwise noted, $\rho = 0$ must be read as $\rho = \rho_{min}$, where ρ_{min} is a small number.

between nodal variables and element densities is a filter in itself and hence prevents formation of checkerboard instabilities.

3 Topology optimization approaches

This section reviews topology optimization approaches that solve the discretized topology optimization formulation (3) with continuous design variables.

3.1 Density approach

Shortly after the homogenization approach to topology optimization was introduced, Bendsøe (1989) and later others (Zhou and Rozvany 1991; Mlejnek 1992) suggested the so-called SIMP (Simplified Isotropic Material with Penalization) or power-law approach, which first was meant as an easy but artificial way of reducing the complexity of the homogenization approach and improving the convergence to 0-1 solutions. Later a physical justification of SIMP was provided in Bendsøe and Sigmund (1999). In the SIMP approach the relation between the density design variable and the material property is given by the power-law, e.g.

$$E(\rho_i) = g(\rho_i)E_0 = \rho_i^p E_0, \qquad g(\rho_i) = \rho_i^p, \tag{4}$$

where p is the penalization parameter and E_0 is the Young's modulus of solid material. For $p = 1$ the optimization problem corresponds to the so-called "variable-thickness-sheet" problem which actually, for the compliance objective, is a convex problem with a unique solution (Petersson 1999; Bendsøe and Sigmund 2004). However, for the same objective, $p > 1$ penalizes intermediate thicknesses or densities and hence favors 0-1 solutions. Choosing p too low or too high either causes too much grey scale or too fast convergence to local minima; the "magic number" that ensures good convergence to almost 0-1 solutions is $p = 3$. This number was later confirmed as the number that ensures physical realizability of elements with intermediate densities (Bendsøe and Sigmund 1999) and was recently also associated with the exponent that makes density gradients equal to topological derivatives for elasticity (Amstutz 2011). It is important to point out that the penalization effect only works in the presence of a volume constraint or some other constraint that indirectly limits volume.

Quite similar to the SIMP interpolation is the RAMP (Rational Approximation of Material Properties) of Stolpe and Svanberg (2001a)

$$E(\rho_i) = \frac{\rho_i}{1+q(1-\rho_i)}E_0, \qquad g(\rho_i) = \frac{\rho_i}{1+q(1-\rho_i)}, \tag{5}$$

where q is the penalization parameter. The reason for introducing the RAMP approach was to alleviate non-concavity of the original SIMP interpolation scheme (Stolpe and Svanberg 2001b) and hence ensure convergence to 0-1 solutions. However, this feature does not seem to play a strong role for practical problems. The main difference between SIMP and RAMP is rather that the latter has a non-zero gradient for $\rho_i = 0$ which may have an influence on convergence properties and may alleviate problems with spurious low density modes in dynamic problems (Pedersen 2000).

It should be noted that the proper choice of the penalization factors p or q is non-unique and depends strongly on the physical problem being solved. In the compliance minimization case, the parameters of SIMP and RAMP need to be chosen such that a sufficient penalization is obtained. For fluid problems considering minimization of pressure loss, the interpolation function needs to be tuned to smoothen the influence of the material distribution on the flow and to prevent too rapid convergence to (locally optimal) discrete designs (Borrvall and Petersson 2003). For complex multiphysics and multimaterial problems one may select appropriate penalization factors based on theoretical material bounds (c.f. Sigmund 2001b) but there also exist cases where the penalization concept is insufficient and other means must be introduced to ensure discrete solutions (c.f. Jensen and Sigmund 2005). Knowing that the compliance minimization problem is convex for $p = 1$ motivates the use of continuation approaches that start with $p = 1$ and gradually increases the penalization during the optimization process. This will often, and even for non-compliance objective functions, result in convergence to better designs. In general, however, the success of this continuation approach, cannot be guaranteed.

There exist a number of other one-variable material interpolation schemes that all serve the same purpose: to provide a continuous interpolation between solid and void with a penalization of intermediate density values. Henceforth, we denote all such interpolation schemes under the same abbreviation: SIMP.

An alternative to the implicit SIMP penalization scheme is to use explicit penalization. This approach entails adding the term

$$\alpha \int_\Omega \rho(1-\rho)dV \tag{6}$$

to the objective function (Allaire and Francfort 1993; Allaire and Kohn 1993), where α is a weighting factor. Alternatively, the term may be added as an explicit constraint to the topology optimization problem. In both cases, however, it is difficult to determine the weighting factor or constraint values that ensure smooth convergence to 0-1 designs and hence this idea has not become very popular (although it

is an integral part of phase field approaches to topology optimization as discussed in Section 3.4).

As mentioned in the introduction, one needs to introduce restriction in order to ensure well-posed and mesh-independent solutions. It is here important to emphasize that well-posedness and mesh-independency is not only of mathematical importance. Certainly, a solution needs to converge with mesh refinement in order to represent good engineering practise. Structural features should never be represented by just one or two low-order elements—everybody who has performed mesh-convergence studies for simple beam bending FE problems will agree on this.

Restriction methods for SIMP problems have developed through the years and we here divide the different methods into one, two and three-field approaches.

3.1.1 One-field SIMP

Ways to introduce restriction directly on the density variables of the SIMP formulation encompass sensitivity filtering, perimeter and gradient control.

The sensitivity filter modifies element sensitivity values to be weighted averages of their neighbors within a mesh-independent radius r_{min} (Sigmund 1997). For a long time the sensitivity filtering scheme has been considered heuristic and it has seemed impossible to show what objective function is actually being optimized. Nevertheless, the concept has proven its worth and is widely used in commercial and academic codes. In recent work (Sigmund and Maute 2012), however, the authors have proven that sensitivity filtering for compliance problems corresponds to minimizing the compliance of a problem modeled by nonlocal elasticity theory, well-known from continuum mechanics (Eringen 1983; Ru and Aifantis 1993). Hence, the sensitivity filter has a sound physical basis and is a well-proven way to introduce length-scale in mechanical problems. Actually, the concept is also quite similar to the different stabilization techniques applied in fluid mechanics problems and hence again, "stabilization by filtering" is not uncommon in other physical disciplines where the search direction computed on the discretized problem is altered to improve convergence toward the continuous optimum.

Disadvantages of the sensitivity filter are partly that it leaves a grey zone of width r_{min} between solid and void regions and partly that it works best for active volume constraints.[3]

Alternatives to the sensitivity filter are explicit constraints and penalty schemes in the form of gradient or perimeter constraints as well as regularized explicit penalization. The two former can in continuous form be written as

$$\alpha \int_\Omega ||\nabla \rho||_q dV, \tag{7}$$

i.e. as q-norms of the gradient of the density field. In case of $q = 1$ we measure the perimeter (at least in the discrete case) (Ambrosio and Buttazzo 1993; Haber et al. 1996), for $q = 2$ we have the total variation and for $q = \infty$ we have the local gradient constraint (Niordson 1983; Petersson and Sigmund 1998). Experiments in Borrvall (2001) show mesh-independence for all values of q and demonstrate how increasing q-norms result in more and more grey transition regions between solid and void regions. Non-linear diffusive terms that partially circumvent the grey transition regions have been investigated (Wang et al. 2004a).

A variation of the explicit penalization (6) is the regularizing explicit penalization (Borrvall and Petersson 2001)

$$\alpha \int_\Omega \overline{\rho}(1 - \overline{\rho}) dV \tag{8}$$

where $\overline{\rho}$ refers to a density average in a fixed neighborhood of radius r_{min} and hence, this constraint or penalization ensures crisp, mesh-independent 0-1 designs.

The two explicit schemes above ((7) and (8)) may be implemented as extra constraints or as weighted penalty terms to the objective function. Common for both approaches is again that it is difficult to select constraint or weight values a priori.

Before ending this subsection on one-field SIMP methods we also briefly mention so-called cellular automatas (CAs) (Kita and Toyoda 2000; Missoum et al. 2005). Although often claimed to be an independent and local update scheme, it is clear that these approaches are extremely similar to standard SIMP schemes, make use of the same interpolation and filtering schemes with the only difference being the update strategy that replace the standard optimality criterion scheme. It seems that most CA schemes can be implemented by changing or modifying a few lines in the publicly available 99-line Matlab code (Sigmund 2001a) and hence, the approach does not motivate a separate name.

3.1.2 Two-field SIMP

An alternative to the sensitivity filter is the density filter (Bruns and Tortorelli 2001; Bourdin 2001) which defines the physical density of an element $\overline{\rho}$ as a weighted average of the design variables in a neighborhood of radius r_{min}. This filter can also be written as a Helmholtz type diffusion operation

$$-r^2 \Delta \overline{\rho} + \overline{\rho} = \rho, \tag{9}$$

[3] Actually, topology optimization approaches often work best with active volume constraints. Depending on the physical problem considered, superfluous material may create non-physical effects or may obstruct the free movement of structural boundaries in turn restricting convergence to (near)global minima.

where r is a length parameter that can be related to the radius r_{min} of the image-processing inspired sensitivity and density filters (Lazarov and Sigmund 2011; Kawamoto et al. 2011). Again, the density filter has the problem of generating grey transitions regions between solid and void regions.

A way to overcome the grey transition regions associated with density gradient constraints, sensitivity as well as density filtering is to diminish the gradient constraint or reduce the filter radius in a continuation approach. If well-penalized, the solution will end up being (nearly) discrete, however, at the risk of ending up with designs that in the best case violate the originally imposed minimum length scale constraints and in the worst case exhibit checkerboarding. Although the continuation strategy may work well in some cases, there are also many examples of failure—especially when considering more advanced objective or constraint functions than compliance.

