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A PENALIZED NEUMANN CONTROL APPROACH FOR SOLVING AN OPTIMAL DIRICHLET CONTROL PROBLEM FOR THE NAVIER–STOKES EQUATIONS*

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Abstract. We introduce a penalized Neumann boundary control approach for solving an optimal Dirichlet boundary control problem associated with the two- or three-dimensional steady-state Navier–Stokes equations. We prove the convergence of the solutions of the penalized Neumann control problem, the suboptimality of the limit, and the optimality of the limit under further restrictions on the data. We describe the numerical algorithm for solving the penalized Neumann control problem and report some numerical results.

Key words. optimal control, Neumann control, Dirichlet control, Navier–Stokes equations, finite element method

AMS subject classifications. 35B40, 35B37, 35Q30, 65M60

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1. Introduction.

Optimal control for the Navier–Stokes equations has been the subject of extensive study in recent years and much progress has been made both mathematically and computationally; see, e.g., [AT], [FS1], [FS2], [FS3], [Fu1], [Fu2], [Fu3], [Gun], [GHS1], [GHS2], [GHS3], [HS], [HY1], [HY2], [HYR], [Li], [S1], and [S2]. In this work we confine ourselves to optimal Dirichlet control problems for the steady-state Navier–Stokes equations. Dirichlet controls, i.e., boundary velocity controls or boundary mass flux controls, are common in applications. For instance, one often attempts, through the suction and injection of fluid through orifices on the boundary to reduce the drag on a body moving through a fluid. Optimal Dirichlet control problems for time-dependent Navier–Stokes equations were studied in [FGH] for general Dirichlet controls and in [FS1], [FS2], [FS3] and [S1], [S2] for Dirichlet controls in a special case, namely, when the control is of the separation-of-variable type. Optimal Dirichlet control problems for steady-state Navier–Stokes equations were studied in [GHS2], [GHS3], and [HS]. In [GHS3], optimal Dirichlet controls of finite dimensions were analyzed and some numerical results presented. In [GHS2], the existence and regularity of optimal solutions for optimal Dirichlet control problems were proved; an optimality system of equations was derived; and finite element approximations were defined and optimal error estimates established. In [HS], optimal control problems with smooth Dirichlet controls were studied; in particular, an optimality system of equations was derived. The optimality systems in [GHS2] and [HS] involve a boundary Laplacian or a boundary biharmonic equation that complicates the numerical resolution of the optimality systems. In finite element approximations of (uncontrolled)

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boundary value problems for partial differential equations, Neumann boundary conditions are generally easier to handle than the Dirichlet ones; and the same is true of optimal boundary control problems. Inspired by the penalty method for solving Dirichlet problems for (uncontrolled) elliptic partial differential equations (see [Ba]), we propose in this article a penalty method for solving the optimal Dirichlet control problem. The proposed penalty approach avoids the boundary Laplacian or boundary biharmonic equations that appeared in [GHS2] and [HS]. The advantages (as well as disadvantages) of the penalty method in solving uncontrolled Dirichlet boundary value problems essentially hold true in solving optimal Dirichlet boundary control problems.

The optimal Dirichlet control problem we consider is to minimize the vorticity of viscous, incompressible flow by choosing an appropriate boundary velocity. Precisely, we will study the following optimal control problem: find a triplet $(\mathbf{u}, p, \mathbf{g})$ such that the functional

$$\mathcal{J}(\mathbf{u},\mathbf{g}) = \frac{\alpha}{2}\int_\Omega |\mathrm{curl}\,\mathbf{u}|^2\,d\mathbf{x} + \frac{\beta}{2}\int_\Gamma |\mathbf{g}|^2\,ds \tag{1.1}$$

is minimized subject to the steady-state Navier–Stokes equations

$$-\nu\,\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega\,, \tag{1.2}$$

$$\mathrm{div}\,\mathbf{u} = 0 \qquad \text{in } \Omega, \tag{1.3}$$

and

$$\mathbf{u} = \mathbf{g} \qquad \text{on } \Gamma\,. \tag{1.4}$$

Here, Ω is a two- or three-dimensional bounded and simply connected flow domain (Ω is assumed to be of class $C^{1,1}$ or convex in $\mathbb{R}^2$ and of class $C^{1,1}$ in $\mathbb{R}^3$); Γ denotes the boundary of Ω; $\nu > 0$ denotes the constant viscosity; $\mathbf{u}$ and p denote the velocity field and the pressure field, respectively; $\mathbf{f}$ is a prescribed forcing term; and $\mathbf{g}$ is the boundary velocity—the control field. Because of the divergence-free condition on $\mathbf{u}$, $\mathbf{g}$ must necessarily satisfy $\int_\Gamma \mathbf{g}\cdot\mathbf{n}\,ds = 0$. The constants α and β appearing in the functional (1.1) are two positive parameters that adjust the relative weights of the two terms in the functional. Note that we use the same notation curl to denote the curl operators in two dimensions and three dimensions, although they are defined differently. The choice of the functional is motivated by the fact that irrotational flows have no local flow recirculations. We hope that minimizing the L^2-norm of the vorticity will lead to reduction in flow recirculations.

The plan of the paper is as follows. In section 2, we review mathematical background materials related to the steady-state Navier–Stokes equations and give a precise description of the optimal control problem we consider. In section 3, we introduce the penalized Neumann control approach and prove the existence of an optimal solution for the penalized Neumann control problem. In section 4, we demonstrate the convergence of the penalized optimal boundary control solutions and show that the limit is suboptimal. In section 5, we show that the limit found in section 4 is indeed an optimal solution for the optimal Dirichlet control problem. Finally in section 6, we describe the formal procedures for computing an approximate optimal solution and present some numerical results.

2. Preliminaries. Throughout, C or C_i (where i is any subscript) denotes a constant depending only on the domain Ω. We denote by $L^2(\Omega)$ the collection of Lebesgue square-integrable functions defined on Ω. Let $H^1(\Omega) = \big\{v \in L^2(\Omega) : \frac{\partial v}{\partial x_i} \in L^2(\Omega) \text{ for } i = 1, \ldots, d\big\}$, where $d = 2$ or 3; $H^1_0(\Omega) = \{v \in H^1(\Omega) : v|_\Gamma = 0\}$; $L^2_0(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q\, d\mathbf{x} = 0\}$; and $H^m(\Omega) = \big\{v \in L^2(\Omega) : \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \in L^2(\Omega) \text{ for all } \alpha = (\alpha_1, \ldots, \alpha_d) \text{ with } |\alpha| \le m\big\}$, where $d = 2$ or 3. Here $m > 0$ is an integer. For the definition of fractional ordered Sobolev spaces $H^s(\Omega)$ (s noninteger), see [Ad]. Negative ordered Sobolev spaces $H^{-s}(\Omega)$ ($s > 0$) are defined as the dual space, i.e., $H^{-s}(\Omega) = \big\{H^s_0(\Omega)\big\}^*$. Vector-valued counterparts of these spaces are denoted by boldface symbols, e.g., $\mathbf{H}^1(\Omega) = [H^1(\Omega)]^d$, where $d = 2$ or 3. The trace spaces $H^r(\Gamma)$ are the restriction to the boundary of $H^{r+1/2}(\Omega)$. We denote the norms and inner products for $H^s(\Omega)$ or $\mathbf{H}^s(\Omega)$ by $\|\cdot\|_s$ and $(\cdot,\cdot)_s$, respectively. The $L^2(\Omega)$ or $\mathbf{L}^2(\Omega)$ inner product is denoted by $(\cdot,\cdot)$. We denote the norms and inner products for $H^r(\Gamma)$ or $\mathbf{H}^r(\Gamma)$ by $\|\cdot\|_{r,\Gamma}$ and $(\cdot,\cdot)_{r,\Gamma}$, respectively. The $L^2(\Gamma)$ or $\mathbf{L}^2(\Gamma)$ inner product is denoted by $(\cdot,\cdot)_\Gamma$. The duality pairing between a Sobolev space $H^s(\Omega)$ ($s > 0$) and its dual space is denoted by $\langle\cdot,\cdot\rangle$. The duality pairing between a trace space $H^r(\Gamma)$ ($r > 0$) and its dual space is denoted by $\langle\cdot,\cdot\rangle_\Gamma$.

We define the following standard bilinear, trilinear forms associated with the Navier–Stokes equations

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega (\nabla\mathbf{u}) : (\nabla\mathbf{v})\, d\mathbf{x} \qquad \forall\, \mathbf{u},\mathbf{v} \in \mathbf{H}^1(\Omega)\,,$$

$$b(\mathbf{u},q) = -\int_\Omega q \operatorname{div}\mathbf{u}\, d\mathbf{x} \qquad \forall\, \mathbf{u} \in \mathbf{H}^1(\Omega), \forall\, q \in L^2(\Omega),$$

and

$$c(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\, d\mathbf{x} \qquad \forall\, \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{H}^1(\Omega)\,.$$

We now summarize some properties of these linear forms. We have the coercivity relations associated with $a(\cdot,\cdot)$:

$$a(\mathbf{u},\mathbf{u}) = \|\nabla\mathbf{u}\|_0^2 \ge C_0\,\|\mathbf{u}\|_1^2 \qquad \forall\, \mathbf{u} \in \mathbf{H}^1_0(\Omega) \tag{2.1}$$

(which is a direct consequence of Poincaré inequality) and

$$\int_\Gamma |\mathbf{v}|^2\, ds + \int_\Omega \big|\nabla\mathbf{v}\big|^2\, d\mathbf{x} \ge C_1\|\mathbf{v}\|_1^2 \qquad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega) \tag{2.2}$$

(whose proof can be found in [Ne]). The forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, and $c(\cdot,\cdot,\cdot)$ are all continuous; in particular, we have

$$\big|c(\mathbf{u},\mathbf{v},\mathbf{w})\big| \le C_2\,\|\mathbf{u}\|_1\,\|\mathbf{v}\|_1\,\|\mathbf{w}\|_1\,. \tag{2.3}$$

The bilinear form $b(\cdot,\cdot)$ satisfies the following inf-sup conditions:

$$\inf_{q\in L^2(\Omega)}\ \sup_{\mathbf{v}\in\mathbf{H}^1(\Omega)} \frac{\int_\Omega q \operatorname{div}\mathbf{v}\, d\mathbf{x}}{\|q\|_0\,\|\mathbf{v}\|_1} \ge C_3 \tag{2.4}$$

and

$$\inf_{q\in L^2_0(\Omega)} \sup_{\mathbf{v}\in\mathbf{H}^1_0(\Omega)} \frac{\int_\Omega q\,\mathrm{div}\,\mathbf{v}\,d\mathbf{x}}{\|q\|_0\,\|\mathbf{v}\|_1} \geq C_3\,. \tag{2.5}$$

The proof of (2.5) can be found in [GR], and that of (2.4) in [Ma]. Using integration-by-parts techniques we may deduce

$$\int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{v}\,d\mathbf{x} = \frac{1}{2}\int_\Gamma (\mathbf{v}\cdot\mathbf{n})|\mathbf{v}|^2\,ds \qquad \forall\,\mathbf{v}\in\mathbf{H}^1(\Omega) \text{ with } \mathrm{div}\,\mathbf{v}=0\,. \tag{2.6}$$

We now give the definition of a solution for the Navier–Stokes equations with a Dirichlet boundary condition. Throughout, we assume $\mathbf{f}\in\mathbf{L}^2(\Omega)$.

Definition 2.1. *Let* $\mathbf{g}\in\mathbf{H}^{1/2}(\Gamma)$*. A pair* $(\mathbf{u},p)\in\mathbf{H}^1(\Omega)\times L^2_0(\Omega)$ *is said to be a solution of the Navier–Stokes equations* (1.2)–(1.4) *iff*

$$\nu\,a(\mathbf{u},\mathbf{v}) + c(\mathbf{u},\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\mathbf{x} \qquad \forall\,\mathbf{v}\in\mathbf{H}^1_0(\Omega)\,, \tag{2.7}$$

$$b(\mathbf{u},q) = 0 \qquad \forall\,q\in L^2_0(\Omega), \tag{2.8}$$

and

$$\mathbf{u}\big|_\Gamma = \mathbf{g}\,. \tag{2.9}$$

A proof of the existence of a solution in the sense of Definition 2.1 can be found in [GR] and [Te].

