Math. Nachr. **171** (1995) 111–148

Strong Solutions of the Navier-Stokes System in Lipschitz Bounded Domains

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(Received April 5, 1994)

Abstract. Let Ω be a bounded domain in $\mathbb{R}^3$ with a connected Lipschitz boundary $\partial\Omega$. Consider the Navier-Stokes system

$$u_t(x, t) - v \cdot \Delta_x u(x, t) + u(x, t) \cdot \nabla_x u(x, t) + \nabla_x \pi(x, t) = f(x, t), \tag{7.29}$$

$$\operatorname{div}_x u(x, t) = 0 \quad \text{for} \quad (x, t) \in \Omega \times ((0, T] \cap \mathbb{R}),$$

with Dirichlet boundary conditions

$$u(x, t) = 0 \quad \text{for} \quad (x, t) \in \partial\Omega \times ((0, T] \cap \mathbb{R}), \tag{7.30}$$

and with initial conditions

$$u(x, 0) = u_0(x), \qquad x \in \Omega. \tag{7.31}$$

Let $H_2(\Omega)$ denote the space of solenoidal functions in $L^2(\Omega)^3$, and let P_2 be the projection of $L^2(\Omega)^3$ onto $H_2(\Omega)$. Then there is a self-adjoint operator $A: \mathfrak{D}(A) \mapsto H_2(\Omega)$ such that $Au = -P_2(v \cdot \Delta u)$ for $u \in C_0^\infty(\Omega)^3$ with $\operatorname{div} u = 0$ ("Stokes-operator").

Take $\varepsilon \in (0, 1/4)$, and assume $u_0 \in \mathfrak{D}(A^{1/4+\varepsilon})$. Let $T_0 \in (0, \infty]$, and assume that

$$f: \Omega \times ((0, T_0] \cap \mathbb{R}) \mapsto \mathbb{C}^3$$

is a bounded measurable function, which, in addition, is locally Hölder continuous in $t \in (0, T_0] \cap \mathbb{R}$.

Then there is some $T \in (0, T_0]$ and a solution (u, π) of (1)–(3) on $\Omega \times ((0, T] \cap \mathbb{R})$ such that

$$u \in C^0([0, T] \cap \mathbb{R}, \mathfrak{D}(A^{1/4+\varepsilon})) \cap C^1((0, T] \cap \mathbb{R}, H_2(\Omega)).$$

$$u(t) \in W_{\mathrm{loc}}^{2,2}(\Omega)^3 \cap W_0^{1,2}(\Omega)^3 \cap H_2(\Omega) \cap W^{3/2-\delta,2}(\Omega)^3,$$

$$\pi(t) \in L^2(\Omega)^3 \cap W_{\mathrm{loc}}^{1,2}(\Omega) \cap W^{1/2-\delta,2}(\Omega) \quad \text{for} \quad \delta \in (0, 1/2), \qquad t \in (0, T] \cap \mathbb{R}.$$

The condition $u_0 \in \mathfrak{D}(A^{1/4+\varepsilon})$ is satisfied if, for example, $u_0 \in W_0^{1,2}(\Omega)^3 \cap H_2(\Omega)$.

If $T_0 = \infty$, and if u_0 and f are small in a suitable sense, then the preceding results hold true for $T = \infty$ ("global solution").

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with connected boundary. Consider the Navier-Stokes system

$$u_t(x, t) - \nu \cdot \Delta_x u(x, t) + u(x, t) \cdot \nabla_x u(x, t) + \nabla_x \pi(x, t) = f(x, t),$$
$$\operatorname{div}_x u(x, t) = 0 \quad \text{for} \quad (x, t) \in \Omega \times ((0, T] \cap \mathbb{R}), \tag{1.1}$$

with Dirichlet boundary conditions

$$u(x, t) = 0 \quad \text{for} \quad (x, t) \in \partial\Omega \times ((0, T_0] \cap \mathbb{R}), \tag{1.2}$$

and with initial conditions

$$u(x, 0) = u_0(x), \qquad x \in \Omega; \tag{1.3}$$

see Section 2 for the notations used here. The functions

$$u_0 : \Omega \mapsto \mathbb{C}^3, \qquad f : \Omega \times ((0, T] \cap \mathbb{R}) \mapsto \mathbb{C}^3,$$

with $T \in (0, \infty]$, and the constant $\nu \in (0, \infty)$ are given, whereas the velocity

$$u : \Omega \times ([0, T_0] \cap \mathbb{R}) \mapsto \mathbb{C}^3$$

and the pressure

$$\pi : \Omega \times ((0, T_0] \cap \mathbb{R}) \mapsto \mathbb{C},$$

defined for some $T_0 \in (0, T]$, are unknown. The terms "velocity" and "pressure" are used because the Navier-Stokes system models the flow of a viscous, incompressible fluid, with u describing the velocity of the fluid and π its pressure times a certain constant. For more details about the physical meaning of problem (1.1)–(1.3) we refer to CHORIN, MARSDEN [7, p. 43ff.].

In the work at hand, it will be supposed that the domain Ω has a general Lipschitz boundary. This means that $\partial\Omega$ is locally given by the graph of an arbitrary Lipschitz function. In addition we shall assume that Ω does not simultaneously lie on both sides of any part of its boundary; see Section 3 for more details.

When partial differential equations are considered on this type of domain, it is often easy to obtain a weak solution by solving the corresponding variational problem. Typically a weak solution of an elliptic equation belongs to the space $W^{1,2}(\Omega)$. Recall for example that Poisson's equation may be solved in this space by applying the Lax-Milgram theorem. Concerning problem (1.1)–(1.3), a corresponding result is more difficult to obtain, due to the nonlinearity in (1.1). However, a suitable variant of the variational method again yields a weak solution (u, π), which is even global in time, that is, it exists on $\Omega \times (0, T)$ with $T = \infty$. Regarding space regularity of the velocity part u of this solution, it holds $u(\cdot, t) \in W_0^{1,2}(\Omega)^3$ for a.e. $t \in (0, \infty)$. These celebrated results are due to HOPF [34].

However, much less is known concerning more regular ("strong") solutions of partial differential equation in domains with a general Lipschitz boundary. In particular, no theory could be derived which covers the class of elliptic equations and systems studied by AGMON, DOUGLIS, NIRENBERG [2], [3] under the assumption that a smooth domain is given. Instead, as concerns strong solution on Lipschitz domains, each partial differential equation had to be investigated separately. It was Poisson's equation which was considered first; see

VERCHOTA [51], DAHLBERG, KENIG [11], COSTABEL [9]. A salient feature of the theory emerging from these papers is the following:

For any $g \in L^2(\Omega)$, there exists a uniquely determined function u belonging to $W^{2,2}_{\mathrm{loc}} \cap W^{1,2}_0(\Omega) \cap W^{3/2-\delta,2}(\Omega)$ for $\delta > 0$ and satisfying the equation $\Delta u = g$.

In general, this result cannot be improved, even if f is assumed to be arbitrarily smooth.

In recent years, the preceding theory was generalized to some other equations and systems. Thus the heat equation was studied by BROWN [5], [6] and COSTABEL [10], the Stokes system by FABES, KENIG, VERCHOTA [24] and DAHLBERG, KENIG, VERCHOTA [13], the Lamé system also by the previous authors [13], the time-dependent Stokes and Lamé system by SHEN [44], and the biharmonic equation by DAHLBERG, KENIG, VERCHOTA [12] and PIPHER, VERCHOTA [43]. All these articles are based on the method of layer potentials.

It is the aim of the present paper to extend the preceding results to the nonlinear, time-dependent Navier-Stokes system. In fact, we shall construct a strong solution of problem (1.1)–(1.3), with the somewhat fuzzy notion of "strong solution" referring to functions which exhibit at least two features: Such solutions should satisfy (1.1) pointwise almost everywhere, and they should belong to a uniqueness class of problem (1.1)–(1.3). Actually, we shall obtain much more, namely a solution (u, π) with $u(\cdot, t)$ contained in $W^{2,2}_{\mathrm{loc}}(\Omega)^3 \cap W^{1,2}_0(\Omega)^3 \cap W^{3/2-\delta,2}(\Omega)^3$, and $\pi(\cdot, t) \in W^{1,1}_{\mathrm{loc}}(\Omega) \cap W^{1/2-\delta,2}(\Omega)$, where $\delta > 0$, $t \in (0, T)$. Thus, concerning regularity in space, our solutions are as good as those of Poisson's equation, and in this sense they can be considered as best possible. Regularity in time will turn out to be optimal too.

It should be remarked that a completely different situation arises when special geometrical assumptions are imposed on $\partial\Omega$. For example, the domain Ω may have corners or edges of prescribed geometrical shape. A large number of articles has been devoted to the study of partial differential equations in such special Lipschitz domains. Any reader who wishes to gain an impression of this extensive literature is referred to the books by GRISVARD [33] and MAZ'YA, NAZAROV, PLAMENEVSKI [39], [40]. To cite some examples of more recent papers dealing with the Stokes system on special Lipschitz domains, we mention DAUGE [14], DEURING [20], FARWIG, SOHR [25], GALDI, SIMADER, SOHR [30], and KOZLOV, MAZYA, SCHWAB [36]. We finally remark that strongly elliptic equations in convex domains admit a L^2-theory similar to the one derived in the case of smooth domains; see GRISVARD [33, Chapter 3]. Related results for the 2-D Stokes system in convex polygons were proved by KELLOG, OSBORNE [35]; see also GRISVARD [33, Section 7.3.3]. Amazingly, however, there is only one reference pertaining to strong solutions of the time-dependent, nonlinear Navier-Stokes system (1.1) in Lipschitz domains of any kind. In fact, in the book [La] by Ladyzhenskaya, strong solutions of (1.1) are constructed in a way which may be carried over to a general Lipschitz domain ([La]), p. 146Ff). However, the assumptions needed for the approach in [La] are rather strong, and the problem of maximal regularity is not studied.

In the present paper, we intend to solve problem (1.1)–(1.3) by the functional analytical approach which was introduced by FUJITA and KATO [28] in order to treat the case of Ω smoothly bounded. Let us keep this assumption on Ω for a moment in order to review the method from [28]. It is based on the fact that the space $L^2(\Omega)^3$ may be represented as a direct sum of $H_2(\Omega)$ and $\{\nabla g : g \in W^{1,2}(\Omega)\}$, where $H_2(\Omega)$ denotes the space of solenoidal functions in $L^2(\Omega)^3$ (see Section 2). Let P_2 denote the projection of $L^2(\Omega)^3$ onto $H_2(\Omega)$. By formally applying P_2 to both sides of (1.1), we are led to the following initial value problem in $H_2(\Omega)$:

$$(1.4) \qquad u_t - Au(t) + P_2(u(t) \cdot \nabla u(t)) = P_2(f(t)) \quad \text{for} \quad t \in (0, T_0] \cap \mathbb{R},$$

$$(1.5) \qquad u(0) = u_0,$$

with an operator $A:\mathfrak{D}(A) \to H_2(\Omega)$ ("Stokes operator") defined by

$$\mathfrak{D}(A) := W^{2,2}(\Omega)^3 \cap W_0^{1,2}(\Omega)^3 \cap H_2(\Omega), \qquad Au := -\nu \cdot P_2(\Delta u).$$

The Stokes operator is self-adjoint – a fact which may be deduced from regularity results pertaining to solutions (u, π) of the Stokes system

$$(1.6) \qquad -\Delta u + \nabla\pi = f, \qquad \operatorname{div} u = 0,$$

in Ω under Dirichlet boundary conditions

$$(1.7) \qquad u \mid \partial\Omega = 0.$$

Combining the Heinz-Kato theorem ([48, p. 44]) with certain results on fractional powers of the Laplace operator yields an estimate concerning fractional powers of the Stokes operator:

$$(1.8) \qquad \|u\|_{2\cdot\alpha,2} \leq C(\Omega, \alpha, \nu) \cdot \|A^\alpha u\|_2$$

for $\alpha \in [0, 1]$, $u \in \mathfrak{D}(A^\alpha)$, where $\mathfrak{D}(A^\alpha)$ denotes the domain of definition of the fractional power A^α of A. By referring to inequality (1.8) with $\alpha = 1/4$ and $\alpha = 3/4$, FUJITA and KATO [28] could show that for small $T_0 > 0$, the theory of self-adjoint operators yields existence of a solution of (1.4), (1.5). This fact implies existence of a strong solution of (1.1)–(1.3). Since T_0 is assumed to be small, such a solution is called "local". The smallness condition on T_0 may be dropped if instead the data f and u_0 are small in a suitable sense. Then it may be shown that a solution of (1.4), (1.5) exists even for $T_0 = \infty$ ("global" solution). The approach by FUJITA and KATO was generalized to a L^p-framework, and to the case that Ω is replaced by $\mathbb{R}^3 \setminus \overline{\Omega}$ ("exterior" instead of "interior" domains).

Let us name some references besides [28] pertaining to the preceding results. We mention TEMAM [49], SIMADER, SOHR [45], and VON WAHL [53] concerning the Helmholtz decomposition. The articles [15]–[22], [25], [26], [29]–[31], [37], [46], and the books [Ga], [Va] among many other references, treat the Stokes system (1.6) or the related resolvent problem. For a proof of (1.8), we refer to VON WAHL [52, p. 95–97]. Among the numerous papers dealing with generalizations of the Fujita-Kato method, we mention VON WAHL [52], [54] and GIGA, SOHR [32].

Now let us suppose that the domain Ω has a general Lipschitz boundary. Then we observe that the Helmholtz decomposition is still valid. This fact and some related ones needed later on are stated in Theorem 4.1 and 4.2 below. For a proof we shall refer to TEMAM [49].

However, with Ω Lipschitz bounded, it is no longer clear how the define a self-adjoint Stokes operator. One way to achieve this consists in setting

$$\tilde{A}\varphi := -\nu \cdot P_2(\Delta\varphi) \quad \text{for} \quad \varphi \in C_0^\infty(\Omega)^3 \quad \text{with} \quad \operatorname{div} \varphi = 0,$$

and then defining $A:\mathfrak{D}(A) \mapsto H_2(\Omega)$ as Friedrich's extension of $\tilde{A}$. But then the question arises of how to describe $\mathfrak{D}(A)$ in terms of Sobolev's spaces. In order to avoid this difficulty, we shall instead choose an ad hoc definition of A (see Definition 6.1), and then refer to a deep-lying result by FABES, KENIG, VERCHOTA [24] concerning regularity of solutions of the Stokes problem (1.6), (1.7) on Lipschitz bounded domains. Reported in Theorem 4.3 below, this result implies that our definition of A in fact yields a self-adjoint operator.

This leaves us to prove inequality (1.8) for $\alpha = 1/4$ and $\alpha = 3/4$. Since we shall be able to show that

$$\mathfrak{D}(A^{1/2}) = W_0^{1,2}(\Omega)^3 \cap H_2(\Omega),$$

$$\|A^{1/2}u\|_2 = \left(\nu \cdot \sum_{k=1}^{3} \|\nabla u_k\|^2\right)^{1/2} \quad \text{for} \quad u \in \mathfrak{D}(A^{1/2})$$

(see Lemma 6.3), the case $\alpha = 1/4$ in (1.8) may easily be treated by means of interpolation (Lemma 6.5). Thus our approach ultimately reduces to a proof of (1.8) for $\alpha = 3/4$.

In order to settle this point, we consider the resolvent problem for the Stokes system,

$$(1.9) \qquad -\Delta u + i \cdot \tau \cdot u + \nabla \pi = f, \qquad \operatorname{div} u = 0 \quad \text{in } \Omega,$$

under Dirichlet boundary condition (1.7), and under the assumptions $\tau \in \mathbb{R}$, $f \in L^2(\Omega)^3$. Starting from a technical lemma by Shen ([44, p. 364, Lemma 5.2.11]; see Lemma 4.1), we are able to prove the inequality

$$|\tau|^{1/4-\delta_1} \cdot \|u\|_{3/2-\delta_2, 2} \leq C(\Omega, \delta_1, \delta_2, \nu) \cdot \|f\|_2,$$

for $\delta_1, \delta_2 \in (0, 1/2)$, $\tau \in \mathbb{R}$ with $|\tau| \geq 1$, $f \in L^2(\Omega)^3$, and for the velocity part u of the corresponding solution (u, π) of (1.9), (1.7) (see Theorem 5.2). This result, which is of interest in its own right, leads to the inequality

$$(1.10) \qquad \|u\|_{3/2-\varepsilon_1, 2} \leq C(\Omega, \varepsilon_1, \varepsilon_2, \nu) \cdot \|A^{3/4+\varepsilon_2} u_2\|_2$$

for $\varepsilon_1 \in (0, 1/2)$, $\varepsilon_2 \in (0, 1/4)$, $u \in \mathfrak{D}(A^{3/4+\varepsilon_2})$; see Corollary 6.1. For the proof of (1.10), we shall use a suitable representation of $A^{3/4+\varepsilon_2}$ involving the resolvent of A.

Of course, inequality (1.10) is somewhat weaker than the estimate given by (1.8) with $\alpha = 3/4$. To compensate, we shall require slightly stronger assumptions on the initial data u_0, namely $u_0 \in \mathfrak{D}(A^{1/4+\varepsilon})$ for some $\varepsilon \in (0, 1/4)$, instead of $u_0 \in \mathfrak{D}(A^{1/4})$. Then we shall be able to modify the Fujita-Kato method in such a way that it may be reduced to the weaker estimate (1.10).

Ultimately, we shall arrive at the following result:

Fix $\varepsilon \in (0, 1/4)$, $T \in (0, \infty]$. Assume that $u_0 \in \mathfrak{D}(A^{1/4+\varepsilon})$, and require in addition that $f: (0, T] \cap \mathbb{R} \mapsto L^2(\Omega)^3$ is a measurable and locally Hölder continuous function which does not grow too rapidly near $t = 0$. Then there is some $T_0 \in (0, T]$ such that a strong solution (u, π) of (1.1)–(1.3) exists on $\Omega \times ([0, T_0] \cap \mathbb{R})$. If $T = \infty$, and if the functions u_0 and f satisfy certain smallness conditions, then we shall obtain a strong solution even on $\Omega \times [0, \infty)$.

As mentioned before, the behaviour of the solution (u, π) near the boundary $\partial\Omega$ is optimal in the sense that $u(\cdot, t) \in W^{3/2-\delta, 2}(\Omega)^3$ and $\pi(\cdot, t) \in W^{1/2-\delta, 2}(\Omega)$ for $t \in (0, T_0] \cap \mathbb{R}$, $\delta \in (0, 1/2)$. For more details we refer to Theorem 7.1 and 7.2.

2. Some notations

Let $n \in \mathbb{N}$. We write $| \; |$ for the euclidean norm on $\mathbb{R}^n$. If $A \subset \mathbb{R}^n$, then ∂A denotes the boundary and $\overline{A}$ the closure of A.

Consider a Banach space B, with norm $\| \;\|$. We equip B^n with the norm

$$\|b\|^{(n)} := (\|b_1\|^2 + \ldots + \|b_n\|^2)^{1/2}, \qquad b \in B^n .$$

In particular, if B is a Hilbert space, then B^n will also be a Hilbert space, with a scalar product defined in an obvious way. For simplicity, we shall drop the subscript "(n)" and write $\| \;\|$ instead of $\| \;\|^{(n)}$.

Let U be an open subset of $\mathbb{R}^n$. For $p \in [1, \infty)$, we denote by $L^p(U)$ the space of all p-integrable functions from U into $\mathbb{C}$. We shall use the symbol $\| \;\|_p$ for the norm of this space.

By $C_0^\infty(U)$ we mean the set of all C^∞-functions $f: U \to \mathbb{C}$ which have compact support in U. Moreover, $C^{0,1}(\overline{U})$ denotes the set of all Lipschitz-continuous functions from $\overline{U}$ into $\mathbb{R}$:

$$C^{0,1}(\overline{U}) := \left\{ f: \overline{U} \mapsto \mathbb{R} : \sup_{x \in \overline{U}} |f(x)| + \sup_{x, x' \in \overline{U}, x \neq x'} |f(x) - f(x')| \cdot |x - x'|^{-1} < \infty \right\}.$$

We further define

$$L^2_{\mathrm{loc}}(U) := \{ f: U \mapsto \mathbb{C} : f \,|\, K \in L^2(K) \text{ for any measurable set } K \subset \mathbb{R}^n \text{ with } \overline{K} \subset U \}.$$

Let f be a function from U into $\mathbb{C}$. Then the symbols $D_l f$, $D_m D_l f$, $D_l^k f$, $D^a f$, for $l, m \in \{1, \ldots, n\}$, $k \in \mathbb{N}$, $a \in \mathbb{N}_0^n$, are used in an obvious way in order to denote partial derivatives of f (weak or classical).

