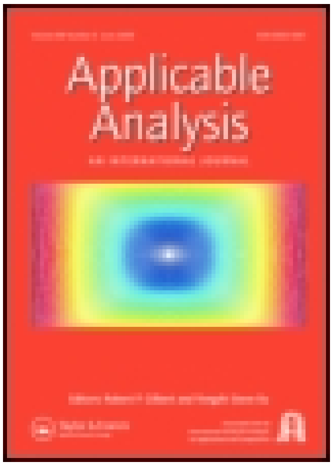

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Existence of solutions of quenching problems

Daniel Philips [a]

[a] Purdue University , West Lafayette, Indiana, U.S.A

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Existence of Solutions of Quenching Problems

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DANIEL PHILLIPS
Purdue University, West Lafayette, Indiana, U.S.A.

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Abstract We find continuous nonnegative solutions of the equation $\Delta u - u_t = u^{-p}\chi(\{u > 0\})$ when $0 < p < 1$ for the Cauchy problem on $\mathbb{R}^n$ and the initial value-Dirichlet problem on bounded domains. The motivation for this work comes from reaction diffusion models, $\Delta u - u_t = f(\varepsilon,u)$, where for $\varepsilon > 0$, $f(\varepsilon,u)$ is smooth and $f(\varepsilon,u) \to u^{-p}$ as $\varepsilon \downarrow 0$ for $u > 0$. Such a limiting process is used here.

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INTRODUCTION

We construct a solution to the Cauchy problem for a semilinear parabolic equation with a singular absorption term. More precisely we find a nonnegative solution to

$$\begin{cases} \Delta u - u_t = u^{-p}\chi(\{u > 0\}) & \text{in } \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^+) \\ u(x,0) = u_0(x) \end{cases} \tag{0.1}$$

where the parameter p is such that $0 < p < 1$ and $\chi(E)$ is the *characteristic function for the set E.*

Theorem 1. Let $u_0(x) \in C_c(\mathbb{R}^n)$, $u_0 \geq 0$. There exists at least one solution to (0.1) that is continuous for $t \geq 0$ and Hölder continuous for $t \geq t_0 > 0$.

The solution that we find does indeed quench, in the sense

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that u vanishes outside of a bounded set in $\mathbb{R}^{n+1}_+$ and certain derivatives of u blow up near $\partial\{u > 0\}$ for $t > 0$. Nevertheless, $u(x,t)$ is as regular in space as the elliptic counterpart to (0.1) studied in [4], $(u(\cdot,t))^{1/\beta} \in C^{0,1}(\mathbb{R}^n)$ for $t > 0$, where $\beta := \frac{2}{1+p}$. Also $\nabla_x u(x,t)$ is continuous for $t > 0$.

I have not been able to resolve the question of uniqueness for (0.1). Also it should be pointed out that the nonlinearity has a mild singularity, i.e.,

$$\int_0^1 u^{-p}\,du < \infty$$

for the range of the parameter we consider and this is critical for out approach. Some interesting work where a more singular nonlinearity is allowed is done in [1]

A similar existence theorem is true for bounded domains.

<u>Theorem 2</u>. Let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega$ of class $C^{2+\alpha}$, let $\psi(x,t) \in C^{2+\alpha}(\bar{\Omega} \times [0,\infty))$ with $\psi \geq 0$ and $\psi(x,t) \geq \delta > 0$ for $(x,t) \in \partial\Omega \times [0,\infty)$ for some constant δ, then there exists a continuous nonnegative solution to

$$\begin{aligned} &\Delta u - u_t = u^{-p}\chi(\{u > 0\}) \quad \text{in } \Omega \times [0,\infty), \\ &\qquad u = \psi \text{ for } (x,t) \in \Gamma := \Omega \times \{0\} \cup \partial\Omega \times [0,\infty). \end{aligned} \tag{0.2}$$

where $0 < p < 1$.

