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On the Navier–Stokes system with pressure boundary condition

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Abstract In this paper we prove the local existence and uniqueness of the weak solution to the non-stationary 2D Navier–Stokes system with pressure boundary condition in a bounded domain.

Keywords Navier–Stokes system · Oseen system · Pressure boundary condition · Existence · Uniqueness

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1 Introduction

It is well known that the Navier–Stokes system, in a bounded two or three dimensional domain, with prescribed velocity on the boundary has a solution (see, e.g. [15], tough main ideas go back to Leray [8]). In some real-life situations it is natural to prescribe the value of the pressure on some part of the boundary, as for instance in case of pipelines, blood vessels, different hydraulic systems involving pumps etc. Pressure boundary conditions represent such things as confined reservoirs of fluid, ambient laboratory conditions and applied pressures arising from mechanical devices. Thus, it seams perfectly reasonable to impose the pressure boundary condition for the Navier–Stokes system.

Unfortunately one cannot prescribe only the value of the pressure on the boundary, since such problem is known to be ill-possed. This is no surprise since, contrary to the above physically motivated arguments, prescribing the pressure on the boundary does

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not seem to be reasonable from mathematical point of view. Indeed, mathematically, the role of the pressure in an incompressible fluid is rather peculiar, as it is only a Lagrange multiplier in the incompressible Navier–Stokes system, allowing to keep the velocity divergence free at any point. Yet, it is a quantity that has a clear physical meaning.

In addition to the value of the pressure, we need to prescribe something more on the boundary. At least two mathematically acceptable boundary conditions, that include prescribing the pressure, can be found in the literature. First condition is prescribing the whole normal stress, including the viscous part, (see e.g. [6]) on the boundary. Second is to prescribe the pressure and the tangential part of the velocity on the boundary, at the same time. In the present paper we treat only the second case. More precisely, we prescribe the pressure and take the tangential velocity to be zero on one part the boundary, while we keep the no-slip condition on the rest of the boundary. The first case (prescribing the normal stress) can be treated using the same methods, with a slight change of functional space in variational formulation. The boundary condition that we have chosen has been used in many papers (both numerical and theoretical) as for instance in [2,3,5,7,11,12] or [14] (the list is by no means exhaustive).

However, for now, there are only few theoretical papers on the solvability for such problem and the full proof of existence and uniqueness for the Navier–Stokes system with such boundary data is unknown. The difficulties come from the fact that, since we only have that the tangential part of the velocity vanishes on the boundary and we have no information about the normal part, we only get

$$\int_{\Omega} (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{u} = \frac{1}{2} \int_{\partial\Omega} |\mathbf{u} \cdot \mathbf{n}|^2 \, (\mathbf{u} \cdot \mathbf{n})$$

where the last integral is not given and we do not know how to control it.Thus we loose the a priori estimates that are essential for the Leray's approach.

One way out was offered by Conca, Murat and Pirroneau in [2] by imposing the value of the, so called, dynamic pressure $p_d = p + |\mathbf{u}|^2/2$ in stead of the pressure p. This solves all the mathematical difficulties, but, from the physical point of view,it would be more interesting to prescribe the pressure itself.

If the velocity is not too large then another way out can be found by considering the linearized (Stokes) equations, obtained by neglecting the effects of inertia. Then the problem is easy to solve (see [2] for the stationary system or [11] for the evolutional system in non-cylindric domain).

Our first goal here is to prove the existence under the small data condition (or, equivalently, for short time).

On the other hand, it is well-known, in 2-dimensional evolutional case, J.L. Lions and G. Prodi [10] were able to prove the general uniqueness of the solution for the Navier–Stokes system with Dirichlet's boundary data, due to the fact that we can control the inertial term (see, [10] and also [15]). This technique is almost independent on the boundary condition and can be applied here, with small modifications, to extend the result on pressure boundary condition. In case of small data our result means that the solution whose existence we have proved is, in fact, the only one. The same kind of difficulties appear in case when the whole normal stress is prescribed on the boundary or in case of flux boundary condition studied in [6]. The same method works in those situations as well.

2 Existence and uniqueness of a weak solution

2.1 The problem

Let Ω be a bounded locally Lipschitz domain in $\mathbf{R}^2$, with boundary $\Gamma = \Gamma^1 \cup \Gamma^2$ and let $T > 0$. We assume that $\Gamma^1 \neq \emptyset$ has a positive 1 dimensional measure and that $\Gamma^1 \cap \Gamma^2 = \emptyset$. We assume in addition that Γ^i, $i = 1, 2$ are both locally Lipschitz hyper-manifolds in $\mathbf{R}^2$. We denote by $\mathbf{n}$ a unit exterior normal on $\partial\Omega$.