The optimal Dirichlet control problem we consider can be stated as:

(P) seek a $(\mathbf{u},p,\mathbf{g})\in\mathbf{H}^1(\Omega)\times L^2_0(\Omega)\times\mathbf{L}^2(\Gamma)$ such that (1.1) is minimized subject to (2.7)–(2.9).

We define the admissible set $\mathcal{U}_{ad}$ for (P) by

$$\mathcal{U}_{ad} = \left\{(\mathbf{u},p,\mathbf{g})\in\mathbf{H}^1(\Omega)\times L^2_0(\Omega)\times\mathbf{L}^2(\Gamma) : (\mathbf{u},p,\mathbf{g}) \text{ satisfies (2.7)–(2.9)}\right\}.$$

3. Penalized optimal Neumann control problems. For each $\epsilon\in(0,1/\nu)$, we consider the following Neumann control problem: find a $(\mathbf{u}_\epsilon,p_\epsilon,\mathbf{g}_\epsilon)\in\mathbf{H}^1(\Omega)\times L^2(\Omega)\times\mathbf{L}^2(\Gamma)$ such that the functional

$$\mathcal{J}(\mathbf{u},\mathbf{g}) = \frac{\alpha}{2}\int_\Omega |\mathrm{curl}\,\mathbf{u}|^2\,d\mathbf{x} + \frac{\beta}{2}\int_\Gamma |\mathbf{g}|^2\,ds \tag{3.1}$$

is minimized subject to the steady-state Navier–Stokes equations (1.2)–(1.3) with the nonlinear Neumann (or Robin)-type boundary condition

$$-p\mathbf{n} + \nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - \frac{1}{2}(\mathbf{u}\cdot\mathbf{n})\mathbf{u} + \frac{1}{\epsilon}\,\mathbf{u} = \frac{1}{\epsilon}\,\mathbf{g} \qquad \text{on } \Gamma\,. \tag{3.2}$$

Formally, we see that as $\epsilon\to 0$, the Neumann boundary condition (3.2) reduces to the Dirichlet boundary condition (1.4), and therefore we expect that optimal solutions for the Neumann boundary control problems approach an optimal solution for the

Dirichlet boundary control problem. Here, ϵ acts as a penalty constant. By formally multiplying (1.2) by a test function $\mathbf{v}$ and integrating by parts, we obtain

$$\nu\int_\Omega (\nabla\mathbf{u}):(\nabla\mathbf{v})\,d\mathbf{x} - \nu\int_\Gamma \frac{\partial\mathbf{u}}{\partial\mathbf{n}}\cdot\mathbf{v}\,ds + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,d\mathbf{x}$$
$$-\int_\Omega p\,\mathrm{div}\,\mathbf{v}\,d\mathbf{x} + \int_\Gamma p\mathbf{n}\cdot\mathbf{v}\,ds = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\mathbf{x} \qquad \forall\,\mathbf{v}\in\mathbf{H}^1(\Omega)\,.$$

Eliminating $-p\mathbf{n}+\nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}}$ in the boundary integrals using (3.2), we are led to the following definition of a (weak) solution for the Navier–Stokes equations with the Neumann boundary condition (3.2).

DEFINITION 3.1. *Let* $\mathbf{g}\in\mathbf{L}^2(\Gamma)$. *A pair* $(\mathbf{u},p)\in\mathbf{H}^1(\Omega)\times L^2(\Omega)$ *is said to be a solution of* (1.2)–(1.3) *with the Neumann condition* (3.2) *iff* $(\mathbf{u},p)$ *satisfies*

$$\nu\int_\Omega (\nabla\mathbf{u}):(\nabla\mathbf{v})\,d\mathbf{x} + \frac{1}{\epsilon}\int_\Gamma \mathbf{u}\cdot\mathbf{v}\,ds - \frac{1}{2}\int_\Gamma (\mathbf{u}\cdot\mathbf{n})\mathbf{u}\cdot\mathbf{v}\,ds + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,d\mathbf{x}$$
$$-\int_\Omega p\,\mathrm{div}\,\mathbf{v}\,d\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\mathbf{x} + \frac{1}{\epsilon}\int_\Gamma \mathbf{g}\cdot\mathbf{v}\,ds \qquad \forall\,\mathbf{v}\in\mathbf{H}^1(\Omega) \tag{3.3}$$

and

$$-\int_\Omega q\,\mathrm{div}\,\mathbf{u}\,d\mathbf{x} = 0 \qquad \forall\,q\in L^2(\Omega)\,. \tag{3.4}$$

For each $\epsilon>0$, the penalized optimal Neumann control problems we consider can be stated as follows:

$$\text{seek a } (\mathbf{u}_\epsilon,p_\epsilon,\mathbf{g}_\epsilon)\in\mathbf{H}^1(\Omega)\times L^2(\Omega)\times\mathbf{L}^2(\Gamma) \text{ such that (3.1) is minimized subject to (3.3)–(3.4)}\,. \tag{P$_\epsilon$}$$

In this section we will derive an estimate for solutions of the constraint equations (3.3)–(3.4) and then prove the existence of a solution for the optimal control problem (P)$_\epsilon$.

LEMMA 3.2. *Assume* $\epsilon\in(0,1/\nu)$ *and* $\mathbf{g}\in\mathbf{L}^2(\Gamma)$. *Then there exists a* $(\mathbf{u},p)\in\mathbf{H}^1(\Omega)\times L^2(\Omega)$ *satisfying* (3.3)–(3.4)*; furthermore,*

$$\frac{\nu}{2}\int_\Omega |\nabla\mathbf{u}|^2\,d\mathbf{x} + \frac{1}{4\epsilon}\int_\Gamma |\mathbf{u}|^2\,ds \le \frac{1}{2\nu C_1}\int_\Omega |\mathbf{f}|^2\,d\mathbf{x} + \frac{1}{\epsilon}\int_\Gamma |\mathbf{g}|^2\,ds \tag{3.5}$$

and

$$\|\overline{p}\|_0 \le \frac{1}{C_3}\big(\nu\,\|\mathbf{u}\|_1 + C_2\|\mathbf{u}\|_1^2 + \|\mathbf{f}\|_0\big)\,, \tag{3.6}$$

where $\overline{p}=p-\big(1/|\Omega|\big)\int_\Omega p\,d\mathbf{x}$.

Proof. Since $\epsilon\in(0,1/\nu)$, we may use (2.2) to obtain

$$\nu\int_\Omega |\nabla\mathbf{v}|\,d\mathbf{x} + \frac{1}{\epsilon}\int_\Gamma |\mathbf{v}|^2\,ds \ge \nu\Big(\int_\Omega |\nabla\mathbf{v}|\,d\mathbf{x} + \int_\Gamma |\mathbf{v}|^2\,ds\Big) \ge \nu C_1\|\mathbf{v}\|_1^2 \tag{3.7}$$

for every $\mathbf{v}\in\mathbf{H}^1(\Omega)$. This coercivity relation together with the inf-sup condition (2.4) allow us to prove the existence of a solution for (3.3)–(3.4) by using standard techniques for proving the existence of a solution for the Navier–Stokes equations with homogeneous Dirichlet conditions (see [Te] or [GR]). Here we also used the fact that $\mathbf{H}^1(\Omega)\big|_\Gamma = \mathbf{H}^{1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$ is continuously embedded into $\mathbf{L}^3(\Gamma)$ so that we have the continuity of the trilinear term $\int_\Gamma (\mathbf{u}\cdot\mathbf{n})\mathbf{w}\cdot\mathbf{v}\,ds$ on $\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)$. It remains to show that estimates (3.5)–(3.6) hold. Setting $\mathbf{v}=\mathbf{u}$ in (3.3) and using (3.4) we obtain

$$\begin{aligned}\nu\int_\Omega |\nabla\mathbf{u}|^2\,d\mathbf{x} + \frac{1}{\epsilon}\int_\Gamma |\mathbf{u}|^2\,ds \\ \le \frac{1}{2\nu C_1}\int_\Omega |\mathbf{f}|^2\,d\mathbf{x} + \frac{\nu C_1}{2}\int_\Omega |\mathbf{u}|^2\,d\mathbf{x} + \frac{1}{\epsilon}\int_\Gamma |\mathbf{g}|^2\,ds + \frac{1}{4\epsilon}\int_\Gamma |\mathbf{u}|^2\,ds,\end{aligned}$$

so that using (3.7) we are led to

$$\frac{\nu}{2}\int_\Omega |\nabla\mathbf{u}|^2\,d\mathbf{x} + \frac{1}{4\epsilon}\int_\Gamma |\mathbf{u}|^2\,ds \le \frac{1}{2\nu C_1}\int_\Omega |\mathbf{f}|^2\,d\mathbf{x} + \frac{1}{\epsilon}\int_\Gamma |\mathbf{g}|^2\,ds\,;$$

i.e., (3.5) is proved. For test functions $\mathbf{v}\in\mathbf{H}^1_0(\Omega)$, equation (3.3) reduces to

$$\nu\, a(\mathbf{u},\mathbf{v}) + c(\mathbf{u},\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = (\mathbf{f},\mathbf{v}) \qquad \forall\,\mathbf{v}\in\mathbf{H}^1_0(\Omega)\,.$$

Note that $\overline{p}\in L^2_0(\Omega)$, where $\overline{p} = p - \big(1/|\Omega|\big)\int_\Omega p\,d\mathbf{x}$, and

$$\begin{aligned} b(\mathbf{v},\overline{p}) &= -\int_\Omega \Big(p - \frac{1}{|\Omega|}\int_\Omega p\,d\mathbf{x}\Big)\operatorname{div}\mathbf{v}\,d\mathbf{x} = -\int_\Omega p\operatorname{div}\mathbf{v}\,d\mathbf{x} + \frac{1}{|\Omega|}\int_\Omega p\,d\mathbf{x}\int_\Omega \operatorname{div}\mathbf{v}\,d\mathbf{x} \\ &= b(\mathbf{v},p) + \frac{1}{|\Omega|}\int_\Omega p\,d\mathbf{x}\int_\Gamma \mathbf{v}\cdot\mathbf{n}\,ds = b(\mathbf{v},p) \qquad \forall\,\mathbf{v}\in\mathbf{H}^1_0(\Omega)\,.\end{aligned}$$

Using the last two relations and the second inf-sup condition (2.5) we easily obtain the estimate for $\overline{p}$:

$$\|\overline{p}\|_0 \le \frac{1}{C_3}\big(\nu\,\|\mathbf{u}\|_1 + C_2\|\mathbf{u}\|_1^2 + \|\mathbf{f}\|_0\big)\,. \qquad \square$$

We will make use of the following two lemmas to prove the existence of a solution for $(\mathrm{P})_\epsilon$.