Certain differential operators coming up frequently will be abbreviated in the usual way:

$$\Delta f := \sum_{k=1}^{3} D_k^2 f, \qquad \nabla f := (D_1 f, D_2 f, D_3 f) \quad \text{for functions} \quad f: V \mapsto \mathbb{C},$$

$$\operatorname{div} g := \sum_{k=1}^{3} D_k g_k, \qquad g \cdot \nabla g := \left(\sum_{k=1}^{3} g_k \cdot D_k g_i \right)_{1 \le i \le 3}$$

for functions $g: V \mapsto \mathbb{C}^3$,

with $V \subset \mathbb{R}^3$ open. Let $J \subset \mathbb{R}$ be an interval, and $h, \tilde{h}$ functions from $V \times J$ into $\mathbb{C}, \mathbb{C}^3$, respectively. Then we set

$$\Delta_x h := \sum_{k=1}^{3} D_k h, \qquad \nabla_x h := (D_1 h, D_2 h, D_3 h), \qquad \operatorname{div}_x \tilde{h} := \sum_{k=1}^{3} D_k \tilde{h}_k,$$

$$\tilde{h} \cdot \nabla_x \tilde{h} := \left(\sum_{k=1}^{3} \tilde{h}_k \cdot D_k \tilde{h}_i \right)_{1 \le i \le 3}, \qquad \tilde{h}_t := D_4 \tilde{h}.$$

Of course, these definitions only make sense under appropriate conditions on differentiability of f, g, h, $\tilde{h}$, but we do not mention these conditions explicitly here.

We define the space $H_2(V)$ as the closure of the set $\{\varphi \in C_0^\infty(V)^3 : \operatorname{div} \varphi = 0\}$ with respect to the norm $\| \;\|_2$ of $L^2(V)^3$. Of course, $H_2(V)$ is a Hilbert space when equipped with the scalar product of $L^2(V)^3$, restricted to $H_2(V) \times H_2(V)$.

For $p \in (1, \infty)$, $s \in (0, \infty)$, we shall consider the usual Sobolev spaces $W^{s,p}(V)$; see [50, p. 310, (4.2.1/3)]. As a norm on $W^{s,p}(V)$, we choose the mapping $\| \ \|_{s,p}$ defined by

$$\|u\|_{s,p} := \left(\sum_{\alpha \in \mathbb{N}_0^3, \alpha_1 + \alpha_2 + \alpha_3 \le s} \|D^\alpha u\|_p^p \right)^{1/p} \quad \text{in the case} \quad s \in \mathbb{N},$$

$$\|u\|_{s,p} := \left(\|u\|_{[s],p}^p + \int_V \int_V |u(x) - u(y)|^p \cdot |x - y|^{-3 + p \cdot (s - [s])} \, dx \, dy \right)^{1/p} \quad \text{else},$$

where $[s] := \max \{k \in \mathbb{Z} : k < s\}$, and $\| \ \|_{0,p} := \| \ \|_p$. The space $W^{k,p}_{\text{loc}}(V)$ is defined by

$$W^{k,p}_{\text{loc}}(V) := \{f : V \mapsto \mathbb{C} : f \mid K \in W^{k,p}(K) \text{ for } K \subset \mathbb{R}^3 \text{ open with } \overline{K} \subset V\}.$$

The closure of the set $C_0^\infty(V)$ with respect to the norm $\| \ \|_{1,p}$ will be denoted by $W_0^{1,p}(V)$. Now assume in addition that ∂V is bounded, and consider a function $f : V \mapsto \mathbb{C}$. If V is bounded, we suppose that $f \in W^{1,1}(V)$. Otherwise, we assume that there is some $R > 0$ such that

$$\partial V \subset K_R(0) := \{x \in \mathbb{R}^3 : |x| < R\},$$

and

$$f \mid (K_R(0) \cap V) \in W^{1,1}(K_R(0) \cap V).$$

Then we write trace (f) for the trace of f on ∂V.

Consider a Banach space E. Let $J \subset \mathbb{R}$ be an interval. A mapping $f : J \to E$ will be called „measurable", if it is strongly E-measurable in the sense of [55, p. 130]. If the norm of E is denoted by $\| \ \|$, and if $f : J \to E$ is measurable with $\int_J \|f(t)\| \, dt < \infty$, then we write $E - \int_J f(t) \, dt$ for Bochner's integral of f over J ([55, p. 132ff.]).

3. Some properties of Lipschitz bounded domains

For the rest of this paper, let Ω denote a bounded domain with connected Lipschitz boundary $\partial\Omega$. This domain will be kept fixed.

Following [27, p. 269/270, 305/306], we choose $k(\Omega) \in \mathbb{N}$, $\alpha(\Omega) \in (0, \infty)$, orthonormal matrices $A_1^{(\Omega)}, \ldots, A_{k(\Omega)}^{(\Omega)} \in \mathbb{R}^{3 \times 3}$, vectors $C_1^{(\Omega)}, \ldots, C_{k(\Omega)}^{(\Omega)} \in \mathbb{R}^3$, and functions $a_1^{(\Omega)}, \ldots, a_{k(\Omega)}^{(\Omega)} \in C^{0,1}([-\alpha(\Omega), \alpha(\Omega)]^2)$ such that the following properties hold true:

Defining for $i \in \{1, \ldots, k(\Omega)\}$, $\gamma \in (0, 1]$, the sets Δ^γ, Λ_i^γ, U_i^γ by

$$\Delta^\gamma := (-\gamma \cdot \alpha(\Omega), \gamma \cdot \alpha(\Omega))^2,$$

$$\Lambda_i^\gamma := \{A_i^{(\Omega)} \cdot (\eta, a_i^{(\Omega)}(\eta)) + C_i^{(\Omega)} : \eta \in \Delta^\gamma\},$$

$$U_i^\gamma := \{A_i^{(\Omega)} \cdot (\eta, a_i^{(\Omega)}(\eta) + r) + C_i^{(\Omega)} : \eta \in \Delta^\gamma, r \in (-\gamma \cdot \alpha(\Omega), \gamma \cdot \alpha(\Omega))\},$$

and the function

$$H^{(i)} : \Delta^1 \times (-\alpha(\Omega), \alpha(\Omega)) \mapsto U_i^1$$

by

$$H^{(i)}(\eta, r) := A_i^{(\Omega)} \cdot (\eta, a_i^{(\Omega)}(\eta) + r) + C_i^{(\Omega)} \quad \text{for} \quad \eta \in \Delta^1, \qquad r \in (-\alpha(\Omega), \alpha(\Omega)),$$

we have

$$(3.1) \qquad U_i^1 \cap \Omega = H^{(i)}(\Delta^1 \times (-\alpha(\Omega), 0)),$$

$$U_i^1 \cap (\mathbb{R}^3 \setminus \overline{\Omega}) = H^i(\Delta^1 \times (0, \alpha(\Omega))),$$

$$U_i^1 \cap \partial\Omega = \Lambda_i^1 \quad \text{for} \quad i \in \{1, \ldots, k(\Omega)\},$$

and

$$\partial\Omega = \bigcup_{i=1}^{k(\Omega)} \Lambda_i^{1/4}.$$

By the letter n, we denote the outward unit normal to Ω, which is defined almost everywhere on $\partial\Omega$; see [41, p. 88/89].

Let us note some simple but important consequences of the preceding definitions:

$$(3.2) \qquad \min\{\operatorname{dist}(U_i^{\gamma_1}, \mathbb{R}^3 \setminus U_i^{\gamma_2}) : i \in \{1, \ldots, k(\Omega)\} > 0 \quad \text{for} \quad \gamma, \delta \in (0, 1] \quad \text{with} \quad \gamma < \delta,$$

$$(3.3) \qquad \operatorname{dist}\left(\overline{\Omega}, (\mathbb{R}^3 \setminus \Omega) \Big\backslash \left(\bigcup_{i=1}^{k(\Omega)} U_i^{1/4}\right)\right) > 0, \qquad \operatorname{dist}\left(\mathbb{R}^3 \setminus \Omega, \overline{\Omega} \Big\backslash \left(\bigcup_{i=1}^{k(\Omega)} U_i^{1/4}\right)\right) > 0,$$

$$(3.4) \qquad \operatorname{dist}\left(\partial\Omega, \mathbb{R}^3 \Big\backslash \left(\bigcup_{i=1}^{k(\Omega)} U_i^{1/4}\right)\right) > 0.$$

Furthermore, there is a constant $\mathscr{D}_1 > 0$ such that

$$(3.5) \qquad |H^{(i)}(\varrho, \varkappa) - H^{(i)}(\eta, \varkappa')| \geq \mathscr{D}_1 \cdot (|\varrho - \eta| + |\varkappa - \varkappa'|)$$

for $\varrho, \eta \in \Delta^1$, $\varkappa, \varkappa' \in (-\alpha(\Omega), \alpha(\Omega))$, $i \in \{1, \ldots, k(\Omega)\}$.

From (3.5) it follows that the mapping $H^{(i)}$ is one-to-one ($i \in \{1, \ldots, k(\Omega)\}$). Thus $H^{(i)}$ is a bijective Lipschitz function. In particular, its Jacobian $(\mathrm{d}H^{(i)}/\mathrm{d}X)(\eta, r)$ exists at almost every point $(\eta, r) \in \Delta^1 \times (-\alpha(\Omega), \alpha(\Omega))$. A short calculation shows that $(H^{(i)})^{-1}$ is Lipschitz continuous too, and

$$\det(\mathrm{d}H^{(i)}/\mathrm{d}X)(\eta, r) = 1 \quad \text{for a.e.} \quad (\eta, r) \in \Delta^1 \times (-\alpha(\Omega), \alpha(\Omega)), \quad i \in \{1, \ldots, k(\Omega)\}.$$

This implies

$$(3.6) \qquad \int_{U_i^1} f(x)\,\mathrm{d}x = \int_{-\alpha(\Omega)}^{\alpha(\Omega)} \int_{\Delta^1} (f \circ H^{(i)})(\eta, s)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for} \quad f \in L^1(U_i^1).$$

For $i \in \{1, \ldots, k(\Omega)\}$, the mapping

$$h^{(i)} : \Delta^1 \mapsto \Lambda_i^1, \qquad h^{(i)}(\eta) := A_i^{(\Omega)} \cdot (\eta, a_i^{(\Omega)}(\eta)) + C_i^{(\Omega)} \quad (\eta \in \Delta^1),$$

is a local parameter of $\partial\Omega$, that is, $h^{(i)}$ is a bijective Lipschitz function with range in $\partial\Omega$, and its Jacobian $(\mathrm{d}h^{(i)}/\mathrm{d}X)(\eta)$ has maximal rank, for almost every $\eta \in \Delta^1$. Set

$$J^{(i)}(\eta) := \left(1 + \sum_{r=1}^{2} |D_r h^{(i)}(\eta)|^2\right)^{1/2} \quad \text{for a.e.} \quad \eta \in \Delta^1, \qquad i \in \{1, \ldots, k(\Omega)\}.$$

Then for any integrable function $f:\partial\Omega \mapsto \mathbb{C}$ it holds

$$(3.7) \qquad \int_{\Delta_i^1} f\,\mathrm{d}\Omega = \int_{\Delta^1} (f \circ h^{(i)}) \cdot J^{(i)}\,\mathrm{d}\eta\,.$$

Because of (3.1), we may choose functions $\Psi_1^{(\Omega)}, \ldots, \Psi_{k(\Omega)}^{(\Omega)} \in C_0^\infty(\mathbb{R}^3)$ with

$$0 \le \Psi_i^{(\Omega)} \le 1\,, \qquad \operatorname{supp}(\Psi_i^{(\Omega)}) \subset U_i^{1/4} \quad \text{for} \quad i \in \{1, \ldots, k(\Omega)\}\,,$$

$$\sum_{i=1}^{k(\Omega)} \Psi_i^{(\Omega)} \mid \partial\Omega = 1\,.$$

Moreover, we define

$$\tilde{m}(x) := \sum_{i=1}^{k(\Omega)} \Psi_i^{(\Omega)}(x) \cdot A_i^{(\Omega)} \cdot (0,0,1) \quad \text{for} \quad x \in \partial\Omega\,.$$

Then $\tilde{m}(x) \neq 0$ for $x \in \partial\Omega$. Hence we may set

$$m(x) := |\tilde{m}(x)|^{-1} \cdot \tilde{m}(x) \quad \text{for} \quad x \in \partial\Omega\,.$$

For $y, z \in \mathbb{R}^3$ with $|z| = 1$, $\varepsilon, \delta \in (0, \infty)$, we define the cone $K(y, z, \delta, \varepsilon) \subset \mathbb{R}^3$ by

$$K(y, z, \delta, \varepsilon) := \{y + t \cdot b : t \in (0, \delta)\,,\ b \in \mathbb{R}^3 \text{ with } |b| = 1\,,\ |b - z| < \varepsilon\}\,.$$

It turns out that there are constants $\mathscr{D}_2, \mathscr{D}_3, \mathscr{D}_4, \mathscr{D}_5 \in (0, \infty)$ with

$$(3.8) \qquad \begin{aligned} &K(x, m(x), \mathscr{D}_2, \mathscr{D}_3) \subset \mathbb{R}^3 \setminus \overline{\Omega}\,,\\ &K(x, -m(x), \mathscr{D}_2, \mathscr{D}_3) \subset \Omega \quad \text{for} \quad x \in \partial\Omega\,,\\ &|x + \varkappa \cdot m(x) - x' - \varkappa' \cdot m(x')| \ge \mathscr{D}_5 \cdot (|x - x'| + |\varkappa - \varkappa'|) \end{aligned}$$

for $x, x' \in \partial\Omega$, $\varkappa, \varkappa' \in (-\mathscr{D}_4, \mathscr{D}_4)$. The proof of these results is by no means trivial. However, it presents no fundamental difficulties, and we may skip it here.

For $p \in [1, \infty)$, we define $L^p(\partial\Omega)$ as the space of all p-integrable functions from $\partial\Omega$ into $\mathbb{C}$. As a norm on $L^p(\partial\Omega)$, we choose the mapping

$$\|\phi\|_p := \left(\int_{\partial\Omega} |\phi(x)|^p\,\mathrm{d}\Omega(x) \right)^{1/p}, \qquad \phi \in L^2(\partial\Omega)\,.$$

Consider the space $W^{1,2}(\partial\Omega)$, defined as in [27, p. 327/328]. Set

$$\|\phi\|_{1,2}^{(\partial\Omega)} := \left(\sum_{i=1}^{k(\Omega)} \|\phi \circ h^{(i)}\|_{1,2}^2 \right)^{1/2} \quad \text{for} \quad \phi \in W^{1,2}(\partial\Omega)\,.$$

Then $\|\ \|_{1,2}^{(\partial\Omega)}$ is a norm on $W^{1,2}(\partial\Omega)$ ([27, p. 327/328]). For shortness, we shall only write $\|\ \|_{1,2}$ instead of $\|\ \|_{1,2}^{(\partial\Omega)}$.

Let us precisely state the version of Gauss' theorem which will be used in the following.

Theorem 3.1. *Let $\Lambda \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary, and with outward unit normal $n^{(\Lambda)}$. Let $h \in W^{1,1}(\Lambda)^3$. Then*

$$\int_{\partial\Lambda} \operatorname{trace}(h) \cdot n^{(\Lambda)}\,\mathrm{d}\Lambda = \int_{\Lambda} \operatorname{div} h\,\mathrm{d}x\,.$$

For a proof, we refer to [4, p. 193/194].

4. Definition of potential functions; auxiliary results

In this section we shall introduce certain layer and volume potentials, and then recall some well-known facts pertaining to these potentials. In addition, we shall list certain well-know results concerning self-adjoint operators, Sobolev spaces, and interpolation theory. But first let us state the Helmholtz decomposition of $L^2(\Omega)^3$.

Theorem 4.1. *The orthogonal complement of $H_2(\Omega)$ in $L^2(\Omega)^3$ coincides with the space $\{\nabla p : p \in W^{1,2}(\Omega)\}$. This means in particular that there exists a linear operator $G : L^2(\Omega)^3 \mapsto W^{1,2}(\Omega)$ and a constant $C_1(\Omega) > 0$ such that for $f \in L^2(\Omega)^3$, it holds*

$$f = P_2 f + \nabla(Gf), \qquad \int_\Omega (Gf)(x)\,dx = 0,$$

$$\|\nabla(Gf)\|_2 \le C_1(\Omega) \cdot \|f\|_2,$$

where P_2 denotes the projection from $L^2(\Omega)^3$ onto $H_2(\Omega)$.

Theorem 4.2. *The closure of the set $\{g \in C_0^\infty(\Omega)^3 : \operatorname{div} g = 0\}$ in $W^{1,2}(\Omega)^3$ is equal to $\{g \in W_0^{1,2}(\Omega)^3 : \operatorname{div} g = 0\}$. This means in particular that for every $u \in W_0^{1,2}(\Omega)^3 \cap H_2(\Omega)$, there is a sequence (g_n) in $C_0^\infty(\Omega)$ with*

$$\|u - g_n\|_{1,2} \to 0 \quad (n \to \infty), \qquad \operatorname{div} g_n = 0 \quad \text{for} \quad n \in \mathbb{N}.$$

For a proof of these theorems, we refer to [49, p. 15, Theorem 1.4; p. 18, Theorem 1.6; p. 19/20, Remark 1.9].

Next we define

$$g_1(r) := e^{-r} + r^{-2} \cdot (r \cdot e^{-r} + e^{-r} - 1),$$

$$g_1(r) := e^{-r} + 3 \cdot r^{-2} \cdot (r \cdot e^{-r} + e^{-r} - 1), \quad \text{for} \quad r \in \mathbb{C} \setminus \{0\},$$

$$E^\lambda(z) := (4 \cdot \pi \cdot |z|)^{-1} \cdot e^{-\sqrt{\lambda} \cdot |z|},$$

$$\tilde{E}^\lambda_{jk}(z) := (4 \cdot \pi \cdot |z|)^{-1} \cdot \delta_{jk} \cdot g_1(\sqrt{\lambda} \cdot |z|) - (4 \cdot \pi \cdot |z|^3)^{-1} \cdot z_j \cdot z_k \cdot g_2(\sqrt{\lambda} \cdot |z|),$$

$$\tilde{E}^0_{jk}(z) := (8 \cdot \pi)^{-1} \cdot (|z|^{-1} \cdot \delta_{jk} + z_j \cdot z_k \cdot |z|^{-3}),$$

$$\tilde{E}_{4k}(z) := (4 \cdot \pi)^{-1} \cdot z_k \cdot |z|^{-3} \quad \text{for} \;\; j, k \in \{1, 2, 3\}, \;\; z \in \mathbb{R}^3 \setminus \{0\}, \;\; \lambda \in \mathbb{C} \setminus (-\infty, 0].$$

For $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, the function E^λ is a fundamental solution of the Helmholtz equation

$$-\Delta u + \lambda \cdot u = f.$$

Furthermore, for $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ the matrix-valued function $(\tilde{E}^\lambda_{1k}, \tilde{E}^\lambda_{2k}, \tilde{E}^\lambda_{3k}, \tilde{E}_{4k})_{1 \le k \le 3}$ is a fundamental solution of the resolvent problem for the Stokes system,

$$-\Delta u + \lambda \cdot u + \nabla \pi = f, \qquad \operatorname{div} u = 0; \tag{4.1}$$

see [38, p. 205/206] for this result. If $\lambda = 0$, then the preceding function is a fundamental solution of the Stokes system (1.6). We note the ensuing estimates of $\tilde{E}^\lambda_{jk}$ and E^λ (compare [15, p. 342, (3.2)]:

$$|E^\lambda(z)| \le (4 \cdot \pi)^{-1} \cdot |z|^{-1} \cdot e^{-|\lambda|^{1/2} \cdot \cos(\vartheta/2) \cdot |z|}, \tag{4.2}$$

$$|D^\alpha \tilde{E}^\lambda_{jk}(z)|, \qquad |D^\alpha E^\lambda(z)| \le C_2(\vartheta) \cdot |\lambda|^{-\gamma} \cdot |z|^{-1-\alpha_1-\alpha_2-\alpha_3-2\cdot\gamma} \tag{4.3}$$

for $\gamma \in [0, 1]$, $1 \leq j, k \leq 3$, $z \in \mathbb{R}^3 \setminus \{0\}$, $\vartheta \in [0, \pi)$, $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\arg \lambda| \leq \vartheta$, $a \in \mathbb{N}_0^3$ with $a_1 + a_2 + a_3 \leq 3$. We insist on the fact that the constant $C_2(\vartheta)$ only depends on ϑ.