Let $T > 0$ and let $\Omega_T = \Omega \times]0, T[$ Our system then reads:

$$\frac{\partial \mathbf{u}}{\partial t} - \mu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega_T \tag{2.1}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega_T \tag{2.2}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma^1_T, \tag{2.3}$$

$$p = q, \ \mathbf{u} \wedge \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma^2_T, \tag{2.4}$$

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0 \quad \text{in } \Omega, \tag{2.5}$$

were $\Gamma^i_T = \Gamma^i \times]0, T[$, $i = 1, 2$ $\mu > 0$.

We assume that $q \in L^2(0, T; H^{-1/2}(\Gamma^2))$, $\mathbf{u}_0 \in L^2(\Omega)^2$ and $\mathbf{f} \in L^2(0, T; V')$, where we have denoted by

$$V = \{\mathbf{z} \in H^1(\Omega)^2;\ \operatorname{div} \mathbf{z} = \mathbf{0} \text{ in } \Omega,\ \mathbf{z} = \mathbf{0} \text{ on } \Gamma^1,\ \mathbf{z} \wedge \mathbf{n} = \mathbf{0} \text{ on } \Gamma^2 \}$$

and by V' it's dual space. Here and in the sequel, for two vectors $\mathbf{a}, \mathbf{b} \in \mathbf{R}^2$ the symbol $\wedge$ denotes

$$\mathbf{a} \wedge \mathbf{b} = a_1 b_2 - a_2 b_1.$$

The space V is equipped by the norm

$$|\mathbf{z}|_V = |\nabla \mathbf{z}|_{L^2(\Omega)},$$

equivalent to the standard $H^1(\Omega)^2$-norm, due to the homogeneous boundary condition on Γ^1.

2.2 Preliminary notations and technical results

Before we proceed we need to introduce some notations convenient for our situation and to recall some simple formulae from vector calculus.

We denote for a vector function $\mathbf{v}$ a scalar value

$$\nabla \wedge \mathbf{v} = \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}, \quad \mathbf{v} = (v_1, v_2).$$

We also denote for a scalar function φ a vector value

$$\mathbf{curl}\, \varphi = \left(-\frac{\partial \varphi}{\partial x_2}, \frac{\partial \varphi}{\partial x_1} \right).$$

Using the above notations we easily verify that

$$\mathbf{curl}\, (\nabla \wedge \mathbf{v}) = -\Delta \mathbf{v} + \nabla(\operatorname{div} \mathbf{v}). \tag{2.6}$$

We also need a partial integration formulae

$$\int_{\Omega} \mathbf{w} \cdot \mathbf{curl}\, u = \int_{\Omega} u \, (\nabla \wedge \mathbf{w}) - \int_{\partial\Omega} u \, (\mathbf{w} \wedge \mathbf{n}). \tag{2.7}$$

That formula is valid for all functions $\mathbf{w} \in H^1(\Omega)^2$ and $u \in H^1(\Omega)$. It's weaker version, where the integrals are replaced by the appropriate duality brackets, holds for $\mathbf{w} \in V$ and $u \in L^2(\Omega)$

$$_V\langle \mathbf{w} \,|\, \mathbf{curl}\, u \rangle_{V'} = \int_{\Omega} u \, (\nabla \wedge \mathbf{w}). \tag{2.8}$$

Using (2.6), the system (2.1)–(2.5) can now be written as

$$\frac{\partial \mathbf{u}}{\partial t} - \mu \, \mathbf{curl}\, (\nabla \wedge \mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega_T \tag{2.9}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega_T. \tag{2.10}$$

Before we proceed we recall the well-known technical result:

Lemma 2.1 *For any $\mathbf{z} \in V$ there exists a constant $C(L^4, H^1) > 0$ such that*

$$|\mathbf{z}|_{L^4(\Omega)} \leq C(L^4, H^1) |\nabla \mathbf{z}|_{L^2(\Omega)}^{1/2} \, |\mathbf{z}|_{L^2(\Omega)}^{1/2}. \tag{2.11}$$

For the proof of (2.11) see e.g. Exercise II 2.9. in [4].

It is also convenient to introduce an equivalent norm on V. It is easy to see that the norm

$$\|\mathbf{v}\|_{H^1(\Omega)} = |\mathbf{v}|_{L^2(\Omega)} + |\operatorname{div} \mathbf{v}|_{L^2(\Omega)} + |\, \nabla \wedge \mathbf{v}|_{L^2(\Omega)}$$

is equivalent to the standard $H^1(\Omega)^2$ norm. Furthermore, since a function $\mathbf{w} \in V$ is divergence free and equals zero on Γ^1, there exists a constant $C_\# > 0$ such that

$$\frac{1}{C_\#}|\nabla \mathbf{w}|_{L^2(\Omega)} \le |\nabla \wedge \mathbf{w}|_{L^2(\Omega)} \le C_\# |\nabla \mathbf{w}|_{L^2(\Omega)}. \tag{2.12}$$

We also need the following lemma:

Lemma 2.2 *There exists a constant $C_\mu > 0$ such that for any $\mathbf{u}, \mathbf{v} \in H^1(\Omega)^2$ the following inequality holds*

$$\left| \int_\Omega (\mathbf{v} \cdot \nabla) \mathbf{u} \cdot \mathbf{u} \right| \le \frac{\mu}{2\,C_\#}|\nabla \mathbf{u}|^2_{L^2(\Omega)} + C_\mu |\mathbf{v}|^4_{L^4(\Omega)}\ |\mathbf{u}|^2_{L^2(\Omega)}, \tag{2.13}$$

with $C_\# > 0$ the constant from (2.12).