Problem (0.2) is related to models from chemical engineering used to understand the behavior of a chemical species within a porous body Ω having concentration u, see [2].

Generally the model is not singular as in (0.2) but of the form

$$\Delta u^\varepsilon - u_t^\varepsilon = f(\varepsilon,u^\varepsilon)$$

where $f(\varepsilon,u)$ is smooth and nonnegative for $\varepsilon > 0$, $u \geq 0$, and $f(\varepsilon,0) = 0$. This can be the situation for Langmuir-Hinshelwood kinetics. We are interested in the special case when $f(\varepsilon,u) \to u^{-p}$ for $u > 0$ as $\varepsilon \downarrow 0$, e.g.,

$$f(\varepsilon,u) = \frac{u}{\varepsilon + u^{1+p}} . \tag{0.3}$$

Thus a solution of (0.2) takes the form of a singular limit. With example (0.3) for f is not hard to show that the sequence $\{u^\varepsilon(x,t) | u^\varepsilon = \psi \text{ on } \Gamma\}$ converges pointwise as $\varepsilon \downarrow 0$. It is not directly evident though that the limit u satisfies (0.2).

What is required are estimates on $|\nabla u^\varepsilon|$ and the modulus of continuity for the u^ε independent of ε and this is what we provide. These estimates imply the uniform convergence of $\{u^\varepsilon\}$ and thus makes u more relevant as an approximation.

We will work with the nonlinearily (0.3) which satisfies $f(\varepsilon,u) \uparrow u^{-p}$ as $\varepsilon \downarrow 0$. Existence results for more general f follows in a similar manner. The idea of studying (0.1) as a singular limit was motivated by the work in [3].

1. THE CAUCHY PROBLEM

Consider

$$P_\varepsilon \quad \begin{cases} \Delta u^\varepsilon - u_t^\varepsilon = f(\varepsilon,u^\varepsilon) & \text{for } x \in \mathbb{R}^n, \ t > 0 \\ u^\varepsilon(x,0) = u_0(x) & \text{for } x \in \mathbb{R}^n. \end{cases}$$

If $u_0 \in C_c(\mathbb{R}^n)$ then P_ε with $\varepsilon > 0$ has a unique nonnegative solution that is C^2 for $t > 0$ and continuous for $t \geq 0$, and $\nabla_x u$ is C^2 for $t > 0$.

Moreover by the maximum principle

$$0 \le u^{\varepsilon}(x,t) \le \|u_0\|_{L^\infty} \quad \text{for } x \in \mathbb{R}^n,\ t \ge 0, \tag{1.1}$$

$$u^{\varepsilon_1} \le u^{\varepsilon_2} \quad \text{if } \varepsilon_1 < \varepsilon_2 \tag{1.2}$$

and if $u_0 \in C^{2+\alpha}(\mathbb{R}^n)$ then

$$|u^{\varepsilon}(x,t)|, |D_t u^{\varepsilon}(x,t)|, |D_x u^{\varepsilon}(x,t)|, |D_x^2 u^{\varepsilon}(x,t)| \tag{1.3}$$
$$\le C_{\varepsilon}(T)\exp(-b_{\varepsilon}|x|)$$

for $0 < t \le T$, $x \in \mathbb{R}^n$ where $0 < C_{\varepsilon}(T)$, $b_{\varepsilon} < \infty$. The solution we are after is $u(x,t) \equiv \lim_{\varepsilon \downarrow 0} u^{\varepsilon}(x,t)$. We shall obtain a gradient estimate for the u^{ε}. For this we need the following estimate on $u_0(x)$.

<u>Lemma 1</u>. Let $u_0(x) \in C^2(\mathbb{R}^n)$, $u_0 \ge 0$, with $\|D^2 u_0\|_{L^\infty} = M < \infty$ then $|\nabla u_0(x)|^2 \le (M+1)^2 u_0(x) = \overline{M}\, u_0(x)$ for $x \in \mathbb{R}^n$.

