CHAPTER 5

Stationary Boundary Value Problems for Compressible Navier–Stokes Equations

P.I. Plotnikov[a], J. Sokolowski[b]

[a] *Lavryentyev Institute of Hydrodynamics, Siberian Division of Russian Academy of Sciences, Lavryentyev pr. 15, Novosibirsk 630090, Russia*
E-mail: plotnikov@hydro.nsc.ru

[b] *Institut Elie Cartan, Laboratoire de Mathématiques, Université Henri Poincaré, Nancy 1, B.P. 239, 54506 Vandoeuvre lés Nancy Cedex, France*
E-mail: Jan.Sokolowski@iecn.u-nancy.fr

Abstract

We give an overview of available results on the well-posedness of basic boundary value problem for equations of viscous compressible fluids.

Keywords: Navier–Stokes equations, Compressible fluids, Shape optimization, Bergman projection, Incompressible limit, Transport equations

Contents

HANDBOOK OF DIFFERENTIAL EQUATIONS
Stationary Partial Differential Equations, volume 6
Edited by M. Chipot

1. Introduction

Compressible Navier–Stokes equations are the subject of current studies. We refer the reader to the books by Lions [34], Feireisl [16], Novotný and Straškraba [41] for the state of the art in the domain. The related references are listed at the end of the chapter.

Inhomogeneous boundary value problems for the equations are of some importance in applications, including optimal control theory, shape optimization and inverse problems. Such results, however, are not available in the literature on the subject in the full range of physical parameters. In our joint research, see e.g., [50–53], the specific problem studied is the minimization of the drag functional which is a representative shape optimization problem for the mathematical models in the form of compressible Navier–Stokes equations.

In this chapter, we present the results and the techniques which can be used to study not only the existence and uniqueness of weak solutions, but also the compactness of the set of solutions with respect to the boundary perturbations, and the differentiability of solutions with respect to the coefficients of differential operators. The class of mathematical models is of elliptic–hyperbolic types. We briefly describe the modelisation issue, define all the physical constants which are related to compressible Navier–Stokes equations. One of the constants, which is called the adiabatic constant $\gamma \geq 1$, which is present in the formula for the pressure in terms of the density, is quite important from the mathematical point of view. We present some results for the range of constants which includes the diatomic gases. The case of the ideal gas with $\gamma = 1$ is considered e.g. in [51] and leads to some singularities in three spatial dimensions, such a singularity is absent however in two spatial dimensions [50].

The content of the chapter can be described as follows.

In Section 2 the steady state equations of compressible fluid dynamics are introduced for the state variables including the density, the velocity field and the temperature. The boundary value problems considered are of elliptic–hyperbolic type.

In Section 3 the general properties of the boundary value problems are described. Local existence and uniqueness results for classical solutions known in the literature are recalled. The weak solutions to compressible Navier–Stokes equations are defined, and the existence of such solutions is discussed.

In Section 4 the mathematical tools including the interpolation theory, the Young measures and the Sobolev spaces are introduced.

In Section 5 the framework for the transport equations which constitute the hyperbolic component of the boundary value problems is established. The so-called emergent vector field conditions are given in order to assure the appropriate solvability of the transport equations.

In Section 6 important properties of the transport equations with discontinuous coefficients are analysed, among others the normalization procedure, the kinetic form of equations, the compactness of renormalized solutions, and the so-called oscillation defect measure. The strong convergence of solutions to the transport equation is shown in Theorem 6.4.

In Section 7 the existence of weak solutions for some range of adiabatic constant is shown, which is the original contribution.

In Sections 8–10, Appendices A and B all proofs of the technical results of the chapter are provided.

2. Equations of viscous compressible fluid dynamics

Let a compressible fluid occupy the domain Ω in Euclidian space $\mathbb{R}^3$. The steady state of a fluid at point $x \in \Omega$ is completely characterized by the macroscopic quantities: the *density* $\varrho(x)$, the *velocity* $\mathbf{u}(x)$, and the *temperature* $\vartheta(x)$. These quantities are called state variables in the sequel. The governing equations represent three basic principles of fluid mechanics: the mass balance

$$\operatorname{div}(\varrho\mathbf{u}) = 0 \quad \text{in } \Omega, \tag{2.1a}$$

the balance of momentum

$$\operatorname{div}(\varrho\mathbf{u} \otimes \mathbf{u}) + \nabla p = \varrho\mathbf{f} + \mathbf{h} + \operatorname{div}\mathbb{S} \quad \text{in } \Omega, \tag{2.1b}$$

energy conservation law

$$\operatorname{div}((E + p)\mathbf{u}) = \operatorname{div}\mathbb{S}(\mathbf{u}) + \operatorname{div}(\kappa\nabla\vartheta) + (\varrho\mathbf{f} + \mathbf{h})\mathbf{u}. \tag{2.1c}$$

Here, given that the vector fields $\mathbf{f}$ and $\mathbf{h}$ denote the densities of external mass and volume forces, the *heat conduction coefficient* κ is a positive constant, the viscous stress tensor $\mathbb{S}$ has the form

$$\mathbb{S}(\mathbf{u}) = \nu_1\left(\nabla\mathbf{u} + \nabla\mathbf{u}^\top - \frac{2}{3}\operatorname{div}\mathbf{v}\,\mathbf{I}\right) + \nu_2\operatorname{div}\mathbf{v}\,\mathbf{I}, \tag{2.1d}$$

in which the viscous coefficients ν_i satisfy the inequality $\nu_1 4/3 + \nu_2 > 0$, the energy density E is given by

$$E = \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e,$$

where e is the density of *internal energy*. The physical properties of a gas are reflected through constitutive equations relating the state variables to the pressure and the internal energy density. We restrict our considerations to the classic case of *perfect polytropic* gases with the pressure and the internal energy density defined by the formulae

$$p = R_m\varrho\vartheta \qquad e = c_v\vartheta. \tag{2.1e}$$

Here R_m is a positive constant inversely proportional to the molecular weight of the gas, that is

$$R_m = c_p - c_v, \quad \text{with } \gamma =: c_p/c_v > 1,$$

where c_v is the specific heat at constant volume and c_p is the specific heat at constant pressure, are positive constants. In this case the entropy density S takes the form

$$S = \log e - (\gamma - 1)\log\varrho. \tag{2.2}$$

The system of partial differential equations (2.1) is called *compressible Navier–Stokes–Fourier* equations.

It is useful to rewrite the governing equations in the dimensionless form, which is widely accepted in applications. To this end we denote by u_c, ϱ_c, and ϑ_c the characteristic values of the velocity, the density, and the temperature, and by l_c and $\Delta\vartheta_c$ the characteristic values of the length and the temperature oscillation. They form five dimensionless combinations: the Reynolds number, the Prandtl number, the Mach number, the viscosity ratio, and the relative temperature oscillation defined by the formulae, see [54],

$$\mathbb{R}\mathrm{e} = \frac{\varrho_c u_c l_c}{\nu_1}, \quad \mathbb{P}\mathrm{r} = \frac{\nu_1 c_p}{\kappa}, \quad \mathbb{M}\mathrm{a}^2 = \frac{u_c^2}{c_p \vartheta_c(\gamma - 1)},$$
$$\lambda = \frac{1}{3} + \frac{\nu_2}{\nu_1}, \quad b = \frac{\Delta\vartheta_c}{\vartheta_c}.$$

Note that the specific values of the constants γ, λ, and $\mathbb{P}\mathrm{r}$ depend only on physical properties of a fluid. For example, for the air under standard conditions, we have $\gamma = 7/5$, $\lambda = 1/3$, and $\mathrm{Pr} = 7/10$. The passage to the dimensionless variables is defined as follows

$$x \to l_c x, \quad \mathbf{u} \to u_c \mathbf{u}, \quad \varrho \to \varrho_c \varrho,$$
$$\vartheta \to \vartheta_c + \Delta\vartheta_c \vartheta, \quad \varrho\,\mathbf{f} \to \frac{\varrho_c l_c^2}{\nu_1 u_c} \varrho\,\mathbf{f}, \quad \mathbf{h} \to \frac{l_c^2}{\nu_1 u_c}\mathbf{h}$$

and (2.1) performed leading to the following system of differential equations for dimensionless quantities in the scaled domain $l_c^{-1}\Omega$, which we still denote by Ω,

$$\Delta\mathbf{u} + \lambda\nabla\operatorname{div}\mathbf{u} = k\varrho\mathbf{u}\cdot\nabla\mathbf{u} + \sigma\nabla(\varrho(1 + b\vartheta)) - \varrho\mathbf{f} - \mathbf{h} \quad \text{in } \Omega, \tag{2.3a}$$
$$\operatorname{div}(\varrho\mathbf{u}) = 0 \quad \text{in } \Omega, \tag{2.3b}$$
$$\begin{aligned}\Delta\vartheta &= k b_1(\varrho\mathbf{u}\nabla\vartheta + (\gamma - 1)(b^{-1} + \vartheta)\varrho\operatorname{div}\mathbf{u}) \\ &\quad - \frac{k}{\sigma} b_2((\nabla\mathbf{u} + \nabla\mathbf{u}^*)^2 + (\lambda - 1)\operatorname{div}\mathbf{u}^2),\end{aligned} \tag{2.3c}$$

where

$$k = \mathbb{R}\mathrm{e}, \quad \sigma = \frac{\mathbb{R}\mathrm{e}}{\gamma\mathbb{M}\mathrm{a}^2}, \quad b_1 = \frac{\mathbb{P}\mathrm{r}}{\gamma}, \quad b_2 = \frac{\mathbb{P}\mathrm{r}}{2b}(\gamma - 1).$$

The asymptotic analysis of solutions is of mathematical and practical importance. In theoretical hydrodynamics the following cases are distinguished:

- the low compressible and hypersonic limits $\mathbb{M}\mathrm{a} \to 0, \infty$;
- the Stokes and the Euler limits $\mathbb{R}\mathrm{e} \to 0, \infty$;
- the elastic low compressible limit $\lambda \to \infty$.

From this point of view the quantities b, b_i do not play any important role in the theory, and further we shall assume that

$$b = b_1 = b_2 = 1.$$

2.1. *Barotropic flows*

The flow is *barotropic* if the pressure depends only on the density. The most important example of such flows are *isentropic flows*. In order to deduce the governing equations for isentropic flows we note that for perfect fluid with $\nu_i = \kappa = 0$, the entropy takes a constant value in each material point. Hence in this case the governing equations have a family of explicit solutions with the entropy $S = \text{const}$. By virtue of (2.1e) and (2.2) in this case we have

$$p(\varrho) = (\gamma - 1)\exp(S_c)\varrho^\gamma,$$

where a positive constant S_c is a characteristic value of the entropy (without loss of generality we can take $(\gamma - 1)\exp(S_c) = 1$). Assuming that this relation holds for $\nu_i \neq 0$ we arrive at the system of *compressible Navier–Stokes equations* for isentropic flows of viscous compressible fluid

$$\Delta \mathbf{u} + \lambda \nabla \operatorname{div} \mathbf{u} = k\varrho \mathbf{u} \cdot \nabla \mathbf{u} + \sigma \nabla \varrho^\gamma - \varrho \mathbf{f} - \mathbf{h} \quad \text{in } \Omega, \tag{2.4a}$$

$$\operatorname{div}(\varrho \mathbf{u}) = 0 \quad \text{in } \Omega. \tag{2.4b}$$

Recall that the exponent γ depends on the physical properties of the fluid. In particular, $\gamma = 5/3$ for mono-atomic, $\gamma = 7/5$ for diatomic and $\gamma = 4/3$ for polyatomic gases, [14]. It is worthy of note that equations (2.4) are not compatible with (2.3), and that they are not *thermodynamically consistent*. Nevertheless, compressible Navier–Stokes equations play an important role in the theory as the only example of physically relevant equations of compressible fluid dynamics for which we have nonlocal existence result.

2.2. *Boundary conditions*

The governing equations should be supplemented with the boundary conditions. The typical boundary conditions for the velocity are: the first boundary condition (Dirichlet-type condition)

$$\mathbf{u} = \mathbf{U} \quad \text{on } \partial\Omega, \tag{2.5}$$

the second boundary condition (Neumann-type condition)

$$(\mathbb{S}(\mathbf{u}) - p\,\mathbf{I})\mathbf{n} = \mathbf{S}_n \quad \text{on } \partial\Omega, \tag{2.6}$$

where $\mathbf{n}$ is the outward normal vector to $\partial\Omega$, $\mathbf{U}$ and $\mathbf{S}_n$ are given vector fields. The important particular cases are the *no-slip boundary condition* with $\mathbf{U} = 0$, and zero normal stress condition with $\mathbf{S}_n = 0$. The third physically and mathematically reasonable condition is the *no-stick boundary condition*

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad ((\mathbb{S}(\mathbf{u}) - p\,\mathbf{I})\mathbf{n}) \times \mathbf{n} = 0 \quad \text{on } \partial\Omega,$$

which corresponds to the case of frictionless boundary.

The typical boundary conditions for the temperature are the Dirichlet boundary condition

$$\vartheta = g \quad \text{on } \partial\Omega, \tag{2.7}$$

the Neumann boundary condition

$$\nabla \vartheta \cdot \mathbf{n} = g \quad \text{on } \partial\Omega,$$

and the third boundary condition

$$\nabla \vartheta \cdot \mathbf{n} + \mathbb{N}\mathrm{u}\, \vartheta = g \quad \text{on } \partial\Omega.$$

Here g is a given function, the dimensionless Nusselt number defined by the equality $\mathbb{N}\mathrm{u} = l_c \alpha/\kappa$, in which α is the positive heat transfer coefficient.

The formulation of boundary conditions for the density is a more delicate task. Assume that the velocity $\mathbf{u}$ satisfies the first boundary condition (2.5), and split the boundary of flow region into three disjoint sets called the inlet Σ_{in}, the outgoing set Σ_{out}, and the characteristic set Σ_0, and defined by the relations

$$\begin{aligned} \Sigma_{\mathrm{in}} = \{x \in \partial\Omega : \mathbf{U} \cdot \mathbf{n} < 0\}, \quad \Sigma_{\mathrm{out}} &= \{x \in \partial\Omega : \mathbf{U} \cdot \mathbf{n} > 0\}, \\ \Sigma_0 &= \{x \in \partial\Omega : \mathbf{U} \cdot \mathbf{n} = 0\}. \end{aligned} \tag{2.8}$$

The density distribution must be given on the inlet

$$\varrho = \varrho_b \quad \text{on } \Sigma_{\mathrm{in}}. \tag{2.9}$$

The boundary conditions for the density are not needed in the case $\Sigma_{\mathrm{in}} = \emptyset$. In particular, there are no boundary conditions for the density if the velocity satisfies the no-slip and no-stick conditions when $\Sigma_{\mathrm{in}} = \Sigma_{\mathrm{out}} = \emptyset$. But in this case one extra scalar condition is required to fix the average density m of the fluid

$$\frac{1}{|\Omega|} \int_{\Omega} \varrho \, dx = m. \tag{2.10}$$

2.3. *Bibliographical comments*

The mathematical theory of compressible viscous flows is covered in the books by Lions [34], Feireisl [16], and Novotný and Straškraba [41]. The statement of basic principles of the fluid mechanics can be found in monographs by Landau and Lifschitz [35], and Serrin [55].

3. Mathematical aspects of the problem

In general the mathematical analysis of boundary value problems includes the following steps, with the mathematical proofs of the required facts,

- existence of solutions for the appropriate given data;
- uniqueness of solutions,
- stability of solutions with respect to perturbations of the given data, including the flow domain, and coefficients of the governing equations.

In spite of considerable progress having been made in the past two decades, the theory of compressible viscous flows is far from being complete. In this section we give a brief overview of available results. First note that governing equations (2.3) form a hyperbolic–elliptic system of differential equations, which include the Lame-type equation for the velocity, the transport equation for the density, and the Poisson-type equation for the temperature. There is a significant disparity between stationary and nonstationary problems:

- in contrast to the nonstationary case, for stationary problems the energy conservation law does not imply the boundedness of the total energy, and the derivation of the first energy estimate becomes nontrivial;
- there are no estimates for the total mass of the gas in the inflow and/or outflow problems, the absence of the mass control is the main difficulty because of which these problems remain essentially unsolved;
- the equation for the density is degenerate in points where $\mathbf{u} = 0$, and the mathematical treatment of the problem requires more extensive mathematical theory of transport equations;
- the governing equations do not guarantee automatically the nonnegativity of the density, and this question needs further consideration.

3.1. *Local existence and uniqueness results*

The local theory deals with *strong solutions* which are close, in an appropriate metric, to some given explicit or approximate solutions (e.g., the equilibrium rest state). By a strong solution is meant a solution which has locally integrable generalized derivatives satisfying the equations almost everywhere in the sense of the Lebesgue measure. The minimal smoothness properties which are usually required for strong solutions in the theory of compressible viscous flows are $\mathbf{u}, \vartheta \in C^1(\Omega)$, $\varrho \in C(\Omega)$. Since the governing equations involves the elliptic component, the scale of Sobolev spaces $H^{s,r}$ can be considered as the most suitable framework for the mathematical treatment of the problem.

In this frame the considerations are focused on the detailed analysis of the linearized problem. The main goal is in one way or another to eliminate the divergence of the velocity field, and to obtain the transport equation for the density distribution. Roughly speaking, there are three different approaches to this problem. The first is the simple algebraic scheme proposed in the pioneering paper by Padula [48] for the isothermal problem with no-slip boundary condition:

$$\Delta \mathbf{u} + \lambda \nabla \operatorname{div} \mathbf{u} = k\varrho \mathbf{u} \cdot \nabla \mathbf{u} + \sigma \nabla \varrho - \varrho \mathbf{f} \quad \text{in } \Omega, \tag{3.1a}$$

$$\operatorname{div}(\varrho \mathbf{u}) = 0 \quad \text{in } \Omega, \tag{3.1b}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \qquad \frac{1}{|\Omega|} \int_\Omega \varrho\, dx = m. \tag{3.1c}$$

The key observation employed in the scheme is that governing equations can be replaced by the Stokes-type equation for the velocity and for *effective viscous pressure* $q = \sigma \varrho - \lambda \operatorname{div} \mathbf{u}$:

$$\Delta \mathbf{u} - \nabla q = k\varrho \mathbf{u} \nabla \mathbf{u} - \varrho \mathbf{f} \quad \text{in } \Omega,$$

$$\operatorname{div}\mathbf{u} = \sigma_\lambda\varrho - \frac{1}{\lambda}q \quad \text{in } \Omega,$$
$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \qquad \sigma_\lambda = \sigma/\lambda,$$

and by the transport equation for the density:

$$\mathbf{u}\cdot\nabla\varrho + \sigma_\lambda\varrho^2 = \frac{q\varrho}{\lambda} \quad \text{in } \Omega, \qquad \frac{1}{|\Omega|}\int_\Omega \varrho = m.$$

This scheme is working properly for large σ_λ, λ and small k. In particular, the approach leads to the following existence and uniqueness result shown in [48], see also [49] and [20] for further details.

THEOREM 3.1. *Let Ω be a bounded domain with the smooth boundary,* $\mathbf{f} \in C(\Omega)$ *and* $m > 0$. *Then there exists $\sigma^* > 0$, depending only on Ω, m and* $\mathbf{f}$, *and positive λ^*, k^*, and R, depending only on Ω,* $\mathbf{f}$, *m and σ_λ such that for all $\sigma_\lambda > \sigma^*$, $\lambda > \lambda^*(\sigma_\lambda, m, \mathbf{f}, \Omega)$ and $0 < k < k^*(\sigma_\lambda, m, \mathbf{f}, \Omega)$, problem* (3.1) *has a unique solution in the ball*

$$\|\mathbf{u}\|_{H^{2,4}(\Omega)} + \|\varrho - m\|_{H^{1,4}(\Omega)} \le R.$$

The second approach is proposed in the papers [7] and [8] by Beirao da Veiga which are devoted to studying the local existence theory for the general boundary value problem

$$\Delta\mathbf{u} + \lambda\nabla\operatorname{div}\mathbf{u} = k\varrho\mathbf{u}\cdot\nabla\mathbf{u} + \sigma\nabla p(\varrho,\vartheta) - \varrho\mathbf{f} \quad \text{in } \Omega, \tag{3.2a}$$
$$\operatorname{div}(\varrho\mathbf{u}) = g \quad \text{in } \Omega, \tag{3.2b}$$
$$\Delta\vartheta = c_1 p'_\vartheta(\varrho,\vartheta)\operatorname{div}\mathbf{u} + c_2\varrho\mathbf{u}\nabla\vartheta \tag{3.2c}$$
$$- \psi(\nabla\mathbf{u},\nabla\mathbf{u}) + \varrho h, \tag{3.2d}$$
$$\mathbf{u} = 0, \quad \vartheta = 0 \quad \text{on } \partial\Omega, \quad \frac{1}{|\Omega|}\int_\Omega \varrho\,dx = m, \tag{3.2e}$$

where c_i, $i = 1, 2$ are positive constants, $\psi(\cdot,\cdot)$ is a quadratic form, whose properties are not essential, and h is a given function. It is assumed the a function $p(\varphi,\vartheta)$ is sufficiently smooth and $p'_\varrho(m, 0) > 0$. In this setting the principal part of linearized equations reads

$$\Delta\mathbf{u} + \lambda\nabla\operatorname{div}\mathbf{u} - \sigma p'(m,0)\varphi = \mathbf{F} \quad \text{in } \Omega,$$
$$\mathbf{u}\cdot\nabla\varphi + \varphi\operatorname{div}\mathbf{u} + m\operatorname{div}\mathbf{u} = g \quad \text{in } \Omega,$$
$$\Delta\vartheta - c_1 p'_\vartheta(m,0)\operatorname{div}\mathbf{u} = h,$$
$$\mathbf{u} = 0 \quad \text{on } \partial\Omega,$$

where $\varphi = \varrho - m$ is the density perturbation. The key observation in papers [7,8] is that this system can be reduced to a Poisson-type equation for $\mathbf{u}$ and the transport equation for the Laplacian of the density perturbation

$$\mathbf{u}\nabla(\Delta\varphi) + \frac{\sigma m}{1+\lambda}(\Delta\varphi) = F(\mathbf{f}, g, h),$$

in which F is some linear operator. Note that this method requires the existence and uniqueness results for a transport equation in negative Sobolev spaces, see [9]. Using this approach Beirao da Veiga proved the following existence result.

THEOREM 3.2. *Let $r \in (1, \infty)$ and $j \geq -1$. Assume that $\partial\Omega \in C^{3+j}$ and $p \in C^{3+j}(\mathbb{R}^2)$. Then there exist constants c_0', c_1' depending only on Ω, λ, σ, m and c_i, $i = 1, 2$ such that if $(\mathbf{f}, g, h)$ verifies the condition*

$$\|\mathbf{f}\|_{H^{1+j,r}(\Omega)} + \|g\|_{H^{2+j,r}(\Omega)} + \|h\|_{H^{1+j,r}(\Omega)} \leq c_0', \quad \langle, g, 1\rangle = 0,$$

then there exists a unique solution to problem (3.2) *in the ball*

$$\|\mathbf{u}\|_{H^{3+j,r}(\Omega)} + \|\varrho - m\|_{H^{2+j,r}(\Omega)} + \|\vartheta\|_{H^{3+j,r}(\Omega)} \leq c_1'.$$

It was also shown that in the case of barotropic flow a solution to the problem tends to a solution of incompressible Stokes equations as $\sigma \to \infty$.

If $\mathbf{h} = 0$ and the mass force is potential, i.e. $\mathbf{f} = \nabla\Phi(x)$, then equations (2.3) endowed with the boundary conditions

$$\mathbf{u} = 0, \quad \vartheta = 0 \quad \text{on } \Omega$$

has the only solution

$$\mathbf{u}_0 = 0, \quad \vartheta_0 = 0, \quad \varrho_0(x) = c(m)\exp(\sigma^{-1}\Phi(x)).$$

The equilibrium solutions of this type also exist for general constitutive law $p = p(\varrho, \vartheta)$ provided that the total mass $\mathcal{M}$ is sufficiently large. The existence of solutions to boundary value problems close to $(\mathbf{u}_0, \varrho_0, \vartheta_0)$ was established by the application of the decomposition method proposed in the paper by Novotny and Padula [42]. The main idea of the method is to split the velocity field $\mathbf{u}$ into an incompressible part $\mathbf{u}_{sol}$, satisfying the relation $\operatorname{div}(\varrho_0\mathbf{u}_{sol}) = 0$, and a compressible irrotational part $\mathbf{u}_{pot} = \nabla\phi$. In such a way the original mixed elliptic–hyperbolic system is split into several equations: a Stokes-type system for $\mathbf{u}_{sol}$, a Poisson-type equation for ϕ and the transport equation for the density ϱ.

In [43] and [44] the results were applied over the case of flow in exterior domains which has practical importance. In this case the problem can be formulated as follows.

Let $\Omega \subset \mathbb{R}^3$ be an exterior domain such that $\Omega_c = \mathbb{R}^3 \setminus \Omega$ is a compact which has a boundary of a class C^{k+2}, $k \geq 0$. It is necessary to find the velocity field $\mathbf{u}$ and the density distribution ϱ satisfying equations (2.4) with $\gamma = 1$ (the extension to the case $\gamma > 1$ is obvious) and the boundary conditions

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \quad \varrho(x) \to \varrho_\infty, \quad \mathbf{u}(x) \to \mathbf{u}_\infty \quad \text{as } |x| \to \infty, \tag{3.3}$$

where the given vector $\mathbf{u}_\infty = (u_\infty, 0, 0)$, $u_\infty \neq 0$. The following existence and local uniqueness result is due to Novotny and Padula.

Let $\mathcal{W}^{k,r}(\Omega)$ be a completion of $C_0^\infty(\Omega)$ in the norm $\|\nabla \cdot\|_{H^{k-1,r}(\Omega)}$. Introduce the Banach spaces $W^{k,q,r} = H^{1,q}(\Omega) \cap H^{k,r}(\Omega)$ and $\mathbb{D}^{k,q,r} = \mathcal{W}^{2,q} \cap \mathcal{W}^{k,r}$ endowed with the standard norm of intersections of Banach spaces. Finally introduce

$$\mathbb{Q}^{k+1,q,r} = W^{k,q,r} \times \left(\mathbb{D}^{k,q,r} \cap L^{\frac{4q}{4-q}}(\Omega)\right)^3.$$

Endowed with the norm

$$|||(\varphi, \mathbf{v})|||_{k+1,q,r,u_\infty} = \|\varphi\|_{W^{k,q,r}} + \|\mathbf{v}\|_{\mathbb{D}^{k+1,q,r}} + |u_\infty|^{1/4}\|\mathbf{v}\|_{L^{\frac{4q}{4-q}}(\Omega)}$$

$\mathbb{Q}^{k+1,q,r}$ becomes the Banach space. The set

$$\mathbb{M}^{k,q,r}_{u_\infty} = \{(\varphi, \mathbf{v}) \in \mathbb{Q}^{k+1,q,r} : \ \mathbf{v} = -\mathbf{u}_\infty \text{ on } \partial\Omega\}$$

is the convex closed subset of $\mathbb{Q}^{k+1,q,r}$.

THEOREM 3.3. *Let $r > 3$, $1 < s < 5/6$, integer $k \geq 0$, $\partial\Omega \in C^{3+k}$ and $\mathbf{f} \in L^s(\Omega) \cap H^{k,r}(\Omega)$. Let q be an arbitrary such that $3s/(3-s) \leq q \leq 2$. Then there exists a positive $k_0 < 1$ which depends only on $s, q, k, r, \partial\Omega$ such that for any u_∞ with*

$$0 < |u_\infty| \leq k_0$$

there exist positive constants γ_0, γ_1 (depending only on $s, q, k, r, \partial\Omega$ and $|u_\infty|$) with the property: If

$$\|\mathbf{f}\|_{L^s(\Omega)} + \|\mathbf{f}\|_{H^{k,r}(\Omega)} \leq \gamma_1,$$

then in the set

$$\{(\varrho, \mathbf{u}) : \ (\varrho - \varrho_\infty, \mathbf{u} - \mathbf{u}_\infty) \in \mathbb{M}^{k+2,q,r}_{u_\infty}, \ |||(\varrho - \varrho_\infty, \mathbf{u} - \mathbf{u}_\infty)|||_{k+2,q,r,u_\infty} \leq \gamma_0\}$$

there exists just one solution to problem (2.4), (3.3).

3.1.1. *Inhomogeneous boundary value problems*

It is worthy of note that there are only a few kinds of physically reasonable external forces: the gravitational force, the centrifugal and centripetal forces, the electromagnetic forces. Therefore the stationary boundary value problems for compressible Navier–Stokes equations in *bounded domains* with no-slip and no-stick boundary conditions are primarily of academic interest. The inflow–outflow boundary value problems for $2D$-compressible Navier–Stokes and compressible Navier–Stokes–Fourier equations were considered in papers [23,24] by Kellogg and Kweon. It was shown that the properties of solutions are very sensitive to the geometry of the flow domain Ω and behavior of the velocity vector field at the boundary of Ω. It turns out that in the case of a smooth domain Ω the density has a weak singularity on the characteristic manifold $\overline{\Sigma_{\text{in}}} \cap \overline{\Sigma_{\text{out}}}$. It was shown that if $\partial\Omega$ and $\mathbf{U}$ satisfy the *emergent field condition* **(H1)**–**(H2)** in Section 5.1, then the problem has a continuous solution close to the explicit solution $\mathbf{u}_0 = (u_0, 0)$, $\varrho = \text{const}$. The flow on polygon was investigated in [24]. We consider this problem in more detail in Section 9.
The qualitative properties of solutions and the propagation of corner singularities were considered in the papers [22,25] by Kellogg and Kellogg and Kweon, and in the paper [56] by Xinfu Chen and Xie. It was shown that jump of discontinuity in the velocity on the boundary produces a discontinuous density across the streamline emanating from the point of discontinuity. Moreover, in polygons the interior singularities do occur even for continuous boundary data.

3.2. *Global existence of generalized solutions*

By now there are no results on the existence of strong solutions to compressible Navier–Stokes–Fourier equations for large data. The available results concern the existence of weak solutions to the boundary value problem for the barotropic compressible Navier–Stokes equations with no-slip conditions at the boundary:

$$\Delta \mathbf{u} + \lambda \nabla \operatorname{div} \mathbf{u} = k\varrho \mathbf{u} \otimes \mathbf{u} + \sigma \nabla p(\rho) - \varrho \mathbf{f} - \mathbf{h} \quad \text{in } \Omega, \tag{3.4a}$$

$$\operatorname{div}(\varrho \mathbf{u}) = 0 \quad \text{in } \Omega, \tag{3.4b}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \tag{3.4c}$$

$$\int_\Omega \varrho \, dx = \mathcal{M}, \tag{3.4d}$$

where $p(\cdot)$ is a smooth strictly monotone function. Along with problem (3.4), in mathematical literature its different regularization is also considered. The most distributed is the *relaxed* boundary value problem, [34],

$$a\varrho \mathbf{u} + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = \varrho \mathbf{f} + \mathbf{h} + \operatorname{div} \mathbb{S}, \tag{3.5}$$

$$a\varrho + \operatorname{div}(\varrho \mathbf{u}) = h \quad \text{in } \Omega, \tag{3.6}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega,$$

where $a(x)$ and $h(x)$ are given positive functions. This problem is of mathematical interest since for a = const., equations (3.5) can be regarded as the time discretization of the evolutionary problem.

The definition of a weak solution to compressible Navier–Stokes equations differs from a standard that is caused by the specific character of the transport equation, see [34] and [16].

DEFINITION 3.4. We say that $(\varrho, \mathbf{u})$ is a weak solution to equations (3.4a)–(3.4c) if $\mathbf{u} \in H_0^{1,2}(\Omega)$, $\varrho \in L^1(\Omega)$, nonnegative functions ϱ, $\varrho|\mathbf{u}|^2$, and $p(\varrho)$ are locally integrable and equations (3.4a)–(3.4b) are satisfied in the sense of distributions: for all vector fields $\varphi \in C_0^\infty(\Omega)$ and functions $G \in C^1(\mathbb{R})$, $\psi \in C^1(\Omega)$,

$$\int_\Omega \nabla\varphi : (\varrho \mathbf{u} \otimes \mathbf{u} + p\,\mathbf{I})\, dx = \int_\Omega (\nabla\varphi : \nabla \mathbf{u} + \lambda \operatorname{div} \varphi \operatorname{div} \mathbf{u} - (\varrho \mathbf{f} + \mathbf{h}) \cdot \varphi)\, dx \tag{3.7a}$$

$$\int_\Omega (G(\varrho)\mathbf{u} \cdot \nabla\psi + (G(\varrho) - G'(\varrho)\varrho)\psi \operatorname{div} \mathbf{u})\, dx = 0. \tag{3.7b}$$

Note that weak solutions whose definition involves arbitrary functions of unknown quantities are generally referred to as *renormalized* solutions.

The principal question in the theory concerns the compactness properties of the generalized solution. Suppose for a moment that we have a sequence of explicit or approximate solutions $(\mathbf{u}_\varepsilon, \varrho_\varepsilon)_{\varepsilon>0}$ to problem (3.4) and the energies of these solutions are uniformly bounded

$$\|\mathbf{u}_\varepsilon\|_{H^{1,2}(\Omega)} + \|\varrho_\varepsilon\|_{L^\gamma(\Omega)} + \|\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2\|_{L^1(\Omega)} < C < \infty.$$

After passing to a subsequence we can assume that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \text{weakly in } H_0^{1,2}(\Omega), \qquad \varrho_\varepsilon \to \varrho \quad \text{weakly in } L^\gamma(\Omega),$$

$$\varrho_\varepsilon \mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon \to \mathbb{M} \quad \text{star weakly in the space of Radon measures as } \varepsilon \to 0.$$

The question is under what conditions will a weak limit $(\mathbf{u}, \varrho)$ be a solution to the original equation. In order to answer this question we have to solve two problems:
First we have to resolve the *oscillation* problem, i.e. to prove the equality $w\text{-lim } p(\varrho_\varepsilon) = p(\varrho)$.
Next we have to prove that the *weak star defect measure* $\mathcal{S} =: \mathbb{M} - \varrho\mathbf{v}\otimes\mathbf{v} = 0$ and thereby to resolve the *concentration* problem.

The mathematical analysis of these problems in 2 and 3 dimensions originated in papers [32,33] by Lions, where the weak regularity of the *effective viscous flux* was established. This result allows to prove the compactness properties of the density and hence gives a way around the oscillation problem. The detailed mathematical treatment of the problems (3.4) and (3.7) was undertaken in the monograph [34] by Lions. In particular, there the following global result on solvability of problem (3.4) was proved.

THEOREM 3.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^1-boundary. Assume that the C^1-function p is strictly monotone and satisfies the following conditions: there exist $\gamma > 5/3$ and $c_0 > 0$ such that*

$$c_0^{-1} s^{\gamma-1} < p'(s) < c_0 s^{\gamma-1}, \quad c_0^{-1} s^\gamma < p(s) < c_0 s^\gamma \quad \textit{for all sufficiently large } s.$$

Then for each $\mathcal{M} > 0$, $\mathbf{f}, \mathbf{h} \in C(\Omega)$ boundary value problem (3.4) *has a generalized solution satisfying the energy inequality*

$$\int_\Omega (|\nabla \mathbf{u}|^2 + \lambda |\operatorname{div} \mathbf{u}|^2)\, dx \le \int_\Omega (\varrho\mathbf{f} + \mathbf{h}) \cdot \mathbf{u}\, dx.$$

The restrictions on the adiabatic exponent were weakened by Novo and Novotny in [39] and Novotny and Straškraba in [41] in the case of *potential mass forces* $\mathbf{f}$. Using the Feireisl theory of the oscillation defect measure, [16,17], they proved that for potential mass forces the statement of the Lions Theorem remains true for all adiabatic constant $\gamma > 3/2$. The extension of these results on the case of the exterior boundary value problem and boundary value problems in domains with noncompact boundary was made by Novo and Novotny in [40].

It is easy to see that in the three-dimensional case the energy estimates guarantee the inclusion of $\varrho|\mathbf{v}|^2 \in L^r(\Omega)$ with $r > 1$ if and only if $\gamma > 3/2$. Hence, for $\gamma \le 3/2$ we have only an L^1 estimate for the density of the kinetic energy, and the concentrations problem becomes nontrivial. In this sense the critical value $\gamma = 3/2$ is the block for the existing nonlocal theory. Steps to overcome this threshold were taken by Frehse *et al.* in [19] and by Plotnikov and Sokolowski in [53], where it was proved that the boundedness of the total energy implies the compactness properties of solutions to problem (3.4) for all $\gamma > 1$. The proof of this result is based on pointwise estimates of the Newtonian potential of the pressure. Using this approach Plotnikov and Sokolowski have proved the existence of solutions to the relaxed problem (3.5) for all $\gamma > 1$. In Section 7 we show that this approach also leads to the existence theory for barotropic flows for all $\gamma > 4/3$.

4. Mathematical preliminaries

In this paragraph we assemble some technical results which are used throughout of the paper. Function spaces play a central role, and we recall some notations, fundamental definitions, and properties, which can be found in [1] and [10]. For the convenience of readers we also collect the basic facts from the theory of interpolation spaces and the Young measures. For our applications we need the results in three spatial dimensions, however the results are presented for the dimension $d \geq 2$.

4.1. *Interpolation theory*

In this paragraph we recall some results from the interpolation theory, see [10] for the proofs. Let A_0 and A_1 be Banach spaces. For simplicity assume that $A_1 \subset A_0$. For $t > 0$ introduce two nonnegative functions $K : A_0 \mapsto \mathbb{R}$ and $J : A_1 \mapsto \mathbb{R}$ defined by

$$K(t, u, A_0, A_1) = \inf_{\substack{u=u_0+u_1 \\ u_i \in A_i}} \|u_0\|_{A_0} + t\|u_1\|_{A_1}$$

and

$$J(t, u, A_0, A_1) = \max\{\|u\|_{A_0}, t\|u\|_{A_1}\}.$$

For each $s \in (0, 1)$, $1 < r < \infty$, the K-interpolation space $[A_0, A_1]_{s,r,K}$ consists of all elements $u \in A_0$, having the finite norm

$$\|u\|_{[A_0,A_1]_{s,r,K}} = \left(\int_0^\infty t^{-1-sr} K(t, u, A_0, A_1)^r \, dt\right)^{1/r}. \tag{4.1}$$

On the other hand, J-interpolation space $[A_0, A_1]_{s,r,J}$ consists of all elements $u \in A_0+A_1$ which admit the representation

$$u = \int_0^\infty \frac{v(t)}{t} \, dt, \quad v(t) \in A_1 \quad \text{for } t \in (0, \infty) \tag{4.2}$$

and have the finite norm

$$\|u\|_{[A_0,A_1]_{s,r,J}} = \inf_{v(t)} \left(\int_0^\infty t^{-1-sr} J(t, v(t), A_0, A_1)^r \, dt\right)^{1/r} < \infty, \tag{4.3}$$

where the infimum is taken over the set of all $v(t)$ satisfying (4.2). The first main result of the interpolation theory reads: For all $s \in (0, 1)$ and $r \in (1, \infty)$ the spaces $[A_0, A_1]_{s,r,K}$ and $[A_0, A_1]_{s,r,J}$ are isomorphic, topologically and algebraically. Hence the introduced norms are equivalent, and we can omit indices J and K. The following simple properties of interpolation spaces directly follows from definitions.

(1) If $A_1 \subset A_0$ is dense in A_0, then $[A_0, A_1]_{s,r} \subset A_0$ is dense in A_0. To show this fix an arbitrary $u \in [A_0, A_1]_{s,r}$ and choose the v in representation (4.2) such that

$\|t^{-s}v\|_{L^r(0,\infty;dt/t)} < \infty$. It is easy to see that $u_n = \int_{n^{-1}}^{n} v(t)t^{-1}dt \in A_1$ and

$$\|u_n - u\|^r_{[A_0,A_1]_{s,r,J}} \\ \le \left(\int_0^{n^{-1}} + \int_n^{\infty}\right) t^{-1-sr} J(t, v(t), A_0, A_1)^r\, dt \to 0 \quad \text{as } n \to \infty.$$

(2) If $\tilde{A}_i$, $i = 0, 1$, are closed subspaces of A_i, then $[\tilde{A}_0, \tilde{A}_1]_{s,r} \subset [A_0, A_1]_{s,r}$ and $\|u\|_{[A_0,A_1]_{s,r}} \le \|u\|_{[\tilde{A}_0,\tilde{A}_1]_{s,r}}$.

One of the important results of the interpolation theory is the following representation for the interpolation of dual spaces. Let A_i be Banach spaces such that $A_1 \cap A_0$ is dense in $A_0 + A_1$. Then the Banach spaces $[(A_0)', (A_1)']_{s,r'}$ and $([A_0, A_1]_{s,r})'$ are isomorphic topologically and algebraically. Hence the spaces can be identified with equivalent norms.

In particular, if $A_1 \subset A_0$ $A'_0 \subset A'_1$ are dense in A_0 and A'_1, then $([A_0, A_1]_{s,r})'$ is the completion of A'_0 in $([A_0, A_1]_{s,r})'$-norm.

The following lemma is the central result of the interpolation theory.

LEMMA 4.1. *Let A_i, B_i, $i = 0, 1$, be Banach spaces and let $T : A_i \mapsto B_i$, be a bounded linear operator. Then for all $s \in (0, 1)$ and $r \in (1, \infty)$, the operator $T : [A_0, A_1]_{s,r} \mapsto [B_0, B_1]_{s,r}$ is bounded and*

$$\|T\|_{\mathcal{L}([A_0,A_1]_{s,r},[X_0,Y_1]_{s,r})} \le \|T\|^s_{\mathcal{L}(A_0,B_0)} \|T\|^{1-s}_{\mathcal{L}(A_1,B_1)}.$$

4.2. *Function spaces*

Let Ω be the whole space $\mathbb{R}^d$ or a bounded domain in $\mathbb{R}^d$ with the boundary $\partial\Omega$ of class C^1. For an integer $l \ge 0$ and for an exponent $r \in [1, \infty)$, we denote by $H^{l,r}(\Omega)$ the Sobolev space endowed with the norm $\|u\|_{H^{l,r}(\Omega)} = \sup_{|\alpha| \le l} \|\partial^\alpha u\|_{L^r(\Omega)}$.

For real $0 < s < 1$, the fractional Sobolev space $H^{s,r}(\Omega) = [H^{0,r}(\Omega), H^{1,r}(\Omega)]_{s,r}$ endowed with one of the equivalent norms (4.1) or (4.3) is obtained by the interpolation between $L^r(\Omega)$ and $H^{1,r}(\Omega)$. It consists of all measurable functions with the finite norm

$$\|u\|_{H^{s,r}(\Omega)} = \|u\|_{L^r(\Omega)} + |u|_{s,r,\Omega},$$

where

$$|u|^r_{s,r,\Omega} = \int_{\Omega\times\Omega} |x - y|^{-d-rs} |u(x) - u(y)|^r\, dxdy. \tag{4.4}$$

In the general case, the Sobolev space $H^{l+s,r}(\Omega)$ is defined as the space of measurable functions with the finite norm $\|u\|_{H^{l+s,r}(\Omega)} = \sup_{|\alpha| \le l} \|\partial^\alpha u\|_{H^{s,r}(\Omega)}$. For $0 < s < 1$, the Sobolev space $H^{s,r}(\Omega)$ is, in fact [10], the interpolation space $[L^r(\Omega), H^{1,r}(\Omega)]_{s,r}$.

Furthermore, the notation $H^{l,r}_0(\Omega)$, with an integer l, stands for the closed subspace of the space $H^{l,r}(\Omega)$ of all functions $u \in H^{l,r}(\Omega)$ which are being extended by zero outside Ω, and which belong to $H^{l,r}(\mathbb{R}^d)$.

Denote by $\mathcal{H}_0^{0,r}(\Omega)$ and $\mathcal{H}_0^{1,r}(\Omega)$ the subspaces of $L^r(\mathbb{R}^d)$ and $H^{1,r}(\mathbb{R}^d)$, respectively, of all functions vanishing outside of Ω. Obviously $\mathcal{H}_0^{1,r}(\Omega)$ and $H_0^{1,r}(\Omega)$ are isomorphic topologically and algebraically and we can identify them. However, we need the interpolation spaces $\mathcal{H}_0^{s,r}(\Omega)$ for nonintegers, in particular for $s = 1/r$.

DEFINITION 4.2. For all $0 < s \le 1$ and $1 < r < \infty$, we denote by $\mathcal{H}_0^{s,r}(\Omega)$ the interpolation space $[\mathcal{H}_0^{0,r}(\Omega), \mathcal{H}_0^{1,r}(\Omega)]_{s,r}$ endowed with one of the equivalent norms (4.1) or (4.3) defined by the interpolation method.