Proof Using (2.11) and Young's inequality we obtain

$$\begin{aligned} \left| \int_\Omega (\mathbf{v} \cdot \nabla) \mathbf{u} \cdot \mathbf{u} \right| &\le |\mathbf{v}|_{L^4(\Omega)} |\nabla \mathbf{u}|_{L^2(\Omega)} |\mathbf{u}|_{L^4(\Omega)} \le |\mathbf{v}|_{L^4(\Omega)} |\nabla \mathbf{u}|^{3/2}_{L^2(\Omega)} |\mathbf{u}|^{1/2}_{L^2(\Omega)} \\ &\le \frac{\mu}{2C_\#}|\nabla \mathbf{u}|^2_{L^2(\Omega)} + C_\mu |\mathbf{v}|^4_{L^4(\Omega)}\ |\mathbf{u}|^2_{L^2(\Omega)}. \end{aligned}$$

□

2.3 The main result

We arrive at following weak formulation of the problem:

Definition 2.1 We say that

$$\mathbf{u} \in L^2(0,T;V) \cap L^\infty(0,T;L^2(\Omega))^2$$

is a weak solution to the problem (2.1)–(2.5) if

$$\begin{aligned} &-\int_{\Omega_T} \mathbf{u} \cdot \frac{\partial \mathbf{w}}{\partial t} + \mu \int_{\Omega_T} (\nabla \wedge \mathbf{u}) \cdot (\nabla \wedge \mathbf{w}) + \int_{\Omega_T} (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{w} = \int_0^T {}_{V'}\left\langle f \,\middle|\, \mathbf{w} \right\rangle_V \\ &+ \int_0^T {}_{H^{-1/2}(\Gamma^2)}\left\langle q \,\middle|\, (\mathbf{w} \cdot \mathbf{n}) \right\rangle_{H^{1/2}(\Gamma^2)} + \int_\Omega \mathbf{u}_0 \cdot \mathbf{w}(\cdot, 0) \\ &\forall\, \mathbf{w} \in L^2(0,T;V) \cap H^1(0,T;L^2(\Omega))^2 \quad \text{such that } \ \mathbf{w}(\cdot, T) = \mathbf{0}. \end{aligned}$$

We prove that:

Theorem 2.1 *Let*

$$q \in L^2(0, T; H^{-1/2}(\Gamma^2)), \quad \mathbf{u}_0 \in L^2(\Omega)^2 \quad and \quad \mathbf{f} \in L^2(0, T; V'). \tag{2.14}$$

Then

- *The weak solution to the problem* (2.1)–(2.5) *is unique.*
- *Let* $C(H^{1/2}(\Gamma^2), V) > 0$ *be such that*

$$|z|_{H^{1/2}(\Gamma^2)} \leq C(H^{1/2}(\Gamma^2), V)\, |z|_V, \quad \forall z \in V$$

and let $C_\# > 0$ *be the constant from* (2.12). *Then there exists a constant* $M > 0$ *such that if*

$$\frac{1}{2}|u_0|^2_{L^2(\Omega)} + \frac{C_\#}{\mu}|f|_{L^2(0,T;V')} + C(H^{1/2}(\Gamma^2), V)\, |q|_{L^2(0,T;H^{-1/2}(\Gamma^2))} \leq M, \tag{2.15}$$

the problem (2.1)–(2.5) *has a weak solution*

$$\mathbf{u} \in L^2(0, T; V) \cap L^\infty(0, T; L^2(\Omega))^2.$$

We separate the proof in two parts, the existence (Proposition 3.1) and the uniqueness (Proposition 4.1).

3 Existence of the solution

As explained, we first prove the existence of a weak solution. We prove the following result:

Proposition 3.1 *Assume that the conditions of Theorem* 2.1 *hold, with*

$$M = \min\{2\, C(L^4, H^1)^{-2}\, C_\#^{-2}\, \mu^2 3^{-3/2}, 4mC(L^4, H^1)^{-4} 3^{-2}\},$$

where $C(L^2, H^1) > 0$ *is the embedding constant from* (2.11) *and* $m = min\{\frac{\mu}{C_\#}, 1\}$. *Then the problem* (2.1)–(2.5) *admits at least one weak solution.*

