Towards the computation of minimum drag profiles in viscous laminar flow

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A numerical method is given for the solution of certain optimum design problems of fluid mechanics. The profile of given area and smallest drag in a uniform laminar flow is computed. This profile is long and slim, its front end is shaped like a wedge of angle 90° and its rear end is shaped like a cusp. Owing to the numerical complexity of the problem the precision of the results is average (around 5%). However, this work is a good illustration of the theoretical method exposed previously and it shows how good precision can be obtained if one is prepared to pay for it. A numerical solution of the adjoint system of the stationary Navier–Stokes equation is also given; this equation will play an important role in optimum design in fluid mechanics.

Introduction

Optimum design of wings, boats, car bodies etc. are all important problems that hydro- and aero-dynamic engineers are faced with daily. These problems are generally very difficult to model mathematically because many 'external' parameters such as the cost of the study, the strength of the design, are involved. In this paper these problems are modelled from a technological viewpoint; we shall look for the designs that minimize the drag. Thus the purpose of this study is to solve an abstract problem of fluid mechanics so as to exhibit some important parameters that might be of use to the engineer as guides for his intuitive solution to some of the more complex problems.

Until now, little has been known about those problems (see for example Landau and Lifchitz[1] for Reynolds numbers less that 10^5); the flow should remain laminar, and the boundary layer should not separate. For these types of flow, Pironneau[2] gave certain conditions to be satisfied by the optimum profiles, and a numerical method to construct them. The flow is incompressible and governed by the stationary Navier–Stokes equations (laminar flow)[2]; this implies average Reynolds numbers and aerodynamical profiles. The optimum conditions[2] were established by the methods of optimum control of Lions[3] for systems governed by partial differential equations.

These optimum conditions are hard to interpret physically since they involve the co-state Q, of the system of Navier–Stokes, a quantity which to our knowledge has no physical interpretation. Our purpose, in doing this study was to illustrate the above method and to give a computation of this new quantity, Q.

It was established[2] that the change in drag due to a small deformation of a profile is connected with the skin friction and the normal derivative of Q. Thus the method used to compute the profile of given area and smallest drag is essentially the following: given the exterior boundary $\hat{C}_0$ of the boundary layer of a profile C_0, the pressure on C_0 is computed by assuming a potential flow. The skin friction $\partial u/\partial n$ and the co-state friction $\partial Q/\partial n$ are then computed by solving Prandtl's equations in the boundary layer (wake included) as well as the co-state system in the boundary layer. Then C_0 is modified[2] into $\hat{C}_1$; the corresponding profile C_1 has same area and smaller drag. The pressure on C_0 is computed by a method of singularity, adapted by Luu[4]. Prandtl's equations are discretized by a finite difference scheme related to those used by Keller[5] and the co-state system is integrated by using a discretization scheme adjoint to the one used on the boundary layer equations. As one may expect the resulting algorithm is quite complex; each iteration takes 30 sec of IBM 370/168 time. The profile obtained satisfies the optimum conditions with an error of 10^{-2}; it is long and slim, with a sharp nose (90°)* and a cusp at the rear end. To improve on the precision one needs more powerful computational facilities in order to use a more refined discretization grid and a more elaborate way of treating the singularity of the boundary layer equation at the tip of the rear end of the profile. Nevertheless this study shows that the method[2] is feasible and that a fluid mechanics laboratory could use it if needed. Besides, as stated earlier, since it is shown how to compute the co-state Q and since an example is given, it is possible that someone will find a physical meaning for it. This is an important problem since the generalization of this method to the case of turbulent boundary layer requires a computation of Q also.

Statement of the problem and recalls

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Let us consider the optimum design problem:

$$\min_{C\in\mathscr{S}}\left\{\frac{2}{\nu}\int_{\Omega-C} e_{ij}e_{ij}\,\mathrm{d}x\right\} \tag{1}$$

where $\mathscr{S}$ is a family of subsets C of Ω of given area a and regular boundary ∂C and where:

$$e_{ij}=\frac{\nu}{2}\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right),\quad 1\leqslant i,j\leqslant 2 \tag{2}$$

$u=(u_1,u_2)$ is the solution of the stationary Navier–Stokes equations:

$$\begin{aligned}&\nu\nabla^2 u-u\cdot\nabla u=\nabla p\quad\text{in}\quad\Omega \dot{-} C\\&\nabla\cdot u=0\qquad\qquad\quad\;\;\text{in}\quad\Omega\dot{-} C\\&u=u_0 \text{ on } \partial\Omega,\\&u=o\text{ on }\partial C,\end{aligned} \tag{3}$$

p being the pressure, ν the kinematic viscosity.

Strictly speaking the above optimization problem is well posed only if equation (3) has a unique solution; this is the case only if ν is not too small and C is slender.

From a physical viewpoint this problem (1) corresponds to the search of the profile of given area and smallest drag in a laminar flow. Indeed, if $\Omega=\mathbf{R}^2$, the drag $F(C)$ of C is such that:

$$u_0\cdot F(C)=\rho\int_{\mathbf{R}^2-C} e_{ij}e_{ij}\,\mathrm{d}\Omega$$

where ρ is the density of the fluid.

Existence results (recalls)

Following Garabedian[7] one can show that if $\{\partial C|C\in\mathscr{S}\}$ is a set of uniformly lipschitz curves without double branches, such that for each of them equation (3) has a unique solution, then equation (1) has a unique solution.

The uniqueness of the solution of equation (1) and the dependence of the solution upon the lipschitz constant are open.

Necessary conditions for optimality (recalls)[2]

Theorem. Let the contours ∂C of $\mathscr{S}$ be piecewise C^1. If ∂C in $\mathscr{S}$ is parameterized by σ with:

$$\partial C=\{\xi(\sigma)|\sigma\in[0,1],\quad \xi(0)=\xi(1)\}$$

if $\partial C'$, is an 'α-neighbour' of ∂C, in the sense:

$$\begin{aligned}\partial C'&=\{\xi'(\sigma)|\xi'(\sigma)\\&=\xi(\sigma)+\alpha(\sigma)n(\sigma),\quad\sigma\in[0,1],\quad\alpha(0)=\alpha(1)\}\end{aligned}$$

where $n(\sigma)$ is the exterior normal of C at $\xi(\sigma)$.

Then the change δE in energy dissipated in the fluid

$$E(C)=\frac{2}{\nu}\int_{\Omega-C} e_{ij}e_{ij}\,\mathrm{d}x \tag{4}$$

is given by:

$$\delta E=\int_{\partial C}\alpha(\sigma)\frac{\partial u}{\partial n}\cdot\left(\frac{\partial u}{\partial n}+2\frac{\partial w}{\partial n}\right)\mathrm{d}(\partial C)+o(\alpha) \tag{5}$$

where $o(\alpha)$ is such that $\lim_{\lambda\to 0}\lambda^{-1}\,o(\alpha\lambda)=0$ and w is the solution of the adjoint of the stationary Navier–Stokes equations:

$$\begin{aligned}&\nu\nabla^2 w+u\cdot\nabla w-\nabla u\cdot w=-u\cdot\nabla u+\nabla q\quad &&\text{in }\Omega-C\\&\nabla\cdot w=0 &&\text{in }\Omega-C\\&w|\partial C=0\\&w|\partial\Omega=0\end{aligned} \tag{6}$$

From theorem (1) the following corollary was shown.