LEMMA 3.3. *There exists a positive constant C_4 such that*

$$\|\mathbf{w}\|_1^2 \le C_4\Big(\int_\Omega |\operatorname{div}\mathbf{w}|^2\,d\mathbf{x} + \int_\Omega |\operatorname{curl}\mathbf{w}|^2\,d\mathbf{x} + \int_\Gamma |\mathbf{w}|^2\,ds\Big) \qquad \forall\,\mathbf{w}\in\mathbf{H}^1(\Omega)\,.$$

Proof. The proof follows standard techniques dealing with norm equivalence on Sobolev spaces (see, e.g., [Ne]). It proceeds as follows. Assume Lemma 3.3 is false. Then we may choose a sequence $\{\mathbf{w}^{(n)}\}_{n=1}^\infty \subset \mathbf{H}^1(\Omega)$ such that $\|\mathbf{w}^{(n)}\|_1 = 1$ for all n and

$$\int_\Omega |\operatorname{curl}\mathbf{w}^{(n)}|^2\,d\mathbf{x} + \int_\Omega |\operatorname{div}\mathbf{w}^{(n)}|^2\,d\mathbf{x} + \int_\Gamma |\mathbf{w}^{(n)}|^2\,ds < \frac{1}{n}\,. \tag{3.8}$$

The boundedness of $\{\mathbf{w}^{(n)}\}$ in $\mathbf{H}^1(\Omega)$ implies that there exists a $\mathbf{w}\in\mathbf{H}^1(\Omega)$ and a subsequence of $\{\mathbf{w}^{(n)}\}$, still denoted by $\{\mathbf{w}^{(n)}\}$, such that as $n\to\infty$,

$$\mathbf{w}^{(n)} \rightharpoonup \mathbf{w} \quad \text{in } \mathbf{H}^1(\Omega)\,, \qquad \mathbf{w}^{(n)} \to \mathbf{w} \quad \text{in } \mathbf{L}^2(\Omega)\,,$$

$$\operatorname{curl} \mathbf{w}^{(n)} \rightharpoonup \operatorname{curl} \mathbf{w} \quad \text{in } L^2(\Omega) \qquad \text{and} \qquad \operatorname{div} \mathbf{w}^{(n)} \rightharpoonup \operatorname{div} \mathbf{w}^{(n)} \quad \text{in } L^2(\Omega)\,.$$

Also, since the trace of $\mathbf{H}^1(\Omega)$ equals $\mathbf{H}^{1/2}(\Gamma)$ and the space $\mathbf{H}^{1/2}(\Gamma)$ is continuously imbedded into $\mathbf{L}^2(\Gamma)$, we have that

$$\mathbf{w}^{(n)} \rightharpoonup \mathbf{w} \quad \text{in } \mathbf{L}^2(\Gamma)\,.$$

From (3.8) we deduce that

$$\operatorname{curl} \mathbf{w}^{(n)} \to 0 \quad \text{in } L^2(\Omega)\,, \qquad \operatorname{div} \mathbf{w}^{(n)} \to 0 \quad \text{in } L^2(\Omega)\,,$$

and

$$\mathbf{w}^{(n)}|_\Gamma \to \mathbf{0} \quad \text{in } \mathbf{L}^2(\Gamma)\,.$$

By uniqueness of weak limits we have that

$$\operatorname{curl} \mathbf{w} = 0\,, \qquad \operatorname{div} \mathbf{w} = 0, \qquad \text{and} \qquad \mathbf{w}\big|_\Gamma = \mathbf{0}\,.$$

Since the boundary value problem $\operatorname{curl} \mathbf{w} = 0$, $\operatorname{div} \mathbf{w} = 0$, and $(\mathbf{w} \cdot \mathbf{n})|_\Gamma = 0$ admits a unique trivial solution (see [GR, Theorem I.3.6, p. 48]), we conclude $\mathbf{w} = \mathbf{0}$. This, of course, contradicts $\|\mathbf{w}\|_1 \geq \liminf_{n\to\infty} \|\mathbf{w}^{(n)}\|_1 = 1$. Thus the lemma is proved. □

LEMMA 3.4. *Assume* $\mathbf{u} \in \mathbf{V} \equiv \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \operatorname{div} \mathbf{v} = 0\}$ *is a solution of*

$$\begin{aligned} \nu \int_\Omega (\nabla \mathbf{u}) : (\nabla \mathbf{v})\, d\mathbf{x} + \frac{1}{\epsilon} \int_\Gamma \mathbf{u} \cdot \mathbf{v}\, ds - \frac{1}{2} \int_\Gamma (\mathbf{u} \cdot \mathbf{n}) \mathbf{u} \cdot \mathbf{v}\, ds + \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v}\, d\mathbf{x} \\ = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} + \frac{1}{\epsilon} \int_\Gamma \mathbf{g} \cdot \mathbf{v}\, ds \qquad \forall\, \mathbf{v} \in \mathbf{V}. \end{aligned}$$

Then, there exists a $p \in L^2(\Omega)$ *such that* (3.3)–(3.4) *hold.*

Proof. The result follows directly from the first inf-sup condition (2.4) and [GR, Theorem IV.1.4, p. 283]. □

We are now in a position to prove the existence of a solution to $(\mathrm{P})_\epsilon$.

THEOREM 3.5. *Assume* $\epsilon \in (0, 1/\nu)$. *Then there exists a solution* $(\mathbf{u}_\epsilon, p_\epsilon, \mathbf{g}_\epsilon) \in \mathbf{H}^1(\Omega) \times L^2(\Omega) \times \mathbf{L}^2(\Gamma)$ *for the optimal control problem* $(\mathrm{P})_\epsilon$.

Proof. From Lemma 3.2 it is obvious that there exists a $(\mathbf{u}, p, \mathbf{g}) \in \mathbf{H}^1(\Omega) \times L^2(\Omega) \times \mathbf{L}^2(\Gamma)$ such that (3.3)–(3.4) holds. Hence we may choose a minimizing sequence $\{(\mathbf{u}_m, p_m, \mathbf{g}_m)\} \subset \mathbf{H}^1(\Omega) \times L^2(\Omega) \times \mathbf{L}^2(\Gamma)$ such that

$$\begin{aligned} \nu \int_\Omega \nabla \mathbf{u}_m : \nabla \mathbf{v}\, d\mathbf{x} + \frac{1}{\epsilon} \int_\Gamma \mathbf{u}_m \cdot \mathbf{v}\, ds - \frac{1}{2} \int_\Gamma (\mathbf{u}_m \cdot \mathbf{n}) \mathbf{u}_m \cdot \mathbf{v}\, ds - \int_\Omega p_m \operatorname{div} \mathbf{v}\, d\mathbf{x} \\ + \int_\Omega (\mathbf{u}_m \cdot \nabla) \mathbf{u}_m \cdot \mathbf{v}\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} + \frac{1}{\epsilon} \int_\Gamma \mathbf{g}_m \cdot \mathbf{v}\, ds \qquad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega)\,, \end{aligned} \tag{3.9}$$

$$-\int_\Omega q \operatorname{div} \mathbf{u}_m\, d\mathbf{x} = 0 \qquad \forall\, q \in L^2(\Omega)\,, \tag{3.10}$$

and

$$\begin{aligned} \lim_{m\to\infty} \mathcal{J}(\mathbf{u}_m, \mathbf{g}_m) = \inf \big\{ \mathcal{J}(\mathbf{u}, \mathbf{g}) : (\mathbf{u}, p, \mathbf{g}) \in \mathbf{H}^1(\Omega) \times L^2(\Omega) \times \mathbf{L}^2(\Gamma) \\ \text{and } (\mathbf{u}, p, \mathbf{g}) \text{ satisfies (3.3)–(3.4)} \big\}\,. \end{aligned}$$

The boundedness of $\{\mathcal{J}(\mathbf{u}_m, \mathbf{g}_m)\}$ implies the boundedness of $\{\|\mathbf{g}_m\|_{0,\Gamma}\}$. Then using (3.5) we see that the set $\{\|\mathbf{u}_m\|_1\}$ is also bounded independent of m (although the bound depends on ϵ, which is fixed). Hence we may extract subsequences (still denoted by $\mathbf{u}_m$ and $\mathbf{g}_m$, respectively) such that

$$\mathbf{u}_m \rightharpoonup \mathbf{u}_\epsilon \quad \text{in } \mathbf{H}^1(\Omega)\,, \qquad \text{and} \qquad \mathbf{g}_m \rightharpoonup \mathbf{g}_\epsilon \quad \text{in } \mathbf{L}^2(\Gamma)$$

for some $(\mathbf{u}_\epsilon, \mathbf{g}_\epsilon) \in \mathbf{H}^1(\Omega) \times \mathbf{L}^2(\Gamma)$, as $m \to \infty$. Compact imbedding results imply the strong convergence $\mathbf{u}_m \to \mathbf{u}_\epsilon$ in $\mathbf{L}^4(\Omega)$ as $m \to \infty$. Using standard techniques in proving the existence of a solution to the steady-state Navier–Stokes equations, we may pass to the limit in (3.9)–(3.10) as $m \to \infty$ to conclude that $\mathbf{u}_\epsilon \in \mathbf{V}$ and $(\mathbf{u}_\epsilon, \mathbf{g}_\epsilon)$ satisfies

$$\begin{aligned} \nu \int_\Omega (\nabla \mathbf{u}_\epsilon) : (\nabla \mathbf{v})\, d\mathbf{x} + \frac{1}{\epsilon} \int_\Gamma \mathbf{u}_\epsilon \cdot \mathbf{v}\, ds - \frac{1}{2} \int_\Gamma (\mathbf{u}_\epsilon \cdot \mathbf{n}) \mathbf{u}_\epsilon \cdot \mathbf{v}\, ds & \\ + \int_\Omega (\mathbf{u}_\epsilon \cdot \nabla) \mathbf{u}_\epsilon \cdot \mathbf{v}\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} + \frac{1}{\epsilon} \int_\Gamma \mathbf{g}_\epsilon \cdot \mathbf{v}\, ds & \qquad \forall\, \mathbf{v} \in \mathbf{V}\,. \end{aligned}$$

The last equations and Lemma 3.4 imply that there exists a $p_\epsilon \in L^2(\Omega)$ such that

$$\begin{aligned} \nu \int_\Omega (\nabla \mathbf{u}_\epsilon) : (\nabla \mathbf{v})\, d\mathbf{x} + \frac{1}{\epsilon} \int_\Gamma \mathbf{u}_\epsilon \cdot \mathbf{v}\, ds - \frac{1}{2} \int_\Gamma (\mathbf{u}_\epsilon \cdot \mathbf{n}) \mathbf{u}_\epsilon \cdot \mathbf{v}\, ds - \int_\Omega p_\epsilon \operatorname{div} \mathbf{v}\, d\mathbf{x} & \\ + \int_\Omega (\mathbf{u}_\epsilon \cdot \nabla) \mathbf{u}_\epsilon \cdot \mathbf{v}\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} + \frac{1}{\epsilon} \int_\Gamma \mathbf{g}_\epsilon \cdot \mathbf{v}\, ds & \qquad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega) \end{aligned} \tag{3.11}$$

and

$$\int_\Omega q \operatorname{div} \mathbf{u}_\epsilon\, d\mathbf{x} = 0 \qquad \forall\, q \in L^2(\Omega)\,; \tag{3.12}$$

i.e., $(\mathbf{u}_\epsilon, p_\epsilon, \mathbf{g}_\epsilon) \in \mathbf{H}^1(\Omega) \times L^2(\Omega) \times \mathbf{L}^2(\Gamma)$ satisfies the constraint equations (3.3)–(3.4). Finally, using the sequential weak lower semicontinuity of the functional $\mathcal{J}(\cdot,\cdot)$ we obtain

$$\begin{aligned} \mathcal{J}(\mathbf{u}_\epsilon, \mathbf{g}_\epsilon) &\le \liminf_{m\to\infty} \mathcal{J}(\mathbf{u}_m, \mathbf{g}_m) \\ &= \inf \big\{ \mathcal{J}(\mathbf{u}, \mathbf{g}) : (\mathbf{u}, p, \mathbf{g}) \in \mathbf{H}^1(\Omega) \times L^2(\Omega) \times \mathbf{L}^2(\Gamma) \\ &\qquad \text{and } (\mathbf{u}, p, \mathbf{g}) \text{ satisfies (3.3)–(3.4)} \big\}\,. \end{aligned}$$

Hence, we have shown that $(\mathbf{u}_\epsilon, p_\epsilon, \mathbf{g}_\epsilon)$ is a solution for problem $(\mathrm{P})_\epsilon$. □

4. Convergence of solutions of Neumann control problems and suboptimality of the limit. Having shown the existence of a solution for $(\mathrm{P})_\epsilon$ for each ϵ, we now examine the convergence of $(\mathbf{u}_\epsilon, p_\epsilon, \mathbf{g}_\epsilon)$ as $\epsilon \to 0$.