In general, we categorize density filtering as a two-field approach since it uses two separate fields: the design variable field ρ and the physical density field $\overline{\rho}$. Consequently, consistent derivatives have to be derived using the chain rule (Bruns and Tortorelli 2001; Bourdin 2001; Lazarov and Sigmund 2011; Kawamoto et al. 2011; Andreassen et al. 2011).

3.1.3 Three-field SIMP

More recently, so-called projection schemes (Guest et al. 2004; Sigmund 2007; Xu et al. 2010) have been proposed in order to circumvent the grey-scale issue associated with the different filtering strategies discussed above. These schemes can be categorized as three-field approaches since they operate with the design variable field ρ, the filtered field $\overline{\rho}$ (9) and the projected field $\hat{\overline{\rho}}$—the latter usually obtained by a smoothed Heaviside projection, i.e. $H(\overline{\rho})$.[4] In order to ensure differentiability and good convergence, the smoothness of the Heaviside projection is controlled by a continuation approach that gradually increases the steepness through iterations. Some papers, however, report stable convergence and near discrete designs with fixed high values of the continuation parameter (Guest et al. 2011; Dühring et al. 2010).

It is important to notice that simple projection schemes in themselves do not guaranty convergence to solid-void designs. For cases where the grey scale is a result of the density filtering, projection will tend to ensure discrete solutions. However for problems where the grey regions are due to shortcomings of the penalization, simple projection schemes will probably also fail in providing discrete solutions. Interestingly, however, robust approaches, to be discussed below, do seem to ensure (near) discrete solutions—even eliminating the need for implicit or explicit penalization.

Extensions of the projection concept are on-going and include imposing both minimum and maximum length-scales (Guest 2009a), robustness to under- and overetching (Wang et al. 2011; Sigmund 2009) (which ensures minimum length-scale of both phases if the topology remains the same during under- or over-etching) as well as robustness with respect to random shape variations (Schevenels et al. 2011; Lazarov et al. 2012a, b).

It is interesting to note that the original projection filter (Guest et al. 2004) provides a minimum length-scale on the material region; that the modified projection filter (Sigmund 2007) provides a minimum length-scale on the void regions; but that the intermediate projection filter (Xu et al. 2010) theoretically provides neither of the two (Wang et al. 2011). However, the intermediate filter (Xu et al. 2010) does in some numerical experiments seem to provide (global) mesh-convergence which is attributed to the fact that it starts out as a density filter and only gradually becomes a sharp Heaviside projection. Whereas this may work for simple compliance optimization problems, there are cases that prove the opposite when the filter is applied to more complex physics problems (Wang et al. 2011; Qian and Sigmund 2012). These observations will be of importance when we review other topology optimization approaches later.

It is also important to notice that the projection techniques are implemented as implicit schemes, meaning that length scale is controlled directly (if applicable) through the filter radius and projection value. This is different from the use of global control methods like curvature, perimeter and total gradient approaches that do not ensure local feature size control, c.f. discussions in Sigmund (2007), Wang et al. (2011).

As we shall discuss later, the three-field projection schemes can be seen as a kind of parameterized level set approach: we have the design variable field ρ which controls the filtered field $\overline{\rho}$ that can be interpreted as the level set function, which in turn is projected to the physical density distribution—just as in the (modified) level set approach using ersatz materials. Also, both approaches make use of smoothed Heaviside functions to determine the physical material distribution, but they differ in the update schemes used to advance the designs.

Finally, we point to recent findings on the interpretation of the design variable field in projection methods (Jansen et al. 2013). Originally, the design variable field was merely seen as a mathematical abstraction with no physical

[4] Note that there exist approaches that use multiple projections, e.g. multiphase projection (Guest 2009b) and advanced morphology filtering (Sigmund 2007), however, we include them under "three-field approaches" by counting the projection steps as one, no matter how many times they are applied.

meaning. However, as demonstrated in Jansen et al. (2013) the design variable field can be seen as the input electron beam dose in a micro/nano lithography process. The incoming beam is scattered by a resist layer and the substrate resulting in a smoothed exposure which corresponds to the filtered density field. Finally, an etching process etches away material that has received a certain exposure corresponding to the Heaviside projection step. By addition of regularization to the design variable field one may optimize electron-beam manufacturing costs and speed. This insight is valuable in future uses of topology optimization for micro and nano systems designs where the design variable distributions may be sent directly to the manufacturing step eliminating the need for complex proximity correction steps and ensuring exact replication of the optimized design.

3.1.4 General comments about the density approach

Common for all the density-based approaches described in this subsection is that they represent smooth, differentiable problems that can efficiently be solved by well-proven, gradient-based optimization approaches such as Optimality Criteria methods (e.g. used in public Matlab codes (Sigmund 2001a; Andreassen et al. 2011)), the Method of Moving Asymptotes (MMA) (Svanberg 1987) or by other mathematical programming-based optimization algorithms. Apart from OC methods, these optimizers also immediately allow for systematic and straight forward inclusion of additional global constraints as required by formulation (3). However, while formally it is easy to include local constraints as well, parameterization issues as seen in stress constrained problems (Duysinx and Bendsøe 1998; Cheng and Guo 1997) may render such problems quite hard to solve in practise (see also Section 6.10).

3.2 Topological derivatives

Use of topological derivatives in topology and shape optimization approaches was initiated by Eschenauer et al. (1994) but back then named the bubble-method. The basic idea is to predict the influence (derivative) of introducing an infinitesimal hole at any point $\mathbf{x}$ in the design domain Ω and use this as the driver for generation of new holes.

The derivation of the topological derivatives requires rather complex mathematics (Sokołowski and Zochowski 1999; Novotny et al. 2003). For general choices of material properties (i.e. Poisson's ratio $\nu = 1/3$), Amstutz (2011) argues that the topological gradients for compliance minimization correspond to the standard density gradients as used in the SIMP scheme. In particular he notices that the two schemes give equal results for the choice of penalization exponents $p = 2$ (for 2D thermal problems) and $p = 3$ (for 2D elastic problems with Poisson's ratio 1/3). As noted in the paper by Amstutz (2011), this observation is in perfect correspondence with the recommendations for the penalty exponents given in Bendsøe and Sigmund (1999) which should not be surprising. The bounds for physical realizability of the SIMP law was in Bendsøe and Sigmund (1999) determined by the SIMP stiffness having to be less or equal to the Hashin-Strikhman upper bound in the high volume fraction (small voids) limit. The upper Hashin-Strikhman bound is in turn realized by a periodic composite with small circular holes (Vigdergauz 1999) and hence the sensitivity of introducing a small circular hole (i.e. the topological derivative) can be directly found from the study of optimal material property bounds.

Above discussion naturally leads to a discussion of the relation between topological derivatives and the original homogenization approach to topology optimization (Bendsøe and Kikuchi 1988). Roughly stated, topological derivatives are a special case of homogenization. Where the homogenization approach represents a general framework for interpolation of density between 0 and 1, the topological derivatives only represent the limit of density going to 0, i.e. topological derivatives should be covered by a micro-hole homogenization framework evaluated at vanishing void regions. The authors find that more work is needed in order to clarify these aspects.

The topological derivatives may either be used to indicate the best place for introducing a new hole and then subsequently optimizing the shape of this hole together with existing boundaries (by regular shape optimization (Eschenauer et al. 1994) or as a part of a level set approach (Allaire et al. 2004; Burger et al. 2004)), or the topological derivatives may be directly used in element-based update schemes (Cea et al. 2000; Norato et al. 2007; Novotny et al. 2007; Amstutz 2011). An unresolved question is the following (Bonnet and Guzina 2004): topological derivatives are derived for infinitesimal holes but in practise finite (element) size holes are introduced in numerical implementations. It is yet unclear whether the computed derivatives are useful; whether they are better than simply using the local strain energy density as a measure for where to introduce holes; or if simply placing holes at random points may work just as well in topological derivative aided level set and shape optimization approaches.

3.3 Level set approach

In the level set (LS) approach (Osher and Sethian 1988; Sethian 1999; Allaire et al. 2002, 2004; Wang et al. 2003), the boundary of the design is defined by the zero level contour of the level set function $\phi(\mathbf{x})$ and the structure is

defined by the domain where the level set function takes positive values, i.e.

$$\rho = \begin{cases} 0 : \forall\, \mathbf{x} \in \Omega : \phi < 0 \\ 1 : \forall\, \mathbf{x} \in \Omega : \phi \geq 0. \end{cases} \tag{10}$$

In the past decade numerous level set methods have emerged which can be classified, for example, by the approach for discretizing the level set function, the approach for mapping the level set field onto the mechanical model, and the approach for updating the level set field in the optimization process. In the following we briefly summarize the essential features of the most prominent level set methods. This discussion focuses on level set approaches where the geometry can only evolve from existing boundaries. So-called level set methods that allow for the nucleation of new holes are discussed separately in Section 3.4.2.