It follows from the property (2) of interpolation spaces that $\mathcal{H}_0^{s,r}(\Omega) \subset H^{s,r}(\mathbb{R}^d)$ and for all $u \in \mathcal{H}_0^{s,r}(\Omega)$,

$$\|u\|_{H^{s,r}(\mathbb{R}^d)} \le c(r, s)\|u\|_{\mathcal{H}_0^{s,r}(\Omega)}, \quad u = 0 \quad \text{outside } \Omega. \tag{4.5}$$

In other words, $\mathcal{H}_0^{s,r}(\Omega)$ consists of all elements $u \in H^{s,r}(\Omega)$ such that the extension $\overline{u}$ of u by 0 outside of Ω have the finite $[\mathcal{H}_0^{0,r}(\Omega), \mathcal{H}_0^{1,r}(\Omega)]_{s,r}$-norm. We identify u and $\overline{u}$ for the elements $u \in \mathcal{H}_0^{s,r}(\Omega)$. With this identification it follows that $H_0^{1,r}(\Omega) \subset \mathcal{H}_0^{s,r}(\Omega)$ and the space $C_0^\infty(\Omega)$ is dense in $\mathcal{H}_0^{s,r}(\Omega)$.

It is worthy of note that a function u belongs to the space $\mathcal{H}_0^{s,r}(\Omega)$, $0 \le s \le 1$, $1 < r < \infty$ if and only if

$$\operatorname{dist}(x, \partial\Omega)^{-s}|u| \in L^r(\Omega).$$

In particular $\mathcal{H}_0^{s,r}(\Omega) = H^{s,r}(\Omega)$ for $0 \le s < r^{-1}$. We also point out that the interpolation space $\mathcal{H}_0^{s,r}(\Omega)$ coincides with the Sobolev space $H_0^{s,r}(\Omega)$ for $s \ne 1/r$. Recall that the standard space $H_0^{s,r}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the $H^{s,r}(\Omega)$-norm.

Embedding of Sobolev spaces. For $sr > d$ and $0 \le \alpha < s - r/d$, the embedding $H^{s,r}(\Omega) \hookrightarrow C^\alpha(\Omega)$ is continuous and compact. In particular, for $sr > d$, the Sobolev space $H^{s,r}(\Omega)$ is a commutative Banach algebra, i.e. for all $u, v \in H^{s,r}(\Omega)$,

$$\|uv\|_{H^{s,r}(\Omega)} \le c(r, s)\|u\|_{H^{s,r}(\Omega)}\|v\|_{H^{s,r}(\Omega)}. \tag{4.6}$$

If $sr < d$ and $t^{-1} = r^{-1} - d^{-1}s$, then the embedding $H^{s,r}(\Omega) \hookrightarrow L^t(\Omega)$ is continuous. In particular, for $\alpha \le s$, $(s - \alpha)r < d$ and $\beta^{-1} = r^{-1} - d^{-1}(s - \alpha)$,

$$\|u\|_{H^{\alpha,\beta}(\Omega)} \le c(r, s, \alpha, \beta, \Omega)\|u\|_{H^{s,r}(\Omega)}. \tag{4.7}$$

If $(s - \alpha)r \ge d$, then estimate (4.7) holds true for all $\beta \in (1, \infty)$. It follows from (4.5) that all the embedding inequalities remain true for the elements of the interpolation space $\mathcal{H}_0^{s,r}(\Omega)$.

Duality. We define

$$\langle u, v\rangle = \int_\Omega u\, v\, dx \tag{4.8}$$

for any function such that the right-hand side make sense. For $r \in (1, \infty)$, each element $v \in L^{r'}(\Omega)$, $r' = r/(r-1)$, determines the functional L_v of $(\mathcal{H}_0^{s,r}(\Omega))'$ by the identity $L_v(u) = \langle u, v\rangle$. We introduce the $(-s, r')$-norm of an element $v \in L^{r'}(\Omega)$ to be by definition the norm of the functional L_v, that is

$$\|v\|_{\mathcal{H}^{-s,r'}(\Omega)} = \sup_{\substack{u \in \mathcal{H}_0^{s,r}(\Omega) \\ \|u\|_{\mathcal{H}_0^{s,r}(\Omega)}=1}} |\langle u, v\rangle|. \tag{4.9}$$

Let $\mathcal{H}^{-s,r'}(\Omega)$ denote the completion of the space $L^{r'}(\Omega)$ with respect to $(-s, r')$-norm. By virtue of pairing (4.8), the space $L^{r'}(\Omega)$ can be identified with $(\mathcal{H}_0^{0,r})'$, which is dense in $\mathcal{H}^{-1,r}(\Omega) = (\mathcal{H}_0^{1,r}(\Omega))'$. Therefore, the space $(\mathcal{H}_0^{s,r})'$ is the completion of $L^{r'}(\Omega)$ in the norm of $(\mathcal{H}_0^{s,r}(\Omega))'$, which is exactly equal to the norm of $\mathcal{H}^{-s,r'}(\Omega)$. Hence $(\mathcal{H}_0^{s,r}(\Omega))' = \mathcal{H}^{-s,r'}(\Omega)$ which leads to the duality principle

$$\|u\|_{\mathcal{H}_0^{s,r}(\Omega)} = \sup_{\substack{v \in C_0^{\infty}(\Omega) \\ \|v\|_{\mathcal{H}^{-s,r'}(\Omega)}=1}} |\langle u, v\rangle|. \tag{4.10}$$

Moreover, we can identify $\mathcal{H}^{-s,r'}(\Omega)$ with the interpolation space $[L^{r'}(\Omega), H_0^{-1,r'}(\Omega)]_{s,r}$.

With applications to the theory of Navier–Stokes equations in mind, we introduce the smaller dual space defined as follows. We identify the function $v \in L^{r'}(\Omega)$ with the functional $L_v \in (H^{s,r}(\Omega))'$ and denote by $\mathbb{H}^{-s,r'}(\Omega)$ the completion of $L^{r'}(\Omega)$ in the norm

$$\|v\|_{\mathbb{H}^{-s,r'}(\Omega)} := \sup_{\substack{u \in H^{s,r}(\Omega) \\ \|u\|_{H^{s,r}(\Omega)}=1}} |\langle u, v\rangle|. \tag{4.11}$$

In the sense of this identification the space $C_0^{\infty}(\Omega)$ is dense in the interpolation space $\mathbb{H}^{-s,r}(\Omega)$. It follows immediately from the definition that

$$\mathbb{H}^{-s,r'}(\Omega) \subset (H^{s,r}(\Omega))' \subset \mathcal{H}^{-s,r'}(\Omega).$$

4.3. *Embedding $H_0^{1,2}(\Omega)$ into $L^2(\Omega, d\mu)$*

The question of integrability of elements of the Sobolev spaces with respect to general Borel measures μ is more difficult. The following theorem belonging to Adams [2] and Mazjia [37], see also [3] for the further discussion, gives necessary and sufficient conditions for the continuity of the embedding $H_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega, d\mu)$

LEMMA 4.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^2-boundary. Then the embedding $H_0^{1,2}(\Omega) \hookrightarrow L^r(\Omega, d\mu)$ is continuous if and only if there is a constant C such that the inequality $\mu(K) \le C\text{cap}\, K$ holds true for any compact $K \Subset \Omega$.*

Further we shall use the following consequence of this result.

COROLLARY 4.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^1 boundary and $\mathfrak{G}(x, y)$ is the Green of domain Ω. Assume that a Radon measure μ satisfies the inequality*

$$M =: ess \sup_{y\in\mathbb{R}^3} \int_\Omega \mathfrak{G}(x, y)\, d\mu < \infty. \tag{4.12}$$

Then there exists a constant c, depending only on Ω, such that inequality

$$\int_\Omega |v|^2\, d\mu \le cM\|v\|^2_{H^{1,2}(\Omega)} \tag{4.13}$$

holds true for all $v \in H^{1,2}_0(D)$.

PROOF. Recall that the capacity of the compact $K \subset D$ is defined by the equality

$$\operatorname{cap} K = \inf\left\{\int_D |\nabla\varphi|^2\, dx : \ \varphi \in C_0^\infty(D), \varphi \ge 0\ \varphi \ge 1 \text{ on } K\right\}.$$

Choose an arbitrary admissible function φ. It is easily seen that

$$\int_K d\mu \le \int_K \varphi\, d\mu \le \|\Delta^{-1/2} 1_K \mu\|_{L^2(D)} \left(\int_D |\nabla\varphi|^2\, dx\right)^{1/2}.$$

On the other hand, we have

$$\|\Delta^{-1/2} 1_K\mu\|^2_{L^2(D)} = \int_D\int_D \mathfrak{G}(x, y) 1_K(x) 1_K(y)\, d\mu(x) d\mu(y) \le M \int_K d\mu(y)$$

which leads to

$$\int_K d\mu \le M^{1/2} \left(\int_K d\mu\right)^{1/2} \left(\int_D |\nabla\varphi|^2\, dx\right)^{1/2}.$$

Hence for any compact $K \subset D$,

$$\int_K d\mu \le M \inf_\varphi \int_D |\nabla\varphi|^2\, dx = M \operatorname{cap} K,$$

which along with Lemma 4.3 yields the desired estimate. □

4.4. *Di-Perna–Lions Lemma*

Let θ be a standard mollifying kernel in $\mathbb{R}$,

$$\theta \in \mathcal{D}_+(\mathbb{R}), \quad \int_{\mathbb{R}} \theta(t)\, dt = 1, \quad \operatorname{spt}\theta \Subset \{|t| \le 1\}. \tag{4.14}$$

For any compactly-supported and locally-integrable function $f : \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}$ define the mollifiers

$$[f]_{,k}(x,\lambda) := k \int_{\mathbb{R}} \theta(k(\lambda - t)) f(x,t)dt, \tag{4.15}$$

$$[f]_{m,}(x,\lambda) := m^n \int_{\mathbb{R}^n} \Theta(m(x-y)) f(y,\lambda)\,dy \quad \Theta(z) = \prod_{i=1}^n \theta(z_i). \tag{4.16}$$

We shall write simply $[f]_{m,k}$ instead of $[[f]_{m,}]_{,k}$. The following Lemma is due to Di-Perna and Lions [29].

LEMMA 4.5. *Let $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R})$ and $\mathbf{v} \in H^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$. Then for any bounded measurable set $E \subset \mathbb{R}^n \times \mathbb{R}$,*

$$\operatorname{div}[f\mathbf{v}]_{m,} - \operatorname{div}([f]_{m,}\mathbf{v}) \to 0 \quad \textit{in } L^1(E) \textit{ as } m \to \infty. \tag{4.17}$$

The kinetic theory of transport equations operates with the extended velocity field

$$\mathbf{V}(x,\lambda) = (\mathbf{v}(x), -\lambda \operatorname{div} \mathbf{v}(x)),$$

which does not meet the requirements of the previous Lemma. Nevertheless, because of the specific structure of $\mathbf{V}$, the following assertion holds true, see [5].

LEMMA 4.6. *Assume that all the assumptions of Lemma 4.5 are satisfied, $f \in L^\infty(\mathbb{R}^n \times \mathbb{R})$ and*

$$I^{k,m}(x,\lambda) = \operatorname{div}[f\mathbf{V}]_{m,k} - \operatorname{div}([f]_{m,k}\mathbf{V}). \tag{4.18}$$

Then, for any bounded measurable set $E \subset \mathbb{R}^n \times \mathbb{R}$,

$$\lim_{k\to\infty}\{\lim_{m\to\infty} I^{m,k}\} = 0 \quad \textit{in } L^1(E). \tag{4.19}$$

4.5. *Young measures*

Since the notion of weak limits plays a crucial role in our analysis, we begin with a short description of some basic facts concerning weak convergence and weak compactness. We refer the reader to [36] for the proofs of basic results.

Let A be an arbitrary bounded, measurable subset of $\mathbb{R}^3$ and $1 < r \leq \infty$. Then for every bounded sequence $\{g_n\}_{n\geq 1} \subset L^r(A)$ there exists a subsequence, still denoted by $\{g_n\}$, and a function $g \in L^r(A)$ such that for $n \to \infty$,

$$\int_A g_n(x)h(x)dx \to \int_A g(x)h(x)dx \quad \text{for all } h \in L^{r/(r-1)}(A).$$

We say the sequence converges $g_n \to g$ weakly in $L^r(A)$ for $r < \infty$, and converges star-weakly in $L^\infty(A)$ in the limit case of $r = \infty$. In the very special case of $r = 1$ it is known that the sequence of g_n contains a weakly convergent subsequence in $L^1(A)$, if and only if there is a continuous function $\Phi : \mathbb{R} \to \mathbb{R}^+$ such that

$$\lim_{s\to\infty} \Phi(s)/s = \infty \quad \text{and} \quad \sup_{n\geq 1} \|\Phi(g_n)\|_{L^1(A)} < \infty.$$

If the sequence of g_n is only bounded in $L^1(A)$ and A is open, then after passing to a subsequence we can assume that g_n converges star-weakly to a bounded Radon measure μ_g i.e.,

$$\begin{aligned}\lim_{n\to\infty} &\int_A g_n(x)h(x)dx \\ &= \int_A h(x)d\mu_g(x) \quad \text{for all compactly supported } h \in C(A).\end{aligned}$$

In the sequel, the linear space of compactly supported functions on a set A is denoted by $C_0(A)$, and its dual by $C_0(A)^*$.

The Ball's version [6], see also [36], of the fundamental Tartar Theorem on Young measures gives a simple and effective representation of weak limits in the form of integrals over families of probabilities measures. The following lemma is a consequence of Ball's theorem.

LEMMA 4.7. *Suppose that a sequence $\{g_n\}_{n\geq 1}$ is bounded in $L^r(A)$, $1 < r \leq \infty$, where A is an open, bounded subset of $\mathbb{R}^3$. Then we have the following characterizations of weak limits.*

(i) *There exists a subsequence, still denoted by $\{g_n\}_{n\geq 1}$, and a family of probability measures $\sigma_x \in C_0(\mathbb{R})^*$, $x \in A$, with a measurable distribution function $f(x,\lambda) := \sigma_x(-\infty,\lambda]$ so that the function $\lambda \mapsto f(x,\lambda)$ is monotone and continuous from the right, and admits the limits $1, 0$ for $\lambda \to \pm\infty$. Furthermore, for any continuous function $G : A \times \mathbb{R} \mapsto \mathbb{R}$ such that*

$$\lim_{|\lambda|\to\infty} \|G(\cdot,\lambda)\|_{C(A)}/|\lambda|^r = 0 \quad \textit{for } r < \infty \quad \textit{and}$$
$$\sup_{|\lambda|} \|G(\cdot,\lambda)\|_{C(A)} < \infty \quad \textit{for } r = \infty,$$

the sequence of $G(\cdot, g_n)$ converges weakly in $L^1(A)$ to a function

$$\overline{G}(x) = \int_{\mathbb{R}} G(x,\lambda)d_\lambda f(x,\lambda). \tag{4.20}$$

Moreover, for $r < \infty$, the function

$$A \ni x \to \int_{\mathbb{R}} |\lambda|^r \, d_\lambda f(x,\lambda) \in \mathbb{R}$$

belongs to $L^1(A)$.

(ii) *If $G(x,\cdot)$ is convex and the sequence g_n converges weakly (star-weakly for $r = \infty$) to $g \in L^r(A)$, then $\overline{G}(x) \leq G(x, g(x))$. If the functions g_n satisfy the inequalities $g_n \leq M$ (resp. $g_n \geq m$), then $f(x,\lambda) = 1$ for $\lambda \geq M$ (resp. $f(x,\lambda) = 0$ for $\lambda < m$).*

(iii) *If* $f(1-f) = 0$ *a.e. in* A, *then the sequence* g_n *converges to* g *in measure, and hence in* $L^s(A)$ *for positive* $s < r$. *Moreover, in this case* $f(x, \lambda) = 0$ *for* $\lambda < g(x)$ *and* $f(x, \lambda) = 1$ *for* $\lambda \geq g(x)$.

5. Boundary value problems for transport equations

The first-order scalar differential equation

$$\mathbf{u} \cdot \nabla \varrho + c\varrho = f, \tag{5.1}$$

which is called the transport equation, is one of the basic equations of mathematical physics. It is widely used for mathematical modeling of the mass and heat transfer and plays an important role in the kinetic theory of such phenomena. In the frame of theory of the compressible Navier–Stokes equations the most important examples of transport equations are: the steady mass balance equation,

$$\operatorname{div}(\varrho \mathbf{u}) = 0, \tag{5.2}$$

and "relaxed" mass balance equation

$$\operatorname{div}(\varphi \mathbf{u}) + \alpha\varphi = h, \tag{5.3}$$

where a given vector field $\mathbf{u}$ belongs to the Sobolev space $H^{1,2}(\Omega)$ and satisfies the boundary condition

$$\mathbf{u} = \mathbf{U} \quad \text{on } \partial\Omega. \tag{5.4}$$

The typical boundary value problem for the transport equation can be formulated as follows. Split the boundary of Ω into three disjoint parts: inlet Σ_{in}, outgoing set Σ_{out}, and the characteristic set Σ^0 defined by inequalities (2.8). The boundary value problem is to find a solution to differential equation (5.1) which takes the prescribed value φ_b at the inlet Σ_{in}.

Suppose for a moment that the vector field $\mathbf{u}$ has continuous derivatives and does not vanish in Ω. If a C^1 curve $l : x = x(s)$, $0 \leq s \leq s^*$ is the integral line of $\mathbf{u}$, i.e., solution of ODE

$$\frac{dx}{ds} = \mathbf{u}(x),$$

then equation (5.1) can be rewritten as the ordinary differential equation

$$\frac{d\varphi(x(s))}{ds} + c(x(s))\varphi(x(s)) = h(x(s))$$

along the line l. If in addition for each point $x^* \in \Omega$, there is a unique integral line such that $x(0) \in \Sigma_{\text{in}}$ and $x(s^*) = x^*$, then any solution of the transport equation is completely defined by the boundary data. Therefore we can apply the classical method of characteristics for solving (5.1). Note that if inlet Σ^+ is not an isolated component of $\partial\Omega$ ($\operatorname{cl} \Sigma^+ \cap \operatorname{cl}(\partial\Omega \setminus \Sigma^+) \neq \emptyset$), then in the general case a solution to the boundary value problem for transport equation is not smooth. Moreover it is easy to construct an example of plane domain Ω such that a solution of simplest transport equation $\partial_{x_1}\varrho = 1$ with zero boundary data has a jump of discontinuity at $\mathrm{cl}\,\Sigma_{\mathrm{in}} \cap \mathrm{cl}\,\Sigma_{\mathrm{out}}$.

The method of characteristics does not work if the totality of integral lines has a complicated structure, for example if $\mathbf{u}$ has rest points within Ω, and when the velocity field is not smooth and therefore integral lines are not well defined.

To address the first issue note that the theory of linear transport equations is an integral part of the general theory of elliptic–parabolic equations known also as the theory of second-order equations with nonnegative quadratic form. Such a theory deals with the general second-order partial differential equation

$$-\sum_{i,j=1}^{n} a_{ij}\partial_{x_i}\partial_{x_j}\varphi + \mathbf{v}\cdot\nabla\varphi + c\varphi = h \tag{5.5}$$

under the assumption

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq 0 \quad \text{for all } x \in \Omega \quad \text{and} \quad \xi \in \mathbb{R}^n.$$

Boundary value problems for second-order PDE with nonnegative quadratic form were studied by many authors starting from the pioneering papers of Fichera [15], Kohn and Nirenberg [26], and Oleinik and Radkevich [47]. In this paragraph we give a short review of available results.

Weak solutions. The first result on the existence of weak solutions to the general equation (5.5) is due to Fichera. Note that the definition of weak solutions proposed by Fichera is not standard since the used set of test functions does not admit the algorithmic description. The complete theory of integrable weak solutions to elliptic–parabolic equations was developed by Oleinik and Radkevich. The following theorem on existence and uniqueness of weak solutions to boundary value problems

$$\varphi = 0 \quad \text{on } \partial\Omega \tag{5.6}$$

for equation (5.1) is a particular case of the Oleinik results, we refer to Theorems 1.5.1 and 1.6.2 in [47].

Recall that a function $\varphi \in L^1(\Omega)$ is a weak solution to problem (5.1), (5.6) if the integral identity

$$\int_\Omega (\varphi\mathcal{L}^*\zeta - f\zeta)\,dx = 0 \tag{5.7}$$

holds true for all test functions $\zeta \in C^1(\Omega)$ vanishing on Σ_{out}. Here the adjoint operator $\mathcal{L}^*$ are defined by

$$\mathcal{L}^*\varphi := -\operatorname{div}(\mathbf{u}\varphi) + c\varphi. \tag{5.8}$$

THEOREM 5.1. *Let Ω be a bounded domain with the boundary of a class C^2 and $1 < r \le \infty$. Assume that the vector field $\mathbf{u} \in C^1(\Omega)$ and a function $c \in C(\Omega)$ satisfy the condition*

$$\delta = \inf_{x\in\Omega} (c(x) - r^{-1} \operatorname{div} \mathbf{u}(x)) > 0.$$

Then problem (5.1), (5.6) *has a weak solution $\varphi \in L^r(\Omega)$ satisfying the inequality*

$$\|\varphi\|_{L^r(\Omega)} \le \delta^{-1} \|f\|_{L^r(\Omega)}.$$

If, in addition, $r > 3$ and the intersection $\Gamma = cl\,(\Sigma_{\text{out}} \cup \Sigma_0) \cap cl\, \Sigma_{\text{in}}$ is a smooth one-dimensional manifold, then a weak solution $\varphi \in L^r(\Omega)$ to problem (5.1), (5.6) *is unique.*

Moreover, in [47] it was shown that weak solutions are continuous at interior points of Σ_{in} and take the boundary value in a classic sense.

It is worthy of note that, under the assumptions of the theorem, the operator $\mathfrak{L}$ with the domain of definition $D(\mathfrak{L}) = \{\varphi \in L^r(\Omega) : \mathbf{u}\nabla\varphi \in L^r(\Omega),\ \varphi = 0 \text{ on } \Sigma_{\text{in}}\}$ is m-accretive.

Strong solutions. The question on the regularity of solutions to boundary value problems for transport equations is more difficult. All known results [26,47] related to the case of multi-connected domains with isolated inlet. We illustrate the theory by two theorems. The first is a consequence of the general result of Kohn and Nirenberg, see [26], on solvability boundary value problems for elliptic–parabolic equations.

THEOREM 5.2. *Let $\Omega \in \mathbb{R}^d$ be a bounded domain with a boundary $\partial\Omega \in C^\infty$, $\mathbf{u}, c \in C^\infty(\Omega)$, and $k \ge 1$ be an arbitrary integer. Furthermore assume that*

$$\text{cl}\ \Sigma_{\text{in}} \cap (\Sigma_{\text{out}} \cup \Sigma_0) = \emptyset, \tag{5.9}$$

and $c > c_0 > 0$, where c_0 is sufficiently large constant depending only on Ω $\|\mathbf{u}\|_{C^3(\Omega)}$, $\|c\|_{C^3(\Omega)}$, and k. Then for any $f \in H^{2,k}(\Omega)$ problem (5.1), (5.6) *has a unique solution satisfying the inequality*

$$\|\varphi\|_{H^{2,k}(\Omega)} \le C(k, \Omega, \mathbf{u}, c) \|f\|_{H^{2,k}(\Omega)}.$$

The second result is a consequence of Theorem 1.8.1 in the monograph [47].

THEOREM 5.3. *Assume that Ω is a bounded domain with the boundary of the class C^2 and $\mathbf{u}, c, f \in C^1(\mathbb{R}^d)$. Furthermore, let the following conditions hold.*

(1) *The vector field $\mathbf{U} = \mathbf{u}\big|_{\partial\Omega}$ and the manifold Ω satisfy condition* (5.9).

(2) *There is $\Omega' \Supset \Omega$ such that the inequality*

$$c(x) - \sup_{\Omega'} \left\{ |\operatorname{div} \mathbf{u}| - \frac{1}{2} \sup_i \sum_{j\ne i} \left|\frac{\partial u_i}{\partial x_j}\right| - \frac{1}{2} \sup_j \sum_{j\ne i} \left|\frac{\partial u_j}{\partial x_i}\right| \right\} > 0$$

is fulfilled. Then a weak solution to problem (5.1), (5.6) *satisfies the Lipschitz condition in* cl Ω.

If the vector field $\mathbf{u}$ satisfies the nonpermeability condition

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega,$$

then $\Sigma_{\text{in}} = \Sigma_{\text{out}} = \emptyset$ and we do not need the boundary conditions. This particular case was investigated in detail by Beirao Da Veiga in [9] and Novotny in [45], [46]. The following theorem is due to Berirao da Veiga

THEOREM 5.4. *Let $1 \le l \le k$ be arbitrary integers, $r \in (1, \infty)$, and $\Omega \subset \mathbb{R}^d$ be a bounded domain with the boundary $\partial\Omega \in C^k$. Let*

$$\mathbf{u}, c \in C^k(\Omega), \quad f \in H^{k,r}(\Omega) \cap H_0^{l,r}(\Omega), \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Then there exists a constant c_Ω depending only on k,r, and Ω such that for all

$$\sigma > \sigma^* \equiv c_\Omega(\|\mathbf{u}\|_{C^k(\Omega)} + \|c\|_{C^k(\Omega)})$$

the equation

$$\mathfrak{L}\varphi + \sigma\varphi \equiv \mathbf{u}\nabla\varphi + c(x)\varphi + \sigma\varphi = f$$

has a unique solution $\varphi \in H^{k,r}(\Omega)$ satisfying the inequality

$$\|\varphi\|_{H^{k,r}(\Omega)} \le (\sigma - \sigma^*)^{-1}\|f\|_{H^{k,r}(\Omega)}.$$

In [45] these results were extended to a broad class of domains including $\mathbb{R}^d$, $\mathbb{R}_+^d$, and exterior domains with compact complements. The case of noninteger k was covered in [46].

5.1. *Strong solutions. General case*

As was mentioned above, there are no results on regularity of solutions to transport equations in the general case of nonempty intersections of inlet and outgoing set. In this paragraph we formulate the theorem which partially fills this gap. With application to compressible Navier–Stokes equations in mind we restrict our considerations by the case when $\Omega \subset \mathbb{R}^3$ is a bounded domain and

$$\Gamma := \text{cl } \Sigma_{\text{in}} \cap \text{cl } \Sigma_{\text{out}} = \text{cl } \Sigma_{\text{in}} \cap \text{cl } \Sigma_0 \tag{5.10}$$

is a smooth one-dimensional manifold. For simplicity we shall assume that $\partial\Omega \in C^\infty$, $c(x) = \sigma = \text{const.}$, and consider the boundary value problem

$$\mathfrak{L}\varphi := \mathbf{u}\nabla\varphi + \sigma\varphi = f \quad \text{in } \Omega, \quad \varphi = 0 \quad \text{on } \Sigma_{\text{in}}. \tag{5.11}$$

The main difficulty of the problem is that the smoothness properties of solutions are very sensitive to the behavior of the vector field $\mathbf{u}$ near the characteristic manifold Γ. Further we shall assume that a characteristic manifold and a vector field $\mathbf{U} = \mathbf{u}|_{\partial\Omega}$ satisfy the following conditions, referred to as the *emergent vector field conditions.*

(H1) The boundary of Ω belongs to class C^∞. For each point $P \in \Gamma$ there exists the local Cartesian coordinates (x_1, x_2, x_3) with the origin at P such that in the new coordinates $\mathbf{U}(P) = (U, 0, 0)$ with $U = |\mathbf{U}(P)|$, and $\mathbf{n}(P) = (0, 0, -1)$. Moreover, there is a neighborhood $\mathcal{O} = [-k, k]^2 \times [-t, t]$ of P such that the intersections $\Sigma \cap \mathcal{O}$ and $\Gamma \cap \mathcal{O}$ are defined by the equations

$$F_0(x) \equiv x_3 - F(x_1, x_2) = 0, \quad \nabla F_0(x) \cdot \mathbf{U}(x) = 0,$$

and $\Omega \cap \mathcal{O}$ is the epigraph $\{F_0 > 0\} \cap \mathcal{O}$. The function F satisfies the conditions

$$\|F\|_{C^2([-k,k]^2)} \le K, \quad F(0, 0) = 0, \quad \nabla F(0, 0) = 0, \tag{5.12}$$

where the constants $k, t < 1$ and $K > 1$ depend only on the curvature of Σ and are independent of the point P.

(H2) The manifold $\Gamma \cap \mathcal{O}$ admits the parameterization

$$x = \mathbf{x}^0(x_2) := (\Upsilon(x_2), x_2, F(\Upsilon(x_2), x_2)), \tag{5.13}$$

such that $\Upsilon(0) = 0$ and $\|\Upsilon\|_{C^2([-k,k])} \le C_\Gamma$, where the constant $C_\Gamma > 1$ depends only on Σ and $\mathbf{U}$.

(H3) There are positive constants $N^\pm$ independent of P such that for $x \in \Sigma \cap \mathcal{O}$ we have

$$\begin{aligned} N^-(x_1 - \Upsilon(x_2)) &\le -\nabla F_0(x) \cdot \mathbf{U}(x) \\ &\le N^+(x_1 - \Upsilon(x_2)) \quad \text{for } x_1 > \Upsilon(x_2), \\ -N^-(x_1 - \Upsilon(x_2)) &\le \nabla F_0(x) \cdot \mathbf{U}(x) \\ &\le -N^+(x_1 - \Upsilon(x_2)) \quad \text{for } x_1 < \Upsilon(x_2). \end{aligned} \tag{5.14}$$

These conditions are obviously fulfilled for all strictly convex domains and constant vector fields. They have simple geometric interpretation, that $\mathbf{U} \cdot \mathbf{n}$ only vanishes up to the first order at Γ, and for each point $P \in \Gamma$, the vector $\mathbf{U}(P)$ points to the part of Σ where $\mathbf{U}$ is an exterior vector field. Note that *emergent vector field* condition plays an important role in the theory of the classical oblique derivative problem, see [21]. It seems that this condition is necessary for continuity of solutions to problem (5.11).

THEOREM 5.5. *Assume that Σ and $\mathbf{U}$ satisfy conditions* (**H1**)–(**H3**), *the vector field* $\mathbf{u}$ *belongs to the class $C^1(\Omega)$, and satisfies the boundary condition*

$$\mathbf{u} = \mathbf{U} \quad \text{on } \Sigma, \qquad \mathbf{u} = 0 \quad \text{on } \partial S. \tag{5.15}$$

Furthermore, let s, r and α be the constants satisfying

$$\begin{aligned} 0 < s \le 1, \quad 1 < r < \infty, \quad 2s - 3/r < 1, \\ \max\{s, 2s - 3/r\} < \alpha < \min\{2s, 1\} \quad \textit{for } 0 < s < 1, \\ \alpha = 1 - 1/r \quad \textit{for } s = 1 \quad \textit{and} \quad 1 < r < 2, \\ 2 - 3/r < \alpha < 1 \quad \textit{for } s = 1 \quad \textit{and} \quad 2 \le r < 3r < 2. \end{aligned} \tag{5.16}$$

Then there are positive constants $\sigma^* > 1$, and C depending only on Σ, $\mathbf{U}$, s, r, α, and $\|\mathbf{u}\|_{C^1(\Omega)}$ such that: for any $\sigma > \sigma^*$ and $f \in H^{s,r}(\Omega) \cap L^\infty(\Omega)$ problem (5.11) *has a unique solution $\varphi \in H^{s,r}(\Omega) \cap L^\infty(\Omega)$, which admits the estimates*

$$\|\varphi\|_{H^{s,r}(\Omega)} \le C\sigma^{-1}\|f\|_{H^{s,r}(\Omega)} + \sigma^{-1+\alpha}C\|f\|_{L^\infty(\Omega)}. \tag{5.17}$$

PROOF. The proof is given in Section 10. □

Since for $sr > 3$ the Sobolev space $H^{s,r}(\Omega)$ is the Banach algebra, Theorem 5.5 along with the contraction mapping principle implies the following result on solvability of the adjoint problem

$$\mathcal{L}^*\varphi := -\operatorname{div}(\varphi\mathbf{u}) + \sigma\varphi = f \quad \text{in } \Omega, \qquad \varphi = 0 \quad \text{on } \Sigma_{\text{out}}. \tag{5.18}$$

THEOREM 5.6. *Let Σ and $\mathbf{U}$ comply with hypotheses* (**H1**)–(**H3**), *the vector field $\mathbf{u} \in C^1(\Omega)$ satisfies boundary condition* (5.15), *exponents s, r, α satisfy conditions* (5.16). *Moreover, assume that $sr > 3$ and* $\operatorname{div}\mathbf{u} \in H^{s,r}(\Omega)$. *Then there are positive constants $\sigma^* > 1$, C depending only on Σ, $\mathbf{U}$, s, r, α, $\|\mathbf{u}\|_{C^1(\Omega)}$, and $\|\operatorname{div}\mathbf{u}\|_{H^{s,r}(\Omega)}$ such that: for any $\sigma > \sigma^*$ and $f \in H^{s,r}(\Omega) \cap L^\infty(\Omega)$ problem* (5.18) *has a unique solution $\varphi \in H^{s,r}(\Omega) \cap L^\infty(\Omega)$, which admits the estimates*

$$\|\varphi\|_{H^{s,r}(\Omega)} \le C\sigma^{-1}\|f\|_{H^{s,r}(\Omega)} + \sigma^{-1+\alpha}C\|f\|_{L^\infty(\Omega)}. \tag{5.19}$$

Solution to problem (5.11) is continuous if $sr > 3$, which along with (5.16) yields inequality $s < 1$. Therefore in this case φ is only a weak solution. Under the assumptions of Theorem 5.5, a solution to problem (5.11) is strong, if $s = 1$. But in this case inequalities (5.16) yield $r < 3$, which does not guarantee the continuity property. In order to obtain strong continuous solutions we introduce the Banach space $X^{s,r} = H^{s,r}(\Omega) \cap H^{1,r_0}(\Omega)$ equipped with the norm

$$\|u\|_{X^{s,r}} = \|u\|_{H^{s,r}(\Omega)} + \|u\|_{H^{1,r_0}(\Omega)}.$$

We shall assume that exponents s, r, r_0, and α satisfy conditions (5.16) and the inequalities

$$rs > 3, \quad 0 < \alpha = 1 - 1/r_0 < 1, \quad r_0 < 3. \tag{5.20}$$

It follows from the embedding theory that in this case $X^{s,r}$ is a Banach algebra. Theorems 5.5 and 5.6 imply the following result.

THEOREM 5.7. *Let Σ and $\mathbf{U}$ comply with hypotheses* (**H1**)–(**H3**), *a vector field $\mathbf{u}$ satisfies boundary condition* (5.15), *exponents s, r, r_0, α satisfy conditions* (5.16) *and* (5.20). *Moreover, assume that $\|\mathbf{u}\|_{H^{1+s,r}(\Omega)} < R < \infty$. Then there are positive constants $\sigma^* > 1$ and C depending only on Σ, $\mathbf{U}$, s, r, r_0, α, and R such that: for any $\sigma > \sigma^*$ and $f \in X^{s,r}$, each of the problems* (5.11), (5.18) *has a unique solution $\varphi \in X^{s,r}$ which admits the estimate*

$$\|\varphi\|_{X^{s,r}} \le C\sigma^{-1}\|f\|_{X^{s,r}} + \sigma^{-1+\alpha}C\|f\|_{L^\infty(\Omega)}, \tag{5.21}$$

$$\|\varphi\|_{C(\Omega)} \le (\sigma - cR)^{-1}\|f\|_{C(\Omega)}. \tag{5.22}$$

6. Transport equations with discontinuous coefficient

The theory of linear transport equations with discontinuous coefficients originated in the pioneering paper [29] by Di-Perna and Lions. The key point of the theory is the concept of *renormalized* solutions as a new class of generalized solutions to linear problems which play a similar role as the Kruzhkov entropy solutions in the theory of scalar nonlinear conservation laws. The main result obtained in [29] is the existence and uniqueness of renormalized solutions to the Cauchy problem for transport equations generated by vector fields with integrable derivatives and bounded divergence. Note that the Cauchy problem for nonstationary transport equations is a particular case of boundary value problem (5.1), (5.6) with the domain $\Omega = \mathbb{R}^{d+1}$ the vector field $\mathbf{u} = (1, u_1, \dots, u_d))$. Nowadays we have the developed theory of the Cauchy problem for transport equations with discontinuous coefficients, see [5] and [28] for the overview. But all available results related to transport equations generated by vector fields $\mathbf{u}$ with $\operatorname{div} \mathbf{u}$ bounded from below, while the theory of compressible Navier–Stokes equations operates with vector fields whose divergence only is integrable with square. In this section we prove the compactness of a totality renormalized solution's relaxed mass balance equation

$$\operatorname{div}(\mathbf{u}\varrho) = h \in L^r(\Omega) \quad \text{in } \Omega \tag{6.1}$$

with a vector field $\mathbf{u}$ satisfying the conditions

$$\mathbf{u} \in H_0^{1,2}(\Omega). \tag{6.2}$$

Before formulation of the results we recall some basic facts and definitions.

6.1. *Renormalized solutions*

We begin with the definition of renormalized solution to equation (6.1).

DEFINITION 6.1. For a given $\mathbf{u} \in H^{1,2}(\Omega)$ satisfying conditions (6.2) and $h \in L^1(\Omega)$, a renormalized solution to equation (6.1), is a function $\varrho \in L^1(\Omega)$, with $\varrho \mathbf{v} \in L^1(\Omega)$ such that the integral identity

$$\int_\Omega (G(\varrho)\mathbf{u} \cdot \nabla \psi + (G(\varrho) - G'(\varrho)\rho)\psi \operatorname{div} \mathbf{u})\, dx + \int_\Omega G'(\varrho) h(x) \psi \, dx = 0 \tag{6.3}$$

holds for any functions $\psi \in C^1(\Omega)$, and any function $G \in C^1_{\mathrm{loc}}[0, \infty)$ with the properties

$$\limsup_{|s| \to \infty} |G'(s)| < \infty,$$
$$[0, \infty) \ni s \mapsto G(s) - G'(s)s \in \mathbb{R} \quad \text{is continuous and bounded.}$$

The generic property of equation (6.1) is extendability of renormalized solutions through $\partial\Omega$ onto $\mathbb{R}^3$. Define the extensions of the vector field $\mathbf{u}(x)$ and the function h onto $\mathbb{R}^3$ by the equalities,

$$\mathbf{u}(x) = 0, \quad h(x) = 0 \quad \text{for } x \in \mathbb{R}^3 \setminus \Omega. \tag{6.4}$$

LEMMA 6.2. *Let ϱ be a renormalized solution to* (6.1). *Then the extended functions serves as renormalized solution to equation* (6.1) *in* $\mathbb{R}^3$.

Proof obviously follows from Definition 6.1. □

It is clear that each renormalized solution to transport equation is a weak solution. The inverse is also true if, for instance $\varrho \in L^2(\Omega)$. By virtue of the extension principle it is sufficient to prove this fact in the case when $\Omega = \mathbb{R}^3$ and $\mathbf{u}$, h and ϱ are compactly supported in $\mathbb{R}^3$. Fix an arbitrary point $x_0 \in \mathbb{R}^3$. Denote by Θ the mollification kernel in $\mathbb{R}^3$ and recall formula (4.16). Substituting $\psi(x) = m^n\Theta(m(x_0 - x))$ into integral identity (6.3) with $G(\varrho) = \varrho$ leads to the equality

$$\operatorname{div}[\varrho\mathbf{u}]_{m,}(x_0) = [h]_{m,}(x_0),$$

which holds true for all $x_0 \in \mathbb{R}^3$. Now fix an arbitrary function G satisfying (6.4). Multiplying both the sides of the last identity by $G'([\varrho]_{m,})$ and noting that the function $[\varrho]_{m,}$ has continuous derivatives of all orders we arrive at the equality

$$\begin{aligned}&\operatorname{div}(G([\varrho]_{m,})\mathbf{u}) + (G'([\varrho]_{m,})[\varrho]_{m,} - G([\varrho]_{m,})\operatorname{div}\mathbf{u}\\&\quad + G'([\varrho]_{m,})(\mathfrak{r}^m - [h]_{m,}) = 0,\end{aligned}$$

where

$$\mathfrak{r}^m = \operatorname{div}[\varrho\mathbf{u}]_{m,} - \operatorname{div}([\varrho]_{m,}\mathbf{u}).$$

Multiplying both the sides by a function $\psi \in C_0^\infty(\mathbb{R}^3)$ and integrating the result over $\mathbb{R}^3$ we conclude that

$$\begin{aligned}&\int_{\mathbb{R}^3}(G([\varrho]_{m,})\mathbf{u}\cdot\nabla\psi - (G'([\varrho]_{m,})[\varrho]_{m,} - G([\varrho]_{m,})\operatorname{div}\mathbf{v}\psi)\,dx\\&\quad + \int_{\mathbb{R}^3} G'([\varrho]_{m,})([h]_{m,} - \mathfrak{r}^m)\psi\,dx = 0.\end{aligned}$$

Since ψ is compactly supported, Lemma 4.5 yields

$$\lim_{m\to\infty}\int_{\mathbb{R}^3}\mathfrak{r}^m\psi\,dx = 0.$$

Letting $m \to \infty$ and noting that $[\varrho]_{m,}$ tends to ϱ in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ we conclude that ϱ satisfies integral identity (6.3). Therefore, ϱ is the renormalized solution to equation (6.1) which is the desired conclusion.

6.2. *Kinetic formulation of a transport equation*

If for a differential equation, linear or nonlinear, the renormalization procedure can be performed, then the equation can be equivalently rewritten as a linear differential equation for the so-called *distribution function* in the extended space of (x, λ), where λ is the extra *kinetic variable*. This fact underlies the *kinetic equation method*, which is one of the most powerful methods in the modern PDE theory. The method of kinetic equations has been created and applied recently to study a wide range of problems, for example, to study the equations of isentropic gas dynamics and p-systems and the first- and second-order quasilinear conservation laws [11,30,31]. In this paragraph we explain how the kinetic formulation of transport equations can be obtained from the definition of renormalized solution.