For $f \in L^2(\Omega)^3$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, we define the volume potentials

$$R^\lambda(f)(x) := \left(\int_\Omega E^\lambda(x-y) \cdot (P_2 f)_l(y)\, dy \right)_{1 \leq l \leq 3},$$

$$R^0(f)(x) := \left(\int_\Omega \sum_{k=1}^3 \tilde{E}^0_{kl}(x-y) \cdot f_k(y)\, dy \right)_{1 \leq l \leq 3} \quad \text{for} \quad x \in \mathbb{R}^3,$$

$$\Pi(f)(x) := \int_\Omega \sum_{k=1}^3 \tilde{E}_{4k}(x-y) \cdot f_k(y)\, dy, \qquad \text{for a.e.} \quad x \in \mathbb{R}^3.$$

Using Theorem 4.1 and the multiplier theorem from [47, p. 94–96], we may conclude (compare [38, p. 207, Theorem 3.10]): For any $\vartheta \in [0, \pi)$, there exists a constant $C_3(\Omega, \vartheta) > 0$ such that

$$|\lambda| \cdot \|R^\lambda(f)\|_2 + |\lambda|^{1/2} \cdot \sum_{l=1}^3 \|D_l R^\lambda(f)\|_2 + \sum_{l,m=1}^3 \|D_m D_l R^\lambda(f)\|_p + \sum_{l=1}^3 \|D_l(Gf)\|_2 \tag{4.4}$$
$$\leq C_3(\Omega, \vartheta) \cdot \|f\|_2 \quad \text{for} \quad f \in L^2(\Omega)^3, \qquad \lambda \in \mathbb{C} \setminus (-\infty, 0] \quad \text{with} \quad |\arg \lambda| \leq \vartheta.$$

It was shown in [21, p. 33–35, Satz 1.4] that there is another constant $C_4(\Omega) > 0$ such that

$$\|R^0(f)\|_{2,2} + \|\Pi(f)\|_{1,2} \leq C_4(\Omega) \cdot \|f\|_2 \quad \text{for} \quad f \in L^2(\Omega)^3. \tag{4.5}$$

After some calculations, we obtain from (4.3) that for any $\vartheta \in [0, \pi)$, there is some $C_5(\Omega, \vartheta) > 0$ with

$$\|R^\lambda(f) \mid \partial\Omega\|_2 \leq C_5(\Omega, \vartheta) \cdot |\lambda|^{-3/4} \cdot \|f\|_2, \tag{4.6}$$

$$\|R^\lambda(f) \mid \partial\Omega\|_{1,2} \leq C_5(\Omega, \vartheta) \cdot |\lambda|^{-1/4} \cdot \|f\|_2 \tag{4.7}$$

$$\text{for} \quad f \in L^2(\Omega)^3, \qquad \lambda \in \mathbb{C} \quad \text{with} \quad |\arg \lambda| \leq \vartheta, \qquad |\lambda| \geq 1.$$

Let $\phi \in L^2(\partial\Omega)^3$, $\psi \in L^2(\partial\Omega)$, $\lambda \in \mathbb{C} \setminus (-\infty, 0)$. Then we define the single layer potentials $V^\lambda(\phi)$, $S(\psi)$, $Q(\phi)$ by setting

$$V^\lambda(\phi)(x) := \left(\int_{\partial\Omega} \sum_{k=1}^3 \tilde{E}^\lambda_{jk}(x-y) \cdot \phi_k(y)\, d\Omega(y) \right)_{1 \leq j \leq 3},$$

$$S(\psi)(x) := (4 \cdot \pi)^{-1} \cdot \int_{\partial\Omega} |x-y|^{-1} \cdot \Psi(y)\, d\Omega(y)$$

for $x \in \mathbb{R}^3 \setminus \partial\Omega$, and for a.e. $x \in \partial\Omega$,

$$Q(\phi)(x) := \int_{\partial\Omega} \sum_{k=1}^3 \tilde{E}_{4k}(x-y) \cdot \phi_k(y)\, d\Omega(y) \quad \text{for} \quad x \in \mathbb{R}^3 \setminus \partial\Omega.$$

Obviously, the functions $V^\lambda_j(\phi) \mid \mathbb{R}^3 \setminus \partial\Omega$ and $S(\psi) \mid \mathbb{R}^3 \setminus \partial\Omega$ belong to $C^\infty(\mathbb{R}^3 \setminus \partial\Omega)$, for λ, ϕ, ψ as before, and for $j \in \{1, 2, 3\}$.

If $f \in L^2(\Omega)^3$, and if λ, ϕ are given as before, we set

$$u(\lambda, f, \phi) := (R^\lambda(f) + V^\lambda(\phi)) \mid \Omega ,$$

$$\pi(\lambda, f, \phi) := \begin{cases} (\Pi(f) + Q(\phi)) \mid \Omega - \left(\int\limits_\Omega \mathrm{d}x\right)^{-1} \cdot \int\limits_\Omega (\Pi(f) + Q(\phi))(x)\,\mathrm{d}x , & \text{if } \lambda = 0 , \\ (Gf + Q(\phi)) \mid \Omega - \left(\int\limits_\Omega \mathrm{d}x\right)^{-1} \cdot \int\limits_\Omega (Gf + Q(\phi))(x)\,\mathrm{d}x & \text{if } \lambda \neq 0 , \end{cases}$$

where Gf was introduced in Theorem 4.1.

By referring to (4.4), (4.5) for f, λ, ϕ as before we obtain

$$u(\lambda, f, \phi) \in W^{2,2}_{\mathrm{loc}}(\Omega)^3 , \qquad \pi(\lambda, f, \phi) \in W^{1,2}_{\mathrm{loc}}(\Omega) .$$

In the case $\lambda \neq 0$, the pair of function $(u(\lambda, f, \phi), \pi(\lambda, f, \phi))$ solves the resolvent problem (4.1) on Ω. Furthermore, the pair of functions $(u(0, f, \phi), \pi(0, f, \phi))$ a is solution of the Stokes system (1.6) on Ω. For a proof of these facts we refer to [38, p. 204], [21, p. 33–35, Satz 1.4].

The preceding remarks hold for any function $\phi \in L^2(\partial\Omega)^3$. Using this fact will allow us to prescribe the boundary value of $u(\lambda, f, \phi)$ on $\partial\Omega$, see Corollary 5.1, Lemma 5.5. (method of integral equations).

Let $\lambda \in \mathbb{C} \setminus (-\infty, 0)$, $\phi \in L^2(\partial\Omega)^3$. Then for $j, k \{1, 2, 3\}$ there are functions $B^+_{jk}(\lambda, \phi)$, $B^-_{jk}(\lambda, \phi)$, $A^+(\phi)$, $A^-(\phi) \in L^2(\partial\Omega)$ such that

$$\int\limits_{\partial\Omega} |D_j(V^\lambda_k(\phi) | \mathbb{R}^3 \setminus \partial\Omega)(x \pm \varepsilon \cdot m(x)) - B^\pm_{jk}(\lambda, \phi)(x)|^2 \,\mathrm{d}\Omega(x) \to 0 \qquad (\varepsilon \downarrow 0) ,$$

$$\int\limits_{\partial\Omega} |Q(\phi)(x \pm \varepsilon \cdot m(x)) - A^\pm(\phi)(x)|^2 \,\mathrm{d}\Omega(x) \to 0 \qquad (\varepsilon \downarrow 0) .$$

In addition, the equations

$$\text{(4.8)} \qquad B^+_{jk}(\lambda, \phi)(x) - \sum_{s=1}^3 n_s(x) \cdot B^+_{sk}(\lambda, \phi)(x) \cdot n_j(x)$$
$$= B^-_{jk}(\lambda, \phi)(x) - \sum_{s=1}^3 n_s(x) \cdot B^-_{sk}(\lambda, \phi)(x) \cdot n_j(x) \qquad (j, k \in \{1, 2, 3\}) ,$$

$$\text{(4.9)} \qquad \left(\sum_{j=1}^3 B^\pm_{jk}(\lambda, \phi)(x) \cdot n_j(x)\right)_{1 \le k \le 3} - A^\pm(\phi)(x) \cdot n(x) = \mp (1/2)\,\phi(x) + K(\lambda, \phi)(x)$$

("jump relation"),

hold for a.e. $x \in \partial\Omega$ where $K(\lambda, \phi)$ denotes a principal value integral defined by

$$\text{(4.10)} \qquad K(\lambda, \phi) := L^2(\partial\Omega)^3 - \lim_{\varepsilon \downarrow 0} \left[\int\limits_{\partial\Omega} \chi_{(\varepsilon, \infty)}(|\mathrm{id}\,(\partial\Omega) - y|) \right.$$
$$\times \left(\sum_{j,l=1}^3 D_j \tilde{E}^\lambda_{kl}(\mathrm{id}\,(\partial\Omega) - y) \cdot n_j \right.$$
$$\left. - n_k \cdot \sum_{l=1}^3 \tilde{E}_{4l}(\mathrm{id}\,(\partial\Omega) - y)) \cdot \phi_l(y)\,\mathrm{d}\Omega(y) \right)_{1 \le k \le 3} .$$

The symbol id $(\partial\Omega)$ in (4.10) denotes the identity mapping of $\partial\Omega$ onto itself.

In the case $\lambda = 0$, these results may be deduced from COIFMAN, MCINTOSH, MEYER [8]; compare the remarks in [24, p. 773/774], and recall the cone property stated in (3.8). Concerning the case $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ the preceding relations follow from the corresponding results in the case $\lambda = 0$, and from the inequality

$$(4.11) \qquad |D^b(\tilde{E}^\lambda_{jk} - \tilde{E}^0_{jk}(z)| \le C_6(\vartheta) \cdot |\lambda| \cdot |z|^{-b_1-b_2-b_3+1},$$

which holds for $\vartheta \in [0, \pi)$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ with $|\arg \lambda| \le \vartheta$, $b \in \mathbb{N}_0^3$ with $1 \le b_1 + b_2 + b_3 \le 2$, $z \in \mathbb{R}^3 \setminus \{0\}$; compare [15, p. 342, (3.3)].

Let us denote the limit on the left- and right-hand side of (4.8) by $T_{jk}(\lambda, \phi)(x)$, for $\phi \in L^2(\partial\Omega)^3$, $\lambda \in \mathbb{C} \setminus (-\infty, 0)$, $j, k \in \{1, 2, 3\}$, $x \in \partial\Omega$. This means that $(T_{jk}(\lambda, \phi))_{1 \le j \le 3}$ is the tangential gradient of $V^\lambda_k(\phi)$ on $\partial\Omega$ ([24, p. 774]).

We further mention that $V^\lambda(\phi) \mid \partial\Omega \in W^{1,2}(\partial\Omega)^3$ for $\lambda \in \mathbb{C} \setminus (-\infty, 0)$, $\phi \in L^2(\partial\Omega)^3$. In addition, there is a constant $C_7(\Omega) > 0$ such that

$$(4.12) \qquad \begin{aligned} (C_7(\Omega))^{-1} \cdot \|V^\lambda(\phi) \mid \partial\Omega\|_{1,2} &\le \|V^\lambda(\phi) \mid \partial\Omega\|_2 + \sum_{k,l=1}^{3} \|T_{kl}(\lambda, \phi)\|_2 \\ &\le C_7(\Omega) \cdot \|V^\lambda(\phi) \mid \partial\Omega\|_{1,2} \\ &\qquad \text{for } \lambda \in \mathbb{C} \setminus (-\infty, 0), \quad \phi \in L^2(\partial\Omega)^3. \end{aligned}$$

This result may be proved by transforming $T_{jk}(\lambda, \phi)$ into local coordinates; compare the remarks in [51, p. 580]. For $\phi \in L^2(\Omega)^3$ we set

$$(4.13) \qquad c(\phi) := \left(\int_{\partial\Omega} d\Omega \right)^{-1} \cdot \int_{\partial\Omega} A^-(\phi)\, d\Omega .$$

Then the following estimates of $V^\lambda(\phi)$ and $Q(\phi)$ hold true:

Lemma 4.1. *There is a constant* $C_8(\Omega) > 0$ *such that*

$$\begin{aligned} &\sum_{j,k=1}^{3} (\|B^+_{jk}(i \cdot \tau, \phi)\|_2 + \|B^-_{jk}(i \cdot \tau, \phi)\|_2) + \|A^-(\phi) - c(\phi)\|_2 + \|A^+(\phi)\|_2 \\ &\le C_8(\Omega) \cdot (\|V^{i\cdot\tau}(\phi) \mid \partial\Omega\|_{1,2} + |\tau|^{1/2} \cdot \|V^{i\cdot\tau}(\phi) \mid \partial\Omega\|_2 + |\tau| \cdot \|S(n \cdot V^{i\cdot\tau}(\phi) \mid \partial\Omega)\|_2) \\ &\qquad \textit{for } \tau \in \mathbb{R}, \quad \phi \in L^2(\partial\Omega)^3. \end{aligned}$$

For a proof we refer to [44, p. 364, Lemma 5.2.11] and to (4.12).

Let

$$(4.14) \qquad L^2_n(\partial\Omega) := \left\{ f \in L^2(\partial\Omega)^3 : \int_{\partial\Omega} f \cdot n \, d\Omega = 0 \right\}.$$

Then the following theorem holds true:

Theorem 4.3. *The mapping*

$$\mathfrak{S}^0 : L^2_n(\partial\Omega) \mapsto L^2_n(\partial\Omega) \cap W^{1,2}(\partial\Omega)^3, \qquad \mathfrak{S}^0(\phi) := V^0(\phi) \mid \partial\Omega$$

is bounded, one-to-one, and onto. In particular, there is a constant $C_9(\Omega) > 0$ *such that*

$$\|\phi\|_2 \le C_9(\Omega) \cdot \|\mathfrak{S}^0(\phi)\|_{1,2} \quad \textit{for} \quad \phi \in L^2_n(\partial\Omega).$$

A proof of this theorem may be found in [24, p. 792].

Let us now list some functional analytic tools which we shall need later on. We begin with some elements of the theory of self-adjoint operators. Let H be a Hilbert space with norm $\| \;\|$, and let $K : \mathfrak{D}(K) \mapsto H$ be a positive, self-adjoint operator in H with domain of definition $\mathfrak{D}(K)$. Let $\mathfrak{J}$ denote the identity operator on H, and $(E(\lambda))_{\lambda \in \mathbb{R}}$ the spectral resolution of K ([55, p. 309–315]). Assume that the set $\mathbb{C} \setminus (0, \infty)$ belongs to the resolvent set

$$\varrho(K) := \{\lambda \in \mathbb{C} : (\lambda \cdot \mathfrak{J} - K)(H) \text{ is dense in } H; \lambda \cdot \mathfrak{J} - K \text{ has a bounded inverse}\}$$

of K. Since $\varrho(K)$ is open, there is some $\varepsilon \in (0, \infty)$ such that $\mathbb{C} \setminus [\varepsilon, \infty) \subset \varrho(K)$. We further require that $\lambda \cdot \mathfrak{J} - K$ is onto, for $\lambda \in \mathbb{C} \setminus (0, \infty)$.

For any continuous mapping $\varphi : [\varepsilon, \infty] \mapsto \mathbb{C}$, the set $\mathfrak{D}(\varphi(K))$ and the operator $\varphi(K) : \mathfrak{D}(\varphi(K)) \mapsto H$ are to be understood in the usual way, that is (see [55, p. 309ff, p. 338 ff]),

$$\mathfrak{D}(\varphi(K)) := \left\{ x \in H : \int_{[\varepsilon, \infty)} |\varphi(\lambda)|^2 \, \mathrm{d} \, \|E(\lambda)\, x\|^2 < \infty \right\},$$

$$\varphi(K)\,(x) := \int_{[\varepsilon, \infty)} \varphi(\lambda) \, \mathrm{d}E(\lambda)\, x \quad \text{for} \quad x \in \mathfrak{D}(\varphi(K)).$$

These definitions give rise to a well-known operator calculus ([55, p. 343]), which we shall use extensively.

For $\alpha \in (0, 1)$, we have $\mathfrak{D}(K^{-\alpha}) = H$, and

$$(4.15) \qquad K^{-\alpha}(x) = \pi^{-1} \cdot \sin(\pi \cdot \alpha) \cdot H - \int_0^\infty \lambda^{-\alpha} \cdot (\lambda \cdot \mathfrak{J} + K)^{-1}(x) \, \mathrm{d}\lambda \quad \text{for} \quad x \in H.$$

Note that $K^{-\alpha}$ is bounded and one-to-one for $\alpha \in (0, 1]$. We further mention that

$$(4.16) \qquad \|(K^\alpha \, \mathrm{e}^{-(t+h) \cdot K} - K^\alpha \, \mathrm{e}^{-t \cdot K})\, x\| \le 4 \cdot h \cdot t^{-1-\alpha} \cdot \|x\|$$
$$\text{for} \quad x \in H, \qquad h, t \in (0, \infty), \qquad \alpha \in [0, 1],$$

$$(4.17) \qquad \mathrm{e}^{-t \cdot K}\, x \in \mathfrak{D}(K^\alpha), \qquad K^\alpha \, \mathrm{e}^{-t \cdot K}\, y = \mathrm{e}^{-t \cdot K} K^\alpha y$$
$$\text{for} \quad \alpha \in [0, 1], \qquad t \in (0, \infty), \qquad x \in H, \qquad y \in \mathfrak{D}(K^\alpha).$$

In addition, for any $\alpha \in [0, 1]$ there is a constant $C_{10}(K, \alpha) > 0$ such that

$$(4.18) \qquad \|K^\alpha \, \mathrm{e}^{-t \cdot K}\, x\| \le C_{10}(K, \alpha) \cdot t^{-\alpha} \cdot \|x\| \quad \text{for} \quad t \in (0, \infty), \qquad x \in H.$$

The preceding results, starting with (4.15), may all be proved by using the spectral resolution of K. Let us note a consequence of (4.18), namely,

$$(4.19) \qquad \|t^\alpha \cdot K^\alpha \, \mathrm{e}^{-t \cdot K}\, x\| \to 0 \; (t \downarrow 0) \quad \text{for} \quad x \in H, \qquad \alpha \in (0, 1].$$

In fact, take $\alpha \in (0, 1]$, $x \in H$, and fix $\varepsilon \in (0, \infty)$. Then by (4.18), we may choose $y \in \mathfrak{D}(K)$ with $\|t^\alpha \cdot K^\alpha \, \mathrm{e}^{-t \cdot K}\,(x - y)\|_2 < \varepsilon$. But by (4.17), $t^\alpha \cdot K^\alpha \, \mathrm{e}^{-t \cdot K}\, y \to 0 \; (t \downarrow 0)$, and (4.19) follows.

Since $K^{-\alpha}$ is one-to-one and bounded, there is a constant $C_{11}(K, \alpha) > 0$ such that

$$(4.20) \qquad \|x\| \le C_{11}(K, \alpha) \cdot \|K^\alpha x\| \quad \text{for} \quad x \in \mathfrak{D}(K^\alpha), \alpha \in (0, 1].$$

Therefore, by assigning the value $\|K^\alpha x\|$ to any $x \in \mathfrak{D}(K^\alpha)$, we obtain a norm on $\mathfrak{D}(K^\alpha)$. In this way, $\mathfrak{D}(K^\alpha)$ becomes a Banach space.

Now let us recall some facts from interpolation theory. Consider two Banach spaces $\mathscr{A}_0$, A_1 with norm $\| \;\|_{\mathscr{A}_0}$, $\| \;\|_{A_1}$. Assume that the spaces $\mathscr{A}_0$, $\mathscr{A}_1$ constitute an interpolation couple ([50, p. 18/19, 1.2.1]). For $\theta \in (0, 1)$, let $(A_0, \mathscr{A}_1)_{\theta, 2}$ denote the corresponding real interpolation space ([50, p. 23–25, 1.3.1, 1.3.2]). The norm of $(\mathscr{A}_0, \mathscr{A}_1)_{\theta,2}$, defined as in [50, p. 24, 1.3.2], will be denoted by $\| \ \|_{\mathscr{A}_0,\mathscr{A}_1,\theta,2}$.

Returning to the self-adjoint operator K introduced before, we observe that

$$\mathfrak{D}(K^{(1-\theta)\cdot\alpha+\theta\cdot\beta}) = (\mathfrak{D}(K^\alpha), \mathfrak{D}(K^\beta))_{\theta,2} \quad \text{for} \quad \alpha, \beta \in [0,1], \qquad \theta \in (0,1), \tag{4.21}$$

and the norms of the spaces appearing in (4.21) are equivalent ([50, p. 141–143, 1.18.10]). Combining this result with the inequality in [50, p. 25, (1.3.3/5)], we arrive at the following estimate: Let $\alpha, \beta \in [0, 1]$, $\theta \in (0, 1)$. Then there is a constant $C_{12}(K, \alpha, \beta, \theta) > 0$ such that

$$\|K^{(1-\theta)\cdot\alpha+\theta\cdot\beta}x\| \leq C_{12}(K,\alpha,\beta,\theta)\cdot\|K^\alpha x\|^{1-\theta}\cdot\|K^\beta x\|^\theta \qquad \text{for} \quad x \in \mathfrak{D}(K^\alpha) \cap \mathfrak{D}(K^\beta). \tag{4.22}$$

We recall that spaces $W^{s,2}(\Omega)$, $W^{r,2}(\Omega)$ constitute an interpolation couple, with

$$W^{(1-\theta)\cdot s+\theta\cdot r,2}(\Omega) = (W^{s,2}(\Omega), W^{r,2}(\Omega))_{\theta,2} \qquad \text{for} \quad r, s \in (1,\infty), \qquad r \neq s, \qquad \theta \in (0,1). \tag{4.23}$$

The norms of the spaces appearing in (4.23) are equivalent. For these results we refer to [50, p. 317, Theorem 4.3.1.2]. By [50, p. 25, (1.3.3/5)] this implies that for r, s, θ as before, there is constant $C_{13}(\Omega, r, s, \theta) > 0$ with

$$\|u\|_{(1-\theta)\cdot s+\theta\cdot r,2} \leq C_{13}(\Omega,r,s,\theta)\cdot\|u\|_{s,2}^{1-\theta}\cdot\|u\|_{r,2}^{\theta} \quad \text{for } u \in W^{s,2}(\Omega) \cap W^{r,2}(\Omega). \tag{4.24}$$

Let us mention another constant which will enter into our estimates. It arises in a version of Sobolev's inequality, which reads as follows (see [50, p. 328, (4.6.1/8)]):

Theorem 4.4. *For $s \in (1, 3/2)$, there is a constant $C_{14}(\Omega, s)$ such that*

$$\|u\|_{(1/2-s/3)^{-1}} \leq C_{14}(\Omega,s)\cdot\|u\|_{s,2} \quad \textit{for} \quad u \in W^{s,2}(\Omega).$$

5. L^2-estimates of solutions of the resolvent problem for the Stokes system

In this section we shall give a L^2-theory for solutions of (4.1) on Ω under Dirichlet boundary conditions $u \mid \partial\Omega = 0$.