<u>Proof</u>. Suppose $|\nabla u_0(\bar{x})|^2 = C^2 u_0(\bar{x}) > 0$ with $C > M + 1$ for some $\bar{x}$ and let ν be in the direction of $-\nabla u_0(\bar{x})$. Then $D_\nu u_0(\bar{x} + s\nu) \le -(C - M)(u_0(\bar{x}))^{1/2}$ for $0 \le s \le (u_0(\bar{x}))^{1/2} \equiv h$. Hence $u_0(\bar{x} + h\nu) \le h^2(1 + M - C) < 0$. ///

<u>Lemma 2.</u> Let $u_0(x) \in C^{2+\alpha}(\mathbb{R}^n)$ and u^{ε} satisfy P_{ε} then $|\nabla_x u^{\varepsilon}(x,t)|^2 \le 2F_{\varepsilon}(u^{\varepsilon}(x,t)) + \overline{M}u^{\varepsilon}(x,t)$ where

$$F_{\varepsilon}(s) = \int_0^s f(\varepsilon,\tau)\,d\tau.$$

<u>Proof</u>. Let $u^{\varepsilon} = u$ and set $w(x,t) = |\nabla u|^2 - 2F_{\varepsilon}(u) - \overline{M}u$. If w is positive somewhere in the strip $\{x \in \mathbb{R}^n,\ 0 \le t \le T\}$ then using (1.3) and lemma 1 it follows that w has a positive maximum at a point (x_0,t_0) with $t_0 > 0$; and thus $\nabla u(x_0,t_0) \ne 0$. If we take $\frac{\partial}{\partial x_1}$ in the direction of $\nabla u(x_0,t_0)$ we obtain from $D_1 w(x_0,t_0) = 0$ that $2D_{11}u(x_0,t_0) = 2f(\varepsilon,u(x_0,t_0)) + \overline{M}$. Computing $\Delta w - w_t$ at

(x_0,t_0) we find $\Delta w - w_t \geq 2(D_{11}u)^2 - 2(f(\varepsilon,u))^2 - \bar{M}f(\varepsilon,u) \geq \bar{M}^2/2$ a contradiction.///

We can get an estimate on $|\nabla u^\varepsilon|$ independent of the smoothness of u_0.

<u>Lemma 3</u>. Let $u_0 \in C_c(\mathbb{R}^n)$ with $\|u_0\|_{L^\infty} = M_0$ and u^ε a solution to P_ε then $|\nabla u^\varepsilon(x,t)|^2 \leq 2F_\varepsilon(u^\varepsilon(x,t)) + 2M_0u^\varepsilon(x,t)/t$ for $t > 0$, $x \in \mathbb{R}^n$.

<u>Proof</u>. First let u_0 be $C^{2+\alpha}$ and set $u^\varepsilon = u$, $w = |\nabla u|^2 - 2F_\varepsilon(u) - 2M_0u/t$. If w were positive somewhere in the strip $\{x \in \mathbb{R}^n, 0 < \sigma \leq t \leq T\}$ then by the same argument as in lemma 2 $w(x,\sigma) > 0$ for some $x \in \mathbb{R}^n$. Using (1.3) and lemma 1 we see that $w(x,\sigma) \leq 0$ for σ small.

Now let $u_{0_j}(x) \to u_0(x) \in C_c(\mathbb{R}^n)$ where the $u_{0_j}(x)$ are smooth. The corresponding solutions to P_ε converge with their first derivatives uniformly to u^ε and ∇u^ε on compact sets bounded away from $t = 0$. Thus the lemma holds in general.///

Since $F_\varepsilon(s) \leq Cs^{1-p}$ for $s \geq 0$ we see that $u^\varepsilon(x,t)$ and $(u^\varepsilon(x,t))^{1/\beta}$, $\beta = \frac{2}{1+p}$ are Lipschitz continuous in x, uniformly in ε and t, $t \geq t_0 > 0$ for $u_0 \in C_c(\mathbb{R}^n)$, $t \geq 0$ for $u_0 \in C_c^{2+\alpha}(\mathbb{R}^n)$. This information implies uniform Hölder continuity in x and t.