3.3.1 Hamilton-Jacobi equation

Most often the level set function is updated via the solution of the Hamilton-Jacobi equation:

$$\frac{\partial \phi}{\partial t} = -V\mathbf{n} \cdot \nabla \phi, \tag{11}$$

or with $\mathbf{n} = (\nabla\phi)/(|\nabla\phi|)$:

$$\frac{\partial \phi}{\partial t} + V|\nabla \phi| = 0, \tag{12}$$

where t is a pseudo-time representing the evolution of the design in the optimization process and V is the so-called speed function, or velocity field, advecting the level set function. The speed function $V(x)$ at $x : \phi(x) = 0$ represents the sensitivity of moving the interface in the normal direction $\mathbf{n}$ with respect to some form of merit function, scaled by the spatial gradient of the level set function. To account for design constraints, penalty or augmented Lagrange multiplier formulations are typically used for constructing merit functions (Allaire et al. 2004; Mei 2004; Luo et al. 2008b). To ensure that the evolution of the level set function is efficiently driven by the shape sensitivities, the gradients of the level set function should be uniform along the interface, for example $|\nabla\phi| \approx 1$ (Burger and Osher 2005). As the Hamilton-Jacobi equation (12) does not maintain uniform spatial gradients of the level set function, re-initialization schemes are often used to prevent the spatial gradients $\nabla\phi$ becoming too steep or too flat along the interface (Osher and Fedkiw 2003; Wang et al. 2003). The speed function can only be derived from the underlying optimization problem at the interface but in (12) is defined in the entire design domain. Therefore an additional model or method needs to be introduced that governs the speed function inside the solid and void domains (Sethian and Wiegmann 2000; Allaire et al. 2004; Wang and Wang 2004b).

As the level set function evolves only along interfaces that exist in the initial design, updating the level set function via the Hamilton-Jacobi equation does not allow for the creation of new holes. Furthermore, additional regularization techniques need to be introduced to prevent rapidly oscillatory level set field and interface geometries. The basic form of the Hamilton-Jacobi equation (11) is purely convective and therefore stabilization schemes need to be introduced for solving it numerically, using either finite element, finite volume or finite difference methods (Burger and Osher 2005). The level set field is advanced in time either via explicit or semi-implicit time-marching schemes, facing additional numerical stability issues. Over the past decade, numerous variations of the Hamilton-Jacobi equation and associated updating schemes have been introduced into level set topology optimization methods. Some of these approaches will be discussed further below.

To regularize the optimization problem, to add further control over the shape of the level set function, and to nucleate new holes, the original form of the Hamilton-Jacobi equation (11) has been augmented via diffusive and reactive terms.

$$\frac{\partial \phi}{\partial t} + V|\nabla \phi| - \mathcal{D}(\phi) - \mathcal{R}(\phi) = 0, \tag{13}$$

where $\mathcal{D}$ denotes a diffusive operator and $\mathcal{R}$ a source/sink term.

In general, the diffusive term smooths the level set field using an isotropic or anistropic, linear or nonlinear diffusion model. In the linear case, for example $\mathcal{D} = \nabla \cdot \nabla\phi$, one can typically associate a potential with the diffusive term (Allaire et al. 2004; Wang and Wang 2004b; Liu et al. 2005) which is minimized in the course of the updating procedure. This diffusion model shares great similarities with the standard relaxed phase field approach discussed in Section 3.4.1. An interesting example for a non-linear diffusion model is suggested in Olsson and Kreiss (2005) and also adopted by the commercial FE-solver Comsol:

$$\mathcal{D} = \gamma \nabla \cdot \left(\epsilon \nabla\phi - \frac{1}{\epsilon}(1+\phi)(1-\phi)\frac{\nabla\phi}{||\nabla\phi||} \right). \tag{14}$$

The first term represents isotropic diffusion. The second nonlinear term favors gradients along the normal direction, promoting a convergence of the level set function to either 1 or -1 (Olsson et al. 2007). The slope of the level set function can be controlled approximately via the regularization parameters γ and ϵ (14). Interestingly, it is not immediately obvious if there exists a potential resulting in the regularization term of (14).

The reactive term $\mathcal{R}(\phi)$ can be used to nucleate holes using, for example, topological derivatives (Burger et al. 2004; Challis and Guest 2009). In some approaches, also labeled level set methods (Yamada et al. 2010; Otomori

et al. 2011), the convective term is omitted and the reactive term is used solely to drive the evolution of the level set function. As these approaches are not based on shape sensitivities but are closer to phase field methods, we do not consider them in the context of level set methods but revisit them in Section 3.4.2. Alternatively to using a reaction term in the Hamilton-Jacobi equation, holes are often generated in a separate, occasionally evoked step of the update procedure (Burger et al. 2004; Mei 2004; Allaire et al. 2005).

3.3.2 Velocity field expansion

The role of the velocity field V in (13) is two-fold: Along the interface it advects the zero level set contour, i.e. the interface, in the steepest descent direction of a merit function; within the solid and void domains it governs the evolution of the level set function. The latter is mainly of importance in the vicinity of the interface where the spatial gradient of the level set function need to be controlled and the velocity field should be sufficiently smooth to prevent the formation of irregular shapes.

At the zero level set contour the velocity field can be constructed from the shape sensitivities of the merit function. To construct a velocity field in the entire domain, the velocity field can be expanded in normal direction by solving an additional PDE (Wang and Wang 2004a, b). In the particular case that the merit function involves only integrals defined over the entire domain, such as compliance and mass, one can construct a velocity field within the material and void domains by evaluating particular terms of the volume integral, which defines the shape derivatives, over the entire domain. To construct a non-zero velocity field just in the vicinity of the interface, the speed function values at the interface can be smeared into the solid and void domains via smoothing methods. Similar to the sensitivity filtering discussed in Section 3.1.1, these methods include direct image-like filtering (Challis 2010) and Helmholtz-like filtering (9) (Allaire et al. 2004; Yamasaki et al. 2011). The same smoothing methods can also be applied to filter velocity fields constructed by normal expansion.

Note that in contrast to SIMP methods, where smoothing methods introducing a mesh-independent length scale lead to mesh-independent optimization results, such smoothing methods applied to the velocity fields do not guarantee mesh-independency. In other words, smoothing the velocity field cannot prevent the formation of ever smaller geometric features as the mesh is refined. However, for strict level set methods, mesh-independency is ensured in other ways since no new holes can be introduced (at least in 2D) and hence a mesh independent starting guess ensures global mesh-independency. It is interesting to note that the level set approach also makes use of heuristic filtering to expand and smooth the velocity field. The difference to the sensitivity filter used in the density approach (Section 3.1) is that the level set filtering does not result in grey scale since the design update is (semi)discrete.

3.3.3 Solution methods for Hamilton-Jacobi equation

In the case that the convective term in (13) is dominant, the Hamilton-Jacobi equation needs to be stabilized to prevent spurious node-to-node oscillations. Such stabilization methods introduce some form of dissipation in the streamline direction, i.e. upwinding, and include essentially non-oscillatory (ENO) and weighted essentially non-oscillatory (WENO) schemes, stream-line diffusion and streamline upwinding Petrov-Galerkin finite element methods (Sethian 1999; Burger and Osher 2005).

To advance the Hamilton-Jacobi equation in time, computationally cheap explicit and semi-implicit methods are typically preferred over fully implicit time marching schemes. In the explicit case the Courant-Friedrichs-Levy (CFL) condition needs to be satisfied, limiting the time-step as a function of mesh size. To relax the CFL constraint semi-implicit schemes based on operator splitting methods can be used, which either operate separately on spatial dimensions of the problem or on the convective and diffusive terms (Luo et al. 2008a; Wei and Wang 2009). To improve the accuracy of the time-marching scheme, typically several time steps of the level set update are run in between solutions of the state equation. Depending on the formulation of the Hamilton-Jacobi equation (13), re-initialization steps need to be applied to control the spatial gradients of the level set function in the vicinity of the interface.

3.3.4 Level set methods using mathematical programming

Alternatively to updating the level set function via the solution of the Hamilton-Jacobi equation, one can consider the parameters of the discretized level set function as optimization variables of a parameter optimization problem which can be solved by standard nonlinear programming methods. This approach allows for the treatment of multiple constraints in a standardized manner. However, various regularization techniques need to be explicitly introduced to control the level set function. For example, to obtain a smooth level set function, alternative approaches for discretizing the level set function are often used, such as radial basis function, high-order polynomial functions with local support, and spectral methods (Norato et al. 2004; de Ruiter and van Keulen 2004; Wang and Wang 2006b; Luo et al. 2007; Gomes and Suleman 2006). The gradient of the level set function along the interface can be controlled, for example, through blending radial basis functions (Pingen et al. 2010). Alternatively, van Dijk et al. (2012) propose a

pre-conditioning scheme to mitigate the effects of degenerating gradients. In addition to the challenges of efficiently controlling the shape of the level set function in the vicinity of the zero level set contour, the convergence of mathematical programming schemes is hampered by vanishing sensitivities away from the boundaries, leading to an ill-conditioned parameter optimization problem.

3.3.5 Mechanical modeling

The geometry of the structure is represented implicity by the level set function through the zero level set contour. There are three approaches to represent the boundary geometry in the discretized mechanical model: fictitious material, immersed boundary techniques, and geometry aligned discretization. The first two approaches allow operating on fixed meshes while the latter approach requires some form of re-meshing. In this subsection we will only consider ersatz material methods as they are closely related to the density method. Immersed boundary and re-meshing schemes can be considered Lagrangian methods and are discussed in Section 5. We note that these methods allow omitting the void domain in the mechanical model which may lead to significantly reduced computational costs.

In the fictitious material approaches, elements in the void domain receive a small density and elements in the solid domain the bulk density. Elements in the vicinity of the boundary can be treated in one of the following ways: In the original work of Allaire et al. (2004), following an ersatz material approach, elements that are intersected by the level set function are given material properties in proportions to the solid area. Alternatively, using a smoothed Heaviside projection the level set function can be directly mapped into an elemental density (Wang et al. 2003; Pingen et al. 2010). The latter approach spreads the transition from solid to void over a narrow band of elements; the width of the band depends on the sharpness of the Heaviside projection and the shape of the level set function in the vicinity of the interface. Furthermore, it is important to control the shape of the level set function in the entire design domain, for example as a strict signed distance function; otherwise the optimization process may take advantage of the flaw and produce large grey regions. This requirement is satisfied in e.g. Wang et al. (2003, 2004b).

