ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS 74

CONTROL THEORY FOR PARTIAL DIFFERENTIAL EQUATIONS: CONTINUOUS AND APPROXIMATION THEORIES

I ABSTRACT PARABOLIC SYSTEMS

IRENA LASIECKA

ROBERTO TRIGGIANI

CAMBRIDGE

more information - www.cambridge.org/9780521434089

Control Theory for Partial Differential Equations: Continuous and Approximation Theories

This is the first volume of a comprehensive and up-to-date three-volume treatment of quadratic optimal control theory for partial differential equations over a finite or infinite time horizon and related differential (integral) and algebraic Riccati equations. Both continuous theory and numerical approximation theory are included. An abstract space, operator theoretic treatment is provided, which is based on semigroup methods, and which is unifying across a few basic classes of evolution. A key feature of this treatise is the wealth of concrete multidimensional PDE illustrations, which fit naturally into the abstract theory, with no artificial assumptions imposed, at both the continuous and numerical level.

Throughout these volumes, emphasis is placed on unbounded control operators or on unbounded observation operators as they arise in the context of various abstract frameworks that are motivated by partial differential equations with boundary/point control. Relevant classes of PDEs include: parabolic or parabolic-like equations, hyperbolic and Petrowski-type equations (such as plate equations and the Schrödinger equation), and hybrid systems of coupled PDEs of the type that arise in modern thermo-elastic and smart material applications. Purely PDE dynamical properties are critical in motivating the various abstract settings and in applying the corresponding theories to concrete PDEs arising in mathematical physics and in other recent technological applications.

Volume I covers the abstract parabolic theory, including both the finite and infinite horizon optimal control problems, as well as the corresponding min–max theory with nondefinite quadratic cost. A lengthy chapter presents many multidimensional PDE illustrations with boundary/point control and observation. These include not only the traditional parabolic equations, such as the heat equation, but also second-order equations with structural ("high") damping, as well as thermo-elastic plate equations. Recently discovered, critical dynamical properties are provided in detail. Many of these new results are appearing here in print for the first time.

Volume II is focused on the optimal control problem over a finite time interval for hyperbolic dynamical systems including second-order hyperbolic equations with Dirichlet boundaries, plate equations and the Schrödinger equation under a variety of boundary controls, and structural acoustic models coupling two hyperbolic equations.

Volume III is in preparation.

Irena Lasiecka is Professor of Mathematics at the University of Virginia, Charlottesville. She has held positions at the Control Theory Institute of the Polish Academy of Sciences, the University of California, Los Angeles, and the University of Florida, Gainesville. She has authored or coauthored over 150 research papers and one other book in the area of PDEs and their control theoretic properties. She serves on the editorial boards of *Applied Mathematics and Optimization*, *Journal of Mathematical Analysis and Applications*, and the *IEEE Transactions on Automatic Control*, among others, and she holds, or has held, numerous offices in the professional societies SIAM, IFIP, and the AMS.

Roberto Triggiani is Professor of Mathematics at the University of Virginia at Charlottesville. He has held regular and visiting academic positions at several institutions in the United States and Europe, including Iowa State University, Ames, and the University of Florida, Gainesville. He has authored or coauthored over 140 research papers and one other book in PDEs and their control theoretic properties. He currently serves on the editorial boards of *Applied Mathematics and Optimization*, *Abstract and Applied Analysis*, *Systems and Control Letters*.

Volume 75
Control Theory for Partial Differential Equations II

6 H. Minc *Permanents*
19 G. G. Lorentz, K. Jetter, and S. D. Riemenschneider *Birkhoff Interpolation*
22 J. R. Bastida *Field Extensions and Galois Theory*
23 J. R. Cannon *The One-Dimensional Heat Equation*
24 S. Wagon *The Banach–Tarski Paradox*
25 A. Salomaa *Computation and Automata*
27 N. H. Bingham, C. M. Goldie, and J. L. Teugels *Regular Variation*
28 P. P. Petrushev and V. A. Popov *Rational Approximation of Real Functions*
29 N. White (ed.) *Combinatorial Geometries*
30 M. Pohst and H. Zassenhaus *Algorithmic Algebraic Number Theory*
31 J. Aczel and J. Dhombres *Functional Equations in Several Variables*
32 M. Kuczma, B. Choczewski, and R. Ger *Iterative Functional Equations*
33 R. V. Ambartzumian *Factorization Calculus and Geometric Probability*
34 G. Gripenberg, S.-O. Londen, and O. Staffans *Volterra Integral and Functional Equations*
35 G. Gasper and M. Rahman *Basic Hypergeometric Series*
36 E. Torgersen *Comparison of Statistical Experiments*
37 A. Neumaier *Interval Methods for Systems of Equations*
38 N. Korneichuk *Exact Constants in Approximation Theory*
39 R. Brualdi and H. Ryser *Combinatorial Matrix Theory*
40 N. White (ed.) *Matroid Applications*
41 S. Sakai *Operator Algebras in Dynamical Systems*
42 W. Hodges *Basic Model Theory*
43 H. Stahl and V. Totik *General Orthogonal Polynomials*
44 R. Schneider *Convex Bodies*
45 G. Da Prato and J. Zabczyk *Stochastic Equations in Infinite Dimensions*
46 A. Björner et al. *Oriented Matroids*
47 G. Edgar and L. Sucheston *Stopping Times and Directed Processes*
48 C. Sims *Computation with Finitely Presented Groups*
49 T. Palmer *Banach Algebras and the General Theory of *-Algebras*
50 F. Borceux *Handbook of Categorical Algebra 1*
51 F. Borceux *Handbook of Categorical Algebra 2*
52 F. Borceux *Handbook of Categorical Algebra 3*
53 V. F. Kolchin *Random Graphs*
54 A. Katok and B. Hasselblatt *Introduction to the Modern Theory of Dynamical Systems*
55 V. N. Sachkov *Combinatorial Methods in Discrete Mathematics*
56 V. N. Sachkov *Probabilistic Methods in Discrete Mathematics*
57 P. M. Cohn *Skew Fields*
58 R. Gardner *Geometric Topography*
59 G. A. Baker Jr. and P. Graves-Morris *Pade Approximants*
60 J. Krajicek *Bounded Arithmetic, Propositional Logic, and Complexity Theory*
61 H. Groemer *Geometric Applications of Fourier Series and Spherical Harmonics*
62 H. O. Fattorini *Infinite Dimensional Optimization and Control Theory*
63 A. C. Thompson *Minkowski Geometry*
64 R. B. Bapat and T. E. S. Raghavan *Nonnegative Matrices with Applications*
65 K. Engel *Sperner Theory*
66 D. Cvetkovic, P. Rowlinson, and S. Simic *Eigenspaces of Graphs*
67 F. Bergeron, G. Labelle, and P. Leroux *Combinatorial Species and Tree-Like Structures*
68 R. Goodman and N. Wallach *Representations and Invariants of the Classical Groups*
70 A. Pietsch and J. Wenzel *Orthonormal Systems and Banach Space Geometry*
71 G. E. Andrews, R. Askey, and R. Roy *Special Functions*

To Maria Ugenti, Janina Krzeminska, Antoni Lech,
and Giuseppe Triggiani

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Control Theory for Partial Differential Equations: Continuous and Approximation Theories

I: Abstract Parabolic Systems

IRENA LASIECKA   ROBERTO TRIGGIANI

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,
São Paulo, Delhi, Dubai, Tokyo, Mexico City

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521434089

First published 2000

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing-in-Publication Data

Lasiecka, I. (Irena), 1948–
Control theory for partial differential equations: continuous and
approximation theories / Irena Lasiecka, Roberto Triggiani.
p. cm. –(Encyclopedia of mathematics and its applications: v.74–75)
Contents: I. Abstract parabolic systems – II. Abstract hyperbolic-like
systems over a finite time horizon
ISBN 0-521-43408-4 (v. 1). – ISBN 0-521-58401-9 (v. 2)
1. Differential equations, Partial. 2. Control theory.
I. Triggiani, R. (Roberto), 1942– II. Title. III. Series.
QA377.L37 1999 99-11617
515'.353 -dc21 CIP

ISBN 978-0-521-43408-9 Hardback
ISBN 978-0-521-15567-0 Paperback

Contents

Preface

This three-volume treatise presents, in a unified framework, a comprehensive, in-depth, and up-to-date treatment of quadratic optimal control theory for (linear) partial differential equations (PDEs) over a finite or infinite time horizon and related differential (integral) and algebraic Riccati equations. Both continuous theory and numerical approximation theory are included. An abstract space, operator theoretic treatment is provided, which is based on semigroup methods, and which is unifying across a few basic classes of evolution.

While addressing all three volumes regarding the basic, broad-range theme covered and the philosophy of approach followed, this preface focuses mostly on Volumes I and II for specific details. Indeed, driven also by recent, new PDE models such as they arise in modern technological applications, the treatment of this work has grown far beyond the original intentions and the anticipated plan. As a result, two volumes now appear in print, with a third one in preparation. A justification for the criteria that have dictated the selection of a natural subdivision of the entire work into three volumes is given below.

This treatise is a much expanded outgrowth, at least in the ratio 1 to 10, of the authors' Springer-Verlag Lectures Notes in Control and Information Sciences, Volume 164, entitled: *Differential and Algebraic Riccati Equations with Applications to Boundary-Point Control Problems: Continuous and Approximation Theory.* These Lecture Notes, published in 1991, contained a comprehensive account of the theories that were available at that time, along with an array of numerous illustrative PDE applications with boundary/point control. However, most technical proofs were referred to the literature. A completion of these Lectures Notes was therefore called for, which inevitably stimulated an extension of their range of coverage with the addition of both new theoretical topics, as well as new PDE models and applications of modern technological origin. These, in turn, required further still theoretical analysis.

The basic dynamics is an abstract equation $\dot{y} = Ay + Bu$, where A (free dynamic operator) is the generator of a strongly continuous (s.c.) semigroup on the Hilbert (state) space Y, and where B (control operator) is an unbounded operator with a degree of unboundedness up to the degree of unboundedness of A. Moreover, u is the control function, which runs over the class of L_2-functions in time, with values in a Hilbert space U. All the boundary/point control problems for PDEs can be modeled by this abstract equation, for specific choices of the operators A, B and of the spaces U, Y. The dynamics is further penalized by a (quadratic) functional cost, containing an observation operator R, to be minimized over a preassigned finite or infinite time horizon. The theory of this problem culminates with the analysis of the corresponding differential or algebraic Riccati (operator) equations, which arise in the (pointwise) feedback synthesis of the optimal solution pair $\{u^0, y^0\}$. This problem, which originated in the late 1950s in the context of ordinary differential equations (with A, B, R matrices of appropriate size) has long been considered a truly central issue – a "battle-horse" – in deterministic optimal control theory, and related stochastic filtering theory, of dynamical systems. In the finite dimensional context, the solution in pointwise feedback form, via Riccati equations, of both the deterministic and the stochastic versions of this problem, has been known since the 1960s, through the work of Kalman and Kalman-Bucy, respectively.

These volumes present the far-reaching, technical extension of the deterministic problem, aimed at accommodating and encompassing multidimensional PDEs with boundary/point control and/or observation, in a natural way. Thus, throughout this work, emphasis is placed on unbounded control operators and/or, possibly, on unbounded observation operators as well, as they arise in the context of various abstract frameworks that are motivated by, and ultimately directed to, PDEs with boundary/point control and observation. A key feature of the entire treatise is then a wealth throughout of concrete, multidimensional PDE illustrations, which naturally fit into the abstract theory, with no artificial assumptions imposed, at both the continuous and numerical level. Justification of the abstract models adopted rests, unequivocally, with their intrinsic ability of capturing the characterizing dynamical properties of specific, relevant classes of PDEs, which motivate them in the first place. Regarding abstract modeling, the flow runs unmistakenly from an understanding of the concrete into the proper abstract.

Naturally, to extract best possible results and tune the technical tools to the problem at hand, it is necessary to distinguish at the outset between different types of PDE classes: primarily, parabolic-like dynamics versus hyperbolic-like dynamics, with further subdistinctions in the latter class. This is due to well-known, intrinsically different dynamical properties between these two classes. As a consequence, they lead to two drastically different basic abstract models, whose defining, characterizing features set them apart. Accordingly, these two abstract models need, therefore, to be investigated by correspondingly different technical strategies and tools. As a consequence, different types of distinctive results are achieved to characterize the two classes. All this dictates that the abstract theory needs to bifurcate at the very outset into a parabolic-like model and hyperbolic-like basic models; moreover, in the latter class, a further distinction into finite and infinite time horizon is called for, to account for different, critical properties between these two cases.

Thus, Volume I contains the optimal control theory for the parabolic-like class, over both the finite and the infinite time horizon, where the s.c. semigroup generated by A is, moreover, analytic; while Volumes II and III refer to the optimal control theory for the hyperbolic-like class over a finite or, respectively, infinite time horizon. This includes hyperbolic dynamics as well as Petrowski-type PDEs such as platelike models, Schrödinger equation, etc.

As already emphasized, purely PDE dynamical properties are critical in motivating the various abstract settings, as well as in applying the corresponding theories to concrete PDEs arising in mathematical physics and in other technological endeavors. This is particularly true in the case of hyperbolic-like dynamics. Unlike the parabolic-like class, which offers a certain degree of flexibility in the choice of the abstract space setting (subject to established parabolic regularity theory), by contrast, the framework in the case of hyperbolic-like dynamics is far more rigid. It requires a preliminary knowledge of the space of optimal regularity theory – a purely PDEs problem – and thus leaves no choice. Moreover, regarding the infinite time optimal control problem, the most complete theory is achieved in the cases (which occur most often, but by no means always) where the space of optimal regularity of the solution under L_2-control coincides with the space of exact controllability (or of uniform stabilization) – in other words, where the map from the class of admissible L_2-controls to the state space is surjective at some finite time. In short: In the hyperbolic-like case, optimal regularity theory is an intrinsic, critical, essential prerequisite factor in the analysis of the corresponding optimal control problem, which rigidly depends upon it (while a margin of latitude exists in the parabolic-like case, once parabolicity has been established). Accordingly, optimal regularity theory of many hyperbolic-like dynamical equations considered in the illustrations is an intrinsic part of the present volumes. A more detailed description is given below in the synopsis of Volume II. The inclusion, on the one hand, of this massive regularity theory and, on the other hand, of new PDE dynamics such as thermo-elastic plate equations and various models of coupled PDEs arising in structural acoustics, helps explain the explosion of this subject matter into three volumes.

Throughout this work, special emphasis is paid to the following topics:

(i) Abstract operator models for boundary/point control and observation problems for PDEs.

(ii) Identification of the space of optimal regularity of the solutions, typically under L_2-controls in time, and particularly for the class of hyperbolic and Petrowski-type systems or coupled PDEs problems; it is with respect to the norm of this space that the solution is then penalized in the cost functional.

(iii) Identification of the regularity properties of the optimal pair of the optimal control problem, particularly, in the parabolic-like case over a finite or infinite interval, and in the hyperbolic-like case over a finite interval. In the hyperbolic-like case over an infinite time horizon, the optimal pair need not be better than the original L_2 regularity in time, inherited from the optimization problem.

(iv) Verification of what we call the "finite cost condition" (F.C.C.) in the infinite time horizon problem and related algebraic Riccati equations, which guarantees the existence of at least one admissible control yielding a finite cost functional. In the case of parabolic-like dynamics, the F.C.C. is most readily verified via uniform feedback stabilization, as the unstable space of the dynamics is, at most, finite dimensional. By contrast, in the case of hyperbolic-like dynamics, the F.C.C. is verified via a study of the related exact controllability problem, or of the related (generally, more challenging) uniform stabilization problem, by means of an explicit, dissipative, boundary, velocity feedback operator. Exact controllability/uniform stabilization of hyperbolic-like dynamics is a topic in its own right, intimately connected with, yet distinct from, the main thrust of the optimal control problem of the present volumes. A vast literature exists, including treatments in book form. We shall return to these topics in Volume III.

(v) Constructive variational approach to the issue of existence of a solution (Riccati operator), and possibly uniqueness of a corresponding differential or algebraic Riccati operator equation.

(vi) Development of numerical algorithms that reproduce numerically the key properties of the continuous problems. This can be done directly in the parabolic-like case. By contrast, the hyperbolic-like (conservative) case requires that a regularization procedure be performed first, before passing to the approximation analysis.

A brief description of the contents of the first two volumes follows.

Volume I focuses on abstract parabolic systems (continuous and approximation theory), where the s.c. semigroup of the free dynamics is, moreover, analytic. Save perhaps for some possible refinements, the overall theory in this chapter and companion notes is essentially optimal. This includes both the finite (Chapter 1) and infinite horizon (Chapter 2) optimal control problems, as well as the corresponding min–max theory with nondefinite quadratic cost (Chapter 6). Here, both control operator and disturbance operator are of the same "maximal" degree of unboundedness allowed with respect to the free dynamics operator. A treatment of the optimal control problem with nondefinite quadratic cost and coercive penalization on the control is included in the appendix to Chapter 6. A lengthy Chapter 3 presents many multidimensional PDE illustrations with boundary/point control and observation. They include not only traditional, classical parabolic equations such as the heat equation with Dirichlet- or Neumann-boundary control, or point control, but also second-order equations with "structural" or "high" damping, as well as thermo-elastic plate equations with no rotational inertia term. For the latter two classes, recently discovered, critical dynamical properties are proved in details. These include "parabolicity" (analyticity of the corresponding semigroup) and uniform stability. Various appendices in Chapter 3, taken cumulatively, provide a self-contained subvolume focused on thermo-elastic parabolic plate equations, whose theory has become available only over the past year or so. Chapter 4 provides a detailed numerical approximation treatment, with appropriate convergence properties (possibly, with optimal rates of convergence) of all the quantities of interest: optimal control, optimal solution, Riccati operator, gain operator, optimal cost, etc. Finally, Chapter 5 provides detailed PDE illustrations of numerical schemes that fit into the theory of Chapter 4. Regarding the theoretical treatment, the analysis in Volume I is almost exclusively operator-theoretic and is based on singular integrals as they arise in the description of the control-solution (state) map, by virtue of the key property of analyticity of the free dynamics semigroup (generated by the operator A). As it turns out, analyticity of the free dynamics compensates, in this case, for the unboundedness of the control operator or of the disturbance operator. Indeed, such analyticity yields a controlled smoothing of the control-solution map and of its adjoint. Once applied to the optimality conditions characterizing the optimal pair, such double smoothing snowballs into a boot-strap argument, which eventually leads to higher regularity of the optimal pair (over the initial regularity inherited from the optimization problem) and – finally – to a smoothing property of the Riccati operator. As a consequence, the gain operator is bounded from the state to the control space, a distinctive, critical property of the parabolic-like class. In applications to concrete PDEs, elliptic theory and identification of domains of appropriate fractional powers with Sobolev spaces play a critical role.

Volume II considers the optimal control theory for hyperbolic or Petrowski-type PDEs over a finite time horizon. It begins with an introductory chapter (Chapter 7) that collects relevant abstract settings and abstract properties of these dynamics that are to be used in subsequent chapters. It then considers three different abstract frameworks. The abstract model of Chapter 8 is motivated by the optimal control problem for second-order hyperbolic equations with Neumann-boundary control and Dirichlet-trace observation. The abstract model of Chapter 9 is motivated by wave and Kirchoff elastic plate equations, under the action of point control. It also includes two models of coupled PDE systems, such as they arise in noise reduction problems in structural acoustics. Both systems are subject to point control, which models the action of smart material technology. One couples the wave equation for the pressure in the acoustic chamber with a Kirchoff equation for the elastic displacement of the moving wall. It is an example of hyperbolic/hyperbolic coupling. Instead, in the second system, the elastic wall is modeled by an Euler–Bernoulli equation with structural damping, thus giving rise to a hyperbolic/parabolic coupling. Finally, the abstract model of Chapter 10, which further builds on that of Chapter 9, looks at first artificial and complicated. Actually, it is a natural framework, which simply extracts the correct settings for problems such as second-order hyperbolic equations with Dirichlet-boundary control, numerous other plate equations with a variety of boundary control, as well as the Schrödinger equation with Dirichlet-boundary control. All the relevant regularity theory, some of which is new, of these dynamical PDEs is provided in detail, subject to the exclusions noted below. Indeed, in contrast with parabolic theory, the regularity theory of hyperbolic and Petrowski-type equations (such as plate equations and

Schrödinger equations) demand a broader array of purely PDE techniques to obtain sharp/optimal interior and trace regularity properties. They include energy methods, or multipliers methods, at the differential level or pseudo-differential/microlocal analysis level, which were discovered much more recently than parabolic techniques. This contrast between the two basic classes of dynamical systems – parabolic-like versus hyperbolic or Petrowski-type equations – was already emphasized in the preface to the authors' Lectures Notes. Accordingly, Volume II contains in detail most of the needed regularity theory (both interior and trace regularity) of the many hyperbolic-like PDE systems here considered. Exceptions include the more recent regularity theory of first-order hyperbolic systems and of second-order hyperbolic equations with Neumann boundary datum, which require a treatment based on the technical apparatus of pseudo-differential operators and microlocal analysis. For these, appropriate references to the recent literature are given.

As already noted, Volume III (in preparation) will cover optimal control problems for hyperbolic-like dynamics (both continuous and numerical approximation theory) and for coupled PDE systems, over an infinite time horizon.

Further information on this treatise in the context of available books is contained in the introductory section of Chapter 0.

Acknowledgments for the First Two Volumes

Though almost all of the results presented in the first two volumes are taken from the authors' original research work in both optimal control theory and PDE theory, there are a numbers of friends whom we wish to thank very warmly, and to whom we wish to express our gratitude. In chronological order, first A.V. Balakrishnan, who, with his original work of the mid-1970s, provided the initial spur into abstract modeling for parabolic equations with Dirichlet boundary control, and who soon thereafter graciously introduced us to it. Next, we wish to thank some of our coauthors, from whom and with whom we have learnt about this fascinating subject, for joint work reported in these first two volumes: G. Da Prato on the abstract differential Riccati equation in the hyperbolic case, via a direct approach, and J. L. Lions on the regularity of second-order hyperbolic equations with Dirichlet-boundary datum. Our frequent visits over the years to the mathematically rewarding environment of the Scuola Normale Superiore di Pisa, Italy, and the stimulating exchange of correspondence with J. L. Lions on a priori inequalities of wave equations are also greatly appreciated. Special thanks for reading portions of the manuscript and offering their comments are due to P. Acquistapace (Chapter 1), A. Ichikawa (Chapter 6), S. K. Chang (Chapter 10), and, particularly, to L. Pandolfi, for his insight on the entire first draft of these two Volumes.

In addition, we wish to express our deep appreciation and gratitude to Ms. J. Riddleberger for her superb, accurate, fast, and dedicated work in typing the various drafts of the chapters of these volumes; for keeping track of them; for safeguarding the resulting disks, etc.; and, beyond all that, for typing many of our papers that have

entered these volumes. Since joining the University of Virginia, our research output would have been much hindered without her highly professional services.

It is a pleasure to acknowledge the understanding and assistance in this project of Ms. Lauren Cowles, Editor of Mathematics and Computer Science at Cambridge University Press.

During the many years in which these volumes were written, along with the papers on which they are based, our work has been continuously sponsored by the National Science Foundation, Division of Mathematical Sciences, whose support is greatly appreciated. In the early years, our work was also supported by the Air Force Office of Scientific Research. Over the past few years, we were fortunate to receive much appreciated support by the Army Research Office, in the program managed by Dr. L. Bushnell. Occasional sponsorship over the years by the Italian Consiglio Nazionale delle Ricerche and by the Scuola Normale Superiore di Pisa is acknowledged with gratitude.

0

Background

Throughout this treatise, Y (state space), U (control space), Z (observation space), and Z_f (final state observation space) are (separable) Hilbert spaces. Moreover, $A : Y \supset \mathcal{D}(A) \to Y$ is the infinitesimal generator of a strongly continuous (s.c.) semigroup e^{At}, $t \geq 0$ on Y.

Following the pioneering work of Kalman in 1960 in the finite-dimensional (matrix) case, the Linear Quadratic (LQ) optimal control problem, and related Riccati theory, for the dynamics

$$\dot{y} = Ay + Bu, \qquad y(0) = y_0 \in Y, \tag{0.1}$$

with quadratic cost functional

$$J(u, y) \equiv \int_0^T \left[\|Ry(t)\|_Z^2 + \|u(t)\|_U^2\right] dt + \|Gy(T)\|_{Z_f}^2 \tag{0.2}$$

to be minimized over all $u \in L_2(0, T; U)$, has already received several treatments in book form, in the most amenable case where the control operator B, the observation operator R, and the final state operator G are all bounded:

$$B \in \mathcal{L}(U; Y); \quad R \in \mathcal{L}(Y; Z); \quad G \in \mathcal{L}(Y; Z_f). \tag{0.3}$$

In (0.2), T may be finite: $0 < T < \infty$, or else infinite: $T = \infty$, in which case we take $G = 0$. See, for example, J. L. Lions [1970], Curtain and Pritchard [1978], Balakrishnan [1981], Bensoussan et al. [1993], and Curtain and Zwart [1995] for a sample of more recent books. Thus, this preliminary topic, under assumptions (0.3), in particular, with *distributed* control (i.e., with $B \in \mathcal{L}(U; Y)$), is well covered in book form. Hence, we shall not pursue it here directly, but rather refer to any of the above references. The treatment in [Balakrishnan, 1981, Chapter 5] and the two-volume treatise [Bensoussan et al., 1993] are the most appropriate prerequisite for the present treatise, along with the Partial Differential Equations–oriented book [Lions, 1970]. Reference [Bensoussan et al., 1993] is also the most useful as a complementary work for topics such as periodic control, strict and classical solutions of the Riccati equations when

$B \in \mathcal{L}(U; Y)$, which are not touched upon in the present treatise, and also for its emphasis on Da Prato's *direct method* on the study of the Riccati equations, followed by the dynamic programming approach to solve the optimal control problem. In contrast, the present treatise will emphasize the more powerful, reverse approach, the variational approach, from the control problem to the Riccati equations pursued by the authors.

Moreover, references [Bensoussan et al., 1993] and [Lions, 1970] are, by far, the most mathematically advanced of the above group, as they set the focus on an in-depth, modern treatment of general PDEs (Partial Differential Equations), which are also the deliberate objective of the present treatise.

Instead, the massive recent reference [Curtain, Zwart, 1995] is useful on a different part of the spectrum, as an introductory textbook, which seeks to present a link between the finite-dimensional theory and the most amenable part of an "infinite-dimensional theory," that is, the one that deals with all bounded operators, except for the free dynamics operator. The corresponding theory is then illustrated by a wealth of detailed, much-appreciated examples, typically involving the basic heat and wave equations in one dimension with distributed control, as well as delay-differential equations.

By contrast, in the present treatise, we shall give emphasis throughout to the case where the control operator B is genuinely and "fully unbounded," such as it arises in the study of *boundary* or *point control* problems for general PDEs. While our approach is abstract, the setting is motivated by, and ultimately directed to, concrete classes of PDEs on a general multidimensional domain. Transition from the abstract to the concrete and conversely requires full familiarity with modern PDE theories and treatments. These will be assumed here as *prerequisites*. [The abstract treatment of the present books will naturally include, as a very special case, other differential dynamics, such as functional differential equations (hereditary or delay-equations), with delay in the state and/or control. However, we shall not explicitly focus on this functional differential subclass in our applications. As abstract systems, functional differential equations are far more amenable than any class of PDEs with boundary/point control. However, as delay-differential equations are defined in terms of matrices and "delay" constants, the final results need to be expressed in terms of these data. Thus, they deserve a special treatment of their own; see Manitius [1976] and [Bensoussan et al., 1993, Vol. I, Chapter 4] for two book-form expositions, along with [Curtain, Zwart, 1995, and many specialized articles.] Accordingly, in the present treatise, all the applications of the abstract theory – continuous as well as numerical theory – will refer to mixed (initial-boundary value) problems for PDEs. More precisely, the concrete PDE classes – parabolic, hyperbolic/Petrowski-type mixed problems – will motivate, and lead to, corresponding abstract mathematical settings in the chapters below.

In Volume I (Chapters 1 through 6), we shall consider the class of "abstract parabolic" problems. This is characterized by the property that the s.c. (free dynamics) semigroup e^{At} is *analytic* on Y, $t > 0$. Chapter 6 extends problem (0.1), (0.2) to a min–max game theory problem, with indefinite cost and to the indefinite cost optimal control case. Li and Yong [1995, Chapter 9] have recently studied the optimal control problem for abstract parabolic equations, with R^*R replaced by a nonnecessarily positive semidefinite operator, along with other mathematically advanced optimal control topics. This book [Li, Yong, 1995] also serves as an attractive companion to the present treatise, not least because Chapter 9 of [Li, Yong, 1995] on the LQ-problem adopts the abstract approach of the present authors of their original papers, which is the thrust of the present treatise.

In Volume II (with Chapters 7 through 10) pertaining to the optimal control problem for (0.1), (0.2) over a finite time interval, $T < \infty$, we shall consider mixed hyperbolic and Petrowski-type PDE problems as well.

0.1 Some Function Spaces Used in Chapter 1

To facilitate the reading of the statements of Theorems 1.2.1.1, 1.2.2.1, and 1.2.2.2 of Chapter 1, we shall list here for convenience a few function spaces used there.

If X is a Hilbert space and r any real number, then:

(i) ${}_rC([s,T];X)$ denotes the Banach space defined by

$$
\begin{aligned}
&{}_rC([s,T];X)\\
&\equiv \left\{ f(t) \in C((s,T];X) : \|f\|_{{}_rC([s,T];X)} = \sup_{s<t\leq T} (t-s)^r \|f(t)\|_X < \infty \right\};
\end{aligned}
\tag{0.4}
$$

(ii) $C_r([s,t];X)$ denotes the Banach space defined by

$$
\begin{aligned}
&C_r([s,T];X)\\
&\equiv \left\{ f(t) \in C([s,T);X) : \|f\|_{C_r([s,T];X)} = \sup_{s\leq t<T} (T-t)^r \|f(t)\|_X < \infty \right\}.
\end{aligned}
\tag{0.5}
$$

(iii) If r_1 and r_2 are real numbers, then ${}_{r_1}C_{r_2}([0,T];X)$ denotes the Banach space defined by

$$
\begin{aligned}
{}_{r_1}C_{r_2}([0,T];X) \equiv \Big\{ f(t) \in C((0,T);X) : &\|f\|_{{}_{r_1}C_{r_2}([0,T];X)}\\
&= \sup_{0<t<T} t^{r_1}(T-t)^{r_2}\|f(t)\|_X < \infty \Big\}.
\end{aligned}
\tag{0.6}
$$

The above spaces measure, when $r,\ r_1,\ r_2 > 0$, the singularity of $f(t)$, as $t \to T$ or $t \to 0$.

0.2 Regularity of the Variation of Parameter Formula When e^{At} Is a s.c. Analytic Semigroup

Prerequisites to the present treatise include the *general theory of s.c. semigroups*, and operator theory in general, which is available in numerous excellent references, some of which are cited throughout this exposition, as needed. However, to facilitate the reading of the abstract parabolic Volume I, we shall collect here a few well-known results (which are scattered in the literature) that are often invoked in Volume I.

Proposition 0.1 *Let $A : Y \supset \mathcal{D}(A) \to Y$ be the infinitesimal generator of a s.c. analytic semigroup e^{At} on the Hilbert space Y, $t > 0$. Then, the Cauchy problem*

$$\dot{y} = Ay + f, \qquad y(0) = 0 \tag{0.7}$$

admits the following regularity properties, for any $0 < T < \infty$:

$$y(t) = (Lf)(t) = \int_0^t e^{A(t-\tau)} f(\tau)\, d\tau: \tag{0.8}$$

$$(i)\ \text{continuous } L_2(0, T; Y) \to L_2(0, T; \mathcal{D}(A)), \tag{0.9}$$

$$(ii)\ \text{continuous } L_2(0, T; Y) \to C\big([0, T]; [\mathcal{D}(A), Y]_{\frac{1}{2}}\big), \tag{0.10}$$

$$(iii)\ \text{continuous } L_2(0, T; Y) \to C\big([0, T]; \mathcal{D}\big((-A)^{\frac{1}{2}-\epsilon}\big)\big), \quad \forall\, \epsilon > 0, \tag{0.11}$$

$$(iv)\ \text{continuous } C([0, T]; Y) \to C\big([0, T]; \mathcal{D}\big((-A)^{1-\epsilon}\big)\big), \quad \forall\, \epsilon > 0 \tag{0.12}$$

[assuming, in (0.11) and (0.12), that the fractional powers are well-defined],

$$(v)\ \text{continuous } L_p(0, T; Y) \to L_p(0, T; \mathcal{D}(A)), \quad 1 < p < \infty, \tag{0.13}$$

generalizing (0.9).

Moreover, via (0.7), property (0.9) is equivalent to the following regularity properties:

$$f \to Ay, \dot{y} : \text{continuous } L_2(0, T; Y) \to L_2(0, T; Y). \tag{0.14}$$

0.2.1 Comments on the Space $[X, Y]_{\frac{1}{2}}$

In (0.10), $[X, Y]_{\frac{1}{2}}$ is the complex interpolation, or intermediate space, geometrically defined as in [Lions, Magenes, 1972, Vol. 1, Eqn. (2.7), p. 10], for X and Y Hilbert spaces, $X \subset Y$, X dense in Y with continuous injection, by means of domains of positive self-adjoint operations (or, equivalently, by the "complex interpolation method" as in [Lions, Magenes, 1972, Vol. 1, Section 14, pp. 91–94]). In fact, this definition is all that is needed in the Hilbertian setting of the present treatise. However, to embed this definition in the broader setting of interpolation theory, we recall that an equivalent definition, which is valid also in the Banach setting, is given in [Lions, Magenes, 1972, Vol. 1, Chapter 1, Section 15, p. 98] by the "real interpolation method" and leads to the spaces of averages $(X, Y)_{\theta,p}$ [Triebel, 1978]. When X and Y are Hilbert, and $p = 2$, then $[X, Y]_\theta = (X, Y)_{\theta,2}$ [Triebel, 1978, Remarks 3 and 4, p. 143]. A useful summary of this theory is given in [Bensoussan et al., 1993, Chapter 1, Section 4].

0.2.2 Cases Where $[\mathcal{D}(A), Y]_{\frac{1}{2}} = \mathcal{D}((-A)^{\frac{1}{2}})$

With reference to (0.10), we have that

$$[\mathcal{D}(A), Y]_{\frac{1}{2}} = \mathcal{D}\big((-A)^{\frac{1}{2}}\big); \text{ indeed, } \quad [\mathcal{D}(A), Y]_{1-\theta} = \mathcal{D}(A^{\theta}), \quad 0 < \theta < 1 \tag{0.15a}$$

(equivalent norms), in the following explicit and progressively more general cases:

(a) when A is a strictly negative, self-adjoint operator (in which case e^{At} is of negative type, and one may take $T = \infty$ in (0.9), (0.10), (0.14));

(b) when A is a normal generator, or it possesses a Riesz basis, on Y, and the fractional powers are well defined;

(c) when the closed operator A is maximal dissipative and $A^{-1} \in \mathcal{L}(Y)$ [Bensoussan et al., 1993, Proposition 6.1, p. 113]; thus identity (0.15) holds true for any generator of a s.c. contraction semigroup with $A^{-1} \in \mathcal{L}(Y)$. See Theorem A.2 in Appendix A of Chapter 3;

(d) when the closed operator $(-A)$ is positive, in the sense of [Triebel, 1978, Definition 1.14.1, p. 91], (in particular, A is the generator of a s.c. semigroup of negative type, by the Hille–Yosida theorem) and $(-A)$ has, locally, bounded imaginary powers: There exist two positive numbers ϵ and C such that $\|(-A)^{it}\|_{\mathcal{L}(Y)} \leq C$, for $-\epsilon \leq t \leq \epsilon$. For a proof that (0.15a) then holds true in this case, we refer to [Triebel, 1978, Theorem 1.15.3, p. 103]. One may also give a proof of (0.15a) by following the arguments of [Lions, Magenes, 1972, Theorem 14.1, p. 92], where one replaces the positive self-adjoint operator Λ in that proof with the present positive operator $(-A)$. Once (0.15a) is established, one obtains as a consequence, via the reinterpolation theorem, still for $0 < \theta < 1$:

$$[\mathcal{D}(A^{\alpha}), \mathcal{D}(A^{\beta})]_{\theta} = \mathcal{D}(A^{\gamma}), \quad \gamma = \alpha(1-\theta) + \beta\theta, \quad 0 \leq \alpha < \beta. \tag{0.15b}$$

The technical condition of local boundedness of imaginary powers is actually satisfied, in a general L_p context, in many cases of relevant classes of operators, including differential operators. The classes include:

(i) realizations of an elliptic differential operator, whose domain is defined by well-posed boundary conditions, with smooth coefficients [Seeley, 1971], [Fujiwara, 1969];

(ii) second-order elliptic operators with Holder-continuous coefficients [Prüss, Sohr, 1990];

(iii) negative generators of special semigroups [Clement, Pruss, 1990];

(iv) vector-valued ordinary differential operators of general order [Fuhrman, 1992];

(v) Stokes operator [Prüss, Sohr, 1991, Section 2];

(vi) m-dissipative operators A in Hilbert space with null space $\mathcal{N}(A) = 0$ [Prüss, Sohr, 1991, Section 2], using Nagy–Foias functional calculus [Sz-Nagy, Foias, 1970]; etc.

Cases of unbounded domains are also included. See also [Triebel, 1978].

0.2.3 Comments on the Proof of Proposition 0.1

Properties (0.9), (0.14)

Property (0.9) is most readily proved (in fact, with $T = \infty$), in the case where e^{At} is a s.c. analytic semigroup of negative type, by using: the Laplace (or Fourier) transform, the well-known λ-characterization $\|\lambda R(\lambda, A)\| \leq$ const of analyticity of the semigroup, and the Plancherel theorem. See [Lasiecka, 1980, Appendix A] for details, which are also reproduced in the proof of [Bensoussan et al., 1993, Lemma 3.3, p. 78]. This way, (0.9), and hence (0.14), are proved.

Property (0.10)

To prove property (0.10), one simply uses the "intermediate derivative theorem" [Lions, Magenes, 1972, Vol. 1, Theorem 2.3, p. 15] between $y \in L_2(0, T; \mathcal{D}(A))$ and $\dot{y} \in L_2(0, T; Y)$, previously established, to obtain (0.10).

Properties (0.11), (0.12)

These are easy and can be proved directly by making use of the analyticity property $\|(-A)^\theta e^{At}\| \leq c_T t^{-\theta}$, $0 < \theta < 1$, in the $\mathcal{L}(Y)$-norm, specialized to $\theta = 1/2 - \epsilon$ (L_2-kernel), and to $\theta = 1 - \epsilon$ (L_1-kernel), respectively.

Properties (0.13)

This is a highly nontrivial generalization of the case $p = 2$ in (0.9) and can be proved by using singular integrals [de Simon, 1964].

Remark 0.1 An additional case where *property* (0.11) *holds true with* $\epsilon = 0$ is when

$$\mathcal{D}\big((-A)^{\frac{1}{2}}\big) = \mathcal{D}\big((-A^*)^{\frac{1}{2}}\big).$$

Equivalently, one proves that $f \in L_2(0, T; [\mathcal{D}((-A)^{\frac{1}{2}})]')$ yields $y \in C([0, T]; Y)$ for (0.7). This is done by taking the Y-inner product of (0.7) with y, integrating over $[0, t]$, and using property (0.9), thus obtaining $y \in L_\infty(0, T; Y)$. By approximation, this is then boosted to $y \in C([0, T]; Y)$.

0.3 The Extrapolation Space $[\mathcal{D}(A^*)]'$

Let Y be a Hilbert space, as in this treatise, and let $A : Y \supset \mathcal{D}(A) \to Y$ be a closed operator, which is boundedly invertible: $A^{-1} \in \mathcal{L}(Y)$. Then $Y_1 \equiv \mathcal{D}(A)$ is a Hilbert space under the norm $\|x\|_{\mathcal{D}(A)} = \|Ax\|_Y$, which is equivalent to the graph norm. Set

$$Y_{-1} \equiv \text{completion of } Y \text{ under the (weaker) norm } \|A^{-1}x\|_Y. \tag{0.16}$$

Then, $Y_1 \subset Y \subset Y_{-1}$, with dense and continuous embeddings. We note that A is an isomorphism of $\mathcal{D}(A)$ onto Y, and A extends to an isomorphism of Y onto Y_{-1}. Moreover, the Y-adjoint A^* is an isomorphism from $\mathcal{D}(A^*)$ onto Y. By transposition, $(A^*)^*$ is an isomorphism of Y onto $[\mathcal{D}(A^*)]'$ where duality is with respect to the pivot space Y. Then, thanks to $(Ax, y)_Y = (x, A^*y)_Y$, we have $(A^*)^*x = Ax$, for $x \in \mathcal{D}(A)$, and $(A^*)^*$ is an *extension* of A, which we shall continue to denote by A: $(A^*)^* = A$: isomorphism from Y onto $[\mathcal{D}(A^*)]'$. Thus, we have $Y_{-1} = [\mathcal{D}(A^*)]'$. Throughout this treatise, we shall use the notation $\mathcal{D}(A)$ and $[\mathcal{D}(A^*)]'$ as in [Lions, Magenes, 1972], instead of Y_1 and Y_{-1}. The space $Y_{-1} = [\mathcal{D}(A^*)]'$ is an instance of an *extrapolation space* generated by A. A general theory for extrapolation spaces in a Banach space setting (replacing the Hilbert space Y) was introduced by Da Prato and Grisvard [1982; 1984] in 1982. See also Da Prato [1983].

0.4 Abstract Setting for Volume I. The Operator L_T in (1.1.9), or L_{sT} in (1.4.1.6), of Chapter 1

With reference to the abstract analytic (parabolic) class of Volume I (Chapters 1 through 6), our setting will essentially be as follows (modulo a translation of the generator):

(a) Let $(-A) : Y \supset \mathcal{D}(-A) \to Y$ (note the deliberate change of sign with respect to (0.7), and the remainder of this treatise, as well) be the generator of a s.c. analytic semigroup e^{-At} on Y, whose fractional powers A^θ, $0 < \theta < 1$, are well defined;

(b) If $[\ ]'$ denotes duality with respect to the pivot space Y, let, for some $0 < \gamma < 1$:

$$B \in \mathcal{L}(U; [\mathcal{D}(A^{*\gamma})]'), \quad \text{or} \quad \text{equivalently, } B_1 \equiv A^{-\gamma} B \in \mathcal{L}(U; Y), \tag{0.17}$$

where $[\mathcal{D}(A^{*\gamma})]'$ is an extrapolation space (see point 0.3 above). We consider

$$y(t) = e^{-At} y_0 + (Lu)(t); \tag{0.18}$$

$$(Lu)(t) = \int_0^t e^{-A(t-\tau)} Bu(\tau)\, d\tau = \int_0^t A^\gamma e^{-A(t-\tau)} B_1 u(\tau)\, d\tau \tag{0.19a}$$

$$: \text{continuous } L_2(0, T; U) \to L_2(0, T; \mathcal{D}(A^{1-\gamma})), \tag{0.19b}$$

by (0.9), recalling (0.17). The $L_2(0, T; \cdot)$-adjoint L^* of L in (0.19) is

$$(L^* f)(t) = \int_t^T B^* e^{-A^*(\tau-t)} f(\tau)\, d\tau = \int_t^T B_1^* A^{*\gamma} e^{-A^*(\tau-t)} f(\tau)\, d\tau \tag{0.20a}$$

$$: \text{continuous } L_2(0, T; [\mathcal{D}(A^{1-\gamma})]') \to L_2(0, T; U). \tag{0.20b}$$

It readily follows from (0.19) (as in (0.12)) that, a fortiori,

$$u \in C([0, T]; U) \to Lu \in C([0, T]; Y). \tag{0.21}$$

However, if $1/2 \le \gamma < 1$ and $u \in L_2(0, T; U)$, then $(Lu)(t)$ in (0.19) need *not* be in $C([0, T]; Y)$ in general; see an explicit, and readily constructed, counterexample (e.g., in [Li, Yong, 1995, p. 367]). Thus, in particular, $(Lu)(T)$ is generally nonmeaningful as an element of Y. There are two approaches that may be pursued.

(a) We may require that the time $t = T$ be a Lebesgue point for $(Lu)(t)$ and accordingly define

$$\begin{cases} \tilde{L}_T u \equiv \lim_{r \downarrow 0} \dfrac{1}{r} \displaystyle\int_{T-r}^{T} (Lu)(t)\, dt, \quad u \in \mathcal{D}(\tilde{L}_T); & (0.22) \\ \mathcal{D}(\tilde{L}_T) = \left\{ u \in L_2(0, T; U) : \lim_{r \downarrow 0} \dfrac{1}{r} \displaystyle\int_{T-r}^{T} (Lu)(t)\, dt \text{ exists in } Y \right\}. & (0.23) \end{cases}$$

Recalling (0.21), and recalling that if T is a point of continuity then a fortiori T is a Lebesgue point, we see that $C([0, T]; U) \subset \mathcal{D}(\tilde{L}_T)$, and so $\mathcal{D}(\tilde{L}_T)$ is dense in $L_2(0, T; U)$. The adjoint operator $\tilde{L}_T^*$ of $\tilde{L}_T$ is given by (recalling $B_1 = A^{-\gamma} B \in \mathcal{L}(U; Y)$ from (0.17):

$$\begin{cases} (\tilde{L}_T^* y)(t) = B^* e^{-A^*(T-t)} y = B_1^* A^{*\gamma} e^{-A^*(T-t)} y; & (0.24) \\ \mathcal{D}(\tilde{L}^*) = \{ y \in Y : \tilde{L}_T^* y \in L_2(0, T; U) \} \begin{cases} = Y & \text{if } \gamma < \frac{1}{2}, \\ \supset \mathcal{D}(A^{*\gamma}) & \text{if } \frac{1}{2} \le \gamma < 1, \end{cases} & (0.25) \end{cases}$$

Since $\tilde{L}_T^*$ is densely defined, then $\tilde{L}_T$ is closable [Kato, 1996, p. 168]. The closure $\bar{\tilde{L}}_T$ of $\tilde{L}_T$ is the operator

$$\begin{cases} \bar{\tilde{L}}_T u = A^{\gamma - (\frac{1}{2} - \epsilon)} \displaystyle\int_0^T A^{\frac{1}{2} - \epsilon} e^{-A(T-t)} A^{-\gamma} B u(\tau)\, d\tau, & (0.26) \\ \mathcal{D}(\bar{\tilde{L}}_T) = \left\{ u \in L_2(0, T; U) : \displaystyle\int_0^T A^{\frac{1}{2} - \epsilon} e^{-A(T-t)} A^{-\gamma} B u(\tau)\, d\tau \right. \\ \qquad \left. \in \mathcal{D}\left(A^{\gamma - (\frac{1}{2} - \epsilon)}\right) \right\}, & (0.27) \end{cases}$$

where $\epsilon > 0$ and the integral in (0.26), or (0.27), is well defined by (0.11).

(b) Recalling (0.21), one may require more and set

$$\begin{cases} L_T u = \lim_{t \uparrow T} (Lu)(t) = \lim_{t \uparrow T} \displaystyle\int_0^t e^{-A(t-\tau)} B u(\tau)\, d\tau, & (0.28) \\ \mathcal{D}(L_T) = \left\{ u \in L_2(0, T; U) : \lim_{t \uparrow T} (Lu)(t) \text{ exists in } Y \right\}. & (0.29) \end{cases}$$

By (0.21), $C([0, T]; U) \subset \mathcal{D}(L_T)$ and thus L_T is *densely defined* in $L_2(0, T; U)$. The adjoint L_T^* of L_T is the same as $\tilde{L}_T^*$ in (0.24), (0.25): $L_T^* \equiv \tilde{L}_T^*$. Consider the following operator

$$\begin{cases} A^{\gamma} K_T u, & (0.30) \\ \mathcal{D}(A^{\gamma} K_T) = \Big\{ u \in L_2(0, T; U) : K_T u \\ \qquad \equiv \int_0^T e^{-A(T-t)} A^{-\gamma} B u(\tau)\, d\tau \in \mathcal{D}(A^{\gamma}) \Big\}. & (0.31) \end{cases}$$

Notice that $A^{\gamma} K_T$ is closed, being the composition of the bounded operator K_T in (0.31), followed by the closed operator A^{γ} with bounded inverse $A^{-\gamma} \in \mathcal{L}(Y)$ [Kato, 1996, p. 164]. Moreover, we have

$$L_T u = A^{\gamma} K_T u, \quad u \in \mathcal{D}(L_T), \tag{0.32}$$

so that L_T is closed and $A^{\gamma} K_T$ is an extension of L_T. To establish (0.32), one way is to apply $A^{-\gamma} \in \mathcal{L}(Y)$ to both sides of definition (0.28) with $u \in \mathcal{D}(L_T)$, move $A^{-\gamma}$ inside the limit and the integral, and obtain

$$\begin{aligned} A^{-\gamma} L_T u &= \lim_{t \uparrow T} \int_0^t e^{-A(t-\tau)} A^{-\gamma} B u(\tau)\, d\tau \\ &= \lim_{t \uparrow T} K_t u = K_T u \in Y, \quad u \in \mathcal{D}(L_T), \end{aligned} \tag{0.33}$$

where K_T is defined in (0.32), and where

$$K_t u \equiv \int_0^t e^{-A(t-\tau)} A^{-\gamma} B u(\tau)\, d\tau \in C([0, T]; Y). \tag{0.34}$$

Then (0.33) leads to (0.32).

In the treatment of the optimal control problem in Chapter 1, one may pursue either one of the above approaches (a) or (b). We shall explicitly follow (b) and use definition (0.28), (0.29) for L_T.

References and Bibliography

A. V. Balakrishnan, *Applied Functional Analysis*, Springer-Verlag, 1981.

A. Bensoussan, G. Da Prato, M. Delfour, and S. Mitter, *Representation and Control of Infinite Dimensional Systems*, Vols. I and II, Birkhäuser, 1993.

Ph. Clement and J. Pruss, Completely positive measures and Feller semigroups, *Math. Ann.* **287** (1990), 73–105.

R. F. Curtain and A. J. Pritchard, *Infinite Dimensional Linear Systems Theory*, Lecture Notes in Control and Information Sciences, Vol. 8, Springer-Verlag, 1978.

R. F. Curtain and H. J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer-Verlag, 1995.

G. Da Prato, Abstract differential equations and extrapolation spaces, in *Infinite Dimensional Systems*, Retzhof 1983, Springer-Verlag Lecture Notes in Mathematics #1076, 53–61.

G. Da Prato and P. Grisvard, On extrapolation spaces, *Rendiconti Accademia Nazionale dei Lincei, Classe di Scienze fisiche, matematiche, naturali* **LXXII** (1982), 330–332.

G. Da Prato and P. Grisvard, Maximal regularity for evolution equations by interpolation and extrapolation, *J. Funct. Anal.* **58** (1984), 107–124.

L. de Simon, Un' applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineari astratte del primo ordine, *Rend. Sem. Matem. Unversitá di Padova* **34** (1964), 205–223.

M. Fuhrman, Bounded imaginary powers of abstract differential operators, Report of Politecnico di Milano, 1992.

D. Fujiwara, Fractional powers of second order elliptic operators, *J. Math. Soc. Japan* **21** (1969), 481–522.

R. E. Kalman, Contributions to the theory of optimal control, *Bol. Soc. Matem. Mex.* **5** (1960), 102–119.

T. Kato, *Perturbation Theory of Linear Operators*, Springer-Verlag, 1996.

I. Lasiecka, Unified theory for abstract parabolic boundary problems: A semigroup approach, *Appl. Math. Optim.* **6** (1980), 287–333.

J. L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, 1970.

J. L. Lions and E. Magenes, *Nonhomogeneous Boundary Value Problems and Applications*, Vol. 1, Springer-Verlag, 1972.

X. Li and J. Yong, *Optimal Control Theory for Infinite Dimensional Systems*, Birhhäuser, 1995.

A. Manitius, Optimal control of hereditary systems, in *Control Theory and Topics in Functional Analysis*, Vol. 3, pp. 43–178, International Atomic Energy Agency, Vienna, 1976.

Sz-Nagy and B. Foias, *Harmonic Analysis of Operators in Hilbert Space*, North Holland, Amsterdam, 1970.

J. Prüss and H. Sohr, On operators with bounded imaginary powers in Banach spaces, *Math. Zeit.* **203** (1990), 429–452.

J. Prüss and H. Sohr, Imaginary powers of elliptic second order differential operators in L^p-spaces, 1991.

R. Seeley, Norms and domains of the complex powers A^z_B, *Am. J. Math.* **93** (1971), 299–309.

H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland Publishing Company, 1978.

1

Optimal Quadratic Cost Problem Over a Preassigned Finite Time Interval: Differential Riccati Equation

This chapter studies the optimal quadratic cost problem over a fixed preassigned time interval $[0, T]$, $T < \infty$, with nonsmoothing terminal condition at $t = T$, for a linear abstract dynamics $\dot{y} = Ay + Bu$ in Hilbert space. The free dynamics operator A generates a s.c. analytic semigroup, while the control operator B has a degree of unboundedness up to that of A. The variational approach of this chapter provides, constructively, an explicit solution (Riccati operator) to the corresponding operator differential Riccati equation, which arises in the synthesis of the optimal pair (optimal control and optimal trajectory). More precisely, (i) first the optimal pair $\{u^0, y^0\}$ is characterized solely in terms of the data of the problem; (ii) next, an operator $P(t), 0 \leq t \leq T$, is constructed in terms of the original and optimal evolution, hence ultimately in terms of the original data of the problem; (iii) finally, the operator $P(t)$ is shown to satisfy the differential Riccati equation and its limiting condition as $t \uparrow T$. The results are sharp as illustrated by counterexamples. Regularity properties of all the quantities involved are also given. In the general case, these may display a singularity of the terminal time $t = T$, which can be measured quantitatively. Uniqueness of the solution to the differential Riccati equation (i.e., the constructed Riccati operator) is then asserted under some additional assumptions of smoothing by the operator of penalization of the terminal condition, in which case all the relevant quantities become more regular; in particular, (i) the optimal trajectory becomes continuous also at the terminal time $t = T$, and, as a consequence, (ii) the differential Riccati equation is satisfied in the classical sense. Finally, if the smoothing property of the operator of penalization of the terminal condition is further increased, then the optimal control as well becomes continuous at $t = T$.

Applications of the abstract theory include: (i) parabolic equations with Dirichlet or Neumann (Robin) boundary control, or else with point control, and (ii) platelike equations with a high degree of internal damping with boundary/point control. These illustrations will be analyzed in Chapter 3.

1.1 Mathematical Setting and Formulation of the Problem

Dynamical Model In this chapter, we consider the following abstract differential equation:

$$\dot{y} = Ay + Bu \text{ on, say, } [\mathcal{D}(A^*)]', \quad y(0) = y_0 \in Y, \tag{1.1.1}$$

subject to the following assumptions, to be maintained throughout the chapter.

(i) A is the infinitesimal generator of a strongly continuous analytic semigroup, denoted by e^{At}, on the Hilbert space Y. Without loss of generality for the problem here considered, where the dynamics (1.1.1) is studied over a finite interval $[0, T]$, $T < \infty$, we may assume that the semigroup is of negative type so that A is boundedly invertible, that is, $A^{-1} \in \mathcal{L}(Y)$. Then, the fractional powers $(-A)^\theta, 0 < \theta < 1$, are well defined, and we have $\|(-A)^\theta e^{At}\|_{\mathcal{L}(Y)} \leq \mathcal{O}_T(t^{-\theta})$, $0 < t$ [Pazy, 1983, p. 74], a property to be used freely in Volume 1, where $\mathcal{O}_T$ denotes a constant depending on T.

(ii) B is a linear continuous operator: $U = \mathcal{D}(B) \to [\mathcal{D}(A^*)]'$, where U is another Hilbert space (B is generally unbounded as an operator from U to Y), such that

$$A^{-\gamma}B \in \mathcal{L}(U; Y) \quad \text{or} \quad \|A^{-\gamma}B\|_{\mathcal{L}(U;Y)} = \|B^*A^{*-\gamma}\|_{\mathcal{L}(Y;U)} \leq c_\gamma$$
$$\text{for some fixed } \gamma, \quad 0 \leq \gamma < 1, \tag{1.1.2}$$

where $(Bu, v)_Y = (u, B^*v)_U$. Generally, dependence on γ will not necessarily be explicitly noted in this Volume 1. In (1.1.1), A^* is the Y-adjoint of A, and $[\mathcal{D}(A^*)]'$ denotes the dual space of $D(A^*)$ with respect to the Y-inner product, so that $\|y\|_{[\mathcal{D}(A^*)]'} = \|A^{-1}y\|_Y$.

Remark 1.1.1 As will be pointed out in Chapter 3, in the case of a second-order parabolic equation defined on a bounded domain $\Omega \subset R^n$, the relevant values of the constant γ are as follows: $\gamma = 3/4 + \epsilon, \forall \epsilon > 0$, for Dirichlet boundary control with $U = L_2(\Gamma), Y = L_2(\Omega)$, or else for Neumann boundary control with $U = L_2(\Gamma)$ and $Y = H^1(\Omega)$; while $\gamma = 1/4 + \epsilon$, for the Neumann boundary control with $U = L_2(\Gamma)$ and $Y = L_2(\Omega)$. For a more detailed discussion of these and other examples of partial differential equations with boundary/point control that fit into the present abstract theory, we refer to Chapter 3.

Optimal Control Problem With the dynamics (1.1.1), we associate the following quadratic functional cost over a preassigned fixed time interval $[0, T]$, $0 < T < \infty$:

$$J(u, y) \equiv \int_0^T \left[\|Ry(t)\|_Z^2 + \|u(t)\|_U^2\right] dt + \|Gy(T)\|_{Z_f}^2, \tag{1.1.3}$$

where in (1.1.3), $y(t) = y(t; y_0)$ and, because of the term $Gy(T)$, the functional (1.1.3) is defined initially only for those u where it makes sense (say, $u \in \mathcal{D}(L_T)$

defined in (1.1.9) below); Z, Z_f are other Hilbert spaces; and

$$R \in \mathcal{L}(Y; Z) \quad \text{and} \quad G \in \mathcal{L}(Y; Z_f). \tag{1.1.4}$$

The corresponding optimal control problem is

$$\begin{aligned} &\text{minimize } J(u, y) \text{ over all } u \in L_2(0, T; U), \text{ where } y \\ &\qquad\qquad\text{is the solution of (1.1.1) due to } u. \end{aligned} \tag{1.1.5}$$

Preliminaries The solution to (1.1.1) is

$$y(t) = e^{At} y_0 + (Lu)(t), \tag{1.1.6}$$

$$(Lu)(t) = \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau \tag{1.1.7a}$$

$$: \text{ continuous } L_2(0, T; U) \to L_2(0, T; \mathcal{D}((-A)^{1-\gamma})) \tag{1.1.7b}$$

by (1.1.2) and Eqn. (0.9) of Chapter 0. The adjoint operator L^* of L is defined by $(Lu, f)_{L_2(0,T;Y)} = (u, L^* f)_{L_2(0,T;U)}$ and is explicitly given by

$$(L^* f)(t) = \int_t^T B^* e^{A^*(\tau - t)} f(\tau)\, d\tau \tag{1.1.8a}$$

$$: \text{ continuous } L_2(0, T; [\mathcal{D}((-A)^{1-\gamma})]') \to L_2(0, T; U), \tag{1.1.8b}$$

and likewise by (1.1.2) or (1.1.7b). We recall now [Chapter 0, Section 0.2] that for $\gamma \geq 1/2$ and for $u \in L_2(0, T; U)$, we have $(Lu)(t) \in L_2(0, T; Y)$, and so $(Lu)(t)$ has values in Y only a.e. Thus, the evaluation $(Lu)(T)$ at $t = T$ is generally not meaningful as an element of Y. Therefore, we set (see (1.4.1.6) below)

$$L_T u \equiv \lim_{t \uparrow T} \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau = \lim_{t \uparrow T} (Lu)(t), \tag{1.1.9a}$$

$$\mathcal{D}(L_T) = \left\{ u \in L_2(0, T; U) : \lim_{t \uparrow T} (Lu)(t) \text{ exists in } Y \right\}. \tag{1.1.9b}$$

Since

$$u \in H^1(0, T; U) \quad \text{or} \quad u \in C([0, T]; U) \Rightarrow Lu \in C([0, T]; Y),$$

we have that $\mathcal{D}(L_T)$ is dense in $L_2(0, T; U)$. Moreover, L_T is a closed operator, see Chapter 0 [below Eqn. (0.32)]. Its adjoint L_T^*, defined by $(L_T x, y)_Y = (u, L_T^* y)_{L_2(0,T;U)}$ is the closed operator

$$\{L_T^* y\}(t) = B^* e^{A^*(T-t)} y, \quad 0 \leq t \leq T, \ \ y \in \mathcal{D}(L_T^*), \tag{1.1.10a}$$

$$\mathcal{D}(L_T^*) = \{ y \in Y : L_T^* y \in L_2(0, T; U) \}. \tag{1.1.10b}$$

1.2 Statement of Main Results

1.2.1 The Nonsmoothing Case. Theorem 1.2.1.1: Existence of a Riccati Operator

The main result of this chapter is the following theorem, which refers to the general case where the operator G of the terminal condition is nonsmoothing, that is, satisfies only (1.1.4). Accordingly, the optimal control, the optimal trajectory, the gain operator, and other relevant quantities all display a singularity at the terminal time $t = T$, which can be quantitatively measured in terms of the Banach spaces $C_r([s, T]; X)$ defined in Eqn. (1.2.1.17) below.

Theorem 1.2.1.1 *Let the (densely defined) operator GL_T be closed (or closable), as an operator $L_2(0, T; U) \supset \mathcal{D}(GL_T) \to Z_f$, a condition satisfied if e.g., $G = kI$.*

Then, there exists a unique optimal pair $\{u^0(t, 0; y_0), y^0(t, 0; y_0)\}$ of the optimal control problem (1.1.1)–(1.1.5), with $T < \infty$, that is explicitly given in terms of the data of the problem by (see (1.4.1.23))

$$-u^0(t, 0; y_0) = \left\{\Lambda_{0T}^{-1}[L_T^* G^* G e^{AT} y_0 + L^* R^* R(e^{A\cdot} y_0)]\right\}(t), \quad y_0 \in Y, \quad (1.2.1.1)$$

$$y^0(t, 0; y_0) = e^{At} y_0 + (Lu^0)(t), \quad (1.2.1.2)$$

where we have set Λ_{0T} to be the self-adjoint operator on $L_2(0, T; U)$,

$$\Lambda_{0T} = I + L^* R^* RL + L_T^* G^* GL_T, \quad (1.2.1.3)$$

which defines an isomorphism between $\mathcal{D}(GL_T)$, topologized by the graph norm, and its dual with respect to the pivot space Y (see (1.4.1.14)). Moreover, L, L^* are defined in (1.1.7), (1.1.8), and L_T, L_T^* are defined by (1.1.9), (1.1.10). The optimality condition yields the optimal feedback control

$$\begin{aligned} -u^0(\,\cdot\,, 0; y_0) &= L^* R^* R y^0(\,\cdot\,, 0; y_0) + L_T^* G^* G y^0(T, 0; y_0) \\ &\in \mathcal{D}(GL_T) \subset L_2(0, T; U) \end{aligned} \quad (1.2.1.4)$$

(see (1.4.1.21a).

Moreover, there exists a nonnegative, self-adjoint operator $P(t) = P^*(t) \geq 0$ (see (1.4.4.41) of Proposition 1.4.4.8), defined explicitly in terms of the data in (vii) = (1.2.1.10) below, such that:

(i) For all $\epsilon > 0$, and actually $\epsilon = 0$ if $\gamma < 1/2$:

$$P(\,\cdot\,) \in \mathcal{L}(Y; C([0, T - \epsilon]; Y)) \cap \mathcal{L}(Y; L_\infty(0, T; Y)) \quad (1.2.1.5)$$

(see (1.4.3.16) of Proposition 1.4.3.2).

(ii) For $0 \leq \theta < 1$, and $x \in Y$, we have continuously

$$(-A^*)^\theta P(\,\cdot\,)x \in C_\theta([0, T]; Y) \quad (1.2.1.6)$$

(see Remark 1.4.4.1), where C_θ is the space defined by Eqn. (0.4) of Chapter 0, so that (1.2.1.6) for $0 < \theta < 1$ *means that*

$$\|(-A^*)^\theta P(t)\|_{\mathcal{L}(Y)} \le \frac{C_{T\gamma\theta}}{(T-t)^\theta}, \quad 0 \le t < T \tag{1.2.1.7a}$$

(see (1.4.4.35) of Corollary 1.4.4.7); and moreover, for any $0 < \epsilon \le T$,

$$(-A^*)^\theta P(\,\cdot\,) \in \mathcal{L}(Y; C[0, T-\epsilon]; Y)), \quad 0 < \theta < 1, \tag{1.2.1.7b}$$

(see (1.4.6.31) of Proposition 1.4.6.5).

(iii) For $x \in Y$,

$$B^* P(t)x \in C_\gamma([0,T]; U) \tag{1.2.1.8}$$

(see Remark 1.4.4.1) where C_γ is the space defined by Eqn. (0.4) of Chapter 0, so that (1.2.1.7) means

$$\|B^* P(t)\|_{\mathcal{L}(Y;U)} \le \frac{C_{T\gamma}}{(T-t)^\gamma}, \quad 0 \le t < T \tag{1.2.1.9a}$$

(see (1.4.4.36) of Corollary 1.4.4.7); and moreover, for any $0 < \epsilon \le T$,

$$B^* P(\,\cdot\,) \in \mathcal{L}(Y; C([0, T-\epsilon]; Y)) \tag{1.2.1.9b}$$

(see (1.4.6.32) of Proposition 1.4.6.5).

(iv) For each $y_0 \in Y$, the optimal control $u^0(t,0;y_0)$ is given in pointwise feedback form by

$$u^0(t,0;y_0) = -B^* P(t) y^0(t,0;y_0), \quad 0 \le t < T \tag{1.2.1.10}$$

(see (1.4.3.17) of Proposition 1.4.3.2).

(v) The operator $P(t)$ is given (explicitly) by (see (1.4.3.15))

$$P(t)x = \int_t^T e^{A^*(\tau-t)} R^* R y^0(\tau,t;x)\,d\tau + e^{A^*(T-t)} G^* G y^0(T,t;x), \quad x \in Y, \tag{1.2.1.11}$$

where the expression in (1.2.1.11) defines $P(t)$ constructively solely in terms of the data of the problem, via formulas (1.2.1.1) and (1.2.1.2), and $Gy^0(T,t;x)$ is well defined; see (1.2.1.15) below.

(vi) The optimal cost of the corresponding optimal control problem on $[t,T]$ initiating at the time t at the point $x \in Y$ is

$$J^0(t,x) = J(u^0(\,\cdot\,,t;x), y^0(\,\cdot\,,t;x)) = (P(t)x, x)_Y, \quad 0 \le t < T \tag{1.2.1.12}$$

(see (1.4.4.42) of Proposition 1.4.4.8).

(vii) For all $0 < t < T$, and for all $x, y \in \mathcal{D}((-A)^\epsilon)$, $\forall \epsilon > 0$, the operator $P(t)$

satisfies the following differential Riccati equation (see Theorem 1.4.6.4):

$$(\dot{P}(t)x, y)_Y = -(R^*Rx, y)_Y - (P(t)_x, Ay)_Y \\ - (P(t)Ax, y)_Y + (B^*P(t)x, B^*P(t)y)_U. \quad (1.2.1.13)$$

(viii) *The following regularity properties hold true for the optimal pair with* $0 \leq s < T$ *and* $x \in Y$:

(viii₁)

$$\|u^0(\cdot, s; x)\|_{L_2(s,T;U)} + \|y^0(\cdot, s; x)\|_{L_2(s,T;Y)} \leq c_T \|x\|_Y \quad (1.2.1.14)$$

(see (1.4.2.1), (1.4.2.2) of Proposition 1.4.2.1);

(viii₂)

$$\|Gy^0(T, s; x)\|_{Z_f} \leq c_T \|x\|_Y \quad (1.2.1.15)$$

(see (1.4.2.3) of Proposition 1.4.2.1);

(viii₃)

$$\|u^0(\cdot, s; x)\|_{C_\gamma([s,T];U)} \leq c_{T\gamma} \|x\|_Y \quad (1.2.1.16)$$

(see (1.4.4.32) of Proposition 1.4.2.6);

(viii₄)

$$\begin{cases} \|y^0(\cdot, s; x)\|_{C([s,T];Y)} \leq c_{T\gamma} \|x\|_Y, & \text{if } 0 \leq \gamma < \frac{1}{2}; \quad (1.2.1.17) \\ \|y^0(\cdot, s; x)\|_{C_{2\gamma-1}([s,T];Y)} \leq c_{T\gamma} \|x\|_Y, & \text{if } \frac{1}{2} < \gamma < 1 \quad (1.2.1.18) \end{cases}$$

(see (1.4.4.33) of Theorem 1.4.4.6), where C_r *is the space defined by Eqn. (0.4).*

(ix) *Moreover (see Theorem 1.4.5.1), for* $x \in Y$*, and for each s fixed,* $0 \leq s < T$*, the optimal control* $u^0(t, s; x)$ *and the optimal solution* $y^0(t, s; x)$ *are respectively U-valued and Y-valued functions that are differentiable in* $t \in (s, T)$ *with*

$$\frac{du^0(t, s; x)}{dt} \in {}_{(\gamma+\epsilon_1)}C([s, T - \epsilon]; U), \quad \forall \epsilon, \epsilon_1 > 0; \quad (1.2.1.19)$$

$$\left[\frac{dy^0(t, s; x)}{dt} - Ae^{A(t-s)}x\right] \in {}_\gamma C([s, T - \epsilon]; U), \quad \forall \epsilon, \epsilon_1 > 0, \quad (1.2.1.20)$$

[see (1.4.5.3) and (1.4.5.7), respectively], where ${}_rC$ *is the space defined by Eqn. (0.5) of Chapter 0. (For* $\gamma < 1/2$*, the order of singularity of both* $du^0(t, s; x)/dt$ *and* $dy^0(t, s; x)/dt$ *can also be given at* $t = T$*; see Remark 1.4.5.1, Eqns. (1.4.5.8), (1.4.5.9)). This result will be extended in Theorem 1.2.2.1(v), (vi), Eqns. (1.2.2.8), (1.2.2.9) for* $1/2 \leq \gamma < 1$ *subject to the smoothing assumption (1.2.2.1) for G below.*

(x) *In fact, for $x \in Y$,*

$$\left\{\begin{array}{l}\textit{these } U\textit{-valued and } Y\textit{-valued functions } u^0(t,s;x) \textit{ and}\\ y^0(t,s;x) \textit{ are analytic in } t\in(s,T), \textit{ if either the operator}\\ A \textit{ has compact resolvent in } Y \textit{ or if } (-A)^{-p}B \textit{ is compact}\\ \textit{as an operator in } \mathcal{L}(U;Y), \textit{ for some } 0<p<1;\end{array}\right. \tag{1.2.1.21}$$

see Theorem 1.4.8.1.

(xi) *If we define the evolution operator*

$$\Phi(t,\tau)x = y^0(t,\tau;x), \quad x \in Y, \tag{1.2.1.22}$$

the following strong convergence results hold true (see (1.4.2.7) of Proposition 1.4.2.2(ii) and (1.4.7.3) of Proposition 1.4.7.2, respectively):

$$\lim_{t\uparrow T} \|G\Phi(T,t)x - Gx\|_{Z_f} = 0, \qquad \lim_{t\uparrow T} \|P(t)x - G^*Gx\|_Y = 0, \quad \forall x \in Y. \tag{1.2.1.23}$$

(xii) *The results of (xi) are a consequence of the following one:*

$$\lim_{s\uparrow T} \|u^0(\,\cdot\,,s;x)\|_{V(s,T;U)} = 0, \quad \forall x \in Y \tag{1.2.1.24}$$

(see (1.4.2.6) of Proposition 1.4.2.2(i)).

Remark 1.2.1.1 An example in Section 1.7.1 will show that the assumption that GL_T be closed (closable) cannot be dispensed with, for otherwise the optimal control may not exist. If GL_T is only closable, one takes the closure $\overline{GL_T}$.

Remark 1.2.1.2 With reference to the assumption on GL_T of Theorem 1.2.1.1, and to claim (a) at the end of Section 0, we have

$$\begin{array}{lcl}\text{closed operator } (GL_T)^* = L_T^*G^*, & & \text{densely defined operator } GL_T\\ \text{be densely defined as an operator} & \Longleftrightarrow & \text{be closable as an operator}\\ Z_f \supset \mathcal{D}((GL_T)^*) \to L_2(0,T;U) & & L_2(0,T;U) \supset \mathcal{D}(GL_T) \to Z_f\end{array} \tag{1.2.1.25}$$

$$\begin{array}{l}\text{(in which case } (GL_T)^{**} = \text{closure of } GL_T)\\ \Uparrow\\ (-A^*)^{\frac{\beta}{2}}G^* \text{ be densely defined as an}\\ \text{operator } Z_f \supset \mathcal{D}((-A^*)^{\beta/2}G^*) \to\\ Y, \text{ for some } \beta > 2\gamma - 1.\end{array} \tag{1.2.1.26}$$

The equivalence in (1.2.1.25) is a standard result [Kato, 1966, Theorems 5.28 and 5.29, p. 168]. To see the sufficient condition (1.2.1.26) → (1.2.1.25), we compute from (1.1.10)

$$\begin{aligned}\{L_T^*G^*z\}(t) &= B^*e^{A^*(T-t)}G^*z\\ &= B^*(-A^*)^{-\gamma}(-A^*)^{\gamma-\beta/2}e^{A^*(T-t)}(-A^*)^{\beta/2}G^*z,\end{aligned} \tag{1.2.1.27}$$

use assumption (1.1.2), and notice that $(-A^*)^{\gamma-\beta/2}e^{A^*(T-t)} \in \mathcal{L}(Y; L_2(0, T; Y))$ for $2\gamma - \beta < 1$ by analyticity of the semigroup [Pazy, 1983, p. 74].

We emphasize that condition (1.2.1.26) on $(-A^*)^{\beta/2}G^*$ – which does not involve B – is only sufficient for the ultimate requirement that GL_T be closable, the latter being instead a condition that also involves B. This will be seen in one example in Section 1.7.2 below.

Remark 1.2.1.3 (Case $G = 0$) When $G = 0$, we can complement Eqns. (1.2.1.1), (1.2.1.4) with the following additional formulas:

$$y^0(t, 0; y_0) = \{[I + LL^*R^*R]^{-1}e^{A\cdot}y_0\}(t) \in L_2(0, T; Y); \tag{1.2.1.28}$$

$$Ry^0(t, 0; y_0) = \{[I + RLL^*R^*]^{-1}Re^{A\cdot}y_0\}(t) \in L_2(0, T; Z), \tag{1.2.1.29}$$

which yield the optimal solution explicitly in terms of the problem data. Moreover, the optimal cost is given by

$$\begin{aligned} J^0(y_0) &= ([I + RLL^*R^*]^{-1}Re^{A\cdot}y_0, Re^{A\cdot}y_0)_{L_2(0,T;Z)} \\ &= (P(0)y_0, y_0)_Y, \end{aligned} \tag{1.2.1.30}$$

and likewise explicitly in terms of the problem data. One can show that the inverse operator $[I + LL^*R^*R]^{-1}$ in (1.2.1.28) is well defined and bounded as an operator on $L_2(0, T; Y)$; see Chapter 2, Appendix 2A. We shall not use formulas (1.2.1.28)–(1.2.1.30) in the present chapter, where the emphasis is on an operator $G \neq 0$ that is nonsmoothing. Formulas (1.2.1.28) and (1.2.1.30) will be far more important in Volume 2, which deals with the hyperbolic case (see, e.g., Chapter 8, Theorem 8.2.1.1, Chapter 9, Theorem 9.4.1, and implications in Chapter 11, Volume 3).

1.2.2 Two Smoothing Cases. Theorem 1.2.2.1: Classical Differential Riccati Equation and Uniqueness of the Riccati Operator. Theorem 1.2.2.2

In this subsection we assume that the operator G of the terminal condition in (1.1.3) has smoothing properties. Two progressively stronger cases will be singled out, each of which is responsible for a remarkable enrichment of the theory. The first case, which falls under the weaker assumption (1.2.2.1) below, produces a few significant consequences: (i) the solution operator $P(t)$ of the differential Riccati equation (DRE) (1.2.1.13) guaranteed and constructed by Theorem 1.2.1.1 is, in fact, a *classical* solution and, moreover, is unique (within a natural class); and (ii) the optimal trajectory y^0 becomes continuous also at the terminal time $t = T$. Moreover, all the quantities involved become more regular. The second case, which falls under the stronger assumption (1.2.2.16) below, guarantees in addition that the optimal control u^0 becomes continuous at the terminal time $t = T$ and, moreover, it implies additional regularity on all other quantities involved.

First Case Here we shall assume that G is a smoothing operator in the sense that

$$(-A^*)^\beta G^* G \in \mathcal{L}(Y), \quad \text{for some } \beta > 2\gamma - 1, \text{ if } \frac{1}{2} \leq \gamma < 1;$$

$$\text{while this condition always holds true with } \beta = 0 \quad \text{if } 0 \leq \gamma < \frac{1}{2} \tag{1.2.2.1}$$

(the latter case $0 \leq \gamma < 1/2$ is already singled out in Theorem 1.2.1.1(i), (ii), (1.2.1.17), etc.).

Claim Assumption (1.2.2.1) implies that GL_T is, in fact, bounded:

$$GL_T \in \mathcal{L}(L_2(0, T; U); Z_f),$$

so that a fortiori the assumption of Theorem 1.2.1.1 is satisfied. The justification of the above claim will be given at the beginning of Subsection 1.5.1.

Thus, if condition (1.2.2.1) is assumed, then, accordingly, stronger results follow.

Theorem 1.2.2.1 *Assume condition (1.2.2.1). Then:*

(i) (Regularity of optimal pair) For $x \in Y$ and any $\epsilon > 0$,

$$\|u^0(\,\cdot\,, s; x)\|_{C_{\gamma-\beta}([s,T];U)} + \|y^0(\,\cdot\,, s; x)\|_{C([s,T];Y)} \leq c_{T\gamma} \|x\|_Y, \tag{1.2.2.2}$$

$$\gamma - \beta = 1 - \gamma - \epsilon, \quad \text{if } \frac{1}{2} \leq \gamma < 1; \quad \text{and} \quad \beta = 0 \text{ so that } \gamma - \beta = \gamma,$$

$$\text{if } 0 \leq \gamma < \frac{1}{2}; \quad y^0(T, \,\cdot\,; x) = \Phi(T, \,\cdot\,)x \in C([s, T]; Y) \tag{1.2.2.3}$$

(see (1.5.2.1b) and (1.5.2.2b) of Theorem 1.5.2.1 from which, in particular, it follows (see (1.5.2.3b)) that

$$\lim_{t \uparrow T} \Phi(T, t)x = x, \quad x \in Y. \tag{1.2.2.4}$$

(ii) For any $0 \leq \theta < 1$, $x \in Y$, we have

$$\begin{cases} (-A^*)^\theta P(t)x \in C_{\theta-\beta}([0, T]; Y), \\ \|(-A^*)^\theta P(t)x\|_U \leq \frac{C_{T\gamma}}{1-\theta} \frac{1}{(T-t)^{\theta-\beta}}, \quad 0 \leq t < T, \end{cases} \tag{1.2.2.5}$$

$$\theta - \beta = \theta + 1 - 2\gamma + \epsilon \quad \text{if } \frac{1}{2} \leq \gamma < 1; \quad \theta - \beta = \theta \quad \text{if } 0 \leq \gamma < \frac{1}{2}$$

(see (1.5.2.4b) of Theorem 1.5.2.1).

(iii)

$$\begin{cases} B^* P(\,\cdot\,) \in \mathcal{L}(Y; C_{\gamma-\beta}([0, T]; U)), \\ \text{i.e., } \|B^* P(t)x\|_U \leq \frac{C_T}{1-\gamma} \frac{1}{(T-t)^{\gamma-\beta}} \|x\|_Y, \quad 0 \leq t < T \end{cases} \tag{1.2.2.6}$$

(see (1.5.2.5) of Theorem 1.5.2.1).

(iv)

$$\lim_{t \uparrow T} P(t)x = G^*Gx, \quad x \in Y \tag{1.2.2.7}$$

(see (1.5.2.6) of Theorem 1.5.2.1).

(v)

$$\frac{du^0(t,s;x)}{dt} \in {}_{(\gamma+\epsilon)}C_{1+\gamma-\beta}([s,T];U), \quad x \in Y \tag{1.2.2.8}$$

(see (1.5.2.12) of Theorem 1.5.2.2 and Chapter 0, Eqn. (0.6), or Eqn. (1.4.5.2), for the definition of the space in (1.2.2.8)).

(vi)

$$\left[\frac{dy^0(t,s;x)}{dt} - Ae^{A(t-s)}x\right] \in {}_{\gamma}C_{2\gamma-\beta}([s,T];Y), \quad x \in Y \tag{1.2.2.9}$$

(see (1.5.2.14) of Theorem 1.5.2.2; and Chapter 0, Eqn. (0.6), or Eqn. (1.4.5.2), for the definition of the space in (1.2.2.9)).

(vii) (Classical DRE) Let t be fixed and satisfy $0 \le t < T$. Let $P(t)$ be the operator defined by (1.2.1.10) which is a solution of (1.2.1.12). Then, the bilinear form

$$x, y \to (P(t)x, Ay)_Y + (Ax, P(t)y)_Y, \quad x, y \in \mathcal{D}(A) \tag{1.2.2.10}$$

admits a unique continuous extension that is bounded on $Y \times Y$. Moreover, for $x \in \mathcal{D}(A)$, we have $P(t)x \in \mathcal{D}(A^*)$, and thus the operator

$$\Gamma(P(t)) \equiv A^*P(t) + P(t)A \tag{1.2.2.11}$$

originally well defined on $\mathcal{D}(A)$, as an element of $\mathcal{L}(\mathcal{D}(A);Y)$, admits a unique continuous extension in $\mathcal{L}(Y)$, still denoted by $\Gamma(P(t))$, satisfying

$$\|\Gamma(P(t))\|_{\mathcal{L}(Y)} \le \frac{C_T}{(T-t)^\alpha}, \quad 0 \le t < T, \tag{1.2.2.12}$$

$\alpha = \max\{1-\beta, 3\gamma-1-\beta+\epsilon\}$, where $\beta = 2\gamma - 1 + 2\epsilon$ for $1/2 \le \gamma < 1$, $\beta = 0$ for $0 \le \gamma < 1/2$; moreover,

$$\left\|\frac{dP(t)}{dt}\right\|_{\mathcal{L}(Y)} \le \frac{C_T}{(T-t)^\alpha}, \quad 0 \le t < T \tag{1.2.2.13}$$

(see Theorem 1.5.3.1). Finally, $P(t)$ solves the DRE in a classical sense:

$$\begin{cases} \dot{P}(t) = -R^*R - \Gamma(P(t)) + (B^*P(t))^*B^*P(t), & 0 \le t < T, \\ P(T) = G^*G \end{cases} \tag{1.2.2.14}$$

(see Corollary 1.5.3.2).

(viii) (Uniqueness, see Theorem 1.5.3.3) The operator $P(t)$, given constructively by Eqn. (1.2.1.10), is the unique solution of the DRE (1.2.1.12) and of the terminal

condition (1.2.2.7) – indeed, the unique classical solution of (1.2.2.14) – within the class of self-adjoint operators $\bar{P}(t)$ satisfying

$$B^* \bar{P}(t)x \in C_{\gamma-\beta}([0, T]; Y), \quad x \in Y, \tag{1.2.2.15}$$

a property satisfied by $P(t)$ by (1.2.2.6) (see Theorem 1.5.3.3).

Second Case We now assume a stronger smoothing property on G in the sense that

$$(-A^*)^\gamma G^* G \in \mathcal{L}(Y). \tag{1.2.2.16}$$

Assumption (1.2.2.16) is stronger than assumption (1.2.2.1) since $2\gamma - 1 < \gamma$. Thus, a fortiori, (1.2.2.16) implies that $GL_T \in \mathcal{L}(L_2(0, T; U); Z_f))$; see *Claim* below (1.2.2.1). Under (1.2.2.16), additional regularity results hold true.

Theorem 1.2.2.2 *Assume condition (1.2.2.16). Then (see (1.6.1.3) of Corollary 1.6.4)*

(i)

$$\|u^0(\,\cdot\,, s; x)\|_{C([s,T];U)} \le c_T \|x\|_Y, \quad x \in Y; \tag{1.2.2.17}$$

(ii) for any $0 \le \theta < 1$, $(-A^*)^\theta P(\,\cdot\,) \in \mathcal{L}(Y; C_{\theta-\gamma}([0, T]; Y)$ (see (1.6.14)),

$$\|(-A^*)^\theta P(t)\|_{\mathcal{L}(Y)} \le \frac{C_T}{1-\theta} \frac{1}{(T-t)^{\theta-\gamma}}, \quad 0 \le t < T; \tag{1.2.2.18}$$

(iii)

$$B^* P(\,\cdot\,) \in \mathcal{L}(Y; C([0, T]; U)) \tag{1.2.2.19}$$

(see (1.6.15)).

Remark 1.2.2.1 Under (1.2.2.16), the results of Theorem 1.2.2.2 are precisely those that hold as if $G = 0$; in other words, G under (1.2.2.16) puts no additional constraints on the optimal control problem (1.1.5).

1.3 Orientation

The variational approach of the present chapter is explicit and constructive in the sense that: (i) first, the optimal pair $\{u^0, y^0\}$ is characterized solely in terms of the data of the problem (see (1.2.1.1)–(1.2.1.3) in Section 1.2.1); (ii) next, an operator $P(t)$ is constructed (see (1.2.1.10)) in terms of original and optimal evolution, hence ultimately in terms of the original data of the problem (in Section 1.3); and (iii) finally, the operator $P(t)$ is shown to satisfy the differential Riccati equation (in Section 1.4.6) and its limiting condition as $t \uparrow T$ (in Section 1.4.7). The presence of the penalization operator G in (1.1.3) introduces additional genuine difficulties into the problem, unless assumption (1.2.2.16) is satisfied. Qualitatively, the analyticity of

e^{At} tends "to compensate" the effects of the unboundedness of B on any interval of the type $[0, T-\epsilon]$, $\forall\epsilon > 0$ small. Instead, the presence of a *nonsmoothing* operator G produces a singularity at $t = T$ for $\{L_T^* G^* G e^{AT} x\}(t) = B^* e^{A^*(T-t)} G^* G e^{AT} x$, which occurs in the explicit formula (1.2.1.1) for the optimal control $u^0(t, 0; x)$, $x \in Y$. This is reflected by the quantitative statements of Theorem 1.2.1.1: (1.2.1.16) for u^0; and (1.2.1.18) for y^0 when $1/2 \leq \gamma < 1$, where the singularity is measured by the spaces (0.5). This singularity is progressively reduced in Theorem 1.2.2.1 under the smoothing assumption (1.2.2.1) on G (vacuous if $0 \leq \gamma < 1/2$) and finally eliminated [see (1.2.2.17)], if further smoothing is imposed on G as in (1.2.2.16) of Theorem 1.2.2.2. Likewise, it is instructive to compare statements (1.2.1.8), (1.2.2.6), and (1.2.2.19) of increasing regularity of the gain operator $B^*P(t)$ under progressively stronger smoothing assumptions on G. The above considerations, in particular (1.2.1.18) and (1.2.2.2), show that, for $0 \leq \gamma < 1/2$, the optimal trajectory y^0 is in $C([0, T]; Y)$. This is not surprising. In fact, standard regularity properties on analytic semigroup theory [Eqns. (0.4) and (0.5) of Chapter 0] yield the well-known result that if $\gamma < 1/2$ in (1.1.2) (or even $\gamma = 1/2$ if A is self-adjoint or normal or similar to a normal operator) then the operator L in (1.1.7) is continuous $L_2(0, T; U)) \to C([0, T]; Y)$ and thus *every* solution of (1.1.1) with $y_0 \in Y$ – not only the optimal solution y^0 – lies in $C([0, T]; Y)$! Thus, the value $\gamma = 1/2$ gives the natural "cutting line" in the range of values of γ, which crucially bears on the degree of technical difficulties present in the analysis. The case $\gamma < 1/2$ behaves like the "B-bounded" case and one has at the outset the important property that any solution $y(t)$, in particular the optimal solution $y^0(t, 0; y_0)$, belongs to $C([0, T]; Y)$. The situation is more demanding in general if instead $1/2 \leq \gamma < 1$. We have, directly from (1.2.1.1)–(1.2.1.3),

$$-u^0(\cdot\,, s; x) = [I_s + L_s^* R^* R L_s]^{-1}\left\{L_s^* R^* e^{A(\cdot - s)} x + L_{sT}^* G^* G y^0(T, s; x)\right\} \quad (1.3.1)$$

for the optimal control problem on $[s, T]$, $0 \leq s < T$, where L_s, L_{sT} are the operators L in (1.1.7) and L_T in (1.1.9) starting now from s rather than 0.

Crucial to the proof of statements (1.2.1.16) for u^0 and (1.2.1.18) for y^0 is the key property that $[I_s + L_s^* R^* R L_s]^{-1} \in \mathcal{L}(C_\gamma([s, T]; U))$ with uniform bound that may be taken independent of s (Theorem 1.4.4.4). This is accomplished via a bootstrap argument starting from the a priori L_2-regularity and using the smoothing properties of regularity of the operators L and L^* in Section 1.4.4.

A similar key invertibility property, this time on the space ${}_{(\gamma+\epsilon_1)}C_{(\gamma+\epsilon_1)}([s, T-\epsilon]; U)$ [see Theorem 1.4.5.10, Eqn. (1.4.5.57)] is behind the proof of the important statements (1.2.1.19), (1.2.1.20), concerning existence and regularity of the time derivatives $du^0(t, s; x)/dt$ and $dy^0(t, s; x)/dt$. Again, the proof (in Section 1.4.5) is done by a bootstrap argument, which exploits the smoothing regularity properties of L and L^*.

Similarly, crucial to the proof of Theorem 2.2.1 in the smoothing case is the key fact that the operator Λ_{sT} (same as Λ_{0T} in (1.2.1.3), except that the process starts now at s rather than 0), satisfies $\Lambda_{sT}^{-1} \in \mathcal{L}(C_{\gamma-\beta}([s,T];U))$ with a uniform bound that may be taken independent of s. This is also done by a bootstrap argument in Theorem 1.5.1.7. A bootstrap technique is also behind the proof in Corollary 1.4.8.5, which shows that the operator $[I + L^*R^*RL]$ is boundedly invertible in the space $\mathcal{A}(\mathcal{F};U)$ of U-valued functions that are analytic on $\mathcal{F}$ and continuous on $\bar{\mathcal{F}}$, where $\mathcal{F}$ is an open symmetric set, in the sector of analyticity of $\exp(At)$, based on the interval $[0,T]$. This step is crucial in obtaining the analyticity properties of the optimal pair ((x) of Theorem 1.2.1.1).

The regularity properties of $\{u^0, y^0\}$ [see (ix) of Theorem 1.2.1.1], as well as those of $P(t)$ [see (1.2.1.6), (1.2.1.7)], are *distinctive* of the class of analytic semigroups e^{At}. They should be contrasted with those available in Chapters 8 through 10 for different classes. A common goal – a key fact in establishing well posedness of the Riccati equation – is that the gain operator $B^*P(t)$ be well defined, which is not a priori clear when B is unbounded: Eqns. (1.2.1.8), (1.2.1.9) in the general case, and (1.2.2.6) as well as (1.2.2.19) in the smoothing case, are statements of these facts.

Table 1.1 provides, at a glance, the main results of the theory of Chapter 1, under progressively stronger assumptions on the observation operator G.

1.4 Proof of Theorem 1.2.1.1 with GL_T Closed

1.4.1 Optimality. Explicit Representation Formulas for the Optimal Pair $\{u^0, y^0\}$

Preliminaries We begin by collecting here some preliminary results that will culminate with the representation formulas of the unique optimal pair $\{u^0, y^0\}$.

The solution to Eqn. (1.1.1) with initial datum $y_0 \in Y$ at the initial time s, $0 \le s \le t \le T$, is given by (see (1.1.6), (1.1.7) for $s=0$, where $L = L_0$)

$$y(t,s;y_0) = e^{A(t-s)}y_0 + (L_s u)(t), \tag{1.4.1.1}$$

$$(L_s u)(t) = \int_s^t e^{A(t-\tau)}Bu(\tau)\,d\tau \tag{1.4.1.2}$$

: continuous $L_2(s,T;U) \to L_2(s,T;\mathcal{D}((-A)^{1-\gamma}))$ with operator norm uniform with respect to s, $0 \le s \le T$; (1.4.1.3a)

that is,

$$\|L_s u\|_{L_2(s,T;\mathcal{D}((-A)^{1-\gamma})} \le K_{T\gamma}\|u\|_{L_2(s,T;U)} \tag{1.4.1.3b}$$

: continuous $L_\infty(s,T;U) \to C([s,T];(\mathcal{D}((-A)^{\theta}))$ for $\gamma + \theta < 1$, with operator norm uniform with respect to s, $0 \le s \le T$ [Sadosky, 1979, Ex. 1.3, p. 29], (1.4.1.3c)

$$\|L_s u\|_{C([s,T];\mathcal{D}((-A)^{\theta}))} \le K_{T\gamma\theta}\|u\|_{L_\infty(s,T;U)}, \tag{1.4.1.3d}$$

where $K_{T\gamma}$ and $K_{T\gamma\theta}$ do not depend on s, $0 \le s \le T$. The continuity expressed by (1.4.1.3a–b) is a consequence of the basic assumption (1.1.2) on B, as well as of the

Table 1.1. *Basic Theory of Chapter 1 at a glance. Moreover,* $y^0(t, s; x)$ *and* $u^0(t, s; x)$ *are, respectively,* Y*-analytic and* U*-analytic functions for* $s < t < T$, *if either* A *has a compact resolvent or if* $(-A)^{-p}B$ *is compact in* $\mathcal{L}(U; Y)$ *for some* $0 < p < 1$ *[Theorem 1.2.1.1(x)].*

Assumptions	$\begin{cases} G \in \mathcal{L}(Y; Z_f) \\ GL_T \text{ closed} \end{cases}$	(1.2.2.1): $(-A^*)^\beta G^*G \in \mathcal{L}(Y)$ $\begin{cases} \beta = 0, & 0 \le \gamma \le \frac{1}{2} \\ \beta > 2\gamma - 1, & \frac{1}{2} \le \gamma < 1 \end{cases}$	(1.2.2.16): $(-A^*)^\gamma G^*G \in \mathcal{L}(Y)$ [same as $G = 0$]	
$y^0(\cdot, s; x)$:	$\begin{cases} C([s,T]; Y) \\ C_{2\gamma-1}([s,T]; Y) \end{cases}$ (1.2.1.17)–(1.2.1.18)	C($[s,T]$;Y) (1.2.2.2)		$0 \le \gamma < \frac{1}{2}$ $\frac{1}{2} < \gamma < 1$
$Gy^0(T, s; x)$	Z_f (1.2.1.15)			
$\lim_{s \uparrow T} Gy^0(T, s; x)$ $= Gx$	(1.2.1.23)			
$u^0(\cdot, s; x)$:	$C_\gamma([s,T]; U)$ (1.2.1.16)	$C_{\gamma-\beta}([s,T]; U)$ (1.2.2.2)	$C([s,T]; U)$ (1.2.2.11)	
$\lim_{s \uparrow T} u^0(\cdot, s; x)$ $= 0$	(1.2.1.24)			
$P(t)x$	$\begin{cases} C([0,T]; Y) \\ \text{C}([0,T); \text{Y}) \\ \quad \cap L_\infty(0,T; Y) \end{cases}$ (1.2.1.5)	C($[0,T]$; Y) (1.2.2.5)		$0 \le \gamma < \frac{1}{2}$ $\frac{1}{2} \le \gamma < 1$
$(-A^*)^\theta P(t)x$ $0 < \theta < 1$	$C_\theta([0,T]; Y)$ (1.2.1.6) (1.4.4.35)	$C_{\theta-\beta}([0,T]; Y)$ (1.2.2.5)	$C_{\theta-\gamma}([0,T]; U)$ (1.2.2.12)	
$B^*P(t)x$	$C_\gamma([0,T]; U)$ (1.2.1.8) (1.4.6.36)	$C_{\gamma-\beta}([0,T]; U)$ (1.2.2.6) (1.5.1.41)	$C([0,T]; U)$ (1.2.2.13)	
DRE	(1.2.2.13) for $x, y \in \mathcal{D}((-A)^\epsilon)$	classical sense and uniqueness		
$P(t)$ at $t = T$:	(1.2.1.23) $\lim_{t \uparrow T} P(t)x$ $= G^*Gx$	$P(T) = G^*G$ (1.2.2.7)		
	Theorem 1.2.1.1	Theorem 1.2.2.1	Theorem 1.2.2.2	

standard result

$$f \to \int_0^t Ae^{A(t-\tau)} f(\tau)\, d\tau : L_2(0, T; Y) \to L_2(0, T; Y)$$

reported in Eqn. (0.4) of Chapter 0. Instead, (1.4.1.3c–d) uses (1.1.2) and the usual analyticity bound [Pazy, 1983, p. 74]. The adjoint operator L_s^* defined by $(L_s u, f)_{L_2(s,T;Y)} = (u, L_s^* f)_{L_2(s,T;U)}$ is given explicitly by (see (1.1.8) for $s = 0$, where $L^* = L_0^*$)

$$(L_s^* f)(t) = \int_0^T B^* e^{A^*(\tau - t)} f(\tau)\, d\tau, \quad s \le t \le T \tag{1.4.1.4}$$

$$: \text{ continuous } L_2(s, T; [\mathcal{D}((-A)^{1-\gamma})]') \to L_2(s, T; U)$$

with operator norm uniform with in $s : 0 \le s \le T$;

that is,

$$\|L_s^* f\|_{L_2(s,T;U)} \leq C_{T\gamma}\|f\|_{L_2(s,T;[\mathcal{D}((-A)^{1-\gamma})]')} \tag{1.4.1.5a}$$

: continuous $L_\infty(s, T; [\mathcal{D}((-A)^\theta)]') \to C([s, T]; U)$ for $\gamma + \theta < 1$, with operator norm uniform with respect to $s, 0 \leq s \leq T$ [Sadosky, 1979, Ex. 1.3, p. 29], (1.4.1.5b)

$$\|L_s^* f\|_{C([s,T];U))} \leq K_{T\gamma\theta}\|f\|_{L_\infty(s,T;[\mathcal{D}((-A)^\theta)]')}. \tag{1.4.1.5c}$$

Other regularity results for L_s and L_s^* will be given in Theorem 1.4.4.3 below. We shall also need the (unbounded) operator L_{sT} (see (1.1.9) for $s = 0$, where then $L_T = L_{0T}$), defined in accordance with (1.1.9) as

$$L_{sT}u = (L_s u)(T) = \lim_{t\uparrow T}\int_s^t e^{A(t-\tau)}Bu(\tau)\,d\tau, \tag{1.4.1.6}$$

with domain $\mathcal{D}(L_{sT}) = \{u \in L_2(s, T; U) : L_{sT}u \in Y\}$, and its adjoint $L_{sT}^* : (L_{sT}u, y)_Y = (u, L_{sT}^* y)_{L_2(s,T;U)}$ given by (see (1.1.10) for $s = 0 : L_T^* = L_{0T}^*$)

$$\{L_{sT}^* y\}(t) = B^* e^{A^*(T-t)}y, \quad s \leq t \leq T, \;\; y \in \mathcal{D}(L_{sT}^*), \tag{1.4.1.7a}$$

with domain $\mathcal{D}(L_{sT}^*) = \{y \in Y : L_{sT}^* y \in L_2(s, T; U)\}$, which is unbounded from $Y \supset \mathcal{D}(L_{sT}^*)$ into $L_2(s, T; U)$. We note that $H^1(s, T; U) \subset \mathcal{D}(L_{sT})$, so that $\mathcal{D}(L_{sT})$ is dense in $L_2(s, T; U)$, and that L_{sT} is a closed operator by Claim (b) at the end of Chapter 0 below (0.32), since

$$L_{sT}u = (-A)^\gamma \int_s^T e^{A(T-t)}(-A)^{-\gamma}Bu(\tau)\,d\tau$$

is the composition of the closed, boundedly invertible operator $(-A)^\gamma$ and a bounded operator by (1.1.2). We note, moreover, for future use in (1.4.2.20) that

$$\lim_{s\uparrow T}\|L_{sT}^* z\|_{L_2(s,T;U)} = 0, \quad z \in \mathcal{D}(L_{0T}^*) = \mathcal{D}(L_{sT}^*). \tag{1.4.1.7b}$$

Next, by using the assumption that GL_T is closed (closable), we shall convert $\mathcal{D}(GL_{sT})$ into a Hilbert space $V(s, T; U)$ equipped with the following (graph) inner product:

$$(u, v)_{V(s,T;U)} = (u, v)_{L_2(s,T;U)} + (GL_{sT}u, GL_{sT}v)_{Z_f} \tag{1.4.1.8}$$

for $u, v \in \mathcal{D}(GL_{sT})$. Let $[V(s, T; U)]'$ denote the dual space of $V(s, T; U)$ with respect to $L_2(s, T; U)$ as pivot space:

$$V(s, T; U) \subset L_2(s, T; U) \subset [V(s, T; U)]', \tag{1.4.1.9a}$$

with continuous injections

$$\|u\|_{[V(s,T;U)]'} \leq \|u\|_{L_2(s,T;U)} \leq \|u\|_{V(s,T;U)}, \tag{1.4.1.9b}$$

where the right-hand inequality follows from (1.4.1.8) and in turn implies the left-hand inequality. Thus by definition of $V(s, T; U)$, we have

$$GL_{sT} : \text{continuous } V(s, T; U) \to Z_f, \tag{1.4.1.10a}$$

$$(GL_{sT})^* : \text{continuous } Z_f \to [V(s, T; U)]'. \tag{1.4.1.10b}$$

We shall write $L_{sT}^* G^* = (GL_{sT})^*$, since G is bounded. We introduce the operator

$$\Lambda_{sT} \equiv I_s + L_s^* R^* R L_s + L_{sT}^* G^* G L_{sT} \tag{1.4.1.11a}$$

$$: \text{continuous } L_2(s, T; U) \supset \mathcal{D}(\Lambda_{sT}) \to [V(s, T; U)]'. \tag{1.4.1.11b}$$

Since GL_{sT} is densely defined and closed from $L_2(s, T; U) \supset \mathcal{D}(GL_{sT}) \to Z_f$, then Λ_{sT} is a (generally unbounded) positive self-adjoint operator on $L_2(s, T; U)$, by the von Neumann theorem [Kato, 1966, p. 275].

Using (1.4.1.11a), (1.4.1.8), (1.4.1.3b), and (1.4.1.9b), we readily verify that

$$|(\Lambda_{sT} u, v)_{L_2(s,T;U)}| \leq M_{T\gamma} \|u\|_{V(s,T;U)} \|v\|_{V(s,T;U)}, \tag{1.4.1.12}$$

$$(\Lambda_{sT} u, u)_{L_2(s,T;U)} \geq \|u\|^2_{V(s,T;U)}, \tag{1.4.1.13}$$

with constant $M_{T\gamma} = 1 + \|R\|^2 K_{T\gamma}^2$ independent on s, $0 \leq s \leq T$. Then, by (1.4.1.12), (1.4.1.13), the Lax–Milgram theorem applies, and we can extend Λ_{sT} as

$$\Lambda_{sT} : \text{isomorphism } V(s, T; U) \text{ onto } [V(s, T; U)]', \tag{1.4.1.14}$$

so that, in particular,

$$\left\| \Lambda_{sT}^{-1} v \right\|_{V(s,T;U)} \leq C_T \|v\|_{[V(s,T;U)]'}, \tag{1.4.1.15}$$

with constant C_T independent of s, $0 \leq s \leq T$. From (1.4.1.8), we obtain

$$\|GL_{sT}\|_{\mathcal{L}(V(s,T;U);Z_f)} = \|L_{sT}^* G^*\|_{\mathcal{L}(Z_f;[V(s,T;U)]')} \leq 1. \tag{1.4.1.16}$$

Optimal Control Problem. Explicit Formulas. Problem on [0, T] Having introduced the space $V(0, T; U)$, we are thus led to minimize the quadratic cost $J(u, y)$ in (1.1.3) over all $u \in V(0, T; U)$ rather than over all $u \in L_2(0, T; U)$, thereby obtaining a new problem, which is a classical quadratic problem with continuous, strictly convex $J(u, y)$. By standard optimization theory [Ekeland, Teman, 1976; Luenberger, 1969], the new problem has a unique optimal solution (control) $u^0 = u^0(\,\cdot\,, 0; y_0)$ with corresponding optimal trajectory $y^0 = y^0(\,\cdot\,, 0; y_0)$. As for $u \in L_2(0, T; U)$ but $u \notin V(0, T; U)$, the corresponding cost value $J(u, y(u)) = \infty$. We now characterize the optimal pair $\{u^0, y^0\}$ via Lagrange multiplier theory. For $u \in V(0, T; U)$, $y = L_2(0, T; Y)$, $p \in L_2(0, T; Y)$, we define the Lagrangean

$$\mathcal{L}(u, y, p) = \frac{1}{2}\left\{\|u\|^2_{L_2(0,T;U)} + \|Ry\|^2_{L_2(0,T;Z)} + \|G(e^{AT} y_0 + L_T u)\|^2_{Z_f}\right\}$$
$$+ (p, y - e^{A\,\cdot} y_0 - Lu)_{L_2(0,T;Y)}.$$

Since $y - e^{A\cdot}y_0 - Lu$ obviously maps $(y, u) \in L_2(0, T; Y) \times V(0, T; U)$ onto $L_2(0, T; Y)$ (for any $g \in L_2(0, T; Y)$ take $u = 0$ and $y = e^{A\cdot}y_0 + g$), Liusternik's Lagrange multiplier theorem [Luenberger, 1969, Theorem 1, p. 243] applies and gives: There exist $u^0 \in V(0, T; U)$, $y^0 \in L_2(0, T; Y)$, $p^0 \in L_2(0, T; Y)$ such that $\mathcal{L}_u = \mathcal{L}_y = \mathcal{L}_p = 0$ at (u^0, y^0, p^0). From the Lagrangean we compute $\mathcal{L}_y = \mathcal{L}_u = 0$ to obtain, respectively,

$$(R^*Ry^0, \delta y)_{L_2(0,T;Y)} + (p^0, \delta y)_{\mathcal{L}_2(0,T;Y)} = 0, \quad \forall \delta y \in L_2(0, T; Y),$$
$$\text{or } p^0 = -R^*Ry^0; \tag{1.4.1.17}$$

$$(u^0 - L^*p^0, \delta u)_{L_2(0,T;U)} = -(G(e^{AT}y_0 + L_Tu^0), GL_T\delta u)_{L_2(0,T;Z_f)},$$
$$\forall \delta u \in V(0, T; U), \tag{1.4.1.18a}$$

from which we deduce via (1.4.1.17) that

$$u^0 - L^*p^0 = u^0 + L^*R^*Ry^0 = -L_T^*G^*G(e^{AT}y_0 + L_Tu^0) \in [V(0, T; U)]'. \tag{1.4.1.18b}$$

Moreover,

$$L_T^*G^*GL_Tu^0 \in [V(0, T; U)]'; \qquad L_T^*G^*Ge^{AT}y_0 \in [V(0, T; U)]'; \tag{1.4.1.19}$$

the inclusion on the left is a consequence of (1.4.1.10), while the inclusion on the right follows directly from the definition of the space V. Using the optimal dynamics

$$y^0(t, 0; y_0) = e^{At}y_0 + \{Lu^0(\,\cdot\,, 0, y_0)\}(t) \tag{1.4.1.20}$$

in (1.4.1.18b) yields the optimality condition

$$-u^0(\,\cdot\,, 0; y_0) = L^*R^*Ry^0(\,\cdot\,, 0; y_0) + L_T^*G^*Gy^0(T, 0; y_0) \in V(0, T; U), \tag{1.4.1.21a}$$

as well as

$$u^0(\,\cdot\,, 0; y_0) + L^*R^*RLu^0(\,\cdot\,, 0; y_0)$$
$$= -L_T^*G^*Gy^0(T, 0; y_0) - L^*R^*Re^{A\cdot}y_0 \in [V(0, T; U)]'. \tag{1.4.1.21b}$$

Finally,

$$[I + L^*R^*RL + L_T^*G^*GL_T]u^0$$
$$= -L_T^*G^*Ge^{AT}y_0 - L^*R^*R(e^{A\cdot}y_0) \in [V(0, T; U)]', \tag{1.4.1.22}$$

recalling (1.4.1.19). Invoking (1.4.1.15) yields

$$-u^0(\,\cdot\,, 0; y_0) = \Lambda_{0T}^{-1}[L_T^*G^*Ge^{AT}y_0 + L^*R^*Re^{A\cdot}y_0] \in V(0, T; U). \tag{1.4.1.23}$$

We note that (1.4.1.23) defines the optimal control explicitly in terms of the data of the problem, after which $y^0(t, 0, y_0)$ is obtained by the optimal dynamics (1.4.1.20).

Problem on [s, T] We now return to the optimal control problem (1.1.3) except that we now consider it over the time interval $[s, T]$, with initial time $t = s$, rather than $t = 0$, $0 \le s < T$, and initial datum x. We call $\{u^0(\cdot, s; x)$ and $y^0(\cdot, s; x)\}$ the corresponding unique optimal pair. Repeating the above procedure, we arrive at the following explicit characterization of the optimal pair:

$$\begin{aligned} -u^0(\cdot, s; x) &= L_s^* R^* R y^0(\cdot, s; x) + L_{sT}^* G^* G y^0(T, s; x) \\ &\in [V(s, T; U)]', \end{aligned} \tag{1.4.1.24}$$

$$\begin{aligned} -u^0(\cdot, s; x) &= \Lambda_{sT}^{-1}\big[L_{sT}^* G^* G e^{A(T-s)} x + L_s^* R^* R e^{A(\cdot - s)} x\big] \\ &\in V(s, T; U), \end{aligned} \tag{1.4.1.25}$$

which is the counterpart of Eqns. (1.4.1.21) and (1.4.1.23), where we note that the element in the square brackets in (1.4.1.25) belongs, say, to $[V(s, T; U)]'$ so that u^0 in (1.4.1.25) is well defined by (1.4.1.14). In going from (1.4.1.24) to (1.4.1.25) we have used the optimal dynamics

$$y^0(t, s; x) = e^{A(t-s)} x + \{L_s u^0(\cdot, s; x)\}(t) \tag{1.4.1.26}$$

for both $y^0(\cdot, s; x)$ and $y^0(T, s; x)$ in (1.4.1.24). We note again that u^0 in (1.4.1.25), and hence y^0 in (1.4.1.26), are given explicitly in terms of the data of the problem. We shall often use the following version of (1.4.1.24):

$$\begin{aligned} &u^0(\cdot, s; x) + L_s^* R^* R L_s u^0(\cdot, s; x) \\ &\quad = -L_{sT}^* G^* G y^0(T, s; x) - L_s^* R^* R\big(e^{A(\cdot - s)} x\big). \end{aligned} \tag{1.4.1.27}$$

Remark 1.4.1.1 Use of the Liusternik's Lagrange multiplier theorem is potentially applicable to more general situations where the cost is not quadratic. In the quadratic cost case, it is possible to give a direct, elementary proof of the explicit formulas (1.2.1.1), (1.4.1.21) by "completing the square." This will be done explicitly in the abstract hyperbolic case of Lemma 8.3.1.3.

1.4.2 L_2-Estimates for $\{u^0, y^0\}$ and Z_f-Estimate for $Gy^0(T; \cdot\,; x)$. Limit Relations as $s \uparrow T$

Proposition 1.4.2.1 *With reference to the optimal pair $\{u^0(\cdot, s; x), y^0(\cdot, s; x)\}$, we have, for $x \in Y$, and all $0 \le s < T$:*

(i)

$$\|u^0(\cdot, s; x)\|_{L_2(s,T;U)} \le \|u^0(\cdot, s; x)\|_{V(s,T;U)} \le C_T \|x\|_Y; \tag{1.4.2.1}$$

(ii)

$$\|y^0(\cdot, s; x)\|_{L_2(s,T;Y)} \le C_T \|x\|_Y; \tag{1.4.2.2}$$

(iii)

$$\|Gy^0(T,s;x)\|_{Z_f} \leq C_T\|x\|_Y, \tag{1.4.2.3}$$

with C_T a generic constant independent of s, $0 \leq s < T$.

Proof. (i) From (1.4.1.25) we compute via (1.4.1.9b), (1.4.1.15), (1.4.1.16), (1.4.1.5a), and (1.1.4),

$$\begin{aligned}
&\|u^0(\,\cdot\,,s;x)\|_{L_2(s,T;U)} \leq \|u^0(\,\cdot\,,s;x)\|_{V(s,T;U)} \\
&\text{(by (1.4.1.15), (1.4.1.25))} \quad \leq C_T\left\|L_{sT}^*G^*Ge^{A(T-s)}x + L_s^*R^*Re^{A(\cdot-s)}x\right\|_{[V(s,T;U)]'} \\
&\text{(by (1.4.1.16), (1.4.1.9b))} \quad \leq C_T\left\{\left\|Ge^{A(T-s)}x\right\|_{Z_f} + \left\|L_s^*R^*Re^{A(\cdot-s)}x\right\|_{L_2(s,T;U)}\right\} \\
&\text{(by (1.4.1.5a), (1.1.4))} \quad \leq C_T\|x\|_Y,
\end{aligned} \tag{1.4.2.4}$$

and (1.4.2.1) is proved.

(ii) Inequality (1.4.2.2) follows then from the optimal dynamics (1.4.1.26) via (1.4.1.3) and inequality (1.4.2.1) established in (i).

(iii) From (1.4.1.26) we have, for $t = T$ via (1.4.1.6),

$$Gy^0(T,s;x) = Ge^{A(T-s)}x + GL_{sT}u^0(\,\cdot\,,s;x). \tag{1.4.2.5}$$

Then, inequality (1.4.2.3) follows from (1.4.2.5) via (1.4.1.16) (left) and (1.4.2.1) (right), with u^0 in $V(s,T;U)$, and (1.1.4). □

We now pass to some important limit relations, as $s \uparrow T$.

Proposition 1.4.2.2 *With reference to (1.4.1.25) and (1.4.2.5), we have*

(i)

$$\lim_{s\uparrow T}\|u^0(\,\cdot\,,s;x)\|_{V(s,T;U)} = 0, \quad x \in Y; \tag{1.4.2.6}$$

(ii)

$$\lim_{s\uparrow T}\|Gy^0(T,s;x) - Gx\|_{Z_f} = 0, \quad x \in Y. \tag{1.4.2.7}$$

Proof. (i)

Step 1 Adding and subtracting, we write for $x \in Y$,

$$u^0(\,\cdot\,,s;x) = u^0(\,\cdot\,,s;x) + \Lambda_{sT}^{-1}L_{sT}^*G^*Gx - \Lambda_{sT}^{-1}L_{sT}^*G^*Gx, \tag{1.4.2.8}$$

so that (1.4.2.8) implies, recalling (1.4.1.25),

$$\begin{aligned}
\|u^0(\,\cdot\,,s;x)\|_{V(s,T;U)} \leq{}& \left\|u^0(\,\cdot\,,s;x) + \Lambda_{sT}^{-1}L_{sT}^*G^*Gx\right\|_{V(s,T;U)} \\
&+ \left\|\Lambda_{sT}^{-1}L_{sT}^*G^*Gx\right\|_{V(s,T;U)}.
\end{aligned} \tag{1.4.2.9}$$

Step 2 Regarding the first term on the right-hand side of (1.4.2.9), we now establish that

$$\left\| u^0(\,\cdot\,, s; x) + \Lambda_{sT}^{-1} L_{sT}^* G^* G x \right\|_{V(s,T;U)} \to 0 \quad \text{as } s \uparrow T,\ x \in Y. \tag{1.4.2.10}$$

In fact, recalling at first (1.4.1.25) and (1.4.1.15), we estimate

$$\begin{aligned}
&\left\| u^0(\,\cdot\,, s; x) + \Lambda_{sT}^{-1} L_{sT}^* G^* G x \right\|_{V(s,T;U)} \\
\text{(by (1.4.1.25))} \quad &= \left\| -\Lambda_{sT}^{-1}\left[L_{sT}^* G^* G e^{A(T-s)}x + L_s^* R^* R e^{A(\cdot-s)}x\right] \right. \\
&\quad \left. + \Lambda_{sT}^{-1} L_{sT}^* G^* G x \right\|_{V(s,T;U)} &\text{(1.4.2.11)}\\
&\le \left\| -\Lambda_{sT}^{-1} L_{sT}^* G^* G\left[e^{A(T-s)}x - x\right] \right\|_{V(s,T;U)} \\
&\quad + \left\| \Lambda_{sT}^{-1} L_s^* R^* R e^{A(\cdot-s)}x \right\|_{V(s,T;U)} &\text{(1.4.2.12)}\\
\text{(by (1.4.1.15))} \quad &\le \left\| L_{sT}^* G^* G\left[e^{A(T-s)}x - x\right] \right\|_{[V(s,T;U)]'} \\
&\quad + \left\| L_s^* R^* R e^{A(\cdot-s)}x \right\|_{[V(s,T;U)]'}. &\text{(1.4.2.13)}
\end{aligned}$$

As to the first term on the right-hand side of (1.4.2.13), we recall (1.4.1.10b) and obtain

$$\left\| L_{sT}^* G^* G\left[e^{A(T-s)}x - x\right] \right\|_{[V(s,T;U)]'} \le c \left\| G\left[e^{A(T-s)}x - x\right] \right\|_{Z_f} \to 0 \quad \text{as } s \uparrow T. \tag{1.4.2.14}$$

As to the second term on the right-hand side of (1.4.2.13), we recall the inclusions with continuous injections in (1.4.1.9a), and obtain

$$\left\| L_s^* R^* R e^{A(\cdot-s)}x \right\|_{[V(s,T;U)]'} \le c \left\| L_s^* R^* R e^{A(\cdot-s)}x \right\|_{L_2(s,T;U)} \to 0 \quad \text{as } s \uparrow T. \tag{1.4.2.15}$$

Thus, (1.4.2.14) and (1.4.2.15), used in (1.4.2.13), yield (1.4.2.10), as desired.

Step 3 Regarding the second term on the right-hand side of (1.4.2.9), we next establish that

$$\left\| \Lambda_{sT}^{-1} L_{sT}^* G^* G x \right\|_{V(s,T;U)} \to 0 \quad \text{as } s \uparrow T,\ x \in Y. \tag{1.4.2.16}$$

Indeed, we first recall that $\mathcal{D}((GL_T)^*) = \mathcal{D}((GL_{sT})^*)$ is dense in Z_f by the assumption that GL_{0T} be closed (closable) via the equivalence in (1.2.1.25). Thus, with $x \in Y$ and hence $Gx \in Z_f$, given $\epsilon > 0$, we can select a point $z \in \mathcal{D}(L_{sT}^* G^*) \subset Z_f$ (z depends on $\epsilon > 0$ and Gx) such that

$$L_{sT}^* G^* z \in L_2(s, T; U) \subset [V(s, T; U)]' \quad \text{and} \quad \|Gx - z\|_{Z_f} < \epsilon. \tag{1.4.2.17}$$

Next, recalling (1.4.1.15) and (1.4.1.10b) again, we estimate with reference to such point z:

$$\begin{aligned}
\left\| \Lambda_{sT}^{-1} L_{sT}^* G^* G x \right\|_{V(s,T;U)} &= \left\| \Lambda_{sT}^{-1} L_{sT}^* G^* [Gx - z] \right\|_{V(s,T;U)} \\
&\quad + \left\| \Lambda_{sT}^{-1} L_{sT}^* G^* G z \right\|_{V(s,T;U)} &\text{(1.4.2.18)}
\end{aligned}$$

$$\text{(by (1.4.1.15))} \qquad \le \|L_{sT}^* G^*[Gx - z]\|_{[V(s,T;U)]'} + \|L_{sT}^* G^* G z\|_{[V(s,T;U)]'} \tag{1.4.2.19}$$

$$\text{(by (1.4.1.10b))} \qquad \le c\|Gx - z\|_{Z_f} + \|L_{sT}^* G^* G z\|_{L_2(s,T;U)}, \tag{1.4.2.20}$$

recalling also (1.4.1.9a) in the last step. But the second term on the right-hand side goes to zero as $s \uparrow T$: $\|L_{sT}^* G^* G z\| \to 0$ [see (1.4.1.7b)]. This combined with (1.4.2.17) yields (1.4.2.16), as desired.

Step 4 Using (1.4.2.10) and (1.4.2.16) in (1.4.2.9) yields (1.4.2.6), as desired. Part (i) is proved.

(ii) Returning to (1.4.2.5), we estimate for $x \in Y$, recalling (1.4.1.10a),

$$\|Gy^0(T, s; x) - Gx\|_{Z_f} = \left\|Ge^{A(T-s)}x + GL_{sT}u^0(\,\cdot\,, s; x) - Gx\right\|_{Z_f} \tag{1.4.2.21}$$

$$\le \left\|G\left[e^{A(T-s)}x - x\right]\right\|_{Z_f} + \|GL_{sT}u^0(\,\cdot\,, s; x)\|_{Z_f} \tag{1.4.2.22}$$

$$\text{(by (1.4.1.10a))} \qquad \le c\left[\left\|e^{A(T-s)}x - x\right\|_Y + \|u^0(\,\cdot\,, s; x)\|_{V(s,T;U)}\right] \tag{1.4.2.23}$$

$$\to 0 \ \text{ as } s \uparrow T, \tag{1.4.2.24}$$

recalling in the last step (1.4.2.6) of part (i). Thus, (1.4.2.7) of part (ii) is proved by (1.4.2.24). $\Box$

1.4.3 Definition of Operators $\Phi(T, s)$ and $P(t)$ and First Properties

The (Evolution) Operator $\Phi(t, s)$ It is convenient to introduce the (evolution) operator $\Phi(t, s)$ defined by

$$\Phi(t, s)x \equiv y^0(t, s; x) \in L_2(s, T; Y), \quad x \in Y, \tag{1.4.3.1}$$

where the claimed regularity is due to (1.4.2.2). We next collect some preliminary transition properties for the optimal pair $\{u^0(t, s; x), y^0(t, s; x) \equiv \Phi(t, s)x\}$, for now in the a.e. sense in $[s, T]$: After Theorem 1.4.4.6(i), Eqn. (1.4.4.32), and after Lemma 1.4.6.2(ii), these transition properties may be boosted to a pointwise sense, as long as $t < T$, respectively.

Proposition 1.4.3.1 *For s fixed, the following transition properties hold true for the optimal pair:*

(i)

$$u^0(t, \tau; \Phi(\tau, s)x) = u^0(t, s; x) \text{ a.e. in } \tau \text{ and } t,$$
$$0 \le s \le \tau \le t \le T; \quad x \in Y. \tag{1.4.3.2}$$

(ii)

$$\Phi(t,t) = \textit{identity on } Y; \quad \Phi(t,s) = \Phi(t,\tau)\Phi(\tau,s), \ \textit{a.e. in } \tau \textit{ and } t, \quad 0 \le s \le \tau \le t \le T. \quad (1.4.3.3)$$

At $t = T$, we moreover have, a.e. in τ,

(iii)

$$G\Phi(T,s)x = G\Phi(T,\tau)\Phi(\tau,s)\,x, \quad x \in Y. \quad (1.4.3.4)$$

Proof.

(i) = (1.4.3.2) We return to Eqn. (1.4.1.25), rewritten by (1.4.1.11a) and for $x \in Y$ as

$$\begin{aligned} &u^0(t,s;x) + \{L_s^*R^*RL_su^0(\,\cdot\,,s;x)\}(t) + \{L_{sT}^*G^*GL_{sT}u^0(\,\cdot\,,s;x)\}(t) \\ &\quad = -\left\{L_s^*R^*RL_se^{A(\cdot-s)}x\right\}(t) - \left\{L_{sT}^*G^*Ge^{A(T-s)}x\right\}(t) \in [V(s,T;U)]' \quad (1.4.3.5) \end{aligned}$$

[see also (1.4.1.21b)] or, explicitly via (1.4.1.3)–(1.4.1.7),

$$\begin{aligned} u^0(t,s;x) &+ \int_t^T B^*e^{A^*(\sigma-t)}R^*R\int_s^\sigma e^{A(\sigma-r)}Bu^0(r,s;x)\,dr\,d\sigma \\ &+ B^*e^{A^*(T-t)}G^*G\int_s^T e^{A(T-r)}Bu^0(r,s;x)\,dr \\ &= -\int_t^T B^*e^{A^*(\sigma-t)}R^*Re^{A(\sigma-s)}x\,d\sigma \\ &- B^*e^{A^*(T-t)}G^*Ge^{A(T-s)}x \in [V(s,T;U)]'. \quad (1.4.3.6) \end{aligned}$$

We now rewrite (1.4.3.6) with s replaced by τ, and with $x \in Y$ replaced by $\Phi(\tau,s)x \in Y$ a.e. in τ, in view of (1.4.2.2), (1.4.3.1). We obtain:

$$\begin{aligned} &u^0(t,\tau;\Phi(\tau,s)x) + \int_t^T B^*e^{A^*(\sigma-t)}R^*R\int_\tau^\sigma e^{A(\sigma-r)}Bu^0(r,\tau;\Phi(\tau,s)x)\,dr\,d\sigma \\ &+ B^*e^{A^*(T-t)}G^*G\int_\tau^T e^{A(T-r)}Bu^0(r,\tau;\Phi(\tau,s)x)\,dr \\ &= -\int_t^T B^*e^{A^*(\sigma-t)}R^*Re^{A(\sigma-\tau)}\Phi(\tau,s)x\,d\sigma \\ &-B^*e^{A^*(T-t)}G^*Ge^{A(T-\tau)}\Phi(\tau,s)x, \quad (1.4.3.7) \end{aligned}$$

valid in $[V(s,T;U)]'$, a.e. in τ and t. Next, we return to (1.4.3.6), split the integrals $\int_s^\sigma = \int_s^\tau + \int_\tau^\sigma$ and likewise $\int_s^T = \int_s^\tau + \int_\tau^T$ in the two integral terms on its left-hand side, and finally subtract (1.4.3.7) from (1.4.3.6). We obtain, still in $[V(s, T; V)]'$, a.e. in τ and t,

$$\begin{aligned}
&u^0(t, s; x) - u^0(t, \tau; \Phi(\tau, s)x) \\
&\quad + \int_t^T B^* e^{A^*(\sigma - t)} R^* R \int_\tau^\sigma e^{A(\sigma - r)} B[u^0(r, s; x) - u^0(r, \tau; \Phi(\tau, s)x)]\, dr\, d\sigma \\
&\quad + \int_t^T B^* e^{A^*(\sigma - t)} R^* R \int_s^\tau e^{A(\sigma - r)} B u^0(r, s; x)\, dr\, d\sigma \\
&\quad + B^* e^{A^*(T-t)} G^* G \int_\tau^T e^{A(T-r)} B[u^0(r, s; x) - u^0(r, \tau; \Phi(\tau, s)x)]\, dr \\
&\quad + B^* e^{A^*(T-t)} G^* G \int_s^\tau e^{A(T-r)} B u^0(r, s; x)\, dr \\
&= \int_t^T B^* e^{A^*(\sigma - t)} R^* R \big[e^{A(\sigma - \tau)} \Phi(\tau, s)x - e^{A(\sigma - s)} x\big]\, d\sigma \\
&\quad + B^* e^{A^*(T-t)} G^* G \big[e^{A(T-\tau)} \Phi(\tau, s)x - e^{A(T-s)} x\big]. \qquad (1.4.3.8)
\end{aligned}$$

Finally, we apply $e^{A(\cdot - \tau)} t_0$, the optimal dynamics (1.4.1.26) with t replaced by τ, which yields via (1.4.3.2), a.e. in τ and σ,

$$e^{A(\sigma - \tau)} \Phi(\tau, s)x - e^{A(\sigma - s)} x = \int_s^\tau e^{A(\sigma - r)} B u^0(r, s; x)\, dr, \qquad (1.4.3.9)$$

$$e^{A(T-\tau)} \Phi(\tau, s)x - e^{A(T-s)} x = \int_s^\tau e^{A(T-r)} B u^0(r, s; x)\, dr. \qquad (1.4.3.10)$$

Inserting (1.4.3.9) and (1.4.3.10) in the last two terms on the right-hand side of (1.4.3.8) produces a cancellation of these terms against like terms, the third and the fifth, on the left-hand side of (1.4.3.8). Thus, (1.4.3.8) simplifies to

$$\begin{aligned}
&u^0(t, s; x) - u^0(t, \tau; \Phi(\tau, s)x) \\
&\quad + \int_t^T B^* e^{A^*(\sigma - t)} R^* R \int_\tau^\sigma e^{A(\sigma - r)} B[u^0(r, s; x) - u^0(r, \tau; \Phi(\tau, s)x)]\, dr\, d\sigma \\
&\quad + B^* e^{A^*(T-t)} G^* G \int_\tau^T e^{A(T-r)} B[u^0(r, s; x) - u^0(r, \tau; \Phi(\tau, s)x)]\, dr, \qquad (1.4.3.11)
\end{aligned}$$

in $[V(s, T; U)]'$, a.e. in τ and t; or, recalling (1.4.1.3)–(1.4.1.7) and (1.4.1.11) this simplifies to

$$\begin{aligned}
&[I_\tau + L_\tau^* R^* R L_\tau + L_{\tau T}^* G^* G L_{\tau T}][u^0(\cdot, s; x) - u^0(\cdot, \tau; \Phi(\tau, s)x)] \\
&\quad = \Lambda_{\tau T}[u^0(\cdot, s; x) - u^0(\cdot, \tau; \Phi(\tau, s)x)] = 0 \qquad (1.4.3.12)
\end{aligned}$$

in $[V(s, T; U)]'$, a.e. in τ and t. Since $\Lambda_{\tau T}$ is an isomorphism as in (1.4.1.14), then (1.4.3.12) yields (1.4.3.2), as desired.

(ii)* $\Rightarrow$ *(1.4.3.3) We now use the transition property (1.4.3.2) for u^0 to show the transition property (1.4.3.3) for y^0. The optimal dynamics (1.4.1.26) expressed for $\Phi(\tau, s)x$ in (1.4.3.1) yields, a.e. in τ and t,

$$\begin{aligned}\Phi(t,\tau)\Phi(\tau,s)x &= e^{A(t-\tau)}\Phi(\tau,s)\,x + \int_\tau^t e^{A(t-\sigma)}Bu^0(\sigma,\tau;\Phi(\tau,s)x)\,d\sigma \\ &= e^{A(t-\tau)}\left[e^{A(\tau-s)}x + \int_s^\tau e^{A(\tau-\sigma)}Bu^0(\sigma,s;x)\,d\sigma\right] \\ &\quad + \int_\tau^t e^{A(t-\sigma)}Bu^0(\sigma,s;x)\,d\sigma, \qquad (1.4.3.13)\end{aligned}$$

where in the last step we have used (1.4.3.2). Thus (1.4.3.13) yields

$$\Phi(t,\tau)\Phi(\tau,s)x = e^{A(t-s)}x + \int_s^t e^{A(t-\sigma)}Bu^0(\sigma,s;x)\,d\sigma = \Phi(t,s)x,$$
$$\text{a.e. in } \tau \text{ and } t, \quad 0 \le s \le \tau \le t \le T, \quad x \in Y, \qquad (1.4.3.14)$$

and property (1.4.3.3) is proved.

(iii) Property (1.4.3.3) yields (1.4.3.4), via the regularity (1.4.2.2) for $Gy^0(T, s; x)$. □

The Operator $P(t)$ We next define the operator $P(t) \in \mathcal{L}(Y)$, $0 \le t < T$, by

$$P(t)x = \int_t^T e^{A^*(\tau-t)}R^*Ry^0(\tau,t;x)\,d\tau + e^{A^*(T-t)}G^*Gy^0(T,t;x) \qquad (1.4.3.15a)$$

$$= \int_t^T e^{A^*(\tau-t)}R^*R\Phi(\tau,t)x\,d\tau + e^{A^*(T-t)}G^*G\Phi(T,t)x, \quad x \in Y. \qquad (1.4.3.15b)$$

$P(t)$ in (1.4.3.15) is defined explicitly in terms of the data of the problem, since so is y^0, as remarked below (1.4.1.26).

Proposition 1.4.3.2 *With reference to (1.4.3.15), we have*

(i)

$$P(t) \in \mathcal{L}(Y; L_\infty(0,T;Y)), \qquad P(t) \in \mathcal{L}(Y), \quad t < T; \qquad (1.4.3.16)$$

(ii)

$$-u^0(t,s;x) = B^*P(t)y^0(t,s;x), \qquad 0 \le s \le t < T; \quad x \in Y. \quad (1.4.3.17)$$

Proof. (i) Property (1.4.3.16) follows from (1.4.3.15) via the regularity properties (1.4.2.2) and (1.4.2.3) of Proposition 1.4.2.1. [It will be boosted to $P(t) \in \mathcal{L}(Y;$

$C([0, T-\epsilon]; Y), \forall\, \epsilon > 0$ small, and $\epsilon = 0$ for $\gamma < 1/2$ using the properties of the subsequent Lemma 1.4.6.2(iii) in the proof of (the stronger) Proposition 1.4.6.5.]

(ii) We rewrite (1.4.1.24) explicitly by virtue of (1.4.1.4), (1.4.1.7), and (1.4.3.1) to get

$$\begin{aligned} -u^0(t,s;x) &= \int_t^T B^* e^{A^*(\tau - t)} R^* R \Phi(\tau, s) x \, d\tau \\ &\quad + B^* e^{A^*(T-t)} G^* G \Phi(T,s) x \quad \text{a.e. in } t. \end{aligned} \tag{1.4.3.18}$$

Writing in (1.4.3.18): $\Phi(\tau,s)x = \Phi(\tau,t)\Phi(t,s)x$ a.e. in τ, t by (1.4.3.3), and $G\Phi(T,s)x = G\Phi(T,t)\Phi(t,s)x$ a.e. in t by (1.4.3.4), we obtain (1.4.3.17) from (1.4.3.18) via (1.4.3.15) and (1.4.3.1). $\square$

We note that, by virtue of (1.4.3.17), the optimal dynamics (1.4.1.26) can be explicitly rewritten as

$$y^0(t,s;x) = e^{A(t-s)}x - \int_s^t e^{A(t-\tau)} BB^* P(\tau) y^0(\tau,s;x)\, d\tau, \tag{1.4.3.19a}$$

$$\Phi(t,s)x = e^{A(t-s)}x - \int_s^t e^{A(t-\tau)} BB^* P(\tau) \Phi(\tau,s)x\, d\tau. \tag{1.4.3.19b}$$

1.4.4 Smoothing Properties of L_s and L_s^* at $t = T$, and on $L_p(s,T;\ \cdot\ )$-Spaces. Pointwise Estimates for $u^0(t,s;x)$, $y^0(t,s;x)$, and $P(t)$

Smoothing Properties of L_s and L_s^* at $t = T$ Let X be a Hilbert space, and let r be a real number. We recall for convenience the Banach space $C_r([s,T]; X)$ from Chapter 0, Eqn. (0.4):

$$C_r([s,T]; X) = \left\{ f(t) \in C([s,T); X) : \|f\|_{C_r([s,T];X)} = \sup_{s \le t < T} (T-t)^r \|f(t)\|_X < \infty \right\}. \tag{1.4.4.1}$$

In the interesting case $r > 0$, r measures the singularity of $f(t)$ as $t \to T$. Note that $C_r([s,T]; X) \subset L_q(s,T; X)$ for $rq < 1$.

Proposition 1.4.4.1 *With reference to the operators L_s and L_s^* defined in (1.4.1.2) and (1.4.1.4) and $0 \le \gamma < 1$ in (1.1.2), we have*

(i) For $r + \gamma < 1$,

$$L_s : \textit{continuous } C_r([s,T]; U) \to C([s,T]; Y); \tag{1.4.4.2a}$$

$$\|L_s u\|_{C([s,T];Y)} \le \frac{C_{T,\gamma}}{1-(\gamma+r)} (T-s)^{1-(\gamma+r)} \|u\|_{C_r([s,T];U)}, \tag{1.4.4.2b}$$

so that the bound in (1.4.4.2) may be made independent of s, $0 \le s \le T$;

(ii) For $r+\gamma > 1$,

$$L_s : \textit{continuous } C_r([s,T];U) \to C_{r+\gamma-1}([s,T];Y); \tag{1.4.4.3a}$$

$$\|L_s u\|_{C_r+\gamma-1([s,T];Y)} \le C_{T\gamma} \|u\|_{C_r([s,T];U)}, \tag{1.4.4.3b}$$

so that the bound in (1.4.4.3) may be made independent of s, $0 \le s \le T$ *(for the case* $r+\gamma = 1$, *see (1.4.4.8b) below).*

(iii) For $0 \le r < 1$,

$$L_s^* : \textit{continuous } C_r([s,T];Y) \to C_{r+\gamma-1}([s,T];U); \tag{1.4.4.4a}$$

$$\|L_s^* f\|_{C_r+\gamma-1([s,T];U)} \le C_{T\gamma} 2^{r+\gamma-1} \max\left\{\frac{1}{1-r}, \frac{1}{1-\gamma}\right\} \|f\|_{C_r([s,T];Y)}, \tag{1.4.4.4b}$$

so that the bound in (1.4.4.4) is independent of s, $0 \le s \le T$.

Proof. We first note directly the required regularity for $s \le t < T$; that is, that $(L_s u)(t) \in C([s,T);Y)$ and $(L_s^* f)(t) \in C([s,T);U)$ for $u \in C_r([s,T];U)$ and $f \in C_r([s,T];Y)$ respectively. Thus, it remains to prove the appropriate potential singularity at $t = T$.

(i) Let $u \in C_r([s,T];U)$. By (1.4.1.2), assumption (1.1.2) on B, and analyticity of e^{At} [Pazy, 1983, p. 74], we have

$$\begin{aligned}
\|(L_s u)(t)\|_Y &= \left\| \int_s^t (-A)^\gamma e^{A(t-\tau)} (-A)^{-\gamma} B u(\tau)\, d\tau \right\|_Y \\
\text{(by (1.1.2))} \quad &\le C_{T\gamma} \int_s^t \frac{\|u(\tau)\|_U (T-\tau)^r}{(t-\tau)^\gamma (T-\tau)^r}\, d\tau \\
\text{(by (1.4.4.1))} \quad &\le C_{T\gamma} \|u\|_{C_r([s,T];U)} \int_s^t \frac{d\tau}{(t-\tau)^\gamma (T-\tau)^r}.
\end{aligned} \tag{1.4.4.5}$$

For any $r \ge 0$, we use $(T-\tau)^r \ge (t-\tau)^r$ in the last integral in (1.4.4.5) so that

$$\|(L_s u)(t)\|_Y \le C_{T\gamma} \|u\|_{C_r([s,T];U)} \frac{(t-s)^{1-(\gamma+r)}}{1-(\gamma+r)}, \tag{1.4.4.6}$$

for $r+\gamma < 1$ as assumed, and (1.4.4.6) yields the desired bound. Conclusion (1.4.4.2) is established.

(ii) For $r+\gamma \ge 1$, $0 \le \gamma < 1$, we integrate by parts, thus obtaining

$$\int_s^t \frac{d\tau}{(t-\tau)^\gamma (T-\tau)^r} = \frac{(t-s)^{1-\gamma}}{1-\gamma} \frac{1}{(T-s)^r} + \frac{r}{1-\gamma} \int_s^t \frac{(t-\tau)^{1-\gamma}}{(T-\tau)^{1+r}}\, d\tau. \tag{1.4.4.7}$$

The first term is continuous in t. As to the second, since $\frac{t-\tau}{T-\tau} \leq 1$, we obtain

$$\int_s^t \frac{(t-\tau)^{1-\gamma}}{(T-\tau)^{1+r}}\,d\tau = \int_s^t \left(\frac{t-\tau}{T-\tau}\right)^{1-\gamma} \frac{d\tau}{(T-\tau)^{r+\gamma}} \leq \int_s^t \frac{d\tau}{(T-\tau)^{r+\gamma}}$$

$$\leq \begin{cases} \frac{1}{r+\gamma-1}\frac{1}{(T-t)^{r+\gamma-1}}, & r+\gamma > 1; \quad (1.4.4.8a) \\ \ln\left(\frac{T-s}{T-t}\right), & r+\gamma = 1. \quad (1.4.4.8b) \end{cases}$$

Then (1.4.4.8a), together with (1.4.4.7), used in (1.4.4.5) yields (1.4.4.3a).

(iii) Let $f \in C_r([s,T];Y)$. By (1.4.1.4), assumption (1.1.2) on B, and analyticity of e^{At} [Pazy, 1983, p. 74], we get

$$\begin{aligned}
\|(L_s^* f)(t)\|_U &= \left\| \int_t^T B^*(-A^*)^{-\gamma}(-A^*)^{\gamma} e^{A^*(\tau-t)} f(\tau)\,d\tau \right\|_U \\
\text{(by (1.1.2) and (1.4.4.1))} \quad &\leq C_{T\gamma} \int_t^T \frac{\|f(\tau)\|_Y (T-\tau)^r}{(\tau-t)^{\gamma}(T-\tau)^r}\,d\tau \\
&\leq C_{T\gamma} \|f\|_{C_r([s,T];Y)} \int_t^T \frac{1\,d\tau}{(\tau-t)^{\gamma}(T-\tau)^r} \\
&\leq C_{T\gamma} \|f\|_{C_r([s,T];Y)} \left\{ \int_t^{t+\frac{T-t}{2}} \frac{d\tau}{(\tau-t)^{\gamma}(T-\tau)^r} \right. \\
&\quad \left. + \int_{t+\frac{T-t}{2}}^T \frac{d\tau}{(\tau-t)^{\gamma}(T-\tau)^r} \right\} \qquad (1.4.4.9)
\end{aligned}$$

(using $T-\tau \geq (T-t)/2$ in the first integral, and $(\tau-t) \geq (T-t)/2$ in the second integral with $0 \leq r < 1$)

$$\leq C_{T\gamma}\|f\|_{C_r([s,T];Y)} \left\{ \left(\frac{2}{T-t}\right)^r \left(\frac{T-t}{2}\right)^{1-\gamma} \frac{1}{1-\gamma} + \left(\frac{2}{T-t}\right)^{\gamma} \left(\frac{T-t}{2}\right)^{1-r} \frac{1}{1-r} \right\}$$

$$\|(L_s^* f)(t)\|_U \leq C_{T\gamma} 2^{r+\gamma-1} \max\left\{\frac{1}{1-\gamma}, \frac{1}{1-r}\right\} \frac{1}{(T-t)^{r+\gamma-1}} \|f\|_{Cr([s,T];Y)}, \qquad (1.4.4.10)$$

and (1.4.4.10) yields (1.4.4.4), as desired. □

The above Proposition 1.4.4.1 says that L_s and L_s^* are smoothing operators: L_s reduces the order of the singularity at $t = T$ from r to 0 if $r+\gamma < 1$ and from r to $r + (\gamma - 1)$ if $r + \gamma > 1$. Similarly, L_s^* reduces the order of the singularity at $t = T$ from r to $r + (\gamma - 1)$.

Corollary 1.4.4.2 *Given $0 < r < 1$ with $r + \gamma < 2$, there exists a positive integer $n_0 = n_0(r)$ such that for all positive integers $n \geq n_0(r)$ we have*

$$(L_s^* R^* R L_s)^n : \text{continuous } C_r([s, T]; U) \to C([s, T]; U), \tag{1.4.4.11}$$

that is,

$$\|(L_s^* R^* R L_s)^n v\|_{C([s,T];U)} \leq C_T \|v\|_{C_r([s,T];U)}, \tag{1.4.4.12}$$

with uniform norm bound that may be taken independent of s, $0 \leq s \leq T$.

Proof. One applies the results of Proposition 1.4.4.1, recursively, with R^* and R bounded as assumed in (1.1.4):

$$\begin{array}{ccccc} r & \xrightarrow{L_s} & r + (\gamma - 1) & \xrightarrow{L_s^*} & r + 2(\gamma - 1) \\ \text{for } n_0(r) = 1, \text{ require } r + 2(\gamma - 1) \leq 0 & & & & \Big\downarrow L_s \\ & & r + 4(\gamma - 1) & \xleftarrow{L_s^*} & r + 3(\gamma - 1) \end{array}$$

for $n_0(r) = 2$, require $r + 4(\gamma - 1) \leq 0$; etc.

After n_0 iterations, one obtains a space $C_k([s, T]; U)$ with $k \leq 0$. For $r = \gamma$, we may take $n_0(r) = 1$ if $0 \leq r < 2/3$; $n_0(r) = 2$ if $2/3 \leq r < 4/5$, $n_0(r) = 3$ if $4/5 \leq r < 6/7$; etc. Details may be omitted. □

Smoothing Properties of L_s and L_s^* on $L_p(s, T; \cdot)$-Spaces Another result showing that L_s and L_s^* are smoothing operators, this time on the spaces $L_p(s, T; \cdot)$, is given next.

Theorem 1.4.4.3 *With reference to the operators L_s and L_s^* defined in (1.4.1.2), (1.4.1.4), and $0 \leq \gamma < 1$ in (1.1.2), we have:*

(i)

$$L_s : \text{continuous } L_2(s, T; U) \to L_{r_1}(s, T; Y), \quad r_1 > 2, \tag{1.4.4.13}$$

where r_1 is an arbitrary positive number satisfying $r_1 < 2/(2\gamma - 1)$, where $2/(2\gamma - 1) > 2$ for $1/2 \leq \gamma < 1$; for $0 \leq \gamma < 1/2$, we may take $r_1 = \infty$.

(ii)

$$L_s^* : \text{continuous } L_{r_1}(s, T; Y) \to L_{r_2}(s, T; U), \tag{1.4.4.14}$$

where r_1 is as in (i), and r_2 is any positive number satisfying $r_2 < 2/(4\gamma - 3)$, where $2/(4\gamma - 3) > r_1$ for $3/4 \leq \gamma < 1$; for $0 < \gamma < 3/4$, we may take $r_2 = \infty$.

(iii) *Generally, let* $r_0 = 2$, *and let* r_n, $n = 1, 2, \ldots$ *be arbitrary positive number such that*

$$2 < r_1 < \cdots < r_n < \frac{2}{2n\gamma - (2n-1)}, \quad n = 1, 2, \ldots \quad \textit{for } \frac{2n-1}{2n} \le \gamma < 1. \tag{1.4.4.15}$$

Then, for $n = 0, 2, 4, \ldots$ *we have*

$$L_s : \textit{continuous } L_{r_n}(s, T; U) \to L_{r_{n+1}}(s, T; Y), \tag{1.4.4.16}$$

where for $0 \le \gamma < [2(n+1) - 1]/[2(n+1)]$ *we may take* $r_{n+1} = \infty$;

$$L_s^* : \textit{continuous } L_{r_{n+1}}(s, T; Y) \to L_{r_{n+2}}(s, T; U), \tag{1.4.4.17}$$

where r_{n+1} *in (1.4.4.17) is the same as in (1.4.4.16), and where for* $0 \le \gamma < [2(n+2) - 1]/[2(n+2)]$ *we may take* $r_{n+2} = \infty$.

(iv) *for* $p > 1/(1-\gamma)$,

$$L_s : \textit{continuous } L_p(s, T; U) \to C([s, T]; Y). \tag{1.4.4.18}$$

(v) *Thus a fortiori, there exists a positive integer* $n_1 = n_1(\gamma)$ *such that*

$$(L_s^* R^* R L_s)^{n_1} : \textit{continuous } L_2(s, T; U) \to C([s, T]; U). \tag{1.4.4.19}$$

We may take $n_1(\gamma) = 1$ *if* $\gamma < 3/4$; *and* $n_1(\gamma) = 2$ *if* $3/4 \le \gamma < 5/6$; $n_1(\gamma) = 3$ *if* $5/6 \le \gamma < 7/8$; *etc.*

In all cases the operator norm has a bound that may be taken to be independent of s, $0 \le s \le T$.

Proof. The proof uses assumption (1.1.2) on B, analyticity of e^{At} [Pazy, 1983, p. 74], and the Young's inequality [Sadosky, 1979, p. 149].

(i) By (1.4.1.2) we compute

$$\begin{aligned} \|(L_s u)(t)\|_Y &= \left\| \int_s^t (-A)^\gamma e^{A(t-\tau)} (-A)^{-\gamma} B u(\tau)\, d\tau \right\|_Y \\ \text{(by (1.1.2))} \qquad &\le C_T \int_0^t \frac{\|u(\tau)\|_U \, d\tau}{(t-\tau)^\gamma} \in L_{r_1}(0, T), \end{aligned} \tag{1.4.4.20a}$$

after using the Young's inequality with $\|u(t)\|_U \in L_2(0, T)$ and $1/t^\gamma \in L_q(0, T)$ for any q such that $\gamma q < 1$, so that

$$\frac{1}{r_1} = \frac{1}{q} + \frac{1}{2} - 1 = \frac{2-q}{2q}, \quad \text{or } r_1 = \frac{2}{\frac{2}{q} - 1} < \frac{2}{2\gamma - 1}, \quad \text{for } 2\gamma > 1, \tag{1.4.4.20b}$$

and $r_1 \uparrow 2/(2\gamma - 1)$ as $1/q \downarrow \gamma$; while r_1 may be taken $r_1 = \infty$ if $0 \le \gamma < 1/2$. Part (i) is proved.

(iv) We return to the convolution integral (1.4.4.20) where we now take $\|u(t)\|_U \in L_p(0, T)$, $p > 1/(1-\gamma)$ so that

$$\frac{1}{r_1} = \frac{1}{q} + \frac{1}{p} - 1 < \frac{1}{q} + 1 - \gamma - 1 \downarrow 0 \quad \text{as } \frac{1}{q} \downarrow \gamma.$$

Thus, we now have $r_1 = \infty$, and then by convolution $\|(L_s u)(t)\| \in C([s, T])$ [Sadosky, 1979, p. 29], with a bound on the operator norm that may be taken independent of s. Hence

$$\|L_s u\|_{L_\infty(s,T;Y)} \le C_T \|u\|_{L_p(s,T;U)}. \tag{1.4.4.21}$$

To boost (1.4.4.21) to (1.4.4.18), we may use an approximation argument. Given $u \in L_p(s, T; U)$, we pick $u_n \in C^1([s, T]; U)$ such that $u_n \to u$ in $L_p(s, T; U)$. By integration by parts on $L_s u$ in (1.4.1.2), we see directly that $(Lu_n)(t) \in C([s, T]; Y)$.

By (1.4.4.21) applied to $[u_n - u]$, we then obtain that $Lu_n \to Lu$ in $L_\infty(s, T; Y)$, and so $Lu \in C([s, T]; Y)$, as desired, and (1.4.4.18) is proved.

In the future, we shall not repeat this type of approximating argument to boost L_∞ in time to C-in time, while preserving the space regularity.

Finally, one may also prove (1.4.4.18) directly, $\|(Lu)(t_1) - (Lu)(t_2)\|_Y \to 0$ as $[t_1 - t_2] \to 0$ without passing through (1.4.4.21), by splitting the interval and using the Lebesgue dominated convergence theorem.

(ii) Similarly, by (1.4.1.4), we compute with $s \le t \le T$,

$$\begin{aligned} \|(L_s^* f)(t)\|_U &= \left\| \int_t^T B^*(-A^*)^{-\gamma}(-A^*)^\gamma e^{A^*(\tau - t)} f(\tau)\, d\tau \right\|_U \\ &\le C_T \int_t^T \frac{\|f(\tau)\|_Y \, d\tau}{(\tau - t)^\gamma} \in L_{r_2}(0, T), \end{aligned} \tag{1.4.4.22}$$

after using Young's inequality with $\|f(t)\|_Y \in L_{r_1}(0, T)$, $1/r_1 = 1/q - 1/2$ as in (1.4.4.20b) of part (i), $1/t^\gamma \in L_q(0, T)$ for any q such that $q\gamma < 1$, so that

$$\frac{1}{r_2} = \frac{1}{q} + \frac{1}{r_1} - 1 = \frac{1}{q} + \left(\frac{1}{q} - \frac{1}{2}\right) - 1 = \frac{4 - 3q}{2q},$$

or

$$r_2 = \frac{2}{\frac{4}{q} - 3} < \frac{2}{4\gamma - 3} \quad \text{for } 4\gamma > 3, \tag{1.4.4.23}$$

and $r_2 \uparrow 2/(4\gamma - 3)$ as $1/q \downarrow \gamma$; while r_2 may be taken $r_2 = \infty$ if $0 \le \gamma \le 3/4$. Part (ii) is proved.

The proof of part (iii) is similar.

(v) One applies the smoothing properties of L_s and L_s^* of parts (i)–(iv) recursively, with R and R^* bounded as assumed in (1.1.4) as in Corollary 1.4.4.2. Details are omitted. □

Theorem 1.4.4.4 *The operator $[I_s + L_s^* R^* R L_s]$ is boundedly invertible on the space $C_\gamma([s, T]; U)$ defined by (1.4.4.1):*

$$[I_s + L_s^* R^* R L_s]^{-1} \in \mathcal{L}(C_\gamma([s, T]; U)), \tag{1.4.4.24}$$

with uniform norm bound that depends on T and γ, but may be taken to be independent of s, $0 \leq s \leq T$.

Proof. Let $h \in C_\gamma([s,T];U)$. We seek a unique $g \in C_\gamma([s,T];U)$ such that

$$g + L_s^* R^* R L_s g = h. \tag{1.4.4.25}$$

To simplify the notation, we may take $R = I$ in the argument below.

Step 1 Given such h, if $n_0 = n_0(\gamma)$ is the positive integer of Corollary 1.4.4.2, Eqn. (1.4.4.11) for $r = \gamma < 1$, then there exists a unique $v \in L_2(s,T;U)$ such that

$$v + L_s^* L_s v = (L_s^* L_s)^{n_0} h \in C([s,T];U) \subset L_2(s,T;U), \tag{1.4.4.26}$$

since $I_s + L_s^* L_s$ is boundedly invertible on $L_2(s,T;U)$.

Step 2 We shall show that, in fact,

$$v \in C([s,T];U). \tag{1.4.4.27}$$

In fact, if $n_1 = 1$ (i.e., $0 \leq \gamma \leq 3/4$) in (1.4.4.19) of Theorem 1.4.4.3, then $L_s^* L_s v \in C([s,T];U)$, and then (1.4.4.26) yields (1.4.4.27). In general, we apply $(L_s^* L_s)^r$ to (1.4.4.26), with $r = 0, 1, \ldots, n_1 - 1$, thus obtaining

$$(L_s^* L_s)^r v + (L_s^* L_s)^{r+1} v = (L_s^* L_s)^{n_0 + r} h \in C([s,T];U), \quad r = 0, 1, \ldots, n_1 - 1, \tag{1.4.4.28}$$

where the regularity on the right of (1.4.4.28) is a consequence of (1.4.1.3c–d) for L_s and (1.4.1.5b–c) for L_s^* conservatively with $\theta = 0$, via the regularity of (1.4.4.26) (right). Starting from $r = n_1 - 1$ in (1.4.4.28), and $v \in L_2(s,T;U)$, we first obtain $(L_s^* L_s)^{r+1} v = (L_s^* L_s)^{n_1} v \in C([s,T];U)$ by (1.4.4.19); hence $(L_s^* L_s)^{n_1 - 1} v \in C([s,T];U)$ by (1.4.4.28). Next, using this latter information in (1.4.4.28), this time with $r = n_1 - 2$, leads to $(L_s^* L_s)^{n_1 - 2} v \in C([s,T];U)$. By repeating this procedure a finite number of times, we arrive at (1.4.4.27), as desired.

Step 3 Starting from the given h and v obtained in (1.4.4.26) satisfying (1.4.4.27), we shall finally define a finite sequence of functions called $g_{n_0-1}, g_{n_0-2}, g_{n_0-3}, \ldots, g_1, g$, whose last element g will be precisely the sought-after unique solution of (1.4.4.25). We define recursively

$$g_{n_0-1} = (L_s^* L_s)^{n_0-1} h - v \in C_\gamma([s,T];U), \tag{1.4.4.29$_{n_0-1}$}$$

$$g_{n_0-2} = (L_s^* L_s)^{n_0-2} h - g_{n_0-1} \in C_\gamma([s,T];U), \tag{1.4.4.29$_{n_0-2}$}$$

$$\vdots$$

$$g_1 = (L_s^* L_s) h - g_2 \in C_\gamma([s,T];U), \tag{1.4.4.29$_1$}$$

$$g = h - g_1 \in C_\gamma([s,T];U). \tag{1.4.4.29$_0$}$$

The (conservative) regularity noted on the right of ($1.4.4.29_{n_0-1}$) is a consequence of Proposition 1.4.4.1 applied to $h \in C_\gamma([s, T]; U)$ and of (1.4.4.27). Then, recursively, the other regularity statements follow down to ($1.4.4.29_0$). In particular, $g \in C_\gamma([s, T]; U)$. It is now an easy matter to show that such g is the unique sought-after solution of (1.4.4.25). (To gain understanding of this procedure, one may apply $L_s^* L_s, (L_s^* L_s)^2, \dots, (L_s^* L_s)^{n_0}$ to Eqn. (1.4.4.25) and compare the resulting n_0 identities with the ones in (1.4.4.29); in particular, $v = (L_s^* L_s)^{n_0} g$, $g_{n_0-1} = (L_s^* L_s)^{n_0-1} g$, etc.) Moreover, the bound on the uniform norm in (1.4.4.24) may be taken not to depend on s, $0 \le s \le T$ since this is the case for L_s and L_s^* in Proposition 1.4.4.1, Theorem 1.4.4.3, and Remark 1.4.4.1. Theorem 1.4.4.4 is proved. □

Pointwise Estimates for $u^0(t,s;x)$, $y^0(t,s;x)$, and $P(t)$ The above results are now used to obtain pointwise estimates.

Lemma 1.4.4.5 *With reference to the last term in (1.4.1.24), we have*

$$\|L_{sT}^* G^* G y^0(T, s; x)\|_{C_\gamma([s,T];U)} \le C_{T\gamma} \|x\|_Y, \quad x \in Y, \tag{1.4.4.30}$$

where the constant $C_{T\gamma}$ does not depend on s, $0 \le s \le T$.

Proof. From the definition (1.4.1.7) of L_{sT}^*, the assumptions (1.1.2) on B and (1.1.4) on G, and analyticity [Pazy, 1983, p. 74], we readily obtain

$$\begin{aligned}\|L_{sT}^* G^* G y^0(T, s; x)\|_U &= \left\| B^*(-A^*)^{-\gamma}(-A^*)^\gamma e^{A^*(T-t)} G^* G y^0(T, s; x) \right\|_U \\ &\le \frac{C_{T\gamma}}{(T-t)^\gamma} \|G y^0(T, s; x)\|_{Z_f} \le \frac{C_{T,\gamma}}{(T-t)^\gamma} \|x\|_Y,\end{aligned} \tag{1.4.4.31}$$

where in the last step we have used (1.4.2.3). Then (1.4.4.31) yields (1.4.4.30) via the definition (1.4.4.1). □

The major result of this section is the following:

Theorem 1.4.4.6 *For the optimal pair $\{u^0, y^0\}$, we have the following estimates with generic constant $C_{T\gamma}$ independent of s, $0 \le s \le T$, and for $x \in Y$:*

(i)

$$\|u^0(\cdot\,, s; x)\|_{C_\gamma([s,T];U)} \le C_{T\gamma} \|x\|_Y; \tag{1.4.4.32}$$

(ii) if $0 \le \gamma < \frac{1}{2}$,

$$\|y^0(\cdot\,, s; x)\|_{C([s,T];Y)} \le C_{T\gamma} \|x\|_Y; \tag{1.4.4.33a}$$

(iii) if $\frac{1}{2} < \gamma < 1$,

$$\|y^0(\cdot\,, s; x)\|_{C_{2\gamma-1}([s,T];Y)} \le C_{T\gamma} \|x\|_Y. \tag{1.4.4.33b}$$

Proof. (i) We return to (1.4.1.27), thus obtaining

$$-u^0(\,\cdot\,,s;x) = [I_s + L_s^* R^* R L_s]^{-1}\left\{L_s^* R^* R e^{A(\cdot - s)}x + L_{sT}^* G^* G y^0(T,s;x)\right\} \in C_\gamma([s,T];U). \tag{1.4.4.34}$$

In fact, the expression in the curly braces in (1.4.4.34) is a well-defined element of $C_\gamma([s,T];U)$ for $x \in Y$: This is so by (1.4.4.30) for its second term and is plainly true (a fortiori) from (1.4.1.5) for its first term. Moreover, the inverse in (1.4.4.34) is well defined in $C_\gamma([s,T];U)$ by Theorem 1.4.4.4. Indeed, these results collectively yield the bound (1.4.4.32), independent of s.

(ii) and (iii) Estimates (1.4.4.33a,b) now follow from estimate (1.4.4.32), via the optimal dynamics (1.4.1.26), where we use property (1.4.4.2) with $r = \gamma$ and $r + \gamma < 1$ for (1.4.4.33a) and property (1.4.4.3) with $r = \gamma$ and $r + \gamma > 1$ for (1.4.4.33b). □

Corollary 1.4.4.7 *With reference to the operator $P(t)$ introduced in (1.4.3.15), we have the following pointwise estimates for $s \le t < T$, $x \in Y$:*

(i) For $0 \le \theta < 1$

$$\|(-A^*)^\theta P(t)x\|_Y \le \frac{C_{T\gamma\theta}}{(T-t)^\theta}\|x\|_Y, \tag{1.4.4.35}$$

(ii)

$$\|B^* P(t)x\|_U \le \frac{C_{T\gamma}}{(T-t)^\gamma}\|x\|_Y, \tag{1.4.4.36}$$

where the constant C_T does not depend on s, $0 \le s \le T$.

Proof. (i) Recalling (1.4.3.15) we compute by analyticity of e^{At} [Pazy, 1983, p. 74] and by (1.1.4) on R, G:

$$\begin{aligned}\|(-A^*)^\theta P(t)x\|_Y &= \left\| \int_t^T (-A^*)^\theta e^{A^*(\tau - t)} R^* R y^0(\tau,t;x)\,d\tau \right.\\ &\quad \left. + (-A^*)^\theta e^{A^*(T-t)} G^* G y^0(T,t;x)\right\|_Y \\ &\le C_T \int_t^T \frac{1}{(\tau - t)^\theta}\|y^0(\tau,t;x)\|_Y\,d\tau + \frac{C_T}{(T-t)^\theta}\|Gy^0(T,t;x)\|_{Z_f} \\ &\le C_T\left\{\int_t^T \frac{1}{(\tau-t)^\theta}\|y^0(\tau,t;x)\|_Y\,d\tau + \frac{1}{(T-t)^\theta}\|x\|_Y\right\},\end{aligned} \tag{1.4.4.37}$$

where in the last step we have used estimate (1.4.2.3) for the second term. We now distinguish two cases according to (1.4.4.33a,b). If $\gamma < 1/2$, then (1.4.4.33a) applies in (1.4.4.37), and (1.4.4.37) yields (1.4.4.35). If, instead, $1/2 \le \gamma < 1$, then (1.4.4.33b) applies in (1.4.4.37), and we obtain for the right-hand side (R.H.S.) of (1.4.4.37) via definition (1.4.4.1), conservatively

$$\text{R.H.S. of (1.4.4.37)} \leq C_T \left\{ \int_t^T \frac{d\tau}{(\tau - t)^\theta (T-\tau)^{2\gamma - 1 + \epsilon}} + \frac{1}{(T-t)^\theta} \right\} \|x\|_Y. \tag{1.4.4.38}$$

But the integral in (1.4.4.38) is of the same type as the one occurring in (1.4.4.9) in estimating L_s^* in the proof of Proposition 1.4.4.1(iii), which culminates with the bound in (1.4.4.10), with γ, r there now replaced by θ, $2\gamma - 1 + \epsilon$. Thus, we obtain by (1.4.4.37), (1.4.4.38), via (1.4.4.10),

$$\begin{aligned} &\|(-A^*)^\theta P(t)x\|_Y \\ &\leq C_{T\gamma\theta} \left\{ \max\left\{ \frac{1}{1-\theta}, \frac{1}{2-2\gamma-\epsilon} \right\} \frac{1}{(T-t)^{2(\gamma-1)+\theta+\epsilon}} + \frac{1}{(T-t)^\theta} \right\} \|x\|_Y, \end{aligned} \tag{1.4.4.39}$$

from which (1.4.4.35b) follows, since $\gamma - 1 < 0$.

(ii) Estimate (1.4.4.36) now follows from estimate (1.4.4.35) with $\theta = \gamma < 1$, via hypothesis (1.1.2) on B, using $B^* P(t) = B^*(-A^*)^{-\gamma}(-A^*)^\gamma P(t)$. □

Remark 1.4.4.1 After Lemma 1.4.6.2(iii) below, we can return to the identity in (1.4.4.37) for $(-A^*)^\theta P(t)x$ (without norm) and conclude that it is continuous in t, for $0 \leq t < T$. Thus, via (1.4.4.1), we can upgrade (1.4.4.35) and write

$$(-A^*)^\theta P(t)x \in C_\theta([0, T]; Y).$$

Similarly, we upgrade (1.4.4.36) into

$$B^* P(t)x \in C_\gamma([0, T]; U).$$

Further properties of $P(t)$ are obtained next, as a consequence of (1.4.4.36).

Proposition 1.4.4.8

(i) For $0 \leq t < T$, the following identity, symmetric in $x, y \in Y$, holds true (recall (1.4.3.17)):

$$\begin{aligned} (P(t)x, y)_Y = &\int_t^T (R\Phi(\tau, t)x, R\Phi(\tau, t)y)_Z \, d\tau + (G\Phi(T, t)x, G\Phi(T, t)y)_{Z_f} \\ &+ \int_t^T (B^* P(\tau)\Phi(\tau, t)x, B^* P(\tau)\Phi(\tau, t)y)_U \, d\tau; \end{aligned} \tag{1.4.4.40}$$

(ii) as a consequence,

$$P(t) = P^*(t) \geq 0; \tag{1.4.4.41}$$

(iii) the optimal cost of the optimal control problem on $[t, T]$ *initiating at the point* $x \in Y$ *at the initial time* t *is*

$$\begin{aligned} J^0 &= J(u^0(\,\cdot\,, t; x), y^0(\,\cdot\,, t; x)) \\ &= \int_t^T \|R\Phi(\tau, t)x\|_Z^2 + \|B^*P(\tau)\Phi(\tau, t)x\|_U^2 \, d\tau + \|G\Phi(T, t)x\|_{Z_f}^2 \\ &= (P(t)x, x)_Y. \end{aligned} \tag{1.4.4.42}$$

Proof. (i) From the definition of $P(t)$ in (1.4.3.3), we have for $x, y \in Y$:

$$\begin{aligned} (P(t)x, y)_Y = &\int_t^T \left(R\Phi(\tau, t)x, Re^{A(\tau - t)}y\right)_Z d\tau \\ &+ \left(G\Phi(T, t)x, Ge^{A(T-t)}y\right)_{Z_f}. \end{aligned} \tag{1.4.4.43}$$

We next substitute for $e^{A(\tau-t)}y$ and $e^{A(T-t)}$ in (1.4.4.43), using the optimal dynamics (1.4.1.26). We obtain, after recalling also (1.4.1.6) and (1.4.3.1),

$$\begin{aligned} (P(t)x, y)_Y = &\int_t^T (R\Phi(\tau, t)x, R\Phi(\tau, t)y)_Z \, d\tau \\ &- (R\Phi(\cdot, t)x, RL_t u^0(\cdot, t; y))_{L_2(t,T;Z)} + (G\Phi(T, t)x, G\Phi(T, t)y)_{Z_f} \\ &- (G\Phi(T, t)x, GL_{tT}u^0(\cdot, t; s))_{Z_f} \end{aligned} \tag{1.4.4.44}$$

$$\begin{aligned} = &\int_t^T (R\Phi(\tau, t)x, R\Phi(\tau, t)y)_Z \, d\tau \\ &- (L_t^*R^*R\Phi(\cdot, t)x + L_{tT}^*G^*G\Phi(T, t)x, u^0(\cdot, t; y))_{L_2(t,T;U)} \\ &+ (G\Phi(T, t)x, G\Phi(T, t)y)_{Z_f} \end{aligned} \tag{1.4.4.45}$$

$$\begin{aligned} = &\int_t^T (R\Phi(\tau, t)x, R\Phi(\tau, t)y)_Z \, d\tau \\ &+ (u^0(\cdot, t; x), u^0(\cdot, t; y))_{L_2(t,T;U)} \\ &+ (G\Phi(T, t)x, G\Phi(T, t)y)_{Z_f}, \end{aligned} \tag{1.4.4.46}$$

where in going from (1.4.4.45) to (1.4.4.46) we have recalled the optimality relation (1.4.1.24). Equation (1.4.4.46) shows (1.4.4.40) as desired; via (1.4.3.17), (ii) and (iii) are immediate consequences of (i). □

1.4.5 Smoothing Properties of L_s and L_s^* at $t = s$. Pointwise Regularity of $du^0(t, s; x)/dt$ and $dy^0(t, s; x)/dt$ for $s < t < T$, $x \in Y$

The main goal of this section is to show the pointwise existence of the time derivatives $du^0(t, s; x)/dt$ and $dy^0(t, s; x)/dt$ of the optimal pair for $s < t < T$, $x \in Y$. Indeed, the next theorem shows more: Not only does it show that these two quantities are, respectively, U-valued and Y-valued functions continuous in t for $s < t < T$, but it also measures their order of singularity at $t = T$. The pointwise existence of

$dy^0(t,s;x)/dt$ is needed in the forthcoming subsection devoted to the derivation of the Riccati equation. To express quantitatively our results, we need two additional spaces to measure the order of singularity at $t = s$, and also at both $t = s$ and $t = T$. They complement the space defined in (1.4.4.1), which measures the singularity at $t = T$ only. Their definition is given below; see also (0.5), (0.6) of Chapter 0. If x is a Hilbert space and r, r_1, r_2 are real numbers, we define the following two Banach spaces:

$$ {}_rC([s,T];X) = \Big\{ f(t) \in C((s,T];X) : \|f\|_{{}_rC([s,T];X)} = \sup_{s\le t\le T} (t-s)^r \|f(t)\|_X < \infty \Big\}; \tag{1.4.5.1} $$

$$ {}_{r_1}C_{r_2}([s,T];X) = \Big\{ f(t) \in C((s,T);X) : \|f\|_{{}_{r_1}C_{r_2}([s,T];X)} = \sup_{s\le t\le T} (t-s)^{r_1}(T-t)^{r_2} \|f(t)\|_X < \infty \Big\}. \tag{1.4.5.2} $$

Theorem 1.4.5.1 *For $x \in Y$, the time derivatives $du^0(t,s;x)/dt$ and $dy^0(t,s;x)/dt$ exist as, respectively, U-valued and Y-valued functions, continuous in $s < t < T$. Moreover,*

(i)

$$ \left\| \frac{du^0(t,s;x)}{dt} \right\|_U \le \frac{C_{T\epsilon_1\gamma}}{(t-s)^{\gamma+\epsilon_1}} \|x\|_Y, \quad s < t < T, \ \forall \epsilon_1 > 0, \tag{1.4.5.3a} $$

so that recalling (1.4.5.1),

$$ \frac{du^0(t,s;x)}{dt} \in {}_{(\gamma+\epsilon_1)}C([s,T-\epsilon];U), \quad \forall \epsilon, \epsilon_1 > 0. \tag{1.4.5.3b} $$

Furthermore,

(ii)

$$ \frac{dy^0(t,s;x)}{dt} = Ae^{A(t-s)}x + e^{A(t-s)}Bu^0(s,s;x) + \left\{ L_s\left(\frac{du^0(\sigma,s;x)}{d\sigma} \right) \right\}(t), \tag{1.4.5.4} $$

(ii_1) where

$$ \left\| L_s\left(\frac{du^0(\sigma,s;x)}{d\sigma} \right) \right\|_Y \le \frac{C_{T\epsilon_1\gamma}}{(t-s)^{2\gamma-1+\epsilon_1}} \|x\|_Y, \quad \epsilon_1 \textit{ as in (1.4.5.3)}, \ s < t < T, \tag{1.4.5.5a} $$

so that recalling (1.4.5.1),

$$ L_s\left(\frac{du^0(\sigma,s;x)}{d\sigma} \right) \in {}_{(2\gamma-1+\epsilon_1)}C([s,T-\epsilon];Y), \quad \forall \epsilon, \epsilon_1 > 0. \tag{1.4.5.5b} $$

(ii₂)

$$\left\| e^{A(t-s)} B u^0(s,s;x) \right\|_Y \le \frac{C_T}{(t-s)^\gamma} \|u^0(s,s;x)\|_U \le \frac{C_{1T}}{(t-s)^\gamma} \|x\|_Y, \quad s < t \le T. \tag{1.4.5.6}$$

(ii_3) Since $2\gamma - 1 + \epsilon_1 < \gamma$, then

$$\left\| \frac{dy^0(t,s;x)}{dt} - A e^{A(t-s)} x \right\|_Y \le \frac{C_{T\epsilon_1\gamma}}{(t-s)^\gamma} \|x\|_Y, \quad s < t < T, \ \epsilon_1 \text{ as in (1.4.5.3)}, \tag{1.4.5.7a}$$

so that recalling (1.4.5.1),

$$\left[\frac{dy^0(t,s;x)}{dt} - A e^{A(t-s)} x \right] \in {}_\gamma C([s, T-\epsilon]; Y), \quad \forall \epsilon > 0. \tag{1.4.5.7b}$$

We note that the order ($\gamma + \epsilon_1$) of singularity of u^0 at the left-hand side $t = s$ is reduced by ($1-\gamma$) to the order ($2\gamma - 1 + \epsilon_1$) of singularity of the term $\{L_s(d/d\sigma)\, u^0(\sigma, s; x)\}(t)$ in (1.4.5.5), but this is dominated by the order of singularity γ of $e^{A(t-s)} Bu^0(s,s;x)$ at $t = s$, leading to (1.4.5.7).

Remark 1.4.5.1 In the special case $\gamma < 1/2$, we can also obtain a measure of the order of singularity at $t = T$; more precisely, for $x \in Y$,

$$\left\| \frac{du^0(t,s;x)}{dt} \right\|_U \le \frac{C_T}{(T-t)^{1+\gamma}} \|x\|_Y, \quad s < t < T; \tag{1.4.5.8a}$$

hence via (1.4.5.3b) and (1.4.5.2),

$$\frac{du^0(t,s;x)}{dt} \in {}_{(\gamma+\epsilon_1)}C_{1+\gamma}([s,T]; U), \quad \forall \epsilon_1 > 0. \tag{1.4.5.8b}$$

Moreover,

$$\left\| \frac{dy^0(t,s;x)}{dt} \right\|_Y \le \frac{C_T}{(T-t)^{2\gamma}} \|x\|_Y, \quad \forall \epsilon > 0, \ s < t < T; \tag{1.4.5.9a}$$

hence via (1.4.4.1),

$$\frac{dy^0(t,s;x)}{dt} \in C_{2\gamma}([s+\epsilon_1, T]; Y), \quad \forall \epsilon_1 > 0, \tag{1.4.5.9b}$$

$$\left[\frac{dy^0(t,s;x)}{dt} - A e^{A(t-s)} x \right] \in {}_\gamma C_{2\gamma}([s,T]; Y). \tag{1.4.5.9c}$$

Proof of Theorem 1.4.5.1 It proceeds through several steps.

Step 1 We return to the characterization (1.4.1.27) of the optimal control, which we rewrite here as

$$\begin{aligned} & u^0(t,s;x) + \{L_s^* R^* R L_s u^0(\,\cdot\,,s;x)\}(t) \\ & \quad = -\{L_{sT}^* G^* G y^0(T,s;x)\}(t) - \left\{L_s^* R^* R e^{A(\cdot - s)} x\right\}(t). \qquad (1.4.5.10) \end{aligned}$$

Our first task is to analyze the time derivative of the right-hand side of (1.4.5.10).

Step 2 As to the first term at the right-hand side of (1.4.5.10), recalling (1.4.1.7), we compute for $x \in Y$,

$$\frac{d}{dt}\{L_{sT}^* G^* G y^0(T,s;x)\}(t) = -B^* A^* e^{A^*(T-t)} G^* G y^0(T,s;x) \qquad (1.4.5.11a)$$

$$\in C_{1+\gamma}([s,T];U); \qquad (1.4.5.11b)$$

that is, the U-function in (1.4.5.11a) is continuous for $s \le t < T$ (recall from (1.4.2.3) that $Gy^0(T,s;x) \in Z_f$) and is norm-bounded by $C/(T-t)^{1+\gamma}$ for t near T, as seen from (1.4.5.11a) by use of (1.1.2) on B and analyticity of e^{A^*t} [Pazy, 1983, p. 74].

Step 3. Lemma 1.4.5.2 *For $x \in Y$, we have for the second term at the right-hand side of (1.4.5.10)*

$$\frac{d}{dt}\left\{L_s^* R^* R e^{A(\cdot - s)} x\right\}(t) = f_1(t;x) + f_2(t;x), \qquad (1.4.5.12)$$

$$f_1(t;x) \equiv B^* e^{A^*(T-t)} R^* R e^{A(T-s)} x \in C_\gamma([s,T];U), \qquad (1.4.5.13)$$

$$f_2(t;x) \equiv -\int_t^T B^* e^{A^*(\tau - t)} R^* R A e^{A(\tau - s)} x \, d\tau \qquad (1.4.5.14a)$$

$$\equiv -\left\{L_s^* R^* R A e^{A(\cdot - s)} x\right\}(t) \in {}_{(\gamma+\epsilon)}C([s,T];U). \qquad (1.4.5.14b)$$

Proof. By recalling (1.4.1.4) and changing variables with $\tau - t = \sigma$, we write

$$\left\{L_s^* R^* R e^{A(\cdot - s)} x\right\}(t) = \int_0^{T-t} B^* e^{A^*\sigma} R^* R e^{A(\sigma + t - s)} x \, d\sigma, \quad x \in Y, \qquad (1.4.5.15)$$

whose time derivative then produces the expression given by (1.4.5.12)–(1.4.5.14a). The regularity of f_1 in (1.4.5.13) follows from (1.1.2) and analyticity of e^{A^*t} [Pazy, 1983, p. 74], as usual. As to the regularity of f_2 described by (1.4.5.15b), U-continuity for $s < t < T$ is obvious by analyticity, while for the singularity at $t = s$ we compute from (1.4.5.14a) via (1.1.2) and analyticity of e^{At} [Pazy, 1983, p. 74]:

$$\begin{aligned} \|f_2(t;x)\| &\le C_T \|x\|_Y \int_t^T \frac{d\tau}{(\tau - t)^\gamma (\tau - s)} \\ &= C_T \|x\|_Y \int_t^T \frac{d\tau}{(\tau - t)^\gamma (\tau - s)^{1-\gamma-\epsilon} (\tau - s)^{\gamma+\epsilon}}, \quad s < t < T, \end{aligned}$$

which, using $(\tau - s)^{1-\gamma-\epsilon} \geq (\tau - t)^{1-\gamma-\epsilon}$ and $(\tau - s)^{\gamma+\epsilon} \geq (t-s)^{\gamma+\epsilon}$, becomes

$$\|f_2(t;x)\| \leq \frac{C_T}{(t-s)^{\gamma+\epsilon}}\|x\|_Y \int_t^T \frac{d\tau}{(\tau-t)^{1-\epsilon}} = \frac{C_T(T-t)^\epsilon}{\epsilon(t-s)^{\gamma+\epsilon}}\|x\|_Y, \quad (1.4.5.16)$$

which, in view of (1.4.5.14b) is the required bound via (1.4.5.1). (The above procedure is similar to the one leading to (1.4.4.7).) □

Using the regularity results in (1.4.5.11b), (1.4.5.13), and (1.4.5.14b), we obtain

Corollary 1.4.5.3 *For $x \in Y$, the following result follows for the right-hand side (R.H.S.) of (1.4.5.10):*

$$\frac{d}{dt}\{\text{R.H.S. of (1.4.5.10)}\}(t)$$
$$= B^*A^*e^{A^*(T-t)}G^*Gy^0(T,s;x) - f_1(t;x) - f_2(t;x) \equiv \rho(t;x); \quad (1.4.5.17a)$$
$$\rho(t;x) \in {}_{(\gamma+\epsilon)}C_{1+\gamma}([s,T];U). \quad (1.4.5.17b)$$

Step 4 Before taking the time derivative of $\{L_s^*R^*RL_su^0(\,\cdot\,,s;x)\}(t)$ for $s < t < T-\epsilon$, we need the following result, which is the counterpart of Proposition 1.4.4.1: Whereas the latter showed that the operators L_s and L_s^* have a smoothing effect that reduces the singularity at the right-hand point $t = T$, we now need similar results that L_s and L_s^* reduce the singularity at the left-hand point $t = s$ as well.

Proposition 1.4.5.4 (Counterpart of Proposition 1.4.4.1) *With reference to the operators L_s and L_s^* defined in (1.4.1.2) and (1.4.1.4), $0 \leq \gamma < 1$ in (1.1.2), and the spaces defined in (1.4.5.1), we have*

(i) Let $0 < r < 1$. Then L_s : continuous ${}_rC([s,T];U) \to {}_{(r+\gamma-1)}C([s,T];Y)$

$$\|L_su\|_{{}_{(r+\gamma-1)}C([s,T];Y)} \leq C_{T\gamma r}\|u\|_{{}_rC([s,T];U)}, \quad (1.4.5.18)$$

so that the bound in (1.4.5.18) may be made independent of s, $0 \leq s \leq T$.

(ii) Let $r > 0$. Then L_s^* : continuous ${}_rC([s,T];Y) \to {}_{(r+\gamma-1+\epsilon)}C([s,T];U)$

$$\|L_s^*f\|_{{}_{(r+\gamma-1+\epsilon)}C([s,T];U)} \leq C_{T\epsilon\gamma r}\|f\|_{{}_rC([s,T];Y)}. \quad (1.4.5.19)$$

(iii) Thus, a fortiori if $0 < r < 1$, there exists a positive integer $m = m(r)$ such that

$$(L_s^*R^*RL_s)^m : \text{continuous } {}_rC([s,t];U) \to C([s,T];Y), \quad (1.4.5.20)$$

with bound on the uniform norm that may be taken independent of s, $0 \leq s \leq T$. We can take the smallest integer m such that $2(1-\gamma)m > r$. □

Proof (similar to the proof of Proposition 1.4.4.1). Continuity of $(L_su)(t)$ and $(L_s^*f)(t)$ for $s < t \leq T$ is seen directly. We need to show the appropriate order of singularity at $t = s$.

(i) Let $u \in {}_rC([s,T];U)$. By (1.1.2) on B and analyticity of e^{At} [Pazy, 1983, p. 74], we compute from (1.4.1.2) and (1.4.5.1)

$$
\begin{aligned}
\|(L_s u)(t)\|_Y &\le C_{T\gamma} \int_s^t \frac{\|u(\tau)\|_U (\tau - s)^r d\tau}{(t-\tau)^\gamma (\tau - s)^r} \\
&\le C_{T\gamma r} \|u\|_{{}_rC([s,T];U)} \left\{ \int_s^{s+\frac{(t-s)}{2}} \frac{d\tau}{(t-\tau)^\gamma (\tau - s)^r} \right. \\
&\qquad \left. + \int_{s+\frac{(t-s)}{2}}^t \frac{d\tau}{(t-\tau)^\gamma (\tau - s)^r} \right\} && (1.4.5.21) \\
&\le C_{T\gamma r} \|u\|_{{}_rC([s,T];U)} \left\{ \left(\frac{2}{t-s}\right)^\gamma \int_s^{s+\frac{(t-s)}{2}} \frac{d\tau}{(\tau - s)^r} \right. \\
&\qquad \left. + \left(\frac{2}{t-s}\right)^r \int_{s+\frac{(t-s)}{2}}^t \frac{1}{(t-\tau)^\gamma} d\tau \right\} && (1.4.5.22) \\
&\le C_{T\gamma r} \|u\|_{{}_rC([s,T];U)} \left\{ \left(\frac{2}{t-s}\right)^{\gamma+r-1} \frac{1}{1-r} + \left(\frac{2}{t-s}\right)^{\gamma+r-1} \frac{1}{1-\gamma} \right\}. \\
& && (1.4.5.23)
\end{aligned}
$$

In the first integral in (1.4.5.21) we have used $(t-\tau) \ge (t-s)/2$, and $r < 1$; in the second integral we have used $(\tau - s) \ge (t-s)/2$. Equation (1.4.5.23) shows (1.4.5.18) via the definition in (1.4.5.1).

(ii) Similarly, from (1.4.1.4) we compute with $f \in {}_rC([s,T];Y)$:

$$
\begin{aligned}
\|(L_s^* f)(t)\|_U &\le C_{T\gamma} \int_t^T \frac{\|f(\tau)\|_Y (\tau - s)^r \, d\tau}{(\tau - t)^\gamma (\tau - s)^r}, \quad s \le t \le T \\
&\le C_{T\gamma r} \|f\|_{{}_rC([s,T];Y)} I(t), && (1.4.5.24)
\end{aligned}
$$

where with $r_1 + r_2 = r$, $\gamma + r_1 = 1 - \epsilon$, $\epsilon > 0$, we have

$$
\begin{aligned}
I(t) &= \int_t^T \frac{d\tau}{(\tau - t)^\gamma (\tau - s)^{r_1} (\tau - s)^{r_2}} \\
&\le \frac{1}{(t-s)^{r_2}} \int_t^T \frac{d\tau}{(\tau - t)^{1-\epsilon}} = \frac{(T-t)^\epsilon}{\epsilon (t-s)^{r_2}}, && (1.4.5.25)
\end{aligned}
$$

upon using $(\tau - s)^{r_1} \ge (\tau - t)^{r_1}$ and $(\tau - s)^{r_2} \ge (t-s)^{r_2}$. Estimate (1.4.5.25) used in (1.4.5.24) yields (1.4.5.19) as desired, with $r_2 = r - r_1 = r + \gamma - 1 + \epsilon$, via the definition (1.4.5.1).

(iii) Iteration of parts (i) and (ii) with R bounded as assumed in (1.1.4) yields part (iii), Eqn. (1.4.5.20). As integer m, we can take the smallest integer such that $2(1-\gamma)m > r$, $(1-\gamma)$ or $(1-\gamma-\epsilon)$ being the reduction of singularity at each application of L_s or L_s^*. $\square$

Step 5 With reference to (1.4.5.10), we next take the time derivative of

$$\{L_s^* R^* R L_s u^0(\,\cdot\,, s; x)\}(t) \quad \text{for } s < t < T - \epsilon, \;\; \epsilon > 0 \text{ arbitrary.}$$

To express the final result we note the following: The operators L_s and L_s^* in (1.4.1.2) and (1.4.1.4) are based on the time interval $s \le t \le T$. In the next result, we shall need to take these same operators and restrict them, however, as to act on the time interval $s < t \le T - \epsilon, \epsilon > 0$. These restrictions are needed only in this step. They will be denoted by $L_{[s,T-\epsilon]}$ and $L_{[s,T-\epsilon]}^*$. Thus by definition

$$\big(L_{[s,T-\epsilon]} u\big)(t) = \int_s^t e^{A(t-\tau)} B u(\tau)\, d\tau, \quad s \le t \le T - \epsilon, \tag{1.4.5.26}$$

whose adjoint is then

$$\big(L_{[s,T-\epsilon]}^* f\big)(t) = \int_t^{T-\epsilon} B^* e^{A^*(\tau - t)} f(\tau)\, d\tau, \quad s \le t \le T - \epsilon. \tag{1.4.5.27}$$

We note from (1.4.1.2) and (1.4.5.27) that plainly

$$\big(L_{[s,T-\epsilon]} u\big)(t) \equiv (L_s u)(t), \quad s \le t \le T - \epsilon. \tag{1.4.5.28}$$

With the above notation we have:

Proposition 1.4.5.5 *Let $T > \epsilon > 0$ arbitrary, and let $\mu(t) \in C^1([s, T-\epsilon]; U) \cap C_\gamma([s, T]; U)$. Then*

$$\begin{aligned} &\frac{d}{dt}\{L_s^* R^* R L_s \mu(\,\cdot\,)\}(t) \\ &\quad = \left\{ L_{[s,T-\epsilon]}^* R^* R L_{[s,T-\epsilon]} \left(\frac{d}{d\sigma} u(\sigma) \right) \right\}(t) + w_\epsilon(t) \in H^{-1}(s, T-\epsilon; U), \end{aligned} \tag{1.4.5.29}$$

where we have set

$$\begin{aligned} w_\epsilon(t) \equiv {} & \big\{ L_{[s,T-\epsilon]}^* R^* R e^{A(\cdot - s)} B \mu(s) \big\}(t) + B^* e^{A^*(T-\epsilon-t)} R^* R\{[L_s \mu(\,\cdot\,)](T-\epsilon)\} \\ & - B^* A^* e^{A^*(T-\epsilon-t)} \int_{T-\epsilon}^T e^{A^*(\tau - (T-\epsilon))} R^* R L_s \mu(\tau)\, d\tau, \end{aligned} \tag{1.4.5.30}$$

and where we note that $[L_s \mu(\,\cdot\,)](T-\epsilon) \in Y$. Also, $\forall \epsilon_1 > 0$:

$$w_\epsilon(t) \in {}_{(2\gamma - 1 + \epsilon_1)}C_{\gamma + \epsilon_1}([s, T-\epsilon]; U) \subset L_1(s, T-\epsilon; U). \tag{1.4.5.31}$$

[We shall later extend the results to $\mu \in C_\gamma([s, T]; U)$ after Lemma 1.4.5.7 below.]

Proof. We take $\mu \in C^1([s; T-\epsilon]; U) \cap C_\gamma([s, T]; U), \epsilon > 0$. If we set for convenience

$$v(t) \equiv \{R^* R L_s \mu(\,\cdot\,)\}(t) \tag{1.4.5.32}$$

we rewrite (1.4.1.4) for $s < t \le T - \epsilon$ as

$$\{L_s^* R^* R L_s \mu(\,\cdot\,)\}(t) = \{L_s^* v(\,\cdot\,)\}(t)$$
$$= \int_t^{T-\epsilon} B^* e^{A^*(\tau - t)} v(\tau)\, d\tau + \int_{T-\epsilon}^{T} B^* e^{A^*(\tau - t)} v(\tau)\, d\tau \tag{1.4.5.33a}$$

(by (1.4.5.27))
$$= \left\{L_{[s,T-\epsilon]}^* v\right\}(t) + \int_{T-\epsilon}^{T} B^* e^{A^*(\tau - t)} v(\tau)\, d\tau, \tag{1.4.5.33b}$$

and after changing variable $\tau - t = \sigma$ in the first integral we get

$$\{L_s^* v(\,\cdot\,)\}(t) = \int_0^{T-\epsilon-t} B^* e^{A^*\sigma} v(t+\sigma)\, d\sigma$$
$$+ B^* e^{A^*(T-\epsilon-t)} \int_{T-\epsilon}^{T} e^{A^*(\tau-(T-\epsilon))} v(\tau)\, d\tau, \quad s < t < T - \epsilon. \tag{1.4.5.34}$$

Taking the time derivative of (1.4.5.34) yields, via (1.4.5.32),

$$\frac{d}{dt}\{L_s^* R^* R L_s \mu(\,\cdot\,)\}(t) = \frac{d}{dt}\{L_s^* v(\,\cdot\,)\}(t)$$
$$= B^* e^{A^*(T-\epsilon-t)} v(T-\epsilon) + \int_t^{T-\epsilon} B^* e^{A^*(\tau-t)} \frac{dv(\tau)}{d\tau}\, d\tau$$
$$- B^* A^* e^{A^*(T-\epsilon-t)} \int_{T-\epsilon}^{T} e^{A^*(\tau-(T-\epsilon))} v(\tau)\, d\tau. \tag{1.4.5.35}$$

We next compute $dv(t)/dt$, as required by the second term in (1.4.5.35). By (1.4.5.32) and (1.4.1.2), we compute, after a change of variable $t - \tau = \sigma$,

$$\frac{dv(t)}{dt} = \frac{d}{dt}\{R^* R L_s \mu(\,\cdot\,)\}(t)$$
$$= R^* R \frac{d}{dt} \int_0^{t-s} e^{A\sigma} B \mu(t-\sigma)\, d\sigma$$
$$= R^* R e^{A(t-s)} B \mu(s) + \left\{ R^* R L_s \left[\frac{d\mu(\sigma)}{d\sigma}\right]\right\}(t), \quad s < t < T - \epsilon. \tag{1.4.5.36}$$

Inserting (1.4.5.36) into (1.4.5.35) yields (1.4.5.29) as desired, via (1.4.5.30), (1.4.5.26)–(1.4.5.28), and (1.4.5.32). Finally, we show the regularity of $w_\epsilon(t)$ noted in (1.4.5.31) as a consequence of the definition of $w_\epsilon(t)$ in (1.4.5.30) and $\mu \in C_\gamma([s, T]; U)$, which is true by assumption.

For the first term in the definition of $w_\epsilon(t)$, we show that

$$L_{[s,T-\epsilon]}^* R^* R e^{(A\cdot - s)} B \mu(s) \in {}_{(2\gamma-1+\epsilon_1)}C([s, T-\epsilon]; U), \quad \forall \epsilon, \epsilon_1 > 0. \tag{1.4.5.37}$$

This is so since $\mu(s) \in Y$ and since

$$e^{A(\cdot - s)} B \mu(s) \in {}_\gamma C([s, T-\epsilon]; Y) \tag{1.4.5.38}$$

by (1.1.2) on B and analyticity of e^{At} [Pazy, 1983, p. 74]. Next, one applies Proposition 1.4.5.4(ii), Eqn. (1.4.5.19) whereby the operator L_s^*, equivalently $L^*_{[s,T-\epsilon]}$, reduces the order of singularity of $t = s$ by $(1-\gamma-\epsilon_1)$ from γ to $\gamma+\gamma-1+\epsilon_1 = 2\gamma-1+\epsilon_1$, and (1.4.5.37) follows from (1.4.5.38).

The second term in the definition of $w_\epsilon(t)$ in (1.4.5.30) is plainly in $C_\gamma([s, T-\epsilon]; U)$, $\forall\epsilon, \epsilon_1 > 0$, by (1.1.2), since $[L_s\mu(\,\cdot\,)](T-\epsilon) \in Y$:

$$B^*e^{A^*(T-\epsilon-t)}R^*R\{[L_s\mu(\,\cdot\,)](T-\epsilon)\} \in C_\gamma([s, T-\epsilon]; U). \tag{1.4.5.39}$$

Finally, the third term in the definition of $w_\epsilon(t)$ in (1.4.5.30) gives us

$$\begin{aligned} B^*A^*e^{A^*(T-\epsilon-t)} & \int_{T-\epsilon}^T e^{A^*(\tau-(T-\epsilon))}R^*RL_s\mu(\tau)\,d\tau \\ &= B^*(-A^*)^{-\gamma}(-A^*)^{1-r+\gamma}e^{A^*(T-\epsilon-t)}z_r && (1.4.5.40)\\ &\in C_{\gamma+\epsilon_1}([s, T-\epsilon]; U), \quad \forall\epsilon, \epsilon_1 > 0, && (1.4.5.41)\end{aligned}$$

since we *claim* that: For any $r < 1$, the vector

$$z_r \equiv \int_{T-\epsilon}^T (-A^*)^r e^{A^*(\tau-(T-\epsilon))}R^*RL_s\mu(\tau)\,d\tau \in Y \tag{1.4.5.42}$$

is a well-defined vector in Y.

Indeed, by assumption, $\mu \in C_\gamma([s, T]; U)$, and hence by Proposition 1.4.4.1(ii), Eqn. (1.4.4.3) with $r = \gamma$ we have that $L_s\mu \in C_{2\gamma-1+\epsilon_2}C([s, T]; Y)$, $\forall\epsilon_2 > 0$. Therefore since $2\gamma - 1 + \epsilon_2 < 1$ and $r < 1$, we have from (1.4.5.41):

$$\|z_r\|_Y \le C_T \int_{T-\epsilon}^T \frac{d\tau}{(\tau-(T-\epsilon))^r(T-\tau)^{2\gamma-1+\epsilon_2}} < \infty \tag{1.4.5.43}$$

as desired, and our *claim* follows. Then, (1.4.5.41) is obtained by (1.1.2) and analyticity [Pazy, 1983, p. 74].

Thus, the regularity of $w_\epsilon(t)$ in (1.4.5.31) is proved, via (1.4.5.37), (1.4.5.39), and (1.4.5.41). □

If now we return to (1.4.5.33a), and we consider only $s < t < T-\epsilon$, then the second term on the right-hand side of (1.4.5.33a) is missing, and so the (consequent) last term on the right-hand side of (1.4.5.35), or (1.4.5.41), is missing. Thus, specializing the above proof of Proposition 1.4.5.5, we obtain on $s < t < T-\epsilon$:

Proposition 1.4.5.6 *Let $T > \epsilon > 0$ arbitrary, and let $\mu \in C^1([s, T-\epsilon]; U)$. Then for $s < t < T-\epsilon$,*

$$\begin{aligned} \frac{d}{dt}\big\{L^*_{[s,T-\epsilon]}R^*RL_{[s,T-\epsilon]}\mu(\,\cdot\,)\big\}(t) &= \frac{d}{dt}\big\{L^*_{[s,T-\epsilon]}v\big\}(t) \\ &= \left\{L^*_{[s,T-\epsilon]}R^*RL_{[s,T-\epsilon]}\left(\frac{d}{d\sigma}\mu(\sigma)\right)\right\}(t) + \tilde{w}_\epsilon(t), && (1.4.5.44)\end{aligned}$$

$$\begin{aligned} \tilde{w}_\epsilon &= \big\{L^*_{[s,T-\epsilon]}R^*Re^{A(\cdot-s)}B\mu(s)\big\}(t) \\ &\quad + B^*e^{A^*(T-\epsilon-t)}R^*R\{[L_s\mu(\,\cdot\,)](T-\epsilon)\} && (1.4.5.45)\\ &\in {}_{(2\gamma-1+\epsilon_1)}C_\gamma([s, T-\epsilon]; U) \subset L_1(s, T-\epsilon; U), && (1.4.5.46)\end{aligned}$$

that is, $\tilde{w}_\epsilon(t)$ is $w_\epsilon(t)$ in (1.4.5.31) without the third term, so that only the regularity in (1.4.5.37) and in (1.4.5.39) count now, producing (1.4.5.46) via (1.4.5.2). Moreover,

$$\|\tilde{w}_\epsilon\|_{L_1(s,T-\epsilon;U)} \leq C_{T-\epsilon}\|\mu\|_{C([s,T-\epsilon];U)}, \quad \forall\epsilon, \epsilon_1 > 0. \tag{1.4.5.47}$$

We next extend the validity of identity (1.4.5.30) to all μ just in $C_\gamma([s, T]; U)$. To this end, we need the following consequence of Proposition 1.4.5.6.

Lemma 1.4.5.7 *Let $\epsilon > 0$. With reference to (1.4.5.26), (1.4.5.27), we have*

$$L^*_{[s,T-\epsilon]}R^*RL_{[s,T-\epsilon]}\frac{d}{dt} : \textit{continuous } C([s, T-\epsilon]; U) \to H^{-1}(s, T; U). \tag{1.4.5.48}$$

Proof. We first take, say, $C^1[s, T-\epsilon], U)$ to be the domain of the operator in (1.4.5.48), dense in $C([s, T-\epsilon]; U)$. If $\mu \in C^1([s, T-\epsilon]; U)$, then by (1.4.5.44) and (1.4.5.46), we obtain

$$\begin{aligned}
&\left\| L^*_{[s,T-\epsilon]}R^*RL_{[s,T-\epsilon]}\left(\frac{d}{dt}\mu(\,\cdot\,)\right)\right\|_{H^{-1}(s,T-\epsilon;U)} \\
&\quad \leq \left\| L^*_{[s,T-\epsilon]}R^*RL_{[s,T-\epsilon]}\mu(\,\cdot\,)\right\|_{L_2(s,T-\epsilon;U)} + \|\tilde{w}_\epsilon\|_{H^{-1}(s,T-\epsilon;U)}.
\end{aligned} \tag{1.4.5.49}$$

But

$$L_1(s, T-\epsilon; U) \hookrightarrow H^{-1}(s, T-\epsilon; U), \tag{1.4.5.50a}$$

as it follows by the following duality relation:

$$\begin{aligned}
\|f\|_{H^{-1}(s,T_1)} &= \sup_\phi \frac{|(f,\phi)|}{\|\phi\|_{H_0^1(s,T_1)}}, \quad \phi \in H_0^1(s, T_1) \\
&\leq \sup_\phi \frac{\|\phi\|_{L_\infty(s,T_1)}\|f\|_{L_1(s,T_1)}}{\|\phi\|_{H_0^1(s,T_1)}} \leq C_T\|f\|_{L_1(s,T_1)},
\end{aligned} \tag{1.4.5.50b}$$

after using the one-dimensional embedding $\|\phi\|_{L_\infty(s,T_1)} \leq C_T\|\phi\|_{H_0^1(s,T_1)}$, for $\phi \in H_0^1(s, T_1)$. Then using (1.4.5.50) and (1.4.5.47) in (1.4.5.49) yields

$$\left\| L^*_{[s,T-\epsilon]}R^*RL_{[s,T-\epsilon]}\frac{d}{dt}\mu(\,\cdot\,)\right\|_{H^{-1}(s,T-\epsilon;U)} \leq C_{T-\epsilon}\|u\|_{C([s,T-\epsilon];U)}, \tag{1.4.5.51}$$

$\forall\mu \in C^1([s, T-\epsilon]; U)$, since $L^*_{[s,T-\epsilon]}R^*RL_{[s,T-\epsilon]}$ is a fortiori continuous, $C([s, T-\epsilon); U] \to L_2(s, T-\epsilon; U)$. Inequality (1.4.5.51) says that the operator in (1.4.5.48) with domain $C^1([s, T-\epsilon]; U)$ is continuous. Taking closure, we can extend inequality (1.4.5.51) to all of $C([s, T-\epsilon]; U)$ and conclusion (1.4.5.48) follows. □

We now return to Proposition 1.4.5.5: Identity (1.4.5.29) was shown there for $\mu \in C^1([s, T-\epsilon]; U) \cap C_\gamma([s, T]; U)$, while the term $w_\epsilon(t)$ in (1.4.5.30) requires, however, only $\mu \in C_\gamma([s, T]; U)$. By virtue of Lemma 1.4.5.7 we can now extend the validity of identity (1.4.5.29) to μ just in $C_\gamma([s, T]; U)$. We obtain

Theorem 1.4.5.8 *(i) Identity (1.4.5.29) in $H^{-1}(s, T-\epsilon; U)$ is extended to all $\mu \in C_\gamma([s,T]; U)$.*

(ii) In particular, since $\mu(t) = u^0(t, s; x) \in C_\gamma([s,T]; U)$ by (1.4.4.32), we obtain that identity (1.4.5.29) of Proposition 1.4.5.5 holds true in $H^{-1}(s, T-\epsilon; U)$ for $\mu(\cdot) = u^0(\cdot, s; x)$. Thus, we have the following identity for any $x \in Y$:

$$\frac{d}{dt}\{L_s^* R^* R L_s u^0(\cdot, s; x)\}(t)$$

$$= \left\{L^*_{[s,T-\epsilon]} R^* R L_{[s,T-\epsilon]} \left(\frac{d}{d\sigma} u^0(\sigma, s; x)\right)\right\}(t) + w_\epsilon(t; x) \tag{1.4.5.52a}$$

$$\in H^{-1}(s, T-\epsilon; U), \tag{1.4.5.52b}$$

where $w_\epsilon(t; x)$ is equal to the function $w_\epsilon(t)$ in (1.4.5.24) with $\mu(\cdot)$ there replaced by $u^0(\cdot, s; x)$ now, so that by (1.4.5.31) and (1.4.5.50), we have

$$w_\epsilon(t; x) \in {}_{(2\gamma-1+\epsilon_1)}C_{\gamma+\epsilon_1}([s, T-\epsilon]; U) \tag{1.4.5.53}$$

$$\subset L_1(s, T-\epsilon; U) \subset H^{-1}(s, T-\epsilon; U). \tag{1.4.5.54}$$

Step 6 Using Corollary 1.4.5.3 and Theorem 1.4.5.8 we obtain directly from identity (1.4.5.10) the following result:

Theorem 1.4.5.9 *Let $T > \epsilon > 0$ be arbitrary, and let $x \in Y$. Then, for $s < t < T - \epsilon$,*

$$\frac{du^0(t,s;x)}{dt} + \left\{L^*_{[s,T-\epsilon]} R^* R L_{[s,T-\epsilon]} \frac{du^0}{d\sigma}(\sigma, s; x)\right\}(t) = \nu_\epsilon(t; x), \tag{1.4.5.55}$$

where $\frac{du^0}{dt}(t, s; x) \in H^{-1}(s, T-\epsilon; U)$ and where recalling (1.4.5.53) for $w_\epsilon(t; x)$ and (1.4.5.17), which gives $\rho(t; x)$ continuous for $s < t < T$, $\nu(t; x)$ satisfies

$$\nu_\epsilon(t; x) \equiv -w_\epsilon(t; x) + \rho(t; x) \in {}_{(\gamma+\epsilon_1)}C_{\gamma+\epsilon_1}([s, T-\epsilon]; U), \quad \forall \epsilon_1 > 0, \tag{1.4.5.56}$$

since $2\gamma - 1 + \epsilon_1 < \gamma + \epsilon_1$.

Step 7 Equation (1.4.5.53) poses the (usual) problem of the inversion of the operator $I + L^*_{[s,T-\epsilon]} R^* R L_{[s,T-\epsilon]}$. This operator is clearly boundedly invertible on $L_2(s, T-\epsilon; U)$. The crux now is to assert that it is likewise boundedly invertible on the space identified in (1.4.5.56) of functions that are continuous in $s < t < T-\epsilon$ and that have the indicated order of singularity at $t = s$ and $t = T - \epsilon$. After this inversion has been justified, (1.4.5.55) then yields $du^0(t, s; x)/dt \in {}_{(\gamma+\epsilon_1)}C_{\gamma+\epsilon_1}([s, T-\epsilon]; U)$, hence a U-valued function continuous in t, for $s < t < T - \epsilon$, $\epsilon > 0$ arbitrary, as desired. To show such an inversion, we employ the same strategy used to prove Theorem 1.4.4.4 when only singularity at the right-hand point occurred. Accordingly, we shall invoke Proposition 1.4.5.4, which states that the operators L_s and L_s^*, or equivalently, the operators $L_{[s,T-\epsilon]}$ and $L^*_{[s,T-\epsilon]}$, have a smoothing effect that reduces the singularity at the left-hand point $t = s$. The next result is the counterpart of Theorem 1.4.4.4 and provides the critical inversion.

Theorem 1.4.5.10 *Let $\epsilon, \epsilon_1 > 0$. The operator (see (1.4.5.26) and (1.4.5.27))*

$$\left[I + L^*_{[s,T-\epsilon]} R^* R L_{[s,T-\epsilon]}\right]$$

is boundedly invertible on the space ${}_{(\gamma+\epsilon_1)}C_{\gamma+\epsilon_1}([s, T-\epsilon]; U)$:

$$\left[I + L^*_{[s,T-\epsilon]} R^* R L_{[s,T-\epsilon]}\right]^{-1} \in \mathcal{L}\left({}_{(\gamma+\epsilon_1)}C_{\gamma+\epsilon_1}([s, T-\epsilon]; U)\right), \tag{1.4.5.57}$$

with uniform norm bound that depends on ϵ, ϵ_1, T, but which may be taken independent of s, $0 \le s \le T - \epsilon$.

Proof (Similar to the proof of Theorem 1.4.4.4.) Let $h \in {}_{(\gamma+\epsilon_1)}C_{\gamma+\epsilon_1}([s, T-\epsilon]; U)$. We seek a unique $g \in {}_{(\gamma+\epsilon_1)}C_{\gamma+\epsilon_1}([s, T-\epsilon]; U)$ such that

$$g + L^*_{[s,T-\epsilon]} R^* R L_{[s,T-\epsilon]} g = h. \tag{1.4.5.58}$$

To simplify notation we may take $R = I$ below. Since $\gamma + \epsilon_1 < 1$, we may apply both Proposition 1.4.5.4 for the left-hand singularity at $t = s$ and Corollary 1.4.4.2 for the right-hand singularity at $t = T - \epsilon$. Let N be a positive integer greater than both m and n_0, where m is the positive integer provided by Proposition 1.4.5.4(iii), Eqn. (1.4.5.20), while n_0 is the positive integer provided by Corollary 1.4.4.2, Eqn. (1.4.4.11). Then there exists a unique $v \in L_2(s, T-\epsilon; U)$ such that

$$v + L^*_{[s,T-\epsilon]} L_{[s,T-\epsilon]} v = \left(L^*_{[s,T-\epsilon]} L_{[s,T-\epsilon]}\right)^N h \in C([s, T]; U) \subset L_2(s, T; U). \tag{1.4.5.59}$$

As in the proof of Step 2 of Theorem 1.4.4.4, we know that, in fact, $v \in C([s, T-\epsilon]; U)$, by relying on Theorem 1.4.4.3. Finally, we proceed as in the last steps of the proof of Theorem 1.4.4.4 by defining recursively the vectors $g_{n_0-1}, \dots, g_1, g$ which now belong all to ${}_{(\gamma+\epsilon_1)}C_{(\gamma+\epsilon_1)}([s, T-\epsilon]; U)$, precisely as h in the present case. The last vector g is the unique sought-after solution of (1.4.5.58). □

Step 8 We return to identity (1.4.5.55) with the right-hand side as in (1.4.5.56), and we apply Theorem 1.4.5.10. We then obtain with $x \in Y$, and $\forall \epsilon, \epsilon_1 > 0$:

$$\frac{du^0(t, s; x)}{dt} \in {}_{(\gamma+\epsilon_1)}C_{\gamma+\epsilon_1}([s, T-\epsilon]; U); \tag{1.4.5.60}$$

explicitly

$$\frac{du^0(t, s; x)}{dt} \in C((s, T-\epsilon); U), \quad \forall \epsilon > 0, \tag{1.4.5.61a}$$

and

$$\left\| \frac{du^0(t, s; x)}{dt} \right\|_U \le \frac{C_{T\gamma\epsilon_1\epsilon}}{[(t-s)(t-(T-\epsilon))]^{\gamma+\epsilon_1}}, \quad s < t < T - \epsilon. \tag{1.4.5.61b}$$

Since in (1.4.5.61a), $\epsilon > 0$ is arbitrary, we then have $\frac{du^0}{dt}(t, s; x) \in C(s, T); U)$ as desired. The statement (1.4.5.3) of Theorem 1.4.5.1 concerning $du^0(t, s; x)/dt$ is thus proved. □

Step 9 (Proof of (ii), (iii) of Theorem 1.4.5.1.) With the regularity (1.4.5.3) of $du^0(t, s; x)/dt$ at hand, we apply Proposition 1.4.5.4(i) with $r = \gamma + \epsilon < 1$ and obtain statement (1.4.5.5) for $L_s(du^0(\sigma, s; x)/d\sigma)$. Moreover, identity (1.4.5.4) is an immediate conseqence of the optimal dynamics (1.4.1.26), differentiating in t. Also, (1.4.5.6) follows at once from (1.1.2) on B and analyticity [Pazy, 1983, p. 74]:

$$\begin{aligned}\left\|e^{A(t-s)}Bu^0(s, s; x)\right\|_Y &= \left\|(-A)^\gamma e^{A(t-s)}(-A)^{-\gamma}Bu^0(s, s; x)\right\| \\ &\le \frac{C_T}{(t-s)^\gamma}\|u^0(s, s; x)\|_U \\ &\le \frac{C_{1T}}{(t-s)^\gamma}\|x\|_Y, \quad t > s,\end{aligned} \tag{1.4.5.62}$$

where $u^0(s, s; x) \in U$ by (1.4.4.32). Then, finally, part (ii$_3$), Eqns. (1.4.5.7) follow at once from (1.4.5.4), (1.4.5.5), and (1.4.5.6). Theorem 1.4.5.1 is fully proved. □

Remark 1.4.5.2 We now prove the content of Remark 1.4.5.1 in the special case $\gamma < 1/2$ noted there. In this case,

$$L_{sT}u^0(\,\cdot\,, s; x) = y^0(T, s; x) - e^{A(T-s)}x \in Y \tag{1.4.5.63}$$

is well defined as a vector in Y, in (1.4.4.433a). Setting $\epsilon = 0$ in the preceding computations we obtain first

$$\frac{d}{dt}\{L_s^*R^*RL_su^0(\,\cdot\,, s; x)\} = \left\{L_s^*R^*RL_s\left(\frac{d}{d\sigma}u^0(\sigma, s; x)\right)\right\}(t) + w_0(t; x) \tag{1.4.5.64}$$

from (1.4.5.52), with

$$w_0(t, x) = [w_\epsilon(t)|_{\epsilon=0} \text{ in } (1.4.5.30)] = [\tilde{w}_\epsilon(t)|_{\epsilon=0} \text{ in } (1.4.5.45)], \tag{1.4.5.65}$$

when $\mu(\,\cdot\,)$ there is replaced by $u^0(\,\cdot\,, s; x)$, so that

$$w_0(t; x) \in {}_{(2\gamma-1+\epsilon_1)}C_\gamma([s, T]; U), \tag{1.4.5.66}$$

by (1.4.5.46). Next, from (1.4.5.55) with $\epsilon = 0$ we obtain

$$\frac{du^0(t, s; x)}{dt} + L_s^*R^*RL_s\frac{du^0}{d\sigma}(\sigma, s; x) = v_0(t; x), \tag{1.4.5.67}$$

where, recalling (1.4.5.56) with $\epsilon = 0$,

$$v_0(t; x) = -w_0(t, x) + \rho(t, x) \in {}_{(\gamma+\epsilon)}C_{1+\gamma}([s, T]; U), \quad \forall\epsilon > 0. \tag{1.4.5.68}$$

The regularity in (1.4.5.68) attains by comparing that in (1.4.5.66) for $w_0(t; x)$ and that in (1.4.5.17) for $\rho(t, x)$, and using $\gamma + \epsilon > 2\gamma - 1 + \epsilon$. So far, the assumption that $\gamma < 1/2$ to obtain (1.4.5.67), (1.4.5.68) is not really crucial. Several other cases with $1/2 \le \gamma < 1$ may still lead to $L_{sT}u^0(\cdot, s; x) \in Y$ as in (1.4.5.63), hence to (1.4.5.67), (1.4.5.68) (e.g., in the smoothing case of Section 1.5). However, it is at the level of asserting, from (1.4.5.67), that

$$[I_s + L_s^* R^* R L_s]^{-1} \in \mathcal{L}\big({}_{(\gamma+\epsilon)}C_{1+\gamma}([s, T]; U)\big) \tag{1.4.5.69}$$

that the assumption $\gamma < 1/2$ is needed. Indeed, (1.4.5.69) can be shown then by the same bootstrap argument employed in the proof of Theorem 1.4.5.10; this requires applying $(L_s^* R^* R L_s)$ repeatedly to v_0 on Eqn. (1.4.5.67), until the singularities at $t = s$ and $t = T$ are eliminated, by virtue of Propositions 1.4.5.4 and 1.4.4.1, respectively [Eqns. (1.4.5.18), (1.4.5.19), (1.4.5.20), and (1.4.4.3), (1.4.4.4), (1.4.4.11), respectively]. At this stage, one then uses the bounded invertibility of $[I + L_s^* R^* R L_s]$ on $L_2(s, T; U)$, followed by the backward recursive procedure of Theorem 1.4.4.4 (Step 3) [or Theorem 1.4.5.10, below (1.4.5.59)], to obtain (1.4.5.69)

$$v_0 \in {}_{(\gamma+\epsilon)}C_{1+\gamma}([s, T]; U) \xrightarrow{R^* R L_s} R^* R L_s v_0 \in {}_{(2\gamma-1)}C_{2\gamma}([s, T]; U)$$

$$\text{[by (1.4.5.18) at } t = s\text{, and (1.4.4.3) at } t = T]$$

$$\begin{bmatrix} by\ (1.4.4.4) & \text{at } t = T, \text{ provided } 2\gamma < 1; \\ by\ (1.4.5.19) & \text{at } t = s \end{bmatrix} \Big\downarrow L_s^*$$

$$L_s^* R^* R L_s v_0 \in C_{3\gamma-1}([s, T]; U).$$

But with v_0 as in (1.4.5.68) we find that the order of singularity of $L_s v_0$ at $t = T$ is $(1 + \gamma) + (\gamma - 1) = 2\gamma$ by Proposition 1.4.4.1(ii), Eqn. (1.4.4.3). However, further application of L_s^* to $R^* R L_s v_0$ requires that $2\gamma < 1$, that is, $\gamma < 1/2$, in accordance with the requirement of Proposition 1.4.4.1(iii), Eqn. (1.4.4.4). Once (1.4.5.69) is established, that is, for $\gamma < 1/2$, then from (1.4.5.67) we obtain

$$\frac{du^0}{dt}(t, s; x) \in {}_{(\gamma+\epsilon)}C_{(1+\gamma)}([s, T]; U), \tag{1.4.5.70}$$

and estimate (1.4.5.8) is proved. Then, it follows from Proposition 1.4.4.1(ii), Eqn. (1.4.4.3), since $1 + \gamma > 1$, that $\{L_s(\frac{du^0}{d\sigma}(\sigma, s; x))\}(t)$ has a singularity at $t = T$ of order $(1 + \gamma) + (\gamma - 1) = 2\gamma$. The other two terms in formula (1.4.5.4) produce no singularity at $t = T$. Thus, conclusion (1.4.5.9) for $dy^0(t, s; x)/dt$ follows. Finally, we point out that the results of this Remark 1.4.5.2 will be generalized in Section 1.5.2, Theorem 1.5.2.2 to the case $1/2 \le \gamma < 1$ under the smoothing assumption (1.2.2.1) on G. $\square$

1.4.6 Derivation of the Riccati Equation (1.2.1.12)

In this section our main goal is to show that the operator $P(t) \in \mathcal{L}(Y; L_\infty(0, T; Y))$ [see (1.4.3.16)], explicitly defined by (1.4.3.15) in terms of the data of the problem, is a Riccati operator; that is, it satisfies the Riccati equation (1.2.1.12). To accomplish this, we need Theorem 1.4.5.1 for $dy^0(t, s; x)/dt$.

We begin by introducing the operator

$$A_P(t) = A - BB^*P(t) : Y \supset \mathcal{D}(A_P(t)) \to Y, \tag{1.4.6.1}$$

where $P(t)$ is defined by (1.4.3.1), and show that

Lemma 1.4.6.1 *With reference to (1.4.6.1), we have for $s \le t < T$:*

$$\mathcal{D}(A_P(t)) \subset \mathcal{D}((-A)^{1-\gamma}). \tag{1.4.6.2}$$

Proof. Let $x \in \mathcal{D}(A_P(t))$, that is, $z(t) = A_P(t)x = [A - BB^*P(t)]x \in Y$,

$$z(t) = -(-A)^{\gamma}[(-A)^{1-\gamma} + (-A)^{-\gamma}BB^*P(t)]x \in Y. \tag{1.4.6.3}$$

But $(-A)^{-\gamma}BB^*P(t)x \in Y$ for $t < T$ by assumption (1.1.2) on B and by property (1.4.4.36) of Corollary 1.4.4.7. From here and (1.4.6.3), we then obtain

$$(-A)^{1-\gamma}x = -(-A)^{-\gamma}z(t) - (-A)^{-\gamma}BB^*P(t)x \in Y, \tag{1.4.6.4}$$

which means that $x \in \mathcal{D}((-A)^{1-\gamma})$, as desired. □

We next recall the operator $\Phi(t,s) \in \mathcal{L}(Y)$, $0 \le s \le t < T$, in (1.4.3.1),

$$\Phi(t,s)x = y^0(t,s;x), \quad x \in Y, \tag{1.4.6.5}$$

which is defined via (1.4.1.26), (1.4.1.25), solely in terms of the data of the problem. All the preceding results on $y^0(t,s;x)$ can be expressed in terms of $\Phi(t,s)$ via (1.4.6.5). In particular, we recall that the optimal dynamics is rewritten (see (1.4.3.19)) as

$$\Phi(t,s)x = e^{A(t-s)}x + \int_s^t e^{A(t-\tau)}Bu^0(\tau,s;x)\,d\tau \tag{1.4.6.6a}$$

$$= e^{A(t-s)}x - \int_s^t e^{A(t-\tau)}BB^*P(\tau)\Phi(\tau,s)x\,d\tau. \tag{1.4.6.6b}$$

In the next two lemmas we collect some properties of Φ, the first of which improves upon (1.4.3.3) of Proposition 1.4.3.1, by removing the a.e. sense for $t < T$.

Lemma 1.4.6.2 *With reference to (1.4.6.5) we have*

(i) $\Phi(t,t) =$ identity, and $\Phi(t,\tau)\Phi(\tau,s) = \Phi(t,s)$ for $0 \le s \le \tau \le t < T$ (transitivity).

(ii) For s fixed, the map $t \to \Phi(t,s)x$ is continuous in Y, $s \le t < T$, and indeed $\Phi(t,s)x \in C_{2\gamma-1}([s,T];Y)$, $\gamma \ne 1/2$, so that, if $\gamma < 1/2$, then continuity in t holds up to $t = T$.

(iii) For $t < T$ fixed, the map $s \to \Phi(t,s)x$ is continuous in Y, for $0 \le s \le t < T$, $x \in Y$; while, this map is continuous for $s \le t \le T$, if $\gamma < 1/2$,

(iv) The map $s \to G\Phi(T,s)x$ is continuous in Z_f, $0 \le s < T$, $x \in Y$, or else $0 \le s \le T$ if $\gamma < 1/2$.

(v) For $0 \le s < t < \tau < T$, the following identity holds for $x \in Y$:

$$\frac{\partial \Phi(\tau,t)}{\partial t}\Phi(t,s)x = -\Phi(\tau,t)\frac{\partial \Phi}{\partial t}(t,s)x \in Y; \tag{1.4.6.7}$$

(vi) For $0 \le s < t < T$, the following identity holds for $x \in Y$:

$$\frac{\partial G\Phi(T,t)}{\partial t}\Phi(t,s)x = -G\Phi(T,t)\frac{\partial \Phi}{\partial t}(t,s)x \in Y. \tag{1.4.6.8}$$

[We note that (1.4.6.7), (1.4.6.8) reduce the derivative of Φ and $G\Phi$ in the second argument, computed along the optimal trajectory, $\Phi(t,s)x$, in terms of the derivative of Φ in the first argument.]

(vii) For $0 \le s < t < T$ and $x \in Y$ we have $\Phi(t,s)x \in \mathcal{D}((-A)^\theta)$ with $\theta < 1-\gamma$; moreover, for $x \in \mathcal{D}((-A)^\theta)$ we have

$$\lim_{s\uparrow t}(-A)^\theta\Phi(t,s)x = (-A)^\theta x. \tag{1.4.6.9}$$

Proof. (i), (ii) For $t < T$, the a.e. sense in t of (1.4.3.3) of Proposition 1.4.3.1 is improved to all t by recalling the pointwise regularity (1.4.4.33). This is then a restatement of the result contained in (1.4.4.33) of Theorem 1.4.4.6 [and a fortiori in Theorem 1.4.5.1 for $s < t < T$].

(ii) For right continuity, we choose $h > 0$ such that $s < s+h \le t < T$:

$$\begin{aligned}\|\Phi(t,s+h)x - \Phi(t,s)x\|_Y &= \|\Phi(t,s+h)[x-\Phi(s+h,s)x]\|_Y\\ &\le \frac{C_T}{(T-t)^r}\|\Phi(s+h,s)x - x\|_Y \to 0,\end{aligned} \tag{1.4.6.10}$$

where in the last step we have used (1.4.4.33) via (1.4.6.5), with $r = 0$ if $\gamma < 1/2$ in which case $t = T$ is also allowed; and otherwise $r = 2\gamma - 1$ for $1/2 < \gamma < 1$; and $r = 2\gamma - 1 + \epsilon$ for $\gamma = 1/2$. Then the right-hand side of (1.4.6.10) goes to zero as $h \downarrow 0$ by (ii). As to the left continuity, we compute for $h > 0$, again by (1.4.4.33) via (1.4.6.5):

$$\begin{aligned}\|\Phi(t,s-h)x - \Phi(t,s)x\|_Y &= \|\Phi(t,s)[\Phi(s,s-h)x - x]\|_Y\\ &\le \frac{C_T}{(T-t)^r}\|\Phi(s,s-h)x - x\|_Y \to 0,\end{aligned} \tag{1.4.6.11}$$

with r as above, where the right-hand side of (1.4.6.11) goes to zero as $h \downarrow 0$ by

$$\begin{aligned}\|\Phi(s,s-h)x - x\|_Y &\le \|e^{Ah}x - x\|_Y\\ &\quad + \int_{s-h}^{s}\left\|e^{A(s-\tau)}(-A)^\gamma(-A)^{-\gamma}Bu^0(\tau,s-h;x)\right\|_Y d\tau\\ &\le \|e^{Ah}x - x\|_Y + \frac{C_{T\gamma}}{(T-s)^\gamma}\|x\|_Y\int_{s-h}^{s}\frac{d\tau}{(s-\tau)} \to 0,\end{aligned} \tag{1.4.6.12}$$

which follows from (1.4.6.6a), where (1.4.4.32) of Theorem 1.4.4.5 is used for u^0 in the last step, with $s < T$, along with (1.1.2) and analyticity [Pazy, 1983, p. 74].

(iv) The case $t = T$ is reduced to case (iii); for $|h|$ sufficiently small and $s < t$, $s + h < t < T$, by (i) we have

$$\begin{aligned}\|G\Phi(T, s+h)x - G\Phi(T, s)x\|_{Z_f} &= \|G\Phi(T, t)[\Phi(t, s+h)x - \Phi(t, s)x]\|_{Z_f} \\ &\le C_T\|\Phi(t, s+h)x - \Phi(t, s)x\|_{Z_f} \to 0,\end{aligned} \tag{1.4.6.13}$$

where we have used (1.4.2.3) of Proposition 1.4.2.1, and the right-hand side of (1.4.6.13) goes to zero as $h \to 0$ by part (iii).

(v) For $|h|$ sufficiently small so that $0 \le s < t + h < \tau < T$ and $s < t < \tau$, we compute for $x \in Y$, by (i),

$$\begin{aligned}&\frac{1}{h}[\Phi(\tau, t+h)\Phi(t, s)x - \Phi(\tau, t)\Phi(t, s)x] \\ &\quad = \Phi(\tau, t+h)\frac{1}{h}[\Phi(t, s)x - \Phi(t+h, s)x].\end{aligned} \tag{1.4.6.14}$$

But since $\partial\Phi(t, s)x/\partial t$ exists as a Y-valued, continuous function in t, for $s < t < T$, by Theorem 1.4.5.1, with reference to the right-hand side of (1.4.6.14) we compute

$$\begin{aligned}&\Phi(\tau, t+h)\frac{1}{h}[\Phi(t+h, s)x - \Phi(t, s)x] - \Phi(\tau, t)\frac{\partial\Phi}{\partial t}(t, s)x \\ &= \Phi(\tau, t+h)\left\{\frac{1}{h}[\Phi(t+h, s)x - \Phi(t, s)x] - \frac{\partial\Phi}{\partial t}(t, s)x\right\} \\ &\quad + [\Phi(\tau, t+h) - \Phi(\tau, t)]\frac{\partial\Phi}{\partial t}(t, s)x \to 0 \text{ as } h \to 0.\end{aligned} \tag{1.4.6.15}$$

But $\Phi(\tau, t+h)$ is strongly continuous in h (property (iii)) and hence uniformly bounded in h in the $\mathcal{L}(Y)$-norm by the Principle of Uniform Boundedness. Then the right-hand side of (1.4.6.15) goes to zero as $h \to 0$, and (1.4.6.7) is proved via (1.4.6.14).

(vi) The proof of (1.4.6.8) is similar to the one in (1.4.6.14), (1.4.6.15) and uses the fact that $G\Phi(T, t+h)$ is now strongly continuous in h for $t < T$ (property (iv)) and hence uniformly bounded in h in the $\mathcal{L}(Z_f)$-norm.

(vii) From (1.4.6.6) we have, for $s < t < T$,

$$\begin{aligned}(-A)^\theta\Phi(t, s)x &= (-A)^\theta e^{A(t-s)}x \\ &\quad + \int_s^t (-A)^\theta e^{A(t-\tau)}(-A)^\gamma(-A)^{-\gamma}Bu^0(\tau, s; x)\, d\tau,\end{aligned} \tag{1.4.6.16}$$

and the integral in (1.4.6.16) is bounded in norm by the expression

$$C_T\int_s^t \frac{d\tau}{(t-\tau)^{\theta+\gamma}(T-\tau)^\gamma} \le \frac{C_T}{(T-t)^\gamma}\int_s^t \frac{d\tau}{(t-\tau)^{\theta+\gamma}}, \tag{1.4.6.17}$$

by virtue of property (1.1.2) on B, and analyticity [Pazy, 1983, p. 74], and of property (1.4.4.32) of Theorem 1.4.4.6 on u^0. This expression (1.4.6.17) is well defined and converges to zero as $s \uparrow t$ if $\theta + \gamma < 1$. Thus Eqn. (1.4.6.16) is well defined in Y, if $x \in Y, s < t$ or if $x \in \mathcal{D}((-A)^\theta), s = t$. □

Lemma 1.4.6.3

(i) For any $x \in Y$ and any $t, s < t < T$, we have the following results with reference to the operator $A_P(t)$ in (1.4.6.1):

$$\frac{\partial \Phi(t,s)x}{\partial t} = A_P(t)\Phi(t,s)x \in Y. \tag{1.4.6.18}$$

(ii) For $0 \le s < t < \tau < T$ and $x \in Y$,

$$\frac{\partial \Phi(\tau,t)}{\partial t}\Phi(t,s)x = -\Phi(\tau,t)A_P(t)\Phi(t,s)x \in Y. \tag{1.4.6.19}$$

(iii) For $0 \le s < t < T$ and $x \in Y$,

$$\frac{\partial G\Phi(T,t)}{\partial t}\Phi(t,s)x = -G\Phi(T,t)A_P(t)\Phi(t,s)x \in Y. \tag{1.4.6.20}$$

Proof. (i) We start from (1.4.6.6b), differentiate in t for $s < t < T$ (as guaranteed by Theorem 1.4.5.1) after taking the Y-inner product with $y \in \mathcal{D}(A^*)$, and obtain for $s < t < T$

$$\begin{aligned}\left(\frac{\partial \Phi(t,s)}{\partial t}x, y\right)_Y &= \left(e^{A(t-s)}x, A^*y\right)_Y - (A^{-1}BB^*P(t)\Phi(t,s)x, A^*y)_Y \\ &\quad - \left(\int_s^t e^{A(t-\tau)}BB^*P(\tau)\Phi(\tau,s)x\,d\tau, A^*y\right)_Y \\ \text{(by (1.4.6.6))} \quad &= ([I - A^{-1}BB^*P(t)]\Phi(t,s)x, A^*y)_Y, \quad y \in \mathcal{D}(A^*),\end{aligned} \tag{1.4.6.21}$$

where all the terms are well defined by property (1.1.2) on B and property (1.4.4.36) on $B^*P(t)$, $t < T$. By Theorem 1.4.5.1, the left-hand side of (1.4.6.21) is a well-defined Y-inner product $\forall x, y \in Y$, $s < t < T$; therefore so is the right-hand side extended as a duality pairing. Thus, A^* can be moved to the left and (1.4.6.18) follows.

Properties (ii) and (iii) are a direct consequence of (1.4.6.7) and (1.4.6.8) of Lemma 1.4.6.2 via (1.4.6.18). □

The main result of this section is the following:

Theorem 1.4.6.4 *The operator $P(t)$ defined by (1.4.3.15) satisfies the following Riccati equation for $0 \le t < T$ and $x, y \in \mathcal{D}(A)$; indeed, for $x, y \in \mathcal{D}((-A)^\epsilon)$, $\forall \epsilon$ with $0 < \epsilon < 1 - \gamma$:*

$$\begin{aligned}(\dot{P}(t)x, y)_Y &= -(R^*Rx, y)_Y - (P(t)x, Ay)_Y - (P(t)Ax, y)_Y \\ &\quad + (B^*P(t)x, B^*P(t)y)_U,\end{aligned} \tag{1.4.6.22}$$

where $\dot{P}(t)$ is a closed operator and $(-A^*)^{-\epsilon}\dot{P}(t)(-A)^{-\epsilon}$ can be extended to a bounded operator in $\mathcal{L}(Y)$.

Proof. With, say, $x \in Y$ and $y \in \mathcal{D}(A)$ and $0 \le s < t < T$, we differentiate in t, Eqn. (1.4.3.15), rewritten now via (1.4.6.5) as

$$(P(t)x, y)_Y = \left(\int_t^T e^{A^*(\tau-t)} R^* R\Phi(\tau, t)x\, d\tau, y\right)_Y + \left(e^{A^*(T-t)} G^* G\Phi(T, t)x, y\right)_Y, \tag{1.4.6.23}$$

and then replace x with $\Phi(t, s)x$ (all inner products are in Y) to obtain

$$\begin{aligned}(\dot{P}(t)\Phi(t, s)x, y) = &-(R^* R\Phi(t, s)x, y)\\ &-\left(\int_t^T e^{A^*(\tau-t)} R^* R\Phi(\tau, t)\Phi(t, s)x\, d\tau, Ay\right)\\ &+\left(\int_t^T e^{A^*(\tau-t)} R^* R\frac{\partial\Phi(\tau, t)}{\partial t}\Phi(t, s)x\, d\tau, y\right)\\ &-\left(e^{A^*(T-t)} G^* G\Phi(T, t)\Phi(t, s)x, Ay\right)\\ &+\left(e^{A^*(T-t)} G^*\frac{\partial G\Phi(T, t)}{\partial t}\Phi(t, s)x, y\right);\end{aligned} \tag{1.4.6.24}$$

using (1.4.6.23) with x replaced by $\Phi(t, s)x$ for the second and fourth terms in (1.4.6.24) we have

$$\begin{aligned}(\dot{P}(t)\Phi(t, s)x, y) = &-(R^* R\Phi(t, s)x, y)_Y - (P(t)\Phi(t, s)x, Ay)\\ &+\left(\int_t^T e^{A^*(\tau-t)} R^* R\frac{\partial\Phi(\tau, t)}{\partial t}\Phi(t, s)x\, d\tau, y\right)\\ &+\left(e^{A^*(T-t)} G^*\frac{\partial G\Phi(T, t)}{\partial t}\Phi(t, s)x, y\right),\end{aligned}$$

which, using identities (1.4.6.19) and (1.4.6.20) in the last two terms, respectively, gives

$$\begin{aligned}(\dot{P}(t)\Phi(t, s)x, y) = &-(R^* R\Phi(t, s)x, y)_Y - (P(t)\Phi(t, s)x, Ay)\\ &-\left(\int_t^T e^{A^*(\tau-t)} R^* R\Phi(\tau, t)A_P(t)\Phi(t, s)x\, d\tau, y\right)\\ &-\left(e^{A^*(T-t)} G^* G\Phi(T, t)A_P(t)\Phi(t, s)x, y\right);\end{aligned}$$

using again (1.4.6.23) with x replaced by $\Phi(t, s)x$ we get

$$\begin{aligned}(\dot{P}(t)\Phi(t, s)x, y) = &-(R^* R\Phi(t, s)x, y) - (P(t)\Phi(t, s)x, Ay)\\ &-(P(t)A_P(t)\Phi(t, s)x, y), \quad x \in Y, y \in \mathcal{D}(A).\end{aligned}$$

From here, recalling the definition of $A_P(t)$ in (1.4.6.1), we obtain

$$\begin{aligned}(\dot{P}(t)\Phi(t, s)x, y) = &-(R^* R\Phi(t, s)x, y) - (P(t)\Phi(t, s)x, Ay)\\ &-(P(t)[A - BB^* P(t)]\Phi(t, s)x, y).\end{aligned} \tag{1.4.6.25}$$

We first note that the second term at the right-hand side of (1.4.6.25) is well defined for all $x \in Y$ and all $y \in \mathcal{D}((-A)^\epsilon)$, $\forall \epsilon > 0$, upon recalling (1.4.4.35) for $(-A^*)^{1-\epsilon} P(t) \in \mathcal{L}(Y)$, $t < T$.

Next we verify the following *claim*: That the term $P(t)[A - BB^*P(t)]\Phi(t, s)x$ is well defined in Y for all $x \in Y$. After this we have that: *The right-hand side of (1.4.6.25) is well defined with $s < t < T$ for all $x \in Y$ and all $y \in \mathcal{D}((-A)^\epsilon)$, $\forall \epsilon > 0$.*

To show the claim, we first note that from (1.4.4.36) of Corollary 1.4.4.7, we have, for $t < T$,

$$B^*P(t) \in \mathcal{L}(Y, U), \quad \text{and hence } P(t)BB^*P(t) \in \mathcal{L}(Y), \tag{1.4.6.26}$$

since $P(t)$ is self-adjoint (see (1.4.4.41) in Proposition 1.4.4.8(ii)). Moreover, for $t < T : (-A^*)^{1-\epsilon} P(t) \in \mathcal{L}(Y), \forall \epsilon > 0$, by (1.4.4.35) of Corollary 1.4.4.7, and likewise, since $P(t)$ is self-adjoint, $P(t)(-A)^{1-\epsilon}$ can be extended to an operator in $\mathcal{L}(Y)$. Thus, we decompose the operator in the last term of (1.4.6.25) as

$$P(t)[A - BB^*P(t)] = -P(t)(-A)^{1-\epsilon}(-A)^\epsilon - P(t)BB^*P(t), \tag{1.4.6.27}$$

so that (1.4.6.27) is well defined on $\mathcal{D}((-A)^\epsilon)$, $\forall \epsilon > 0$, by (1.4.6.26). But, recalling Lemma 1.4.6.2(vii), we then see that $\Phi(t, s)x \in \mathcal{D}((-A)^\epsilon)$ for all $0 < \epsilon < 1 - \gamma$, $x \in Y$, $s < t < T$, so that the corresponding term $P(t)[A - BB^*P(t)]\Phi(t, s)x$ is well defined for all $x \in Y$. Thus, our claim has been verified.

We now restrict to $x \in \mathcal{D}((-A)^\epsilon)$ with any $\epsilon < 1 - \gamma$ and, after recalling Lemma 1.4.6.2(iii), $t < T$, (1.4.6.26), and (1.4.6.9), we obtain from (1.4.6.27)

$$\begin{aligned} &\lim_{s \uparrow t} P(t)[A - BB^*P(t)]\Phi(t, s)x \\ &\quad = -P(t)(-A)^{1-\epsilon} \lim_{s \uparrow t} (-A)^\epsilon \Phi(t, s)x - P(t)BB^*P(t) \lim_{s \uparrow t} \Phi(t, s)x \\ \text{(by (1.4.6.9))} &\quad = P(t)[A - BB^*P(t)]x, \quad x \in \mathcal{D}((-A)^\epsilon). \end{aligned} \tag{1.4.6.28}$$

Thus, taking the limit of the right-hand side of (1.4.6.25) as $s \uparrow t$, $t < T$, and using (1.4.6.28) and Lemma 1.4.6.2(iii), we obtain the right-hand side of (1.4.6.22), well defined $\forall x, y \in \mathcal{D}((-A)^\epsilon)$, $\forall \epsilon > 0$, as desired.

As to the left-hand side of (1.4.6.25), we may consider the operator $\dot{P}(t)$ to be well defined at least on the set $\mathcal{M}_t$ defined by

$$\mathcal{M}_t \equiv \bigcup_{0 \le s < t} M_{ts} \subset \mathcal{D}((-A)^\theta), \quad \text{where } \mathcal{M}_{ts} = \Phi(t, s)Y \subset \mathcal{D}((-A)^\theta), \tag{1.4.6.29}$$

for θ fixed, $\theta < 1 - \gamma$ (by Lemma 1.4.6.2(vii)), with $s < t < T$. By using the transitivity property of $\Phi(\,\cdot\,, \,\cdot\,)$, one then sees that $s_1 < s_2$ implies $\mathcal{M}_{s_1 t} \subset \mathcal{M}_{s_2 t}$ and that $\mathcal{M}_t$ is actually a subspace of Y, which is dense in Y by Lemma 1.4.6.2(ii), (iii). We next show that $\dot{P}(t)$ with domain $\mathcal{M}_t$ is *closable*. In fact, let $\mathcal{M}_t \ni x_n = \Phi(t, s_n)y_n \to 0$, $s_n < t < T$, $y_n \in Y$, and let $\dot{P}(t)x_n \to v$ in Y. Then, $v = 0$. In fact, identity (1.4.6.25) with $\Phi(t, s)x$ replaced now by $\Phi(t, s_n)y_n$ implies as $n \to \infty$ that

$(v, y) = 0$ for all $y \in \mathcal{D}((-A)^{\epsilon})$, hence for all $y \in Y$, and then $v = 0$. We denote the *closure* of $\dot{P}(t)$ (smallest closed extension) still by $\dot{P}(t)$. Moreover, $\dot{P}(t)$ is also self-adjoint and thus, recalling Lemma 1.4.6.2(ii), we have for $t < T$:

$$\begin{aligned}\lim_{s \uparrow t}(\dot{P}(t)\Phi(t,s)x, y)_Y &= \lim_{s \uparrow t}(\Phi(t,s)x, \dot{P}(t)y)_Y \\ &= (x, \dot{P}(t)y)_Y = (\dot{P}(t)x, y)_Y, \qquad (1.4.6.30)\end{aligned}$$

at least $\forall x \in Y$ and $\forall y \in \mathcal{M}_t$. Thus, at this point, we have that the differential Riccati equation (1.4.6.22) holds true for all $x \in \mathcal{D}((-A)^{\epsilon})$, $\forall \epsilon$ with $0 < \epsilon < 1 - \gamma$, and for all $y \in \mathcal{M}_t$. But, as seen above, the right-hand side of (1.4.6.22) is well defined also for all $y \in \mathcal{D}((-A)^{\epsilon})$. Moreover, $\mathcal{M}_t$ is dense in $\mathcal{D}((-A)^{\epsilon})$ in the $\mathcal{D}((-A)^{\epsilon})$-topology, as it follows a fortiori from (1.4.6.9) of Lemma 1.4.6.2. Then, the left-hand side of (1.4.6.22) can be extended likewise to all $y \in \mathcal{D}((-A)^{\epsilon})$. This implies that $(-A^*)^{-\epsilon}\dot{P}(t)(-A)^{-\epsilon}$ can be extended to a bounded operator in $\mathcal{L}(Y)$. Thus, the DRE (1.4.6.22) holds true for all $x, y \in \mathcal{D}((-A)^{\epsilon})$, as desired. The proof of Theorem 1.4.6.4 is complete. □

A final property of $P(t)$ which complements property (1.4.3.16) and property (1.4.4.35) is the following:

Proposition 1.4.6.5 *For any $\epsilon > 0$ small we have*

$$(-A^*)^{\theta}P(t) \in \mathcal{L}(Y; C([0, T-\epsilon]; Y)), \quad 0 \le \theta < 1; \qquad (1.4.6.31a)$$

$$P(t) \in \mathcal{L}(Y; C([0,T]; Y)) \quad \text{if } \gamma < \frac{1}{2}; \qquad (1.4.6.31b)$$

$$B^*P(t) \in \mathcal{L}(Y; C([0, T-\epsilon]; Y)). \qquad (1.4.6.32)$$

Proof. (i) From (1.4.3.3) with $x \in Y$, we have

$$\begin{aligned}(-A^*)^{\theta}P(t)x &= \int_t^T (-A^*)^{\theta}e^{A^*(\tau-t)}R^*R\Phi(\tau,t)x\,d\tau \\ &\quad + (-A^*)^{\theta}e^{A^*(T-t)}G^*G\Phi(T,t)x, \qquad (1.4.6.33)\end{aligned}$$

and conclusion (1.4.6.31) follows from (1.4.6.33) using the properties of Lemma 1.4.6.2(ii) and (iv) with $t \le T - \epsilon$, if $\gamma \ge 1/2$, and $t \le T$ for $\gamma < 1/2$, in the latter case, if $\theta = 0$, $P(t)x$ is $C([0,T]; Y)$, $x \in Y$.

(ii) Then (1.4.6.32) follows from (1.4.6.31) via assumption (1.1.2) on B. □

1.4.7 The Issue of the Limit of $P(t)$ as $t \uparrow T$

We rewrite (1.4.3.15) for convenience,

$$P(t)x = \int_t^T e^{A^*(\tau-t)}R^*R\Phi(\tau,t)x\,d\tau + e^{A^*(T-t)}G^*G\Phi(T,t)x, \qquad (1.4.7.1)$$

and see that its first term satisfies the following result.

Lemma 1.4.7.1 *For $x \in Y$, we have*

$$\lim_{t \uparrow T} \int_t^T e^{A^*(\tau - t)} R^* R \Phi(\tau, t) x \, d\tau = 0, \quad x \in Y. \tag{1.4.7.2}$$

Proof. Conclusion (1.4.7.2) follows just by invoking the L_2-estimate (1.4.2.2) for $y^0(\tau, t; x) = \Phi(\tau, t)x$ and using Schwarz's inequality, or else by invoking the sharper estimate in (1.4.4.33). □

Lemma 1.4.7.1, Eqn. (1.4.7.2), along with Proposition 1.4.2.7(ii), Eqn. (1.4.2.7), yields:

Proposition 1.4.7.2 *For $x \in Y$, we have*

$$\lim_{t \uparrow T} P(t)x = G^* G x, \quad x \in Y. \tag{1.4.7.3}$$

Proof. For the first and second terms on the right-hand side of (1.4.7.1), we use (1.4.7.2) of Lemma 1.4.7.1, and (1.4.2.7) of Proposition 1.4.2.2(ii), respectively. This way (1.4.7.3) follows. □

1.4.8 Analyticity of the Optimal Pair: Proof of (xi) in Theorem 1.2.1.1

The aim of this section is to show the following result, noted as property (xi) in the statement of Theorem 1.2.1.1.

Theorem 1.4.8.1 *For $x \in Y$, the optimal control $u^0(t, s; x)$ and corresponding optimal trajectory $y^0(t, s; x)$ are, respectively, U-valued and Y-valued functions that are analytic in t for $s < t < T$, provided that either (i) the generator A has the k^{th} power $R^k(\lambda, A)$ of the resolvent operator compact as an operator on Y for some positive integer k; or else (ii) the operator $(-A)^{-p}B$ is compact as an operator $U \to Y$ for some $0 < p < 1$.*

Proof. It suffices to take the case of the initial time $s = 0$, as the case $s \neq 0$ is similar. Thus, we shall write $u^0(t; x)$ and $y^0(t; x)$ instead of $u^0(t, 0; x)$ and $y^0(t, 0; x)$. We return to the characterization (1.4.5.10) of the optimal control, which we rewrite here for $s = 0$:

$$\begin{aligned} &u^0(t; x) + \{L^* R^* R L u^0(\,\cdot\,; x)\}(t) \\ &\quad = -\{L_T^* G^* G y^0(T; x)\}(t) - \{L^* R^* R e^{A\cdot} x\}(t), \quad 0 \le t < T, \end{aligned} \tag{1.4.8.1}$$

suppressing indication of $s = 0$.

Step 1 We extend the definition of the quantities that enter into (1.4.8.1) from the real variable $t \in (0, T)$ to the complex variable $z \in \mathcal{F}$. Here $\mathcal{F}$ is the open symmetric sector of $\mathbb{C}$ based on $(0, T)$ and delimited by the four line segments $\rho e^{\pm i\theta_0}$, $\rho e^{\pm i(\pi - \theta_0)} + T$, $0 \le \rho \le \rho_{\max}$ for some $\rho_0 : 0 < \theta_0 < \pi/2$ chosen so that $\bar{\mathcal{F}}$ (the closure of $\mathcal{F}$) lies

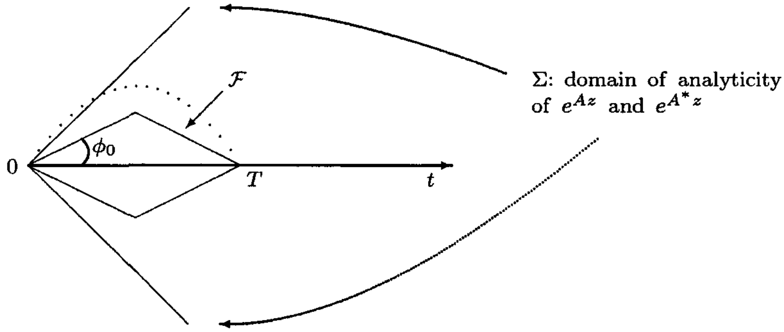


Figure 1.1

entirely in the infinite triangular sector Σ of analyticity of the semigroups e^{Az} and e^{A^*z} (Figure 1.1).

We note that the transformation $z \to T - z$ (needed below) maps $\mathcal{F}$ onto itself. We shall now introduce a function space based on $\mathcal{F}$.

Definition 1.4.8.1 *The space $\mathcal{A}(\mathcal{F}; U)$ [resp. $\mathcal{A}(\mathcal{F}; Y)$] consists of all U-valued [resp. Y-valued] functions $f(z)$ that are: (i) analytic (holomorphic) on $\mathcal{F}$ and (ii) continuous on $\bar{\mathcal{F}}$. Equipped with the norm*

$$\|f\|_{\mathcal{A},U} \equiv \max_{z\in\bar{\mathcal{F}}} \|f(z)\|_U \quad \left[\text{resp. } \|f\|_{\mathcal{A},Y} \equiv \max_{z\in\bar{\mathcal{F}}} \|f(z)\|_Y\right], \tag{1.4.8.2}$$

the space $\mathcal{A}(\mathcal{F}; U)$ is a Banach space (completeness being a consequence of Weierstrass's uniform convergence theorem[1]). □

We note that for $z \neq 0$ in the sector $\mathcal{F}$ of analyticity, we have [Friedman, 1976, p. 101]

$$\|(-A)^\alpha e^{Az}\|_{\mathcal{L}(Y)} = \|(-A^*)^\alpha e^{A^*z}\|_{\mathcal{L}(Y)} \le \frac{M}{|z|^\alpha}, \quad 0 \le \alpha \le 1, \tag{1.4.8.3}$$

with constant M depending on $\mathcal{F}$. Next, let $u(z)$ be an analytic U-function in $\mathcal{F}$. With $z = re^{i\theta} \in \mathcal{F}$, we let $\zeta = \rho e^{i\theta}$, $0 \le \rho \le r$, see Figure 1.2, be the line segment from 0 to z in $\mathcal{F}$, and we extend the definition of L in (1.1.7) by

$$(Lu)(z) = \int_0^z e^{A(z-\zeta)} Bu(\zeta)\, d\zeta. \tag{1.4.8.4}$$

Similarly, if ξ is the running point from z to T along the line segment joining them, we extend the definition of L^* in (1.1.8) by

$$(L^*f)(z) = \int_z^T B^* e^{A^*(\xi - z)} f(\xi)\, d\xi, \tag{1.4.8.5}$$

where $f(\cdot)$ is analytic in $\mathcal{F}$.

[1]The real version of this theorem does not hold true and for this reason extension from t to z is essential.

We shall, in effect, show below that $u^0(z;x)$ *and* $y^0(z;x)$ *are U-valued and Y-valued functions analytic in* $\mathcal{F}$ *in terms of the complex variable* z*, and hence are real analytic on (0, T).*

Step 2 The following properties hold true for the operators L and L^* in (1.4.8.4) and (1.4.8.5), once extended in a natural way to $z \in \mathcal{F}$.

Lemma 1.4.8.2

(i) Let $u(z)$ *be U-analytic in* $\mathcal{F}$*. Then* $(Lu)(z)$ *defined by (1.4.8.4) is likewise Y-analytic in* $\mathcal{F}$*.*

(ii) With reference to Definition 1.4.8.1, we have

$$L: \textit{continuous } \mathcal{A}(\mathcal{F}; U) \to \mathcal{A}(\mathcal{F}; Y), \tag{1.4.8.6a}$$

$$\|Lu\|_{\mathcal{A},Y} \leq \frac{T^{1-\gamma}}{1-\gamma} C_\gamma \|u\|_{\mathcal{A},U}. \tag{1.4.8.6b}$$

(iii) Let $f(z)$ *be Y-analytic in* $\mathcal{F}$*. Then* $(L^*f)(z)$ *defined by (1.4.8.5) is likewise U-analytic in* $\mathcal{F}$*.*

(iv) With reference to Definition 1.4.8.1, we have

$$L^*: \textit{continuous } \mathcal{A}(\mathcal{F}; Y) \to \mathcal{A}(\mathcal{F}; U), \tag{1.4.8.7a}$$

$$\|L^* f\|_{\mathcal{A},U} \leq \frac{T^{1-\gamma}}{1-\gamma} C_\gamma \|f\|_{\mathcal{A},Y}. \tag{1.4.8.7b}$$

Proof. (i) With $z = re^{i\theta} \in \mathcal{F}$ and $u(z)$ and U-analytic function in $\mathcal{F}$, we write from (1.4.8.4)

$$\begin{aligned}(Lu)(z) &= \int_0^{\epsilon e^{i\theta}} e^{A(z-\epsilon e^{i\theta})} e^{A(\epsilon e^{i\theta} - \zeta)} Bu(\zeta)\, d\zeta \\ &\quad - \int_{\epsilon e^{i\theta}}^{z} \frac{de^{A(z-\zeta)}}{d\zeta} A^{-1} Bu(\zeta)\, d\zeta,\end{aligned} \tag{1.4.8.8}$$

where integrations are along line segments, and $0 < \epsilon < r$. Thus, integrating the second integral in (1.4.8.8) by parts, we obtain

$$\begin{aligned}(Lu)(z) &= Ae^{A(z-\epsilon e^{i\theta})} \int_0^{\epsilon e^{i\theta}} e^{A(\epsilon e^{i\theta} - \zeta)} A^{-1} Bu(\zeta)\, d\zeta \\ &\quad - A^{-1}Bu(z) + e^{A(z-\epsilon e^{i\theta})} A^{-1} Bu(\epsilon e^{i\theta}) \\ &\quad + \int_{\epsilon e^{i\theta}}^{z} e^{A(z-\zeta)} A^{-1} B \frac{du(\zeta)}{d\zeta}\, d\zeta.\end{aligned} \tag{1.4.8.9}$$

Recalling (1.1.2), and that $u(z)$ is U-analytic in $\mathcal{F}$, we see that (1.4.8.9) exhibits

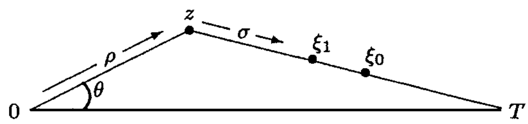


Figure 1.2

$(Lu)(z)$ as the Y-analytic sum of four Y-analytic terms in $\mathcal{F}$, as desired. (Note $Ae^{A(z-\epsilon e^{i\theta})} = e^{A(z-3/2\epsilon e^{i\theta})}Ae^{A(1/2\epsilon e^{i\theta})}$ with $\epsilon > 0$ suitably small.)