Assume that a renormalized solution to (6.1) meets all requirements of Definition 6.1. For each $(x, \lambda) \in \Omega \times \mathbb{R}$, define the distribution function $f(x, \lambda)$ by the equalities

$$f(x, \lambda) = 0 \quad \text{for } \lambda < \varrho(x), \qquad f(x, \lambda) = 1 \quad \text{otherwise.} \tag{6.5}$$

Obviously, for a.e. $x \in \Omega$, the distribution function $f(x, \cdot)$ is monotone and continuous from the right with respect to λ, and the associated Stieltjes measure is given by

$$d_\lambda f(x, \cdot) = \partial_\lambda f(x, \lambda) = \delta(\lambda - \varrho(x)),$$

where $\delta(\lambda - \varrho(x))$ is the Dirac measure concentrated at $\varrho(x)$. In particular, for each continuous function $G : \mathbb{R} \mapsto \mathbb{R}$,

$$G(\varrho(x)) = \int_{\mathbb{R}} G(\lambda) d_\lambda f(x, \lambda) \quad \text{a.e. in } \Omega.$$

LEMMA 6.3. *Under the above assumptions the function $f(x, \lambda)$ satisfies the extended transport equation* (6.6)

$$\Omega : \ \operatorname{div}(f(x, \lambda)\mathbf{V}(x, \lambda)) + h(x) f(x, \lambda) = 0, \tag{6.6}$$

which is understood in the sense of distributions. Here, the velocity field $\mathbf{V} : \Omega \times \mathbb{R} \mapsto \mathbb{R}^4$ is defined by

$$\mathbf{V}(x, \lambda) = (\mathbf{u}(x), -\lambda \operatorname{div} \mathbf{u}(x)). \tag{6.7}$$

PROOF. Choose an arbitrary $\psi \in C^\infty(\Omega)$ vanishing near $\partial D \setminus \Sigma^+$ and a function $\eta \in C^\infty(\mathbb{R})$. Set

$$G(\lambda) = \int_\lambda^\infty \eta(s)\, ds.$$

Note that

$$\begin{aligned} G(\varrho(x)) &= \int_{\mathbb{R}} \left(\int_s^\infty \eta(\lambda) d\lambda \right) d_s f(x, s) \\ &= \int_{\mathbb{R}} \eta(\lambda) \left(\int_{(-\infty, \lambda]} d_s f(x, s) \right) d\lambda \\ &= \int_{\mathbb{R}} \eta(\lambda) f(x, \lambda)\, d\lambda. \end{aligned}$$

Repeating these arguments gives

$$\begin{aligned} G(\varrho) - G'(\varrho)\varrho &= \int_{\mathbb{R}} \eta(\lambda) f(x, \lambda)\, d\lambda + \int_{\mathbb{R}} \lambda \eta(\lambda) d_\lambda f(x, \lambda) \\ &= -\int_{\mathbb{R}} \lambda f(x, \lambda) \eta'(\lambda)\, d\lambda, \end{aligned}$$

$$G(\varrho)a(x,\varrho) = \int_{\mathbb{R}} \eta(\lambda)a(x,\lambda)f(x,\lambda)\,d\lambda \\ - \int_{\mathbb{R}} \eta(\lambda)\left(\int_{-\infty}^{\lambda} a(x,s)f(x,s)\,ds\right)d\lambda.$$

Substituting these inequalities into integral identity (6.3) we arrive at

$$\int_{\Omega\times\mathbb{R}} f(x,\lambda)(\mathbf{u}(x)\cdot\nabla_x\psi\eta - \lambda\,\mathrm{div}\,\mathbf{u}(x)\psi\partial_\lambda\eta + \alpha(h(x)-a(x,\lambda))\psi\eta)\,d\lambda\,dx \\ + \int_{\Omega\times\mathbb{R}} \psi\eta(\lambda)\left(\int_{-\infty}^{\lambda} \partial_s a(x,s)f(x,s)\,ds\right)d\lambda\,dx \\ + \int_{\Sigma^+\times\mathbb{R}} f_\infty(\lambda)\psi\eta(\lambda)\mathbf{U}_\infty\cdot\mathbf{n}\,d\lambda\,d\Sigma.$$

Noting that an arbitrary function $\zeta \in C^\infty(\Omega\times\mathbb{R})$ vanishing for all sufficiently large λ can be approximate in the norm $C^1(\Omega\times\mathbb{R})$ by linear combinations of functions $\psi(x)\eta(\lambda)$ we arrive at the integral identity

$$\int_{\Omega\times\mathbb{R}} f(x,\lambda)(\mathbf{u}(x)\cdot\nabla_x\zeta - \lambda\,\mathrm{div}\,\mathbf{u}(x)\partial_\lambda\zeta + \alpha(h(x)-a(x,\lambda))\zeta)\,d\lambda\,dx \\ + \int_{\Omega\times\mathbb{R}} \zeta(x,\lambda)\left(\int_{-\infty}^{\lambda} \partial_s a(x,s)f(x,s)\,ds\right)d\lambda\,dx \\ + \int_{\Sigma^+\times\mathbb{R}} f_\infty(\lambda)\zeta(x,\lambda)\mathbf{U}_\infty\cdot\mathbf{n}\,d\lambda\,d\Sigma,$$

which yields (6.6). □

The preferences of the kinetic formulation of the equation are:

- kinetic equation (6.6) is a linear equation in any case;
- the unknown function is a priori uniformly bounded and monotone with respect to kinetic variable;
- the velocity field $\mathbf{V}$ is divergence-free i.e., $\mathrm{div}\,\mathbf{V} = 0$ in any case and the kinetic transport equation is a *Liouville*-type equation.

6.3. *Compactness of solutions to mass conservation equation*

In this paragraph we prove the main theorem on compactness of renormalized solutions to equation (6.1). Assume that sequences of a vector fields $\mathbf{u}_n \in H^{1,2}(\mathbb{R}^3)$ and nonnegative functions $\varrho_n : \mathbb{R}^3 \mapsto \mathbb{R}^+$ satisfy the following conditions.

(H4) Vector fields $\mathbf{u}_n$ and functions ϱ_n vanish outside of bounded domain $\Omega \Subset \mathbb{R}^3$. There exist positive κ and c such that for all $n \ge 1$,

$$\|\mathbf{u}_n\|_{H^{1,2}(\mathbb{R}^3)} \le c, \qquad \int_{\mathbb{R}^3} p_n\,dx + \int_{\mathbb{R}^3} |\varrho_n\mathbf{u}_n|^{1+\kappa}\,dx \le c.$$

Here the functions $p_n = p(\varrho_n)$ are defined by the equalities

$$p_n(\varrho) = \varrho_n^{\gamma} + \varepsilon_n \varrho_n^{k}, \tag{6.8}$$

in which $1 < \gamma < k$ are given constants and a sequence of positive numbers ε_n tends to zero as $n \to \infty$. Moreover, for each compact $\Omega' \Subset \Omega$,

$$\int_{\Omega'} p_n^{1+\kappa}\, dx + \int_{\Omega'} (\varrho_n |\mathbf{u}_n|^2)^{1+\kappa}\, dx \le c(\Omega'),$$

where $c(\Omega')$ does not depend on n.

(H5) The vector fields $\mathbf{u}_n$ converges weakly in $H^{1,2}(\mathbb{R}^3)$ to a vector field $\mathbf{u}$. For each compact $E \subset \mathbb{R}^3$ and an arbitrary function $G : \mathbb{R} \to \mathbb{R}$ satisfying the condition $\lim_{\varrho\to\infty} \varrho^{-\gamma} G(\varrho) = 0$, the functions $G(\varrho_n))$ converges weakly in $L^1(E)$ to a function $\overline{G} \in L^1_{\text{loc}}(\mathbb{R}^3)$. Moreover, if G satisfies the more weak condition $\limsup_{\varrho\to\infty} \varrho^{-\gamma}|G(\varrho)| < \infty$, then the sequence $G(\varrho_n))$ converges weakly in $L^1(\Omega')$ the function $\overline{G}$ in any subdomain $\Omega' \Subset \Omega$. In particular,

$$\varrho_n^{\gamma} \to \overline{p} \in L^1(\Omega) \cap L^{1+\kappa}_{\text{loc}}(\Omega) \quad \text{weakly in } L^1(\Omega') \quad \text{for all } \Omega' \Subset \Omega.$$

(H6) There exists a function $h \in C_0(\mathbb{R}^3)$ such that the limiting relation

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} (G(\varrho_n)\mathbf{u}_n \cdot \nabla\psi + (G(\varrho_n) \\ -G'(\varrho_n)\varrho_n)\psi \operatorname{div} \mathbf{u}_n + G'(\varrho_n) h\psi)\, dx = 0 \tag{6.9}$$

holds true for any function $G \in C^2(\mathbb{R})$ with $\lim_{\varrho\to\infty} G''(\varrho) \to 0$, and any nonnegative function $\psi \in C_0^{\infty}(\mathbb{R}^3)$.

(H7) There exists a constant $\mu \neq 0$ such that for all functions $\psi \in C_0(\Omega)$ and $g \in C(\mathbb{R})$,

$$\liminf_{n\to\infty} \int_{\Omega} \psi g(\varrho_n)(p(\varrho_n) - \mu \operatorname{div} \mathbf{v}_n)\, dx = \int_{\Omega} \psi \overline{g}(\overline{p} - \mu \operatorname{div} \mathbf{v})\, dx.$$

THEOREM 6.4. *If sequences $\mathbf{u}_n$, ϱ_n satisfy conditions* **(H4)**–**(H7)**, *then $\varrho_n \to \varrho$ strongly in $L^r_{\text{loc}}(\mathbb{R}^3)$ for any $r \in [1, 1+\kappa)$.*

The rest of this section is devoted to the proof of Theorem 6.4. We begin with the following lemma which describes the properties of the Young measure associated with the sequence of ϱ_n.

LEMMA 6.5. *Under the assumptions of Theorem* 6.4 *there exists a distribution function $f : \mathbb{R}^n \times \mathbb{R} \mapsto [0,1]$ such that:*

(i)

$$f(x,\lambda) = 0 \quad \text{for } \lambda < 0 \quad \text{a.e. in } \mathbb{R}^3,$$

$$f(x,\lambda) = 1 \quad \text{for } \lambda \ge 0 \quad \text{a.e. in } \mathbb{R}^3 \setminus \Omega.$$

(ii) *f meets all requirements of Lemma* 4.7. *For any bounded set $E \subset \mathbb{R}^3$ and a continuous function $G : \mathbb{R}^3 \times \mathbb{R}$ with $\lim_{\rho\to\infty} \rho^{-\gamma} \|G(\cdot,\rho)\|_{C(E)} = 0$, the sequence $G(\cdot, \varrho_n)$ converges weakly in $L^1(A)$ to the function*

$$\overline{G}(x) = \int_{[0,\infty)} G(x,\lambda)\, d_\lambda f(x,\lambda) \quad \text{a.e. in } \mathbb{R}^n. \tag{6.10}$$

In particular, the weak limit of the sequence ϱ_n admits the representation

$$\rho(x) = \int_{[0,\infty)} \lambda\, d_\lambda f(x,\lambda) \equiv \int_{[0,\infty)} (1 - f(x,\lambda))\, d\lambda \quad \text{a.e. in } \mathbb{R}^3. \tag{6.11}$$

(iii) *There is a function* $\overline{p} \in L^1(\Omega) \cap L^{1+\kappa}_{\mathrm{loc}}(\Omega)$, *such that for any compact* $\Omega' \Subset \Omega$, *the sequence* ϱ^γ *converges weakly to* $\overline{p}$ *in* $L^{1+\kappa}_{\mathrm{loc}}(\Omega')$. *The function* $\overline{p}$ *vanishes outside* Ω *and admits the representation*

$$\overline{p}(x) = \int_{[0,\infty)} \lambda^\gamma \, d_\lambda f(x,\lambda) \equiv \gamma \int_{[0,\infty)} \lambda^{\gamma-1}(1 - f(x,\lambda))\, d\lambda. \tag{6.12}$$

PROOF. Assertions (i)–(ii) are the obvious consequence of Lemma 4.7 and conditions **(H4)**–**(H5)**. Recall that the sequence $\{p(\rho_n)\}$ is bounded in $L^{1+\kappa}(E)$ for any compact $E \Subset \mathbb{R}^3 \setminus \partial\Omega$. Hence the function $\overline{p}$ is well defined and belongs to the class $L^{1+\kappa}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \partial\Omega)$. Since $\|\overline{p}\|_{L^1(K)} \le \liminf_{n\to\infty} \|p(\rho_n)\|_{L^1(K)} \le c$, the function $\overline{p}$ is integrable in $\mathbb{R}^3$. □

6.4. *Kinetic equation*

In this paragraph we deduce the kinetic equation for the distribution function $f(x,\lambda)$ pointed out in Lemma 6.5.

LEMMA 6.6. *Under the assumption 6.4 the distribution function* $f(x,\lambda)$ *satisfies the equation*

$$\operatorname{div}(f\mathbf{V}) + \partial_\lambda(\lambda\mathcal{M}) + h\partial_\lambda f = 0 \quad \text{in } D'(\mathbb{R}^4). \tag{6.13}$$

Here bilinear operator $\mathcal{M}$ *and the divergence-free vector field* $\mathbf{V} : \mathbb{R}^4 \mapsto \mathbb{R}^4$ *defined by the equalities*

$$\begin{aligned}
\mathcal{M}(x,\lambda) &= -\frac{1}{\mu}\int_{(-\infty,\lambda)} (s^\gamma - \overline{p})\, d_s f(x,s) \\
&= \frac{1}{\mu}\int_{[\lambda,\infty)} (s^\gamma - \overline{p})\, d_s f(x,s),
\end{aligned} \tag{6.14}$$

$$\mathbf{V}(x,\lambda) = (\mathbf{u}(x), -\lambda \operatorname{div}\mathbf{u}(x)).$$

PROOF. Choose arbitrary functions $\psi \in C_0^\infty(\mathbb{R}^3)$, $\eta \in C_0^\infty(\mathbb{R})$ and set $G(\lambda) = \int_\lambda^\infty \eta(s)\, ds$. Substituting G and ψ into (6.9) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} (\overline{G}\mathbf{v}\cdot\nabla\psi + \overline{G'}h\psi)\, dx \\
&\quad + \lim_{n\to\infty}\int_{\mathbb{R}^3} (G(\varrho_n) - G'(\varrho_n)\varrho_n)\psi \operatorname{div}\mathbf{v}_n\, dx = 0.
\end{aligned} \tag{6.15}$$

Since the function G is continuous and vanishes near ∞, it follows from Lemma 6.5 that

$$\begin{aligned}
\overline{G}(x) &= \int_{\mathbb{R}} \left(\int_s^\infty \eta(\lambda) d\lambda\right) d_s f(x,s) = \int_{\mathbb{R}} \eta(\lambda) f(x,\lambda)\, d\lambda, \\
\overline{G'}(x) &= -\int_{\mathbb{R}} \eta(\lambda) d_\lambda f(x,\lambda).
\end{aligned}$$

Substituting these relations into (7.22) we arrive at the integral identity

$$\int_{\mathbb{R}^4} f(x,\lambda)\eta(\lambda)\mathbf{v}\cdot\nabla\psi\,dx d\lambda - \int_{\mathbb{R}^4} h(x))\eta\psi\,d_\lambda f(x,\lambda)\,dx \\ + \lim_{n\to\infty}\int_{\mathbb{R}^3}(G(\varrho_n)-G'(\varrho_n)\varrho_n)\psi\,\operatorname{div}\mathbf{u}_n\,dx = 0. \tag{6.16}$$

Denote by O_δ δ-neighborhood of $\partial\Omega$. Since the sequence $(G - G'\varrho_n)\operatorname{div}\mathbf{u}_n$ is bounded $L^2(\mathbb{R}^3)$, we can apply Condition **(H7)** to obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}(G(\varrho_n)-G'(\varrho_n)\varrho_n)\psi\,\operatorname{div}\mathbf{u}_n\,dx \\ = \lim_{\delta\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3\setminus O_\delta}(G(\varrho_n)-G'(\varrho_n)\varrho_n)\psi\,\operatorname{div}\mathbf{u}_n\,dx \\ = \lim_{\delta\to 0}\int_{\mathbb{R}_3\setminus O_\delta}\overline{(G-G'\varrho)\operatorname{div}\mathbf{u}}\psi\,dx \\ + \frac{1}{\mu}\lim_{\delta\to 0}\int_{\mathbb{R}_3\setminus O_\delta}(\overline{(G-G'\varrho)p} - \overline{(G-G'\varrho)}\overline{p})\,dx \\ = \int_{\mathbb{R}_3}\overline{(G-G'\varrho)\operatorname{div}\mathbf{u}}\psi\,dx + \frac{1}{\mu}\int_{\mathbb{R}_3}(\overline{(G-G'\varrho)p} - \overline{(G-G'\varrho)}\overline{p}). \tag{6.17}$$

Convergence of the integrals in this relation follows from integrability of the function λ^γ with respect to the measure $d_\lambda f\,dx$. Thus we get

$$\int_{\mathbb{R}^3}\psi(\overline{(G-G'\varrho)p} - \overline{(G-G'\varrho)}\overline{p})dx = \int_{\mathbb{R}^4}\psi\eta\lambda(\lambda^\gamma - \overline{p})d_\lambda f(x,\lambda)dx \\ + \int_{\mathbb{R}^4}\psi\left\{\int_\lambda^\infty \eta(s)ds\right\}(\lambda^\gamma - \overline{p})d_\lambda f(x,\lambda)dx = -\mu\int_{\mathbb{R}^4}\psi\eta\,d_\lambda(\lambda\mathcal{M}(x,\lambda))dx$$

which leads to the limiting relation

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}(G(\varrho_n)-G'(\varrho_n)\varrho_n)\operatorname{div}\mathbf{u}_n\psi\,dx \\ = -\int_{\mathbb{R}^3}\operatorname{div}\mathbf{v}f(x,\lambda)\lambda\eta'(\lambda)\psi\,dx + \int_{\mathbb{R}^4}\psi\eta'(\lambda)(\lambda\mathcal{M}(x,\lambda))\,dx d\lambda.$$

Substituting this result into (6.16) gives the integral identity

$$\int_{\mathbb{R}^4} f\mathbf{V}\cdot\nabla_{x,\lambda}(\eta\psi)\,dx d\lambda - \int_{\mathbb{R}^4} h\eta\psi\,d_\lambda f\,dx + \int_{\mathbb{R}^4}(\psi\eta')(\lambda\mathcal{M})\,dx d\lambda = 0,$$

which is equivalent to equation (6.13). $\square$

Our next task is to justify the renormalization procedure for kinetic equation (6.13). To this end introduce the concave function

$$\Psi(f) := f(1-f), \tag{6.18}$$

and note that Theorem 6.4 will be proved if we show that $\Psi(f) = 0$. The following lemma gives the kinetic equation for the function $\Psi(f)$.

THEOREM 6.7. *Under the assumptions of Theorem* 6.4 *the function* $\Psi(f)$ *satisfies the equation*

$$\operatorname*{div}_{x,\lambda}(\Psi\mathbf{V}) + 2\lambda\mathfrak{M}\partial_\lambda f - \partial_\lambda(\Psi'(f)\lambda\mathfrak{M}) + \partial_\lambda h\Psi(f) = 0 \quad in\ \mathcal{D}'(\mathbb{R}^4), \tag{6.19}$$

in which the function $\mathfrak{M}$ *is defined by*

$$\mathfrak{M}(x, .\lambda) = \frac{1}{2}\lim_{s\to\lambda+0}\mathcal{M}(s) + \frac{1}{2}\lim_{s\to\lambda-0}\mathcal{M}(s). \tag{6.20}$$

PROOF. The technical difficulty which arises in the proof of the theorem concerns the integration of discontinuous function with respect to a Stieltjes measure. The following lemma alleviates the problem.

LEMMA 6.8. *If a function* $g:\mathbb{R}\mapsto\mathbb{R}$ *has the bounded variation, then for any* $\chi\in C_0(\mathbb{R})$,

$$\lim_{k\to\infty}[\chi[g]_{,k}]_{,k} = \lim_{k\to\infty}[\chi g]_{,k} = \chi\tilde{g},$$
$$where\ \tilde{g}(\lambda) = \frac{1}{2}\lim_{s\to\lambda+0} g(s) + \frac{1}{2}\lim_{s\to\lambda-0} g(s).$$

Moreover, for any $F\in C^1(\mathbb{R})$ *the function* $F'(\tilde{g})$ *is integrable with respect to the Stieltjes measure* $d_\lambda g(\lambda)$ *and*

$$\int_{\mathbb{R}}\chi(\lambda)F'(\tilde{g}(\lambda))d_\lambda g = \int_{\mathbb{R}}\chi(\lambda)d_\lambda F(g(\lambda)).$$

In particular, the function $\mathfrak{M}$ *defined by formula* (6.20) *is integrable over* $\mathbb{R}^3$ *with respect to the measure* $d_\lambda f(x,\lambda)dx$. *Recall that mollifier* $[]_{,k}$ *is defined by formula* (4.15).

PROOF. The first assertion is obvious, and the second follows from the first. Next note that for almost each $x\in\mathbb{R}^3$, the function $\mathfrak{M}(x,\cdot)$ is the pointwise limit of the sequence of the continuous functions $[\mathcal{M}(x,\cdot)]_k$ and hence is measurable with respect to any finite Borel measure. It remain to note that the nonnegative function $\mathcal{M}$ has the integrable majorant $\overline{p}$. $\square$

Let us turn to the proof of Theorem 6.7. Applying mollifiers (4.15), (4.16) to both the sides of equation (6.13) we arrive at the equality

$$\operatorname{div}([f]_{m,k}\mathbf{V}) + \partial_\lambda[\lambda\mathcal{M}]_{m,k} + [h\partial_\lambda f]_{m,k} = I^{m,k},$$

where $I^{m,k}$ denotes the commutator $\operatorname{div}([f]_{m,k}\mathbf{V}) - \operatorname{div}[f\mathbf{V}]_{m,k}$. Multiplying both the sides by $\Psi'([f]_{m,k})$ and noting that $\operatorname{div}\mathbf{V}=0$ we arrive at

$$\begin{aligned}&\operatorname{div}(\Psi([f]_{m,k})\mathbf{V}) + \Psi'([f]_{m,k})\partial_\lambda[\lambda\mathcal{M}]_{m,k}\\&\quad + \Psi'([f]_{m,k})[h\partial_\lambda f]_{m,k} = \Psi'([f]_{m,k})I^{k,m}.\end{aligned}$$

Next multiplying both the sides of this equality by a test function $\zeta \in C_0^\infty(\mathbb{R}^3)$ and integrating the result over $\mathbb{R}^4$ we obtain the integral identity

$$
\begin{aligned}
&-\int_{\mathbb{R}^4} \Psi([f]_{m,k})\mathbf{V}\cdot\nabla_{x,\lambda}\zeta\,dx\,d\lambda + 2\int_{\mathbb{R}^4} \zeta[\lambda\mathcal{M}]_{m,k}\partial_\lambda[f]_{m,k}\,dx\,d\lambda \\
&-\int_{\mathbb{R}^4} \partial_\lambda\zeta\,\Psi'([f]_{m,k})[\lambda\mathcal{M}]_{m,k}\,dx\,d\lambda + \int_{\mathbb{R}^4} \zeta\,\Psi'([f]_{m,k})[h\partial_\lambda f]_{m,k}\,dx\,d\lambda \\
&= \int_{\mathbb{R}^4} \zeta\,\Psi'([f]_{m,k})I^{m,k}\,dx\,d\lambda.
\end{aligned}
$$

Now our task is to pass to the limit as $n, k \to \infty$. To this end note that the nonnegative function $\mathcal{M}$ vanishes outside of the cylinder $\Omega\times\mathbb{R}^+$ in which it has the integrable majorant $\mathcal{M} \le \overline{p}$. Therefore, the sequence $[\lambda\mathcal{M}]_{m,k}$ converges to the function $\lambda\mathcal{M}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^4)$ as $m, k \to \infty$. From this and Lemma 4.6 we conclude that

$$
\begin{aligned}
&-\int_{\mathbb{R}^4} \Psi(f)\mathbf{V}\cdot\nabla_{x,\lambda}\zeta\,dx\,d\lambda - \int_{\mathbb{R}^4} \partial_\lambda\zeta\,\Psi'(f)\lambda\mathcal{M}\,dx\,d\lambda \\
&+ \lim_{k\to\infty}\lim_{m\to\infty}\left\{\int_{\mathbb{R}^4} \zeta[\lambda\mathcal{M}]_{m,k}\partial_\lambda[f]_{m,k}\,dx\,d\lambda\right. \\
&\left.+ \int_{\mathbb{R}^4} \zeta\,\Psi'([f]_{m,k})[h\partial_\lambda f]_{m,k}\,dx\,d\lambda\right\} = 0.
\end{aligned}
\tag{6.21}
$$

Note that for any positive k, the function $\partial_\lambda[f]_{,k}$ is a smooth function of the variable λ, the function $\mathcal{M} \le \overline{p}(x)$ belongs to the class $L^1(\mathbb{R}^4)$ and

$$
[\lambda\mathcal{M}]_{m,k} \to [\lambda\mathcal{M}]_{,k} \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^4) \quad \text{as } m \to \infty.
$$

From this we conclude that

$$
\lim_{m\to\infty}\int_{\mathbb{R}^4} \zeta[\lambda\mathcal{M}]_{m,k}\partial_\lambda[f]_{m,k}\,dx\,d\lambda = \int_{\mathbb{R}^4}\left(\int_{\mathbb{R}} \zeta[\lambda\mathcal{M}]_{,k}\partial_\lambda[f]_{,k}\,d\lambda\right)dx.
$$

On the other hand, for each fixed x, the function $\zeta[\lambda\mathcal{M}]_{,k}(x,\cdot)$ is compactly supported and smooth in λ which leads to the identity

$$
\int_{\mathbb{R}} \zeta[\lambda\mathcal{M}]_{,k}\partial_\lambda[f]_{,k}\,d\lambda = \int_{\mathbb{R}} [\zeta[\lambda\mathcal{M}]_{,k}]_{,k}\,d_\lambda f(x,\lambda).
$$

It is easy to see that the functions $[\zeta[\lambda\mathcal{M}]_{,k}]_{,k}(x,\cdot)$ have the common integrable majorant

$$
|[\zeta[\lambda\mathcal{M}]_{,k}]_{,k}(x,\cdot)| \le \overline{p}(x)\sup_{\lambda}\zeta(\lambda)\sup_{\mathrm{spt}\,\eta}\{\lambda\},
$$

and for each fixed x, they converge to the function $\zeta\lambda\mathfrak{M}(x,\cdot)$ everywhere in $\mathbb{R}$. From this and the Lebesgue dominant convergence theorem we conclude that for a.e. fixed $x \in \mathbb{R}^3$,

$$
\lim_{k\to\infty}\int_{\mathbb{R}} \zeta[[\lambda\mathcal{M}]_{,k}]_{,k}(x,\lambda)\,d_\lambda f(x,\lambda) = \int_{\mathbb{R}} \zeta\lambda\mathfrak{M}\,d_\lambda f(x,\lambda).
$$

Applying once again the Lebesgue dominant convergence theorem we finally obtain

$$\lim_{k\to\infty}\lim_{m\to\infty}\int_{\mathbb{R}^4}\zeta[\lambda\mathcal{M}]_{m,k}\partial_\lambda[f]_{m,k}\,dx\,d\lambda=\int_{\mathbb{R}^4}\zeta\lambda\mathfrak{M}\,d_\lambda f(x,\lambda)\,dx. \tag{6.22}$$

Next note that for a fixed positive k, the functions $\zeta[h\partial_\lambda f]_{,k}$ are continuous with respect to λ and uniformly bounded in $L^2(\mathbb{R}^4)$ which along with the identity $\Psi'([f]_{,k})=[\Psi'(f)]_{,k}$ yields the relation

$$\lim_{m\to\infty}\int_{\mathbb{R}^4}\zeta\Psi'([f]_{m,k})[h\partial_\lambda f]_{m,k}\,dx\,d\lambda=\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}}h[\zeta\Psi'(f)]_{,k}\,d_\lambda f\right)dx.$$

On the other hand, for a.e. x, the functions $h[\zeta[\Psi'(f)]_{,k}]_{,k}(x,\cdot)$ have the common integrable majorant $c|h|\sup_{\mathrm{spt}\zeta}|\lambda|$, and converge on the number axis to the function $h\zeta\Psi'(\tilde{f})(x,\cdot)$. Applying the Lebesgue dominant convergence theorem and invoking Lemma 6.8 we finally obtain

$$\lim_{k\to\infty}\lim_{m\to\infty}\int_{\mathbb{R}^4}\zeta\Psi'([f]_{m,k})[h\partial_\lambda f]_{m,k}\,dx\,d\lambda=\int_{\mathbb{R}^4}\zeta\,h\,d_\lambda\Psi(f)\,dx. \tag{6.23}$$

Finally, substituting (6.22), (6.23) into (6.21) we obtain the integral identity

$$\begin{aligned}2&\int_{\mathbb{R}^4}\zeta\lambda\mathfrak{M}\,d_\lambda f(x,\lambda)\,dx\\&=-\int_{\mathbb{R}^4}\partial_\lambda\zeta\Psi'(f)\lambda\mathcal{M}\,dx\,d\lambda+\int_{\mathbb{R}^4}\partial_\lambda\zeta\,h\,\Psi(f)\,dx\,d\lambda\\&\quad+\int_{\mathbb{R}^4}\Psi(f)\mathbf{V}\cdot\nabla_{x,\lambda}\zeta\,dx\,d\lambda\end{aligned} \tag{6.24}$$

which is equivalent to the renormalized equation (6.19). □

6.5. *The oscillation defect measure*

The notion of oscillation defect measure was introduced in [17] in order to justify the existence theory for isentropic flows with *small* values of the adiabatic constant γ. Following [17,16] the r-oscillation defect measure associated with the sequence $\{\varrho_n\}_{n\le 1}$ is defined as follows

$$\mathbf{osc}_r[\varrho_n\to\varrho](K):=\sup_{k\ge 1}\limsup_{n\to\infty}\|T_k(\varrho_n)-T_k(\varrho)\|^r_{L^r(K)},$$

where $T_k(z)=kT(z/k)$, $T(z)$ is a smooth concave function, which is equal to z for $z\le 1$ and is a constant for $z\ge 3$. The smoothness properties of T_k are not important and we can take the simplest form $T_k(z)=\min\{z,k\}$. The unexpected result was obtained by Feireisl et al. in papers [17,18], where it was shown that $(1+\gamma)$-oscillation defect measure associated with the sequence $\{\varrho_n\}$ of solutions to compressible Navier–Stokes equations is uniformly bounded on all compact subsets of Ω.

Note that in the assumptions of Theorem 6.4 we cannot replace the compact subsets $K \Subset \Omega$ by the domain Ω itself, since the oscillation defect measure is not any regular set additive function on the family of compact subsets of Ω, i.e., it is not any measure in the sense of measure theory. In order to bypass this difficulty we observe that the finiteness of the oscillation defect measure on compacts gives some additional information on the properties of the distribution function Γ. Our task is to extract this information and then to use in the proof of Theorem 6.4. In order to formulate the appropriate auxiliary result we define the function $\mathcal{T}_\theta(x)$ by the equality

$$\mathcal{T}_\theta(x) = \overline{\min\{\varrho, \theta\}}(x) - \min\{\varrho(x), \theta(x)\} \quad \text{for each } \theta \in C(\Omega).$$

LEMMA 6.9. *Under the assumptions of Theorem* 6.4, *there is a constant c independent of θ and K such that the inequalities*

$$\|\mathcal{T}_\theta\|^{1+\gamma}_{L^{1+\gamma}(K)} \le \lim_{n\to\infty} \int_\Omega |\min\{\varrho_n(x), \theta(x)\} - \min\{\varrho(x), \theta(x)\}|^{1+\gamma} dx \le c \tag{6.25}$$

hold for all $\theta \in C(\Omega)$ and $K \Subset \Omega$. We point out that the limit in (6.25) *does exist by the choice of the sequence ϱ_n.*

PROOF. The proof imitates the proof of Lemma 4.3 from [18]. It can be easily seen that

$$\|\mathcal{T}_\theta\|^{1+\gamma}_{L^{1+\gamma}(K)} \le \limsup_{n\to\infty} \int_K |\min\{\varrho_n(x), \theta(x)\} - \min\{\varrho(x), \theta(x)\}|^{1+\gamma}\, dx. \tag{6.26}$$

Hence, it suffices to show that the right-hand side of this inequality admits a bound independent of θ. From the properties of $\min\{\cdot,\cdot\}$ it follows that

$$\begin{aligned} &|\min\{s', \theta\} - \min\{s'', \theta\}|^{1+\gamma} \\ &\quad \le (\min\{s', \theta\} - \min\{s'', \theta\})(s'^\gamma - s''^\gamma) \quad \text{for all } s', s'' \in \mathbb{R}^+, \end{aligned}$$

furthermore, for the weak limits we have the inequalities $\overline{\varrho^\gamma} \ge \varrho^\gamma$, and $\overline{\min\{\varrho, \theta\}} \le \min\{\varrho, \theta\}$, therefore, for any compactly supported, nonnegative function $h \in C(\Omega)$, we get

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega h|\min\{\varrho_n, \theta\} - \min\{\varrho, \theta\}|^{1+\gamma}\, dx \\ &\le \lim_{n\to\infty} \int_\Omega h(\min\{\varrho_n, \theta\} - \min\{\varrho, \theta\})(\varrho_n^\gamma - \varrho^\gamma)\, dx \\ &\quad \le \lim_{n\to\infty} \int_\Omega h(\min\{\varrho_n, \theta\} - \min\{\varrho, \theta\})(\varrho_n^\gamma - \varrho^\gamma)\, dx \\ &\qquad + \int_\Omega (\overline{\varrho^\gamma} - \varrho^\gamma)(\min\{\varrho, \theta\} - \overline{\min\{\varrho, \theta\}})\, dx \\ &\quad = \lim_{n\to\infty} \int_\Omega h(\varrho_n^\gamma \min\{\varrho_n, \theta\} - \overline{\varrho^\gamma}\,\overline{\min\{\varrho, \theta\}})\, dx \\ &\quad = \lim_{n\to\infty} \int_\Omega h(p(\varrho_n) \min\{\varrho_n, \theta\} - \overline{p}\,\overline{\min\{\varrho, \theta\}})\, dx. \end{aligned} \tag{6.27}$$

By Condition (**H7**), the right-hand side of (6.27), divided by μ, is equal to

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega h(\min\{\varrho_n,\theta\}\operatorname{div}\mathbf{u}_n-\overline{\min\{\varrho,\theta\}}\operatorname{div}\mathbf{u})\,dx\\
&\quad=\lim_{n\to\infty}\int_\Omega h(\min\{\varrho_n,\theta\}-\min\{\varrho,\theta\})\operatorname{div}\mathbf{u}_n dx\\
&\qquad-\lim_{n\to\infty}\int_\Omega h(\min\{\varrho_n,\theta\}-\min\{\varrho,\theta\})\operatorname{div}\mathbf{u}\,dx\\
&\quad\le\delta\limsup_{n\to\infty}\int_\Omega h|\min\{\varrho_n,\theta\}-\min\{\varrho,\theta\}|^{1+\gamma}\,dx\\
&\qquad+\delta^{-\gamma}\lim_{n\to\infty}\int_\Omega h(|\operatorname{div}\mathbf{u}_n|+|\operatorname{div}\mathbf{u}|)^{(1+\gamma)/\gamma}\\
&\quad\le\delta\lim_{n\to\infty}\int_\Omega h|\min\{\varrho_n,\theta\}-\min\{\varrho,\theta\}|^{1+\gamma}\,dx+c\delta^{-\gamma}\|h\|_{C(\Omega)}.
\end{aligned}\tag{6.28}$$

Combining (6.28) and (6.27), choosing $h=1$ on K, and $\delta>0$, δ sufficiently small we obtain (6.25). □

We reformulate this result in terms of the distribution function f. Recall that the functions $\min\{\varrho_n,\lambda\}$ are uniformly bounded in $\mathbb{R}^3$ and $\min\{\varrho_n,\lambda\}\operatorname{div}\mathbf{u}_n$ converges weakly in $L^2(D)$ for all nonnegative λ. Introduce the functions

$$\mathcal{V}_\lambda=\overline{(\min\{\varrho,\lambda\}\operatorname{div}\mathbf{u})}-\overline{\min\{\varrho,\lambda\}}\operatorname{div}\mathbf{u}\in L^2(D),\tag{6.29}$$

$$\mathfrak{H}(x)=\int_{[0,\infty)}f(x,s)(1-f(x,s))\,ds,\qquad\mathfrak{H}\in L^\gamma(D).$$

LEMMA 6.10. *There is a constant c independent of λ such that*

$$\|\mathfrak{H}\|_{L^{1+\gamma}(D\setminus S)}+\sup_\lambda\|\mathcal{V}_\lambda\|_{L^1(D\setminus S)}\le c.\tag{6.30}$$

PROOF. Recall that $\mathfrak{H}=\mathcal{V}_\lambda=0$ on $\mathbb{R}^3\setminus\Omega$. Hence, it is sufficient to prove that for all compacts $K\Subset\Omega$, we have

$$\|\mathfrak{H}\|_{L^{1+\gamma}(K)}+\sup_\lambda\|\mathcal{V}_\lambda\|_{L^1(K)}\le c\tag{6.31}$$

with the constant c independent of K. We begin with the observation that by Lemma 6.5,

$$\mathcal{T}_\theta(x)=\int_{[0,\infty)}\min\{\lambda,\theta(x)\}\,d_\lambda f(x,\lambda)-\min\left\{\int_{[0,\infty)}\lambda\,d_\lambda f(x,\lambda),\theta(x)\right\}\tag{6.32}$$

for all functions $\theta\in C(D)$. From this and the identity $\varrho(x)=\int_{[0,\infty)}(1-f(x,\lambda))\,d\lambda$ we conclude that

$$\mathcal{T}_\theta(x)=\int_0^{\theta(x)}f(x,s)\,ds\quad\text{for }\theta(x)\ge\varrho(x)$$

$$\text{and}\quad\mathcal{T}_\theta(x)=\int_{\theta(x)}^\infty(1-f(x,s))\,ds\quad\text{otherwise.}\tag{6.33}$$

Next, choose a sequence of continuous nonnegative functions $\{\theta_k\}_{k\geq 1}$ which converges for $k \to \infty$ to ϱ a.e. in RR^3. By Lemma 6.9 the norms in $L^{1+\gamma}(K)$ of functions $\mathcal{T}_{\theta_k}$ are uniformly bounded by a constant independent of k and K. Moreover, $\mathcal{T}_{\theta_k}$ converges a.e. in K to the function

$$\mathcal{T}_\varrho(x) = \int_0^{\varrho(x)} f(x,s)\,ds = \int_{\varrho(x)}^{\infty} (1-f)\,ds,$$

which yields the estimates $\|\mathcal{T}_\varrho\|_{L^{1+\gamma}(K)} \leq c$ with the constant c independent of K. It remains to note that estimate (6.31) for $\mathfrak{H}$ obviously follows from the inequality $\mathfrak{H} \leq 2\mathcal{T}_\varrho$.

In order to estimate $\mathcal{V}_\lambda$ note that

$$\begin{aligned}\mathcal{V}_\lambda &= \text{w-}\lim_{n\to\infty}\big((\min\{\varrho_n,\lambda\} - \min\{\varrho,\lambda\})\operatorname{div}\mathbf{u}_n\big)\\ &\quad - \Big(\text{w-}\lim_{n\to\infty}\min\{\varrho_n,\lambda\} - \min\{\varrho,\lambda\}\Big)\operatorname{div}\mathbf{u},\end{aligned}$$

where by w-lim is denoted the weak limit in $L^1(\mathbb{R}^3)$. From this and the boundedness of norms $\|\operatorname{div}\mathbf{u}_n\|_{L^2(\mathbb{R}^3)}$ hence, we obtain

$$\begin{aligned}\|\mathcal{V}_\lambda\|_{L^1(K)} \leq{}& \limsup_{n\to\infty}(\|\operatorname{div}\mathbf{u}_n\|_{L^2(K)} + \|\operatorname{div}\mathbf{u}\|_{L^2(K)})\\ &\times \|\min\{\varrho_n(x),\lambda\} - \min\{\varrho(x),\lambda\}\|_{L^2(K)},\end{aligned}$$

which along with (6.25) implies (6.31) and the proof of Lemma 6.10 is completed. □

At the end of this paragraph we describe the basic properties of the function $\mathfrak{M}(x,\lambda)$, which are important for the further analysis.

LEMMA 6.11. *For a.e. $x \in \Omega$,* (i) *$\mathcal{M}(x,\cdot)$ is nonnegative and vanishes on $\mathbb{R}^-$. Moreover, if the Borel function $\mathfrak{M}(x,.)$ given by* (6.20) *vanishes $d_\lambda f(x,\cdot)$-almost everywhere on the interval (ω,∞) with $\omega = \overline{p}(x)^{1/\gamma}$, then $d_\lambda f(x,\cdot)$ is a Dirac measure and*

$$f(x,\lambda) = 0 \quad \text{for } \lambda < \overline{p}(x)^{1/\gamma}, \qquad f(x,\lambda) = 1 \quad \text{for } \lambda \geq \overline{p}(x)^{1/\gamma}.$$

(ii) *For all $g \in C_0^\infty(0,\infty)$,*

$$\int_{\mathbb{R}} g(\lambda)\mathcal{M}(x,\lambda)d\lambda = -\int_{[0,\infty)} g'(\lambda)\mathcal{V}_\lambda(x)d\lambda, \tag{6.34}$$

where $\mathcal{V}_\lambda$ is defined by (6.29).

PROOF. By abuse of notations we will write simply f_k instead of $[f]_{,k}$. The mollified distribution function $f_k(x,\cdot)$ belongs to the class $C^\infty(\mathbb{R})$ and generates the absolutely continuous Stieltjes measure σ_{kx} of the form $d\sigma_{kx} = \partial_\lambda f_k d\lambda$. It is easy to see that for $k \to \infty$ the sequence of measures σ_{kx} converges star-weakly to the measure $\sigma_x = d_\lambda f$ in the space of Radon's measures on $\mathbb{R}$. In particular, for all λ with $d_\lambda f(x,\cdot)\{\lambda\} := \lim_{s\to\lambda+0} f(x,s) - \lim_{s\to\lambda-0} f(x,s) = 0$, we can pass to the limit, to obtain

$$\int_{[0,\lambda)} (t^\gamma - \overline{p})\partial_t f_k(x,t)\,dt \to \int_{[0,\lambda)} (t^\gamma - \overline{p})\,d_t f(x,t) \quad \text{for } k \to \infty. \tag{6.35}$$

In other words, relation (6.35) holds true for all λ, possibly except for some countable set. Since $\partial_\lambda f_k \geq 0$, the function on the left-hand side of (6.35) increases on $(-\infty,\omega)$ and decreases on (ω, ∞). From this and (6.35) we conclude that $\mathcal{M}(x, \cdot)$ does not decrease for $\lambda < \omega$ and does not increase for $\lambda > \omega$, which along with the obvious relations $\lim_{\lambda\to\pm\infty} \mathcal{M}(x, \lambda) = 0$ yields the nonnegativity of $\mathcal{M}$.

In order to prove the second part of *(i)* note that $\mathfrak{M}(x, \lambda) = \lim_{k\to\infty} \mathbf{S}_k\mathcal{M}(x, \lambda)$ belongs to the first Baire class, and hence is measurable in σ_x. It follows from the monotonicity of $\mathfrak{M}(x, \cdot)$ on the interval (ω, ∞) that if $\mathfrak{M}(x, \alpha) = 0$ for some $\alpha > \omega$, then $\mathfrak{M}(x, \lambda) = 0$ and $f(x, \lambda) = 1$ on (α, ∞). Assume that $\mathfrak{M}(x, \cdot)$ vanishes $d_\lambda f(x, \cdot)$-almost everywhere on (ω, ∞), and consider the set

$$\mathcal{O} = \left\{\alpha > \omega : \sigma_x(\omega, \alpha) \equiv \lim_{s\to\alpha-0} f(x, s) - \lim_{s\to\omega+0} f(x, s) = 0\right\}.$$

Let us prove that $\mathcal{O} = (\omega, \infty)$. If the set $\mathcal{O}$ is empty, then there is a sequence of points $\lambda_k \searrow \omega$ with $\mathfrak{M}(x, \lambda_k) = 0$, which yields $f(x, \cdot) = 1$ on (ω, ∞) thus $\mathcal{O} = (\omega, \infty)$. Hence $\mathcal{O} \neq \emptyset$. If $m = \sup \mathcal{O} < \infty$, then there is a sequence $\lambda_k \searrow m$ with $\mathfrak{M}(x, \lambda_k) = 0$, which yields $f(x, \cdot) = 1$ on (m, ∞). By construction, $f(x, \lambda) = c =$ constant on (ω, m). In other words, restriction of the Stieltjes measure $d_\lambda f(x, \cdot)$ to (ω, ∞) is the mono-atomic measure $(1-c)\delta(\cdot - m)$. Hence $\mathfrak{M}(x, m) = 2^{-1}(1-c)(m^\gamma - \omega^\gamma) = 0$ which yields $c = 1$. From this we can conclude that $f(x, \cdot) = 1$ on (ω, ∞), and $d_\lambda f(x, \cdot)$ is a probability measure concentrated on $[0, \omega]$. Recalling that $\omega^\gamma = \overline{p}(x)$ we obtain

$$\mu\mathcal{M}(x, 0) = \int_{[0,\omega]} (\lambda^\gamma - \omega^\gamma)\, d_\lambda f(x, \lambda) \geq 0.$$

Hence $d_\lambda f(x, \lambda)$ is the Dirac measure concentrated at ω, which implies (i).