Lemma 5.1. *Let $\lambda \in \mathbb{C} \setminus (-\infty, 0)$, $\phi \in L^2(\partial\Omega)^3$. Then $V^\lambda(\phi) \mid \Omega$ belongs to $W^{1,2}(\Omega)^3$, and*

$$\operatorname{trace}(V^\lambda(\phi) \mid \Omega) = V^\lambda(\phi) \mid \partial\Omega. \tag{5.1}$$

In the case $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, we have in addition

$$V^\lambda(\phi) \mid \mathbb{R}^3 \setminus \overline{\Omega} \in W^{1,2}(\mathbb{R}^3 \setminus \overline{\Omega})^3,$$

$$\operatorname{trace}(V^\lambda(\phi) \mid \mathbb{R}^3 \setminus \overline{\Omega}) = V^\lambda(\phi) \mid \partial\Omega. \tag{5.2}$$

Equation (5.2) also holds in the case $\lambda = 0$, but we shall not need this fact.

Proof. By using (3.8) and (4.3), it may be shown that there is a constant $\mathscr{C} > 0$ such that

$$|D^a \tilde{E}^\lambda_{jk}(x - y - \varkappa\cdot m(y))| \leq \mathscr{C}\cdot|x-y|^{-1-a_1-a_2-a_3} \tag{5.3}$$

for $a \in \mathbb{N}_0^3$ with $a_1 + a_2 + a_3 \leq 1$, $x \in \overline{\Omega}$, $y \in \partial\Omega$, $\varkappa \in [0, \mathscr{D}_4)$.

This inequality with $\varkappa = 0$ implies that $V^\lambda(\phi) \mid \Omega \in W^{1,2}(\Omega)^3$. For $\varkappa \in (0, \mathscr{D}_4)$, we define the function $V^{\lambda,\varkappa}(\phi) : \overline{\Omega} \mapsto \mathbb{C}^3$ by

$$V^{\lambda,\varkappa}(\phi)(x) := \left(\int_{\partial\Omega} \sum_{k=1}^{3} \tilde{E}^\lambda_{jk}(x - y - \varkappa \cdot m(y)) \cdot \phi_j(y) \, d\Omega(y) \right)_{1 \le j \le 3} \quad \text{for} \quad x \in \overline{\Omega}.$$

Note that by (3.8), $V^{\lambda,\varkappa}(\phi) \in C^\infty(\overline{\Omega})^3$ for $\varkappa \in (0, \mathscr{D}_4)$. On the other hand, it follows from (5.3) and the theorem on dominated convergence that

$$\| V^{\lambda,\varkappa}(\phi) \,|\partial\Omega - V^\lambda(\phi)|\, \partial\Omega \|_2 \to 0, \qquad \| V^{\lambda,\varkappa}(\phi) \,|\Omega - V^\lambda(\phi)|\, \Omega \|_{1,2} \to 0$$
$$\text{for} \quad \varkappa \downarrow 0.$$

This implies (5.1). Now assume that $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. By (3.8), (4.3), there exists a constant $\mathscr{C} > 0$ such that

$$|D^\alpha \tilde{E}^\lambda(x - y + \varkappa \cdot m(y))| \le \mathscr{C} \cdot \min \{ |x - y|^{-1-a_1-a_2-a_3}, |x - y|^{-3-a_1-a_2-a_3} \}$$

for $a \in \mathbb{N}_0^3$ with $a_1 + a_2 + a_3 \le 1$, $x \in \mathbb{R}^3 \setminus \overline{\Omega}$, $y \in \partial\Omega$, $\varkappa \in (0, \mathscr{D}_4)$.

Now the second part of the lemma may be shown by proceding in a similar way as in the proof of the first part. □

Lemma 5.2. *Let $\phi \in L^2(\partial\Omega)^3$. Then $Q(\phi) \in L^2(\mathbb{R}^3 \setminus \partial\Omega)$.*

The proof of this lemma is straightforward. We omit the details.

Corollary 5.1. *Let $\lambda \in \mathbb{C} \setminus (-\infty, 0)$, $f \in L^2(\Omega)^3$, $\phi \in L^2(\partial\Omega)^3$. Then*

$$u(\lambda, f, \phi) \in W^{1,2}(\Omega)^3, \ \pi(\lambda, f, \phi) \in L^2(\Omega),$$
$$\operatorname{trace}(u(\lambda, f, \phi)) = V^\lambda(\phi) \,|\partial\Omega + R^\lambda(f)|\, \partial\Omega.$$

In particular, if $V^\lambda(\phi) \,|\partial\Omega + R^\lambda(f)|\, \partial\Omega = 0$, then $u(\lambda, f, \phi) \in W_0^{1,2}(\Omega)$.

Proof. According to (4.4), (4.5) and Sobolev's lemma, the function $R^\lambda(f)$ belongs to $C^0(\mathbb{R}^3)^3$ so that trace $(R^\lambda(f) \mid \Omega) = R^\lambda(f)|\, \partial\Omega$. Thus the corollary follows from Lemma 5.1, 5.2, (4.4), (4.5), and Theorem 4.1. □

Let us now introduce the type of solutions of (4.1) which will be considered in the following. Let $A \subset \mathbb{R}^3$ be a domain with Lipschitz boundary. For $f \in L^2(A)^3$, $\lambda \in \mathbb{C} \setminus (-\infty, 0)$, we set

$$\text{(5.4)} \qquad SOL(A, \lambda, f) := \Big\{ \mathrm{D}(u, \pi) \in W^{2,2}_{\mathrm{loc}}(A)^3 \times W^{1,2}_{\mathrm{loc}}(A) : u \in W_0^{1,2}(A)^3, \pi \in L^2(A);$$
$$-\Delta u + \lambda \cdot u + \nabla \pi = f, \ \operatorname{div} u = 0 \text{ in } A; \ \int_A \pi(x)\, dx = 0,$$
$$\text{if } A \text{ is bounded} \Big\}.$$

Corollary 5.2. *Take $\lambda \in \mathbb{C} \setminus (-\infty, 0)$, $f \in L^2(\Omega)^3$, $\phi \in L^2(\partial\Omega)^3$, and assume*

$$V^\lambda(\phi) \,|\partial\Omega + R^\lambda(f)|\, \partial\Omega = 0.$$

Then $(u(\lambda, f, \phi), \pi(\lambda, f, \phi)) \in SOL(\Omega, \lambda, f)$.

Proof. Recall Corollary 5.1 and the remarks following the definition of $u(\lambda, f, \phi)$ and $\pi(\lambda, f, \phi)$ in Section 4.

Lemma 5.3. *Take* $\vartheta \in [0, \pi)$, $\lambda \in \mathbb{C} \setminus \{0\}$ *with* $|\arg \lambda| \leq \vartheta$, $A \in \{\Omega, \mathbb{R}^3 \setminus \bar{\Omega}\}$, $f \in L^2(A)^3$, $(u, \pi) \in SOL(A, \lambda, f)$. *Then*

$$|\lambda| \cdot \|u\|_2 \leq \max\{\sqrt{2}, \sin^{-1}(\max\{\pi/2, \vartheta\})\} \cdot \|f\|_2 . \tag{5.5}$$

Furthermore, for $(u, \pi) \in SOL(\Omega, 0, f)$, *we have*

$$\sum_{r=1}^{3} \|\nabla u_r\|_2^2 \leq \|f\|_2 \cdot \|u\|_2 . \tag{5.6}$$

Proof. Since $u \in W_0^{1,2}(A)^3$, there is a sequence (g_k) in $C_0^\infty(A)^3$ with $\|g_k - u\|_{1,2} \to 0$ $(k \to \infty)$ (Note that in the case $A = \mathbb{R}^3 \setminus \bar{\Omega}$, we cannot additionally require $\operatorname{div} g_k = 0$; see [49, p. 19]). After an integration by parts, it follows

$$\int_A \left(\sum_{l=1}^{3} \nabla u_l \cdot \nabla g_{k,l} + \lambda \cdot u \cdot g_k - \pi \cdot \operatorname{div} g_k \right) dx = \int_A f \cdot g_k \, dx \; (k \in \mathbb{N}) .$$

Since $D_r u_l, u_l, \pi \in L^2(A)$ (see (5.4)), we obtain, by letting k tend to infinity,

$$\sum_{r=1}^{3} \|\nabla u_r\|_2^2 + \lambda \cdot \|u\|_2^2 = \int_A f \cdot u \, dx .$$

This implies (5.5). The estimate in (5.6) is shown by a similar reasoning. □

Lemma 5.4 *Let* $\phi \in L^2(\partial\Omega)^3$, $\lambda \in \mathbb{C} \setminus (-\infty, 0)$. *Then*

$$Q(n) \,|\, \Omega = -1 , \qquad c(\phi + c(\phi) \cdot n) = 0 , \qquad V^\lambda(\phi + c(\phi) \cdot n) = V^\lambda(\phi) ,$$

where $c(\phi)$ *was defined in* (4.13).

The proof is an easy application of Theorem 3.1.

Lemma 5.5. *Let* $\lambda \in \mathbb{C} \setminus (-\infty, 0)$. *Define the mapping*

$$\mathfrak{S}^\lambda : L_n^2(\partial\Omega) \mapsto L_n^2(\partial\Omega) \cap W^{1,2}(\partial\Omega)^3$$

by

$$\mathfrak{S}^\lambda(\phi) := V^\lambda(\phi) \,|\, \partial\Omega \quad \textit{for} \quad \phi \in L_n^2(\partial\Omega) .$$

Then $\mathfrak{S}^\lambda$ *is one-to-one and onto (The space* $L_n^2(\partial\Omega)$ *was defined in* (4.14)).

Proof. (compare [44, p. 366]): In the case $\lambda = 0$ we refer to Theorem 4.3. Thus we may assume $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. Because of (4.11), the operator $\mathfrak{S}^\lambda - \mathfrak{S}^0$ is a compact mapping from $L_n^2(\partial\Omega)$ into the subspace $L_n^2(\partial\Omega) \cap W^{1,2}(\partial\Omega)^3$ of $W^{1,2}(\partial\Omega)^3$. Therefore $\mathfrak{S}^\lambda$ is a Fredholm operator with index 0, and the lemma is proved if we can show that $\mathfrak{S}^\lambda$ is one-to-one. To this end, consider $\phi \in L_n^2(\partial\Omega)$ with $\mathfrak{S}^\lambda(\phi) = 0$. Recalling Lemma 5.1, 5.2, and the fact that $V_j^\lambda(\phi) \,|\, \Omega$, $Q(\phi) \,|\, \Omega \in C^\infty(\Omega)$ $(1 \leq j \leq 3)$, we obtain

$$\left(V^\lambda(\phi) \,|\, \Omega,\, Q(\phi) \,|\, \Omega - \left(\int_\Omega dx \right)^{-1} \cdot \int_\Omega Q(\phi)(x) \, dx \right) \in SOL(\Omega, \lambda, 0) .$$

This implies by Lemma 5.3 that $\|V^\lambda(\phi) \mid \Omega\|_2 = 0$. Since $V_j^\lambda(\phi) \mid \Omega$, $Q(\phi) \mid \Omega \in C^\infty(\Omega)$ for $j \in \{1, 2, 3\}$, it follows that $V^\lambda(\phi) \mid \Omega = 0$, $Q(\phi) \mid \Omega = \gamma$ for some $\gamma \in \mathbb{C}$. Now set $\psi := \phi + \gamma \cdot n$. Lemma 5.4 yields that $V^\lambda(\psi) \mid \Omega$ and $Q(\psi) \mid \Omega$ are both vanishing on Ω. It follows from (4.9) that

$$(5.7) \qquad (1/2) \cdot \psi + K(\lambda, \psi) = 0\,,$$

where $K(\lambda, \psi)$ was defined in (4.10). On the other hand, we conclude from Lemma 5.1, 5.2

$$(V^\lambda(\psi) \mid \mathbb{R}^3 \setminus \overline{\Omega},\, Q(\psi) \mid \mathbb{R}^3 \setminus \overline{\Omega}) \in SOL(\mathbb{R}^3 \setminus \overline{\Omega}, \lambda, 0)\,.$$

Now we may apply Lemma 5.3 once more to obtain

$$V^\lambda(\psi) \mid \mathbb{R}^3 \setminus \overline{\Omega} = 0\,, \qquad Q(\psi) \mid \mathbb{R}^3 \setminus \Omega = \sigma \quad \text{for some} \quad \sigma \in \mathbb{C}\,.$$

But $Q(\psi)(x)$ tends to zero for $|x| \to \infty$. Thus $Q(\psi) \mid \mathbb{R} \setminus \overline{\Omega} = 0$, and from (4.9) it follows

$$(-1/2) \cdot \psi + K(\lambda, \psi) = 0\,.$$

Combining this equation with (5.7), we obtain $\psi = 0$. Since $\psi = \phi + \gamma \cdot n$ and $\phi \in L_n^2(\partial\Omega)$, we may conclude that $\gamma = 0$. Thus we finally obtain $\phi = 0$. □

The following theorem is the key result in our attempt to prove L^2-estimates of solutions of (4.1).

Theorem 5.1. *Let $\vartheta \in [0, \pi)$, $\varepsilon \in (0, 1/2)$. The there is a constant $C_{15}(\Omega, \vartheta, \varepsilon) > 0$ such that*

$$\|S(n \cdot R^\lambda(f) \mid \partial\Omega)\|_2 \le C_{15}(\Omega, \vartheta, \varepsilon) \cdot |\lambda|^{-5/4+\varepsilon} \cdot \|f\|_2$$

for $\lambda \in \mathbb{C}$ with $|\arg \lambda| \le \vartheta$, $|\lambda| \ge 1$, $f \in L^2(\Omega)^3$.

Proof. In the following, the letters $\mathscr{C}$, $\tilde{\mathscr{C}}$ will denote constants which only depend on Ω, ϑ or ε.

Assume that λ and f are given as in the lemma. Then Theorem 3.1 yields

$$\|S(n \cdot R^\lambda(f) \mid \partial\Omega)\|_2 = \left(\int_{\partial\Omega} \left| \int_{\mathbb{R}^3 \setminus \bar{\Omega}} \sum_{l=1}^{3} (x-y)_l \cdot |x-y|^{-3} \cdot R_l^\lambda(f)(y)\, dy \right|^2 d\Omega(x) \right)^{1/2}.$$

Thus we have by (3.1)

$$(5.8) \qquad \|S(n \cdot R^\lambda(f) \mid \partial Q)\|_2 \le \mathscr{C} \cdot \sum_{i=1}^{k(\Omega)} \sum_{\sigma=1}^{3} \left(\int_{\underline{A}(\sigma, i)} \left(\int_{(\mathbb{R}^3 \setminus \bar{\Omega}) \cap U_i^{1/2}} |x-y|^{-2} \right.\right.$$
$$\left.\left. \times \int_{\underline{B}(\sigma, i)} E^\lambda(y-z) \cdot |(P_2 f)(z)|\, dz\, dy \right)^2 d\Omega(x) \right)^{1/2}$$
$$+ \mathscr{C} \cdot \left\{ \int_{\partial\Omega} \left[\int_{(\mathbb{R}^3 \setminus \bar{\Omega}) \setminus \bigcup_{i=1}^{k(\Omega)} U_i^{1/2}} |x-y|^{-2} \cdot \int_{\Omega} E^\lambda(y-z) \cdot |(P_2 f)(z)|\, dz\, dy \right]^2 d\Omega(x) \right\}^{1/2},$$

with

$$\underline{A}(1, i) := A_i^1\,, \qquad \underline{A}(2, i) := \partial\Omega \setminus A_i^1\,, \qquad \underline{A}(3, i) := \partial\Omega\,,$$
$$\underline{B}(1, i) := \underline{B}(2, i) := \Omega \cap U_i^1\,, \qquad \underline{B}(3, i) := \Omega \setminus U_i^1\,.$$

Let us estimate the summands appearing in the sum on the right-hand side of (5.8). To this end, take $i \in \{1, \dots, k(\Omega)\}$ and consider the summand corresponding to i and to $\sigma = 1$, that is,

$$\alpha(i) := \left(\int_{A_i^1} \left(\int_{U_i^{1/2} \cap (\mathbb{R}^3 \setminus (\bar{\Omega})} |x - y|^{-2} \int_{\Omega \cap U_i^1} E^\lambda(y - z) \cdot |(P_2 f)(z)| \, dz \, dy \right)^{1/2} .$$

For brevity, we set $\tilde{f} := (P_2 f) \circ H^{(i)}$. Recalling (3.5), (3.6), (3.7), and (4.2), we obtain

$$(5.9) \qquad |\alpha(i)| \le \mathscr{C} \cdot \left(\int_{\Delta^1} \left(\int_0^{\alpha(\Omega)} \int_{\Delta^{1/2}} (|\sigma - \varrho| + |r|)^{-2} \cdot \int_{-\alpha(\Omega)}^{0} \int_{\Delta^1} (|\varrho - \eta| + |r - s|)^{-1} \right.\right.$$
$$\left.\left. \times \exp\left(- |\lambda|^{1/2} \cdot \tilde{\mathscr{C}} \cdot (|\varrho - \eta| + |r - s|)\right) \cdot \tilde{f}(\eta, s) \, d\eta \, ds \, d\varrho \, dr \right)^2 d\sigma \right)^{1/2} .$$

Observe that

$$\exp\left(- |\lambda|^{1/2} \cdot \tilde{\mathscr{C}} \cdot (|\varrho - \eta| + |r - s|)\right) \le \exp\left(- |\lambda|^{1/2} \cdot \tilde{\mathscr{C}} \cdot r/3\right)$$
$$\times \exp\left(- |\lambda|^{1/2} \cdot \tilde{\mathscr{C}} \cdot |s|/3\right) \cdot \exp\left(- |\lambda|^{1/2} \cdot \tilde{\mathscr{C}} \cdot |\varrho - \eta|/3\right)$$
$$\text{for} \quad r \in (0, \alpha(\Omega)), \quad s \in (-\alpha(\Omega), 0), \quad \varrho, \eta \in \Delta^1;$$

$$(5.11) \qquad (|\sigma - \varrho| + r)^{-2} \ge |\sigma - \varrho|^{-2 + 2 \cdot \varepsilon} \cdot r^{-2 \cdot \varepsilon} \qquad \text{for} \quad \sigma, \varrho \in \Delta^1, \quad r \in (0, \alpha(\Omega));$$

$$(5.12) \qquad \int_{-\alpha(\Omega)}^{0} \exp\left(- |\lambda|^{1/2} \cdot \tilde{\mathscr{C}} \cdot |s|/3\right) \cdot |\tilde{f}(\eta, s)| \, ds \le \mathscr{C} \cdot |\lambda|^{-1/4} \cdot \left(\int_{-\alpha(\Omega)}^{0} |\tilde{f}(\eta, s)|^2 ds \right)^{1/2}$$
$$\text{for} \quad \eta \in \Delta^1.$$

After inserting (5.10)–(5.12) into (5.9) and integrating in r and s, we obtain

$$|\alpha(i)| \le C \cdot |\lambda|^{-3/4 + \varepsilon} \cdot \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \chi_{(0, 2 \cdot \sqrt{2} \cdot \alpha(\Omega))} (|\sigma - \varrho|) \cdot |\sigma - \varrho|^{-2 + 2 \cdot \varepsilon} \right.\right.$$
$$\left.\left. \times \int_{\Delta^1} |\varrho - \eta|^{-1} \cdot \exp\left(-|\lambda|^{1/2} \cdot \tilde{\mathscr{C}} \cdot |\varrho - \eta|/3\right) \cdot \left(\int_{-\alpha(\Omega)}^{0} |\tilde{f}(\eta, s)|^2 ds \right)^{1/2} d\eta \, d\varrho \right)^2 d\sigma \right)^{1/2} .$$

By twice applying Young's inequality ([1, p. 90, 4.30]), we conclude that

$$|\alpha(i)| \le \mathscr{C} \cdot |\lambda|^{-3/4 + \varepsilon} \cdot \left(\int_{\mathbb{R}^2} \left(\int_{\Delta^1} |\varrho - \eta|^{-1} \cdot \exp\left(- |\lambda|^{1/2} \cdot \tilde{\mathscr{C}} \cdot |\varrho - \eta|/3\right) \right.\right.$$
$$\left.\left. \times \left(\int_{-\alpha(\Omega)}^{0} |\tilde{f}(\eta, s)|^2 \, ds \right)^{1/2} d\eta \right)^2 d\varrho \right)^{1/2}$$
$$\le \mathscr{C} \cdot |\lambda|^{-3/4 + \varepsilon} \cdot \int_{\mathbb{R}^2} |\gamma|^{-1} \cdot \exp\left(- |\lambda|^{1/2} \cdot \tilde{\mathscr{C}} \cdot |\gamma|/3\right) d\gamma \cdot \|\tilde{f}\|_2$$
$$\le \mathscr{C} \cdot |\lambda|^{-5/4 + \varepsilon} \cdot \|f\|_2 .$$

Note that according to (3.6) and Theorem 4.1, we have $\|\tilde{f}\|_2 \le \|f\|_2$.