<u>Lemma 4</u>. Let $u_0 \in C_c(\mathbb{R}^n)$ then for each $\tau > 0$ there is a constant $C(\tau)$ so that

$$|u^\varepsilon(x,t) - u^\varepsilon(y,s)| \leq C\cdot(|x-y| + |t-s|^{1/3n} + |t-s|^{1/3})$$

for $\tau \leq s,t$ and $0 < \varepsilon \leq 1$. If $u_0 \in C_c^{2+\alpha}(\mathbb{R}^n)$ then we take $\tau = 0$.

<u>Proof</u>. Multiplying the equation in P_ε by D_tu^ε we get

$$\int_\tau^T\int_{\mathbb{R}^n} |D_tu^\varepsilon|^2 dxdt \leq \int_{\mathbb{R}^n}\left(\frac{|\nabla u^\varepsilon(x,\tau)|^2}{2} + F_\varepsilon(u^\varepsilon(x,\tau))\right) dx$$

If u_0 is smooth we can take $\tau \geq 0$ and then the left hand side is bounded uniformly in T and ε, $0 < \varepsilon \leq 1$. If $u_0 \in C_c(\mathbb{R}^n)$ then the right hand side is dominated by

$$C(1 + \frac{1}{\tau}) \int_{\mathbb{R}^n} (u^\varepsilon(x,\tau))^{1-p} dx.$$

Since $\Delta u^\varepsilon - u_t^\varepsilon \geq 0$ we know that $u^\varepsilon \leq h$ when $\Delta h - h_t = 0$ and $h(x,0) = u_0$. It is easily checked that

$$\int_{\mathbb{R}^n} (h(x,\tau))^{1-p} < \infty \quad \text{for any } p < 1.$$

Hence

$$\int_\tau^T |D_t u^\varepsilon|^2 dxdt < \overline{C}$$

where $\overline{C}$ is independent of T and ε.

Suppose $t \geq s$, and set $r = |x - y| + |t - s|^{1/3n}$. Then for some $\overline{x} \in B_r(x)$ we get

$$|u^\varepsilon(\overline{x},t) - u^\varepsilon(\overline{x},s)|^2 \leq (t - s) \int_s^t |D_t u^\varepsilon(\overline{x},h)|^2 dh$$

$$= \frac{(t-s)}{|B_r|} \int_s^t \int_{B_r(x)} |D_t u^\varepsilon(z,h)|^2 dzdh$$

$$\leq C\,\overline{C}\,\frac{(t-s)}{r^n} \leq \overline{\overline{C}}\,(t - s)^{2/3}.$$

Since $|\nabla u^\varepsilon| \leq C_1$ uniformly in ε and $t \geq \tau$ we get $|u^\varepsilon(x,t) - u^\varepsilon(y,s)| \leq C(t - s)^{1/3} + 3C_1 r.$///

Proof of Theorem 1. We see that the limit u is Hölder continuous for $t \geq \tau$. Now we show that u is continuous at t = 0 if all that is assumed is $u_0 \in C_c(\mathbb{R}^n)$. Let $\underline{u}_{0_j}$, $\bar{u}_{0_j} \in C_c^\infty(\mathbb{R}^n)$ with $0 \leq \underline{u}_{0_j} \leq u_0 \leq \bar{u}_{0_j}$ and $\bar{u}_{0_j} - \underline{u}_{0_j} \leq \frac{1}{j}$. Let $\underline{u}_j^\varepsilon$, $\bar{u}_j^\varepsilon$ be the correspond- solutions to P_ε. We have $\underline{u}_j^\varepsilon \leq u^\varepsilon \leq \bar{u}_j^\varepsilon$ Hence in the limit $\underline{u}_j \leq u \leq \bar{u}_j$. But from lemma 4 $\underline{u}_j$ and $\bar{u}_j$ are continuous at $t = 0$. Since this is true for each j we see that u is continuous at $t = 0$.