Corollary. Under the hypothesis of theorem (1), each solution C of problem (1) must satisfy:

$$\frac{\partial u}{\partial n}\cdot\left(\frac{\partial u}{\partial n}+2\frac{\partial w}{\partial n}\right)=\text{constant almost everywhere on }\partial C$$

We recall that it was shown[2] that this condition cannot be satisfied at the front end of the C unless it is tangent to a wedge of angle 90°. Indeed a change of coordinate of the Falkner–Skan type[8] shows that equations (3) and (6) have a parabolic character and that the time-like derivatives disappear at the front end; the remaining equations being satisfied by a 90° wedge only.

Transformation of the adjoint system (6) (recalls)

From Prandtl we know that under certain conditions (C slender, Reynold number R not too large) the solution of equation (3) can be reasonably approximated as follows: if $\Omega=R^2$, $\Omega-C$ can be decomposed into two regions: $\Omega-\hat{C}$, and $\hat{C}-C$ (see *Figure 1*).

In $\Omega-\hat{C}$ u derive from a potential φ:

$$u=(u_1,u_2)=\left(\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y}\right) \tag{7}$$

*This result was found by Pironneau[2]. A similar result was found by Pironneau[6] also for the case of Stokes flow (120° for 3-dimensional bodies, 90° for profiles). It seems, thus, that the front end of the profile depends very little on the Reynolds number: 90° for the profiles, and between 110° and 120° for the 3-dimensional bodies.

where φ is the solution of:

$$\nabla^2\varphi = 0 \text{ in } \Omega - C, \quad \varphi_\infty = u_0 x, \quad \left.\frac{\partial\varphi}{\partial n}\right|_C = 0 \tag{8}$$

$\hat{C} - C$ is a very thin domain (thickness of the order of $R^{-\frac{1}{2}}$) on each side of C and in its wake. Let (s, n) be the tangential and normal coordinates in $\hat{C} - C$. Then u can be approximated by (u_s, u_n) solution of

$$\begin{aligned} -\nu\frac{\partial^2 u_s}{\partial n} + u_s\frac{\partial u_s}{\partial s} + u_n\frac{\partial u_s}{\partial n} &= U(s)\frac{\mathrm{d}U(s)}{\mathrm{d}s} \\ \frac{\partial u_s}{\partial s} + \frac{\partial u_n}{\partial n} &= 0 \end{aligned} \tag{9}$$

where $U(s) = \left.\frac{\partial\varphi}{\partial s}\right|_{\partial\hat{C}}$ and with $u_s = u_n = 0$ on $\partial C(n = 0)$ and $u_s = U(s)$ on $\partial\hat{C}$. In the wake the condition $u_s = 0$ is replaced by $(\partial u_s/\partial n)(s, 0) = 0$.

Proposition 1. In $\hat{C} - C$, $w = (w_s, w_n)$ can be approximated by the solution of:

$$\left.\begin{aligned} &\nu\frac{\partial^2 w_s}{\partial n^2} + u_s\frac{\partial w_s}{\partial s} + u_n\frac{\partial w_s}{\partial n} - \frac{\partial u_s}{\partial s}w_s = \frac{\partial q}{\partial s} - \nu\frac{\partial^2 u_s}{\partial n^2} \\ &-\frac{\partial u_s}{\partial n}w_s = \frac{\partial q}{\partial n} \\ &\frac{\partial w_s}{\partial s} + \frac{\partial w_u}{\partial n} = 0 \end{aligned}\right\} \tag{10}$$

with the boundary conditions:

$w_n = 0$ on ∂C, $w_s = 0$ on ∂C or $[(\partial w_s/\partial n) = 0$ on the axis of the wake] $w_s = 0$, $q = 0$ on $\partial\hat{C}$. In $\Omega - \hat{C}$, $w = 0$.

Analysis of the singularity at the rear end of C

If the rear end is blunt or conical, the potential speed $U(s)$ on $\partial\hat{C}$ tends to zero when s tends to s_1; s_1 being the curvilinear abscissa of the rear end. This causes the boundary layer to separate at a point $s_d < s_1$, and Landau and Lifchitz[1] showed that up to higher order terms in $(s_d - s)^{\frac{1}{2}}$:

$$u_n = \frac{A}{2}\frac{U_s(s_d, n)}{(s_d - s)^{\frac{1}{2}}} \tag{11}$$

$$u_s = u_s(s_d, n) + A\frac{\partial u_s}{\partial n}(s_d, n)(s_d - s)^{\frac{1}{2}} \tag{12}$$

If q is eliminated in equation (10), we obtain:

$$\nu\frac{\partial^3 w_s}{\partial n^3} + \frac{\partial^3 w_s}{\partial n^2}u_n + \frac{\partial^2 w_s}{\partial n\partial_s} + 2\frac{\partial u_n}{\partial n}\frac{\partial w_s}{\partial n} + 2\frac{\partial u_s}{\partial n}\frac{\partial w_s}{\partial s} = -\nu\frac{\partial^3 u_s}{\partial n^3} \tag{13}$$

If, in the neighbourhood of s_d, w_s can be expanded as:

$$w_s = \sum_{i=1}^{+\infty} a_i(n)(s_d - s)^{i/2} + a_0(n) + \sum_{j=1}^{m} a_{-j}(n)(s_d - s)^{-j/2} \tag{14}$$

we deduce from equations (11)–(14) by identification that:

$$a_{-j} = 0, \quad j = 1, \ldots, m$$

and

$$\frac{A}{2}a_o'' u_s + \frac{A}{2}u_s' a_o' - \frac{1}{2}a_1' u_s - a_1 u_s' = 0 \tag{15}$$

In the neighbourhood of $n = 0$, $u_s = u_s''(s_d, 0)(n^2/2)$. Therefore equation (15) implies that:

$$Aa_o'' n + Aa_o' - a_1' n - 2a_1 = 0$$

which, together with the boundary condition $w(s, 0) = 0$ leads to:

$$w_s(s, n) = \alpha n(s_d - s)^{\frac{1}{2}} + o(n) + o(s_d - s)^{\frac{1}{2}}$$

Hence

$$\lim_{s\to s_d}\frac{\partial u_s}{\partial n}\left(\frac{\partial u_s}{\partial n} + 2\frac{\partial w_s}{\partial n}\right) = 0$$

Thus no separation can occur; it implies that the rear end must be a cusp. Nevertheless the change of limit conditions on u and w at $s = s_1$ makes $\partial u/\partial n, \partial w/\partial n$ discontinuous at $s = s_1$. This makes the computation of w very difficult.