The proof of (ii) is straightforward. It is easily seen that

$$\begin{aligned}
&-\mu \int_{\mathbb{R}} g(\lambda)\mathcal{M}(x, \lambda)\, d\lambda \\
&= \int_{[0,\infty)} \left(\int_{[\lambda,\infty)} g'(s)\, ds\right)\left(\int_{[\lambda,\infty)} (t^\gamma - \overline{p}) d_t f(x, t)\right) d\lambda \\
&= \int_{[0,\infty)} g'(s) \left(\int_{[0,s)} d\lambda \int_{[\lambda,\infty)} (t^\gamma - \overline{p})\, d_t f(x, t)\right) ds \\
&= \int_{[0,\infty)} g'(s) \left(\int_{[0,\infty)} \min\{t, s\}(t^\gamma - \overline{p})\, d_t \Gamma(x, t)\right) ds \\
&= \int_{[0,\infty)} g'(s)(\overline{\min\{\varrho, s\}p} - \overline{\min\{\varrho, s\}\overline{p}}))\, ds.
\end{aligned}$$

On the other hand, Condition **(H7)** of Theorem 6.4 yields $\overline{\min\{\varrho, \lambda\}p} - \overline{\min\{\varrho, \lambda\}\overline{p}} = \mu \mathcal{V}_\lambda(x)$, and the proof of Lemma 6.11 is completed. □

6.6. *Proof of Theorem 6.4*

We are now in a position to complete the proof of Theorem 6.4. We begin with the observation that all terms in equation (6.19) vanish outside of the slab $\Omega \times \mathbb{R}^+$, and integral identity (6.24) holds true after substituting any smooth function $\zeta = \eta(\lambda)$ vanishing near $+\infty$. Set $\eta(\lambda) = \int_\lambda^\infty \theta(s-t)\,ds$, where θ is defined by relations (4.14) and t is an arbitrary number from the interval $(2, \infty)$. Note that η' vanishes outside of the segment $[t-1, t+1]$, and $\eta(\lambda) = 1$ on the interval $(-\infty, t-1)$. Substituting η into integral identity (6.24) and noting that $\eta' \le 0$ we arrive at the integral inequality

$$2\int_{\mathbb{R}^3\times(-\infty,t]} \eta\lambda\mathfrak{M}\,d_\lambda f(x,\lambda)\,dx \le -(t+1)\int_{\mathbb{R}^4} \eta'\mathcal{M}\,dx\,d\lambda$$

$$-(t+1)\int_{\mathbb{R}^4} \eta'\Psi(f)(|h| + |\operatorname{div}\mathbf{v}|)\,dx\,d\lambda.$$

Identity (6.34) gives the representation for the first integral on the right-hand side of this inequality

$$-\int_{\mathbb{R}^4} \eta'(\lambda)\mathcal{M}(x,\lambda)d\lambda = \int_{[0,\infty)} \eta''(\lambda)\wp_1(\lambda)\,d\lambda, \quad \text{where } \wp_1(\lambda) = \int_{\mathbb{R}^3} \mathfrak{V}_\lambda(x).$$

The second admits the similar representation

$$-\int_{\mathbb{R}^4} \eta'\Psi(f)(|h| + |\operatorname{div}\mathbf{v}|)\,dx = \int_{[0,\infty)} \eta''(\lambda)\wp_2(\lambda)\,d\lambda,$$

$$\wp_2(\lambda) = \int_{\mathbb{R}^3} (|h| + |\operatorname{div}\mathbf{v}|)\left\{\int_0^\lambda \Psi(f(x,s))\,ds\right\}\,dx.$$

Using these relations we can rewrite (6.6) in the form

$$2\int_{(D\setminus S)\times(-\infty,t]} \eta\lambda\mathfrak{M}\,d_\lambda f(x,\lambda)\,dx \le (1+t)\int_{[1,\infty)} \eta''(\lambda)\wp(\lambda)d\lambda, \tag{6.36}$$

where $\wp = \wp_1 + \wp_2$. Let us prove that the function $\wp$ is bounded on the positive semi-axis. By virtue of Lemma 6.10 we have $|\wp_1(\lambda)| \le \|\mathfrak{V}_\lambda\|_{L^1(\mathbb{R}^3)} \le c$. On the other hand, the obvious inequality $\int_0^\lambda \Psi(f(x,s))\,ds \le \mathfrak{H}(x)$ along with Lemma 6.10 implies the estimate

$$|\wp_2(\lambda)| \le c\||\operatorname{div}\mathbf{v}| + a + h\|_{L^2(\mathbb{R}^3)}\|\mathfrak{H}\|_{L^2(\mathbb{R}^3)} \le c, \tag{6.37}$$

which yields the boundedness of $\wp$.

Recalling the expression for $\eta(\lambda)$ we can rewrite inequality (6.36) in the form

$$2\int_{\mathbb{R}^3}\left(\int_{[0,t-1)} \lambda\mathfrak{M}d_\lambda f(x,\lambda)\right)dx \le (1+t)\frac{d}{dt}(\theta * \wp)(t). \tag{6.38}$$

Since the smooth function $(\theta * \wp)(t)$ is uniformly bounded on the positive semi-axis, there exist a sequence $t_k \to \infty$ such that $\lim_{k\to\infty}(t_k + 1)(\theta * \wp)'(t_k) \le 0$. Substituting $t = t_k$ into (6.38) and letting $k \to \infty$, we finally obtain

$$\int_{\mathbb{R}^3}\left\{\int_{[0,\infty)} \lambda\mathfrak{M}\,d_\lambda f(x,\lambda)\right\}\,dx = 0.$$

Therefore, for almost every $x \in \mathbb{R}^3$, the functions $\mathcal{M}(x,\cdot)$ is equal to zero $d_\lambda f(x,\lambda)$-a.e. on $(0,\infty)$. From this and Lemma 6.11 we conclude $f(1-f) = 0$ a.e. in $\mathbb{R}^4$, which implies the strong convergence of the sequence ϱ_n. □

7. Isentropic flows with adiabatic constants $\gamma \in (4/3, 5/3]$

In this section we prove the existence of a solution to the boundary value problem for compressible Navier–Stokes equations

$$-\Delta\mathbf{u} - \lambda\,\mathrm{div}\,\mathbf{u} + \mathrm{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla p(\rho) = \varrho\mathbf{f} + \mathbf{h} \quad \text{in } \Omega, \tag{7.1a}$$

$$\mathrm{div}(\varrho\mathbf{u}) = 0 \quad \text{in } \Omega, \tag{7.1b}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega. \tag{7.1c}$$

Here $p = \rho^\gamma$, where $\gamma > 1$- is the adiabatic constant, and **f**, **h** are given continuous vector fields. Furthermore, we shall assume that the density satisfies the weighted-mass condition

$$\int_\Omega d(x)^{-s}\varrho\,dx = \mathbf{M}, \tag{7.1d}$$

in which **M** is a given constant, $d(x)$ is the distance between a point $x \in \Omega$ and the boundary of Ω, exponent $s \in (0, 1/2)$, depending only on γ will be specified below. The flow is characterized by the *internal energy* **E** and the *energy dissipation rate* **D**, given by the formulae

$$\mathbf{E} = \int_\Omega P(\varrho(x))\,dx \quad \mathbf{D} = \int_\Omega (|\nabla\mathbf{u}|^2 + \lambda\,\mathrm{div}\,\mathbf{u}^2)\,dx, \tag{7.2}$$

where a nonnegative function P is defined with an accuracy to the inessential linear function from the equation $sP'(s) - P(s) = p(s)$. The typical form of P is

$$P(\varrho) = \varrho\int_{\varrho_e}^{\varrho} s^{-2}p(s)\,ds \equiv \varrho e(\varrho),$$

where e is the specific internal energy and ϱ_e is some equilibrium value of the density. In the case of the isentropic flow with $p = \varrho^\gamma$ we can take $P(\varrho) = (\gamma-1)^{-1}\varrho^\gamma$. Introduce also the weighted kinetic energy

$$\mathbf{K} = \int_\Omega d(x)^s\varrho|\mathbf{u}|^2\,dx. \tag{7.3}$$

Taking formally the product of the moment equation with **u** and integrating the result by parts we obtain the integral identity

$$\int_\Omega (|\mathrm{rot}\,\mathbf{u}|^2 + (1+\lambda)|\,\mathrm{div}\,\mathbf{u}|^2)dx = \int_\Omega (\varrho\mathbf{f} + \mathbf{h})\cdot\mathbf{u}dx, \tag{7.4}$$

which express the energy balance law. The energy identity (7.4) implies the energy inequality

$$\int_\Omega |\nabla\mathbf{u}|^2\,dx \le c\int_\Omega \varrho|\mathbf{f}||\mathbf{u}|\,dx + c(\Omega)\|\mathbf{h}\|^2_{C(\Omega)},$$

which, in its turn, yields the estimate

$$\mathbf{D} \le c\|\mathbf{f}\|_{C(\Omega)}\sqrt{\mathbf{MK}} + c(\Omega)\|\mathbf{h}\|^2_{C(\Omega)}$$

of the energy dissipation rate via the weighted mass and the kinetic energy. It is important to note that, in contrast with the nonstationary problems, the energy identity do not imply the boundedness of the total energy. In what follows, we shall give an outline of the nonlocal existence theory in the class of weak solutions having a finite weighted energy and, in particular, shall prove the following theorem on solvability of problem (7.1).

THEOREM 7.1. *Let the adiabatic constant γ and the exponent s satisfy the inequalities*

$$\gamma > 4/3, \quad s \in ((5\gamma - 4)^{-1}, 2^{-1}). \tag{7.5}$$

Furthermore assume that the Ω is a bounded region with the boundary $\partial\Omega \in C^3$. Then for any $\mathbf{f}, \mathbf{h} \in C(\Omega)$, problem (7.1) *has a generalized solution which meets all requirements of Definition* 3.4 *and satisfies the inequalities*

$$\|\mathbf{u}\|_{H_0^{1,2}(\Omega)} + \int_\Omega d^{-s}(p(\varrho) + \varrho|\mathbf{u}\cdot\nabla d|^2)\,dx + \int_\Omega d\varrho|\mathbf{u}|^2\,dx \le c, \tag{7.6}$$

where c depends only on γ, s, Ω and $\mathbf{M}$, $\|\mathbf{f}\|_{C(\Omega)}$, $\|\mathbf{h}\|_{C(\Omega)}$.

Inequalities (7.6) show that the internal energy and the normal component of the kinetic energy tensor $\varrho\mathbf{u}\otimes\mathbf{u}$ take moderate values in the vicinity of the boundary, while the tangential components of the kinetic energy tensor may concentrate near the wall. Does this phenomena really take place or is it a peculiarity of our method is the question which we cannot decide with certainty.

The rest of the section is devoted to the proof of this theorem. Recall that the scheme for solving nonlinear problems consists of the following steps:

- First of all, we have to choose an approximate scheme and to construct a family of approximate solutions.
- Next we find *a priori estimates* which guarantee the uniform boundedness of the family of approximate solutions in the suitable norms.
- Finally we have to prove that any limiting points of the set of approximation solutions in suitable topology is a solution to the original problem.

Hence our first task is to construct approximate solutions to problem (7.1).

7.1. *Approximate solutions*

There are several possibilities on how to construct the family of approximate solutions. Here, we use a two-level approximate scheme based on solving the momentum equation with artificial pressure $p = \varrho^\gamma + \epsilon\varrho^3$ and continuity equation with vanishing diffusion and homogeneous boundary conditions

$$\Delta\mathbf{u} + \lambda\nabla\operatorname{div}\mathbf{u} = \operatorname{div}((t\varrho\mathbf{u} - \varepsilon\nabla\varrho)\otimes\mathbf{u}) + \nabla p(\varrho) - t\varrho\mathbf{f} - t\mathbf{h} \quad \text{in } \Omega, \tag{7.7a}$$

$$-\varepsilon\Delta\varrho + t\operatorname{div}(\varrho\mathbf{u}) = 0 \quad \text{in } \Omega, \tag{7.7b}$$

$$\mathbf{u} = 0, \quad \partial_n\varrho = 0 \quad \text{on } \partial\Omega, \tag{7.7c}$$

$$\int_\Omega d^{-s}\varrho\,dx = \mathbf{M}. \tag{7.7d}$$

Here $t \in [0, 1]$ is an artificial parameter needed for application of the Leray–Schauder fixed-point theorem. The following theorem establishes the existence of a solution to this problem for $p \sim \varrho^3$.

THEOREM 7.2. *Let Ω be a bounded domain with the boundary of the class C^3 and $\gamma > 1$, $0 < \beta < \gamma - 1$. Furthermore assume that a function $p \in C^\gamma(\mathbb{R}^1)$ satisfies the conditions*

$$\begin{aligned} &c_0^{-1}\varrho^3 \le p(\varrho) \le c_0\varrho^3, \quad c_0^{-1}\varrho^2 \le p'(\varrho) \le c_0\varrho^2 \quad \text{for } \varrho \ge 1, \\ &c_0^{-1}\varrho^\gamma \le p(\varrho) \le c_0\varrho^\gamma, \quad c_0^{-1}\varrho^{\gamma-1} \le p'(\varrho) \le c_0\varrho^{\gamma-1} \quad \text{for } \varrho \in [0,1]. \end{aligned} \tag{7.8}$$

Then for any $\mathbf{f}, \mathbf{h} \in C^\infty(\Omega)$, problem (7.1) *has a solution $(\mathbf{u}, \varrho) \in C^{2+\beta}(\Omega)^3 \times C^{2+\beta}(\Omega)$, $\varrho \ge 0$ satisfying the inequalities*

$$\mathbf{D} \le c_{ab}(1 + \mathbf{M}^2 + \sqrt{\mathbf{MK}}), \tag{7.9a}$$

$$\|\mathbf{u}\|_{H^{1,2}(\Omega)} + \|\varrho\|_{L^{9/2}(\Omega)} + \sqrt{\varepsilon}\|\varrho\|_{H^{1,2}(\Omega)} \le C(1 + \mathbf{M}), \tag{7.9b}$$

$$\|\mathbf{u}\|_{C^{2+\beta}(\Omega)} + \|\varrho\|_{C^{2+\beta}(\Omega)} \le C_\varepsilon, \tag{7.9c}$$

where a constant c_{ab} depends only on Ω, $\|\mathbf{f}, \mathbf{h}\|_{C(\Omega)}$, a constant C depends only on Ω, $\mathbf{f}$, $\mathbf{h}$, c_0, and a constant C_ε does not depend on t, The energy dissipation rate and the weighted kinetic energy are defined by the equalities (7.2) *and* (7.3).

PROOF. We begin with proving a priori estimates. Assume that $(\mathbf{u}, \varrho)$ be a C^2-solution to the problem with a nonnegative density ϱ. Introduce a convex differentiable function $g(\varrho) = \varrho \int_0^\varrho s^{-2}p(s)ds$ satisfying the equation $sg'(s) - g(s) = p(s)$. It is clear that

$$0 < c^{-1} \le g''(\varrho) \le c(1 + \varrho).$$

Multiplying both the sides of equation (7.7a) by $\mathbf{u}$ and integrating by parts we obtain the integral identity

$$\int_\Omega (|\nabla \mathbf{u}|^2 + \lambda \operatorname{div} \mathbf{u}^2 + \varepsilon g''(\varrho)|\nabla\varrho|^2)\, dx = t \int_\Omega (\varrho\mathbf{f} + \mathbf{h}) \cdot \mathbf{u}\, dx,$$

which along with the Poincare inequality leads to the estimate

$$\|\mathbf{u}\|_{H^{1,2}(\Omega)} + \sqrt{\varepsilon}\|\varrho\|_{H^{1,2}(\Omega)} \le C(1 + \sqrt{\varepsilon}\mathbf{M}) + C\|\varrho\mathbf{u}\|_{L^1(\Omega)}{}^{1/2}. \tag{7.10}$$

Moreover, since

$$\int_\Omega \varrho\mathbf{f}\mathbf{u}\, dx \le C\int_\Omega \varrho|\mathbf{u}|\, dx \le 2\sqrt{\mathbf{MK}},$$

the energy identity obviously yields the energy estimate (7.9a). $\square$

Our next task is to obtain the estimate for ϱ. Recall that by virtue of the Bogovskii Lemma for any $\psi \in C^\infty(\Omega)$ with

$$\psi_{av} = \frac{1}{|\Omega|} \int_\Omega \psi \, dx = 0,$$

there exists a vector field $\mathbf{q} \in H_0^{1,3}(\Omega)$ satisfying the conditions

$$\operatorname{div} \mathbf{q} = \psi \quad \text{in } \Omega, \qquad \|\mathbf{q}\|_{H^{1,3}(\Omega)} \le c\|\psi\|_{H^{1,3}(\Omega)}.$$

Multiplying both the sides of equation (7.7a) by $\mathbf{q}$ and integrating by parts we obtain

$$\begin{aligned}\int_\Omega p(\varrho)\psi \, dx = {}& \int_\Omega (\nabla \mathbf{u} : \nabla \mathbf{q} + \lambda \operatorname{div} \mathbf{u}\psi) \, dx \\ & - \int_\Omega (t\varrho \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{q} + \varepsilon \nabla \varrho \otimes \mathbf{u} : \nabla \mathbf{q}) \, dx - \int_\Omega (\varrho \mathbf{f} + \mathbf{h})\mathbf{q} dx,\end{aligned}$$

which along with the standard duality arguments leads to the estimate

$$\begin{aligned}\|p - p_{av}\|_{L^{3/2}(\Omega)} \le {}& C(\|\mathbf{u}\|_{H^{1,2}(\Omega)} + \|\varrho|\mathbf{u}|^2\|_{L^{3/2}(\Omega)} \\ & + \varepsilon\| |\mathbf{u}| \nabla \varrho\|_{L^{3/2}(\Omega)} + \|\varrho\|_{L^{3/2}(\Omega)} + 1).\end{aligned} \tag{7.11}$$

Since by the embedding theorem $\|\mathbf{u}\|_{L^6(\Omega)} \le c\|\mathbf{u}\|_{H^{1,2}(\Omega)}$, we have

$$\begin{aligned}&\|\varrho|\mathbf{u}|^2\|_{L^{3/2}(\Omega)} \le c\|\varrho\|_{L^3(\Omega)}\|\mathbf{u}\|^2_{H^{1,2}(\Omega)}, \\ &\| |\mathbf{u}| \nabla \varrho\|_{L^{3/2}(\Omega)} \le c\|\nabla \varrho\|_{L^2(\Omega)}\|\mathbf{u}\|_{H^{1,2}(\Omega)}.\end{aligned}$$

Combining this result with (7.11) and using the obvious inequalities

$$p_{av} \le c\|\varrho\|^3_{L^3(\Omega)}, \qquad \|\varrho\|^3_{L^{9/2}(\Omega)} \le c\|p\|_{L^{3/2}(\Omega)}$$

we obtain the estimate

$$\begin{aligned}\|\varrho\|^3_{L^{9/2}(\Omega)} \le {}& C(\|\mathbf{u}\|_{H^{1,2}(\Omega)} + \|\varrho\|_{L^3(\Omega)}\|\mathbf{u}\|^2_{H^{1,2}(\Omega)} \\ & + \varepsilon\|\nabla \varrho\|_{L^2(\Omega)}\|\mathbf{u}\|_{H^{1,2}(\Omega)} + \|\varrho\|^3_{L^3(\Omega)} + 1).\end{aligned} \tag{7.12}$$

Next note that the Holder inequality implies the estimate

$$\|\varrho\|_{L^3(\Omega)} \le \left(\int_\Omega \varrho \, dx\right)^{1/7} \left(\int_\Omega \varrho^{9/2}\right)^{4/21} \le C\mathbf{M}^{1/7}\|\varrho\|^{6/7}_{L^{9/2}(\Omega)},$$

which along with the Young inequality yields the estimates

$$\begin{aligned}&\|\varrho\|_{L^3(\Omega)}\|\mathbf{u}\|^2_{H^{1,2}(\Omega)} \le \delta\|\varrho\|^3_{L^{9/2}(\Omega)} + C(\delta)(\mathbf{M}^3 + \|\mathbf{u}\|^3_{H^{1,2}(\Omega)}), \\ &\|\varrho\|^3_{L^3(\Omega)} \le \delta\|\varrho\|^3_{L^{9/2}(\Omega)} + C(\delta)(\mathbf{M}^3 + \|\mathbf{u}\|^3_{H^{1,2}(\Omega)}),\end{aligned}$$

where δ is an arbitrary positive number. Substituting these inequalities into the right-hand side of (7.12) we arrive at the estimate

$$\|\varrho\|_{L^{9/2}(\Omega)} \le C(\|\mathbf{u}\|_{H^{1,2}(\Omega)} + \varepsilon\|\nabla \varrho\|_{L^2(\Omega)} + \mathbf{M} + 1).$$

Combining this estimate with the energy inequality (7.10) and choosing δ sufficiently small we obtain the estimate

$$\|\varrho\|_{L^{9/2}(\Omega)} + \|\mathbf{u}\|_{H^{1,2}(\Omega)} + \sqrt{\varepsilon}\|\varrho\|_{H^{1,2}(\Omega)} \le C(1 + \mathbf{M} + \|\varrho\mathbf{u}\|_{L^1(\Omega)}{}^{1/2}). \quad (7.13)$$

Applying the Holder inequality we can obtain the following estimate for the last term on the right-hand side of

$$\|\varrho\mathbf{u}\|_{L^1(\Omega)} \le \|\mathbf{u}\|_{L^6(\Omega)} \|\varrho\|_{L^1(\Omega)}^{11/14} \|\varrho\|_{L^{9/2}(\Omega)}^{3/14}$$

which together with the Young inequality leads to the estimate

$$\|\varrho\mathbf{u}\|_{L^1(\Omega)}{}^{1/2} \le \delta(\|\varrho\|_{L^{9/2}(\Omega)} + \|\mathbf{u}\|_{H^{1,2}(\Omega)}) + C(\delta)\mathbf{M}.$$

Substituting this result in (7.13) we obtain estimate (7.9b), which along with the standard bootstrap arguments, see for example [34], leads to the estimate

$$\|\varrho\|_{L^\infty(\Omega)} + \|\mathbf{u}\|_{L^\infty(\Omega)} \le C(\varepsilon, \mathbf{M}, \Omega, p, \|\mathbf{f}, \mathbf{h}\|_{L^\infty(\Omega)}).$$

From this and the results from the theory of weakly nonlinear elliptic equations, see Theorem 13.1 in [4], we conclude that the inequality

$$\|(\varrho, \mathbf{v})\|_{C^{2+\beta}(\Omega)} < C(\varepsilon, \Omega, \|(\mathbf{f}, \mathbf{h})\|_{C^\beta(\Omega)}, \mathbf{M}, p) \quad (7.14)$$

holds for every solution $(\varrho, \mathbf{u}) \in C^{1+\beta}(\Omega)$, $\varrho > 0$, to problem (7.1).

To tackle the existence question we need to reformulate problem (7.1) as a nonlinear operator equation in the form $(\varrho, \mathbf{v}) = \Phi_t(\varrho, \mathbf{u})$. Introduce the mapping $\Phi_t : (\hat{\varrho}, \hat{\mathbf{u}}) \mapsto (\varrho, \mathbf{u})$ defined as a solution to the boundary value problems

$$\Delta\mathbf{u} + \lambda\nabla \operatorname{div} \mathbf{u} = \operatorname{div}((t\varrho\hat{\mathbf{u}} - \varepsilon\nabla\hat{\varrho}) \otimes \hat{\mathbf{u}}) + \nabla p(\hat{\varrho}) - t\hat{\varrho}\,\mathbf{f} - t\mathbf{h} \quad \text{in } \Omega, \quad (7.15a)$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \quad (7.15b)$$

$$-\varepsilon\Delta\varrho + t \operatorname{div}(\varrho\,\hat{\mathbf{u}}) = 0 \quad \text{in } \Omega, \quad (7.15c)$$

$$\partial_n\varrho = 0 \quad \text{on } \partial\Omega, \quad (7.15d)$$

$$\int_\Omega d^{-s}\varrho\, dx = \mathbf{M}. \quad (7.15e)$$

The solvability of boundary value problem (7.15a)–(7.15b) is a well-known fact from the theory of Lame equation. The question on existence of *positive* nontrivial solution to equations (7.15c) is less trivial. This fact is a consequence of the general theory of positive solutions to linear operator equations and results from the following lemma.

LEMMA 7.3. *Let Ω be a bounded region in $\mathbb{R}^d$ with the boundary $\partial\Omega \in C^{2+\beta}$, $0 < \beta < 1$, and a vector field $\mathbf{u} \in C^{1+\beta}(\Omega)$ vanishing at $\partial\Omega$. Then for any positive $\mathbf{M}$ and $s < 1$, there exists a unique positive solution to the problem* (7.7b)–(7.7d) *which satisfies*

$$\int_\Omega d^{-s}\varrho(x)dx = \mathbf{M}.$$

We refer to [50] for the proof. Note that only the existence of nontrivial solutions follows from the Fredholm theory since 0 is the simple eigenvalue of the adjoint problem

$$-\varepsilon\Delta\varrho - t\hat{\mathbf{u}}\nabla\varrho = 0 \quad \text{in } \Omega, \qquad \partial_n\varrho = 0 \quad \text{on } \Omega.$$

The positivity of ϱ follows from the positivity of the first eigenfunction of a second-order elliptic operator.

Hence the mapping $(\hat{\varrho}, \hat{\mathbf{u}}, t) \mapsto (\varrho, \mathbf{v})$ denoted by $\Phi : C^{1+\beta}(\Omega)^4 \times [0, 1] \mapsto C^{2+\beta}(\Omega)^4$ is well defined and continuous. Denote by $\mathcal{J}$ a subset of $C^{1+\beta}(\Omega)^4$ defined by the inequalities $\{(\varrho, \mathbf{v}) : \varrho \geq 0, \|(\varrho, \mathbf{v})\|_{C^{1+\beta}(\Omega)} \leq C_\varepsilon\}$, where C_ε is a constant from estimate (7.9c). It follows from Lemma 7.3 that every fixed point $(\varrho, \mathbf{u})$ of Φ_t satisfies inequality (7.9c). Moreover, ϱ is strictly positive. Hence there are no fixed points of Φ_t at $\partial\mathcal{J}$ for all $t \in [0, 1]$. On the other hand, the mapping Φ_0 has the unique fixed point inside $\mathcal{J}$. By the Leray–Schauder fixed-point theorem, problem (7.7) has a solution $(\varrho, \mathbf{u}) \in \operatorname{int}\mathcal{J}$ and the proof of Theorem 7.2 is completed. □

7.2. *Weak convergence results*

Theorem 7.2 guarantees the existence of smooth vector fields $\mathbf{u}_\varepsilon$ and positive functions ϱ_ε satisfying conditions (7.7b)–(7.7d) and the equations

$$\Delta\mathbf{u}_\varepsilon + \lambda\nabla \operatorname{div}\mathbf{u}_\varepsilon - \operatorname{div}(\varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \nabla p(\varrho_\varepsilon) + \varrho_\varepsilon\mathbf{f} + \mathbf{h} = \mathbf{O}_\varepsilon \tag{7.16a}$$

$$\operatorname{div}(\varrho\mathbf{u}) = \mathbf{o}_\varepsilon \quad \text{in } \Omega, \tag{7.16b}$$

where

$$\mathbf{O}_\varepsilon = \varepsilon \operatorname{div}(\nabla\varrho_\varepsilon \otimes \mathbf{u}_\varepsilon), \qquad \mathbf{o}_\varepsilon = \varepsilon\Delta\varrho_\varepsilon.$$

It follows from a priori estimates that there exist a subsequence, still denoted by $\mathbf{u}_\varepsilon$, ϱ_ε, and functions $\mathbf{u} \in H_0^{1,2}(\Omega)$, $\varrho \in L^{9/2}(\Omega)$ $\overline{p} \in L^{3/2}(\Omega)$, such that

$$\begin{aligned} \mathbf{u}_\varepsilon &\to \mathbf{u} \quad \text{weakly in } H^{1,2}(\Omega), \\ \varrho_\varepsilon &\to \varrho \quad \text{weakly in } L^{9/2}(\Omega), \\ \varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon &\to \varrho\mathbf{u} \otimes \mathbf{u} \quad \text{weakly in } L^{3/2}(\Omega), \\ p(\varrho_\varepsilon) &\to \overline{p} \quad \text{weakly in } L^{3/2}(\Omega). \end{aligned} \tag{7.17}$$

Moreover, estimate (7.9b) for $\nabla\varrho_\varepsilon$ implies the limiting relations

$$\mathbf{O}_\varepsilon \to 0 \quad \text{in } H^{-1,3/2}(\Omega), \qquad \mathbf{o}_\varepsilon \to 0 \quad \text{in } H^{-1,2}(\Omega). \tag{7.18}$$

Letting $\varepsilon \to 0$ in equations (7.16) we obtain

$$\Delta\mathbf{u} + \lambda\nabla \operatorname{div}\mathbf{u} - \operatorname{div}(\varrho\mathbf{u} \otimes \mathbf{u}) - \nabla\overline{p} + \varrho\mathbf{f} + \mathbf{h} = 0 \tag{7.19a}$$

$$\operatorname{div}(\varrho\mathbf{u}) = 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{7.19b}$$

Applying the renormalization procedure, see Section 6.1, to (7.19b) we conclude that ϱ serves as the renormalized solution to the mass balance equation. Hence in order to show

that $(\mathbf{u}, \varrho)$ is a weak solution to the original problem in the sense of Definition 3.4 we have to establish the equality $\overline{p} = p(\varrho)$, which, in fact is equivalent to strong convergence of the sequence ϱ_ε.
The proof of this fact is the heart of the theory. It is based on the detailed consideration of the properties of the so-called *effective viscous flux*.

7.3. *Effective viscous flux*

Following [34] we defined the effective viscous flux by the equalities

$$V(\varrho, \mathbf{u}) =: p - \nabla\Delta^{-1}\nabla : \mathbb{S} = p - (1+\lambda)\operatorname{div}\mathbf{u}.$$

As was shown in [34,17,18] the effective viscous flux enjoys many remarkable properties. The most important is the multiplicative relation

$$\overline{bV} = \overline{b}\,\overline{V}, \quad \text{where } \overline{b} = \text{w-}\lim_{\varepsilon\to 0} b(\varrho_\varepsilon), \quad \overline{bV} = \text{w-}\lim_{\varepsilon\to 0} b(\varrho_\varepsilon)V(\varrho_\varepsilon, \mathbf{u}_\varepsilon), \tag{7.20}$$

which was discovered in [34]. The simple and effective proof of this relation, based on the new version of *compensated compactness principle*, was given in papers [17,18]. The following result is due to Feireisl [16].

LEMMA 7.4. *Let the quantities $\mathbf{u}_\varepsilon$, ϱ_ε satisfy equations* (7.16) *and for some $q > 3/2$, $r > 1$,*

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \textit{weakly in } H^{1,2}(\Omega), \qquad \varrho_\varepsilon \to \varrho \quad \textit{weakly in } L^q(\Omega),$$
$$\mathbf{O}_\varepsilon \to 0, \quad \mathbf{o}_\varepsilon \to 0 \quad \textit{in } H^{-1,r'}(\Omega).$$

Then equality (7.20) *holds true for any continuously differentiable function $b : \mathbb{R} \mapsto \mathbb{R}$ with $|b'| \le c$.*

This result can be easy generalized if we take into account that:

- the assertion of the lemma is local and the behavior of the quantities near the boundary does not play an important role,
- we need the restriction $q > 3/2$ only to guarantee the integrability of the energy density with exponent greater than one, and the lemma remains true if we require the boundedness of $\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2$ in $L^r_{\text{loc}}(\Omega)$ with $r > 1$.

Thus we come to the following version of Lemma 7.4, [51,52].

LEMMA 7.5. *Let the quantities $\mathbf{u}_\varepsilon$, ϱ_ε satisfy equations* (7.16) *with $\mathbf{O}_\varepsilon = 0$, $\mathbf{o}_\varepsilon = 0$. Furthermore assume that they satisfy the following conditions:*
(1) *There exist positive κ and c such that for all $n \ge 1$,*

$$\int_\Omega p_\varepsilon\,dx + \int_\Omega |\varrho_\varepsilon \mathbf{u}_\varepsilon|^{1+\kappa}\,dx \le c.$$

Moreover, for each compact $\Omega' \Subset \Omega$,

$$\int_{\Omega'} p_\varepsilon^{1+\kappa}\,dx + \int_{\Omega'} (\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2)^{1+\kappa}\,dx \le c(\Omega'),$$

where $c(\Omega')$ does not depend on n.
(2) *For each compact $E \subset \mathbb{R}^3$ and an arbitrary function $G : \mathbb{R} \to \mathbb{R}$ satisfying the condition $\lim_{\varrho\to\infty} \varrho^{-\gamma} G(\varrho) = 0$, the function $G(\varrho_\varepsilon))$ converges weakly in $L^1(E)$ to a function $\overline{G} \in L^1_{\mathrm{loc}}(\Omega)$. Moreover, if G satisfies more weak condition $\limsup_{\varrho\to\infty} \varrho^{-\gamma}|G(\varrho)| < \infty$, then the sequence $G(\varrho_\varepsilon))$ converges weakly in $L^1(\Omega')$ to the function $\overline{G}$ in any subdomain $\Omega' \Subset \Omega$.*
(3) *For some $r > 1$,*

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \textit{weakly in } H^{1,2}(\Omega),$$
$$\varrho_\varepsilon \to \varrho \quad \textit{weakly in } L^r(\Omega).$$

Then the limiting relation

$$\int_\Omega \overline{\Phi(\cdot,\varrho)V(\varrho,\mathbf{v})}dx = \int_\Omega \overline{\Phi}\,\overline{V}dx, \quad \textit{where } \overline{V} = \overline{p} - (2+\nu)\operatorname{div}\mathbf{v} \tag{7.21}$$

holds true for any function $\Phi \in C(\Omega \times \mathbb{R})$ satisfying the conditions

$$\Phi(\cdot,\lambda) \in C_0(\Omega) \quad \textit{for all } \lambda \in \mathbb{R}^+,$$
$$\Phi(\cdot,\lambda) = \Phi_\infty(\cdot) \in C_0(\Omega) \quad \textit{for all } \lambda > N > 0.$$

7.4. *Proof of Theorem 7.1*

We begin with the observation that for any $\gamma > 1$ and $\epsilon > 0$, the artificial pressure $p_\epsilon(\varrho) = \varrho^\gamma + \epsilon\varrho^3$ meets all requirements of Theorem 7.2. Therefore, the corresponding boundary value problem (7.7), with $t = 1$ and $p = p_\epsilon$ has a family of strong solutions $(\mathbf{u}_\varepsilon, \varrho_\varepsilon)$ which admit estimates (7.9). After passing to a subsequence we can assume that this sequence satisfies limiting relations (7.17) and its weak limit $(\mathbf{u}, \varrho)$ serves as a generalized solution to equations (7.19). In particular for any function G with $G' \in C^1(\Omega)$ and any function $\psi \in C^1(\Omega)$ we have

$$\begin{aligned}&\int_\Omega (G(\varrho_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla\psi + (G(\varrho_\varepsilon) - G'(\varrho_\varepsilon)\varrho_\varepsilon)\psi \operatorname{div}\mathbf{u}_\varepsilon)\,dx \\ &\quad = \varepsilon\int_\Omega (G'(\varrho_\varepsilon)\nabla\varrho_\varepsilon\nabla\psi + G'(\varrho_\varepsilon)|\nabla\varrho_\varepsilon|^2\psi)\,dx \to 0 \quad \text{as } \varepsilon \to 0.\end{aligned} \tag{7.22}$$

Moreover, estimate (7.9b) implies that $(\mathbf{u}_\varepsilon, \varrho_\varepsilon)$ satisfy all assumptions of Lemma 7.4. It follows from this, inequalities (7.9b), and limiting relations (7.22) that being extended by zero outside of Ω the functions $(\mathbf{u}_\varepsilon, \varrho_\varepsilon)$ satisfy conditions **(H3)**–**(H7)** of Theorem 6.4. Applying this theorem we conclude that ϱ_ε converge to ϱ strongly in any space $L^r(\Omega)$ with $r < 9/2$. Therefore the limiting functions $(\mathbf{u}, \varrho)$ serve as a weak solution to problem 7.1 with the artificial pressure function $p = p_\epsilon(\varrho)$.

Our next task is to pass to the limit as $\epsilon \to 0$. Note that estimate (7.9b), which plays the key role in the previous considerations, depends on ϵ and cannot be used in the analysis of the second-level approximation. The only estimate which does not depend on ϵ is the weak energy inequality (7.17), but it does not guarantee the boundedness of $\|\mathbf{u}\|_{H^{1,2}(\Omega)}$ and $\|p\|_{L^1(\Omega)}$. The following theorem whose proof is given in the next section allows us to obviate the difficulty.

Let us consider the general moment relation

$$\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})+\nabla p=\mathbf{F}+\operatorname{div}\mathbb{S}\quad\text{in } \mathcal{D}'(\Omega), \tag{7.23}$$

which links nonnegative functions ϱ, p, vector fields $\mathbf{u}$, $\mathbf{F}$, and a tensor field $\mathbb{S}$. We do not suppose the existence of any other relations between these quantities. The unexpected result is that this relation along with the weak energy inequality implies the effective estimates for p and $\mathbf{u}$.

THEOREM 7.6. *Let nonnegative functions $\varrho\in L^\gamma(\Omega)$, $p\in L^1(\Omega)$, vector fields $\mathbf{u}\in H_0^{1,2}(\Omega)$, $\mathbf{F}\in L^1(\Omega)$, and a tensor field $\mathbb{S}\in L^2(\Omega)$ satisfy relation* (7.23) *and the inequalities*

$$\|\mathbb{S}\|^2_{L^2(\Omega)}\le c_e\mathbf{D}\le c_e(\sqrt{\mathbf{K}}+1),\qquad p\ge\varrho^\gamma, \tag{7.24}$$

where the energy dissipation rate $\mathbf{D}$ and the weighted kinetic energy $\mathbf{K}$ are defined by the formulae (7.2) *and* (7.3). *Furthermore assume that*

$$\frac{1}{5\gamma-4}<s<\frac{1}{2},\quad \gamma>\frac{4}{3},\quad \textit{and}\quad t>0.$$

Then there exist constants c and $\sigma>1$, depending only on Ω, $\|\mathbf{F}\|_{L^1(\Omega)}$, s, c_e, γ, a constant $c(t)$ depending only on Ω, $\|\mathbf{F}\|_{L^1(\Omega)}$, s,γ c_e, t

$$\mathbf{D}+\|d^s\varrho|\mathbf{v}|^2\|_{L^\sigma(\Omega)}+\|p\|_{L^1(\Omega)}\le c,\qquad \|p\|_{L^\sigma(\Omega_t)}\le c(t). \tag{7.25}$$

Denote by $(\mathbf{u}_\epsilon,\varrho_\epsilon)$ a sequence of generalized solutions to problem (7.1) with p replaced by p_ϵ. It follows from (7.1a) that the functions ϱ_ϵ, p_ϵ, and the vector fields $\mathbf{u}_\epsilon$ satisfy relation (7.23) with

$$\mathbf{F}=\mathbf{f}\varrho_\epsilon+\mathbf{h},\qquad \mathbb{S}=-(\nabla\mathbf{u}_\epsilon+\nabla\mathbf{u}_\epsilon{}^*)+(\lambda-1)\operatorname{div}\mathbf{u}_\epsilon\mathbf{I}.$$

Obviously

$$\|\mathbf{F}\|_{L^1(\Omega)}\le C(\mathbf{M}+1),\qquad \|\mathbb{S}\|^2_{L^2(\Omega)}\le c_e\mathbf{D}_\epsilon,$$

where $\mathbf{D}_\epsilon$ is given by formula (7.2) with $\mathbf{u}$ replaced by $\mathbf{u}_\epsilon$. On the other hand, by virtue of (7.17), ϱ_ϵ and $\mathbf{u}_\epsilon$ satisfy the weak energy inequality (7.24) with the constant c_e depending only on $\mathbf{M}$ and $\|\mathbf{f},\mathbf{h}\|_{C(\Omega)}$. Hence ϱ_ϵ, p_ϵ, and $\mathbf{u}_\epsilon$ meet all requirements of Theorem 7.6

and satisfy inequalities (7.25) with a constant c, $c(t)$ independent on ϵ. After passing to a subsequence we can assume that

$$\begin{aligned} \mathbf{u}_\epsilon \to \mathbf{u} \quad & \text{weakly in } H^{1,2}(\Omega), \\ \mathbf{u}_\epsilon \to \mathbf{u} \quad & \text{strongly in } L(\Omega), \quad r < 6, \\ \varrho_\epsilon \to \varrho \quad & \text{weakly in } L^\gamma(\Omega). \end{aligned} \tag{7.26}$$

Moreover, the limiting relation

$$G(\varrho_\epsilon) \to \overline{G} \quad \text{weakly in } L^1(\Omega) \tag{7.27}$$

holds true for any continuous function G satisfying the condition $\lim_{\varrho\to\infty} |G(\varrho)|/\varrho = 0$. Since $\gamma > 4/3$ the sequence $\varrho_\epsilon \mathbf{u}_\epsilon$ is bounded in $L^{12/11}(\Omega)$ we also have

$$\varrho_\epsilon \mathbf{u}_\epsilon \to \varrho \mathbf{u} \quad \text{weakly in } L^{12/11}(\Omega). \tag{7.28}$$

Next, it follows from inequalities (7.25) that for any subdomain $\Omega' \Subset \Omega$,

$$\begin{aligned} \varrho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon \to \varrho \mathbf{u} \otimes \mathbf{u} \quad & \text{weakly in } L^\sigma(\Omega'), \\ p(\varrho_\epsilon) \to \overline{p} \quad & \text{weakly in } L^\sigma(\Omega'). \end{aligned} \tag{7.29}$$

Let us show that for any positive $\kappa < \sigma - 1$,

$$v_\epsilon \equiv \epsilon \varrho_\epsilon^3 \to 0 \quad \text{in } L^{1+\kappa}(\Omega') \quad \text{as } \epsilon \to 0. \tag{7.30}$$

To this end note that the sequence v_ϵ is bounded in $L^\sigma(\Omega')$, and hence the inequality

$$\int_E v_\epsilon^{1+\kappa}\, dx \le \left(\int_{\Omega'} v_\epsilon^\sigma\, dx\right)^{\frac{1+\kappa}{\sigma}} \left(\int_E dx\right)^{1-\frac{1+\kappa}{\sigma}} \le c(\Omega')(\text{meas } E)^{1-\frac{1+\kappa}{\sigma}}$$

holds true for any measurable set $E \subset \Omega$. On the other hand, by virtue of the Chebyshev inequality we have

$$\text{meas } \{\varrho_\epsilon \ge N\} \le \mathbf{M} N^{-1} \quad \text{and} \quad \int_{\{\varrho_\epsilon \le N\}} v_\epsilon^{1+\kappa}\, dx \le \epsilon^{1+\kappa} N^{1+\kappa}.$$

Combining these results we obtain

$$\begin{aligned} &\limsup_{\epsilon\to 0} \int_{\Omega'} v_\epsilon^{1+\kappa}\, dx \le \limsup_{\epsilon\to 0} (\epsilon^{1+\kappa} N^{1+\kappa} + (\mathbf{M} N^{-1})^{1-\frac{1+\kappa}{\sigma}}) \le \\ &c(\mathbf{M}, \Omega') N^{-1+\frac{1+\kappa}{\sigma}}) \to 0 \quad \text{as } N \to \infty, \end{aligned}$$

which gives (7.30). It follows from (7.29) and (7.30) that for any $\Omega' \Subset \Omega$,

$$\varrho_\epsilon^\gamma \to \overline{p} \quad \text{weakly in } L^\sigma(\Omega'). \tag{7.31}$$

Finally note that functions ϱ_ϵ serve as renormalized solutions to mass balance equations associated with vector fields $\mathbf{u}_\epsilon$. It follows from this that ϱ_ϵ, $p(\varrho_\epsilon)$, and $\mathbf{u}_\epsilon$ meet all requirements of Lemma 7.5 and hence satisfy all assumptions of Theorem 6.4. Applying this theorem we conclude that the sequence ϱ_ϵ converges to ϱ a.e. in Ω and $p(\varrho_\epsilon) \to \varrho^\gamma$ in $L^1_{\text{loc}}(\Omega)$. Therefore the functions $(\mathbf{u}, \varrho)$ serve as the generalized solution to problem (7.1), which completes the proof of Theorem 7.1. $\square$

8. Estimate of a Green potential. Proof of Theorem 7.6

This section is devoted to the proof of Theorem 7.6. Before the presentation of the formal proof we outline the basic ideas of our method. The key observation is that in compressible fluids the energy tensor $\varrho\mathbf{u}\otimes\mathbf{u}+p\,\mathbf{I}$ is a nonnegative symmetric matrix. Using this property we can obtain a priori estimates by means of the choice of a test vector field φ in the integral identity (3.7a) determining a weak solution to the moment balance equation. Calculations show that in order to get the optimal result one should take $\varphi(x) = \Phi(x)$ with a convex "potential" $\Phi(x) = |x - y|$ depending of the current point $y \in \Omega$. As was shown in [19] and [51] these simple arguments lead to new internal a priori estimates for solutions to compressible Navier–Stokes equations.