The other summands appearing in the sum on the right-hand side of (5.8) may be dealt with in a similar way. In fact, they are somewhat less difficult to handle since they may be estimated by using (3.2) or (3.3). $\square$

Lemma 5.6. *For $f \in L^2(\Omega)^3$, $\lambda \in \mathbb{C} \setminus (-\infty, 0)$, the function $R^\lambda(f) \mid \partial\Omega$ belongs to $L^2_n(\partial\Omega) \cap W^{1,2}(\partial\Omega)^3$. Let $\varepsilon \in (0, 1/2)$. Then there is a constant $C_{16}(\Omega, \varepsilon) > 0$ such that*

(5.13) $$\|(\mathfrak{S}^{i\cdot\tau})^{-1}(-R^{i\cdot\tau}(f) \mid \partial\Omega)\|_2 \le C_{16}(\Omega, \varepsilon) \cdot |\tau|^{-1/4+\varepsilon} \cdot \|f\|_2$$

for $f \in L^2(\Omega)^3$, $\tau \in \mathbb{R}$ with $|\tau| \ge 1$. (The operator $\mathfrak{S}^{i\cdot\tau}$ was introduced in Lemma 5.5.)

Proof. The relation $R^\lambda(f) \mid \partial\Omega \in W^{1,2}(\partial\Omega)^3$ can be derived by a calculation based on (4.6). We recall that for the case $|\lambda| \ge 1$, a sharper result was stated in (4.8). Since $\operatorname{div} R^\lambda(f) = 0$, the function $R^\lambda(f) \mid \partial\Omega$ belongs to $L^2_n(\partial\Omega)$.

Turning to the proff of (5.13), we assume that $\tau \in \mathbb{R}$ with $|\tau| \ge 1$, and $f \in L^2(\Omega)^3$. For brevity, we set

$$\phi^{i\cdot\tau} := (\mathfrak{S}^{i\cdot\tau})^{-1}(-R^{i\cdot\tau}(f) \mid \partial\Omega)\,.$$

Then we may deduce from the jump relation (4.9)

$$\|\phi^{i\cdot\tau}\|^2 \le \left\| \left(\sum_{j=1}^{3} B^+_{jk}(i\cdot\tau, \phi^{i\cdot\tau}) \cdot n_j \right)_{1\le k\le 3} - n \cdot A^+(\phi^{i\cdot\tau}) \right\|_2 + \left\| \left(\sum_{j=1}^{3} B^-_{jk}(i\cdot\tau, \phi^{i\cdot\tau}) \cdot n_j \right)_{1\le k\le 3} - n \cdot A^-(\phi^{i\cdot\tau}) \right\|_2 .$$

Now we obtain from Lemma 4.1

$$\|\phi^{i\cdot\tau}\|_2 \le C_8(\Omega) \cdot (\|R^\lambda(f) \mid \partial\Omega\|_{1,2} + |\tau|^{1/2} \cdot \|R^\lambda(f) \mid \partial\Omega\|_2 + |\tau| \cdot \|S(n \cdot R^\lambda(f) \mid \partial\Omega)\|_2)\,,$$

where we used the fact that $V^{i\cdot\tau}(\phi^{i\cdot\tau}) \mid \partial\Omega = -R^{i\cdot\tau}(f) \mid \partial\Omega$. The second part of the lemma now follows from (4.6), (4.7) and Theorem 5.1. $\square$

Lemma 5.7. *Let $\varepsilon \in (0, 1/2)$, $E \in (0, \infty)$. Denote the identity mapping of Ω by* id (Ω). *Then there exists a constant $C_{17}(\Omega, E, \varepsilon) > 0$ such that the estimate*

(5.14) $$\left\| \int_{\partial\Omega} K(\mathrm{id}\,(\Omega), y) \cdot \phi(y)\, d\Omega(y) \right\|_{3/2-\varepsilon, 2} \le C_{17}(\Omega, E, \varepsilon) \cdot \|\phi\|_2$$

holds true for any $\phi \in L^2(\partial\Omega)^3$ and for any measurable function $K : \mathbb{R}^3 \times \partial\Omega \mapsto \mathbb{C}^{3\times 3}$ satisfying the relations $K(\cdot, y) \in C^2(\mathbb{R}^3)^3$ for $y \in \partial\Omega$,

$$|K(x, y)| \cdot |x-y| + \sum_{l=1}^{3} |D_l K(x, y)| \cdot |x-y|^2 + \sum_{k,l=1}^{3} |D_k D_l K(x, y)| \cdot |x-y|^3 \le E$$

$$\textit{for} \quad x \in \mathbb{R}^3, \qquad y \in \partial\Omega\,, \qquad x \neq y\,.$$

Proof. Let ϕ and K be given as in the lemma. We shall introduce certain integrals, the sum of which will yield an upper bound for the left-hand side of (5.14). First we set

(5.15) $$A := \int_\Omega \left| \int_{\partial\Omega} K(x, z) \cdot \phi(z)\, d\Omega(z) \right|^2 dx + \sum_{k=1}^{3} \int_\Omega \left| \int_{\partial\Omega} D_k K(x, y) \cdot \phi(z)\, d\Omega(z) \right|^2 dx\,,$$

$$B_i := \int_{-\alpha(\Omega)/4}^{0} \int_{\Delta^{1/4}} \int_{-\alpha(\Omega)/2}^{0} \int_{\Delta^{1/2}} |H^{(i)}(\varrho, r) - H^{(i)}(\sigma, s)|^{-4+2\cdot\varepsilon} \times \sum_{k=1}^{3} \left| \int_{\{\eta\in\Delta^1 : |\varrho-\eta| \le 2\cdot|\sigma-\varrho|\}} D_k K(H^{(i)}(\sigma, s), h^{(i)}(\eta)) \cdot (\phi \circ h^{(i)})(\eta) \cdot J^{(i)}(\eta)\, d\eta \right|^2 d\sigma\, ds\, d\varrho\, dr\,,$$

for $i \in \{1, \ldots, k(\Omega)\}$. Furthermore, let C_i be defined in an analogous way as B_i, but with the factor $D_k K(H^{(i)}(\sigma, s), h^{(i)}(\eta))$ in the innermost integral replaced by $D_k K(H^{(i)}(\varrho, r), h^{(i)}(\eta))$. Let N_i also denote an integral as in (5.15), but now the innermost integral is to be replaced by the integral

$$\int_{\{\eta \in \Delta^1 : |\varrho - \eta| \geq 2 \cdot |\sigma - \varrho|\}} (D_k K(H^{(i)}(\sigma, s), h^{(i)}(\eta)) - D_k K(H^{(i)}(\varrho, r), h^{(i)}(\eta))) \cdot (\phi \circ h^{(i)})(\eta) \cdot J^{(i)}(\eta)\, \mathrm{d}\eta .$$

We further define for $i \in \{1, \ldots, k(\Omega)\}$

$$F_i := \int_{U_i^{1/4} \cap \Omega} \int_{U_i^{1/2} \cap \Omega} |x - y|^{-4+2\cdot\varepsilon} \sum_{k=1}^{3} \left| \int_{\partial\Omega \setminus \Delta_i^1} (D_k K(x, z) - D_k K(y, z)) \cdot \phi(z)\, \mathrm{d}\Omega(z) \right|^2 \mathrm{d}x\, \mathrm{d}y ,$$

$$G_i := \int_{U_i^{1/4} \cap \Omega} \int_{\Omega \setminus U_i^{1/2}} |x - y|^{-4+2\cdot\varepsilon}$$

$$\times \sum_{k=1}^{3} \left| \int_{\partial\Omega} (D_k K(x, z) - D_k(K(y, z)) \cdot \phi(z)\, \mathrm{d}\Omega(z) \right|^2 \mathrm{d}x\, \mathrm{d}y .$$

In addition, we set

$$M_1 := \bigcup_{i=1}^{k(\Omega)} U_i^{1/4}, \qquad \delta := \operatorname{dist}(\partial\Omega, \mathbb{R}^3 \setminus M_1),$$

$$M_2 := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta/2\},$$

$$H_1 := \int_{\Omega \setminus M_1} \int_{M_2} |x - y|^{-4+2\cdot\varepsilon}$$

$$\times \sum_{k=1}^{3} \left| \int_{\partial\Omega} (D_k K(x, z) - D_k K(y, z)) \cdot \phi(z)\, \mathrm{d}\Omega(z) \right|^2 \mathrm{d}x\, \mathrm{d}y .$$

As mentioned in (3.4), the constant δ is positive. Let H_2 be defined in the same manner as H_1, with the only difference that the domain of integration of the second integral is replaced by $\Omega \setminus M_2$. Then, according to (3.6), (3.7), the left-hand side of (5.14) is bounded by

$$\left(A + \sum_{i=1}^{k(\Omega)} (B_i + C_i + N_i + F_i + G_i) + H_1 + H_2 \right)^{1/2} . \tag{5.16}$$

We claim that all the summands appearing in (5.16) may be estimated against $\mathscr{C} \cdot \|\phi\|_2$. Here and in the following, the letter $\mathscr{C}$ denotes constants which only depend on Ω, E, or ε. To prove our claim, let us consider the integral N_i ($i \in \{1, \ldots, k(\Omega)\}$), which is perhaps the most difficult to estimate among the summands listed in (5.16).

We first note the following inequality, which holds for $k \in \{1, 2, 3\}$, $\varrho, \sigma, \eta \in \Delta^1$ with $|\varrho - \eta| \geq 2 \cdot |\varrho - \sigma|$:

$$|D_k K(H^{(i)}(\varrho, r), h^{(i)}(\eta)) - D_k K(H^{(i)}(\sigma, s), h^{(i)}(\eta))|$$

$$= \left| \int_0^1 \sum_{j=1}^{3} (D_j D_k K)(H^{(i)}(S(\vartheta)), h^{(i)}(\eta)) \cdot ((\varrho, r) - (\sigma, s))_j\, \mathrm{d}\vartheta \right|$$

$$\leq \mathscr{C} \cdot \int_0^1 |H^{(i)}(S(\vartheta)) - h^{(i)}(\eta)|^{-3}\, \mathrm{d}\vartheta \cdot (|\varrho - \sigma| + |r - s|)$$

$$\leq \mathscr{C} \cdot \int_0^1 (|\varrho - \eta| + |s + \vartheta \cdot (r - s)|)^{-3}\, \mathrm{d}\vartheta \cdot (|\varrho - \sigma| + |r - s|),$$

where we used the abbreviation $S(\vartheta) := (\sigma, s) + \vartheta \cdot ((\varrho, r) - (\sigma, s))$, for $\vartheta \in [0, 1]$. Note that the last inequality is implied by (3.5).

Set $\varkappa := 1/2 - \varepsilon/2$. Then it follows from Hölder's inequality, for $\varrho, \sigma \in \Delta^1$, $r, s \in (0, \alpha(\Omega))$, $k \in \{1, 2, 3\}$, that

$$(5.17)\qquad \left(\int_{\{\eta \in \Delta^1 : |\varrho - \eta| \geq 2 \cdot |\varrho - \sigma|\}} |D_k K(H^{(i)}(\varrho, r), h^{(i)}(\eta)) - D_k K(H^{(i)}(\sigma, s), h^{(i)}(\eta))| \cdot |(\phi \circ h^{(i)})(\eta)| \, d\eta\right)^2$$

$$\leq (|\varrho - \sigma| + |r - s|)^2 \cdot \int_{\{\eta \in \Delta^1 : |\varrho - \eta| \geq 2 \cdot |\varrho - \sigma|\}} |\varrho - \eta|^{-4 + 2 \cdot \varkappa} \, d\eta$$

$$\times \int_{\Delta^1} \int_0^1 (|\varrho - \eta| + |s + \vartheta \cdot (r - s)|)^{-2 - 2 \cdot \varkappa} \cdot |(\phi \circ h^{(i)})(\eta)|^2 \, d\eta \, .$$

After computing the first integral on the right-hand side of (5.17), we obtain by (3.5)

$$N_i \leq \mathscr{C} \cdot \int_0^{\alpha(\Omega)} \int_{\Delta^1} \int_0^{\alpha(\Omega)} \int_{\Delta^1} \int_{\Delta^1} \int_0^1 |\varrho - \sigma|^{-5/2 + \varepsilon + \varkappa}$$

$$\times |r - s|^{-3/2 + \varepsilon + \varkappa} \cdot |\varrho - \eta|^{-3/2 - \varkappa}$$

$$\times |s + \vartheta(r - s)|^{-1/2 - \varkappa} \cdot |(\phi \circ h^{(i)})(\eta)|^2 \, d\vartheta \, d\eta \, d\varrho \, dr \, d\sigma \, ds \, .$$

Next we split the integration in ϑ into an integral over $[0, 1/2]$ and another one over $[1/2, 1]$. Then we integrate in ϱ, afterwards in s and r, and finally in σ, to obtain

$$N_i \leq \mathscr{C} \cdot \|\phi\|_2^2 \, .$$

The terms B_i and C_i are treated in a similar way ($i \in \{1, \ldots, k(\Omega)\}$). For an estimate of A, we apply Hölder's inequality in the same way as in (5.17). In order to deal with the terms F_i ($i \in \{1, \ldots, k(\Omega)\}$) and H_2, we intend to use the mean value theorem. To this end, we first note that there is a constant $\tilde{\mathscr{C}} > 0$ with

$$|x - z|, |y - z| \geq \tilde{\mathscr{C}}$$

for $(x, y, z) \in (U_i^{1/4} \cap \Omega) \times (U_i^{1/2} \cap \Omega) \times (\partial\Omega \setminus \Lambda_i^1)$ and for $(x, y, z) \in (\Omega \setminus M_1) \times (\Omega \setminus M_2) \times \partial\Omega$, with $i \in \{1, \ldots, k(\Omega)\}$; see (3.2), (3.4). However, for x, y, z as before, there may be some $\vartheta \in (0, 1)$ such that $x + \vartheta(y - x) - z = 0$. Therefore the mean value theorem cannot be applied directly. Instead, for $x, y, z \in \mathbb{R}^3$ with $x \neq y$, we set

$$s_0 := s_0(x, y, z) := |x - y|^{-2} \cdot ((z - x) \cdot (y - x))^{-2} ,$$

$$a := a(x, y, z) := x + s_0 \cdot (y - x) - z \, ,$$

$$b := b(x, y, z) := x + (1/2) \cdot (y - x) + |y - x| \cdot |a|^{-1} \cdot a \, .$$

Then, observing that $a \cdot (x - y) = 0$, we obtain after some computations

$$|v + \vartheta \cdot (b - v) - z| \geq (1/2) \cdot |v - z| \quad \text{for} \quad v \in \{x, y\} \, , \qquad \vartheta \in [0, 1] \, .$$

Thus, for $x, y, z \in \mathbb{R}^3$ with $x \neq y$, $k \in \{1, 2, 3\}$, we have

$$
\begin{aligned}
&|D_k K(x, z) - D_k K(y, z)| \\
&\leq \sum_{v \in \{x, y\}} |D_k K(v, z) - D_k K(b, z)| \\
&= \sum_{v \in \{x, y\}} \left| \int_0^1 \sum_{j=1}^{3} (D_j D_k K)(v + \vartheta \cdot (b - v), z) \cdot (b - v)_j \, d\vartheta \right| \\
&\leq \mathscr{C} \cdot |x - y| \cdot \sum_{v \in \{x, y\}} |v - z|^{-3} .
\end{aligned}
$$

Using (3.2), (3.4), it is now easy to show that $F_i, H_2 \leq \mathscr{C} \cdot \|\phi\|_2$ for $i \in \{1, \ldots, k(\Omega)\}$. The term G_i, may be estimated by a direct application of (3.2). Finally, it is obvious how to handle H_1. $\square$

Let us now collect our results and draw some conclusions.

Theorem 5.2. *For $\lambda \in \mathbb{C} \setminus (-\infty, 0)$, $f \in L^2(\Omega)^3$, the set $SOL(\Omega, \lambda, f)$, defined in (5.4), contains one and only one element (u, π). For any $\varepsilon \in (0, 1/2)$, there exists a constant $C_{18}(\Omega, \varepsilon) > 0$ such that*

$$
\|u\|_{3/2-\varepsilon,2} + \|\pi\|_{1/2-\varepsilon,2} \leq C_{18}(\Omega, \varepsilon) \cdot \|f\|_2 \qquad \text{for} \quad f \in L^2(\Omega)^3 , \quad (u, \pi) \in SOL(\Omega, 0, f) . \tag{5.18}
$$

Furthermore, for $\vartheta \in [0, \pi)$, inequality (5.5) holds for $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\arg \lambda| \leq \vartheta$, $f \in L^2(\Omega)^3$, $(u, \pi) \in SOL(\Omega, \lambda, f)$.

In addition, for $\vartheta \in [0, \pi)$, $\varepsilon \in (0, 1/2)$, there is a constant $C_{19}(\Omega, \vartheta, \varepsilon) > 0$ such that

$$
\|u\|_{3/2-\varepsilon,2} + \|\pi\|_{1/2-\varepsilon,2} \leq C_{19}(\Omega, \vartheta, \varepsilon) \cdot \|f\|_2 \qquad \text{for} \quad \lambda \in \mathbb{C} \setminus \{0\} \quad \text{with} \quad |\arg \lambda| \leq \vartheta , \quad f \in L^2(\Omega)^3 , \quad (u, \pi) \in SOL(\Omega, \lambda, f) . \tag{5.19}
$$

Finally, for $\delta_1, \delta_2 \in (0, 1/2)$, there is a constant $C_{20}(\Omega, \delta_1, \delta_2) > 0$ such that

$$
|\tau|^{1/4-\delta_1} \cdot \|u\|_{3/2-\delta_2,2} \leq C_{20}(\Omega, \delta_1, \delta_2) \cdot \|f\|_2 \qquad \text{for} \quad \tau \in \mathbb{R} , \quad f \in L^2(\Omega)^3 , \quad \text{and for} \quad (u, \pi) \in SOL(\Omega, i \cdot \tau, f) . \tag{5.20}
$$

Proof. Let $f \in L^2(\Omega)^3$, $\lambda \in \mathbb{C} \setminus (-\infty, 0)$. It is clear from (5.5), (5.6) that $SOL(\Omega, \lambda, f)$ contains at most one element.

Let us show that $SOL(\Omega, \lambda, f) \neq \phi$. To this end, we note that the function $R^\lambda(f) \,|\, \partial\Omega$ belongs to $L_n^2(\partial\Omega) \cap W^{1,2}(\partial\Omega)^3$ (Lemma 5.6). Thus, due to Lemma 5.5, the mapping $(\mathfrak{S}^\lambda)^{-1}$ may be applied to $R^\lambda(f) \,|\, \partial\Omega$. For shortness we set

$$
\phi^\lambda := (\mathfrak{S}^\lambda)^{-1} (-R^\lambda(f) \,|\, \partial\Omega) \quad \text{for} \quad \lambda \in \mathbb{C} \setminus (-\infty, 0) . \tag{5.21}
$$

Then we obtain by Corollary 5.2

$$
(u(\lambda, f, \phi^\lambda), \pi(\lambda, f, \phi^\lambda)) \in SOL(\Omega, \lambda, f) .
$$

Next we want to prove (5.18). By referring to (4.5) and a trace theorem, or by a direct calculation, we find

$$
\|R^0(f) \,|\, \partial\Omega\|_{1,2} \leq \mathscr{C} \cdot \|f\|_2 ,
$$

where the constant $\mathscr{C}$ only depends on Ω. On the other hand, Theorem 4.3 yields

$$\|\phi^0\|_2 \leq C_9(\Omega) \cdot \|R^0(f) \mid \partial\Omega\|_{1,2},$$

with ϕ^0 defined in (5.21). Finally, we deduce from Lemma 5.7 that

$$\|V^0(\phi^0) \mid \Omega\|_{3/2-\varepsilon,2} + \|Q(\phi^0) \mid \Omega\|_{1/2-\varepsilon,2} \leq \mathscr{C} \cdot \|\phi^0\|_2$$

for $\varepsilon \in (0, 1/2)$, with $\mathscr{C}$ only depending on ε and Ω. After combining the previous estimates with (4.5), we obtain (5.18).