THEOREM 4.1. *For each $\epsilon \in (0, 1/\nu)$, let $(\mathbf{u}_\epsilon, p_\epsilon, \mathbf{g}_\epsilon) \in \mathbf{H}^1(\Omega) \times L^2(\Omega) \times \mathbf{L}^2(\Gamma)$ be a solution of the optimal Neumann control problem $(\mathrm{P})_\epsilon$. Then there exists a $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{\mathbf{g}}) \in \mathcal{U}_{ad}$ and a subsequence $\{\epsilon_k\}_{k=1}^\infty$ such that as $k \to \infty$,*

$$\mathbf{u}_{\epsilon_k} \rightharpoonup \widehat{\mathbf{u}} \quad \text{in } \mathbf{H}^1(\Omega)\,, \qquad \overline{p_{\epsilon_k}} \rightharpoonup \widehat{p} \quad \text{in } L^2_0(\Omega) \qquad \text{and} \qquad \mathbf{g}_{\epsilon_k} \rightharpoonup \widehat{\mathbf{g}} \quad \text{in } \mathbf{L}^2(\Gamma)\,,$$

where $\overline{p_{\epsilon_k}} = p_{\epsilon_k} - (1/|\Omega|)\int_\Omega p_{\epsilon_k}\, d\mathbf{x}$. Moreover,

$$\mathbf{u}_{\epsilon_k} \to \widehat{\mathbf{u}} \quad \text{in } \mathbf{L}^2(\Omega)\,.$$

Proof. We first prove that the sets $\{\|\mathbf{u}_\epsilon\|_1\}$, $\{\|\overline{p_\epsilon}\|_0\}$, and $\{\|\mathbf{g}_\epsilon\|_{0,\Gamma}\}$ are all bounded independent of ϵ. Let $(\widetilde{\mathbf{u}}_\epsilon, \widetilde{p}_\epsilon)$ be the solution of (3.3)–(3.4) with $\mathbf{g} = \mathbf{0}$, i.e.,

$$\nu \int_\Omega (\nabla \widetilde{\mathbf{u}}_\epsilon) : (\nabla \mathbf{v})\, d\mathbf{x} + \frac{1}{\epsilon} \int_\Gamma \widetilde{\mathbf{u}}_\epsilon \cdot \mathbf{v}\, ds - \frac{1}{2} \int_\Gamma (\widetilde{\mathbf{u}}_\epsilon \cdot \mathbf{n}) \widetilde{\mathbf{u}}_\epsilon \cdot \mathbf{v}\, ds$$
$$+ \int_\Omega (\widetilde{\mathbf{u}}_\epsilon \cdot \nabla) \widetilde{\mathbf{u}}_\epsilon \cdot \mathbf{v}\, d\mathbf{x} - \int_\Omega \widetilde{p}_\epsilon \operatorname{div} \mathbf{v}\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} \qquad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega)$$

and

$$-\int_\Omega q \operatorname{div} \widetilde{\mathbf{u}}_\epsilon \, d\mathbf{x} = 0 \qquad \forall\, q \in L^2(\Omega)\,.$$

Lemma 3.2 gives us the estimate

$$\frac{\nu}{2} \int_\Omega |\nabla \widetilde{\mathbf{u}}_\epsilon|^2 \, d\mathbf{x} + \frac{1}{4\epsilon} \int_\Gamma |\widetilde{\mathbf{u}}_\epsilon|^2 \, ds \le \frac{1}{2\nu C_1} \int_\Omega |\mathbf{f}|^2 \, d\mathbf{x}\,.$$

Since $(\widetilde{\mathbf{u}}_\epsilon, \widetilde{p}_\epsilon, \mathbf{0})$ is an admissible element for $(\mathrm{P})_\epsilon$, we have that

$$\mathcal{J}(\mathbf{u}_\epsilon, \mathbf{g}_\epsilon) \le \mathcal{J}(\widetilde{\mathbf{u}}_\epsilon, \mathbf{0})\,,$$

so that

$$\frac{\alpha}{2} \int_\Omega |\operatorname{curl} \mathbf{u}_\epsilon|^2 \, d\mathbf{x} + \frac{\beta}{2} \int_\Gamma |\mathbf{g}_\epsilon|^2 \, ds \le \frac{\alpha}{2} \int_\Omega |\operatorname{curl} \widetilde{\mathbf{u}}_\epsilon|^2 \, d\mathbf{x}$$
$$\le \frac{\alpha}{2} \int_\Omega |\nabla \widetilde{\mathbf{u}}_\epsilon|^2 \, d\mathbf{x} \le \frac{\alpha}{2\nu^2 C_1} \int_\Omega |\mathbf{f}|^2 \, d\mathbf{x}\,,$$

which implies

$$\int_\Omega |\operatorname{curl} \mathbf{u}_\epsilon|^2 \, d\mathbf{x} \le \frac{1}{\nu^2 C_1} \int_\Omega |\mathbf{f}|^2 \, d\mathbf{x}$$

and

$$\int_\Gamma |\mathbf{g}_\epsilon|^2 \, ds \le \frac{\alpha}{\beta \nu^2 C_1} \int_\Omega |\mathbf{f}|^2 \, d\mathbf{x}\,.$$

Since $(\mathbf{u}_\epsilon, p_\epsilon, \mathbf{g}_\epsilon)$ satisfies (3.3)–(3.4), we have the estimate (from Lemma 3.2)

$$\frac{\nu}{2} \int_\Omega |\nabla \mathbf{u}_\epsilon|^2 \, d\mathbf{x} + \frac{1}{4\epsilon} \int_\Gamma |\mathbf{u}_\epsilon|^2 \, ds \le \frac{1}{2\nu C_1} \int_\Omega |\mathbf{f}|^2 \, d\mathbf{x} + \frac{1}{\epsilon} \int_\Gamma |\mathbf{g}_\epsilon|^2 \, ds$$

so that

$$\int_\Gamma |\mathbf{u}_\epsilon|^2 \, ds \le \frac{2\epsilon}{\nu C_1} \int_\Omega |\mathbf{f}|^2 \, d\mathbf{x} + 4 \int_\Gamma |\mathbf{g}_\epsilon|^2 \, ds \le \frac{2 + 4\alpha/\beta}{\nu^2 C_1} \int_\Omega |\mathbf{f}|^2 \, d\mathbf{x}\,.$$

Using Lemma 3.3 and the divergence-free condition of $\mathbf{u}_\epsilon$ we easily deduce that

$$\|\mathbf{u}_\epsilon\|_1^2 \le C_4 \Big(\int_\Omega |\operatorname{curl} \mathbf{u}_\epsilon|^2 \, d\mathbf{x} + \int_\Gamma |\mathbf{u}_\epsilon|^2 \, ds \Big) \le \frac{C_4 \big(3 + 4\alpha/\beta\big)}{\nu^2 C_1} \int_\Omega |\mathbf{f}|^2 \, d\mathbf{x}\,.$$

Combining this last estimate with (3.6) we easily see that

$$\|\overline{p_\epsilon}\|_0 \le C_3\left(\sqrt{\frac{C_4(3+4\alpha/\beta)}{\nu C_1}}\|\mathbf{f}\|_0 + \frac{C_2C_4(3+4\alpha/\beta)}{\nu^2 C_1}\|\mathbf{f}\|_0^2 + \|\mathbf{f}\|_0\right),$$

where $\overline{p_\epsilon} = p_\epsilon - (1/|\Omega|)\int_\Omega p_\epsilon\,d\mathbf{x}$. Thus we may extract a subsequence $\{\mathbf{u}_{\epsilon_k}\}$, $\{\overline{p_{\epsilon_k}}\}$, and $\{\mathbf{g}_{\epsilon_k}\}$ such that as $k \to \infty$,

$$\epsilon_k \to 0\,, \quad \mathbf{u}_{\epsilon_k} \rightharpoonup \widehat{\mathbf{u}} \quad \text{in } \mathbf{H}^1(\Omega)\,, \quad \overline{p_{\epsilon_k}} \rightharpoonup \widehat{p} \quad \text{in } L^2_0(\Omega)\,, \quad \text{and } \mathbf{g}_{\epsilon_k} \rightharpoonup \widehat{\mathbf{g}} \quad \text{in } \mathbf{L}^2(\Gamma)$$

for some $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{\mathbf{g}}) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{L}^2(\Gamma)$. Compact imbedding implies $\mathbf{u}_{\epsilon_k} \to \widehat{\mathbf{u}}$ in $\mathbf{L}^4(\Omega)$. We recall that $(\mathbf{u}_\epsilon, p_\epsilon, \mathbf{g}_\epsilon)$ satisfies equations (3.11)–(3.12). For each $\mathbf{v} \in \mathbf{H}^1_0(\Omega)$ and when $\epsilon = \epsilon_k$, equation (3.11) reduces to

$$\nu\int_\Omega \nabla\mathbf{u}_{\epsilon_k} : \nabla\mathbf{v}\,d\mathbf{x} + \int_\Omega (\mathbf{u}_{\epsilon_k}\cdot\nabla)\mathbf{u}_{\epsilon_k}\cdot\mathbf{v}\,d\mathbf{x} - \int_\Omega p_{\epsilon_k}\,\mathrm{div}\,\mathbf{v}\,d\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}\,.$$

Letting $k \to \infty$ yields

$$\nu\int_\Omega \nabla\widehat{\mathbf{u}} : \nabla\mathbf{v}\,d\mathbf{x} + \int_\Omega (\widehat{\mathbf{u}}\cdot\nabla)\widehat{\mathbf{u}}\cdot\mathbf{v}\,d\mathbf{x} - \int_\Omega \widehat{p}\,\mathrm{div}\,\mathbf{v}\,d\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\mathbf{x} \qquad \forall\,\mathbf{v}\in\mathbf{H}^1_0(\Omega)\,.$$

Letting $k \to \infty$ in (3.12) yields

$$-\int_\Omega q\,\mathrm{div}\,\widehat{\mathbf{u}}\,d\mathbf{x} = 0 \qquad \forall\,q\in L^2_0(\Omega)\,.$$

Multiplying (3.11) (where we set $\epsilon = \epsilon_k$) by ϵ_k and letting $k \to \infty$ we obtain

$$\int_\Gamma \widehat{\mathbf{u}}\cdot\mathbf{v}\,ds = \int_\Gamma \widehat{\mathbf{g}}\cdot\mathbf{v}\,ds \qquad \forall\,\mathbf{v}\in\mathbf{H}^1(\Omega)\,,$$

which implies $\widehat{\mathbf{u}}\big|_\Gamma = \widehat{\mathbf{g}}$. This last relation and trace theorems imply $\widehat{\mathbf{g}} \in \mathbf{H}^{1/2}(\Gamma)$. Hence we have shown that $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{\mathbf{g}})$ satisfies (2.7)–(2.9), i.e., that $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{\mathbf{g}})$ is an admissible element for the optimal control problem (P). The strong convergence $\mathbf{u}_{\epsilon_k} \to \widehat{\mathbf{u}}$ in $\mathbf{L}^2(\Omega)$ follows from the compact imbedding $\mathbf{H}^1(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^2(\Omega)$. ∎

REMARK. *In the Neumann control problem* $(\mathrm{P})_\epsilon$ *we do not require* $\int_\Gamma \mathbf{g}_\epsilon\cdot\mathbf{n}\,ds = 0$. *However, the limit* $\widehat{\mathbf{g}}$ *automatically satisfies* $\int_\Gamma \widehat{\mathbf{g}}\cdot\mathbf{n}\,ds = 0$ *from the fact that* $\mathrm{div}\,\widehat{\mathbf{u}} = 0$ *and* $\widehat{\mathbf{u}}\big|_\Gamma = \widehat{\mathbf{g}}$. *The fact that* $\widehat{\mathbf{u}}\big|_\Gamma = \widehat{\mathbf{g}}$ *also implies* $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$, *although each* $\mathbf{g}_\epsilon$ *is merely in* $\mathbf{L}^2(\Gamma)$. ∎

We wish to show that the limit $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{\mathbf{g}})$ is indeed a solution of the optimal Dirichlet control problem (P); namely, we will verify that

$$\mathcal{J}(\widehat{\mathbf{u}}, \widehat{\mathbf{g}}) \le \mathcal{J}(\mathbf{w}, \mathbf{z}) \qquad \forall\,(\mathbf{w}, r, \mathbf{z}) \in \mathcal{U}_{ad}. \tag{4.1}$$

In the remainder of this section we will prove that $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{\mathbf{g}})$ is suboptimal in the sense that (4.1) is satisfied if $(\mathbf{w}, r)$ satisfies the additional condition

$$-r\mathbf{n} + \nu\frac{\partial\mathbf{w}}{\partial\mathbf{n}} \in \mathbf{L}^2(\Gamma)\,. \tag{4.2}$$

The optimality of $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{\mathbf{g}})$ will be studied in the next section.