The question on existence of a priori estimates near the boundary is more difficult since there are no nontrivial convex functions with gradients vanishing at the boundary. Nevertheless, we show that the choice of a potential Φ in the form $\Phi(x) = |x - y| + \Phi_0(x, y)$, where y is an arbitrary point of Ω and Φ_0 is some regular function, leads to pointwise estimates for Newtonian potential of p, which, together with the Corollary 4.4, give the efficient estimates for the density of the kinetic energy. The following theorem gives the explicit formulation of this result

THEOREM 8.1. *Assume that nonnegative functions $\varrho \in L^\gamma(\Omega)$, $p \in L^1(\Omega)$, vector fields $\mathbf{u} \in H_0^{1,2}(\Omega)$, $\mathbf{F} \in L^1(\Omega)$, and a tensor field $\mathbb{S} \in L^2(\Omega)$ satisfy the equation*

$$\operatorname{div}(\varrho\mathbf{u} \otimes \mathbf{u}) + \nabla p = \mathbf{F} + \operatorname{div}\mathbb{S} \quad \text{in } \mathcal{D}'(\Omega), \tag{8.1}$$

and inequality $p \geq \varrho^\gamma$. Then for any $\iota \in [0, 1/2)$, there exist positive constants r and c depending only on ι, γ, Ω, and $\|\mathbb{S}\|_{L^2(\Omega)}$ such that for all $y \in \Omega$,

$$\int_\Omega d(x)\mathfrak{G}(x, y)p(x)\,dx \leq c\left(\int_{\Omega_r} p\,dx + \int_\Omega d^\iota\varrho|\mathbf{u}|^2 dx + \sqrt{\mathbf{D}} + 1 + \|\mathbf{F}\|_{L^1(\Omega)}\right). \tag{8.2}$$

We emphasize that ϱ, p, $\mathbf{u}$, and $\mathbb{S}$ are independent and connected only by equation (8.1).

PROOF. In order to avoid technicalities we give the complete proof for the case when a domain Ω is a ball in $\mathbb{R}^3$. The extension to the case of bounded domains with smooth boundary is obvious. The proof is based on the following construction which reduces the original problem to an auxiliary problem in the half-space.

Further we shall assume that all coordinates are contravariant. Note that in the original Cartesian coordinate system x_i covariant and contravariant components of any object are coincident but after passing to the general curvilinear coordinates they become different. For any $t > 0$, denote the subdomain by Ω_t

$$\Omega_t = \{x \in \Omega : \ d(x) =: \operatorname{dist}(x, \partial\Omega) \geq t\}.$$

Since Ω is conformally equivalent to a half-space, for each point of $\partial\Omega$ there exists a standard neighborhood $\mathcal{U}_0$ and a conformal mapping $\mathcal{O} : y = y(x)$, which takes $\Omega \cap \mathcal{U}_0$ onto the cylinder

$$\mathcal{V} = \mathcal{D} \times [0, 2R], \quad \text{where } \mathcal{D} = \{(y^1, y^2) : |y^1|^2 + |y^2|^2 < R\}$$

such that $c^{-1}d(x) < y^3(x) < cd(x)$. A finite collection of standard neighborhoods $\mathcal{U}_k$ covers the set $\Omega \setminus \Omega_{2R}$. There exists also a finite collection of smooth cut-off functions $\chi_k : \mathcal{V} \to [0, 1]$ such that χ_k vanishes in a vicinity of the set $\partial\mathcal{V} \setminus \{y^3 = 0\}$. Moreover, the functions $\chi_k \circ \mathcal{O}$, being extended by zero onto the set $\Omega \setminus \mathcal{U}_k$, belong to the class $C^\infty(\Omega)$ and satisfy the equality

$$\sum_k \chi_k \circ \mathcal{O}_k = 1 \quad \text{in } \Omega \setminus \Omega_R.$$

Fix an arbitrary standard neighborhood and write $\mathcal{U}$ and χ instead of $\mathcal{U}_k$ and χ_k. We shall consider the velocity vector field $\mathbf{u}$ and the force vector field $\mathbf{F}$ as contravariant vector fields, and a test vector field φ as a covariant vector field. In the conformal coordinate system y^i their components are defined by the equalities

$$\begin{aligned}
&\bar{\mathbf{u}} = (\bar{u}^1, \bar{u}^2, \bar{u}^3), \qquad \bar{\varphi} = (\bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3),\\
&\bar{u}^i(y) = \frac{\partial y^i}{\partial x^j}(x(y))u^j(x(y)), \qquad \bar{\varphi}_i(y) = \frac{\partial x^j}{\partial y^i}(y)\varphi_j(x(y)).
\end{aligned}$$

If $\mathbf{u}$ and ϱ, p satisfy equation (8.1) and $\operatorname{spt}\varphi \subset \mathcal{U}$, $\varphi|_{\partial\Omega} = 0$, then $\bar{\mathbf{u}}$, ϱ and $\bar{\varphi}$ satisfy the integral identity

$$\begin{aligned}
&\int_{\mathcal{V}} \varrho \bar{u}^i \bar{u}^j \nabla_j \bar{\varphi}_i g^{1/2}\, dy + \int_{\mathcal{V}} p \operatorname{div}(g^{1/6}\bar{\varphi})\, dy - \int_{\mathcal{V}} g^{1/2} \nabla_i \bar{\varphi}_j\, \bar{\mathbb{S}}^{ij}\, dy\\
&\quad + \int_{\mathcal{V}} g^{1/2} \bar{\mathbf{F}} \bar{\varphi}\, dy = 0. \qquad (8.3)
\end{aligned}$$

Here $\sqrt{g(y)} = \det x'(y)$, the covariant derivatives ∇_i are defined by

$$\nabla_i \bar{\varphi}_j = \frac{\partial \bar{\varphi}_j}{\partial y^i} - \Gamma^k_{ij}\bar{\varphi}_k, \qquad \Gamma^k_{ij} = \frac{1}{6g}\left(\frac{\partial g}{\partial y^i}\delta_{jk} + \frac{\partial g}{\partial y^j}\delta_{ik} - \frac{\partial g}{\partial y^k}\delta_{ij}\right).$$

Let us turn to the proof of Theorem 8.1. It naturally falls into three steps. □

The first step. First we show that the function p does not oscillate near the boundary of Ω.

LEMMA 8.2. *Under the assumptions of Theorem* 8.1 *for any* $\iota < 1/2$ *there exist constants* c *and* r_0, *depending only on* Ω *and* s *such that for all* $r \in (0, r_0]$,

$$\begin{aligned}
\int_{\{0<d<r\}} d(x)^{-\iota}(\varrho|\mathbf{u}_n|^2 + p)\, dx &\le c \int_\Omega d^{1/2}(\varrho|\mathbf{u}|^2 + p)\, dx\\
&\quad + c(\|\mathbb{S}\|_{L^2(\Omega)} + \|\mathbf{F}\|_{L^1(\Omega)}), \qquad (8.4)
\end{aligned}$$

where $\mathbf{u}_n = \nabla d \cdot \mathbf{u}$.

PROOF. Introduce the family of functions depending on a parameter $0 < \delta < t$ and given by the formulae

$$\eta(y) = 1 \quad \text{for } y^3 < t, \quad \eta(y) = \frac{t + \delta - y^3}{\delta}$$
$$\text{for } t \le y^3 \le t + \delta, \quad \eta(y) = 0 \quad \text{for } t > 0.$$

Next set $\bar{\varphi}_1 = \bar{\varphi}_2 = 0, \quad \bar{\varphi}_3 = \eta(y)\chi(y)y^3$. Substituting $\bar{\varphi}$ into (8.3) leads to the equality

$$-\frac{1}{\delta}\int_{\mathcal{D}\times[t,t+\delta]} y^3\Psi\,dy + \int_{\mathcal{D}\times[0,t+\delta]} \eta(y)\Psi dy + \int_{\mathcal{D}\times[0,t+\delta]} y^3\eta(y)\Xi\,dy = 0, \tag{8.5}$$

where

$$\Psi = \chi(g^{1/2}\varrho(\bar{u}^3)^2 + pg^{1/6} - \overline{\mathbb{S}}^{33}g^{1/2}),$$
$$\Xi = \chi\Gamma^3_{ij}(\varrho\bar{u}^i\bar{u}^j - \overline{\mathbb{S}^{ij}})g^{1/2} + p\frac{\partial}{\partial y^3}(g^{1/6}\chi) + \chi\bar{F}^3 g^{1/2} + \frac{\chi}{\partial y^i}(\overline{\mathbb{S}}^{3i} - \varrho\bar{u}^3\bar{u}^i)g^{1/2}.$$

It is easy to see that

$$c\chi(p + \varrho(\bar{u}^3)^2 - |\overline{\mathbb{S}}|) \le \Psi \le C\chi(p + \varrho(\bar{u}^3)^2 + |\overline{\mathbb{S}}|), \tag{8.6}$$
$$|\Xi| \le c(p + |\overline{\mathbf{F}}| + \varrho + |\overline{\mathbb{S}}| + \varrho|\bar{u}|^2), \tag{8.7}$$

where c, C depends only on Ω. Letting $\delta \to 0$ in integral identity (8.5), multiplying both sides of the obtained relation by t^{-2}, and integrating the result with respect to t over the interval $(r, 2R)$ we arrive at the identity

$$\frac{1}{r}\int_{\mathcal{D}\times[0,r]} \Psi\,dy = \frac{1}{2R}\int_{\mathcal{D}\times[0,2R]} \Psi\,dy - \int_r^{2R} \frac{1}{t^2}\left(\int_{\mathcal{D}\times[0,t]} y^3\Xi\,dy\right)dt.$$

Since

$$\int_r^{2R} \frac{1}{t^2}\left(\int_{\mathcal{D}\times[0,t]} y^3\Xi\,dy\right)dt = \int_{\mathcal{D}\times[0,r]} y^3\left(\frac{1}{r} - \frac{1}{2R}\right)\Xi\,dy + \int_{\mathcal{D}\times[r,2R]}\left(1 - \frac{y^3}{2R}\right)\Xi\,dy,$$

we conclude from this that

$$\frac{1}{r}\int_{\mathcal{D}\times[0,r]} |\Psi|\,dy \le \frac{1}{2R}\int_{\mathcal{D}\times[0,2R]} |\Psi|\,dy + \int_{\mathcal{D}\times[0,r]} y^3\left(\frac{1}{r} - \frac{1}{2R}\right)|\Xi|\,dy + \int_{\mathcal{D}\times[r,2R]} y^3\left(1 - \frac{y^3}{2R}\right)|\Xi|\,dy.$$

Using the obvious inequality

$$\frac{1}{\sqrt{r}} \int_{\mathcal{D}\times[0,r]} |\overline{\mathbb{S}}|\,dy \le c\|\overline{\mathbb{S}}\|_{L^2(\mathcal{V})} \le c\|\mathbb{S}\|_{L^2(\Omega)},$$

and invoking estimate (8.6) we obtain the inequality

$$\frac{1}{\sqrt{r}} \int_{\mathcal{D}\times[0,r]} \chi(p+\varrho(\bar{u}^3)^2)\,dy \le c\|\mathbb{S}\|_{L^2(\Omega)} + \sqrt{r}\int_{\mathcal{D}\times[r,2R]} (p+\varrho(\bar{u}^3)^2)\,dy$$
$$+ c\sqrt{r}\int_{\mathcal{D}\times[0,r]} y^3\left(\frac{1}{r}-\frac{1}{2R}\right)|\Xi|\,dy + c\sqrt{r}\int_{\mathcal{D}\times[r,2R]}\left(1-\frac{y^3}{2R}\right)|\Xi|\,dy,$$

which along with the inequalities

$$\sqrt{r}y^3\left(\frac{1}{r}-\frac{1}{2R}\right) \le \sqrt{y^3} \quad \text{for } 0 < y^3 < r,$$
$$\sqrt{r}\left(1-\frac{y^3}{2R}\right) \le \sqrt{y^3} \quad \text{for } r < y^3 < 2R$$

implies the estimate

$$\frac{1}{\sqrt{r}} \int_{\mathcal{D}\times[0,r]} \chi(p+\varrho(\bar{u}^3)^2)\,dy \le c\|\mathbb{S}\|_{L^2(\Omega)} + \sqrt{r}c\int_{\mathcal{D}\times[r,2R]} (p+\varrho(\bar{u}^3)^2)\,dy$$
$$+ c\int_{\mathcal{V}} \sqrt{y^3}|\Xi|\,dy.$$

From this and (8.7) we obtain

$$\frac{1}{\sqrt{r}} \int_{\mathcal{D}\times[0,r]} \chi(p+\varrho(\bar{u}^3)^2)\,dy$$
$$\le c\|\mathbb{S}\|_{L^2(\Omega)} + c\int_{\mathcal{V}} \sqrt{y^3}(1+|\overline{\mathbf{F}}|+p+\varrho|\bar{\mathbf{u}}|^2)\,dy. \tag{8.8}$$

Next note that for $\iota < 1/2$ any nonnegative function $f \in L^1(\mathcal{V})$, satisfies the inequality

$$\int_{\mathcal{D}\times[0,2R]} (y^3)^{-\iota} f\,dy \le \sup_{(0,2R)} \frac{c(\iota)}{\sqrt{r}} \int_{\mathcal{D}\times[0,r]} f\,dy + c(\iota,R)\int_{\mathcal{D}\times[0,2R]} f\,dy,$$

which together with (8.8) yields the estimate

$$\int_{\mathcal{D}\times[0,2R]} (y^3)^{-\iota}\chi(p+\varrho(\bar{u}^3)^2)\,dy$$
$$\le c\|\mathbb{S}\|_{L^2(\Omega)} + c\int_{\mathcal{V}} \sqrt{y^3}(1+p+\varrho|\bar{\mathbf{u}}|^2+|\overline{\mathbf{F}}|)\,dy.$$

After return to the original Cartesian coordinates x we finally obtain

$$\int_{\mathcal{U}} d^{-\iota}\tilde{\chi}(p+\varrho|\mathbf{u}_n|^2)\,dx \le c\|\mathbb{S}\|_{L^2(\Omega)} + c\int_{\Omega} \sqrt{d}(1+|\overline{\mathbf{F}}|+p+\varrho|\mathbf{u}|^2)\,dx.$$

Here $\tilde{\chi} = \chi \circ \mathcal{O}$ and $r \le R$. It remains to note that the functions $\tilde{\chi}$ form a partition of unity in the domain $\Omega \setminus \Omega_R$ and the lemma follows. $\square$

COROLLARY 8.3. *Inequality* (8.4) *can be rewritten in the equivalent form*

$$\int_{\{0<d<r\}} d(x)^{-\iota}(\varrho|\mathbf{u}_n|^2 + p)\,dx \tag{8.9}$$
$$\leq c\int_\Omega d^{1/2}\varrho|\mathbf{u}|^2\,dx + c\int_{\Omega_r} p\,dx + c(\|\mathbb{S}\|_{L^2(\Omega)} + \|\mathbf{F}\|_{L^1(\Omega)} + 1).$$

The second step. Our next task is to estimate for the convolution of p and the green function of domain Ω near the boundary. To this end fix an arbitrary standard neighborhood $\mathcal{U}$ and corresponding diffeomorphism $\mathcal{O} : \mathcal{U} \mapsto \mathcal{V}$. For any $t \in \mathcal{V}$, set $\mathfrak{r}_- = |y - t|$,

$$\mathfrak{r}_+ = \sqrt{(y^1 - t^1)^2 + (y^2 - t^2)^2 + (y^3 + t^3)^2},$$
$$\mathfrak{r} = \sqrt{(y^1 - t^1)^2 + (y^2 - t^2)^2 + (t^3)^2}.$$

LEMMA 8.4. *Under the assumptions of Theorem* 8.1 *there exists a constant c depending only on Ω such that*

$$\int_\mathcal{V}\left(\frac{1}{\mathfrak{r}_-} - \frac{1}{\mathfrak{r}_+}\right)\chi y^3 p\,dy \leq c\int_\Omega d^{-\iota}\varrho(\nabla d\cdot\mathbf{u})^2\,dx + c\|\mathbb{S}\|_{L^2(\Omega)}$$
$$+ c\int_\Omega d^\iota(1 + |\mathbf{F}| + p + \varrho|\mathbf{u}|^2)\,dx. \tag{8.10}$$

PROOF. Introduce the auxiliary potentials given by the formulae

$$\Phi_t(y) = \mathfrak{r}_- + \mathfrak{r}_+,\quad \Phi_0(y) = h_b\left(\frac{y^3}{t^3}\right)\mathfrak{r},\quad \Phi = \Phi_t - 2\Phi_0,$$

in which

$$h(z) = 1 - bz\quad\text{when } 0 \leq z \leq b^{-1},\qquad h(z) = 0\quad\text{when } z > b^{-1},$$

a positive constant $b > 4$ will be specified below. Introduce also the covariant vector field $\bar\varphi$ defined by the equalities

$$\bar\varphi_1(y) = \frac{\partial\Phi}{\partial y^1},\quad \bar\varphi_2(y) = \frac{\partial\Phi}{\partial y^2},\quad \bar\varphi_3(y) = \frac{\partial\Phi_t}{\partial y^3}. \tag{8.11}$$

It is clear that $\bar\varphi$ vanishes for $y^3 = 0$. Substituting $\chi(y)\bar\varphi(y)$ into (8.3) we obtain the integral identity

$$\int_\mathcal{V}\chi(\Psi_0(y) + \Psi_1(y))\,dy + \int_\mathcal{V}(\Theta_0(y) + \Theta_1(y))\,dy = 0, \tag{8.12}$$

where

$$\Psi_0 = \frac{\partial \bar{\varphi}_j}{\partial y^i} \varrho \bar{u}^j \bar{u}^i g^{1/2}, \quad \Psi_1 = p g^{1/6} \operatorname{div} \bar{\varphi}, \tag{8.13}$$

$$\Theta_0 = -\frac{\partial \bar{\varphi}_j}{\partial y^i} \overline{\mathbb{S}}^{ij} g^{1/2}, \tag{8.14}$$

$$\begin{aligned} \Theta_1 = {} & g^{1/2} (\varrho \bar{u}^i \bar{u}^j - \overline{\mathbb{S}}^{ij}) \left(\frac{\partial \chi}{\partial y^i} \bar{\varphi}_j - \chi \Gamma^k_{ij} \bar{\varphi}_k \right) \\ & + p \nabla (\chi g^{1/6}) \bar{\varphi} + g^{1/2} \chi \overline{\mathbf{F}} \bar{\varphi}. \end{aligned} \tag{8.15}$$

The further considerations are based on the following proposition.

PROPOSITION 8.5. *There are absolute positive constants b and c such that for all $y, t \in \mathbb{R}^2 \times \mathbb{R}^+$,*

$$|\bar{\varphi}| \le c \frac{y^3}{t^3}, \quad \left| \frac{\partial \bar{\varphi}_i}{\partial y^j} \right| \le c \left(\frac{1}{\mathfrak{r}_-} + \frac{1}{\mathfrak{r}_+} + \frac{1}{t^3} \right), \quad \operatorname{div} \bar{\varphi} \ge c^{-1} \frac{y^3}{t^3} \left(\frac{1}{\mathfrak{r}_-} - \frac{1}{\mathfrak{r}_+} \right), \tag{8.16}$$

and the quadratic form

$$\mathfrak{S}(y, t) z \cdot z := \sum_{i,j=1}^{3} \frac{\partial^2 \Phi_t}{\partial y^i \partial y^j} z^i z^j - 2 \sum_{i,j=1}^{2} \frac{\partial^2 \Phi_0}{\partial y^i \partial y^j} z^i z^j, \quad z \in \mathbb{R}^3,$$

is nonnegative.

PROOF. Note that the potentials Φ_i are homogeneous functions. Moreover, they are invariant with rotation around the vertical axis and shift in the horizontal direction. Therefore it suffices to prove the proposition for

$$\begin{aligned} & y = (y^1, 0, y^3), \quad t = (0, 0, 1), \quad \mathfrak{r}_\pm = \sqrt{1 + (y^1)^2 + (y^3)^2 \pm 2y^3}, \\ & \mathfrak{r} = \sqrt{1 + (y^1)^2}. \end{aligned}$$

In this case the first two inequalities (8.16) are obviously true. In order to prove the third note that

$$\operatorname{div} \bar{\varphi} = \frac{2}{\mathfrak{r}_-} + \frac{2}{\mathfrak{r}_-} - h(y^3) \frac{4}{\mathfrak{r}} + \frac{2(y^1)^2}{\mathfrak{r}^3} \ge \frac{2}{\mathfrak{r}_-} + \frac{2}{\mathfrak{r}_-} - h(y^3) \frac{4}{\mathfrak{r}}.$$

Hence inequality (8.16) is trivial for $y^3 > b^{-1}$ when $h(y^3) = 0$. In the strip $0 < y^3 < b^{-1} < 1/4$ we have the identity

$$\operatorname{div} \bar{\varphi} - y^3 \left(\frac{1}{\mathfrak{r}_-} - \frac{1}{\mathfrak{r}_+} \right) \ge \frac{1}{\mathfrak{r}} \left(4by^3 + \frac{1}{\mathfrak{r}_-} + \frac{3}{\mathfrak{r}_+} - 4 \right) = \frac{y^3}{\mathfrak{r}} (4b + O(1)),$$

in which the quantity $O(1)$ is bounded by an absolute constant. It follows from this that inequalities (8.16) hold true for all sufficiently large b. In order to prove the nonnegativity of the quadratic form (8.5) note that under the above assumptions its coefficients are given by the formulae

$$\mathfrak{S}_{11}=\frac{(y^3-1)^2}{\mathfrak{r}_-^3}+\frac{(y^3+1)^2}{\mathfrak{r}_+^3}-2h(y^3)\frac{1}{\mathfrak{r}^3},\quad \mathfrak{S}_{22}=\frac{1}{\mathfrak{r}_-}+\frac{1}{\mathfrak{r}_+}-2h(y^3)\frac{1}{\mathfrak{r}},$$
$$\mathfrak{S}_{33}=\frac{(y^1)^2}{\mathfrak{r}_-^3}+\frac{(y^1)^2}{\mathfrak{r}_+^3},\quad \mathfrak{S}_{13}=-\frac{y^1(y^3-1)}{\mathfrak{r}_-^3}-\frac{y^1(y^3+1)}{\mathfrak{r}_+^3},$$

$\mathfrak{S}_{12}=\mathfrak{S}_{23}=0$. It is clear that the quadratic form $\mathfrak{S}$ is defined positive when $h=0$. On the other hand, in the strip $0<y^3<b^{-1}$, where h is not equal to zero, we have the inequality

$$\mathfrak{S}_{22}=\frac{1}{\mathfrak{r}}\left(2by^3+\frac{\mathfrak{r}}{\mathfrak{r}_-}+\frac{\mathfrak{r}}{\mathfrak{r}_+}-2\right),$$
$$\mathfrak{S}_{11}\mathfrak{S}_{33}-\mathfrak{S}_{13}^2=\frac{2(y^1)^2(\mathfrak{r}_-^3+\mathfrak{r}_+^3)}{\mathfrak{r}_-^3\mathfrak{r}_+^3\mathfrak{r}^3}\left(by^3+\frac{2\mathfrak{r}^3}{\mathfrak{r}_-^3+\mathfrak{r}_-^3}-1\right),$$

which yields the positivity of $\mathfrak{S}$ for all sufficiently large b. □

Let us turn to the proof of Lemma 8.4. Assume that the vector field $\bar{\varphi}$ meets all requirements of Proposition 8.5. Our task is to estimate Ψ_i from below and Θ_i from above. The expression for $\bar{\varphi}$ implies the identity

$$\frac{\partial\bar{\varphi}_\mu}{\partial y^\tau}\varrho\bar{u}^\mu\bar{u}^\tau=\varrho\mathfrak{S}\bar{\mathbf{u}}\cdot\bar{\mathbf{u}}-\frac{4}{t^3}h'\left(\frac{y^3}{t^3}\right)\left(\frac{y^1-t^1}{\mathfrak{r}}\bar{u}^1+\frac{y^2-t^2}{\mathfrak{r}}\bar{u}^2\right)\bar{u}^3\varrho$$

which along with Proposition 8.5 yields the estimate

$$\frac{\partial\bar{\varphi}_\mu}{\partial y^\tau}\varrho\bar{u}^\mu\bar{u}^\tau\geq-\frac{c}{t^3}\varrho|\bar{\mathbf{u}}||\bar{u}^3|\geq-\frac{c}{t^3}(y^3)^\iota\varrho|\bar{\mathbf{u}}|^2-\frac{c}{t^3}(y^3)^{-\iota}\varrho|\bar{u}^3|^2.$$

Combining this result with inequalities (8.16) we obtain

$$\begin{aligned}\int_{\mathcal{V}}\chi(\Psi_0(y)+\Psi_1(y))\,dy&\geq\frac{c}{t^3}\int_{\mathcal{V}}\left(\frac{1}{\mathfrak{r}_-}-\frac{1}{\mathfrak{r}_+}\right)y^3p\chi\,dy\\&-\frac{c}{t^3}\int_{\mathcal{V}}((y^3)^\iota\varrho|\bar{\mathbf{u}}|^2+(y^3)^{-\iota}\varrho|\bar{u}^3|^2)\chi\,dy.\end{aligned}\tag{8.17}$$

Next estimates (8.14) and (8.16) implies the inequality

$$|\Theta_0|\leq c\left(\frac{1}{\mathfrak{r}_-}+\frac{1}{\mathfrak{r}_+}+\frac{1}{t^3}\right)|\mathbb{S}|,$$

which together with the Cauchy inequality implies the estimate

$$\begin{aligned}\int_{\mathcal{V}}|\Theta_0|\,dy&\leq c\left(\int_{\mathcal{V}}(\mathfrak{r}_-^{-2}+\mathfrak{r}_+^{-2}+(t^3)^{-2})\,dy\int_{\mathcal{V}}|\mathbb{S}|^2\,dy\right)^{1/2}\\&\leq\frac{c}{t^3}\|\mathbb{S}\|_{L^2(\Omega)}.\end{aligned}\tag{8.18}$$

In its turn, from (8.15), (8.16), and the obvious inequality

$$|\Theta_1| \le \frac{cy^3}{t^3}(|\overline{\mathbb{S}}| + \varrho|\bar{\mathbf{u}}|^2 + p + |\overline{\mathbf{F}}| + 1),$$

we obtain the estimate

$$\int_{\mathcal{V}} |\Theta_1|\,dy \le \frac{c}{t^3}\|\overline{\mathbb{S}}\|_{L^2(\Omega)} + \frac{c}{t^3}\int_{\mathcal{V}} y^3(\varrho|\bar{\mathbf{u}}|^2 + p + |\overline{\mathbf{F}}| + 1)\,dy. \tag{8.19}$$

It remains to note that substituting (8.17)–(8.19) into (8.12) leads to the desired estimate

$$\begin{aligned}\int_{\mathcal{V}}\left(\frac{1}{\mathfrak{t}_-} - \frac{1}{\mathfrak{t}_+}\right) y^3 p\chi\,dy &\le c\int_{\mathcal{V}} (y^3)^{-\iota}\varrho|\bar{u}^3|^2\chi\,dy + c\|\overline{\mathbb{S}}\|_{L^2(\Omega)} \\ &\quad + \int_{\mathcal{V}} y^3(\varrho|\bar{\mathbf{u}}|^2 + p + |\overline{\mathbf{F}}| + 1)\,dy \\ &\le c\int_{\Omega} d^{-\iota}\varrho|\bar{\mathbf{u}}_n|^2\,dx + c\|\overline{\mathbb{S}}\|_{L^2(\Omega)} \\ &\quad + c\int_{\Omega} d(\varrho|\mathbf{u}|^2 + p + |\mathbf{F}| + 1)\,dx. \quad \square\end{aligned}$$

Lemmas 8.2 and 8.4 imply the following estimate for the convolution of the Green function $\mathfrak{G}$ of domain Ω and the function p.

LEMMA 8.6. *Let R be a diameter of standard neighborhood depending only on Ω. Then there exist positive constants c, r depending only on Ω such that for any $z \in \Omega \setminus \Omega_{R/2}$,*

$$\begin{aligned}\int_{\Omega} d(x)\mathfrak{G}(x,z)p(x)\,dx &\le c\|\overline{\mathbb{S}}\|_{L^2(\Omega)} + c\int_{\Omega} d^{\iota}\varrho|\mathbf{u}|^2\,dx \\ &\quad + c\int_{\Omega_r} p\,dy + c\int_{\Omega}(|\mathbf{F}| + 1)\,dx.\end{aligned}$$

PROOF. Recalling Lemma 8.4 and Corollary 8.3 we get

$$\begin{aligned}\int_{\mathcal{V}}\left(\frac{1}{\mathfrak{t}_-} - \frac{1}{\mathfrak{t}_+}\right)\chi y^3 p\,dy &\le c\int_{\Omega} d^{\iota}\varrho|\mathbf{u}|^2\,dx \\ &\quad + c\int_{\Omega_r} p\,dx + c(\|\overline{\mathbb{S}}\|_{L^2(\Omega)} + \|\mathbf{F}\|_{L^1(\Omega)} + 1).\end{aligned}$$

Next note that for all $x, z \in \mathcal{U}$ and $y = y(x)$, $t = y(z) \in \mathcal{V}$,

$$c^{-1}\mathfrak{G}(x,z) \le \left(\frac{1}{\mathfrak{t}_-} - \frac{1}{\mathfrak{t}_+}\right) \le c\,\mathfrak{G}(x,z).$$

Hence the estimate

$$\begin{aligned}\int_{\mathcal{U}} \mathfrak{G}(x,z)d(x)p(x)\tilde{\chi}(x)\,dx &\le c\int_{\Omega} d^{\iota}(\varrho|\mathbf{u}|^2\,dx \\ &\quad + c\int_{\Omega_r} p\,dx + c(\|\overline{\mathbb{S}}\|_{L^2(\Omega)} + \|\mathbf{F}\|_{L^1(\Omega)} + 1)\end{aligned}$$

holds true for all $z \in \mathcal{U}$. Now choose an arbitrary point $z \in \Omega \setminus \Omega_{R/2}$. Since the functions $\tilde{\chi}$ form the partition of unity in the domain $\Omega \setminus \Omega_R$ we have

$$\int_{\Omega\setminus\Omega_R} \mathfrak{G}(x,z)d(x)p(x)\,dx \le c\int_\Omega d^\iota(\varrho|\mathbf{u}|^2\,dx \\ + c\int_{\Omega_r} p\,dx + c(\|\mathbb{S}\|_{L^2(\Omega)} + \|\mathbf{F}\|_{L^1(\Omega)} + 1).$$

It remains to note that $x \in \Omega_R$ and $z \in \Omega \setminus \Omega_{R/2}$, $\mathfrak{G}(x,z) \le c$ and hence

$$\int_{\Omega_R} \mathfrak{G}(x,z)d(x)p(x)\,dx \le c\int_\Omega d^\iota\varrho|\mathbf{u}|^2\,dx + c\int_{\Omega_r} p\,dx \\ + c(\|\mathbb{S}\|_{L^2(\Omega)} + \|\mathbf{F}\|_{L^1(\Omega)} + 1). \quad \square$$

The third step. In order to complete the proof of Theorem 8.1 it remains to deduce the internal estimates for the Green potential of p. They result from the following

LEMMA 8.7. *Under the assumptions of Theorem* 8.1 *there is* $r > 0$, *depending only on* Ω *such that for any* $z \in \Omega_{R/2}$,

$$\int_\Omega \mathfrak{G}(x,z)p(x)\,dx \le c\|\mathbb{S}\|_{L^2(\Omega)} + c\int_\Omega d^\iota\varrho|\mathbf{u}|^2\,dx \\ + c\int_\Omega (p + 1 + |\mathbf{F}|)\,dx. \tag{8.20}$$

PROOF. Recall that $\Omega_{R/2} = \{\text{dist}\,(x,\partial\Omega) > r/2\}$. Choose an arbitrary $z \in \Omega_{R/2}$ and set $t = R/4$. Since $\mathfrak{G}(x,z) \le c|x-z|^{-1}$, we have

$$\int_\Omega \mathfrak{G}(x,z)p\,dx \le c\int_{B(z,t)} \frac{p\,dx}{|x-z|} + \frac{c}{t}\int_{\Omega\setminus B(z,t)} p\,dx \quad \text{and} \\ \int_\Omega \mathfrak{G}(x,z)|\mathbb{S}|\,dx \le c\|\mathbb{S}\|_{L^2(\Omega)}.$$

Next we use the identity

$$\int_{B(z,t)} \frac{\varrho}{|x-z|}(I - \mathbf{n}\otimes\mathbf{n}) : \mathbf{u}\otimes\mathbf{u}\,dx + \int_{B(z,t)} \frac{2}{|x-z|}p\,dx \\ = \int_{B(z,t)} \frac{1}{|x-z|}(I - \mathbf{n}\otimes\mathbf{n}) : \mathbb{S}\mathbf{u}\,dx \\ + \frac{1}{t}\int_{B(z,t)} (\varrho\mathbf{u}^2 + 3p - \operatorname{Tr}\mathbb{S})\,dx + \int_{B(z,t)} \varrho\left(\mathbf{n} - \frac{1}{t}(x-z)\right)\cdot\mathbf{F}\,dx = 0,$$

with $\mathbf{n} = (x-z)/|x-z|$, which yields the estimate

$$\int_{B(z,t),} \frac{\varrho}{|x-z|}(I - \mathbf{n}\otimes\mathbf{n}) : \mathbf{u}\otimes\mathbf{u}\,dx + \int_\Omega \frac{p}{|x-z|}\,dx \\ \le \frac{1}{t}\int_{B(y,t)} \varrho|\mathbf{u}|^2\,dx + c\int_\Omega p\,dx + c(\|\mathbb{S}\|_{L^2(\Omega)} + \|\mathbf{F}\|_{L^1(\Omega)} + 1). \tag{8.21}$$

Since $(I - \mathbf{n} \otimes \mathbf{n}) : \mathbf{u} \otimes \mathbf{u} \geq 0$ and we have $t^{-1} \leq 16d(x)R^{-2}$ for all $x \in B(z, t)$, the desired inequality (8.20) is the straightforward consequence of (8.21). □

In conclusion we note that the statement of Theorem 8.1 is a consequence of Lemma 8.6, 8.7 and Corollary 8.3.

Theorem 8.1 and Lemma 4.4 imply the following corollary which plays the key role in the proof of Theorem 7.6.

COROLLARY 8.8. *Let all the assumptions of Theorem* 8.1 *be satisfied. Then for any* $0 < \iota < 1/2$ *and* $v \in H_0^{1,2}(\Omega)$,

$$\int_\Omega d(x)p(x)|v|^2\,dx \leq c\biggl(\int_\Omega d(x)^\iota \varrho|\mathbf{u}|^2\,dx + \int_{\Omega_r} p\,dx + \|\mathbb{S}\|_{L^2(\Omega)} + \|\mathbf{F}\|_{L^1(\Omega)} + 1\biggr)\|v\|^2_{H^{1,2}(\Omega)}. \tag{8.22}$$

8.1. *Proof of Theorem 7.6*

We are now in a position to complete the proof of Theorem 7.6. It is based on the following lemma, which gives the auxiliary estimate for the pressure.

LEMMA 8.9. *Under the assumptions Theorem* 7.6 *for any* $t \geq 0$ $1 < \sigma < 3/2$,

$$\|p\|_{L^\sigma(\Omega_t)} \leq c(\|\varrho|\mathbf{u}|^2\|_{L^\sigma(\Omega_t)} + \sqrt{\mathbf{D}} + \|\mathbf{F}\|_{L^1(\Omega_t)} + \mathbf{M}). \tag{8.23}$$

PROOF. Recall that for any $\psi \in L^{\sigma/(\sigma-1)}(\Omega_t)$ with $\int_{\Omega_t} \psi\,dx = 0$ there exists a vector field φ satisfying the conditions

$$\begin{aligned} &\Omega_t : \operatorname{div}\varphi = \psi, \qquad \partial\Omega_t : \varphi = 0\\ &\|\nabla\varphi\|_{L^{\sigma/(\sigma-1)}(\Omega_t)} \leq c(\Omega_r)\|\psi\|_{L^{\sigma/(\sigma-1)}(\Omega_t)}, \qquad |\varphi| \leq c\|\psi\|_{L^{\sigma/(\sigma-1)}(\Omega_t)}. \end{aligned}$$

Multiplying both the sides of relation (8.1) by φ and integrating the result over Ω_t we obtain the integral identity

$$-\int_{\Omega_t} ((\varrho\mathbf{u} \otimes \mathbf{u} - \mathbb{S}) : \nabla\varphi + \mathbf{F}\varphi) = \int_{\Omega_r} p\psi\,dx,$$

which along with the obvious inequality

$$\int_{\Omega_t} (|\varrho\mathbf{u} \otimes \mathbf{u} : \nabla\varphi| + |\mathbb{S}||\nabla\varphi|)\,dx \leq (\|\varrho|\mathbf{u}|^2\|_{L^\sigma(\Omega_t)} + \sqrt{\mathbf{D}})\|\nabla\varphi\|_{L^{\sigma/(\sigma-1)}(\Omega_t)}$$

yields the estimate

$$\int_{\Omega_t} p\psi\,dx \leq c(\|\varrho|\mathbf{u}|^2\|_{L^\sigma(\Omega_t)} + \sqrt{\mathbf{D}} + 1)c\|\psi\|_{L^{\sigma/(\sigma-1)}(\Omega_t)}.$$

Thus we get

$$\|p - (\text{ meas } \Omega_t)^{-1} \int_{\Omega_t} p\, dx\|_{L^\sigma(\Omega_r)} \le (\|\varrho|\mathbf{v}|^2\|_{L^\sigma(\Omega_t)} + \sqrt{\mathbf{D}} + \|\mathbf{F}\|_{L^1(\Omega_t)}).$$

It remains to note that for any $\delta > 0$,

$$\int_{\Omega_t} p\, dx \le \delta \|p\|_{L^\sigma(\Omega_r)} + c(\delta) \int_\Omega \varrho\, dx \le c\delta \|p\|_{L^\sigma(\Omega_t)} + c(\delta)\mathbf{M}. \quad \square$$

Let us turn to the proof of Theorem 7.6. Choose a number σ satisfying the inequalities

$$1 < \sigma < 6/(10 - 3\gamma) \quad \text{for } 4/3 < \gamma < 2, \quad 1 < \sigma < 3/2 \quad \text{for } 2 < \gamma, \tag{8.24}$$

and set

$$\alpha = \frac{4\sigma - 3}{3\gamma - 2}, \quad \beta = \frac{\gamma\sigma - 2\sigma + 1}{3\gamma - 2}, \quad \tau = \frac{3\gamma - 2\sigma - \gamma\sigma}{3\gamma - 2}, \quad \iota = \frac{\alpha}{\sigma + \tau}. \tag{8.25}$$

It follows from (8.24) that

$$\alpha + \beta + \tau = 1, \quad (2\sigma - 2\alpha)/\beta = 6, \quad \alpha/\sigma < 1/2. \tag{8.26}$$

From this and the Holder inequality we obtain

$$\begin{aligned} &\left(\int_\Omega (d^\iota \varrho |\mathbf{u}|^2)^\sigma\, dx\right)^{1/\sigma} \\ &\quad \le \left(\int_\Omega d^{-\iota} \varrho\, dx\right)^\kappa \left(\int_\Omega dp|\mathbf{u}|^2\, dx\right)^a \left(\int_\Omega |\mathbf{u}|^6\, dx\right)^{2(1-a)/6}, \end{aligned} \tag{8.27}$$

where $\kappa = \tau/\sigma$, $a = \alpha/\sigma$. It is easily seen that $\iota = (5\gamma - 4)^{-1} > s$ for $\sigma = 1$. Hence we can choose σ so close to 1 that $\iota \le s$. For such choice of σ, inequality (8.27) yields the estimate

$$\|d^\iota \varrho |\mathbf{u}|^2\|_{L^\sigma(\Omega)} \le c\mathbf{M}^\kappa \left(\int_\Omega dp|\mathbf{u}|^2\, dx\right)^a \mathbf{D}^{(1-a)}. \tag{8.28}$$

By virtue of weak energy estimate (7.24), we have $\|\overline{\mathbb{S}}\|_{L^2(\Omega)} \le \sqrt{\mathbb{D}}$. From this, Corollary 8.8 and Lemma 8.9 we obtain

$$\begin{aligned} \|dp|\mathbf{u}|^2\|_{L^1(\Omega)} &\le c(\|d^\iota \varrho|\mathbf{u}|^2\|_{L^\sigma(\Omega)} + \|p\|_{L^\sigma(\Omega_r)} + \sqrt{\mathbf{D}} + 1)\mathbf{D} \\ &\le c(\|d^\iota \varrho|\mathbf{u}|^2\|_{L^\sigma(\Omega)} + \|\varrho|\mathbf{u}|^2\|_{L^\sigma(\Omega_r)} + \sqrt{\mathbf{D}} + 1 + \|\mathbf{F}\|_{L^1(\Omega)} + \mathbf{M})\mathbf{D} \\ &\le c(\|d^\iota \varrho|\mathbf{u}|^2\|_{L^\sigma(\Omega)} + \sqrt{\mathbf{D}} + 1 + \|\mathbf{F}\|_{L^1(\Omega)} + \mathbf{M})\mathbf{D}. \end{aligned}$$

Substituting this inequality into (8.28) leads to the estimate

$$\mathcal{K} \le c(\mathcal{K} + \mathbf{D}^{1/2} + \|\mathbf{F}\|_{L^1(\Omega)} + \mathbf{M} + 1)^a \mathbf{D}, \tag{8.29}$$

where $\mathcal{K} = \|d^{\iota}\varrho|\mathbf{u}|^2\|_{L^{\sigma}(\Omega)}$. Since $\iota \le s$, we have

$$\mathbf{K} \le c\|d^{s}\varrho|\mathbf{u}|^2\|_{L^{\sigma}(\Omega)} \le c\mathcal{K}.$$

From this and the weak energy estimate (7.24) we conclude that

$$\mathbf{D} \le c\sqrt{\mathcal{K}} + c\|\mathbf{F}\|_{L^1(\Omega)}. \tag{8.30}$$

Combining inequalities (8.29) and (8.30) we get

$$\mathcal{K} \le c(\mathcal{K} + \|\mathbf{F}\|_{L^1(\Omega)} + \mathbf{M} + 1)^{a}(\sqrt{\mathcal{K}} + \|\mathbf{F}\|_{L^1(\Omega)}).$$

Since, by virtue of (8.26), the exponent a is less than $1/2$, we finally obtain estimate

$$\|d^{s}\varrho|\mathbf{u}|^2\|_{L^{\sigma}(\Omega)} + \mathbf{D} \le \mathcal{K} + \mathbf{D} \le C(\Omega, s, \gamma, \mathbf{M}, \|\mathbf{F}\|_{L^1(\Omega)}). \tag{8.31}$$

To complete the proof of estimate (7.25) we note that the estimate for $\|p\|_{L^{\sigma}(\Omega_t)}$ follows from (8.31) and Lemma 8.9, and the estimate for $\|p\|_{L^1}(\Omega)$ follows from (8.31), estimate for $\|p\|_{L^{\sigma}(\Omega_t)}$, and inequality (8.9). □

9. Outflow–inflow problem

In this section we consider the questions on the local existence and uniqueness of the outflow–inflow problem in a smooth domain. For simplicity we restrict our considerations to the case of isothermal flow without mass and volume forces. The extension of the results on the case of barotropic flows is obvious. Note also that in the local theory the heat transfer does not create principal mathematical difficulties and the result also holds true for the Navier–Stokes–Fourier equations. Under these assumptions the problem can be formulated as follows. Let $\Omega \Subset \mathbb{R}^3$ be a bounded domain with the boundary $\partial\Omega \in C^{\infty}$, and $\mathbf{U} : \partial\Omega \mapsto \mathbb{R}^3$ be a given smooth vector field satisfying the compatibility condition

$$\int_{\partial\Omega} \mathbf{U} \cdot \mathbf{n}\, ds = 0. \tag{9.1}$$

The problem is to find the velocity field $\mathbf{u}$ and the density distribution ϱ satisfying the following equation and the boundary conditions:

$$\begin{aligned}
&\Delta\mathbf{u} + \lambda\nabla \operatorname{div}\mathbf{u} = k\varrho\mathbf{u}\cdot\nabla\mathbf{u} + \sigma\nabla\varrho \quad \text{in } \Omega,\\
&\operatorname{div}(\varrho\mathbf{u}) = 0 \quad \text{in } \Omega,\\
&\mathbf{u} = \mathbf{U} \quad \text{on } \partial\Omega, \qquad \varrho = 1 \quad \text{on } \Sigma_{\text{in}}.
\end{aligned}$$

Before formulation of the main results we write the governing equation in more transparent form using the change of unknown functions proposed in [48]. To do so we introduce *the* *effective viscous pressure* $q = \sigma\varrho - \lambda \operatorname{div} \mathbf{u}$, and rewrite equations (9.2) in the equivalent form

$$\Delta \mathbf{u} - \nabla q = k\varrho \mathbf{u}\nabla \mathbf{u} \quad \text{in } \Omega, \tag{9.2a}$$

$$\operatorname{div} \mathbf{u} = \frac{1}{\lambda}(\sigma\varrho - q) \quad \text{in } \Omega, \tag{9.2b}$$

$$\mathbf{u} = \mathbf{U} \quad \text{on } \partial\Omega, \tag{9.2c}$$

$$\mathbf{u} \cdot \nabla\varrho + \frac{\sigma}{\lambda}\varrho^2 = \frac{q\varrho}{\lambda} \quad \text{in } \Omega, \tag{9.2d}$$

$$\varrho = 1 \quad \text{on } \Sigma_{\text{in}}. \tag{9.2e}$$

We assume that $\lambda \gg 1$ and $k \ll 1$. In such a case, the *approximate solutions* to problem (9.2) can be chosen in the form $(1, \mathbf{u}_0, q_0)$, where $(\mathbf{u}_0, q_0)$ is a solution to the boundary value problem for the Stokes equations,

$$\begin{aligned} &\Delta \mathbf{u}_0 - \nabla q_0 = 0, \quad \operatorname{div} \mathbf{u}_0 = 0 \quad \text{in } \Omega, \\ &\mathbf{u}_0 = \mathbf{U} \quad \text{on } \partial\Omega, \quad \Pi q_0 = q_0. \end{aligned} \tag{9.3}$$

In our notations Π is the projector,

$$\Pi u = u - \frac{1}{\operatorname{meas} \Omega} \int_\Omega u \, dx.$$

Equations (9.3) can be obtained as the limit of equations (9.2) for the passage $\lambda \to \infty$, $k \to 0$. It follows from the standard elliptic theory that for the boundary $\partial\Omega \in C^\infty$, we have $(\mathbf{u}_0, q_0) \in C^\infty(\Omega)$. We look for solutions to problem (9.2) in the form

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{v}, \quad \varrho = 1 + \varphi, \quad q = q_0 + \sigma + \pi + \lambda m, \tag{9.4}$$

with the unknown functions $\Theta = (\mathbf{v}, \pi, \varphi)$ and the unknown constant m. Substituting (9.4) into (9.2) we obtain the following boundary problem for the vector-function Θ,

$$\begin{aligned} \Delta \mathbf{v} - \nabla\pi &= k\mathcal{B}(\varrho, \mathbf{u}, \mathbf{u}) \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{v} &= \sigma_\lambda\varphi - \Psi(\Theta) - m \quad \text{in } \Omega, \\ \mathbf{u} \cdot \nabla\varphi + \sigma_\lambda\varphi &= \Psi_1(\Theta) + \varrho m \quad \text{in } \Omega, \\ \mathbf{v} = 0 \quad \text{on } \partial\Omega, \quad \varphi = 0 \quad &\text{on } \Sigma_{\text{in}}, \quad \Pi\pi = \pi, \end{aligned} \tag{9.5a}$$

where

$$B(\varrho, \mathbf{v}, \mathbf{w}) = \varrho \mathbf{v}\nabla \mathbf{w}, \quad \varrho_\lambda = \varrho/\lambda,$$

$$\Psi_1[\Theta] = \varrho\Psi[\mu] - \sigma_\lambda\varphi^2, \quad \Psi[\Theta] = \frac{q_0 + \pi}{\lambda},$$

the vector field $\mathbf{u}$ and the function ϱ are given by (9.4). Finally, we specify the constant m. In our framework, in contrast to the case of homogeneous boundary problem, the solution to such a problem is not trivial since this problem is connected with the difficult question of mass control in outflow and/or inflow problems. Recall that the absence of mass control is the main obstacle for proving the global solvability of inhomogeneous boundary problems for compressible Navier–Stokes equations, we refer to [34] for a discussion. In order to cope with this difficulty we write the compatibility condition in a sophisticated form, which allows us to control the total mass of the gas. To this end we introduce the auxiliary function ζ satisfying the equations

$$-\operatorname{div}(\mathbf{u}\zeta) + \sigma_\lambda \zeta = \sigma_\lambda \quad \text{in } \Omega, \quad \zeta = 0 \quad \text{on } \Sigma_{\text{out}}, \tag{9.5b}$$

and fix the constant m as follows

$$m = \varkappa \int_\Omega (\Psi_1[\Theta]\zeta - \Psi[\Theta])\, dx, \quad \varkappa = \left(\int_\Omega (1 - \zeta - \zeta\varphi)\, dx\right)^{-1}. \tag{9.5c}$$

In this way the auxiliary function ζ becomes an integral part of the solution to the problem. Now, our aim is to prove the existence and uniqueness of solutions to problem (9.5).