Of course, any element of $SOL(\Omega, \lambda, f)$ is contained in $SOL(\Omega, 0, f - \lambda \cdot u)$. Thus inequality (5.19) follows from (5.5) and (5.18).

Finally we remark that (5.20) is a consequence of (4.4), (4.24), (4.3), Lemma 5.7, and (5.13). □

6. The Stokes operator

For the rest of this paper, let $\nu \in (0, \infty)$ be fixed.

Lemma 6.1. *Let $f \in L^2_{\text{loc}}(\Omega)^3$. Assume that there are functions $p, \tilde{p} \in W^{1,2}_{\text{loc}}(\Omega) \cap L^2(\Omega)$ with $f - \nabla p, f - \nabla\tilde{p} \in H_2(\Omega)$. Then it follows $\nabla p = \nabla\tilde{p}$, and hence $f - \nabla p = f - \nabla\tilde{p}$.*

Proof. Our assumptions yield

$$\nabla(p - \tilde{p}) = (f - \nabla p) - (f - \nabla\tilde{p}) \in H_2(\Omega).$$

The lemma now follows from Theorem 4.1. □

Due to Lemma 6.1, we may introduce the following operators.

Definition 6.1. Set

$$R(\Omega) := \{u \in L^2_{\text{loc}}(\Omega)^3 : \text{There is some} \quad p \in W^{1,2}_{\text{loc}}(\Omega) \cap L^2(\Omega) \\ \text{with } u - \nabla p \in H_2(\Omega)\}.$$

For $u \in R(\Omega)$, we set

$$\tilde{P}_2 := u - \nabla p, \qquad \tilde{G}u := p,$$

where $p \in W^{1,2}_{\text{loc}}(\Omega) \cap L^2(\Omega)$ with

$$u - \nabla p \in H_2(\Omega), \qquad \int_\Omega p \, dx = 0.$$

We further define the set $\mathfrak{D}(A)$ by

$$\mathfrak{D}(A) := \{u \in W^{2,2}_{\text{loc}}(\Omega)^3 \cap W^{1,2}_0(\Omega)^3 \cap H_2(\Omega) : \Delta u \in R(\Omega)\},$$

and the operator

$$A : \mathfrak{D}(A) \mapsto H_2(\Omega) \quad \text{by} \quad Au := -\nu \cdot \tilde{P}_2(\Delta u) \quad \text{for} \quad u \in \mathfrak{D}(A).$$

Lemma 6.2. Let $u \in \mathfrak{D}(A)$. *Then we have*

$$\int_\Omega Au \cdot \bar{u}\, dx = \int_\Omega \nu \cdot \sum_{k=1}^{3} |D_k u|^2\, dx\,.$$

Proof. Observe that $u \in H_2(\Omega) \cap W_0^{1,2}(\Omega)^3$. Thus, according to Theorem 4.2, we may choose a sequence (g_n) in $C_0^\infty(\Omega)^3$ such that $\|u - \varphi_n\|_{1,2} \to 0$ $(n \to \infty)$, and $\operatorname{div} \varphi_n = 0$ for $n \in \mathbb{N}$. Take $p \in W_{\mathrm{loc}}^{1,2}(\Omega)$ with $\Delta u - \nabla p \in H_2(\Omega)$. Then

$$\begin{aligned}\int_\Omega Au \cdot \bar{u}\, dx &= \lim_{n\to\infty} \int_\Omega Au \cdot \bar{\phi}_n\, dx \\ &= \nu \cdot \lim_{n\to\infty} \left(-\int_\Omega \Delta u \cdot \bar{\phi}_n\, dx + \int_\Omega \nabla p \cdot \bar{\phi}_n\, dx \right) \\ &= \nu \cdot \lim_{n\to\infty} \int_\Omega \sum_{k=1}^{3} D_k u \cdot D_k \bar{\phi}_n\, dx = \nu \cdot \sum_{k=1}^{3} \|D_k u\|_2^2\,. \quad \square\end{aligned}$$

Theorem 6.1. *The set $\mathfrak{D}(A)$ is dense in $H_2(\Omega)$. The operator A is positive, hermitian and onto. In particular, it is self-adjoint in $H_2(\Omega)$. The set $\mathbb{C} \setminus (0, \infty)$ belongs to the resolvent set $\varrho(A)$ of A.*

Let id_2 denote the identity mapping of $H_2(\Omega)$ onto itself. Then, for any $\lambda \in \mathbb{C} \setminus (0, \infty)$, the mapping $\lambda \cdot \mathrm{id}_2 - A$ is onto.

For $\vartheta \in [0, \pi)$, $\varepsilon, \delta_1, \delta_2 \in (0, 1/2)$, there are constants $C_{21}(\Omega, \vartheta, \nu)$, $C_{22}(\Omega, \vartheta, \varepsilon, \nu)$, $C_{23}(\Omega, \delta_1, \delta_2, \nu) > 0$ such that

$$\|(\lambda \cdot \mathrm{id}_2 + A)^{-1}(f)\|_2 \le C_{21}(\Omega, \vartheta, \nu) \cdot (|\lambda| + 1)^{-1} \|f\|_2 \quad \text{for } \lambda \in \mathbb{C} \setminus \{0\} \text{ with } |\arg \lambda| \le \vartheta\,, \quad f \in H_2(\Omega)\,; \tag{6.1}$$

$$\|(\lambda \cdot \mathrm{id}_2 + A)^{-1}(f)\|_{3/2-\varepsilon,2} \le C_{22}(\Omega, \vartheta, \varepsilon, \nu) \cdot \|f\|_2 \quad \text{for } \lambda \in \mathbb{C} \setminus \{0\} \text{ with } |\arg \lambda| \le \vartheta\,, \quad f \in H_2(\Omega)\,; \tag{6.2}$$

$$\|(i \cdot \tau \cdot \mathrm{id}_2 + A)^{-1}(f)\|_{3/2-\delta_2,2} \le C_{23}(\Omega, \delta_1, \delta_2, \nu) \cdot |\tau|^{-1/4+\delta_1} \cdot \|f\|_2 \quad \text{for } \tau \in \mathbb{R} \text{ with } |\tau| \ge 1\,, \quad f \in H_2(\Omega)\,. \tag{6.3}$$

Proof. The set $\mathfrak{D}(A)$ contains any function $\varphi \in C_0^\infty(\Omega)^3$ with $\operatorname{div} \varphi = 0$. Thus $\mathfrak{D}(A)$ is dense in $H_2(\Omega)$. It readily follows from Lemma 6.2 that A is hermitian and positive. All the other claims stated in Theorem 6.1 hold true according to Theorem 5.2. $\square$

Lemma 6.3. *The sets $\mathfrak{D}(A^{1/2})$ and $W_0^{1,2}(\Omega)^3 \cap H_2(\Omega)$ coincide, and it holds*

$$\nu \cdot \sum_{k=1}^{3} \|D_k u\|_2^2 = \|A^{1/2} u\|_2^2 \quad \text{for} \quad u \in \mathfrak{D}(A^{1/2})\,. \tag{6.4}$$

Proof. We may proceed as in the case of a smoothly bounded domain. Since $A^{1/2}$ is self-adjoint, we have

$$\int_\Omega Au \cdot \bar{v}\, dx = \int_\Omega (A^{1/2} u) \cdot \overline{(A^{1/2} v)}\, dx \quad \text{for} \quad u, v \in \mathfrak{D}(A)\,. \tag{6.5}$$

The inclusion $\mathfrak{D}(A^{1/2}) \subset W_0^{1,2}(\Omega)^3 \cap H_2(\Omega)$ and equation (6.4) follow from (6.5), (4.17), and Lemma 6.2. The inverse inclusion is a consequence of Theorem 4.2, equation (6.5) and Lemma 6.2. $\square$

Lemma 6.4. *Let Γ be the curve in $\mathbb{R}^2$ which is defined by the paths*

$$\gamma_1 := [0, 1] \mapsto \mathbb{R}^2, \qquad \gamma_2 := [0, \pi/2] \mapsto \mathbb{R}^2, \qquad \gamma_3 : [1, \infty] \mapsto \mathbb{R}^2,$$

with

$$\gamma_1(t) := (t, 0) \quad \textit{for} \quad t \in [0, 1],$$

$$\gamma_2 := (\cos\theta, \sin\theta) \quad \textit{for} \quad \theta \in [0, \pi/2],$$

$$\gamma_3(t) := (0, t) \quad \textit{for} \quad t \in [1, \infty).$$

Then it holds for $\varepsilon \in (0, 1/4)$, $f \in H_2(\Omega)$,

$$A^{-3/4-\varepsilon}(f) = \pi^{-1} \cdot \sin(\pi \cdot (3/4 + \varepsilon))$$

$$\cdot H_2(\Omega) - \int_\Gamma \lambda^{-3/4-\varepsilon} \cdot (\lambda \cdot \mathrm{id}_2 + A)^{-1}(f)\, d\lambda.$$

Proof. According to Theorem 6.1, the set $\mathbb{C} \setminus (0, \infty)$ belongs to the resolvent set of A. Thus the mapping

$$\phi := \{\lambda \in \mathbb{C} \setminus (-\infty, 0) : |\lambda| > 1/2\} \mapsto H_2(\Omega),$$

$$\phi(z) := z^{-3/4-\varepsilon} \cdot (z \cdot \mathrm{id}_2 + A)^{-1}(f)$$

is holomorphic. Defining the curve Γ_R by the path

$$\gamma_R : [0, \pi/2] \mapsto \mathbb{C}, \qquad \gamma_R(\theta) := R \cdot (\cos\theta, \sin\theta) \quad (\theta \in [0, \pi/2]),$$

for $R \in (0, \infty)$, we obtain from (6.1)

$$H_2(\Omega) - \int_{\Gamma_R} \lambda^{-3/4-\varepsilon} \cdot (\lambda \cdot \mathrm{id}_2 + A)^{-1}(f)\, d\lambda \to 0 \quad (R \uparrow \infty).$$

Thus the lemma follows from (4.15) and Cauchy's theorem. $\square$

Corollary 6.1. *Let $\varepsilon_1 \in (0, 1/4)$, $\varepsilon_2 \in (0, 1/2)$. Then $\mathfrak{D}(A^{3/4+\varepsilon_1}) \subset W^{3/2-\varepsilon_2, 2}(\Omega)^3$, and there is a constant $C_{24}(\Omega, \varepsilon_1, \varepsilon_2, \nu) > 0$ such that*

$$\|u\|_{3/2-\varepsilon_2, 2} \le C_{24}(\Omega, \varepsilon_1, \varepsilon_2, \nu) \cdot \|A^{3/4+\varepsilon_1} u\|_2 \quad \textit{for} \quad u \in D(A^{3/4+\varepsilon_1}).$$

Proof. We shall use the notations from Lemma 6.4. Furthermore, let $\mathscr{C}$ denote constants which only depend on Ω, ε_1, ε_2, or v.

From (6.2) we conclude for $j \in \{1, 2\}$, $f \in H_2(\Omega)$, that

$$\int_{J_j} \|\gamma_j'(t) \cdot (\gamma_j(t))^{-3/4-\varepsilon_1} \cdot (\gamma_j(t) \cdot \mathrm{id}_2 + A)^{-1}(f)\|_{3/2-\varepsilon_2, 2}\, dt \le \mathscr{C} \cdot \|f\|_2,$$

with $J_j := [0, 1]$, if $j = 1$, and $J_j := [0, \pi/2]$, if $j = 2$. Moreover, using (6.3), we find

$$\int_1^\infty \|\gamma_3'(t) \cdot (\gamma_3(t))^{-3/4-\varepsilon_1} \cdot (\gamma_3(t) \cdot \mathrm{id}_2 + A)^{-1}(f)\|_{3/2-\varepsilon_2,2}\,dt$$

$$\leq C_{23}(\Omega, \varepsilon_1/2, \varepsilon_2, \nu) \cdot \|f\|_2 \cdot \int_1^\infty t^{-3/4-\varepsilon_1} \cdot t^{-1/4+\varepsilon_1/2}\,dt \leq \mathscr{C} \cdot \|f\|_2 .$$

Now the corollary follows from Lemma 6.4. □

Lemma 6.5. *Let* $\alpha \in (0, 1/2)$. *Then* $\mathfrak{D}(A^\alpha) \subset W^{2\cdot\alpha,2}(\Omega)$, *and there is a constant* $C_{25}(\Omega, \alpha, \nu) > 0$ *such that*

$$\|u\|_{2\cdot\alpha,2} \leq C_{25}(\Omega, \alpha, \nu) \cdot \|A^\alpha u\|_2 \quad \textit{for} \quad u \in D(A^\alpha).$$

Proof. Let $J: H_2(\Omega) \mapsto L^2(\Omega)^3$ denote the canonical imbedding of $H_2(\Omega)$ in $L^2(\Omega)^3$. Clearly J is continuous. Due to Lemma 6.3 and (4.20), the restriction

$$J \mid \mathfrak{D}(A^{1/2}): \mathfrak{D}(A^{1/2}) \mapsto W^{1,2}(\Omega)^3$$

is well defined and continuous too. Recalling (4.21) and (4.23), we may now derive the lemma by an interpolation argument. □

7. Strong solutions of the Navier-Stokes equations

In this section we shall adapt the approach of Fujita and Kato — in the version of [52, p. 117–120] and [54, p. 331–336] — to the case of a Lipschitz bounded domain. Our main tool will be the following lemma which gives an estimate of the nonlinearity appearing in the Navier-Stokes system (1.1).

Lemma 7.1. *Let* $\varepsilon, \delta \in (0, 1/4)$. *Then there is a constant* $C_{26}(\Omega, \varepsilon, \delta, \nu) > 0$ *such that it holds for* $u, v \in \mathfrak{D}(A^{3/4+\delta})$

(7.1)
$$\begin{aligned} &\|u \cdot \nabla u - v \cdot \nabla v\|_2 \\ &\leq C_{26}(\Omega, \varepsilon, \delta, \nu) \cdot (\|A^{1/4+\varepsilon}(u - v)\|_2^{1/2} \cdot \|A^{3/4+\delta}(u - v)\|_2^{1/2} \cdot \|A^{3/4+\delta}u\|_2 \\ &\qquad + \|A^{1/4+\varepsilon}v\|_2^{1/2} \cdot \|A^{3/4+\delta}v\|_2^{1/2} \cdot \|A^{3/4+\delta}(u - v)\|_2), \end{aligned}$$

(7.2)
$$\|u \cdot \nabla u\|_2 \leq C_{26}(\Omega, \varepsilon, \delta, \nu) \cdot \|A^{1/4+\varepsilon}u\|_2^{1/2} \cdot \|A^{3/4+\delta}v\|_2^{3/2} .$$

Proof. We shall show the first equation. The second one follows with a similar reasoning. We begin by defining

$$t := 3/2 - 2 \cdot \varepsilon/3, \qquad \tilde{t} := 1/2 - 2 \cdot \varepsilon/3, \qquad \gamma := (1/3 + 2 \cdot \varepsilon/9)^{-1},$$

$$\tilde{\gamma} := (1/6 - 2 \cdot \varepsilon/9)^{-1}, \qquad s := 1 + 2 \cdot \varepsilon/3,$$

This implies

$$1/\gamma + 1/\tilde{\gamma} = 1/2, \qquad (1/2 - \tilde{t}/3)^{-1} = \gamma, \qquad (1/2 - s/3)^{-1} = \tilde{\gamma}.$$

It follows from Hölder's inequality, for $u, v \in \mathfrak{D}(A^{3/4+\delta})$,

$$\|u \cdot \nabla u - v \cdot \nabla v\|_2 \leq \|(u - v) \cdot \nabla u\|_2 + \|v \cdot (\nabla u - \nabla v)\|_2$$
$$\leq \|u - v\|_{\bar{\gamma}} \cdot \|\nabla u\|_\gamma + \|v\|_{\bar{\gamma}} \cdot \|\nabla(u - v)\|_\gamma .$$

Now we apply Sobolev's inequality, in the version of Theorem 4.4, as well as inequality (4.24). It follows for $u, v \in \mathfrak{D}(A^{3/4+\varepsilon})$ that

$$\|u \cdot \nabla u - v \cdot \nabla v\|_2 \leq \mathscr{C} \cdot (\|(u - v)\|_{s,2} \cdot \|u\|_{t,2} + \|v\|_{s,2} \cdot \|u - v\|_{t,2})$$
$$\leq \mathscr{C} \cdot (\|u - v\|_{1/2+2\cdot\varepsilon,2}^{1/2} \cdot \|u - v\|_{t,2}^{1/2} \cdot \|u\|_{t,2}$$
$$+ \|v\|_{1/2+2\cdot\varepsilon,2}^{1/2} \cdot \|v\|_{t,2}^{1/2} \cdot \|u - v\|_{t,2}),$$

where we used the letter $\mathscr{C}$ to denote constants which only depend on Ω, ε, or ν. Now estimate (7.1) may be deduced from Corollary 6.1 and Lemma 6.5. □

From now on, let $\varepsilon \in (0, 1/4)$ be fixed. We shall use the following notations:

Definition 7.1. For $u \in \mathfrak{D}(A^{3/4+\varepsilon/3})$, we set

$$M(u) := P_2(u \cdot \nabla u).$$

Consider $u_0 \in \mathfrak{D}(A^{1/4+\varepsilon})$, $T \in (0, \infty]$ and a measurable mapping $f: (0, T) \mapsto H_2(\Omega)$. Then we define

$$\sigma(1/4 + \varepsilon, f) := \sup \left\{ \int_0^t \|A^{1/4+\varepsilon} \mathrm{e}^{-(t-s)\cdot A} (f(s))\|_2 \, \mathrm{d}s : t \in (0, T) \right\},$$

$$\sigma(3/4 + \varepsilon/3, f) := \sup \left\{ t^{1/2-2\cdot\varepsilon/3} \cdot \int_0^t \|A^{3/4+\varepsilon/3} \mathrm{e}^{-(t-s)\cdot A} (f(s))\|_2 \, \mathrm{d}s : t \in (0, T) \right\},$$

$$a(u_0, T, f) := \sup \{ t^{1/2-2\cdot\varepsilon/3} \cdot \|A^{3/4+\varepsilon/3} \mathrm{e}^{-t\cdot A} u_0\|_2 : t \in (0, T) \} + \sigma(3/4 + \varepsilon/3, f).$$

Assuming that $\sigma(1/4 + \varepsilon, f) < \infty$ and $\sigma(3/4 + \varepsilon/3, f) < \infty$, we further define

$$\begin{aligned} \mathfrak{M}(u_0, T, f) := \Big\{ & w: (0, T) \mapsto H_2(\Omega): w \text{ measurable}, \; w(t) \in \mathfrak{D}(A^{3/4+\varepsilon/3}) \\ & \text{for } t \in (0, T); \\ & \sup \{ \|A^{1/4+\varepsilon} w(t)\|_2 : t \in (0, T) \} \\ & \leq 1 + \sup \{ \|A^{1/4+\varepsilon} \mathrm{e}^{-t\cdot A} u_0\|_2 : t \in (0, T) \} + \sigma(1/4 + \varepsilon, f); \\ & \sup \{ t^{1/2-2\cdot\varepsilon/3} \cdot \|A^{3/4+\varepsilon/3} w(t)\|_2 : t \in (0, T) \} \\ & \leq 2 \cdot a(u_0, T, f) \Big\}. \end{aligned}$$

We finally set

$$\begin{aligned} B(T) := \Big\{ & w: (0, T) \mapsto H_2(\Omega): w \text{ measurable}, \\ & w(t) \in \mathfrak{D}(A^{3/4+\varepsilon/3}) \text{ for } t \in (0, T); \\ & \sup \{ \|A^{1/4+\varepsilon}(w(t))\|_2 : t \in (0, T) \} < \infty, \\ & \sup \{ t^{1/2-2\cdot\varepsilon/3} \cdot \|A^{3/4+\varepsilon/3}(w(t))\|_2 : t \in (0, T) \} < \infty \Big\}, \end{aligned}$$

$$\begin{aligned} \mu(T)(w) := {} & \sup \{ \|A^{1/4+\varepsilon}(w(t))\|_2 : t \in (0, T) \} \\ & + \sup \{ t^{1/2-2\cdot\varepsilon/3} \cdot \|A^{3/4+\varepsilon/3}(w(t))\|_2 : t \in (0, T) \} \quad \text{for} \quad w \in B(T). \end{aligned}$$

Note that $\mu(T)$ is a norm on $B(T)$. Furthermore, $B(T)$ is a Banach space with this norm, see [23, p. 13, (7.1.1)].