Description of the optimization algorithm in the continuous case

Although $\mathscr{S}$ has no linear structure we see from equation (5) that the quantity:

$$\left.\frac{\partial u}{\partial n}\left(\frac{\partial u}{\partial n} + 2\frac{\partial w}{\partial n}\right)\right|_{\partial C}$$

will play the role of the gradient of E. For this reason we set

$$\operatorname{grad} E = \left.\frac{\partial u}{\partial n}\left(\frac{\partial u}{\partial n} + 2\frac{\partial u}{\partial n}\right)\right|_{\partial C} \tag{16}$$

Let

$$K_1 = \int_{\partial C}\operatorname{grad} E(\sigma)\,\mathrm{d}(\partial C) \tag{17}$$

Note that from equation (5):

$$\int_0^1 \alpha(\sigma)\,\mathrm{d}(\partial C) = 0 \Rightarrow \delta E = \int_{\partial C}\alpha(\sigma)\,[\operatorname{grad} E(\sigma) - K_1]\,\mathrm{d}(\partial C) + o(\alpha)$$

Consequently, if C is not a solution of equation (1), we can obtain a new profile with equal area and smaller drag, by choosing:

$$\alpha(\sigma) = -\lambda(\operatorname{grad} E(\sigma) - K_1), \quad \lambda \text{ 'small'} \tag{18}$$

This is so because:

$$\delta E = -\lambda\int_{\partial C}[\operatorname{grad} E(\sigma) - K_1]^2\,\mathrm{d}(\partial C) + o(\alpha) < 0 \tag{19}$$

Hence this leads to the following algorithm:

choose C_0 of area a; set $i = 0$; choose λ small (20)

compute $\frac{\partial u_i}{\partial n}$, u_i solution of equation (3) with
$C = C_i$ (21)

compute $\frac{\partial w_i}{\partial n}$, w_i solution of equation (6) with
$C = C_i$ (22)

Construct C_{i+1} by modifying C_i normally or a quantity (23)

$$-\lambda\left\{\frac{\partial u_i}{\partial n}\left[\frac{\partial u_i}{\partial n} + 2\frac{\partial w_i}{\partial n}\right] - \int_{\partial C_i}\frac{\partial u_i}{\partial n} \times \left(\frac{\partial u_i}{\partial n} + 2\frac{\partial w_i}{\partial n}\right) \mathrm{d}(\partial C_i) \int_{\partial C_i} \mathrm{d}(\partial C_i)\right\}$$

Let $i = i + 1$ and go to equation (21) (24)

Algorithm (20)–(24) is nothing but a gradient method with constant step size. In fact to ensure convergence of the algorithm one should use a variable step size (a two line method, for instance, see Polak[9]). But this would be costly; in practice it is more feasible to keep λ fixed unless $E(C_{i+1}) > E(C_i)$ in which case λ should be halved.

For Reynolds number not too large ($100 < R < 100\,000$), one may use Prandtl's equations to compute u. The system (3) is then replaced by equations (8) and (9). Similarly w_i is computed from equation (10) instead of equation (6).

Discretization of problem (1)

Preliminaries

Remark 1. If C is slender, more precisely if its boundary layer does not separate ($\partial u_s/\partial n > 0$) the region $\hat{C} - C$ of $\Omega - C$ reduces to a thin layer (thickness $\sim R$) around C and in its wake. Since $\hat{C}$ is difficult to compute from C it is easier to iterate on $\hat{C}$ instead of C. The profile C can be deduced from C by subtracting normally the thickness of the boundary layer (a quantity which is easy to compute from a knowledge of u_s and u_n).

Remark 2. The constraint 'C slender' is hard to translate quantitatively. It imposes strong restrictions on C_0.

Remark 3. It is easier to compute (Q, r) instead of (w, q) where (Q, r) is defined by:

$$Q = u + 2w \quad r = p + 2q - \frac{1}{2}u\cdot u$$

it is easy to show that (Q, r) are solutions of:

$$\left.\begin{aligned}
&\nu\frac{\partial^2 Q_s}{\partial n^2} + u_s\frac{\partial Q_s}{\partial s} + u_n\frac{\partial Q_s}{\partial n} - \frac{\partial u_s}{\partial s}Q_s = \frac{\partial r}{\partial s}\\
&-\frac{\partial u_s}{\partial n}Q_s = \frac{\partial r}{\partial n}\\
&\frac{\partial Q_s}{\partial s} + \frac{\partial Q_n}{\partial n} = 0 \quad \text{in} \quad \hat{C} - C,\\
&Q_s = Q_n = 0 \quad \text{on} \quad \partial C, \quad Q_s = \frac{\mathrm{d}U}{\mathrm{d}s},\\
&\qquad\qquad r = 0 \quad \text{on} \quad \partial\hat{C}
\end{aligned}\right\} \qquad (25)$$

Proof. By adding twice equation (6) to (3) we deduce that:

$$\nu\nabla^2 Q + u\cdot\nabla(Q - u) - \nabla u\cdot(Q - u) + u\cdot\nabla u = \nabla(p + 2q)$$

which can be rewritten as:

$$\begin{aligned}\nu\nabla^2 Q + u\cdot\nabla Q - \nabla u\cdot Q &= \nabla(p + 2q) - \nabla u\cdot u\\ &= \nabla(p + 2q - \tfrac{1}{2}|u|^2) \qquad (26)\end{aligned}$$

Equation (25) can be deduced from equation (26) by the same singular perturbation technique which led to equation (10) from (6).

Discretization of the profile

The set $\mathscr{S}$ of admissible profiles is approached by a subset $\mathscr{S}_h$ as follows: $\sigma_1 = 0, \sigma_2, \ldots, \sigma_{N+1} = 1$, being $N + 1$ distinct points of $[0, 1]$ we set:

$$\partial C_h = \{\xi_h(\sigma) | \sigma \in [0,1], \quad \xi_h \in C^\circ[0,1]xC^\circ[0,1], \; \xi_h \text{ affine on } [\sigma_j, \sigma_{j+1}]\, j = 1, \ldots, N\}$$

where $h = \max_{j=1,\ldots,N} |\sigma_{j+1} - \sigma_j|$

It is also convenient to introduce: $\xi_j = \xi_h(\sigma_j)$. Note that $\xi_1 = \xi_{N+1}$ and that the arc $\xi_j\xi_{j+1}$ of ∂C_h is a straight segment. The normal of ∂C_h at ξ_j is approximated by the perpendicular at $\xi_{j-1}\xi_{j+1}$ running through ξ_j (see *Figure 1*); and we set:

$$s_j = \sum_{i=1}^{j-1} |\xi_i\xi_{i+1}|, \quad s_1 = 0$$

Discretization of the potential flow

The computation of the flow in $\Omega - \hat{C}$ is only an intermediary step for the computation of u in $\hat{C} - C$. In fact one must note that only the trace $\varphi|_{\partial\hat{C}}$ of φ on $\partial\hat{C}$ is needed [φ solution of equation (8)]. For this we have used the adaptation of Luu[4] of the method of singularities. We recall that the method is based on the use of Green's formula in $\Omega - C$ with the Green function of ∇^2, i.e. any function harmonic outside C and zero at infinity (so the following should be done with $\varphi - u_0 x$ and not φ) satisfies

$$2\pi\varphi(z) = \int_{\partial C}\left\{\frac{\partial\varphi}{\partial n}\log|z - \xi(\sigma)| - \varphi[\xi(\sigma)]\frac{\partial}{\partial n_\sigma}\log|z - \xi(\sigma)|\right\}\mathrm{d}(\partial C)$$