In many level set papers authors have chosen to show the zero-level contour to representat the optimized design. This results in a seemingly very smooth and crisp boundary descriptions. In subsequent papers these smooth and crisp boundaries have been used as arguments for using level set approaches as opposed to density approaches that "suffer from jagged edges and blurred grey regions". However, only in few cases does the zero-level contour represent what is actually used in the finite element analysis. In most cases, these contour plots disguise the underlying jagged edges and blurred transition regions that are present for non-discrete level set approaches as well. It is in general strongly recommended that graphics representing topology results show the mesh and element stiffnesses that are used in the finite element analysis and not some kind of smoothed or processed idealization. Otherwise meaningful comparisons between methods is obscured.

3.3.6 General comments about the level set approach

In contrast to density methods, level set approaches define the geometry of the structure via the definition of a solid-void interface. This difference is somewhat blurred when fictitious materials techniques are used to represent the geometry in the mechanical model. However, the geometry of this interface is clearly defined throughout the optimization process and thus in principle allows explicitly formulating objectives and constraints on the interface and describing boundary conditions at the interface (see more in Section 6.6). Level set methods are also well suited for capturing stochastic shape variations for robust design optimization (Chen and Chen 2011; Guo et al. 2013).

Conceptual drawbacks of level set methods are the restriction of the geometry that only can evolve from existing boundaries and the inability to generate new holes at points surrounded by solid material (in 2D). While hole nucleation based on heuristic and topological sensitivity information mitigate these drawbacks, the introduction of a new hole is typically done in a separate step of the optimization procedure, affecting the convergence of the optimization process. The convergence is further strongly influenced by spatial gradients of the level set function in the vicinity of the boundary. Typically, the results of level set methods strongly depend on starting guess.

While updating the level set function via the solution of the Hamilton-Jacobi equation has shown great promise, the large variety of formulations of the Hamilton-Jacobi equations and the need for re-initialization indicate the presence of unresolved challenges, such as regularization, control of the spatial gradients of the level set function, and size control of geometric features. Furthermore, the construction of the speed function implies additional complexity and issues, such as the definition of a merit function in the case of multiple constraints and the expansion of the velocity field into the solid and void domains.

Using fictitious materials approaches to represent the boundary geometry in the mechanical model ensures differentiability and allows for an easy implementation of level set methods into finite element solvers. However, at least from a theoretical perspective, level set methods using fictitious materials face the same challenges as density methods regarding the (non)physical behavior of elements

with intermediate densities. Furthermore, the enforcement of boundary conditions and the evaluation of objectives and constraints defined on boundaries might be complicated. Levelset methods using mathematical programming methods have recently gained popularity. Whether these methods or Hamilton-Jacobi schemes are more efficient is an open question since they have not yet been compared directly against each other.

3.4 Phase field approach

The phase field approach works directly on the density variables and considers minimization of the functional

$$\overline{F}(\mathbf{u}(\rho), \rho) = \int_\Omega \left(\frac{1}{\varepsilon} w(\rho) + \varepsilon ||\nabla \rho||^2 \right) dV + \eta F, \qquad (15)$$

where $w(\rho)$ is a double well function that takes the value 0 for $\rho = 0$ and 1. The use of the double well function penalizes intermediate density values and hence the density interpolation function $g(\rho)$ (see (3)) can be a linear function of ρ. The parameter ε determines the interfacial thickness (i.e. the transition region between solid and void) and η is a weight factor. Note that the two explicit penalty terms correspond to the explicit penalization (6) and the gradient norm (7) sometimes used in the standard density approach. The functional (15) has the derivative

$$\overline{F}_{,\rho} = \frac{1}{\varepsilon} w_{,\rho} - \varepsilon \Delta \rho + \eta F_{,\rho}, \qquad (16)$$

where $_{,\rho}$ means differentiation with respect to ρ. The above functional is minimized based on the Cahn-Hilliard equation

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (M \nabla \bar{F}_{,\rho}), \qquad (17)$$

where M is a diffusion coefficient. As (17) is a fourth order differential equation it is common to simplify the implementation by splitting it into two coupled second order equations

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (M \nabla \mu), \qquad (18)$$

$$\mu = \frac{1}{\varepsilon} w_{,\rho} - \varepsilon \Delta \rho + F_{,\rho}, \qquad (19)$$

where μ is an auxiliary field. It is remarkable that the update of the density field ρ by the Cahn-Hilliard equation (19) is divergence free, i.e. the volume stays constant through the iteration procedure and hence no separate volume constraint is needed when using this "true" phase field approach. So far this possibility has only been pursued in Wallin et al. (2012).

3.4.1 Relaxed phase field approaches

A number of other approaches (Bourdin and Chambolle 2003; Wang and Zhou 2004; Burger and Stainko 2006) minimize the functional (15) directly, without the use of the auxiliary field μ but with an added volume constraint, hence they cannot be categorized as "true" phase field solutions. In this way, the approaches look more like standard density approaches with a 2-norm gradient regularization (7) and an explicit penalization term like (8).

In Yamada et al. (2010) the explicit penalization term (the double well potential) is dropped in favor of a standard SIMP penalization. The authors show by examples that their approach can introduce new holes whereas the approach becomes more like a shape optimization (boundary capturing) approach if the explicit penalization term is included. Essentially, this approach is equal to the standard SIMP approach with a penalization of the total variation.

Slow convergence rates are commonly reported for phase field approaches.

3.4.2 Relaxed level set or reaction-diffusion approaches

Recently, a number of papers have introduced level set approaches that allow for introduction of holes. This is done by replacing $-V|\nabla \phi|$ in the level set equation (12) with a factor times the gradient of the functional to be minimized

$$\frac{\partial \phi}{\partial t} = -\alpha \frac{\partial \overline{F}}{\partial \phi} \qquad (20)$$

In Yamada et al. (2010) this functional is defined as the sum of the elastic energy and a diffusive term

$$\overline{F} = \int_\Omega f H(\phi) dV - \int_\Omega \frac{1}{2} \tau ||\nabla \phi||^2 dV. \qquad (21)$$

Here, black and white solutions are obtained by a discrete update strategy. It is debateable whether (20) is still a level set equation or if it should rather be denoted as a reaction-diffusion approach, or even just a steepest descent method in disguise. Seen in the light of the discussions in Section 3.3.1, equation (20) together with (21) does not contain a convective term and hence does not resemble the original Hamilton-Jacobi equation (11). What the approach still has in common with the level set approach is that the design is described by a projection of the level set field ϕ. Since (20) neither depends on the gradient of the level set function nor on the maintenance of a signed-distance function (compare to (12)), holes can be introduced at any time. In a follow up work, Yamada et al. (2011) discuss their update strategy as a reaction-diffusion problem but still coin the overall approach a level set approach. In a later paper, some of the same authors (Kawamoto et al. 2013) follow a very similar approach but here they actually call it a

"time-dependent diffusion equation" although the terminology "reaction-diffusion" seems to be even more appropriate. The difference between the two papers is that the former uses a sharp Heaviside projection for the material distribution whereas the latter uses a smoothed projection together with a SIMP penalization scheme.

The above discussion on the classification of topology optimization approaches clearly shows the close proximity of phase field, level set methods using ersatz material, and SIMP-type methods. The difficulty of differentiating these methods increases further when different time-marching schemes are used to update the design, such as the Hamilton-Jacobi or the Cahn-Hilliard equation. In the authors' eyes, less emphasis should be put on claiming novelty of method variations and more devoted to the theoretical and numerical comparisons of existing and new methods in terms of convergence behavior and speed for complex test problems.

4 Discrete approaches

Since the original topology optimization problem (2) uses discrete variables it seems natural to try solving it using a discrete optimization approach. Formulated directly in discrete variables, the problem becomes extremely hard to solve and so far only very small problems have been solved to global optimality (Stolpe and Bendsøe 2010). Relaxing requirements to global optimality, discrete mathematical approaches based on sensitivity analysis (efficient reanalysis) can solve larger although still rather restricted problem sizes (Svanberg and Werme 2006). As argued in Sigmund (2011) using non-gradient, nature inspired methods like Genetic Algorithms, swarms and differential evolution techniques are not viable alternatives for the vast majority of topology optimization problems. Nevertheless there exist a number of approaches that actually solve discrete topology optimization problems with surprising efficiency.

Many discrete topology optimization approaches can actually be seen as continuous problems, i.e., they use gradients derived from continuous variable assumptions but make use of discrete design changes or sharp projections. To this category we can include several of the approaches discussed above like the level set approaches with sharp interfaces as discussed in Section 3.3.5 (although an occasional grey element may turn up at boundaries), the relaxed level set approach that makes use of a sharp projection (Yamada et al. 2010), and also the density projection approaches attain nearly discrete designs after the continuation approach (Section 3.1.3). However, there also exist dedicated discrete approaches, such as the discrete level set approach (Challis 2010), discrete density approaches making use of convex approximations and mathematical programming techniques (Beckers 1999, 2000), as well as the evolutionary methods discussed in the next subsection.