(ii) Let now $u \in \mathcal{A}(\mathcal{F}; U)$. Then, from (1.4.8.4), (1.4.8.3), with $\alpha = \gamma$, and (1.1.2), we obtain

$$\begin{aligned} \|(Lu)(z)\|_Y &\le \int_0^r \left\|(-A)^\gamma e^{A(z-\zeta)}\right\| \left\|(-A)^{-\gamma}B\right\| \|u(\zeta)\|_U \, d\zeta \\ \text{(by (1.4.8.2))} \qquad &\le C_\gamma \left(\int_0^r \frac{d\rho}{|z-\zeta|^\gamma}\right) \|u\|_{\mathcal{A},U}, \end{aligned} \tag{1.4.8.10}$$

with $|z-\zeta|^\gamma = (r-\rho)^\gamma$, and (1.4.8.6) follows from (1.4.8.10) since $r \le T$, see Figure 1.2.

(iii) With $z \in \mathcal{F}$ and $f(z)$ a Y-analytic function in $\mathcal{F}$, we write, from (1.4.8.5) with ξ_0 an interior point on the line segment from z to T, see Figure 1.2

$$\begin{aligned} (L^* f)(z) = &- \int_z^{\xi_0} B^*(-A^*)^{-1} \frac{de^{A^*(\xi-z)}}{d\xi} f(\xi)\, d\xi \\ &+ \int_{\xi_0}^T B^* e^{A^*(\xi-\xi_0)} e^{A^*(\xi_0-z)} f(\xi)\, d\xi, \end{aligned} \tag{1.4.8.11}$$

where integrations are along line segments. Thus, integrating by parts the first integral on the right of (1.4.8.11) we obtain

$$\begin{aligned} (L^* f)(z) = &-B^*(-A^*)^{-1} e^{A^*(\xi_0-z)} f(\xi_0) + B^*(-A^*)^{-1} f(z) \\ &+ B^*(-A^*)^{-1} \int_z^{\xi_0} e^{A^*(\xi-z)} \frac{df(\xi)}{d\xi}\, d\xi \\ &- B^*(-A^*)^{-1}(-A^*) e^{A^*(\xi_0-z)} \int_{\xi_0}^T e^{A^*(\xi-\xi_0)} f(\xi)\, d\xi. \end{aligned} \tag{1.4.8.12}$$

Equation (1.4.8.12) exhibits $(L^* f)(z)$ as the U-analytic sum of four U-analytic terms in $\mathcal{F}$, after recalling both (1.1.2) and that $f(z)$ is Y-analytic in $\mathcal{F}$ (and noting that $(-A^*)e^{A^*(\xi_0-z)} = e^{A^*(\xi_1-z)}(-A^*)e^{A^*(\xi_0-\xi_1)}$, where ξ_1 is an interior point on the line segment from z to ξ_0), see Figure 1.2.

(iv) Let now $f \in \mathcal{A}(\mathcal{F}; Y)$. Then, from (1.4.8.5), (1.4.8.3) with $\alpha = \gamma$, and (1.1.2) we obtain

$$\begin{aligned} \|(L^* f)(z)\|_U &\le \int_{|z|}^{T} \|B^*(-A^*)^{-\gamma}\| \, \|(-A^*)^{\gamma} e^{A^*(\xi - z)}\| \, \|f(\xi)\|_Y d|\xi| \\ \text{(by (1.4.8.2))} \qquad &\le C_\gamma \left(\int_r^T \frac{d|\xi|}{|\xi - z|^{\gamma}} \right) \|f\|_{\mathcal{A},Y} \\ &\le C_\gamma \left(\int_0^T \frac{d\sigma}{\sigma^{\gamma}} \right) \|f\|_{\mathcal{A},Y}, \end{aligned} \tag{1.4.8.13}$$

and (1.4.8.7) follows from (1.4.8.13). Lemma 1.4.8.2 is proved. □

Step 3. Lemma 1.4.8.3 *Let $x \in Y$. With reference to (1.1.10) and (1.1.8), we have the following properties for the right-hand side of (1.4.8.1):*

(i) the ("bad") U-function defined below,

$$-b(z) \equiv \{L_T^* G^* G y^0(T; x)\}(z) = B^* e^{A^*(T-z)} G^* G y^0(T; x), \tag{1.4.8.14}$$

is continuous on $\bar{\mathcal{F}} - \{T\}$, analytic in $\mathcal{F}$, and of class $C_\gamma([0, T]; U)$;

(ii)

$$-g(z) \equiv \{L^* R^* R e^{A \cdot} x\}(z) = \int_z^T B^* e^{A^*(\xi - z)} R^* R e^{A\xi} x \, d\xi \in \mathcal{A}(\mathcal{F}; U) \tag{1.4.8.15}$$

belongs to $\mathcal{A}$ ($\mathcal{F}$; U).

Proof. (i) We recall both (1.1.2) and that $Gy^0(T; x) \in Z_f$ by (1.2.1.14) = (1.4.2.3); then, the listed properties of $b(z)$ are immediate, via (1.4.4.1).

(ii) We apply Lemma 1.4.8.2(iv) to $e^{Az}x \in \mathcal{A}(\mathcal{F}; Y)$ for $x \in Y$. □

Remark 1.4.8.1 It is clear from (1.4.8.14) that the U-function $b(z)$ has a singularity of order γ at $t = T$ also as a function of the complex variable z; more precisely,

$$\sup_{z \in \mathcal{F}} |T - z|^{\gamma} \|b(z)\|_U < \infty. \tag{1.4.8.16}$$

The quantity in (1.4.8.16) defines a norm on the space of U-continuous functions on $\bar{\mathcal{F}} - \{T\}$. We shall denote this space with the notation $C_\gamma^{\bar{\mathcal{F}}}([0, T]; U)$. Thus $b(z)$ is an element in this space.

Step 4. Lemma 1.4.8.4 *Let either (i) the operator A have the k^{th} power $R^k(\lambda, A)$ of the resolvent operator compact as an operator on Y, for some positive integer k or else (ii) let $(-A)^{-p}B$ be compact as an operator $U \to Y$ for some $0 < p < 1$. Then, with reference to (1.4.8.4),*

$$L \textit{ is compact as an operator} : \mathcal{A}(\mathcal{F}; U) \to \mathcal{A}(\mathcal{F}; Y). \tag{1.4.8.17}$$

Proof. For $0 < \delta < r$, we introduce the operators L_δ by

$$(L_\delta u)(z) = \int_0^{z-\delta e^{i\theta}} e^{A(z-\zeta)} Bu(\zeta)\, d\zeta, \quad z = re^{i\theta}, \; \zeta = \rho e^{i\theta} \text{ in } \mathcal{F}. \tag{1.4.8.18}$$

Then, by (1.4.8.4), (1.4.8.18), (1.1.2), and (1.4.8.3) with $\alpha = \gamma$ and (1.4.8.2):

$$\begin{aligned} \|[(L_\delta - L)u](z)\|_Y &\le \int_{r-\delta}^{r} \left\|(-A)^\gamma e^{A(z-\zeta)}\right\| \left\|(-A)^{-\gamma} B\right\| \|u(\zeta)\|_U \, d\rho \\ \text{(by (1.4.8.2))} \qquad &\le C_\gamma \left(\int_{r-\delta}^{r} \frac{d\rho}{(r-\rho)^\gamma}\right) \|u\|_{\mathcal{A},U} \\ &\le C_\gamma \frac{\delta^{1-\delta}}{1-\gamma} \|u\|_{\mathcal{A},U}, \end{aligned} \tag{1.4.8.19}$$

from which it follows via (1.4.8.2) that

$$\|(L_\delta - L)u\|_{\mathcal{A},Y} \le \frac{C_\gamma}{1-\gamma} \delta^{1-\gamma} \|u\|_{\mathcal{A},U}, \tag{1.4.8.20}$$

and L_δ converges to L in $\mathcal{L}(\mathcal{A}(\mathcal{F}; U); \mathcal{A}(\mathcal{F}; Y))$ as $\delta \downarrow 0$. Thus, it remains to show that L_δ is compact for fixed δ. To this end, we apply the generalized Ascoli's theorem [Royden, 1968, p. 179] to the family of functions $\{L_\delta u_n\}$ from $\mathcal{F}$ to Y, where the $\mathcal{A}(\mathcal{F}; U)$-norm of the u_n is taken equal to one. This family $\{L_\delta u_n\}$ is plainly equicontinuous in n, the kernel of L_δ being continuous. Moreover, for each fixed $z \in \bar{\mathcal{F}}$, if A^{-k} is compact as assumed in the first case (i), the totality of points

$$(L_\delta u_n)(z) = (-A)^{-k} \int_0^{z-\delta e^{i\theta}} (-A)^{k+1} e^{A(z-\zeta)} (-A)^{-1} B u_n(\zeta)\, d\zeta \tag{1.4.8.21}$$

clearly lies in a precompact set of Y, since the points defined by the integral in (1.4.8.21) lie in a bounded set of Y. Ascoli's theorem then guarantees uniform convergence on $\bar{\mathcal{F}}$, that is convergence in $\mathcal{A}(\mathcal{F}; Y)$, of a subsequence $\{L_\delta u_n\}$, so that L_δ is compact as desired.

If, instead, $(-A)^{-p} B$ is a compact operator $U \to Y$ for some $0 < p < 1$, as assumed in the second case (ii), then we write

$$(L_\delta u_n)(z) = \int_0^{z-\delta e^{i\theta}} (-A)^p e^{A(z-\zeta)} (-A)^{-p} B u_n(\zeta)\, d\zeta,$$

and again the totality of points $(L_\delta u_n)(z)$ lies in a precompact set of Y, since the closed convex hull of a compact set is compact (Mazur theorem). Thus, L_δ is again compact.

Moreover, in place of the Ascoli-Arzelá Theorem invoked above, one could use Vitali's theorem, as in [Hille, Phyllips, 1957, p. 104].

Remark 1.4.8.2 Essentially the same proof shows compactness of L as an operator from $C([0, T]; U)$ to $C([0, T]; Y)$, under the same assumptions. $\square$

Step 5. Corollary 1.4.8.5 *Assume the hypotheses of Lemma 1.4.8.4. Then,* $I + L^*R^*RL$ *is boundedly invertible on the space* $\mathcal{A}(\mathcal{F}; U)$:

$$[I + L^*R^*RL]^{-1} \in \mathcal{L}(\mathcal{A}(\mathcal{F}; U)). \tag{1.4.8.22}$$

Proof. First, the value $\lambda = -1$ cannot be an eigenvalue of the operator L^*R^*RL on $\mathcal{A}(\mathcal{F}; U)$ since, as we know, it is not an eigenvalue of L^*R^*RL on $L_2(\mathcal{F}; U)$. Next, to show (1.4.8.22), it is sufficient to have L compact as in (1.4.8.17), a property guaranteed by Lemma 1.4.8.4. □

Step 6 We now show that for $x \in Y$,

$$u^0(z; x) \text{ is a } U\text{-function analytic on } \mathcal{F}, \tag{1.4.8.23}$$

thus a fortiori real analytic on $(0, T)$, as desired. To this end, we rewrite identity (1.4.8.1) as

$$u^0(z; x) + \{L^*R^*RLu^0(\,\cdot\,, x)\}(z) = b(z) + g(z), \tag{1.4.8.24}$$

after recalling the definitions of b and g in (1.4.8.14), (1.4.8.15), whose properties are listed in Lemma 1.4.8.3. Accordingly, we now apply a bootstrap argument (as in the proof of Theorem 1.4.4.4, Step 3; or of Theorem 1.4.5.10), which for sake of simplicity of notation we explain below with $R = I$. We apply the operator (L^*L) recursively n times to Eqn. (1.4.8.24), thereby obtaining $(n + 1)$ equations:

$$\begin{aligned}
u^0 + L^*Lu^0 &= b + g, && (1.4.8.25_0)\\
(L^*L)u^0 + (L^*L)^2u^0 &= (L^*L)b + (L^*L)g, && (1.4.8.25_1)\\
(L^*L)^2u^0 + (L^*L)^3u^0 &= (L^*L)^2b + (L^*L)^2g, && (1.4.8.25_2)\\
&\;\;\vdots\\
(L^*L)^{n-1}u^0 + (L^*L)^nu^0 &= (L^*L)^{n-1}b + (L^*L)^{n-1}g, && (1.4.8.25_{n-1})\\
[I + L^*L](L^*L)^nu^0 &= (L^*L)^nb + (L^*L)^ng. && (1.4.8.25_n)
\end{aligned}$$

Here n is chosen so that, according to Corollary 1.4.4.2, Eqn. (1.4.4.11), we have

$$b \in C_\gamma([0, T]; U) \to (L^*L)^nb \in C([0, T]; U) \tag{1.4.8.26}$$

by Lemma 1.4.8.3(i) on b.

Actually, as noted in Remark 1.4.8.1, the function $b(z)$ belongs to the space $C_\gamma^{\bar{\mathcal{F}}}([0, T]; U)$ with norm (1.4.8.16) describing the singularity at $t = T$ of order γ as a function of the complex variable $z \in \mathcal{F}$. The proof of Corollary 1.4.4.2 holds also in this context with z complex. This, together with the properties of $b(z)$ noted in Lemma 1.4.8.3(i), produces first that

$$(L^*L)^nb \in C(\bar{\mathcal{F}}; U), \tag{1.4.8.27}$$

and next that, in fact, via Lemma 1.4.8.2(i),

$$(L^*L)^n b \in \mathcal{A}(\mathcal{F}; U). \tag{1.4.8.28}$$

However, since $g \in \mathcal{A}(\mathcal{F}; U)$ by Lemma 1.4.8.3(ii), it follows via Lemma 1.4.8.2 that

$$(L^*L)^k g \in \mathcal{A}(\mathcal{F}; U), \quad \forall k = 1, 2, \ldots. \tag{1.4.8.29}$$

Thus, by (1.4.8.28) and (1.4.8.29), the right-hand side $(L^*L)^n(b+g)$ of Eqn. (1.4.8.25$_n$) is in $\mathcal{A}(\mathcal{F}; U)$. An application of Corollary 1.4.8.5, Eqn. (1.4.8.22) yields then that $(L^*L)^n u^0 \in \mathcal{A}(\mathcal{F}; U)$. Using this information in Eqn. (1.4.8.25$_{n-1}$), where the right-hand side r_{n-1} is analytic in $\mathcal{F}$ by Lemma 1.4.8.3(i) and Lemma 1.4.8.2(i) on b, and by (1.4.8.28) on g, we now obtain that

$$\begin{aligned}(L^*L)^{n-1}u^0 &= [(L^*L)^{n-1}b + (L^*L)^{n-1}g] - (L^*L)^n u^0 \\ &= r_{n-1} - (L^*L)^n u^0 \\ &= \text{analytic in } \mathcal{F}.\end{aligned} \tag{1.4.8.30}$$

Proceeding this way backward along the equations from (1.4.8.25$_n$) down to (1.4.8.25$_0$), where the right-hand side r_i of each of them is always analytic in $\mathcal{F}$, we obtain finally from (1.4.8.25$_0$) that $u^0(z, x)$ is analytic in $\mathcal{F}$, and (1.4.8.23) is proved as desired. Theorem 1.4.8.1 for u^0 is thus proved.

Step 7 The conclusion that the Y-function $y^0(z; x)$, $x \in Y$, is likewise analytic on $\mathcal{F}$, hence on $(0, T)$, follows now from the analyticity of u^0 just proved, via the optimal dynamics

$$y^0(z; x) = e^{Az}x + (Lu^0)(z), \tag{1.4.8.31}$$

after invoking Lemma 1.4.8.2(i) for L. The proof of Theorem 1.4.8.1 is now complete. □

Remark 1.4.8.3 In "concrete" mixed problems for partial differential equations, which fit naturally into the abstract framework of the present chapter [see Chapter 3], one may further improve the local regularity results of analyticity of the present abstract section. For instance, consider a parabolic equation, canonically the heat equation, defined on a bounded domain $\Omega \subset R^n$ with smooth boundary $\Gamma = \partial\Omega$, acted upon by a Dirichlet-boundary control $u \in L_2(0, T; L_2(\Gamma))$:

$$\begin{cases} y_t = \Delta y, & \text{in } (0, T; \times\Omega; & (1.4.8.32a) \\ y(0, \cdot\,) = y_0, & \text{in } \Omega; & (1.4.8.32b) \\ y|_\Sigma = u, & \text{in } (0, T] \times \Gamma \equiv \Sigma. & (1.4.8.32c) \end{cases}$$

It will be justified in Chapter 3 [Section 3.1] that the mixed problem (1.4.8.30) may be written in the abstract form (1.1.1), for instance with the following (natural) choice of spaces:

$$U = L_2(\Gamma); \qquad Y = L_2(\Omega); \quad \text{and with parameter } \gamma = \frac{3}{4} + \epsilon. \tag{1.4.8.33}$$

Moreover, the resolvent $R(\lambda, A)$ is compact on Y. Accordingly, Theorem 1.4.8.1 yields the following local regularity results of the optimal control and corresponding optimal solution for problem (1.4.8.32), where $y_0 \in Y = L_2(\Omega)$:

$$\begin{cases} u^0(t, 0; y_0) \text{ and } y^0(t, 0; y_0) \text{ are } L_2(\Gamma) = H^0(\Gamma)\text{-valued, respectively} \\ \quad L_2(\Omega) = H^0(\Omega)\text{-valued, analytic functions in } 0 < t < T, \end{cases} \tag{1.4.8.34}$$

where $H^s(\Gamma)$ and $H^s(\Omega)$ are the usual Sobolev spaces. However, once analyticity is achieved on $L_2(\Gamma)$ for u^0, or $L_2(\Omega)$ for y^0, it can readily be extended to higher order Sobolev spaces based on Γ or Ω. To this end, and with reference to problem (1.4.8.32), we introduce the Dirichlet map D (harmonic extension into the interior Ω of the boundary datum on Γ),

$$h = Dg \Longleftrightarrow \begin{cases} \Delta h = 0 & \text{in } \Omega; \\ h = g & \text{in } \Gamma. \end{cases} \tag{1.4.8.35}$$

Conservatively, elliptic theory yields

$$D \in \mathcal{L}(L_2(\Gamma); L_2(\Omega)) \quad \text{and} \quad D^* \in \mathcal{L}(L_2(\Omega); L_2(\Gamma)),$$

where D^* denotes the adjoint of D in the sense

$$(Dg, f)_{L_2(\Omega)} = (g, D^* f)_{L_2(\Gamma)}, \quad g \in L_2(\Gamma), \quad f \in L_2(\Omega). \tag{1.4.8.36}$$

Indeed, the higher regularity of the assumption below holds true [Chapter 3, Eqn. (3.1.7)]. The following result is given in [Lasiecka, Triggiani, 1983, Corollary 1.1.2] (see also Seidman [1982] and the Notes at the end of Chapter 1).

Theorem 1.4.8.6 *With reference to (1.4.8.35), assume that the boundary Γ is such that the Dirichlet map D satisfies the following regularity property:*

$$D : \text{continuous } H^s(\Gamma) \to H^{s+\frac{1}{2}}(\Omega), \quad \forall s \geq 0, \tag{1.4.8.37}$$

and likewise D^*.

Then, $u^0(t, 0; x)$ and $y^0(t, 0; x)$ are $H^s(\Gamma)$-valued, respectively $H^s(\Omega)$-valued analytic functions in $0 < t < T$, for all $s \geq 0$, where $y_0 \in Y = L_2(\Omega)$. Regarding (1.4.8.37), we point out that this assumption on Γ holds for all smooth domains Ω [Lions, Magenes, Vol. I, 1972, p. 188] and also for all Ω with conical points, provided that certain relations between the dimension of Ω and the solid angles of the conical *points hold, as specified in [Kondratiev, 1967, Theorems 4.1, 4.2]. As to the fulfillment of the assumption on D^*, however, the literature that we are aware of assumes smooth Γ.*

1.5 First Smoothing Case of the Operator G: The Case $(-A^*)^\beta G^*G \in \mathcal{L}(Y), \beta > 2\gamma - 1$. Proof of Theorem 1.2.2.1

1.5.1 Smoothing Properties of $L_{sT}^*G^*GL_{sT}$ and Bounded Inversion of Λ_{sT} on $C_{\gamma-\beta}([s,T];U)$. An Abstract Lemma

In this section, we point out a more regular theory that becomes available under the stronger assumption (which is always satisfied with $\beta = 0$ if $\lambda < 1/2$) that

$$(-A^*)^\beta G^*G \in \mathcal{L}(Y), \quad \beta > 2\gamma - 1, \quad \text{if } \frac{1}{2} \le \gamma < 1; \; \beta = 0 \quad \text{if } 0 \le \gamma < \frac{1}{2}, \tag{1.5.1.1}$$

that is, when G^*G maps all of Y into $D((-A^*)^\beta)$, so that (1.5.1.1) holds true by the closed graph theorem. In this case, Theorem 1.2.2.1 holds true. Its statement reveals, as already noted, that its main features include: (i) the assertion of higher regularity properties of all quantities involved, in particular continuity also at the terminal time $t = T$ of the optimal trajectory y^0; (ii) the assertion that the operator $P(t)$ defined constructively by (1.4.3.15) satisfies now the differential Riccati equation in a classical sense as in (1.2.2.14) (stronger than (1.2.1.12)); (iii) the assertion that such $P(t)$ is, in fact, the unique solution (within a natural class) of the integral Riccati equation, hence of the differential Riccati equation (1.2.1.2) = (1.4.6.22), hence of the classical version (1.2.2.14).

We begin with a lemma, which, once specialized to our needs, plays a crucial role under the smoothing assumption (1.5.1.1). Parts (i) and (ii) are taken from Flandoli [1984].

Lemma 1.5.1.1 *Let $\mathcal{A}$ be the generator of a strongly continuous analytic semigroup $e^{\mathcal{A}t}$ of negative type on the Hilbert space $\mathcal{Y}$. Let $\mathcal{G} \in \mathcal{L}(\mathcal{Y})$ be a bounded self-adjoint operator on $\mathcal{Y}$ such that $\mathcal{A}^*\mathcal{G} \in \mathcal{L}(\mathcal{Y})$. Then*

(i)

$$\int_0^\infty e^{\mathcal{A}^*t}[(-\mathcal{A}^*)\mathcal{G} + \mathcal{G}(-\mathcal{A})]e^{\mathcal{A}t}\,dt = \mathcal{G}. \tag{1.5.1.2}$$

(ii) For any $0 < \epsilon < r < 1$, we have

$$\int_0^\infty (-\mathcal{A}^*)^{1-r} e^{\mathcal{A}^*t}[(-\mathcal{A}^*)\mathcal{G} + \mathcal{G}(-\mathcal{A})]e^{\mathcal{A}t}(-\mathcal{A})^{r-\epsilon}\,dt = (-\mathcal{A}^*)^{1-r}\mathcal{G}(-\mathcal{A})^{r-\epsilon} \tag{1.5.1.3a}$$

in the sense that the integral on the left-hand side of (1.5.1.3) defines an element of $\mathcal{L}(\mathcal{Y})$ *that coincides with the closure (extension) of* $(-\mathcal{A}^*)^{1-\theta}\mathcal{G}(-\mathcal{A})^{\theta-\epsilon}$ *(denoted by the same symbol). Moreover,*

$$\|(-\mathcal{A}^*)^{1-r}\mathcal{G}(-\mathcal{A})^{r-\epsilon}\|_{\mathcal{L}(\mathcal{Y})} \leq C_{r,\epsilon}\|\mathcal{A}^*\mathcal{G}\|_{\mathcal{L}(\mathcal{Y})}. \tag{1.5.1.3b}$$

(iii) If, in addition, $\mathcal{A}$ *is self-adjoint on* $\mathcal{Y}$*, then we have*

$$(-\mathcal{A})^{\frac{1}{2}}\mathcal{G}(-\mathcal{A})^{\frac{1}{2}} \in \mathcal{L}(\mathcal{Y}); \tag{1.5.1.4a}$$

more precisely,

$$\left\|(-\mathcal{A})^{\frac{1}{2}}\mathcal{G}(-\mathcal{A})^{\frac{1}{2}}\right\|_{\mathcal{L}(\mathcal{Y})} \leq \|\mathcal{A}\mathcal{G}\|_{\mathcal{L}(\mathcal{Y})}, \tag{1.5.1.4b}$$

that is, if $\mathcal{A}$ *is also self-adjoint, then (1.5.1.3) holds true for* $r = 1/2$ *with no loss of* ϵ *(i.e.,* $\epsilon = 0$*).*

Proof. (i) We first note that the assumptions on $\mathcal{G}$ guarantee that $\mathcal{G}\mathcal{A}$ extends to a bounded operator in $\mathcal{L}(\mathcal{Y})$. We next compute by integrating by parts

$$\begin{aligned}\int_0^\infty e^{\mathcal{A}^*t}\mathcal{A}^*\mathcal{G}e^{\mathcal{A}t}dt &= \int_0^\infty \frac{de^{\mathcal{A}^*t}}{dt}\mathcal{G}e^{\mathcal{A}t}dt \\ &= [e^{\mathcal{A}^*t}\mathcal{G}e^{\mathcal{A}t}]_{t=0}^{t=\infty} - \int_0^\infty e^{\mathcal{A}^*t}\mathcal{G}\frac{de^{\mathcal{A}t}}{dt}dt \\ &= -\mathcal{G} - \int_0^\infty e^{\mathcal{A}^*t}\mathcal{G}\mathcal{A}e^{\mathcal{A}t}dt,\end{aligned} \tag{1.5.1.5a}$$

after using the fact that $e^{\mathcal{A}t}$ and $e^{\mathcal{A}^*t}$ are of negative type, and (1.5.1.2) is proved.

(ii) By analyticity [Pazy, 1983, p. 74], the integral, say $\mathcal{T}$, on the left of (1.5.1.3) is norm-bounded as follows, since $0 < \epsilon < r < 1$:

$$\|\mathcal{T}\|_{\mathcal{L}(\mathcal{Y})} \leq C\int_0^\infty \frac{e^{-\omega t}}{t^{1-r}}\frac{e^{-\omega t}}{t^{r-\epsilon}}\,dt < \infty, \tag{1.5.1.5b}$$

where the constant C includes the bounded norm of $(-\mathcal{A}^*)\mathcal{G} + \mathcal{G}(-\mathcal{A})$, in $\mathcal{L}(\mathcal{Y})$, and $(-\omega) < 0$ is the negative type. The boundedness in (1.5.1.5) proves part (ii), via part (i).

(iii) The special case of (5.1.4) (which will not be used in this section) is proved in Appendix 1A. □

As a corollary, we obtain the situation that interests us.

Corollary 1.5.1.2

(i) For $1/2 \leq \gamma < 1$*, assumption (1.5.1.1) impies that for any* ρ *and* θ *such that* $0 < \rho < \theta < \beta$*, the operator* $(-A^*)^{\beta-\theta}G^*G(-A)^{\theta-\rho}$ *admits a bounded extension in* $\mathcal{L}(Y)$*, a condition that we simply write as*

$$(-A^*)^{\beta-\theta}G^*G(-A)^{\theta-\rho} \in \mathcal{L}(Y), \quad 0 < \rho < \theta < \beta; \tag{1.5.1.6a}$$

$$\|(-A^*)^{\beta-\theta}G^*G(-A)^{\theta-\rho}\|_{\mathcal{L}(Y)} \leq C_{\beta,\theta,\rho}\|(-A)^\beta G^*G\|_{\mathcal{L}(Y)}. \tag{1.5.1.6b}$$

(ii) *If, in addition, A is self-adjoint on Y, then (1.5.1.6) holds true with $\theta = \beta/2$ and $\rho = 0$:*

$$\left\|(-A)^{\frac{\beta}{2}} G^* G(-A)^{\frac{\beta}{2}}\right\|_{\mathcal{L}(Y)} \leq \|(-A)^{\beta} G^* G\|_{\mathcal{L}(Y)}, \tag{1.5.1.6c}$$

so that, in this case, (1.5.1.1) implies $(-A)^{\beta/2}G^* \in \mathcal{L}(Z_f; Y)$ [cf. (1.2.1.26)].

Proof. We specialize Lemma 1.5.1.1 with

$$\mathcal{Y} = Y, \quad \mathcal{A} = -(-A)^{\beta}, \quad \mathcal{G} = G^*G; \tag{1.5.1.7}$$

where the operator $-(-A)^{\beta}$ generates, as is well known [Pazy, 1983, p. 74, Krein, 1971, p. 123], a s.c. analytic semigroup on Y of negative type, since, as assumed, A is the generator of a s.c. analytic semigroup on Y of negative type. Moreover, $\mathcal{G} = G^*G$ satisfies the assumptions of Lemma 1.5.1.1, also by virtue of (1.5.1.1). Then, the conclusion of Lemma 1.5.1.1 applies to the operator

$$(-\mathcal{A}^*)^{1-r}\mathcal{G}(-\mathcal{A})^{r-\epsilon} = (-A^*)^{\beta-\theta}G^*G(-A)^{\theta-\rho}, \quad \theta = \beta r < \beta; \quad \rho = \beta\epsilon < \theta, \tag{1.5.1.8}$$

and part (i) is proved. For part (ii) we notice that $\mathcal{A}$ is, moreover, self-adjoint, if A is self-adjoint. A. Then Lemma 1.5.1.1(iii) yields (1.5.1.6c), as desired. □

As a second corollary we now substantiate the *claim* on GL_T being bounded, made just below (1.2.2.1).

Corollary 1.5.1.3

(i) *For $0 \leq \gamma < 1/2$, the operator L_T is bounded: $L_2(0, T; U) \to Y$.*

(ii) *For $1/2 \leq \gamma < 1$, assume (1.5.1.1). Then, the operator GL_T is bounded: $L_2(0, T; U) \to Z_f$ In either case, a fortiori, the assumption of Theorem 1.2.1.1 is satisfied.*

Proof. (ii) By specializing Corollary 1.5.1.2 with $\theta = \beta/2$ we see that the operator $(-A^*)^{\beta/2}G^*G(-A)^{\beta/2-\epsilon}$ admits a bounded extension in $\mathcal{L}(Y)$, $\beta/2 > \epsilon > 0$. Thus, $(-A^*)^{\beta/2-\epsilon}G^*G(-A)^{\beta/2-\epsilon}$ is self-adjoint and in $\mathcal{L}(Y)$. Then, $(-A^*)^{\beta/2-\epsilon}G^* \in \mathcal{L}(Z_f; Y)$. Then, as in (1.2.1.26) and the line below it, we see that $L_T^*G^*$ is bounded: $L_T^*G^* \in \mathcal{L}(Z_f; L_2(0, T; U))$, as desired:

$$\{L_T^*G^*z\}(t) = B^*(-A^*)^{-\gamma}(-A^*)^{\gamma-\beta/2+\epsilon}e^{A^*(T-t)}(-A^*)^{\beta/2-\epsilon}G^*z,$$

since $(-A^*)^{\gamma-\beta/2+\epsilon}e^{A^*(T-t)} \in \mathcal{L}(Y; L_2(0, T; Y))$ with $2\gamma - \beta + 2\epsilon < 1$ by analyticity of the semigroup [Pazy, 1983, p. 74]. □

Next, we return to Eqn. (1.4.1.25) rewritten here for convenience: For $x \in Y$,

$$-u^0(\cdot\,, s; x) = \Lambda_{sT}^{-1}\left[L_{sT}^*G^*Ge^{A(T-s)}x + L_s^*R^*Re^{A(\cdot-s)}x\right], \tag{1.5.1.9}$$

$$\Lambda_{sT} = I_s + L_s^*R^*RL_s + L_{sT}^*G^*GL_{sT}, \tag{1.5.1.10}$$

which provides $u^0(\cdot\,, s; x)$ explicitly in terms of the data of the problem.

Lemma 1.5.1.4 *Assume hypothesis (1.5.1.1). Then, for any $x \in Y$, the term in the square brackets of (1.5.1.9) satifies (recall the space of definition (1.4.4.1))*

$$L_{sT}^* G^* G e^{A(T-s)} x + L_s^* R^* R e^{A(\cdot - s)} x \in C_{\gamma-\beta}([s,T];U) \tag{1.5.1.11}$$

Proof. Assume first $1/2 \le \gamma < 1$. Since $L_s^* R^* R e^{A(\cdot - s)} x \in C([s,T];U)$ a fortiori from (1.4.1.5b), and since $L_{sT}^* G^* G e^{A(T-s)} x \in C([s,T];U)$ from (1.4.1.7), it remains to show that

$$\left\| \left(L_{sT}^* G^* G e^{A(T-s)} x \right)(t) \right\|_U \le \frac{C_{T\gamma}}{(T-t)^{\gamma-\beta}} \|x\|_Y, \quad s \le t < T. \tag{1.5.1.12}$$

But (1.5.1.12) follows readily via (1.4.1.7), (1.5.1.1), (1.1.2), and analyticity of $e^{A^* t}$ [Pazy, 1983, p. 74] from

$$\begin{aligned} &\left\| \left(L_{sT}^* G^* G e^{A(T-s)} x \right)(t) \right\|_U \\ &\quad = \left\| B^* (-A^*)^{-\gamma} (-A^*)^{\gamma-\beta} e^{A^*(T-t)} (-A^*)^{\beta} G^* G e^{A(T-s)} x \right\|_U. \end{aligned} \tag{1.5.1.13}$$

The other case $0 \le \gamma < 1/2$ ($\beta = 0$) follows from (1.5.1.13) to yield (1.5.1.11) with $\beta = 0$ as required. □

The crucial result of this section is Theorem 1.5.1.7, below, concerning Λ_{sT} in (1.5.1.10).

Before presenting it, we need to establish appropriate regularity properties of the operator $L_{sT}^* G^* G L_{sT}$.

Lemma 1.5.1.5 *Assume hypothesis (1.5.1.1). Then*

$$L_{sT}^* G^* G L_{sT} : \text{continuous } C_{\gamma-\beta}([s,T];U) \to C_{\gamma-\beta}([s,T];U). \tag{1.5.1.14}$$

Proof. Assume first $1/2 \le \gamma < 1$, so that $\gamma - \beta < 1/2$. Let $u \in C_{\gamma-\beta}([s,T];U) \subset L_2(s,T;U)$. Then by (1.4.1.7) and (1.4.1.6), we see, via Corollary 1.5.1.3 on GL_{sT}, that $\{L_{sT}^* G^* G L_{sT} u\}(t) \in C([s,T];U)$. Thus, we only need to show the required singularity at $t = T$. We compute via (1.4.1.6) and (1.4.1.7), by virtue of (1.1.2), analyticity [Pazy, 1983, p. 74], (1.5.1.1), and (1.4.4.1):

$$\begin{aligned} \|\{L_{sT}^* G^* G L_{sT} u\}(t)\|_U &= \left\| B^* (-A^*)^{-\gamma} (-A^*)^{\gamma-\beta} e^{A^*(T-t)} (-A^*)^{\beta} G^* G L_{sT} u \right\|_U \\ \text{(by (1.5.1.1))} \qquad &\le \frac{C_T}{(T-t)^{\gamma-\beta}} \left\| \int_s^T (-A)^{\gamma} e^{A(T-\tau)} (-A)^{-\gamma} B u(\tau) d\tau \right\|_Y \\ &\le \frac{C_T}{(T-t)^{\gamma-\beta}} \int_s^T \frac{1}{(T-\tau)^{\gamma}} \frac{\|u(\tau)\|_U (T-\tau)^{\gamma-\beta}}{(T-\tau)^{\gamma-\beta}} d\tau \\ \text{(by (1.4.4.1))} \qquad &\le \frac{C_T}{(T-t)^{\gamma-\beta}} \left\{ \int_s^T \frac{d\tau}{(T-\tau)^{2\gamma-\beta}} \right\} \|u\|_{C_{\gamma-\beta}([s,T];U)} \end{aligned} \tag{1.5.1.15}$$

$$\le \frac{C_T}{(T-t)^{\gamma-\beta}} \|u\|_{C_{\gamma-\beta}([s,T];U)}, \quad s \le t < T, \tag{1.5.1.16}$$

where, in the last step, we have used the fact that $2\gamma - \beta < 1$ by (1.5.1.1), so that the integral in (1.5.1.15) is convergent. This yields (1.5.1.16), as desired. So (1.5.1.14) is proved for $1/2 \le \gamma < 1$.

The case $0 \le \gamma < 1/2$ ($\beta = 0$) proceeds exactly the same as the above, by taking $\beta = 0$ all along. □

Theorem 1.5.1.6 *(a) Assume (1.5.1.1) with $1/2 \le \gamma < 1$. Then*

(i)

$$L_{sT}^* G^* G L_{sT} : \textit{continuous } L_2(s, T; U) \to C_{\frac{1}{2}-\epsilon_0}([s, T]; U), \tag{1.5.1.17}$$

with $\epsilon_0 > 0$ such that $\beta > 2\gamma - 1 + 2\epsilon_0$;

(ii)

$$L_{sT}^* G^* G L_{sT} : \textit{continuous } C_{\frac{1}{2}-\epsilon_0}([s, T]; U) \to C_{\frac{1}{2}-2\epsilon_0}([s, T]; U), \tag{1.5.1.18}$$

so that

$$(L_{sT}^* G^* G L_{sT})^2 : \textit{continuous } L_2(s, T; U) \to C_{\frac{1}{2}-2\epsilon_0}([s, T]; U); \tag{1.5.1.19}$$

(iii) generally, for $k = 1, 2, \dots, (k+1)\epsilon_0 \le 1/2$,

$$L_{sT}^* G^* G L_{sT} : \textit{continuous } C_{\frac{1}{2}-k\epsilon_0}([s, T]; U) \to C_{\frac{1}{2}-(k+1)\epsilon_0}([s, T]; U), \tag{1.5.1.20}$$

so that for $n = 1, 2, 3, \dots, n\epsilon_0 \le 1/2$,

$$(L_{sT}^* G^* G L_{sT})^n : \textit{continuous } L_2(s, T; U) \to C_{\frac{1}{2}-n\epsilon_0}([s, T]; U); \tag{1.5.1.21}$$

(iv) for an integer $n_1 > [1/2 - (\gamma - \beta)]/\epsilon_0$, so that $\gamma - \beta > 1/2 - n_1\epsilon_0$, we then have

$$(L_{sT}^* G^* G L_{sT})^{n_1} : \textit{continuous } L_2(s, T; U) \to C_{\gamma-\beta}([s, T]; U). \tag{1.5.1.22}$$

(b) However, if $0 \le \gamma < 1/2$, then (1.5.1.22) holds true with $\beta = 0$ and $n_1 = 1$.

Proof. The proof will resemble the arguments of Section 1.4.4 leading to Theorem 1.4.4.4.

(a) We first assume (1.5.1.1) with $1/2 \le \gamma < 1$.

(i) Proof of (1.5.1.17).

Step 1. Claim Let $v_0 \in L_2(s, T; U)$; then

$$\|(-A)^{-\sigma_0} L_{sT} v_0\|_Y \le \frac{C_T}{\sqrt{\epsilon_0}}(T - s)^{\epsilon_0} \|v_0\|_{L_2(s,T;U)}; \tag{1.5.1.23}$$

$$\forall \sigma_0 > \gamma - \frac{1}{2}, \quad \text{so that} \quad \sigma_0 = \gamma - \frac{1}{2} + \epsilon_0, \ \epsilon_0 > 0. \tag{1.5.1.24}$$

In fact, we compute, from (1.4.1.6) via (1.1.2) and analyticity of e^{At} [Pazy, 1983, p. 74],

$$\begin{aligned}\|(-A)^{-\sigma_0}L_{sT}v_0\|_Y &= \left\|(-A)^{-\sigma_0}\int_s^T(-A)^{\gamma}e^{A(T-\tau)}(-A)^{-\gamma}Bv_0(\tau)\,d\tau\right\|_Y\\ &\le C_T\int_s^T\frac{\|v_0(\tau)\|_U d\tau}{(T-\tau)^{\gamma-\sigma_0}}\\ &\le C_T\left\{\int_s^T\frac{d\tau}{(T-\tau)^{2(\gamma-\sigma_0)}}\right\}^{\frac{1}{2}}\|v_0\|_{L_2(s,T;U)}. \qquad (1.5.1.25)\end{aligned}$$

Thus, by Schwarz's inequality and for $2(\gamma-\sigma_0)<1$ in (1.5.1.25), we obtain (1.5.1.23), as desired, via (1.5.1.24).

Step 2 Claim Setting, still with $v_0\in L_2(s,T;U)$,

$$v_1(t)=(L_{sT}^*G^*GL_{sT}v_0)(t), \qquad (1.5.1.26)$$

we now have, with $\beta>2\sigma_0=2\gamma-1+2\epsilon_0$,

$$\begin{aligned}\|v_1(t)\|_U &= \|(L_{sT}^*G^*GL_{sT}v_0)(t)\|_U\\ &\le \frac{C_T}{\sqrt{\epsilon_0}}\frac{(T-s)^{\epsilon_0}}{(T-t)^{\frac{1}{2}-\epsilon_0}}\|v_0\|_{L_2(s,T;U)}, \quad s\le t<T. \qquad (1.5.1.27)\end{aligned}$$

In fact, we compute, from (1.4.1.7) via (1.1.2) and analyticity [Pazy, 1983, p. 74],

$$\begin{aligned}&\|(L_{sT}^*G^*GL_{sT}v_0)(t)\|_U\\ &\quad=\|B^*(-A^*)^{-\gamma}(-A^*)^{\gamma-\sigma_0}e^{A^*(T-t)}(-A^*)^{\sigma_0}G^*G(-A)^{\sigma_0}(-A)^{-\sigma_0}L_{sT}v_0\|_U\\ &(\text{by } (1.5.1.6))\quad \le C_T\frac{1}{(T-t)^{\gamma-\sigma_0}}\|(-A)^{-\sigma_0}L_{sT}v_0\|_Y, \quad s\le t<T \qquad (1.5.1.28)\\ &(\text{by } (1.5.1.23))\quad \le \frac{C_T}{\sqrt{\epsilon_0}}\frac{(T-s)^{\epsilon_0}}{(T-t)^{\gamma-\sigma_0}}\|v_0\|_{L_2(s,T;U)}, \quad s\le t<T, \qquad (1.5.1.29)\end{aligned}$$

and (1.5.1.29) shows (1.5.1.27) since $\gamma-\sigma_0=1/2-\epsilon_0$ by (1.5.1.24). In obtaining (1.5.1.28) we have used Corollary 1.5.1.2, Eqn. (1.5.1.6) for asserting $(-A^*)^{\sigma_0}G^*G(-A)^{\sigma_0}\in\mathcal{L}(Y)$, which is justified since $\beta>2\sigma_0>2\gamma-1=2(\sigma_0-\epsilon_0)$ from (1.5.1.1) and (1.5.1.24). Moreover, (1.5.1.28) implies (1.5.1.29) by (1.5.1.23). Finally, since $\{L_{sT}^*G^*GL_{sT}v_0\}(t)\in C([s,T);U)$, via (1.4.1.7) and Corollary 1.5.1.3 (or the identity, without norm, leading to (1.5.1.28)), we see that (1.5.1.27) yields (1.5.1.17) via (1.4.4.1).

(ii) Proof of (1.5.1.18). We reiterate the procedure of Steps 1–2 above.

Step 1. Claim With $v_1\in C_\rho([s,T];U)$, $\rho=\gamma-\sigma_0=1/2-\epsilon_0$, then

$$\|(-A)^{-\sigma_1}L_{sT}v_1\|_Y\le\frac{C_T}{1-\epsilon_0}(T-s)^{1-\epsilon_0}\|v_1\|_{C_\rho([s,T];U)}; \qquad (1.5.1.30)$$

$$\sigma_1=\sigma_0-\epsilon_0=\gamma-\frac{1}{2}<\sigma_0. \qquad (1.5.1.31)$$

In fact, from (1.4.1.6), we compute, via (1.1.2), analyticity [Pazy, 1983, p. 74], and (1.4.4.1),

$$\begin{aligned}
\|(-A)^{-\sigma_1} L_{sT} v_1\|_Y &= \left\| (-A)^{-\sigma_1} \int_s^T (-A)^{\gamma} e^{A(T-\tau)} (-A)^{-\gamma} B v_1(\tau)\, d\tau \right\|_Y \\
&\leq C_T \int_s^T \frac{\|v_1(\tau)\|_U (T-\tau)^{\rho} d\tau}{(T-\tau)^{\gamma-\sigma_1} (T-\tau)^{\rho}} \\
&\leq C_T \left(\int_s^T \frac{d\tau}{(T-\tau)^{\gamma-\sigma_1+\rho}} \right) \|v_1\|_{C_\rho([s,T];U)}, \qquad (1.5.1.32)
\end{aligned}$$

and (1.5.1.32) yields (1.5.1.30), since $\gamma - \sigma_1 + \rho = \gamma - \sigma_1 + \gamma - \sigma_0 = 1 - \epsilon_0 < 1$ from (1.5.1.31).

Step 2. Claim Setting, still with $v_1 \in C_\rho([s, T]; U)$, $\rho = \gamma - \sigma_0 = 1/2 - \epsilon_0$,

$$v_2(t) = (L_{sT}^* G^* G L_{sT} v_1)(t), \qquad (1.5.1.33)$$

we have that

$$\|v_2(t)\|_U = \|(L_{sT}^* G^* G L_{sT} v_1)(t)\|_U \leq \frac{C}{1-\epsilon_0} \frac{(T-s)^{1-\epsilon_0}}{(T-t)^{\frac{1}{2}-2\epsilon_0}} \|v_1\|_{C_{\frac{1}{2}-\epsilon_0}([s,T];U)}. \qquad (1.5.1.34)$$

In fact, since $\sigma_1 = \sigma_0 - \epsilon_0 < \sigma_0$, we rewrite the counterpart of (1.5.1.28) as

$$\begin{aligned}
\|(L_{sT}^* G^* G L_{sT} v_1)(t)\|_U &= \big\| B^*(-A^*)^{-\gamma} (-A^*)^{\gamma-(\sigma_0+\epsilon_0)} e^{A^*(T-t)} \\
&\qquad \times (-A^*)^{\sigma_0+\epsilon_0} G^* G (-A)^{\sigma_0-\epsilon_0} (-A)^{-\sigma_1} L_{sT} v_1 \big\|_U \\
\text{(by (1.5.1.6))} \quad &\leq C_T \frac{1}{(T-t)^{\gamma-(\sigma_0+\epsilon_0)}} \|(-A)^{-\sigma_1} L_{sT} v_1\|_Y \qquad (1.5.1.35) \\
\text{(by (1.5.1.30))} \quad &\leq \frac{C_T}{1-\epsilon_0} \frac{(T-s)^{1-\epsilon_0}}{(T-t)^{\frac{1}{2}-2\epsilon_0}} \|v_1\|_{C_\rho([s,T];U)}, \quad s \leq t < T,
\end{aligned}$$

$$(1.5.1.36)$$

and (1.5.1.36) proves (1.5.1.34) since $\rho = 1/2 - \epsilon_0$. In obtaining (1.5.1.35), we have used Corollary 1.5.1.2, Eqn. (1.5.1.6), for asserting $(-A^*)^{\sigma_0+\epsilon_0} G^* G (-A)^{\sigma_0-\epsilon_0} \in \mathcal{L}(Y)$, which is justified since $\beta > (\sigma_0 + \epsilon_0) + (\sigma_0 - \epsilon_0) = 2\sigma_0 > 2\gamma - 1 = 2(\sigma_0 - \epsilon_0)$ by (1.5.1.1), (1.5.1.31). Moreover, (1.5.1.35) implies (1.5.1.36) by (1.5.1.30) and $\gamma - (\sigma_0 + \epsilon_0) = 1/2 - \epsilon_0 - \epsilon_0 = 1/2 - 2\epsilon_0$ via (1.5.1.31). Finally, since $\{L_{sT}^* G^* G L_{sT} v_1\}(t) \in C([s, T); U)$ via the identity, without norm, leading to (1.5.1.35), we see that (1.5.1.34) yields (1.5.1.18).

(iii) Proof of (1.5.1.20) We repeat the argument of (ii), with

$$v_k \in C_{\frac{1}{2}-k\epsilon_0}([s, T]; U), \quad \sigma_k = \sigma_0 - k\epsilon_0, \quad k = 1, 2, \ldots,$$

so that

$$\|(-A)^{-\sigma_k} L_{sT} v_k\|_Y = \frac{C_T}{1-\epsilon_0} (T-s)^{1-\epsilon_0} \|v_k\|C_{\frac{1}{2}-k\epsilon_0([s,T];U)}, \qquad (1.5.1.37)$$

by the argument leading to (1.5.1.32), this time with $\rho = 1/2 - k\epsilon_0$, so that $\gamma - \sigma_k + \rho = \gamma - \sigma_0 + \frac{1}{2} = 1 - \epsilon_0$, again by (1.5.1.31). Moreover, now the counterpart of (1.5.1.35) (without norm) is

$$(L_{sT}^* G^* G L_{sT} v_k)(t) = B^*(-A^*)^{-\gamma}(-A^*)^{\gamma-(\sigma_0+k\epsilon_0)} e^{A^*(T-t)}(-A^*)^{\sigma_0+k\epsilon_0} G^* G \times (-A)^{\sigma_0-k\epsilon_0}(-A)^{-\sigma_k} L_{sT} v_k, \quad (1.5.1.38)$$

with $\sigma_k = \sigma_0 - k\epsilon_0$, and $(-A^*)^{\sigma_0+k\epsilon_0} G^* G(-A)^{\sigma_0-k\epsilon_0} \in \mathcal{L}(Y)$ again by Corollary 1.5.1.2, Eqn. (1.5.1.6), with $\beta > (\sigma_0 + k\epsilon_0) + (\sigma_0 - k\epsilon_0) = 2\sigma_0 > 2\gamma - 1$ by (1.5.1.1), (1.5.1.31). Thus, (1.5.1.38) yields $(L_{sT}^* G^* G L_{sT} v_k)(t) \in C([0, T]; U)$, as well as

$$\begin{aligned} \|(L_{sT}^* G^* G L_{sT} v_k)(t)\| &\leq \frac{C_T}{(T-t)^{\gamma-(\sigma_0+k\epsilon_0)}} \|(-A)^{-\sigma_k} L_{sT} v_k\|_Y \\ \text{(by (1.5.1.37))} \qquad &\leq \frac{C_T}{1-\epsilon_0} \frac{(T-s)^{1-\epsilon_0}}{(T-t)^{\frac{1}{2}-(k+1)\epsilon_0}} \|v_k\|_{C_{\frac{1}{2}-k\epsilon_0}([s,T];U)}, \quad s \leq t < T, \end{aligned} \quad (1.5.1.39)$$

with $\gamma - \sigma_0 = 1/2 - \epsilon_0$, and (1.5.1.20) is proved. Thus, Theorem 1.5.1.6(a) is established, at least when $1/2 \leq \gamma < 1$.

(b) If $0 \leq \gamma < 1/2$ ($\beta = 0$), then part (i) can be repeated with $\beta = 0$, yielding (1.5.1.22) with $n_1 = 1$. □

The main result of this section is the following:

Theorem 1.5.1.7 *Assume hypothesis (1.5.1.1). Then the operator Λ_{sT} satisfies*

$$\Lambda_{sT}^{-1} = [I_s + L_s^* R^* R L_s + L_{sT}^* G^* G L_{sT}]^{-1} \in \mathcal{L}(C_{\gamma-\beta}([s, T]; U)), \quad (1.5.1.40)$$

with uniform bound that may be taken independent of s, $s \leq t \leq T$.

Proof. We return to identity (1.5.1.9) and apply a bootstrap argument (similar to the one carried out in Theorem 1.4.4.4). Let $w \in C_{\gamma-\beta}([s, T]; U)$. We seek a unique $g \in C_{\gamma-\beta}([s, T]; U)$ such that

$$g + L_s^* R^* R L_s g + L_{sT}^* G^* G L_{sT} g = w \in C_{\gamma-\beta}([s, T]; U). \quad (1.5.1.41)$$

Since $2(\gamma - \beta) < 1$, a fortiori from $\beta > 2\gamma - 1$ if $1/2 \leq \gamma < 1$, and for $\beta = 0$ if $0 \leq \gamma < 1/2$, we have that $w \in L_2(s, T; U)$. Then, there exists a unique $g \in L_2(s, T; U)$ such that (1.5.1.41) holds true by (1.4.1.14), with $V(s, T; U) = L_2(0, T; U)$ now, by Corollary 1.5.1.3. We next show that, in fact, $g \in C_{\gamma-\beta}([s, T]; U)$, as desired. To this end, we need two bootstrap procedures, which are carried out in the next two steps.

Step 1 We first show that

$$L_s^* R^* R L_s g \in C([s, T]; U). \quad (1.5.1.42)$$

In fact, with

$$L_{sT}^* G^* G L_{sT} g \in C_{\frac{1}{2}-\epsilon_0}([s, T]; U) \quad (1.5.1.43)$$

from Eqn. (1.5.1.17) of Theorem 1.5.1.6, we have

$$L_s^* R^* R L_s [L_{sT}^* G^* G L_{sT} g] \in C([s, T]; U), \tag{1.5.1.44}$$

$$L_s^* R^* R L_s w \in C([s, T]; U), \tag{1.5.1.45}$$

by invoking (1.4.4.11) of Corollary 1.4.4.2, or better, its proof, which shows one iteration $n_0(r) = 1$ for $r < 2/3$, where $r = 1/2 - \epsilon_0$ via (1.5.1.43) for the term in (1.5.1.44), and $r = \gamma - \beta < 1/2$ by assumption on w for the term in (1.5.1.45). A fortiori from (1.5.1.44) and (1.5.1.45), via the regularity of L_s in (1.4.4.18) and L_s^* in (1.4.1.5b), we obtain, for $r = 1, 2, 3, \ldots$,

$$(L_s^* R^* R L_s)^r [L_{sT}^* G^* G L_{sT} g] \in C([s, T]; U), \tag{1.5.1.46}$$

$$(L_s^* R^* R L_s)^r w \in C([s, T]; U). \tag{1.5.1.47}$$

After these preliminaries, we let $n_1(\gamma)$ be the positive integer of Theorem 1.4.4.3(v), Eqn. (1.4.4.19), so that

$$(L_s^* R^* R L_s)^{n_1} g \in C([s, T]; U). \tag{1.5.1.48}$$

We then apply $L_s^* R^* R L_s$ recursively $(n_1 - 1)$ times on Eqn. (1.5.1.41), thereby obtaining (we take here $R = I$ for simplicity of notation)

$$L_s^* L_s g + L_s^* L_s [L_{sT}^* G^* G L_{sT} g] = L_s^* L_s w - (L_s^* L_s)^2 g, \tag{1.5.1.49$_1$}$$

$$\vdots$$

$$(L_s^* L_s)^{n_1 - 2} g + (L_s^* L_s)^{n_1 - 2} [L_{sT}^* G^* G L_{sT} g] = (L_s^* L_s)^{n_1 - 2} w - (L_s^* L_s)^{n_1 - 1} g, \tag{1.5.1.49$_{n_1-2}$}$$

$$(L_s^* L_s)^{n_1 - 1} g + (L_s^* L_s)^{n_1 - 1} [L_{sT}^* G^* G L_{sT} g] = (L_s^* L_s)^{n_1 - 1} w - (L_s^* L_s)^{n_1} g. \tag{1.5.1.49$_{n_1-1}$}$$

Beginning with (1.5.1.49$_{n_1-1}$), we see via (1.5.1.46)–(1.5.1.48) that $(L_s^* L_s)^{n_1 - 1} g \in C([s, T]; U)$. Using this information on the right-hand side of (1.5.1.49$_{n_1-2}$), along with (1.5.1.46) and (1.5.1.47), we then deduce that $(L_s^* L_s)^{n_1 - 2} g \in C([s, T]; U)$. Continuing this way down to Eqn. (1.5.1.49$_1$), we obtain $L_s^* L g \in C([s.T]; U)$, and (1.5.1.42) is proved.

Step 2 We next show that, as desired,

$$g \in C_{\gamma - \beta}([s, T]; U). \tag{1.5.1.50}$$

Let now n be the positive integer of Theorem 1.5.1.6(iv), Eqn. (1.5.1.22), so that

$$(L_{sT}^* G^* G L_{sT})^n g \in C_{\gamma - \beta}([s, T]; U). \tag{1.5.1.51}$$

Moreover, by the assumption on w and by (1.5.1.42) we have

$$w - L_s^* R^* R L_s g \equiv f \in C_{\gamma - \beta}([s, T]; U), \tag{1.5.1.52}$$

and by Lemma 1.5.1.5, Eqn. (1.5.1.14),

$$(L_{sT}^* G^* G L_{sT})^r f \in C_{\gamma-\beta}([s,T];U), \quad r = 1, 2, \dots. \tag{1.5.1.53}$$

Next, applying $L_{sT}^* G^* G L_{sT}$ recursively $(n-1)$ times on (1.5.1.41) we obtain, via (1.5.1.52) and (1.5.1.53),

$$g + L_{sT}^* G^* G L_{sT} g = f \in C_{\gamma-\beta}([s,T];U); \tag{1.5.1.54$_0$}$$

$$L_{sT}^* G^* G L_{sT} g + (L_{sT}^* G^* G L_{sT})^2 g$$
$$= L_{sT}^* G^* G L_{sT} f \in C_{\gamma-\beta}([s,T];U); \tag{1.5.1.54$_1$}$$

$$(L_{sT}^* G^* G L_{sT})^{n-2} g + (L_{sT}^* G^* G L_{sT})^{n-1} g$$
$$= (L_{sT}^* G^* G L_{sT})^{n-2} f \in C_{\gamma-\beta}([s,T];U); \tag{1.5.1.54$_{n-2}$}$$

$$(L_{sT}^* G^* G L_{sT})^{n-1} g + (L_{sT}^* G^* G L_{sT})^{n} g$$
$$= (L_{sT}^* G^* G L_{sT})^{n-1} f \in C_{\gamma-\beta}([s,T];U). \tag{1.5.1.54$_{n-1}$}$$

Beginning with Eqn. (1.5.1.54$_{n-1}$), we invoke (1.5.1.51) and obtain $(L_{sT}^* G^* G L_{sT})^{n-1} g \in C_{\gamma-\beta}([s,T];U)$. Using this information in Eqn. (1.5.1.54$_{n-2}$), we obtain $(L_{sT}^* G^* \, G L_{sT})^{n-2} g \in C_{\gamma-\beta}([s,T];U)$. Continuing this way down to Eqn. (1.5.1.54$_0$), we obtain $g \in C_{\gamma-\beta}([s,T];U)$, and (1.5.1.50) is proved. Theorem 1.5.1.7 is thus established. □

1.5.2 Improved Regularity Properties of Control Quantities at $t = T$

In this subsection we collect one set of benefits to the control problem accured under assumption (1.5.1.1): improved regularity properties of all control quantities at $t = T$. In the next subsection, we shall examine a second set of benefits, concerning the differential Riccati equation.

In the next theorem – a corollary of Theorem 1.5.1.7 – we obtain, under (1.5.1.1), an improvement of the regularity properties of all quantities involved, including the continuity of y^0 at $t = T$.

Theorem 1.5.2.1 *Assume hypothesis (1.5.1.1), and set $\beta = 2\gamma - 1 + \epsilon$, if $1/2 \le \gamma < 1$, and $\beta = 0$, if $0 \le \gamma < 1/2$. Then, for $x \in Y$:*

(i)

$$u^0(\,\cdot\,, s; x) \in C_{\gamma-\beta}([s,T];U), \tag{1.5.2.1a}$$

$$\|u^0(\,\cdot\,, s; x)\|_{C_{\gamma-\beta}([s,T];U)} \le C_{T\gamma} \|x\|_Y,$$
$$\gamma - \beta = \begin{cases} 1 - \gamma - \epsilon, & \text{if } \frac{1}{2} \le \gamma < 1, \\ \gamma, & \text{if } 0 \le \gamma < \frac{1}{2} \end{cases} \tag{1.5.2.1b}$$

with bound that may be taken independent of s;

(ii)

$$y^0(\,\cdot\,, s; x) = \Phi(\,\cdot\,, s)x \in C([s, T]; Y), \tag{1.5.2.2a}$$

$$\|y^0(\,\cdot\,, s; x)\|_{C([s,T];Y)} \leq C_{T\gamma}\|x\|_Y, \tag{1.5.2.2b}$$

with bound that may be taken independent of s;

(iii)

$$y^0(T, \,\cdot\,; x) = \Phi(T, \,\cdot\,)x \in C([s, T]; Y), \tag{1.5.2.3a}$$

in particular,

$$\lim_{t \uparrow T} \Phi(T, t)x = x; \tag{1.5.2.3b}$$

(iv) for any $0 \leq \theta < 1$, and

$$\theta - \beta = \begin{cases} \theta + 1 - 2\gamma - \epsilon, & \text{if } \frac{1}{2} \leq \gamma < 1, \\ \theta, & \text{if } 0 \leq \gamma < \frac{1}{2} \end{cases}$$

then

$$(-A^*)^\theta P(t)x \in C_{\theta-\beta}([0, T]; Y) \tag{1.5.2.4a}$$

and

$$\|(-A^*)^\theta P(t)x\|_Y \leq \frac{C_{T\gamma}}{1-\theta} \frac{1}{(T-t)^{\theta-\beta}} \|x\|_Y, \quad 0 \leq t < T; \tag{1.5.2.4b}$$

(v) with $\gamma - \beta$ as in (1.5.2.1),

$$B^* P(t)x \in C_{\gamma-\beta}([0, T]; U), \tag{1.5.2.5a}$$

$$\|B^* P(t)x\|_U \leq \frac{C_T}{1-\gamma} \frac{1}{(T-t)^{\gamma-\beta}} \|x\|_Y, \quad 0 \leq t < T; \tag{1.5.2.5b}$$

(vi)

$$\lim_{t \uparrow T} P(t)x = \lim_{t \uparrow T} e^{A^*(T-t)} G^* G \Phi(T, t)x = G^* G x \tag{1.5.2.6}$$

(this property was already noted in Proposition 1.4.7.2, Eqn. (1.4.7.3) in full generality, that is, without (1.5.1.1)).

Proof. (i) Conclusion (1.5.2.1) is a consequence of Theorem 1.5.1.7, Eqn. (1.5.1.40), via (1.5.1.9).

(ii) Conclusion (1.5.2.2) follows from (1.5.2.1) via the optimal dynamics (1.4.1.26) and property (1.4.4.2) for the operator L_s with $r = \gamma - \beta$, so that $r + \gamma = 2\gamma - \beta < 1$, as required there.

(iii) Conclusion (1.5.2.3) follows from (1.4.1.26) with $t = T$, that is, from

$$y^0(T, t; x) = \Phi(T, t)x = e^{A(T-t)}x + L_{tT} u^0(\,\cdot\,, t; x), \tag{1.5.2.7}$$

where by virtue of (1.4.1.6) (1.5.2.1) and (1.1.2), we obtain, as desired, since $2\gamma - \beta < 1$,

$$\begin{aligned}\|L_{tT}u^0(\,\cdot\,,t;x)\|_Y &= \left\| \int_t^T (-A)^\gamma e^{A(T-\tau)}(-A)^{-\gamma}Bu^0(\tau,t;x)\,d\tau \right\|_Y \\ &\leq C_{T\gamma}\int_t^T \frac{d\tau}{(T-\tau)^\gamma (T-\tau)^{\tau-\beta}}\|x\|_Y \\ &= C_{T\gamma}(T-t)^{(\beta+1-2\gamma)}\|x\|_Y \to 0 \quad \text{as } t\uparrow T. \qquad (1.5.2.8)\end{aligned}$$

Thus, $L_{tT}u^0(\,\cdot\,,t;x) \in L_\infty(s,T;Y)$ in t. This can be readily boosted to $C([s,T];Y)$ by direct computation.

(iv) Recalling (1.4.3.15), we compute, via (1.5.2.2), (1.5.2.3), and (1.5.1.1),

$$\begin{aligned}(-A^*)^\theta P(t)x &= \int_t^T (-A^*)^\theta e^{A^*(\tau-t)}R^*Ry^0(\tau,t;x)\,d\tau \\ &\quad + (-A^*)^{\theta-\beta}e^{A^*(T-t)}(-A^*)^\beta G^*Gy^0(T,t;x) \\ &\in C([0,T);Y) \qquad (1.5.2.9)\end{aligned}$$

and

$$\|(-A^*)^\theta P(t)x\|_Y \leq C_{T\gamma}\left\{\left(\int_t^T \frac{d\tau}{(\tau-t)^\theta}\right) + \frac{C_T}{(T-t)^{\theta-\beta}}\right\}\|x\|_Y, \quad 0 \leq t < T, \qquad (1.5.2.10)$$

so that (1.5.2.4) follows.

(v) By (1.5.2.4) with $\theta = \gamma$ and (1.1.2),

$$\|B^*P(t)x\|_U = \|B^*(-A^*)^{-\gamma}(-A^*)^\gamma P(t)x\|_U = \frac{C_{T\gamma}}{1-\gamma}\frac{1}{(T-t)^{\gamma-\beta}}\|x\|_Y, \quad 0 \leq t < T, \qquad (1.5.2.11)$$

while $B^*P(t)x \in C([0,T);U)$, and (1.5.2.5) is proved.

(vi) Conclusion (1.5.2.6) follows from (1.5.2.3b) via (1.4.7.1) and (1.4.7.2). □

We next provide improved regularity properties of the time derivatives $du^0(t,s;x)/dt$ and $[dy^0(t,s;x)/dt - Ae^{A(t-s)}x]$, which become available under assumption (1.5.1.1), over the results of Section 1.4.5. We recall that Theorem 1.4.5.1 gives a controlled degree of singularity of these time derivatives at $t = s$; while Remark 1.4.5.1 provides a controlled degree of singularity also at $t = T$, however, only under the assumption $0 \leq \gamma < 1/2$. We next extend these latter results at $t = T$, also to the case $1/2 \leq \gamma < 1$, under assumption (1.5.1.1), which is an assumption effecting the behavior at $t = T$. The new results reduce to those of Theorem 1.4.5.1

at $t = s$ and to those of Remark 1.4.5.1 at $t = T$, when $0 \le \gamma < 1/2$, that is, when $\beta = 0$.

Theorem 1.5.2.2 *Assume hypothesis (1.5.1.1). Then for $x \in Y$:*

(i)

$$\frac{du^0(t,s;x)}{dt} \in {}_{(\gamma+\epsilon)}C_{1+\gamma-\beta}([s,T];U); \tag{1.5.2.12a}$$

$$\left\| \frac{du^0(t,s;x)}{dt} \right\|_U \le \frac{C_T}{(T-t)^{1+\gamma-\beta}(t-s)^{\gamma+\epsilon}} \|x\|_Y, \quad \epsilon > 0, \ s < t < T; \tag{1.5.2.12b}$$

(ii)

$$L_s\left(\frac{du^0(\sigma,s;x)}{d\sigma}\right) \in C_{2\gamma-\beta}([s+\epsilon_1,T];U), \quad \epsilon, \epsilon_1 > 0; \tag{1.5.2.13}$$

(iii)

$$\left[\frac{dy^0(t,s;x)}{dt} - Ae^{A(t-s)}x\right] \in {}_{\gamma}C_{2\gamma-\beta}([s,T];Y), \tag{1.5.2.14a}$$

$$\left\| \frac{dy^0(t,s;x)}{dt} - Ae^{A(t-s)}x \right\|_Y \le \frac{C_T}{(T-t)^{2\gamma-\beta}(t-s)^{\gamma}} \|x\|_Y, \quad s < t < T. \tag{1.5.2.14b}$$

Proof. The proof follows the pattern of that of Theorem 1.4.5.1 and Remark 1.4.5.2 while taking into account the improved regularity properties of Theorem 1.5.2.1, in particular (1.5.2.2).

(i) Indeed, we return to (1.4.5.11a) and use (1.1.2), (1.5.1.1), and (1.5.2.2) to obtain

$$\begin{aligned} &\frac{d}{dt}\{L_{sT}^* G^* G y^0(T,s;x)\}(t) \\ &\quad = -B^*(-A^*)^{-\gamma}(-A^*)^{1+\gamma-\beta} e^{A^*(T-t)}(-A^*)^{\beta} G^* G y^0(T,s;x) && (1.5.2.15a) \\ &\quad \in C_{1+\gamma-\beta}([s,T];U), && (1.5.2.15b) \end{aligned}$$

a reduction of singularity by β over (1.4.5.11b).

Next, with reference to Corollary 1.4.5.3, Eqn. (1.4.5.17), we obtain, by virtue of (1.5.2.15b), (1.4.5.13), and (1.4.5.14b),

$$\frac{d}{dt}\{\text{R.H.S. of (1.4.5.7)}\} = \rho(t,x) \in {}_{(\gamma+\epsilon)}C_{1+\gamma-\beta}([s,T];U), \tag{1.5.2.16}$$

for all $\epsilon > 0$. We then return to Eqn. (1.4.5.67), use (1.5.2.16) and (1.4.5.65) and obtain

$$v_0(t; x) = -w_0(t, x) + \rho(t, x) \in {}_{(\gamma+\epsilon)}C_{1+\gamma-\beta}([s, T]; U). \tag{1.5.2.17}$$

With the reduced singularity at $t = T$ by β, as given by (1.5.2.17) over (1.4.5.68), we return to (1.4.5.66), rewritten here for convenience,

$$\frac{du^0(t, s; x)}{dt} + L_s^* R^* R L_s \frac{du^0(\sigma, s; x)}{d\sigma} = v_0(t; x) \in {}_{(\gamma+\epsilon)}C_{1+\gamma-\beta}([s, T]; U), \tag{1.5.2.18}$$

and apply the same bootstrap argument described below (1.4.5.68) in the case $0 \le \gamma < 1/2$ of Remark 1.4.5.2. Namely we apply $(L_s^* R^* R L_s)$ on Eqn. (1.5.2.18) repeatedly, until the singularities at $t = s$ and $t = T$ are eliminated, by virtue of Propositions 1.4.5.4 and 1.4.4.1, respectively [Eqns. (1.4.5.18), (1.4.5.19), (1.4.5.20) and (1.4.4.3), (1.4.4.4), (1.4.4.11), respectively]:

$$v_0 \in {}_{(\gamma+\epsilon)}C_{1+\gamma-\beta}([s, T]; U) \xrightarrow{R^*RL_s} R^* R L_s v_0 \in {}_{(2\gamma-1+\epsilon)}C_{2\gamma-\beta}([s, T]; U)$$

[by (1.4.5.18) at $t = s$, and (1.4.4.3) at $t = T$]

$$\begin{bmatrix} \text{by (1.4.4.4)} & \text{at } t = T, \text{ provided } 2\gamma - \beta < 1; \\ \text{by (1.4.5.19)} & \text{at } t = s \end{bmatrix} \Big\downarrow L_s^*$$

$$L_s^* R^* R L_s v_0 \in {}_{3\gamma-2+\epsilon}C_{3\gamma-1-\beta}([s, T]; U).$$

But with v_0 as in (1.5.2.17), we find that the order of singularity of $L_s v_0$ at $t = T$ is $(1 + \gamma - \beta) + (\gamma - 1) = 2\gamma - \beta < 1$, by Proposition 1.4.4.1(ii), Eqn. (1.4.4.3). Since $2\gamma - \beta < 1$, we can further apply L_s^* to $R^* R L_s v_0$, in accordance with the requirement of Proposition 1.4.4.1(iii), Eqn. (1.4.4.4) at $t = T$, and Proposition 1.4.5.4(ii), Eq. (4.5.19) at $t = s$. Once both singularities at $t = s$ and $t = T$ are eliminated through the repeated application of $(L_s^* R^* R L_s)$ to (1.5.2.18), one uses the bounded invertibility of $[I + L_s^* R^* R L_s]$ on $L_2(s, T; U)$, followed by the backward recursive procedure of Theorem 1.4.4.4 (Step 3) [or Theorem 1.4.5.10, below (1.4.5.59)] to obtain

$$[I_s + L_s^* R^* R L_s]^{-1} \in \mathcal{L}({}_{\gamma+\epsilon}C_{1+\gamma-\beta}([s, T]; Y)). \tag{1.5.2.19}$$

Then, (1.5.2.19) used in (1.5.2.18) yields (1.5.2.12a), as desired in (i).

(ii) By applying Proposition 1.4.4.1(ii), Eqn. (1.4.4.3) to (1.5.2.12a) [since $(1 + \gamma - \beta) + \gamma > 1$], we find that the degree of singularity of $L_s\, du^0/dt(t, s; x)$ at $t = T$ is $[(1 + \gamma - \beta) + (\gamma - 1)] = 2\gamma - \beta$, and (1.5.2.13) is proved.

(iii) We return to identity (1.4.5.4) and use here (1.5.2.13) and (1.1.2) and obtain (1.5.2.14). Theorem 1.5.5.2 is proved. □

1.5.3 The Differential Riccati Equation Is Classical. Uniqueness

In this subsection, under hypothesis (1.5.1.1), we prove two results: First, that for $0 \le t < T$, the differential Riccati equation (1.2.1.12) is classical and holds true on the whole space Y, in the sense that the operator $\Gamma(P(t)) \equiv A^*P(t) + P(t)A$ may be given a meaning and has a bounded extension as an operator, on $\mathcal{L}(Y)$. To this end, we use the continuity up to T of $y^0(t, s; x)$ as in (1.5.2.2) of Theorem 1.5.2.1. Next, we show uniqueness of the Riccati operator $P(t)$, within a suitable class.

1.5.3.1 The Differential Riccati Equation Is Classical: Theorem 1.2.2.1(vii)

The following result is a restatement of Theorem 1.2.2.1(vii).

Theorem 1.5.3.1 *Assume (1.5.1.1). Then, with reference to the operator $P(t)$ defined by (1.4.3.15), we have:*

(i) the bilinear form

$$\phi_{P(t)}(x, y) = (P(t)x, Ay)_Y + (Ax, P(t)y)_Y, \quad x, y \in \mathcal{D}(A), \tag{1.5.3.1}$$

$0 \le t < T$, originally well defined on $x, y \in \mathcal{D}(A)$, satisfies the bound

$$|\phi_{P(t)}(x, y)| \le \frac{C_T}{(T-t)^\alpha}\|x\|_Y\|y\|_Y, \quad 0 \le t < T, \ x, y \in \mathcal{D}(A), \tag{1.5.3.2}$$

and hence admits a unique extension, denoted by the same symbol $\phi_{P(t)}$, satisfying the same estimate (1.5.3.2), for all $x, y \in Y$. Moreover, for $x \in \mathcal{D}(A)$, we have $P(t)x \in \mathcal{D}(A^*)$, and thus the operator

$$\Gamma(P(t)) \equiv A^*P(t) + P(t)A, \tag{1.5.3.3}$$

originally well defined on $\mathcal{D}(A)$, as an element of $\mathcal{L}(\mathcal{D}(A); Y)$, admits, a unique continuous extension in $\mathcal{L}(Y)$, still denoted by $\Gamma(P(t))$. In conclusion, we have

$$\|\Gamma(P(t))\|_{\mathcal{L}(Y)} = \|A^*P(t) + P(t)A\|_{\mathcal{L}(Y)} \le \frac{C_T}{(T-t)^\alpha}, \quad 0 \le t < T; \tag{1.5.3.4}$$

$$\alpha = \max\{1-\beta, 3\gamma - 1 - \beta\} = \begin{cases} \gamma - \epsilon, & \frac{2}{3} \le \gamma < 1, \\ 2(1-\gamma) - \epsilon, & \frac{1}{2} \le \gamma < \frac{2}{3}, \\ 1, & 0 \le \gamma < \frac{1}{2}, \end{cases} \tag{1.5.3.5}$$

where $\beta = 2\gamma - 1 + \epsilon$, for $1/2 \le \gamma < 1$; $\beta = 0$ for $0 \le \gamma < 1/2$.

(ii)

$$\left\|\frac{dP(t)}{dt}\right\|_{\mathcal{L}(Y)} \le \frac{C_T}{(T-t)^\alpha}, \quad 0 \le t < T. \tag{1.5.3.6}$$

Proof. (i) We return to the defining formula (1.4.3.15), for $P(t)$ and compute with $x \in Y, 0 \leq t < T$:

$$A^*P(t)x = \int_t^T A^*e^{A^*(\tau-t)}R^*Ry^0(\tau,t;x)\,d\tau \\ + A^*e^{A^*(T-t)}G^*Gy^0(T,t;x). \tag{1.5.3.7}$$

As to the second term in (1.5.3.4), we have by (1.5.1.1), analyticity, and $Gy^0(T,t;x) \in Z_f$ by (1.4.2.3) of Proposition 1.4.2.1 [the stronger result $y^0(T,t;x) \in Y$ as in (1.5.2.2) of Theorem 1.5.2.1 is not critical here]:

$$\begin{aligned} &\left\| A^*e^{A^*(T-t)}G^*Gy^0(T,t;x)\right\|_Y \\ &= \|(-A^*)^{1-\beta}e^{A^*(T-t)}(-A^*)^{\beta}G^*Gy^0(T;t;x)\| \\ &\leq \frac{C_T}{(T-t)^{1-\beta}}\|x\|_Y, \quad 0 \leq t < T; \end{aligned} \tag{1.5.3.8a}$$

$$A^*e^{A^*(T-t)}G^*Gy^0(T,t;x) \in C_{1-\beta}([0,T];Y). \tag{1.5.3.8b}$$

The first term in (1.5.3.7) can be integrated by parts:

$$\begin{aligned} \int_t^T A^*e^{A^*(\tau-t)}R^*Ry^0(\tau.t;x)\,d\tau &= e^{A^*(T-t)}R^*Ry^0(T,t;x) - R^*Rx \\ &- \int_t^T e^{A^*(\tau-t)}R^*R\left[\frac{dy^0(\tau,t;x)}{d\tau} - Ae^{A(\tau-t)}x\right]d\tau \\ &- \int_t^T e^{A^*(\tau-t)}R^*RAe^{A(\tau-t)}x\,d\tau. \end{aligned} \tag{1.5.3.9}$$

Using (this time critically) (1.5.2.2) for $y^0(T,t;x)$, we estimate

$$\left\|e^{A^*(T-t)}R^*Ry^0(T,t;x) - R^*Rx\right\|_Y \leq C_T\|x\|_Y. \tag{1.5.3.10}$$

Next, recalling estimate (1.5.2.14), we have

$$\begin{aligned} &\left\|\int_t^T e^{A^*(\tau-t)}R^*R\left[\frac{dy^0(\tau,t;x)}{d\tau} - Ae^{A(\tau-t)}x\right]d\tau\right\|_Y \\ &\leq C_T\|x\|_Y\int_t^T \frac{d\tau}{(T-\tau)^{2\gamma-\beta}(\tau-t)^{\gamma}} \qquad (1.5.3.11)\\ &\leq \frac{C_TC_{\gamma,\beta,\epsilon}\|x\|_Y}{(T-t)^{3\gamma-\beta-1}}, \quad 0 \leq t < T, \qquad (1.5.3.12) \end{aligned}$$

where in the last step from (1.5.3.11) to (1.5.3.12) we have invoked the integral in (1.4.4.9) with $r = 2\gamma - \beta < 1$, as required by Proposition 1.4.4.1(iii), and the subsequent estimates leading to (1.4.4.10) with power $r + \gamma - 1 = 3\gamma - \beta - 1$ for $(T-t)$ in the denominator (and yielding $C_{\gamma,\beta,\epsilon}$ explicitly). Combining

(1.5.3.8)–(1.5.3.12) in (1.5.3.7), we obtain

$$\left\| A^*P(t)x + \int_t^T e^{A^*(\tau-t)}R^*RAe^{A(\tau-t)}x\,d\tau \right\|_Y \leq \frac{C_T\|x\|_Y}{(T-t)^\alpha}, \quad 0 \leq t < T, \tag{1.5.3.13}$$

with $\alpha = \max\{1-\beta, 3\gamma-\beta-1\}$, as in (1.5.3.5).

Let now, at first, $x, y \in \mathcal{D}(A)$. Then, from (1.5.3.13) we obtain after adding and subtracting same quantities:

$$\begin{aligned}
&|(A^*P(t)x, y)_Y + (x, A^*P(t)y)_Y| \\
&\leq \frac{C_T\|x\|_Y\|y\|_Y}{(T-t)^\alpha} + \left| \left(\int_t^T e^{A^*(\tau-t)}R^*Re^{A(\tau-t)}Ax\,d\tau, y \right)_Y \right. \\
&\quad \left. + \left(x, \int_t^T e^{A^*(\tau-t)}R^*Re^{A(\tau-t)}Ay\,d\tau \right)_Y \right| \\
&= \frac{C_T\|x\|_Y\|y\|_Y}{(T-t)^\alpha} + \left| \left(\int_t^T \frac{d}{d\tau}\left[e^{A^*(\tau-t)}R^*Re^{A(\tau-t)}x\right]d\tau, y \right)_Y \right| \qquad (1.5.3.14) \\
&= \frac{C_T\|x\|_Y\|y\|_Y}{(T-t)^\alpha} + \left| \left(e^{A^*(T-t)}R^*Re^{A(T-t)}x, y\right)_Y - (R^*Rx, y)_Y \right| \\
&\leq C_T\|x\|_Y\|y\|_Y \left\{ \frac{1}{(T-t)^\alpha} + 1 \right\}, \quad x, y \in \mathcal{D}(A),\ 0 \leq t < T, \qquad (1.5.3.15)
\end{aligned}$$

and (1.5.3.2) is proved. Thus, (1.5.3.15) says that the bilinear form in (1.5.3.1),

$$\phi_{P(t)}(x, y) = (P(t)x, Ay)_Y + (Ax, P(t)y)_Y, \tag{1.5.3.16}$$

$0 \leq t < T$, originally well defined on $x, y \in \mathcal{D}(A)$ admits a continuous extension (denoted by same symbol) to all $x, y \in Y$. Thus, for $x \in \mathcal{D}(A)$ fixed, the map

$$y \in \mathcal{D}(A) \to (P(t)x, Ay)_Y = (\phi_{P(t)}x, y)_Y - (Ax, P(t)y)_Y \tag{1.5.3.17}$$

is continuous, and thus

$$\begin{gathered}
P(t)x \in \mathcal{D}(A^*) \text{ and the operator:} \\
\Gamma(P(t))) \equiv [A^*P(t) + P(t)A] : \text{continuous } \mathcal{D}(A) \to Y \qquad (1.5.3.18)
\end{gathered}$$

in (1.5.3.3) admits a unique continuous extension $\Gamma(P(t)) \in \mathcal{L}(Y)$, which we shall continue to denote by $\Gamma(P(t)) = [A^*P(t) + P(t)A]$. Then, (1.5.3.15) yields the desired estimate (1.5.3.4).

(ii) We now recall from (1.5.2.5) that $B^*P(t)$ has a singularity of degree $(\gamma - \beta)$ at $t = T$, and we verify that $(\gamma - \beta) < \alpha$ for all $0 \leq \gamma < 1$, via (1.5.3.5). Thus, returning to the differential Riccati equation (1.2.1.12) = (1.4.6.22), and using the bounded extension (1.5.3.4) and (1.5.2.5), we then obtain (1.5.3.6) for $\dot{P}(t)$. The proof of Theorem 1.5.3.1 is complete. □

Remark 1.5.3.1 If, in addition, A is self-adjoint ($A = A^*$) and say $R = cI$, then

$$\int_t^T e^{A^*(\tau-t)} R^* R A e^{A(\tau-t)} x \, d\tau$$
$$= \frac{1}{2} \int_t^T \frac{d}{d\tau} \left[e^{A(\tau-t)} R^* R e^{A(\tau-t)} x \right] d\tau$$
$$= \frac{1}{2} e^{A(T-t)} R^* R e^{A(T-t)} x - \frac{1}{2} R^* R x = \mathcal{O}_T(\|x\|). \tag{1.5.3.19}$$

Using (1.5.3.19) in (1.5.3.13) then yields

$$\|A^* P(t)\|_{\mathcal{L}(Y)} \le \frac{C_T}{(T-t)^\alpha}, \quad 0 \le t < T, \tag{1.5.3.20}$$

and hence $P(t)A$ has a bounded extension

$$\|P(t)A\|_{\mathcal{L}(Y)} \le \frac{C_T}{(T-t)^\alpha}, \quad 0 \le t < T. \tag{1.5.3.21}$$

The same conclusion (1.5.3.21) is obtained if, more generally, $A = A^*$, $R = R^*$, and R commutes with A in the sense of [Kato, 1966, p. 171] $RA \subset AR$, meaning that whenever $x \in \mathcal{D}(A)$, then $Rx \in \mathcal{D}(A)$ as well and $ARx = RAx$.

Corollary 1.5.3.2 *Assume (1.5.1.1). Then, with reference to the operator $P(t)$ defined by (1.4.3.15), we have that it satisfies the DRE (1.2.1.12) in a classical sense:*

$$\begin{cases} \dot{P}(t) = -R^*R - \Gamma(P(t)) + (P(t)B^*)^* B^* P(t), \quad 0 \le t < T, & (1.5.3.22) \\ P(T) = G^*G, & (1.5.3.23) \end{cases}$$

with $\Gamma(P(t))$ the bounded extension of $[A^*P(t) + P(t)A]$ in (1.5.3.3) in $\mathcal{L}(Y)$.

Proof. First $P(t)$ satisfies the DRE in the form of (1.2.1.12) = (1.4.6.22). However, due to Theorem 1.5.3.1, as well as to Eqn. (1.5.2.5) of Theorem (1.5.2.1), $P(t)$ then solves (1.5.3.22). Finally, (1.5.3.23) was already observed in (1.5.2.6), with $P(t)x \in C([0,T]; Y)$ by (1.5.2.4). □

1.5.3.2 Uniqueness of the Riccati Equation. IRE and DRE

Theorem 1.5.3.3 (Uniqueness of Riccati operator) *Assume (1.5.1.1). Then, the operator $P(t)$ defined constructively in Eqn. (1.4.3.15) in terms of the data of the problem is the unique solution to the differential Riccati equation (1.2.1.12) = (1.4.6.22), indeed of its classical version (1.5.3.22), and its terminal condition (1.5.3.23), of Corollary 1.5.3.2, within the class of self-adjoint operator $\bar{P}(t) \in \mathcal{L}(Y)$ that satisfy the regularity property*

$$B^* \bar{P}(t) x \in C_{\gamma-\beta}([0,T]; Y), \quad x \in Y \quad \left(\beta = 0, \text{if } 0 \le \gamma < \frac{1}{2}\right) \tag{1.5.3.24}$$

[a property that is fulfilled by $P(t)$ in (1.4.3.15) by Theorem 1.5.2.1(v), Eqn. (1.5.2.5)].

Proof.

Step 1 It suffices to show uniqueness within the specified class for the corresponding integral Riccati equation (IRE):

$$(P(t)x, y)_Y = \left(Ge^{A(T-t)}x, Ge^{A(T-t)}y\right)_{Z_f} + \int_t^T \left(Re^{A(\tau-t)}x, Re^{A(\tau-t)}y\right)_Z d\tau$$
$$- \int_t^T \left(B^* P(\tau)e^{A(\tau-t)}x, B^* P(\tau)e^{A(\tau-t)}y\right)_U d\tau, \qquad (1.5.3.25)$$

$x, y \in Y$. This will be justified in Step 2 at the end of the proof, below.

Proposition 1.5.3.3 *Under (1.5.1.1), there is a unique, self-adjoint solution of the IRE (1.5.3.25) within the class satisfying property (1.5.3.24).*

Proof of Proposition 1.5.3.3. Let $P_1(t)$ and $P_2(t)$ be two solutions of (1.5.3.25) within the specified class, and let $Q(t) = P_1(t) - P_2(t)$. Then, for $x \in Y$, $Q(t)$ satisfies

$$B^* Q(t)x \in C_{\gamma-\beta}([0, T]; U) \quad \left(\beta = 0 \text{ for } 0 \le \gamma < \frac{1}{2}\right), \qquad (1.5.3.26)$$

as well as, for $0 < t < T$ and $x, y \in Y$,

$$(Q(t)x, y)_Y = \int_t^T \left(B^* P_2(\tau)e^{A(\tau-t)}x, B^* Q(\tau)e^{A(\tau-t)}y\right)_U d\tau$$
$$- \int_t^T (B^* Q(\tau)e^{A(\tau-t)}x, B^* P_1(\tau)e^{A(\tau-t)}y)_U \, d\tau, \qquad (1.5.3.27)$$

as one readily sees by subtracting (1.5.3.25) for P_2 from (1.5.3.25) for P_1, using self-adjointness. Set

$$y \equiv Bv, \quad v \in U, \quad \text{and} \quad B^* Q(t) \equiv V(t), \qquad (1.5.3.28)$$

so that $V(t)$, for $0 \le t < T$, is a solution to

$$V(t)x = \int_t^T B^*(-A^*)^{-\gamma}(-A^*)^{\gamma} e^{A^*(\tau-t)} V^*(\tau) B^* P_2(\tau) e^{A(\tau-t)}x \, d\tau$$
$$- \int_t^T B^*(-A^*)^{-\gamma}(-A^*)^{\gamma} e^{A^*(\tau-t)} (B^* P_1(\tau))^* V(\tau) e^{A(\tau-t)}x \, d\tau. \qquad (1.5.3.29)$$

We seek to establish uniqueness of the solution $V(t)$ of (1.5.3.29) within the class $V(t)x \in C_{\gamma-\beta}([0, T]; U)$ as specified by (1.5.3.26), via (1.5.3.28). We do this first locally, near T, say on $t_0 \le t \le T$ for some $0 < t_0 < T$, and then extend globally to all of $P(t)$. After uniqueness of $B^* P(t)$ has been established, then uniqueness of $P(t)$ will follow from (1.5.3.25).

We multiply (1.5.3.29) across by $(T-t)^{\gamma-\beta}$, use (1.1.2), analyticity, and

$$\|B^* P_i(t)y\|_U \leq \frac{C_T}{(T-t)^{\gamma-\beta}}\|y\|_Y, \quad 0 < t < T,$$

as dictated by (1.5.3.24), and obtain

$$\begin{aligned}(T-t)^{\gamma-\beta}\|V(t)x\|_U &\leq (T-t)^{\gamma-\beta}C_T\int_t^T \frac{(T-\tau)^{\gamma-\beta}\|V^*(\tau)\|\,d\tau}{(\tau-t)^{\gamma}(T-\tau)^{2(\gamma-\beta)}}\|x\|_Y \\ &\leq (T-t)^{\gamma-\beta}C_T\left\{\int_t^T \frac{d\tau}{(\tau-t)^{\gamma}(T-\tau)^{2(\gamma-\beta)}}\right\} \\ &\quad\times\left\{\sup_{t\leq\tau\leq T}(T-\tau)^{\gamma-\beta}\|V(\tau)\|\right\}\|x\|_Y, \end{aligned} \tag{1.5.3.30}$$

where $V(\tau)$ is given in the $\mathcal{L}(Y;U)$-norm. Since $r = 2(\gamma-\beta)$ for all $0 \leq \gamma < 1$ by (1.5.1.1), the computations of case (iii) in the proof of Proposition 1.4.4.1, Eqn. (1.4.4.8) leading to Eqn. (1.4.4.10) can be applied to the integral in (1.5.3.30). We thus obtain, with $r+\gamma-1 = 2(\gamma-\beta)+\gamma-1$,

$$\begin{aligned}&(T-t)^{\gamma-\beta}\|V(t)x\|_U \\ &\quad\leq C_T(T-t)^{\gamma-\beta}\frac{1}{(T-t)^{2(\gamma-\beta)+\gamma-1}}\left\{\sup_{t\leq\tau\leq T}(T-\tau)^{\gamma-\beta}\|V(\tau)\|\right\}\|x\|_Y \\ &\quad\leq C_T(T-t)^{\beta-2\gamma+1}\left\{\sup_{t\leq\tau\leq T}(T-\tau)^{\gamma-\beta}\|V(\tau)\|\right\}\|x\|_Y, \end{aligned} \tag{1.5.3.31}$$

where $\beta-2\gamma+1 > 0$ by (1.5.1.1). Letting $t_0 \leq t \leq T$, we obtain from (1.5.3.31)

$$\sup_{t_0\leq t\leq T}(T-t)^{\gamma-\beta}\|V(t)\| \leq C_T(T-t_0)^{\beta-2\gamma+1}\left\{\sup_{t\leq\tau\leq T}(T-\tau)^{\gamma-\beta}\|V(\tau)\|\right\}, \tag{1.5.3.32}$$

and selecting $(T-t_0)$ sufficiently small, we obtain $C_T(T-t_0)^{\gamma-\beta} < 1$, and hence uniqueness of $V(t)$ on $[t_0, T]$ is established within the class $V(t)x \in C_{\gamma-\beta}([t_0, T]; U)$, by (1.5.3.32).

Finally, after a finite number of steps, we obtain uniqueness of $V(t)$ on all $[0, T]$, within the class $V(t)x \in C_{\gamma-\beta}([0, T]; U)$, as desired, and $B^*P_1(t)x \equiv B^*P_2(t)x$. From here, via (1.5.3.25), we obtain uniqueness: $P_1(t) \equiv P_2(t)$ within the self-adjoint class defined by (1.5.3.24). Proposition 1.5.3.2 is proved. □

Step 2. Proposition 1.5.3.4 *Let $\bar{P}(t)$ be a self-adjoint operator in $\mathcal{L}(Y; C([0, T]; Y)$, with $\bar{P}(T) = G^*G$, satisfying the regularity property (1.5.3.24) for a constant $\beta > 2\gamma-1$ if $1/2 \leq \gamma < 1$, and $\beta = 0$ if $0 \leq \gamma < 1/2$. Let such $\bar{P}(t)$ be a solution of the DRE (1.2.1.12) = (1.4.6.22), for $0 \leq t < T$, and, say, for all $x, y \in \mathcal{D}(A)$. [All this is the case a fortiori for the operator $P(t)$ defined by (1.4.3.15), by virtue of Theorem 1.5.2.1(v) (regularity) and Theorem 1.4.6.4 (existence), let alone Corollary 1.5.3.2* *(classical existence), under assumption (1.5.1.1).] Then, such $\bar{P}(t)$ also satisfies the IRE (1.5.3.25).*

Proof. For t fixed (whose dependence is then omitted), we define the operator

$$M(\tau) \equiv e^{A^*(\tau-t)}\bar{P}(\tau)e^{A(\tau-t)} - e^{A^*(\tau-t)}G^*Ge^{A(\tau-t)}$$
$$: \text{continuous } Y \to C([t,T];Y). \tag{1.5.3.33}$$

For $x, y \in Y$ we compute, with $t < \tau < T$,

$$\begin{aligned}
&\frac{d}{d\tau}(M(\tau)x, y)_Y \\
&= \frac{d}{d\tau}\big(\bar{P}(\tau)e^{A(\tau-t)}x, e^{A(\tau-t)}y\big)_Y \\
&= \big(\dot{\bar{P}}(\tau)e^{A(\tau-t)}x, e^{A(\tau-t)}y\big)_Y + \big(\bar{P}(\tau)Ae^{A(\tau-t)}x, e^{A(\tau-t)}y\big)_Y \\
&\quad + \big(\bar{P}(\tau)e^{A(\tau-t)}x, Ae^{A(\tau-t)}y\big)_Y - \frac{d}{d\tau}\big(Ge^{A(\tau-t)}x, Ge^{A(\tau-t)}y\big)_{Z_f},
\end{aligned} \tag{1.5.3.34}$$

where $e^{A(\tau-t)}x, e^{A(\tau-t)}y \in \mathcal{D}(A)$ by analyticity. Moreover, since $\bar{P}(\tau)$ satisfies the DRE (1.2.1.12) = (1.4.6.22), on these vectors, by assumption, we may rewrite the first term on the right-hand side of (1.5.3.34) as

$$\begin{aligned}
&\left(\frac{d}{dt}\bar{P}(\tau)e^{A(\tau-t)}x, e^{A(\tau-t)}y\right)_Y = -\big(Re^{A(\tau-t)}x, Re^{A(\tau-t)}y\big)_Z \\
&\quad - \big(\bar{P}(\tau)e^{A(\tau-t)}x, Ae^{A(\tau-t)}y\big)_Y - \big(\bar{P}(\tau)Ae^{A(\tau-t)}x, e^{A(\tau-t)}y\big)_Y \\
&\quad + (B^*\bar{P}(\tau)e^{A(\tau-t)}x, B^*\bar{P}(\tau)e^{A(\tau-t)}y)_U, \quad x, y \in Y, \quad t < \tau < T.
\end{aligned} \tag{1.5.3.35}$$

Each term of (1.5.3.35) is well defined by assumption (1.5.3.24) on $\bar{P}(t)$. Inserting the right-hand side of (1.5.3.35) into the first term on the right-hand side of (1.5.3.34) results in a cancellation of the last two terms of (1.5.3.34), so that we obtain for $x, y \in Y, t < \tau < T$:

$$\begin{aligned}
\frac{d}{d\tau}(M(\tau)x, y)_Y = &-\big(Re^{A(\tau-t)}x, Re^{A(\tau-t)}y\big)_Z \\
&+ \big(B^*\bar{P}(\tau)e^{A(\tau-t)}x, B^*\bar{P}(\tau)e^{A(\tau-t)}y\big)_U \\
&- \frac{d}{d\tau}\big(Ge^{A(\tau-t)}x, Ge^{A(\tau-t)}y\big)_{Z_f}.
\end{aligned} \tag{1.5.3.36}$$

In view of hypothesis (1.5.3.24), we have, for any $x \in Y$,

$$B^*P(\tau)e^{A(\tau-t)}x \in C_{\gamma-\beta}([t,T];U) \subset L_2(t,T;U), \tag{1.5.3.37}$$

since $\gamma - \beta < 1/2$ for any $0 \le \gamma < 1$, so that the quadratic term in (1.5.3.36) is integrable over $[t,T]$. Integrating (1.5.3.36) in τ over $[t,T]$, and recalling

$\bar{P}(T) = G^*G$ by assumption, we obtain by (1.5.3.33):

$$\begin{aligned}
&\left(Ge^{A(T-t)}x, Ge^{A(T-t)}y\right)_{Z_f} - \left(Ge^{A(T-t)}x, Ge^{A(T-t)}y\right)_{Z_f} - (\bar{P}(t)x, y) + (Gx, Gy)_{Z_f} \\
&\quad = -\int_t^T \left(Re^{A(\tau-t)}x, Re^{A(\tau-t)}y\right)_Z d\tau \\
&\qquad + \int_t^T \left(B^*\bar{P}(\tau)e^{A(\tau-t)}x, B^*\bar{P}(\tau)e^{A(\tau-t)}y\right) d\tau \\
&\qquad - \left(Ge^{A(T-t)}x, Ge^{A(T-t)}y\right)_{Z_f} + (Gx, Gy)_{Z_f}. \qquad (1.5.3.38)
\end{aligned}$$

After cancellations (1.5.3.38) becomes the IRE (1.5.3.25) for $\bar{P}(t)$, as desired. Proposition 1.5.3.4 is proved. □

Combining Propositions 1.5.3.3 and 1.5.3.4, we obtain Theorem 1.5.3.2. □

For completeness, we observe that Proposition 1.5.3.4 admits a converse.

Proposition 1.5.3.5 *Let $P(t) \in \mathcal{L}(Y)$ be [as in the case of the operator $P(t)$ defined by (1.4.3.15) under assumption (1.5.1.1)] a solution of the IRE (1.5.3.25) that satisfies property (1.5.3.24), so that $P(T) = G^*G$. Then, $P(t)$ satisfies the DRE for all $x, y \in \mathcal{D}(A)$.*

Proof. With $x, y \in \mathcal{D}(A)$ we differentiate both sides of (1.5.3.25) for $0 < t < T$, thereby obtaining

$$\begin{aligned}
\frac{d}{dt}(P(t)x, y)_Y &= -\left(Ge^{A(T-t)}Ax, Ge^{A(T-t)}y\right)_{Z_f} \\
&\quad - \left(Ge^{A(T-t)}x, Ge^{A(T-t)}Ay\right)_{Z_f} - (Rx, Ry)_Z + (B^*P(t)x, B^*P(t)y)_U \\
&\quad - \int_t^T \left(Re^{A(\tau-t)}Ax, Re^{A(\tau-t)}y\right)_Z d\tau \\
&\quad + \int_t^T \left(B^*P(\tau)e^{A(\tau-t)}Ax, B^*P(\tau)e^{A(\tau-t)}y\right)_U d\tau \\
&\quad - \int_t^T \left(Re^{A(\tau-t)}x, Re^{A(\tau-t)}Ay\right)_Z d\tau \\
&\quad + \int_t^T \left(B^*P(\tau)e^{A(\tau-t)}x, B^*P(\tau)e^{A(\tau-t)}Ay\right)_U d\tau, \qquad (1.5.3.39)
\end{aligned}$$

where all terms are well defined by the assumed property (1.5.3.24) on $P(t)$. Using again (1.5.3.25) once with x replaced by Ax for the first, fifth, and sixth terms on the right-hand side of (1.5.3.39), and once with y replaced by Ay for the second, seventh, and eighth terms, we obtain from (1.5.3.39)

$$\begin{aligned}
\frac{d}{dt}(P(t)x, y)_Y &= -(Rx, Ry)_Z + (B^*P(t)x, B^*P(t)y)_U - (P(t)Ax, y)_Y \\
&\quad - (P(t)x, Ay)_Y, \quad 0 \le t < T, \quad x, y \in \mathcal{D}(A), \qquad (1.5.3.40)
\end{aligned}$$

as desired. □

1.6 A Second Smoothing Case of the Operator G: The Case $(-A^*)^\gamma G^*G \in \mathcal{L}(Y)$. Proof of Theorem 1.2.2.2

In this Section 1.6 we assume that

$$(-A^*)^\gamma G^*G \in \mathcal{L}(Y), \tag{1.6.1}$$

which is a stronger condition than assumption (1.5.1.1), since $2\gamma - 1 < \gamma$. A fortiori, Theorems 1.2.2.1 and 1.2.1.1 apply. However, under hypothesis (1.6.1), additional regularity results hold true; in particular, the optimal control u^0 becomes continuous at the terminal time $t = T$. This is stated in Theorem 1.2.2.2.

Proof of Theorem 1.2.2.2 We begin with

Lemma 1.6.1 *Assume hypothesis (1.6.1). Then, for any $x \in Y$, the term in the square brackets of (1.5.1.9) satifies*

$$L_{sT}^* G^* G e^{A(T-s)} x + L_s^* R^* R e^{A(\cdot - s)} x \in C([s, T]; U). \tag{1.6.2}$$

Proof. The proof is the same as the proof of Lemma 1.5.1.4 except that (1.5.1.13) is now replaced, via (1.6.1), by

$$\left(L_{sT}^* G^* G e^{A(T-s)} x\right)(t) = B^*(-A^*)^{-\gamma} e^{A^*(T-t)} (-A^*)^\gamma G^* G e^{A(T-s)} x \in C([s, T]; U). \tag{1.6.3}$$

Lemma 1.6.2 *Assume hypothesis (1.6.1). Then:*

(i)

$$L_{sT}^* G^* G (-A)^\rho L_{sT} : \textit{continuous } L_\infty(s, T; U) \to C([s, T]; Y), \tag{1.6.4}$$

for all $\rho > 0$ such that $(\rho + \gamma) < 1$;

(ii)

$$L_{sT}^* G^* G L_{sT} : \textit{continuous } L_2(s, T; U) \to C_{\frac{1}{2}-\epsilon_0}([s, T]; U), \tag{1.6.5}$$

with $\epsilon_0 > 0$ satisfying $\gamma > 2\gamma - 1 + 2\epsilon_0$;

(iii) there is a positive integer n_1, in fact, $n_1 \geq 1/2\epsilon_0$, such that

$$(L_{sT}^* G^* G L_{sT})^{n_1} : L_2(s, T; U) \to C([s, T]; U). \tag{1.6.6}$$

In all three cases, (1.6.4), (1.6.5), and (1.6.6), the norm bound may be taken independent of s.

Proof. (i) If $f \in L_\infty(s, T; U)$, then, in fact,

$$\begin{aligned} &\{L_{sT}^* G^* G (-A)^\rho L_{sT} f\}(t) \\ &= B^*(-A^*)^{-\gamma} e^{A^*(T-t)} (-A^*)^\gamma G^* G \int_s^T (-A)^{\rho+\gamma} e^{A(T-\tau)} (-A)^{-\gamma} B f(\tau)\, d\tau \end{aligned} \tag{1.6.7a}$$

$$\in C([s, T]; U), \quad \text{if } \rho + \gamma < 1, \tag{1.6.7b}$$

by (1.1.2), (1.6.1), and analyticity [Pazy, 1983, p. 74], and (1.6.4) is proved.

(ii) This part is essentially contained in the proof of Theorem 1.5.1.6(i). Claim (1.5.1.23)–(1.5.1.24) in Step 1 remains true, of course. Then, the validity of inequality (1.5.1.27) holds true now, with ϵ_0 satisfying $\gamma > 2\sigma_0 = 2\gamma - 1 + 2\epsilon_0$ by (1.5.1.24) (which is possible since $\gamma > 2\gamma - 1$).

In fact, returning to (1.5.1.28) we must require that $\gamma > 2\sigma_0$ in order to invoke Corollary 1.5.1.2, Eqn. (1.5.1.6), and assert that $(-A^*)^{\sigma_0} G^* G (-A)^{\sigma_0} \in \mathcal{L}(Y)$, in addition to the requirement $\sigma_0 > \gamma - 1/2$ of Step 1.

(iii) Reiterating as in the proof of Theorem 1.5.1.6(iii), we find that

$$(L_{sT}^* G^* G L_{sT})^n : \text{continuous } L_2(s,T;U) \to C_{\frac{1}{2}-n\epsilon_0}([s,T];U), \tag{1.6.8}$$

as in (1.5.1.21). Thus, if $n_1\epsilon_0 \geq 1/2$, then (1.6.6) follows from (1.6.8).

In all cases, the norm bound can be taken independent of s. □

Returning to (1.5.1.9) and (1.5.1.10), we state the crucial result of the present section.

Theorem 1.6.3 *Assume hypothesis (1.6.1). Then the operator Λ_{sT} defined by (1.5.1.10) satisfies*

$$\Lambda_{sT}^{-1} = [I_s + L_s^* R^* R L_s + L_{sT}^* G^* G L_{sT}]^{-1} \in \mathcal{L}(C([s,T];U)), \tag{1.6.9}$$

with uniform bound that may be taken independent of s, $s \leq t \leq T$.

Proof. It is the same as the proof of Theorem 1.5.1.7, except that now it uses Lemma 1.6.2, instead of Lemma 1.5.1.5 and Theorem 1.5.1.6. Given $w \in C([s,T];U) \subset L_2(s,T;U)$, there exists a unique $g \in L_2(s,T;U)$, such that

$$g + L_{sT}^* G^* G L_{sT} g = w - L_s^* R^* R L_s g \tag{1.6.10}$$

[counterpart of (1.5.1.41)]. Next, exactly as in Step 1, Eqn. (1.5.1.42) of Theorem 1.5.1.7, we obtain

$$L_s^* R^* R L_s g \in C([s,T];U) \tag{1.6.11}$$

by bootstrapping (1.6.10) in $L_s^* R^* R L_s$ via (1.6.5) of Lemma 1.6.2 [same as (1.5.1.43)]. Thus, $[w - L_s^* R^* R L_s g] \equiv f \in C([s,T];U)]$ on the right-hand side of (1.6.10) [counterpart of (1.5.1.52)], by (1.6.11). As a final step, we bootstrap (1.6.10) this time in $L_{sT}^* G^* G L_{sT}$ as in Step 2 in the proof of Theorem 1.5.1.7, now using (1.6.4) (with $\rho = 0$) and (1.6.6) of Lemma 1.6.2, and obtain, as desired,

$$g \in C([s,T];U). \tag{1.6.12}$$

Thus, (1.6.9) is proved. □

Corollary 1.6.4 *Assume hypothesis (1.6.1). Then, the following results hold true:*

(i)

$$u^0(\,\cdot\,, s; x) \in C([s, T]; U), \quad x \in Y, \tag{1.6.13a}$$

$$\max_{s \le t \le T} \|u^0(t, s; x)\|_U \le C_T \|x\|_Y, \quad x \in Y, \tag{1.6.13b}$$

with C_T independent of s;

(ii) for any $0 \le \theta < 1$,

$$(-A^*)^\theta P(t)x \in C_{\theta-\gamma}([0, T]; Y), \tag{1.6.14a}$$

$$\|(-A^*)^\theta P(t)x\|_Y \in \frac{C_T}{1-\theta} \frac{1}{(T-t)^{\theta-\gamma}} \|x\|_Y; \tag{1.6.14b}$$

(iii)

$$B^* P(t)x \in C([0, T]; U), \quad x \in Y, \tag{1.6.15a}$$

$$\max_{0 \le t \le T} \|B^* P(t)x\|_U \le C_T \|x\|_Y, \quad x \in Y. \tag{1.6.15b}$$

Proof. (i) Conclusion (1.6.13) is a consequence of Theorem 1.6.3, Eqn. (1.6.9), via (1.6.2) and (1.5.1.9).

Parts (ii) and (iii) follow now as in the proof of Theorem 1.5.2.1, in effect replacing β there by γ now. □

1.7 The Theory of Theorem 1.2.1.1 Is Sharp. Counterexamples When GL_T Is Not Closable

In this section we show that suitable one-dimensional range (finite range) operators G furnish examples that illustrate the sharpness of the theory presented. In particular, Section 1.7.1 shows that the assumption that GL_T is closable cannot be dispensed with in the fundamental Theorem 1.2.1.1, for otherwise the optimal control u^0 may not exist.

1.7.1 Counterexample to the Existence of the Optimal Control u^0 When GL_T Is Not Closable

In this example, the operator GL_T is not closable and the optimal control does not exist. This shows that our Theorem 1.2.1.1 is sharp. We recall from Section 1.4.1 that the assumption that GL_T be closable is used to obtain a *complete* inner product space (Hilbert) $V(s, T; U)$ defined by (the extension of) $\mathcal{D}(GL_T)$ with respect to the inner product in (1.4.1.8), and that moreover the optimal control is characterized by (1.4.1.25), with Λ_{sT}^{-1} an isomorphism from $[V(s, T; U)]'$ onto $V(s, T; U)$.

The Example Consider, say, the heat equation defined on a (smooth) bounded domain $\Omega \subset R^n$ with $L_2(0, T; L_2(\Gamma))$-control in the Dirichlet boundary conditions

$$\begin{cases} y_t = \Delta y & \text{in } Q = (0, T] \times \Omega; & (7.1.1a) \\ y(0, \cdot) = y_0 & \text{in } \Omega; & (7.1.1b) \\ y|_\Sigma = u & \text{in } \Sigma = (0, T] \times \Gamma. & (7.1.1c) \end{cases}$$

Here $Y = L_2(\Omega)$, $U = L_2(\Gamma)$. There exists $\phi \in Y$, $\|\phi\|_Y = 1$, $\phi \notin \mathcal{D}(L_T^*)$, such that

$$\int_0^T \left\| B^* e^{A^*(T-t)} \phi \right\|_U^2 dt = \infty, \quad \phi \notin \mathcal{D}(L_T^*), \tag{1.7.1.2}$$

for then, otherwise, by transposition, the map $u \to y(T)$ (where $y_0 = 0$) would be continuous, $L_2(0, T; L_2(\Gamma)) \to L_2(\Omega) = Y$, which is false even in the one-dimensional case [e.g. Lions, 1971, p. 217]. This duality or transposition result will be given emphasis in Chapter 7 [Theorem 7.2.1]. We consider the associated optimal control problem (1.1.3) with

$$Z_f = Y; \quad R = 0; \quad Gy = (y, \phi)_Y \phi; \quad \phi \in Y; \quad G^* = G = G^*G. \tag{1.7.1.3}$$

Note that we have, by (1.1.9) and (1.7.1.3),

$$GL_T u = \left(\int_0^T e^{A(T-t)} Bu(t)\, dt, \phi \right)_Y \phi = \left(u, B^* e^{A^*(T-\cdot)} \phi \right)_{L_2(0,T;U)} \phi, \tag{1.7.1.4}$$

so that GL_T is finite rank and unbounded by (1.7.1.2) and hence is *unclosable* [Kato, 1966, p. 166].

Claim #1 *There is no optimal control in this case.*

In fact, following Section 1.4.1, if an optimal control $u^0(\cdot, 0; x) = u^0 \in L_2(0, T; U)$ exists, it satisfies the present version of (1.4.1.25), (or (1.2.1.1)), that is, since $R = 0$,

$$-[u^0 + L_T^* G^* G L_T u^0] = L_T^* G^* G e^{AT} x = (e^{AT} x, \phi)_Y B^* e^{A^*(T-t)} \phi, \tag{1.7.1.5}$$

where we have used (1.7.1.3) on G^*G and (1.1.10) for L_T^*. Moreover, by (1.7.1.3), (1.1.9), and (1.1.10),

$$L_T^* G^* G L_T u^0 = L_T^* \{(L_T u, \phi)_Y \phi\} = \left(\int_0^T e^{A(T-t)} B u^0(t)\, dt, \phi \right)_Y B^* e^{A^*(T-t)} \phi. \tag{1.7.1.6}$$

Using (1.7.1.6) and (1.7.1.4) in (1.7.1.5) yields

$$-u^0 = \left\{ \left(u^0, B^* e^{A^*(T-\cdot)} \phi \right)_{L_2(0,T;U)} + (e^{AT} x, \phi)_Y \right\} B^* e^{A^*(T-t)} \phi. \tag{1.7.1.7}$$

Since $B^* e^{A^*(T-t)} \phi \notin L_2(0, T; U)$ by (1.7.1.2), then (1.7.1.7) yields that $u^0 \notin L_2(0, T; U)$, a contradiction.

Claim #2 *The operator Λ_{0T} in (1.2.1.3) is neither self-adjoint, nor boundedly invertible, on $L_2(0, T; U)$ in the present case.*

Indeed, by (1.7.1.3), we obtain

$$G^*GL_T u = (L_T u, \phi)_Y \phi, \quad \phi \notin \mathcal{D}(L_T^*). \tag{1.7.1.8}$$

Since $\phi \notin \mathcal{D}(L_T^*)$, then (1.7.1.8) implies

$$\begin{cases} \mathcal{D}(L_T^* G^* G L_T) = \{u \in \mathcal{D}(L_T) \subset L_2(0, T; U) : (L_T u, \phi)_Y = 0\}, & \text{(1.7.1.9a)} \\ L_T^* G^* G L_T u = 0, \quad \forall u \in \mathcal{D}(L_T^* G^* G L_T). & \text{(1.7.1.9b)} \end{cases}$$

Thus, by (1.2.1.3) with $R = 0$ as in (1.7.1.3), we obtain, via (1.7.1.9),

$$\Lambda_{0T} u = (I + L_T^* G^* G L_T) u = u, \quad \forall u \in \mathcal{D}(\Lambda_{0T}) = \mathcal{D}(L_T^* G^* G L_T). \tag{1.7.1.10}$$

Then, (1.7.1.10) says that the range of Λ_{0T} is equal to the domain of Λ_{0T} and is contained in $\mathcal{D}(L_T)$ and thus is not all of $L_2(0, T; U)$. We conclude that Λ_{0T} cannot be either boundedly invertible, or self-adjoint, on $L_2(0, T; U)$.

Plainly, the present example can be generalized to any case where L_T^* is genuinely unbounded, under (1.7.1.3).

Remark 1.7.1 We note that the choice (1.7.1.2) for ϕ implies that $\phi \notin \mathcal{D}((-A^*)^{\beta/2})$ for all $\beta > 2\gamma - 1$. In fact, if we had $\phi \in \mathcal{D}((-A^*)^{\beta/2})$ we would find that

$$B^* e^{A^*(t-t)} \phi = B^*(-A^*)^{-\gamma} (-A^*)^{\gamma - \beta/2} e^{A^*(T-t)} (-A^*)^{\beta/2} \phi \tag{1.7.1.11}$$

would belong to $L_2(0, T; U)$ by (1.1.2) and analyticity of e^{A^*t} with $2\gamma - \beta < 1$, thus contradicting (1.7.1.2). With reference to (1.2.1.26), we note that, in this case, we have $\mathcal{D}((-A^*)^{\beta/2} G^*) = \{0\}$, $\forall \beta > 2\gamma - 1$. This fact is consistent with the implication (1.2.1.25) $\Rightarrow$ (1.2.1.24), since GL_T is not closable in this case.

1.7.2 Assumption (1.2.1.26) Is Only Sufficient for GL_T to Be Closed

We shall provide a class of examples where condition (1.2.1.26) is *violated*, yet GL_T is *closed*. We notice that condition (1.2.1.26) – unlike GL_T – does not involve B.

Let the generator A be negative, self-adjoint, say with compact resolvent. (We shall, however, maintain the notation A^*.) Let $\{e_n, n = 1, 2, \ldots\}$ be the corresponding orthonormal basis of eigenvectors of A on Y with eigenvalues $\{-\mu_n\}$, $\mu_n > 0$. Let $\mathcal{S}_i, i = 1, 2$, be two infinite, disjoint sequences of positive integers that exhaust all of the positive integers $\mathbb{Z} : \mathcal{S}_1 \cup \mathcal{S}_2 = \mathbb{Z}; \mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$. For example: $\mathcal{S}_1 = \{n = 2, 4, 6, \ldots\}$, $\mathcal{S}_2 = \{n = 1, 3, 5, \ldots\}$. Consider the orthogonal decomposition of Y,

$$Y = Y_1 + Y_2, \quad Y_i = \overline{\text{span}}\{e_n, n \in \mathcal{S}_i\}, \quad i = 1, 2. \tag{1.7.2.1}$$

Let Π_i be the orthogonal projection of Y onto Y_i, so that Π_i commutes with A, hence with the semigroup e^{At}, and Y_i are invariant under e^{At}. Define a vector $b \in Y_1$ by setting

$$(b, e_n)_Y = \begin{cases} \text{sequence in } n \in \mathcal{S}_1, \text{ such that } \sum\limits_{n \in \mathcal{S}_1} \mu_n^\beta |(b, e_n)_Y|^2 = \infty, \\ 0, n \in \mathcal{S}_2 \end{cases} \tag{1.7.2.2}$$

for all $\beta > 2\gamma - 1$, so that

$$b \notin \mathcal{D}\big((-A^*)^{\frac{\beta}{2}}\big), \quad \forall \beta > 2\gamma - 1. \tag{1.7.2.3}$$

Next, with $U = Y = Z_f$, define the bounded operators G^*, G and the unbounded operators B^*, B by

$$G^* y = (y_1, a)_Y b + y_2, \quad Gy = (y_1, b)_Y a + y_2, \quad y_i = \Pi_i y \in Y_i, \ a \in Y; \tag{1.7.2.4}$$

$$\begin{cases} By_1 = 0, \\ By_2 = (-A)^\gamma y_2, \end{cases} \quad \begin{cases} B^* y_1 = 0, & y_1 = \Pi_1 y \in Y_1, \quad (1.7.2.5a) \\ B^* y_2 = (-A)^\gamma y_2, & y_2 = \Pi_2 y \in Y_2 \cap \mathcal{D}((-A^*)^\gamma). \quad (1.7.2.5b) \end{cases}$$

One readily obtains by (1.7.2.4), (1.7.2.3) that $y \in \mathcal{D}((-A^*)^{\beta/2} G^*)$ if and only if $y_1 = 0$ and $y_2 \in \mathcal{D}((-A^*)^{\beta/2} G^*)$, so that

$$\mathcal{D}\big((-A^*)^{\beta/2} G^*\big) = \mathcal{D}\big((-A^*)^{\beta/2}\big) \cap Y_2; \tag{1.7.2.6a}$$

$$(-A^*)^{\beta/2} G^* y = (-A^*)^{\beta/2} y_2, \quad y \in \mathcal{D}\big((-A^*)^{\beta/2} G^*\big). \tag{1.7.2.6b}$$

Thus, $\mathcal{D}((-A^*)^{\beta/2} G^*)$ is *not dense in* Y, and condition (1.2.1.26) is *violated.*

In contrast, since $B\Pi_1 u(t) \equiv 0$ by (1.7.2.5a) and Y_2 is invariant under A and e^{At}, we obtain, by (1.1.9), (1.7.2.4), and (1.7.2.5b),

$$\begin{aligned} GL_T u &= G \int_0^T e^{A(T-t)} Bu(t)\, dt \\ &= G \int_0^T e^{A(T-t)} B\Pi_1 u(t)\, dt + G \int_0^T e^{A(T-t)} B\Pi_2 u(t)\, dt \\ \text{(by (1.7.2.5b))} \qquad &= G \int_0^T e^{A(T-t)} (-A)^\gamma \Pi_2 u(t)\, dt \\ &= (-A)^\gamma \int_0^T e^{A(T-t)} \Pi_2 u(t)\, dt, \end{aligned} \tag{1.7.2.7}$$

where in the last step we have used that, by (1.7.2.4), G is the identity on Y_2, and that the integral term is in Y_2. Thus, GL_T is a *closed* operator (being the product of a closed, boundedly invertible operator $(-A)^\gamma$ and of a bounded operator [Kato, 1966, p. 164]). Our claim is proved. Note that, by (1.1.10), one likewise has

$$\{L_T^* G^* y\}(t) = (-A^*)^\gamma e^{A^*(T-t)} y_2, \quad y_2 = \Pi_2 y \in Y_2.$$

1.8 Extension to Unbounded Operators *R* and *G*

In this section – in addition to the standing hypotheses (i) and (ii) = (1.1.2) on the model (1.1.1) of Section 1 – we assume that the observation operators R and G have both a controlled degree of unboundedness. Then in Subsection 1.8.1, we reduce this setting to the case considered in Section 1.2, by means of suitable "change of (operator) variables." Next, in Subsection 1.8.2, we discuss the case where the final state operator G has a degree of unboundedness of any order with respect to A.

To quantify the controlled degree of unboundedness of R, we let $G = 0$ and refer to the optimality formulas (1.2.1.1)–(1.2.1.3), (1.2.1.28)–(1.2.1.30) (directly in terms of the data), which require that:

(i)

$$Re^{At} : \text{continuous } Y \to L_2(0, T; Z); \tag{1.8.1}$$

(ii)

$$RL : \text{continuous } L_2(0, T; U) \to L_2(0, T; Z). \tag{1.8.2}$$

As to (i), by analyticity on $R(-A)^{-\delta}(-A)^{\delta}e^{At}$, we see that the regularity (1.8.1) is satisfied with

$$R \in \mathcal{L}(\mathcal{D}((-A)^{\delta}); Z), \quad 0 \le \delta < \frac{1}{2}; \tag{1.8.3}$$

or even $\delta = 1/2$ if A is self-adjoint (see Appendix 1A, Eqn. (1A.2)).

As to (ii), writing

$$(RLu)(t) = R(-A)^{\gamma-1} \int_0^t Ae^{A(t-\tau)}(-A)^{-\gamma} Bu(\tau)\, d\tau, \tag{1.8.4}$$

and invoking (1.1.2) and the standard regularity (0.4) of Chapter 0, we see that the regularity (1.8.2) is satisfied with

$$R \in \mathcal{L}(D((-A)^{1-\gamma}); Z). \tag{1.8.5}$$

The constrains imposed by (1.8.3) and (1.8.5), via these preliminary considerations aimed at achieving (1.8.1) and (1.8.2), are precisely those that are assumed in Eqn. (1.8.1.1b) of Section 1.8.1, a section primarily focused on the controlled unboundedness of R.

1.8.1 The Case Where $R \in \mathcal{L}(\mathcal{D}((-A)^{\delta}); Z)$ and $G \in \mathcal{L}(\mathcal{D}((-A)^{\delta}); Z_f), \delta < \min\{1-\gamma, \frac{1}{2}\}$

New hypotheses Throughout this subsection we assume (in addition to (i) and (ii) = (1.1.2) of Section 1.1) that

$$\begin{cases} R \in \mathcal{L}(\mathcal{D}((-A)^{\delta}); Z) \text{ and } G \in \mathcal{L}(\mathcal{D}((-A)^{\delta}); Z_f); & (1.8.1.1a) \\ \delta < \min\left\{1-\gamma, \frac{1}{2}\right\}, & (1.8.1.1a) \end{cases}$$

thereby relaxing (1.1.4).

Change of Variables We now introduce the following new operators:

(i)

$$\bar{R} \equiv R(-A)^{-\delta} \in \mathcal{L}(Y; Z); \qquad \bar{G} \equiv G(-A)^{-\delta} \in \mathcal{L}(Y; Z_f); \tag{1.8.1.2}$$

(ii)

$$\begin{cases} \bar{B} \equiv (-A)^{\delta} B : \text{continuous } U \to [\mathcal{D}((-A^*)^{1+\delta})]' & (1.8.1.3a) \\ \text{so that, in view of (1.8.1.1b) and of the standing hypothesis (1.1.2)} & \\ (-A)^{-\bar{\gamma}} \bar{B} = (-A)^{-\bar{\gamma}+\delta} B = (-A)^{-\gamma} B \in \mathcal{L}(U; Y); & (1.8.1.3b) \\ \bar{\gamma} \equiv \gamma + \delta < 1. & (1.8.1.3c) \end{cases}$$

Consequences (a) We define the operator $\bar{L}_T$ by

$$\bar{L}_T u = \int_0^T e^{A(T-t)} \bar{B} u(\tau)\, d\tau; \tag{1.8.1.4a}$$

$$\mathcal{D}(\bar{L}_T) = \{u \in L_2(0, T; U) : \bar{L}_T u \in Y\}. \tag{1.8.1.4b}$$

in the limit sense, as in (1.1.9). We then verify that

$$\bar{G}\bar{L}_T = G L_T; \tag{1.8.1.5a}$$

$$\mathcal{D}(\bar{G}\bar{L}_T) = \mathcal{D}(G L_T) = \{u : \bar{G}\bar{L}_T u = G L_T u \in Z_f\}, \tag{1.8.1.5b}$$

so that

$$\bar{G}\bar{L}_T \text{ (see (1.8.1.4)) is closed (closable)} \Longleftrightarrow \text{so is } G L_T \text{ (see (1.1.9)).} \tag{1.8.1.6}$$

Indeed, to check (1.8.1.5), we compute via (1.8.1.2)–(1.8.1.4):

$$\begin{aligned} \bar{G}\bar{L}_T u &= G(-A)^{-\delta} \int_0^T e^{A(T-t)} (-A)^{\delta} B u(\tau)\, d\tau \\ &= G \int_0^T e^{A(T-t)} B u(\tau)\, d\tau = G L_T u, \end{aligned} \tag{1.8.1.7}$$

as desired, for $u \in \mathcal{D}(\bar{G}\bar{L}_T) = \mathcal{D}(G L_T) = \{u \in L_2(0, T; U) : \bar{G}\bar{L}_T u = G L_T u \in Z_f\}$.

(b) We likewise define the operator $\bar{L}$ by

$$\{\bar{L}u\}(t) = \int_0^t e^{A(t-\tau)} \bar{B} u(\tau)\, d\tau \tag{1.8.1.8a}$$

$$\text{(by (1.8.1.3))} \qquad = (-A)^{\delta+\gamma} \int_0^t e^{A(t-\tau)} (-A)^{-\gamma} B u(\tau)\, d\tau \tag{1.8.1.8b}$$

$$\begin{aligned} &: \text{continuous } L_2(0, T; U) \to L_2(0, T; \mathcal{D}((-A)^{1-\delta-\gamma})) \\ &\subset L_2(0, T; Y), \end{aligned} \tag{1.8.1.8c}$$

where, in (1.8.1.8c), we have recalled the standard regularity result (0.4) of Chapter 0, via (1.1.2). We then similarly verify that

$$\bar{R}\bar{L} = RL \tag{1.8.1.9a}$$

$$: \text{continuous } L_2(0,T;U) \to L_2(0,T;Z_f), \tag{1.8.1.9b}$$

where, in (1.8.1.9b), we have recalled (1.8.1.2) on $\bar{R}$ and (1.8.1.8c) on $\bar{L}$. To check (1.8.1.9a), we compute, via (1.8.1.2), (1.8.1.3), (1.8.1.8a), and (1.1.6),

$$\begin{aligned}\{\bar{R}\bar{L}u\}(t) &= R(-A)^{-\delta}\int_0^t e^{A(t-\tau)}(-A)^{\delta}Bu(\tau)\,d\tau \\ &= R\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau = \{RLu\}(t),\end{aligned}$$

as desired.

(c) As already observed, we have, since $\delta < 1/2$,

$$Re^{At} = R(-A)^{-\delta}(-A)^{\delta}e^{At} : \text{continuous } Y \to L_2(0,T;Z). \tag{1.8.1.10}$$

(d) We introduce the *new dynamics*

$$\dot{y} = Ay + \bar{B}u \in [\mathcal{D}((A^*)^{1+\delta})]', \quad y(0) = y_0 \in Y, \tag{1.8.1.11}$$

and the *new cost functional*

$$\bar{J}(u,y) = \int_0^T \left[\|\bar{R}y(t)\|_Z^2 + \|u(t)\|_U^2\right]dt + \|\bar{G}y(T)\|_{Z_f}^2. \tag{1.8.1.12}$$

Then:

$$\left\{\begin{array}{l}\text{The system } \{A,\bar{B}\} \text{ in (1.8.1.11), with parameter } \bar{\gamma} \\ \text{in (1.8.1.3c), and bounded observation operators } \bar{R} \text{ and } \bar{G} \\ \text{in (1.8.1.12) satisfies the setting of Section 1.1 (hypotheses} \\ \text{(1.1.1) through (1.1.4)) by virtue of (1.8.1.3b), (1.8.1.2).}\end{array}\right. \tag{1.8.1.13}$$

It follows that, if the original operator GL_T is closed (closable), then – by (1.8.1.6) and (1.8.1.13) – we can apply Theorem 1.2.1.1 to problem (1.8.1.11), (1.8.1.12). We obtain

Theorem 1.1.8.1 *Assume (i), (ii) = (1.1.2) of Section 1.1, as well as (1.8.1.1). Suppose further that GL_T is closed (closable). Then:*

(i) For each $y_0 \in Y$, there exists a unique optimal pair $\{u^0(t,0.y_0), y^0(t,0;y_0)\}$ of the optimal control problem (1.1.1)–(1.1.3), which is explicitly given in terms

of the data of the problem by

$$-u^0(t,0;y_0) = \left\{\Lambda_{0T}^{-1}[(GL_T)^* Ge^{AT}y_0 + (RL)^* R(e^{A\cdot}y_0)]\right\}(t) \in L_2(0,T;U), \tag{1.8.1.14}$$

$$y^0(t,0;y_0) = e^{At}y_0 + \{Lu^0\}(t) \in L_2(0,T;Y), \tag{1.8.1.15}$$

$$\Lambda_{0T} = I + (RL)^* RL + (GL_T)^* GL_T, \tag{1.8.1.16}$$

where Λ_{0T} defines an isomorphism between $\mathcal{D}(GL_T)$, topologized by the graph norm, and its dual with respect to the pivot space Y. Moreover, L, L^* are defined by (1.1.7), (1.1.8), and L_T, L_T^* are defined by (1.1.9), (1.1.10). The optimality condition is

$$-u^0(\,\cdot\,,0;y_0) = (RL)^* Ry^0(\,\cdot\,,0;y_0) + (GL_T)^* Gy^0(T,0;y_0) \tag{1.8.1.17a}$$

$$\in \mathcal{D}(GL_T) \subset L_2(0,T;U). \tag{1.8.1.17b}$$

(ii) *Let $\{\bar{u}^0(t,0;y_0), \bar{y}^0(t,0;y_0)\}$ denote the unique optimal pair of the optimal control problem (1.8.1.11), (1.8.1.2), guaranteed by Theorem 1.2.1.1 via (1.8.1.2), with $y_0 \in Y$:*

$$-\bar{u}^0(t,0;y_0) = \left\{\bar{\Lambda}_{0T}^{-1}[\bar{L}_T^* \bar{G}^* \bar{G} e^{AT}y_0 + \bar{L}^* \bar{R}^* \bar{R}(e^{A\cdot}y_0)]\right\}(t), \tag{1.8.1.18}$$

$$\bar{y}^0(t,0;y_0) = e^{At}y_0 + \{\bar{L}\bar{u}^0\}(t) \in L_2(0,T;Y), \tag{1.8.1.19}$$

$$\bar{\Lambda}_{0T} = I + \bar{L}^* \bar{R}^* \bar{R}\bar{L} + \bar{L}_T^* \bar{G}^* \bar{G}\bar{L}_T. \tag{1.8.1.20}$$

The optimality condition is

$$-\bar{u}^0(\,\cdot\,,0;y_0) = \bar{L}^* \bar{R}^* \bar{R}\bar{y}^0(\,\cdot\,,0;y_0) + \bar{L}_T^* \bar{G}^* \bar{G}\bar{y}^0(T,0;y_0)$$
$$\subset \mathcal{D}(\bar{G}\bar{L}_T) \subset L_2(0,T;U). \tag{1.8.1.21}$$

We have, with $\bar{\gamma}$ as in (1.8.1.3c):

(ii_1)

$$\bar{\Lambda}_{0T} = \Lambda_{0T}; \tag{1.8.1.22}$$

(ii_2)

$$\bar{u}^0(\,\cdot\,,0;(-A^\delta)y_0)) = u^0(\,\cdot\,,0;y_0) \in C_{\bar{\gamma}}([0,T];U), \quad y_0 \in \mathcal{D}((-A)^\delta); \tag{1.8.1.23}$$

(ii_3)

$$(-A)^{-\delta}\bar{y}^0(t,0;(-A)^\delta y_0) = y^0(t,0;y_0), \quad y_0 \in \mathcal{D}((-A)^\delta) \tag{1.8.1.24a}$$

$$\in \begin{cases} C([0,T];\mathcal{D}((-A)^\delta)) & \text{if } 0 \le \bar{\gamma} < \frac{1}{2}, \quad (1.8.1.24b) \\ C_{2\bar{\gamma}-1}([0,T];\mathcal{D}((-A)^\delta)) & \text{if } \frac{1}{2} < \bar{\gamma} < 1; \quad (1.8.1.24c) \end{cases}$$

$$\bar{R}\bar{y}^0(t,0;(-A)^\delta y_0) = Ry^0(t,0;y_0), \quad y_0 \in \mathcal{D}((-A)^\delta) \tag{1.8.1.25a}$$
$$\in \begin{cases} C([0,T];Z) & 0 \le \bar{\gamma} < \frac{1}{2}, \quad (1.8.1.25b)\\ C_{2\bar{\gamma}-1}([0,T];Z) & \frac{1}{2} < \bar{\gamma} < 1. \quad (1.8.1.25c)\end{cases}$$

(ii₄)

$$\bar{G}\bar{y}^0(T,0;(-A)^\delta y_0) = Gy^0(T,0;y_0), \quad y_0 \in \mathcal{D}((-A)^\delta); \tag{1.8.1.26a}$$
$$Gy^0(T,0;y_0) \in Z_f, \quad y_0 \in Y. \tag{1.8.1.26b}$$

(iii) Let $\bar{P}(t)$ be the nonnegative, self-adjoint Riccati operator for problem (1.8.1.11), (1.8.1.12) guaranteed by Theorem 1.2.1.1, given for $x \in Y$ by

$$\bar{P}(t)x = \int_t^T e^{A^*(\tau-t)}\bar{R}^*\bar{R}\bar{y}^0(\tau,t;x)\,d\tau + e^{A^*(T-t)}\bar{G}^*\bar{G}\bar{y}^0(T,t;x), \tag{1.8.1.27}$$

so that for $x \in Y$,

$$\bar{J}(t;x) = \bar{J}(\bar{u}^0(\,\cdot\,,t;x),\bar{y}^0(\,\cdot\,,t;x)) = (\bar{P}(t)x,x)_Y, \quad 0 \le t < T. \tag{1.8.1.28}$$

Then

(iii₁) the operator

$$P(t)x \equiv (-A^*)^\delta \bar{P}(t)(-A)^\delta x, \quad 0 \le t < T, \; x \in Y, \tag{1.8.1.29}$$

is a bounded, self-adjoint operator on Y; indeed, more precisely,

$$P(t) \in \mathcal{L}(Y; C([0,T-\epsilon];Y)), \quad 0 < \epsilon < T. \tag{1.8.1.30}$$

(iii₂)

$$P(t)x = \int_t^T e^{A^*(\tau-t)}R^*Ry^0(\tau,t;x)\,d\tau + e^{A^*(T-t)}G^*Gy^0(T,t;x), \quad x \in Y. \tag{1.8.1.31}$$

(iii₃) For $x \in Y$,

$$J^0(t;x) = J(u^0(\,\cdot\,,t;x),y^0(\,\cdot\,,t;x)) = (P(t)x,x)_Y. \tag{1.8.1.32}$$

(iii₄) The following estimate holds true:

$$\|(-A^*)^r P(t)(-A)^{-q}\|_{\mathcal{L}(Y)} \le \frac{C_{rq\delta\gamma}}{(T-t)^{1-r+q-2\delta}},$$
$$0 \le t < T; \; r,q \ge 0; \; \delta - q \ge 0; \; r + 2\delta - q < 1. \tag{1.8.1.33}$$

Moreover,

$$(-A^*)^r P(t)(-A)^{-q} \in \mathcal{L}(Y; C([0,T-\epsilon];Y)), \quad 0 < \epsilon < T. \tag{1.8.1.34}$$

In short,

$$(-A^*)^r P(t)(-A)^{-q}x \in C_{1-r+q-2\delta}([0,T];Y), \quad x \in Y. \tag{1.8.1.35}$$

The following two special cases are notable: For $q = 0$, $0 < r < 1 - 2\delta$:

$$(-A^*)^r P(t)x \in C_{1-r-2\delta}([0,T];Y), \quad x \in Y; \tag{1.8.1.36}$$

and for $q = \delta$, $r < 1 - \delta$:

$$(-A^*)^r P(t)(-A)^{-\delta}x \in C_{1-\delta-r}([0,T];Y), \quad x \in Y. \tag{1.8.1.37}$$

(iv)

$$B^*P(t)(-A)^{-\delta}x = \bar{B}^*\bar{P}(t)x \in C_{\bar{\gamma}}([0,T];U), \quad x \in Y; \tag{1.8.1.38}$$

$$u^0(t,0;y_0) = -B^*Py^0(t,0;y_0) \in L_2(0,T;U), \quad y_0 \in Y. \tag{1.8.1.39}$$

(v) $P(t)$ *satisfies the following DRE for all* $0 < t < T$, *and all* $x, y \in \mathcal{D}((-A)^{\delta+\epsilon})$, δ *as in (1.8.1.1b),* $\forall \epsilon > 0$:

$$\begin{aligned}(\dot{P}(t)x, y)_Y = &-(Rx, Ry)_Z - (P(t)x, Ay)_Y - (P(t)Ax, y)_Y \\ &+ (B^*P(t)x, B^*P(t)y)_U.\end{aligned} \tag{1.8.1.40}$$

(vi) For all $x \in Y$*; via (8.1.29)*

$$\begin{aligned}\lim_{t\uparrow T} P(t)x &= \lim_{t\uparrow T}(-A^*)^\delta \bar{P}(t)(-A)^\delta x \\ &= (-A^*)^\delta \bar{G}^*\bar{G}(-A)^\delta x \\ &= G^*\bar{G}x\end{aligned} \tag{1.8.1.41}$$

(vii) The operator

$$A_P(t) = A - BB^*P(t) \tag{1.8.1.42a}$$

$$= (-A)^{-\delta}[A - \bar{B}\bar{B}^*\bar{P}(t)](-A)^\delta \tag{1.8.1.42b}$$

is the generator of the evolution operator

$$\Phi(t,s)x = (-A)^{-\delta}\bar{\Phi}(t,s)(-A)^\delta x, \quad x \in \mathcal{D}((-A)^\delta) \tag{1.8.1.43}$$

on the space $\mathcal{D}((-A)^\delta)$,

$$\Phi(t,s)x = y^0(t,s;x), \quad \bar{\Phi}(t,s)x = \bar{y}^0(t,s;x). \tag{1.8.1.44}$$

Proof. The statement of Theorem 1.8.1.1 down to Eqn. (1.8.1.21) is contained in (the proof of) Theorem 1.2.1.1 since RL satisfies (1.8.1.9) and Re^{At} satisfies (1.8.1.10).

(ii) Property $(\text{ii}_1) = (1.8.1.22)$ is readily verified via (1.8.1.5) and (1.8.1.9), on the basis of the definitions (1.8.1.16) and (1.8.1.20).

As to (ii_2), starting from (1.8.1.18), then $(\text{ii}_1) = (1.8.1.22)$ implies $(\text{ii}_2) = (1.8.1.23)$, via (1.8.1.5), (1.8.1.9), (1.8.1.2), and (1.8.1.14):

$$\begin{aligned}-\bar{u}^0(\,\cdot\,,0;(-A)^\delta y_0) &= \bar{\Lambda}_{0T}^{-1}[(\bar{G}\bar{L}_T)^*\bar{G}e^{AT}(-A)^\delta y_0 + (\bar{R}\bar{L})^*\bar{R}(e^{A\cdot}(-A)^\delta y_0)] \\ &= \Lambda_{0T}^{-1}[(GL_T)^*Ge^{AT}y_0 + (RL)^*Re^{A\cdot}y_0] \\ \text{(by (1.8.1.14))} \qquad &= -u^0(\,\cdot\,,0;y_0) \in C_{\bar{\gamma}}([0,T];U), \quad y_0 \in \mathcal{D}((-A)^\delta),\end{aligned} \tag{1.8.1.45}$$

as desired, where the regularity in (1.8.1.45) stems from (1.2.1.16) applied to $\bar{u}^0(\cdot\,, 0; (-A)^\delta y_0)$, $y_0 \in \mathcal{D}((-A)^\delta y_0)$with $\bar{y}$ as in (1.8.1.3c).

Similarly, for (ii$_3$): Starting from (1.8.1.19) and using (1.8.1.3) and (ii$_2$) = (1.8.1.23), we obtain

$$\begin{aligned}
(-A)^{-\delta}\bar{y}^0(t, 0; (-A)^\delta y_0) &= (-A)^{-\delta}e^{At}(-A)^\delta y_0 \\
&\quad + (-A)^{-\delta}\int_0^t e^{A(t-\tau)}\bar{B}\bar{u}^0(\tau, 0; (-A_0)^\delta y_0)\,d\tau \\
\text{(by (1.8.1.23))} \qquad &= e^{At}y_0 + \int_0^t e^{A(t-\tau)}Bu^0(\tau, 0; y_0)\,d\tau \\
\text{(by (1.8.1.15))} \qquad &= y^0(t, 0; y_0), \quad y_0 \in \mathcal{D}((-A)^\delta), \qquad (1.8.1.46)
\end{aligned}$$

as desired and (1.8.1.24a) is proved. The regularity noted in (1.8.1.24b,c) stems from (1.2.1.17), (1.2.1.18) of Theorem 1.2.1.1. Then (1.8.1.2b,c) implies (1.8.1.25a,b) by (1.8.1.1) on R. Next, (1.8.1.24) implies (1.8.1.26a) by (1.8.1.1) on G. Finally, (1.8.1.26b) follows from (1.8.1.17b), (1.8.1.5b) as in Proposition 1.4.2.1(iii).

(iii) Equations (1.8.1.27) and (1.8.1.28) are guaranteed by Theorem 1.2.1.1. The theorem also shows that (see (1.2.1.7b))

$$(-A^*)^\theta \bar{P}(t)x \in C([0, T); Y), \quad x \in Y, \; 0 < \theta < 1. \qquad (1.8.1.47)$$

(iii$_1$) To show that the operator $P(t)$ defined by (1.8.1.29) is bounded on Y, as in (1.8.1.30), we invoke on (1.8.1.47) the critical Lemma 1.5.1.1, as in the proof of Corollary 1.5.1.2, since $\bar{P}(t)$ is self-adjoint and $2\delta < 1$ by assumption (1.8.1.1b). Such $P(t)$ is plainly self-adjoint, nonnegative definite on Y.

(iii$_2$) Starting from (1.8.1.27), the validity of (1.8.1.31) is readily verified by use of definition (1.8.1.29), (1.8.1.27), (1.8.1.2), as well as (1.8.1.24):

$$\begin{aligned}
P(t)x &= (-A^*)^\delta \bar{P}(t)(-A)^\delta x \\
\text{(by (1.8.1.27))} \qquad &= (-A^*)^\delta \int_t^T e^{A^*(\tau-t)}\bar{R}^*\bar{R}\bar{y}^0(\tau, t; (-A)^\delta x)\,d\tau \\
&\quad + (-A^*)^\delta e^{A^*(T-t)}\bar{G}^*\bar{G}\bar{y}^0(T, t; (-A)^\delta x) \\
\text{(by (1.8.1.2))} \qquad &= \int_t^T e^{A^*(\tau-t)}R^*R(-A)^{-\delta}\bar{y}^0(\tau, t; (-A)^\delta x)\,d\tau \\
&\quad + e^{A^*(T-t)}G^*G(-A)^{-\delta}\bar{y}^0(T, t; (-A)^\delta x) \\
\text{(by (1.8.1.24))} \qquad &= \int_t^T e^{A^*(\tau-t)}R^*Ry^0(\tau, t; x)\,d\tau \\
&\quad + e^{A^*(T-t)}G^*Gy^0(T, t; x), \qquad (1.8.1.48)
\end{aligned}$$

recalling (1.8.1.24) in the last step. Thus, (1.8.1.31) is verified.

(iii$_3$) Equation (1.8.1.31) implies (1.8.1.32) as in the proof of Proposition 1.4.4.8.

(iii$_4$) By (1.8.1.29) we compute by critically invoking Lemma 1.5.1.1, as in the proof of Corollary 1.5.1.2, as well as estimate (1.2.1.7a) for $\bar{P}(t)$ (guaranteed by

Theorem 1.2.1.1):

$$\begin{aligned}
\|(-A^*)^r P(t)(-A)^{-q}\|_{\mathcal{L}(Y)} &= \|(-A^*)^r(-A^*)^\delta \bar{P}(t)(-A)^{\delta-q}\|_{\mathcal{L}(Y)} \\
\text{(by Lemma 1.5.1.1)} \qquad &\le C_{r\delta q}\|(-A^*)^\theta \bar{P}(t)\|_{\mathcal{L}(Y)} \\
\text{(by (1.2.1.7a))} \qquad &\le \frac{C_{r\delta q\theta\gamma}}{(T-t)^\theta}, \quad 0 \le t < T, \ \theta < 1 \\
&\le \frac{C_{r\delta q\theta\gamma}}{(T-t)^{1-r+q-2\delta}}, \quad 0 \le t < T, \qquad (1.8.1.49)
\end{aligned}$$

where application of Lemma 1.5.1.1, or Corollary 1.5.1.2, is legitimate with

$$0 \le q, \quad \delta - q \ge 0, \quad \text{and} \quad r + 2\delta - q < \theta < 1. \qquad (1.8.1.50)$$

Thus, (1.8.1.49) shows (1.8.1.33), as desired, while (1.8.1.34) follows from (1.8.1.47), as well.

(iv) The validity of (1.8.1.38) is a consequence of (1.8.1.3), (1.8.1.29) and (1.2.1.8) for $\bar{P}(t)$, guaranteed by Theorem 1.2.1.1. Next, (1.8.1.39) follows from $\bar{u}^0(t, 0; y_0) = -\bar{B}^*\bar{P}\bar{y}^0(t, 0; y_0)$, (see (1.2.1.10)), and use of (1.8.1.23), (1.8.1.24), and (1.8.1.38).

(v) The operator $\bar{P}(t)$ satisfies, by Theorem 1.2.1.1(vii), the DRE

$$\begin{aligned}
(\dot{\bar{P}}(t)x_1, y_1)_Y &= -(\bar{R}^*\bar{R}x_1, y_1)_Y - (\bar{P}(t)x_1, Ay_1)_Y \\
&\quad - (\bar{P}(t)Ax_1, y_1)_Y - (\bar{B}^*\bar{P}(t)x_1, \bar{B}^*\bar{P}(t)y_1)_U, \qquad (1.8.1.51)
\end{aligned}$$

for $0 \le t < T$ and $x_1, y_1 \in \mathcal{D}((-A)^\epsilon)$, $\forall \epsilon > 0$.

Recalling (1.8.1.29), we then obtain the desired DRE for $P(t)$ from (1.8.1.51) with $x = (-A)^{-\delta}x_1 \in \mathcal{D}((-A)^{\delta+\epsilon})$ and $y = (-A)^{-\delta}y_1 \in \mathcal{D}((-A)^{\delta+\epsilon})$. We point out explicitly that: (a) The constant term $(Rx, Ry)_Z$ in (1.8.1.40) is a fortiori well defined by assumption (1.8.1.1); (b) the quadratic (last) term in (1.8.1.40) is well defined (a fortiori) with $x, y \in \mathcal{D}((-A)^{\delta+\epsilon})$ by (1.8.1.39); (c) finally, the linear terms in (1.8.1.40) are also well defined; e.g., for $x \in \mathcal{D}((-A)^\delta)$ and $y \in \mathcal{D}((-A)^{\delta+\epsilon})$, we have, via (1.8.1.28),

$$\begin{aligned}
(P(t)x, Ay)_Y &= -\left((-A^*)^{1-(\delta+\epsilon)}P(t)(-A)^{-\delta}(-A)^\delta x, (-A)^{\delta+\epsilon}y\right)_Y \\
&\in C[0, T), \qquad (1.8.1.52)
\end{aligned}$$

a fortiori by (1.8.1.37).

(vi) The validity of (1.8.1.41) follows from (1.8.1.29); (1.2.1.23) applied to $\bar{P}(t)$ as guaranteed by Theorem 1.2.1.1(xi); as well as (1.8.1.1).

(vii) The validity of (1.8.1.42) follows from (1.8.1.1) and (1.8.1.29). Equation (1.8.1.43) is a restatement of (1.8.1.24a), initiating at s, rather than zero, via (1.8.1.44).

For the statement that A_P generates the evolution operator $\Phi(t, s)$, we use the same statement of (1.4.6.18) in Lemma 1.4.6.3 that $[A - \bar{B}\bar{B}^*\bar{P}(t)]$ generates the evolution operator $\bar{\Phi}(t, s)$. Theorem 1.8.1.1 is proved. $\square$

We conclude this subsection by explicitly pointing out the corresponding extensions of Theorems 1.2.2.1 and 1.2.2.2, under the relaxed assumptions (1.8.1.1) for R and

G. More precisely,

(a) Let (recall (1.8.1.1))

$$\begin{cases} (-A^*)^{\beta-\delta}G^*G(-A)^{-\delta} = (-A^*)^{\beta}\bar{G}^*\bar{G} \in \mathcal{L}(Y) \\ \text{for } \beta > 2\bar{\gamma} - 1, \quad \frac{1}{2} \le \bar{\gamma} < 1, \end{cases} \tag{1.8.1.53}$$

be the counterpart of (1.2.2.1). Then, the optimal pair is more regular:

$$u^0(\,\cdot\,, 0; y_0) = \bar{u}^0(\,\cdot\,, 0; (-A)^{\delta}y_0) \in C_{\bar{\gamma}-\beta}([0, T]; \mathcal{D}((-A)^{\delta}), \quad y_0 \in \mathcal{D}((-A)^{\delta}) \tag{1.8.1.54}$$

(recall (1.8.1.23)),

$$y^0(\,\cdot\,, 0; y_0) = (-A)^{-\delta}\bar{y}^0(t, 0; (-A)^{\delta}y_0) \in C([0, T]; \mathcal{D}((-A)^{\delta}), \quad y_0 \in \mathcal{D}((-A)^{\delta}) \tag{1.8.1.55}$$

(recall (1.8.1.24)) by (1.2.2.2) of Theorem 1.2.2.1, applied to $\bar{u}^0$ and $\bar{y}^0$. Moreover, by (1.2.2.6) of Theorem 1.2.2.1,

$$B^*P(t)(-A)^{-\delta}x = \bar{B}^*\bar{P}(t)x \in C_{\bar{\gamma}-\beta}([0, T]; U]), \quad x \in Y. \tag{1.8.1.56}$$

Finally, $P(t)$ is the unique solution, within the class of solutions that are nonnegative, self-adjoint on Y, and that satisfy the regularity property (1.8.1.51) of the DRE (1.8.1.38); indeed, as a classical solution [in the sense of (1.2.2.14) of Theorem 1.2.2.1] on the space $\mathcal{D}((-A)^{\delta})$.

(b) Let (recall (1.8.1.1))

$$(-A^*)^{\bar{\gamma}-\delta}G^*G(-A)^{-\delta} = (-A^*)^{\bar{\gamma}}\bar{G}^*\bar{G} \in \mathcal{L}(Y). \tag{1.8.1.57}$$

Then, recalling (1.8.1.23),

$$u^0(\,\cdot\,, 0, y_0) = \bar{u}^0(\,\cdot\,, 0; (-A)^{\delta}y_0) \in C([0, T]; \mathcal{D}((-A)^{\delta})), \quad y_0 \in \mathcal{D}((-A)^{\delta}), \tag{1.8.1.58}$$

by (1.2.2.17) of Theorem 1.2.2.2. Moreover,

$$B^*P(t)(-A)^{-\delta}x = \bar{B}^*\bar{P}(t)x \in C([0, T]; U), \quad x \in Y, \tag{1.8.1.59}$$

by (1.2.2.19) of Theorem 1.2.2.2.

Further results and details are omitted.

1.8.2 The Case Where $G \in \mathcal{L}(\mathcal{D}((-A)^{\rho}); Z_f)$, for any $\rho > 0$

The variational approach of the present chapter can be readily extended to cover the case where G has an unlimited degree of unboundedness with respect to A, more precisely, to the case where G satisfies the requirement that

$$G \in \mathcal{L}(\mathcal{D}((-A)^{\rho}); Z_f), \quad \text{or } G(-A)^{-\rho} \in \mathcal{L}(Y; Z_f) \quad \text{for any } \rho > 0. \tag{1.8.2.1}$$

Recalling the explicit formula (1.2.1.1) for the optimal control $u^0(t, 0; x)$, we write, accordingly,

$$\{L_T^* G^* G e^{AT} x\}(t) = B^*(-A^*)^{-\gamma}(-A^*)^{\gamma} e^{A^*(T-t)}(-A^*)^{\rho} G^* G(-A)^{-\rho}(-A)^{\rho} e^{AT} x,$$

with $(-A)^{\rho} e^{AT} x \in Y$ for $x \in Y$, whereby the original γ deteriorates to $\gamma + \rho$ in terms of singularity. This issue will not be pursued further here and is left to the reader.

1A Proof of Lemma 1.5.1.1(iii)

In this appendix we prove the specialized part (iii) of Lemma 1.5.1.1.

Lemma 1.A.1 *Let $\mathcal{A}$ be a self-adjoint operator on the Hilbert space $\mathcal{Y}$, which is the generator of a strongly continuous, self-adjoint, analytic semigroup $e^{\mathcal{A}t}$ of negative type on $\mathcal{Y}$. Let $\mathcal{G} \in \mathcal{L}(\mathcal{Y})$ be a bounded, self-adjoint operator on $\mathcal{Y}$ such that $\mathcal{A}\mathcal{G} \in \mathcal{L}(\mathcal{Y})$. Then*

$$(-\mathcal{A})^{\frac{1}{2}} \mathcal{G} (-\mathcal{A})^{\frac{1}{2}} \in \mathcal{L}(\mathcal{Y}); \tag{1A.1a}$$

more precisely,

$$\left\| (-\mathcal{A})^{\frac{1}{2}} \mathcal{G} (-\mathcal{A})^{\frac{1}{2}} \right\|_{\mathcal{L}(\mathcal{Y})} \leq \|\mathcal{A}\mathcal{G}\|_{\mathcal{L}(\mathcal{Y})}. \tag{1A.1b}$$

Proof.

Step 1 First we notice that

$$\int_0^\infty \left\| (-\mathcal{A})^{\frac{1}{2}} e^{\mathcal{A}t} x \right\|_{\mathcal{Y}}^2 dt = \frac{1}{2} \|x\|_{\mathcal{Y}}^2, \quad x \in \mathcal{Y}. \tag{1A.2}$$

In fact, since $\mathcal{A}$ is self-adjoint,

$$\begin{aligned} \int_0^\infty \left\| (-\mathcal{A})^{\frac{1}{2}} e^{\mathcal{A}t} x \right\|_{\mathcal{Y}}^2 dt &= \int_0^\infty (-\mathcal{A} e^{\mathcal{A}t} x, e^{\mathcal{A}t} x)_{\mathcal{Y}}\, dt \\ &= -\frac{1}{2} \int_0^\infty \frac{d}{dt} (e^{\mathcal{A}t} x, e^{\mathcal{A}t} x)_{\mathcal{Y}}\, dt \\ &= -\frac{1}{2} \left[\|e^{\mathcal{A}t} x\|_{\mathcal{Y}}^2 \right]_{t=0}^{t=\infty} = \frac{1}{2} \|x\|_{\mathcal{Y}}^2, \end{aligned} \tag{1A.3}$$

as desired, where in the last step we have used that $e^{\mathcal{A}t}$ is of negative type.

Step 2 The operator

$$\mathcal{S} = (-\mathcal{A})\mathcal{G} + \mathcal{G}(-\mathcal{A}) \in \mathcal{L}(\mathcal{Y}) \tag{1A.4}$$

is bounded and self-adjoint on $\mathcal{Y}$. Applying $(-\mathcal{A})^{1/2}$ to the left and to the right of $\mathcal{G}$ given by Eqn. (1.5.1.2) we obtain via (1A.4)

$$\int_0^\infty (-\mathcal{A})^{\frac{1}{2}} e^{\mathcal{A}t} \mathcal{S} e^{\mathcal{A}t} (-\mathcal{A})^{\frac{1}{2}} x\, dt = (-\mathcal{A})^{\frac{1}{2}} \mathcal{G} (-\mathcal{A})^{\frac{1}{2}} x \in \mathcal{Y}, \quad x \in \mathcal{Y}, \tag{1A.5}$$

which is well defined in $\mathcal{Y}$. In fact, for $x \in \mathcal{D}((-\mathcal{A})^{1/2}))$ we compute via (1A.5)

$$\begin{aligned}\left|\left((-\mathcal{A})^{\frac{1}{2}}\mathcal{G}(-\mathcal{A})^{\frac{1}{2}}x, x\right)_{\mathcal{Y}}\right| &= \left|\int_0^\infty \left(\mathcal{S}e^{\mathcal{A}t}(-\mathcal{A})^{\frac{1}{2}}x, e^{\mathcal{A}t}(-\mathcal{A})^{\frac{1}{2}}x\right)_{\mathcal{Y}} dt\right| \\ &\le \|\mathcal{S}\| \int_0^\infty \left\|e^{\mathcal{A}t}(-\mathcal{A})^{\frac{1}{2}}x\right\|_{\mathcal{Y}}^2 dt \\ &= \frac{1}{2}\|\mathcal{S}\|\|x\|_{\mathcal{Y}}^2, \end{aligned} \tag{1A.6}$$

where in the last step we have invoked (1A.2). Then (1A.6) is extended to all $x \in \mathcal{Y}$, and (1A.5) follows. Indeed, for the self-adjoint operator $(-\mathcal{A})^{\frac{1}{2}}\mathcal{G}(-\mathcal{A})^{\frac{1}{2}}$, estimate (1A.6) – extended to all $x \in \mathcal{Y}$ – shows (1A.1b), since $\|\mathcal{S}\| = 2\|\mathcal{A}\mathcal{G}\|$ from (1A.4). $\square$

Notes on Chapter 1

Sections 1.1 Through 1.6: The Variational Versus the Direct Method. Case $G \neq 0$

By *variational approach* one means the approach that proceeds from the optimal control problem (hence, from the corresponding optimality conditions) to the corresponding differential Riccati equation (DRE). The converse approach, which proceeds from the well-posedness of the DRE to the optimal control problem via dynamic programming is usually referred to as the *direct approach*; see Da Prato [1973], Temam [1971], Tartar [1974], and J. L. Lions [1971, p. 153]. In the direct approach, well-posedness of the DRE is established by a local contraction argument combined with global a priori bounds. Thus, the direct approach – when successful – provides *existence* and *uniqueness* of the solution of the DRE (or of the corresponding integral Riccati equation), within a certain natural class.

Variational Approach

The variational approach of Sections 1.1 through 1.6 follows closely that of Lasiecka and Triggiani [1983; 1992], as augmented by Lasiecka and Triggiani [1993]. In the generality of the sharp Theorem 1.2.1.1, it provides existence, but does not claim uniqueness, of the DRE (or of the IRE). The general case of Theorem 1.2.1.1 given in Lasiecka and Triggiani [1992] with GL_T closed (closable) is, in turn, an improved extension of methods and results of Lasiecka and Triggiani [1983]. This latter reference treated, by abstract operator methods such as those of the present chapter, the "concrete" case of parabolic equations with Dirichlet boundary control on $Y = L_2(\Omega)$ – in which the parameter $\gamma = 3/4 + \epsilon, \epsilon > 0$ – in the canonical situation of a non-smoothing final state penalization $G = k^2 I$, k real. Moreover, Lasiecka and Triggiani [1992] incorporated the spaces $C_r([0, T]; \cdot)$, previously introduced by Da Prato and Ichikawa [1985], to measure quantitatively the singularities occurring at $t = T$ of the various control quantities, due to the presence of $G \neq 0$. Finally, the following topics are taken directly from Lasiecka and Triggiani [1993]: (i) the pointwise regularity of the time derivatives of $u^0(t, s; x)$ and of $y^0(t, s; x)$ in $0 < t < T$ in Section 1.4.5; (ii) the corresponding Theorem 1.5.2.2; and (iii) the statements and proofs on the classical version of the DRE in Theorem 1.5.3.1 and in Corollary 1.5.3.2, both in the first smoothing case; and (iv) uniqueness of the DRE, Theorem 1.5.3.3.

We also note that Delfour and Sorine [1982] treat the case of parabolic equations with Dirichlet control, by making a change of variable $\bar{y}(t) = A^{-1/2}y(t)$ to fall into the treatment of J. L. Lions [1971, Chapter 3]. This way, however, requires a strong assumption on the final state penalization ($\|y(T)\|_{H^{-1}(\Omega)}$), rather than $\|y(T)\|_{L_2(\Omega)}$ as in the present treatment.

Direct Approach

The first smoothing case of Section 1.5 under assumption (1.5.1.1) for G was previously studied by Da Prato and Ichikawa [1985] using the direct method. This work, in turn, extended the original work of Flandoli [1984], also based on the direct method, which referred to the second smoothing case of Section 1.6, under assumption (1.6.1) for G. For the smoothing cases of Sections 1.5 and 1.6, both variational and direct approaches also assert the *uniqueness* of the DRE.

The Work of F. Flandoli

(Flandoli [1993]; see also Flandoli [1989].) Independent from, and contemporaneous to, the work of Lasiecka and Triggiani [1992] is the work [Flandoli, 1993] of Flandoli, reported also in book form in Bensoussan et al. [1992, Vol. 2, Part II, Chapter 2]. We concentrate (for the time being) to the case of observation operators R and G both bounded to describe it. This work may be divided in two parts.

In the first part, Flandoli extends the *direct approach*, thereby obtaining existence and uniqueness of the DRE (as well as of the IRE), under the following symmetric regularity assumption (S.R.A.) on G:

$$(-A^*)^{\beta/2}G^* \in \mathcal{L}(Z_f; Y); \quad \text{equivalently } (-A^*)^{\beta/2}G^*G(-A)^{\beta/2} \in \mathcal{L}(Y), \tag{1N.1}$$

for $2\gamma - 1 < \beta < 1$, which is the same as (1.2.1.26). This condition should be compared with assumption (5.1.1): $(-A^*)^{\beta}G^*G \in \mathcal{L}(Y)$, $\beta > 2\gamma - 1$, $1/2 \leq \gamma < 1$,of the first smoothing case in Section 1.5. First, if $(-A^*)^{\beta}G^*G \in \mathcal{L}(Y)$, then Corollary 1.5.1.2 – which, along with Lemma 1.5.1.1 (i), (ii), is due to Flandoli [1984] – implies that $(-A^*)^{\beta/2-\epsilon}G^*G(-A)^{\beta/2-\epsilon} \in \mathcal{L}(Y))$ [with $\epsilon = 0$, if A is self-adjoint, by Lemma 1.5.1.1(iii)] and the S.R.A. in (1N.1) holds true. The converse is false. Thus, Flandoli [1993] slightly relaxes the *direct* treatment of Da Prato and Ichikawa [1985] and the *variational* treatment of Lasiecka and Triggiani [1992], here reported in Section 1.5, under which existence and uniqueness of the DRE is asserted. (To date, we have not investigated the variational treatment under the S.R.A. in (1N.1).) However, as pointed out in Section 1.7.2, assumption (1.2.1.26) (i.e., (1N.1)) is only sufficient for GL_T to be closed.

A second justification for adopting the S.R.A., is that the S.R.A. is the minimal assumption, at least in terms of fractional powers of $(-A)$, to achieve a cost functional (1.1.3) that depends continuously on the control u in the $L_2(0, T; U)$-topology.

Indeed, we first observe that, setting $\beta = 2\gamma - 1 + 2\epsilon$, we have that the solution of (1.1.1) satisfies

$$\begin{aligned}(-A)^{-\beta/2}y(t) &= e^{At}(-A)^{-\beta/2}y_0 + (-A)^{\frac{1}{2}-\epsilon}\int_0^t e^{A(t-\tau)}(-A)^{-\gamma}Bu(\tau)\,d\tau \\ &\in C([0,T]; Y), \end{aligned} \tag{1N.2}$$

for

$$y_0 \in Y\big(\text{even } y_0 \in \big[D(-A^*)^{\beta/2}\big]'\big) \quad \text{and} \quad u \in L_2(0, T; U), \tag{1N.3}$$

as it follows from the standard regularity result in (0.5) in Chapter 0. As a consequence of (1N.1) and (1N.2), we then find that

$$Gy(T) = G(-A)^{\beta/2}(-A)^{-\beta/2}y(T) \in Z_f \tag{1N.4}$$

for all data as in (1N.3). Thus, the cost J in (1.1.3) is well defined and finite for data as in (1N.3), and it depends continuously on these, as claimed.

In the second part of his paper [Flandoli, 1993], Flandoli extends the *existence* part (not the uniqueness) of the DRE in the nonsmoothing case, by an approximating argument from below, as follows. Instead of assuming that GL_T be closed (closable) – a natural hypothesis on G in the variational approach of Lasiecka and Triggiani [1983; 1992], followed in the present chapter – Flandoli [1993] makes the following assumptions on G, say in the case $G \in \mathcal{L}(Y; Z_f)$:

There exists a sequence $G_n \in \mathcal{L}(Y; Z_f)$ of operators such that:

(a) There exists a common $\beta > 2\gamma - 1$, such that each G_n satisfies the symmetric regularity property (1N.1) (same as (1.2.1.26))

$$G_n(-A)^{\beta/2} \in \mathcal{L}(Y; Z_f), \quad n = 1, 2, \ldots. \tag{1N.5}$$

(b) $\{G_n^*G_n\}$ is a nondecreasing family of self-adjoint operators that converges monotonically to G^*G in the sense that

$$(G_n^*G_nx, x)_Y = \|G_nx\|_{Z_f} \uparrow (G^*Gx, x)_Y = \|Gx\|^2_{Z_f}, \quad \forall x \in Y. \tag{1N.6}$$

Under these assumptions, Flandoli [1993, Theorem 1.3.2] shows *existence* of a solution $P(t)$ of the DRE (1.2.1.13) [except with $d/dt(P(t)x, y)_Y$ on its left-hand side], which satisfies the regularity properties (1.2.1.5), (1.2.1.7a), among others, as in Theorem 1.2.1.1. In addition, Flandoli [1993] also obtains the strong convergence of $P(t) \to G^*G$ as $t \uparrow T$ (as in (1.4.7.3) of Proposition 1.4.7.2) – because of the postulated monotonic approximation property (b) above.

Flandoli's framework in [Flandoli, 1993] includes also the case of the observation $G = k^2 I$, k real for the heat equation with Dirichlet control, previously contained in Lasiecka and Triggiani [1983]. We have seen in Section 1.7.2 that Flandoli's assumptions, in particular (1N.5) in (a) – which do not involve the operator B, only A and G – are *stronger* than the assumption that GL_T be closed of the variational approach of Lasiecka and Triggiani [1983; 1992] given in the present chapter, which instead involves also B. As discussed in Section 1.7.1, the assumption that GL_T be closed is *sharp*: If omitted, the optimal control may not exist.

We finally point out that a preliminary study of parabolic equations with Dirichlet boundary control and $G = 0$ is contained in Balakrishnan [1977].

Regularity Properties

Regularity properties of the optimal control u^0 over all of $[0, T]$ were first given in Lasiecka [1978; 1980(b), Section 6] in the case of $G = 0$ for parabolic equations with Dirichlet boundary control ($\gamma = 3/4 + \epsilon$) by abstract operator methods. Her results are:

(i) $u^0 \in H^{2-\epsilon, 1-\epsilon}(\Gamma \times [0, T]) \cap C^\infty(\Gamma \times \{0 < t < T\})$, $\epsilon > 0$, in the case of smooth boundary Γ;
(ii) $u^0 \in H^{1-2\epsilon, 1/2-\epsilon}(\Gamma \times [0, T])$ in the case of conical domains as in Kondratiev [1967].

Later, still in the case of parabolic equations with Dirichlet boundary control and, this time, $G =$ Identity, Seidman [1982] proved that, in the case of a smooth boundary Γ, the optimal control u^0 is an $H^s(\Gamma)$-analytic function on $(0, T)$, for any $s \geq 0$. This result is reproved in Lasiecka and Triggiani [1983] with an operator approach such as the one in the present chapter (see Remark 1.4.8.3 and, particularly, Theorem 1.4.8.6) and with some technical simplifications over the PDE approach of Seidman [1982]. For example, the operator analysis of Section 1.4.8 reveals the perturbation of the identity factorized as $L^* R^* R L$; thus in the inversion result of Corollary 1.4.8.5, Eqn. (1.4.8.22), only the compactness of L is needed (Lemma 1.4.8.4, Eqn. (1.4.8.17)).

Section 1.7

The parabolic example of Section 1.7.1 was introduced by Flandoli [1990] where it was argued that it is *not possible* for the corresponding optimal problem (1.1.3), (1.1.5), (1.7.1.3) to satisfy the following three desirable, classical properties:

(i) that there exists a unique optimal control u^0;
(ii) that there exists $P(t), 0 \leq t \leq T$, nonnegative, self-adjoint, such that identity (1.2.1.11) holds;
(iii) that for every $0 \leq t < T$ and $x \in D(A)$, $(P(t)x, x)$ is differentiable, $P(t)x \in D((-A^*)^\gamma)$, and the DRE (1.2.1.12) is satisfied.

This example was reanalyzed in Lasiecka and Triggiani [1992] where it was shown, more fundamentally, that (i) the optimal control does not exist in this case through the argument reported in Section 1.7.1, with no need to involve the DRE, and, moreover, (ii) that the operator GL_T is not closable. This makes Theorem 1.2.1.2 sharp, at least regarding the existence of the DRE.

This very same example (in the abstract version of Eqn. (1.7.1.3)) has been later reused in the book by Li and Yong [1995, p. 373], to illustrate that the operator Λ_{0T} in (2.1.3) is (strictly) coercive, but not boundedly invertible (our Claim #2).

The classes of examples of Section 1.7.2 are taken from Lasiecka and Triggiani [1992].

Section 1.8

The paper by Da Prato and Ichikawa [1985] treats also the case of unbounded R, under hypothesis (1.5.1.1). It was noted in Flandoli [1993] and Lasiecka and Triggiani [1992, p. 455; Remark 7.3, p. 480] that each respective method – the direct and approximating approach of Flandoli recalled above and the variational approach of Lasiecka and Triggiani of the present chapter – can also cover the cases where the observation operator R and the final state operator G are unbounded. More precisely, they cover cases where R has a controlled degree of unboundedness as in (1.8.1.1), and G has a degree of unboundedness of any order with respect to A, as in (1.8.2.1). The details of the variational approach of Section 1.8.1.1 are given here for the first time. A corresponding treatment in the direct/approximating approach is given in Flandoli [1993, Section 2.3 and Section 3.1] and in Da Prato and Ichikawa under assumption (1.5.1.1). See also the book by Bensoussan et al. [1992].

The Nonautonomous Case: The Work of Acquistapace and Terreni [1999; to appear]

Very recently, the abstract variational approach of the present chapter, extensively developed in Lasiecka and Triggiani [1983; 1992; 1993], has served as a close guide for the extension in Acquistapace and Terreni [1999; to appear] to the *nonautonomous* abstract parabolic case: Here, the operators entering into the description of the optimal control problem – the operators A, B, R, G, and the strictly coervice operator of penalization of the control function – are all *time dependent*. Indeed, the work of Acquistapace and Terreni [1999; to appear] contains some refinements over the autonomous treatment of the present chapter. Notably:

(i) The operator $P(t)$ explicitly constructed from the optimality conditions (see (1.2.1.11)) satisfies the differential Riccati equation as a *classical solution*, in the sense of Eqn. (1.2.2.14), in the full generality of Theorem 1.2.1.1, that is under the sole assumption that GL_T be closed, thereby, dispensing with assumption (1.2.2.1) in Theorem 1.2.2.1, (vii) of this chapter.

(ii) The regularity of the optimal pair $\{u^0(\,\cdot\,, s; x), y^0(\,\cdot\,, s; x)\}$ is described by spaces Z_{r_1,r_2} (previously introduced by Acquistapace and Terreni in Acquistapace and Terreni [1987] in the study of nonautonomous abstract parabolic equations), whose first subindex r_1 measures the "blowing up" at $t = T$ (as the space $C_{r_1}([0, T];\ \cdot\ )$ of the present chapter), while the second subindex r_2 describes local r_2-Holder continuity in [s, T) (while our space $C_{r_1}([0, T];\ \cdot\ )$ includes only continuity in [s, T)). "Morally," $Z_{\gamma,0}$ coincides with C_γ. The regularity results in Acquistapace and Terreni [1999] on the optimal pair are of maximal regularity (in the sense of abstract parabolic equations): As in our Theorem 1.4.4.6 of the present chapter, Acquistapace and Terreni reobtain $u^0(\,\cdot\,, s; x) \in C_\gamma([s, T]; U)$ and $y^0(\,\cdot\,, s; x) \in C_{2\gamma-1}([s, T]; Y)$, that is, *the same order of blowing-up at* $t = T$; but, in addition, they show that $u^0(\,\cdot\,, s; x)$ is *locally δ-Holder continuous in [s, T)* (where the exponent $\delta > 1 - \gamma$ depends on the regularity in t of the operators $A(t)$), while $y^0(\,\cdot\,, \epsilon; x)$ is locally r_2-Holder continuous in (s, T), with $r_2 = 1 - \gamma$.

However, there is no counterpart in Acquistapace and Terreni [1999] of the regularity results of the time-derivative $du^0(t, s; x)/dt$ and $dy^0(t, s; x)/dt$ of our Section 1.4.5 nor of the analyticity regularity of our Section 1.4.8, which is a phenomenon of the autonomous case. Papers Acquistapace and Terreni [1999] and Acquistapace and Terreni [to appear], closely based on the variational approach of this chapter, supersede these authors' previous effort [Acquistapace et al., 1991] (with F. Flandoli) and Acquistapace and Terreni [1996], where, instead, a direct approach was followed. Acquistapace and Terreni [to appear] also includes uniqueness in the nonautonomous case of the differential Riccati equation under the same smoothing hypothesis (1.2.2.1) as Theorem 1.2.2.1 (or Theorem 1.5.3.3) of the present chapter, through a derivation that is likewise based on the arguments of Lasiecka and Triggiani [1992], given in Section 1.5.3.2.

Glossary of Symbols for Chapter 1

A, B, γ	(1.1.1), (1.1.2)
J, R, G	(1.1.3)
L, L^*, L_T, L_T^*	(1.1.7)–(1.1.10)
$u^0(t, 0; y_0), y^0(t, 0; y_0), \Lambda_{0T}$	(1.2.1.1)–(1.2.1.3)
$P(t), J^0(t; x), \Phi(t, \tau)$	(1.2.1.11), (1.2.1.12), (1.2.1.22)
$\beta, \Gamma(P(t))$	(1.2.2.1), (1.2.2.11)
$L_s, L_s^*, L_{sT}, L_{sT}^*, \Lambda_{sT}$	(1.4.1.1), (1.4.1.4), (1.4.1.6), (1.4.1.7), (1.4.1.11)
$V(s, T; U)$	(1.4.1.8)
$C_r([0, T]; \cdot), {}_rC_r([0, T]; \cdot), {}_{r_1}C_{r_2}([0, T]; \cdot)$	(1.4.4.1), (1.4.5.1), (1.4.5.2)

$\rho(t,x)$, $L_{[s,T-\epsilon]}$, $w_\epsilon(t)$	(1.4.5.17a), (1.4.5.26), (1.4.5.30)
$v_\epsilon(t;x)$, $w_0(t;x)$	(1.4.5.56), (1.4.5.65)
$A_P(t) = A - BB^*P(t)$	(1.4.6.1)
$\mathcal{M}_t$, $\mathcal{M}_{ts}$	(1.4.6.29)
$\mathcal{F}$, $\mathcal{A}(\mathcal{F};U)$, $\mathcal{A}(\mathcal{F};Y)$	(1.4.8.1), (1.4.8.2)
$\mathcal{L}_\delta$, D	(1.4.8.18), (1.4.8.35)
$\mathcal{A}$, $\mathcal{G}$	Lemma 1.1.5.1
$\Phi_{P(t)}$, $V(t)$, $M(\tau)$	(1.5.3.1), (1.5.3.28), (1.5.3.33)
ζ_i, Π_i	above and below (1.7.2.1)

References and Bibliography

P. Acquistapace, F. Flandoli, and B. Terreni, Initial boundary value problems and optimal control for non-autonomous parabolic systems, *SIAM J. Control Optim.* **29** (1991), 89–118.

P. Acquistapace and B. Terreni, Infinite horizon LQR problems for non-autonomous parabolic systems with boundary control, *SIAM J. Control Optim.* **34** (1996), 1–30.

P. Acquistapace and B. Terreni, Classical Solutions of non-autonomous Riccati equations arising in parabolic boundary control problems, Part I, *Appl. Math. Optim.* Vol. 39, N3, May/June 1999, p. 361–410.

P. Acquistapace and B. Terreni, Classical Solutions of non-autonomous Riccati equations arising in parabolic boundary control problems, Part II, *Appl. Math. Optim.*, to appear.

P. Acquistapace and B. Terreni, A unified approach to abstract linear non-autonomous parabolic equations, *Remd. Sem. Matem. Universitá di Padova*, **78** (1987), 47–107.

A. V. Balakrishnan, *Applied Functional Analysis,* Springer-Verlag, 1981.

A. V. Balakrishnan, Boundary control of parabolic equations. L-Q-R theory. Proc. V. International Summer School, Control Inst. Math. Mech. Acad. Sci. GDR, Berlin (1977).

A. Bensoussan, G. Da Prato, M. Delfour, and S. Mitter, *Representation and Control of Infinite Dimensional Systems*, Vol. II, Birkhäuser, 1992.

G. Da Prato, Quelques resultats d'existence, unicité et regularité pour un probleme de la theorie du control, *J. Math. Pures Appl.*, 1973. Vol. 52 pp. 353–375.

G. Da Prato and A. Ichikawa, Riccati equations with unbounded coefficients, *Ann. Math. Pura Appl.* **140** (1985), 209–221.

M. C. Delfour and M. Sorine, The linear quadratic optimal control problem for parabolic systems with boundary control through a Dirichlet boundary condition, in *Control of Distributed Parameter Systems*, J. P. Barbary and L. Le Letty, eds., Chap. I, pp. 13–16, Pergamon Press, Oxford, England, 1982. (Proc. 3rd IFAC Symp. on Control of Distributed Parameter Systems, Toulouse, France, June 29–July 2, 1982; see also CRMA report no. 1051, Université de Montréal, 1981).

I. Ekeland and R. Teman, *Convex Analysis and Variational Problems*, North-Holland, 1976.

F. Flandoli, Riccati equations arising in a boundary control problem, with distributed parameters, *SIAM J. Control Optim.* **22** (1984), 76–86.

F. Flandoli, A new proof of an a priori estimate arising in boundary control theory, *Appl. Math. Lett.* **2**(4) (1989), 341–343.

F. Flandoli, A counterexample in the boundary control of parabolic systems, *Appl. Math. Lett.* **3** (1990), 47–50.

F. Flandoli, On the direct solutions of Riccati equations arising in boundary control theory, *Annali di Matematica Pura e Applicata* (iv), **CLXIII** (1993), 93–131.

A. Friedman, *Partial Differential Equations*, Robert Krieger Publ. Co. 1976.

E. Hille and R. Phyllips, *Functional Analysis and Semigroups, Amer. Math. Soc. Collog. Publ.*, Vol. 31, Providence R.I. (1957).

T. Kato, *Perturbation Theory of Linear Operators*, Springer-Verlag, 1966.

V. A. Kondratiev, Boundary problems for elliptic equations in domains with conical or angular points, *Trans. Moscow Math. Soc.* **16** (1967).

S. G. Krein, *Linear Differential Equations in Banach Space*, Vol. 29, Translations of Mathematical Monographs, Amer. Math. Soc., 1971.

I. Lasiecka, Boundary control of parabolic systems: Regularity of optimal solutions, *Appl. Math. Optim.* **4** (1978), 301–327.

I. Lasiecka, Boundary control of parabolic systems: Finite element approximation, *Appl. Math. Optim.* **6** (1980)(a), 31–62.

I. Lasiecka, Unified theory for abstract parabolic boundary problems: A semigroup approach, *Appl. Math. Optim.* **6** (1980)(b), 287–333.

I. Lasiecka and R. Triggiani, Dirichlet boundary control problem for parabolic equations with quadratic cost: Analyticity and Riccati's feedback synthesis, *SIAM J. Control Optim.* **21**(1) (1983), 41–67.

I. Lasiecka and R. Triggiani, Riccati Differential equations with unbounded coefficients and non-smoothing terminal condition. The case of analytic semigroups, *SIAM J. Math. Anal.* **23** (1992), 449–481.

I. Lasiecka and R. Triggiani, *Differential and Algebraic Riccati Equations with Applications to Boundary/Point Control Problems: Continuous Theory and Approximation Theory*, Springer-Verlag Lecture Notes, LNCIS, Vol. 164, 1991(a), pp. 160.

I. Lasiecka and R. Triggiani, Further regularity properties in quadratic cost problems for parabolic equations with boundary control and non-smoothing finite state penalization, Lecture Notes in Pure and Applied Mathematics, Vol. 160, Optimal control of differential equations, N. H. Pavel (ed.), A Festschrift in honor of C. Cordumeanu, Conference held at the Department of Mathematics, Ohio University, Athens, Ohio, March 25–27, 1993.

J. L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, 1971.

X. Li and J. Yong, *Optimal Control Theory for Infinite Dimensional Systems*, Birkhäuser, 1995.

D. G. Luenberger, *Optimization by Vector Space Methods*, Wiley, 1969.

A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, 1983.

H. L. Royden, *Real Analysis*, MacMillan, 2nd ed., 1968.

C. Sadosky, *Interpolation of Operators and Singular Integrals*, Marcel Dekker, Monographs in Pure and Applied Mathematics, 1979.

T. Seidman, Regularity of optimal boundary control for parabolic equations: Semigroup methods and analyticity, *SIAM J. Contr. Optim.* **20** (1982).

L. Tartar, Sur l'etude directe d'equations non lineaires interenant en theorie du controle optimal, *J. Funct. Anal.* **6** (1974), 1–47.

R. Temam, Sur l'equation de Riccati associée á des opérateurs non bornés en dimension infinie, *J. Funct. Anal.* **7** (1971), 85–116.

J. L. Lions, E. Magenes, *Non-homogeneous Boundary Value Problems* Vols. I, II, (1972), Springer-Verglag.

2

Optimal Quadratic Cost Problem over an Infinite Time Interval: Algebraic Riccati Equation

This chapter studies the optimal quadratic cost problem over the infinite time horizon, $T = \infty$: $0 \le t \le \infty$, for the same abstract dynamics $\dot{y} = Ay + Bu$ of the preceding chapter. Thus, the free dynamics operator A generates a s.c. analytic semigroup on the Hilbert space Y, which now, however, is generally unstable, while the control operator B has, as before, a degree of unboundedness up to that of A. The treatment of this chapter relies, as a starting point, on the theory of the preceding Chapter 1 for the finite time ($T < \infty$) optimal control problem, in the simplified version where the finite state penalization is absent; that is, $G = 0$ in [Chapter 1, Eqn. (1.1.3)]. Due to the infinite time horizon, two additional hypotheses are needed now for a complete theory: (i) the Finite Cost Condition, which guarantees existence of a unique optimal pair $\{u^0, y^0\}$, and (ii) the Detectability Condition, which guarantees two desirable features. They are: (ii_1) uniform (exponential) stability of the resulting feedback semigroup and, as a consequence, (ii_2) uniqueness of the solution operator of the corresponding algebraic Riccati equation, which arises in the synthesis of the optimal pair. The variational approach of this chapter is based on a limit process as $T \uparrow \infty$ of selected results of Chapter 1. As a consequence, one obtains: (i) the *candidate* (algebraic) Riccati operator P as well as its basic relation (which may well be considered a defining formula for P, when the original semigroup e^{At} is stable), and (ii) basic convergence results of the optimal control and optimal trajectory of the finite time problem ($T < \infty$) to the corresponding quantities of the infinite horizon problem ($T = \infty$) in the topology of the spaces $C_{ub}([0, \infty]; U)$ and $C_{ub}([0, \infty]; Y)$, respectively. This sharp convergence is the result of regularizing properties of the input-solution operator $\hat{L}$ in (2.1.17) below and its adjoint $\hat{L}^*$ in (2.1.18), along with a related bootstrap argument in the style of Chapter 1. (As in Chapter 1, it is again this bootstrap argument that produces continuity in time of the optimal control in the critical case where the parameter γ in Eqn. (1.1.7) below is $1/2 \le \gamma < 1$. [For $\gamma < 1/2$, as observed in Chapter 1, all solutions, in particular the optimal solution, are in $C([0, T]; Y)$, by virtue of the regularity result (0.11) in Chapter 0.] The optimal control defines an operator $\Phi(t)y_0 = y^0(t; y_0)$, $y_0 \in Y$, which is a s.c., analytic semigroup on Y. Moreover, the optimal synthesis $u^0(t; y_0) = -B^* P y^0(t; y_0)$ is attained,

$0 \le t < \infty$. The variational approach concludes by showing, constructively, that the *candidate* (algebraic) Riccati operator P does satisfy the algebraic Riccati equation and that, in fact, under the Detectability Condition such P is the unique, nonnegative, self-adjoint solution with the property that $(A^*)^\gamma P \in \mathcal{L}(Y)$; hence $B^*P \in \mathcal{L}(Y; U)$. In this case, the semigroup $\Phi(t)$ is also uniformly (exponentially) stable on Y. [The non-definite cost case is treated in the appendix of Chapter 6.]

2.1 Mathematical Setting and Formulation of the Problem

Dynamical Model In this chapter we consider the following abstract differential equation:

$$\dot{y} = Ay + Bu \quad \text{on, say } [\mathcal{D}(A^*)]', \quad y(0) = y_0 \in Y, \tag{2.1.1}$$

where A^* is the Y-adjoint of A, and $[\mathcal{D}(A^*)]'$ denotes the dual space of $\mathcal{D}(A^*)$ with respect to the Y-topology, so that $\|y\|_{[\mathcal{D}(A^*)]'} = \|(\lambda_0 I - A)^{-1} y\|_Y$, for any λ_0 in the resolvent set of A. The dynamics in (2.1.1) is subject to the following *assumptions* to be maintained throughout the chapter.

(H.1): A is the infinitesimal generator of a strongly continuous, analytic semigroup, denoted by e^{At}, on the Hilbert space Y. Unlike the situation in the preceding chapter where the finite time horizon was considered, now e^{At} is generally unstable on Y, that is, it satisfies

$$\omega_0 \equiv \lim_{t\to\infty} \frac{\ln\|e^{At}\|_{\mathcal{L}(Y)}}{t} > 0, \quad \text{so that} \tag{2.1.2a}$$

$$\|e^{At}\|_{\mathcal{L}(Y)} \le M e^{(\omega_0+\epsilon)t}, \quad t \ge 0, \ \forall\, \epsilon > 0, \tag{2.1.2b}$$

with M depending on $\omega_0 + \epsilon$. We shall then consider throughout the chapter the translation

$$\hat{A} = -A + \omega I, \quad \omega \text{ fixed } > \omega_0, \tag{2.1.3}$$

so that the fractional powers $\hat{A}^\theta$ of $\hat{A}$ are well defined [Pazy, 1983] and $-\hat{A}$ is the generator of a strongly continuous, analytic semigroup $e^{-\hat{A}t}$ on Y, $t \ge 0$, which satisfies [Pazy, 1983, p. 61],

$$\|e^{-\hat{A}t}\|_{\mathcal{L}(Y)} \le \hat{M} e^{-\hat{\omega}t}, \quad t \ge 0, \ \hat{\omega} = \omega - \omega_0 - \epsilon > 0, \tag{2.1.4a}$$

and more generally,

$$\|\hat{A}^\theta e^{-\hat{A}t}\|_{\mathcal{L}(Y)} \le \frac{\hat{M} e^{-\hat{\omega}t}}{t^\theta}, \quad 0 < t, \ 0 \le \theta. \tag{2.1.4b}$$

The λ version of analyticity is that the spectrum $\sigma(-\hat{A})$ of $-\hat{A}$ satisfies

$$\sigma(-\hat{A}) \subset \Sigma(-\hat{A}) \equiv \Sigma(-\hat{A}; -\hat{\omega}; \theta_0); \tag{2.1.5a}$$

$$\Sigma(-\hat{A}; -\hat{\omega}; \theta_0) = \text{(closed) triangular sector containing the axis } [-\infty, -\hat{\omega}] \text{ and delimited by the two rays } -\hat{\omega} + \rho e^{\pm i\theta_0},\ 0 \le \rho \le \infty, \text{ for some } \theta_0 \text{ with } \tfrac{\pi}{2} < \theta_0 \le \pi. \tag{2.1.5b}$$

Thus $\Sigma(-\hat{A}; -\hat{\omega}; \theta_0)$ is the complement in $\mathbb{C}$ of the set

$$\Sigma^c(-\hat{A}; -\hat{\omega}; \theta_0) = \{\lambda \in \mathbb{C} : |\arg(\lambda + \hat{\omega})| < \theta_0\}, \tag{2.1.5c}$$

and the sector $\{\lambda \in \mathbb{C} : |\arg \lambda| < \theta_0 - \pi/2\}$ is in the domain of holomorphicity of e^{Az}, $z \in \mathbb{C}$. We have

$$\|\hat{A}^\theta R(\lambda, -\hat{A})\|_{\mathcal{L}(Y)} \le \frac{c}{|\lambda + \hat{\omega}|^{1-\theta}}, \quad 0 \le \theta \le 1, \ \lambda \in \Sigma^c(-\hat{A}; -\hat{\omega}; \theta_0). \tag{2.1.6}$$

(H.2): B is, as in Chapter 1, a linear continuous operator $U = \mathcal{D}(B) \to [\mathcal{D}(A^*)]'$, where U is another Hilbert space (while B is generally unbounded as an operator $U \to Y$), such that

$$A^{-\gamma} B \in \mathcal{L}(U; Y), \quad \text{or} \quad \|A^{-\gamma} B\|_{\mathcal{L}(U;Y)} = \|B^* A^{*-\gamma}\|_{\mathcal{L}(Y;U)} \le c_\gamma \quad \text{for some fixed constant } \gamma,\ 0 \le \gamma < 1. \tag{2.1.7}$$

where $B^* \in \mathcal{L}(\mathcal{D}(A^*); U)$, is defined by $(Bu, v)_Y = (u, B^* v)_U$.

Generally, dependence on γ will not necessarily be explicitly noted in this chapter.

Optimal Control Problem With the dynamics (2.1.1), we associate the following quadratic cost functional over an infinite time horizon:

$$J(u, y) \equiv \int_0^\infty \left[\|Ry(t)\|_Z^2 + \|u(t)\|_U^2\right] dt, \tag{2.1.8}$$

where $y(t) = y(t; y_0)$ is the solution of (2.1.1) due to $u(t)$ and, moreover,

(H.3):

$$R \in \mathcal{L}(Y; Z), \tag{2.1.9}$$

with Z another Hilbert space. The corresponding optimal control problem is:

$$\text{Minimize } J(u, y) \text{ over all } u \in L_2(0, \infty; U), \text{ where } y \text{ is the solution of (2.1.1) due to } u. \tag{2.1.10}$$

Since R^*R is a nonnegative, self-adjoint, bounded operator on Y, we shall find it useful to write

$$\|Ry(t)\|_Z^2 = (R^*Ry(t), y(t))_Y = \left\|(R^*R)^{\frac{1}{2}} y(t)\right\|_Y^2, \tag{2.1.11}$$

where the square root is well defined.

Within the above abstract setting, to obtain a complete theory, we shall need two basic control-theoretic *assumptions*. They are:

$$\text{(H.4)}: \begin{cases} \textit{Finite Cost Condition:} \text{ For each } y_0 \in Y, \text{ there exists} \\ \bar{u} \in L_2(0,\infty;U) \text{ such that the corresponding solution } y \\ \text{to (2.1.1) satisfies } R\bar{y} \in L_2(0,\infty;Z) \text{ so that } J(\bar{u},\bar{y}) < \infty. \end{cases} \tag{2.1.12}$$

$$\text{(H.5)}: \begin{cases} \textit{Detectability Condition:} \text{ There exists an operator } K \in \mathcal{L}(Y) \\ \text{such that the strongly continuous, analytic semigroup } e^{(A+KR)t} \\ \text{is exponentially stable on } Y\text{: } \|e^{(A+KR)t}\|_{\mathcal{L}(Y)} \le M_K e^{-\omega_K t}, \\ t \ge 0, \text{ for some constant } \omega_K > 0. \end{cases} \tag{2.1.13}$$

[In the above definition (2.1.13), we could equally well substitute $(R^*R)^{\frac{1}{2}}$ for R, and hence replace the operator $A + KR$ with the operator $A + K(R^*R)^{\frac{1}{2}}$. This is so because of (2.1.11): under the Detectability Condition, $Ry^0(t; y_0) \in L_2(0, \infty; Z)$ [equivalently by (2.1.11) $(R^*R)^{\frac{1}{2}} y^0(t; y_0) \in L_2(0, \infty; Y)$] will imply that $y^0(t; y_0) \in L_2(0, \infty; Y)$; see Theorem 2.4.1, where $y^0(t; y_0)$ is the optimal solution with $y_0 \in Y$].

Remark 2.1.1 The Detectability Condition is automatically satisfied if $(R^*R)^{\frac{1}{2}} \ge \rho I$, $\rho > 0$; indeed in this case we simply take $K = -k^2(R^*R)^{-\frac{1}{2}} \in \mathcal{L}(Y)$ with positive constant k^2 sufficiently large, in fact, $k^2 > \omega_0$. Comments on the verifiability of the Finite Cost Condition (2.1.12) as well as of the Detectability Condition (2.1.13) are relegated to the Notes at the end of Chapter 2.

The Finite Cost Condition will guarantee existence of a unique optimal pair $\{u^0, y^0\}$ of the optimal control problem (2.1.10) for (2.1.1), via standard convex optimization theory. Instead, the Detectability Condition will guarantee (i) uniform (exponential) stability of the resulting feedback semigroup and, as a consequence, (ii) the uniqueness of the solution to the corresponding algebraic Riccati equation, which arises in the pointwise feedback synthesis of the optimal pair.

Preliminaries The solution to problem (2.1.1) is given by

$$y(t) = e^{At} y_0 + (Lu)(t), \tag{2.1.14}$$

where, by (2.1.7) and Eqn. (0.4) of Chapter 0, we have (sharp results in time were given in Chapter 1, Theorem 1.4.4.3, and will also be given in subsequent Theorem 2.3.5.1 below)

$$\begin{aligned} (Lu)(t) &= \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau \\ &: \text{ continuous } L_2(0,T;U) \to L_2(0,T;\mathcal{D}(\hat{A}^{1-\gamma})) \subset L_2(0,T;Y). \end{aligned} \tag{2.1.15}$$

Its adjoint operator L^* in the sense that $(Lu, v)_{L_2(0,T;Y)} = (u, L^*v)_{L_2(0,T;U)}$ is given by

$$(L^*v)(t) = \int_t^T B^* e^{A^*(\tau - t)} v(\tau)\, d\tau$$

$$: \text{ continuous } L_2(0,T;[\mathcal{D}(\hat{A}^{1-\gamma})]') \subset L_2(0,T;Y) \to L_2(0,T;U) \tag{2.1.16}$$

for B^*, see below (2.1.7); (sharp results in time are given in Chapter 1, Theorem 1.4.4.3 to be also given in Theorem 2.3.5.1 below). We shall similarly introduce the corresponding operators related to the generator $-\hat{A} = A - \omega I$ in (2.1.3), rather than to the generator A (with conservative regularity):

$$(\hat{L}u)(t) = \int_0^t e^{\hat{A}(t-\tau)} Bu(\tau)\, d\tau$$

$$: \text{ continuous } L_2(0,\infty;U) \to L_2(0,\infty;Y), \tag{2.1.17}$$

$$(\hat{L}^*v)(t) = \int_t^\infty B^* e^{\hat{A}^*(\tau - t)} v(\tau)\, d\tau$$

$$: \text{ continuous } L_2(0,\infty;Y) \to L_2(0,\infty;U), \tag{2.1.18}$$

with sharp results in time to be given below in Theorem 2.3.5.1.

2.2 Statement of Main Results

The main results of this chapter are the following theorems.

Theorem 2.2.1 *Assume (H.1), (H.2) = (2.1.7), (H.3) = (2.1.9), and the Finite Cost Condition (H.4) = (2.1.12). Then:*

(a_1) For each $y_0 \in Y$, there exists a unique optimal pair $\{u^0(t; y_0), y^0(t; y_0)\}$ of the optimal control problem (2.1.10), (2.1.8) for the dynamics (2.1.1) (see Theorem 2.3.3.1(i)).

(a_2) There exists a nonnegative, self-adjoint operator $0 \le P = P^* \in \mathcal{L}(Y)$ such that

$$u^0(t; y_0) = -B^*Py^0(t; y_0) \in L_2(0,\infty;U) \cap C([0,T_0];U) \quad \textit{for all } T_0 < \infty \tag{2.2.1}$$

(see Theorem 2.3.3.1(ii), Corollary 2.3.7.2), where $(Bu, v)_Y = (u, B^*v)_U$ as below (2.1.7), and where P satisfies the following algebraic Riccati equation (ARE):

$$(A^*Px, y)_Y + (PAx, y)_Y + (R^*Rx, y)_Y = (B^*Px, B^*Py)_U,$$

$$\forall\, x, y \in \mathcal{D}(\hat{A}^\epsilon),\ \textit{any } \epsilon > 0; \quad \textit{in particular, } x, y \in \mathcal{D}(A_P) \subset \mathcal{D}(\hat{A}^{1-\gamma}) \tag{2.2.2}$$

(see Theorem 2.3.9.1), where A_P is defined in (2.2.6) below.

(a_3) *For any* $0 \le \theta < 1$:

$$(\hat{A}^*)^\theta P \in \mathcal{L}(Y) \tag{2.2.3a}$$

(see Theorem 2.3.7.1(ii)), so that P is compact, if A has compact resolvent; moreover, one can take $\theta = 1$, if A is self-adjoint and $R = I$, or, more generally, R^*R commutes with A (see Appendix 2B). As a consequence,

$$(\hat{A}^*)^\alpha P \hat{A}^\beta \in \mathcal{L}(Y), \quad \alpha + \beta < 1;\ \alpha, \beta \ge 0;\ (\hat{A}^*)^r P \hat{A}^r \in \mathcal{L}(Y), \quad r < \frac{1}{2} \tag{2.2.3b}$$

(see Theorem 2.3.7.1(ii)).

(a_4) *The optimal cost is given by*

$$J^0(y_0) \equiv J(u^0(\,\cdot\,; y_0), y^0(\,\cdot\,; y_0)) = (Py_0, y_0)_Y, \quad y_0 \in Y \tag{2.2.4}$$

(see Theorem 2.3.3.1(iii)).

(a_5) *The gain operator B^*P is bounded $Y \to U$:*

$$B^*P \in \mathcal{L}(Y; U) \tag{2.2.5}$$

(see Theorem 2.3.7.1(iii)).

(a_6) *Setting $\Phi(t)x \equiv y^0(t; x)$, $x \in Y$, $t \ge 0$, then $\Phi(t)$ is a strongly continuous analytic semigroup on Y, with infinitesimal generator*

$$A_P = A - BB^*P; \quad Y \supset \mathcal{D}(A_P) \to Y, \tag{2.2.6a}$$

$$\mathcal{D}(A_P) = \{x \in \mathcal{D}(\hat{A}^{1-\gamma}) : \hat{A}^{1-\gamma}x - \hat{A}^{-\gamma}BB^*Px \in \mathcal{D}(\hat{A}^\gamma)\} \subset \mathcal{D}(\hat{A}^{1-\gamma}) \tag{2.2.6b}$$

(see (2.3.8.11)) so that $\Phi(t) = e^{A_P t} = e^{(A-BB^*P)t}$ (see Theorems 2.3.6.1 and 2.3.8.1).

(a_7) *Setting $\hat{y}^0(t; x) = e^{-\omega t}y^0(t; x)$, $x \in Y$, ω as in (2.1.3), and $\hat{u}^0(t; x) = e^{-\omega t}u^0(t; x)$, then the optimal control and corresponding optimal trajectory are given by the following formulas:*

$$\hat{y}^0(\,\cdot\,; y_0) = [I + \hat{L}\hat{L}^*(R^*R + 2\omega P)]^{-1}\{e^{-\hat{A}\,\cdot}\, y_0\} \in L_2(0, \infty; Y), \tag{2.2.7}$$

$$-\hat{u}^0(\,\cdot\,; y_0) = \hat{L}^*(R^*R + 2\omega P)\hat{y}^0(\,\cdot\,; y_0) \tag{2.2.8a}$$

$$= [I + \hat{L}^*(R^*R + 2\omega P)\hat{L}]^{-1}\hat{L}^*(R^*R + 2\omega P)\{e^{-\hat{A}\,\cdot}\, y_0\} \in L_2(0, \infty; U), \tag{2.2.8b}$$

with inverses well defined as bounded operators in $L_2(0, \infty; \cdot)$, $\cdot = Y$ or U (see Theorem 2.3.4.1; Appendix 2A).

If the original semigroup e^{At} is stable to begin with, so that $\omega_0 < 0$ in (2.1.2a), we may take $\omega = 0$ in the above formulas (2.2.7) and (2.2.8), thereby obtaining the optimal pair $\{u^0, y^0\}$ explicitly in terms of the data of the problem, via formulas that are the direct counterparts of formulas (1.2.1.1), (1.2.1.4) of Chapter 1.

(a_8) *The operator P in (a_2) satisfies the following relation:*

$$Px = \int_0^\infty e^{-\hat{A}^* t}(R^*R + 2\omega P)e^{-\omega t} y^0(t; x)\, dt, \quad x \in Y, \tag{2.2.9}$$

(see Theorem 2.3.7.1), which becomes – via the considerations of (a_7) above – a defining formula for P solely in terms of the data of the problem when $\omega_0 < 0$ in (2.1.2a) so that we may take $\omega = 0$ in (2.2.9) [as in (1.2.1.11) of Chapter 1].

Theorem 2.2.2 (Uniqueness) *Assume (H.1), (H.2) = (2.1.7), (H.3) = (2.1.8), the Finite Cost Condition (H.4) = (2.1.12), and, in addition, the Detectability Condition (H.5) = (2.1.13). Then the strongly continuous, analytic semigroup $\Phi(t) = e^{A_P t}$ of Theorem 2.2.1 (a_6) is also exponentially stable: There exist constant $M_P \geq 1$ and $\omega_P > 0$, such that*

(b_1)

$$\|e^{A_P t}\|_{\mathcal{L}(Y)} = \left\|e^{(A - BB^*P)t}\right\|_{\mathcal{L}(Y)} \leq M_P e^{-\omega_P t}, \quad t \geq 0, \tag{2.2.10}$$

and likewise there exists a constant C_P such that

$$\|u^0(t; x)\|_U \leq C_P e^{-\omega_P t}\|x\|_Y, \quad t \geq 0, \; x \in Y, \tag{2.2.11}$$

so that now

$$u^0(t, x) \in C_{ub}([0, \infty]; U), \tag{2.2.12a}$$

or equivalently,

$$B^* P \Phi(\,\cdot\,) : \textit{continuous } Y \to C_{ub}([0, \infty]; U) \tag{2.2.12b}$$

(see Theorem 2.4.1 and Corollary 2.4.2).

(b_2) *Moreover, the operator P of Theorem 2.2.1 is the unique solution of the ARE (2.2.2), within the class of self-adjoint operators $\bar{P}$ such that $\hat{A}^Y \bar{P} \in \mathcal{L}(Y)$; hence $B^* \bar{P} \in \mathcal{L}(Y; U)$ (see Theorem 2.4.5).*

Remark 2.2.1 Assume that the original generator A is stable, that is, that ω_0 in (2.1.2a) is negative: $\omega_0 < 0$. This case includes several *canonical* parabolic partial differential equations and so is of intrinsic interest. Then, we may take $\omega = 0$ throughout, and the subsequent analysis greatly simplifies. In particular, there is no need to take the limit process $T \uparrow \infty$, as in Section 3.3, on the optimal control problem on $[0, T]$ (see Chapter 6). Rather, the infinite time control problem can be analyzed directly. All of the above formulas in Theorem 2.2.1, specialized with $\omega = 0$, become explicit solely in terms of the data. We invite the reader to carry out this exercise. Moreover, we may express the optimal cost $J^0(y_0)$, equivalently P, by the following intrinsic formula:

$$\begin{aligned} J^0(y_0) &= J(u^0(\cdot\,; y_0), y^0(\cdot\,; y_0)) = (Py_0, y_0)_Y \\ &= ([I + RLL^*R^*]^{-1} Re^{A\cdot} y_0, Re^{A\cdot} y_0)_{L_2(0,\infty;Z)}. \end{aligned} \tag{2.2.13}$$

Actually, as is readily seen, (2.2.13) holds true, if

$$Re^{At}: \text{continuous } Y \to L_2(0, \infty; Z), \tag{2.2.14}$$

$$RL: \text{continuous } L_2(0, \infty; U) \to L_2(0, \infty; Z). \tag{2.2.15}$$

We shall consider this case in the more complicated situation of min–max problems in Chapter 6. A formula such as (2.2.13) will play an important role in the (abstract) *hyperbolic* case [Chapter 9, Sections 9.4 and 9.6.7, as well as Chapter 11] in obtaining that, under a natural and verifiable (exact controllability) condition, the Riccati operator is an *isomorphism* on Y. By contrast, in the present *parabolic* case, P is smoothing and cannot be an isomorphism.

Remark 2.2.2 (Conditions for P to be Hilbert–Schmidt) The regularity result for the Riccati operator P given by Theorem 2.2.1(a_3), Eqn. (2.2.3a), readily allows one to infer *when P is a Hilbert–Schmidt (H–S) operator on Y*. Indeed, since the class of H–S operators is 2-sided ideal within the class of all bounded operators [Weidmann, 1980, p. 138], then, via (2.2.3a), we see that:

$$\begin{cases} P = (\hat{A}^*)^{-\theta}[(\hat{A}^*)^{\theta} P] \text{ is H–S on } Y \text{ provided that} \\ (\hat{A}^*)^{-\theta}, \text{ equivalently } (\hat{A})^{-\theta}, \text{ is H–S on } Y, \theta < 1, \text{ as in (2.2.3a).} \end{cases} \tag{2.2.16}$$

Case of Classical Parabolic Equations on $Y = L_2(\Omega)$ Given a smooth bounded domain $\Omega \in \mathbb{R}^N$, let A be the realization in $L_2(\Omega)$ of an elliptic operator of order $2m$, subject to appropriate boundary conditions (see Chapter 3, Appendix 3A). Then A generates a s.c. analytic semigroup on $L_2(\Omega)$. Moreover, $\mathcal{D}(A^*) \subset H^{2m}(\Omega)$ (Sobolev space).

Claim In this case,

$$\begin{cases} P \textit{ is H–S on } Y = L_2(\Omega), \textit{ provided that } 4m > N; \\ \textit{in particular, if } m = 1 \textit{ (second-order operator such as the Laplacian),} \\ \textit{then } P \textit{ is H–S on } L_2(\Omega), \textit{ provided that } N = \dim \Omega = 1, 2, 3. \end{cases} \tag{2.2.17}$$

First Proof. Since Ω is bounded, A has compact resolvent in $L_2(\Omega)$. The eigenvalues $\{\lambda_n\}$ of A satisfy the well-known asymptotic estimates: $\lambda_n \sim n^{2m/N}$ [Courant, Hilbert, 1953; Ch. VI; Triebel, 1978, p. 395]. Thus, $(\hat{A})^{-\theta}$ is H–S on $Y = L_2(\Omega)$, as required by (2.2.16) in case

$$\sum_n 1/(|\lambda_n|^{2\theta}) \sim \sum_n 1/\left(n^{4m\theta/N}\right) < \infty, \text{ i.e., in case } 4m\theta > N, \tag{2.2.18}$$

since we may take $\theta < 1$ arbitrarily close to 1 by (2.2.3a) (and even $\theta = 1$ in the special cases noted there), and then (2.2.17) follows from (2.2.18). $\square$

Second Proof. Here we use the following well-known result [Dunford, Schwartz, 1963, Vol. II, p. 1742 bottom] (see also [Weidmann, 1980, Theorem 6.12, p. 140]):

Let M be a continuous mapping of $L_2(\Omega)$ into $C(\bar{\Omega})$. Then M is H–S.

We apply this result to $M = P$, the Riccati operator where, by (2.2.3a), $P : L_2(\Omega) \to \mathcal{D}((\hat{A}^*)^\theta) \subset H^{2m\theta}(\Omega) \to C(\bar{\Omega})$, is H–S, provided that $2m\theta > N/2$ by standard embedding on the last step, and (2.2.17) follows again. $\square$

The advantages of a Hilbert–Schmidt operator, with its integral representation with L_2-kernels, is well known [Weidmann, 1980, §6.2]. The examples in Chapter 3 (Sections 3.1 and 3.3) follow into the framework of claim (2.2.17) with $m = 1$. Other examples, where P is H–S following condition (2.2.16) are given in Chapter 3 (Sections 3.4 and 3.6).

2.3 Proof of Theorem 2.2.1

2.3.1 Orientation and Summary of the Corresponding Results for the Quadratic Cost Problem on $[0, T]$, $T < \infty$

Orientation Consideration of the general case with $\omega_0 \geq 0$ in (2.1.2a) is responsible not only for some annoying notation (e.g., at the level of taking fractional powers $(-A + \omega I)^\theta$ of a translation $\hat{A} = -A + \omega I$ in (2.1.3) of $-A$), but also for some technical and conceptual complications and implications as noted in the statements of (a_7) and (a_8) of Theorem 2.2.1. The strategy consists in starting with the corresponding optimal control problem on a finite time interval $[0, T]$ with $G = 0$ in [Chapter 1, Eqn. (1.1.3)], and then taking the limit as $T \uparrow \infty$ on the corresponding relevant quantities: the differential Riccati operator and the optimal control and optimal trajectory. Our variational approach, however, avoids the classical approach of taking the limit as $T \uparrow \infty$ on the differential Riccati equation to obtain the corresponding algebraic Riccati equation. This can be done (see Notes at the end of Chapter 2), but it is now a delicate issue owing to the unboundedness of the operator B. Rather, our strategy is this: The candidate (algebraic) Riccati operator obtained through the limit process is now shown directly to verify the algebraic Riccati equation.

Summary of Results of the Corresponding Quadratic Cost Problem on $[0, T]$, $T < \infty$, ***and Preliminaries*** We shall hereafter indicate with a subscript "T" quantities corresponding to the quadratic cost problem for the dynamics (2.1.1) over the time interval $[0, T]$, $T < \infty$, that is,

$$\text{minimize } J_T(u, y) = \int_0^T \left[\|Ry(t)\|_Z^2 + \|u(t)\|_U^2 \right] dt \tag{2.3.1.1}$$

over all $u \in L_2(0, T; U)$, where $y(t) = y(t, y_0)$ is the solution of (2.1.1) due to $u(t)$. This is precisely the problem of Chapter 1 with $G = 0$ in the final state penalization, [see Chapter 1, Eqn. (1.1.3)]. Thus we can invoke the results of Chapter 1. It will be convenient to collect here below those that will be relevant to our present problem. Accordingly:

(i) there exists a nonnegative, self-adjoint (differential) Riccati operator $P_T(\cdot) \in \mathcal{L}(Y; C([0, T]; Y)$ for the optimal quadratic cost problem (2.3.1.1) for the dynamics (2.1.1), given explicitly by

$$P_T(t)x = \int_t^T e^{A^*(\tau - t)} R^* R \Phi_T(\tau, t)x \, d\tau \in C([0, T]; Y), \quad x \in Y \tag{2.3.1.2}$$

[see Chapter 1, Theorem 1.2.1.1, Eqn. (1.2.1.11)], where

(i_1)

$$\Phi_T(\tau, t)x \equiv y_T^0(\tau, t; x) \in C([t, T]; Y) \tag{2.3.1.3a}$$

[see Chapter 1, Theorem 1.2.2.1, Eqn. (1.2.2.2)] for the regularity, $y_T^0(\tau, t; x)$ being the optimal trajectory at time τ of the optimal problem corresponding to (2.3.1.1) with starting point $x \in Y$ and the initial time t:

(i_2)

$$\Phi_T(\tau, t)\Phi_T(t, s) = \Phi_T(\tau, s), \quad 0 \le s \le t \le \tau \le T, \tag{2.3.1.3b}$$

(see Chapter 1, Eqn. (1.4.3.3)).

(ii) We have, [see Chapter 1, Lemma 1.4.6.3, Eqn. (1.4.6.18)]

$$\frac{\partial \Phi_T(t, s)x}{\partial t} = (A - BB^* P_T(t))\Phi_T(t, s)x, \quad x \in Y. \tag{2.3.1.4}$$

(iii) $P_T(t)$ is the unique, nonnegative, self-adjoint solution of the differential Riccati equation [see Chapter 1, Theorem 1.2.1.1, Eqn. (1.2.1.13) and Theorem 1.2.2.1, Eqn. (1.2.2.7)] with $G = 0$:

$$\begin{cases} (\dot{P}_T(t)x, y)_Y = -(R^* Rx, y)_Y - (A^* P_T(t)x, y)_Y - (P_T(t)Ax, y)_Y \\ \qquad\qquad + (B^* P_T(t)x, B^* P_T(t)y)_U, \ \forall\, x, y \in \mathcal{D}(\hat{A}^\epsilon), \\ \lim_{t \uparrow T} P_T(t)x = 0. \end{cases} \tag{2.3.1.5}$$

Adding and subtracting $(2\omega P_T(t)x, y)_Y$, with ω as the constant in (2.1.3), we obtain

$$\begin{aligned} (\dot{P}_T(t)x, y)_Y &= -((R^* R + 2\omega P_T(t))x, y)_Y - ((A^* - \omega I)P_T(t)x, y)_Y \\ &\quad - (P_T(t)(A - \omega I)x, y)_Y + (B^* P_T(t)x, B^* P_T(t)y)_U. \end{aligned} \tag{2.3.1.6}$$

(iv) Consistently with (2.1.3) and (a_7) in Theorem 2.2.1, we set

$$\hat{\Phi}_T(t,s) \equiv e^{-\omega(t-s)}\Phi_T(t,s) \tag{2.3.1.7}$$

and obtain, by (2.3.1.4) and $\hat{A} = -A + \omega I$ in (2.1.3),

$$\frac{\partial \hat{\Phi}_T(t,s)x}{\partial t} = (-\hat{A} - BB^* P_T(t))\hat{\Phi}_T(t,s)x, \quad x \in Y. \tag{2.3.1.8}$$

Bearing in mind that $P_T(t)$ given by the formula in (2.3.1.2) is the solution to the DRE (2.3.1.5), we then deduce that the solution to Eqn. (2.3.1.6) (still $P_T(t)$) can also be written as

$$P_T(t)x = \int_t^T e^{-\hat{A}^*(\tau - t)}[R^*R + 2\omega P_T(\tau)]\hat{\Phi}_T(\tau,t)x\,d\tau, \quad x \in Y. \tag{2.3.1.9}$$

Indeed, the following correspondence exists between quantities in (2.3.1.5) and quantities in (2.3.1.6): The operators R^*R and A in (2.3.1.5) correspond to the operators $(R^*R + 2\omega P_T(t))$ and $-\hat{A} = A - \omega I$ in (2.3.1.6); while the evolution operator $\Phi_T(t)$ of $(A - BB^*P_T(t))$ corresponds by (2.3.1.8) to the evolution operator $\hat{\Phi}_T(t)$ of $(-\hat{A} - BB^*P_T(t))$.

(v) The optimal control $u_T^0(t, 0; y_0)$ of the optimal problem (2.3.1.1) for the dynamics (2.1.1) is given by [see Chapter 1, Theorem 1.2.1.1, Eqn. (1.2.1.10)]

$$u_T^0(t,0;y_0) = -B^*P_T(t)y_T^0(t,0;y_0) = -B^*P_T(t)\Phi_T(t,0)y_0, \quad 0 < t \le T. \tag{2.3.1.10}$$

Hence, setting consistently with (2.3.1.7)

$$\hat{u}_T^0(t,0;x) \equiv e^{-\omega t}u_T^0(t,0;x), \tag{2.3.1.11}$$

we obtain by (2.3.1.10), (2.3.1.7), and $P_T(t)$ in (2.3.1.9)

$$\begin{aligned}\hat{u}_T^0(t,0;x) &= -B^*P_T(t)\hat{\Phi}_T(t,0)x \\ &= -B^*\int_t^T e^{-\hat{A}^*(\tau - t)}[R^*R + 2\omega P_T(\tau)]\hat{\Phi}_T(\tau,0)x\,d\tau, \end{aligned} \tag{2.3.1.12}$$

after recalling (2.3.1.3b), and hence $\hat{\Phi}_T(\tau,t)\hat{\Phi}_T(t,0) = \hat{\Phi}_T(\tau,0)$. If, as in Chapter 1, Eqn. (1.1.8a) we introduce

$$\hat{L}^*_{(T)}(t) = \int_t^T B^*e^{-\hat{A}^*(\tau - t)}v(\tau)\,d\tau : \text{continuous } L_2(0,T;Y) \to L_2(0,T;U), \tag{2.3.1.13}$$

which is the $L_2(0,T;\,\cdot\,)$-adjoint of the operator $\hat{L}$ introduced in (2.1.17), but viewed now as an operator $L_2(0,T;U) \to L_2(0,T;Y)$ [the subscript "T" in this section should not be confused with the subscript "T" in Chapter 1, Eqn. (1.1.9) or (1.1.10)], then (2.3.1.12) can be rewritten as

$$\hat{u}_T^0(t, 0; x) = -\left\{\hat{L}_{(T)}^*[R^*R + 2\omega P_T(\cdot)]\hat{\Phi}_T(\cdot, 0)x\right\}(t), \quad 0 < t \le T. \tag{2.3.1.14}$$

(vi) The optimal cost of the optimal problem (2.3.1.1) for the dynamics (2.1.1) is

$$J_T^0(y_0) \equiv J_T\left(u_T^0(\cdot, 0; y_0), y_T^0(\cdot, 0; y_0)\right) = (P_T(0)y_0, y_0)_Y, \quad y_0 \in Y \tag{2.3.1.15}$$

[Chapter 1, Theorem 1.2.1.1, Eqn. (1.2.1.12)].

(vii) The optimal dynamics for problem (2.3.1.1) for (2.1.1) is

$$y_T^0(t, 0; y_0) = e^{At}y_0 + \left\{Lu_T^0(\cdot, 0; x)\right\}(t). \tag{2.3.1.16}$$

2.3.2 A Preparatory Lemma for Φ_T and P_T

We need a new result over those in Chapter 1.

Lemma 2.3.2.1 *Assume (H.1)–(H.3). With reference to Φ_T in (2.3.1.3) and P_T in (2.3.1.2), we have*

(i) For $x \in Y$, $0 \le t \le T$; $0 \le \sigma \le T - t$:

$$\Phi_{T-t}(\sigma, 0)x = \Phi_T(t + \sigma, t)x \underset{(\text{in } \sigma)}{\in} C([0, T - t]; Y); \tag{2.3.2.1}$$

(ii)

$$P_{T-t}(0) = P_T(t), \quad 0 \le t < T. \tag{2.3.2.2}$$

Proof. We use the optimal dynamics [see Chapter 1, Eqn. (1.4.1.26) and Eqn. (1.4.1.24) with $G = 0$], in the notation of L_s and L_s^* of Chapter 1, that is, with $s \le t \le T$:

$$\Phi_T(t, s)x = e^{A(t-s)}x + \left\{L_s u_T^0(\cdot, s; x)\right\}(t) \tag{2.3.2.3}$$

$$= e^{A(t-s)}x - \{L_s L_s^* R^* R \Phi_T(\cdot, s)x\}(t) \tag{2.3.2.4}$$

or explicitly by Chapter 1, Eqn. (1.4.1.2), for L_s, and Eqn. (1.4.1.4) for L_s^*,

$$\Phi_T(t, s)x = e^{A(t-s)}x - \int_s^t e^{A(T-\tau)}B\left(\int_\tau^T B^* e^{A^*(r-\tau)}R^*R\Phi_T(r, s)x\,dr\right)d\tau. \tag{2.3.2.5}$$

Specializing (2.3.2.5) we find that

$$\Phi_{T-t}(\sigma, 0)x = e^{A\sigma}x - \int_0^\sigma e^{A(\sigma-\tau)}B\left(\int_\tau^{T-t} B^* e^{A^*(r-\tau)}R^*R\Phi_{T-t}(r, 0)x\,dr\right)d\tau, \tag{2.3.2.6}$$

and next

$$\Phi_T(t+\sigma,t)x = e^{A(t+\sigma-t)}x - \int_t^{t+\sigma} e^{A(t+\sigma-\tau)}B\left(\int_\tau^T B^* e^{A^*(\alpha-\tau)}R^*R\Phi_T(\alpha,t)x\,d\alpha\right)d\tau. \tag{2.3.2.7}$$

We now set $\tau - t = \beta$ in the first itegral of (2.3.2.7), and then set $\alpha - t = r$ in the second integral of (2.3.2.7), thus obtaining

$$\Phi_T(t+\sigma,t)x = e^{A\sigma}x + \int_0^{\sigma} e^{A(\sigma-\beta)}B\left(\int_\beta^{T-t} B^* e^{A^*(r-\beta)}R^*R\Phi_T(t+r,t)x\,dr\right)d\beta. \tag{2.3.2.8}$$

Comparison between (2.3.2.6) and (2.3.2.8) reveals that both $\Phi_{T-t}(\sigma,0)$ and $\Phi_T(t+\sigma,t)$ satisfy the same equation. But then the difference

$$z(\sigma,t;x) \equiv \Phi_T(t+\sigma,t)x - \Phi_{T-t}(\sigma,0)x \underset{(\text{in }\sigma)}{\in} C([0,T-t];Y) \tag{2.3.2.9}$$

satisfies

$$z(\sigma,t;x) + \int_0^{\sigma} e^{A(\sigma-\tau)}B\left(\int_\tau^{T-t} B^* e^{A^*(r-\tau)}R^*Rz(r,t;x)\,dr\right)d\tau, \tag{2.3.2.10}$$

or, in the notation L and L^* of Chapter 1, that is, with $0 \le \sigma \le T-t$:

$$[I + LL^*R^*R]z(\,\cdot\,,t;x) = 0. \tag{2.3.2.11}$$

But

$$[I + LL^*R^*R]^{-1} \in \mathcal{L}(L_2(0,T;U) \tag{2.3.2.12}$$

for any $T < \infty$. The key for the validity of (2.3.2.12) is that both LL^* and R^*R are self-adjoint, nonnegative (see the general result in Appendix 2A, at the end of the chapter).

By (2.3.2.12) applied to (2.3.2.11), we deduce that $z(\sigma,t;x)$ is the zero element in $L_2(0,T-t;Y)$ and by the regularity in (2.3.2.9) we conclude that, in fact, $z(\sigma,t;x) \equiv 0$, $0 \le \sigma \le T-t$, as desired. Part (i) is proved.

(ii) Part (ii) follows now by applying part (i) to (2.3.1.2): Changing here variable $\tau - t = \sigma$ yields

$$\begin{aligned} P_T(t)x &= \int_0^{T-t} e^{A^*\sigma}R^*R\Phi_T(t+\sigma,t)x\,d\sigma \\ \text{(by (2.3.2.1))} \qquad &= \int_0^{T-t} e^{A^*\sigma}R^*R\Phi_{T-t}(\sigma,0)x = P_{T-t}(0)x, \end{aligned} \tag{2.3.2.13}$$

where in the last step we have invoked again (2.3.1.2). Part (ii) is proved.

Remark 2.3.2.1 Both conclusions of Lemma 2.3.2.1 also hold true in the setting of Chapter 1, Eqn. (1.1.3), where the functional cost also contains the final state operator $G \in \mathcal{L}(Y; Z_f)$. Indeed, by invoking the full statement of Chapter 1, Eqn. (1.4.1.24) in (2.3.2.3), we obtain in the notation for L_s, L_s^*, and L_{sT}^* in Chapter 1, Eqn. (1.4.1.2), (1.4.1.4), and (1.4.1.7),

$$\begin{aligned}\Phi_T(t,s)x &= e^{A(t-s)}x - \{L_s L_s^* R^* R \Phi_T(\cdot, s)x\}(t) \\ &\quad - \left\{L_s L_{sT}^* G^* G \Phi_T(T,s)x\right\}(t),\end{aligned} \tag{2.3.2.14}$$

generalizing (2.3.2.4) to the case $G \neq 0$; hence, explicitly,

$$\begin{aligned}\Phi_T(t,s)x &= e^{A(t-s)}x - \int_s^t e^{A(t-\tau)}B\left(\int_\tau^T B^* e^{A^*(r-\tau)}R^* R \Phi_T(r,s)x\,dr\right)ds \\ &\quad - \int_s^t e^{A(t-\tau)}BB^* e^{A^*(T-\tau)}G^* G \Phi_T(t,s)x\,d\tau,\end{aligned} \tag{2.3.2.15}$$

generalizing (2.3.2.5) to the case $G \neq 0$. The same argument below (2.3.2.5) then yields

$$[I + LL^* R^* R + L_T L_T^* G^* G]z(\cdot, t; x) = 0 \tag{2.3.2.16}$$

in the notation of Chapter 1, generalizing (2.3.2.11). But the operator in the square brackets is likewise boundedly invertible on $L_2(0, T; U)$ for any $T < \infty$ (see Appendix 2A). Hence, $z(\sigma, t; x) \equiv 0$ as before, as desired. We, however, shall not need these extensions to the case $G \neq 0$ in this chapter.

2.3.3 The Limit Process $T \uparrow \infty$

The limit process as $T \uparrow \infty$ on the theory of Section 2.3.2 produces the following result for the original problem (2.1.10), (2.1.8).

Theorem 2.3.3.1 *Assume (H.1)–(H.3) and the Finite Cost Condition (H.4) = (2.1.12). With reference to the original optimal quadratic cost problem (2.1.10), (2.1.8) for the dynamics (2.1.1), we have:*

(i) There exists a unique optimal pair $\{u^0(t; y_0), y^0(t; y_0)\}$ *with*

$$\begin{aligned}&u^0(\cdot\,; y_0) \in L_2(0,\infty; U); \quad R y^0(\cdot\,; y_0) \in L_2(0,\infty; Z); \\ &(R^* R)^{\frac{1}{2}} y^0(\cdot\,; y_0) \in L_2(0,\infty; Y).\end{aligned} \tag{2.3.3.1}$$

(ii) There exists a nonnegative, self-adjoint operator $0 \leq P = P^* \in \mathcal{L}(Y)$ *defined by*

$$Px = \lim_{T\uparrow\infty} P_T(0)x; \ \textit{or alternatively, } Px = \lim_{T\uparrow\infty} P_T(t)x, \tag{2.3.3.2a}$$
$$x \in Y, t \textit{ fixed and arbitrary } < T \uparrow \infty,$$

and, in fact, uniformly on compact sets $0 \le t \le T_0 < T \uparrow \infty$. *Moreover,*

$$\sup_{T} \sup_{0 \le t \le T} \|P_T(t)\|_{\mathcal{L}(Y)} \le M < \infty. \tag{2.3.3.2b}$$

(iii) *The optimal cost of problem (2.1.10), (2.1.8) for the dynamics (2.1.1) is*

$$J^0(y_0) \equiv J(u^0(\cdot\,; y_0), y^0(\cdot\,, y_0)) = (Py_0, y_0)_Y, \quad y_0 \in Y. \tag{2.3.3.3}$$

(iv) *If we extend the optimal pair functions* $u_T^0(\cdot\,, 0; x)$ *and* $y_T^0(\cdot\,, 0; x)$ *of problem (2.3.1.1) by setting them equal to zero for all* $t > T$ *(while keeping the same symbols), then for* $x \in Y$:

$$\left.\begin{array}{l} u_T^0(\cdot\,, 0; x) \to u^0(\cdot\,; x) \\ (R^*R)^{\frac{1}{2}} y_T^0(\cdot\,, 0; x) \to (R^*R)^{\frac{1}{2}} y^0(\cdot\,; x) \\ Ry_T^0(\cdot\,, 0; x) \to Ry^0(\cdot\,; x) \end{array}\right\} \quad \text{strongly in} \quad \left\{\begin{array}{ll} L_2(0, \infty; U) & (2.3.3.4a) \\ L_2(0, \infty; Y) & (2.3.3.4b) \\ L_2(0, \infty; Z) & (2.3.3.4c) \end{array}\right.$$

and

$$\left.\begin{array}{l} \hat{u}_T^0(\cdot\,, 0; x) \to u^0(\cdot\,; x) \\ \hat{y}_T^0(\cdot\,, 0; x) = \hat{\Phi}_T(\cdot\,, 0)x \to \hat{y}^0(\cdot\,; x) = \hat{\Phi}(\cdot)x \end{array}\right\} \quad \text{strongly in} \quad \left\{\begin{array}{ll} L_2(0, \infty; U) & (2.3.3.5a) \\ L_2(0, \infty; Y) & (2.3.3.5b) \end{array}\right.$$

where we have set

$$\Phi(t)x \equiv y^0(t; x); \quad \hat{\Phi}(t)x \equiv e^{-\omega t}\Phi(t)x; \quad \hat{u}^0(t; x) \equiv e^{-\omega t}u^0(t; x). \tag{2.3.3.5c}$$

(v) *The optimal dynamics is*

$$\hat{y}^0(t; x) = \hat{\Phi}(t)x = e^{-\hat{A}t}x + \{\hat{L}\hat{u}^0(\cdot\,; x)\}(t) \in L_2(0, \infty; Y). \tag{2.3.3.6}$$

Proof. (i) The existence of a unique optimal pair $\{u^0(t; y_0), y^0(t; y_0)\}$ for the quadratic functional J in (2.1.8) under the Finite Cost Condition stems from convex optimization theory [Ekeland, Teman, 1976]. For sake of clarity we shall append the subscript "∞" to these optimal functions and call them $u_\infty^0(t; y_0)$ and $y_\infty^0(t; y_0)$ in the present proof. Similarly we shall call J_∞ the cost in (2.1.8).

(ii) Equation (2.3.1.15) reveals that the family of nonnegative, self-adjoint operators $\{P_T(0)\}$ is monotone nondecreasing. Moreover, by the Finite Cost Condition (2.1.12), this family is bounded above. In fact,

$$\begin{aligned} (P_T(0)x, x)_Y &= J_T\big(u_T^0(\cdot\,, 0; x), y_T^0(\cdot\,, 0; x)\big) \le J_\infty\big(u_\infty^0(\cdot\,; x), y_\infty^0(\cdot\,, x)\big) \\ &\equiv J_\infty^0(x) < \infty, \end{aligned} \tag{2.3.3.7a}$$

as the function $u^0_\infty(t; x)$ and $y^0_\infty(t; x)$, restricted on $0 \le t \le T$, form a competing pair over the interval $[0, T]$ for problem (2.1.1), (2.3.1.1). Then, as is well known [Taylor, Lay, 1980, p. 353], there exists a nonnegative, self-adjoint operator P on Y such that $Px = \lim P_T(0)$, $x \in Y$, as $T \uparrow \infty$, and (2.3.3.2a) on the left is proved. Then (2.3.3.2a) on the right follows immediately at least for t fixed $< T \uparrow \infty$, via the identity $P_{T-t}(0) = P_T(t)$ in (2.3.2.2) of Lemma 2.3.2.1. Uniform convergence in (2.3.3.2a) (right) for t over a fixed interval is readily shown. (However, we note that

$$\sup_{0 \le t \le T} \|P_T(t)x - Px\|_Y$$

does *not* converge to zero, as $T \uparrow \infty$; in fact, $P_T(T) = 0$ by the end condition in (2.3.1.5).) To show the uniform bound (2.3.3.2b), we note that, as in (2.3.3.7a), we generally have for $0 \le t \le T$:

$$(P_T(t)x, x)_Y = J_T\big(u^0_T(\cdot\,, t; x), y^0_T(\cdot\,, t; x)\big) \le J^0_\infty(x), \tag{2.3.3.7b}$$

as the functions $u^0_\infty(\cdot\,; x)$ and $y^0_\infty(\cdot\,; x)$, restricted now over $[t, T]$ form a competing pair for the optimal control problem over $[t, T]$. Then (2.3.3.7b) yields the uniform bound (2.3.3.2b) since P_T is nonnegative self-adjoint.

(iii) Taking the limit on (2.3.3.7a) as $T \uparrow \infty$, we obtain by (2.3.3.2a):

$$(Px, x)_Y \le J_\infty\big(u^0_\infty(\cdot\,; x), y^0_\infty(\cdot\,; x)\big) \equiv J^0_\infty(x) < \infty. \tag{2.3.3.8}$$

We now obtain the opposite inequality. We first observe that the bound in (2.3.3.7a) along with (2.3.1.1) and (2.1.11) show that the extended functions

$$u^0_{T,\text{ext}}(t, 0; x) = \begin{cases} u^0_T(t, 0; x), & 0 \le t \le T; \\ 0, & t > T, \end{cases} \tag{2.3.3.9}$$

and $(R^*R)^{\frac{1}{2}} y^0_{T,\text{ext}}(t, 0; x)$, where

$$y^0_{T,\text{ext}}(t, 0; x) = \begin{cases} y^0_T(t, 0; x), & 0 \le t \le T; \\ 0, & t > T, \end{cases} \tag{2.3.3.10}$$

are contained in a fixed ball (depending on x fixed) of $L_2(0,\infty; U)$ and $L_2(0, \infty; Y)$, respectively, for all T. Hence, one can extract subsequences

$$u^0_{T,\text{ext}}(\cdot\,, 0; x) \text{ converging weakly to, say, a function } \tilde{u} \in L_2(0, \infty; U); \tag{2.3.3.11}$$

$$Ry^0_{T,\text{ext}}(\cdot\,, 0; x) \text{ converging weakly to, say, a function } \tilde{\zeta}(\cdot\,; x) \in L_2(0, \infty; Z). \tag{2.3.3.12}$$

Next, we see that the above limits $\tilde{u}$ and $\tilde{\zeta}$ are connected by the underlying dynamics; that is, for any $0 < T_0 < \infty$,

$$\tilde{\zeta}(t; x) = R\tilde{y}(t; x) = Re^{At}x + R\{L\tilde{u}\}(t) \in L_2(0, T_0; Z), \tag{2.3.3.13}$$

where $\tilde{y}(t; x)$ is defined by

$$\tilde{y}(t; x) = e^{At}x + \{L\tilde{u}\}(t) \in L_2(0, T_0; Y), \tag{2.3.3.14}$$

in other words, as the weak limit of the optimal dynamics (2.3.1.16) in $L_2(0, T_0; Y)$: This is attained by the limit (2.3.3.11) and by the continuity of L as in (2.1.15). Thus, $Ry^0_{T,\text{ext}} \to R\tilde{y}$ weakly in $L_2(0, T_0; Z)$, and by (2.3.3.12) and the uniqueness of the weak limit, we obtain $\tilde{\zeta}(\cdot\,; x) = R\tilde{y}(\cdot\,; x)$ in $L_2(0, T_0; Z)$ as desired, for any T_0. Thus, (2.3.3.12) is refined to

$$Ry^0_{T,\text{ext}}(\cdot\,, 0; x) \to R\tilde{y}(\cdot\,; x) \text{ weakly in } L_2(0, \infty; Z). \tag{2.3.3.15}$$

Moreover, from (2.1.15) for L, (2.1.17) for $\hat{L}$, and (2.1.3) for $\hat{A}$, we readily verify that

$$e^{-\omega t}\{Lu\}(t) = \{\hat{L}\hat{u}\}(t), \quad \text{where } \hat{u}(t) = e^{-\omega t}u(t) \tag{2.3.3.16}$$

for any $u(t) \in L_2(0, \infty; U)$. Thus, multiplying (2.3.1.16) by $e^{-\omega t}$ yields, by virtue of (2.3.3.16),

$$\hat{y}^0_T(t, 0; x) = e^{-\omega t}y^0_T(t, 0; x) = e^{-\hat{A}t}x + \left\{\hat{L}\widehat{u^0_T}(\cdot\,, 0; x)\right\}(t), \quad 0 \le t \le T, \tag{2.3.3.17}$$

for the optimal trajectory of the cost (2.3.1.1) on $[0, T]$. Next, we extend (2.3.3.17) by zero for $t > T$ and take the $L_2(0, \infty; Y)$-weak limit with $\hat{L}$ bounded as in (2.1.17) to obtain, by (2.3.3.11) and (2.3.3.14),

$$\begin{cases} \hat{\tilde{y}}(t; x) = e^{-\hat{A}t}x + \{\hat{L}\hat{\tilde{u}}(\cdot\,; x)\}(t) \in L_2(0, \infty; Y), \\ \hat{\tilde{u}}(t; x) = e^{-\omega t}\tilde{u}(t; x); \quad \hat{\tilde{y}}(t; x) = e^{-\omega t}\tilde{y}(t; x). \end{cases} \tag{2.3.3.18}$$

In contrast, the quadratic cost (2.1.8) is *a fortiori* lower semicontinuous on the space $V \equiv L_2(0, \infty; Y) \times L_2(0, \infty; U)$ and remains lower semicontinuous in the weak topology of V [Ekeland, Teman, 1976, Corollary 2.2, p. 11; Balakrishnan, 1981, p. 30]. But the weak limits $\tilde{u}(\cdot\,; x)$ and $\tilde{y}(\cdot\,; x)$ in (2.3.3.11), (2.3.3.14) form a competing pair for the cost (2.1.8) by (2.3.3.14). Hence, the just recalled result yields the first inequality in (2.3.3.19) below:

$$\begin{aligned} (P_T(0)x, x)_Y &= J_T\big(u^0_T(\cdot\,, 0; x), y^0(\cdot\,, 0; x)\big) \\ &= J_T\big(u^0_{T,\text{ext}}(\cdot\,, 0; x), y^0_{T,\text{ext}}(\cdot\,, 0; x)\big) \\ &\ge J_\infty(\tilde{u}(\cdot\,; x), \tilde{y}(\cdot\,; x)) \ge J^0_\infty(x). \end{aligned} \tag{2.3.3.19}$$

Taking the limit as $T \uparrow \infty$ in inequality (2.3.3.19) yields

$$(Px, x)_Y \ge J^0_\infty(x) \tag{2.3.3.20}$$

by (2.3.3.2). Then (2.3.3.8) and (2.3.3.20) combined yield

$$(Px, x)_Y = J_\infty(\tilde{u}(\,\cdot\,; x), \tilde{y}(\,\cdot\,; x)) = J\big(u^0_\infty(\,\cdot\,; x), y^0_\infty(\,\cdot\,; x)\big) = J^0_\infty(x), \tag{2.3.3.21}$$

and part (iii) is proved.

(iv) By uniqueness of the optimal pair (part (i)) we deduce from (2.3.3.21) that

$$\tilde{u}(\,\cdot\,; x) = u^0_\infty(\,\cdot\,; x) \quad \text{in } L_2(0, \infty; U); \tag{2.3.3.22a}$$

$$\tilde{y}(\,\cdot\,; x) = y^0_\infty(\,\cdot\,; x) \quad \text{in } L_2(0, T; Y), \text{ any } T < \infty; \tag{2.3.3.22b}$$

$$R\tilde{y}(\,\cdot\,; x) = Ry^0_\infty(\,\cdot\,; x) \quad \text{in } L_2(0, \infty; Z). \tag{2.3.3.22c}$$

Thus (2.3.3.11) and (2.3.3.15) can be rewritten via (2.3.3.22) as

$$u^0_{T,\text{ext}} \quad \text{converges weakly to } u^0_\infty \text{ in } L_2(0, \infty; U), \tag{2.3.3.23}$$

$$Ry^0_{T,\text{ext}} \quad \text{converges weakly to } Ry^0_\infty \text{ in } L_2(0, \infty; Y). \tag{2.3.3.24}$$

However, the established convergence $J^0_T(x) \to J^0_\infty(x)$ as $T \uparrow \infty$ in part (iii) provides norm-convergence:

$$\begin{aligned} &\left\|u^0_{T,\text{ext}}(\,\cdot\,; x)\right\|^2_{L_2(0,\infty;U)} + \left\|Ry^0_{T,\text{ext}}(\,\cdot\,; x)\right\|^2_{L_2(0,\infty;Y)} \\ &\quad \to \left\|u^0_\infty(\,\cdot\,; x)\right\|^2_{L_2(0,\infty;U)} + \left\|Ry^0_\infty(\,\cdot\,; x)\right\|^2_{L_2(0,\infty;Y)}. \end{aligned} \tag{2.3.3.25}$$

Thus, weak convergence in (2.3.3.23), (2.3.3.24), combined with norm-convergence in (2.3.3.25), provide strong convergence:

$$\begin{aligned} u^0_{T,\text{ext}}(\,\cdot\,, 0; x) &\to u^0_\infty(\,\cdot\,; x) \quad \text{in } L_2(0, \infty; U), \\ Ry^0_{T,\text{ext}}(\,\cdot\,, 0; x) &\to Ry^0(\,\cdot\,; x) \quad \text{in } L_2(0, \infty; Z). \end{aligned} \tag{2.3.3.26}$$

Thus, returning to (2.3.3.17) and taking this time the $L_2(0, \infty; Y)$-strong limit, we obtain in view of (2.3.3.22) and (2.1.17) that, in fact,

$$\hat{y}^0_{T,\text{ext}}(\,\cdot\,, 0; x) \to \hat{y}^0(\,\cdot\,; x) \quad \text{in } L_2(0, \infty; U), \tag{2.3.3.27}$$

which is precisely (2.3.3.5b), in view of (2.3.1.3a) and (2.3.3.6).

(v) Equation (2.3.3.6) follows now from (2.3.3.18) via (2.3.3.22). The proof of Theorem 2.3.3.1 is complete. □

Lemma 2.3.3.2 *Let $f_T(t) \in L_2(0, T; Y)$ for all $T > 0$ and extend $f_T(t)$ by zero for $t > T$. Let $f(t) \in L_2(0, \infty; Y)$ and assume that*

$$\|f_T(\,\cdot\,) - f(\,\cdot\,)\|_{L_2(0,\infty;Y)} \to 0 \quad \text{as } T \uparrow \infty. \tag{2.3.3.28}$$

Then, with reference to the operators $\hat{L}^*$ and $\hat{L}^*_{(T)}$ defined by (2.1.18) and (2.3.1.13) respectively, we have

$$\left\|\hat{L}^* f - \hat{L}^*_{(T)} f_T\right\|_{L_2(0,\infty;U)} \to 0 \quad \text{as } T \uparrow \infty, \tag{2.3.3.29}$$

where $(\hat{L}^*_{(T)} f_T)(t)$ is extended by zero for $t > T$.

Proof.

Step 1 By the definitions (2.1.18) and (2.3.1.13) we obtain

$$\left\| \hat{L}^* f - \hat{L}^*_{(T)} f_T \right\|_{L_2(0,\infty;U)} = \int_0^\infty \left\| (\hat{L}^* f)(t) - \left(\hat{L}^*_{(T)} f_T\right)(t) \right\|_U^2 dt$$
$$= (1) + (2), \tag{2.3.3.30}$$

where

$$(1) = \int_0^T \left\| (\hat{L}^* f)(t) - \left(\hat{L}^*_{(T)} f_T\right)(t) \right\|_U^2 dt$$
$$= \int_0^T \left\| \int_t^\infty B^* e^{-\hat{A}^*(\tau - t)} f(\tau)\, d\tau - \int_t^T B^* e^{-\hat{A}^*(\tau - t)} f_T(\tau)\, d\tau \right\|_U^2 dt$$
$$= \int_0^T \left\| \int_t^T B^* e^{-\hat{A}^*(\tau - t)} (f(\tau) - f_T(\tau))\, d\tau + \int_T^\infty B^* e^{-\hat{A}^*(\tau - t)} f(\tau)\, d\tau \right\|_U^2 dt; \tag{2.3.3.31}$$

$$(1) \leq 2 \int_0^T \left\| \int_t^T B^* e^{-\hat{A}^*(\tau - t)} (f(\tau) - f_T(\tau))\, d\tau \right\|_U^2 dt$$
$$+ 2 \int_0^T \left\| \int_T^\infty B^* e^{-\hat{A}^*(\tau - t)} f(\tau)\, d\tau \right\|_U^2 dt; \tag{2.3.3.32}$$

$$(2) = \int_T^\infty \|(\hat{L}^* f)(t)\|_U^2\, dt \to 0 \quad \text{as } T \uparrow \infty; \tag{2.3.3.33}$$

since, by (2.1.18), $\hat{L}^* f \in L_2(0, \infty; U)$ when $f \in L_2(0, \infty; Y)$, as assumed.

Step 2 In preparation to the analysis of the two terms on the right of (2.3.3.32), we note that:

(i) for $t \leq T$, setting $\tau = T + \sigma$, and recalling $\hat{\omega} > 0$ and $\gamma < 1$ from (2.1.4a) and (2.1.7), we have

$$\int_T^\infty \frac{e^{-\hat{\omega}(\tau - t)}\, d\tau}{(\tau - t)^\gamma} = \int_0^\infty \frac{e^{-\hat{\omega}(\sigma + T - t)}\, d\sigma}{(\sigma + T - t)^\gamma} \leq \int_0^\infty \frac{e^{-\hat{\omega}\sigma}}{\sigma^\gamma}\, d\sigma = c < \infty; \tag{2.3.3.34}$$

(ii) for $\tau \geq T$, we similarly obtain

$$\int_0^T \frac{e^{-\hat{\omega}(\tau - t)}}{(\tau - t)^\gamma}\, dt = \int_{\tau - T}^\tau \frac{e^{-\hat{\omega}\sigma}}{\sigma^\gamma}\, d\sigma \leq \int_0^\infty \frac{e^{-\hat{\omega}\sigma}}{\sigma^\gamma}\, d\sigma = c < \infty; \tag{2.3.3.35}$$

(iii) for $t \leq T$,

$$\int_t^T \frac{e^{-\hat{\omega}(\tau - t)}\, d\tau}{(\tau - t)^\gamma} = \int_0^{T - t} \frac{e^{-\hat{\omega}\sigma}}{\sigma^\gamma}\, d\sigma \leq \int_0^\infty \frac{e^{-\hat{\omega}\sigma}}{\sigma^\gamma}\, d\sigma = c < \infty. \tag{2.3.3.36}$$

Step 3 We estimate the second term on the right of (2.3.3.2). By (2.1.7) and (2.1.4b) with $\theta = \gamma$ we have

$$\int_0^T \left\| \int_T^\infty B^* e^{-\hat{A}^*(\tau - t)} f(\tau)\, d\tau \right\|_U^2 dt$$

$$\leq C_\gamma \int_0^T \left[\int_T^\infty \frac{e^{-\hat{\omega}(\tau - t)}}{(\tau - t)^\gamma} \|f(\tau)\|_Y\, d\tau \right]^2 dt, \tag{2.3.3.37}$$

(splitting $\gamma = \gamma/2 + \gamma/2$ and using Schwarz's inequality)

$$\leq C_\gamma \int_0^T \left(\int_T^\infty \frac{e^{-\hat{\omega}(\tau - t)}\, d\tau}{(\tau - t)^\gamma} \right) \left(\int_T^\infty \frac{e^{-\hat{\omega}(\tau - t)}}{(\tau - t)^\gamma} \|f(\tau)\|_Y^2\, d\tau \right) dt, \tag{2.3.3.38}$$

(using (2.3.3.34) and changing the order of integration)

$$\leq C_\gamma \int_0^T \left(\int_T^\infty \frac{e^{-\hat{\omega}(\tau - t)}}{(\tau - t)^\gamma} \right) \|f(\tau)\|_Y^2\, d\tau\, dt$$

$$\leq C_\gamma \int_T^\infty \|f(\tau)\|_Y^2 \left(\int_0^T \frac{e^{-\hat{\omega}(\tau - t)}\, dt}{(\tau - t)^\gamma} \right) d\tau, \tag{2.3.3.39}$$

(using (2.3.3.35))

$$\leq C_\gamma \int_T^\infty \|f(\tau)\|_Y^2\, d\tau \to 0 \quad \text{as } T \uparrow \infty, \tag{2.3.3.40}$$

as desired.

Step 4 We estimate the first term on the right of (2.3.3.32) in a manner similar to Step 3:

$$\int_0^T \left\| \int_t^T B^* e^{-\hat{A}^*(\tau - t)} (f(\tau) - f_T(\tau))\, d\tau \right\|_U^2 dt$$

$$\leq C_\gamma \int_0^T \left[\int_t^T \frac{e^{-\hat{\omega}(\tau - t)}}{(\tau - t)^\gamma} \|(f(\tau) - f_T(\tau)\|_Y\, d\tau \right]^2 dt \tag{2.3.3.41}$$

$$\leq C_\gamma \int_0^T \left(\int_t^T \frac{e^{-\hat{\omega}(\tau - t)}\, d\tau}{(\tau - t)^\gamma} \right) \left(\int_t^T \frac{e^{-\hat{\omega}(\tau - t)}}{(\tau - t)^\gamma} \|(f(\tau) - f_T(\tau)\|_Y^2\, d\tau \right)^2 dt, \tag{2.3.3.42}$$

(using (2.3.3.36) and changing the order of integration)

$$\leq C_\gamma \int_0^T \|f(\tau) - f_T(\tau)\|_Y^2 \left(\int_0^\tau \frac{e^{-\hat{\omega}(\tau - t)}}{(\tau - t)^\gamma}\, dt \right) d\tau, \tag{2.3.3.43}$$

(using (2.3.3.35) with $T = \tau$)

$$\leq C_Y \int_0^T \|f(\tau) - f_T(\tau)\|_Y^2 \, d\tau \to 0 \quad \text{as } T \uparrow \infty, \tag{2.3.3.44}$$

upon invoking assumption (2.3.3.28).

Step 5 We use (2.3.3.40) and (2.3.3.44) in (2.3.3.32) along with (2.3.3.33) and (2.3.3.30) to obtain the desired conclusion (2.3.3.29). □

Theorem 2.3.3.3 *Assume (H.1)–(H.3) and the Finite Cost Condition (FCC) (H.4) = (2.1.12). For the optimal control $u^0(\cdot\,; x)$ claimed in Theorem 2.3.3.1(i), $x \in Y$, the following feedback form holds true in $L_2(0, \infty; U)$:*

$$-\hat{u}^0(t; x) = -e^{-\omega t} u^0(t; x) = \{\hat{L}^*(R^*R + 2\omega P)\hat{\Phi}(\cdot)x\}(t) \tag{2.3.3.45a}$$

$$\equiv \int_t^\infty B^* e^{-\hat{A}^*(\tau - t)}(R^*R + 2\omega P)\hat{\Phi}(\tau)x \, d\tau. \tag{2.3.3.45b}$$

Proof.

Step 1 We return to (2.3.1.14) giving $\hat{u}_T^0$ for the optimal problem (2.3.1.1) on $[0, T]$ and take here the $L_2(0, \infty; U)$-limit; on the left-hand side we use the $L_2(0, \infty; U)$-convergence of $\hat{u}_T^0$ to $\hat{u}^0$ as in (2.3.3.5a). The right-hand side of (2.3.1.14) is handled below.