It follows easily that $u^{-p}\chi(\{u > 0\}) \in L^1_{loc}(\mathbb{R}^n \times [0,\infty))$. Indeed if $\zeta(x,t) \geq 0, \zeta \in C_c^\infty(\mathbb{R}^n \times [0,\infty))$ then from P_ε we get

$$\int_0^\infty\int_{\mathbb{R}^n} f(\varepsilon,u^\varepsilon)\zeta \, dxdt \leq C(\zeta)$$

and the result follows from Fatou's lemma.

Next we point out that u satisfies $\Delta u - u_t = u^{-p}$ on $\{u > 0\} \cap \{t > 0\}$ in the classical sense. Indeed if $u(x_0,t_0) > 0$ for some $t_0 > 0$ then from the uniform continuity of the family $\{u^\varepsilon\}$ we get $0 < c_1 \leq u^\varepsilon \leq c_2 < \infty$ in some neighborhood of (x_0,t_0) for $0 \leq \varepsilon < 1$ where c_i are constants depending on $u(x_0,t_0)$. Since $f(\varepsilon,u) \to u^{-p}$ as $\varepsilon \downarrow 0$ uniformly on the set $\{u: c_1 \leq u\}$ the assertion follows from local parabolic estimates. Again using parabolic estimates and the fact that

$$\left|\frac{\partial f}{\partial u}(\varepsilon,u)\right| \leq C(c_1)$$

for $0 < \varepsilon \leq 1$ and $u \geq c_1$ one sees that ∇u^ε converge uniformly to ∇u on compact subsets of $\{u > 0\} \cap \{t > 0\}$. Thus the estimates from lemmas 2 and 3 hold on $\{u > 0\}$ for $\varepsilon = 0$. Also if at some point (x_0,t_0), $u(x_0,t_0) = 0$ for $t_0 > 0$ we have seen that $(u)^{1/\beta}$ is Lipschitz continuous is x so $0 \leq u(x,t_0) \leq C|x - x_0|^\beta$ with $\beta > 1$. Thus $\nabla u(x_0,t_0)$ exists and vanishes. In short the inequalities in lemmas 2 and 3 hold in $\mathbb{R}^n \times (0,\infty)$ and $\nabla u(x,t)$ is a continuous function for $t > 0$.

Finally we show that u satisfies (0.1). Consider a function $\varphi(s) \in C^\infty(\mathbb{R}^n)$ such that

$$\varphi(s) = s - 1 \qquad \text{for } s \geq 2,$$
$$\qquad = 0 \qquad \text{for } s < 1/2,$$

and $\varphi'(s), \varphi''(s) \geq 0$. Define for $h > 0$, $\varphi_h(s) := h\varphi(s/h)$. Now let $\zeta \in C_c^\infty(\mathbb{R}^n \times (0,\infty))$ and fix $h > 0$. Then

$$\int_{\mathbb{R}^n\times(0,\infty)} \varphi_h(u)\ (\Delta\zeta + \zeta_t)dxdt \underset{\varepsilon\downarrow 0}{\leftarrow}$$

$$\int_{\mathbb{R}^n\times(0,\infty)} \varphi_h(u^\varepsilon)\ (\Delta\zeta + \zeta_t)dxdt$$

$$\int_{\mathbb{R}^n\times(0,\infty)} \left(\varphi_h'(u^\varepsilon)f(\varepsilon,u^\varepsilon) + \varphi_h''(u^\varepsilon)\ |\nabla u^\varepsilon|^2\right)\zeta dxdt.$$

From the preceding estimates we have that the term in brackets converges uniformly on supp ζ as $\varepsilon \downarrow 0$ yielding

$$\int_{\{u>0\}} \varphi_h'(u)u^{-p}\zeta dxdt + \int_{\{0<u\leq 2h\}} \varphi_h''(u)\ |\nabla u|^2\zeta dxdt.$$