Common for all discrete or near-discrete approaches is, that they become extremely sensitive to parameter variations and they may often end up in oscillating and non-converging solutions. According to the authors' experience—both from reading the literature and from numerical experiments—it is very easy to suggest schemes that can perform discrete updates, however, it is difficult to find parameters and update strategies that make them converge to good designs in stable and efficient manners. Even more important, it is very difficult to find optimization algorithms for discrete problems that can treat multiple non-trivial constraints. So far, we have only been able to locate a few papers that treat large scale discrete topology optimization problems with multiple (non volume) constraints in a systematic manner (Challis et al. 2008; Zuo et al. 2012).

4.1 Evolutionary approaches

An important branch of discrete approaches for topology optimization is the evolutionary structural optimization approach (Mattheck and Burkhardt 1990; Xie and Steven 1993). The idea has developed from simple hard-kill strategies (elements with lowest strain energy density are removed) to bi-directional schemes (Young et al. 1999) where elements can be reintroduced if considered rewarding. Although initially solely based on intuition, the concept by now uses standard adjoint gradient analysis and filtering techniques similar to those used in the density approach in order to stabilize algorithms and results (Ansola et al. 2007, 2010; Huang and Xie 2010b). In fact, the implementation of the BESO scheme in Matlab essentially only requires changing a handful of lines (Huang and Xie 2010a) in the 99-line code provided in Sigmund (2001a) and hence the BESO approach, as it is presently defined, should not be categorized as a separate approach but rather as a discrete update version of the standard SIMP scheme. This claim is further supported by the fact that newer BESO approaches use a power law (SIMP) interpolation parametrization (with $p = 3$) for the computation of the discrete gradients (Huang and Xie 2010a).

The ESO/BESO approaches have been criticized for failing in certain situations pertaining to flaws in only making discrete density updates (Zhou and Rozvany 2001; Rozvany 2009). This is an important point and other test examples like compliant mechanism design problems with thin hinges will most probably also make the discrete procedure fail. However, such problems will be challenging for all the discrete fixed-mesh methods discussed in this review and probably also for several of the continuous variable schemes.

A problem with the discrete ESO/BESO approaches and variants thereof is lack of algorithmic convergence and selection of appropriate stopping criteria. In its early days, the stopping criteria for the method was a degenerated structure. The "optimal" structure was then defined as the one with the best compliance to volume ratio before degeneracy. This concept has then been modified to simply stopping the algorithm when the desired volume fraction has been reached (e.g. Ansola et al. 2007, 2010). Obviously, a stopping criterion like that is unable to monitor convergence and one can easily imagine that the structure (and thereby the objective function) could change a lot after the prescribed volume fraction has been reached. In the Matlab code presented in Huang and Xie (2010a), the stopping criterion is modified to be the change of the average of the objective function values over the past 6 to 10 iterations and the average over the five last objective function values. When the difference between these two averaged values is small enough the algorithm is stopped (the same concept is used for a discrete level set approach (Challis 2010)). Again, this has nothing to do with convergence—it rather tells that the algorithm has run into an oscillating state or it may accidentally stop the algorithm a few iterations after the volume fraction has been satisfied.[5]

As noted in the previous subsection, another problem with ESO/BESO approaches is their lack of mathematical generality for including multiple non-linear constraints in a systematic manner.

As a conclusion of this short ESO/BESO review, the authors suggest to recoin BESO methods to "discrete density approaches" or "discrete SIMP approaches" and recommend that further research is devoted to the establishment of better convergence, more sensible convergence criteria as well as systematic and flexible ways to include multiple constraints for such kinds of procedures.

5 Lagrangian approaches and combined shape and topology optimization

Soon after the introduction of topology optimization, the method was combined with shape optimization in a post processing step (Olhoff et al. 1992; Papalambros and Chirehdast 1993; Bletzinger and Maute 1997). The material layout obtained via topology optimization was manually or automatically converted into a CAD-type shape optimization model; the optimum shape was then optimized without the ability to subsequently change the topology. This concept was extended into a multi-step approach iterating between topology and shape optimization in Maute and Ramm (1995). Due to lack of efficient size control in the shape optimization step convergence of the multi-step approach could not be guaranteed. An alternative approach to combining shape and topology optimization became known as the bubble-method (Eschenauer et al. 1994; Kim et al. 2008), resolving the conventional shape optimization methods' inability of introducing topological changes. As discussed previously in Section 3.2, the bubble-method allows for introduction of holes (bubbles) based on a strain energy criterion at certain steps during the shape optimization process. The bubble idea can be considered to be the forerunner for the topological derivatives concept although back then it was combined with a shape optimization concept rather than the now more popular level set approach on fixed meshes.

Probably, partly due to the much higher complexity of shape optimization approaches, partly due to the complexity of constant remeshing, and partly due to the rapid advances in the quality of topology optimization results, shape optimization and shape-based topology optimization approaches in particular have led a rather silent life for almost two decades. However, recently, shape optimization has been revived, partly by introduction of so-called non-parametric approaches where boundary nodes are allowed to move independently (with some regularization) (Le et al. 2011; Arnout et al. 2012), partly due to great interest in applications of isogeometric approaches (Wall et al. 2008; Cho and Ha 2009; Seo et al. 2010) and partly due to the need of explicit mesh-resolved boundary conditions.

An often seen argument for using a level set approach is that it operates with smooth and well-defined boundaries. Still, however, most (including level set as well as density) topology optimization approaches work with fixed meshes that inevitably result in jagged edges that may obscure physics in certain cases (like stresses in elasticity, boundary layers in fluid flow or skin depth issues in electromagnetics (Erentok and Sigmund 2011; Yamasaki et al. 2011)). In order to work with meshes that can represent smooth and accurate boundaries, several researchers have applied various kinds of generalized and adaptive finite element schemes, such as the extended finite element method (xFEM) (Van Miegroet and Duysinx 2007; Kreissl and Maute 2012; Wei et al. 2010), the super-imposed finite element method (Wang and Wang 2006a; Wang et al. 2007), and local remeshing schemes (Yamasaki et al. 2011), within level set methods. These methods overcome the need of extensive remeshing and shape parameterizing, as required by the bubble-method.

An interesting new approach that combines non-parametric shape optimization approaches with the ability to introduce and remove holes is a scheme inspired by

[5] The compliance increases until the volume fraction has been reached and decreases after. Hence, if the average energy before and after feasibility becomes equal the algorithm terminates prematurely.

computer graphics called Deformable Simplicial Complex (DSC) (Misztal and Bærentzen 2012; Christiansen et al. 2013).

Given the very recent revival of Lagrangian approaches the (dis)advantages of level set methods versus non-parameteric shape optimization approaches are not clear yet and further research is needed to compare the features of both approaches. However, both methods are fundamentally challenged by the discontinuous nature of changing the topology in a mechanical model. The crisp description of the structural boundaries may lead to an abrupt change of the structural behavior as geometric features merge or separate. For example, consider a bar with a symmetric notch at its center; two axial forces are applied at its ends. As size of the notch grows, the stiffness of the bar continuously decreases until the bar is cut into two segments. The topological change abruptly alters the structural response and the shape sensitivities of the separated structures do not predict the response of the single-bar configuration. Thus topological changes are non-differentable in Lagrangean approaches, independent of the geometry description, while such changes are typically continuous and differentiable in most Eulerian methods, such as SIMP and level set method using ersatz material concepts. Future research will have to address this challenge and study for which problems Lagrangean methods provide a distinct advantage over Eulerian approaches. Above challenges are especially pronounced for more advanced objectives as seen for example in compliant mechanism design (Sigmund 1997).

6 Comparison of methods

Researches working in topology optimization appear to be splitting into different main directions, favoring SIMP, level set and BESO approaches. In our review of the various approaches we have become increasingly aware that the various approaches are not so different after all. In fact, we will claim that all fixed mesh, Eulerian approaches (holds for probably 90 % of topology optimization approaches) are just variations of the same theme. Be it density, phase field or BESO approaches, they all make use of the same sensitivity information and they almost all make use of filtering techniques to ensure smooth convergence and mesh-independency. Even in the case of level set methods using shape sensitivities, the derivative information closely resembles the one obtained from varying the material properties along the structural boundary. Hence, the only differences are seen in the types of update schemes (continuous/discrete, heuristic/mathematically derived, unconstrained/constrained linear/nonlinear programming, etc.); whether they allow control of either local or global length scales; and in whether they impose the length scale by implicit or explicit means. Even the relatively few works on Lagrangian approaches make use of many of the tools from the Eulerian approaches. In the following, some of these observations are further elaborated upon.

6.1 Use of filtering and smoothing operators

Without smoothing or filtering, topology optimization approaches typically converge to local minima with poor performance and show mesh-dependent solutions.[6] In the standard density approach as well as for BESO approaches, sensitivity and density filters regularize and smooth the optimization problem. In level set approaches using Hamilton-Jacobi update schemes, velocity extension and/or diffusive terms are needed to mitigate convergence issues and to discourage the formation of ever smaller features as the mesh is refined. Here it is important to note that filters for velocity smoothing do not impose a length-scale on the geometry since they merely control the advancement of the zero level set—not its relations to other boundaries. Phase field approaches smooth the design field by adding the total density variation to the objective, and topological derivative approaches indirectly include filtering by mapping between nodal and element (or subelement) based design variables. Hence, we conclude that filtering is imperative for the successful solution of density-based topology optimization problems. Here, it is remarkable that a large part of the filtering and regularization schemes are heuristic in nature and the exact functional for what is being optimized is unknown. However, stabilization schemes like these are well-known from many methods of numerically solving PDEs and the need for such schemes should not be considered a fundamental weakness of topology optimization. In some cases it is possible to give physical motivations for the filtering scheme such as the relation between the original sensitivity filter and non-local elasticity models (Sigmund and Maute 2012).