Let ∂C be as in the previous section and let ξ_j^c be the middle point of $\xi_j\xi_{j+1}$ then the formula above leads to:

$$\frac{\partial\varphi}{\partial s}(\xi_j^C) - i\frac{\partial\varphi}{\partial n}(\xi_j^C) = \sum_{l=1}^{N}\frac{q_l + i\gamma_l}{2\pi}\exp[i(\beta_j - \beta_l)]\log\left(\frac{\xi_j^C - \xi_l}{\xi_j^C - \xi_{l+1}}\right) \qquad (27)$$

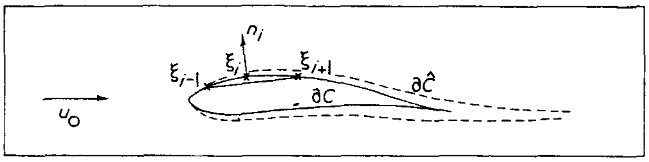

Figure 1

where β_j is the angle $(0x, \xi_j\xi_{j+1})$ and q_l, γ_l are constants that can be computed by solving the linear system formed by the imaginary part of equation (27) where $\partial\varphi/\partial n \cdot (\xi_j^C)$ is replaced by its value [given by equation (8)].

For symmetric profiles we take $\gamma_l = 0$*. Equation (27) becomes:

$$\mathscr{I} \sum_{l=1}^{N} \frac{q_l}{2\pi}\left(\exp[i(\beta_j - \beta_l)] \log\frac{\xi_j^C - \xi_l}{\xi_j^C - \xi_{l+1}}\right) = u_0 \sin\beta_j, \quad j = 1,\dots N \tag{28}$$

and

$$\frac{\partial\varphi}{\partial s}(\xi_j^C) = \sum_{l=1}^{N} \frac{q_l}{2\pi} Re\left[\exp[i(\beta_j - \beta_l)] \times \log\left(\frac{\xi_j^C - \xi_l}{\xi_j^C - \xi_{l+1}}\right)\right] + u_0 \cos\beta_j \tag{29}$$

Discretization of Prandtl's equations

In equation (9) we can use a reduced system of variable (see for example, Jones and Watson in Rosenhead[10]); thus it can be rewritten as:

$$\left.\begin{aligned} &-\frac{\partial^2 u_s}{\partial n^2} + u_s\frac{\partial u_s}{\partial s} + u_n\frac{\partial u_s}{\partial n} \\ &= U(s)\frac{\mathrm{d}U(s)}{s} \quad \text{in} \quad]0, +\infty[x]0, +\infty[\\ &\frac{\partial u_s}{\partial s} + \frac{\partial u_n}{\partial n} = 0 \quad \text{in} \quad]0, +\infty[x]0, +\infty[\end{aligned}\right\} \tag{30}$$

with the following boundary conditions:

$$\begin{aligned} &u(s) = U(s) \quad \text{if} \quad n = +\infty \\ &u_s = u_n = 0 \quad \text{at} \quad n = 0, \quad \text{if} \quad s \leqslant L \\ &u_s = 0, \quad \frac{\partial u_s}{\partial n} = 0 \quad \text{at} \quad n = 0 \quad \text{if} \quad s > L \end{aligned} \tag{31}$$

where L is the length at the upper edge of the profile $\partial\hat{C}$, wake not included. The true friction can be: recovered by multiplying the friction obtained from equation (30) by v.

For non-symmetrical profiles equation (30) must be solved on each side of C.

From a Falkner–Skan transformation (see Rosenhead[10], for example) one can show that the system (30) has a parabolic character in s and that it is self starting at $s = 0$. However, for stability it is better to add an extra boundary condition at $s = s_0 > 0$

$$u(n, s_0) = u_d(n) \qquad n \in]o, +\infty[$$

where u_d is computed from the Falkner–Skan equation (it can be found in Rosenhead[10] for a conical leading edge of angle 90°).

Then $]0, +\infty[x]0, +\infty[$ is approximated by $]0, \infty[]0, H[$, H sufficiently large and equations (30) and (31) are discretized by finite differences. Let M be a large integer; let $k = H/M$, k will be the discretization step in the normal direction. At the point $\{s_j, ik\}$ equations (30) are approximated by:

$$\left.\begin{aligned} &-\frac{u_{j+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{k2} + \frac{1}{2}\frac{(u_i^{j+1})^2 - (u_i^j)^2}{s_{j+1} - s_j} + \\ &v_i^{j+1}\frac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2k} = \frac{1}{2}\frac{(U^{j+1})^2 - (U^j)^2}{s_{j+1} - s_j} \\ &v_i^{j+1} = v_{i-1}^{j+1} - k\frac{u_i^{j+1} - u_i^j}{s_{j+1} - s_j}; \quad i = 2,\dots M-1, \\ &\qquad j \geqslant j_0 \end{aligned}\right\} \tag{32}$$

with the boundary conditions:

$$\begin{aligned} &v_1^j = 0, \quad u_M^j = U^j; \qquad u_1^j = 0 \quad \text{if} \quad s_j \leqslant L, \\ &\qquad\qquad\qquad\qquad\quad u_1^j = u_2^j \quad \text{if} \quad s_j > L \\ &u_i^{j_0} = u_d^i, \quad i = 1,\dots M \end{aligned} \tag{33}$$

Iterative solution of systems (32) and (33)

Owing to the particular form of systems (32) and (33) and to the fact that $\{u_i^1\}_{1\leqslant i\leqslant M}$ is known, it is natural to solve it by a step-by-step method in which $\{u_i^j\}, 1 \leqslant i \leqslant M, j \geqslant 1$ being known, $\{u_i^{j+1}\}_{1\leqslant i\leqslant M}$ is determined by the following iterative process:

With self explanatory notations, for a given $\{u_i^j\}_{1\leqslant i\leqslant M}$, the system (32) and (33) is of the form:

$$\left.\begin{aligned} g_1(x, y) = 0 \\ g_2(x, y) = 0 \end{aligned}\right\} \tag{34}$$

with $x, y, \in R^M$, $\{u_i^{j+1}\}$ being x, $\{v_i^{j+1}\}$ being y.