To prove that result, we start by discussing the existence of the appropriate linearized system of the Oseen type:

$$\frac{\partial \mathbf{u}}{\partial t} - \mu \Delta \mathbf{u} + (\mathbf{v} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega_T \tag{3.1}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega_T \tag{3.2}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_T^1, \tag{3.3}$$

$$p = q, \ \mathbf{u} \wedge \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_T^2, \tag{3.4}$$

$$\mathbf{u}(\,\cdot\,, 0) = \mathbf{u}_0 \quad \text{in } \Omega, \tag{3.5}$$

Proposition 3.2 *Let the conditions (2.14) be satisfied and let* $\mathbf{v} \in L^4(\Omega^T)$. *Then the problem (3.1)–(3.5) has a unique weak solution*

$$\mathbf{u} \in L^2(0,T;V) \cap L^\infty(0,T;L^2(\Omega))^2.$$

Proof Let $C_\mu > 0$ be the constant given by Lemma 2.2 and let

$$\lambda(t) = C_\mu \int_0^t |\mathbf{v}(s)|^4_{L^4(\Omega)}\, ds.$$

We now define $\mathbf{w}(x,t) = e^{-\lambda(t)}\, \mathbf{u}(x,t)$, $\pi(x,t) = e^{-\lambda(t)}\, p(x,t)$. Now $(\mathbf{u}, p)$ solves (2.1)–(2.5) if and only if $(\mathbf{w}, \pi)$ solves the system

$$\frac{\partial \mathbf{w}}{\partial t} + \lambda'\, \mathbf{w} - \mu \Delta \mathbf{w} + (\mathbf{v} \cdot \nabla)\mathbf{w} + \nabla \pi = e^{-\lambda}\, \mathbf{f} \quad \text{in } \Omega_T \tag{3.6}$$

$$\operatorname{div} \mathbf{w} = 0 \quad \text{in } \Omega_T \tag{3.7}$$

$$\mathbf{w} = \mathbf{0} \quad \text{on } \Gamma_T^1, \tag{3.8}$$

$$\pi = e^{-\lambda}\, q, \;\; \mathbf{w} \wedge \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_T^2, \tag{3.9}$$

$$\mathbf{w}(\,\cdot\,, 0) = \mathbf{u}_0 \quad \text{in } \Omega, \tag{3.10}$$

Thus, proving the existence and uniqueness of the solution for the problem (2.1)–(2.5) is equivalent to the existence and uniqueness of the solution for (3.6)–(3.10). The choice of lambda will assure an a priori estimate for such a system and enable us to prove the existence of the solution using the Galerkin procedure. Let $\{\mathbf{z}_i\}$ be a basis in V. We choose it to be orthonormal in $L^2(\Omega)^2$. We seek a Galerkin's approximation for $\mathbf{w}$ in the form

$$\mathbf{w}_m(x,t) = \sum_{i=1}^m h_{mi}(t)\, \mathbf{z}_i(x),$$

chosen such that

$$\begin{aligned} &h'_{mj}(t) + \sum_{i=1}^m h_{mi} \int_\Omega (\mu\, (\nabla \wedge \mathbf{z}_i) \cdot (\nabla \wedge \mathbf{z}_j) + (\mathbf{v} \cdot \nabla)\mathbf{z}_i \cdot \mathbf{z}_j) + \lambda'\, h_{mj} \\ &\quad = e^{-\lambda(t)} \left(\langle \mathbf{f} \,|\, \mathbf{z}_j \rangle + \int_{\Gamma^1} q(\mathbf{n} \cdot \mathbf{z}_j) \right) h_{mj}(0) = h^0_{mj} = \int_\Omega \mathbf{u}_0 \cdot \mathbf{z}_j, \quad j = 1, \ldots, m. \end{aligned} \tag{3.11}$$

System (3.11) is linear. If we can prove that the matrix $\mathbf{A} = [a_{ij}]$ with

$$a_{ij} = \int_{\Omega} (\mu\ (\nabla \wedge \mathbf{z}_i) \cdot (\nabla \wedge \mathbf{z}_j) + (\mathbf{v} \cdot \nabla)\mathbf{z}_i \cdot \mathbf{z}_j) + \lambda'\ \delta_{ij}$$

is regular, then it admits a unique C^1 solution $h_{m1}, \ldots, h_{mm}$. Due to the choice of λ we are able to prove that $\mathbf{A}$ is positive definite. Indeed for arbitrary $\xi \in \mathbf{R}^m$ we have

$$\xi \cdot \mathbf{A}\,\xi = \int_{\Omega} \left(\mu\ |\nabla \wedge \Phi_\xi|^2 + \lambda'|\Phi_\xi|^2 + (\mathbf{v}(t) \cdot \nabla)\Phi_\xi \cdot \Phi_\xi\right),$$

where

$$\Phi_\xi(x) = \sum_{i=1}^{m} \xi_i\ \mathbf{z}_i(x).$$

Using Lemma 2.2, we have

$$\begin{aligned}\left|\int_{\Omega} (\mathbf{v}(t) \cdot \nabla)\Phi_\xi \cdot \Phi_\xi\right| &\le \frac{\mu}{2\,C_\#}|\nabla\Phi_\xi|^2_{L^2(\Omega)} + C_\mu\ |\mathbf{v}(t)|^4_{L^4(\Omega)}|\Phi_\xi|^2_{L^2(\Omega)} \\ &= \frac{\mu}{2\,C_\#}|\nabla\Phi_\xi|^2_{L^2(\Omega)} + \lambda'(t)\ |\Phi_\xi|^2_{L^2(\Omega)}.\end{aligned}$$