6.2 Convergence and density vs phasefield updates

Many papers conclude examples by statements like: “the optimal design is visually similar to results published in the literature”. As the following example will show this is not a satisfactory conclusion. Also, one should always avoid using the term “optimal” in the field of structural optimization—unless one can prove that the problem is convex.

[6] These problems can partially be avoided by performing the optimization on consecutively refined meshes, however, for many physical problems that are more complex than simple compliance minimization (c.f. wave propagation problems as e.g. reviewed in Jensen and Sigmund 2011) and electrostatic actuators (Qian and Sigmund 2012) this is not a viable approach.

Figure 1 shows an example of mechanism design as described in Sigmund (1997, 2007). The two designs in Fig. 1a and b are "visually similar, however, they represent snapshots at the 100s and the 1562nd iteration. The corresponding objective functions (output displacements) are −1.46 and −1.54, respectively, i.e. the final design is 5.5 % better than the design obtained after 100 iterations. Figure 1c shows the difference in densities between the two designs. Seen from an engineering perspective the 5.5 % may be unimportant for a specific application. However, seen from an optimization perspective, an algorithm should always be able to converge to the better design. It must be the user who stops the procedure prematurely due to time constraints—not convergence flaws of the algorithm. Hence, when testing an algorithm, quantitative and not only qualitative comparisons to results from the literature should be performed. For the same reason one should always include all details of optimization settings and results and preferably use benchmark examples from the literature rather than invent new ones.

The example in Fig. 1 also highlights another issue. The example was run using a SIMP approach with density filtering but a similar behavior would have been observed for sensitivity filtering (and probably many other approaches). The optimization rapidly finds a fairly good design but requires a very large number of iterations for just slight improvements in objective function but rather large changes in geometry. To illustrate this problem more clearly one may use the very simple 2-bar compliance minimization example shown in Fig. 2. The design domain has size 2.5 by 1, is fully supported at the lower edge and is subjected to a horizontal point load at the center of the top surface.

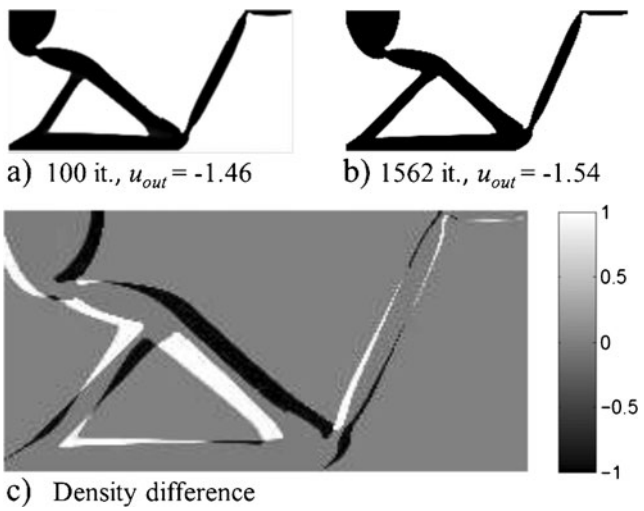

Fig. 1 Snapshots of compliant inverter design at iteration 100 and 1562. The objective function of the final design (**a**) is 5.5 % better than at 100 iterations (**b**). **c** Density difference between the two designs

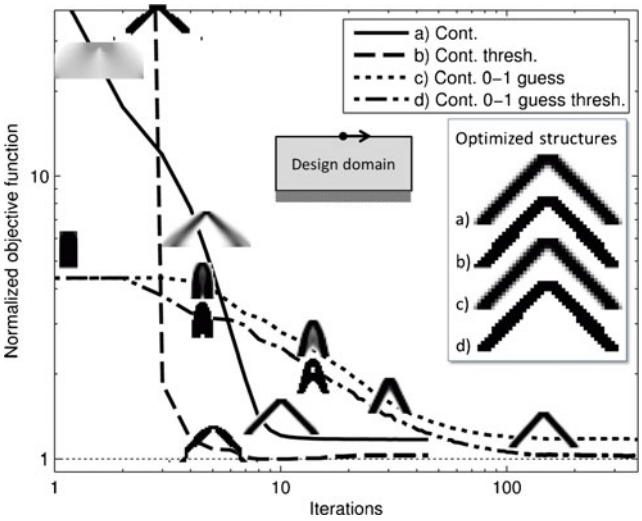

Fig. 2 Convergence curves for 2-bar compliance minimization problem for two different starting guesses. The compliance normalization value corresponds to the best discrete design which has the compliance value 8.84

The problem is solved using the default 99-line code (Sigmund 2001a) using 50 by 20 elements, volume fraction of 0.2, filtersize 1.5 times element size and the convergence criterion (maximum design variable change between iterations) modified from 0.01 to 0.001. Curves a and c show the normalized compliance as function of iterations for a starting guess with uniform density (0.2) and for a starting guess corresponding to a straight, solid, vertical bar satisfying the volume constraint, respectively. Curves b and d show objective functions of the corresponding discrete designs obtained by thresholding the continuous designs from a and c during each iteration, respectively (using the code-snippet in the Appendix). The appendix shows code snippets on how to perform this thresholding for the 88 and 99-line Matlab codes.

Despite its simplicity, a number of interesting conclusions can be drawn from this test example. First of all, it is interesting to observe that the problem converges in just 45 iterations for a uniform starting guess, whereas it needs 380 iterations for the solid bar starting guess. This shows that the SIMP approach is very good in locating a good design for a uniform grey starting guess but that it behaves similar to a phase field method, i.e. it operates with a solid and void phase and changes shape by moving boundaries, when the design has reached or is initialized in a solid-void state. This observation also explains the slow convergence of the inverter example discussed above. The phase field like behavior is easily understandable. When the design reaches a solid-void state, it is (due to the penalization) very expensive for a region to change from solid to void or vise versa. Hence, only grey elements governed

by the filter can change density and therefore the approach becomes a boundary translation approach.[7] The boundary translation process is unfortunately extremely slow (as also reported in phase field papers). Independent of the particular optimization algorithm used (the problem holds for both OC and MMA optimizers) the design changes are small to continuously satisfy the optimality criterion for the grey transition elements. To the best of our knowledge remedies for this issues are unknown and we pose it as a challenge to the community to come up with more efficient updates for continuous variable approaches. In the quest for solving this problem the simple 2-bar test case as stated above with a straight bar starting guess is a good test example. It is clear that discrete variable approaches have an advantage here because they can change densities much more rapidly. However, even the discrete approaches rely on filtering—without, they also get stuck with the straight bar starting design.

Curves b and d in Fig. 1 show the compliance for thresholded versions of designs a and c. As expected the discrete designs are better than the continuous ones (except for the first few iterations) since they do not "loose" stiffness in penalized intermediate density elements. The thresholded designs may be used as benchmarks for discrete algorithms as will be discussed in Section 6.4.

6.3 Local vs global regularization

Mesh-convergence for topology optimization problems can be achieved in several different ways. In many circumstances, one may be satisfied with global convergence, i.e. the topology of the optimized structure converges with mesh refinement. Also, for simple compliance minimization problems similar behavior and results are seen for both global and local schemes. However, for more complex physics problems including hinges in compliant mechanism problems and small gaps as in electrostatic actuator problems (Yoon and Sigmund 2008; Qian and Sigmund 2012), global regularization schemes will fail and there is a need for better local length-scale control. As the authors see it, so far only the robust optimization schemes (Sigmund 2009; Wang et al. 2011; Schevenels et al. 2011; Chen and Chen 2011; Guo et al. 2013) are able to provide full length-scale control without running into problems with favorization of grey scale regions. The robust optimization schemes are computationally rather intensive and hence research in provision of more efficient local control schemes is still needed.

[7] Note that without filtering the boundaries will not move and hence the design cannot move away from the solid bar starting guess.

6.4 Continuous vs. discrete design variables

Clearly, there are many design problems where purely discrete "solid-void" designs are needed. Partially, grey scale transition regions cannot easily be manufactured, but worse, there may be cases where it is difficult or impossible to provide interpolation schemes that result in correct physical modeling of grey scale regions (Qian and Sigmund 2012; Kreissl and Maute 2012; Yoon and Sigmund 2008). Hence, it is important to develop discrete variable approaches, or alternatively to develop schemes with boundary tracking capabilities as discussed in Section 5.

Discrete topology optimization approaches can reasonably be tested on simple compliance problems. However, here, it is important to compare results with grey-scale density approaches in a meaningful way. Many papers on discrete methods criticize continuum variable schemes for their grey scale regions and proceed to compare their compliance values obtained for discrete designs with grey scale results from continuous variable methods (c.f. Huang and Xie 2010a; Wu and Tseng 2010). However, it was hardly the intention of continuous schemes to suggest actual manufacturing of the perforated grey scale regions. In contrary, one would typically take the design with grey scale transition regions, impose a threshold satisfying the volume constraint, and use this design for the manufacturing process. Taking outset in the publicly available Matlab codes (Sigmund 2001a; Andreassen et al. 2011), one may add extra 14–20 lines of Matlab code (see Appendix A) as a post-processing step to get a discrete design that satisfies the volume constraint. Hence, this manufacturable design should be used as reference for comparison against the results of discrete approaches.