To solve equation (34) we may use the following method of surrelaxation:

$$(x^o, y^o) \text{ given } \omega_1 > 0 \tag{35}$$

(x^m, y^m) given, compute (x^{m+1}, y^{m+1}) by:

$$g_1^i(\bar{x}_1^{m+1}, \dots, \bar{x}_i^{m+1}, \dots, y^m) = 0 \quad 1 \leqslant i \leqslant M \tag{36}$$

$$x_i^{m+1} = x_i^m + \omega_1(\bar{x}_i^{m+1} - x_i^m) \tag{37}$$

$$g_2(x^{m+1}, y^{m+1}) = 0 \tag{38}$$

Note that the computation of $\bar{x}_i^{m+1}$ from equation (36) amounts to finding the largest root of an equation of degree 2. Note also that the computation of y^{m+1} from equation (38) is also straightforward because it amounts to the resolution of a didiagonal triangular linear system.

The initialization in equation (35) is made by using u_d^i to compute v_i^1 according to equation (38).

The convergence of the method (35)–(38) is established theoretically only for certain monotone operators (see Cea–Glowinski[11]).

Discretization of the adjoint system (25)

With the same reduced system of coordinates, v can be eliminated and $\hat{C} - C$ transformed into a quarter of a plane. System (25) becomes.

$$\begin{aligned} &\frac{\partial^2 Q_s}{\partial n^2} + u_s\frac{\partial Q_s}{\partial s} + u_u\frac{\partial Q_s}{\partial n} + \frac{\partial u_n}{\partial n}Q_s = \frac{\partial_r}{\partial s} \\ &-\frac{\partial w}{\partial n}Q_s = \frac{\partial r}{\partial n} \quad \text{in} \quad]0, +\infty[x]0, +\infty[\end{aligned} \tag{39}$$

*In the general case γ_l is chosen so as to satisfy the Kutta-Joukowsky condition, at the cusp.

with the boundary conditions:

$$\left.\begin{aligned} &Q_s(n=0)=0 \quad \text{if} \quad s<L, \quad \frac{\partial Q_s}{\partial n}(n=0)=0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad s>L \\ &Q_s(n=+\infty)=\frac{\partial\varphi}{\partial s}(s), \quad r(n=+\infty)=0 \\ &Q_s(s=+\infty)=u_s(s=+\infty) \end{aligned}\right\} \quad (40)$$

A discretization of system (39) independent of the one of system (40) may introduce serious numerical instabilities. To discretize system (40) we note that system (39) is the adjoint (in the sense of differential operators) of equation (30); this is not obvious *a priori* since it was their 'mother' equations (3) and (6) that were adjoint systems. To obtain the discrete adjoint of equation (32) we introduce μ_i^j (i.e. $\partial u_s/\partial n$ in the continuous case) and this is equivalent to:

$$\left.\begin{aligned} &\frac{1}{k}(\mu_{i+1}^{j+1}-\mu_i^{j+1})=v_i^{j+1}\frac{\mu_{i+1}^{j+1}+\mu_i^{j+1}}{2}+ \\ &\qquad \frac{1}{2}\,\frac{(u_i^{j+1})^2-(u_i^j)^2-(U^{j+1})^2+(U^j)^2}{s_{j+1}-s_j} \\ &\frac{1}{k}(u_i^{j+1}-u_{i-1}^{j+1})=\mu_i^{j+1} \\ &\frac{1}{k}(v_i^{j+1}-v_{i-1}^{j+1})=\frac{u_i^{j+1}-u_i^j}{s_{j+1}-s_j}, \quad i=2,\ldots \\ &\qquad\qquad\qquad\qquad M-1, \quad j>j_0 \end{aligned}\right\} \quad (41)$$

The Hamiltonian of equation (41) is:

$$\left.\begin{aligned} \mathscr{H}=\sum_{i,j}Q_i^j&\left[\frac{1}{k}(\mu_{i+1}^{j+1}-\mu_i^{j+1})-v_i^{j+1}(\mu_{i+1}^{j+1}+\mu_i^{j+1})-\right. \\ &\left.\frac{(u_i^{j+1})^2-(u_i^j)^2+(U^j)^2-(U^{j+1})^2}{2(s_{j1}-s_j)}\right]+ \\ &g_i^j\left[\frac{1}{k}(u_i^{j+1}-u_{i-1}^{j+1})-\mu_i^{j+1}\right]+ \\ &r_i^j\left[\frac{1}{k}(v_i^{j+1}-v_i^{j+1})+\frac{(u_i^{j+1}-u_i^j)}{s_{j+1}-s_j}\right] \end{aligned}\right\} \quad (42)$$

By partial derivation of $\mathscr{H}$ we obtain:

$$\begin{aligned} \frac{\partial\mathscr{H}}{\partial\mu_i^{j+1}}=-\frac{1}{k}Q_i^j+\frac{1}{k}Q_{i-1}^j-\frac{1}{2}Q_i^jv_i^{j+1}-& \\ \frac{1}{2}Q_{i-1}^jv_{i+1}^{j+1}-g_i^j& \end{aligned} \quad (43)$$

$$\frac{\partial\mathscr{H}}{\partial v_i^{j+1}}=-\frac{1}{2}Q_i^j(\mu_{i+1}^{j+1}+\mu_i^{j+1})+\frac{1}{k}(r_i^j-r_{i+1}^j) \quad (44)$$

$$\begin{aligned} \frac{\partial\mathscr{H}}{\partial u_i^{j+1}}=\frac{Q_i^ju_i^{j+1}}{s_{j+1}-s_j}+\frac{Q_i^{j+1}u_i^{j+1}}{s_{j+2}-s_{j+1}}+& \\ \frac{g_i^j-g_{i+1}^j}{k}+\frac{r_i^j}{s_{j+1}-s_j}-\frac{r_i^{j+1}}{s_{j+2}-s_{j+1}}& \end{aligned} \quad (45)$$

The discrete adjoint of equation (42) is obtained from equations (43)–(45) by:

$$\frac{\partial\mathscr{H}}{\partial\mu_i^{j+1}}=0, \quad \frac{\partial\mathscr{H}}{\partial v_i^{j+1}}=0, \quad \frac{\partial\mathscr{H}}{\partial u_i^{j+1}}=0 \quad (46)$$

Hence, after elimination of μ_i^j:

$$\left.\begin{aligned} &\frac{Q_{i-1}^j-2Q_i^j+Q_{i+1}^j}{k^2}+\left(\frac{Q_i^{j+1}}{s_{j+2}-s_{j+1}}-\right. \\ &\left.\frac{Q_i^j}{s_{j+1}-s_j}\right)u_i^{j+1}+\frac{Q_{i+1}^jv_{i+1}^{j+1}-Q_{i-1}^jv_{i-1}^{j+1}}{2k}=0 \\ &\frac{r_i^{j+1}}{s_{j+2}-s_{j+1}}-\frac{r_i^j}{s_{j+1}-s_j}-Q_i^j\frac{u_{i+1}^{j+1}-u_{i-1}^{j+1}}{2k} \\ &=\frac{r_{i+1}^j-r^j}{k} \end{aligned}\right\} \quad (47)$$

with the boundary conditions:

$$\begin{aligned} &Q_M^j=U_M^j, \quad r_M^j=0; \quad Q_1^j=0 \quad \text{if} \quad s_j\leqslant L, \\ &\qquad\qquad\qquad\qquad\quad Q_1^j=Q_2^j \quad \text{if} \quad s_j>L \\ &Q_i^J=u_i^J \quad (J \text{ large positive integer}) \end{aligned}$$

Iterative solution of system (46) and (47)

The system (46) and (47) is similar to (32); it is a parabolic linear discrete system which can by integrated step by step in the direction of *decreasing j only*, starting from $j=J$.