Thus

$$\xi \cdot \mathbf{A}\xi \ge \frac{\mu}{2\,C_\#}|\nabla\Phi_\xi|^2_{L^2(\Omega)} \ge 0$$

and

$$\xi \cdot \mathbf{A}\xi = 0 \quad\Leftrightarrow\quad \nabla\Phi_\xi = \mathbf{0} \quad\Leftrightarrow\quad \Phi_\xi = \mathbf{0} \quad\Leftrightarrow\quad \xi = \mathbf{0}.$$

Now, for any $\mathbf{z} \in Span\{\mathbf{z}_1, \ldots, \mathbf{z}_m\}$ we have, at any time $t \in]0, T[$.

$$\begin{aligned}&\left(\frac{\partial \mathbf{w}_m}{\partial t} + \lambda'\mathbf{w}_m, \mathbf{z}\right)_{L^2(\Omega)} + \mu(\nabla \wedge \mathbf{w}_m, \nabla \wedge \mathbf{z})_{L^2(\Omega)} + \int_{\Omega} (\mathbf{v} \cdot \nabla)\mathbf{w}_m\ \mathbf{z} \\ &\quad = e^{-\lambda}(\ {}_{V'}\langle \mathbf{f}\,|\,\mathbf{z}\rangle_V + {}_{H^{-1/2}(\Gamma^2)}\langle q\,|\,(\mathbf{z}\cdot\mathbf{n})\ \rangle_{H^{1/2}(\Gamma^2)}\ ).\end{aligned} \tag{3.12}$$

Taking $\mathbf{z} = \mathbf{w}_m$ in (3.12) and using the fact that, due to the Lemma 2.2 we have

$$\mu|\nabla \wedge \mathbf{w}_m|^2_{L^2(\Omega)} + \int_{\Omega} (\mathbf{v} \cdot \nabla)\mathbf{w}_m \cdot \mathbf{w}_m + \lambda'\ |\mathbf{w}_m|^2_{L^2(\Omega)} \ge \frac{\mu}{2C_\#}\ |\nabla\mathbf{w}_m|^2_{L^2(\Omega)}$$

we obtain the a priori estimates

$$|\mathbf{w}_m|_{L^2(0,T;V)} \leq C(|e^{-\lambda}\mathbf{f}|_{L^2(0,T;V')} + |e^{-\lambda}q|_{L^2(0,T;H^{-1/2}(\Gamma^2)} + |\mathbf{u}_0^m|_{L^2(\Omega)}) \tag{3.13}$$

$$|\mathbf{w}_m|_{L^\infty(0,T;L^2(\Omega))} \leq C(|e^{-\lambda}\mathbf{f}|_{L^2(0,T;V')} + |e^{-\lambda}q|_{L^2(0,T;H^{-1/2}(\Gamma^2)} + |\mathbf{u}_0^m|_{L^2(\Omega)}), \tag{3.14}$$

with $\mathbf{u}_0^m = \sum_{i=0}^m h_{mi}^0\, \mathbf{z}_i$. Thus, there exists a subsequence of $\{\mathbf{w}_m\}_{m\in\mathbf{N}}$, denoted for simplicity by the same symbol and a function $\mathbf{w} \in L^2(0,T;V)\cap L^\infty(0,T;L^2(\Omega))$ such that

$$\mathbf{w}_m \rightharpoonup \mathbf{w} \;\text{ weakly in }\; L^2(0,T;V) \;\text{ and weak* in }\; L^\infty(0,T;L^2(\Omega)).$$

Passing to the limit in (3.12) we get

$$\left(\frac{\partial \mathbf{w}}{\partial t} + \lambda'\mathbf{w}, \mathbf{z}\right)_{L^2(\Omega)} + \mu(\nabla\wedge\mathbf{w}, \nabla\wedge\mathbf{z})_{L^2(\Omega)} + \int\limits_\Omega (\mathbf{v}\cdot\nabla)\mathbf{w}\,\phi$$
$$= e^{-\lambda}({}_{V'}\langle\, \mathbf{f} \mid \mathbf{z}\,\rangle_V + {}_{H^{-1/2}(\Gamma^2)}\langle\, q \mid (\mathbf{z}\cdot\mathbf{n})\,\rangle_{H^{1/2}(\Gamma^2)}\,). \tag{3.15}$$

for any $\mathbf{z} \in V$. Thus $\mathbf{w}$ is the weak solution to (3.6)–(3.10). The existence of π follows from the De Rham theorem (see e.g. [15]). Note that (3.13) and (3.14) imply the estimates

$$|\mathbf{w}|_{L^2(0,T;V)} \leq C(|e^{-\lambda}\mathbf{f}|_{L^2(0,T;V')} + |e^{-\lambda}q|_{L^2(0,T;H^{-1/2}(\Gamma^2))} + |\mathbf{u}_0|_{L^2(\Omega)}) \tag{3.16}$$

$$|\mathbf{w}|_{L^\infty(0,T;L^2(\Omega))} \leq C(|e^{-\lambda}\mathbf{f}|_{L^2(0,T;V')} + |e^{-\lambda}q|_{L^2(0,T;H^{-1/2}(\Gamma^2))} + |\mathbf{u}_0|_{L^2(\Omega)}). \tag{3.17}$$