Lemma 7.2. *Let* $\alpha \in [0, 1)$. *Then there is a constant* $C_{27}(\Omega, \varepsilon, \alpha, \nu) > 0$ *such that the following inequalities hold for* $u_0 \in \mathfrak{D}(A^{1/4+\varepsilon})$, $T \in (0, \infty]$, $f : (0, T) \mapsto H_2(\Omega)$ *measurable with* $\sigma(1/4 + \varepsilon, f) < \infty$, $\sigma(3/4 + \varepsilon/3, f) < \infty$, *and for* $w_1, w_2, w_3 \in \mathfrak{M}(u_0, T, f)$, $t \in (0, T)$:

$$\int_0^t \|A^\alpha e^{-(t-s)\cdot A}(M(w_1(s)) - M(w_2(s)))\|_2 \, ds \tag{7.3}$$

$$\leq C_{27}(\Omega, \varepsilon, \alpha, \nu) \cdot (1 + \|A^{1/4+\varepsilon}u_0\|_2 + \sigma(1/4 + \varepsilon, f) + \sigma(3/4 + \varepsilon/3, f))^{1/2}$$
$$\times (a(u_0, T, f))^{1/2} \cdot \mu(T)(w_1 - w_2) \cdot t^{1/4-\alpha+\varepsilon};$$

$$\int_0^t \|A^\alpha e^{-(t-s)\cdot A} M(w(s))\|_2 \, ds \tag{7.4}$$

$$\leq \mathscr{C}_{27}(\Omega, \varepsilon, \alpha, \nu) \cdot (1 + \|A^{1/4+\varepsilon}u_0\|_2 + \sigma(1/4 + \varepsilon, f))^{1/2}$$
$$\times (a(u_0, T, f))^{3/2} \cdot t^{1/4-\alpha+\varepsilon}.$$

Proof. We shall give some indications on the proof of (7.3). The estimate in (7.4) may be shown with similar arguments.

Constants which only depend on Ω, ε, α, or ν will be denoted by $\mathscr{C}$.

Let u_0, T, f, w_1, w_2, t be given as in the lemma, and let $\mathscr{A}$ denote the left-hand side in (7.3). By (4.18) and Lemma 7.1 we obtain

$$\mathscr{A} \leq \mathscr{C} \cdot \int_0^t (t-s)^{-\alpha} \cdot \big(\|A^{1/4+\varepsilon}(w_1(s) - w_2(s))\|_2^{1/2}$$
$$\times \|A^{3/4+\varepsilon/3}(w_1(s) - w_2(s))\|_2^{1/2} \cdot \|A^{3/4+\varepsilon/3}(w_1(s))\|_2$$
$$+ \|A^{1/4+\varepsilon}(w_2(s))\|_2^{1/2} \cdot \|A^{3/4+\varepsilon/3}(w_2(s))\|_2^{1/2}$$
$$\times \|A^{3/4+\varepsilon/3}(w_1(s) - w_2(s))\|_2\big) \, ds.$$

It follows by the definition of $\mu(T)$ that

$$\mathscr{A} \leq \mathscr{C} \cdot \mu(T)(w_1 - w_2) \cdot \int_0^t (t-s)^{-\alpha} \cdot s^{-3/4+\varepsilon} \big(s^{1/2-2\cdot\varepsilon/3} \cdot \|A^{3/4+\varepsilon/3}(w_1(s))\|_2$$
$$+ \|A^{1/4+\varepsilon}(w_2(s))\|_2^{1/2} \cdot (s^{1/2-2\cdot\varepsilon/3} \cdot \|A^{3/4+\varepsilon/3}(w_2(s))\|_2)^{1/2}\big) \, ds.$$

However, for $s \in (0, T)$, $\tilde{w} \in \mathfrak{M}(u_0, T, f)$, we have

$$s^{1/2-2\cdot\varepsilon/3} \cdot \|A^{3/4+\varepsilon/3}(\tilde{w}(s))\|_2 \leq 2 \cdot a(u_0, T, f)$$
$$\leq \mathscr{C} \cdot (\|A^{1/4+\varepsilon}u_0\|_2 + \sigma(3/4 + \varepsilon/3, f)),$$

where we used (4.17), (4.18) and the definition of $\mathfrak{M}(u_0, T, f)$. It may be shown in a similar way that

$$\|A^{1/4+\varepsilon}(w(s))\|_2 \leq \mathscr{C}\,(\|A^{1/4+\varepsilon}u_0\|_2 + \sigma(1/4 + \varepsilon, f)).$$

Inequality (7.3) now follows by combining the preceding estimates. $\square$

Definition 7.2. For $u_0 \in D(A^{1/4+\varepsilon})$, $T \in (0, \infty]$, $f: (0, T) \mapsto H_2(\Omega)$ measurable with $\sigma(1/4 + \varepsilon, f) < \infty$, $\sigma(3/4 + \varepsilon/3, f) < \infty$, and for $w \in \mathfrak{M}(u_0, T, f)$, we define a mapping $\mathcal{T}(u_0, T, f): (0, T) \mapsto H_2(\Omega)$ by setting for $t \in (0, T)$

$$\mathcal{T}(u_0, T, f)(w)(t) := e^{-t \cdot A} u_0 - H_2(\Omega) - \int_0^t e^{-(t-s) \cdot A} (M(w(s)) - f(s)) \, ds .$$

The next result readily follows from Lemma 7.2.

Corollary 7.1. *There is a constant $C_{28}(\Omega, \varepsilon, \nu) > 0$ such that for $u_0 \in D(A^{1/4+\varepsilon})$, $T \in (0, \infty]$, $f: (0, T) \mapsto H_2(\Omega)$ measurable with $\sigma(1/4 + \varepsilon, f) < \infty$, $\sigma(3/4 + \varepsilon/3, f) < \infty$, and for $w_1, w_2 \in \mathfrak{M}(u_0, T, f)$, the following inequalities hold true:*

$$\begin{aligned}
&\mu(T)(\mathcal{T}(u_0, T, f)(w_1) - \mathcal{T}(u_0, T, f)(w_2)) \\
&\le C_{28}(\Omega, \varepsilon, \nu) \cdot (1 + \|A^{1/4+\varepsilon} u_0\|_2 + \sigma(1/4 + \varepsilon, f) + \sigma(3/4 + \varepsilon/3, f))^{1/2} \\
&\quad \times (a(u_0, T, f))^{1/2} \cdot \mu(T)(w_1 - w_2) ;
\end{aligned}$$

$$\begin{aligned}
&\sup \{\|A^{1/4+\varepsilon}(\mathcal{T}(u_0, T, f)(w)(t))\|_2 : t \in (0, T)\} \\
&\le \sup \{\|A^{1/4+\varepsilon} e^{-t \cdot A} u_0\|_2 : t \in (0, T)\} \\
&\quad + C_{28}(\Omega, \varepsilon, \nu) \cdot (1 + \|A^{1/4+\varepsilon} u_0\|_2 + \sigma(1/4 + \varepsilon, f))^{1/2} \cdot (a(u_0, T, f))^{3/2} \\
&\quad + \sigma(1/4 + \varepsilon, f) ;
\end{aligned}$$

$$\begin{aligned}
&\sup \{t^{1/2 - 2 \cdot \varepsilon/3} \cdot \|A^{3/4+\varepsilon/3}(\mathcal{T}(u_0, T, f)(w)(t))\|_2 : t \in (0, T)\} \\
&\le a(u_0, T, f) + C_{28}(\Omega, \varepsilon, \nu) \cdot (1 + \|A^{1/4+\varepsilon} u_0\|_2 + \sigma(1/4 + \varepsilon, f))^{1/2} \\
&\quad \times (a(u_0, T, f))^{3/2} .
\end{aligned}$$

Now we are able to establish our existence results. First we consider local strong solutions.

Theorem 7.1. *Let $u_0 \in D(A^{1/4+\varepsilon})$, $T \in (0, \infty)$. Assume that $f: (0, T] \mapsto L^2(\Omega)^3$ is a mapping with $(P_2 \circ f) \mid [\delta, T]$ Hölder continuous for $\delta \in (0, T)$, and with*

$$(P_2 \circ f) \mid (0, T) \in L^p((0, T), H_2(\Omega))$$

for some $p \in ((1/4 - \varepsilon/3)^{-1}, \infty)$. Then there exists some $T_0 \in (0, T]$ and a mapping $u: [0, T_0] \mapsto H_2(\Omega)$ with

(7.5) $\quad u \mid (0, T_0) \in B(T_0) , \qquad u \in C^0([0, T_0], \mathfrak{D}(A^{1/4+\varepsilon})) ,$

(7.6) $\quad u(0) = u_0 ,$

(7.7) $\quad u \mid (0, T_0] \in C^0((0, T_0], \mathfrak{D}(A)) \cap C^1((0, T_0], H_2(\Omega)),$

(7.8) $\quad (u \mid [\delta, T_0])'$ *and* $A \circ u \mid [\delta, T_0]$ *are Hölder continuous for* $\delta \in (0, T_0)$,

(7.9) $\quad u'(t) + A(u(t)) + M(u(t)) = (P_2 \circ f)(t)$ *for* $t \in (0, T_0]$.

This means in particular, for $t \in (0, T_0]$, that

(7.10) $\quad u(t) \in W^{2,2}_{\mathrm{loc}}(\Omega)^3 \cap W^{1,2}_0(\Omega)^3 \cap H_2(\Omega) \cap W^{3/2-\delta, 2}(\Omega)^3$ *for* $\delta \in (0, 1/2)$.

The mapping u is uniquely determined in the sense that it is the only mapping from $[0, T_0]$ into $H_2(\Omega)$ which satisfies (7.5)–(7.7) as well as (7.9).

Define for $t \in (0, T_0]$

$$\pi(t) := G(-u(t) \cdot \nabla(u(t)) + f(t)) + \tilde{G}(v \cdot \Delta(u(t))),$$

where we used notations from Definiton 6.1 and Theorem 4.1. Then it holds for $t \in (0, T_0]$ that

$$\pi(t) \in L^2(\Omega) \cap W^{1,2}_{\mathrm{loc}}(\Omega) \cap W^{1/2-\delta,2}(\Omega) \quad \textit{for} \quad \delta \in (0, 1/2), \tag{7.11}$$

$$u'(t) - v \cdot \Delta(u(t)) + u(t) \cdot \nabla(u(t)) + \nabla(\pi(t)) = f(t), \qquad \operatorname{div}(u(t)) = 0. \tag{7.12}$$

For $\sigma \in (0, 1/2)$, $\delta \in (0, T_0)$, the mappings $u \mid [\delta, T_0]$ and $\pi \mid [\delta, T_0]$ are Hölder continuous with respect to the norm $\| \;\|_{3/2-\sigma,2}$ and $\| \;\|_{1/2-\sigma,2}$, respectively.

Proof. First we remark that

$$\begin{aligned} &\sigma(3/4 + \varepsilon/3, (P_2 \circ f) \mid (0, R)) \\ &\leq \sup \left\{ t^{1/2-2\cdot\varepsilon/3} \cdot C_{10}(A, 3/4 + \varepsilon/3) \cdot \left(\int_0^t (t-s)^{(-3/4-\varepsilon/3)/(1-1/p)} \, dt \right)^{1-1/p} \right. \\ &\qquad \left. \times \left(\int_0^R \|(P_2 \circ f)\|_2^p \, ds \right)^{1/p} : t \in (0, R) \right\} \end{aligned} \tag{7.13}$$

for $R \in (0, T]$, as follows from (4.18) and Hölder's inequality. This implies, since $p > (1/4 - \varepsilon/3)^{-1}$ that

$$\sigma(3/4 + \varepsilon/3, P_2 \circ f) < \infty, \qquad \sigma(3/4 + \varepsilon/3, (P_2 \circ f) \mid (0, R)) \to 0 \qquad (R \downarrow 0). \tag{7.14}$$

It may be shown in a similar way that

$$\sigma(1/4 + \varepsilon, P_2 \circ f) < \infty, \qquad \sigma(1/4 + \varepsilon, (P_2 \circ f) \mid (0, R)) \to 0 \qquad (R \downarrow 0). \tag{7.15}$$

Combining (7.14), (4.17), and (4.19) yields

$$a(u_0, R, (P_2 \circ f) \mid (0, R)) \to 0 \qquad (R \downarrow 0). \tag{7.16}$$

Therefore we may choose $T_0 \in (0, T]$ such that

$$a(u_0, T_0, (P_0 \circ f) \mid (0, T_0)) < 1,$$

$$\begin{aligned} (a(u_0, T_0, (P_2 \circ f) \mid (0, T_0)))^{1/2} \leq \big[2 \cdot C_{28}(\Omega, \varepsilon, v) \cdot (1 + \|A^{1/4+\varepsilon} u_0\|_2 \\ + \sigma(1/4 + \varepsilon, P_2 \circ f) + \sigma(3/4 + \varepsilon/3, P_2 \circ f))^{1/2} \big]^{-1}, \end{aligned}$$

with $C_{28}(\Omega, \varepsilon, v)$ from Corollary 7.1.

For brevity we set $g := (P_2 \circ f) \mid (0, T_0)$. Corollary 7.1 then yields

$$\mathscr{T}(u_0, T_0, g)\, (\mathfrak{M}(u_0, T_0, g)) \subset \mathfrak{M}(u_0, T_0, g),$$

$$\mu(T_0)\, (\mathscr{T}(u_0, T_0, g)\, (w)) \leq (1/2) \cdot \mu(T_0)\, (w) \quad \text{for} \quad w \in \mathfrak{M}(u_0, T_0, g).$$

Thus, applying Banach's fixed point theorem, we conclude that there exists a function $\tilde{u} \in \mathfrak{M}(u_0, T_0, g)$ with $\mathscr{T}(u_0, T_0, g)\, (\tilde{u}) = \tilde{u}$. This implies for $t \in (0, T_0)$

$$\tilde{u}(t) = e^{-t \cdot A} u_0 - H_2(\Omega) - \int_0^t e^{-(t-s) \cdot A} (M(\tilde{u}(s)) - g(s)) \, ds. \tag{7.17}$$

Define the mappings $v_1, v_2 : [0, T_0] \mapsto H_2(\Omega)$ by

$$v_1(t) := e^{-A \cdot t} u_0 + H_2(\Omega) - \int_0^t e^{-(t-s) \cdot A} g(s)\, ds\,,$$

$$v_2(t) := H_2(\Omega) - \int_0^t e^{-(t-s) \cdot A}(-M(\tilde{u}(s)))\, ds \quad \text{for} \quad t \in [0, T_0]\,,$$

so that by (7.17) we obtain

(7.18) $$\tilde{u}(t) = v_1(t) + v_2(t) \quad \text{for} \quad t \in (0, T_0)\,.$$

For any $\delta \in (0, T_0)$, it follows from [42, p, 106, Corollary 4.2.2; p. 113, Corollary 4.3.3; p. 114, Theorem 4.3.5] that for $j = 1$, the relations

(7.19) $$v_j \mid [\delta, T_0] \in C^1([\delta, T_0], H_2(\Omega))\,,$$

(7.20) $$v_j(t) \in D(A) \quad \text{for} \quad t \in [\delta, T_0]$$

hold true, and where
the mappings

(7.21) $(v_j \mid [\delta, T_0])'$ and $A \circ v_j \mid [\delta, T_0]$ are Hölder continuous.
In addition, the preceding references imply

(7.22) $$v_1'(t) + Av_1(t) = g(t) \quad \text{for} \quad t \in (0, T_0]\,.$$

Concerning v_2, we may use (4.18), (4.16), (7.2) and an argument as in [42, p. 113] in order to show that for any $\alpha \in [0, 1)$, there is a constant $\mathscr{C}(\alpha) > 0$ with

$$\|A^\alpha \circ v_2(t) - A^\alpha \circ v_2(s)\|_2 \leq \mathscr{C}(\alpha) \cdot |s - t|^{1-\alpha} \quad \text{for} \quad s, t \in [0, T_0]\,.$$

Thus, for any $\alpha \in [0, 1)$, the mapping $A^\alpha \circ v_2$ is Hölder continuous on $[0, T_0]$. This result, equation (7.18), and (7.21) with $j = 1$ may be combined to imply that $A^\alpha \circ \tilde{u} \mid [\delta, T_0)$ is Hölder continuous too ($\alpha \in [0, 1)$, $\delta \in (0, T_0)$). (Note that the mapping $\tilde{u}$ is defined on the open interval $(0, T_0)$ only.) Hence, recalling Lemma 7.1, we see that the mapping $M \circ \tilde{u}$ is Hölder continuous on $[\delta, T_0)$, for $\delta \in (0, T_0)$. In particular, the latter mapping may be continuously extended to $[\delta, T_0]$, and this extension is Hölder continuous as well. Thus we are able to refer to [42] in the same way as we did before, and we obtain that the relations in (7.19)–(7.21) hold in the case $j = 2$ too. In addition, it follows

(7.23) $$v_2'(t) + Av_2(t) = -M(\tilde{u}(t)) \quad \text{for} \quad t \in (0, T_0)\,.$$

Now we set

$$u(t) := v_1(t) + v_2(t) \quad \text{for} \quad t \in [0, T_0]\,.$$

Then the mapping

(7.24) $M \circ u \mid [\delta, T_0]$ is Hölder continuous if $\delta \in (0, T_0)$,

and the assertions in (7.7)–(7.9) follow from (7.18), (7.19)–(7.22), and (7.23). Concerning (7.10), we refer to Definition 6.1, Corollary 6.1, and to (7.7). Furthermore, for $t \in (0, T_0]$, the function $\pi(t)$ is well defined and belongs to $L^2(\Omega) \cap W^{1,2}_{\mathrm{loc}}(\Omega)$; see Theorem 4.1, Definition 6.1, and (7.2). We further note that equation (7.12) follows from (7.9) and Theorem 4.2. In addition, we observe that

$$(v \cdot u(t), \pi(t)) \in SOL(\Omega, 0, f(t) - u(t) \cdot \nabla(u(t)) - u'(t)) \quad \text{for} \quad t \in (0, T_0];$$

see (5.4), (7.2), (7.7). Due to the relation in the preceding line, it is easy to deduce (7.11) from Theorem 5.2, and the last statement of Theorem 7.1 from (7.24), (7.8) and Theorem 5.2. Next we shall establish (7.5) and (7.6). To this end, we remark that

$$\|A^{1/4+\varepsilon}(e^{-t\cdot A}u_0 - u_0)\|_2 \to 0 \qquad (t \downarrow 0),$$

since $u_0 \in \mathfrak{D}(A^{1/4+\varepsilon})$; see (4.17). On the other hand, from (7.4), (7.16) and (7.15) it follows

$$\int_0^t \|A^{1/4+\varepsilon}\, e^{-(t-s)\cdot A}(M(u(s)) - g(s))\|_2 \, ds \to 0 \qquad (t \downarrow 0).$$

Thus $\|A^{1/4+\varepsilon}(u(t) - u_0)\|_2 \to 0$ for $t \downarrow 0$. Since $u \mid (0, T_0] \in C^0((0, T_0], \mathfrak{D}(A))$, we have in particular that $u \mid (0, T_0] \in C^0((0, T_0], \mathfrak{D}(A^{1/4+\varepsilon}))$. By combining these results, we obtain (7.5) and (7.6).

Let us finally prove the uniqueness result stated in Theorem 7.1. To this end, suppose that for $j \in \{1, 2\}$, the mapping $w_j : [0, T_0] \mapsto H_2(\Omega)$ satisfies (7.5)–(7.7) and (7.9). Due to (7.6), the set $\mathfrak{K} := \{s \in [0, T_0] : w_1(r) = w_2(r) \text{ for } r \in [0, s]\}$ is not empty, so the definition $S_0 := \sup \mathscr{K}$ yields a number in $[0, T_0]$.