To solve system (46) and (47), we use a method of surrelaxation analogous to the one used for systems (32) and (33) with a surrelaxation parameter ω_2.

Description of the descent method for the discretized problem

Choose C_0 slender, axisymmetric, of given area, a, with angular leading edge of angle 90° and with a rear end shape like a cusp, set $i_g=0$ (48)

Compute $U^j=\frac{\partial\varphi}{\partial s}(\xi_j^c)$ by equation (29), q_1 being determined by solving the linear equation (28). Compute $\{s_j\}$ and $L=s_{N+1}$ (49)

Compute $\{u_i^j\}$, $\{v_i^j\}$, $i=1,\ldots,M$, $j=j_0,\ldots,J$ by solving equations (32) and (33) by surrelaxation of parameter ω_1 (50)

Compute $Q_i^j i=1,\ldots,M$ $j=j_0,\ldots,J$ by solving equation (46), by surrelaxation of paramater ω_2 (51)

$$\left.\begin{aligned} &\text{Compute grad}\,E_j=\frac{u_2^j-u_1^j}{k}\times\frac{Q_2^j-Q_1^j}{k} \quad \text{and} \\ &K_1=\frac{1}{L}\sum_{j=1}^{N'}(s_{j+1}-s_j)\,\text{grad}\,E_j, \\ &K_2=\sum_{j=1}^{N'}(s_{j+1}-s_j)(\text{grad}\,E_j-K_1)^2 \\ &\text{and} \quad F=\sum_{j=1}^{N'}(s_{j+1}-s_j)\left(\frac{u_2^j-u_1^j}{k}\right) \end{aligned}\right\} \quad (52)$$

if $K_2\leqslant\varepsilon'$ stop. (53)

if $K_2>\varepsilon'$ compute C_{i_g+1} by normal displacement of C_{i_g} of a quantity $\lambda(\text{grad}\,E_j-K_1)$ (see earlier section set $i_g=i_g+1$ and return to (49) (54)

Remark. In algorithm (48) and (54) the leading edge of the profile is maintained angular of angle 90°. Similarly the cusp at the rear end is maintained artificially.

Description of the results

Description of the example treated

We looked for the axisymmetric profile of given area and smallest drag in a uniform incompressible laminar viscous flow at Reynolds numbers between 100 and 10^5.

Numerical value of the parameters

From the homogeneity of the formula we can choose $u_0 = 1$, $\nu = 1$, $\rho = 1$.

The initial profile chosen is an approximation of C_0 defined by:

$$\partial C_0 = \{\xi \in R^2 | \xi_2 = 0.1(1 - \xi_1^2)(0.9\xi_1 + 1.1) \quad \text{if} -1 < \xi_1 < 0.9$$

and a 90° wedge connected smoothly if $\xi_1 \in [0.9, 0.95]$. The profile ∂C_0 was approximated by the polygonal line passing through the points of ∂C_0 with abscissa:

$$\xi_1 = 1 - 2\left(\frac{i-1}{N'}\right), \quad i = 1, \ldots, N'$$

where

$$\alpha_i = 1.7 - 0.7\left(\frac{i-1}{N'}\right) \quad i = 1, \ldots, N'$$

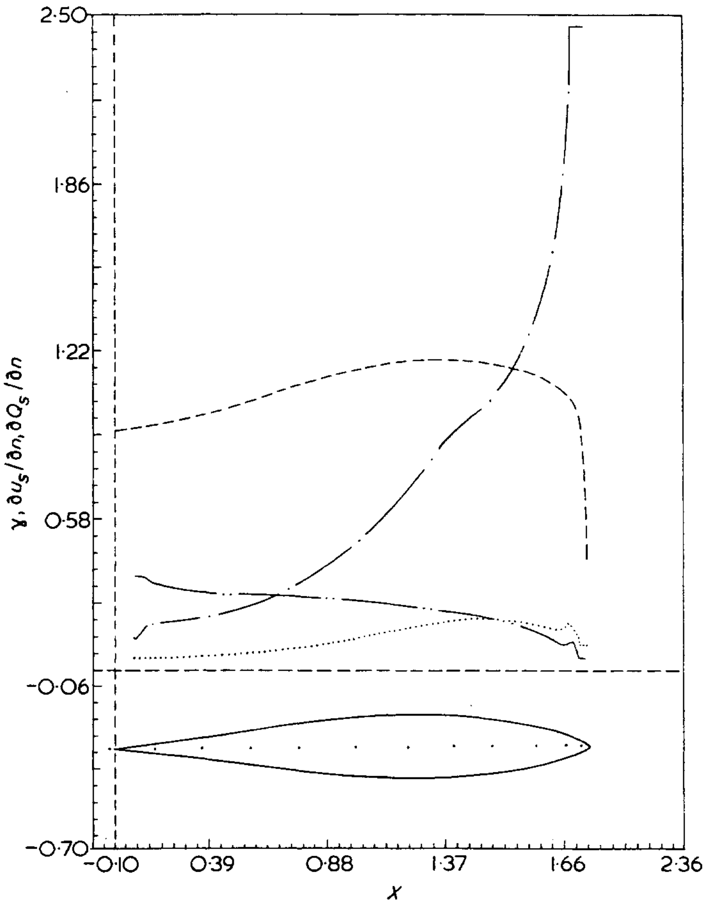

Figure 2 The tangential speed $U(s)$ (– – –) in the potential region on C_3, the skin friction $\partial u_s/\partial n$ (—·—·—) the adjoint skin friction $\partial Q_s/Qn$ (—··—··—) and the gradient $(\partial u_s/\partial n)$ $(\partial Q_s/\partial n)$ (······) on the profile C_3 (———) are shown. The profile C_3 is drawn on the bottom part of the Figure

and $N' = 47$. The area of C_0 is $a = 0.147$ and its length $L = 1.98$. Its drag, $(\rho/u_0)\cdot[E(C_0)]$ is 2.66 and we have $K_1 = 0.102$ and $K_2 = 0.5 \times 10^{-2}$.