We will need the following lemma on integration by parts for functions in the space $H(\mathrm{div}\,,\Omega) \equiv \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\}$.

LEMMA 4.2. *Let* $\mathbf{w} \in H(\mathrm{div}\,,\Omega)$. *Then* $(\mathbf{w}\cdot\mathbf{n})\big|_\Gamma \in H^{-1/2}(\Gamma)$ *and*

$$\langle \mathbf{w}\cdot\mathbf{n}, v\rangle_\Gamma = \int_\Omega v\,\mathrm{div}\,\mathbf{w}\,d\mathbf{x} + \int_\Omega \mathbf{w}\cdot\nabla v\,d\mathbf{x} \qquad \forall\, v \in H^1(\Omega)\,,$$

where $\langle\cdot,\cdot\rangle_\Gamma$ *is the duality pairing between* $H^{-1/2}(\Gamma)$ *and* $H^{1/2}(\Gamma)$.

Proof. See [GR, equation (I.2.17), p. 28]. ∎

THEOREM 4.3. *Assume that* $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{\mathbf{g}}) \in \mathcal{U}_{ad}$ *is the limit defined in Theorem* 4.1. *Then*

$$\mathcal{J}(\widehat{\mathbf{u}}, \widehat{\mathbf{g}}) \le \mathcal{J}(\mathbf{w},\mathbf{z}) \qquad \forall\,(\mathbf{w}, r, \mathbf{z}) \in \mathcal{U}_{ad} \textit{ satisfying } (4.2).$$

Proof. Let $(\mathbf{w}, r, \mathbf{z})$ be an arbitrary element in $\mathcal{U}_{ad}$ satisfying (4.2). By the definition of $\mathcal{U}_{ad}$, $(\mathbf{w}, r, \mathbf{z})$ is a solution of

$$-\nu\Delta\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{w} + \nabla r = \mathbf{f} \qquad \text{in } \Omega\,, \tag{4.3}$$

$$\mathrm{div}\,\mathbf{w} = 0 \qquad \text{in } \Omega, \tag{4.4}$$

and

$$\mathbf{w}\big|_\Gamma = \mathbf{z}\,. \tag{4.5}$$

From (4.2) and the regularity results for the Navier–Stokes equations we obtain $(\mathbf{w}, r) \in \mathbf{H}^{3/2}(\Omega)\times H^{1/2}(\Omega)$ and $-r\mathbf{n} + \nu\frac{\partial\mathbf{w}}{\partial\mathbf{n}} \in \mathbf{L}^2(\Gamma)$. Using (4.3) and the imbedding results for Sobolev spaces we obtain

$$\mathrm{div}\,\big(-rI + \nu\nabla\mathbf{w}\big) = \nu\Delta\mathbf{w} - \nabla r = -\mathbf{f} + (\mathbf{w}\cdot\nabla)\mathbf{w} \in \mathbf{L}^2(\Omega)\,.$$

By the integration-by-parts formula (Lemma 4.2) we have

$$\begin{aligned}\int_\Gamma [(-rI + \nu\nabla\mathbf{w})\cdot\mathbf{n}]\cdot\mathbf{v}\,ds &= \int_\Omega [-\nabla r + \nu\Delta\mathbf{w}]\cdot\mathbf{v}\,d\mathbf{x} + \int_\Omega [-rI + \nu\nabla\mathbf{w}] : \nabla\mathbf{v}\,d\mathbf{x}\\ &= \int_\Omega [-\mathbf{f} + (\mathbf{w}\cdot\nabla)\mathbf{w}]\cdot\mathbf{v}\,d\mathbf{x} - \int_\Omega r\,\mathrm{div}\,\mathbf{v}\,d\mathbf{x} + \nu\int_\Omega \nabla\mathbf{w} : \nabla\mathbf{v}\,d\mathbf{x},\end{aligned}$$

so that using (4.5) and adding/subtracting terms, we are led to

$$\begin{aligned}\nu\int_\Omega \nabla\mathbf{w} : \nabla\mathbf{v}\,d\mathbf{x} + \frac{1}{\epsilon}\int_\Gamma \mathbf{w}\cdot\mathbf{v}\,ds - \frac{1}{2}\int_\Gamma (\mathbf{w}\cdot\mathbf{n})\mathbf{w}\cdot\mathbf{v}\,ds + \int_\Omega (\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}\,d\mathbf{x}&\\ - \int_\Omega r\,\mathrm{div}\,\mathbf{v}\,d\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\mathbf{x} + \frac{1}{\epsilon}\int_\Gamma \mathbf{z}_\epsilon\cdot\mathbf{v}\,ds \qquad \forall\,\mathbf{v}\in\mathbf{H}^1(\Omega)&\end{aligned}$$

where

$$\mathbf{z}_\epsilon \equiv \mathbf{z} + \epsilon\Big(-r\mathbf{n} + \nu\frac{\partial\mathbf{w}}{\partial\mathbf{n}}\Big) - \frac{\epsilon}{2}(\mathbf{w}\cdot\mathbf{n})\mathbf{w} \in \mathbf{L}^2(\Gamma)\,.$$

Thus, $(\mathbf{w}, r, \mathbf{z}_\epsilon)$ is an admissible element for $(\mathrm{P})_\epsilon$, so that

$$\mathcal{J}(\mathbf{w},\mathbf{z}_\epsilon) \ge \mathcal{J}(\mathbf{u}_\epsilon, \mathbf{g}_\epsilon)\,.$$

Combining the last inequality with

$$\begin{aligned}\mathcal{J}(\mathbf{w},\mathbf{z}_\epsilon) &\equiv \frac{\alpha}{2}\int_\Omega |\mathrm{curl}\,\mathbf{w}|^2\,d\mathbf{x} + \frac{\beta}{2}\int_\Gamma \left|\mathbf{z} - \epsilon r\mathbf{n} + \epsilon\nu\frac{\partial\mathbf{w}}{\partial\mathbf{n}} - \frac{\epsilon}{2}(\mathbf{w}\cdot\mathbf{n})\mathbf{w}\right|^2 ds\\ &= \mathcal{J}(\mathbf{w},\mathbf{z}) + \frac{\epsilon^2\beta}{2}\int_\Gamma \left|-r\mathbf{n} + \nu\frac{\partial\mathbf{w}}{\partial\mathbf{n}} - \frac{1}{2}(\mathbf{w}\cdot\mathbf{n})\mathbf{w}\right|^2 ds\\ &\quad + \epsilon\beta\int_\Gamma \mathbf{z}\cdot\left(-r\mathbf{n} + \nu\frac{\partial\mathbf{w}}{\partial\mathbf{n}} - \frac{1}{2}(\mathbf{w}\cdot\mathbf{n})\mathbf{w}\right) ds,\end{aligned}$$

we obtain

$$\begin{aligned}\mathcal{J}(\mathbf{w},\mathbf{z}) \geq \mathcal{J}(\mathbf{u}_\epsilon,\mathbf{g}_\epsilon) &- \frac{\epsilon^2\beta}{2}\int_\Gamma \left|-r\mathbf{n} + \nu\frac{\partial\mathbf{w}}{\partial\mathbf{n}} - \frac{1}{2}(\mathbf{w}\cdot\mathbf{n})\mathbf{w}\right|^2 ds\\ &- \epsilon\beta\int_\Gamma \mathbf{z}\cdot\left(-r\mathbf{n} + \nu\frac{\partial\mathbf{v}}{\partial\mathbf{n}} - \frac{1}{2}(\mathbf{w}\cdot\mathbf{n})\mathbf{w}\right) ds\,.\end{aligned}$$

Setting $\epsilon = \epsilon_k$ in the above relation (where ϵ_k is as defined in Theorem 4.1) and letting $k\to\infty$ we obtain

$$\mathcal{J}(\mathbf{w},\mathbf{z}) \geq \liminf_{k\to\infty}\mathcal{J}(\mathbf{u}_{\epsilon_k},\mathbf{g}_{\epsilon_k}) \geq \mathcal{J}(\widehat{\mathbf{u}},\widehat{\mathbf{g}})\,. \qquad \square$$

5. Optimality of the limit. In this section we will show that under certain restrictions on the data ν, $\mathbf{f}$, etc., the limit $(\widehat{\mathbf{u}},\widehat{p},\widehat{\mathbf{g}})$ defined in Theorem 4.1 is indeed a solution of the optimal Dirichlet control problem (P).

LEMMA 5.1. *Assume that* $(\widehat{\mathbf{u}},\widehat{p},\widehat{\mathbf{g}})$ *is the limit defined in Theorem* 4.1. *Then*

$$\mathcal{J}(\widehat{\mathbf{u}},\widehat{\mathbf{g}}) \leq \frac{\alpha}{2\nu^2 C_1}\int_\Omega |\mathbf{f}|^2\,d\mathbf{x}$$

Proof. Let $(\mathbf{u}_0,p_0)\in\mathbf{H}^1(\Omega)\times L^2(\Omega)$ be the solution of the Navier–Stokes equations (2.7)–(2.9) with the zero Dirichlet condition. Then $(\mathbf{u}_0,p_0,\mathbf{0})\in\mathcal{U}_{ad}$. Lemma 3.2 gives us the estimate

$$\frac{\nu}{2}\int_\Omega |\nabla\mathbf{u}_0|^2\,d\mathbf{x} \leq \frac{1}{2\nu C_1}\int_\Omega |\mathbf{f}|^2\,d\mathbf{x}\,.$$

The regularity theory for the Navier–Stokes equations implies $(\mathbf{u}_0,p_0)\in\mathbf{H}^2(\Omega)\times H^1(\Omega)$ so that $-p_0\mathbf{n}+\nu\frac{\partial\mathbf{u}_0}{\partial\mathbf{n}}\in\mathbf{L}^2(\Gamma)$. Hence by Theorem 4.3,

$$\mathcal{J}(\widehat{\mathbf{u}},\widehat{\mathbf{g}}) \leq \mathcal{J}(\mathbf{u}_0,\mathbf{0}) = \frac{\alpha}{2}\int_\Omega |\mathrm{curl}\,\mathbf{u}_0|^2\,d\mathbf{x} \leq \frac{\alpha}{2}\int_\Omega |\nabla\mathbf{u}_0|^2\,d\mathbf{x} \leq \frac{\alpha}{2\nu^2 C_1}\int_\Omega |\mathbf{f}|^2\,d\mathbf{x}\,. \qquad \square$$

LEMMA 5.2. *Define* $\mathbf{H}^{1/2}_n(\Gamma)\equiv\left\{\mathbf{z}\in\mathbf{H}^{1/2}(\Gamma) : \int_\Gamma \mathbf{z}\cdot\mathbf{n}\,ds = 0\right\}$. *Then there exist a constant* $C_5>0$ *(depending on* Ω *only) and an extension operator* $E:\mathbf{H}^{1/2}_n(\Gamma)\to\mathbf{V}$ *such that* $\|E\mathbf{z}\|_1\leq C_5\|\mathbf{z}\|_{1/2,\Gamma}$ *for every* $\mathbf{z}\in\mathbf{H}^{1/2}_n(\Gamma)$.