Step 2 We apply Lemma 2.3.3.2 with

$$f_T(t) = R^*R\hat{\Phi}_T(t, 0)x; \qquad f(t) = R^*R\hat{\Phi}(t)x, \quad x \in Y, \tag{2.3.3.46}$$

and note that with such a choice, the assumptions of Lemma 2.3.3.2 are fulfilled, in particular (2.3.3.28) is guaranteed by (2.3.3.5b). From Lemma 2.3.3.2, we obtain

$$\left\| \hat{L}^*R^*R\hat{\Phi}(\cdot)x - \hat{L}_{(T)}^*R^*R\hat{\Phi}_T(\cdot\,, 0)x \right\|_{L_2(0,\infty;U)} \to 0 \quad \text{as } T \uparrow \infty. \tag{2.3.3.47}$$

Step 3 We again apply Lemma 2.3.3.2, this time with

$$f_T(t) = P_T(t)\hat{\Phi}_T(t, 0)x; \quad f(t) = P\hat{\Phi}(t)x, \quad x \in Y. \tag{2.3.3.48}$$

To verify the validity of assumption (2.3.3.28) of Lemma 2.3.3.2 in this case, we estimate, after adding and subtracting and recalling (2.3.3.2b),

$$\begin{aligned} \|f_T(t) - f(t)\|_Y &= \|P_T(t)\hat{\Phi}_T(t, 0)x - P\hat{\Phi}(t)x\|_Y \\ &= \|P_T(t)[\hat{\Phi}_T(t, 0)x - \hat{\Phi}(t)x] + [P_T(t) - P]\hat{\Phi}(t)x\|_Y \\ \text{(by (2.3.3.2b))} \qquad &\leq M\|\hat{\Phi}_T(t, 0)x - \hat{\Phi}(t)x\|_Y + \|[P_T(t) - P]\hat{\Phi}(t)x\|_Y. \end{aligned} \tag{2.3.3.49}$$

Thus, from (2.3.3.48) and (2.3.3.49), we obtain

$$\begin{aligned}\int_0^T \|f_T(t)-f(t)\|_Y^2 &= \int_0^T \|P_T(t)\hat{\phi}_T(t,0)x - P\hat{\phi}(t)x\|_Y^2\,dt \\ &\leq C\left\{\int_0^T \|\hat{\phi}_T(t,0)x-\hat{\phi}(t)x\|_Y^2\,dt\right. \\ &\quad \left.+\int_0^T \|[P_T(t)-P]\hat{\phi}(t)x\|_Y^2\,dt\right\} \qquad (2.3.3.50)\\ &\to 0 \quad \text{as } T\uparrow\infty, \qquad (2.3.3.51)\end{aligned}$$

where the limit in (2.3.3.51) goes to zero as $T \uparrow \infty$, since

$$\int_0^T \|\hat{\phi}_T(t,0)x-\hat{\phi}(t)x\|_Y^2\,dt \to 0 \quad \text{as } T\uparrow\infty, \tag{2.3.3.52}$$

by (2.3.3.5b), while

$$\int_0^T \|[P_T(t)-P]\hat{\phi}(t)x\|_Y^2\,dt \to 0 \quad \text{as } T\uparrow\infty, \tag{2.3.3.53}$$

by an application of the Lebesgue dominated convergence theorem, via the convergence (2.3.3.2a) (right), so that

$$\lim_{T\uparrow\infty} \|(P_T(t)-P)\hat{\Phi}(t)x\|_Y = 0, \quad \text{a.e. in } t \text{ fixed } < T\uparrow\infty, \tag{2.3.3.54}$$

and, moreover, recalling the uniform bound (2.3.3.2b), we have, as required

$$\|(P_T(t)-P)\hat{\Phi}(t)x\|_Y^2 \leq C\|\hat{\Phi}(t)x\|^2 \in L_1(0,\infty). \tag{2.3.3.55}$$

Then, the convergence in (2.3.3.51) verifies assumption (2.3.3.28). Thus, Lemma 2.3.3.2 yields, as desired,

$$\|\hat{L}_{(T)}P_T(\cdot)\hat{\Phi}_T(\cdot,0)x - \hat{L}^*P\hat{\Phi}(\cdot)x\|_{L_2(0,\infty;U)} \to 0 \quad \text{as } T\to\infty. \tag{2.3.3.56}$$

Step 4 We combine (2.3.3.47) and (2.3.3.56) to obtain

$$\lim_{T\uparrow\infty} \left\|\hat{L}^*_{(T)}[R^*R+2\omega P_T(\cdot)]\hat{\phi}_T(\cdot,0)x - \hat{L}^*[R^*R+2\omega P]\hat{\phi}(\cdot)x\right\|_{L_2(0,\infty;U)} = 0, \tag{2.3.3.57}$$

which is the desired convergence result on the right-hand side of (2.3.1.14). Then, (2.3.3.5a) as in Step 1, combined with (2.3.3.57) of Step 3, yields (2.3.3.45), as the $L_2(0,\infty;U)$-limit of (2.3.1.14). The proof of Theorem 2.3.3.3 is complete. □

2.3.4 Explicit Expressions for the Optimal u^0 and y^0

Theorem 2.3.4.1 *Assume (H.1)–(H.3) and the FCC (H.4) = (2.1.12). Explicit expressions of the optimal pair $\hat{u}^0$ and $\hat{y}^0$ are given, for $x \in Y$, by*

(i)

$$\hat{y}^0(\,\cdot\,;x) = \hat{\Phi}(\,\cdot\,)x = [I + \hat{L}\hat{L}^*(R^*R + 2\omega P)]^{-1}\{e^{-\hat{A}\,\cdot}\,x\} \in L_2(0,\infty;Y), \tag{2.3.4.1}$$

(ii)

$$\hat{u}^0(\,\cdot\,;x) = -[I + \hat{L}^*(R^*R + 2\omega P)\hat{L}]^{-1}[R^*R + 2\omega P]\{e^{-\hat{A}\,\cdot}\,x\} \in L_2(0,\infty;U). \tag{2.3.4.2}$$

Proof. We have already remarked in (2.3.3.6) that the optimal dynamics is

$$\hat{y}^0(t;x) = \hat{\Phi}(t)x = e^{-\hat{A}t}x + \{\hat{L}\hat{u}^0(\,\cdot\,;x)\}(t) \in L_2(0,\infty;Y). \tag{2.3.4.3}$$

Inserting $\hat{u}^0$ from (2.3.3.45a) into (2.3.4.3) yields

$$\{[I + \hat{L}\hat{L}^*(R^*R + 2\omega P)]\hat{\Phi}(\,\cdot\,)x\}(t) = e^{-\hat{A}t}x \in L_2(0,\infty;Y). \tag{2.3.4.4}$$

Since $R^*R + 2\omega P$ is a nonnegative, self-adjoint operator on Y, the result of Appendix 2A, applies, and thus the operator in square brackets in (2.3.4.4) is boundedly invertible on $L_2(0,\infty;Y)$. We thus obtain part (i), Eqn. (2.3.4.1). Part (ii), Eqn. (2.3.4.2) follows by inserting (2.3.4.3) into (2.3.3.45a). $\square$

2.3.5 Smoothing Properties of the Operators $\hat{L}$ and $\hat{L}^*$

The next theorem provides smoothing properties of the operators $\hat{L}$ and $\hat{L}^*$ defined by (2.1.17) and (2.1.18). It is the counterpart of Chapter 1, Theorem 1.4.4.3. The proof is essentially the same and is sketched here for convenience only.

Theorem 2.3.5.1 *Assume (H.1) and (H.2). With reference to the operators $\hat{L}$ and $\hat{L}^*$ defined by (2.1.17) and (2.1.18) we have:*

(i)

$$\hat{L} : \textit{continuous } L_2(0,\infty;U) \to L_{r_1}(0,\infty;Y), \tag{2.3.5.1}$$

where r_1 is an arbitrary positive number satisfying $r_1 < 2/(2\gamma - 1)$, where $2/(2\gamma - 1) > 2$ for $1/2 \le \gamma < 1$; for $0 \le \gamma < 1/2$ we may take $r_1 = \infty$;

(ii)

$$\hat{L}^* : \textit{continuous } L_{r_1}(0,\infty;Y) \to L_{r_2}(0,\infty;U), \tag{2.3.5.2}$$

where r_1 is as in (i) and r_2 is any positive number satisfying $r_2 < 2/(4\gamma - 3)$, where $2/(4\gamma - 3) > r_1$ for $3/4 \le \gamma < 1$; for $0 \le \gamma < 3/4$ we may take $r_2 = \infty$.

(iii) Generally, let $r_0 = 2$, and let r_n, $n = 1, 2, \dots$ be arbitrary positive numbers such that

$$2 < r_1 \cdots < r_n < \frac{2}{2n\gamma - (2n-1)}, \quad n = 1, 2, \dots \textit{ for } \frac{2n-1}{2n} \le \gamma < 1. \tag{2.3.5.3}$$

Then, for $n = 0, 2, 4, \dots$, we have

$$\hat{L} : \textit{continuous } L_{r_n}(0, \infty; U) \to L_{r_{n+1}}(0, \infty; Y), \tag{2.3.5.4}$$

where for $0 \le \gamma < \frac{2(n+1)-1}{2(n+1)}$ we may take $r_{n+1} = \infty$ and

$$\hat{L}^* : \textit{continuous } L_{r_{n+1}}(0, \infty; Y) \to L_{r_{n+2}}(0, \infty; U), \tag{2.3.5.5}$$

where r_{n+1} in (2.3.5.5) is the same as in (2.3.5.4), and where for $0 \le \gamma < \frac{2(n+2)-1}{2(n+2)}$ we may take $r_{n+2} = \infty$.

(iv) *For $p > \frac{1}{1-\gamma}$,*

$$\hat{L} : \textit{continuous } L_p(0, \infty; U) \to C_{ub}([0, \infty]; Y), \tag{2.3.5.6}$$

where $C_{ub}([0, \infty]; Y)$ denotes the space of Y-valued continuous functions on $0 \le t < \infty$, uniformly bounded on $[0, \infty]$.

(v)

$$\hat{L}^* : \textit{ continuous } C_{ub}([0, \infty]; Y) \to C_{ub}([0, \infty]; U). \tag{2.3.5.7}$$

Proof. It is closely related to the proof of Theorem 1.4.4.3 of Chapter 1 and uses assumption (2.1.7) on B, (2.1.4b) for $\theta = \gamma$, and the Young's inequality [Sadosky, 1979, p. 149].

(i) By (2.1.7) and (2.1.4b), we have from (2.1.17)

$$\begin{aligned} \|(\hat{L}u)(t)\|_Y &= \left\| \int_0^t (\hat{A})^\gamma e^{-\hat{A}(t-\tau)} (\hat{A})^{-\gamma} Bu(\tau)\, d\tau \right\|_Y \\ &\le C \int_0^t \frac{\hat{M} e^{-\hat{\omega}(t-\tau)} \|u(\tau)\|_U \, d\tau}{(t-\tau)^\gamma} \in L_{r_1}(0, \infty), \end{aligned} \tag{2.3.5.8}$$

by the same argument as in the proof of Theorem 1.4.4.3 of Chapter 1 based on the Young's inequality with $\|f(t)\|_Y \in L_{r_1}(0, \infty)$, $1/r_1 = 1/q + 1/2 - 1$ where $\|u(t)\|_U \in L_2(0, \infty)$, $e^{-\hat{\omega} t}/t^\gamma \in L_q(0, \infty)$ with $\gamma q < 1$.

(ii) Similarly, by (2.1.18), (2.1.4b), and (2.1.7),

$$\begin{aligned} \|(\hat{L}^* f)(t)\|_U &= \left\| \int_t^\infty B^* (\hat{A}^*)^{-\gamma} e^{-\hat{A}^*(\tau - t)} (\hat{A}^*)^\gamma f(\tau)\, d\tau \right\|_U \\ &\le C \int_t^\infty \frac{e^{-\hat{\omega}(\tau - t)} \|f(\tau)\|_Y \, d\tau}{(\tau - t)^\gamma} \in L_{r_2}(0, \infty), \end{aligned} \tag{2.3.5.9}$$

again by using Young's inequality with $\|f(t)\|_Y \in L_{r_1}(0, \infty)$, $1/r_2 = 1/q + 1/r_1 - 1$, $\gamma q < 1$, as in the proof of Theorem 1.4.4.3 of Chapter 1.

For the proof of parts (iii) and (iv) we refer to the arguments in the corresponding parts (iii) and (iv) of Theorem 1.4.4.3, which use [Sadosky, 1979, p. 29]. The argument for the estimates of part (v) is contained in the argument for the estimates of part (iv). □

As a corollary of Theorem 2.3.5.1 we obtain the desired improvement from $L_2(0,\infty;\,\cdot\,)$ to $C_{ub}([0,\infty];\,\cdot\,)$ for both $\hat{u}^0(t;x)$ and $\hat{y}^0(t;x)$.

Corollary 2.3.5.2 *Assume (H.1)–(H.3) and the FCC (H.4) = (2.1.12). With reference to the optimal pair $\{u^0(t;x), y^0(t;x)\}$, $x \in Y$ guaranteed by Theorem 2.3.3.1(i) we have:*

(i)

$$\begin{cases} \hat{y}^0(t;x) \equiv \hat{\Phi}(t)x \in C_{ub}([0,\infty];Y), & (2.3.5.10a)\\ y^0(t;x) = \Phi(t)x \in C([0,T_0];Y), \quad \forall\, T_0 < \infty; & (2.3.5.10b)\end{cases}$$

(ii)

$$\begin{cases} \hat{u}^0(t;x) \in C_{ub}([0,\infty];U), & (2.3.5.11a)\\ u^0(t;x) \in C([0,T_0];U), \quad \forall\, T_0 < \infty; & (2.3.5.11b)\end{cases}$$

(iii)

$$\begin{cases} \hat{y}^0_T(t,0;x) = \hat{\Phi}_T(t,0)x \to \hat{y}^0(t;x) & \\ \qquad = \hat{\Phi}(t)x \textit{ in } C_{ub}([0,\infty];Y), & (2.3.5.12a)\\ y^0_T(t,0;x) = \Phi_T(t,0)x \to y^0(t;x) & \\ \qquad = \Phi(t)x \textit{ in } C([0,T_0];Y), \quad \forall\, T_0 < \infty; & (2.3.5.12b)\end{cases}$$

(iv)

$$\begin{cases} \hat{u}^0_T(t,0;x) \to \hat{u}^0(t;x) \textit{ in } C_{ub}([0,\infty];U), & (2.3.5.13a)\\ u^0(t,0;x) \to u^0(t;x) \textit{ in } C([0,T_0];U), \quad \forall\, T_0 < \infty; & (2.3.5.13b)\end{cases}$$

(v) for $\gamma + \epsilon < 1$, and $x \in Y$,

$$\{\hat{A}^{\epsilon}\hat{L}u^0(\,\cdot\,;x)\}(t) = \int_0^t \hat{A}^{\gamma+\epsilon}e^{-\hat{A}(t-\tau)}\hat{A}^{-\gamma}Bu^0(\tau;x)\,d\tau \qquad (2.3.5.14)$$

$$\in C([0,T_0];Y), \quad \forall\, T_0 < \infty. \qquad (2.3.5.15)$$

Proof. (i) and (ii). We already know that $\hat{u}^0(\,\cdot\,;x) \in L_2(0,\infty;U)$ a fortiori from Theorem 2.3.1.1. We then apply Theorem 2.3.5.1(i), Eqn. (2.3.5.1) to get $\hat{L}\hat{u}^0 \in L_{r_1}(0,\infty;Y)$. Since $e^{-\hat{A}t}$ is exponentially stable as in (2.1.4a), we obtain from the optimal dynamics (2.3.4.3) that $\hat{\Phi}(t)x \in L_{r_1}(0,\infty;Y)$ as well. From here we apply now Theorem 2.3.5.1(ii), Eqn. (2.3.5.2), to $\hat{L}^*$ in (2.3.3.45a) and find that $\hat{u}^0(\,\cdot\,;x) \in L_{r_2}(0,\infty;U)$, an improvement over the a priori regularity of $\hat{u}^0(\,\cdot\,;x)$ at the start. We repeat this bootstrap argument on $\hat{L}$ and $\hat{L}^*$ in (2.3.4.3) and (2.3.3.45a) using Theorem 2.3.5.1(iii), (iv), and (v) and readily obtain (2.3.5.10a) for $\hat{y}^0(\,\cdot\,;x)$ and (2.3.5.11a) for $\hat{u}^0(\,\cdot\,;x)$, from which (2.3.5.10b) and (2.3.5.11b) follow at once via (2.3.3.5c).

The same bootstrap argument, starting from the L_2-convergence in (2.3.3.4a) for $u_T^0 \to u^0$, yields the C-convergence in parts (iii) and (iv), Eqns. (2.3.5.12) and (2.3.5.13). □

(v) By (2.1.7) and (2.3.5.11) we have

$$\hat{A}^{-\gamma} B u^0(\cdot\,; x) \in C([0, T_0]; Y), \quad x \in Y, \ \forall\, T_0 < \infty. \tag{2.3.5.16}$$

Moreover, by analyticity as in (2.1.4b),

$$\|\hat{A}^{\gamma+\epsilon} e^{-\hat{A}t}\|_{\mathcal{L}(Y)} \leq \frac{\hat{M} e^{-\hat{\omega} t}}{t^{\gamma+\epsilon}} \in L_1(0, \infty). \tag{2.3.5.17}$$

By Young inequality [Sadosky, 1979, p. 29] applied on (2.3.5.14) via (2.3.5.16) and (2.3.5.17), we obtain (2.3.5.15). □

Remark 2.3.5.1 Under the Detectability Condition $(H.5) = (2.1.13)$, we shall have that $u^0(t; x) \in C_{ub}([0, \infty]; U)$ for $x \in Y$ [see (2.2.12) of Theorem 2.2.1]. In this case, then, we may take $T_0 = \infty$ in (2.3.5.15).

2.3.6 The Operators $\Phi(t)$ and $\hat{\Phi}(t)$ Are Strongly Continuous Semigroups on Y

Theorem 2.3.6.1 *Assume (H.1)–(H.3) and the FCC (H.4) = (2.1.12). The operators $\Phi(t)$ and $\hat{\Phi}(t)$ in (2.3.3.5c) are strongly continuous semigroups on Y. Moreover, $\hat{\Phi}(t)$ is exponentially stable on Y, and so is $\phi(t)$ if $R^*R \geq \rho I$, $\rho > 0$.*

Proof. Strong continuity at the origin is proved in (2.3.5.10) of Corollary 2.3.5.2. We next show that the semigroup property of $\Phi(t)$ is inherited from $\Phi_T(t, 0)$ in (2.3.1.3a) via its evolution property (2.3.1.3b) and Lemma 2.3.2.1(i), Eqn. (2.3.2.1). Accordingly, with $x \in Y$, we write for fixed $0 < \tau \leq T$ and $0 < t \leq T$:

$$\begin{aligned}\Phi_T(t+\tau, 0)x &= \Phi_T(t+\tau, \tau)\Phi_T(\tau, 0)x = \Phi_{T-\tau}(t, 0)\Phi_T(\tau, 0)x \\ &= \Phi_{T-\tau}(t, 0)[\Phi_T(\tau, 0)x - \Phi(\tau)x] + \Phi_{T-\tau}(t, 0)\Phi(\tau)x,\end{aligned} \tag{2.3.6.1}$$

using (2.3.1.3b) and (2.3.2.1). But

$$\lim_{T\uparrow\infty} \Phi_{T-\tau}(t, 0)\Phi(\tau)x = \Phi(t)\Phi(\tau)x \tag{2.3.6.2}$$

by (2.3.5.12) and τ fixed. Moreover, (2.3.5.12) implies that, for t fixed,

$$\|\Phi_{T-\tau}(t, 0)\|_{\mathcal{L}(Y)} \leq C_t, \quad \text{as } T \uparrow \infty,$$

via the Principle of Uniform Boundedness. Hence

$$\begin{aligned}&\|\Phi_{T-\tau}(t, 0)[\Phi_T(\tau, 0)x - \Phi(\tau)x]\|_Y \\ &\quad \leq C_t \|\Phi_T(\tau, 0)x - \Phi(\tau)x\|_Y \to 0, \quad \text{as } T \uparrow \infty,\end{aligned} \tag{2.3.6.3}$$

again by (2.3.5.12). Taking the limit in (2.3.6.1) and using (2.3.6.2), (2.3.6.3), and (2.3.5.12), we obtain

$$\Phi(t+\tau)x = \Phi(t)\Phi(\tau)x, \quad x \in Y, \tag{2.3.6.4}$$

which is the desired semigroup property for $\Phi(t)$. Recalling (2.3.3.5c), we multiply (2.3.6.4) by $e^{-\omega(t+\tau)}$ and obtain the semigroup property for $\hat{\Phi}(t)$:

$$\hat{\Phi}(t+\tau)x = \hat{\Phi}(t)\hat{\Phi}(\tau)x, \quad x \in Y. \tag{2.3.6.5}$$

Thus both $\Phi(t)$ and $\hat{\Phi}(t)$ are strongly continuous semigroups on Y. Therefore, the exponential stability of $\hat{\Phi}(t)$, and (if $R^*R \geq \rho I,\ p > 0$) of $\Phi(t)$, follows now via a well-known result [Datko, 1970], since $(R^*R)^{1/2}\Phi(t)x \in L_2(0,\infty;Y),\ \forall\, x \in Y$, by (2.3.3.4b) and $\hat{\Phi}(t)x \in L_2(0,\infty;Y),\ \forall\, x \in Y$ by (2.3.3.5b) of Theorem 2.3.3.1.

Remark 2.3.6.1 The exponential stability of $\Phi(t),\ \hat{\Phi}(t)$ will be greatly generalized in Section 2.4.

2.3.7 A Relation for P and a Pointwise Feedback Expression for $u^0(t;x)$

Theorem 2.3.7.1 *Assume (H.1)–(H.3) and the FCC (H.4) = (2.1.12). For $x \in Y$ we have:*

(i)

$$Px = \int_0^\infty e^{-\hat{A}^*\sigma}[R^*R + 2\omega P]\hat{\Phi}(\sigma)x\,d\sigma \tag{2.3.7.1a}$$

$$= \int_t^\infty e^{-\hat{A}^*(\tau-t)}[R^*R + 2\omega P]\hat{\Phi}(\tau - t)x\,d\tau. \tag{2.3.7.1b}$$

(ii)

$$\textit{Range of } P = PY \subset \mathcal{D}((\hat{A}^*)^\theta), \quad \forall\, 0 \leq \theta < 1, \tag{2.3.7.2}$$

and

$$(\hat{A}^*)^\theta Px = \int_0^\infty (\hat{A}^*)^\theta e^{-\hat{A}^*\sigma}[R^*R + 2\omega P]\hat{\Phi}(\sigma)x\,d\sigma, \tag{2.3.7.3}$$

$$(\hat{A}^*)^\theta P \in \mathcal{L}(Y); \quad 0 \leq \theta < 1 \tag{2.3.7.4a}$$

so that P is compact in $\mathcal{L}(Y)$, if A has, moreover, compact resolvent; in (2.3.7.4a), one can also take $\theta = 1$, if A is self-adjoint and $R = I$, or else R^*R commutes with A (see Appendix 2B):

$$(\hat{A}^*)^\alpha P\hat{A}^\beta \in \mathcal{L}(Y), \alpha + \beta < 1; \quad \alpha, \beta \geq 0; \quad (\hat{A}^*)^r P\hat{A}^r \in \mathcal{L}(Y),$$
$$0 \leq r < \frac{1}{2}. \tag{2.3.7.4b}$$

(iii)

$$B^*P \in \mathcal{L}(Y;U). \tag{2.3.7.5}$$

Proof. (i) We return to Eqn. (2.3.1.9) for $P_T(t)x$, change variables by setting $\sigma = \tau - t$, and invoke Lemma 2.3.2.1, Eqn. (2.3.2.1) for Φ_T and Eqn. (2.3.2.2) for P_T.

We thus obtain for $x \in Y$:

$$P_T(t)x = \int_0^{T-t} e^{-\hat{A}\sigma}[R^*R + 2\omega P_T(t+\sigma)]\Phi_T(\sigma+t,t)x\,d\sigma$$
$$= \int_0^{T-t} e^{-\hat{A}^*\sigma}[R^*R + 2\omega P_{T-t-\sigma}(0)]\hat{\Phi}_{T-t}(\sigma,0)x\,d\sigma. \qquad (2.3.7.6)$$

We now take the limit in (2.3.7.6) as $T \uparrow \infty$. On the left-hand side we obtain Px by (2.3.3.2a). On the right-hand side we find ourself in a situation more amenable than that in Theorem 2.3.3.3 for $\hat{u}^0$.

Step 1 As in Step 3, Eqn. (2.3.3.49), of the proof of Theorem 2.3.3.3, we write for $x \in Y$:

$$P_{T-t-\sigma}(0)\hat{\Phi}_{T-t}(\sigma,0)x - P\hat{\Phi}(\sigma)x$$
$$= P_{T-t-\sigma}(0)[\hat{\Phi}_{T-t}(\sigma,0)x - \hat{\Phi}(\sigma)x] + [P_{T-t-\sigma}(0) - P]\hat{\Phi}(\sigma)x, \qquad (2.3.7.7)$$

where in view of the uniform bound (2.3.3.2b) we have

$$\|P_{T-t-\sigma}(0)[\hat{\Phi}_{T-t}(\sigma,0)x - \hat{\Phi}(\sigma)x]\|_{C_{ub}([0,\infty];Y)}$$
$$\leq M\|\hat{\Phi}_{T-t}(\cdot\,,0)x - \hat{\Phi}(\cdot\,)x\|_{C_{ub}([0,\infty];Y)} \to 0, \qquad (2.3.7.8)$$

as $T \to \infty$, and t fixed $< T$, upon invoking the convergence in (2.3.5.12a).

Step 2 With reference to (2.3.7.6) we estimate by (2.1.4a):

$$\left\| \int_0^{T-t} e^{-\hat{A}^*\sigma} R^*R[\hat{\Phi}_{T-t}(\sigma,0)x - \hat{\Phi}(\sigma)x]\,d\sigma \right\|_Y$$
$$\leq C\left(\int_0^{T-t} e^{-\hat{\omega}\sigma}d\sigma\right) \|\hat{\Phi}_{T-t}(\cdot\,,0)x - \hat{\Phi}(\cdot\,)x\|_{C_{ub}([0,\infty];Y)}$$
$$\to 0 \text{ as } T \uparrow \infty, \qquad (2.3.7.9)$$

invoking again (2.3.5.12a) with t fixed $<T$.

Step 3 As to the first term on the right of (2.3.7.7), we estimate

$$\left\| \int_0^{T-t} e^{-\hat{A}^*\sigma} P_{T-t-\sigma}(0)[\hat{\Phi}_{T-t}(\sigma,0)x - \hat{\Phi}(\sigma)x]\,d\sigma \right\|_Y$$
$$\leq C\left(\int_0^{T-t} e^{-\hat{\omega}\sigma}d\sigma\right) \|\hat{\Phi}_{T-t}(\cdot\,,0)x - \hat{\Phi}(\cdot\,)x\|_{C_{ub}([0,\infty];Y)}$$
$$\to 0 \text{ as } T \uparrow \infty, \qquad (2.3.7.10)$$

by (2.3.3.2b), (2.1.4a), and (2.3.5.12a).

Step 4 As to the second term on the right of (2.3.7.7), we use

$$\lim_{T\uparrow\infty}\int_0^{T-t} e^{-\hat{A}^*\sigma}[(P_{T-t-\sigma}(0)-P)\hat{\Phi}(\sigma)x]\,d\sigma$$
$$=\int_0^{\infty}\lim_{T\uparrow\infty}\left\{e^{-\hat{A}^*\sigma}[(P_{T-t-\sigma}(0)-P)\hat{\Phi}(\sigma)x]\mathcal{X}_{[0,T-t]}(\sigma)\right\}d\sigma=0, \tag{2.3.7.11}$$

where $\mathcal{X}$ is the characteristic function on $[0, T-t]$, upon using the pointwise limit (2.3.3.2a) and the Lebesgue dominated convergence theorem, which is legal since the integrand is dominated in norm uniformly in T by the scalar function $Ce^{-\hat{\omega}\sigma}\|\hat{\Phi}(\sigma)x\|_Y$ via (2.3.3.2b), which is an $L_1(0,\infty)$-function, by $\hat{\omega}>0$ via (2.1.4a), and $\hat{\phi}(\sigma)x\in C_{ub}([0,\infty]; Y)$ by (2.3.5.12a). But the $L_2(0,\infty; Y)$ result of (2.3.3.6) suffices.

Step 5 We return to (2.3.7.6) and take the limit as $T\uparrow\infty$. Recalling (2.3.7.7), (2.3.7.10), (2.3.7.11), as well as (2.3.7.9) on the right-hand side of (2.3.7.6), we obtain (2.3.7.1a), as desired. Then, (2.3.7.1b) follows from (2.3.7.1a) via $\sigma=\tau-t$. Part (i) is proved.

(ii) Equation (2.3.7.3) is a consequence of part (i): To show that the integral in (2.3.7.3) is well defined in Y, we note that, via (2.1.4b), the integrand is majorized in the Y-norm by the scalar function

$$f(\sigma)=C\,\frac{e^{-\hat{\omega}\sigma}}{\sigma^\theta}\,\|\hat{\Phi}(\sigma)x\|_Y,$$

where $f\in L_1(0,\infty)$ for $\theta<1$ since $\hat{\Phi}(\cdot)x\in C_{ub}([0,\infty]; Y)$ by (2.3.5.12a). The closed graph theorem then yields (2.3.7.4a) from (2.3.7.3). Instead, to obtain (2.3.7.4b), we use (2.3.7.4a) and invoke Lemma 1.5.1.1 and Corollary 1.5.1.2 of Chapter 1.

(iii) With $B^*(\hat{A}^*)^{-\theta}\in\mathcal{L}(Y;U)$ by (2.1.7) with $\gamma\le\theta<1$, we obtain

$$B^*P=B^*(\hat{A}^*)^{-\theta}(\hat{A}^*)^\theta P\in\mathcal{L}(Y;U), \tag{2.3.7.12}$$

via (2.3.7.4) of part (ii), and (2.3.7.5) is proved. □

Corollary 2.3.7.2 *Assume (H.1)–(H.3) and the FCC (H.4) = (2.1.12). For $x\in Y$ we have*

(i)

$$-\hat{u}^0(t;x)=-e^{-\omega t}u^0(t;x)=\{\hat{L}^*[R^*R+2\omega P]\hat{\Phi}(\cdot)x\}(t) \tag{2.3.7.13}$$
$$=B^*P\hat{\Phi}(t)x\in C_{ub}([0,\infty]; U); \tag{2.3.7.14}$$

(ii)

$$u^0(t;x)=-B^*P\Phi(t)x\in L_2(0,\infty; U)\cap C([0,T_0]; U),\quad \forall\, T_0<\infty. \tag{2.3.7.15}$$

Equation (2.3.7.14) expresses the desired pointwise relation in feedback form.

(iii) Assume that $R^*R \geq \rho I,\ \rho > 0$. *Then*

$$\begin{cases} \|u^0(t;x)\|_U \leq M_\delta e^{-\delta t}\|x\|_Y, & \delta > 0,\ t \geq 0,\ x \in Y, \\ u^0(t;x) \in C_{ub}([0,\infty];U) \end{cases} \tag{2.3.7.16}$$

in this case (this result will be greatly generalized in Section 2.4). Thus from (ii) and (iii), we have

$$B^*P\Phi(t) : \textit{continuous} \begin{cases} Y \to L_2(0,\infty;U) \cap C([0,T_0];U), & (2.3.7.17a) \\ Y \to C_{ub}([0,\infty];U), \quad \textit{if } R^*R \geq \rho I. & (2.3.7.17b) \end{cases}$$

Proof. We return to (2.3.3.45b): To obtain (2.3.7.13), it then suffices to invoke (2.1.18) on $\hat{L}^*$; instead, to obtain (2.3.7.14), we write $\hat{\phi}(\tau)x = \hat{\phi}(\tau - t)\hat{\phi}(t)x$ by (2.3.6.5) and then invoke (2.3.7.1b) for P. The regularity in (2.3.7.14) was already obtained in (2.3.5.11a).

That $u^0(t;x) \in C_{ub}([0,\infty];U)$ (improving upon (2.3.5.11b)) is a consequence of (2.3.7.5) of Theorem 2.3.7.1 for B^*P and of Φ being a exponentially stable semigroup by Theorem 2.3.6.1 when $R^*R \geq \rho I,\ \rho > 0$.

Remark 2.3.7.1 Exponential stability of $\Phi(t)$, hence the indicated regularity $u^0(\cdot\,;x) \in C_{ub}([0,\infty];U)$ in (2.3.7.15), will be achieved in Section 2.4 under far greater generality than $R^*R \geq \rho I,\ \rho > 0$. □

2.3.8 $\Phi(t)$ Is an Analytic Semigroup on Y for $t > 0$

Theorem 2.3.8.1 *Assume (H.1)–(H.3) and the FCC (H.4) = (2.1.12).*

(i) The infinitesimal generators A_P *and* $\hat{A}_P$ *of the s.c. semigroups* $\Phi(t)$ *and* $\hat{\Phi}(t)$ *guaranteed by Theorem 2.3.6.1 are the operators*

$$A_P \equiv A - BB^*P, \quad \hat{A}_P = -\hat{A} - BB^*P, \tag{2.3.8.1}$$

respectively with maximal domains, so that

$$\frac{d\Phi(t)x}{dt} = (A - BB^*P)\Phi(t)x = \Phi(t)(A - BB^*P)x \in Y, \quad t > 0,\ x \in \mathcal{D}(A_P); \tag{2.3.8.2a}$$

$$\frac{d\hat{\Phi}(t)x}{dt} = (-\hat{A} - BB^*P)\hat{\Phi}(t)x = \hat{\Phi}(t)(-\hat{A} - BB^*P)x \in Y, \quad t > 0,\ x \in \mathcal{D}(A_P). \tag{2.3.8.2b}$$

(ii) The s.c. semigroups $\Phi(t)$ *and* $\hat{\Phi}(t)$ *are analytic on* Y *for* $t > 0$. *Thus, the validity of the identities on the left of (2.3.8.2a,b) is extended to all* $x \in Y$.

Proof. (i) We insert (2.3.7.15) for u^0 into the optimal dynamics

$$y^0(t;x) = \Phi(t)x = e^{At}x + \int_0^t e^{A(t-\tau)}Bu^0(\tau;x)\,d\tau, \tag{2.3.8.3}$$

differentiate in $t > 0$ for $x \in \mathcal{D}(A_P)$ after taking the inner product with $y \in \mathcal{D}(A^*)$, and obtain

$$\begin{aligned}
\left(\frac{dy^0(t;x)}{dt}, y\right)_Y &= \left(\frac{d\Phi(t)x}{dt}, y\right)_Y = (\Phi(t)A_P x, y)_Y \\
&= (\Phi(t)x, A^*y) + (Bu^0(t;x), y)_Y \\
\text{(by (2.3.7.15))} \qquad &= ([A - BB^*P]\Phi(t)x, y)_Y, \;\; x \in \mathcal{D}(A_P), \;\; y \in \mathcal{D}(A^*),
\end{aligned} \tag{2.3.8.4}$$

where we notice that each term above is well defined as a duality pairing; in particular, $(Bu^0(t;x), y)_Y = (A^{-1}Bu^0(t;x), A^*y)_Y$ is well defined by assumption (ii), that is, (2.1.7). Then (2.3.8.4) shows the identity: $\Phi(t)A_P x = [A - BB^*P]\Phi(t)x$, first in $[\mathcal{D}(A^*)]'$, next in Y, for $x \in \mathcal{D}(A_P)$. Then, the left-hand side of (2.3.8.1) and the left-hand side identity of (2.3.8.2a) are proved. By a standard semigroup result (2.3.8.2a) (right) also follows. Then, (2.3.8.2b) follows at once recalling (2.1.3) and (2.3.3.5c). These lead to (2.3.8.1) (right).

(ii) First Proof We shall show that the generator $A_P = A - BB^*P$ of the s.c. semigroup $\Phi(t)$ satisfies the resolvent condition for analyticity of $\Phi(t)$. From

$$\lambda I - A_P = (\lambda I - A)[I + R(\lambda, A)BB^*P], \quad \operatorname{Re}\lambda > \omega_0,$$

via (2.3.8.1), we obtain

$$(\lambda I - A_P)^{-1} = R(\lambda, A_P) = [I + R(\lambda, A)BB^*P]^{-1}R(\lambda, A), \tag{2.3.8.5}$$

whenever, in addition, the inverse of the operator in the square brackets is well defined. To this end, we recall (2.1.6) for $\theta = \gamma < 1$,

$$\|\hat{A}^\gamma R(\lambda, -\hat{A})\|_{\mathcal{L}(Y)} \le \frac{C}{|\lambda + \hat{\omega}|^{1-\gamma}} \to 0 \text{ as } \operatorname{Re}\lambda \to \infty, \;\; \lambda \in \Sigma^c(-\hat{A}; -\hat{\omega}; \theta_0). \tag{2.3.8.6}$$

Thus, invoking (2.3.8.6), (2.1.7), and (2.3.7.5), on B^*P, we obtain

$$\begin{aligned}
\|R(\lambda, -\hat{A})BB^*P\|_{\mathcal{L}(Y)} &= \|R(\lambda, -\hat{A})\hat{A}^\gamma \hat{A}^{-\gamma}BB^*P\|_{\mathcal{L}(Y)} \\
&\le C\|R(\lambda, -\hat{A})\hat{A}^\gamma\|_{\mathcal{L}(Y)} \to 0, \\
&\quad \text{as } \operatorname{Re}\lambda \to \infty, \;\; \lambda \in \Sigma^c(-\hat{A}; -\hat{\omega}; \theta_0).
\end{aligned} \tag{2.3.8.7}$$

Thus, in view of (2.3.8.7) and (2.1.3), identity (2.3.8.5) holds true surely for all Re λ sufficiently large, and then by (2.3.8.7) and (2.1.6)

$$\|R(\lambda, A_P)\|_{\mathcal{L}(Y)} \leq C_{r_0} \|R(\lambda, A)\|_{\mathcal{L}(Y)} \leq \frac{C_{r_0}}{|\lambda - \omega_0|}$$

$$\text{for all } \lambda \text{ with Re } \lambda \geq r_0 > 0, \quad (2.3.8.8)$$

for a suitable constant $r_0 > 0$. By a well-known result (e.g. [Fattorini, 1983, p. 185]), estimate (2.3.8.8) implies that A_P is the generator of a s.c. semigroup, which, in addition, is also analytic on Y for $t > 0$. Then the validity of the identities on the left of (2.3.8.2a,b) is extended to all $x \in Y$.

(ii) Second Proof Alternatively, one may prove (ii) by perturbation theory. Consider the Y-adjoint $\hat{A}_P^*$ of the operator $\hat{A}_P$ in (2.3.8.1):

$$\hat{A}_P^* = \hat{A}^* - (B^* P)^* B^*. \tag{2.3.8.9}$$

Let $x \in \mathcal{D}((\hat{A}^*)^\gamma)$. Then, by (2.3.7.5) and (2.1.7), we obtain

$$\begin{aligned} \|(B^* P)^* B^* x\|_Y &\leq C \|B^* (\hat{A}^*)^{-\gamma} (\hat{A}^*)^\gamma x\| \\ &\leq C \|(\hat{A}^*)^\gamma x\|_Y. \end{aligned} \tag{2.3.8.10}$$

Equation (2.3.8.10) says that: $(B^* P)^* B^*$ is $(\hat{A}^*)^\gamma$-bounded, with $\gamma < 1$. By a standard perturbation result [Pazy, 1983, p. 81], we then obtain that $\hat{A}_P^*$ in (2.3.8.9) is the generator of a s.c., analytic semigroup on Y, since so is $\hat{A}^*$. Then, $\hat{A}_P$ and A_P likewise generate s.c. semigroups that are analytic on Y. $\square$

Corollary 2.3.8.2 *Assume (H.1)–(H.3) and the FCC (H.4) = (2.1.12). Then, with reference to (2.3.8.1), we have*

(i)

$$\mathcal{D}(A_P) = \mathcal{D}(\hat{A}_P) = \{x \in \mathcal{D}(\hat{A}^{1-\gamma}) : \hat{A}^{1-\gamma} x - \hat{A}^{-\gamma} B B^* P x \in \mathcal{D}(\hat{A}^\gamma)\} \tag{2.3.8.11a}$$

$$\subset \mathcal{D}(\hat{A}^{1-\gamma}); \tag{2.3.8.11b}$$

(ii) for $x \in Y$ and $t > 0$,

$$e^{A_P t} x, \ e^{\hat{A}_P t} x \in \mathcal{D}(\hat{A}^{1-\gamma}). \tag{2.3.8.12}$$

Proof. (i) (as in Lemma 1.4.6.1 of Chapter 1). Let $x \in \mathcal{D}(A_P) = \mathcal{D}(\hat{A}_P)$ so that

$$\hat{A}_P x = [-\hat{A} - B B^* P] x = \hat{A}^\gamma [-\hat{A}^{1-\gamma} - \hat{A}^{-\gamma} B B^* P] x = z \in Y. \tag{2.3.8.13}$$

But, by (2.1.7) and (2.3.7.5) on $B^* P$, we have

$$\hat{A}^{-\gamma} B B^* P x \in Y. \tag{2.3.8.14}$$

Thus, by (2.3.8.13) and (2.3.8.14), we deduce that

$$-\hat{A}^{1-\gamma}x = \hat{A}^{-\gamma}BB^*Px + \hat{A}^{-\gamma}z \in Y, \tag{2.3.8.15}$$

which means that $x \in \mathcal{D}(\hat{A}^{1-\gamma})$, as desired, and (2.3.8.11) is proved.

(ii) By analyticity of $e^{A_P t}$ established in Theorem 2.3.8.1, we have $e^{\hat{A}_P t}x \in \mathcal{D}(\hat{A}_P)$, $e^{A_P t}x \in \mathcal{D}(\hat{A}_P)$ for $x \in Y$ and $t > 0$, and then (2.3.8.12) follows from (2.3.8.11). $\square$

Corollary 2.3.8.3 *Assume (H.1)–(H.2). Suppose that $P_1 \in \mathcal{L}(Y)$ satisfies the assumption that*

$$B^*P_1 \in \mathcal{L}(Y;U). \tag{2.3.8.16}$$

Then:

(i) The same proof of Corollary 2.3.8.2(i) yields that

$$\mathcal{D}(A_{P_1}) \subset \mathcal{D}(\hat{A}^{1-\gamma}), \tag{2.3.8.17}$$

where we have set

$$A_{P_1} = A - BB^*P_1 : Y \supset \mathcal{D}(A_{P_1}) \to Y \tag{2.3.8.18}$$

(with maximal domain as in (2.3.8.11b)) so that, in particular, $\mathcal{D}(A_{P_1})$ is dense in Y.

(ii) The same proof of Theorem 2.3.8.1(ii) provides the key estimate that

$$\|R(\lambda, A_{P_1})\|_{\mathcal{L}(Y)} \le \frac{C_{r_0}}{|\lambda - \omega_0|} \tag{2.3.8.19}$$
$$\textit{for all } \lambda \textit{ with } \operatorname{Re}\lambda \ge r_0 > 0,$$

for a suitable constant $r_0 > 0$.

(iii) By (2.3.8.19) and the denseness of $\mathcal{D}(A_{P_1})$ observed in (i), we may still appeal to the well-known result of [Fattorini, 1983, p. 185] and conclude that:
A_{P_1} generates a s.c. analytic semigroup,

$$\Phi_1(t) = e^{A_{P_1}t} \ \textit{on } Y. \tag{2.3.8.20}$$

(iv) For $x \in Y$ and $t > 0$, parts (i) and (iii) imply – as in Corollary 2.3.8.2(ii) – that

$$\Phi_1(t)x = e^{A_{P_1}t}x \in \mathcal{D}(\hat{A}^{1-\gamma}). \tag{2.3.8.21}$$

2.3.9 The Operator P Satisfies the Algebraic Riccati Equation (2.2.2)

Theorem 2.3.9.1 *Assume (H.1)–(H.3) and the FCC (H.4) = (2.1.12). The operator P defined by Theorem 2.3.3.1(ii), Eqn. (2.3.3.2), is a solution of the algebraic Riccati* *equation*

$$(A^*Px, y)_Y + (PAx, y)_Y + (R^*Rx, y)_Y = (B^*Px, B^*Py)_U,$$
$$\forall\, x, y \in \mathcal{D}(\hat{A}^{\epsilon}),\ \textit{any}\ \epsilon > 0;\ \textit{in particular},\ x, y \in \mathcal{D}(A_P) \subset \mathcal{D}(\hat{A}^{1-\gamma}). \tag{2.3.9.1}$$

Proof. We return to Eqn. (2.3.7.1b) and write for $x, y \in Y$

$$(Px, y)_Y = \int_t^{\infty} \left([R^*R + 2\omega P]\hat{\Phi}(\tau - t)x, e^{-\hat{A}(\tau-t)}y\right)_Y d\tau. \tag{2.3.9.2}$$

Step 1 We first show that P satisfies (2.3.9.1) for all $x \in \mathcal{D}(A_P)$, $y \in \mathcal{D}(A)$. For $x \in \mathcal{D}(A_P)$ and $y \in Y$ we differentiate identity (2.3.9.2) in t using

$$\frac{d\hat{\Phi}(\tau - t)x}{dt} = -\hat{\Phi}(\tau - t)(A - \omega I - BB^*P)x, \quad \tau > t,\ x \in \mathcal{D}(A_P), \tag{2.3.9.3}$$

by (2.3.8.2b) (right) and (2.1.3). We obtain for $x \in \mathcal{D}(A_P)$, $y \in Y$,

$$\begin{aligned} 0 = &-([R^*R + 2\omega P]x, y)_Y \\ &+ \int_t^{\infty} \left([R^*R + 2\omega P]\frac{d\hat{\Phi}(\tau - t)x}{dt} x, e^{-\hat{A}(\tau-t)}y\right)_Y d\tau \\ &+ \int_t^{\infty} \left([R^*R + 2\omega P]\hat{\Phi}(\tau - t)x, \hat{A}e^{-\hat{A}(\tau-t)}y\right)_Y d\tau, \end{aligned} \tag{2.3.9.4}$$

and upon using (2.3.9.3), and specializing to $y \in \mathcal{D}(A) = \mathcal{D}(\hat{A})$, we get

$$\begin{aligned} 0 = &-(R^*Rx, y)_Y - (2\omega Px, y)_Y \\ &- \int_t^{\infty} \left(e^{-\hat{A}^*(\tau-t)}[R^*R + 2\omega P]\hat{\Phi}(\tau - t)(A - \omega I - BB^*P)x, y\right)_Y d\tau \\ &+ \int_t^{\infty} \hat{A}^*\left(e^{-\hat{A}^*(\tau-t)}(R^*R + 2\omega P)\hat{\Phi}(\tau - t)x, y\right)_Y d\tau, \quad x \in \mathcal{D}(A_P),\ y \in \mathcal{D}(A). \end{aligned} \tag{2.3.9.5}$$

Invoking the relation (2.3.7.1b) for P in both of the last two terms of (2.3.9.5), we find

$$0 = -(R^*Rx, y)_Y - (2\omega Px, y)_Y - (P[A - \omega I - BB^*P]x, y)_Y + (\hat{A}^*Px, y)_Y,$$
$$x \in \mathcal{D}(A_P),\ y \in \mathcal{D}(A). \tag{2.3.9.6}$$

Recalling $\hat{A}^* = -A^* + \omega I$ from (2.1.3), we finally obtain, after a cancellation of the term involving $(2\omega Px, y)_Y$,

$$(A^*Px, y)_Y + (P[A - BB^*P]x, y)_Y + (R^*Rx, y)_Y = 0, \tag{2.3.9.7}$$

at least for $x \in \mathcal{D}(A_P)$ and $y \in \mathcal{D}(A)$. We recall that $\mathcal{D}(A_P) \subset \mathcal{D}(\hat{A}^{1-\gamma})$ by (2.3.8.11).

Step 2 We now extend the validity of (2.3.9.7) by continuity to all $x, y \in \mathcal{D}(\hat{A}^\epsilon)$, any $\epsilon > 0$. Indeed, the first term in (2.3.9.7) can be extended to all $x \in Y$ and $y \in \mathcal{D}(\hat{A}^\epsilon)$ by invoking (2.3.7.4) with $\theta = 1 - \epsilon$:

$$(A^* Px, y)_Y = \text{ well defined for all } x \in Y, \; y \in \mathcal{D}(\hat{A}^\epsilon). \tag{2.3.9.8}$$

Next, we can write the second term with A replaced by $-\hat{A}$ as

$$\begin{aligned}(P[\hat{A} - BB^*P]x, y)_Y &= (P\hat{A}[I - \hat{A}^{-1}BB^*P]x, y)_Y \\ &= (\hat{A}^\epsilon[I - \hat{A}^{-1}BB^*P]x, (\hat{A}^*)^{1-\epsilon}Py)_Y,\end{aligned} \tag{2.3.9.9}$$

where the right-hand side of (2.3.9.9) holds true for $y \in Y$ (by (2.3.7.4)) and for $x \in \mathcal{D}(\hat{A}^\epsilon)$. This is so, since $\hat{A}^{-1+\epsilon}B \in \mathcal{L}(U; Y)$ by (2.1.7) with $\gamma \leq 1 - \epsilon < 1$ and $B^*P \in \mathcal{L}(Y; U)$ by (2.3.7.5). Thus, in conclusion, (2.3.9.7) can be extended to all $x, y \in \mathcal{D}(\hat{A}^\epsilon)$, as claimed, as elements of $\mathcal{D}(\hat{A}^\epsilon)$ can be arbitrarily approximated in the Y-norm by elements of $\mathcal{D}(A_P)$ and $\mathcal{D}(A)$. Statement (2.3.9.1) for $\mathcal{D}(A_P)$ follows from (2.3.8.11). $\square$

2.4 Proof of Theorem 2.2.2: Exponential Stability of $\Phi(t)$ and Uniqueness of the Solution of the Algebraic Riccati Equation under the Detectability Condition (2.1.13)

Exponential Stability So far the Detectability Condition (DC) (H.5) = (2.1.13) has not been used. We begin by showing that under the Detectability Condition, the s.c. analytic semigroup $\Phi(t)$ is also exponentially stable on Y. This generalizes the corresponding statement of Theorem 2.3.6.1; see Remark 2.3.6.1.

Theorem 2.4.1 *Assume (H.1)–(H.3) the FCC (H.4) = (2.1.12), as well as the DC (H.5) = (2.1.13). Then, the s.c. analytic semigroup $\Phi(t) = e^{A_P t}$ is exponentially stable on Y: There exist constants $M_P \geq 1$ and $\omega_P > 0$ such that*

$$\|\Phi(t)\|_{\mathcal{L}(Y)} \leq M_P e^{-\omega_P t}, \quad t \geq 0. \tag{2.4.1}$$

Proof. In the proof below, S may be either R or else $(R^*R)^{\frac{1}{2}}$. By assumption, we know that there exists an operator $K \in \mathcal{L}(Y)$ such that the s.c. analytic semigroup $e^{(A+KS)t}$ is exponentially stable on Y: There exist constants $M_K \geq 1$ and $\omega_K > 0$ such that

$$\left\| e^{(A+KS)t} \right\|_{\mathcal{L}(Y)} \leq M_K e^{-\omega_K t}, \quad t \geq 0. \tag{2.4.2}$$

Since we already know by Theorem 2.3.6.1 that $\Phi(t)$ is a s.c. semigroup on Y, to show its exponential decay, as in (2.4.1), it suffices to establish that

$$\int_0^\infty \|\Phi(t)x\|_Y^2 \, dt < \infty, \quad \forall x \in Y, \tag{2.4.3}$$

and then invoke a well-known result [Datko, 1970]. To establish (2.4.3) we write after adding and subtracting

$$\frac{d\Phi(t)x}{dt} = (A+KS)\Phi(t)x - KS\Phi(t)x - BB^*P\Phi(t)x, \quad x \in Y, \ t > 0, \tag{2.4.4}$$

after recalling (2.3.8.2a) and Theorem 3.8.1(ii). We set for convenience

$$A_K = A + KS, \quad S = R, \quad \text{or else } S = (R^*R)^{\frac{1}{2}} \in \mathcal{L}(Y). \tag{2.4.5}$$

The solution to (2.4.4) is thus, by (2.4.5),

$$\Phi(t)x = e^{A_K t}x - \int_0^t e^{A_K(t-\tau)}KS\Phi(\tau)x - \int_0^t e^{A_K(t-\tau)}BB^*P\Phi(\tau)x. \tag{2.4.6}$$

With reference to the three terms of (2.4.6), we readily have

(i)

$$e^{A_K t} : \text{continuous } Y \to L_2(0, \infty; Y) \tag{2.4.7}$$

by assumption (2.4.2) and (2.4.5).

(ii) Conservatively,

$$f(\cdot) \to \int_0^t e^{A_K(t-\tau)} f(\tau)\, d\tau : \text{continuous } L_2(0, \infty; Y) \to L_2(0, \infty; Y), \tag{2.4.8}$$

a fortiori from (2.4.2). We now critically recall that, by optimality as in (2.3.3.1), with S either of the two choices in (2.4.5) [see (2.1.11)], we already know via (2.3.5.10b) that

$$f(t) = S\Phi(t)x \in L_2(0, \infty; Y), \quad \forall\, x \in Y. \tag{2.4.9}$$

Combining (2.4.8) with (2.4.9), we then obtain: the map

$$x \to \textstyle\int_0^t e^{A_K(t-\tau)}KS\Phi(\tau)x : \text{continuous } Y \to L_2(0, \infty; Y). \tag{2.4.10}$$

(iii)

$$u(\cdot) \to \int_0^t e^{A_K(t-\tau)}Bu(\tau)\, d\tau$$
$$: \text{continuous } L_2(0, \infty; U) \to L_2(0, \infty; Y) \tag{2.4.11}$$

(same regularity as that of $\hat{L}$) a fortiori from Theorem 2.3.5.1, Eqn. (2.3.5.1), via (2.4.2). But, recalling (2.3.7.15) or (2.3.7.17a),

$$B^*P\Phi(\cdot) : \text{continuous } Y \to L_2(0, \infty; U), \tag{2.4.12}$$

such that, combining (2.4.11) with (2.4.12), we obtain that the map

$$x \to \int_0^t e^{A_K(t-\tau)}BB^*P\Phi(\tau)x\, d\tau : \text{continuous } Y \to L_2(0, \infty; Y). \tag{2.4.13}$$

Using now (2.4.7), (2.4.10), and (2.4.13) for the three terms of $\Phi(t)x$ in (2.4.6), we readily obtain (2.4.3), as desired. □

We now transfer the result to $u^0(\cdot\,; x)$.

Corollary 2.4.2 *Assume (H.1)–(H.3), the FCC (H.4) = (2.1.12), as well as the DC (H.5) = (2.1.13). Then, recalling (2.3.7.15),*

$$\|u^0(t;x)\|_U = \|B^*P\Phi(t)x\|_U \le C_P e^{-\omega_P t}\|x\|_Y, \quad t \ge 0, \ x \in Y; \tag{2.4.14}$$

$$u^0(\cdot\,;x) = -B^*P\Phi(\cdot) : \textit{continuous } Y \to C_{ub}([0,\infty]; U). \tag{2.4.15}$$

Proof. We use (2.4.1) and (2.3.7.5) on B^*P. □

Uniqueness Finally, we consider the issue of uniqueness of the solution to the algebraic Riccati equation. We begin with a result that will be needed in the proof of Theorem 2.4.4, which is, moreover, of independent interest. It points out an interesting *minimality property* enjoyed by the operator P in (2.3.3.2), (2.3.3.3), among all nonnegative, self-adjoint solutions of the ARE (2.3.9.1).

Proposition 2.4.3 Assume (H.1)–(H.3). Let $0 \le P_1 = P_1^* \in \mathcal{L}(Y)$ be a nonnegative, self-adjoint operator such that

(i)

$$(\hat{A}^*)^\gamma P_1 \in \mathcal{L}(Y), \quad \text{hence } B^*P_1 \in \mathcal{L}(Y;U); \tag{2.4.16}$$

(ii) P_1 is a solution of the ARE (2.3.9.1):

$$(A^*P_1x, y)_Y + (P_1Ax, y)_Y + (R^*Rx, y)_Y = (B^*P_1x, B^*P_1y)_U \tag{2.4.17}$$

for all $x, y \in \mathcal{D}(\hat{A}^\epsilon)$, $\quad \epsilon > 0$.

Let $A_{P_1} = (A - BB^*P_1)$, with maximal domain as in (2.3.8.11b), be the generator of a s.c. analytic semigroup $\Phi_1(t) \equiv e^{A_{P_1}t} \subset \mathcal{D}(\hat{A}^{1-\gamma})$ on Y, as guaranteed by Corollary 2.3.8.3.

For $x \in Y$, set

$$u_1(t;x) = -B^*P_1\Phi_1(t)x \in C([0,T];U), \quad \forall\, T < \infty, \tag{2.4.18}$$

so that

$$\frac{d\Phi_1(t)x}{dt} = A_{P_1}\Phi_1(t)x = A\Phi_1(t)x + Bu_1(t;x), \quad t > 0, \ x \in Y. \tag{2.4.19}$$

Then

(a_1) for $x \in Y$,

$$(P_1 x, x)_Y \geq \int_0^\infty \left[\|R\Phi_1(t)x\|_Z^2 + \|u_1(t;x)\|_U^2\right] dt = J(u_1(\cdot\,;x), \Phi_1(t)x); \tag{2.4.20}$$

(a_2)

$$P_1 \geq P \geq 0, \tag{2.4.21}$$

where P is (as usual) the nonnegative, self-adjoint operator defined by (2.3.3.2).

Proof. (a_1) Let $x \in Y$, $t > 0$. By the analyticity of $\Phi_1(t)x$, guaranteed by Corollary 2.3.8.3, we have $\Phi_1(t)x \in \mathcal{D}(A_{P_1})$. We then compute, since P_1 is self-adjoint,

$$\begin{aligned}\frac{d}{dt}(P_1\Phi_1(t)x, \Phi_1(t)x)_Y &= 2(P_1 A_{P_1}\Phi_1(t)x, \Phi_1(t)x)_Y \\ &= 2(P_1[A - BB^*P_1]\Phi_1(t)x, \Phi_1(t)x)_Y \qquad (2.4.22)\\ &= 2(P_1 A\Phi_1(t)x, \Phi_1(t)x)_Y \\ &\quad -2\|B^*P_1\Phi_1(t)x\|_U^2. \qquad (2.4.23)\end{aligned}$$

Notice that each term above is well defined; in particular, (innocuously) replacing A with $\hat{A}$ in the first term of (2.4.23), we have

$$\begin{aligned}(P_1\hat{A}\Phi_1(t)x, \Phi_1(t)x)_Y &= (\hat{A}^{1-\gamma}\Phi_1(t)x, \hat{A}^{*\gamma}P_1\Phi_1(t)x)_Y \\ &= \text{well defined for } x \in Y,\ t > 0, \qquad (2.4.24)\end{aligned}$$

by (2.3.8.21) of Corollary 2.3.8.3 and assumption (2.4.16). Invoking the ARE (2.4.17) for P_1, with x and y both replaced by $\Phi_1(t)x \in \mathcal{D}(\hat{A}^\epsilon)$, $0 < \epsilon \leq 1 - \gamma$ by (2.3.8.21), we obtain

$$2(P_1 A\Phi_1(t)x, \Phi_1(t)x)_Y = -\|R\Phi_1(t)x\|_Z^2 + \|B^*P_1\Phi_1(t)x\|_U^2. \tag{2.4.25}$$

Inserting (2.4.25) in the right-hand side of (2.4.23) yields, for $x \in Y$, $t > 0$,

$$\frac{d}{dt}(P_1\Phi_1(t)x, \Phi_1(t)x)_Y = -\|R\Phi_1(t)x\|_Z^2 - \|B^*P_1\Phi_1(t)x\|_U^2. \tag{2.4.26}$$

Integrating (2.4.26) over $[0, T]$ gives

$$\begin{aligned}\int_0^T \left[\|R\Phi_1(t)x\|_Z^2 + \|B^*P_1\Phi_1(t)x\|_U^2\right] dt &= (P_1x, x)_Y - (P_1\Phi_1(T)x, \Phi_1(T)x)_Y \qquad (2.4.27)\\ &\leq (P_1x, x)_Y, \qquad (2.4.28)\end{aligned}$$

since P_1 is nonnegative. Letting $T \uparrow \infty$ in (2.4.27), (2.4.28) then yields (2.4.20), as desired, by recalling (2.4.18) and that $\{u_1(\cdot\,;x), \Phi_1(t)x\}$ forms a corresponding pair of control and related solution by (2.4.19).

(a_2) Since, as noted above, $\{u_1(\cdot\,;x), \Phi_1(t)x\}$ forms a competitive pair, we then have by (2.3.3.3)

$$J(u_1(\cdot\,;x), \Phi_1(\cdot)x) \geq J(u^0(\cdot\,;x), y^0(\cdot\,;x)) = (Px, x)_Y, \tag{2.4.29}$$

and (2.4.21) follows by combining (2.4.20) with (2.4.29). □

Corollary 2.4.4 *Without assuming nonnegativity of $P_1 = P_1^*$, and assuming instead the Detectability Condition (H.5) = (2.1.13), in addition to (H.1), (H.3), we obtain*

(i) the s.c. analytic semigroup $\Phi_1(t) = e^{A_{P_1}t}$ is, moreover, uniformly stable: There exists constants $M_1 \geq 1$ and $\omega_1 > 0$ such that

$$\|e^{A_{P_1}t}\|_{\mathcal{L}(Y)} \leq M_1 e^{-\omega_1 t}, \quad t \geq 0 \tag{2.4.30}$$

[same proof as in Theorem 2.4.1];

(ii) (2.4.20) is replaced now by

$$\begin{aligned}(P_1 x, x)_Y &= \int_0^\infty \left[\|R\Phi_1(t)x\|_Z^2 + \|u_1(t;x)\|_U^2\right] dt \\ &= J(u_1(\cdot\,;x), \Phi_1(t)x) &(2.4.31)\\ &\geq (Px, x)_Y, &(2.4.32)\end{aligned}$$

as one obtains by letting $T \uparrow \infty$ in (2.4.27) and invoking (4.30).

We can now show the desired uniqueness result.

Theorem 2.4.5 *Assume (H.1)–(H.3), the FCC (H.4) = (2.1.12), as well as the DC (H.5) = (2.1.13). The solution operator P of the algebraic Riccati equation (2.3.9.1) asserted by Theorem 2.3.9.1 is unique within the class of self-adjoint operators $\bar{P}$ such that $(\hat{A}^*)^\gamma \bar{P} \in \mathcal{L}(Y)$ (a property satisfied by P defined by (2.3.3.2), by virtue of (2.3.7.4a)).*

Proof.

Step 1 Let $P_1 \in \mathcal{L}(Y)$ be another self-adjoint solution of the algebraic Riccati equation (2.3.9.1) such that $(\hat{A}^*)^\gamma P_1 \in \mathcal{L}(Y)$; hence $B^* P_1 \in \mathcal{L}(Y; U)$. We set

$$Q = P_1 - P \in \mathcal{L}(Y) \quad \text{so that} \quad Q = Q^* \quad \text{and} \quad B^* Q \in \mathcal{L}(Y; U). \tag{2.4.33}$$

Subtracting the corresponding algebraic Riccati equations for P_1 and P, we obtain by (2.3.9.1)

$$\begin{aligned}&(A^*(P_1 - P)x, y)_Y + ((P_1 - P)Ax, y)_Y = (B^* P_1 x, B^* P_1 y)_U - (B^* Px, B^* Py)_U, \\ &\qquad x, y \in \mathcal{D}(\hat{A}^\epsilon),\ \forall\, \epsilon > 0 \text{ in particular } x, y \in \mathcal{D}(A_P) \subset \mathcal{D}(\hat{A}^{1-\gamma}). \qquad (2.4.34)\end{aligned}$$

By adding and subtracting $(B^*Qx, B^*Qy)_U$, we find by (2.4.33)

$$\begin{aligned}(B^*P_1x, B^*P_1y)_U &- (B^*Px, B^*Py)_U \\ &= (B^*Qx, B^*Qy)_U + (B^*Qx, B^*Py)_U + (B^*Px, B^*Qy)_U. \quad (2.4.35)\end{aligned}$$

Inserting (2.4.35) into (2.4.34), we obtain by self-adjointness of Q

$$([A - BB^*P]x, Qy)_Y + (x, Q[A - BB^*P]y)_Y = (B^*Qx, B^*Qy)_U,$$
$$x, y \in \mathcal{D}(\hat{A}^{\epsilon}), \ \forall \epsilon > 0. \quad (2.4.36)$$

Step 2 With $A_P = A - BB^*P$ the generator of a s.c. analytic semigroup $\Phi(t) = e^{A_P t}$ by Theorem 2.3.8.1, we compute by self-adjointness of Q, with $x \in Y$, $t > 0$:

$$\begin{aligned}\frac{d}{dt}(Q\Phi(t)x, \Phi(t)x)_Y &= (A_P\Phi(t)x, Q\Phi(t)x)_Y + (\Phi(t)x, QA_P\Phi(t)x)_Y \\ &= \|B^*Q\Phi(t)x\|_U^2, \quad (2.4.37)\end{aligned}$$

where in the last step we have invoked (2.4.35) with x, y replaced by $e^{A_P t}x \in \mathcal{D}(A_P) \subset \mathcal{D}(\hat{A}^{1-\gamma})$, $x \in Y$, $t > 0$. Integrating (2.4.37) in t over $[0, T]$ leads to

$$\begin{aligned}\left(Qe^{A_P T}x, e^{A_P T}x\right)_Y &= (Qx, x)_Y + \int_0^T \left\|B^*Qe^{A_P t}x\right\|_U^2 dt, \quad x \in Y \quad (2.4.38) \\ &\geq (Qx, x)_Y \geq 0, \quad (2.4.39)\end{aligned}$$

by the critical result $Q = P_1 - P \geq 0$ guaranteed by Corollary 2.4.4, Eqn. (2.4.32).

But as a consequence of the detectability assumption (2.1.13), we have that $\Phi(t) = e^{A_P t} = e^{(A-BB^*P)t}$ is an exponentially stable semigroup via Theorem 2.4.1. Hence, as $T \uparrow \infty$, the left-hand side of inequality (2.4.38) goes to zero, leading to

$$(Qx, x)_Y = 0, \quad \forall\, x \in Y. \quad (2.4.40)$$

Since Q is self-adjoint, it then follows that $Q = 0$ or $P = P_1$, and uniqueness is proved. □

Remark 2.4.1 *If we compare Eqn. (2.4.26) involving P_1 with the corresponding Eqn. (2.4.37) involving Q, we see that the right-hand side of Eqn. (2.4.26) is nonpositive, while the right-hand side of Eqn. (2.4.37) is nonnegative. In the subsequent corresponding arguments, the noted signs are critical.*

2.5 Extensions to Unbounded R: $R \in \mathcal{L}(\mathcal{D}(\hat{A}^{\delta}); Z), \delta < \min\{1 - \gamma, \frac{1}{2}\}$

This section is the counterpart to the case $T = \infty$ of Chapter 1, Section 1.8 for the case $T < \infty$. It briefly points out the extension of the preceding theory of this chapter to the case where the observation operator R is unbounded, with a controlled degree of unboundedness, as in the new hypothesis (2.5.5) below. The rationale for the limitation on the constant δ in (2.5.5) is the same as in Chapter 1, Section 1.8. The optimality formulas (2.2.7) and (2.2.8) [and (2.2.13) in the stable case] require that the following two regularity properties be satisfied:

(i)

$$Re^{-\hat{A}t} = (R\hat{A}^{-\delta})\hat{A}^\delta e^{-\hat{A}t} : \text{continuous } Y \to L_2(0, \infty; Z), \tag{2.5.1}$$

(ii)

$$(R\hat{L}u)(t) = R\hat{A}^{\gamma-1} \int_0^t \hat{A}e^{-\hat{A}(t-\tau)}\hat{A}^{-\gamma}Bu(\tau)\,d\tau$$
$$: \text{ continuous } L_2(0, \infty; U) \to L_2(0, \infty; Z). \tag{2.5.2}$$

Property (2.5.1) is satisfied with

$$\begin{cases} R \in \mathcal{L}(\mathcal{D}(\hat{A}^\delta); Z), \quad 0 \le \delta < \frac{1}{2} \\ \text{or even } \delta = \frac{1}{2} \text{ if } A \text{ is self-adjoint} \\ \text{(see Chapter 1, Appendix 1A, Eqn. (1A.2))} \end{cases} \tag{2.5.3}$$

by the analyticity estimate (2.1.4b).

Property (2.5.2) is satisfied with

$$R \in \mathcal{L}(\mathcal{D}(\hat{A}^{1-\gamma}); Z) \tag{2.5.4}$$

by application of hypothesis (2.1.7) and the standard regularity (0.4) in Chapter 0 to formula (2.5.2).

Accordingly, we introduce the following:

New Hypothesis (H.3′) Throughout this subsection we assume (in addition to (H.1), (H.2) = (2.1.7), and the FCC (H.3) = (2.1.12) of Section 2.1) that
(H.3′):

$$\begin{cases} R \in \mathcal{L}(\mathcal{D}(\hat{A}^\delta); Z), \quad \delta < \min\left\{1-\gamma, \frac{1}{2}\right\} \\ \text{or } \delta = \frac{1}{2} \text{ if } A \text{ is self-adjoint and } \gamma < \frac{1}{2}, \end{cases} \tag{2.5.5}$$

thus relaxing hypothesis (H.3) = (2.1.9) of the preceding treatment in Sections 2.1 through 2.4.

Change of "Variables" As in Section 1.8, we introduce the following new operators:

(i)

$$\bar{R} \equiv R\hat{A}^{-\delta} \in \mathcal{L}(Y; Z), \tag{2.5.6}$$

(ii)

$$\left\{\begin{array}{ll} \bar{B} \equiv \hat{A}^{\delta} B : \text{continuous } U \to [\mathcal{D}((\hat{A}^*)^{1+\delta})]' & (2.5.7\text{a}) \\ \text{so that, in view of (2.5.7a), and the standing hypothesis (2.1.7)} & \\ \hat{A}^{-\bar{\gamma}} \bar{B} = \hat{A}^{-(\bar{\gamma}-\delta)} B = \hat{A}^{-\gamma} B \in \mathcal{L}(U; Y), & (2.5.7\text{b}) \\ \qquad \bar{\gamma} = \gamma + \delta < 1. & (2.5.7\text{c}) \end{array}\right.$$

Consequences (a) We define the operator $\bar{L}$ by

$$\{\bar{L}u\}(t) = \int_0^t e^{A(t-\tau)} \bar{B} u(\tau)\, d\tau = \hat{A}^{\bar{\gamma}} \int_0^t e^{A(t-\tau)} \hat{A}^{-\bar{\gamma}} \bar{B} u(\tau)\, d\tau \tag{2.5.8a}$$

$$= \hat{A}^{\delta} \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau = \hat{A}^{\delta} \{Lu\}(t) \tag{2.5.8b}$$

$$: \text{ continuous } L_2(0, T; U) \to L_2(0, T; \mathcal{D}(\hat{A}^{1-\bar{\gamma}})) \subset L_2(0, T; Y), \tag{2.5.8c}$$

via (2.5.7a), the counterpart of Eqn. (1.8.1.8). Thus, by (2.5.8b) and (2.5.6), we obtain

$$\bar{R}\bar{L} = RL : \text{continuous } L_2(0, T; U) \to L_2(0, T; Z) \tag{2.5.9}$$

which is the counterpart of Eqn. (1.8.1.9) of Chapter 1.

(b) We similarly define the operator $\hat{\bar{L}}$ by

$$\{\hat{\bar{L}}u\}(t) = \int_0^t e^{-\hat{A}(t-\tau)} \bar{B} u(\tau)\, d\tau = \hat{A}^{\delta} \{\hat{L}u\}(t) \tag{2.5.10a}$$

$$: \text{ continuous } L_2(0, T; U) \to L_2(0, \infty; Y), \tag{2.5.10b}$$

the counterpart of (2.1.17). By (2.5.10a) and (2.5.6),

$$\bar{R}\hat{\bar{L}} = R\hat{L} : \text{continuous } L_2(0, \infty; U) \to L_2(0, \infty; Z), \tag{2.5.11}$$

complementing (2.5.9).

(c) We introduce the *new dynamics*

$$\dot{\bar{y}} = A\bar{y} + \bar{B}u, \quad \bar{y}(0) = \zeta_0 \tag{2.5.12}$$

and the *new cost functional*

$$\bar{J}(u; \zeta_0) = \int_0^{\infty} \left[\|\bar{R}\bar{y}(t)\|_Z^2 + \|u(t)\|_U^2 \right] dt. \tag{2.5.13}$$

Then, comparing the solution to (2.1.1) with the solution to (2.5.12), we obtain by (2.5.8b):

$$\begin{aligned} y(t; y_0, u) &= e^{At} y_0 + \{Lu\}(t) \\ &= \hat{A}^{-\delta}[e^{At} \hat{A}^{\delta} y_0 + \{\bar{L}u\}(t)] = \hat{A}^{-\delta} \bar{y}(t; \hat{A}^{\delta} y_0, u); \end{aligned} \tag{2.5.14}$$

moreover, by (2.5.6) and (2.5.14),

$$Ry(t; y_0, u) = \bar{R}\bar{y}(t; \hat{A}^\delta y_0, u) = Re^{At}y_0 + (\bar{R}\bar{L}u)(t), \tag{2.5.15}$$

so that

$$J(u; y_0) = \bar{J}(u; \zeta_0), \quad \zeta_0 = \hat{A}^\delta y_0. \tag{2.5.16}$$

Thus, we conclude: *If the original problem (2.1.1), (2.1.8) satisfies the Finite Cost Condition (H.4) = (2.1.12) for $y_0 \in Y$, so does the new problem (2.5.12), (2.5.13), where we can take $\zeta_0 \in Y$ as well, and*

$$\begin{cases} \textit{interpret the term } \bar{R}e^{At}\hat{A}^\delta y_0 \textit{ of } \bar{R}\bar{y}(t; \hat{A}^\delta y_0, u) \\ \textit{as meaning } \bar{R}\hat{A}^\delta e^{At}y_0 \in L_2(0, T; Z) \textit{ for } y_0 \in Y \textit{ by (2.5.5).} \end{cases} \tag{2.5.17}$$

Then, the system $\{A, \bar{B}\}$ in (2.5.12), with parameter $\bar{\gamma}$ in (2.5.7c) and with observation $\bar{R}$ in (2.5.13) bounded by (2.5.6), satisfies the setting of the preceding theory in Section 2.1 by (2.5.7) as well, that is, assumptions (H.1) through (H.4) of Section 2.1. Accordingly, we may apply Theorem 2.2.1 to the new problem $\{A, \bar{B}, \bar{R}\}$ in (2.5.12), (2.5.13) with $\zeta_0 \in Y$, as noted above in (2.5.17). We obtain

Theorem 2.5.1 *Assume (H.1), (H.2) = (1.7), the FCC (H.4) = (2.1.12) of Section 2.1, as well as assumption (H.3′) = (2.5.5) [relaxing (H.3) = (2.1.9)]. Then:*

(a) For each $y_0 \in Y$, there exists a unique optimal pair $\{u^0(t; y_0), y^0(t; y_0\}$ of the optimal control problem (2.1.1), (2.1.8), satisfying

$$u^0(t; y_0) \in L_2(0, \infty; U); \quad Ry^0(t; y_0) \in L_2(0, \infty; Z). \tag{2.5.18}$$

(b) For each $y_0 \in Y$, let $\{\bar{u}^0(\cdot\,; y_0), \bar{y}^0(\cdot\,; y_0)\}$ denote the unique optimal pair of the optimal control problem (2.5.12), (2.5.13) with $y_0 = \zeta_0$. Theorem 2.2.1 holds true in this case, as noted below (2.5.17) for the corresponding optimal pair $\{\bar{u}^0, \bar{y}^0\}$, the optimal cost $\bar{J}^0(y_0)$, the operators $\hat{\bar{L}}$ and $\hat{\bar{L}}^*$, and the Riccati operator $\bar{P}$. Thus, we have explicitly, as in (2.2.7), (2.2.8):

(b_1)

$$-\hat{\bar{u}}^0(\cdot\,; y_0) = \hat{\bar{L}}^*(\bar{R}^*\bar{R} + 2\omega\bar{P})\hat{\bar{y}}^0(\cdot\,; y_0) \tag{2.5.19}$$

$$= [I + \hat{\bar{L}}^*(\bar{R}^*\bar{R} + 2\omega\bar{P})\hat{\bar{L}}]^{-1}\hat{\bar{L}}^*(\bar{R}^*\bar{R} + 2\omega\bar{P})\{e^{-\hat{A}\cdot}y_0\} \in L_2(0, \infty; U); \tag{2.5.20}$$

(b_2)

$$\hat{\bar{y}}^0(\cdot\,; y_0) = [I + \hat{\bar{L}}\hat{\bar{L}}^*(\bar{R}^*\bar{R} + 2\omega\bar{P})]^{-1}\{e^{-\hat{A}\cdot}y_0\} \in L_2(0, \infty; Y). \tag{2.5.21}$$

In particular, the nonnegative, self-adjoint operator $\bar{P}$ provided by Theorem 2.2.1 satisfies

(b₃)

$$(\hat{A}^*)^\theta \bar{P} \in \mathcal{L}(Y), \quad 0 \le \theta < 1 \tag{2.5.22}$$

[and $\theta = 1$ as well, if A is self-adjoint and $\bar{R} = I$, or $\bar{R}^*\bar{R}$ commutes with A]. Hence

$$\bar{B}^*\bar{P} \in \mathcal{L}(Y; U), \tag{2.5.23}$$

which is the counterpart of (2.2.3a), (2.2.5).

(c) Define the bounded, self-adjoint operator P by

$$P = (\hat{A}^*)^\delta \bar{P} \hat{A}^\delta \in \mathcal{L}(Y), \quad \delta < \frac{1}{2}, \tag{2.5.24a}$$

whose boundedness on Y is the counterpart of (2.2.3b) with $r = \delta < 1/2$ by (2.5.5), so that

$$(\hat{A}^*)^\sigma P = (\hat{A}^*)^{\sigma+\delta} \bar{P} \hat{A}^\delta \in \mathcal{L}(Y), \quad \sigma < 1 - 2\delta; \tag{2.5.24b}$$

or $\sigma = 0$ if $\delta = 1/2$ with A self-adjoint; ultimately (2.5.24) follows from (2.5.22) by Lemma 1.5.1.1 (see also the proof of Corollary 1.5.1.2) of Chapter 1. We may take $\delta = 1/2$ in (2.5.24) whenever (2.5.22) holds true with $\theta = 1$ and A is self-adjoint, as in Lemma 1.5.1.1(iii). Then we have the following relations between the two problems:

(c_1)

$$y^0(\cdot\,; y_0) = \hat{A}^{-\delta}\bar{y}^0(\cdot\,; \hat{A}^\delta y_0) \tag{2.5.25}$$

$$= e^{A\cdot} y_0 + Lu^0(\cdot\,; y_0) = e^{A\cdot} y_0 + \hat{A}^{-\delta}\bar{L}\bar{u}^0(\cdot\,; \hat{A}^\delta y_0) \tag{2.5.26}$$

$$\in C([0, T_0]; \mathcal{D}(\hat{A}^\delta)), \quad y_0 \in \mathcal{D}(\hat{A}^\delta), \ \forall\, T_0 < \infty; \tag{2.5.27}$$

(c_2)

$$u^0(\cdot\,; y_0) = \bar{u}^0(\cdot\,; \hat{A}^\delta y_0) \in L_2(0, \infty; U) \cap C([0, T_0]; U),$$
$$y_0 \in \mathcal{D}(\hat{A}^\delta), \ \forall\, T_0 < \infty. \tag{2.5.28}$$

The same formulas (2.2.7) and (2.2.8) hold true for the optimal pair of problem (2.1.1), (2.1.8); that is explicitly,

(c_3)

$$-\hat{u}^0(\cdot\,; y_0) = \hat{L}^*(R^*R + 2\omega P)\hat{y}^0(\cdot\,; y_0)$$
$$= [I + \hat{L}^*(R^*R + 2\omega P)\hat{L}]^{-1}\hat{L}^*(R^*R + 2\omega P)\{e^{-\hat{A}\cdot} y_0\}$$
$$\in L_2(0, \infty; U); \tag{2.5.29}$$

(c_4)

$$\hat{y}^0(\,\cdot\,; y_0) = [I + \hat{L}\hat{L}^*(R^*R + 2\omega P)]^{-1}\{e^{-\hat{A}\,\cdot}\, y_0\} \in L_2(0, \infty; Y); \quad (2.5.30)$$

where, as usual, we have set

$$\hat{u}^0(t; y_0) = e^{-\omega t} u^0(t; y_0); \qquad \hat{y}^0(t; y_0) = e^{-\omega t} y^0(t; y_0); \quad (2.5.31)$$

(c_5)

$$Px = \int_0^\infty e^{-\hat{A}^* t}[R^*R + 2\omega P] y^0(t; x)\, dt, \quad x \in Y; \quad (2.5.32)$$

(c_6)

$$B^* P \hat{A}^{-\delta} x = \bar{B}^* \bar{P} x; \quad x \in Y, \quad B^* P \in \mathcal{L}(\mathcal{D}(\hat{A}^{\delta}); U), \quad (2.5.33)$$

complementing (2.5.23);

(c_7)

$$u^0(t; y_0) = -B^* P y^0(t; y_0) \in L_2(0, \infty; U), \quad y_0 \in Y; \quad (2.5.34)$$

(c_8)

$$J^0(y_0) = J(u^0(\,\cdot\,; y_0), y^0(\,\cdot\,; y_0)) = (Py_0, y_0)_Y, \quad y_0 \in Y; \quad (2.5.35)$$

(c_9) the operator P is a solution of the following ARE for all $x, y \in \mathcal{D}(\hat{A}^{\delta+\epsilon})$:

$$(A^*Px, y)_Y + (PAx, y)_Y + (Rx, Ry)_Z = (B^*Px, B^*Py)_U; \quad (2.5.36)$$

(c_{10}) the operator

$$A_P = A - BB^*P = \hat{A}^{-\delta} A_{\bar{P}} \hat{A}^{\delta} \quad (2.5.37a)$$

$$: \mathcal{D}(A_P) = \hat{A}^{-\delta}\mathcal{D}(A_{\bar{P}}) \to \mathcal{D}(\hat{A}^{\delta}), \quad (2.5.37b)$$

is the generator of a s.c. analytic semigroup $e^{A_P t}$ on $\mathcal{D}(\hat{A}^{\delta})$, given by

$$e^{A_P t} = \hat{A}^{-\delta} e^{A_{\bar{P}} t} \hat{A}^{\delta}, \quad (2.5.38)$$

and we have

$$\|A_P x\|_{\mathcal{D}(\hat{A}^{\delta})} = \|A_{\bar{P}} \hat{A}^{\delta} x\|_Y = \|A_{\bar{P}} y\|_Y; \quad (2.5.39)$$

$$x \in \mathcal{D}(A_P) \Longleftrightarrow y = \hat{A}^{\delta} x \in \mathcal{D}(A_{\bar{P}}); \quad (2.5.40)$$

$$\mathcal{D}(A_{\bar{P}}) = \{z \in \mathcal{D}(\hat{A}^{1-\bar{\gamma}}) : \hat{A}^{1-\bar{\gamma}} z - \hat{A}^{-\bar{\gamma}} \bar{B}\bar{B}^* \bar{P} z \in \mathcal{D}(\hat{A}^{\gamma})\} \quad (2.5.41)$$

$$\subset \mathcal{D}(\hat{A}^{1-\gamma}); \quad (2.5.42)$$

$$\|e^{A_P t} x\|_{\mathcal{D}(\hat{A}^{\delta})} = \|e^{A_{\bar{P}} t} y\|_Y, \quad y = \hat{A}^{\delta} x \in Y. \quad (2.5.43)$$

Proof. (a) Part (a) follows as in the proof of Theorem 2.2.1, by virtue of properties (2.5.1), (2.5.11), and (2.5.16).

(b) Part (b) is an application of Theorem 2.2.1 to problem (2.5.12), (2.5.13), as justified in the conclusive statement below (2.5.16).

(c) Identity (2.5.28) in (c_1) for the optimal controls of the two problems follows from (2.5.16) on their respective cost functionals. As a consequence, identity (2.5.25) in (c_2) on their respective optimal solutions is obtained by (2.5.28) and (2.5.14).

To show (c_3), (c_4) we start from (2.5.19)–(2.5.21); verify by (2.5.11), (2.5.10), and (2.5.24) that

$$\hat{\bar{L}}^*(\bar{R}^*\bar{R} + 2\omega\bar{P})\hat{\bar{L}} = \hat{L}^*(R^*R + 2\omega P)\hat{L}, \tag{2.5.44}$$

$$\hat{\bar{L}}^*(\bar{R}^*\bar{R} + 2\omega\bar{P})e^{-\hat{A}\cdot}y_0 = \hat{L}^*(R^*R + 2\omega P)\hat{A}^{-\delta}e^{-\hat{A}\cdot}y_0; \tag{2.5.45}$$

and obtain (2.5.29), (2.5.30), at least for $y_0 \in \mathcal{D}(\hat{A}^\delta)$, by using (2.5.28), (2.5.29). Then, (2.5.29) and (2.5.30) are extended to all $y_0 \in Y$.

Verification of identity (2.5.32) in (c_5) follows as in the proof of Theorem 1.8.1.1 (iii), Eqn. (1.8.1.27): We start from (2.2.9) written for $\bar{P}$, $\bar{y}^0(t;x)$ and use (2.5.24).

Similarly, identity (2.5.33) in (c_6) is verified by means of (2.5.7) and (2.5.24).

Then (2.5.33) proves (2.5.34) in (c_7) via $\bar{u}^0(t;y_0) = -\bar{B}^*\bar{P}\bar{y}^0(t;y_0)$ as in (2.2.1), and (2.5.25), (2.5.28). Moreover, (2.5.25), (2.5.28), and (2.5.24) prove identity (2.5.35) by using $\bar{J}^0(\zeta_0) = (\bar{P}\zeta_0, \zeta_0)$ as in (2.2.4), along with (2.5.24), first for $\zeta_0 \in \mathcal{D}(\hat{A}^\delta)$, and then by extension to all $\zeta_0 \in Y$, since $P \in \mathcal{L}(Y)$ in (2.5.24).

To show (c_9), we first write the ARE (2.2.2) for $\bar{P}$, $\bar{B}^*$, $\bar{R}^*$ on $\mathcal{D}(\hat{A}^\epsilon)$ and then use (2.5.6), (2.5.7), and (2.5.24) to obtain (2.5.36) on $\mathcal{D}(\hat{A}^{\epsilon+\delta})$. Identity (2.5.37) in ($c_{10}$) can be similarly verified directly starting from $A_{\bar{P}} = A - \bar{B}\bar{B}^*\bar{P}$ and invoking (2.5.6), (2.5.7), and (2.5.24). Equations (2.5.41), (2.5.42) are nothing but Eqns. (2.3.8.11a,b) corresponding to $A_{\bar{P}} = A - \bar{B}\bar{B}^*\bar{P}$. □

We finally turn to the issue of uniqueness of P as a solution of (2.5.36) within a natural class. Accordingly, we introduce the following:

(H.5′): Detectability Condition (DC′) There exists an operator $K \in \mathcal{L}(Z; \mathcal{D}(\hat{A}^\delta))$, such that the s.c. analytic semigroup $e^{(A+KR)t}$ on $\mathcal{D}(\hat{A}^\delta)$, generated by

$$A + KR = \hat{A}^{-\delta}(A + \bar{K}\bar{R})\hat{A}^\delta, \tag{2.5.46}$$

$$\bar{K} \equiv \hat{A}^\delta K \in \mathcal{L}(Z; Y), \tag{2.5.47}$$

is exponentially stable: There exist constants $M_K \geq 1$ and $\omega_K > 0$, such that

$$\left\| e^{(A+KR)t}x \right\|_{\mathcal{D}(\hat{A}^\delta)} = \left\| e^{(A+\bar{K}\bar{R})t}y \right\|_Y \tag{2.5.48}$$

$$\leq M_K e^{-\omega_K t}\|y\|_Y = M_K e^{-\omega_K t}\|x\|_{\mathcal{D}(\hat{A}^\delta)}, \tag{2.5.49}$$

$$y = \hat{A}^\delta x \in Y. \tag{2.5.50}$$

Theorem 2.5.2 (Uniqueness) *In addition to hypotheses (H.1), (H.2) = (2.1.7), the FCC (H.4) = (2.1.12), and (H.3′) = (2.5.5) on R, assume the above DC′ (H.5′). Then:*

(a) The s.c. analytic semigroup $e^{A_P t}$ is also exponentially stable on $\mathcal{D}(\hat{A}^\delta)$: There exist constants $M_P \geq 1$ and $\omega_P > 0$ such that

$$\|e^{A_P t} x\|_{\mathcal{D}(\hat{A}^\delta)} = \|e^{A_{\bar{P}} t} y\|_Y \tag{2.5.51}$$

$$\leq M_P e^{-\omega_P t} \|y\|_Y = M_P e^{-\omega_P t} \|x\|_{\mathcal{D}(\hat{A}^\delta)}, \tag{2.5.52}$$

and there is a constant $C_P > 0$ such that

$$\|u^0(t; x)\|_U = \|B^* P e^{A_P t} x\|_U \leq C_P e^{-\omega_P t} \|x\|_{\mathcal{D}(\hat{A}^\delta)}. \tag{2.5.53}$$

(b) Moreover, the operator P defined by (2.5.24) is the unique solution of the ARE (2.5.36) within the class of nonnegative, self-adjoint operators Π such that $\hat{A}^{*\gamma} \Pi \hat{A}^{-\delta} \in \mathcal{L}(Y)$; hence $B^*\Pi \in \mathcal{L}(\mathcal{D}(\hat{A}^\delta); U)$, properties that are satisfied by P.

Proof. First, the above Detectability Condition (H.5′) for the pair $\{A, R\}$ on $\mathcal{D}(\hat{A}^\delta)$ via the operator $K \in \mathcal{L}(Z; \mathcal{D}(\hat{A}^\delta))$ is equivalent to the DC (H.5) = (2.1.13) for the pair $\{A, \bar{R}\}$, $\bar{R}$ as in (2.5.6) via the operator $\bar{K}$ in (2.5.47).

(a) Then, Theorem 2.2.1(a) applied to problem $\{A, \bar{B}, \bar{R}\}$ (i.e., to $e^{A_{\bar{P}} t}$) yields (2.5.52) of part (a) by recalling (2.5.43), from which (2.5.53) follows via (2.5.33).

(b) The uniqueness of $\bar{P}$ as a nonnegative, self-adjoint solution of the ARE (2.2.2) within the class with the property $\hat{A}^{*\bar{\gamma}} \bar{P} \in \mathcal{L}(Y)$ is guaranteed by Theorem 2.2.1(b). This then, in turn, yields the desired conclusion for P in (2.5.24) solution of (2.5.36). □

2A Bounded Inversion of $[I + SV]$, $S, V \geq 0$

In (2.3.2.11) of Lemma 2.3.2.1, as well as in (2.3.4.4) of Theorem 2.3.4.1, we need a specialization of the following inversion result. This result will likewise be needed in Chapter 9, Lemma 9.3.1.2.

Lemma 2A.1 *Let X be a Hilbert space and let S, V be two nonnegative, self-adjoint, bounded operators in $\mathcal{L}(X)$. Then:*

$$[I + SV]^{-1} \in \mathcal{L}(X), \tag{2A.1}$$

$$[I + SV]^{-1} = I - S^{\frac{1}{2}}\left[I + S^{\frac{1}{2}} V S^{\frac{1}{2}}\right]^{-1} S^{\frac{1}{2}} V, \tag{2A.2}$$

[the inverse of the positive, self-adjoint operator on the right-hand side of (2.3.1.2) plainly exists as a bounded operator on X], and

$$\|[I + SV]^{-1}\| \leq 1 + \|S\|\|V\| \tag{2A.3}$$

in the norm of $\mathcal{L}(X)$.

Proof of Lemma 2A.1 (i) First, $[I + SV]$ is injective on X:

$$[I + SV]x = 0 \Rightarrow (x, Vx) + (SVx, Vx) = 0. \tag{2A.4}$$

Since V and S are nonnegative, the second identity in (2A.4) implies $(x, Vx) = 0$; hence $Vx = 0$; Using this in the first identity of (2A.4) yields $x = 0$, as desired.

(ii) Second, the range $\mathcal{R} = [I + SV]X$ of $[I + SV]$ is dense in X, since the adjoint $[I + SV]^* = [I + VS]$ is injective by the same argument of part (i).

(iii) Third, the identity

$$\begin{aligned} I &= I + SV - S^{\frac{1}{2}}\left[I + S^{\frac{1}{2}}VS^{\frac{1}{2}}\right]^{-1}\left[I + S^{\frac{1}{2}}VS^{\frac{1}{2}}\right]S^{\frac{1}{2}}V \\ &= [I + SV] - S^{\frac{1}{2}}\left[I + S^{\frac{1}{2}}VS^{\frac{1}{2}}\right]^{-1}S^{\frac{1}{2}}V[I + SV] \\ &= \left\{I - S^{\frac{1}{2}}\left[I + S^{\frac{1}{2}}VS^{\frac{1}{2}}\right]^{-1}S^{\frac{1}{2}}V\right\}[I + SV] \end{aligned} \tag{2A.5}$$

implies by the injectivity of part (i) that identity (A.2) holds true at least on the range $\mathcal{R}$ of $[I + SV]$; that is, for all $r \in [I + SV]X = \mathcal{R}$, we have

$$[I + SV]^{-1}r = \left\{I - S^{\frac{1}{2}}\left[I + S^{\frac{1}{2}}VS^{\frac{1}{2}}\right]^{-1}S^{\frac{1}{2}}\right]^{-1}V\Big\}r. \tag{2A.6}$$

But the range $\mathcal{R}$ is dense in X by (ii), and the operator on the right-hand side of (2A.2), or of (2A.6), is bounded on X. Thus, identity (2A.6) can be extended to all of X and (2.3.1.2) holds true. It then follows from (2A.2) that, in the norm of $\mathcal{L}(X)$, we have

$$\|[I + SV]^{-1}\| \leq 1 + \left\|S^{\frac{1}{2}}\right\| \cdot 1 \cdot \left\|S^{\frac{1}{2}}\right\| \|V\| = 1 + \|S\| \|V\|, \tag{2A.7}$$

since for the nonnegative, self-adjoint S we have

$$\left\|\left[I + S^{\frac{1}{2}}VS^{\frac{1}{2}}\right]^{-1}\right\| \leq 1, \quad \left\|S^{\frac{1}{2}}\right\| \left\|S^{\frac{1}{2}}\right\| = \|S\|. \tag{2A.8}$$

Then (2A.7) proves (2A.3). $\square$

2B The Case $\theta = 1$ in (2.3.7.4) When A is Self-Adjoint and $R = I$

Lemma 2B.1 *Let the generator A be self-adjoint and $R = I$. Then, the regularity property (2.3.7.4) holds true also for $\theta = 1$: $AP \in \mathcal{L}(Y)$.*

Proof. We may, without loss of generality, assume that A is stable, in which case we may take $\omega = 0$ in (2.3.7.1) for P. Since then $\hat{\Phi}(t) = \Phi(t)$ and $-\hat{A} = A$, our task is to show that the operator AP given by

$$APx = \int_0^\infty Ae^{At}\Phi(t)x\,dt \tag{2B.1}$$

is well defined for all $x \in Y$, when A is self-adjoint and $R = I$. Using the optimal dynamics (2.3.3.6) [or (2.3.8.3)], we rewrite (2B.1) as

$$APx = \int_0^\infty Ae^{At}e^{At}x\,dt + \int_0^\infty (-A)^{1-\epsilon}e^{At}(-A)^\epsilon\{Lu^0(t;x)\}(t)\,dt. \tag{2B.2}$$

But, by stability, the first term in (2B.2) is well defined:

$$\int_0^\infty Ae^{A2t}x\,dt = \frac{1}{2}\int_0^\infty \frac{de^{A2t}x}{dt}\,dt = -\frac{1}{2}x \in Y. \tag{2B.3}$$

Moreover, the second term in (2B.2) is also well defined, since

$$\begin{aligned}(-A)^\epsilon\{Lu^0(\,\cdot\,;x)\}(t) &= \int_0^t (-A)^{\gamma+\epsilon}e^{A(t-\tau)}(-A)^{-\gamma}Bu^0(\tau;x)\,d\tau \\ &\in L_2(0,\infty;Y) \cap C([0,T_0];Y) \quad \text{for } 0 < \epsilon \le 1-\gamma.\end{aligned} \tag{2B.4}$$

The $L_2(0,\infty;Y)$-regularity stems from (i) the standard regularity property (0.4) of Chapter 0 with A stable and self-adjoint. For the $C([0,T_0];Y)$-regularity we invoke Corollary 2.3.5.2(v), Eqn. (2.3.5.15). Thus, the closed operator [Kato, 1966, p. 167] AP is well defined on all of Y, and by the closed graph theorem we obtain

$$AP \in \mathcal{L}(Y), \tag{2B.5}$$

as desired.

Remark 2B.1 Lemma 2B.1 plainly continues to hold true if R^*R commutes with A. □

A useful extension, to be invoked in Chapter 3, is given next.

Lemma 2B.2 *Still with $R = I$, Lemma 2B.1 can be extended, so that $P \in \mathcal{L}(Y;\mathcal{D}(A))$ as before in (2B.5), under the following relaxed assumptions that the generator A of the s.c., analytic semigroup e^{At} be normal so that $\mathcal{D}(A) = \mathcal{D}(A^*) = \mathcal{D}(A + A^*)$ (see Remark 2B.2 below).*

Proof. With A stable, without loss of generality, we have now

$$Px = \int_0^\infty e^{A^*t}\Phi(t)x\,dt. \tag{2B.6}$$

We only need to extend the analysis of (2B.3), by showing that

$$P_1x \equiv \int_0^\infty e^{A^*t}e^{At}x\,dt \in \mathcal{D}(A). \tag{2B.7}$$

Indeed, since A is normal, we have $e^{A^*t}e^{At} = e^{(A^*+A)t}$, with $(A^* + A)$ self-adjoint. Then, the analysis in (2B.3), applied to the present case, yields

$$(A^* + A)P_1 x = \int_0^\infty (A^* + A)e^{(A^*+A)t}x \in Y \tag{2B.8}$$

or $P_1 x \in \mathcal{D}(A) = \mathcal{D}(A^*) = \mathcal{D}(A^* + A)$ by assumption, as desired. Equation (2B.7) is proved. $\square$

Remark 2B.2 The property $\mathcal{D}(A) = \mathcal{D}(A^*)$ holds true for any normal operator A [Kato, 1966, p. 276; Weidmann, 1990, p. 125]. Moreover, if, say, A has compact resolvent, normality yields that A has an orthonormal basis of eigenvectors $\{\phi_n\}$ with corresponding eigenvalues $\{\lambda_n\}$, so that the expansions in Y,

$$Ax = \sum_{n=1}^{\infty} \lambda_n (x, \phi_n)\phi_n, \quad A^*x = \sum_{n=1}^{\infty} \bar{\lambda}_n (x, \phi_n)\phi_n; \tag{2B.9}$$

$$(A + A^*)x = \sum_{n=1}^{\infty} 2(\operatorname{Re} \lambda_n)(x, \phi_n)\phi_n, \tag{2B.10}$$

hold true in the respective domains. Finally, analyticity of e^{At} implies that $|\operatorname{Im} \lambda_n| \le k|\operatorname{Re} \lambda_n|$, for some constant $k > 0$, uniformly in $n = 1, 2 \dots$. Thus,

$$\begin{aligned} \mathcal{D}(A) = \mathcal{D}(A^*) = \mathcal{D}(A + A^*) &= \{x \in Y : \Sigma|\lambda_n|^2|(x, \phi_n)|^2 < \infty\} \\ &= \{x \in Y : \Sigma|\operatorname{Re} \lambda_n|^2|(x, \phi_n)|^2 < \infty\}. \end{aligned} \tag{2B.11}$$

Remark 2B.3 The assumptions of Lemma 2B.2 hold true in the case of the heat equation with normal generators and Dirichlet boundary conditions, in which case $\mathcal{D}(A) = \mathcal{D}(A^*)$ [Lions, Magenes, 1972, Vol. I, p. 187]. Moreover, the parabolic examples of Chapter 3 (Sections 3.4–3.8, 3.10], dealing with second-order equations (in time) with "structural damping," though not normal, are typically the direct (nonorthogonal) sum of two normal operators (see reference by Chen and Triggiani listed as [Chen, Triggiani, 1989] at the end of Chapter 3), so that conclusion (2B.5) may be extended to these cases as well.

Remark 2B.4 In the *hyperbolic* case, say canonically $A^* = -A$, we have $\mathcal{D}(A^* + A) = \mathcal{D}(0) = Y$, the entire space, while $\mathcal{D}(A^*) \subsetneq \mathcal{D}(A + A^*)$. Thus, in this case, (2B.8) does *not* imply $P_1 \in \mathcal{D}(A^*) = \mathcal{D}(A)$, that is, (2B.7).

Notes on Chapter 2

The variational approach presented in this chapter follows closely the treatment of Lasiecka and Triggiani [1987], who studied, by abstract methods, the corresponding optimal control problem for parabolic partial differential equations, defined on a bounded domain $\Omega \subset R^n$ with boundary $\Gamma = \partial\Omega$, subject to the action of Dirichlet-boundary control. In this case [to be examined in detail in Chapter 3, Sections 3.1 and 3.2], we have, for example $U = L_2(\Gamma)$, $Y = L_2(\Omega)$, and consequently $\gamma = 3/4 + \epsilon$, $\forall\, \epsilon > 0$ [or else, we could take: $Y = H^s(\Omega)$, $s < 1/2$, and $\gamma = 1 - \epsilon_s \uparrow 1$ as $s \uparrow 1/2$]. In this "concrete" parabolic problem, Lasiecka and Triggiani [1987] provided, in addition, regularity results of the optimal pair $\{u^0(\cdot\,; y_0), y^0(\cdot\,; y_0)\}$, to be given in Chapter 3, Section 3.2, Theorem 3.2.1 in terms of explicit Sobolev spaces. The useful Lemma 2A.1 in Appendix 2A is taken from Lasiecka and Triggiani [1986, p. 891].

The treatment given here, after that of Lasiecka and Triggiani [1987], is fully general. In particular, the s.c. analytic semigroup e^{At} is assumed unstable. We have already noted in Remark 2.2.1 that the present variational treatment greatly simplifies if e^{At} is stable, so that $\omega_0 = 0$ and $\omega = 0$. The limit process $T \uparrow \infty$ of Section 2.3.3 is dispensed with, and all formulas (with $\omega = 0$) become *explicit*. The case $T < \infty$ with $G = 0$ in Chapter 1, and the case $T = \infty$ with e^{At} stable, are conceptually and technically the same. See Chapter 6 in the more general setting of a min-max problem.

Clarity is only gained if we keep, initially, R bounded and then, as in Section 2.5, reduce the case of R unbounded to R bounded as is done in Chapter 1, Section 1.8. The unboundedness of B is far more critical, and technically more demanding, than the unboundedness of R. After all, the quadratic term in the ARE is B^*P and involves B.

Variational Versus Direct Methods

As in the case $T < \infty$ of Chapter 1, two distinct, yet complementary, approaches are available to study the optimal control problem of the present chapter, in particular, the existence and uniqueness of the corresponding algebraic Riccati equation:

(i) a variational approach, as in Lasiecka and Triggiani [1987] (and its numerical version Lasiecka and Triggiani [1983] to be used in Chapter 4) and
(ii) a direct method as in Flandoli [1987; 1993] and in Da Prato and Ichikawa [1985], reported in the book [Bensoussan et al., 1992] by Bensoussan et al. The variational argument starts from the control problem as the primary issue and constructs an explicit candidate for the Riccati operator (in terms of the data of the problem with the help of the optimal solution; see below (2.2.9)), which is then shown to satisfy the ARE (2.2.2). This is done in the present chapter. All the formulas of the relevant quantities are truly explicit in terms of the data in the stable case, where we can then take $\omega = 0$ throughout, which greatly simplifies the analysis. A treatment, in the min–max case, will be given in Chapter 6.

In contrast, the direct approach takes the direct study of well-posedness (existence and uniqueness) of the ARE as the primary object and only subsequently recovers the control problem (via dynamic programming), which generates the original ARE.

In carrying out its task, the direct method begins actually with a direct study of the corresponding differential (or integral) Riccati equation of the optimal problem over a finite interval $[0, T]$, $T < \infty$, and operates a limit process as $T \to \infty$, on the differential Riccati equation (in line with a classical approach, which now, however, has to overcome new technical difficulties, particularly the strong convergence of $B^* P_T(0)$ to $B^* P$).

In both approaches, a key point consists in establishing that $(\hat{A}^*)^\gamma P \in \mathcal{L}(Y)$, and hence that the gain operator $B^* P$ (a priori not necessarily well defined) is, in fact, a bounded operator, $B^* P \in \mathcal{L}(Y; U)$; see (2.2.5).

In the challenging case $1 > \gamma \geq 1/2$, the variational approach of this chapter achieves the boundedness of $B^* P$ by using analyticity of the free dynamics, together with a bootstrap argument (as in Chapter 1), based on the Young inequality, to show that the optimal pair is more regular; indeed $e^{-\omega t} u^0(t; y_0) \in C([0, \infty]; U)$ and $e^{-\omega t} y^0(t; y_0) \in C([0, \infty]; Y)$ for $y_0 \in Y$ (Theorem 2.3.5.1). (A priori, we only know that $u^0 \in L_2(0, \infty; U)$, whereas a general control $u \in L_2(0, T; U)$ need *not* produce in general a corresponding solution $y \in C([0, T]; Y)$ unless $\gamma < 1/2$. As noted in Chapter 1, a counterexample is obtained by a parabolic equation, even in one dimension, with Dirichlet boundary control where $U = L_2(\Gamma)$, and $Y = L_2(\Omega)$ [Lions, 1971, p. 217].) All this leads to the regularity property on $(\hat{A}^*)^\theta P \in \mathcal{L}(Y)$, $\theta < 1$, via the explicit representation of P in terms of the optimal solution, which in turn leads to property (2.2.5) on $B^* P \in \mathcal{L}(Y; U)$.

Instead, in the direct approach of Da Prato and Ichikawa [1985] and Flandoli [1987], the boundedness (2.2.5) of the gain operator $B^* P$ is established by proving first that the solution of the corresponding differential Riccati equation for the problem on $[0, T]$ possesses the corresponding desired regularity properties, and then by passing to the limit as $T \to \infty$. This, in turn, is accomplished in Flandoli [1987] by repeated applications of the Young's inequality to prove that the optimal trajectory is in $C([0, T]; Y)$ for any $T > 0$, or in Da Prato and Ichikawa [1985] by a direct study of the evolution equation via a fixed point argument.

Subsequently, in Flandoli [1993], Flandoli has considerably simplified his earlier treatment (while removing the assumption that A has compact resolvent) in taking the limit on the differential Riccati equations to obtain the algebraic Riccati equation, thanks to his Lemma 2.2.1 in Flandoli [1993], the difficult new step in the treatment of Flandoli [1993].

A Frequency Domain Approach

Recently, in Pandolfi [1996], Pandolfi has proposed to rederive some of the existing results for the problem of the present chapter "following a route which is largely (but not completely) in the frequency domain," that is, by Fourier transform analysis, coupled with the Plancherel Theorem. This treatment, which assumes that the operator A is stable, recovers existence of the ARE. Uniqueness is not studied.

Verification of the Finite Cost Condition (2.1.12)

Some canonical, classical parabolic partial differential equations – such as the heat equation with homogeneous Dirichlet boundary conditions – yield an operator A that is already stable, and thus the Finite Cost Condition is automatically satisfied (with $K = 0$). In the general case of parabolic PDEs defined on a bounded domain Ω of R^n – where at most, finitely many eigenvalues are unstable – the Finite Cost Condition (2.1.12) is readily verified by appealing to one of the numerous (boundary) stabilization results of the literature. These, moreover, prescribe a priori the structure of the stabilizing feedback operator K. References will be provided in (the Notes of) Chapter 3, which is devoted to illustrative parabolic PDE problems. These are covered by the abstract treatment of the present chapter.

Verification of the Detectability Condition (2.1.13)

Detectability of the pair $\{A, R\}$ [or $\{A, (R^*R)^{1/2}\}$] is equivalent to stabilizability of the dual pair $\{A^*, R^*\}$. Thus, the above comments on stabilization apply regarding its verifiability, with the added simplification when R is bounded.

Detectability is Not a Necessary Condition for the Feedback Generator $A_P = A - BB^*P$ to be Stable

The Detectability Condition (2.1.13) – apparently first introduced in Wonham [1968] in the finite dimensional case, and subsequently used by Zabczyk [1976] in an infinite dimensional context – is generally only sufficient, but not necessary, for the feedback generator $A_P = A - BB^*P$, to generate a uniformly stable semigroup. Here P is the Riccati operator (2.3.3.2), defining the optimal cost by (2.3.3.3), which is the *minimal* nonnegative, self-adjoint solution of the ARE (2.2.2), as seen in Proposition 2.4.3, Eqn. (2.4.21). A counterexample is provided in Da Prato and Delfour [1988] and is reported in [Bensoussan et al., 1992, p. 271]. Here, moreover, a necessary and sufficient condition is given for the Detectability Condition to be equivalent to the stability of the generator $A_P = A - BB^*P$; see Bensoussan et al. [1992, p. 277].

Spectral Factorization Approach to Algebraic Riccati Equations: $T = \infty$, and Exponentially Stable e^{At}

The Riccati theory of Chapter 1 ($T < \infty$) and Chapter 2 ($T = \infty$) – whether direct or variational – has been essentially in place since approximately 1985. Over the past few years, system theoretic circles, in an attempt to mimic and extend frequency domain theories available in finite dimension, have resurrected an old idea, that of "spectral factorization" (see e.g. [Balakrishnan, 1981, Theorem 2.3.4.5], which reports at p. 132 and uses at p. 327 M. G. Krein's factorization theorem in the filtering problem).

In these more recent efforts, spectral factorization is applied to Toeplitz operators [Rosenblum, Rovnyak, 1985]. At end of the fifth paragraph in the Preface to these volumes, we wrote: "Regarding abstract modeling, the flow runs unmistakenly from an understanding of the concrete into the proper abstract." That is, it is a proper analysis of boundary/point control models for PDE's that yields the abstract models which we consider. There are no artificial assumptions. The spectral factorization approach, instead, follows, apparently, the opposite direction: it begins at the level of an axiomatic definition of a linear system in Hilbert space and introduces whatever definitions of subclasses are convenient to arrive at some results. At present, this top down attempt does not include boundary/point control problems of multidimensional PDE's. Thus, we shall not pursue this topic in these first two volumes beyond the present comments.

In synthesis, the spectral factorization approach is as follows (see [Curtain, Staffans, 1998, p. 90]):

(i) the first step is to reduce the regulator problem for $T = \infty$ (and exponentially stable free dynamics e^{At}) to an associated spectral factorization problem;
(ii) next, one appeals to known results for the existence of a spectral factor, e.g., in Rosenblum, Rovnyak [1985];
(iii) next again, one uses "realization theory" to construct a realization of this spectral factor from a given realization of the original system;
(iv) at this point, one introduces definitions and makes assumptions to require certain critical properties of the spectral factor;
(v) using the above postulated critical properties, the appropriate algebraic Riccati equation can be derived, and the optimal solution can be constructed.

The following comments seem appropriate:

(a) This approach cannot work for finite horizon problems ($T < \infty$) as the operator of interest is not Toeplitz: thus, Differential (Integral) Riccati equations such as in Chapter 1 (parabolic) and Chapters 8 through 10 (hyperbolic) are ruled out.
(b) In the infinite horizon problem ($T = \infty$), this approach requires, at present at least, exponential stability of the original semigroup e^{At} (see Staffans [1998] and its reference "Staffans 1998e").
(c) In the hyperbolic case (which will be reviewed in Volume 3), this approach is presently stuck with "two gaps" described in [Curtain, Staffans, 1998, p. 90]: while, in general, the required properties of the spectral factor in (iv) need not hold true, verifiable conditions under which they are true are presently lacking. "In particular, it is not known to what extent systems modelled as boundary control problem for the wave equation in several space dimensions, possess these spectral properties" [Curtain, Staffans, 1998, p. 91].
(d) In the parabolic case, according to [Staffans, 1998, Section 1, 4] the key ingredient [needed to establish these spectral properties] is the boot-strap argument

introduced in Lasiecka and Triggiani (this reference is Lasiecka, Triggiani [1983]), which is reported in Chapters 1 and 2. Moreover, in addition to exponential stability of the original semigroup, the approach in Staffans [1998] and its reference "Staffans 1998e" requires a space X such that the solution $y(t)$ subject to an L_2-control, satisfies the property $y(t) \in C([0, T]; X)$. This assumption, which is natural for classes of hyperbolic problems (see [Chapters 7, 9, 10]), requires instead a severe penalization in the space variable for parabolic problems as well-documented in the subsequent Chapter 3. Why, then, not take $X = [\mathcal{D}(A^*)]'$, with duality with respect to the state space Y selected in this Volume 1? However, one surely wants a final theory in a much more regular state space – Y, not $[\mathcal{D}(A^*)]'$ – where the above continuity property in time does not hold true for parabolic problems, (see [Chapter 3, e.g., Sections 3.1–3.3]). No example of a PDE boundary problem is given in Staffans [1998] and its reference "Staffans 1998e," to ascertain if this is the case, and more generally, to assess the scope of these results. In the latter reference "Staffans 1998e," which is not confined to parabolic problems, the framework requires for its main result, a long list of assumptions, including the hypothesis on the existence of a boot-strap property from L_2 to C for the optimal pair.

(e) In the present Chapter 2 – which, unlike Staffans [1998], treats the fully unstable case – the boot-strap argument (leading to the regularity results of the optimal pair in $C([0, T]; \cdot)$ in Corollary 2.3.5.2) represents the bulk of the proof. By contrast, in the spectral factorization approach, the boot-strap argument is merely an ingredient used to verify the spectral factor properties in (ii) above. In addition, further machinery such as Toeplitz operators [Rosenblum, Rovnyak, 1985], realization theory, analysis of the dynamics for negative time, etc. – of which there is no trace or use in these volumes – need to be employed, yielding a lengthy process, which is presently dispersed in many papers.

In conclusion, these volumes attempt to follow the credo that the problems should dictate the techniques. Accordingly, we shall not consider the proposed spectral factorization scheme.

Glossary of Symbols for Chapter 2

$A, B, \omega_0, \hat{A}, \hat{\omega}, \gamma$	(2.1.1)–(2.1.4), (2.1.7)
J, R	(2.1.8)
K, ω_K, M_K	(2.1.13)
$L, L^*, \hat{L}, \hat{L}^*$	(2.1.15)–(2.1.18)
$u^0(t; y_0), y^0(t; y_0)$	Theorem 2.2.1, (2.2.1)
P	(2.2.4), (2.2.9)

$A_P = A - BB^*P$	(2.2.6)
$\Phi(t)x = y^0(t;x) = e^{A_P t}$	above and below (2.2.6), (2.3.3.5c)
$\hat{u}^0(t;y_0),\ \hat{y}^0(t;y_0)$	above (2.2.7), (2.3.3.5c)
$J^0(y_0)$	(2.2.13)
$J_T,\ P_T(t),\ \Phi_T(\tau,t)$	(2.3.1.1)–(2.3.1.3)
$\hat{\Phi}_T(t,s)$	(2.3.1.7)
$u_T^0(t,0;y_0),\ y_T^0(t,0;y_0),$	
$\hat{u}_T^0(t,0;y_0),\ \hat{y}_T^0(t,0;y_0)$	(2.3.1.10), (2.3.1.11)
$\hat{L}_{(T)}^*,\ J_T^0(y_0)$	(2.3.1.13), (2.3.1.15)
$\hat{\Phi}(t);\ \hat{y}^0(t;x),\ \hat{u}^0(t;x)$	(2.3.3.5c)
$u_{T,\text{ext}}^0,\ y_{T,\text{ext}}^0$	(2.3.3.10)
$\tilde{u}(t;x),\ \tilde{y}(t;x)$	(2.3.3.11), (2.3.3.14)
$\hat{y}_T^0(t,0;x),\ \hat{\tilde{u}}(t;x),\ \hat{\tilde{y}}(t;x)$	(2.3.3.17), (2.3.3.18)
$C_{ub}([0,\infty];\ \cdot)$	below (2.3.5.6)
$A_P,\ \hat{A}_P,\ \hat{A}_P^*$	(2.3.8.1), (2.3.8.9)
$R(\lambda,A_P),\ R(\lambda,A)$	(2.3.8.5)
$P_1,\ A_{P_1},\ \Phi_1(t) = e^{A_{P_1}t}$	(2.3.8.16), (2.3.8.18),(2.3.8.20)
$A_K,\ e^{A_K t}$	(2.4.5), (2.4.6)
Q	(2.4.33)

References and Bibliography

A. V. Balakrishnan, *Applied Functional Analysis*, 2nd ed., Springer-Verlag, 1981.

A. Bensoussan, G. Da Prato, M. Delfour, and S. Mitter, *Representation and Control of Infinite Dimensional Systems*, Vol. II, Birkhauser, 1992.

R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Wiley, 1953.

R. Curtain and O. J. Staffans, Regular spectral factorizations, in *Open Problems in Mathematical Systems and Control Theory,* V. Blondel. etc., Editors, Springer, 1998.

R. Datko, Extending a theorem of Liapunov to Hilbert spaces, *J. Math. Anal. Appl.* **32** (1970), 610–616.

G. Da Prato and M. C. Delfour, Stabilization and unbounded solutions of the Riccati equation, in *Proc. 27th IEEE Conference on Decision and Control*, pp. 352–357, IEEE Publications, New York, 1988.

G. Da Prato and M. C. Delfour, Unbounded solutions to the linear quadratic control problem, *SIAM J. Control Optim.* **30** (1992), 31–48.

G. Da Prato and A. Ichikawa, Riccati equations with unbounded coefficients, *Ann. Matem. Pura Appl.* **140** (1985), 209–221.

N. Dunford and J. Schwartz, *Linear Operators, Part II*, Interscience Publishers, Wiley, 1963.

I. Ekeland and R. Teman, *Convex Analysis and Variational Problems*, North-Holland, 1976.

H. O. Fattorini, *The Cauchy Problem*, Encyclopedia of Mathematics and Its Applications, Vol. 18, Addison-Wesley, 1983.

F. Flandoli, Algebraic Riccati equations arising in boundary control problems, *SIAM J. Control Optim.* **25** (1987), 612–636.

F. Flandoli, On the direct solutions of Riccati equations arising in boundary control theory, *Ann. Matem. Pura Appl.* (iv) **CLXIII** (1993), 93–131.

T. Kato, *Perturbation Theory of Linear Operators*, Springer-Verlag, 1966.

J. L. Lions and E. Magenes, *Nonhomogeneous Boundary Value Problems, I*, Springer-Verlag, 1972.

I. Lasiecka and R. Triggiani, The regulator problem for parabolic equations with Dirichlet boundary control, Part I: Riccati's feedback synthesis and regularity of optimal solution, *Appl. Math. Optim.* **16** (1986), 147–168.

I. Lasiecka and R. Triggiani, Riccati equations for hyperbolic partial differential equations with $L_2(0, T; L_2(\Gamma))$-Dirichlet boundary terms, *SIAM J. Control Optim.* **24** (1986), 884–925.

I. Lasiecka and R. Triggiani, Dirichlet boundary control problem for parabolic equations with quadratic cost: analyticity and Riccati's feedback synthesis, *SIAM J. Control & Optimiz.* **21** (1983), 41–67.

J. L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, 1971.

L. Pandolfi, The standard regulator problem for systems with input delays: An approach through singular control theory, *Appl. Math. Optim.* **31** (1995), 119–136.

L. Pandolfi, A frequency domain approach to the boundary control problem of parabolic equations, in *Modelling and Optimization of Distributed Parameter Systems*, K. Malanowski, Z. Nahorski, and M. Peszynska, eds., Chapman and Hall, 1996. pp. 149–158.

L. Pandolfi, From singular to regular control problems, *Marcel Dekker Lecture Notes in Pure and Applied Mathematics*, Vol. 165, pp. 153–166, *Control of Partial Differential Equations*, G. Da Prato and L. Tubaro, eds.

A. Pazy, *Semigroups of Operators and Applications to Partial Differential Equations*, *Appl. Math. Sci.* **44**, Springer-Verlag, 1983.

M. Rosenblum and J. Rovnyak, *Hardy Classes and Operator Theory*, Oxford Mathematical Monographs, 1985.

C. Sadosky, *Interpolation of Operators and Singular Integrals*, Marcel Dekker, Monographs in Pure and Applied Mathematics, 1979.

M. Sorine, Un résultat d'existence et d'unicité pour l'équation de Riccati stationaire, INRIA Report 55, Rocquencourt, France 1981 (also CRMA report no. 984, Université de Montréal, 1980).

O. J. Staffans, Quadratic Control of a parabolic equation, Proceedings of MTNS (*Mathem. Theory of Network and Systems*), held at University of Padova, Italy, June 1998.

A. E. Taylor and D. C. Lay, *Introduction to Functional Analysis*, 2nd ed., 1980.

H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, 1978.

J. Weidmann, *Linear Operators in Hilbert Spaces*, Springer-Verlag, 1980.

W. M. Wonham, On a matrix Riccati equation of stochastic control, *SIAM J. Control Optim.* **6** (1968), 681–697.

J. Zabczyk, Remarks on the algebraic Riccati equation in Hilbert spaces, *Appl. Math. Optim.* **2** (1976), 251–258.

3

Illustrations of the Abstract Theory of Chapters 1 and 2 to Partial Differential Equations with Boundary/Point Controls

The goal of the present chapter is to illustrate the applicability of the abstract theory of Chapters 1 and 2 to physically significant parabolic, or parabolic-like, partial differential equations that are subject to boundary controls or to point controls, and – possibly – with boundary observation as well. We shall see that the generality of the abstract theory of Chapters 1 and 2 readily accommodates boundary/point control actions on parabolic-like partial differential equations, and, in fact, on optimal function spaces, which are consistent with the regularity theory of such dynamics. No artificial assumptions are introduced on the PDE problems. Rather, the PDE formulation and the intrinsic properties of its solutions are captured by the general abstract model and by the general abstract theory of Chapters 1 and 2. Our illustrations include the following broad classes:

(a) Classical parabolic equations, such as
  - (i) parabolic PDEs with Dirichlet boundary control and interior observation (Sections 3.1 and 3.2);
  - (ii) parabolic PDEs with Neumann boundary control and interior or boundary observation (Section 3.3).

(b) The class of second-order (in time) platelike or wavelike partial differential equations, with high internal damping ("structural damping"), which display a parabolic behavior, such as:
  - (iii) a structurally damped Euler–Bernoulli-type equation with (interior) point control (Sections 3.4 and 3.7) and with boundary control (Section 3.6);
  - (iv) a Kelvin–Voight platelike equation with point control (Section 3.5);
  - (v) a strongly damped wave equation with point control (Section 3.8);
  - (vi) a structurally damped Kirchhoff equation with (interior) point control exercised through $\delta(\cdot - x^0)$ (Section 3.9) or through $\delta'(\cdot - x^0)$ (Section 3.10).

(c) The class of systems of two PDEs that couple a mechanical equation (Euler–Bernoulli type) with a thermal equation (heat equation), such as they arise in modeling thermo-elastic plates. They include:

(vii) thermo-elastic plates with thermal control in the Dirichlet Boundary Conditions (BC) and homogeneous clamped mechanical BC (Section 3.11.1);

(viii) thermo-elastic plates with thermal control in the Neumann/Robin BC, homogeneous clamped BC, and boundary observation (Section 3.11.2);

(ix) thermo-elastic plates with mechanical control in the bending moment and homogeneous Neumann thermal BC (Section 3.12);

(x) thermo-elastic plates with mechanical control as a shear force, in the free BC (Section 3.13).

It should be noted that the mathematical analysis of certain key properties of thermo-elastic plates – such as the analyticity of the underlying s.c. contraction semigroup and its stability in the uniform operator topology – is a research topic of very recent origin, with key new results on the analyticity issue, due to the authors, appearing in print for the first time in this volume. Accordingly, all of the related control problems in Sections 3.11 through 3.13 are new. More on this topic is contained in the Notes at the end of Chapter 3.

(d) Structurally damped Euler–Bernoulli equations with damped free BC and point control; or else with boundary control in the shear forces, or in the moment BC (Section 3.14).

(e) A physically interesting linearized model of well/reservoir coupling for a monophasic flow, with boundary control (Section 3.15). This model also appears to be of recent origin; see the Notes at the end of Chapter 3.

Finally, in Section 3.16, we include additional parabolic examples (besides the thermo-elastic plate of Section 3.11.2) with a genuinely unbounded control operator B, and a genuinely unbounded observation operator R, illustrating the theory of Chapter 1, Section 1.8 and Chapter 2, Section 2.5. They include: (i) the heat equation with Neumann boundary control and point observation and (ii) the heat equation with point control and point observation when $\dim \Omega = 1$.

3.0 Examples of Partial Differential Equation Problems Satisfying Chapters 1 and 2

In this chapter, we illustrate the applicability of the main results of Chapter 1 (Theorem 1.2.1.1 for $T < \infty$) and of Chapter 2 (Theorem 2.2.1 and Theorem 2.2.2, for $T = \infty$) to the "analytic" class subject to the standing hypothesis $A^{-\gamma}B \in \mathcal{L}(U; Y)$, $\gamma < 1$. Obvious candidates for the analytic class are heat or diffusion problems. A few illustrations thereof, with Dirichlet and Neumann boundary control, are treated below in Sections 3.1 through 3.3. Less standard illustrations involving platelike or wavelike equations with a strong degree of damping ("structural damping"), such as may arise in the study of flexible structures, are given in Sections 3.4 through 3.10 (and Appendix 3B). Even less traditional illustrations, which have become available only very recently, on the topic of thermo-elastic plates are included in Sections 3.11 through 3.13 (and Appendices 3D–3J). The key proofs of *analyticity*, due to the authors, for most of the thermoelastic models of Sections 3.11 through 3.13 are new and are provided in the relative appendices. Section 3.14 treats Euler–Bernoulli equations with special damping in the shear, or forces, BC, which yield attractive and unexpected abstract (new) models. A well/reservoir coupling model with Dirichlet boundary control is analyzed in Section 3.15. The case of a genuinely unbounded control operator B and a genuinely unbounded observation operator R is covered in Section 3.11.2 and Section 3.16. Finally, we emphasize that for sake of simplicity of exposition, our illustrations below will be explicitly confined to canonical cases involving the Laplacian Δ or the biharmonic operator Δ^2. Appropriate remarks, however, will point out the applicability of the theory to elliptic differential operators with smooth coefficients depending on the space variable.

3.1 Heat Equation with Dirichlet Boundary Control: Riccati Theory

Let $\Omega \subset R^n$ be an open bounded domain with sufficiently smooth boundary Γ (corners may be allowed; see Remark 3.1.2 below). In Ω, we consider the Dirichlet mixed problem for the heat equation in the unknown $y(t, x)$:

$$\begin{cases} y_t = \Delta y + c^2 y & \text{in } (0, T] \times \Omega \equiv Q, & (3.1.1a)\\ y(0, \cdot) = y_0 & \text{in } \Omega, & (3.1.1b)\\ y|_\Sigma = u & \text{in } (0, T] \times \Gamma \equiv \Sigma, & (3.1.1c)\end{cases}$$

with boundary control $u \in L_2(\Sigma)$ and $y_0 \in L_2(\Omega)$. We consider both cases $T < \infty$ and $T = \infty$. The cost functional we seek to minimize is then

$$J(u, y) = \int_0^T \left\{ \|y(t)\|^2_{L_2(\Omega)} + \|u(t)\|^2_{L_2(\Gamma)} \right\} dt + \alpha \|y(T)\|^2_{L_2(\Omega)}, \tag{3.1.2}$$

where $\alpha = 0$ if $T = \infty$, and $\alpha = 1$ for $T < \infty$. The following Regularity Result of problem (3.1.1) is well known [Lions, Magenes, 1972, Vol. II, p. 81], and, for completeness, a proof of it will be given at the end of this section:

$$(\text{R.R.}) = \begin{cases} \text{Let, say, } y_0 = 0 \text{ and } u \in L_2(\Sigma); \text{ then:} \\ \Rightarrow \\ y \in H^{\frac{1}{2}, \frac{1}{4}}(Q) = L_2\big(0, T; H^{\frac{1}{2}}(\Omega)\big) \cap H^{\frac{1}{4}}(0, T; L_2(\Omega)), \end{cases}$$

for any $T < \infty$, but y does *not* belong to $C([0, T]; L_2(\Omega))$, even in the one-dimensional case [Lions, 1971, p. 217]. (Thus, the abstract model of the subsequent Volume 2 is not applicable to problem (3.1.1), (3.1.2), where $Y = L_2(\Omega)$.)

Abstract Setting [Balakrishnan, 1981; Washburn, 1974; 1979; Lasiecka, 1978; 1980a, b; Triggiani, 1979; 1980a, b; see also Notes at the end of this chapter.]

To put problems (3.1.1), (3.1.2) into the abstract setting of the preceding Chapters 1 and 2, we introduce the self-adjoint operator

$$Ah = \Delta h + c^2 h : \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega) \to L_2(\Omega), \tag{3.1.3}$$

select the spaces

$$Z_f = Z = Y = L_2(\Omega); \quad U = L_2(\Gamma), \tag{3.1.4}$$

and finally, define the operators

$$Bu = -ADu; \quad B : L_2(\Gamma) \to [\mathcal{D}(A)]'; \quad R = I, \tag{3.1.5}$$

where D (Dirichlet map) is defined by

$$h = Dg \quad \text{iff} \quad (\Delta + c^2)h = 0 \text{ in } \Omega; \quad h|_\Gamma = g \text{ on } \Gamma. \tag{3.1.6}$$

In (3.1.5), A is the isomorphic extension, say, $L_2(\Omega) \to [\mathcal{D}(A)]'$ (generally, $L_2(\Omega) \to [\mathcal{D}(A^*)]'$) of the operator A in (3.1.3). For simplicity of notation, we shall use the same symbol A in all cases, with no fear of confusion. By elliptic theory [Lions, Magenes, 1972, Vol. I, p. 187] and known identifications [Grisvard, 1967; Lasiecka, 1980(b), Appendix B; Fujiwara, 1967; Kondratiev, 1967, p. 227; Lions, Magenes, 1972, Vol. 1, p. 107] (see also Appendix 3A)

$$D : \text{continuous } L_2(\Gamma) \to H^{\frac{1}{2}}(\Omega) \subset H^{\frac{1}{2}-2\epsilon}(\Omega) \equiv \mathcal{D}\big(A_D^{\frac{1}{4}-\epsilon}\big), \quad \forall \epsilon > 0, \tag{3.1.7a}$$

or, more generally [Lions, Magenes, 1972, p. 187],

$$D : \text{continuous } H^s(\Gamma) \to H^{s+\frac{1}{2}}(\Omega), \quad s \in \mathbb{R}, \tag{3.1.7b}$$

where A_D is the positive, self-adjoint operator

$$A_D h = -\Delta h, \quad \mathcal{D}(A_D) = H^2(\Omega) \cap H_0^1(\Omega). \tag{3.1.8}$$

We note that, from (3.1.7a), we have that $A_D^{\frac{1}{4}-\epsilon} D \in \mathcal{L}(L_2(\Gamma); L_2(\Omega))$, $\forall \epsilon > 0$, where we *cannot* take $\epsilon = 0$; see Remark 3.1.4 at the end of this section.

Lemma 3.1.1 *In the notation introduced above, we have*

$$B^*\phi = -D^* A\phi|_\Gamma = -\frac{\partial \phi}{\partial \nu}, \quad \phi \in \mathcal{D}(A) \tag{3.1.9}$$

and (3.1.9) can be extended to all $\phi \in H^{\frac{3}{2}+\epsilon}(\Omega) \cap H_0^1(\Omega)$, $\epsilon > 0$ *(see generalization in Remark 3.1.2 below).*

Proof. Let $\phi \in \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $u \in L_2(\Gamma)$. Application of Green's second theorem yields, by (3.1.5) and (3.1.3),

$$\begin{aligned}(Bu, \phi)_{L_2(\Omega)} &= -(Du, A\phi)_{L_2(\Omega)} = -(Du, (\Delta + c^2)\phi)_{L_2(\Omega)} \\ &= -(\cancel{(\Delta + c^2)Du}, \phi)_{L_2(\Omega)} - \int_\Gamma Du\frac{\partial\phi}{\partial\nu}\,d\Gamma + \cancel{\int_\Gamma \frac{\partial Du}{\partial\nu}\phi\,d\Gamma} \\ \text{(by (3.1.6))} \quad &= -\left(u, \frac{\partial\phi}{\partial\nu}\right)_{L_2(\Gamma)},\end{aligned}$$

and (3.1.9) follows. The noted cancellations are due to (3.1.6) and to $\phi|_\Gamma = 0$, while $Du|_\Gamma = u$ follows again by (3.1.6). □

Notice that the fractional powers of the translation $\hat{A} = -A + \omega I$ of $-A$ or of A_D are well defined.

Assumption (1.1.2) or (2.1.7) $(\hat{A})^{-\gamma}B \in \mathcal{L}(U; Y)$ This assumption is satisfied in our present case with $\gamma = 3/4 + \epsilon, \forall\, \epsilon > 0$. In fact, we may take $\hat{A} = A_D$. From (3.1.7a), we have via (3.1.5)

$$-B = AD\colon \text{continuous } L_2(\Gamma) \to [\mathcal{D}(\hat{A}^{\frac{3}{4}+\epsilon})]' = [\mathcal{D}(A_D^{\frac{3}{4}+\epsilon})]', \tag{3.1.10}$$

and we then have, with $\gamma = 3/4 + \epsilon$, that our claim is verified:

$$\hat{A}^{-\gamma}B = -A_D^{-\gamma}AD \in \mathcal{L}(L_2(\Gamma); L_2(\Omega)) = \mathcal{L}(U; Y). \tag{3.1.11}$$

Assumption of Analyticity The operator A in (3.1.3) generates a s.c. semigroup e^{At} on $L_2(\Omega)$, which is, moreover, analytic here for $t > 0$ (and contraction after a suitable translation of the generator). [This holds true also if $(-\Delta)$ is replaced by a more general second-order, uniformly elliptic operator [Fattorini, 1983; Friedman, 1976; Pazy, 1983], as in Remark 3.1.2 below.]

Finite Cost Condition (2.1.12) Since its resolvent is compact, the generator A has (for suitably large constant c^2 in (3.1.3)) only finitely many unstable eigenvalues of finite multiplicity. Moreover, e^{At} is analytic. Thus, the stabilization theory as in [Triggiani, 1976; 1979; 1980a, b; Lasiecka, Triggiani, 1983(a), Manitius, Triggiani, 1978a, b, Appendix], etc. applies: Problem (3.1.1) is stabilizable on $L_2(\Omega)$ if and only if its projection onto the finite-dimensional unstable subspace is controllable. In particular, as shown in Triggiani [1980(b)] and Lasiecka and Triggiani [1983(a)] for general second-order elliptic operators, one may prescribe the stabilizing feedback to be of the form

$$u(t) = \sum_{k=1}^{N}(y(t), w_k)_{L_2(\Omega)}g_k \tag{3.1.12}$$

for suitable vectors $w_k \in L_2(\Omega)$ and $g_k \in L_2(\Gamma)$ and minimal N equal to the largest multiplicity of the unstable eigenvalues, in order to stabilize uniformly the corresponding feedback system, in the norm of $H^{\frac{1}{2}-\epsilon}(\Omega)$. Thus, a fortiori the Finite Cost Condition (2.1.12) of Chapter 2 on $L_2(\Omega)$ is satisfied.

Detectability Condition (2.1.13) This is automatically satisfied since, in our case, $R = I$; see Remark 2.1.1 of Chapter 2.

Conclusion: Case $T = \infty$ Theorems 2.2.1 and 2.2.2 of Chapter 2 apply to problem (3.1.1) and (3.1.2), asserting, in particular, the existence and uniqueness (in the sense of [Chapter 2, Theorem 2.2.2, below (2.2.12)]) of the Riccati operator $P \in \mathcal{L}(L_2(\Omega); H^{2-\epsilon}(\Omega)), \forall\, \epsilon > 0$ [see Chapter 2, Eqn. (2.2.3a)]. (Actually, in the canonical case where A is self-adjoint, one may take $\epsilon = 0$; see Chapter 2, Appendix 2B.) The Riccati operator P is Hilbert–Schmidt on $L_2(\Omega)$ for dim $\Omega = 1, 2, 3$ [see Chapter 2, Remark 2.2.2, Eqn. (2.2.17)].

Conclusion: Case $T < \infty$ Theorem 1.2.1.1 of Chapter 1 applies to problem (3.1.1) and (3.1.2) with $T < \infty$ for any final state operator G that makes GL_T a closed (closable) operator. In our present case in (3.1.2), $G =$ Identity on $L_2(\Omega)$, $Z = Z_f = Y$, while the operator L_T in Chapter 1 [Eqn. (1.1.9)]:

$$L_T(u) = (Lu)(T) = \hat{A}^{\gamma} \int_0^T e^{A(T-\tau)} \hat{A}^{-\gamma} Bu(\tau)\, d\tau \tag{3.1.13}$$

is closed as an operator $L_2(0, T; U) \supset \mathcal{D}(L_T) \to Y$, being the composition of the closed operator $\hat{A}^{\gamma}$ and a bounded operator, with $\hat{A}^{-\gamma}$ bounded [Kato, 1966, p. 164]. Thus, Theorem 1.2.2.1 of Chapter 1 applies. (One may also check the sufficient condition (1.2.1.26) of Chapter 1, that is, that $(-A^*)^{\beta/2} G^* = (-A^*)^{\beta/2}$ is densely defined $L_2(\Omega) \to L_2(\Omega)$, which is true for all $\beta > 0$.)

Remark 3.1.1 The above analysis applies, with no essential change, to $y(t)$ penalized in $L_2(0, T; H^{\frac{1}{2}-\epsilon})$), with $y_0 \in H^{\frac{1}{2}-\epsilon}(\Omega)$ [rather than in $L_2(0, T; L_2(\Omega))$ as in (3.1.2)], where then $\gamma = 1 - \epsilon/2$.

Remark 3.1.2 The above analysis applies, with no essential changes, to the case where $(-\Delta)$ is replaced by a general second-order operator with uniformly elliptic principal part and with variable coefficients (smoothly depending on the space variable):

$$-A(x, \partial) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + \sum_{j=1}^{n} b_j(x)\frac{\partial}{\partial x_j} + c, \tag{3.1.14}$$

where the matrix $[a_{ij}]$ is symmetric. Self-adjointness of the realization of $A(x, \partial)$ on $L_2(\Omega)$ is not critical: For example, Eqn. (3.1.9) of Lemma 3.1.1 reads, in general,

$$B^*\phi = -D^*A^*\phi|_\Gamma = -\frac{\partial \phi}{\partial \nu_{A^*}} = \sum_{ij} a_{ij} \frac{\partial}{\partial x_j} \nu_i, \quad \phi \in \mathcal{D}(A^*), \tag{3.1.15a}$$

where because of the smoothness of the coefficients

$$c\left\|\frac{\partial\phi}{\partial\nu}\right\|_{L_2(\Gamma)} \le \|D^*A^*\phi\|_{L_2(\Gamma)}$$

$$\le C\left\|\frac{\partial\phi}{\partial\nu}\right\|_{L_2(\Gamma)}, \quad \phi\in\mathcal{D}(A^*), 0<c<C<\infty, \tag{3.1.15b}$$

where $\partial/\partial\nu_{A^*}$ is the co-normal derivative relative to A^* [Lasiecka et al., 1986, Eqns. (4.2)–(4.3)]. One may also consider domains Ω with conical or angular points as in [Kondratiev, 1967, p. 227; Grisvard, 1972; Necas, 1967, p. 252], etc.

Finally, under homogeneous Dirichlet boundary conditions one has [Lions, Magenes, 1972, p. 187]

$$\mathcal{D}(A) = \mathcal{D}(A^*), \tag{3.1.16}$$

and hence by interpolation

$$\mathcal{D}(\hat{A}^\theta) = \mathcal{D}(\hat{A}^{*\theta}), \quad 0\le\theta\le 1. \tag{3.1.17}$$

Proof of the Regularity Result (R.R.) for Parabolic Mixed Problems with Dirichlet Control $u\in L_2(\Sigma)$. We provide here a proof of the regularity result (R.R.), given below (3.1.2), that is

$$y_0 = 0, \quad u\in L_2(\Sigma) \Rightarrow y\in H^{\frac{1}{2},\frac{1}{4}}(Q) \tag{3.1.18}$$

for problem (3.1.1), but the proof actually *applies to a general strongly elliptic operator of order 2* in place of $(-\Delta)$; here we recall [Lions, Magenes, 1972, Vol. II, pp. 6–8]

$$H^{r,s}(Q) \equiv L_2(0,T;H^r(\Omega))\cap H^s(0,T;L_2(\Omega));$$
$$H^{r,s}(\Sigma) \equiv L_2(0,T;H^r(\Gamma))\cap H^s(0,T;L_2(\Gamma)) \tag{3.1.19}$$

for r and s two nonnegative real numbers.

Step 1 With reference to problem (3.1.1), say with $c=0$, we have:

$$u\in H^{\frac{3}{2},\frac{3}{4}}(\Sigma), \qquad u|_{t=0} = y_0|_\Gamma \Rightarrow y\in H^{2,1}(Q). \tag{3.1.20}$$

To prove (3.1.20), we begin by extending the given $u\in H^{\frac{3}{2},\frac{3}{4}}(\Sigma)$ from the boundary to the interior to obtain a function $\tilde{u}\in H^{2,1}(Q)$, so that $\tilde{u}|_{t=0} = y_0$; this is possible by trace theory [Lions, Magenes, 1972, Vol. II, p. 9]. Next, with the y solution of problem (3.1.1), say with $c=0$, we introduce a new variable $\tilde{y}$,

$$\tilde{y}\equiv y-\tilde{u}, \quad \text{so that} \begin{cases} \tilde{y}_t = \Delta\tilde{y}+\mu & \text{in } Q,\ \mu\equiv\Delta\tilde{u}-\tilde{u}_t\in L_2(\Sigma); \\ \tilde{y}|_{t=0} = 0 & \text{in } \Omega; \\ \tilde{y}|_\Sigma \equiv 0 & \text{in } \Sigma, \end{cases} \tag{3.1.21}$$

using the Compatibility Condition $\tilde{u}|_{t=0} = y_0$. Standard parabolic estimates, with $\mu \in L_2(\Sigma)$ (via the regularity of $\tilde{u} \in H^{2,1}(Q)$), yield

$$\tilde{y} \in H^{2,1}(Q), \qquad \text{hence } y = \tilde{y} + \tilde{u} \in H^{2,1}(Q), \tag{3.1.22}$$

and (3.1.20) is proved, as desired. [The conclusion in (3.1.22) about $\tilde{y}$ also follows by semigroup methods, using the regularity properties in (0.4) of Chapter 0.]

Step 2 With reference to problem (3.1.1), we now prove that

$$y_0 = 0, \quad u \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) = \left[H^{\frac{1}{2},\frac{1}{4}}_{,0}(\Sigma)\right]' \Rightarrow Lu = y \in L_2(Q); \tag{3.1.23}$$

see [Lions, Magenes, 1972, Vol. II, p. 41] for the duality. Equivalently, we shall show that

$$L^*: L_2(Q) \Rightarrow H^{\frac{1}{2},\frac{1}{4}}_{,0}(\Sigma), \tag{3.1.24}$$

where the subzero means "vanishing at $t = T$." Indeed, the regularity (3.1.24) can be shown either by PDE methods, using the PDE dual problem corresponding to L^*, or else by operator methods. Choosing the latter route, we have, by recalling (3.1.9) [or (3.1.15a)],

$$(L^*f)(t) = B^* \int_t^T e^{A^*(\tau-t)} f(\tau)\,d\tau = B^*(K^*f)(t) = -\frac{\partial}{\partial\nu}(K^*f)(t), \tag{3.1.25}$$

where, as in obtaining (3.1.21) for $\tilde{y}$, we have

$$(K^*f)(t) = \int_t^T e^{A^*(\tau-t)} f(\tau)\,d\tau : \text{ continuous } L_2(Q) \to H^{2,1}(\Sigma). \tag{3.1.26}$$

Applying trace theory [Lions, Magenes, 1972, Vol. II, p. 9] to (3.1.25) and (3.1.26), we obtain

$$f \in L_2(Q) \Rightarrow (L^*f) \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \quad \text{and} \quad (L^*f)(T) = 0. \tag{3.1.27}$$

Thus, (3.1.27) proves (3.1.24) as desired.

Step 3 We finally interpolate [Lions, Magenes, 1972, Vol. I, p. 27] between the regularity in (3.1.20) and the regularity in (3.1.23), where the compatibility condition is irrelevant at $\theta = 3/4$; that is, $u \in L_2(\Sigma)$:

$$\begin{aligned} u \in L_2(\Sigma) &\equiv \left[H^{\frac{3}{2},\frac{3}{4}}(\Sigma), H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)\right]_{\theta=\frac{3}{4}} \\ &\Rightarrow Lu = [H^{2,1}(Q), H^{0,0}(Q)]_{\theta=\frac{3}{4}} = H^{\frac{1}{2},\frac{1}{4}}(Q), \end{aligned} \tag{3.1.28}$$

as desired. The proof of the regularity result (3.1.18) is complete. □

Remark 3.1.3 The above regularity result (3.1.28) for problem (3.1.1) does not quite follow by operator/semigroups methods, which instead lose ϵ-regularity. To illustrate the point, if we apply the regularity result of (0.9) of Chapter 0 to

$$(Lu)(t) = \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau = -A \int_0^t e^{A(t-\tau)} Du(\tau)\, d\tau, \tag{3.1.29}$$

recalling (3.1.5), we only get

$$\big((-A)^{\frac{1}{4}-\frac{\epsilon}{2}} Lu\big)(t) = -A \int_0^t e^{A(t-\tau)} (-A)^{\frac{1}{4}-\frac{\epsilon}{2}} Du(\tau)\, d\tau \in L_2(0, T; L_2(\Omega)), \tag{3.1.30}$$

or

$$(Lu)(t) \in L_2\big(0, T; \mathcal{D}\big((-A)^{\frac{1}{4}-\frac{\epsilon}{2}}\big) = H^{1-\epsilon}(\Omega)\big), \tag{3.1.31}$$

an ϵ-*loss* over the space regularity in (3.1.28) or (3.1.18).

Remark 3.1.4 With reference to (3.1.7a), we have: $A_D^{\frac{1}{4}-\epsilon} D \in \mathcal{L}(L_2(\Gamma); L_2(\Omega))$, $\forall\, \epsilon > 0$. We show now that we cannot take $\epsilon = 0$, not even in the one-dimensional case. In other words: Claim: Let $\dim \Omega = 1$; then

$$\text{the statement } D : L_2(\Gamma) \to \mathcal{D}\big(A_D^{\frac{1}{4}}\big) \text{ is false.} \tag{3.1.32}$$

Proof. We deliberately avoid eigenfunction expansion and use instead the characterization [Fujiwara, 1967; Lions, Magenes, 1972, Eqn. (3.11.52), p. 66]

$$\mathcal{D}\big(A_D^{\frac{1}{4}}\big) \equiv H_{00}^{\frac{1}{2}}(\Omega), \qquad \text{where } f \in H_{00}^{\frac{1}{2}}(\Omega) \iff \begin{cases} \text{(i) } f \in H_0^{\frac{1}{2}}(\Omega) = H^{\frac{1}{2}}(\Omega) \\ \text{(ii) } \dfrac{f(x)}{\rho^{\frac{1}{2}}(x)} \in L_2(\Omega), \end{cases} \tag{3.1.33}$$

where, with no loss of generality, we may take $\rho(x) = \operatorname{dist}(x, \Gamma) =$ distance from $x \in \Omega$ to the boundary [Lions, Magenes, 1972, Eqn. (3.11.16), p. 57]. For $\Omega = (0, 1)$, we have

$$\rho(x) = \begin{cases} x, & 0 < x \le \dfrac{1}{2} \\ 1 - x, & \dfrac{1}{2} \le x < 1 \end{cases}; \quad g = \begin{bmatrix} 0 \\ g_2 \end{bmatrix}, \quad \text{so that } (Dg)(x) = xg_2, \quad 0 \le x \le 1, \tag{3.1.34}$$

where g_2 is a number $\neq 0$, say $g_2 = 1$. Then, $Dg \in \mathcal{D}(A_D^{\frac{1}{4}}) = H_{00}^{\frac{1}{2}}(\Omega)$ requires, by (3.1.33) and (3.1.34), that

$$\frac{Dg}{\rho^{\frac{1}{2}}} \in L_2(\Omega), \qquad \text{which is false, since } \int_{\frac{1}{2}}^1 \frac{x^2}{1-x}\, dx = \infty. \tag{3.1.35}$$

We conclude that $Dg \notin H_{00}^{\frac{1}{2}}(\Omega) = \mathcal{D}(A_D^{\frac{1}{4}})$, and (3.1.32) is established. $\square$

3.2 Heat Equation with Dirichlet Boundary Control: Regularity Theory of the Optimal Pair

With reference to problems (3.1.1) and (3.1.2) – indeed, with $(-\Delta)$ in (3.1.1a) replaced by the general operator in (3.1.14) – the aim of this section is to prove the following regularity result of the optimal pair $\{u^0(t; y_0), y^0(t; y_0)\}$. To this end, we recall the spaces [Lions, Magenes, 1972, Vol. II, p. 6]

$$H^{r,s}(Q_\infty) = L_2(0, \infty; H^r(\Omega)) \cap H^s(0, \infty; L_2(\Omega)) \tag{3.2.1}$$

for $Q_\infty = (0, \infty) \times \Omega$, and similarly for $\Sigma_\infty = (0, \infty) \times \Gamma$.

Theorem 3.2.1 *Let $\{u^0(t; y_0), y^0(t; y_0)\}$ be the optimal pair of the optimal problems (3.1.1) and (3.1.2) – with $(-\Delta)$ in (3.1.1a) replaced by the general second-order operator in (3.1.14) with uniformly elliptic principal part – initiating at the point y_0. Then, in the notation of (3.2.1), we have for all $\epsilon > 0$:*

(i)

$$y_0 \in L_2(\Omega) \Rightarrow \begin{cases} \hat{y}^0(\,\cdot\,; y_0) \in H^{1-2\epsilon, \frac{1}{2}-\epsilon}(Q_\infty), & (3.2.2) \\ \hat{u}^0(\,\cdot\,; y_0) \in H^{\frac{1}{2}-2\epsilon', \frac{1}{4}-\epsilon'}(\Sigma_\infty), \quad \forall\, \epsilon' > \epsilon > 0, & (3.2.3) \end{cases}$$

where $\hat{y}^0(t; y_0) = e^{-\omega t} y^0(t; y_0)$, and similarly, for $\hat{u}^0(t; y_0)$, as defined below Eqn. (2.2.6) of Chapter 2.

(ii) Moreover, recalling A_D from (3.1.8), we have

$$y_0 \in \mathcal{D}\big(A_D^{\frac{1}{4}-\frac{\rho}{2}}\big) = H^{\frac{1}{2}-\rho}(\Omega)$$
$$\Rightarrow \begin{cases} \hat{y}^0(\,\cdot\,; y_0) \in H^{\frac{3}{2}-2\rho, \frac{3}{4}-\rho}(Q_\infty), & (3.2.4) \\ \hat{u}^0(\,\cdot\,; y_0) \in H^{2-2\rho', 1-\rho'}(\Sigma_\infty), \quad \forall\, \rho' > \rho > 0. & (3.2.5) \end{cases}$$

Proof of Theorem 3.2.1. The proof is similar to the proof of Corollary 2.3.5.2 in Chapter 2 and is similarly based on the following lemmas [to be compared with Theorem 2.3.5.1 of Chapter 2].

Step 1 First, specialize the operators $\hat{L}$ and $\hat{L}^*$ in Eqns. (2.1.17) and (2.1.18) of Chapter 2 to the present case, where $B = -AD$ from (3.1.5), to obtain

$$(\hat{L}u)(t) = \int_0^t e^{-\hat{A}(t-\tau)} Bu(\tau)\, d\tau = -A \int_0^t e^{-\hat{A}(t-\tau)} Du(\tau)\, d\tau, \tag{3.2.6}$$

$$(\hat{L}^*v)(t) = \int_t^\infty B^* e^{-\hat{A}^*(\tau-t)} v(\tau)\, d\tau = -D^*A^* \int_t^\infty e^{-\hat{A}^*(\tau-t)} v(\tau)\, d\tau. \tag{3.2.7}$$

Lemma 3.2.2 *For the operator $\hat{L}$ defined by (3.2.6), we have for $\epsilon > 0$:*

(i)

$$\hat{L} : \textit{continuous } L_2(0, \infty; L_2(\Gamma)) = H^{0,0}(\Sigma_\infty) \Rightarrow H^{\frac{1}{2}-2\epsilon, \frac{1}{4}-\epsilon}(Q_\infty), \tag{3.2.8}$$

(ii)

$$\left.\begin{array}{l} u \in H^{2,1}(\Sigma_\infty) \\ u(0) = 0 \end{array}\right\} \Rightarrow \hat{L}u \in H^{\frac{5}{2}-2\epsilon,\frac{5}{4}-\epsilon}(Q_\infty), \tag{3.2.9}$$

where $u \in H^{2,1}(\Sigma_\infty)$ implies [Lions, Magenes, 1972, Vol. I, Theorem 3.1, p. 19] $u \in C([0,\infty]; H^1(\Gamma))$ = space of continuous bounded functions with values in $H^1(\Gamma)$, so that, in particular, $u(0) \in H^1(\Gamma)$ is well defined.

(iii) For $0 \le \theta < \frac{1}{2}$,

$$\hat{L} : \textit{continuous } H^{2\theta,\theta}(\Sigma_\infty) \Rightarrow H^{2\theta+\frac{1}{2}-2\epsilon,\theta+\frac{1}{4}-\epsilon}(Q_\infty). \tag{3.2.10}$$

Lemma 3.2.3 *For the operator $\hat{L}^*$ defined by (3.2.7), we have for $\epsilon > 0$:*

(i)

$$\hat{L}^* : \textit{continuous } L_2(0,\infty; L_2(\Omega)) = H^{0,0}(Q_\infty) \Rightarrow H^{\frac{1}{2}-2\epsilon,\frac{1}{4}-\epsilon}(\Sigma_\infty). \tag{3.2.11}$$

(ii)

$$\hat{L}^* : \textit{continuous } H^{2,1}(Q_\infty) \Rightarrow H^{\frac{5}{2}-2\epsilon,\frac{5}{4}-\epsilon}(\Sigma_\infty). \tag{3.2.12}$$

(iii) For $0 \le \theta \le 1$,

$$\hat{L}^* : \textit{continuous } H^{2\theta,\theta}(\Sigma_\infty) \Rightarrow H^{2\theta+\frac{1}{2}-2\epsilon,\theta+\frac{1}{4}-\epsilon}(Q_\infty). \quad \Box \tag{3.2.13}$$

Lemma 3.2.4 *For the operator $\hat{A}$ [which can be taken to be equal to A_D in (3.1.8) in the canonical case; see Eqn. (2.1.3) in Chapter 2], we have for $\rho > 0$:*

(i)

$$e^{-\hat{A}t} : \textit{continuous } L_2(\Omega) \to H^{1-2\rho,\frac{1}{2}-\rho}(Q_\infty); \tag{3.2.14}$$

(ii)

$$e^{-\hat{A}t} : \textit{continuous } H^{\frac{1}{2}-\rho}(\Omega) \Rightarrow H^{\frac{3}{2}-2\rho,\frac{3}{4}-\rho}(Q_\infty), \tag{3.2.15}$$

where we can take $\rho = 0$ in case A is a self-adjoint, or a normal, operator. $\Box$

Assuming for the time being the validity of the three lemmas above, we now proceed to complete the proof of Theorem 3.2.1.

(i) We know at the outset that $\hat{y}^0 \in H^{0,0}(Q_\infty)$ and hence that $[I + 2\omega P]\hat{y}^0 \in H^{0,0}(Q_\infty)$, by Theorem 2.2.1 of Chapter 2. Then, applying Lemma 3.2.3(i) to Eqn. (2.2.8a) of Chapter 2, we deduce that $\hat{u}^0 \in H^{\frac{1}{2}-2\epsilon,\frac{1}{4}-\epsilon}(\Sigma_\infty)$. From here, using Lemma 3.2.2(iii) with $\theta = 1/4 - \epsilon$, as well as Lemma 3.2.4(i), we obtain from the optimal dynamics [Chapter 2, Eqn. (2.3.4.3)] that $\hat{y}^0 \in H^{1-4\epsilon,\frac{1}{2}-2\epsilon}(Q_\infty)$. A further application of Lemma 3.2.3(iii), this time with $\theta = 1/2 - 2\epsilon$, yields $\hat{u}^0 \in H^{\frac{3}{2}-6\epsilon,\frac{3}{4}-3\epsilon}(\Sigma_\infty)$, again via (2.2.8a) of Chapter 2. With y_0 only in $L_2(\Omega)$, this bootstrap argument cannot continue, and part (i) of Theorem 3.2.1 is proved.

(ii) Similarly, we know at the outset that $\hat{u}^0 \in H^{0,0}(\Sigma_\infty)$, by Theorem 2.2.1 of Chapter 2. Applying Lemma 3.2.2(i) and Lemma 3.2.4(ii) to the optimal dynamics [Chapter 2, Eqn. (2.3.4.3))], we deduce that $\hat{y}^0 \in H^{\frac{1}{2}-2\epsilon,\frac{1}{4}-\epsilon}(Q_\infty)$. Thus, $[I + 2\omega P]\hat{y}^0 \in H^{\frac{1}{2}-2\epsilon,\frac{1}{4}-\epsilon}(Q_\infty)$ by Theorem 2.2.1 of Chapter 2. Hence, applying Lemma 3.2.3(iii), with $\theta = 1/4 - \epsilon$, we obtain via (2.2.8a) of Chapter 2 that $\hat{u}^0 \in H^{1-4\epsilon,\frac{1}{2}-2\epsilon}(\Sigma_\infty)$. From here, a further application of Lemma 3.2.2(iii), this time with $\theta = 1/2 - 2\epsilon$, and Lemma 3.2.4(ii) gives $\hat{y}^0 \in H^{\frac{3}{2}-6\epsilon,\frac{3}{4}-3\epsilon}(Q_\infty)$ via the optimal dynamics [Chapter 2, Eqn. (2.3.4.3)], where $2\rho = 6\epsilon$. Finally, again by (2.2.8a) of Chapter 2, we find that $\hat{u}^0 \in H^{2-8\epsilon,1-4\epsilon}(\Sigma_\infty)$, by applying Lemma 3.2.3(iii) with $\theta = 3/4 - 3\epsilon$. Part (ii) is also proved.

Step 2 It remains to prove the lemmas in order to establish Theorem 3.2.1.

Proof of Lemma 3.2.2 It suffices to show the validity of this lemma with the operator $\hat{L}$ in (3.2.6) replaced by the operator $\hat{L}_1$ defined by

$$\begin{aligned}(\hat{L}_1 u)(t) &\equiv \int_0^t \hat{A}e^{-\hat{A}(t-\tau)} Du(\tau)\,d\tau \\ &= \int_0^t \hat{A}^{\frac{3}{4}+\frac{\epsilon}{2}} e^{-\hat{A}(t-\tau)} \hat{A}^{\frac{1}{4}-\frac{\epsilon}{2}} Du(\tau)\,d\tau, \end{aligned} \tag{3.2.16}$$

with $\hat{A}$ defined by Eqn. (2.1.3) of Chapter 2.

(i) With $u \in H^{0,0}(\Sigma_\infty)$, that is, $\hat{A}^{\frac{1}{4}-\epsilon/2} Du \in L_2(0,\infty; L_2(\Omega))$ by (3.1.7a), we have, recalling Eqn. (2.1.4b) of Chapter 2,

$$\begin{aligned}\left\|\left(\hat{A}^{\frac{1}{4}-\epsilon}\hat{L}_1 u\right)(t)\right\|_{L_2(\Omega)} &= \left\| \int_0^t \hat{A}^{1-\frac{\epsilon}{2}} e^{-\hat{A}(t-\tau)} \hat{A}^{\frac{1}{4}-\frac{\epsilon}{2}} Du(\tau)\,d\tau \right\|_{L_2(\Omega)} \\ &\le \int_0^t \frac{\hat{M}e^{-\hat{\omega}(t-\tau)}}{(t-\tau)^{1-\frac{\epsilon}{2}}} \left\|\hat{A}^{\frac{1}{4}-\frac{\epsilon}{2}} Du(\tau)\right\|_{L_2(\Omega)} d\tau \in L_2(0,\infty), \end{aligned} \tag{3.2.17}$$

by using the convolution theorem (Young's inequality) between $L_1(0,\infty)$ and $L_2(0,\infty)$ [Sadosky, 1979, p. 26], or even better Eqn. (3.0.4) of Chapter 0. Thus, (3.2.17) says

$$\hat{A}^{\frac{1}{4}-\epsilon}\hat{L}_1 u \in L_2(0,\infty; L_2(\Omega)), \tag{3.2.18a}$$

or equivalently,

$$\hat{L}_1 u \in L_2\left(0,\infty; \mathcal{D}\left(\hat{A}^{\frac{1}{4}-\epsilon}\right) = H^{\frac{1}{2}-2\epsilon}(\Omega)\right), \tag{3.2.18b}$$

by (3.1.7a). Next, differentiating (3.2.16) in time yields, by (3.2.18a) and (3.1.7a) and (3.1.17),

$$\frac{d(\hat{L}_1 u)(t)}{dt} = \hat{A}^{\frac{3}{4}+\epsilon}\hat{A}^{\frac{1}{4}-\epsilon} Du(t) - \left(\hat{A}^{\frac{3}{4}+\epsilon}\hat{A}^{\frac{1}{4}-\epsilon}\hat{L}_1 u\right)(t) \in L_2\left(0,\infty; \left[\mathcal{D}\left(\hat{A}^{\frac{3}{4}+\epsilon}\right)\right]'\right). \tag{3.2.19}$$

Then (3.2.19) implies

$$D_t\big(\hat{A}^{\frac{1}{4}-\epsilon}\hat{L}_1 u\big) \in L_2(0,\infty;[\mathcal{D}(\hat{A})]'). \tag{3.2.20}$$

Applying the intermediate derivative theorem [Lions, Magenes, 1972, Vol. I, p. 15] between (3.2.18a) and (3.2.20), we obtain

$$D_t^{\theta}\big(\hat{A}^{\frac{1}{4}-\epsilon}\hat{L}_1 u\big) \in L_2\big(0,\infty;\big[\mathcal{D}(\hat{A}^{\theta})\big]'\big), \tag{3.2.21}$$

since recalling [Lions, Magenes, 1972, Vol. I, p. 26] and [Triebel, 1978, p. 103] we have

$$\big[L_2(\Omega);[\mathcal{D}(\hat{A})]'\big]_{\theta} = \big[[\mathcal{D}(\hat{A}),L_2(\Omega)]_{1-\theta}\big]' = [\mathcal{D}(\hat{A}^{\theta})]', \quad 0 \le \theta \le 1. \tag{3.2.22}$$

Setting $\theta = 1/4 - \epsilon$ in (3.2.21) and applying $\hat{A}^{-\frac{1}{4}+\epsilon}$, we obtain

$$D_t^{\frac{1}{4}-\epsilon}\hat{L}_1 u \in L_2(0,\infty;L_2(\Omega)). \tag{3.2.23}$$

Then, (3.2.18b) and (3.2.23) prove, via (3.2.1), the required regularity (3.2.8) of part (i) for the operator $\hat{L}_1$, and hence for the operator $\hat{L}$.

(ii) With $u \in H^{2,1}(\Sigma_\infty)$, $u(0) = 0$, we integrate (3.2.16) by parts, obtaining

$$(\hat{L}_1 u)(t) = Du(t) - \cancel{e^{-\hat{A}t}Du(0)} - \int_0^t e^{-\hat{A}(t-\tau)}D\dot{u}(\tau)\,d\tau, \tag{3.2.24}$$

where then $u \in L_2(0,\infty;H^2(\Gamma))$ and (3.1.7b) with $s = 2$ imply

$$Du(t) \in L_2(0,\infty;H^{\frac{5}{2}}(\Omega)), \tag{3.2.25}$$

while $\dot{u} \in L_2(0,\infty;L_2(\Gamma))$ and (3.2.18a) and (3.2.16) yield

$$\int_0^t e^{-\hat{A}(t-\tau)}D\dot{u}(\tau)\,d\tau \in L_2\big(0,\infty;\mathcal{D}\big(\hat{A}^{1+\frac{1}{4}-\epsilon}\big)\big) \subset L_2\big(0,\infty;H^{\frac{5}{2}-2\epsilon}(\Omega)\big). \tag{3.2.26}$$

Thus, (3.2.25) and (3.2.26), inserted in (3.2.24), give

$$(\hat{L}_1 u)(t) \in L_2\big(0,\infty;H^{\frac{5}{2}-2\epsilon}(\Omega)\big). \tag{3.2.27}$$

Next, differentiating (3.2.24) in time, we obtain

$$\frac{d(\hat{L}_1 u)(t)}{dt} = \cancel{D\dot{u}(t)} - \cancel{D\dot{u}(t)} + (\hat{L}_1\dot{u})(t) \in H^{\frac{1}{2}-2\epsilon,\frac{1}{4}-\epsilon}(Q_\infty) \tag{3.2.28}$$

by application of part (i) of Lemma 3.2.2, since $\dot{u} \in L_2(0,\infty;L_2(\Gamma))$. A fortiori, we obtain from (3.2.28) that

$$(\hat{L}_1 u)(t) \in H^{1+\frac{1}{4}-\epsilon}(0,\infty;L_2(\Omega)). \tag{3.2.29}$$

Then, (3.2.27) and (3.2.29) prove, via (3.2.1), the required regularity (3.2.9) of part (ii) for the operator $\hat{L}_1$, and hence for the operator $\hat{L}$.

(iii) Part (iii), Eqn. (3.2.10), follows by interpolation between parts (i) and (ii), since for $\theta < 1/2$, the compatibility relation $u(0) = 0$ does not interfere. Lemma 3.2.2 is proved. $\square$

Proof of Lemma 3.2.3. It suffices to show the validity of this lemma with the operator $\hat{L}^*$ in (3.2.7) replaced by the operator $\hat{L}_1^*$ defined by

$$(\hat{L}_1^* v)(t) = D^* \hat{A}^* \int_t^\infty e^{-\hat{A}^*(\tau - t)} v(\tau)\, d\tau \tag{3.2.30}$$

[where $\hat{A}$ is self-adjoint in our canonical case (3.1.1)].

(i) With $v \in L_2(Q_\infty)$ our first task is to show that

$$\int_t^\infty e^{-\hat{A}^*(\tau - t)} v(\tau)\, dt \in H^{2-2\epsilon, 1-\epsilon}(Q_\infty). \tag{3.2.31}$$

In fact, by the analyticity property in Eqn. (2.1.4b) of Chapter 2

$$\int_t^\infty e^{-\hat{A}^*(\tau - t)} v(\tau)\, d\tau \in L_2\big(0, \infty; \mathcal{D}\big(\hat{A}^{*1-\epsilon}\big)\big) \subset L_2\left(0, \infty; H^{2-2\epsilon}(\Omega)\right), \tag{3.2.32a}$$

or equivalently,

$$\hat{A}^{*1-\epsilon} \int_t^\infty e^{-\hat{A}^*(\tau - t)} v(\tau)\, d\tau \in L_2(0, \infty; L_2(\Omega)), \tag{3.2.32b}$$

which is again true by Young's inequality [Sadosky, 1979, p. 26], that is, convolution between L_1 and L_2.

Next, from (3.2.32b), we obtain

$$\begin{aligned} &\frac{d}{dt} \int_t^\infty e^{-\hat{A}^*(\tau - t)} v(\tau)\, d\tau \\ &\quad = -v(t) + \hat{A}^{*\epsilon} \int_t^\infty \hat{A}^{*1-\epsilon} e^{-\hat{A}^*(\tau - t)} v(\tau) d\tau \in L_2(0, \infty; [\mathcal{D}(\hat{A}^\epsilon)]'), \end{aligned} \tag{3.2.33}$$

from which, applying $(\hat{A}^*)$, we obtain that

$$D_t \left(\hat{A}^{*1-\epsilon} \int_t^\infty e^{-\hat{A}^*(\tau - t)} v(\tau) d\tau \in L_2(0, \infty; [\mathcal{D}(\hat{A})]' \right). \tag{3.2.34}$$

The intermediate derivative theorem between (3.2.32b) and (3.2.34) gives, as in the proof of (3.2.21) above, that

$$D_t^\theta \int_t^\infty e^{-\hat{A}^*(\tau - t)} v(\tau)\, d\tau \in L_2\big(0, \infty; \mathcal{D}\big(\hat{A}^{*-\theta+(1-\epsilon)}\big)\big). \tag{3.2.35a}$$

Setting $\theta = 1 - \epsilon$ in (3.2.35a) yields, as desired,

$$D_t^{1-\epsilon} \left(\int_t^\infty e^{-\hat{A}(\tau - t)} v(\tau)\, d\tau \right) \in L_2(0, \infty; L_2(\Omega)). \tag{3.2.35b}$$

Then, (3.2.35b) and (3.2.32a) prove the preliminary conclusion (3.2.31).

Next, by trace theory on $H^{r,s}$-spaces applied to (3.2.31), we obtain

$$\frac{\partial}{\partial\nu}\int_t^\infty e^{-\hat{A}^*(\tau-t)}v(\tau)\,d\tau \in H^{\frac{1}{2}-2\epsilon,\frac{1}{4}-\epsilon}(\Sigma_\infty) \tag{3.2.36}$$

[Lions, Magenes, 1972, Vol. II, Theorem 2.1, p. 9]. Invoking (3.1.9) and (3.1.15), we obtain from (3.2.36)

$$(\hat{L}_1^*v)(t) = D^*\hat{A}^*\int_t^\infty e^{-\hat{A}^*(\tau-t)}v(\tau)\,d\tau \in H^{\frac{1}{2}-2\epsilon,\frac{1}{4}-\epsilon}(\Sigma_\infty), \tag{3.2.37}$$

and (3.2.37) proves the required regularity (3.2.11) of part (i) for $\hat{L}_1^*$, and hence for $\hat{L}^*$. [For the general case we recall Remark 3.1.2, in particular (3.1.15) and (3.1.17).]

(ii) Let $v \in H^{2,1}(Q_\infty)$. Integrating by parts on (3.2.37) yields

$$(\hat{L}_1^*v)(t) = D^*v(t) + D^*\int_t^\infty e^{-\hat{A}^*(\tau-t)}\dot{v}(\tau)\,d\tau, \tag{3.2.38}$$

since $\lim_{t\to\infty} v(t) = 0$. But $v \in L_2(0,\infty; H^2(\Omega))$ implies $A^{-1}v \in L_2(0,\infty; H^4(\Omega))$. [$v = AA^{-1}v = -A(\xi,\partial)A^{-1}v$; setting $w = A^{-1}v$ yields $-A(\xi,\partial)w = v \in H^2(\Omega)$ and $w = 0$ on Γ; elliptic theory then gives $w \in H^4(\Omega)$.] Thus, by trace theory, recalling (3.1.15) on D^*A^*, we have

$$D^*v(t) = D^*A^*A^{*-1}v(t) \in L_2(0,\infty; H^{4-\frac{3}{2}}(\Gamma)). \tag{3.2.39}$$

Moreover, with $\dot{v} \in L_2(Q_\infty)$, we obtain from (3.2.32a) and (3.1.17)

$$\hat{A}^{*-1}\int_t^\infty e^{-\hat{A}^*(\tau-t)}\dot{v}(\tau)\,d\tau \in L_2\big(0,\infty; \mathcal{D}\big(\hat{A}^{2-\epsilon}\big)\big) \subset L_2\left(0,\infty; H^{4-2\epsilon}(\Omega)\right), \tag{3.2.40}$$

and by trace theory, recalling (3.1.15), we get

$$\begin{aligned} D^*&\int_t^\infty e^{-\hat{A}^*(\tau-t)}\dot{v}(\tau)\,d\tau \\ &= D^*\hat{A}^*\hat{A}^{*-1}\int_t^\infty e^{-\hat{A}^*(\tau-t)}\dot{v}(\tau)\,d\tau \in L_2\big(0,\infty; H^{4-\frac{3}{2}-2\epsilon}(\Gamma)\big). \end{aligned} \tag{3.2.41}$$

Then (3.2.39) and (3.2.41), used in (3.2.38), yield the desired space regularity conclusion

$$\hat{L}_1^*v \in L_2\big(0,\infty; H^{\frac{5}{2}-2\epsilon}(\Gamma)\big), \tag{3.2.42}$$

as required by (3.2.12). As to the time regularity, we have by differentiating (3.2.38) in t

$$\begin{aligned} \frac{d(\hat{L}_1^*v)(t)}{dt} &= \cancel{D^*\dot{v}(t)} - \cancel{D^*\dot{v}(t)} + D^*\hat{A}^*\int_t^\infty e^{-\hat{A}^*(\tau-t)}\dot{v}(\tau)\,d\tau \\ \text{(by (3.2.30))} \quad &= (\hat{L}_1^*\dot{v})(t) \in H^{\frac{1}{2}-2\epsilon,\frac{1}{4}-\epsilon}(\Sigma_\infty), \end{aligned} \tag{3.2.43}$$

by (3.2.37) of part (i), since $\dot{v} \in L_2(Q_\infty)$. A fortiori, we obtain from (3.2.43) that

$$(L_1^* v)(t) \in H^{1+\frac{1}{4}-\epsilon}(0, \infty; L_2(\Gamma)). \tag{3.2.44}$$

Then, (3.2.42) and (3.2.44) prove the required regularity (3.2.12) of part (ii) for the operator $\hat{L}_1^*$, and hence for the operator $\hat{L}^*$.

(iii) Part (iii), Eqn. (3.2.13), follows by interpolation between parts (i) and (ii). Lemma 3.2.3 is proved. □

Proof of Lemma 3.2.4. (i) If $x \in L_2(\Omega)$, then the analyticity property [Chapter 2, Eqn. (2.1.4b)] yields that for $\rho > 0$ sufficiently small:

$$\hat{A}^{\frac{1}{2}-\rho} e^{-\hat{A}t} x \in L_2(0, \infty; L_2(\Omega)), \tag{3.2.45}$$

or

$$e^{-\hat{A}t} x \in L_2(0, \infty; \mathcal{D}(\hat{A}^{\frac{1}{2}-\rho}) = H_0^{1-2\rho}(\Omega)), \tag{3.2.46}$$

whereby, since $\mathcal{D}(\hat{A}^{\frac{1}{2}+\rho}) = H_0^{1+2\rho}(\Omega)$ [see Fujiwara [1967], Grisvard [1967], Lasiecka [1980(b)], Appendix 3B; and Appendix 3A], then

$$\begin{aligned} \frac{d}{dt}(e^{-\hat{A}t} x) &= -\hat{A}^{\frac{1}{2}+\rho} \hat{A}^{\frac{1}{2}-\rho} e^{-\hat{A}t} x \in L_2\big(0, \infty; \big[\mathcal{D}\big((\hat{A}^*)^{\frac{1}{2}+\rho}\big)\big]'\big) \\ &= L_2(0, \infty; H^{-1-2\rho}(\Omega)). \end{aligned} \tag{3.2.47}$$

The intermediate derivative theorem between (3.2.46) and (3.2.47) yields (recalling [Lions, Magenes, 1972, Vol. I, Theorem 12.3, p. 72])

$$\begin{aligned} D_t^\theta(e^{-\hat{A}t} x) &\in L_2\big(0, \infty; \big[H_0^{1-2\rho}(\Omega), H^{-1-2\rho}(\Omega)\big]_\theta\big) \\ &= L_2\big(0, \infty; H_0^{1-2\rho-2\theta}(\Omega)\big), \end{aligned} \tag{3.2.48}$$

for $(1-2\rho)(1-\theta)-\theta(1+2\rho) = 1-2\rho-2\theta \geq 0$, so that specializing to $2\theta = 1-2\rho$, we obtain from (3.2.48)

$$D_t^{\frac{1}{2}-\rho}(e^{-\hat{A}t} x) \in L_2(0, \infty; L_2(\Omega)). \tag{3.2.49}$$

Then, (3.2.46) and (3.2.49) prove (3.2.14), as desired.

(ii) The proof of part (ii) is exactly the same, using now $H^{\frac{1}{2}-\rho}(\Omega) = \mathcal{D}(\hat{A}^{\frac{1}{4}-\rho/2})$, by (3.1.7a). Thus, for $x \in \mathcal{D}(\hat{A}^{\frac{1}{4}-\rho/2})$ (see Appendix 3A)

$$e^{-\hat{A}t} x \in L_2\big(0, \infty; \mathcal{D}\big(\hat{A}^{\frac{3}{4}-\rho}\big) = H_0^{\frac{3}{2}-2\rho}(\Omega)\big), \tag{3.2.50}$$

$$\frac{d}{dt}(e^{-\hat{A}t} x) = -\hat{A} e^{-\hat{A}t} x \in L_2(\big(0, \infty; \big[\mathcal{D}\big((\hat{A}^*)^{\frac{1}{4}+\rho}\big)\big]' = H^{-\frac{1}{2}-2\rho}(\Omega)\big), \tag{3.2.51}$$

so that the intermediate derivative theorem between (3.2.50) and (3.2.51) yields

$$D_t^\theta(e^{-\hat{A}t} x) \in L_2\left(0, \infty; H_0^{\frac{3}{2}-2\rho-2\theta}(\Omega)\right) \tag{3.2.52}$$

for $(3/2 - 2\rho)(1-\theta) - \theta(1/2 + 2\rho) = 3/2 - 2\rho - 2\theta \geq 0$, whereby specializing to $2\theta = 3/2 - 2\rho$, we obtain from (3.2.52)

$$D_t^{\frac{3}{4}-\rho}(e^{-\hat{A}t}x) \in L_2(0, \infty; L_2(\Omega)). \tag{3.2.53}$$

Then, (3.2.50) and (3.2.53) prove (3.2.15), as desired. The case where $\hat{A}$ is normal can be obtained by direct eigenvector expansion, which shows that one can take $\rho = 0$. Details are omitted. Lemma 3.2.4 is proved. □

3.3 Heat Equation with Neumann Boundary Control

3.3.1 Interior State Penalization

We return to the mixed problem (3.1.1), where, however, now the control function u acts in the Neumann BC:

$$\begin{cases} y_t = \Delta y + c^2 y & \text{in } (0, T] \times \Omega = Q, & (3.3.1.1\text{a}) \\ y(0, \cdot) = y_0 & \text{in } \Omega, & (3.3.1.1\text{b}) \\ \left.\dfrac{\partial y}{\partial \nu}\right|_{\Sigma} = u & \text{in } (0, T] \times \Gamma = \Sigma; & (3.3.1.1\text{c}) \end{cases}$$

and cost functional is

$$J(u, y) = \int_0^T \left\{ \|y(t)\|^2_{L_2(\Omega)} + \||\nabla y(t)|\|^2_{L_2(\Omega)} + \|u(t)\|^2_{L_2(\Gamma)} \right\} dt \\ + \alpha \||\nabla y(T)|\|^2_{L_2(\Omega)}, \tag{3.3.1.2}$$

where $\alpha = 0$ if $T = \infty$, and $\alpha = 1$ if $T < \infty$, instead of (3.1.2), where we now take $y_0 \in H^1(\Omega)$. The following regularity result of problem (3.3.1.1) is well known [Lions, Magenes, 1972, Vol. II, p. 81] (for which a proof may be given as at the end of Section 3.1, in the Dirichlet case, mutatis mutandis):

$$\begin{cases} \text{Let, say, } y_0 = 0 \text{ and } u \in L_2(\Sigma), \text{ then} \\ \Rightarrow \\ y \in H^{\frac{3}{2},\frac{3}{4}}(Q) = L_2\left(0, T; H^{\frac{3}{2}}(\Omega)\right) \cap H^{\frac{3}{4}}(0, T; L_2(\Omega)); \end{cases}$$

hence [Lions, Magenes, 1972, Vol. I, Theorem 3.3.1, with $j = 0$, $m = 1$, $X = H^{\frac{3}{2}}(\Omega)$, $Y = H^{-\frac{1}{2}}(\Omega)$]

$$y \in C\left([0, T]; H^{\frac{1}{2}}(\Omega)\right), \tag{3.3.1.3}$$

for any $T < \infty$, but y does *not* belong to $C([0, T]; H^1(\Omega))$. (Thus, the abstract model of the forthcoming Volume 2 is not applicable to problem (3.3.1.1), (3.3.1.2), where $Y = H^1(\Omega)$.)

Abstract Setting To put problem (3.3.1.1), (3.3.1.2) into the abstract setting of the preceding Chapters 1 and 2, we introduce the self-adjoint operator

$$Ah = \Delta h + c^2 h, \qquad \mathcal{D}(A) = \left\{ h \in H^2(\Omega) : \left.\frac{\partial h}{\partial \nu}\right|_\Gamma = 0 \right\}, \tag{3.3.1.4a}$$

which we shall consider as lifted to

$$A : \mathcal{D}(\hat{A}^{\frac{3}{2}}) \to \mathcal{D}(\hat{A}^{\frac{1}{2}}) = H^1(\Omega), \tag{3.3.1.4b}$$

in the notation $\hat{A} = -A + \omega I$, as in Chapter 1 [Eqn. (1.1.3)] on $L_2(\Omega)$. Moreover, we select the spaces and operators

$$Z = Y = H^1(\Omega) = \mathcal{D}(\hat{A}^{\frac{1}{2}}), \quad Z_f = [L_2(\Omega)]^n, \quad U = L_2(\Gamma), \tag{3.3.1.5}$$

$$Bu = -ANu, \quad B : L_2(\Gamma) \to [\mathcal{D}(A)]', \quad R = I, \tag{3.3.1.6}$$

where A in (3.3.1.6) is an isomorphic extension, say, $L_2(\Omega) \to [\mathcal{D}(A)]'$ of the operator A in (3.3.1.4). Here, without loss of generality, we assume that $-c^2$ is not an eigenvalue of Δ with homogeneous Neumann BC, so that A is boundedly invertible on $L_2(\Omega)$ and the Neumann map N is well defined by

$$h = Ng \quad \text{iff} \quad (\Delta + c^2)h = 0 \quad \text{in } \Omega, \qquad \left.\frac{\partial h}{\partial \nu}\right|_\Gamma = g. \tag{3.3.1.7}$$

We have from elliptic theory [Lions, Magenes, 1972, Vol. I, p. 187] and known identifications [Grisvard, 1967; Lasiecka, 1980(a), Appendix B; Fujiwara, 1967]

$$N : \text{ continuous } L_2(\Gamma) \to H^{\frac{3}{2}}(\Omega) \subset H^{\frac{3}{2}-2\epsilon}(\Omega) = \mathcal{D}(\hat{A}^{\frac{3}{4}-\epsilon}), \quad \forall \epsilon > 0, \tag{3.3.1.8a}$$

or, more generally,

$$N : \text{continuous } H^s(\Gamma) \to H^{s+\frac{3}{2}}(\Omega), \quad s \in \mathbb{R}. \tag{3.3.1.8b}$$

We next define the $L_2(\Omega)$-adjoint $N^* \in \mathcal{L}(L_2(\Omega); L_2(\Gamma))$ of N by

$$(Nu, y)_{L_2(\Omega)} = (u, N^* y)_{L_2(\Gamma)}, \quad u \in L_2(\Gamma), \quad y \in L_2(\Omega). \tag{3.3.1.9}$$

Instead, regarding the operator B in (3.3.1.6), we shall have to distinguish the $L_2(\Omega)$-adjoint $B^{\#} \in \mathcal{L}(\mathcal{D}(A); L_2(\Gamma))$ defined by

$$(Bu, \phi)_{L_2(\Omega)} = (u, B^{\#}\phi)_{L_2(\Omega)}, \quad u \in L_2(\Gamma), \phi \in \mathcal{D}(A), \tag{3.3.1.10}$$

as well as the Y-adjoint, consistently denoted by B^*, $B^* \in \mathcal{L}(\mathcal{D}(A^{\frac{3}{2}}); L_2(\Gamma))$, defined with respect to the $\mathcal{D}(\hat{A}^{\frac{1}{2}})$-topology by

$$(Bu, \phi)_Y = (Bu, \phi)_{\mathcal{D}(\hat{A}^{\frac{1}{2}})} = (u, B^*\phi)_{L_2(\Gamma)}, \quad u \in L_2(\Gamma), \phi \in \mathcal{D}(\hat{A}^{\frac{3}{2}}). \tag{3.3.1.11}$$

[We keep the $*$ for the adjoint of B, which, with the selection of Y as in (3.3.1.5), is the one that enters into the Riccati theory, in particular with the property $B^*P \in \mathcal{L}(Y; U)$.] We have

Lemma 3.3.1.1 *In the notation introduced above in (3.3.1.9)–(3.3.1.11), we have*

(i)

$$B^{\#}\phi = -N^*A\phi = \phi|_\Gamma, \quad \phi \in \mathcal{D}(A), \tag{3.3.1.12}$$

and (3.3.1.12) can be extended to all $\phi \in H^{\frac{3}{2}+\epsilon}(\Omega)$, $\epsilon > 0$ *with* $\frac{\partial\phi}{\partial\nu}|_\Gamma = 0$.

(ii)

$$B^*\phi = B^{\#}\hat{A}\phi = -N^*A\hat{A}\phi = \hat{A}\phi|_\Gamma, \quad \phi \in \mathcal{D}(\hat{A}^{\frac{3}{2}}). \tag{3.3.1.13}$$

[Generalization to the general non–self-adjoint case is given in Remark 3.3.1.5 below.]

Proof. (i) As in the proof of Lemma 3.1.1, we obtain from (3.3.1.6) and (3.3.1.4), via Green's second theorem with $\phi \in \mathcal{D}(A)$, and (3.3.1.9) and (3.3.1.10):

$$\begin{aligned}
(Bu, \phi)_{L_2(\Omega)} &= -(Nu, A\phi)_{L_2(\Omega)} = -(Nu, (\Delta + c^2)\phi)_{L_2(\Omega)} \\
&= -(\cancel{(\Delta + c^2)Nu}, \phi)_{L_2(\Omega)} - \int_\Gamma \cancel{Nu\frac{\partial\phi}{\partial\nu}}\, d\Gamma + \int_\Gamma \frac{\partial Nu}{\partial\nu}\,\phi\, d\Gamma \\
\text{(by (3.3.1.7))} \quad &= (u, \phi)_{L_2(\Gamma)} = (u, B^{\#}\phi)_{L_2(\Gamma)},
\end{aligned}$$

and (3.3.1.12) follows. The noted cancellations are due to (3.3.1.7) and $\phi \in \mathcal{D}(A)$ in (3.3.1.4a), while $\partial Nu/\partial\nu = u$, again by (3.3.1.7).

(ii) Similarly, using this time (3.3.1.11),

$$\begin{aligned}
(Bu, \phi)_{\mathcal{D}(\hat{A}^{\frac{1}{2}})} &= (Bu, \hat{A}\phi)_{L_2(\Omega)} = (u, B^{\#}\hat{A}\phi)_{L_2(\Gamma)} \\
&= (u, B^*\phi)_{L_2(\Gamma)},
\end{aligned}$$

and (3.3.1.13) follows also via (3.3.1.12). □

We remark that again, fractional powers of the translation $\hat{A}$ of $-A$ are well defined.

Assumption (1.1.2) or (2.1.7): $\hat{A}^{-\gamma}B \in \mathcal{L}(U; Y)$ This assumption holds true in the present case with $\gamma = 3/4 + \epsilon$, $\forall \epsilon > 0$. In fact, with $\gamma = 3/4 + \epsilon$, we need to show that, via (3.3.1.5),

$$\hat{A}^{-\gamma}B \in \mathcal{L}(U; Y) = \mathcal{L}\big(L_2(\Gamma); \mathcal{D}(\hat{A}^{\frac{1}{2}})\big), \tag{3.3.1.14}$$

or equivalently that (see (3.3.1.6))

$$\hat{A}^{\frac{1}{2}}\hat{A}^{-\gamma}\hat{A}N = \hat{A}^{\frac{1}{2}-\gamma+\frac{1}{4}+\epsilon}\hat{A}^{\frac{3}{4}-\epsilon}N \in \mathcal{L}(L_2(\Gamma); L_2(\Omega)), \tag{3.3.1.15}$$

which is precisely true in view of (3.3.1.8a), since $1/2 - \gamma + 1/4 + \epsilon = 0$.

Assumption of Analyticity Since A defined in (3.3.1.4a) generates an s.c. analytic semigroup on $L_2(\Omega)$, then its lifting as in (3.3.1.4b) generates an s.c. analytic semigroup on $\mathcal{D}(\hat{A}^{\frac{1}{2}}) = H^1(\Omega) = Y$, as desired. The same holds true if $(-\Delta)$ is replaced by a more general second-order, uniformly elliptic operator; see Fattorini [1983], Friedman [1976], and Remark 3.3.1.6 below.

Finite Cost Condition (2.1.12) Considerations similar to those made for the Dirichlet case in Section 3.1 apply now; see, for example, Triggiani [1980(b)], Lasiecka and Triggiani [1982; 1983b, c; 1986] for uniform feedback stabilization results of problem (3.3.1.1), achieved in the norm of $H^{\frac{3}{2}-\epsilon}(\Omega)$. Thus, a fortiori the Finite Cost Condition (2.1.12) of Chapter 2 on $H^1(\Omega)$ holds true for problem (3.3.1) and (3.3.2).

Detectability Condition (2.1.13) With $R = I$, this is automatically satisfied.

Conclusion: Case $T = \infty$ Theorems 2.2.1 and 2.2.2 of Chapter 2 apply to problems (3.3.1.1) and (3.3.1.2). The Riccati operator P is Hilbert–Schmidt on $Y = H^1(\Omega)$ for dim $\Omega = 1, 2, 3$ [see Chapter 2, Remark 2.2.2, Eqn. (2.2.17)]. Further details are provided at the end of this subsection.

Conclusion: Case $T < \infty$ We shall see that Theorem 1.2.1.1 of Chapter 1 applies to problems (3.3.1.1) and (3.3.1.2).

In our present case, we have by the last term in (3.3.2):

$$G \equiv \nabla : \text{continuous } H^1(\Omega) \equiv Y \to [L_2(\Omega)]^n \equiv Z_f. \tag{3.3.1.16}$$

We then have

Lemma 3.3.1.2 *(i) For the constant β satisfying $2\gamma - 1 = 2(3/4 + \epsilon) - 1 = 1/2 + \epsilon < \beta \leq 1$, we have with reference to (3.3.1.16):*

$$\begin{aligned} &(\hat{A}^*)^{\frac{\beta}{2}} G^* \textit{ densely defined as an operator} \\ &[L_2(\Omega)]^n \equiv Z_f \supset \mathcal{D}\big((-\hat{A}^*)^{\frac{\beta}{2}} G^*\big) \to Y \equiv H^1(\Omega). \end{aligned} \tag{3.3.1.17}$$

Proof. (i) If $g \in [H_0^1(\Omega)]^n$ and $f \in H^1(\Omega)$, then the divergence theorem yields by (3.3.1.12):

$$\begin{aligned} (Gf, g)_{Z_f} &= (f, G^* g)_{H^1(\Omega) = \mathcal{D}(\hat{A}^{\frac{1}{2}})} = (\hat{A} f, G^* g)_{L_2(\Omega)} \\ &= \int_\Omega \nabla f \cdot g \, d\Omega = \int_\Gamma f \cancel{g \cdot \nu} \, d\Gamma - \int_\Omega f \operatorname{div} g \, d\Omega \\ &= -(f, \operatorname{div} g)_{L_2(\Omega)}, \end{aligned} \tag{3.3.1.18}$$

and thus ($\hat{A} = \hat{A}^*$ in our canonical case)

$$G^* g = -(\hat{A}^*)^{-1} \operatorname{div} g, \qquad g \in \left[H_0^1(\Omega)\right]^n. \tag{3.3.1.19}$$

Hence, with reference to the sufficient condition in Chapter 1, Eqn. (1.2.1.26), we have

$$(\hat{A}^*)^{\frac{\beta}{2}} G^* g = \hat{A}^{\frac{\beta}{2}} G^* g = -\hat{A}^{\frac{\beta}{2}-1} \operatorname{div} g, \qquad g \in [H_0^1(\Omega)]^n, \tag{3.3.1.20}$$

where $\hat{A}^{\beta/2-1} \in \mathcal{L}(L_2(\Omega); \mathcal{D}(\hat{A}^{\frac{1}{2}}) = Y)$ provided $\beta \leq 1$, in which case (3.3.1.16) shows that $(\hat{A}^*)^{\beta/2} G^*$ is densely defined on Y to itself, as desired.

Thus, by Eqn. (1.2.1.26) of Chapter 1, GL_T is closable, as required. □

Remark 3.3.1.1 If the last term of final state penalization in (3.3.2) were

$$\|y(T)\|^2_{H^1(\Omega)} = \||\nabla y(T)|\|^2_{L_2(\Omega)} + \|y(T)\|^2_{L_2(\Omega)}, \tag{3.3.1.21}$$

instead of $\||\nabla y(T)|\|^2_{L_2(\Omega)}$ alone, then one would take $Z_f = H^1(\Omega) = Y, G =$ Identity, and thus $GL_T = L_T$ is closed, as desired, as an operator $L_2(0, T; U) \to Y = \mathcal{D}(\hat{A}^{\frac{1}{2}})$ by essentially the same argument made in the Dirichlet case, in connection with (3.1.13)

Remark 3.3.1.2 We also remark that the above analysis applies, with no essential change, to $y(t)$ penalized in $L_2(0, T; H^{\frac{3}{2}-\epsilon}(\Omega))$ with $y_0 \in H^{\frac{3}{2}-\epsilon}(\Omega)$ [rather than in $L_2(0, T; H^1(\Omega))$ as in (3.3.1.2)], where then $\gamma = 1 - \epsilon/2$. Here, since $H^{\frac{3}{2}-\epsilon}(\Omega) = \mathcal{D}(\hat{A}^{\frac{3}{4}-\epsilon/2})$, we can then take, for example, $G = \hat{A}^{\frac{3}{4}-\epsilon/2}$ with $Z_f = L_2(\Omega)$, to obtain $G \in \mathcal{L}(Y; Z_f)$.

We now provide further details of the established theory. We shall limit ourselves to the case $T = \infty$, that is, to the application of Theorems 2.2.1 and 2.2.2 of Chapter 2 to problem (3.3.1.1) and (3.3.1.2). The case $T < \infty$ is similar.

Case $T = \infty$. Results Application of Theorems 2.2.1 and 2.2.2 of Chapter 2 to problem (3.3.1.1) and (3.3.1.2) on the spaces Z, Y, U in (3.3.1.5) yields the following results concerning the corresponding Riccati operator P. We begin with the regularity of P.

Theorem 3.3.1.3 *With reference to problem (3.3.1.1) and (3.3.1.2), with $T = \infty$, Theorem 2.2.1 of Chapter 2 provides a nonnegative, self-adjoint Riccati operator $P = P^* \in \mathcal{L}(\mathcal{D}(\hat{A}^{\frac{1}{2}})) = \mathcal{L}(H^1(\Omega))$, such that*

(i)

$$\hat{A}P, AP \in \mathcal{L}\big(\mathcal{D}(\hat{A}^{\frac{1}{2}})\big) = \mathcal{L}(H^1(\Omega)); \tag{3.3.1.22}$$

(ii)

$$\hat{A}^{\frac{1}{2}} P \hat{A}^{\frac{1}{2}} \in \mathcal{L}\big(\mathcal{D}(\hat{A}^{\frac{1}{2}})\big) = \mathcal{L}(H^1(\Omega)). \tag{3.3.1.23}$$

Moreover,

(iii)

$$\hat{A}P,\ AP \in \mathcal{L}(L_2(\Omega)). \tag{3.3.1.24}$$

Finally, for $0 \le s \le 1$*:*

(iv)

$$AP \in \mathcal{L}(\mathcal{D}(\hat{A}^{\frac{s}{2}})) = \mathcal{L}(H^s(\Omega)). \tag{3.3.1.25}$$

Proof. (i) Since $R = I$ [see (3.3.1.6)], and A in (3.3.1.4) is self-adjoint, then we may take $\theta = 1$ in Eqn. (2.2.3a), as justified by Chapter 2, Appendix 2B. We thus obtain (3.3.1.22).

(ii) We apply Lemma 1.5.1.1(iii) of Chapter 1 to (3.3.1.18) with $\mathcal{Y} = Y = \mathcal{D}(\hat{A}^{\frac{1}{2}})$, $\mathcal{G} = P$ bounded, self-adjoint on $\mathcal{Y}$, and $\mathcal{A} = -\hat{A}^{\frac{1}{2}}$ self-adjoint on $\mathcal{Y}$, which generates a s.c. analytic semigroup of negative type on $\mathcal{Y}$. In this way we obtain (3.3.1.23) from (3.3.1.22).

(iii) By (3.3.1.23), if $x \in \mathcal{D}(\hat{A}^{\frac{1}{2}})$, $y = \hat{A}^{\frac{1}{2}}x \in L_2(\Omega)$, we have $\hat{A}^{\frac{1}{2}}Py = \hat{A}^{\frac{1}{2}}P\hat{A}^{\frac{1}{2}}x \in \mathcal{D}(\hat{A}^{\frac{1}{2}})$, that is, $\hat{A}Py \in L_2(\Omega)$, as desired, where y runs over all of $L_2(\Omega)$. By the closed graph theorem, we then obtain (3.3.1.24).

(iv) Equation (3.3.1.25) follows by interpolating between (3.3.1.22) and (3.3.1.24). □

Remark 3.3.1.3 The operator $\hat{A}^{\frac{1}{2}}P\hat{A}^{-\frac{1}{2}}$ is nonnegative, self-adjoint on $L_2(\Omega)$.

Theorem 3.3.1.4 *The Riccati operator P of Theorem 3.3.1.3 satisfies*

(i)

$$B^*P = -N^*A\hat{A}P = [\hat{A}P\cdot]|_\Gamma \tag{3.3.1.26a}$$

$$= \textit{continuous } Y = \mathcal{D}\big(\hat{A}^{\frac{1}{2}}\big) \to L_2(\Gamma), \tag{3.3.1.26b}$$

(ii) as well as the ARE on $Y = \mathcal{D}(\hat{A}^{\frac{1}{2}}) = H^1(\Omega)$*:*

$$(Px, Ay)_Y + (Ax, Py)_Y + (x, y)_Y = \big([\hat{A}Px]|_\Gamma, [\hat{A}Py]|_\Gamma\big)_{L_2(\Gamma)},$$
$$\forall\, x, y \in \mathcal{D}(\hat{A}^{\frac{1}{2}+\epsilon}). \tag{3.3.1.27}$$

The ARE (3.3.1.27) has a unique solution within the class of nonnegative, self-adjoint solutions P on $\mathcal{D}(\hat{A}^{\frac{1}{2}})$ such that $\hat{A}^\gamma P \in \mathcal{L}(\mathcal{D}(\hat{A}^{\frac{1}{2}}))$, $\gamma = 3/4 + \epsilon$, that is, $\hat{A}^{\frac{5}{4}+\epsilon}P\hat{A}^{-\frac{1}{2}} \in \mathcal{L}(L_2(\Omega))$.

Remark 3.3.1.4 The choice of the functional, say, for $T = \infty$:

$$J(u, y) = \int_0^\infty \big[\|y(t)\|^2_{L_2(\Omega)} + \|u(t)\|^2_{L_2(\Gamma)}\big]\,dt, \tag{3.3.1.28}$$

in place of (3.3.1.2), considerably simplifies the analysis, since with $Y = L_2(\Omega)$ one easily sees now that in this case we have that assumption (2.1.7) of Chapter 2 holds true with $\gamma = 1/4 + \epsilon < 1/2$. The considerations made in the Orientation of Chapter 2 for the case $\gamma < 1/2$ apply. [This easier case, with $Y = L_2(\Omega)$, also fits into the abstract model of Volume 2, but the results of Chapters 1 and 2 are definitely preferable.] The Dirichlet mixed problem of Section 3.1 with $Y = L_2(\Omega)$, and the Neumann mixed problem of the present Section 3.3 with $Y = H^1(\Omega)$ are, abstractly, "the same problem," in the sense that the critical parameter γ is equal to $3/4 + \epsilon$ in both cases.

Remark 3.3.1.5 This is the counterpart of Remark 3.1.2 in the Dirichlet case. As there, we notice that the present section applies to general second-order differential operators with uniformly elliptic principal part as in (3.1.14). The extended version of Eqn. (3.3.1.9) of Lemma 3.3.1.1 is now

$$B^{\#}\phi = -N^*A^*\phi = \phi|_\Gamma, \quad \phi \in \mathcal{D}(A^*) \tag{3.3.1.29}$$

for the $L_2(\Omega)$-adjoint of B defined by (3.3.1.10), and

$$B^*\phi = B^{\#}\hat{A}^*\phi = -N^*A^*\hat{A}^*\phi, \quad \phi \in \mathcal{D}\big(\hat{A}^{*\frac{3}{2}}\big) \tag{3.3.1.30}$$

for the $Y = \mathcal{D}(\hat{A}^{\frac{1}{2}})$-adjoint of B defined by (3.3.1.11).

Now, however, under homogeneous Neumann boundary conditions, we have only

$$\mathcal{D}(\hat{A}^s) = \mathcal{D}(\hat{A}^{*s}) = H^{2s}(\Omega), \quad s < \frac{3}{4}; \tag{3.3.1.31}$$

see Appendix 3A. One could also provide a regularity theorem for the Neumann case corresponding to Theorem 3.2.1 in the Dirichlet case, *mutatis mutandis*.

3.3.2 The Purely Boundary Case: Boundary Control and Boundary Observation

Once we have solved in $H^1(\Omega)$ the quadratic cost problem for the heat equation problem (3.3.1.1), (3.3.1.2) [or (3.3.1.21)] (indeed, as remarked, in $H^{\frac{3}{2}-\epsilon}(\Omega)$, if we like), we can then obtain as a consequence a solution to the easier "purely boundary" quadratic cost problem, which penalizes the cost functional

$$J(u, y) = \int_0^\infty \left\{ \|y(t)|_\Gamma\|^2_{H^{\frac{1}{2}}(\Gamma)} + \|u(t)\|^2_{L_2(\Gamma)} \right\} dt, \quad y_0 \in H^1(\Omega) \tag{3.3.2.1a}$$

[respectively,

$$J(u, y) = \int_0^\infty \left\{ \|y(t)|_\Gamma\|^2_{H^{1-\epsilon}(\Gamma)} + \|u(t)\|^2_{L_2(\Gamma)} \right\} dt, \quad y_0 \in H^{\frac{3}{2}-\epsilon}(\Omega)] \tag{3.3.2.1b}$$

over all $u \in L_2(0, \infty; L_2(\Gamma))$ with y solution to (3.3.1.1), and $y_0 \in H^1(\Omega)$ in (3.3.2.1a) [respectively, $y_0 \in H^{\frac{3}{2}-\epsilon}(\Omega)$ in (3.3.2.1b)]. Now we take, for the cost in (3.3.2.1a):

$$Y = H^1(\Omega); \quad Z = H^{\frac{1}{2}}(\Gamma); \quad R \text{ the (Dirichlet) trace operator;}$$
$$Ry = y|_\Gamma : \text{ continuous } H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma) \tag{3.3.2.2a}$$

[respectively, for the cost in (3.3.2.1b):

$$\begin{aligned} &Y = H^{\frac{3}{2}-\epsilon}(\Omega); \quad Z = H^{1-\epsilon}(\Gamma); \\ &Ry = y|_\Gamma : \text{ continuous } H^{\frac{3}{2}-\epsilon}(\Omega) \to H^{1-\epsilon}(\Gamma)]. \end{aligned} \tag{3.3.2.2b}$$

Finite Cost Condition The previously recalled uniform stabilization results [Triggiani, 1980(b); Lasiecka, Triggiani, 1982; 1983b, c; 1986] for the solution y in $H^{\frac{3}{2}-\epsilon}(\Omega)$ of the corresponding feedback closed loop problem with u, say, of the form as in (3.1.12), guarantees a fortiori exponential uniform decay of $y(t)|_\Gamma$ in $H^{1-\epsilon}(\Gamma)$. Thus, the required Finite Cost Condition (2.1.12) of Chapter 2 for (3.3.2.1) is satisfied.

Detectability Condition In order to satisfy the Detectability Condition (2.1.13) of Chapter 2, we appeal to the stabilization results as in Triggiani [1979] (see also Lasiecka and Triggiani [1983(b)]) to obtain the required "stabilizing" operator $K \in \mathcal{L}(L_2(\Gamma); H^s(\Omega))$ in (2.1.13) of Chapter 2, with $s = 1$ for the cost in (3.3.2.1a) [respectively, $s = 3/2 - \epsilon$ for the cost in (3.3.2.1b)], which may be taken of the form

$$KRy = Ky|_\Gamma = \sum_{n=1}^{N} (y|_\Gamma, w_n)_{L_2(\Gamma)}\, g_n \tag{3.3.2.3}$$

for suitable $w_n \in L_2(\Gamma)$ and $g_n \in H^1(\Omega)$ in the case of (3.3.2.1a) [respectively, $g_n \in H^{\frac{3}{2}-\epsilon}(\Omega)$, in the case of (3.3.2.1b)]. Thus, the abstract problem required by the Detectability Condition, that is,

$$\dot{y} = Ay + KRy, \quad y(0) = y_0 \tag{3.3.2.4}$$

corresponds to the following heat equation with homogeneous boundary conditions:

$$\begin{cases} y_t = (\Delta + c^2)y + \displaystyle\sum_{n=1}^{N} (y|_\Gamma, w_n)_{L_2(\Gamma)}\, g_n & \text{in } Q, \quad (3.3.2.5a) \\ y(0, \cdot) = y_0 & \text{in } \Omega, \quad (3.3.2.5b) \\ \left.\dfrac{\partial y}{\partial \nu}\right|_\Sigma \equiv 0 & \text{in } \Sigma. \quad (3.3.2.5c) \end{cases}$$

Then, under suitable algebraic (rank) conditions on w_n, g_n, uniform decay of the solution y of (3.3.2.5) in (at least) $H^2(\Omega)$ will result [Triggiani, 1979], as desired.

Finally, if $T < \infty$ is the cost functional (3.3.2.1a), we may add the purely boundary final state penalization term

$$\|y(T)|_\Gamma\|^2_{H^{\frac{1}{2}}(\Gamma)} \quad \left[\text{respectively, } \|y(T)|_\Gamma\|^2_{H^{1-\epsilon}(\Gamma)} \text{ in (3.3.2.1b)}\right],$$

and Theorem 1.2.1.1 of Chapter 1 applies.

3.3.3 The Interior State Penalization Revisited: The (Unwise) Choice of Lower γ and R Unbounded

Orientation Given a parabolic partial differential equation with, say, boundary control and an assigned cost functional, how do we choose the abstract setting among possibly infinite choices? In principle, we may range between, say, the choice of $\{\gamma$ larger and R bounded$\}$ and the choice of $\{\gamma$ smaller and R unbounded$\}$, with γ restricted to $0 < \gamma < 1$. Of course, the larger the γ, the higher the topology of the space Y. In the first case, we may appeal to the theory of Chapter 1, Section 1.1 through 1.7 for $T < \infty$ and of Chapter 2, Section 2.1 through 2.4 for $T = \infty$. In the second case, we may instead invoke Section 1.8 of Chapter 1 for $T < \infty$ and Section 2.5 of Chapter 2 for $T = \infty$. For sake of simplicity, we shall confine our considerations to the case $T = \infty$.

We shall illustrate the following principle (to be expected): *Given a parabolic mixed problem with preassigned cost functional, one obtains a richer Riccati theory by choosing the "largest possible" γ within the range $0 \le \gamma < 1$, in particular, a more regular Riccati operator, and an ARE satisfied on a space with higher topology.* Physical considerations may slightly tame the above principle. This means that, in the abstract parabolic setting of Chapters 1 and 2 for the equation $\dot{y} = Ay + Bu$ [and its PDE illustrations with boundary/point control], one has to accept, as a starting point of a sharp quadratic optimization theory, the *parabolic* regularity: With, say, $y_0 = 0$, this produces a space Y such that $y \in L_2(0, T; Y)$ for $u \in L_2(0, T; U)$, that is, a "large" γ. In this way, a richer Riccati theory will result over the selection of a space Y_0, larger and with weaker topology than Y, that is, small γ, where instead $y \in C([0, T]; Y_0)$ for $u \in L_2(0, T; U)$, $y_0 = 0$.

In the case of the Neumann mixed parabolic problem (3.3.1.1), one has, by (3.3.1.3):

(i) $Y_0 = H^{\frac{1}{2}}(\Omega)$, which by physical considerations may reduce to $Y_0 = L_2(\Omega)$, and
(ii) $Y = H^{\frac{3}{2}-\epsilon}(\Omega)$ [a loss of ϵ over (3.3.1.3) to comply with the requirement $\gamma < 1$], which in turn, by physical considerations may reduce to $Y = H^1(\Omega)$, a full unit higher in Sobolev space regularity.

By contrast, in the "hyperbolic" case, if $y \in L_2(0, T; Y)$ for $u \in L_2(0, T; U)$, then, in fact, $y \in C([0, T]; Y)$, a boosting of the time regularity that preserves the space regularity, see Chapter 7, Section 7.3.

Here below we shall provide some quantitative convincing illustrations of the aforementioned principle: in the present section, for the interior state penalization case, and in Section 3.3.4 for the purely boundary penalization case.

Case $T = \infty$ for Problems (3.3.1.1) and (3.3.1.2) Revisited. We return to problems (3.3.1.1) and (3.3.1.2) with $T = \infty$ (hence $\alpha = 0$ in (3.3.1.2)), that is,

$$J(u, y) = \int_0^\infty \left[\|y(t)\|^2_{H^1(\Omega)} + \|u(t)\|^2_{L_2(\Gamma)} \right] dt. \tag{3.3.3.1}$$

In the present treatment we deliberately choose a space Y with weaker topology than $H^1(\Omega)$ as in (3.3.1.1), say,

$$Y = L_2(\Omega), \quad Z = H^1(\Omega) \tag{3.3.3.2}$$

[recall the regularity (3.3.1.3)]. We accordingly define R as the injection

$$R = \text{injection: } Y \subset \mathcal{D}(R) = \mathcal{D}\big(\hat{A}^{\frac{1}{2}}\big) = H^1(\Omega) \to Z = H^1(\Omega), \tag{3.3.3.3}$$

unbounded as an operator from Y to Z. In this way we (deliberately) fall into the setting of Section 2.5, with

$$\gamma = \frac{1}{4} + \epsilon, \quad \delta = \frac{1}{2} < 1 - \gamma; \quad A \text{ self-adjoint} \tag{3.3.3.4}$$

[recall from Remark 3.3.1.4 that $\gamma = 1/4 + \epsilon$ for $Y = L_2(\Omega)$], in line with Chapter 2, Eqn. (2.5.5). Accordingly, Theorem 2.5.1 applies. In particular, it yields [see Chapter 2, Eqn. (2.5.24a] that there is a Riccati operator

$$P = P^* \geq 0, \quad P \in \mathcal{L}(L_2(\Omega)), \tag{3.3.3.5}$$

satisfying [Chapter 2, Eqn. (2.5.33)]

$$B^{\#}P = -N^*AP : \text{ continuous } \mathcal{D}(\hat{A}^{\delta}) = \mathcal{D}\big(\hat{A}^{\frac{1}{2}}\big) = H^1(\Omega) \to L_2(\Gamma), \tag{3.3.3.6}$$

where $B^{\#}$ is the $L_2(\Omega)$-adjoint of B defined by (3.3.1.1), and characterized by (3.3.1.12). Since $\sigma = 0$ in Eqn. (2.5.24b) of Chapter 2, no further regularity is available for P beyond (3.3.3.5). We are using, as usual, the letter P for the Riccati operator, but the P here is not the P found in Section 3.3.1.1 in Theorems 3.3.1.3 and 3.3.1.4. A comparison between (3.3.1.24) and (3.3.3.5) shows that the Riccati operator P obtained in Theorem 3.3.1.3 is much more regular than the Riccati operator P in the setting of the present section. The present Riccati operator P satisfies the ARE on $L_2(\Omega)$ for $x, y \in \mathcal{D}(\hat{A}^{\epsilon})$, whereas the Riccati operator P of Theorem 3.3.1.4 satisfies the ARE on $H^1(\Omega)$ for $x, y \in \mathcal{D}(\hat{A}^{\frac{1}{2}+\epsilon})$.

3.3.4 The Purely Boundary Case Revisited: Lower γ and R Unbounded

We return to the purely boundary case for the Neumann mixed problem (3.3.1.1), with, say, physical $L_2(\Gamma)$-boundary state penalization on $T = \infty$, that is,

$$J(u, y) = \int_0^{\infty} \left[\|y(t)|_{\Gamma}\|^2_{L_2(\Gamma)} + \|u(t)\|^2_{L_2(\Gamma)} \right] dt. \tag{3.3.4.1}$$

In the present treatment we deliberately choose a space Y with weaker topology than in (3.3.1.2), say,

$$Y = L_2(\Omega), \qquad \text{so that } \gamma = \frac{1}{4} + \epsilon; \quad Z = L_2(\Gamma), \tag{3.3.4.2}$$

$$R : Y \supset \mathcal{D}(R) = H^{\frac{1}{2}+2\epsilon}(\Omega) = \mathcal{D}\big(\hat{A}^{\frac{1}{4}+\epsilon}\big) \to Z = L_2(\Gamma); \qquad (3.3.4.3)$$
$$Ry = y|_\Gamma$$

unbounded, as an operator from Y to Z. In this way, we (deliberately) fall into the setting of Chapter 2, Section 2.5, with

$$\delta = \frac{1}{4} + \epsilon < \min\left\{1 - \gamma, \frac{1}{2}\right\}. \qquad (3.3.4.4)$$

Accordingly, Theorem 2.5.1 of Chapter 2 applies, and, in particular, yields that there is a Riccati operator

$$P \in \mathcal{L}(L_2(\Omega)), \quad P = P^* \geq 0, \qquad (3.3.4.5)$$

satisfying

(i) [Chapter 2, Eqn. (2.5.24a)]:

$$\hat{A}^\sigma P \in \mathcal{L}(L_2(\Omega)), \quad \sigma < 1 - 2\delta = \frac{1}{2} - 2\epsilon; \qquad (3.3.4.6)$$

(ii) (recalling (3.3.1.12) and [Chapter 2, Eqn. (2.5.33)])

$$B^\# P = -N^* A P : \text{continuous } \mathcal{D}(\hat{A}^\delta) = \mathcal{D}\big(\hat{A}^{\frac{1}{4}+\epsilon}\big) \to L_2(\Gamma); \qquad (3.3.4.7)$$

(iii) as well as the ARE on $L_2(\Omega)$ for $x, y \in \mathcal{D}(\hat{A}^\epsilon)$.

The above results under the present setting (3.3.4.1), (3.3.4.2) show that the present Riccati operator P is *less* regular than the Riccati operator of the interior state penalization in $H^1(\Omega)$ obtained in Section 3.3.1, by Theorem 3.3.1.3 and Theorem 3.3.1.4, which would then correspond to a *higher* boundary state penalization in $H^{\frac{1}{2}}(\Gamma)$, as in (3.3.2.1a).

3.4 A Structurally Damped Platelike Equation with Point Control and Simplified Hinged BC

Consider the following model of a platelike equation in the deflection $w(t, x)$, where $\rho > 0$ is any constant [the case $\alpha = 1/2$ in Chen and Russell [1982] and Chen and Triggiani [1987; 1989(a)] (see Appendix 3B at the end of this chapter)]:

$$\begin{cases} w_{tt} + \Delta^2 w - \rho\Delta w_t = \delta(x - x^0)u(t) & \text{in } (0, T] \times \Omega = Q, \quad (3.4.1a) \\ w(0, \cdot) = w_0; \ w_t(0, \cdot) = w_1 & \text{in } \Omega, \quad (3.4.1b) \\ w|_\Sigma \equiv \Delta w|_\Sigma \equiv 0 & \text{in } (0, T] \times \Gamma = \Sigma, \quad (3.4.1c) \end{cases}$$

with load concentrated at the interior point x^0 of an open bounded (smooth) domain Ω of $R^n, n \leq 3$. Regularity results for problem (3.4.1), and other problems of this type, are given in Triggiani [1991]. Consistently with these results, the cost functional we wish to minimize is

$$J(u, w) = \int_0^T \left\{ \|w(t)\|^2_{H^2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} + |u(t)|^2 \right\} dt$$
$$+ \alpha \|\{w(T), w_t(T)\}\|^2_{H^2(\Omega)\times L_2(\Omega)}, \tag{3.4.2}$$

where $\{w_0, w_1\} \in [H^2(\Omega) \cap H^1_0(\Omega)] \times L_2(\Omega)$; moreover, $\alpha = 0$ if $T = \infty$ and $\alpha = 1$ for $T < \infty$.

Abstract Setting To put problems (3.4.1) and (3.4.2) into the abstract setting (3.1.1) and (3.1.2) of the preceding Chapters 1 and 2, we introduce the strictly positive definite self-adjoint operator on $L_2(\Omega)$:

$$\mathcal{A}h = \Delta^2 h, \quad \mathcal{D}(\mathcal{A}) = \{h \in H^4(\Omega) : h|_\Gamma = \Delta h|_\Gamma = 0\}, \tag{3.4.3}$$

where we notice that

$$\mathcal{A} = A_D^2, \quad \mathcal{A}^{\frac{1}{2}} = A_D, \tag{3.4.4}$$

where A_D is the operator defined by (3.1.8) in Section 3.1, and we select the spaces and operators

$$Y \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) = \left[H^2(\Omega) \cap H^1_0(\Omega)\right] \times L_2(\Omega); \quad U = R^1, \tag{3.4.5}$$

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\rho\mathcal{A}^{\frac{1}{2}} \end{bmatrix}, \quad B : R^1 \to [\mathcal{D}(A)]', \quad Bu = \begin{bmatrix} 0 \\ \delta(x - x^0)u \end{bmatrix}, \quad R = I. \tag{3.4.6}$$

$\mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$. We need to verify a few assumptions.

Assumptions (1.1.2) or (2.1.7): $(-A)^{-\gamma}B \in \mathcal{L}(U; Y)$ It is easy to verify that this assumption is satisfied with $\gamma = 1$. Indeed, from (3.4.6), we require that

$$(-A)^{-1}Bu = \begin{bmatrix} \rho\mathcal{A}^{-\frac{1}{2}} & \mathcal{A}^{-1} \\ -I & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \delta(x - x^0)u \end{bmatrix} = \begin{bmatrix} \mathcal{A}^{-1}\delta(x - x^0)u \\ 0 \end{bmatrix} \in Y, \tag{3.4.7}$$

that is, from (3.4.5), we require that $\mathcal{A}^{-\frac{1}{2}}\delta(x - x^0) \in L_2(\Omega)$, or that

$$\delta(x - x^0) \in \left[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})\right]', \text{ the dual of } \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \text{ with respect to } L_2(\Omega). \tag{3.4.8}$$

Since it is true that $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset H^2(\Omega)$ for the fourth-order operator $\mathcal{A}$ in (3.4.3) (in fact, regardless of the particular boundary conditions), and thus $[H^2(\Omega)]' \subset [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, then condition (3.4.8) is satisfied provided $\delta(x - x^0) \in [H^2(\Omega)]'$, that is, provided $H^2(\Omega) \subset C(\bar{\Omega})$, which is indeed the case by Sobolev embedding provided $2 > n/2$, or $n < 4$, as required.

However, the above result is not sufficient for our purposes as – according to assumption (3.1.1.2) of Chapter 1 – we need to show that we can take $\gamma < 1$.

Claim: We now show that this assumption holds true for any $\gamma > n/4$, which then for $n \leq 3$ yields $\gamma < 1$ as desired. To this end, we note that

$$(-A)^{-\gamma} B \in \mathcal{L}(U; Y) \quad \text{if and only if} \quad B \in \mathcal{L}(U; [\mathcal{D}((-A^*)^{\gamma}]' \tag{3.4.9}$$

with duality with respect to Y. But $\mathcal{D}((-A^*)^{\gamma}) = \mathcal{D}((-A)^{\gamma})$. This follows since A is the direct sum of two normal operators on Y, when $\rho \neq 2$ (see Chen and Triggiani [1989(a)], Lemma 3A.1, case (v)a with $\alpha = 1/2$, where ρ in this section corresponds to 2ρ in this reference), and in any case $A_{\rho\alpha}$ is a spectral operator. Moreover, [Chen, Triggiani, 1990(b), with $\alpha = \frac{1}{2}$; see Appendix 3B, Theorem 3B.2], we have

$$\mathcal{D}((-A^*)^{\gamma}) = \mathcal{D}((-A)^{\gamma}) = \mathcal{D}(\mathcal{A}^{\frac{1}{2}+\frac{\gamma}{2}}) \times \mathcal{D}(\mathcal{A}^{\frac{\gamma}{2}}), \quad 0 < \gamma < 1 \tag{3.4.10}$$

(the first component does not really matter in the argument below). Thus, from (3.4.10) and B as in (3.4.6), it follows that (3.4.9) holds true, provided $\delta(x - x^0) \in [\mathcal{D}(\mathcal{A}^{\gamma/2})]'$ (duality with respect to $L_2(\Omega)$), where $\mathcal{D}(\mathcal{A}^{\gamma/2}) \subset H^{2\gamma}(\Omega)$, and hence, provided $\delta(x - x^0) \in [H^{2\gamma}(\Omega)]' \subset [\mathcal{D}(\mathcal{A}^{\gamma/2})]'$. But this in turn is the case, provided $H^{2\gamma}(\Omega) \subset C(\bar{\Omega})$; that is, by Sobolev embedding provided $2\gamma > n/2$, as desired. We could, alternatively, write $(-A)^{-\gamma} Bu = (-A)^{1-\gamma}(-A)^{-1} Bu$ and use (3.4.5), as well as (3.4.10) with γ there replaced by $1 - \gamma$ now, to obtain the required condition $\delta(\cdot - x^0) \in [\mathcal{D}(\mathcal{A}^{\gamma/2})]'$.

We conclude: *Assumption (1.1.2) of Chapter 1: $(-A)^{-\gamma} B \in \mathcal{L}(U; Y)$ holds true for problem (3.4.1) with $n/4 < \gamma < 1$, $n \leq 3$, in which case we can take $\gamma = n/4 + \epsilon$.*

Assumption of Analyticity The operator A in (3.4.6) generates an s.c. contraction semigroup e^{At} on Y, which, moreover, is analytic here for $t > 0$. (This is a special case of a much more general result [Chen, Triggiani, 1987; 1989(a), $\alpha = 1/2$]), reported in Appendix 3B as Theorem 3B.1(b).)

Remark 3.4.1 Since the semigroup e^{At} is analytic on Y and also uniformly stable [Chen, Triggiani, 1989(a); Appendix 3B, Theorem 3B.1(d) at the end of Chapter 3], then, by the just-verified property (1.1.2) of Chapter 1, we have in the norm of $\mathcal{L}(Y; U)$:

$$\|B^* e^{A^* t}\| = \left\| B^*(-A^*)^{-\gamma}(-A^*)^{\gamma} e^{A^* t} \right\| = \mathcal{O}\left(\frac{1}{t^{\gamma}}\right), \quad 0 < t, \tag{3.4.11}$$

with $n/4 < \gamma < 1, n \leq 3$. For $n = 1$, so that we can take $\gamma = 1/4 + \epsilon$, it allows one to assert that the basic assumption of "abstract trace theory" of the forthcoming Volume 2 is satisfied.

Finite Cost Condition (2.1.12) We have just noted in Remark 3.4.1 above that with A as in (3.4.6), the semigroup e^{At} is uniformly (exponentially) stable in Y [Chen, Triggiani, 1989(a)], and thus the Finite Cost Condition (2.1.12) of Chapter 2 holds true with $u \equiv 0$.

Remark 3.4.2 Suppose that instead of Eqn. (3.4.1a), one has

$$w_{tt} + (\Delta^2 + k_1)w - (\Delta + k_2)w_t = \delta(x - x^0)u(t) \quad \text{in } Q, \tag{3.4.12}$$

along with (3.4.1b,c). Then, if $0 < k_1 + k_2$ is sufficiently large, the generator A, which has compact resolvent, has finitely many unstable eigenvalues in $\{\operatorname{Re}\lambda > 0\}$. Since e^{At} is analytic on Y, the usual theory [Triggiani, 1975] applies: The problem is stabilizable on Y if [Triggiani, 1975] and only if [Manitius, Triggiani, 1978(b), Appendix] its projection onto the finite-dimensional unstable subspace is controllable.

For instance, if $\lambda_1, \ldots, \lambda_K$ are the unstable eigenvalues of A, assumed at first to be simple, and $\Phi_1, \ldots, \Phi_K$ are the corresponding eigenfunctions in Y, then the necessary and sufficient condition for stabilization is that $\Phi_k(x^0) \neq 0, k = 1, \ldots, K$.

If $\lambda_1, \ldots, \lambda_K$ are not simple, then their largest multiplicity M determines the smallest number of scalar controls needed for the stabilization of (3.4.12), where now the right-hand side is replaced by $\sum_{i=1}^{M} \delta(x - x^i)u_i(t)$, x^i interior points, along with (3.4.1b, c). The necessary and sufficient condition for stabilization is now a well-known full rank condition [Triggiani, 1975].

Detectability Condition (3.2.1.13) With $R = I$, this is satisfied.

Conclusion: Case $T = \infty$ Theorems 2.2.1 and 2.2.2 of Chapter 2 apply to problems (3.4.1)–(3.4.2), $n \leq 3$, and provide existence and uniqueness of the solution to the ARE (2.2.2) of Chapter 2, with Riccati operator $P \in \mathcal{L}(Y; \mathcal{D}(A^{1-\epsilon}))$, $\forall\, \epsilon > 0$; indeed, for $\rho \neq 2$, $P \in \mathcal{L}(Y; \mathcal{D}(A))$ (since A, as remarked above (3.4.10), is the direct sum of two normal operators on Y so that Lemma 2B.2 applies with $R = I$). Here $\mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$; see (3.4.3) and (3.4.5) for the characterizations of these spaces. Thus, in particular, we have $B^*P \in \mathcal{L}(Y; U)$, where $B^*\left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right] = v_2(x^0)$.

According to Remark 2.2.2, Eqn. (2.2.16) of Chapter 2, the Riccati operator P is Hilbert–Schmidt (H–S) on Y in (3.4.5), provided that $A^{-\theta}$ is H–S on Y, where $\theta < 1$ may be taken arbitrarily close to 1. From the spectral analysis of the operator A in (3.4.6) conducted in [Chen, Triggiani, 1989(a), Appendix 3A] one has that the eigenvalues of A have two branches. Let λ_k be the eigenvalues of the elliptic realization $\mathcal{A}$ in (3.4.3), so that $\lambda_k \sim k^{4/n}$, $n = \dim \Omega$. Then, the two branches of eigenvalues of A are of the order $\lambda_k^{\frac{1}{2}} \sim k^{2/n}$ [Chen, Triggiani, 1989(a), (3A.3) for $\alpha = 1/2$]. Thus, according to Remark 2.2.2, first proof of Claim (2.2.17), we have that P is H–S provided that $A^{-\theta}$ is H–S, that is, provided that $\sum_k 1/(k^{4/n}) < \infty$, for $n = \dim \Omega = 1, 2, 3$. These considerations apply also to Section 3.6 and will not be repeated.

Conclusion: Case $T < \infty$ Theorem 1.2.1.1 of Chapter 1 applies, since $G =$ Identity, and the operator L_T is closed (see (3.1.13) and below). Any final state penalization operator G that makes GL closed (closable) is acceptable; see also the sufficient condition (1.2.1.26) of Chapter 1.

Remark 3.4.3 Essentially the same analysis with minimal changes applies also to problem (3.4.1a, b), with the BC (3.4.1c) replaced now by $\frac{\partial w}{\partial \nu}|_\Sigma \equiv \frac{\partial \Delta w}{\partial \nu}|_\Sigma \equiv 0$. The new definition of $\mathcal{A}$ incorporates, of course, these boundary conditions, and it is still true that the damping operator is precisely $\mathcal{A}^{\frac{1}{2}}$, so that A now has the same form (3.4.6) as before. The main difference is that the present $\mathcal{A}$ is nonnegative self-adjoint and has $\mu = 0$ as an eigenvalue with corresponding one-dimensional eigenspace, spanned by the nonzero constant function. Thus, the new operator A has $\lambda = 0$ as an eigenvalue with corresponding eigenfunction $\Phi = [\Phi_1, \Phi_2]$, $\Phi_1 = \text{const}$, $\Phi_2 = 0$. Then, Remark 3.4.2 applies to stabilize the system, as the condition $\Phi(x^0) \neq 0$ is satisfied. (With no harm, one may choose to work on the space $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2^0(\Omega)$, where $L_2^0(\Omega)$ is the quotient space $L_2(\Omega)/\mathcal{N}(A)$, with $\mathcal{N}(A)$ the null space of $\mathcal{A}$.)

Remark 3.4.4 A similar analysis applies also to problem (3.4.1a, b), with genuinely hinged BC, when dim $\Omega = 2$:

$$w|_\Sigma \equiv 0, \quad [\Delta w + (1-\mu)B_1 w]_\Sigma \equiv 0 \quad \text{in } (0,T] \times \Gamma \equiv \Sigma, \tag{3.4.13}$$

or even with the BC (3.4.1c) replaced by the following free BC involving shears and forces, still when dim $\Omega = 2$:

$$[\Delta w + (1-\mu)B_1 w]_\Sigma = 0 \quad \text{in } (0,T] \times \Gamma = \Sigma, \tag{3.4.14a}$$

$$\left[\frac{\partial \Delta w}{\partial \nu} + (1-\mu)B_2 w - w\right]_\Sigma = 0 \quad \text{in } \Sigma. \tag{3.4.14b}$$

These boundary conditions are defined and analyzed in the context of the next example, which, however, refers to an equation with stronger damping than (3.4.1a). An analysis of the boundary operators B_1 and B_2, and of the properties of the corresponding differential operator Δ^2 with these new BC is given in Appendix 3C.

3.5 Kelvin–Voight Platelike Equation with Point Control with Free BC

The Kelvin–Voight model for a plate equation in the deflection $w(t,x)$ is [the case $\alpha = 1$ in [Chen, Triggiani, 1987; 1989(a); see Appendix 3B Chapter 3]:

$$\begin{cases} w_{tt} + \Delta^2 w + \rho \Delta^2 w_t = \delta(x - x^0)u(t) & \text{in } (0,T] \times \Omega = Q, & (3.5.1a) \\ w(0,\cdot) = w_0; \ w_t(0,\cdot) = w_1 & \text{in } \Omega, & (3.5.1b) \\ \Delta w|_\Sigma + (1-\mu)B_1 w \equiv 0 & \text{in } (0,T] \times \Gamma = \Sigma, & (3.5.1c) \\ \left.\dfrac{\partial \Delta w}{\partial \nu}\right|_\Sigma + (1-\mu)B_2 w - w \equiv 0 & \text{in } \Sigma, & (3.5.1d) \end{cases}$$

with $0 < \mu < 1/2$ the Poisson modulus and $\rho > 0$ any constant. The boundary operators B_1 and B_2 are zero for $n = 1$, and for $n = 2$ [Lagnese, 1987] (see Appendix 3C):

$$B_1 w = 2\nu_1 \nu_2 w_{xy} - \nu_1^2 w_{yy} - \nu_2^2 w_{xx}, \tag{3.5.2a}$$

$$B_2 w = \frac{\partial}{\partial \tau}\left[\left(\nu_1^2 - \nu_2^2\right) w_{xy} + \nu_1 \nu_2 \left(w_{yy} - w_{xx}\right)\right], \tag{3.5.2b}$$

where again x^0 is an interior point of the open bounded $\Omega \subset R^n$, $n \le 2$. Regularity results for problem (3.5.1) are given in Triggiani [1991]. Consistently with these, we take the cost functional J to be the same as in (3.4.2) with $\{w_0, w_1\} \in H^2(\Omega) \times L_2(\Omega)$.

Abstract Setting We introduce the positive, self-adjoint operator (see Appendix 3C, Proposition 3C.4)

$$\mathcal{A}h = \Delta^2 h,$$
$$\mathcal{D}(\mathcal{A}) = \left\{ h \in H^4(\Omega) : \Delta h + (1-\mu)B_1 h|_\Gamma = 0; \frac{\partial \Delta h}{\partial \nu} + (1-\mu)B_2 h - h|_\Gamma = 0 \right\}, \tag{3.5.3}$$

and select the spaces and operators

$$Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) = H^2(\Omega) \times L_2(\Omega), \quad U = \mathbb{R}^1, \quad \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega), \tag{3.5.4}$$

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\rho\mathcal{A} \end{bmatrix}, \quad Bu = \begin{bmatrix} 0 \\ \delta(x - x^0)u \end{bmatrix}, \quad R = I \tag{3.5.5a}$$

to obtain the abstract model (2.1.1), (2.1.2) of Chapter 2. To get that A is closed, and the generator of a s.c. semigroup, we take

$$\mathcal{D}(A) = \{x_1 \in \mathcal{D}(\mathcal{A}^{1/2}), x_2 \in L_2(\Omega); x_1 + \rho x_2 \in \mathcal{D}(\mathcal{A})\} \subset \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}) \equiv \mathcal{S} \tag{3.5.5b}$$

We note that $\mathcal{S}$ is invariant under the resolvent $R(\lambda, A)$ of A, hence under the semigroup $e^{(At)}$. Thus, if $\{w_0, w_1\} \in \mathcal{S}$, then the solution of problem (3.5.1) with $u \equiv 0$ satisfies $\{w(t), w_t(t)\} \in \mathcal{S}$ as well, for $t > 0$. We note explicitly that it can be shown that [Lagnese, 1987; Appendix 3C, Eqn. (3C.21)]

$$\|\mathcal{A}^{\frac{1}{2}} v\|^2_{L_2(\Omega)} = \|v\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} = a(v, v), \tag{3.5.6}$$

where for $n = 2$ with x and y the space variables:

$$a(v, v) = \int_\Omega |\Delta v|^2 + 2(1-\eta)\left[v_{xy}^2 - v_{xx}^2 v_{yy}^2\right] d\Omega + \int_\Gamma v^2 \, d\Gamma \tag{3.5.7}$$

is equivalent to the squared norm $\|v\|^2_{H^2(\Omega)}$ of $H^2(\Omega)$; in fact, $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega)$, as noted in (3.5.4), by Appendix 3A.

Assumption (1.1.2) or (2.1.7): $(-A)^{-\gamma}B \in \mathcal{L}(U; Y)$ Again, it is straightforward to verify that this assumption is satisfied with $\gamma = 1$: From (3.5.5), we require that

$$(-A)^{-1}Bu = \begin{bmatrix} \rho I & \mathcal{A}^{-1} \\ -I & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \delta(x - x^0)u \end{bmatrix} = \begin{bmatrix} \mathcal{A}^{-1}\delta(x - x^0)u \\ 0 \end{bmatrix} \in Y, \tag{3.5.8}$$

that is, from (3.5.4) we require that $\mathcal{A}^{-\frac{1}{2}}\delta(x - x^0) \in L_2(\Omega)$. The same argument below (3.4.7) then applies yielding that (3.5.8) holds true if $n \le 3$.

However, to verify assumption (1.1.2) of Chapter 1, which requires that γ should be less than 1, the most elementary way is to check that this assumption holds with $\gamma = 1/2$. In this case, we can in fact rely on the direct computation of $(-A)^{-\frac{1}{2}}$ (for simplicity of notation, we take henceforth $\rho = 1$) [see Triggiani, 1991, Eqn. (3.2.10) with $\alpha = 1$]:

$$(-A)^{-\frac{1}{2}} = \begin{bmatrix} (1) & \mathcal{A}^{-\frac{3}{4}}\left(2I + \mathcal{A}^{\frac{1}{2}}\right)^{-\frac{1}{2}} \\ (2) & \mathcal{A}^{-\frac{1}{4}}\left(2I + \mathcal{A}^{\frac{1}{2}}\right)^{-\frac{1}{2}} \end{bmatrix} \tag{3.5.9}$$

(where the entries $(1) = \mathcal{A}^{-\frac{1}{4}}(2I + \mathcal{A}^{\frac{1}{2}})^{-\frac{1}{2}}(I + \mathcal{A}^{\frac{1}{2}})$ and $(2) = -\mathcal{A}^{-\frac{1}{4}}(2I + \mathcal{A}^{\frac{1}{2}})^{-\frac{1}{2}}$ do not really count in the present analysis), and avoid the domain of fractional powers as in [Chen, Triggiani, 1990(b), Theorem 3B.2 of Appendix 3B Chapter 3].

We need to compute

$$(-A)^{-\frac{1}{2}} Bu = \begin{vmatrix} \mathcal{A}^{-\frac{3}{4}}\left(2I + \mathcal{A}^{\frac{1}{2}}\right)^{-\frac{1}{2}} \delta(x - x^0)u \\ \mathcal{A}^{-\frac{1}{4}}\left(2I + \mathcal{A}^{\frac{1}{2}}\right)^{-\frac{1}{2}} \delta(x - x^0)u \end{vmatrix}. \tag{3.5.10}$$

From (3.5.10), we then readily see that $(-A)^{-\frac{1}{2}} Bu \in Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$ provided

$$\mathcal{A}^{-\frac{1}{2}}\delta(\cdot - x^0) \in L_2(\Omega) \quad \text{or} \quad \delta(\cdot - x^0) \in \left[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})\right]'. \tag{3.5.11}$$

But $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega)$ (see Appendix 3A) (and, in fact, only $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset H^2(\Omega)$ suffices for the present analysis) so that condition (3.5.11) is satisfied provided $\delta(x - x^0) \in [H^2(\Omega)]'$ (duality with respect to $L_2(\Omega)$), that is, provided $H^2(\Omega) \subset C(\bar{\Omega})$, i.e., by Sobolev embedding provided $2 > n/2$ or $n < 4$, as desired.

We have shown: *Assumption (1.1.2) of Chapter 1: $(-A)^{-\gamma} B \in \mathcal{L}(U; Y)$ holds true for problem (3.5.1) with $n \leq 3$, and $\gamma = 1/2$.* The above argument shows some "leverage." Indeed, $\gamma = 1/2$ is not the least γ for which assumption (1.1.2) holds true. To obtain the least γ for which assumption (1.1.2) holds true, we proceed as in the case of problem (3.4.1) above, in the argument that begins with (3.4.9) and uses the domains of fractional powers $\mathcal{D}((-A)^{\gamma})$ in (3.4.10). As was done below (3.4.9) we need to show that

$$Bu \in [\mathcal{D}((-A)^{\gamma})]', \qquad \text{duality with respect to } Y. \tag{3.5.12}$$

But for $0 < \gamma \leq 1/2$, we have from [Chen, Triggiani, 1990(b), with $\alpha = 1$; Theorem 3B.2(i) of Appendix 3B] that

$$\mathcal{D}((-A)^{\gamma}) = \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right) \times \mathcal{D}\left(\mathcal{A}^{\gamma}\right). \tag{3.5.13}$$

Thus, from B as in (3.5.5), we see that condition (3.5.12) above holds true, provided $\delta(x - x^0) \subset [\mathcal{D}(\mathcal{A}^{\gamma})]'$, duality with respect to $L_2(\Omega)$, that is, provided $\delta(x - x^0) \subset [H^{4\gamma}(\Omega)]'$, since $\mathcal{D}(\mathcal{A}^{\gamma}) \subset H^{4\gamma}(\Omega)$ for the fourth-order operator in (3.5.3), that is, provided $H^{4\gamma}(\Omega) \subset C(\bar{\Omega})$, which in turn is the case, provided $4\gamma > n/2$, or $1/2 \geq \gamma > n/8$.

We conclude: *Assumption (1.1.2):* $(-A)^{-\gamma}B \in \mathcal{L}(U; Y)$ *holds true for problem (3.5.1) provided* $n/8 < \gamma \leq 1/2$, $n \leq 3$, in which case we can take $\gamma = n/8 + \epsilon$.

Assumption of Analyticity The operator A in (3.5.5) generates an s.c. uniformly stable semigroup e^{At} on Y, which, moreover, is analytic here for $t > 0$. (This is a special case of a much more general result [Chen, Triggiani, 1989(a), $\alpha = 1$] (see Theorem 3B.1(b), (d) of Appendix 3B.)

Remark 3.5.1 Since the semigroup e^{At} is analytic and uniformly stable on Y [Theorem 3B.1(b),(d) of Appendix 3B Chapter 3], then, by the just-verified property (1.1.2) of Chapter 1, we have in the norm of $\mathcal{L}(Y; U)$:

$$\left\|B^* e^{A^* t}\right\| = \left\|B^*(-A^*)^{-\gamma}(-A^*)^{\gamma} e^{A^* t}\right\| \leq O\left(\frac{1}{t^{\gamma}}\right), \quad 0 < t,$$

with $n/8 < \gamma < 1/2$, $n \leq 3$, so that the "abstract trace theory" assumption of Volume 2 is satisfied.

Finite Cost Condition (2.1.12) We have just noted in Remark 3.5.1 above, that with A as in (3.5.5), the semigroup e^{At} is uniformly (exponentially) stable in Y, and thus the Finite Cost Condition (2.1.12) of Chapter 2 is automatically satisfied on this space with $u \equiv 0$.

Detectability Condition (2.1.13) With $R = I$, this is satisfied.

Conclusion: Case $T = \infty$ Theorems 2.2.1 and 2.2.2 of Chapter 2 apply to problem (3.5.1), (3.4.2) for $n \leq 3$.

Conclusion: Case $T < \infty$ Theorem 1.2.1.1 of Chapter 1 also applies to (3.5.1), (3.4.2) for $n \leq 3$, as in the conclusion of Section 3.4.

3.6 A Structurally Damped Platelike Equation with Boundary Control in the Simplified Moment BC

We consider the platelike problem

$$\begin{cases} w_{tt} + \Delta^2 w - \rho \Delta w_t = 0 & \text{in } (0, T] \times \Omega \equiv Q, & (3.6.1a) \\ w(0, \cdot) = w_0; \ w_t(0, \cdot) = w_1 & \text{in } \Omega, & (3.6.1b) \\ w|_{\Sigma} \equiv 0 & \text{in } (0, T] \times \Gamma \equiv \Sigma, & (3.6.1c) \\ \Delta w|_{\Sigma} \equiv u & \text{in } \Sigma, & (3.6.1d) \end{cases}$$

$\{w_0, w_1\} \in [H^2(\Omega) \cap H_0^1(\Omega)] \cap L_2(\Omega)$, which is the same model as the one in (3.4.1), except that it is acted upon by a boundary control $u \in L_2(0, T; L_2(\Gamma)) \equiv L_2(\Sigma)$, rather than by a point control as in (3.4.1a). Accordingly, the dimension $\dim \Omega = n$ of the (smooth) bounded domain Ω is arbitrary. We take the same functional J as in

(3.4.2) except that now u is penalized in the $L_2(\Gamma)$-norm:

$$J(u,w) = \int_0^T \left\{ \|w(t)\|^2_{H^2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} + \|u(t)\|^2_{L_2(\Gamma)} \right\} dt \\ + \alpha \|\{w(T), w_t(T)\}\|^2_{H^2(\Omega)\times L_2(\Omega)}, \tag{3.6.2}$$

where $\alpha = 0$ if $T = \infty$ and $\alpha = 1$ for $T < \infty$. Following Lasiecka and Triggiani [1989], we introduce the Green map G_2 defined by

$$y = G_2 v \Longleftrightarrow \{\Delta^2 y = 0 \text{ in } \Omega;\ y|_\Gamma = 0;\ \Delta y|_\Gamma = v\}. \tag{3.6.3}$$

Then, if $\mathcal{A}$ is the same operator defined in (3.4.3), it is rather straightforward to see that the abstract representation of problem (3.6.1) is given by the equation

$$w_{tt} + \mathcal{A}w + \rho\mathcal{A}^{\frac{1}{2}} w_t = \mathcal{A}G_2 u. \tag{3.6.4}$$

Indeed, problem (3.6.1) can be rewritten first as

$$w_{tt} + \Delta^2(w - G_2 u) - \rho\Delta w_t = 0 \text{ in } Q, \quad (w - G_2 v)|_\Sigma = \Delta(w - G_2 u)|_\Sigma = 0, \tag{3.6.5a}$$

by (3.6.3), and hence abstractly by

$$w_{tt} + \mathcal{A}(w - G_2 u) + \rho\mathcal{A}^{\frac{1}{2}} w_t = 0, \tag{3.6.5b}$$

because of the BC, since now $\mathcal{A}^{\frac{1}{2}}h = -\Delta h, \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega) \cap H^1_0(\Omega)$. From (3.6.5b), we have that (3.6.4) follows by extending the original $\mathcal{A}$ in (3.4.3), as usual, by isomorphism to, say, $\mathcal{A}: L_2(\Omega) \to [\mathcal{D}(\mathcal{A})]'$. It is shown below that the Green map G_2 can be expressed in terms of the Dirichlet map D defined below, as follows:

$$G_2 = -\mathcal{A}^{-\frac{1}{2}} D, \qquad \text{where } y = Dv \Longleftrightarrow \{\Delta y = 0 \text{ in } \Omega;\ y|_\Gamma = v\}, \tag{3.6.6}$$

where D satisfies [Fujiwara, 1967; Grisvard, 1967; Lasiecka, 1980(b), Appendix 3B]; see also Appendix 3A at the end of this chapter]

$$D: \text{ continuous } L_2(\Gamma) \to H^{\frac{1}{2}}(\Omega) \subset H^{\frac{1}{2}-2\epsilon}(\Omega) \equiv \mathcal{D}\big(\mathcal{A}^{\frac{1}{8}-\frac{\epsilon}{2}}\big), \quad \epsilon > 0, \tag{3.6.7}$$

[recall (3.1.7)]. Indeed, setting $\zeta = \Delta y$, with y as in (3.6.3), we obtain from (3.6.3): $\Delta\zeta = 0$ in $\Omega, \zeta|_\Gamma = v$; hence $\zeta = Dv$ according to (3.6.6). Moreover, recalling (3.4.4) and (3.1.8), we obtain $-\mathcal{A}^{\frac{1}{2}}y = \Delta y = \zeta = Dv$; hence $y = -\mathcal{A}^{-\frac{1}{2}}Dv$ and (3.6.3) yields $G_2 = -\mathcal{A}^{-\frac{1}{2}}D$, as claimed in (3.6.6).

Abstract Setting Thus, by (3.6.6), we have that (3.6.4) becomes the abstract equation

$$w_{tt} + \mathcal{A}w + \rho\mathcal{A}^{\frac{1}{2}} w_t = -\mathcal{A}^{\frac{1}{2}} Du, \tag{3.6.8}$$

or

$$\frac{d}{dt}\begin{bmatrix} w \\ w_t \end{bmatrix} = A \begin{bmatrix} w \\ w_t \end{bmatrix} + Bu, \quad A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\rho\mathcal{A}^{\frac{1}{2}} \end{bmatrix}, \quad Bu = \begin{bmatrix} 0 \\ -\mathcal{A}^{\frac{1}{2}} Du \end{bmatrix} \tag{3.6.9}$$

with $\mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ on the spaces

$$Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) = \left[H^2(\Omega) \cap H_0^1(\Omega)\right] \times L_2(\Omega); \qquad U = L_2(\Gamma).$$

Assumption (1.1.2) or (2.1.7): $(-A)^{-\gamma}B \in \mathcal{L}(U;Y)$ Again, it is elementary to verify that this assumption is satisfied with $\gamma = 1$: Indeed, from (3.6.9) we require

$$(-A)^{-1}Bu = \begin{bmatrix} \rho\mathcal{A}^{-\frac{1}{2}} & \mathcal{A}^{-1} \\ -I & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -\mathcal{A}^{\frac{1}{2}}Du \end{bmatrix}$$

$$= \begin{bmatrix} -\mathcal{A}^{-\frac{1}{2}}Du \\ 0 \end{bmatrix} \in Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega), \tag{3.6.10}$$

which certainly holds true by (3.6.7). We may also verify that the value $\gamma = 1/2$ fails: From direct computations (as in (3.5.9), with A as in (3.4.3) however), or from [Triggiani, 1991, Eqn. (3.2.10)], we obtain (say with $\rho = 1$)

$$(-A)^{-\frac{1}{2}}Bu = \frac{-1}{\sqrt{3}} \begin{bmatrix} \mathcal{A}^{-\frac{3}{4}}\mathcal{A}^{\frac{1}{2}}Du \\ \mathcal{A}^{-\frac{1}{4}}\mathcal{A}^{\frac{1}{2}}Du \end{bmatrix} = \frac{-1}{\sqrt{3}} \begin{bmatrix} \mathcal{A}^{-\frac{1}{4}}Du \\ \mathcal{A}^{\frac{1}{4}}Du \end{bmatrix}, \tag{3.6.11}$$

and from (3.6.7) we see that $(-A)^{-\frac{1}{2}}Bu$ in (3.6.11) fails by $1/8 + \epsilon/2$, to be in Y.

Indeed, we have the following claim: *Assumption (1.1.2) of Chapter 1 holds true for all* $3/4 < \gamma < 1$.

The above claim can be verified by an argument similar to the ones of the preceding two examples, based on the domain of fractional power [Chen, Triggiani, 1990(b), with $\alpha = 1/2$, Theorem 3B.2(ii) of Appendix 3B]

$$\mathcal{D}((-A)^{\gamma}) = \mathcal{D}(\mathcal{A}^{\frac{1}{2}+\frac{\gamma}{2}}) \times \mathcal{D}(\mathcal{A}^{\frac{\gamma}{2}}) \times \mathcal{D}(\mathcal{A}^{\frac{\gamma}{2}}), \quad \frac{1}{2} \leq \gamma < 1 \tag{3.6.12}$$

(only the second component is needed in our argument), whereby the usual condition $Bu \in [\mathcal{D}((-A)^{\gamma})]'$, duality with respect to Y, is satisfied provided, from (3.6.9) and (3.6.12), $\mathcal{A}^{\frac{1}{2}}Du \in [\mathcal{D}(\mathcal{A}^{\gamma/2})]'$, duality with respect to $L_2(\Omega)$, that is, from (3.6.7) provided $1/2 - \gamma/2 = 1/8 - \epsilon/2$ or $\gamma = 3/4 + \epsilon, \forall\, \epsilon$, and the claim is proved.

Assumption of Analyticity It was already asserted, at the end of Section 3.4, that the operator A defined by (3.4.6), that is, by (3.6.9), is the generator of an s.c. contraction semigroup e^{At} on $Y \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$, $\mathcal{A}$ as in (3.4.3), which, moreover, is analytic here for $t > 0$. This is a special case of a more general result [Chen, Triggiani, 1987, 1989(a)]; see Theorem 3B.1(b) of Appendix 3B, with $\alpha = 1/2$ of Chapter 3.

Finite Cost Condition (2.1.12) and Detectability Condition (2.1.13) Likewise, it was already asserted in Section 3.4 that e^{At} is uniformly (exponentially) stable on Y [Chen, Triggiani, 1989(a), Theorem 3B.1(d) of Appendix 3B of Chapter 3], and thus the Finite Cost Condition (2.1.12) and the Detectability Condition (2.1.13) of Chapter 2 are automatically satisfied with $u \equiv 0$, and $R = I$, respectively.

Conclusion: Case $T = \infty$ Theorems 2.2.1 and 2.2.2 of Chapter 2, with $R = I$, apply to problem (3.6.1) for any n. The Riccati operator P is Hilbert–Schmidt for $n = \dim \Omega = 1, 2, 3$; see Section 3.4.

Conclusion: Case $T < \infty$ Theorem 1.2.1.1 of Chapter 1 applies to problem (3.6.1) and (3.6.2) (for any n), as in the conclusion of Sections 3.4 or 3.5.

Remark 3.6.1 A generalization of the above analysis to the case where the control u acts as a genuine bending moment,

$$w|_\Sigma \equiv 0, \quad [\Delta w + (1-\mu)B_1 w]_\Sigma = u \quad \text{in } (0, T] \times \Gamma \equiv \Sigma, \tag{3.6.13}$$

can be given, with the help of Appendix 3C, which analyzes the boundary operator B_1, as well as the operator Δ^2 subject to the BC (3.6.13). A similar analysis applies if the Boundary Conditions (3.6.1c, d) are replaced by

$$\left.\frac{\partial w}{\partial \nu}\right|_\Sigma \equiv 0, \qquad \left.\frac{\partial \Delta w}{\partial \nu}\right|_\Sigma = u;$$

refer to Remark 3.4.3.

3.7 Another Platelike Equation with Point Control and Clamped BC

So far, we have considered examples of damped plates where the damping operator is equal to the αth power of the original elastic differential operator ($\alpha = 1/2$ in Sections 3.4 and 3.6 and $\alpha = 1$ in Section 3.5). This was due to the special choice of boundary conditions. In the next example, we return to the same Eqn. (3.4.1a), which we now complement with boundary conditions that make the damping operator only comparable, in a technical sense [Chen, Triggiani, 1987; 1989(a), Appendix 3B, to the αth power of the elastic operator ($\alpha = 1/2$). On some smooth Ω, $\dim \Omega = n \le 3$, consider the plate like equation with $\rho > 0$ any constant:

$$\begin{cases} w_{tt} + \Delta^2 w - \rho \Delta w_t = \delta(x - x^0)u(t) & \text{in } (0, T] \times \Omega = Q, \quad (3.7.1\text{a}) \\ w(0, \cdot) = w_0; \ w_t(0, \cdot) = w_1 & \text{in } \Omega, \quad (3.7.1\text{b}) \\ w|_\Sigma \equiv \left.\dfrac{\partial w}{\partial \nu}\right|_\Sigma \equiv 0 & \text{in } (0, T] \times \Gamma \equiv \Sigma. \quad (3.7.1\text{c}) \end{cases}$$

Regularity results for problem (3.7.1) are given in Triggiani [1991]. Consistently with these, we associate with (3.7.1) the same cost functional J as in (3.4.2):

$$J(u, w) = \int_0^T \left\{ \|w(t)\|^2_{H^2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} + |u(t)|^2 \right\} dt$$
$$+ \alpha \|\{w(T), w_t(T)\}\|^2_{H^2(\Omega) \times L_2(\Omega)}, \tag{3.7.2}$$

where $\{w_0, w_1\} \in H_0^2(\Omega) \times L_2(\Omega)$, and where $\alpha = 0$ if $T = \infty$ and $\alpha = 1$ if $T < \infty$.

Abstract Setting Now, however, we introduce the positive self-adjoint operator

$$\mathcal{A}h = \Delta^2 h, \quad \mathcal{D}(\mathcal{A}) = \left\{ h \in H^4(\Omega) : h|_\Gamma = \left.\frac{\partial h}{\partial \nu}\right|_\Gamma = 0 \right\}, \tag{3.7.3}$$

and the positive self-adjoint operator

$$\mathcal{B}h = -\Delta h, \quad \mathcal{D}(\mathcal{B}) = H^2(\Omega) \cap H_0^1(\Omega) \supset H_0^2(\Omega) = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}), \tag{3.7.4}$$

where the equality with $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ in (3.7.4) (equivalent norms) is standard [Grisvard, 1967, Appendix 3A]. Thus, problem (3.7.1) admits the second-order abstract version

$$w_{tt} + \mathcal{A}w + \rho\mathcal{B}w_t = \delta(x - x^0)u, \tag{3.7.5}$$

which fits the abstract model (1.1.1), (1.1.2) of Chapter 1 with

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\rho\mathcal{B} \end{bmatrix}, \quad Bu = \begin{bmatrix} 0 \\ \delta(x - x^0)u \end{bmatrix}, \quad R = I \tag{3.7.6}$$

on the spaces $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$, $u = \mathbb{R}^1$. From (3.7.3) and (3.7.4), we have [Grisvard, 1967, Appendix 3A]

$$\mathcal{D}(\mathcal{A}^{\frac{\alpha}{2}}) = \mathcal{D}(\mathcal{A}^{\frac{1}{4}}) = \mathcal{D}(\mathcal{B}^{\frac{1}{2}}) = H_0^1(\Omega), \quad \alpha = \frac{1}{2}. \tag{3.7.7}$$

Thus, in view of (3.7.7), [Chen, Triggiani, 1989(a)], or Theorem 3B.1(b) of Appendix 3B Chapter 3, applies with $\alpha = 1/2$: This, in particular, establishes that the operator A in (3.7.6) *generates an s.c., analytic semigroup* e^{At} *on* Y, with $\mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$.