9.1. *Existence result*

For an arbitrary bounded domain $\Omega \subset \mathbb{R}^3$ with a Lipschitz boundary, we introduce the Banach spaces

$$X^{s,r} = H^{s,r}(\Omega) \cap H^{1,2}(\Omega), \quad Y^{s,r} = H^{s+1,r}(\Omega) \cap H^{2,2}(\Omega),$$
$$Z^{s,r} = \mathcal{H}^{s-1,r}(\Omega) \cap L^2(\Omega)$$

equipped with the norms

$$\|u\|_{X^{s,r}} = \|u\|_{H^{s,r}(\Omega)} + \|u\|_{H^{1,2}(\Omega)}, \quad \|u\|_{Y^{s,r}} = \|u\|_{H^{1+s,r}(\Omega)} + \|u\|_{H^{2,2}(\Omega)},$$
$$\|u\|_{Z^{s,r}} = \|u\|_{\mathcal{H}^{s-1,r}(\Omega)} + \|u\|_{L^2(\Omega)}.$$

It can be easily seen that the embeddings $Y^{s,r} \hookrightarrow X^{s,r} \hookrightarrow Z^{s,r}$ are compact and for $sr > 3$, each of the spaces $X^{s,r}$ and $Y^{s,r}$ is a commutative Banach algebra. Denote by E the closed subspace of the Banach space $Y^{s,r}(\Omega)^3 \times X^{s,r}(\Omega)^2$ in the following form

$$E = \{\vartheta = (\mathbf{v}, \pi, \varphi) : \mathbf{v} = 0 \text{ on } \partial\Omega, \varphi = 0 \text{ on } \Sigma_{\text{in}}, \Pi\pi = \pi\}, \tag{9.6}$$

and denote by $\mathcal{B}_\tau \subset E$ the closed ball of radius τ centered at 0. Next, note that for $sr > 3$, elements of the ball $\mathcal{B}_\tau$ satisfy the inequality

$$\|\mathbf{v}\|_{C^1(\Omega)} + \|\pi\|_{C(\Omega)} + \|\varphi\|_{C(\Omega)} \le c_e(r, s, \Omega)\|\Theta\|_E \le c_e\tau, \tag{9.7}$$

where the norm in E is defined by

$$\|\Theta\|_E = \|\mathbf{v}\|_{Y^{s,r}(\Omega)} + \|\pi\|_{X^{s,r}(\Omega)} + \|\varphi\|_{X^{s,r}(\Omega)}.$$

THEOREM 9.1. *Assume that the surface $\partial\Omega$ and given vector field $\mathbf{U}$ satisfy the emergent field condition* (**H1**)–(**H3**). *Furthermore, let σ^*, R be constants given by Theorem 5.7,* *and let positive numbers r, s, σ_λ, r_0 meet all requirements of this theorem and satisfy the inequalities*

$$1/2 < s \le 1, \qquad sr > 3, \sigma_\lambda > \sigma^*. \tag{9.8}$$

Then there exists $\tau_0 \in (0, R]$, depending only on $\mathbf{U}, \Omega, r, s, \sigma_\lambda$, such that for all

$$\tau \in (0, \tau_0], \quad \lambda^{-1}, k \in (0, \tau^2], \tag{9.9}$$

problem (9.5), with $\mathbf{u}_0$ given by (9.3), has a solution $\Theta \in B_\tau$. Moreover, the auxiliary function ζ and the constants $\varkappa, m$ admit the estimates

$$\|\zeta\|_{X^{s,r}} + |\varkappa| \le c, \quad |m| \le c\tau < 1, \tag{9.10}$$

where the constant c depends only on $\mathbf{U}, \Omega, r, s$ and σ_λ.

The proof is based on the following lemma which is a straightforward consequence of the classical results on solvability of first boundary value problem for the Stokes equations (see [13]) and the interpolation theory.

LEMMA 9.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C^2$ and $(F, G) \in \mathcal{H}^{s-1,r}(\Omega) \times H^{s,r}(\Omega)$ $(0 \le s \le 1, 1 < r < \infty)$ Then the boundary value problem*

$$\begin{aligned} &\Delta \mathbf{v} - \nabla\pi = F, \quad \operatorname{div} \mathbf{v} = \Pi G \quad in\ \Omega,\\ &\mathbf{u} = 0 \ on\ \partial\Omega, \quad \Pi\pi = \pi, \end{aligned} \tag{9.11}$$

has a unique solution $(\mathbf{v}, \pi) \in H^{s+1,r}(\Omega) \times H^{s,r}(\Omega)$ such that

$$\|\mathbf{v}\|_{H^{s+1,r}(\Omega)} + \|\pi\|_{H^{s,r}(\Omega)} \le c(\Omega, r, s)(\|F\|_{\mathcal{H}^{s-1,r}(\Omega)} + \|G\|_{H^{s,r}(\Omega)}). \tag{9.12}$$

In particular, we have

$$\|\mathbf{v}\|_{Y^{s,r}} + \|\pi\|_{X^{s,r}} \le c(\Omega, r, s)(\|F\|_{Z^{s,r}} + \|G\|_{X^{s,r}}).$$

PROOF. Note that, by virtue of Theorem 6.1 in [13], for any $\mathbf{F} \in \mathcal{H}^{s-1,r}(\Omega)$ and $G \in H^{s,r}(\Omega)$ with $s = 0, 1$, problem (9.11) has a unique solution $\mathbf{v}, \pi$ satisfying inequality

$$\|\mathbf{v}\|_{H^{s+1,r}(\Omega)} + \|\pi\|_{H^{s,r}(\Omega)} \le c(\Omega, r, s)(\|\mathbf{F}\|_{\mathcal{H}^{s-1,r}(\Omega)} + \|G\|_{H^{s,r}(\Omega)}).$$

Thus the relation $(F, G) \mapsto (\mathbf{v}, \pi)$ determines the linear operator $T : \mathcal{H}^{s-1,r}(\Omega) \times H^{s,r}(\Omega) \mapsto H^{s+1,r}(\Omega) \times H^{s,r}(\Omega)$. Therefore, Lemma 9.2 is a consequence of Lemma 4.1. □

Let us turn to the proof of Theorem 9.1. We solve problem (9.5) by an application of the Schauder fixed-point theorem in the following framework. Fix $\sigma_\lambda > \sigma^*$. Next choose an arbitrary element $\Theta \in \mathcal{B}_\tau$. Since $\tau < R$, Theorem 5.7 implies that the problem

$$\mathbf{u} \cdot \nabla \varphi_1 + \sigma_\lambda \varphi_1 = \Psi_1[\Theta] + m\varrho \quad \text{in } \Omega, \qquad \varphi_1 = 0 \quad \text{on } \Sigma_{\text{in}} \tag{9.13}$$

has a unique solution satisfying the inequality

$$\|\varphi_1\|_{X^{s,r}} \le c(\Omega, \mathbf{U}, \sigma, r, s)(\|\Psi_1[\vartheta]\|_{X^{s,r}} + |m|). \tag{9.14}$$

Next, define $\mathbf{v}_1$ and π_1 to be the solutions of the boundary problem for the Stokes equations

$$\begin{aligned} \Delta \mathbf{v}_1 - \nabla \pi_1 &= k\,\mathcal{B}(\varrho, \mathbf{u}, \mathbf{u}) \equiv F[\Theta] \quad \text{in } \Omega \\ \operatorname{div} \mathbf{v}_1 &= \Pi(\sigma_\lambda \varphi_1 - \Psi[\Theta] - m) \quad \text{in } \Omega, \\ \mathbf{v}_1 &= 0 \quad \text{on } \partial\Omega, \qquad \pi_1 - \Pi\pi_1 = 0, \end{aligned} \tag{9.15}$$

where m is given by (9.5c). By Lemma 9.2, this problem admits a unique solution such that

$$\|\mathbf{v}_1\|_{Y^{s,r}} + \|\pi_1\|_{X^{s,r}} \le c(\|F[\Theta]\|_{Z^{s,r}} + |\Psi[\Theta]|_{X^{s,r}} + \|\varphi_1\|_{X^{s,r}} + |m|). \tag{9.16}$$

Equations (9.13), (9.15), (9.5c), define the mapping $\Xi : \Theta \to \Theta_1 = (\mathbf{v}_1, \pi_1, \varphi_1)$. We claim that for a suitable choice of the constant τ, Ξ is a weakly continuous automorphism of the ball $\mathcal{B}_\tau$. We begin with the estimates for nonlinear operators present in (9.13). Fix an arbitrary $\Theta \in \mathcal{B}_\tau$. We have

$$\|\Psi[\Theta]\|_{X^{s,r}} \le \frac{c}{\lambda}(\|q_0\|_{C^1(\Omega)} + \|\pi\|_{X^{s,r}}) \le c/\lambda \le c\tau^2. \tag{9.17}$$

Since, under assumptions of Theorem 9.1, $X^{s,r}(\Omega)$ is a Banach algebra and $\|\varrho\|_{X^{s,r}} \le c + \|\varphi\|_{X^{s,r}} \le$ const., we conclude from this and (9.14) that

$$\|\varphi_1\|_{X^{s,r}} \le c/\lambda + c\tau^2 + c|m| \le c\tau^2 + c|m|. \tag{9.18}$$

Recall that the operator $\mathcal{B}$ constitutes a cubic polynomial of $\mathbf{u}$, $\nabla\mathbf{u}$ and ϱ, which along with the inequalities $k < \tau^2 < 1$ yields

$$\|\mathcal{B}(\varrho, \mathbf{u}, \mathbf{u})\|_{X^{s,r}} \le ck(1 + \|\varrho\|_{X^{s,r}} + \|\mathbf{u}\|_{Y^{s,r}})^3 \le c\tau^2 \quad \text{in } \mathcal{B}_\tau. \tag{9.19}$$

Inequalities (9.9) and (9.19) imply

$$\|F[\vartheta]\|_{Z^{s,r}} \le c\tau^2(1+\tau) \quad \text{in } \mathcal{B}_\tau. \tag{9.20}$$

Combining inequalities (9.17) and (9.18) we get the estimate

$$\|\sigma_\lambda \varphi_1 + \Psi[\Theta]\|_{X^{s,r}} \le c\tau^2 + c|m|.$$

From this, (9.20), (9.16) and Lemma 9.2 we finally obtain

$$\|\mathbf{v}_1\|_{Y^{s,r}} + |\pi_1|_{X^{s,r}} \le c\tau^2 + c|m|. \tag{9.21}$$

It remains to estimate m. Recall that the vector field $\mathbf{u}$ and parameter σ_λ meet all requirements of Theorem 5.7. Therefore, problem (9.5b) has the unique solution $\zeta \in$

$H^{s,r}(\Omega)$. In particular, inequalities (5.21) yield the estimate $\|\zeta\|_{X^{s,r}} \le c$. Since $sr > 3$, by virtue of the embedding theory the embedding $H^{s,r}(\Omega) \hookrightarrow C^\beta(\Omega)$ is bounded for some $\beta \in (0,1)$. Hence estimates (5.17) and (5.22) for $rs > 3$ yield

$$\|\zeta\|_{C^\beta(\Omega)} + \|\zeta\|_{H^{1,r_0}(\Omega)} \le C(\mathbf{U}, \Omega, \sigma_\lambda). \tag{9.22}$$

Recalling that div $\mathbf{u}$ = div $\mathbf{v}$, we obtain $|\operatorname{div}\mathbf{u}| \le c\tau$. From this and the maximum principle (5.22) we conclude that

$$\|\zeta\|_{C(\Omega)} \le (1 - \sigma_\lambda^{-1} c\tau)^{-1} \le (1 - c\tau)^{-1}, \tag{9.23}$$

which leads to the following estimate

$$|1 - \zeta| \le c\tau(1 - c\tau)^{-1}.$$

Now we can estimate the right-hand side of (9.5c). Rewrite the first integral in the form

$$\int_\Omega (1 - \zeta - \zeta\varphi)\,dx = \int_\Omega (1-\zeta)^+\,dx - \int_\Omega ((1-\zeta)^- \zeta\varphi)\,dx.$$

We have

$$|(1-\zeta)^- + \zeta\varphi)| \le c_e\tau + c\tau(1 - c\tau)^{-1}.$$

On the other hand, we have $\|(1-\zeta)^+\|_{C^\beta(\Omega)} \le c(\mathbf{U}, \Omega, \sigma)$ and $(1-\zeta)^+ = 1$ on Σ_{out}. Hence

$$\int_\Omega (1-\zeta)^+\,dx > \kappa(\mathbf{U}, \Omega, \sigma) > 0.$$

Thus, we get

$$\varkappa^{-1} \ge \kappa(1 - c\kappa^{-1}\tau(1 - c\tau)^{-1}).$$

In particular, there is a positive τ_0 depending only on $\mathbf{U}$, Ω and σ_λ, such that

$$|\varkappa^{\pm 1}| \le c \quad \text{for all } \tau \le \tau_0.$$

Repeating these arguments and using inequality (9.17), and relation (9.5c), we arrive at $|m| \le c\tau^2$. Combining this estimate with (9.18) and (9.21), we finally obtain $\|\Theta_1\|_{X^{s,r}} \le c\tau^2$. Choose sufficiently small $\tau_0 = \tau_0(\mathbf{U}, \Omega, \sigma_\lambda)$, so that $c\tau_0^2 < \tau_0$. Thus, for all $\tau \le \tau_0$, Ξ maps the ball $\mathcal{B}_\tau$ into itself. Let us show that Ξ is weakly continuous. Choose an arbitrary sequence $\Theta_n \in \mathcal{B}_\tau$ such that $\Theta_n = (\mathbf{v}_n, \pi_n, \varphi_n)$ converges weakly in E to some Θ. Since the ball $\mathcal{B}_\tau$ is closed and convex, Θ belongs to $\mathcal{B}_\tau$. Let us consider the corresponding sequences of the elements $\Theta_{1,n} = \Xi(\Theta_n) \in \mathcal{B}_\tau$ and functions ζ_n. There are subsequences $\{\Theta_{1,j}\} \subset \{\Theta_{1,n}\}$ and $\{\zeta_j\} \subset \{\zeta_n\}$ such that $\Theta_{1,j}$ converges weakly in E to some element $\Theta_1 \in \mathcal{B}_\tau$ and ζ_j converges weakly in $X^{s,r}$ to some function $\zeta \in X^{s,r}$. Since the embedding $E \hookrightarrow C(\Omega)^5$ is compact, we have $\Theta_n \to \Theta$, $\Theta_{1j} \to \Theta_1$ in $C(\Omega)^5$, and

$$\nabla\zeta_j \rightharpoonup \nabla\zeta \quad \text{weakly in } L^{r_0}(\Omega), \qquad \zeta_j \to \zeta \quad \text{in } C(\Omega).$$

Substituting Θ_j and $\Theta_{1,j}$ into equations (9.13), (9.15), (9.5c) and letting $j \to \infty$ we obtain that the limits Θ and Θ_1 also satisfy (9.13), (9.15), (9.5c). Thus, we get $\Theta_1 = \Xi(\vartheta)$. Since for given Θ, a solution to equations (9.13), (9.15) is unique, we conclude from this that all weakly convergent subsequences of $\Theta_{1,n}$ have the unique limit Θ_1. Therefore, the whole sequence $\Theta_{1,n} = \Xi(\Theta_n)$ converges weakly to $\Xi(\vartheta)$. Hence the mapping $\Xi : \mathcal{B}_\tau \mapsto \mathcal{B}_\tau$ is weakly continuous and, by virtue of the Schauder fixed-point theorem, there is $\Theta \in \mathcal{B}(\tau)$ such that $\Theta = \Xi(\vartheta)$.

It remains to prove that Θ is given by a solution to problem (9.5a). For $\Theta_1 = \Theta$, the only difference between problems (9.5a) and (9.15), (9.5c) is the presence of the projector Π on the right-hand side of (9.15). Hence, it suffices to show that

$$\Pi(\sigma_\lambda \varphi - \Psi[\Theta] - m) = \sigma_\lambda \varphi - \Psi[\Theta] - m. \tag{9.24}$$

To this end we note that φ is a strong continuous solution to the transport equation

$$\mathbf{u} \cdot \nabla \varphi + \sigma_\lambda \varphi = \Psi_1[\Theta] + m\varrho.$$

Multiplying both the sides by ζ and integrating by parts we obtain

$$\sigma_\lambda \int_\Omega \varphi \, dx = \int_\Omega \zeta(\Psi_1[\Theta] + m\varrho) \, dx.$$

On the other hand, equality (9.5c) reads

$$\int_\Omega \zeta(\Psi_1[\Theta] + m(1 + \varphi)) \, dx = \int_\Omega (\Psi[\Theta] + m) \, dx.$$

Combining these equalities and noting that $1 + \varphi = \varrho$ we obtain

$$\int_\Omega (\sigma_\lambda \varphi - \Psi[\Theta] - m) \, dx = 0$$

which yields (9.24) and the proof of Theorem 9.1 is completed. □

9.2. *Uniqueness*

In this paragraph we prove that under the assumptions of Theorem 9.1 a solution to problem (9.5) is unique, in the ball $\mathcal{B}_\tau$. Assume, contrary to our claim that there exist two different solutions $(\Theta_i, \zeta_i, m_i) \in E \times X^{s,r} \times \mathbb{R}$, $i = 1, 2$, with $\Theta_i \in \mathcal{B}_\tau$ to problem (9.5) with $\Theta_i \in \mathcal{B}_\tau$. Recall that they, together with the constants $\varkappa_i$, satisfy the inequalities

$$|m_i| + \|\Theta_i\|_E \le c\tau, \quad |\varkappa_i| + \|\zeta\|_{X^{s,r}} \le c, \tag{9.25}$$

where the constant c depends only on $\mathbf{U}, \Omega, r, s$, and σ_λ. We denote $\mathbf{u}_i = \mathbf{u}_0 + \mathbf{v}_i$, $i = 1, 2$, the corresponding solutions to (9.2) $i = 1, 2$. Now set

$$\mathbf{w} = \mathbf{v}_1 - \mathbf{v}_2, \quad \omega = \pi_1 - \pi_2, \quad \psi = \varphi_1 - \varphi_2, \quad \xi = \zeta_1 - \zeta_2,$$
$$n = m_1 - m_2.$$

It follows from (9.5) that these quantities satisfy the linear equations

$$
\begin{aligned}
\mathbf{u}_1\nabla\psi+\sigma_\lambda\psi &= -\mathbf{w}\cdot\nabla\varphi_2+b_{11}\psi+b_{12}\omega+b_{13}n \quad \text{in } \Omega,\\
\Delta\mathbf{w}-\nabla\omega &= k\mathcal{C}_1(\psi,\mathbf{w}) \quad \text{in } \Omega,\\
\operatorname{div}\mathbf{w} &= b_{21}\psi+b_{22}\omega+b_{23}n \quad \text{in } \Omega,\\
-\operatorname{div}(\mathbf{u}_1\xi)+\sigma_\lambda\xi &= \operatorname{div}(\zeta_2\mathbf{w})+ \quad \text{in } \Omega,\\
\mathbf{w}=0 \quad \text{on } \partial\Omega, \quad \psi=0 &\quad \text{on } \Sigma_{\text{in}}, \quad \xi=0 \quad \text{on } \Sigma_{\text{out}},\\
\omega-\Pi\omega=0, \quad n &= \varkappa\int_\Omega(b_{31}\psi+b_{32}\omega+b_{34}\xi)\,dx.
\end{aligned}
\tag{9.26}
$$

Here the coefficients are given by the formulae

$$
\begin{aligned}
&b_{11}=\Psi[\Theta_1]+m_2-\sigma_\lambda(\varphi_1+\varphi_2),\\
&b_{12}=\lambda^{-1}\varrho_2, \quad b_{13}=\varrho_1, \quad b_{21}=\sigma_\lambda, \quad b_{22}=-1/\lambda, \quad b_{23}=-1,\\
&b_{31}=\zeta_1\Psi[\Theta_1]-\sigma_\lambda\zeta_1(\varphi_1+\varphi_2)+m_2\zeta_2,\\
&b_{32}=\zeta_1 b_{12}-b_{22}, \quad b_{34}=\Psi_1[\Theta_2]+m_2(1+\varphi_1),
\end{aligned}
\tag{9.27}
$$

and the operator $\mathcal{C}$ is defined by the equalities

$$
\mathcal{C}(\psi,\mathbf{w})=\mathcal{B}(\psi,\mathbf{u}_1,\mathbf{u}_1)+\mathcal{B}(\varrho_2,\mathbf{w},\mathbf{u}_1)+\mathcal{B}(\varrho_2,\mathbf{u}_2,\mathbf{w}).
$$

We consider relations (9.26) as the system of equations and boundary conditions for unknowns $\mathbf{w}$, ψ, ξ, and n. The next step is crucial for further analysis. We replace equations (9.26) by an integral identity, which leads to the notion of a very weak solution of problem (9.26). To this end choose an arbitrary function $(\mathbf{H},G,F,M)\in C^\infty(\Omega)^6$ such that $G-\Pi G=0$, and consider the auxiliary boundary value problems

$$
\mathcal{L}^*\varsigma=F, \quad \mathcal{L}\upsilon=M \quad \text{in } \Omega, \quad \varsigma=0 \quad \text{on } \Sigma_{\text{out}}, \quad \upsilon=0 \quad \text{on } \Sigma_{\text{in}}. \tag{9.28}
$$

$$
\Delta\mathbf{h}-\nabla g=\mathbf{H}, \quad \operatorname{div}\mathbf{h}=\Pi G \quad \text{in } \Omega, \quad \mathbf{h}=0 \quad \text{on } \partial\Omega, \quad \Pi g=g, \tag{9.29}
$$

where $\mathcal{L} =: \mathbf{u}\nabla+\sigma_\lambda$. Since under the assumptions of Theorem 9.1, $\mathbf{u}$ and σ_λ meet all requirements of Theorem 5.7, each of problems (9.28) has a unique solution, such that

$$
\|\varsigma\|_{H^{s,r}(\Omega)}\le c\|F\|_{H^{s,r}(\Omega)}, \qquad \|\upsilon\|_{H^{s,r}(\Omega)}\le c\|M\|_{H^{s,r}(\Omega)}, \tag{9.30}
$$

where c depends only on $\mathbf{U}$, Ω, r, s, and σ_λ. On the other hand, by virtue of Lemma 9.2, problem (9.29) has the unique solution satisfying the inequality

$$
\|\mathbf{h}\|_{H^{1+s,r}(\Omega)}+\|g\|_{H^{s,r}(\Omega)}\le c\|\mathbf{H}\|_{H^{1+s,r}(\Omega)}+c\|G\|_{H^{s,r}(\Omega)}. \tag{9.31}
$$

Recall that $\mathbf{w}\in H^{2,r_0}(\Omega)^3\cap C^1(\Omega)^3$ vanishes on $\partial\Omega$, and $(\omega,\psi,\xi)\in H^{1,r_0}(\Omega)^3\cap C(\Omega)^3$, where $1<r_0<\infty$ is the exponent in the definition of $X^{s,r}$. Multiplying both sides of the first equation in system (9.26) by ς, both sides of the fourth equation in (9.26) by υ, integrating the results over Ω and using the Green formula for the Stokes equations we obtain the system of integral equalities

$$
\begin{aligned}
&\int_\Omega \psi F\,dx = \int_\Omega (-\mathbf{w}\cdot\nabla\varphi_2 + b_{11}\psi + b_{12}\omega + b_{13}n)\varsigma\,dx,\\
&\int_\Omega \mathbf{w}\mathbf{H}\,dx + \int_\Omega \omega G\,dx = \int_\Omega (b_{21}\psi + b_{22}\omega + b_{23}n)g\,dx\\
&\quad + \int_\Omega k\mathcal{C}(\mathbf{w},\psi)\mathbf{h}\,dx, \qquad \int_\Omega \xi M\,dx = \int_\Omega \operatorname{div}(\zeta_2\mathbf{w})\upsilon\,dx.
\end{aligned}
\tag{9.32}
$$

Next, since $\operatorname{div}(\varrho_2\mathbf{u}_2) = 0$, we have

$$
\begin{aligned}
&\int_\Omega (\mathcal{B}(\varrho_2,\mathbf{w},\mathbf{u}_1) + \mathcal{B}(\varrho_2,\mathbf{u}_2,\mathbf{w}))\cdot\mathbf{h}\,dx\\
&\quad = \int_\Omega \varrho_2\mathbf{w}\cdot(\nabla\mathbf{u}_1\mathbf{h} - (\nabla\mathbf{h})^*\mathbf{u}_2)\,dx.
\end{aligned}
$$

On the other hand, integration by parts gives

$$
\int_\Omega \operatorname{div}(\zeta_2\mathbf{w})\upsilon\,dx = -\int_\Omega \zeta_2\mathbf{w}\nabla\upsilon\,dx.
$$

Using these identities and recalling the duality pairing we can collect relations (9.32), together with the expression for n, in one integral identity

$$
\begin{aligned}
&\int_\Omega \mathbf{w}(\mathbf{H} - k\varrho_2\nabla\mathbf{u}_1\cdot\mathbf{h} + k\varrho_2(\nabla\mathbf{h})^*\mathbf{u}_2)\,dx\\
&\quad - \mathfrak{B}(\mathbf{w},\varphi_2,\varsigma) - \mathfrak{B}(\mathbf{w},\upsilon,\zeta_2) + \langle\omega, G - b_{12}\varsigma - b_{22}g - \varkappa b_{32}\rangle\\
&+ \langle\psi, F - b_{11}\varsigma - b_{21}g - \varkappa b_{31} - k\mathbf{u}_1\cdot\nabla\mathbf{u}_1\cdot\mathbf{h}\rangle\\
&\quad + \langle\xi, M - \varkappa b_{34}\rangle + n - n\langle 1, b_{13}\varsigma + b_{23}g\rangle = 0.
\end{aligned}
\tag{9.33}
$$

Here, the trilinear form $\mathfrak{B}$ is defined by the equality

$$
\mathfrak{B}(\mathbf{w},\varphi_2,\varsigma) = -\int_\Omega \varsigma\mathbf{w}\cdot\nabla\varphi_2\,dx.
$$

Note, that relations (9.33) are well defined for all $\mathbf{w}\in\mathcal{H}_0^{1-s,r'}(\Omega)$ and $\psi,\xi\in\mathbb{H}^{-s,r'}(\Omega)$. It is obviously true for all terms, possibly except of $\mathfrak{B}$. Well-posedness of the form $\mathfrak{B}$ follows the next lemma, the proof is at the end of this section. The lemma is given in $\mathbb{R}^d$, for our application $d = 3$.

LEMMA 9.3. *Let $\Omega\subset\mathbb{R}^d$ be a bounded domain with the Lipschitz boundary, let exponents s and r satisfy the inequalities $sr > d$, $1/2 \le s \le 1$ and $\varphi,\varsigma\in H^{s,r}(\Omega)\cap H^{1,r_0}(\Omega)$,*

$\mathbf{w} \in \mathcal{H}_0^{1-s,r'}(\Omega) \cap H_0^{1,r_0}(\Omega)$, $1 < r_0 < \infty$. *Then there is a constant* c *depending only on* s, r *and* Ω, *such that the trilinear form*

$$\mathfrak{B}(\mathbf{w}, \varphi, \varsigma) = -\int_\Omega \varsigma \mathbf{w} \cdot \nabla\varphi \, dx$$

satisfies the inequality

$$|\mathfrak{B}(\mathbf{w}, \varphi, \varsigma)| \le c\|\mathbf{w}\|_{\mathcal{H}_0^{1-s,r'}(\Omega)} \|\varphi\|_{H^{s,r}(\Omega)} \|\varsigma\|_{H^{s,r}(\Omega)}, \tag{9.34}$$

and can be continuously extended to $\mathfrak{B} : \mathcal{H}_0^{1-s,r'}(\Omega)^d \times H^{s,r}(\Omega)^2 \mapsto \mathbb{R}$. *In particular, we have* $\varsigma\nabla\varphi \in \mathcal{H}^{s-1,r}(\Omega)$ *and* $\|\varsigma\nabla\varphi\|_{H^{1-s,r}(\Omega)} \le c\|\varphi\|_{H^{s,r}(\Omega)}\|\varsigma\|_{H^{s,r}(\Omega)}$.

Thus, relations (9.33) are well defined for all $(\mathbf{w}, \psi, \omega, \xi) \in \mathcal{H}_0^{1-s,r'}(\Omega)^3 \times \mathbb{H}^{-s,r'}(\Omega)^3$. Equalities (9.33) along with equations (9.28), (9.29) are called the very weak formulation of problem (9.26). Note that the notion of *very weak solutions* to incompressible Stokes equations was introduced in [13], see also [12] for generalizations. The natural question is the uniqueness of solutions to such weak formulation. The following theorem, which is the second main result of this section, guarantees the uniqueness of very weak solutions for sufficiently small τ.

THEOREM 9.4. *Let* s, r, *and parameters* λ, k, σ_λ, *and positive number* τ *meet all requirements of Theorem* 9.1 *and the solutions* (Θ_i, ζ_i, m_i), $i = 1, 2$, *to problem* (9.5) *belong to* $\mathcal{B}_\tau \times X^{s,r} \times \mathbb{R}$. *Furthermore, assume that for any* $(\mathbf{H}, G, F, M) \in C^\infty(\Omega)^6$ *and for* $(\varsigma, \upsilon, \mathbf{h}, g)$ *satisfying* (9.28), (9.29), *the elements* $(\mathbf{w}, \omega, \psi, \xi) \in \mathcal{H}_0^{1-s,r'}(\Omega)^3 \times \mathbb{H}^{-s,r'}(\Omega)^3$ *and the constant* n *satisfy identity* (9.33). *Then* $\mathbf{w} = 0$, $\psi = \xi = n = 0$.

PROOF. The proof is based upon two auxiliary lemmas, the first lemma establishes the bounds for coefficients of problem (9.26). □

LEMMA 9.5. *Under the assumptions of Theorem* 9.4, *all the coefficients of identity* (9.33) *satisfy the inequalities* $\|b_{ij}\|_{X^{s,r}} \le c$, *furthermore*

$$\begin{aligned} &\|b_{12}\|_{X^{s,r}} + \|b_{22}\|_{X^{s,r}} + \|b_{11}\|_{X^{s,r}} \le c\tau, \\ &\|b_{31}\|_{X^{s,r}} + \|b_{32}\|_{X^{s,r}} \le c\tau. \end{aligned} \tag{9.35}$$

PROOF. Since $X^{s,r}$ is a Banach algebra, estates (9.35) follows from formulae (9.27). □

In order to formulate the second auxiliary result we introduce the following denotations.

$$\begin{aligned} &\mathfrak{I}_1 = \langle\psi, b_{11}\varsigma\rangle + \langle\omega, b_{12}\varsigma\rangle, \quad \mathfrak{I}_2 = \langle\psi, b_{21}g\rangle + \langle\omega, b_{22}g\rangle, \\ &\mathfrak{I}_3 = \varkappa(\langle\psi, b_{31}\rangle + \langle\omega, b_{32}\rangle + \langle\xi, b_{34}\rangle), \ \mathfrak{I}_4 = \langle\psi, \mathbf{u}_1 \cdot \nabla\mathbf{u}_1 \cdot \mathbf{h}\rangle, \\ &\mathfrak{I}_5 = \int_\Omega \varrho_2 \mathbf{w} \cdot (\nabla\mathbf{u}_1 \mathbf{h} - \nabla h^* \mathbf{u}_2)\, dx, \\ &\mathfrak{G} = \|\mathbf{w}\|_{\mathcal{H}_0^{1-s,r'}(\Omega)} + \|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} + \|\omega\|_{\mathbb{H}^{-s,r'}(\Omega)} + \|\xi\|_{\mathbb{H}^{-s,r'}(\Omega)}, \\ &\mathfrak{Q} = \|\mathbf{H}\|_{\mathcal{H}^{s-1,r}(\Omega)} + \|G\|_{H^{s,r}(\Omega)} + \|F\|_{H^{s,r}(\Omega)} + \|M\|_{H^{s,r}(\Omega)}. \end{aligned}$$

LEMMA 9.6. *Under the assumptions of Theorem 9.4, there is a constant c, depending only on $\mathbf{U}$, Ω, s, r, and σ, such that*

$$\mathfrak{I}_1 \le c\tau\mathfrak{Q}\mathfrak{G} \tag{9.36}$$

$$\mathfrak{I}_2 \le c\mathfrak{Q}(\tau\mathfrak{G} + \|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)}) \tag{9.37}$$

$$\mathfrak{I}_3 \le c\tau\,\mathfrak{G}, \quad \mathfrak{I}_4 + \mathfrak{I}_5 \le c\mathfrak{Q}\,\mathfrak{G}. \tag{9.38}$$

PROOF. We have

$$\begin{aligned}\langle\psi, b_{11}\varsigma\rangle + \langle\omega, b_{12}\varsigma\rangle + \langle\mathfrak{d}, b_{10}\varsigma\rangle \le\ & \|b_{11}\varsigma\|_{H^{s,r}(\Omega)}\|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} \\ & + \|b_{12}\varsigma\|_{H^{s,r}(\Omega)}\|\omega\|_{\mathbb{H}^{-s,r'}(\Omega)}.\end{aligned}$$

Recall that for $rs > 3$, $H^{s,r}(\Omega)$ is a Banach algebra. From this, estimate (9.30), and inequalities (9.35) we obtain

$$\begin{aligned}&\|b_{11}\varsigma\|_{H^{s,r}(\Omega)}\|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} + \|b_{12}\varsigma\|_{H^{s,r}(\Omega)}\|\omega\|_{\mathbb{H}^{-s,r'}(\Omega)} \\ &\quad \le c\|\varsigma\|_{H^{s,r}(\Omega)}(\|b_{11}\|_{H^{s,r}(\Omega)}\|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} + \|b_{12}\|_{H^{s,r}(\Omega)}\|\omega\|_{\mathbb{H}^{-s,r'}(\Omega)} \\ &\quad \le c\tau\|F\|_{H^{s,r}(\Omega)}(\|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} + \|\omega\|_{\mathbb{H}^{-s,r'}(\Omega)}),\end{aligned}$$

which gives (9.36). Repeating these arguments and using inequality (9.30) we obtain the estimates for $\mathfrak{I}_2$ and $\mathfrak{I}_3$. Next, we have

$$\|\mathbf{u}_1 \cdot \nabla\mathbf{u}_1 \cdot \mathbf{h}\|_{H^{s,r}(\Omega)} \le c\|\mathbf{u}_1\|_{H^{s,r}(\Omega)}\|\mathbf{u}_1\|_{H^{1+s,r}(\Omega)}\|\mathbf{h}\|_{H^{1+s,r}(\Omega)} \le c\|\mathbf{H}\|_{H^{1+s,r}(\Omega)}$$

which gives the estimate for $\mathfrak{I}_4$. Since the embeddings $H^{s,r}(\Omega) \hookrightarrow C(\Omega)$, $H^{1+s,r}(\Omega) \hookrightarrow C^1(\Omega)$ are bounded, we have

$$\varrho_2|\nabla\mathbf{u}_1||\mathbf{h}| + \varrho_2|\mathbf{u}_2||\nabla\mathbf{h}| \le c\|\mathbf{h}\|_{H^{1+s,r}(\Omega)},$$

which leads to the inequality

$$\mathfrak{I}_5 \le c\|\mathbf{h}\|_{H^{1+s,r}(\Omega)}\|\mathbf{w}\|_{L^1(\Omega)} \le c(\|\mathbf{H}\|_{\mathcal{H}^{s-1,r}(\Omega)} + \|G\|_{H^{s,r}(\Omega)})\|\mathbf{w}\|_{\mathcal{H}^{1-s,r'}(\Omega)},$$

and the proof of Lemma 9.6 is completed. □

Let us return to the proof of Theorem 9.4. It follows from the duality principle that the theorem is proved provided we show that, under the assumptions of Theorem 9.1, the following inequality holds

$$\begin{aligned}&\sup_{\mathfrak{Q}(\mathbf{H},G,F,M)=1} (\langle\mathbf{w},\mathbf{H}\rangle + \langle\omega, G\rangle + \langle\psi, F\rangle + \langle\xi, M\rangle) + |n| \\ &\quad \le c\tau\,(\,\mathfrak{G}(\mathbf{w},\omega,\psi,\xi) + |n|)+,\end{aligned} \tag{9.39}$$

where the constant c depends only on Ω, $\mathbf{U}$ and r, s, σ_λ. Therefore, our task is to estimate step by step all terms on the left-hand side of (9.39). We begin with an estimate for the term $\langle\psi, F\rangle$. To this end, take $\mathbf{H} = \mathbf{h} = 0$, $G = g = 0$, $M = \upsilon = 0$, and rewrite identity (9.33) in the form

$$\langle\psi, F\rangle = \mathfrak{B}(\mathbf{w}, \varphi_2, \varsigma) + \mathfrak{I}_1 + \mathfrak{I}_3 + n\langle 1, b_{13}\varsigma\rangle - n.$$

By virtue of Lemma 9.3 and estimate (9.31) we have

$$\mathfrak{B}(\mathbf{w}, \varphi_2, \varsigma) \le c\tau\|\mathbf{w}\|_{\mathcal{H}_0^{1-s,r'}(\Omega)}\|\varsigma\|_{H^{s,r}(\Omega)} \le c\tau\|\mathbf{w}\|_{H^{1-s,r'}(\Omega)}\|F\|_{H^{s,r}(\Omega)}. \tag{9.40}$$

On the other hand, Lemma 9.5 and inequality (9.30) yield $|\langle 1, b_{13}\varsigma\rangle| \le c\|F\|_{H^{s,r}(\Omega)}$. From this and (9.36), (9.38) we finally obtain

$$\langle\psi, F\rangle \le |n| + c\|F\|_{H^{s,r}(\Omega)}\,(\,\tau\mathfrak{G} + |n|\,). \tag{9.41}$$

Moreover, by virtue of the duality principle

$$\|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} = \sup_{\|F\|_{H^{s,r}(\Omega)}=1} \big|\langle\psi, F\rangle\big|,$$

we have the following estimate for ψ

$$\|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} \le c\tau\mathfrak{G} + c|n|. \tag{9.42}$$

Let us estimate $\mathbf{w}$ and ω. Substituting $F = \varsigma = 0$ and $M = \upsilon = 0$ into (9.33) we obtain

$$\langle\mathbf{w}, \mathbf{H}\rangle + \langle\omega, G\rangle = \mathfrak{I}_2 + \mathfrak{I}_3 + k\mathfrak{I}_4 + k\mathfrak{I}_5 + n\langle 1, b_{23}g\rangle - n.$$

Next we have

$$|n\langle 1, b_{23}g\rangle| \le c(\|\mathbf{H}\|_{\mathcal{H}^{s-1,r}(\Omega)} + \|G\|_{H^{s,r}(\Omega)})|n|.$$

This inequality together with estimates (9.37)–(9.38) and inequality $k \le \tau^2$ imply

$$\begin{aligned}&\langle\mathbf{w}, \mathbf{H}\rangle + \langle\omega, G\rangle \le |n| + c\tau\mathfrak{Q}\,\mathfrak{G} +\\ &c\mathfrak{Q}(\|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} + |n|).\end{aligned}$$

Combining this result with (9.42) we obtain

$$\langle\mathbf{w}, \mathbf{H}\rangle + \langle\omega, G\rangle \le |n| + c\tau\mathfrak{Q}\,\mathfrak{G} + c\mathfrak{Q}\,|n|, \tag{9.43}$$

where $\mathfrak{Q} = \mathfrak{Q}(\mathbf{H}, G, 0, 0)$. For $G = 0$ and by the duality principle

$$\|\mathbf{w}\|_{\mathcal{H}_0^{1-s,r'}(\Omega)} = \sup_{\|\mathbf{H}\|_{\mathcal{H}^{s-1,r}(\Omega)}=1} \langle\mathbf{H}, \mathbf{w}\rangle,$$

we conclude from this that

$$\|\mathbf{w}\|_{\mathcal{H}_0^{1-s,r'}(\Omega)} \le c|n| + c\tau\mathfrak{G}. \tag{9.44}$$

Next, substituting $\mathbf{H} = \mathbf{h} = 0$, $G = g = F = \varsigma = 0$ into identity (9.33), we arrive at

$$\langle \xi, M \rangle = \mathfrak{B}(\mathbf{w}, \zeta_2, \upsilon) + \mathfrak{I}_3 - n.$$