Now we assume that there is some $s \in [0, T_0]$ such that $w_1(s) \neq w_2(s)$. Then $S_0 < T_0$, and we may choose a number $\varepsilon \in (0, T_0 - S_0]$ with

$$\text{(7.25)} \qquad w_1(s) \neq w_2(s) \quad \text{for} \quad s \in (S_0, S_0 + \varepsilon].$$

Set $v(t) := (w_1 - w_2)(t)$ for $t \in [0, T_0]$, and take $\delta \in (S_0, T_0)$. Multiplying equation (7.9) by $v(s)$, and integrating with respect to s, we obtain for $t \in [\delta, T_0]$, due to (7.7)

$$\text{(7.26)} \qquad \begin{aligned} \mathfrak{L}(t, \delta) &:= (\|v(t)\|_2^2 - \|v(\delta)\|_2^2)/2 + v \cdot \sum_{j=1}^{3} \int_\delta^t \|D_j(v(s))\|_2^2 \, ds \\ &= -\int_\delta^t \int_\Omega (v(s) \cdot \nabla(w_2(s)))(x) \cdot v(s)(x) \, dx \, ds \\ &\quad 4- \int_\delta^t \int_\Omega (w_1(s) \cdot \nabla(v(s)))(x) \cdot v(s)(x) \, dx \, ds. \end{aligned}$$

Concerning the first integral on the right-hand side of (7.26), we observe for $s \in (0, T_0)$

$$\int_\Omega |(v(s) \cdot \nabla(w_2(s)))(x) \cdot v(s)(x)| \, dx \le \sum_{j=1}^{3} \|D_j(w_2(s))\|_2 \cdot \|v(s)_6 \cdot \|v(s)\|_3,$$

$$\|v(s)\|_6 \le \mathscr{C}_1 \cdot \sum_{j=1}^{3} \|D_j(v(s))\|_2,$$

$$\|v(s)\|_3 \le \mathscr{C}_2 \cdot \|A^{1/4}(v(s))\|_2 \le \mathscr{C}_3 \cdot \|v(s)\|_2^{1/2} \cdot \sum_{j=1}^{3} \|D_j(v(s))\|_2^{1/2},$$

with constants $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$ independent of s. The second estimate is a consequence of Theorem 4.4, (4.20), and (6.4), whereas the third one follows from Theorem 4.4, Lemma

6.5, (4.22), and (6.4). Analogous inequalities hold true with respect to the second integral on the right-hand side of (7.26). Collecting our results, we see that there is a constant $\mathscr{C}_4 > 0$ such that for $\delta \in (S_0, T_0)$, $t \in [\delta, T_0]$, we have

$$\mathscr{L}(t, \delta) \leq \mathscr{C}_4 \cdot \int_\delta^t \left(\sum_{\sigma=1}^{2} \sum_{j=1}^{3} \|D_j(w_\sigma(s))\|_2 \right) \cdot \left(\sum_{j=1}^{3} \|D_j(v(s))\|_2^{3/2} \right) \cdot \|v(s)\|^{1/2} \, ds \,,$$

hence by Young's inequality

$$\mathscr{L}(t, \delta) \leq (\nu/2) \cdot \int_\delta^t \sum_{j=1}^{3} \|D_j(v(s))\|_2^2 \, ds + \mathscr{C}_5 \cdot \int_\delta^t \sum_{\sigma=1}^{2} \sum_{j=1}^{3} \|D_j(w_\sigma(s))\|_2^4 \cdot \|v(s)\|_2^2 \, ds \,,$$

with $\mathscr{C}_5$ independent of t and δ. Since $w_1(S_0) = w_2(S_0)$, it follows by letting δ tend to S_0

$$\|v(t)\|_2^2 \leq \mathscr{C}_5 \cdot \int_{S_0}^t \sum_{\sigma=1}^{2} \sum_{j=1}^{3} \|D_j(w_\sigma(s))\|_2^4 \cdot \|v(s)\|_2^2 \, ds \quad \text{for} \quad t \in [S_0, T_0] \,. \tag{7.27}$$

On the other hand, we have for $j \in \{1, 2, 3\}$, $s \in (0, T_0]$, $\sigma \in \{1, 2\}$

$$\begin{aligned} \|D_j(w_\sigma(s))\|_2 &\leq \mathscr{C}_6 \cdot \|A^{3/4+\varepsilon/3}(w_\sigma(s))\|_2^{(3-12\cdot\varepsilon)/(6-8\cdot\varepsilon)} \\ &\quad \times \|A^{1/4+\varepsilon}(w_\sigma(s))\|_2^{(3+4\cdot\varepsilon)/(6-8\cdot\varepsilon)} \\ &\leq \mathscr{C}_7 \cdot s^{(-1/2+2\cdot\varepsilon/3)\cdot(3-12\cdot\varepsilon)/(6-8\cdot\varepsilon)} \,. \end{aligned}$$

Here we exploited (6.4), (4.22), and the fact that $w_2 \mid (0, T_0) \in B(T_0)$. The preceding estimate implies

$$\int_0^{T_0} \|D_j(w_\sigma(s))\|_2^4 \, ds < \infty \quad \text{for} \quad j \in \{1, 2, 3\} \,, \quad \sigma \in \{1, 2\}$$

hence we may conclude from (7.27) and Gronwall's lemma that $w_1(s) = w_2(s)$ for $s \in [S_0, T_0]$, a contradiction to (7.25). Therefore the functions w_1 and w_2 must coincide. □

Our next result concerns global strong solutions for small data.

Theorem 7.2. *There exists a constant $C_{29}(\Omega, \varepsilon, \nu) > 0$ with the properties to follow: Let $u_0 \in D(A^{1/4+\varepsilon})$, and let f be a mapping from $(0, \infty)$ into $L^2(\Omega)^3$. Assume that*

$$\|A^{1/4+\varepsilon} u_0\|_0 \leq C_{29}(\Omega, \varepsilon, \nu) \,,$$

$P_2 \circ f \mid (0, 1) \in L^p((0, 1), H_2(\Omega))$ *for some* $p \in ((1/4 - \varepsilon/3)^{-1}), \infty)$,
$P_2 \circ f \mid (\delta_1, \delta_2)$ *is Hölder continuous for* $\delta_1, \delta_2 \in (0, \infty)$ *with* $\delta_1 < \delta_2$,

$$\sigma(3/4 + \varepsilon/3, P_2 \circ f) \leq C_{29}(\Omega, \varepsilon, \nu) \,, \qquad \sigma(1/4 + \varepsilon, P_2 \circ f) \leq 1 \,.$$

Then there exists a uniquely determined mapping $u : (0, \infty) \mapsto H_2(\Omega)$ such that

$$u \in C^0([0, \infty), \mathfrak{D}(A^{1/4+\varepsilon})) \cap L^\infty([0, \infty), D(A^{1/4+\varepsilon})) \,,$$

$$u(0) = u_0 \,, \qquad u \mid (0, \infty) \in C^0((0, \infty), \mathfrak{D}(A)) \cap C^1((0, \infty), H_2(\Omega)) \cap B(\infty) \,,$$

$$u'(t) + A(u(t)) + M(u(t)) = (P_2 \circ f)(t) \quad \textit{for} \quad t \in (0, \infty) \,.$$

In addition, the relation (7.10) *is valid for* $t \in (0, \infty)$, *and the mappings* $(u \mid (\delta_1, \delta_2))'$ *and* $A \circ u \mid (\delta_1, \delta_2)$ *are Hölder continuous for* $\delta_1, \delta_2 \in (0, \infty)$ *with* $\delta_1 < \delta_2$.

For $t \in (0, \infty)$, *define* $\pi(t)$ *as in Theorem 7.1. Then the relations in* (7.11) *and* (7.12) *hold true for* $t \in (0, \infty)$.

If $\sigma \in (0, 1/2)$ *and* $\delta_1, \delta_2 \in (0, \infty)$ *with* $\delta_1 < \delta_2$, *then the mappings* $u \mid [\delta_1, \delta_2]$ *and* $\pi \mid [\delta_1, \delta_2]$ *are Hölder continuous with respect to the norm* $\| \ \|_{3/2-\sigma, 2}$ *and* $\| \ \|_{1/2-\sigma, 2}$, *respectively.*

Proof. For $u_0 \in \mathfrak{D}(A^{1/4+\varepsilon})$, and for measurable mappings $g : (0, \infty) \mapsto H_2(\Omega)$ with $\sigma(3/4 + \varepsilon/3, g) < \infty$, we obtain, using (4.18)

$$a(u_0, \infty, P_2 \circ f) \leq \mathscr{C}_{10}(A, 1/2 - 2 \cdot \varepsilon/3) \cdot \|A^{1/4+\varepsilon} u_0\|_2 + \sigma(3/4 + \varepsilon/3, g) .$$

Now define

$$C_{29}(\Omega, \varepsilon, \nu) := \min \{1/2, [6 \cdot C_{28}(\Omega, \varepsilon\, \nu) \cdot (C_{10}(A, 1/2 - 2 \cdot \varepsilon/3) + 1)^{3/2}]^{-2}\} ,$$

with $C_{28}(\Omega, \varepsilon, \nu)$ from Corollary 7.1, and with $C_{10}(A, 1/2 - 2 \cdot \varepsilon/3)$ from (4.18). Then Theorem 7.2 may be proved in the same way as Theorem 7.1, apart from some obvious modifications. □

In the following corollary, the smallness conditions on the data stated in Theorem 7.2 are written in a more explicit but less general way.

Corollary 7.2. *Let* $p \in ((1/4 - \varepsilon/3)^{-1}, \infty)$, *Then there is a constant* $C_{30}(\Omega, \varepsilon, \nu, p) > 0$ *with the properties to follow:*

If $u_0 \in W_0^{1,2}(\Omega)^3$ *with*

$$\|u_0\|_2^{1/2+2\varepsilon} \cdot \left(\sum_{k=1}^{3} \|D_k u_0\|_2^2 \right)^{1/2-2\varepsilon} \leq C_{30}(\Omega, \varepsilon, \nu, p) ,$$

if $f : (0, \infty) \mapsto L^2(\Omega)^3$ *is a mapping such that* $(P_2 \circ f) \mid (\delta_1, \delta_2)$ *is Hölder continuous for* $\delta_1, \delta_2 \in (0, \infty)$ *with* $\delta_1 < \delta_2$, *and if in addition*

$$\left(\int_0^1 \|(P_2 \circ f)\,(s)\|_2^p \,\mathrm{d}s \right)^{1/p} \leq C_{30}(\Omega, \varepsilon, \nu, p) ,$$

$$\sup \{t^{3/4-\varepsilon} \cdot \|(P_2 \circ f)\,(t)\|_2 : s \in (1/2, \infty)\} \leq C_{30}(\Omega, \varepsilon, \nu, p) ,$$

then the conclusions of Theorem 7.2 hold true.

Proof. For $u_0 \in W_0^{1,2}(\Omega)^3$, by (4.22) and Lemma 6.3 we find

$$\|A^{1/4+\varepsilon} u_0\|_2 \leq \mathscr{C} \cdot \|u_0\|_2^{1/2+2\cdot\varepsilon} \cdot \left(\sum_{k=1}^{3} \|D_k u_0\|_2^2 \right)^{1/2-2\cdot\varepsilon} .$$

Here and below, the letter $\mathscr{C}$ denotes constants which only depend on Ω, ε, ν, or p.

Let $g : (0, \infty) \mapsto H_2(\Omega)$ be a measurable mapping. Then, proceeding in the same way as in (7.13), we obtain from (4.17), (4.18) for $t \in (0, 1)$, that

$$t^{1/2-2\cdot\varepsilon/3} \cdot \int_0^t \|A^{3/4+\varepsilon/3}\, \mathrm{e}^{-(t-s)\cdot A}\, g(s)\|_2 \,\mathrm{d}s \leq \mathscr{C} \cdot \left(\int_0^1 \|g(s)\|_2^p \,\mathrm{d}s \right)^{1/p} . \tag{7.28}$$

If $t \in (1, \infty)$, then the left-hand side of (7.28) is bounded by

$$\mathscr{C} \cdot \sup \{s^{3/4-\varepsilon} \cdot \|g(s)\|_2 : s \in (1/2, \infty)\} + \mathscr{C} \cdot \left(\int_0^1 \|g(s)\|_2^p \, ds \right)^{1/p} .$$

Now it is obvious how to derive the corollary from Theorem 7.2. □

References

[1] ADAMS, R. A.: Sobolev spaces. Academic Press, New York e.a., 1975
[2] AGMON, S., DOUGLIS, A., NIRENBERG, L.: Estimates near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions, I. Comm. Pure Appl. Math. **12** (1959), 623–727
[3] AGMON, S., DOUGLIS, A., NIRENBERG, L.: Estimates near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions, II. Comm. Pure Appl. Math. **17** (1964), 35–92
[4] ALT, H. W.: Lineare Funktionalanalysis. Springer, Berlin e.a., 1991
[5] BROWN, R. M.: The Method of Layer Potentials for the Heat Equation in Lipschitz Cylinders. Amer. J. Math. **111** (1989), 339–379
[6] BROWN, R. M.: The Initial-Neumann Problem for the Heat Equation in Lipschitz Cylinders. Trans. Amer. Math. Soc. **320** (1990), 1–52
[7] CHORIN, A. J., MARSDEN, J. E.: A Mathematical Introduction to Fluid Mechanics. Springer, New York e.a., 1979
[8] COIFMAN, R. R., MCINTOSH, A., MEYER, J.: L'Intégrale de Chauchy Définit un Opérateur Borné sur L^2 pour les Courbes Lipschitziennes. Ann. of Math. **116** (1982), 361–388
[9] COSTABEL, M.: Boundary Integral Operators on Lipschitz Domains: Elementary Results. SIAM J. Math. Anal. **19** (1988), 613–626
[10] COSTABEL, M.: Boundary Integral Operators for the Heat Equation. Integral Equations Operator Theory **13** (1990), 498–552
[11] DAHLBERG, D. E. J., KENIG, C. E.: Hardy Spaces and the Neumann Problem in L^p for Laplace's Equation in Lipschitz Domains. Ann. of Math. **125** (1987), 437–465
[12] DAHLBERG, B. E. J., KENIG, C. E., VERCHOTA, G. C.: The Dirichlet Problem for the Biharmonic Equation in a Lipschitz Domain. Ann. Inst. Fourier (Grenoble) **36** (1986), 109–135
[13] DAHLBERG, B. E. J., KENIG, C. E., VERCHOTA, G. C.: Boundary Value Problems for the System of Elastostatics in Lipschitz Domains. Duke Math. J. **57** (1988), 795–818
[14] DAUGE, M.: Stationary Stokes and Navier-Stokes Systems on Two-or Three-dimensional Domains with Corners. Part I. Linearized Equations. SIAM J. Math. Anal. **20** (1989), 74–97
[15] DEURING, P.: The Resolvent Problem for the Stokes System in Exterior Domains: An Elementary Approach. Math. Meth. Appl. Sci. **13** (1990), 335–349
[16] DEURING, P.: An Integral Operator Related to the Stokes System in Exterior Domains. Math. Meth. Appl. Sci. **13** (1990), 323–333 Addendum, Math. Meth. Appl. Sci. **14** (1991), 445
[17] DEURING, P.: The Stokes-system in Exterior Domains: L^p-estimates for Small Values of a Resolvent Parameter. J. Appl. Math. Phys. (ZAMP) **41** (1990), 829–842
[18] DEURING, P.: The Stokes-system in Exterior Domains: Existence, Uniqueness and Regularity of Solutions in L^p-spaces. Commun. Part. Diff. Equ. **16** (1991), 1513–1528
[19] DEURING, P.: The Resolvent Problem for the Stokes System in Exterior Domains: Uniqueness and Non-regularity in Hölder Spaces. Proc. Royal Soc. Edinburgh **122A** (1992), 1–10
[20] DEURING, P.: The Stokes System in an Infinite Cone. Akademie Verlag, Berlin 1994
[21] DEURING, P., WAHL, W. VON, WEIDEMAIER, P.: Das lineare Stokes System im $\mathbb{R}^3$ (I. Vorlesungen über das Innenraumproblem). Bayreuther Mathematische Schriften **27** (1988), 1–252
[22] DEURING, P., WAHL, W. VON: Das lineare Stokes System im $\mathbb{R}^3$ (II. Das Außenraumproblem). Bayreuther Mathematische Schriften **28** (1989), 1–109
[23] DIEUDONNÉ, J.: Eléments d'Analyse I. Gauthiers-Villars, Paris, 1972

[24] Fabes, E. B., Kenig, C. E., Verchota, G. C.: The Dirichlet Problem for the Stokes System on Lipschitz Domains. Duke Math. J. **57** (1988), 769–793

[25] Farwig, R., Sohr, H.: Generalized Resolvent Estimates in Bounded and Unbounded Domains. Preprint no. 249, SFB 256, R.F.W.-Universität Bonn

[26] Farwig, R., Sohr, H.: Helmholtz Decomposition and Stokes Resolvent System for Aperture Domains in L^q-spaces. Preprint no. 294, SFB 256, R.F.W.-Universität Bonn

[27] Fucik, S., John, O., Kufner, A.: Function Spaces. Noordhoff, Leyden, 1977

[28] Fujita, H., Kato, T.: On the Navier-Stokes Initial Value Problem, I. Arch. Rat. Mech. Anal. **16**, (1964), 269–315

Galdi, G. P.: An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Volume I, II. Springer, New York e.a., 1994

[29] Galdi, G. P., Simader, C. G.: Existence, Uniqueness and L^q-estimates for the Stokes Problem in an Exterior Domain. Arch. Rational Mech. Anal. **112** (1990), 291–318

[30] Galdi, G. P., Simader, C. G., Sohr, H.: On the Stokes Problem in Lipschitz Domains. To appear in: Ann. Mat. Pura Appl

[31] Giga, Y.: Analyticity of the Semigroup Generated by the Stokes Operator in L_r Spaces. Math. Z. **178** (1981), 297–329

[32] Giga, Y., Sohr, H.: Abstract L^p-estimates for the Cauchy Problem with Applications to the Navier Stokes Equations in Exterior Domains. J. Funct. Anal. **102** (1991), 72–94

[33] Grisvard, P.: Elliptic Problems in Nonsmooth Domains. Pitmann, Boston e.a. 1985

[34] Hopf, E.: Über die Anfangswertaufgaben für die hydrodynamischen Grundgleichungen. Math. Nachr. **4** (1951), 213–231

[35] Kellog, R. B., Osborn, J. E.: A Regularity Result for the Stokes Problem in a Convex Polygon. J. Funct. Anal. **21** (1976), 397–431

[36] Kozlov, V. A., Mazya, V. G., Schwab, C.: On Singularities of Solutions to the Dirichlet Problem of Hydrodynamics near the Vertex of a Cone. To appear in J. Reine Angewandte Mathematik

[37] Ladyzhenskaya, O. A.: The Mathematical Theory of Viscous Incompressible Flow. Gordon and Breach, New York e.a., 1969

[38] McCracken, M.: The Resolvent Problem for the Stokes Equations on Halfspace in L_p. Siam J. Math. Anal. **12** (1981), 201–228

[39] Mazja, W. G., Nasarow, S. A., Plamenewski, B. A.: Asymptotische Theorie elliptischer Randwertaufgaben in singulär gestörten Gebieten. I. Akademie-Verlag, Berlin 1991

[40] Mazja, W. G., Nasarow, S. A., Plamenewski, B. A.: Asymptotische Theorie elliptischer Randwertaufgaben in singulär gestörten Gebieten. II. Akademie-Verlag, Berlin 1991

[41] Nečas, J.: Les Méthodes Directes en Théorie des Équations Elliptiques. Masson, Paris 1967

[42] Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, New York e.a. 1983

[43] Pipher, J., Verchota, G. C.: The Dirichlet Problem in L^p for the Biharmonic Equation on Lipschitz Domains. Amer. J. Math. **114** (1992), 923–972

[44] Shen, Zhongwei: Boundary Value Problems for Parabolic Lamé System and a Nonstationary Linerarized System of Navier-Stokes Equations in Lipschitz Cylinders. American J. Math. **113** (1991), 293–373

[45] Simader, C. G., Sohr, H.: A new Approach to the Helmholtz Decomposition and the Neumann Problem in L^q-Spaces for Bounded and Exterior Domains. Series on Advances in Mathematics for Applied Science, Vol. II, World Scietific, (1992), 1–35

[46] Solonnikov, V. A.: Estimates for Solutions of Nonstationary Navier-Stokes Equations. J. Sov. Math. **8** (1977), 467–529

[47] Stein, E. M.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton, N. J., 1970

[48] Tanabe, H.: Equations of Evolution. Pitman, London e.a., 1979

[49] Temam, R.: Navier-Stokes Equations. North Holland, Amsterdam e.a., 1977

[50] Triebel, H.: Interpolation Theory, Function Spaces, Differential Operators. North Holland, Amsterdam e.a., 1978

Vanhorn, W.: The Stokes Equations, Akademie Verlag, Berlin 1994

[51] Verchota, G.: Layer Potentials and Regularity for the Dirichlet Problem for Laplace's Equation in Lipschitz Domains. J. Funct. Anal. **59** (1984), 572–611

WAHL, W. VON: The Equations of Navier-Stokes and Abstract Parabolic Equations. Vieweg, Braunschweig, 1985

[53] WAHL, W. VON: Abschätzungen für das Neumann-Problem und die Helmholtz-Zerlegung von L^p. Nachr. Akad. d. Wiss. Göttingen, II. Math.-Phys. Klasse. Jahrgang 1990, Nr. 2

[54] WAHL, W. VON: Vorlesung über das Außenraumproblem für die instationären Gleichungen von Navier-Stokes. Vorlesungsreihe Nr. 11 (Rudolph-Lipschitz-Vorlesung) SFB 256, Bonn, 1990

[55] YOSIDA, K.: Functional Analysis. Springer, Berlin e.a., 1978

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