Proof. For each $\mathbf{z}\in\mathbf{H}^{1/2}_n(\Gamma)$ we define $\mathbf{w}=E\mathbf{z}\in\mathbf{V}$ as the unique solution of the Stokes problem

$$-\Delta\mathbf{w}+\nabla r = \mathbf{0} \qquad \text{in } \Omega\,,$$

$$\mathrm{div}\,\mathbf{w} = 0 \qquad \text{in } \Omega,$$

and

$$\mathbf{w}\big|_\Gamma = \mathbf{z}\,.$$

Clearly E maps $\mathbf{H}_n^{1/2}(\Gamma)$ into $\mathbf{V}$ linearly. The estimate $\|E\mathbf{z}\|_1 = \|\mathbf{w}\|_1 \leq C_5\|\mathbf{z}\|_{1/2,\Gamma}$ follows from the estimates for Dirichlet boundary value problems for the steady-state Stokes equations (see [Te]). ∎

LEMMA 5.3. *Assume that* $(\mathbf{w}, r, \mathbf{z}) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}_n^{1/2}(\Gamma)$ *and* $(\widetilde{\mathbf{w}}, \widetilde{r}, \widetilde{\mathbf{z}}) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}_n^{1/2}(\Gamma)$ *satisfy, respectively,*

$$\begin{cases} -\nu\Delta\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{w} + \nabla r = \mathbf{f} \qquad \text{in } \Omega\,,\\ \qquad\qquad \operatorname{div}\mathbf{w} = 0 \qquad \text{in } \Omega,\\ \qquad\qquad \mathbf{w} = \mathbf{z} \qquad \text{on } \Gamma, \end{cases}$$

and

$$\begin{cases} -\nu\Delta\widetilde{\mathbf{w}} + (\widetilde{\mathbf{w}}\cdot\nabla)\widetilde{\mathbf{w}} + \nabla\widetilde{r} = \mathbf{f} \qquad \text{in } \Omega\\ \qquad\qquad \operatorname{div}\widetilde{\mathbf{w}} = 0 \qquad \text{in } \Omega,\\ \qquad\qquad \widetilde{\mathbf{w}} = \widetilde{\mathbf{z}} \qquad \text{on } \Gamma\,. \end{cases}$$

Assume further that $\|\mathbf{w}\|_1 < \frac{\nu C_0}{4C_2}$ *and* $\|\mathbf{z}-\widetilde{\mathbf{z}}\|_{1/2,\Gamma} \leq \frac{\nu C_0}{4C_2C_5}$. *Then*

$$\|\widetilde{\mathbf{w}}-\mathbf{w}\|_1 \leq \Big(\frac{4C_5}{C_0} + \sqrt{2}C_5 + C_5\Big)\|\widetilde{\mathbf{z}}-\mathbf{z}\|_{1/2,\Gamma} + \frac{4C_2C_5^2}{\nu C_0}\|\widetilde{\mathbf{z}}-\mathbf{z}\|_{1/2,\Gamma}^2\,.$$

Proof. Set $\boldsymbol{\xi} = \widetilde{\mathbf{w}} - \mathbf{w}$ and $\sigma = \widetilde{r} - r$. Then, by subtracting the relevant equations for $(\widetilde{\mathbf{w}}, \widetilde{r})$ and $(\mathbf{w}, r)$, we see that $(\boldsymbol{\xi}, \sigma) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ satisfies

$$-\nu\Delta\boldsymbol{\xi} + (\boldsymbol{\xi}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\boldsymbol{\xi} + (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{\xi} + \nabla\sigma = \mathbf{0} \qquad \text{in } \Omega\,,$$

$$\operatorname{div}\boldsymbol{\xi} = 0 \qquad \text{in } \Omega,$$

and

$$\boldsymbol{\xi} = \widetilde{\mathbf{z}} - \mathbf{z} \qquad \text{on } \Gamma\,.$$

Put $\boldsymbol{\eta} = E(\widetilde{\mathbf{z}} - \mathbf{z})$, where E is the extension operator defined in Lemma 5.2. Then $\boldsymbol{\eta} \in \mathbf{H}^1(\Omega)$, $\boldsymbol{\eta}\big|_\Gamma = \widetilde{\mathbf{z}} - \mathbf{z}$, and $\|\boldsymbol{\eta}\|_1 \leq C_5\|\widetilde{\mathbf{z}} - \mathbf{z}\|_{1/2,\Gamma}$. Setting $\boldsymbol{\zeta} = \boldsymbol{\xi} - \boldsymbol{\eta}$ we see that

$$\begin{aligned} -\nu\Delta\boldsymbol{\zeta} &+ (\boldsymbol{\zeta}\cdot\nabla)\mathbf{w} + (\boldsymbol{\eta}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\boldsymbol{\zeta} + (\mathbf{w}\cdot\nabla)\boldsymbol{\eta} + (\boldsymbol{\eta}\cdot\nabla)\boldsymbol{\zeta}\\ &+ (\boldsymbol{\zeta}\cdot\nabla)\boldsymbol{\eta} + (\boldsymbol{\eta}\cdot\nabla)\boldsymbol{\eta} + (\boldsymbol{\zeta}\cdot\nabla)\boldsymbol{\zeta} + \nabla\sigma = \nu\Delta\boldsymbol{\eta} \qquad \text{in } \Omega\,, \end{aligned}$$

$$\operatorname{div}\boldsymbol{\zeta} = 0 \qquad \text{in } \Omega,$$

and

$$\boldsymbol{\zeta} = \mathbf{0} \qquad \text{on } \Gamma\,.$$

Using the weak form of these equations (see Definition 2.1) and the fact that $c(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0$ for all $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ and all $\mathbf{u} \in \mathbf{V}$, we obtain

$$\nu\|\nabla\boldsymbol{\zeta}\|_0^2 + c(\boldsymbol{\zeta}, \mathbf{w}, \boldsymbol{\zeta}) + c(\boldsymbol{\eta}, \mathbf{w}, \boldsymbol{\zeta}) + c(\mathbf{w}, \boldsymbol{\eta}, \boldsymbol{\zeta}) + c(\boldsymbol{\zeta}, \boldsymbol{\eta}, \boldsymbol{\zeta}) + c(\boldsymbol{\eta}, \boldsymbol{\eta}, \boldsymbol{\zeta}) = -\nu\int_\Omega (\nabla\boldsymbol{\eta}) : (\nabla\boldsymbol{\zeta})\,d\mathbf{x}$$

so that using inequalities (2.1), (2.3), and $rs \le \delta r^2 + \frac{1}{4\delta}s^2$, we are led to (for any $\delta > 0$)

$$\big(\nu C_0 - C_2\|\mathbf{w}\|_1 - C_2\|\boldsymbol{\eta}\|_1 - 3\delta\big)\|\boldsymbol{\zeta}\|_1^2 - \frac{C_2^2}{4\delta}\big(2\|\boldsymbol{\eta}\|_1^2\,\|\mathbf{w}\|_1^2 + \|\boldsymbol{\eta}\|_1^4\big) \le \frac{\nu^2}{4\delta}\|\nabla\boldsymbol{\eta}\|_0^2 + \delta\|\boldsymbol{\zeta}\|_1^2\,.$$

Choosing $\delta = \frac{\nu C_0}{16}$ and noting that $C_2\|\boldsymbol{\eta}\|_1 \le C_2C_5\|\widetilde{\mathbf{z}} - \mathbf{z}\|_{1/2,\Gamma} \le \frac{\nu C_0}{4}$, we obtain

$$\frac{\nu C_0}{4}\|\boldsymbol{\zeta}\|_1^2 \le \frac{\nu^2 C_5^2}{4\delta}\|\widetilde{\mathbf{z}} - \mathbf{z}\|_{1/2,\Gamma}^2 + \frac{2C_2^2C_5^2}{4\delta}\Big(\frac{\nu C_0}{4C_2}\Big)^2\|\widetilde{\mathbf{z}} - \mathbf{z}\|_{1/2,\Gamma}^2 + \frac{C_2^2C_5^4}{4\delta}\|\widetilde{\mathbf{z}} - \mathbf{z}\|_{1/2,\Gamma}^4\,,$$

so that

$$\|\boldsymbol{\zeta}\|_1^2 \le \Big(\frac{16C_5^2}{C_0^2} + 2C_5^2\Big)\|\widetilde{\mathbf{z}} - \mathbf{z}\|_{1/2,\Gamma}^2 + \frac{16C_2^2C_5^4}{\nu^2C_0^2}\|\widetilde{\mathbf{z}} - \mathbf{z}\|_{1/2,\Gamma}^4\,.$$

Hence, using the inequality $(r^2+s^2) \le (r+s)^2$ we are led to

$$\|\boldsymbol{\zeta}\|_1 \le \Big(\frac{4C_5}{C_0} + \sqrt{2}C_5\Big)\|\widetilde{\mathbf{z}} - \mathbf{z}\|_{1/2,\Gamma} + \frac{4C_2C_5^2}{\nu C_0}\|\widetilde{\mathbf{z}} - \mathbf{z}\|_{1/2,\Gamma}^2\,.$$

Finally, we use the triangle inequality to derive the estimate for $\boldsymbol{\xi}$:

$$\|\boldsymbol{\xi}\|_1 \le \|\boldsymbol{\eta}\|_1 + \|\boldsymbol{\zeta}\|_1 \le \Big(\frac{4C_5}{C_0} + \sqrt{2}C_5 + C_5\Big)\|\widetilde{\mathbf{z}} - \mathbf{z}\|_{1/2,\Gamma} + \frac{4C_2C_5^2}{\nu C_0}\|\widetilde{\mathbf{z}} - \mathbf{z}\|_{1/2,\Gamma}^2\,. \qquad \square$$

THEOREM 5.4. *Assume that*

$$\frac{\|\mathbf{f}\|_0}{\nu^2}\sqrt{\frac{\alpha}{\min\{\alpha,\beta\}}} < \frac{1}{4}\,\frac{C_0}{C_2}\sqrt{\frac{C_1}{C_4}} \tag{5.1}$$

and let $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{\mathbf{g}})$ *be the limit defined in Theorem* 4.1. *Then* $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{\mathbf{g}})$ *is a solution of the optimal Dirichlet control problem* (P).