Assumption (1.1.2) or (2.1.7): $(-A)^{-\gamma}B \in \mathcal{L}(U; Y)$ Again, it is immediate to see that this assumption holds true for $\gamma = 1$. In fact, as in the argument below (3.4.7), we find that

$$(-A)^{-1}Bu = \begin{bmatrix} \mathcal{A}^{-1}\mathcal{B} & \mathcal{A}^{-1} \\ -I & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \delta(x - x^0) \end{bmatrix} = \begin{bmatrix} \mathcal{A}^{-1}\delta(x - x^0) \\ 0 \end{bmatrix} \in Y. \tag{3.7.8}$$

In effect, *Assumption (1.1.2) of Chapter 1 holds true for problem (3.7.1) for all* γ *with* $n/4 < \gamma < 1$, $n \le 3$, exactly as in the case of problem (3.4.1). To see this, with $\mathcal{A}$ and $\mathcal{B}$ as in (3.7.3) and (3.7.4), we denote for convenience

$$A_{\mathcal{B}} = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\mathcal{B} \end{bmatrix}, \quad A_{\mathcal{B}}^* = \begin{bmatrix} 0 & -I \\ \mathcal{A} & -\mathcal{B} \end{bmatrix}, \tag{3.7.9}$$

$$A_{\frac{1}{2}} = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\mathcal{A}^{\frac{1}{2}} \end{bmatrix}, \quad A_{\frac{1}{2}}^* = \begin{bmatrix} 0 & -I \\ \mathcal{A} & -\mathcal{A}^{\frac{1}{2}} \end{bmatrix}, \tag{3.7.10}$$

where the adjoints are with respect to Y. Regarding the form of A_B^*, notice that if $[x_1, x_2] \in \mathcal{D}(A_B^*)$, then $x_2 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ [second component of Y], so that $\mathcal{B}x_2 = \mathcal{B}^* x_2$ by (3.7.4). We have

$$\mathcal{D}(A_B^*) = \mathcal{D}(A_B) = \mathcal{D}\big(A_{\frac{1}{2}}\big) = \mathcal{D}\big(A_{\frac{1}{2}}^*\big). \tag{3.7.11}$$

As a consequence of (3.7.7), we have in our present case [Chen, Triggiani, 1990(b), Remark 5.3, p. 291; Corollary 3B.4 of Appendix 3B of Chapter 3]

$$\mathcal{D}\big((-A_{\frac{1}{2}})^{\gamma+\epsilon}\big) \subset \mathcal{D}\big((-A_B^*)^{\gamma}\big) \subset \mathcal{D}\big((-A_{\frac{1}{2}})^{\gamma-\epsilon}\big), \quad 0 < \gamma < 1, \tag{3.7.12}$$

and $\gamma + \epsilon < 1$. Then, to obtain $(-A_{\mathcal{B}})^{-\gamma} B \in \mathcal{L}(U; Y)$, $n/4 < \gamma < 1$, as desired (i.e., $Bu \in [\mathcal{D}((-A_{\mathcal{B}}^*)^{\gamma})]'$), duality with respect to Y, it suffices via the right-hand side of (3.7.12) to have $Bu \in [\mathcal{D}((-A_{\frac{1}{2}})^{\gamma-\epsilon})]'$. But this was shown to be true in Section 3.4, precisely for $n/4 < \gamma - \epsilon < 1$, in the argument below (3.4.9).

Alternatively, we may write, using Section 3.4, that

$$(-A_{\mathcal{B}})^{-\gamma} B = (-A_{\mathcal{B}})^{-\gamma} \big(-A_{\frac{1}{2}}\big)^{\gamma-\epsilon} \big(-A_{\frac{1}{2}}\big)^{-(\gamma-\epsilon)} B \in \mathcal{L}(U; Y), \tag{3.7.13}$$

since $(-A_{\mathcal{B}})^{-\gamma}(-A_{\frac{1}{2}})^{\gamma-\epsilon}$ is bounded, because $(-A_{\frac{1}{2}}^*)^{\gamma-\epsilon}(-A_{\mathcal{B}}^*)^{-\gamma}$ is bounded by (3.7.12) and the closed graph theorem.

Assumption of Analyticity It was already noted below (3.7.7) that the operator A in (3.7.6) generates an s.c., analytic semigroup e^{At} on $Y, t > 0$, [Chen, Triggiani, 1989(a), $\alpha = 1/2$; Theorem 3B.1(b) of Appendix 3B of Chapter 3].

The Finite Cost Condition (2.1.12) and the Detectability Condition (2.1.13) These also hold true, since e^{At} is uniformly stable [Chen, Triggiani, 1989(a), Theorem 3B.1(d) of Appendix 3B of Chapter 3], and $R = I$.

Conclusion: Case $T = \infty$ Theorems 2.2.1 and 2.2.2 of Chapter 2 apply to problem (3.7.1), (3.7.2).

Conclusion: Case $T < \infty$ Theorem 1.2.1.1 of Chapter 1 applies to problem (3.7.1), (3.7.2) (with $n \leq 3$) as in the conclusion of Sections 3.4 and 3.5.

3.8 The Strongly Damped Wave Equation with Point Control and Dirichlet BC

On some smooth bounded domain $\Omega \subset R^n$ we consider the wave equation with a strong degree of damping and constant $\rho > 0$ (the case $\alpha = 1$ in Chen and Triggiani

[1987; 1989(a)], see Appendix 3B):

$$\begin{cases} w_{tt} - \Delta w - \rho \Delta w_t = \delta(x - x^0) u(t) & \text{in } (0, T] \times \Omega = Q, \quad (3.8.1a) \\ w(0, \cdot) = w_0;\ w_t(0, \cdot) = w_1 & \text{in } \Omega, \quad (3.8.1b) \\ w|_\Sigma \equiv 0 & \text{in } (0, T] \times \Gamma = \Sigma \quad (3.8.1c) \end{cases}$$

and point control acting at an interior point x^0. Problem (3.8.1) can be written abstractly as

$$w_{tt} + \mathcal{A} w + \rho \mathcal{A} w_t = \delta(\cdot - x^0) u(t), \tag{3.8.2}$$

that is, precisely as problem (3.5.1), except that now $\mathcal{A}$ is defined by

$$\mathcal{A} = -\Delta, \quad \mathcal{D}(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega). \tag{3.8.3}$$

Case 1 We begin by selecting

$$Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) = H_0^1(\Omega) \times L_2(\Omega), \quad U = R^1. \tag{3.8.4}$$

The cost functional for $T = \infty$ is then

$$J(u, w) = \int_0^\infty \left\{ \|w(t)\|^2_{H^1(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} + |u(t)|^2 \right\} dt. \tag{3.8.5}$$

Assumption (1.1.2) or (2.1.7): $(-A)^{-\gamma} B \in \mathcal{L}(U; Y)$ The case $\gamma = 1$ follows as in (3.5.8), since A and B are the same expressions as in (3.5.5):

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\rho \mathcal{A} \end{bmatrix}, \quad Bu = \begin{bmatrix} 0 \\ \delta(x - x^0) u \end{bmatrix}, \tag{3.8.6}$$

with $\mathcal{A}$ now given by (3.8.3). We require $\mathcal{A}^{-1}\delta \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$, that is, $\mathcal{A}^{-\frac{1}{2}}\delta \in L_2(\Omega)$, or $\delta \in H^{-1}(\Omega)$. And this is true only for dim $\Omega = n = 1$. Indeed, $\gamma = 1/2$ works as in (3.5.9), (3.5.10) and (3.5.11): We have $(-A)^{-\frac{1}{2}} Bu \in Y$ provided $\mathcal{A}^{-\frac{1}{2}}\delta \in L_2(\Omega)$, that is, $H_0^1(\Omega) \subset C(\bar{\Omega})$, for $n = 1$.

Assumption of Analyticity The operator A in (3.8.6) generates a contraction, analytic semigroup e^{At} on Y [Chen, Triggiani, 1989(a), see Theorem 3B.1(b) of Appendix 3B of Chapter 3].

To remove the limitation $n = 1$ of Case 1, we now consider a second case.

Case 2 We choose now the following spaces and cost functional for $T = \infty$:

$$Y = L_2(\Omega) \times L_2(\Omega), \quad U = R^1, \tag{3.8.7}$$

$$J(u, w) = \int_0^\infty \left\{ \|w(t)\|^2_{L_2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} + |u(t)|^2 \right\} dt. \tag{3.8.8}$$

Assumption (1.1.2) or (2.1.7): $(-A)^{-\gamma}B \in \mathcal{L}(U;Y)$ As before, using (3.5.8), we see that the case $\gamma = 1$ requires now that $\mathcal{A}^{-1}\delta \in \mathcal{L}_2(\Omega)$. Since $\mathcal{D}(A) \subset H^2(\Omega)$, this requirement is then satisfied provided $\delta \in [H^2(\Omega)]'$, which in turn is true provided $H^2(\Omega) \subset C(\bar{\Omega})$, that is, provided $2 > n/2$ or $n < 4$. Indeed, we shall show the *Claim*: $(-A)^{-\gamma}B \in \mathcal{L}(U;Y)$ *holds true for* $\gamma = 3/4 + \epsilon$ *and* $n \leq 3$. In fact, to this end, it suffices to show that

$$(-A)^{-1}Bu \in \mathcal{D}((-A)^{\theta}), \quad u \in R^1, \quad \theta < \frac{1}{4}, \quad n \leq 3, \tag{3.8.9}$$

where $\theta = 1 - \gamma < 1/4$. Indeed, we have that

$$\mathcal{D}((-A)^{\theta}) = \{x, y \in L_2(\Omega) : x + \rho y \in \mathcal{D}(\mathcal{A}^{\theta})\}, \quad 0 \leq \theta \leq 1; \tag{3.8.10}$$

see Chen and Triggiani [1990(b)], Remark 3.5.4, pp. 291–2]; see also Triggiani [1991, Eqn. (3.2.51), p. 314] or Theorem 3B.4(b) of Appendix 3B of Chapter 3. Consequently, recalling (3.5.8) with A in (3.8.6) of the same form as in (3.5.5), we see that condition (3.8.9) holds true if and only if $\mathcal{A}^{-1}\delta \in \mathcal{D}(\mathcal{A}^{\theta})$, or $\mathcal{A}^{-(1-\theta)}\delta \in L_2(\Omega)$. Since $\mathcal{D}(\mathcal{A}^{1-\theta}) \subset H^{2-2\theta}(\Omega)$, we see that this latter condition holds true in case $\delta \in [H^{2-2\theta}(\Omega)]'$, that is, provided $H^{2-2\theta}(\Omega) \subset C(\bar{\Omega})$ (i.e., provided $2 - 2\theta > n/2$). For $\theta < 1/4$ we have $4 - 4\theta > 3 \geq n$, as desired.

Assumption of Analyticity The operator A generates an s.c., analytic semigroup e^{At} on Y (not contraction now, however), since, as one sees readily, $\|R(\lambda, A)\| \leq C/|\lambda|$ for $\operatorname{Re}\lambda > 0$ in the norm of $\mathcal{L}(Y)$ [Bucci, 1992; Triggiani, 1991, p. 314], or Theorem 3B.6(a) of Appendix 3B of Chapter 3]. Same comments as below (3.5.5) apply.

Finite Cost Condition (2.1.12) and Detectability Condition (2.1.13) The semigroup e^{At} is uniformly stable with either choice of Y, (3.8.4) or (3.8.7) [Theorem 3B.1(d) of Appendix 3B of Chapter 3], and thus the required conditions are satisfied with $u \equiv 0$, or $R = I$, respectively.

Conclusion: Case $T = \infty$ Theorems 2.2.1 and 2.2.2 of Chapter 2 apply to the cost (3.8.5) for $n = 1$ and to the cost (3.8.8) for $n \leq 3$.

Conclusion: Case $T < \infty$ Theorem 1.2.1.1 of Chapter 1 applies with Y as in (3.8.4) for $n = 1$, and to Y as in (3.8.7) for $n \leq 3$; see the conclusion of Sections 3.4, 3.5, and 3.7.

3.9 A Structurally Damped Kirchhoff Equation with Point Control Acting through $\delta(\cdot - x^0)$ and Simplified Hinged BC

On some smooth bounded domain Ω of R^n, $n \leq 3$, we consider the following Kirchhoff equation with a strong degree of damping in the deflection $w(t, x)$ and constants

$k > 0$ and $\rho > 0$:

$$\begin{cases} w_{tt} - k\Delta w_{tt} + \Delta^2 w + \rho\Delta^2 w_t = \delta(x - x^0)u(t) & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0, T] \times \Omega \equiv Q, & (3.9.1a) \\ w(0, \cdot) = w_0; \ w_t(0, \cdot) = w_1 \quad \text{in } \Omega, & (3.9.1b) \\ w|_\Sigma \equiv \Delta w|_\Sigma \equiv 0 \qquad\qquad \text{in } (0, T] \times \Gamma \equiv \Sigma & (3.9.1c) \end{cases}$$

and point control acting at an interior point x^0 of Ω. We take $\{w_0, w_1\} \in [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$ and associate with problem (3.9.1) the cost functional (which represents the typical energy functional of a Kirchhoff equation):

$$J(w, u) = \int_0^T \left\{\|\Delta w(t)\|^2_{L_2(\Omega)} + \||\nabla w_t(t)|\|^2_{L_2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} + |u(t)|^2\right\} dt, \tag{3.9.2}$$

where $T = \infty$ or $T < \infty$.

Abstract Setting We recall the positive self-adjoint operator $\mathcal{A}$ in (3.4.3) and its relationship to the operator A_D defined in (3.1.8), expressed by (3.4.4):

$$\mathcal{A}h = \Delta^2 h; \quad \mathcal{D}(\mathcal{A}) = \left\{h \in H^4(\Omega) : h|_\Gamma = \Delta h|_\Gamma = 0\right\}; \tag{3.9.3}$$

$$\mathcal{A}^{\frac{1}{2}} = A_D. \tag{3.9.4}$$

According to (3.9.3) and (3.9.4), problem (3.9.1) admits the following abstract second-order version:

$$w_{tt} + k\mathcal{A}^{\frac{1}{2}} w_{tt} + \mathcal{A}w + \rho\mathcal{A}w_t = \delta(\cdot - x^0)u(t), \tag{3.9.5}$$

or

$$w_{tt} + \mathbb{A}w + \rho\mathbb{A}w_t = \left(I + k\mathcal{A}^{\frac{1}{2}}\right)^{-1}\delta(\cdot - x^0)u(t), \tag{3.9.6}$$

where we have set

$$\mathbb{A} = \left(I + k\mathcal{A}^{\frac{1}{2}}\right)^{-1}\mathcal{A}, \quad \mathcal{D}(\mathbb{A}) = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega) \cap H_0^1(\Omega) \tag{3.9.7}$$

which is a positive, self-adjoint operator on the space $\mathcal{D}(\mathcal{A}_k^{\frac{1}{4}})$, endowed with the following inner product:

$$(x_1, x_2)_{\mathcal{D}(\mathcal{A}_k^{\frac{1}{4}})} = \left((I + k\mathcal{A}^{\frac{1}{2}})x_1, x_2\right)_{L_2(\Omega)}, \quad x_1, x_2 \in \mathcal{D}\left(\mathcal{A}^{\frac{1}{4}}\right) \equiv H_0^1(\Omega). \tag{3.9.8}$$

We note that [Grisvard, 1967, Appendix 3A],

$$\mathcal{D}\left(\mathcal{A}^{\frac{1}{4}}\right) = H_0^1(\Omega) \quad \text{and} \quad \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right) = H^2(\Omega) \cap H_0^1(\Omega), \tag{3.9.9}$$

where, for $g \in H_0^1(\Omega)$,

$$\|g\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{4}})} = \|\mathcal{A}^{\frac{1}{4}} g\|_{L_2(\Omega)} = \left\{ \int_\Omega |\nabla g|^2 \, d\Omega \right\}^{\frac{1}{2}}, \tag{3.9.10a}$$

which is equivalent to the norm $\|g\|_{H_0^1(\Omega)}$, which is in turn equivalent to the norm corresponding to (3.9.8):

$$\begin{aligned} \|g\|_{\mathcal{D}(\mathcal{A}_k^{\frac{1}{4}})} &= \left\{ \|g\|^2_{L_2(\Omega)} + k\|\mathcal{A}^{\frac{1}{4}} g\|^2_{L_2(\Omega)} \right\}^{\frac{1}{2}} \\ &= \left\{ \int_\Omega \left[|g|^2 + k|\nabla g|^2 \right] d\Omega \right\}^{\frac{1}{2}}. \end{aligned} \tag{3.9.10b}$$

With reference to (3.9.6), problem (3.9.1) can then be put into the abstract model (1.1.1), (1.1.2) of Chapter 1, by introducing the operators A, B, and R,

$$A = \begin{bmatrix} 0 & I \\ -\mathbb{A} & -\rho\mathbb{A} \end{bmatrix}, \quad Bu = \begin{bmatrix} 0 \\ (I + k\mathcal{A}^{\frac{1}{2}})^{-1} \delta(\cdot - x^0) u \end{bmatrix}, \quad R = I, \tag{3.9.11}$$

and selecting the spaces $U = R^1$ and

$$\begin{aligned} Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{4}}) &= \left[H^2(\Omega) \cap H_0^1(\Omega) \right] \times H_0^1(\Omega) && (3.9.12a) \\ &= \mathcal{D}(\mathbb{A}) \times \mathcal{D}(\mathbb{A}^{\frac{1}{2}}). && (3.9.12b) \end{aligned}$$

Assumption (1.1.2) or (2.1.7): $(-A)^{-\gamma} B \in \mathcal{L}(U; Y)$ We shall show below that: *this assumption with* $\gamma < 1$ *holds true, provided* $3 > 1 + 2\gamma > n/2$*, and* $n \le 3$*, in which case* $\gamma > n/4 - 1/2$. Thus, for $n = 1, 2, 3$, we may take $\gamma = 0$, $\gamma = \epsilon$, $\gamma = 1/4 + \epsilon$, respectively. We substantiate our claim through the following steps:

Step 1 Starting from (3.9.11), we first compute [as in (3.5.8)]:

$$\begin{aligned} (-A)^{-1} Bu &= \begin{bmatrix} \rho I & \mathbb{A}^{-1} \\ -I & 0 \end{bmatrix} \begin{bmatrix} 0 \\ (I + k\mathcal{A}^{\frac{1}{2}})^{-1} \delta(\cdot - x^0) u \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{A}^{-1} \delta(\cdot - x^0) u \\ 0 \end{bmatrix}, \end{aligned} \tag{3.9.13}$$

after a cancellation, recalling (3.9.7).

Step 2 We rewrite Y in (3.9.12) as an energy space as follows:

$$Y = \mathcal{D}(\bar{\mathbb{A}}^{\frac{1}{2}}) \times H, \quad H = \mathcal{D}(\mathbb{A}^{\frac{1}{2}}), \tag{3.9.14}$$

where $\bar{\mathbb{A}}$ denotes the operator $\mathbb{A}$ lifted as acting on the space $H = \mathcal{D}(\mathbb{A}^{\frac{1}{2}})$: $\bar{\mathbb{A}}^{\frac{1}{2}}$: $\mathcal{D}(\mathbb{A}) \to \mathcal{D}(\mathbb{A}^{\frac{1}{2}})$.

Step 3 From [Chen, Triggiani, 1990(b), Theorem 1.1, p. 280, with $\alpha = 1$ and $1/2 \le \theta \le 1$] or Theorem 3B.2(ii) of Appendix 3B of Chapter 3, we obtain via (3.9.14) that, for $1/2 \le \theta \le 1$,

$$\begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathcal{D}((-A)^{\theta}) \Longleftrightarrow x \in \mathcal{D}(\bar{\mathbb{A}}^{\theta}) = \mathcal{D}(\mathbb{A}^{\theta+\frac{1}{2}}) = \mathcal{D}(\mathcal{A}^{\frac{\theta}{2}+\frac{1}{4}}) \tag{3.9.15}$$

by the definition of $\bar{\mathbb{A}}$ and (3.9.7). In fact, Theorem 3B.2(ii) of Appendix 3B, Chapter 3 yields the requirements $x \in \mathcal{D}(\bar{A}^{\frac{1}{2}})$ as well as $x + 0 \in \mathcal{D}(\bar{A}^{\theta})$.

Step 4 From (3.9.13) we have that

$$(-A)^{-\gamma} Bu = (-A)^{1-\gamma}(-A)^{-1} Bu = (-A)^{1-\gamma} \begin{bmatrix} \mathcal{A}^{-1}\delta(\cdot - x^0)u \\ 0 \end{bmatrix} \in Y, \tag{3.9.16}$$

if and only if, via (3.9.15) with $\theta = 1 - \gamma \ge 1/2$, or $\gamma \le 1/2$,

$$\mathcal{A}^{-1}\delta(\cdot - x^0) \in \mathcal{D}(\mathcal{A}^{\frac{1-\gamma}{2}+\frac{1}{4}}) \Longleftrightarrow \delta(\cdot - x^0) \in [\mathcal{D}(\mathcal{A}^{\frac{1}{4}+\frac{\gamma}{2}})]' \tag{3.9.17}$$

duality with respect to $L_2(\Omega)$. Since $\mathcal{D}(\mathcal{A}^{\frac{1}{4}+\gamma/2}) \subset H^{1+2\gamma}(\Omega)$ for the fourth-order operator $\mathcal{A}$ in (3.9.3), we see that (3.9.17) holds true provided $H^{1+2\gamma}(\Omega) \subset C(\bar{\Omega})$, that is, provided $1 + 2\gamma > n/2$ (and $\gamma \le 1/2$), by Sobolev embedding, as desired, and our claim is proved. The restriction $\gamma \le 1/2$ limits n to $n \le 3$.

Assumption of Analyticity The operator A in (3.9.11) generates an s.c. semigroup e^{At} on Y, which is, moreover, analytic here for $t > 0$ [Chen, Triggiani, 1989(a), Theorem 3B.1(b) of Appendix 3B of Chapter 3, with $\alpha = 1$]. Same comments as below (3.5.5) apply.

Finite Cost Condition (2.1.12) and Detectability Condition (2.1.13) The s.c. analytic semigroup e^{At} is uniformly stable on Y [Chen, Triggiani, 1989(a), Theorem 3B.1(d)], and thus these two conditions are satisfied with $u \equiv 0$ and $R = I$, respectively.

Conclusion: Case $T = \infty$ Theorems 2.2.1 and 2.2.2 of Chapter 2 apply to problem (3.9.1) and (3.9.2) for $n \le 3$.

Conclusion: Case $T < \infty$ Theorem 1.2.1.1 of Chapter 1 applies to problem (3.9.1) and (3.9.2), for $n \le 3$.

Remark 3.9.1 When $\dim \Omega = 2$, the analysis of the present section can be extended to the case of hinged BC as in (3.4.13), rather than (3.9.1c), by use of Appendix 3C and Appendix 3D (Application # 2) of Chapter 3.

3.10 A Structurally Damped Kirchhoff Equation (Revisited) with Point Control Acting through $\delta'(\cdot - x^0)$ and Simplified Hinged BC

In this section we consider a modification of problem (3.9.1), in the sense that the control $u(t)$ now acts through the derivative of the Dirac measure δ concentrated at an interior point x^0, and where $k > 0$, $\rho > 0$ are constants as before:

$$\begin{cases} w_{tt} - k\Delta w_{tt} + \Delta^2 w + \rho\Delta^2 w_t = \delta'(x - x^0)u(t) & \text{in } Q, \quad (3.10.1a) \\ w(0, \cdot) = w_0; \; w_t(0, \cdot) = w_1 & \text{in } \Omega, \quad (3.10.1b) \\ w|_\Sigma = \Delta w|_\Sigma \equiv 0 & \text{in } \Sigma. \quad (3.10.1c) \end{cases}$$

Case 1 We begin by selecting the same space Y as in (3.9.12) and (3.9.14):

$$Y = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega) = \mathcal{D}(\bar{\mathbb{A}}^{\frac{1}{2}}) \times H, \quad H = \mathcal{D}(\mathbb{A}^{\frac{1}{2}}), \tag{3.10.2}$$

and hence the same cost functional as in (3.9.2):

$$J(w, u) = \int_0^T \left\{ \|\Delta w(t)\|^2_{L_2(\Omega)} + k\||\nabla w_t(t)|\|^2_{L_2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} + |u(t)|^2 \right\} dt. \tag{3.10.3}$$

The operator A is the same as the one in (3.9.11), whereas B is now

$$A = \begin{bmatrix} 0 & I \\ -\mathbb{A} & -\rho\mathbb{A} \end{bmatrix}; \quad Bu = \begin{bmatrix} 0 \\ \left(I + k\mathcal{A}^{\frac{1}{2}}\right)^{-1} \delta'(\cdot - x^0)u \end{bmatrix}, \tag{3.10.4}$$

and then problem (3.10.1) fits into model (1.1.1) of Chapter 1.

Assumption (1.1.2) or (2.1.7): $(-A)^{-\gamma}B \in \mathcal{L}(U; Y)$ We shall show that this *assumption with* $\gamma < 1$ *holds true, provided* $1/2 \geq \gamma > n/4$, that is, for $n = 1$, in which case we can take $\gamma = 1/4 + \epsilon$.

Indeed, as in the argument of Section 3.9, from (3.9.13) to (3.9.17), in view of the present B given by (3.10.4), we have that

$$(-A)^{-\gamma}Bu = (-A)^{1-\gamma} \begin{bmatrix} \mathcal{A}^{-1}\delta'(\cdot - x^0)u \\ 0 \end{bmatrix} \in Y, \tag{3.10.5}$$

if and only if, via (3.9.15) with $\theta = 1 - \gamma \geq 1/2$, or $\gamma \leq 1/2$,

$$\mathcal{A}^{-1}\delta'(\cdot - x^0) \in \mathcal{D}(\mathcal{A}^{\frac{1-\gamma}{2}+\frac{1}{4}}) \iff \delta'(\cdot - x^0) \in [\mathcal{D}(\mathcal{A}^{\frac{1}{4}+\frac{\gamma}{2}})]' \tag{3.10.6}$$

where the indicated duality is with respect to $L_2(\Omega)$. (3.10.6) is a counterpart of (3.9.17). Since $\mathcal{D}(\mathcal{A}^{\frac{1}{4}+\gamma/2}) \subset H^{1+2\gamma}(\Omega)$, we see that (3.10.6) holds true provided $H^{1+2\gamma}(\Omega) \subset C^1(\bar{\Omega})$, that is, provided $1 + 2\gamma > n/2 + 1$ (and $\gamma \leq 1/2$) by Sobolev embedding as desired, or $1/2 \geq \gamma > n/4$, which yields only $n = 1$, in which case we can take $\gamma = 1/4 + \epsilon$.

Assumptions of Analyticity, Finite Cost Condition, and Detectability The operator A in (3.10.4) generates a s.c. analytic semigroup e^{At}, $t > 0$ on Y, defined by (3.10.2), which is uniformly stable here, as noted at the end of Section 3.9. So, the Finite Cost Condition (2.1.12) and the Detectability Condition (2.1.13) of Chapter 2 hold true for problems (3.10.1) and (3.10.2), with $R = I$.

To remove the limitation $n = 1$ of Case 1, we now consider (as in Section 3.8) a second alternative.

Case 2 We choose the following space Y and corresponding cost functional:

$$Y = H \times H, \qquad H = \mathcal{D}(\mathbb{A}^{\frac{1}{2}}) = \mathcal{D}(\mathcal{A}^{\frac{1}{4}}) = H_0^1(\Omega); \tag{3.10.7}$$

$$J(u, w) = \int_0^T \left\{ |||\nabla w(t)|||^2_{L_2(\Omega)} + |||\nabla w_t(t)|||^2_{L_2(\Omega)} + |u(t)|^2 \right\} dt. \tag{3.10.8}$$

Assumption (1.1.2) or (2.1.7): $(-A)^{-\gamma}B \in \mathcal{L}(U; Y)$ With the choice of Y as in (3.10.7), hence of J as in (3.10.8), we shall *now show that this assumption with* $\gamma < 1$ *holds true, provided* $1 > \gamma > n/4$, that is, $n \leq 3$, in which case we can take $\gamma = n/4 + \epsilon$. Indeed, now, with Y as in (3.10.7), condition (3.10.5) holds true (with A viewed on $H \times H$) provided

$$\mathcal{A}^{-1}\delta'(\cdot - x^0) \in \mathcal{D}(\tilde{\mathbb{A}}^{\theta}) = \mathcal{D}(\mathbb{A}^{\theta+\frac{1}{2}}) = \mathcal{D}(\mathcal{A}^{\frac{\theta}{2}+\frac{1}{4}}), \quad 0 < \theta \leq 1, \tag{3.10.9}$$

with $0 < \theta = 1 - \gamma < 1$ (see [Chen, Triggiani, 1990(b), Eqn. (3.5.17), p. 292] or Theorem 3B.6(b) of Appendix 3B [i.e., this time, for $0 \leq \gamma < 1$, without the limitation $\theta = 1 - \gamma \geq 1/2$ as in (3.9.17) and (3.10.6)]. The same argument below (3.10.6) then yields that (3.10.9) is indeed the case provided $1 + 2\gamma > n/2 + 1$ or $1 > \gamma > n/4$ [this time with no limitation $\gamma \leq 1/2$ as before]. Thus, $n \leq 3$ is allowed and then γ can be taken to be $1 > \gamma = n/4 + \epsilon$, as claimed.

Assumptions of Analyticity, Finite Cost Condition, and Detectability It was already asserted at the end of Section 3.8, Case 2, as well as in Theorem 3B.6(a) of Appendix 3B, that an operator A such as the one in (3.10.4) is the generator of a s.c. analytic semigroup $e^{At}, t > 0$ on $Y = H \times H$ [see (3.10.7)] [since $\mathbb{A}$ is positive, self-adjoint on $H = \mathcal{D}(\mathbb{A}^{\frac{1}{2}})$], which, moreover, is uniformly (exponentially) stable on $Y = H \times H$. Thus, the Finite Cost Condition (2.1.12) and the Detectability Condition (2.1.13) of Chapter 2 hold true for problems (3.10.1) and (3.10.8) on Y as in (3.10.7), with $R = I$.

Conclusion: Case $T = \infty$ Theorems 2.2.1 and 2.2.2 of Chapter 2 apply to problems (3.10.1) and (3.10.3) on the space Y as in (3.10.2) for $n = 1$ and to problems (3.10.1) and (3.10.8) on the space Y as in (3.10.7) for $n = 1, 2, 3$.

Conclusion: Case $T < \infty$ Theorems 1.2.1.1 of Chapter 1 applies to problems (3.10.1) and (3.10.3) on the space Y as in (3.10.2) for $n = 1$ and to problems (3.10.1) and (3.10.8) on the space Y as in (3.10.7) for $n = 1, 2, 3$.

3.11 Thermo-Elastic Plates with Thermal Control and Homogeneous Clamped Mechanical BC

In the present section, we shall consider a (linear) thermo-elastic plate with boundary control exercised as thermal control, either of the temperature (Dirichlet control) or else of the flux of the temperature (Neumann or Robin control). The case of a thermo-elastic plate with mechanical control acting as a bending moment applied on the edges of the plate is treated in Section 3.12 (hinged BC). Finally, Section 3.13 treats a thermo-elastic plate with mechanical control as a shear force (free BC). The models are two dimensional, and so are the mathematical models (boundary operators) in Sections 3.12 and 3.13. Instead, for the models of Section 3.11, the analysis works for any dimension. Thermo-elastic plates are examples of models governed by a system of PDEs with prescribed transmission conditions. More precisely, in Sections 3.11–3.13, the PDE system couples a parabolic dynamics with an Euler–Bernoulli dynamics (Petrowski-type equation with infinite speed of propagation). The overall effect of the coupling between these two dynamics is that the parabolic component prevails on the combined system, in the sense that the overall *free* system generates a s.c. analytic semigroup.

Other examples of thermo-elastic plates accounting for rotational forces, which therefore couple a Kirchoff equation (hyperbolic, with finite speed of propagation) with a heat equation where the free dynamics generates a s.c. semigroup, which, however, is not analytic (see Appendix 3J of Chapter 3, Remark 3J.1), will be considered in Appendix 3J regarding the stability issue, as well as in Volume II.

3.11.1 Thermal Control in the Dirichlet BC

Let Ω be a two-dimensional bounded open region (of $\mathbb{R}^2$) with smooth boundary Γ. It is assumed that, in addition to mechanical strains and stresses, the dynamics of the plate is affected by thermal stresses, which result from variations of the temperature. The following PDE model of a thermo-elastic plate (without accounting for rotational forces) is taken from Lagnese [1989] in the variables $w(t,x)$ (vertical displacement) and $\theta(t,x)$ (temperature):

$$\begin{cases} w_{tt} + \Delta^2 w + \alpha\Delta\theta = 0 & \text{in } (0,T]\times\Omega \equiv Q, & (3.11.1.1\text{a}) \\ \theta_t - \eta\Delta\theta + \sigma\theta - \alpha\Delta w_t = 0 & \text{in } Q, & (3.11.1.1\text{b}) \\ w(0,\,\cdot\,) = w_0,\ w_t(0,\,\cdot\,) = w_1, & & \\ \quad \theta(0,\,\cdot\,) = \theta_0 & \text{in } \Omega; & (3.11.1.1\text{c}) \\ w \equiv \dfrac{\partial w}{\partial \nu} \equiv 0 \text{ (clamped BC)} & \text{on } (0,T]\times\Gamma \equiv \Sigma; & (3.11.1.1\text{d}) \\ \theta \equiv u & \text{on } \Sigma. & (3.11.1.1\text{e}) \end{cases}$$

[The case where Eqn. (3.11.1.1a) also contains the term $-k\Delta w_{tt}$ (where $k > 0$ is a constant accounting for the rotational forces) that is, the case where the w equation is a Kirchoff equation, will be treated in Volume II; see also Appendix 3J of Chapter 3.]

All constants α, η, σ are positive. For their physical interpretation, we refer to Lagnese [1989]. Problem (3.11.1.1) is affected by the Dirichlet-boundary control $u \in L_2(0, T; L_2(\Gamma)) \equiv L_2(\Sigma)$ acting on the temperature θ (thermal control). We notice that the coupling between the w equation (3.11.1.1a) and the θ equation (3.11.1.1b) is strong, in the sense that: (i) the coupling term Δw_t in (3.11.1.1b) has anisotropic order equal to 4 (for the Euler–Bernoulli equation, one time derivative has order two), the same order of the principal part of the w equation (3.11.1.1a), and (ii) the coupling term $\Delta\theta$ in (3.11.1.1a) is the same as the principal part of the θ equation (3.11.1.1b). Consistently with the regularity theory (Appendices 3E, 3F, 3G), we select the following cost functional to be minimized:

$$J(u, w, \theta) = \int_0^T \left\{ \|w(t)\|^2_{H^2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} + \|\theta(t)\|^2_{L_2(\Omega)} + \|u(t)\|^2_{L_2(\Gamma)} \right\} dt$$
$$+ a\|\{w(T), w_t(T), \theta(T)\}\|^2_{H^2(\Omega)\times L_2(\Omega)\times L_2(\Omega)}, \tag{3.11.1.2}$$

or, equivalently, as we shall see below (3.11.1.9),

$$J(u, w, \theta) = \int_0^T \left\{ \|\Delta w(t)\|^2_{L_2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} + \|\theta(t)\|^2_{L_2(\Omega)} + \|u(t)\|^2_{L_2(\Gamma)} \right\} dt$$
$$+ a\|\{\Delta w(T), w_t(T), \theta(T)\}\|^2_{[L_2(\Omega)]^3}, \tag{3.11.1.3}$$

where, in both cases, $\{w_0, w_1, \theta_0\} \in H_0^2(\Omega) \times L_2(\Omega) \times L_2(\Omega)$; moreover, $a = 0$ if $T = \infty$ and $a = 1$ if $T < \infty$.

Abstract Setting To put problem (3.11.1.1), (3.11.1.2) [or (3.11.1.3)] into the abstract setting (1.1.1) and (1.1.2) of the preceding Chapters 1 and 2, we introduce the following operators and spaces. First, we define the following two strictly positive definite self-adjoint operators:

$$\mathcal{A}h = \Delta^2 h; \quad \mathcal{D}(\mathcal{A}) = \left\{ h \in H^4(\Omega) : h|_\Gamma = \left.\frac{\partial h}{\partial \nu}\right|_\Gamma = 0 \right\} = H^4(\Omega) \cap H_0^2(\Omega); \tag{3.11.1.4}$$

$$\mathcal{A}_D h = -\Delta h; \quad \mathcal{D}(\mathcal{A}_D) = \left\{ h \in H^2(\Omega) : h|_\Gamma = 0 \right\} = H^2(\Omega) \cap H_0^1(\Omega), \tag{3.11.1.5}$$

on $L_2(\Omega)$ [same $\mathcal{A}_D$ as in (3.1.8) of Section 3.1]. From (3.11.1.4), we have [Grisvard, 1967; see Appendix 3A of Chapter 3],

$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = \left\{ h \in H^2(\Omega) : h|_\Gamma = \left.\frac{\partial h}{\partial \nu}\right|_\Gamma = 0 \right\} = H_0^2(\Omega) \quad \text{(equivalent norms).} \tag{3.11.1.6}$$

Let D (Dirichlet map) be the harmonic extension of a boundary datum (recall (3.1.6) and (3.1.7)):

$$h \equiv Dg \iff \{\Delta h = 0 \text{ in } \Omega, h|_\Gamma = g \text{ on } \Gamma\}; \tag{3.11.1.7a}$$

$$D: \text{ continuous } L_2(\Gamma) \to H^{\frac{1}{2}}(\Omega) \subset H^{\frac{1}{2}-2\epsilon}(\Omega) = \mathcal{D}\big(\mathcal{A}_D^{\frac{1}{4}-\epsilon}\big), \quad 0 < \epsilon < \tfrac{1}{4}. \tag{3.11.1.7b}$$

From (3.11.1.5) and (3.11.1.6), we obtain via the closed graph theorem, and self-adjointness of $\mathcal{A}$ and $\mathcal{A}_D$,

$$\begin{aligned}\mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \subset \mathcal{D}(\mathcal{A}_D) &\Leftrightarrow \mathcal{A}_D\mathcal{A}^{-\frac{1}{2}} \in \mathcal{L}(L_2(\Omega))\\ &\Leftrightarrow \mathcal{A}^{-\frac{1}{2}}\mathcal{A}_D \quad \text{has a bounded extension on } \mathcal{L}(L_2(\Omega)).\end{aligned} \tag{3.11.1.8}$$

Accordingly, if $w \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$, then Green's second theorem yields via (3.11.1.4) and (3.11.1.6):

$$\begin{aligned}\|w\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} &= \big\|\mathcal{A}^{\frac{1}{2}}w\big\|^2_{L_2(\Omega)} = (\mathcal{A}w, w)_{L_2(\Omega)} = \int_\Omega \Delta^2 w\, w\, d\Omega\\ &= \int_\Omega |\Delta w|^2 d\Omega + \int_\Gamma \frac{\partial \Delta w}{\partial \nu}\cancel{w}\, d\Gamma - \int_\Gamma \Delta w \cancel{\frac{\partial w}{\partial \nu}}\, d\Gamma\\ &= \|\Delta w\|^2_{L_2(\Omega)}, \quad \text{which is equivalent to } \|w\|^2_{H_0^2(\Omega)},\end{aligned} \tag{3.11.1.9}$$

where $w \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H_0^2(\Omega)$ implies $\Delta^2 w \in H^{-2}(\Omega)$ so that the integral over Ω in the top line is well defined. [Alternatively, first obtain (3.11.1.9) for $w \in \mathcal{D}(\mathcal{A})$ and then extend to $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$.] Thus, by (3.11.1.9), we have that the cost functionals in (3.11.1.2) and (3.11.1.3) are *equivalent*, as it was anticipated just below (3.11.1.2), for the solutions of problem (3.11.1.1), due to the BC (3.11.1.1d). Accordingly, we take the following spaces, Y, U, Z, Z_f, and operators R, G:

$$Y \equiv H_0^2(\Omega) \times L_2(\Omega) \times L_2(\Omega) = \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times L_2(\Omega) \times L_2(\Omega) \quad \text{(equivalent norms);} \tag{3.11.1.10}$$

$$U = L_2(\Gamma); \quad Z = Z_f = Y; \quad R = G = I \quad \text{for } J \text{ as in (3.11.1.2) or (3.11.1.3).} \tag{3.11.1.11}$$

Using the definition of D in (3.11.1.7a), we rewrite (3.11.1.1) as

$$\begin{cases} w_{tt} + \Delta^2 w + \alpha\Delta(\theta - Du) = 0 & \text{in } Q, & (3.11.1.12a)\\ \theta_t - \eta\Delta(\theta - Du) + \sigma\theta - \alpha\Delta w_t = 0 & \text{in } Q, & (3.11.1.12b)\\ w \equiv \dfrac{\partial w}{\partial \nu} \equiv 0 & \text{in } \Sigma, & (3.11.1.12c)\\ [\theta - Du]_\Sigma \equiv 0 & \text{in } \Sigma. & (3.11.1.12d)\end{cases}$$

Recalling (3.11.1.4) and (3.11.1.5), we rewrite (3.11.1.12) abstractly as

$$\begin{cases} w_{tt} + \mathcal{A}w - \alpha\mathcal{A}_D(\theta - Du) = 0, & (3.11.1.13)\\ \theta_t + \eta\mathcal{A}_D(\theta - Du) + \sigma\theta + \alpha\mathcal{A}_D w_t = 0. & (3.11.1.14)\end{cases}$$

Finally, extending the original $\mathcal{A}_D$ to $\mathcal{A}_D : L_2(\Omega) \to [\mathcal{D}(\mathcal{A}_D^*)]' = [\mathcal{D}(\mathcal{A}_D)]'$ by isomorphism, where the duality is with respect to $L_2(\Omega)$ as a pivot space, we rewrite (3.11.1.13), (3.11.1.14) as the following second-order system:

$$\begin{cases} w_{tt} + \mathcal{A}w - \alpha\mathcal{A}_D\theta = -\alpha\mathcal{A}_D Du \in [\mathcal{D}(\mathcal{A}_D)]', & (3.11.1.15) \\ \theta_t + \eta\mathcal{A}_D\theta + \sigma\theta + \alpha\mathcal{A}_D w_t = \eta\mathcal{A}_D Du \in [\mathcal{D}(\mathcal{A}_D)]'. & (3.11.1.16) \end{cases}$$

Setting $y = [w, w_t, \theta]$, we thus rewrite the above second-order system as the following first-order equation $\dot{y} = Ay + Bu$, as in Eqn. (1.1.1) of Chapter 1:

$$\frac{d}{dt}\begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix} = A\begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix} + Bu, \quad y = \begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix}, \tag{3.11.1.17}$$

$$A = \begin{bmatrix} 0 & I & 0 \\ -\mathcal{A} & 0 & \alpha\mathcal{A}_D \\ 0 & -\alpha\mathcal{A}_D & -(\eta\mathcal{A}_D + \sigma I) \end{bmatrix} : Y \supset \mathcal{D}(A) \to Y, \tag{3.11.1.18}$$

$$\mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right) \times \mathcal{D}(\mathcal{A}_D), \tag{3.11.1.19}$$

where, in identifying the second component of $\mathcal{D}(A)$, we have recalled $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset \mathcal{D}(\mathcal{A}_D)$ from (3.11.1.8) and $w_t \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ by the BC (3.11.1.1d); moreover,

$$Bu = \begin{bmatrix} 0 \\ -\alpha\mathcal{A}_D Du \\ \eta\mathcal{A}_D Du \end{bmatrix}, \quad u \in U; \quad B : \text{ continuous } U \to [\mathcal{D}(A^*)]', \tag{3.11.1.20}$$

where the continuity asserted in (3.11.1.20), with duality with respect to Y as a pivot space, will be a fortiori proved after (3.11.2.25). We readily find that the Y-adjoint of A, with $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega)$, is

$$A^* = \begin{bmatrix} 0 & -I & 0 \\ \mathcal{A} & 0 & -\alpha\mathcal{A}_D \\ 0 & \alpha\mathcal{A}_D & -(\eta\mathcal{A}_D + \sigma I) \end{bmatrix} : Y \supset \mathcal{D}(A^*) = \mathcal{D}(A) \to Y. \tag{3.11.1.21}$$

It is readily seen from (3.11.1.18) and (3.11.1.21) that: *Both A and A^* are dissipative* on $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega)$: For $x = [x_1, x_2, x_3] \in \mathcal{D}(A) = \mathcal{D}(A^*)$, we have

$$\mathrm{Re}(Ax, x)_Y = \mathrm{Re}(A^*x, x)_Y = -([\eta\mathcal{A}_D + \sigma I]x_3, x_3)_{L_2(\Omega)} \leq 0. \tag{3.11.1.22}$$

Indeed, by (3.11.1.22), the operator obtained from A (or A^*) by omitting the corner entry $-(\eta\mathcal{A}_D + \sigma I)$ is skew-adjoint on Y via a standard result [Balakrishnan, 1981, p. 188]. Thus, A, being dissipative, and densely defined, is closeable on Y [Pazy, 1983, p. 15]. Indeed, one can show, either directly or by observing that $A = A^{**}$, that

Lemma 3.11.1.1 *The operator A in (3.11.1.18) and (3.11.1.19) is closed on Y.*

Assumption of Generation by A of a s.c. Semigroup on Y We next obtain that A generates a s.c. contraction semigroup on Y.

Proposition 3.11.1.2 *The operators A and A^* in (3.11.1.18), (3.11.1.19), and in (3.11.1.21) respectively, generate s.c. contraction semigroups e^{At} and e^{A^*t} on $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega)$.*

Proof. We have that A is densely defined and closed by Lemma 3.11.1.1, and, moreover, A and A^* are both dissipative. Thus [Pazy, 1983, p. 15; Balakrishnan, 1981, p. 188], A is maximal dissipative and the Lumer–Phillips Theorem yields the conclusion. [Alternatively, one may check maximal dissipativity of A directly, in which case closedness of A is a consequence.] □

Next, we interpolate between (3.11.1.19) ($r = 1$) and (3.11.1.10) ($r = 0$), to obtain

$$\mathcal{D}((-A)^r) = \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}+\frac{r}{2}}\big) \times \mathcal{D}\big(\mathcal{A}^{\frac{r}{2}}\big) \times \mathcal{D}\left(\mathcal{A}_D^r\right), \quad 0 \le r \le 1. \tag{3.11.1.23}$$

Assumption (1.1.2) or (2.1.7): $A^{-\gamma}B \in \mathcal{L}(U;Y)$ First, it suffices to work with the lower order term $\sigma = 0$. Thus, let $A_0 = A|_{\sigma=0}$. Then, one computes or verifies that

$$A_0^{-1} = \begin{bmatrix} -\dfrac{\alpha^2}{\eta}\mathcal{A}^{-1}\mathcal{A}_D & -\mathcal{A}^{-1} & -\dfrac{\alpha}{\eta}\mathcal{A}^{-1} \\ I & 0 & 0 \\ -\dfrac{\alpha}{\eta} I & 0 & -\dfrac{\mathcal{A}_D^{-1}}{\eta} \end{bmatrix}, \tag{3.11.1.24}$$

where we note, by (3.11.1.8), that

$$\mathcal{A}^{-1}\mathcal{A}_D : \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \to \mathcal{D}(\mathcal{A}), \qquad \text{or equivalently } \mathcal{A}_D\mathcal{A}^{-\frac{1}{2}} : L_2(\Omega) \to L_2(\Omega),$$

as required from the first component of Y in (3.11.1.10) and the first component of $\mathcal{D}(A)$ in (3.11.1.19). Thus, by (3.11.1.24) and (3.11.1.20), we obtain with $u \in U$:

$$A_0^{-1}Bu = \begin{bmatrix} -\dfrac{\alpha^2}{\eta}\mathcal{A}^{-1}\mathcal{A}_D & -\mathcal{A}^{-1} & -\dfrac{\alpha}{\eta}\mathcal{A}^{-1} \\ I & 0 & 0 \\ -\dfrac{\alpha}{\eta} I & 0 & -\dfrac{\mathcal{A}_D^{-1}}{\eta} \end{bmatrix} \begin{bmatrix} 0 \\ -\alpha\mathcal{A}_D Du \\ \eta\mathcal{A}_D Du \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -Du \end{bmatrix}. \tag{3.11.1.25}$$

Since by (3.11.1.7b) $Du \in \mathcal{D}(\mathcal{A}_D^{\frac{1}{4}-\epsilon})$, which is the third component of $\mathcal{D}((-A)^{\frac{1}{4}-\epsilon})$ $= \mathcal{D}((-A_0)^{\frac{1}{4}-\epsilon})$ by (3.11.1.23) with $r = 1/4 - \epsilon$, we finally obtain

$$A_0^{-1}Bu \in \mathcal{D}\big((-A_0)^{\frac{1}{4}-\epsilon}\big), \quad u \in U, \tag{3.11.1.26}$$

and by the closed graph theorem, since A differs from A_0 by the lower order term σI,

$$A_0^{-\gamma} B \in \mathcal{L}(U;Y), \quad \text{i.e., } A^{-\gamma} B \in \mathcal{L}(U;Y), \quad \gamma = \frac{3}{4} + \epsilon. \qquad (3.11.1.27)$$

Thus, (3.11.1.27) proves assumption (1.1.2) of Chapter 1 or (2.1.7) of Chapter 2 with $\gamma = 3/4 + \epsilon$ (compare with Section 3.1).

Assumption of Analyticity of e^{At} It was already asserted in Proposition 3.11.1.2 that the operator A in (3.11.1.18), (3.11.1.19) generates a s.c. contraction semigroup e^{At} on $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega)$. In addition, such semigroup e^{At} is *analytic* on Y, for $t > 0$. This result was first proved in Liu and Renardy [1995] through a technical PDE proof.

Several direct operator theoretic proofs of analyticity of the underlying semigroup for *abstract* thermo-elastic models, which in particular include problem (3.11.1.1), were given in Lasiecka and Triggiani [1998a, b, c]. They are reproduced in Appendices 3E (see its part 3E.4), 3F, and 3G at the end of the present chapter.

Finite Cost Condition (2.1.12) and Detectability Condition (2.1.13) So far we have established that the operator $A_0 = A|_{\sigma=0}$ generates a s.c. contraction, analytic semigroup $e^{A_0 t}$ on Y. We shall now prove that, moreover, $e^{A_0 t}$ is uniformly stable on Y and so is, a fortiori, e^{At}.

Proposition 3.11.1.3 *With reference to the operator $A_0 = A|_{\sigma=0}$ introduced before (3.11.1.24), we have:*

(i) Its inverse A_0^{-1} in (3.11.1.24) is compact on Y, so that A_0 has compact resolvent on Y, and the spectrum $\sigma(A_0)$ of A_0 is only a point spectrum: $\sigma(A_0) = \sigma_p(A_0)$, and thus consists of eigenvalues of A_0.

(ii) There is no spectrum of A_0 in the closed right-hand complex plane $\mathbb{C}^+ = \{\lambda : \operatorname{Re}\lambda \geq 0\}$; that is, $\sigma(A_0) \cap \mathbb{C}^+ = \emptyset$.

(iii) For constants $\omega_0, \omega > 0$, we have:

$$\sup \operatorname{Re} \sigma(A_0) \equiv -\omega_0 < 0, \quad \sup \operatorname{Re} \sigma(A) \equiv -\omega < 0. \qquad (3.11.1.28)$$

(iv) The s.c. contraction, analytic semigroups $e^{A_0 t}$ and e^{At}, guaranteed by the preceding analysis, are, moreover, uniformly stable on Y: With ω_0 and ω the constants in (3.11.1.28) and $\epsilon > 0$, there exists $M_0 \geq 1$, $M \geq 1$, depending on $[-\omega_0 + \epsilon]$ and on $[-\omega + \epsilon]$ respectively, such that

$$\|e^{A_0 t}\|_{\mathcal{L}(Y)} \leq M_0 e^{-(\omega_0 - \epsilon)t}, \quad \|e^{At}\|_{\mathcal{L}(Y)} \leq M e^{-(\omega-\epsilon)t}, \quad t \geq 0. \qquad (3.11.1.29)$$

Proof. (i) We have [see (3.11.1.19) and (3.11.1.10)]

$$A_0^{-1} : Y \to \text{onto } \mathcal{D}(A_0) = \mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times \mathcal{D}(\mathcal{A}_D)$$
$$\hookrightarrow Y = \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times L_2(\Omega) \times L_2(\Omega), \qquad (3.11.1.30)$$

where $\hookrightarrow$ denotes the injection, which is compact, since $\mathcal{A}^{-1}$ and $\mathcal{A}_D^{-1}$ are compact on, say, $L_2(\Omega)$, Ω being bounded.

(ii) Since the s.c. semigroup e^{At} is contraction, it suffices to show the following *Claim: There is no eigenvalue of A_0, or of A, on the imaginary axis.*

Proof for A_0. For r real and $[w_1, w_2, \theta] \in \mathcal{D}(A_0)$, let

$$A_0 \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix} = ir \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix}, \quad \text{or} \quad \begin{cases} w_2 = irw_1, & (3.11.1.31a) \\ -\mathcal{A}w_1 + \alpha\mathcal{A}_D\theta = irw_2 & \\ \qquad = -r^2 w_1, & (3.11.1.31b) \\ -ir\alpha\mathcal{A}_D w_1 - \eta\mathcal{A}_D\theta = ir\theta. & (3.11.1.31c) \end{cases}$$

We must show that $w_1 = w_2 = \theta = 0$. This is trivial for $r = 0$ since $\mathcal{A}_D$ and $\mathcal{A}$ are strictly positive, self-adjoint, and so we assume $r \neq 0$. Recalling identity (3.11.1.22), that is, that the operator obtained from A_0 by omitting the bottom-right corner entry $-\eta\mathcal{A}_D$ is skew-adjoint on Y, we obtain via (3.11.1.31) with $y = [w_1, w_2, \theta]$:

$$\operatorname{Re}\left(A_0 \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix} \right)_Y = -\eta(\mathcal{A}_D\theta, \theta)_{L_2(\Omega)} = \operatorname{Re} ir\|y\|_Y^2 = 0. \qquad (3.11.1.32)$$

Then, since $\mathcal{A}_D$ is strictly positive self-adjoint, we obtain $\theta = 0$ by (3.11.1.32). Hence, $w_1 = 0$ by (3.11.1.31c) with $r \neq 0$; finally $w_2 = 0$ by (3.11.1.31a) [or else, for any r real, $\theta = 0$ by (3.11.1.32) implies then $w_2 = 0$ by (3.11.1.31c), and hence $w_1 = 0$ by (3.11.1.31b)].

(iii) The only possible accumulation point of the eigenvalues of the unbounded operators A_0, or A with compact resolvents, is infinity; thus (3.11.1.28) follows from parts (i) and (ii).

(iv) As the s.c. semigroups $e^{A_0 t}$ and e^{At} are analytic on Y, they satisfy the spectrum determined growth condition [Triggiani, 1975, 1976].

Thus, the uniform decay in (3.11.1.29) follows from (3.11.1.28). □

Remark 3.11.1.1 The proof of part (ii) works for any two strictly positive self-adjoint operators in place of $\mathcal{A}$ and $\mathcal{A}_D$.

Remark 3.11.1.2 A different proof of the uniform stability (3.11.1.29) will be indicated in Appendix 3J for a more general model (possibly a Kirchoff equation); this proof is by energy methods and does *not* use the analyticity of the semigroups $e^{A_0 t}$ or e^{At} on Y.

Remark 3.11.1.3 The exponential uniform decay (3.11.1.29) for e^{At} can be equivalently rewritten in the following form:

$$E(t; y_0) \leq Me^{-(\omega-\epsilon)t} E(0; y_0), \quad y_0 = [w_0, w_1, \theta_0] \in Y, \qquad (3.11.1.33)$$

where the energy $E(t; y_0)$ of the homogeneous problem resulting from (3.11.1.1) by setting $u \equiv 0$ in (3.11.1.1e) is defined by

$$E(t; y_0) = \int_\Omega \left[|\Delta w(t; y_0)|^2 + |w_t(t; y_0)|^2 + |\theta(t; y_0)|^2 \right] d\Omega \qquad (3.11.1.34)$$

[we recall (3.11.1.9) for its first addendum].

A fortiori, then (3.11.1.29) implies that the Finite Cost Condition (2.1.12) and the Detectability Condition (3.1.13) of Chapter 2 are both automatically satisfied, with control $u \equiv 0$ and operator $K = 0$, respectively.

Conclusion: Case $T = \infty$ Theorems 2.2.1 and 2.2.2 of Chapter 2 apply to problem (3.11.1.1), (3.11.1.2) with $a = 0$, where $R = I$, within the above framework.

Conclusion: Case $T < \infty$ Theorem 1.2.1.1 of Chapter 1 applies to problem (3.11.1.1), (3.11.1.2) with $a = 1$, where $G = I$.

3.11.2 Thermal Control in the Neumann/Robin BC and Boundary Observation: Control Operator B and Observation Operator R Both Genuinely Unbounded

Let Ω be a two-dimensional bounded open region (of $\mathbb{R}^2$) with smooth boundary Γ. In this section we consider the following thermo-elastic plate in the variables $w(t, x)$ (vertical displacement) and $\theta(t, x)$ (temperature):

$$\begin{cases} w_{tt} + \Delta^2 w + \alpha \Delta \theta = 0 & \text{in } (0, T] \times \Omega \equiv Q; & (3.11.2.1a) \\ \theta_t - \eta \Delta \theta + \sigma \theta - \alpha \Delta w_t = 0 & \text{in } Q; & (3.11.2.1b) \\ w(0, \cdot) = w_0, \, w_t(0, \cdot) = w_1; & & \\ \quad \theta(0, \cdot) = \theta_0 & \text{in } \Omega; & (3.11.2.1c) \\ w \equiv \dfrac{\partial w}{\partial \nu} \equiv 0 \text{ (clamped BC)} & \text{on } (0, T] \times \Gamma \equiv \Sigma; & (3.11.2.1d) \\ \dfrac{\partial \theta}{\partial \nu} + b\theta = u & \text{on } \Sigma, b \geq 0; & (3.11.2.1e) \end{cases}$$

that is, the same dynamics of Section 3.11.1 with the Dirichlet boundary thermal control (3.11.1.1e) replaced now by the Neumann/Robin boundary thermal control (3.11.2.1e). The constants α, η, σ are all positive as in Section 3.11.1, whereas we explicitly take below the Robin case with constant $b > 0$. The Neumann case $b = 0$ requires only minor modifications: The operator $\mathcal{A}_N$ below in (3.11.2.3) should, in this case, be viewed on the space $L_2(\Omega)$ quotient the one-dimensional null space of $\mathcal{A}_N$. See comments below (3.11.2.3). The present problem is affected by a thermal boundary control $u \in L_2(0, T; L_2(\Gamma)) \equiv L_2(\Sigma)$ acting on a suitable linear combination of the temperature θ and its flux $\partial\theta/\partial\nu$. Consistently with the regularity theory (Appendices 3E, 3F, and 3G), we select now the following cost functional to be minimized:

$$J(u, w, \theta) = \int_0^T \left\{ \|\Delta w(t)|_\Gamma\|^2_{L_2(\Gamma)} + \|w_t(t)|_\Gamma\|^2_{L_2(\Gamma)} + \|\theta(t)|_\Gamma\|^2_{L_2(\Gamma)} \right.$$
$$\left. + \|u(t)\|^2_{L_2(\Gamma)} \right\} dt + a\|\{w(T), w_t(T), \theta(T)\}\|^2_{H_0^2(\Omega)\times L_2(\Omega)\times L_2(\Omega)}, \tag{3.11.2.2}$$

where $\{w_0, w_1, \theta_0\} \in H_0^2(\Omega) \times L_2(\Omega) \times L_2(\Omega)$; moreover, $a = 0$ if $T = \infty$ and $a = 1$ if $T < \infty$. For $T = \infty$, problem (3.11.2.1), (3.11.2.2) is a purely boundary problem, with boundary control and boundary observation.

Abstract Setting We now put problem (3.11.2.1), (3.11.2.2) into the abstract setting of Chapter 1 or Chapter 2. To this end, we let $\mathcal{A}$ be the same operator as in (3.11.1.4), so that the norm-equivalence relationships (3.11.1.6) and (3.11.1.9) still hold true. We now introduce [instead of the operator $\mathcal{A}_D$ in (3.11.1.5)] the operator $\mathcal{A}_N$, still strictly positive self-adjoint on $L_2(\Omega)$, which is defined by

$$\mathcal{A}_N h = -\Delta h; \qquad \mathcal{D}(\mathcal{A}_N) = \left\{ h \in H^2(\Omega) : \left[\frac{\partial h}{\partial \nu} + bh\right]_\Gamma = 0 \right\}, \quad b > 0. \tag{3.11.2.3}$$

[If $b = 0$, then $\mathcal{A}_N$ is only nonnegative, self-adjoint on $L_2(\Omega)$ and becomes positive on $L_2^0(\Omega) \equiv L_2(\Omega)/\mathcal{N}(\mathcal{A}_N)$, where $\mathcal{N}(\mathcal{A}_N)$ is the one-dimensional null space of $\mathcal{A}_N$ given by the constant functions on Ω. The same analysis with $b > 0$ as below then applies if $b = 0$ by working with $L_2^0(\Omega)$ instead of $L_2(\Omega)$]. Comparing (3.11.1.6) with (3.11.2.3) we obtain, via the closed graph theorem and self-adjointness of $\mathcal{A}$ and $\mathcal{A}_N$:

$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset \mathcal{D}(\mathcal{A}_N) \iff \mathcal{A}_N\mathcal{A}^{-\frac{1}{2}} \in \mathcal{L}(L_2(\Omega))$$
$$\iff \mathcal{A}^{-\frac{1}{2}}\mathcal{A}_N \quad \text{bounded extension on } \mathcal{L}(L_2(\Omega)). \tag{3.11.2.4}$$

Let N (Neumann/Robin map) be the harmonic extension of a boundary datum (recall (3.1.7) and (3.1.8)),

$$h \equiv Ng \iff \left\{ \Delta h = 0 \text{ in } \Omega, \left[\frac{\partial h}{\partial \nu} + bh\right]_\Gamma = g \right\}, \quad b > 0, \tag{3.11.2.5a}$$

where N is uniquely defined if $b > 0$,

$$N : \text{continuous } L_2(\Gamma) \to H^{\frac{3}{2}}(\Omega) \subset H^{\frac{3}{2}-2\epsilon}(\Omega) = \mathcal{D}(\mathcal{A}_N^{\frac{3}{4}-\epsilon}), \quad 0 < \epsilon < \frac{3}{4} \tag{3.11.2.5b}$$

[for $b = 0$, we must take $H^{\frac{3}{2}}(\Omega)/\mathcal{N}(\mathcal{A}_N)$].

Accordingly, we take the following spaces Y, U, Z, Z_f:

$$Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega) = H_0^2(\Omega) \times L_2(\Omega) \times L_2(\Omega) \quad \text{(norm equivalence);} \tag{3.11.2.6}$$

$$U = L_2(\Gamma); \quad Z = L_2(\Gamma) \times L_2(\Gamma) \times L_2(\Gamma); \quad Z_f = Y. \tag{3.11.2.7}$$

Moreover, in view of the cost in (3.11.2.2) and Y in (3.11.2.6), we take R to be the following (Dirichlet) trace operator:

$$Rf = \{\Delta f_1|_\Gamma, f_2|_\Gamma, f_3|_\Gamma\}, \quad f = \{f_1, f_2, f_3\}, \tag{3.11.2.8a}$$

$$\begin{aligned} R: \text{ continuous } & H_0^{\frac{5}{2}+\epsilon}(\Omega) \times H_0^{\frac{1}{2}+\epsilon}(\Omega) \times H^{\frac{1}{2}+\epsilon}(\Omega) \\ & \to Z = L_2(\Gamma) \times L_2(\Gamma) \times L_2(\Gamma), \end{aligned} \tag{3.11.2.8b}$$

with $\epsilon > 0$ arbitrary, and

$$G = I \quad \left(\text{on } H_0^2(\Omega) \times L_2(\Omega) \times L_2(\Omega)\right). \tag{3.11.2.9}$$

Using the definition of N in (3.11.2.5a), we rewrite (3.11.2.1) as

$$\begin{cases} w_{tt} + \Delta^2 w + \alpha\Delta(\theta - Nu) = 0 & \text{in } Q, \quad (3.11.2.10a) \\ \theta_t - \eta\Delta(\theta - Nu) + \sigma\theta - \alpha\Delta w_t = 0 & \text{in } Q, \quad (3.11.2.10b) \\ w = \dfrac{\partial w}{\partial \nu} \equiv 0 \text{ in } \Sigma, & \quad (3.11.2.10c) \\ \left[\dfrac{\partial}{\partial \nu}(\theta - Nu) + b(\theta - Nu)\right]_\Sigma \equiv 0 & \text{in } \Sigma. \quad (3.11.2.10d) \end{cases}$$

Hence, recalling (3.11.1.4) for $\mathcal{A}$ and (3.11.2.3) for $\mathcal{A}_N$, we rewrite (3.11.2.10) abstractly as

$$\begin{cases} w_{tt} + \mathcal{A}w - \alpha\mathcal{A}_N(\theta - Nu) = 0, & (3.11.2.11) \\ \theta_t + \eta\mathcal{A}_N(\theta - Nu) + \sigma\theta + \alpha\mathcal{A}_N w_t = 0. & (3.11.2.12) \end{cases}$$

Proceeding with the extension $\mathcal{A}_N : L_2(\Omega) \to [\mathcal{D}(\mathcal{A}_N^*)]' = [\mathcal{D}(\mathcal{A}_N)]'$ of the original self-adjoint operator $\mathcal{A}_N$ in (3.11.2.3), by isomorphism techniques as below (3.11.1.4), where $[\mathcal{D}(\mathcal{A}_N)]'$ is the dual of $\mathcal{D}(\mathcal{A}_N)$ with respect to the pivot space $L_2(\Omega)$, we rewrite (3.11.1.11), (3.11.1.12) abstractly as

$$\begin{cases} w_{tt} + \mathcal{A}w - \alpha\mathcal{A}_N\theta = -\alpha\mathcal{A}_N Nu \in [\mathcal{D}(\mathcal{A}_N)]', & (3.11.2.13) \\ \theta_t + \eta\mathcal{A}_N\theta + \sigma\theta + \alpha\mathcal{A}_N w_t = \eta\mathcal{A}_N Nu \in [\mathcal{D}(\mathcal{A}_N)]'. & (3.11.2.14) \end{cases}$$

Hence, we obtain the form $\dot{y} = Ay + Bu$ in (1.1.1) of Chapter 1:

$$\frac{d}{dt}\begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix} = A\begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix} + Bu, \quad y = \begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix}, \tag{3.11.2.15}$$

where

$$A = \begin{bmatrix} 0 & I & 0 \\ -\mathcal{A} & 0 & \alpha\mathcal{A}_N \\ 0 & -\alpha\mathcal{A}_N & -(\eta\mathcal{A}_N + \sigma I) \end{bmatrix} : Y \supset \mathcal{D}(A) \to Y; \tag{3.11.2.16}$$

$$\mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times \mathcal{D}(\mathcal{A}_N), \quad \mathcal{D}(\mathcal{A}_N) \text{ in (3.11.2.3)}, \tag{3.11.2.17}$$

and where, in identifying the second component of $\mathcal{D}(A)$, we have recalled $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset \mathcal{D}(\mathcal{A}_N)$ from (3.11.2.4) and $w_t \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ by the BC (3.11.1.1d); moreover,

$$Bu = \begin{bmatrix} 0 \\ -\alpha\mathcal{A}_N N u \\ \eta\mathcal{A}_N N u \end{bmatrix}, \quad u \in U; \quad B : \text{ continuous } U \to [\mathcal{D}(A^*)]', \tag{3.11.2.18}$$

where the continuity of B asserted in (3.11.2.18), with duality with respect to Y as a pivot space, will be a fortiori proved after (3.11.2.23). The above model (3.11.2.15)–(3.11.2.18) is the counterpart of the model (3.11.1.17)–(3.11.1.20) in the Dirichlet case of Section 3.11.1. We proceed similarly to establish the required properties. First, we readily find that the Y-adjoint of A, with $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega)$, is

$$A^* = \begin{bmatrix} 0 & -I & 0 \\ \mathcal{A} & 0 & -\alpha\mathcal{A}_N \\ 0 & \alpha\mathcal{A}_N & -(\eta\mathcal{A}_N + \sigma I) \end{bmatrix} : \mathcal{D}(A^*) = \mathcal{D}(A) \to Y, \tag{3.11.2.19}$$

which is the counterpart of (3.11.1.21). Moreover, it is readily seen from (3.11.2.16) and (3.11.2.19) that *both A and A^* are dissipative on $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega)$*:

$$\text{Re}(Ax, x)_Y = \text{Re}(A^*x, x)_Y = -([\eta\mathcal{A}_N + \sigma I]x_3, x_3)_{L_2(\Omega)} \le 0,$$
$$\forall\, x = [x_1, x_2, x_3] \in \mathcal{D}(A) = \mathcal{D}(A^*), \tag{3.11.2.20}$$

which is the counterpart of (3.11.1.22). In fact, by (3.11.2.20), the operator obtained from A (or A^*) by omitting the corner entry $-(\eta\mathcal{A}_N + \sigma I)$ is skew-adjoint on Y [Balakrishnan, 1981, p. 188]. As in Lemma 3.11.1.1, we can claim that A in (3.11.2.16), (3.11.2.17) is closed, and as in Proposition 3.11.1.2, we can claim the desired result that A generates a s.c. contraction semigroup on Y. We combine all this in the next result.

Assumption of Generation by A of a s.c. Semigroup on Y We have

Proposition 3.11.2.1

(i) *The operator A in (3.11.2.16), (3.11.2.17) is closed, and in fact $A = A^{**}$.*
(ii) *The operators A and A^* in (3.11.2.16), (3.11.2.17), and in (3.11.2.19), respectively, generate s.c. contraction semigroups e^{At} and e^{A^*t} on $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega)$.*

Proof. (i) Closedness can also be shown directly.

(ii) We use part (i) and the dissipativity of A and A^* in (3.11.2.20) to obtain the desired conclusion in (ii) by a standard corollary of the Lumer–Phillips theorem, e.g., [Pazy, 1983, p. 15], [Balakrishnan, 1981, p. 188] on e^{At}. □

We next interpolate (3.11.2.17) ($r = 1$) and (3.11.2.6) ($r = 0$), to obtain

$$\mathcal{D}((-A)^r) = \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}+\frac{r}{2}}\right) \times \mathcal{D}\left(\mathcal{A}^{\frac{r}{2}}\right) \times \mathcal{D}\left(\mathcal{A}_N^r\right), \quad 0 \le r \le 1, \tag{3.11.2.21}$$

which is the counterpart of (3.11.1.23).

Assumption (1.1.2) or (2.1.7): $A^{-\gamma}B \in \mathcal{L}(U;Y)$ It suffices to work with the lower order term $\sigma = 0$. Thus, let $A_0 = A|_{\sigma=0}$. Then

$$A_0^{-1} = \begin{bmatrix} -\dfrac{\alpha^2}{\eta}\mathcal{A}^{-1}\mathcal{A}_N & -\mathcal{A}^{-1} & -\dfrac{\alpha}{\eta}\mathcal{A}^{-1} \\ I & 0 & 0 \\ -\dfrac{\alpha}{\eta} I & 0 & -\dfrac{\mathcal{A}_N^{-1}}{\eta} \end{bmatrix}, \tag{3.11.2.22}$$

which is the counterpart of (3.11.1.24), where we note that by (3.11.2.4): $\mathcal{A}^{-1}\mathcal{A}_N : \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \to \mathcal{D}(\mathcal{A})$; equivalently, $\mathcal{A}_N\mathcal{A}^{-\frac{1}{2}} : L_2(\Omega) \to L_2(\Omega)$ as required, by the first component of Y in (3.11.2.6) and the first component of $\mathcal{D}(A)$ in (3.11.2.17). Thus, (3.11.2.22) and (3.11.2.18) yield for $u \in U$:

$$A_0^{-1}Bu = \begin{bmatrix} 0 \\ 0 \\ -Nu \end{bmatrix} \in \mathcal{D}\left((-A)^{\frac{3}{4}-\epsilon}\right), \quad \text{since } Nu \in \mathcal{D}\left(\mathcal{A}_N^{\frac{3}{4}-\epsilon}\right), \tag{3.11.2.23}$$

recalling (3.11.2.5b) and (3.11.2.21) with $r = 3/4-\epsilon$, the counterpart of (3.11.1.26). By the closed graph theorem, we conclude from (3.11.2.23) that

$$A_0^{-\gamma}B \in \mathcal{L}(U;Y); \quad \text{hence } A^{-\gamma}B \in \mathcal{L}(U;Y); \quad \gamma = \frac{1}{4}+\epsilon, \tag{3.11.2.24}$$

which is the counterpart of (3.11.2.27). Thus, (3.11.2.24) proves assumption (3.1.1.2) of Chapter 1 or (2.1.7) of Chapter 2 with $\gamma = 1/4+\epsilon$ (compare with Remark 3.3.1.4 for the purely heat equation problem with Neumann control).

Assumption of Analyticity of e^{At} The s.c. contraction semigroup e^{At} asserted by Proposition 3.11.2.1 must be, in fact, analytic on Y, $t > 0$. This result is a special case of the analyticity theorem for abstract thermo-elastic systems given in Appendices 3E (see its part 3E.4), 3F, and 3G, which in particular contains problem (3.11.2.1).

Finite Cost Condition (2.1.12) and Detectability Condition (2.1.13) The counterpart of Proposition 3.11.1.3 holds true for the homogeneous problem (3.11.2.1) with $u \equiv 0$, thus yielding uniform stability of the s.c. contraction analytic semigroup e^{At}.

Proposition 3.11.2.2 *With reference to the operator $A_0 = A|_{\sigma=0}$ introduced before (3.11.2.22), we have:*

(i) Its inverse A_0^{-1} in (3.11.2.22) is compact on Y, so that A_0 has compact resolvent on Y and the spectrum $\sigma(A_0)$ of A_0 is only a point spectrum, $\sigma(A_0) = \sigma_p(A_0)$, and thus consists of eigenvalues of A_0.
(ii) There is no spectrum of A_0 in the closed right-hand complex plane $\mathbb{C}^+ = \{\lambda : \text{Re}\,\lambda \geq 0\}$; that is, $\sigma(A_0) \cap \mathbb{C}^+ = \emptyset$.
(iii) For constants $\omega_0, \omega > 0$, we have:

$$\sup \text{Re}\,\sigma(A_0) \equiv -\omega_0 > 0, \quad \sup \text{Re}\,\sigma(A) \equiv -\omega < 0. \tag{3.11.2.25}$$

(iv) The s.c. contraction, analytic semigroups e^{A_0t} and e^{At}, guaranteed by the preceding analysis, are, moreover, uniformly stable on Y: With ω_0 and ω the constants in (3.11.2.25) and $\epsilon > 0$, there exist $M_0 \geq 1$, $M \geq 1$ depending on $[-\omega_0 + \epsilon]$ and on $[-\omega + \epsilon]$ respectively such that

$$\left\| e^{A_0t} \right\|_{\mathcal{L}(Y)} \leq M_0 e^{-(\omega_0-\epsilon)t}, \quad \left\| e^{At} \right\|_{\mathcal{L}(Y)} \leq M e^{-(\omega-\epsilon)t}, \quad t \geq 0. \tag{3.11.2.26}$$

Proof. The proof is the same as the proof of Proposition 3.11.1.3, where we recall, by Remark 3.11.1.1, that the proof of part (ii) – which amounts to showing that no point spectrum of A_0 lies on the imaginary axis – applies also with the present operator $\mathcal{A}_N$, which is strictly positive on $L_2(\Omega)$ if $b > 0$, or else on $L_2(\Omega)/\mathcal{N}(\mathcal{A}_N)$ if $b = 0$. See the identity in (3.11.2.20). □

Remark 3.11.2.1 A different proof of the uniform stability (3.11.2.26) will be given in Appendix 3J for a more general model (possibly a Kirchoff equation); this proof is by energy methods and does *not* use the analyticity of the semigroups e^{A_0t} and e^{At}.

Remark 3.11.2.2 The exponential uniform decay (3.11.2.26) for e^{At} can be equivalently rewritten in the following form:

$$E(t; y_0) \leq M e^{-(\omega-\epsilon)t} E(0; y_0), \quad y_0 = [w_0, w_1, \theta_0] \in Y, \tag{3.11.2.27}$$

where the energy $E(t; y_0)$ of the homogeneous problem resulting from (3.11.2.1) by setting $u \equiv 0$ in (3.11.2.1e) is defined by (3.11.1.34).

A fortiori, then (3.11.2.26) implies that the Finite Cost Condition (2.1.12) and the Detectability Condition (2.1.13) of Chapter 2 are both automatically satisfied, with control $u \equiv 0$ and operator $K = 0$, respectively.

Assumption (1.8.1.1) or (2.5.5) on the Unbounded Observation R: $R \in \mathcal{L}(D((-A)^{\delta}); Z)$ We return to the regularity (3.11.2.8b) of R, which we rewrite by noticing that

$$H_0^{\frac{5}{2}+\epsilon}(\Omega) = \mathcal{D}\big(\mathcal{A}^{\frac{5}{8}+\frac{\epsilon}{4}}\big); \quad H_0^{\frac{1}{2}+\epsilon}(\Omega) = \mathcal{D}\big(\mathcal{A}^{\frac{1}{8}+\frac{\epsilon}{4}}\big);$$
$$H^{\frac{1}{2}+2\epsilon}(\Omega) = \mathcal{D}\big(\mathcal{A}_N^{\frac{1}{4}+\epsilon}\big), \quad \epsilon > 0. \tag{3.11.2.28}$$

All three identifications in (3.11.2.28) are obtained from recalling Appendix 3A, Eqn. (3A.10); the third is explicitly noted in (3.11.2.5b). Thus, via (3.11.2.28), we see that (3.11.2.21) is rewritten for $r = \delta = 1/4 + \epsilon/2$ as

$$\mathcal{D}((-A)^{\delta}) = H_0^{\frac{5}{2}+\epsilon}(\Omega) \times H_0^{\frac{1}{2}+\epsilon}(\Omega) \times H^{\frac{1}{2}+\epsilon}(\Omega), \quad \delta = \frac{1}{4} + \frac{\epsilon}{2}, \tag{3.11.2.29}$$

and then (3.11.2.8b) is rewritten as

$$R: \text{ continuous } \mathcal{D}((-A)^{\delta}) \to Z, \quad \delta = \frac{1}{4} + \frac{\epsilon}{2}, \tag{3.11.2.30}$$

where, since $\gamma = 1/4 + \epsilon$ by (3.11.2.24), we have

$$\frac{1}{4} + \frac{\epsilon}{2} = \delta < \left\{\frac{1}{2}, 1 - \gamma\right\}, \quad 1 - \gamma = \frac{3}{4} + \epsilon, \tag{3.11.2.31}$$

as required. We conclude that assumption (1.8.1.1) of Chapter 1, or (2.5.5) of Chapter 2, is satisfied by the observation R in (3.11.2.8a) with $\delta = 1/4 + \epsilon/2$.

Conclusion: Case $T = \infty$ Theorems 2.5.1 and 2.5.2 of Chapter 2, which extend Theorems 2.2.1 and 2.2.2 of Chapter 2 to R unbounded as in (3.11.2.30), apply to problem (3.11.2.1), (3.11.2.2) with $a = 0$, within the above framework.

Conclusion: Case $T < \infty$ Theorem 1.8.1.1 of Chapter 1 applies to problem (3.11.2.1), (3.11.2.2), with R as in (3.11.2.8), (3.11.2.30) and $G = I$ as in (3.11.2.9).

3.12 Thermo-Elastic Plates with Mechanical Control in the Bending Moment (Hinged BC) and Homogeneous Neumann Thermal BC

Let Ω be a two-dimensional bounded open region (of $\mathbb{R}^2$) with smooth boundary Γ. In this section we consider the following thermo-elastic plate in the variables $w(t, x)$

(vertical displacement) and $\theta(t, x)$ (temperature):

$$\begin{cases} w_{tt} + \Delta^2 w + \alpha\Delta\theta = 0 & \text{in } (0, T] \times \Omega \equiv Q; & (3.12.1\text{a}) \\ \theta_t - \eta\Delta\theta + \sigma\theta - \alpha\Delta w_t = 0 & \text{in } Q; & (3.12.1\text{b}) \\ w(0, \cdot) = w_0, w_t(0, \cdot) = w_1; & & \\ \quad \theta(0, \cdot) = \theta_0 & \text{in } \Omega; & (3.12.1\text{c}) \\ w \equiv 0, \Delta w + (1-\mu)B_1 w + \alpha\theta \equiv u & \text{on } \Sigma; & (3.12.1\text{d}) \\ \dfrac{\partial\theta}{\partial\nu} + b\theta \equiv 0 & \text{on } \Sigma; & (3.12.1\text{e}) \end{cases}$$

that is, with mechanical control $u \in L_2(0, T; L_2(\Gamma))$ acting as a (physical) bending moment. [The case $u \equiv 0$ corresponds to the so-called hinged BC in w.] The constants α, η, σ, b are all positive, as in Section 3.11.2. In (3.1.2.1d), we have denoted by B_1 the boundary operator (see Appendix 3C, Proposition 3C.1):

$$B_1 w \equiv \left[-\frac{\partial^2 w}{\partial\tau^2} - c(x)\frac{\partial w}{\partial\nu} \right] \equiv -c(x)\frac{\partial w}{\partial\nu} \quad \text{on } \Sigma. \tag{3.12.1f}$$

In (3.12.1f), $c(x) \equiv \operatorname{div} \nu(x)$ is the *mean curvature* at $x \in \Gamma$, while $\partial/\partial\tau$ denotes the tangential derivative to Γ (counter clockwise), so that, in view of the BC, $w|_\Sigma \equiv 0$ in (3.12.1c), one obtains $\partial w/\partial\tau \equiv 0$ on Σ, and thus $B_1 w$ becomes the right-hand side of (3.12.1f). Moreover, the constant $\mu \in (0, 1)$ is the Poisson's modulus (physically $0 < \mu < 1/2$). The mathematics works for $0 < \mu < 1$. Appendix 3C of Chapter 3 shows, among other things, the required equality between the expression of the boundary operator B_1 in Eqn. (3.12.1f) and the expression of the boundary operator B_1 in Eqn. (3.5.2) of Section 3.5.

Consistently with the regularity theory (Appendix 3H Chapter 3), we select the following cost functional to be minimized:

$$\begin{aligned} J(u, w, \theta) = \int_0^T & \left\{ \|w(t)\|^2_{H^2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} + \|\theta(t)\|^2_{L_2(\Omega)} + \|u(t)\|^2_{L_2(\Gamma)} \right\} dt \\ & + a\|\{w(T), w_t(T), \theta(T)\}\|^2_{H^2(\Omega)\times L_2(\Omega)\times L_2(\Omega)}, \end{aligned} \tag{3.12.2}$$

where $\{w_0, w_1, \theta_0\} \in [H^2(\Omega) \cap H_0^1(\Omega)] \times L_2(\Omega) \times L_2(\Omega)$; moreover, $a = 0$ if $T = \infty$ and $a = 1$ if $T < \infty$.

Abstract Setting To put problem (3.12.1), (3.12.2) into the abstract form (1.1.1) of Chapter 1, we introduce the following spaces and operators. First, we let $\mathcal{A}$ be the following positive, self-adjoint operator on $L_2(\Omega)$ (see Appendix 3C, Proposition 3C.4 of Chapter 3):

$$\mathcal{A}h = \Delta^2 h; \qquad \mathcal{D}(\mathcal{A}) = \{h \in H^4(\Omega) \cap H_0^1(\Omega) : \Delta h + (1-\mu)B_1 h = 0 \text{ on } \Gamma\}. \tag{3.12.3}$$

Next, we introduce the same operators $\mathcal{A}_D$ and $\mathcal{A}_N$ of (3.11.1.5) and (3.11.2.3), respectively:

$$\mathcal{A}_D h = -\Delta h; \qquad \mathcal{D}(\mathcal{A}_D) = H^2(\Omega) \cap H_0^1(\Omega), \tag{3.12.4}$$

$$\mathcal{A}_N h = -\Delta h; \qquad \mathcal{D}(\mathcal{A}_N) = \left\{ h \in H^2(\Omega) : \left[\frac{\partial h}{\partial \nu} + bh \right]_\Gamma = 0 \right\}, \quad k > 0. \tag{3.12.5}$$

[The case $b = 0$ can be similarly handled as in Section 3.11.2.] We have [Grisvard, 1967, see Appendix 3A, Theorem 3A.1]

$$\mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) = H^2(\Omega) \cap H_0^1(\Omega) = \mathcal{D}(\mathcal{A}_D) \quad \text{(equivalent norms)}. \tag{3.12.6}$$

Accordingly, we take the following spaces and operators:

$$\begin{aligned} Y = \big[H^2(\Omega) \cap H_0^1(\Omega)\big] \times L_2(\Omega) \times L_2(\Omega) &= \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times L_2(\Omega) \times L_2(\Omega) \\ &= \mathcal{D}(\mathcal{A}_D) \times L_2(\Omega) \times L_2(\Omega) \\ &\qquad \text{(equivalent norms);} \end{aligned} \tag{3.12.7}$$

$$U = L_2(\Gamma), \quad Z = Z_f = Y, \quad R = G = I. \tag{3.12.8}$$

Next, we introduce the Green map G, defined by

$$h = Gg \iff \begin{cases} \Delta^2 h = 0 \text{ in } \Omega, \\ h|_\Gamma = 0, (\Delta + (1-\mu)B_1)h|_\Gamma = g. \end{cases} \tag{3.12.9}$$

Elliptic regularity [Lions, Magenes, 1972, p. 188–9] gives

$$\begin{aligned} G \ : \ \text{continuous } L_2(\Gamma) \to H^{\frac{5}{2}}(\Omega) \cap H_0^1(\Omega) &\subset H^{\frac{5}{2}-4\epsilon}(\Omega) \cap H_0^1(\Omega) \\ &= \mathcal{D}\big(\mathcal{A}^{\frac{5}{8}-\epsilon}\big), \quad \epsilon > 0, \end{aligned} \tag{3.12.10a}$$

where the identification in (3.12.10a) follows from Appendix 3A, Eqn. (3A.11). Equivalently,

$$\mathcal{A}^{\frac{5}{8}-\epsilon} G : \text{continuous } L_2(\Gamma) \to L_2(\Omega). \tag{3.12.10b}$$

For future use in Lemma 3.12.1 below, we note that if $g \in L_2(\Gamma)$ and $f \in H_0^1(\Omega)$, then the first BC in (3.12.9) gives $(Gg)|_\Gamma = 0$, and then (3.12.1f) yields

$$-B_1(Gg) = c(x)\frac{\partial(Gg)}{\partial \nu}, \quad B_1 f = -c(x)\frac{\partial f}{\partial \nu}, \tag{3.12.11}$$

and hence

$$\begin{aligned} &\int_\Gamma \frac{\partial f}{\partial \nu}(-B_1(Gg))\, d\Gamma + \int_\Gamma (B_1 f)\frac{\partial(Gg)}{\partial \nu}\, d\Gamma \\ &\quad = \int_\Gamma c(x)\frac{\partial f}{\partial \nu}\,\frac{\partial(Gg)}{\partial \nu}\, d\Gamma - \int_\Gamma c(x)\frac{\partial f}{\partial \nu}\,\frac{\partial(Gg)}{\partial \nu}\, d\Gamma = 0. \end{aligned} \tag{3.12.12}$$

Lemma 3.12.1 *Let* $(Gg, y)_{L_2(\Omega)} = (g, G^*y)_{L_2(\Gamma)}$, $g \in L_2(\Gamma)$, $y \in L_2(\Omega)$. *Then*

$$G^*\mathcal{A}f = \frac{\partial f}{\partial \nu}, \quad f \in \mathcal{D}(\mathcal{A}). \tag{3.12.13}$$

Proof. Let $g \in L_2(\Gamma)$ and $f \in \mathcal{D}(\mathcal{A})$. Applying Green's second theorem twice, we obtain via (3.12.9):

$$\begin{aligned}
(G^*\mathcal{A}f, g)_{L_2(\Gamma)} &= (\mathcal{A}f, Gg)_{L_2(\Omega)} = (\Delta^2 f, Gg)_{L_2(\Omega)} \\
&= \int_\Omega \cancel{f(\Delta^2 Gg)}\, d\Omega + \int_\Gamma \frac{\partial f}{\partial \nu}(\Delta Gg)\, d\Gamma \\
&\quad + \int_\Gamma \cancel{f \frac{\partial(\Delta Gg)}{\partial \nu}}\, d\Gamma + \int_\Gamma \cancel{\frac{\partial \Delta f}{\partial \nu}(Gg)}\, d\Gamma - \int_\Gamma \Delta f \frac{\partial Gg}{\partial \nu}\, d\Gamma
\end{aligned} \tag{3.12.14}$$

$$\text{(by (3.12.9))} \qquad = \int_\Gamma \frac{\partial f}{\partial \nu}(g - (1-\mu)B_1(Gg))\, d\Gamma + (1-\mu)\int_\Gamma (B_1 f)\frac{\partial Gg}{\partial \nu}\, d\Gamma \tag{3.12.15}$$

$$\text{(by 3.12.12)} \qquad = \int_\Gamma \frac{\partial f}{\partial \nu} g\, d\Gamma, \quad \forall\, g \in L_2(\Gamma). \tag{3.12.16}$$

Indeed, in (3.12.14), $\Delta^2(Gg) \equiv 0$ in Ω and $(Gg)|_\Gamma \equiv 0$ by (3.12.9), while $f|_\Gamma = 0$ by (3.12.3). Next, (3.12.14) yields (3.12.15) since $\Delta(Gg) = g - (1-\mu)B_1(Gg)$ on Γ by (3.12.9) and $-\Delta f = (1-\mu)B_1 f$ on Γ by (3.12.3). Finally, (3.12.15) yields (3.12.16) by invoking identity (3.12.12). Then (3.12.16), which is valid for all $g \in L_2(\Gamma)$, proves (3.12.13). □

Using the definition of G in (3.12.9), we rewrite (3.12.1) as

$$\begin{cases}
w_{tt} + \Delta^2[w - G(u - \alpha(\theta|_\Gamma))] + \alpha\Delta\theta = 0 & \text{in } Q; \quad (3.12.17\text{a}) \\
\theta_t - \eta\Delta\theta + \sigma\theta - \alpha\Delta w_t = 0 & \text{in } Q; \quad (3.12.17\text{b}) \\
w \equiv 0,\ (\Delta + (1-\mu)B_1)[w - G(u - \alpha(\theta|_\Gamma))] = 0 & \text{on } \Sigma; \quad (3.12.17\text{c}) \\
\dfrac{\partial\theta}{\partial\nu} + b\theta = 0 & \text{on } \Sigma. \quad (3.12.17\text{d})
\end{cases}$$

Recalling (3.12.3)–(3.12.5), we rewrite the PDE problem (3.12.7) abstractly as

$$\begin{cases}
w_{tt} + \mathcal{A}[w - G(u - \alpha(\theta|_\Gamma))] - \alpha\mathcal{A}_N\theta = 0, & (3.12.18) \\
\theta_t + \eta\mathcal{A}_N\theta + \sigma\theta + \alpha\mathcal{A}_D w_t = 0. & (3.12.19)
\end{cases}$$

We now recall from Eqn. (3.3.1.12) of Section 3.3 that in the present notation

$$\theta|_\Gamma = N^*\mathcal{A}_N\theta, \quad \theta \in \mathcal{D}(\mathcal{A}_N), \tag{3.12.20}$$

for $\mathcal{A}_N$ defined by (3.12.5) (with sign opposite to the definition in (3.1.4a)). Finally, using (3.12.20) and extending the original $\mathcal{A}$ in (3.12.3) to $\mathcal{A} : L_2(\Omega) \to [\mathcal{D}(\mathcal{A}^*)]' =$

$[\mathcal{D}(\mathcal{A})]'$ by isomorphism, where the duality is with respect to $L_2(\Omega)$ as a pivot space, we rewrite (3.12.8), (3.12.9) as the following second-order system:

$$\begin{cases} w_{tt} + \mathcal{A}w + \alpha\mathcal{A}GN^*\mathcal{A}_N\theta - \alpha\mathcal{A}_N\theta = \mathcal{A}Gu \in [\mathcal{D}(\mathcal{A})]', & (3.12.21) \\ \theta_t + \eta\mathcal{A}_N\theta + \sigma\theta + \alpha\mathcal{A}_D w_t = 0. & (3.12.22) \end{cases}$$

[If $B_1 = 0$, then $\mathcal{A}G = -\mathcal{A}_D D$, where $\mathcal{A}_D = \mathcal{A}^{\frac{1}{2}}$ [see (3.6.6) of Section 3.6]. Thus, in this case, $\mathcal{A}GN^*\mathcal{A}_N = -\mathcal{A}_D DN^*\mathcal{A}_N$.] Setting $y = [w, w_t, \theta]$, we then rewrite the above second-order system as the first-order equation $\dot{y} = Ay + Bu$, as in Eqn. (1.1.1) of Chapter 1:

$$\frac{d}{dt}\begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix} = A\begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix} + Bu, \quad y = \begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix}, \tag{3.12.23}$$

$$A = \begin{bmatrix} 0 & I & 0 \\ -\mathcal{A} & 0 & \alpha(\mathcal{A}_N - \mathcal{A}GN^*\mathcal{A}_N) \\ 0 & -\alpha\mathcal{A}_D & -(\eta\mathcal{A}_N + \sigma I) \end{bmatrix} : Y \supset \mathcal{D}(A) \to Y, \tag{3.12.24a}$$

to be interpreted in the sense that

$$A\begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix} = \begin{bmatrix} w_2 \\ -\mathcal{A}(\alpha G(\theta|_\Gamma) + w_1) + \alpha\mathcal{A}_N\theta \\ -\alpha\mathcal{A}_D w_2 - (\eta\mathcal{A}_N + \sigma I)\theta \end{bmatrix}, \quad \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix} \in \mathcal{D}(A), \tag{3.12.24b}$$

where, recalling Y in (3.12.7), we obtain from (3.12.24b)

$$\begin{aligned} \mathcal{D}(A) = \big\{ & w_1 \in \mathcal{D}(\mathcal{A}_D) = \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big);\ w_2 \in \mathcal{D}(\mathcal{A}_D);\ \theta \in \mathcal{D}(\mathcal{A}_N); \\ & w_1 + \alpha G(\theta|_\Gamma) \in \mathcal{D}(\mathcal{A})\big\}; \end{aligned} \tag{3.12.25}$$

$$Bu = \begin{bmatrix} 0 \\ \mathcal{A}Gu \\ 0 \end{bmatrix}, \quad u \in U; \quad B : \text{continuous } U \to [\mathcal{D}(A^*)]'. \tag{3.12.26}$$

The continuity of B, asserted in (3.12.26), with duality with respect to Y as a pivot space, will be a fortiori proved in (3.12.48).

The Y-adjoint of A in (3.12.24), with $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega)$, is given by

$$A^* = \begin{bmatrix} 0 & -I & 0 \\ \mathcal{A} & 0 & -\alpha\mathcal{A}_D \\ 0 & \alpha(\mathcal{A}_N - \mathcal{A}_N NG^*\mathcal{A}) & -(\eta\mathcal{A}_N + \sigma I) \end{bmatrix} : \ Y \supset \mathcal{D}(A^*) \to Y, \tag{3.12.27a}$$

to be interpreted in the sense that

$$A^* \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix} = \begin{bmatrix} -w_2 \\ \mathcal{A}w_1 - \alpha \mathcal{A}_D \theta \\ \alpha \mathcal{A}_N (w_2 - NG^* \mathcal{A} w_2) - (\eta \mathcal{A}_N + \sigma I)\theta \end{bmatrix}, \quad \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix} \in \mathcal{D}(A^*), \tag{3.12.27b}$$

$$\mathcal{D}(A^*) = \{w_1 \in \mathcal{D}(\mathcal{A}), w_2 \in \mathcal{D}(\mathcal{A}_D), \theta \in \mathcal{D}(\mathcal{A}_D) : [w_2 - NG^*\mathcal{A}w_2] \in \mathcal{D}(\mathcal{A}_N)\}, \tag{3.12.28}$$

where we notice that, by Appendix 3A via (3.12.3),

$$\mathcal{D}(\mathcal{A}_D) = H^2(\Omega) \cap H_0^1(\Omega) \subset \mathcal{D}\big(\mathcal{A}^{\frac{3}{8}+\epsilon}\big) = H^{\frac{3}{2}+4\epsilon}(\Omega) \cap H_0^1(\Omega), \tag{3.12.29}$$

and hence, by duality on (3.12.10),

$$\begin{cases} G^*\mathcal{A}w_2 = G^*\mathcal{A}^{\frac{5}{8}-\epsilon}\mathcal{A}^{\frac{3}{8}+\epsilon}w_2 \in L_2(\Gamma) \quad \text{for } w_2 \in \mathcal{D}(\mathcal{A}_D) = \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big), & (3.12.30a) \\ G^*\mathcal{A} : \text{ continuous } \mathcal{D}(\mathcal{A}_D) = \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \to U = L_2(\Gamma), & (3.12.30b) \end{cases}$$

so that $NG^*\mathcal{A}w_2 \in \mathcal{D}(\mathcal{A}_N^{\frac{3}{4}-\epsilon})$ by (3.1.18) for the term in (3.12.28).

Lemma 3.12.2 *With reference to (3.12.24) and (3.12.25) and with $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega)$ as in (3.12.7), we have that A is dissipative:*

$$\operatorname{Re}\left(A \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix} \right)_Y$$

$$= \operatorname{Re}\left(\begin{bmatrix} 0 & \alpha(\mathcal{A}_N - \mathcal{A}GN^*\mathcal{A}_N) \\ -\alpha\mathcal{A}_D & -(\eta\mathcal{A}_N + \sigma I) \end{bmatrix} \begin{bmatrix} w_2 \\ \theta \end{bmatrix}, \begin{bmatrix} w_2 \\ \theta \end{bmatrix} \right)_{L_2(\Omega)\times L_2(\Omega)} \tag{3.12.31}$$

$$= -((\eta\mathcal{A}_N + \sigma I)\theta, \theta)_{L_2(\Omega)} \le 0, \quad \forall\, [w_1, w_2, \theta] \in \mathcal{D}(A). \tag{3.12.32}$$

Proof. Since $\left[\begin{smallmatrix} 0 & I \\ -\mathcal{A} & 0 \end{smallmatrix}\right]$ is skew-adjoint on $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$, it remains to show that

$$\operatorname{Re}\left(\begin{bmatrix} 0 & \alpha(\mathcal{A}_N - \mathcal{A}GN^*\mathcal{A}_N) \\ -\alpha\mathcal{A}_D & 0 \end{bmatrix} \begin{bmatrix} w_2 \\ \theta \end{bmatrix}, \begin{bmatrix} w_2 \\ \theta \end{bmatrix} \right)_{L_2(\Omega)\times L_2(\Omega)} = 0,$$

$$\forall\, [w_2, \theta] \in \mathcal{D}(\mathcal{A}_D) \times \mathcal{D}(\mathcal{A}_N). \tag{3.12.33}$$

To prove (3.12.33), we compute with $[w_2, \theta] \in \mathcal{D}(\mathcal{A}_D) \times \mathcal{D}(\mathcal{A}_N)$, via (3.12.4), (3.12.5), and (3.12.20),

$$\operatorname{Re}\left(\begin{bmatrix} 0 & \alpha(\mathcal{A}_N - \mathcal{A}GN^*\mathcal{A}_N) \\ -\alpha\mathcal{A}_D & 0 \end{bmatrix} \begin{bmatrix} w_2 \\ \theta \end{bmatrix}, \begin{bmatrix} w_2 \\ \theta \end{bmatrix} \right)_{L_2(\Omega)\times L_2(\Omega)}$$

$$= \operatorname{Re}\left\{ \alpha(\mathcal{A}_N\theta, w_2)_{L_2(\Omega)} - \alpha(\theta|_\Gamma, G^*\mathcal{A}w_2)_{L_2(\Gamma)} - \alpha(\mathcal{A}_D w_2, \theta)_{L_2(\Omega)} \right\} \tag{3.12.34}$$

$$\text{(by (3.12.13))} \qquad = \operatorname{Re}\left\{-\alpha \int_\Omega \Delta\theta \bar{w}_2 \, d\Omega - \alpha \int_\Gamma \theta \frac{\partial \bar{w}_2}{\partial \nu} \, d\Gamma + \alpha \int_\Omega \Delta w_2 \bar{\theta} \, d\Omega\right\} \tag{3.12.35}$$

(using Green's second theorem on the last integral term in (3.12.35))

$$= \operatorname{Re}\left\{-\alpha \int_\Omega \cancel{\Delta\bar{\theta} w_2 \, d\Omega} - \alpha \int_\Gamma \theta \cancel{\frac{\partial \bar{w}_2}{\partial \nu}} \, d\Gamma + \alpha \int_\Omega \cancel{w_2 \Delta\bar{\theta}} \, d\Omega \right.$$
$$\left. + \, \alpha \int_\Gamma \cancel{\frac{\partial w_2}{\partial \nu}} \bar{\theta} \, d\Gamma - \alpha \int_\Gamma w_2 \frac{\partial \bar{\theta}}{\partial \nu} \, d\Gamma \right\} = 0, \tag{3.12.36}$$

where, on the left-hand side of (3.12.36), the first and third integral terms, as well as the second and fourth, cancel out; while the last term vanishes, since $w_2|_\Gamma = 0$ due to $w_2 \in \mathcal{D}(\mathcal{A}_D)$. We note that all terms in the above computations are well defined, in particular in (3.12.34), via (3.12.30). Thus (3.12.36) proves (3.12.33), and Lemma 3.12.2 is established. $\square$

Lemma 3.12.3

(i) With reference to (3.12.27) and (3.12.28), and with $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega)$ *as in (3.12.7), we have that* A^* *is dissipative:*

$$\operatorname{Re}\left(A^* \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix}\right)_Y$$
$$= \operatorname{Re}\left(\begin{bmatrix} 0 & -\alpha\mathcal{A}_D \\ \alpha(\mathcal{A}_N - \mathcal{A}_N N G^* \mathcal{A}) & -(\eta\mathcal{A}_N + \sigma I) \end{bmatrix} \begin{bmatrix} w_2 \\ \theta \end{bmatrix}, \begin{bmatrix} w_2 \\ \theta \end{bmatrix}\right)_{L_2(\Omega)\times L_2(\Omega)} \tag{3.12.37}$$

$$= -((\eta\mathcal{A}_N + \sigma I)\theta, \theta)_{L_2(\Omega)} \le 0, \quad \forall\, [w_1, w_2, \theta] \in \mathcal{D}(A^*). \tag{3.12.38}$$

(ii) The operator obtained from A by omitting the bottom-right corner term $-(\eta\mathcal{A}_N + \sigma I)$ *is skew-adjoint on Y and generates a s.c. unitary group on Y.*

Proof. (i) The proof is similar to that of Lemma 3.12.2. By (3.12.27a), we need to verify that

$$\operatorname{Re}\left(\begin{bmatrix} 0 & -\alpha\mathcal{A}_D \\ \alpha(\mathcal{A}_N - \mathcal{A}_N N G^* \mathcal{A}) & 0 \end{bmatrix} \begin{bmatrix} w_2 \\ \theta \end{bmatrix}, \begin{bmatrix} w_2 \\ \theta \end{bmatrix}\right)_{L_2(\Omega)\times L_2(\Omega)}$$

$$= -\alpha(\mathcal{A}_D\theta, w_2)_{L_2(\Omega)} + \alpha(\mathcal{A}_N w_2, \theta)_{L_2(\Omega)} - \cancel{(G^*\mathcal{A} w_2, \theta|_\Gamma)_{L_2(\Gamma)}} \tag{3.12.39}$$

$$= \alpha \int_\Omega \Delta\theta w_2 \, d\Omega - \alpha \int_\Omega \Delta w_2 \theta \, d\Omega = 0, \quad \forall\, [w_2, \theta] \in \mathcal{D}(\mathcal{A}_D) \times \mathcal{D}(\mathcal{A}_D). \tag{3.12.40}$$

Cancellation in the last term of (3.12.39) follows since $\theta|_\Gamma = 0$ for $\theta \in \mathcal{D}(\mathcal{A}_D)$. This and the similar trace vanishing $w_2|_\Gamma = 0$ for $w_2 \in \mathcal{D}(\mathcal{A}_D)$ produce zero in (3.12.40), by use of Green's second theorem.

(ii) This follows from the identity in (3.12.32) and the identity in (3.12.38) via a standard result [Balakrishnan, 1981, p. 188]. □

Thus, the operator A, being dissipative (by Lemma 3.12.2) and densely defined, is closable on Y [Pazy, 1983, p. 15]. Indeed, A is closed.

Proposition 3.12.4

(i) The operator A in (3.12.24) and (3.12.25) is densely defined and closed.
(ii) The operators A and A^* (in (3.12.27) and (3.12.28)) generate s.c. contraction semigroups e^{At} and e^{A^*t} on $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega)$.

Proof. (i) can be shown directly or by observing that $A^{**} = A$.

(ii) We recall by Lemmas 3.12.2 and 3.12.3 that A and A^* are both dissipative. Thus, by [Pazy, 1983, p. 15; Balakrishnan, 1981, p. 188], A is maximal dissipative and the Lumer–Phillips theorem yields the conclusion for e^{At}. [Alternatively, one may check maximal dissipativity of A directly, in which case closedness of A is a consequence.] □

Next, we note the following *Claim*:

$$\mathcal{D}((-A^r)) = \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}+\frac{r}{2}}\big) \times \mathcal{D}\big(\mathcal{A}_D^r\big) \times \mathcal{D}\big(\mathcal{A}_N^r\big), \quad 0 \le r < \frac{1}{4}, \tag{3.12.41}$$

for the operator A in (3.12.24), with domain $\mathcal{D}(A)$ in (3.12.25) over the space Y in (3.12.7).

Proof of Claim. With reference to the operator $\mathcal{A}$ in (3.12.3), we have via Appendix 3A, Theorem 3A.1:

$$\mathcal{D}(\mathcal{A}^s) = \begin{cases} H^{4s}(\Omega), & 0 \le s < \frac{1}{8}, \quad (3.12.42a) \\ H^{4s}(\Omega) \cap H_0^1(\Omega), & \frac{1}{8} < s < \frac{5}{8}, \quad (3.12.42b) \\ \{h \in H^{4s}(\Omega) \cap H_0^1(\Omega) : & \\ \quad \Delta h + (1-\mu)B_1 h|_\Gamma = 0\}, & \frac{5}{8} < s \le 1. \quad (3.12.42c) \end{cases}$$

The ranges for s follow from: $4s - 1/2 < 0$ for (3.12.42a); $0 < 4s - 1/2 < 2$ for (3.12.42b); and $2 < 4s - 1/2$ for (3.12.42c) for $\mathcal{A}$ defined by (3.12.3). But $[w_1, w_2, \theta] \in Y$ already implies the first BC $w_1|_\Gamma = 0$ of $\mathcal{D}(\mathcal{A})$, while the second BC of $\mathcal{D}(\mathcal{A})$ does not occur in $\mathcal{D}((-A)^s)$ of (3.12.41) as long as $s = 1/2 + r/2 < 5/8$, or $r < 1/4$, as desired.

Assumption (1.1.2) or (2.1.7): $A^{-\gamma}B \in \mathcal{L}(U;Y)$ First, it suffices to work with the lower order term $\sigma = 0$. Thus, let $A_0 = A|_{\sigma=0}$ for A as in (3.12.24). One may then compute, or verify, that

$$A_0^{-1} = \begin{bmatrix} -\frac{\alpha^2}{\eta}\mathcal{A}^{-1}(I-\mathcal{A}GN^*)\mathcal{A}_D & -\mathcal{A}^{-1} & -\frac{\alpha}{\eta}\mathcal{A}^{-1}(I-\mathcal{A}GN^*) \\ I & 0 & 0 \\ -\frac{\alpha}{\eta}\mathcal{A}_N^{-1}\mathcal{A}_D & 0 & -\frac{\mathcal{A}_N^{-1}}{\eta} \end{bmatrix}$$
$$: Y \to \mathcal{D}(A_0) = \mathcal{D}(A), \tag{3.12.43a}$$

and we verify (3.12.43): If $[y_1, y_2, y_3] \in Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega)$, $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = \mathcal{D}(\mathcal{A}_D)$, then $A_0^{-1}[y_1, y_2, y_3] = [w_1, w_2, \theta]$, where

$$\begin{cases} w_1 = -\frac{\alpha^2}{\eta}[\mathcal{A}^{-1}\mathcal{A}_D y_1 - GN^*\mathcal{A}_D y_1] - \mathcal{A}^{-1}y_2 - \frac{\alpha}{\eta}(\mathcal{A}^{-1}y_3 - GN^*y_3) & (3.12.44a) \\ (\text{by } (3.12.10)) \quad \in \mathcal{D}\big(\mathcal{A}^{\frac{5}{8}-\epsilon}\big) \subset \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) = \mathcal{D}(\mathcal{A}_D), & (3.12.44b) \\ w_2 = y_1 \in \mathcal{D}(\mathcal{A}_D), & (3.12.45) \\ \theta = -\frac{\alpha}{\eta}\mathcal{A}_N^{-1}\mathcal{A}_D y_1 - \frac{\mathcal{A}_N^{-1}}{\eta} y_3 \in \mathcal{D}(\mathcal{A}_N), & (3.12.46) \end{cases}$$

where, as required by (3.12.25), the additional condition,

$$w_1 + \alpha GN^*\mathcal{A}_N\theta = -\frac{\alpha}{\eta}\mathcal{A}^{-1}\mathcal{A}_D y_1 - \mathcal{A}^{-1}y_2 - \frac{\alpha}{\eta}\mathcal{A}^{-1}y_3 \in \mathcal{D}(\mathcal{A}), \tag{3.12.47}$$

holds true after a cancellation of four terms.

A fortiori from (3.12.43) and (3.12.26), we obtain

$$A_0^{-1}Bu = \begin{bmatrix} -Gu \\ 0 \\ 0 \end{bmatrix}, \quad \text{where } Gu \in \mathcal{D}\big(\mathcal{A}^{\frac{5}{8}-\epsilon}\big) \subset \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \quad \text{for } u \in L_2(\Gamma), \tag{3.12.48}$$

and the regularity property in (3.12.26) is proved. In fact, more precisely, recalling (3.12.41), we see that $\mathcal{D}(\mathcal{A}^{\frac{1}{2}+\frac{r}{2}}) = \mathcal{D}(\mathcal{A}^{\frac{5}{8}-\epsilon})$ for the first component of $\mathcal{D}((-A_0)^r)$, when $r = 1/4 - 2\epsilon < 1/4$, so that

$$A_0^{-1}Bu \in \mathcal{D}\big((-A_0)^{\frac{1}{4}-2\epsilon}\big), \quad u \in U, \tag{3.12.49}$$

and by the closed graph theorem, since A differs from A_0 by the lower order term σI,

$$A_0^{-\gamma}B \in \mathcal{L}(U;Y), \qquad \text{that is, } A^{-\gamma}B \in \mathcal{L}(U;Y), \quad \gamma = \frac{3}{4} + 2\epsilon. \tag{3.12.50}$$

Thus, (3.12.50) proves assumption (1.1.2) of Chapter 1 or (2.1.7) of Chapter 2, with $\gamma = 3/4 + \epsilon, \forall\, \epsilon > 0$ (compare with Section 3.11.1).

Assumption of Analyticity of e^{At} It was already asserted in Proposition 3.12.4 that the operator A in (3.12.24) generates a s.c. contraction semigroup e^{At} on $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega)$. In addition, such semigroup e^{At} is *analytic* on Y, for $t > 0$. This is proved in Appendix 3H.

Remark 3.12.1 The far easier case where the Neumann/Robin BC (3.12.1e) is replaced by a corresponding Dirichlet BC for θ [as to match the first BC for w in (3.12.1d)] is treated explicitly in Appendix 3D: Theorem 3D.5 provides analyticity of the corresponding s.c. contraction semigroup, while Proposition 3D.6 asserts uniform stability. The corresponding value of the parameter γ is: $\gamma = 3/4 + \epsilon$, $\epsilon > 0$ [Triggiani, 1997].

Finite Cost Condition (2.1.12) and Detectability Condition (2.1.13) As in Sections 3.11.1 and 3.11.2, we can again establish that the s.c. contraction analytic semigroup e^{At} on Y guaranteed by Proposition 3.12.4 and Appendix 3H is, moreover, uniformly stable on Y. The counterpart of Proposition 3.11.1.3 and Proposition 3.11.2.2 is

Proposition 3.12.5 *With reference to the operator $A_0 = A|_{\sigma=0}$ introduced before (3.12.42a), we have:*

(i) Its inverse A_0^{-1} in (3.12.43a) is compact on Y, so that A_0 has compact resolvent on Y, and the spectrum $\sigma(A_0)$ of A_0 is only a point spectrum, $\sigma(A_0) = \sigma_p(A_0)$, and thus consists of eigenvalues of A_0.

(ii) There is no spectrum of A_0 in the closed right-hand complex plane $\mathbb{C}^+ = \{\lambda : \operatorname{Re}\lambda \geq 0\}$; that is, $\sigma(A_0) \cap \mathbb{C}^+ = \emptyset$.

(iii) For constants $\omega_0, \omega > 0$, we have

$$\sup \operatorname{Re} \sigma(A_0) \equiv -\omega_0 < 0, \quad \sup \operatorname{Re} \sigma(A) \equiv -\omega < 0, \tag{3.12.51}$$

(iv) The s.c. contraction, analytic semigroups $e^{A_0 t}$ and e^{At}, guaranteed by the preceding analysis, are, moreover, uniformly stable on Y: With ω_0 and ω the constants in (3.12.51) and $\epsilon > 0$, there exist $M_0 \geq 1$, $M \geq 1$ depending on $[-\omega_0 + \epsilon]$ and on $[-\omega + \epsilon]$ respectively, such that

$$\|e^{A_0 t}\|_{\mathcal{L}(Y)} \leq M_0 e^{-(\omega_0 - \epsilon)t}, \qquad \|e^{At}\|_{\mathcal{L}(Y)} \leq M e^{-(\omega-\epsilon)t}, \quad t \geq 0. \tag{3.12.52}$$

Proof. The proof is the same as the proof of Proposition 3.11.1.3 or Proposition 3.11.2.2. Regarding part (ii) we provide the relevant details.

(ii) By contraction of $e^{A_0 t}$, it suffices to show the following *Claim*: There is no eigenvalue of A_0, or of A, on the imaginary axis.

Proof of Claim for A_0. For r real and $[w_1, w_2, \theta] \in \mathcal{D}(A_0) = \mathcal{D}(A)$, let

$$A \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix} = ir \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix},$$

$$\text{or} \quad \begin{cases} w_2 = ir\, w_1, & (3.12.53a) \\ -\mathcal{A}w_1 + \alpha(\mathcal{A}_N - \mathcal{A}GN^*\mathcal{A}_N)\theta = ir\, w_2 = -r^2 w_1, & (3.12.53b) \\ -ir\,\alpha\mathcal{A}_D w_1 - \eta\mathcal{A}_N\theta = ir\,\theta. & (3.12.53c) \end{cases}$$

We must show that $w_1 = w_2 = \theta = 0$. This is trivial for $r = 0$, since $\mathcal{A}_N$ and $\mathcal{A}$ are strictly positive self-adjoint, and so we assume $r \neq 0$. Recalling identity (3.12.32), we obtain, via (3.12.53) with $y = [w_1, w_2, \theta]$,

$$\operatorname{Re}\left(A_0 \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix} \right)_Y = -\eta(\mathcal{A}_N\theta, \theta)_{L_2(\Omega)} = \operatorname{Re} ir\|y\|_Y^2 = 0. \quad (3.12.54)$$

Then, since $\mathcal{A}_N$ is strictly positive self-adjoint, we obtain $\theta = 0$ by (3.12.54). Hence $w_1 = 0$ by (3.12.53c) with $r \neq 0$ and $\mathcal{A}_D$ is likewise strictly positive self-adjoint; finally $w_2 = 0$ by (3.12.53a). □

Remark 3.12.2 The uniform stability of e^{At} in Eqn. (3.12.52) is reproved in Appendix 3J, even for a more general model (possibly, a Kirchoff equation), by a very different approach, based on energy methods. This proof does *not* use the analyticity of the s.c. contraction semigroup e^{At} on Y.

Remark 3.12.3 The uniform stability of e^{At} in Eqn. (3.12.52) can be equivalently rewritten in the following form:

$$E(t; y_0) \leq Me^{-(\omega-\epsilon)t}E(0; y_0), \qquad y_0 = [w_0, w_1, \theta_0] \in Y, \quad (3.12.55)$$

where $E(t; y_0)$ is the energy of the homogeneous problem, which results from (3.12.1) by setting $u \equiv 0$ in (3.12.1d), and which is defined by

$$E(t; y_0) \equiv \int_\Omega |\Delta w(t; y_0)|^2 + 2(1-\mu)\left[w_{xy}^2(t; y_0) - w_{xx}(t; y_0)w_{yy}(t; y_0)\right] d\Omega$$

$$+ \int_\Omega |w_t(t; y_0)|^2\, d\Omega + \int_\Omega |\theta(t; y_0)|^2\, dt \quad (3.12.56)$$

$$= \|\{w(t; y_0), w_t(t; y_0), \theta(t; y_0)\}\|_Y^2 \quad (3.12.57)$$

$$= \left\| e^{At} \begin{bmatrix} w_0 \\ w_1 \\ \theta_0 \end{bmatrix} \right\|_Y^2, \quad (3.12.58)$$

recalling Y in (3.12.7) and Appendix 3C, Eqn. (3C.21) for the $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$-norm of $w(t; y_0)$.

A fortiori, then (3.12.52) implies that the Finite Cost Condition (2.1.12) and the Detectability Condition (2.1.13) of Chapter 2 are both automatically satisfied, with control $u \equiv 0$ and operator $K = 0$, respectively.

Conclusion: Case $T = \infty$ Theorems 2.2.1 and 2.2.2 apply to problem (3.12.1), (3.12.2) with $a = 0$, where $R = I$ by (3.12.8), within the above framework.

Conclusion: Case $T < \infty$ Theorem 1.2.1.1 of Chapter 1 applies to problem (3.12.1), (3.12.2) with $a = 1$, where $G = I$.

Remark 3.12.4 If one replaces the Neumann/Robin BC (3.12.1e) with the corresponding Dirichlet BC

$$\theta|_\Sigma \equiv 0, \tag{3.12.59}$$

as done in Appendix 3D, Application #2, Problem (3D.53), one finds the following corresponding operator A:

$$A = \begin{bmatrix} 0 & I & 0 \\ -\mathcal{A} & 0 & \alpha\mathcal{A}_D \\ 0 & -\alpha\mathcal{A}_D & -\eta\mathcal{A}_D \end{bmatrix} : Y \supset \mathcal{D}(A) \to Y; \tag{3.12.60a}$$

$$\mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}_D) \times \mathcal{D}(\mathcal{A}_D), \quad Y \text{ as in (3.12.7)}, \tag{3.12.60b}$$

in place of (3.12.24), but of course the same operator B in (3.12.26); in (3.12.60), $\mathcal{A}$ and $\mathcal{A}_D$ are the operators defined in (3.12.3) and (3.12.4). Then, one readily obtains

$$A^{-1} = \begin{bmatrix} -\dfrac{\alpha^2}{\eta}\mathcal{A}^{-1}\mathcal{A}_D & -\mathcal{A}^{-1} & -\dfrac{\alpha}{\eta}\mathcal{A}^{-1} \\ I & 0 & 0 \\ -\dfrac{\alpha}{\eta} I & 0 & -\dfrac{\mathcal{A}_D^{-1}}{\eta} \end{bmatrix} : Y \to \mathcal{D}(A); \quad A^{-1}Bu = \begin{bmatrix} -Gu \\ 0 \\ 0 \end{bmatrix}. \tag{3.12.61}$$

It is shown in Appendix 3D, Theorem 3D.5 (with a different notation) that the operator A in (3.12.60) *generates a s.c. contraction analytic semigroup e^{At} on Y*. The parameter γ, where $A^{-\gamma}B \in \mathcal{L}(U; Y)$, is still $\gamma = 3/4 + 2\epsilon$, by the same argument leading to (3.12.50).

3.13 Thermo-Elastic Plates with Mechanical Control as a Shear Force (Free BC)

Let Ω be a two-dimensional bounded open region (of $\mathbb{R}^2$) with smooth boundary Γ. In this section we consider the following thermo-elastic plate in the variables $w(t, x)$

(vertical displacement) and $\theta(t, x)$ (temperature):

$$
\begin{cases}
w_{tt} + \Delta^2 w + \alpha\Delta\theta = 0 & \text{in } (0, T] \times \Omega \equiv Q; \quad (3.13.1a)\\
\theta_t - \eta\Delta\theta + \sigma\theta - \alpha\Delta w_t = 0 & \text{in } Q; \quad (3.13.1b)\\
w(0, \cdot) = w_0,\ w_t(0, \cdot) = w_1;\ \theta(0, \cdot) = \theta_0 & \text{in } \Omega; \quad (3.13.1c)\\
\Delta w + (1-\mu)B_1 w + \alpha\theta = 0 & \text{in } (0, T] \times \Gamma \equiv \Sigma; \quad (3.13.1d)\\
\dfrac{\partial \Delta w}{\partial \nu} + (1-\mu)B_2 w - w + \alpha\dfrac{\partial\theta}{\partial\nu} = u & \text{in } \Sigma; \quad (3.13.1e)\\
\dfrac{\partial\theta}{\partial\nu} + b\theta \equiv 0, \quad b > 0 & \text{in } \Sigma; \quad (3.13.1f)
\end{cases}
$$

that is, with mechanical control $u \in L_2(0, T; L_2(\Gamma))$ acting as a shear force. [The case $u \equiv 0$ corresponds to the so-called free BC in w.] As in the preceding Sections 3.11 and 3.12, the constants α, η, σ, b are all positive while $0 < \mu < 1$. The boundary operators B_1 and B_2 are the same as in Section 3.5, Eqns. (3.5.2a, b), or as in Appendix 3C of Chapter 3, Eqns. (3C.11) and (3C.12), that is,

$$B_1 w \equiv 2\nu_1\nu_2 w_{xy} - \nu_1^2 w_{yy} - \nu_2^2 w_{xx} \quad \text{on } \Gamma, \tag{3.13.2}$$

$$B_2 w \equiv \frac{\partial}{\partial\tau}\left[\left(\nu_1^2 - \nu_2^2\right)w_{xy} + \nu_1\nu_2(w_{yy} - w_{xx})\right] \quad \text{on } \Gamma, \tag{3.13.3}$$

with $\nu(x) = [\nu_1(x), \nu_2(x)]$ the unit outward normal at $x \in \Gamma$. Consistently with regularity theory, we select the following cost functional to be minimized:

$$
\begin{aligned}
&J(u, w, \theta)\\
&= \int_0^T \left\{ \left\|\mathcal{A}^{\frac{1}{2}} w(t)\right\|^2_{L_2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} + \|\theta(t)\|^2_{L_2(\Omega)} + \|u(t)\|^2_{L_2(\Gamma)} \right\} dt\\
&\quad + a\|\{w(T), w_t(T), \theta(T)\}\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})\times L_2(\Omega)\times L_2(\Omega)},
\end{aligned}
\tag{3.13.4}
$$

where $\{w_0, w_1, \theta_0\} \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega)$; moreover, $a = 0$ if $T = \infty$ and $a = 1$ if $T < \infty$. In (3.13.4), we have denoted by $\mathcal{A}$ the same strictly positive self-adjoint operator introduced in Section 3.5, Eqn. (3.5.3), or in Appendix 3C of Chapter 3, Eqn. (3C.26), whose strict positive self-adjointness is proved in Appendix 3C, Proposition 3C.5:

$$
\begin{aligned}
\mathcal{A}h &= \Delta^2 h,\\
\mathcal{D}(\mathcal{A}) &= \Bigg\{ h \in H^4(\Omega) : [\Delta h + (1-\mu)B_1 h]_\Gamma = 0;\\
&\qquad \left[\frac{\partial \Delta h}{\partial\nu} + (1-\mu)B_2 h - h\right]_\Gamma = 0 \Bigg\},
\end{aligned}
\tag{3.13.5}
$$

whereby, via [Grisvard, 1967] and Appendix 3A,

$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega), \tag{3.13.6a}$$

$$\begin{aligned}\|\mathcal{A}^{\frac{1}{2}} w(t)\|^2_{L_2(\Omega)} &= \int_\Omega |\Delta w(t)|^2 \, d\Omega + 2(1-\mu) \int_\Omega \left[w_{xy}^2(t) - w_{xx}(t) w_{yy}(t)\right] d\Omega \\ &\quad + \int_\Gamma |w(t)|^2 d\Gamma, \qquad (3.13.6b) \\ &= \int_\Omega \left\{\mu |\Delta w|^2 + (1-\mu)\left(w_{xx}^2 + w_{yy}^2\right) \right. \\ &\quad \left. + 2(1-\mu) w_{xy}^2 \right\} d\Omega + \int_\Gamma w^2 \, d\Gamma \qquad (3.13.6c)\end{aligned}$$

recalling Appendix 3C, Eqn. (3C.27) of Proposition 3C.5 for (3.13.6b).

Abstract Setting To put problem (3.13.1)–(3.13.4) into the abstract form (1.1.1) of the preceding Chapters 1 and 2, we introduce, in addition to the operator $\mathcal{A}$ above in (3.13.5), the following spaces and operators:

(i)

$$\begin{aligned} Y &\equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega) \\ &\equiv H^2(\Omega) \times L_2(\Omega) \times L_2(\Omega) \quad \text{(equivalent norms);} \qquad (3.13.7) \end{aligned}$$

$$Z = Z_f = Y; \quad U = L_2(\Gamma); \quad R = G = I. \tag{3.13.8}$$

(ii) The positive self-adjoint operator as in Chapter 12, Eqn. (3.12.5):

$$\mathcal{A}_N h = -\Delta h;$$

$$\mathcal{D}(\mathcal{A}_N) = \left\{ h \in H^2(\Omega) : \left[\frac{\partial h}{\partial \nu} + bh\right]_\Gamma = 0 \right\} \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}}); \tag{3.13.9}$$

$$\mathcal{A}^{\frac{1}{2}} \mathcal{A}_N^{-1} \in \mathcal{L}(L_2(\Omega)). \tag{3.13.10}$$

(iii) The Green operator G_1 corresponding to the first mechanical BC (3.13.1d):

$$h \equiv G_1 g \Longleftrightarrow \begin{cases} \Delta^2 h = 0 \quad \text{in } \Omega; & (3.13.11a) \\ [\Delta h + (1-\mu) B_1 h]_\Gamma = g; & (3.13.11b) \\ \left[\dfrac{\partial \Delta h}{\partial \nu} + (1-\mu) B_2 h - h\right]_\Gamma = 0. & (3.13.11c) \end{cases}$$

which is a regular elliptic problem for $0 < \mu < 1$. (The Lopatinski–Shapiro condition [Lions, Magenes, 1972], [Wlokea, 1987, p. 148] is satisfied for $\mu \neq 1$). Elliptic regularity [Lions, Magenes, 1972, pp. 188–9] gives

$$G_1 : \text{ continuous } L_2(\Gamma) \to H^{\frac{5}{2}}(\Omega) \subset H^{\frac{5}{2}-4\epsilon}(\Omega) \equiv \mathcal{D}(\mathcal{A}^{\frac{5}{8}-\epsilon}), \quad \epsilon > 0, \tag{3.13.12a}$$

where the identification in (3.13.12a) follows from Grisvard [1967] and Appendix 3A, Eqn. (3A.11). Equivalently,

$$\mathcal{A}^{\frac{5}{8}-\epsilon}G_1 : \text{ continuous } L_2(\Gamma) \to L_2(\Omega). \tag{3.13.12b}$$

(iv) The Green operator G_2 corresponding to the second mechanical BC (3.13.1e):

$$h \equiv G_2 g \iff \begin{cases} \Delta^2 h = 0 \quad \text{in } \Omega; & (3.13.13a) \\ [\Delta h + (1-\mu)B_1 h]_\Gamma = 0; & (3.13.13b) \\ \left[\dfrac{\partial \Delta h}{\partial \nu} + (1-\mu)B_2 h - h\right]_\Gamma = g. & (3.13.13c) \end{cases}$$

which is likewise a regular elliptic problem for $0 < \mu < 1$. Elliptic regularity [Lions, Magenes, 1972, pp. 188–9] gives

$$\begin{cases} G_2 : \text{ continuous } L_2(\Gamma) \to H^{\frac{7}{2}}(\Omega) \subset H^{\frac{7}{2}-4\epsilon}(\Omega) = \mathcal{D}\big(\mathcal{A}^{\frac{7}{8}-\epsilon}\big) & (3.13.14a) \\ \quad = \big\{h \in H^{\frac{7}{2}-4\epsilon}(\Omega) : [\Delta h + (1-\mu)B_1 h]_\Gamma = 0\big\}; & (3.13.14b) \\ \mathcal{A}^{\frac{7}{8}-\epsilon}G_2 : \text{ continuous } L_2(\Gamma) \to L_2(\Omega), & (3.13.14c) \end{cases}$$

invoking again Appendix 3A, Eqn. (3A.11) for the identification in (3.13.14a, b).

Next, using the definitions of G_1 and G_2 in (3.13.11) and (3.13.13), we may rewrite Eqns. (3.13.1a) and (3.13.1d, e) for w as:

$$\begin{cases} w_{tt} + \Delta^2\left[w - G_1(-\alpha(\theta|_\Gamma)) - G_2\left(u - \alpha\dfrac{\partial\theta}{\partial\nu}\right)\right] + \alpha\Delta\theta = 0 \text{ in } Q, & (3.13.15) \\ \left[[\Delta + (1-\mu)B_1][w + G_1(\alpha\theta|_\Gamma) - G_2\left(u - \alpha\dfrac{\partial\theta}{\partial\nu}\right)\right] \equiv 0 \text{ on } \Sigma, & (3.13.16) \\ \left[\left[\dfrac{\partial\Delta}{\partial\nu} + (1-\mu)B_1 - 1\right][w + G_1(\alpha\theta|_\Gamma) - G_2\left(u - \alpha\dfrac{\partial\theta}{\partial\nu}\right)\right] \equiv 0 \text{ on } \Sigma. & (3.13.17) \end{cases}$$

Hence, using the definition (3.13.5) of $\mathcal{A}$ on problem (3.13.15)–(3.13.17) and the definition (3.13.9) of $\mathcal{A}_N$ on the θ component of Eq. (3.13.15), we may rewrite problem (3.3.15)–(3.13.17) in the following abstract form:

$$w_{tt} + \mathcal{A}\left[w + G_1(\alpha\theta|_\Gamma) - G_2\left(u - \alpha\frac{\partial\theta}{\partial\nu}\right)\right] - \alpha\mathcal{A}_N\theta = 0. \tag{3.13.18}$$

Finally, returning to Eqs. (3.13.1b), (3.13.1f) for θ, we rewrite problem (3.13.1) abstractly via (3.13.18) as

$$\begin{cases} w_{tt} + \mathcal{A}w + \mathcal{A}G_1(\alpha(\theta|_\Gamma) + \mathcal{A}G_2\left(\alpha\dfrac{\partial\theta}{\partial\nu}\right) - \alpha\mathcal{A}_N\theta = \mathcal{A}G_2u \in [\mathcal{D}(\mathcal{A})]', & (3.13.19) \\ \theta_t + \eta\mathcal{A}_N\theta + \sigma\theta - \alpha\Delta w_t = 0, & (3.13.20) \end{cases}$$

where in going from (3.13.18) to (3.13.19), we have, as usual, extended the original operator $\mathcal{A}$ in (3.13.5) to $\mathcal{A}: L_2(\Omega) \to [\mathcal{D}(\mathcal{A}^*)]' = [\mathcal{D}(\mathcal{A})]'$, by isomorphism, where the duality is with respect to $L_2(\Omega)$, as a pivot space. Setting $y = [w, w_t, \theta]$, we then rewrite the second-order system in (3.13.19), (3.13.20) as the first-order equation $\dot{y} = Ay + Bu$ as in Eqn. (1.1.1) of Chapter 1:

$$\frac{d}{dt}\begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix} = A\begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix} + Bu, \quad y = \begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix}, \tag{3.13.21}$$

$$A = \begin{bmatrix} 0 & I & 0 \\ -\mathcal{A} & 0 & -\alpha\mathcal{A}G_1(\cdot|_\Gamma) - \alpha\mathcal{A}G_2\dfrac{\partial\,\cdot}{\partial\nu} + \alpha\mathcal{A}_N \\ 0 & \alpha\Delta & -(\eta\mathcal{A}_N + \sigma I) \end{bmatrix} : Y \supset \mathcal{D}(A) \to Y, \tag{3.13.22a}$$

to be interpreted in the sense that

$$A\begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix} = \begin{bmatrix} w_2 \\ -\mathcal{A}\left[w_1 + \alpha G_1(\theta|_\Gamma) + \alpha G_2\dfrac{\partial\theta}{\partial\nu}\right] + \alpha\mathcal{A}_N\theta \\ \alpha\Delta w_2 - \eta\mathcal{A}_N\theta - \sigma\theta \end{bmatrix}, \quad \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix} \in \mathcal{D}(A), \tag{3.13.22b}$$

where, recalling Y in (3.13.7) and $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega)$ from (3.13.6a), we have from (3.13.22b)

$$\mathcal{D}(A) = \left\{ w_1 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}});\ w_2 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}});\ \theta \in \mathcal{D}(\mathcal{A}_N) : \left[w_1 + \alpha G_1(\theta|_\Gamma) + \alpha G_2\frac{\partial\theta}{\partial\nu}\right] \in \mathcal{D}(\mathcal{A}) \right\}, \tag{3.13.23}$$

$$Bu = \begin{bmatrix} 0 \\ \mathcal{A}G_2u \\ 0 \end{bmatrix}, \quad u \in U = L_2(\Gamma); \quad B: \text{ continuous } U \to [\mathcal{D}(A^*)]'. \tag{3.13.24}$$

The continuity of B, asserted in (3.13.24), with duality with respect to Y as a pivot space, will be a fortiori proved in (3.13.58) and (3.13.59) below.

The Operators $G_1^*\mathcal{A}$ and $G_2^*\mathcal{A}$ We now provide two results, to be needed below, describing the operators $G_1^*\mathcal{A}$ and $G_2^*\mathcal{A}$, where

$$(G_i u, y)_{L_2(\Omega)} \equiv (u, G_i^* y)_{L_2(\Gamma)}, \quad \forall\, u \in L_2(\Gamma),\, y \in L_2(\Omega), \tag{3.13.25}$$

since $G_i u \in L_2(\Omega)$, a fortiori by (3.13.12) and (3.13.14). These results are the present counterpart of Section 3.12, Lemma 3.12.1; Section 3.1, Lemma 3.1.1; Section 3.3, Lemma 3.3.1.1; etc.

Lemma 3.13.1

(i) With reference to G_1 in (3.13.11), to (3.13.25), and to $\mathcal{A}$ in (3.13.5), we have

$$G_1^*\mathcal{A} f = \frac{\partial f}{\partial \nu}, \qquad f \in \mathcal{D}(\mathcal{A}). \tag{3.13.26}$$

(ii) Since $G_1^*\mathcal{A}^{\frac{5}{8}-\epsilon} \in \mathcal{L}(L_2(\Omega); L_2(\Gamma))$ by (3.13.12b), Eqn. (3.13.26) says that $G_1^*\mathcal{A}$ extends the normal derivative $\partial/\partial\nu$ to all $f \in \mathcal{D}(\mathcal{A}^{\frac{3}{8}+\epsilon}) = H^{\frac{3}{2}+4\epsilon}(\Omega)$.

Proof. Let $g \in L_2(\Gamma)$ and $f \in \mathcal{D}(\mathcal{A})$; see (3.13.5). Applying Green's second theorem twice, we obtain via (3.13.25)

$$\begin{aligned}
(G_1^*\mathcal{A} f, g)_{L_2(\Gamma)} &= (\mathcal{A} f, G_1 g)_{L_2(\Omega)} = \int_\Omega (\Delta^2 f)(G_1 g)\, d\Omega \\
&= \int_\Omega (\Delta f)(\Delta G_1 g)\, d\Omega + \int_\Gamma \frac{\partial \Delta f}{\partial \nu}(G_1 g)\, d\Gamma - \int_\Gamma (\Delta f)\frac{\partial G_1 g}{\partial \nu}\, d\Gamma \\
\text{(by (3.13.11a))} \quad &= \int_\Omega f\cancel{(\Delta^2 G_1 g)}\, d\Omega + \int_\Gamma \frac{\partial f}{\partial \nu}(\Delta G_1 g)\, d\Gamma - \int_\Gamma f\, \frac{\partial(\Delta G_1 g)}{\partial \nu}\, d\Gamma \\
&\quad + \int_\Gamma \frac{\partial \Delta f}{\partial \nu}(G_1 g) d\Gamma - \int_\Gamma (\Delta f)\frac{\partial G_1 g}{\partial \nu}\, d\Gamma.
\end{aligned} \tag{3.13.27}$$

We now use $\Delta G_1 g = g - (1-\mu) B_1 G_1 g$ on Γ and $\partial(\Delta G_1 g)/\partial\nu = G_1 g - (1-\mu)$ $B_2 G_1 g$ on Γ, from (3.13.11b, c), respectively; moreover, we use $\partial\Delta f/\partial\nu = f - (1-\mu)B_2 f$ on Γ and $\Delta f = -(1-\mu)B_1 f$ on Γ, from (3.13.5), since $f \in \mathcal{D}(\mathcal{A})$. We then obtain from (3.13.27):

$$\begin{aligned}
(G_1^*\mathcal{A} f, g)_{L_2(\Gamma)} &= \int_\Gamma \frac{\partial f}{\partial \nu}\, g\, d\Gamma - (1-\mu)\int_\Gamma \frac{\partial f}{\partial \nu}(B_1 G_1 g)\, d\Gamma \\
&\quad - \int_\Gamma \cancel{f(G_1 g)\, d\Gamma} + (1-\mu)\int_\Gamma f(B_2 G_1 g)\, d\Gamma \\
&\quad + \int_\Gamma \cancel{f(G_1 g)\, d\Gamma} - (1-\mu)\int_\Gamma (B_2 f)(G_1 g)\, d\Gamma \\
&\quad + (1-\mu)\int_\Gamma (B_1 f)\frac{\partial G_1 g}{\partial \nu}\, d\Gamma.
\end{aligned} \tag{3.13.28}$$

Next, we invoke Appendix 3C, Corollary 3C.3, Eqn. (3C.19a) with $w = G_1 g$ and $v = f$, and we obtain that the four integral terms in (3.13.28) involving the boundary operators B_1 and B_2 sum to zero. Thus we finally obtain from (3.13.28):

$$(G_1^* \mathcal{A} f, g)_{L_2(\Gamma)} = \int_\Gamma \frac{\partial f}{\partial \nu} g \, d\Gamma, \quad \forall\, g \in L_2(\Gamma),\ f \in \mathcal{D}(\mathcal{A}), \tag{3.13.29}$$

as desired. Equation (3.13.29) proves (3.13.26).

Lemma 3.13.2

(i) With reference to G_2 in (3.13.13), to (3.13.25), and to $\mathcal{A}$ in (3.13.5), we have

$$G_2^* \mathcal{A} f = -f|_\Gamma, \quad f \in \mathcal{D}(\mathcal{A}). \tag{3.13.30}$$

(ii) Since $G_2^* \mathcal{A}^{\frac{7}{8}-\epsilon} \in \mathcal{L}(L_2(\Omega); L_2(\Gamma))$ by (3.13.14c), Eqn. (3.13.30) says that $G_2^* \mathcal{A}$ extends the negative $-(\cdot|_\Gamma)$ of the Dirichlet trace to all $f \in \mathcal{D}(\mathcal{A}^{\frac{1}{8}+\epsilon}) = H^{\frac{1}{2}+4\epsilon}(\Omega)$.

Proof. Let $G \in L_2(\Gamma)$ and $f \in \mathcal{D}(\mathcal{A})$; see (3.13.5). We similarly apply Green's second theorem twice, to obtain the counterpart of (3.13.27), where G_1 is now replaced by G_2:

$$\begin{aligned}
(G_2^* \mathcal{A} f, g)_{L_2(\Gamma)} &= (\mathcal{A} f, G_2 g)_{L_2(\Omega)} = \int_\Omega (\Delta^2 f)(G_2 g)\, d\Omega \\
\text{(by (3.13.13a))} \quad &= \int_\Omega f(\cancel{\Delta^2} G_2 g)\, d\Omega + \int_\Gamma \frac{\partial f}{\partial \nu}(\Delta G_2 g)\, d\Gamma - \int_\Gamma f \frac{\partial(\Delta G_2 g)}{\partial \nu}\, d\Gamma \\
&\quad + \int_\Gamma \frac{\partial \Delta f}{\partial \nu}(G_2 g)\, d\Gamma - \int_\Gamma \Delta f \frac{\partial G_2 g}{\partial \nu}\, d\Gamma.
\end{aligned} \tag{3.13.31}$$

We now use $\Delta G_2 g = -(1-\mu) B_1 G_2 g$ on Γ and $\partial(\Delta G_2 g)/\partial \nu = g + G_2 g - (1 - \mu) B_2 G_2 g$ on Γ, from (3.13.13b, c), respectively; moreover, we use $\partial \Delta f / \partial \nu = f - (1-\mu) B_2 f$ on Γ and $\Delta f = -(1-\mu) B_1 f$ on Γ, by (3.13.5), since $f \in \mathcal{D}(\mathcal{A})$. We then obtain from (3.13.31):

$$\begin{aligned}
(G_2^* \mathcal{A} f, g)_{L_2(\Gamma)} &= -(1-\mu) \int_\Gamma \frac{\partial f}{\partial \nu}(B_1 G_2 g)\, d\Gamma - \int_\Gamma f g \, d\Gamma \\
&\quad - \int_\Gamma f(\cancel{G_2} g)\, d\Gamma + (1-\mu) \int_\Gamma f(B_2 G_2 g)\, d\Gamma \\
&\quad + \int_\Gamma f(\cancel{G_2} g)\, d\Gamma - (1-\mu) \int_\Gamma (B_2 f)(G_2 g)\, d\Gamma \\
&\quad + (1-\mu) \int_\Gamma (B_1 f) \frac{\partial G_2 g}{\partial \nu}\, d\Gamma.
\end{aligned} \tag{3.13.32}$$

Next, we invoke once more Appendix 3C, Corollary 3C.3, Eqn. (3C.19a), this time with $w = G_2 g$ and $v = f$, to obtain again that the four integral terms in (3.13.32) involving the boundary operators B_1 and B_2 sum to zero. We thus finally obtain from (3.13.32):

$$(G_2^* \mathcal{A} f, g)_{L_2(\Gamma)} = -\int_\Gamma fg \, d\Gamma, \quad \forall\, g \in L_2(\Gamma),\ f \in \mathcal{D}(\mathcal{A}), \tag{3.13.33}$$

as desired. Equation (3.13.33) proves (3.13.30). □

Dissipativity of A. Lemma 3.13.3 *With reference to the operator A in (3.13.22) and with $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega)$ as in (3.13.7), we have*

$$\operatorname{Re}\left(A \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix} \right)_Y$$

$$= \operatorname{Re}\left(\begin{bmatrix} 0 & -\alpha \mathcal{A} G_1(\cdot|_\Gamma) - \alpha \mathcal{A} G_2 \frac{\partial \cdot}{\partial \nu} + \alpha \mathcal{A}_N \\ \alpha \Delta & -(\eta \mathcal{A}_N + \sigma I) \end{bmatrix} \begin{bmatrix} w_2 \\ \theta \end{bmatrix}, \begin{bmatrix} w_2 \\ \theta \end{bmatrix} \right)_Y \tag{3.13.34}$$

$$= -((\eta \mathcal{A}_N + \sigma I)\theta, \theta)_{L_2(\Omega)} \le 0, \quad \forall\, [w_1, w_2, \theta] \in \mathcal{D}(A). \tag{3.13.35}$$

Proof. Since $\begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix}$ is skew-adjoint on $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$, it remains to show that

$$\operatorname{Re}\left(\begin{bmatrix} 0 & -\alpha \mathcal{A} G_1(\cdot|_\Gamma) - \alpha \mathcal{A} G_2 \frac{\partial \cdot}{\partial \nu} + \alpha \mathcal{A}_N \\ \alpha \Delta & 0 \end{bmatrix} \begin{bmatrix} w_2 \\ \theta \end{bmatrix}, \begin{bmatrix} w_2 \\ \theta \end{bmatrix} \right)_{L_2(\Omega) \times L_2(\Omega)} = 0,$$

$$\forall\, [w_2, \theta] \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(\mathcal{A}_N). \tag{3.13.36}$$

To prove (3.13.36), we compute with $[w_2, \theta] \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(\mathcal{A}_N)$, via (3.13.9) on $\mathcal{A}_N$:

$$\operatorname{Re}\left(\begin{bmatrix} 0 & -\alpha \mathcal{A} G_1(\cdot|_\Gamma) - \alpha \mathcal{A} G_2 \frac{\partial \cdot}{\partial \nu} + \alpha \mathcal{A}_N \\ \alpha \Delta & 0 \end{bmatrix} \begin{bmatrix} w_2 \\ \theta \end{bmatrix}, \begin{bmatrix} w_2 \\ \theta \end{bmatrix} \right)_{L_2(\Omega) \times L_2(\Omega)}$$

$$= \operatorname{Re}\left\{ -\alpha \left(\theta|_\Gamma, G_1^* \mathcal{A} w_2\right)_{L_2(\Gamma)} - \alpha \left(\frac{\partial \theta}{\partial \nu}, G_2^* \mathcal{A} w_2 \right)_{L_2(\Gamma)} \right.$$

$$\left. - \alpha \int_\Omega (\Delta \theta) w_2 \, d\Omega + \alpha \int_\Omega (\Delta w_2) \theta \, d\Omega \right\}. \tag{3.13.37}$$

Recalling (3.13.26) on $G_1^* \mathcal{A}$ and (3.13.30) on $G_2^* \mathcal{A}$ and using Green's second theorem on the last integral term of (3.13.37),

$$
\begin{aligned}
&= \operatorname{Re}\left\{ -\alpha \left(\theta|_\Gamma, \frac{\partial \bar{w}_2}{\partial \nu} \right)_{L_2(\Gamma)} - \alpha \left(\frac{\partial \theta}{\partial \nu}, -\bar{w}_2|_\Gamma \right)_{L_2(\Gamma)} - \alpha \int_\Omega (\Delta\theta)\bar{w}_2 \, d\Omega \right. \\
&\quad \left. + \alpha \int_\Omega w_2 (\Delta\bar{\theta}) \, d\Omega + \alpha \left(\left.\frac{\partial w_2}{\partial \nu}\right|_\Gamma, \bar{\theta}|_\Gamma \right)_{L_2(\Gamma)} - \alpha \left(w_2|_\Gamma, \left.\frac{\partial \bar{\theta}}{\partial \nu}\right|_\Gamma \right)_{L_2(\Gamma)} \right\} = 0,
\end{aligned}
\tag{3.13.38}
$$

as desired. Thus, (3.13.38) proves (3.13.36). □

Adjoint A^* of A and Its Dissipativity For convenience, we define the operator A_c by omitting the bottom-right corner element $-(\eta \mathcal{A}_N + \sigma I)$ from A in (3.13.22a), that is, by setting

$$
A_c = \begin{bmatrix} 0 & I & 0 \\ -\mathcal{A} & 0 & -\alpha \mathcal{A} G_1(\cdot|_\Gamma) - \alpha \mathcal{A} G_2 \dfrac{\partial \cdot}{\partial \nu} + \alpha \mathcal{A}_N \\ 0 & \alpha\Delta & 0 \end{bmatrix} : Y \supset \mathcal{D}(A_c) = \mathcal{D}(A) \to Y.
\tag{3.13.39}
$$

Lemma 3.13.4

(i) *The Y-adjoint A^* of the operator A defined in (3.13.22) and (3.13.23), $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega)$ as in (3.13.7), is*

$$
A_c^* = \begin{bmatrix} 0 & -I & 0 \\ \mathcal{A} & 0 & \alpha\Delta + \alpha \mathcal{A} G_1(\cdot|_\Gamma) + \alpha \mathcal{A} G_2 \dfrac{\partial \cdot}{\partial \nu} \\ 0 & -\alpha\Delta & -\eta \mathcal{A}_N - \sigma I \end{bmatrix}
\tag{3.13.40a}
$$

$$
= \begin{bmatrix} 0 & -I & 0 \\ \mathcal{A} & 0 & \alpha\Delta - \alpha \mathcal{A} G_1 G_2^* \mathcal{A} + \alpha \mathcal{A} G_2 G_1^* \mathcal{A} \\ 0 & -\alpha\Delta & -\eta \mathcal{A}_N - \sigma I \end{bmatrix},
\tag{3.13.40b}
$$

to be interpreted as

$$
A^* \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix} = \begin{bmatrix} -w_2 \\ \mathcal{A}\left[w_1 + \alpha G_1(\theta|_\Gamma) + \alpha G_2 \dfrac{\partial \theta}{\partial \nu} \right] + \alpha \Delta\theta \\ -\alpha\Delta w_2 - \eta \mathcal{A}_N \theta - \sigma\theta \end{bmatrix} \in Y,
\tag{3.13.40c}
$$

$$
\begin{aligned}
\mathcal{D}(A^*) = \Big\{ & w_1 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}});\ w_2 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega);\ \theta \in \mathcal{D}(\mathcal{A}_N): \\
& \left[w_1 + \alpha G_1(\theta|_\Gamma) + \alpha G_2 \frac{\partial \theta}{\partial \nu} \right] \in \mathcal{D}(\mathcal{A}) \Big\} = \mathcal{D}(A).
\end{aligned}
\tag{3.13.41}
$$

(ii) With reference to A_c in (3.13.39), we have for $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega)$,

$$\operatorname{Re}\left(A_c^* \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix}\right)_Y = 0, \quad \forall\, [w_1, w_2, \theta] \in \mathcal{D}(A_c^*) = \mathcal{D}(A^*). \tag{3.13.42}$$

(iii) The operator A_c in (3.13.39) is skew-adjoint on Y, and thus it generates a s.c. unitary group on Y.

(iv) The Y-adjoint A^* of the operator A defined by (3.13.22) is

$$A^* = A_c^* + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -(\eta \mathcal{A}_N + \sigma I) \end{bmatrix}, \quad \mathcal{D}(A^*) = \mathcal{D}(A_c^*) = \mathcal{D}(A), \tag{3.13.43}$$

so that A^* is dissipative on Y:

$$\operatorname{Re}\left(A^* \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix}\right)_Y = -((\eta \mathcal{A}_N + \sigma I)\theta, \theta)_{L_2(\Omega)} \leq 0,$$
$$\forall\, [w_1, w_2, \theta] \in \mathcal{D}(A^*). \tag{3.13.44}$$

Proof. (i) The adjoint of $\left[\begin{smallmatrix} 0 & I \\ -\mathcal{A} & 0 \end{smallmatrix}\right]$ is $\left[\begin{smallmatrix} 0 & -I \\ \mathcal{A} & 0 \end{smallmatrix}\right]$ with respect to the space $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$. Thus, to obtain the Y-adjoint A_c^* of A_c defined by (3.13.39), with Y as in (3.13.7), it suffices to take $[w_2, \theta]$ in the last two components of $\mathcal{D}(A_c)$, $[\bar{w}_2, \bar{\theta}]$ in the last two components of $\mathcal{D}(A_c^*)$ defined by (3.13.41) and to verify that

$$\left(\begin{bmatrix} 0 & -\alpha \mathcal{A} G_1(\cdot|_\Gamma) - \alpha \mathcal{A} G_2 \frac{\partial \cdot}{\partial \nu} + \alpha \mathcal{A}_N \\ \alpha \Delta & 0 \end{bmatrix} \begin{bmatrix} w_2 \\ \theta \end{bmatrix}, \begin{bmatrix} \bar{w}_2 \\ \bar{\theta} \end{bmatrix}\right)_{L_2(\Omega) \times L_2(\Omega)}$$
$$= \left(\begin{bmatrix} w_2 \\ \theta \end{bmatrix}, \begin{bmatrix} 0 & \alpha \Delta + \alpha \mathcal{A} G_1(\cdot|_\Gamma) + \alpha \mathcal{A} G_2 \frac{\partial \cdot}{\partial \nu} \\ -\alpha \Delta & 0 \end{bmatrix} \begin{bmatrix} \bar{w}_2 \\ \bar{\theta} \end{bmatrix}\right)_{L_2(\Omega) \times L_2(\Omega)} \tag{3.13.45}$$

In fact, taking such $[w_2, \theta]$, $[\bar{w}_2, \bar{\theta}]$, we compute

$$\left(\begin{bmatrix} 0 & -\alpha \mathcal{A} G_1(\cdot|_\Gamma) - \alpha \mathcal{A} G_2 \frac{\partial \cdot}{\partial \nu} + \alpha \mathcal{A}_N \\ \alpha \Delta & 0 \end{bmatrix} \begin{bmatrix} w_2 \\ \theta \end{bmatrix}, \begin{bmatrix} \bar{w}_2 \\ \bar{\theta} \end{bmatrix}\right)_{L_2(\Omega) \times L_2(\Omega)}$$
$$= -\alpha(\theta|_\Gamma, G_1^* \mathcal{A} \bar{w}_2)_{L_2(\Gamma)} - \alpha \left(\frac{\partial \theta}{\partial \nu}, G_2^* \mathcal{A} \bar{w}_2\right)_{L_2(\Gamma)}$$
$$-\alpha \int_\Omega (\Delta \theta) \bar{w}_2 \, d\Omega + \alpha \int_\Omega (\Delta w_2) \bar{\theta} \, d\Omega \tag{3.13.46}$$

[recalling (3.13.26) on $G_1^*\mathcal{A}$; (3.13.30) on $G_2^*\mathcal{A}$, and applying Green's second theorem to both integrals over Ω in (3.13.46)]

$$
\begin{aligned}
&= -\alpha \cancel{\left(\theta|_\Gamma, \frac{\partial \bar{w}_2}{\partial \nu}\right)_{L_2(\Gamma)}} + \alpha \cancel{\left(\frac{\partial \theta}{\partial \nu}, \bar{w}_2|_\Gamma\right)_{L_2(\Gamma)}} - \alpha \int_\Omega \theta(\Delta \bar{w}_2)\, d\Omega \\
&\quad + \alpha \cancel{\left(\theta|_\Gamma, \frac{\partial \bar{w}_2}{\partial \nu}\right)_{L_2(\Gamma)}} - \alpha \cancel{\left(\frac{\partial \theta}{\partial \nu}, \bar{w}_2|_\Gamma\right)_{L_2(\Gamma)}} + \alpha (w_2, \Delta \bar{\theta})_{L_2(\Omega)} \\
&\quad + \alpha \left(\frac{\partial w_2}{\partial \nu}, \bar{\theta}|_\Gamma,\right)_{L_2(\Gamma)} - \alpha \left(w_2, \frac{\partial \bar{\theta}}{\partial \nu}\right)_{L_2(\Gamma)}
\end{aligned}
\tag{3.13.47}
$$

[invoking (3.13.26) and (3.13.30) once more]

$$
\begin{aligned}
&= -\alpha(\theta, \Delta \bar{w}_2)_{L_2(\Omega)} + \alpha(w_2, \Delta\bar{\theta})_{L_2(\Omega)} \\
&\quad + \alpha(G_1^* \mathcal{A} w_2, \bar{\theta}|_\Gamma)_{L_2(\Gamma)} + \alpha \left(G_2^* \mathcal{A} w_2, \frac{\partial \bar{\theta}}{\partial \nu}\right)_{L_2(\Gamma)}
\end{aligned}
\tag{3.13.48}
$$

$$
\begin{aligned}
&= -\alpha(\theta, \Delta \bar{w}_2)_{L_2(\Omega)} + \alpha(w_2, \Delta\bar{\theta})_{L_2(\Omega)} \\
&\quad + \alpha\big(w_2, \mathcal{A} G_1(\bar{\theta}|_\Gamma)\big)_{L_2(\Omega)} + \alpha \left(w_2, \mathcal{A} G_2 \frac{\partial \theta}{\partial \nu}\right)_{L_2(\Omega)}
\end{aligned}
\tag{3.13.49}
$$

$$
= \left(\begin{bmatrix} w_2 \\ \theta \end{bmatrix}, \begin{bmatrix} 0 & \alpha\Delta + \alpha \mathcal{A} G_1(\cdot|_\Gamma) + \alpha \mathcal{A} G_2 \frac{\partial \cdot}{\partial \nu} \\ -\alpha \Delta & 0 \end{bmatrix} \begin{bmatrix} \bar{w}_2 \\ \bar{\theta} \end{bmatrix} \right)_{L_2(\Omega)\times L_2(\Omega)}, \tag{3.13.50}
$$

as desired, and (3.13.45) is proved. Then, (3.13.45) leads to (3.13.40a), from which (3.13.40b) follows via (3.13.26) and (3.13.30).

(ii) Equation (3.13.42) is obtained by noticing that, when $w_2 = \bar{w}_2, \theta = \bar{\theta}$ in $\mathcal{D}(A_c^*)$, the expression in (3.13.47) becomes zero by Green's second theorem.

(iii) The identity in Eqn. (3.13.34) means $\mathrm{Re}(A_c x, x)_Y = 0, \forall\, x \in \mathcal{D}(A_c)$, with A_c defined by (3.13.39). This, combined with Eqn. (3.13.42), yields part (iii) by a standard result [Balakrishnan, 1981, p. 188].

(iv) This is an immediate consequence of parts (i) and (ii). □

A is a Generator of a s.c. Semigroup. Proposition 3.13.5 *The operator A defined by (3.13.22) is the generator of a s.c. contraction semigroup e^{At} on $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega)$; see (3.13.7).*

Proof. A is closed and A and A^* are both dissipative by (3.13.35) and (3.13.44). Thus [Pazy, 1983, p. 15; Balakrishnan, 1981, p. 188], A is maximal dissipative and the Lumer–Phillips theorem applies, yielding the desired conclusion for e^{At}. [Alternatively, one may check maximal dissipativity of A directly.] □

Proposition 3.13.6 *The operator A in (3.13.22) has compact resolvent on Y and has no (point) spectrum $\sigma(A)$ on $\mathbb{C}^+ = \{\lambda : \operatorname{Re}\lambda \geq 0\} : \sigma(A) \cap \mathbb{C}^+ = \emptyset$.*

Proof. (i) We first show that there is no point spectrum on the imaginary axis. Let r be real, $[w_1, w_2, \theta] \in \mathcal{D}(A)$, and let

$$A \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix} = ir \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix}, \tag{3.13.51}$$

or by (3.13.22b),

$$\begin{cases} w_2 = ir\, w_1, & \text{(3.13.52a)} \\ -\mathcal{A}\left[w_1 + \alpha G_1(\theta|_\Gamma) + \alpha G_2 \dfrac{\partial \theta}{\partial \nu} \right] + \alpha \mathcal{A}_N \theta = ir\, w_2 = -r^2 w_1, & \text{(3.13.52b)} \\ i\alpha r \Delta w_1 - \eta(\mathcal{A}_N + \sigma I)\theta = ir\theta. & \text{(3.13.52c)} \end{cases}$$

We must show that $w_1 = w_2 = \theta = 0$. This is trivial for $r = 0$ since $\mathcal{A}_N$ and $\mathcal{A}$ are strictly positive self-adjoint, and so we assume $r \neq 0$. Recalling identity (3.13.35) [i.e., that the operator A_c defined by (3.13.39) satisfies $\operatorname{Re}(A_c x, x) = 0$, $x \in \mathcal{D}(A)$], we obtain via (3.12.51), with $y = [w_1, w_2, \theta]$:

$$\operatorname{Re} \left(A \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ \theta \end{bmatrix} \right)_Y = ((\eta \mathcal{A}_N + \sigma I)\theta, \theta)_{L_2(\Omega)} = \operatorname{Re}\big(ir\|y\|_Y^2\big) = 0. \tag{3.13.53}$$

Then, (3.13.53) implies $\theta = 0$, since $\mathcal{A}_N$ is strictly positive. Then, with $r \neq 0$, Eqns. (3.13.52b, c) imply

$$\Delta w_1 \equiv 0 \text{ in } \Omega \quad \text{and} \quad \mathcal{A} w_1 = r^2 w_1. \tag{3.13.54}$$

From the right-hand side of (3.13.54), using (3.13.5) and Green's second theorem, we obtain

$$\begin{aligned} (\mathcal{A} w_1, w_1)_{L_2(\Omega)} &= \int_\Omega (\Delta^2 w_1) w_1 \, d\Omega \\ &= \int_\Omega \cancel{(\Delta w_1)}(\Delta w_1)\, d\Omega + \int_\Gamma \cancel{\frac{\partial \Delta w_1}{\partial \nu}}\, w_1 \, d\Gamma - \int_\Gamma \cancel{(\Delta w_1)} \frac{\partial w_1}{\partial \nu} \, d\Gamma \\ &= r^2 \int_\Omega w_1^2 \, d\Omega = 0, \end{aligned} \tag{3.13.55}$$

via the left-hand side of (3.13.54). Hence, with $r \neq 0$ (3.13.55) implies

$$\int_\Omega w_1^2 \, d\Omega = 0 \quad \text{and} \quad w_1 = 0 \text{ in } L_2(\Omega). \tag{3.13.56}$$

Finally, (3.13.52a) gives $w_2 = 0$, as desired. Thus, there is no point spectrum of A on the imaginary axis.

(ii) We shall show that A^{-1} is compact on Y, so that A has compact resolvent on Y, and [by (i)] A has no spectrum on the imaginary axis. It suffices to take $\sigma = 0$ in (3.13.22). Thus, let $A_0 = A|_{\sigma=0}$ for A as in (3.13.22). One then readily computes that

$$A_0^{-1} = \begin{bmatrix} \frac{\alpha^2}{\eta}[\quad]\Delta & -\mathcal{A}^{-1} & -\frac{\alpha}{\eta}[\quad] \\ I & 0 & 0 \\ \frac{\alpha}{\eta}\mathcal{A}_N^{-1}\Delta & 0 & -\frac{1}{\eta}\mathcal{A}_N^{-1} \end{bmatrix} : Y \to \mathcal{D}(A_0) = \mathcal{D}(A), \tag{3.13.57a}$$

where [ ] stands for

$$[\quad] = \left[\mathcal{A}^{-1} + G_1 G_2^* \mathcal{A}\mathcal{A}_N^{-1} - G_2 G_1^* \mathcal{A}\mathcal{A}_N^{-1}\right]. \tag{3.13.57b}$$

Thus, A_0^{-1} is compact on Y, via also the compactness of G_1 and $G_2 : L_2(\Gamma) \to \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ [see (3.13.12a) and (3.13.14a)].

Analyticity The s.c. contraction semigroup e^{At} of Proposition 3.13.5 is, moreover, analytic on Y. This is proved in Appendix 3I. Then, stability of e^{At} can be proved via Proposition 3.13.6, as was done in Proposition 3.11.1.3; Proposition 3.11.2.2; and Proposition 3.12.5 in the other cases. Uniform stability of e^{At} is also shown directly by energy methods [Avalos, Lasiecka, 1997; 1998], as is done in Appendix 3J of Chapter 3 for other boundary conditions. □

The Operator B By direct computation, writing $A[w_1, w_2, \theta] = Bu$ and recalling (3.13.22b) and (3.13.24), we readily obtain

$$A^{-1}Bu = \begin{bmatrix} -G_2 u \\ 0 \\ 0 \end{bmatrix} \in \mathcal{D}\left(\mathcal{A}^{\frac{7}{8}-\epsilon}\right) \times L_2(\Omega) \times L_2(\Omega), \quad u \in U, \tag{3.13.58a}$$

after recalling the regularity of G_2 in (3.13.14). It follows that

$$A^{-1}Bu \in \mathcal{D}\left((-A)^{\frac{3}{4}-2\epsilon}\right), \tag{3.13.58b}$$

and hence by the closed graph theorem

$$A^{-\gamma}B \in \mathcal{L}(U; Y), \quad \gamma = \frac{1}{4} + 2\epsilon, \tag{3.13.59}$$

In fact; by Grisvard [1967] and Appendix 3A, Theorem 3A.1:

$$\mathcal{D}(\mathcal{A}^s) = H^{4s}(\Omega), \quad 0 \le s < \frac{5}{8}, \tag{3.13.60}$$

$$\mathcal{D}(\mathcal{A}^s) = \left\{h \in H^{4s}(\Omega) : [\Delta h + (1-\mu)B_1 h]_\Gamma = 0\right\}, \quad \frac{5}{8} < s < \frac{7}{8}, \tag{3.13.61}$$

so that the boundary conditions do not interfere up to $\mathcal{D}(\mathcal{A}^s)$ with $s < 5/8$. Then, from (3.13.7) ($r = 0$) and (3.13.23) ($r = 1$), we obtain

$$\mathcal{D}(A^r) = \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}+\frac{r}{2}}\big) \times \mathcal{D}\big(\mathcal{A}^{\frac{r}{2}}\big) \times \mathcal{D}\big(\mathcal{A}_N^r\big) \quad \text{for } \frac{1}{2} + \frac{r}{2} < \frac{5}{8}, \text{ that is, for } r < \frac{1}{4}. \tag{3.13.62}$$

However, the vector $A^{-1}Bu$ in (3.13.57) has second and third component zero, so that BCs do not interfere. Thus, matching $7/8 - \epsilon = 1/2 + r/2$ from (3.13.57) and the first component in (3.13.62) yields $r = 3/4 - 2\epsilon$, and (3.13.58) follows.

3.14 Structurally Damped Euler–Bernoulli Equations with Damped Free BC and Point Control or Boundary Control

In this section we consider the Euler–Bernoulli equation defined on an open bounded domain $\Omega \subset \mathbb{R}^n$, with smooth boundary Γ, subject to two damping terms: (i) one term, Δw_t, in the interior, and (ii) another in the free boundary conditions. The latter may act either (ii_1) in the shear forces BC (Subsection 3.14.1 with point control, and Subsection 3.14.2 with boundary control) or (ii_2) in the moment BC (Subsection 3.14.3).

3.14.1 Damping Term in the Shear Forces BC and Point Control

In the present subsection, we consider the following dynamics in the unknown displacement $w(t, x)$:

$$\begin{cases} w_{tt} + \Delta^2 w - \Delta w_t = \delta(x - x^0)u(t) & \text{in } (0, T] \times \Omega \equiv Q; & (3.14.1.1\text{a}) \\ w(0, \cdot) = w_0, \ w_t(0, \cdot) = w_1 & \text{in } \Omega; & (3.14.1.1\text{b}) \\ \Delta w + (1 - \mu)B_1 w \equiv 0 & \text{in } (0, T] \times \Gamma \equiv \Sigma; & (3.14.1.1\text{c}) \\ \dfrac{\partial \Delta w}{\partial \nu} + (1 - \mu)B_2 w - w = \dfrac{\partial w_t}{\partial \nu} & \text{in } \Sigma, & (3.14.1.1\text{d}) \end{cases}$$

where B_1 and B_2 are the operators in, say, (3.13.2), (3.13.3) [or in (3.5.2a, b)], with damping terms in the interior and in the shear forces BC and point control acting at an interior point x^0 of Ω. We take $\{w_0, w_1\} \in H^2(\Omega) \times L_2(\Omega)$ and associate with problem (3.14.1.1) the cost functional

$$\begin{aligned} J(w, u) = &\int_0^T \left\{ \|w(t)\|^2_{H^2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} + |u(t)|^2 \right\} dt \\ &+ a\|\{w(T), w_t(T)\}\|^2_{H^2(\Omega)\times L_2(\Omega)}, \end{aligned} \tag{3.14.1.2}$$

where $a = 0$ if $T = \infty$, and $a = 1$ if $T < \infty$.

Abstract Setting. Case u ≡ 0 As in (3.5.3), we introduce the positive, self-adjoint operator (see Appendix 3C, Proposition 3C.4)

$$\mathcal{A}h = \Delta^2 h,$$
$$\mathcal{D}(\mathcal{A}) = \left\{ h \in H^4(\Omega) : \Delta h + (1-\mu)B_1 h|_\Gamma = 0; \right.$$
$$\left. \frac{\partial \Delta h}{\partial \nu} + (1-\mu)B_2 h - h|_\Gamma = 0 \right\}, \tag{3.14.1.3}$$

as well as the positive, self-adjoint operator $\mathcal{A}_N$ [see e.g., (3.11.2.3) or $L_2^0(\Omega) = L_2(\Omega)/\mathcal{N}(\mathcal{A}_N)$],

$$\mathcal{A}_N h = -\Delta h, \qquad \mathcal{D}(\mathcal{A}_N) = \left\{ h \in H^2(\Omega) : \left.\frac{\partial h}{\partial \nu}\right|_\Gamma = 0 \right\}, \tag{3.14.1.4}$$

and we have [Appendix 3A, Theorem 3A.1]

$$\mathcal{D}(\mathcal{A}_N) \subset \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) = H^2(\Omega); \quad \mathcal{D}\big(\mathcal{A}^{\frac{1}{4}}\big) = H^1(\Omega) = \mathcal{D}\big(\mathcal{A}_N^{\frac{1}{2}}\big). \tag{3.14.1.5}$$

Proposition 3.14.1.1 *With reference to (3.14.1.3), (3.14.1.4), the abstract model of problem (3.14.1.1) with $u \equiv 0$ is*

$$w_{tt} + \mathcal{A}w + \mathcal{A}_N w_t = 0, \quad w(0) = w_0, \quad w_t(0) = w_1. \tag{3.14.1.6}$$

Proof.

Step 1 Let G_2 be the map introduced, say, in (3.13.13),

$$h = G_2 g \iff \begin{cases} \Delta^2 h = 0 \text{ in } \Omega, & (3.14.1.7a) \\ [\Delta h + (1-\mu)B_1 h]_\Gamma = 0, & (3.14.1.7b) \\ \left[\dfrac{\partial \Delta h}{\partial \nu} + (1-\mu)B_2 h - h\right]_\Gamma = g, & (3.14.1.7c) \end{cases}$$

satisfying the regularity property (3.13.14c),

$$\mathcal{A}^{\frac{7}{8}-\epsilon} G_2 : \text{continuous } L_2(\Gamma) \to L_2(\Omega), \quad \epsilon > 0. \tag{3.14.1.8}$$

Similarly, let N be the Neumann map as defined, say, in (3.11.2.5),

$$h = Ng \iff \left\{ \Delta h = 0 \text{ in } \Omega : \frac{\partial h}{\partial \nu} = g \text{ in } \Gamma;\ h \in L_2^0(\Omega) \right\}, \tag{3.14.1.9}$$

satisfying the regularity property (3.11.2.6),

$$\mathcal{A}_N^{\frac{3}{4}-\epsilon} N : \text{ continuous } L_2(\Gamma) \to L_2^0(\Omega), \epsilon > 0. \tag{3.14.1.10}$$

With reference to (3.14.1.1d), calling by g the common value

$$\frac{\partial \Delta w}{\partial \nu} + (1-\mu)B_2 w - w = \frac{\partial w_t}{\partial \nu} = g, \tag{3.14.1.11}$$

and using the definitions of G_2 and N above in problem (3.14.1.1) with $u \equiv 0$, we obtain via (3.14.1.11)

$$\begin{cases} w_{tt} + \Delta^2(w - G_2 g) - \Delta(w_t - Ng) = 0 & \text{on } Q, \quad (3.14.1.12a) \\ \Delta(w - G_2 g) + (1-\mu)B_1(w - G_2 g) = 0 & \text{on } \Sigma, \quad (3.14.1.12b) \\ \dfrac{\partial \Delta}{\partial \nu}(w - G_2 g) + (1-\mu)B_2(w - G_2 g) - (w - G_2 g) = 0 & \text{on } \Sigma, \quad (3.14.1.12c) \\ \dfrac{\partial}{\partial \nu}(w_t - Ng) = \dfrac{\partial w_t}{\partial \nu} - g = 0 & \text{on } \Sigma. \quad (3.14.1.12d) \end{cases}$$

Thus, by use of (3.14.1.3) and (3.14.1.4), we can rewrite problem (3.14.1.12) abstractly as

$$w_{tt} + \mathcal{A}(w - G_2 g) + \mathcal{A}_N(w_t - Ng) = 0 \quad \text{in } L_2(\Omega). \tag{3.14.1.13}$$

Step 2 Regarding $\mathcal{A}$ and $\mathcal{A}_N$ in (3.14.1.3) and (3.14.1.4), we have [Grisvard, 1967]; [Appendix 3A, Theorem 3A.1]

$$\begin{cases} \mathcal{D}(\mathcal{A}^{\frac{1}{8}+\frac{\epsilon}{2}}) = H^{\frac{1}{2}+2\epsilon}(\Omega) = \mathcal{D}(\mathcal{A}_N^{\frac{1}{4}+\epsilon}), \quad \mathcal{D}(\mathcal{A}^{\frac{1}{4}}) = \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}) = H^1(\Omega), \\ \mathcal{D}(\mathcal{A}^{\frac{3}{8}+\frac{\epsilon}{2}}) = H^{\frac{3}{2}+2\epsilon}(\Omega), \quad \mathcal{D}(\mathcal{A}_N) \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega). \end{cases} \tag{3.14.1.14}$$

Thus, if $g \in L_2(\Gamma)$, then by use of (3.14.1.8), (3.14.1.10), and (3.14.1.14), we obtain

$$\mathcal{A}G_2 g = \mathcal{A}^{\frac{1}{8}+\frac{\epsilon}{2}}\mathcal{A}^{\frac{7}{8}-\frac{\epsilon}{2}}G_2 g \in \left[\mathcal{D}(\mathcal{A}^{\frac{1}{8}+\frac{\epsilon}{2}})\right]' = [H^{\frac{1}{2}+2\epsilon}(\Omega)]', \tag{3.14.1.15}$$

$$\mathcal{A}_N N g = \mathcal{A}_N^{\frac{1}{4}+\epsilon}\mathcal{A}_N^{\frac{3}{4}-\epsilon}Ng \in \left[\mathcal{D}(\mathcal{A}_N^{\frac{1}{4}+\epsilon})\right]' = \left[H^{\frac{1}{2}+2\epsilon}(\Omega)\right]', \tag{3.14.1.16}$$

where the indicated duality is with respect to the pivot space $L_2(\Omega)$. Thus, via (3.14.1.15), (3.14.1.16), we may rewrite (3.14.1.13) as

$$w_t + \mathcal{A}w + \mathcal{A}_N w_t - \mathcal{A}G_2 g - \mathcal{A}_N N g = 0 \quad \text{in } \left[H^{\frac{1}{2}+2\epsilon}(\Omega)\right]'. \tag{3.14.1.17}$$

Step 3. Lemma 3.14.1.2 *With reference to (3.14.1.3), (3.14.1.4), (3.14.1.7), (3.14.1.9), and (3.14.1.15), (3.14.1.16), we have for any $g \in L_2(\Gamma)$:*

$$\mathcal{A}G_2 g + \mathcal{A}_N N g = 0 \quad \text{in } \left[H^{\frac{1}{2}+2\epsilon}(\Omega)\right]'. \tag{3.14.1.18}$$

Proof. The proof, based on Section 3.13, Lemma 3.13.2, Eqn. (3.13.30), as well as on Section 3.3, Eqn. (3.3.1.29), yields

$$G_2^*\mathcal{A}f = -f|_\Gamma, \quad f \in \mathcal{D}(\mathcal{A}^{\frac{1}{8}+\frac{\epsilon}{2}}); \quad N^*\mathcal{A}_N f = f|_\Gamma, \quad f \in \mathcal{D}(\mathcal{A}_N^{\frac{1}{4}+\epsilon}). \tag{3.14.1.19}$$

Then, let $f \in H^{\frac{1}{2}+2\epsilon}(\Omega)$ as in (3.14.1.14), and let $g \in L_2(\Gamma)$, and compute

$$\begin{aligned} (\mathcal{A}G_2 g + \mathcal{A}_N N g, f)_{L_2(\Omega)} &= (g, G_2^*\mathcal{A}f)_{L_2(\Gamma)} + (g, N^*\mathcal{A}_N f)_{L_2(\Gamma)} \\ \text{(by (3.14.1.19))} \qquad &= -(g, f|_\Gamma)_{L_2(\Gamma)} + (g, f|_\Gamma)_{L_2(\Gamma)} = 0. \end{aligned} \tag{3.14.1.20}$$

Thus (3.14.1.20) proves (3.14.1.18), as desired. □

Step 4 Using (3.14.1.18) in (3.14.1.17) yields (3.14.1.6), as desired. Proposition 3.14.1.1 is established. □

Case u ≠ 0. Proposition 3.14.1.3 *(i) Problem (3.14.1.1) can be rewritten abstractly as the following second-order equation:*

$$w_{tt} + \mathcal{A}w + \mathcal{A}_N w_t = \delta(\cdot - x^0)u, \qquad w(0) = w_0, \qquad w_t(0) = w_1, \quad (3.14.1.21)$$

or as the following first-order equation:

$$\dot{y} = Ay + Bu, \quad y(t) = [w(t), w_t(t)]; \qquad (3.14.1.22)$$

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\mathcal{A}_N \end{bmatrix} : Y \supset \mathcal{D}(A) \to Y; \quad B = \begin{bmatrix} 0 \\ \delta(\cdot - x^0) \end{bmatrix}; \qquad (3.14.1.23)$$

$$Y \equiv \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times L_2(\Omega) \equiv H^2(\Omega) \times L_2(\Omega) \quad \textit{(norm equivalent)}; \qquad (3.14.1.24)$$

$$\mathcal{D}(A) = \Big\{(x_1, x_2) \in Y : x_1 \in \mathcal{D}\big(\mathcal{A}^{\frac{3}{4}}\big), x_2 \in \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \subset \mathcal{D}\big(\mathcal{A}^{\frac{1}{4}}\big),$$
$$\mathcal{A}^{\frac{3}{4}}x_1 + \big(\mathcal{A}^{-\frac{1}{4}}\mathcal{A}_N\mathcal{A}^{-\frac{1}{4}}\big)\mathcal{A}^{\frac{1}{4}}x_2 \in \mathcal{D}\big(\mathcal{A}^{\frac{1}{4}}\big)\Big\}, \qquad (3.14.1.25)$$

where $\mathcal{A}^{-\frac{1}{4}}\mathcal{A}_N\mathcal{A}^{-\frac{1}{4}} \in \mathcal{L}(L_2(\Omega))$ by (3.14.1.14).

(ii) The operator A generates a s.c. contraction, uniformly stable semigroup e^{At} on Y, which moreover is analytic here for $t > 0$.

(iii) Finally, for $\epsilon > 0$,

$$A^{-\gamma}B \in \mathcal{L}(U; Y); \quad U = \mathbb{R}; \quad \gamma = \begin{cases} \frac{1}{4} + \epsilon, & \textit{if} \dim \Omega = 1; & (3.14.1.26) \\ \frac{1}{2} + \epsilon, & \textit{if} \dim \Omega = 2; & (3.14.1.27) \\ \frac{3}{4} + \epsilon, & \textit{if} \dim \Omega = 3. & (3.14.1.28) \end{cases}$$

Proof. (i) Equation (3.14.1.21) follows from (3.14.1.1), (3.14.1.6) when $u \neq 0$. (ii) Equation (3.14.1.21) can be readily rewritten as in (3.14.1.23) where we may apply Appendix 3B, Theorem 3B.1, with $\mathcal{B} = \mathcal{A}_N$, and hence $\alpha = 1/2$ via $\mathcal{D}(\mathcal{A}^{\frac{1}{4}}) = \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}})$ as in (3.14.1.14). Thus, A is the generator of a s.c. contraction, analytic, uniformly stable semigroup e^{At} on Y.

(iii) We have, where $n = \dim \Omega$,

$$\delta \in \begin{cases} \big[H^{\frac{1}{2}+2\epsilon}(\Omega)\big]' = \big[\mathcal{D}\big(\mathcal{A}^{\frac{1}{8}+\frac{\epsilon}{2}}\big)\big]' \iff \mathcal{A}^{-\frac{1}{8}-\frac{\epsilon}{2}}\delta \in L_2(\Omega), \quad n = 1; & (3.14.1.29) \\ [H^{1+2\epsilon}(\Omega)]' = \big[\mathcal{D}\big(\mathcal{A}^{\frac{1}{4}+\frac{\epsilon}{2}}\big)\big]' \iff \mathcal{A}^{-\frac{1}{4}-\frac{\epsilon}{2}}\delta \in L_2(\Omega), \quad n = 2; & (3.14.1.30) \\ \big[H^{\frac{3}{2}+2\epsilon}(\Omega)\big]' = \big[\mathcal{D}\big(\mathcal{A}^{\frac{3}{8}+\frac{\epsilon}{2}}\big)\big]' \iff \mathcal{A}^{-\frac{3}{8}-\frac{\epsilon}{2}}\delta \in L_2(\Omega), \quad n = 3, & (3.14.1.31) \end{cases}$$

by using again (3.14.1.14) and [Grisvard, 1967] and Appendix 3A, Theorem 3A.1. Next, consider the operator

$$A_{\frac{1}{2}} = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\mathcal{A}^{\frac{1}{2}} \end{bmatrix} : Y \supset \mathcal{D}(A_{\frac{1}{2}}) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}), \tag{3.14.1.32}$$

obtained from A in (3.14.1.23) by replacing $\mathcal{A}_N$ with $\mathcal{A}^{\frac{1}{2}}$, where $\mathcal{A}_N$ is comparable to $\mathcal{A}^{\frac{1}{2}}$ in the sense of Appendix 3B, as noted, and used, in part (ii).

By applying Appendix 3B, Theorem 3B.2, Eqn. (3B.17) with $\alpha = 1/2$ as well as Appendix 3B, Corollary 3B.4, Eqn. (3B.19), we obtain, for any $0 < \theta_2 < \theta_1 < \theta < 1$:

$$\mathcal{D}(A_{\frac{1}{2}}^{\theta}) = \mathcal{D}(\mathcal{A}^{\frac{1}{2}+\frac{\theta}{2}}) \times \mathcal{D}(\mathcal{A}^{\frac{\theta}{2}}) \subset \mathcal{D}(A^{\theta_1}) \subset \mathcal{D}(A_{\frac{1}{2}}^{\theta_2}) = \mathcal{D}(\mathcal{A}^{\frac{1}{2}+\frac{\theta_2}{2}}) \times \mathcal{D}(\mathcal{A}^{\frac{\theta_2}{2}}). \tag{3.14.1.33}$$

Thus, by (3.14.1.23) on B, (3.14.1.33), and (3.14.1.29)–(3.14.1.31), we have

$$B = \begin{bmatrix} 0 \\ \delta(\cdot - x^0) \end{bmatrix} \in [\mathcal{D}(A_{\frac{1}{2}}^{\theta_2})]' \subset [\mathcal{D}(A^{\theta_1})]'; \quad \text{hence } A^{-\theta_1} B \in \mathcal{L}(U; Y), \tag{3.14.1.34}$$

where the indicated duality is with respect to Y, provided that

$$\frac{\theta_2}{2} = \begin{cases} \dfrac{1}{8} + \dfrac{\epsilon}{2} \\[2mm] \dfrac{1}{4} + \dfrac{\epsilon}{2}, \\[2mm] \dfrac{3}{8} + \dfrac{\epsilon}{2} \end{cases} \quad \text{or} \quad \theta_2 = \begin{cases} \dfrac{1}{4} + \epsilon, & \dim \Omega = 1; & (3.14.1.35) \\[2mm] \dfrac{1}{2} + \epsilon, & \dim \Omega = 2; & (3.14.1.36) \\[2mm] \dfrac{3}{4} + \epsilon, & \dim \Omega = 3. & (3.14.1.37) \end{cases}$$

Since θ_1 may be taken to be an arbitrary number greater than θ_2, say $\theta_1 = \theta_2 + \epsilon$, then (3.14.1.34) with $\gamma = \theta_1 = \theta_2 + \epsilon$ yields (3.14.1.26)–(3.14.1.28), as desired, via (3.14.1.35)–(3.14.1.37). □

Conclusion The above results verify the assumption (1.1.2) of Chapter 1, or (2.1.7) of Chapter 2: $A^{-\gamma} B \in \mathcal{L}(U; Y)$, as well as analyticity and uniform stability of the semigroup e^{At} on Y. Accordingly, Theorems 2.2.1 and 2.2.2 of Chapter 2 apply to problem (3.14.1.1), (3.14.1.2) for $T = \infty$ with $a = 0$, where $R = I$; while Theorem 1.2.1.1 of Chapter 1 applies for $T < \infty$ with $a = 1$ and $G = I$.

3.14.2 Damping Term in the Shear Forces BC and Boundary Control

In the present subsection we consider the same dynamics as in (3.14.1.1):

$$\begin{cases} w_{tt} + \Delta^2 w - \Delta w_t = 0 & \text{in } (0, T] \times \Omega \equiv Q; & (3.14.2.1a) \\ w(0, \cdot) = w_0, \quad w_t(0, \cdot) = w_1 & \text{in } \Omega; & (3.14.2.1b) \\ \Delta w + (1-\mu) B_1 w = u & \text{in } (0, T] \times \Gamma \equiv \Sigma; & (3.14.2.1c) \\ \dfrac{\partial \Delta w}{\partial \nu} + (1-\mu) B_2 w - w = \dfrac{\partial w_t}{\partial \nu} & \text{in } \Sigma, & (3.14.2.1d) \end{cases}$$

except that it is now acted upon by boundary control $u \in L_2(0,T;L_2(\Gamma))$ rather than by point control as in Section 3.14.1. To take into account the boundary control u in (3.14.2.1c), we introduce the Green operator G_1 as in, say, Eqn. (3.13.11) of Section 3.13,

$$h = G_1 g \iff \begin{cases} \Delta^2 h = 0 & \text{in } \Omega, & (3.14.2.2) \\ [\Delta h + (1-\mu)B_1 h]_\Gamma = g & \text{in } \Gamma, & (3.14.2.3) \\ \left[\dfrac{\partial \Delta h}{\partial \nu} + (1-\mu)B_2 h - h\right]_\Gamma = 0 & \text{in } \Gamma, & (3.14.2.4) \end{cases}$$

with regularity (3.13.12) of Section 3.13,

$$\mathcal{A}^{\frac{5}{8}-\epsilon} G_1 \; : \; \text{continuous } L_2(\Gamma) \to L_2(\Omega), \tag{3.14.2.5}$$

and proceed as in Section 3.14.1. We find that the abstract model of problem (3.14.2.1) is now

$$w_{tt} + \mathcal{A}w + \mathcal{A}_N w_t = \mathcal{A}G_1 u \in [\mathcal{D}(\mathcal{A})]', \tag{3.14.2.6}$$

or, in first-order form,

$$\dot{y} = Ay + Bu, \quad y(t) = [w(t), w_t(t)], \tag{3.14.2.7}$$

where A is the same operator as in (3.14.1.23)–(3.14.1.25), and B satisfies

$$B = \begin{bmatrix} 0 \\ \mathcal{A}G_1 \end{bmatrix} \; : \; \text{continuous } U = L_2(\Gamma) \to L_2(\Omega) \times \left[\mathcal{D}\left(\mathcal{A}^{\frac{3}{8}+\epsilon}\right)\right]', \tag{3.14.2.8}$$

by virtue of (3.14.2.5), where the indicated duality is with respect to $L_2(\Omega)$ as a pivot space. If we compare (3.14.2.8) with (3.14.1.31), for $\dim \Omega = 3$, we see that the present operator B has the same regularity of the operator B in (3.14.1.23) for $\dim \Omega = 3$. Thus, the very same argument from (3.14.1.32) to (3.14.1.37) for $\dim \Omega = 3$ yields that

$$A^{-\gamma} B \in \mathcal{L}(U;Y), \quad \gamma = \frac{3}{4} + \epsilon, \quad Y = \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right) \times L_2(\Omega), \tag{3.14.2.9}$$

as in (3.14.1.28) for $\dim \Omega = 3$. Thus, to problem (3.14.2.1) we can associate the cost (3.14.1.2) and conclude that Theorem 1.2.1.1 of Chapter 1 ($T < \infty$), and Theorems 2.2.1 and 2.2.2 of Chapter 2 ($T = \infty$) apply [see (3.14.2.9)], as in Section 3.14.1.

3.14.3 Damping Term in the Moment BC and Point Control

In the present subsection we consider the same dynamics:

$$\begin{cases} w_{tt} + \Delta^2 w - \Delta w_t = \delta(x - x^0)u(t) & \text{in } (0, T] \times \Omega = Q; & (3.14.3.1a) \\ w(0, \cdot) = w_0, \; w_t(0, \cdot) = w_1 & \text{in } \Omega; & (3.14.3.1b) \\ \Delta w + (1-\mu)B_1 w = w_t & \text{in } (0, T] \times \Gamma = \Sigma; & \\ \dfrac{\partial \Delta w}{\partial \nu} + (1-\mu)B_2 w - w = 0 & \text{in } \Sigma, & (3.14.3.1c) \end{cases}$$

with the usual boundary operators B_1 and B_2 as in (3.13.2), (3.13.3), with damping terms in the interior and in the moment BC and point control acting at the interior point x^0 of Ω. We take $\{w_0, w_1\} \in H^2(\Omega) \times L_2(\Omega)$ and associate with problem (3.14.3.1) the same cost functional as in (3.14.1.2),

$$J(w, u) = \int_0^T \left[\|w(t)\|^2_{H^2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} + |u(t)|^2 \right] dt + a\|\{w(T), w_t(T)\}\|^2_{H^2(\Omega)\times L_2(\Omega)}, \tag{3.14.3.2}$$

where $a = 0$ if $T = \infty$, and $a = 1$ if $T < \infty$.

Abstract Setting. Case u ≡ 0 Let $\mathcal{A}$ be the same operator as in (3.14.1.3) and let $\mathcal{A}_D$ be the positive, self-adjoint operator as in, say, (3.11.1.4),

$$\mathcal{A}_D h = -\Delta h, \quad \mathcal{D}(\mathcal{A}_D) = H^2(\Omega) \cap H_0^1(\Omega), \tag{3.14.3.3}$$

so that we have

$$\begin{cases} \mathcal{D}(\mathcal{A}_D) \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega), \quad \mathcal{D}(\mathcal{A}_D^{\frac{1}{2}}) = H_0^1(\Omega) \subset \mathcal{D}(\mathcal{A}^{\frac{1}{4}}) = H^1(\Omega); \\ \text{or } \mathcal{A}^{\frac{1}{4}}\mathcal{A}_D^{-\frac{1}{2}} \in \mathcal{L}(L_2(\Omega)); \quad \text{or} \quad \|\mathcal{A}^{\frac{1}{4}}x\|_{L_2(\Omega)} \le c\|\mathcal{A}_D^{\frac{1}{2}}x\|_{L_2(\Omega)}, \quad x \in \mathcal{D}(\mathcal{A}_D^{\frac{1}{2}}). \end{cases} \tag{3.14.3.4}$$

Proposition 3.14.3.1 *With reference to (3.14.1.3) and (3.14.3.4), the abstract model of problem (3.14.3.1) with $u \equiv 0$ is*

$$w_{tt} + \mathcal{A}w + \mathcal{A}_D w_t = 0, \quad w(0) = w_0, \quad w_t(0) = w_1. \tag{3.14.3.5}$$

Proof. The proof is similar to that of Proposition 3.14.1.1, mutatis mutandis. It will only be sketched.

Step 1 We call by g the common value

$$\Delta w + (1-\mu)B_1 w = w_t = g \quad \text{on } \Sigma. \tag{3.14.3.6}$$

Moreover, we recall the Green operator G_1 in (3.14.2.2)–(3.14.2.3), as well as the Dirichlet map D as in, say, (3.11.1.7),

$$h \equiv Dg \iff \{\Delta h = 0 \text{ in } \Omega, h|_\Gamma = g\}, \quad \mathcal{A}_D^{\frac{1}{4}-\epsilon} D : \text{ continuous } L_2(\Gamma) \to L_2(\Omega). \tag{3.14.3.7}$$

We may then rewrite problem (3.14.3.1) with $u \equiv 0$ as

$$\begin{cases} w_{tt} + \Delta^2(w - G_1 g) - \Delta(w_t - Dg) = 0 & \text{in } Q, \quad (3.14.3.8a) \\ \Delta(w - G_1 g) + (1-\mu)B_1(w - G_1 g) = 0 & \text{on } \Sigma, \quad (3.14.3.8b) \\ \dfrac{\partial \Delta}{\partial \nu}(w - G_1 g) + (1-\mu)B_2(w - G_1 g) - (w - G_1 g) = 0 & \text{on } \Sigma, \quad (3.14.3.8c) \\ (w_t - Dg)|_\Sigma = w_t - g = 0 & \text{on } \Sigma, \quad (3.14.3.8d) \end{cases}$$

or, recalling $\mathcal{A}$ from (3.14.1.3) and $\mathcal{A}_D$ from (3.14.3.3),

$$w_{tt} + \mathcal{A}(w - G_1 g) + \mathcal{A}_D(w_t - Dg) = 0 \quad \text{in } L_2(\Omega). \tag{3.14.3.9}$$

Step 2 For $g \in L_2(\Gamma)$ we have by (3.14.2.5), (3.14.3.7), and Appendix 3A, Theorem 3A.1,

$$\begin{aligned} \mathcal{A}G_1 g &= \mathcal{A}^{\frac{3}{8}+\frac{\epsilon}{2}}\mathcal{A}^{\frac{5}{8}-\frac{\epsilon}{2}}G_1 g \in \left[\mathcal{D}(\mathcal{A}^{\frac{3}{8}+\frac{\epsilon}{2}})\right]' \\ &= \left[H^{\frac{3}{2}+2\epsilon}(\Omega)\right]' \subset \left[\mathcal{D}\big(\mathcal{A}_D^{\frac{3}{4}+\epsilon}\big)\right]', \end{aligned} \tag{3.14.3.10}$$

$$\mathcal{A}_D Dg = \mathcal{A}_D^{\frac{3}{4}+\epsilon}\mathcal{A}_D^{\frac{1}{4}-\epsilon}Dg \in \left[\mathcal{D}\big(\mathcal{A}_D^{\frac{3}{4}+\epsilon}\big)\right]'. \tag{3.14.3.11}$$

Thus, via (3.14.3.10), (3.14.3.11), we can rewrite (3.14.3.9) as

$$w_{tt} + \mathcal{A}w + \mathcal{A}_D w_t - \mathcal{A}G_1 g - \mathcal{A}_D Dg = 0 \quad \text{in } \left[\mathcal{D}\big(\mathcal{A}_D^{\frac{3}{4}+\epsilon}\big)\right]'. \tag{3.14.3.12}$$

Step 3. Lemma 3.14.3.2 *With reference to (3.14.1.3), (3.14.3.3), (3.14.2.2)–(3.14.2.3), and (3.14.3.7), we have for any $g \in L_2(\Gamma)$:*

$$-\mathcal{A}G_1 g - \mathcal{A}_D Dg = 0 \quad in \left[\mathcal{D}\big(\mathcal{A}_D^{\frac{3}{4}+\epsilon}\big)\right]'. \tag{3.14.3.13}$$

Proof. The proof is based on Section 3.13, Lemma 3.13.1, Eqn. (3.13.26), as well as Section 3.1, Eqn. (3.1.9) yielding

$$G_1^*\mathcal{A}f = \frac{\partial f}{\partial \nu}, \qquad D^*\mathcal{A}_D f = -\frac{\partial f}{\partial \nu}, \qquad f \in \mathcal{D}\big(\mathcal{A}_D^{\frac{3}{4}+\epsilon}\big). \tag{3.14.3.14}$$

Then, let $f \in \mathcal{D}(\mathcal{A}_D^{\frac{3}{4}+\epsilon})$, $g \in L_2(\Gamma)$ and compute

$$\begin{aligned} (-\mathcal{A}G_1 g - \mathcal{A}_D Dg, f)_{L_2(\Omega)} &= -(g, G_1^*\mathcal{A}f)_{L_2(\Gamma)} - (g, D^*\mathcal{A}_D f)_{L_2(\Gamma)} \\ \text{(by (3.14.3.14))} \qquad &= -\left(g, \frac{\partial f}{\partial \nu}\right)_{L_2(\Gamma)} + \left(g, \frac{\partial f}{\partial \nu}\right)_{L_2(\Gamma)} = 0. \end{aligned} \tag{3.14.3.15}$$

Thus, (3.14.3.15) proves (3.14.3.13), as desired. □

Step 4 We use (3.14.3.13) in (3.14.3.12) and obtain (3.14.3.5), as desired. Proposition 3.14.3.1 is established. □

Case $u \not\equiv 0$. Proposition 3.14.3.3

(i) Problem (3.14.3.1) can be rewritten abstractly as the following second-order equation:

$$w_{tt} + \mathcal{A}w + \mathcal{A}_D w_t = \delta(\cdot - x^0)u, \quad w(0) = w_0, \quad w_t(0) = w_1, \tag{3.14.3.16}$$

or as the following first-order equation:

$$\dot{y} = Ay + Bu, \quad y(t) = [w(t), w_t(t)]; \tag{3.14.3.17}$$

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\mathcal{A}_D \end{bmatrix} : Y \supset \mathcal{D}(A) \to Y; \quad B = \begin{bmatrix} 0 \\ \delta(\cdot - x^0) \end{bmatrix}; \tag{3.14.3.18}$$

$$Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \equiv H^2(\Omega) \times L_2(\Omega) \quad \textit{(norm equivalence)}; \tag{3.14.3.19}$$

$$\begin{aligned} \mathcal{D}(A) = \big\{&(x_1, x_2) \in Y : x_1 \in \mathcal{D}(\mathcal{A}^{\frac{3}{4}}),\ x_2 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset \mathcal{D}(\mathcal{A}^{\frac{1}{4}}), \\ &\mathcal{A}^{\frac{3}{4}}x_1 + (\mathcal{A}^{-\frac{1}{4}}\mathcal{A}_D\mathcal{A}^{-\frac{1}{4}})\mathcal{A}^{\frac{1}{4}}x_2 \in \mathcal{D}(\mathcal{A}^{\frac{1}{4}})\big\}, \end{aligned} \tag{3.14.3.20}$$

where $\mathcal{A}^{-\frac{1}{4}}\mathcal{A}_D\mathcal{A}^{-\frac{1}{4}} \in \mathcal{L}(L_2(\Omega))$.

(ii) The operator A generates a s.c. contraction, analytic, uniformly stable semigroup e^{At} on Y.

Proof. (i) Equation (3.14.3.16) follows from (3.14.3.1) and (3.14.3.5) when $u \neq 0$.

(ii) As usual, we invoke the Lumer–Phillips theorem or corollary thereof [Pazy, 1983, p. 15], to claim semigroup generation. In order to obtain analyticity and exponential stability of the semigroup, we invoke Theorem 3B.1′ in Appendix 3B, with $\mathcal{B} = \mathcal{B}^* = \mathcal{A}_B^{\frac{1}{2}}$, so that the properties in (3.14.3.4) fit assumptions (3B.14). □

Conclusion. Problem (3.14.3.1) fits assumption (1.1.2) of Chapter 1, or (2.1.7) of Chapter 2: $A^{-\gamma}B \in \mathcal{L}(U; Y)$ with γ as in (3.14.1.26), as well as the analyticity and uniform stability of the semigroup. Accordingly, Theorem 2.2.1 and Theorem 2.2.2 of Chapter 2 apply to problem (3.14.3.1), (3.14.3.2) for $T = \infty$ with $a = 0$, where $R = I$; while Theorem 1.2.1.1 of Chapter 1 applies for $T < \infty$ and with $a = 1$ and $G = I$.

3.15 A Linearized Model of Well/Reservoir Coupling for a Monophasic Flow with Boundary Control

The Physical Model In this section we consider the following linearized model of a well/reservoir coupling for a monophasic flow. It is a mathematical generalization of the model with a two-dimensional square reservoir Ω and a one-dimensional well considered in Bourgeat [1992]. Let Ω be a bounded domain in $\mathbb{R}^n$: we think of Ω as either (i) a parallelopiped, in which case we call Γ_0 one of its faces, and Γ_1 the union of all other faces so that $\Gamma = \partial\Omega = \Gamma_0 \cup \Gamma_1$, or else as (ii) a smooth domain with a flat portion of the boundary, which we call Γ_0, and $\Gamma_1 = \partial\Omega \setminus \Gamma_0$, $\Gamma = \partial\Omega$. See Figure 3.1.

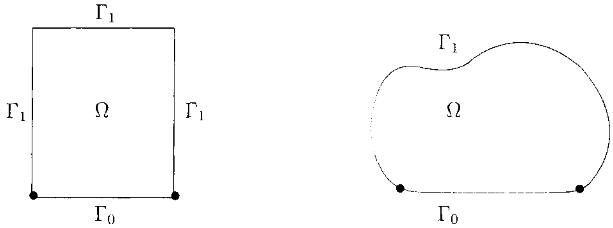


Figure 3.1

On Ω we consider the following problem in $\{w, h\}$:

$$\begin{cases} w_{tt} = \Delta_{\Gamma_0} w - \left.\dfrac{\partial h}{\partial \nu}\right|_{\Gamma_0} & \text{in } (0, T] \times \Gamma_0; & (3.15.1a) \\ w|_{\partial\Gamma_0} = u & \text{in } (0, T] \times \partial\Gamma_0; & (3.15.1b) \\ h_t = \Delta_\Omega h & \text{in } (0, T] \times \Omega; & (3.15.1c) \\ h|_{\Sigma_1} \equiv 0, \ h|_{\Sigma_0} = w_t & \text{in } (0, T] \times \Gamma_i = \Sigma_i, i = 0, 1. & (3.15.1d) \end{cases}$$

In (3.15.1), Δ_{Γ_0} and Δ_Ω denote the Laplacian with respect to the space variables of Γ_0 and of Ω, respectively. Also, ν denotes the unit outward normal defined on Γ_0. Thus, problem (3.15.1) couples the heat equation in h on all of Ω with the wave equation in w on the flat portion Γ_0 of the boundary $\partial\Omega$ of Ω. The control function u acts in the Dirichlet BC of the boundary $\partial\Gamma_0$ of Γ_0. Physically, when the reservoir $\Omega = [0, 1] \times [0, 1]$ is a square and the well $\Gamma_0 = \{0 \le x \le 1; y = 0\}$ is a one-dimensional segment, so that $\Delta_{\Gamma_0} w = w_{xx}$, this model has been considered in Bourgeat [1992], where $w(t, \cdot)$ is the well pressure, while h is the time derivative of the reservoir pressure. We learnt of this model through [Benabdallah, 1998], see Notes. With problem (3.15.1), we associate the following cost functional:

$$J(u; w, h) \equiv \int_0^T \left\{ \|\nabla_{\Gamma_0} w(t)\|^2_{L_2(\Gamma_0)} + \|w_t(t)\|^2_{L_2(\Gamma_0)} + \|h(t)\|^2_{L_2(\Omega)} \right.$$
$$\left. + \|u(t)\|^2_{L_2(\partial\Gamma_0)} \right\} dt + a \|\{w(T), w_t(T), h(T)\}\|^2_{H_0^1(\Gamma_0) \times L_2(\Gamma_0) \times L_2(\Omega)}, \tag{3.15.2}$$

where the initial condition $\{w_0, w_1, h_0\} \in H_0^1(\Gamma_0) \times L_2(\Gamma_0) \times L_2(\Omega)$. Moreover, $a = 0$ if $T = \infty$, and $a = 1$ if $T < \infty$.

Abstract Setting We put problem (3.15.1), (3.15.2) into the abstract setting of Chapter 1 or 2. To this end, we introduce the following spaces and operators:

$$\mathcal{A}_\Omega = -\Delta_\Omega : L_2(\Omega) \supset \mathcal{D}(\mathcal{A}_\Omega) = H^2(\Omega) \cap H_0^1(\Omega) \to L_2(\Omega), \quad \mathcal{D}\big(\mathcal{A}_\Omega^{\frac{1}{2}}\big) \equiv H_0^1(\Omega), \tag{3.15.3}$$

$$\mathcal{A}_{\Gamma_0} = -\Delta_{\Gamma_0} : L_2(\Gamma_0) \supset \mathcal{D}(\mathcal{A}_{\Gamma_0}) = H^2(\Gamma_0) \cap H_0^1(\Gamma_0) \to L_2(\Gamma_0),$$

$$\mathcal{D}\big(\mathcal{A}_{\Gamma_0}^{\frac{1}{2}}\big) = H_0^1(\Gamma_0), \quad \mathcal{D}\big(\mathcal{A}_{\Gamma_0}^{\frac{1}{4}}\big) = H^{\frac{1}{2}}(\Gamma_0), \tag{3.15.4}$$

where both $\mathcal{A}_\Omega$ and $\mathcal{A}_\Gamma$ are positive self-adjoint operators, and

$$\begin{cases} \tilde{D}_\Omega g \equiv v \iff \{\Delta_\Omega v = 0 \text{ in } \Omega;\quad v|_{\Gamma_1} = 0, v|_{\Gamma_0} = g\}; & (3.15.5a) \\ \tilde{D}_\Omega : H^s(\Gamma_0) \to H^{s+\frac{1}{2}}(\Omega);\, L_2(\Gamma_0) \to H^{\frac{1}{2}}(\Omega) \subset H^{\frac{1}{2}-2\epsilon}(\Omega) & \\ \qquad = \mathcal{D}\big(\mathcal{A}_\Omega^{\frac{1}{4}-\epsilon}\big), \quad \epsilon > 0, s \text{ real}; & (3.15.5b) \end{cases}$$

$$\begin{cases} D_{\Gamma_0}\mu = z \iff \{\Delta_\Gamma z = 0 \text{ in } \Gamma_0;\quad z|_{\partial\Gamma_0} = \mu\}; & (3.15.6a) \\ D_{\Gamma_0} : L_2(\partial\Gamma_0) \to H^{\frac{1}{2}}(\Gamma_0) \subset H^{\frac{1}{2}-2\epsilon}(\Gamma_0) = \mathcal{D}\big(\mathcal{A}_{\Gamma_0}^{\frac{1}{4}-\epsilon}\big), \quad \epsilon > 0; & (3.15.6b) \end{cases}$$

$$\tilde{D}_\Omega^* \mathcal{A}_\Omega f = -\left.\frac{\partial f}{\partial \nu}\right|_{\Gamma_0}, \quad f \in H^{\frac{3}{2}+2\epsilon}(\Omega) \cap H_0^1(\Omega) = \mathcal{D}\big(\mathcal{A}_\Omega^{\frac{3}{4}+\epsilon}\big). \tag{3.15.7}$$

The above relations are the usual ones as, say, in Section 3.1: (3.1.7), (3.1.9), etc. By virtue of (3.15.3)–(3.15.7), we rewrite problem (3.15.1) abstractly, as usual, first as

$$\begin{cases} w_{tt} = -\mathcal{A}_{\Gamma_0} w + \mathcal{A}_{\Gamma_0} D_{\Gamma_0} u - \left.\dfrac{\partial h}{\partial \nu}\right|_{\Gamma_0} & \text{in } [\mathcal{D}(\mathcal{A}_{\Gamma_0})]', \quad (3.15.8) \\ h_t = -\mathcal{A}_\Omega h + \mathcal{A}_\Omega \tilde{D}_\Omega w_t & \text{in } [\mathcal{D}(\mathcal{A}_\Omega)]' \quad (3.15.9) \end{cases}$$

[we cannot write $\tilde{D}_\Omega^* \mathcal{A}_\Omega h$, since h does not satisfy $h|_{\Gamma_0} = 0$], and next, as the first-order abstract equation with $y = [w, w_t, h]$,

$$\frac{d}{dt}\begin{bmatrix} w \\ w_t \\ h \end{bmatrix} = A \begin{bmatrix} w \\ w_t \\ h \end{bmatrix} + Bu, \quad A = \begin{bmatrix} 0 & I & 0 \\ -\mathcal{A}_{\Gamma_0} & 0 & -\left.\dfrac{\partial}{\partial \nu}\cdot\right|_{\Gamma_0} \\ 0 & \mathcal{A}_\Omega \tilde{D}_\Omega & -\mathcal{A}_\Omega \end{bmatrix},$$

$$Y \supset \mathcal{D}(A) \to Y; \tag{3.15.10a}$$

$$A^* = \begin{bmatrix} 0 & -I & 0 \\ \mathcal{A}_{\Gamma_0} & 0 & \left.\dfrac{\partial}{\partial \nu}\cdot\right|_{\Gamma_0} \\ 0 & -\mathcal{A}_\Omega \tilde{D}_\Omega & -\mathcal{A}_\Omega \end{bmatrix}, \quad \mathcal{D}(A^*) = \mathcal{D}(A), \tag{3.15.10b}$$

to be interpreted in the sense that (similarly for A^*x):

$$A\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_2\\-\mathcal{A}_{\Gamma_0}x_1-\left.\dfrac{\partial}{\partial\nu}x_3\right|_{\Gamma_0}\\\mathcal{A}_\Omega(\tilde{D}_\Omega x_2-x_3)\end{bmatrix};\qquad Bu=\begin{bmatrix}0\\\mathcal{A}_{\Gamma_0}D_{\Gamma_0}u\\0\end{bmatrix};\tag{3.15.11}$$

$$\begin{aligned}Y\equiv\mathcal{D}\big(\mathcal{A}_{\Gamma_0}^{\frac{1}{2}}\big)\times L_2(\Gamma_0)\times L_2(\Omega)\equiv H_0^1(\Gamma_0)\times L_2(\Gamma_0)\times L_2(\Omega);\\U=L_2(\partial\Gamma_0);\end{aligned}\tag{3.15.12}$$

$$\begin{aligned}\mathcal{D}(A)=\Big\{[x_1,x_2,x_3]\in Y:x_1\in\mathcal{D}(\mathcal{A}_{\Gamma_0});\,x_2\in\mathcal{D}\big(\mathcal{A}_{\Gamma_0}^{\frac{1}{2}}\big);\,\Delta x_3\in L_2(\Omega);\\x_3|_{\Gamma_1}=0,\,x_3|_{\Gamma_0}=x_2\Big\}.\end{aligned}\tag{3.15.13}$$

Notice that the requirements on x_3 in the definition of $\mathcal{D}(A)$ in (3.15.13) may be simply expressed by writing that $[\tilde{D}_\Omega x_2-x_3]\in\mathcal{D}(\mathcal{A}_\Omega)$. Moreover, the definition of $\mathcal{D}(A)$ automatically guarantees that $\frac{\partial x_3}{\partial\nu}\big|_{\Gamma_0}\in L_2(\Gamma_0)$, as needed. In fact, write $x_3=\bar{x}_3+\tilde{x}_3$, where: (i) $\Delta\bar{x}_3\in L_2(\Omega)$, $\bar{x}_3|_\Gamma=0$, so that $\bar{x}_3\in H^2(\Omega)$ by elliptic theory, and $\frac{\partial\bar{x}_3}{\partial\nu}\big|_{\Gamma_0}\in L_2(\Gamma_0)$ by trace theory; (ii) $\Delta\tilde{x}_3=0$ in Ω, $\tilde{x}_3|_{\Gamma_1}=0$, $\tilde{x}_3|_{\Gamma_0}=x_2\in H_0^1(\Gamma_0)=\mathcal{D}(\mathcal{A}_{\Gamma_0}^{\frac{1}{2}})$, so that $\tilde{x}_3=\tilde{D}_\Omega x_2\in H^{\frac{3}{2}}(\Omega)$ by (3.15.5b), with $s=1$. Since $\tilde{x}_3$ is harmonic, elliptic theory yields [Kellog, 1972] $\frac{\partial\tilde{x}_3}{\partial\nu}\big|_{\Gamma_0}\in L_2(\Gamma_0)$. Thus, in conclusion, $\frac{\partial x_3}{\partial\nu}\big|_{\Gamma_0}\in L_2(\Gamma_0)$, as asserted.

Claim The operator A in (3.15.11), (3.15.13) is closed.

Proof of Claim. Let $x_n=[x_{1n},x_{2n},x_{3n}]\in\mathcal{D}(A)\to x=[x_1,x_2,x_3]\in Y$ in Y, and let Ax_n (see (3.15.11)) $\to\ell=[\ell_1,\ell_2,\ell_3]\in Y$ in Y. We must show that $x\in\mathcal{D}(A)$ (see (3.15.13)) and $Ax=\ell$. Indeed, from the assumptions, we obtain: $x_{2n}\to x_2=\ell_1$ in $\mathcal{D}(\mathcal{A}_{\Gamma_0}^{\frac{1}{2}})=H_0^1(\Gamma_0)$, and hence $[\tilde{D}_\Omega x_{2n}-x_{3n}]\to[\tilde{D}_\Omega\ell_1-x_3]$ in $L_2(\Omega)$, which together with the assumed $\mathcal{A}_\Omega[\tilde{D}_\Omega x_{2n}-x_{3n}]\to\ell_3$ in $L_2(\Omega)$ yields $[\tilde{D}_\Omega\ell_1-x_3]\in\mathcal{D}(\mathcal{A}_\Omega)$ and $\mathcal{A}_\Omega[\tilde{D}_\Omega\ell_1-x_3]=\ell_3$, since $\mathcal{A}_\Omega$ is closed. Moreover, we obtain $\tilde{D}_\Omega x_{2n}\to\tilde{D}_\Omega\ell_1$ in $H^{\frac{3}{2}}(\Omega)$, and $x_{3n}\to x_3=\tilde{D}_\Omega\ell_1-\mathcal{A}_\Omega^{-1}\ell_3$ in $H^{\frac{3}{2}}(\Omega)$. Thus, by using again [Kellog, 1972] as in the argument below (3.15.13), we obtain $\frac{\partial x_{3n}}{\partial\nu}\big|_{\Gamma_0}\to\frac{\partial x_3}{\partial\nu}\big|_{\Gamma_0}\in L_2(\Gamma_0)$. Hence $x_{1n}\to x_1\in\mathcal{D}(\mathcal{A}_{\Gamma_0})$ and $\mathcal{A}_{\Gamma_0}x_1=\ell_2-\frac{\partial x_3}{\partial\nu}\big|_{\Gamma_0}$ since $\mathcal{A}_{\Gamma_0}$ is closed. The claim is proved.

Assumption of Generation by A of a s.c. Contraction Semigroup on Y By the Lumer–Phillips theorem, as usual, we obtain (as in the case of thermo-elastic equations) the following:

Proposition 3.15.1 *The operator A in (3.15.11), (3.15.13) is dissipative,*

$$\mathrm{Re}(Ax,x)_Y=-\|\,|\nabla x_3|\,\|_{L_2(\Omega)}^2,\qquad x=[x_1,x_2,x_3]\in\mathcal{D}(A);\tag{3.15.14}$$

in fact, it is maximal dissipative; hence, it is the generator of a s.c. contraction semigroup e^{At} on $Y: y_0 = [w_0, w_1, h_0] \to e^{At}y_0 = [w(t), w_t(t), h(t)] \in C([0,T]; Y)$. Moreover, $|\nabla h| \in L_2(0,T; L_2(\Omega))$.

Proof. We limit ourselves to showing (3.15.14): By (3.15.11)–(3.15.14), with $x = [x_1, x_2, x_3] \in \mathcal{D}(A)$, we compute since $\mathcal{A}_D[\tilde{D}_\Omega x_2 - x_3] = \Delta x_3$ by (3.15.5a):

$$
\begin{aligned}
\operatorname{Re}(Ax, x)_Y - 0 + \left(\left.\frac{\partial x_3}{\partial \nu}\right|_{\Gamma_0}, x_2 \right)_{L_2(\Gamma_0)} &= (\Delta x_3, x_3)_{L_2(\Omega)} \\
&= \int_\Gamma \frac{\partial x_3}{\partial \nu} x_3 \, d\Gamma - \int_\Omega |\nabla x_3|^2 \, d\Omega \\
\text{(by (3.15.13))} \qquad &= \int_{\Gamma_0} \frac{\partial x_3}{\partial \nu} x_2 \, d\Gamma_0 - \int_\Omega |\nabla x_3|^2 \, d\Omega,
\end{aligned}
\tag{3.15.15}
$$

and (3.15.14) follows from (3.15.15) [in the last step we have used $x_3|_{\Gamma_1} = 0$ and $x_3|_{\Gamma_0} = x_2$ from (3.15.13)]. $\square$

Assumption (3.1.1.2) or (3.2.1.7): $A^{-\gamma}B \in \mathcal{L}(U; Y)$ From (3.15.11), one readily verifies, or computes, that

$$
A^{-1} = \begin{bmatrix} -\mathcal{A}_{\Gamma_0}^{-1}\frac{\partial}{\partial\nu}(\tilde{D}_\Omega \cdot)|_{\Gamma_0} & -\mathcal{A}_{\Gamma_0}^{-1} & \mathcal{A}_{\Gamma_0}^{-1}\frac{\partial}{\partial\nu}\mathcal{A}_\Omega^{-1} \cdot |_{\Gamma_0} \\ I & 0 & 0 \\ \tilde{D}_\Omega & 0 & \mathcal{A}_\Omega^{-1} \end{bmatrix}, \quad A^{-1}Bu = \begin{bmatrix} -\tilde{D}_{\Gamma_0}u \\ 0 \\ 0 \end{bmatrix}
\tag{3.15.16}
$$

($A^{-1}Bu$ can, of course, be computed at once, from (3.15.11), without A^{-1}, but we need A^{-1} in Proposition 3.15.5(i) below). From (3.15.6b) we have that $D_{\Gamma_0}u \in \mathcal{D}(\mathcal{A}_{\Gamma_0}^{\frac{1}{4}-\epsilon})$, for $u \in L_2(\partial\Gamma_0)$. Moreover, from (3.15.13), we see that $\mathcal{D}(\mathcal{A}_{\Gamma_0}^r)$ is the first component space of the domain of fractional power $\mathcal{D}(A^r)$, for $r < 1/4$, where compatibility conditions do not interfere, containing $A^{-1}Bu$, which has second and third components zero. Thus, for $u \in L_2(\partial\Gamma_0) = U$, we have

$$
A^{-1}Bu \in \mathcal{D}\big(A^{\frac{1}{4}-\epsilon}\big), \quad \text{or} \quad A^{-\gamma}B \in \mathcal{L}(U; Y), \quad \gamma = \frac{3}{4} + \epsilon. \tag{3.15.17}
$$

Thus, (3.15.17) proves assumption (1.1.2) of Chapter 1 or (2.1.7) of Chapter 2, with $\gamma = 3/4 + \epsilon$, $\epsilon > 0$.

Assumption of Analyticity of e^{At} on Y Showing analyticity of the s.c. contraction semigroup e^{At} on Y, guaranteed by Propositions 3.15.1, in the present case of model (3.15.1) is a much simpler task than showing analyticity in the case of thermoelastic equations, such as in Section 3.11.1, once the critical Lemma 3.15.3 below is established. Compare the simple proof in Step 3 below with the more technical proof of Appendix 3E. See Notes for references.

Theorem 3.15.2 *The s.c. contraction semigroup e^{At} on Y, guaranteed by Proposition 3.15.1, is, moreover, analytic on Y.*

Proof of Theorem 3.15.2.

Step 1 Inserting $h = -\mathcal{A}_\Omega^{-1} h_t + \tilde{D}_\Omega w_t$ from (3.15.9) into (3.15.8) with control $u \equiv 0$ (homogeneous case) yields the following damped second-order equation:

$$w_{tt} + \mathcal{A}_{\Gamma_0} w + \frac{\partial}{\partial \nu} \tilde{D}_\Omega w_t \bigg|_{\Gamma_0} = \frac{\partial}{\partial \nu} \mathcal{A}_\Omega^{-1} h_t \bigg|_{\Gamma_0}. \tag{3.15.18}$$

Motivated by (3.15.18), we thus introduce the capacity operator K [Dautray, Lions, 1987, p. 477] (which acts in (3.15.18) as a damping or friction operator)

$$K = \frac{\partial}{\partial \nu} \tilde{D}_\Omega \cdot \bigg|_{\Gamma_0} : L_2(\Gamma_0) \supset \mathcal{D}(K) \to L_2(\Gamma_0), \tag{3.15.19a}$$

$$\mathcal{D}(K) = \left\{ g \in L_2(\Gamma_0) : \frac{\partial}{\partial \nu} \tilde{D}_\Omega g \bigg|_{\Gamma_0} \in L_2(\Gamma_0) \right\}. \tag{3.15.19b}$$

The following result is critical in our analysis, see Notes.

Lemma 3.15.3 *(i) With reference to (3.15.19), we have*

$$\mathcal{D}(K) = \mathcal{D}\left(\frac{\partial}{\partial \nu} \tilde{D}_\Omega \cdot \bigg|_{\Gamma_0} \right) = H^1(\Gamma_0). \tag{3.15.20}$$

(ii) K is positive self-adjoint: $Kg = 0 \Rightarrow g = 0$ and

$$(Kg, g)_{L_2(\Gamma_0)} = \int_\Omega |\nabla g|^2 \, d\Omega, \quad g \in H^1(\Gamma_0). \tag{3.15.21}$$

Proof of Lemma 3.15.3. (i) We first show that $H^1(\Gamma_0) \subset \mathcal{D}(K)$. Let $g \in H^1(\Gamma_0)$. Then $\tilde{D}_\Omega g \in H^{\frac{3}{2}}(\Omega)$ by (3.15.5b) with $s = 1$, and $\tilde{D}_\Omega g$ is a harmonic function defined as in (3.15.5a), thus with $\tilde{D}_\Omega g|_{\Gamma_1} = 0$. By elliptic theory [Kellog, 1972], we then have that $\frac{\partial}{\partial \nu}(\tilde{D}_\Omega g)|_{\Gamma_0} \in L_2(\Gamma_0)$. Thus, $g \in \mathcal{D}(K)$.

We next show that $\mathcal{D}(K) \subset H^1(\Gamma_0)$. Let $g \in \mathcal{D}(K)$ so that, by (3.15.5a) and (3.15.19b), we have

$$\Delta(\tilde{D}_\Omega g) \equiv 0 \text{ in } \Omega, \quad \tilde{D}_\Omega g|_{\Gamma_1} \equiv 0, \quad \frac{\partial}{\partial \nu}(\tilde{D}_\Omega g)|_{\Gamma_0} \in L_2(\Gamma_0). \tag{3.15.22}$$

By elliptic theory [Lions, Magenes, 1972, p. 187], it follows from (3.15.22) that $\tilde{D}_\Omega g \in H^{\frac{3}{2}}(\Omega)$. Hence, by trace theory, $g = \tilde{D} g|_{\Gamma_0} \in H^1(\Gamma_0)$, as desired. Thus, (3.15.20) is established.

(ii) Initially, for $g \in H^{\frac{3}{2}}(\Gamma_0)$ we compute, since $(\Delta \tilde{D}_\Omega g, g)_{L_2(\Omega)} = 0$ via (3.15.5a), by virtue of Green's first theorem

$$(Kg, g)_{L_2(\Gamma_0)} = \left(\frac{\partial \tilde{D}_\Omega g}{\partial \nu}\bigg|_{\Gamma_0}, g \right)_{L_2(\Gamma_0)} - (\Delta \tilde{D}_\Omega g, g)_{L_2(\Omega)} = \int_\Omega |\nabla g|^2 \, d\Omega. \tag{3.15.23}$$

Then, (3.15.23) is extended to all $g \in H^1(\Gamma_0) = \mathcal{D}(K)$. Finally, let $Kg = \frac{\partial \tilde{D}_\Omega g}{\partial \nu}\big|_{\Gamma_0} = 0$, where $\tilde{D}_\Omega g|_{\Gamma_1} = 0$ by (3.15.5a). Then, $\tilde{D}_\Omega g = 0$ in Ω, and hence $g = \tilde{D}_\Omega g|_{\Gamma_0} = 0$, as asserted. □

Step 2. Proposition 3.15.4. With reference to (3.15.19), the dynamical operator yielding the solution of problem (3.15.17),

$$A_1 = \begin{bmatrix} 0 & I \\ -\mathcal{A}_{\Gamma_0} & -K \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mathcal{A}_{\Gamma_0} & -\dfrac{\partial}{\partial \nu} \tilde{D}_\Omega \cdot \bigg|_{\Gamma_0} \end{bmatrix},$$

$$Y_1 \equiv \mathcal{D}\big(\mathcal{A}_{\Gamma_0}^{\frac{1}{2}}\big) \times L_2(\Gamma_0) \supset \mathcal{D}(A_1) \to Y_1; \tag{3.15.24a}$$

$$\mathcal{D}(A_1) = \mathcal{D}(\mathcal{A}_{\Gamma_0}) \times \mathcal{D}\big(\mathcal{A}_{\Gamma_0}^{\frac{1}{2}}\big); \quad \mathcal{D}\big(\mathcal{A}_{\Gamma_0}^{\frac{1}{2}}\big) = H_0^1(\Gamma_0) \subset \mathcal{D}(K) = H^1(\Gamma_0), \tag{3.15.24b}$$

is the generator of a s.c. semigroup $e^{A_1 t}$ on Y_1, which moreover is analytic here.

Proof of Proposition 3.15.4 We use Theorem 3B.1, Appendix 3B, with $\alpha = 1/2$. It follows from Lemma 3.15.3 via Lowner's theorem [Krein, 1988, Corollary 7.1, p. 146] that

$$\mathcal{D}(K^{\frac{1}{2}}) = \mathcal{D}\left(\left(\frac{\partial}{\partial \nu} \tilde{D}_\Omega \cdot \bigg|_{\Gamma_0} \right)^{\frac{1}{2}} \right) = H^{\frac{1}{2}}(\Gamma_0) = \mathcal{D}\big(\mathcal{A}_{\Gamma_0}^{\frac{1}{4}}\big), \tag{3.15.25}$$

recalling also (3.15.4). Thus, Theorem 3B.1, Appendix 3B, with $\alpha = 1/2$ applied to (3.15.24) yields the desired conclusion via (3.15.25). □

Step 3 We rewrite problem (3.15.18) via (3.15.24) as

$$\frac{d}{dt} \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} = A_1 \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{\partial}{\partial \nu} \mathcal{A}_\Omega^{-1} h_t(t) \bigg|_{\Gamma_0} \end{bmatrix}, \tag{3.15.26}$$

so that, by (3.15.9), and by Propositions 3.15.1 and 3.15.4, its solution is

$$\begin{bmatrix} \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} \\ h(t) \end{bmatrix} = e^{At} \begin{bmatrix} w_0 \\ w_1 \\ h_0 \end{bmatrix}$$

$$= \begin{bmatrix} \left[e^{A_1 t} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} + \int_0^t e^{A_1(t-\tau)} \begin{bmatrix} 0 \\ \frac{\partial}{\partial \nu} \mathcal{A}_\Omega^{-1} h_t(\tau) \Big|_{\Gamma_0} \end{bmatrix} dt \right] \\ e^{-\mathcal{A}_\Omega t} h_0 + \int_0^t e^{-\mathcal{A}_\Omega (t-\tau)} \mathcal{A}_\Omega \tilde{D}_\Omega w_t(\tau)\, d\tau \end{bmatrix}. \qquad (3.15.27)$$

We note that because of the analyticity of the s.c. semigroup $e^{A_1 t}$ (Proposition 3.15.4), the first integral is well defined, at least in the sense that, after integration by parts,

$$\int_0^t e^{A_1(t-\tau)} \begin{bmatrix} 0 \\ \frac{\partial}{\partial \nu} \mathcal{A}_\Omega^{-1} h_t(\tau) \Big|_{\Gamma_0} \end{bmatrix} d\tau = \begin{bmatrix} 0 \\ \frac{\partial}{\partial \nu} \mathcal{A}_\Omega^{-1} h(t) \Big|_{\Gamma_0} \end{bmatrix} - e^{A_1 t} \begin{bmatrix} 0 \\ \frac{\partial}{\partial \nu} \mathcal{A}_\Omega^{-1} h(0) \end{bmatrix}$$

$$+ \int_0^t e^{A_1(t-\tau)} A_1 \begin{bmatrix} 0 \\ \frac{\partial}{\partial \nu} \mathcal{A}_\Omega^{-1} h(\tau) \Big|_{\Gamma_0} \end{bmatrix} d\tau \in L_2(0, T; Y_1), \qquad (3.15.28)$$

since $\frac{\partial}{\partial \nu} \mathcal{A}_\Omega^{-1} h \big|_{\Gamma_0} \in C([0, T]; H^{\frac{1}{2}}(\Gamma_0))$ by the a priori regularity of $h \in C([0, T]; L_2(\Omega))$ via Proposition 3.15.1 and by trace theory and property (0.9) of Chapter 0. Moreover, again by analyticity of the s.c. semigroup $e^{-\mathcal{A}_\Omega t}$, the second integral in (3.15.27) is well defined in $C([0, T]; L_2(\Omega))$ by convolution between $\|\mathcal{A}_\Omega^{\frac{3}{4}-\epsilon} e^{-\mathcal{A}_\Omega t}\| \leq c/t^{\frac{3}{4}-\epsilon} \in L_1(0, T)$ and $\|\mathcal{A}_\Omega^{\frac{1}{4}-\epsilon} \tilde{D}_\Omega w_t\| \in C([0, T]; L_2(\Gamma_0))$, via (3.15.5b) and Proposition 3.15.1 on $w_t \in C([0, T]; L_2(\Gamma_0))$. Continuing with the proof, we see that for $y_0 = [w_0, w_1, h_0] \in Y$, we have $[\hat{w}(\lambda), \hat{w}_t(\lambda), \hat{h}(\lambda)] = R(\lambda, A) y_0 \in \mathcal{D}(A)$, for $\text{Re}\,\lambda > 0$ by (3.15.27) (left), where $\hat{}$ denotes the Laplace transform. A fortiori, via (3.15.13), we have $w(\lambda) \in \mathcal{D}(\mathcal{A}_{\Gamma_0})$, $\hat{w}_t(\lambda) \in \mathcal{D}(\mathcal{A}_{\Gamma_0}^{\frac{1}{2}})$, $\hat{h}(\lambda) \in L_2(\Omega)$. Thus, the Laplace transform version of (3.15.27) for $\text{Re}\,\lambda > 0$ is

$$\begin{bmatrix} \hat{w}(\lambda) \\ \hat{w}_t(\lambda) \end{bmatrix} = R(\lambda, A_1) \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} - \lambda R(\lambda, \lambda_1) \begin{bmatrix} 0 \\ \tilde{D}_\Omega^* \hat{h}(\lambda) \end{bmatrix} + R(\lambda, A_1) \begin{bmatrix} 0 \\ \tilde{D}_\Omega^* h_0 \end{bmatrix}, \qquad (3.15.29)$$

$$\hat{h}(\lambda) = R(\lambda, -\mathcal{A}_\Omega) h_0 + \left[R(\lambda, -\mathcal{A}_\Omega) \mathcal{A}_\Omega^{\frac{3}{4}+\epsilon} \right] \mathcal{A}_\Omega^{\frac{1}{4}-\epsilon} \tilde{D}_\Omega \hat{w}_t(\lambda), \qquad (3.15.30)$$

where in (3.15.29) we have used $(\partial/\partial \nu) \mathcal{A}_\Omega^{-1} \hat{h}(\lambda)|_{\Gamma_0} = (D_\Omega^* \mathcal{A}_\Omega) \mathcal{A}_\Omega^{-1} \hat{h}(\lambda)$ via (3.15.7), since $\mathcal{A}_\Omega^{-1} \hat{h}(\lambda) \in \mathcal{D}(\mathcal{A}_\Omega)$. Using, this time in λ, the analyticity of the (self-adjoint) s.c. semigroup $e^{-\mathcal{A}_\Omega t}$, we have, for $\text{Re}\,\lambda > 0$,

$$\left\| R(\lambda, -\mathcal{A}_\Omega) \mathcal{A}_\Omega^{\frac{3}{4}+\epsilon} \right\|_{\mathcal{L}(L_2(\Omega))} \leq \frac{k}{|\lambda|^{1-\beta}}, \quad \beta = \frac{3}{4} + \epsilon,$$

while $\mathcal{A}_\Omega^{\frac{1}{4}-\epsilon} \tilde{D}_\Omega = \mathcal{A}_\Omega^{1-\beta} \tilde{D}_\Omega \in \mathcal{L}(L_2(\Gamma_0); L_2(\Omega))$, (3.15.31) recalling (3.15.5b). Using (3.15.31) in (3.15.30) and the usual resolvent estimates for the analytic generators $-\mathcal{A}_\Omega$ and A_1 (from Proposition 3.15.4) in (3.15.30) and (3.15.29) respectively, we obtain since $\tilde{D}_\Omega^* \in \mathcal{L}(L_2(\Omega); L_2(\Gamma_0))$:

$$\|\hat{h}(\lambda)\|_{L_2(\Omega)} \leq \frac{k_1}{|\lambda|}\|h_0\|_{L_2(\Omega)} + \frac{c_1}{|\lambda|^{1-\beta}}\|\hat{w}_t(\lambda)\|_{L_2(\Gamma_0)}, \quad \text{Re}\,\lambda > 0; \tag{3.15.32}$$

$$\left\|\begin{bmatrix}\hat{w}(\lambda)\\ \hat{w}_t(\lambda)\end{bmatrix}\right\|_{Y_1} \leq \frac{k_2}{|\lambda|}\|y_0\|_Y + c_2\|\hat{h}(\lambda)\|_{L_2(\Omega)}, \quad \text{Re}\,\lambda > 0, \quad y_0 = [w_0, w_1, h_0] \in Y, \tag{3.15.33}$$

where k_i and c_i are generic constants. Now we insert (3.15.32) into the right-hand side of (3.15.33) and specialize to all λ with Re λ sufficiently large, say Re $\lambda > r_0$ for some suitable $r_0 > 0$, as to take advantage of $1/|\lambda|^{1-\beta} \searrow 0$ as $|\lambda| \nearrow \infty$, $0 < \beta < 1$, so that moving $\|\hat{w}_t(\lambda)\|$ to the left-hand side, we obtain (recalling Y_1 in (3.15.24a):

$$\frac{1}{2}\left\|\begin{bmatrix}\hat{w}(\lambda)\\ \hat{w}_t(\lambda)\end{bmatrix}\right\|_{Y_1} \leq \left(1 - \frac{c_1c_2}{|\lambda|^{1-\beta}}\right)\left\|\begin{bmatrix}\hat{w}(\lambda)\\ \hat{w}_t(\lambda)\end{bmatrix}\right\|_{Y_1} \leq \frac{k_3}{|\lambda|}\|y_0\|_Y,$$

$$\forall\, \lambda \text{ with Re}\,\lambda > r_0 > 0, \tag{3.15.34}$$

with, say, $r_0 \equiv (2c_1c_2)^{1/1-\beta}$. Substituting (3.15.34) back into (3.15.32), we obtain

$$\|\hat{h}(\lambda)\|_{L_2(\Omega)} \leq \frac{k_4}{|\lambda|}\|y_0\|_Y, \quad \forall\, \lambda \text{ with Re}\,\lambda > r_0 > 0. \tag{3.15.35}$$

Once combined inequalities (3.15.34) and (3.15.35), via the left-hand side of (3.15.27), say (recalling Y from (3.15.12)) that

$$\|\{\hat{w}(\lambda), \hat{w}_t(\lambda), \hat{h}(\lambda)\}\|_Y = \|R(\lambda, A)y_0\|_Y \leq \frac{k_5}{|\lambda|}\|y_0\|_Y, \quad \forall\lambda \text{ with Re}\,\lambda > r_0 > 0. \tag{3.15.36}$$

Inequality (3.15.36) then yields that the operator A is the generator of a s.c. analytic semigroup e^{At} on Y [Fattorini, 1983, p. 185]. The proof of Theorem 3.15.2 is complete. □

Finite Cost Condition (2.1.12) and Detectability Condition (2.1.13) As in the case of the thermo-elastic plates of Sections 3.11 and 3.12, and essentially for the same reasons, the s.c. contraction semigroup e^{At} is also uniformly stable on Y. Thus, both the Finite Cost Condition (2.1.12) and the Detectability Condition (2.1.13) of Chapter 2 are automatically satisfied.

Proposition 3.15.5 *With reference to the operator A defined in (3.15.11), (3.15.13), we have:*

(i) Its inverse A^{-1} in (3.15.16) is compact on Y; thus, A has a compact resolvent on Y, and the sepctrum $\sigma(A)$ of A is only a point spectrum (eigenvalues).

(ii) There is no spectrum of A on the closed right-hand complex plane $\mathbb{C}^+ = \{\lambda : Re\,\lambda \geq 0\}$; *that is,* $\sigma(A) \cap \mathbb{C}^+ = \emptyset$,

(iii) The s.c. contraction, analytic semigroup e^{At} *on Y is, moreover, uniformly stable here: There exist constants* $M \geq 1$, $-\omega = \sup\ \mathrm{Re}\,\sigma(A) < 0$ *such that*

$$\|e^{At}\|_{\mathcal{L}(Y)} \leq Me^{-(\omega-\epsilon)t}, \quad t \geq 0, \tag{3.15.37}$$

with M depending on $(-\omega + \epsilon)$, $\epsilon > 0$.

Proof. Recalling the proof of Proposition 3.11.1.3, we see that it suffices to show that A has no eigenvalues on the imaginary axis. To establish this, we proceed as in that proof as well. For r real and $[w_1, w_2, h_0] \in \mathcal{D}(A)$, let

$$A\begin{bmatrix} w_1 \\ w_2 \\ h_0 \end{bmatrix} = ir\begin{bmatrix} w_1 \\ w_2 \\ h_0 \end{bmatrix}; \quad \text{or} \quad \begin{cases} w_2 = irw_1, & (3.15.38a) \\ -\mathcal{A}_{\Gamma_0} w_1 - \dfrac{\partial}{\partial\nu} h_0\Big|_{\Gamma_0} = irw_2 = -r^2 w_1, & (3.15.38b) \\ \mathcal{A}_\Omega(\tilde{D}_\Omega w_2 - h_0) = irh_0. & (3.15.38c) \end{cases}$$

We must show that $w_1 = w_2 = h_0 = 0$. This is trivial if $r = 0$. Let $r \neq 0$. Recalling (3.15.14), we obtain, with $y = [w_1, w_2, h_0] \in \mathcal{D}(A)$,

$$\mathrm{Re}(Ay, y)_Y = -\||\nabla h_0|\|^2_{L_2(\Omega)} = \mathrm{Re}\, ir\|y\|_Y^2 = 0. \tag{3.15.39}$$

Thus, (3.15.39) implies first $h_0 =$ const. in Ω, and then $h_0 = 0$ in Ω, since $h_0|_{\Gamma_1} = 0$ by (3.15.13). Then, (3.15.38c) yields $\tilde{D}_\Omega w_2 = 0$, where $\tilde{D}_\Omega w_2$ is harmonic in Ω [see (3.15.5a)]. Then, $w_2 = 0$ for its boundary value on Γ_0. Finally, $w_1 = 0$ by (3.15.38a). □

Conclusion. Case $T = \infty$ Theorems 2.2.1 and 2.2.2 of Chapter 2 apply to problem (3.15.1), (3.15.2), with $a = 0$, where $R = I$, within the above framework.

Conclusion. Case $T < \infty$ Theorem 1.2.2.1 of Chapter 1 applies to problem (3.15.1), (3.15.2), with $a = 1$ and $G = I$.

3.16 Additional Illustrations with Control Operator *B* and Observation Operator *R* Both Genuinely Unbounded

In Section 3.11.2 we have already presented a thermo-elastic PDE problem, where both the control operator B and the observation operator R are genuinely unbounded, and fit the abstract theory of Chapter 1, Section 1.8 or Chapter 2, Section 2.5. In this section, we shall present a few more of such examples.

3.16.1 Heat Equation with Neumann Boundary Control and Point Observation

Let Ω be an open bounded domain in $\mathbb{R}^n$. We consider the following parabolic dynamics (which is essentially that of Section 3.3, except that we now make the problem stable for notational convenience):

$$\begin{cases} y_t = \Delta y - c^2 y & \text{in } Q, \quad (3.16.1.1a) \\ y(0, \cdot\,) = y_0 & \text{in } \Omega, \quad (3.16.1.1b) \\ \left.\dfrac{\partial y}{\partial \nu}\right|_{\Sigma} = u & \text{in } \Sigma. \quad (3.16.1.1c) \end{cases}$$

This time, however, we have the following cost functional:

$$J(u, y) = \int_0^\infty \left[|y(t, \xi)|^2 + \|u(t)\|^2_{L_2(\Gamma)}\right] dt, \tag{3.16.1.2}$$

where ξ is a fixed (interior) point of Ω: $\xi \in \Omega$. Thus, the functional J penalizes the boundary control u and the point observation $y(\,\cdot\,, \xi)$, $\xi \in \Omega$. Here below we shall only limit ourselves to note the modifications that are necessary over the analysis of Section 3.3.

***Case 1:* dim $\Omega = 2$** Let A be the operator (similarly to Eqn. (3.3.1.4) of Section 3.3):

$$-Ah = \Delta h + c^2 h; \quad \mathcal{D}(A) = \left\{ h \in H^2(\Omega) : \left.\frac{\partial h}{\partial \nu}\right|_{\Gamma} = 0 \right\} \tag{3.16.1.3}$$

considered, as in Section 3.3.1, as lifted to the self-adjoint operator

$$A : \mathcal{D}\big(\hat{A}^{\frac{3}{2}}\big) \to \mathcal{D}\big(\hat{A}^{\frac{1}{2}}\big) = H^1(\Omega) \tag{3.16.1.4}$$

with $\hat{A} = A$ defined on $L_2(\Omega)$. We take

$$Y = \mathcal{D}\big(\hat{A}^{\frac{1}{2}}\big) = H^1(\Omega), \quad Z = \mathbb{R}. \tag{3.16.1.5}$$

Under this choice of Y, the analysis of Section 3.3.1 shows that the corresponding value of the parameter γ is (see Eqn. (3.1.14))

$$\gamma = \frac{3}{4} + \epsilon > 0, \quad A^{-\gamma} B \in \mathcal{L}(U; Y). \tag{3.16.1.6}$$

By the two-dimensional Sobolev embedding, we have

$$f \to f(\xi) : \text{continuous } \mathcal{D}\big(\hat{A}^{\frac{1}{2}+\epsilon}\big) = H^{1+2\epsilon}(\Omega) \to \mathbb{R}. \tag{3.16.1.7}$$

Thus, if we introduce the operator R by

$$Rf = f(\xi), \tag{3.16.1.8}$$

and view it as an operator on $Y = \mathcal{D}(\hat{A}^{\frac{1}{2}}) = H^1(\Omega)$, we have by (3.16.1.7):

$$\begin{cases} R: \text{continuous } Y \supset \mathcal{D}(R) \to \mathbb{R} = Z; \\ \mathcal{D}(R) = \mathcal{D}(A^{\epsilon}),\ A \text{ as in (3.16.1.4)}. \end{cases} \tag{3.16.1.9}$$

Thus, in the notation of Chapter 1, Section 1.8 or of Chapter 2, Section 2.5, we have, via (3.16.1.9) and (3.16.1.6), that we can take

$$\delta = \epsilon < \min\left\{1 - \gamma, \frac{1}{2}\right\}. \tag{3.16.1.10}$$

The theory of Section 2.5 applies to problem (3.16.1.1), (3.16.1.2) and yields, in particular, the Riccati operator $P = P^* \in \mathcal{L}(\mathcal{D}(\hat{A}^{\frac{1}{2}}))$ such that [Chapter 2, Eqn. (2.5.24b)]:

$$A^{\theta} P \in \mathcal{L}\big(\mathcal{D}(\hat{A}^{\frac{1}{2}})\big) \iff A^{\frac{1}{2}+\theta} P A^{-\frac{1}{2}} \in \mathcal{L}(L_2(\Omega)), \quad 0 \le \theta < 1 - 2\delta = 1 - 2\epsilon, \tag{3.16.1.11}$$

since A is self-adjoint on $\mathcal{D}(\hat{A}^{\frac{1}{2}}) = H^1(\Omega)$. By applying Lemma 1.5.1.1 of Chapter 1 with $\mathcal{Y} = \mathcal{D}(\hat{A}^{\frac{1}{2}})$, $\mathcal{G} = P$, $\mathcal{A} = A$, we obtain from (3.16.1.11)

$$A^r P A^s \in \mathcal{L}\big(\mathcal{D}(\hat{A}^{\frac{1}{2}})\big) \iff A^{r+\frac{1}{2}} P \hat{A}^{s-\frac{1}{2}} \in \mathcal{L}(L_2(\Omega)),$$
$$0 < r, s; \quad r + s < \theta < 1 - 2\epsilon. \tag{3.16.1.12}$$

Selecting $s = 1/2$, hence $r < 1/2 - 2\epsilon$, or $r + 1/2 < 1 - 2\epsilon$, we obtain

$$\hat{A}^{1-\epsilon} P \in \mathcal{L}(L_2(\Omega)). \tag{3.16.1.13}$$

***Case 2:* dim $\Omega = 1$** An alternative setting for problem (3.16.1.1), (3.16.1.2) when dim $\Omega = 1$ is as follows

$$Y = L_2(\Omega); \quad Z = \mathbb{R}; \quad R \in \mathcal{L}\big(\mathcal{D}(\hat{A}^{\frac{1}{4}-\epsilon});\ Z\big) \tag{3.16.1.14}$$

with

$$f \to f(\xi) \text{ continuous } \mathcal{D}\big(\hat{A}^{\frac{1}{4}+\epsilon}\big) = H^{\frac{1}{2}+2\epsilon}(\Omega) \to \mathbb{R} \tag{3.16.1.15}$$

by the one-dimensional Sobolev embedding. With this choice $Y = L_2(\Omega)$, Remark 3.3.1.4 provides $\gamma = 1/4 + \epsilon$. Moreover, by (3.16.1.14), in the notation of Section 1.8 of Chapter 1 or Section 2.5 of Chapter 2, we have

$$\delta = \frac{1}{4} + \epsilon < \min\left\{1 - \gamma, \frac{1}{2}\right\}. \tag{3.16.1.16}$$

Thus, the theory of Section 2.5 applies to problem (3.16.1.1), (3.16.1.2) in the present setting and yields, in particular, the Riccati operator $P = P^* \in \mathcal{L}(L_2(\Omega))$ such that

$$\hat{A}^{\theta} P \in \mathcal{L}(L_2(\Omega)), \quad 0 \le \theta < 1 - 2\theta = \frac{1}{2} - 2\epsilon. \tag{3.16.1.17}$$

Remark 3.16.1 With the choice $Y = L_2(\Omega)$, the case $\dim \Omega = 2$, where then $R \in \mathcal{L}(\mathcal{D}(\hat{A}^{\frac{1}{2}+\epsilon}); Z)$, $\mathcal{D}(\hat{A}^{\frac{1}{2}+\epsilon}) = H^{1+2\epsilon}(\Omega)$, is excluded for problem (3.16.1.1), (3.16.1.2), since it leads to the candidate value $\delta = 1/2 + \epsilon$, which violates the requirement $\delta < \min\{1-\gamma, 1/2\}$ of the theory of Section 1.8 or Section 2.5.

3.16.2 Heat Equation with Point Control (and Point Observation when $\dim \Omega = 1$*)*

With $\Omega \subset \mathbb{R}^n$, we consider the parabolic problem

$$\begin{cases} y_t = \Delta y + \delta(\cdot - x^0)u & \text{in } (0,T] \times \Omega \equiv Q, & (3.16.2.1a) \\ y(0, \cdot) = y_0 & \text{in } \Omega, & (3.16.2.1b) \\ y|_\Sigma \equiv 0 & \text{in } (0,T] \times \Gamma \equiv \Sigma, & (3.16.2.1c) \end{cases}$$

with point control $u(t)$ acting at the interior point $x^0 \in \Omega$. Let

$$Ah = -\Delta h, \quad h \in \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega);$$
$$B = \delta(\cdot - x^0); \quad Y = L_2(\Omega), \quad U = \mathbb{R}. \tag{3.16.2.2}$$

We have

$$A^{-\gamma}B \in \mathcal{L}(U;Y), \quad \gamma = \frac{n}{4} + \epsilon, \quad n = \dim \Omega; \quad \epsilon > 0. \tag{3.16.2.3}$$

In fact, we have that $\delta(\cdot - x^0) \in [H^s(\Omega)]'$, where the duality is with respect to $L_2(\Omega)$, provided $H^s(\Omega) \subset C(\bar{\Omega})$, that is, provided $s > n/2$, $n = \dim \Omega$. Moreover, $\mathcal{D}(A^{\frac{s}{2}}) \subset H^s(\Omega)$ for the second-order differential operator in (3.16.2.2). Hence $[H^s(\Omega)]' \subset [\mathcal{D}(A^{s/2})]'$, and then

$$\delta(\cdot - x^0) \in \left[\mathcal{D}(A)^{\frac{s}{2}}\right]' \iff A^{-\frac{s}{2}}\delta(\cdot - x^0) \in L_2(\Omega) = Y, \tag{3.16.2.4}$$

and (3.16.2.3) follows from (3.16.2.4) with $B = \delta(\cdot - x^0)$ as in (3.16.2.2), and $\gamma = s/2 > n/4$. Thus we take $n = \dim \Omega \leq 3$. Thus: *For $n \leq 3$, (3.16.2.3) fulfills the basic condition of Chapters 1 and 2 that $\gamma < 1$.*

Thus, we now associate to Eqn. (3.16.2.1), the cost functional

$$J(u,y) = \int_0^\infty \left[\|Ry(t)\|_Z^2 + |u(t)|^2\right] dt, \tag{3.16.2.5}$$

with

$$R \in \mathcal{L}(\mathcal{D}(A^\delta); Z), \tag{3.16.2.6}$$

where δ satisfies, with $n \leq 3$,

$$\delta < \min\left\{1 - \gamma, \frac{1}{2}\right\} = \min\left\{1 - \frac{n}{4} - \epsilon, \frac{1}{2}\right\}; \tag{3.16.2.7}$$

$$\delta < \frac{1}{4} - \epsilon, \text{ for } n = 3; \; \delta < \frac{1}{2} - \epsilon, \quad \text{for } n = 2; \quad \delta < \frac{1}{2}, \text{ for } n = 1. \tag{3.16.2.8}$$

***Case n* = 1** For instance, for $n = 1$ we can take Z and R as in Eqn. (3.16.1.15) (point observation), which results in $\delta = 1/4 + \epsilon < 1/2$ [see (3.16.1.16)]. The corresponding cost functional is then

$$J(u, y) = \int_0^\infty [|y(t, \xi)|^2 + |u(t)|^2]\, dt \tag{3.16.2.9}$$

for a fixed (interior) point $\xi \in \Omega$. Thus, problem (3.16.2.1), (3.16.2.9) is point control at x^0 and point observation at ξ.

3A Interpolation (Intermediate) Sobolev Spaces and Their Identification with Domains of Fractional Powers of Elliptic Operators

This appendix recalls fundamental and by now classical results, which, in particular, provide the identification of Sobolev spaces with domain of fractional powers of the realization of a properly elliptic operator on an open bounded domain $\Omega \subset \mathbb{R}^n$ with a normal system of boundary operators on its boundary Γ, covering it [Lions, Magenes, 1972, Vol. 1, p. 113].

This result is repeatedly invoked in the present Chapter 3 dealing with illustrative parabolic PDE examples. It will also be invoked for the hyperbolic illustrative examples, e.g., in Chapters 8 and 10.

Realization of Properly Elliptic Operators with a Normal System of Boundary Operators Let Ω be an open bounded domain in $\mathbb{R}^n$ with boundary Γ, an $(n-1)$-dimensional infinitely differentiable variety, where Ω is locally on one side of Γ. Henceforth, $x = \{x_1, \dots, x_n\} \in \Omega$. Let

$$A(x, D)u = \sum_{|p|,|q| \le m} (-1)^{|p|} D^p (a_{pq}(x) D^q u); \tag{3A.1}$$

$$D^\alpha = \frac{\partial^{\alpha_1 + \cdots \alpha_n}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad \alpha = \{\alpha_1, \dots, \alpha_n\},\ |\alpha| = \alpha_1 + \cdots \alpha_n, \tag{3A.2}$$

be a differential operator of order $2m$, $m = 1, 2, \dots$ with infinitely differentiable coefficients in Ω. The characteristic form of A is

$$A_0(x, \xi) = \sum_{|p|,|q| = m} (-1)^m a_{pq}(x) \xi^{p+q}, \quad \text{where } \xi^{p+q} = \xi_1^{p_1 + q_1} \cdots \xi_n^{p_n + q_n}. \tag{3A.3}$$

Let

$$B_j(x, D)\phi = \sum_{|h| \le m_j} b_{jh}(x) D^h \phi, \quad j = 1, 2, \dots, m, \tag{3A.4}$$

be m boundary differential operators of order $m_j \le 2m - 1$, with infinitely differentiable coefficients on Γ.

We assume the following conditions:

(a) $A(x, D)$ is *properly elliptic* in $\bar{\Omega}$. In the present setting, we obtain that $A(x, D)$ is *uniformly elliptic* on $\bar{\Omega}$, that is, there exists a constant $c > 0$, independent of

x, such that

$$\frac{1}{c}|\xi|^{2m} \le |A_0(x,\xi)| \le c|\xi|^{2m}, \quad \forall\, x \in \bar{\Omega} \text{ and } \forall\, \xi \in \mathbb{R}^n \tag{3A.5}$$

(see [Lions, Magenes, 1972, Vol. 1, Def. 1.2, p. 110 and p. 112).

(b) $\{B_j(x,D)\}$, of order $m_j \le 2m-1$, $1 \le j \le m$, is a normal system on Γ; that is, [Lions, Magenes, 1972, Vol. I, Def. 1.4, p. 114],

(b_1)

$$\sum_{|h|=m_j} b_{jh}(x)\xi^h \ne 0, \quad \forall\, x \in \Gamma \text{ and } \forall\, \xi \ne 0 \text{ and normal to } \Gamma \text{ at } x, \tag{3A.6}$$

thus, Γ is not characteristic with respect to $B_j(x,D)$.

(b_2) $m_j \ne m_i$ for $j \ne i$.

(c) $A(x;D)$ and $\{B_j(x;D)\}_{1\le j\le m}$ verify the covering or complementing conditions, (see [Lions, Magenes, 1972, Vol. I, p. 113], H. Tanabe [1979] [1, Chapter 3, Theorem 3.8.1]) for $\theta = \pi$ [Triebel, 1978, pp. 333–4].

Under the above assumptions define an operator A in $L_2(\Omega)$ as follows:

$$\begin{cases} \mathcal{D}(A) = \{u \in H^{2m}(\Omega) : B_j(x;D)u = 0 \text{ on } \Gamma,\ 1 \le j \le m\}, \\ \quad Au = A(x;D)u. \end{cases} \tag{3A.7}$$

A is called a *realization* of $A(x;D)$ in $L_2(\Omega)$ under the boundary conditions $\{B_j(x;D)\}_{1\le j\le m}$. Its adjoint is given by

$$\begin{cases} \mathcal{D}(A^*) = \{v \in H^{2m}(\Omega) : C_j(x;D)v = 0 \text{ on } \Gamma,\ 1 \le j \le m\}, \\ \quad A^*v = A(x;D)^*v, \end{cases} \tag{3A.8}$$

where $\{C_j(x;D)\}_{1\le j\le m}$ is the adjoint system of $\{B_j(x;D)\}_{1\le j\le m}$ [Lions, Magenes, 1972, Vol. I, p. 115], so that Green's formula holds true. The space $\mathcal{D}(A)$ will also be denoted by $H_B^{2m}(\Omega)$, and similarly the space $\mathcal{D}(A^*)$ will also be denoted by $H_C^{2m}(\Omega)$:

$$\mathcal{D}(A) = H_B^{2m}(\Omega), \quad \mathcal{D}(A^*) = H_C^{2m}(\Omega), \tag{3A.9}$$

when these spaces are identified with a subspace of $H^{2m}(\Omega)$. In fact, from the a priori estimates for elliptic operators [Lions, Magenes, 1972, Vol. I, Section 7.3, pp. 187–90] the spaces in (3A.9) coincide since the graph norm of $\mathcal{D}(A)$ [respectively, $\mathcal{D}(A^*)$] is equivalent to the norm of $H^{2m}(\Omega)$.