To prove that $(\mathbf{w}, \pi)$ is unique it is sufficient to see that any solution of (3.6)–(3.10) with homogeneous data $\mathbf{f} = \mathbf{u}_0 = \mathbf{0},\; q = 0$ is necessarily trivial $\mathbf{w} = \mathbf{0},\; \pi = 0$. But that follows from the estimates (3.16) and (3.17). □

Remark 3.1 The above proposition is not only a technical step in the proof of the existence of the solution for the Navier–Stokes system. It proves the well-posedness of the Oseen equations with boundary conditions involving the pressure. Since the Oseen model can be used as a linear approximation for the Navier–Stokes system in cases where the Stokes linearization is inappropriate (see e.g. [1,9,13] or [16]) such result is of some interest by itself.

We define the space $X = L^2(0,T;V)\cap L^\infty(0,T;L^2(\Omega))$, with the norm

$$|\mathbf{u}|_X = \sqrt{|\mathbf{u}|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{\mu}{C_\#}|\nabla\mathbf{u}|^2_{L^2(\Omega^T)}}.$$

We now define the mapping $T : X \to X$, by $T(\mathbf{v}) = \mathbf{u}$ where $\mathbf{u}$ is the solution to the Oseen system (3.1)–(3.5). Using the Banach contraction theorem we prove that, under the condition (2.15), the Navier–Stokes system (2.1)–(2.5) has a solution.

Lemma 3.1 *Let $T : X \to X$ be the above defined mapping and let $\frac{1}{2}|u_0|^2_{L^2(\Omega)} + \frac{C_\#}{\mu}|f|_{L^2(0,T;V')} + C(H^{1/2}(\Gamma^2), V)\, |q|_{L^2(0,T;H^{-1/2}(\Gamma^2))} \le M$ for*

$$M = \min\{2\, C(L^4, H^1)^{-2}\, C_\#^{-2}\, \mu^2 3^{-3/2}, 4mC(L^4, H^1)^{-4} 3^{-2}\}$$

with $C(L^2, H^1) > 0$ the embedding constant from (2.11) and $m = min\{\frac{\mu}{C_\#}, 1\}$. Let $B = \{\mathbf{v} \in X;\ |\mathbf{v}|_X \le M\}$. Then $T(B) \subset B$ and T is a contraction on B.

Proof For $\mathbf{v} \in B$ we define $\mathbf{u} = T(\mathbf{v})$. The standard a priori estimate implies

$$\begin{aligned}
&\frac{1}{2}\ \frac{d}{dt}\ |\mathbf{u}(t)|^2_{L^2(\Omega)} + \mu|\nabla\ \wedge\ \mathbf{u}(t)|^2_{L^2(\Omega)} \\
&\quad = -\int_\Omega (\mathbf{v}(t)\cdot\nabla)\mathbf{u}(t)\,\mathbf{u}(t) + {}_{V'}\langle \mathbf{f}(t) \mid \mathbf{u}(t)\rangle_V + {}_{H^{-1/2}(\Gamma^2)}\langle\, q(t) \mid \mathbf{u}(t)\cdot\mathbf{n}\,\rangle_{H^{1/2}(\Gamma^2)} \\
&\quad \le |\mathbf{f}(t)|_{V'}|\ |\mathbf{u}(t)|_V + |q(t)|_{H^{-1/2}(\Gamma^2)}\ C(H^{1/2}(\Gamma^2), V)|\mathbf{u}(t)|_V \\
&\qquad + |\mathbf{v}|^{1/2}_{L^2(\Omega)}|\nabla\mathbf{v}|^{1/2}_{L^2(\Omega)}\ |\mathbf{u}|^{1/2}_{L^2(\Omega)}|\nabla\mathbf{u}|^{3/2}_{L^2(\Omega)}.
\end{aligned}$$