Since $|\varphi_h'(u)| \leq 1$ and $\varphi_h'(u) \to \chi(\{u > 0\})$ as $h \downarrow 0$ the local integrability of $u^{-p}\chi(\{u > 0\})$ implies that the first term goes to

$$\int_{\{0<u\}} u^{-p}\zeta dxdt$$

The second term is dominated by

$$\frac{c_1}{h}\int_{\{0<u<2h\}} |\nabla u|^2|\zeta|dxdt \leq c_2 \int_{\{0<u<2h\}} |\zeta|(u^{-p} + c_3)dxdt$$

where we have used lemma 3. The local integrability of $u^{-p}\chi(\{u > 0\})$ implies this term goes to zero as $h \to 0$. ///

Corollary 1. $u(x,t)$ has compact support.

Proof. Let $M_0 = \| u_0 \|_{L^\infty}$ and consider the o.d.e.s.

$$\frac{d}{dt} v^\varepsilon(t) = -f(\varepsilon, v^\varepsilon) \qquad \text{for } t > 0$$

$$v^\varepsilon(0) = M_0$$

$$\frac{d^2}{dx^2} w^\varepsilon(x) = f(\varepsilon, w^\varepsilon) \qquad \text{for } x > 0$$

$$w^\varepsilon(0) = M_0$$

$$\lim_{x\to\infty} w^\varepsilon(x) = 0.$$

It is straightforward to show that

$$v^\varepsilon(t) \to v(t) = [M_0^{1+p} - (1+p)t]^{+\,\frac{1}{1+p}} \qquad \text{for } t > 0$$

$$w^\varepsilon(x) \to w(x) = c_2 \;\; [c_1 M_0^{\frac{1+p}{2}} - x]^{+\,\frac{2}{1+p}} \qquad \text{for } x > 0$$

where c_1, c_2 are positive constants depending on p and the convergence is uniform as $\varepsilon \downarrow 0$.

Let supp $u_0 \subseteq \{x: |x_i| \leq R \;\; i = 1,\cdots,n\}$. Then from comparison when $\varepsilon > 0$ we have that $u_\varepsilon^\varepsilon(x,t) \leq w^\varepsilon(x_i - R)$ for $x_i \geq R$. Passing to the limit as $\varepsilon \downarrow 0$ we see that $u(x,t) \equiv 0$ for $x_i \geq R_1(R, M_0 . p)$. A similar argument yields the result for x_i negative and t. ///

Remark. The nonlinearity (0.3) was used mainly because it allowed us to find the limit u directly. If one has a nonlinearity $f(\varepsilon,s)$ satisfying:

1) $f(\varepsilon,s) \geq 0$ for $s \geq 0$, $\varepsilon \geq 0$, $f(\varepsilon,0) = 0$,
2) $f(\varepsilon,\cdot) \in C^2(\mathbb{R})$ for $\varepsilon > 0$,

3) $f(\varepsilon,s) \xrightarrow[\varepsilon\to 0]{} f(0,s)$ uniformly on $[h,\infty)$ for each $h > 0$,

4) $\left|\frac{d}{ds} f(\varepsilon,s)\right| \leq C(h) < \infty$ for $s \geq h > 0$, $\varepsilon > 0$.

5) $0 \leq \int_0^s f(\varepsilon,\tau)d\tau \leq Cs^{1-p}$ for $0 \leq s \leq 1$, c independent of ε.

6) $f(0,s) = s^{-p}\hat{f}(s)$ where $\hat{f}(s) \in C^1(0,\infty))$ and $0 < \hat{f}(s)$ for $s \geq 0$,

then the preceding a-priori estimates allow us to find a subsequence $\{u^{\varepsilon_i}\}$ with $u^{\varepsilon_i} \to \tilde{u}$ where $\tilde{u}(x,t)$ is a solution to (0.1). We point out that we have not shown the solution to (0.1) is unique; this is an open question.