The parameters of the discretization grid for the boundary layer are: $M = 250$, $k = 0.035$, $J = 58$, $J_0 = 4$. We choose the surrelaxation parameter to be $w_1 = 1.5$, $w_2 = 1.2$ and for the stopping rule: $\varepsilon = 1.5 \times 10^{-5}$. The step size was chosen equal to 0.04.

Description of the results

After 1 min and 30 sec on an IBM 370/168 (3 iterations of the algorithm!) we obtained the profile C_3 and the curves of *Figures 2, 3,* and *4*. The value of

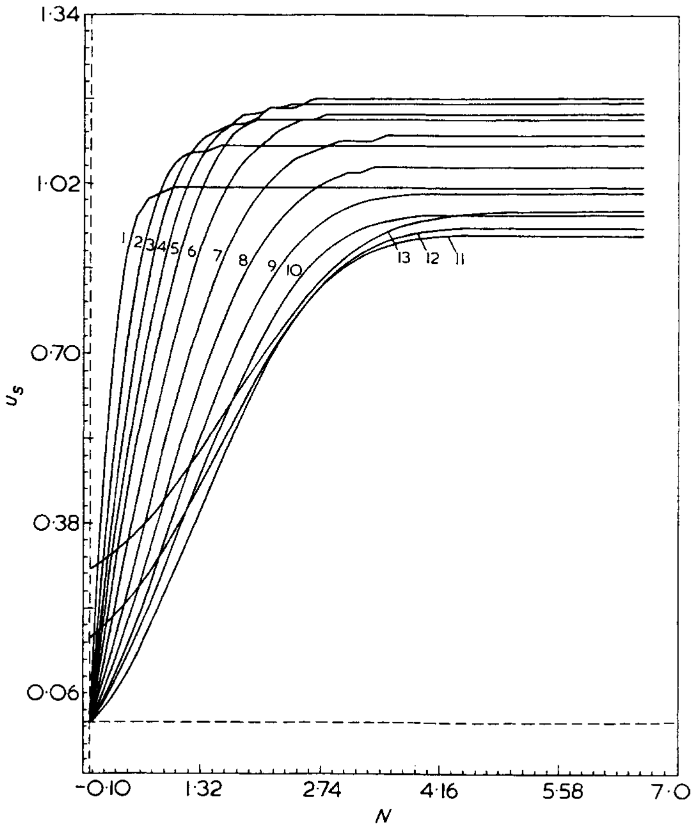

Figure 3 The tangential $u_s(n, s)$ in the boundary layer of C_3 is shown for values of s corresponding to • of *Figure 2*

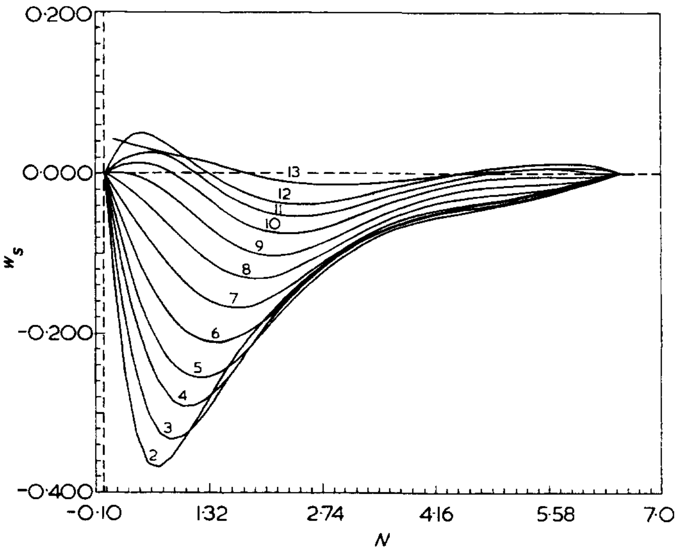

Figure 4 The tangential co-state $w_s(n, s)$ in the boundary layer of C_3 is shown for 12 values of s corresponding to the 12 first • of *Figure 2*

$E(C_3)$ is not significantly smaller than $E(C_0)$ (we shall come back to this point in the last section). However, grad E_j is more uniform (especially towards the front end, cf. *Figures 2* and *5*), and we now have $K_1 = 0.109$ and $K_2 = 0.4 \times 10^{-2}$.

Another computation was done on a profile obtained from C_0 by adding at its rear end a semi-infinite flat plate in order to study the effect of the discontinuity at the cusp. The results are shown in *Figures 5, 6* and *7*.

Analysis of the results

With the parameter chosen, we can estimate that the precision on $u, \partial u/\partial n$ is around 10^{-3} which gives a precision of 2×10^{-2} for the drag. The precision on $Q, \partial Q/\partial n$ is around 5×10^{-2} (this can be seen in *Figure 4* because w should be asymptotic to zero; it is not so because the relation $Q = 2w + u$ is not preserved by the discretization); therefore, the precision on K_1 is 5×10^{-2}. Those precisions are not sufficient to improve on C_3, in particular at the rear end where $(\partial u/\partial n) \times (\partial Q/\partial n)$ is small.

Nevertheless we note the following:

(1) If the initial profile is longer than C_3 the algorithm generates shorter profiles.
(2) The profiles C_3 is longer than C_0.
(3) Grad $E(\sigma)|_{C_3}$ is almost constant at the front of C_3 ($\xi_1 > 0.30$).