Theorem 3A.1 *In the preceding setting we have the following characterizations, for* $0 < \theta < 1$*:*

(i)

$$\mathcal{D}(A^\theta) = [\mathcal{D}(A), L_2(\Omega)]_{1-\theta} = [H_B^{2m}(\Omega), H^0(\Omega)]_{1-\theta} \tag{3A.10}$$

$$= \left\{ \begin{array}{ll} u \in H^{2m\theta}(\Omega) : B_j(x, D)u = 0 \ on \ \Gamma, & if \, m_j < 2m\theta - \dfrac{1}{2} \\ \dfrac{1}{\sqrt{\rho}} B_j(x, D)u \in L_2(\Omega), & if \, m_j = 2m\theta - \dfrac{1}{2} \end{array} \right\}, \tag{3A.11}$$

where $\rho(x) = d(x, \Gamma) \equiv$ distance from x to Γ; or, more generally [Lions, Magenes, 1972, Vol. I, p. 27] $\rho(x)$ is an infinitely differentiable function on $\bar{\Omega}$, positive on Ω, and vanishing on Γ of the order of $d(x, \Gamma)$, that is, such that

$$\lim_{x \to x_0} \frac{\rho(x)}{d(x, \Gamma)} = d \neq 0 \quad if \, x_0 \in \Gamma. \tag{3A.12}$$

(ii)

$$\mathcal{D}(A^{*\theta}) = [\mathcal{D}(A^*), L_2(\Omega)]_{1-\theta} = [H_C^{2m}(\Omega), H^0(\Omega)]_{1-\theta} \tag{3A.13}$$

$$= \left\{ \begin{array}{ll} u \in H^{2m\theta}(\Omega) : C_j(x, D)u = 0 \ on \ \Gamma, & if \, m_j < 2m\theta - \dfrac{1}{2} \\ \dfrac{1}{\sqrt{\rho}} C_j(x, D)u \in L_2(\Omega), & if \, m_j = 2m\theta - \dfrac{1}{2} \end{array} \right\}. \tag{3A.14}$$

(iii) In particular, for all θ, $0 < \theta < 1/4m$,

$$\mathcal{D}(A^\theta) = \left[H_B^{2m}(\Omega), H^0(\Omega)\right]_{1-\theta} = H^{2m\theta}(\Omega) \tag{3A.15}$$

$$= \left[H_C^{2m}(\Omega), H^0(\Omega)\right]_{1-\theta} = \mathcal{D}(A^{*\theta}). \tag{3A.16}$$

(iv) In the Dirichlet problem, that is, when $B_j = \partial^j/\partial\nu^j$, with ν normal to Γ, oriented toward the interior of Ω [Lions, Magenes, 1972, Vol. I, p. 114] (a special system of boundary operators that satisfy the above assumptions for every properly *elliptic operator A), we have $C_j = \partial^j/\partial\nu^j$, as well [Lions, Magenes, 1972, Vol. I, p. 121], and*

$$\mathcal{D}(A) = \mathcal{D}(A^*) = H_B^{2m}(\Omega) = H_C^{2m}(\Omega) = H^{2m}(\Omega) \cap H_0^m(\Omega) \tag{3A.17}$$

[Lions, Magenes, 1972, Vol. I, p. 199].

In the described generality the above Theorem (i)–(iii) is a combination of results of Grisvard [1967] and Yagi [1984]; see [Yagi, 1984, pp. 175–6].

Grisvard's highly technical results are described, or reported, also in [Lions, Magenes, 1972, Vol. I, p. 107, and Vol. II, Theorem 14.4, p. 75, and Remark 14.5, p. 77]. An enlightening account based on Yagi [1984] is given in [Bensoussan et al., 1992, Vol. I, pp. 111–15], which has served as a basis of our exposition. In the case of second-order elliptic operators with Dirichlet or Neumann (Robin) boundary conditions, independent proofs of Theorem 3A.1(i) are given in Fujiwara [1967] and in [Lasiecka, 1980(b), Appendix 3B].

Closed Maximal Accretive Operators

Theorem 3A.2 *Let A be a closed maximal accretive operator on the Hilbert space H. Then:*

(i) A^θ *is a closed, maximal accretive operator (for* $0 \le \theta \le 1$*, in fact), and for* $0 \le \theta < 1/2$*:*

$$\mathcal{D}(A^\theta) = \mathcal{D}(A^{*\theta}), \quad 0 \le \theta < \frac{1}{2} \tag{3A.18}$$

[Kato, 1961, Theorem 1.1, p. 249].

(ii) For $1/2 \le \theta \le 1$*, identity (3A.18) is generally false. See [Kato, 1961, Example, pp. 252–3] for* $\frac{1}{2} < \theta \le 1$ *and [Lions, 1962, p. 240] for* $\theta = 1/2$*. [Sufficient conditions for (3A.18) to be true also for* $\theta = 1/2$ *are given in Lions [1962] in important cases where A is a partial differential operator of elliptic type. Some of these results are re-proved in Kato [1962]].*

(iii)

$$\mathcal{D}(A^\theta) = [\mathcal{D}(A), H]_{1-\theta}, \quad 0 < \theta < 1 \tag{3A.19}$$

[Lions, 1962].

(iv) Assume further that $A^{-1} \in \mathcal{L}(H)$*. Then (see also Chapter 0, Section 0.2)*

$$\mathcal{D}(A^\theta) = [\mathcal{D}(A), H]_{1-\theta}; \quad \mathcal{D}(A^{*\theta}) = [\mathcal{D}(A^*), H]_{1-\theta}, \quad 0 \le \theta \le 1, \tag{3A.20}$$

and

$$[\mathcal{D}(A), H]_{1-\theta} = \mathcal{D}(A^\theta) = \mathcal{D}(A^{*\theta}) = [\mathcal{D}(A^*), H]_{1-\theta}, \quad 0 \le \theta < \frac{1}{2}. \tag{3A.21}$$

[Part (iii) follows from part (i) via results of Yagi [1984]; see, for example, [Bensoussan et al., 1992, Vol. I, Proposition 6.1, p. 113].

Thus, Theorem 3A.1 is a generalization of Theorem 3A.2 for operators that are not necessarily maximal accretive.

3B Damped Elastic Operators

Throughout this appendix, X is a Hilbert space. On it, we consider two operators $\mathcal{A}$ and $\mathcal{B}$ subject to the following assumptions:

(H.1) $\mathcal{A}$ (the elastic operator): $X \supset \mathcal{D}(\mathcal{A}) \to X$, with $\mathcal{D}(\mathcal{A})$ dense in X, is a strictly postive, self-adjoint operator.

(H.2) $\mathcal{B}$ (the dissipation operator): $X \supset \mathcal{D}(\mathcal{B}) \to X$, with $\mathcal{D}(\mathcal{B})$ dense in X, is a positive, self-adjoint operator.

(H.3) There exist a constant $0 < \alpha \leq 1$ and two constants $0 < \rho_1 < \rho_2 < \infty$, such that

$$\rho_1 \mathcal{A}^\alpha \leq \mathcal{B} \leq \rho_2 \mathcal{A}^\alpha, \text{ on } \mathcal{D}(\mathcal{A}^{\frac{\alpha}{2}}) \tag{3B.1a}$$

that is, explicitly

$$\rho_1 (\mathcal{A}^\alpha x, x)_X \leq (\mathcal{B}x, x)_X \leq \rho_2 (\mathcal{A}^\alpha x, x)_X, \quad x \in \mathcal{D}\left(\mathcal{A}^{\frac{\alpha}{2}}\right) \subset \mathcal{D}(\mathcal{B}^{\frac{1}{2}}), \tag{3B.1b}$$

or equivalently, since $\mathcal{A}$ and $\mathcal{B}$ are self-adjoint,

$$\sqrt{\rho_1}\left\|\mathcal{A}^{\frac{\alpha}{2}}x\right\|_X \leq \left\|\mathcal{B}^{\frac{1}{2}}x\right\|_X \leq \sqrt{\rho_2}\left\|\mathcal{A}^{\frac{\alpha}{2}}x\right\|_X, \quad x \in \mathcal{D}(\mathcal{A}^{\frac{\alpha}{2}}), \tag{3B.1c}$$

or, with $y = \mathcal{A}^{\frac{\alpha}{2}}x \in X$ for $x \in \mathcal{D}(\mathcal{A}^{\alpha/2})$:

$$\rho_1 \|y\|_X^2 \leq \left(\mathcal{A}^{-\frac{\alpha}{2}}\mathcal{B}\mathcal{A}^{-\frac{\alpha}{2}}y, y\right)_X \leq \rho_2\|y\|_X^2, \quad y \in X, \tag{3B.1d}$$

and the self-adjoint operator $\mathcal{A}^{-\alpha/2}\mathcal{B}\mathcal{A}^{-\alpha/2}$ is bounded and boundedly invertible on X.

Remark 3B.0 A sufficient (but not necessary) condition for (3B.1b) to hold true is

$$\mathcal{D}(\mathcal{B}^{\frac{1}{2}}) = \mathcal{D}\left(\mathcal{A}^{\frac{\alpha}{2}}\right), \tag{3B.1e}$$

since then $\mathcal{B}^{\frac{1}{2}}\mathcal{A}^{-\alpha/2}$ and $\mathcal{A}^{\alpha/2}\mathcal{B}^{-\frac{1}{2}}$ are bounded operators by the closed graph theorem. Then, in the norm of X, we have $\|\mathcal{B}^{\frac{1}{2}}\mathcal{A}^{-\alpha/2}y\| \leq c\|y\|$ for all $y \in X$, as well as $\|\mathcal{A}^{\alpha/2}\mathcal{B}^{-\frac{1}{2}}z\| \leq c\|z\|$ for all $z \in X$. Setting $x = \mathcal{A}^{-\alpha/2}y \in \mathcal{D}(\mathcal{A}^{\alpha/2})$ and $x = \mathcal{B}^{-\frac{1}{2}}z \in \mathcal{D}(\mathcal{B}^{\frac{1}{2}}) = \mathcal{D}(\mathcal{A}^{\alpha/2})$ yields then (3B.1c). In this case, by self-adjointness, $\mathcal{A}^{-\alpha/2}\mathcal{B}^{\frac{1}{2}}$ and $\mathcal{B}^{-\frac{1}{2}}\mathcal{A}^{\alpha/2}$ have bounded extensions on X.

In contrast, an example with $\alpha = 1/2$, where (3B.1c) holds true, yet one only has $\mathcal{D}(\mathcal{A}^{\alpha/2}) \subsetneq \mathcal{D}(\mathcal{B}^{\frac{1}{2}})$ so that (3B.1e) is false, is given as follows: $\mathcal{A}$ is the elastic operator Δ^2 with clamped BC $f|_\Gamma = \frac{\partial f}{\partial \nu}|_\Gamma = 0$, while $\mathcal{B}$ is $(-\Delta)$ with Neumann BC; see details in Appendix 3E, Example 3E.4.2. □

The object of our interest is the second-order abstract equation

$$\ddot{x} + \mathcal{B}\dot{x} + \mathcal{A}x = 0 \quad \text{on } X. \tag{3B.2}$$

The concept of "parabolicity" can be introduced at the level of the second-order equation (3B.2) [Yakubov, 1983; Favini, Obrecht, 1991]. Alternatively, one rewrites (3B.2) in first-order form and then "parabolicity" results from the abstract property that the resulting matrix operator, A_B below in (3B.4) (once closed by taking the domain as in (3B.5)), generates a s.c. *analytic* semigroup on a suitable cross product space.

On the Energy Space $E = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times X$ We rewrite (3B.2) as a first-order equation

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = A_{\mathcal{B}}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} \quad \text{on the space } E = \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times X, \tag{3B.3}$$

$$A_{\mathcal{B}} = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\mathcal{B} \end{bmatrix} \quad \text{with domain } \mathcal{D}(A_{\mathcal{B}}) \text{ containing } \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{B}). \tag{3B.4}$$

$A_{\mathcal{B}}$ is densely defined and dissipative, and hence closable [Pazy, 1983, p. 15]. However, $A_{\mathcal{B}}$ need not be closed, for example, when $\mathcal{B} = \mathcal{A}$ [Chen, Russell, 1982, p. 436]. Equipped with the domain

$$\begin{aligned} \mathcal{D}(A_{\mathcal{B}}) = \Big\{&[x_1, x_2] \in E : x_1 \in \mathcal{D}\big(\mathcal{A}^{1-\frac{\alpha}{2}}\big);\ x_2 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset \mathcal{D}(\mathcal{A}^{\frac{\alpha}{2}}); \\ &\mathcal{A}^{\frac{\alpha}{2}}\left[\mathcal{A}^{1-\frac{\alpha}{2}}x_1 + \mathcal{A}^{-\frac{\alpha}{2}}\mathcal{B}x_2\right] \\ &= \mathcal{A}^{\frac{\alpha}{2}}\left[\mathcal{A}^{1-\frac{\alpha}{2}}x_1 + \big(\mathcal{A}^{-\frac{\alpha}{2}}\mathcal{B}\mathcal{A}^{-\frac{\alpha}{2}}\big)\big(\mathcal{A}^{\frac{\alpha}{2}}x_2\big)\right] \in X\Big\}, \end{aligned} \tag{3B.5a}$$

so that $A_{\mathcal{B}}x$, $x = [x_1, x_2] \in \mathcal{D}(A_{\mathcal{B}})$ means

$$A_{\mathcal{B}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\mathcal{A}^{\frac{\alpha}{2}}\left[\mathcal{A}^{1-\frac{\alpha}{2}}x_1 + \mathcal{A}^{-\frac{\alpha}{2}}\mathcal{B}x_2\right] \end{bmatrix}. \tag{3B.5b}$$

$A_{\mathcal{B}}$ is densely defined and closed, using (3B.1b).

Claim Under (H.1), (H.2), and (H.3) above, we have that $A_{\mathcal{B}}$ in (3B.4) and (3B.5) is closed.

Proof. Let (i) $\xi_n = [x_n, y_n] \in \mathcal{D}(A_{\mathcal{B}})$ with (ii) $\xi_n \to \xi = [x, y]$ in E, and (iii) $A_{\mathcal{B}}\xi_n \to v$ in E. We must show that $\xi \in \mathcal{D}(A_{\mathcal{B}})$ and $A_{\mathcal{B}}\xi = v$. In fact, (i) means by (3B.5) that $y_n \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset \mathcal{D}(\mathcal{A}^{\alpha/2}) \subset \mathcal{D}(\mathcal{B}^{\frac{1}{2}})$ by (3B.1b) – and thus $\mathcal{A}^{-\alpha/2}\mathcal{B}y_n = \mathcal{A}^{-\alpha/2}\mathcal{B}^{\frac{1}{2}}\mathcal{B}^{\frac{1}{2}}y_n \in X$ – as well as $x_n \in \mathcal{D}(\mathcal{A}^{1-\alpha/2})$ and $\mathcal{A}^{1-\alpha/2}x_n + \mathcal{A}^{-\alpha/2}\mathcal{B}y_n \in \mathcal{D}(\mathcal{A}^{\alpha/2})$. Also, (ii) means by (3B.3) that $x_n \to x$ in $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ and $y_n \to y$ in X. Finally, (iii) means by (3B.4) that $y_n \to v_1$ in $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$; thus, $y = v_1 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$, as desired, as well as

$$-\mathcal{A}^{\frac{\alpha}{2}}\left[\mathcal{A}^{1-\frac{\alpha}{2}}x_n + \mathcal{A}^{-\frac{\alpha}{2}}\mathcal{B}y_n\right] \to v_2 \text{ in } X. \tag{3B.6}$$

By (3B.1b), $\mathcal{B}^{\frac{1}{2}}\mathcal{A}^{-\frac{1}{2}} = \mathcal{B}^{\frac{1}{2}}\mathcal{A}^{-\alpha/2}\mathcal{A}^{(\alpha-1)/2} \in \mathcal{L}(X)$ and, as noted below (3B.1e), $\mathcal{A}^{-\alpha/2}\mathcal{B}^{\frac{1}{2}}$ has a bounded extension on X. Thus

$$\mathcal{A}^{-\frac{\alpha}{2}}\mathcal{B}y_n = \mathcal{A}^{-\frac{\alpha}{2}}\mathcal{B}^{\frac{1}{2}}\mathcal{B}^{\frac{1}{2}}\mathcal{A}^{-\frac{1}{2}}\mathcal{A}^{\frac{1}{2}}y_n \to \mathcal{A}^{-\frac{\alpha}{2}}\mathcal{B}v_1 \text{ in } X. \tag{3B.7}$$

[In (3B.7) we could also use $\mathcal{A}^{-\alpha/2}\mathcal{B}y_n = (\mathcal{A}^{-\alpha/2}\mathcal{B}\mathcal{A}^{-\alpha/2})\mathcal{A}^{\alpha/2}y_n$ with $(\mathcal{A}^{-\alpha/2}\mathcal{B}\mathcal{A}^{-\alpha/2}) \in \mathcal{L}(X)$ by (3B.1d); and $\mathcal{A}^{\alpha/2}y_n = \mathcal{A}^{(\alpha-1)/2}\mathcal{A}^{\frac{1}{2}}y_n \to \mathcal{A}^{\alpha/2}v_1$ in X by (iii) above with $0 \le \alpha \le 1$.]

By (3B.6) and (3B.7), a fortiori

$$-\mathcal{A}^{1-\frac{\alpha}{2}}x_n \to \mathcal{A}^{-\frac{\alpha}{2}}v_2 + \mathcal{A}^{-\frac{\alpha}{2}}\mathcal{B}v_1 \quad \text{in } X. \tag{3B.8}$$

Since $\mathcal{A}^{1-\alpha/2}$ is closed, it follows also via (3B.8) that $x \in \mathcal{D}(\mathcal{A}^{1-\alpha/2})$ and $\mathcal{A}^{1-\alpha/2}x = -\mathcal{A}^{-\alpha/2}v_2 - \mathcal{A}^{-\alpha/2}\mathcal{B}v_1$, or since $y = v_1$,

$$-\mathcal{A}^{\frac{\alpha}{2}}\left[\mathcal{A}^{1-\frac{\alpha}{2}}x + \mathcal{A}^{-\frac{\alpha}{2}}\mathcal{B}y\right] = v_2,$$

so that $\xi \in \mathcal{D}(A_{\mathcal{B}})$ and $A_{\mathcal{B}}\xi = v$, as desired. $\square$

Assumption (H.3) expresses, by (3B.1), the technical notion that $\mathcal{B}$ *is comparable to* $\mathcal{A}^\alpha$, a canonical case of which is obtained when

$$\mathcal{B} = \rho\mathcal{A}^\alpha, \quad \text{and hence } A_{\rho\alpha} = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\rho\mathcal{A}^\alpha \end{bmatrix}; \tag{3B.9}$$

$$\mathcal{D}(A_{\rho\alpha}) = \left\{[x_1, x_2] \in E : x_1 \in \mathcal{D}(\mathcal{A}^{1-\alpha});\ x_2 \in \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big);\right.$$
$$\left.\mathcal{A}^\alpha\left[\mathcal{A}^{1-\alpha}x_1 + \rho x_2\right] \in X\right\}. \tag{3B.10}$$

The operator $A_{\rho\alpha}$, whose domain includes $\mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^\alpha)$, is densely defined and closed.

Remark 3B.1 The following condition:

$$\rho_1^2(\mathcal{A}^{2\alpha}x, x)_Y \le (\mathcal{B}^2x, x)_X \le \rho_2^2(\mathcal{A}^{2\alpha}x, x)_X, \quad x \in \mathcal{D}(\mathcal{A}^\alpha), \tag{3B.11a}$$

for which, as in Remark 3B.0, a sufficient condition is

$$\mathcal{D}(\mathcal{B}) = \mathcal{D}(\mathcal{A}^\alpha), \tag{3B.11b}$$

which may be easier to verify than condition (3B.1e) (e.g., if $\alpha = 1/2$), does imply (3B.1b) (Lowner's Theorem, [Krein, 1988, Corollary 7.1, p. 146]). The converse is generally false, that is, (3B.1b) generally does not imply (3B.11a) unless $\mathcal{A}$ and $\mathcal{B}$ commute. An investigation of the operator $A_{\mathcal{B}}$ in (3B.4) in terms of the parameter $\alpha, 0 \le \alpha \le 1$, was carried out in Chen and Triggiani [1987; 1989a, b; 1990a, b]. For the purposes of the present chapter, the following results, which are contained in these references, may suffice.

Theorem 3B.1 [Chen, Triggiani, 1987; 1989(a); 1990(a)]

(a) *(generation) Assume the standing hypotheses (H.1), (H.2), and (H.3). Then, the operator* $A_{\mathcal{B}}$ *in (3B.4) is maximal dissipative, and thus (Lumer–Phillips) it generates a s.c. semigroup of contractions* $e^{A_{\mathcal{B}}t}$ *on* $E = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times X$.

(b) *(*$1/2 \le \alpha \le 1$ *: analyticity) If the parameter* α *in (H.3) satisfies* $1/2 \le \alpha \le 1$, *then the s.c. semigroup* $e^{A_{\mathcal{B}}t}$ *of part (a), is, moreover, analytic on* E.

(c) *($0 < \alpha < 1/2$: Gevrey class) If the parameter α in (H.3) satisfies $0 < \alpha < 1/2$, then the resolvent $R(\lambda, A_{\mathcal{B}})$ satisfies*

$$\lim_{|\tau|\to\infty} |\tau|^{2\alpha} \|R(i\tau, A_{\mathcal{B}})\|_{\mathcal{L}(E)} = c < \infty, \tag{3B.12}$$

and hence [Triggiani, 1980(b), Theorem 4; Pazy, 1983, Theorem 4.9, p. 57] the s.c. semigroup $e^{A_{\mathcal{B}}t}$ of part (a) is of Gevrey class $\delta > 1/(2\alpha)$, and hence is a fortiori differentiable for all $t > 0$ on E.

(d) *For all $0 < \alpha \leq 1$, the s.c. semigroup $e^{A_{\mathcal{B}}t}$ is uniformly stable on E: There exist constants $M \geq 1$ and $a > 0$ [indeed $-a = \sup \operatorname{Re} \sigma(A_{\mathcal{B}})$] such that*

$$\|e^{A_{\mathcal{B}}t}\|_{\mathcal{L}(E)} \leq Me^{-at}, \quad t \geq 0. \tag{3B.13}$$

(e) *Let $\mathcal{A}$ have compact resolvent in $\mathcal{L}(X)$. Then $A_{\rho\alpha}$ has compact resolvent in $\mathcal{L}(E)$ if $\alpha < 1$; but not for $\alpha = 1$, as the point $\lambda = -1/(2\rho)$ is in this case in the continuous spectrum of $A_{\rho\alpha}$.*

Remark B.2 In all PDE illustrations of the present Chapter 3 – see Sections 3.4 through 3.13, plus Section 15 – it is Theorem B.1 in the range $\frac{1}{2} \leq \alpha \leq 1$ which is invoked to claim analyticity of the semigroup. However, there is one illustration, in Section 3.14.3, where only domain containment

$$\mathcal{D}(\mathcal{B}) = H^2(\Omega) \cap H_0^1(\Omega) \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}}), \quad \text{hence } \mathcal{D}(\mathcal{B}^{\frac{1}{2}}) = H_0^1(\Omega) \subset \mathcal{D}(\mathcal{A}^{\frac{1}{4}}) = H^1(\Omega)$$

holds true; see Eqn. (3.14.3.4). To handle this case and still claim analyticity of the semigroup, we shall invoke the following variation of Theorem B.1, which we shall state in the general non–self-adjoint case of $\mathcal{B}$, after Chen and Triggiani [1989(b)].

Theorem B.1′ [Chen, Triggiani, 1989(b)] Let $\mathcal{A}$ be a densely defined, strictly positive, self-adjoint operator as in (H.1). Let $\mathcal{B}$ be a densely defined closed operator with $\mathcal{D}(\mathcal{B}) = \mathcal{D}(\mathcal{B}^*)$. Further, assume that

(i) for some $\frac{1}{2} \leq \alpha \leq 1$ and some $\rho > 0$, we have

$$\rho(\mathcal{A}^\alpha x, x)_X \leq \operatorname{Re}(\mathcal{B}x, x)_X = \operatorname{Re}(\mathcal{B}^* x, x)_X, \quad x \in \mathcal{D}(\mathcal{B}) = \mathcal{D}(\mathcal{B}^*); \tag{3B.14a}$$

(ii) there is a constant K such that

$$|\operatorname{Im}(\mathcal{B}x, x)_X| \leq K \operatorname{Re}(\mathcal{B}x, x)_X, \quad x \in \mathcal{D}(\mathcal{B}). \tag{3B.14b}$$

Then, (the closure of) the operator $\mathcal{A}_{\mathcal{B}}$ in (3B.4) generates a s.c. semigroup of contractions $e^{A_{\mathcal{B}}t}$ which is analytic on the space E in (3B.3), and exponentially stable here as in (3B.13).

Remark B.3 Hypothesis (B.14b) is necessary. Let $\mathcal{A}$ be a self-adjoint, strictly positive operator with compact resolvent, and let $\mathcal{B} = \mathcal{A}^{\alpha} + i\mathcal{A}^{\beta}$, with $\frac{1}{2} \leq \alpha \leq \beta$. Then one readily shows [Chen, Triggiani, 1989(b), Remark 2.3] that the point spectrum of the operator $A_{\mathcal{B}}$ in (B.4) cannot be contained in any triangular sector typical of analytic semigroup generators; in the sense that if $\{\lambda_n\}$ are the eigenvalues of $A_{\mathcal{B}}$, then $|(\mathrm{Re}\,\lambda_n)/\mathrm{Im}\lambda_n)| \to 0$ as $n \to \infty$.

Domains of Fractional Powers $\mathcal{D}((-A_{\rho\alpha})^{\theta})$ The next result characterizes the domains of fractional power $\mathcal{D}((-A_{\rho\alpha})^{\theta})$ of the operator in (3B.9) for $1/2 \leq \alpha \leq 1$, in which case (3B.10) can be rewritten as

$$\mathcal{D}(A_{\rho\alpha}) = \Big\{[x_1, x_2] \in E : x_1 \in \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}+1-\alpha}\big), x_2 \in \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big),$$
$$\mathcal{A}^{1-\alpha}x_1 + \rho x_2 \in \mathcal{D}(\mathcal{A}^{\alpha})\Big\}, \qquad (3B.15)$$

since now $x_1 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset \mathcal{D}(\mathcal{A}^{1-\alpha})$, $x_2 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$, and $\mathcal{A}^{1-\alpha}x_1 + \rho x_2 \subset \mathcal{D}(\mathcal{A}^{\alpha}) \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ imply $\mathcal{A}^{1-\alpha}x_1 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$, as desired.

Theorem 3B.2 [Chen, Triggiani, 1990(b)] *Consider the operator $A_{\rho\alpha} : E \supset \mathcal{D}(A_{\rho\alpha}) \to E$ in (3B.9), (3B.10), or (3B.15), with $\rho > 0$ and $1/2 \leq \alpha \leq 1$.*

(i) Let $0 \leq \theta \leq 1/2$. Then

$$\mathcal{D}((-A_{\rho\alpha})^{\theta}) = \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}+\theta(1-\alpha)}\big) \times \mathcal{D}(\mathcal{A}^{\alpha\theta}). \qquad (3B.16)$$

(ii) Let $1/2 \leq \theta \leq 1$. Then

$$\mathcal{D}((-A_{\rho\alpha})^{\theta}) = \left\{ \begin{vmatrix} x \\ y \end{vmatrix} \in E : x \in \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}+\theta(1-\alpha)}\big);\right.$$
$$\left. y \in \mathcal{D}(\mathcal{A}^{\alpha-\frac{1}{2}+\theta(1-\alpha)});\ \mathcal{A}^{1-\alpha}x + \rho y \in \mathcal{D}(\mathcal{A}^{\alpha\theta}) \right\}. \qquad (3B.17)$$

For $\theta = 1/2$, or $\alpha = 1/2$, the third requirement in (3B.17) is automatically satisfied.

Various representations may be given for $\mathcal{D}((-A_{\rho\alpha})$, equivalent to (3B.17) for $\theta = 1$; see below in (3B.35), (3B.36), and (3B.39).

Corollary 3B.3 [Chen, Triggiani, 1990(b), Corollary 4.1, p. 288] *Let $\mathcal{B} = \rho\mathcal{A}^{\alpha} + \mathcal{B}_1$, $\rho > 0$, $1/2 \leq \alpha \leq 1$, where $\mathcal{B}_1$ is a (nonnecessarily accretive) closed operator, and*

$$\mathcal{D}\,(\mathcal{A}^{\alpha_1}) \subset \mathcal{D}(\mathcal{B}_1) \text{ for some } \alpha_1 < \frac{1}{2} \quad \text{if } \alpha < 1,$$

$$\text{or else for } \alpha_1 = \frac{1}{2} \quad \text{if } \alpha = 1. \qquad (3B.18)$$

Then, the operator $A_{\mathcal{B}}$ in (3B.4) and (3B.5) generates a s.c. analytic semigroup on $E = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times X$ (of contractions if $\mathcal{B}_1$ is also accretive).

***Domains of Fractional Powers* $\mathcal{D}((-A_{\mathcal{B}})^{\theta})$** The next result extends the usefulness of Theorem 3B.2 to obtain information on the domains of fractional powers $\mathcal{D}((-A_{\mathcal{B}})^{\theta})$ of $(-A_{\mathcal{B}})$ in (3B.4) and (3B.5).

Corollary 3B.4 *Assume that the operator $A_{\mathcal{B}}$ in (3B.4) and (3B.5) generates a s.c., analytic semigroup in the situation of Theorem 3B.1(b) for $1/2 \le \alpha \le 1$, or else of Corollary 3B.3, so that $\mathcal{D}(A_{\rho\alpha}) = \mathcal{D}(A_{\mathcal{B}})$. Then, for $0 < \theta_2 < \theta_1 < \theta < 1$, we have*

$$\mathcal{D}((-A_{\rho\alpha})^{\theta}) \subset \mathcal{D}((-A_{\mathcal{B}})^{\theta_1}) \subset ((-A_{\rho\alpha})^{\theta_2}). \tag{3B.19}$$

Proof. Let $[\ ,\ ]_\theta$ be the usual (Hilbert or complex) interpolation space [Lions, Magenes, 1972]. We have

$$[\mathcal{D}(A_{\rho\alpha}), E]_\theta = \mathcal{D}((-A_{\rho\alpha})^{1-\theta}), \quad 0 < \theta < 1. \tag{3B.20}$$

This follows from [Triebel, 1978, p. 203, Theorem 1.15.3]. Indeed, $(-A_{\rho\alpha})$ is a positive operator in the terminology of [Triebel, 1978, p. 91], being the generator of a s.c. semigroup; moreover, the assumption required in [Triebel, 1978, p. 103] that $\|(-A_{\rho\alpha})^{it}\|_{\mathcal{L}(E)} \le C$, for $-\epsilon \le t \le \epsilon$, can be readily verified in our case, by using the spectral properties of $A_{\rho\alpha}$ on E: $A_{\rho\alpha}$ is the direct sum of two normal operators, except for some exceptional cases of ρ, α, where $A_{\rho\alpha}$ is a spectral operator. Since $\mathcal{D}(A_{\rho\alpha}) = \mathcal{D}(A_{\mathcal{B}})$, then (3B.20) implies, in the notation of Triebel [1978], that, for $0 < \theta < 1$, we have

$$\mathcal{D}((-A_{\rho\alpha})^{\theta}) = ((\mathcal{D}(A_{\rho\alpha}), E)_{1-\theta,p=2} = (\mathcal{D}(A_{\mathcal{B}}), E)_{1-\theta,p=2}, \tag{3B.21}$$

where by [Triebel, 1978, p. 25, Theorem 1.3.3, Eqn. (4)] we have, for $0 < \theta_1 < \theta < 1$,

$$(\mathcal{D}(A_{\mathcal{B}}), E)_{1-\theta,p=2} \subset (\mathcal{D}(A_{\mathcal{B}}), E)_{1-\theta,p=1}. \tag{3B.22}$$

However, the operator $(-A_{\mathcal{B}})$ is likewise positive on E in the terminology of [Triebel, 1978, p. 31], since $A_{\mathcal{B}}$ is a generator of a s.c. semigroup. Thus, [Triebel, 1978, p. 101, Theorem 1.15.2] applies and gives

$$(\mathcal{D}(A_{\mathcal{B}}), E)_{1-\theta_1,p=1} \subset \mathcal{D}((-A_{\mathcal{B}})^{\theta_1}) \subset (\mathcal{D}(A_{\mathcal{B}}), E)_{1-\theta_1,p=\infty}. \tag{3B.23}$$

Finally, again by [Triebel, 1978, p. 25, Theorem 1.3.3] with $0 < \theta_2 < \theta_1 < 1$ and (3B.21),

$$(\mathcal{D}(A_{\mathcal{B}}), E)_{1-\theta_1,p=\infty} \subset (\mathcal{D}(A_{\mathcal{B}}), E)_{1-\theta_2,p=2} \subset \mathcal{D}((-A_{\mathcal{B}})^{\theta_2}). \tag{3B.24}$$

Thus, for $0 < \theta_2 < \theta_1 < \theta < 1$, we finally have, by (3B.21)–(3B.24),

$$\mathcal{D}((-A_{\rho\alpha})^{\theta}) \subset \mathcal{D}((-A_{\mathcal{B}})^{\theta_1}) \subset \mathcal{D}((-A_{\rho\alpha})^{\theta_2}), \tag{3B.25}$$

and (3B.20) is proved. □

On the Space $X \times X$ We follow Balakrishnan and Triggiani [1993]. Let $\mathcal{A}$ be a positive, self-adjoint, unbounded operator on the Hilbert space H as in (H.1). Let $\rho \geq 0$ and $0 \leq \alpha \leq 1$ be given constants. We return to the operator in (3B.9),

$$A_{\rho\alpha} = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\rho\mathcal{A}^{\alpha} \end{bmatrix}, \quad \text{and its special case } A_0 = \begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix}, \tag{3B.26}$$

which arise in the usual way, when writing the second-order equation

$$\ddot{x} + \rho\mathcal{A}^{\alpha}\dot{x} + \mathcal{A}x = 0, \quad \text{on } X, \tag{3B.27}$$

as a first-order system in the new variable $[x(t), \dot{x}(t)]$. What is a correct choice of the underlying space on which $A_{\rho\alpha}$ acts?

(a) If we choose the energy space

$$E \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times X, \quad \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right)_E = \left(\mathcal{A}^{\frac{1}{2}}x_1, \mathcal{A}^{\frac{1}{2}}y_1\right)_X + (x_2, y_2)_X, \tag{3B.28}$$

then a rather complete description is given by the much more general Theorem 3B.1 above, at least for $\rho > 0$, $0 \leq \alpha \leq 1$:

(i) If $\rho = 0$, then the operator A_0 in (3B.26) is skew-adjoint on E and thus generates a strongly continuous *unitary* group on E.
(ii) If $0 < \rho$, then $A_{\rho\alpha}$ is maximal dissipative on E and thus (Lumer–Phillips) generates a s.c. *contraction* semigroup $e^{A_{\rho\alpha}t}$ on E; furthermore, $e^{A_{\rho\alpha}t}$ is of *Gevrey class* $\delta > (2\alpha)$ if $0 < \alpha < \frac{1}{2}$, whereas $e^{A_{\rho\alpha}t}$ is *analytic* if $1/2 \leq \alpha \leq 1$.

(b) We now report what happens, however, if – on the basis of what one would do if $\mathcal{A}$ were bounded – one insists on choosing the space $X \times X$ for $A_{\rho\alpha}$, equipped then with the following domain:

$$\mathcal{D}(A_{\rho\alpha}) = \left\{[x_1, x_2] : x_2 \in X, x_1 \in \mathcal{D}(\mathcal{A}^{1-\alpha}), \mathcal{A}^{1-\alpha}x_1 + \rho x_2 \in \mathcal{D}(\mathcal{A}^{\alpha})\right\}. \tag{3B.29}$$

Notice that the domain in (3B.29) contains $\{[x_1, x_2] : x_1 \in \mathcal{D}(\mathcal{A}), x_2 \in \mathcal{D}(\mathcal{A}^{\alpha})\}$, and, moreover, $A_{\rho\alpha}$ is readily seen to be densely defined and closed (see Claim below (3B.5)).

Cases: ($\rho = 0$, or else $\rho > 0$ and $0 \leq \alpha < 1$) If $\rho = 0$, or else $\rho > 0$ and $\alpha < 1$, a negative answer is contained in the following result.

Theorem 3B.5 [Balakrishnan, Triggiani, 1993] *Let $\rho \geq 0$ and $\alpha < 1$, and let $\mathcal{A}$ be an unbounded, positive, self-adjoint operator. Let $\lambda > 0$. Then the resolvent operator $R(\lambda, A_{\rho\alpha})$ of $A_{\rho\alpha}$ in (3B.26) satisfies the lower bound*

$$\|R(\lambda, A_{\rho\alpha})\|_{\mathcal{L}(X\times X)} \geq \frac{1}{2+\rho}, \quad \forall\, \lambda > 0. \tag{3B.30}$$

A fortiori Eqn. (3B.30) violates the necessary condition of generation of a s.c. semigroup by $A_{\rho\alpha}$ on $X \times X$ [Fattorini, 1983; Pazy, 1983]. Thus, $A_{\rho\alpha}$ does not generate a s.c. semigroup on $X \times X$ in these cases.

Remark 3B.4 It is noted in [Balakrishnan, Triggiani, 1993, Remark 3.2, p. 36] that if $\alpha \geq 1/2$, we then have

$$\|(R(\lambda, A_{\rho\alpha})\|_{\mathcal{L}(X\times X)} \leq \text{const}_{\rho\alpha r_0}, \quad \forall\, \lambda \text{ with } \operatorname{Re}\lambda > r_0 > 0, \tag{3B.31}$$

and so [Arendt, 1987, p. 341] $A_{\rho\alpha}$ is the generator of a so-called integrated semigroup (or distribution semigroup [Lions, 1960]) on $X \times X$.

Case: $\rho > 0$ and $\alpha = 1$ In this case, $A_{\rho\alpha}$ does generate a s.c., even analytic semigroup on $X \times X$. A characterization of the domain $\mathcal{D}((-A_{\rho\alpha})^\theta)$, $\alpha = 1$, of fractional powers is also provided.

Theorem 3B.6 *Let $\rho > 0$ and $\alpha = 1$. Consider the operator $A_{\rho\alpha} : X \times X \supset \mathcal{D}(-A_{\rho\alpha}) \to X \times X$, where*

$$\mathcal{D}(A_{\rho\alpha}) = \{x, y \in X : x + \rho y \in X\}. \tag{3B.32}$$

Then:

(a) *[Bucci, 1992], [Triggiani, 1980(b), p. 314] The operator $A_{\rho,\alpha=1}$ generates a s.c. semigroup on $X \times X$ (noncontractive), which, moreover, satisfies the estimate*

$$\|R(\lambda, A_{\rho,\alpha=1})\|_{\mathcal{L}(X\times X)} \leq \frac{C}{|\lambda|}, \quad \operatorname{Re}\lambda > 0. \tag{3B.33}$$

Hence, such s.c. semigroup is, moreover, analytic on $X \times X, t > 0$.

(b) *[Chen, Triggiani, 1990(b), p. 292] For $0 \leq \theta \leq 1$, the domains of fractional powers $\mathcal{D}((-A_{\rho\alpha})^\theta)$, $\alpha = 1$, are given by*

$$\mathcal{D}((-A_{\rho\alpha})^\theta) = \{x, y \in X : x + \rho y \in \mathcal{D}(\mathcal{A}^\theta)\}, \quad \rho > 0, \alpha = 1. \tag{3B.34}$$

Representations of $\mathcal{D}(A_{\rho\alpha})$, $0 < \alpha \leq 1$ We note that formula (3B.5), once specialized to $\mathcal{B} = \rho\mathcal{A}^\alpha$, yields

$$\begin{aligned}\mathcal{D}(A_{\rho\alpha}) = \Big\{[x, y] \in E : x \in \mathcal{D}\big(\mathcal{A}^{1-\frac{\alpha}{2}}\big), \quad y \in \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \subset \mathcal{D}\big(\mathcal{A}^{\frac{\alpha}{2}}\big) :\\ \big[\mathcal{A}^{1-\frac{\alpha}{2}}x + \rho\mathcal{A}^{\frac{\alpha}{2}}y\big] \in \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big)\Big\}, \quad 0 < \alpha \leq 1.\end{aligned} \tag{3B.35}$$

However, formula (3B.17), once specialized to $\theta = 1$, yields

$$\begin{aligned}\mathcal{D}(A_{\rho\alpha}) = \Big\{[x, y] \in E : x \in \mathcal{D}\big(\mathcal{A}^{\frac{3}{2}-\alpha}\big) \subset \mathcal{D}(\mathcal{A}^{1-\alpha}), y \in \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) :\\ [\mathcal{A}^{1-\alpha}x + \rho y] \in \mathcal{D}(\mathcal{A}^\alpha)\Big\}, \quad 0 < \alpha \leq 1.\end{aligned} \tag{3B.36}$$

We readily see the following *Claim: The right-hand sides of* (3B.35) *and of* (3B.36) *coincide; they are representations of the same entity, precisely $\mathcal{D}(A_{\rho\alpha})$, and so the notation is consistent.*

To see this, we preliminarily observe that

$$\mathcal{D}(\mathcal{A}^{\frac{3}{2}-\alpha}) \subset \mathcal{D}(\mathcal{A}^{1-\frac{\alpha}{2}}) \subset \mathcal{D}(\mathcal{A}^{1-\alpha}) \quad \text{and} \quad \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset \mathcal{D}(\mathcal{A}^{\frac{\alpha}{2}}), \quad 0 < \alpha \leq 1. \tag{3B.37}$$

(3B.36) $\Rightarrow$ (3B.35). Let $[x, y] \in E$ satisfy the right-hand side of (3B.36). Then, by (3B.37), it follows that

$$x \in \mathcal{D}(\mathcal{A}^{1-\frac{\alpha}{2}}), \text{ equivalently } \mathcal{A}^{1-\alpha}x \in \mathcal{D}(\mathcal{A}^{\frac{\alpha}{2}}); \text{ as well as } y \in \mathcal{D}(\mathcal{A}^{\frac{\alpha}{2}}). \tag{3B.38}$$

Then, by (3B.38), the condition for the square brackets in (3B.36) implies, by transferring $\mathcal{A}^{\alpha/2}$ from the right to the left side, the condition for the brackets in (3B.35), and thus $[x, y]$ satisfies the right-hand side of (3B.35).

(3B.35) $\Rightarrow$ (3B.36). Conversely, let $[x, y] \in E$ satisfy the right-hand side of (3B.35). Then, by (3B.37), it follows that $x \in \mathcal{D}(\mathcal{A}^{1-\alpha})$, and so the condition for the square brackets in (3B.35) implies, by pulling out $\mathcal{A}^{\alpha/2}$ from the left side, the condition for the brackets in (3B.36), and thus $[x, y]$ satisfies the right-hand side of (3B.36). The *Claim* is proved.

Another Representation of $\mathcal{D}(A_{\rho\alpha})$, $1/2 \leq \alpha \leq 1$ In the range of analyticity $1/2 \leq \alpha \leq 1$, we may give yet another representation of $\mathcal{D}(A_{\rho\alpha})$ as

$$\mathcal{D}(A_{\rho\alpha}) = \Big\{x \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset \mathcal{D}(\mathcal{A}^{1-\alpha}),\ y \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset \mathcal{D}(\mathcal{A}^{\alpha-\frac{1}{2}}) : \\ \big[\mathcal{A}^{\frac{1}{2}}x + \rho\mathcal{A}^{\alpha-\frac{1}{2}}y\big] \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})\Big\}, \quad \frac{1}{2} \leq \alpha \leq 1, \tag{3B.39}$$

with the cross condition for the bracketed term in terms of $\mathcal{A}^{\frac{1}{2}}$.

Claim: for $1/2 \leq \alpha \leq 1$, the right-hand side of (3B.39) *coincides with the right-hand side of* (3B.36), *and so the notation for $\mathcal{D}(A_{\rho\alpha})$ is consistent.*

(3B.39) $\Rightarrow$ (3B.36). Rewrite the condition for the bracketed term in (3B.39) as $\mathcal{A}^{\alpha}\mathcal{A}^{\frac{1}{2}-\alpha}[\mathcal{A}^{\frac{1}{2}}x + \rho\mathcal{A}^{\alpha-\frac{1}{2}}y] \in X$, and since $1/2 - \alpha \leq 0$, we can move the bounded operator $\mathcal{A}^{\frac{1}{2}-\alpha}$ inside the brackets, and thus obtain the condition for the bracketed term in (3B.36). The latter implies a fortiori $[\mathcal{A}^{1-\alpha}x + \rho y] \in \mathcal{D}(\mathcal{A}^{\alpha}) \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$, again since $\alpha \geq 1/2$, a condition that along with $y \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ from (3B.39) implies $\mathcal{A}^{1-\alpha}x \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$. Thus, $[x, y]$ satisfies the right-hand side of (3B.36).

(3B.36) $\Rightarrow$ (3B.39). First, since $\mathcal{D}(\mathcal{A}^{\frac{3}{2}-\alpha}) \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ for $\alpha \leq 1$, we have that $x \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$. Next, rewrite the condition for the bracketed term in (3B.36) as $\mathcal{A}^{\frac{1}{2}}\mathcal{A}^{\alpha-\frac{1}{2}}[\mathcal{A}^{1-\alpha}x + \rho y] \in X$, and since $1/2 - \alpha \leq 0$, we can move the bounded operator $\mathcal{A}^{\alpha-\frac{1}{2}}$ inside the brackets and thus obtain the condition for the bracketed term in (3B.39). Thus, $[x, y]$ satisfies the right-hand side of (3B.39). The *Claim* is proved. $\square$

Plainly, when $\mathcal{B} = \rho\mathcal{A}^{\alpha}$, many equivalent representations of $\mathcal{D}(A_{\rho\alpha})$ may be written.

Case $1/2 \le \alpha \le 1$***: Explicit Conditions of Analyticity of*** $e^{A_{\mathcal{B}}t}$ ***on E, in Terms of the Resolvent Operator*** $R(\lambda, A_{\mathcal{B}})$ [Chen, Triggiani, 1987; 1989(a)]. We enlighten the statement of Theorem 3B.1(b), case $1/2 \le \alpha \le 1$, by pointing out that the various proofs of analyticity given in Chen and Triggiani [1987; 1989(a)] establish the following *characterization* of analyticity of the s.c. contraction semigroup $e^{A_{\mathcal{B}}t}$, in terms of the resolvent condition [Fattorini, 1983, pp. 180–5]:

$$\|R(\lambda, A_{\mathcal{B}})\|_{\mathcal{L}(E)} \le \frac{C}{|\lambda|}, \quad \forall\, \lambda \text{ with } \operatorname{Re} \lambda > 0. \tag{3B.40}$$

We have, explicitly, via a direct computation,

$$R(\lambda, A_{\mathcal{B}}) = \begin{bmatrix} R_1(\lambda) & R_2(\lambda) \\ R_3(\lambda) & R_4(\lambda) \end{bmatrix}, \tag{3B.41}$$

where, after setting

$$V_{\mathcal{B}}(\lambda) = \lambda^2 I + \lambda \mathcal{B} + \mathcal{A}; \quad [V_{\mathcal{B}}(\lambda)]^* = V_{\mathcal{B}}(\bar{\lambda}); \quad [V_{\mathcal{B}}^{-1}(\lambda)]^* = V_{\mathcal{B}}^{-1}(\bar{\lambda}), \tag{3B.42}$$

we find

$$R_1(\lambda) = \tfrac{I - V_{\mathcal{B}}^{-1}(\lambda)\mathcal{A}}{\lambda} = V_{\mathcal{B}}^{-1}(\lambda)(\lambda I + \mathcal{B}); \quad R_2(\lambda) = V_{\mathcal{B}}^{-1}(\lambda); \tag{3B.43}$$

$$R_3(\lambda) = -V_{\mathcal{B}}^{-1}(\lambda)\mathcal{A}; \quad R_4(\lambda) = \lambda V_{\mathcal{B}}^{-1}(\lambda). \tag{3B.44}$$

Thus, under the assumptions of Theorem 3B.1(b), the characterization (3B.40) is established by showing the following uniform estimates, for all λ with $\operatorname{Re} \lambda > 0$, which are cumulatively equivalent to (3B.40) [Chen, Triggiani, 1989(a), Eqns. (4.2), (4.3), (4.4), p. 30]:

$$\begin{cases} \|\mathcal{A}^{\frac{1}{2}} V_{\mathcal{B}}^{-1} \mathcal{A}^{\frac{1}{2}}\|_{\mathcal{L}(X)} \le M; & \quad (3B.45) \\ \left\|\lambda \mathcal{A}^{\frac{1}{2}} V_{\mathcal{B}}^{-1}(\lambda)\right\|_{\mathcal{L}(X)} \le M; \ \begin{cases} \text{equivalently by (3B.42)} \\ \left\|\lambda V_{\mathcal{B}}^{-1}(\lambda)\mathcal{A}^{\frac{1}{2}}\right\|_{\mathcal{L}(X)} \le M, \end{cases} & \quad (3B.46) \\ \left\|\lambda^2 V_{\mathcal{B}}^{-1}(\lambda)\right\|_{\mathcal{L}(X)} \le M; & \quad (3B.47) \end{cases}$$

Indeed, [Chen, Triggiani, 1989(a)] establishes (3B.40) by proving (3B.45)–(3B.47) individually.

Moreover, still under the assumptions of Theorem 3B.1(b) yielding analyticity of $e^{A_{\mathcal{B}}t}$ via (3B.40), the following estimates hold true, uniformly for all λ with $\operatorname{Re} \lambda > 0$ [Lasiecka, Triggiani, 1998(a), Appendix to Section 4]:

(a_1):

$$\|[\mathcal{A}R_1(\lambda) + \mathcal{B}R_3(\lambda)]\|_{\mathcal{L}\left(\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right); X\right)} \le C, \tag{3B.48a}$$

explicitly, by (3B.43) and (3B.44),

$$\left\|\left[\mathcal{A}V_{\mathcal{B}}^{-1}(\lambda)(\lambda I + \mathcal{B})\mathcal{A}^{-\frac{1}{2}} - \mathcal{B}V_{\mathcal{B}}^{-1}(\lambda)\mathcal{A}^{\frac{1}{2}}\right]\right\|_{\mathcal{L}(X)} \le C. \tag{3B.48b}$$

(a$_2$):

$$\|[\mathcal{A}R_2(\lambda) + \mathcal{B}R_4(\lambda)]\|_{\mathcal{L}(X)} \leq C, \tag{3B.49a}$$

explicitly, by (3B.43) and (3B.44),

$$\left\|\left[\mathcal{A}V_{\mathcal{B}}^{-1}(\lambda) + \lambda\mathcal{B}V_{\mathcal{B}}^{-1}(\lambda)\right]\right\|_{\mathcal{L}(X)} \leq C. \tag{3B.49b}$$

Further Consequences under the Additional Assumption We have

$$\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right) \subset \mathcal{D}(\mathcal{B}), \quad \text{equivalently } \mathcal{B}\mathcal{A}^{-\frac{1}{2}} \in \mathcal{L}(X), \tag{3B.50}$$

which is often fulfilled [see Appendices 3E, 3F, and 3G, related to thermo-elastic systems]. Then, the following estimates, uniform in λ for Re $\lambda > 0$, are fulfilled:

$$\|\mathcal{B}R_3(\lambda)\|_{\mathcal{L}\left(\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right);X\right)} = \left\|\mathcal{B}V_{\mathcal{B}}^{-1}(\lambda)\mathcal{A}^{\frac{1}{2}}\right\|_{\mathcal{L}(X)} \leq C, \tag{3B.51}$$

$$\|\mathcal{B}R_4(\lambda)\|_{\mathcal{L}(X)} = \left\|\lambda\mathcal{B}V_{\mathcal{B}}^{-1}(\lambda)\right\|_{\mathcal{L}(X)} \leq C, \tag{3B.52}$$

$$\|\mathcal{A}R_2(\lambda)\|_{\mathcal{L}(X)} = \left\|\mathcal{A}V_{\mathcal{B}}^{-1}(\lambda)\right\|_{\mathcal{L}(X)} \leq C, \tag{3B.53}$$

$$\|\mathcal{A}R_1(\lambda)\|_{\mathcal{L}\left(\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right);X\right)} \leq C. \tag{3B.54}$$

Conditions (3B.53) and (3B.54) will be invoked in the proof of Proposition 3G.2.4 in Appendix 3G, of Chapter 3.

3C Boundary Operators for Bending Moments and Shear Forces on Two-Dimensional Domains

In this appendix we collect a number of important results regarding two-dimensional plate equations (Euler–Bernoulli equations, possibly with high damping as in Sections 3.4 or 3.5; or Kirchoff equations) with physical boundary conditions involving *bending moments* and *shear forces.* When homogeneous, these boundary conditions (BC) are referred to as *free* BC. They involve the boundary operators B_1 and B_2 defined by the expressions in Eqns. (3.5.2a, b) such as they arise in the variational derivation of these two-dimensional models; see Lagnese [1989]. These rather complicated expresssions are not necessarily insightful in several cases, for example, in the analysis of trace regularity. Here, an expression of these boundary operators solely in terms of normal and tangential derivatives is more desirable, at least for B_1. Examples of this situation are given in Eqn. (3.12.1f) of Section 3.12 and of Eqn. (3I.103) of Appendix I, critical to the proof of Theorem 3I.1 on analyticity of the thermo-elastic semigroup under free BC. However, expressions of the boundary operators B_1 and B_2 – ultimately, of the free BC consisting of the bending moment BC and the shear force BC – solely in terms of normal and tangential derivatives are not readily available in the mathematical literature of these plate equations, and indeed not well-known, apparently. For all these reasons we derive and collect in this appendix the expressions of B_1, B_2, $\Delta w|_\Gamma$ in terms of normal and tangential derivatives, beyond the need in this book. Points 1 through 4 below – in particular, results such as Proposition 3C.1, Lemma 3C.2, Corollary 3C.3, and Proposition 3C.4 – are needed in various sections, such as Sections 3.5 and 3.12 of the present Chapter 3; Appendix 3I, Eqn. (3I.103); as well as in extending Section 3.4, Section 3.6, Section 3.9, and Section 3.10 to the case of the physical bending moment.

Instead, points 5 through 8 below are included for completeness and future reference. The main results are Proposition 3C.6 through Proposition 3C.8. Expressions (3C.49) and (3C.50) recover formulas listed in Stahel [1987].

3C.0 Preliminary Considerations

We consider throughout a two-dimensional bounded domain Ω with C^1-boundary Γ, so that the unit outward normal vector $\nu(\eta) = [\nu_1(\eta), \nu_2(\eta)]$ and its corresponding unit tangent vector $\tau(\eta) = [-\nu_2(\eta), \nu_1(\eta)]$ vary smoothly with the point η moving over the boundary Γ. Actually, since the following considerations are local in character, we may as well focus on a subportion $\tilde{\Gamma}$ of Γ, and let simply η be an interior point of $\tilde{\Gamma}$: $\eta \in \tilde{\Gamma}$. We shall then define the vector or point in Ω,

$$\xi = r(t; \eta) = \eta + t\nu(\eta), \quad -t_0 < t < 0, \eta \in \tilde{\Gamma},$$

$|t_0|$ sufficiently small, that, for t fixed and η running over $\tilde{\Gamma}$, describes the parallel translation curve $\tilde{\Gamma}_t$ in Ω of $\tilde{\Gamma}$; moreover, as t runs over $(-t_0, 0)$, then the family $\tilde{\Gamma}_t$ of curves sweep a collar, or strip, S_{t_0} of $\tilde{\Gamma}$:

$$\tilde{\Gamma}_t = \bigcup_{\eta \in \tilde{\Gamma}} r(t; \eta), \quad S_{t_0} = \bigcup_{-t_0 < t < 0} \tilde{\Gamma}_t$$

(See Figure 3C.1.)

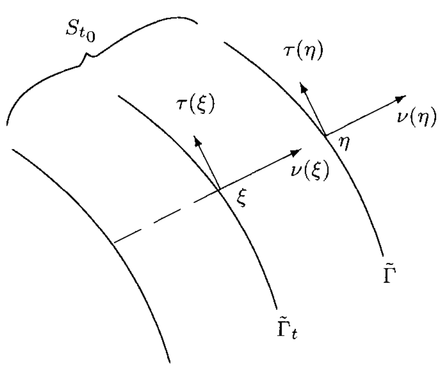


Figure 3C.1.

The map $\eta \in \tilde{\Gamma} \to \xi \in S_{t_0}$ is one-to-one (the Jacobian is $\neq 0$). For each $\eta \in \tilde{\Gamma}$ and corresponding $\xi = \eta + t\nu(\eta) \in S_{t_0}$, we let $\nu(\xi)$ be the unit outward normal to the curve $\tilde{\Gamma}_t$ passing through ξ and let $\tau(\xi) = [-\nu_2(\xi), \nu_1(\xi)]$ be the corresponding unit tangent vector. Thus, we have

$$\nu(\xi) \equiv \nu(\eta) \quad \text{and} \quad \tau(\xi) \equiv \tau(\eta), \quad \text{for all } \xi = \eta + t\nu(\eta), \quad -t_0 < t < 0, \tag{3C.0a}$$

that is, the normal unit vector $\nu(\eta)$ at the boundary point $\eta \in \tilde{\Gamma}$ generates a *constant* vector field $\nu(\xi)$ along each point ξ of *the normal line* to η in the collar; and similarly for the tangent vector $\tau(\eta)$. In this way, two smooth vector fields $\nu(\xi)$ and $\tau(\xi)$ are defined on all points ξ of the collar, by a parallel translation of the pair $\nu(\eta)$ and $\tau(\eta)$, $\eta \in \tilde{\Gamma}$, along the normal line to η. Thus, we may define the normal derivative and tangential derivative

$$\frac{\partial w(\xi)}{\partial \nu} = \nabla w(\xi) \cdot \nu(\xi), \quad \frac{\partial w(\xi)}{\partial \tau} = \nabla w(\xi) \cdot \tau(\xi) \tag{3C.0b}$$

to $\tilde{\Gamma}_t$ for each point $\xi = \eta + t\nu(\eta)$ of the collar S_{t_0}, as well as, of course, at $\eta \in \tilde{\Gamma}$. These considerations form the preliminary background for the arguments below.

3C.1 The Bending Moment Boundary Operator B_1

Our first result shows that the two expressions – in Eqn. (3.5.2a) of Section 3.5 and in Eqn. (3.12.1f) in Section 3.12 – for the bending moment boundary operator B_1 do coincide, as expected.

Proposition 3C.1 *Let Ω be a two-dimensional domain with C^1-boundary Γ. Let $w \in H^2(\Omega)$. Denote by*

$$\nu = [\nu_1, \nu_2], \quad \tau \equiv [-\nu_2, \nu_1], \quad c(\eta) \equiv \operatorname{div} \nu(\eta) \tag{3C.1}$$

the unit outward normal ν to the boundary Γ, the unit tangent vector τ along Γ accordingly oriented, and the mean curvature *$c(\eta)$ at the point $\eta \in \Gamma$. Then:*

$$B_1 w \equiv -\frac{\partial^2 w}{\partial \tau^2} - c(\eta)\frac{\partial w}{\partial \nu} = 2\nu_1\nu_2 w_{xy} - \nu_1^2 w_{yy} - \nu_2^2 w_{xx} \quad \text{on } \Gamma. \tag{3C.2}$$

Proof.

Step 1 With reference to the preliminary considerations above, we readily relate the normal and tangential derivatives $\partial w/\partial \nu$ and $\partial w/\partial \tau$ to the partial derivatives w_x and w_y for each point $\xi = (x, y)$ of the collar S_{t_0} as well as for $\eta \in \Gamma$:

$$\begin{cases} \dfrac{\partial w}{\partial \nu} = \nabla w \cdot \nu = w_x \nu_1 + w_y \nu_2, \\ \dfrac{\partial w}{\partial \tau} = \nabla w \cdot \tau = -w_x \nu_2 + w_y \nu_1, \end{cases} \quad \text{and hence} \quad \begin{cases} w_x = \nu_1 \dfrac{\partial w}{\partial \nu} - \nu_2 \dfrac{\partial w}{\partial \tau}, & \text{(3C.3)} \\ w_y = \nu_2 \dfrac{\partial w}{\partial \nu} + \nu_1 \dfrac{\partial w}{\partial \tau}. & \text{(3C.4)} \end{cases}$$

Moreover, differentiating in x and y the condition of unity for ν, in the collar we obtain

$$\nu_1^2 + \nu_2^2 \equiv 1 \Rightarrow \nu_1 \nu_{1x} + \nu_2 \nu_{2x} \equiv 0, \quad \nu_1 \nu_{1y} + \nu_2 \nu_{2y} \equiv 0, \quad \text{in } \bar{S}_{t_0}, \tag{3C.5}$$

in the closure of S_{t_0} (i.e., in particular, also on Γ).

Step 2 From Eqn. (3C.4) (on the left) and (3C.1), we obtain

$$\begin{aligned} \frac{\partial^2 w}{\partial \tau^2} &= \nabla \left(\frac{\partial w}{\partial \tau} \right) \cdot \tau = [(-\nu_2 w_x + \nu_1 w_y)_x, (-\nu_2 w_x + \nu_1 w_y)_y] \cdot [-\nu_2, \nu_1] \\ &= \nu_2^2 w_{xx} - 2\nu_1 \nu_2 w_{xy} + \nu_1^2 w_{yy} + (\nu_2 \nu_{2x} - \nu_1 \nu_{2y}) w_x + (\nu_1 \nu_{1y} - \nu_2 \nu_{1x}) w_y \\ & \qquad \text{on } \Gamma. \end{aligned} \tag{3C.6}$$

Step 3 Substituting for w_x and w_y in (3C.6) the expressions in (3C.3), (3C.4) in terms of $\partial w / \partial \nu$, $\partial w / \partial \tau$, we obtain

$$\begin{aligned} &(\nu_2 \nu_{2x} - \nu_1 \nu_{2y}) w_x + (\nu_1 \nu_{1y} - \nu_2 \nu_{1x}) w_y \\ &\quad = \left[\nu_1 (\nu_2 \nu_{2x}) - \nu_1^2 \nu_{2y} + \nu_2 (\nu_1 \nu_{1y}) - \nu_2^2 \nu_{1x} \right] \frac{\partial w}{\partial \nu} \\ &\qquad + \left[\nu_1 (\nu_1 \nu_{1y} + \nu_2 \nu_{2y}) - \nu_2 (\nu_1 \nu_{1x} + \nu_2 \nu_{2x}) \right] \frac{\partial w}{\partial \tau} \\ &\quad = \left[\nu_1 (\nu_2 \nu_{2x}) - \nu_1^2 \nu_{2y} + \nu_2 (\nu_1 \nu_{1y}) - \nu_2^2 \nu_{1x} \right] \frac{\partial w}{\partial \nu}, \end{aligned} \tag{3C.7}$$

since the coefficient of $\partial w / \partial \tau$ vanishes by use of the two identities in (3C.5) in Γ. With reference to the coefficient of $\partial w / \partial \nu$ in (3C.7), adding and subtracting $\nu_1^2 \nu_{1x}$ and $\nu_2^2 \nu_{2y}$ and using (3C.5), we obtain:

$$\begin{aligned} & -\nu_1 (\nu_2 \nu_{2x}) + \nu_1^2 \nu_{2y} - \nu_2 (\nu_1 \nu_{1y}) + \nu_2^2 \nu_{1x} \\ &\quad = \left(\nu_1^2 + \nu_2^2 \right) \nu_{1x} + \left(\nu_1^2 + \nu_2^2 \right) \nu_{2y} \\ &\qquad + \left[-\nu_1^2 \nu_{1x} - \nu_2 (\nu_2 \nu_{2y}) - \nu_1 (\nu_2 \nu_{2x}) - \nu_2 \nu_1 \nu_{1y} \right] \\ \text{(by (3C.5))} &\quad = \nu_{1x} + \nu_{2y} \\ &\qquad + \left[-\nu_1^2 \nu_{1x} - \nu_2 (-\nu_1 \nu_{1y}) - \nu_1 (-\nu_1 \nu_{1x}) - \nu_2 \nu_1 \nu_{1y} \right] \quad \text{(3C.8)} \\ &\quad = \operatorname{div} \nu(\eta) = c(\eta), \quad \text{(3C.9)} \end{aligned}$$

as desired, as the term in the square brackets in (3C.8) vanishes. Thus, (3C.6) is rewritten via Eqn. (3.5.2a) of Section 3.5, (3C.7) and (3C.9) as

$$\frac{\partial^2 w}{\partial \tau^2} = \left[\nu_2^2 w_{xx} - 2\nu_1\nu_2 w_{xy} + \nu_1^2 w_{yy}\right] - c(\eta)\frac{\partial w}{\partial \nu} = -B_1 w - c(\eta)\frac{\partial w}{\partial \nu}, \quad \eta \in \Gamma, \tag{3C.10}$$

and (3C.10) proves (3C.2), as desired. □

3C.2 A Critical Lemma Involving the Bending Moments Operator B_1 and the Shear Forces Operator B_2

We rewrite the boundary operators B_1 and B_2 from Eqn. (3.5.2):

$$B_1 w \equiv 2\nu_1\nu_2 w_{xy} - \nu_1^2 w_{yy} - \nu_2^2 w_{xx} \text{ on } \Gamma, \tag{3C.11}$$

$$B_2 w \equiv \frac{\partial}{\partial \tau}\left[\left(\nu_1^2 - \nu_2^2\right) w_{xy} + \nu_1\nu_2(w_{yy} - w_{xx})\right] \text{ on } \Gamma. \tag{3C.12}$$

Lemma 3C.2 *For w and v sufficiently smooth on Ω, say in $C^3(\Omega)$, the following identity holds true, with Γ oriented along τ:*

$$\int_\Omega [2w_{xy}v_{xy} - w_{xx}v_{yy} - w_{yy}v_{xx}]\, d\Omega = \int_\Gamma \left[(B_1 w)\frac{\partial v}{\partial \nu} - (B_2 w) v\right] d\Gamma, \tag{3C.13}$$

which can then be extended to $w, v \in H^2(\Omega)$.

Proof. We first verify, after two cancellations, that

$$\frac{\partial}{\partial x}(w_{xy}v_y - w_{yy}v_x) + \frac{\partial}{\partial y}(w_{xy}v_x - w_{xx}v_y) = 2w_{xy}v_{xy} - w_{xx}v_{yy} - w_{yy}v_{xx}. \tag{3C.14}$$

Next, invoking Green's theorem in the plane, we obtain from (3C.14), via $dx/ds = -\nu_2$ and $dy/ds = \nu_1$ for τ, where s is the arc length on Γ:

$$\begin{aligned}
&\int_\Omega [2w_{xy}v_{xy} - w_{xx}v_{yy} - w_{yy}v_{xx}]\, d\Omega \\
&\quad = \int_\Omega \left[\frac{\partial}{\partial x}(w_{xy}v_y - w_{yy}v_x) + \frac{\partial}{\partial y}(w_{xy}v_x - w_{xx}v_y)\right] d\Omega \\
&\quad = \int_\Gamma -(w_{xy}v_x - w_{xx}v_y)\, dx + (w_{xy}v_y - w_{yy}v_x)\, dy \\
&\quad = \int_\Gamma [\nu_1(w_{xy}v_y - w_{yy}v_x) + \nu_2(w_{xy}v_x - w_{xx}v_y)]\, d\Gamma
\end{aligned} \tag{3C.15}$$

(using (3C.4) and (3C.5) for v_x and v_y in terms of $\partial v/\partial/\nu$ and $\partial v/\partial \tau$)

$$\begin{aligned}
&= \int_\Gamma \left(2\nu_1\nu_2 w_{xy} - \nu_1^2 w_{yy} - \nu_2^2 w_{xx}\right)\frac{\partial v}{\partial \nu}\, d\Gamma \\
&\quad + \int_\Gamma \left[\left(\nu_1^2 - \nu_2^2\right) w_{xy} + \nu_1\nu_2(w_{yy} - w_{xx})\right]\frac{\partial v}{\partial \tau}\, d\Gamma.
\end{aligned} \tag{3C.16}$$

Finally, (3C.16) becomes (3C.13), as desired, via (3C.11) and (3C.12), after using the identity

$$\int_\Gamma \left[(\nu_1^2 - \nu_2^2)w_{xy} + \nu_1\nu_2(w_{yy} - w_{xx})\right]\frac{dv}{d\tau}\,d\Gamma$$

$$= \int_\Gamma -\frac{\partial}{\partial\tau}\left[(\nu_1^2 - \nu_2^2)w_{xy} + \nu_1\nu_2(w_{yy} - w_{xx})\right] v\,d\Gamma \qquad (3C.17)$$

$$\text{(by (3C.12))} \qquad = -\int_\Gamma (B_2 w)v\,d\Gamma, \qquad (3C.18)$$

which is obtained by integration by parts with Γ a closed curve. □

Corollary 3C.3

(i) Let $w, v \in H^2(\Omega)$. Then

$$\int_\Gamma \left[(B_1 w)\frac{\partial v}{\partial\nu} - (B_2 w)v\right] d\Gamma = \int_\Gamma \left[(B_1 v)\frac{\partial w}{\partial\nu} - (B_2 v)w\right] d\Gamma$$

$$= \int_\Omega [2w_{xy}v_{xy} - w_{xx}v_{yy} - w_{yy}v_{xx}]\,d\Omega, \qquad (3C.19a)$$

where one boundary integral is obtained from the other by interchanging w with v.

(ii) Let w be sufficiently smooth on Ω with $w|_\Gamma \equiv 0$, say $w \in H^2(\Omega) \cap H_0^1(\Omega)$. Then, for Γ oriented along τ, the following identity holds true:

$$\int_\Omega 2\left[w_{xy}^2 - w_{xx}w_{yy}\right] d\Omega = \int_\Gamma (B_1 w)\frac{\partial w}{\partial\nu}\,d\Gamma. \qquad (3C.19b)$$

Proof. (i) If on the right-hand side of (3C.13) over Γ we interchange w with v, the left-hand side over Ω does not change, and (3C.19a) is obtained.

(ii) Set $v = w$, with $v = w|_\Gamma = 0$ in (3C.19a) to obtain (3C.19b). □

3C.3 Self-Adjointness and Positivity of the Biharmonic Operator in Eqn. (3.12.3)

We recall the operator $\mathcal{A}$ in Eqn. (3.12.3):

$$\mathcal{A}h = \Delta^2 h,$$

$$\mathcal{D}(\mathcal{A}) = \left\{h \in H^4(\Omega) \cap H_0^1(\Omega) : \Delta h + (1-\mu)B_1 h = 0 \text{ on } \Gamma\right\}, \qquad (3C.20)$$

for a two-dimensional domain Ω with boundary Γ, $0 < \mu < 1$.

Proposition 3C.4 *With reference to the operator $\mathcal{A}$ in (3C.20), we have: (i) $\mathcal{A}$ is self-adjoint on $L_2(\Omega)$ and, in fact, (ii) strictly positive for $0 < \mu < 1$: For*

$w \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega) \cap H_0^1(\Omega)$, we have

$$\|\mathcal{A}^{\frac{1}{2}} w\|^2_{L_2(\Omega)} = (\mathcal{A}w, w)_{L_2(\Omega)} = \int_\Omega |\Delta w|^2 \, d\Omega + 2(1-\mu) \int_\Omega \left[w_{xy}^2 - w_{xx} w_{yy} \right] d\Omega \tag{3C.21}$$

$$= \int_\Omega \left\{ \mu |\Delta w|^2 + (1-\mu)\left(w_{xx}^2 + w_{yy}^2\right) + 2(1-\mu) w_{xy}^2 \right\} d\Omega > 0. \tag{3C.22}$$

Proof. (i) Let $w, v \in \mathcal{D}(\mathcal{A})$ in (3C.20). By Green's second theorem we compute, with $v|_\Gamma = 0$ and $-\Delta w|_\Gamma = (1-\mu) B_1 w$ by (3C.20):

$$\begin{aligned}
(\mathcal{A}w, w)_{L_2(\Omega)} &= \int_\Omega \Delta\Delta w \, v \, d\Omega = \int_\Omega \Delta w \, \Delta v \, d\Omega \\
&\quad + \cancel{\int_\Gamma \frac{\partial \Delta w}{\partial \nu} v \, d\Gamma} - \int_\Gamma \Delta w \frac{\partial v}{\partial \nu} \, d\Gamma \\
\text{(by (3C.20))} \qquad &= \int_\Omega \Delta w \, \Delta v \, d\Omega + (1-\mu) \int_\Gamma B_1 w \frac{\partial v}{\partial \nu} \, d\Gamma \qquad (3C.23) \\
\text{(by (3C.3))} \qquad &= \int_\Omega \Delta w \, \Delta v \, d\Omega - (1-\mu) \int_\Gamma c(x) \frac{\partial w}{\partial \nu} \frac{dv}{\partial \nu} \, d\Gamma. \qquad (3C.24)
\end{aligned}$$

Identities (3C.23) or (3C.24) may be extended to all $w, v \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega) \cap H_0^1(\Omega)$. From (3C.24) one deduces that $\mathcal{A}$ is self-adjoint [but not that $\mathcal{A}$ is positive!].

(ii) To establish that $\mathcal{A}$ is actually positive, we need to use that $c(x)$ is the mean curvature div $\nu(x)$ as in (3C.2): Let $w \in \mathcal{D}(\mathcal{A})$. Then, returning to (3C.23) with $v = w$ and invoking identity (3C.19) of Corollary 3C.3, we obtain (3C.21).

Moreover, we compute after a cancellation

$$\begin{aligned}
&(w_{xx} + w_{yy})^2 + 2(1-\mu)\left(w_{xy}^2 - w_{xx} w_{yy}\right) \\
&= w_{xx}^2 + w_{yy}^2 + 2\mu w_{xx} w_{yy} + 2(1-\mu) w_{xy}^2 \\
&= \mu(w_{xx} + w_{yy})^2 + (1-\mu)\left(w_{xx}^2 + w_{yy}^2\right) + 2(1-\mu) w_{xy}^2. \qquad (3C.25)
\end{aligned}$$

Identity (3C.25) permits us to go from (3C.21) to (3C.22). Finally, (3C.22) shows positivity for $0 \le \mu \le 1$, as desired. □

3C.4 Self-Adjointness and Positivity of the Biharmonic Operator in Eqn. (3.5.3)

We return to the operator $\mathcal{A}$ in Eqn. (3.5.3) of Section 3.5:

$$\mathcal{A}h = \Delta^2 h,$$

$$\mathcal{D}(\mathcal{A}) = \left\{ h \in H^4(\Omega) : [\Delta h + (1-\mu)B_1 h]_\Gamma = 0, \right.$$
$$\left. \left[\frac{\partial \Delta h}{\partial \nu} + (1-\mu)B_2 h - h \right]_\Gamma = 0 \right\}, \tag{3C.26}$$

for a two-dimensional domain Ω with boundary Γ, $0 < \mu < 1$.

Proposition 3C.5 *With reference to the operator $\mathcal{A}$ in (3C.26), we have: (i) $\mathcal{A}$ is self-adjoint on $L_2(\Omega)$ and, in fact, (ii) positive for $0 < \mu < 1$: For $w \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega)$, which satisfies the BC (3C.26), we have*

$$\begin{aligned}
&\|\mathcal{A}^{\frac{1}{2}} w\|^2_{L_2(\Omega)} \\
&\quad = (\mathcal{A}w, w)_{L_2(\Omega)} \\
&\quad = \int_\Omega |\Delta w|^2 \, d\Omega + 2(1-\mu) \int_\Omega \left[w_{xy}^2 - w_{xx} w_{yy} \right] d\Omega + \int_\Gamma w^2 \, d\Gamma \qquad (3C.27) \\
&\quad = \int_\Omega \left\{ \mu |\Delta w|^2 + (1-\mu)\left(w_{xx}^2 + w_{yy}^2\right) + 2(1-\mu) w_{xy}^2 \right\} d\Omega + \int_\Gamma w^2 \, d\Gamma.
\end{aligned} \tag{3C.28}$$

Proof. (i) Let $w, v \in \mathcal{D}(\mathcal{A})$ in (3C.26). By Green's second theorem we compute via (3C.13) of Lemma 3C.2, with $-\Delta w|_\Gamma = (1-\mu)B_1 w$ and $\frac{\partial \Delta w}{\partial \nu}|_\Gamma = [-(1-\mu) B_2 w + w]_\Gamma$ by (3C.26):

$$\begin{aligned}
(\mathcal{A}w, v)_{L_2(\Omega)} &= \int_\Omega \Delta\Delta w \, v \, d\Omega \\
&= \int_\Omega \Delta w \Delta v \, d\Omega + \int_\Gamma \frac{\partial \Delta w}{\partial \nu} v \, d\Gamma - \int_\Gamma \Delta w \frac{\partial v}{\partial \nu} \, d\Gamma \\
\text{(by (3C.26))} \quad &= \int_\Omega \Delta w \Delta v \, d\Omega + \int_\Gamma w v \, d\Gamma \\
&\quad + (1-\mu) \int_\Gamma \left[(B_1 w) \frac{\partial v}{\partial \nu} - (B_2 w) v \right] d\Gamma \qquad (3C.29) \\
\text{(by (3C.13))} \quad &= \int_\Omega \Delta w \Delta v \, d\Omega + \int_\Gamma w v \, d\Gamma \\
&\quad + (1-\mu) \int_\Omega [2 w_{xy} v_{xy} - w_{xx} v_{yy} - w_{yy} v_{xx}] \, d\Omega. \qquad (3C.30)
\end{aligned}$$

Identities (3C.29) and (3C.30) may be extended to $w, v \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega)$, provided that they satisfy the BC.

From (3C.30) we deduce self-adjointness of $\mathcal{A}$ [but not positivity of $\mathcal{A}$].

(ii) To establish that $\mathcal{A}$ is actually positive, we let $w \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega)$ satisfy the B.C. and return to (3C.30) with $w = v$, whereby we obtain

$$\|\mathcal{A}^{\frac{1}{2}} w\|^2_{L_2(\Omega)} = (\mathcal{A}w, w)_{L_2(\Omega)}$$

$$= \int_\Omega |\Delta w|^2 d\Omega + \int_\Gamma w^2 d\Gamma + 2(1-\mu) \int_\Omega \left[w_{xy}^2 - w_{xx}^2 w_{yy}^2 \right] d\Omega, \tag{3C.31}$$

and (3C.27) is proved. Then (3C.28) follows from (3C.27), via identity (3C.25). □

3C.5 The Trace $\Delta w|_\Gamma$ in Terms of Normal and Tangential Derivatives

Henceforth we provide a series of results that, though not used anywhere in this book, are important in their own right and thus are included for completeness and future reference. First, the preliminary considerations in the introductory point 3C.0 above offer a strategy for defining higher order normal derivatives. Thus, with $\eta \in \Gamma$ fixed, in the notation set there, we introduce the scalar function:

$$g(t) = w(\eta + t\nu(\eta)), \quad w \in C^2(\bar{\Omega}), \eta \in \Gamma, \quad -t_0 < t < 0, \tag{3C.32}$$

$|t_0|$ sufficiently small, so that, with $\xi = \eta + t\nu(\eta) = (\xi_1, \xi_2) \in \Omega$, we have

(i)

$$g'(t) = \nabla w(\eta + t\nu(\eta)) \cdot \nu(\eta) = \sum_{i=1}^2 \frac{\partial w}{\partial \xi_i}(\eta + t\nu(\eta))\nu_i(\eta), \quad t < 0 \tag{3C.33}$$

$$= \sum_{i=1}^2 \frac{\partial w}{\partial \xi_i}(\xi)\nu_i(\xi) = \nabla w(\xi) \cdot \nu(\xi) = \frac{\partial w(\xi)}{\partial \nu}, \quad \xi \in \Omega \tag{3C.34}$$

by recalling that $\nu(\xi) \equiv \nu(\eta)$ by (3C.0a) and by (3C.0b);

(ii) by (3C.33), and again by $\nu(\xi) \equiv \nu(\eta)$, we obtain

$$g''(t) = \sum_{i,j=1}^2 \frac{\partial^2 w}{\partial \xi_i \partial \xi_j}(\eta + t\nu(\eta))\nu_i(\eta)\nu_j(\eta), \quad t < 0 \tag{3C.35}$$

$$= \nabla(\nabla w(\xi) \cdot \nu(\eta)) \cdot \nu(\eta) = \nabla(\nabla w(\xi) \cdot \nu(\xi)) \cdot \nu(\xi)$$

$$= \sum_{i,j=1}^2 \frac{\partial^2 w}{\partial \xi_i \partial \xi_j}(\xi)\nu_i(\xi)\nu_j(\xi) = \frac{\partial^2 w}{\partial \nu^2}(\xi), \quad \xi \in \Omega. \tag{3C.36}$$

Then, by definition, for $\eta \in \Gamma$, recalling (3C.33) and (3C.35):

$$\frac{\partial w}{\partial \nu}(\eta) = \lim_{t \uparrow 0} g'(t) = g'(0) = \nabla w(\eta) \cdot \nu(\eta) = \sum_{i=1}^2 \frac{\partial w}{\partial x_i}(\eta)\nu_i(\eta), \tag{3C.37}$$

$$\frac{\partial^2 w}{\partial \nu^2}(\eta) = \lim_{t \uparrow 0} g''(t) = g''(0) = \sum_{i,j=1}^2 \frac{\partial^2 w}{\partial x_i \partial x_j}(\eta)\nu_i(\eta)\nu_j(\eta). \tag{3C.38}$$

In the above notation we have

Proposition 3C.6 *Let Ω be a two-dimensional domain with a C^1-boundary Γ. Let $w \in C^2(\bar{\Omega})$. Then*

$$\Delta w|_\Gamma = \frac{\partial^2 w}{\partial \nu^2} + \frac{\partial^2 w}{\partial \tau^2} + \left(\frac{\partial w}{\partial \nu}\right) \text{div } \nu. \tag{3C.39}$$

Proof.

Step 1 With reference to the preliminary considerations of point 3C.0 above, for each point ξ in Ω close to Γ, we may write

$$\begin{aligned} \nabla w(\xi) &= (\nabla w(\xi) \cdot \nu(\xi))\nu(\xi) + (\nabla w(\xi) \cdot \tau(\xi))\tau(\xi) \\ &= \frac{\partial w}{\partial \nu}(\xi)\nu(\xi) + \frac{\partial w}{\partial \tau}(\xi)\tau(\xi), \end{aligned} \tag{3C.40}$$

with $\nu(\xi)$ and $\tau(\xi)$ defined there in (3C.0b). We next compute, using the identity $\text{div}(\alpha h) = \alpha \text{ div } h + \nabla\alpha \cdot h$, valid for a scalar α and a vector field h, and recalling (3C.34), (3C.36), and (3C.40):

$$\Delta w(\xi) = \text{div } \nabla w(\xi) = \text{div}[(\nabla w(\xi) \cdot \nu(\xi))\nu(\xi) + (\nabla w(\xi) \cdot \tau(\xi))\tau(\xi)] \tag{3C.41}$$

$$\begin{aligned} \text{(by (3C.40))} \quad &= (\nabla w(\xi) \cdot \nu(\xi))\text{div } \nu(\xi) + \nabla(\nabla w(\xi) \cdot \nu(\xi)) \cdot \nu(\xi) \\ &\quad + (\nabla w(\xi) \cdot \tau(\xi))\text{div } \tau(\xi) + \nabla(\nabla w(\xi) \cdot \tau(\xi)) \cdot \tau(\xi) \end{aligned} \tag{3C.42}$$

$$\begin{aligned} \text{(by (3C.34) and (3C.36))} \quad &= \frac{\partial w}{\partial \nu}(\xi)\text{div } \nu(\xi) + \frac{\partial^2 w}{\partial \nu^2}(\xi) \\ &\quad + \frac{\partial w}{\partial \tau}(\xi) \text{ div } \tau(\xi) + \frac{\partial^2 w}{\partial \tau^2}(\xi). \end{aligned} \tag{3C.43}$$

Step 2 We shall next show that, with $\xi = [\xi_1, \xi_2] \in \Omega$:

$$\text{div } \tau(\xi) = \text{div}[-\nu_2(\xi), \nu_1(\xi)] = -\nu_{2\xi_1}(\xi) + \nu_{1\xi_2}(\xi) \equiv 0. \tag{3C.44}$$

In fact, the unit condition for $\nu(\xi)$ gives (3C.5), rewritten now with x and y there replaced by the current notation ξ_1 and ξ_2 respectively:

$$\nu_1\nu_{1\xi_1} + \nu_2\nu_{2\xi_1} \equiv 0, \quad \nu_1\nu_{1\xi_2} + \nu_2\nu_{2\xi_2} \equiv 0. \tag{3C.45}$$

Moreover, since $\nu(\xi) = [\nu_1(\xi), \nu_2(\xi)]$ is constant along normal lines to Γ, $\nu(\xi) \equiv \nu(\eta)$, [see (3C.0a)], we obtain

$$\begin{aligned} \frac{\partial \nu_1(\xi)}{\partial \nu} &= \nabla\nu_1 \cdot \nu = \nu_{1\xi_1}\nu_1 + \nu_{1\xi_2}\nu_2 \equiv 0, \\ \frac{\partial \nu_2(\xi)}{\partial \nu} &= \nabla\nu_2 \cdot \nu = \nu_{2\xi_1}\nu_1 + \nu_{2\xi_2}\nu_2 \equiv 0. \end{aligned} \tag{3C.46}$$

Comparing the first two identities in (3C.45) and (3C.46), and likewise the second two identities there, we obtain

$$\nu_2[\nu_{1\xi_2} - \nu_{2\xi_1}] \equiv 0 \quad \text{and} \quad \nu_1[\nu_{2\xi_1} - \nu_{1\xi_2}] \equiv 0, \tag{3C.47}$$

respectively. Since ν is nowhere a zero vector by the assumption of smoothness of Γ, we conclude that (3C.47) implies (3C.44), as desired.

Step 3 Using (3C.44) in (3C.43) yields

$$\Delta w(\xi) = \frac{\partial^2 w}{\partial \nu^2}(\xi) + \frac{\partial^2 w}{\partial \tau^2}(\xi) + \frac{\partial w}{\partial \nu}(\xi) \operatorname{div} \nu(\xi) \tag{3C.48}$$

for any point $\xi = \eta + t\nu(\eta)$ near $\eta \in \Gamma$ (collar of Γ). By (3C.37) and (3C.38), then (3C.48) holds true also at $\eta \in \Gamma$, and (3C.39) is proved.

Remark 3C.1 It is possible to give a differential (Riemann) geometry proof of Proposition 3C.6 (personal communication by P. F. Yao, Academia Sinica, Beijing). □

3C.6 First (Bending Moment) BC in Terms of Normal and Tangential Derivatives

As a corollary of Proposition 3C.1 and of Proposition 3C.6, we can now express the first BC (of the free BC), modeling the bending moment, soley in terms of normal and tangential derivatives.

Proposition 3C.7 *Let Ω be a two-dimensional domain with C^1-boundary Γ. Let $w \in C^2(\bar{\Omega})$. Then, with reference to (3C.2), we have for $\eta \in \Gamma$, $0 < \mu < 1$:*

$$[\Delta w + (1-\mu)B_1 w]_\Gamma = \frac{\partial^2 w}{\partial \nu^2} + \mu \frac{\partial^2 w}{\partial \tau^2} + \mu c(\eta) \frac{\partial w}{\partial \nu}, \qquad c(\eta) = \operatorname{div} \nu(\eta). \tag{3C.49}$$

Proof. Combine (3C.2) of Proposition 3C.1 with (3C.39) of Proposition 3C.6 to obtain (3C.49) after a cancellation.

3C.7. The Shear Force Boundary Operator B_2

Our goal now is to express the boundary operator B_2, originally defined by (3.5.2b), or (3C.12) – as it arises in the variational formulation of the model [Lagnese, 1989] – solely in terms of normal and tangential derivatives. This is accomplished in the following proposition.

Proposition 3C.8 *Let Ω be a two-dimensional domain with C^1-boundary Γ. Let $w \in C^2(\bar{\Omega})$. Then, with reference to (3.5.2b), or (3C.12), we have*

$$B_2 w|_\Gamma = \frac{\partial}{\partial \tau}\left[\left(\nu_1^2 - \nu_2^2\right)w_{xy} + \nu_1\nu_2(w_{yy} - w_{xx})\right] = \frac{\partial}{\partial \tau}\frac{\partial}{\partial \nu}\frac{\partial w}{\partial \tau}. \tag{3C.50}$$

Proof. Using (3C.4) (left) and $\partial\nu_1/\partial\nu \equiv \partial\nu_2/\partial\nu \equiv 0$ from (3C.0a), we compute at each point $\xi = \eta + t\nu(\eta)$ of Ω near Γ, $t < 0$:

$$\frac{\partial}{\partial\nu}\frac{\partial w}{\partial\tau} = \frac{\partial}{\partial\nu}[-w_x\nu_2 + w_y\nu_1] = -\frac{\partial w_x}{\partial\nu}\nu_2 + \frac{\partial w_y}{\partial\nu}\nu_1 \tag{3C.51}$$

$$\text{(by (3C.3))} \quad = [-w_{xx}\nu_1 - w_{xy}\nu_2]\nu_2 + [w_{yx}\nu_1 + w_{yy}\nu_2]\nu_1 \tag{3C.52}$$

$$= \nu_1\nu_2(w_{yy} - w_{xx}) + \left(\nu_1^2 - \nu_2^2\right)w_{xy}, \tag{3C.53}$$

where to go from (3C.51) to (3C.52) we have used identity (3C.3) (left) with w there replaced by w_x and w_y, respectively; moreover, to go from (3C.52) to (3C.53), we have used $w_{xy} = w_{yx}$. Then, (3C.53) yields (3C.50). □

We next provide another expression for $B_2 w|_\Gamma$.

Proposition 3C.9 *Let Ω be a two-dimensional domain with C^1-boundary Γ. Let $w \in C^2(\bar{\Omega})$. Then, with reference to (3.5.2b), or (3C.12), we have*

$$B_2 w|_\Gamma = \frac{\partial}{\partial\tau}\left[\left(\nu_1^2 - \nu_2^2\right)w_{xy} + \nu_1\nu_2(w_{yy} - w_{xx})\right] = \frac{\partial}{\partial\tau}\left[\frac{\partial}{\partial\tau}\frac{\partial w}{\partial\nu} - (\operatorname{div}\nu)\frac{\partial w}{\partial\tau}\right]. \tag{3C.54}$$

Proof.

Step 1 Equation (3C.4) (left) with w there replaced by w_x and w_y respectively yields

$$\frac{\partial w_x}{\partial\tau} = -w_{xx}\nu_2 + w_{xy}\nu_1, \qquad \frac{\partial w_y}{\partial\tau} = -w_{yx}\nu_2 + w_{yy}\nu_1. \tag{3C.55}$$

Hence multiplying the first identity in (3C.55) by ν_1, the second by ν_2, and adding up yields

$$\frac{\partial w_x}{\partial\tau}\nu_1 + \frac{\partial w_y}{\partial\tau}\nu_2 = \left[\left(\nu_1^2 - \nu_2^2\right)w_{xy} + \nu_1\nu_2(w_{yy} - w_{xx})\right]. \tag{3C.56}$$

By comparison with (3.5.2b), or (3C.12), that is, with (3C.54) (left), we see that (3C.56) gives the expression which, after application of $\partial/\partial\tau$, yields $B_2 w$. However, the expression of (3C.56) in the left-hand side is easier to handle than that on the right-hand side. Using (3C.3) and (3C.4) on the right, we obtain

$$\frac{\partial w_x}{\partial\tau} = \frac{\partial}{\partial\tau}\left[\nu_1\frac{\partial w}{\partial\nu} - \nu_2\frac{\partial w}{\partial\tau}\right], \qquad \frac{\partial w_y}{\partial\tau} = \frac{\partial}{\partial\tau}\left[\nu_2\frac{\partial w}{\partial\nu} + \nu_1\frac{\partial w}{\partial\tau}\right]. \tag{3C.57}$$

Next, we multiply the first identity in (3C.57) by ν_1, the second by ν_2, add up, and perform the indicated differentiation $\partial/\partial\tau$. We thus obtain, after a cancellation of the term $\nu_1\nu_2\,\partial^2 w/\partial\tau^2$, where $\nu_1^2 + \nu_2^2 \equiv 1$,

$$\left[\frac{\partial w_x}{\partial\tau}\nu_1 + \frac{\partial w_y}{\partial\tau}\nu_2\right] = \left(\nu_1^2 + \nu_2^2\right)\frac{\partial^2 w}{\partial\tau\partial\nu} + \left[\nu_1\frac{\partial\nu_1}{\partial\tau} + \nu_2\frac{\partial\nu_2}{\partial\tau}\right]\frac{\partial w}{\partial\nu}$$

$$+\left[\nu_2 \frac{\partial \nu_1}{\partial \tau} - \nu_1 \frac{\partial \nu_2}{\partial \tau}\right] \frac{\partial w}{\partial \nu}. \tag{3C.58}$$

Step 2 Regarding the coefficients of $\partial w/\partial \nu$ and $\partial w/\partial \tau$, we now establish that

$$\left[\nu_1 \frac{\partial \nu_1}{\partial \tau} + \nu_2 \frac{\partial \nu_2}{\partial \tau}\right] \equiv 0, \quad \text{while} \left[\nu_2 \frac{\partial \nu_1}{\partial \tau} - \nu_1 \frac{\partial \nu_2}{\partial \tau}\right] = -\text{div } \nu. \tag{3C.59}$$

In fact, the first identity on the left of (3C.59) follows from differentiating the unit condition $\nu_1^2 + \nu_2^2 \equiv 1$ with respect to τ. As to the second identity in (3C.59), we compute by using (3C.3) and (3C.4) (on the right), with w replaced by ν_1 and ν_2, respectively,

$$\begin{aligned}
\text{div } \nu &= \text{div}[\nu_1, \nu_2] = \nu_{1x} + \nu_{2y} \\
&= \left(\nu_1 \frac{\partial \nu_1}{\partial \nu} - \nu_2 \frac{\partial \nu_1}{\partial \tau}\right) + \left(\nu_2 \frac{\partial \nu_2}{\partial \nu} + \nu_1 \frac{\partial \nu_2}{\partial \tau}\right) && (3C.60) \\
&= \left(\nu_1 \frac{\partial \nu_1}{\partial \nu} + \nu_2 \frac{\partial \nu_2}{\partial \nu}\right) + \left(\nu_1 \frac{\partial \nu_2}{\partial \tau} - \nu_2 \frac{\partial \nu_1}{\partial \tau}\right) && (3C.61) \\
&= -\left(\nu_2 \frac{\partial \nu_1}{\partial \tau} - \nu_1 \frac{\partial \nu_2}{\partial \tau}\right), && (3C.62)
\end{aligned}$$

since differentiating the unit condition $\nu_1^2 + \nu_2^2 \equiv 1$ with respect to ν yields $(\nu_1\, \partial \nu_1/\, \partial \nu - \nu_2\, \partial \nu_2/\partial \nu) \equiv 0$, which is used in (3C.61). Then (3C.62) shows the second identity in (3C.59), as desired.

Step 2 Using (3C.59) in (3C.58), we see that the coefficient of $\partial w/\partial \nu$ vanishes, whereas that of $\partial w/\partial \tau$ is $[-\text{div } \nu]$, and we thus obtain

$$\left[\frac{\partial w_x}{\partial \tau} \nu_1 + \frac{\partial w_y}{\partial \tau} \nu_2\right] = \frac{\partial^2 w}{\partial \tau \partial \nu} - (\text{div } \nu) \frac{\partial w}{\partial \tau}. \tag{3C.63}$$

Combining (3C.63) with (3C.56) yields finally

$$\begin{aligned}
B_2 w &= \frac{\partial}{\partial \tau}\left[(\nu_1^2 - \nu_2^2) w_{xy} + \nu_1 \nu_2 (w_{yy} - w_{xx})\right] = \frac{\partial}{\partial \tau}\left[\frac{w_x}{\partial \tau} \nu_1 + \frac{\partial w_y}{\partial \tau} \nu_2\right] && (3C.64) \\
&= \frac{\partial}{\partial \tau}\left[\frac{\partial^2 w}{\partial \tau \partial \nu} - (\text{div } \nu) \frac{\partial w}{\partial \nu}\right], && (3C.65)
\end{aligned}$$

and (3C.54) is proved. □

Corollary 3C.10 *Let Ω be a two-dimensional domain, with C^1-boundary Γ. Let $w \in C^2(\bar{\Omega})$. Then we have*

(i)

$$B_2 w|_\Gamma = \frac{\partial}{\partial \tau} \frac{\partial}{\partial \nu} \frac{\partial w}{\partial \tau} = \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau} \frac{\partial w}{\partial \nu} - \frac{\partial}{\partial \tau}\left[(\text{div } \nu) \frac{\partial w}{\partial \tau}\right]; \tag{3C.66}$$

(ii)

$$\frac{\partial}{\partial \nu}\frac{\partial w}{\partial \tau}\bigg|_\Gamma = \left[\nu_1\nu_2(w_{yy} - w_{xx}) + \left(\nu_1^2 - \nu_2^2\right)w_{xy}\right]_\Gamma = \left[-\frac{\partial w_x}{\partial \nu}\nu_2 + \frac{\partial w_y}{\partial \nu}\nu_1\right]_\Gamma \tag{3C.67}$$

$$= \left[\frac{\partial w_x}{\partial \tau}\nu_1 + \frac{\partial w_y}{\partial \tau}\nu_2\right]_\Gamma = \left[\frac{\partial}{\partial \tau}\frac{\partial w}{\partial \nu} - (\operatorname{div} \nu)\frac{\partial w}{\partial \tau}\right]_\Gamma. \tag{3C.68}$$

Proof. Part (i) is obtained by combining (3C.50) with (3C.54). Part (ii) combines (3C.53) and (3C.51) for (3C.67); and (3C.56) and (3C.63) for (3C.68). □

Corollary 3C.10 (ii) shows lack of commutativity for $\partial/\partial\nu$ and $\partial/\partial\tau$, unless $w|_\Gamma \equiv 0$, or else the mean curvature div $\nu = 0$ (Γ is locally straight).

3C.8 Second (Shear Force) BC in Terms of Normal and Tangential Derivatives

As a corollary of identity (3C.48) for Δw in a collar of Γ, and of identity (3C.50) of Proposition 3C.8 for B_2, we can now express the second BC (of the free BC), modeling the shear force, solely in terms of normal and tangential derivatives.

Proposition 3C.11 *Let Ω be a two-dimensional domain with C^1-boundary Γ. Let $w \in C^2(\bar{\Omega})$. Then, with reference to (3.5.1d) we have for $0 < \mu < 1$:*

$$\left[\frac{\partial \Delta w}{\partial \nu} + (1-\mu)B_2 w\right]_\Gamma = \frac{\partial}{\partial \nu}\left[\frac{\partial^2 w}{\partial \nu^2} + \frac{\partial^2 w}{\partial \tau^2}\right] + (1-\mu)\frac{\partial}{\partial \tau}\frac{\partial}{\partial \nu}\frac{\partial w}{\partial \tau} + \frac{\partial}{\partial \nu}\left[\frac{\partial w}{\partial \nu}\operatorname{div} \nu\right]. \tag{3C.69}$$

Proof. Combine (3C.48) [leading to (3C.39)] with (3C.50) for $B_2 w|_\Gamma$. □

Final Remark 3C.2 Our goal in Proposition 3C.7, Eqn. (3C.49), for the first (bending moment) BC, and in Proposition 3C.8, Eqn. (3C.50), for the boundary operator B_2, was to recover expressions for these quantities that are listed in a Ph.D. thesis Stahel [1987] of 1987 (written in German), in particular, the top of p. 2, and Eqn. (7), p. 39 of Stahel [1987], for whose derivation Stahel [1987] seems to refer to Landau and Lifschitz [1975] (also written in German). Being too rusty in German, we found it easier, and quicker, to prove these expressions ourselves, using as a starting point the definitions for B_1 and B_2 given by (3.5.2a, b), or (3C.11) and (3C.12), which are taken from the variational approach of Lagnese [1989].

A Green formula We conclude the treatment of this Appendix 3C by providing a useful Green formula, see e.g., Lagnese [1989], even though we shall not specifically need it in this Volume 1.

Proposition C.12 *For w, v smooth, say $w \in H^4(\Omega)$, $v \in H^2(\Omega)$ we have*

$$\int_\Omega \Delta^2 w v \, d\Omega = a(w, v) + \int_\Gamma \left[\frac{\partial \Delta w}{\partial \nu} + (1-\mu) B_2 w\right] v \, d\Gamma - \int_\Gamma [\Delta w + (1-\mu) B_1 w] \frac{\partial v}{\partial \nu} \, d\Gamma, \tag{3C.70}$$

where $a(w, v)$ is the bilinear form

$$a(w, v) = \int_\Omega \left[w_{xx} v_{xx} + w_{yy} v_{yy} + 2(1-\mu) w_{xy} v_{xy} + \mu(w_{xx} v_{yy} + w_{yy} v_{xx})\right] d\Omega. \tag{3C.71}$$

Proof. All the ingredients were already established for the purpose of proving Proposition 3C.5. Indeed, as in that proof we have by means of Green's formula

$$\begin{aligned}\int_\Omega \Delta^2 w v \, d\Omega &= \int_\Omega \Delta w \Delta v \, d\Omega + \int_\Gamma \frac{\partial \Delta w}{\partial \nu} v \, d\Gamma - \int_\Gamma \Delta w \frac{\partial v}{\partial \nu} \, d\Gamma \\ &= \int_\Omega [w_{xx} v_{xx} + w_{yy} v_{yy} + w_{xx} v_{yy} + w_{yy} v_{xx}] \, d\Omega \\ &\quad + \int_\Gamma \frac{\partial \Delta w}{\partial \nu} v \, d\Gamma - \int_\Gamma \Delta w \frac{\partial v}{\partial \nu} \, d\Gamma.\end{aligned} \tag{3C.72}$$

As in (3C.25) we obtain after two cancellations

$$\begin{aligned}&(w_{xx} + w_{yy})(v_{xx} + v_{yy}) + 2(1-\mu)\left[w_{xy} v_{xy} - \frac{1}{2} w_{xx} v_{yy} - \frac{1}{2} w_{yy} v_{xx}\right] \\ &= w_{xx} v_{xx} + w_{yy} v_{yy} + 2(1-\mu) w_{xy} v_{xy} + \mu(w_{xx} v_{yy} + w_{yy} v_{xx}).\end{aligned} \tag{3C.73}$$

By definition (3C.71), we see that (3C.73) yields

$$\begin{aligned}a(w, v) &= \int_\Omega [w_{xx} v_{xx} + w_{yy} v_{yy} + w_{xx} v_{yy} + w_{yy} v_{xx}] \, d\Omega \\ &\quad + (1-\mu) \int_\Omega [2 w_{xy} v_{xy} - w_{xx} v_{yy} - w_{yy} v_{xx}] \, d\Omega \\ \text{(by (3C.19a))} \quad &= \int_\Omega [w_{xx} v_{xx} + w_{yy} v_{yy} + w_{xx} v_{yy} + w_{yy} v_{xx}] \, d\Omega \\ &\quad + (1-\mu) \int_\Gamma \left[(B_1 w) \frac{\partial v}{\partial \nu} - (B_2 w) v\right] d\Gamma,\end{aligned} \tag{3C.74}$$

after recalling identity (3C.19a) in the last step. Finally, substituting (3C.75) into the right-hand side of (3C.72) yields (3C.70), as desired. □

3D C_0-Semigroup/Analytic Semigroup Generation when $A = \mathcal{A}M$, $\mathcal{A}$ Positive Self-Adjoint, M Matrix. Applications to Thermo-Elastic Equations with Hinged Mechanical BC and Dirichlet Thermal BC

Assumptions Throughout this Appendix 3D, we let X be a Hilbert space and $\mathcal{A} : X \supset \mathcal{D}(\mathcal{A}) \to X$ be a strictly positive, self-adjoint, unbounded operator, $\overline{\mathcal{D}(\mathcal{A})} = X$. Moreover, we let $M = [m_{ij}]$, $i, j = 1, \ldots, n$ be a $n \times n$ complex matrix.

Problem We define the operator A in factor form by

$$A = \mathcal{A}M = \begin{bmatrix} m_{11}\mathcal{A} & \cdots & m_{1n}\mathcal{A} \\ \vdots & & \vdots \\ m_{n1}\mathcal{A} & \cdots & m_{nn}\mathcal{A} \end{bmatrix} : H \supset \mathcal{D}(A) \to H, \tag{3D.1}$$

where H and $\mathcal{D}(A)$ are the Cartesian products

$$H = X^n = X \times \cdots \times X, \qquad \mathcal{D}(A) = [\mathcal{D}(\mathcal{A})]^n = \mathcal{D}(\mathcal{A}) \times \cdots \times \mathcal{D}(\mathcal{A}), n \text{ times}. \tag{3D.2}$$

Then, H is a Hilbert space, A is densely defined $\overline{\mathcal{D}(A)} = H$, and A is closed. We seek to determine if, and when, A is the generator of a s.c. semigroup, that is moreover analytic on H. The next result provides a characterization, which in particular has an impact on the stability properties of the corresponding analytic semigroup.

Theorem 3D.1 *Under the preceding assumptions, we distinguish two cases:*

(a) Let the $n \times n$ matrix M be diagonalizable. Then:

(i) The operator A in (3D.1) generates a s.c. semigroup e^{At} on H if and only if the eigenvalues λ_i of the matrix M satisfy the condition:

$$\text{Re}\,\lambda_i \leq 0, \quad i = 1, 2, \ldots, n. \tag{3D.3}$$

In this case, e^{At} is uniformly bounded:

$$\|e^{At}\|_{\mathcal{L}(H)} \leq const, \quad t \geq 0. \tag{3D.4}$$

(ii) The operator A in (3D.1) generates a s.c. analytic semigroup e^{At} on H if and only if, in particular, for $i = 1, 2, \ldots, n$,

$$\text{Re}\,\lambda_i < 0 \quad \textit{if } \lambda_i = \textit{complex}, \quad \lambda_i \leq 0 \textit{ if } \lambda_i = \textit{real}. \tag{3D.5}$$

Moreover, if (3D.5) is satisfied with all Re $\lambda_i < 0$, *then, in this case, e^{At} is uniformly stable:*

$$\begin{cases} \|e^{At}\|_{\mathcal{L}(H)} \leq Ce^{(-\omega+\epsilon)t}, \quad t \geq 0, \forall\, \epsilon > 0, -\omega + \epsilon < 0; & \text{(3D.6a)} \\ \mu_1 \left[\sup_i(\text{Re}\,\lambda_i)\right] \equiv -\omega < 0, & \text{(3D.6b)} \end{cases}$$

where the constant $C \geq 1$ depends on $(-\omega + \epsilon)$, and where $\mu_1 > 0$ is the lowest point in the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ in $\mathbb{R}^+$.

(b) For a general M, then the operator A in (3D.1) generates a s.c. semigroup e^{At} on H if and only if

$$\operatorname{Re} \lambda_i < 0, \quad i = 1, \ldots, n, \tag{3D.7}$$

in which case the semigroup is, moreover, analytic on H, and (3D.6) holds true.

Remark 3D.1 The characterizations of Theorem 3D.1 are illustrated in Example 3D.1 just below the end of the proof.

Proof.

Step 1 There exists a nonsingular matrix $P : n \times n$,

$$P = [p_{ij}] \text{ with inverse } P^{-1} = Q = [q_{ij}], \quad i, j = 1, \ldots, n, \tag{3D.8}$$

such that M is brought into its Jordan canonical form, that is,

$$M = PJP^{-1}, \quad J = \begin{bmatrix} J_{k_1}(\lambda_1) & & & 0 \\ & J_{k_2}(\lambda_2) & & \\ & & \ddots & \\ 0 & & & J_{k_r}(\lambda_r) \end{bmatrix}, \tag{3D.9}$$

where $J_k(\lambda)$ is a $k \times k$ matrix

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}, \quad k \geq 2; \;\; J_1(\lambda) = [\lambda], \text{ for } k = 1, \tag{3D.10}$$

with $k_1 + k_2 + \cdots + k_r = n$, while $\lambda_1 \cdots \lambda_r$ are the eigenvalues of M, not necessarily distinct. M is in particular diagonalizable if and only if $k_1 = \cdots = k_r = 1$.

Step 2 Let I denote the identity operator on X. Define via (3D.8) the operator

$$\Pi = IP = [p_{ij}I] = \begin{bmatrix} p_{11}I & \cdots & p_{1n}I \\ \vdots & & \vdots \\ p_{n1}I & \cdots & p_{nn}I \end{bmatrix} \in \mathcal{L}(H), \tag{3D.11}$$

which is boundedly invertible on H, with inverse via (3D.8)

$$\Pi^{-1} = IQ = [q_{ij}I] = \begin{bmatrix} q_{11}I & \cdots & q_{1n}I \\ \vdots & & \vdots \\ q_{n1}I & \cdots & q_{nn}I \end{bmatrix} \in \mathcal{L}(H). \tag{3D.12}$$

Then, from $M = PJP^{-1}$ in (3D.9), and from (3D.11) and (3D.12), we obtain via (3D.1)

$$A = \Pi \begin{bmatrix} \mathcal{A}J_{k_1}(\lambda_1) & 0 & \cdots & 0 \\ 0 & \mathcal{A}J_{k_2}(\lambda_2) & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \mathcal{A}J_{k_r}(\lambda_r) \end{bmatrix} \Pi^{-1}; \tag{3D.13}$$

$$\mathcal{A}J_k(\lambda) = \begin{bmatrix} \lambda\mathcal{A} & \mathcal{A} & & & 0 \\ & \lambda\mathcal{A} & \mathcal{A} & & \\ & & \ddots & & \\ & & & \ddots & \mathcal{A} \\ 0 & & & & \lambda\mathcal{A} \end{bmatrix}, \quad k \geq 2; \mathcal{A}J_1(\lambda) = [\lambda\mathcal{A}], k = 1. \tag{3D.14}$$

Step 3 From (3D.13) and (3D.14), we obtain that

$$\begin{cases} \text{the operator } A \text{ is the generator of a s.c. semigroup (in} \\ \text{particular, of a s.c. analytic semigroup) on } H = X^n \end{cases} \tag{3D.15}$$

if and only if

$$\begin{cases} \text{all diagonal operators } \mathcal{A}J_{k_1}(\lambda_1), \ldots, \mathcal{A}J_{k_r}(\lambda_r) \text{ are} \\ \text{generators of s.c. semigroups (in particular, of s.c. analytic} \\ \text{semigroups) on, respectively, } H_{k_1} \equiv X^{k_1}, \ldots, H_{k_r} \equiv X^{k_r}. \end{cases} \tag{3D.16}$$

(a) If M is diagonalizable, that is, $k_1 = k_2 = \cdots = k_r = 1$, then conditions (3D.16) specialize, via (3D.14) (on the right), to hold true if and only if

$$\begin{cases} \lambda_1\mathcal{A}, \ldots, \lambda_r\mathcal{A} \text{ are generators of s.c. semigroups} \\ \text{(in particular, of s.c. analytic semigroups) on } X. \end{cases} \tag{3D.17}$$

Finally, $\mathcal{A}$ is, by assumption, a strictly positive unbounded self-adjoint operator on X, thus with spectrum $\sigma(\mathcal{A})$ *not* contained in any finite interval of $\mathbb{R}^+ :\ \sup \sigma(\mathcal{A}) = +\infty$. Moreover, $\lambda_j\mathcal{A}$ is a normal operator on X with spectrum

$$\sigma(\lambda_j\mathcal{A}) = (\operatorname{Re} \lambda_j)\sigma(\mathcal{A}) + \sqrt{-1}(\operatorname{Im} \lambda_j)\sigma(\mathcal{A}). \tag{3D.18}$$

Combining these last properties (where in particular we use that $\mathcal{A}$ is unbounded) we conclude that statement (3D.17) holds true if and only if

$$\text{Re}\,\lambda_j \le 0 \quad (\text{in particular, Re}\,\lambda_j < 0 \text{ unless } \lambda_j = 0), \quad j = 1, \dots, r, \tag{3D.19}$$

that is, if and only if condition (3D.3) (in particular, condition (3D.5)) holds true. In this case, the s.c. semigroup (in particular, the s.c. analytic semigroup) $e^{(\lambda_j \mathcal{A})t}$ is given by

$$e^{(\lambda_j \mathcal{A})t} = \int_{\mu_1}^{\infty} e^{\lambda_j \mu t} E_\mu \, d\mu, \quad \mu_1 > 0, \tag{3D.20}$$

where $\{E_\mu\}$ is the resolution of the identity of $\mathcal{A}$. Then, from (3D.20), we obtain

$$\left\| e^{(\lambda_j \mathcal{A})t} \right\|_{\mathcal{L}(X)} \le C_j e^{(\text{Re}\,\lambda_j)\mu_1 t}, \quad t \ge 0. \tag{3D.21}$$

(b) In the general case where the matrix M is not diagonalizable, and so at least one $k_j \ge 2$, $j = 1, \dots, r$, then condition (3D.16) that each diagonal operator $\mathcal{A}J_{k_j}(\lambda_j)$ is the generator of a s.c. semigroup on its respective space H_{k_j} holds true if and only if Re $\lambda_j < 0$ as in (3D.7), in which case each said semigroup is, in fact, analytic on H_{k_j}. For details on this step, we refer to Example 3D.1 below, which deals with a Jordan cell.

Step 4 Under the necessary and sufficient condition in (3D.7), we return to $\mathcal{A}J_{k_\rho}(\lambda_\rho)$, $\rho = 1, \dots, r$, given by (3D.14), and obtain that $\mathcal{A}J_{k_\rho}(\lambda_\rho)$ likewise generates a s.c. semigroup on $H_{k_\rho} = X^{k_\rho}$ given by

$$e^{(\mathcal{A}J_{k_\rho}(\lambda_\rho))t} = \begin{bmatrix} e^{(\lambda_\rho \mathcal{A})t} & t\mathcal{A}e^{(\lambda_\rho \mathcal{A})t} & \dfrac{t^2}{2!}\mathcal{A}^2 e^{(\lambda_\rho \mathcal{A})t} & \cdots & \dfrac{t^{k-1}}{(k-1)!}\mathcal{A}^{k-1}e^{(\lambda_\rho \mathcal{A})t} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \dfrac{t^2}{2!}\mathcal{A}^2 e^{(\lambda_\rho \mathcal{A})t} \\ & 0 & & \ddots & t\mathcal{A}e^{(\lambda_\rho \mathcal{A})t} \\ & & & & e^{(\lambda_\rho \mathcal{A})t} \end{bmatrix}, \tag{3D.22}$$

so that $\forall\, \epsilon > 0$ there is M_{k_ρ} depending also on ϵ such that

$$\left\| e^{(\mathcal{A}J_{k_\rho}(\lambda_\rho))t} \right\|_{\mathcal{L}(H_k)} \le M_{k_\rho} e^{((\text{Re}\,\lambda_\rho)\mu_1 + \epsilon)t}, \quad t \ge 0. \tag{3D.23}$$

See more details in Example 3D.1 below.

Step 5 Finally, returning to (3D.13) we obtain that, under the conditions in (3D.7), the operator A generates a s.c. analytic semigroup e^{At} on H given by

$$e^{At} = \Pi \begin{bmatrix} e^{(\mathcal{A}J_{k_1}(\lambda_1))t} & & 0 \\ & e^{(\mathcal{A}J_{k_2}(\lambda_2))t} & \\ 0 & \ddots & \\ & & e^{(\mathcal{A}J_{k_r}(\lambda_r))t} \end{bmatrix} \Pi^{-1}, \tag{3D.24a}$$

and by virtue of (3D.23), we see that e^{At} satisfies the bound in (3D.6) and (3D.7).

In the diagonalizable case, (3D.24a) specializes to

$$e^{At} = \Pi \begin{bmatrix} e^{\lambda_1 \mathcal{A}t} & & 0 \\ & e^{\lambda_2 \mathcal{A}t} & \\ 0 & \ddots & \\ & & e^{\lambda_n \mathcal{A}t} \end{bmatrix} \Pi^{-1}. \tag{3D.24b}$$

□

Remark 3D.2 The above proof shows also that

$$\sigma(A) = \bigcup_{j=1}^{n} [\lambda_j \sigma(\mathcal{A})]. \tag{3D.25}$$

Example 3D.1 Let M be the simplest Jordan cell

$$M = J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad \text{so that } A = \mathcal{A}M = \begin{bmatrix} \lambda\mathcal{A} & \mathcal{A} \\ 0 & \lambda\mathcal{A} \end{bmatrix} : \mathcal{D}(A) \to Y, \tag{3D.26}$$

where λ is a complex number, $\mathcal{A}$ a positive, unbounded, self-adjoint operator on the Hilbert space X, and

$$Y \equiv X \times X, \quad \mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}). \tag{3D.27}$$

Analysis of this example illustrates the equivalence between condition (3D.16) and condition (3D.7) in the proof of Theorem 3D.1(b).

(1) The differential equation,

$$\dot{y}(t) = Ay(t), \text{ that is, } \begin{cases} \dot{y}_1(t) = \lambda\mathcal{A}y_1(t) + \mathcal{A}y_2(t), & (3D.28) \\ \dot{y}_2(t) = \lambda\mathcal{A}y_2 t, & (3D.29) \end{cases} \quad y(0) = y_0,$$

admits a unique solution, which is, in fact, explicitly given by

$$\begin{cases} y_1(t) = e^{(\lambda\mathcal{A})t} y_{10}(t) + t\mathcal{A}e^{(\lambda\mathcal{A})t} y_{20}, & (3D.30) \\ y_2(t) = e^{(\lambda\mathcal{A})t} y_{20}, & (3D.31) \end{cases}$$

if and only if

$$\operatorname{Re} \lambda \leq 0, \tag{3D.32}$$

in which case $e^{(\lambda\mathcal{A})t}$ is a s.c. semigroup on X, which – moreover – is analytic on X if and only if

$$\operatorname{Re} \lambda < 0. \tag{3D.33}$$

Indeed, under assumption (3D.32), we first solve equation (3D.29) and then substitute in (3D.28) and use the variation of parameter formula here.

(2) From (3D.30) and (3D.31), we have that

$$e^{At} = \begin{bmatrix} e^{(\lambda\mathcal{A})t} & t\mathcal{A}e^{(\lambda\mathcal{A})t} \\ 0 & e^{(\lambda\mathcal{A})t} \end{bmatrix} \tag{3D.34}$$

is the s.c. semigroup generated by the operator A in Y if and only if (3D.33) holds true. See Point 4 below.

(3) Under condition (3D.33), and only in this case, we have, moreover, that

$$\|Ae^{At}\|_{\mathcal{L}(Y)} \leq \frac{C_\lambda}{t}, \quad 0 < t, \tag{3D.35}$$

in which case e^{At} is, in addition, analytic on Y.

In fact, from (3D.26) and (3D.34) we have

$$Ae^{At} = \begin{bmatrix} (\lambda\mathcal{A})e^{(\lambda\mathcal{A})t} & t(\lambda\mathcal{A})\mathcal{A}e^{(\lambda\mathcal{A})t} + \mathcal{A}e^{(\lambda\mathcal{A})t} \\ 0 & (\lambda\mathcal{A})e^{(\lambda\mathcal{A})t} \end{bmatrix}. \tag{3D.36}$$

For $\operatorname{Re} \lambda < 0$ as in (3D.33) so that $e^{(\lambda\mathcal{A})t}$ is an analytic semigroup on X, we have the well-known estimate [Pazy, 1983]

$$\left\|(\lambda\mathcal{A})^n e^{(\lambda\mathcal{A})t}\right\|_{\mathcal{L}(X)} \leq \frac{c_\lambda}{t^n}, \quad 0 < t, \quad n = 1, 2, \ldots. \tag{3D.37}$$

Considering the "worst" term in (3D.36), we obtain by (3D.37)

$$\begin{aligned} \left\|t(\lambda\mathcal{A})\mathcal{A}e^{(\lambda\mathcal{A})t}\right\|_{\mathcal{L}(X)} &= \frac{t}{|\lambda|}\left\|(\lambda\mathcal{A})^2 e^{(\lambda\mathcal{A})t}\right\|_{\mathcal{L}(X)} \\ \text{(by (3D.37))} \qquad &\leq \frac{t}{|\lambda|}\frac{c_\lambda}{t^2} = \frac{C_\lambda}{t}, \quad 0 < t. \end{aligned} \tag{3D.38}$$

From (3D.38), we obtain (3D.35) via (3D.36), as desired.

(4) Returning to (3D.30), we see that

$$\text{for } \operatorname{Re} \lambda = 0, \quad \text{in particular for } \lambda = 0, \tag{3D.39}$$

so that $e^{(\lambda\mathcal{A})t}$ is a unitary group on X, we need $y_{20} \in \mathcal{D}(\mathcal{A})$ to claim that $y_1(t) \in X$. Thus, *under condition (3D.39), problem (3D.28), (3D.29) cannot be described by* *a s.c. semigroup on* Y and thus, in this case, the operator A does not generate a s.c. semigroup on Y. We conclude:

Proposition 3D.2 *The operator A in (3D.26) generates a s.c. semigroup e^{At} on Y, given in fact by (3D.34), if and only if condition (3D.33) holds true, in which case e^{At} is, moreover, analytic on Y, and uniformly stable here.* □

This result can be readily generalized for the operator $\mathcal{A}J_k(\lambda)$ in (3D.14), and this generalization leads to the equivalence between condition (3D.16) and condition (3D.7) in the proof of Theorem 3D.1(b). The above proposition expresses this equivalence in the special case of the operator A in (3D.26).

Application #1: A Stable, Analytic, Contraction, s.c Semigroup Arising from a Thermo-Elastic PDE Problem with Simplified "Hinged" Homogeneous BC and Dirichlet Thermal BC

We modify the thermo-elastic problem (3.12.1) in Section 3.12 by taking $\sigma \equiv 0$ in Eqn. (3.12.1b), an inessential simplification, and then $B_1 \equiv 0$ and $\alpha = 0$ in the BC (3.12.1d). The case $B_1 \neq 0$ will be taken up below in Application #2. Moreover, we take homogeneous Dirichlet BC for θ, instead of (3.12.1e). This is a serious simplification. Thus, we consider the following homogeneous problem on any bounded $\Omega \subset R^n$ with smooth boundary Γ:

$$\begin{cases} w_{tt} + \Delta^2 w + \alpha\Delta\theta = 0 & \text{in } Q; & \text{(3D.40a)} \\ \theta_t - \eta\Delta\theta - \alpha\Delta w_t = 0 & \text{in } Q; & \text{(3D.40b)} \\ w(0,\,\cdot\,) = w_0,\ w_t(0,\,\cdot\,) = w_1, \theta(0,\,\cdot\,) = \theta_0 & \text{in } \Omega; & \text{(3D.40c)} \\ w = \Delta w \equiv 0 & \text{in } \Sigma; & \text{(3D.40d)} \\ \theta = 0 & \text{in } \Sigma. & \text{(3D.40e)} \end{cases}$$

Abstract Model We define the positive self-adjoint operator

$$\mathcal{A}h = -\Delta h; \quad \mathcal{D}(\mathcal{A}) = H^2(\Omega) \cap H^1(\Omega) \to L_2(\Omega), \tag{3D.41}$$

so that

$$\mathcal{A}^2 h = \Delta^2 h, \quad \mathcal{D}(\mathcal{A}^2) = \{h \in H^4(\Omega) : h|_\Gamma = \Delta h|_\Gamma = 0\} \to L_2(\Omega). \tag{3D.42}$$

Then the abstract model of problem (3D.40) is

$$\begin{cases} w_{tt} + \mathcal{A}^2 w - \alpha\mathcal{A}\theta = 0, & \text{(3D.43)} \\ \theta_t + \eta\mathcal{A}\theta + \alpha\mathcal{A}w_t = 0, & \text{(3D.44)} \end{cases}$$

or in first-order form,

$$\dot{y} = A_e y; \quad A_e = \begin{bmatrix} 0 & I & 0 \\ -\mathcal{A}^2 & 0 & \alpha\mathcal{A} \\ 0 & -\alpha\mathcal{A} & -\eta\mathcal{A} \end{bmatrix} : Y \supset \mathcal{D}(A_e) \to Y; \quad y = \begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix}; \tag{3D.45}$$

$$Y \equiv \mathcal{D}(\mathcal{A}) \times L_2(\Omega) \times L_2(\Omega); \quad \mathcal{D}(A_e) = \mathcal{D}(\mathcal{A}^2) \times \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}). \tag{3D.46}$$

[The subindex "e" stands to remind us that A_e is defined on the energy space in $\{w, w_t\}$.] Then:

The operator A_e in (3D.45) is the generator of a s.c. semigroup (in particular, of a s.c. analytic semigroup) on Y if and only if the operator

$$\begin{aligned} A &= \begin{bmatrix} 0 & \mathcal{A} & 0 \\ -\mathcal{A} & 0 & \alpha\mathcal{A} \\ 0 & -\alpha\mathcal{A} & -\eta\mathcal{A} \end{bmatrix} \\ &= \left\{ \begin{bmatrix} \mathcal{A} & & 0 \\ & I & \\ 0 & & I \end{bmatrix} \begin{bmatrix} 0 & I & 0 \\ -\mathcal{A}^2 & 0 & \alpha\mathcal{A} \\ 0 & -\alpha\mathcal{A} & -\eta\mathcal{A} \end{bmatrix} \begin{bmatrix} \mathcal{A}^{-1} & & 0 \\ & I & \\ 0 & & I \end{bmatrix} \right\} \end{aligned} \tag{3D.47}$$

generates a s.c. semigroup (in particular, a s.c. analytic semigroup) on the space

$$H = X \times X \times X, \qquad X = L_2(\Omega). \tag{3D.48}$$

The operator A in (3D.47) and (3D.48) satisfies

$$A = \mathcal{A}M, \quad M = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \alpha \\ 0 & -\alpha & -\eta \end{bmatrix}. \tag{3D.49}$$

The characteristic equation of M is

$$f(\lambda) = \det[\lambda - M] = \lambda^3 + \eta\lambda^2 + (\alpha^2 + 1)\lambda + \eta = 0, \tag{3D.50}$$

whose roots have all negative real parts:

$$\lambda_1 = -a < 0, \quad \lambda_{2,3} = \frac{(a-\eta) \pm \sqrt{(a-\eta)^2 - \frac{4\eta}{a}}}{2}, \tag{3D.51}$$

where $0 < a < \eta$. In fact, there is at least one real root, which is indeed negative: $\lambda_1 = -a < 0$, since $f(0) = \eta > 0$ and $f(\lambda) \to -\infty$ as $\lambda \to -\infty$. Moreover, since $f'(\lambda) > 0$ on $(-\infty, -\eta]$, and $f(-\eta) < 0$, there is no root in $(-\infty, -\eta]$. The other two roots are complex conjugate.

Conclusion By virtue of (3D.49)–(3D.51), we can apply Theorem 3D.1 and conclude that:

Theorem 3D.3

(i) The operator A in H [respectively, the operator A_e in Y] generates a s.c. analytic semigroup e^{At} on H [respectively, $e^{A_e t}$ on Y];

(ii) such semigroups are uniformly stable: There are constants $C \geq 1$, $\omega > 0$ such that

$$\|e^{At}\|_{\mathcal{L}(H)} = \|e^{A_e t}\|_{\mathcal{L}(Y)} \leq Ce^{-(\omega+\epsilon)t}, \tag{3D.52}$$

where $-\omega = \mu_1 \sup_i(\text{Re}\, \lambda_i) < 0$ by (3D.6b).

Remark 3D.3 The above s.c. analytic semigroups e^{At} on H and $e^{A_e t}$ on Y are, moreover, *contraction*: The operators A in (3D.47) and A_e in (3D.46) are maximal dissipative; or, alternatively, the H-adjoint A^* and the Y-adjoint A_e^* are likewise dissipative, whereas A and A_e are closed, as in the cases considered in Sections 3.11 and 3.12.

Application #2: A Stable, Analytic, Contraction, s.c. Semigroup Arising from a Thermo-Elastic PDE Problem with Hinged Homogeneous BC and Thermal Dirichlet BC

Let Ω be a two-dimensional bounded open region with smooth boundary Γ. Refining model (3D.40), we now consider the same problem with hinged boundary conditions:

$$\begin{cases} w_{tt} + \Delta^2 w + \alpha\Delta\theta = 0 & \text{in } Q; & \text{(3D.53a)} \\ \theta_t - \eta\Delta\theta - \alpha\Delta w_t = 0 & \text{in } Q; & \text{(3D.53b)} \\ w(0,\,\cdot) = w_0,\ w_t(0,\,\cdot) = w_1, \theta(0,\,\cdot) = \theta_0 & \text{in } \Omega; & \text{(3D.53c)} \\ w \equiv 0, \quad \Delta w + B_1 w \equiv 0 & \text{on } \Sigma; & \text{(3D.53d)} \\ \theta \equiv 0 & \text{on } \Sigma, & \text{(3D.53e)} \end{cases}$$

where B_1 is the boundary operator introduced in (3.12.1f)

$$B_1 w = (1-\mu)\left[-\frac{\partial^2 w}{\partial\tau^2} - c(x)\,\frac{\partial w}{\partial\nu}\right] = b(x)\,\frac{\partial w}{\partial\nu}, \tag{3D.54}$$

using (3D.53d) (left), and where $b(x) = -(1-\mu)c(x) \in L_\infty(\Gamma)$.

Abstract Model Let $\mathcal{A}$, $\mathcal{A}^2$ be the positive, self-adjoint operators defined in (3D.41), (3D.42). Let now G_2 be the Green operator introduced in Eqn. (3.6.3) of Section 3.6:

$$y = G_2 v \iff \{\Delta^2 y = 0 \text{ in } \Omega;\ y|_\Gamma = 0;\ \Delta y|_\Gamma = v\}, \tag{3D.55}$$

so that Eqns. (3.6.6), (3.6.7) hold true [where the operator $\mathcal{A}^{\frac{1}{2}}$ of Section 3.6 has been denoted by $\mathcal{A}$ in the present Appendix 3D]:

$$G_2 = -\mathcal{A}^{-1}D,\ D: \text{continuous } L_2(\Gamma) \to \mathcal{D}\big(\mathcal{A}^{\frac{1}{4}-\epsilon}\big), \quad \epsilon > 0. \tag{3D.56}$$

Proceeding as usual from (3D.53) by use of (3D.55), we find that the abstract model of the PDE problem (3D.53) is

$$\begin{cases} w_{tt} + \mathcal{A}^2(w + G_2B_1w) - \alpha\mathcal{A}\theta = 0, & \text{(3D.57)} \\ \theta_t + \eta\mathcal{A}\theta + \alpha\mathcal{A}w_t = 0, & \text{(3D.58)} \end{cases}$$

or in first-order form,

$$\dot{y} = \tilde{A}_e y; \quad \tilde{A}_e : Y \supset \mathcal{D}(\tilde{A}_e) \to Y, \quad y = \begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix}, \tag{3D.59}$$

on the same space $Y = \mathcal{D}(\mathcal{A}) \times L_2(\Omega) \times L_2(\Omega)$ as in (3D.46), with

$$\tilde{A}_e = \begin{bmatrix} 0 & I & 0 \\ -\mathcal{A}^2[I + G_2B_1] & 0 & \alpha\mathcal{A} \\ 0 & -\alpha\mathcal{A} & -\eta\mathcal{A} \end{bmatrix} = A_e[I + P] = A_e + \tilde{P}; \tag{3D.60}$$

$\mathcal{D}(\tilde{A}_e) = \{[w_1, w_2, \theta] : w_1 \in \mathcal{D}(\mathcal{A}) : w_1 + G_2B_1w_1 \in \mathcal{D}(\mathcal{A}^2);\ w_2 \in \mathcal{D}(\mathcal{A}), \theta \in \mathcal{D}(\mathcal{A})\}$, where the operator A_e is given by (3D.45), while

$$P = \begin{bmatrix} G_2B_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad G_2B_1 = -\mathcal{A}^{-1}D\left(b\frac{\partial}{\partial\nu}\right) : H^{\frac{3}{2}+\epsilon}(\Omega) \to \mathcal{D}(\mathcal{A}), \tag{3D.61}$$

recalling (3D.56) and (3D.54);

$$\tilde{P} = \begin{bmatrix} 0 & 0 & 0 \\ -\mathcal{A}^2G_2B_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \equiv A_eP; \tag{3D.62a}$$

$$-\mathcal{A}^2G_2B_1 = \mathcal{A}D\left(b\frac{\partial}{\partial\nu}\right) : H^{\frac{3}{2}+\epsilon}(\Omega) \to \left[\mathcal{D}\big(\mathcal{A}^{\frac{3}{4}+\epsilon}\big)\right]'. \tag{3D.62b}$$

[Recall from (3.1.9) that $\partial/\partial\nu = D^*\mathcal{A}$ in the present notation.] From (3D.46) we obtain

$$\mathcal{D}((-A_e)^r) = \mathcal{D}(\mathcal{A}^{1+r}) \times \mathcal{D}(\mathcal{A}^r) \times \mathcal{D}(\mathcal{A}^r), \quad 0 \le r \le 1. \tag{3D.63}$$

Lemma 3D.4 *With reference to (3D.62), and for $y = [y_1, y_2, y_3] \in Y$, Y defined by (3D.46), we obtain*

(i)

$$A_e^{-1}\tilde{P}y = Py = \begin{bmatrix} G_2 B_1 y_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\mathcal{A}^{-\frac{5}{4}+\epsilon}\mathcal{A}^{\frac{1}{4}-\epsilon} D B_1 y_1 \\ 0 \\ 0 \end{bmatrix} \tag{3D.64}$$

$$\in \mathcal{D}\big((-A_e)^{\frac{1}{4}-\epsilon}\big); \tag{3D.65}$$

(ii)

$$A_e^{-s}\tilde{P} \in \mathcal{L}(Y), \qquad s = \frac{3}{4} + \epsilon. \tag{3D.66}$$

Proof. (i) The identities in (3D.64) follow from $\tilde{P} = A_e P$ in (3D.62) and from (3D.61), (3D.56). To show (3D.65), we must establish, according to (3D.63) with $r = 1/4 - \epsilon$ and (3D.64), that

$$\mathcal{A}^{-\frac{5}{4}+\epsilon}\mathcal{A}^{\frac{1}{4}-\epsilon} D B_1 y_1 \in \mathcal{D}\big(\mathcal{A}^{\frac{5}{4}-\epsilon}\big), \quad y_1 \in \mathcal{D}(\mathcal{A}). \tag{3D.67}$$

Indeed, (3D.67) holds true. In fact, let $y \in Y$. Then $y_1 \in \mathcal{D}(\mathcal{A}) \subset H^2(\Omega)$ (by (3D.41)) implies that $\partial y_1/\partial\nu \in H^{\frac{1}{2}}(\Gamma)$; hence $B_1 y_1 \in H^{\frac{1}{2}}(\Gamma) \subset L_2(\Gamma)$ by (3D.54); finally, we invoke $\mathcal{A}^{\frac{1}{4}-\epsilon} D \in \mathcal{L}(L_2(\Gamma); L_2(\Omega))$ from (3D.56), and thus (3D.67) and (3D.65) are proved.

(ii) (3D.66) follows at once from (3D.65). □

Theorem 3D.5 *With reference to (3D.60) we have that the operator $\tilde{A}_e$ is the generator of a s.c. analytic semigroup $e^{\tilde{A}_e t}$ on Y.*

Proof. By (3D.66) we have $\tilde{P}^* A_e^{*-s} \in \mathcal{L}(Y)$, that is, $\tilde{P}^*$ is A_e^{*s}-bounded ($\tilde{P}^*$ is relatively bounded with respect to A_e^{*s}). But A_e, and hence A_e^*, are generators of s.c. analytic semigroups on Y, and, moreover, $s < 1$ by (3D.66). These two latter ingredients then permit us to invoke a standard perturbation result [Pazy, 1983, p. 81] and conclude that $[A_e^* + \tilde{P}^*]$ generates a s.c analytic semigroup. Then, the same conclusion holds for $\tilde{A}_e = [A_e + \tilde{P}]$, as desired. □

However, to show that the s.c. analytic semigroup $e^{\tilde{A}_e t}$ of Theorem 3D.5 is, moreover, contraction and uniformly stable, it is more convenient to rewrite the basic elastic operator as

$$-\mathbb{A} = -\mathcal{A}^2[I + G_2 B_1], \tag{3D.68}$$

where $\mathbb{A}$ is the same operator defined by Eqn. (3.12.3) (with a different notation):

$$\mathbb{A}h = \Delta^2 h; \quad \mathcal{D}(\mathbb{A}) = \{h \in H^4(\Omega) \cap H_0^1(\Omega) : \Delta h + (1-\mu)B_1 h = 0 \text{ on } \Gamma\}, \tag{3D.69}$$

which we have seen in Appendix 3C, Proposition 3C.4 to be positive self-adjoint on $L_2(\Omega)$. Then, via (3D.68), we rewrite $\tilde{A}_e$ in (3D.60) as

$$\tilde{A}_e = \begin{bmatrix} 0 & I & 0 \\ -\mathbb{A} & 0 & \alpha\mathcal{A} \\ 0 & -\alpha\mathcal{A} & -\eta\mathcal{A} \end{bmatrix} : Y \supset \mathcal{D}(\tilde{A}_e) \to Y, \tag{3D.70}$$

while, via (3D.69) and Appendix 3A, and (3D.41), we have

$$\mathcal{D}\big(\mathbb{A}^{\frac{1}{2}}\big) = \mathcal{D}(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega); \qquad \mathcal{D}(\tilde{A}_e) = \mathcal{D}(\mathbb{A}) \times \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}), \tag{3D.71}$$

so that Y in (3D.46) may be rewritten as

$$Y = \mathcal{D}\big(\mathbb{A}^{\frac{1}{2}}\big) \times L_2(\Omega) \times L_2(\Omega). \tag{3D.72}$$

Having rewritten the abstract model as in (3D.70) through (3D.71), we can then obtain the counterpart of Proposition 3.12.4 and Proposition 3.12.5 to the present case of problem (3D.53) with Dirichlet BC for θ as in (3D.40e), by essentially the same proofs.

Proposition 3D.6 *With reference to the operator $\tilde{A}_e$ in (3D.70), we have:*

(i) $\tilde{A}_e$ is the generator of a s.c. contraction semigroup $e^{\tilde{A}_e t}$ on Y in (3D.72).
(ii) The inverse $\tilde{A}_e^{-1}$ is compact on Y, and thus the spectrum $\sigma(\tilde{A}_e)$ of $\tilde{A}_e$ is only a point spectrum.
(iii) There is no spectrum of $\tilde{A}_e$ in the closed right-hand complex plane $\mathbb{C}^+ = \{\lambda :$ $\text{Re}\,\lambda \geq 0\}$; that is, $\sigma(\tilde{A}_e) \cap \mathbb{C}^+ = \emptyset$.
(iv) For a constant $\omega > 0$, we have

$$\sup \text{Re}\,\sigma(\tilde{A}_e) = -\omega < 0. \tag{3D.73}$$

(v) The s.c. contraction analytic semigroup $e^{\tilde{A}_e t}$ is, moreover, uniformly stable: With ω as in (3D.73) and $\epsilon > 0$, there is a constant $M \geq 1$, depending on $[-\omega + \epsilon]$, such that

$$\|e^{\tilde{A}_e t}\|_{\mathcal{L}(Y)} \leq M e^{-(\omega-\epsilon)t}, \quad t \geq 0, \tag{3D.74}$$

or equivalently

$$E(t; y_0) \leq M e^{-(\omega-\epsilon)t} E(0; y_0), \qquad y_0 = [w_0, w_1, \theta_0], \tag{3D.75}$$

where $E(t; y_0)$ is the energy of problem (3D.53) and is defined as in (3.12.56).

Remark 3D.4 An alternative proof, by energy methods, of the uniform stability property (3D.74) will be given in Appendix 3J, without using analyticity of the s.c. semigroup $e^{\tilde{A}_e t}$.

Application #3: A Special Case of Structural Damping in Appendix 3B

The above Theorem 3D.1 recovers the special case of Appendix 3B, Theorem 3B.1 when $\mathcal{B} = \rho\mathcal{A}^\alpha$, $\rho > 0$, $\alpha = 1/2$ in (3B.9), thus concerning the operator $A_{\rho\alpha}, \alpha = 1/2$, that is,

$$A_{\rho\frac{1}{2}} = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\rho\mathcal{A}^{\frac{1}{2}} \end{bmatrix} : \mathcal{D}(A_{\rho\frac{1}{2}}) \to E = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times X \tag{3D.76}$$

on the energy space E, with $\mathcal{A}$ (strictly) positive self-adjoint on the Hilbert space X. The operator $A_{\rho\frac{1}{2}}$ generates a s.c. analytic semigroup on E if and only if the operator

$$A = \begin{bmatrix} 0 & \mathcal{A}^{\frac{1}{2}} \\ -\mathcal{A}^{\frac{1}{2}} & -\rho\mathcal{A}^{\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} \mathcal{A}^{\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\rho\mathcal{A}^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \mathcal{A}^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} \tag{3D.77}$$

$$: \mathcal{D}(A) = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \to H = X \times X \tag{3D.78}$$

generates a s.c. analytic semigroup on $H = X \times X$. From (3D.77) we have

$$A = \mathcal{A}^{\frac{1}{2}} M, \quad M = \begin{bmatrix} 0 & 1 \\ -1 & -\rho \end{bmatrix}, \tag{3D.79}$$

with $\mathcal{A}^{\frac{1}{2}}$ (strictly) positive self-adjoint on X. The characteristic equation of M and its characteristic roots are given below, along with the diagonalizing matrix P [see (3D.8)]:

$$\det[\lambda - M] = \lambda^2 + \rho\lambda + 1 = 0, \quad \lambda_{1,2} = \frac{-\rho \pm \sqrt{\rho^2 - 4}}{2}, \quad P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}. \tag{3D.80}$$

For $\rho \neq 2$, we have the s.c. analytic semigroup

$$e^{At} = \Pi \begin{bmatrix} e^{\lambda_1 \mathcal{A}^{\frac{1}{2}} t} & 0 \\ 0 & e^{\lambda_2 \mathcal{A}^{\frac{1}{2}} t} \end{bmatrix} \Pi^{-1}, \quad \Pi = \begin{bmatrix} I & I \\ \lambda_2 I & \lambda_2 I \end{bmatrix}. \tag{3D.81}$$

For $\rho = 2$, we are in the situation of Example 3D.1 with $\lambda = -\rho/2$.

Conclusion Theorem 3D.1 applies by virtue of (3D.80) and we conclude that the operator $A_{\rho\frac{1}{2}}$ in (3D.76) [respectively, the operator A in (3D.77), or (3D.79)] generates a s.c. analytic semigroup on the energy space E [respectively, on the space $H = X \times X$], which, moreover, is uniformly stable here:

$$\left\| e^{A_{\rho\frac{1}{2}} t} \right\|_{\mathcal{L}(E)} = \| e^{At} \|_{\mathcal{L}(H)} \leq C e^{-(\omega + \epsilon)t}, \tag{3D.82}$$

where $\omega = \mu_1 \rho/2$ for $0 < \rho \leq 2$, and $\omega = \mu_1 \lambda_2$ for $\rho > 2$; see (3D.6b).

Application #4

(Complementing (3D.77)). If the operator A in (3D.77), say with $\rho = 1$, is replaced by the operator A_ϵ on $H = X \times X$:

$$A_\epsilon = \begin{bmatrix} \epsilon \mathcal{A}^{\frac{1}{2}} & \mathcal{A}^{\frac{1}{2}} \\ -\mathcal{A}^{\frac{1}{2}} & -\mathcal{A}^{\frac{1}{2}} \end{bmatrix} = \mathcal{A}^{\frac{1}{2}} M_\epsilon, \quad M_\epsilon = \begin{bmatrix} \epsilon & 1 \\ -1 & -1 \end{bmatrix}, \tag{3D.83}$$

where $\det(\lambda - M_\epsilon) = \lambda^2 + (1-\epsilon)\lambda + (1-\epsilon) = 0$, we have that: (i) For $0 \le \epsilon < 1$, the matrix M_ϵ has two distinct, complex conjugate roots λ_1 and λ_2, with negative real part. Thus, by Theorem 3D.1(a), A_ϵ generates a s.c. analytic semigroup on H. (ii) For $\epsilon = 1$, M_ϵ has the double root $\lambda_1 = \lambda_2 = 0$; thus, by Theorem 3D.1(b), A_ϵ does not generate a s.c. semigroup on H.

3E Analyticity of the s.c. Semigroups Arising from Abstract Thermo-Elastic Equations. First Proof

3E.1 Abstract Thermo-Elastic Equations. Analyticity

Beginning with the present Appendix 3E, and continuing through Appendices 3F, 3G, 3H, and 3I, we proceed to provide the required proofs that the s.c. contraction semigroups corresponding to the thermo-elastic problems in Sections 3.11.1–3.11.2, 3.12, and 3.13 are, in fact, *analytic*. The present Section 3E deals with an *abstract* thermo-elastic model, which, in particular, encompasses the concrete thermo-elastic PDEs (with hinged BC as in Appendix 3D, Application #1, Eqn. (3D.40) as well as) with clamped mechanical BC and either Dirichlet or Neumann (Robin) thermal BC, as the ones considered in Section 3.11.1 and Section 3.11.2, respectively. Additional examples are provided at the end of this appendix.

Mathematical Setting Let X be a Hilbert space with norm $\| \ \|_X$ and inner product $(\cdot\ ,\ \cdot)_X$. On it, we consider two operators $\mathcal{A}$ and $\mathcal{B}$ subject to the following set of assumptions:

(H.1) $\mathcal{A} : X \supset \mathcal{D}(\mathcal{A}) \to X$ and $\mathcal{B} : X \supset \mathcal{D}(\mathcal{B}) \to X$ are two strictly positive self-adjoint operators, with compact resolvent;

(H.2)

$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset \mathcal{D}(\mathcal{B}), \quad \text{equivalently } \mathcal{B}\mathcal{A}^{-\frac{1}{2}} \in \mathcal{L}(X) \tag{3E.1}$$

(the implications $\Rightarrow$ follows by the closed graph theorem);

(H.3) there is a constant $c > 0$ such that

$$c\left\|\mathcal{A}^{\frac{1}{4}}x\right\|_X \le \left\|\mathcal{B}^{\frac{1}{2}}x\right\|_X, \quad \forall\, x \in \mathcal{D}(\mathcal{A}^{\frac{1}{4}}). \tag{3E.2}$$

Remark 3E.1 By Lowner's theorem [Kato, 1966, Corollary 7.1, p. 146; Xia, 1983, p. 5], condition (3E.1) of assumption (H.2) implies:

(H.2w)

$$\mathcal{D}(\mathcal{A}^{\frac{1}{4}}) \subset \mathcal{D}(\mathcal{B}^{\frac{1}{2}}), \quad \text{equivalently } \mathcal{B}^{\frac{1}{2}}\mathcal{A}^{-\frac{1}{4}} \in \mathcal{L}(X),$$
$$\text{equivalently } \|\mathcal{B}^{\frac{1}{2}}x\|_X \leq C\|\mathcal{A}^{\frac{1}{4}}x\|_X, \quad \forall\, x \in \mathcal{D}(\mathcal{A}^{\frac{1}{4}}). \tag{3E.3}$$

Thus, assumption (H.3) = (3E.2) reverses inequality (3E.3) – which is implied by assumption (H.2) in (3E.1) – however, in a relaxed form, that is, only for smoother elements $x \in \mathcal{D}(\mathcal{A}^{\frac{1}{4}})$, not necessarily for all $x \in \mathcal{D}(\mathcal{B}^{\frac{1}{2}})$. Thus, (3E.2) is generally weaker than the requirement $\mathcal{D}(\mathcal{B}^{\frac{1}{2}}) \subset \mathcal{D}(\mathcal{A}^{\frac{1}{4}})$. This distinction is important in the case of Example 3E.4.2 at the end of Appendix 3E (clamped BC on the mechanical variable; Neumann BC on the thermal variable), which refers to the thermo-elastic problem of Section 3.11.2. Notice that (H.1), (H.2w), and (H.3) yield that $\mathcal{A}^{-\frac{1}{4}}\mathcal{B}\mathcal{A}^{-\frac{1}{4}}$ is a bounded, boundedly invertible, self-adjoint operator on $\mathcal{L}(X)$.

Abstract Thermo-Elastic System The abstract thermo-elastic system considered in this appendix is

$$\begin{cases} w_{tt} + \mathcal{A}w - \mathcal{B}\theta = 0, & (3E.4)\\ \theta_t + \mathcal{B}\theta + \mathcal{B}w_t = 0, & (3E.5)\end{cases}$$

or, in first-order form, with $y(t) = [w(t), w_t(t), \theta(t)]$,

$$\dot{y} = Ay; \qquad y(0) = y_0 = [w_0, w_1, \theta_0] \in Y; \tag{3E.6}$$

$$A = \begin{bmatrix} 0 & I & 0 \\ -\mathcal{A} & 0 & \mathcal{B} \\ 0 & -\mathcal{B} & -\mathcal{B}\end{bmatrix} : Y \supset \mathcal{D}(A) \to Y; \quad A^* = \begin{bmatrix} 0 & -I & 0 \\ \mathcal{A} & 0 & -\mathcal{B} \\ 0 & \mathcal{B} & -\mathcal{B}\end{bmatrix};$$
$$\mathcal{D}(A^*) = \mathcal{D}(A), \tag{3E.7}$$

where A^* is the Y-adjoint of A; moreover,

$$Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times X \times X; \quad \mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(\mathcal{B}), \tag{3E.8}$$

by use of (H.2) = (3E.1). The following result is a straightforward application of the Lumer–Phillips theorem [Pazy, 1983, p. 14] directly or of its standard corollary [Pazy, 1983, p. 15] involving A and A^*. It contains Propositions 3.11.1.2 and 3.11.2.1 of Sections 3.11.1 and 3.11.2, which are specializations thereof.

Remark 3E.2 If one substitutes $\mathcal{B}\theta = -\mathcal{B}w_t - \theta_t$ from (3E.5) into (3E.4), one obtains the equation $w_{tt} + \mathcal{A}w + \mathcal{B}w_t = -\theta_t$, which is "structurally damped" in the mechanical variable w; refer to Appendix 3B. A proof of Theorem 3E.2 below, specifically based on this observation will be given in Appendix 3F.

Proposition 3E.1 *Assume (H.1), without the compact resolvent part, and (H.2) = (3E.1). Then:*

(i) *The operators A and A^* in (3E.7) are dissipative,*

$$\operatorname{Re}(Ax, x)_Y = \operatorname{Re}(A^*x, x)_Y = -(\mathcal{B}x_3, x_3)_X, \quad \forall\, x = [x_1, x_2, x_3] \in \mathcal{D}(A) = \mathcal{D}(A^*), \tag{3E.9}$$

and, in fact, maximally dissipative.

(ii) *Thus, A and A^* generate s.c. contraction semigroups e^{At} and e^{A^*t} on Y, $t \geq 0$.*

(iii) *The operator obtained from A in (3E.7) by omitting the bottom-right corner element $-\mathcal{B}$ is skew-adjoint on Y, and hence it generates a s.c. unitary group on Y.*

(iv) *The inverse of A is given by*

$$A^{-1} = \begin{bmatrix} -\mathcal{A}^{-1}\mathcal{B} & -\mathcal{A}^{-1} & -\mathcal{A}^{-1} \\ I & 0 & 0 \\ -I & 0 & -\mathcal{B}^{-1} \end{bmatrix} : Y \to \mathcal{D}(A) \tag{3E.10}$$

and is compact as an operator of $\mathcal{L}(Y)$, if and only if both $\mathcal{A}^{-1}$ and $\mathcal{B}^{-1}$ are compact on X, in which case the spectrum $\sigma(A)$ of A is only a point spectrum.

(v) *There is no point spectrum of A, or of A^*, on the imaginary axis, and hence there is no residual spectrum of A, or of A^*, on it.*

Proof. For (i)–(iii), see Proposition 3.11.1.2 or Proposition 3.11.2.1.

For (v), see Proposition 3.11.1.3, parts (i) and (ii), or Proposition 3.11.2.2, parts (i) and (ii), with the remark that the null space $\mathcal{N}(ir - A^*) = \{0\}$ in case $\overline{\operatorname{Range}(-ir - A)} = Y$, $r \in \mathbb{R}$. $\square$

The main result of the present appendix is the following Theorem 3E.2, which gives sufficient conditions under which the s.c. contraction semigroup e^{At} is, moreover, analytic on Y, $t > 0$.

Theorem 3E.2 [Lasiecka, Triggiani, 1998(a)] *Assume hypotheses (H.1), (H.2) = (3E.1), and (H.3) = (3E.2). Then, the s.c. contraction semigroup e^{At} guaranteed by Proposition 3E.1(ii) is analytic on Y, $t > 0$.*

3E.2 Proof of Theorem 3E.2

3E.2.1 General Strategy and Preliminaries

General Strategy With reference to Y in (3E.8), let $f_0 \in Y$ be arbitrary:

$$f_0 = [u_0, v_0, \theta_0] \in Y \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times X \times X. \tag{3E.11}$$

With reference to the operator A in (3E.7), let ω be real, $\omega \in \mathbb{R}$, and define

$$y(\omega) = \begin{bmatrix} u(\omega) \\ v(\omega) \\ \theta(\omega) \end{bmatrix} = (i\omega I - A)^{-1} f_0 = R(i\omega, A) f_0 \in \mathcal{D}(A), \tag{3E.12}$$

where the resolvent is well defined on the imaginary axis by Proposition 3E.1(iv), (v).

Our goal is to show that the following characterization of analyticity holds true for the s.c. semigroup e^{At} guaranteed by Proposition 3E.1, whose generator A has no spectrum on the imaginary axis, as asserted by (what should be a well-known result) Theorem 3E.3 at the end of this appendix: that there exists a constant $C > 0$ such that for all $\omega \in \mathbb{R}$, with, say, $|\omega| \geq 1$, the following uniform estimate is fulfilled:

$$\left\| \begin{bmatrix} u(\omega) \\ v(\omega) \\ \theta(\omega) \end{bmatrix} \right\|_Y = \|y(\omega)\|_Y = \|R(i\omega, A) f_0\|_Y \leq \frac{C}{|\omega|} \|f_0\|_Y. \tag{3E.13}$$

Once estimate (3E.13) has been established, we then appeal to the aforementioned Theorem 3E.3 and conclude that the s.c. contraction semigroup e^{At} of Proposition 3E.1 is analytic on $Y, t > 0$. Thus, the goal is to prove (3E.13). To this end, we shall pursue the following strategy, which consists in proving the following three simultaneous estimates for the components of the vector $y(\omega) = [u(\omega), v(\omega), \theta(\omega)]$ in (3E.12): that for all $\epsilon > 0$, there exists a constant $C_\epsilon > 0$, such that for all $\omega \in \mathbb{R}$ with $|\omega| \geq 1$, the vector $y(\omega) = [u(\omega), v(\omega), \theta(\omega)]$ in (3E.12) satisfies

$$\begin{cases} \|u(\omega)\|^2_{\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)} \leq \epsilon \|y(\omega)\|^2_Y + C_\epsilon \left\| \frac{f_0}{\omega} \right\|^2_Y, & \text{(3E.14)} \\ \|v(\omega)\|^2_X \leq \epsilon \|y(\omega)\|^2_Y + C_\epsilon \left\| \frac{f_0}{\omega} \right\|^2_Y, & \text{(3E.15)} \\ \|\theta(\omega)\|^2_X \leq \epsilon \|y(\omega)\|^2_Y + C_\epsilon \left\| \frac{f_0}{\omega} \right\|^2_Y. & \text{(3E.16)} \end{cases}$$

Henceforth, we shall drop the dependence on ω and simply write $[u, v, \theta] = y$. Estimates (3E.14), (3E.15), and (3E.16) are proved below, in Proposition 3E.2.7.1, Proposition 3E.2.6.1, and Proposition 3E.2.4.1, respectively. Clearly, summing up estimates (3E.14) through (3E.16) (once established) yields the final desired estimate (3E.13) with constant $C = [3C_\epsilon/(1 - 3\epsilon)]^{\frac{1}{2}}$, which then proves Theorem 3E.2.

Preliminaries By (3E.7) for A, we obtain explicitly from (3E.12):

$$(i\omega I - A) \begin{bmatrix} u \\ v \\ \theta \end{bmatrix} = \begin{bmatrix} i\omega u - v \\ i\omega v + \mathcal{A}u - \mathcal{B}\theta \\ i\omega\theta + \mathcal{B}v + \mathcal{B}\theta \end{bmatrix} = \begin{bmatrix} u_0 \\ v_0 \\ \theta_0 \end{bmatrix} = f_0, \tag{3E.17}$$

or, upon dividing by ω, $|\omega| \geq 1$,

$$\begin{cases} I: \quad iu - \dfrac{v}{\omega} = \dfrac{u_0}{\omega}, & \text{(3E.18)} \\ II: \quad iv + \dfrac{1}{\omega}\mathcal{A}u - \dfrac{1}{\omega}\mathcal{B}\theta = \dfrac{v_0}{\omega}, & \text{(3E.19)} \\ III: \quad i\theta + \dfrac{1}{\omega}\mathcal{B}v + \dfrac{1}{\omega}\mathcal{B}\theta = \dfrac{\theta_0}{\omega}, & \text{(3E.20)} \end{cases}$$

where, recalling the definition of $\mathcal{D}(A)$ in (3E.8), we see that we have the following regularity properties, to be freely used below:

$$y = [u, v, \theta] \in \mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times \mathcal{D}(\mathcal{B}). \tag{3E.21}$$

Orientation The basic "driver," or "driving" term, in the present proof is the thermal estimate (3E.25) below for θ, which follows at once from the basic a priori dissipativity condition (3E.9). To propagate such a key estimate (3E.25) to other terms, and eventually achieve the desired estimates (3E.14) through (3E.16), we shall seek to employ the "driver" (3E.25) repeatedly. This will be done in conjunction with a priori bounds in the right norms, to dominate each norm quantity $\|q\|$ of interest, as follows:

$$\|q\| \leq [a+b][\epsilon a + k_\epsilon b] \leq 2\epsilon a^2 + C_\epsilon b^2, \quad a, b \geq 0 \tag{3E.22}$$

to be specialized with $a = \|y\|_Y$ and $b = \|\frac{f_0}{\omega}\|_Y$. [Inequality (3E.23) is established with $C_\epsilon = (k_\epsilon^2/(2\epsilon) + k_\epsilon + \epsilon/2)$, by using in the expansion of the product $\epsilon ab \leq \frac{\epsilon}{2}(a^2+b^2)$ and

$$k_\epsilon ab \leq \frac{k_\epsilon}{2}\left(\frac{\epsilon}{k_\epsilon}a^2 + \frac{k_\epsilon}{\epsilon}b^2\right).$$

3E.2.2 A Priori Bounds for θ, v, and u

Recalling Proposition 3E.1(i), (iii), we have at once via (3E.17):

Lemma 3E.2.2.1 (preliminary a priori bounds for θ) *With reference to (3E.11), (3E.12), we have for $\omega \in \mathbb{R}$:*

(i)

$$(\mathcal{B}\theta, \theta)_X = \operatorname{Re}\left([i\omega - A]\begin{bmatrix} u \\ v \\ \theta \end{bmatrix}, \begin{bmatrix} u \\ v \\ \theta \end{bmatrix}\right)_Y = \operatorname{Re}(f_0, y)_Y; \tag{3E.23}$$

(ii)

$$\left\|\mathcal{B}^{\frac{1}{2}}\theta\right\|_X^2 \leq \|f_0\|_Y \|y\|_Y; \tag{3E.24}$$

(iii) for any $\epsilon > 0$ and $\omega \in \mathbb{R}$, $\omega \neq 0$,

$$\frac{1}{|\omega|}\left\|\mathcal{B}^{\frac{1}{2}}\theta\right\|_X^2 \leq \frac{\epsilon}{2}\|y\|_Y^2 + \frac{1}{2\epsilon}\left\|\frac{f_0}{\omega}\right\|_Y^2. \tag{3E.25}$$

Lemma 3E.2.2.2 (a priori bounds for v) *With reference to (3E.11), (3E.12), we* *have for $\omega \in \mathbb{R}$, $|\omega| \geq 1$:*

(i)

$$\frac{1}{|\omega|} \left\|\mathcal{A}^{\frac{1}{2}} v\right\|_X \leq \left\|\mathcal{A}^{\frac{1}{2}} u\right\|_X + \left\|\frac{\mathcal{A}^{\frac{1}{2}} u_0}{\omega}\right\|_X \tag{3E.26}$$

$$\leq \|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y; \tag{3E.27}$$

(ii)

$$\frac{1}{\sqrt{|\omega|}} \left\|\mathcal{A}^{\frac{1}{4}} v\right\|_X \leq \frac{3}{2}\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{3E.28}$$

Proof. (i) The validity of estimate (3E.26) stems at once from Eqn. I = (3E.18). Then (3E.26) yields (3E.27) by majorizing $\mathcal{A}^{\frac{1}{2}} u$ and $\mathcal{A}^{\frac{1}{2}} u_0/\omega$ in X by y and f_0/ω in Y, via (3E.11), (3E.12).

(ii) Using henceforth freely $\sqrt{a^2+b^2} \leq a+b$, $a, b \geq 0$, we compute by (3E.27), and majorizing v by y via (3E.11), (3E.12):

$$\left\|\mathcal{A}^{\frac{1}{4}} v\right\|_X = \left[\left(\mathcal{A}^{\frac{1}{2}} v, v\right)_X\right]^{\frac{1}{2}} \leq \left\|\mathcal{A}^{\frac{1}{2}} v\right\|_X^{\frac{1}{2}} \|v\|_X^{\frac{1}{2}} \tag{3E.29}$$

$$\text{(by (3E.27))} \qquad \leq |\omega|^{\frac{1}{2}}\left[\|y\|_Y^{\frac{1}{2}} + \left\|\frac{f_0}{\omega}\right\|^{\frac{1}{2}}\right]\|y\|_Y^{\frac{1}{2}} \tag{3E.30}$$

$$\leq |\omega|^{\frac{1}{2}}\left[\|y\|_Y + \frac{1}{2}\|y\|_Y + \frac{1}{2}\left\|\frac{f_0}{\omega}\right\|_Y\right], \tag{3E.31}$$

and (3E.31) proves estimates (3E.28), as desired. □

Lemma 3E.2.2.3 (further a priori bound for θ) *With reference to (3E.11), (3E.12), we have for $\omega \in \mathbb{R}$, $|\omega| \geq 1$, with a constant C related to the $\mathcal{L}(X)$-norm of $\mathcal{B}\mathcal{A}^{-\frac{1}{2}}$ (see assumption (H.2) = (3E.1)):*

$$\frac{1}{|\omega|} \|\mathcal{B}\theta\|_X \leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{3E.32}$$

Proof. We return to Eqn. III = (3E.20), where we use hypothesis (H.2) = (3E.1) as well as estimate (3E.27) for v, thus obtaining

$$\frac{1}{|\omega|} \|\mathcal{B}\theta\|_X = \left\|\frac{\theta_0}{\omega} - i\theta - \frac{1}{\omega}\mathcal{B}v\right\|_X \tag{3E.33}$$

$$\leq \left\|\frac{\theta_0}{\omega}\right\|_X + \|\theta\|_X + \frac{1}{|\omega|}\left\|\left(\mathcal{B}\mathcal{A}^{-\frac{1}{2}}\right)\mathcal{A}^{\frac{1}{2}} v\right\|_X \tag{3E.34}$$

$$\text{(by (3E.1))} \qquad \leq C\left[\left\|\frac{f_0}{\omega}\right\|_Y + \|y\|_Y + \frac{1}{|\omega|}\left\|\mathcal{A}^{\frac{1}{2}}v\right\|_X\right] \tag{3E.35}$$

$$\text{(by (3E.27))} \qquad \leq C\left\{\left\|\frac{f_0}{\omega}\right\|_Y + \|y\|_Y + \left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]\right\}, \tag{3E.36}$$

majorizing also θ_0 and θ by f_0 and y via (3E.11), (3E.12). Then, (3E.36) proves (3E.32). □

Lemma 3E.2.2.4 (a priori bounds for u) *With reference to (3E.11), (3E.12), we have for $\omega \in \mathbb{R}$, $|\omega| \geq 1$:*

(i)

$$\frac{1}{|\omega|}\|\mathcal{A}u\|_X \leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]; \tag{3E.37}$$

(ii)

$$\frac{1}{\sqrt{|\omega|}}\left\|\mathcal{A}^{\frac{3}{4}}u\right\|_X \leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{3E.38}$$

Proof. (i) Returning to Eqn. II = (3E.19), we use (3E.32) and majorize v_0 and v by f_0 and y, via (3E.11), (3E.12), thus obtaining

$$\frac{1}{|\omega|}\|\mathcal{A}u\|_X = \left\|\frac{1}{\omega}\mathcal{B}\theta + \frac{v_0}{\omega} - iv\right\|_X \tag{3E.39}$$

$$\text{(by (3E.32))} \qquad \leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right] + \left\|\frac{f_0}{\omega}\right\|_Y + \|y\|_Y, \tag{3E.40}$$

and (3E.40) proves (3E.37).

(ii) Invoking (3E.37) and majorizing $\mathcal{A}^{\frac{1}{2}}u$ by y, via (3E.11), (3E.12), we get

$$\left\|\mathcal{A}^{\frac{3}{4}}u\right\|_X^2 = \left(\mathcal{A}u, \mathcal{A}^{\frac{1}{2}}u\right)_X \leq \|\mathcal{A}u\|_X\left\|\mathcal{A}^{\frac{1}{2}}u\right\|_X \tag{3E.41}$$

$$\text{(by (3E.37))} \qquad \leq C|\omega|\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]\|y\|_Y \tag{3E.42}$$

$$\leq C|\omega|\left[\|y\|_Y^2 + \frac{1}{2}\|y\|^2 + \frac{1}{2}\left\|\frac{f_0}{\omega}\right\|_Y^2\right], \tag{3E.43}$$

and (3E.43) proves (3E.38), as desired. □

3E.2.3 A Fundamental Estimate on $\frac{1}{\omega}(\mathcal{B}v, \theta)_X$

To obtain the next critical results, we implement for the first time the strategy described in the *Orientation*, by combining the "driver" (3E.25) with the a priori bound (3E.28).

Proposition 3E.2.3.1 *With reference to (3E.11), (3E.12), given* $\epsilon > 0$, *there exists* $C_\epsilon > 0$, *such that for all* $\omega \in \mathbb{R}$, $|\omega| \geq 1$, *we have*

$$\left| \frac{1}{\omega} (\mathcal{B}v, \theta)_X \right| \leq \epsilon \|y\|_Y^2 + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y^2. \tag{3E.44}$$

Proof. We invoke the consequence (H.2w) = (3E.3) of hypothesis (H.2) = (3E.1), the a priori bound (3E.28) for v, and the "driving" estimate (3E.25) to obtain

$$\left| \frac{1}{\omega} (\mathcal{B}v, \theta)_X \right| = \left| \left((\mathcal{B}^{\frac{1}{2}} \mathcal{A}^{-\frac{1}{4}}) \frac{\mathcal{A}^{\frac{1}{4}} v}{\sqrt{|\omega|}}, \frac{\mathcal{B}^{\frac{1}{2}} \theta}{\sqrt{|\omega|}} \right)_X \right| \tag{3E.45}$$

$$\text{(by (3E.3))} \qquad \leq C \, \frac{\|\mathcal{A}^{\frac{1}{4}} v\|_X}{\sqrt{|\omega|}} \, \frac{\|\mathcal{B}^{\frac{1}{2}} \theta\|_X}{\sqrt{|\omega|}} \tag{3E.46}$$

$$\text{(by (3E.28) and (3E.25))} \qquad \leq C \, \frac{3}{4} \left[\|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right] \left[\epsilon_1 \|y\|_Y + \frac{1}{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y \right] \tag{3E.47}$$

$$\text{(by (3E.22))} \qquad \leq C \, \frac{3}{2} \epsilon_1 \|y\|_Y^2 + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y^2, \tag{3E.48}$$

and (3E.48) proves (3E.44), as desired. □

3E.2.4 Proof of Estimate (3E.16) for θ

As a corollary of the "driving" estimate (3E.25) for θ, as well as of Proposition 3E.2.3.1 (which, in turn, also stems from (3E.25)), we obtain the desired inequality (3E.16) for θ.

Proposition 3E.2.4.1 *With reference to (3E.11), (3E.12), given* $\epsilon > 0$, *there exists* $C_\epsilon > 0$ *such that for all* $\omega \in \mathbb{R}$, $|\omega| \geq 1$, *we have*

$$\|\theta\|_X^2 \leq \epsilon \|y\|_Y^2 + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y^2. \tag{3E.49}$$

Proof. We return to Eqn. (3E.20), take here the X-inner product with θ, use estimate (3E.44) of Proposition 3E.2.3.1 and (3E.25) for θ, and obtain

$$\|\theta\|_X^2 \leq \left| \frac{1}{\omega} (\mathcal{B}v, \theta)_X \right| + \frac{1}{|\omega|} \|\mathcal{B}^{\frac{1}{2}} \theta\|_X^2 + \left| \left(\frac{\theta_0}{\omega}, \theta \right)_X \right| \tag{3E.50}$$

$$\text{(by (3E.44) and (3E.25))} \leq \left[\epsilon \|y\|_Y^2 + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y^2 \right] + \left[\frac{\epsilon}{2} \|y\|_Y^2 + \frac{1}{2\epsilon} \left\| \frac{f_0}{\omega} \right\|_Y^2 \right]$$

$$+ \frac{\epsilon}{2} \|\theta\|_X^2 + \frac{1}{2\epsilon} \left\| \frac{\theta_0}{\omega} \right\|_X^2. \tag{3E.51}$$

Then, the desired inequality (3E.49) readily follows from (3E.51), majorizing θ by y and θ_0 by f_0, via (3E.11), (3E.12). □

3E.2.5 Improving upon an A Priori Bound on v

The following result – a corollary of Proposition 3E.2.4.1, Eqn. (3E.49), for θ – improves upon the a priori bound (3E.28) for v.

Lemma 3E.2.5.1 *With reference to (3E.11), (3E.12), given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for all $\omega \in \mathbb{R}$, $|\omega| \geq 1$, we have*

$$\frac{c}{|\omega|}\left\|\mathcal{A}^{\frac{1}{4}}v\right\|_X^2 \leq \frac{1}{|\omega|}\left\|\mathcal{B}^{\frac{1}{2}}v\right\|_X^2 \leq \epsilon\|y\|_Y^2 + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y^2. \tag{3E.52}$$

Proof. Since $v \in \mathcal{D}(\mathcal{A}^{\frac{1}{4}})$ [see (3E.28)], then the left-hand side inequality with $c > 0$ is due to hypothesis (H.3) = (3E.2) (which is used here for the first time). We now prove the right-hand side inequality in (3E.52). We return to Eqn. III = (3E.20), take here the X-inner product with v, and use here Eqn. (3E.44) of Proposition 3E.2.3.1 and Eqn. (3E.49) of Proposition 3E.2.4.1 on θ, while majorizing v by y via (3E.11), (3E.12), to obtain

$$\frac{1}{|\omega|}\left\|\mathcal{B}^{\frac{1}{2}}v\right\|_X^2 = \left|\frac{1}{\omega}(\mathcal{B}v, v)_X\right| = \left|\left(\frac{\theta_0}{\omega}, v\right)_X - i(\theta, v)_X - \frac{1}{\omega}(\mathcal{B}\theta, v)_X\right| \tag{3E.53}$$

$$\text{(by (3E.44))} \quad \leq \left\|\frac{\theta_0}{\omega}\right\|_X \|v\|_X + \|\theta\|_X\|v\|_X + \left[\epsilon_1\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2\right] \tag{3E.54}$$

$$\text{(by (3E.49))} \quad \leq \left[\frac{1}{2\epsilon_1}\left\|\frac{\theta_0}{\omega}\right\|_X^2 + \frac{\epsilon_1}{2}\|v\|_X^2\right] + \left[\epsilon_1\|y\|_Y + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y\right]\|y\|_Y$$
$$+ \left[\epsilon_1\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2\right] \tag{3E.55}$$

$$\leq 3\epsilon_1\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2, \tag{3E.56}$$

majorizing, in the last step, θ_0 by f_0, and v by y, via (3E.11), (3E.12). Equation (3E.56) then proves the right-hand side inequality of (3E.52), as desired. □

3E.2.6 Proof of Estimate (3E.15) for v

As a corollary of the critical Lemma 3E.2.5.1 and Proposition 3E.2.3.1, as well as of the a priori bound (3E.38) of Lemma 3E.2.2.4, we obtain the desired estimate (3E.15) for v.

Proposition 3E.2.6.1 *With reference to (3E.11), (3E.12), given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for all $\omega \in \mathbb{R}$, $|\omega| \geq 1$,*

$$\|v\|_X^2 \leq \epsilon\|y\|_Y^2 + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y^2. \tag{3E.57}$$

Proof. We return to Eqn. II = (3E.19) and obtain after the X-inner product with v:

$$\|v\|_X^2 \leq \left|\frac{1}{\omega}(\mathcal{A}u, v)_X\right| + \left|\frac{1}{\omega}(\mathcal{B}\theta, v)_X\right| + \left|\left(\frac{v_0}{\omega}, v\right)_X\right|. \tag{3E.58}$$

By use of the critical estimate (3E.52) for v along with the a priori bound (3E.38) for u, we obtain

$$\left|\frac{1}{\omega}(\mathcal{A}u, v)_X\right| \leq \frac{\|\mathcal{A}^{\frac{3}{4}}u\|}{\sqrt{|\omega|}} \frac{\|\mathcal{A}^{\frac{1}{4}}v\|}{\sqrt{|\omega|}} \tag{3E.59}$$

$$\text{(by (3E.38), (3E.52))} \quad \leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]\left[\epsilon_1\|y\|_Y + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y\right] \tag{3E.60}$$

$$\text{(by (3E.22))} \quad \leq C\epsilon_1\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2, \tag{3E.61}$$

recalling (3E.22) in the last step.

Moreover, recalling the critical estimate (3E.44), we obtain since $\mathcal{B}$ is self-adjoint

$$\left|\frac{1}{\omega}(\mathcal{B}\theta, v)_X\right| + \left|\left(\frac{v_0}{\omega}, v\right)_X\right|$$

$$\text{(by (3E.44))} \quad \leq \left[\epsilon_1\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2\right] + \left[\frac{\epsilon_1}{2}\|v\|_X^2 + \frac{1}{2\epsilon_1}\left\|\frac{v_0}{\omega}\right\|_Y^2\right] \tag{3E.62}$$

$$\leq \frac{3}{2}\epsilon_1\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2, \tag{3E.63}$$

majorizing v by y and v_0 by f_0 via (3E.11), (3E.12). Combining (3E.61) and (3E.63) in (3E.58) yields (3E.57), as desired. □

3E.2.7 Proof of Estimate (3E.14) for u

We can finally establish also estimate (3E.14) for u.

Proposition 3E.2.7.1 *With reference to (3E.11), (3E.12), given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for all $\omega \in \mathbb{R}$, $|\omega| \geq 1$,*

$$\left\|\mathcal{A}^{\frac{1}{2}}u\right\|_X^2 \leq \epsilon\|y\|_Y^2 + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y^2. \tag{3E.64}$$

Proof. Estimate (3E.61) gives preliminarily

$$\left|\frac{1}{\omega}(\mathcal{A}u, v)_X\right| \leq \epsilon_1\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2. \tag{3E.65}$$

However, by substituting $v/\omega = iu - u_0/\omega$ from Eqn. I = (3E.18), we obtain

$$\frac{1}{\omega}(\mathcal{A}u, v)_X = \left(\mathcal{A}u, iu - \frac{u_0}{\omega}\right)_X = -i\left\|\mathcal{A}^{\frac{1}{2}}u\right\|_X^2 - \left(\mathcal{A}^{\frac{1}{2}}u, \frac{\mathcal{A}^{\frac{1}{2}}u_0}{\omega}\right)_X. \tag{3E.66}$$

Then (3E.65) and (3E.66) yield

$$\|\mathcal{A}^{\frac{1}{2}}u\|_X^2 \le \left|\frac{1}{\omega}(\mathcal{A}u, v)_X\right| + \left|\left(\mathcal{A}^{\frac{1}{2}}u, \frac{\mathcal{A}^{\frac{1}{2}}u_0}{\omega}\right)_X\right| \tag{3E.67}$$

$$(\text{by (3E.65)}) \quad \le \left[\epsilon_1\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2\right] + \frac{\epsilon_1}{2}\|\mathcal{A}^{\frac{1}{2}}u\|_X^2 + \frac{1}{2\epsilon_1}\left\|\frac{\mathcal{A}^{\frac{1}{2}}u_0}{\omega}\right\|_X^2 \tag{3E.68}$$

$$\le \frac{3}{2}\epsilon_1\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2, \tag{3E.69}$$

by majorizing $\mathcal{A}^{\frac{1}{2}}u$ and $\mathcal{A}^{\frac{1}{2}}u_0$ by y and f_0, via (3E.11), (3E.12). Equation (3E.69) proves (3E.64), as desired. □

The proof of estimates (3E.14), (3E.15), (3E.16) for u, v, θ is complete. Thus, the resolvent estimate (3E.13) has been established: The s.c. semigroup e^{At} is then analytic on Y, $t > 0$, by virtue of the result given below as Theorem 3E.3. □

3E.3 A Characterization for the Analyticity of a s.c. Uniformly Bounded Semigroup, with No Spectrum on the Imaginary Axis

We next provide the result on analyticity of semigroups, which was invoked on estimate (3E.13). While this *should* be a well-known result, we are unable to find an explicit statement in many standard, comprehensive monographs on semigroup theory (save for the *original* Lecture Notes #10 of A. Pazy, Department of Mathematics, University of Maryland, January 1974; see Theorem 5.2, part (a) for the closest version). Inquiries among experts in the field have convinced us that it is desirable to provide here an explicit statement and proof.

In seeking to show that an operator A is the generator of a s.c. analytic semigroup, one has often at the outset – and by simple arguments – two preliminary pieces of information: (i) that A is the generator of a s.c. contraction semigroup; and (ii) that A has no spectrum on the imaginary axis. This is the case for the class of thermo-elastic systems. Then, the following criterion is handier to use than the usual characterization [Fattorini, 1983; Pazy, 1983]. It will be employed in the next two appendices as well.

Theorem 3E.3 *Let the (linear) operator A on the Banach space Y satisfy the following three hypotheses:*

(h.1) A generates a s.c. contraction semigroup e^{At} on Y.
(h.2) A has no spectrum on the imaginary axis (so that the spectrum $\sigma(A)$ of A is contained in the open half-complex plane: $\sigma(A) \subset \{\lambda \in \mathbb{C} : \text{Re}\,\lambda < 0\}$).
(h.3) There exists a positive constant $C > 0$ such that for all ω real, with $|\omega| \ge \omega_0 > 0$

for some ω_0, the following uniform estimate on the imaginary axis holds true:

$$\|R(i\omega, A)\|_{\mathcal{L}(Y)} \leq \frac{C}{|\omega|}, \quad \forall\, |\omega| \geq \omega_0 > 0. \tag{3E.70}$$

Then, in fact, for a suitable constant $M > 0$, we have

$$\|R(\lambda, A)\|_{\mathcal{L}(Y)} \leq \tfrac{M}{|\lambda|}, \quad \lambda \neq 0, \quad \forall\, \lambda \in \Sigma_{\theta_1}, \tag{3E.71}$$

$$\Sigma_{\theta_1} = \left\{\lambda \in \mathbb{C} : 0 \leq |arg\, \lambda| \leq \tfrac{\pi}{2} + \theta_1\right\}, \tag{3E.72}$$

where one may take the angle $0 < \theta_2 < \pi/2$ *such that* $\tan(\pi/2 - \theta_1) = C/\rho$, *with C as in (3E.70), for an arbitrary fixed constant* $0 < \rho < 1$. *Thus, via (3E.71), the s.c. contraction semigroup e^{At} is, moreover, analytic on Y, $t > 0$, by the well-known characterization [Pazy, 1983; Fattorini, 1983].*

Proof.

Step 1 By use of assumption (h.3), Eqn. (3E.70), we shall show that there exists a cone K_{θ_1} in the complex plane,

$$K_{\theta_1} = \left\{\lambda \in \mathbb{C} : \frac{\pi}{2} - \theta_1 \leq |\arg \lambda| \leq \frac{\pi}{2} + \theta_1\right\}, \tag{3E.73}$$

such that (we drop henceforth the subindex $\mathcal{L}(Y)$ for the uniform norm)

$$\|R(\lambda, A)\| \leq \frac{C}{(1-\rho)\cos\theta_1}\, \frac{1}{|\lambda|}, \quad \forall \lambda \in K_{\theta_1}, \quad |\mathrm{Im}\, \lambda| \geq \omega_0. \tag{3E.74}$$

Indeed, we write the Taylor expansion of $R(\lambda, A)$ around the point $i\omega$, $|\omega| \geq \omega_0 > 0$ [Fattorini, 1983; Pazy, 1983]:

$$R(\lambda, A) = \sum_{k=0}^{\infty} R^{k+1}(i\omega, A)(i\omega - \lambda)^k. \tag{3E.75}$$

This series converges in the uniform norm, if

$$\|R(i\omega, A)\|\; |i\omega - \lambda| \leq \rho < 1, \tag{3E.76}$$

in which case we obtain from (3E.75) and (3E.70) the estimate

$$\|R(\lambda, A)\| \leq \|R(i\omega, A)\| \sum_{k=0}^{\infty} \|R(i\omega, A)\|^k |i\omega - \lambda|^k$$

$$\text{(by (3E.70), (3E.76))} \qquad \leq \frac{C}{|\omega|} \sum_{k=0}^{\infty} \rho^k = \frac{C}{1-\rho}\, \frac{1}{|\omega|}. \tag{3E.77}$$

With $\lambda = \sigma + i\omega$, $|\omega| \geq \omega_0 > 0$, we estimate by use of hypothesis (3E.70) and obtain, as required,

$$\begin{cases} \|(R(i\omega, A)\| \, |i\omega - \lambda| \leq \dfrac{C}{|\omega|}|\sigma| \leq \rho < 1 \\ \text{provided that } |\omega| \geq \dfrac{C}{\rho}|\sigma|, \text{ that is, provided that such } \lambda \in K_{\theta_1}. \end{cases} \tag{3E.78}$$

Thus, if λ satisfies condition (3E.78), whereby $|\omega| \geq |\lambda| \cos\theta_1$, we have that estimate (3E.77) holds true, and hence

$$\|R(\lambda, A)\| \leq \frac{C}{1-\rho}\frac{1}{|\omega|} \leq \frac{C}{(1-\rho)\cos\theta_1}\frac{1}{|\lambda|}, \quad \lambda \in K_{\theta_1}, \quad |\mathrm{Im}\,\lambda| \geq \omega_0, \tag{3E.79}$$

which proves (3E.74).

Step 2 Hypothesis (h.1) implies, by the Hille–Yosida theorem, the estimate

$$\|R(\lambda, A)\| \leq \frac{1}{\mathrm{Re}\,\lambda}, \quad \forall\, \lambda, \mathrm{Re}\,\lambda > 0. \tag{3E.80}$$

But for all $\lambda \in \Sigma_{\theta_2} \equiv \{\lambda \in \mathbb{C} : 0 \leq |\arg\lambda| \leq \theta_2\}$, where $\theta_1 + \theta_2 = \pi/2$, we have $\mathrm{Re}\,\lambda \geq |\lambda| \cos\theta_2$, so that (3E.80) yields

$$\|R(\lambda, A)\| \leq \frac{1}{\mathrm{Re}\,\lambda} \leq \frac{1}{\cos\theta_2}\frac{1}{|\lambda|}, \quad \forall\, \lambda \in \Sigma_{\theta_2}. \tag{3E.81}$$

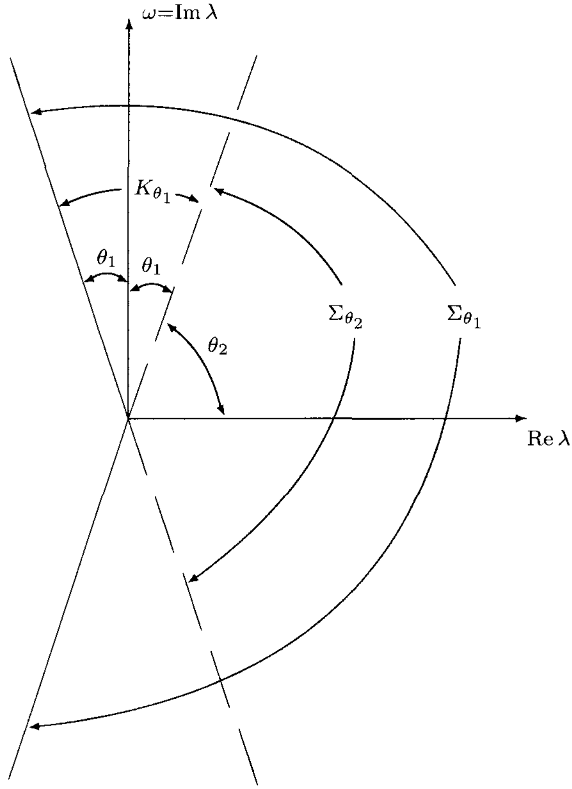
By combining (3E.74) and (3E.81), we obtain (3E.71), as the resolvent operator is bounded on any compact set of $\mathbb{C}$. Refer to Figure 3E.1. □

3E.4 Application of the Abstract Theorem 3E.2 to the Thermo-Elastic Examples of Sections 3.11.1, 3.11.2 (and others)

In this subsection we apply the abstract Theorem 3E.2 to several "concrete" thermo-elastic problems, including those dealt with in Sections 3.11.1 and 3.11.2. Henceforth, let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ (in physical applications $n = 1, 2$), be an open bounded domain with sufficiently smooth boundary Γ. Stripped from lower-order terms and with physical constants normalized to one, the equations of a thermo-elastic "plate" in the vertical displacement $w(t, x)$ and in the temperature $\theta(t, x)$ are Lagnese [1987]:

$$\begin{cases} w_{tt} + \Delta^2 w + \Delta\theta \equiv 0 & \text{in } (0, T] \times \Omega \equiv Q; & \text{(3E.82a)} \\ \theta_t - \Delta\theta - \Delta w_t \equiv 0 & \text{in } Q; & \text{(3E.82b)} \\ w(0, \cdot) = w_0, \; w_t(0, \cdot) = w_1, \; \theta(0, \cdot) = \theta_0 & \text{in } \Omega, & \text{(3E.82c)} \end{cases}$$

which will be supplemented by appropriate boundary conditions (BC) in the examples below.



$$\Sigma_{\theta_1} = K_{\theta_1} \cup \Sigma_{\theta_2}$$

Figure 3E.1.

Example 3E.4.1 (Clamped mechanical BC and Dirichlet thermal BC) This is the concrete problem explicitly considered in Section 3.11.1. We supplement Eqns. (3E.82a,b,c) with the following BC;

$$w \equiv \frac{\partial w}{\partial \nu} \equiv 0, \quad \theta \equiv 0 \text{ in } (0, T] \times \Gamma \equiv \Sigma. \tag{3E.83}$$

To put problem (3E.82a,b,c), (3E.83) into the abstract setting of Section 3E.1 of the present appendix, we set

(i)

$$X \equiv L_2(\Omega); \quad \mathcal{A}f = \Delta^2 f, \quad \mathcal{D}(\mathcal{A}) = \left\{ f \in H^4(\Omega) : f|_\Gamma = \left.\frac{\partial f}{\partial \nu}\right|_\Gamma \equiv 0 \right\}; \tag{3E.84}$$

(ii)

$$\mathcal{B} = \mathcal{A}_D, \quad \text{where } \mathcal{A}_D f = -\Delta f; \quad \mathcal{D}(\mathcal{A}_D) = H^2(\Omega) \cap H_0^1(\Omega). \tag{3E.85}$$

Then, $\mathcal{A}$ and $\mathcal{B}$ are positive, self-adjoint operators on $L_2(\Omega)$, and hypothesis (H.1) is verified. Moreover,

$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv H_0^2(\Omega) \subset \mathcal{D}(\mathcal{B}) \equiv H^2(\Omega) \cap H_0^1(\Omega); \tag{3E.86}$$

$$\mathcal{D}(\mathcal{A}^{\frac{1}{4}}) \equiv H_0^1(\Omega) = \mathcal{D}(\mathcal{B}^{\frac{1}{2}}) = \mathcal{D}(\mathcal{A}_D^{\frac{1}{2}}). \tag{3E.87}$$

Thus, (3E.86) verifies hypothesis (H.2) = (3E.1), whereas (3E.87) is stronger than (H.3) = (3E.2). Thus, Proposition 3E.1 and Theorem 3E.2 apply under the set (H.1), (H.2), and (H.3). The first proof of analyticity of the semigroup for this case (3E.83) was given in Liu and Renardy [1995], by a different, ad hoc, PDE proof.

Example 3E.4.2 (Clamped mechanical BC and Neuman/Robin thermal BC) This is the concrete problem explicitly considered in Section 3.11.2. We supplement Eqns. (3E.82a,b,c) with the following BC:

$$w \equiv \frac{\partial w}{\partial \nu} \equiv 0; \quad \frac{\partial \theta}{\partial \nu} + b\theta \equiv 0, \quad b \geq 0, \quad \text{in } (0, T] \times \Gamma \equiv \Sigma. \tag{3E.88}$$

Henceforth, we take explicitly the constant $b > 0$ (Robin BC). [The Neumann case $b = 0$ is similar and is obtained by replacing the space X below with the space $X = L_2^0(\Omega) = L_2(\Omega)/\mathcal{N}(\mathcal{A}_N)$, where $\mathcal{N}(\mathcal{A}_N)$ is the one-dimensional null space of the operator $\mathcal{A}_N$ below for $b = 0$.] Let $X = L_2(\Omega)$ and $\mathcal{A}$ be as in (3E.84). Select

$$\mathcal{B} = \mathcal{A}_N; \quad \mathcal{A}_N f = -\Delta f, \quad \mathcal{D}(\mathcal{A}_N) = \left\{ f \in H^2(\Omega) : \left[\frac{\partial f}{\partial \nu} + bf \right]_\Gamma \equiv 0 \right\}. \tag{3E.89}$$

Then, hypothesis (H.1) still holds true. Moreover, now, the following properties hold true:

(i)

$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv H_0^2(\Omega) \subset \mathcal{D}(\mathcal{B}) = \mathcal{D}(\mathcal{A}_N), \tag{3E.90}$$

so that hypothesis (H.2) = (3E.1) is verified;

(ii)

$$\|\mathcal{A}^{\frac{1}{2}} f\|_X^2 = \|\mathcal{A}_N f\|_X^2 = \|\mathcal{B} f\|_X^2 = \int_\Omega |\Delta f|^2 d\Omega, \quad \forall f \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset \mathcal{D}(\mathcal{B}); \tag{3E.91}$$

(iii)

$$\mathcal{D}(\mathcal{A}^{\frac{1}{4}}) = H_0^1(\Omega) \subset \mathcal{D}(\mathcal{B}^{\frac{1}{2}}) = \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}) = H^1(\Omega). \tag{3E.92}$$

Hence, by the Poincaré inequality (with equivalent norms)

$$\|\mathcal{A}^{\frac{1}{4}} f\|_X^2 = \|f\|_{H_0^1(\Omega)}^2 = \int_\Omega |\nabla f|^2 d\Omega, \quad \forall f \in \mathcal{D}(\mathcal{A}^{\frac{1}{4}}), \tag{3E.93}$$

while first for $f \in \mathcal{D}(\mathcal{A}_N)$ via Green's first theorem, and then extending to all $f \in \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}})$, we obtain

$$(\mathcal{A}_N f, f)_X = \|\mathcal{A}_N^{\frac{1}{2}} f\|_X^2 = \int_\Omega |\nabla f|^2 \, d\Omega, \quad \forall\, f \in \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}). \tag{3E.94}$$

Thus, (3E.93) and (3E.94) yield the norm equivalence

$$\|\mathcal{A}^{\frac{1}{4}} f\|_X = \|\mathcal{A}_N^{\frac{1}{2}} f\|_X = \|\mathcal{B}^{\frac{1}{2}} f\|_X, \quad \forall\, f \in \mathcal{D}(\mathcal{A}^{\frac{1}{4}}) \subset \mathcal{D}(\mathcal{B}^{\frac{1}{2}}), \tag{3E.95}$$

and hypothesis (H.3) = (3E.2) is verified. [Equation (3E.91), which is first verified for $(\mathcal{A}f, f)_X$ with $f \in \mathcal{D}(\mathcal{A})$ by Green's second theorem, and then extended to all $f \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H_0^2(\Omega)$, is not strictly needed; it is included here for completeness of the analysis.] We conclude that Proposition 3E.1 and Theorem 3E.2 apply, under the set of assumptions (H.1), (H.2), (H.3).

Example 3E.4.3 (Hinged mechanical BC/Dirichlet thermal BC) (a) We begin with the most amenable set of BC, *simplified hinged* (or *simply supported*) BC associated with Eqns. (3E.82a,b,c), and studied directly in Appendix 3D,

$$w|_\Sigma \equiv 0, \quad \Delta w|_\Sigma \equiv 0; \quad \theta|_\Sigma \equiv 0. \tag{3E.96}$$

This case is readily included in the set of assumptions (H.1), (H.2), (H.3) in Section 3E.1, with

$$\mathcal{B} = \mathcal{A}_D \text{ as in (3E.85)}; \quad \mathcal{A} = \mathcal{A}_D^2; \quad \mathcal{B} = \mathcal{A}^{\frac{1}{2}}; \tag{3E.97}$$

$$\mathcal{A}f = \Delta^2 f, \quad \mathcal{D}(\mathcal{A}) = \{f \in H^4(\Omega) : f|_\Gamma = \Delta f|_\Gamma = 0\}. \tag{3E.98}$$

Thus, Proposition 3E.1 and Theorem 3E.2 apply. For this very special set of BC, the direct, ad hoc proof, which consists in factoring out $\mathcal{A}^{\frac{1}{2}}$ from the operator matrix A in (3E.7) and using diagonalization as in Appendix 3D, Theorem 3D.1, and Application #1, is the most elementary [Liu, Renardy, 1995].

(b) When $\dim \Omega = 2$ a further refinement of model (3E.96) takes the *physical bending moment*,

$$w|_\Sigma \equiv 0; \quad [\Delta w + (1-\mu) B_1 w]_\Sigma \equiv 0; \quad \theta|_\Sigma \equiv 0, \tag{3E.99}$$

where B_1 is the boundary operator in Eqn. (3.5.2a) or Eqn. (3C.11) of Appendix 3C, which may be rewritten as [Appendix 3C, Eqn. (3C.2)]

$$B_1 w = -c(x) \left. \frac{\partial w}{\partial \nu} \right|_\Gamma, \quad c(x) = \operatorname{div} \nu(x) = \text{mean curvature on } \Gamma, \tag{3E.100}$$

with $\nu(x)$ = unit normal vector, and $0 < \mu < 1$, the Poisson's modulus. Now we take the positive self-adjoint [Appendix 3C, Proposition 3C.4] operator $\mathcal{A}$ defined by

$$\mathcal{A}f = \Delta^2 f, \quad \mathcal{D}(\mathcal{A}) = \{f \in H^4(\Omega) : f|_\Gamma = [\Delta f + (1-\mu)B_1 w]_\Sigma = 0\}; \tag{3E.101}$$

$$\mathcal{B} = \mathcal{A}_D \text{ as in (3E.85)}; \quad \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = \mathcal{D}(\mathcal{B}) = H^2(\Omega) \cap H_0^1(\Omega); \tag{3E.102}$$

$$\mathcal{D}(\mathcal{A}^{\frac{1}{4}}) = \mathcal{D}(\mathcal{B}^{\frac{1}{2}}) = H_0^1(\Omega),$$

and the set of assumptions (H.1), (H.2), (H.3) in Section 3E.1 are satisfied. Thus, Proposition 3E.1 and Theorem 3E.2 apply. An ad hoc proof for analyticity may be given in this case of BC (3E.99), as a perturbation of the simplified hinged BC case (3E.96); see Appendix 3D, Application #2.

Example 3E.4.4 (Partially clamped/partially hinged mechanical BC and Dirichlet thermal BC) This is the (only) thermo-elastic example given in Liu and Liu [1997] to illustrate the applicability of their abstract result for analyticity. With dim $\Omega = 2$, $\partial\Omega = \Gamma = \Gamma_0 \cup \Gamma_1$, and Γ_0 and Γ_1 both nonempty, we supplement Eqns. (3E.82a,b,c) with the following BC:

$$w|_\Sigma \equiv 0; \quad \theta|_\Sigma \equiv 0; \quad \left.\frac{\partial w}{\partial \nu}\right|_{\Sigma_0} \equiv 0; \quad [\Delta w + (1-\mu)B_1 w]_{\Sigma_1} \equiv 0, \tag{3E.103}$$

with B_1 and μ as in Example 3E.4.3; see (3E.100). Thus, the present set of BC combines the clamped BC of Example 3E.4.1 on Γ_0 with the hinged BC of Example 3E.4.3 on Γ_1. Define the positive self-adjoint operator $\mathcal{A}$ (see Appendix 3C, Theorem 3C.4) by

$$\mathcal{A}f = \Delta^2 f,$$
$$\mathcal{D}(\mathcal{A}) = \left\{ f \in H^4(\Omega) : f|_\Gamma = 0; \left.\frac{\partial f}{\partial \nu}\right|_{\Gamma_0} = 0; \left.[\Delta f + (1-\mu)B_1 f]\right|_{\Gamma_1} = 0 \right\}; \tag{3E.104}$$

$$\mathcal{B} = \mathcal{A}_D \text{ as in (3E.85)};$$
$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = \left\{ f \in H^2(\Omega) : f|_\Gamma = 0, \left.\frac{\partial f}{\partial \nu}\right|_{\Gamma_0} = 0 \right\} \subset \mathcal{D}(\mathcal{B}) = H^2(\Omega) \cap H_0^1(\Omega); \tag{3E.105}$$

$$\mathcal{D}(\mathcal{A}^{\frac{1}{4}}) = H_0^1(\Omega) = \mathcal{D}(\mathcal{B}^{\frac{1}{2}}). \tag{3E.106}$$

Thus, set of assumptions (H.1), (H.2), and (H.3) in Section 3E.1 is satisfied, and Proposition 3E.1 and Theorem 3E.2 apply.

3E.5 An Abstract Generalization of the Analyticity Result of Theorem 3E.2

As evidenced by the applications in Section 3E.4, the setting of Section 3E.1 – in particular, the abstract model (3E.6)–(3E.8), subject to the assumptions (H.1), (H.2) = (3E.1), (H.3) = (3E.2) – is surely adequate to describe, and precisely tuned to encompass, thermo-elastic plate equations with *uncoupled* BC. On the other hand, the basic argument in the proof of Theorem 3E.2 in Section 3E.2 allows for further generalizations, through essentially rather cosmetic changes. In the present section, we shall provide a resulting abstract extension for the sake of completeness, even though, when tested to thermo-elastic plate equations with various sets of canonical BC, it does not include any more examples than those already covered by Theorem 3E.2. We leave to future investigations the possible applicability of the present generalization, Theorem 3E.5.2 below, to interesting PDE problems that do not fit into Theorem 3E.2.

Mathematical Setting Let X_1 and X_2 be two Hilbert spaces, with norms $\|\cdot\|_{X_i}$ and inner product $(\cdot,\cdot)_{X_i}$, respectively. We consider three operators, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, subject to the following assumptions (which are labeled as to make transparent their relationship to the corresponding assumptions of Section 3E.1):

(h.1a)

$$\mathcal{A} : X_1 \supset \mathcal{D}(\mathcal{A}) \to X_1 \quad \text{and} \quad \mathcal{C} : X_2 \supset \mathcal{D}(\mathcal{C}) \to X_2$$

are two strictly positive, self-adjoint operators; while $\mathcal{B} : X_2 \supset \mathcal{D}(\mathcal{B}) \to X_1$ is a densely defined, closed operator with adoint $\mathcal{B}^* : X_1 \supset \mathcal{D}(\mathcal{B}^*) \to X_2$, so that $\mathcal{B} = \mathcal{B}^{**}$;

(h.1b)

$$\mathcal{D}(\mathcal{C}) \subset \mathcal{D}(\mathcal{B}), \text{ equivalently } \mathcal{B}\mathcal{C}^{-1} \in \mathcal{L}(X_2; X_1),$$
$$\text{and } \mathcal{C}^{-1}\mathcal{B}^* \text{has a bounded extension in } \mathcal{L}(X_1; X_2); \tag{3E.107}$$

(h.2)

$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset \mathcal{D}(\mathcal{B}^*), \text{ equivalently } \mathcal{B}^*\mathcal{A}^{-\frac{1}{2}} \in \mathcal{L}(X_2),$$
$$\text{and } \mathcal{A}^{-\frac{1}{2}}\mathcal{B} \quad \text{has a bounded extension in } \mathcal{L}(X_2); \tag{3E.108}$$

(h.3) there is a constant $\rho_1 > 0$ such that

$$\rho_1 \left\|\mathcal{A}^{\frac{1}{4}} x\right\|_{X_1}^2 \le (\mathcal{B}\mathcal{C}^{-1}\mathcal{B}^* x, x)_{X_1} = \left\|(\mathcal{B}\mathcal{C}^{-1}\mathcal{B}^*)^{\frac{1}{2}} x\right\|_{X_1}^2, \quad \forall\, x \in \mathcal{D}(\mathcal{A}^{\frac{1}{4}}), \tag{3E.109}$$

where $\mathcal{B}\mathcal{C}^{-1}\mathcal{B}^*$ is a positive, self-adjoint operator.

Remark 3E.5.1 By (h.1b) and (h.2), we have $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset \mathcal{D}(\mathcal{B}\mathcal{C}^{-1}\mathcal{B}^*)$. By Lowner's theorem [Kato, 1966, Corollary 7.1, p. 146], it follows that $\mathcal{D}(\mathcal{A}^{\frac{1}{4}}) \subset \mathcal{D}((\mathcal{B}\mathcal{C}^{-1}\mathcal{B}^*)^{\frac{1}{2}})$, equivalently $(\mathcal{B}\mathcal{C}^{-1}\mathcal{B}^*)^{\frac{1}{2}}\mathcal{A}^{-\frac{1}{4}} \in \mathcal{L}(X_1)$, or

$$\left\|(\mathcal{B}\mathcal{C}^{-1}\mathcal{B}^*)^{\frac{1}{2}}x\right\|_{X_1} \leq \sqrt{\rho_2}\left\|\mathcal{A}^{\frac{1}{4}}x\right\|_{X_1}, \quad \forall x \in \mathcal{D}(\mathcal{A}^{\frac{1}{4}}), \tag{3E.110}$$

which is the reverse inequality of (3E.109). Thus, under (h.1), (h.2), and (h.3), we combine (3E.109) and (3E.110) and write equivalently: There are constants $0 < \rho_1 < \rho_2 < \infty$, such that

$$\rho_1(\mathcal{A}^{\frac{1}{2}}x, x)_{X_1} \leq ((\mathcal{B}\mathcal{C}^{-1}\mathcal{B}^*)x, x)_{X_1} \leq \rho_2(\mathcal{A}^{\frac{1}{2}}x, x)_{X_1}, \quad x \in \mathcal{D}(\mathcal{A}^{\frac{1}{4}}). \tag{3E.111}$$

For the meaning of the double inequality (3E.111) in the present framework, we recall Appendix 3B, Eqn. (3B.1a, b, c) with $\alpha = 1/2$ and related Theorem 3B.1(a),(b) and refer to Remark 3E.5.3 below.

Remark 3E.5.2 The present setting encompassing assumptions (h.1), (h.2), and (h.3) above plainly generalizes the setting encompassing assumptions (H.1), (H.2), and (H.3) of Section 3E.1, to which it reduces when $X_1 = X_2 = X$ and $\mathcal{B} = \mathcal{B}^* = \mathcal{C}$ a positive self-adjoint operator on X.

In the present degree of generality, the following consequence of assumptions (h.1) and (h.2) is useful; it extends (3E.3) in the setting of Section 3E.1.

Lemma 3E.5.1 *Let assumptions (h.1) and (h.2) above hold true. Then*

$$\mathcal{C}^{-\frac{1}{2}}\mathcal{B}^*\mathcal{A}^{-\frac{1}{4}} \in \mathcal{L}(X_1; X_2). \tag{3E.112}$$

Proof.

Step 1 First, we claim that

$$\mathcal{C}^{-\frac{1}{2}}\mathcal{B}^*\mathcal{A}^{-\frac{1}{4}} : \text{continuous } \mathcal{D}(\mathcal{A}^{\frac{1}{4}}) \to \mathcal{D}(\mathcal{C}^{\frac{1}{2}}), \tag{3E.113}$$

since, equivalently,

$$\mathcal{C}^{\frac{1}{2}}(\mathcal{C}^{-\frac{1}{2}}\mathcal{B}^*\mathcal{A}^{-\frac{1}{4}})\mathcal{A}^{-\frac{1}{4}} = \mathcal{B}^*\mathcal{A}^{-\frac{1}{2}} : \text{continuous } X_1 \to X_2, \tag{3E.114}$$

which is true by assumption (h.2) = (3E.108).

Step 2 Next, we claim that

$$\mathcal{C}^{-\frac{1}{2}}\mathcal{B}^*\mathcal{A}^{-\frac{1}{4}} : \text{continuous } [\mathcal{D}(\mathcal{A}^{\frac{1}{4}})]' \to [\mathcal{D}(\mathcal{C}^{\frac{1}{2}})]' \tag{3E.115}$$

(where $[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$ denotes the dual space of $\mathcal{D}(\mathcal{A}^{\frac{1}{4}})$ with respect to the pivot space X_1, while $[\mathcal{D}(\mathcal{C}^{\frac{1}{2}})]'$ denotes the dual space of $\mathcal{D}(\mathcal{C}^{\frac{1}{2}})$ with respect to the pivot space X_2), since equivalently

$$\mathcal{C}^{-\frac{1}{2}}(\mathcal{C}^{-\frac{1}{2}}\mathcal{B}^*\mathcal{A}^{-\frac{1}{4}})\mathcal{A}^{\frac{1}{4}} = \mathcal{C}^{-1}\mathcal{B}^* : \text{continuous } X_1 \to X_2, \tag{3E.116}$$

which is true by assumption (h.1b) = (3E.107).

Step 3 By interpolation [Lions, Magenes, 1972, Vol. 1, p. 27] between (3E.113) and (3E.116), we obtain (3E.112), as desired. □

Abstract System The present abstract system is

$$\begin{cases} w_{tt} + \mathcal{A}w - \mathcal{B}\theta = 0, & \text{(3E.117)} \\ \theta_t + \mathcal{C}\theta + \mathcal{B}^* w_t = 0, & \text{(3E.118)} \end{cases}$$

which extends (3E.4) and (3E.5), or in first-order form with $y(t) = [w(t), w_t(t), \theta(t)]$:

$$\dot{y} = Ay; \quad y(0) = y_0 = [w_0, w_1, \theta_0] \in Y; \tag{3E.119}$$

$$A = \begin{bmatrix} 0 & I & 0 \\ -\mathcal{A} & 0 & \mathcal{B} \\ 0 & -\mathcal{B}^* & -\mathcal{C} \end{bmatrix} : Y \supset \mathcal{D}(A) \to Y;$$

$$A^* = \begin{bmatrix} 0 & -I & 0 \\ \mathcal{A} & 0 & -\mathcal{B} \\ 0 & \mathcal{B}^* & -\mathcal{C} \end{bmatrix}; \quad \mathcal{D}(A^*) = \mathcal{D}(A), \tag{3E.120}$$

where A^* is the Y-adjoint of A; moreover,

$$Y = \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times X_1 \times X_2; \qquad \mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times \mathcal{D}(\mathcal{C}). \tag{3E.121}$$

Notice that in writing down $\mathcal{D}(A)$ in (3E.121), we have made use of assumptions (h.1b) and (h.2).

Remark 3E.5.3 If one substitutes $\theta = -\mathcal{C}^{-1}\mathcal{B}^* w_t - \mathcal{C}^{-1}\theta_t$ from (3E.118) into (3E.117), one obtains the equation $w_{tt} + \mathcal{A}w + \mathcal{B}\mathcal{C}^{-1}\mathcal{B}^* w_t = -\mathcal{B}\mathcal{C}^{-1}\theta_t$, which is a "structurally damped" (analytic semigroup, in the sense of Appendix 3B, Theorem 3B.1(a), (b) with $\alpha = 1/2$ in the mechanical variable w, by virtue of the assumptions (h.1), (h.2), and (h.3) leading to (3E.11). A proof of Theorem 3E.5.2 below, specifically based on this observation, will be given in Section 3F.4 of Appendix 3F. The following is the announced abstract generalization of Theorem 3E.2, whose proof follows closely that of Theorem 3E.2, given in the preceding Section 3E.2. Therefore, only the main differences will be noted in the sketch below. A different, indirect proof by contradiction, which however requires $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = \mathcal{D}(\mathcal{B}^*)$ rather than the relaxed condition (h.2), is given in Liu and Liu [1997]. Another direct, vastly different proof will be given in Section 3F.4 of the subsequent Appendix 3F.

Theorem 3E.5.2 *(i) Under (h.1) and (h.2) = (3E.108), the operators A and A^* in (3E.120) and (3E.121) are dissipative:*

$$\text{Re}(Ax, x)_Y = \text{Re}(A^*x, x)_Y = -(\mathcal{C}x_3, x_3)_{X_2}, \quad \forall\, x = [x_1, x_2, x_3] \in \mathcal{D}(A) = \mathcal{D}(A^*), \tag{3E.122}$$

and in fact maximally dissipative. Thus, A and A^* generate s.c. contraction semigroups e^{At} and e^{A^*t} on Y, $t \geq 0$.

(ii) Under the additional assumption (h.3) = (3E.109), the s.c. semigroup e^{At} is analytic on Y, $t > 0$, and likewise for e^{A^*t}.

Sketch of a Proof of Analyticity of e^{At}, Which Closely Parallels the Proof of Theorem 3E.2 The general strategy is the one described in Section 3E.2.1, from (3E.11) through (3E.22). In particular, the goal is to establish inequalities (3E.14), (3E.15) with X_1 in place of X, and (3E.16) with X_2 in place of X, by relying on the present counterpart of Eqns. (3E.18), (3E.19), and (3E.20), that is:

$$\mathrm{I}: \qquad iu - \frac{v}{\omega} = \frac{u_0}{\omega}, \tag{3E.123}$$

$$\mathrm{II}: \quad iv + \frac{1}{\omega}\mathcal{A}u - \frac{1}{\omega}\mathcal{B}\theta = \frac{v_0}{\omega}, \tag{3E.124}$$

$$\mathrm{III}: \quad i\theta + \frac{1}{\omega}\mathcal{B}^*v + \frac{1}{\omega}\mathcal{C}\theta = \frac{\theta_0}{\omega}, \tag{3E.125}$$

where $y = [u, v, \theta] \in \mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(\mathcal{C})$, by (3E.121).

Step 1. Counterpart of Lemma 3E.2.2.1 through Lemma 3E.2.2.4 Using (3E.122) instead of (3E.9), one obtains now

$$(\mathcal{C}\theta, \theta)_{X_2} = \mathrm{Re}(f_0, y)_Y, \quad \frac{1}{|\omega|}\left\|\mathcal{C}^{\frac{1}{2}}\theta\right\|_{X_2}^2 \le \frac{\epsilon}{2}\|y\|_Y^2 + \frac{1}{2\epsilon}\left\|\frac{f_0}{\omega}\right\|_Y^2, \tag{3E.126}$$

which is the counterpart of (3E.23)–(3E.25), while Lemma 3E.2.2.2 is unchanged:

$$\frac{1}{|\omega|}\left\|\mathcal{A}^{\frac{1}{2}}v\right\|_{X_1}, \quad \frac{1}{\sqrt{|\omega|}}\left\|\mathcal{A}^{\frac{1}{4}}v\right\|_{X_1} \le \mathrm{const}\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{3E.127}$$

Lemma 3E.2.2.3 holds true with $\mathcal{B}$ replaced by $\mathcal{C}$:

$$\frac{1}{|\omega|}\|\mathcal{C}\theta\|_{X_2} \le \mathrm{const}\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right] \tag{3E.128}$$

as, in that proof, $\mathcal{B}v$ in (3E.33) is replaced now by $\mathcal{B}^*v = (\mathcal{B}^*\mathcal{A}^{-\frac{1}{2}})\mathcal{A}^{\frac{1}{2}}v$, and (3E.35) still follows by (h.2) = (3E.108). Lemma 3E.2.2.4 continues to hold true:

$$\frac{1}{|\omega|}\|\mathcal{A}u\|_{X_1}, \qquad \frac{1}{\sqrt{|\omega|}}\left\|\mathcal{A}^{\frac{3}{4}}u\right\|_{X_1} \le \mathrm{const}\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right] \tag{3E.129}$$

as, in that proof, $\mathcal{B}\theta$ in (3E.39) is rewritten now as $(\mathcal{B}\mathcal{C}^{-1})\mathcal{C}\theta$, so that $\|\mathcal{B}\theta\|_{X_1} \le \mathrm{const}\|\mathcal{C}\theta\|_{X_2}$, to which we apply (3E.128) to obtain again (3E.40).

Step 2. Counterpart of Proposition 3E.2.3.1 The counterpart of Proposition 3E.2.3.1 is now

$$\left|\frac{1}{\omega}(\mathcal{B}^* v, \theta)_{X_2}\right| \leq \epsilon \|y\|_Y^2 + C_\epsilon \left\|\frac{f_0}{\omega}\right\|_Y^2, \tag{3E.130}$$

as one obtains now by estimating as follows, recalling (3E.112):

$$\left|\frac{1}{\omega}(\mathcal{B}^* v, \theta)_{X_2}\right| = \left|\frac{1}{\omega}\left((\mathcal{C}^{-\frac{1}{2}}\mathcal{B}^*\mathcal{A}^{-\frac{1}{4}})\mathcal{A}^{\frac{1}{4}} v, \mathcal{C}^{\frac{1}{2}}\theta\right)_{X_2}\right|$$

$$\text{(by (3E.112))} \qquad \leq \text{const}\frac{\left\|\mathcal{A}^{\frac{1}{4}} v\right\|_{X_1}}{\sqrt{|\omega|}} \frac{\left\|\mathcal{C}^{\frac{1}{2}}\theta\right\|_{X_2}}{\sqrt{|\omega|}}, \tag{3E.131}$$

to which we then apply (3E.127) for v and (3E.126) for θ hereby reobtaining (3E.48).

Step 3. Counterpart of Proposition 3E.2.4.1: Required Estimate (3E.16) for θ Proposition 3E.2.4.1 continues to hold true:

$$\|\theta\|_{X_2}^2 \leq \epsilon \|y\|_Y^2 + C_\epsilon \left\|\frac{f_0}{\omega}\right\|_Y^2 \tag{3E.132}$$

as, in proof, Eqn. III = (3E.125) now yields after X_2-inner product with θ:

$$\|\theta\|_{X_2}^2 \leq \left|\frac{1}{\omega}(\mathcal{B}^* v, \theta)_{X_2}\right| + \frac{1}{|\omega|}\left\|\mathcal{C}^{\frac{1}{2}}\theta\right\|_{X_2}^2 + \left|\left(\frac{\theta_0}{\omega}, \theta\right)_{X_2}\right|, \tag{3E.133}$$

to whose first two terms we now apply (3E.130) and (3E.126), respectively, thus re-obtaining (3E.51).

Step 4. Counterpart of Lemma 3E.2.5.1 The counterpart of Lemma 3E.2.5.1 is now

$$\frac{c}{|\omega|}\left\|\mathcal{A}^{\frac{1}{4}} v\right\|_{X_1}^2 \leq \left|\frac{1}{\omega}(\mathcal{C}^{-1}\mathcal{B}^* v, \mathcal{B}^* v)_{X_2}\right| \leq \epsilon \|y\|_Y^2 + C_\epsilon \left\|\frac{f_0}{\omega}\right\|_Y^2. \tag{3E.134}$$

In fact, the left-hand side inequality of (3E.134) is nothing but assumption (h.3)–(3E.109), since $v \in \mathcal{D}(\mathcal{A}^{\frac{1}{4}})$; see (3E.127). To prove the right-hand side inequality of (3E.134), we proceed as in (3E.53) through (3E.56). Taking the X_2-inner product of Eqn. III = (3E.125) with $\mathcal{C}^{-1}\mathcal{B}^* v$ (where (h.1b) = (3E.107) is used, so that $\mathcal{C}^{-1}\mathcal{B}^* \in \mathcal{L}(X_1; X_2)$), we obtain:

$$\left|\frac{1}{\omega}(\mathcal{B}^* v, \mathcal{C}^{-1}\mathcal{B}^* v)_{X_2}\right| = \left|\left(\frac{\theta_0}{\omega}, \mathcal{C}^{-1}\mathcal{B}^* v\right)_{X_2} - i(\theta, \mathcal{C}^{-1}\mathcal{B}^* v)_{X_2} - \frac{1}{\omega}(\theta, \mathcal{B}^* v)_{X_2}\right| \tag{3E.135}$$

$$\leq \left\|\frac{\theta_0}{\omega}\right\| \|\mathcal{C}^{-1}\mathcal{B}^*\| \|v\| + \|\theta\| \|\mathcal{C}^{-1}\mathcal{B}^*\| \|v\| + \left|\frac{1}{\omega}(\theta, \mathcal{B}^* v)_{X_2}\right|, \tag{3E.136}$$

in the appropriate norms. Equations (3E.135), (3E.136) are the counterparts of (3E.53), (3E.54), respectively. Using now (3E.132) for $\|\theta\|$ in (3E.136), and (3E.130) for the last term of (3E.136), we readily reobtain (3E.56), that is, (3E.134).

Step 5. Counterpart of Proposition 3E.2.5.1: Required Estimate for v Proposition 3E.2.6.1 remains unchanged:

$$\|v\|_{X_1}^2 \le \epsilon\|y\|_Y^2 + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y^2 \tag{3E.137}$$

by virtue of the same proof based on (3E.127) for $\|\mathcal{A}^{\frac{1}{4}} v\|/\sqrt{|\omega|}$ and (3E.129) for $\|\mathcal{A}^{\frac{3}{4}} u\|/\sqrt{|\omega|}$, as well as (3E.130) for $(\mathcal{B}\theta, v)_{X_1}/\omega$.

Step 6. Counterpart of Proposition 3E.2.7.1: Required Estimate for u Proposition E3.2.7.1 remains unchanged:

$$\left\|\mathcal{A}^{\frac{1}{4}} u\right\|_{X_1}^2 \le \epsilon\|y\|_Y^2 + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y^2, \tag{3E.138}$$

by virtue of the same proof.

Thus, estimates (3E.132) for θ, (3E.137) for v, and (3E.138) for u prove the required inequalities (counterparts of (3E.14), (3E.15), and (3E.16)) for analyticity of e^{At} on Y. The proof of the analyticity claim in Theorem 3E.5.2 is complete. □

3F Analyticity of the s.c. Semigroup Arising from Abstract Thermo-Elastic Equations. Second Proof

3F.1 Abstract Thermo-Elastic Equations. Analyticity

In the present Appendix 3F, as well as in the forthcoming Appendix 3G, we return to the abstract thermo-elastic system considered in Appendix 3E, Eqns. (3E.4) and (3E.5), under two abstract sets of hypotheses:

(i) Set #1, which is the same as the set of assumptions (H.1), (H.2), (H.3) of Appendix 3E;
(ii) a variation thereof, labeled here Set # 2.

Our goal is to provide two additional proofs of the analyticity of the thermo-elastic semigroup (one here, the other in Appendix 3G) that are quite different in conception and technicalities from the proof of Appendix 3E. Moreover, these two proofs share one common idea: that of reducing the analyticity of the 3×3 operator matrix generator A arising from thermo-elastic equations (see A in Appendix 3E, Eqns. (3E.6), (3E.7) to the analyticity of the 2×2 operator matrix generator, arising in structurally damped elastic equations, and dealt with in Appendix 3B; see Appendix 3B, Eqns. (3B.4), (3B.5). Otherwise, the two proofs are also quite different from each other, in both approach and technicalities (with some variations on the abstract setting pertaining to each of them). For convenience, we repeat the abstract setting of the present appendix.

Mathematical Setting Let X be a Hilbert space with norm $\| \ \|_X$ and inner product $(\ , \ )_X$. On it, we consider two operators $\mathcal{A}$ and $\mathcal{B}$ subject to the following two sets of assumptions.

Set # 1 It consists of assumptions (H.1), (H.2) [or (H.2 weak)], and (H.3) below.

(H.1) $\mathcal{A}: X \supset \mathcal{D}(\mathcal{A}) \to X$ and $\mathcal{B}: X \supset \mathcal{D}(\mathcal{B}) \to X$ are two strictly positive self-adjoint operators.

(H.2)

$$\mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \subset \mathcal{D}(\mathcal{B}), \quad \text{equivalently } \mathcal{B}\mathcal{A}^{-\frac{1}{2}} \in \mathcal{L}(X) \tag{3F.1}$$

(the implications $\Rightarrow$ follows by the closed graph theorem).

(H.3) There is a constant $c > 0$ such that

$$c\big\|\mathcal{A}^{\frac{1}{4}}x\big\|_X \le \big\|\mathcal{B}^{\frac{1}{2}}x\big\|_X, \quad \forall\, x \in \mathcal{D}\big(\mathcal{A}^{\frac{1}{4}}\big), \tag{3F.2}$$

and we have already noted in Appendix 3E, Remark 3E.1, that by Lowner's theorem [Kato, 1966, Corollary 7.1, p. 146; Xia, 1983, p. 5], condition (3F.1) of assumption (3H.2) implies:

(H.2w)

$$\mathcal{D}\big(\mathcal{A}^{\frac{1}{4}}\big) \subset \mathcal{D}\big(\mathcal{B}^{\frac{1}{2}}\big), \quad \text{equivalently } \mathcal{B}^{\frac{1}{2}}\mathcal{A}^{-\frac{1}{4}} \in \mathcal{L}(X),$$

$$\text{equivalently } \big\|\mathcal{B}^{\frac{1}{2}}x\big\|_X \le C\big\|\mathcal{A}^{\frac{1}{4}}x\big\|_X, \quad \forall\, x \in \mathcal{D}\big(\mathcal{A}^{\frac{1}{4}}\big). \tag{3F.3}$$

We notice again that (H.1), (H.2w), and (H.3) yield that $\mathcal{A}^{-\frac{1}{4}}\mathcal{B}\mathcal{A}^{-\frac{1}{4}}$ is a bounded, boundedly invertible, self-adjoint operator on $\mathcal{L}(X)$.

Set # 2 It consists of assumption (H.1) above, as well as assumptions (A.2) and (A.3) below.

(A.2)

$$\mathcal{D}(\mathcal{B}) \subset \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big). \tag{3F.4}$$

(A.3)

$$\mathcal{D}\big(\mathcal{B}^{\frac{1}{2}}\big) = \mathcal{D}\big(\mathcal{A}^{\frac{1}{4}}\big). \tag{3F.5}$$

Thus, a fortiori, (A.3) implies both (H.2w) = (3F.3) and (H.3) = (3F.2) above.

The thermo-elastic operator is [Appendix 3E, Eqns. (3E.6), (3E.7)]

$$A = \begin{bmatrix} 0 & I & 0 \\ -\mathcal{A} & 0 & \mathcal{B} \\ 0 & -\mathcal{B} & -\mathcal{B} \end{bmatrix} : Y \supset \mathcal{D}(A) \to Y; \tag{3F.6}$$

$$Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times X \times X;$$

$$\mathcal{D}(A) = \begin{cases} \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(\mathcal{B}), & \text{under (H.2);} \quad (3F.7a) \\ \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{B}) \times \mathcal{D}(\mathcal{B}), & \text{under (A.2)} \quad (3F.7b) \end{cases}$$

and is the *generator of a s.c. contraction semigroup* e^{At} *on* Y [Appendix 3E, Proposition 3E.1]. It corresponds to the abstract thermo-elastic system [Appendix 3E, Eqns. (3E.4), (3E.5)],

$$\begin{cases} w_{tt} + \mathcal{A}w - \mathcal{B}\theta = 0, & (3F.8) \\ \theta_t + \mathcal{B}\theta + \mathcal{B}w_t = 0, & (3F.9) \end{cases}$$

in first-order form $\dot{y} = Ay$, with $y(t) = [w(t), w_t(t), \theta(t)]$.

The main result of the present appendix is:

Theorem 3F.1 *Assume either one of the two sets of assumptions:*

Set #1 [(H.1), (H.2) = (3F.1), (H.3) = (3F.2)], or
Set #2 [(3H.1), (A.2) = (3F.4), (A.3) = (3F.5)].

Then, the resolvent $R(\lambda, A)$ *of* A *in (3F.6) satisfies the estimate*

$$\|R(\lambda, A)\|_{\mathcal{L}(Y)} \leq \frac{C}{|\lambda|}, \quad \forall\, \lambda \textit{ with } \operatorname{Re}\lambda > 0. \tag{3F.10}$$

Hence, A generates a s.c. contraction semigroup, which, moreover, is analytic (holomorphic) on Y, $t > 0$ [Fattorini, 1983, pp. 180–5].

Of course, Appendix 3E provided a proof of the above Theorem 3F.1 for the Set #1 of assumptions. The proof of the present appendix, which covers both Set #1 and Set #2 of assumptions, is quite different from that of Appendix 3E: it is based on the observation of Remark 3E.2, whereby the analyticity of the s.c. contraction semigroup generated by the 3×3 operator A in (3F.6) is reduced to the analyticity of the s.c. contraction semigroup generated by the following 2×2 operator $(-A_1)$ [see Appendix 3B, Theorem 3B.1(a), (B), case $\alpha = 1/2$]. Under the preliminary assumption (H.1), supplemented by either (H.2) or (A.2), consider the operator $-A_1$ and its Y_1-adjoint $-A_1^*$, given by

$$-A_1 = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\mathcal{B} \end{bmatrix} : Y_1 \supset \mathcal{D}(A_1) \to Y_1; \quad -A_1^* = \begin{bmatrix} 0 & -I \\ \mathcal{A} & -\mathcal{B} \end{bmatrix}; \quad \mathcal{D}(A_1^*) = \mathcal{D}(A_1); \tag{3F.11}$$

$$Y_1 = \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times X; \quad \mathcal{D}(A_1) = \begin{cases} \mathcal{D}(\mathcal{A}) \times \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big), & \text{under (H.2);} \quad (3F.12a) \\ \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{B}), & \text{under (A.2).} \quad (3F.12b) \end{cases}$$

Theorem 3F.2 [Appendix 3B, Theorem 3B.1(a), (b)] *Assume hypotheses (H.1), (H.2w) = (3F.3), (H.3) = (3F.2), so that $\mathcal{A}^{-\frac{1}{4}}\mathcal{B}\mathcal{A}^{-\frac{1}{4}}$ is a bounded, boundedly invertible self-adjoint operator in $\mathcal{L}(X)$. Then the operator $(-A_1)$ in (3F.9) satisfies the following estimate: There is $C > 0$ such that*

$$\|R(\lambda, -A_1)\|_{\mathcal{L}(Y_1)} \le \frac{C}{|\lambda|} \quad \textit{for all } \lambda \textit{ with } \operatorname{Re}\lambda > 0, \tag{3F.13}$$

and hence [Fattorini, 1983, pp. 180–5] $(-A_1)$, with domain

$$\mathcal{D}(A_1) = \big\{x_1 \in \mathcal{D}(\mathcal{A}^{\frac{3}{4}}) \subset \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big); x_2 \in \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \subset \mathcal{D}\big(\mathcal{A}^{\frac{1}{4}}\big) : \\ \mathcal{A}^{\frac{3}{4}}x_1 + \big(\mathcal{A}^{-\frac{1}{4}}\mathcal{B}\mathcal{A}^{-\frac{1}{4}}\big)\mathcal{A}^{\frac{1}{4}}x_2 \in \mathcal{D}\big(\mathcal{A}^{\frac{1}{4}}\big)\big\}, \tag{3F.14}$$

generates a s.c. contraction, analytic semigroup $e^{-A_1 t}$ on Y_1.

3F.2 Proof of Theorem 3F.1 under Set #1 and Set #2 of Assumptions

Let $y_0 = [w_0, w_1, \theta_0] \in Y$ be the fixed initial condition. Recalling that A generates a s.c. contraction semigroup e^{At} on Y (see below (3F.7b)), we set for $\operatorname{Re}\lambda > 0$:

$$y(t) = \begin{bmatrix} w(t) \\ w_t(t) \\ \theta(t) \end{bmatrix} = e^{At}y_0; \quad \hat{y}(\lambda) = \begin{bmatrix} \hat{w}(\lambda) \\ \hat{w}_t(\lambda) \\ \hat{\theta}(\lambda) \end{bmatrix} = R(\lambda, A)y_0, \tag{3F.15}$$

where reference to the initial condition is omitted.

Step 1 [Assumptions (H.1), (H.2w) = (3F.3), (H.3) = (3F.2).] After substituting $\mathcal{B}\theta$ from (3F.9) into (3F.8), the original abstract system (3F.8), (3F.9) may be alternatively rewritten as [recall Appendix 3E, Remark 3E.2]

$$w_{tt} + \mathcal{A}w + \mathcal{B}w_t = -\theta_t; \quad \text{or} \quad \frac{d}{dt}\begin{bmatrix} w \\ w_t \end{bmatrix} = -A_1\begin{bmatrix} w \\ w_t \end{bmatrix} + \begin{bmatrix} 0 \\ \theta_t \end{bmatrix}, \tag{3F.16}$$

where the operator $(-A_1)$ is defined by (3F.11), (3F.12). On the strength of assumptions (H.1), (H.2w) = (3F.3), (H.3) = (3F.2), used in Theorem 3F.2, $(-A_1)$ is the generator of a s.c. analytic semigroup $e^{-A_1 t}$ on the space Y_1 defined by (3F.12). Notice that $Y \equiv Y_1 \times X$; see (3F.7). Thus, the solution of (3F.16), initiating at $y_0 = [w_0, w_1, \theta_0]$, is

$$\begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} = e^{-A_1 t}\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} + \int_0^t e^{-A_1(t-\tau)}\begin{bmatrix} 0 \\ -\theta_t(\tau) \end{bmatrix} d\tau, \tag{3F.17}$$

whose Laplace transform version, via (3F.15), is

$$\begin{bmatrix} \hat{w}(\lambda) \\ \hat{w}_t(\lambda) \end{bmatrix} = R(\lambda, -A_1)\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} + R(\lambda, -A_1)\begin{bmatrix} 0 \\ \theta_0 \end{bmatrix} - R(\lambda, -A_1)\begin{bmatrix} 0 \\ \lambda\hat{\theta}(\lambda) \end{bmatrix}. \tag{3F.18}$$

Equation (3F.17) [or Eqn. (3F.18)] expresses the mechanical variables $\{w, w_t\}$ in terms of the thermal variable θ.

Step 2. Lemma 3F.2.1 (energy estimate) Assume (H.1). Then, with reference to (3F.15), we have for $y_0 \in Y$:

$$\left\|\mathcal{B}^{\frac{1}{2}}\hat{\theta}(\lambda)\right\|_X^2 \le C\left\|\mathcal{B}^{-\frac{1}{2}}\theta_0\right\|_X^2 + \left|\left(\begin{bmatrix} w_0 \\ w_1 \end{bmatrix}, \begin{bmatrix} \hat{w}(\lambda) \\ \hat{w}_t(\lambda) \end{bmatrix}\right)_{Y_1}\right|. \tag{3F.19}$$

Proof. We return to the operator A in (3F.6) and obtain by the dissipativity of A (and A^*) [see Appendix 3E, Eqn. (3E.9)]:

$$\operatorname{Re}(Ax, x)_Y = \operatorname{Re}(A^*x, x)_Y = -(\mathcal{B}x_3, x_3)_X, \quad \forall x = [x_1, x_2, x_3] \in \mathcal{D}(A) = \mathcal{D}(A^*), \tag{3F.20}$$

such that for $x = [x_1, x_2, x_3] \in \mathcal{D}(A)$ and $\operatorname{Re}\lambda > 0$, we have

$$\operatorname{Re}\left((\lambda - A)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)_Y = (\operatorname{Re}\lambda)\|x\|_Y^2 + (\mathcal{B}x_3, x_3)_X. \tag{3F.21}$$

We now use (3F.21) with $x = \hat{y}(\lambda) = R(\lambda, A)y_0 \in \mathcal{D}(A)$, $y_0 \in Y$, with Y defined by (3F.7), so that in particular $\hat{\theta}(\lambda) \subset \mathcal{D}(\mathcal{B})$ by (3F.7), and $(\lambda - A)\hat{y}(\lambda) = y_0$; drop the positive term $(\operatorname{Re}\lambda)\|x\|_Y^2$ in (3F.21); recall that $Y \equiv Y_1 \times X$; and obtain

$$\begin{aligned} \left\|\mathcal{B}^{\frac{1}{2}}\hat{\theta}(\lambda)\right\|_X^2 &\le \operatorname{Re}((\lambda - A)\hat{y}(\lambda), \hat{y}(\lambda))_Y = \operatorname{Re}(y_0, \hat{y}(\lambda))_Y \\ &= \operatorname{Re}\left(\mathcal{B}^{-\frac{1}{2}}\theta_0, \mathcal{B}^{\frac{1}{2}}\hat{\theta}(\lambda)\right)_X + \operatorname{Re}\left(\begin{bmatrix} w_0 \\ w_1 \end{bmatrix}, \begin{bmatrix} \hat{w}(\lambda) \\ \hat{w}_t(\lambda) \end{bmatrix}\right)_{Y_1} \\ &\le \frac{\epsilon}{2}\left\|\mathcal{B}^{\frac{1}{2}}\hat{\theta}(\lambda)\right\|_X^2 + \frac{1}{2\epsilon}\left\|\mathcal{B}^{-\frac{1}{2}}\theta_0\right\|_X^2 + \left|\left(\begin{bmatrix} w_0 \\ w_1 \end{bmatrix}, \begin{bmatrix} \hat{w}(\lambda) \\ \hat{w}_t(\lambda) \end{bmatrix}\right)_{Y_1}\right|. \end{aligned} \tag{3F.22}$$

Selecting $0 < \epsilon < 1$, we see that (3F.22) yields (3F.19), as desired. □

Thus, Eqn. (3F.19) complements Eqn. (3F.18), in the sense that (3F.19) expresses the thermal variable θ in the norm of $\mathcal{D}(\mathcal{B}^{\frac{1}{2}})$ in terms of the mechanical variables $\{w, w_t\}$.

Step 3 (domains of fractional powers under assumptions (H.1), (H.2) or (H.1), (A.2), (A.3))

Lemma 3F.2.2 *Either (i) under assumptions (H.1), (H.2) = (3F.1) or else (ii) under assumptions (H.1), (A.2) = (3F.4), (A.3) = (3F.5), we have:*

(a)

$$\mathcal{D}\big(A_1^{\frac{1}{2}}\big) = \mathcal{D}\big(A_1^{*\frac{1}{2}}\big) = \mathcal{D}\big(\mathcal{A}^{\frac{3}{4}}\big) \times \mathcal{D}\big(\mathcal{A}^{\frac{1}{4}}\big); \tag{3F.23}$$

(b)

$$\mathcal{D}\big(A^{\frac{1}{2}}\big) = \mathcal{D}\big(\mathcal{A}^{\frac{3}{4}}\big) \times \mathcal{D}\big(\mathcal{A}^{\frac{1}{4}}\big) \times \mathcal{D}\big(\mathcal{B}^{\frac{1}{2}}\big) \tag{3F.24a}$$

$$= \mathcal{D}\big(A_1^{\frac{1}{2}}\big) \times \mathcal{D}\big(\mathcal{B}^{\frac{1}{2}}\big) = \mathcal{D}\big(\mathcal{A}_1^{*\frac{1}{2}}\big) \times \mathcal{D}\big(\mathcal{B}^{\frac{1}{2}}\big); \tag{3F.24b}$$

(c) for $y_0 \in Y$, then $\hat{y}(\lambda) = [\hat{w}(\lambda), \hat{w}_t(\lambda), \hat{\theta}(\lambda)] = R(\lambda, A)y_0 \in \mathcal{D}(A)$, so that by (3F.7),

$$\left\{\begin{array}{ll} \hat{w}(\lambda) \in \mathcal{D}(\mathcal{A}), \quad \hat{\theta}(\lambda) \in \mathcal{D}(\mathcal{B}), & \\ \left\{\begin{array}{ll} \hat{w}_t(\lambda) \in \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \subset \mathcal{D}\big(\mathcal{A}^{\frac{1}{4}}\big) \subset \mathcal{D}\big(\mathcal{B}^{\frac{1}{2}}\big) & \textit{under (H.2)}, \\ \hat{w}_t(\lambda) \in \mathcal{D}(\mathcal{B}) \subset \mathcal{D}\big(\mathcal{B}^{\frac{1}{2}}\big) = \mathcal{D}\big(\mathcal{A}^{\frac{1}{4}}\big) & \textit{under (A.2), (A.3)}, \end{array}\right. & \\ [\hat{w}(\lambda), \hat{w}_t(\lambda)] \in \mathcal{D}\big(A_1^{\frac{1}{2}}\big) \textit{ in either case by (3F.23)} & \end{array}\right\}, \tag{3F.25}$$

and

$$\big\|\mathcal{B}^{\frac{1}{2}}\hat{w}_t(\lambda)\big\|_X \le C\big\|\mathcal{A}^{\frac{1}{4}}\hat{w}_t(\lambda)\big\|_X \le c\left\|A_1^{\frac{1}{2}}\begin{bmatrix}\hat{w}(\lambda)\\ \hat{w}_t(\lambda)\end{bmatrix}\right\|_{Y_1}; \tag{3F.26}$$

(d) for $[w_0, w_1] \in Y_1$,

$$\left\|A_1^{*-\frac{1}{2}}\begin{bmatrix}w_0\\ w_1\end{bmatrix}\right\|_{Y_1}^2 = \left\|A_1^{-\frac{1}{2}}\begin{bmatrix}w_0\\ w_1\end{bmatrix}\right\|_{Y_1}^2 = \left\|\begin{bmatrix}\mathcal{A}^{-\frac{3}{4}} & 0\\ 0 & \mathcal{A}^{-\frac{1}{4}}\end{bmatrix}\begin{bmatrix}w_0\\ w_1\end{bmatrix}\right\|_{Y_1}^2$$

$$= \big\|\mathcal{A}^{-\frac{3}{4}}w_0\big\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} + \big\|\mathcal{A}^{-\frac{1}{4}}w_1\big\|_X^2; \tag{3F.27}$$

(e)

$$\big\|\mathcal{A}^{-\frac{1}{4}}\theta_0\big\|_X \le C\big\|\mathcal{B}^{-\frac{1}{2}}\theta_0\big\|_X, \quad \theta_0 \in X; \tag{3F.28}$$

(f)

$$\left\|A_1^{-\frac{1}{2}}\begin{bmatrix}0\\ \theta_0\end{bmatrix}\right\|_{Y_1} = \big\|\mathcal{A}^{-\frac{1}{4}}\theta_0\big\|_X \le C\big\|\mathcal{B}^{-\frac{1}{2}}\theta_0\big\|_X, \quad \theta_0 \in X; \tag{3F.29}$$

(g) *for* $[w_0, w_1, \theta_0] \in Y$,

$$\left\| A^{-\frac{1}{2}} \begin{bmatrix} w_0 \\ w_1 \\ \theta_0 \end{bmatrix} \right\|_Y^2 = \left\| A_1^{-\frac{1}{2}} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_{Y_1}^2 + \left\| \mathcal{B}^{-\frac{1}{2}} \theta_0 \right\|_X^2$$

$$= \left\| \begin{bmatrix} \mathcal{A}^{-\frac{3}{4}} & 0 & 0 \\ 0 & \mathcal{A}^{-\frac{1}{4}} & 0 \\ 0 & 0 & \mathcal{B}^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \theta_0 \end{bmatrix} \right\|_Y^2 \tag{3F.30}$$

$$= \left\| \mathcal{A}^{-\frac{3}{4}} w_0 \right\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})}^2 + \left\| \mathcal{A}^{-\frac{1}{4}} w_1 \right\|_X^2 + \left\| \mathcal{B}^{-\frac{1}{2}} \theta_0 \right\|_X^2. \tag{3F.31}$$

Proof. (a), (b) Under (H.1), (H.2) = (3F.1), then (3F.23) is obtained by interpolation between $\mathcal{D}(A_1^0) = Y_1$ and $\mathcal{D}(A_1) = \mathcal{D}(A_1^*)$ in (3F.12a), whereas (3F.24a) is obtained by interpolation between $\mathcal{D}(A^0) = Y$ and $\mathcal{D}(A)$ in (3F.7a). Under (A.2) = (3F.4), the same argument starting this time from (3F.12b) and (3F.7b) yields $\mathcal{D}(\mathcal{B}^{\frac{1}{2}})$ for the second component space of both $\mathcal{D}(A_1^{\frac{1}{2}}) = \mathcal{D}(A_1^{*\frac{1}{2}})$, as well as of $\mathcal{D}(A^{\frac{1}{2}})$. But $\mathcal{D}(\mathcal{B}^{\frac{1}{2}}) = \mathcal{D}(\mathcal{A}^{\frac{1}{4}})$ by (A.3) = (3F.5), so that (3F.23), (3F.24a), and (3F.24b) hold true in both cases.

(c) The validity of (3F.25) is self-explanatory. The left-hand side inequality of (3F.26) is assumption (H.2w) or (A.3), while its right-hand side inequality uses (3F.23) in both cases.

(d) Equation (3F.27) follows by duality on (3F.23).

(e), (f) By assumption (H.2w) = (3F.3) or by (A.3) = (3F.5), we have $\mathcal{B}^{\frac{1}{2}}\mathcal{A}^{-\frac{1}{4}} \in \mathcal{L}(X)$, so that, by adjointness, the operator $\mathcal{A}^{-\frac{1}{4}}\mathcal{B}^{\frac{1}{2}}$ admits a bounded extension in $\mathcal{L}(X)$. Then, writing $\mathcal{A}^{-\frac{1}{4}}\theta_0 = (\mathcal{A}^{-\frac{1}{4}}\mathcal{B}^{\frac{1}{2}})\mathcal{B}^{-\frac{1}{2}}\theta_0$ yields (3F.28). Then, (3F.27) and (3F.28) imply (3F.29).

(g) Equations (3F.30), (3F.31) are obtained by duality on (3F.24a), (3F.24b).

Step 4. Lemma 3F.2.3 Assume (H.1) and then either (H.2) = (3F.1) or else (A.2) = (3F.4) and (A.3) = (3F.5). Then, for $y_0 \in Y$, and for all λ with Re $\lambda > 0$, the following relations hold true:

(i)

$$\hat{\theta}(\lambda) = -R(\lambda, -\mathcal{B})\mathcal{B}\hat{w}_t(\lambda) + R(\lambda, -\mathcal{B})\theta_0; \tag{3F.32}$$

(ii)

$$\left\| \mathcal{B}^{-\frac{1}{2}}\hat{\theta}(\lambda) \right\|_X = \left\| -R(\lambda, -\mathcal{B})\mathcal{B}^{\frac{1}{2}}\hat{w}_t(\lambda) + R(\lambda, -\mathcal{B})\mathcal{B}^{-\frac{1}{2}}\theta_0 \right\|_X \tag{3F.33}$$

$$\leq \frac{C}{|\lambda|}\left\{ \left\| \mathcal{B}^{\frac{1}{2}}\hat{w}_t(\lambda) \right\|_X + \left\| \mathcal{B}^{-\frac{1}{2}}\theta_0 \right\|_X \right\}; \tag{3F.34}$$

(iii)

$$\|\hat{\theta}(\lambda)\|_X^2 = \left(\mathcal{B}^{\frac{1}{2}}\hat{\theta}(\lambda), \mathcal{B}^{-\frac{1}{2}}\hat{\theta}(\lambda)\right)_X \tag{3F.35}$$

$$\leq \left\|\mathcal{B}^{\frac{1}{2}}\hat{\theta}(\lambda)\right\|_X \frac{C}{|\lambda|}\left\{\left\|\mathcal{B}^{\frac{1}{2}}\hat{w}_t(\lambda)\right\|_X + \left\|\mathcal{B}^{-\frac{1}{2}}\theta_0\right\|_X\right\} \tag{3F.36}$$

$$|\lambda|\|\hat{\theta}(\lambda)\|_X^2 \leq C\left\|\mathcal{B}^{\frac{1}{2}}\hat{\theta}(\lambda)\right\|_X \left\{\left\|A_1^{\frac{1}{2}}\begin{bmatrix}\hat{w}(\lambda)\\ \hat{w}_t(\lambda)\end{bmatrix}\right\|_{Y_1} + \left\|\mathcal{B}^{-\frac{1}{2}}\theta_0\right\|_X\right\}. \tag{3F.37}$$

Proof. Equation (3F.32) is readily obtained by taking the Laplace transform of the solution to (3F.9), with $\mathcal{B}$ positive self-adjoint, hence with $(-\mathcal{B})$ the generator of a s.c. analytic (self-adjoint) semigroup. This latter property, plus application of $\mathcal{B}^{-\frac{1}{2}}$ on (3F.32) yields (3F.33) and (3F.34) from (3F.32) via (3F.25) for $\mathcal{B}\hat{w}_t(\lambda)$. Then, (3F.35) yields (3F.36) by use of (3F.34). Finally, (3F.36) yields (3F.37) by recalling (3F.3) [or (3F.5)] and (3F.24). □

Step 5. Lemma 3F.2.4 Assume either Set #1 [hypotheses (H.1), (H.2), (H.3)] or Set #2 [hypotheses (H.1), (A.2), (A.3)]. Then, for $y_0 = [w_0, w_1, \theta_0] \in Y$ and Re $\lambda > 0$, the following estimate holds true:

$$\left\|A_1^{\frac{1}{2}}\begin{bmatrix}\hat{w}(\lambda)\\ \hat{w}_t(\lambda)\end{bmatrix}\right\|_{Y_1}^2 \leq C\left[\left\|A_1^{-\frac{1}{2}}\begin{bmatrix}w_0\\ w_1\end{bmatrix}\right\|_{Y_1}^2 + \left\|\mathcal{B}^{-\frac{1}{2}}\theta_0\right\|_X^2 + \left\|\mathcal{B}^{\frac{1}{2}}\hat{\theta}(\lambda)\right\|_X^2\right]. \tag{3F.38}$$

Proof. We return to Eqn. (3F.18), apply $A_1^{\frac{1}{2}}$ to both sides, which is allowed by (3F.25), and obtain

$$\begin{aligned}A_1^{\frac{1}{2}}\begin{bmatrix}\hat{w}(\lambda)\\ \hat{w}_t(\lambda)\end{bmatrix} &= A_1 R(\lambda, -A_1)A_1^{-\frac{1}{2}}\begin{bmatrix}w_0\\ w_1\end{bmatrix} + A_1 R(\lambda, -A_1)A_1^{-\frac{1}{2}}\begin{bmatrix}0\\ \theta_0\end{bmatrix}\\ &\quad + \lambda^{\frac{1}{2}}A_1^{\frac{1}{2}}R(\lambda, -A_1)\begin{bmatrix}0\\ \lambda^{\frac{1}{2}}\hat{\theta}(\lambda)\end{bmatrix}.\end{aligned} \tag{3F.39}$$

Since $(-A_1)$ is the generator of a s.c. analytic contraction semigroup on Y_1, by Theorem 3F.2, under both Set #1 and Set #2 of assumptions, we have the uniform bounds [Fattorini, 1983, pp. 180–5; Kato, 1966, p. 115]

$$\|A_1 R(\lambda, -A_1)\|_{\mathcal{L}(Y_1)} + \left\|\lambda^{\frac{1}{2}}A_1^{\frac{1}{2}}R(\lambda, -A_1)\right\|_{\mathcal{L}(Y_1)} \leq \text{const.}, \quad \text{Re}\,\lambda > 0, \tag{3F.40}$$

which used in (3F.39) yield

$$\left\|A_1^{\frac{1}{2}}\begin{bmatrix}\hat{w}(\lambda)\\ \hat{w}_t(\lambda)\end{bmatrix}\right\|_{Y_1}^2 \leq C\left[\left\|A_1^{-\frac{1}{2}}\begin{bmatrix}w_0\\ w_1\end{bmatrix}\right\|_{Y_1}^2 + \left\|A_1^{-\frac{1}{2}}\begin{bmatrix}0\\ \theta_0\end{bmatrix}\right\|_{Y_1}^2 + |\lambda|\|\hat{\theta}(\lambda)\|_X^2\right] \tag{3F.41}$$

$$\text{(by (3F.29) and (3F.37))} \quad \leq C\left[\left\|A_1^{-\frac{1}{2}}\begin{bmatrix}w_0\\ w_1\end{bmatrix}\right\|_{Y_1}^2 + \|\mathcal{B}^{-\frac{1}{2}}\theta_0\|_X^2\right]$$

$$+ C\left\|\mathcal{B}^{\frac{1}{2}}\hat{\theta}(\lambda)\right\|_X\left[\left\|A_1^{\frac{1}{2}}\begin{bmatrix}\hat{w}(\lambda)\\ \hat{w}_t(\lambda)\end{bmatrix}\right\|_{Y_1} + \|\mathcal{B}^{-\frac{1}{2}}\theta_0\|_X\right]. \tag{3F.42}$$

For $0 < \epsilon < 1$, using

$$\left\|\mathcal{B}^{\frac{1}{2}}\hat{\theta}(\lambda)\right\|_X\left\|A_1^{\frac{1}{2}}\begin{bmatrix}\hat{w}(\lambda)\\ \hat{w}_t(\lambda)\end{bmatrix}\right\|_{Y_1} \le \frac{\epsilon}{2}\left\|A_1^{\frac{1}{2}}\begin{bmatrix}\hat{w}(\lambda)\\ \hat{w}_t(\lambda)\end{bmatrix}\right\|_{Y_1}^2 + \frac{1}{2\epsilon}\left\|\mathcal{B}^{\frac{1}{2}}\hat{\theta}(\lambda)\right\|_X^2 \tag{3F.43}$$

in (3F.42) yields (3F.38).

Step 6. Lemma 3F.2.5 *Assume either Set #1 [hypotheses (H.1), (H.2), (H.3)] or Set #2 [hypotheses (H.1), (A.2), (A.3)]. Then, the following basic estimate for the mechanical variables holds true for* $\operatorname{Re}\lambda > 0$*:*

$$\left\|A_1^{\frac{1}{2}}\begin{bmatrix}\hat{w}(\lambda)\\ \hat{w}_t(\lambda)\end{bmatrix}\right\|_{Y_1} \le C\left[\left\|A_1^{-\frac{1}{2}}\begin{bmatrix}w_0\\ w_1\end{bmatrix}\right\|_{Y_1} + \left\|\mathcal{B}^{-\frac{1}{2}}\theta_0\right\|_X^2\right]. \tag{3F.44}$$

Proof. First, we return to Eqn. (3F.19), which we rewrite and estimate accordingly, recalling also (3F.25):

$$\begin{aligned}\left\|\mathcal{B}^{\frac{1}{2}}\hat{\theta}(\lambda)\right\|_X^2 &\le C\left\|\mathcal{B}^{-\frac{1}{2}}\theta_0\right\|_X^2 + \left|\left(A_1^{*-\frac{1}{2}}\begin{bmatrix}w_0\\ w_1\end{bmatrix}, A_1^{\frac{1}{2}}\begin{bmatrix}\hat{w}(\lambda)\\ \hat{w}_t(\lambda)\end{bmatrix}\right)_{Y_1}\right|\\ &\le C\left\|\mathcal{B}^{-\frac{1}{2}}\theta_0\right\|_X^2 + \frac{\epsilon}{2}\left\|A_1^{\frac{1}{2}}\begin{bmatrix}\hat{w}(\lambda)\\ \hat{w}_t(\lambda)\end{bmatrix}\right\|_{Y_1}^2 + \frac{1}{2\epsilon}\left\|A_1^{*-\frac{1}{2}}\begin{bmatrix}w_0\\ w_1\end{bmatrix}\right\|_{Y_1}^2, \quad \operatorname{Re}\lambda > 0.\end{aligned} \tag{3F.45}$$

Inserting estimate (3F.45) in the last term of inequality (3F.38) along with (3F.27) yields (3F.44). □

Step 7. Lemma 3F.2.6 *Assume either Set #1 [hypotheses (H.1), (H.2), (H.3)] or Set #2 [hypotheses (H.1), (A.2), (A.3)]. Then the following basic estimate holds true for* $\operatorname{Re}\lambda > 0$*, and* $y_0 \in Y$*:*

$$\left\|A_1^{\frac{1}{2}}\begin{bmatrix}\hat{w}(\lambda)\\ \hat{w}_t(\lambda)\end{bmatrix}\right\|_{Y_1}^2 + \left\|\mathcal{B}^{\frac{1}{2}}\hat{\theta}(\lambda)\right\|_X^2 \le C\left[\left\|A_1^{-\frac{1}{2}}\begin{bmatrix}w_0\\ w_1\end{bmatrix}\right\|_{Y_1}^2 + \left\|\mathcal{B}^{-\frac{1}{2}}\theta_0\right\|_X^2\right]. \tag{3F.46}$$

Proof. We add $\|\mathcal{B}^{\frac{1}{2}}\hat{\theta}(\lambda)\|_X^2$ to both sides of Eqn. (3F.44) and, on the right-hand side, we use estimate (3F.45) for it, along with (3F.27). We thus obtain (3F.46). □

Step 8 Return to topologies based on A and Y.