2. THE INITIAL VALUE-BOUNDARY VALUE PROBLEM

The proof of Theorem 2 follows along the same lines as that of Theorem 1. The only point that needs comment is the analogue of lemma 2.

Lemma 5. Let u^ε satisfy

$$\Delta u^\varepsilon - u^\varepsilon_t = f(\varepsilon,u^\varepsilon) \qquad \text{in } \Omega \times (0,\infty)$$

$$u^\varepsilon = \psi \qquad \text{on } \Gamma$$

where ψ is as in theorem 2. Then there is a constant $\overline{M} = \overline{M}(\psi)$ so that $|\nabla_x u^\varepsilon(x,t)|^2 \leq 2F_\varepsilon(u^\varepsilon(x,t)) + \overline{M}u^\varepsilon(x,t)$ for $0 \leq t$, $x \in \overline{\Omega}$.

Proof. Since $\psi \geq \delta > 0$ on $\partial\Omega \times [0,\infty)$ and in light of the proof of lemma 2 it suffices to show that

$$|\nabla u^\varepsilon(x,t)| \leq C < \infty \qquad \text{for } x \in \partial\Omega,\ 0 \leq t.$$

To do this we first show near $\partial\Omega$ that $u^\varepsilon \geq \tilde{\delta} > 0$ uniformly in ε.

Consider the set $A_r(x_0) := \{x: r < |x - x_0|) < 2r\}$ when r is so small that for each $p \in \partial\Omega$ a translate of $B_r(0)$ satisfies the exterior sphere condition at p and that if $B_r(\tilde{p})$ is this translate then $\psi(x,0) \geq \delta/2$ for $x \in A_r(\tilde{p}) \cap \Omega$.

Now let $v^\varepsilon(x)$ be a solution to

$$\Delta v^\varepsilon = f(\varepsilon, v^\varepsilon) \qquad \text{in } A_r(0) \quad ,$$

$$v^\varepsilon = \delta/2 \qquad \text{if } |x| = r \quad ,$$

$$v^\varepsilon = 0 \qquad \text{if } |x| = 2r \ ,$$

obtained by minimizing $J(w) := \int_{A_r} (|\nabla w|^2/2 + F_\varepsilon(w))dx$ over the set

$$K := \{w : w \in H^1(A_r),\ w = \delta/2 \text{ on } |x| = r,\ w = 0$$
$$\text{on } |x| = 2r\}.$$

Such a minimizer can be found that is radial, $0 \leq v^\varepsilon \leq \delta/2$, and from the definition of $J(\cdot)$ we see that

$$\int_r^{2r} \left| \frac{d}{ds} v^\varepsilon(s) \right|^2 ds \leq C < \infty$$

where C is independent of ε. This implies that there exists $\tilde{r}$, $r < \tilde{r} < 2r$, and a constant $\tilde{\delta} > 0$ so that $\tilde{\delta} \leq v^\varepsilon(x)$ for $r \leq |x| \leq \tilde{r}$ and $0 < \varepsilon \leq 1$.

We can use a translate of v^ε in comparison with u^ε to obtain $\tilde{\delta} \leq u^\varepsilon(x,t)$ for $0 \leq t$ and $x \in \Omega$ so that $\text{dist.}(x,\partial\Omega) \leq \tilde{r} - r$. Setting $w^\varepsilon = u^\varepsilon - \psi$ we have $|w^\varepsilon| \leq C_1 < \infty$, $|\Delta w^\varepsilon - w_t^\varepsilon| \leq C_2 < \infty$ for $0 \leq t$, x near $\partial\Omega$, and $w^\varepsilon = 0$ on Γ where C_i are independent of ε. Using barriers it follows that

$$|\nabla u^\varepsilon(x,t)| \leq C \qquad \text{for } 0 \leq t\ ,\quad x \in \partial\Omega. ///$$

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