Proof. Let $(\mathbf{w}, r, \mathbf{z}) \in \mathcal{U}_{ad}$ be given. We need to prove that $\mathcal{J}(\widehat{\mathbf{u}}, \widehat{\mathbf{g}}) \le \mathcal{J}(\mathbf{w}, \mathbf{z})$. Using Lemma 3.3 and the facts that $\operatorname{div}\mathbf{w} = 0$ and $\mathbf{w}\big|_\Gamma = \mathbf{z}$, we obtain

$$\mathcal{J}(\mathbf{w}, \mathbf{z}) \ge \frac{1}{2C_4}\min\{\alpha,\beta\}\|\mathbf{w}\|_1^2\,.$$

Hence, if $\frac{1}{2C_4}\min\{\alpha,\beta\}\|\mathbf{w}\|_1^2 > \mathcal{J}(\widehat{\mathbf{u}}, \widehat{\mathbf{g}})$, then $\mathcal{J}(\mathbf{w}, \mathbf{z}) > \mathcal{J}(\widehat{\mathbf{u}}, \widehat{\mathbf{g}})$. So we only need to consider the case where

$$\frac{1}{2C_4}\min\{\alpha,\beta\}\|\mathbf{w}\|_1^2 \le \mathcal{J}(\widehat{\mathbf{u}}, \widehat{\mathbf{g}})\,. \tag{5.2}$$

We assume (5.2) holds. Then using (5.1)–(5.2) and Lemma 5.1 we obtain

$$\begin{aligned}\|\mathbf{w}\|_1 \le \Big\{\frac{2C_4}{\min\{\alpha,\beta\}}\mathcal{J}(\widehat{\mathbf{u}}, \widehat{\mathbf{g}})\Big\}^{1/2} &\le \Big\{\frac{2C_4}{\min\{\alpha,\beta\}}\,\frac{\alpha}{2\nu^2C_1}\int_\Omega |\mathbf{f}|^2\,d\mathbf{x}\Big\}^{1/2} \\ &\le \Big\{\frac{2\alpha C_4}{\nu^2C_1\min\{\alpha,\beta\}}\Big\}^{1/2}\|\mathbf{f}\|_0 \le \frac{\nu C_0}{4C_2}\,.\end{aligned}$$

Using the denseness of $\mathbf{C}^\infty(\Gamma)\cap\mathbf{H}_n^{1/2}(\Gamma)$ in $\mathbf{H}_n^{1/2}(\Gamma)$ we may choose a sequence $\{\mathbf{z}_m\} \subset \mathbf{C}^\infty(\Gamma)\cap\mathbf{H}_n^{1/2}(\Gamma)$ such that $\|\mathbf{z}_m - \mathbf{z}\|_{1/2,\Gamma} \to 0$ as $m \to \infty$. For sufficiently large m, we have

$$\|\mathbf{z}_m - \mathbf{z}\|_{1/2,\Gamma} < \frac{\nu C_0}{4C_2C_5}\,.$$

Let $(\mathbf{w}_m, r_m) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega)$ be the solution of (2.7)–(2.8) with the Dirichlet condition $\mathbf{w}_m\big|_\Gamma = \mathbf{z}_m$. Using Lemma 5.3 we obtain $\|\mathbf{w}_m - \mathbf{w}\|_1 \to 0$ as $m \to \infty$. The regularity theories for the Navier–Stokes equations imply $(\mathbf{w}_m, r_m) \in \mathbf{H}^{3/2}(\Omega) \times H^{1/2}(\Omega)$ and $-r_m\mathbf{n} + \nu\frac{\partial \mathbf{w}_m}{\partial \mathbf{n}} \in \mathbf{L}^2(\Gamma)$. From Theorem 4.3 we obtain $\mathcal{J}(\mathbf{w}_m, \mathbf{z}_m) \geq \mathcal{J}(\widehat{\mathbf{u}}, \widehat{\mathbf{g}})$. Upon letting $m \to \infty$ we conclude that

$$\mathcal{J}(\mathbf{w}, \mathbf{z}) = \lim_{m\to\infty} \mathcal{J}(\mathbf{w}_m, \mathbf{z}_m) \geq \mathcal{J}(\widehat{\mathbf{u}}, \widehat{\mathbf{g}}) . \qquad \Box$$

REMARK. *Combining Theorems* 4.1 *and* 5.4 *we see that if the solution for* (P) *is unique, then we have the convergence as* $\epsilon \to 0$ *(instead of merely a subsequence convergence):*

$$\mathbf{u}_\epsilon \rightharpoonup \widehat{\mathbf{u}}\,, \quad \overline{p_\epsilon} \rightharpoonup \widehat{p}\,, \quad \textit{and } \mathbf{g}_\epsilon \rightharpoonup \widehat{\mathbf{g}}\,. \qquad \Box$$

REMARK. *The small data requirement* (5.1) *is due to the small data requirements in Lemma* 5.3. *On the other hand, the small data requirements in Lemma* 5.3 *are those that are needed in order to show the continuous dependence of solutions on Dirichlet data for the Navier–Stokes equations. It is well known that the Navier–Stokes equations do not always have a unique solution; thus it seems hopeless to prove, without the small data requirement, the continuous dependence on Dirichlet data of solutions of the Navier–Stokes equations. This in turn suggests that it seems hopeless to prove Theorem* 5.4 *for arbitrary data. But for most practical purposes, one should be content with the suboptimal result of Theorem* 4.3 (*which does not require the smallness of data*). $\Box$

6. Finite element approximations and numerical results. We have shown that the optimal solutions of Neumann control problems $(\mathrm{P})_\epsilon$ converge to an optimal solution of the Dirichlet control problem (P). Thus, we may choose a sufficiently small ϵ and solve $(\mathrm{P})_\epsilon$ to obtain an approximate solution for (P). In this section we briefly describe the solution procedures for $(\mathrm{P})_\epsilon$ with a fixed ϵ and present some numerical results. The purpose of this section is merely to confirm numerically the convergence of the optimal Neumann boundary control solutions which we have proven rigorously. Thus the presentation of this section is mostly formal.

The solution procedures for $(\mathrm{P})_\epsilon$ are as follows. First, by introducing the Lagrangian for $(\mathrm{P})_\epsilon$,

$$\begin{aligned}\mathcal{L}(\mathbf{u}, p, \mathbf{g}, \boldsymbol{\mu}, \rho) = \mathcal{J}(\mathbf{u}, \mathbf{g}) - \Big(\nu\, a(\mathbf{u}, \boldsymbol{\mu}) + \frac{1}{\epsilon}\int_\Gamma \mathbf{u}\cdot\boldsymbol{\mu}\, ds - \frac{1}{2}\int_\Gamma (\mathbf{u}\cdot\mathbf{n})\mathbf{u}\cdot\boldsymbol{\mu}\, ds \\ + c(\mathbf{u}, \mathbf{u}, \boldsymbol{\mu}) + b(\boldsymbol{\mu}, p) + b(\mathbf{u}, \rho) - \int_\Omega \mathbf{f}\cdot\boldsymbol{\mu}\, d\mathbf{x} - \frac{1}{\epsilon}\int_\Gamma \mathbf{g}\cdot\boldsymbol{\mu}\, ds\Big),\end{aligned}$$

and differentiating the Lagrangian with respect to each of its arguments we obtain the following optimality system of equations that the optimal solution for $(\mathrm{P})_\epsilon$ must satisfy:

$$\begin{aligned}\nu\int_\Omega (\nabla\mathbf{u}) : (\nabla\mathbf{v})\, d\mathbf{x} + \frac{1}{\epsilon}\int_\Gamma \mathbf{u}\cdot\mathbf{v}\, ds - \frac{1}{2}\int_\Gamma (\mathbf{u}\cdot\mathbf{n})\mathbf{u}\cdot\mathbf{v}\, ds + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\, d\mathbf{x} \\ -\int_\Omega p\,\mathrm{div}\,\mathbf{v}\, d\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\mathbf{x} + \frac{1}{\epsilon}\int_\Gamma \mathbf{g}\cdot\mathbf{v}\, ds \qquad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega)\,,\end{aligned} \tag{6.1}$$

$$-\int_\Omega q\,\mathrm{div}\,\mathbf{u}\, d\mathbf{x} = 0 \qquad \forall\, q \in L^2(\Omega)\,, \tag{6.2}$$

$$
\begin{aligned}
&\nu\int_\Omega(\nabla\boldsymbol{\mu}):(\nabla\mathbf{w})\,d\mathbf{x}+\frac{1}{\epsilon}\int_\Gamma\boldsymbol{\mu}\cdot\mathbf{w}\,ds-\frac{1}{2}\int_\Gamma(\mathbf{w}\cdot\mathbf{n})\mathbf{u}\cdot\boldsymbol{\mu}\,ds\\
&\quad-\frac{1}{2}\int_\Gamma(\mathbf{u}\cdot\mathbf{n})\mathbf{w}\cdot\boldsymbol{\mu}\,ds+\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\boldsymbol{\mu}\,d\mathbf{x}+\int_\Omega(\mathbf{w}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\mu}\,d\mathbf{x}\\
&\quad-\int_\Omega\rho\,\mathrm{div}\,\mathbf{w}\,d\mathbf{x}=\int_\Omega(\mathrm{curl}\,\mathbf{u})\cdot(\mathrm{curl}\,\mathbf{w})\,d\mathbf{x}\qquad\forall\,\mathbf{w}\in\mathbf{H}^1(\Omega)\,,
\end{aligned}\tag{6.3}
$$

$$
-\int_\Omega r\,\mathrm{div}\,\boldsymbol{\mu}\,d\mathbf{x}=0\qquad\forall\,r\in L^2(\Omega),\tag{6.4}
$$

and

$$
\int_\Gamma\Big(\beta\mathbf{g}+\frac{1}{\epsilon}\boldsymbol{\mu}\Big)\cdot\mathbf{z}\,ds=0\qquad\forall\,\mathbf{z}\in\mathbf{L}^2(\Gamma)\,.
$$

Note that we may use the last relation to eliminate $\mathbf{g}$ in (6.1) to obtain

$$
\begin{aligned}
&\nu\int_\Omega\nabla\mathbf{u}:\nabla\mathbf{v}\,d\mathbf{x}+\frac{1}{\epsilon}\int_\Gamma\mathbf{u}\cdot\mathbf{v}\,ds-\frac{1}{2}\int_\Gamma(\mathbf{u}\cdot\mathbf{n})\mathbf{u}\cdot\mathbf{v}\,ds+\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,d\mathbf{x}\\
&\quad-\int_\Omega p\,\mathrm{div}\,\mathbf{v}\,d\mathbf{x}=\int_\Omega\mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}-\frac{1}{\epsilon^2\beta}\int_\Gamma\boldsymbol{\mu}\cdot\mathbf{v}\,ds\qquad\forall\,\mathbf{v}\in\mathbf{H}^1(\Omega).
\end{aligned}\tag{6.5}
$$

The system formed by (6.2)–(6.5) will be called *an optimality system of equations.*

Next we choose finite element subspaces and define finite element approximations of the optimality system. The finite element spaces $\mathbf{X}_h\subset\mathbf{H}^1(\Omega)$ and $S_h\subset L^2(\Omega)$ are chosen such that

$$
\inf_{\mathbf{v}_h\in\mathbf{X}_h}\|\mathbf{v}_h-\mathbf{v}\|_1\le Ch^m\|\mathbf{v}\|_{m+1}\qquad\forall\,\mathbf{v}\in\mathbf{H}^{m+1}(\Omega)\,,
$$

$$
\inf_{q_h\in S_h}\|q_h-q\|_1\le Ch^m\|q\|_m\qquad\forall\,q\in H^m(\Omega)\,,
$$

and

$$
\inf_{q_h\in S_h}\sup_{\mathbf{v}_h\in\mathbf{X}_h}\frac{b(\mathbf{v}_h,q_h)}{\|\mathbf{v}_h\|_1\|q_h\|_0}\ge C_6\,.
$$

The last discrete inf-sup condition is needed in finite element approximations of the Navier–Stokes equations (see, e.g., [GR]) and naturally is also needed in the approximations of the optimality system of equations. We define finite element approximations of the optimality system (6.2)–(6.5) as follows:

$$
\begin{aligned}
&\nu\int_\Omega\nabla\mathbf{u}_h:\nabla\mathbf{v}_h\,d\mathbf{x}+\frac{1}{\epsilon}\int_\Gamma\mathbf{u}_h\cdot\mathbf{v}_h\,ds-\frac{1}{2}\int_\Gamma(\mathbf{u}_h\cdot\mathbf{n})\mathbf{u}_h\cdot\mathbf{v}_h\,ds-\int_\Omega p_h\,\mathrm{div}\,\mathbf{v}_h\,d\mathbf{x}\\
&\quad+\int_\Omega(\mathbf{u}_h\cdot\nabla)\mathbf{u}_h\cdot\mathbf{v}_h\,d\mathbf{x}=\int_\Omega\mathbf{f}\cdot\mathbf{v}_h\,d\mathbf{x}-\frac{1}{\epsilon^2\beta}\int_\Gamma\boldsymbol{\mu}_h\cdot\mathbf{v}_h\,ds\qquad\forall\,\mathbf{v}_h\in\mathbf{X}_h\,,
\end{aligned}\tag{6.6}
$$

$$
-\int_\Omega q_h\,\mathrm{div}\,\mathbf{u}_h\,d\mathbf{x}=0\qquad\forall\,q_h\in S_h\,,\tag{6.7}
$$

$$
\begin{aligned}
&\nu\int_\Omega(\nabla\boldsymbol{\mu}_h):(\nabla\mathbf{w}_h)\,d\mathbf{x}+\frac{1}{\epsilon}\int_\Gamma\boldsymbol{\mu}_h\cdot\mathbf{w}_h\,ds-\frac{1}{2}\int_\Gamma(\mathbf{w}_h\cdot\mathbf{n})\mathbf{u}_h\cdot\boldsymbol{\mu}_h\,ds\\
&\quad-\frac{1}{2}\int_\Gamma(\mathbf{u}_h\cdot\mathbf{n})\mathbf{w}_h\cdot\boldsymbol{\mu}_h\,ds+\int_\Omega(\mathbf{u}_h\cdot\nabla)\mathbf{w}_h\cdot\boldsymbol{\mu}_h\,d\mathbf{x}-\int_\Omega\rho_h\,\mathrm{div}\,\mathbf{w}_h\,d\mathbf{x}\\
&\quad+\int_\Omega(\mathbf{w}_h\cdot\nabla)\mathbf{u}_h\cdot\boldsymbol{\mu}_h\,d\mathbf{x}=\int_\Omega(\mathrm{curl}\,\mathbf{u}_h)\cdot(\mathrm{curl}\,\mathbf{w}_h)\,d\mathbf{x}\qquad\forall\,\mathbf{w}_h\in\mathbf{X}_h,
\end{aligned}\tag{6.8}
$$