Theorem 3F.2.7 *Assume either Set #1 [hypotheses (H.1), (H.2), (H.3)] or Set #2 [hypotheses (H.1), (A.2), (A.3)]. Then, estimate (3F.46) for* $\operatorname{Re}\lambda > 0$ *and* $y_0 \in Y$ *is equivalently rewritten as*

$$\left\| A^{\frac{1}{2}} \begin{bmatrix} \hat{w}(\lambda) \\ \hat{w}_t(\lambda) \\ \hat{\theta}(\lambda) \end{bmatrix} \right\|_Y^2 = \left\| A^{\frac{1}{2}} R(\lambda, A) y_0 \right\|_Y^2 = \left\| A R(\lambda, A) A^{-\frac{1}{2}} y_0 \right\|_Y^2$$

$$\leq C \left\| A^{-\frac{1}{2}} \begin{bmatrix} w_0 \\ w_1 \\ \theta_0 \end{bmatrix} \right\|_Y^2 = C \left\| A^{-\frac{1}{2}} y_0 \right\|_Y^2. \tag{3F.47}$$

Proof. We invoke (3F.24b) on the left-hand side of (3F.46), invoke (3F.31) on the right-hand side of (3F.46), and recall (3F.15), thereby obtaining estimate (3F.47). □

Step 9 (Extension of (3F.47)) Estimate (3F.47) can be equivalently rewritten as

$$\|AR(\lambda, A) z_0\|_{\mathcal{L}(Y)} \leq C \|z_0\|_Y, \quad \operatorname{Re}\lambda > 0, \tag{3F.48}$$

for all $z_0 = A^{-\frac{1}{2}} y_0 \in \mathcal{D}(A^{\frac{1}{2}})$, $y_0 \in Y$. But then (3F.48) is extended to all $z_0 \in Y$, and estimate (3F.10) is then proved.

Theorem 3F.1 under either Set #1 [hypotheses (H.1), (H.2), (H.3)] or under Set #2 [hypotheses (H.1), (A.2), (A.3)] is established.

Remark 3F.2.1 Assume either Set #2 [hypotheses (H.1), (A.2), (A.3)] or hypotheses (H.1), (H.2), and (A.3), so that $\mathcal{A}$ and $\mathcal{B}$ are positive self-adjoint and

$$\mathcal{D}(\mathcal{B}^{\frac{1}{2}}) = \mathcal{D}(\mathcal{A}^{\frac{1}{4}}) \quad \text{and} \quad \begin{cases} \text{either} & \mathcal{D}(\mathcal{B}) \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \\ \text{or else} & \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset \mathcal{D}(\mathcal{B}). \end{cases} \tag{3F.49}$$

The first case occurs in Example 3F.3.1 at the end of this appendix, the second case in all examples in Appendix 3E, Section 3E.4. Under (3F.49) the preceding proof leading to the key estimate (3F.38) of Lemma 3F.2.4 can be simplified. More precisely we do not need Lemma 3F.2.3 in this case. Indeed, if $y_0 \in Y$, then under (3F.49) we have $\hat{\theta}(\lambda) \subset \mathcal{D}(\mathcal{B}) \subset \mathcal{D}(\mathcal{B}^{\frac{1}{2}}) = \mathcal{D}(\mathcal{A}^{\frac{1}{4}})$ [see top line of (3F.25)]. Then, we may streamline the argument and proceed as follows: First

$$\left\| \begin{bmatrix} 0 \\ \hat{\theta}(\lambda) \end{bmatrix} \right\|_{\mathcal{D}(A_1^{\frac{1}{2}})}^2 = \left\| A_1^{\frac{1}{2}} \begin{bmatrix} 0 \\ \hat{\theta}(\lambda) \end{bmatrix} \right\|_{Y_1}^2 = \left\| \mathcal{A}^{\frac{1}{4}} \hat{\theta}(\lambda) \right\|_X^2$$

$$\text{(by (3F.49))} \qquad \leq C \left\| \mathcal{B}^{\frac{1}{2}} \hat{\theta}(\lambda) \right\|_X^2. \tag{3F.50}$$

Then, estimate (3F.38) of Lemma 3F.2.4 can be proved as follows, by just using (3F.50) in place of Lemma 3F.2.3. Indeed, we return to Eqn. (3F.18) and write accordingly

$$A_1^{\frac{1}{2}}\begin{bmatrix}\hat{w}(\lambda)\\ \hat{w}_t(\lambda)\end{bmatrix} = A_1 R(\lambda, -A_1)A_1^{-\frac{1}{2}}\begin{bmatrix}w_0\\ w_1\end{bmatrix} + A_1 R(\lambda, -A_1)A_1^{-\frac{1}{2}}\begin{bmatrix}0\\ \theta_0\end{bmatrix} - \lambda R(\lambda, -A_1)A_1^{\frac{1}{2}}\begin{bmatrix}0\\ \hat{\theta}(\lambda)\end{bmatrix}, \tag{3F.51}$$

in place of (3F.39). Using now that $(-A_1)$ is the generator of a s.c. analytic semigroup by Theorem 3F.2, and, moreover, recalling estimates (3F.29) on $\|A_1^{-\frac{1}{2}}[0, \theta_0]\|_{Y_1}$ and estimate (3F.50), we readily obtain from (3F.51)

$$\left\|A_1^{\frac{1}{2}}\begin{bmatrix}\hat{w}(\lambda)\\ \hat{w}_t(\lambda)\end{bmatrix}\right\|_{Y_1}^2 \leq C\left[\left\|A_1^{-\frac{1}{2}}\begin{bmatrix}w_0\\ w_1\end{bmatrix}\right\|_{Y_1}^2 + \left\|\mathcal{B}^{-\frac{1}{2}}\theta_0\right\|_X^2 + \left\|\mathcal{B}^{\frac{1}{2}}\hat{\theta}(\lambda)\right\|_X^2\right]. \tag{3F.52}$$

This way, we reobtain (3F.38) via (3F.52).

3F.3 Applications

Section 3E.4 of Appendix 3E provided several examples of concrete thermo-elastic problems covered under Set #1 of assumptions. Here we illustrate an application covered by Set #2: (H.1), (A.2), (A.3).

Example 3F.3.1 (damped free BC) Here we take $\dim\Omega = 2$, as the model below is two dimensional. We consider the following problem:

$$\begin{cases} w_{tt} + \Delta^2 w + \Delta\theta \equiv 0 & \text{in } (0, T] \times \Omega \equiv Q; & \text{(3F.53a)}\\ \theta_t - \Delta\theta - \Delta w_t \equiv 0 & \text{in } Q; & \text{(3F.53b)}\\ w(0, \cdot) = w_0, \;\; w_t(0, \cdot) = w_1, \;\; \theta(0, \cdot) = \theta_0 & \text{in } \Omega; & \text{(3F.53c)}\\ \Delta w + (1-\mu)B_1 w = 0 & \text{on } (0, T] \times \Gamma \equiv \Sigma; & \text{(3F.53d)}\\ \dfrac{\partial\Delta w}{\partial\nu} + (1-\mu)B_2 w - w = -\dfrac{\partial}{\partial\nu}\theta & \text{on } \Sigma; & \text{(3F.53e)}\\ \dfrac{\partial\theta}{\partial\nu} = -\dfrac{\partial w_t}{\partial\nu} & \text{on } \Sigma. & \text{(3F.53f)} \end{cases}$$

Here, B_1 and B_2 are the usual boundary operators arising in "free boundary conditions" [Appendix 3C, Eqns. (3C.11) and (3C.12)]:

$$\begin{cases} B_1 w \equiv 2\nu_1\nu_2 w_{xy} - \nu_1^2 w_{yy} - \nu_2^2 w_{xx}, & \text{(3F.54)}\\ B_2 w \equiv \dfrac{\partial}{\partial\tau}\left[\left(\nu_1^2 - \nu_2^2\right)w_{xy} + \nu_1\nu_2(w_{yy} - w_{xx})\right], & \text{(3F.55)} \end{cases}$$

where $\nu = [\nu_1, \nu_2]$ is the unit outer normal to the boundary $\Gamma = \partial\Omega$ and $\tau = [-\nu_2, \nu_1]$ is a unit tangent vector along Γ. Moreover, $0 < \mu < 1$ mathematically [Appendix 3C]. Introduce the biharmonic operator $\mathcal{A}$ with free BC and the harmonic operator $\mathcal{A}_N$

with Neumann BC:

$$\mathcal{A}f = \Delta^2 f,$$

$$\mathcal{D}(\mathcal{A}) = \Big\{ f \in H^4(\Omega) : [\Delta f + (1-\mu)B_1 f]_\Gamma = \left[\frac{\partial \Delta f}{\partial \nu} + (1-\mu)B_2 w - w\right]_\Gamma = 0 \Big\}; \tag{3F.56}$$

$$\mathcal{A}_N f = -\Delta f, \quad \mathcal{D}(\mathcal{A}_N) = \left\{ f \in H^2(\Omega) : \left.\frac{\partial f}{\partial \nu}\right|_\Gamma = 0 \right\}. \tag{3F.57}$$

Then, $\mathcal{A}: L_2(\Omega) \supset \mathcal{D}(\mathcal{A}) \to L_2(\Omega)$ and $\mathcal{A}_N : L_2^0(\Omega) \supset \mathcal{D}(\mathcal{A}_N) \to L_2^0(\Omega)$ are strictly positive self-adjoint operators [Lasiecka, Triggiani, 1982; Chapter 3, Appendix 3C, Propositions 3C.4 and 3C.5], where $L_2^0(\Omega) = L_2(\Omega)/\mathcal{N}(\mathcal{A}_N)$, and where $\mathcal{N}(\mathcal{A}_N)$ is the one-dimensional null space of $\mathcal{A}_N$ of constant functions.

Proposition 3F.3.1 *With reference to (3F.56), (3F.57), the abstract model for problem (3F.53a–f) is given on $X = L_2^0(\Omega)$ by*

$$\begin{cases} w_{tt} + \mathcal{A}w - \mathcal{A}_N \theta = 0, & (3\text{F}.58) \\ \theta_t + \mathcal{A}_N \theta + \mathcal{A}_N w_t = 0. & (3\text{F}.59) \end{cases}$$

The proof of Proposition 3F.3.1 is given at the end of this section. Here, we analyze its consequences. Set

$$X = L_2^0(\Omega), \quad \mathcal{B} = \mathcal{A}_N, \tag{3F.60}$$

so that assumption (H.1) is verified. We have [Grisvard, 1967; Appendix 3A, Theorem 3A.1]

$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega) \supset \mathcal{D}(\mathcal{B}) = \mathcal{D}(\mathcal{A}_N); \tag{3F.61}$$

$$\mathcal{D}(\mathcal{A}^{\frac{1}{4}}) = H^1(\Omega) = \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}) = \mathcal{D}(\mathcal{B}^{\frac{1}{2}}). \tag{3F.62}$$

Then, (3F.61) verifies assumption (A.2) = (3F.4), while (3F.62) verifies assumption (A.3) = (3F.5). We conclude that Theorem 3F.1 holds true under the Set #2 of assumptions [(H.1), (A.2), (A.3)].

Proof of Proposition 3F.3.1.

Step 1 Let G_2 be the Green operator corresponding to the second mechanical BC in (3F.53b) [see Section 3.13, Eqn. (3.13.13)]:

$$h \equiv G_2 g \iff \begin{cases} \Delta^2 h = 0, & \text{in } \Omega; & (3\text{F}.63\text{a}) \\ [\Delta h + (1-\mu)B_1 h]_\Gamma = 0, & \text{in } \Gamma; & (3\text{F}.63\text{b}) \\ \left[\dfrac{\partial \Delta h}{\partial \nu} + (1-\mu)B_2 h - h\right]_\Gamma = g, & \text{in } \Gamma, & (3\text{F}.63\text{c}) \end{cases}$$

where we have set

$$g = -\frac{\partial\theta}{\partial\nu} = \frac{\partial w_t}{\partial\nu}. \tag{3F.64}$$

Elliptic regularity [Lions, Magenes, 1972, pp. 188–9] and [Grisvard, 1967] and Appendix 3A, Theorem 3A.1 give

$$\begin{cases} G_2 : \text{continuous } L_2(\Gamma) \to H^{\frac{7}{2}}(\Omega) \subset H^{\frac{7}{2}-4\epsilon}(\Omega) = \mathcal{D}\big(\mathcal{A}^{\frac{7}{8}-\epsilon}\big) & \text{(3F.65a)} \\ = \big\{h \in H^{\frac{7}{2}-4\epsilon}(\Omega) : [\Delta h + (1-\mu)B_1 h]_\Gamma = 0\big\}, & \text{(3F.65b)} \\ \mathcal{A}^{\frac{7}{8}-\epsilon} G_2 : \text{ continuous } L_2(\Gamma) \to L_2(\Omega), & \text{(3F.65c)} \end{cases}$$

The following key result [Section 3.13, Lemma 3.13.2, Eqn. (3.13.30)] will be invoked below:

$$G_2^* \mathcal{A} f = -f|_\Gamma, \qquad f \in \mathcal{D}\big(\mathcal{A}^{\frac{1}{8}+\epsilon}\big), \tag{3F.66}$$

where $(G_2 u, y)_{L_2(\Omega)} = (u, G_2^* y)_{L_2(\Gamma)}$.

Step 2 Let $\mathcal{N}$ be the Neumann map [Section 3.3, Eqn. (3.3.1.7)] defined by

$$h \equiv N v \Longleftrightarrow \left\{ \Delta h = 0 \text{ in } \Omega;\ \frac{\partial h}{\partial \nu}\bigg|_\Gamma = v \right\}, \tag{3F.67}$$

which uniquely defines $h \in L_2^0(\Omega) = X$. Elliptic regularity [Lions, Magenes, 1972] and [Grisvard, 1967] and Appendix 3A, Theorem 3A.1 yield

$$\begin{cases} N : \text{ continuous } L_2(\Gamma) \to H^{\frac{3}{2}}(\Omega) \subset H^{\frac{3}{2}-2\epsilon}(\Omega) = \mathcal{D}\big(\mathcal{A}_N^{\frac{3}{4}-\epsilon}\big), & \text{(3F.68a)} \\ \mathcal{A}_N^{\frac{3}{4}-\epsilon} N : \text{ continuous } L_2(\Gamma) \to L_2^0(\Omega). & \text{(3F.68b)} \end{cases}$$

The following key result, Eqn. (3.1.29) of Section 3.3, will be invoked below:

$$N^* \mathcal{A}_N f = f|_\Gamma, \quad f \in \mathcal{D}\big(\mathcal{A}_N^{\frac{1}{4}+\epsilon}\big), \tag{3F.69}$$

where $(Nu, y)_{L_2(\Omega)} = (u, N^* y)_{L_2(\Gamma)}$.

Step 3 Using the definitions of G_2 and N in (3F.63) and (3F.67), we may rewrite problem (3F.53a–f) with g as in (3F.64) as:

$$\begin{cases} w_{tt} + \Delta^2(w - G_2 g) + \Delta(\theta - N(-g)) \equiv 0 & \text{in } Q; & \text{(3F.70a)} \\ \theta_t - \Delta(\theta - N(-g)) - \Delta(w_t - Ng) \equiv 0 & \text{in } Q; & \text{(3F.70b)} \\ \Delta(w - G_2 g) + (1-\mu)B_1(w - G_2 g) = 0 & \text{on } \Sigma; & \text{(3F.70c)} \\ \dfrac{\partial\Delta(w - G_2 g)}{\partial\nu} + (1-\mu)B_2(w - G_2 g) - (w - G_2 g) \equiv 0 & \text{on } \Sigma; & \text{(3F.70d)} \\ \dfrac{\partial(\theta - N(-g))}{\partial\nu} \equiv 0, \quad \dfrac{\partial(w_t - Ng)}{\partial\nu} \equiv 0 & \text{on } \Sigma. & \text{(3F.70e)} \end{cases}$$

Thus, invoking the definition of $\mathcal{A}$ and $\mathcal{A}_N$ in (3F.56) and (3F.57), we rewrite problem (3F.70) as

$$\begin{cases} w_{tt} + \mathcal{A}(w - G_2 g) - \mathcal{A}_N(\theta + Ng) = 0 \quad \text{in } L_2(\Omega); & \text{(3F.71)} \\ \theta_t + \mathcal{A}_N(\theta + Ng) + \mathcal{A}_N(w_t - Ng) = 0 \quad \text{in } L_2(\Omega). & \text{(3F.72)} \end{cases}$$

Extending the original operators $\mathcal{A}$ and $\mathcal{A}_N$ as $\mathcal{A}: L_2(\Omega) \to [\mathcal{D}(\mathcal{A})]'$ and $\mathcal{A}_N : L_2(\Omega) \to [\mathcal{D}(\mathcal{A}_N)]'$, as usual by isomorphism, we rewrite (3F.71), (3F.72) as

$$\begin{cases} w_{tt} + \mathcal{A}w - \mathcal{A}_N\theta - \mathcal{A}G_2 g - \mathcal{A}_N Ng = 0, & \text{(3F.73)} \\ \theta_t + \mathcal{A}_N\theta + \mathcal{A}_N w_t = 0, & \text{(3F.74)} \end{cases}$$

after a cancellation in (3F.73).

Step 4. Lemma 3F.3.2 For $g \in L_2(\Gamma)$, we have

$$\mathcal{A}G_2 g + \mathcal{A}_N Ng = 0 \quad \text{in } \left[H^{\frac{1}{2}+2\epsilon}(\Omega)\right]'; \tag{3F.75}$$

$$\mathcal{D}\left(\mathcal{A}^{\frac{1}{8}+\frac{\epsilon}{2}}\right) = \mathcal{D}\left(\mathcal{A}_N^{\frac{1}{4}+\epsilon}\right) = H^{\frac{1}{2}+2\epsilon}(\Omega). \tag{3F.76}$$

Proof. Let $f \in H^{\frac{1}{2}+2\epsilon}(\Omega)$ as in (3F.76) and $g \in L_2(\Gamma)$, and compute

$$\begin{aligned} (\mathcal{A}G_2 g + \mathcal{A}_N Ng, f)_{L_2(\Omega)} &= \left(g, G_2^*\mathcal{A}f\right)_{L_2(\Gamma)} + (g, N^*\mathcal{A}_N f)_{L_2(\Gamma)} \\ \text{(by (3F.66) and (3F.69))} \quad &= -(g, f|_\Gamma)_{L_2(\Gamma)} + (g, f|_\Gamma)_{L_2(\Gamma)} = 0, \end{aligned} \tag{3F.77}$$

after critically invoking (3F.66) and (3F.69). Thus, (3F.76) proves (3F.74), as desired. □

Step 5 Using (3F.75) in (3F.73) yields (3F.58), (3F.59), as desired.

3F.4 An Abstract Generalization of the Analyticity Result of Theorem 3F.1 under Set #1 (a Second Proof of Theorem 3E.5.2.)

In this section we provide a sketch of a second proof of Theorem 3E.5.2 – the abstract generalization of the analyticity result of Theorem 3E.2 or of Theorem 3F.1, under Set #1. The present proof closely follows this time the proof of Theorem 3F.1 given in Section 3F.2. For convenience, we recall the setting and the statement from Section 3E.5.

Mathematical Setting Let X_1 and X_2 be two Hilbert spaces, with norms $\|\cdot\|_{X_i}$ and inner product $(\cdot\,,\,\cdot)_{X_i}$, respectively. We consider three operators, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, subject to the following assumptions:

(h.1a) $\mathcal{A}: X_1 \supset \mathcal{D}(\mathcal{A}) \to X_1$ and $\mathcal{C}: X_2 \supset \mathcal{D}(\mathcal{C}) \to X_2$ are two strictly positive, self-adjoint operators; while $\mathcal{B}: X_2 \supset \mathcal{D}(\mathcal{B}) \to X_1$ is a densely defined, closed operator with adjoint $\mathcal{B}^* : X_1 \supset \mathcal{D}(\mathcal{B}^*) \to X_2$, so that $\mathcal{B}^{**} = \mathcal{B}$;

(h.1b)

$$\mathcal{D}(\mathcal{C}) \subset \mathcal{D}(\mathcal{B}), \text{ equivalently } \mathcal{B}\mathcal{C}^{-1} \in \mathcal{L}(X_2; X_1),$$
$$\text{and } \mathcal{C}^{-1}\mathcal{B}^* \text{has a bounded extension in } \mathcal{L}(X_1; X_2); \tag{3F.78}$$

(h.2)

$$\mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \subset \mathcal{D}(\mathcal{B}^*), \text{ equivalently } \mathcal{B}^*\mathcal{A}^{-\frac{1}{2}} \in \mathcal{L}(X_2),$$
$$\text{and } \mathcal{A}^{-\frac{1}{2}}\mathcal{B} \text{ has a bounded extension in } \mathcal{L}(X_2); \tag{3F.79}$$

(h.3) there is a constant $\rho_1 > 0$ such that

$$\rho_1 \big\|\mathcal{A}^{\frac{1}{4}}x\big\|_{X_1}^2 \le (\mathcal{B}\mathcal{C}^{-1}\mathcal{B}^*x, x)_{X_1} = \big\|(\mathcal{B}\mathcal{C}^{-1}\mathcal{B}^*)^{\frac{1}{2}}x\big\|_{X_1}^2, \quad \forall\, x \in \mathcal{D}\big(\mathcal{A}^{\frac{1}{4}}\big), \tag{3F.80}$$

where $\mathcal{B}\mathcal{C}^{-1}\mathcal{B}^*$ is a positive, self-adjoint operator. The following consequence of (h.1), (h.2), and (h.3) above was noted in (3E.111), Section 3E.5:

$$\rho_1\big(\mathcal{A}^{\frac{1}{2}}x, x\big)_{X_1} \le ((\mathcal{B}\mathcal{C}^{-1}\mathcal{B}^*)x, x)_{X_1} \le \rho_2\big(\mathcal{A}^{\frac{1}{2}}x, x\big)_{X_1}, \quad x \in \mathcal{D}\big(\mathcal{A}^{\frac{1}{4}}\big), \tag{3F.81}$$

along with the following consequence of (h.1) and (h.2) noted in Lemma 3E.5.1, Eqn. (3E.112);

$$\mathcal{C}^{-\frac{1}{2}}\mathcal{B}^*\mathcal{A}^{-\frac{1}{4}} \in \mathcal{L}(X_1; X_2). \tag{3F.82}$$

With reference to system (3E.117), (3E.118), (3E.119), (3E.120), (3E.121) of Section 3E.5, we introduce two operators A and $-A_1$:

$$A = \begin{bmatrix} 0 & I & 0 \\ -\mathcal{A} & 0 & \mathcal{B} \\ 0 & -\mathcal{B}^* & -\mathcal{C} \end{bmatrix} : Y = \mathcal{D}(A) \to Y;$$
$$-A_1 = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\mathcal{B}\mathcal{C}^{-1}\mathcal{B}^* \end{bmatrix} : Y_1 \supset \mathcal{D}(A_1) \to Y_1; \tag{3F.83}$$

$$Y = \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times X_1 \times X_2; \quad Y_1 = \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times X_1;$$
$$\mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times \mathcal{D}(\mathcal{C}); \quad \mathcal{D}(A_1) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big). \tag{3F.84}$$

We restate Theorem 3E.5.2 as

Theorem 3F.4.1 *(i) Under (h.1) and (h.2) = (3F.79), the operator A in (3F.83), (3F.84) is dissipative, in fact maximally dissipative. Thus, A generates a s.c. contraction semigroup e^{At} on Y.*

(ii) Under the additional assumption (h.3) = (3F.80), the s.c. semigroup e^{At} is analytic on Y, $t > 0$.

Sketch of a Proof of Analyticity of e^{At}, Which Closely Parallels the Proof of Theorem 3F.1, Set #1 The general strategy is the same as in Section 3F.2, and rests upon the following result (extension of Theorem 3F.2):

Theorem 3F.4.2 *Under (h.1), (h.2), (h.3), the operator $-A_1$ in (3F.83), (3F.84) generates a s.c. contraction semigroup on Y_1, which, moreover, is analytic here for $t > 0$.*

The proof is an application of Appendix 3B, Theorem 3B.1(a),(b) with $\alpha = 1/2$ on the basis of the double inequality (3F.80).

We now parallel the proof of Section 3F.2.

Step 1 With reference to Remark 3E.5.3, the counterpart of (3F.17), (3F.18) is now

$$\begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} = e^{-A_1 t} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} + \int_0^t e^{-A_1(t-\tau)} \begin{bmatrix} 0 \\ -\mathcal{B}\mathcal{C}^{-1}\theta_t(\tau) \end{bmatrix} d\tau, \tag{3F.85}$$

$$\begin{bmatrix} \hat{w}(\lambda) \\ \hat{w}_t(\lambda) \end{bmatrix} = R(\lambda, -A_1) \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} + R(\lambda, -A_1) \begin{bmatrix} 0 \\ \mathcal{B}\mathcal{C}^{-1}\theta_0 \end{bmatrix} - R(\lambda, -A_1) \begin{bmatrix} 0 \\ \lambda \mathcal{B}\mathcal{C}^{-1}\hat{\theta}(\lambda) \end{bmatrix}. \tag{3F.86}$$

Step 2 The counterpart of Lemma 3F.2.1 (3F.19) is now (using the same proof based on the dissipativity of A):

$$\|\mathcal{C}^{\frac{1}{2}}\hat{\theta}(\lambda)\|^2_{X_2} \leq c\|\mathcal{C}^{-\frac{1}{2}}\theta_0\|^2_{X_2} + \left| \left(\begin{bmatrix} w_0 \\ w_1 \end{bmatrix}, \begin{bmatrix} \hat{w}(\lambda) \\ \hat{w}_t(\lambda) \end{bmatrix} \right)_{Y_1} \right|. \tag{3F.87}$$

Step 3 Counterparts of (3F.23)–(3F.31) include now:

$$\mathcal{D}(A^{\frac{1}{2}}) = \mathcal{D}(A_1^{\frac{1}{2}}) \times \mathcal{D}(\mathcal{C}^{\frac{1}{2}}); \quad \mathcal{D}(A_1^{\frac{1}{2}}) = \mathcal{D}(A_1^{*\frac{1}{2}}) = \mathcal{D}(\mathcal{A}^{\frac{3}{4}}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{4}}); \tag{3F.88}$$

$$\left\| A_1^{*-\frac{1}{2}} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|^2_{Y_1} = \left\| A_1^{-\frac{1}{2}} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|^2_{Y_1} = \|\mathcal{A}^{-\frac{3}{4}} w_0\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} + \|\mathcal{A}^{-\frac{1}{4}} w_1\|^2_{X_1}; \tag{3F.89}$$

$$\begin{aligned} \left\| A_1^{-\frac{1}{2}} \begin{bmatrix} 0 \\ \mathcal{B}\mathcal{C}^{-1}\theta_0 \end{bmatrix} \right\|_{Y_1} &= \|\mathcal{A}^{-\frac{1}{4}}\mathcal{B}\mathcal{C}^{-1}\theta_0\|_{X_1} = \|(\mathcal{A}^{-\frac{1}{4}}\mathcal{B}\mathcal{C}^{-\frac{1}{2}})\mathcal{C}^{-\frac{1}{2}}\theta_0\|_{X_1} \\ &\leq \text{const} \|\mathcal{C}^{-\frac{1}{2}}\theta_0\|_{X_2}, \end{aligned} \tag{3F.90}$$

recalling (3F.82), which is the counterpart of (3F.28), (3F.29);

$$\left\| A^{-\frac{1}{2}} \begin{bmatrix} w_0 \\ w_1 \\ \theta_0 \end{bmatrix} \right\|^2_{Y} = \|\mathcal{A}^{-\frac{3}{4}} w_0\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} + \|\mathcal{A}^{-\frac{1}{4}} w_1\|^2_{X_1} + \|\mathcal{C}^{-\frac{1}{2}}\theta_0\|^2_{X_2}. \tag{3F.91}$$

Step 4 The counterpart of Lemma 3F.2.3 is now, by the same proof:

(i)

$$\hat{\theta}(\lambda) = -R(\lambda, -\mathcal{C})\mathcal{B}^*\hat{w}_t(\lambda) + R(\lambda, -\mathcal{C})\theta_0 \tag{3F.92}$$

by Laplace transforming $\theta_t = -\mathcal{C}\theta - \mathcal{B}^* w_t$;

(ii)

$$\|\mathcal{C}^{-1}\hat{\theta}(\lambda)\|_{X_2} \leq \frac{\text{const}}{|\lambda|}\left\{\left\|\mathcal{C}^{-\frac{1}{2}}\mathcal{B}^*\hat{w}_t(\lambda)\right\|_{X_2} + \left\|\mathcal{C}^{-\frac{1}{2}}\theta_0\right\|_{X_2}\right\}; \tag{3F.93}$$

(iii)

$$|\lambda|\|\hat{\theta}(\lambda)\|_{X_2} = |\lambda|\left(\mathcal{C}^{\frac{1}{2}}\hat{\theta}(\lambda), \mathcal{C}^{-\frac{1}{2}}\hat{\theta}(\lambda)\right)_{X_2} \leq |\lambda|\left\|\mathcal{C}^{\frac{1}{2}}\hat{\theta}(\lambda)\right\|_{X_2}\left\|\mathcal{C}^{-\frac{1}{2}}\hat{\theta}(\lambda)\right\|_{X_2} \tag{3F.94}$$

(by (3F.93))
$$\leq c\left\|\mathcal{C}^{\frac{1}{2}}\hat{\theta}(\lambda)\right\|_{X_2}\left\{\left\|(\mathcal{C}^{-\frac{1}{2}}\mathcal{B}^*\mathcal{A}^{-\frac{1}{4}})\mathcal{A}^{\frac{1}{4}}\hat{w}_t(\lambda)\right\|_{X_2} + \left\|\mathcal{C}^{-\frac{1}{2}}\theta_0\right\|_{X_2}\right\} \tag{3F.95}$$

(by (3F.82))
$$\leq c\left\|\mathcal{C}^{\frac{1}{2}}\hat{\theta}(\lambda)\right\|_{X_2}\left\{\|\mathcal{A}^{\frac{1}{4}}\hat{w}_t(\lambda)\|_{X_2} + \left\|\mathcal{C}^{-\frac{1}{2}}\theta_0\right\|_{X_2}\right\} \tag{3F.96}$$

(by (3F.88))
$$\leq c\left\|\mathcal{C}^{\frac{1}{2}}\hat{\theta}(\lambda)\right\|_{X_2}\left\{\left\|A_1^{\frac{1}{2}}\begin{bmatrix}\hat{w}(\lambda)\\ \hat{w}_t(\lambda)\end{bmatrix}\right\|_{Y_1} + \left\|\mathcal{C}^{-\frac{1}{2}}\theta_0\right\|_{X_2}\right\}. \tag{3F.97}$$

Step 5 The counterpart of Lemma 3F.2.4 is now

$$\left\|A_1^{\frac{1}{2}}\begin{bmatrix}\hat{w}(\lambda)\\ \hat{w}_t(\lambda)\end{bmatrix}\right\|_{Y_1}^2 \leq c\left[\left\|A_1^{-\frac{1}{2}}\begin{bmatrix}w_0\\ w_1\end{bmatrix}\right\|_{Y_1}^2 + \left\|\mathcal{C}^{-\frac{1}{2}}\theta_0\right\|_{X_2}^2 + \left\|\mathcal{C}^{\frac{1}{2}}\hat{\theta}(\lambda)\right\|_{X_2}^2\right]. \tag{3F.98}$$

This follows by using the same proof as the one given in (3F.39) through (3F.43), which is based on applying $A_1^{\frac{1}{2}}$ to (3F.86), whereby one now writes

$$A_1^{\frac{1}{2}}\begin{bmatrix}\hat{w}(\lambda)\\ \hat{w}_t(\lambda)\end{bmatrix} = A_1 R(\lambda, -A_1)A_1^{-\frac{1}{2}}\begin{bmatrix}w_0\\ w_1\end{bmatrix} + A_1 R(\lambda, -A_1)A_1^{-\frac{1}{2}}\begin{bmatrix}0\\ \mathcal{B}\mathcal{C}^{-1}\theta_0\end{bmatrix}$$
$$- \lambda^{\frac{1}{2}}R(\lambda, -A_1)A_1^{\frac{1}{2}}\begin{bmatrix}0\\ \lambda^{\frac{1}{2}}\mathcal{B}\mathcal{C}^{-1}\hat{\theta}(\lambda)\end{bmatrix}. \tag{3F.99}$$

To the second term in (3F.99) we apply (3F.90), while for $|\lambda|\|\mathcal{B}\mathcal{C}^{-1}\hat{\theta}(\lambda)\|_{X_2}^2 \leq c|\lambda|\|\hat{\theta}(\lambda)\|^2$, via (h.1b) = (3F.78), we apply (3F.97). This way we obtain (3F.98) as in (3F.42), (3F.43).

Steps 6–7 The counterpart of Lemma 3F.2.6 and Lemma 3F.2.7 is now (by the same proof)

$$\left\|A_1^{\frac{1}{2}}\begin{bmatrix}\hat{w}(\lambda)\\ \hat{w}_t(\lambda)\end{bmatrix}\right\|_{Y_1}^2 + \|\mathcal{C}^{\frac{1}{2}}\hat{\theta}(\lambda)\|_{X_2}^2 \leq c\left[\left\|A_1^{-\frac{1}{2}}\begin{bmatrix}w_0\\ w_1\end{bmatrix}\right\|_{Y_1}^2 + \left\|\mathcal{C}^{-\frac{1}{2}}\theta_0\right\|_{X_2}^2\right]. \tag{3F.100}$$

Step 8 By (3F.88), and (3F.91), we have that (3F.100) is rewritten as

$$\left\| A^{\frac{1}{2}}[\hat{w}(\lambda), \hat{w}_t(\lambda), \hat{\theta}(\lambda)] \right\|_Y^2 \leq c \left\| A^{-\frac{1}{2}} y_0 \right\|_Y^2, \quad y_0 = [w_0, w_1, \theta_0], \tag{3F.101}$$

which is the desired result, the counterpart of (3F.47).

The proof of Theorem 3F.4.2 is complete. □

3G Analyticity of the s.c. Semigroup Arising from Abstract Thermo-Elastic Equations. Third Proof

In the present Appendix 3G, we provide a third abstract proof of the analyticity of the thermo-elastic semigroup under abstract assumptions that generalize Set #1 of hypotheses in Appendix 3F, that is, the hypotheses of Appendix 3E. The present proof shares with the proof of Appendix 3F the common idea of reducing the analyticity of the s.c. semigroup generated by the 3×3 thermo-elastic operator in Appendix 3F, Eqn. (3F.6), (3F.7a) to the analyticity of the s.c. semigroup generated by the 2×2 structurally damped operator of Appendix 3B, Eqn. (3B.4), (3B.5). However, the present proof is purely operator-theoretic and applies to the case $1/2 \leq \alpha \leq 1$, and not only to the case $\alpha = 1/2$ as in Appendix 3E or 3F.

Mathematical Setting Let X be a Hilbert space with norm $\| \quad \|_X$ and inner product $(\ , \ )_X$. On it, we consider two operators $\mathcal{A}$ and $\mathcal{B}$ subject to the following set of assumptions, which generalize Set #1 of Appendix 3F (i.e., hypotheses of Appendix 3E).

(H.1) $\mathcal{A}: X \supset \mathcal{D}(\mathcal{A}) \to X$ and $\mathcal{B}: X \supset \mathcal{D}(\mathcal{B}) \to X$ are two strictly positive self-adjoint operators;

(H.2)

$$\mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \subset \mathcal{D}(\mathcal{B}), \quad \text{equivalently } \mathcal{B}\mathcal{A}^{-\frac{1}{2}} \in \mathcal{L}(X) \tag{3G.1}$$

(the implications $\Rightarrow$ follows by the closed graph theorem);

(SD) ("structural damping," see Appendix 3B) for constants $1/2 \leq \alpha \leq 1$, and $0 < c < C < \infty$, we have:

$$c \left\| \mathcal{A}^{\frac{\alpha}{2}} x \right\|_X \leq \left\| \mathcal{B}^{\frac{1}{2}} x \right\|_X \leq C \left\| \mathcal{A}^{\frac{\alpha}{2}} x \right\|, \quad \forall\, x \in \mathcal{D}\big(\mathcal{A}^{\frac{\alpha}{2}}\big), \tag{3G.2a}$$

equivalently,

$$c^2(\mathcal{A}^{\alpha} x, x) \leq (\mathcal{B}x, x)_X \leq C^2(\mathcal{A}^{\alpha} x, x) \quad \forall\, x \in \mathcal{D}\big(\mathcal{A}^{\frac{\alpha}{2}}\big). \tag{3G.2b}$$

A sufficient condition for (3G.2) to hold is (as noted in Appendix 3B)

$$\mathcal{D}\big(\mathcal{B}^{\frac{1}{2}}\big) = \mathcal{D}\big(\mathcal{A}^{\frac{\alpha}{2}}\big). \tag{3G.3}$$

For $\alpha = 1/2$, assumption (3G.2) reduces to the combination of (H.2w) = (3F.3) and (H.3) = (3F.2) together. Thus, the present set generalizes Set #1 to which it reduces when $\alpha = 1/2$. Notice that (H.1) and (SD) above yield that $\mathcal{A}^{-\frac{\alpha}{2}}\mathcal{B}\mathcal{A}^{-\frac{\alpha}{2}}$ is a bounded, boundedly invertible, self-adjoint operator in $\mathcal{L}(X)$ as noted in Appendix 3B.

As noted in Appendix 3F and in Appendix 3E, by Lowner's theorem [Kato, 1966, Corollary 7.1, p. 146; Xia, 1983, p. 5], condition (3G.1) of assumption (H.2) implies:

(H.2w)

$$\mathcal{D}(\mathcal{A}^{\frac{1}{4}}) \subset \mathcal{D}(\mathcal{B}^{\frac{1}{2}}), \quad \text{equivalently } \mathcal{B}^{\frac{1}{2}}\mathcal{A}^{-\frac{1}{4}} \in \mathcal{L}(X),$$

$$\text{equivalently } \left\|\mathcal{B}^{\frac{1}{2}}x\right\|_X \leq C\left\|\mathcal{A}^{\frac{1}{2}}x\right\|_X, \quad \forall\, x \in \mathcal{D}(\mathcal{A}^{\frac{1}{4}}). \tag{3G.4}$$

We rewrite for convenience the thermo-elastic operator A (Appendix 3F, Eqn. (3F.6), (3F.7a)) as

$$A = \begin{bmatrix} 0 & I & 0 \\ -\mathcal{A} & 0 & \mathcal{B} \\ 0 & -\mathcal{B} & -\mathcal{B} \end{bmatrix} : Y \supset \mathcal{D}(A) \to Y; \tag{3G.5a}$$

$$Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times X \times X; \qquad \mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(\mathcal{B}), \tag{3G.5b}$$

and A is *the generator of a s.c. contraction semigroup* e^{At} *on* Y [Appendix 3E, Proposition 3E.1]. The main result of the present appendix is

Theorem 3G.1 *Assume the set (H.1), (H.2) = (3G.1), and SD = (3G.2). Then, the resolvent* $R(\lambda, A)$ *of* A *in (3G.5) satisfies the estimate*

$$\|R(\lambda, A)\|_{\mathcal{L}(Y)} \leq \frac{C}{|\lambda|}, \quad \forall\, \lambda \quad \textit{with } \operatorname{Re}\lambda > 0. \tag{3G.6}$$

Hence, A generates a s.c. contraction semigroup, which, moreover, is analytic (holomorphic) on Y, $t > 0$ *[Fattorini, 1983, pp. 180–5].*

Our present third proof of analyticity, this time with $1/2 \leq \alpha \leq 1$, will rely on the following result from Appendix 3B, Theorem 3B.1(a), (b), Theorem 3B.1′, plus Remark 3B.3].

Theorem 3G.2 *Under assumptions (H.1), (H.2) = (3G.1), and (H.3) = (3G.2) with* λ_1 *a complex number with* $\operatorname{Re}\lambda_1 < 0$, *the operator*

$$\mathbb{A} = \begin{bmatrix} 0 & \frac{1}{2}I \\ -\mathcal{A} & \lambda_1\mathcal{B} \end{bmatrix}, \qquad Y_1 \supset \mathcal{D}(\mathbb{A}) \to Y_1 \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times X, \quad \operatorname{Re}\lambda_1 < 0; \tag{3G.7a}$$

$$\mathcal{D}(\mathbb{A}) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}), \tag{3G.7b}$$

generates a s.c. contraction, analytic semigroup $e^{\mathbb{A}t}$ *on* Y_1, $t > 0$.

3G.2 Proof of Theorem 3G.1

Proof.

Step 1 With no assumptions whatsoever, we decompose the (original) operator A in (3G.5) as follows:

$$A = \begin{bmatrix} 0 & I & 0 \\ -\mathcal{A} & 0 & \mathcal{B} \\ 0 & -\mathcal{B} & -\mathcal{B} \end{bmatrix} = A_1 + A_2 : Y \supset \mathcal{D}(A) = \mathcal{D}(A_1) \cap \mathcal{D}(A_2) \to Y; \quad (3G.8)$$

$$A_1 = \begin{bmatrix} 0 & I & 0 \\ -\mathcal{A} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathcal{B} \\ 0 & -\mathcal{B} & -\mathcal{B} \end{bmatrix}; \quad (3G.9)$$

$$\mathcal{D}(A_1) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times X; \quad \mathcal{D}(A_2) = X \times \mathcal{D}(\mathcal{B}) \times \mathcal{D}(\mathcal{B}); \quad (3G.10)$$

$$Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times X \times X. \quad (3G.11)$$

In this step, we diagonalize the bottom-right 2×2 operator submatrix of A_2, following Appendix 3D, in particular, Application #3. We first factor it as follows:

$$\begin{bmatrix} 0 & \mathcal{B} \\ -\mathcal{B} & -\mathcal{B} \end{bmatrix} = \mathcal{B}M; \quad M = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}; \quad \det(\lambda - M) = \lambda^2 + \lambda + 1 = 0. \quad (3G.12)$$

The eigenvalues λ_1 and λ_2 and the corresponding eigenvectors e_1 and e_2, of the 2×2 matrix M are:

$$\lambda_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \lambda_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}; \quad e_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}. \quad (3G.13)$$

The matrix P that diagonalizes M and its inverse $Q = P^{-1}$ are:

$$M = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}; \quad P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}; \quad Q = P^{-1} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}; \quad (3G.14)$$

$$q_{11} = \frac{\lambda_2}{\lambda_2 - \lambda_1} = \frac{1}{2} - i\frac{1}{2\sqrt{3}}; \quad q_{21} = -\frac{\lambda_1}{\lambda_2 - \lambda_1}; \quad q \equiv -\frac{q_{21}}{q_{11}} = \frac{\lambda_1}{\lambda_2} = \frac{\lambda_1}{\bar{\lambda}_1}, \quad (3G.15)$$

where we have shown only the quantities of interest below. Let

$$\Pi = PI = \begin{bmatrix} I & I \\ \lambda_1 I & \lambda_2 I \end{bmatrix}; \quad \Pi^{-1} = QI = \begin{bmatrix} q_{11}I & q_{12}I \\ q_{21}I & q_{22}I \end{bmatrix}, \quad \Pi, \Pi^{-1} \in \mathcal{L}(X \times X). \quad (3G.16)$$

Using these preliminaries, we have the following diagonalization result [Appendix 3D].

Lemma 3G.2.1 *With reference to (3G.13) and (3G.16), we have*

$$\begin{bmatrix} 0 & \mathcal{B} \\ -\mathcal{B} & -\mathcal{B} \end{bmatrix} = \Pi \begin{bmatrix} \lambda_1 \mathcal{B} & 0 \\ 0 & \lambda_2 \mathcal{B} \end{bmatrix} \Pi^{-1}. \tag{3G.17}$$

Step 2 We introduce the transformation $\mathcal{T}$ and its inverse $\mathcal{T}^{-1}$ by

$$\mathcal{T} = \left[\begin{array}{c|c} \frac{1}{q_{11}} I & 0 \quad 0 \\ \hline 0 & \\ & \Pi \\ 0 & \end{array}\right]; \quad \mathcal{T}^{-1} = \left[\begin{array}{c|c} q_{11} I & 0 \quad 0 \\ \hline 0 & \\ & \Pi^{-1} \\ 0 & \end{array}\right]; \quad \mathcal{T}, \mathcal{T}^{-1} \in \mathcal{L}(Y), \tag{3G.18}$$

via (3G.15) and (3G.16). We now use $\mathcal{T}$ as a similarity transformation to transform the original A into a more amenable form $\tilde{A}$.

Proposition 3G.2.2 *With reference to (3G.8) and (3G.18), we have*

$$A = \mathcal{T} \tilde{A} \mathcal{T}^{-1}; \quad \tilde{A} = \begin{bmatrix} 0 & q_{11} I & q_{11} I \\ -\mathcal{A} & \lambda_1 \mathcal{B} & 0 \\ q\mathcal{A} & 0 & \lambda_2 \mathcal{B} \end{bmatrix}, \quad q \equiv -\frac{q_{21}}{q_{11}} = \frac{\lambda_1}{\bar{\lambda}_1}. \tag{3G.19}$$

Proof. First, we diagonalize A_2 in (3G.9). By Lemma 3G.2.1 and (3G.18), we have

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathcal{B} \\ 0 & -\mathcal{B} & -\mathcal{B} \end{bmatrix} = \mathcal{T} \tilde{A}_2 \mathcal{T}^{-1}; \quad \tilde{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_1 \mathcal{B} & 0 \\ 0 & 0 & \lambda_2 \mathcal{B} \end{bmatrix}. \tag{3G.20}$$

Next, we similarly define the transformed version $\tilde{A}_1$ of the operator A_1 in (3G.9) by

$$\tilde{A}_1 \equiv \mathcal{T}^{-1} A_1 \mathcal{T} = \mathcal{T}^{-1} \begin{bmatrix} 0 & I & 0 \\ -\mathcal{A} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathcal{T} \tag{3G.21}$$

[recalling (3G.18) and (3G.16)]

$$= \begin{bmatrix} q_{11} I & 0 & 0 \\ 0 & q_{11} I & q_{12} I \\ 0 & q_{21} I & q_{22} I \end{bmatrix} \begin{bmatrix} 0 & I & 0 \\ -\mathcal{A} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{q_{11}} & 0 & 0 \\ 0 & I & I \\ 0 & \lambda_1 I & \lambda_2 I \end{bmatrix} \tag{3G.22}$$

$$= \begin{bmatrix} q_{11} I & 0 & 0 \\ 0 & q_{11} I & q_{12} I \\ 0 & q_{21} I & q_{22} I \end{bmatrix} \begin{bmatrix} 0 & I & I \\ -\frac{1}{q_{11}} \mathcal{A} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & q_{11} I & q_{11} I \\ -\mathcal{A} & 0 & 0 \\ -\frac{q_{21}}{q_{11}} \mathcal{A} & 0 & 0 \end{bmatrix}. \tag{3G.23}$$

We have provided the straightforward details from (3G.21) to (3G.23) to emphasize that the desirable coefficient (-1) in front of the operator $\mathcal{A}$ in the first column/second row entry of $\tilde{A}_1$ in (3G.23) is the result of the choice $\frac{1}{q_{11}}I$ in the top-left corner entry of $\mathcal{T}$ in (3G.18). Combining (3G.8), (3G.20), and (3G.21), and (3G.23), we compute

$$A = A_1 + A_2 = \mathcal{T}\{\mathcal{T}^{-1}A_1\mathcal{T} + \tilde{A}_2\}\mathcal{T}^{-1} = \mathcal{T}\{\tilde{A}_1 + \tilde{A}_2\}\mathcal{T}^{-1} = \mathcal{T}\tilde{A}\mathcal{T}^{-1}; \qquad (3G.24)$$

$$\tilde{A} = \tilde{A}_1 + \tilde{A}_2 = \begin{bmatrix} 0 & q_{11}I & q_{11}I \\ -\mathcal{A} & \lambda_1\mathcal{B} & 0 \\ -\frac{q_{21}}{q_{11}}\mathcal{A} & 0 & \lambda_2\mathcal{B} \end{bmatrix}, \quad \mathcal{D}(\tilde{A}) = \mathcal{D}(A). \qquad (3G.25)$$

Then, (3G.24) and (3G.25) prove (3G.19), as desired. □

Step 3 Proposition 3G.2.2 was obtained without any assumptions on $\mathcal{A}$ and $\mathcal{B}$, just the structure of A: It produces the operator $\tilde{A}$ that is similar to A, via the bounded, boundedly invertible (explicit) similarity transformation $\mathcal{T} \in \mathcal{L}(Y)$. In the present step, we shall begin to use the assumptions on $\mathcal{A}$ and $\mathcal{B}$ for the purpose of showing that the s.c. contraction semigroup generated by A on Y (see below (3G.5)) is, in fact, analytic on Y – more precisely, that the resolvent estimate (3G.6) on $R(\lambda, A)$ holds true. To this end, we shall show, equivalently, that the (uniformly bounded) s.c. semigroup $e^{\tilde{A}t}$ is analytic on Y, where $\tilde{A}$ given by (3G.19) is similar to A. To begin with, since $\mathrm{Re}(q_{11}) = 1/2$ from (3G.15), we shall remove bounded perturbations from $\tilde{A}$ and equivalently show that the following critical component $\tilde{A}_{cr}$ of the operator $\tilde{A}$ is the generator of a s.c. analytic semigroup on Y:

$$\tilde{A}_{cr} = \begin{bmatrix} 0 & \frac{1}{2}I & 0 \\ -\mathcal{A} & \lambda_1\mathcal{B} & 0 \\ q\mathcal{A} & 0 & \lambda_2\mathcal{B} \end{bmatrix}, \quad \mathcal{D}(\tilde{A}_{cr}) = \mathcal{D}(\tilde{A}) = \mathcal{D}(A),$$

$$\lambda_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}; \quad q = \frac{\lambda_1}{\bar{\lambda}_1}, \qquad (3G.26a)$$

where by assumption (H.2) = (3G.1),

$$\mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times \mathcal{D}(\mathcal{B}). \qquad (3G.26b)$$

To this end, we decompose further $\tilde{A}_{cr}$ as follows, by extracting its diagonal component $\tilde{A}_d$:

$$\tilde{A}_{cr} = \tilde{A}_d + \tilde{P}; \qquad (3G.27)$$

$$\tilde{A}_d = \begin{bmatrix} 0 & \frac{1}{2}I & 0 \\ -\mathcal{A} & \lambda_1\mathcal{B} & 0 \\ 0 & 0 & \lambda_2\mathcal{B} \end{bmatrix}; \quad \tilde{P} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ q\mathcal{A} & 0 & 0 \end{bmatrix}, \quad q = \frac{\lambda_1}{\bar{\lambda}_1}; \qquad (3G.28)$$

$$\mathcal{D}(\tilde{A}_d) = \mathcal{D}(\tilde{A}_{cr}) = \mathcal{D}(A) \subset \mathcal{D}(\tilde{P}) = \mathcal{D}(\mathcal{A}) \times X \times X \subset Y. \qquad (3G.29)$$

Proposition 3G.2.3. *Assume the set of hypotheses (H.1), (H.2) = (3G.1), and (SD) = (3G.2). Then, the operator $\tilde{A}_d$ in (3G.28) is the generator of a s.c. contraction, analytic semigroup $e^{\tilde{A}_d t}$ on Y, $t > 0$. Accordingly, the following resolvent characterization holds true: There is a constant $C > 0$, such that*

$$\|R(\lambda, \tilde{A}_d)\|_{\mathcal{L}(Y)} \leq \frac{C}{|\lambda|}, \quad \forall\, \lambda \quad \textit{with}\ \operatorname{Re}\lambda > 0. \tag{3G.30}$$

Proof. (i) First, by use of set (H.1), (H.2), (SD) of assumptions and of $\operatorname{Re}\lambda_1 = -1/2 < 0$ by (3G.13), we invoke Theorem 3G.2 and conclude that the operator

$$\mathbb{A} = \begin{bmatrix} 0 & \frac{1}{2}I \\ -\mathcal{A} & \lambda_1 \mathcal{B} \end{bmatrix}, \quad Y_1 \supset \mathcal{D}(\mathbb{A}) \to Y_1 \equiv \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times X, \tag{3G.31a}$$

$$\mathcal{D}(\mathbb{A}) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big), \tag{3G.31b}$$

generates a s.c. contraction, analytic semigroup $e^{\mathbb{A}t}$ on Y_1, $t > 0$.

(ii) Next, since $\mathcal{B}$ is a positive, self-adjoint operator on X by (H.1) and $\operatorname{Re}\lambda_2 = -1/2 < 0$ by (3G.13), we have that the bottom-right term $\lambda_2\mathcal{B}$ of $\tilde{A}_d$ in (3G.28) generates a s.c. analytic contraction semigroup

$$e^{\lambda_2 \mathcal{B}t} = e^{-\frac{1}{2}\mathcal{B}t} e^{-i\frac{\sqrt{3}}{2}\mathcal{B}t} \quad \text{on } X, t > 0. \tag{3G.32}$$

(iii) By block-diagonalization of $\tilde{A}_d$ in (3G.28), we conclude that the operator $\tilde{A}_d$ is, as desired, the generator of a s.c. contraction, analytic semigroup $e^{\tilde{A}_d t}$ on $Y = Y_1 \times X = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times X \times X, t > 0$. □

Step 4 Having established in Proposition 3G.2.3 that $\tilde{A}_d$ is an analytic semigroup generator, we shall transfer this same property to $\tilde{A}_{cr}$ in (3G.27) by perturbation.

Remark 3G.2.1 Up to this point, assumption (H.2) = (3G.1) that $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset \mathcal{D}(\mathcal{B})$ is not essential. If, instead, $\mathcal{D}(\mathcal{B}) \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ as in Appendix 3F, Example 3F.3.1, Eqn. (3F.53), then the only change is that $\mathcal{D}(\mathbb{A}) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{B})$, instead of (3G.31b). However, assumption (H.2) will be seriously used in the proof of Proposition 3G.2.4.

Proposition 3G.2.4 *Assume the set of hypotheses (H.1), (H.2) = (3G.1), (SD) = (3G.2). Let $R(\lambda, \tilde{A}_d)$ and $R(\lambda, \tilde{A}_{cr})$ be the resolvent operators of $\tilde{A}_d$ and $\tilde{A}_{cr}$ in (3G.28) and (3G.26), respectively. Then:*

(i) With reference to $\tilde{P}$ in (3G.27), we have

$$\|[I - \tilde{P}R(\lambda, \tilde{A}_d)]^{-1}\|_{\mathcal{L}(Y)} \leq \textit{const.}, \quad \forall\, \lambda\ \textit{with}\ \operatorname{Re}\lambda > 0; \tag{3G.33}$$

(ii)

$$\|R(\lambda, \tilde{A}_{cr})\|_{\mathcal{L}(Y)} = \|R(\lambda, \tilde{A}_d)[I - \tilde{P}R(\lambda, \tilde{A}_d)]^{-1}\|_{\mathcal{L}(Y)} \leq \frac{c}{|\lambda|},$$

$$\forall\, \lambda\ \textit{with}\ \operatorname{Re}\lambda > 0. \tag{3G.34}$$

(iii) Accordingly, the operator $\tilde{A}_{cr}$ in (3G.26) is the generator of a s.c. uniformly bounded analytic semigroup $e^{\tilde{A}_{cr}t}$ on Y, $t > 0$. □

Proof. (i) First, let

$$R(\lambda, \mathbb{A}) = \begin{bmatrix} \mathbb{R}_1(\lambda) & \mathbb{R}_2(\lambda) \\ \mathbb{R}_3(\lambda) & \mathbb{R}_4(\lambda) \end{bmatrix}, \tag{3G.35}$$

so that, by (3G.35) and (3G.28), on $\tilde{A}_d$ and $\tilde{P}$, we obtain

$$R(\lambda, \tilde{A}_d) = \begin{bmatrix} \mathbb{R}_1(\lambda) & \mathbb{R}_2(\lambda) & 0 \\ \mathbb{R}_3(\lambda) & \mathbb{R}_4(\lambda) & 0 \\ 0 & 0 & (\lambda - \lambda_2 \mathcal{B})^{-1} \end{bmatrix}; \tag{3G.36}$$

$$[I - \tilde{P}R(\lambda, \tilde{A}_d)] = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -q\mathcal{A}\mathbb{R}_1(\lambda) & -q\mathcal{A}\mathbb{R}_2(\lambda) & I \end{bmatrix}. \tag{3G.37}$$

Thus, if $g = [g_1, g_2, g_3] \in Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times X \times X$, we seek to solve $[I - \tilde{P}R(\lambda, \tilde{A}_d)]f = g$ for $f = [f_1, f_2, f_3] \in \mathcal{D}(\tilde{A}_d)$, and thus obtain

$$[I - \tilde{P}R(\lambda, \tilde{A}_d)]^{-1} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ q\mathcal{A}\mathbb{R}_1(\lambda) & q\mathcal{A}\mathbb{R}_2(\lambda) & I \end{bmatrix}. \tag{3G.38}$$

Invoking the estimates (3B.53), (3B.54) from Appendix 3B [which require the Set #3 of hypotheses: (H.1), (H.2) = (3G.1), (SD) = (3G.2)]:

$$\|\mathcal{A}\mathbb{R}_1(\lambda)\|_{\mathcal{L}(\mathcal{D}(\mathcal{A}^{\frac{1}{2}});X)} + \|\mathcal{A}\mathbb{R}_2(\lambda)\|_{\mathcal{L}(X)} \le C, \quad \forall\, \lambda \quad \text{with } \operatorname{Re}\lambda > 0, \tag{3G.39}$$

and hence (3G.33) follows at once.

(ii) The usual perturbation formula, as applied to (3G.27), is

$$R(\lambda, \tilde{A}_{cr}) = R(\lambda, \tilde{A}_d)[I - \tilde{P}R(\lambda, \tilde{A}_d)]^{-1}, \quad \operatorname{Re}\lambda > 0. \tag{3G.40}$$

Applying the uniform bound (3G.33) and the characterization (3G.30) to (3G.40) yields the resolvent estimate (3G.34).

(iii) By standard results [Fattorini, 1983, pp. 180–5], estimate (3G.34) is a characterization for $\tilde{A}_{cr}$ to generate a s.c. uniformly bounded analytic semigroup $e^{\tilde{A}_{cr}t}$ on $Y, t > 0$. □

Remark 3G.2.2 Instead of using perturbation theory as in Proposition 3G.2.4, one could readily find $R(\lambda, \tilde{A}_{cr})$ directly, and use the estimates of the Appendix 3B to obtain (3.6.34).

Step 5 Since $\tilde{A}$ differs from $\tilde{A}_{cr}$ by bounded perturbations, we obtain that $\tilde{A}$ in (3G.25) likewise generates a s.c. analytic semigroup. Thus, the same property applies to A in (3G.24), which is similar to $\tilde{A}$. The proof of Theorem 3G.1 is now complete. □

3H Analyticity of the s.c. Semigroup Arising from Problem (3.12.1) (Hinged Mechanical BC/Neumann (Robin) Thermal BC)

3H.1 Review of Problem. Statement of Main Results

In this appendix we return to the thermo-elastic problem (3.12.1) of Section 3.12, which we now strip of lower-order terms, while normalizing the (inessential) constants to 1. The key homogeneous model is then

$$\begin{cases} w_{tt} + \Delta^2 w + \Delta\theta = 0 & \text{in } (0, T] \times \Omega = Q; & \text{(3H.1a)} \\ \theta_t - \Delta\theta - \Delta w_t = 0 & \text{in } Q; & \text{(3H.1b)} \\ w(0, \cdot) = w_0, \ \ w_t(0, \cdot) = w_1, \ \ \theta(0, \cdot) = \theta_0 & \text{in } \Omega; & \text{(3H.1c)} \\ w \equiv 0, \ \ \Delta w + (1-\mu)B_1 w + \theta = 0 & \text{in } (0, T] \times \Gamma = \Sigma; & \text{(3H.1d)} \\ \dfrac{\partial\theta}{\partial\nu} + b\theta = 0 & \text{on } \Sigma; & \text{(3H.1e)} \\ B_1 w = -c(x)\dfrac{\partial w}{\partial\nu} & \text{on } \Sigma. & \text{(3H.1f)} \end{cases}$$

In (3H.1), Ω is a two-dimensional bounded open region with smooth boundary Γ, so that, in particular, the mean curvature $c(\cdot) \in L_\infty(\Gamma)$. The operator B_1 was defined in (3.12.1f) of Section 3.12. We have seen in Proposition 3.12.4 that the operator A in (3.12.24) and (3.12.25), now simplified for problem (3H.1) as

$$A\begin{bmatrix} u \\ v \\ \theta \end{bmatrix} = \begin{bmatrix} v \\ -\mathcal{A}[G(\theta|_\Gamma) + u] + \mathcal{A}_N\theta \\ -\mathcal{A}_D v - \mathcal{A}_N\theta \end{bmatrix}; \tag{3H.2}$$

$$\begin{aligned} [u, v, \theta] &\in \mathcal{D}(A) \subset \mathcal{D}(\mathcal{A}_D) \times \mathcal{D}(\mathcal{A}_D) \times \mathcal{D}(\mathcal{A}_N); \\ \mathcal{D}(\mathcal{A}_D) &= \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) = H^2(\Omega) \cap H_0^1(\Omega) \end{aligned} \tag{3H.3}$$

generates a s.c. contraction semigroup e^{At} on the space (see (3.12.7))

$$Y = \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times L_2(\Omega) \times L_2(\Omega). \tag{3H.4}$$

The operators $\mathcal{A}$, $\mathcal{A}_D$, $\mathcal{A}_N$, and G are defined in (3.12.3)–(3.12.5), and (3.12.10), respectively.

We now prove the following key result.

Theorem 3H.1 *The above s.c. contraction semigroup e^{At}:*

$$y_0 = [w_0, w_1, \theta_0] \in Y \to e^{At}y_0 = [w(t), w_t(t), \theta(t)] \tag{3H.5}$$

is, moreover, analytic on Y, $t > 0$.

3H.2 Proof of Theorem 3H.1

3H.2.1 General Strategy and Preliminaries

General Strategy With reference to the space Y in (3H.4), let $f_0 \in Y$ be arbitrary:

$$\begin{cases} f_0 = [u_0, v_0, \theta_0] \in Y \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega), \\ \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = \mathcal{D}(\mathcal{A}_D) = H^2(\Omega) \cap H_0^1(\Omega) \text{ (equivalent norms).} \end{cases} \tag{3H.6}$$

With reference to the operator A in (3H.2), let ω be real, $\omega \in \mathbb{R}$, and define

$$y(\omega) = [u(\omega), v(\omega), \theta(\omega)] = [i\omega I - A]^{-1} f_0 = R(i\omega, A) f_0 \in \mathcal{D}(A), \tag{3H.7}$$

where the resolvent of A is well defined on the imaginary axis; see Proposition 3.12.5(ii).

Our goal is to show that the following uniform estimate holds true: There exists a constant $C > 0$ such that for all $\omega \in \mathbb{R}$, with say $|\omega| \geq 1$,

$$\left\| \begin{bmatrix} u(\omega) \\ v(\omega) \\ \theta(\omega) \end{bmatrix} \right\|_Y = \|y(\omega)\|_Y = \|R(i\omega, A) f_0\|_Y \leq \frac{C}{|\omega|} \|f_0\|_Y. \tag{3H.8}$$

Once estimate (3H.8) has been established for the generator A of the s.c. contraction semigroup e^{At} asserted by Proposition 3.12.4, we can invoke Theorem 3E.3 of Appendix 3E and show that the s.c. semigroup e^{At} is, in fact, analytic on Y, $t > 0$. To establish estimate (3H.8) – and hence prove Theorem 3H.1 – we shall pursue the following strategy of Appendix 3E, which consists in proving the following three simultaneous estimates for the components of $y(\omega)$ in (3H.7): For all $\epsilon > 0$ there exists a constant $C_\epsilon > 0$, such that for all $\omega \in \mathbb{R}$, with $|\omega| > 1$, the vector $y(\omega) = [u(\omega), v(\omega), \theta(\omega)]$ in (3H.7) satisfies

$$\begin{cases} \|u(\omega)\|^2_{H^2(\Omega)} \leq \epsilon \|y(\omega)\|^2_Y + C_\epsilon \left\| \dfrac{f_0}{\omega} \right\|^2_Y; & \text{(3H.9)} \\[2ex] \|v(\omega)\|^2_{L_2(\Omega)} \leq \epsilon \|y(\omega)\|^2_Y + C_\epsilon \left\| \dfrac{f_0}{\omega} \right\|^2_Y; & \text{(3H.10)} \\[2ex] \|\theta(\omega)\|^2_{L_2(\Omega)} \leq \epsilon \|y(\omega)\|^2_Y + C_\epsilon \left\| \dfrac{f_0}{\omega} \right\|^2_Y. & \text{(3H.11)} \end{cases}$$

Hereafter, we drop from $y = [u, v, \theta]$ the explicit dependence on ω. Estimates (3H.9)–(3H.11) are proved below, in Proposition 3H.2.4.1, Eqn. (3H.59); Proposition 3H.2.6.1, Eqn. (3H.72); and Proposition 3H.2.7.1, Eqn. (3H.80), respectively. Clearly, summing up estimates (3H.9) through (3H.11) (once established) yields the final desired estimate (3H.8) with constant $C = [3C_\epsilon/(1 - 3\epsilon)]^{\frac{1}{2}}$.

Preliminaries By (3H.2), we obtain explicitly from (3H.7),

$$(i\omega I - A)\begin{bmatrix} u \\ v \\ \theta \end{bmatrix} = \begin{bmatrix} i\omega u - v \\ i\omega v + \mathcal{A}[u + G(\theta|_\Gamma)] - \mathcal{A}_N\theta \\ i\omega\theta + \mathcal{A}_D v + \mathcal{A}_N\theta \end{bmatrix} = \begin{bmatrix} u_0 \\ v_0 \\ \theta_0 \end{bmatrix} = f_0 \in Y, \quad \text{(3H.12)}$$

or, upon dividing by ω, $|\omega| \geq 1$:

$$\begin{cases} I: & iu - \dfrac{v}{\omega} = \dfrac{u_0}{\omega}; & \text{(3H.13)} \\ II: & iv + \dfrac{1}{\omega}\mathcal{A}[u + G(\theta|_\Gamma)] - \dfrac{1}{\omega}\mathcal{A}_N\theta = \dfrac{v_0}{\omega}; & \text{(3H.14)} \\ III: & i\theta + \dfrac{1}{\omega}\mathcal{A}_D v + \dfrac{1}{\omega}\mathcal{A}_N\theta = \dfrac{\theta_0}{\omega}, & \text{(3H.15)} \end{cases}$$

where, recalling (3H.3) we have a fortiori the following regularity properties:

$$y = [u, v, \theta] \in \mathcal{D}(\mathcal{A}_D) \times \mathcal{D}(\mathcal{A}_D) \times \mathcal{D}(\mathcal{A}_N), \quad \text{(3H.16)}$$

which we shall freely use below.

Orientation The basic "driving" term in the present proof is the thermal estimate (3H.20) below for θ, which follows at once from the basic a priori dissipativity condition (3H.19); see also Eqn. (3.12.32) of Section 3.12. To achieve the desired estimates (3H.9) through (3H.11), we shall (as in Appendix 3E) employ the "driving" estimate (3H.20) repeatedly, along with a priori bounds in the right norms, to *dominate* each norm-quantity $\|q\|$ of interest, as follows:

$$\|q\| \leq [a + b][\epsilon a + k_\epsilon b] \leq 2\epsilon a^2 + C_\epsilon b^2, \quad a, b \geq 0, \quad \text{(3H.17)}$$

to be specialized with $a = \|y\|_Y$ and $b = \|\frac{f_0}{\omega}\|_Y$ [inequality (3H.17) is obtained with $c_\epsilon = (\frac{k_\epsilon^2}{2\epsilon} + k_\epsilon + \frac{\epsilon}{2})$ by using $\epsilon ab \leq \frac{\epsilon}{2}(a^2 + b^2)$ and $k_\epsilon ab \leq \frac{k_\epsilon}{2}(\frac{\epsilon}{k_\epsilon} a^2 + \frac{k_\epsilon}{\epsilon} b^2)$].

3H.2.2 A Priori Bounds for θ, v, and u

We begin by recalling Lemma 3.12.2, Eqn. (3.12.32) or Lemma 3.12.3(ii), in (3H.18) below.

Lemma 3H.2.2.1 (preliminary a priori bounds for θ) *With reference to (3H.6) and (3H.7), we have with $\omega \in \mathbb{R}$:*

(i)

$$(\mathcal{A}_N\theta, \theta)_{L_2(\Omega)} = \text{Re}\left([i\omega I - A]\begin{bmatrix} u \\ v \\ \theta \end{bmatrix}, \begin{bmatrix} u \\ v \\ \theta \end{bmatrix}\right)_Y = \text{Re}(f_0, y)_Y; \quad \text{(3H.18)}$$

(ii)

$$\|\theta\|^2_{H^1(\Omega)} \doteq \left\|\mathcal{A}_N^{\frac{1}{2}}\theta\right\|^2_{L_2(\Omega)} \leq \|f_0\|_Y \|y\|_Y, \quad \text{(3H.19)}$$

where here and henceforth $\doteq$ denotes equivalence of norms;

(iii) for any $\epsilon > 0$ and $\omega \in \mathbb{R}$, $\omega \neq 0$,

$$\frac{1}{|\omega|}\|\theta\|^2_{H^1(\Omega)} \doteq \frac{1}{|\omega|}\left\|\mathcal{A}_N^{\frac{1}{2}}\theta\right\|^2_{L_2(\Omega)} \leq \frac{\epsilon}{2}\|y\|_Y^2 + \frac{1}{2\epsilon}\left\|\frac{f_0}{\omega}\right\|_Y^2. \tag{3H.20}$$

Lemma 3H.2.2.2 (a priori bounds for v) *With reference to (3H.6) and (3H.7), we have for $\omega \in \mathbb{R}$, $|\omega| \geq 1$:*

(i)

$$\frac{1}{|\omega|}\|\mathcal{A}_D v\|_{L_2(\Omega)} \doteq \frac{1}{|\omega|}\|v\|_{H^2(\Omega)} \leq \|u\|_{H^2(\Omega)} + \left\|\frac{u_0}{\omega}\right\|_{H^2(\Omega)} \tag{3H.21}$$

$$\leq \|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y; \tag{3H.22}$$

(ii)

$$\frac{1}{\sqrt{|\omega|}}\left\|\mathcal{A}_D^{\frac{1}{2}}v\right\|_{L_2(\Omega)} \doteq \frac{1}{\sqrt{|\omega|}}\|v\|_{H^1(\Omega)} \leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{3H.23}$$

Proof. (i) The validity of estimate (3H.21) stems at once from Eqn. I = (3H.13) and the norm equivalence in (3H.6). Then, (3H.21) implies at once (3H.22) by majorizing u and u_0/ω in $H^2(\Omega)$ by y and f_0/ω in Y, via (3H.6) and (3H.7).

(ii) By interpolation (moment inequality [Krain, 1988; Lions, Magenes, 1972]), we compute [henceforth we use freely $\sqrt{a^2+b^2} \leq a + b$ for $a, b \geq 0$ throughout this appendix], via (3H.22) and majorizing v by y, by (3H.6), (3H.7):

$$\|v\|_{H^1(\Omega)} \leq C\|v\|^{\frac{1}{2}}_{H^2(\Omega)}\|v\|^{\frac{1}{2}}_{L_2(\Omega)} \tag{3H.24}$$

$$\text{(by (3.2.2.5))} \qquad \leq C|\omega|^{\frac{1}{2}}\left[\|y\|_Y^{\frac{1}{2}} + \left\|\frac{f_0}{\omega}\right\|^{\frac{1}{2}}\right]\|y\|_Y^{\frac{1}{2}} \tag{3H.25}$$

$$\leq C|\omega|^{\frac{1}{2}}\left[\|y\|_Y + \frac{1}{2}\left\|\frac{f_0}{\omega}\right\| + \frac{1}{2}\|y\|_Y\right], \tag{3H.26}$$

and (3H.26) proves estimate (3H.23), as desired. □

Lemma 3H.2.2.3 (further a priori bound for θ) *With reference to (3H.6) and (3H.7), we have for $\omega \in \mathbb{R}$, $|\omega| \geq 1$:*

$$\frac{1}{|\omega|}\|\theta\|_{H^2(\Omega)} \doteq \frac{1}{|\omega|}\|\mathcal{A}_N\theta\|_{L_2(\Omega)} \leq 2\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{3H.27}$$

Proof. We return to Eqn. III = (3H.15), where we use estimate (3.2.2.5) for v, thus obtaining

$$\left\|\frac{1}{\omega}\mathcal{A}_N\theta\right\|_{L_2(\Omega)} = \left\|\frac{\theta_0}{\omega} - i\theta - \frac{1}{\omega}\mathcal{A}_D v\right\|_{L_2(\Omega)} \tag{3H.28}$$

$$\leq \left\| \frac{\theta_0}{\omega} \right\|_{L_2(\Omega)} + \|\theta\|_{L_2(\Omega)} + \frac{1}{|\omega|} \|\mathcal{A}_D v\|_{L_2(\Omega)} \tag{3H.29}$$

$$\text{(by (3H.22))} \qquad \leq \left\| \frac{f_0}{\omega} \right\|_Y + \|y\|_Y + \left[\|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right], \tag{3H.30}$$

majorizing, in the last step, θ_0 and θ by f_0 and y via (3H.6) and (3H.7). Then, (3H.30) proves (3H.27). □

Lemma 3H.2.2.4 (a priori bounds for u) *With reference to (3H.6) and (3H.7), we have for $\omega \in \mathbb{R}$, $|\omega| \geq 1$:*

(i)

$$\frac{1}{|\omega|} \|u\|_{H^4(\Omega)} \leq C \left[\|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right]; \tag{3H.31}$$

(ii)

$$\frac{1}{\sqrt{|\omega|}} \|u\|_{H^3(\Omega)} \leq C \left[\|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right]. \tag{3H.32}$$

Proof. (i) Eqn. II = (3H.14) rewritten abstractly as

$$\frac{1}{\omega} \mathcal{A}[u + G(\theta|_\Gamma)] = -iv + \frac{1}{\omega} \mathcal{A}_N \theta + \frac{v_0}{\omega} \tag{3H.33}$$

is equivalent, via the definition (3.12.9) of the Green operator G, to the following elliptic boundary value problem (i.e., the original elliptic problem (3.12.9) [see also (3.12.17a)], of which (3H.33) is the abstract version, in the first place):

$$\begin{cases} \Delta^2 \left(\dfrac{u}{\omega} \right) = -iv + \dfrac{1}{\omega} \mathcal{A}_N \theta + \dfrac{v_0}{\omega} & \text{in } \Omega; \qquad (3H.34) \\ u|_\Gamma \equiv 0; \quad [\Delta u + (1-\mu) B_1 u|_\Gamma] = -\theta|_\Gamma & \text{on } \Gamma. \qquad (3H.35) \end{cases}$$

From the right-hand side of (3H.34), we readily estimate, by virtue of (3H.27) for $\mathcal{A}_N \theta/\omega$, majorizing v and θ_0 by y and f_0 via (3H.6) and (3H.7),

$$\left\| \Delta^2 \left(\frac{u}{\omega} \right) \right\|_{L_2(\Omega)} \leq 3 \left[\|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right]. \tag{3H.36}$$

Moreover, from the second BC in (3H.35), we readily estimate for $|\omega| \geq 1$, by virtue of trace theory, (3H.1f) for B_1, and again (3H.27),

$$\left\| \Delta \left(\frac{u}{\omega} \right) \right\|_{H^{\frac{3}{2}}(\Gamma)} \leq \frac{C}{|\omega|} \|u\|_{H^2(\Omega)} + \left\| \frac{\theta}{\omega} \right\|_{H^{\frac{3}{2}}(\Gamma)}$$

$$\leq C \left[\|u\|_{H^2(\Omega)} + \frac{1}{|\omega|} \|\theta\|_{H^2(\Omega)} \right] \tag{3H.37}$$

$$\text{(by (3H.27))} \qquad \leq C \left[\|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right], \tag{3H.38}$$

majorizing u in $H^2(\Omega)$ by y in Y, via (3H.6) and (3H.7). Thus, standard elliptic theory [Lions, Magenes, 1972] applied to the right-hand side estimate (3H.36), the first BC in (3H.35), and the boundary estimate (3H.38) produces a gain of $2\frac{1}{2}$ Sobolev units from boundary to interior ($\frac{3}{2}+2\frac{1}{2}=4$) and a gain of 4 Sobolev units from right-hand side to interior ($0+4=4$), thus yielding

$$\left\|\frac{u}{\omega}\right\|_{H^4(\Omega)} \leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right], \tag{3H.39}$$

which proves (3H.31). [We note that right-hand side and boundary estimates produce, *independently*, the same interior regularity of u/ω in $H^4(\Omega)$.]

(ii) By interpolation (moment inequality), we estimate via (3H.31), and majorizing u by y, by (3H.6) and (3H.7):

$$\|u\|_{H^3(\Omega)} \leq C\|u\|_{H^4(\Omega)}^{\frac{1}{2}}\|u\|_{H^2(\Omega)}^{\frac{1}{2}} \tag{3H.40}$$

$$\text{(by (3H.31))} \qquad \leq C|\omega|^{\frac{1}{2}}\left[\|y\|_Y^{\frac{1}{2}} + \left\|\frac{f_0}{\omega}\right\|_Y^{\frac{1}{2}}\right]\|y\|_Y^{\frac{1}{2}} \tag{3H.41}$$

$$\leq C|\omega|^{\frac{1}{2}}\left[\|y\|_Y + \frac{1}{2}\left\|\frac{f_0}{\omega}\right\|_Y + \frac{1}{2}\|y\|_Y\right], \tag{3H.42}$$

and (3H.42) proves estimate (3H.32), as desired. □

Remark 3H.2.2.1 One could, alternatively, sum up Eqns. II = (3H.14) and Eqn. III = (3H.15) to eliminate $[\mathcal{A}_N\theta/\omega]$, and then use estimate (3H.22) for $[\mathcal{A}_D v/w]$ (rather than estimate (3H.27) on $[\mathcal{A}_N\theta/\omega]$), which is a consequence of (3H.22), to obtain likewise the interior bound (3H.36). It is, however, at the level of obtaining the boundary estimate (3H.38) that estimate (3H.27) is needed.

3H.2.3 A Fundamental Estimate on $\frac{1}{\omega}(\mathcal{A}_D v, \theta)_{L_2(\Omega)}$

The following result – a consequence of the driving term (3H.20) – is fundamental for the subsequent development.

Proposition 3H.2.3.1. *With reference to (3H.6) and (3H.7), given* $\epsilon > 0$, *there exists* $C_\epsilon > 0$ *such that for all* $\omega \in \mathbb{R}$, $|\omega| \geq 1$, *we have*

$$\left|\frac{1}{\omega}(\mathcal{A}_D v, \theta)_{L_2(\Omega)}\right| \leq \epsilon\|y\|_Y^2 + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y^2. \tag{3H.43}$$

Proof.

Step 1 By Green's first theorem with $v \in \mathcal{D}(\mathcal{A}_D) = H^2(\Omega) \cap H_0^1(\Omega)$ and $\theta \in \mathcal{D}(\mathcal{A}_N)$, we have

$$-\frac{1}{\omega}(\mathcal{A}_D v, \theta)_{L_2(\Omega)} = \frac{1}{\omega}\int_\Omega \Delta v \bar{\theta}\, d\Omega$$
$$= \frac{1}{\omega}\int_\Gamma \frac{\partial v}{\partial \nu}\bar{\theta}\, d\Gamma - \frac{1}{\omega}\int_\Omega \nabla v \cdot \nabla\bar{\theta}\, d\Omega. \tag{3H.44}$$

Step 2. Lemma 3H.2.3.2. *In the same notation of Proposition 3H.2.3.1, we have*

(i)

$$\left|\frac{1}{\omega}\int_\Omega \nabla v \cdot \nabla\bar{\theta}\, d\Omega\right| \le \epsilon \|y\|_Y^2 + C_\epsilon \left\|\frac{f_0}{\omega}\right\|_Y^2; \tag{3H.45}$$

(ii)

$$\left|\frac{1}{\omega}\int_\Gamma \frac{\partial v}{\partial \nu}\bar{\theta}\, d\Gamma\right| \le \epsilon \|y\|_Y^2 + C_\epsilon \left\|\frac{f_0}{\omega}\right\|_Y^2. \tag{3H.46}$$

Assuming for the moment the validity of Lemma 3H.2.3.2, we see that the desired conclusion (3H.43) follows by use of estimates (3H.45) and (3H.46) in identity (3H.44).

To prove Lemma 3H.2.3.2, we shall use, for each part, inequality (3H.17) plus a priori bounds.

Step 3

Proof of Inequality (3H.45). By the a priori bound (3H.23) and the driving bound (3H.20), we estimate

$$\left|\frac{1}{\omega}\int_\Omega \nabla v \cdot \nabla\bar{\theta}\, d\Omega\right| \le \left(\frac{1}{\sqrt{|\omega|}}\|v\|_{H^1(\Omega)}\right)\left(\frac{1}{\sqrt{|\omega|}}\|\theta\|_{H^1(\Omega)}\right) \tag{3H.47}$$

$$\text{(by (3H.23) and (3H.20))} \quad \le C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]\left[\sqrt{\epsilon_1}\|y\|_Y + \frac{1}{\sqrt{\epsilon_1}}\left\|\frac{f_0}{\omega}\right\|_Y\right] \tag{3H.48}$$

$$\text{(by (3H.17))} \quad \le C\sqrt{\epsilon_1}\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2, \tag{3H.49}$$

with $\epsilon_1 > 0$ arbitrary, and (3H.49) proves (3H.45).

Step 4

Proof of Inequality (3H.46). We recall the a priori inequalities [Brenner, Scott, 1994, p. 39]; [Thomee, 1984, 1997, p. 26]:

$$\left\|\frac{\partial v}{\partial \nu}\right\|_{L_2(\Gamma)} \le C\|v\|_{H^2(\Omega)}^{\frac{1}{2}}\|v\|_{H^1(\Omega)}^{\frac{1}{2}}, \tag{3H.50}$$

$$\|\theta|_\Gamma\|_{L_2(\Gamma)} \le C\|\theta\|_{H^1(\Omega)}^{\frac{1}{2}}\|\theta\|_{L_2(\Omega)}^{\frac{1}{2}}. \tag{3H.51}$$

Then, using (3H.50) and (3H.51),

$$\left|\frac{1}{\omega}\int_\Gamma \frac{\partial v}{\partial \nu}\bar{\theta}\,d\Gamma\right| \leq \frac{1}{|\omega|}\left\|\frac{\partial v}{\partial \nu}\Big|_\Gamma\right\|_{L_2(\Gamma)} \|\theta|_\Gamma\|_{L_2(\Gamma)}$$

$$\leq C\frac{\|v\|^{\frac{1}{2}}_{H^2(\Omega)}}{\sqrt{|\omega|}}\,\frac{\|v\|^{\frac{1}{2}}_{H^1(\Omega)}}{|\omega|^{\frac{1}{4}}}\,\frac{\|\theta\|^{\frac{1}{2}}_{H^1(\Omega)}}{|\omega|^{\frac{1}{4}}}\,\|\theta\|^{\frac{1}{2}}_{L_2(\Omega)}. \tag{3H.52}$$

Taking the one-half power of the a priori bounds (3H.22) and (3H.23), we obtain the following uniform bound for $|\omega| \geq 1$:

$$\frac{\|v\|^{\frac{1}{2}}_{H^2(\Omega)}}{\sqrt{|\omega|}}\,\frac{\|v\|^{\frac{1}{2}}_{H^1(\Omega)}}{|\omega|^{\frac{1}{4}}} \leq C\left[\|y\|^{\frac{1}{2}}_Y + \left\|\frac{f_0}{\omega}\right\|^{\frac{1}{2}}_Y\right]\left[\|y\|^{\frac{1}{2}}_Y + \left\|\frac{f_0}{\omega}\right\|^{\frac{1}{2}}_Y\right] \tag{3H.53}$$

$$\leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{3H.54}$$

However, taking the one-quarter power of the driving bound (3H.20) and majorizing θ by y by (3H.6) and (3H.7), we obtain the following uniform bound for $|\omega| \geq 1$:

$$\frac{\|\theta\|^{\frac{1}{2}}_{H^1(\Omega)}}{|\omega|^{\frac{1}{4}}}\,\|\theta\|^{\frac{1}{2}}_{L_2(\Omega)} \leq 2^{-\frac{1}{4}}\left[\epsilon_1\|y\|^{\frac{1}{2}}_Y + \frac{1}{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|^{\frac{1}{2}}_Y\right]\|y\|^{\frac{1}{2}}_Y \tag{3H.55}$$

$$\leq 2^{-\frac{5}{4}}\left[3\epsilon_1\|y\|_Y + \frac{1}{\epsilon_1^3}\left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{3H.56}$$

Using estimates (3H.54) and (3H.56) on the right-hand side of (3H.52), we obtain

$$\left|\frac{1}{\omega}\int_\Gamma \frac{\partial v}{\partial \nu}\bar{\theta}\,d\Gamma\right| \leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]\left[\epsilon_1\|y\|_Y + \frac{1}{\epsilon_1^3}\left\|\frac{f_0}{\omega}\right\|_Y\right] \tag{3H.57}$$

$$\text{(by (3H.17))} \qquad \leq C\left[2\epsilon_1\|y\|^2_Y + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|^2_Y\right], \tag{3H.58}$$

recalling (3H.17) in the last step, where $\epsilon_1 > 0$ is arbitrary. Then, inequality (3H.58) proves (3H.46), as desired. The proof of Lemma 3H.2.3.2 is complete, and so is the proof of Proposition 3H.2.3.1.

3H.2.4 Proof of Estimate (3H.11) for θ

As a corollary of the driving estimate (3H.20) for θ, as well as of Proposition 3H.2.3.1 (which also stems from (3H.20), we obtain the desired inequality (3H.11) for θ.

Proposition 3H.2.4.1. *With reference to (3H.6) and (3H.7), given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for all $\omega \in \mathbb{R}$, $|\omega| \geq 1$, we have*

$$\|\theta\|^2_{L_2(\Omega)} \leq \epsilon\|y\|^2_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|^2_Y. \tag{3H.59}$$

Proof. We return to Eqn. III = (3H.15), take here the $L_2(\Omega)$-inner product with θ, use estimates (3H.43) and (3H.20), and obtain

$$\|\theta\|^2_{L_2(\Omega)} \leq \left| \frac{1}{\omega} (\mathcal{A}_D v, \theta)_{L_2(\Omega)} \right| + \frac{1}{|\omega|} \left\| \mathcal{A}_N^{\frac{1}{2}} \theta \right\|^2_{L_2(\Omega)} + \left| \left(\frac{\theta_0}{\omega}, \theta \right)_{L_2(\Omega)} \right| \tag{3H.60}$$

$$(\text{by (3H.43) and (3H.20)}) \leq \left[\epsilon \|y\|^2_Y + C_\epsilon \left\| \frac{f_0}{\omega} \right\|^2_Y \right] + \frac{1}{2} \left[\epsilon \|y\|^2_Y + \frac{1}{\epsilon} \left\| \frac{f_0}{\omega} \right\|^2_Y \right]$$
$$+ \frac{\epsilon}{2} \|\theta\|^2_{L_2(\Omega)} + \frac{1}{2\epsilon} \left\| \frac{\theta_0}{\omega} \right\|^2_Y . \tag{3H.61}$$

Then, the desired inequality (3H.59) readily follows from (3H.61), by majorizing θ_0 by f_0 and θ by y, via (3H.6) and (3H.7).

3H.2.5 Improving Upon A Priori Bounds

The driving estimate (3H.20) and the a priori bound (3H.23) for $[\mathcal{A}_D^{\frac{1}{2}} v / \sqrt{|\omega|}]$ yield

Lemma 3H.2.5.1 *With reference to (3H.6) and (3H.7), given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for all $\omega \in \mathbb{R}$, $|\omega| \geq 1$,*

$$\left| \frac{1}{\omega} (\mathcal{A}_N \theta, v)_{L_2(\Omega)} \right| \leq \epsilon \|y\|^2_Y + C_\epsilon \left\| \frac{f_0}{\omega} \right\|^2_Y . \tag{3H.62}$$

Proof. With $\theta \in \mathcal{D}(\mathcal{A}_N)$ and $v \in \mathcal{D}(\mathcal{A}_D) \subset H^1(\Omega) = \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}})$, we estimate by (3H.20) and (3H.23), since $\mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}) = H^1(\Omega)$:

$$\left| \frac{1}{\omega} (\mathcal{A}_N \theta, v)_{L_2(\Omega)} \right| = \left| \left(\frac{\mathcal{A}_N^{\frac{1}{2}} \theta}{\sqrt{|\omega|}}, \frac{\mathcal{A}_N^{\frac{1}{2}} v}{\sqrt{|\omega|}} \right)_{L_2(\Omega)} \right| \tag{3H.63}$$

$$\leq \frac{\left\| \mathcal{A}_N^{\frac{1}{2}} \theta \right\|_{L_2(\Omega)}}{\sqrt{|\omega|}} \frac{\left\| \mathcal{A}_N^{\frac{1}{2}} v \right\|_{L_2(\Omega)}}{\sqrt{|\omega|}} \tag{3H.64}$$

$$(\text{by (3H.20) and (3H.23)}) \quad \leq C \left[\epsilon_1 \|y\|_Y + \frac{1}{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y \right] \left[\|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right] \tag{3H.65}$$

$$(\text{by (3H.17)}) \quad \leq C\epsilon_1 \|y\|^2_Y + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|^2_Y , \tag{3H.66}$$

after invoking (3H.17) in the last step, for an arbitrary $\epsilon_1 > 0$. Then, (3H.66) proves (3H.62), as desired. □

The following result – a corollary of Proposition 3H.2.4.1, Eqn. (3H.59), for θ – improves upon the a priori bound (3H.23) for v.

Lemma 3H.2.5.2. *With reference to (3H.6) and (3H.7), given* $\epsilon > 0$, *there exists* $C_\epsilon > 0$ *such that for all* $\omega \in \mathbb{R}$, $|\omega| \geq 1$, *we have*

$$\frac{1}{|\omega|}\|v\|^2_{H^1(\Omega)} \doteq \frac{1}{|\omega|}\left\|\mathcal{A}_D^{\frac{1}{2}} v\right\|^2_{L_2(\Omega)} \leq \epsilon\|y\|_Y^2 + C_\epsilon \left\|\frac{f_0}{\omega}\right\|_Y^2. \tag{3H.67}$$

Proof. We return to Eqn. III = (3H.15), take here the $L_2(\Omega)$-inner product with v, and invoke estimates (3H.59) and (3H.62), thereby obtaining

$$\frac{1}{|\omega|}\left\|\mathcal{A}_D^{\frac{1}{2}} v\right\|^2_{L_2(\Omega)} = \left|\frac{1}{\omega}(\mathcal{A}_D v, v)_{L_2(\Omega)}\right| = \left|\left(\frac{\theta_0}{\omega} - i\theta - \frac{1}{\omega}\mathcal{A}_N\theta, v\right)_{L_2(\Omega)}\right| \tag{3H.68}$$

$$\leq \left\|\frac{\theta_0}{\omega}\right\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)} + \|\theta\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)} + \left|\frac{1}{\omega}(\mathcal{A}_N\theta, v)_{L_2(\Omega)}\right| \tag{3H.69}$$

(by (3H.59), (3H.62))
$$\leq \left[\frac{\epsilon_1}{2}\|v\|^2_{L_2(\Omega)} + \frac{1}{2\epsilon_1}\left\|\frac{\theta_0}{\omega}\right\|^2_{L_2(\Omega)}\right] + \left[\epsilon_1\|y\|_Y + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y\right]\|y\|_Y + \left[\epsilon_1\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2\right] \tag{3H.70}$$

$$\leq \epsilon\|y\|_Y^2 + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y^2, \tag{3H.71}$$

majorizing v by y twice via (3H.6) and (3H.7), once from (3H.69) to (3H.70), and once from (3H.69) to (3H.71). Equation (3H.71) proves (3H.67).

3H.2.6 Proof of Estimate (3H.10) for v

As a corollary of Lemma 3H.2.5.2 and of the a priori bound Eqn. (H.32), Lemma 3H.2.2.4 on u, we obtain the desired estimate (3H.10) for v.

Proposition 3H.2.6.1. *With reference to (3H.6) and (3H.7), given* $\epsilon > 0$, *there exists* $C_\epsilon > 0$ *such that for all* $\omega \in \mathbb{R}$, $|\omega| \geq 1$,

$$\|v\|^2_{L_2(\Omega)} \leq \epsilon\|y\|_Y^2 + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y^2. \tag{3H.72}$$

Proof.

Step 1 We return to Eqn. II = (3H.14) and take here the $L_2(\Omega)$-inner product with v, thereby obtaining [see definitions (3.12.3) and (3.12.9) of the operators $\mathcal{A}$ and G]:

$$\|v\|^2_{L_2(\Omega)} \leq \left| \frac{1}{\omega} \int_\Omega \Delta^2 u \bar{v}\, d\Omega \right| + \left| \frac{1}{\omega} (\mathcal{A}_N \theta, v)_{L_2(\Omega)} \right| + \left| \left(\frac{\theta_0}{\omega}, v \right)_{L_2(\Omega)} \right|. \tag{3H.73}$$

Step 2. Lemma 3H.2.6.2 In the notation of Proposition 3H.2.6.1, we have

$$\left| \frac{1}{\omega} \int_\Omega \Delta^2 u \bar{v}\, d\Omega \right| \leq \epsilon \|y\|^2_Y + C_\epsilon \left\| \frac{f_0}{\omega} \right\|^2_Y. \tag{3H.74}$$

Proof of Lemma 3H.2.6.2 Since $v \in \mathcal{D}(\mathcal{A}_D) = H^2(\Omega) \cap H^1_0(\Omega)$, then $\bar{v}|_\Gamma = 0$ and Green's first theorem yields, by virtue of (3H.32) and (3H.67),

$$\left| \frac{1}{\omega} \int_\Omega \Delta^2 u \bar{v}\, d\Omega \right| = \left| \frac{1}{\omega} \int \nabla \Delta u \cdot \nabla \bar{v}\, d\Omega \right| \tag{3H.75}$$

$$\leq \left(\frac{1}{\sqrt{|\omega|}} \|u\|_{H^3(\Omega)} \right) \left(\frac{1}{\sqrt{|\omega|}} \|v\|_{H^1(\Omega)} \right) \tag{3H.76}$$

$$\text{(by (3H.32) and (3H.67))} \quad \leq C \left[\|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right] \left[\epsilon_1 \|y\|_Y + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y \right] \tag{3H.77}$$

$$\text{(by (3H.17))} \quad \leq C \left[\epsilon_1 \|y\|^2_Y + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|^2_Y \right], \tag{3H.78}$$

invoking (3H.17) in the last step, with $\epsilon_1 > 0$ arbitrary. Then (3H.78) proves (3H.74).

□

Step 3 We use estimate (3H.74) and estimate (3H.62) on the right-hand side of (3H.73) to obtain

$$\|v\|^2_{L_2(\Omega)} \leq 2 \left[\epsilon_1 \|y\|^2_Y + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|^2_Y \right] + \left[\frac{\epsilon_1}{2} \|v\|^2_{L_2(\Omega)} + \frac{1}{2\epsilon_1} \left\| \frac{\theta_0}{\omega} \right\|^2_{L_2(\Omega)} \right], \tag{3H.79}$$

from which the desired estimate (3H.72) follows at once, majorizing v and θ_0 by y and f_0, via (3H.6) and (3H.7). □

3H.2.7 Proof of Estimate (3H.9) for u

Proposition 3H.2.7.1 *With reference to (3H.6) and (3H.7), given $\epsilon > 0$, there is $C_\epsilon > 0$ such that for all $\omega \in \mathbb{R}$, $|\omega| \geq 1$,*

$$\|u\|^2_{H^2(\Omega)} \leq \epsilon \|y\|^2_Y + C_\epsilon \left\| \frac{f_0}{\omega} \right\|^2_Y. \tag{3H.80}$$

Proof.

Step 1 We have already noted in Lemma 3H.2.6.2, recalling the definitions (3.12.3) and (3.12.9) in Section 12 of the operators $\mathcal{A}$ and G, that

$$\left|\frac{1}{\omega}(\mathcal{A}[u+G(\theta|_\Gamma)], v)_{L_2(\Omega)}\right| = \left|\frac{1}{\omega}\int_\Omega \Delta^2 u\bar{v}\, d\Omega\right| \le \epsilon_1\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2. \tag{3H.81}$$

Step 2 However, by substituting $v/w = iu - u_0/\omega$ from Eqn. I = (3H.13), we obtain

$$\frac{1}{\omega}(\mathcal{A}[u+G(\theta|_\Gamma)], v)_{L_2(\Omega)} = \left(\mathcal{A}u, iu - \frac{u_0}{\omega}\right)_{L_2(\Omega)} + \frac{1}{\omega}(\theta|_\Gamma, G^*\mathcal{A}v)_{L_2(\Gamma)} \tag{3H.82}$$

$$\text{(by (3.12.13))} \qquad = -i\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2_{L_2(\Omega)} - \left(\mathcal{A}^{\frac{1}{2}}u, \frac{\mathcal{A}^{\frac{1}{2}}u_0}{\omega}\right)_{L_2(\Omega)} + \frac{1}{\omega}\left(\theta|_\Gamma, \left.\frac{\partial v}{\partial \nu}\right|_\Gamma\right)_{L_2(\Gamma)}, \tag{3H.83}$$

recalling $G^*\mathcal{A}v = \partial v/\partial\nu$ from (3.12.13) of Lemma 3.12.1, in the last step.

Step 3 Combining (3H.82) with (3H.81), we obtain via the norm equivalence in (3H.6):

$$\|u\|^2_{H^2(\Omega)} \doteq \left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2_{L_2(\Omega)} \le \left[\epsilon_1\|y\|_Y^2 + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y^2\right] + \left[\frac{\epsilon_1}{2}\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2_{L_2(\Omega)} + \frac{1}{2\epsilon_1}\left\|\frac{\mathcal{A}^{\frac{1}{2}}u_0}{\omega}\right\|_Y^2\right] + \left|\frac{1}{\omega}\left(\theta|_\Gamma, \left.\frac{\partial v}{\partial \nu}\right|_\Gamma\right)_{L_2(\Gamma)}\right| \tag{3H.84}$$

$$\text{(by (3H.46))} \quad \le \epsilon_2\|y\|_Y^2 + C_{\epsilon_2}\left\|\frac{f_0}{\omega}\right\|_Y^2 + \frac{\epsilon_1}{2}\|\mathcal{A}^{\frac{1}{2}}u\|^2_{L_2(\Omega)}, \tag{3H.85}$$

where in going from (3H.84) to (3H.85) we have invoked the boundary estimate (3H.46) of Lemma 3H.2.3.2 and have majorized $\mathcal{A}^{\frac{1}{2}}u_0$ by f_0 by (3H.6). Estimate (3H.85) readily leads to the desired conclusion (3H.80). □

The required estimates (3H.9), (3H.10), and (3H.11) are all proved. □

Final Remark Having shown the required estimate (3H.80) for u, we return to Eqn. I = (3H.13) and obtain: Given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for all $\omega \in \mathbb{R}$,

$|\omega| \geq 1$, we have

$$\frac{1}{|\omega|}\|\mathcal{A}_D v\|_{L_2(\Omega)} \doteq \frac{1}{|\omega|}\|v\|_{H^2(\Omega)} \leq \epsilon\|y\|_Y + C_\epsilon \left\|\frac{f_0}{\omega}\right\|_Y . \tag{3H.86}$$

Moreover, armed with estimate (3H.86), we return to Eqn. III = (3H.15), use here also estimate (3H.59) for θ, and obtain

$$\frac{1}{|\omega|}\|\theta\|_{H^2(\Omega)} \doteq \frac{1}{|\omega|}\|\mathcal{A}_N\theta\|_{L_2(\Omega)} = \left\|\frac{\theta_0}{\omega} - i\theta - \frac{1}{\omega}\mathcal{A}_D v\right\|_{L_2(\Omega)} \tag{3H.87}$$

$$\text{(by (3H.59) and (3H.86))} \qquad \leq \epsilon\|y\|_Y + C_\epsilon \left\|\frac{f_0}{\omega}\right\|_Y , \tag{3H.88}$$

majorizing also θ_0 by f_0 via (3H.6). Inequality (3H.88) complements the driving estimate (3H.20) as well as its consequence (3H.59).

3I Analyticity of the s.c. Semigroup Arising from Problem (3.13.1) of Section 13 (Free Mechanical BC/Neumann (Robin) Thermal BC)

3I.1 Review of Problem. Statement of Main Results

In this appendix we return to the thermo-elastic problem (3.13.1), which we now strip of lower-order terms, while normalizing the (inessential) constants to 1. The key homogeneous model on a two-dimensional bounded domain Ω is then

$$\begin{cases} w_{tt} + \Delta^2 w + \Delta\theta = 0 & \text{in } (0, T] \times \Omega = Q; & \text{(3I.1a)} \\ \theta_t - \Delta\theta - \Delta w_t = 0 & \text{in } Q; & \text{(3I.1b)} \\ w(0, \cdot) = w_0,\ w_t(0, \cdot) = w_1,\ \theta(0, \cdot) = \theta_0 & \text{in } \Omega; & \text{(3I.1c)} \\ \Delta w + (1-\mu)B_1 w + \theta = 0 & \text{in } (0, T] \times \Gamma \equiv \Sigma; & \text{(3I.1d)} \\ \dfrac{\partial \Delta w}{\partial \nu} + (1-\mu)B_2 w - w + \dfrac{\partial\theta}{\partial\nu} = 0 & \text{on } \Sigma; & \text{(3I.1e)} \\ \dfrac{\partial\theta}{\partial\nu} + b\theta = 0, b > 0 & \text{on } \Sigma; & \text{(3I.1f)} \end{cases}$$

$$\text{on } \Sigma : B_1 w = 2\nu_1\nu_2 w_{xy} - \nu_1^2 w_{yy} - \nu_2^2 w_{xx}; \tag{3I.1g}$$

$$\text{on } \Sigma : B_2 w = \frac{\partial}{\partial\tau}\left[\left(\nu_1^2 - \nu_2^2\right)w_{xy} + \nu_1\nu_2(w_{yy} - w_{xx})\right], \tag{3I.1h}$$

where the expression for B_1 and B_2 in (3I.1g, h) are taken from (3.13.2) and (3.13.3). Here, $0 < \mu < 1$, and $\nu = [\nu_1, \nu_2]$ is the unit outward normal to Γ, whereas τ is the unit tangential vector along Γ, oriented counter clockwise. Thus, $\frac{\partial}{\partial\nu}$ and $\frac{\partial}{\partial\tau}$ are the corresponding normal and tangential derivatives.

We have seen in Proposition 3.13.5 that the operator A in (3.13.22) and (3.13.23), now simplified for problem (3I.1) as

$$A \begin{bmatrix} u \\ v \\ \theta \end{bmatrix} = \begin{bmatrix} v \\ -\mathcal{A}\left[u + G_1(\theta|_\Gamma) + G_2 \dfrac{\partial \theta}{\partial \nu}\right] + \mathcal{A}_N \theta \\ \Delta v - \mathcal{A}_N \theta \end{bmatrix}; \tag{3I.2}$$

$$[u, v, \theta] \in \mathcal{D}(A) \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(\mathcal{A}_N), \quad \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega) \tag{3I.3}$$

generates a s.c. contraction semigroup e^{At} on the space (see (3.13.7))

$$Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega). \tag{3I.4}$$

The operators $\mathcal{A}$, $\mathcal{A}_N$, G_1, and G_2 are defined in (3.13.5), (3.13.9), (3.13.11), and (3.13.13) of Section 3.13, respectively. We now prove the following key result.

Theorem 3I.1. *The above s.c. contraction semigroup e^{At}:*

$$y_0 = [w_0, w_1, \theta_0] \in Y \to e^{At} y_0 = [w(t), w_t(t), \theta(t)] \tag{3I.5}$$

is, moreover, analytic on Y, $t > 0$.

The proof follows the approach of Appendix 3H (which, in turn, expanded further the approach of the abstract, operator-theoretic strategy of Appendix 3E, by injecting a direct PDE analysis, in conjunction with the coupled hinged/Neumann BC for the mechanical and thermal variables w and θ). However, the present case with free BC offers additional serious difficulties over the case of hinged/Neumann BC of Appendix 3H, owing to the present high level (second-order and third-order) BC for the mechanical variable, which leave the action Δv on $H^2(\Omega)$ for the third component of A in (3I.2) with no BC attached. See *Orientation* below Eqn. (3I.16), as well as Remark 3I.3.1 below.

3I.2 Proof of Theorem 3.I.1

3I.2.1 General Strategy and Preliminaries

General Strategy With reference to the space Y in (3H.4), let $f_0 \in Y$ be arbitrary:

$$\begin{cases} f_0 = [u_0, v_0, \theta_0] \in Y \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \times L_2(\Omega), \\ \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega) \text{ (equivalent norms).} \end{cases} \tag{3I.6}$$

With reference to the operator A in (3I.2), let ω be real, $\omega \in \mathbb{R}$, and define

$$y(\omega) = [u(\omega), v(\omega), \theta(\omega)] = [i\omega I - A]^{-1} f_0 = R(i\omega, A) f_0 \in \mathcal{D}(A), \tag{3I.7}$$

where the resolvent of A is well-defined on the imaginary axis; see Proposition 3.13.6.

Our goal is to show that the following uniform estimate holds true: There exists a constant $C > 0$ such that for all $\omega \in \mathbb{R}$, with $|\omega| > \omega_0 > 0$ for some suitable ω_0,

$$\left\| \begin{bmatrix} u(\omega) \\ v(\omega) \\ \theta(\omega) \end{bmatrix} \right\|_Y = \|y(\omega)\|_Y = \|R(i\omega, A)f_0\|_Y \le \frac{C}{|\omega|} \|f_0\|_Y. \tag{3I.8}$$

Once estimate (3I.8) has been established for the generator A of the s.c. contraction semigroup e^{At} asserted by Proposition 3.13.5, we can invoke Theorem 3E.3 to find that the s.c. semigroup e^{At} is, in fact, analytic on $Y, t > 0$. As in Appendix 3H (and 3E), in order to prove (I.8) we then seek to establish the following three simultaneous estimates for the components of $y(\omega)$ in (3I.7): there exists a suitable $\omega_0 > 0$, such that for all $\epsilon > 0$, there exists a constant $C_\epsilon > 0$, such that for all $\omega \in \mathbb{R}$, with $|\omega| > \omega_0 > 0$ the vector $y(\omega) = [u(\omega), v(\omega), \theta(\omega)]$ in (3I.7) satisfies

$$\begin{cases} \|u(\omega)\|^2_{H^2(\Omega)} \le \epsilon \|y(\omega)\|^2_Y + C_\epsilon \left\| \dfrac{f_0}{\omega} \right\|^2_Y; & (3I.9) \\[2ex] \|v(\omega)\|^2_{L_2(\Omega)} \le \epsilon \|y(\omega)\|^2_Y + C_\epsilon \left\| \dfrac{f_0}{\omega} \right\|^2_Y; & (3I.10) \\[2ex] \|\theta(\omega)\|^2_{L_2(\Omega)} \le \epsilon \|y(\omega)\|^2_Y + C_\epsilon \left\| \dfrac{f_0}{\omega} \right\|^2_Y. & (3I.11) \end{cases}$$

As in Appendix 3H (and 3E), we drop hereafter the explicit dependence of ω from $y(\omega) = [u(\omega), v(\omega), \theta(\omega)]$. Estimates (3I.9)–(3I.11) are proved below, in Proposition 3I.2.6, Eqn. (3I.35) for θ, and Proposition 3I.4.3, Eqns. (3I.120), (3I.121) for u and v.

Preliminaries By (3I.2), we obtain explicitly from (3I.7)

$$(i\omega - A) \begin{bmatrix} u \\ u \\ \theta \end{bmatrix} = \begin{bmatrix} i\omega u - v \\ i\omega v + \mathcal{A}\left[u + G_1(\theta|_\Gamma) + G_2 \dfrac{\partial \theta}{\partial \nu}\right] - \mathcal{A}_N \theta \\ i\omega\theta - \Delta v + \mathcal{A}_N \theta \end{bmatrix} = \begin{bmatrix} u_0 \\ v_0 \\ \theta_0 \end{bmatrix} = f_0 \in Y, \tag{3I.12}$$

or upon dividing by $\omega \neq 0$,

$$\begin{cases} I: & iu - \dfrac{v}{\omega} = \dfrac{u_0}{\omega}; & (3I.13) \\[2ex] II: & iv + \dfrac{1}{\omega}\mathcal{A}\left[u + G_1(\theta|_\Gamma) + G_2 \dfrac{\partial \theta}{\partial \nu}\right] - \dfrac{1}{\omega}\mathcal{A}_N\theta = \dfrac{v_0}{\omega}; & (3I.14) \\[2ex] III: & i\theta - \dfrac{1}{\omega}\Delta v + \dfrac{1}{\omega}\mathcal{A}_N\theta = \dfrac{\theta_0}{\omega}, & (3I.15) \end{cases}$$

where, recalling (3I.3), we have a fortiori the following regularity properties:

$$y = [u, v, \theta] \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(\mathcal{A}_N), \quad \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega), \tag{3I.16}$$

to be used freely below.

We shall break the proof of the present free BC case into three parts. Part I, dealt with in Section 3I.2, follows closely the proof given in Appendix 3H of the case of coupled hinged mechanical BC and Neumann thermal BC, up to the breaking point of that proof (which will be duly noted; see Remark 3I.3.1 below). It collects the "driving" estimate (3I.18), as well as the a priori bounds on u, v, θ. With Part II, expounded in Section 3I.3, we begin a radical departure from the proof of Appendix 3H, to compensate for the lack, at this stage, of the "good" ϵ-estimate for $\|v\|_{H^1(\Omega)}$, such as in [Appendix 3H, Eqn. (3H.2.5.6)]. More precisely, Part II collects all those new results that can be obtained without making explicit use of the *structure* of the boundary operators B_1 and B_2 with (3I.1g,h). This includes the required estimate (3I.11) for θ (see (3I.35) of Proposition 3I.2.6 below), as well as the "right," desired ϵ-estimate for the *difference* $[\|v\|^2_{L_2(\Omega)} - \|u\|^2_{H^2(\Omega)}]$ of the first two variables; see (3I.94) of Proposition 3I.3.6 below. Finally, we complete the proof in Part III (Section 3I.4), by showing simultaneously the required estimates (3I.10) for v and (3I.11) for u. To this end, we shall exploit the special structure of the boundary operator B_1: see Eqn. (3I.97) in terms of tangential and normal derivatives (rather than in terms of the original x and y variables).

3I.2.2 The "Driving" Estimate for θ, and A Priori Bounds for u, v, θ

In this section we collect results on Eqns. I = (3I.13), II = (3I.14), III = (3I.15), that can be proved exactly as in the case of hinged/Neumann BC in Appendix 3H. Accordingly, they will only be listed, with due reference for a proof to the corresponding result in Appendix 3H.

Lemma 3I.2.1 (preliminary A Priori Bounds for θ) *With reference to (3I.6) and (3I.7), we have*

(i)

$$(\mathcal{A}_N\theta, \theta)_{L_2(\Omega)} = \mathrm{Re}\left([i\omega I - A]\begin{bmatrix} u \\ v \\ \theta \end{bmatrix}, \begin{bmatrix} u \\ v \\ \theta \end{bmatrix}\right)_Y = \mathrm{Re}(f_0, y)_Y; \tag{3I.17}$$

(ii) for any $\epsilon > 0$ and $\omega \in \mathbb{R}$, $\omega \neq 0$,

$$\frac{1}{|\omega|}\|\theta\|^2_{H^1(\Omega)} \doteq \frac{1}{|\omega|}\left\|\mathcal{A}_N^{\frac{1}{2}}\theta\right\|^2_{L_2(\Omega)} \le \frac{\epsilon}{2}\|y\|^2_Y + \frac{1}{2\epsilon}\left\|\frac{f_0}{\omega}\right\|^2_Y. \tag{3I.18}$$

Throughout, equivalence in norm is denoted by $\doteq$.

Proof. Part (i) follows from Lemma 3.13.3, Eqn. (3.13.35), or Lemma 3.13.4(iii). □

Lemma 3I.2.2 (a priori bounds for v) *For f_0 and y as in (3I.6) and (3I.7), we have*

(i)

$$\frac{1}{|\omega|}\|v\|_{H^2(\Omega)} \leq \|u\|_{H^2(\Omega)} + \left\|\frac{u_0}{\omega}\right\|_{H^2(\Omega)} \leq \|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y; \tag{3I.19}$$

(ii)

$$\frac{1}{\sqrt{|\omega|}}\|v\|^2_{H^1(\Omega)} \leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{3I.20}$$

Proof. See proof of Lemma 3H.2.2.2. □

Lemma 3I.2.3 (further A Priori Bound for θ) *For f_0 and y as in (3I.6), (3I.7), we have*

$$\frac{1}{|\omega|}\|\theta\|^2_{H^2(\Omega)} \doteq \frac{1}{|\omega|}\|\mathcal{A}_N\theta\|_{L_2(\Omega)} \leq 2\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{3I.21}$$

Proof. See proof of Lemma 3H.2.2.3. By use of Eqn. III = (3I.15), the only difference with that proof is that $\|\mathcal{A}_D v\|_{L_2(\Omega)}$ there is replaced now by $\|\Delta v\|_{L_2(\Omega)}$, which is still estimated by (3I.19), as before. □

Lemma 3I.2.4 (A Priori Bounds for u) *With reference to (3I.6) and (3I.7), we have, for $\omega \in \mathbb{R}$,*

(i)

$$\frac{1}{|\omega|}\|u\|_{H^4(\Omega)} \leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]; \tag{3I.22}$$

(ii)

$$\frac{1}{\sqrt{|\omega|}}\|u\|_{H^3(\Omega)} \leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{3I.23}$$

Proof. (i) As in the proof of Lemma 3H.2.2.4, we shall obtain (3I.22) by elliptic regularity, except that now the elliptic problem has different BC. Referring to the definition of $\mathcal{A}$ in (3.13.5) of Section 3.13, we have that

$$\frac{1}{\omega}\mathcal{A}\left[u + G_1(\theta|_\Gamma) + G_2\left(\frac{\partial\theta}{\partial\nu}\right)\right] = -iv + \frac{1}{\omega}\mathcal{A}_N\theta + \frac{v_0}{\omega} \tag{3I.24}$$

is equivalent, via the definitions of the Green operators G_1 and G_2, given in (3.13.12) and (3.13.13), to the following elliptic boundary value problem (i.e., the original elliptic problem (3.13.12) [see also (3.I.1)], of which (3.I.24) is the abstract version, in the first place):

$$\begin{cases} \Delta^2\left(\dfrac{u}{\omega}\right) = -iv + \dfrac{1}{\omega}\mathcal{A}_N\theta + \dfrac{v_0}{\omega} & \text{in } \Omega; & (3I.25)\\ \Delta\left(\dfrac{u}{\omega}\right) + (1-\mu)B_1\left(\dfrac{u}{\omega}\right) = -\dfrac{1}{\omega}\theta & \text{on } \Gamma; & (3I.26)\\ \dfrac{\partial\Delta}{\partial\nu}\left(\dfrac{u}{\omega}\right) + (1-\mu)B_2\left(\dfrac{u}{\omega}\right) - \left(\dfrac{u}{\omega}\right) = -\dfrac{1}{\omega}\dfrac{\partial\theta}{\partial\nu} & \text{on } \Gamma. & (3I.27)\end{cases}$$

From the right-hand side of (3I.25), we readily estimate, by virtue of (3I.21) for $\mathcal{A}_N\theta/\omega$, majorizing v and θ_0 by y and f_0, via (3I.6) and (3I.7),

$$\left\|\Delta^2\left(\frac{u}{\omega}\right)\right\|_{L_2(\Omega)} \leq 3\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{3I.28}$$

(This step is the same as in Appendix 3H, Eqn. (3H.36).) Moreover, from the first BC in (3I.26) we estimate, by trace theory on θ, followed by estimate (3I.21),

$$\left\|\Delta\left(\frac{u}{\omega}\right) + (1-\mu)B_1\left(\frac{u}{\omega}\right)\right\|_{H^{\frac{3}{2}}(\Gamma)} = \frac{1}{|\omega|}\|\theta|_\Gamma\|_{H^{\frac{3}{2}}(\Gamma)} \tag{3I.29}$$

$$(\text{by (3I.21)}) \qquad \leq \frac{C}{|\omega|}\|\theta\|_{H^2(\Omega)} \leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{3I.30}$$

Finally, we likewise estimate the second BC (3I.27), via trace theory on θ, and (3I.21):

$$\left\|\frac{\partial\Delta}{\partial\nu}\left(\frac{u}{\omega}\right) + (1-\mu)B_2\left(\frac{u}{\omega}\right) - \left(\frac{u}{\omega}\right)\right\|_{H^{\frac{1}{2}}(\Gamma)} = \frac{1}{|\omega|}\left\|\frac{\partial\theta}{\partial\nu}\Big|_\Gamma\right\|_{H^{\frac{1}{2}}(\Gamma)} \tag{3I.31}$$

$$(\text{by (3I.21)}) \qquad \leq \frac{C}{|\omega|}\|\theta\|_{H^2(\Omega)} \leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{3I.32}$$

We can then apply elliptic regularity theory on problem (3I.25), (3I.26), (3I.27), satisfying estimates (3I.28), (3I.30), (3I.32), thus obtaining

$$\left\|\frac{u}{\omega}\right\|_{H^4(\Omega)} \leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right], \tag{3I.33}$$

and (3I.22) is proved. [We note that the right-hand side estimate (3I.28) and the boundary estimates (3I.30), (3I.31) produce, *independently*, the same regularity of u/ω in $H^4(\Omega)$ for the corresponding elliptic problem in u/w.] Part (ii), Eqn. (3I.23) then follows by interpolation, as in (3H.40) through (3H.42). □

Lemma 3I.2.5 *For f_0 and y as in (3I.6) and (3I.7), given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for all $\omega \in \mathbb{R}$, $|\omega| \geq 1$, we have*

$$\left|\frac{1}{\omega}(\Delta v, \theta)_{L_2(\Omega)}\right| \leq \epsilon\|y\|_Y^2 + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y^2. \tag{3I.34}$$

Proof. The proof is the same as in Appendix 3H, Proposition 3H.2.3.1; see (3H.43). □

We can then obtain the desired estimate (3I.11) for θ.

Proposition 3I.2.6 *For f_0 and y as in (3I.6) and (3I.7), given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for all $\omega \in \mathbb{R}$, $|\omega| \geq 1$, we have*

$$\|\theta\|^2_{L_2(\Omega)} \leq \epsilon \|y\|^2_Y + C_\epsilon \left\| \frac{f_0}{\omega} \right\|^2_Y . \tag{3I.35}$$

Proof. As in Proposition 3H.2.4.1, we return to Eqn. III = (3I.15), take here the $L_2(\Omega)$-inner product with θ, use estimates (3I.34) and (3I.18), and obtain (3I.35). □

Lemma 3I.2.7 *For f_0 and y as in (3I.6) and (3I.7), given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for all $\omega \in \mathbb{R}$, $|\omega| \geq 1$, we have*

$$\left| \frac{1}{\omega} (\mathcal{A}_N \theta, v)_{L_2(\Omega)} \right| \leq \epsilon \|y\|^2_Y + C_\epsilon \left\| \frac{f_0}{\omega} \right\|^2_Y . \tag{3I.36}$$

Proof. The proof is the same as the proof of Lemma 3H.2.5.1. □

3I.3 Desired ϵ-Estimates for θ, Δu, and $[\|v\|^2 - \|\mathcal{A}^{\frac{1}{2}} u\|]$

In the case of hinged mechanical/Neumann thermal BC of Appendix 3H, we had $v \in \mathcal{D}(\mathcal{A}_D) = H^2(\Omega) \cap H^1_0(\Omega)$. In the present case, we only have $v \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega)$. The consequence is that, although in the case of hinged/Neumann BC where $v|_\Gamma = 0$, we could get at this stage the good ϵ-estimate for $\frac{1}{|\omega|} \|v\|^2_{H^1(\Omega)}$ as in Appendix 3H, Eqn. (3H.2.5.6)], instead, in the present development, we obtain at this stage only a *weaker* result, as follows.

Lemma 3I.3.1 *With reference to (3I.6) and (3I.7), given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for all $\omega \in \mathbb{R}$, $|\omega| \geq 1$, we have*

$$\left| \frac{1}{\omega} (\Delta v, v)_{L_2(\Omega)} \right| \leq \epsilon \|y\|^2_Y + C_\epsilon \left\| \frac{f_0}{\omega} \right\|^2_Y . \tag{3I.37}$$

Proof. The proof is the same as the one in Appendix 3H, Lemma 3H.2.5.2. We return to Eqn. III = (3I.11), take here the $L_2(\Omega)$-inner product with v, and obtain

$$\left| \frac{1}{\omega} (\Delta v, v)_{L_2(\Omega)} \right| = \left| \left(\frac{\theta_0}{\omega} - i\theta - \frac{1}{\omega} \mathcal{A}_N \theta, v \right)_{L_2(\Omega)} \right| , \tag{3I.38}$$

where the right-hand side of (3I.38) is evaluated exactly as in Appendix 3H, from (3H.68) through (3H.71). □

Remark 3I.3.1 In Appendix 3H, Lemma 3H.2.5.2, for the left-hand side of (3I.38), hence of (3I.37), we obtained, instead:

$$\frac{1}{|\omega|}\|v\|^2_{H^1(\Omega)} \doteq \frac{1}{|\omega|}\left\|\mathcal{A}_D^{\frac{1}{2}}v\right\|^2_{L_2(\Omega)} = \left|\frac{1}{\omega}(\mathcal{A}_D v, v)_{L_2(\Omega)}\right| \leq \epsilon\|y\|_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y, \tag{3I.39}$$

which is a stronger result over (3I.37). Equation (3I.39) was then used in the next step of the proof in Proposition 3H.2.6.1 after the $L_2(\Omega)$-inner product of Eqn. II with v, in combination with the *a priori* bound (3I.23) for u. In the present development, where (3I.37) represents a *loss* over (3I.39), the variable v is still not good enough. Thus a major departure from the proof of Appendix 3H takes place here: We must carry out still with the 'good' variable θ (satisfying the 'driving' estimate (3I.18)). Accordingly, in our next step, we take the $L_2(\Omega)$-inner product of Eqn. II with θ, not with v as in Appendix 3H. In the present case, the proof of the required estimates (3I.10) and (3I.11) for u and v is much more complicated.

Lemma 3I.3.2 *With reference to (3I.6) and (3I.7), given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for all $\omega \in \mathbb{R}$, $|\omega| \geq 1$, we have:*

(i)

$$\left|\frac{1}{\omega}(\Delta^2 u, \theta)_{L_2(\Omega)}\right| = \left|\frac{1}{\omega}(\Delta u, \Delta\theta)_{L_2(\Omega)} + \frac{1}{\omega}\int_\Gamma \frac{\partial \Delta u}{\partial \nu}\bar{\theta}\, d\Gamma - \frac{1}{\omega}\int_\Gamma \Delta u \frac{\partial\bar{\theta}}{\partial\nu}\, d\Gamma\right| \tag{3I.40}$$

$$\leq \epsilon\|y\|_Y^2 + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y^2. \tag{3I.41}$$

(ii) Similarly,

$$\left|\frac{1}{\omega}\int_\Gamma \frac{\partial\Delta u}{\partial\nu}\bar{\theta}\, d\Gamma\right| \leq \epsilon\|y\|_Y^2 + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y^2; \tag{3I.42}$$

(iii)

$$\left|\frac{1}{\omega}\int_\Gamma \Delta u\frac{\partial\bar{\theta}}{\partial\nu}\, d\Gamma\right| \leq \epsilon\|y\|_Y^2 + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y^2. \tag{3I.43}$$

(iv) Finally,

$$\left|\frac{1}{\omega}(\Delta u, \Delta\theta)_{L_2(\Omega)}\right| \leq \epsilon\|y\|_Y^2 + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y^2. \tag{3I.44}$$

Proof. (i) We return to Eqn. II = (3I.14); take here the $L_2(\Omega)$-inner product with θ and, recalling (3.13.5) for $\mathcal{A}$, (3.13.1) for G_1, and (3.13.3) for G_2, and use the estimates (3I.35) on θ and (3I.18) on $\mathcal{A}_N^{\frac{1}{2}}\theta$ to obtain

$$\left|\frac{1}{\omega}(\Delta^2 u, \theta)_{L_2(\Omega)}\right| \leq \left|\left(\frac{v_0}{\omega}, \theta\right)_{L_2(\Omega)}\right| + \frac{1}{|\omega|}\left\|\mathcal{A}_N^{\frac{1}{2}}\theta\right\|^2_{L_2(\Omega)} + |i(v,\theta)_{L_2(\Omega)}| \tag{3I.45}$$

$$
\begin{aligned}
\text{(by (3I.35) and (3I.18))} \quad &\leq \left\| \frac{v_0}{\omega} \right\|_{L_2(\Omega)} \left[\epsilon_1 \|y\|_Y + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y \right] \\
&+ \left[\frac{\epsilon_1}{2} \|y\|_Y^2 + \frac{1}{2\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y^2 \right] \\
&+ \|v\|_{L_2(\Omega)} \left[\epsilon_1 \|y\|_Y + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y \right] \qquad (3I.46) \\
&\leq 3\epsilon_1 \|y\|_Y^2 + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y^2, \qquad (3I.47)
\end{aligned}
$$

majorizing, in the last step, v_0 by f_0 and v by y, via (3I.6) and (3I.7). Then, (3I.47) proves estimate (3I.41), whose left-hand side (3I.40) is just an application of Green's second theorem.

(ii) We estimate by [Brenner, Scott, 1994, p. 39], [Thomee, 1984, 1997, p. 26]

$$
\begin{aligned}
\left| \frac{1}{\omega} \int_\Gamma \frac{\partial \Delta u}{\partial \nu} \bar{\theta}\, d\Gamma \right| &\leq \frac{1}{|\omega|} \left\| \frac{\partial \Delta u}{\partial \nu} \Big|_\Gamma \right\|_{L_2(\Gamma)} \|\theta|_\Gamma\|_{L_2(\Gamma)} \qquad (3I.48) \\
&\leq \frac{1}{|\omega|} \left[\|\Delta u\|_{H^2(\Omega)}^{\frac{1}{2}} \|\Delta u\|_{H^1(\Omega)}^{\frac{1}{2}} \right] \left[\|\theta\|_{H^1(\Omega)}^{\frac{1}{2}} \|\theta\|_{L_2(\Omega)}^{\frac{1}{2}} \right] \qquad (3I.49) \\
&\leq C \left(\frac{\|u\|_{H^4(\Omega)}^{\frac{1}{2}}}{|\omega|^{\frac{1}{2}}} \right) \left(\frac{\|u\|_{H^3(\Omega)}^{\frac{1}{2}}}{|\omega|^{\frac{1}{4}}} \right) \left(\frac{\|\theta\|_{H^1(\Omega)}^{\frac{1}{2}}}{|\omega|^{\frac{1}{4}}} \right) \|\theta\|_{L_2(\Omega)}^{\frac{1}{2}}. \qquad (3I.50)
\end{aligned}
$$

But, invoking inequalities (3I.22) and (3I.23), we estimate

$$
\begin{aligned}
\frac{\|u\|_{H^4(\Omega)}^{\frac{1}{2}}}{|\omega|^{\frac{1}{2}}} \frac{\|u\|_{H^3(\Omega)}^{\frac{1}{2}}}{|\omega|^{\frac{1}{4}}} &\leq C \left[\|y\|_Y^{\frac{1}{2}} + \left\| \frac{f_0}{\omega} \right\|_Y^{\frac{1}{2}} \right] \left[\|y\|_Y^{\frac{1}{2}} + \left\| \frac{f_0}{\omega} \right\|_Y^{\frac{1}{2}} \right] \qquad (3I.51) \\
&\leq C \left[\|y\|_Y + \left\| \frac{f_0}{\omega} \right\|_Y \right]. \qquad (3I.52)
\end{aligned}
$$

Next, invoking the fourth root estimate of (3I.18) and majorizing θ by y via (3I.6) and (3I.7), we obtain

$$
\begin{aligned}
\frac{\|\theta\|_{H^1(\Omega)}^{\frac{1}{2}}}{|\omega|^{\frac{1}{4}}} \|\theta\|_{L_2(\Omega)}^{\frac{1}{2}} &\leq \left[\left(\frac{\epsilon_1}{2} \right)^{\frac{1}{4}} \|y\|_Y^{\frac{1}{2}} + \left(\frac{1}{2\epsilon_1} \right)^{\frac{1}{4}} \left\| \frac{f_0}{\omega} \right\|_Y^{\frac{1}{2}} \right] \|y\|_Y^{\frac{1}{2}} \qquad (3I.53) \\
&\leq \epsilon_2 \|y\|_Y + C_{\epsilon_2} \left\| \frac{f_0}{\omega} \right\|_Y. \qquad (3I.54)
\end{aligned}
$$

Putting together (3I.52) and (3I.54), we obtain, via the inequality in Appendix 3H, (3H.17),

$$C\left(\frac{\|u\|^{\frac{1}{2}}_{H^4(\Omega)}}{|\omega|^{\frac{1}{2}}}\right)\left(\frac{\|u\|^{\frac{1}{2}}_{H^3(\Omega)}}{|\omega|^{\frac{1}{4}}}\right)\left(\frac{\|\theta\|^{\frac{1}{2}}_{H^1(\Omega)}}{|\omega|^{\frac{1}{4}}}\right)\|\theta\|^{\frac{1}{2}}_{L_2(\Omega)}$$

$$\leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]\left[\epsilon_2\|y\|_Y + C_{\epsilon_2}\left\|\frac{f_0}{\omega}\right\|_Y\right] \tag{3I.55}$$

$$\text{(by (3H.17))} \qquad \leq \epsilon\|y\|^2_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|^2_Y. \tag{3I.56}$$

Then, (3I.56) used in (3I.50) yields (3I.42), as desired.

(iii) This is similar to the proof of part (ii). We likewise estimate by [Brenner, Scott, 1994, p. 39], [Thomee, 1984, p. 26]

$$\left|\frac{1}{\omega}\int_\Gamma \Delta u\,\frac{\partial\bar{\theta}}{\partial\nu}\,d\Gamma\right| \leq \frac{1}{|\omega|}\,\|\Delta u|_\Gamma\|_{L_2(\Gamma)}\left\|\frac{\partial\theta}{\partial\nu}\Big|_\Gamma\right\|_{L_2(\Gamma)} \tag{3I.57}$$

$$\leq \frac{C}{|\omega|}\left[\|\Delta u\|^{\frac{1}{2}}_{H^1(\Omega)}\|\Delta u\|^{\frac{1}{2}}_{L_2(\Omega)}\right]\left[\|\theta\|^{\frac{1}{2}}_{H^2(\Omega)}\|\theta\|^{\frac{1}{2}}_{H^1(\Omega)}\right] \tag{3I.58}$$

$$\leq C\left(\frac{\|u\|^{\frac{1}{2}}_{H^3(\Omega)}}{|\omega|^{\frac{1}{4}}}\right)\left(\|u\|^{\frac{1}{2}}_{H^2(\Omega)}\right)\left(\frac{\|\theta\|^{\frac{1}{2}}_{H^2(\Omega)}}{\sqrt{|\omega|}}\right)\left(\frac{\|\theta\|^{\frac{1}{2}}_{H^1(\Omega)}}{|\omega|^{\frac{1}{4}}}\right). \tag{3I.59}$$

Recalling again estimate (3I.23) and majorizing u by y via (3I.6) and (3I.7), we obtain

$$\frac{\|u\|^{\frac{1}{2}}_{H^3(\Omega)}}{|\omega|^{\frac{1}{4}}}\,\|u\|^{\frac{1}{2}}_{H^2(\Omega)} \leq C\left[\|y\|^{\frac{1}{2}}_Y + \left\|\frac{f_0}{\omega}\right\|^{\frac{1}{2}}_Y\right]\|y\|^{\frac{1}{2}}_Y \tag{3I.60}$$

$$\leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{3I.61}$$

Moreover, recalling estimate (3I.21) and (3I.18) on θ, we obtain, via the inequality in Appendix 3H, (3H.17),

$$\frac{\|\theta\|^{\frac{1}{2}}_{H^2(\Omega)}}{\sqrt{|\omega|}}\;\frac{\|\theta\|^{\frac{1}{2}}_{H^1(\Omega)}}{|\omega|^{\frac{1}{4}}}$$

$$\leq C\sqrt{2}\left[\|y\|^{\frac{1}{2}}_Y + \left\|\frac{f_0}{\omega}\right\|^{\frac{1}{2}}_Y\right]\left[\left(\frac{\epsilon_1}{2}\right)^{\frac{1}{4}}\|y\|^{\frac{1}{2}}_Y + \left(\frac{1}{2\epsilon_1}\right)^{\frac{1}{4}}\left\|\frac{f_0}{\omega}\right\|^{\frac{1}{2}}_Y\right] \tag{3I.62}$$

$$\text{(by (3H.17))} \qquad \leq \epsilon_2\|y\|_Y + C_{\epsilon_2}\left\|\frac{f_0}{\omega}\right\|_Y. \tag{3I.63}$$

Using both (3I.61) and (3I.63) in (3I.59), we obtain

$$C\left(\frac{\|u\|_{H^3(\Omega)}^{\frac{1}{2}}}{|\omega|^{\frac{1}{4}}}\right)\left(\|u\|_{H^2(\Omega)}^{\frac{1}{2}}\right)\left(\frac{\|\theta\|_{H^2(\Omega)}^{\frac{1}{2}}}{\sqrt{|\omega|}}\right)\left(\frac{\|\theta\|_{H^1(\Omega)}^{\frac{1}{2}}}{|\omega|^{\frac{1}{4}}}\right)$$

$$\leq C\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]\left[\epsilon_2\|y\|_Y + C_{\epsilon_2}\left\|\frac{f_0}{\omega}\right\|_Y\right] \tag{3I.64}$$

$$(\text{by (3H.17)}) \quad \leq \epsilon\|y\|^2 + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y^2, \tag{3I.65}$$

invoking once more the inequality in Appendix 3H, (3H.17) in the last step.

Finally, (3I.65) used in (3I.59) yields (3I.43), as desired.

(iv) Equation (3I.44) is an immediate consequence of estimates (3I.42) and (3I.43) used in (3I.41). □

The next result is a first serious step in achieving the desired estimates (3I.10) and (3I.11) for u and v. Part (ii) thereof improves upon estimate (3I.37) of Lemma 3I.3.1.

Lemma 3I.3.3 *With reference to (3I.6) and (3I.7), given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that*

(i)

$$\int_\Omega |\Delta u|^2\, d\Omega \leq \epsilon\|y\|_Y^2 + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y^2. \tag{3I.66}$$

[See Proposition 3C.5, Eqn. (3C.27), or (3C.28) for the difference between $\|\mathcal{A}^{\frac{1}{2}}u\|_{L_2(\Omega)}^2$ and $\int_\Omega |\Delta u|^2\, d\Omega$.]

(ii)

$$\frac{1}{|\omega|^2}\int_\Omega |\Delta v|^2\, d\Omega \leq \epsilon\|y\|_Y^2 + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y^2. \tag{3I.67}$$

Proof. (i) We return to Eqn. III = (3I.15), take here the $L_2(\Omega)$-inner product with Δu, and by use of (3I.35) and (3I.44), after majorizing Δu in $L_2(\Omega)$ by y in Y via (3I.6) and (3I.7), obtain

$$\left|\frac{1}{\omega}\int_\Omega \Delta v\Delta\bar{u}\, d\Omega\right| = \left|(i\theta, \Delta u)_{L_2(\Omega)} - \frac{1}{\omega}(\Delta\theta, \Delta u)_{L_2(\Omega)} - \left(\frac{\theta_0}{\omega}, \Delta u\right)_{L_2(\Omega)}\right| \tag{3I.68}$$

$$\leq \|\theta\|_{L_2(\Omega)}\|\Delta u\|_{L_2(\Omega)} + \frac{1}{|\omega|}\left|(\Delta\theta, \Delta u)_{L_2(\Omega)}\right| + \left\|\frac{\theta_0}{\omega}\right\|_{L_2(\Omega)}\|\Delta u\|_{L_2(\Omega)} \tag{3I.69}$$

$$\text{(by (3I.35), (3I.44))} \quad \leq \left[\epsilon_1 \|y\|_Y + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y \right] C \|y\|_Y$$

$$+ \left[\epsilon_2 \|y\|_Y^2 + C_{\epsilon_2} \left\| \frac{f_0}{\omega} \right\|_Y^2 \right] + C \left\| \frac{f_0}{\omega} \right\|_Y \|y\|_Y. \tag{3I.70}$$

Hence, (3I.70) yields

$$\left| \frac{1}{\omega} \int_\Omega \Delta v \Delta \bar{u} \, d\Omega \right| \leq \epsilon \|y\|_Y^2 + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y^2 . \tag{3I.71}$$

Next, we recall Eqn. I $=$ (3I.13), apply Δ throughout, and take the $L_2(\Omega)$-inner product with Δu to obtain the identity

$$i \int_\Omega \Delta u \Delta \bar{u} \, d\Omega = \frac{1}{\omega} \int_\Omega \Delta v \Delta \bar{u} \, d\Omega + \int_\Omega \Delta \left(\frac{u_0}{\omega} \right) \Delta \bar{u} \, d\Omega, \tag{3I.72}$$

from which we estimate by use of (3I.71) and majorizing Δu in $L_2(\Omega)$ by y in Y via (3I.6) and (3I.7):

$$\int_\Omega |\Delta u|^2 d\Omega \leq \left| \frac{1}{\omega} \int_\Omega \Delta v \Delta \bar{u} \, d\Omega \right| + \left\| \Delta \left(\frac{u_0}{\omega} \right) \right\|_{L_2(\Omega)} \|\Delta u\|_{L_2(\Omega)} \tag{3I.73}$$

$$\text{(by (3I.71))} \quad \leq \left[\epsilon_1 \|y\|_Y^2 + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y^2 \right] + C \left\| \frac{f_0}{\omega} \right\|_Y \|y\|_Y \tag{3I.74}$$

$$\leq \epsilon \|y\|_Y^2 + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y^2 . \tag{3I.75}$$

Thus (3I.75) proves (3I.66), as desired.

(ii) A further use of Eqn. I $=$ (3I.13) gives, via (3I.66), majorizing Δu_0 in $L_2(\Omega)$ by f_0 in Y via (3I.6) and (3I.7):

$$\int_\Omega \left| \frac{\Delta v}{\omega} \right|^2 d\Omega \leq \int_\Omega |\Delta u|^2 d\Omega + \int_\Omega \left| \frac{\Delta u_0}{\omega} \right|^2 d\Omega \tag{3I.76}$$

$$\text{(by (3I.66))} \quad \leq \left[\epsilon_1 \|y\|_Y^2 + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y^2 \right] + C \left\| \frac{f_0}{\omega} \right\|_Y^2 , \tag{3I.77}$$

and (3I.77) proves (3I.67), as desired. □

As a corollary we obtain

Lemma 3I.3.4 *With reference to (3I.6) and (3I.7), given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that*

$$\frac{1}{|\omega|} \|\mathcal{A}_N \theta\|_{L_2(\Omega)} \leq \epsilon \|y\|_Y + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y . \tag{3I.78}$$

Proof. We return to Eqn. III $=$ (3I.15) and estimate by use of (3I.67) and (3I.35) and majorizing θ_0 by f_0 via (3I.6) and (3I.7):

$$\frac{1}{|\omega|}\|\mathcal{A}_N\theta\|_{L_2(\Omega)} \le \frac{1}{|\omega|}\|\Delta v\|_{L_2(\Omega)} + \|i\theta\|_{L_2(\Omega)} + \left\|\frac{\theta_0}{\omega}\right\|_{L_2(\Omega)} \tag{3I.79}$$

$$\text{(by (3I.66), (3I.35))} \qquad \le \left[\epsilon_1\|y\|_Y + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y\right]$$

$$+\left[\epsilon\|y\|_Y + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|_Y\right] + \left\|\frac{f_0}{\omega}\right\|_Y \tag{3I.80}$$

$$\le \epsilon\|y\|_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y, \tag{3I.81}$$

and (3I.81) proves (3I.78), as desired. □

Lemma 3I.3.5 *With reference to (3I.6) and (3I.7), we have*

(i)

$$i\|v\|^2_{L_2(\Omega)} - i\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2_{L_2(\Omega)} + \frac{1}{\omega}\left(\theta|_\Gamma, \left.\frac{\partial v}{\partial \nu}\right|_\Gamma\right)_{L_2(\Gamma)} - \frac{1}{\omega}\left(\left.\frac{\partial \theta}{\partial \nu}\right|_\Gamma, v|_\Gamma\right)_{L_2(\Gamma)}$$

$$= \left(\mathcal{A}^{\frac{1}{2}}u, \frac{\mathcal{A}^{\frac{1}{2}}u_0}{\omega}\right)_{L_2(\Omega)} + \left(\frac{1}{\omega}\mathcal{A}_N\theta, v\right)_{L_2(\Omega)} + \left(\frac{v_0}{\omega}, v\right)_{L_2(\Omega)}. \tag{3I.82}$$

(ii) Given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\left|\frac{1}{\omega}(\mathcal{A}_N\theta, v)_{L_2(\Omega)}\right| = \left|\frac{1}{\omega}\int_\Omega \Delta\theta\bar{v}\, d\Omega\right| \le \epsilon\|y\|^2_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|^2_Y. \tag{3I.83}$$

(iii) Given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\left|\frac{1}{\omega}\int_\Gamma \frac{\partial v}{\partial \nu}\bar{\theta}\, d\Gamma - \frac{1}{\omega}\int_\Gamma v\frac{\partial\bar{\theta}}{\partial \nu}d\Gamma\right| \le \epsilon\|y\|^2_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|^2_Y. \tag{3I.84}$$

Proof. (i) We return to Eqn. II = (3I.14), and take here the $L_2(\Omega)$-inner product with v, thereby obtaining

$$i\|v\|^2_{L_2(\Omega)} + \left(\mathcal{A}u, \frac{v}{\omega}\right)_{L_2(\Omega)} + \frac{1}{\omega}(\theta|_\Gamma, G_1^*\mathcal{A}v)_{L_2(\Gamma)} + \frac{1}{\omega}\left(\left.\frac{\partial\theta}{\partial\nu}\right|_\Gamma, G_2^*\mathcal{A}v\right)_{L_2(\Gamma)}$$

$$= \left(\frac{1}{\omega}\mathcal{A}_N\theta, v\right)_{L_2(\Omega)} + \left(\frac{v_0}{\omega}, v\right)_{L_2(\Omega)}. \tag{3I.85}$$

Next, we substitute $v/\omega = iu - u_0/\omega$ from Eqn. I = (3I.13) into the second term on the left-hand side of (3I.85), and we recall that

$$G_1^*\mathcal{A}v = \frac{\partial v}{\partial\nu}, \quad G_2^*\mathcal{A}v = -v|_\Gamma, \quad v\in H^2(\Omega) \tag{3I.86}$$

from Lemma 3.13.1, Eqn. (3.13.26) and Lemma 3.13.2, Eqn. (3.13.30), respectively, to obtain (3I.82), as desired from (3I.85).

(ii) By (3I.78) we estimate, majorizing also v by y via (3I.6) and (3I.7),

$$\left| \left(\frac{1}{\omega} \mathcal{A}_N \theta, v \right)_{L_2(\Omega)} \right| \leq \frac{1}{|\omega|} \|\mathcal{A}_N \theta\|_{L_2(\Omega)} \|v\|_{L_2(\Omega)} \tag{3I.87}$$

$$\text{(by (3I.78))} \qquad \leq \left[\epsilon_1 \|y\|_Y + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y \right] \|y\|_Y, \tag{3I.88}$$

$$\leq \epsilon \|y\|_Y^2 + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y^2, \tag{3I.89}$$

and (3I.89) proves (3I.83), as desired.

(iii) By Green's second theorem we compute

$$\frac{1}{\omega} \int_\Omega \Delta v \bar{\theta} \, d\Omega = \frac{1}{\omega} \int_\Omega v \Delta \bar{\theta} \, d\Omega + \frac{1}{\omega} \int_\Gamma \frac{\partial v}{\partial \nu} \bar{\theta} \, d\Gamma - \frac{1}{\omega} \int_\Gamma v \frac{\partial \bar{\theta}}{\partial \nu} \, d\Gamma. \tag{3I.90}$$

Thus, by (3I.90), recalling (3I.67) and (3I.83), we estimate

$$\left| \frac{1}{\omega} \int_\Gamma \frac{\partial v}{\partial \nu} \bar{\theta} \, d\Gamma - \frac{1}{\omega} \int_\Gamma v \frac{\partial \bar{\theta}}{\partial \nu} \, d\Gamma \right| \leq \left\| \frac{1}{\omega} \Delta v \right\|_{L_2(\Omega)} \|\theta\|_{L_2(\Omega)} + \left| \frac{1}{\omega} (v, \Delta \theta)_{L_2(\Omega)} \right| \tag{3I.91}$$

$$\text{(by (3I.67), (3I.83))} \qquad \leq \left[\epsilon_1 \|y\|_Y + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y \right] \|y\|_Y + \left[\epsilon_1 \|y\|_Y^2 + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y^2 \right] \tag{3I.92}$$

$$\leq \epsilon \|y\|_Y^2 + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y^2, \tag{3I.93}$$

and (3I.93) proves (3I.84), as desired. [In going from (3I.91) to (3I.92), there was no need to use the "good" estimate (3I.35) for θ, just majorization of θ by y via (3I.6) and (3I.7) as $\Delta v / \omega$ already has the "good" estimate (3I.67).] □

As a corollary to Lemma 3I.3.5, we obtain the desired good estimate for the difference $[\|v\|_{L_2(\Omega)}^2 - \|\mathcal{A}^{\frac{1}{2}} u\|_{L_2(\Omega)}^2]$. This is a second serious step (the first was Lemma 3I.3.3) in achieving the final desired estimates (3I.9) and (3I.10) for u and v.

Proposition 3I.3.6 *With reference to (3I.6) and (3I.7), given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that*

$$\left| \|v\|_{L_2(\Omega)}^2 - \left\| \mathcal{A}^{\frac{1}{2}} u \right\|_{L_2(\Omega)}^2 \right| \leq \epsilon \|y\|_Y^2 + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y^2. \tag{3I.94}$$

Proof. We return to identity (3I.82), and use here estimates (3I.83) and (3I.84), obtaining

$$\begin{aligned}\left|\|v\|^2_{L_2(\Omega)} - \|\mathcal{A}^{\frac{1}{2}}u\|^2_{L_2(\Omega)}\right| \leq & \left|\frac{1}{\omega}\left(\theta|_\Gamma, \left.\frac{\partial v}{\partial \nu}\right|_\Gamma\right)_{L_2(\Gamma)} - \frac{1}{\omega}\left(\left.\frac{\partial \theta}{\partial \nu}\right|_\Gamma, v|_\Gamma\right)_{L_2(\Gamma)}\right| \\ & + \left\|\mathcal{A}^{\frac{1}{2}}u\right\|_{L_2(\Omega)} \left\|\mathcal{A}^{\frac{1}{2}}\left(\frac{u_0}{\omega}\right)\right\|_{L_2(\Omega)} + \left|\frac{1}{\omega}(\mathcal{A}_N\theta, v)_{L_2(\Omega)}\right| \\ & + \left\|\frac{v_0}{\omega}\right\|_{L_2(\Omega)} \|v\|_{L_2(\Omega)} \end{aligned} \tag{3I.95}$$

(by (3I.83), (3I.84))

$$\leq \left[\epsilon_1 \|y\|^2_Y + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|^2_Y\right] + \left[\frac{\epsilon_1}{2}\|y\|^2_Y + \frac{1}{2\epsilon_1}\left\|\frac{f_0}{\omega}\right\|^2_Y\right], \tag{3I.96}$$

majorizing $\mathcal{A}^{\frac{1}{2}}u$, v, and $\mathcal{A}^{\frac{1}{2}}u_0$, v_0 in $L_2(\Omega)$ by y and f_0 in Y, respectively, via (3I.6) and (3I.7). Then, (3I.96) yields (3I.94), as desired.

3I.4 Proof of Estimates (3I.9) and (3I.10) for u and v

Orientation So far, throughout the arguments of Sections 3I.2 and 3I.3, we have made no use of the special structure of the boundary operators B_1 and B_2; see (3I.1g) and (3I.1h).

In this way, we have achieved only the "right" ϵ-estimates for the following quantities: for θ in (3I.35); for Δu, or $\frac{1}{\omega}\,\Delta v$, in (3I.66) and (3I.67); and finally, for the difference $[\|v\|^2_{L_2(\Omega)} - \|\mathcal{A}^{\frac{1}{2}}u\|^2_{L_2(\Omega)}]$ in (3I.94). However, formula (3C.27) in Appendix 3C shows the relationship between $\|\mathcal{A}^{\frac{1}{2}}u\|^2_{L_2(\Omega)} = \|u\|^2_{H^2(\Omega)}$ and $\|\Delta u\|^2_{L_2(\Omega)}$. In the present section, we shall finally complete the proof, by achieving the desired estimates (3I.10) and (3I.9) for $\|v\|^2_{L_2(\Omega)}$ and $\|u\|^2_{H^2(\Omega)}$, in fact simultaneously. To this end, we need to work with a corresponding elliptic problem: We already know by (3I.66) that

$$\|\Delta u\|^2_{L_2(\Omega)} \leq \epsilon\|y\|^2_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|^2_Y, \tag{3I.97}$$

and by (3I.67) that, likewise,

$$\left\|\Delta\left(\frac{v}{\omega}\right)\right\|^2_{L_2(\Omega)} \leq \epsilon\|y\|^2_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|^2_Y. \tag{3I.98}$$

Therefore, if we manage to show that either

$$\|u|_\Gamma\|^2_{H^{\frac{3}{2}}(\Gamma)} \leq \epsilon\|y\|^2_Y + C_\epsilon \left\|\frac{f_0}{\omega}\right\|^2_Y, \tag{3I.99}$$

or else

$$\left\|\left(\frac{v}{\omega}\right)\Big|_\Gamma\right\|^2_{H^{\frac{3}{2}}(\Gamma)} \leq \epsilon\|y\|^2_Y + C_\epsilon \left\|\frac{f_0}{\omega}\right\|^2_Y, \tag{3I.100}$$

then we can appeal to elliptic theory either for the u-problem (3I.97), (3I.99) or for the $\left(\frac{v}{\omega}\right)$-problem (3I.98), (3I.100) and obtain, respectively, either

$$\|\mathcal{A}^{\frac{1}{2}}u|_\Gamma\|^2_{L_2(\Omega)} = \|u\|^2_{H^2(\Omega)} \leq \epsilon\|y\|^2_Y + C_\epsilon \left\|\frac{f_0}{\omega}\right\|^2_Y \tag{3I.101}$$

or

$$\left\|\frac{v}{\omega}\right\|^2_{H^2(\Omega)} \leq \epsilon\|y\|^2_Y + C_\epsilon \left\|\frac{f_0}{\omega}\right\|^2_Y. \tag{3I.102}$$

Once either one of the estimates (3I.101) or (3I.102) has been established, the other readily follows via I = (3I.13). Then, (3I.101) proves (3I.9), as desired. Moreover, (3I.101) used in (3I.94), proves (3I.10), as well, and the proof of Theorem 3I.1 is complete. Thus, the remaining key estimate to prove is either estimate (3I.99) for u or estimate (3I.100) for $\left(\frac{v}{\omega}\right)$. To this end, we shall take advantage, for the first time, of the special structure of the boundary operator B_1, rewritten as [Appendix 3C, Proposition 3C.1, Eqn. (3C.2)]

$$B_1 = -\left[D^2_\tau + k\frac{\partial}{\partial\nu}\right], \tag{3I.103}$$

where D^2_τ denotes the second tangential derivative, and $-k(x) = \operatorname{div}\nu(x)$ is the mean curvature at the point $x \in \Gamma$. Because of the required smoothness of Γ, we may assume that $k \in L_\infty(\Gamma)$. A first step is the following result on $u|_\Gamma$ in $H^2(\Gamma)$:

Lemma 3I.4.1 *With reference to (3I.6) and (3I.7), given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for all $\omega \in \mathbb{R}$, say $|\omega| \geq 1$, we have*

(i)

$$\|\Delta u|_\Gamma\|_{L_2(\Gamma)} \leq |\omega|^{\frac{1}{4}}\left[\epsilon\|y\|_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y\right]; \tag{3I.104}$$

(ii)

$$\|\theta|_\Gamma\|_{L_2(\Gamma)} \leq |\omega|^{\frac{1}{4}}\left[\epsilon\|y\|_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|_Y\right]; \tag{3I.105}$$

(iii)

$$\|u|_\Gamma\|_{H^2(\Gamma)} \le |\omega|^{\frac{1}{4}}\left[\epsilon\|y\|_Y + C_\epsilon \left\|\frac{f_0}{\omega}\right\|_Y\right] + C_\mu \left\|k\,\frac{\partial u}{\partial \nu}\right\|_{L_2(\Gamma)}; \tag{3I.106}$$

(iv) moreover, given $\epsilon > 0$, *there exists* $C_\epsilon > 0$ *such that for all* $|\omega| \ge \omega_0 = C\left(\max_{x\in\Gamma}|k|\right)/[\epsilon(1-\mu)]$, *we have*

$$\|u|_\Gamma\|_{H^2(\Gamma)} \le |\omega|^{\frac{1}{4}}\left[\epsilon\|y\|_Y + C_\epsilon \left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{3I.107}$$

Proof. (i) By use of the usual trace estimates [Brenner, Scott, 1994, p. 39], [Thomee, 1984, 1997, p. 26], of estimate (3I.23) and of estimate (3I.66), we obtain, say, for $|\omega| \ge 1$:

$$\|\Delta u|_\Gamma\|_{L_2(\Gamma)} \le C\|\Delta u\|^{\frac{1}{2}}_{H^1(\Gamma)}\|\Delta u\|^{\frac{1}{2}}_{L_2(\Omega)} \tag{3I.108}$$

$$\le C\|u\|^{\frac{1}{2}}_{H^3(\Omega)}\|\Delta u\|^{\frac{1}{2}}_{L_2(\Omega)} \tag{3I.109}$$

$$\text{(by (3I.23) and (3I.66))}\quad \le C|\omega|^{\frac{1}{4}}\left[\|y|^{\frac{1}{2}}_Y + \left\|\frac{f_0}{\omega}\right\|^{\frac{1}{2}}_Y\right]\left[\epsilon_1\|y\|^{\frac{1}{2}}_Y + C_{\epsilon_1}\left\|\frac{f_0}{\omega}\right\|^{\frac{1}{2}}_Y\right] \tag{3I.110}$$

$$\le |\omega|^{\frac{1}{4}}\left[\epsilon\|y\|_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|\right], \tag{3I.111}$$

and (3I.111) proves (3I.104), as desired.

(ii) Similarly, by the trace estimates [Brenner, Scott, 1994, p. 39], [Thomee, 1984, 1997, p. 26], recalling the driving estimate (3I.18), we obtain

$$\|\theta|_\Gamma\|_{L_2(\Gamma)} \le C\|\theta\|^{\frac{1}{2}}_{H^1(\Omega)}\|\theta\|^{\frac{1}{2}}_{L_2(\Omega)} \tag{3I.112}$$

$$\text{(by (3I.18))}\quad \le |\omega|^{\frac{1}{4}}\left[\left(\frac{\epsilon_1}{2}\right)^{\frac{1}{4}}\|y\|^{\frac{1}{2}}_Y + \left(\frac{1}{2\epsilon_1}\right)^{\frac{1}{4}}\left\|\frac{f_0}{\omega}\right\|^{\frac{1}{2}}_Y\right]\|y\|^{\frac{1}{2}}_Y \tag{3I.113}$$

$$\le |\omega|^{\frac{1}{4}}\left[\epsilon\|y\|_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|\right], \tag{3I.114}$$

and (3I.114) proves (3I.105), as desired. [In going from (3I.112) to (3I.113), we have simply majorized θ by y, with no need of invoking the finer estimate (3I.35).]

(iii) We use the first BC (3I.1d) for u and (for the first time) the structure (3I.103) for the boundary operator B_1, thus obtaining

$$\text{on } \Gamma: \Delta u + (1-\mu)B_1 u + \theta = \Delta u - (1-\mu)\left[D^2_\tau u + k\,\frac{\partial u}{\partial \nu}\right] + \theta = 0. \tag{3I.115}$$

Thus (with $0 < \mu < 1$), by (3I.115), recalling (3I.104) and (3I.105), we estimate

$$\left\| D_\tau^2 u|_\Gamma \right\|_{L_2(\Gamma)} \leq \frac{1}{1-\mu}\Big[\|\Delta u|_\Gamma\|_{L_2(\Gamma)} + \|\theta|_\Gamma\|_{L_2(\Gamma)} \Big] + \left\| k \frac{\partial u}{\partial \nu} \right\|_{L_2(\Gamma)} \tag{3I.116}$$

$$\text{(by (3I.104) and (3I.105))} \quad \leq \frac{1}{1-\mu}|\omega|^{\frac{1}{4}} \left[\epsilon_1 \|y\|_Y + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y \right] + \left\| k \frac{\partial u}{\partial \nu} \right\|_{L_2(\Gamma)}, \tag{3I.117}$$

and (3I.117) yields (3I.106), as desired.

(iv) To prove (3I.107) from (3I.106), we may use trace theory, with $k \in L_\infty(\Gamma)$, and majorize u by y via (3I.6) and (3I.7):

$$\left\| k \frac{\partial u}{\partial \nu} \right\|_{L_2(\Gamma)} \leq \left(\max_{x\in\Gamma} |k| \right) C\|u\|_{H^2(\Omega)} \leq C_k \|y\|_Y \tag{3I.118}$$

$$\leq \frac{C_k}{|\omega|^{\frac{1}{4}}} |\omega|^{\frac{1}{4}} \|y\|_Y \leq C_\mu \epsilon |\omega|^{\frac{1}{4}} \|y\|_Y, \tag{3I.119}$$

for all $\omega \in \mathbb{R}$ with $|\omega|^{\frac{1}{4}} \geq C_k/[(C_\mu)\epsilon]$, and (3I.107) follows from (3I.106), by use of (3I.119). $\Box$

It remains to establish the desired estimate (3I.99) for u [or (3I.100) for (v/ω)] in $H^{\frac{3}{2}}(\Gamma)$ from inequality (3I.107) in $H^2(\Gamma)$; this requires getting rid of the factor $|\omega|^{\frac{1}{4}}$ while lowering the boundary norm of u from $H^2(\Gamma)$ to $H^{\frac{3}{2}}(\Gamma)$. We have two options:

(a) either we use the following estimate for u in connection with (3I.99):

$$\|u|_\Gamma\|_{H^{\frac{3}{2}}(\Gamma)} \leq \left(\|u|_\Gamma\|^{\frac{1}{2}}_{H^2(\Gamma)} \right) \left(\|u|_\Gamma\|^{\frac{1}{2}}_{H^1(\Gamma)} \right); \tag{3I.120}$$

(b) or else we use the following estimate for (v/ω), in connection with (3I.100):

$$\left\| \frac{v}{\omega}\Big|_\Gamma \right\|_{H^{\frac{3}{2}}(\Gamma)} \leq \left(\left\| \frac{v}{\omega}\Big|_\Gamma \right\|^{\frac{1}{2}}_{H^2(\Gamma)} \right) \left(\left\| \frac{v}{\omega}\Big|_\Gamma \right\|^{\frac{1}{2}}_{H^1(\Gamma)} \right). \tag{3I.121}$$

With reference to (a), that is, (3I.120): To obtain the desired estimate (3I.99) from inequality (3I.120), the first term is "good" by (3I.107) just proved, so that we obtain for $|\omega| > \omega_0$:

$$\|u|_\Gamma\|_{H^{\frac{3}{2}}(\Gamma)} \leq C|\omega|^{\frac{1}{8}} \left[\epsilon_1 \|y\|_Y^{\frac{1}{2}} + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y^{\frac{1}{2}} \right] \|u|_\Gamma\|^{\frac{1}{2}}_{H^1(\Gamma)}, \tag{3I.122}$$

where by trace theory

$$\|u|_\Gamma\|^{\frac{1}{2}}_{H^1(\Gamma)} \leq C\|u\|^{\frac{1}{2}}_{H^{\frac{3}{2}}(\Omega)}. \tag{3I.123}$$

We then see, by using (3I.123) in (3I.122), that to obtain the desired estimate (3I.99), we would like to show the estimate, say, for $|\omega| \geq \omega_0$:

$$\|u\|^{\frac{1}{2}}_{H^{\frac{3}{2}}(\Omega)} \leq \frac{C}{|\omega|^{\frac{1}{8}}}\left[\|y\|^{\frac{1}{2}}_Y + \left\|\frac{f_0}{\omega}\right\|^{\frac{1}{2}}_Y\right]. \tag{3I.124}$$

However, there appear to be technical difficulties in showing inequality (3I.124), since the $H^{\frac{3}{2}}(\Omega)$-level for u is "too low" (i.e., below the required $H^2(\Omega)$-level, even though estimate (3I.124) is philosophically consistent with the two bonafide, established, a priori bounds (3I.22) and (3I.23) for u [in the sense that lowering the interior Sobolev norm of u by $1/2$ unit corresponds to a decrease of power of $|\omega|$ by $|\omega|^{\frac{1}{4}}$]. Thus, with reference to the right-hand side of (3I.120), we have that the first factor is "good" [see (3I.122)], but the second factor is "bad."

With reference to (b), that is, (3I.121): In contrast, the corresponding version of (3I.124), this time for (v/ω), rather than u, is an easy matter as shown by the next result.

Lemma 3I.4.2 *With reference to (3I.6) and (3I.7) we have, say, for $|\omega| \geq 1$:*

$$\left\|\frac{v}{\omega}\Big|_\Gamma\right\|_{H^1(\Gamma)} \leq C\left\|\frac{v}{\omega}\right\|_{H^{3/2}(\Omega)} \leq \frac{C}{|\omega|^{\frac{1}{4}}}\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right]. \tag{3I.125}$$

Proof. We interpolate (moment inequality) between estimate (3I.19) and (3I.20), rewritten here as

$$\left\|\frac{v}{\omega}\right\|_{H^2(\Omega)} \leq \|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y \quad \text{and} \quad \left\|\frac{v}{\omega}\right\|_{H^1(\Omega)} \leq \frac{C}{|\omega|^{\frac{1}{2}}}\left[\|y\|_Y + \left\|\frac{f_0}{\omega}\right\|_Y\right], \tag{3I.126}$$

to obtain (3I.125). □

Thus, with reference to the right-hand side of (3I.121), we have that the second factor is good by (3I.125), whereas the first factor is bad, an interchange of the situation that occurs in (3I.120).

However, estimate (3I.125) for (v/ω) does not yield the same estimate for u (to produce the desired estimate (3I.124)), because of the datum u_0/ω, via Eqn. I = (3I.13), again since the $H^{\frac{3}{2}}(\Omega)$-norm for u_0 is too low, that is, below the $H^2(\Omega)$-level. Accordingly, we shall proceed by taking appropriate initial conditions u_0 in a dense set of $H^2(\Omega)$, prove inequalities (3I.9) and (3I.10) in this case, and then extend them to all u_0 in $H^2(\Omega)$ by density.

Proposition 3I.4.3 *With reference to (3I.6) and (3I.7), given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for all $|\omega| > \omega_0$, $w_0 > 0$ as in Lemma 3I.4.1(iii), we have*

$$\|u\|^2_{H^2(\Omega)} \leq \epsilon\|y\|^2_Y + C_\epsilon\left\|\frac{f_0}{\omega}\right\|^2_Y; \tag{3I.127}$$

$$\|v\|^2_{L_2(\Omega)} \le \epsilon \|y\|^2_Y + C_\epsilon \left\| \frac{f_0}{\omega} \right\|^2_Y. \tag{3I.128}$$

First Proof. We define the subspace S_0 of $H^2(\Omega)$ of initial data

$$S_0 = \left\{ u_0 \in H^3(\Omega) : D^2_\tau u_0 = 0 \right\}, \tag{3I.129}$$

which is dense in $H^2(\Omega)$. Let $u_0 \in S_0$; then $i D^2_\tau u = D^2_\tau \left(\frac{v}{\omega}\right)$ via Eqn. I = (3I.13), and hence we obtain the equivalence $\|u|_\Gamma\|_{H^2(\Gamma)} \doteq \|\frac{v}{\omega}|_\Gamma\|_{H^2(\Gamma)}$, which when used in (3I.121) yields

$$\left\| \frac{v}{\omega}\Big|_\Gamma \right\|_{H^{\frac{3}{2}}(\Gamma)} \le C \|u|_\Gamma\|^{\frac{1}{2}}_{H^2(\Gamma)} \left\| \frac{v}{\omega}\Big|_\Gamma \right\|^{\frac{1}{2}}_{H^{\frac{1}{2}}(\Gamma)} \tag{3I.130}$$

(by (3I.107) and (3I.125))

$$\le C|\omega|^{\frac{1}{8}} \left[\epsilon_1 \|y\|^{\frac{1}{2}}_Y + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|^{\frac{1}{2}}_Y \right] \frac{C}{|\omega|^{\frac{1}{8}}} \left[\|y\|^{\frac{1}{2}}_Y + \left\| \frac{f_0}{\omega} \right\|^{\frac{1}{2}}_Y \right] \tag{3I.131}$$

$$\le \epsilon \|y\|_Y + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y, \qquad u_0 \in S_0, \tag{3I.132}$$

and the desired estimate (3I.100) is established. Then, as noted in the *Orientation* of Section 3I.4, elliptic estimates applied to (3I.98) = (3I.67) and (3I.100) = (3I.132) produce the desired estimate (3I.102) for $(v/\omega) = iu - u_0/\omega$ (see Eqn. I = (3I.13)), and hence the desired estimate (3I.127) for u, at least for $u_0 \in S_0$. Then (3I.127), used in inequality (3I.94), proves (3I.128), still at least for $u_0 \in S_0$. Finally, by the denseness of S_0 in $H^2(\Omega)$, we extend the validity of (3I.127) and (3I.128) to all $u_0 \in H^2(\Omega)$. Proposition 3I.4.3 is proved.

Second Proof. We define the subspace S_0' of $H^2(\Omega)$ of initial data

$$S_0' = \left\{ u_0 \in H^2(\Omega) : D^{\frac{3}{2}}_\tau u_0 = 0 \right\} \subset S_0, \tag{3I.133}$$

where S_0' is still dense in $H^2(\Omega)$. Let $u_0 \in S_0'$; then by Eqn. I = (3I.13), we have the equivalences

$$\|u|_\Gamma\|_{H^{\frac{3}{2}}(\Gamma)} \doteq \left\| \frac{v}{\omega}\Big|_\Gamma \right\|_{H^{\frac{3}{2}}(\Gamma)}, \qquad \|u|_\Gamma\|_{H^2(\Gamma)} \doteq \left\| \frac{v}{\omega}\Big|_\Gamma \right\|_{H^2(\Gamma)}. \tag{3I.134}$$

Then, recalling (3I.121), we estimate via (3I.134)

$$\|u|_\Gamma\|_{H^{\frac{3}{2}}(\Gamma)} \doteq \left\| \frac{v}{\omega}\Big|_\Gamma \right\|_{H^{\frac{3}{2}}(\Gamma)} \le \left\| \frac{v}{\omega}\Big|_\Gamma \right\|^{\frac{1}{2}}_{H^2(\Gamma)} \left\| \frac{v}{\omega}\Big|_\Gamma \right\|^{\frac{1}{2}}_{H^1(\Gamma)} \tag{3I.135}$$

$$\text{(by (3I.134))} \qquad \le C \|u|_\Gamma\|^{\frac{1}{2}}_{H^2(\Gamma)} \left\| \frac{v}{\omega}\Big|_\Gamma \right\|^{\frac{1}{2}}_{H^1(\Gamma)} \tag{3I.136}$$

$$\text{(by (3I.107), (3I.125))} \quad \leq C|\omega|^{\frac{1}{8}} \left[\epsilon_1 \|y\|_Y^{\frac{1}{2}} + C_{\epsilon_1} \left\| \frac{f_0}{\omega} \right\|_Y^{\frac{1}{2}} \right] \frac{C}{|\omega|^{\frac{1}{8}}} \left[\|y\|_Y^{\frac{1}{2}} + \left\| \frac{f_0}{\omega} \right\|_Y^{\frac{1}{2}} \right] \tag{3I.137}$$

$$\leq \epsilon \|y\|_Y + C_\epsilon \left\| \frac{f_0}{\omega} \right\|_Y, \quad u_0 \in S_0'. \tag{3I.138}$$

Thus, Eqn. (3I.138) establishes both estimates (3I.99) and (3I.100), and hence, by elliptic estimates on (3I.97) and (3I.98) as well, both inequalities (3I.101) = (3I.127) and (3I.102) = (3I.128) are established, at least for $u_0 \in S_0'$. But S_0' is dense in $H^2(\Omega)$, and so (3I.127) and (3I.128) are extended to all $u_0 \in H^2(\Omega)$. Proposition 3I.4.3 is proved. □

3J Uniform Exponential Energy Decay of Thermo-Elastic Equations with, or without, Rotational Term. Energy Methods

The present appendix addresses the issue of (exponential) decay of the energy of homogeneous thermo-elastic equations, with, or without, a rotational term in the elastic equation, thus irrespective of whether the underlying s.c. semigroup is analytic or not. Equivalently, we investigate the stability of the corresponding s.c. semigroup in the uniform operator topology. Unlike Section 3.11, Section 3.12, and Appendices 3D through 3I, the PDE models considered here include in the equation, possibly, also the term $[-k\Delta w_{tt}]$, which for two-dimensional plate models accounts for rotational forces, where the positive constant k is proportional to the square of the thickness. Thus, in the presence of this term, the w-equation becomes the (hyperbolic) Kirchoff equation (as in Sections 3.9 and 3.10, however, with no damping now: $\rho = 0$ in (3.9.1a) or (3.10.1a)). The present appendix will treat both cases $k = 0$ and $k > 0$ simultaneously by energy methods.

The core of this section treats simplified hinged BC: Thus, for $k = 0$, this is the model (3D.40) of Appendix 3D, which we have seen there to generate a s.c. contraction semigroup, which is also *analytic*, and hence *uniformly stable* (Theorem 3D.3). For $k > 0$, it will be established in Proposition 3J.1 below that this thermo-elastic model with a Kirchoff equation in w continues to generate a s.c. contraction semigroup (on a different space). However, this semigroup is no longer analytic for $k > 0$, unlike the case $k = 0$; see Remark 3J.1 below. For this reason, this Appendix 3J presents a proof by *energy methods* – which works for $k = 0$ as well as for $k > 0$ – yielding uniform stability of the underlying semigroup, indeed, uniformly in the parameter k varying on any finite interval $[0, K]$. In this way one recovers, for $k = 0$, the stability result of Theorem 3D.3 of Appendix 3D, which was obtained, instead, via analyticity of the semigroup, a property now absent for $k > 0$; see Chang and Triggiani [1998] and Triggiani [1997] for precise results in this direction for these specific hinged BC. See also Lasiecka and Triggiani [1999] for a general structural decomposition of thermo-elastic plates in the case $k > 0$ for all canonical BC.

Remark 3J.3 refines the model and the analysis of the present Appendix 3J by extending the (exponential) energy decay to a two-dimensional model with hinged boundary conditions as in Eqn. (3D.53) of Appendix 3D: Here the case $k = 0$ was treated in Proposition 3D.6 of Appendix 3D, via analyticity of the corresponding semigroup in Theorem 3D.5, a property that is now false for $k > 0$ [Triggiani, 1976; Chang, Triggiani, 1998, Lasiecka, Triggiani, 1999].

Finally, Remark 3J.4 points out that the energy method, here presented in a canonical simplified BC case, actually generalizes, with additional serious technical difficulties, to cover also more complicated BC, such as clamped BC, and free BC as in Avalos and Lasiecka [1997; 1998]. It is based on an operator multiplier such as $\mathcal{A}^{-1/2}\theta$ below (3J.36), first introduced in Avalos and Lasiecka [1997; 1998].

PDE Model. The Case of Simplified "Hinged" BC Let Ω be an open bounded domain of $\mathbb{R}^n$ with smooth boundary Γ. In this subsection we consider the following thermo-elastic homogeneous problem:

$$\begin{cases} w_{tt} - k\Delta w_{tt} + \Delta^2 w + \alpha\Delta\theta = 0 & \text{in } (0, T] \times \Omega \equiv Q; \quad (3J.1a) \\ \theta_t - \eta\Delta\theta - \alpha\Delta w_t = 0 & \text{in } Q; \quad (3J.1b) \\ w(0, \cdot) = w_0, \quad w_t(0, \cdot) = w_1, \quad \theta(0, \cdot) = \theta_0 & \text{in } \Omega; \quad (3J.1c) \\ w \equiv \Delta w \equiv 0 & \text{in } \Sigma; \quad (3J.1d) \\ \theta \equiv 0 & \text{in } \Sigma, \quad (3J.1e) \end{cases}$$

where $\alpha > 0$, $\eta > 0$ are constants, while the constant k is either $k = 0$ or $k > 0$ (as in Sections 3.9 and 3.10). The case $k = 0$ was treated ad hoc in Theorem 3D.3 (see also Theorem 3D.5 and Proposition 3D.6 of Appendix 3D). Here, we shall analyze problem (3J.1) for $k = 0$ and $k > 0$ simultaneously.

Abstract Model Consistently with the notation of Sections 3.9 and 3.10, we recall the positive self-adjoint operators on $L_2(\Omega)$ of Eqns. (3.9.3) and (3.9.4):

$$\mathcal{A}h = \Delta^2 h; \quad \mathcal{D}(\mathcal{A}) = \{h \in H^4(\Omega) : h|_\Gamma = \Delta h|_\Gamma = 0\}, \tag{3J.2}$$

$$\mathcal{A}^{\frac{1}{2}}h = \mathcal{A}_D h = -\Delta h; \quad \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) = H^2(\Omega) \cap H_0^1(\Omega), \tag{3J.3}$$

as well as the operator of Eqn. (3.9.7),

$$\mathbb{A}_k = \big(I + k\mathcal{A}^{\frac{1}{2}}\big)^{-1}\mathcal{A}, \quad \mathcal{D}(\mathbb{A}) = \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big), \tag{3J.4}$$

which is positive, self-adjoint on the space $\mathcal{D}(\mathcal{A}_k^{\frac{1}{4}})$ topologized by Eqn. (3.9.8):

$$\|x\|_{\mathcal{D}(\mathcal{A}_k^{\frac{1}{4}})} = \left\|\big(I + k\mathcal{A}^{\frac{1}{2}}\big)^{\frac{1}{2}}x\right\|_{L_2(\Omega)}, \quad x \in \mathcal{D}\big(\mathcal{A}^{\frac{1}{4}}\big) \equiv H_0^1(\Omega); \tag{3J.5}$$

$$(x_1, x_2)_{\mathcal{D}(\mathcal{A}_k^{\frac{1}{4}})} = \big(\big(I + k\mathcal{A}^{\frac{1}{2}}\big)x_1, x_2\big)_{L_2(\Omega)}, \quad x_1, x_2 \in \mathcal{D}\big(\mathcal{A}^{\frac{1}{4}}\big) \equiv H_0^1(\Omega); \tag{3J.6}$$

whose norm is equivalent to the $H_0^1(\Omega)$-norm by Eqn. (3.9.10b) for $k > 0$. Consequently, in view of (3J.2)–(3J.4), the first equation (3J.1a) of the PDE problem can be rewritten abstractly as

$$w_{tt} + k\mathcal{A}^{\frac{1}{2}} w_{tt} + \mathcal{A}w - \alpha\mathcal{A}^{\frac{1}{2}}\theta = 0. \tag{3J.7}$$

Thus, the entire problem (3J.1) is rewritten then as either a second-order system

$$\begin{cases} w_{tt} + \mathbb{A}_k w - \alpha\left(I + k\mathcal{A}^{\frac{1}{2}}\right)^{-1}\mathcal{A}^{\frac{1}{2}}\theta = 0, & (3\text{J}.8) \\ \theta_t + \eta\mathcal{A}^{\frac{1}{2}}\theta + \alpha\mathcal{A}^{\frac{1}{2}} w_t = 0 & (3\text{J}.9) \end{cases}$$

or, setting $y = [w, w_t, \theta]$, as a first-order equation

$$\frac{d}{dt}\begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix} = A_k \begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix}; \quad y = \begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix}; \tag{3J.10}$$

$$A_k = \begin{bmatrix} 0 & I & 0 \\ -\left(I + k\mathcal{A}^{\frac{1}{2}}\right)^{-1}\mathcal{A} & 0 & \alpha\left(I + k\mathcal{A}^{\frac{1}{2}}\right)^{-1}\mathcal{A}^{\frac{1}{2}} \\ 0 & -\alpha\mathcal{A}^{\frac{1}{2}} & -\eta\mathcal{A}^{\frac{1}{2}} \end{bmatrix} : Y_k \supset \mathcal{D}(A_k) \to Y_k; \tag{3J.11}$$

$$\begin{cases} Y_0 = \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right) \times L_2(\Omega) \times L_2(\Omega), & (3\text{J}.12) \\ \mathcal{D}(A_0) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right) \times \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right), & (3\text{J}.13) \end{cases} \quad k = 0$$

[which is the same as in (3D.46), however with a different notation; in a choice consistent with Sections 3.9 and 3.10, the operator $\mathcal{A}$ in this Appendix 3J corresponds to $\mathcal{A}^2$ in Appendix 3D];

$$\begin{cases} Y_k = \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right) \times \mathcal{D}\left(\mathcal{A}_k^{\frac{1}{4}}\right) \times L_2(\Omega), & (3\text{J}.14) \\ \mathcal{D}(A_k) = \mathcal{D}\left(\mathcal{A}^{\frac{3}{4}}\right) \times \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right) \times \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right), & (3\text{J}.15) \end{cases} \quad k > 0.$$

The Y_k-adjoint operator A_k^* of A_k is

$$A_k^* = \begin{bmatrix} 0 & -I & 0 \\ \left(I + k\mathcal{A}^{\frac{1}{2}}\right)^{-1}\mathcal{A} & 0 & -\alpha\left(I + k\mathcal{A}^{\frac{1}{2}}\right)^{-1}\mathcal{A}^{\frac{1}{2}} \\ 0 & \alpha\mathcal{A}^{\frac{1}{2}} & -\eta\mathcal{A}^{\frac{1}{2}} \end{bmatrix}$$

$$: Y_k \supset \mathcal{D}(A_k^*) = \mathcal{D}(A_k) \to Y_k. \tag{3J.16}$$

Thus, comparing (3J.11) with (3J.16), we see that the operator obtained from A_k by omitting the corner entry $-\eta\mathcal{A}^{\frac{1}{2}}$ is skew-adjoint on Y_k; hence

$$A_k = iS_k + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\eta\mathcal{A}^{\frac{1}{2}} \end{bmatrix}, \quad S_k \text{ self-adjoint on } Y_k. \tag{3J.17}$$

[The same property was noted to hold true for the thermo-elastic problems of Section 3.11.1, Section 3.11.2, Section 3.12, and Appendix 3D.] Thus, from (3J.17), we obtain via [Pazy, 1983, p. 15] and [Balakrishnan, 1981, p. 188], and (3J.14), (3J.12):

Proposition 3J.1 *(i) The operators A_k and A_k^* are dissipative on Y_k, $k \geq 0$:*

$$\operatorname{Re}(A_k x, x)_{Y_k} = \operatorname{Re}(A_k^* x, x)_{Y_k} = \operatorname{Re}\left(\begin{bmatrix} 0 & I & 0 \\ -\mathcal{A} & 0 & \alpha\mathcal{A}^{\frac{1}{2}} \\ 0 & -\alpha\mathcal{A}^{\frac{1}{2}} & -\eta\mathcal{A}^{\frac{1}{2}} \end{bmatrix} x, x\right)_{Y_0} \tag{3J.18}$$

$$= -\eta\left(\mathcal{A}^{\frac{1}{2}} x_3, x_3\right)_{L_2(\Omega)} \leq 0 \quad \forall x = [x_1, x_2, x_3] \in \mathcal{D}(A_k) = \mathcal{D}(A_k^*). \tag{3J.19}$$

(ii) Thus, A_k and A_k^* generate s.c. contraction semigroups $e^{A_k t}$ and $e^{A_k^* t}$ on Y_k, $k \geq 0$.

Remark 3J.1 (Case $k > 0$) Whereas, for $k = 0$, the s.c. semigroup $e^{A_0 t}$ is, moreover, analytic on Y_0 for $t > 0$ (Theorem 3D.5 of Appendix 3D), the situation is drastically different for $k > 0$. Indeed, for $k > 0$, one can show that the s.c. contraction semigroup $e^{A_k t}$ on Y_k is, *a fortiori*, neither continuous in the uniform operator topology, nor compact, for all $t > 0$, let alone analytic Chang and Triggiani [1998]. Indeed, more is true. It is possible to give a very precise spectral analysis description Chang and Triggiani [1998] of the operator A_k on Y_k [see (3J.11) and (3J.14)], or, equivalently, of the operator

$$\mathcal{G}_k = \begin{bmatrix} 0 & \mathcal{A}^{\frac{1}{2}}\left(I + k\mathcal{A}^{\frac{1}{2}}\right)^{-\frac{1}{2}} & 0 \\ -\left(I + k\mathcal{A}^{\frac{1}{2}}\right)^{-\frac{1}{2}}\mathcal{A}^{\frac{1}{2}} & 0 & \alpha\left(I + k\mathcal{A}^{\frac{1}{2}}\right)^{-\frac{1}{2}}\mathcal{A}^{\frac{1}{2}} \\ 0 & -\alpha\mathcal{A}^{\frac{1}{2}}\left(I + k\mathcal{A}^{\frac{1}{2}}\right)^{-\frac{1}{2}} & -\eta\mathcal{A}^{\frac{1}{2}} \end{bmatrix} \tag{3J.20}$$

$$= \begin{bmatrix} \mathcal{A}^{\frac{1}{2}} & 0 & 0 \\ 0 & \mathcal{A}_k^{\frac{1}{4}} & 0 \\ 0 & 0 & I \end{bmatrix} A_k \begin{bmatrix} \mathcal{A}^{-\frac{1}{2}} & 0 & 0 \\ 0 & \mathcal{A}_k^{-\frac{1}{4}} & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$: H = L_2(\Omega) \times L_2(\Omega) \times L_2(\Omega) \supset \mathcal{D}(\mathcal{G}_k) \to H; \tag{3J.21}$$

$\mathcal{A}_k^{\frac{1}{4}} = (I + k\mathcal{A}^{\frac{1}{2}})^{\frac{1}{2}}$. In particular: For $k > 0$, there exist three explicitly identified infinite dimensional subspaces, H_1, H_2, H_3 such that $H_1 \oplus H_2 \oplus H_3 = H$ (direct, nonorthogonal sum). Moreover, the restriction $e^{\mathcal{G}_k t}|_{H_1}$ of the s.c. contraction semigroup $e^{\mathcal{G}_k t}$ over the invariant subspace H_1 is *analytic* and *self-adjoint*, whereas the restriction $e^{\mathcal{G}_k t}|_{H_i}$, $i = 2, 3$ of the s.c. contraction semigroup $e^{\mathcal{G}_k t}$ over the invariant subspace H_i is a s.c. *group*. This, and more, is proved in Chang and Triggiani [1998] (see also Hansen and Zhang [1997]). To appreciate the difference between the case $k = 0$ and the case $k > 0$, we return to Eqns. (3J.8) and (3J.9), say with $\alpha = \eta = 1$, and substitute $\mathcal{A}^{\frac{1}{2}}\theta$ from (3J.9) into (3J.8), thus obtaining

$$w_{tt} + \mathbb{A}_k w + \left(I + k\mathcal{A}^{\frac{1}{2}}\right)^{-1}\mathcal{A}^{\frac{1}{2}} w_t = -\left(I + k\mathcal{A}^{\frac{1}{2}}\right)^{-1}\theta_t. \tag{3J.22}$$

Thus, recalling $\mathbb{A}_k$ in (3J.4), we see that the damping term w_t in (3J.22) provides a "structurally damped" term (In the sense of Appendix 3B, case $\alpha = 1/2$) if $k = 0$ and a "viscous damping" term if $k > 0$. This suggests analyticity for $k > 0$ (see proofs in Appendices 3F and 3G, based on these considerations) and group-dominated dynamical behavior for $k > 0$ [Chang, Triggiani, 1998; Lasiecka, Triggiani, 1999].

Energy Decay In the special case $k = 0$ in (3J.1a), whereby the Kirchoff equation becomes the Euler-Bernoulli equation, we have seen in Appendix 3D that the s.c. contraction semigroup $e^{A_0 t}$ generated by the operator A_0 in (3J.11) on the space Y_0 in (3J.12) is actually analytic and hence uniformly stable on Y_0; see Theorem 3D.3 and Proposition 3D.6 of Appendix 3D.

However, in the Kirchoff case with $k > 0$, the s.c. contraction semigroup $e^{A_k t}$ on Y_k, guaranteed by Proposition 3J.1 is *not* analytic [by Remark 3J.1]. Nevertheless, such $e^{A_k t}$ continues to be stable in the uniform topology of $\mathcal{L}(Y_k)$, indeed, uniformly with respect to the constant $k \in [0, K]$, for some positive constant $K < \infty$. This will be shown in Theorem 3J.2 below, by purely energy methods.

With reference to problem (3J.1) or its abstract versions (3J.8)–(3J.9), or (3J.10), we introduce the energy $E_k(t)$ by setting

$$\begin{aligned} E_k(t) = E_k(t; y_0) &\equiv \left\|\mathcal{A}^{\frac{1}{2}} w(t)\right\|^2_{L_2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} \\ &\quad + k\left\|\mathcal{A}^{\frac{1}{4}} w_t(t)\right\|^2_{L_2(\Omega)} + \|\theta(t)\|^2_{L_2(\Omega)} && \text{(3J.23a)} \\ \text{(by (3J.5))} \quad &= \left\|\mathcal{A}^{\frac{1}{2}} w(t)\right\|^2_{L_2(\Omega)} + \|w_t(t)\|^2_{\mathcal{D}(\mathcal{A}_k^{\frac{1}{4}})} + \|\theta(t)\|^2_{L_2(\Omega)} && \text{(3J.23b)} \\ \text{(by (3J.14))} \quad &= \|\{w(t), w_t(t), \theta(t)\}\|^2_{Y_k} \equiv \|e^{A_k t} y_0\|^2_{Y_k}, \quad y_0 = [w_0, w_1, \theta_0], && \text{(3J.23c)} \end{aligned}$$

where we have recalled Proposition 3J.1(ii) in the last step. The main result of the present Section 3J.1 is

Theorem 3J.2 *With reference to problem (3J.1), or (3J.10), we have: Given a number $K > 0$, there exists constants $M_K \geq 1$ and $a_K > 0$ such that, for all $0 \leq k \leq K < \infty$, the s.c. contraction semigroup guaranteed by Proposition 3J.1 satisfies the following uniform stability estimate:*

$$\|e^{A_k t}\|_{\mathcal{L}(Y_k)} \leq M_K e^{-a_K t}, \qquad \forall\, 0 \leq k \leq K;\ t \geq 0, \tag{3J.24a}$$

equivalently, by (3J.23),

$$E_k(t; y_0) \leq M_K e^{-a_K t} E_k(0; y_0), \qquad \forall\, y_0 \in Y_k,\ 0 \leq k \leq K;\ t \geq 0. \tag{3J.24b}$$

Proof of Theorem 3J.2. The proof is broken down into several partial results.

Step 1. Lemma 3J.3 *(dissipativity identity) With reference to the energy $E_k(t)$ defined in (3J.23) for problem (3J.10), we have:*

(i)

$$E_k(t) + \int_0^t \eta \left\|\mathcal{A}^{\frac{1}{4}}\theta(\tau)\right\|^2_{L_2(\Omega)} d\tau \equiv E_k(0), \quad \forall\, t \geq 0, \tag{3J.25}$$

where the notation of the initial point y_0 has been omitted;

(ii)

$$\frac{dE_k(t)}{dt} = -\eta \left\|\mathcal{A}^{\frac{1}{4}}\theta(t)\right\|^2_{L_2(\Omega)} \leq 0; \quad E_k(t_2) \leq E_k(t_1), \quad 0 < t_1 < t_2. \tag{3J.26}$$

Remark 3J.2 Lemma 3J.3 is an energy version of the dissipative result in Proposition 3J.1.

Proof. We take the $L_2(\Omega)$-inner product $(\,\cdot\,,\,\cdot\,)$ with norm $\|\cdot\|$ of the (i) w-equation (3J.7) with w_t and (ii) of the θ-equation (3J.9) with θ, and readily obtain, since $\mathcal{A}$ is self-adjoint:

$$\frac{1}{2}\frac{d}{dt}\left[(w_t, w_t) + k\left(\mathcal{A}^{\frac{1}{4}} w_t, \mathcal{A}^{\frac{1}{4}} w_t\right) + \left(\mathcal{A}^{\frac{1}{2}} w, \mathcal{A}^{\frac{1}{2}} w\right)\right] - \alpha\left(\mathcal{A}^{\frac{1}{2}}\theta, w_t\right) \equiv 0, \tag{3J.27}$$

$$\frac{1}{2}\frac{d}{dt}(\theta, \theta) + \eta\left(\mathcal{A}^{\frac{1}{4}}\theta, \mathcal{A}^{\frac{1}{4}}\theta\right) + \alpha\left(\mathcal{A}^{\frac{1}{2}} w_t, \theta\right) \equiv 0. \tag{3J.28}$$

Adding (3J.27) and (3J.28) results in a cancellation of the term $\alpha(\mathcal{A}^{\frac{1}{2}}\theta, w_t)$ since $\mathcal{A}$ is self-adjoint, and we obtain, recalling (3J.23a):

$$\frac{1}{2}\frac{d}{dt}E_k(t) + \eta\left\|\mathcal{A}^{\frac{1}{4}}\theta(t)\right\| \equiv 0, \quad t \geq 0, \tag{3J.29}$$

which is (3J.26). Integrating (3J.29) in t yields (3J.25), as desired. □

Step 2 By standard semigroup theory [Balakrishnan, 1981; Pazy, 1983], uniform stability of $e^{A_k t}$ in the norm of $\mathcal{L}(Y_k)$, as in (3J.24), holds true if and only if: Given $K > 0$, there exist a time $T > 0$ and a constant $0 < r_{T,K} < 1$ such that

$$\|e^{A_k T}\|_{\mathcal{L}(Y_k)} \le r_{T,K} < 1, \quad \text{or} \quad E_k(T) \le r_{T,K} E_k(0), \quad 0 \le k \le K. \tag{3J.30}$$

For problems satisfying a dissipativity identity such as (3J.25), an equivalent but more convenient characterization may be given.

Lemma 3J.4 *In view of Lemma 3J.3, a necessary and sufficient condition for characterization (3J.30) to hold true is: Given $K > 0$, there exist a time $T > 0$ and a constant $c_{T,K} > 0$ such that*

$$E_k(T) \le c_{T,K} \int_0^T \eta \big\|\mathcal{A}^{\frac{1}{4}}\theta(t)\big\|^2_{L_2(\Omega)} dt, \quad 0 \le k \le K, \tag{3J.31a}$$

equivalently, via (3J.25),

$$E_k(0) \le (c_{T,K} + 1) \int_0^T \eta \big\|\mathcal{A}^{\frac{1}{4}}\theta(t)\big\|^2_{L_2(\Omega)} dt, \quad 0 \le k \le K. \tag{3J.31b}$$

Proof. If. Assume (3J.31a). Then, in view of the validity of (3J.25) in Lemma 3J.3, we have

$$E_k(T) \le c_{T,K} \int_0^T \eta \big\|\mathcal{A}^{\frac{1}{4}}\theta(t)\big\|^2_{L_2(\Omega)} dt = c_{T,K}[E_k(0) - E_k(T)], \tag{3J.32}$$

and (3J.30) follows from (3J.32), with $r_{T,K} = c_{T,K}/(c_{T,K} + 1) < 1$.

Only if. Conversely, assume (3J.30). Then, again by (3J.25):

$$E_k(T) \le r_{T,K} E_k(0) = r_{T,K} E_k(T) + r_{T,K} \int_0^T \eta \big\|\mathcal{A}^{\frac{1}{4}}\theta(t)\big\|^2_{L_2(\Omega)} dt, \tag{3J.33}$$

and (3J.31a) follows from (3J.33) with $c_{T,K} = r_{T,K}/(1 - r_{T,K})$. $\square$

Step 3 Thus, the crux of the proof of Theorem 3J.2 consists in establishing the validity of the characterization (3J.31a): This says, qualitatively, that the energy at $t = T$ is dominated by the dissipation over $[0, T]$. The remaining part of the proof is aimed at establishing (3J.31a).

Lemma 3J.5 *With reference to problem (3J.1) or (3J.7), (3J.9), the following estimate holds true for any T, where $\|\cdot\|$ is the $L_2(\Omega)$-norm:*

$$\begin{aligned} &\frac{\alpha}{2} \int_0^T \left[\|w_t(t)\|^2 + k\big\|\mathcal{A}^{\frac{1}{4}} w_t(t)\big\|^2\right] dt \\ &\quad \le C_{1k} \int_0^T \big\|\mathcal{A}^{\frac{1}{4}}\theta(t)\big\|^2 dt + \frac{\epsilon}{2} \int_0^T \big\|\mathcal{A}^{\frac{1}{4}} w(t)\big\|^2 dt + C_{2k}[E_k(T) + E_k(0)], \end{aligned} \tag{3J.34}$$

where $\epsilon > 0$ *arbitrary, and*

$$\begin{cases} C_{1k} = \max\left\{\dfrac{k\eta^2}{2\alpha} + \dfrac{1}{2\epsilon}, \left(\dfrac{\eta^2}{2\alpha} + \alpha\right)\|\mathcal{A}^{-\frac{1}{4}}\|\right\}, \\ 2C_{2k} = \max\left\{1,\ k\|\mathcal{A}^{-\frac{1}{2}}\|, k\|\mathcal{A}^{-\frac{1}{4}}\|\right\}. \end{cases} \tag{3J.35}$$

Proof.

Step (i) We first establish the following identity, which is valid for all T, where $(\cdot\,,\,\cdot)$ and $\|\cdot\|$ refer to $L_2(\Omega)$:

$$\begin{aligned} \alpha\int_0^T \left[\|w_t\|^2 + k\|\mathcal{A}^{\frac{1}{4}} w_t\|^2\right] dt = &-\eta\int_0^T (w_t, \theta)dt - k\eta\int_0^T \left(\mathcal{A}^{\frac{1}{4}} w_t, \mathcal{A}^{\frac{1}{4}}\theta\right) dt \\ &- \int_0^T \left(\mathcal{A}^{\frac{1}{4}} w, \mathcal{A}^{\frac{1}{4}}\theta\right)dt + \alpha\int_0^T \|\theta\|^2\, dt \\ &- \left[(w_t, \mathcal{A}^{-\frac{1}{2}}\theta)\right]_0^T - k\left[(\mathcal{A}^{\frac{1}{4}} w_t, \mathcal{A}^{-\frac{1}{4}}\theta)\right]_0^T. \end{aligned} \tag{3J.36}$$

To prove (3J.36), following Avalos and Lasiecka [1997; 1998], we begin by taking the $L_2(\Omega)$-inner product of the w-equation (3J.7) with $\mathcal{A}^{-\frac{1}{2}}\theta(t)$ and integrate by parts in time over $[0, T]$ to yield, since $\mathcal{A}$ is self-adjoint,

$$\begin{aligned} &\left[(w_t, \mathcal{A}^{-\frac{1}{2}}\theta)\right]_0^T - \int_0^T \left(w_t, \mathcal{A}^{-\frac{1}{2}}\theta_t\right) dt + k\int_0^T (w_{tt}, \theta)\, dt \\ &+ \int_0^T \left(\mathcal{A}^{\frac{1}{2}} w, \theta\right) dt - \alpha\int_0^T \|\theta\|^2\, dt \equiv 0. \end{aligned} \tag{3J.37}$$

We next replace $\mathcal{A}^{-\frac{1}{2}}\theta_t = -\alpha w_t - \eta\theta$ from (3J.9) into the first integral term of (3J.37) and integrate by parts in time (w_{tt}, θ) in the second integral term of (3J.37), thus obtaining

$$\begin{aligned} &\left[(w_t, \mathcal{A}^{-\frac{1}{2}}\theta)\right]_0^T + \alpha\int_0^T (w_t, w_t)\, dt + \eta\int_0^T (w_t, \theta)\, dt \\ &+ k[(w_t, \theta)]_0^T - k\int_0^T (w_t, \theta_t)\, dt + \int_0^T \left(\mathcal{A}^{\frac{1}{4}} w, \mathcal{A}^{\frac{1}{4}}\theta\right) dt - \alpha\int_0^T \|\theta\|^2\, dt \equiv 0. \end{aligned} \tag{3J.38}$$

Using (3J.9) again to replace θ_t in the third integral term of (3J.38) and rearranging terms yields (3J.36), as desired.

Step (ii) We notice that the left-hand side of (3J.36) contains the "kinetic part" of the energy $E_k(t)$ in (3J.23a). We now estimate the right-hand side of identity (3J.36) to obtain (3J.34). We use

$$\begin{cases} 2|(w_t,\theta)| \le \epsilon_1 \|w_t\|^2 + \frac{1}{\epsilon_1}\|\theta\|^2, \\ 2\left|\left(\mathcal{A}^{\frac14} w_t, \mathcal{A}^{\frac14}\theta\right)\right| \le \epsilon_1 \left\|\mathcal{A}^{\frac14} w_t\right\|^2 + \frac{1}{\epsilon_1}\left\|\mathcal{A}^{\frac14}\theta\right\|^2, \\ 2\left|\left(\mathcal{A}^{\frac14} w, \mathcal{A}^{\frac14}\theta\right)\right| \le \epsilon \left\|\mathcal{A}^{\frac14} w\right\|^2 + \frac{1}{\epsilon}\left\|\mathcal{A}^{\frac14}\theta\right\|^2 \end{cases} \tag{3J.39}$$

(i.e., penalize by ϵ the kinetic terms and by $1/\epsilon$ the dissipation term $\|\mathcal{A}^{\frac14}\theta\|$), move the kinetic terms to the left, and obtain

$$\begin{aligned} &\int_0^T \left\{\left[\alpha - \frac{\epsilon_1\eta}{2}\right]\|w_t\|^2 + k\left[\alpha - \frac{\eta\epsilon_1}{2}\right]\left\|\mathcal{A}^{\frac14} w_t\right\|^2\right\} dt \\ &\le \left(\frac{\eta}{2\epsilon_1} + \alpha\right)\int_0^T \|\theta\|^2\, dt + \left(\frac{k\eta}{2\epsilon_1} + \frac{1}{2\epsilon}\right)\int_0^T \left\|\mathcal{A}^{\frac14}\theta\right\|^2 dt \\ &\quad + \frac{\epsilon}{2}\int_0^T \left\|\mathcal{A}^{\frac14} w\right\|^2 dt + \frac12\left[\|w_t(T)\|^2 + k\left\|\mathcal{A}^{\frac14} w_t(T)\right\|^2\right] \\ &\quad + \frac12\left[\|w_t(0)\|^2 + k\left\|\mathcal{A}^{\frac14} w_t(0)\right\|^2\right] + \frac12\left[\left\|\mathcal{A}^{-\frac12}\theta(T)\right\|^2 + k\left\|\mathcal{A}^{-\frac14}\theta(T)\right\|^2\right] \\ &\quad + \frac12\left[\left\|\mathcal{A}^{-\frac12}\theta(0)\right\|^2 + k\left\|\mathcal{A}^{-\frac14}\theta(0)\right\|^2\right]. \end{aligned} \tag{3J.40}$$

We now choose $\epsilon_1 = \alpha/\eta$ and (3J.40) readily yields (3J.34) via (3J.35), by recalling E_k in (3J.23). Lemma 3J.5 is proved. □

Step 4. Lemma 3J.6 *With reference to problem (3J.1) or (3J.8), (3J.9), we have the following identity in the norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$ of $L_2(\Omega)$:*

$$\begin{aligned} \int_0^T \left\|\mathcal{A}^{\frac12} w\right\|^2 dt = \int_0^T \left[\|w_t\|^2 + k\left\|\mathcal{A}^{\frac14} w_t\right\|^2\right] dt + \alpha\int_0^T \left(\mathcal{A}^{\frac14}\theta, \mathcal{A}^{\frac14} w\right) dt \\ - \left[(w_t, w)\right]_0^T - k\left[\left(\mathcal{A}^{\frac14} w_t, \mathcal{A}^{\frac14} w\right)\right]_0^T. \end{aligned} \tag{3J.41}$$

Proof. We take the $L_2(\Omega)$-inner product of the w-equation (3J.7) with w and integrate by parts in t to obtain (3J.41), since $\mathcal{A}$ is self-adjoint. □

Step 5 Denoting for convenience by $E_{k,c}(t)$ the kinetic part of the energy $E_k(t)$ in (3J.23a), that is, setting

$$E_{k,c}(t) \equiv E_{k,c}(t; y_0) \equiv \left[\|w_t(t)\|^2 + k\left\|\mathcal{A}^{\frac14} w_t(t)\right\|\right]^2, \tag{3J.42}$$

we obtain from Lemma 3J.6 the following:

Corollary 3J.7 *With reference to problem (3J.1) or (3J.8), (3J.9), we have*

(i) The following identity holds true:

$$\int_0^T E_k(t)\,dt \equiv \int_0^T \left[\left\|\mathcal{A}^{\frac{1}{2}} w(t)\right\|^2 + \|w_t(t)\|^2 + k\left\|\mathcal{A}^{\frac{1}{4}} w_t(t)\right\|^2\right] dt$$
$$= 2\int_0^T E_{k,c}(t)\,dt + \alpha \int_0^T \left(\theta(t), \mathcal{A}^{\frac{1}{2}} w(t)\right) dt$$
$$- [(w_t, w)]_0^T - k\left[\left(\mathcal{A}^{\frac{1}{4}} w_t, \mathcal{A}^{\frac{1}{4}} w\right)\right]_0^T. \tag{3J.43}$$

(ii) Hence, the following estimate holds true for all $\epsilon_0 > 0$:

$$\int_0^T E_k(t)\,dt \equiv \int_0^T \left[\left\|\mathcal{A}^{\frac{1}{2}} w(t)\right\|^2 + \|w_t(t)\|^2 + k\left\|\mathcal{A}^{\frac{1}{4}} w_t(t)\right\|^2\right] dt$$
$$\leq 2\int_0^T E_{k,c}(t)dt + \frac{\epsilon_0\alpha}{2}\int_0^T \left\|\mathcal{A}^{\frac{1}{2}} w(t)\right\|^2 dt$$
$$+ \frac{\alpha}{2\epsilon_0}\int_0^T \|\theta(t)\|^2 dt + \bar{C}_{1,k}[E_k(T) + E_k(0)]; \tag{3J.44}$$

$$2\bar{C}_{1,k} = \max\left\{1, \left\|\mathcal{A}^{-\frac{1}{2}}\right\| + k\left\|\mathcal{A}^{-\frac{1}{4}}\right\|\right\}. \tag{3J.45}$$

Proof. (i) Identity (3J.43) is obtained by simply adding $\int_0^T E_{k,c}(t)\,dt$ to both sides of identity (3J.41) and invoking (3J.23) and (3J.42).

(ii) Identity (3J.43) readily implies inequality (3J.44) via (3J.45), recalling (3J.23). □

Step 5 We next recall estimate (3J.34) for $\int_0^T E_{k,c}(t)\,dt$ on the first integral term on the right-hand side of estimate (3J.44) and obtain

Lemma 3J.8 *With reference to problem (3J.1) or (3J.8), (3J.9), we have the following key estimate: Given $K > 0$, there exist positive constants p_K and q_K, depending on K (but also on α, η, but not on T, though we emphasize only the dependence on K) such that, for all $0 \leq k \leq K < \infty$, the following estimate uniform in k holds true:*

$$\int_0^T E_k(t)\,dt = \int_0^T \left[\left\|\mathcal{A}^{\frac{1}{2}} w(t)\right\|^2 + \|w_t(t)\|^2 + k\left\|\mathcal{A}^{\frac{1}{4}} w_t(t)\right\|^2\right] dt$$
$$\leq p_K \int_0^T \eta\left\|\mathcal{A}^{\frac{1}{4}}\theta(t)\right\|^2 dt + q_K[E_k(T) + E_k(0)]. \tag{3J.46}$$

Proof. We return to the right-hand side of estimate (3J.44) and substitute inequality (3J.34) for its first integral term, while we move the two energy terms: $\epsilon_0 \int_0^T \|\mathcal{A}^{\frac{1}{2}} w(t)\|^2 dt$ [see (3J.44)] and

$$\epsilon \int_0^T \left\|\mathcal{A}^{\frac{1}{4}} w(t)\right\|^2 dt \leq \epsilon\left\|\mathcal{A}^{-\frac{1}{4}}\right\| \int_0^T \left\|\mathcal{A}^{\frac{1}{2}} w(t)\right\|^2 dt \tag{3J.47}$$

[see (3J.34)], to the left-hand side, with $\epsilon_0, \epsilon > 0$ arbitrary. Keeping track of the various constants, we readily obtain estimate (3J.46), uniformly in $0 \le k \le K$, with constants p_K, q_K independent of T. $\square$

Step 6 We finally obtain the sought-after estimate (3J.31a), as a corollary of both inequality (3J.46) and the monotone decreasing property (3J.26) of $E_k(t)$.

Corollary 3J.9 *With reference to problem (3J.1), or (3J.8), (3J.9), or (3J.10), the following estimate holds true: Given $K > 0$, there exists a constant $const_{T,K} > 0$ such that*

$$E_k(T) \le const_{T,K} \int_0^T \eta \left\| A^{\frac{1}{4}} \theta(t) \right\|^2 dt, \quad \textit{uniformly in } k \in [0, K]. \tag{3J.48}$$

Proof. We return to (3J.46). On the left-hand side, we obtain

$$T E_k(T) \le \int_0^T E_k(t)\, dt, \tag{3J.49}$$

since $E_k(t)$ is decreasing by (3J.26). On the right-hand side, we use identity (3J.25) for $t = T$, to eliminate $E_K(0)$, and thus obtain

$$T E_k(T) \le 2 p_K \int_0^T \eta \left\| A^{\frac{1}{4}} \theta(t) \right\|^2 dt + 2 q_K E_K(T). \tag{3J.50}$$

Since q_K does not depend on T, we take $T > 2q_K$, and then (3J.50) yields (3J.48) with $\text{const}_{T,K} = 2p_K/(T - 2q_K)$. $\square$

Corollary 3J.9 proves (3J.31a). Thus, via Lemma 3J.4, we see that Theorem 3J.2 is proved.

Remark 3J.3 The above proof of Theorem 3J.2 readily extends to cover the case of hinged BC, that is, problem (3J.1a,b,c,e), with the BC (3J.1d) replaced by

$$w|_\Sigma \equiv 0; \quad [\Delta w + (1 - \mu) B_1 w]_\Sigma \equiv 0; \tag{3J.51}$$

$$B_1 w \equiv -c(x) \frac{\partial w}{\partial \nu}, \tag{3J.52}$$

according to Appendix 3C, Eqn. (3C.2). In fact, the boundary term $\partial w / \partial \nu$ is then of lower order with respect to the $H^2(\Omega)$-energy level. Details are given in Avalos and Lasiecka [1997].

Remark 3J.4 The energy method of Theorem 3J.2 extends, with additional technical difficulties, to cover more complicated BC, such as clamped BC ($w \equiv \partial w / \partial \nu \equiv 0$ on Σ) and *coupled* BC, in particular hinged/Neumann BC, and free BC as in Section 3.13.

Details are given in Avalos and Lasiecka [1997; 1998] on which the present appendix is based.

Notes on Chapter 3

This chapter presents a much extended version of the collection of illustrative parabolic PDE examples already given in Lasiecka and Triggiani [1991]. New additions include: (i) the examples of Sections 3.9 and 3.10 on the Kirchhoff equation; (ii) the thermo-elastic plate systems with various BC of Sections 3.11 through 3.13; (iii) the structurally damped Euler–Bernoulli equations with boundary damping in the free BC (either shear forces BC or moment BC) of Section 3.14; (iv) the well/reservoir coupling of Section 3.15; and (v) the examples with unbounded control operator B and unbounded observation operator R of Section 3.16, some being as in Da Prato and Ichikawa [1985] or Bensoussan et al. [1992].

Sections 3.1–3.10

As mentioned above, the examples in Sections 3.1–3.8 are taken from Lasiecka and Triggiani [1991], while those in Sections 3.9 and 3.13 are new. However, Section 3.2 – which concerns the regularity theory of the optimal pair in explicit Sobolev spaces, in the case of the heat equation with Dirichlet boundary control – is taken from Lasiecka and Triggiani [1987]. Undoubtedly, this theory may be adapted to the other examples mutatis mutandis. The purely boundary case of Remark 3.3.4, however, with trace of the solution $y(t)|_\Gamma$ penalized in $L_2(\Gamma)$, rather than in $H^{1-\epsilon}(\Gamma)$ as in (3.3.2.1b), was previously studied in Sorine [1977] by variational PDE methods.

Sections 3.11–3.13

Sections 3.11 through 3.13, along with the corresponding Appendices 3D, 3E, 3F, 3G, 3H, 3I, and 3J contain several key results by the authors on the analyticity and on the uniform stability of the semigroups arising from thermo-elastic plate systems under various canonical BC, which are new. See "Analyticity" below.

Uniform Stability

In the approach given in Sections 3.11 – 3.13, stability of the thermo-elastic semigroup in the uniform norm $\mathcal{L}(Y)$ is an immediate consequence of analyticity, as it is a ready task to ascertain directly that the generator has no spectrum (= point spectrum) on the imaginary axis. The historical approach to stability of thermo-elastic plates was instead direct, not through the (at that time unknown) property of analyticity. It is interesting to note, in fact, that, though thermo-elastic systems have been extensively investigated in both the mechanical and the mathematical literature, much of the research efforts in the linear case were concentrated on establishing uniform stability (uniform decay rates), or even strong stability of the solutions under specific, case-by-case BC. This also included some cases where the elastic equation in w is of Kirchoff type, rather than of Euler–Bernoulli type, and thus accounts also for the term $-k\Delta w_{tt}$ (rotational forces), as in Section 3.10, with $k > 0$ proportional to the square of the thickness of the two-dimensional plate; and, moreover, the wavelike systems of n-dimensional thermo-elasticity. For $k = 0$, uniform stability results for thermo-elastic plates were proved directly in Kim [1992] (clamped in w Dirichlet in θ BC), in Liu and Zheng [1993] for more general BC, and Shibata [1994] (Neumann BC). [For a recent treatment on uniform (exponential) stability of the wavelike system of n-dimensional thermo-elasticity (which is not analytic, of course), see Henry et al. [1993], which follows in time the strong stability result in Dafermos [1968] and the exponential decay with respect to a higher norm in the one-dimensional, variable coefficient case of Slemrod [1981]. See also Hansen [1992] for uniform stability of a one-dimensional rod.

For $k > 0$, uniform stability for thermo-elastic plates was first shown in Lagnese [1989], however under additional mechanical dissipation in the free BC (of Section 3.13). This result raised the interesting question as to whether mechanical dissipation is really necessary for uniform stability. The answer, in the negative, came in the papers of Avalos and Lasiecka [1997; 1998], where they showed that (i) uniform stability attains without mechanical dissipation for all BC with $k > 0$; (ii) moreover, the decay is uniform with respect to k, and thus holds also for $k = 0$, *for all* BC except for the case of free BC where the case $k = 0$ was excluded. An account of the *energy method* employed by Avalos and Lasiecka is given in Appendix 3J, at least for simplified hinged in w/Dirichlet in θ BC, the easiest case. A key ingredient in the stability proof of Avalos and Lasiecka [1997; 1998] is the selection of the "right" multiplier $\mathcal{A}_D^{-1}\theta$ (which is novel when compared to the standard differential multipliers used in plate theory). Thus, our present result of Appendix 3I after Lasiecka and Triggiani [1998(c)], which shows analyticity of problem (3.13.1) of Section 3.13 with free BC and $k = 0$, establishes as a consequence, also the property of uniform stability for it, thus covering the limit case ($k = 0$) with free BC, which was excluded from the work of Avalos and Lasiecka [1997; 1998].

Very precise uniform stability results for thermo-elastic plates with $k > 0$ fixed (and not uniformly in k up to $k = 0$), under all BC except, so far, the free BC case, follow also from the sharp structural decomposition results in Lasiecka and Triggiani [1999], which are akin to, but distinct from and surely not derivable from, those in Henry et al. [1993] for wave like systems of n-dimensional thermo-elasticity. A rich theory for the special case of hinged BC with $k > 0$ is given in Chang and Triggiani [1998].

Analyticity

It was the paper by Liu and Renardy [1995], which appeared in late 1995, that first addressed, and proved, the much stronger and more desirable property of analyticity of the s.c. contraction thermo-elastic semigroup arising in the demanding case of Section 3.11.1 (clamped in w/Dirichlet in θ BC) – and in the amenable, simplified hinged/Dirichlet BC as in Appendix 3D, Eqn. (3D.40) – with no rotational term ($k = 0$). Their proof is technical, as the problem is challenging. The issue of analyticity of the thermo-elastic semigroup, under different canonical BC such as those of Section 3.11.2, and particularly of the coupled BC such as those of Section 3.12 and of Section 3.13 was open as of, say, late 1996. The result in the paper of Liu and Renardy [1995] influenced the present authors and stimulated these authors' interest in this area, in seeking to establish analyticity of the thermo-elastic semigroup (for $k = 0$, i.e., with Euler–Bernoulli elastic equation in w), for all canonical BC, beyond the Liu–Renardy's clamped/Dirichlet BC case. The goal was to include the attractive examples of optimal control problems given in Sections 3.11.2, 3.12, and 3.13 (and similar ones) in the present Chapter 3! The authors' main results on analyticity are contained in Lasiecka and Triggiani [1998a, b, c], and are reproduced here in Appendices 3E through 3I. Lasiecka and Triggiani [1998(a)] show analyticity of an abstract thermo-elastic model, which in particular covers the concrete cases of Section 3.11.1 and Section 3.11.2 (as well as of Appendix 3D, Eqn. (3D.40) and Eqn. (3D.53) of course). Two *direct* proofs are given, both by purely operator-theoretic methods, and both relying on the structural damped result of analyticity of Appendix 3B, Theorem 3B.1(b), $1/2 \leq \alpha \leq 1$. These proofs are reproduced in Appendices 3F and 3G. Analyticity of the thermo-elastic semigroup arising in the progressively more demanding cases of *coupled* BC of Section 3.12 and Section 3.13 remained open, and was first settled in the affirmative by Lasiecka and Triggiani [1998b, c] by June 1997. These results were then presented by the authors at numerous conferences, both in Europe and in the U.S., in the second half of 1997. The authors' proofs are again *direct*. However, in these coupled BC cases, a major novelty of these proofs is that they use critically ad hoc PDE methods and PDE estimates, such as the trace moment inequalities (3H.50), (3H.51), (3I.49), (3I.57), (3I.112). Thus, these proofs are definitely more technical, particularly for the free BC case of Section 3.13. A purely operator-theoretic, simplified version of the underlying idea (given in Appendix 3E) produces yet another proof of analyticity for the abstract thermo-elastic system of Appendices 3F and 3G as well. The authors' various proofs are all direct. Instead, a proof by contradiction (on the "standard" resolvent characterization of Appendix 3E, Theorem 3E.3) is given in Liu and Liu [1997], for an abstract system of the type as in Appendices 3E, 3F, and 3G however under the stronger assumption $\mathcal{D}(\mathcal{A}^{1/2}) = \mathcal{D}(\mathcal{B}^*)$, see Remark 3E.5.3. The only analytic thermo-elastic example given in Liu and Liu [1997] is Example 3E.4.4 in Appendix 3E, Eqn. (3E.103).

During the final stages of publication of this Volume, the present authors became aware of the just published (1999) book Liu and Zheng [1999] by Z. Liu and S. Zheng, which deals – among other things – with thermo-elastic problems. In particular, Theorem 2.5.4 of Section 2.5 of Liu and Zheng [1999] presents the analyticity results of the hinged/Neumann B.C. of our Section 12, and of the free B.C. of our Section 13, which were first proved in Lasiecka and Triggiani [1998b, c]. These proofs are reproduced in Appendix H and I, respectively of the present Volume. The authors of Lasiecka and Triggiani [1998b, c], having sent, privately, in 1998, these reprints to the authors of Liu and Zheng [1999], are pleased to see that the proof of analyticity of Theorem 2.5.4 given in Liu and Zheng [1999] follows very closely – one may say, step by step – both the sequence of the analysis and the technical details of the proofs in Lasiecka and Triggiani [1998b, c]. This includes: the several trace moment inequalities such as (3H.50), (3H.51), (3I.49), (3I.57), (3I.112); the several interior moment inequalities; elliptic theory; the representation of the first boundary operator as in (3I.115) in terms of tangential and normal derivatives; etc., introduced in Lasiecka and Triggiani [1998b, c]. The only exception is the fact that the authors of Liu and Zheng [1999] have chosen to turn the direct proofs of Lasiecka and Triggiani [1998b, c] into indirect proofs by *contradiction*, and this of course results in some streamlining. Generally, direct proofs are more desirable than proofs by contradiction, in that the former provides more insight than the latter. In the present case in question, direct proofs have the advantage of admitting *semi-discrete counterparts* of interest in the numerical analysis of these equations, with control on the constants arising in the relevant estimates.

Analyticity of the thermo-elastic semigroup plays a critical role in the recent uniform stability result of a non-linear thermo-elastic plate with free BC, as in Avalos et al. [1999].

Lack of Analyticity (and Much More)

By contrast, if the elastic model for w is of Kirchoff type (as in Section 3.10), and thus accounts for rotational forces ($k > 0$), then Lasiecka and Triggiani [1999] show that under all BC the corresponding s.c. contraction semigroup (which has markedly different properties from the case $k = 0$, indeed) admits a structural decomposition, which is sharp under all BC except, so far, the case of free BC: the thermo-elastic semigroup for $k > 0$ decomposes at each $t > 0$, as the sum of a *uniformly stable group* corresponding only to a damped Kirchoff elastic equation (no thermal component) plus a *compact operator*. Thus, the case $k > 0$ is group dominated, whereas the case $k = 0$ is analytic. A fortiori, for $k > 0$, the thermo-elastic semigroup is *neither compact nor continuous in the uniform operator topology for all* $t > 0$ on the corresponding state space (which for $k > 0$ is different from that of the case $k = 0$). In particular, in the amenable case of hinged/Dirichlet BC, an attractive and precise spectral theory is available, as shown in Chang and Triggiani [1998], following the original announcement in Triggiani [1975, 1976]; there exists an infinite dimensional invariant subspace (based on the eigenvectors) such that, on it, the s.c. semigroup becomes a group. Here, another relevant reference is the paper by Hansen and Zhang [1997], which treats one-dimensional thermo-elastic equations.

The following considerations are worth pointing out in the 'hyperbolic,' group-dominated Kirchoff case with $k > 0$. That, while the statements of the decomposition results of the thermo-elastic semigroup in Lasiecka and Triggiani [1999] are functional analytic/operator-theoretic in flavor, their proofs are not entirely in this mode. In fact, they rely critically – as typically for hyperbolic problems – on sharp trace/regularity results for elastic and thermo-elastic mixed problems, which are obtained by purely PDE energy methods. These may be either at the differential level, as in Lasiecka and Triggiani [1999] for coupled hinged/Neumann B.C., or else at the pseudo-differential level as in Lasiecka and Triggiani [1999] in the case of free B.C.

A related result with the semigroup decomposed as a simpler semigroup plus a compact perturbation was previously shown in Henry et al. [1993], for an abstract system which is however motivated by n-dimensional systems of thermo-elasticity. When applied to thermo-elastic *plate* equations, only the Hinged/Dirichlet BC case is covered. See a comparison pointed out in Lasiecka and Triggiani [1998(d)]. Sharp regularity theory of mixed problem are given in Hansen and Zhang [1997] in one-dimension, and in Triggiani [1999] in any dimension.

Section 3.14

The examples of this section are new. The abstract models corresponding to the "concrete" PDE mixed problem (3.14.1.1), (3.14.2.1), and (3.14.3.1) – and based on the "cancellation" properties (3.14.1.18) of Lemma 3.14.1.2 and (3.14.3.13) of Lemma 3.14.3.2 – are interesting in their own rights. Analyticity of problem (3.14.3.1) is based on Theorem B.1'. to be discussed below.

Section 3.15

For the well/reservoir model (3.15.1) of Section 3.15, when specialized to a two-dimensional square reservoir $\Omega : 0 \leq x, y \leq 1$, with a one-dimensional well Γ_0: $0 \leq x \leq 1$, we refer to A. Benabdallah [1998]. She in turn credits A. Bourgeat [1992] for the derivation of the model and A. Khodja (*Prépublcation, Besanson, 1997*) for the statement of analyticity of the s.c. contraction semigroup generated by the dynamical operator A in the case above with a square Ω, where his specialization of our Lemma 3.15.3 to the one-dimensional case for Γ_0 uses the available one-dimensional eigenfunction expansion. By contrast, our proof of Lemma 3.15.3 is PDE based and holds true for any dimension. It was Benabdallah [1998] that generated our interest in this problem. Our simple proof of analyticity of A in Section 3.15 reveals (since the parameter $\beta < 1$; see below (3.15.33)) that this model fits, really a "perturbation framework." This simple proof plainly breaks down for $\beta = 1$ and should be compared with the more technical proof of the abstract thermo-elastic model of Appendix 3E, which corresponds to the case $\beta = 1$.

Section 3.16

This section contains several examples of genuinely unbounded control/unbounded observation, in the style of Lasiecka and Triggiani [1991] and Da Prato and Ichikawa [1985]. See also Bensoussan et al. [1992].

Finite Cost Condition and Detectability Condition

As pointed out in the Notes to Chapter 2, in the case of parabolic problems, the Finite Cost Condition and the Detectability Condition are most readily verified by establishing the (uniform) exponential decay of feedback solutions, as in the related uniform stabilization problem. This is so, since *generally* the original generator A has compact resolvent, with at most finitely many unstable eigenvalues. The case of the unstable heat equation in Section 3.1 (see Eqn. (3.1.12)) may serve as a guiding illustration. An exception where A does not have compact resolvent arises in the case of elastic equations, or operators, such as (3B.9) in Appendix 3B, with $\alpha = 1$. Here, the spectrum of $A_{\rho\alpha}$ with $\alpha = 1$ contains also the point $\lambda = -1/(2\rho)$ in its continuous spectrum; see [Chen, Triggiani, 1989(a), Lemma A.1(I), p. 46]. However, all these operators $A_{\rho\alpha}$, across the entire range $0 < \alpha \leq 1$, are intrinsically stable; see Theorem 3B.1(d) in Appendix 3B.

The problems of *uniform stabilization* and *structural assignment* (assignment of eigenvalues of the feedback operator and coupled with the requirement that its eigenvectors form a Riesz basis) are of interest in themselves, for both parabolic and hyperbolic problems. The structure of the feedback operator may be preassigned in advance, through all possible combinations of observations and actuators in the interior and/or on the boundary. The general strategy on the stabilization of parabolic (or retarded functional differential equations) was set up in Triggiani [1975, 1976], extending finite-dimensional results (but also providing new pathological phenomena). The works of Triggiani [1979, 1980a, b] and Lasiecka and Triggiani [1982, 1983a, b, c] solve essentially all possible cases of boundary actuators and/or observation with preassigned structure of the finite range feedback operator. The most demanding case is (canonically) the uniform stabilization of the unstable heat equation with a finite range feedback operator acting on the boundary and based on boundary *traces*. This was treated in Lasiecka and Triggiani [1983(b)], where a general abstract condition is provided, which is then verified to hold true in special geometries (spheres and parallelopipeds, and for different reasons). This problem still awaits a full solution. The contribution of Amann [1987], which follows the strategy of Lasiecka and Triggiani [1983(a)] in general spaces $W^{s,p}$ (Ω) rather than $H^s(\Omega)$, reobtains a general condition related to that of Lasiecka and Triggiani [1983(b)].

In principle, the Finite Cost Condition could also be asserted, once *exact-null controllability* is established. For parabolic equations, however, it is nicer, and easier, to use uniform stabilization by virtue of a suitable feedback operator, rather than the more difficult property of exact null-controllability. Nevertheless, the notion of exact null-controllability for deterministic parabolic-like PDEs plays an essential, critical role in connection with corresponding stochastic parabolic differential equations. In this context, it is known, in fact, that the notion of exact null-controllability is equivalent to the strong Feller property of the semigroup of transition of the corresponding stochastic differential equation, which is obtained from the deterministic one by simply replacing the deterministic control with stochastic noise. For classical parabolic equations, that is, heat-type equations, we refer to the review paper by [Russell, 1978, Sections 7 and 8] for results on exact null-controllability. For some preliminary results on exact null-controllability of abstract structurally damped or thermo-elastic parabolic models, we refer to Lasiecka and Triggiani [1998(e)], in the case of distributed control; and for a one-dimensional thermo-elastic equation to Hansen and Zhang [1997], in the case of boundary control.

Appendix 3A

The presentation of these standard important results follows Bensoussan et al. [1992].

Appendix 3B

The material on elastic systems reported in Appendix 3B, and extensively used in Sections 3.4 through 3.15 is mostly taken from S. Chen and Triggiani [1987, 1989a, b; 1990a, b] and Triggiani [1991], whose work was stimulated by G. Chen and Russell [1982], who raised a few conjectures, a-fortiori established in Theorem 3B.1.

Cumulatively, papers [Chen, Triggiani, 1987; 1989(a)] give several proofs of Theorem 3B.1, and the authors have additional unpublished proofs as well.

Theorem 3B.1′ – which is invoked only in Section 3.14.3 of this Chapter 3 – is given in. S. Chen and Triggiani [1989(b)], which was submitted to Scientia Sinica in February 1989: its proof is based on the ideas and technicalities contained in a prior paper by [S. Chen, Triggiani, 1989(a), Section 5]. See also Huang [1985; 1988] for some related results on analytic semigroups in Hilbert space and Favini and Obrecht [1991] for results on Banach spaces under different conditions. Theorem 3B.5 is taken from Balakrishnan and Triggiani [1993].

Below we provide some historical, as well as technical, comments on abstract models of parabolic mixed problems, by selecting the heat equation with Dirichlet control as a guiding case study.

Abstract Operator Models of Mixed Parabolic Problems

Models in Integral Form: A Canonical Example

We consider, for simplicity of notation, the canonical heat equation in the unknown $y(t, x)$:

$$\begin{cases} y_t = \Delta y & \text{in } (0, T] \times \Omega = Q, & \text{(3N.1a)} \\ y(0, \cdot) = y_0 & \text{in } \Omega, & \text{(3N.1b)} \\ y|_\Sigma = u & \text{in } (0, T] \times \Gamma = \Sigma, & \text{(3N.1c)} \end{cases}$$

defined on a bounded domain $\Omega \subset R^n$ with sufficiently smooth boundary Γ. Let A be the negative, self-adjoint operator defined by

$$Af = \Delta f : L_2(\Omega) \supset \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega) \to L_2(\Omega) \tag{3N.2}$$

(see [Friedman, 1976, Theorem 17.2, p. 67] for a useful claim for general elliptic smooth problems, which, once specialized to our case, states that: $f \in H_0^1(\Omega)$, $\Delta f \in L_2(\Omega)$ in the sense of distributions imply that $f \in H^2(\Omega)$). Then, A generates a s.c., analytic semigroup e^{At} on $L_2(\Omega)$. Let D be the Dirichlet map:

$$v = Dh \iff \{\Delta v = 0 \text{ in } \Omega, v|_\Gamma = h \text{ on } \Gamma\}. \tag{3N.3}$$

Phase 1

Let, at first, choose $u \in C^1([0, T]; C(\Gamma))$, the class of continuously differentiable functions on $[0, T]$, with values in $C(\Gamma)$. For u in such class, using (3N.3) so that $\Delta Du = 0$ in Ω and $Du|_\Gamma = u$ on Γ, we rewrite (3N.1) as

$$\begin{cases} (y - Du)_t = \Delta(y - Du) - Du_t & \text{in } Q, & \text{(3N.4a)} \\ (y - Du)(0, \cdot) = y_0 - Du(0) & \text{on } \Omega, & \text{(3N.4b)} \\ [y - Du]|_\Sigma = 0 & \text{on } \Sigma. & \text{(3N.4c)} \end{cases}$$

By (3N.2), problem (3N.4) can be written as an abstract equation on $Y = L_2(\Omega)$ as

$$\begin{aligned} (y - Du)_t &= A(y - Du) - Du_t, \\ (y - Du)(0) &= y_0 - Du(0) \in Y. \end{aligned} \tag{3N.5}$$

The solution of (3N.5), that is, of problem (3N.4), is

$$y(t) - Du(t) = e^{At}[y_0 - Du(0)] - \int_0^t e^{A(t-\tau)} Du_t(\tau)\, d\tau. \tag{3N.6}$$

The above semigroup integral formula for problem (3N.1), as well as its derivation, are essentially due to Fattorini [1968]. This procedure is an abstract version, in operator setting, of the very classical and elementary approach of reducing a boundary nonhomogeneous problem (3N.1) to a boundary homogeneous problem (3N.4), by subtracting off the "steady state" v, that is, the solutions of the corresponding elliptic equation (3N.3), when the boundary term u is C^1 in time.

Phase 2

Starting from the preliminary formula (3N.6), [Balakrishnan, 1981, Section 3.4.11] proceeded with an integration by parts to obtain

$$y(t) - Du(t) = e^{At}[y_0 - Du(0)] - \left[e^{A(t-\tau)}Du(\tau)\right]_{\tau=0}^{\tau=t} + \int_0^t \frac{de^{A(t-\tau)}}{d\tau} Du(\tau)\,d\tau \tag{3N.7}$$

$$y(t) - \cancel{Du(t)} = e^{At}y_0 - e^{At}\cancel{Du(0)} - \cancel{Du(t)} + e^{At}\cancel{Du(0)} - A\int_0^t e^{A(t-\tau)}Du(\tau)\,d\tau, \tag{3N.8}$$

that is, the following explicit input $\to$ solution formula

$$y(t) = e^{At}y_0 - A\int_0^t e^{A(t-\tau)}Du(\tau)\,d\tau \tag{3N.9}$$

for $u \in C^1([0, T]; C(\Gamma))$. Under the regularity assumption

$$D : \text{continuous } L_2(\Gamma) \to L_2(\Omega), \tag{3N.10}$$

a mild restriction for the elliptic problem (3N.3), which in particular allows corners (see [Necas, 1967, p. 250]), it then follows that the analyticity of the semigroup e^{At} allows one to obtain the extension [recall the standard regularity result of Chapter 0, Eqn. (0.4)]

$$u \to A\int_0^t e^{A(t-\tau)}Du(\tau)\,d\tau : \text{continuous } L_2(0, T; L_2(\Gamma)) \to L_2(0, T; L_2(\Omega)). \tag{3N.11}$$

Thus, accordingly, the validity of (3N.9) can be extended as a map

$$\{u, y_0\} \to y(t) = y(t; y_0; u) : \text{continuous } L_2(0, T; L_2(\Gamma)) \times L_2(\Omega) \to L_2(0, T; L_2(\Omega)). \tag{3N.12}$$

To obtain sharper results, one needs to dig further into the regularity properties of the Dirichlet map D.

In the case of a two-dimensional square Ω, [Balakrishnan, 1981, Section 4.11] gives the following bound,

$$\|Ae^{At}D\| \le Ct^{-(\frac{3}{4}+\epsilon)}, \quad 0 < t \le 1; \quad \forall\, \epsilon > 0, \tag{3N.13}$$

in the uniform norm of $L_2(\Gamma) \to L_2(\Omega)$. [Actually, Balakrishnan [1981] and later Washburn [1974; 1977; 1979] write incorrectly the above bound with $\epsilon = 0$, and conjectured its validity in "the general case."]

Indeed, in his Ph.D. thesis at UCLA, Balakrishnan's student, Washburn [1974; 1977; 1979] established the validity of the bound (3N.13) for $(-\Delta)$ replaced by a general second-order uniformly strongly elliptic operators with real $C^\infty(\bar{\Omega})$ coefficients and Dirichlet BC, on smooth domains $\Omega \subset R^n$ (i.e., with C^∞-boundary and such that Ω lies on one side of Γ). Using the apparatus on intermediate spaces in [Lions, Magenes, 1972, Vol. I] and in Butzer and Berens [1967] as they pertain to analytic semigroups and the regularity of the elliptic problem, Washburn established the following two results (among others):

(a) the regularity

$$D : \text{continuous } L_2(\Gamma) \to [\mathcal{D}(A), L_2(\Omega)]_{1-\theta}, \quad 0 < \theta < \frac{1}{4}; \tag{3N.14}$$

(b) and the implication

$$f \in [\mathcal{D}(A), L_2(\Omega)]_{1-\theta} \Rightarrow \|Ae^{At} f\|_{L_2(\Omega)} = \mathcal{O}(t^{\theta-1}) \tag{3N.15}$$

for some $\theta \in (0, 1)$.

Then, the two above results (3N.14) and (3N.15) readily yield (3N.13) [with $\epsilon > 0$].

Phase 3

In Triggiani [1979] (and also in Triggiani [1980a, b]), R. Triggiani provided the following substantial simplification in the derivation of the key bound (3N.13), over the important work of Washburn, while offering an intrinsic enlightenment of (3N.13). R. Triggiani [1979] introduces two new ingredients over the work of Washburn. They are:

(i) domains of fractional powers $\mathcal{D}((-A)^\theta)$, $0 < \theta < 1$, for $(-A)$ of problem (3N.1) [or a suitable translation thereof, in general];
(ii) along with their identification with Sobolev spaces in the range of interest [Fujiwara, 1967; Grisvard, 1967; Lasiecka, 1978, Appendix B]; see Appendix 3A, that is, in the case of homogeneous Dirichlet BC as in (3N.2),

$$\mathcal{D}((-A)^\theta) = H^{2\theta}(\Omega) \text{ (equivalent norms)}, \quad 0 \le \theta < \frac{1}{4}. \tag{3N.16}$$

These two facts are then combined in Triggiani [1979] with

(iii) the regularity of the elliptic problem

$$D : \text{continuous } L_2(\Gamma) \to H^{\frac{1}{2}}(\Omega) \tag{3N.17}$$

to obtain:

$$D : \text{continuous } L_2(\Gamma) \to H^{\frac{1}{2}}(\Omega) \subset H^{2\theta}(\Omega) = \mathcal{D}((-A)^{\theta})), \quad 0 < \theta < \frac{1}{4}, \tag{3N.18}$$

equivalently,

$$(-A)^{\theta} D : \text{continuous } L_2(\Gamma) \to L_2(\Gamma), \quad 0 \le \theta < \frac{1}{4}, \tag{3N.19}$$

so that

$$-Ae^{At}D = (-A)^{1-\theta}(-A)^{\theta}e^{At}D = (-A)^{1-\theta}e^{At}(-A)^{\theta}D, \tag{3N.20}$$

and hence, by (3N.19) and standard analyticity of e^{At} (see [Pazy, 1983, p. 74]),

$$\|Ae^{At}D\| = \|(-A)^{1-\theta}e^{At}(-A)^{\theta}D\| \tag{3N.21}$$

$$\le \|(-A)^{1-\theta}e^{At}\| \, \|(-A)^{\theta}D\|$$

$$\le C t^{-(1-\theta)}, \quad 0 < t \le 1, \quad 0 \le \theta < \frac{1}{4}. \tag{3N.22}$$

Thus, (3N.13) is proved with $\theta = 1/4 - \epsilon$ and $1 - \theta = 3/4 + \epsilon$. The factoring as in (3N.20) readily permits one to improve (3N.11) to obtain

$$u \to -A \int_0^t e^{A(t-\tau)} Du(\tau)\, d\tau$$

$$= (-A)^{1-\theta} \int_0^t e^{A(t-\tau)}(-A)^{\theta} Du(\tau)\, d\tau \tag{3N.23}$$

$$: \text{continuous } L_2(0, T; L_2(\Gamma)) \to L_2(0, T; \mathcal{D}((-A)^{\theta}), \quad 0 \le \theta < \frac{1}{4}, \tag{3N.24}$$

by invoking the standard result in Chapter 0, Eqn. (0.4). Use of (3N.17)–(3N.24) has since become standard in mixed parabolic problems. It has been extensively used by the authors, for example, in regularity issues [Lasiecka, 1978; 1980a, b], in uniform stabilization and structural assignment problems [Lasiecka, Triggiani, 1982; 1983a, b, c; 1986; 1987; 1989; 1991], [Triggiani, 1975; 1976; 1979; 1980a, b; 1991] and also [Amann, 1989], etc.

Differential Abstract Models in Factor or Additive Form

The Fattorini–Balakrishnan–Washburn approach in (3N.1) through (3N.9) leads to an abstract model in *integral form*, such as (3N.9). We now present an alternative route (employed by the authors of this book since 1979–1980), which yields quickly abstract models in *differential* forms, first as "factor models" and then as "additive models." With reference to (3N.1) and (3N.3), we may write now for any $u \in L_2(\Gamma)$,

$$\begin{cases} y_t = \Delta(y - Du) & \text{in } \Omega, \\ [y - Du]|_\Gamma = 0 & \text{on } \Gamma, \end{cases} \tag{3N.25}$$

or, in *abstract factor form*, recalling (3N.2),

$$y_t = A(y - Du) \in Y, \quad y(0) = y_0. \tag{3N.26}$$

We now extend the original operator in (3N.2) as $A : L_2(\Omega) \to [\mathcal{D}(A^*)]'$ by isomorphism techniques while preserving the same notation; in short, we extend the original A to $(A^*)^*$. Then, with such extension, (3N.24) becomes the abstract *model in additive form*

$$y_t = Ay - ADu \in [\mathcal{D}(A^*)]', \quad y(0) = y_0 \in Y, \tag{3N.27}$$

so that

$$B = -AD = (-A)^{1-\theta}(-A)^\theta D, \quad \theta < \frac{1}{4}, \tag{3N.28}$$

and then

$$(-A)^{-\gamma} B = (-A)^\theta D \in \mathcal{L}(L_2(\Gamma); L_2(\Omega)), \quad \theta = \frac{1}{4} - \epsilon, \quad \gamma = \frac{3}{4} + \epsilon. \tag{3N.29}$$

This canonical example may serve as a guideline for any other parabolic mixed problem, from the heat equation with Neumann boundary control as in Section 3.3 (see Triggiani [1979; 1980a, b]) to other parabolic problems as in Sections 3.4 through 3.13.

Appendix 3C

Our starting reference for the form of the boundary operators B_1 and B_2, as defined by (3.5.2a,b) or (3C.11), (3C.12) is Lagnese [1987; 1989]. In this form, B_1 and B_2 are defined by (messy) expressions in d/dx and d/dy, that is, in terms of the Cartesian variables. As mentioned in the text, this form is not suitable for trace regularity analysis, see e.g., Eqn. (3I.103) in the proof of Theorem 3I.1 (analyticity of the thermo-elastic semigroup under free BC). Rather, an expression of B_1 (particularly) and B_2 in terms of tangential and normal derivatives is needed. We are not aware of references (even after asking workers in the field) that perform a translation of the expressions of B_1 and B_2 from the Cartesian (x, y) variables as in (3C.11), (3C.12) to the normal/tangential (ν, τ) variables. Proposition 3C.1 on the boundary operator B_1, Proposition 3C.6 on $\Delta w|_\Gamma$, Proposition 3C.7 on the first BC, Proposition 3C.8 (and Proposition 3C.9) on the boundary operator B_2, and Proposition 3C.11 on the second BC all respond to this expressed need. We had some target formulas in the Ph.D. thesis Stahel [1987] by Stahel to aim at, and these appear to be taken from

Landau and Lifschitz [1975], (both references in German). The results of Proposition 3C.7 and of Proposition 3C.8 recover these formulas in $\partial/\partial\nu$ and $\partial/\partial\tau$. Instead, for the critical Lemma 3C.2 and consequences: Corollary 3C.3, Proposition 3C.4, Proposition 3C.5, we have relied on Lagnese [1987]. As mentioned in the text, all results in Appendix 3C from Proposition 3C.6 on are actually not needed in the book and are included only for completeness and future reference.

Appendix 3D

Here the results on generation follow Triggiani [1975; 1976], with some remarks from Liu and Renardy [1995].

Appendices 3E, 3F, 3G, 3H, and 3I

The results on analyticity of thermo-elastic semigroups (with no rotational forces) are taken from Lasiecka and Triggiani [1998a, b, c].

Appendix 3J

Here the result and the technique follow – in a special case of more amenable BC – the general approach of Avalos and Lasiecka [1997; 1998], where the novel multiplier $\mathcal{A}_D^{-1}\theta$ is introduced ($\mathcal{A}_D$ is the Laplacian with Dirichlet BC.)

References and Bibliography

H. Amann, Feedback stabilization of linear and semilinear parabolic systems, in *Proceedings of Trends in Semigroup Theory and Applications*, pp. 21–57 (Trieste, 1987), Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York, 1989.

W. Arendt, Vector-valued Laplace transforms and Cauchy problems, *Israel J. Math.* **59** (1987), 327–352.

G. Avalos and I. Lasiecka, Differential Riccati equation for the active control of a problem of structural acoustics. *J. Optim. Theory Appli.* **91** (1996), 695–728.

G. Avalos and I. Lasiecka, Exponential stability of a thermo-elastic system without mechanical dissipation, IMA preprint #1357, November 1995, Institute for Mathematics and Its Applications, University of Minnesota. Rendiconti dell'Istituto di Matematica dell'Universitā di Trieste, Italy, Suppl. Vol. XXVIII (1997), 1–28; invited paper, special volume dedicated in memory of P. Grisvard.

G. Avalos and I. Lasiecka, The strong stability of a semigroup arising from a coupled hyperbolic/parabolic system, Semigroup Forum, Vol. 57 (1998), 278–292, IMA Preprint nr. 1347, 1995, Institute for Mathematics and its Applications, University of Minnesota.

G. Avalos and I. Lasiecka, Exponential stability of a thermo-elastic system with free boundary conditions without mechanical dissipation, IMA preprint nr. 1397, March 1996, Institute for Mathematics and Its Applications, University of Minnesota, *SIAM J. Math. Anal.*, **29** (1998), 155–182.

G. Avalos, I. Lasiecka, and R. Triggiani, Uniform stability of non-linear thermo-elastic plates with free boundary conditions, Proceedings of IFIP conference held at Chemmitz, Germany, April 1998, International Series of Numerical Mathematics, Vol. 133, pp. 13–32, 1999 by Birkhauser.

A. V. Balakrishnan, *Applied Functional Analysis*, Springer-Verlag, 2nd ed., 1981.

A. V. Balakrishnan and R. Triggiani, Lack of generation of strongly continuous semigroups by the damped wave operator on $H \times H$, *Appl. Math. Lett.* **6** (1993), 33–37.

A. Benabdallah, presentation (and private communication) at P.D.E. Workshop on Visco-elasticity and Thermo-elasticity, held at LNPC Rio de Janeiro, Brazil, March 1998.

A. Bensoussan, G. Da Prato, M. Delfour, and S. Mitter, *Representation and Control of Infinite Dimensional Systems,* Vol. II, Birkhäuser, 1992.

A. Bourgeat, Simulating gas-liquid flow in a well-reservoir system, in *Numerical Methods in Engineering and Applied Sciences*, H. Adler, J. C. Heinrich, S. Lavanchy, E. Onate, and B. Suarez (eds.), CIMNE, Barcelona (1992).

S. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, TAM 15, Springer-Verlag, 1994. Translation Americ. Math. Soc.

F. Bucci, A Dirichlet boundary control problem for the strongly damped wave equation, *SIAM J. Control Optim.* **30** (5) (1992), 1092–1100.

P. L. Butzer and H. Berens, *Semigroups of Operators and Approximation*, Springer-Verlag, 1967.

S. K. Chang, I. Lasiecka, and R. Triggiani, Finite element compensators for thermoelastic systems with boundary control and point observation, *Numer. Funct. Anal. and Optimiz.*, **20** (5&6) (1999), 419–435.

G. Chen and D. L. Russell, A mathematical model for linear elastic systems with structural damping. *Quart. Appl. Math.*, January (1982), 433–454.

S. Chen and R. Triggiani, Proof of two conjectures of G. Chen and D. L. Russell on structural damping for elastic systems: The case $\alpha = 1/2$, Proceedings of the Seminar in Approximation and Optimization held at University of Havana, Cuba, January 12–14, 1987, Lecture Notes in Mathematics, 1354, Springer-Verlag, 234–256.

S. Chen and R. Triggiani, Proof of extension of two conjectures on structural damping for elastic systems: The case $1/2 \leq \alpha \leq 1$, *Pac. J. Math.* **136**, N1 (1989(a)), 15–55.

S. Chen and R. Triggiani, Analyticity for elastic systems with non–self-adjoint dissipation: the general case, submitted to Scientia Sinica in February 1989(b).

S. Chen and R. Triggiani, Gevrey class semigroups arising from elastic systems: The case $0 < \alpha < 1/2$, *Proc. Am. Math. Soc.* **110** (1990(a)), 401–415.

S. Chen and R. Triggiani, Characterization of domains of fractional powers of certain operators arising in elastic systems, and applications, *J. Diff. Eqn.* **88** (1990(b)), 279–293.

S. K. Chang and R. Triggiani, Spectral analysis of thermo-elastic plates with rotational forces, in *Optimal Control: Theory, Algorithms and Applications*, pp. 84–115, W. W. Hager and P. M. Pardalos, Eds., Kluwer, 1998. Proceedings of IFIP Conference, University of Florida, Geinesville, February 1997.

C. M. Dafermos, On the existence and the asymptotic stability of solutions to the equations of linear thermo-elasticity, *Arch. Rat. Mech. Anal.* **29** (1968), 241–271.

G. Da Prato and A. Ichikawa, Riccati equations with unbounded coefficients, *Ann. Mat. Pura Appl.* **140** (1985), 209–221.

R. Dautray and J. L. Lions, *Analyse Mathématique et Calcul Numerique*, Vol. 2, Masson (1987).

H. O. Fattorini, Boundary control systems, *SIAM J. Control Optima.* **6** (1968), 349–385.

H. O. Fattorini, *The Cauchy Problem*, Encyclopedia of Mathematics and Its Applications, Addison-Wesley, 1983.

A. Favini and E. Obrecht, Conditions for parabolicity of second-order abstract differential equations, *Differential and Integral Equations* **4** (1991), 1005–1022.

A. Friedman, *Partial Differential Equations*, reprinted by Robert E. Krieger Publ., Huntington, NY, 1976.

D. Fujiwara, Concrete characterization of the domains of fractional powers of some elliptic differential operators of the second order, *Proc. Japan Acad.* **43** (1967), 82–86.

P. Grisvard, Caracterization de quelques espaces d'interpolation, *Arch. Rational Mech. Anal.* **25** (1967), 40–63.

P. Grisvard, Spazi di tracce e applicazioni, *Rendiconti di Matematica* (4), **5** (VI) (1972), 657–729.

S. Hansen, Exponential energy decay in a linear thermo-elastic rod, *J. Math. Anal. Appl.* **167** (1992), 429–442.

S. Hansen and B.-Y. Zhang, Boundary control of a linear thermo-elastic beam, *J. Math. Anal. Appl.* **210** (1997), 182–205.

F. Huang, On the holomorphic property of the semigroup associated with linear elastic systems with structural damping, *Acta. Math. Sci.* (in Chinese) **55** (1985), 271–277.

F. Huang, On the mathematical model for linear elastic systems with analytic damping, *SIAM J. Control Optim.* **26** (1988). 715–724.

D. Henry, O. Lupes, and A. Perrissinotto, On the essential spectrum of a semigroup of thermo-elasticity, Non Linear Analysis, T.M.A. (1993) 65–75.

G. Ji and I. Lasiecka, Partially observed analytic systems with fully unbounded actuators and sensors – FEM algorithms, in *Optimal Control: Theory, Algorithms and Applications*, W. W. Hager and P. M. Pardalos, eds., Kluwer Academic Publishers.

T. Kato, Fractional powers of dissipative operators, *J. Math. Soc. Japan* **13** (1961), 246–274.

T. Kato, Fractional powers of dissipative operators, II, *J. Math. Soc. Japan* **14** (1962), 243–248.

T. Kato, *Perturbation Theory of Linear Operators*, Springer-Verlag, 1966.

B. Kellog, Properties of elliptic B.V.P., Chapter 3, in *The Mathematical Foundation of the Finite Elements Method*, Academic Press, New York, London, 1972.

J. U. Kim, On the energy decay of a linear thermoelastic bar and plate, *SIAM J. Math. Anal.* **23** (1992), 889–899.

V. A. Kondratiev, *Boundary Problems for Elliptic Equations in Domains with Conical or Angular Points, Trans. Moscow Math. Soc.* **16** (1967).

S. G. Krein, *Linear Differential Equations in Banach space*, Trans. of Math. Monographs 29, Am. Math. Soc., Providence, RI, 1988.

J. Lagnese, Uniform boundary stabilization of homogeneous isotropic plates, Springer-Verlag Lecture Notes in Control and Information Sciences LNCIS vol. 102 (1987), pp. 204–215; Proceedings of the 3rd International Conference held at Vorau, Austria, July 1986.

J. Lagnese, *Boundary Stabilization of Thin Plates*, SIAM, 1989.

J. Lagnese and J. L. Lions, *Modelling Analysis and Control of Thin Plates*, Masson, 1988.

L. D. Landau and E. M. Lifschitz: *Lehrbuch der Theoretischen Physik vii.* Akademie-Verlag, Berlin, 1975.

I. Lasiecka, Boundary control of parabolic systems, *Appl. Math. Optim.* **4** (1978), 301–327.

I. Lasiecka, Boundary control of parabolic systems: Finite-element approximation, *Appl. Math. Optim.* **6** (1980(a)), 31–62.

I. Lasiecka, Unified theory for abstract parabolic boundary problems: A semigroup approach, *Appl. Math. Optim.* **6** (1980(b)), 287–333.

I. Lasiecka, Finite element approximations of compensator design for analytic generators with fully unbounded control-observation operators, *SIAM J. Control and Optimiz.*, **33** (1995), 67–68.

I. Lasiecka, Control and stabilization of interactive structures, in *Systems and Control in the 21st Century*, Birkhäuser (1997), 245–262.

I. Lasiecka, J. L. Lions, and R. Triggiani, Non-homogeneous boundary value problems for second order hyperbolic operators, *J. Mathem. Pure Appl.*, **65** (1986), 149–192.

I. Lasiecka and R. Triggiani, Structural assignment of Neumann boundary feedback parabolic equations: The case of trace in the feedback loop, *Ann. Matem. Pura Appl.* (IV) **XXXII** (1982), 131–175.

I. Lasiecka and R. Triggiani, Stabilization and structural assignment of Dirichlet boundary feedback parabolic equations, *SIAM J. Control Optim.* **21** (1983(a)), 766–803.

I. Lasiecka and R. Triggiani, Stabilization of Neumann boundary feedback parabolic equations: The case of trace in the feedback loop, *Appl. Math. Optim.* **19** (1983(b)), 307–350. (Preliminary version in Lecture Notes Springer Verlag, **54** (1983(b)), 238–246.)

I. Lasiecka and R. Triggiani, Feedback semigroups and cosine operators for boundary feedback parabolic and hyperbolic equations, *J. Diff. Eqn.* **47** (1983(c)), 246–272.

I. Lasiecka and R. Triggiani, Finite rank, relatively bounded perturbations of C_0-semigroups, Part II: Spectrum allocation and Riesz basis in parabolic and hyperbolic feedback systems, *Ann. Matem. Pura Appl.* IV, **CXLIII** (1986), 47–100.

I. Lasiecka and R. Triggiani, The regulator problem for parabolic equations with Dirichlet boundary control; Part I: Riccati's feedback synthesis and regularity of optimal solutions, *Appl. Math. Optim.* **16** (1987), 147–168.

I. Lasiecka and R. Triggiani, Regularity theory for a class of nonhomogeneous Euler–Bernoulli equations: A cosine operator approach, *Bollett. Unione Matem. Italiana UMI* (7), **3-B** (1989), 199–228.

I. Lasiecka and R. Triggiani, *Differential and Algebraic Riccati Equations with Applications to Boundary/Point Control Problems: Continuous Theory and Approximation Theory*, Springer-Verlag, Lecture Notes in Control and Information Sciences #164, 1991, 160 pp.

I. Lasiecka and R. Triggiani, Two direct proofs of analyticity of the s.c. semigroup arising in abstract thermo-elastic equations, presented at IFIP conference, University of Floriday, Gainesville, February 1997, *Advances in Differential Equations*, Vol. 3, n.3, May 1998(a), pp. 387–416.

I. Lasiecka and R. Triggiani, Analyticity of thermo-elastic semigroups with coupled BC. Part I: The case of hinged thermal Neumann BC., *Abstract Appl. Anal.*, Vol. 3, n.1–2. (1998(b)), 153–169; presented at "Workshop on Deterministic and Stochastic Evolution Equations," held at Scuola Normale di Pisa, July 1997; IFIP Conference held in Detroit, MI, July 1997; Control Conference MMAR'97, held at Miedzyzdroje, Poland, August 1997.

I. Lasiecka and R. Triggiani, Analyticity of thermo-elastic semigroups with coupled BC. Part II: The case of free BC., *Annali Scuola Normale di Pisa* Classe Science (Serie IV, Fascicolo 3–4), Vol. XXVII (1998(c)), 457–497, presented at "Workshop on Deterministic and Stochastic Evolution Equations," held at Scuola Normale di Pisa, July 1997; IFIP Conference held in Detroit, MI, July 1997; Control Conference MMAR'97, held at Miedzyzdroje, Poland, August 1997.

I. Lasiecka and R. Triggiani, Analyticity, and lack thereof, of thermo-elastic semigroups, *Esaim*, Vol. 4 (1998(d)), pp. 199–222.

I. Lasiecka and R. Triggiani, Exact null-controllability of structurally damped and thermo-elastic parabolic models, invited Memoria for Accademia Nazionale dei Lincei, Mathematics Section, Rome, Italy, Series IX Vol. IX, N. 1 (1998(e)), 43–69.

I. Lasiecka and R. Triggiani, Structural decomposition of thermo-elastic semigroups with rotational forces, *Semigroup Forum*, **5** (1999), 585–599.

I. Lasiecka and R. Triggiani, A sharp trace result on a thermo-elastic plate equation with coupled Hinged/Neumann Boundary Conditions, *Discrete & Continuous Dynamical Systems*, Vol. 5, N. 3, July 1999, 585–598.

I. Lasiecka and R. Triggiani, Sharp regularity theory for elastic and thermo-elastic Kirchoff equations with free boundary conditions, *Rocky Mountain J. of Mathematics*, to appear.

J. L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, New York 1971.

J. L. Lions and E. Magenes, *Nonhomogeneous Boundary Value Problems and Applications I, II*, 1972, Springer-Verlag.

J. L. Lions, Les semigroupes distributions, *Portug. Math.* **19** (1960), 141–164.

J. L. Lions, Espaces d'interpolation et domaines de puissances fractionnaires d'opérateurs, *J. Math. Soc. Japan* **14** (2) (1962), 233–241.