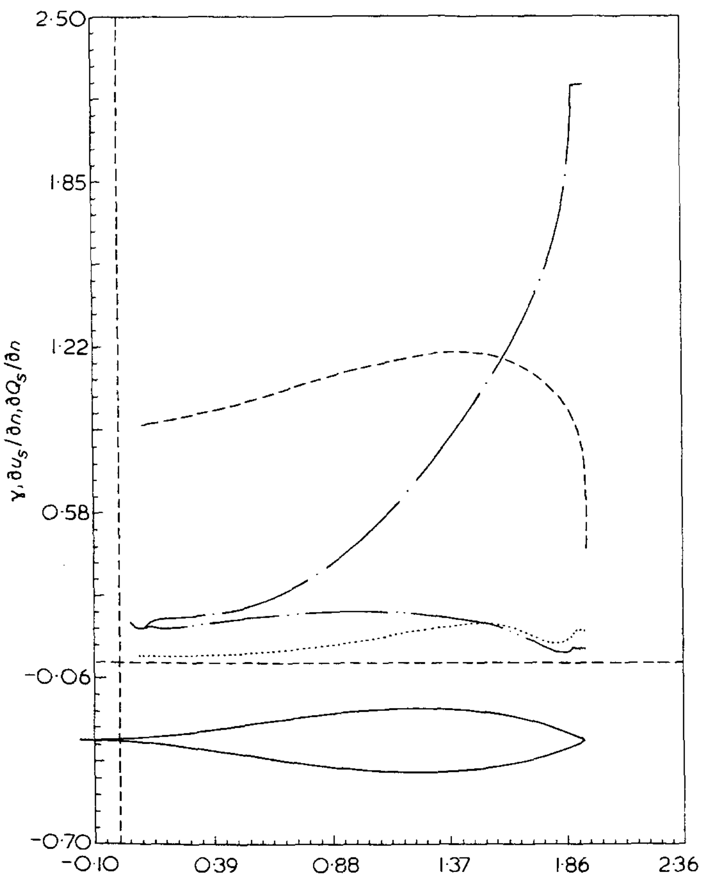

Figure 5 Tangential speed $U(s)$ (————) in the potential region on C_0 augmented with a flat plate in the axis of its wake, the skin friction $\partial u_s/\partial n$ (—·—·), the adjoint skin friction $\partial Q_s/\partial n$ (— · — · —) and the gradient $(\partial u_s/\partial n)$ $(\partial Q_s/\partial n)$ (··· ··) on the profile C_0 (————) are shown

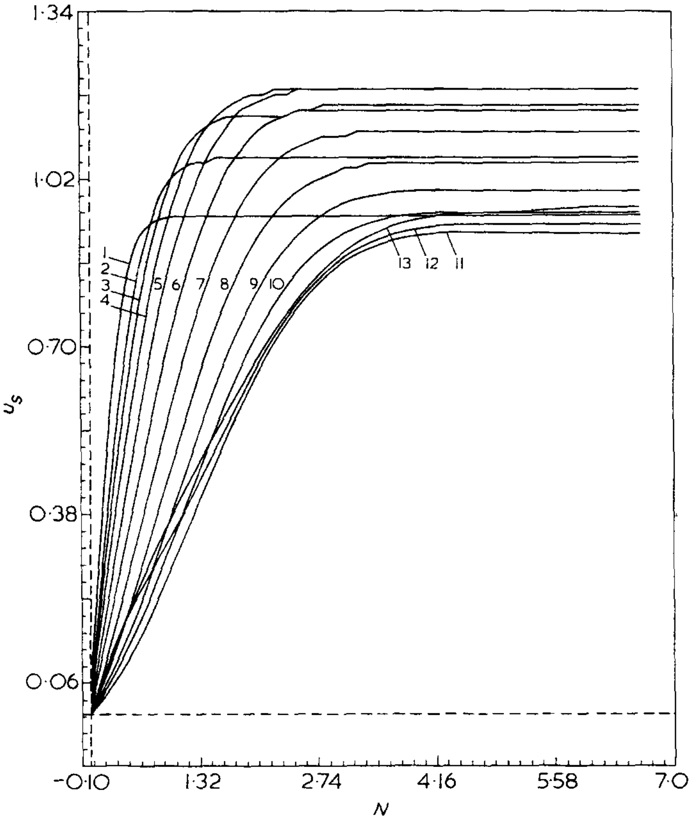

Figure 6 Tangential $u_s(ns)$ in the boundary layer of C_0 is shown for values of s corresponding to the • of *Figure 5*

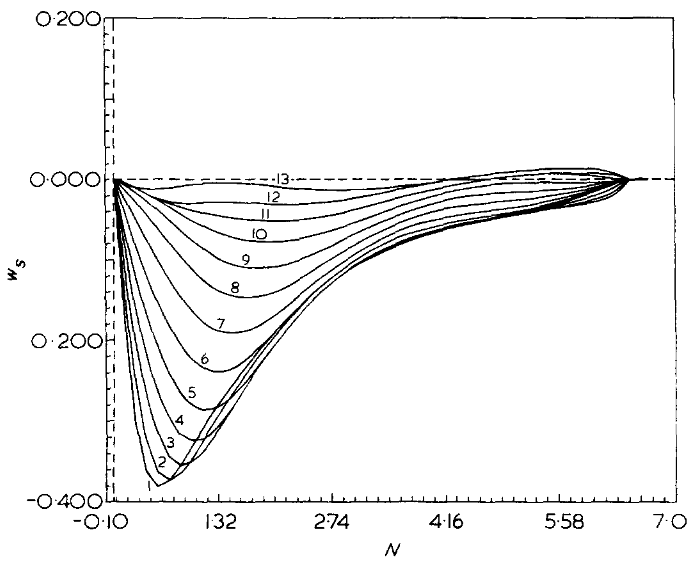

Figure 7 Tangential co-state w_s (n, s) in the boundary layer of C_0 is shown for 13 values of s corresponding to the 13 first • of *Figure 5*

From this we estimate that it is not possible to improve on C_3 (C_3 is already in the 'numerical noise') and that C_3 does not differ by more than 5% from the true optimum.

Discussion

Computation of the optimum profiles for other surface areas and other Reynolds numbers

Let $C_3'(z, \gamma)$ be the transform of C_3 by a scaling of centre z and ratio γ. One can show that there exists $T(z, z')$ such that $C_3'(z, \gamma)$ can be obtained by translating

$C'_3(z,\gamma)$ of a vector T. Therefore the potential flows around $C'_3(z',\gamma)$ and around $C'_3(z,\gamma)$ are identical and furthermore by homogeneity the tangential speed $U(s)$ along $C'_3(z,\gamma)$ is γ times the tangential speed along C_3.

From the homogeneity of the boundary layer equations we deduce that *if* C_3 *is optimal then all* $C'_3(z,\gamma)$ *are optimal in their class*, and their drag is:

$$F(C'_3) = 1.33\left(\frac{Lu_0}{\nu}\right)\rho L u_0^2 \tag{55}$$

where L is the length of C'_3. The surface area of C'_3 is $0.035\,L^2$. Equation (55) should be compared with equation V.12.89 of Rosenhead[10]. Note that the drag of C_3 is very close to the drag of the flat plate of same length L (1.328 instead of 1.33).

The profiles C_3 do not depend on the Reynolds number $R = Lu_0/\nu$. However, the boundary layer thickness depends on R; hence C_3 depends on R. Note, however, that since the boundary layer is very thin at the front end, the shape of the front ends of the optimum profiles do not depend on R or on a; it is a wedge of angle 90°. This result is also true for three-dimensional axisymmetric profiles; their front end is conical of angle 111°.

Conclusion

Owing to the lack of powerful means of computation we have not been able to improve noticeably on the initial profile C_0, obtained by guessing. The numerical complexity makes each iteration of the algorithm very costly and it was also not possible to demonstrate the descent method. Thus our work was rather to show that C_0 is close to the optimum, that it can be improved slightly, and that there is no hope of finding other shapes dramatically different from C_0 and with a much smaller drag in laminar flow.

We have also shown that the method is feasible and how to implement it. Lastly, a computation of the costate vector w (or Q) was done. Any fluid mechanics laboratory now has all the necessary information to decide on the feasibility of the method for the solution of other optimum design problems they might be faced with.

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