TABLE 1
$\mathbf{L}^2(\Omega)$ *errors of each two consecutive optimal solutions.*

i	1	2	3	4	5	6	7	8
ϵ_i	10	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
$\left\|\mathbf{u}_i - \mathbf{u}_{i+1}\right\|_2^2$	.3553	.04725	.01763	.002506	.0002618	.00002627	.000009	

and

$$-\int_\Omega r_h \operatorname{div} \boldsymbol{\mu}_h \, d\mathbf{x} = 0 \qquad \forall\, r_h \in S_h\,. \tag{6.9}$$

We solve the discrete, nonlinear system of equations (6.6)–(6.9) by Newton's method with the initial guess obtained from solving the corresponding linear system of equations (simply dropping all nonlinear terms in the optimality system).

It is possible to use the techniques of [GHS1] to mathematically justify these solution procedures, e.g., to prove the existence of a solution $(\mathbf{u}, p, \boldsymbol{\mu}, \rho)$ for (6.2)–(6.5) such that $(\mathbf{u}, p, -\boldsymbol{\mu}/(\beta\epsilon))$ gives a solution of $(\mathrm{P})_\epsilon$; to prove that for each solution of (6.2)–(6.5) and for each sufficiently small h, there exists a solution $(\mathbf{u}_h, p_h, \boldsymbol{\mu}_h, \rho_h)$ such that as $h \to 0$,

$$\mathbf{u}_h \to \mathbf{u}, \quad p_h \to p, \quad \boldsymbol{\mu}_h \to \boldsymbol{\mu}, \quad \text{and} \quad \rho_h \to \rho\,;$$

and to prove that if $(\mathbf{u}_h, p_h, \boldsymbol{\mu}_h, \rho_h) \in \mathbf{H}^{m+1}(\Omega) \times \mathbf{H}^m(\Omega) \times \mathbf{H}^{m+1}(\Omega) \times \mathbf{H}^m(\Omega)$, then

$$\begin{aligned} &\|\mathbf{u}_h - \mathbf{u}\|_1 + \|p_h - p\|_0 + \|\boldsymbol{\mu}_h - \boldsymbol{\mu}\|_1 + \|\rho_h - \rho\|_0 \\ &\qquad \le C h^m \left(\|\mathbf{u}\|_{m+1} + \|p\|_m + \|\boldsymbol{\mu}\|_{m+1} + \|\rho\|_m \right). \end{aligned}$$

However, the detailed justification of these results are beyond the scope of this paper.

We conclude this paper by presenting some numerical results for two test problems. These results confirm numerically the convergence results we established, i.e., Theorem 4.1. Further computational studies of the proposed method will be reported elsewhere.

In the first example we consider a Dirichlet optimal control problem for the Navier–Stokes equations (1.2)–(1.4) with the following data: Ω is the unit square; $\nu = 0.1$; and the prescribed body force $\mathbf{f} = (f_1, f_2)^T$, where $f_1 = 0.8\pi^2 \sin(2\pi x)\, \cos(2\pi y) + 2\pi \sin(2\pi x)\, \cos(2\pi x)$ and $f_2 = -0.8\pi^2 \cos(2\pi x)\, \sin(2\pi y) + 2\pi \sin(2\pi y)\, \cos(2\pi y)$. The functional is given by (1.1), wherein we choose $\alpha = \beta = 1$. For each sufficiently small ϵ, we can compute an optimal solution for $(\mathrm{P})_\epsilon$ by solving the discrete optimality system (6.6)–(6.9). We used a uniform mesh in Ω with 162 triangles and chose the finite element spaces to be continuous piecewise quadratics for the velocity/adjoint velocity and continuous piecewise linear functions for the pressure/adjoint pressure. We computed the optimal solutions for a sequence of ϵ values: $\epsilon_1 = 10$, $\epsilon_2 = 1$, $\epsilon_3 = 10^{-1}$, $\epsilon_4 = 10^{-2}$, $\epsilon_5 = 10^{-3}$, $\epsilon_6 = 10^{-4}$, $\epsilon_7 = 10^{-5}$, and $\epsilon_8 = 10^{-6}$. We also computed the $\mathbf{L}^2(\Omega)$ norms of $\widehat{\mathbf{u}}_{\epsilon_{i+1}} - \widehat{\mathbf{u}}_{\epsilon_i}$, and these are summarized in Table 1.

The computational results in Table 1 are consistent with the convergence results of Theorem 4.1. The solution $(\mathbf{u}_0, p_0)$ of the equations with $\mathbf{g} = \mathbf{0}$ is given by $\mathbf{u}_0 = \left(\sin(2\pi x)\, \cos(2\pi y), -\cos(2\pi x)\, \sin(2\pi y)\right)^T$ and $p_0 = 0$, and $\int_\Omega |\operatorname{curl} \mathbf{u}_0|^2 \, d\mathbf{x} = 38.8605$. The values of $\int_\Omega |\operatorname{curl} \mathbf{u}_\epsilon|^2 \, d\mathbf{x}$ for $\epsilon \le 10^{-5}$ are around 18.

In the second example we consider a Dirichlet optimal control problem for the Navier–Stokes equations (1.2)–(1.4) with the following data: Ω is the unit square;

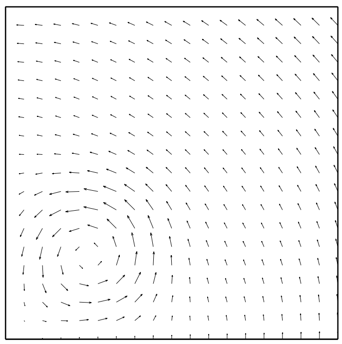

FIG. 1. *Uncontrolled velocity field.*

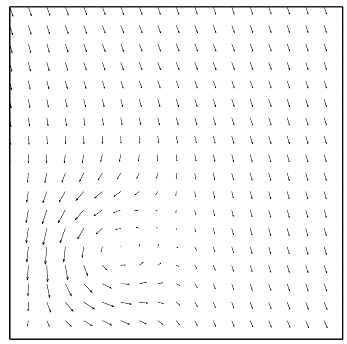

FIG. 2. *Optimal velocity field* ($\epsilon = 1$).

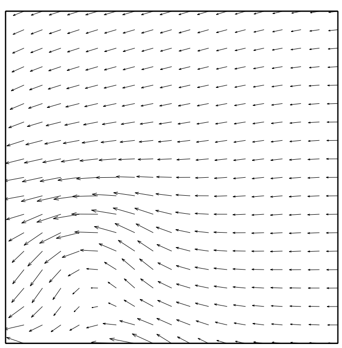

FIG. 3. *Optimal velocity field* ($\epsilon = 10^{-1}$).

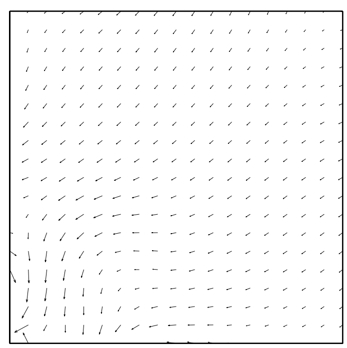

FIG. 4. *Optimal velocity field* ($\epsilon = 10^{-3}$).

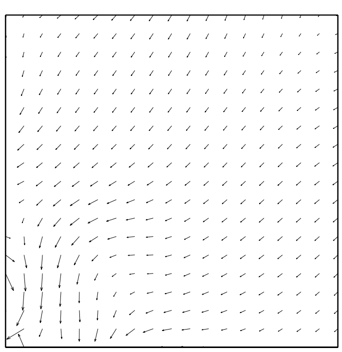

FIG. 5. *Optimal velocity field* ($\epsilon = 10^{-5}$).

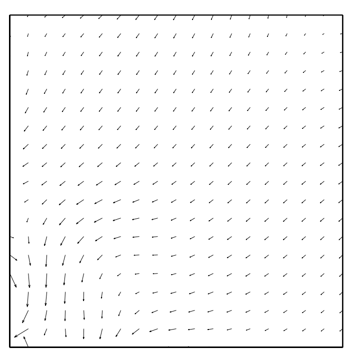

FIG. 6. *Optimal velocity field* ($\epsilon = 10^{-7}$).

Table 2
The $\mathbf{L}^2(\Omega)$-norm of the vorticity of optimal solutions.

ϵ_i	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
$\left\|\operatorname{curl}\mathbf{u}_\epsilon\right\|_2^2$	12.20	11.81	11.64	11.63	11.62	11.62	11.62	11.62

$\nu = 0.1$; and the prescribed body force $\mathbf{f} = (f_1, f_2)^T$, where

$$\mathbf{f} = \begin{pmatrix} -50\nu\pi\,\cos((x-0.25)\pi/0.4)\,\sin((y-0.25)\pi/0.4) - \frac{20}{\pi}\,\sin((x-0.25)\pi/0.2) \\ 50\nu\pi\,\sin((x-0.25)\pi/0.4)\,\cos((y-0.25)\pi/0.4) - \frac{20}{\pi}\,\sin((y-0.25)\pi/0.2) \end{pmatrix}$$

in the region $\{(x,y) : |x-0.25| \le 0.2, |y-0.25| \le 0.2\}$ and $\mathbf{f} = (-0.25x, -0.25y)^T$ elsewhere on the unit square. The functional is given by (1.1), wherein we choose $\alpha = 100$ and $\beta = 1$. We used the same mesh as in the first example. We computed the optimal solutions for a sequence of ϵ values by solving the discrete optimality system (6.6)–(6.9): $\epsilon_1 = 1$, $\epsilon_2 = 10^{-1}$, $\epsilon_3 = 10^{-2}$, $\epsilon_4 = 10^{-3}$, $\epsilon_5 = 10^{-4}$, $\epsilon_6 = 10^{-5}$, $\epsilon_7 = 10^{-6}$, and $\epsilon_8 = 10^{-7}$. We also computed the $\mathbf{L}^2(\Omega)$-norms of $\operatorname{curl}\widehat{\mathbf{u}}_{\epsilon_i}$ as shown in Table 2. The $\mathbf{L}^2(\Omega)$-norm of the the vorticity of the uncontrolled velocity field is 39.77.

Figure 1 shows the uncontrolled flow field. Figures 2–6 depict the optimal velocity fields for various ϵ values we tested. We could easily visualize from these figures the convergence of the optimal solutions as $\epsilon \to 0$. Also, by comparing the uncontrolled flow field with the optimal flow fields (for small ϵ), we clearly see the reduction in recirculation in the optimal control solutions.

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