Obviously, one should also make sure that similar regularization techniques have been used in order to ensure that the different approaches impose the same length scale and thus the convergence of the designs is comparable as the mesh is refined. Using post-processing step mentioned above it turns out that the continuous variable approach (as for example represented by the 99-line Matlab code) becomes very hard to beat. For example, this approach produces a better thresholded design after 60 iterations than obtained after 15,730 evaluations for a non-gradient approach (Sigmund 2011). Also, this simple approach produces a discrete design that is only a fraction of a percent worse than results obtained by a discrete BESO approach (Huang and Xie 2010a). Here, the discrepancy is due to the fact that the continuous approach followed by a thresholding satisfies a minimum member size constraint which is not the case for the considered BESO design approach.

6.5 Optimizers

A large number of different design update schemes have been suggested for topology optimization problems. For simple compliance minimization problems with a single volume constraint, basically any rigorous mathematical approach as well as intuition-based schemes will work fine and produce nice plots of material distributions in limited number of iterations. The simplicity of compliance problems is due to all gradients having the same sign and hence adding material will always decrease compliance. Therefore, simple schemes that add material where strain energy densities are high and subtract material where they are low will always work fine. Clearly, however, some approaches will be more efficient than others. For example, it is well-known that steepest descent-like approaches are inefficient (see e.g. Klarbring and Torstenfelt (2010) for a recent investigation on this aspect). Nevertheless, steepest descent-type approaches have recently seen a revival—especially within level set approaches using Hamilton-Jacobi update schemes. The authors feel that using rudimentary optimization algorithms is the wrong direction to take and that efforts should be concentrated on the use and development of more efficient and generally applicable optimization approaches.

As soon as more difficult problems like non-selfadjoint problems with varying signs on gradients or problems involving more than one, possibly non-linear, constraint are considered, the majority of the intuition-based approaches fail.

If the goal is to work with a wide palette of multi-physics optimization problems with non-trivial and multiple constraints (as is the case for both authors' research groups), the number of applicable optimizers becomes very limited. The Method of Moving Asymptotes, MMA (Svanberg 1987), and its globally convergent counterpart GCMMA (Svanberg 2002) have been extremely popular and reliable optimization engines for many years. More advanced mathematical programming tools like SNOPT (Gill et al. 2005) and IPOPT (Wächter and Biegler 2006) have appeared and have been tested, however, so far, they have not proven more efficient or more reliable than the MMA approaches. The authors find that research is missing on the issue of solving large scale, complex topology optimization problems efficiently using modern mathematical programming techniques. Likewise, it will be interesting to see new developments in mathematical programming-based optimization approaches for discrete variables and multiple constraints in the spirit of Beckers (2000).

6.6 Boundary dependent loads and critical boundary conditions

Boundary dependent loads should be conveniently implemented using the level set approach since boundaries here are well defined. However, only few works address this issue. In their original work on the level set approach Allaire and co-workers (2004) converted boundary loads to volume loads by a smoothed Dirac function. The same approach is used by Yamada and co-workers (2011) for thermal convection loads in their phase field-like level set scheme. A similar scheme combined with a boundary following mesh adaption approach has been used to resolve skin-depth issues for electromagnetic problems (Yamasaki et al. 2011).

Since boundary loads for level set approaches seem to be implemented by use of the smoothed Dirac functions associated with the smoothed Heaviside functions describing the design, it should be possible to use the same approach using density projection and robust approaches, however, this has so far not been seen in the literature. Instead, density approaches have used ad-hoc solutions to solve e.g. pressure load problems (e.g. Hammer and Olhoff 2000; Chen and Kikuchi 2001; Bourdin and Chambolle 2003; Du and Olhoff 2004a, b; Fuchs and Shemesh 2004) or have been based on monolithic formulations for solving multiphysics problems like pressure loads (Sigmund and Clausen 2007; Kreissl et al. 2010), acoustic-structure interaction (Yoon et al. 2007) and electrostatic actuation (Yoon and Sigmund 2008; Raulli and Maute 2005).

There exist physically hard problems like boundary layers in fluids and skin-depth issues in electromagnetics that require accurate geometric descriptions of boundaries where xFEM, adaptive meshing or other boundary tracking methods are the only ones that will work. All interpolation methods (including SIMP and ersatz material approaches in level sets) will fail (Aage and Sigmund 2013). This is an emerging area of research; some relevant references are given in Section 5.

6.7 Body fitted meshes

To ensure positive definiteness of system matrices, most topology optimization approaches operate with finite material properties for void regions. For compliance optimization problems, bounds on minimum density are often defined as 10^{-3}, resulting in a ratio between stiff and background material of 10^{-9} (for $p = 3$ in the SIMP approach) or by an explicit value for the minimum stiffness E_{min}. In many cases the specific value of E_{min} is not important and the design does not change even for rather high values.

However, in other cases even small values of E_{min} may have a significant influence on both performance and design. This is for example seen for fluid problems (e.g. Gersborg-Hansen et al. 2005; Kreissl and Maute 2012) and for non-linear elasticity problems (e.g. Pedersen et al. (2001)). To ensure that passive regions indeed do not influence the performance and design, it is highly recommended to check solutions' quality by an analysis on a "body fitted mesh". In its simplest form this corresponds to performing an analysis where passive elements have been eliminated from the FE mesh. In its more advanced form one should define a new mesh with smooth boundaries based on a contour plot or another boundary extraction method. For both approaches, performance of the body-fitted model should be very close to the one predicted from the topology optimization. Discrepancies may indicate flaws in the mechanical model or unacceptable sensitivity to boundary perturbations.

6.8 Provision of research codes to the community

Since its publication in 2001, the 99-line Matlab code (Sigmund 2001a) has been very popular, both as an educational tool in courses on topology optimization, but also as a platform for research and development of other topology optimization approaches. Meanwhile, the code has been optimized for speed and compactness (Andreassen et al. 2011), modified for a discrete level set approach (Challis 2010), continuous level set approaches www.cmap.polytechnique.fr/~allaire/levelset_en.html and www2.mae.cuhk.edu.hk/~cmdl/download.htm, as well as for the evolutionary BESO approach (Huang and Xie 2010a). Also extensions to Pareto strategies (Suresh 2010) and alternative element discretizations (Talischi et al. 2012) have appeared.

The provision of clear and well-written codes is a great service to the community and the inclusion of codes or code snippets in research papers is strongly recommend. However, we also suggest that published codes should represent procedures that work well on a range of test problems without further parameter tuning. For example, we do not recommend code extractions from bigger research codes that clearly are missing important parts. Codes that are too complex, require too much parameter tuning or that exhibit bad or oscillatory behavior for simple test problems will scare away students and potential users, rather than attracting them to an interesting approach.

6.9 Benchmark problems

For compliance minimization problems, the topology optimization community has more or less agreed on two standard benchmark examples, i.e. the MBB beam and the 5 by 8 cantilever. Also, the L-shaped design domain is a must for showing problems with local stress constraints (see Section 6.10). However, when it comes to compliant mechanism design this general agreement on benchmarks seems to end. Clearly, there are different ways to formulate the compliant mechanism modeling and optimization problems but nevertheless, it would be good to agree on a few challenging standard test cases. This is especially important since there are huge differences in convergence behavior for various kinds of test problems. Probably the most challenging but still simple to implement compliant mechanism benchmark is the inverter example (c.f. Fig. 1) first proposed in Sigmund (1997) but "standardized" as benchmark example in a number of recent papers by the first author and co-workers (Sigmund 2007, 2009; Wang et al. 2011). Partly, the appearance (or avoidance) of one-node connected hinges in the optimized topology represents a challenge, but the main challenge lies in the strong influence of the initial design on the optimization process and the first few iterations. There is a strong local minimum represented by a structure that disconnects the input point and stiffens the output point. Many algorithms will get stuck in this solution and never produce an inverter mechanism. This is especially critical for methods that operate with boundary control, like level set methods. Hence, it is interesting to note that few, if any, true level set papers show solutions for the inverter example. Based on this observation, the authors recommend that the inverter example (c.f. Sigmund 2007) is chosen as a standard benchmark problem in future publications.

When presenting the results for benchmark problems using a new optimization schemes, rigorous studies should be performed that show the sensitivity of the proposed approach with respect to key algorithmic parameters and parameters in the problem formulation. The evolution of the design in the optimization process should be clearly documented by monitoring objectives, constraints, and optimality criteria. Only by providing this information the robustness and efficiency of a new optimization scheme can be objectively assessed.

6.10 Local constraints

One of the main strengths of topology optimization is the efficiency of computing gradients for objective and constraint functions even for huge problems with millions of design variables. Often the gradients are obtained for almost free (compliance); with effort corresponding to the solution of an extra right hand side (e.g. compliant mechanism); or at most at the cost of one extra analysis (e.g. for problems with non-symmetric system matrix or for non-conservative or transient problems) per objective and constraint function. This efficiency applies to problems with few objective and constraint functions. Also, problems with many simple constraints (i.e. with sparse gradient matrices) like slope constrained topology optimization problems

(Petersson and Sigmund 1998) can be solved rather efficiently with selected optimizers (Schury et al. 2012).

A problem that has been challenging for many years is the stress constraint problem. The challenges are threefold: a) the so-called stress singularity problem (Cheng and Jiang 1992; Cheng and Guo 1997) prevents smooth convergence to zero density; b) if one wants to minimize stress, the best structure is no structure; and c) the number of constraints equals the number of elements or nodes, depending on the formulation of the constraint. The stress singularity problem was partially alleviated with the $\epsilon-$relaxation approach for truss (Cheng and Guo 1997) and continuum problems (Duysinx and Bendsøe 1998; Duysinx and Sigmund 1998); the "no structure" problem is often alleviated by combining the stress constraint with an compliance objective or simply ignored; and the large number of variables can be collected in a single constraint by smooth envelope functions or constraint aggregations like the p-norm (Duysinx and Sigmund 1998; Le et al. 2010) or Kreisselmeier-Steinhauser (K-S) functions (Rozvany and Sobieszczanski-Sobieski 1992; Luo et al. 2013).