Lemma 9.3 and (9.30) give the estimate for the first term

$$|\mathfrak{B}(\mathbf{w}, \zeta_2, \upsilon)| \le c\|\mathbf{w}\|_{\mathcal{H}_0^{1-s,r'}(\Omega)} \|\upsilon\|_{H^{s,r}(\Omega)} \le c\|\mathbf{w}\|_{\mathcal{H}_0^{1-s,r'}(\Omega)} \|M\|_{H^{s,r}(\Omega)}.$$

From this and estimates (9.38), (9.30), we obtain

$$\langle \xi, M \rangle \le c\tau \mathfrak{Q}\,\mathfrak{G} + c\mathfrak{Q}\,\|\mathbf{w}\|_{\mathcal{H}_0^{1-s,r'}(\Omega)} + |n|.$$

Combining this result with inequality (9.44) we arrive at

$$\langle \xi, M \rangle \le c\mathfrak{Q}(\tau \mathfrak{G} + |n|) + c|n|. \tag{9.45}$$

Finally, choosing all test functions in (9.33) equal to 0 we obtain $n = \mathfrak{I}_3$ which together with (9.38) yields

$$|n| \le c\tau \mathfrak{Q}\,\mathfrak{G}. \tag{9.46}$$

From (9.41), (9.43), (9.45), combined with (9.46), it follows (9.39) and the proof of Theorem 9.4 is completed. □

9.3. *Proof of Lemma 9.3*

Since $\partial\Omega$ belongs to the class C^1, functions φ, ς have the extensions $\overline{\varphi}, \overline{\varsigma} \in H^{s,r}(\Omega) \cap H^{1,2}(\Omega)$, such that $\overline{\varphi}, \overline{\varsigma}$ are compactly supported in $\mathbb{R}^d$ and

$$\|\overline{\varphi}\|_{H^{s,r}(\mathbb{R}^d)} \le c\|\varphi\|_{H^{s,r}(\Omega)}, \quad \|\overline{\varsigma}\|_{H^{s,r}(\mathbb{R}^d)} \le c\|\varsigma\|_{H^{s,r}(\Omega)}.$$

By virtue of Definition 4.2 and inequality (4.5), function $\mathbf{w}$ has the extension by 0 outside Ω, denoted by $\overline{\mathbf{w}}$, such that

$$\|\overline{\mathbf{w}}\|_{H^{1-s,r'}(\mathbb{R}^d)} \le c\|\mathbf{w}\|_{\mathcal{H}_0^{1-s,r'}(\Omega)}.$$

Obviously we have

$$\mathfrak{B}(\mathbf{w}, \varphi, \varsigma) = -\int_{\mathbb{R}^d} \overline{\mathbf{w}} \cdot \nabla\overline{\varphi}\,\overline{\varsigma}\,dx.$$

The following multiplicative inequality is due to Mazja [38]. For all $s > 0$, $r > 1$ and $rs < d$,

$$\|uv\|_{H^{s,r}(\mathbb{R}^d)} \le c(r, s, d)(\|v\|_{H^{s,s/d}(\mathbb{R}^d)} + \|v\|_{L^\infty(\mathbb{R}^d)})\|u\|_{H^{s,r}(\mathbb{R}^d)}. \tag{9.47}$$

By virtue of (9.47), we have

$$\|\overline{\mathbf{w}}\,\overline{\varsigma}\|_{H^{1-s,r'}(\mathbb{R}^d)} \le c\|\overline{\mathbf{w}}\|_{H^{1-s,r'}(\mathbb{R}^d)}(\|\overline{\varsigma}\|_{H^{1-s,d/(1-s)}(\mathbb{R}^d)} + \|\overline{\varsigma}\|_{L^\infty(\mathbb{R}^d)}).$$

On the other hand, since $r^{-1} - (s - (1 - s))/d \le (1 - s)/d$ for $sr > d$, embedding inequality (4.7) yields

$$\|\varsigma\|_{H^{1-s,d/(1-s)}(\mathbb{R}^d)} \le c\|\varsigma\|_{H^{s,r}(\mathbb{R}^d)}, \qquad \|\varsigma\|_{L^\infty(\mathbb{R}^d)} \le c\|\varsigma\|_{H^{s,r}(\mathbb{R}^d)}.$$

Thus we get

$$\|\overline{\mathbf{w}}\,\varsigma\|_{H^{1-s,r'}(\mathbb{R}^d)} \le c\|\overline{\mathbf{w}}\|_{H^{1-s,r'}(\mathbb{R}^d)}\|\varsigma\|_{H^{s,r}(\mathbb{R}^d)}.$$

It is well known that elements of the fractional Sobolev spaces can be represented via Liouville potentials

$$\overline{\mathbf{w}\varsigma} = (1-\Delta)^{-(1-s)/2} w, \qquad \overline{\varphi} = (1-\Delta)^{-s/2}\phi,$$

with

$$\|w\|_{L_{r'}(\mathbb{R}^d)} \le c\|\overline{\mathbf{w}\varsigma}\|_{H^{1-s,r'}(\mathbb{R}^d)}, \qquad \|\phi\|_{L_r(\mathbb{R}^d)} \le c\|\overline{\varphi}\|_{H^{1-s,r'}(\mathbb{R}^d)}.$$

Thus we get

$$\begin{aligned}\mathfrak{B}(\mathbf{w}, \varphi, \varsigma) &= -\int_{\mathbb{R}^d} (1-\Delta)^{-(1-s)/2} w \cdot \nabla(1-\Delta)^{-s/2}\phi\, dx \\ &= -\int_{\mathbb{R}^d} w \cdot \nabla(1-\Delta)^{-1/2}\phi\, dx.\end{aligned}$$

Since the Riesz operator $(1-\Delta)^{-1/2}\nabla$ is bounded in $L^r(\mathbb{R}^d)$, we conclude from this and the Holder inequality that

$$|\mathfrak{B}(\mathbf{w}, \varphi, \varsigma)| \le c\|w\|_{L^{r'}(\mathbb{R}^d)}\|\phi\|_{L^r(\mathbb{R}^d)} \le c\|\mathbf{w}\|_{H^{1-s,r'}(\Omega)}\|\varphi\|_{H^{s,r}(\Omega)}\|\varsigma\|_{H^{s,r}(\Omega)},$$

and the lemma follows. $\square$

10. Proof of Theorem 5.5

Our strategy is the following. First we show that in the vicinity of each point $P \in \Sigma_{\text{in}} \cup \Gamma$ there exist normal coordinates (y_1, y_2, y_3) such that $\mathbf{u}\nabla_x = \mathbf{e}_1\nabla_y$. Hence problem of existence of solutions to the transport equation in the neighborhood of $\Sigma_{\text{in}} \cap \Gamma$ is reduced to boundary problem for the model equation $\partial_{y_1}\varphi + \sigma\varphi = f$ in a parabolic domain. Next we prove that the boundary value problem for the model equations has a unique solution in fractional Sobolev space, which leads to the existence and uniqueness of solutions in the neighborhood of the inlet set. Using the existence of local solution we reduce problem (5.11) to the problem for modified equation, which does not require the boundary data. Application of well-known results on solvability of elliptic–hyperbolic equations in the case $\Gamma = \emptyset$ finally gives the existence and uniqueness of solutions to problems (5.11) and (5.18).

LEMMA 10.1. *Assume that the C^2-manifold $\Sigma = \partial B$ and the vector field $\mathbf{U} \in C^2(\Sigma)^3$ satisfy conditions* (**H1**)–(**H3**). *Let $\mathbf{u} \in C^1(\mathbb{R}^3)^3$ be a compactly supported vector field such that*

$$\mathbf{u} = \mathbf{U} \quad \text{on } \Sigma, \qquad \mathbf{u} = 0 \quad \text{on } S,$$

and denote $M = \|\mathbf{u}\|_{C^1(\mathbb{R}^3)}$. Then there is $a > 0$, depending only on M and Σ, with the properties:

(**P1**) *For any point $P \in \Gamma$ there exists a mapping $y \to \mathbf{x}(y)$ which takes diffeomorphically the cube $Q_a = [-a, a]^3$ onto a neighborhood $\mathcal{O}_P$ of P and satisfies the equations*

$$\partial_{y_1}\mathbf{x}(y) = \mathbf{u}(\mathbf{x}(y)) \quad \text{in } Q_a, \quad \mathbf{x}(0, y_2, 0) \in \Gamma \cap \mathcal{O}_P \quad \text{for } |y_2| \le a, \tag{10.1}$$

and the inequalities

$$\|\mathbf{x}\|_{C^1(Q_a)} + \|\mathbf{x}^{-1}\|_{C^1(\mathcal{O}_P)} \le C_M, \qquad |\mathbf{x}(y)| \le C_M|y|, \tag{10.2}$$

where $C_M = 3(1 + M^{-1})(M^2 + C_\Gamma^2 + 2)^{1/2}$ and C_Γ is the constant in condition (**H2**).
(**P2**) *There is a C^1 function $\Phi(y_1, y_2)$ defined in the square $[-a, a]^2$ such that $\Phi(0, y_2) = 0$, and*

$$\mathbf{x}(\{y_3 = \Phi\}) = \Sigma \cap \mathcal{O}_P, \qquad \mathbf{x}(\{y_3 > \Phi\}) = \Sigma \cap \mathcal{O}_P. \tag{10.3}$$

Moreover Φ is strictly decreasing in y_1 for $y_1 < 0$, is strictly increasing in y_1 for $y_1 > 0$, and satisfies the inequalities

$$C^- y_1^2 \le \Phi(y_1, y_2) \le C^+ y_1^2, \tag{10.4}$$

where the constants $C^- = |\mathbf{U}(P)|N^-/12$ and $C^+ = 12|\mathbf{U}(P)|N^+$ depend only on $\mathbf{U}$ and Σ, where $N^\pm$ are defined in Condition (**H2**).
(**P3**) *Introduce the sets*

$$\Sigma_{\text{in}}^y = \{(y_2, y_3) : |y_2| \le a, 0 < y_3 < \Phi(-a, y_2)\},$$
$$\Sigma_{\text{out}}^y = \{(y_2, y_3) : |y_2| \le a, 0 < y_3 < \Phi(a, y_2)\}.$$

For every $(y_2, y_3) \in \Sigma_{\text{in}}^y$ (resp. $(y_2, y_3) \in \Sigma_{\text{out}}^y$), the equation $y_3 = \Phi(y_1, y_2)$ has a unique negative (resp. positive) solution $y_1 = a^-(y_2, y_3)$, (resp. $y_1 = a^+(y_2, y_3)$) such that

$$|\partial_{y_j} a^\pm(y_2, y_3)| \le C/\sqrt{y_3},$$
$$|a^\pm(y_2, y_3) - a^\pm(y_2', y_3')| \le C(|y_2 - y_2'| + |y_3 - y_3'|)^{1/2}. \tag{10.5}$$

(**P4**) *Denote by $G_a \subset Q_a$ the domain*

$$G_a = \{y \in Q_a : \Phi(y_1, y_2) < y_3 < \Phi(-a, y_2)\}, \tag{10.6}$$

and by $B_P(\rho)$ the ball $|x - P| \le \rho$. Then we have the inclusions

$$B_P(\rho_c) \subset \mathbf{x}(G_a) \subset \mathcal{O}_P \subset B_P(R_c), \tag{10.7}$$

where the constants $\rho_c = a^2 C_M^{-1} C^-$, $R_c = aC_M$.

The next lemma constitutes the existence of the normal coordinates in the vicinity of points of the inlet Σ_{in}.

LEMMA 10.2. *Let vector fields $\mathbf{u}$ and $\mathbf{U}$ meet all requirements of Lemma* 10.1 *and $U_n = -\mathbf{U}(P) \cdot \mathbf{n} > N > 0$. Then there is $b > 0$, depending only on N, Σ and $M = \|\mathbf{u}\|_{C^1(\Omega)}$, with the following properties. There exists a mapping $y \to \mathbf{x}(y)$,* *which takes diffeomorphically the cube* $Q_b = [-b,b]^3$ *onto a neighborhood* $\mathcal{O}_P$ *of* P *and satisfies the equations*

$$\partial_{y_3}\mathbf{x}(y) = \mathbf{u}(\mathbf{x}(y)) \quad \text{in } Q_b, \quad \mathbf{x}(y_1,y_2,0) \in \Sigma \cap \mathcal{O}_P \quad \textit{for } |y_2| \le a, \tag{10.8}$$

and the inequalities

$$\|\mathbf{x}\|_{C^1(Q_b)} + \|\mathbf{x}^{-1}\|_{C^1(\mathcal{O}_P)} \le C_{M,N} \qquad |\mathbf{x}(y)| \le C_M|y|, \tag{10.9}$$

where $C_{M,N} = 3(1+N^{-1})(M^2+2)^{1/2}$. *The inclusions*

$$B_P(\rho_i) \cap \Omega \subset \mathbf{x}(Q_b \cap \{y_3 > 0\}) \subset B_P(R_i) \cap \Omega, \tag{10.10}$$

hold true for $\rho_i = C_{M,N}^{-1}b$ *and* $R_i = C_{M,N}b$.

Model equation. Assume that the function $\Phi : [-a,a]^2 \mapsto \mathbb{R}$ and the constant $a > 0$ meet all requirements of Lemma 10.1. Recall that for each y satisfying the conditions $\Phi(y_1,y_2) < y_3 < \Phi(-a,y_2)$ (resp. $\Phi(y_1,y_2) < y_3 < \Phi(a,y_2)$), equation $y_3 = \Phi(y_1,y_2)$ has the solutions $y_1 = a^-(y_2,y_3)$ (resp. $y_1 = a^+(y_3,y_3)$). The functions $a^\pm$ vanish for $y_3 = 0$ and satisfy the inequalities

$$\begin{aligned} & -a < a^-(y_2,y_3) \le 0 \le a^+(y_2,y_3) \le a, \\ & |a^\pm(z_2,y_3) - a^\pm(y_2,y_3)| \le c|z_2 - y_2|, \\ & |a^\pm(y_2,z_3) - a^\pm(y_2,y_3)| \le c|\sqrt{z_3} - \sqrt{y_3}|, \end{aligned} \tag{10.11}$$

where c depends only on Σ, and $\mathbf{U}$. We assume that the functions $a^\pm$ are extended on the rectangle $[-a,a] \times [0,a]$ by the equalities $a^\pm(y_2,y_3) = \pm a$ for $y^3 > \Phi(\pm a, y_2)$. It is clear that the extended functions satisfy (10.11) and

$$Q_a^\phi := \{y_3 > \Phi(y_1,y_2)\} = \{y : a^-(y_2,y_3) \le y_1 \le a^+(y_2,y_3)\}.$$

Let us consider the boundary value problem

$$\begin{aligned} & \partial_{y_1}\varphi(y) + \sigma\varphi(y) = f(y) \quad \text{in } Q_a^\phi, \\ & \varphi(y) = 0 \quad \text{for } y_1 = a^-(y_2,y_3). \end{aligned} \tag{10.12}$$

LEMMA 10.3. *For any* $f \in L^r(Q_a)$, $1 < r \le \infty$, *problem* (10.12) *has a unique generalized solution such that*

$$\|\varphi\|_{L^r(Q_a^\phi)} \le \sigma^{-1}\|f\|_{L^r(Q_a)}, \quad 1 \le r \le \infty. \tag{10.13}$$

Moreover, if $\sigma > 1$, *exponents* r, s, α *satisfy the inequalities* (5.16) *and* f *belongs to the space* $H^{s,r}(Q_a^\phi) \cap L^\infty(Q_a^\phi)$, *then a solution to problem* (10.12) *admits the estimate*

$$\|\varphi\|_{H^{s,r}(Q_a^\phi)} \le c(a,r,s,\alpha)(\sigma^{-1}\|f\|_{H^{s,r}(Q_a^\phi)} + \sigma^{-1+\alpha}\|f\|_{L^\infty(Q_a^\phi)}). \tag{10.14}$$

Let us consider the following boundary value problem

$$\begin{aligned} & \partial_{y_3}\varphi(y) + \sigma\varphi(y) = f(y) \quad \text{in } Q_a^+ = [-a,a]^2 \times [0,a], \\ & \varphi(y) = 0 \quad \text{for } y_3 = 0. \end{aligned} \tag{10.15}$$

LEMMA 10.4. *Let $\sigma > 1$, and exponents r, s, α satisfy* (5.16). *Then problem* (10.15) *has a unique solution satisfying the inequality*

$$\|\varphi\|_{H^{s,r}(Q_a^+)} \le c(r,s,\alpha,a)(\sigma^{-1}\|f\|_{H^{s,r}(Q_a^+)} + \sigma^{-1+\alpha}\|f\|_{L^\infty(Q_a^+)}). \tag{10.16}$$

PROOF. The proof of Lemma 10.3 can be used also in this case. □

Local existence results. It follows from the conditions of Theorem 5.5 that the vector field $\mathbf{u}$ and the manifold Σ satisfy all assumptions of Lemma 10.1. Therefore, there exist positive numbers a, ρ_c and R_c, depending only on Σ and $\|\mathbf{u}\|_{C^1(\Omega)}$, such that for all $P \in \Gamma$, the canonical diffeomorphism $\mathbf{x} : Q_a \mapsto \mathcal{O}_P$ is well defined and meet all requirements of Lemma 10.1. Fix an arbitrary point $P \in \Gamma$ and consider the boundary value problem

$$\mathbf{u}\cdot\nabla\varphi + \sigma\varphi = f \quad \text{in } \mathcal{O}_P, \qquad \varphi = 0 \quad \text{on } \Sigma_{\text{in}} \cap \mathcal{O}_P. \tag{10.17}$$

LEMMA 10.5. *Suppose that the exponents s, r, α, satisfy condition* (5.16) *and $\|u\|_{C^1(\Omega)} \le M$. Then for any $f \in C^1(\Omega)$ and $\sigma > 1$, problem* (10.17) *has a unique solution satisfying the inequalities*

$$\begin{aligned} |\varphi|_{s,r,B_P(\rho_c)} &\le c(\sigma^{-1+\alpha}\|f\|_{C(B_P(R_c))} + \sigma^{-1}|f|_{s,r,B_P(R_c)}), \ \|\varphi\|_{C(B_P(\rho_c))} \\ &\le \sigma^{-1}\|f\|_{C(B_P(R_c))}, \end{aligned} \tag{10.18}$$

where the constant c depends only on Σ, M, s, r, α, and ρ_c is determined by Lemma 10.1.

PROOF. We transform equation (10.18) using the normal coordinates (y_1, y_2, y_3) given by Lemma 10.1. Set $\overline{\varphi}(y) = \varphi(\mathbf{x}(y))$ and $\overline{f}(y) = f(\mathbf{x}(y))$. Next note that equation (10.1) implies the identity $\mathbf{u}\nabla_x\varphi = \partial_{y_1}\overline{\varphi}(y)$. Therefore the function $\overline{\varphi}(y)$ satisfies the following equation and boundary conditions

$$\partial_{y_1}\overline{\varphi} + \sigma\overline{\varphi} = \overline{f} \quad \text{in } Q_a \cap \{y_3 > \Phi\}, \qquad \overline{\varphi} = 0 \quad \text{for } y_3 = \Phi(y_1, y_2),\ y_1 < 0. \tag{10.19}$$

It follows from Lemma 10.3 that for all $\sigma > 1$, problem (10.19) has a unique solution $\overline{\varphi} \in H^{s,r}(G_a)$ satisfying the inequality

$$\begin{aligned} &|\overline{\varphi}|_{s,r,G_a} \le c(\sigma^{-1+\alpha}\|\overline{f}\|_{C(Q_a)} + \sigma^{-1}|\overline{f}|_{s,r,Q_a}), \\ &\|\overline{\varphi}\|_{C(G_a)} \le \sigma^{-1}\|f\|_{C(Q_a)}, \end{aligned} \tag{10.20}$$

where the domain G_a is defined by (10.6). It remains to note that, by estimate (10.2), the mappings $\mathbf{x}^{\pm 1}$ are uniformly Lipschitz, which along with inclusions (10.7) implies the estimates

$$|\varphi|_{s,r,B_P(\rho_c)} \le c|\overline{\varphi}|_{s,r,G_a}, \qquad |\overline{f}|_{s,r,Q_a} \le c|f|_{s,r,B_P(R_c)}.$$

Combining these results with (10.20) we finally obtain (10.18) and the lemma follows. □

In order to formulate the similar result for interior points of inlet we introduce the set

$$\Sigma'_{\rm in} = \{x \in \Sigma_{\rm in} : \quad \mathrm{dist}\,(x, \Gamma) \geq \rho_c/3), \tag{10.21}$$

where the constant ρ_c is given by Lemma 10.1. It is clear that

$$\inf_{P\in\Sigma'_{\rm in}} \mathbf{U}(P)\cdot\mathbf{n}(P) \geq N > 0,$$

where the constant N depends only on M, $\mathbf{U}$, and Σ. It follows from Lemma 10.2 that there are positive numbers b, ρ_i, and R_i such that for each $P \in \Sigma'_{\rm in}$, the canonical diffeomorphism $\mathbf{x} : Q_b \mapsto \mathcal{O}_P$ is well defined and satisfies the hypotheses of Lemma 10.2. The following lemma gives the local existence and uniqueness of solutions to the boundary value problem

$$\mathbf{u}\cdot\nabla\varphi + \sigma\varphi = f \quad \text{in } \mathcal{O}_P, \qquad \varphi = 0 \quad \text{on } \Sigma_{\rm in}\cap\mathcal{O}_P. \tag{10.22}$$

LEMMA 10.6. *Suppose that the exponents s, r, α satisfy condition* (5.16). *Then for any $f \in C^1(\Omega)$, $\sigma > 1$ and $P \in \Sigma'_{\rm in}$, problem* (10.17) *has a unique solution satisfying the inequalities*

$$\begin{aligned} |\varphi|_{s,r,B_P(\rho_i)} \leq \quad & c(\sigma^{-1+\alpha}\|f\|_{C(B_P(R_i))} + \sigma^{-1}|f|_{s,r,B_P(R_i)}), \\ & \|\varphi\|_{C(B_P(\rho_i))} \leq \sigma^{-1}\|f\|_{C(B_P(R_i))}, \end{aligned} \tag{10.23}$$

where c depends on Σ, M, $\mathbf{U}$ and exponents s, r, α.

PROOF. Using the normal coordinates given by Lemma 10.2 we rewrite equation (10.22) in the form.

$$\partial_{y_3}\overline{\varphi} + \sigma\overline{\varphi} = \overline{f} \quad \text{in } Q_b, \qquad \overline{\varphi} = 0 \quad \text{for } y_3 = 0.$$

Applying Lemma 10.3 and arguing as in the proof of Lemma 10.5 we obtain (10.23). □

Existence of solutions near inlet. The next step is based on the well-known geometric lemma (see Ch. 3 in [27]).

LEMMA 10.7. *Suppose that a given set $A \subset \mathbb{R}^d$ is covered by balls such that each point $x \in A$ is the center of a certain ball $B_x(r(x))$ of radius $r(x)$. If $\sup r(x) < \infty$, then from the system of the balls $\{B_x(r(x))\}$ it is possible to select a countable system $B_{x_k}(r(x_k))$ covering the entire set A and having multiplicity not greater than a certain number $n(d)$ depending only on the dimension d.*

The following lemma gives the dependence of the multiplicity of radii of the covering balls.

LEMMA 10.8. *Assume that a collection of balls $B_{x_k}(r) \subset \mathbb{R}^3$ of constant radius r has the multiplicity n_r. Then the multiplicity of the collections of the balls $B_{x_k}(R)$, $r < R$, is bounded by the constant $27(R/r)^3 n_r$.*

PROOF. Let n_R be a multiplicity of the system $\{B_{x_k}(R)\}$. This means that at least n_R balls, say $B_{x_1}(R), \dots, B_{x_{n_R}}(R)$, have the common point P. In particular, we have $B_{x_i}(r) \subset B_P(3R)$ for all $i \le n_R$. Introduce the counting function $\iota(x)$ for the collection of balls $B_{x_i}(r)$, defined by

$$\iota(x) = \operatorname{card}\{i : x \in B_{x_i}(r), 1 \le i \le n_r\}.$$

Note that $\iota(x) \le n_r$. We have

$$\begin{aligned}\frac{4\pi}{3} n_R r^3 &= \sum_{i=1}^{n_R} \operatorname{meas} B_{x_i}(r)\\ &= \int_{\bigcup_i B_{x_i}(r)} \iota(x)\,dx \le n_r \int_{\bigcup_i B_{x_i}(r)} dx \le \frac{4\pi}{3}(3R)^3 n_r,\end{aligned}$$

and the lemma follows. □

We are now in a position to prove the local existence and uniqueness of solution for the first boundary value problem for the transport equation in the neighborhood of the inlet. Let Ω_t be the t-neighborhood of the set Σ_{in},

$$\Omega_t = \{x \in \Omega : \operatorname{dist}(x, \Sigma_{\text{in}}) < t\}.$$

LEMMA 10.9. *Let $t = \min\{\rho_c/2, \rho_i/2\}$ and $T = \max\{R_c, R_i\}$, where the constants ρ_α, R_α are defined by Lemmas* 10.1 *and* 10.2. *Then there exists a constant C depending only on M, Σ and σ, such that for any $f \in C^1(\Omega)$, the boundary value problem*

$$\mathbf{u} \cdot \nabla\varphi + \sigma\varphi = f \quad \text{in } \Omega_t, \qquad \varphi = 0 \quad \text{on } \Sigma_{\text{in}} \tag{10.24}$$

has a unique solution satisfying the inequalities

$$|\varphi|_{s,r,\Omega_t} \le C(\sigma^{-1+\alpha}\|f\|_{C(\Omega_T)} + \sigma^{-1}|f|_{s,r,\Omega_T}), \quad \|\varphi\|_{C(\Omega_t)} \le \sigma^{-1}\|f\|_{C(\Omega_T)}. \tag{10.25}$$

PROOF. It follows from Lemma 10.7 that there is a covering of the characteristic manifold Γ by the finite collection of balls $B_{P_i}(\rho_c/4)$, $1 \le i \le m$, $P_i \in \Gamma$, of the multiplicity n. The cardinality m of this collection does not exceed $4n(\rho_c)^{-1}L$, where L is the length of Γ. Obviously, the balls $B_{P_i}(\rho_c)$ cover the set

$$\mathcal{V}_\Gamma = \{x \in \Omega : \operatorname{dist}(x, \Gamma) < \rho_c/2\}.$$

By virtue of Lemma 10.5, in each of such balls the solution to problem (10.24) satisfies inequalities (10.18), which leads to the estimate

$$\begin{aligned}|\varphi|^r_{s,r,\mathcal{V}_\Gamma} &\le \sum_i |\varphi|^r_{s,r,B_{P_i}(\rho_c)}\\ &\le c\sigma^{-1+\alpha}\sum_i \|f\|^r_{C(B_{P_i}(R_c))} + c\sigma^{-1}\sum_i |f|^r_{s,r,B_{P_i}(R_c)},\end{aligned} \tag{10.26}$$

where c depends only on M, Σ and $\mathbf{U}$. By Lemma 10.8, the multiplicity of the system of balls $B_{P_i}(R_c)$ is bounded from above by $12^3(R_c/\rho_c)^3$, which along with the inclusion $\cup_i B_{P_i}(R_c) \subset \Omega_T$ yields

$$\sum_{i=1}^{m} |f|^r_{s,r,B_{P_i}(R_c)} \le 12^3(R_c/\rho_c)^3 |f|^r_{s,r,\Omega_T}.$$

Obviously we have

$$\sum_i \|f\|^r_{C(B_{P_i}(R_c))} \le m\|f\|^r_{C(\Omega_T)} \le 4n(\rho_c)^{-1}L\|f\|^r_{C(\Omega_T)}.$$

Combining these results with (10.26) we obtain the estimates for solution to problem (10.24) in the neighborhood of the characteristic manifold Γ,

$$|\varphi|_{s,r,\mathcal{V}_\Gamma} \le c\sigma^{-1+\alpha}\|f\|_{C(\Omega_T)} + c\sigma^{-1}|f|_{s,r,\Omega_T}. \tag{10.27}$$

Our next task is to obtain the similar estimate in the neighborhood of the compact $\Sigma'_{\text{in}} \subset \Sigma_{\text{in}}$. To this end, we introduce the set

$$\mathcal{V}_{\text{in}} = \{x \in \Omega : \text{ dist }(x, \Sigma'_{\text{in}}) < \rho_i/2\},$$

where Σ'_{in} is given by (10.21). By virtue of Lemma 10.7, there exists the finite collection of balls $B_{P_k}(\rho_i/4)$, $1 \le k \le m$, $P_k \in \Sigma'_{\text{in}}$, of the multiplicity n which covers Σ'_{in}. Obviously $m \le 16n(\rho_i)^{-2}\text{meas } \Sigma_{\text{in}}$, and the balls $B_{P_k}(\rho_i)$ cover the set $\mathcal{V}_{\text{in}}$. From this and Lemma 10.6 we conclude that

$$\begin{aligned}|\varphi|^r_{s,r,\mathcal{V}_{\text{in}}} &\le \sum_k |\varphi|^r_{s,r,B_{P_k}(\rho_i)} \\ &\le c\sigma^{-1+\alpha}\sum_k \|f\|^r_{C(B_{P_k}(R_i))} + c\sigma^{-1}\sum_k |f|^r_{s,r,B_{P_k}(R_i)}.\end{aligned}$$

By virtue of Lemma 10.8, the multiplicity of the system of balls $B_{P_i}(R_i)$ is not greater than $12^3(R_i/\rho_i)^3$, which yields

$$\sum_i |f|^r_{s,r,B_{P_i}(R_i)} \le 12^3(R_i/\rho_i)^3|f|^r_{s,r,\Omega_T}.$$

Obviously we have

$$\sum_k \|f\|^r_{C(B_{P_k}(R_i))} \le m\|f\|^r_{C(\Omega_T)} \le 16n(\rho_k)^{-2}\text{meas } \Sigma_{\text{in}}\|f\|^r_{C(\Omega_T)}.$$

Thus we get

$$|\varphi|_{s,r,\mathcal{V}_{in}} \le c\sigma^{-1+\alpha}\|f\|_{C(\Omega_T)} + c\sigma^{-1}|f|_{s,r,\Omega_T}. \tag{10.28}$$

Since $\mathcal{V}_\Gamma$ and $\mathcal{V}_{\text{in}}$ cover Ω_t, this inequality along with inequalities (10.27) yields (10.25), and the lemma follows. $\square$

Partition of unity. Let us turn to the analysis of the general problem

$$\mathfrak{L}\varphi := \mathbf{u} \cdot \nabla\varphi + \sigma\varphi = f \quad \text{in } \Omega, \qquad \varphi = 0 \quad \text{on } \Sigma_{\text{in}}. \tag{10.29}$$

Recall that by virtue of Theorem 5.3, for any $f \in C^1(\Omega)$, problem (10.29) has a unique strong solution defined in neighborhood Ω_t of the inlet Σ_{in}. On the other hand, Theorem 5.1 guarantees the existence and uniqueness of the bounded weak solution to problem (10.29). The following lemma shows that both the solutions coincide in Ω_t.

LEMMA 10.10. *Under the assumptions of Theorem* 5.5 *each bounded generalized solution to problem* (10.29) *coincides in Ω_t with the local solution φ_t.*

PROOF. Let $\varphi \in L^\infty(\Omega)$ be a weak solution to problem (10.29). Recall that each point $P \in \Gamma$ has a canonical neighborhood $\mathcal{O}_P := \mathbf{x}(Q_a)$, where canonical diffeomorphism $\mathbf{x} : Q_a \mapsto \mathcal{O}_P$ is defined by Lemma 10.1. Choose an arbitrary function $\zeta \in C^1(\Omega)$ vanishing on Σ_{in} and outside of $\mathcal{O}_P$ and set

$$\overline{\varphi}(y) = \varphi(\mathbf{x}(y)), \quad \overline{f}(y) = f(\mathbf{x}(y)), \quad \overline{\zeta}(y) = \zeta(\mathbf{x}(y)), \quad y \in Q_a \cap \{y_3 > \Phi\}.$$

By definition of the weak solution to the transport equation we have

$$\int_{\mathcal{O}_P \cap \Omega} (\sigma\varphi\zeta - \varphi \operatorname{div}(\zeta\mathbf{u}) - f\zeta)\, dx = 0.$$

Direct calculations lead to the identity $\operatorname{div}_x(\zeta\mathbf{u}) = \det\mathfrak{F}^{-1} \operatorname{div}_y(\overline{\zeta}\det\mathfrak{F}\,\mathfrak{F}^{-1}\overline{\mathbf{u}})$, in which the notation $\mathfrak{F}$ stands for the Jacobi matrix $\mathfrak{F} = D_y\mathbf{x}(y)$. On the other hand, equation (10.1) implies the equality $\mathfrak{F}^{-1}\overline{\mathbf{u}} = \mathbf{e}_1$. From this we conclude that

$$\int_{Q_a \cap \{y_3 > \Phi\}} \left((\det\mathfrak{F}\,\overline{\zeta})(\sigma\overline{\varphi} - \overline{f}) - \overline{\varphi}\frac{\partial}{\partial y_1}(\det\mathfrak{F}\,\overline{\zeta}) \right) dy = 0.$$

Recall that, by Lemma 10.1, $\partial_{y_1}\mathfrak{F}$ is continuous and $\det\mathfrak{F}$ is strictly positive in the cube Q_a. Setting $\xi = \det\mathfrak{F}\zeta$ we conclude that the integral identity

$$\int_{Q_a \cap \{y_3 > \Phi\}} \left(\xi(\sigma\overline{\varphi} - \overline{f}) - \overline{\varphi}\frac{\partial\xi}{\partial y_1} \right) dy = 0$$

holds true for all functions $\xi \in C_0(Q_a)$ having continuous derivative $\partial_{y_1}\zeta \in C(Q_a)$ and vanishing for $y_3 = \Phi(y_1, y_2)$, $y_1 < 0$. Since $\overline{f}$ is continuously differentiable, $\overline{\varphi}$ belongs to the class $C^1_{\text{loc}}(Q_a) \cap \{y_3 > \Phi\}$, and satisfies equations (10.19). On the other hand, $\overline{\varphi_t}$ also satisfies (10.19). Obviously, all solutions to problem (10.19) coincides in the domain G_a and hence $\overline{\varphi}_t = \overline{\varphi}$ in this domain. Recalling that $B_P(\rho_c) \subset \mathbf{x}(G_a)$ we obtain that $\varphi_t = \varphi$ in the ball $B_P(\rho_c)$. The same arguments show that for any $P \in \Sigma'_{\text{in}}$, the function φ_t is equal to φ in the ball $B_P(\rho_i)$. It remains to note that the balls $B_P(\rho_c)$ and $B_P(\rho_i)$ cover Ω_t and the lemma follows. $\square$

Now we split the weak solution $\varphi \in L^\infty(\Omega)$ to problem (10.29) into two parts, namely the local solution φ_t and the remainder vanishing near the inlet. To this end fix a function $\Lambda \in C^\infty(\mathbb{R})$ such that

$$0 \le \Lambda' \le 3, \quad \Lambda(u) = 0 \quad \text{for } u \le 1 \quad \text{and} \quad \Lambda(u) = 1 \quad \text{for } u \ge 3/2, \tag{10.30}$$

and introduce the one-parametric family of smooth functions

$$\chi_t(x) = \frac{1}{t^3}\int_{\mathbb{R}^3} \Theta\left(\frac{2(x-y)}{t}\right)\Lambda\left(\frac{\text{dist }(y, \Sigma_{\text{in}})}{t}\right)dy \tag{10.31}$$

where $\Theta \in C^\infty(\mathbb{R}^3)$ is a standard mollifying kernel supported in the unit ball. It follows that

$$\begin{aligned} &\chi_t(x) = 0 \quad \text{for dist } (x, \Sigma_{\text{in}}) \le t/2, \quad \chi_t(x) = 1 \quad \text{for dist } (x, \Sigma_{\text{in}}) \ge 2t, \\ &|\partial^l \chi_t(x)| \le \varpi(l) t^{-l} \quad \text{for all } l \ge 0, \end{aligned} \tag{10.32}$$

where $\varpi(l)$ is a constant. Now fix a number $t = t(\Sigma, M)$ satisfying all assumptions of Lemma 10.9 and set

$$\varphi(x) = (1 - \chi_{t/2}(x))\varphi_t(x) + \phi(x). \tag{10.33}$$

By virtue of (10.32) and Lemma 10.10, the function $\phi \in L^\infty(\Omega)$ vanishes in $\Omega_{t/4}$ and satisfies in a weak sense the equations

$$\mathbf{u}\nabla\phi + \sigma\phi = \chi_{t/2} f + \varphi_t \mathbf{u}\nabla\chi_{t/2} =: F \quad \text{in } \Omega, \qquad \phi = 0 \quad \text{on } \Sigma_{\text{in}}.$$

Next introduce new vector field $\tilde{\mathbf{u}}(x) = \chi_{t/8}(x)\mathbf{u}(x)$. It easy to see that $\chi_{t/8} = 1$ on the support of ϕ and hence the function ϕ is also a weak solution to the modified transport equation

$$\tilde{\mathcal{L}}\phi := \tilde{\mathbf{u}}\nabla\phi + \sigma\phi = F \quad \text{in } \Omega. \tag{10.34}$$

The advantage of such an approach is that the topology of integral lines of the modified vector field $\tilde{\mathbf{u}}$ drastically differs from the topology of integral lines of $\mathbf{u}$. The corresponding inlet, outgoing set, and characteristic set have the other structure and $\tilde{\Sigma}_{\text{in}} = \emptyset$. In particular, equation (10.34) does not require boundary conditions. Finally note that the C^1-norm of the modified vector fields has the majorant

$$\|\tilde{\mathbf{u}}\|_{C^1(\Omega)} \le M(1 + 16\varpi(1)t^{-1}), \tag{10.35}$$

where $\varpi(1)$ is a constant from (10.32). The following lemma constitutes the existence and uniqueness of solutions to the modified equation.

LEMMA 10.11. *Suppose that*

$$\begin{aligned} &\sigma > \sigma^*(M, \Sigma) + 1, \quad \sigma^* = 4M(1 + 16\varpi(1)t^{-1}) + 1, \\ &M = \|\mathbf{u}\|_{C^1(\Omega)}, \end{aligned} \tag{10.36}$$

and $0 \le s \le 1$, $r > 1$. *Then for any* $F \in H^{s,r}(\Omega) \cap L^\infty(\Omega)$, *equation* (10.34) *has a unique weak solution* $\phi \in H^{s,r}(\Omega) \cap L^\infty(\Omega)$ *such that*

$$\|\phi\|_{H^{s,r}(\Omega)} \le c\sigma^{-1}\|F\|_{H^{s,r}(\Omega)}, \quad \|\phi\|_{L^\infty(\Omega)} \le \sigma^{-1}\|F\|_{L^\infty(\Omega)}, \tag{10.37}$$

where c *depends only on* r.

PROOF. Without any loss of generality we can assume that $F \in C^1(\Omega)$. By virtue of (10.35) and (10.36), the vector field $\tilde{\mathbf{u}}$ and σ meet all requirements of Lemma 5.3. Hence equation (10.34) has a unique solution $\phi \in H^{1,\infty}(\Omega)$. For $i = 1, 2, 3$ and $\tau > 0$, define the finite difference operator

$$\delta_{i\tau}\phi = \frac{1}{\tau}(\phi(x + \tau\mathbf{e}_i) - \phi(x)).$$

It is easy to see that

$$\tilde{\mathbf{u}}\nabla\delta_{i\tau}\phi + \sigma\delta_{i\tau}\phi = \delta_{i\tau}F - \delta_{i\tau}\tilde{\mathbf{u}}\nabla\phi(x + \tau\mathbf{e}_i) \quad \text{in } \Omega \cap (\Omega - \tau\mathbf{e}_i). \tag{10.38}$$

Next introduce the function $\eta \in C^\infty(\mathbb{R})$ such that $\eta' \geq 0$, $\eta(u) = 0$ for $u \leq 1$ and $\eta(u) = 1$ for $u \geq 1$, and set $\eta_h(x) = \eta(\text{dist }(x, \partial\Omega)/h)$. Since $\tilde{\Sigma}_{\text{in}} = \emptyset$, the inequality

$$\limsup_{h\to 0} \int_\Omega g\tilde{\mathbf{u}} \cdot \nabla\eta_h(x)\,dx \leq 0 \tag{10.39}$$

holds true for all nonnegative functions $g \in L^\infty(\Omega)$. Choosing $h > \tau$, multiplying both the sides of equation (10.38) by $\eta_h|\delta_{i\tau}\phi|^{r-2}\delta_{i\tau}\phi$ and integrating the result over $\Omega \cap (\Omega - \tau\mathbf{e}_i)$ we obtain

$$\begin{aligned}
&\int_{\Omega\cap(\Omega-\tau\mathbf{e}_i)} \eta_h|\delta_{i\tau}\phi|^r \left(\sigma - \frac{1}{r}\operatorname{div}\tilde{\mathbf{u}}\right) dx - \int_{\Omega\cap(\Omega-\tau\mathbf{e}_i)} |\delta_{i\tau}\phi|^r \tilde{\mathbf{u}}\nabla\eta_h\,dx \\
&\quad = \int_{\Omega\cap(\Omega-\tau\mathbf{e}_i)} (\delta_{i\tau}F - \delta_{i\tau}\tilde{\mathbf{u}}\nabla\phi(x + \tau\mathbf{e}_i))\eta_h|\delta_{i\tau}\phi|^{r-2}\delta_{i\tau}\phi\,dx.
\end{aligned}$$

Letting $\tau \to 0$ and then $h \to 0$ and using inequality (10.39) we obtain

$$\int_\Omega |\partial_{x_i}\phi|^r \left(\sigma - \frac{1}{r}\operatorname{div}\tilde{\mathbf{u}}\right) dx \leq \int_\Omega (\partial_{x_i}F - \partial_{x_i}\tilde{\mathbf{u}}\nabla\phi)|\partial_{x_i}\phi|^{r-2}\partial_{x_i}\phi\,dx. \tag{10.40}$$

Next note that

$$\sum_i \partial_{x_i}\tilde{\mathbf{u}}\nabla\phi\left|\partial_{x_i}\phi\right|^{r-2}\partial_{x_i}\phi \leq 3\|\tilde{\mathbf{u}}\|_{C^1(\Omega)} \sum_i |\partial_{x_i}\phi|^r.$$

On the other hand, since $1/r + 3 \leq 4$, inequalities (10.35) and (10.36) imply

$$\sigma - \left(\frac{1}{r} + 3\right)\|\tilde{\mathbf{u}}\|_{C^1(\Omega)} \geq \sigma - \sigma^* \geq 1.$$

From this we conclude that

$$\begin{aligned}
(\sigma - \sigma^*)\sum_i \int_\Omega |\partial_{x_i}\phi|^r\,dx &\leq \sum_i \int_\Omega |\partial_{x_i}\phi|^{r-1}|\partial_{x_i}F|\,dx \\
&\leq \left(\sum_i \int_\Omega |\partial_{x_i}\phi|^{r/(r-1)}\,dx\right)^{(r-1)/r} \left(\sum_i \int_\Omega |\partial_{x_i}F|^r\,dx\right)^{1/r},
\end{aligned}$$

which leads to the estimate

$$\|\nabla\phi\|_{L^r(\Omega)} \leq c(r)\sigma^{-1}\|\nabla F\|_{L^r(\Omega)} \quad \text{for } \sigma > \sigma^*(M, r). \tag{10.41}$$

Next multiplying both the sides of (10.34) by $|\phi|^{r-2}\eta_h$ and integrating the result over Ω we get the identity

$$\int_\Omega (\sigma - r^{-1}\operatorname{div}\tilde{\mathbf{u}})\eta_h|\phi|^r\,dx - \int_\Omega |\phi|^r\tilde{\mathbf{u}}\nabla\eta_h\,dx = \int_\Omega F\eta_h|\phi|^{r-2}\phi\,dx.$$

The passage $h \to 0$ gives the inequality

$$\int_\Omega (\sigma - r^{-1}\operatorname{div}\tilde{\mathbf{u}})|\phi|^r\,dx \le \int_\Omega |F||\phi|^{r-1}\,dx.$$

Recalling that $\sigma - r^{-1}\operatorname{div}\tilde{\mathbf{u}} \ge \sigma - \sigma^*$ we finally obtain

$$\|\phi\|_{L^r(\Omega)} \le c(r)\sigma^{-1}\|F\|_{L^r(\Omega)}. \tag{10.42}$$

Inequalities (10.41) and (10.42) imply estimate (10.37) for $s = 0, 1$. Hence for $\sigma > \sigma^*$, the linear operator $\tilde{\mathfrak{L}}^{-1} : F \mapsto \phi$ is continuous in the Banach spaces $H^{0,r}(\Omega)$ and $H^{1,r}(\Omega)$ and its norm does not exceed $c(r)\sigma^{-1}$. Recall that $H^{s,r}(\Omega)$ is the interpolation space $[L^r(\Omega), H^{1,r}(\Omega)]_{s,r}$. From this and Lemma 4.1 we conclude that inequality (10.37) is fulfilled for all $s \in [0, 1]$, which completes the proof. □

PROOF OF THEOREM 5.5. *Fix $\sigma > \sigma^*$, where the constant σ^* depends only on Σ, $\mathbf{U}$ and $\|\mathbf{u}\|_{C^1(\Omega)}$, and it is defined by* (10.36). *Without any loss of generality we can assume that $f \in C^1(\Omega)$. The existence and uniqueness of a weak bounded solution for $\sigma > \sigma^*$, follows from Lemma* 5.1. *Therefore, it suffices to prove estimate* (5.17) *for $\|\varphi\|_{H^{s,r}(\Omega)}$. Since $H^{s,r}(\Omega) \cap L^\infty(\Omega)$ is the Banach algebra, representation* (10.33) *together with inequality* (10.32) *implies*

$$\|\varphi\|_{H^{s,r}(\Omega)} \le c(1 + t^{-1})(\|\varphi_t\|_{H^{s,r}(\Omega_t)} + \|\varphi_t\|_{L^\infty(\Omega_t)}) + c\|\phi\|_{H^{s,r}(\Omega)}. \tag{10.43}$$

On the other hand, Lemma 10.11 *along with* (10.34) *yields*

$$\|\phi\|_{H^{s,r}(\Omega)} \le c\sigma^{-1}\|F\|_{H^{s,r}(\Omega)} \le c\sigma^{-1}\|\chi_{t/2}f\|_{H^{s,r}(\Omega)} + \sigma^{-1}\|\varphi_t\mathbf{u}\nabla\chi_{t/2}\|_{H^{s,r}(\Omega)}.$$

The first terms on the right-hand side is bounded,

$$\|\chi_{t/2}f\|_{H^{s,r}(\Omega)} \le c(1 + t^{-1})\|f\|_{H^{s,r}(\Omega)}.$$

In order to estimate the second term we note that, by virtue of (10.32), $\|\mathbf{u}\nabla\chi_{t/2}\|_{C^1(\Omega)} \le cM(1 + t^{-2})$ *which gives*

$$\|\varphi_t\mathbf{u}\nabla\chi_{t/2}\|_{H^{s,r}(\Omega)} \le cM(1 + t^{-2})(\|\varphi_t\|_{H^{s,r}(\Omega_t)} + \|\varphi_t\|_{L^\infty(\Omega_t)}).$$

Substituting the obtained estimates into (10.43) *we arrive at the inequality*

$$\begin{aligned}\|\varphi\|_{H^{s,r}(\Omega)} \le c(M + 1)(1 + t^{-2})\big((1 + \sigma^{-1})\|\varphi_t\|_{H^{s,r}(\Omega)} + \|\varphi_t\|_{L^\infty(\Omega_t)} \\ + \sigma^{-1}\|f\|_{H^{s,r}(\Omega_t)} + \sigma^{-1}\|f\|_{L^\infty(\Omega)}\big),\end{aligned}$$

which along with (10.25) *leads to the estimate*

$$\|\varphi\|_{H^{s,r}(\Omega)} \le c(\sigma^{-1}\|f\|_{H^{s,r}(\Omega)} + \sigma^{-1+\alpha}\|f\|_{L^\infty(\Omega)}), \tag{10.44}$$

which holds true for all $\sigma > \sigma^* + 1$. It remains to note that c and σ^* depend only on Σ, $\mathbf{U}$, $\|\mathbf{u}\|_{C^1(\Omega)}$, r, s, α and do not depend on σ, and the theorem follows.