The definition of M now guarantees that for $\mathbf{v} \in B$ we necessarily have $\mathbf{u} \in B$. Thus $T(B) \subset B$. It remains to prove that T is a contraction. Let $\mathbf{v}, \mathbf{z} \in B$ and let $\mathbf{w} = T(\mathbf{v})$, $\mathbf{u} = T(\mathbf{z})$. Subtracting equations for $\mathbf{w}$ and $\mathbf{u}$, multiplying by $\mathbf{w} - \mathbf{u}$ and integrating over Ω^T we obtain

$$\begin{aligned}
&\frac{1}{2}|\mathbf{u}-\mathbf{w}|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{\mu}{C_\#}|\nabla(\mathbf{u}-\mathbf{w})|^2_{L^2(\Omega^T)} \\
&\quad \le C(L^4, H^1)^2\ |\mathbf{v}|^{1/2}_{L^\infty(0,T;L^2(\Omega))}|\nabla\mathbf{v}|^{1/2}_{L^2(\Omega^T)}\ \ |\nabla(\mathbf{u}-\mathbf{v})|^{3/2}_{L^2(\Omega^T)}|\mathbf{u}-\mathbf{v}|_{L^\infty(0,T;L^2(\Omega))} \\
&\qquad + \le C(L^4, H^1)^2\ |\nabla\mathbf{v}|_{L^2(\Omega^T)}\ |\mathbf{v}-\mathbf{z}|^{1/2}_{L^\infty(0,T;L^2(\Omega))}\ |\nabla(\mathbf{v}-\mathbf{z})|^{1/2}_{L^2(\Omega^T)} \\
&\qquad \times |\mathbf{u}-\mathbf{w}|^{1/2}_{L^\infty(0,T;L^2(\Omega))}\ |\nabla(\mathbf{u}-\mathbf{w})|^{1/2}_{L^2(\Omega^T)} \\
&\quad \le \frac{C(L^4, H^1)^2}{2}\ \left(\frac{C_\#}{\mu}\right)^{1/4}\ |\mathbf{v}|_X\ \left(\frac{3}{4}|\nabla(\mathbf{u}-\mathbf{v})|^2_{L^2(\Omega^T)} + \frac{1}{4}|\mathbf{u}-\mathbf{v}|^2_{L^\infty(0,T;L^2(\Omega))}\right) \\
&\qquad + \frac{1}{4}C(L^4, H^1)^2\sqrt{\frac{C_\#}{\mu}}\ |\mathbf{u}|_X\ \Big(|\mathbf{v}-\mathbf{z}|^2_{L^\infty(0,T;L^2(\Omega))} + |\nabla(\mathbf{v}-\mathbf{z})|^2_{L^2(\Omega^T)} \\
&\qquad + |\mathbf{u}-\mathbf{w}|^2_{L^\infty(0,T;L^2(\Omega))} + |\nabla(\mathbf{u}-\mathbf{w})|^2_{L^2(\Omega^T)}\Big).
\end{aligned}$$

Since $\mathbf{u}, \mathbf{v} \in B$, due to the choice of M we get

$$|\mathbf{u}-\mathbf{w}|_X \le q|\mathbf{v}-\mathbf{z}|_X, \quad 0 < q < 1,$$

proving that T is a contraction. □

That also ends the proof of the existence of solution under the small data assumption via Banach contraction theorem, i.e. that completes the proof of Proposition 3.1.

Remark 3.2 It should be noticed that the small data assumption (2.15) is needed for the global (in time) existence. We can get the short time existence for arbitrary data. In fact, such result can be proved using the same method.

4 Uniqueness of the solution

In this section we prove the uniqueness of the weak solution of the 2D evolutional Navier–Stokes system with pressure boundary data (2.1)–(2.5). We do not make the small data assumption. Thus, we have the existence of the solution only for small data but we have the general uniqueness theorem. It guarantees that the solution, whose existence we have obtained, assuming that the given data is not too large, is the only one. We knew from the contraction method that the solution is unique in some ball around 0. Our goal here is to prove that there could be no other solutions with large norm as well. Besides, it also implies that, even in case of large data where our method does not give the global existence of the solution, if the solution exists, it is unique.

Proposition 4.1 *Let $\Omega \subset \mathbf{R}^2$ be a bounded locally Lipschitz 2D domain and let*

$$q \in L^2(0, T; H^{-1/2}(\Gamma^2)), \quad \mathbf{u}_0 \in L^2(\Omega)^2 \quad \textit{and} \ \ \mathbf{f} \in L^2(0, T; V'). \tag{4.1}$$

Then the problem (2.1)–(2.5) can have at most one solution.

Proof Suppose that (2.1)–(2.5) admits two solutions $u^i \in L^2(0, T; V) \cap L^\infty(0, T; L^2(\Omega))^2$ and $p^i \in L^2(\Omega_T)$, $i = 1, 2$. Then the difference $R = u^1 - u^2$, $r = p^1 - p^2$ satisfies the system

$$\begin{gathered}
\frac{\partial R}{\partial t} - \mu \Delta R + (u^1 \cdot \nabla) u^1 - (u^2 \cdot \nabla) u^2 + \nabla r = 0 \quad \text{in } \Omega_T \\
\operatorname{div} R = 0 \quad \text{in } \Omega_T \\
R = 0 \quad \text{on } \Gamma_T^1, \\
r = 0, \ R \times \mathbf{n} = 0 \quad \text{on } \Gamma_T^2, \\
R(\cdot, 0) = 0 \quad \text{in } \Omega.
\end{gathered}$$