For geometrically unconstrained problems, the compliance optimized structure will also be optimal with respect to stress (Michell 1904; Bendsøe et al. 1993). For geometrically constrained problems, there are often no significant differences between compliance and stress optimized structures and hence roughly speaking, stress constraints may not be very important in topology optimization. Even for the standard test case for stress constrained problems: the L-shaped beam (Duysinx and Bendsøe 1998) (also called the "Norwegian sock" problem), a simple fillet in the sharp inner corner will alleviate most of the stress concentration. Nevertheless, it is still, both from an academic and a practical perspective, interesting to propose efficient methods that convincingly solve the stress minimization problem. A minimum requirement for such algorithms is that they produce a rounded corner for the L-shape test problem. This seems to have been fulfilled for a number of density approaches (Le et al. 2010; Luo et al. 2013), level set approaches (Allaire and Jouve 2008) as well as for topological derivatives (Amstutz and Novotny 2010). Key components of these methods seem to be high resolution or adaptive mesh strategies (Xia et al. 2012) combined with intricate continuation schemes and parameter tuning. Also the "minimize volume with stress constraint" formulation seems to be preferred to the opposite formulation (minimize stress with volume constraint)). Whether obtaining more successful results for this way of formulating the problem is a coincidence or due to something intrinsically related to the physical problem has yet to be resolved. The commercial software TOSCA has an option for both 2D and 3D stress constrained topology optimization (FE-Design 2011) with an approach that seems to be fairly similar to Le et al. (2010). In connection with the stress constraint problem it is interesting to note that all results presented so far seem to be different and no unique design or approach stands out as being the optimal. It is clear that efficient algorithms for stress constraints have yet to be developed and that optimized structures should be compared to designs obtained for shape optimization and/or boundary adaptive approaches.

6.11 On the need for level set approaches

Level set methods have significantly gained in popularity due to their promise to operate on clearly defined boundaries throughout the optimization process. While Hamilton-Jacobi update schemes have been predominately used to advance the level set field, more recently mathematical programming methods are applied to level set methods. As the majority of level set methods employ smoothed Heaviside functions and ersatz material approaches the crispness is lost when mapping the geometry onto the mechanical model. Independently of the update scheme, level set methods rely on a tight control of the level set field, in particular on the slope in the vicinity of the boundary. The many flavors of velocity field smoothing and dissipation built into Hamilton-Jacobi update schemes, as well as the need for re-initializing the level set field, demonstrate the still unresolved challenge of efficiently controlling the level set field. Similar to density methods, level set approaches require regularization methods to mitigate the mesh-dependency of the optimization results. While global approaches, such as perimeter constraints, have been successfully applied in (Yamasaki et al. 2010; Maute et al. 2011; van Dijk et al. 2012), efficient local regularization techniques are lacking. Note that filtering does not guarantee mesh-independency even when the formulation of the filter involves a mesh-independent length scale. As discussed in Section 6.2 for density methods, significantly changing the geometry of an initial design in the optimization process solely through evolution of shape of boundaries typically requires a large number of iterations. This also holds for level set methods which, similar to density methods, rely on sensitivity information of boundary variations. While level set methods have been applied to a broad range of problems, it is not clear yet whether and for which problems they provide an advantage over density methods. In particular issues related to efficiency, i.e. number of iterations until convergence, and local mesh-dependency need to be addressed.

6.12 Lagrangian methods

Lagrangian and boundary following topology optimization approaches have recently been revived, partly inspired by new non-parameterized shape optimization approaches and partly by the need for solving problems with physical

responses that require exact boundary descriptions. Whereas these approaches may resolve the boundary description problem, other issues may appear concerning robustness to topological changes. Hence to really gain momentum new approaches should be rigourously compared to standard approaches and tested on challenging problems like the inverter problem and others.

7 Conclusions and challenges

In the course of working on this review, the authors have become increasingly aware of how small the differences are between various topology optimization approaches. In many cases it is even difficult to identify the novelty of a supposedly new approach. Evolutionary methods are converging towards discrete SIMP schemes (although lacking the mathematical framework for multiple constraint handling), density projection methods look more and more like parameterized level set approaches, phase field methods correspond to density approaches with explicit penalization and regularization, topological derivatives can be derived from material bounds that also are used to devise bounds for the SIMP approach etc.

Nevertheless, the topology optimization community seems to be splitting up and is generating individual research paths in various directions. Instead of comparing approaches and looking for the most efficient ways to solve general topology optimization problems, there seems to be a trend in redoing old applications with alternative strategies. Whether these new strategies are better, comparable or worse than existing ones is not obvious since few, if any, research groups perform direct and meaningful comparisons between approaches or (re)use standard benchmarks.

In our eyes, the topology optimization community should reunite, get together in joint ventures in the search for the "optimal optimization approach" and in this process use standard benchmarks as well as insightful and expert-based comparisons between methods.

As we see it, the nine main challenges (not prioritized) of topology optimization are

1. Efficiency:
   Many papers test algorithms on coarsely discretized 2D problems. To have real potential the numerical and computational performance of an algorithm should be demonstrated for large scale 3D problems.
2. General applicability:
   A large number of approaches are applied only to simple compliance minimization problems. To have real potential an algorithm must be applicable to arbitrary physics problems.
3. Multiple constraints:
   Many approaches only work with simple volume constraints and some even work with fixed Lagrange multipliers. To show any promise, algorithms must be able to handle at least a small number of geometric and physical constraints.
4. Complex boundary conditions:
   A broad range of physics problems require accurate modeling of boundary conditions and are not solvable using standard regular mesh approaches. More research and insight is needed in this area.
5. Independence on starting guess:
   Convergence to good solutions is often associated with large number of iterations and continuation approaches. Some methods are sensitive to starting guesses but are very efficient in later stages in the design process. Hybrid approaches could combine the benefits of the different approaches but are largely lacking.
6. Few tuning parameters:
   Some approaches require problem dependent tuning parameters. To be applicable to a broad range of problems, optimization schemes should work for all problems with fixed parameter sets.
7. Mesh-independent convergence:
   Ideally the number of iterations in the optimization process should be independent of the mesh refinement used to discretize the density or level set fields. This challenge has received little, if any, attention in the past. The step size in level set methods is typically restricted by the CFL condition and the information contained in the shape sensitivities, a similar slow down of convergence is observed for density and phase field approaches in particular. More research in alternative design parameterizations or multi-level approaches is needed.
8. Ease of use:
   Most commercial as well as academic codes are tightly linked with finite element software, hence requiring expert knowledge of the user. Spreading the concept of topology optimization to a broader audience (like e.g. architects and designers) requires intuitive user interfaces and real time interaction (Aage et al. 2013).
9. Alternatives to finite element analysis:
   While the finite element method is the numerical modeling framework of choice for a broad class of engineering problems, other methods provide great efficiency and robustness for particular problems and therefore are more commonly used in practice, such as finite volume methods for compressible flow problems or finite difference methods for nano-optical problems. To make topology optimization a practical design tool for such problems, non-FE based topology optimization methods should be further explored.

Appendix

A Matlab threshold code

The Matlab script shown below is intended as a post-processing step that converts a grey scale design obtained with the 99-line code (Sigmund 2001a) to a discrete design satisfying the volume fraction constraint.

```
% THRESHOLD DESIGN SATISFYING VOLUME CONSTRAINT
[Y,I]=sort(x(:),'descend');
vt = floor(((volfrac-0.001)*nelx*nely)/(1-0.001));
xd(I(1:vt))=1;
xd(I(vt+1:end))=0.001;
xd=reshape(xd,nely,nelx);
imagesc(-xd); axis equal; axis off;
% FE-ANALYSIS
[U]=FE(nelx,nely,xd,penal);
% CALCULATE DISCRETE OBJECTIVE FUNCTION
cd = 0.;
for ely = 1:nely
 for elx = 1:nelx
  n1 = (nely+1)*(elx-1)+ely;
  n2 = (nely+1)* elx   +ely;
  Ue = U([2*n1-1;2*n1; 2*n2-1;2*n2;
          2*n2+1;2*n2+2; 2*n1+1;2*n1+2],1);
  cd = cd + xd(ely,elx)^penal*Ue'*KE*Ue;
 end
end
disp(['  Discrete Obj.: ' sprintf('%10.4f',cd) ...
 ' Vol.: ' sprintf('%6.3f',sum(xd(:))/(nelx*nely))])
```

In the script, the total volume includes the volume taken up by low-density elements. If the discrete approach does not include low-density elements, the third line above can simply be changed to

```
vt = floor(volfrac*nelx*nely);
```

'For the more compact 88-line code (Andreassen et al. 2011) the FE-part of above script should be substituted with the following

```
%% FE-ANALYSIS
sK = reshape(KE(:)*(0.001+xd(:)'*(1-0.001)),64*nelx*nely,1);
K = sparse(iK,jK,sK); K = (K+K')/2;
U(freedofs) = K(freedofs,freedofs)\F(freedofs);
ce = reshape(sum((U(edofMat)*KE).*U(edofMat),2),nely,nelx);
cd = sum(sum((Emin+xd.^penal*(E0-0.001)).*ce))
```

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