K. Liu and Z. Liu, Exponential stability and analyticity of abstract linear thermo-elastic systems, *ZAMP* **48** (1997), 885–904.
Z. Liu and M. Renardy. A note on the equations of thermo-elastic plate, *Appl. Math. Lett.* **8** (1995), 1–6.
Z. Liu and S. Zheng, Exponential stability of a semigroups associated with a thermo-elastic system, *Quart. Appl. Math.* LI (3) (1993), 535–545.
Z. Liu and S. Zheng, Exponential stability of the Kirchoff plate with thermal or viscoelastic dampings, *Quart. Appl. Math.* LV (1997), 551–564.
Z. Liu and S. Zheng, *Semigroups associated with dissipative systems*, Chapman & Hall/CRC, 1999.
A. Manitius and R. Triggiani, Sufficient conditions for function space controllability and feedback stabilizability of linear retarded system, *IEEE Trans. Automatic Control* (1978(a)), 659–665. Also, announcement in Proceedings of the Decision and Control Conference, Florida, December 1976.
A. Manitius and R. Triggiani, Function space controllability of linear retarded systems: A derivation from abstract operator conditions, *SIAM J. Control Optim.* **16** (1978(b)), 599–645.
J. Necas, *Les Méthodes Directes en Théorie des Équations Élliptiques*, Masson et Cie, Paris, 1967.
A. Pazy, *Semigroups of Linear Operators and Application to Partial Differential Equations*, Springer-Verlag, 1983.
D. L. Russell, Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions, *SIAM Review*, **20** (1978), 639–739.
C. Sadosky, *Interpolation of Operators and Singular Integrals*, Marcel Dekker, Series of Monographs and Textbooks, Pure and Applied Mathematics, 1979.
M. Sorine, Sur les equations de Chandrasekhar associees au probleme de contrôle d'un systeme parabolique: un example avec controle et observation frontiere, *C. R. Acad. Sc. Paris*, **285** (21 November 1977).
Y. Shibata, On the exponential decay of the energy of a linear thermoelastic plate, *Comp. Appl. Math.* **13** (1994), 81–102.
M. Slemrod, Global existence, uniqueness, and asymptotic stability of classical smooth solutions in one-dimensional thermoelasticity, *Arch. Rational Mech. Anal.* **76** (1981), 97–134.
A. Stahel, Nichtlineare randbedingungen bei problemen vom typ der Wellengleichung, Inaugural Dissertation, Zurich, 1987.
H. Tanabe, *Equations of evolutions*, Pitman 1979.
V. Thomee, *Galerkin Finite Element Methods for Parabolic Problems*, Lectures Notes in Mathem. Vol. 1054 Springer, Berlin 1984. Expanded version reprinted as a Springer-Verlag book, 1997.
H. Triebel, *Interpolation Theory and Function Spaces, Differential Operators*, North Holland Math Library 18, 1978, North Holland, Amsterdam.
R. Triggiani, On the stabilizability problem in Banach space, *J. Math. Anal. Appl.* **52** (1975), 383–403. Addendum *J.M.A.A.* **56** (1976), 492–493.
R. Triggiani, On Nambu's boundary stabilizability problem for diffusion processes, *J. Diff. Eqn.* **33** (1979), 189–200. Preliminary version as, Boundary stabilization of parabolic equations, International Conference on Recent Advances in Differential Equations, Miramare-Trieste (Italy), August 1978, Academic Press; R. Conti Edit., 431–447. (Also, Proceedings International Conference on Systems Analysis, I.R.I.A., Paris, 1978, Springer-Verlag LNCIS 14, 305–313.)
R. Triggiani, Well-posedness and regularity of boundary feedback parabolic systems, *J. Diff. Eqn.* **36** (1980(a)), 347–362.
R. Triggiani, Boundary feedback stabilizability of parabolic equations, *Appl. Math. Optim.* **6** (1980(b)), 201–220.

R. Triggiani, Regularity of structurally damped systems with point/ boundary control, *J. Math. Anal. Appl.* **161** (1991), 299–331.

R. Triggiani, Analyticity, and lack thereof, of semigroups arising from thermo-elastic plates, special volume entitled *Computational Science for the 21 Century*, Chapter on Control: Theory and Numerics, Wiley 1997; Proceedings of a Conference in honor of R. Glowinski, May 1997.

R. Triggiani, Sharp regularity theory for thermo-elastic mixed problems, *Applicable Analysis*, to appear in 1999.

D. C. Washburn, *A semigroup approach to time optimal boundary control of diffusion processes*, Ph.D. dissertation, Univ. California, Los Angeles, 1974.

D. C. Washburn, A semi-group theoretic approach to modeling of boundary input problems, Proceedings of IFIP Working Conference (University of Rome), Lecture Notes in Control and Information Science, Springer-Verlag, 1977.

D. C. Washburn, A bound on the boundary input map for parabolic equations with application to time optimal control, *SIAM J. Control* **17** (1979), 652–671.

J. Wloka, *Partial differential equations* Cambridge University Press 1987.

D. Xia, *Spectral Theory of Hyponormal Operators*, Birkhäuser, book 1983.

A. Yagi, Concidence entre des espaces d'interpolation et des domaines du puissances fractionnaires d'opérateurs, *C. R. Acad. Sc. Paris*, t **299** Sér. I, no. 6 (1984), 173–176.

S. Ya. Yakubov, A non-local boundary value problem for a class of Petrovski well posed equations, *Math. Sb.* **118** (1982), 252–261; *Math. USSR-Sb* (1983), 255–265.

4

Numerical Approximations of Algebraic Riccati Equations

This chapter provides, in the analytic case, a numerical approximation theory of the optimal quadratic cost problem over an infinite time interval whose continuous counterpart was studied in Chapter 2. Assumptions and notation of Chapter 2 are essentially retained. Because of the additional complexity in notation in the discrete case, a glossary of symbols is provided at the end of the chapter. Particular emphasis is placed on approximating the Riccati operator P, the gain operator B^*P, the optimal controls u^0, the optimal solution y^0; etc. This is provided by the main Theorems 4.1.4.1 and 4.1.4.2, which apply to any consistent approximation scheme. They provide, generally, rates of convergence. Under the general setting of Theorems 4.1.4.1 and 4.1.4.2, the rates of convergence are not optimal, however. A more specialized theory, to be presented in Section 4.6 (Theorems 4.6.2.1 and 4.6.2.2), provides optimal rates of convergence. Applications of these results to several examples of partial differential equations of parabolic type with boundary/point control will be given in the subsequent Chapter 5, where numerical schemes will be presented where all the assumptions required by Theorems 4.1.4.1 and 4.1.4.2 will be satisfied. In addition, however, Chapter 5 will provide numerical schemes that will guarantee *optimal* rates of convergence in the case of the heat equation with Dirichlet or Neumann boundary control according to Theorems 4.6.2.1 and 4.6.2.2.

4.1 Introduction: Continuous and Discrete Optimal Control Problems

4.1.1 Mathematical Setting of the Continuous Problem

Dynamical Model In this chapter we return to the abstract differential equation

$$\dot{y} = Ay + Bu \quad \text{on, say, } [\mathcal{D}(A^*)]', \qquad y(0) = y_0 \in Y, \tag{4.1.1.1}$$

where A^* is the Y-adjoint of A and Y is a Hilbert space, subject to the assumptions of Chapter 2, Eqn. (2.1.1) through Eqn. (2.1.7), which we recall here concisely for future convenience:

(i) $A : Y \supset \mathcal{D}(A) \to Y$ is the generator of a s.c., analytic semigroup e^{At} on

$Y,\ t > 0$, and is generally unstable. If ω_0 is given by Eqn. (2.1.2a) in Chapter 2, we consider throughout the translation

$$\hat{A} = -A + \omega I, \qquad \omega = \text{fixed} > \omega_0, \tag{4.1.1.2}$$

so that $-\hat{A}$ is the generator of a s.c. analytic semigroup $e^{-\hat{A}t}$ on Y satisfying

$$\|e^{-\hat{A}t}\|_{\mathcal{L}(Y)} \leq \hat{M}e^{-\hat{\omega}t}, \quad t \geq 0; \quad \hat{\omega} = \omega - \omega_0 - \epsilon > 0; \tag{4.1.1.3}$$

(ii) with U another Hilbert space, $B : U \to [\mathcal{D}(A^*)]'$ is a linear operator such that

$$A^{-\gamma}B;\ (\hat{A})^{-\gamma}B \in \mathcal{L}(U; Y) \quad \text{for some constant } \gamma, \quad 0 \leq \gamma < 1. \tag{4.1.1.4}$$

Optimal Control Problem As in Chapter 2, we associate with the dynamics (4.1.1.1) the quadratic cost functional

$$J(u, y) = \int_0^\infty \left[\|Ry(t)\|_Z^2 + \|u(t)\|_U^2\right] dt, \tag{4.1.1.5}$$

where Z is another (output) Hilbert space and

(iii)

$$R \in \mathcal{L}(Y; Z). \tag{4.1.1.6}$$

The corresponding optimal control problem is:

$$\begin{aligned} &\text{Minimize} \quad J(u, y) \text{ over all } u \in L_2(0, \infty; U),\\ &\qquad\qquad\ \ \text{where } y \text{ is the solution of (4.1.1.1) due to } u. \end{aligned} \tag{4.1.1.7}$$

Assumptions (i), (ii), and (iii) are in force throughout the chapter. We shall next rephrase the Finite Cost Condition (2.1.12) of Chapter 2 in a form more suitable for the present chapter. To this end, we begin with a preparatory observation.

Remark 4.1.1.1 Let $F \in \mathcal{L}(Y; U)$. Then:

$$\text{the operator } F^*B^* = F^*B^*(\hat{A}^*)^{-\gamma}(\hat{A}^*)^{\gamma} \text{ is } (\hat{A}^*)^{\gamma}\text{-bounded}$$

by virtue of assumption (4.1.1.4). Since $\gamma < 1$, a standard perturbation result [Pazy, 1983, p. 81] yields that: $A^* + F^*B^*$ is the generator of a s.c. analytic semigroup on Y. Then $A + BF$ is the generator of a s.c. analytic semigroup on Y. This conclusion will be freely used in this chapter.

Throughout this chapter, we shall assume the

(iv) **Stabilizability Condition (SC):**

$$\begin{cases} \text{There exists an operator } F \in \mathcal{L}(Y; U) \text{ such that the s.c. analytic semigroup} \\ e^{(A+BF)t} \text{ (as guaranteed by Remark 4.1.1.1 above) is exponentially} \\ \text{stable on } Y, \text{ i.e., } \left\|e^{(A+BF)t}\right\|_{\mathcal{L}(Y)} \leq M_F e^{-\omega_F t}, \quad \text{for some } \omega_F > 0,\ M_F \geq 1. \end{cases} \tag{4.1.1.8}$$

We notice that: Given A and B, the stabilizability condition (4.1.1.8) implies the Finite Cost Condition (2.1.12) of Chapter 2, If we take, in fact, $\bar{u} = F\bar{y}$ where $\bar{y}$ solves $\dot{\bar{y}} = (A + BF)\bar{y}$, $\bar{y}(0) = y_0$, then $\bar{y} \in L_2(0, \infty; Y)$, $\bar{u} \in L_2(0, \infty; U)$. Conversely, under the Detectability Condition (2.1.13) of Chapter 2, repeated below, we see from Theorem 2.2.2, property (b), Eqn. (2.2.10), that the Finite Cost Condition implies the stabilizability condition with $F = P$, the Riccati operator of Chapter 2. Thus, the Stabilization Condition (4.1.1.8) guarantees existence of a unique optimal pair $\{u^0, y^0\}$ of the optimal control problem (4.1.1.7) for (4.1.1.1).

Finally, to guarantee the exponential decay of $e^{A_P t}$ in (2.2.10) of Chapter 2, as well as uniqueness of the solution of the corresponding algebraic Riccati equation, we know from Chapter 2 that we need to assume the Detectability Condition, which we repeat below.

(iv) **Detectability Condition (DC):**

$$\begin{cases} \text{There exists an operator } K \in \mathcal{L}(Z; Y) \text{ such that the s.c.} \\ \text{analytic semigroup } e^{(A+KR)t} \text{ is exponentially stable on } Y, \text{ i.e.,} \\ \left\| e^{(A+KR)t} \right\|_{\mathcal{L}(Y)} \leq M_K e^{-\omega_K t}, \quad \text{for some } \omega_K > 0,\ M_K \geq 1 \end{cases} \tag{4.1.1.9}$$

(or similarly for $A + K(R^*R)^{\frac{1}{2}}$). Thus, the present optimal control problem (4.1.1.7) for the dynamics (4.1.1.1) is covered by Theorems 2.2.1 and 2.2.2 in Chapter 2. The goal of the present chapter is to provide an approximation theory for all the relevant quantities of the problem (optimal control u^0; optimal solution y^0; Riccati operator P; gain operator B^*P; etc.), that is consistent with the continuous theory of Chapter 2, Theorems 2.2.1 and 2.2.2. To this end, we shall first summarize for convenience the results of the continuous theory of Chapter 2. Next, we shall introduce an approximating framework for both the dynamics (4.1.1.1) and the optimal control problem (4.1.1.7), in particular for the corresponding algebraic Riccati equation.

Summary of Results of Chapter 2, Theorems 2.2.1 and 2.2.2

(1) Under the Stabilizability Condition (SC) = (4.1.1.8), there exists a unique solution $\{u^0, y^0\}$ of the optimal control problem (4.1.1.7) for (4.1.1.1).

(2) Under the additional Detectability Condition (DC) = (4.1.1.9), there is a unique nonnegative operator $P = P^* \in \mathcal{L}(Y)$ such that with $u^0(t) = u^0(t; y_0)$ and $y^0(t) = y^0(t; y_0)$, $y_0 \in Y$, we have

$$u^0(t) = -B^*Py^0(t), \quad 0 < t < \infty, \tag{4.1.1.10}$$

where $(Bu, v)_Y = (u, B^*v)_U$, and P satisfies the following algebraic Riccati equation (ARE):

$$(A^*Px, y)_Y + (PAx, y)_Y + (R^*Rx, y)_Y - (B^*Px, B^*Py)_U = 0,$$
$$\forall x, y \in \mathcal{D}(\hat{A}^\epsilon), \text{ any } \epsilon > 0. \tag{4.1.1.11}$$

Moreover,

(3)

$$(\hat{A}^*)^{1-\epsilon} P \in \mathcal{L}(Y), \quad \forall \epsilon > 0; \tag{4.1.1.12}$$

(4)

$$J(u^0, y^0) = (Py_0, y_0)_Y; \quad y_0 \in Y; \tag{4.1.1.13}$$

(5)

$$B^* P \in \mathcal{L}(Y; U); \tag{4.1.1.14}$$

(6) the s.c. semigroup $\Phi(t) = e^{A_P t} = e^{(A-BB^*P)t}$ generated by $A_P = A - BB^*P$ is analytic and exponentially stable,

$$\left\| e^{(A-BB^*P)t} \right\|_{\mathcal{L}(Y)} \le M_P e^{-\omega_P t}, \quad \text{for some } \omega_P > 0, \tag{4.1.1.15}$$

(7) and finally,

$$y^0(t; y_0) = e^{A_P t} y_0; \qquad u^0(t; y_0) = -B^* P e^{A_P t} y_0. \tag{4.1.1.16}$$

Further properties are collected in Section 4.2.1; see in particular identity (4.2.1.10) for P.

4.1.2 Approximation of Continuous Dynamics and Related Properties

4.1.2.1 Approximation Assumptions

Approximating Subspaces We introduce a family of finite-dimensional approximating subspaces $V_h \subset Y \cap \mathcal{D}(B^*)$, where h is a parameter of discretization that tends to zero, $0 < h \le h_0$, and $B^* \in \mathcal{L}(\mathcal{D}((\hat{A}^*)^\gamma); U)$ by (4.1.1.4). Let Π_h be the Y-orthogonal projection of Y onto V_h with the usual approximating property:

$$\|\Pi_h x - x\|_Y \to 0, \quad \text{as } h \downarrow 0, \text{ for all } x \in Y. \tag{4.1.2.1}$$

Approximation of A Let $A_h : V_h \to V_h$ be a finite-dimensional approximation of A that satisfies the following requirements (A.1) and (A.2):

(A.1) *uniform analyticity.* For the formulation in the t-domain (refer to Chapter 2, Eqn. (2.1.4) for $\hat{A}$):

$$\left\| A_h^\theta e^{A_h t} \right\|_{\mathcal{L}(Y)} \le \frac{C_\theta e^{(\omega_0+\epsilon)t}}{t^\theta}, \quad t > 0; \ 0 \le \theta \le 1. \tag{4.1.2.2}$$

(the cases $0 < \theta < 1$ follow by interpolation from the endpoint cases $\theta = 0$, $\theta = 1$) with constant C_θ independent of h.

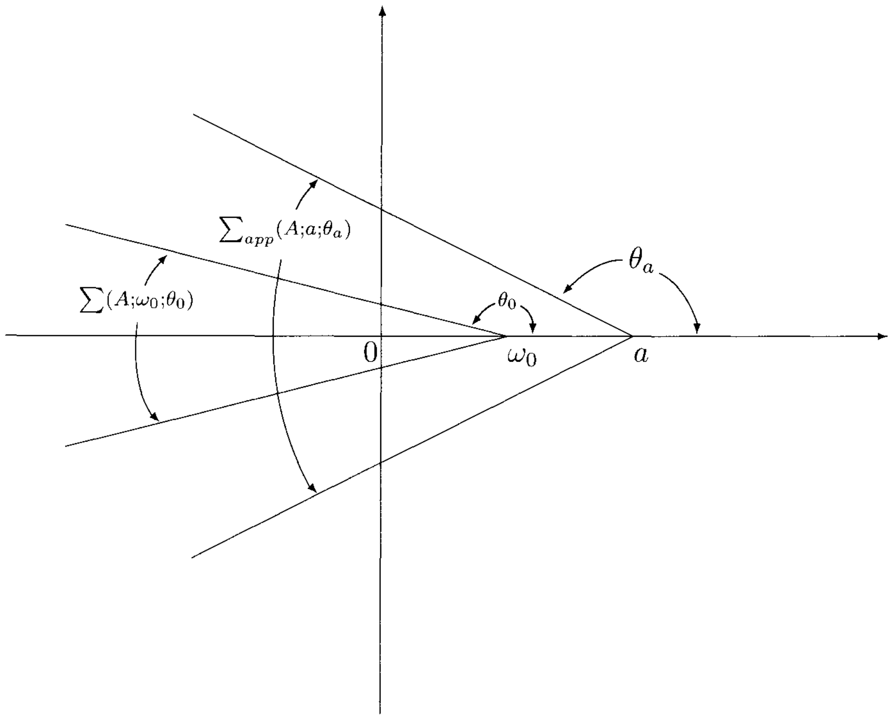


Figure 4.1

The equivalent formulation in the λ-domain (refer to Chapter 2, Eqn. (2.1.5), (2.1.6) for $\hat{A}$) is as follows: For $a > \omega_0$ (ω_0 given by Eqn. (2.1.2) in Chapter 2), there exists

$$\begin{aligned}\Sigma_{\text{app}}(A) &= \Sigma_{\text{app}}(A;\, a;\, \theta_a)\\ &= \text{closed triangular sector containing the axis}\\ &\quad [-\infty, a] \text{ and delimited by the two rays } a + \rho e^{\pm i\theta_a},\\ &\quad \text{for some } \pi/2 < \theta_a \le \theta_0 < \pi, \end{aligned} \tag{4.1.2.3a}$$

(see Figure 4.1) associated with the analytic semigroup e^{At}, and there exists h_a such that, if Σ^c denotes the complement of $\Sigma \subset \mathbb{C}$, then for all $0 < h \le h_a$ we have

$$\sigma(A_h) = \text{spectrum of } A_h \subset \Sigma_{\text{app}}(A), \tag{4.1.2.3b}$$

and

$$\left\| R(\lambda, A_h) A_h^{\theta} \right\|_{\mathcal{L}(Y)} \le \frac{C}{|\lambda - a|^{1-\theta}}, \quad \forall \lambda \in \Sigma^c_{\text{app}}(A), \quad 0 \le \theta \le 1. \tag{4.1.2.3c}$$

(The cases $0 < \theta < 1$ follow by interpolation from $\theta = 0$ and $\theta = 1$.)

(A.2)

$$\|\hat{A}^{-1} - \hat{A}_h^{-1}\Pi_h\|_{\mathcal{L}(Y)} \le Ch^s, \quad \text{for some } s > 0 \text{ independent of } h, \tag{4.1.2.4}$$

and constant C independent of h, where by analogy with (4.1.1.2) we have defined

$$\hat{A}_h = -A_h + \omega I : V_h \to V_h. \tag{4.1.2.5}$$

(Condition (4.1.2.4) may be equivalently formulated at any point of the common resolvent of $\hat{A}$ and of $\hat{A}_h$; see Eqn. (4.1.2.11) below.)

Remark 4.1.2.1 The proof of equivalence of the t-domain version (4.1.2.2) and of the λ-domain version (4.1.2.3) follows as in the continuous case, with constants that are now independent of h; see [Ladas and Lakshmikantham, 1972, p. 49; Kato, 1966, p. 489; Pazy, 1983, p. 63], or [Fattorini, 1983, §4.1 and 4.2], for the implications from the λ-version to the t-version, and conversely.

Approximation of B We shall assume that the operators $B : U \to [\mathcal{D}(A^*)]'$ and $B_h : U \to V_h$ satisfy the following approximating properties, where γ and s are defined by (4.1.1.4) and (4.1.2.4), respectively:

(A.3) (*inverse approximation property*)

$$\|B^* x_h\|_U + \|B_h^* x_h\|_U \le Ch^{-\gamma s}\|x_h\|_Y, \quad \forall x_h \in V_h; \tag{4.1.2.6}$$

(A.4)

$$\|B^*(\Pi_h - I)x\|_U \le Ch^{s(1-\gamma)}\|x\|_{\mathcal{D}(A^*)}, \quad x \in \mathcal{D}(A^*); \tag{4.1.2.7}$$

(A.5)

$$\|B^* x - B_h^*\Pi_h x\|_U \le Ch^{s(1-\gamma)}\|x\|_{\mathcal{D}(A^*)}, \quad x \in \mathcal{D}(A^*). \tag{4.1.2.8}$$

(If, in particular, we take $B_h = \Pi_h B$, then (A.5) is contained in (A.4).)

(A.6)

$$\|B^*\Pi_h x\|_U \le C\|(\hat{A}^*)^\gamma x\|_Y, \quad x \in \mathcal{D}((\hat{A}^*)^\gamma). \tag{4.1.2.9}$$

Remark 4.1.2.2 We notice that assumptions (A.2) through (A.6) are standard approximation properties, where, moreover, in the case of spline approximations, s is the order of the differential operator A. They are consistent with the regularity of the original operators A and B. Moreover, they are satisfied by typical schemes (finite elements, finite differences, mixed methods, spectral approximations) defined on a quasi-uniform grid. In contrast, the property of uniform analyticity (A.1) is not a standard assumption and needs to be verified in each case. However, it is satisfied in many of the schemes and examples that arise from analytic semigroup problems. For instance, a sufficient condition for (A.1) to hold true is the uniform coercivity of the bilinear form associated with A_h (see Lemma 4.2 in Lasiecka [1984]). There are, however, a number of significant physical examples (e.g., structurally damped elastic systems) in which the bilinear form is not coercive, while the underlying semigroups $e^{A_h t}$ are uniformly analytic (see Chapter 5).

4.1.2.2 Consequences of Approximating Assumptions on A

From (A.1) and (A.2), the following rough data estimates follow.

Proposition 4.1.2.1 Assume (A.1) and (A.2). Then for $0 \le \theta \le 1$:

(i)

$$\|e^{A_h t}\Pi_h - e^{At}\|_{\mathcal{L}(Y)} = \|e^{A_h^* t}\Pi_h - e^{A^* t}\|_{\mathcal{L}(Y)} \le c\, \frac{h^{s\theta} e^{(\omega_0+\epsilon)t}}{t^\theta}, \quad t > 0, \ \forall \epsilon > 0; \tag{4.1.2.10a}$$

with ω_0 as in Chapter 2, (2.1.2), C depending on $\omega_0 + \epsilon$; equivalently, by (4.1.1.2), (4.1.1.3), and (4.1.2.5),

$$\|e^{-\hat{A}_h t}\Pi_h - e^{-\hat{A} t}\|_{\mathcal{L}(Y)} = \|e^{-\hat{A}_h^* t}\Pi_h - e^{-\hat{A}^* t}\|_{\mathcal{L}(Y)} \le \frac{\hat{c} h^{s\theta} e^{-\hat{\omega} t}}{t^\theta}, \quad t > 0; \tag{4.1.2.10b}$$

(ii)

$$\|R(\lambda, A) - R(\lambda, A_h)\Pi_h\|_{\mathcal{L}(Y)} \le Ch^s, \quad s > 0, \tag{4.1.2.11a}$$

uniformly in $\lambda \in \Sigma^c_{\text{app}}(A)$ [see definition of $\Sigma^c_{\text{app}}(A)$ in (4.1.2.3a)]; equivalently, by (4.1.1.2) and (4.1.2.5),

$$\|R(\lambda, -\hat{A}) - R(\lambda, -\hat{A}_h)\Pi_h\|_{\mathcal{L}(Y)} \le Ch^s, \quad s > 0, \tag{4.1.2.11b}$$

uniformly in λ, which runs over the translation

$$\Sigma^c_{\text{app}}(A; a; \theta_a) - \omega = \Sigma^c_{\text{app}}(A; a - \omega; \theta_a)$$

of $\Sigma^c_{\text{app}}(A; a; \theta_a)$ by ω to the left.

(iii)

$$\|e^{A_h t}\Pi_h - e^{At}\|_{\mathcal{L}(\mathcal{D}(A);Y)} \le Ch^s e^{(\omega_0+\epsilon)t}, \tag{4.1.2.12a}$$

$$\|e^{A_h^* t}\Pi_h - e^{A^* t}\|_{\mathcal{L}(\mathcal{D}(A^*);Y)} \le Ch^s e^{(\omega_0+\epsilon)t}, \quad t \ge 0, \tag{4.1.2.12b}$$

with C depending on $\omega_0 + \epsilon$.

(iv)

$$\|R(\lambda, A) - R(\lambda, A_h)\Pi_h\|_{\mathcal{L}(\mathcal{D}(A);Y)} \le \frac{Ch^s}{|\lambda|} \tag{4.1.2.13a}$$

uniformly in $\lambda \in \Sigma^c_{\text{app}}(A; a; \theta_a)$, that is,

$$\|R(\lambda, -\hat{A}) - R(\lambda, -\hat{A}_h)\Pi_h\|_{\mathcal{L}(\mathcal{D}(\hat{A});Y)} \le \frac{Ch^s}{|\lambda|} \tag{4.1.2.13b}$$

uniformly in $\lambda \in \Sigma^c_{\text{app}}(A; a - \omega; \theta_a)$.

Proof. (ii) We first prove (4.1.2.11b), which is assumption (A.2) = (4.1.2.4) expressed at a different point $\lambda \in \Sigma^c_{\text{app}}(A; a-\omega; \theta_a)$ of the respective resolvent operators. We use the first resolvent equation to get:

$$R(\lambda, -\hat{A}) = R(0, -\hat{A}) - \lambda R(\lambda, -\hat{A}) R(0, -\hat{A}), \tag{4.1.2.14}$$

$$R(\lambda, -\hat{A}_h)\Pi_h = R(0, -\hat{A}_h)\Pi_h - \lambda R(\lambda, -\hat{A}_h) R(0, -\hat{A}_h)\Pi_h. \tag{4.1.2.15}$$

Subtracting (4.1.2.15) from (4.1.2.14), and adding and subtracting $\lambda R(\lambda, -\hat{A}_h)$ $\Pi_h R(0, -\hat{A})$ to the resulting expression, yields (since Π_h is the identity on V_h):

$$\begin{aligned}&[R(\lambda, -\hat{A}) - R(\lambda, -\hat{A}_h)\Pi_h][I + \lambda R(0, -\hat{A})] \\ &\quad = [I - \lambda R(\lambda, -\hat{A}_h)\Pi_h][R(0, -\hat{A}) - R(0, -\hat{A}_h)\Pi_h].\end{aligned} \tag{4.1.2.16}$$

Next, recalling (A.1) = (4.1.2.3c) for $\theta = 0$, we obtain

$$\|\lambda R(\lambda, -\hat{A}_h)\Pi_h\|_{\mathcal{L}(Y)} \le \text{const.}, \quad \forall h, \ \forall \lambda \in \Sigma^c_{\text{app}}(A; a-\omega; \theta_a). \tag{4.1.2.17}$$

Moreover, since

$$[I + \lambda \hat{A}^{-1}] = [\lambda I + \hat{A}]\hat{A}^{-1}$$

we obtain by analyticity

$$\|[I + \lambda \hat{A}^{-1}]^{-1}\|_{\mathcal{L}(Y)} = \|\hat{A} R(\lambda, \hat{A})\|_{\mathcal{L}(Y)} \le \text{const.}, \quad \lambda \in \Sigma^c_{\text{app}}(A; a-\omega; \theta_a). \tag{4.1.2.18}$$

Thus, using (4.1.2.17) and (4.1.2.18) in (4.1.2.16) yields

$$\begin{aligned}\|R(\lambda, -\hat{A}) - R(\lambda, -\hat{A}_h)\Pi_h\|_{\mathcal{L}(Y)} &\le C\left\|\hat{A}^{-1} - \hat{A}_h^{-1}\Pi_h\right\|_{\mathcal{L}(Y)}, \\ &\lambda \in \Sigma^c_{\text{app}}(A; a-\omega; \theta_a).\end{aligned} \tag{4.1.2.19}$$

Recalling now (A.2) = (4.1.2.4), we then obtain the desired estimate from (4.1.2.19),

$$\|R(\lambda, -\hat{A}) - R(\lambda, -\hat{A}_h)\Pi_h\|_{\mathcal{L}(Y)} \le Ch^s, \quad \lambda \in \Sigma^c_{\text{app}}(A; a-\omega; \theta_a), \tag{4.1.2.20}$$

and (4.1.2.11b) is proved. The same proof gives (4.1.2.11a).

(i) We now prove (4.1.2.10b) for $\theta = 1$. We let Γ be a path running in $\Sigma^c_{\text{app}}(A; a-\omega; \theta_a)$ consisting of three sides: the two infinite rays $\rho e^{\pm i\hat{\theta}}$, where $\pi/2 < \hat{\theta} < \pi$ and $0 < \rho_0 \le \rho < \infty$ with $-\infty < \text{Re } \lambda \le -\hat{\omega}$, as well as the finite vertical segment $\text{Re } \lambda \equiv -\hat{\omega}$ intersecting with these two rays and determining $\rho_0 > 0$. We then use the representation

$$e^{-\hat{A}t} - e^{-\hat{A}_h t}\Pi_h = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}[R(\lambda, -\hat{A}) - R(\lambda, -\hat{A}_h)\Pi_h]\, d\lambda. \tag{4.1.2.21}$$

Using now estimate (4.1.2.20) in (4.1.2.21) and noticing that $\text{Re } \lambda = -\beta \le -\hat{\omega}$ on Γ, we obtain

$$\|e^{-\hat{A}t} - e^{-\hat{A}_h t}\Pi_h\|_{\mathcal{L}(Y)} \le Ch^s \int_{\hat{\omega}}^\infty e^{-\beta t}\, d\beta = \frac{ch^s e^{-\hat{\omega}t}}{t}, \quad t > 0, \tag{4.1.2.22}$$

which proves (4.1.2.10b) for $\theta = 1$. Next, the case $\theta = 0$:

$$\|e^{-\hat{A}t} - e^{-\hat{A}_h t}\Pi_h\|_{\mathcal{L}(Y)} \le Ce^{-\hat{\omega}t}, \quad t \ge 0, \tag{4.1.2.23}$$

follows from assumption (A.1) = (4.1.2.2) via the translation (4.1.2.5), as well as from (4.1.1.3). Finally, for $0 < \theta < 1$, we raise (4.1.2.22) to power θ, we raise (4.1.2.23) to power $(1 - \theta)$, and we multiply the results to obtain (4.1.2.10b), as desired.

(iv) Let $x \in \mathcal{D}(\hat{A})$. Then we obtain the following modification of (4.1.2.18) via analyticity:

$$\begin{aligned}\|[I + \lambda\hat{A}^{-1}]^{-1}x\|_Y &= \|R(\lambda, \hat{A})\hat{A}x\|_Y \\ &\le \|R(\lambda, \hat{A})\|_{\mathcal{L}(Y)}\|\hat{A}x\|_Y \\ &\le \frac{C}{|\lambda|}\|\hat{A}x\|_Y, \quad x \in \mathcal{D}(\hat{A}), \ \lambda \in \Sigma^c_{\text{app}}(A; a - \omega; \theta_a).\end{aligned} \tag{4.1.2.24}$$

Returning to identity (4.1.2.16), we repeat the same argument as the one leading to (4.1.2.19), except that we now use (4.1.2.24) instead of (4.1.2.18). We now obtain

$$\begin{aligned}\|[R(\lambda, -\hat{A}) - R(\lambda, -\hat{A}_h)\Pi_h]x\|_Y &\le C\left\|\hat{A}^{-1} - \hat{A}_h^{-1}\Pi_h\right\|_{\mathcal{L}(Y)}\frac{C}{|\lambda|}\|\hat{A}x\|_Y \\ \text{(by (4.1.2.4))} \qquad &\le \frac{ch^s}{|\lambda|}\|\hat{A}x\|_Y, \\ &\qquad x \in \mathcal{D}(\hat{A}), \ \lambda \in \Sigma^c_{\text{app}}(A; a - \omega; \theta_a),\end{aligned} \tag{4.1.2.25}$$

using (4.1.2.4) on the last step. Thus, (4.1.2.25) proves (4.1.2.13b). The proof of (4.1.2.13a) is similar, working with A, A_h, rather than $-\hat{A}$, $-\hat{A}_h$.

(iii) We now prove (4.1.2.12a) for $[e^{At} - e^{A_h t}\Pi_h]$. We first obtain the required estimate for t bounded away from zero, say for $t \ge 1$. Using estimate (4.1.2.13b) = (4.1.2.25) in the representation formula (4.1.2.21), we obtain now with $x \in \mathcal{D}(\hat{A})$ and still $\text{Re}\,\lambda \equiv -\beta \le -\hat{\omega}$:

$$\begin{aligned}\|[e^{-\hat{A}t} - e^{-\hat{A}_h t}\Pi_h]x\|_Y &\le Ch^s\|\hat{A}x\|_Y\int_{\hat{\omega}}^{\infty}\frac{e^{-\beta t}}{\beta}\,d\beta \\ &\le Ch^s\|\hat{A}x\|_Y\frac{e^{-\hat{\omega}t}}{\hat{\omega}t} \le \frac{Ch^s}{\hat{\omega}}e^{-\hat{\omega}t}\|\hat{A}x\|, \quad t \ge 1,\end{aligned} \tag{4.1.2.26}$$

in place of (4.1.2.22). Next, it suffices to show the uniform bound

$$\|[e^{-\hat{A}t} - e^{-\hat{A}_h t}\Pi_h]x\|_Y \le Ch^s\|\hat{A}x\|_Y, \quad 0 \le t \le 1, \tag{4.1.2.27}$$

for then (4.1.2.27) and (4.1.2.26) combined yield

$$\|[e^{-\hat{A}t} - e^{-\hat{A}_h t}\Pi_h]x\|_Y \le Ch^s e^{-\hat{\omega}t}\|\hat{A}x\|_Y, \quad t \ge 0, \ x \in \mathcal{D}(\hat{A}). \tag{4.1.2.28}$$

To show (4.1.2.27), we again invoke formula (4.1.2.21), where we now use a standard device [Kato, 1966, p. 489; Ladas, Lakshmikantham, 1972, p. 50]. We introduce a new variable $\lambda' = \lambda t$, which sends the original Γ into $\Gamma' = \Gamma t$ (still in $\Sigma^c_{\text{app}}(A; a - \omega; \theta_a)$ for t small, say $t \leq 1$). However, after the change of variable $\lambda \to \lambda'$, we may perform integration over the original path Γ by Cauchy's theorem. We obtain then for $x \in \mathcal{D}(\hat{A})$, invoking again estimate (4.1.2.25),

$$
\begin{aligned}
&2\pi \|[e^{-\hat{A}t} - e^{-\hat{A}_h t}\Pi_h]x\|_Y \\
&= \left\| \int_\Gamma e^{\lambda'} \left[R\left(\frac{\lambda'}{t}, -\hat{A}\right) - R\left(\frac{\lambda'}{t}, -\hat{A}_h\right)\Pi_h \right] x \, \frac{d\lambda'}{t} \right\|_Y \\
\text{(by (4.1.2.25))} \quad &\leq C\, h^s \|\hat{A}x\|_Y \left[\int_\Gamma |e^{\lambda'}| \frac{t}{|\lambda'|} \frac{|d\lambda'|}{t} \right] \\
&\leq C\, h^s \|\hat{A}x\|_Y, \quad 0 < t \leq 1,
\end{aligned}
\tag{4.1.2.29}
$$

and (4.1.2.27) is proved. Thus, (4.1.2.28) has been established. This then gives (4.1.2.12a), by translation (4.1.1.2) and (4.1.2.5). The proof of (4.1.2.12b) for $[e^{A^*t} - e^{A_h^* t}\Pi_h]$ is the same, using now the adjoints rather than the original operators. □

Remark 4.1.2.3 All of the estimates (4.1.2.10a), (4.1.2.12a), and (4.1.2.12b), hold a fortiori with Π_h in front of e^{At} or e^{A^*t}, since Π_h is the identity on V_h. This fact will be used freely below.

4.1.3 Approximation of the Continuous Dynamics and Control Problem. Related Riccati Equation

Control Problem Given the approximating dynamics $y_h(t) \subset V_h$ satisfying

$$
\dot{y}_h(t) = A_h y_h(t) + B_h u(t), \qquad y_h(0) = \Pi_h y_0 \tag{4.1.3.1}
$$

minimize over all $u \in L_2(0, \infty; U)$ the cost

$$
J(u, y_h) = \int_0^\infty \left[\|R y_h(t)\|_Z^2 + \|u(t)\|_U^2 \right] dt. \tag{4.1.3.2}
$$

It will be shown in Sections 4.4.1 and 4.4.2 that the approximating dynamics (4.1.3.1) is stabilizable and detectable, in fact, uniformly in h. Thus, it is a standard finite-dimensional result (on V_h) [which is, in fact, contained in Chapter 2, Theorems 2.2.1 and 2.2.2 when specialized to V_h] that there exists a unique, nonnegative, self-adjoint Riccati approximating operator P_h, associated with (4.1.3.1), (4.1.3.2), which is the solution of the following algebraic Riccati equation, ARE$_h$,

$$
\begin{aligned}
&(A_h^* P_h x_h, y_h)_Y + (P_h A_h x_h, y_h)_Y + (R x_h, R y_h)_Z \\
&= (B_h^* P_h x_h, B_h^* P_h y_h)_U, \quad \forall x_h, y_h \in V_h.
\end{aligned}
\tag{4.1.3.3}
$$

Further properties of the approximating problem will be collected in Section 4.2.2 below; see, in particular, identity (4.2.2.10) for P_h.

4.1.4 Main Results of Approximating Schemes

Our main results of this chapter are formulated in the two theorems below.

Theorem 4.1.4.1 *Assume*

I. the continuous hypotheses (4.1.1.4), (SC) = (4.1.1.8), (DC) = (4.1.1.9), and, in addition,

$$\begin{cases} \text{(a) either } R^*R \geq rI, \quad r > 0, & (4.1.4.1) \\ \text{(b) or else } \hat{A}^{-1}KR : Y \to Y \text{ compact;} & (4.1.4.2) \end{cases}$$

$$\begin{cases} \text{(a) either } B^*\hat{A}^{*-1} : Y \to U \text{ compact,} & (4.1.4.3) \\ \text{(b) or else } F : Y \to U \text{ compact.} & (4.1.4.4) \end{cases}$$

II. The approximation properties (4.1.2.1) and (A.1) = (4.1.2.2) through (A.6) = (4.1.2.9).

Then there exists $h_0 > 0$, such that for all $h < h_0$, the $(ARE)_h$ in (4.1.3.3) admits a unique, nonnegative, self-adjoint solution P_h, and the following uniform bounds and convergence properties hold true:

(i)

$$\|e^{A_{h,P_h}t}\|_{\mathcal{L}(Y)} \leq \bar{M}_P e^{-\bar{\omega}_P t}, \quad \text{for some } \bar{\omega}_P > 0, \ \bar{M}_P \geq 1 \text{ uniformly in } h \tag{4.1.4.5}$$

(see Theorem 4.4.3.1, Eqn. (4.4.3.3)), where

$$A_{h,P_h} = A_h - B_h B_h^* P_h : V_h \to V_h. \tag{4.1.4.6}$$

Moreover, still uniformly in h, we have

(ii)

$$\|(\hat{A}_h^*)^{1-\epsilon} P_h\|_{\mathcal{L}(Y)} + \left\|(\hat{A}_h^*)^{\frac{1}{2}-\epsilon} P_h \hat{A}_h^{\frac{1}{2}-\epsilon}\right\|_{\mathcal{L}(Y)} \leq C_\epsilon, \quad \forall \epsilon > 0 \tag{4.1.4.7}$$

(see Theorem 4.4.4.1, Eqns. (4.4.4.1) and (4.4.4.3)).

(iii)

$$\|P_h\Pi_h - P\|_{\mathcal{L}(Y)} \leq C\, h^{\epsilon_o} \to 0 \text{ as } h \downarrow 0, \quad \forall \epsilon_o < s(1-\gamma) \tag{4.1.4.8}$$

(see Theorem 4.5.1.1, Eqn. (4.5.1.1)).

(iv)

$$\|B_h^* P_h \Pi_h - B^* P\|_{\mathcal{L}(Y;U)} \to 0 \text{ as } h \downarrow 0 \tag{4.1.4.9}$$

with rate $h^{s\theta}$, $\forall \theta < 1/2$, if $\gamma < 1/2$ (see Theorem 4.5.4.1, Eqn. (4.5.4.1)).

(v) For all $\epsilon_o < s(1-\gamma)$,

$$\sup_{t \geq 0} \left\{ e^{\bar{\omega}_P t} \left\| u_h^0(t; \Pi_h x) - u^0(t; x) \right\|_{\mathcal{L}(Y;U)} \right\} \leq C h^{\epsilon_o} \to 0 \text{ as } h \downarrow 0, \quad x \in Y \tag{4.1.4.10a}$$

(see Theorem 4.5.2.1, Eqn. (4.5.2.1)),

$$\left\| u_h^0(\,\cdot\,; \Pi_h x) - u^0(\,\cdot\,; x) \right\|_{L_2(0,\infty;U)} \leq C h^r, \tag{4.1.4.10b}$$

$$r = m\left(\frac{1}{\frac{\omega}{\tilde{\omega}_P} + 1}\right); \qquad m = \begin{cases} \epsilon_0, & \frac{1}{2} \leq \gamma < 1 \\ s\theta, \forall \theta < \frac{1}{2}, & 0 \leq \gamma \leq \frac{1}{2} \end{cases}$$

(see Theorem 4.5.3.1, Eqn. (4.5.3.4), and Eqn. (4.5.3.14) of Remark 4.5.3.1).

(vi) For all $\epsilon_0 < s(1-\gamma)$, and for $h \downarrow 0$:

$$\left\| y_h^0(\,\cdot\,; \Pi_h x) - y^0(\,\cdot\,; x) \right\|_{\mathcal{L}(Y; L_2(0,\infty;Y))}$$

$$\leq \begin{cases} C\, h^{\epsilon_o} \to 0, & \text{if } \frac{1}{2} < \gamma < 1 \qquad (4.1.4.11a) \\ C_\theta\, h^{s\theta} \to 0, \ \forall \theta < \frac{1}{2}, & \text{if } 0 \leq \gamma \leq \frac{1}{2} \qquad (4.1.4.11b) \end{cases}$$

(see Theorem 4.5.3.1, Eqn. (4.5.3.3)), where

$$y_h^0(t; \Pi_h x) = e^{A_{h,P_h} t}\Pi_h x = e^{(A_h - B_h B_h^* P_h \Pi_h)t}\Pi_h x \quad \text{and} \quad y^0(t; x) = e^{A_P t} x,$$

by (4.5.1.27).

(vii)

$$\left\| y_h^0(\,\cdot\,; \Pi_h x) - y^0(\,\cdot\,; x) \right\|_{C([0,\infty];U)} \to 0 \text{ as } h \downarrow 0, \quad x \in Y \tag{4.1.4.12}$$

(see Theorem 4.5.3.1, Eqn. (4.5.3.5)).

(viii) For all $\epsilon_o < s(1-\gamma)$ and for all $1 \geq \epsilon \geq 0$,

$$\sup_{t \geq 0} \left\{ t^\epsilon e^{\tilde{\omega}_P t} \left\| y_h^0(t; \Pi_h x) - y^0(t; x) \right\|_{\mathcal{L}(Y)} \right\} \leq C\, h^{\epsilon_o \epsilon} \to 0 \text{ as } h \downarrow 0 \tag{4.1.4.13}$$

that is, uniformly in $x \in Y$ (see Theorem 4.5.1.4, Eqn. (4.5.1.29)).

(ix) For all $\epsilon_o < s(1-\gamma)$,

$$\begin{aligned} &\left| J\big(u_h^0(\,\cdot\,; \Pi_h x), y_h^0(\,\cdot\,; \Pi_h x)\big) - J(u^0(\,\cdot\,; x), y^0(\,\cdot\,; x)) \right| \\ &\leq C\, h^{\epsilon_o} \to 0 \text{ as } h \downarrow 0 \end{aligned} \tag{4.1.4.14}$$

(which is a consequence of property (iii) in (4.1.4.8) and of (4.1.1.13) and (4.2.2.10)).

(x) Moreover, if in addition, for some $0 < \theta < 1$,

$$\|(\hat{A}^*)^\theta x_h\|_Y \leq C_\theta \|(\hat{A}_h^*)^\theta x_h\|_Y, \quad \text{or} \quad (\hat{A}^*)^\theta \big(\hat{A}_h^{*-1}\big)^\theta \in \mathcal{L}(V_h; Y), \tag{4.1.4.15}$$

then (see Proposition 4.5.5.1, Eqns. (4.5.5.2), (4.5.5.3))

$$(\mathrm{x}_1) \quad \|(\hat{A}^*)^\theta (P_h \Pi_h - P)x\|_Y \to 0 \text{ as } h \downarrow 0, \quad x \in Y, \ 0 \leq \theta < 1; \tag{4.1.4.16}$$

$$(\mathrm{x}_2) \quad \|(\hat{A}^*)^\theta (P_h \Pi_h - P)\hat{A}^\theta x\|_Y \to 0 \text{ as } h \downarrow 0, \quad x \in Y, \ 0 \leq \theta < \frac{1}{2}. \tag{4.1.4.17}$$

Remark 4.1.4.1 Assumption (4.1.4.15) typically holds true with $\theta = 1/2$. This is certainly the case when A is coercive and A_h is a standard Galerkin approximation of A: that is, $(A_h x_h, y_h)_Y = (A x_h, y_h)_Y$.

Remark 4.1.4.2 If A is self-adjoint (or, more generally, if $A = A_1 + A_2$, with A_1 self-adjoint and $A_2 : Y \supset \mathcal{D}((-A_1)^{1-\epsilon}) \to Y$ is bounded), one can take $\theta = 1/2$ in (4.1.4.17).

Theorem 4.1.4.2 *(i) Under the same assumptions of Theorem 4.1.4.1, the following uniform exponential stability holds true (see Section 4.5.6):*

$$\left\| e^{(A-BB_h^* P_h \Pi_h)t} \right\|_{\mathcal{L}(Y)} \leq \hat{C} e^{-\hat{\omega}_P t}, \quad \textit{for some } \hat{\omega}_P > 0. \tag{4.1.4.18}$$

(ii) Moreover (see (4.5.6.6)),

$$\sup_{t \geq 0} \left\{ e^{\hat{\omega}_P t} \left\| e^{(A-BB_h^* P_h \Pi_h)t} - e^{(A-BB^* P)t} \right\|_{\mathcal{L}(Y)} \right\} \to 0 \textit{ as } h \downarrow 0. \tag{4.1.4.19}$$

Theorem 4.1.4.1 provides the basic convergence results (with rates) for the optimal solutions of the approximating problem (4.1.3.1), (4.1.3.2), the corresponding Riccati operators and gain operators to the same quantities of the original continuous problem (4.1.1.1), (4.1.1.2).

The advantage of Theorem 4.1.4.2 is this: It states that the original *system, once acted upon by the discrete feedback control law given by* $u_h^*(t; \Pi_h x) = -B_h^* P_h y_h^*(t; x)$ *yields (uniformly) exponentially stable solutions; see Figure 4.2.*

Remark 4.1.4.3 Instead of the original inner product $(x_h, y_h)_Y$, one can introduce an equivalent inner product $(x_h, y_h)_{Y_h}$, where

$$c_1 \|x_h\|_Y \leq \|x_h\|_{Y_h} \leq c_2 \|x_h\|_Y, \quad 0 < c_1 < c_2 < \infty.$$

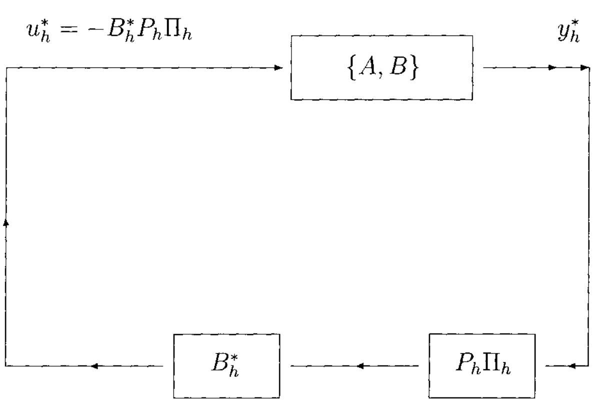


Figure 4.2

In some situations, it is more convenient to work with a discrete inner product $(\ ,\ )_{Y_h}$ as to simplify the computations for the adjoint operators for the discrete problem.

Remark 4.1.4.4 The theory providing optimal rates of convergence is deferred to Section 4.6. It will be a specialization of the theory of Section 4.1.

4.2 Background Material

4.2.1 Continuous Problem

In order to prove the main results, Theorems 4.1.4.1 and 4.1.4.2, we shall proceed along a track, that may be viewed as a discrete counterpart of the approach followed in the continuous case in Chapter 2. To this end, it shall be expedient to collect here relevant quantities and relations from Chapter 2 to be used in this chapter. We first recall the operators L and its adjoint L^* from Chapter 2, Eqns. (2.1.15), (2.1.16):

$$(Lu)(t) = \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau$$

$$: \text{continuous } L_2(0, T; U) \to L_2(0, T; Y) \tag{4.2.1.1a}$$

$$: \text{continuous } L_\infty(0, T; U) \to C([0, T]; \mathcal{D}(\hat{A}^{1-\gamma})), \tag{4.2.1.1b}$$

$$(L^*v)(t) = B^* \int_t^T e^{A^*(\tau-t)} v(\tau)\, d\tau$$

$$: \text{continuous } L_2(0, T; Y) \to L_2(0, T; U), \tag{4.2.1.2}$$

as well as the corresponding operators related to the generator $-\hat{A} = A - \omega I$ in (4.1.1.2) [Chapter 2, Eqns. (2.1.17), (2.1.18)]:

$$(\hat{L}u)(t) = \int_0^t e^{-\hat{A}(t-\tau)} Bu(\tau)\, d\tau : \text{continuous } L_2(0, \infty; U) \to L_2(0, \infty; Y), \tag{4.2.1.3}$$

$$(\hat{L}^*v)(t) = B^* \int_t^\infty e^{-\hat{A}^*(\tau-t)} v(\tau)\, d\tau : \text{continuous } L_2(0, \infty; Y) \to L_2(0, \infty; U). \tag{4.2.1.4}$$

The optimal control problem (4.1.1.7) for (4.1.1.1) is covered by Chapter 2, Theorem 2.2.1, which is summarized in Section 4.1.1. With ω fixed once and for all by (4.1.1.2), we recall the notation of Chapter 2, above Eqn. (2.2.7),

$$\hat{u}^0(t; y_0) = e^{-\omega t} u^0(t; y_0); \qquad \hat{y}^0(t; y_0) = e^{-\omega t} y^0(t; y_0), \tag{4.2.1.5}$$

where $u^0(t; y_0)$ and $y^0(t; y_0)$ comprise the optimal pair of problem (4.1.1), (4.1.2), which originates at the point y_0 at time $t = 0$. We set, as in Chapter 2, Eqn. (2.3.3.5c),

$$\hat{\Phi}(t)x = \hat{y}^0(t; x) = e^{-\omega t} y^0(t; x) = e^{-\omega t} \Phi(t)x, \quad x \in Y. \tag{4.2.1.6}$$

Then, the optimal control and the corresponding optimal trajectory are given by the following explicit formulas (see Chapter 2, Eqns. (2.2.7), (2.2.8))

$$\hat{y}^0(\,\cdot\,; y_0) = \hat{\Phi}(\cdot)y_0 = [I + \hat{L}\hat{L}^*(R^*R + 2\omega P)]^{-1}\{e^{-\hat{A}\,\cdot}\, y_0\} \in L_2(0, \infty; U), \tag{4.2.1.7}$$

$$-\hat{u}^0(\,\cdot\,; y_0) = [I + \hat{L}^*(R^*R + 2\omega P)\hat{L}]^{-1}\hat{L}^*[R^*R + 2\omega P]\{e^{-\hat{A}\,\cdot}\, y_0\} \tag{4.2.1.8}$$

$$= \{\hat{L}^*[R^*R + 2\omega P]\hat{y}^0(\,\cdot\,; x)\}(t) \in L_2(0, \infty; U), \tag{4.2.1.9}$$

with inverses well defined in $L_2(0, \infty; \cdot)$, $\cdot = Y$ or U by Chapter 2, Appendix 2A. The solution P to the ARE in (4.1.1.11) satisfies the relation [Chapter 2, Eqn. (2.2.9)]

$$Px = \int_0^\infty e^{-\hat{A}^*t}[R^*R + 2\omega P]\hat{\Phi}(t)x\, dt, \quad x \in Y. \tag{4.2.1.10}$$

4.2.2 Discrete Problem

To describe the solution to the discrete problem (4.1.3.1), (4.1.3.2), we similarly introduce the operators

$$(L_h u)(t) = \int_0^t e^{A_h(t-\tau)} B_h u(\tau)\, d\tau$$

$$: \text{continuous } L_2(0, T; U) \to L_2(0, T; V_h), \tag{4.2.2.1}$$

$$(L_h^* v)(t) = B_h^* \int_t^T e^{A_h^*(\tau-t)} \Pi_h v(\tau)\, d\tau$$

$$: \text{continuous } L_2(0, T; Y) \to L_2(0, T; U), \tag{4.2.2.2}$$

with L_h^* being the L_2-adjoint of L_h, and finally,

$$(\hat{L}_h u)(t) = \int_0^t e^{-\hat{A}_h(t-\tau)} B_h u(\tau)\, d\tau$$

$$: \text{continuous } L_2(0, \infty; U) \to L_2(0, \infty; V_h), \tag{4.2.2.3}$$

$$(\hat{L}_h^* v)(t) = B_h^* \int_t^\infty e^{-\hat{A}_h^*(\tau-t)} \Pi_h v(\tau)\, d\tau$$

$$: \text{continuous } L_2(0, \infty; Y) \to L_2(0, \infty; U), \tag{4.2.2.4}$$

where $\hat{A}_h = -A_h + \omega I$; see (4.1.2.5). Let now $\{u_h^0(t; x), y_h^0(t; x)\}$ be the optimal pair of the discrete optimal problem (4.1.3.1), (4.1.3.2), originating at the point $x \in V_h$ at the time $t = 0$, and set

$$\hat{u}_h^0(t; x) = e^{-\omega t} u_h^0(t; x), \qquad \hat{y}_h^0(t; x) = e^{-\omega t} y_h^0(t; x), \tag{4.2.2.5}$$

$$\hat{\Phi}_h(t)x = \hat{y}_h^0(t; x) = e^{-\omega t}\Phi_h(t)x, \quad x \in V_h, \tag{4.2.2.6}$$

consistently with (4.2.1.5), (4.2.1.6). Then, the discrete optimal pair of problem (4.1.3.1), (4.1.3.2) is given by the following explicit formulas with $y_{0h} = \Pi_h y_0 \in V_h$, which are the counterpart of formulas (4.2.1.7)–(4.2.1.9) in the continuous case:

$$\hat{y}_h^0(\,\cdot\,; y_{0h}) = \hat{\Phi}_h(\,\cdot\,)y_{0h} = [I + \hat{L}_h\hat{L}_h^*\Pi_h(R^*R + 2\omega P_h)]^{-1}\{e^{-\hat{A}_h\cdot}y_{0h}\} \in L_2(0,\infty; V_h); \tag{4.2.2.7}$$

$$-\hat{u}_h^0(\,\cdot\,; y_{0h}) = [I + \hat{L}_h^*\Pi_h(R^*R + 2\omega P_h)\hat{L}_h]^{-1} \hat{L}_h\Pi_h[R^*R + 2\omega P_h]\{e^{-\hat{A}_h\cdot}y_{0h}\} \in L_2(0,\infty; U), \tag{4.2.2.8}$$

along with the optimality condition (see Chapter 2, Eqn. (2.2.8a))

$$-\hat{u}_h^0(t; x) = \hat{L}_h^*[R^*R + 2\omega P_h]\hat{y}_h^0(\,\cdot\,; x)(t), \quad x \in V_h. \tag{4.2.2.9}$$

The corresponding Riccati operator P_h satisfies (counterpart of (4.2.1.10), (4.1.1.13)),

$$P_h x = \int_0^\infty e^{-\hat{A}_h^* t}\Pi_h[R^*R + 2\omega P_h]\hat{\Phi}_h(t)x\,dt, \quad x \in V_h;$$

$$J\big(u_h^0(\,\cdot\,; x), y_h^0(\,\cdot\,; x) = (P_h x, x)_Y, \quad x \in V_h, \tag{4.2.2.10}$$

and the optimal discrete dynamics satisfies (counterpart of Chapter 2, Eqn. (2.3.3.6))

$$\hat{y}_h^0(t; x) = e^{-\hat{A}_h t}x + \{\hat{L}_h\hat{u}_h^0(\,\cdot\,; x)\}(t), \quad x \in V_h. \tag{4.2.2.11}$$

The proofs of our main convergence results are based on a careful analysis of the convergence properties of the basic operator $\hat{L}_h$ and $\hat{L}_h^*$ to be given in Section 4.3.

4.3 Convergence Properties of the Operators L_h and L_h^*; $\hat{L}_h$ and $\hat{L}_h^*$

Lemma 4.3.1 *Let assumptions (A.1) = (4.1.2.2) through (A.6) = (4.1.2.9) hold true. Then, we have for all* $0 < h \le h_0$, *with constants independent on h:*

(i)

$$\|B_h^* e^{A_h^* t}\Pi_h - B^* e^{A^* t}\|_{\mathcal{L}(Y;U)} \le C\,\frac{h^{s(1-\gamma)}}{t}e^{(\omega_0+\epsilon)t}, \quad t > 0; \tag{4.3.1}$$

(ii)

$$\|B_h^* e^{A_h^* t}\Pi_h - B^* e^{A^* t}\|_{\mathcal{L}(\mathcal{D}(\hat{A}^*);U)} \le C\,h^{s(1-\gamma)}e^{(\omega_0+\epsilon)t}, \quad t > 0; \tag{4.3.2}$$

and by interpolation between (4.3.1) and (4.3.2), with $0 \le \theta \le 1$,

(iii)

$$\|B_h^* e^{A_h^* t}\Pi_h - B^* e^{A^* t}\|_{\mathcal{L}(\mathcal{D}((\hat{A}^*)^\theta);U)} \le C_\theta\,\frac{h^{s(1-\gamma)}}{t^{1-\theta}}e^{(\omega_0+\epsilon)t}, \quad t > 0; \tag{4.3.3}$$

moreover,

(iv)

$$\|B_h^* e^{A_h^* t}\Pi_h\|_{\mathcal{L}(Y;U)} \le C\,\frac{e^{(\omega_0+\epsilon)t}}{t^\gamma}, \quad t > 0; \tag{4.3.4}$$

(v)

$$\|B_h^* e^{A_h^* t}\Pi_h - B^* e^{A^* t}\|_{\mathcal{L}(Y;U)} \le C\, \frac{h^{s(1-\gamma)\theta}}{t^{\theta+(1-\theta)\gamma}} e^{(\omega_0+\epsilon)t},$$
$$t > 0,\ 0 \le \theta \le 1; \qquad (4.3.5)$$

and by interpolation between (4.3.5) and (4.3.2) with $0 \le r \le 1$,

(vi)

$$\|B_h^* e^{A_h^* t}\Pi_h - B^* e^{A^* t}\|_{\mathcal{L}(\mathcal{D}((\hat{A}^*)^r);U)} \le C\, \frac{h^{s(1-\gamma)[r+(1-r)\theta]}}{t^{(1-r)[\theta+\gamma(1-\theta)]}}\, e^{(\omega_0+\epsilon)t},$$
$$t > 0,\ 0 \le \theta \le 1. \qquad (4.3.6)$$

Proof. (i) With Π_h the orthogonal projection of Y onto V_h, we compute, after adding and subtracting $B_h^* \Pi_h e^{A^* t}$,

$$\|B_h^* e^{A_h^* t}\Pi_h - B^* e^{A^* t}\|_{\mathcal{L}(Y;U)} \le \|B_h^* \left(e^{A_h^* t}\Pi_h - \Pi_h e^{A^* t}\right)\|_{\mathcal{L}(Y;U)}$$
$$+ \|B^* e^{A^* t} - B_h^* \Pi_h e^{A^* t}\|_{\mathcal{L}(Y;U)}. \qquad (4.3.7)$$

Now, on the first term on the right-hand side of (4.3.7), we use (A.3) = (4.1.2.6), as well as the rough data estimate (4.1.2.10) for $\theta = 1$ (a consequence of (A.1) and (A.2)). Instead, on the second term on the right-hand side of (4.3.7), we use (A.5) = (4.1.2.8). We thus obtain from (4.3.7)

$$\|B_h^* e^{A_h^* t}\Pi_h - B^* e^{A^* t}\|_{\mathcal{L}(Y;U)} \le C\, h^{-\gamma s} \frac{h^s e^{(\omega_0+\epsilon)t}}{t} + C\, h^{s(1-\gamma)} \|A^* e^{A^* t}\|_{\mathcal{L}(Y)}$$
$$\le C\, h^{s(1-\gamma)} \frac{e^{(\omega_0+\epsilon)t}}{t}, \quad t > 0, \qquad (4.3.8)$$

where in the last step we have used the analyticity of $e^{A^* t}$. Thus (4.3.1) is proved.

(ii) Similarly,

$$\|B_h^* e^{A_h^* t}\Pi_h - B^* e^{A^* t}\|_{\mathcal{L}(\mathcal{D}(A^*);U)} \le \|B_h^* \left(e^{A_h^* t}\Pi_h - \Pi_h e^{A^* t}\right)\|_{\mathcal{L}(\mathcal{D}(A^*);U)}$$
$$+ \|B^* e^{A^* t} - B_h^* \Pi_h e^{A^* t}\|_{\mathcal{L}(\mathcal{D}(A^*);U)} \qquad (4.3.9)$$

(using (A.3) = (4.1.2.6) and (4.1.2.12), and Remark 4.1.2.3, a consequence of (A.1) and (A.2), on the first term of (4.3.9). Using (A.5) = (4.1.2.8) on the second term of (4.3.9), we have)

$$\le \left[C\, h^{-\gamma s} h^s + C\, h^{s(1-\gamma)}\right] e^{(\omega_0+\epsilon)t}, \quad t > 0, \qquad (4.3.10)$$

and conclusion (4.3.2) follows from (4.3.10).

(iii) Equation (4.3.3) follows from (4.3.1) and (4.3.2) by use of the interpolation (moment) inequality [Lions, Magenes, 1972, p. 19].

(iv) First, from the full assumption (A.3) = (4.1.2.6) and the uniform analyticity (A.1) = (4.1.2.2) with $\theta = 0$, we obtain

$$\|B_h^* e^{A_h^* t}\Pi_h - B^* \Pi_h e^{A^* t}\|_{\mathcal{L}(Y;U)} \le C\, h^{-\gamma s} e^{(\omega_0+\epsilon)t}. \qquad (4.3.11)$$

Next, we shall show that

$$\|B_h^* e^{A_h^* t} \Pi_h - B^* \Pi_h e^{A^* t}\|_{\mathcal{L}(Y;U)} \leq C \frac{h^{s(1-\gamma)}}{t} e^{(\omega_0+\epsilon)t}, \quad t > 0, \tag{4.3.12}$$

through a computation similar to the ones above. Indeed, adding and substracting $B^* e^{A^* t}$ we get

$$\begin{aligned} \|B_h^* e^{A_h^* t} \Pi_h - B^* \Pi_h e^{A^* t}\|_{\mathcal{L}(Y;U)} \leq \; & \|B_h^* e^{A_h^* t} \Pi_h - B^* e^{A^* t}\|_{\mathcal{L}(Y;U)} \\ & + \|B^*(I - \Pi_h) e^{A^* t}\|_{\mathcal{L}(Y;U)}. \end{aligned} \tag{4.3.13}$$

Using (4.3.1) of part (i) on the first term of (4.3.13) and (A.4) = (4.1.2.7) on the second term of (4.3.13)) yields

$$\begin{aligned} & \|B_h^* e^{A_h^* t} \Pi_h - B^* \Pi_h e^{A^* t}\|_{\mathcal{L}(Y;U)} \\ & \leq C \frac{h^{s(1-\gamma)}}{t} \left(e^{(\omega_0+\epsilon)t} + \|A^* e^{A^* t}\|_{\mathcal{L}(Y)} \right), \quad t > 0, \end{aligned} \tag{4.3.14}$$

and (4.3.12) follows from (4.3.14) by analyticity of $e^{A^* t}$. Next, we raise (4.3.11) to the power $(1 - \gamma)$, we raise (4.3.12) to the power of γ, and we multiply the resulting expression together. In this way we obtain

$$\|B_h^* e^{A_h^* t} \Pi_h - B^* \Pi_h e^{A^* t}\|_{\mathcal{L}(Y;U)} \leq C \frac{e^{(\omega_0+\epsilon)t}}{t^\gamma}, \quad t > 0. \tag{4.3.15}$$

However, by assumption (A.6) = (4.1.2.9) and analyticity of $e^{A^* t}$, we obtain, recalling the notation $A = -\hat{A} + \omega I$ from (4.1.1.2), and $\hat{\omega} = \omega - \omega_0 - \epsilon$ from (4.1.1.3):

$$\begin{aligned} \|B^* \Pi_h e^{A^* t}\|_{\mathcal{L}(Y;U)} & \leq C \|(\hat{A}^*)^\gamma e^{A^* t}\|_{\mathcal{L}(Y)} \\ & = C\, e^{\omega t} \|(\hat{A}^*)^\gamma e^{-\hat{A} t}\|_{\mathcal{L}(Y)} \leq C \frac{e^{(\omega - \hat{\omega})t}}{t^\gamma} \\ & = C \frac{e^{(\omega_0+\epsilon)t}}{t^\gamma}, \quad t > 0. \end{aligned} \tag{4.3.16}$$

Combining (4.3.15) with (4.3.16), we obtain (4.3.4) as desired.

(v) First, from the standing assumption (1.1.4) and analyticity of $e^{A^* t}$ we have

$$\|B^* e^{A^* t}\|_{\mathcal{L}(Y;U)} \leq C \|(\hat{A}^*)^\gamma e^{A^* t}\| \leq C \frac{e^{(\omega_0+\epsilon)t}}{t^\gamma}, \quad t > 0, \tag{4.3.17}$$

recalling the computations leading to (4.3.16). Then (4.3.17) and (4.3.4) of part (iv) imply

$$\|B_h^* e^{A_h^* t} \Pi_h - B^* e^{A^* t}\|_{\mathcal{L}(Y;U)} \leq C \frac{e^{(\omega_0+\epsilon)t}}{t^\gamma}, \quad t > 0. \tag{4.3.18}$$

Finally, we raise (4.3.1) of part (i) to the power θ, we raise (4.3.18) to power $(1 - \theta)$, and we multiply the resulting expressions together. In this way we obtain (4.3.5).

(vi) Equation (4.3.6) follows from (4.3.5) and (4.3.2) via the interpolation (moment) inequality [Lions, Magenes, 1972, p. 19].

Lemma 4.3.1 is completely proved. □

Remark 4.3.1 We note that the proof of (4.3.4) and (hence of) (4.3.5) requires all assumptions (A.1) through (A.6). Below, (4.3.4) and/or (4.3.5) will be invoked in the proof of all subsequent results.

From the definition of the operators L_h in (4.2.2.1) and L_h^* in (4.2.2.2) and the uniform estimate (4.3.4) of Lemma 4.3.1(iv), we obtain at once the following *stability* result, since $\gamma < 1$, via the Young inequality (convolution between an L_1-function and an L_2-function is L_2) [Sadosky, 1979]. We give the details that identify the constant $C_{T\gamma}$ below.

Corollary 4.3.2 *Assume (A.1) through (A.6). Then we have, uniformly in* $0 < h \leq h_0$ *for* $0 < T < \infty$*:*

(i)

$$\|L_h u\|_{L_2(0,T;Y)} \leq C_{T,\gamma} \|u\|_{L_2(0,T;U)}; \tag{4.3.19}$$

(ii)

$$\|L_h^* u\|_{L_2(0,T;U)} \leq C_{T,\gamma} \|v\|_{L_2(0,T;Y)}. \tag{4.3.20}$$

Proof. We show (ii). Starting from L_h^* in (4.2.2.2) we compute, by using (4.3.4) on the first step, the Schwarz inequality on the second, and a change in the order of integration in the third:

$$\begin{aligned}
\int_0^T \|(L_h^* v)(t)\|_U^2 \, dt &= \int_0^T \left\| \int_t^T B_h^* e^{A_h^*(\tau - t)} \Pi_h v(\tau) \, d\tau \right\|_U^2 dt \\
\text{(by (4.3.4))} \qquad &\leq C_T^2 \int_0^T \left[\int_t^T \frac{1}{(\tau - t)^\gamma} \|v(\tau)\|_Y \, d\tau \right]^2 dt \\
&\leq C_T^2 \int_0^T \left[\int_t^T \frac{1}{(\tau - t)^\gamma} \, d\tau \right] \left[\int_t^T \frac{\|v(\tau)\|_Y^2}{(\tau - t)^\gamma} \, d\tau \right]^2 dt \\
&\leq \frac{C_T^2 T^{1-\gamma}}{1-\gamma} \int_0^T \|v(\tau)\|_Y^2 \left[\int_0^\tau \frac{1}{(\tau - t)^\gamma} dt \right] d\tau \\
&\leq \frac{C_T^2 T^{2(1-\gamma)}}{(1-\gamma)^2} \int_0^T \|v(\tau)\|_Y^2 \, d\tau,
\end{aligned}$$

where $C_T = C \exp[(\omega_0 + \epsilon)T]$ from (4.3.4). The proof of (i) is identical. $\square$

The main results of this subsection are the next two theorems.

Theorem 4.3.3 *Assume (A.1) through (A.6) as in Lemma 4.4.3.1. With reference to the operators L and* L^* *defined by (4.2.1.1) and (4.2.1.2), and* L_h *and* L_h^* *defined by (4.2.2.1) and (4.2.2.2), the following results hold true, where* $0 < \theta < 1$ *arbitrary,* $0 < h \leq h_0$*:*

(i)

$$\|L_h - L\|_{L_2} = \|L_h^* - L^*\|_{L_2} \leq C_{T,\theta} h^{s(1-\gamma)\theta}, \tag{4.3.21}$$

where the first norm is in $\mathcal{L}(L_2(0,T;U); L_2(0,T;Y))$, and similarly for the second norm with U and Y interchanged, where $0 < T < \infty$:

(ii)

$$\|L_h - L\|_{\mathcal{L}(L_\infty(0,T;U);C([0,T];Y))} \leq C_{T,\theta} h^{s(1-\gamma)\theta}. \tag{4.3.22}$$

Proof. (i) We compute from (4.2.1.2) and (4.2.2.2), after recalling the estimate (4.3.5) of Lemma 4.3.1(v), with $\beta = \theta + (1-\theta)\gamma$, where $\gamma \leq \beta < 1$, for any $\theta < 1$ and $\gamma < 1$:

$$\begin{aligned}
\|L_h^* v - L^* v\|^2_{L_2(0,T;U)} &= \left\| \int_t^T \left[B_h^* e^{A_h^*(\tau - t)} \Pi_h - B^* e^{A^*(\tau-t)} \right] v(\tau)\, d\tau \right\|^2_{L_2(0,T;U)} \\
\text{(by (4.3.5))} \qquad &\leq C_T^2 h^{2s(1-\gamma)\theta} \int_0^T \left[\int_t^T \frac{1}{(\tau - t)^\beta} \|v(\tau)\|_Y\, d\tau \right]^2 dt \\
&\leq C_{T,\beta} h^{2s(1-\gamma)\theta} \|v\|^2_{L_2(0,T;Y)},
\end{aligned} \tag{4.3.23}$$

after proceeding as in the proof of Corollary 4.3.2, with $\gamma < 1$ there replaced by $\beta < 1$ now. Then (4.3.23) proves (4.3.21).

(ii) Similarly, from (4.2.1.1) and (4.2.2.1), again by use of the dual version of estimate (4.3.5) with $\gamma \leq \beta = \theta + (1-\theta)\gamma < 1$:

$$\begin{aligned}
\|L_h u - Lu\|_{C([0,T];Y)} &= \left\| \int_0^t \left(e^{A_h(t-\tau)} B_h - e^{A(t-\tau)} B \right) u(\tau)\, d\tau \right\|_{C([0,T];Y)} \\
&\leq C_T h^{s(1-\gamma)\theta} \sup_{0 \leq t \leq T} \left[\int_0^T \frac{1}{(t-\tau)^\beta} \|u(\tau)\|_U\, d\tau \right] \\
&\leq C_{T,\beta} h^{s(1-\gamma)\theta} \|u\|_{L_\infty(0,T;U)},
\end{aligned} \tag{4.3.24}$$

and (4.3.24) proves (4.3.22). □

Theorem 4.3.4 *Assume (A.1) through (A.6) as in Lemma 4.3.1. With reference to the operators $\hat{L}$ and $\hat{L}^*$ defined in (4.2.1.3) and (4.2.1.4), and the operators $\hat{L}_h$ and $\hat{L}_h^*$ defined in (4.2.2.3) and (4.2.2.4), the following convergence results hold true, where $0 \leq \theta < 1$ arbitrary:*

(i)

$$\|\hat{L}_h - \hat{L}\|_{L_2} = \|\hat{L}_h^* - \hat{L}^*\|_{L_2} \leq C_\theta h^{s(1-\gamma)\theta} \to 0 \text{ as } h \downarrow 0, \tag{4.3.25a}$$

where the first norm is in $\mathcal{L}(L_2(0,\infty;U); L_2(0,\infty;Y))$, while the second norm is similar with U and Y interchanged:

(ii)

$$\|\hat{L}_h - \hat{L}\|_{\infty,C} = \|\hat{L}_h^* - \hat{L}^*\|_{\infty,C} \leq C_\theta h^{s(1-\gamma)\theta} \to 0 \quad \text{as } h \downarrow 0, \tag{4.3.25b}$$

where the first norm is in $\mathcal{L}(L_\infty(0,\infty;U); C([0,\infty];Y))$, while the second norm is similar with U and Y interchanged.

Proof. (i) From (4.2.1.4) and (4.2.2.3), we compute with $v \in L_2(0,\infty;Y)$

$$\|\hat{L}_h v - \hat{L}^* v\|^2_{L_2(0,\infty;U)} = \int_0^\infty \left\| \int_t^\infty \left[B_h^* e^{-\hat{A}_h(\tau-t)}\Pi_h - B^* e^{-\hat{A}^*(\tau-t)}\right] v(\tau)\, d\tau \right\|_U^2 dt$$

$$\text{(by (4.3.5))} \qquad \le C^2 h^{2s(1-\gamma)\theta} \int_0^\infty \left[\int_t^\infty \frac{e^{-\hat{\omega}(\tau-t)}}{(\tau-t)^\beta} \|v(\tau)\|_Y\, d\tau \right]^2 dt, \tag{4.3.26}$$

after recalling (4.3.5) with $\gamma \le \beta = \theta + (1-\theta)\gamma < 1$, for any $\theta < 1$ and $\gamma < 1$ (as in the proof of Theorem 4.3.3), as well as $-\hat{A}^* = A^* - \omega I$, $\omega_0 + \epsilon - \omega = -\hat{\omega} < 0$ from (4.1.1.2), (4.1.1.3), and $-\hat{A}_h^* = A_h^* - \omega I$ from (4.1.2.5). Next, as in the proof of Corollary 4.3.2, via Schwarz inequality and a change of order of integration, we have

$$\int_0^\infty \int_t^\infty \frac{e^{-\hat{\omega}(\tau-t)}}{(\tau-t)^\beta} \|v(\tau)\|_Y\, d\tau\, dt$$

$$\le \int_0^\infty \left(\int_t^\infty \frac{e^{-\hat{\omega}(\tau-t)}}{(\tau-t)^\beta} d\tau \right) \left(\int_t^\infty \frac{e^{-\hat{\omega}(\tau-t)}}{(\tau-t)^\beta} \|v(\tau)\|_Y^2\, d\tau \right) dt \tag{4.3.27}$$

$$\le C_\beta \int_0^\infty \left(\int_0^\tau \frac{e^{-\hat{\omega}(\tau-t)}}{(\tau-t)^\beta} dt \right) \|v(\tau)\|_Y^2\, d\tau \le C_\beta \|v\|^2_{L_2(0,\infty;Y)}, \tag{4.3.28}$$

since the first internal integral in (4.3.27) is bounded by a constant. Then (4.3.26) and (4.3.28) prove (4.3.25).

(ii) The proof is similar to that of part (i). □

4.4 Perturbation Results

The goal of the first two subsections is to show that the properties of analyticity and exponential stability of the semigroup $e^{A_F t}$, $A_F = A + BF$ [see (4.1.1.8)], are preserved, uniformly in h, by its approximations.

4.4.1 Uniform Analyticity and Uniform Exponential Stability of the Operators $A_{h,F_h} = A_h + B_h F_h$ and $A_{F_h} = A + BF_h$

4.4.1.1 Uniform Analyticity of A_{h,F_h}

Throughout this subsection, we let $F \in \mathcal{L}(Y;U)$, and we consider the operator

$$A_F \equiv A + BF : Y \supset \mathcal{D}(A_F) \to Y, \tag{4.4.1.1}$$

which, in view of the standing assumption (4.1.1.4), generates likewise a s.c., analytic semigroup $e^{A_F t}$ on Y, $t \ge 0$ (as justified in Remark 4.1.1.1). In later sections, but not in this subsection, we shall consider the case where F is a stabilizing feedback

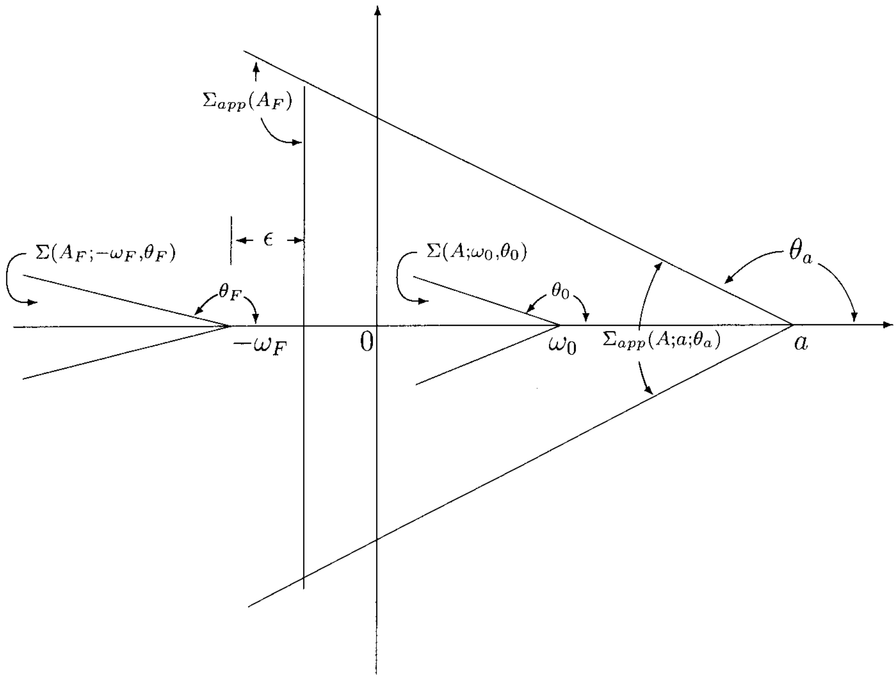


Figure 4.3

operator. We let (see Figure 4.3.)

$$\Sigma(A_F) = \Sigma(A_F; a_F; \theta_F) = \text{(closed) triangular sector containing the axis } [-\infty, a_F] \text{ and delimited by the two rays } a_F + \rho e^{\pm i\theta_F}, \; 0 \leq \rho < \infty, \text{ for some } \theta_F, \; \frac{\pi}{2} < \theta_F < \pi, \tag{4.4.1.2}$$

such that the spectrum $\sigma(A_F) \subset \Sigma(A_F)$. For a stabilizing F as in the assumption (SC) = (4.1.1.8), we may take $a_F = -\omega_F < 0$ in the notation of (4.1.1.8). In comparing $\Sigma(A_F)$ with the sector $\Sigma_{\text{app}}(A) \supset \sigma(A)$ in Subsection 4.1.2.2, we may say that we can always assume without loss of generality that one sector is contained in the other: If $a_F \leq a$, then we can choose $\theta_a < \theta_F$, and then $\Sigma(A_F) \subset \Sigma_{\text{app}}(A)$; instead, if $a_F > a$, we can choose $\theta_a > \theta_F$, and then $\Sigma(A_F) \supset \Sigma_{\text{app}}(A)$. The first instance with $a_F = -\omega_F < a$ occurs if F is a stabilizing feedback operator. For sake of definiteness, in the lemma below *we shall assume that $a_F \leq a$, and so $\Sigma(A_F) \subset \Sigma_{\text{app}}(A)$*, the case that arises with F a stabilizing operator. Next, we consider the approximation of A_F defined by

$$A_{h,F_h} \equiv A_h + B_h F_h : V_h \to V_h \tag{4.4.1.3}$$

with $F_h \in \mathcal{L}(V_h; U)$ and A_h, B_h as in Subsection 4.1.2.1.

The next result shows that if $\{\|F_h\|\}$ is uniformly bounded in h, then the operators A_{h,F_h} defined by (4.4.1.3) generate "uniformly" analytic semigroups on Y. With the stipulation above (4.4.1.3), we have

Theorem 4.4.1.1 *(Uniform analyticity of A_{h,F_h}). Assume (A.1) through (A.6). Let $\|F_h\|_{\mathcal{L}(V_h;U)} \le$ const., uniformly in h. Then, given $1 > \delta > 0$, there exist $r_\delta > 0$ and $h_\delta > 0$ such that for a suitable $\Sigma^c_{\text{app}}(A) = \Sigma^c_{\text{app}}(A; a, \theta_a)$ (recall (4.1.2.3a)), we have*

(i)

$$\left\|R\left(\lambda, A_{h,F_h}\right)\right\|_{\mathcal{L}(V_h)} \le \frac{1}{1-\delta}\|R(\lambda, A_h)\|_{\mathcal{L}(V_h)}, \quad \forall\lambda \in \Sigma^c_{\text{app}}(A),\ |\lambda| \ge r_\delta,$$
$$\forall h,\ 0 < h < h_\delta \le h_a; \tag{4.4.1.4}$$

(ii)

$$\left\|R\left(\lambda, A_{h,F_h}\right)\right\|_{\mathcal{L}(V_h)} \le \frac{C}{1-\delta}\frac{1}{|\lambda - a|}, \quad \lambda \textit{ and } h \textit{ as in } (4.4.1.4); \tag{4.4.1.5}$$

(iii) (since A_{h,F_h} are finite-dimensional operators)

$$\left\|R\left(\lambda, A_{h,F_h}\right)A^\theta_{h,F_h}\right\|_{\mathcal{L}(V_h)} \le \frac{C}{1-\delta}\frac{1}{|\lambda - a|^{1-\theta}},$$
$$0 \le \theta \le 1,\ \lambda \textit{ and } h \textit{ as in } (4.4.1.4). \tag{4.4.1.6}$$

Proof of Theorem 4.4.1.1. We have stipulated above (4.4.1.3) that, for the sake of definiteness, we are taking the case $\Sigma(A_F) \subset \Sigma_{\text{app}}(A)$ (Figure 4.3). Thus, for all $\lambda \in \Sigma^c_{\text{app}}(A) = \Sigma^c_{\text{app}}(A; a; \theta_a)$, the complement of $\Sigma_{\text{app}}(A)$, we have that $R(\lambda, A_F)$ and $R(\lambda, A_h)$ are well-defined, bounded operators on Y and V_h respectively, $\forall h \le h_a$.

Step 1 Given $\delta > 0$, there exist $r_\delta > 0$ and $0 < h_\delta \le h_a$ such that

$$\|R(\lambda, A_h)B_h F_h\|_{\mathcal{L}(V_h)} < \delta, \quad \text{for all } \lambda \text{ and } h \text{ as in (4.4.1.4)}. \tag{4.4.1.7}$$

To show (4.4.1.7), we first notice that the Laplace transform version in the λ-domain of estimate (4.3.4), Lemma 4.3.1(iv), in the t-domain is

$$\|B^*_h R(\bar{\lambda}, A^*_h)\Pi_h\|_{\mathcal{L}(Y;U)} \le \frac{C}{\|\bar{\lambda} - a\|^{1-\gamma}}, \quad \lambda \in \Sigma^c_{\text{app}}(A). \tag{4.4.1.8}$$

In fact, the Laplace transform of $B^*_h R(\lambda, A^*_h)$ yields, by virtue of estimate (4.3.4), the desired bound (4.1.8), first for Re $\lambda > a = \omega_0 + \epsilon$ (see [Doetsch, 1970, p. 319, #26]), next in $\Sigma^c_{\text{app}}(A)$ as usual by analytic continuation. Then, estimate (4.4.1.7) follows from (the dual version of) (4.4.1.8) by taking $\frac{C\|F_h\|}{|\lambda - a|^{1-\gamma}} < \delta$, with $\|F_h\| \le C_F$ uniformly bounded, so that one may take $r_\delta > a + (\frac{CC_F}{\delta})^{\frac{1}{1-\gamma}}$.

Step 2 From (4.4.1.7) it follows that

$$\|[I - R(\lambda, A_h)B_h F_h]^{-1}\|_{\mathcal{L}(V_h)} \leq \frac{1}{1 - \|R(\lambda, A_h)B_h F_h\|_{\mathcal{L}(V_h)}}$$
$$\leq \frac{1}{1-\delta}, \quad \lambda \text{ and } h \text{ as in (4.4.1.4)}, \ \delta < 1. \tag{4.4.1.9}$$

Then, in view of (4.4.1.9), the usual perturbation formula

$$R\left(\lambda, A_{h,A_h}\right) = [I - R(\lambda, A_h)B_h F_h]^{-1} R(\lambda, A_h), \tag{4.4.1.10}$$

holds true for all λ as in (4.4.1.4).

Step 3 The desired estimate (4.4.1.4) of part (i) follows now from (4.4.1.10) and (4.4.1.9). Then (4.4.1.4) implies the desired estimate (4.4.1.5) of part (ii) via $\|R(\lambda, A_h)\|_{\mathcal{L}(V_h)} \leq \frac{C}{|\lambda - a|}$, $\lambda \in \Sigma^c_{\text{app}}(A)$ [see (A.1) = (4.1.2.3c)] for $\theta = 0$.

Finally, the desired estimate (4.4.1.6) of part (iii) follows from (4.4.1.5) in the usual way: We write $R\left(\lambda, A_{h,F_h}\right) A_{h,F_h} = \lambda R\left(\lambda, A_{h,F_h}\right) - I$, and we use on this (4.4.1.5), thereby obtaining (4.4.1.6) for $\theta = 1$. Then the cases $\theta = 0$ and $\theta = 1$ imply the cases $0 < \theta < 1$ via the interpolation (moment) inequality. Theorem 4.4.1.1 is proved. □

Remark 4.4.1.1 Lemma 4.4.1.1 on uniform analyticity holds true also for the operators

$$A_{F_h} \equiv A + BF_h\Pi_h, \qquad \mathcal{D}\left(A^*_{F_h}\right) = \mathcal{D}((\hat{A}^*)^\gamma), \tag{4.4.1.11}$$

in which case the proof is simpler.

4.4.1.2 Uniform Exponential Stability of A_{h,F_h} and A_{F_h}

In this subsection we assume explicitly the stabilizability condition (SC) = (4.1.1.8) that $F \in \mathcal{L}(Y; U)$ is stabilizing for $A_F = A + BF$, that is,

$$\|e^{A_F t}\|_{\mathcal{L}(Y)} \leq M_F e^{-\omega_F t}, \quad t \geq 0, \ \omega_F > 0, \ M_F \geq 1. \tag{4.4.1.12}$$

With reference to (4.4.1.3) and (4.4.1.11), we shall now prove that $e^{A_{h,F_h} t}$ and $e^{A_{F_h} t}$ are uniformly exponentially stable.

Theorem 4.4.1.2 *Assume (A.1) through (A.6). Moreover, assume (SC) = (4.4.1.12). Finally, with $\|F_h\| \leq$ const assume that*

$$\|R(\lambda_0, A)B(F_h\Pi_h - F)\|_{\mathcal{L}(Y)} \to 0 \textit{ as } h \downarrow 0, \quad \textit{for some } \lambda_0 \in \rho(A). \tag{4.4.1.13}$$

Then, given $\epsilon > 0$, there exists $h_\epsilon > 0$ and a three-sided sector $\Sigma_{\text{app}}(A_F)$, which may be taken to be (see Figure 4.3)

$$\Sigma_{\text{app}}(A_F) = \Sigma_{\text{app}}(A) \cap \{Re\, \lambda \leq -\omega_F + \epsilon\}, \tag{4.4.1.14}$$

with $\Sigma_{\text{app}}(A) = \Sigma_{\text{app}}(A; a; \theta_a)$ *defined by (4.1.2.3a), such that for all* $0 < h \le h_\epsilon \le h_a$, *the operators* A_{h,F_h} *in (4.4.1.3) satisfy*

(i)

$$\sigma\left(A_{h,F_h}\right) \subset \Sigma_{\text{app}}(A_F); \tag{4.4.1.15}$$

(ii)

$$\|e^{A_{h,F_h}t}\|_{\mathcal{L}(V_h)} \le M_{1\epsilon} e^{(-\omega_F+\epsilon)t}, \quad t \ge 0; \tag{4.4.1.16}$$

(iii) (since A_{h,F_h} *are finite-dimensional operators)*

$$\left\|R\left(\lambda, A_{h,A_h}\right) A^{\theta}_{h,F_h}\right\|_{\mathcal{L}(V_h)} \le \frac{C}{|\lambda + \omega_F - \epsilon|\|^{1-\theta}}, \quad \lambda \in \Sigma^c_{\text{app}}(A_F), \quad 0 \le \theta \le 1, \tag{4.4.1.17}$$

uniformly in h. Thus, $e^{A_{h,F_h}t}$ *is uniformly exponentially stable. Moreover, a fortiori from Theorem 4.4.1.1, it is uniformly analytic.*

Remark 4.4.1.2 We note explicitly that either one of the following conditions is sufficient for assumption (4.4.1.13) to hold true:

(i) Either $F_h^* \to F^*$ strongly, and $B^*R(\lambda_0 A^*)$ is compact $Y \to U$ (as assumed in (4.1.4.3)); see [Anselone, 1971, p. 8; Kato, 1966, p. 151];
(ii) or else $F_h \Pi_h \to F$ in the uniform operator norm $\mathcal{L}(Y; U)$.

Remark 4.4.1.3 Theorem 4.4.1.2 holds true also for the operators $A_{F_h} = A + BF_h$ in (4.4.1.11), where, in fact, a simpler proof applies.

Remark 4.4.1.4 The role between assumption (4.4.1.12) and conclusion (4.4.1.16) can be reversed. The same proof yields the following result. Let

$$\|e^{A_{h,F_h}t}\|_{\mathcal{L}(Y)} \le Me^{-\delta t}, \quad \delta > 0$$

(instead of (4.4.1.12)), and assume the converence property (4.4.1.13) as before. Then, we obtain the conclusions corresponding to (4.4.1.15), (4.4.1.16), and (4.4.1.17), with A_{h,F_h} replaced by A_F; in particular, we obtain

$$\|e^{A_F t}\|_{\mathcal{L}(Y)} \le \bar{M} e^{-\bar{\omega}_F t}, \quad \bar{\omega}_F > 0.$$

Proof of Theorem 4.4.1.2. Orientation. From Theorem 4.4.1.1 we know, a fortiori, that $e^{A_{h,F_h}t}$ are uniformly (in h) analytic on Y and that the spectrum $\sigma(A_{h,F_h})$ is uniformly (in h) contained in a common sector, which preliminarily can be taken to be $\Sigma^c_{\text{app}}(A) \cap \{|\lambda| > r_\delta\}$. The next step is to show that as a consequence of (4.4.1.12), in fact, $\sigma(A_{h,F_h})$ satisfies (4.4.1.15), that is, via (4.4.1.14), $\sigma(A_{h,F_h})$ is contained on a three-sided sector on the left-hand side of the complex plane. Finally, uniform analyticity combined with the "correct" location of the spectrum will imply the remaining parts (ii) = (4.4.1.16) and (iii) = (4.4.1.17) of Theorem 4.4.1.2 via operator calculus. Details follow. We begin with

Lemma 4.4.1.3 *Assume the hypotheses of Theorem 4.4.1.2.*

(i) Let, for the sake of definiteness, λ_0 be fixed with Re $\lambda_0 > r_\delta$, r_δ as in (4.4.1.4). Then, the following convergence holds true:

$$\left\| R(\lambda_0, A_F) - R\left(\lambda_0, A_{h,F_h}\right)\Pi_h \right\|_{\mathcal{L}(Y)} \to 0 \text{ as } h \downarrow 0. \tag{4.4.1.18}$$

Moreover, if assumption (4.4.1.13) is strengthened to

$$\|R(\lambda_0, A)B(F_h\Pi_h - F)\|_{\mathcal{L}(Y)} \le C\, h^{s(1-\gamma)\theta}, \quad \text{for some } \theta < 1, \tag{4.4.1.19}$$

then conclusion (4.4.1.18) is accordingly strengthened to

$$\left\| R(\lambda_0, A_F) - R\left(\lambda_0, A_{h,F_h}\right)\Pi_h \right\|_{\mathcal{L}(Y)} \le C\, h^{\epsilon_o} \to 0 \text{ as } h \downarrow 0,$$
$$\text{for all } \epsilon_o < s(1-\gamma). \tag{4.4.1.20}$$

See also Remark 4.4.1.5, Eqn. (4.4.1.37) for a similar result.

(ii) In (4.4.1.18) or (4.4.1.19), one may replace the chosen λ_0 with any other

$$\lambda_0 \in \rho(A_F) \cap \left\{ \bigcap_h \rho\left(A_{h,F_h}\right) \right\},$$

with a different constant.

(iii) For any compact set $\mathcal{K} \subset \bigcap_h \rho(A_{h,F_h})$ and $h < h_a$, we have

$$\sup_{\lambda\in\mathcal{K};h} \left\| R\left(\lambda, A_{h,F_h}\right) \right\|_{\mathcal{L}(Y)} \le \text{const}_{\mathcal{K};h_a}. \tag{4.4.1.21}$$

Proof of Lemma 4.4.1.3. (i) From the perturbation formula (4.4.1.10) and its continuous counterpart

$$R(\lambda, A_F) = [I - R(\lambda, A)BF]^{-1}R(\lambda, A), \quad \forall\lambda \in \Sigma^c(A_F),\ |\lambda| \text{ large} \tag{4.4.1.22}$$

(where by the standing assumption (4.1.1.4)

$$\begin{aligned}\|R(\lambda, A)BF\|_{\mathcal{L}(Y)} &= \|R(\lambda, A)\hat{A}^\gamma \hat{A}^{-\gamma}BF\|_{\mathcal{L}(Y)} \\ &\le \frac{C}{|\lambda - \omega_0\|^{1-\gamma}} \to 0 \text{ as } |\lambda| \to \infty, \quad \lambda \in \Sigma^c_{\text{app}}(A)),\end{aligned} \tag{4.4.1.23}$$

we obtain in the $\mathcal{L}(Y)$-norm:

$$\begin{aligned}&\left\| R(\lambda_0, A_F) - R\left(\lambda_0, A_{h,F_h}\right)\Pi_h \right\| \\ &= \|[I - R(\lambda_0, A)BF]^{-1}R(\lambda_0, A) - [I - R(\lambda_0, A_h)B_hF_h]^{-1}R(\lambda_0, A_h)\Pi_h\| \\ &= \|(1) + (2)\|,\end{aligned} \tag{4.4.1.24}$$

where, after adding and subtracting, we get

$$(1) = [I - R(\lambda_0, A)BF]^{-1}(R(\lambda_0, A) - R(\lambda_0, A_h)\Pi_h); \tag{4.4.1.25}$$

$$(2) = \left\{[I - R(\lambda_0, A)BF]^{-1} - [I - R(\lambda_0, A_h)B_hF_h]^{-1}\right\} R(\lambda_0, A_h)\Pi_h. \tag{4.4.1.26}$$

Thus, by assumption (A.2) = (4.1.2.4) (see Remark 4.1.2.2)

$$\|(1)\| \le C\, h^s \|[I - R(\lambda_0, A)BF]^{-1} \to 0 \text{ as } h \downarrow 0. \tag{4.4.1.27}$$

As to (2), we use the identity

$$[I - T_1]^{-1} - [I - T_2]^{-1} = [I - T_1]^{-1}[T_1 - T_2][I - T_2]^{-1} \tag{4.4.1.28}$$

in (4.4.1.26) with $T_1 = R(\lambda_0, A)BF$ and $T_2 = R(\lambda_0, A_h)B_hF_h$. By (4.4.1.14),

$$R\left(\lambda_0, A_{h,F_h}\right) = [I - T_2]^{-1}R(\lambda_0, A_h),$$

and we can rewrite (4.4.1.26) as

$$(2) = [I - T_1]^{-1}[T_1 - T_2]R\left(\lambda_0, A_{h,F_h}\right)\Pi_h. \tag{4.4.1.29}$$

Setting $K = \|[I - T_1]^{-1}\|$, we obtain from (4.4.1.29)

$$\begin{aligned}\|(2)\| &\le cK \left\|R\left(\lambda_0, A_{h,F_h}\right)\right\| \|R(\lambda_0, A)BF - R(\lambda_0, A_h)B_hF_h\| \\ &\le \frac{c}{1-\delta}\,\frac{1}{|\lambda - a|}\,\|R(\lambda_0, A)BF - R(\lambda_0, A_h)B_hF_h\|,\end{aligned} \tag{4.4.1.30}$$

where in the last step we have used (4.4.1.5) of Lemma 4.4.1(ii), with $0 < \delta < 1$, preassigned, and $0 < h \le h_\delta$. We next compute the term in (4.4.1.30),

$$\begin{aligned}\|R(\lambda_0, A)BF - R(\lambda_0, A_h)B_hF_h\| &\le \|[F^* - F_h^*]B^*R(\lambda_0, A^*)\| \\ &\quad + \|F_h^*[B^*R(\lambda_0, A^*) - B_h^*R(\lambda_0, A_h^*)\Pi_h]\|.\end{aligned} \tag{4.4.1.31}$$

As to the first term in (4.4.1.31), we have

$$\|[F^* - F_h^*]B^*R(\lambda_0, A^*)\|$$

$$\begin{cases} \to 0 \text{ as } h \downarrow 0, & \text{under assumption (4.4.1.13);} \quad (4.4.1.32)\\ \le C\, h^{s(1-\gamma)\theta},\ \theta < 1, & \text{under assumption (4.4.1.19).} \quad (4.4.1.33)\end{cases}$$

As to the second term in (4.4.1.31), we have the following estimate:

$$\|F_h^*[B^*R(\lambda_0, A^*) - B_h^*R(\lambda_0, A_h^*)\Pi_h]\| \le \text{ const } h^{s(1-\gamma)\theta} \to 0 \text{ as } h \downarrow 0, \quad \theta < 1. \tag{4.4.1.34}$$

In fact, (4.4.1.34) is obtained by recalling the assumption $\|F_h\| \le$ const, and by taking the Laplace transform at $\lambda_0 > \omega_0 + \epsilon$ of estimate (3.5) in t of Lemma 4.3.1(v), with

$\theta < 1$, $\gamma < 1$, hence with $\theta + (1-\theta)\gamma < 1$. Combining (4.4.1.32), or (4.4.1.33), with (4.4.1.34) in (4.4.1.31), we obtain from (4.4.1.30),

$$\|(2)\| \begin{cases} \to 0 \text{ as } h \downarrow 0, & \text{under assumption (4.4.1.13);} \qquad (4.4.1.35) \\ \le C\, h^{s(1-\gamma)\theta}, \theta < 1, & \text{under assumption (4.4.1.19).} \qquad (4.4.1.36) \end{cases}$$

Then, since $s(1-\gamma)\theta = \epsilon_o < s$, the desired conclusion (4.4.1.18) [respectively (4.4.1.20)] is obtained from (4.4.1.27) and (4.4.1.35) [respectively (4.4.1.36)] under assumption (4.4.1.13) [respectively (4.4.1.19)].

Remark 4.4.1.5 The above proof shows (see particularly (4.4.1.32)) that if (4.4.1.13) is replaced by $\|F_h\Pi_h - F\| = \|F^* - F_h^*\| \to 0$ in the uniform norm, then conclusion (4.4.1.18) becomes

$$\left\|R(\lambda_0, A_F) - R\left(\lambda_0, A_{h,F_h}\right)\Pi_h\right\|_{\mathcal{L}(Y)} \le C[h^{\epsilon_o} + \|F_h - F\|_{\mathcal{L}(U;Y)}] \to 0 \text{ as } h \downarrow 0,$$
$$\text{for all } \epsilon_o < s(1-\gamma). \qquad (4.4.1.37)$$

(ii) The statement for any other $\lambda \in \rho(A_F)$ follows now from standard results [Kato, 1966, Remark 4.3.13, p. 211, Probl. 4.3.14, p. 212].

(iii) Part (iii), Eqn. (4.4.1.21) is a consequence of the joint continuity of the resolvent $R(\lambda, A_{h,F_h})$ in both arguments [Kato, 1966, Theorem 3.15, p. 212]. □

Continuing with the proof of Theorem 4.4.1.2, we return to Theorem 4.4.1.1(i), Eqn. (4.4.1.4): Given $1 > \delta > 0$, there exist r_δ, $h_\delta > 0$ such that for all $0 < h < h_\delta$,

$$\left[\Sigma^c_{\text{app}}(A) \cap \{|\lambda| \ge r_\delta\}\right] \subset \rho\left(A_{h,F_h}\right) = \text{ resolvent set of } A_{h,F_h}, \qquad (4.4.1.38)$$

where $\rho(\cdot)$ denotes the resolvent set; see Figure 4.4.

We next refine the statement (4.4.1.38) with the much more precise

Lemma 4.4.1.4 *Assume the hypotheses of Theorem 4.4.1.2. For any $\epsilon' > 0$ there exists $h_{\epsilon'} > 0$ such that*

$$\sup \operatorname{Re} \sigma\left(A_{h,F_h}\right) \le -\omega_F + \epsilon', \quad 0 < h \le h_{\epsilon'}, \qquad (4.4.1.39)$$

where ϵ' may be taken to be 0 if $-\omega_F \in \rho(A_F)$.

Proof. With reference to Figure 4.5, consider the compact set $\mathcal{K}$ in $\mathbb{C}$ enclosed by the following contour: the vertical segment $\operatorname{Re}\lambda = -\omega_F + \epsilon'$ comprised between the points P_1 and P_2 of intersection with the circle of radius r_δ (obtained in (4.4.1.4), which without loss of generality may be taken sufficiently large) centered at the origin and the arc P_1P_2 of said circle in $\Sigma^c_{\text{app}}(A)$. By construction, $\mathcal{K} \subset \rho(A_F) =$ resolvent set of A_F. We shall now show that

$$\mathcal{K} \subset \rho\left(A_{h,F_h}\right), \quad \forall h,\ 0 < h \le h_{\epsilon'}. \qquad (4.4.1.40)$$

Once (4.4.1.40) is shown, we return to (4.4.1.38) and obtain (4.4.1.39), thereby establishing Lemma 4.4.1.4. To obtain (4.4.1.40), we shall invoke perturbation theory

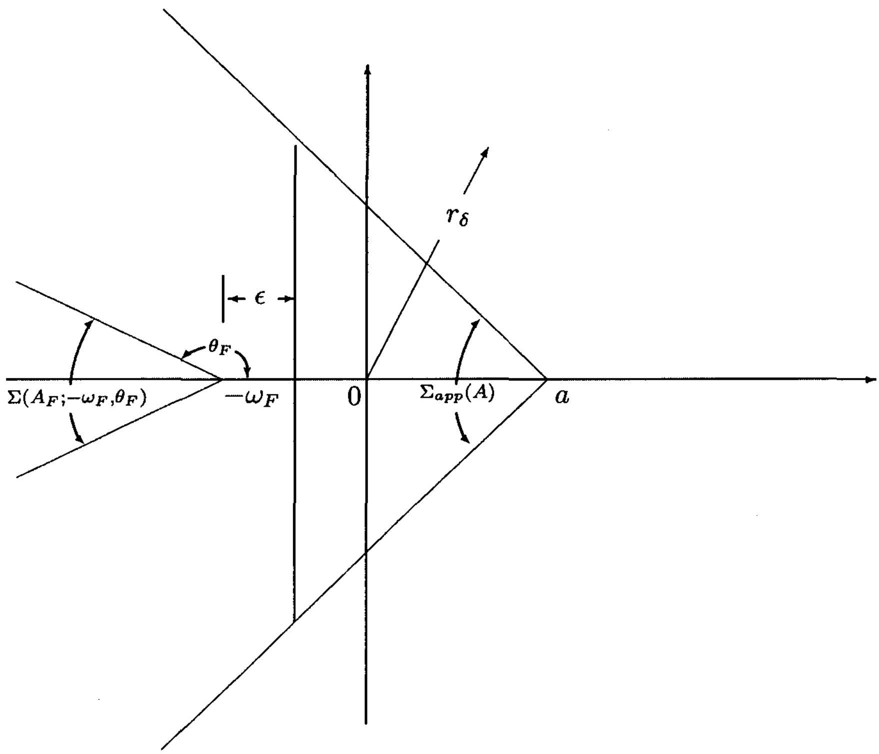


Figure 4.4

with respect to "generalized convergence" as in [Kato, 1966, pp. 206–8]. We proceed along two steps.

First, we show that: A_{h,F_h} converges to A_F in the generalized sense, that is, (in Kato's notation), the gap $\hat{\delta}(A_{h,F_h}, A_F) \to 0$ as $h \downarrow 0$, where $\hat{\delta}$ is the gap between the two closed operators A_{h,F_h} and A_F. Indeed, by virtue of the uniform convergence (4.4.1.18) of Lemma 4.4.1.3, the following three assertions hold true, as $h \downarrow 0$:

(i) $R(\lambda_0, A_{h,F_h})$ converges to $R(\lambda_0, A_F)$ in the generalized sense;
(ii) $[\lambda_0 I - A_{h,F_h}]$ converges to $[\lambda_0 I - A_F]$ in the generalized sense; and
(iii) finally, A_{h,F_h} converges to A_F in the generalized sense; that is, $\hat{\delta}(A_{h,F_h}, A_F) \to 0$ as $h \downarrow 0$.

For the validity of these three assertions, we invoke parts (a), (b), and (c) of [Kato, 1966, Theorem 2.23, p. 206].

Second, having established that $\hat{\delta}(A_{h,F_h}, A_F) \to 0$ as $h \downarrow 0$, we next apply to A_F the perturbation result as in [Kato, 1966, Theorem 3.1, p. 208] to conclude that (4.4.1.40) follows, as desired, since $\mathcal{K} \subset \rho(A_F)$. Lemma 4.4.1.4 is proved. □

We are now in a position to complete the proof of Theorem 4.4.1.2.

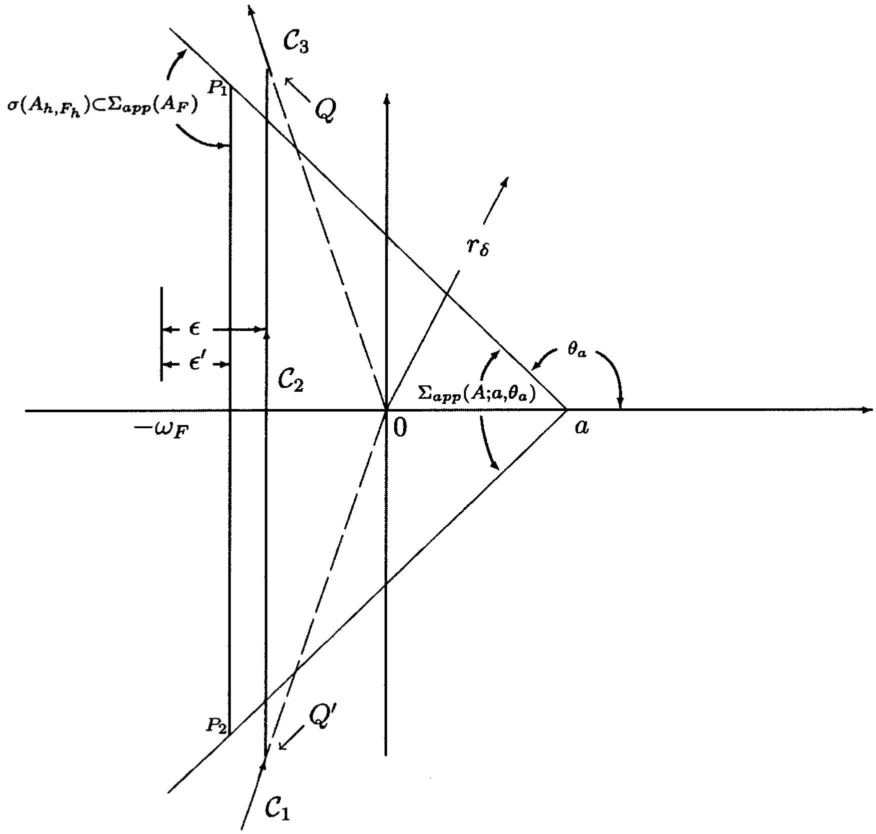


Figure 4.5

(i) First, we return to statement (4.4.1.38). Without loss of generality, we can always redefine $\Sigma_{\text{app}}(A, a; \theta_a)$ and make it larger, if necessary, by taking a new, smaller $\theta_a (> \pi/2)$ in such a way that the new rays $a + \rho e^{\pm i\theta_a}$ meet the points P_1 and P_2 on the circle of radius r_δ; see Figure 4.5. With this understanding, then statement (4.4.1.38) and Lemma 4.4.1.4 taken together say that: Given any $\epsilon' > 0$, there exists $h_{\epsilon'} > 0$ such that for all $0 < h \leq h_{\epsilon'} \leq h_a$,

$$\sigma\left(A_{h,F_h}\right) \subset \Sigma_{\text{app}}(A) \cap \{\text{Re}\,\lambda \leq -\omega_F + \epsilon'\} \equiv \Sigma_{\text{app}}(A_F) \quad \text{(independent of } h\text{)}, \tag{4.4.1.41}$$

and where, by (4.4.1.14), the right-hand side of (4.4.1.41) defines $\Sigma_{\text{app}}(A_F)$. Thus, part (i), Eqn. (4.4.1.15) of Theorem 4.4.1.2 is proved. Moreover, recalling (4.4.1.21), we can extend the estimate of Theorem 4.1.1(ii) to all of $\Sigma^c_{\text{app}}(A_F)$ so defined, by modifying, if necessary, the constant. We thus obtain

$$\left\| R\left(\lambda, A_{h,F_h}\right)\right\|_{\mathcal{L}(Y)} \leq \frac{\text{const}}{|\lambda - a|}, \quad \forall \lambda \in \Sigma^c_{\text{app}}(A_F),\ h \text{ sufficiently small.} \tag{4.4.1.42}$$

(ii) To prove part (ii) of Theorem 4.4.1.2, we now use a standard argument as in, say [Kato, 1966, p. 488]. For $t > 0$, we can write

$$2\pi i\, e^{A_{h,F_h} t} \int_{\mathcal{C}} e^{\lambda t} R\left(\lambda, A_{h,F_h}\right) d\lambda, \quad t > 0, \tag{4.4.1.43}$$

where $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ is a path entirely in $\Sigma^c_{\text{app}}(A_F)$ – hence in $\rho(A_{h,F_h})$ – defined as follows. Choose $\epsilon > 0$ so that the line Re $\lambda = -\omega_F + \epsilon$ is entirely in $\Sigma^c_{\text{app}}(A_F)$ (possible by Lemma 4.4.1.4), and let Q (top) and Q' (bottom) be the points of its intersection with the circle of radius $OP_1 = OP_2$. Let $r =$ ray OQ and $r' =$ ray OQ'. Then, $\mathcal{C}_2 =$ segment $Q'Q$; $\mathcal{C}_3 =$ part of ray r from Q to ∞; $\mathcal{C}_1 =$ part of ray r' from Q' to ∞. The orientation of $\mathcal{C}$ is chosen so that Im λ increases from $-\infty$ to ∞; see Figure 4.5. In computing (4.4.1.33) for say, $0 < t \leq 1$, we perform, as usual [Kato, 1966, p. 489; Ladas, Lakshmikantham, 1972, p. 50], a change of variable $\lambda' = \lambda t$, which sends $\mathcal{C}$ into $\mathcal{C}' = \mathcal{C}t$. However, by Cauchy's theorem the new contour $\mathcal{C}'$ can be deformed back into the original one $\mathcal{C}$, so that the path of integration is actually independent on $0 < t \leq 1$. By employing (4.4.1.42) along $\mathcal{C}$ in (4.4.1.33), we readily obtain a uniform bound for $\exp(A_{h,F_h} t)$:

$$\begin{aligned} 2\pi \| e^{A_{h,F_h} t} \|_{\mathcal{L}(Y)} &= \left\| \int_{\mathcal{C}} e^{\lambda'} R\left(\frac{\lambda'}{t}, A_{h,F_h}\right) \frac{d\lambda'}{t} \right\|_{\mathcal{L}(Y)} \\ &\leq C \int_{\mathcal{C}} |e^{\lambda'}| \frac{t}{|\lambda'|} \frac{|d\lambda'|}{t} < \infty, \quad 0 < t \leq 1. \end{aligned} \tag{4.4.1.44}$$

We now estimate for $t \geq 1$. On the finite segment $\mathcal{C}_2$ where Re $\lambda \equiv -\omega_F + \epsilon$, we plainly have

$$\left\| \int_{\mathcal{C}_2} e^{\lambda t} R\left(\lambda, A_{h,F_h}\right) d\lambda \right\|_{\mathcal{L}(Y)} \leq C\, e^{(-\omega_F + \epsilon)t}, \quad t \geq 1, \tag{4.4.1.45}$$

while in $\mathcal{C}_3$ where Re $\lambda = -\beta < 0$, we have, by (4.4.1.42),

$$\left\| \int_{\mathcal{C}_3} e^{\lambda t} R\left(\lambda, A_{h,F_h}\right) d\lambda \right\|_{\mathcal{L}(Y)} \leq \int_{\omega_F - \epsilon}^{\infty} e^{-\beta t} \frac{C}{\beta}\, d\beta \leq \frac{C}{(\omega_F - \epsilon)} e^{(-\omega_F + \epsilon)t}, \quad t \geq 1. \tag{4.4.1.46}$$

A similar estimate is obtained on $\mathcal{C}_1$ for $t \geq 1$. Then this, along with (4.4.1.45) and (4.4.1.46), provide the desired estimate (4.4.1.16) of part (ii) of Theorem 4.4.1.2, by use of (4.4.1.43) for $t \geq 1$ and of (4.4.1.44) for $0 < t \leq 1$.

Part (ii), Eqn. (4.4.1.16), then implies part (iii), Eqn. (4.4.1.7), for $\theta = 0$ by the usual Laplace transform. The cases $0 < \theta \leq 1$ are then obtained exactly as in Step 3 of the proof of Theorem 4.4.1.1. Namely, the case $\theta = 1$ is reduced to the case $\theta = 0$ via $R(\lambda, A_{h,F_h}) A_{h,F_h} = \lambda R(\lambda, A_{h,F_h}) - I$, whereas the cases $0 < \theta < 1$ are obtained by interpolation. Theorem 4.4.1.2 is fully proved. $\square$

4.4.1.3 Uniform Bounds on $B_h^* R(\,\cdot\,; A_{h,F_h}^*)$ and on $B_h^* e^{A_{h,F_h}^* t}$

The main ingredients of the proof of the next result are already contained in the preceding subsections.

Corollary 4.4.1.4 *Assume (A.1) through (A.6), as well as $\|F_h\|_{\mathcal{L}(V_h;U)} \leq$ const, as in Theorem 4.4.1.1. Then:*

(i) Given $1 > \delta > 0$, there exist $r_\delta > 0$ and $0 < h_\delta \leq h_a$ such that

$$\left\| B_h^* R(\bar{\lambda}, A_{h,F_h}^*) \right\|_{\mathcal{L}(Y;U)} \leq \frac{1}{1-\delta} \frac{C}{|\lambda - a|^{1-\gamma}}, \quad \forall h \leq h_\delta,$$
$$\forall \lambda \in \Sigma_{\text{app}}^c(A), \quad |\lambda| \geq r_\delta. \tag{4.4.1.47}$$

(ii) For any compact set $\mathcal{K}$ in $\bigcap_h \{\rho(A_{h,F_h}^*)\}$ we have

$$\sup_{\mu \in \mathcal{K}; h} \left\| B_h^* R(\mu, A_{h,F_h}^*) \right\|_{\mathcal{L}(Y;U)} = C_{\mathcal{K},h_a} < \infty. \tag{4.4.1.48}$$

(iii) Assume the hypotheses of Theorem 4.4.1.2. Then

$$\| B_h^* e^{A_{h,F_h}^* t} \|_{\mathcal{L}(Y;U)} \leq C \frac{e^{(-\omega_F + \epsilon)t}}{t^\gamma}, \quad t > 0. \tag{4.4.1.49}$$

Proof. (i) We return to (4.4.1.10), take adjoints, and obtain

$$B_h^* R\left(\bar{\lambda}, A_{h,F_h}^*\right) = B_h^* R(\bar{\lambda}, A_h^*)\{[I - R(\lambda, A_h) B_h F_h]^{-1}\}^*. \tag{4.4.1.50}$$

Recalling now estimate (4.4.1.8) for $B_h^* R(\,\cdot\,, A_h^*)$ and estimate (4.4.1.9) for the inverse term $[\ ]^{-1}$ (under the assumptions of Theorem 4.4.1.1), we readily obtain (4.4.1.47) from (4.4.1.50).

(ii) We use the first resolvent equation. Let λ_0 be a fixed point where estimate (4.4.1.47) holds true. Then

$$B_h^* R\left(\mu, A_{h,F_h}^*\right) = B_h^* R\left(\lambda_0, A_{h,F_h}^*\right)$$
$$+ (\lambda_0 - \mu) B_h^* R\left(\lambda_0, A_{h,F_h}^*\right) R\left(\mu, A_{h,F_h}^*\right). \tag{4.4.1.51}$$

Using now estimate (4.4.1.47) at λ_0, as well as a bound like (4.4.1.21), we readily obtain (4.4.1.48) from (4.4.1.51).

(iii) Under the stronger assumptions of Theorem 4.4.1.2, we have $\Sigma_{\text{app}}(A_F)$ defined by (4.4.1.14), and we may proceed as in the last part of the proof of Theorem 4.4.1.2 (following Lemma 4.4.1.4). From (4.4.1.43) we obtain

$$2\pi \| B_h^* e^{A_{h,F_h}^* t} \|_{\mathcal{L}(Y;U)} = \left\| \int_C e^{\lambda t} B_h^* R\left(\lambda, A_{h,F_h}^*\right) d\lambda \right\|_{\mathcal{L}(Y;U)}, \tag{4.4.1.52}$$

where C is the same path defined below (4.4.1.43). For $0 < t \leq 1$, we estimate (4.4.1.52) as in (4.4.1.44), except that now we can only use (4.4.1.47) and (4.4.1.48), instead of (4.4.1.42). We obtain, with $\lambda' = \lambda t$ as in (4.4.1.44),

$$2\pi \|B_h^* e^{A_{h,F_h}^* t}\|_{\mathcal{L}(Y;U)} = \left\| \int_C e^{\lambda'} B_h^* R\left(\frac{\lambda'}{t}, A_{h,F_h}^*\right) \frac{d\lambda'}{t} \right\|_{\mathcal{L}(Y;U)}$$
$$\leq C \int_C |e^{\lambda'}| \left(\frac{t}{\lambda'}\right)^{1-\gamma} \frac{d\lambda'}{t} = \frac{c}{t^\gamma}, \quad 0 < t \leq 1, \qquad (4.4.1.53)$$

after using (4.4.1.47) and (4.4.1.48) in the last step. To estimate (4.4.1.52) for $t \geq 1$, we obtain, by (4.4.1.48),

$$\left\| \int_{C_2} e^{\lambda t} B_h^* R\left(\lambda, A_{h,F_h}^*\right) d\lambda \right\|_{\mathcal{L}(Y;U)} \leq C e^{(-\omega_F + \epsilon)t}, \quad t \geq 1. \qquad (4.4.1.54)$$

Moreover, still for $t \geq 1$, we obtain, by (4.4.1.47) on the ray C_3 where $\operatorname{Re}\lambda = -\beta < 0$,

$$\left\| \int_{C_3} e^{\lambda t} B_h^* R\left(\lambda, A_{h,F_h}^*\right) d\lambda \right\|_{\mathcal{L}(Y;U)} \leq C \int_{\omega_F - \epsilon}^{\infty} \frac{e^{-\beta t}}{\beta^{1-\gamma}} d\beta \leq \frac{C}{(\omega_F - \epsilon)^{1-\gamma}} \frac{e^{(-\omega_F + \epsilon)t}}{t}$$
$$\leq C \frac{e^{(-\omega_F + \epsilon)t}}{t^\gamma}, \quad 1 \leq t. \qquad (4.4.1.55)$$

A similar estimate is obtained on C_1. Then this, along with (4.4.1.55) and (4.4.1.54), provides the desired estimate (4.4.1.49), by use of (4.4.1.52) for $t \geq 1$ and of (4.4.1.53) for $0 < t \leq 1$. $\square$

4.4.2 Uniform Analyticity and Detectability of the Generators $A_{h,K} = A_h + \Pi_h K R \Pi_h$; L_2-Stability of $L_{h,K} B_h$ and $L_{h,K} \Pi_h K$

Throughout this subsection, we consider the operator

$$A_K = A + KR, \quad Y \supset \mathcal{D}(A_K) \to Y, \qquad (4.4.2.1)$$

with $K \in \mathcal{L}(Z; Y)$ satisfying the detectability condition (DC) = (4.1.1.9), so that

$$\|e^{A_K t}\|_{\mathcal{L}(Y)} \leq M_K e^{-\omega_K t}, \quad \text{for some } \omega_K > 0, \ M_K \geq 1. \qquad (4.4.2.2)$$

Moreover, we assume throughout that $R(\lambda_0, A)KR$: compact $Y \to Y$, which is hypotheses (4.1.4.2) of Theorem 4.1.4.1. We take the following approximations:

$$A_{h,K} = A_h + \Pi_h K R \Pi_h = A_h + K_h R_h : V_h \to V_h, \qquad (4.4.2.3a)$$
$$R_h = R\Pi_h \to \text{ strongly}; \qquad K_h = \Pi_h K \to K \text{ strongly}. \qquad (4.4.2.3b)$$

Notice that the operator $A_{h,K}$ in (4.4.2.3) corresponds to the operator A_{h,F_h} in (4.4.1.3) under the following law of correspondence:

$$\{A_{h,K}, K_h, R_h\} \to \left\{A_{h,F_h}, B_h, F_h\right\}, \text{ respectively.} \qquad (4.4.2.4)$$

We want to apply Theorems 4.4.1.1 and 4.4.1.2 and Corollary 4.4.1.4 to $A_{h,K}$. To this end, we check that: (i) assumption (DC) in (4.4.2.2) corresponds to assumption (SC) in (4.4.1.12) for the continuous problems; (ii) $\|R_h\| \leq$ const, uniformly, is the counterpart of $\|F_h\| \leq$ const in Section 4.4.1; and moreover, (iii) that the counterpart of assumption (4.1.13) in Theorem 4.4.1.2 requires now that $R(\lambda_0, A) K R(\Pi_h - I) \to 0$ in the uniform norm of $\mathcal{L}(Y)$. This latter convergence holds indeed true, by invoking the compactness assumption (1.4.2) recalled above on $R^* K^* R (\lambda_0, A^*)$, whereby then the equivalent condition $(\Pi_h - I) R^* K^* R(\lambda_0, A^*) \to 0$ in the uniform norm of $\mathcal{L}(Y)$ is ascertained [Anselone, 1971, p. 8; Kato, 1966, p. 151]. Finally, (iv) $\{K_h\}$ must satisfy the counterparts of assumptions (A.3)–(A.6) for B_h.

In conclusion, we can apply Theorems 4.4.1.1 and 4.4.1.2 and Corollary 4.4.1.4 to $A_{h,K}$ and obtain

Theorem 4.4.2.1 *Assume (A.1)–(A.6) and the (DC) = (4.1.1.9) [or (4.4.2.2)]. Then: The semigroups $e^{A_{h,K}t}$ are uniformly analytic and, moreover, uniformly exponentially stable in h. Given $\epsilon > 0$ there exist $h_\epsilon > 0$ and a three-sided sector*

$$\Sigma_{\text{app}}(A_K) \equiv \Sigma_{\text{app}}(A) \cap \{\operatorname{Re} \lambda \leq -\omega_K + \epsilon\} \tag{4.4.2.5}$$

such that

(i)

$$\sigma(A_{h,K}) \subset \Sigma_{\text{app}}(A_K), \tag{4.4.2.6}$$

(ii)

$$\|e^{A_{h,K}t}\|_{\mathcal{L}(Y)} \leq M_{K,\epsilon} e^{(-\omega_K+\epsilon)t}, \quad t > 0. \tag{4.4.2.7}$$

(iii) Since $A_{h,K}$ are finite-dimensional operators,

$$\left\| R(\lambda, A_{h,K}) A_{h,K}^{\theta} \right\|_{\mathcal{L}(V_h)} \leq \frac{C}{|\lambda + \omega_K - \epsilon|^{1-\theta}}, \quad \lambda \in \Sigma_{\text{app}}^c(A_K), \quad 0 \leq \theta \leq 1. \tag{4.4.2.8}$$

(iv) Given $1 > \delta > 0$, there exist $r_\delta > 0$ and $0 < h_\delta \leq h_a$ such that

$$\|B_h^* R(\lambda, A_{h,K}^*)\|_{\mathcal{L}(Y;U)} \leq \frac{1}{1-\delta} \frac{C}{|\lambda - a|^{1-\gamma}},$$
$$\forall h \leq h_\delta, \quad \forall \lambda \in \Sigma_{\text{app}}^c(A), \quad |\lambda| \geq r_\delta. \tag{4.4.2.9}$$

(v) For any compact set $\mathcal{K}$ in $\Sigma_{\text{app}}^c(A_K)$ we have

$$\sup_{\mu \in \mathcal{K}; h} \|B_h^* R(\mu, A_{h,K}^*)\|_{\mathcal{L}(Y;U)} \leq C_{\mathcal{K},h_a} < \infty; \tag{4.4.2.10}$$

(vi)

$$\|B_h^* e^{A_{h,K}^* t}\|_{\mathcal{L}(Y;U)} \leq C \frac{e^{(-\omega_K+\epsilon)t}}{t^\gamma}, \quad t > 0. \tag{4.4.2.11}$$

As a corollary to Theorem 4.4.2.1(iv), (v), we shall derive a stability result involving the approximating operators

$$(L_{h,K}f_h)(t) \equiv \int_0^t e^{A_{h,K}(t-\tau)} f_h(\tau)\, d\tau : L_2(0,\infty; V_h) \to \text{ itself.} \tag{4.4.2.12}$$

Corollary 4.4.2.2 *Assume (A.1)–(A.6) and the (DC) = (4.1.1.9) [or (4.4.2.2)]. With reference to (4.4.2.12), we have*

(i)

$$\|L_{h,K}\, B_h u\|_{L_2(0,\infty;Y)} \leq C\|u\|_{L_2(0,\infty;U)}; \tag{4.4.2.13}$$

(ii)

$$\|L_{h,K}\Pi_h K z\|_{L_2(0,\infty;Y)} \leq C\|z\|_{L_2(0,\infty;Z)}. \tag{4.4.2.14}$$

Proof. (i) One may use (4.4.2.11), where the bound is an $L_1(0,\infty)$-function, and Young's inequality in the corresponding integral in (4.4.2.12). One may also use Parseval's inequality, after noticing that

$$\|R(\lambda, A_{h,K})B_h\hat{u}(\lambda)\|_Y \leq C\|\hat{u}(\lambda)\|_U \tag{4.4.2.15}$$

holds true for all $\{\lambda : \text{Re}\,\lambda = 0\} \subset \Sigma^c_{\text{app}}(A_K)$, by virtue of (4.4.2.9) for large imaginary part and of (4.4.2.10) on a finite segment containing the origin. Similar considerations apply to (ii). □

4.4.3 Uniform Exponential Stability of the Feedback Semigroup $\exp(A_{h,P_h}t)$

Let P be the Riccati operator asserted by Theorems 2.2.1 and 2.2.2 of Chapter 2 and recalled in Section 4.1.1, below (4.1.1.9). With reference to the approximating optimal control problem (4.1.3.1), (4.1.3.2), we may now say – on the basis of the results of Sections 4.4.1 and 4.4.2 – that the approximating dynamics (4.1.3.1) is stabilizable and detectable, in fact uniformly in h. Thus, it is a standard result (contained in Theorems 2.2.1 and 2.2.2 of Chapter 2, when interpreted on V_h) that there exists a unique, nonnegative, self-adjoint, Riccati (approximating) operator P_h associated with (4.1.3.1), (4.1.3.2), solution of the ARE$_h$ in (4.1.3.3). The goal of this subsection is to prove that the operator

$$A_{h,P_h} \equiv A_h - B_h B_h^* P_h : V_h \to V_h \tag{4.4.3.1}$$

corresponding to such P_h satisfies the uniform exponential stability condition (4.1.4.5) of Theorem 4.1.4.1. Comparing (4.4.3.1) with the optimal solution of the approximating problem (4.1.3.1), we see that the optimal pair is

$$\Phi_h(t)x = y_h^0(t;x) = e^{A_{h,P_h}t}x; \qquad u_h^0(t;x) = -B_h^* P_h e^{A_{h,P_h}t}x, \quad x \in V_h. \tag{4.4.3.2}$$

Theorem 4.4.3.1 *(Uniform stability of $e^{A_{h,P_h}t}$) Under the sets of assumptions I and II of Theorem 4.1.4.1, there exist $\tilde{\omega}_P > 0$, $\bar{M}_P \geq 1$ such that*

$$\|\Phi_h(t)\|_{\mathcal{L}(Y)} = \|e^{A_{h,P_h}t}\|_{\mathcal{L}(Y)} \leq \bar{M}_P e^{-\tilde{\omega}_P t}, \quad t \geq 0, \tag{4.4.3.3}$$

thereby proving (4.1.4.5) of Theorem 4.1.4.1.

Proof of Theorem 4.4.3.1. The first step is a consequence of Theorem 4.4.1.2. In the steps below we shall not repeat the assumptions of Theorem 4.4.3.1.

Step 1. Lemma 4.4.3.2 *We have*

$$\|P_h\|_{\mathcal{L}(V_h)} \leq \text{const}, \quad \textit{uniformly in } h. \quad \square \tag{4.4.3.4}$$

Proof of Lemma 4.4.3.2. We first note that the assumption of uniform convergence (4.4.1.13) in Theorem 4.4.1.2 holds true for the choice $F_h = F\Pi_h$, by virtue of the compactness assumption (4.1.4.3), or else (4.1.4.4) [Anselone, 1971, p. 8; Kato, 1966, p. 151]. Thus, Theorem 4.4.1.2 applies and Eqn. (4.1.16) of part (ii) implies that $\exp\{(A_{h,F_h})t\} = \exp\{(A_h + B_h F_h)t\}$ is uniformly stable. For the approximating optimal control problem (4.1.3.1), (4.1.3.2) on V_h with optimal pair given by (4.4.3.3) and initial point $x \in V_h$, the feedback control $F_h e^{A_{h,F_h}t}x$ and corresponding solution $e^{A_{h,F_h}t}x$ form a competing pair; see (4.4.1.3). Thus, by (4.2.2.10) we get

$$\begin{aligned} J(u_h^0, y_h^0) = (P_h x, x)_Y &= \int_0^\infty \left[\left\|u_h^0(t;x)\right\|_U^2 + \left\|R_h y_h^0(t;x)\right\|_Z^2\right] dt \\ &\leq \int_0^\infty \|F_h e^{A_{h,F_h}t}x\|_U^2 + \|R_h e^{A_{h,F_h}t}x\|_Z^2\, dt \\ &\leq C \int_0^\infty e^{2(-\omega_F+\epsilon)t}\, dt\|x\|^2 \leq C\|x\|_Y^2, \end{aligned} \tag{4.4.3.5}$$

where in the last step we have invoked (4.4.1.18), as well as the uniform boundedness on R_h (see (4.4.2.3)) and $F_h = F\Pi_h$. Since P_h is nonnegative, self-adjoint, (4.4.3.5) yields $\|P_h\|_{\mathcal{L}(V_h)} = \sup(P_h x, x)_Y \leq C$, over all $x \in V_h$ with $\|x\| \leq 1$. $\square$

Step 2. Lemma 4.4.3.3 *We have*

$$\int_0^\infty \left\|e^{A_{h,P_h}t}x\right\|_Y^2 dt \leq C\|x\|_Y^2, \quad x \in Y. \tag{4.4.3.6}$$

Proof of Lemma 4.4.3.3. Suppose first that $R^*R \geq rI$, $r > 0$ as postulated in (4.1.4.1). Then for all h sufficiently small, $((R_h^* R_h)^{\frac{1}{2}}x, x)_Y \geq (r - \epsilon)\|x\|^2$ by (4.4.2.3b), and then (4.4.3.6) is a direct consequence of (4.4.3.5) via (4.4.3.2).

Otherwise, we shall use, as usual, the more general detectability assumption (DC) = (4.1.1.9), which leads to the uniform estimate (4.4.2.7). Writing (4.4.2.3a) and

(4.4.3.1),

$$A_{h,P_h} = A_{h,K} - \Pi_h K R \Pi_h - B_h B_h^* P_h, \tag{4.4.3.7}$$

and recalling from (4.4.3.2) that $\dot{y}_h^0 = A_{h,P_h} y_h^0$ is the approximating optimal problem, we have by (4.4.3.7)

$$\begin{aligned} y_h^0(t;x) = e^{A_{h,P_h}t}x &= e^{A_{h,K}t}x - \int_0^t e^{A_{h,K}(t-\tau)} \Pi_h K R \Pi_h y_h^0(\tau;x)\,d\tau \\ &\quad - \int_0^t e^{A_{h,K}(t-\tau)} B_h B_h^* P_h y_h^0(\tau;x)\,d\tau \\ &= e^{A_{h,K}t}x - \left\{L_{h,K} \Pi_h K R \Pi_h y_h^0(\cdot\,;x)\right\}(t) \\ &\quad - \left\{L_{h,K} B_h B_h^* P_h y_h^0(\cdot\,;x)\right\}(t), \end{aligned} \tag{4.4.3.8}$$

after recalling the operators $L_{h,K}$ in (4.4.2.12), whose regularity properties (4.4.2.13), (4.4.2.14) will now be invoked. In fact, by (4.4.2.13) we have the first step of

$$\begin{aligned} \left\|L_{h,K} \Pi_h K R \Pi_h y_h^0(\cdot\,;x)\right\|_{L_2(0,\infty;Y)} &\leq C \left\| R \Pi_h y_h^0(\cdot\,;x)\right\|_{L_2(0,\infty;Z)} \\ \text{(by (4.4.2.3b))} \qquad &\leq C \left\| y_h^0(\cdot\,;x)\right\|_{L_2(0,\infty;Y)} \\ \text{(by (4.4.3.2))} \qquad &= C \|e^{A_{h,P_h}\cdot} x\|_{L_2(0,\infty;Y)} \\ \text{(by (4.4.3.3))} \qquad &\leq C\|x\|_Y, \end{aligned} \tag{4.4.3.9}$$

as desired, upon using (4.4.2.3b) and (4.4.3.2), and (4.4.3.3). Similarly, by (4.4.2.13) we have the first step of

$$\begin{aligned} \left\|L_{h,K} B_h B_h^* P_h y_h^0(\cdot\,;x)\right\|_{L_2(0,\infty;Y)} &\leq C \left\| B_h^* P_h y_h^0(\cdot\,;x)\right\|_{L_2(0,\infty;U)} \\ \text{(by (4.4.3.2))} \qquad &\leq C \left\| u_h^0(\cdot\,;x)\right\|_{L_2(0,\infty;U)} \\ \text{(by (4.4.3.5))} \qquad &\leq C\|x\|_Y, \end{aligned} \tag{4.4.3.10}$$

as desired. Finally, using (4.4.3.10) and (4.4.3.11), as well as (4.4.2.7) in (4.4.3.9), we obtain (4.4.3.6), as desired. □

Remark 4.4.3.1 In the continuous (nondiscretized) case, a well-known result of Datko [1970] states that a s.c. semigroup $S(t)$ is exponentially stable if and only if $S(t)x \in L_2(0,\infty;Y)$, $\forall x \in Y$; see Theorem 4A.1 in the appendix. Thus, in the continuous case, the counterpart of (4.4.3.6) would imply the counterpart of (4.4.3.3). In the present discretized case where the exponential stability (4.3.6) uniform in h is sought, if we refer to Theorem 4A.2 in the appendix, we see that we need an additional preliminary result. This is given by Proposition 4.4.3.4 below.

Step 3. Proposition 4.4.3.4 *There are numbers $c > 0$ and $\alpha > 0$, independent of h, such that*

$$\|e^{A_{h,P_h}t}\|_{\mathcal{L}(V_h)} \leq ce^{at}, \quad t \geq 0. \tag{4.4.3.11}$$

In fact, we can take $\alpha = \omega$, where ω is defined by (4.1.1.2).

Proof of Proposition 4.4.3.4. We shall prove (4.3.11), in fact, with $\alpha = \omega$, by using a bootstrap argument, as in Chapter 2, Theorem 2.3.5.1, based on the following equations for the optimal pair:

$$e^{-\omega t}e^{A_{h,P_h}t}x_h = \hat{y}_h^0(t; x_h) = e^{-\hat{A}_h t}x_h + \left\{\hat{L}_h\hat{u}_h^0(\,\cdot\,; x)\right\}(t); \tag{4.4.3.12}$$

$$\hat{u}_h^0(t; x_h) = -\left\{\hat{L}_h^*[R^*R + 2\omega P_h]\hat{y}_h^0(\,\cdot\,; x_h)\right\}(t), \tag{4.4.3.13}$$

with $x_h \in V_h$ [see (4.2.2.11) and (4.2.2.9) in Section 4.2]. The bootstrap argument uses the following result, which is of interest only in the more demanding situation where $1/2 \leq \gamma < 1$, as already explained in Chapter 1, Orientation and Section 1.3, and Chapter 2, Notes. For the sake of convenience, we shall state explicitly the counterpart version of Chapter 2, Theorem 2.3.5.1 in the form needed here.

Lemma 4.4.3.5 *For the operators $\hat{L}_h$ and $\hat{L}_h^*$ defined by (4.2.2.3) and (4.2.2.4), we have*

(i)

$$\hat{L}_h : \textit{continuous } L_2(0, \infty; U) \to L_{r_1}(0, \infty; Y) \textit{ uniformly in } h \downarrow 0, \tag{4.4.3.14}$$

$$\textit{that is,} \quad \sup_{h>0} \|\hat{L}_h\|_{2,r_1} \leq \text{const},$$

where r_1 is an arbitrary number satisfying $r_1 < 2/(2\gamma - 1)$, where $2/(2\gamma - 1) > 2$ for $1/2 < \gamma < 1$; for $0 \leq \gamma \leq 1/2$, one can take $r_1 = \infty$.

(ii)

$$\hat{L}_h^* : \textit{continuous } L_{r_1}(0, \infty; Y) \to L_{r_2}(0, \infty; U), \textit{ uniformly in } h \downarrow 0, \tag{4.4.3.15}$$

$$\textit{that is,} \quad \sup_{h>0} \|\hat{L}_h^*\|_{r_1,r_2} \leq \text{const},$$

where r_1 is as in (i), and r_2 is any number satisfying $r_2 < 2/(4\gamma - 3)$, where $3/4 < \gamma < 1, 2/(4\gamma - 3) > r$; for $0 < \gamma < 3/4$, we can take $r_2 = \infty$.

(iii) With $p > 1/(1 - \gamma)$,

$$\hat{L}_h : \textit{continuous } L_p(0, \infty; U) \to C([0, \infty]; Y), \textit{ uniformly in } h \downarrow 0, \tag{4.4.3.16}$$

$$\textit{that is,} \quad \sup_{h>0} \|\hat{L}_h\|_{p,\text{contin}} < \infty.$$

Proof of Lemma 4.4.3.5. As in Theorem 2.3.5.1 of Chapter 2, the proof is based on Young's inequality [Sadosky, 1979, p. 29]. By using inequality (4.3.4) of

Lemma 4.3.1, as well as $-\hat{A}_h = A_h - \omega I$, $-\hat{\omega} = \omega_0 + \epsilon - \omega$, from (4.1.1.2), (4.1.1.3), we have preliminarily

$$\|e^{-\hat{A}_h t} B_h\|_{\mathcal{L}(U;V_h)} = \|B_h^* e^{-\hat{A}_h t}\Pi_h\|_{\mathcal{L}(V_h;U)} = \mathcal{O}\left(\frac{e^{-\hat{\omega}t}}{t^\gamma}\right), \quad t > 0. \tag{4.4.3.17}$$

Thus, it follows from (4.2.2.3), respectively (4.2.2.4), via (4.4.3.17), that

$$\|(\hat{L}_h u)(t)\|_Y \leq C \int_0^t \frac{e^{-\hat{\omega}(t-\tau)}}{(t-\tau)^\gamma}\|u(\tau)\|_U \, d\tau; \tag{4.4.3.18}$$

$$\|(\hat{L}_h^* v)(t)\|_U \leq C \int_t^\infty \frac{e^{-\hat{\omega}(\tau-t)}}{(\tau-t)^\gamma}\|v(\tau)\|_Y \, d\tau. \tag{4.4.3.19}$$

Then parts (i) and (ii) follow immediately from (4.4.3.19) and (4.4.3.20), via Young's inequality. Part (iii) follows likewise with $1/r = 1/q + 1/p - 1 = 0$, where we have $\gamma q < 1$, and $1/q = 1 - 1/p \downarrow \gamma$ as $p \downarrow 1/(1-\gamma)$. The proof of Lemma 4.4.3.5 is complete. □

To complete the proof of Proposition 4.4.3.4, Eqn. (4.4.3.12), we start with $\hat{u}_h^0 \in L_2(0, \infty; U)$ and apply a bootstrap argument on (4.4.3.13), (4.4.3.14), using Lemma 4.4.3.5. As in Theorem 2.3.5.1 of Chapter 2, we first apply (4.4.3.14) to (4.4.3.12) to obtain $\hat{y}_h^0 \in L_{r_1}(0, \infty; Y)$ via also $e^{-\hat{A}_h t} = \mathcal{O}(e^{-\hat{\omega}t})$ by (4.1.2.2) with $\theta = 0$. Next, we apply (4.4.3.15) on (4.4.3.13) to obtain the improvement $\hat{u}_h^0 \in L_{r_2}(0, \infty; U)$; etc. After a finite number of iterations we obtain $\hat{u}_h^0 \in L_\infty(0, \infty; U)$ and $y_h^0 \in C([0, \infty]; Y)$ from which (4.4.3.11) follows from (4.4.3.12) with the constant $a = \omega$ in (4.1.1.2). □

Step 4 We have now achieved (i) the uniform (in h) $L_2(0, \infty; Y)$-property in (4.4.3.6), as well as the preliminary (ii) uniform exponential bound (4.4.3.11). As noted in Remark 4.4.3.1, we can now appeal to Theorem 4A.2 in the appendix, and we see that these two ingredients permit one to conclude that the uniform (in h) exponential stability (4.3.3) holds true. Theorem 4.4.3.1 is proved. □

4.4.4 Uniform Regularity of P_h

We recall that P_h is given, as described at the beginning of Section 4.4.3. In this section we shall show that P_h is uniformly bounded not only in $\mathcal{L}(Y)$, as already claimed by Eqn. (4.4.3.4) of Lemma 4.4.3.2, but in fact in a stronger norm. In particular, we shall prove statement (1.4.7) of the main Theorem 4.1.4.1.

Theorem 4.4.4.1 *Under the sets of assumptions I and II of Theorem 4.1.4.1, we have, uniformly in $h \downarrow 0$,*

(i)

$$\|(\hat{A}_h^*)^\theta P_h\|_{\mathcal{L}(Y)} \leq \text{const}_\theta, \quad 0 \leq \theta < 1; \tag{4.4.4.1}$$

(ii)

$$\|B_h^* P_h\|_{\mathcal{L}(Y;U)} \leq \text{const}; \tag{4.4.4.2}$$

(iii)

$$\left\|(\hat{A}_h^*)^\theta \; P_h \; \hat{A}_h^\theta\right\|_{\mathcal{L}(Y)} \leq \text{const}_\theta, \quad 0 \leq \theta < \frac{1}{2}. \tag{4.4.4.3}$$

Proof of Theorem 4.4.4.1. (i) We return to identity (4.2.2.10) for P_h and obtain, with $x \in V_h$,

$$(\hat{A}_h^*)^\theta P_h x = \int_0^\infty (\hat{A}_h^*)^\theta e^{-\hat{A}_h^* t} \Pi_h [R^* R + 2\omega P_h] \hat{\Phi}_h(t) x \, dt. \tag{4.4.4.4}$$

Invoking the preliminary estimate (4.4.3.4) of Lemma 4.4.3.2 and the analyticity assumption (A.1) = (4.1.2.2) along with (4.1.1.2), we obtain

$$\left\|(\hat{A}_h^*)^\theta P_h\right\|_{\mathcal{L}(Y)} \leq C \int_0^\infty \frac{e^{-\hat{\omega} t}}{t^\theta} \|\hat{\Phi}_h(t)\|_{\mathcal{L}(Y)} \, dt, \quad 0 \leq \theta < 1. \tag{4.4.4.5}$$

Then (4.4.4.1) of part (i) follows immediately from (4.4.4.5) via (4.2.2.5) and the uniform exponential bound (4.4.3.3) of Theorem 4.4.3.1 for $\Phi_h(t)$.

(ii) The proof of (4.4.4.2) is similar: From (4.2.2.10), with $x \in V_h$,

$$B_h^* P_h x = \int_0^\infty B_h^* e^{-\hat{A}_h^* t} \Pi_h [R^* R + 2\omega P_h] \hat{\Phi}_h(t) x \, dt, \tag{4.4.4.6}$$

and invoking estimates (4.3.4) of Lemma 4.3.1 and (4.3.4) of Lemma 4.4.3.2, and $-\hat{A}_h = A_h - \omega I$, $-\hat{\omega} = \omega_0 + \epsilon - \omega$, we obtain from (4.4.4.6)

$$\|B_h^* P_h\|_{\mathcal{L}(Y;U)} \leq C \int_0^\infty \frac{e^{-\hat{\omega} t}}{t^\gamma} \|\hat{\Phi}_h(t)\|_{\mathcal{L}(Y)} \, dt, \tag{4.4.4.7}$$

and (4.4.4.2) follows likewise via (4.2.2.5) and (4.4.3.3) for $\Phi_h(t)$.

(iii) Part (iii) follows from part (i) via self-adjoint calculus as in Chapter 1, Lemma 1.5.1.1 and Corollary 1.5.1.2. □

Corollary 4.4.4.2 *Assume the hypotheses of Theorem 4.4.4.1. If for some $\theta < 1$ we have*

$$\|(\hat{A}^*)^\theta x_h\|_Y \leq C \|(\hat{A}_h^*)^\theta x_h\|_Y, \quad \forall x_h \in V_h, \tag{4.4.4.8}$$

then it follows that

$$\|(\hat{A}^*)^\theta P_h\|_{\mathcal{L}(Y)} \leq \text{const}_\theta \tag{4.4.4.9}$$

and

$$\|(\hat{A}^*)^\theta P_h \hat{A}^\theta\|_{\mathcal{L}(Y)} \leq \text{const}_\theta, \quad 0 \leq \theta < \frac{1}{2}. \tag{4.4.4.10}$$

Proof. Equation (4.4.4.9) follows from (4.4.4.8) and (4.4.4.1). To obtain (4.4.4.10), we use (4.4.4.8) twice, along with $P_h = P_h^*$ and (4.4.4.3):

$$\begin{aligned}\|(\hat{A}^*)^\theta P_h \hat{A}^\theta\|_{\mathcal{L}(Y)} &\le C\|(\hat{A}_h^*)^\theta P_h \hat{A}^\theta\|_{\mathcal{L}(Y)} = C\|(\hat{A}^*)^\theta P_h \hat{A}_h^\theta\|_{\mathcal{L}(Y)} \\ &\le C\left\|(\hat{A}_h^*)^\theta P_h \hat{A}_h^\theta\right\|_{\mathcal{L}(Y)} \le \text{ const}_\theta,\end{aligned}$$

where in the last step the restriction $\theta < 1/2$ of (4.4.4.3) applies. □

Remark 4.4.4.1 Assumption (4.4.4.8) holds true with any $\theta \le 1/2$ for Galerkin approximations.

The operators A_{h,P_h} generate a family of semigroups $e^{A_{h,P_h}t}$ that are uniformly stable by Theorem 4.4.3.1. They are also uniformly analytic by the next corollary.

Corollary 4.4.4.3 *The operators $A_{h,P_h} = A_h - B_h B_h^* P_h$ in (4.4.3.1) generate a uniformly analytic family $e^{A_{h,P_h}t}$ of semigroups (in the sense of Theorem 4.4.1.1).*

Proof. We just apply Theorem 4.4.1.1 with $F_h = B_h^* P_h$, where then the assumption $\|F_h\|_{\mathcal{L}(Y;U)} \le$ const required by it holds true by the uniform estimate (4.4.4.2) of Theorem 4.4.4.1. □

4.5 Uniform Convergence $P_h\Pi_h \to P$ and $B_h^* P_h \Pi_h \to B^* P$

4.5.1 Uniform Convergence $P_h\Pi_h \to P$ of Riccati Operators

Theorem 4.5.1.1 *(Property (4.1.4.8) of main Theorem 4.1.4.1) Assume the sets of assumptions I and II of Theorem 4.1.4.1. For any $\epsilon_o < s(1-\gamma)$,*

$$\|P_h\Pi_h - P\|_{\mathcal{L}(Y)} \le C\, h^{\epsilon_o} \to 0 \text{ as } h \downarrow 0. \tag{4.5.1.1}$$

Proof.

Step 1 The following four operators with maximal domains will play a key role. The first and fourth, defined by Chapter 2, Eqn. (2.2.6) (and recalled in (1.1.15)) and by (4.4.3.1), respectively, refer to the optimal dynamics, continuous and discrete. The second and third are introduced here for the first time. They will define competitive dynamics:

$$A_P = A - BB^*P : Y \to Y; \tag{4.5.1.2}$$

$$A_{h,P} = A_h\Pi_h - B_h B^* P : Y \to Y; \tag{4.5.1.3}$$

$$A_{P_h} = A - BB_h^* P_h \Pi_h : Y \to Y; \tag{4.5.1.4}$$

$$A_{h,P_h} = A_h - B_h B_h^* P_h : V_h \to V_h. \tag{4.5.1.5}$$

Under the present assumptions, in particular the (DC) = (4.1.1.9), the semigroup generated by A_P is analytic and stable; see (4.1.1.15). As to the other operators, we have

Proposition 4.5.1.2 *Under the assumptions of Theorem 4.5.1.1, the semigroups generated by the operators $A_{h,P}$, A_{P_h}, A_{h,P_h} are both uniformly analytic (in the sense of Theorem 4.4.1.1) and uniformly stable.*

Proof. In the case of A_{h,P_h}, uniform analyticity was established in Corollary 4.4.4.3, while uniform stability was established in Theorem 4.4.3.1, Eqn. (4.3.3). The same properties then hold true for $A_{h,P}$ as a special case of the latter, where $F_h \equiv B^*P$. Next, uniform stability of $\exp(A_{P_h}t)$ follows from Remark 4.4.1.4: In fact, we already know that $e^{A_{h,P_h}t}$ is uniformly stable, and thus we take $F = F_h = B_h^* P_h$, so that the required assumption (4.4.1.13) holds true automatically. Finally, uniform analyticity $\exp(A_{P_h}t)$ follows from Theorem 4.4.1.1 with $A_h \equiv A$, $B_h \equiv B$, and $F_h \equiv B_h^* P_h$, the latter being uniformly bounded by (4.4.4.2), as required. □

Step 2. ***Proposition 4.5.1.3*** *Assume the hypotheses of Theorem 4.5.1.1. Let ϵ_o be the same number $\epsilon_o < s(1-\gamma)$ as in Lemma 4.4.1.3, Eqn. (4.4.1.20). Then, with reference to (4.5.1.2)–(4.5.1.5):*

$$\left\| A_P^{-1} - A_{h,P}^{-1} \right\|_{\mathcal{L}(Y)} \leq C\, h^{\epsilon_o} \to 0 \text{ as } h \to 0; \tag{4.5.1.6a}$$

$$\left\| A_{P_h}^{-1} - A_{h,P_h}^{-1} \right\|_{\mathcal{L}(Y)} \leq C\, h^{\epsilon_o} \to 0 \text{ as } h \to 0. \tag{4.5.1.7a}$$

More generally,

$$\|R(\lambda_0, A_P) - R(\lambda_0, A_{h,P})\|_{\mathcal{L}(Y)} \leq C_{\lambda_0} h^{\epsilon_o} \to 0 \text{ as } h \to 0; \tag{4.5.1.6b}$$

$$\|R(\lambda_0, A_{P_h}) - R(\lambda_0, A_{h,P_h})\|_{\mathcal{L}(Y)} \leq C_{\lambda_0} h^{\epsilon_o} \to 0 \text{ as } h \to 0, \tag{4.5.1.7b}$$

where, say, Re λ_0 *is sufficiently large.*

Proof. By use of the first resolvent equation (see below, at the end of this proof), it suffices to show the "b" equations (4.5.1.6b) and (4.5.1.7b). Then, in these cases, the desired bounds follow from Eqn. (4.4.1.20) of Lemma 4.4.1.3, where we note that the required assumption (4.4.1.19) holds true with $F = F_h = B^*P$ in the case of (4.5.1.6b) and with $F = F_h = B_h^* P_h$ in the case of (4.5.1.7b). In the latter situation, we remark that the fact that now F depends on h does not make any difference in the argument of Lemma 4.4.1.3 as long as $\|F_h\| \leq$ const, which is true by (4.4.4.2). Alternatively, we may invoke (4.4.1.37) in Remark 4.4.1.5 with $F \equiv F_h$ in both cases. Equations (4.5.1.6b) and (4.5.1.7b) are then proved. When λ_0 is replaced by $\lambda = 0$ as in Eqns. (4.5.1.6a) and (4.5.1.7a), we may either use the first resolvent equation (as in Remark 4.1.2.2) or invoke [Kato, 1966, Remark 3.13, p. 211 and Probl. 3.14, p. 212]. □

Step 3 By (4.1.1.13) and (4.2.2.10), we have, since $(P_h \Pi_h x, \Pi_h x)_Y = (P_h \Pi_h x, x)_Y$:

$$|([P_h \Pi_h - P]x, x)_Y| = \left| J\left(u_h^0(\,\cdot\,; \Pi_h x), y_h^0(\,\cdot\,; \Pi_h x)\right) - J(u^0(\,\cdot\,; x), y^0(\,\cdot\,; x))\right|. \tag{4.5.1.8}$$

Now, with x and h fixed, if $J(u_h^0, y_h^0) - J(u^0, y^0) > 0$, we introduce the competing pair

$$\tilde{u}_h(t;\Pi_h x) = -B^*Pe^{A_{h.P}t}\Pi_h x; \qquad \tilde{y}_h(t;\Pi_h x) = e^{A_{h.P}t}\Pi_h x, \tag{4.5.1.9}$$

for the approximating problem, after recalling (4.5.1.3), so that in this case (omitting initial conditions)

$$\left|J\left(u_h^0, y_h^0\right) - J(u^0, y^0)\right| \le J(\tilde{u}_h, \tilde{y}_h) - J(u^0, y^0). \tag{4.5.1.10}$$

Instead, if $J(u^0, y^0) - J(u_h^0, y_h^0) > 0$, we introduce the competing pair

$$\bar{u}_h(t;x) = -B_h^*P_h e^{A_{P_h}t}\Pi_h x; \qquad \bar{y}_h(t;x) = e^{A_{P_h}t}\Pi_h x, \tag{4.5.1.11}$$

for the continuous problem (see (4.5.1.4)), so that in this case

$$|J(u^0, y^0) - J\left(u_h^0, y_h^0\right) < J(\bar{u}, \bar{y}) - J\left(u_h^0, y_h^0\right). \tag{4.5.1.12}$$

Thus, in all cases, we have from (4.5.1.10) and (4.5.1.12)

$$\left|J\left(u_h^0, y_h^0\right) - J(u^0, y^0)\right| \le |J(\tilde{u}_h, \tilde{y}_h) - J(u^0, y^0)| + \left|J(\bar{u}_h, \bar{y}_h) - J\left(u_h^0, y_h^0\right)\right|. \tag{4.5.1.13}$$

We now rewrite the right-hand side (R.H.S.) of (4.5.1.13), after recalling the costs (4.1.1.5) and (4.1.3.2), as well as the dynamics (4.5.1.9), (4.5.1.11), and (4.4.3.2):

$$\begin{aligned}
\text{R.H.S. of (4.5.1.13)} \quad &= \Bigg| \int_0^\infty \Big\{ \|Re^{A_P t}x\|_Z^2 - \|Re^{A_{h.P}t}\Pi_h x\|_Z^2 \\
&\quad + \|B^*Pe^{A_P t}x\|_U^2 - \|B^*Pe^{A_{h.P}t}\Pi_h x\|_U^2 \Big\} \Bigg| \\
&\quad + \Bigg| \int_0^\infty \Big\{ \|Re^{A_{P_h}t}\Pi_h x\|_Z^2 - \|Re^{A_{h.P_h}t}\Pi_h x\|_Z^2 \\
&\quad + \|B_h^*P_h e^{A_{P_h}t}\Pi_h x\|_U^2 - \|B_h^*P_h e^{A_{h.P_h}t}\Pi_h x\|_U^2 \Big\}\, dt \Bigg|.
\end{aligned} \tag{4.5.1.14}$$

Next, on the right-hand side of (4.5.1.14), we use $|\|a\|^2 - \|b\|^2| \le \|a-b\|(\|a\| + \|b\|)$, along with the uniform bound $\|B^*P\| + \|B_h^*P_h\| \le \text{const}$ (by (4.1.1.14) and (4.4.4.2)), as well as the uniform (in h) exponential decay of all semigroups involved, as guaranteed by (4.1.1.15) and Proposition 4.5.1.2. We thus obtain from (4.5.1.14)

$$\begin{aligned}
\text{R.H.S. of (4.5.1.13)} \quad &\le C \left\{ \int_0^\infty \|e^{A_P t}x - e^{A_{h.P_h}t}\Pi_h x\|_Y \, dt \right. \\
&\quad \left. + \int_0^\infty \|e^{A_{P_h}t}\Pi_h x - e^{A_{h.P_h}t}\Pi_h x\|_Y dt \right\} \|x\|_Y.
\end{aligned} \tag{4.5.1.15}$$

We now split each integration over $[0, \infty]$ into two parts: one over $[0, h^s]$ and one over $[h^s, \infty]$. Since the integrands are dominated (uniformly in h) by decaying exponentials (by (4.1.1.5) and Proposition 4.5.1.2), the integrals over $[0, h^s]$ are dominated by $C\, h^s \|x\|_Y$. Thus, we obtain from (4.5.1.6)

$$\begin{aligned}\text{R.H.S. of (4.5.1.13)} \quad &\leq C\|x\|_Y \left\{h^s\|x\|_Y + \|e^{A_P t}x - e^{A_{h,P_h}t}\Pi_h x\|_{L_1(h^s,\infty;Y)}\right.\\ &\left.+ \|e^{A_{P_h}t}\Pi_h x - e^{A_{h,P_h}t}\Pi_h x\|_{L_1(h^s,\infty;Y)}\right\}. \qquad (4.5.1.16)\end{aligned}$$

Now, if $\Gamma : \rho e^{\pm i\theta} - k$, $\pi/2 < \theta < \pi$, $0 \leq \rho < \infty$, $k > 0$, is the boundary of a triangular sector containing the spectrum of A_P, A_{h,P_h}, uniformly in h, as guaranteed by Proposition 4.5.1.2, we compute

$$\begin{aligned}&2\pi\|e^{A_P t} - e^{A_{h,P_h}t}\Pi_h\|_{\mathcal{L}(Y)}\\ &= \left\|\int_\Gamma e^{\lambda t}[R(\lambda, A_P) - R(\lambda, A_{h,P_h})]\,d\lambda\right\|_{\mathcal{L}(Y)} \qquad (4.5.1.17)\\ \text{(by (4.1.28))} \quad &= \left\|\int_\Gamma e^{\lambda t}R(\lambda, A_P)(A_P - A_{h,P_h})R(\lambda, A_{h,P_h})\,d\lambda\right\| \qquad (4.5.1.18)\\ &= \left\|\int_\Gamma e^{\lambda t}R(\lambda, A_P)A_P\left(A_P^{-1} - A_{h,P_h}^{-1}\right)A_{h,P_h}R(\lambda, A_{h,P_h})\,d\lambda\right\|_{\mathcal{L}(Y)}\\ &\leq C\left(\int_\Gamma e^{(\operatorname{Re}\lambda)t}|d\lambda|\right)\|A_P^{-1} - A_{h,P_h}^{-1}\|_{\mathcal{L}(Y)}, \qquad (4.5.1.19)\end{aligned}$$

where in going from (4.5.1.17) to (4.5.1.18) we have used an identity such as (4.4.1.28), while in going from (4.5.1.18) to (4.5.1.19), we have used analyticity of $e^{A_P t}$ (hence $A_P R(\lambda, A_P)$ uniformly bounded for all $\lambda \in \Gamma$), as well as uniform analyticity of $e^{A_{h,P_h}t}$ (see Proposition 4.5.1.2) (hence $A_{h,P_h}R(\lambda, A_{h,P_h})$ uniformly bounded in h and in $\lambda \in \Gamma$, in the sense of (4.4.1.6), Theorem 4.4.1.1 with $\theta = 1$, with $k > 0$ sufficiently large). Thus from (4.5.1.19) we obtain, with $k > 0$ and $t > 0$,

$$\|e^{A_P t} - e^{A_{h,P_h}t}\|_{\mathcal{L}(Y)} \leq C\,\frac{e^{-kt}}{t}\|A_P^{-1} - A_{h,P_h}^{-1}\|_{\mathcal{L}(Y)}, \quad t > 0. \qquad (4.5.1.20)$$

By a similar argument,

$$\|e^{A_{P_h}t}\Pi_h - e^{A_{h,P_h}t}\Pi_h\|_{\mathcal{L}(Y)} \leq C\,\frac{e^{-kt}}{t}\|A_{P_h}^{-1} - A_{h,P_h}^{-1}\|_{\mathcal{L}(Y)}, \quad t > 0. \qquad (4.5.1.21)$$

We now return to (4.5.1.20), which we integrate in t over $[h^s, \infty]$. Splitting the integration over $[h^s, 1]$ and $[1, \infty]$ with $h^s < 1$ as $h \downarrow 0$, we obtain from (4.5.1.20):

$$\begin{aligned}&\int_{h^s}^\infty \|e^{A_P t}x - e^{A_{h,P_h}t}\Pi_h x\|_Y\,dt\\ &\leq \left\|A_P^{-1} - A_{h,P_h}^{-1}\right\|_{\mathcal{L}(Y)} C\left[\int_{h^s}^1 \frac{1}{t}\,dt + \int_1^\infty e^{-kt}\right]\|x\|_Y\\ &\leq C\left\|A_P^{-1} - A_{h,P_h}^{-1}\right\|_{\mathcal{L}(Y)}\|x\|_Y\left(s\ln\frac{1}{h}\right). \qquad (4.5.1.22)\end{aligned}$$

We now recall the bound (4.5.1.6) for the first term in (4.5.1.22) and use that $\ln(1/h) \leq C\, h^{\epsilon}$ for $\epsilon > 0$, as $h \downarrow 0$. Thus, we finally obtain from (4.5.1.22)

$$\int_{h^s}^{\infty} \|e^{A_P t}x - e^{A_{h,P_h} t}\Pi_h x\|_Y \, dt \leq C\, h^{\epsilon_o - \epsilon}\|x\|_Y, \tag{4.5.1.23}$$

for all $\epsilon > 0$. The same estimate can be obtained starting from (4.5.1.21) and using (4.5.1.7),

$$\int_{h^s}^{\infty} \|e^{A_{P_h} t}\Pi_h x - e^{A_{h,P_h} t}\Pi_h x\|_Y \, dt \leq C\, h^{\epsilon_o - \epsilon}\|x\|_Y. \tag{4.5.1.24}$$

Using estimates (4.5.1.23), (4.5.1.24) in (4.5.1.16), as well as recalling (4.5.1.8) and (4.5.1.13), we obtain

$$|([P_h\Pi_h - P]x, x)_Y| \leq C\, h^{\epsilon_o}\|x\|_Y^2, \tag{4.5.1.25}$$

with any $\epsilon_o < s(1-\gamma) < s$, as desired. Then (4.5.1.25) implies (4.5.1.1) by taking the sup over all x, $\|x\| \leq 1$, since P_h and P are self-adjoint. Theorem 4.5.1.1 is proved. $\square$

As a corollary of part of the preceding proof, we obtain

Theorem 4.5.1.4 *(Property (4.1.4.13) of main Theorem 4.1.4.1) Assume the sets of Assumptions I and II of Theorem 4.1.4.1. Then for any* $0 \leq \theta \leq 1$, *we have:*

(i)

$$\sup_{0 \leq t}\left\{ e^{\tilde{\omega}_P t} t^{\theta} \|e^{A_{h,P_h} t}\Pi_h - e^{A_P t}\|_{\mathcal{L}(Y)}\right\} \leq C_\theta\, h^{\epsilon_o \theta} \to 0 \text{ as } h \downarrow 0,$$

$$\forall\, \epsilon_o < s(1-\gamma) \text{ as in (4.5.1.6)}; \tag{4.5.1.26}$$

(ii) recalling A_P *from (4.1.1.15) and* A_{h,P_h} *from (4.1.4.6),*

$$\Phi(t)x = e^{A_P t}x = y^0(\cdot; x), \qquad \Phi_h(t)\Pi_h x = e^{A_{h,P_h} t}\Pi_h x = y_h^0(\,\cdot\,; \Pi_h x). \tag{4.5.1.27}$$

Proof. We interpolate between inequality (4.5.1.20) and (see (4.1.1.15) and (4.4.3.3))

$$\|e^{A_P t} - e^{A_{h,P_h} t}\Pi_h\|_{\mathcal{L}(Y)} \leq C\, e^{-kt}, \tag{4.5.1.28}$$

for $k > 0$, defined below (4.5.1.16); that is, we raise (4.5.1.20) to power θ, raise (4.5.1.28) to power $(1-\theta)$, and multiply the resulting expressions to obtain, for any $0 < \theta < 1$,

$$\|e^{A_P t} - e^{A_{h,P_h} t}\Pi_h\|_{\mathcal{L}(Y)} \leq C\, \frac{e^{-kt}}{t^{\theta}} \left\| A_P^{-1} - A_{h,P_h}^{-1} \right\|_{\mathcal{L}(Y)}^{\theta}, \quad t > 0. \tag{4.5.1.29}$$

Then (4.5.1.26) follows from (4.5.1.29) after recalling (4.5.1.6), since we can take $k = \tilde{\omega}_P$, defined by (4.4.3.3). $\square$

Corollary 4.5.1.5 *Under the assumptions of Theorem 4.5.1.4, we have*

(i)

$$\|e^{A_{h,P_h}t}\Pi_h - e^{A_P t}\|_{\mathcal{L}(Y;L_2(0,\infty;Y))} \leq C_{T,\theta}h^{\epsilon_o}, \quad T > 0; \tag{4.5.1.30}$$

(ii)

$$\|e^{A_{h,P_h}t}\Pi_h - e^{A_P t}\|_{\mathcal{L}(Y;L_2(0,\infty;Y))} \leq C_\theta h^{s(1-\gamma)\theta}, \quad \forall\, \theta < \frac{1}{2}; \tag{4.5.1.31}$$

(iii)

$$\|e^{A_{h,P_h}t}\Pi_h - e^{A_P t}\|_{\mathcal{L}(Y;L_1(T,\infty;Y))} \leq C_\theta h^{s(1-\gamma)\theta}, \quad \forall\, \theta < 1. \tag{4.5.1.32}$$

Proof. These are immediate consequences of (4.5.1.26), with $\theta = 1$ in the case of (4.5.1.30). □

4.5.2 Uniform Convergence $u_h^0 \to u^0$

The goal of this section is to provide the convergence $u_h^0 \to u^0$ in the $C([0,\infty];U)$-topology.

Theorem 4.5.2.1 *(Property (4.1.4.10) of main Theorem 4.1.4.1) Assume the sets of Assumptions I and II of Theorem 4.1.4.1. Then we have* $\forall \epsilon_o < s(1-\gamma)$

$$\left\|u_h^0(\cdot\,;\Pi_h x) - u^0(\cdot\,;x)\right\|_{\mathcal{L}(Y;C([0,\infty];U))} \leq C h^{\epsilon_o} e^{-\tilde{\omega}_P t} \text{ as } h \downarrow 0, \quad x \in Y. \tag{4.5.2.1}$$

Proof.

Step 1 We shall first show that (4.5.2.1) holds true with u replaced by $\hat{u}$. Indeed, from (4.2.1.9) and (4.2.2.9) [= (4.4.3.14)]

$$\begin{aligned}\hat{u}_h^0(\cdot\,;\Pi_h x) - \hat{u}^0(\cdot\,;x) &= \hat{L}_h^*(R^*R + 2\omega P_h)\hat{y}_h^0(\cdot\,;\Pi_h x) \\ &\quad - \hat{L}^*(R^*R + 2\omega P)\hat{y}^0(\cdot\,;x) = (1) + (2),\end{aligned} \tag{4.5.2.2}$$

where

$$(1) = [\hat{L}_h^*(R^*R + 2\omega P_h) - \hat{L}^*(R^*R + 2\omega P)]\hat{y}_h^0(\cdot\,;\Pi_h x), \tag{4.5.2.3}$$

$$(2) = \hat{L}^*(R^*R + 2\omega P)\big[\hat{y}_h^0(\cdot\,;\Pi_h x) - \hat{y}^0(\cdot\,;x)\big]. \tag{4.5.2.4}$$

To estimate (1), we recall the convergence with rate $\hat{L}_h \to \hat{L}$ in (4.3.25b) in $\mathcal{L}(L_\infty(0,\infty;U); C([0,\infty];Y))$, the convergence $P_h \to P$ in (4.5.1.5), along with the uniform boundedness of y_h^0 in (4.4.3.3) to obtain from (4.5.2.3)

$$\|(1)\|_{C([0,\infty];U)} \leq C h^{\epsilon_o}\|x\|_Y; \quad \forall \epsilon_o < s(1-\gamma). \tag{4.5.2.5}$$

As to the term (2) in (4.5.2.4), we first invoke the convergence (4.5.1.26) of Theorem 4.5.1.4 for $y_h^0(\cdot\,;\Pi_h x) - y^0(\cdot\,;x)$. This, in turn, requires the estimate

$$\|\hat{L}^*(t^{-\theta}f)\|_{C([0,\infty];U)} = \left\|\int_t^\infty B^* e^{-\hat{A}^*(\tau-t)}\frac{f(\tau)}{\tau^\theta}\,d\tau\right\|_{C([0,\infty];U)}$$

$$\leq C \sup_{t\in[0,\infty]} \int_t^\infty \frac{e^{-\hat{\omega}(\tau-t)}}{(\tau-t)^\gamma \tau^\theta} \|f(\tau)\|_Y \, d\tau$$
$$\leq C\|f\|_{L_\infty(0,\infty;Y)}, \quad \text{for } \gamma+\theta<1, \tag{4.5.2.6}$$

which is obtained by recalling (4.2.1.4), (4.1.1.4), and analyticity of $e^{-\hat{A}t}$. Finally, combining (4.5.1.26) with (4.5.2.6) in (4.5.2.4) yields

$$\|(2)\|_{C([0,\infty];U)} \leq C\, h^{\epsilon_o}\|x\|_Y \quad \text{for } \epsilon_o < s(1-\gamma). \tag{4.5.2.7}$$

Thus, we obtain the desired estimate by using (4.5.2.5) and (4.5.2.7) in (4.5.2.2),

$$\left\|\hat{u}_h^0(\,\cdot\,;\Pi_h x) - \hat{u}^0(\,\cdot\,;x)\right\|_{\mathcal{L}(Y;C([0,\infty];U))} \leq Ch^{\epsilon_o}. \tag{4.5.2.8}$$

Step 2 From (4.5.2.8), (4.2.1.5), and (4.2.2.5), it follows that, for any fixed $T>0$,

$$\left\|u_h^0(\,\cdot\,;\Pi_h x) - u^0(\,\cdot\,;x)\right\|_{\mathcal{L}(Y;C([0,2T];U))} \leq C_T h^{\epsilon_o}, \quad C_T = Ce^{-\omega T},$$

or explicitly recalling (4.4.3.2) and (4.1.1.16),

$$\sup_{0\leq t\leq 2T} \|B_h^* P_h e^{A_{h,P_h}t}\Pi_h - B^* P e^{A_P t}\|_{\mathcal{L}(Y;U)} \leq C_T h^{\epsilon_o}. \tag{4.5.2.9}$$

By the semigroup property, for any $t>2T$ we compute

$$\begin{aligned}
&\left\|u_h^0(t;\Pi_h x) - u^0(t;x)\right\|_{\mathcal{L}(Y;U)}\\
&= \left\|B_h^* P_h e^{A_{h,P_h}T} e^{A_{h,P_h}(t-T)}\Pi_h - B^* P e^{A_P T} e^{A_P(t-T)}\right\|_{\mathcal{L}(Y)}\\
&= \|B_h^* P_h e^{A_{h,P_h}T}\Pi_h - B^* P e^{A_P T}\|_{\mathcal{L}(Y;U)} \left\|e^{A_{h,P_h}(t-T)}\Pi_h\right\|_{\mathcal{L}(Y)}\\
&\quad + \|B^* P e^{A_P T}\|_{\mathcal{L}(Y;U)} \left\|e^{A_{h,P_h}(t-T)}\Pi_h - e^{A_P(t-T)}\right\|_{\mathcal{L}(Y)}\\
&\leq C_T\, h^{\epsilon_o} e^{-\tilde{\omega}_P(t-T)} + C\, e^{-\omega_P T}\frac{h^{\epsilon_o}}{t-T} e^{-\tilde{\omega}_P(t-T)}, \quad t>2T,
\end{aligned} \tag{4.5.2.10}$$

where in the last step we have used (4.5.2.9), (4.4.3.3) for the first term, and (4.1.1.14), (4.1.1.15), and (4.5.1.26) with $\theta=1$ for the second term. We thus obtain the desired result in (4.5.2.1). □

4.5.3 L_2-Convergence $u_h^0 \to u^0$ and $y_h^0 \to y^0$

The main goal of this section is to improve upon estimate (4.5.1.31) for the L_2-convergence $y_h^0 \to y^0$ [see (4.5.1.27)], and to obtain related results for the L_2-convergence $u_h^0 \to u^0$.

Theorem 4.5.3.1 *(Properties (4.1.4.10b), (4.1.4.11), and (4.1.4.12) of the main Theorem 4.1.4.1) Assume the sets of Assumptions I and II of Theorem 4.1.4.1. Then, with $\epsilon_0 < s(1-\gamma)$, we have*

(i)

$$\left\|\hat{\Phi}_h(\cdot)\Pi_h - \hat{\Phi}(\cdot)\right\|_{\mathcal{L}(Y;L_2(0,\infty;Y))}$$
$$\leq \begin{cases} C\, h^{\epsilon_o} \to 0 \text{ as } h \downarrow 0, & \text{if } \frac{1}{2} \leq \gamma < 1, \quad (4.5.3.1a) \\ C_\theta h^{s\theta} \to 0 \text{ as } h \downarrow 0, \forall \theta < \frac{1}{2}, & \text{if } 0 \leq \gamma \leq \frac{1}{2}; \quad (4.5.3.1b) \end{cases}$$

$$\left\|\hat{u}_h^0(\cdot\,;\Pi_h x) - \hat{u}^0(\cdot\,;x)\right\|_{\mathcal{L}(Y;L_2(0,\infty;Y))}$$
$$\leq \begin{cases} Ch^{\epsilon_o} \to 0 \text{ as } h \downarrow 0, & \text{if } \frac{1}{2} \leq \gamma < 1, \quad (4.5.3.2a) \\ C_\theta h^{s\theta} \to 0 \text{ as } h \downarrow 0, \forall \theta < \frac{1}{2}, & \text{if } 0 \leq \gamma \leq \frac{1}{2}. \quad (4.5.3.2b) \end{cases}$$

(ii) With reference to (4.5.1.27), we have

$$\left\|y_h^0(\cdot\,;\Pi_h x) - y^0(\cdot\,;x)\right\|_{\mathcal{L}(Y;L_2(0,\infty;Y))}$$
$$\leq \begin{cases} Ch^{\epsilon_o} \to 0, & \text{for } \frac{1}{2} \leq \gamma < 1, \quad (4.5.3.3a) \\ C_\theta h^{s\theta} \to 0, \forall \theta < \frac{1}{2}, & \text{if } 0 \leq \gamma \leq \frac{1}{2}; \quad (4.5.3.3b) \end{cases}$$

$$\left\|u_h^0(\cdot\,;\Pi_h x) - u^0(\cdot\,;x)\right\|_{\mathcal{L}(Y;L_2(0,\infty;U))} \to 0, \tag{4.5.3.4}$$

with rate of convergence discussed in Remark 4.5.3.1 below.

(iii) The following result, which complements (4.5.1.26), holds true:

$$\|[\Phi_h(\cdot)\Pi_h - \Phi(\cdot)]x\|_{C([0,\infty];Y)} \to 0 \text{ as } h \downarrow 0, \quad x \in Y. \tag{4.5.3.5}$$

Proof. (i) We return to (4.2.2.7) and (4.2.2.8), rewritten here for convenience as

$$\hat{y}_h^0(\cdot\,;\Pi_h x) = [I + \hat{L}_h\hat{L}_h^*\Pi_h(R^*R + 2\omega P_h)]^{-1}\{e^{-\hat{A}_h\cdot}\Pi_h x\}, \quad x \in Y; \tag{4.5.3.6}$$

$$\begin{aligned} -\hat{u}_h^0(\cdot\,;\Pi_h x) &= [I + \hat{L}_h^*\Pi_h(R^*R + 2\omega P_h)\hat{L}_h]^{-1} \\ &\quad \times \hat{L}_h^*\Pi_h[R^*R + 2\omega P_h]\{e^{-\hat{A}_h\cdot}\Pi_h x\}, \end{aligned} \tag{4.5.3.7}$$

and we take the limit as $h \downarrow 0$. First, we recall the uniform convergence results

$$\|P_h\Pi_h - P\|_{\mathcal{L}(Y)} \leq Ch^{\epsilon_o}, \quad \epsilon_o < s(1-\gamma);$$

$$\|\hat{L}_h - \hat{L}\|_{L_2} \leq Ch^{s(1-\gamma)\theta}, \quad \theta < 1,$$

from (4.5.1.1) and (4.3.25a) respectively (the latter in the norm of $\mathcal{L}(L_2(0,\infty;U);L_2(0,\infty;Y))$, to deduce, via an identity such as (4.4.1.29), which preserves the rates of convergence when taking the inverse, that

$$\|[I + \hat{L}_h\hat{L}_h^*\Pi_h(R^*R + 2\omega P_h)]^{-1} - [I + \hat{L}\hat{L}^*\Pi_h(R^*R + 2\omega P)]^{-1}\|_{L_2} \leq C\, h^{\epsilon_o}, \tag{4.5.3.8}$$

$\epsilon_o < s(1-\gamma)$, in the norm of $\mathcal{L}(L_2(0,\infty;Y))$. Next, we note the uniform convergence

$$\|e^{-\hat{A}_h t}\Pi_h - e^{-\hat{A}t}\|_{\mathcal{L}(Y;L_2(0,\infty;Y))} \leq Ch^{s\theta} \to 0 \text{ as } h \downarrow 0, \quad \theta < \frac{1}{2}, \tag{4.5.3.9}$$

which follows from Eqn. (4.1.2.10), used in the following steps:

$$\int_0^\infty \|e^{-\hat{A}_h t}\Pi_h x - e^{-\hat{A}t}x\|_Y^2 dt \le C^2 h^{2s\theta} \int_0^\infty \frac{e^{-2\hat{\omega}t}}{t^{2\theta}}\, dt \|x\|_Y^2, \tag{4.5.3.10}$$

where we also recall (4.1.1.2) and (4.1.1.3). Finally, by (4.5.3.8) and (4.5.3.9), the rate of convergence for $\hat{y}_h^0(\cdot\,;\Pi_h x) = \hat{\Phi}_h(\cdot)\Phi_h x$ in (4.5.3.6) to its continuous version (4.2.1.7) for $\hat{y}^0(\cdot;x) = \hat{\Phi}(\cdot)x$ is determined, as $h \downarrow 0$, by the smaller of the two exponents: ϵ_o and $s\theta$ with $\theta < 1/2$; where $\epsilon_o < s(1-\gamma)$. Thus, conclusions (4.5.3.1a) and (4.5.3.1b) result from (4.5.3.8) and (4.5.3.9), used with (4.5.3.6) and (4.2.1.7).

A similar argument, which starts now from (4.5.3.7) and uses (4.2.1.8), yields likewise (4.5.3.2).

(ii) A fortiori from (4.5.3.1) and (4.5.3.2), passing to the corresponding quantities without ^, defined by (4.2.1.5), (4.2.1.6), (4.2.2.5), and (4.2.2.6), we obtain for any $0 < T < \infty$:

$$\left\|u_h^0(\cdot\,;\Pi_h x) - u^0(\cdot\,;x)\right\|_{\mathcal{L}(Y;L_2(0,T;U))} + \|\Phi_h(\cdot)\Pi_h - \Phi(\cdot)\|_{\mathcal{L}(Y;L_2(0,T;Y))}$$

$$\le \begin{cases} C_T h^{\epsilon_o} \to 0 \text{ as } h \downarrow 0, & \frac{1}{2} \le \gamma < 1, \qquad (4.5.3.11a) \\ C_T h^{s\theta} \to 0, \forall \theta < \frac{1}{2}, & 0 \le \gamma \le \frac{1}{2}, \qquad (4.5.3.11b) \end{cases}$$

$C_T = C_\theta e^{\omega T}$. Then, the statement on $[\Phi_h\Pi_h - \Phi]$ in (4.5.3.11) for $0 \le t \le T$, combined with statement (4.5.1.30) for $t \ge T > 0$, yields [recalling (4.5.1.27)] the desired conclusion (4.5.3.3a) or (4.5.3.3b) for the approximate and continuous optimal trajectories.

To prove (4.5.3.4), we compute by (4.4.3.2) and (4.1.1.16)

$$\begin{aligned} &\int_T^\infty \left[\left\|u_h^0(t;\Pi_h x) - u^0(t;x)\right\|_U^2\right] dt \\ &\le 2\int_T^\infty \left[\left\|u_h^0(t;\Pi_h x)\right\|_U^2 + \|u^0(t;x)\|_U^2\right] dt \\ &= 2\int_T^\infty \|B_h^* P_h e^{A_{h,P_h}t}\Pi_h x\|_U^2\, dt + 2\int_T^\infty \|B^*Pe^{A_P t}x\|_U^2\, dt \qquad (4.5.3.12) \\ &\le \int_T^\infty C(e^{-2\tilde{\omega}_P t} + e^{2\omega_P t})\, dt \|x\|_Y \le Ce^{-\tilde{\omega}_P T}, \qquad (4.5.3.13) \end{aligned}$$

where in the last steps we have invoked the uniform bound (4.4.4.2) on $B_h^*P_h$ and the uniform exponential decay (4.4.3.3) for the first integral in (4.5.3.12) and (4.1.1.14) and (4.1.1.15) for the second integral in (4.5.3.12); moreover, we are taking $\tilde{\omega}_P < \omega_P$. Thus, letting $T \to \infty$ in (4.5.3.13) and combining this with (4.5.3.11) for $[u_h^0 - u^0]$ yields the convergence result in (4.5.3.4).

Remark 4.5.3.1 It is possible also to give a rate of convergence for (4.5.3.4):

$$\left\| u_h^0(\,\cdot\,; \Pi_h x) - u^0(\,\cdot\,; x) \right\|_{\mathcal{L}(Y; L_2(0,\infty;U))} \leq Ch^r, \tag{4.5.3.14}$$

where

$$r \equiv m \left(\frac{1}{\frac{\omega}{\bar{\omega}_P} + 1} \right); \qquad m \equiv \begin{cases} \epsilon_o, & \frac{1}{2} \leq \gamma < 1, \\ s\theta, \theta < \frac{1}{2}, & 0 \leq \gamma \leq \frac{1}{2}. \end{cases} \tag{4.5.3.15}$$

In fact, with reference to the factor $e^{\omega T}$ of $C_T = C_\theta e^{\omega T}$ in (4.5.3.11) we let $T \uparrow \infty$ in the following way, by setting

$$e^{\omega T} \equiv \frac{1}{h^\epsilon}, \quad \text{or} \quad \bar{\omega}_P T = \frac{\bar{\omega}_P}{\omega} \ln \frac{1}{h^\epsilon}, \quad e^{-\bar{\omega}_P T} = h^{\epsilon(\frac{\bar{\omega}_P}{\omega})}. \tag{4.5.3.16}$$

Thus the right-hand side (R.H.S.) of (4.5.3.11) and (4.5.3.13) satisfy

$$\text{R.H.S. of (4.5.3.11)} \leq C\, h^{m-\epsilon}; \quad \text{R.H.S. of (4.5.3.13)} \leq C\, h^{\epsilon(\frac{\bar{\omega}_P}{\omega})}. \tag{4.5.3.17}$$

Imposing $m - \epsilon = \epsilon\,(\bar{\omega}_P/\omega)$ results in $m - \epsilon = r$, r given by (4.5.3.15). Then (4.5.3.14) follows from (4.5.3.17). □

(iii) We first note that from the representation (see (4.4.3.2) or (4.5.1.27))

$$R(\lambda, A_{h,P_h})\Pi_h x - R(\lambda, A_P)x = \int_0^\infty e^{-\lambda t} \left[\Phi_h(t)\Pi_h x - \Phi(t)x\right] dt \tag{4.5.3.18}$$

for Re $\lambda > \min\{-\omega_P, -\bar{\omega}_P\}$ (defined in (4.1.1.15) and (4.4.4.3)), and from (4.5.3.3) of part (ii), we obtain with $\epsilon_o < s(1-\gamma)$:

$$\left\| R\left(\lambda, A_{h,P_h}\right) \Pi_h - R(\lambda, A_P) \right\|_{\mathcal{L}(Y)} \leq \begin{cases} C_\lambda h^{\epsilon_o}, & \frac{1}{2} \leq \gamma < 1, \\ C_\lambda h^{s\theta}, \forall \theta < \frac{1}{2}, & 0 \leq \gamma \leq \frac{1}{2}. \end{cases} \tag{4.5.3.19}$$

Then the convergence in (4.5.3.19) as $h \downarrow 0$, combined with the uniform bounds (4.1.1.15) and (4.4.4.3) for $\Phi(t)$ and $\Phi_h(t)$, allows us to invoke the Trotter–Kato theorem [Pazy, 1983, p. 87] and obtain for any $T < \infty$:

$$\|[\Phi_h(\,\cdot\,)\Pi_h - \Phi(\,\cdot\,)]x\|_{C([0,T];Y)} \to 0 \text{ as } h \downarrow 0, \quad x \in Y. \tag{4.5.3.20}$$

Then (4.5.3.20), combined with (4.5.1.26) for $t \geq T > 0$ or the exponential decay of $\Phi(t)$ and $\Phi_h(t)$ (uniformly in h) from (4.1.1.15) and (4.4.4.3), implies (4.5.3.5). □

4.5.4 Uniform Convergence $B_h^* P_h \Pi_h \to B^* P$ of Gain Operators

Theorem 4.5.4.1 *(Property (4.1.4.9) of main Theorem 4.1.4.1) Assume the sets of assumptions I and II of Theorem 4.1.4.1. We have*

$$\|B_h^* P_h \Pi_h - B^* P\|_{\mathcal{L}(Y;U)} \to 0 \text{ as } h \downarrow 0, \tag{4.5.4.1}$$

where the rate is $h^{s\theta}$, $\forall \theta < 1/2$, if $\gamma < 1/2$.

Proof. From (4.2.2.10) (or (4.4.4.6)) and (4.2.1.0), we compute

$$\begin{aligned} B_h^* P_h \Pi_h - B^* P &= \int_0^\infty B_h^* e^{-\hat{A}_h^* t} \Pi_h [R^* R + 2\omega P_h] \hat{\Phi}_h(t) \Pi_h \, dt \\ &\quad - \int_0^\infty B^* e^{-\hat{A}^* t} [R^* R + 2\omega P] \hat{\Phi}(t) \, dt \\ &= I_{1,h} + I_{2,h} + I_{3,h}, \end{aligned} \tag{4.5.4.2}$$

where, after suitable adding and subtracting,

$$I_{1,h} = \int_0^\infty [B_h^* e^{-\hat{A}_h^* t} \Pi_h - B^* e^{-\hat{A}^* t}][R^* R + 2\omega P_h] \hat{\Phi}_h(t) \Pi_h \, dt, \tag{4.5.4.3}$$

$$I_{2,h} = \int_0^\infty B^* e^{-\hat{A}^* t} 2\omega \, [P_h \Pi_h - P] \, \hat{\Phi}_h(t) \Pi_h \, dt, \tag{4.5.4.4}$$

$$I_{3,h} = \int_0^\infty B^* e^{-\hat{A}^* t} [R^* R + 2\omega P][\hat{\Phi}_h(t) \Pi_h - \hat{\Phi}(t)] \, dt. \tag{4.5.4.5}$$

To handle $I_{1,h}$, we recall that from Lemma 4.3.1(v), Eqn. (4.3.5), applied with $\theta < 1$, we have (by the definitions of $\hat{A}^*$ and $\hat{A}_h^*$ from (4.1.1.2) and (4.1.2.5)):

$$\| B_h^* e^{-\hat{A}_h^* t} \Pi_h - B^* e^{-\hat{A}^* t} \|_{\mathcal{L}(Y;U)} \le \frac{C \, h^{s(1-\gamma)\theta}}{t^{\theta + (1-\theta)\gamma}} \, e^{-\hat{\omega} t}, \quad t > 0. \tag{4.5.4.6}$$

We now insert (4.5.4.6) into the integral of (4.5.4.3); recall that $\theta + (1-\theta)\gamma < 1$; use the uniform bound (4.4.3.4) of Lemma 4.4.3.2 for P_h; use the uniform bound (4.4.3.3) of Theorem 4.4.3.1 for $\Phi_h(t)$; and readily obtain from (4.5.4.3) that

$$\| I_{1,h} \|_{\mathcal{L}(Y;U)} \le C \, h^{s(1-\gamma)\theta} \downarrow 0 \text{ as } h \downarrow 0, \quad \theta < 1. \tag{4.5.4.7}$$

To handle $I_{2,h}$ in (4.5.4.4), we first note the bound

$$\| B^* e^{-\hat{A}^* t} \|_{\mathcal{L}(Y;U)} = \| B^* (\hat{A}^*)^{-\gamma} (\hat{A}^*)^{\gamma} e^{-\hat{A}^* t} \|_{\mathcal{L}(Y;U)} \le C \, \frac{e^{-\hat{\omega} t}}{t^\gamma}, \quad t > 0 \tag{4.5.4.8}$$

(from (4.1.1.4) and analyticity); next recall the convergence result for $[P_h \Pi_h - P]$ in Eqn. (4.5.1.1) of Theorem 4.5.1.1; finally, note the uniform exponential bound on $\hat{\Phi}_h(t) = e^{-\omega t} \Phi_h(t)$ from (4.4.3.3), to conclude from (4.5.4.4) that

$$\| I_{2,h} \|_{\mathcal{L}(Y;U)} \le C \, h^{\epsilon_o} \to 0 \text{ as } h \downarrow 0, \quad \epsilon_o < s(1-\gamma). \tag{4.5.4.9}$$

Finally, to handle $I_{3,h}$ in (4.5.4.5), we use (4.5.4.8) and obtain

$$\| I_{3,h} \|_{\mathcal{L}(Y;U)} \le C \int_0^\infty \frac{e^{-\hat{\omega} t}}{t^\gamma} \| \hat{\Phi}_h(t) \Pi_h - \hat{\Phi}(t) \|_{\mathcal{L}(Y)} \, dt. \tag{4.5.4.10}$$

But the norm inside the integral in (4.5.4.10) is dominated (uniformly in h) by a decaying exponential by (4.1.1.15) and (4.4.4.3). Thus the integrand of (4.5.4.10) is dominated, uniformly in h, by an $L_1(0, \infty)$-function, since $\gamma < 1$. Moreover,

$\hat{\Phi}_h(t)\Pi_h \to \hat{\Phi}(t)$ pointwise by (4.5.3.5). Thus, the Lebesgue's dominated convergence theorem applies in (4.5.4.10) and yields

$$\|I_{3,h}\|_{\mathcal{L}(Y;U)} \to 0 \text{ as } h \downarrow 0. \tag{4.5.4.11}$$

(We note that for $\gamma < 1/2$, one still obtains $\|I_{3,h}\| \le Ch^{s\theta}$, $\forall \theta < 1/2$, by (4.5.3.3b).) Then (4.5.4.7), (4.5.4.9), and (4.5.4.11), used in (4.5.4.2), produce the claimed convergence in (4.5.4.1). Theorem 4.5.4.1 is proved. □

4.5.5 Convergence $(\hat{A}^*)^\theta(P_h\Pi_h - P)x \to 0$

So far, we have shown conclusions (4.1.4.5)–(4.1.4.14) of the main Theorem 4.1.4.1. We now complete the proof of Theorem 4.1.4.1 by establishing properties (4.1.4.16) and (4.1.4.17) as well.

Proposition 4.5.5.1 *(Properties (4.1.4.16) and (4.1.4.17) of Theorem 4.1.4.1) Assume the sets of Assumptions I and II of Theorem 4.1.4.1. Assume further the approximating property (4.1.4.15), rewritten here for convenience as*

$$(\hat{A}^*)^\theta\left(\hat{A}_h^{*-1}\right)^\theta \in \mathcal{L}(V_h; Y); \quad or$$
$$\|(\hat{A}^*)|^\theta x_h\|_Y \le C_\theta\|(\hat{A}_h^*)^\theta x_h\|_Y, \quad 0 \le \theta < 1, \ \forall x_h \in V_h. \tag{4.5.5.1}$$

Then

(i)

$$\|(\hat{A}^*)^\theta(P_h\Pi_h - P)x\|_Y \to 0 \ \ as \ \ h \downarrow 0, \quad x \in Y, \ 0 \le \theta < 1; \tag{4.5.5.2}$$

(ii)

$$\|(\hat{A}^*)^\theta(P_h\Pi_h - P)\hat{A}^\theta x\|_Y \to 0 \ as \ h \downarrow 0, \quad x \in Y, \ 0 \le \theta < \frac{1}{2}. \tag{4.5.5.3}$$

Proof. (i) We return to (4.2.2.10) and get

$$(\hat{A}^*)^\theta P_h\Pi_h x = \int_0^\infty (\hat{A}^*)^\theta e^{-\hat{A}_h^* t}\Pi_h[R^*R + 2\omega P_h]\hat{\Phi}_h(t)x\,dt, \quad x \in Y, \tag{4.5.5.4}$$

where

$$(\hat{A}^*)^\theta e^{-\hat{A}_h^* t} = (\hat{A}^*)^\theta\left(\hat{A}_h^{*-1}\right)^\theta(\hat{A}_h^*)^\theta e^{-\hat{A}_h^* t} = \mathcal{O}\left(\frac{e^{-\hat{\omega}t}}{t^\theta}\right)$$

by (4.5.5.1) and uniform analyticity (4.1.2.2), $\theta < 1$. Moreover, $\hat{\Phi}_h(\cdot)x \to \hat{\Phi}(\cdot)x$ in $C([0,\infty]; Y)$ by (4.5.3.5) of Theorem 4.5.3.1(iii), and $P_h \to P$ by (4.5.1.1). Thus, letting $h \downarrow 0$ in (4.5.5.4) and recalling (4.2.1.10), for P, we obtain the limit $(\hat{A}^*)^\theta Px$ and (4.5.5.2) is proved.

(ii) We recall (4.4.4.10) of Corollary 4.4.4.2 for the discrete problem and the corresponding version for the continuous problem since $P = P^*$ (see Chapter 1, Lemma 1.5.1.1 and Corollary 1.5.1.2) to obtain

$$\|(\hat{A}^*)^\theta (P_h\Pi_h - P)\hat{A}^\theta\|_{\mathcal{L}(Y)} \leq \text{const}_\theta, \quad 0 \leq \theta < \frac{1}{2}. \tag{4.5.5.5}$$

Now, if $x \in \mathcal{D}(\hat{A}^\theta)$, $\theta < 1/2$, the convergence in (4.5.5.3) is a special case of (4.5.5.2), which has already been proved. In other words, $\hat{A}^\theta(P_h\Pi_h - P)\hat{A}^\theta$ converges to zero at least in $\mathcal{D}(\hat{A}^\theta)$, which is dense in Y. But then we recall the uniform bound (4.5.5.5) and obtain (4.5.5.3) for all $x \in Y$ as well. $\square$

The proof of the main Theorem 4.1.4.1 is complete.

4.5.6 Completion of the Proof of Main Theorem 4.1.4.2

Proof of (4.1.4.18) of Theorem 4.4.1.2. This conclusion follows from Theorem 4.4.1.2 applied with $F = -B^*P$, $F_h = -B_h^* P_h \Pi_h$, whose uniform convergence $\|B_h^* P_h\Pi_h - B^*P\| \to 0$ in $\mathcal{L}(Y; U)$ is provided by (4.5.4.1) of Theorem 4.5.4.1. Thus, by Remark 4.4.1.2(ii), assumption (4.4.1.13) of Theorem 4.4.1.2 holds true with $A_F = A_P$ being a uniformly stable generator, as required, by (4.1.1.15). Thus the conclusion (4.4.1.16) of Theorem 4.4.2.1 is precisely the desired conclusion (4.1.4.18).

Proof of (4.1.4.19) From (4.5.1.2) and (4.5.1.4), using an identity such as (4.4.1.28), we obtain

$$R(\lambda, A_P) - R\big(\lambda, A_{P_h}\big) = R(\lambda, A_P)B[B^*P - B_h^* P_h\Pi_h]R(\lambda, A_h), \tag{4.5.6.1}$$

since the term between the resolvents on the right-hand side of (4.5.6.1) is $A_P - A_{P_h}$, where the usual perturbation formula gives

$$R(\lambda, A_P)B = [I + R(\lambda, A)BB^*P]^{-1}R(\lambda, A)B, \quad \text{say } \lambda \in \Gamma_P. \tag{4.5.6.2}$$

It follows that

$$\|R(\lambda, A_P)B\|_{\mathcal{L}(U;Y)} \leq \frac{C}{|\lambda - \omega_0|^{1-\gamma}}, \quad \lambda \in \Gamma_P, \tag{4.5.6.3}$$

where Γ_P is the path $\rho e^{\pm i\theta_P} - \tilde{\omega}_P$, $\pi/2 < \theta_P < \pi$, $0 \leq \rho < \infty$, on which $R(\lambda, A_P)$ is well defined. In fact, first estimate (4.5.6.3) is derived for, say, $\lambda \in \Sigma^c(A)$, $|\lambda|$ sufficiently large, and then extended also on Γ_P by analytic continuation. To obtain (4.5.6.3) preliminarily for $\lambda \in \Sigma^c(A)$, $|\lambda|$ large, we recall $B^*P \in \mathcal{L}(Y; U)$ by (4.1.1.14), and use in (4.5.6.2) the estimate

$$\|R(\lambda, A)B\|_{\mathcal{L}(U;Y)} \leq \frac{C}{|\lambda - \omega_0|^{1-\gamma}}, \quad \lambda \in \Sigma^c(A), \ |\lambda| \text{ large}, \tag{4.5.6.4}$$

which is a consequence of assumption (4.1.1.4) on $A^{-\gamma}B$ bounded and of analyticity of e^{At}. Then, by (4.5.6.4), the inverse in (4.5.6.2) is uniformly bounded for $\lambda \in$

$\Sigma^c(A)$, $|\lambda|$ large, as desired. Having established (4.5.6.3), we then use it in (4.5.6.1) along with estimate of analyticity (4.1.2.3c) for $\theta = 0$ for $R(\lambda, A_h)$. We thus obtain

$$\|R(\lambda, A_P) - R(\lambda, A_{P_h})\|_{\mathcal{L}(U;Y)} \leq \frac{C}{|\lambda - \omega_0|^{2-\gamma}} \|B^*P - B_h^* P_h \Pi_h\|_{\mathcal{L}(Y)}, \quad \lambda \in \Gamma_P. \tag{4.5.6.5}$$

Since $|\lambda| < |\lambda - \omega_0|$ on Γ_P, we then obtain from (4.5.6.5)

$$\begin{aligned} \|e^{A_P t} - e^{A_{P_h} t}\|_{\mathcal{L}(Y)} &\leq C \int_{\Gamma_P} e^{(\mathrm{Re}\,\lambda)t} \frac{C}{|\lambda|^{2-\gamma}} d\lambda \|B^*P - B_h^* P_h \Pi_h\|_{\mathcal{L}(Y;U)}\, d|\lambda| \\ &\leq C\, e^{-\tilde{\omega}_P t} \|B^*P - B_h^* P_h \Pi_h\|_{\mathcal{L}(Y;U)} \to 0 \text{ as } h \downarrow 0, \end{aligned} \tag{4.5.6.6}$$

where the right-hand side of (4.5.6.6) goes to zero by (4.5.4.1). Thus (4.1.4.19) now follows from (4.5.6.6). Theorem 4.1.4.2 is proved. □

4.6 Optimal Rates of Convergence

4.6.1 Introduction

In this section we examine the issue of the "optimal" rates of convergence of the approximating schemes. By "optimal" we mean approximations that reconstruct the optimal regularity of the original continuous solutions, as well as the best approximation properties of the finite-dimensional approximating subspaces.

The main approximating Theorem 4.1.4.1 provides rate of convergence $\mathcal{O}(h^{s(1-\gamma)})$ for the approximating problem; see (4.1.4.8), (4.1.4.10a), (4.1.4.11a), and (4.1.4.14). This rate is, in general, nonoptimal, as it does not reflect the regularity properties of the original continuous problem. More precisely, the regularity properties of the Riccati operator (given by (4.1.1.12)), together with the approximation property (A.2) = (4.1.2.4), suggests that the optimal rate of convergence of Riccati operators reconstructing this regularity should be

$$\|P_h \Pi_h - P\|_{\mathcal{L}(Y)} = \mathcal{O}\big(h^{s(1-\epsilon)}\big). \tag{4.6.1.1}$$

Similarly, because of estimate (4.1.2.10), and because $\exp(A_P)t$ is analytic (see point (6) above (4.1.1.15)), one would expect that the approximating feedback semigroup would retain convergence properties similar to (4.1.2.10), that is,

$$\|e^{A_P t} - e^{A_{h,P_h} t}\|_{\mathcal{L}(Y)} \leq \frac{C\, h^{s\theta}}{t^\theta}\, e^{\omega t}, \quad t > 0, \ 0 \leq \theta \leq 1, \tag{4.6.1.2}$$

where, as in (4.1.4.6), $A_{h,P_h} \equiv A_h - B_h B_h^* P_h$. If the operator B is bounded (i.e., $B \in \mathcal{L}(U; Y)$ and $\gamma = 0$), then the above rates of convergence, (6.1.1) and (6.1.2), are given by Theorem 4.1.4.1; see (4.1.4.8) and (4.1.4.13) [via (4.5.1.27)].

Instead, if $A^{-\gamma}B \in \mathcal{L}(U; Y)$, then (4.1.4.13) of Theorem 4.1.4.1 provides the convergence rates equal to $\mathcal{O}(h^{s(1-\gamma)\theta}/t^\theta)$ that are "nonoptimal" if $\gamma > 0$, instead of (4.6.1.2). Thus, the following question arises: Is it possible to obtain the optimal rates of convergence, (4.6.1.1) and (4.6.1.2) in the unbounded case, that is, when

$A^{-\gamma}B \in \mathcal{L}(U; Y), \gamma > 0$ (particularly, in the interesting case $\gamma > 1/2$)? Below we shall provide a positive answer to the above question, provided, however, very special care is given to the selection of the approximations A_h and B_h. Although the convergence results of Theorem 4.1.4.1 are valid for any consistent approximations A_h and B_h (for instance, $B_h = B$ or $B_h = \Pi_h B$) subject to assumptions (A.1) = (4.1.2.2) through (A.6) = (4.1.2.9), the optimal rates of convergence (4.6.1.1) and (4.6.1.2) will require, in general, additional hypotheses imposed on the approximations of the unbounded operator B.

Finally, it should be noted that in the case of B unbounded, the optimal rates require a more delicate analysis. This is so because of the necessity of "tracing" the singular behavior (at the origin) of the optimal solutions. The crucial role to this end is played by the so-called rough data estimates, together with a perturbation result that asserts, roughly speaking, that relatively bounded perturbations preserve "uniform analyticity" and "uniform stabilizability" with estimates independent of the parameter of discretization.

4.6.2 Main Results on Optimal Rates of Convergence

In this subsection, we shall formulate our main abstract results. Proofs are given in Lasiecka [1992].

The additional approximation properties for the operators B_h and A_h are the following.

(A.7) Let $U_{r_0} \subseteq U \subseteq U_{r_1}$ be two additional Hilbert spaces such that

(i)

$$\left\|[\hat{A}_h^{-1}B_h - \hat{A}^{-1}B]u\right\|_Y \leq C\, h^{\rho}\|u\|_{U_{r_1}}, \tag{4.6.2.1}$$

(ii)

$$\left\|[\hat{A}_h^{-1}B_h - \hat{A}^{-1}B]u\right\|_Y \leq C\, h^{r_0}\|u\|_{U_{r_0}}, \tag{4.6.2.2}$$

where $0 \leq r_0 \leq s$, and $\rho > 0$.

(A.8) Let Y_{r_1} be another Hilbert space such that for some $\epsilon > 0$, $Y_{r_1} \supset \mathcal{D}(\hat{A}^{1-\epsilon}) \cap \mathcal{D}(\hat{A}^{*1-\epsilon})$ and

(i)

$$\hat{A}^{-1+\epsilon}B \in \mathcal{L}\left(U_{r_1}; Y\right), \quad \hat{A}^{-1}B \in \mathcal{L}\left(U_{r_0}; Y_{r_1}\right); \tag{4.6.2.3}$$

(ii)

$$\|[B_h^*(\hat{A}_h^*)^{-1}\Pi_h - B^*(\hat{A}^*)^{-1}]y\|_{U_{r_1}} \leq C\, h^{r_1}\|y\|_{Y_{r_1}}, \tag{4.6.2.4}$$

where $0 \leq r_1 \leq s$.

(A.9)

(i)

$$B^* \hat{A}^{*-2+\epsilon} \in \mathcal{L}\left(Y; U_{r_0}\right), \quad B^* \hat{A}^{*-1+\epsilon} \hat{A}^{-1+\epsilon} \in \mathcal{L}(Y; U). \tag{4.6.2.5}$$

(ii) There exists $n \geq 1$ such that

$$[B^* \hat{A}^{*-1} \hat{A}^{-1} B]^n \in \mathcal{L}\left(U; U_{r_0}\right). \tag{4.6.2.6}$$

Theorem 4.6.2.1 *In addition to the hypotheses of Theorem 4.1.4.1 (in particular (A.1) = (4.1.2.2) through (A.6) = (4.1.2.9)), assume the above hypotheses (A.7)–(A.9). Then, with $\epsilon > 0$ arbitrarily small, and C independent of h and t, we obtain the following estimates:*

(i)

$$\|P - P_h \Pi_h\|_{\mathcal{L}(Y)} \leq C\big[h^{s(1-\epsilon)} + h^{r_0} + h^{r_1}\big], \tag{4.6.2.7}$$

(ii)

$$\|B_h^* P_h \Pi_h - B^* P\|_{\mathcal{L}(Y)} \leq C\, h^{-\gamma(s+\epsilon)}[h^s + h^{r_0} + h^{r_1}]. \tag{4.6.2.8}$$

Theorem 4.6.2.2 *Assume the same hypotheses as in Theorem 4.6.2.1. Then, there exists $\omega_0 > 0$, such that for any $\epsilon > 0,\ t > 0,\ x \in Y$, we have:*

(i)

$$\begin{aligned}\left\|y^0(t;x) - y_h^0(t;\Pi_h x)\right\|_Y &= \|[e^{A_P t} - e^{A_{h,P_h} t}]x\|_Y \\ &\leq \frac{Ce^{-\omega_0 t}}{t^{1-\epsilon}}\|x\|_Y \left[h^{s(1-\epsilon)} + h^{r_0} + h^{r_1}\right],\end{aligned} \tag{4.6.2.9}$$

(ii)

$$\left\|u^0(t;x) - u_h^0(t;\Pi_h x)\right\|_U \leq \frac{Ce^{-\omega_0 t}}{t^{\gamma-\epsilon}}\|x\|_Y \left[h^{s(1-\epsilon)} + h^{r_0} + h^{r_1}\right]. \tag{4.6.2.10}$$

Moreover, recalling $A_{P_h} = A - BB_h^* P_h \Pi_h$ from (4.5.1.4), we have

(iii)

$$\|e^{A_P t} - e^{A_{P_h} t}\|_{\mathcal{L}(Y)} \leq C\, e^{-\omega_0 t} h^{-\gamma(s+\epsilon)}[h^s + h^{r_0} + h^{r_1}], \tag{4.6.2.11}$$

(iv)

$$\|e^{A_P t} - e^{A_{P_h} t}\|_{\mathcal{L}(Y)} \leq \frac{C\, e^{-\omega_0 t}}{t^{\gamma-\epsilon}}\left[h^{s(1-\epsilon)} + h^{r_0} + h^{r_1}\right]. \tag{4.6.2.12}$$

Corollary 4.6.2.3 *Assume the same hypotheses as in Theorem 4.6.2.1. Let $x \in \mathcal{D}(A)$; then*

$$\left\|u^0(t;x) - u_h^0(t;\Pi_h x)\right\|_U \leq C\, e^{-\omega_0 t}\big[h^{s(1-\epsilon)} + h^{r_0} + h^{r_1}\big]\|x\|_{\mathcal{D}(A)}, \tag{4.6.2.13}$$

$$\begin{aligned}\left\|y^0(t;x) - y_h^0(t;\Pi_h x)\right\|_Y &= \|[e^{A_P t} - e^{A_{h,P_h} t}]x\|_Y \\ &\leq C\, e^{-\omega_0 t}\big[h^{s(1-\epsilon)} + h^{r_0} + h^{r_1}\big]\|x\|_{\mathcal{D}(A)}.\end{aligned} \tag{4.6.2.14}$$

Remark 4.6.2.1 Equation (4.6.2.7) of Theorem 4.6.2.1 and Eqns. (4.6.2.9) and (4.6.2.10) of Theorem 4.6.2.2 give the optimal rates of convergence [as in (4.6.1.1) and (4.6.1.2)], if we can take $r_0 = r_1 = s$ in the additional assumptions (A.7) through (A.9). In the next chapter, we shall show that in the "concrete" parabolic-like problems, which motivate this present abstract treatment, suitable approximating schemes can be constructed that do comply with the additional assumptions (A.7) through (A.9) with $r_0 = r_1 = s$. For instance, in the case of parabolic equations with Dirichlet boundary control, the so-called Nitsche scheme provides an example of approximating algorithms that satisfy all the requirements (A.7) through (A.9) with $r_0 = r_1 = s = 2$. Instead, in the case of parabolic equations with Neumann boundary controls, it suffices to consider the usual Galerkin approximation of the basic elliptic operator to fulfill the assumptions (A.7) through (A.9) with $r_0 = r_1 = s = 2$. Additional examples involving strongly damped platelike equations (which have a parabolic behavior) with boundary or point controls are also considered in the next chapter.

The values of r_0 and r_1 in assumptions (A.7) through (A.9) depend on how good the approximations B_h of B are. We shall consider two cases in the next two corollaries.

Corollary 4.6.2.4 *Assume the hypotheses of Theorem 4.1.4.1 and, in addition, that B is bounded, $B \in \mathcal{L}(U; Y)$. Then, if we take*

$$B_h = \Pi_h B \quad or \quad B_h = B, \tag{4.6.2.15}$$

the additional assumptions (A.7) through (A.9) are automatically satisfied with $r_0 = r_1 = s$, and thus Theorems 4.6.2.1and 4.6.2.2 hold true with $r_0 = r_1 = s$.

Proof. It is enough to take $U_{r_0} = U_{r_1} = U$, $Y_{r_1} = Y$, and $r_0 = r_1 = s$ to see that, with B bounded and B_h as in (4.6.2.15), then assumptions (A.7) and (A.8) are a consequence of assumption (A.2) = (4.1.2.4), while properties (A.9) (i), (ii) in (4.6.2.5), (4.6.2.6) are trivially true. $\square$

Corollary 4.6.2.5 *Assume the hypotheses of Theorem 4.1.4.1. Take $B_h = \Pi_h B$. Then, the additional assumptions (A.7) through (A.9) are automatically satisfied with $r_0 = r_1 = s(1-\gamma)$, and thus Theorems 4.6.2.1 and 4.6.2.2 hold true with $r_0 = r_1 = s(1-\gamma)$.*

Proof. We take $U_{r_0} = U_{r_1} = U$; $Y_{r_1} = Y$, with $r_0 = r_1 = s(1-\gamma)$.

(A.8) To verify (A.8), we estimate by first adding and subtracting $B^* \Pi_h \hat{A}^{*-1} y$:

$$\begin{aligned}\left\| B^*(\hat{A}_h^{*-1}\Pi_h - \hat{A}^{*-1})y \right\|_U &\le \left\| B^*(\hat{A}_h^{*-1}\Pi_h - \Pi_h \hat{A}^{*-1})y \right\|_U \\ &\quad + \|B^*(\Pi_h - I)\hat{A}^{*-1})y\|_U. \end{aligned} \tag{4.6.2.16}$$

Next, we invoke (A.3) = (4.1.2.6) followed by (A.2) = (4.1.2.4) (in dual form):

$$\begin{aligned}\left\| B^*(\hat{A}_h^{*-1}\Pi_h - \Pi_h \hat{A}^{*-1})y \right\|_U &\le C\, h^{-\gamma s} \left\| (\hat{A}_h^{*-1}\Pi_h - \Pi_h \hat{A}^{*-1})y \right\|_U \\ &\le C\, h^{-\gamma s} h^s \|y\|_Y, \end{aligned} \tag{4.6.2.17}$$

and, finally, we use (A.4) = (4.1.2.7) to get

$$\|B^*(\Pi_h - I)\hat{A}^{*-1}y\|_U \leq C\, h^{s(1-\gamma)}\|y\|_Y. \tag{4.6.2.18}$$

Using (4.6.2.17) and (4.6.2.18) in (4.6.2.16) yields

$$\left\| B^*\left(\hat{A}_h^{*-1}\Pi_h - \hat{A}^{*-1}\right)y\right\|_U \leq C\, h^{s(1-\gamma)}\|y\|_Y, \tag{4.6.2.19}$$

which proves the approximating assumption (4.6.2.4) in (A.8), with $B_h = \Pi_h B$, as desired.

(A.7) With $Y_{r_1} = Y$, $r_0 = r_1$, then (A.7) follows from (A.8) by duality.

(A.8) The validity of (4.6.2.6) with $n = 1$ and $U_{r_0} = U$ follows from the basic assumption (4.1.1.4) on B. □

4A A Sharp Result on the Exponential Operator-Norm Decay of a Family of Strongly Continuous Semigroups

In the present appendix, after [Triggiani, 1994], given a family $T_h(t)$ of strongly continuous semigroups on a Banach space X, we give a sharp result on the exponential decay in $\mathcal{L}(X)$ of the family, which is uniform in the parameter h. Sharpness is shown by means of counterexamples. This result generalizes a well-known and very useful criterion [Datko, 1970; Pazy, 1983] of stability of a single strongly continuous semigroup, which is invoked, in fact, in every chapter of this book dealing with the infinite time horizon problem. The generalization of the present appendix is likewise critically invoked in the proof of Theorem 4.4.3.1 of the present chapter.

Case of a Single Semigroup T(t) Let X be a Banach space, and let $T(t) \in \mathcal{L}(X)$ be a strongly continuous semigroup of bounded operators on X, $t \geq 0$. The following result is well known.

Theorem 4A.1 $T(t)$ is (exponentially) stable in $\mathcal{L}(X)$:

$$\|T(t)\|_{\mathcal{L}(X)} \leq Me^{-\mu t}, \quad t \geq 0 \text{ for some } M \geq 1,\ \mu > 0, \tag{4A.1}$$

if (and only if) for some p, $1 \leq p < \infty$,

$$\int_0^\infty \|T(t)x\|_X^p\, dt < \infty \text{ for every } x \in X. \tag{4A.2}$$

The "if part" of this result was first established for $p = 2$ and for X a Hilbert space in Datko [1970] and was later generalized to the above statement in Pazy [1983] (see also [Pazy, 1982, Theorem 4.1, p. 116]), with a different proof. Further generalizations of Theorem 4A.1 are possible, and they are noted, for example, in [Triggiani, 1994, p. 388] and in [Bensoussan et al., 1992, p. 21]. However, they are beyond the needs of the present book.

Case of a Family $T_h(t)$ of Semigroups The aim of this appendix is to give the following sharp generalization of Theorem 4A.1 to the case of a family $T_h(t)$ of strongly continuous semigroups on X, depending on the parameter h, as needed in the proof of Theorem 4.4.3.1.

Theorem 4A.2 [Triggiani, 1994] Assume that

(i) there exist constants $M \geq 1$ and $\omega > 0$, independent of h, such that

$$\|T_h(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \quad t \geq 0, \text{ uniformly in } h; \tag{4A.3}$$

(ii) there exists a constant $c > 0$, independent of h, and some p, $1 \leq p < \infty$, such that

$$\sup_h \int_0^\infty \|T_h(t)x\|_X^p \, dt < \infty \quad \text{for all } x \in X, \tag{4A.4a}$$

or, equivalently, by the closed graph theorem or the Principle of Uniform Boundedness,

$$\int_0^\infty \|T_h(t)x\|_X^p \, dt \leq c\|x\|_X^p, \quad \text{for all } x \in X, \text{ uniformly in } h. \tag{4A.4b}$$

Then there exists constants $K \geq 1$ and $\mu > 0$, independent of h, such that

$$\|T_h(t)\|_{\mathcal{L}(X)} \leq Ke^{-\mu t}, \quad t \geq 0, \ \mu > 0 \text{ uniformly in } h. \tag{4A.5}$$

First Proof of Theorem 4A.2 This proof is a suitable variation of that given in [Pazy, 1983, p. 116] in the case of a single s.c. semigroup. The main point of the proof, which requires modifications over the proof of Pazy [1983], is singled out in the following lemma.

Step 1. Lemma 4A.3 Under the assumptions of Theorem 4A.2 we have: There exists a constant $k > 0$, independent of h, such that

$$\|T_h(t)\|_{\mathcal{L}(X)} \leq k, \quad \text{for all } t \geq 0, \text{ uniformly in } h. \tag{4A.6}$$

Proof of Lemma 4A.3 It suffices to show that for each $x \in X$, there exists a constant C_x (independent of h), such that

$$\|T_h(t)x\|_X \leq C_x, \quad \text{for all } t \geq 0; \text{ for all } x \in X; \text{ for all } h, \tag{4A.7}$$

for then (4A.7) implies (4A.6) by the Principle of Uniform Boundedness. We then prove (4A.7). Suppose (4A.7) is false. Then, there exist an $x \in X$, a sequence $h_j \downarrow 0$, and a sequence $t_j \uparrow +\infty$, as $j \to +\infty$, such that

$$\delta_j \equiv \left\|T_{h_j}(t_j)x\right\|_X \to +\infty \quad \text{as } j \to +\infty. \tag{4A.8}$$

Without loss of generality we may take $t_{j+1} - t_j > 1/\omega$, where ω is given by (4A.3). Define

$$\Delta_j = \left[t_j - \frac{1}{\omega}, t_j \right], \tag{4A.9}$$

so that the intervals Δ_j do not overlap. Let now $t \in \Delta_j$; then

$$\delta_j \left\| T_{h_j}(t_j)x \right\|_X = \left\| T_{h_j}(t_j - t) T_{h_j}(t)x \right\|_X \leq \left\| T_{h_j}(t_j - t) \right\|_{\mathcal{L}(X)} \left\| T_{h_j}(t)x \right\|_X. \tag{4A.10}$$

But, by (4A.3), the choice of t, and the size of Δ_j, we obtain

$$\left\| T_{h_j}(t_j - t) \right\|_{\mathcal{L}(X)} \leq M e^{\omega(t_j - t)} \leq M e^{\omega \frac{1}{\omega}} = M\, e, \tag{4A.11}$$

which inserted into (4A.10) yields

$$\left\| T_{h_j}(t)x \right\|_X \geq \frac{\delta_j}{Me}, \quad t \in \Delta_j. \tag{4A.12}$$

Hence, using (4A.12) and the size $1/\omega$ of Δ_j from (4A.9), we obtain by (4A.8)

$$\sup_h \int_0^\infty \|T_h(t)x\|_X^p \, dt \geq \int_{\Delta_j} \left\| T_{h_j}(t)x \right\|_X^p \geq \left(\frac{\delta_j}{Me} \right)^p \frac{1}{\omega} \to +\infty \text{ as } j \to \infty, \tag{4A.13}$$

and (4A.13) contradicts assumption (4A.4). Thus (4A.7) holds true.

Step 2 Once Lemma 4A.3 is proved, we may finish off the proof of Theorem 4A.2, by paralleling Pazy [1983]. For each h fixed and each $x \in X$ fixed, we have

$$T_h(t)x \to 0, \quad \text{as } t \to +\infty, \tag{4A.14}$$

a result established in Pazy [1983], since h is fixed, by arguing by contradiction; indeed, if (4A.14) were false with h fixed, we could find $x \in X$ and $\delta > 0$ and $t_j \to \infty$ (all depending on h fixed) such that $\|T_h(t_j)x\|_X \geq \delta$ and an argument as in (4A.9) through (4A.13) would then lead to a contradiction. Next we define

$$t_{x,h}(\rho) = \max\{t : \|T_h(s)x\|_X \geq \rho\|x\|_X, \text{ for } 0 \leq s \leq t\}. \tag{4A.15}$$

By strong continuity,

$$\|T_h(t_{x,h}(\rho))x\| = \rho\|x\|. \tag{4A.16}$$

Moreover, by (4A.14), $t_{x,h}(\rho)$ is finite and positive for every h and x. By (4A.15),

$$t_{x,h}(\rho)\rho^p \|x\|_X^p \leq \int_0^{t_{x,h}(\rho)} \|T_h(t)x\|_X^p \, dt \leq \int_0^\infty \|T_h(t)x\|_X^p \, dt \leq c\|x\|_X^p, \tag{4A.17}$$

recalling assumption (4A.4) in the last step of (4A.17), whereby

$$t_{x,h}(\rho) \leq \frac{c}{\rho^p} \text{ (independent of } h) \equiv t_0. \tag{4A.18}$$

For $t > t_0$, in the norm of X,

$$\|T_h(t)x\| \le \|T_h(t - t_{x,h}(\rho))\| \, \|T_h(t_{x,h}(\rho))x\| \le k\rho\|x\|, \quad t > t_0, \text{ uniformly in } h, \tag{4A.19}$$

where in the last step of (4A.19), we have recalled (4A.6) of Lemma 4A.3, as well as (4A.16). Given k from (4A.6), we next select $\rho > 0$, so that $k\rho < 1$, and then, via (4A.19):

$$\|T_h(t)x\| \le \beta\|x\| \quad \text{or} \quad \|T_h(t)\|_{\mathcal{L}(X)} < 1, \quad t > t_0, \text{ uniformly in } h, \ \beta = k\rho < 1. \tag{4A.20}$$

Let now $t_1 > t_0$ be fixed, and let $t = nt_1 + s$, $0 \le s < t_1$. Then, in the norm of $\mathcal{L}(X)$, recalling (4A.6) once more:

$$\|T_h(t)\| \le \|T_h(s)\| \, \|T_h(nt_1)\| \le k\|T_h(t_1)\|^n \le k\beta^n. \tag{4A.21}$$

But $n = (t - s/t_1) > (t/t_1) - 1$, and, with $\beta < 1$, then $\beta^n < \frac{1}{\beta}\beta^{t/t_1}$, so that from (4A.21) we obtain

$$\|T_h(t)\| \le \frac{k}{\beta}\,\beta^{t/t_1} = Ke^{-\mu t}, \tag{4A.22}$$

and conclusion (4A.5) is proved, with

$$\mu = \frac{-1}{t_1}\ln\beta > 0, \quad t_1 > t_0; \ K = \frac{k}{\beta}. \ \square \tag{4A.23}$$

Second Proof of Theorem 4A.2 This is a direct proof of both steps, that is, of Lemma 4A.3 and of the key inequality (4A.20). It follows this time the continuous proof as in [Pritchard, Zabczyk, 1981] or [Bensoussan et al., 1992, p. 22].

Second Proof of Lemma 4A.3 We compute with $T(t)x = T(t-r)T(r)x$, $0 \le r \le t$:

$$\frac{1 - e^{-p\omega t}}{p\omega}\|T_h(t)x\|_X^p = \int_0^t e^{-p\omega(t-r)}\|T_h(t)x\|^p\,dr \tag{4A.24}$$

$$\le \int_0^t e^{-p\omega(t-r)}\|T_h(t-r)\|^p\|T_h(r)x\|_X^p\,dr$$

$$\text{(by (4A.3))} \qquad \le M^p\int_0^t \|T_h(r)x\|_X^p\,dr$$

$$\text{(by (4A.4))} \qquad \le M^p c\|x\|_X^p \quad \text{for all } x \in X. \tag{4A.25}$$

Thus, (4A.25) can be rewritten as

$$\|T_h(t)x\|_X^p \le \frac{M^p c\,p\omega}{1 - e^{-p\omega t}}\|x\|_X^p, \quad \text{for all } t > 0. \tag{4A.26}$$

Combining the bound (4A.26) for all $t > 0$ with the assumed bound in (4A.3), say for $0 \le t \le 1$, we obtain the uniform bound (4A.6) once more, and Lemma 4A.3 is reproved.

Second Proof of the Uniform Bound (4A.20) in Step 2 We repeat the computations leading from (4A.24) to (4A.25), this time with $\omega = 0$. We obtain, in the X-norm:

$$t\|T_h(t)x\|^p = \int_0^t \|T_h(t)x\|^p\,dr \leq \int_0^t \|T_h(r)\|^p \|T_h(t-r)x\|^p\,dr \tag{4A.27}$$

$$\text{(by (4A.6))} \qquad \leq k^p \int_0^t \|T_h(t-r)x\|^p\,dr \leq k^p \int_0^\infty \|T_h(\sigma)x\|^p\,d\sigma \tag{4A.28}$$

$$\text{(by (4A.4))} \qquad \leq k^p c\|x\|^p \quad \text{for all } x \in X,\ t \geq 0, \tag{4A.29}$$

where in going from (4A.27) to (4A.28) we have invoked the bound (4A.6) of Lemma 4A.3. Thus, (4A.29) is rewritten as

$$\|T_h(t)\| \leq \frac{k^p c}{t} < 1, \quad \text{for } t > k^p c, \text{ uniformly in } h,$$

which reproves (4A.20), as desired. □

Remark 4A.1 [Triggiani, 1994] Theorem 4A.2 is sharp in the sense that assumption (4A.4) does not imply assumption (4A.3), as the following (classes of) counter-examples from [Triggiani, 1994] show. Thus, the extension of Theorem 4A.1 to a family of s.c. semigroups, based only on assumption (4A.4), is false.

(i) Let $X = L_p(0,1)$, $1 \leq p < \infty$. Let $a_h \geq 1$ be a function of the parameter h such that

$$a_h \uparrow +\infty \text{ as } h \downarrow 0, \tag{4A.30}$$

and define g_h by

$$(a_h)^{g_h} = \ln a_h; \quad \text{or} \quad 1 \geq g_h = \frac{\ln \ln a_h}{\ln a_h} \downarrow 0 \text{ as } h \downarrow 0. \tag{4A.31}$$

(ii) We now define a family $T_h(t)$ of s.c. semigroups on X by setting for $f \in X$ (Figure 4A.1):

$$\{T_h(t)f\}(x) = \begin{cases} (a_h)^{t/p} f\left(x + \frac{t}{g_h}\right), & 0 \leq x \leq 1 - \frac{t}{g_h}, & (4A.32a) \\ 0, & 1 - \frac{t}{g_h} < x \leq 1. & (4A.32b) \end{cases}$$

Thus, the action of T_h on $f \in X$ consists of two main features: translation of f to the left as t increases with speed $1/g_h$ and modulation of its amplitude by $(a_h)^{t/p}$.

(iii) We note that the s.c. semigroups $T_h(t)$ are all nilpotent for $t \geq g_h$ with threshold of nilpotency $g_h \downarrow 0$ as $h \downarrow 0$ from (4A.31). Moreover, the uniform norm is given by

$$\|T_h(t)\|_{\mathcal{L}(X)} = \begin{cases} (a_h)^{t/p}, & 0 \leq t \leq g_h, & (4A.33a) \\ 0, & g_h < t, & (4A.33b) \end{cases}$$

as can be readily seen by taking $L_p(0,1)$-functions f of norm 1, identically 1 near $x = 1$, on progressively smaller intervals with endpoint $x = 1$, and zero elsewhere (Figure 4A.2).

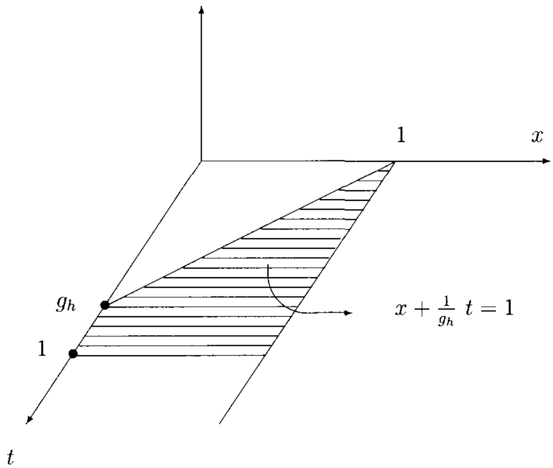


Figure 4A.1

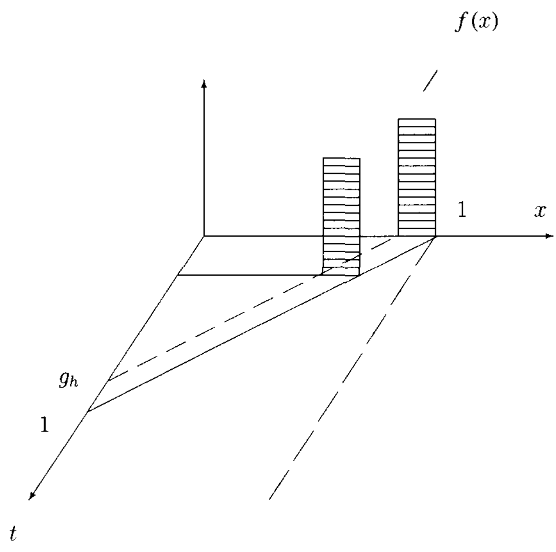


Figure 4A.2

(iv) We now show that the peak (maximum) of $\|T_h(t)\|$, which occurs at $t = g_h \downarrow 0$ by (4A.33), is monotone increasing to $+\infty$ as a function of $h \downarrow 0$, so that, as a consequence, the uniform bound

$$\|T_h(t)\|_{\mathcal{L}(X)} \leq Ce^{at}, \quad t \geq 0 \text{ for some } C \geq 1 \text{ and } a > 0 \tag{4A.34}$$

is violated. In fact, the maximum of $\|T_h(t)\|$ at $t = g_h$ by (4A.33) equals

$$F(h) \equiv \|T_h(g_h)\|_{\mathcal{L}(X)} = (a_h)^{g_h/p}. \tag{4A.35}$$

Thus, recalling (4A.31), we see that (4A.35) implies

$$\ln F(h) = \frac{g_h}{p} \ln a_h = \frac{1}{p} \ln \ln a_h \uparrow +\infty \quad \text{as } h \downarrow 0, \tag{4A.36}$$

so that, as desired

$$\lim_{h \downarrow 0} F(h) = \|T_h(g_h)\|_{\mathcal{L}(X)} = +\infty, \tag{4A.37}$$

and then condition (4A.34) is impossible. Thus, as $h \downarrow 0$, the family $T_h(t)$ has the maximum of its uniform norm, which occurs at $t = g_h \downarrow 0$ and which explodes to $+\infty$ (Figure 4A.3).

(v) Finally, we now prove the following uniform bound:

$$\int_0^\infty \|T_h(t) f\|_X^p \, dt \leq \|f\|_X^p, \quad \forall f \in X, \text{ uniformly in } h. \tag{4A.38}$$

In fact, returning to (4A.33), we compute for $f \in X$, by (4A.32):

$$\begin{aligned} \int_0^\infty \|T_h(t) f\|_X^p \, dt &= \int_0^{g_h} \|T_h(t) f\|_X^p \, dt \\ &= \int_0^{g_h} \left\{ \int_0^{1 - t/g_h} (a_h)^t \left| f\left(x + \frac{t}{g_h}\right) \right|^p dx \right\} dt. \end{aligned} \tag{4A.39}$$

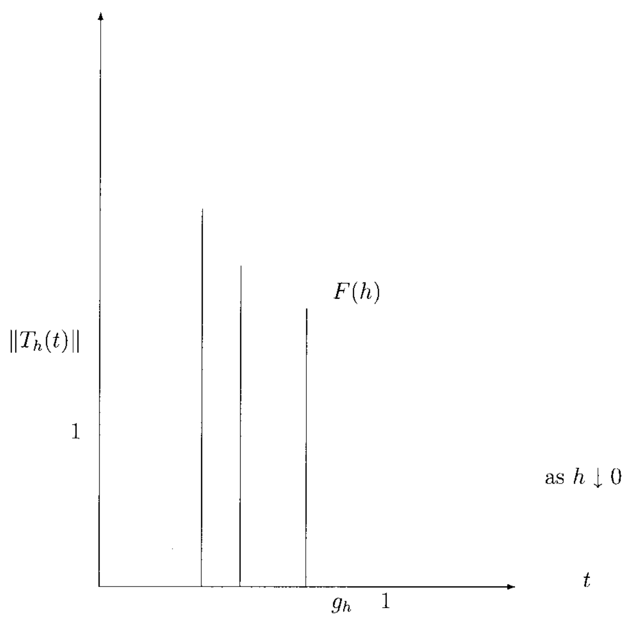


Figure 4A.3

Thus, setting $\sigma = x + t/g_h$ in (4A.39), we have

$$\int_0^\infty \|T_h(t)f\|_X^p\,dt = \int_0^{g_h} (a_h)^t \int_{t/g_h}^1 |f(\sigma)|^p\,d\sigma\,dt$$
$$\le \left(\int_0^{g_h} (a_h)^t dt\right) \int_0^1 |f(\sigma)|^p\,d\sigma$$
$$= \left[\frac{(a_h)^t}{\ln a_h}\right]_{t=0}^{t=g_h} \|f\|_X^p \le \frac{(a_h)^{g_h}}{\ln a_h}\|f\|_X^p \le \|f\|_X^p, \quad (4A.40)$$

after recalling (4A.31) in the last step in (4A.40). Then (4A.40) proves (4A.38).

In Conclusion The family of s.c. semigroups $T_h(t)$ in (4A.32) satisfies condition (4A.4) with $c = 1$ (see (4A.38)) but does not satisfy condition (4A.3) (see point (iv), violation of (4A.34)).

4B Finite Element Approximations of Dynamic Compensators of Luenberger's Type for Partially Observed Analytic Systems with Fully Unbounded Control and Observation Operators

The finite element approximation theory of algebraic Riccati equations with fully unbounded control operator B presented in this chapter (after Lasiecka and Triggiani [1987; 1991(a)] for Sections 4.1 through 4.5, and Lasiecka [1984; 1992] for Section 4.6) is used in an essential way in the study of finite element approximations of dynamic compensators of Luenberger's type for *partially observed systems*, whose free dynamics is described by an analytic semigroup. Both control operator and observation operator are fully unbounded. Accordingly, we shall provide here a brief account and the main highlights of this theory, which has recently reached a high degree of maturity. Appropriate references will be given below. For adherence to these, the observation operator will be denoted by C rather than R as in the previous sections.

Continuous Problem We put ourselves in the setting of Eqn. (4.1.1.1), rewritten here as

$$\begin{cases} \dot y = Ay + Bu \quad \text{on, say, } [\mathcal{D}(A^*)]',\ y(0) = y_0 \in Y; & (4B.1) \\ z = Cy, & (4B.2) \end{cases}$$

subject to the following standing hypotheses:

Hypothesis 1 This collects all standing continuous assumptions of this Chapter 4, regarding the analytic semigroup generator A, the control operator B, the state and control spaces Y and U, as described in properties (i) and (ii), Eqn. (4.1.1.1) through Eqn. (4.1.1.4), of Section 4.1.1. In addition, we have the following assumption on the observation operator C (previously called R) [see Chapter 1, Section 1.8.1, Eqn. (1.8.1.1) and Chapter 2, Section 2.5, Eqn. (2.5.3)]:

$C : Y \supset \mathcal{D}(C) \to Z$ (observation Hilbert space) is densely defined, is closed, and satisfies the following regularity property:
$CA^{-r} \in \mathcal{L}(Y; Z)$, for some $0 \le r < 1$, or $C \in \mathcal{L}(\mathcal{D}(\hat{A}^r; Z)$. (4B.3)

Standard arguments, of the type used in Chapter 2, Section 2.3.8 from perturbation theory of analytic semigroups yield the following

Claim For any operators $F \in \mathcal{L}(Y; U)$ and $K \in \mathcal{L}(Z; Y)$, then the operators $(A + BF)$ and $(A - KC)$ generate s.c. analytic semigroups $e^{(A+BF)t}$ and $e^{(A-KC)t}$ on $Y, t > 0$.

Hypothesis 2 (Stabilizability/Detectability, possibly with preassigned decay) There exist suitable operators $F \in \mathcal{L}(Y; U)$ and $K \in \mathcal{L}(Z; Y)$ such that the corresponding s.c. analytic semigroups of the preceding claim satisfy

$$\left\|e^{(A+BF)t}\right\|_{\mathcal{L}(Y)} + \left\|e^{(A-KC)t}\right\|_{\mathcal{L}(Y)} \le Me^{-kt}, \quad t \ge 0, \tag{4B.4}$$

for constants $k > 0$ and $M \ge 1$, depending on k. In particular, one may preassign $k > 0$ and require F and K to depending on $k > 0$. See Notes on Chapter 2 and Chapter 3 for boundary stabilization results for parabolic problems, which guarantee (4B.4).

Continuous Dynamic Compensator of Luenberger's Type In the continuous problem (4B.1), (4B.2), it is assumed that only the observation Cy is available, not the full state y. Then, under the preceding setting, the continuous problem (4B.1), (4B.2) admits an infinite-dimensional dynamic compensator of Luenberger's type

$$\dot{z} = (A + BF - KC)z + K(Cy) \text{ in } Y, \quad z(0) = z_0 \in Y \tag{4B.5}$$

based only on the observation (Cy), such that the following results hold true.

Proposition 4B.1 [Lasiecka, 1995] Under the above setting, in particular with F and K as in (4B.3), consider the system

$$\begin{cases} \dot{y} = Ay + Bu, \quad & u = Fz \\ \dot{z} = (A + BF - KC)z + K(Cy) \end{cases} \quad \text{or} \quad \frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \mathbb{A}\begin{bmatrix} y \\ z \end{bmatrix}; \tag{4B.6}$$

$$\mathbb{A} = \begin{bmatrix} A & BF \\ KC & A + BF - KC \end{bmatrix} : \mathcal{Y} \supset \mathcal{D}(\mathbb{A}) \to \mathcal{Y}; \quad \mathcal{Y} \equiv Y \times Y; \tag{4B.7}$$

$$\mathcal{D}(\mathbb{A}) = \{(y_1, y_2) \in \mathcal{Y} : Ay_1 + BFy_2 \in Y;\ KCy_1 + (A + BF - KC)y_2 \in Y\}. \tag{4B.8}$$

Then:

(i) $\mathbb{A}$ generates a s.c. analytic, exponentially stable semigroup $e^{\mathbb{A}t}$ on $\mathcal{Y} = Y \times Y$, with the same exponential decay e^{-kt} as in (4B.3):

$$\|e^{\mathbb{A}t}\|_{\mathcal{L}(\mathcal{Y})} \leq me^{-kt}, \quad t \geq 0; \tag{4B.9}$$

(ii) we have

$$\frac{d}{dt}(y - z) = (A - KC)(y - z), \tag{4B.10}$$

$$\|y(t) - z(t)\|_Y = \left\|e^{(A-KC)t}(y_0 - z_0)\right\|_Y \leq ce^{-kt}\|y_0 - z_0\|_Y, \tag{4B.11}$$

and the compensator $z(t)$, based only on the observation Cy, asymptotically approaches the true state y of the original system.

Proof. For part (i), the usual transformation

$$\mathcal{J} = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \text{ with inverse } \mathcal{J}^{-1} = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \tag{4B.12}$$

transforms $\mathbb{A}$ into its similarity form

$$\tilde{\mathbb{A}} = \mathcal{J}^{-1}\mathbb{A}\mathcal{J} = \begin{bmatrix} A - KC & 0 \\ KC & A + BF \end{bmatrix}. \tag{4B.13}$$

We note that the off-diagonal term KC is unbounded. Using a perturbation argument, one can show [Lasiecka, 1995] that the resolvent of $\tilde{\mathbb{A}}$ satisfies the usual characterization of analyticity, whereby $\mathbb{A}$ satisfies the usual characterization of analyticity, whereby $\tilde{\mathbb{A}}$ and hence $\mathbb{A}$ generate similar s.c. analytic semigroups $e^{\tilde{\mathbb{A}}t}$ and $e^{\mathbb{A}t}$; moreover, by (4B.4), (4B.9) follows from (4B.13).

(ii) This part follows after a cancellation of BFz. □

Discrete Version In the present context, the main goal is to construct a *finite-dimensional* compensator z_h, with h a parameter of discretization – which is based on a finite-dimensional approximation of the original continuous problem – such that the finite-dimensional feedback control $u_h = F_h z_h$, once inserted into the original system, will produce solutions that are exponentially decaying with bound $e^{-(k-\epsilon)t}$ (with an ϵ-extra margin over the decay (4B.9) of the continuous system), uniformly in the parameter of discretization $h \searrow 0$.

Approximating Subspaces and Operators

Approximating Subspaces V_h We introduce a family of finite-dimensional approximating subspaces $V_h \subset Y \cap \mathcal{D}(B^*) \cap \mathcal{D}(C)$, where h is a parameter of discretization that tends to zero, $0 < h \leq h_0$. Let Π_h, as before, be the orthogonal projection of Y

onto V_h, with the usual approximating property, for some $s > 0$:

$$\|\Pi_h x - x\|_Y \leq M h^s \|x\|_{\mathcal{D}(A)}, \quad x \in \mathcal{D}(A). \tag{4B.14}$$

Throughout, M is a generic constant independent of h.

Approximation A_h of A Let $A_h : V_h \to V_h$ be a finite-dimensional approximation of A that satisfies *Assumptions 1* and *2* below:

Assumption 1: This is the requirement (A.1) of *uniform analyticity* formulated in (4.1.2.2) in the t-domain or in (4.1.2.3) in the λ-domain.

Assumption 2:

$$\left\|\Pi_h \hat{A}^{-1} - \hat{A}_h^{-1}\Pi_h\right\|_{\mathcal{L}(Y)} \leq M h^s, \tag{4B.15}$$

for the same $s > 0$ as in (4B.14).

Remark 4B.1 *Properties (4B.14) and (4B.15) imply assumption (A.2) = (4.1.2.4). Thus,* all the approximating assumptions of A*:* (A.1), (A.2) of Section 4.1.2, are reassumed here.

Approximation Assumptions on the Operators B and C The following *Assumptions 3(i), 4(i), 5(i)* for B are precisely (A.3) = (4.1.2.6), (A.4) = (4.1.2.7), (A.6) = (4.1.2.9), respectively. Moreover, since we are presently taking $B_h = \Pi_h B$, then (A.5) = (4.1.2.8) is contained in (A.4) = (4.1.2.7), as remarked there. Thus, all *the approximating assumptions* of B: *(A.3) through (A.6) of Section 4.1.2, are reassumed here.*

Moreover, we impose on C analogous assumptions: They are obtained from those in B, by replacing B^* with C and the constant γ for B in (4.1.1.4) (degree of unboundedness of B) with the constant r for C in (4B.3) (degree of unboundedness of C). Throughout here below, s is the same constant as in (4B.14) and (4B.15).

Assumption 3: (Inverse approximation properties)

(i)

$$\|B^* x_h\|_U \leq M h^{-\gamma s} \|x_h\|_Y, \quad \forall\, x_h \in V_h; \tag{4B.16}$$

(ii)

$$\|C x_h\|_Z \leq M h^{-rs} \|x_h\|_Y, \quad \forall\, x_h \in V_h. \tag{4B.17}$$

Assumption 4: (Convergence)

(i)

$$\|B^*(\Pi_h - I)x\|_U \leq M h^{s(1-\gamma)} \|x\|_{\mathcal{D}(A^*)}, \quad x \in \mathcal{D}(A^*); \tag{4B.18}$$

(ii)

$$\|C(\Pi_h - I)x\|_Z \leq M h^{s(1-r)} \|x\|_{\mathcal{D}(A)}, \quad x \in \mathcal{D}(A); \tag{4B.19}$$

Assumption 5: (Stability)

(i)

$$\|B^*\Pi_h x\|_U \le M\|(\hat{A}^*)^\gamma x\|_Y, \quad x \in \mathcal{D}((\hat{A}^*)^\gamma); \tag{4B.20}$$

(ii)

$$\|C\Pi_h x\|_Z \le M\|(\hat{A}^r)x\|_Y, \quad x \in \mathcal{D}(\hat{A}^r). \tag{4B.21}$$

Remark 4B.2 We summarize: When restricted on A, B, V_h, the above assumptions reduce to those (A.1) = (4.1.2.2) through (A.6) = (4.1.2.9), with $B_h = \Pi_h B$ of Section 4.1.2. One could also approximate the operator B by its approximation $B_h \in \mathcal{L}(U; V_h)$ as in (A.3)–(A.6). We have already noted in Remark 4.1.2.2 that all these are standard approximation properties satisfied by typical schemes defined on a quasi-uniform grid, except possibly for (A.1), which needs to be verified in each case.

Approximations F_h and K_h of F and K

Assumption 6: The operator $F_h : V_h \to U$ satisfies one of the following conditions:

(a) either, when $h \searrow 0$,

$$F_h\Pi_h \to F \text{ strongly, and } B^*R(\lambda_0, A^*) \text{ is compact;} \tag{4B.22}$$

(b) or

$$\|F_h\Pi_h - F\|_{\mathcal{L}(Y;U)} \to 0 \text{ as } h \downarrow 0. \tag{4B.23}$$

Assumption 7: Similarly, $K_h : Z \to V_h$ satisfies as $h \downarrow 0$

(a) either

$$K_h \to K \text{ strongly and } CR(\lambda_0, A) \text{ is compact,} \tag{4B.24}$$

(b) or

$$\|K_h - K\|_{\mathcal{L}(Z;Y)} \to 0, \quad \text{as } h \downarrow 0. \tag{4B.25}$$

Use of the Approximating Algebraic Riccati Theory of Chapter 4 (Theorem 4.1.4.1) to Satisfy Constructively Assumption 6 and Assumption 7 in the form of Eqn. (4B.23) and (4B.25)

Step 1 Under Hypotheses 1 and 2, the following dual algebraic Riccati equations,

$$(A^*Px, y)_Y + (PAx, y)_Y + (x, y)_Y = (B^*Px, B^*Py)_U \text{ say,} \quad \text{for } x, y \in \mathcal{D}(A); \tag{4B.26}$$

$$(AQx, y)_Y + (QA^*x, y)_Y + (x, y)_Y = (CQx, CQy)_Y \text{ say,} \quad \text{for } x, y \in \mathcal{D}(A^*), \tag{4B.27}$$

admit, respectively, a unique positive definite solution $P = P^* \in \mathcal{L}(Y)$ and $Q^* = Q \in \mathcal{L}(Y)$, such that

$$\begin{aligned} &B^*P \in \mathcal{L}(U;Y);\\ &\quad e^{(A-BB^*P)t} \text{is a s.c. analytic, exponentially stable semigroup on } Y; \end{aligned} \tag{4B.28}$$

$$\begin{aligned} &CQ \in \mathcal{L}(Y;Z);\\ &\quad e^{(A^*-C^*CQ)t} \text{is a s.c. analytic, exponentially stable semigroup on } Y. \end{aligned} \tag{4B.29}$$

These results are, plainly, an application of Chapter 2, Theorem 2.2.1 with $Z = Y$ and $R =$ Identity, so that the Stabilizability Condition (SC) = (4.1.1.8) and the Detectability Condition (DC) = (4.1.1.9) hold true; see also the Summary (4.1.1.10) through (4.1.1.16) of Section 4.1.1.

Step 2 Consider the discrete version ARE$_h$ (see (4.1.3.3)) of (4B.26) on V_h:

$$A_h^* P_h + P_h A_h + \Pi_h = \Pi_h B B^* P_h, \tag{4B.30}$$

where $B_h = \Pi_h B$. It has a unique positive definite solution $P_h^* = P_h \in \mathcal{L}(V_h)$. We now assume the discrete *Assumption 1* through *Assumption 5* (which, by Remark 4B.2, reduce to assumptions (A.1) = (4.1.2.2) through (A.6) = (4.1.2.9) of Section 4.1.1). Since, presently, $R =$ Identity, $Z = Y$, in Step 1, Theorem 4.1.4.1 is applicable to (4B.29) and (4B.26), provided that we assume, say, (4.1.4.3), that is, the assumption: $B^*R(\lambda_0, A^*)$ is compact, of (4B.22). Thus, Theorem 4.1.4.1 yields, with $B_h = \Pi_h B$ (see Eqn. (4.1.4.8) and Eqn. (4.1.4.9)),

$$\|P_h\Pi_h - P\|_{\mathcal{L}(Y)} + \|B^*(P_h\Pi_h - P)\|_{\mathcal{L}(Y;U)} \to 0 \text{ as } h \downarrow 0. \tag{4B.31}$$

Step 3 We next consider the discrete version of (4B.27) on V_h:

$$A_h Q_h + Q_h A_h^* + \Pi_h = Q_h C^* C Q_h, \tag{4B.32}$$

which has a unique positive definite solution $Q_h^* = Q_h \in \mathcal{L}(V_h)$. Theorem 4.1.4.1 is again applicable to (4B.31) and (4B.27), if we assume the discrete *Assumption 1* through *Assumption 5* with reference to C, as well as the assumption: $CR(\lambda_0, A)$ is compact, in (4B.24), the counterpart of the assumption that $B^*R(\lambda_0 A^*)$ is compact in Step 2. (B^* replaces C; A replaces A^*). As a consequence, Theorem 4.1.4.1 produces the counterpart of (4B.30), that is,

$$\|Q_h\Pi_h - Q\|_{\mathcal{L}(Y)} + \|C(Q_h\Pi_h - Q)\|_{\mathcal{L}(Y)} \to 0 \text{ as } h \downarrow 0. \tag{4B.33}$$

Step 4 By (4B.28) and (4B.29), and by (4B.31) and (4B.33), we see that the above procedure – which critically relies on Theorem 4.1.4.1 – permits us to define constructively operators

$$F = B^*P; \quad F_h \equiv B^*P_h; \quad K = QC^*; \quad K_h = Q_h\Pi_h C^*, \tag{4B.34}$$

so that the decay (4B.4) holds true for some $k > 0$ (and, moreover, the uniform convergence (4B.23) and (4B.25) required by *Assumptions 6* and *7* hold true likewise).

We have proved the following result.