A. Appendix. Proof of Lemmas 10.1 and 10.2

Proof of Lemma 10.1. We start with the proof of (**P1**). Choose the Cartesian coordinate system (x_1, x_2, x_3) associated with the point P and satisfying Condition (**H1**). Let us consider the Cauchy problem.

$$\partial_{y_1}\mathbf{x} = \mathbf{u}(\mathbf{x}(y)) \quad \text{in } Q_a, \qquad \mathbf{x}|_{y_1=0} = \mathbf{x}_0(y_2) + y_3\mathbf{e}_3. \tag{A.1}$$

Here the function $\mathbf{x}_0$ is given by condition (**H2**). Without loss of generality we can assume that $0 < a < k < 1$. It follows from (**H1**) that for any such a, problem (A.1) has a unique solution of class $C^1(Q_a)$. Next note that, by virtue of condition (**H1**), for $y_1 = 0$, we have

$$|\mathbf{x}(y)| \le (C_\Gamma + 1)|y|, \qquad |\mathbf{u}(\mathbf{x}(y)) - \mathbf{u}(0)| \le M(C_\Gamma + 1)|y|. \tag{A.2}$$

Denote by $\mathfrak{F}(y) = D_y\mathbf{x}(y)$. The calculations show that

$$\mathfrak{F}_0 := \mathfrak{F}(y)\Big|_{y_1=0} = \begin{pmatrix} u_1 & \Upsilon'(y_2) & 0 \\ u_2 & 1 & 0 \\ u_3 & \partial_{y_2}F(\Upsilon(y_2), y_2) & 1 \end{pmatrix}$$

and

$$\mathfrak{F}(0) = \begin{pmatrix} U & \Upsilon'(0) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which along with (A.2) implies

$$\|\mathfrak{F}(0)^{\pm 1}\| \le C_M/3, \qquad \|\mathfrak{F}_0(y) - \mathfrak{F}(0)\| \le ca. \tag{A.3}$$

Differentiation of (A.1) with respect to y leads to the ordinary differential equation for $\mathfrak{F}$

$$\partial_{y_1}\mathfrak{F} = D_y\mathbf{u}(\mathbf{x})\mathfrak{F}, \qquad \mathfrak{F}\Big|_{y_1=0} = \mathfrak{F}_0.$$

Noting that $\partial_{y_1}\|\mathfrak{F} - \mathfrak{F}_0\| \le \|\partial_{y_1}\mathfrak{F}\|$ we obtain

$$\partial_{y_1}\|\mathfrak{F} - \mathfrak{F}_0\| \le M(\|\mathfrak{F} - \mathfrak{F}_0\| + \|\mathfrak{F}_0\|),$$

and hence $\|\mathfrak{F} - \mathfrak{F}_0\| \le c(M)\|\mathfrak{F}_0\|a$. Combining this result with (A.3) we finally arrive at

$$\|\mathfrak{F}(y) - \mathfrak{F}(0)\| \le ca. \tag{A.4}$$

From this and the implicit function theorem we conclude that there is a positive constant a, depending only on M and Σ, such that the mapping $x = \mathbf{x}(y)$ takes diffeomorphically the cube Q_a onto some neighborhood of the point P, and satisfy inequalities (10.2).

Let us turn to the proof of **(P2)**. We begin with the observation that the manifold $\mathbf{x}(\Sigma \cap \mathcal{O}_P)$ is defined by the equation

$$\Phi_0(y) := x_3(y) - F(x_1(y), x_2(y)) = 0, \quad y \in Q_a.$$

Let us show that Φ_0 is strictly monotone in y_3 and has the opposite signs on the faces $y_3 = \pm a$. To this end note that the formula for $\mathfrak{F}(0)$ along with (A.4) implies the estimates

$$|\partial_{y_3} x_3(y) - 1| + |\partial_{y_3} x_1(y)| + |\partial_{y_3} x_2(y)| \le ca \quad \text{in } Q_{a_1}.$$

Thus we get

$$1 - ca \le \partial_{y_3} \Phi_0(y) = \partial_{y_3} x_3(y) - \partial_{x_i} F(x_1, x_2) \partial_{y_3} x_i(y) \le 1 + ca. \tag{A.5}$$

On the other hand, by (A.4), we have the inequality $|x_3(y)| \le ca|y|$, which along with (10.2) yields the estimate

$$|\Phi_0(y)| \le |x_3(y)| + |F(x(y))| \le ca|y| + KC_M|y|^2 \le ca^2 \quad \text{for } y_3 = 0. \tag{A.6}$$

Combining (A.5) and (A.6), we conclude that there exists a positive a depending only on M and Σ, such that the inequalities

$$1/2 \le \partial_{y_3} \Phi_0(y) \le 2, \qquad \pm\Phi_0(y_1, y_2, \pm a) > 0, \tag{A.7}$$

hold true for all $y \in Q_a$. Therefore, the equation $\Phi_0(y) = 0$ has a unique solution $y_3 = \Phi(y_1, y_2)$ in the cube Q_a, this solution vanishes for $y_1 = y_3 = 0$. By the implicit function theorem, the function Φ belongs to the class $C^1([-a, a]^2)$.
It remains to prove that Φ admits the both-side estimates (10.3). Note that inequality (10.2) implies the estimate $|\mathbf{u}(\mathbf{x}(y)) - U\mathbf{e}_1| \le M|\mathbf{x}(y)| \le MC_M a$. Therefore, we can choose $a = a(M, \Sigma)$ sufficiently small, such that

$$2U/3 \le u_1 \le 4U/3, \qquad C_\Gamma |u_2| \le U/3.$$

Recall that $x_1(y) - \Upsilon(x_2(y))$ vanishes at the plane $y_1 = 0$ and

$$\partial_{y_1}[x_1(y) - \Upsilon(x_2(y))] = u_1(y) - \Upsilon'(x_2(y))u_2(y).$$

Since $|\Upsilon'| \le C_\Gamma$, we obtain from this that

$$|y_1|U/3 \le \big|x_1(y) - \Upsilon(x_2(y))\big| \le |y_1|5U/3 \quad \text{for } y \in Q_a. \tag{A.8}$$

Equations (10.1) implies the identity

$$\partial_{y_1} \Phi_0(y) \equiv \nabla F_0(\mathbf{x}(y)) \cdot \mathbf{u}(x(y)) = \nabla F_0(\mathbf{x}(y)) \cdot \mathbf{U}(x(y)) \quad \text{for } \Phi_0(y) = 0.$$

Combining this result with (5.14) and (A.8), we finally obtain the inequality,

$$|y_1|N^- U/3 \le |\partial_{y_1} \Phi_0(y)| \le |y_1| N^+ U/5, \tag{A.9}$$

which along with estimate (A.7) and the identity $\partial_{y_1} \Phi = -\partial_{y_1} \Phi_0 (\partial_{y_3} \Phi_0)^{-1}$ yields (10.4). Since the term $x_1(y) - \Upsilon(x_2(y))$ is positive for positive y_1, the function Φ is increasing in y_1 for $y_1 > 0$ and is decreasing for $y_1 < 0$, which implies the existence of the functions

$a^{\pm}$. Next, the identities $\partial_{y_i} a^{\pm} = -\partial_{y_i}\Phi_0/\partial_{y_1}\Phi_0$, $i = 2, 3$, and estimate (A.9) yield the inequality

$$|\partial_{y_i} a^{\pm}(y)| \le c|y_1|^{-1}.$$

On the other hand, for $y_1 = a^{\pm}(y_2, y_3)$, we have $y_3 = |\Phi(y_1, y_2)| \ge c y_1^2$ and hence $|y_1|^{-1} \le c y_3^{-1/2}$, which implies the first estimates in (10.5). The second estimate is obvious.
In order to prove inclusions (10.7) note that $\Phi(-a, y_2) \ge a^2 C^-$ and hence $B_0(r) \cap \{y_3 > \Phi\} \subset G_a \subset Q_a$ for $r = a^2 C^-$. But estimate (10.2) implies that $B_P(\rho_c) \subset \mathbf{x}(B_0(r))$ for $\rho_c = r C_M^{-1}$, which yields the first inclusion in (10.7). It remains to note that the second is a consequence of (10.2) and the lemma follows.

Proof of Lemma 10.2. The proof simulates the proof of the Lemma 10.1. Choose the local Cartesian coordinates (x_1, x_2, x_3) centered at P such that in new coordinates $\mathbf{n} = \mathbf{e}_3$. By the smoothness of Σ, there is a neighborhood $\mathcal{O} = [-k, k]^2 \times [-t, t]$ such that the manifold $\Sigma \cap \mathcal{O}$ is defined by the equation

$$x_3 = F(x_1, x_2), \quad F(0, 0) = 0, \quad |\nabla F(x_1, x_2)| \le K(|x_1| + |x_2|).$$

The constants k, t and K depend only on Σ. Let us consider the initial value problem

$$\partial_{y_3}\mathbf{x} = \mathbf{u}(\mathbf{x}(y)) \quad \text{in } Q_a, \qquad \mathbf{x}|_{y_3=0} = (y_1, y_2, F(y_1, y_2)). \tag{A.10}$$

Without loss of generality we can assume that $0 < b < k < 1$. It follows from **(H1)** that for any such b, problem (A.10) has a unique solution of class $C^1(Q_b)$. Next, note that for $y_3 = 0$ we have

$$|\mathbf{x}(y)| \le (K + 1)|y|, \qquad |\mathbf{u}(\mathbf{x}(y)) - \mathbf{u}(0)| \le M(K + 1)|y|. \tag{A.11}$$

Denote by $\mathfrak{F}(y) = D_y\mathbf{x}(y)$. The calculations show that

$$\mathfrak{F}_0 := \mathfrak{F}(y)\Big|_{y_3=0} = \begin{pmatrix} 1 & 0 & u_1 \\ 0 & 1 & u_2 \\ 0 & 0 & u_3 \end{pmatrix} \qquad \mathfrak{F}(0) = \begin{pmatrix} 1 & 0 & u_1(P) \\ 0 & 1 & u_2(P) \\ 0 & 0 & U_n \end{pmatrix},$$

which along with (A.11) implies

$$\|\mathfrak{F}(0)^{\pm 1}\| \le C_{M,N}/3, \qquad \|\mathfrak{F}_0(y) - \mathfrak{F}(0)\| \le cb. \tag{A.12}$$

Next, differentiation of (A.10) with respect to y leads to the equation

$$\partial_{y_1}\mathfrak{F} = D_y\mathbf{u}(\mathbf{x})\mathfrak{F}, \qquad \mathfrak{F}\Big|_{y_3=0} = \mathfrak{F}_0.$$

Arguing as in the proof of Lemma 10.1 we obtain $\|\mathfrak{F} - \mathfrak{F}_0\| \le c(M)\|\mathfrak{F}_0\| b$. Combining this result with (A.12) we finally arrive at $\|\mathfrak{F}(y) - \mathfrak{F}(0)\| \le cb$. From this and the implicit function theorem we conclude that there is positive b, depending only on M and Σ, such that the mapping $x = \mathbf{x}(y)$ takes diffeomorphically the cube Q_b onto some neighborhood of the point P, and satisfies inequalities (10.9). Inclusions (10.10) easily follows from (10.9).

B. Appendix. Proof of Lemma 10.3

Existence and uniqueness of solution to problem (10.12) is obvious. Multiplying both the sides of equation (10.12) by $|\varphi|^{r-1}\varphi$ and integrating the result over Q_a^ϕ we obtain the inequality

$$\sigma \int_{Q_a^\phi} |\varphi|^r\, dy \le \int_{Q_a^\phi} |\varphi|^{r-1}|f|\, dy \le \Big(\int_{Q_a^\phi} |\varphi|^r\, dy\Big)^{1-1/r} \Big(\int_{Q_a^\phi} |f|^r\, dy\Big)^{1/r},$$

which yields (10.13) for $r < \infty$. Letting $r \to \infty$ we get (10.13) for $r = \infty$.

Let us turn to the proof of inequality (10.14) and begin with the case $s = 1$. For every $y \in \mathbb{R}^3$, we denote by $Y = (y_2, y_3)$. It is easy to see that the function $\partial_{y_3}\varphi$ has the representation $\partial_{y_3}\varphi = \varphi' + \varphi''$, where

$$\varphi'(y) = -e^{\sigma(y_1 - a^-(Y))}\partial_{y_3} a_-(Y) f(a^-(Y), Y),$$

and φ'' is a solution to boundary value problem

$$\partial_{y_1}\varphi'' + \sigma\varphi'' = \partial_{y_3} f \quad \text{in } Q_a^\phi, \qquad \varphi''(y) = 0 \quad \text{for } y_1 = a^-(y_2, y_3).$$

It follows from (10.13) that $\|\varphi''\|_{L^r(Q_a^\phi)} \le \sigma^{-1}\|f\|_{H^{1,r}(Q_a^\phi)}$. On the other hand, inequalities

$$|\partial a^\pm(Y)| \le c y_3^{-1/2}, \qquad a^+(Y) - a^-(Y) \le c y_3^{1/2},$$

yield the estimate

$$\begin{aligned}\int_{a^-(Y)}^{a^+(Y)} |\varphi'(y_1, Y)|^r\, dy_1 &\le c(r)\sigma^{-1}\|f\|^r_{L^\infty(Q_a^\phi)} y_3^{-r/2}(1 - e^{r\sigma(a^-(Y) - a^+(Y))}) \\ &\le c(r,\beta)\sigma^{-1+\beta} y_3^{(\beta - r)/2}\|f\|^r_{L^\infty(Q_a^\phi)},\end{aligned}$$

which holds true for all $\beta \in [0, 1]$. We conclude from this that

$$\|\varphi'\|_{L^r(Q_a^\phi)} \le c(r,\beta)\sigma^{(\beta-1)/r}\|f\|_{L^\infty(Q_a^\phi)} \quad \text{for } r - 2 < \beta < 1.$$

Combining this result with the estimate of φ'' and setting $\alpha = 1 + (\beta - 1)/r$ we finally obtain

$$\|\partial_{y_3}\varphi\|_{L^r(Q_a^\phi)} \le c(r,\alpha)(\sigma^{-1}\|f\|_{H^{1,r}(Q_a^\phi)} + \sigma^{-1+\alpha}\|f\|_{L^\infty(Q_a^\phi)}).$$

The same estimates hold true for $\partial_{y_2}\varphi$ and $\partial_{y_1}\varphi$, which yields (10.14) in the case $s = 1$.

The proof of inequality (10.14) for $0 < s < 1$ is more complicated. By virtue of (10.13), it suffices to estimate the semi-norm $|\varphi|_{s,r,Q_a^\phi}$. For an arbitrary $y, z \in \mathbb{R}^3$, set $Y = (y_2, y_3)$,

$Z = (z_2, z_3)$. Since expression (4.4) for the semi-norm $|\varphi|_{s,r,Q_a^\phi}$ is invariant with respect to the permutation $(Y, Z) \to (Z, Y)$, we have

$$|\varphi|_{s,r,Q_a^\phi} \le (2I)^{1/r}, \qquad I = \int_{D_a} |\varphi(z) - \varphi(y)|^r |z - y|^{-3-rs}\, dx dy, \tag{B.1}$$

where $D_a = \{(y, z) \in (Q_a^\Phi)^2 : a^-(Z) \le a^-(Y)\}$. It is easy to see that

$$\varphi(z) - \varphi(y) = \varphi(z_1, Z) - \varphi(y_1, Z) + \int_{a^-(Z)}^{a^-(Y)} e^{\sigma(x_1 - y_1)} f(x_1, Z)\, dx_1$$
$$+ \int_{a^-(Y)}^{y_1} e^{\sigma(x_1 - y_1)} (f(x_1, Z) - f(x_1, Y))\, dx_1 = I_1 + I_2 + I_3. \tag{B.2}$$

Hence our task is to estimate the quantities

$$J_k = \int_{D_a} |I_k|^r |z - y|^{-3-rs}\, dy dz, \quad k = 1, 2, 3. \tag{B.3}$$

The evaluation falls naturally into three steps and is based on the following proposition

PROPOSITION B.1. *If $r, s > 0$ and $i \ne j \ne k$, $i \ne k$, then*

$$\int_{[-a,a]^2} |z - y|^{-3+rs}\, dy_i dy_j \le c(r, s)|z_k - y_k|^{-1-rs},$$
$$\int_{[-a,a]} |z - y|^{-3+rs}\, dy_i \le c(r, s)(|z_j - y_j|^2 + |z_k - y_k|^2)^{(-2-rs)/2}.$$

PROOF. The left-hand side of the first equality is equal to

$$|z_k - y_k|^{-1-rs} \int_{[-a,a]^2} \left(1 + \frac{|z_i - y_i|^2 + |z_j - y_j|^2}{|z_k - y_k|^2}\right)^{(-2-rs)/2} \frac{dy_i dy_j}{|z_k - y_k|^2}$$
$$\le c(r, s)|z_k - y_k|^{-1-rs} \int_{\mathbb{R}^2} (1 + |y_i|^2 + |y_j|^2)^{-(3+rs)/2}\, dy_i dy_j,$$

which yields the first estimate. Repeating these arguments gives the second, and the proposition follows. □

THE FIRST STEP. We begin with the observation that, by virtue of the extension principle, the right-hand side f has an extension over $\mathbb{R}^3$, which vanishes outside the cube Q_{3a} and satisfies the inequalities

$$\|f\|_{H^{s,r}(\mathbb{R}^3)} \le c(a, r, s)\|f\|_{H^{s,r}(Q_a)}, \qquad \|f\|_{L^\infty(\mathbb{R}^3)} \le \|f\|_{L^\infty(Q_a)}. \tag{B.4}$$

Next recall that $a^-(Z) \le y_1, z_1 \le a$ for all $(y,z) \in D_a$. From this and Proposition B.1 we obtain

$$\begin{aligned}
&\int_{D_a} |I_1|^r |z-y|^{-3-rs}\,dy\,dz \\
&\quad \le \int_{[-a,a]^2} \left\{ \int_{a^-(Z)}^{a} \int_{a^-(Z)}^{a} |\varphi(z_1,Z)-\varphi(y_1,Z)|^r \right. \\
&\qquad \times \left. \left\{ \int_{[-a,a]^3} |z-y|^{-3-rs} dy_2 dy_3 dz_2 \right\} dy_1 dz_1 \right\} dZ \\
&\quad \le \int_{[-a,a]^2} \left\{ \int_{[a^-(Z),a]^2} |\varphi(z_1,Z)-\varphi(y_1,Z)|^r |y_1-z_1|^{-1-rs}\,dy_1 dz_1 \right\} dZ.
\end{aligned}$$

Since the right-hand side of this inequality is invariant with respect to the permutation $(y_1,z_1) \to (z_1,y_1)$, we have

$$J_1 \le c(r,s) \int_{[-a,a]^2} \left\{ \int_{D(Z)} |\varphi(z_1,Z)-\varphi(y_1,Z)|^r |y_1-z_1|^{-1-rs}\,dy_1 dz_1 \right\} dZ, \tag{B.5}$$

where $D(Z) = \{(y_1,z_1) : \; a^-(Z) \le z_1 \le y_1 \le a\}$. Now our task is to estimate the integral over $D(Z)$. Note that for all $(y_1,z_1) \in D(Z)$, we have the identity

$$\begin{aligned}
\varphi(y_1,Z) - \varphi(z_1,Z) &= \int_{a^-(Z)}^{z_1} e^{\sigma(t-z_1)} (f(t+\xi,Z) - f(t,Z))\,dt \\
&\quad + \int_{a^-(Z)-\xi}^{a^-(Z)} e^{\sigma(t-z_1)} f(t+\xi,Z)\,dt \;:=\; I_{11} + I_{12},
\end{aligned} \tag{B.6}$$

where $\xi = y_1 - z_1$.

Since f is extended over $\mathbb{R}^3$, we have the estimate

$$\begin{aligned}
&\int_{D(Z)} |I_{11}|^r |y_1-z_1|^{-1-rs}\,dy_1 dz_1 \\
&\quad \le \int_{a^-(Z)}^{a} \int_0^{2a} \left| \xi^{-s-1/r} \int_{a^-(Z)}^{z_1} e^{\sigma(t-z_1)} (f(t+\xi,Z) - f(t,Z))\,dt \right|^r dz_1 d\xi \\
&\quad = \int_{a^-(Z)}^{a} \int_0^{2a} |M(z_1,\xi,Z)|^r\,dz_1 d\xi.
\end{aligned}$$

It is easy to see that the function M on the right-hand side satisfies the equation and boundary condition

$$\partial_{z_1} M + \sigma M = K \quad \text{for } z_1 \in (a^-(Z),a), \qquad M = 0 \quad \text{for } z_1 = a^-(Z),$$

where $K(z_1,\xi,Z) = \xi^{-s-1/r}(f(z_1+\xi,Z) - f(z_1,Z))$. Multiplying both the sides of this equation by $|M|^{r-2}M$ and integrating the result over the interval $(a^-(Z),a)$ we arrive at the inequality

$$\sigma \int_{a^-(Z)}^{a} |M|^r \, dz_1 \le \int_{a^-(Z)}^{a} |M|^{r-1}|K| \, dz_1$$
$$\le \Big(\int_{a^-(Z)}^{a} |M|^r \, dz_1\Big)^{1-1/r} \Big(\int_{a^-(Z)}^{a} |K|^r \, dz_1\Big)^{1/r},$$

which gives

$$\int_{a^-(Z)}^{a} |M(z_1, \xi, Z)|^r \, dz_1 \le \sigma^{-r} \xi^{-1-rs} \int_{a^-(Z)}^{a} |f(z_1+\xi, Z) - f(z_1, Z)|^r \, dz_1.$$

Recalling that f is extended over $\mathbb{R}^3$ and vanishes outside the cube Q_{3a} we obtain the following estimate for the quantity I_{11} on the right-hand side of (B.6),

$$\begin{aligned}
&\int_{[-a,a]^2} \int_{D(Z)} |I_{11}|^r |y_1 - z_1|^{-1-rs} \, dy_1 dz_1 dZ \qquad\qquad (B.7)\\
&\le \sigma^{-r} \int_{[-a,a]^2} \int_{a^-(Z)}^{a} \int_0^{2a} \xi^{-1-rs} |f(z_1+\xi, Z) - f(z_1, Z)|^r \, dz_1 d\xi dZ\\
&\le \sigma^{-r} \int_{\mathbb{R}^4} |f(y_1, Z) - f(z_1, Z)|^r |y_1 - z_1|^{-1-rs} dy_1 dz_1 dZ\\
&\le c\sigma^{-r} \|f\|^r_{L^r(\mathbb{R}^2; H^{r,s}(\mathbb{R}))}\\
&\le c\sigma^{-r} \|f\|^r_{H^{r,s}(\mathbb{R}^3)}.
\end{aligned}$$

In order to estimate I_{12} note that for any $\beta \in [0, 1]$,

$$\begin{aligned}
|I_{12}| &\le \|f\|_{L^\infty(Q_{2a})} e^{\sigma(a^-(Z)-z_1)} \sigma^{-1}(1 - e^{-\sigma\xi})\\
&\le c(\beta) \|f\|_{L^\infty(Q_{2a})} \sigma^{\beta-1} e^{\sigma(a^-(Z)-z_1)} \xi^\beta.
\end{aligned}$$

Hence for all $\beta \in (s, 1]$ we have

$$\begin{aligned}
&\int_{[-a,a]^2} \int_{D(Z)} |I_{12}|^r |y_1 - z_1|^{-1-rs} \, dy_1 dz_1 dZ\\
&\le c\sigma^{(\beta-1)r} \|f\|^r_{L^\infty(Q_{2a})} \int_{[-a,a]^2} \int_{a^-(Z)}^{a} \int_0^{2a} e^{\sigma(a^-(Z)-z_1)} \xi^{-1+r(\beta-s)} \, dZ dz_1 d\xi\\
&\le c(r, s, \lambda) \sigma^{-1+(\beta-1)r} \|f\|^r_{L^\infty(Q_{2a})}. \qquad\qquad (B.8)
\end{aligned}$$

Substituting inequalities (B.7), (B.8) in (B.5), setting $\alpha = \beta - r^{-1}$, and recalling inequalities (B.4) we conclude that the estimate

$$J_1 \le c(r, s, a, \lambda)(\sigma^{-r} \|f\|^r_{H^{r,s}(Q_a)} + \sigma^{-r+\alpha r} \|f\|^r_{L^\infty(Q_a)}) \qquad (B.9)$$

holds true for each $\alpha \in (s - r^{-1}, 1 - r^{-1}]$. Since $\sigma > 1$, this estimate is obviously fulfilled for all $\alpha > s$.

THE SECOND STEP. Our next task is to estimate J_2. Recall that, by virtue of (10.11) and Lemma 10.1, for all $(y, z) \in D_a$,

$$a^-(Z) \le a^-(Y) \le y_1 \le a, \quad a^-(Z) \le z_1 \le a, \quad cz_1^2 \le z_3 \le a. \tag{B.10}$$

In particular, for any $\alpha \in [0, 1]$, we have

$$\begin{aligned}|I_2| &\le \|f\|_{L^\infty(Q_a)}\sigma^{-1}(1 - e^{\sigma(a^-(Z)-a^-(Y))})\\ &\le \|f\|_{L^\infty(Q_a)}\sigma^{-1+\alpha}|a^-(Z) - a^-(Y))|^\alpha.\end{aligned}$$

Since,

$$|a^-(Z) - a^-(Y))| \le c|z_2 - y_2| + c|\sqrt{z_3} - \sqrt{y_3}|,$$

we conclude from this that

$$\begin{aligned}J_2 &= \int_{D_a} |I_2|^r |z-y|^{-3-rs} \le c\sigma^{(\alpha-1)r}\|f\|^r_{L^\infty(Q_a)}(J_{21} + J_{22}), \quad \text{where} \\ J_{21} &= \int_{D_a} |z_2 - y_2|^{\alpha r}|z-y|^{-3-rs}\,dydz,\\ J_{22} &= \int_{D_a} |\sqrt{z_3} - \sqrt{y_3}|^{\alpha r}|z-y|^{-3-rs}\,dydz.\end{aligned} \tag{B.11}$$

On the other hand, Proposition B.1 yields the estimate

$$\int_{[-a,a]^4} |z-y|^{-3-rs} dy_1 dy_3 dz_3 \le c(r,s)|z_2 - y_2|^{-1-rs},$$

which leads to the inequality

$$J_{21} \le c(r,s)\int |z_2 - y_2|^{-1+r(\alpha-s)} \le c(r,s,\alpha) \quad \text{for all } \alpha \in (s, 1]. \tag{B.12}$$

Next note that

$$\begin{aligned}J_{22} \le \int_a^a dz_1 \left\{\int_{cz_1^2}^a dz_3 \left\{\int_0^a |\sqrt{z_3} - \sqrt{y_3}|^{\alpha r}\right.\right.\\ \left.\left.\left\{\int_{[-a,a]^3} |y-z|^{-3-rs}\,dy_1 dy_2 dz_2\right\} dy_3\right\}\right\}.\end{aligned} \tag{B.13}$$

It follows from Proposition B.1 that the interior integral has the estimate

$$\begin{aligned}&\int_0^a |\sqrt{z_3} - \sqrt{y_3}|^{\alpha r}\left\{\int_{[-a,a]^3} |y-z|^{-3-rs}\,dy_1 dy_2 dz_2\right\} dy_3\\ &\le \int_0^a |\sqrt{z_3} + \sqrt{y_3}|^{-\alpha r}|z_3 - y_3|^{-1+(\alpha-s)r}\,dy_3\\ &= z_3^{r(\alpha/2-s)}\int_0^{a/z_3} (1+\sqrt{t})^{-\alpha r}|1-t|^{-1+(\alpha-s)r}\,dt \le cz_3^{r(\alpha/2-s)}\\ &\quad \text{for } \alpha/2 < s < \alpha.\end{aligned}$$

Substituting this result in (B.13) we arrive at the inequality

$$J_{22} \le c(r, s, \alpha) \int_a^a |z_1|^{r(\alpha-2s)+2} dz_1 \le c \quad \text{for } s \in (\alpha/2, \alpha), \alpha \in (0, 1),$$
$$(2s - \alpha)r < 3.$$

Combining this result with (B.12) we finally obtain that for all exponents λ, r, and s satisfying the inequalities

$$1 < r < \infty, \quad 0 < s < 1, \quad \alpha/2 < s < \alpha < 1, \quad (2s - \alpha)r < 3, \tag{B.14}$$

the integral J_2 has the estimate

$$J_2 \le c(r, s, \alpha, a)\|f\|^r_{L^\infty(Q_a)} \sigma^{r(\alpha-1)}. \tag{B.15}$$

The third step. We begin with the observation that the function $I_3(y_1, Y, Z)$ defined by relation (B.2) satisfies the equation and boundary condition

$$\partial_{y_1} I_3 + \sigma I_3 = K_3 \quad \text{for } a^-(Y) < y_1 < a, \qquad I_3(a^-(Y), Y, Z) = 0,$$

where $K_3(y_1, Y, Z) = f(y_1, Z) - f(y_1, Y)$. Multiplying both the sides of this equation by $|I_3|^{r-2} I_3$ and integrating the result over the interval $(a^-(Y), a)$ we arrive at the inequality

$$\begin{aligned}\sigma \int_{a^-(Y)}^a |I_3|^r \, dy_1 &\le \int_{a^-(Y)}^a |I_3|^{r-1} |K_3| \, dy_1 \\ &\le \left(\int_{a^-(Y)}^a |I_3|^r \, dy_1\right)^{1-1/r} \left(\int_{a^-(Y)}^a |K_3|^r \, dy_1\right)^{1/r},\end{aligned}$$

which leads to the estimate

$$\int_{a^-(Y)}^a |I_3|^r \, dy_1 \le \sigma^{-r} \int_{[-a,a]} |f(y_1, Z) - f(y_1, Y)|^r \, dy_1.$$

Since $a^-(Y) \le y_1$ for all $(y, z) \in D_a$, we conclude from this and the inequality

$$\int_{[-a,a]} |z - y|^{-3-rs} dz_1 \le c|Y - Z|^{-2-rs}$$

that

$$\begin{aligned} J_3 &= \int_{D_a} |I_3|^r |z - y|^{-3-rs} dy dz \\ &\le c\sigma^{-r} \int_{[-a,a]^3} |f(y_1, Z) - f(y_1, Y)|^r |Y - Z|^{-2-rs} dy_1 dY dZ \\ &\le c\sigma^{-r} \|f\|^r_{L^r(-a,a; H^{s,r}([-a,a]^2))} \le c\sigma^{-r} \|f\|^r_{H^{s,r}(Q_a)}. \end{aligned} \tag{B.16}$$

Combining estimates (B.9), (B.15), and (B.16) we conclude that the estimate

$$\|\varphi\|_{s,r Q_a^\phi} \le c(a, r, s, \lambda)(\sigma^{-1}\|f\|_{H^{s,r}(Q_a)} + \sigma^{-r(1-\alpha)}\|f\|_{L^\infty(Q_a)}),$$

holds true for all exponents r, s and α satisfying inequalities (B.14). It remains to note that these inequalities can be written in the form $\max\{2s - 3/r, s\} < \alpha < \min\{2s, 1\}$.

References

[1] R.A. Adams, *Sobolev Spaces*, Academic press, New-York (1975).

[2] D. Adams, *On the existence of capacitary strong type estimates in R^n*, Ark. Mat. **14** (1976), 125–140.

[3] D. Adams and L. Hedberg, *Function Spaces and Potential Theory*, Springer-Verlag, Berlin etc (1995).

[4] S. Agmon, A. Douglis and L. Nirenberg, Comm. Pure Appl. Math. **12** (1959), 623–727.

[5] Ambrosio, L., *Transport Equations and Cauchy Problem for Non-Smooth Vector Fields*, Lecture notes of the CIME course given on Cetraro, June 27–July 2, 2005 (Preprint available at http://cvgmt.sns.it) (2004), 1635–1641.

[6] J.M. Ball, *A version of the fundamental theorem for Young measures, PDEs and Continuum Models of Phase Trans*, Lecture Notes in Physics, 344, (1989), pp. 241–259.

[7] H. Beirao da Veiga, *Stationary motions and the incompressible limit for compressible viscous limit*, Houston J. Math. **13** (14) (1987), 527–544.

[8] H. Beirao da Veiga, *An L^p-Theory for n-Dimensional Stationary Compressible Navier–Stokes equations, and the incompressible Limit for Compressible Fluids. The equilibrium solutions*, Comm. Math. Phys. **109** (1987), 229–248.

[9] H. Beirao da Veiga, *Existence results in Sobolev spaces for a transport equation*, Ricerche Mat. **36** (1987), 173–184. Suppl.

[10] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer-Verlag, Berlin, Heidelberg, New-York (1976).

[11] Y. Brenier, *Resolution d'equations quasiliniaires en dimension N d'espace al'aide d''equations lineaires en dimension $N+1$*, Differ. Equ. **50** (1983), 375–390 (in French).

[12] R. Farwig, G. Galdi and H. Sohr, *A new class of weak solutions of the Navier–Stokes Equations*, J. Math. Fluid Mech. **8** (2006), 423–443.

[13] G. Galdi, *An Introduction to the Mathematical Theory of the Navier–Stokes Equations V.I*, Springer-Verlag, Berlin, Heidelberg, New-York (1998).

[14] H. Ebert, *Physicalisches Tascenbuch*, Friedr, Vieweg & Sohn, Braunschweig (1957).

[15] G. Fichera, *Sulle equazioni differenziali lineari elliptico-paraboliche del secondo ordine*, Atti Accad. naz. Lincey, Mem. Cl. Sci. Fis., Mat. Natur., Sez1 **5** (1) (1956), 30.

[16] E. Feireisl, *Dynamics of Viscous Compressible Fluids*, Oxford University Press, Oxford (2004).

[17] E. Feireisl, *On compactness of solutions to the compressible isentropic Navier–Stokes equations when the density is not square integrable*, Comment. Math. Univ. Carolin. **42** (2001), 83–98.

[18] E. Feireisl, A. Novotný and H. Petzeltová, *On the existence of globally defined weak solutions to the Navier–Stokes equations*, J. Math. Fluid Mech. **3** (2001), 358–392.

[19] J. Frehse, S. Goj and M. Steinhauer, *L^p-estimates for the Navier–Stokes equations for steady compressible flow*, Manuscripta Math. **116** (3) (2005), 265–275.

[20] J.G. Heywood and M. Padula, *On the uniqueness and existence theory for steady compressible viscous flow in Fundamental directions in mathematical fluids mechanics, Adv. Math. Fluids Mech.*, Birkhauser, Basel (2000), pp. 171–189.

[21] L. Hörmander, *Non-elliptic boundary value problems*, Ann. of Math. **83** (1966), 129–209.

[22] R.B. Kellogg, *Discontinuous solutions of the linearized, steady state, compressible viscous Navier–Stokes equations*, SIAM J. Math. Anal. **19** (1988), 567–579.

[23] J.R. Kweon and R.B. Kellogg, *Compressible Navier–Stokes equations in a bounded domain with inflow boundary condition*, SIAM J. Math. Anal. **28** (1997), 94–108.

[24] J.R. Kweon and R.B. Kellogg, *Regularity of solutions to the Navier–Stokes equations for compressible barotropic flows on a polygon*, Arch. Ration. Mech. Anal. **163** (1) (2000), 36–64.

[25] J.R. Kweon and R.B. Kellogg, *Singularity in the density gradient*, J. Math. Fluid Mech. **8** (2006), 445–454.

[26] J.J. Kohn and L. Nirenberg, *Degenerate elliptic-parabolic equations of second order*, Comm. Pure Appl. Math. **20** (4) (1967), 797–872.

[27] N.S. Landkof, *Foundations of Modern Potential Theory*, Springer-Verlag, Berlin, Heidelberg, New-York (1972).

[28] De Lellis, C., *Notes on hyperbolic systems of conservation laws and transport equations*, to appear in Handbook of Evolutionary Differential Equations (Preprint available at http://www.math.unizh.ch).

[29] R.J. Di-Perna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math. **48** (1989), 511–547.

[30] P.-L. Lions, B. Perthame and E. Tudmor, *A kinetic formulation of multidimensional conservation laws and related equations*, J. Amer. Math. Soc. **7** (1994), 415–431.

[31] P.-L. Lions, B. Perthame and P. Souganidis, *Existence and stability of entropy solutions for the hyperbolic systems of isentropic has dynamics in Eulerian and Lagrangian coordinates*, Comm. Pure Appl. Math. **59** (1996), 599–638.

[32] P.-L. Lions, *Compacticite des solutions des equations de Navier–Stokes compressible isentropiques*, C. R. Acad. Sci. Paris, Ser. I **317** (1993), 115–120.

[33] P.-L. Lions, *Bornes sur la densite pour les de Navier–Stokes compressible isentropiques avec conditions aux limits de Dirichlet*, C. R. Acad. Sci. Paris, Ser. I **328** (1999), 659–662.

[34] P.-L. Lions, *Mathematical topics in fluid dynamics*, Compressible Models, Vol. 2, Oxford Science Publication, Oxford (1998).

[35] L.D. Landau and E.M. Lifshitz, *Course of Theoretical Physics. Vol. 6. Fluid Mechanics*, Pergamon Press, Oxford (1987).

[36] J. Malek, J. Nečas, M. Rokyta and M. Ruzicka, *Weak and Measure-Valued Solutions to Evolutionare PDE*, Chapman and Hall, London (1996).

[37] V.G. Maz'ya, *Sobolev Spaces*, Nauka, Leningrad (1985).

[38] V.G. Maz'ya and T.O. Shaposhnikova, *Multipliers in Spaces of Differential Functions*, Leningrad university, Leningrad (1986).

[39] S. Novo and A. Novotný, *On the existence of weak solutions to the steady compressible Navier–Stokes equations when the density is not square integrable*, J. Math. Kyoto Univ. **42** (2002), 531–550.

[40] S. Novo and A. Novotný, *On the existence of weak solutions to the steady compressible Navier–Stokes equations in domains with conical outlets*, J. Math. Fluid Mech. **8** (2006), 187–210.

[41] A. Novotný and I. Straškraba, *Introduction to the mathematical theory of compressible flow*, Oxford Lecture Series in Mathematics and its Applications, vol. 27, Oxford University Press, Oxford (2004).

[42] A. Novotný and M. Padula, *Existence and Uniqueness of Stationary solutions for viscous compressible heat conductive fluid with large potential and small non-potential external forces*, Siberian Math. J. **34** (1993), 120–146.

[43] A. Novotný and M. Padula, *L^p- Approach to Steady flows of Viscous Compressible Fluids in Exterior Domains*, Arch. Ration. Mech. Anal. **126** (1994), 243–297.

[44] A. Novotný and M. Padula, *Physically reasonable solutions to steady compressible Navier–Stokes equations in 3D-Exterior Domains*, Math. Ann. **308** (1997), 438–439.

[45] A. Novotný, *About steady transport equation. I. L^p-approach in domains with smooth boundaries*, Comment. Math. Univ. Carolin. **37** (1) (1996), 43–89.

[46] A. Novotný, *About steady transport equation. II. Schauder estimates in domains with smooth boundaries*, Portugal. Math. **54** (3) (1997), 317–333.

[47] O.A. Oleinik and E.V. Radkevich, *Second Order Equation with Non-Negative Characteristic Form*, American Math. Soc., Providence, Rhode Island (1973), Plenum Press. New York, London.

[48] M. Padula, *Existence and uniqueness for viscous steady compressible motions*, Arch. Ration. Mech. Anal. **97** (1) (1986), 1–20.

[49] Padula, M., Steady flows of barotropic viscous fluids, in Classical Problems of in Mechanics, 1 (1997), Dipartamento di Matematika Seconda Universita di Napoli, Caserta.

[50] P.I. Plotnikov and J. Sokolowski, *On compactness, domain dependence and existence of steady state solutions to compressible isothermal Navier–Stokes equations*, J. Math. Fluid Mech. **7** (4) (2005), 529–573.

[51] P.I. Plotnikov and J. Sokolowski, *Concentrations of solutions to time-discretized compressible Navier–Stokes equations*, Comm. Math. Phys. **258** (3) (2005), 567–608.

[52] P.I. Plotnikov and J. Sokolowski, *Stationary Boundary Value Problems for Navier–Stokes Equations with Adiabatic Index* $\nu < 3/2$, Doklady Mathematics **70** (1) (2004), 535–538. Translated from Doklady Akademii Nauk, Volume 397, Nos. 1-6, 2004.

[53] P.I. Plotnikov and J. Sokolowski, *Domain dependence of solutions to compressible Navier–Stokes equations*, SIAM J. Control Optim. **45** (4) (2006), 1147–1539.

[54] H. Schlichting, *Boundary-layer theory*, McGraw-Hill series in mechanical engineering, McGraw-Hill, New York (1955), p. 535.

[55] J. Serrin, *Mathematical Principles of Classical Fluid Mechanics, Handbook der Physic VIII/I*, Springer, Berlin (1972).

[56] Xinfu Chen and Weiqing Xie, *Discontinuous solutions of steady state viscous compressible Navier–Stokes equations*, J. Differential Equations **115** (1995), 567–579.