Using R as the test function in the above equation yields, for (a.e) $t \in]0, T[$

$$\begin{aligned}
\frac{d}{dt} |R(t)|^2_{L^2(\Omega)} + 2\mu \, |\nabla \wedge R(t)|^2_{L^2(\Omega)} = -2 \int_\Omega [\, & (R(t) \cdot \nabla) u^1(t) \, R(t) \\
& - (u^2(t) \cdot \nabla) R(t) \, R(t) \,],
\end{aligned}$$

for (a.e.) $t \in]0, T[$. To estimate the two integrals on the right-hand side we use the well-known Ladyzhenskaya inequality (2.11). We also recall it's consequence

$$|z|_{L^4(\Omega_T)} \le C(L^4, H^1)|z|^{1/2}_{L^\infty(0,T;L^2(\Omega))}|\nabla z|^{1/2}_{L^2(\Omega_T)},\\ z \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; V), \tag{4.2}$$

Thus, for the first integral we obtain

$$\begin{aligned}\left|\int_\Omega [\,(R(t)\cdot\nabla)u^1(t)\ R(t)\right| &\le |\nabla u^1(t)|_{L^2(\Omega)}|R(t)|^2_{L^4(\Omega)}\\ &\le C(L^4, H^1)^2\ |\nabla u^1(t)|_{L^2(\Omega)}|R(t)|_{L^2(\Omega)}|\nabla R(t)|_{L^2(\Omega)}\\ &\le C^1_\mu\ |\nabla u^1(t)|^2_{L^2(\Omega)}\ |R(t)|^2_{L^2(\Omega)}+\frac{\mu}{2\ C_\#}|\nabla R(t)|^2_{L^2(\Omega)}.\end{aligned}$$

For the second integral, we proceed in the similar way:

$$\begin{aligned}\left|\int_\Omega (u^2(t)\cdot\nabla)R(t)\ R(t)\right| &\le |u^2(t)|_{L^4(\Omega)}|\nabla R(t)|_{L^2(\Omega)}|R(t)|_{L^4(\Omega)}\\ &\le C(L^4, H^1)\ |u^2(t)|_{L^4(\Omega)}|\nabla R(t)|^{3/2}_{L^2(\Omega)}|R(t)|^{1/2}_{L^2(\Omega)}\\ &\quad \text{(Young inequality } r=4, r'=\tfrac{4}{3})\\ &\le C^2_\mu|u^2(t)|^4_{L^4(\Omega)}\ |R(t)|^2_{L^2(\Omega)}+\frac{\mu}{2\ C_\#}|\nabla R(t)|^2_{L^2(\Omega)}.\end{aligned}$$

We notice that $u^2 \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ in 2 dimensional domain implies that $u^2 \in L^4(\Omega_T)$ (see (4.2)) so that the above expression makes sense in 2D but not in 3D.

We now have

$$\begin{aligned}&\frac{d}{dt}|R(t)|^2_{L^2(\Omega)}+\frac{2\mu}{C_\#}\ |\nabla R(t)|^2_{L^2(\Omega)}\\ &\quad\le \frac{d}{dt}|R(t)|^2_{L^2(\Omega)}+2\mu\ |\nabla\wedge R(t)|^2_{L^2(\Omega)}\\ &\quad\le (C^1_\mu|\nabla u^1(t)|^2_{L^2(\Omega)}+C^2_\mu|u^2(t)|^4_{L^4(\Omega)})\ |R(t)|^2_{L^2(\Omega)}+\frac{\mu}{C_\#}|\nabla R|^2_{L^2(\Omega)}.\end{aligned} \tag{4.3}$$

We denote

$$\lambda(t)=\int_0^t (C^1_\mu|\nabla u^1(s)|^2_{L^2(\Omega)}+C^2_\mu|u^2(s)|^4_{L^4(\Omega)})\ ds$$

and multiply (4.3) by $e^{-\lambda(t)}$ to get

$$\begin{aligned}
&\frac{d}{dt}\left(e^{-\lambda(t)}\,|R(t)|^2_{L^2(\Omega)}\right)\\
&\quad= e^{-\lambda(t)}\left(\frac{d}{dt}|R(t)|^2_{L^2(\Omega)} - \lambda'(t)\,|R(t)|^2_{L^2(\Omega)}\right)\\
&\quad\le e^{-\lambda(t)}\left(C^1_\mu|\nabla u^1(t)|^2_{L^2(\Omega)} + C^2_\mu|u^2(t)|^4_{L^4(\Omega)} - \lambda'(t)\right)\,|R(t)|^2_{L^2(\Omega)} = 0. \qquad (4.4)
\end{aligned}$$

Integration of (4.4) with respect to t over $[0, s]$ for $s \in [0, T]$, yields, using the initial condition

$$e^{-\lambda(s)}\,|R(s)|^2_{L^2(\Omega)} \le 0.$$

Thus we have $R \equiv 0$. □

This also ends the proof of Theorem 2.1.

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