Proposition 4B.2 *Assume the continuous Hypotheses 1 and 2, and, moreover, the discrete* Assumption 1 *through* Assumption 5 *of the present Appendix 4B. Finally, let the following conditions hold true:*

$$B^*R(\lambda_0, A^*) \text{ is compact} : Y \to U; \quad CR(\lambda_0, A) \text{ is compact} : Y \to Z \tag{4B.35}$$

(same as the continuous condition in (4B.22) and (4B.24)). Then, the procedure above – based on the critical use of Theorem 4.1.4.1 of this chapter – allows one to define the operators F, K, F_h, and K_h as in (4B.34) such that the standing decay property (4B.4) and the uniform convergence properties (4B.23) and (4B.25) of Assumptions 6 *and* 7 *hold true.*

Discrete Dynamic Compensator of Luenberger's Type Within the approximating setting introduced above, we consider the following finite-dimensional dynamic compensator on V_h:

$$\dot{z}_h = (A_h + \Pi_h BF_h - K_hC)z_h + K_hCy, \quad z_h(0) \in V_h. \tag{4B.36}$$

This is the discrete version of (4B.5), based again on the observation (Cy), which, once inserted into the original continuous dynamics (4B.1) produces the system ($\zeta \in V_h, \eta \in Y$):

$$\begin{cases} \dot{\eta} = A\eta + Bu_h, \quad u_h = F_h\zeta \\ \dot{\zeta} = (A_h + \Pi_h BF_h - K_hC)\zeta + K_hC\eta \end{cases} \quad \text{or} \quad \frac{d}{dt}\begin{bmatrix}\eta\\ \zeta\end{bmatrix} = \mathbb{A}_h\begin{bmatrix}\eta\\ \zeta\end{bmatrix}; \tag{4B.37}$$

$$\mathbb{A}_h = \begin{bmatrix} A\Pi_h & BF_h\Pi_h \\ K_hC & A_h + \Pi_hBF_h\Pi_h - K_hC\end{bmatrix} \quad \text{on } \mathcal{Y} = Y \times Y. \tag{4B.38}$$

Main Results We can finally state the main results of the present Appendix B:

Theorem 4B.3 [Lasiecka, 1995] *Assume the continuous Hypotheses 1 and 2 and the discrete* Assumption 1 *through* Assumption 7 *of the present appendix. Assume, further, that:*

$$\gamma + r < 1, \quad \text{where } A^{-\gamma}B \in \mathcal{L}(U; Y); \quad CA^{-r} \in \mathcal{L}(Y; Z); \tag{4B.39}$$

see (4.1.1.4) and (4B.3). Then:

(i) The operator $\mathbb{A}_h$ in (4B.38) generates a s.c. analytic semigroup $e^{\mathbb{A}_h t}$ on $\mathcal{Y} = Y \times Y$.

(ii) $e^{\mathbb{A}_h t}$ possesses the following property: With $k > 0$ the constant of exponential decay of the continuous dynamics $e^{\mathbb{A}t}$ in (4B.9) [same as in (4B.4)], given $\epsilon > 0$, there exists $h_\epsilon > 0$ such that, for all h with $0 < h \leq h_\epsilon$, the semigroup $e^{\mathbb{A}_h t}$ satisfies the following uniform (in h) exponential decay:

$$\|e^{\mathbb{A}_h t}\|_{\mathcal{L}(\mathcal{Y})} \leq Me^{-(k-\epsilon)t}, \quad \forall\, t \geq 0, \ 0 < h < h_\epsilon, \tag{4B.40}$$

with M independent on h, but dependent on $k - \epsilon$.

(iii) If $y = [y_1, y_2] \in \mathcal{Y}$, then

$$\left\| e^{\mathbb{A}_h t} \begin{bmatrix} \Pi_h y_1 \\ \Pi_h y_2 \end{bmatrix} - e^{\mathbb{A} t} y \right\|_{\mathcal{Y}} \to 0 \textit{ as } h \downarrow 0 \tag{4B.41}$$

uniformly on $0 \leq t \leq \infty$.

Thus, under a very general approximating scheme (see Remark 4.1.2.2), and for fully unbounded control and observation operators B and C such that (4B.39) holds true, Theorem 4B.2 provides a *finite-dimensional* control $u_h = F_h z_h$, with z_h the dynamic compensator in (4B.36), such that when inserted into the original continuous system yields exponential stability, uniformly in h, modulo a penalization $e^{\epsilon t}$ over the corresponding continuous problem in (4B.9).

Using Proposition 4B.1, we obtain the following:

Corollary 4B.4 [Lasiecka, 1995] *Under the assumptions of Proposition 4B.2 to integrate those of Theorem 4B.3, then Theorem 4B.3 holds true with operators F, K, F_h, and K_h constructed by the continuous and approximating algebraic Riccati theory of the present chapter, as in (4B.34).*

Applications of Theorem 4B.3 Papers of the literature dealing with parabolic (analytic semigroups) systems prior to Lasiecka [1995] dealt with the case where the control/observation operators B and C are either *bounded*, or only *mildly unbounded*, that is, satisfying $\gamma + r \leq 1/2$ (see (4B.39)). This is a severe limitation, which then admits a classical variational formulation of the problem; see Notes below. The case $\gamma + r > 1/2$ is both physically very relevant and mathematically much more challenging. Theorem 4B.3 was proved in Lasiecka [1995], where several examples are given [Lasiecka, 1995, Section 6], all satisfying $1/2 < \gamma + r < 1$. They include the following ones:

\# 1: An unstable heat equation in y (temperature) defined on a smooth, or convex, bounded domain $\Omega \subset \mathbb{R}^n$, with Dirichlet-boundary control u, and observation

$$Cy = \left(\int_\Omega y(t, s) w(s)\, d\Omega \right) g(\xi), \quad \xi \in \Gamma,$$

where $g \in L_2(\Gamma)$ and $w \in H^{-\alpha}(\Omega)$, $0 \leq \alpha < 1/2$, so that $C \in \mathcal{L}(H^\alpha(\Omega); L_2(\Gamma))$. Then, $r = 1/4 - \epsilon_1$ and $\gamma = 3/4 + \epsilon_2$, and $1/2 < \gamma + r < 1$.

\# 2: An unstable heat equation in y (temperature) defined on a smooth, or convex, bounded domain $\Omega \subset \mathbb{R}^n$, with Neumann-boundary control u, and observation $Cy = y|_\Gamma$. Here, $r = 1/4 + \epsilon_1$, $\gamma = 1/4 + \epsilon_2$, and $1/2 < \gamma + r < 1$.

\# 3: An unstable structurally damped plate equation in $[w, w_t]$ (displacement and velocity) with Kelvin–Voigt damping on a smooth, or convex, bounded domain $\Omega \subset \mathbb{R}^2$, with free boundary conditions, interior point control and observation $C(w, w_t) = \frac{\partial w_t}{\partial \nu}|_\Gamma$. Here $r = 1/2$, $\gamma = 1/4 + \epsilon$, and $1/2 < \gamma + r < 1$. Moreover, in this case the resolvent of A is *not* compact.

The above examples indicate the desirability to push further Theorem 4B.3 to the maximum degree allowable: $\gamma < 1, r < 1, \gamma + r > 1$, without running into ill-posed problems. This is possible, and an improvement of Theorem 4B.3 from Ji and Lasiecka [1998] is given below. However, it requires an additional more restrictive, more special approximation in the area of support of the unbounded control/observation operators B and C.

Assumption 8: For a constant M independent of h, and λ_0 a fixed point in the resolvent of A and A_h, we have:

$$\|CR(\lambda_0, A)BF_h\Pi_h\|_{\mathcal{L}(Y;Z)} \leq M; \tag{4B.42}$$

$$\|C[R(\lambda_0, A_h)\Pi_h B - R(\lambda_0, B)]F_h\Pi_h\|_{\mathcal{L}(Y;Z)} \leq Mh^{\epsilon_0} \quad \text{for some } \epsilon_0 > 0. \tag{4B.43}$$

Theorem 4B.5 [Ji, Lasiecka, 1998] *Assume the conditions of Theorem 4B.3: Hypotheses 1 and 2 and the discrete* Assumptions 1 *through* Assumptions 7 *of the present appendix. Moreover, assume the* Assumption 8 *in (B.42) and (B.43) as well. Let now*

$$\gamma < 1, \ r < 1, \quad \textit{so that } \gamma + r < 2, \tag{4B.44}$$

thus relaxing (4B.39). Then, all the conclusions of Theorem 4B.3 hold true. Moreover, under the setting of Proposition 4B.2 integrated with the above assumptions, the operators F, K, F_h, and K_h are given constructively by (4B.34), as in Corollary 4B.4.

Applications of Theorem 4B.5. Comments on *Assumption 8:* The paper Ji and Lasiecka [1998], where Theorem 4B.5 is proved, provides several examples where $1 < \gamma + r < 2$.

(a) The first example of Ji and Lasiecka [1998] is an unstable heat equation with Dirichlet boundary control and with Neumann trace of the solution as the observation. Here $3/2 + \gamma < \gamma + r < 2$. *Assumption 8* is satisfied by using the Nitsche, rather than the classical Galerkin, approximation scheme of the elliptic operator with quasi-uniform mesh (see Section 4.6).
(b) The next two examples of Ji and Lasiecka [1998] consider a structurally damped plate with either boundary or point control, and with point observation. In both cases, *Assumption 8* holds trivially true: $CA^{-1}Bu = 0$, $CA_h^{-1}B = 0$. In both cases, $\gamma = 3/4 + \epsilon_1$, $r = 1/2 + \epsilon_2$, so that $5/4 < \gamma + r < 2$.
(c) In another recent paper Chang et al. [1999] prove the same conclusions of Theorem 4B.3 for a thermo-elastic plate equation with boundary control (as a bending moment) and point observation, by verifying the validity of all *Assumptions 1* through *8*. In particular, it turns out that *Assumption 8* is trivially satisfied: $CA^{-1}B = 0$, $CA_h^{-1}B = 0$.

We conclude that the setting of the present Appendix 4B provides very general results, with "right" stability estimates, for the design of a finite-dimensional dynamic compensator of Luenberger's type for control systems generated by analytic semigroups. Chief features include: (i) the setting treats fully unbounded control and observation operators B and C; (ii) the analysis does not require compact resolvent of A nor other assumptions that may well be violated; (iii) the design of the finite-dimensional compensator is based on finite element approximations of the *original continuous* model rather than on modal (eigenfunction) approximations (which is generally not available).

Notes on Chapter 4

Sections 4.1 through 4.5

The treatment of Sections 4.1 through 4.5 follows closely that of Lasiecka and Triggiani [1991(a)]. This paper, in turn, is a fully abstract treatment of an earlier work by Lasiecka and Triggiani [1987] that studied the corresponding numerical problem in the concrete case of a second-order parabolic equation with Dirichlet boundary control on $Y = L_2(\Omega)$ [case $\gamma = 3/4 + \epsilon$], by abstract operator methods. However, we note that Lasiecka and Triggiani [1991(a)], unlike Lasiecka and Triggiani [1987], does not assume that A has compact resolvent (a property automatically true in the concrete PDE setting of Lasiecka and Triggiani [1987]). The treatment of the present chapter uses, as a starting point, two sources: on the one hand, the properties of the continuous optimal control problem and related algebraic Riccati equation following the variational approach of Chapter 2, and, on the other hand, the approximation results for analytic semigroups as in Lasiecka [1984] in the non–self-adjoint case (see also [Bramble et al., 1977] in the self-adjoint case).

Section 4.6

Section 4.6 of the optimal rates of convergence is taken from Lasiecka [1992], where proof of the relevant results are given. Two examples illustrating such theory – the heat equation with boundary control in the Dirichlet or Neumann boundary conditions – will be given in Chapter 5.

Appendix 4A

The appendix is taken from Triggiani [1994], where the sharp Theorem 4A.2 is given, along with classes of counterexamples as in Remark 4A.1. It should be noted that the statement of Theorem 4A.2 was already given and used critically (for $p = 2$) in the numerical analysis of the operator algebraic Riccati equation in [Lasiecka, Triggiani, 1987, p. 202; Lasiecka, Triggiani, 1991(a), p. 22 in Supplement, Proof of Theorem 4.6] in the parabolic case, as well as in [Lasiecka, 1990, p. 340] in the hyperbolic case. In all these references, however, no proof was given owing to space limitations. Instead, it was explicitly noted that a proof could be given as a suitable modification of the proof in [Pazy, 1983, p. 116] in the case of a single s.c. semigroup. Technical details were then given in Triggiani [1994]. This reference (and accordingly the appendix) gives not only an indirect proof patterned after Pazy's but also a direct proof patterned after the one in Pritchard and Zabczyk [1981] for a single s.c. semigroup.

Literature. Philosophy, Insight on the Degree of Unboundedness of B, and Chronology

Philosophy

The literature on approximating schemes of optimal problems and related Riccati equations generally assumed (see Gibson [1979])

(i) convergence properties of the "open loop" solutions, that is, of the maps $u \to y$ of the continuous problem;
(ii) "uniform stabilizability/detectability" hypotheses for the approximating problems.

In contrast, the basic assumptions of the present chapter are (after [Lasiecka, Triggiani, 1987; 1991(a)]):

(a) stabilizability/detectability hypotheses (SC)/(DC) of the continuous system;
(b) a "uniform analyticity" hypothesis (A.1) on the approximations.

Starting from (a) and (b), this chapter then derives both the convergence properties of the open loop and the uniform stabilizability/detectability hypotheses [(i) and (ii) above], which are taken as assumptions in other treatments. Thus, the theory presented here is sharp, in the sense that it assumes the SC, which is also necessary, and the DC, which is close to being necessary (see comments on the Detectability Condition on the Notes of Chapter 2), for the main theorems presented here. These considerations are an important aspect of the entire theory, since, in the case where B is an unbounded operator, the requirement, corresponding to (i) above in other treatments, of convergence $L_h \to L$ of the open loop solutions (where L and L_h are defined in (4.2.1.1) and (4.2.2.1)) is a very strong assumption. Generally, even when L is bounded, and the scheme is consistent, it may well happen that the scheme is not even stable; that is, L_h may not be uniformly bounded in h. The properties of the composition $e^{At}B$ may not be retained in the approximation $e^{A_h t}B_h$. Special care must be exercised in approximating B.

Insight on the Degree of Unboundedness of B and Chronology

There is a rather extensive (and growing) literature concerned with the general issue of approximation schemes for algebraic Riccati equations in infinite-dimensional spaces. Here, we shall concentrate only on works that focus on the case where the original free dynamics is modeled by an analytic semigroup e^{At}, as in the present chapter. Chronologically, the first reference is Banks and Kunish [1984], which presented approximation results for parabolic problems with distributed controls, that is, with the operator B bounded. No rates of convergence were given. Next, Lasiecka and Triggiani [1987] analyzed the case of a parabolic problem with Dirichlet boundary control, via an abstact semigroup approach. The abstract treatment in Lasiecka and Triggiani [1987] of the physically concrete, important problem of a parabolic equation defined on a bounded domain Ω of R^n, with Dirichlet boundary control – where then the operator B^* is $(\hat{A}^*)^{\frac{3}{4}+\epsilon}$-bounded, $\forall\, \epsilon > 0$, or equivalently, the operator $\hat{A}^{-(\frac{3}{4}+\epsilon)}B$ is bounded – may be viewed as a canonical illustration of the purely abstract situation where one has $\hat{A}^{-\gamma}B$ bounded for $\gamma < 1$, and A has compact resolvent (the latter property being automatically satisfied in the parabolic problem over a bounded domain $\Omega \subset R^n$). As emphasized in Chapters 1 and 2, there is a natural "cutting line" in the range of values of γ, which crucially bears on the degree of technical difficulties present in the treatment of the optimal control problem and its algebraic Riccati approximation; this is given by the special value $\gamma = 1/2$.

Indeed, if $\hat{A}^{-\gamma}B$ is bounded, or equivalently B^* is $(\hat{A}^*)^{\gamma}$-bounded, with $\gamma < 1/2$, then the corresponding input $\to$ solution operator L is a priori continuous into $C([0, T]; Y)$, so that all the trajectories of the continuous dynamical system are a priori pointwise continuous in time, and the operator B^*P is then a priori a bounded operator. Thus, in the case $\gamma < 1/2$, a derivation of the ARE may be given that closely parallels the pattern where B is a bounded operator. (The same applies to the case $\gamma = 1/2$ if A is self-adjoint based on the regularity result in (0.4) of Chapter 0.) Instead, if B^* is $(\hat{A}^*)^{\gamma}$-bounded, or equivalently $\hat{A}^{-\gamma}B$ is bounded, with $1/2 < \gamma$, the operator L is not continuous into $C([0, T]; Y)$; that is, the open loop trajectories are generally not pointwise continuous in time. Here, a main technical difficulty is therefore to show that, nevertheless, the gain operator B^*P is bounded. This is done by carefully analyzing the properties of the *optimal solutions* $y^0(t)$ (as distinguished from ordinary solutions $y(t)$) and by eventually showing via a bootstrap argument that the optimal solutions $y^0(t)$, are pointwise continuous in time (unlike ordinary solutions $y(t)$, which are only, say, in $L_2(0, T; Y)$). This strategy then succeeds in proving boundedness of the gain operator B^*P.

The strategy outlined above for the case $\gamma \geq 1/2$ was successfully implemented in Lasiecka and Triggiani [1987] in the canonical case of the parabolic equation with Dirichlet boundary control, where in fact $\gamma = 3/4 + \epsilon$, and later in the fully abstract treatment of Lasiecka and Triggiani [1991(a)], where the assumption that A has a compact resolvent is dispensed with.

More recently, Banks and Ito [1993] considered the approximations of a subclass of analytic problems modeled by strictly coercive bilinear forms. The results in Banks and Ito [1993] assume that the operator B is "$A^{\frac{1}{2}}$" bounded, that the problem is coercive, and that the resolvent $R(\lambda, A)$ is compact. The importance of having a theory of approximation valid for $\gamma > 1/2$ is fully justified by important physical problems, which are not solved by the direct generalization from the case of B bounded to the case of $\hat{A}^{-\gamma}$ bounded with $\gamma \leq 1/2$. Relevant examples where $\gamma > 1/2$ include, in addition to parabolic problems with Dirichlet boundary control, also structurally damped elastic equations considered in Chapter 3. Their numerical treatment will be given in Chapter 5.

Appendix 4B

The concept of dynamic compensator was introduced by Luenberger [1971], in the context of finite-dimensional systems. Most of the subsequent generalizations of the infinite-dimensional setting dealt with parabolic (or analytic semigroups) problems, with control/observations operators B and C that are either bounded (Gibson [1981; 1990]; Schumacher [1983]) or only mildly unbounded, that is, satisfying the constraint $\gamma + r \leq 1/2$ (see (4B.39)) (Curtain [1984]). This is a severe limitation. Moreover, in Schumacher [1983] and Curtain [1984], the construction of the finite-dimensional compensator requires knowledge of the eigenvalues/eigenfunctions of the generator A. This information is generally not available for PDEs on higher dimension (greater than one) defined on arbitrary domains. This was recognized by Gibson, whose finite elements method schemes however require B and C bounded. If the restrictive condition $\gamma + r \leq 1/2$ is assumed, then the problem admits a classical variational formulation within the $V \subset H \subset V'$ framework. Many physically significant examples (see those given in Appendix 4B) require that $\gamma + r > 1/2$. This case of stronger degree of unboundedness for B and C is more technical and more demanding, as classical variational tools are no longer applicable. As repeatedly noted in Chapters 1 and 2, the map: control $\rightarrow$ state ceases then to be pointwise well defined in time, on the basic state space Y. This is a source of major difficulties and requires the use of more technical machinery in the theory of analytic semigroups. The results of Appendix 4B up to Theorem 4B.3 and Corollary 4B.4 are taken from Lasiecka [1995] (case $1/2 < \gamma + r < 1$), while the further improvement of Theorem 4B.5 is taken from Ji and Lasiecka [1998]. The examples mentioned in Appendix 4B are all treated in detail in these references, whereas in a more recent paper, Chang et al. [1999] obtain the same results (by verifying all assumptions of Theorem 4B.5) for a thermo-elastic plate equation with boundary control (as a "bending moment") and point observation.

Parametrized Problems (as in Adaptive Control Theory)

Clearly, the discrete treatment of the present chapter, along with its key convergence results, is adaptable to the description of other problems depending on a scalar, or possibly vector parameters, α, going to zero in norm, such as they arise in adaptive control theory [other than the numerical approximation problem of the present chapter, which depends on a scalar parameter of numerical discretization $h \downarrow 0$].

Qualitative Problem Formulation

Let Y, U, Z be the three Hilbert spaces. Let α be a real scalar, or possibly vector parameter, such that $|\alpha| \downarrow 0$ and $|\alpha| < 1$, in the norm $|\cdot|$. We are given:

(i) a family of generators $\{A_\alpha\}$ of s.c. semigroups $e^{A_\alpha t}$ on Y, such that the resolvent set $\rho(A_\alpha)$ of A_α satisfies:

$$\rho(A_\alpha) \subset [\omega_0, \infty] \text{ for some fixed } \omega_0 > 0;$$

(ii) a family of control operators $\{B_\alpha\}$, $B_\alpha : U \to [\mathcal{D}(A_\alpha^*)]'$, such that

$$\left\| \hat{A}_\alpha^{-1} B_\alpha \right\|_{\mathcal{L}(U;Y)} \leq M \text{ uniformly in } \alpha; \quad \hat{A}_\alpha = -A_\alpha + \omega_0 I. \tag{4N.1}$$

We consider a family of control problems, parametrized by α, with dynamics

$$\dot{y} = A_\alpha y + B_\alpha u, \qquad y(0) = y_0 \in Y, \tag{4N.2}$$

and cost functional

$$J_\alpha(u, y) = \int_0^\infty \left[\|Ry(t)\|_Z^2 + \|u(t)_U^2\right] dt, \quad R \in \mathcal{L}(Y; Z). \tag{4N.3}$$

With (4N.2) and (4N.3) we associate the corresponding algebraic Riccati equation ARE $_\alpha$

$$(A_\alpha^* P_\alpha x, y)_Y + (P_\alpha A_\alpha x, y)_Y + (R^* R x, y)_Y = (B_\alpha^* P_\alpha x, B_\alpha^* P_\alpha y)_U, \tag{4N.4}$$

$\forall\, x, y$ in some suitable subspace of Y, related to $\mathcal{D}(A_\alpha)$. Under suitable conditions such as the stabilization/detectability conditions, there is [as we know, e.g., in the analytic case of Chapter 2] a unique nonnegative, self-adjoint solution $P_\alpha \in \mathcal{L}(Y)$. Set

$$A \equiv A_0, \qquad B \equiv B_0, \qquad P \equiv P_0, \tag{4N.5}$$

as a convenient notation. Then, a general problem of interest is: *Under what conditions imposed on A_α, B_α, and R does one obtain the uniform convergence properties*

$$\|P_\alpha - P\|_{\mathcal{L}(Y)} \to 0, \quad \|B_\alpha^* P_\alpha - B^* P\|_{\mathcal{L}(Y;U)} \to 0 \text{ as } |\alpha| \downarrow 0? \tag{4N.6}$$

Surely the treatment of the present chapter – in the specific direct approximation version arising in the numerical analysis of the ARE – lends itself to obtaining an affirmative answer to the *uniform convergence* properties in (4N.6) [recall Eqns. (4.1.4.8) and (4.1.4.9) of Theorem 4.1.4.1, which give the required *uniform convergence with rates*].

A solution, in the present context, of the above parametrized control problems with the required convergence as in (4N.6) is given in Lasiecka and Triggiani [1991(b)] in the analytic semigroup case and fully unbounded case with $\gamma < 1$, following the treatment of Lasiecka and Triggiani [1991(a)], that is, of the present chapter. The main difference between the present parametrized problems and the discrete approximation problem of the present Chapter 4 is that the latter deals with *finite-dimensional* approximations A_h and B_h of the original operators A and B. However, if we restrict the framework of the present chapter to $V_h \equiv Y$ and $\Pi_h \equiv$ Identity, then the approximating assumptions imposed on A_h on Theorem 4.1.4.1 follow from the assumptions imposed in Lasiecka and Triggiani [1991(b)] as the parametrized problems. We refer to Lasiecka and Triggiani [1991(b)] for further details. This result markedly improves upon the previous result (in adaptive control theory) of Duncan et al. [1996], which, in the analytic case, solves the problem with $\gamma < 1/2$, that is, with only mild unboundedness of B.

Glossary of Symbols for Chapter 4

$\hat{A},\ \hat{M},\ \hat{\omega}$	(4.1.1.2), (4.1.1.3)
(SC), $A_F = A + BF,\ M_F,\ \omega_F$	(4.1.1.8), (4.4.1.1)
(DC), $A_K = A + KR,\ M_K,\ \omega_K$	(4.1.1.9), (4.4.2.1)
$A_P = A - BB^*P,\ M_P,\ \omega_P$	(4.1.1.15)
$V_h,\ \Pi_h,\ A_h, \hat{A}_h$	(4.1.2.1), (4.1.2.2), (4.1.2.5)
$\Sigma_{\text{app}}(A) = \Sigma_{\text{app}}(A; a; \theta_a)$	(4.1.2.3a)
$A_{h,P_h} = A_h - B_h B_h^* P_h,\ \bar{\omega}_P,\ \bar{M}_P$	(4.1.4.5), (4.1.4.6)
$L,\ L^*;\ \hat{L},\ \hat{L}^*;\ L_h,\ L_h^*$	
$\Phi_h,\ \Phi;\ \hat{\Phi}_h,\ \hat{\Phi};\ \hat{y}_h^0,\ \hat{y}^0;\ \hat{u}_h^0,\ \hat{u}^0$	
$\Sigma(A_F) = \Sigma(A_F; a_F; \theta_F)$	(4.4.1.2)
$A_{h,F_h} = A_h + B_h F_h$	(4.4.1.3)
$A_{F_h} = A + BF_h\Pi_h$	(4.4.1.11)
$\Sigma_{\text{app}}(A_F),\ \bar{M},\ \bar{\omega}_F$	(4.4.1.14), Remark 4.4.1.4
$A_{h,K} = A_h + \Pi_h K R \Pi_h$	(4.4.2.3)
$L_{h,K}$	(4.4.2.12)
$\Sigma_{\text{app}}(A_K)$	(4.4.2.5)
$A_{h,P} = A_h\Pi_h - B_h B^* P$	(4.5.1.3)
$A_{P_h} = A - BB_h^* P_h \Pi_h$	(4.5.1.4)

References and Bibliography

P. M. Anselone, *Collectively Compact Operator Approximation Theory,* Prentice-Hall, 1971.

H. T. Banks and K. Kunisch. The linear regulator problem for parabolic systems. *SIAM J. Control Optim* **22** (1984), 684–699.

H. T. Banks and K. Ito, *SIAM Meeting*, San Francisco, 1993.

A. Bensoussan, G. DaPrato, M. Delfour, and S. Mitter, *Representation and Control of Infinite Dimensional Systems*, Vol. I, Birkhäuser, 1992.

S. K. Chang, I. Lasiecka, and R. Triggiani, Finite element compensators for thermo-elastic systems with boundary control and point observation. *Numerical Functional Analysis and Optimization*, **20** (5–6), (1999), 419–435.

R. Curtain, Finite dimensional compensators for parabolic distributed systems with unbounded control and observation, *SIAM J. Control Optim.* **22** (1984), 255–277.

R. Datko, Extending a theorem of Liapunov to Hilbert space, *J. Math. Anal. Appl.* **32** (1970), 610–616.

G. Doetsch, *Introduction to the Theory and Application of the Laplace Transformation*, Springer-Verlag, 1970.

T. Duncan, B. Mastowski, and B. Pasik-Duncan, Adaptive boundary control of linear stochastic distributed parameter systems, *Appl. Math. Optim.* **33** (1996), 107–138.

H. O. Fattorini, *The Cauchy Problem*, Encyclopedia of Mathematics and Its Applications, Vol. 18, Addison-Wesley, 1983.

J. S. Gibson, The Riccati integral equations for optimal control problems on Hilbert spaces, *SIAM J. Control Optim.* **17** (1979), 537–565.

J. S. Gibson, An analysis of optimal model regulation: Convergence and stability, *SIAM J. Control Optim.* **19** (1981), 686–707.

J. S. Gibson, Approximation theory for linear quadratic Gaussian control of flexible structures, *SIAM J. Control Optim* **29** (1990), 1–38.

G. Ji and I. Lasiecka, Partially observed analytic systems with fully unbounded actuators and sensors – FEM Algorithms, in *Optimal Control: Theory, Algorithms, and Applications*, W. W. Hager and P. M. Pardalos, ed., Kluwer, 1998.

T. Kato, *Perturbation Theory for Linear Operators,* Springer-Verlag, 1966.

G. E. Ladas and V. Lakshmikantham, *Differential Equations in Abstract Spaces,* Academic Press, 1972.

I. Lasiecka, Convergence estimates for semidiscrete approximations of nonselfadjoint parabolic equations, *SIAM J. Num. Anal.* **21** (1984), 894–909.

I. Lasiecka, Approximations of solutions to infinite-dimensional algebraic Riccati equations with unbounded input-operators, *Num. Anal. Optim.* **11** (1990), 303–378.

I. Lasiecka, Convergence rates for the approximations of the solutions of algebraic Riccati equations with unbounded coefficients: the case of analytic semigroups, *Num. Math.* **63** (1992), 357–390.

I. Lasiecka, Finite element approximations of compensator design for analytic generators with fully unbounded controls/observations, *SIAM J. Control Optim.* **33** (1995), 67–88.

I. Lasiecka and R. Triggiani, The regulator problem for parabolic equations with Dirichlet boundary control, Part II: Galerkin approximation, *Appl. Math. Optim.* **16** (1987), 187–216.

I. Lasiecka and R. Triggiani, Numerical approximations of algebraic Riccati equations for abstract systems modelled by analytic semigroups and applications, *Math. Comput.* **57** (1991(a)), 639–662, and *Supplement*, 513–537.

I. Lasiecka and R. Triggiani, Uniform convergence of the solutions to Riccati equations arising in boundary/point control problems, *Springer-Verlag Lecture Notes in Control and Information Sciences* 184, Proceedings of a Workshop in Stochastic Theory and Adaptive Control, held at the University of Kansas, Lawrence, Kansas, September 1991(b).

J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems*, Vol. I Springer-Verlag, 1972.

D. G. Luenberger, An introduction to observers, *IEEE TAC* AC-16 (1971), 596–602.

A. Pazy, On the applicability of Lyapunov's theorem in Hilbert space, *SIAM J. Math. Anal.* **3** (1972), 291–294.

A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations,* Springer-Verlag, 1983.

A. J. Pritchard and J. Zabczyk, Stability and stabilizability of infinite dimensional systems, *SIAM Review* **23** (1981), 25–52.

C. Sadosky, *Interpolation of Operators and Singular Integrals*, Marcel Dekker, 1979.

J. M. Schumacher, A direct approach to compensator design for distributed parameter systems, *SIAM J. Control Optim.* **21** (1983), 823–837.

R. Triggiani, A sharp result on the exponential operator-norm decay of a family of s.c. semigroups, *Semigroup Forum*, **49** (1994), 387–395.

5

Illustrations of the Numerical Theory of Chapter 4 to Parabolic-Like Boundary/Point Control PDE Problems

In this chapter we give detailed illustrations of the applicability of the numerical theory of Chapter 4 to various boundary/point control PDE problems of parabolic type, whose continuous analysis was presented in Chapter 3. In particular, we shall provide numerical algorithms, that satisfy all the required assumptions of Chapter 4. This will include both the general assumptions (A.1) = (4.1.2.2) through (A.6) = (4.1.2.9) of stability and consistency leading to the general theory of Theorems 4.1.4.1 and 4.1.4.2 of Chapter 4, as well as the additional and more specialized assumptions (A.7) in (4.6.2.1) and (4.6.2.2) through (A.9) in (4.6.2.5) and (4.6.2.6) of Chapter 4, leading to the sharp results of Theorems 4.6.2.1 and 4.6.2.2 with optimal rates of convergence.

5.1 Introductory Approximation Results

A General Result In this preliminary section, we give a numerical lemma for Galerkin approximations, which will be invoked frequently, in the present chapter, as well as in Chapters 14 and 15. The framework is as follows.

Let A be a positive, self-adjoint operator: $Y \supset \mathcal{D}(A) \to Y$, with Y a Hilbert space. Let $V_h \subset \mathcal{D}(A^{\frac{1}{2}})$ be an approximating (finite-dimensional) subspace, and let Π_h be the orthogonal projection Y onto V_h, so that $(\Pi_h x, \phi_h)_Y = (x, \phi_h)_Y$, $x \in Y$, $\phi_h \in V$, and $A^{\frac{1}{2}}\Pi_h \in \mathcal{L}(Y)$. Finally, let

$$A_h : V_h \to V_h \text{ be a Galerkin approximation of } A, \tag{5.1.1a}$$

that is,

$$A_h x_h = \Pi_h A x_h = \Pi_h A^{\frac{1}{2}} A^{\frac{1}{2}} x_h, \quad \forall x_h \in V_h \tag{5.1.1b}$$

[notice that $\Pi_h A$ is well defined on $V_h \subset \mathcal{D}(A^{\frac{1}{2}})$].

Proposition 5.1.1 *With reference to the above framework and assumptions, we have*

(i)

$$\left\| A_h^{-1}\Pi_h x - A^{-1}x \right\|_{\mathcal{D}(A^{\frac{1}{2}})} \le 2\|(I-\Pi_h)A^{-1}x\|_{\mathcal{D}(A^{\frac{1}{2}})}, \quad x \in \left[\mathcal{D}\left(A^{\frac{1}{2}}\right)\right]' \tag{5.1.2}$$

[notice that Π_h extends from $[\mathcal{D}(A^{\frac{1}{2}})]'$ to Y];

(ii) for $x \in \mathcal{D}(A^{\frac{1}{2}})$,

$$\left\| A_h^{-1}\Pi_h x - A^{-1}x \right\|_Y \le \left\{ \sup_{\substack{\phi\in\mathcal{D}(A)\\ \|\phi\|_{\mathcal{D}(A)}=1}} 2\|(I-\Pi_h)\phi\|_{\mathcal{D}(A^{\frac{1}{2}})} \right\} \|(I-\Pi_h)A^{-1}x\|_{\mathcal{D}(A^{\frac{1}{2}})}. \tag{5.1.3}$$

Proof. (i) Set, for $x \in [\mathcal{D}(A^{\frac{1}{2}})]'$,

$$A_h^{-1}\Pi_h x = y_h \in V_h \subset \mathcal{D}\left(A^{\frac{1}{2}}\right), \quad \text{or} \quad \Pi_h x = A_h y_h; \tag{5.1.4}$$

$$A^{-1}x = y \in \mathcal{D}\left(A^{\frac{1}{2}}\right), \quad \text{or} \quad x = Ay. \tag{5.1.5}$$

Let $v_h \in V_h$. Then, by the right-hand equality in (5.1.4) and by the definition (5.1.1b), we obtain

$$(\Pi_h x, v_h)_Y = (A_h y_h, v_h)_Y = (A y_h, v_h)_Y. \tag{5.1.6}$$

Similarly, by the right-hand equality in (5.1.5), we have

$$(\Pi_h x, v_h)_Y = (\Pi_h A y, v_h)_Y = (Ay, \Pi_h v_h)_Y = (Ay, v_h)_Y. \tag{5.1.7}$$

We subtract (5.1.7) from (5.1.6) to obtain

$$(A(y_h - y), v_h)_Y = 0, \quad v_h \in V_h. \tag{5.1.8}$$

Hence, subtracting (5.1.8), we get

$$(A(y_h - \Pi_h y), v_h)_Y = (A(y - \Pi_h y), v_h)_Y. \tag{5.1.9}$$

We now choose v_h as follows,

$$v_h = y_h - \Pi_h y \in V_h \tag{5.1.10}$$

in (5.1.9), and we thus obtain via the Schwarz inequality on the right of (5.1.9),

$$\left\| A^{\frac{1}{2}}(y_h - \Pi_h y)\right\|_Y^2 \le \left\| A^{\frac{1}{2}}(y - \Pi_h y)\right\|_Y \left\| A^{\frac{1}{2}}(y_h - \Pi_h y)\right\|_Y,$$

or, upon simplifying,

$$\|y_h - \Pi_h y\|_{\mathcal{D}(A^{\frac{1}{2}})} \le \|y - \Pi_h y\|_{\mathcal{D}(A^{\frac{1}{2}})}. \tag{5.1.11}$$

Next, subtracting $\Pi_h y$ and using (5.1.11) we get

$$\begin{aligned} \|y_h - y\|_{\mathcal{D}(A^{\frac{1}{2}})} &\le \|y_h - \Pi_h y\|_{\mathcal{D}(A^{\frac{1}{2}})} + \|\Pi_h y - y\|_{\mathcal{D}(A^{\frac{1}{2}})} \\ &\le 2\|y - \Pi_h y\|_{\mathcal{D}(A^{\frac{1}{2}})}. \end{aligned} \tag{5.1.12}$$

Rewriting inequality (5.1.12) in terms of the original variables in (5.1.4) and (5.1.5), we obtain

$$\left\| A_h^{-1}\Pi_h x - A^{-1}x \right\|_{\mathcal{D}(A^{\frac{1}{2}})} \leq 2\left\| A^{-1}x - \Pi_h A^{-1}x \right\|_{\mathcal{D}(A^{\frac{1}{2}})}, \tag{5.1.13}$$

which proves (5.1.2) of part (i).

(ii) In the notation of part (i), we estimate for $f = A\phi, \phi \in \mathcal{D}(A)$, so that f exhausts all of Y as ϕ runs over all of $\mathcal{D}(A)$:

$$\begin{aligned} \|y_h - y\|_Y &= \sup_{f\in Y} \frac{|(y_h - y, f)_Y|}{\|f\|_Y} \\ &= \sup_{\phi\in\mathcal{D}(A)} \frac{|(y_h - y, A\phi)_Y|}{\|\phi\|_{\mathcal{D}(A)}} \end{aligned}$$

[recalling $(y_h - y, A\phi_h)_Y = 0, \forall\phi_h \in V_h$ by (5.1.8)]

$$= \sup_{\phi\in\mathcal{D}(A)} \frac{|(y_h - y, A(\phi - \phi_h))_Y|}{\|\phi\|_{\mathcal{D}(A)}}. \tag{5.1.14}$$

Thus, from (5.1.14),

$$\|y_h - y\|_Y \leq \left\{ \sup_{\substack{\phi\in\mathcal{D}(A) \\ \|\phi\|_{\mathcal{D}(A)}=1}} \|\phi - \phi_h\|_{\mathcal{D}(A^{\frac{1}{2}})} \right\} \|y_h - y\|_{\mathcal{D}(A^{\frac{1}{2}})}, \tag{5.1.15}$$

and recalling (5.1.12), we get

$$\|y_h - y\|_Y \leq \left\{ \sup_{\substack{\phi\in\mathcal{D}(A) \\ \|\phi\|_{\mathcal{D}(A)}=1}} \|\phi - \phi_h\|_{\mathcal{D}(A^{\frac{1}{2}})} \right\} 2\|y - \Pi_h y\|_{\mathcal{D}(A^{\frac{1}{2}})}, \tag{5.1.16}$$

which proves (5.1.3) of part (ii), as desired, by returning to the original variables in (5.1.4), (5.1.5) as in the proof of part (i) and taking $\phi_h = \Pi_h\phi$. $\square$

Results for Second-Order Equations In Sections 5.4, 5.5, and 5.6 of the present chapter [as well as in Part III of Chapter 14, Sections 21 through 25], dealing with second-order equations in time, we shall encounter the following situation:

(i) $\mathcal{A}$ is a positive, self-adjoint (unbounded) operator $\mathcal{H} : \mathcal{D}(\mathcal{A}) \to \mathcal{H}, \mathcal{H}$ being a Hilbert space [different from Y];
(ii) $\mathcal{V}_h$ is (a finite-dimensional) approximating subspace, $\mathcal{V}_h \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$;
(iii) π_h is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{V}_h$:

$$(\pi_h x, v_h)_{\mathcal{H}} = (x, v_h)_{\mathcal{H}}, \quad x \in \mathcal{H}, \quad v_h \in \mathcal{V}_h, \quad \|\pi_h\|_{\mathcal{L}(\mathcal{H})} \equiv 1, \tag{5.1.17}$$

so that $\mathcal{A}^{\frac{1}{2}}\pi_h, \pi_h\mathcal{A}^{\frac{1}{2}} \in \mathcal{L}(\mathcal{H})$;

(iv) if $\hat{\mathcal{V}}_h$ denotes $\mathcal{V}_h$ topologized by the norm of $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$, π_h^1 is the orthogonal projection of $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ onto $\hat{\mathcal{V}}_h$:

$$\left(\mathcal{A}^{\frac{1}{2}}\left[\pi_h^1 x - x\right], \mathcal{A}^{\frac{1}{2}} v_h\right)_{\mathcal{H}} = 0, \quad x \in \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right), \quad v_h \in \hat{\mathcal{V}}_h. \tag{5.1.18}$$

The following general property then holds true.

Proposition 5.1.2 *Under assumptions (i), (ii), (iii), and (iv), above, we have*

$$\left\|\pi_h^1 x - x\right\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \leq 2\|\pi_h x - x\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})}, \quad x \in \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right). \tag{5.1.19}$$

Proof. First, by adding and subtracting $\pi_h x \in \mathcal{V}_h \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$, we obtain from (5.1.18), with $x \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$, $v_h \in \mathcal{V}_h$:

$$\left(\mathcal{A}^{\frac{1}{2}}\left[\pi_h^1 x - \pi_h x\right], \mathcal{A}^{\frac{1}{2}} v_h\right)_{\mathcal{H}} = -\left(\mathcal{A}^{\frac{1}{2}}[\pi_h x - x], \mathcal{A}^{\frac{1}{2}} v_h\right)_{\mathcal{H}}. \tag{5.1.20}$$

Setting $v_h = \pi_h^1 x - \pi_h x \in \mathcal{V}_h$, in (5.1.20), we obtain by the Schwarz inequality on the right-hand side, followed by a cancellation:

$$\left\|\mathcal{A}^{\frac{1}{2}}\left[\pi_h^1 x - \pi_h x\right]\right\|_{\mathcal{H}} \leq \left\|\mathcal{A}^{\frac{1}{2}}[\pi_h x - x]\right\|_{\mathcal{H}}. \tag{5.1.21}$$

Next, again by adding and subtracting $\pi_h x$, we obtain

$$\begin{aligned} \left\|\mathcal{A}^{\frac{1}{2}}\left[\pi_h^1 x - x\right]\right\|_{\mathcal{H}} &\leq \left\|\mathcal{A}^{\frac{1}{2}}\left[\pi_h^1 x - \pi_h x\right]\right\|_{\mathcal{H}} + \left\|\mathcal{A}^{\frac{1}{2}}[\pi_h x - x]\right\|_{\mathcal{H}} \\ \text{(by (5.1.21))} \qquad &\leq 2\left\|\mathcal{A}^{\frac{1}{2}}[\pi_h x - x]\right\|_{\mathcal{H}}, \end{aligned} \tag{5.1.22}$$

after using (5.1.21), and (5.1.22) proves (5.1.19). □

In the illustrations of the upcoming Sections 5.4 through 5.6 we shall deal with a more specialized situation, where $\mathcal{H} = L_2(\Omega)$, and where the following additional conditions hold true:

(v) $\mathcal{A}_h = \pi_h \mathcal{A} : \mathcal{V}_h \to \mathcal{V}_h$ is the (positive, self-adjoint) Galerkin approximation of $\mathcal{A}$,

$$(\mathcal{A}_h x_h, v_h)_{\mathcal{H}} = (\mathcal{A} x_h, v_h)_{\mathcal{H}}, \quad x_h, v_h \in \mathcal{V}_h, \tag{5.1.23}$$

and $\mathcal{H} = L_2(\Omega)$;

(vi) the subspace $\mathcal{V}_h$ satisfies the following usual approximating properties:

(vi_1)

$$\begin{aligned} \|\pi_h z - z\|_{H^{\ell}(\Omega)} \leq C h^{s-\ell} \|z\|_{H^s(\Omega)}, \quad & z \in H^s(\Omega) \cap H_0^1(\Omega), \\ 0 \leq \ell \leq 2, \quad & \ell \leq s \leq r+1; \end{aligned} \tag{5.1.24}$$

where r is the order of approximation (degree of polynomials);

(vi_2)

$$\|z_h\|_{H^{\alpha}(\Omega)} \leq C h^{-s} \|z_h\|_{H^{\alpha-s}(\Omega)}, \quad 0 \leq \alpha \leq 2; \tag{5.1.25}$$

(vii) the operator $\mathcal{A}$ is defined by

$$\mathcal{A}f = \Delta^2 f, \quad \mathcal{D}(\mathcal{A}) = \{f \in H^4(\Omega) : f|_\Gamma = \Delta f|_\Gamma = 0\}, \tag{5.1.26}$$

$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega) \cap H_0^1(\Omega), \tag{5.1.27a}$$

$$\|z\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \text{ equivalent to } \|z\|_{H^2(\Omega)}, \quad z \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}), \tag{5.1.27b}$$

so that, taking $\ell = s = 2$ in (5.1.24), we have then

$$\|\pi_h\|_{\mathcal{L}(H^2(\Omega))}, \text{ equivalent to } \|\pi_h\|_{\mathcal{L}(\mathcal{D}(\mathcal{A}^{\frac{1}{2}}))} \leq C. \tag{1.27c}$$

Proposition 5.1.3 *Under assumptions (i) through (vii) above, we have for $z \in L_2(\Omega)$:*

(a)

$$\left\| [\mathcal{A}^{-1} - \mathcal{A}_h^{-1}\pi_h]z \right\|_{H^2(\Omega)} \leq Ch^2 \|z\|_{L_2(\Omega)}, \tag{5.1.28}$$

(b)

$$\left\| [\mathcal{A}^{-1} - \mathcal{A}_h^{-1}\pi_h]z \right\|_{L_2(\Omega)} \leq Ch^4 \|z\|_{L_2(\Omega)}, \tag{5.1.29}$$

(c)

$$\left\| \mathcal{A}^{\theta/2}[\mathcal{A}^{-1} - \mathcal{A}_h^{-1}\pi_h]z \right\|_{L_2(\Omega)} \leq Ch^{4-2\theta} \|z\|_{L_2(\Omega)}, \quad 0 \leq \theta \leq 1. \tag{5.1.30}$$

Proof. (a) First, we apply Lemma 5.1.1, Eqn. (5.1.2), to the positive, self-adjoint operator $\mathcal{A}$ on $\mathcal{H}$ to obtain (5.1.28):

$$\begin{aligned}
\left\| (\mathcal{A}^{-1} - \mathcal{A}_h^{-1}\pi_h)z \right\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} &\leq 2\|(I - \pi_h)\mathcal{A}^{-1}z\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} && (5.1.31)\\
\text{(by (5.1.27b))} \qquad &\leq 2\|(I - \pi_h)\mathcal{A}^{-1}z\|_{H^2(\Omega)} && (5.1.32)\\
\text{(by (5.1.24) with } \ell = 2, s = 4) \qquad &\leq Ch^2 \|\mathcal{A}^{-1}z\|_{H^4(\Omega)} && (5.1.33)\\
\text{(by (5.1.26))} \qquad &\leq Ch^2 \|z\|_{L_2(\Omega)}, && (5.1.34)
\end{aligned}$$

where in going from (5.1.32) to (5.1.33) we have recalled (5.1.24) with $\ell = 2, s = 4$ (which is legal, since $\mathcal{A}^{-1}z \in \mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega) \cap H_0^1(\Omega)$ as required); while in going from (5.1.33) to (5.1.34) we have used that $\mathcal{A}^{-1}$ lifts by 4 units in Sobolev regularity. Then, (5.1.34) proves (5.1.28), as desired, since the $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$-norm on the left of (5.1.30) is equivalent to the $H^2(\Omega)$-norm by (5.1.27b).

(b) For part (b) we invoke Lemma 5.1.1(ii), Eqn. (5.1.3), for $\mathcal{A}$, $\mathcal{H}$, and π_h, thus obtaining

$$\left\|(\mathcal{A}^{-1} - \mathcal{A}_h^{-1}\pi_h)z\right\|_{L_2(\Omega)}$$

$$\leq \left\{ \sup_{\substack{\phi\in\mathcal{D}(\mathcal{A}) \\ \|\phi\|_{\mathcal{D}(\mathcal{A})}=1}} 2\|(I-\pi_h)\phi\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \right\} \|(I-\pi_h)\mathcal{A}^{-1}z\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \tag{5.1.35}$$

(setting $\phi = \mathcal{A}^{-1}\psi$, $\|\psi\|_{L_2(\Omega)} = 1$)

(from (5.1.31) to (5.1.34))

$$\leq \left\{ \sup_{\substack{\psi\in L_2(\Omega) \\ \|\psi\|_{L_2(\Omega)}=1}} 2\|(I-\pi_h)\mathcal{A}^{-1}\psi\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \right\} Ch^2\|z\|_{L_2(\Omega)} \tag{5.1.36}$$

(from (5.1.31) to (5.1.33))

$$\leq \left\{ \sup_{\|\psi\|_{L_2(\Omega)}=1} Ch^2\|\mathcal{A}^{-1}\psi\|_{H^4(\Omega)=\mathcal{D}(\mathcal{A})} \right\} Ch^2\|z\|_{L_2(\Omega)} \tag{5.1.37}$$

$$\leq Ch^4\|z\|_{L_2(\Omega)}, \tag{5.1.38}$$

where in going from (5.1.35) to (5.1.36) we have recalled the inequality on z that runs from (5.1.31) to (5.1.34); while in going from (5.1.36) to (5.1.37) we have recalled the inequality in ψ replacing z that runs from (5.1.31) to (5.1.33), where, with $\psi \in L_2(\Omega)$, the $H^4(\Omega)$-norm of $\mathcal{A}^{-1}\psi$ is equivalent to its $\mathcal{D}(\mathcal{A})$-norm by (5.1.26). Then (5.1.38) proves (5.1.29), as desired.

(c) Equation (5.1.30) follows now by the moment inequality [Lions, Magenes, 1967, p. 19]

$$\|\Lambda^\theta u\| \leq \|\Lambda u\|^\theta \|u\|^{1-\theta}, \tag{5.1.39}$$

with $\Lambda = \mathcal{A}^{\frac{1}{2}}$, from (5.1.28) and (5.1.29).

Proposition 5.1.4 *Under assumptions (i) through (vii) above, we have for $z \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ (see (5.1.27b)):*

$$\left\|\left[\mathcal{A}^{-\frac{1}{2}} - \mathcal{A}_h^{-\frac{1}{2}}\pi_h\right]z\right\|_{H^2(\Omega)} \leq Ch^{2-\epsilon}\|z\|_{H^2(\Omega)}. \tag{5.1.40}$$

Proof.

Step 1 Properties (i), (ii), and (v) imply uniform continuity

$$|(\mathcal{A}_h v, w)_{\mathcal{H}}| \leq \left\|\mathcal{A}^{\frac{1}{2}}v\right\|_{\mathcal{H}}\left\|\mathcal{A}^{\frac{1}{2}}w\right\|_{\mathcal{H}}, \quad v, w \in \mathcal{V}_h, \tag{5.1.41}$$

and uniform coercivity

$$(\mathcal{A}_h v, v) = \left\|\mathcal{A}^{\frac{1}{2}}v\right\|_{\mathcal{H}}^2, \quad v \in \mathcal{V}_h \tag{5.1.42}$$

of the form $(\mathcal{A}_h v, v)_{\mathcal{H}}$ on $V_h \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$, uniformity being in h. Thus [Showalter, 1977, pp. 99–100], $e^{\mathcal{A}_h t}$ is a family of s.c., analytic semigroups, uniformly in h, in the sense of Chapter 4, Eqns. (4.1.2.2), (4.1.2.3):

$$\left\|\mathcal{A}_h^{\theta} R(\lambda, -\mathcal{A}_h)\right\|_{\mathcal{L}(\mathcal{H})} \leq \frac{c}{|\lambda - a|^{1-\theta}}, \quad 0 \leq \theta \leq 1, \quad \forall \lambda \in \Sigma_{\text{app}}(\mathcal{A}), \tag{5.1.43}$$

with $a < 0$ sufficiently small and $\Sigma_{\text{app}}(\mathcal{A})$ defined in Chapter 4, Eqn. (4.1.2.3a), and likewise, of course,

$$\|\mathcal{A}^{\theta} R(\lambda, -\mathcal{A})\|_{\mathcal{L}(\mathcal{H})} \leq \frac{c}{|\lambda - a|^{1-\theta}}, \quad 0 \leq \theta \leq 1, \quad \forall \lambda \in \Sigma_{\text{app}}(\mathcal{A}). \tag{5.1.44}$$

Step 2 Adding and subtracting with $z \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$, we get

$$\left\|\mathcal{A}^{-\frac{1}{2}} z - \mathcal{A}_h^{-\frac{1}{2}} \pi_h z\right\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \leq \left\|\mathcal{A}^{-\frac{1}{2}}[I - \pi_h]z\right\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} + \left\|\left[\mathcal{A}^{-\frac{1}{2}} - \mathcal{A}_h^{-\frac{1}{2}}\right]\pi_h z\right\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})}, \tag{5.1.45}$$

where

$$\left\|\mathcal{A}^{-\frac{1}{2}}[I - \pi_h]z\right\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \leq \|[I - \pi_h]z\|_{L_2(\Omega)} \leq Ch^2 \|z\|_{H^2(\Omega)} \leq Ch^2 \|z\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})}, \tag{5.1.46}$$

where we have used (5.1.24) with $\ell = 0, s = 2$, followed by (5.1.27b). Thus, by (5.1.46) used in (5.1.45), and recalling the equivalence (5.1.27b), we see that all we need to complete the proof of (5.1.40) of Proposition 5.1.4 is

Step 3. Lemma 5.1.5 *Under assumptions (i) through (vii) above, we have for $z \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$:*

$$\left\|\left[\mathcal{A}^{-\frac{1}{2}} - \mathcal{A}_h^{-\frac{1}{2}}\right]\pi_h z\right\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \leq Ch^{2-\epsilon} \|z\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})}. \tag{5.1.47}$$

Proof of Lemma 5.1.5.

First Step The standard (Balakrishnan's) formula for fractional powers [Pazy, 1983, Eqn. (6.4), p. 69] gives

$$\left[\mathcal{A}^{-\frac{1}{2}} - \mathcal{A}_h^{-\frac{1}{2}}\right]\pi_h z = \frac{1}{\pi} \int_0^{\infty} t^{-\frac{1}{2}}[(tI + \mathcal{A})^{-1} - (tI + \mathcal{A}_h)^{-1}]\pi_h z \, dt. \tag{5.1.48}$$

The usual second resolvent equation gives

$$\begin{aligned}
&[(tI + \mathcal{A})^{-1} - (tI + \mathcal{A}_h)^{-1}]\pi_h z \\
&= (tI + \mathcal{A})^{-1}[\mathcal{A}_h - \mathcal{A}](tI + \mathcal{A}_h)^{-1}\pi_h z && (5.1.49)\\
&= (tI + \mathcal{A})^{-1}\mathcal{A}\left[\mathcal{A}^{-1} - \mathcal{A}_h^{-1}\right]\mathcal{A}_h(tI + \mathcal{A}_h)^{-1}\pi_h z && (5.1.50)\\
&= (tI + \mathcal{A})^{-1}\mathcal{A}\left[\mathcal{A}^{-1} - \mathcal{A}_h^{-1}\right]\eta_h(t), && (5.1.51)
\end{aligned}$$

where

$$\eta_h(t) = \mathcal{A}_h(tI + \mathcal{A}_h)^{-1}\pi_h z \in \mathcal{V}_h \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}}). \tag{5.1.52}$$

By (5.1.52) and (5.1.44) with $\theta = 1$, we get

$$\begin{aligned}
&\|[(tI + \mathcal{A})^{-1} - (tI + \mathcal{A}_h)^{-1}]\pi_h z\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \\
&\leq \|(tI + \mathcal{A})^{-1}\mathcal{A}\|_{\mathcal{L}(L_2(\Omega))} \left\| [\mathcal{A}^{-1} - \mathcal{A}_h^{-1}]\eta_h \right\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \\
\text{(by (5.1.44) with } \theta = 1) \quad &\leq C \left\| [\mathcal{A}^{-1} - \mathcal{A}_h^{-1}]\eta_h \right\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})}, \quad 0 \leq t < \infty, \qquad (5.1.53) \\
&= C\left\{ [\mathcal{A}^{-1} - \mathcal{A}_h^{-1}\pi_h]\eta_h \right\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})}, \qquad (5.1.54)
\end{aligned}$$

as the vectors (within the norm signs) in (5.1.53) and (5.1.54) are plainly the same. Next, recalling (5.1.28) and again (5.1.27b), we have

$$\left\| [\mathcal{A}^{-1} - \mathcal{A}_h^{-1}\pi_h]\eta_h \right\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \leq Ch^2 \|\eta_h\|_{L_2(\Omega)}. \tag{5.1.55}$$

Substituting (5.1.55) into the right-hand side of (5.1.54) yields

$$\|[(tI + \mathcal{A})^{-1} - (tI + \mathcal{A}_h)^{-1}]\pi_h z\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \leq Ch^2 \|\eta_h(t)\|_{L_2(\Omega)}, \quad 0 \leq t \leq \infty. \tag{5.1.56}$$

Finally, (5.1.56) used on the right-hand side of (5.1.48) gives

$$\left\| [\mathcal{A}^{-\frac{1}{2}} - \mathcal{A}_h^{-\frac{1}{2}}]\pi_h z \right\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \leq Ch^2 \int_0^\infty t^{-\frac{1}{2}} \|\eta_h(t)\|_{L_2(\Omega)}\, dt. \tag{5.1.57}$$

Second Step Next, returning to the definition of $\eta_h(t)$ in (5.1.52), we split the interval of integration in t into $[0, 1]$ and $[1, \infty]$, and we estimate as follows:

(i) For $0 \leq t \leq 1$, we invoke (5.1.43) with $\theta = 1$, followed by (5.1.23), and write

$$\begin{aligned}
\|\eta_h(t)\|_{L_2(\Omega)} &= \|\mathcal{A}_h(tI + \mathcal{A}_h)^{-1}\pi_h z\|_{L_2(\Omega)} \qquad (5.1.58) \\
\text{(by (5.1.43))} \quad &= C\|\pi_h z\|_{L_2(\Omega)} \leq C\|z\|_{L_2(\Omega)}, \quad 0 \leq t \leq 1. \qquad (5.1.59)
\end{aligned}$$

Thus (5.1.59) yields

$$\int_0^1 t^{-\frac{1}{2}} \|\eta_h(t)\|_{L_2(\Omega)}\, dt \leq C\|z\|_{L_2(\Omega)}. \tag{5.1.60}$$

(ii) For $1 \leq t < \infty$, we recall again (5.1.52) and (5.1.43), this time for $\theta = 1/2 - \epsilon$,

$$\begin{aligned}
\|\eta_h(t)\|_{L_2(\Omega)} &= \left\| \mathcal{A}_h^{\frac{1}{2}-\epsilon}(tI + \mathcal{A}_h)^{-1}\mathcal{A}_h^{\frac{1}{2}+\epsilon}\pi_h z \right\|_{L_2(\Omega)} \qquad (5.1.61) \\
\text{(by (5.1.43))} \quad &\leq \frac{C}{t^{\frac{1}{2}+\epsilon}} \left\| \mathcal{A}_h^{\frac{1}{2}+\epsilon}\pi_h z \right\|_{L_2(\Omega)} \\
\text{(by (5.1.23))} \quad &\leq \frac{C}{t^{\frac{1}{2}+\epsilon}} \left\| \mathcal{A}^{\frac{1}{2}}\mathcal{A}_h^{\epsilon}\pi_h z \right\|_{L_2(\Omega)} \qquad (5.1.62) \\
\text{(by (5.1.27b))} \quad &\leq \frac{C}{t^{\frac{1}{2}+\epsilon}} \left\| \mathcal{A}_h^{\epsilon}\pi_h z \right\|_{H^2(\Omega)}, \quad 1 \leq t < \infty. \qquad (5.1.63)
\end{aligned}$$

Third Step. Lemma 5.1.6 *Under assumptions (i) through (vii) above, we have for* $z \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$*, for* $0 < \epsilon \le 1/2$*:*

$$\left\| \mathcal{A}_h^{\epsilon} \pi_h z \right\|_{H^2(\Omega)} \le C h^{-4\epsilon} \|z\|_{H^2(\Omega)}. \tag{5.1.64}$$

Proof of Lemma 5.1.6. We use the moment inequality (5.1.39), this time with $\Lambda = \mathcal{A}_h^{\frac{1}{2}}$ and $\theta = \epsilon$, on the space $H^2(\Omega)$, to obtain

$$\left\| \mathcal{A}_h^{\frac{\epsilon}{2}} \pi_h z \right\|_{H^2(\Omega)} \le \|\Lambda^{\epsilon} \pi_h z\|_{H^2(\Omega)} \le \|\Lambda \pi_h z\|^{\epsilon}_{H^2(\Omega)} \|\pi_h z\|^{1-\epsilon}_{H^2(\Omega)} \tag{5.1.65}$$

$$\text{(by (5.1.27c))} \qquad \le \left\| \mathcal{A}_h^{\frac{1}{2}} \pi_h z \right\|^{\epsilon}_{H^2(\Omega)} \|z\|^{1-\epsilon}_{H^2(\Omega)}. \tag{5.1.66}$$

Next, invoking the inverse approximation property (5.1.25) with $\alpha = 2 = s$, we obtain via (5.1.23)

$$\begin{aligned} \left\| \mathcal{A}_h^{\frac{1}{2}} \pi_h z \right\|_{H^2(\Omega)} &\le C h^{-2} \left\| \mathcal{A}_h^{\frac{1}{2}} \pi_h z \right\|_{L_2(\Omega)} = C h^{-2} \left\| \mathcal{A}^{\frac{1}{2}} \pi_h z \right\|_{L_2(\Omega)} \\ \text{(by (5.1.27c))} \qquad &\le C h^{-2} \|z\|_{H^2(\Omega)}. \end{aligned} \tag{5.1.67}$$

Thus, inserting (5.1.67) into (5.1.66) yields

$$\begin{aligned} \left\| \mathcal{A}_h^{\frac{\epsilon}{2}} \pi_h z \right\|_{H^2(\Omega)} &\le C h^{-2\epsilon} \|z\|^{\epsilon}_{H^2(\Omega)} \|z\|^{1-\epsilon}_{H^2(\Omega)} \\ &\le C h^{-2\epsilon} \|z\|_{H^2(\Omega)}, \end{aligned} \tag{5.1.68}$$

which proves (5.1.64). □

Fourth Step Inserting now (5.1.64) into the right-hand side of (5.1.63) yields, finally,

$$\|\eta_h(t)\|_{L_2(\Omega)} \le \frac{C}{t^{\frac{1}{2}+\epsilon}} h^{-4\epsilon} \|z\|_{H^2(\Omega)}, \quad 1 \le t < \infty. \tag{5.1.69}$$

Thus, by (5.1.69),

$$\begin{aligned} \int_1^{\infty} t^{-\frac{1}{2}} \|\eta_h(t)\|_{L_2(\Omega)}\, dt &\le C h^{-4\epsilon} \left(\int_1^{\infty} \frac{1}{t^{1+\epsilon}}\, dt \right) \|z\|_{H^2(\Omega)} \\ \text{(by (5.1.27b))} \qquad &\le C h^{-4\epsilon} \|z\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})}. \end{aligned} \tag{5.1.70}$$

In conclusion, (5.1.60) and (5.1.70), once used in (5.1.57), yield

$$\begin{aligned} \left\| [\mathcal{A}^{-\frac{1}{2}} - \mathcal{A}_h^{-\frac{1}{2}}] \pi_h z \right\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} &\le C h^2 \left[\int_0^1 t^{-\frac{1}{2}} \|\eta_h(t)\|_{L_2(\Omega)}\, dt \right. \\ &\qquad \left. + \int_1^{\infty} t^{-\frac{1}{2}} \|\eta_h(t)\|_{L_2(\Omega)}\, dt \right] \\ &\le C[h^2 + h^{2-4\epsilon}] \|z\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})}, \end{aligned} \tag{5.1.71}$$

which proves Lemma 5.1.5, Eqn. (5.1.47), as desired. □

A duality argument in the style of Proposition 5.1.1, part (ii), based on the definition (5.1.18) of π_h^1, yields

Proposition 5.1.5 *Under assumptions (i) through (vii) above, we have for* $x \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$*:*

$$\left\|\left(\pi_h^1 - I\right)x\right\|_{L_2(\Omega)} \le Ch^2\left\|\mathcal{A}^{\frac{1}{2}}\left(\pi_h^1 - I\right)x\right\|_{L_2(\Omega)} \tag{5.1.72}$$

$$\le Ch^2\left\|\mathcal{A}^{\frac{1}{2}}(\pi_h - I)x\right\|_{L_2(\Omega)} \tag{5.1.73}$$

$$\le Ch^2\left\|\mathcal{A}^{\frac{1}{2}}x\right\|_{L_2(\Omega)} \le Ch^2\|x\|_{H^2(\Omega)}. \tag{5.1.74}$$

Proof. It suffices to prove (5.1.72), since the inequality from (5.1.72) to (5.1.73) is nothing but (5.1.19) of Proposition 5.1.2, while then (5.1.74) follows from (5.1.73) via (5.1.27c). To prove (5.1.72) we estimate for $x \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$:

$$\left\|\left(\pi_h^1 - I\right)x\right\|_{L_2(\Omega)} = \sup_{y\in L_2(\Omega)} \frac{\left|\left(\mathcal{A}\left(\pi_h^1 - I\right)x, \mathcal{A}^{-1}y\right)_{L_2(\Omega)}\right|}{\|y\|_{L_2(\Omega)}}$$

(setting $\psi = \mathcal{A}^{-1}y$ and recalling (5.1.18))

$$= \sup_{\substack{\psi\in\mathcal{D}(\mathcal{A})\\ \|\mathcal{A}\psi\|_{L_2(\Omega)}=1}} \left\{\frac{\left|\left(\mathcal{A}\left(\pi_h^1 - I\right)x, \psi - \pi_h\psi\right)_{L_2(\Omega)}\right|}{\|\mathcal{A}\psi\|_{L_2(\Omega)}}\right\}$$

$$\le \left\|\mathcal{A}^{\frac{1}{2}}\left(\pi_h^1 - I\right)x\right\|_{L_2(\Omega)} \sup_{\substack{\psi\in\mathcal{D}(\mathcal{A})\\ \|\mathcal{A}\psi\|_{L_2(\Omega)}=1}} \left\{\frac{\left|\left(\mathcal{A}^{\frac{1}{2}}(I - \pi_h)\psi\right)_{L_2(\Omega)}\right|}{\|\mathcal{A}\psi\|_{L_2(\Omega)}}\right\}. \tag{5.1.75}$$

As to the second term on the right-hand side of (5.1.75), we estimate by virtue of (5.1.27c) and of (5.1.24), with $s = 4$, $\ell = 2$:

$$\left\|\mathcal{A}^{\frac{1}{2}}(I - \pi_h)\psi\right\|_{L_2(\Omega)} \le C\left\|(I - \pi_h)\psi\right\|_{H^2(\Omega)}$$

$$\text{(by (1.24))} \qquad \le Ch^2\|\psi\|_{H^4(\Omega)} \le Ch^2\|\psi\|_{\mathcal{D}(\mathcal{A})}, \tag{5.1.76}$$

using (5.1.26) in the last step. Thus, we return to the right-hand side of (5.1.75) and insert (5.1.76) for the second term, thus obtaining

$$\left\|\left(\pi_h^1 - I\right)x\right\|_{L_2(\Omega)} \le Ch^2\left\|\mathcal{A}^{\frac{1}{2}}\left(\pi_h^1 - I\right)x\right\|_{L_2(\Omega)}, \tag{5.1.77}$$

which proves (5.1.72), as desired. □

In the next sections, we shall illustrate the applicability of the numerical theory of Chapter 4 to the parabolic-like PDE problems, both first order and second order in time, with boundary/point control, whose continuous analysis was presented in Chapter 3. We shall concentrate on the more demanding optimal control problem on an infinite horizon, $T = \infty$.

5.2 Heat Equation with Dirichlet Boundary Control

We return to the continuous problem of Chapter 3, Section 3.1, which we rewrite here for convenience:

$$\begin{cases} y_t = \Delta y + c^2 y & \text{in } (0, T] \times \Omega, & (5.2.1a) \\ y(0, \cdot) = y_0 & \text{in } \Omega, & (5.2.1b) \\ y|_\Sigma = u & \text{in } (0, T] \times \Gamma = \Sigma, & (5.2.1c) \end{cases}$$

with boundary control $u \in L_2(0, T; L_2(\Gamma)) = L_2(\Sigma)$ and $y_0 \in L_2(\Omega)$. The cost functional to be minimized is

$$J(u, y) = \int_0^\infty \left[\|y(t)\|^2_{L_2(\Omega)} + \|u(t)\|^2_{L_2(\Gamma)}\right] dt. \tag{5.2.2}$$

In Section 5.2.1 we shall consider a (Galerkin) approximation scheme, which will comply with all approximation assumptions (A.1) = (4.1.2.2) through (A.6) = (4.1.2.9) of stability and consistency in Chapter 4, under which the general approximation theory of Theorems 4.1.4.1 and 4.1.4.2 of Chapter 4 holds true. Next, in Section 5.2.2, we shall consider a more sophisticated (Nitsche) approximation scheme, which will comply with all approximation assumptions (A.1) in (4.6.2.1), (4.6.2.2) through (4.A.9) in (4.6.2.3), (4.6.2.4), under which the sharp results of Theorems 4.6.2.1 and 4.6.2.2 of Chapter 4 hold true, yielding optimal rates of convergence.

5.2.1 Galerkin Approximation: Application of Theorems 4.1.4.1 and 4.1.4.2 with r ≥ 1

Choice of V_h We shall select the approximating space $V_h \subset H_0^1(\Omega)$ to be a space of splines (linear, $r = 1$; quadratic, $r = 2$; etc), which comply with the usual approximation properties, where r is the order of approximation (degree of polynomials):

$$\|\Pi_h y - y\|_{H^\ell(\Omega)} \le ch^{s-\ell}\|y\|_{H^s(\Omega)}, \quad s \le r + 1, \quad s - \ell \ge 0, \quad 0 \le \ell \le 1 \tag{5.2.1.1}$$

and the inverse approximation properties:

$$\|y_h\|_{H^\alpha(\Omega)} \le Ch^{-\alpha}\|y_h\|_{L_2(\Omega)}, \quad 0 \le \alpha \le 1; \tag{5.2.1.2a}$$

$$h^{-1}\|y - \Pi_h y\|_{L_2(\Gamma)} + \left\|\frac{\partial}{\partial \nu}(y - \Pi_h y)\right\|_{L_2(\Gamma)} \le Ch^{s-\frac{3}{2}}\|y\|_{H^s(\Omega)},$$
$$\frac{3}{2} < s \le r + 1, \quad y \in H^s(\Omega); \tag{5.2.1.2b}$$

$$\|y_h\|_{L_2(\Gamma)} + h\left\|\frac{\partial y_h}{\partial \nu}\right\|_{L_2(\Gamma)} \le Ch^{-\frac{1}{2}}\|y_h\|_{L_2(\Omega)}, \quad y_h \in V_h, \tag{5.2.1.2c}$$

where Π_h is the orthogonal projection of $L_2(\Omega)$ onto V_h. The approximating properties (5.2.1.1), (5.2.1.2a), and (5.2.1.2c), are standard [Thomée, 1984]. The approximating property (5.2.1.2b) is likewise standard for values of $s \geq 2$; its validity for $3/2 < s < 2$ was recently shown by T. Peterson (private communication).

Choice of A_h We define $A_h : V_h \to V_h$ as the usual Galerkin approximation where the inner products are in $L_2(\Omega)$:

$$(A_h x_h, y_h)_\Omega = (Ax_h, y_h)_\Omega = -\int_\Omega \nabla x_h \cdot \nabla y_h \, d\Omega + c^2 (x_h, y_h)_\Omega, \quad x_h, y_h \in V_h. \tag{5.2.1.3}$$

Choice of B_h With reference to the notation of Chapter 3, Section 3.1, we define $B_h : U \to V_h$ by

$$B_h = -\Pi_h A D, \tag{5.2.1.4a}$$

with D as in Chapter 3, Eqns. (3.1.6), (3.1.7), and we notice that then (L_2-inner products)

$$(B_h u, y_h)_\Omega = -(ADu, y_h)_\Omega = -(u, D^* A y_h)_\Gamma = -\left(u, \frac{\partial y_h}{\partial \nu}\right)_\Gamma, \tag{5.2.1.4b}$$

recalling Chapter 3, Eqn. (3.1.9). Hence, as in Chapter 3, Eqn. (3.1.9),

$$B_h^* y_h = -\left.\frac{\partial y_h}{\partial \nu}\right|_\Gamma. \tag{5.2.1.5}$$

Approximating Control Problem This is given by the ordinary differential equation problem: $\dot{y}_h = A_h y_h + B_h u$, that is, via (5.2.1.3) and (5.2.1.4):

$$\begin{cases} (\dot{y}_h, \phi_h)_\Omega + \displaystyle\int_\Omega \nabla y_h \cdot \nabla \phi_h \, d\Omega - c^2 \|y_h\|_\Omega^2 = -\left(u, \frac{\partial}{\partial \nu}\phi_h\right)_\Gamma, & \forall\, \phi_h \in V_h; \\ (y_h(0), \phi_h)_\Omega = (\Pi_h y(0), \phi_h)_\Omega. \end{cases} \tag{5.2.1.6}$$

The optimal feedback control for the approximating finite-dimensional problem is

$$u_h^0(t, 0; \Pi_h y_0) = -B_h^* P_h y_h^0(t, 0; \Pi_h y_0) = \left.\frac{\partial}{\partial \nu} P_h y_h^0(t, 0; \Pi_h y_0)\right|_\Gamma, \tag{5.2.1.7}$$

where P_h is the unique, nonnegative, self-adjoint solution of the following discrete algebraic Riccati equation (ARE$_h$):

$$\begin{aligned} &-\int_\Omega \nabla P_h x_h \cdot \nabla y_h \, d\Omega - \int_\Omega \nabla x_h \cdot \nabla P_h y_h \, d\Omega + (x_h, y_h)_\Omega \\ &= \left(\frac{\partial}{\partial \nu} P_h x_h, \frac{\partial}{\partial \nu} P_h y_h\right)_\Gamma, \quad \forall\, x_h, y_h \in V_h. \end{aligned} \tag{5.2.1.8}$$

Verification of Continuous Assumptions (4.1.4.1) and (4.1.4.3) of Theorem 4.1.4.1 These are plainly satisfied since $R = I$ and $\hat{A}^{-\gamma}B \in \mathcal{L}(U;Y)$ with $\gamma = 3/4 + \epsilon$ in our case (see Chapter 3, Section 3.1). Because of the compactness of A^{-1} on $L_2(\Omega) = Y$ (since Ω is bounded), this then implies in turn that $\hat{A}^{-1}B$ is compact $U \to Y$, and thus $B^*(\hat{A}^*)^{-1}$ is compact $Y \to U$, as desired.

Verification of Discrete Assumptions (A.1) through (A.6) of Theorem 4.1.4.1

Assumption (A.1) = (4.1.2.2) (uniform analyticity) That this is satisfied follows from results on Galerkin approximations of elliptic operators [Bramble et al., 1977] for the self-adjoint case and [Lasiecka, 1984] for the general non–self-adjoint case.

Assumption (A.2) = (4.1.2.4) With reference to the translated, positive, self-adjoint operator $\hat{A}$ in (4.1.1.2) of Chapter 4, with domain $\mathcal{D}(\hat{A}) = H^2(\Omega) \cap H_0^1(\Omega)$, we estimate

$$
\begin{aligned}
\|(\Pi_h - I)\hat{A}^{-1}x\|_{\mathcal{D}(\hat{A}^{\frac{1}{2}})} &\le C\|(\Pi_h - I)\hat{A}^{-1}x\|_{H^1(\Omega)} && (5.2.1.9)\\
\text{(by (5.2.1.1))} \qquad &\le Ch\|\hat{A}^{-1}x\|_{H^2(\Omega)} \\
&= Ch\|\hat{A}^{-1}x\|_{\mathcal{D}(\hat{A})} = Ch\|x\|_{L_2(\Omega)}, && (5.2.1.10)
\end{aligned}
$$

where in going from (5.2.1.9) to (5.2.1.10) we have used (5.2.1.1) with $s = 2$, $\ell = 1$. The same argument with $\hat{A}^{-1}x$ replaced by $\psi \in \mathcal{D}(\hat{A})$ yields the first line in the following estimate:

$$
\|(\Pi_h - I)\psi\|_{\mathcal{D}(\hat{A}^{\frac{1}{2}})} \le Ch\|\psi\|_{H^2(\Omega)} \le Ch\|\psi\|_{\mathcal{D}(\hat{A})}. \tag{5.2.1.11}
$$

Hence, using (5.2.1.10) and (5.2.1.11) in Eqn. (5.1.3) of Lemma 5.1.1(ii) yields

$$
\left\|\Pi_h\hat{A}^{-1} - \hat{A}_h^{-1}\Pi_h\right\|_{\mathcal{L}(L_2(\Omega))} \le Ch^2, \tag{5.2.1.12}
$$

and (5.2.1.12) is precisely (A.2) = (4.1.2.4) of Chapter 4 with $s = 2$.

Assumption (A.3) = (4.1.2.6) By Chapter 3, Eqn. (3.1.9), (5.2.1.5), and (5.2.1.2a), we obtain, with $U = L_2(\Gamma)$ and $Y = L_2(\Omega)$,

$$
\|B^*y_h\|_U = \|B_h^*y_h\|_U = \left\|\frac{\partial}{\partial\nu}y_h\right\|_{L_2(\Gamma)} \le Ch^{-\frac{3}{2}}\|y_h\|_{L_2(\Omega)}. \tag{5.2.1.13}
$$

Thus, (A.3) = (4.1.2.6) of Chapter 4 is satisfied (conservatively) for $s = 2$, $\gamma = 3/4 + \epsilon$.

Assumption (A.4) = (4.1.2.7) By Chapter 3, Eqn. (3.1.9) and (5.2.1.2b) applied with $s = 2$,

$$
\begin{aligned}
\|B^*(\Pi_h x - x)\|_{L_2(\Gamma)} = \left\|\frac{\partial}{\partial\nu}(\Pi_h x - x)\right\|_{L_2(\Gamma)} &\le Ch^{\frac{1}{2}}\|x\|_{H^2(\Omega)} && (5.2.1.14a)\\
&\le Ch^{s(1-\gamma)}\|x\|_{\mathcal{D}(A^*)}, && (5.2.1.14b)
\end{aligned}
$$

which implies (A.4) = (4.1.2.7) of Chapter 4, in view of the fact that $\mathcal{D}(A^*) \subset H^2(\Omega)$ and $s(1-\gamma) = 2(1 - 3/4 - \epsilon) = 1/2 - 2\epsilon < 1/2$.

Assumption (A.5) = (4.1.2.8) Since in our case $B_h^*\Pi_h = B^*\Pi_h$, (A.4) coincides with (A.5).

Assumption (A.6) = (4.1.2.9) From Chapter 3, Eqn. (3.1.9), from (2.1.2b) applied with $s = 3/2 + \epsilon$ and from the trace theorem, we obtain

$$\|B^*\Pi_h x\|_{L_2(\Gamma)} = \left\|\frac{\partial}{\partial \nu}\Pi_h x\right\|_{L_2(\Gamma)} \leq \left\|\frac{\partial}{\partial \nu}(\Pi_h - I)x\right\|_{L_2(\Gamma)} + \left\|\frac{\partial}{\partial \nu}x\right\|_{L_2(\Gamma)}$$

$$\leq Ch^\epsilon \|x\|_{H^{\frac{3}{2}+\epsilon}(\Omega)} + C\|x\|_{H^{\frac{3}{2}+\epsilon}(\Omega)}. \tag{5.2.1.15}$$

Thus, (A.6) now follows from $\mathcal{D}(A^{*\frac{3}{4}+\epsilon}) \subset H^{\frac{3}{2}+\epsilon}(\Omega)$, $\gamma = 3/4 + \epsilon$.

Conclusion The continuous analysis of Chapter 3, Section 3.1 and the above discrete analysis show that we have verified all the assumptions of Theorems 4.1.4.1 and 4.1.4.2 in Chapter 4 in the case of the heat equation problem with Dirichlet boundary control as in (5.2.1), (5.2.2), with $r + 1 \geq s = 2$, or order of approximation $r \geq 1$. Then, application of Theorem 4.1.4.1 of Chapter 4 yields the following convergence results, since $s(1-\gamma) = 1/2 - 2\epsilon$.

Theorem 5.2.1.1 *Assume $r \geq 1$ in (5.2.1.1). Then, with reference to problem (5.2.1), (5.2.2), we have:*

I: The unique solution $P_h : V_h \to V_h$ to (ARE$_h$) = (5.2.1.8) satisfies the following estimate:

(i)

$$\|P_h\Pi_h - P\|_{\mathcal{L}(L_2(\Omega))} \leq Ch^{\epsilon_0}, \quad \forall\, \epsilon_0 < \frac{1}{2}; \tag{5.2.1.16}$$

(ii)

$$\left\|\frac{\partial}{\partial \nu}[P_h\Pi_h - P]\right\|_{\mathcal{L}(L_2(\Omega); L_2(\Gamma))} \to 0 \ \text{as}\ h \downarrow 0. \tag{5.2.1.17}$$

II: The optimal finite-dimensional approximations $y_h^0(t; \Pi_h y_0)$ and $u_h^0(t; \Pi_h y_0)$ [see (5.2.1.7)] satisfy

(iii)

$$\left\|y_h^0(\cdot\,; \Pi_h y_0) - y^0(\cdot\,; y_0)\right\|_{\mathcal{L}(L_2(\Omega); L_2(0,\infty; L_2(\Omega)))}$$
$$+ \sup_{t \geq 0}\left\{e^{\tilde{\omega}_P t} t^\epsilon \left\|y_h^0(t; \Pi_h y_0) - y^0(t; y_0)\right\|_{\mathcal{L}(L_2(\Omega))}\right\} \leq Ch^{\epsilon_0\epsilon},$$

$$\forall\, \epsilon_0 < \frac{1}{2}, \quad 0 \leq \epsilon \leq 1, \tag{5.2.1.18}$$

that is, uniformly in $y_0 \in Y = L_2(\Omega)$.

(iv)

$$\sup_{t\geq 0}\left\{e^{\hat{\omega}_P t}\left\|u_h^0(t;\Pi_h y_0)-u^0(t;y_0)\right\|_{\mathcal{L}(L_2(\Omega);L_2(\Gamma))}\right\}\leq Ch^{\epsilon_0},\quad \forall\,\epsilon_0<\frac{1}{2}, \tag{5.2.1.19}$$

that is, uniformly in $y_0 \in L_2(\Omega)$.

(v) Since $\|\hat{A}^{*\theta}x_h\| = \|\hat{A}_h^{*\theta}x_h\|$ *for* $0 \leq \theta \leq 1/2$, *then Theorem 4.1.4.1(x), Eqn. (4.1.4.16) gives*

$$\|(P_h\Pi_h - P)x\|_{H^1(\Omega)} \to 0,\ x \in L_2(\Omega). \tag{5.2.1.20}$$

III: Furthermore, application of Theorem 4.1.4.2 yields the following result: If we consider the problem given by

$$\begin{cases} (y_h^*)_t = (\Delta + c^2)y_h^* & \text{in } (0,T]\times\Omega, & (5.2.1.21a)\\ y_h^*(0,\,\cdot\,) = y_0 & \text{in } \Omega, & (5.2.1.21b)\\ y_h^*|_\Sigma = -\dfrac{\partial}{\partial\nu}P_h\Pi_h y_h^* & \text{in } (0,T]\times\Gamma, & (5.2.1.21c)\end{cases}$$

that is, problem (5.2.1) with feedback law

$$u_h^*(t;y_0) = -\frac{\partial}{\partial\nu}P_h\Pi_h y_h^*(t;y_0), \tag{5.2.1.22}$$

with P_h *a solution of (5.2.1.8), then* $y_h^*(t;y_0)$ *is exponentially stable in* $\mathcal{L}(L_2(\Omega))$ *uniformly in the parameter h,*

$$\|y_h^*(t;y_0)\|_{L_2(\Omega)} \leq \hat{C}e^{-\hat{\omega}_P t}\|y_0\|_{L_2(\Omega)},\quad \forall h,\ t\geq 0. \tag{5.2.1.23}$$

Moreover,

$$\sup_{t\geq 0}\left\{e^{\hat{\omega}_P t}\|y_h^*(t;y_0)-y^0(t;y_0)\|_{\mathcal{L}(L_2(\Omega))}\right\}\to 0, \tag{5.2.1.24}$$

that is, uniformly in $y_0 \in L_2(\Omega)$.

Remark 5.2.1.1 The rate of convergence $\mathcal{O}(h^{\frac{1}{2}-\epsilon})$ guaranteed by (5.2.1.15), and (5.2.1.17) is not optimal. In view of the regularity $P \in \mathcal{L}(L_2(\Omega); H^{2-\epsilon}(\Omega))$ of the Riccati operator (see Chapter 3, Section 3.1), one would expect that the optimal rate of convergence should be of the order of $\mathcal{O}(h^{2(1-\epsilon)})$. Indeed, we shall show that this is possible, but for different, appropriate approximations of A_h and B_h. This will be done in the next subsection, however, with an order of approximation (degree of polynomials) $r \geq 2\frac{1}{2}$, as opposed to $r = 1$ in the present subsection.

5.2.2 Nitsche's Approximation: Application of Theorems 4.6.2.1 and 4.6.2.2. Optimal Rates of Convergence with r $\geq 2\frac{1}{2}$

With reference to the above Remark 5.2.1.1, in order to obtain the optimal rates of convergence ($\mathcal{O}(h^{2(1-\epsilon)})$), care must be exercised in selecting the approximation of the Poisson operator A^{-1}. Since the Dirichlet problem does not admit a natural variational formulation, extra attention must be paid to the approximation of the boundary conditions. Thus, to obtain the optimal rate ($\mathcal{O}(h^{2(1-\epsilon)})$), we need to introduce an approximation that approximates "well" the boundary conditions. For this purpose we shall use the elliptic approximation of the Poisson operator due to Nitsche [1971].

With V_h defined by (5.2.1.1) and (5.2.1.2) with $s > 3/2$, let $A_h : V_h \to V_h$ be defined as (see Nitsche [1971])

$$-(A_h x_h, y_h) \equiv \tilde{a}(x_h, y_h) \equiv a(x_h, y_h) - \left(\frac{\partial}{\partial\nu}x_h, y_h\right)_\Gamma - \left(x_h, \frac{\partial}{\partial\nu}y_h\right)_\Gamma$$
$$+ \beta h^{-1}(x_h, y_h)_\Gamma + c^2(x_h, y_h)_\Omega \tag{5.2.2.1}$$

in the L_2-norms, where $\beta > 0$ is sufficiently large and c^2 as in (2.1a).

The approximating finite-dimensional Riccati operator $P_h : V_h \to V_h$ satisfies the following approximating algebraic Riccati equation (ARE$_h$):

$$-(A_h P_h x_h, y_h) - (P_h A_h x_h, y_h) + (x_h, y_h)$$
$$= \left(\left(\frac{\partial}{\partial\nu} - \beta h^{-1}\right)P_h x_h, \left(\frac{\partial}{\partial\nu} - \beta h^{-1}\right)P_h y_h\right)_\Gamma. \tag{5.2.2.2}$$

We shall now verify the assumptions of Theorems 4.6.2.1 and 4.6.2.2 yielding optimal rates.

Hypotheses (A.1) = (4.1.2.2) and (A.2) = (4.1.2.4) (with $s = 2$) These are well known for the Nitsche's approximation A_h defined in (5.2.2.1) (see Bramble et al. [1977]); in particular,

$$\left\|[\hat{A}^{-1} - \hat{A}_h^{-1}\Pi_h]x\right\|_{L_2(\Omega)} \le Ch^s\|\hat{A}^{-1}x\|_{H^s(\Omega)}, \quad s \le r+1, \tag{5.2.2.3}$$

which then implies (A.2) for $s = 2$.

Hypotheses (A.3) = (4.1.2.6) through (A.5) = (4.1.2.8) For (A.3), in the Dirichlet case we define (see Choudoury and Lasiecka [1991])

$$-B_h^* x_h = \frac{\partial}{\partial\nu}x_h + \beta h^{-1}x_h\Big|_\Gamma, \tag{5.2.2.4}$$

while $B^*x = -\frac{\partial}{\partial\nu}x$ by Chapter 3, Eqn. (3.1.9). Thus, hypothesis (A.3) is the result of the inverse approximation property (5.2.1.2c), applied on (5.2.2.3):

$$\|B^*x_h\|_{L_2(\Gamma)} + \|B_h^*x_h\|_{L_2(\Gamma)} \le Ch^{-\frac{3}{2}}\|x_h\|_{L_2(\Omega)} \tag{5.2.2.5a}$$
$$\le Ch^{-\gamma s}\|x_h\|_{L_2(\Omega)}, \tag{5.2.2.5b}$$

with $\gamma s = (3/4 + \epsilon)\,2 = 3/2 + 2\epsilon$.

For (A.5), $B^*x = -\frac{\partial}{\partial\nu}x$ and for $x \in \mathcal{D}(A)$, we have $x|_\Gamma = 0$, and so we obtain

$$\|B^*x - B_h^*\Pi_h x\|_{L_2(\Gamma)} = \left\| \frac{\partial}{\partial\nu}x + \beta h^{-1}x\Big|_\Gamma - \frac{\partial}{\partial\nu}\Pi_h x - \beta h^{-1}\Pi_h x \right\|_{L_2(\Gamma)}$$

$$\leq \left\| \frac{\partial}{\partial\nu}(\Pi_h - I)x \right\|_{L_2(\Gamma)} + \beta h^{-1}\|(\Pi_h - I)x\|_{L_2(\Gamma)} \tag{5.2.2.6a}$$

(by the approximation property (5.2.1.2b) with $s = 2$)

$$\leq Ch^{-\frac{3}{2}}h^2\|x\|_{H^2(\Omega)} \leq Ch^{2(1-\frac{3}{4})}\|x\|_{\mathcal{D}(A^*)} \leq Ch^{s(1-\gamma)}\|x\|_{\mathcal{D}(A^*)}, \tag{5.2.2.6b}$$

since $\mathcal{D}(A^*) \subset H^2(\Omega)$ and $\gamma = 3/4 + \epsilon$, and thus (A.5) holds true.

As for (A.4), we have as desired

$$\|B^*(I - \Pi_h)x\|_U = \left\| \frac{\partial}{\partial\nu}(I - \Pi_h)x \right\|_U \leq Ch^{\frac{1}{2}}\|x\|_{H^2(\Omega)} \leq Ch^{\frac{1}{2}}\|x\|_{\mathcal{D}(A^*)} \tag{5.2.2.7}$$

by (5.2.1.2b) with $s = 2$ and $U = L_2(\Gamma)$.

Hypothesis (A.6) = (4.1.2.9) It involves only B^* (not A_h, B_h^*) and was verified before in (5.2.1.14).

Hypothesis (A.7) [see (4.6.2.1) and (4.6.2.2)] For part (ii) let $z_h = \hat{A}_h^{-1}B_h u$ and $z = \hat{A}^{-1}Bu$. We then have

$$(\hat{A}_h z_h, x_h)_\Omega = (B_h u, x_h)_\Omega = \big(u, B_h^* x_h\big)_\Gamma, \quad x_h \in V_h \tag{5.2.2.8}$$

in the L_2-norms. Recalling $\hat{A}_h = -A_h + \omega I$ from Chapter 4, Eqn. (4.1.2.5) and (5.2.2.1) on the left-hand side of (5.2.2.8) and the definition (5.2.2.4) on B_h^* on the right-hand side of (5.2.2.8), we then arrive at

$$\tilde{a}(z_h, x_h) + \omega(z_h, x_h) = \left(u, \frac{\partial}{\partial\nu}x_h\right)_\Gamma + \beta h^{-1}(u, x_h)_\Gamma. \tag{5.2.2.9}$$

Similarly, $\hat{A}z = (-A + \omega I)z = Bu = -ADu$ (see Chapter 3, Eqn. (3.1.5)), or $-A(z - Du) = 0$ implies $z - Du \in \mathcal{D}(A)$, and so, upon recalling the definitions of A and D in Chapter 3, Eqns. (3.1.3) and (3.1.6), we have

$$(\Delta + c^2)z - \omega z = 0 \quad \text{in } \Omega; \; z|_\Gamma = u. \tag{5.2.2.10}$$

Since (5.2.2.9) defines an elliptic approximation of z, the convergence results of Nitsche [1971] apply to yield

$$\|z - z_h\|_{L_2(\Omega)} \leq Ch^2\|z\|_{H^2(\Omega)} \leq Ch^2\|u\|_{H^{\frac{3}{2}}(\Gamma)}, \tag{5.2.2.11}$$

so that hypothesis (A.7)(ii) = (4.6.2.2) of Chapter 4 holds true with $r_0 = 2$ and $U_{r_0} = H^{\frac{3}{2}}(\Gamma) \subset L_2(\Gamma)$.

As for (A.7)(i) = (4.6.2.1), we shall prove that

$$\left\| \left[\hat{A}_h^{-1} B_h - \hat{A}^{-1} B \right] \right\|_{\mathcal{L}(L_2(\Gamma); L_2(\Omega))} = \mathcal{O}\left(h^{\frac{1}{2}}\right), \tag{5.2.2.12}$$

and so (A.7)(i) is satisfied with $\rho = 1/2$ and $U_{r_1} = L_2(\Gamma)$.

To assert (5.2.2.12), we use duality: After recalling $B^* x = -\frac{\partial x}{\partial \nu}$, (5.2.2.4) for B_h^*, and that $\hat{A}^{-1} x|_{L_2(\Gamma)} = 0$, since $x \in \mathcal{D}(\hat{A})$ with $\hat{A}$ self-adjoint, we show equivalently

$$\begin{aligned} \left\| \left[B^* \hat{A}^{*-1} - B_h^* \hat{A}_h^{*-1} \Pi_h \right] x \right\|_U &= \left\| \frac{\partial}{\partial \nu} \left[\hat{A}_h^{-1} \Pi_h - \hat{A}^{-1} \right] x \right\|_{L_2(\Gamma)} \\ &\quad + \beta h^{-1} \left\| \left[\hat{A}_h^{-1} \Pi_h - \hat{A}^{-1} \right] x \right\|_{L_2(\Gamma)} \\ \text{(by (5.2.1.2c))} \qquad &\leq C h^{-\frac{3}{2}} \left\| \left[\hat{A}_h^{-1} \Pi_h - \hat{A}^{-1} \right] x \right\|_{L_2(\Omega)} \\ \text{(by (5.2.2.3) with } s = 2) \qquad &\leq C h^{-\frac{3}{2}} h^2 \| \hat{A}^{-1} x \|_{H^2(\Omega)} \leq C h^{\frac{1}{2}} \| x \|_{L_2(\Omega)}, \end{aligned} \tag{5.2.2.13}$$

and (5.2.2.12) is proved.

Hypothesis (A.8) [see (4.6.2.3) and (4.6.2.4)] Here we have $U_{r_1} = U = L_2(\Gamma)$, and $U_{r_0} = H^{\frac{3}{2}}(\Gamma)$, and $Y_{r_1} = H^{\frac{3}{2}}(\Omega) \supset \mathcal{D}(\hat{A}^{\frac{3}{4}})$; $Y = L_2(\Omega)$, $r_0 = r_1 = 2$. Part (i) of (A.8), or (4.6.2.3), is trivially satisfied, since, by Chapter 3, Eqn. (3.1.5), $A^{-1} B = -D$, with the regularity of D being $H^\alpha(\Gamma) \to H^{\alpha + \frac{1}{2}}(\Omega)$, $\forall\, \alpha \in \mathbb{R}$, [see Chapter 3, Eqn. (3.1.7b)]. As to part (ii) of (A.8), or (4.6.2.4), we compute, after adding and subtracting $B_h^* \Pi_h \hat{A}^{*-1} y$:

$$\begin{aligned} \left\| \left[B_h^* \hat{A}_h^{*-1} \Pi_h - B^* \hat{A}^{*-1} \right] y \right\|_{U_{r_1}} &\leq \left\| B_h^* \left[\hat{A}_h^{*-1} \Pi_h - \Pi_h \hat{A}^{*-1} \right] y \right\|_{L_2(\Gamma)} \\ &\quad + \left\| \left[B_h^* \Pi_h \hat{A}^{*-1} - B^* \hat{A}^{*-1} \right] y \right\|_{L_2(\Gamma)}. \end{aligned} \tag{5.2.2.14}$$

As to the first term on the right-hand side of (5.2.2.14), we estimate by recalling (5.2.2.5a) and self-adjointness:

$$\begin{aligned} &\left\| B_h^* \left[\hat{A}_h^{*-1} \Pi_h - \Pi_h \hat{A}^{*-1} \right] y \right\|_{L_2(\Gamma)} \\ &\leq C h^{-\frac{3}{2}} \left\| \left[\hat{A}_h^{-1} \Pi_h - \Pi_h \hat{A}^{-1} \right] y \right\|_{L_2(\Gamma)} \\ &\leq C h^{-\frac{3}{2}} \left\{ \left\| \left[\hat{A}_h^{-1} \Pi_h - \hat{A}^{-1} \right] y \right\|_{L_2(\Gamma)} + \| [I - \Pi_h] \hat{A}^{-1} y \|_{L_2(\Gamma)} \right\} \end{aligned}$$

(using (5.2.2.3) with $s = r + 1$ on the first term, and (5.2.1.1) with $s = r + 1$ and $\ell = 0$ on the second term)

$$\leq C h^{-\frac{3}{2}} h^{r+1} \| \hat{A}^{-1} y \|_{H^{r+1}(\Omega)}. \tag{5.2.2.15}$$

As to the second term on the right-hand side of (5.2.2.14), we estimate by recalling (5.2.2.6a) followed by (5.2.1.2b) with $s = r + 1$, and by using self-adjointness:

$$\|[B_h^* \Pi_h - B^*]\hat{A}^{*-1} y\|_{L_2(\Gamma)} \leq C h^{-\frac{3}{2}} h^{r+1} \|\hat{A}^{-1} y\|_{H^{r+1}(\Omega)}. \tag{5.2.2.16}$$

Thus, inserting (5.2.2.15) and (5.2.2.16) into the right-hand side of (5.2.2.14), we obtain

$$\begin{aligned} \left\|\left[B_h^* \hat{A}_h^{*-1} \Pi_h - B^* \hat{A}^{*-1}\right] y\right\|_{U_{r_1} = L_2(\Gamma)} &\leq C h^{-\frac{3}{2}} h^{r+1} \|\hat{A}^{-1} y\|_{H^{r+1}(\Omega)} \\ &\leq C h^{-\frac{3}{2}} h^{r+1} \|y\|_{H^{r-1}(\Omega)} \qquad (5.2.2.17) \\ \left(\text{taking } r \geq 2\tfrac{1}{2}\right) \qquad &\leq C h^2 \|y\|_{H^{\frac{3}{2}}(\Omega) = Y_{r_1}}, \qquad (5.2.2.18) \end{aligned}$$

provided the order of approximation (degree of polynomials) r is taken $\geq 2\frac{1}{2}$ to justify the last step. Then (5.2.2.18) proves (A.8)(ii) = (4.6.2.4) of Chapter 4, since $r_1 = 2$.

Hypothesis (A.9) [see (4.6.2.5) and (4.6.2.6)] For (A.9)(i) = (4.6.2.5), by Chapter 3, Eqn. (3.1.9) and self-adjointness, we have

$$B^* \hat{A}^{*-2+\epsilon} = -\frac{\partial}{\partial \nu} \hat{A}^{-2+\epsilon}. \tag{5.2.2.19}$$

Since

$$\hat{A}^{-2+\epsilon} \in \mathcal{L}(L_2(\Omega); H^{4-2\epsilon}(\Omega)), \tag{5.2.2.20}$$

the trace theorem implies, via (5.2.2.19) and (5.2.2.10),

$$B^* \hat{A}^{*-2+\epsilon} \in \mathcal{L}\big(L_2(\Omega); H^{\frac{5}{2}-2\epsilon}(\Gamma)\big) \subset \mathcal{L}(Y; U_{r_0}), \tag{5.2.2.21}$$

since $U_{r_0} = H^{\frac{3}{2}}(\Gamma)$, and (A.9)(i), or (4.6.2.5) of Chapter 4 is proved, since the second relation there is implied by the first, that is, (5.2.2.21), for A self-adjoint.

For (A.9)(ii) = (4.6.2.6), by Chapter 3, Eqns. (3.1.9) and (3.1.5) and self-adjointness, we have

$$B^* \hat{A}^{*-1} \hat{A}^{-1} B = \frac{\partial}{\partial \nu} \hat{A}^{-1} \hat{A}^{-1} A D. \tag{5.2.2.22}$$

From elliptic theory [see Chapter 3, Eqn. (3.1.7)] which gives the regularity of D, we obtain

$$\hat{A}^{-1} A D \in \mathcal{L}\big(H^\alpha(\Gamma); H^{\alpha+\frac{1}{2}}(\Omega)\big), \quad \alpha \geq 0, \tag{5.2.2.23}$$

$$\hat{A}^{-1} \hat{A}^{-1} A D \in \mathcal{L}\big(H^\alpha(\Gamma); H^{\alpha+\frac{5}{2}}(\Omega)\big). \tag{5.2.2.24}$$

Thus, setting $\alpha = 0$ in (5.2.2.24), and using the trace theorem, we obtain by (5.2.2.22)

$$B^* \hat{A}^{*-1} \hat{A}^{-1} B \in \mathcal{L}(L_2(\Gamma); H^1(\Gamma)). \tag{5.2.2.25}$$

A repeated application of (5.2.2.24), this time with $\alpha = 1$, gives

$$B^* \hat{A}^{-1} \hat{A}^{-1} B \in \mathcal{L}(H^1(\Gamma); H^2(\Gamma)). \tag{5.2.2.26}$$

Combining (5.2.2.25) with (5.2.2.26) gives the desired result of (A.9)(ii) = (4.6.2.6) with $n = 2$:

$$[B^* \hat{A}^{*-1} \hat{A}^{-1} B]^2 \in \mathcal{L}(L_2(\Gamma); H^2(\Gamma)) \subset \mathcal{L}(U; U_{r_0}), \tag{5.2.2.27}$$

since $U_{r_0} = H^{\frac{3}{2}}(\Gamma)$.

Conclusion The continuous analysis of Chapter 3, Section 3.1 and the above discrete analysis show that we have verified all the assumptions of Theorems 5.6.2.1 and 5.6.2.2 of Chapter 5 with $r_0 = r_1 = s = 2, \gamma = 3/4 + \epsilon$ in the case of the heat equation problem with Dirichlet boundary control as in (5.2.1) and (5.2.2), provided the order of approximation (degree of polynomials) $r \geq 2\frac{1}{2}$ (see (5.2.2.18)) [as opposed to $r \geq 1$ in Subsection 3.2.1]. We then specialize these results to obtain the following theorem.

Theorem 5.2.2.1 *Assume $r \geq 2\frac{1}{2}$ in (5.2.1.1). Then, with reference to the heat equation with Dirichlet boundary control problem (5.2.1), (5.2.2), we have:*

I: The unique (nonnegative, self-adjoint) solution $P_h : V_h \to V_h$ to (ARE_h) = (5.2.2.2) satisfies the following estimates:

$$\|P_h \Pi_h - P\|_{\mathcal{L}(L_2(\Omega))} \leq C h^{2(1-\epsilon)}, \quad \forall\, \epsilon > 0; \tag{5.2.2.28}$$

$$\left\| \frac{\partial}{\partial \nu} [P_h \Pi_h - P] \right\|_{\mathcal{L}(L_2(\Omega); L_2(\Gamma))} \leq C h^{\frac{1}{2}-\delta}, \quad \forall\, \delta > 0. \tag{5.2.2.29}$$

II: There exists $w_0 > 0$ and $C \geq 1$ such that

$$\left\| y_h^0(t; \Pi_h y_0) \right\|_{L_2(\Omega)} \leq C e^{-\omega_0 t} \|y_0\|_{L_2(\Omega)}, \tag{5.2.2.30}$$

where $y_h^0(t) = y_h^0(t; \Pi_h y_0)$ satisfies in the L_2-norms

$$\begin{cases} \left(\dot{y}_h^0(t), x_h\right) + \tilde{a}\left(y_h^0(t), x_h\right) = \left(\left(\frac{\partial}{\partial \nu} - \beta h^{-1}\right) P_h y_h^0(t), x_h\right)_\Gamma, \\ \left(y_h^0(0), x_h\right) = \left(y_0, x_h\right). \end{cases} \tag{5.2.2.31}$$

III:

$$\left\| y_h^0(t; \Pi_h y_0) - y^0(t; y_0) \right\|_{L_2(\Omega)} \leq \frac{C h^{2(1-\epsilon)}}{t^{1-\epsilon}} e^{-\omega_0 t} \|y_0\|_{L_2(\Omega)}; \tag{5.2.2.32}$$

$$\left\| \frac{\partial}{\partial \nu} P_h y_h^0(t; \Pi_h y_0) - \frac{\partial}{\partial \nu} P y^0(t; y_0) \right\|_{L_2(\Gamma)} \leq \frac{C e^{-\omega_0 t}}{t^{\frac{3}{4}}} h^{2(1-\epsilon)} \|y_0\|_{L_2(\Omega)}. \tag{5.2.2.33}$$

IV: If we consider the feedback problem

$$\begin{cases} (y_h^*)_t = (\Delta + c^2) y_h^* & \text{in } (0, T] \times \Omega, & (5.2.2.34a) \\ y_h^*(0, \cdot) = y_0 & \text{in } \Omega, & (5.2.2.34b) \\ y_h^*|_\Sigma = \frac{\partial}{\partial \nu} P_h \Pi_h y_h^* & \text{in } (0, T] \times \Gamma, & (5.2.2.34c) \end{cases}$$

that is, problem (5.2.1) with feedback law

$$u_h^*(t; y_0) = \frac{\partial}{\partial \nu} P_h \Pi_h y_h^*(t; y_0), \tag{5.2.2.35}$$

with P_h a solution of (5.2.2.2), then $y_h^*(t; y_0)$ is exponentially stable in $\mathcal{L}(L_2(\Omega))$, uniformly in h:

$$\|y_h^*(t; y_0)\|_{L_2(\Omega)} \leq \hat{C} e^{-\hat{\omega}_P t} \|y_0\|_{L_2(\Omega)}, \quad \forall h, \ t \geq 0. \tag{5.2.2.36}$$

Moreover,

$$\|y_h^*(t; y_0) - y^0(t; y_0)\|_{L_2(\Omega)} \leq \frac{C h^{2(1-\epsilon)}}{t^{\frac{3}{4}}} e^{-\omega_0 t} \|y_0\|_{L_2(\Omega)}; \tag{5.2.2.37}$$

$$\|y_h^*(t; y_0) - y^0(t; y_0)\|_{L_2(\Omega)} \leq C h^{\frac{1}{2}-\epsilon} e^{-\omega_0 t} \|y_0\|_{L_2(\Omega)}. \tag{5.2.2.38}$$

Remark 5.2.2.1 The rates of convergence provided by Theorem 5.2.2.1 are optimal in the sense that they reconstruct the optimal regularity of the original solutions. The presence of factor $1/t^{1-\epsilon}$ in Part III is consistent with rough data estimates available for parabolic semigroups.

Remark 5.2.2.2 The approximating operator P_h in Section 5.2.1 is the unique (non-negative, self-adjoint) solution of the $(\text{ARE}_h) = (5.2.1.8)$; whereas the approximating operator P_h in Section 5.2.2 is the unique solution of the $(\text{ARE}_h) = (5.2.2.2)$. Thus, $y_h^*(t; y_0)$ in Section 5.2.1, which is the solution of (5.2.1.21), is different from the $y_h^*(t; y_0)$ in Section 5.2.2, which is the solution of (5.2.2.32).

Remark 5.2.2.3 One could replace $-\Delta - c^2$ with any second-order, uniformly elliptic operator with self-adjoint realization on $L_2(\Omega)$.

5.3 Heat Equation with Neumann Boundary Control. Optimal Rates of Convergence with $r \geq 1$ and Galerkin Approximation

To emphasize the contrast with Section 5.2, we shall now consider the heat equation with Neumann control

$$\begin{cases} y_t = \Delta y + c^2 y & \text{in } (0, T] \times \Omega, & (5.3.1a) \\ y(0, \cdot) = y_0 & \text{in } \Omega, & (5.3.1b) \\ \frac{\partial y}{\partial \nu}\Big|_\Sigma = u & \text{in } (0, T] \times \Gamma = \Sigma, & (5.3.1c) \end{cases}$$

with cost functional

$$J(u, y) = \int_0^\infty [\|y(t)\|^2_{L_2(\Omega)} + \|u(t)\|^2_{L_2(\Gamma)}]\,dt, \tag{5.3.2}$$

where then

$$U = L_2(\Gamma), \qquad Y = L_2(\Omega), \tag{5.3.3}$$

rather than the higher cost functional in Chapter 3, Eqn. (3.2.2) with $Y = H^1(\Omega)$ (see Chapter 3, Remark 3.2.1 for the continuous theory). Now, since $\gamma < 1/2$ for (5.3.2), one can take the usual Galerkin approximations for A_h and simply define $B_h = \Pi_h B$. These choices produce optimal rate $\mathcal{O}(h^{2(1-\epsilon)})$ of convergence with any spline approximation of order $r \geq 1$, via Theorems 4.6.2.1 and 4.6.2.2 of Chapter 4.

Choice of V_h We shall select the approximating space $V_h \subset H^1(\Omega)$ to be a space of linear splines ($r = 1$), subject to the usual approximating properties given in (5.2.1.1) and (5.2.1.2a,b,c). In particular, Π_h is the orthogonal projection $L_2(\Omega)$ onto V_h.

Choice of A_h We define $A_h : V_h \to V_h$ as the usual Galerkin approximation as in (5.2.1.3), that is,

$$\begin{aligned}(A_h x_h, y_h)_\Omega &= (A x_h, y_h)_\Omega = -\int_\Omega \nabla x_h \cdot \nabla y_h \, d\Omega + c^2 (x_h, y_h)_\Omega \\ &= -a(x_h, y_h), \quad x_h, y_h \in V_h.\end{aligned} \tag{5.3.4}$$

Choice of B_h With reference to the notation of Chapter 3, Section 3.3, we define $B_h : U \to V_h$ by

$$B_h = -\Pi_h A N, \tag{5.3.5a}$$

with N as in Chapter 3, Eqn. (3.3.7), and we notice that then (in the L_2-norms):

$$(B_h u, y_h)_\Omega = (ANu, y_h)_\Omega = (u, N^* A y_h)_\Gamma = (u, y_h|_\Gamma)_\Gamma, \tag{5.3.5b}$$

recalling N^*A from Chapter 3, Eqn. (3.3.9). Hence, as in Eqn. (3.3.9):

$$B_h^* y_h = y_h|_\Gamma. \tag{5.3.5c}$$

Approximating Control Problem This is given by the ordinary differential equation problem $\dot{y}_h = A_h y_h + B_h u$; that is, via (5.3.4) and (5.3.5),

$$\begin{cases} (\dot{y}_h, \phi_h)_\Omega + \int_\Omega \nabla y_h \cdot \nabla \phi_h \, d\Omega - c^2 \|y_h\|^2_\Omega = (u, \phi_h|_\Gamma)_\Gamma, \quad \forall \phi_h \in V_h; \\ (y_h(0), \phi_h)_\Omega = (\Pi_h y(0), \phi_h)_\Omega. \end{cases} \tag{5.3.6}$$

The optimal feedback control for the approximating finite-dimensional problem is

$$\begin{aligned} u_h^0(t,0;\Pi_h y_0) &= -B_h^* P_h y_h^0(t,0;\Pi_h y_0) \\ &= -P_h y_h^0(t,0;\Pi_h y_0)\big|_\Gamma, \end{aligned} \tag{5.3.7}$$

where P_h satisfies the following discrete algebraic Riccati equation (ARE_h)

$$-\int_\Omega \nabla P_h x_h \cdot \nabla y_h \, d\Omega - \int_\Omega \nabla x_h \cdot \nabla P_h y_h \, d\Omega + (x_h, y_h)_\Omega = (P_h x_h|_\Gamma, P_h y_h|_\Gamma)_\Gamma, \quad \forall\, x_h, y_h \in V_h. \tag{5.3.8}$$

Verification of Continuous Assumptions (4.1.4.1) and (4.1.4.3) These are plainly satisfied since $R = I$ and $\hat{A}^{-\gamma} B \in \mathcal{L}(U; Y)$ [see (5.3.3)], with $\gamma = 1/4 + \epsilon$ in our case (5.3.2) (see Chapter 3, Remark 3.3.3). Because of the compactness of A^{-1} on $L_2(\Omega) = Y$ (since Ω is bounded), this then implies in turn that $\hat{A}^{-1}B$ is compact $U \to Y$, and thus $B^*(\hat{A}^*)^{-1}$ is compact $Y \to U$, as desired.

Verification of Discrete Assumptions (A.1) through (A.9)
of Theorems 4.6.2.1 and 4.6.2.2

Assumption (A.1) = (4.1.2.2) (uniform analyticity) That this is satisfied follows from results on Galerkin approximations of elliptic operators; see Bramble et al. [1977] for the self-adjoint case and [Lasiecka, 1984] for the general non–self-adjoint case.

Assumption (A.2) = (4.1.2.4) This holds true with $s = 2$. Verification is identical to that given in Section 5.2.1, as it is based only on the general properties (5.2.1.1) and (5.2.1.2a,b,c) of the approximating V_h.

Assumption (A.3) = (4.1.2.6) By Chapter 3, Eqn. (3.3.9) for B^* and (3.3.5c) for its counterpart B_h^*, as well as by (5.2.1.2c),

$$\|B^* y_h\|_U = \|B_h^* y_h\|_U = \|y_h|_\Gamma\|_{L_2(\Gamma)} \le Ch^{-\frac{1}{2}}\|y_h\|_{L_2(\Omega)} \tag{5.3.9}$$

$$\le Ch^{-\gamma s}\|y_h\|_{L_2(\Omega)}, \tag{5.3.10}$$

and, by (5.3.3), property (A.3) = (4.1.2.5) is satisfied since $s\gamma = 2(1/4 + \epsilon) = 1/2 + \epsilon$.

Assumption (A.4) = (4.1.2.7) By Chapter 3, Eqn. (3.3.9) and (5.2.1.2b) applied with $s = 2$,

$$\|B^*(\Pi_h x - x)\|_U = \|(\Pi_h - I)x\|_{L_2(\Gamma)} \le Ch^{\frac{3}{2}}\|x\|_{H^2(\Omega)} \tag{5.3.11}$$

$$\le Ch^{\frac{3}{2}}\|x\|_{\mathcal{D}(A^*)} \le Ch^{s(1-\gamma)}\|x\|_{\mathcal{D}(A^*)}, \tag{5.3.12}$$

as desired, since

$$\mathcal{D}(A^*) = \mathcal{D}(A) \subset H^2(\Omega)$$

and

$$s(1-\gamma) = 2\left(1 - \frac{1}{4} - \epsilon\right) = \frac{3}{2} - 2\epsilon.$$

Assumption (A.5) = (4.1.2.8) Since, in our case, $B_h^* \Pi_h = B^* \Pi_h$, (A.4) coincides with (A.5).

Assumption (A.6) = (4.1.2.9) From Chapter 3, Eqn. (3.3.9), from (5.2.1.2b) applied with $s = 1/2 + \epsilon$, and from the trace theorem, we obtain for $x \in \mathcal{D}(\hat{A}^{*\gamma}) = \mathcal{D}(\hat{A}^{\gamma})$:

$$\begin{aligned} \|B^* \Pi_h x\|_U &= \|\Pi_h x\|_{L_2(\Gamma)} \le \|(\Pi_h - I)x\|_{L_2(\Gamma)} + \|x\|_{L_2(\Gamma)} \\ &\le Ch^{\epsilon} \|x\|_{H^{\frac{1}{2}+\epsilon}(\Omega)} + C\|x\|_{H^{\frac{1}{2}+\epsilon}(\Omega)} \qquad (5.3.13) \\ &\le Ch^{\epsilon} \|x\|_{\mathcal{D}(\hat{A}^{*\gamma})} + C\|x\|_{\mathcal{D}(\hat{A}^{*\gamma})}, \qquad (5.3.14) \end{aligned}$$

as desired, since $\mathcal{D}(\hat{A}^{*\gamma}) \subset H^{\frac{1}{2}+2\epsilon}(\Omega)$ for $\gamma = 1/4 + \epsilon$.

Remark 5.3.1 Having verified above the continuous and the discrete assumptions of the approximation Theorems 4.1.4.1 and 4.1.4.2, we could then invoke these results as they apply to our present problem (5.3.1), (5.3.2). However, as we know, Theorems 4.1.4.1 and 4.1.4.2 provide only a rate of convergence of order $s(1-\gamma) = 2(1 - 1/4 - \epsilon) = 3/2 - 2\epsilon$, which is nonoptimal. To obtain the optimal rate of convergence, $2 - \epsilon$, we need to verify the additional assumptions (A.7) through (A.9) of Chapter 4 and then invoke Theorems 4.6.2.1 and 4.6.2.2. This will be done below, where we shall see that assumptions (A.7) through (A.9) hold true with

$$\begin{cases} s = r_0 = r_1 = 2, \ \rho = 1, \quad U_{r_0} = H^{\frac{1}{2}}(\Gamma) \subset U = L_2(\Gamma) \subset U_{r_1} = H^{-\frac{1}{2}}(\Gamma); \\ Y_{r_1} = L_2(\Omega) = Y. \end{cases} \qquad (5.3.15)$$

Assumption (A.7) [see (4.6.2.1) and (4.6.2.2)] Let $z_h = \hat{A}_h^{-1} B_h u$ and $z = \hat{A}^{-1} Bu$. Then, we have

$$(\hat{A}_h z_h, x_h)_\Omega = (B_h u, x_h)_\Omega = (u, B_h^* x_h)_\Gamma, \quad x_h \in V_h \qquad (5.3.16)$$

in the L_2-norms. Recalling $\hat{A}_h = -A_h + \omega I$ from Chapter 4, Eqn. (4.1.2.5) and (5.3.4) on the left-hand side of (5.3.16), and the definition (5.3.5c) on B_h^* on the right-hand side of (5.3.16), we then arrive at

$$a(z_h, x_h) + \omega(z_h, x_h)_\Omega = (u, x_h|_\Gamma)_\Gamma. \qquad (5.3.17)$$

Similarly, $\hat{A}z = (-A + \omega I)z = Bu = -ANu$ [see Chapter 3, Eqn. (3.3.6)], or $-A(z - Nu) = 0$ implies $z - Nu \in \mathcal{D}(A)$, and so, upon recalling the definitions of A and N in Chapter 3, Eqns. (3.3.4) and (3.3.7)

$$(\Delta + c^2)z - \omega z = 0 \text{ in } \Omega; \qquad \frac{\partial z}{\partial \nu} = u \text{ on } \Gamma. \qquad (5.3.18)$$

Standard results for Galerkin approximations, as in Thomée [1984], yield for $0 \leq \alpha \leq 1$:

$$\|z - z_h\|_{L_2(\Omega)} \leq Ch^{1+\alpha}\|z\|_{H^{1+\alpha}(\Omega)} \leq Ch^{1+\alpha}\|u\|_{H^{\alpha-\frac{1}{2}}(\Gamma)}. \tag{5.3.19}$$

Choosing $\alpha = 1$ in (5.3.19) and recalling the definitions of z_h and z, we rewrite it as

$$\left\|\left[\hat{A}_h^{-1}B_h - \hat{A}^{-1}B\right]u\right\|_{L_2(\Omega)=Y} \leq Ch^2\|u\|_{H^{\frac{1}{2}}(\Gamma)=U_{r_1}}, \tag{5.3.20}$$

which proves (A.7)(ii) = (4.6.2.2), with $r_0 = 2$; $U_{r_0} = H^{\frac{1}{2}}(\Gamma) \subset U = L_2(\Gamma)$. Also, setting $\alpha = 0$ in (5.3.19) yields

$$\left\|\left[\hat{A}_h^{-1}B_h - \hat{A}^{-1}B\right]u\right\|_{L_2(\Omega)=Y} \leq Ch\|u\|_{H^{-\frac{1}{2}}(\Gamma)=U_{r_0}}, \tag{5.3.21}$$

which proves (A.7)(i) = (4.6.2.1), with $\rho = 1$; $U_{r_0} = H^{-\frac{1}{2}}(\Gamma) \supset U = L_2(\Gamma)$.

Assumption (A.8) [see (4.6.2.3) and (4.6.2.4)] Here we take $r_1 = 2$; $U_{r_0} = H^{\frac{1}{2}}(\Gamma) \subset U = L_2(\Gamma) \subset U_{r_1} = H^{-\frac{1}{2}}(\Gamma)$; $Y_{r_1} = L_2(\Omega) = Y$.

(A.8)(i): Recalling B from Chapter 3, Eqn. (3.3.6), we have $A^{-1}B = -N$, with the regularity of N being $H^\alpha(\Gamma) \to H^{\alpha+\frac{3}{2}}(\Omega)$, $\forall\, \alpha \in \mathbb{R}$ [see Chapter 3, Eqn. (3.3.8b)]. Thus, (A.8)(i) = (4.6.2.3) is trivially satisfied:

$$A^{-1}B = -N \in \mathcal{L}\left(H^{-\frac{1}{2}}(\Gamma); H^1(\Omega) = \mathcal{D}(\hat{A}^{\frac{1}{2}})\right) \cap \mathcal{L}\left(H^{\frac{1}{2}}(\Gamma); H^2(\Omega)\right), \tag{5.3.22a}$$

and thus

$$\hat{A}^{-1+\epsilon}B \in \mathcal{L}\left(H^{-\frac{1}{2}}(\Gamma); \mathcal{D}(\hat{A}^{\frac{1}{2}-\frac{\epsilon}{2}})\right) \subset \mathcal{L}(U_{r_1}; L_2(\Omega)). \tag{5.3.22b}$$

(A.8)(ii): Duality on (5.3.20) of (A.7) gives

$$\|[B_h^*(\hat{A}_h^*)^{-1}\Pi_h - B^*(\hat{A}^*)^{-1}]y\|_{H^{-\frac{1}{2}}(\Gamma)} \leq Ch^2\|y\|_{L_2(\Omega)}, \tag{5.3.23}$$

which proves (A.8)(ii) = (4.6.2.4), with $r_1 = 2$; $U_{r_1} = H^{-\frac{1}{2}}(\Gamma)$; $Y_{r_1} = L_2(\Omega)$.

Assumption (A.9) [see (4.6.2.5) and (4.6.2.6)] (A.9)(i): Recalling Chapter 3, Eqn. (3.3.9), we compute by self-adjointness, with $y \in L_2(\Omega)$,

$$B^*\hat{A}^{*-2+\epsilon}y = -N^*A\hat{A}^{-2+\epsilon}y = -\hat{A}^{-2+\epsilon}y|_\Gamma \in H^{\frac{1}{2}}(\Gamma) = U_{r_0}, \tag{5.3.24}$$

a fortiori by the trace theorem, since $\hat{A}^{-2+\epsilon} \in \mathcal{L}(L_2(\Omega); H^{4-2\epsilon}(\Omega))$, and (5.3.24) proves (A.9)(i) = (4.6.25), since the second relation there is implied by the first, that is, (5.3.24), for A self-adjoint.

(A.9)(ii): By Chapter 3, Eqns. (3.3.6) and (3.3.9) and self-adjointness, we have

$$B^*\hat{A}^{*-1}\hat{A}^{-1}B = N^*A\hat{A}^{-2}AN \in \mathcal{L}(L_2(\Gamma); H^{3-\epsilon}(\Gamma)) \subset \mathcal{L}\left(L_2(\Gamma); H^{\frac{1}{2}}(\Gamma)\right). \tag{5.3.25}$$

Indeed, $\hat{A}^{\frac{3}{4}-\epsilon}N \in \mathcal{L}(L_2(\Omega))$, and thus

$$\hat{A}^{-2}AN \in \mathcal{L}\big(L_2(\Gamma); \mathcal{D}(\hat{A}^{\frac{7}{4}-\epsilon})\big) \subset \mathcal{L}\big(L_2(\Gamma); H^{\frac{7}{2}-2\epsilon}(\Omega)\big), \tag{5.3.26}$$

and recalling $N^*Ay = -y|_\Gamma$ by Chapter 3, Eqn. (3.3.9), we see that the trace theorem on (5.3.26) yields (5.3.25), which is precisely (A.9)(ii) = (4.6.2.6) with $n = 1$ and $U_{r_0} = H^{\frac{1}{2}}(\Gamma)$.

Conclusion The continuous analysis of Chapter 3, Section 3.3 and the above analysis show that we have verified all the assumptions of Theorems 4.6.2.1 and 4.6.2.2, with parameters and spaces as in (5.3.15), in the case of the heat equation problem with Neumann boundary control as in (5.3.1) and (5.3.2), with order of approximation (degree of polynomials) $r \geq 1$. We then specialize these results to obtain the following theorem.

Theorem 5.3.1 *Assume $r \geq 1$ in (5.2.1.1). Then, with reference to the heat equation with Neumann boundary control problem (5.3.1), (5.3.2), we have:*

I: The unique (nonnegative, self-adjoint) solution $P_h : V_h \to V_h$ of the (ARE_h) = (5.3.8) satisfies the following estimates:

$$\|P_h\Pi_h - P\|_{\mathcal{L}(L_2(\Omega))} \leq Ch^{2(1-\epsilon)}, \quad \forall\, \epsilon > 0; \tag{5.3.27}$$

$$\|P_h\Pi_h x|_\Gamma - Px|_\Gamma\|_{L_2(\Gamma)} \leq Ch^{\frac{3}{2}-\delta}\|x\|_{L_2(\Omega)}, \quad \forall\, \delta > 0. \tag{5.3.28}$$

II: There exist $\omega_0 > 0$ and $C \geq 1$ such that

$$\left\|y_h^0(t; \Pi_h y_0)\right\|_{L_2(\Omega)} \leq Ce^{-\omega_0 t}\|y_0\|_{L_2(\Omega)}, \tag{5.3.29}$$

where $y_h^0(t) = y_h^0(t; \Pi_h y_0)$ satisfies in the L_2-norms

$$\begin{cases} \big(\dot{y}_h^0(t), x_h\big)_\Omega + a\big(y_h^0(t), x_h\big)_\Omega = -\big(P_h y_h^0(t)\big|_\Gamma, x_h|_\Gamma\big)_\Gamma, \\ (y_h(0), x_h)_\Omega = (y_0, x_h)_\Omega, \quad \forall\, x_h \in V_h; \end{cases} \tag{5.3.30}$$

see (5.3.6) with u replaced by $-P_h y_h^0(t)|_\Gamma$.

III:

$$\left\|y_h^0(t; \Pi_h y_0) - y^0(t; y_0)\right\|_{L_2(\Omega)} \leq \frac{Ce^{-\omega_0 t}}{t^{1-\epsilon}}h^{2(1-\epsilon)}\|y_0\|_{L_2(\Omega)}; \tag{5.3.31}$$

$$\begin{aligned} \left\|u_h^0(t; \Pi_h y_0) - u^0(t; y_0)\right\|_{L_2(\Gamma)} &= \left\|P_h y_h^0(t; \Pi_h y_0)\big|_\Gamma - Py^0(t; y_0)\big|_\Gamma\right\|_{L_2(\Gamma)} \\ &\leq \frac{Ce^{-\omega_0 t}}{t^{\frac{1}{4}}}h^{2(1-\epsilon)}\|y_0\|_{L_2(\Omega)}. \end{aligned} \tag{5.3.32}$$

IV: If we consider the feedback problem

$$\begin{cases} (y_h^*)_t = (\Delta + c^2) y_h^* & \textit{in } (0, T] \times \Omega, & (5.3.33a) \\ y_h^*(0, \cdot) = y_0 & \textit{in } \Omega, & (5.3.33b) \\ \left.\dfrac{\partial y_h^*}{\partial \nu}\right|_\Sigma = -P_h \Pi_h y_h^*|_\Sigma & \textit{in } (0, T] \times \Gamma, & (5.3.33c) \end{cases}$$

that is, problem (5.3.1) with feedback law

$$u_h^*(t; y_0) = -P_h \Pi_h y_h^*(t; y_0)|_\Gamma, \tag{5.3.34}$$

with P_h a solution of $(ARE_h) = (5.3.8)$, then $y_h^*(t; y_0)$ is exponentially stable in $\mathcal{L}(L_2(\Omega))$, uniformly in h,

$$\|y_h^*(t; y_0)\|_{L_2(\Omega)} \leq \hat{C} e^{-\hat{\omega}_P t} \|y_0\|_{L_2(\Omega)}, \quad \forall h, \ t \geq 0. \tag{5.3.35}$$

Moreover,

$$\|y_h^*(t; y_0) - y^0(t; y_0)\|_{L_2(\Omega)} \leq \frac{C e^{-\omega_0 t}}{t^{\frac{1}{4}}} h^{2(1-\epsilon)} \|y_0\|_{L_2(\Omega)} \tag{5.3.36}$$

$$\|y_h^*(t; y_0) - y^0(t; y_0)\|_{L_2(\Omega)} \leq C h^{\frac{3}{2}-\epsilon} e^{-\omega_0 t} \|y_0\|_{L_2(\Omega)}. \tag{5.3.37}$$

Remark 5.3.2 The rates of convergence in Theorem 5.3.1 are optimal, as they reconstruct the optimal regularity properties of the original problem Chapter 3, Section 3.3. By using the same arguments as in [Lasiecka, 1984], one could replace $h^{-\epsilon}$ by $\ln h$.

5.4 A Structurally Damped Platelike Equation with Interior Point Control with $r \geq 3$

We return to the platelike equation with point control considered in Chapter 3, Section 3.4 in the "deflection" $w(t, x)$, with $\rho > 0$ a constant:

$$\begin{cases} w_{tt} + \Delta^2 w - \rho \Delta w_t = \delta(x - x^0) u(t) & \text{in } (0, T] \times \Omega; & (5.4.1a) \\ w(0, \cdot) = w_0, \quad w_t(0, \cdot) = w_1 & \text{in } \Omega; & (5.4.1b) \\ w|_\Sigma = \Delta w|_\Sigma \equiv 0 & \text{in } (0, T] \times \Gamma \equiv \Sigma, & (5.4.1c) \end{cases}$$

with scalar control $u(t)$ concentrated at the interior point x^0 of a smooth open bounded domain Ω of $\mathbb{R}^n$, $n \leq 3$, with boundary Γ. The cost functional is

$$J(u, w) = \int_0^\infty [\|w(t)\|^2_{H^2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} + |u(t)|^2]\, dt, \tag{5.4.2}$$

with $\{w_0, w_1\} \in [H^2(\Omega) \cap H_0^1(\Omega)] \times L_2(\Omega)$. In the notation of Chapter 3, Section 3.4, we have

$$\mathcal{A}f = \Delta^2 f, \qquad \mathcal{D}(\mathcal{A}) = \{f \in H^4(\Omega) : f|_\Gamma = \Delta f|_\Gamma = 0\}; \tag{5.4.3}$$

$$\mathcal{A}^{\frac{1}{2}} f = -\Delta f, \qquad \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) = H^2(\Omega) \cap H_0^1(\Omega); \quad U = \mathbb{R}^1; \tag{5.4.4}$$

$$Y = \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times \mathcal{H}; \qquad \mathcal{H} = L_2(\Omega); \quad R = I; \tag{5.4.5}$$

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\rho\mathcal{A}^{\frac{1}{2}} \end{bmatrix}; \qquad Bu = \begin{bmatrix} 0 \\ \delta(x - x^0)u \end{bmatrix} : U \to [\mathcal{D}(A^*)]'; \tag{5.4.6a}$$

$$B^* y = y_2(x^0), \quad y = [y_1, y_2]. \tag{5.4.6b}$$

$\mathcal{A}$ is, of course, a positive, self-adjoint operator on $\mathcal{H}$. Thus, in this case, Section 5.1 applies, in particular in the form (5.1.17) through (5.1.77) for second-order equations in t; we shall briefly review the key ingredients.

Choice of $\mathcal{V}_h$ We select the approximating subspaces $\mathcal{V}_h \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega) \cap H_0^1(\Omega)$ to be a space of splines of order 3 ($r = 3$), at least, which comply with properties (ii) through (vii) of Section 5.1, that is, from (5.1.17) through (5.1.27), with $r = 3$ in (5.1.24).

Choice of $\mathcal{A}_h$ We choose the Galerkin approximation in (5.1.23), which now, because of (5.4.4), becomes

$$\begin{aligned}(\mathcal{A}_h x_h, v_h)_{L_2(\Omega)} &= (\mathcal{A}x_h, v_h)_{L_2(\Omega)} = \big(\mathcal{A}^{\frac{1}{2}} x_h, \mathcal{A}^{\frac{1}{2}} v_h\big)_{L_2(\Omega)} \\ &= (\Delta x_h, \Delta v_h)_{L_2(\Omega)}, \quad x_h, v_h \in \mathcal{V}_h.\end{aligned} \tag{5.4.7a}$$

As in (iv) = (5.1.18), $\hat{\mathcal{V}}_h$ denotes $\mathcal{V}_h$ topologized as follows:

$$\begin{aligned}\|x\|_{\hat{\mathcal{V}}_h} = \big\|\mathcal{A}_h^{\frac{1}{2}} x_h\big\|_{L_2(\Omega)} &= \big\|\mathcal{A}^{\frac{1}{2}} x_h\big\|_{L_2(\Omega)} = \|\Delta x_h\|_{L_2(\Omega)} \\ &\text{equivalent to } \|x_h\|_{H^2(\Omega)}, \ x_h \in \mathcal{V}_h.\end{aligned} \tag{5.4.7b}$$

Propositions 5.1.3 and 5.1.4 and Eqn.(5.1.7) of Section 5.1 hold true, of course.

Choice of A_h and B_h We set

$$V_h = \hat{\mathcal{V}}_h \times \mathcal{V}_h, \tag{5.4.8a}$$

$$\|[v_{h1}, v_{h2}]\|_{V_h}^2 = \big\|\mathcal{A}_h^{\frac{1}{2}} v_{h1}\big\|_{L_2(\Omega)}^2 + \|v_{h2}\|_{L_2(\Omega)}^2, \quad v_{h1} \in \hat{\mathcal{V}}_h, \quad v_{h2} \in \mathcal{V}_h, \tag{5.4.8b}$$

and we define the approximation A_h of A in (5.4.6) by

$$A_h = \begin{bmatrix} 0 & \pi_h \\ -\mathcal{A}_h & -\rho\mathcal{A}_h^{\frac{1}{2}} \end{bmatrix} : V_h \to V_h \tag{5.4.9}$$

and the approximating B_h of B in (5.4.6) by

$$B_h u = \begin{bmatrix} 0 \\ \mathcal{B}_h u \end{bmatrix} = \Pi_h B u : L_2(\Gamma) \to V_h; \tag{5.4.10}$$

$$(\mathcal{B}_h u, v_h)_{L_2(\Omega)} = v_h(x^0)u, \tag{5.4.11}$$

where with π_h the orthogonal projection $L_2(\Omega)$ onto $\mathcal{V}_h$, and π_h^1 the orthogonal projection of $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ onto $\hat{\mathcal{V}}_h$, as in (iii) and (iv) of Section 5.1, we have introduced

$$\Pi_h = \begin{bmatrix} \pi_h^1 & 0 \\ 0 & \pi_h \end{bmatrix} : \text{orthogonal projection } V_h \text{ onto } Y. \tag{5.4.12}$$

Computation of Adjoints A_h^* and B_h^* To compute the adjoints A_h^* and B_h^* of A_h and B_h in (5.4.9) and (5.4.10):

$$(A_h x_h, v_h)_{V_h} = (x_h, A_h^* v_h)_{V_h}; \quad (B_h u, x_h)_{V_h} = (u, B_h^* x_h)_U, \tag{5.4.13}$$

we use the inner products generated by $\hat{\mathcal{V}}_h$ and $\mathcal{V}_h$ [see (5.4.8)] and obtain, as in the continuous case:

$$A_h^* = \begin{bmatrix} 0 & -\pi_h \\ \mathcal{A}_h & \rho \mathcal{A}_h^{\frac{1}{2}} \end{bmatrix} : V_h \to V_h; \quad B_h^* v_h = v_{h2}(x^0), \quad v_h = [v_{h1}, v_{h2}]. \tag{5.4.14}$$

Approximating Control Problem The approximating dynamics $\dot{y}_h = A_h y_h + B_h u$ is given, via (5.4.9) and (5.4.10), (5.4.11), by

$$\begin{cases} (\ddot{w}_h, \phi_h) + (\mathcal{A}_h w_h, \phi_h) + \rho\left(\mathcal{A}_h^{\frac{1}{2}} \dot{w}_h, \phi_h\right) = \phi_h(x^0)u; & \text{(5.4.15a)} \\ (\mathcal{A}_h w_h, \phi_h) = (\Delta w_h, \Delta \phi_h), \quad \forall\, \phi_h \in \mathcal{V}_h; & \text{(5.4.15b)} \\ (w_h(0), \phi_h) = (w_0, \phi_h), \quad (\dot{w}_h(0), \phi_h) = (w_1, \phi_h), & \text{(5.4.15c)} \end{cases}$$

where $y_h = [w_h, \dot{w}_h]$, and where all inner products are in $L_2(\Omega)$. The optimal feedback control for the approximating finite-dimensional problem is

$$u_h^0(t; \Pi_h y_0) = -B_h^* P_h y_h^0(t; \Pi_h y_0) \tag{5.4.16}$$

$$\text{(by (5.4.14))} \qquad = -\left\{\left[P_h y_h^0\right]_2\right\}(x^0), \tag{5.4.17}$$

where we have set

$$P_h z = \begin{bmatrix} P_{h,11} & P_{h,12} \\ P_{h,21} & P_{h,22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} [P_h z]_1 \\ [P_h z]_2 \end{bmatrix}, \quad z = [z_1, z_2], \tag{5.4.18}$$

and P_h is the unique, nonnegative, self-adjoint solution of the following algebraic Riccati equation (ARE$_h$):

$$(A_h^* P_h z_h, v_h)_{V_h} + (z_h, A_h^* P_h v_h)_{V_h} + (z_h, v_h)_{V_h} = (B_h^* P_h z_h, B_h^* P_h v_h)_U. \tag{5.4.19}$$

Hence, via (5.4.8), (5.4.14), and (5.4.18),

$$
\begin{aligned}
&-(\mathcal{A}_h[P_h z_h]_2, v_{h1}) + (\mathcal{A}_h[P_h z_h]_1 + \rho \mathcal{A}_h^{\frac{1}{2}}[P_h z_h]_2, v_{h2}) - (z_{h1}, \mathcal{A}_h[P_h v_h]_2) \\
&\quad + (z_{h2}, \mathcal{A}_h[P_h v_h]_1 + \rho \mathcal{A}_h^{\frac{1}{2}}[P_h v_h]_2) + (\mathcal{A}_h z_{h1}, v_{h1}) + (z_{h2}, v_{h2}) \\
&\quad = (\{[P_h z_h]_2\}(x^0))(\{[P_h v_h]_2\}(x^0)) \qquad (5.4.20)
\end{aligned}
$$

for $z_h, v_h \in \mathcal{V}_h$, where the inner products are all in $L_2(\Omega)$.

Verification of Continuous Assumptions (4.1.4.1) and (4.1.4.3) of Theorem 4.1.4.1 These are plainly satisfied: (4.1.4.1) because $R = I$; (4.1.4.3) follows from Chapter 3, Eqn. (3.4.6), and the argument below it (in essence, $\mathcal{A}^{-(\frac{1}{2}-\epsilon)}\delta \in L_2(\Omega)$), while $\mathcal{A}^{-\epsilon}$ is compact on $L_2(\Omega)$; see also Theorem 3B.1(a) of Appendix 3B to Chapter 3.

Verification of Discrete Assumptions (A.1) through (A.6) of Theorem 4.1.4.1

Assumption (A.1) = (4.1.2.2) This follows by applying the arguments of [Chen, Triggiani, 1989, Section 3] of the continuous case to the finite-dimensional operator A_h given by (5.4.9).

Assumption (A.2) = (4.1.2.4) We shall show that this assumption holds true with $s = 2 - \epsilon$. First, from (5.4.6) and (5.4.9),

$$
A^{-1} = \begin{bmatrix} -\rho \mathcal{A}^{-\frac{1}{2}} & -\mathcal{A}^{-1} \\ I & 0 \end{bmatrix}; \qquad A_h^{-1} = \begin{bmatrix} -\rho \mathcal{A}_h^{-\frac{1}{2}} & -\mathcal{A}_h^{-1} \\ \pi_h & 0 \end{bmatrix}. \qquad (5.4.21)
$$

Next, adding and subtracting, and recalling Π_h from (5.4.12), we get

$$
\left\| A_h^{-1}\Pi_h x - A^{-1}x \right\|_Y \le \left\| [A_h^{-1} - A^{-1}]\Pi_h x \right\|_Y + \|A^{-1}[\Pi_h - I]x\|_Y. \qquad (5.4.22)
$$

As to the first term on the right-hand side of (5.4.22), we estimate via (5.4.21), with $x_h = [x_{h1}, x_{h2}] = \Pi_h x \in V_h$, after recalling (5.1.40), (5.1.28), and (5.1.27b):

$$
\begin{aligned}
\left\| [A_h^{-1} - A^{-1}]\Pi_h x \right\|_Y &= \left\| [A_h^{-1} - A^{-1}]x_h \right\|_Y \\
&\le \rho \left\| [\mathcal{A}^{-\frac{1}{2}} - \mathcal{A}_h^{-\frac{1}{2}}\pi_h]x_{h1} \right\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \\
&\quad + \left\| [\mathcal{A}^{-1} - \mathcal{A}_h^{-1}\pi_h]x_{h2} \right\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \\
\text{(by (5.1.40), (5.1.28))} \quad &\le C\left[h^{2-\epsilon}\|x_{h1}\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} + h^2\|x_{h2}\|_{L_2(\Omega)}\right] \\
&\le Ch^{2-\epsilon}\|x_h\|_Y = Ch^{2-\epsilon}\|\Pi_h x\|_Y. \qquad (5.4.23)
\end{aligned}
$$

As to the second term on the right-hand side of (5.4.22), we compute via (5.4.21)

$$\begin{aligned}\|A^{-1}[\Pi_h - I]x\|_Y &\le \rho \left\| \mathcal{A}^{-\frac{1}{2}}\left(\pi_h^1 - I\right)x_1\right\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} + \|\mathcal{A}^{-1}(\pi_h - I)x_2\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \\ &\quad + \left\|\left(\pi_h^1 - I\right)x_1\right\|_{L_2(\Omega)} \\ &= (\rho+1)\left\|\left(\pi_h^1 - I\right)x_1\right\|_{L_2(\Omega)} + \left\|\mathcal{A}^{-\frac{1}{2}}(\pi_h - I)x_2\right\|_{L_2(\Omega)}.\end{aligned} \tag{5.4.24}$$

As to the second term on the right-hand side of (5.4.24), we have

$$\left\|\mathcal{A}^{-\frac{1}{2}}(\pi_h - I)x_2\right\|_{L_2(\Omega)} \le Ch^2\|x_2\|_{L_2(\Omega)}, \tag{5.4.25}$$

since, equivalently, taking the adjoint and invoking (5.1.24) with $s = 2$, $\ell = 0$, we obtain

$$\left\|(\pi_h - I)\mathcal{A}^{-\frac{1}{2}}z\right\|_{L_2(\Omega)} \le Ch^2\left\|\mathcal{A}^{-\frac{1}{2}}z\right\|_{H^2(\Omega)} \le Ch^2\|z\|_{L_2(\Omega)}, \tag{5.4.26}$$

which precisely proves (5.4.25). As to the first term on the right-hand side of (5.4.24), we recall Proposition 5.1.7, Eqn. (5.1.74), as well as (5.1.27b), to obtain

$$\left\|\left(\pi_h^1 - I\right)x_1\right\|_{L_2(\Omega)} \le Ch^2\|x_1\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})}. \tag{5.4.27}$$

Inserting (5.4.25) and (5.4.27) into (5.4.24) yields

$$\|A^{-1}[\Pi_h - I]x\|_Y \le Ch^2\|x\|_Y. \tag{5.4.28}$$

Finally, using (5.4.23) and (5.4.28) on the right-hand side of (5.4.22) yields

$$\left\|A_h^{-1}\Pi_h x - A^{-1}x\right\|_Y \le Ch^{2-\epsilon}\|x\|_Y, \tag{5.4.29}$$

which proves (A.1) = (4.1.2.2) of Chapter 4, as desired, with $s = 2 - \epsilon$.

The same result holds for the adjoint A^*, in view of its definition.

Assumption (A.3) = (4.1.2.6) By (5.4.6b) on B^*, by Sobolev embedding and by the inverse approximation property (5.1.25) with $\alpha = s = n/2 + \epsilon$, we have, for any $\epsilon > 0$,

$$\begin{aligned}\|B^*x_h\|_U = |x_{h2}(x^0)| &\le C\|x_{h2}\|_{H^{\frac{n}{2}+\epsilon}(\Omega)} \le Ch^{-\frac{n}{2}-\epsilon}\|x_{h2}\|_{L_2(\Omega)} \\ &\le Ch^{-\frac{n}{2}-\epsilon}\|x_h\|_Y \le Ch^{-\gamma s}\|x_h\|_Y.\end{aligned} \tag{5.4.30}$$

To justify the last step in (5.4.30), we recall, from Chapter 3, Section 3.4 that $1 > \gamma > n/4$, $\dim \Omega = n \le 3$, and that from (A.2) just verified above (see (5.4.29)) $s < 2$; we then check that for $\epsilon > 0$ small:

$$\gamma s = \left(\frac{n}{4} + \epsilon\right)(2 - \epsilon) = \frac{n}{2} + 2\epsilon - \frac{n}{4}\epsilon - \epsilon^2 > \frac{n}{2}, \tag{5.4.31}$$

as needed, and (5.4.30) is proved. Then, (5.4.30) verifies (A.3) = (4.1.2.6), as desired.

Assumption (A.4) = (4.1.2.7) By (5.4.6b) on B^*, (5.4.12) on Π_h, Sobolev embedding, and (5.1.24) with $s = 2$, $\ell = n/2 + \epsilon$, we compute

$$\begin{aligned} \|B^*(\Pi_h x - x)\|_U &= |(\pi_h x_2)(x^0) - x_2(x^0)|_{R^1} \leq C\|\pi_h x_2 - x_2\|_{H^{\frac{n}{2}+\epsilon}(\Omega)} \\ \text{(by (5.1.24))} \quad &\leq Ch^{2-\frac{n}{2}-\epsilon}\|x_2\|_{H^2(\Omega)} \leq Ch^{2-\frac{n}{2}-\epsilon}\|x\|_{\mathcal{D}(A^*)} \qquad (5.4.32) \\ &\leq Ch^{s(1-\gamma)}\|x\|, \qquad (5.4.33) \end{aligned}$$

where in going from (5.4.32) to (5.4.33) we have recalled that, as noted above (5.4.31), $1 > \gamma > n/4$, $s < 2$; so that

$$s(1-\gamma) = (2-\epsilon)\left(1 - \frac{n}{4} - \epsilon\right) < 2 - \frac{n}{2} - \epsilon, \qquad (5.4.34)$$

as needed. Then (5.4.33) verifies assumption (A.4) = (4.1.2.7) of Chapter 4.

Assumption (A.5) = (4.1.2.8) This coincides with (A.4) since $B_h^* = B^*\Pi_h$ from (5.4.10).

Assumption (A.6) = (4.1.2.9) By (5.4.6b) on B^*, (5.4.12) on Π_h, and Sobolev embedding, we obtain with $x_h = \Pi_h x = [x_{h1}, x_{h2}]$:

$$\|B^*\Pi_h x\|_U = |x_{h2}(x^0)| \leq C\|x_{h2}\|_{H^{\frac{n}{2}+\epsilon}(\Omega)} \leq C\|x_h\|_{\mathcal{D}((A^*)^\gamma)}, \qquad (5.4.35)$$

as in [Chen, Triggiani, 1990, $\alpha = 1/2$; Chapter 3, Appendix 3B, Theorem 3B.2], $\mathcal{D}((A^*)^\gamma) \subset H^{4\gamma}(\Omega) \times H^{2\gamma}(\Omega)$, and $2\gamma = 2\,(n/4 + \epsilon) = n/2 + 2\epsilon > n/2 + \epsilon$. Then, (5.4.35) verifies assumption (A.6) = (4.1.2.9) of Chapter 4.

Conclusion The continuous analysis of Chapter 3, Section 3.4 and the above discrete analysis show that we have verified all the assumptions of Theorems 4.1.4.1 and 4.1.4.2 of Chapter 4 in the case of the structurally damped problem (5.4.1), (5.4.2) with interior point control, with order of approximation $r \geq 3$. Then, application of Theorems 4.1.4.1 and 4.1.4.2 yields the following convergence results, since $s(1-\gamma) < 2 - n/2 - \epsilon$ from (5.4.34).

Theorem 5.4.3 *Assume $r \geq 3$ in (5.1.24). Then, with reference to problem (5.4.1), (5.4.2), we have:*

I: The unique solution $P_h : V_h \to V_h$ to the (ARE_h) = (5.4.20) satisfies the following estimates:

(i)

$$\|P_h\Pi_h - P\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega))} \leq Ch^{\epsilon_0} \to 0 \text{ as } h \downarrow 0, \quad \epsilon_0 < \frac{4-n}{2}; \qquad (5.4.36)$$

(ii)

$$\begin{aligned} \|B^*P_h\Pi_h - B^*P\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega);\mathbb{R}^1)} &= \|(P_h\Pi_h - P)_{x=x^0}\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega);\mathbb{R}^1)} \\ &\to 0 \text{ as } h \downarrow 0. \qquad (5.4.37) \end{aligned}$$

II: The optimal finite-dimensional approximations $y_h^0(t; \Pi_h y_0)$ and $u_h^0(t; \Pi_h y_0)$ [see (5.4.16)] satisfy still with $\epsilon_0 < (4-n)/2$:

(iii)

$$\sup_{t\geq 0}\left\{e^{\tilde{\omega}_P t}\left\|u_h^0(t; \Pi_h y_0) - u^0(t; y_0)\right\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega);\mathbb{R}^1)}\right\} \leq Ch^{\epsilon_0} \to 0, \quad \text{as } h \downarrow 0; \tag{5.4.38}$$

(iv)

$$\sup_{t\geq 0}\left\{t^{\epsilon} e^{\tilde{\omega}_P t}\left\|y_h^0(t; \Pi_h y_0) - y^0(t; y_0)\right\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega))}\right\} \leq Ch^{\epsilon_0 \epsilon} \to 0, \quad \text{as } h \downarrow 0, \tag{5.4.39}$$

uniformly in $y_0 \in [H^2(\Omega) \cap H_0^1(\Omega)] \times L_2(\Omega)$; and

(v)

$$\left|J\left(u_h^0(\,\cdot\,; \Pi_h y_0), y_h^0(\,\cdot\,; \Pi_h y_0)\right) - J(u^0(\,\cdot\,; y_0), y^0(\,\cdot\,; y_0))\right| \leq Ch^{\epsilon_0} \to 0, \quad \text{as } h \downarrow 0. \tag{5.4.40}$$

III: Application of Theorem 4.1.4.2 of Chapter 4 to our problem yields the following result: Consider the problem given by

$$\begin{cases} \ddot{w}_h^* + \Delta^2 w_h^* - \rho\Delta\dot{w}_h^* \\ \quad = -\delta(x - x^0)\left\{P_h\begin{bmatrix} w_h^*(t; y_0) \\ \dot{w}_h^*(t; y_0)\end{bmatrix}_2\right\}(x^0) & \text{in } (0, T] \times \Omega; \quad (5.4.41\text{a}) \\ w_h^*(0, \,\cdot\,) = w_0, \ \dot{w}_h^*(0, \,\cdot\,) = w_1 & \text{in } \Omega; \quad (5.4.41\text{b}) \\ w_h^*|_\Sigma = \Delta w_h^*|_\Sigma = 0 & \text{in } (0, T] \times \Gamma = \Sigma; \quad (5.4.41\text{c}) \end{cases}$$

with feedback law

$$u_h^*(t; y_0) = -\left\{P_h\begin{bmatrix} w_h^*(t; y_0) \\ \dot{w}_h^*(t; y_0)\end{bmatrix}_2\right\}(x^0), \quad y_0 = \begin{bmatrix} w_0 \\ w_1\end{bmatrix}, \tag{5.4.42}$$

in the notation of (5.4.18), with P_h a solution of (5.4.20). Then $y_h^*(t; y_0) = [w_h^*(t; y_0), \dot{w}_h^*(t; y_0)]$ is exponentially stable in $\mathcal{L}(H^2(\Omega) \times L_2(\Omega))$, uniformly in the parameter h,

$$\|y_h^*(t; y_0)\|_{H^2(\Omega)\times L_2(\Omega)} \leq \hat{C}e^{-\hat{\omega}_P t}\|y_0\|_{H^2(\Omega)\times L_2(\Omega)}, \quad \forall h, \ t \geq 0. \tag{5.4.43}$$

Moreover,

$$\sup_{t\geq 0}\left\{e^{\hat{\omega}_P t}\|y_h^*(t; y_0) - y^0(t; y_0)\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega))}\right\} \to 0, \tag{5.4.44}$$

that is, uniformly in $y_0 \in [H^2(\Omega) \cap H_0^1(\Omega)] \times L_2(\Omega)$.

As we know, convergence results based on Theorems 4.1.4.1 and 4.1.4.2 of Chapter 4 do not provide optimal rates of convergence.

5.5 Kelvin–Voight Platelike Equation with Interior Point Control with $r \geq 3$

We return to the platelike equation with point control considered in Chapter 3, Section 3.5 in the "deflection" $w(t, x)$:

$$\begin{cases} w_{tt} + \Delta^2 w + \rho \Delta^2 w_t = \delta(x - x^0)u(t) & \text{in } (0, T] \times \Omega; & (5.5.1\text{a}) \\ w(0, \cdot) = w_0, \; w_t(0, \cdot) = w_1 & \text{in } \Omega & (5.5.1\text{b}) \\ \Delta w|_\Sigma + (1 - \mu)B_1 w = 0 & \text{in } (0, T] \times \Gamma = \Sigma; & (5.5.1\text{c}) \\ \left.\dfrac{\partial \Delta w}{\partial \nu}\right|_\Sigma + (1 - \mu)B_2 w - w = 0 & \text{in } \Sigma, & (5.5.1\text{d}) \end{cases}$$

with $\rho > 0$ a constant, $0 < \mu < 1/2$ the Poisson modulus, and with scalar control $u(t)$ concentrated at the interior point x^0 of a smooth, open bounded domain Ω of $\mathbb{R}^n$, $n = 1, 2$. The boundary operators B_1 and B_2 are zero for $n = 1$ and are given by Eqn. (3.5.2) of Chapter 3 for $n = 2$. The cost functional is

$$J(u, w) = \int_0^\infty \left[\|w(t)\|^2_{H^2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} + |u(t)|^2 \right] dt, \tag{5.5.2}$$

where $\{w_0, w_1\} \in H^2(\Omega) \times L_2(\Omega)$. In the notation of Chapter 3, Section 3.5, we have:

$$\mathcal{A}f = \Delta^2 f;$$

$$\mathcal{D}(\mathcal{A}) = \left\{ f \in H^4(\Omega) : \Delta f + (1 - \mu)B_1 f|_\Gamma = 0; \; \frac{\partial \Delta f}{\partial \nu} + (1 - \mu)B_2 f|_\Gamma = 0 \right\}; \tag{5.5.3}$$

$$Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) = H^2(\Omega) \times L_2(\Omega); \quad U = \mathbb{R}^1; \tag{5.5.4}$$

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\rho\mathcal{A} \end{bmatrix}; \qquad Bu = \begin{bmatrix} 0 \\ \delta(x - x^0)u \end{bmatrix} : U \to [\mathcal{D}(A^*)]'; \; R = I; \tag{5.5.5}$$

$$B^* y = y_2(x^0); \qquad y = [y_1, y_2]; \tag{5.5.6}$$

$$(\mathcal{A}u, v)_{L_2(\Omega)} = a(u, v)$$
$$= \int_\Omega \{\Delta u \Delta v + (1 - \mu)[2u_{xy}v_{xy} - u_{xx}v_{xx} - u_{yy}v_{yy}]\} d\Omega + \int_\Gamma uv \, d\Gamma; \tag{5.5.7a}$$

$$a(u, u) \text{ being equivalent to } \|u\|^2_{H^2(\Omega)} \tag{5.5.7b}$$

[Lagnese, 1986]. $\mathcal{A}$ is, of course, a positive, self-adjoint operator on $\mathcal{H} = L_2(\Omega)$. Thus, in this case, Section 5.1 applies, in particular in the form (5.1.17) through (5.1.77) for second-order equations in t. A brief review of the key ingredients follows.

Choice of $\mathcal{V}_h$ This is same as in the preceding example of Section 5.4. Properties (ii) through (vii) of Section 5.1 from (5.1.17) through (5.1.27) hold true with $r = 3$.

Choice of $\mathcal{A}_h$ We choose the Galerkin approximation in (5.1.23) with $\mathcal{H} = L_2(\Omega)$:

$$\mathcal{A}_h = \pi_h \mathcal{A} : \mathcal{V}_h \to \mathcal{V}_h;$$

$$(\mathcal{A}_h x_h, v_h)_{\mathcal{H}} = (\mathcal{A} x_h, v_h)_{\mathcal{H}} = a(x_h, v_h), \quad x_h, v_h \in \mathcal{V}_h, \tag{5.5.8}$$

with $a(\cdot, \cdot)$ defined by (5.5.7a), and $\mathcal{A}$ by (5.5.3). Propositions 5.1.3, 5.1.4, and 5.1.7 of Section 5.1 hold true.

Choice of A_h and B_h With V_h and Π_h as in (5.4.8) and (5.4.12), we have now the approximation A_h of A in (5.5.5) given by

$$A_h = \begin{bmatrix} 0 & \pi_h \\ -\mathcal{A}_h & -\rho \mathcal{A}_h \end{bmatrix} : V_h \to V_h \tag{5.5.9}$$

and the approximating B_h of B in (5.5.5) given again by (5.4.10) and (5.4.11), that is,

$$B_h u = \begin{bmatrix} 0 \\ \mathcal{B}_h u \end{bmatrix} = \Pi_h B u : L_2(\Gamma) \to V_h; \quad (\mathcal{B}_h u, v_h)_{L_2(\Omega)} = v_h(x^0) u. \tag{5.5.10}$$

Computation of Adjoints A_h^* and B_h^* As in (5.4.13), we obtain

$$A_h^* = \begin{bmatrix} 0 & -\pi_h \\ \mathcal{A}_h & \rho \mathcal{A}_h \end{bmatrix} : V_h \to V_h; \quad B_h^* v_h = v_{h2}(x^0), \quad v_h = [v_{h1}, v_{h2}]. \tag{5.5.11}$$

Approximating Control Problem The approximating dynamics $\dot{y}_h = A_h y_h + B_h u$ is given, via (5.5.9) and (5.5.10), by

$$\begin{cases} (\ddot{w}_h, \phi_h) + (\mathcal{A}_h w_h, \phi_h) + \rho(\mathcal{A}_h \dot{w}_h, \phi_h) = \phi_h(x^0) u; & (5.5.12a) \\ (\mathcal{A}_h w_h, \phi_h) = a(w_h, \phi_h); & (5.5.12b) \\ (w_h(0), \phi_h) = (w_0, \phi_h), \quad (\dot{w}_h(0), \phi_h) = (w_1, \phi_h), & (5.5.12c) \end{cases}$$

where $y_h = [w_h, \dot{w}_h]$, and where all inner products are in $L_2(\Omega)$. The optimal feedback control for the approximating finite-dimensional problem is

$$u_h^0(t; \Pi_h y_0) = -B_h^* P_h y_h^0(t; \Pi_h y_0) \tag{5.5.13}$$

$$\text{(by (5.5.11))} \qquad = -\{[P_h y_h^0]_2\}(x^0), \tag{5.5.14}$$

in the notation of (5.4.18), where P_h is the unique, nonnegative, self-adjoint solution of the following algebraic Riccati equation (ARE$_h$), as in (5.4.19), that is, explicitly via (5.5.11) and (5.4.8):

$$\begin{aligned} &-(\mathcal{A}_h[P_h z_h]_2, v_{h1}) + (\mathcal{A}_h[P_h z_h]_1 + \rho \mathcal{A}_h[P_h z_h]_2, v_{h2}) - (z_{h1}, \mathcal{A}_h[P_h v_h]_2) \\ &\quad + (z_{h2}, \mathcal{A}_h[P_h v_h]_1 + \rho \mathcal{A}_h[P_h v_h]_2) + (\mathcal{A}_h z_{h1}, v_{h1}) + (z_{h2}, v_{h2}) \\ &\quad = (\{[P_h z_h]_2\}(x^0))(\{[P_h v_h]_2\}(x^0)) \end{aligned} \tag{5.5.15}$$

for $z_h, v_h \in \mathcal{V}_h$, where the inner products are all in $L_2(\Omega)$.

Verification of Continuous Assumptions (4.1.4.1) and (4.1.4.3) of Theorem 4.1.4.1 These are plainly satisfied: (4.1.4.1) follows since $R = I$, whereas (4.1.4.3) follows from the fact that the operator

$$A^{-1}B = \begin{bmatrix} -\rho I & -\mathcal{A}^{-1} \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \delta(x - x^0) \end{bmatrix} = \begin{bmatrix} -\mathcal{A}^{-1}\,\delta(x - x^0) \\ 0 \end{bmatrix}$$
$$: \mathbb{R} \to Y = H^2(\Omega) \cap L_2(\Omega) = \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times L_2(\Omega) \tag{5.5.16}$$

is compact, that is, $\mathcal{A}^{-\frac{1}{2}}\delta : \mathbb{R} \to L_2(\Omega)$ is compact. Indeed, this is true, since: (i) $\mathcal{A}^{-(\frac{1}{2}-\epsilon)}\delta u \in L_2(\Omega)$, or $\delta u \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon}) \subset H^{2-4\epsilon}(\Omega)$ by Sobolev embedding in $n \le 3$, and (ii) $\mathcal{A}^{-1}$ is compact as an operator in $\mathcal{L}(L_2(\Omega))$.

Verification of Discrete Assumptions (A.1) through (A.6) of Theorem 4.1.4.1

Assumption (A.1) = (4.1.2.2) This follows by applying the arguments of [Chen, Triggiani, 1989, Section 3] of the continuous case to the finite-dimensional operator A_h given by (5.9).

Assumption (A.2) = (4.1.2.4) We shall show that this assumption holds true with $s = 2$. From (5.5) and (5.9), we have

$$A^{-1} = \begin{bmatrix} -\rho I & -\mathcal{A}^{-1} \\ I & 0 \end{bmatrix}; \quad A_h^{-1} = \begin{bmatrix} -\rho\pi_h & -\mathcal{A}_h^{-1} \\ \pi_h & 0 \end{bmatrix}, \tag{5.5.17}$$

from which we obtain via Π_h in (5.4.12) and Y in (5.5.4):

$$\begin{aligned}\big\|A_h^{-1}\Pi_h x - A^{-1}x\big\|_Y \le \rho\big\|x_1 - \pi_h\pi_h^1 x_1\big\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} &+ \big\|\mathcal{A}^{-1}x_2 - \mathcal{A}_h^{-1}\pi_h x_2\big\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \\ &+ \big\|\pi_h\pi_h^1 x_1 - x_1\big\|_{L_2(\Omega)}.\end{aligned} \tag{5.5.18}$$

As to the first term on the right-hand side of (5.5.18), we estimate by adding and subtracting

$$\begin{aligned}\big\|x_1 - \pi_h\pi_h^1 x_1\big\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} &\le \|x_1 - \pi_h x_1\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} + \big\|\pi_h\big[x_1 - \pi_h^1\big]x_1\big\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \\ \text{(by (5.1.27c) and (5.1.19))} \quad &\le C\|x_1 - \pi_h x_1\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega)} \\ \text{(by (5.1.24))} \quad &\le Ch^2\|x_1\|_{H^2(\Omega)},\end{aligned} \tag{5.5.19}$$

where in the last step we have invoked (5.1.24) with $s = 4$, $\ell = 2$. As to the second term on the right-hand side of (5.5.18), we recall (5.1.28) and (5.5.4) and obtain

$$\big\|\mathcal{A}^{-1}x_2 - \mathcal{A}_h^{-1}\pi_h x_2\big\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \le Ch^2\|x_2\|_{L_2(\Omega)}. \tag{5.5.20}$$

As for the third term on the right-hand side of (5.5.18), we estimate

$$\big\|\pi_h\pi_h^1 x_1 - x_1\big\|_{L_2(\Omega)} \le \big\|\pi_h\big[\pi_h^1 x_1 - x_1\big]\big\|_{L_2(\Omega)} \le Ch^2\|x_1\|_{H^2(\Omega) = \mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \tag{5.5.21}$$

by applying (5.1.74). Inserting (5.5.19), (5.5.20), and (5.5.21) into (5.5.18), we arrive by (5.5.4) at

$$\left\| A_h^{-1}\Pi_h x - A^{-1}x \right\|_Y \leq Ch^2\|x\|_Y. \tag{5.5.22}$$

Then (5.5.22) proves assumption (A.2) = (4.1.2.4) of Chapter 4, as desired, with $s = 2$.

Assumption (A.3) = (4.1.2.5) The argument is identical to that of the corresponding assumption in Section 5.4, since B^*, and B_h^* are the same in both sections; compare (5.5.6) with (5.4.6b), and (5.5.11) with (5.4.10).

Assumption (A.4) = (4.1.2.7) Precisely as in the corresponding assumption (A.4) in Section 5.4, with the same B, we obtain, as in (5.4.32),

$$\|B^*(\Pi_h x - x)\|_U \leq Ch^{2-\frac{n}{2}-\epsilon}\|x\|_{H^2(\Omega)} \leq Ch^{2-\frac{n}{2}-\epsilon}\|x\|_{\mathcal{D}(A^*)} \tag{5.5.23}$$

$$\leq Ch^{s(1-\gamma)}\|x\|_{\mathcal{D}(A^*)}, \tag{5.5.24}$$

where, in the present case, $s = 2$ (see (5.5.22) and the statement below), and $\gamma = n/8+\epsilon$ (see Chapter 3, Section 3.5, Claim below Eqn. (5.13)), whereby $2 - n/2 - \epsilon > s(1-\gamma) = 2(1 - n/8 - \epsilon)$, and (5.5.23) implies (5.5.24). Then, (5.5.24) proves (A.4) = (5.1.2.7) of Chapter 4.

Assumption (A.5) = (4.1.2.8) This coincides with (A.4) since $B_h^* = B^*\Pi_h$ from (5.5.10).

Assumption (A.6) = (4.1.2.9) Again, precisely as in (5.4.35), with the same B, we obtain with $x_h = \Pi_h x = [x_{h1}, x_{h2}]$ the first inequality of

$$\|B^*\Pi_h x\|_U \leq C\|x_{h2}\|_{H^{\frac{n}{2}+\epsilon}(\Omega)} \leq C\|x_h\|_{\mathcal{D}((A^*)^\gamma)}, \tag{5.5.25}$$

where the second inequality follows by [Chen, Triggiani, 1989, $\alpha = 1$; Chapter 3, Appendix 3B, Theorems 3B.2], which gives $\mathcal{D}((A^*)^\gamma) \subset H^2(\Omega) \times H^{4\gamma}(\Omega)$, and $4\gamma = 4(n/8+\epsilon) = n/2 + 4\epsilon > n/2 + \epsilon$. Thus, (5.5.25) verifies assumption (A.6) = (4.1.2.9) of Chapter 4.

Conclusion The continuous analysis of Chapter 3, Section 3.5 and the above discrete analysis show that we have verified all the assumptions of Theorems 4.1.4.1 and 4.1.4.2 of Chapter 4, in the case of the Kelvin–Voight problem (5.5.1), (5.5.2), with interior point control with order of approximation $r \geq 3$. Then, application of Theorems 4.1.4.1 and 4.1.4.2 yields the following convergence results, since $s(1-\gamma) = 2(1 - n/8 - \epsilon) = 2 - n/4 - 2\epsilon$.

Theorem 5.5.1 *Assume $r \geq 3$ in (5.1.24). Then, with reference to problem (5.5.1), (5.5.2), where $n = 1, 2$, we have:*

I: The unique solution $P_h : V_h \to V_h$ to the (ARE_h) = (5.5.15) satisfies the following estimates, as $h \downarrow 0$:

(i)

$$\|P_h\Pi_h - P\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega))} \le Ch^{\epsilon_0} \to 0, \quad \epsilon_0 < \frac{8-n}{4}; \tag{5.5.26}$$

(ii)

$$\begin{aligned}&\|B^*P_h\Pi_h - B^*P\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega);\mathbb{R}^1)}\\ &\quad = \|(P_h\Pi_h - P)|_{x=x^0}\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega);\mathbb{R}^1)} \to 0, \quad \text{as } h \downarrow 0.\end{aligned} \tag{5.5.27}$$

II: The optimal finite-dimensional approximation $y_h^0(t;\Pi_h y_0)$ and $u_h^0(t;\Pi_h y_0)$ [see (5.5.13)] satisfy, still with $\epsilon_0 < (8-n)/4$:

(iii)

$$\sup_{t\ge 0}\left\{e^{\tilde{\omega}_P t}\left\|u_h^0(t;\Pi_h y_0) - u^0(t;y_0)\right\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega);\mathbb{R}^1)}\right\} \le Ch^{\epsilon_0} \to 0; \tag{5.5.28}$$

(iv)

$$\sup_{t\ge 0}\left\{t^{\epsilon}e^{\tilde{\omega}_P t}\left\|y_h^0(t;\Pi_h y_0) - y^0(t;y_0)\right\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega))}\right\} \le Ch^{\epsilon_0\epsilon} \to 0, \tag{5.5.29}$$

uniformly in $y_0 \in H^2(\Omega)\times L_2(\Omega)$;

(v)

$$\left|J\left(u_h^0(\,\cdot\,;\Pi_h y_0), y_h^0(\,\cdot\,;\Pi_h y_0)\right) - J(u^0(\,\cdot\,;y_0), y^0(\,\cdot\,;y_0))\right| \le Ch^{\epsilon_0} \to 0. \tag{5.5.30}$$

III: Application of Theorem 4.1.4.2 of Chapter 4 to our problem yields the following result: Consider the problem given by

$$\begin{cases} \ddot{w}_h^* + \Delta^2 w_h^* + \rho\Delta^2\ddot{w}_h^* & \\ \quad = -\delta(x-x^0)\left\{P_h\begin{bmatrix} w_h^*(t;y_0)\\ \dot{w}_h^*(t;y_0)\end{bmatrix}_2\right\}(x^0) & \text{in } (0,T]\times\Omega; \quad (5.5.31a)\\ w_h^*(0,\,\cdot\,) = w_0,\ \dot{w}_h^*(0,\,\cdot\,) = w_1 & \Omega; \quad (5.5.31b)\\ [\Delta w_h^* + (1-\mu)B_1 w_h^*]_\Sigma = 0 & \text{in } (0,T]\times\Gamma = \Sigma; \quad (5.5.31c)\\ \left[\dfrac{\partial\Delta w_h^*}{\partial\nu} + (1-\mu)B_2 w_h^* - w_h^*\right]_\Sigma = 0 & \text{in } \Sigma, \quad (5.5.31d)\end{cases}$$

with feedback law

$$u_h^*(t;y_0) = -\left\{P_h\begin{bmatrix} w_h^*(t;y_0)\\ \dot{w}_h^*(t;y_0)\end{bmatrix}_2\right\}(x^0), \quad y_0 = \begin{bmatrix} w_0\\ w_1\end{bmatrix} \tag{5.5.32}$$

in the notation of (5.5.14) and (5.4.18), with P_h a solution of (5.5.15). Then $y_h^*(t;y_0) = [w_h^*(t;y_0), \dot{w}_h^*(t;y_0)]$ is exponentially stable in $\mathcal{L}(H^2(\Omega)\times L_2(\Omega))$, uniformly in the parameter h,

$$\|y_h^*(t;y_0)\|_{H^2(\Omega)\times L_2(\Omega)} \le \hat{C}e^{-\hat{\omega}_P t}\|y_0\|_{H^2(\Omega)\times L_2(\Omega)}, \quad \forall\, h,\ t\ge 0. \tag{5.5.33}$$

Moreover, as $h \downarrow 0$,

$$\sup_{t \geq 0} \{e^{\hat{\omega}_p t} \|y_h^*(t; y_0) - y^0(t; y_0)\|_{\mathcal{L}(H^2(\Omega) \times L_2(\Omega))}\} \to 0, \tag{5.5.34}$$

that is, uniformly in $y_0 \in H^2(\Omega) \times L_2(\Omega)$.

5.6 A Structurally Damped Platelike Equation with Boundary Control with $r \geq 3$

We return to the platelike equation with boundary control considered in Chapter 3, Section 3.6 in the "deflection" $w(t, x)$, with $\rho > 0$ a constant:

$$\begin{cases} w_{tt} + \Delta^2 w - \rho \Delta w_t = 0 & \text{in } (0, T] \times \Omega = Q; & (5.6.1a) \\ w(0, \cdot) = w_0, \ w_t(0, \cdot) = w_1 & \text{in } \Omega; & (5.6.1b) \\ w|_\Sigma \equiv 0 & \text{in } (0, T] \times \Gamma = \Sigma; & (5.6.1c) \\ \Delta w|_\Sigma \equiv u & \text{in } \Sigma, & (5.6.1d) \end{cases}$$

with boundary control $u \in L_2(0, T; L_2(\Gamma)) \equiv L_2(\Sigma)$, and initial data $\{w_0, w_1\} \in [H^2(\Omega) \cap H_0^1(\Omega)] \cap L_2(\Omega)$. The cost functional is

$$J(u, w) = \int_0^\infty \left[\|w(t)\|^2_{H^2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} + \|u(t)\|^2_{L_2(\Gamma)} \right] dt. \tag{5.6.2}$$

In the notation of Chapter 3, Section 3.6 we have

$$\mathcal{A} f = \Delta^2 f, \qquad \mathcal{D}(\mathcal{A}) = \{f \in H^4(\Omega) : f|_\Gamma = \Delta f|_\Gamma = 0\}; \tag{5.6.3}$$

$$\mathcal{A}^{\frac{1}{2}} f = -\Delta f, \qquad \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega) \cap H_0^1(\Omega); \qquad U = L_2(\Gamma); \tag{5.6.4}$$

$$Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{H}, \qquad \mathcal{H} = L_2(\Omega); \quad R = I, \tag{5.6.5}$$

as in Chapter 4, Eqns. (4.4.3)–(4.4.5), along with

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\rho \mathcal{A}^{\frac{1}{2}} \end{bmatrix}; \qquad Bu = \begin{bmatrix} 0 \\ -\mathcal{A}^{\frac{1}{2}} D u \end{bmatrix}; \tag{5.6.6}$$

with D being the Dirichlet map defined by Chapter 3, Eqn. (3.6.6): $y = Dv \iff \{\Delta y = 0 \text{ in } \Omega;\ y|_\Gamma = v\}$ with regularity as in Eqn. (3.6.7). We have already noted in Section 5.4 that Section 5.1 applies, in particular, in the form (5.1.17) through (5.1.77) for second-order equations in t; we shall briefly recall the key ingredients.

Choice of $\mathcal{V}_h, \mathcal{A}_h, V_h$ We take the same approximating spaces $\mathcal{V}_h \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega) \cap H_0^1(\Omega)$ and the same Galerkin approximation $\mathcal{A}_h$ of $\mathcal{A}$ as in Section 5.4, in particular, Eqn. (5.4.7a), as well as the same $V_h = \hat{\mathcal{V}}_h \times \mathcal{V}_h$ as in (5.4.8). As there, the order of approximation is at least $r = 3$.

Choice of A_h and B_h We take the same approximation A_h of A as in (5.4.9),

$$A_h = \begin{bmatrix} 0 & \pi_h \\ -\mathcal{A}_h & -\rho \mathcal{A}_h^{\frac{1}{2}} \end{bmatrix} : V_h \to V_h, \tag{5.6.7}$$

while the approximation B_h of B is now

$$B_h u = \begin{bmatrix} 0 \\ \mathcal{B}_h u \end{bmatrix} : L_2(\Gamma) \to V_h; \quad \mathcal{B}_h u = \pi_h\big(-\mathcal{A}^{\frac{1}{2}} D u\big). \tag{5.6.8}$$

Computation of Adjoints A_h^* and B_h^* The adjoints A_h^* of A_h with respect to the inner product of $V_h = \hat{\mathcal{V}}_h \times \mathcal{V}_h$ and the adjoint B_h^* of B_h in the sense of (5.4.13) are given by

$$A_h^* = \begin{bmatrix} 0 & -\pi_h \\ \mathcal{A}_h & \rho \mathcal{A}_h^{\frac{1}{2}} \end{bmatrix} : V_h \to V_h, \tag{5.6.9}$$

$$B_h^* v_h = \mathcal{B}_h^* v_{h2} = -D^* \mathcal{A}^{\frac{1}{2}} v_{h2} = \frac{\partial v_{h2}}{\partial \nu}, \tag{5.6.10}$$

recalling in the last step [Chapter 3, Eqn. (3.1.9)] that

$$D^* \mathcal{A}^{\frac{1}{2}} f = \frac{\partial f}{\partial \nu}, \quad f \in \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \tag{5.6.11}$$

for $\mathcal{A}^{\frac{1}{2}}$ as in (5.6.4).

Approximating Control Problem The approximating dynamics $\dot{y}_h = A_h y_h + B_h u$ is given, via (5.6.7), (5.6.8), (5.6.10), or (5.6.11), by

$$\begin{cases} (\ddot{w}_h, \phi_h) + (\mathcal{A}_h w_h, \phi_h) + \rho\big(\mathcal{A}_h^{\frac{1}{2}} \dot{w}_h, \phi_h\big) = \left(u, \dfrac{\partial \phi_h}{\partial \nu}\right)_\Gamma; & (5.6.12a) \\ (\mathcal{A}_h w_h, \phi_h) = (\Delta w_h, \Delta \phi_h), \quad \forall\, \phi_h \in \mathcal{V}_h; & (5.6.12b) \\ (w_h(0), \phi_h) = (w_0, \phi_h), \quad (\dot{w}_h(0), \phi_h) = (w_1, \phi_h), & (5.6.12c) \end{cases}$$

where $y_h = [w_h, \dot{w}_h]$, and where all inner products $(\ ,\ )$ are in $L_2(\Omega)$, while $(\ ,\ )_\Gamma$ is the inner product in $L_2(\Gamma)$ [see (5.4.15) for a comparison]. The optimal feedback control for the approximating finite-dimensional problem is, by (5.6.10), given by

$$u_h^0(t, \Pi_h y_0) = -B_h^* P_h y_h^0(t; \Pi_h y_0) = -\frac{\partial}{\partial \nu}\big[P_h y_h^0(t; \Pi_h y_0)\big]_2 \tag{5.6.13}$$

in the same notation for P_h as in (5.4.18), where $\Pi_h = \operatorname{diag}[\pi_h^1, \pi_h]$ is once more given by (5.4.12). Here, P_h is the unique, nonnegative, self-adjoint solution of the algebraic Riccati equation (ARE$_h$) given by (5.4.19); that is, recalling (5.4.8), (5.6.10), and (5.4.18), we have

$$\begin{aligned} &-(\mathcal{A}_h[P_h z_h]_2, v_{h1}) + \big(\mathcal{A}_h[P_h z_h]_1 + \rho \mathcal{A}_h^{\frac{1}{2}}[P_h z_h]_2, v_{h2}\big) - (z_{h1}, \mathcal{A}_h[P_h v_h]_2) \\ &\quad + \big(z_{h2}, \mathcal{A}_h[P_h v_h]_1 + \rho \mathcal{A}_h^{\frac{1}{2}}[P_h v_h]_2\big) + (\mathcal{A}_h z_{h1}, v_{h1}) + (z_{h2}, v_{h2}) \\ &\quad = \left(\frac{\partial}{\partial \nu}[P_h z_h]_2, \frac{\partial}{\partial \nu}[P_h v_h]_2\right)_\Gamma, \end{aligned} \tag{5.6.14}$$

for $z_h, v_h \in V_h$, where $(\ , \ )$ and $(\ , \ )_\Gamma$ are the inner products in $L_2(\Omega)$ and $L_2(\Gamma)$, respectively [(5.6.14) has, of course, the same left-hand side as (5.4.20)].

Verification of Continuous Assumptions (4.1.4.1) and (4.1.4.3) of Theorem 4.1.4.1 These are plainly satisfied: (4.1.4.1) follows since $R = I$; whereas (4.1.4.3), follows from $B^*A^{*-1} = B^*A^{*-\gamma}A^{*\gamma-1}$, since $B^*A^{*-1} = B^*A^{*-\gamma}A^{*\gamma-1}$, with $\gamma = 3/4 + \epsilon$ by Chapter 3, Section 3.6, Claim below (3.6.11), where $B^*A^{*-\gamma} \in \mathcal{L}(Y; U)$, while $A^{*(\gamma-1)}$ with $(\gamma - 1) < 0$ is compact on Y, since $\mathcal{A}$ has compact resolvent on $L_2(\Omega)$ [Chapter 3, Theorem 3B.1(e) of Appendix 3B].

Verification of Discrete Assumptions (A.1) through (A.6) of Thoerem 4.1.4.1

Assumption (A.1) = (4.1.2.2) and (A.2) = (4.1.2.4) These were already verified to hold true in Section 5.4. In particular, we have already seen that (A.2) is fulfilled with $s = 2 - \epsilon$, a fact to be used in (A.3) below.

Assumption (A.3) = (4.1.2.5) Let $v_h = [v_{h1}, v_{h2}] \in V_h$. By (5.6.6), (5.6.10), and (5.6.11), we compute

$$\|B^*v_h\|_{L_2(\Gamma)} = \|B_h^*v_h\|_{L_2(\Gamma)} = \left\|D^*\mathcal{A}^{\frac{1}{2}}v_{h2}\right\|_{L_2(\Gamma)} = \left\|\frac{\partial}{\partial\nu}v_{h2}\right\|_{L_2(\Gamma)} \tag{5.6.15}$$

$$\leq C\|v_{h2}\|_{H^{\frac{3}{2}+\epsilon'}(\Omega)} \leq Ch^{-(\frac{3}{2}+\epsilon')}\|v_{h2}\|_{L_2(\Omega)}, \tag{5.6.16}$$

where in going from (5.6.15) to (5.6.16) we have first used trace theory and then property $(\mathrm{vi}_2) = (5.1.25)$ with $\alpha = s = 3/2 + \epsilon'$. Then, assumption (A.3) = (4.1.2.6) is verified via (5.6.16) with $s = 2 - \epsilon$ [as remarked above for (A.2); see Section 5.4], and $\gamma = 3/4 + \epsilon$ [Chapter 3, Section 3.6, Claim below (3.6.11)], so that $s\gamma = (2 - \epsilon)(3/4 + \epsilon) > 3/2$, and then $s\gamma = 3/2 + \epsilon'$, as required by (A.3) for (5.6.16).

Before proceeding, we note that verification of assumptions (A.4) through (A.6) below is performed exactly as in the case of Section 5.2 (heat equation with Dirichlet boundary control).

Assumption (A.4) = (4.1.27) By (5.6.15), with $\Pi_h = \mathrm{diag}[\pi_h^1, \pi_h]$ as in (5.4.12), we compute with $x = [x_1, x_2] \in \mathcal{D}(A^*)$:

$$\|B^*(\Pi_h x - x)\|_{L_2(\Gamma)} = \left\|\frac{\partial}{\partial\nu}(\pi_h x_2 - x_2)\right\|_{L_2(\Gamma)} \tag{5.6.17}$$

$$\leq Ch^{\frac{1}{2}}\|x_2\|_{H^2(\Omega)} \leq Ch^{\frac{1}{2}}\|x\|_{\mathcal{D}(A^*)} \tag{5.6.18}$$

$$\leq Ch^{s(1-\gamma)}\|x\|_{\mathcal{D}(A^*)}, \tag{5.6.19}$$

where in going from (5.6.17) to (5.6.18) we have used (5.2.1.2b) with $s = 2$ [exactly as in (5.2.1.14a) for the heat equation, where the Π_h in (5.2.1.14a) is the same as π_h now; that is, an orthogonal projection on $L_2(\Omega)$]. Moreover, $A^* = -A$, and $[x_1, x_2] \in \mathcal{D}(A^*)$ means $x_2 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega) \cap H_0^1(\Omega)$. So, (5.6.18) is established. Finally, since $s(1-\gamma) = (2-\epsilon)(1/4-\epsilon) < 1/2$, then (5.6.18) leads to (5.6.19), and assumption (A.4) is verified.

Assumption (A.5) = (4.1.2.8) As in (A.5) of Section 5.2.1, since in our case $B_h^*\Pi_h = B^*\Pi_h$ [see (5.6.15)], then (A.4) coincides with (A.5).

Assumption (A.6) = (4.1.2.9) We compute

$$\|B^*\Pi_h x\|_{L_2(\Gamma)} = \left\| \frac{\partial}{\partial\nu}\pi_h x_2 \right\|_{L_2(\Gamma)} = \left\| \frac{\partial}{\partial\nu}(\pi_h x_2 - x_2) \right\|_{L_2(\Gamma)} + \left\| \frac{\partial}{\partial\nu} x_2 \right\|_{L_2(\Gamma)} \tag{5.6.20}$$

$$\text{(by trace theory)} \qquad \leq C\Big[\|\pi_h x_2 - x_2\|_{H^{\frac{3}{2}+\epsilon}(\Omega)} + \|x_2\|_{H^{\frac{3}{2}+\epsilon}(\Omega)}\Big] \tag{5.6.21}$$

$$\text{(by (5.1.2.4))} \qquad \leq C h^{\epsilon}\|x_2\|_{H^{\frac{3}{2}+2\epsilon}(\Omega)} + C\|x_2\|_{H^{\frac{3}{2}+\epsilon}(\Omega)} \tag{5.6.22}$$

$$\leq C\|x_2\|_{H^{2\gamma}(\Omega)} \leq C\left\|\mathcal{A}^{\frac{\gamma}{2}}x_2\right\|_{L_2(\Omega)} \tag{5.6.23}$$

$$\leq C\|A^{*\gamma}x\|_Y. \tag{5.6.24}$$

Indeed, (5.6.20) yields (5.6.21) by trace theory, whereas (5.6.22) follows from (5.6.21) by property $(\text{vi}_1) = (5.1.24)$ with $\ell = 3/2+\epsilon$, $s = 3/2+2\epsilon$. Since $\gamma = 3/4+\epsilon$, and $\mathcal{A}$ is a fourth-order operator, then (5.6.22) implies (5.6.23). Finally, the passage from (5.6.23) to (5.6.24) uses Theorem 3B.2, Eqn. (3B.17) of Appendix 3B of Chapter 3, with $\alpha = 1/2$, $\theta = \gamma > 1/2$. According to this result, then $x = [x_1, x_2] \in \mathcal{D}(A^{*\gamma})$ implies that $x_2 \in \mathcal{D}(\mathcal{A}^{\alpha-\frac{1}{2}+\gamma(1-\alpha)}) = \mathcal{D}(\mathcal{A}^{\frac{\gamma}{2}}) \subset H^{2\gamma}(\Omega)$, as desired. Equation (5.6.24) is proved.

Conclusion The continuous analysis of Chapter 3, Section 3.6 and the above discrete analysis show that we have verified all the assumptions of Theorems 4.1.4.1 and 4.1.4.2 of Chapter 4 in the case of the structurally damped platelike equation with boundary control as in (5.6.1), with order of approximation $r \geq 3$. Then, application of Theorems 4.1.4.1 and 4.1.4.2 yields the following convergence results, since $s(1-\gamma) = (2-\epsilon)(1/4-\epsilon) < 1/2$.

Theorem 5.6.1 *Assume $r \geq 3$ in (5.1.24). Then, with reference to problem (5.6.1), (5.6.2), we have:*

I: The unique solution $P_h : V_h \to V_h$ to the $(ARE_h) = (5.6.14)$ satisfies the following estimates:

(i)

$$\|P_h\Pi_h - P\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega))} \leq Ch^{\epsilon_0} \to 0 \text{ as } h \downarrow 0, \quad \forall\, \epsilon_0 < \frac{1}{2}; \tag{5.6.25}$$

(ii)

$$\|B^* P_h \Pi_h - B^* P\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega); L_2(\Gamma))} \to 0 \text{ as } h \downarrow 0. \tag{5.6.26}$$

II: The optimal finite-dimensional approximations $y_h^0(t; \Pi_h y_0)$ and $u_h^0(t; \Pi_h y_0)$ [see (5.6.13)] satisfy, still with $\epsilon < 1/2$,

(iii)

$$\sup_{t\geq 0}\left\{e^{\bar{\omega}_P t}\left\|u_h^0(t; \Pi_h y_0) - u^0(t; y_0)\right\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega); L_2(\Gamma))}\right\} \leq C\, h^{\epsilon_0} \to 0 \quad \text{as } h \downarrow 0; \tag{5.6.27}$$

(iv)

$$\sup_{t\geq 0}\left\{t^{\epsilon} e^{\bar{\omega}_P t}\left\|y_h^0(t; \Pi_h y_0) - y^0(t; y_0)\right\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega))}\right\} \leq C h^{\epsilon_0 \epsilon} \to 0 \quad \text{as } h \downarrow 0, \tag{5.6.28}$$

uniformly in $y_0 \in [H^2(\Omega) \cap H_0^1(\Omega)] \times L_2(\Omega)$;

(v)

$$\left|J\big(u_h^0(\,\cdot\,; \Pi_h y_0), y_h^0(\,\cdot\,; \Pi_h y_0) - J(u^0(\,\cdot\,; y_0), y^0(\,\cdot\,; y_0)\big)\right| \leq C h^{\epsilon_0} \to 0 \quad \text{as } h \downarrow 0. \tag{5.6.29}$$

III: Application of Theorem 4.1.4.2 of Chapter 4 to our problem yields the following result: Consider the problem given by

$$\begin{cases} \ddot{w}_h^* + \Delta^2 w_h^* - \rho\Delta \dot{w}_h^* = 0 & \text{in } Q; & (5.6.30a) \\ w_h^*(0, \,\cdot\,) = w_0, \quad \dot{w}_h^*(0, \,\cdot\,) = w_1 & \text{in } \Omega; & (5.6.30b) \\ w_h^*|_\Sigma = 0, \quad \Delta w_h^* = -\dfrac{\partial}{\partial \nu}\left\{P_h \begin{bmatrix} w_h^*(t; y_0) \\ \dot{w}_h^*(t; y_0)\end{bmatrix}_2\right\} & \text{in } \Sigma, & (5.6.30c) \end{cases}$$

with feedback law

$$u_h^*(t; y_0) = -\frac{\partial}{\partial \nu}\left\{P_h \begin{bmatrix} w_h^*(t; y_0) \\ \dot{w}_h^*(t; y_0)\end{bmatrix}_2\right\}, \quad y_0 = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}, \tag{5.6.31}$$

in the notation of (5.4.18), with P_h a solution of (5.6.14). Then $y_h^*(t; y_0) = [w_h^*(t; y_0), \dot{w}_h^*(t; y_0)]$ is exponentially stable in $\mathcal{L}(H^2(\Omega)\times L_2(\Omega))$, uniformly in the parameter h,

$$\|y_h^*(t; y_0)\|_{H^2(\Omega)\times L_2(\Omega)} \leq \hat{C}\, e^{-\hat{\omega}_P t}\|y_0\|_{H^2(\Omega)\times L_2(\Omega)}, \quad \forall\, h,\ t \geq 0. \tag{5.6.32}$$

Moreover,

$$\sup_{t\geq 0}\left\{e^{\hat{\omega}_P t}\|y_h^*(t; y_0) - y^0(t; y_0)\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega))}\right\} \to 0, \tag{5.6.33}$$

that is, uniformly in $y_0 \in H^2(\Omega) \times L_2(\Omega)$.

As we know, convergence results based on Theorems 4.1.4.1 and 4.1.4.2 of Chapter 4 do not provide optimal rates of convergence.

Notes on Chapter 5

Section 5.1

The preliminary Section 5.1 involves duality arguments of the numerical literature (Aubin [1972], Nitsche [1971], etc.).

Sections 5.2–5.6

Section 5.2.1 (Galerkin approximation of the heat equation with Dirichlet control) was originally studied, by abstract operator methods, in Lasiecka and Triggiani [1987]. This paper formed the basis for the full abstract treatment of Lasiecka and Triggiani [1991(a)]. This latter reference contained, in its supplement, several illustrative examples of the theory provided, including Section 5.2.1, Section 5.4 (structurally damped platelike equation with interior point control), and Section 5.5 (Kelvin–Voight platelike equation with interior point control). (Because of its length the original manuscript had to be divided in two parts, a main body and a supplement, under editorial constraints.) The first two examples have also appeared in Lasiecka and Triggiani [1991(b)]. The examples illustrating the *optimal rates of convergence* – Sections 5.2.2 and 5.3 – of the heat equation with Dirichlet, respectively, Neumann-control are taken from Lasiecka [1992], which also provides the theoretical foundation (given in Chapter 4, Section 4.6).

Regarding the Dynamic Compensator of Chapter 4, Appendix 4B Actual numerical schemes for the design of a finite-dimensional dynamic compensator of Luenberger's type for several parabolic problems are given in detail in the references [Lasiecka, 1995], [Ji, Lasiecka, 1988], and [Chang et al., 1999] noted in Chapter 4, Appendix 4B.

Glossary of Symbols for Chapter 5, Section 5.1

$A,\ A_h,\ V_h,\ \Pi_h$	(5.1.1)
$\mathcal{A},\ \mathcal{H},\ \mathcal{V}_h, \hat{\mathcal{V}}_h,\ \pi_h,\ \pi_h^1$	(5.1.17)–(5.1.18)
$\mathcal{A}_h$	(5.1.23)

References and Bibliography

J. P. Aubin, *Approximation of Elliptic Boundary-Value Problems*, Wiley Interscience, 1972.

J. Bramble, A. Shatz, V. Thomee, and L. Wahlbin. Some convergence estimates for semidiscrete Galerkin type approximations for parabolic equations. *SIAM J. Num. Anal.* **14** (1977), 218–241.

S. K. Chang, I. Lasiecka, and R. Triggiani, Finite element compensators for thermoelastic systems with boundary control and point observation, *Numer. Funct. Anal. & Optimiz* **20** (5, 6) (1999), 419–435.

S. Chen and R. Triggiani, Proof of extension of two conjectures on structural damping for elastic systems: The case $1/2 \leq \alpha \leq 1$, *Pacific J. Math.* **136** (N1) (1989), 15–55.

S. Chen and R. Triggiani, Characterization of domains of fractional powers of certain operators arising in elastic systems, and applications, preprint 1989, *J. Diff. Eqn.* **88** (1990), 279–293.

G. Choudoury and I. Lasiecka, Semidiscrete approximations of parabolic boundary value problems with nonsmooth boundary data, *Num. Func. Anal. Optim.* **12** (5, 6) (1991), 463–487.

G. Ji and I. Lasiecka, Partially observed analytic systems with fully unbounded actuators and sensors-FEM algorithms, in *Optimal Control: Theory, Algorithms and Applications*, W. W. Hager and P. M. Pardalos, eds., Kluwer Academic Publishers.

J. Lagnese, Uniform boundary stabilization of homogeneous isotropic plates, Springer-Verlag Lecture Notes, Vol. 102, Proceedings of the 3rd International Conference, Vorau, Austria, July 1986, 204–215.

I. Lasiecka, Convergence estimates for semidiscrete approximations of nonselfadjoint parabolic equations, *SIAM J. Num. Anal.* **21** (5) (1984), 894–908.

I. Lasiecka, Convergence rates for the approximation of solutions to Algebraic Riccati equations with unbounded coefficients: The case of analytic semigroups, *Numerische Mathematik* **63** (1992), 357–390.

I. Lasiecka, Finite element approximations of compensator design for analytic generators with fully unbounded control-observation operators, *SIAM J. Control Optim.* **33** (1995), 67–88.

I. Lasiecka and R. Triggiani, The regulator problem for parabolic equations with Dirichlet boundary control. Part II: Galerkin approximation, *Applied Math. Optim.* **16** (1987), 187–216.

I. Lasiecka and R. Triggiani, Numerical approximations of algebraic Riccati equations for abstract systems modelled by analytic semigroups, and applications, *Math. Comput.* **57** (196) (1991)(a), 639–662 and *Supplement*, 513–537.

I. Lasiecka and R. Triggiani, *Differential and Algebraic Riccati Equations with Applications to Boundary/Point Control Points: Continuous Theory and Approximation Theory*, 164 Springer-Verlag LNCI, 1991(b); 160 pp.

J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Vol. 1, Springer-Verlag, 1967.

J. Nitsche, Uber ein Variationsprinzip zur Lösung von Dirichlet Problemen bei Verwendung von Teilräumen, die Keinen Randbedingungen unterworfen sind, *Abh. Math. Sem. Univ. Hamburg* **36** (1971), 9–15.

A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, 1983.

R. E. Showalter, *Hilbert Space Methods for Partial Differential Equations*, Pitman, 1977.

V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems, Lecture Notes in Mathematics*, Springer-Verlag, 1984.

6

Min–Max Game Theory over an Infinite Time Interval and Algebraic Riccati Equations

In this chapter we augment the abstract dynamics of the preceding chapters by introducing, in addition to the control function u, a deterministic disturbance w, which acts upon the system through a disturbance operator of the same class as the control operator. To this two-player dynamic we associate a quadratic cost functional J over an infinite time horizon, which – unlike in preceding chapters – is nondefinite. The resulting min–max game theory consists in minimizing J over all controls u (good players) for a fixed w, followed by maximization of J over all w (bad players). The min–max theory of this chapter may be viewed as a nontrivial generalization of the quadratic cost optimal control theory of Chapter 2, to which it reduces when $w \equiv 0$.

The first part of the present chapter (Sections 6.1–6.18) treats the general case, while the second part (Sections 6.19–6.26) specializes to the physically significant class where the free dynamics e^{At} is stable. The approach is variational and the resulting theory is sharp. It produces an intrinsically defined sharp value of the parameter γ in the indefinite cost, called here γ_c (critical γ), $\gamma_c \geq 0$, such that a complete theory is available for $\gamma > \gamma_c$, whereas the maximization problem over the disturbances w does not have a finite solution if $0 < \gamma < \gamma_c$, for all initial conditions. In the former case $\gamma > \gamma_c$, explicit formulas are provided for all relevant quantities: the optimal control $u^*(\cdot\,; y_0)$, the worst disturbance $w^*(\cdot\,; y_0)$, and the related solution $y^*(\cdot\,; y_0)$.

In the general case (Sections 6.1–6.18), these formulas require the introduction of the Riccati operator $P_{w=0}$ corresponding to the optimal control problem of Chapter 2 with no disturbance. In the specialized stable case of the rest of the chapter, instead, these formulas can be expressed directly in terms of the problem data, thus dispensing with $P_{w=0}$ altogether. The optimal control and the worst disturbance are both synthesized in a feedback form, pointwise in time, in terms of a Riccati operator, solution of an algebraic Riccati equation, which – unlike the case of Chapter 2 – is nondefinite in its quadratic term. This feature rules out a direct approach.

If successful, a direct approach based on the contraction mapping principle automatically yields a *unique* nonnegative self-adjoint solution of the ARE. However, for $\gamma \neq 0$, the corresponding ARE may well have two distinct positive solutions, even in the scalar case (Remark 6.1.3.2).

The abstract approach of this chapter covers parabolic and parabolic-like PDEs with point/boundary control and disturbance, as in the applications in Chapter 3. The connection of the present min–max game theory to the so-called H^∞-robust stabilization theory is recalled in the Notes at the end of the chapter. A concise treatment of an optimal control problem with non-definite quadratic cost concludes the chapter.

Part I: General Case

6.1 Mathematical Setting; Formulation of the Min–Max Game Problem; Statement of Main Results

6.1.1 Problem Setting

Dynamical Model Let U (control), V (disturbance), and Y (state) be separable Hilbert spaces. We consider the following abstract state equation:

$$\dot{y}(t) = Ay(t) + Bu(t) + Gw(t) \quad \text{in } [\mathcal{D}(A^*)]'; \quad y(0) = y_0 \in Y. \qquad (6.1.1.1)$$

Here, the function $u \in L_2(0, \infty; U)$ is the control and $w \in L_2(0, \infty; V)$ is a deterministic disturbance. The dynamics (6.1.1.1) is subject to the assumptions (H.1)–(H.3) described below, which will be maintained throughout this chapter:

(H.1) $A : Y \supset \mathcal{D}(A) \to Y$ is the infinitesimal generator of a strongly continuous (s.c.) analytic semigroup e^{At} on Y.

As in Chapter 2 (to which the present chapter will collapse if $w \equiv 0$) we shall consider the general case of A being generally unstable, with a margin of stability

$$\omega_0 \equiv \lim_{t\to\infty} \frac{\ell n \|e^{At}\|_{\mathcal{L}(Y)}}{t} > 0, \qquad (6.1.1.2a)$$

so that

$$\|e^{At}\|_{\mathcal{L}(Y)} \le M e^{(\omega_0+\epsilon)t}, \quad t \ge 0, \quad \forall \epsilon > 0, \qquad (6.1.1.2b)$$

with M depending on $(\omega_0 + \epsilon)$. We shall then consider throughout the translation

$$\hat{A} = -A + \omega I, \quad \omega \text{ fixed} > \omega_0, \qquad (6.1.1.3)$$

so that the fractional powers $\hat{A}^\theta$ of $\hat{A}$, $0 < \theta < 1$, are well defined [Pandolfi, 1995] and $-\hat{A}$ is the generator of a strongly continuous, analytic semigroup $e^{-\hat{A}t}$ on Y, $t \ge 0$, satisfying

$$\|e^{-\hat{A}t}\|_{\mathcal{L}(Y)} \le \hat{M} e^{-\hat{\omega}t}, \quad t \ge 0, \quad \hat{\omega} = \omega - \omega_0 - \epsilon > 0, \qquad (6.1.1.4)$$

and more generally,

$$\|\hat{A}^\theta e^{-\hat{A}t}\|_{\mathcal{L}(Y)} \le \frac{\hat{M} e^{-\hat{\omega}t}}{t^\theta}, \quad 0 < t, \quad 0 \le \theta. \qquad (6.1.1.5)$$

(H.2) B is a linear continuous operator $U \to [\mathcal{D}(A^*)]'$, the dual space of $\mathcal{D}(A^*)$, A^* being the adjoint of A in Y, such that

$$A^{-\delta}B \in \mathcal{L}(U; Y) \text{ for some fixed constant } \delta < 1; \tag{6.1.1.6a}$$

$$\|A^{-\delta}B\|_{\mathcal{L}(U;Y)} = \|B^*A^{*-\delta}\|_{\mathcal{L}(Y;U)} = c_\delta < \infty. \tag{6.1.1.6b}$$

(H.3) G is a linear continuous operator $V \to [\mathcal{D}(A^*)]'$ of the same class as B, i.e., such that

$$A^{-\rho}G \in \mathcal{L}(V; Y) \text{ for some fixed constant } \rho < 1; \tag{6.1.1.7a}$$

$$\|A^{-\rho}G\|_{\mathcal{L}(V;Y)} = \|G^*A^{*-\delta}\|_{\mathcal{L}(Y;V)} = c_\rho < \infty. \tag{6.1.1.7b}$$

[The above assumptions are the same as those in Chapter 2 for A and B.]

Remark 6.1.1.1 The above abstract setting includes partial differential equations of parabolic type with boundary/point control u and boundary/point disturbance w, as documented by the illustrative classes of PDE examples in Chapter 3. As we have seen there, these encompass not only mixed problems for heat/diffusion equations but also for wave/plate equations with a sufficiently high degree of internal damping, as well as thermo-elastic plates.

In this chapter, we shall study the min–max game-theoretic problem with indefinite cost described below in Section 6.1.2 for the above abstract dynamics, where the observation operator R in (6.1.2.1) below is subject to the assumption (see Remark 6.1.2.1 below)

(H.4)

$$R \in \mathcal{L}(Y; Z), \tag{6.1.1.8}$$

where Z is another Hilbert space.

The solution to the state equation (6.1.1.1) for $0 \le t \le T$ is given explicitly by

$$y(t) = y(t; y_0) = e^{At}y_0 + (L_T u)(t) + (W_T w)(t), \tag{6.1.1.9}$$

where the operators L_T and W_T are given below in (6.2.3)–(6.2.4).

6.1.2 Game Theory Problem

For a fixed $\gamma > 0$, we associate with (6.1.1.1) or (6.1.1.9) the cost functional

$$J(u, w) = J(u, w, y(u, w)) = \int_0^\infty \left[\|Ry(t)\|_Z^2 + \|u(t)\|_U^2 - \gamma^2\|w(t)\|_V^2\right] dt, \tag{6.1.2.1}$$

where $y(t) = y(t; y_0)$ is given by (6.1.1.9). The aim of this chapter is to study the following game-theory problem:

$$\sup_w \inf_u J(u, w), \tag{6.1.2.2}$$

where the infimum is taken over all $u \in L_2(0, \infty; U)$, for w fixed, and the supremum is taken over all $w \in L_2(0, \infty; V)$.

Remark 6.1.2.1 One may take the output equation

$$z(t) = Ry(t) + Du(t), \tag{6.1.2.3}$$

and use the cost functional

$$J(u, w) = \int_0^\infty \left[\|z(t)\|_Z^2 - \gamma^2 \|w(t)\|_V^2\right] dt. \tag{6.1.2.4}$$

Under the customary assumptions,

$$D \in \mathcal{L}(U; Z) \text{ with } D^*D = I, \quad D^*R = 0, \tag{6.1.2.5}$$

then

$$\|z(t)\|_Z^2 = \|Ry(t)\|_Z^2 + \|u(t)\|_U^2, \tag{6.1.2.6}$$

and the cost in (6.1.2.4) becomes equal to the cost in (6.1.2.1). In this chapter we shall work explicitly with the cost defined by (6.1.2.1).

6.1.3 Statement of Main Results: Theorems 6.1.3.1 and 6.1.3.2

In addition to hypotheses (H.1)–(H.3), which refer to the dynamics, and (H.4), which refers to the observation R, we need in the general case, as usual, the following control-theoretic assumptions:

(H.5) (Finite Cost Condition) With $w \equiv 0$, for any $y_0 \in Y$, there exists $\bar{u} \in L_2(0, \infty; U)$ such that for the corresponding solution $\bar{y}$ of (6.1.1.1), that is, of (6.1.1.9), we have $J(\bar{u}, \bar{y}, w) < \infty$.

(H.6) (Detectability Condition) There exists an operator $K \in \mathcal{L}(Y)$ such that the strongly continuous analytic semigroup $\exp[(A + K(R^*R)^{\frac{1}{2}})t]$ is exponentially stable on Y:

$$\left\| e^{(A+K(R^*R)^{\frac{1}{2}})t} \right\|_{\mathcal{L}(Y)} \leq M_K e^{-\omega_K t}, \quad t \geq 0, \quad \omega_K > 0. \tag{6.1.3.1}$$

Remark 6.1.3.1 As seen in Chapter 2, for parabolic-like dynamics, the Finite Cost Condition is most readily checked by the property of uniform stabilization. The Detectability Condition is automatically satisfied if $(R^*R) \geq cI, c > 0$: indeed, in this case, we simply take $K = -k^2(R^*R)^{-\frac{1}{2}} \in \mathcal{L}(Y)$ with positive constant k^2 sufficiently large, $k^2 > \omega_0$. As seen in Chapter 2 [below (2.1.13)], in the above

Detectability Condition (6.1.3.1), we could equally well replace $(R^*R)^{\frac{1}{2}}$ with R, and hence consider $A + KR$ rather than the operator $A + K(R^*R)^{\frac{1}{2}}$, this time with $K \in \mathcal{L}(Z; Y)$.

In Sections 6.19–6.26, we shall specialize the study of problem (6.1.2.2) under the additional hypothesis – frequently satisfied by parabolic-like dynamics, particularly by canonical models – that e^{At} is exponentially stable, in which case (H.5) and (H.6) are automatically satisfied (with $\bar{u} \equiv 0$ and $K = 0$, respectively). In this case, thus only under assumptions (H.1)–(H.4), it is possible to give a complete solution of the min–max problem (6.1.2.2), which provides a fully explicit characterization of all the relevant optimal quantities directly in terms of the data of the problem, such as it is not possible in the general case. In this chapter we treat the general case. The solution is more complicated and, in particular (unlike the stable case of later sections), it relies on the algebraic Riccati operator, $P_{0,\infty}$ below, of the corresponding linear quadratic regulator problem when the disturbance $w = 0$. However, $P_{0,\infty}$ is uniquely defined by the problem data. Thus, in the general case of the present chapter, all relevant quantities are characterized in terms of the problem data via $P_{0,\infty}$. Finally, we know from Chapter 1, Section 1.8 or Chapter 2, Section 2.5 that the assumption (H.4) on R *can be relaxed to allow R with a controlled degree of unboundedness*. We shall not pursue this generalization explicitly, however.

We now state the main result of the present chapter.

Theorem 6.1.3.1 *Assume (H.1)–(H.6). Then there exists a (critical) value $\gamma_c \geq 0$, defined explicitly in terms of the problem data by Eqn. (6.6.1) below, such that:*

(a) If $0 < \gamma < \gamma_c$, then taking the supremum in w as in (6.1.2.2) leads to $+\infty$ for all initial conditions $y_0 \in Y$; that is, there is no finite solution of the game theory problem (6.1.2.2) (see Proposition 6.17.1 (below)).

(b) If $\gamma > \gamma_c$, then:

(i) There exists a unique solution $\{u^*(\cdot\,; y_0), w^*(\cdot\,; y_0), y^*(\cdot\,; y_0)\}$ of the game theory problem (6.1.2.2) (see Theorem 6.7.1).

(ii) There exists a unique bounded, nonnegative, self-adjoint operator, $P = P^* \in \mathcal{L}(Y)$ (see Eqn. (6.13.1), (6.13.7)), that satisfies the following algebraic Riccati equation (ARE$_\gamma$) for all $x, z \in \mathcal{D}((-\hat{A})^\epsilon)$, $\forall \epsilon > 0$,

$$(PAx, z)_Y + (Px, Az)_Y + (Rx, Rz)_Y = (B^*Px, B^*Pz)_U - \gamma^{-2}(G^*Px, G^*Pz)_V \tag{6.1.3.2}$$

(see Theorem 6.15.1), with the properties that (see Eqns. (6.13.19), (6.13.20))

$$(-\hat{A}^*)^\theta P \in \mathcal{L}(Y), \quad 0 \leq \theta < 1, \tag{6.1.3.3}$$

$$B^*P \in \mathcal{L}(Y; U); \qquad G^*P \in \mathcal{L}(Y; V); \tag{6.1.3.4}$$

(iii) *The following pointwise feedback relations hold:*

$$u^*(t; y_0) = -B^*Py^*(t; y_0) \in L_2(0, \infty; U) \cap C([0, \infty]; U), \quad (6.1.3.5)$$

$$\gamma^2 w^*(t; y_0) = G^*Py^*(t; y_0) \in L_2(0, \infty; U) \cap C([0, \infty]; V) \quad (6.1.3.6)$$

(see Eqns. (6.13.3) and (6.13.4) and Proposition 6.13.1).

(iv) *The operator (F stands for "feedback") with maximal domain*

$$A_F = A - BB^*P + \gamma^{-2}GG^*P : Y \supset \mathcal{D}(A_F) \to Y, \quad (6.1.3.7a)$$

$$\mathcal{D}(A_F) = \{x \in \mathcal{D}(\hat{A}^{1-\sigma}) : [\hat{A}^{1-\sigma}x - \hat{A}^{-\sigma}BB^*Px + \gamma^{-2}\hat{A}^{-\sigma}GG^*Px] \in \mathcal{D}(\hat{A}^{\sigma})\} \subset \mathcal{D}(\hat{A}^{1-\sigma}), \quad (6.1.3.7b)$$

$\sigma = \max\{\rho, \delta\}$, *is the generator of a s.c. semigroup $e^{A_F t}$ on Y, which is, moreover, analytic for $t > 0$ (see Lemma 6.14.1 and Corollary 6.14.2), and, in fact, for $y_0 \in Y$ (see Eqn. (6.14.5)):*

$$y^*(t; y_0) = e^{A_F t}y_0 = e^{(A-BB^*P+\gamma^{-2}GG^*P)t}y_0 \in L_2(0, \infty; Y) \cap C([0, \infty]; Y). \quad (6.1.3.8)$$

Furthermore, the semigroup $e^{A_F t}$ is uniformly stable in Y (see Eqn. (6.12.3));

(v) *For any $y_0 \in Y$ the cost of the game is (see Eqn. (6.13.11))*

$$(Py_0, y_0) = J^*(y_0) \equiv J(u^*(\cdot\,; y_0), w^*(\cdot\,; y_0), y^*(\cdot\,; y_0)) = \sup_w \inf_u J(u, w, y(\cdot\,; y_0)). \quad (6.1.3.9)$$

An explicit relationship expressing P (which depends on γ) as the sum of $P_{0,\infty}$ and a nonnegative, self-adjoint operator (so that, in particular, $P \geq P_{0,\infty}$, as is obvious from (6.1.3.9)) is given in Proposition 6.13.1, Eqn. (6.13.6).

(vi) *The operators $(A - BB^*P)$ and $(A + \gamma^{-2}GG^*P)$ with maximal domains generate s.c. analytic semigroups, the first of which, $e^{(A-BB^*P)t}$, is moreover stable (Proposition 6.16.1).*

Theorem 6.1.3.2 *Conversely, suppose that $P = P^* \geq 0$ is an operator in $\mathcal{L}(Y)$ such that:*

(a) *The operator $A_F = A - BB^*P + \gamma^{-2}GG^*P$ is the generator of a s.c. uniformly stable semigroup on Y for some $\gamma > 0$; and*

(b) *P is a solution of the corresponding ARE_γ in (6.1.3.1), $\forall\, x, z \in D(\hat{A}^\epsilon)$ with the properties that $B^*P \in \mathcal{L}(Y; U)$ and $G^*P \in \mathcal{L}(Y; V)$.*

Then, the operators $(A - BB^*P)$ and $(A - \gamma^{-2}GG^*P)$ are likewise the generators of s.c. semigroups on Y, the first of which is uniformly stable (see Remark 6.16.1), and, moreover, the game problem (6.1.2.2) has a finite optimal cost functional for all $y_0 \in Y$, so that then $\gamma \geq \gamma_c$ (see Section 6.18).

Additional results are given in the treatment below. In short: Although relying on the results of Chapter 2 for the corresponding linear quadratic regulator problems with no disturbance ($w \equiv 0$), the proof of Theorem 6.1.3.1 below needs to overcome additional conceptual and technical difficulties.

Remark 6.1.3.2 Regarding the uniqueness issue of a nonnegative, self-adjoint solution of the ARE (6.1.3.2), one readily sees that it may fail even in the scalar case. More precisely: Given (i) any constants b and $g \neq 0$ and (ii) any constant $q > 0$, then:

(1) for all constants γ such that $\epsilon = 1/\gamma^2 > b^2/g^2$, so that the quadratic term $(b^2 - \epsilon g^2) < 0$ in (6.1.3.2) is negative; and
(2) for all constants $a < 0$ such that $a^2 > q(\epsilon g^2 - b^2)$,

the corresponding scalar ARE in (6.1.3.2),

$$2ax - (b^2 - \epsilon g^2)x^2 + q = 0, \tag{6.1.3.10}$$

has two distinct positive solutions $x_1, x_2 > 0$.

6.2 Minimization of $J_{w,T}$ over $u \in L_2(0, T; U)$ for w Fixed

Let $0 < T < \infty$ be arbitrary and fixed. In this section we consider the functional

$$J_{w,T} = J_T(u, w) \equiv \int_0^T \left[\|Ry(t)\|_Z^2 + \|u(t)\|_U^2 - \gamma^2 \|w(t)\|_V^2 \right] dt \tag{6.2.1}$$

and study the corresponding minimization problem

$$\inf_{u \in L_2(0,T;U)} J_{w,T}, \quad \text{holding } w \in L_2(0, T; V) \text{ fixed.} \tag{6.2.2}$$

For $0 \leq t \leq T$, we write the solution of (6.1.1.1) as in (6.1.1.9), where the operators L_T and W_T are given by

$$(L_T u)(t) = \int_0^t e^{A(T-\tau)} Bu(\tau)\, d\tau : \text{continuous } L_2(0, T; U) \to L_2(0, T; Y), \tag{6.2.3}$$

$$(W_T w)(t) = \int_0^t e^{A(t-\tau)} Gw(\tau)\, d\tau : \text{continuous } L_2(0, T; V) \to L_2(0, T; Y), \tag{6.2.4}$$

and their $L_2(0, T; \cdot)$-adjoints are

$$(L_T^* v)(t) = B^* \int_t^T e^{A^*(\tau-t)} v(\tau)\, d\tau : \text{continuous } L_2(0, T; Y) \to L_2(0, T; U), \tag{6.2.5}$$

$$(W_T^* f)(t) = G^* \int_t^T e^{A(\tau-t)} f(\tau)\, d\tau : \text{continuous } L_2(0, T; Y) \to L_2(0, T; V). \tag{6.2.6}$$

Remark 6.2.1 As we know from Chapter 1, via Eqn. (0.4) of Chapter 0, the above regularity results are conservative. Indeed, all these operators are smoothing or regularizing [e.g., Chapter 1, Section 1.4.4, Chapter 2, Section 2.3.5] (see Section 6.9), and this is a *key feature* of the present parabolic theory.

6.2.1 Existence of a Unique Optimal Pair and Its Characterization

Theorem 6.2.1.1 *Assume hypotheses (H.1)–(H.4) on the dynamics (6.1.1.1).*

(i) With reference to the minimization problem (6.2.2) of $J_{w,T}$ over $[0, T]$, there exists a unique optimal pair denoted by $\{u^0_{w,T}(\cdot\,; y_0), y^0_{w,T}(\cdot\,; y_0)\}$ with corresponding cost denoted by

$$J^0_{w,T}(y_0) = J_T\left(u^0_{w,T}(\cdot\,; y_0), y^0_{w,T}(\cdot\,; y_0)\right) = \int_0^T \left[\left\| R y^0_{w,T}(t; y_0)\right\|^2_Z + \left\|u^0_{w,T}(t; y_0)\right\|^2_U - \gamma^2 \|w(t)\|^2_V\right] dt. \tag{6.2.1.1}$$

(ii) The optimal pair is related by

$$u^0_{w,T}(\cdot\,; y_0) = -L^*_T R^* R y^0_{w,T}(\cdot\,; y_0) \in L_2(0, T; U), \tag{6.2.1.2a}$$

$$u^0_{w,T}(t; y_0) = -B^* \int_t^T e^{A^*(\tau - t)} R^* R y^0_{w,T}(\tau; y_0)\, dt \text{ a.e. in } t, \tag{6.2.1.2b}$$

recalling (6.2.5) in (6.2.1.2a), and is characterized explicitly in terms of the problem data by the following formulae:

$$-u^0_{w,T}(\cdot\,; y_0) = [I + L^*_T R^* R L_T]^{-1} L^*_T R^* R[e^{A\cdot} y_0 + W_T w] \in L_2(0, T; U) \tag{6.2.1.3a}$$

$$= -u^0_{w=0,T}(\cdot\,; y_0) - u^0_{w,T}(\cdot\,; y_0 = 0), \tag{6.2.1.3b}$$

$$y^0_{w,T}(\cdot\,; y_0) = [I + L_T L^*_T R^* R]^{-1}[e^{A\cdot} y_0 + W_T w] \in L_2(0, T; Y) \tag{6.2.1.4a}$$

$$= y^0_{w=0,T}(\cdot\,; y_0) + y^0_{w,T}(\cdot\,; y_0 = 0), \tag{6.2.1.4b}$$

$$R y^0_{w,T}(\cdot\,; y_0) = [I + R L_T L^*_T R^*]^{-1}[R e^{A\cdot} y_0 + R W_T w] \in L_2(0, T; Z), \tag{6.2.1.4c}$$

where the inverse operators in (6.2.1.3a), (6.2.1.4a), and (6.2.1.4c) are well defined as bounded operators on all of $L_2(0, T; U)$ and $L_2(0, T; Y)$, respectively (for the latter, see Chapter 2, Appendix 2A). Moreover, the corresponding optimal dynamics is

$$y^0_{w,T}(t; y_0) = e^{At} y_0 + \left\{L_T u^0_{w,T}(\cdot\,; y_0)\right\}(t) + \{W_T w(\cdot)\}(t) \in L_2(0, T; Y). \tag{6.2.1.5}$$

(iii) We obtain

$$J^0_{w,T}(y_0) = J^0_{w,T}(y_0 = 0) + J^0_{w=0,T}(y_0) + \chi_{w,T,y_0}, \tag{6.2.1.6}$$

$$J^0_{w,T}(y_0 = 0) = (w, [W_T^* R^*(I + RL_T L_T^* R^*)^{-1} RW_T - \gamma^2 I]w)_{L_2(0,T;V)} \tag{6.2.1.7}$$

$$= \textit{quadratic in } w,$$

$$\chi_{w,T,y_0} = 2(Re^{A\cdot} y_0, [I + RL_T L_T^* R^*]^{-1} RW_T w)_{L_2(0,T;Z)} \tag{6.2.1.8a}$$

$$= \textit{ linear in } w, \tag{6.2.1.8b}$$

while the constant term (in w) is (of course, from Chapter 1, Eqn. (1.2.1.30))

$$J^0_{w=0,T}(y_0) = (Re^{A\cdot} y_0, [I + RL_T L_T^* R^*]^{-1} Re^{A\cdot} y_0)_{L_2(0,T;Z)}$$
$$= (P_{0,T}(0)y_0, y_0)_Y. \tag{6.2.1.9}$$

(iv) When $w \equiv 0$:

$$u^0_{w=0,T}(t; y_0) \in C([0,T]; U), \qquad y^0_{w=0,T}(t; y_0) \in C([0,T]; Y). \tag{6.2.1.10}$$

Proof. Parts (i) and (ii) are a slight modification of Chapter 1, Theorem 1.2.1.1, Eqns. (1.2.1.1) and (1.2.1.3), now simplified as the final state penalization operator is zero; and Remark 6.2.1.3.

(iii) We compute (suppressing the inner product specifications)

$$\begin{aligned}
J^0(y_0) &= \left(Ry^0_{w,T}, Ry^0_{w,T}\right) + \left(u^0_{w,T}, u^0_{w,T}\right) - \gamma^2(w,w) \\
\text{(by (6.2.1.2a))} \quad &= \left([I + RL_T L_T^* R^*]Ry^0_{w,T}, Ry^0_{w,T}\right) - \gamma^2(w,w) \\
\text{(by (6.2.1.4c))} \quad &= \left(Re^{A\cdot} y_0 + RW_T w, Ry^0_{w,T}\right) - \gamma^2(w,w) \\
\text{(by (6.2.1.4c))} \quad &= (Re^{A\cdot} y_0, [I + RL_T L_T^* R^*]^{-1}[Re^{A\cdot} y_0 + RW_T w]) \\
&\quad + (RW_T w, [I + RL_T L_T^* R^*]^{-1}[Re^{A\cdot} y_0 + RW_T w]) \\
&\quad - \gamma^2(w,w).
\end{aligned} \tag{6.2.1.11}$$

Expanding (6.2.1.11) readily yields (6.2.1.6) via (6.2.1.7)–(6.2.1.9).

(iv) The regularity (2.1.10) was obtained (even in a more general context) in Chapter 1, Theorem 1.2.2.1, Eqn. (1.2.2.2). □

6.2.2 The Functions $p_{w,T}(\cdot\,; y_0)$ and $r_{w,T}(t)$; the (Differential) Riccati Operator $P_{0,T}(t)$ When $w \equiv 0$

For $y_0 \in Y$ we define

$$p_{w,T}(t; y_0) = \int_t^T e^{A^*(\tau - t)} R^* R y^0_{w,T}(\tau, 0; y_0)\, d\tau \in C([0,T]; Y), \tag{6.2.2.1}$$

where in the present section

$$L_2(0, T; Y) \ni y^0_{w,T}(\,\cdot\,, s; y_0) = \begin{array}{l}\text{optimal control of optimization problem (6.2.2)}\\ \text{except on the interval } [s, T], \text{ rather than } [0, T],\\ \text{originating at the point } y_0 \text{ at the initial time } s,\end{array} \tag{6.2.2.2}$$

whereby $y^0_{w,T}(\,\cdot\,, 0; y_0) \equiv y^0_{w,T}(\,\cdot\,; y_0)$ in the notation of Section 6.2.1. Similarly to (6.2.1.10) and still from Chapter 1, Theorem 1.2.2.1, Eqn. (1.2.2.2), we have that $y^0_{w=0,T}(\,\cdot\,, s; y_0) \in C([s, T]; Y)$ and that

$$\Phi_{0,T}(\tau, t)x \equiv y^0_{0,T}(\tau, t; x) \in C([t, T]; Y), \quad x \in Y, \tag{6.2.2.3}$$

is the evolution operator [Chapter 1, Proposition 1.4.3.1(ii), Eqn. (1.4.3.3)],

$$\Phi_{0,T}(\tau, s)x - \Phi_{0,T}(\tau, t)\Phi_{0,T}(t, s)x, \quad 0 \le s \le t \le \tau \le T, \tag{6.2.2.4}$$

corresponding to the optimal problem (6.2.2) with $w \equiv 0$, and on the interval $[t, T]$ rather than $[0, T]$. We also let $P_{0,T}(t) \in \mathcal{L}(Y)$ be the nonnegative, self-adjoint operator defined as in Chapter 1, Eqn. (1.4.3.15):

$$P_{0,T}(t)x = \int_t^T e^{A^*(\tau-t)} R^* R \Phi_{0,T}(\tau, t)x \, d\tau, \quad x \in Y; \tag{6.2.2.5a}$$

$$P_{0,T}(\cdot) : \text{continuous } Y \to C([0, T]; Y); \tag{6.2.2.5b}$$

see Chapter 1, Eqn. (1.2.2.5).

Lemma 6.2.2.1 *Assume (H.1)–(H.4). With reference to (6.2.2.1)–(6.2.2.5) we have*

(i) $p_{w,T}(t; y_0)$ *is the unique solution of the equation*

$$\begin{cases} \dfrac{d}{dt} p_{w,T}(t; y_0) = -A^* p_{w,T}(t; y_0) - R^* R y^0_{w,T}(t; y_0), & (6.2.2.6a) \\ p_{w,T}(T; y_0) = 0; & (6.2.2.6b) \end{cases}$$

(ii) the following identity a.e. in t holds true:

$$p_{w,T}(t; y_0) = P_{0,T}(t) y^0_{w,T}(t; y_0) + r_{w,T}(t) \in L_2(0, T; Y), \tag{6.2.2.7}$$

where $r_{w,T}(t)$ is defined by

$$r_{w,T}(t) = p_{w,T}(t; y_0 = 0) - P_{0,T}(t) y^0_{w,T}(t; y_0 = 0) \in L_2(0, T; Y); \tag{6.2.2.8}$$

(iii) the optimal control in (6.2.1.2) is rewritten as

$$u^0_{w,T}(t; y_0) = -B^* p_{w,T}(t; y_0) \in L_2(0, T; U) \tag{6.2.2.9a}$$

$$= -B^*[P_{0,T}(t) y^0_{w,T}(t; y_0) + r_{w,T}(t)]. \tag{6.2.2.9b}$$

Proof. (i) Equation (6.2.2.6) follows by differentiation in t of (6.2.2.1), recalling (6.2.2.2) for $s = 0$.

(ii) To prove (6.2.2.7), we return to (6.2.2.1), substitute identity (6.2.1.4b), rewritten in the notation of (6.2.2.2) as $y^0_{w,T}(\tau; y_0) = y^0_{w,T}(\tau, 0; y_0) = y^0_{0,T}(\tau, 0; y_0) + y^0_{w,T}(\tau, 0; y_0 = 0)$, recall (6.2.2.3) and (6.2.2.4), and obtain

$$p_{w,T}(t; y_0) = \int_t^T e^{A^*(\tau-t)} R^* R \Phi_{0,T}(\tau, t)\Phi_{0,T}(t, 0) y_0 \, d\tau + \int_t^T e^{A^*(\tau-t)} R^* R y^0_{w,T}(\tau, 0; y_0 = 0)\, d\tau$$

(recalling (6.2.2.5) for the first term and (6.2.2.1) in the notation of (6.2.2.2) for the second)

$$\begin{aligned} p_{w,T}(t; y_0) &= P_{0,T}(t)\Phi_{0,T}(t, 0)y_0 + p_{w,T}(t; y_0 = 0) \\ \text{(by (6.2.2.3))} \quad &= P_{0,T}(t) y^0_{0,T}(t; y_0) + p_{w,T}(t; y_0 = 0) \\ \text{(by (6.2.1.4b))} \quad &= P_{0,T}(t)\big[y^0_{w,T}(t; y_0) - y^0_{w,T}(t; y_0 = 0)\big] + p_{w,T}(t; y_0 = 0). \end{aligned} \tag{6.2.2.10}$$

Thus, (6.2.2.10) yields (6.2.2.7) and (6.2.2.8), as desired.

(iii) To obtain (6.2.2.9a), we return to (6.2.1.2b) and recall (6.2.2.1) in the notation (6.2.2.2). Then (6.2.2.9b) follows from (6.2.2.9a) via (6.2.2.7). □

We now rewrite (6.2.2.1) in a form useful for future use in Corollary 6.3.2.3.

Proposition 6.2.2.2 *Assume (H.1)–(H.4). We can rewrite $p_{w,T}$ in (6.2.2.1) as*

$$p_{w,T}(t; y_0) = \int_t^{t_0} e^{A^*(\tau-t)} R^* R y^0_{w,T}(\tau; y_0)\, d\tau + e^{A^*(t_0-t)} p_{w,T}(t_0; y_0), \tag{6.2.2.11}$$

where t_0 is an arbitrary point $t < t_0 < T$.

Proof. From (6.2.2.1) we compute, using (6.2.2.2) with $s = 0$,

$$p_{w,T}(t; y_0) = \int_t^{t_0} e^{A^*(\tau-t)} R^* R y^0_{w,T}(\tau; y_0)\, d\tau + e^{A^*(t_0-t)} \int_{t_0}^T e^{A^*(\tau-t_0)} R^* R y^0_{w,T}(\tau; y_0)\, d\tau, \tag{6.2.2.12}$$

and (6.2.2.11) follows from (6.2.2.12) via (6.2.2.1). □

6.2.3 Complementary Results on the Minimization Problem in the Presence of a Fixed Disturbance

The results given in this subsection, which concern the minimization problem (6.2.2) of the functional J over all $u \in L_2(0, T; U)$, under a fixed disturbance $w \in L_2(0, T; V)$

in (6.1.1.1), are *not* strictly needed in our present development of the min–max problem (6.1.2.2). [Indeed, our strategy will be to prove, in Section 6.3.5 and in Section 6.3.3 below, some results that are the counterpart version for $T = \infty$, of Proposition 6.2.3.1 and Proposition 6.2.3.3 below expressed in the case $T < \infty$.] However, they are included here for completeness of the study of the minimization problem (6.2.2) in the presence of a fixed disturbance. To this end, we shall make key use of regularity results established in Chapter 1, Section 1.2.2 and applied here with final state penalization operator (called G in Chapter 1) equal to zero.

Proposition 6.2.3.1 *Assume (H.1)–(H.4). Recall that $w \in L_2(0, T; V)$. Then, with reference to the function $r_{w,T}(t)$ in (6.2.2.8), we have*

$$B^* r_{w,T} \in L_2(0, T; U). \tag{6.2.3.1}$$

Proof. Applying B^* on (6.2.2.8), we obtain

$$B^* r_{w,T}(t) = B^* p_{w,T}(t; y_0 = 0) - B^* P_{0,T}(t) y^0_{w,T}(t; y_0 = 0). \tag{6.2.3.2}$$

For the first term on the right-hand side of (6.2.3.2), we recall the definition of $p_{w,T}$ in (6.2.2.1), the definition of L_T^* and its regularity in (6.2.5), as well as the regularity of $y^0_{w,T}$ in (6.2.1.4a). We thus obtain

$$B^* p_{w,T}(\,\cdot\,; y_0 = 0) = L_T^* R^* R y^0_{w,T}(\,\cdot\,; y_0) \in L_2(0, T; U). \tag{6.2.3.3}$$

For the second term on the right-hand side of (6.2.3.2), we recall from Chapter 1, Theorem 1.2.2.2, Eqn. (1.2.2.19), applied in the present case where the operator of final state penalization (called G in Chapter 1) is zero, that

$$B^* P_{0,T}(\,\cdot\,) \in \mathcal{L}(Y; C([0, T]; U). \tag{6.2.3.4}$$

Thus, by (6.2.3.4) and (6.2.1.4a), we have

$$B^* P_{0,T}(\,\cdot\,) y^0_{w,T}(\,\cdot\,; y_0 = 0) \in L_2(0, T; U). \tag{6.2.3.5}$$

Thus, (6.2.3.3) and (6.2.3.5), used in (6.2.3.2), yield (6.2.3.1), as desired. □

Remark 6.2.3.1 For $w \in L_2(0, T; V)$ given, the regularity of $u^0_{w,T} \in L_2(0, T; U)$ in (6.2.1.2a) and of $y^0_{w,T} \in L_2(0, T; Y)$ in (6.2.1.4a) can be improved in time, by using, in relations (6.2.1.2a) and (6.2.1.4a), the regularizing properties of the operators L_T, L_T^*, W_T, and W_T^*, as given, for example, in Chapter 1, Section 1.4.4 and Chapter 2, Section 2.3.5. We shall single out explicitly only the case where the bootstrap argument of Chapter 1 (see also the forthcoming Section 6.9) works at its best, that is, when $W_T w \in C([0, T]; Y)$. This covers the case where the constant ρ in (6.1.1.7) satisfies $\rho \le 1/2$ and A is self-adjoint (recall Chapter 0), or else $\rho < 1/2$ in general. Thus, this is the case, a fortiori, when the disturbance acts on the right-hand side of Eqn. (6.1.1.1) with $G =$ Identity on $V = Y$, a classical case.

Proposition 6.2.3.2 *Assume (H.1)–(H.4) and that $W_T w \in C([0,T];Y)$ for $w \in L_2(0,T;V)$. Then,*

$$w^0_{w,T}(\,\cdot\,;y_0) \in C([0,T];U); \quad y^0_{w,T}(\,\cdot\,;y_0) \in C([0,T];Y); \tag{6.2.3.6a}$$

$$B^* r_{w,T} \in C([0,T];U). \tag{6.2.3.6b}$$

Proof. The bootstrap argument as in Chapter 1, or in Section 6.9 below, on the relations (6.2.1.2a) and (6.2.1.4a) yields (6.2.3.6a), under the present assumption on W_T. Then, with $y^0_{w,T}$ boosted in time from $L_2(0,T;Y)$ to $C([0,T];Y)$, Eqns. (6.2.3.3) and (6.2.3.5) give now regularity results likewise boosted to $C([0,T];U)$, and thus (6.2.3.6b) follows from (6.2.3.2). □

Proposition 6.2.3.3 *Assume (H.1)–(H.4). Then, $r_{w,T}$ in (6.2.2.8) satisfies the following problem:*

$$\begin{cases} \dot r_{w,T}(t) = -[A - BB^*P_{0,T}(t)]^* r_{w,T}(t) - P_{0,T}(t)Gw(t), \quad 0 \le t < T; & (6.2.3.7a) \\ r_{w,T}(T) = 0. & (6.2.3.7b) \end{cases}$$

Proof. The key of the proof, which proceeds as in the finite-dimensional case, consists in justifying the calculations under the present setting, in particular with A the generator of a s.c. analytic semigroup.

Step 1 We return to (6.2.2.8) and differentiate in t to get

$$\dot r_{w,T}(t) = \dot p_{w,T}(t;y_0=0) - \dot P_{0,T}(t) y^0_{w,T}(t;y_0=0) - P_{0,T}(t)\dot y^0_{w,T}(t;y_0=0), \tag{6.2.3.8}$$

where:

(i) for $\dot p_{w,T}(t;y_0=0)$ we invoke (6.2.2.6a):

$$\dot p_{w,T}(t;y_0=0) = -A^* p_{w,T}(t;y_0=0) - R^*Ry^0_{w,T}(t;y_0=0); \tag{6.2.3.9}$$

(ii) for $\dot P_{0,T}(t)$ we invoke from Chapter 1, Theorem 1.2.2.1, Eqn. (1.2.2.14) the DRE in the classical sense (which is legal, since the finite state penalization operator, denoted by G in Chapter 1, is presently zero, so that the required assumption of Chapter 1, Eqn. (1.2.2.1) holds true):

$$\begin{aligned} -\dot P_{0,T}(t) y^0_{w,T}(t;y_0=0) &= R^*Ry^0_{w,T}(t;y_0=0) \\ &\quad + [A^*P_{0,T}(t) + P_{0,T}(t)A] y^0_{w,T}(t;y_0=0) \\ &\quad - (B^*P_{0,T}(t))^* B^* P_{0,T}(t) y^0_{w,T}(t;y_0=0); \end{aligned} \tag{6.2.3.10}$$

(iii) for $\dot y^0_{w,T}$ we invoke the optimal dynamics, via (6.1.1.1) and (6.2.2.9b), and obtain first

$$\dot y^0_{w,T}(t;y_0=0) = Ay^0_{w,T}(t;y_0=0) + Bu^0_{w,T}(t;y_0=0) + Gw(t) \tag{6.2.3.11}$$

$$\text{(by (6.2.2.9b))} \quad = Ay^0_{w,T}(t;y_0=0) - BB^*[P_{0,T}(t)y^0_{w,T}(t;y_0=0) + r_{w,T}(t)] + Gw(t), \tag{6.2.3.12}$$

and then, recalling the regularity properties in (6.2.3.5) and (6.2.3.1), we obtain from (6.2.3.12):

$$
\begin{aligned}
-P_{0,T}(t)\dot{y}^0_{w,T}(t; y_0 = 0) = & -P_{0,T}(t)Ay^0_{w,T}(t; y_0 = 0) \\
& + (B^*P_{0,T}(t))^*B^*P_{0,T}(t)y^0_{w,T}(t; y_0 = 0) \\
& + (B^*P_{0,T}(t))^*B^*r_{w,T}(t) - P_{0,T}(t)Gw(t).
\end{aligned} \tag{6.2.3.13}
$$

Step 2 We now ascertain that all terms (6.2.3.10) and (6.2.3.13) are well defined for $t < T$, a.e. in t.

(a) First we write [assuming without loss of generality that e^{At} is of negative type so that the fractional powers of $(-A)$ are well defined]

$$
\begin{aligned}
P_{0,T}(t)Ay^0_{w,T}(t; y_0 = 0) &= -P_{0,T}(t)(-A)^{1-\epsilon}(-A)^{\epsilon}y^0_{w,T}(t; y_0 = 0) \\
&= -[(-A^*)^{1-\epsilon}P_{0,T}(t)]^*(-A)^{\epsilon}y^0_{w,T}(t; y_0 = 0) \qquad (6.2.3.14) \\
&= \text{well defined, for a.e. in } t,\ t < T \qquad (6.2.3.15)
\end{aligned}
$$

[compare with Proposition 6.3.1.2 below], where, by Chapter 1, Theorem 1.2.2.2, Eqn. (1.2.2.18) with γ there replaced by δ in (6.1.1.6) now:

$$
\|(-A^*)^{1-\epsilon}P_{0,T}(t)\|_{\mathcal{L}(Y)} \leq \frac{C_T}{\epsilon}\,\frac{1}{(T-t)^{1-\epsilon-\delta}}, \quad 0 \leq t < T, \tag{6.2.3.16}
$$

and, moreover, as in Chapter 1, Lemma 1.4.6.2(vii),

$$
\begin{aligned}
(-A)^{\epsilon}y^0_{w,T}(t; y_0 = 0) = & \int_0^t (-A)^{\delta+\epsilon}e^{A(t-\tau)}(-A)^{-\delta}Bu^0_{w,T}(\tau; y_0 = 0)\,d\tau \\
& + \int_0^t (-A)^{\rho+\epsilon}e^{A(t-\tau)}(-A)^{-\rho}Gw(\tau)\,d\tau \qquad (6.2.3.17) \\
& \in L_2(0, T; Y), \quad \forall \epsilon \text{ s.t., } \epsilon < \min\{1-\delta, 1-\rho\} \qquad (6.2.3.18)
\end{aligned}
$$

(or even $\epsilon = \min\{1-\delta, 1-\rho\}$ if A is self-adjoint [see Chapter 0] by convolution between an L_1-function and an L_2-function [Sadosky, 1979, p. 26]). Thus (6.2.3.15) follows from (6.2.3.16) and (6.2.3.18).

(b) Next, by virtue of Lemma 1.5.1.1 of Chapter 1 we have, since $P_{0,T}(t)$ is self-adjoint in $\mathcal{L}(Y)$ and A is an analytic semigroup generator:

$$
\begin{aligned}
& -A^*P_{0,T}(t)y^0_{w,T}(t; y_0 = 0) \\
& = (-A^*)^{1-\alpha}P_{0,T}(t)(-A)^{\alpha-\beta}y^0_{w,T}(t; y_0 = 0), \quad 0 < \beta < \alpha < 1 \qquad (6.2.3.19) \\
& = \text{well defined for a.e. in } t,\ t < T, \qquad (6.2.3.20)
\end{aligned}
$$

by recalling (6.2.3.16) and (6.2.3.18).

(c) Finally,

$$\begin{cases} (B^* P_{0,T}(t))^* B^* P_{0,T}(t) y^0_{w,T}(t;\, y_0 = 0) \quad \text{and} \quad (B^* P_{0,T}(t))^* B^* r_{w,T}(t) \\ \quad \text{are well defined a.e. in } t, \end{cases} \tag{6.2.3.21}$$

by (6.2.3.4) and (6.2.3.5), (6.2.3.1).

Step 3 We next sum up (6.2.3.9) for $\dot{p}_{w,T}$, (6.2.3.10) for $-\dot{P}_{0,T}(t) y^0_{w,T}$, and (6.2.3.13) for $-P_{0,T}(t)\dot{y}_{w,T}$; perform a cancellation of $R^* R y^0_{w,T}(t;\, y_0 = 0)$, of $P_{0,T}(t) A y^0_{w,T}(t;\, y_0 = 0)$, and of $(B^* P_{0,T}(t))^* B^* P_{0,T}(t) y^0_{w,T}(t;\, y_0 = 0)$; and recall (6.2.3.8), thereby obtaining:

$$\begin{aligned} \dot{r}_{w,T}(t) &= -A^* p_{w,T}(t;\, y_0 = 0) + A^* P_{0,T}(t) y^0_{w,T}(t;\, y_0 = 0) \\ &\quad + (B^* P_{0,T}(t))^* B^* r_{w,T}(t) - P_{0,T}(t) G w(t) \qquad (6.2.3.22) \\ (\text{by } (6.2.2.8)) \quad &= -A^* r_{w,T}(t) + (B^* P_{0,T}(t))^* B^* r_{w,T}(t) - P_{0,T}(t) G w(t), \qquad (6.2.3.23) \end{aligned}$$

and (6.2.3.7a) is proved for $t < T$ via (6.2.3.23). □

6.3 Minimization of $J_{w,\infty}$ over $u \in L_2(0,\infty; U)$ for w Fixed: The Limit Process as $T \uparrow \infty$

In this section, with reference to J in (6.1.2.1), we obtain the solution of the minimization problem

$$\inf_{u \in L_2(0,\infty;U)} J, \text{ holding } w \in L_2(0,\infty;\, U) \text{ fixed}, \tag{6.3.1}$$

over the infinite time horizon process, as $T \uparrow \infty$ on the optimal problem (6.2.2) over $[0, T]$ of Section 6.2.

6.3.1 The Limit Process for $u^0_{w,T}(\,\cdot\,; y_0)$, $y^0_{w,T}(\,\cdot\,; y_0)$, and $P_{0,T}(t)$ as $T \uparrow \infty$

The following result is contained in Chapter 2 and will be repeated here for convenience. In its statement, we shall take care to point out explicitly the role of the various control-theoretic assumptions (H.5)–(H.6).

For Part (1) of Theorem 6.3.1.1 below, see Chapter 2, Theorems 2.2.1 and 2.3.3.1; Part (2) of Theorem 6.3.1.1 coincides with Theorem 2.2.2 of Chapter 2.

Theorem 6.3.3.1 *Assume hypotheses (H.1)–(H.3) on the dynamics (6.1.1.1), and (H.4) on R.*

1. *Parts (i) through (x) below require also the Finite Cost Condition, hypothesis (H.5).*
   (i) There exists a unique solution pair denoted by $\{u^0_{w,\infty}(\,\cdot\,; y_0), y^0_{w,\infty}(\,\cdot\,; y_0)\}$ of the optimal problem (6.3.1) for the dynamics (6.1.1.1) initiating at $t = 0$

at the point $y_0 \in Y$, over all $u \in L_2(0, \infty; U)$ for $w \in L_2(0, \infty; Y)$ fixed, that satisfies

$$u^0_{w,\infty} \in L_2(0, \infty; U); \qquad (R^*R)^{\frac{1}{2}} y^0_{w,\infty} \in L_2(0, \infty; Y); \tag{6.3.1.1}$$
$$R y^0_{w,\infty} \in L_2(0, \infty; Y).$$

(ii) *There exists a nonnegative, self-adjoint operator $0 \leq P_{0,\infty} = P^*_{0,\infty} \in \mathcal{L}(Y)$ for the optimal problem with $w \equiv 0$, defined as in Chapter 2, Theorem 2.3.3.1(ii), Eqn. (2.3.3.2) by*

$$P_{0,\infty} x = \lim_{T \uparrow \infty} P_{0,T}(t)x, \text{ independently of } t \text{ fixed}, \quad 0 \leq t < T, \tag{6.3.1.2}$$

and, in fact, uniformly on compact t-sets, where, moreover,

$$\sup_T \sup_{0 \leq t \leq T} \|P_T(t)\|_{\mathcal{L}(Y)} = M < \infty.$$

(iii) *For any $y_0 \in Y$, let $\tilde{u}^0_{w,T}(\,\cdot\,; y_0)$ and $\tilde{y}^0_{w,T}(\,\cdot\,; y_0)$ denote the extension by zero of $u^0_{w,T}(\,\cdot\,; y_0)$ and $y^0_{w,T}(\,\cdot\,; y_0)$, respectively, for $t > T$. Then*

$$\tilde{u}^0_{w,T} \to u^0_{w,\infty} \quad \text{in } L_2(0, \infty; U), \tag{6.3.1.3}$$

$$(R^*R)^{\frac{1}{2}} \tilde{u}^0_{w,T} \to (R^*R)^{\frac{1}{2}} y^0_{w,\infty} \quad \text{in } L_2(0, \infty; Y). \tag{6.3.1.4}$$

Moreover,

$$y^0_{w,\infty}(t; y_0) = e^{At} y_0 + \int_0^t e^{A(t-\tau)} B u^0_{w,\infty}(\tau; y_0)\, d\tau + \int_0^t e^{A(t-\tau)} G w(\tau; y_0)\, d\tau; \tag{6.3.1.5}$$

$$P_{0,T}(t) y^0_{w,T}(t; y_0) \to P_{0,\infty} y^0_{w,\infty}(t; y_0) \text{ in } L_2(0, \infty; Y). \tag{6.3.1.6}$$

(iv) *The optimal cost for problem (6.3.1) with $w \equiv 0$ is*

$$J^0_{w=0,\infty}(y_0) = \int_0^\infty \left[\|R y_{0,\infty}(t)\|^2_Z + \|u_{0,\infty}(t)\|^2_U\right] dt = (P_{0,\infty} y_0, y_0)_Y.$$

(v) *The operator $P_{0,\infty}$ satisfies the following regularity properties:*

$$(\hat{A}^*)^\theta P_{0,\infty} \in \mathcal{L}(Y),\ 0 \leq \theta < 1; \quad B^* P_{0,\infty} \in \mathcal{L}(Y; U); \tag{6.3.1.7}$$
$$G^* P_{0,\infty} \in \mathcal{L}(Y; V).$$

(vi) *Setting*

$$\Phi_{0,\infty}(t)x = y^0_{w=0,\infty}(t; x) \in C([0, T_0]; Y), \quad \forall\, T_0 < \infty, \quad x \in Y, \tag{6.3.1.8}$$

we have that $\Phi_{0,\infty}(t)$ is a s.c. semigroup on Y, denoted also by

$$\Phi_{0,\infty}(t)x = e^{A_{P_{0,\infty}} t} x = e^{(A - BB^* P_{0,\infty})t} x, \tag{6.3.1.9}$$

where, with maximal domain,

$$A_{P_{0,\infty}} = A - BB^* P_{0,\infty} : Y \supset \mathcal{D}(A_{P_{0,\infty}}) \to Y; \tag{6.3.1.10a}$$

$$\mathcal{D}(A_{P_{0,\infty}}) \subset \mathcal{D}(\hat{A}^{1-\delta}) \quad \textit{[Chapter 2, Eqn. (2.3.8.11)]} \tag{6.3.1.10b}$$

is its infinitesimal generator on Y.

(vii) *For any $x \in Y$ and any T_0 finite, we have that*

$$\begin{aligned} y^0_{w=0,\infty}(t; x) &= e^{A_{P_{0,\infty}} t} x \in C([0, T_0]; Y); \\ u^0_{w=0,\infty}(t; x) &= -B^* P_{0,\infty} e^{A_{P_{0,\infty}} t} x \in C([0, T_0]; U). \end{aligned} \tag{6.3.1.11}$$

(viii) *$P_{0,\infty}$ in (6.3.1.2) solves the Algebraic Riccati Equation (ARE)*

$$\begin{aligned} &(A^* P_{0,\infty} x, z)_Y + (P_{0,\infty} A x, z)_Y + (Rx, Rz)_Z \\ &\quad = (B^* P_{0,\infty} x, B^* P_{0,\infty} z)_U, \end{aligned} \tag{6.3.1.12}$$

$\forall x, z \in \mathcal{D}((\hat{A})^\epsilon)$; in particular, $\forall x, z \in \mathcal{D}(A_{P_{0,\infty}}) \subset \mathcal{D}((\hat{A})^{1-\delta})$.

(ix) *With reference to $J^0_{w,T}(y_0)$ in (6.2.1.6), we have*

$$J^0_{w,T}(y_0) \to J^0_{w,\infty}(y_0) = J^0_{w,\infty}(y_0 = 0) + J^0_{w=0,\infty}(y_0) + \chi_{w,\infty}(y_0); \tag{6.3.1.13a}$$

$$J^0_{w,\infty}(y_0 = 0) = \textit{quadratic in } w; \qquad \chi_{w,\infty}(y_0) = \textit{linear in } w; \tag{6.3.1.13b}$$

$$J^0_{w=0,T}(y_0) = (P_{0,T}(0) y_0, y_0) \to J^0_{w=0,\infty}(y_0) = (P_{0,\infty} y_0, y_0). \tag{6.3.1.13c}$$

2. *Parts (xi) and (xii) below require, in addition, the Detectability Condition, hypothesis (H.6).*

(x) *The semigroup in (6.3.1.9) is exponentially stable: There exist constants $M \geq 1$, $k > 0$ such*

$$\|e^{A_{P_{0,\infty}} t}\|_{\mathcal{L}(Y)} \leq M e^{-kt}, \quad t \geq 0. \tag{6.3.1.14}$$

(xi) *The operator $P_{0,\infty}$ defined by (6.3.1.2) is the unique solution of the ARE (6.3.1.12) within the class of nonnegative, self-adjoint bounded operators on Y that satisfy the property that $(\hat{A}^*)^\gamma P_{0,\infty} \in \mathcal{L}(Y)$, and hence $B^* P_{0,\infty} \in \mathcal{L}(Y; U)$.*

Proposition 6.3.1.2 *Assume (H.1)–(H.6). With reference to $y^0_{w,\infty}$ in (6.3.1.5) and to (6.1.1.6) and (6.1.1.7), we have for all $y_0 \in Y$:*

(i)

$$y^0_{w,\infty}(t; y_0) \in L_2(0, T; \mathcal{D}(\hat{A}^\epsilon)), \quad \forall T < \infty, \ \forall \epsilon \ \textit{s.t.} \ \max\{\rho + \epsilon, \delta + \epsilon\} < 1; \tag{6.3.1.15}$$

(ii)

$$P_{0,\infty} A y^0_{w,\infty}(t; y_0) \in L_2(0, T; Y), \quad \forall T < \infty. \tag{6.3.1.16}$$

Proof. (i) The optimal dynamics (6.3.1.5) yields, recalling (6.1.1.3):

$$\hat{A}^\epsilon y^0_{w,\infty}(t; y_0) = \hat{A}^\epsilon e^{At} y_0 + \int_0^t \hat{A}^\epsilon e^{A(t-\tau)} \hat{A}^\delta \hat{A}^{-\delta} B u^0_{w,\infty}(\tau; y_0)\, d\tau$$

$$+ \int_0^t \hat{A}^\epsilon e^{A(t-\tau)} \hat{A}^\rho \hat{A}^{-\rho} G w(\tau)\, d\tau \tag{6.3.1.17}$$

$$\in L_2(0, T; Y), \quad \forall T < \infty, \tag{6.3.1.18}$$

where (6.3.1.18) follows readily from (6.3.1.17) by using the analytic estimate (6.1.1.5) (note that $e^{A(t-\tau)} = e^{-\hat{A}(t-\tau)} e^{\omega(t-\tau)}$), the assumptions (6.1.1.6) and (6.1.1.7), as well as Young's inequality [Sadosky, 1979, p. 26] on the convolution between $\hat{A}^{-\delta} B u^0_{w,\infty}$, $\hat{A}^{-\rho} G w \in L_2(0, T; Y)$ and an L_1-function.

(ii) Part (ii) in (6.3.1.16) is an immediate consequence of part (i) by use of (6.3.1.7), which yields that $P_{0,\infty} \hat{A}^{1-\epsilon}$ has a bounded extension equal to $\{(\hat{A}^*)^{1-\epsilon} P_{0,\infty}\}^*$. □

6.3.2 The Limit Process for $p_{w,T}(\cdot\,; y_0)$ and $r_{w,T}(\cdot)$ as $T \uparrow \infty$; the Equation for $p_{w,\infty}(\cdot\,; y_0)$

We return to identity (6.2.2.7) where we now wish to take the limit as $T \uparrow \infty$. We invoke (6.3.1.6) for one term, but we need to establish a corresponding limit result for $p_{w,T}(\cdot\,; y_0)$. To this end, we define for $y_0 \in Y$:

$$p_{w,\infty}(t; y_0) \equiv \int_t^\infty e^{A^*_{P_{0,\infty}}(\tau - t)} \left[-P_{0,\infty} B u^0_{w,\infty}(\tau; y_0) + R^* R y^0_{w,\infty}(\tau; y_0) \right] d\tau, \tag{6.3.2.1}$$

and we prove that $p_{w,\infty}(t; y_0)$ is well defined.

Proposition 6.3.2.1 *Assume hypotheses (H.1)–(H.6). Then, with reference to (6.3.2.1), we have*

$$p_{w,\infty}(\cdot\,; y_0) : \text{continuous } Y \to C([0, \infty]; Y); \tag{6.3.2.2a}$$

$$\max_{0 \le t \le \infty} \|p_{w,\infty}(t; y_0)\|_Y = \|p_{w,\infty}(\cdot\,; y_0)\|_{C([0,\infty]; Y)} \le C \|y_0\|_Y. \tag{6.3.2.2b}$$

Proof. The proof follows readily from the convolution [Sadosky, 1979, p. 29] of (6.3.2.1) by use of the properties that (the extension of) $P_{0,\infty} B \in \mathcal{L}(U; Y)$ (by duality on (6.3.1.7) with $P_{0,\infty} = P^*_{0,\infty}$) and the uniform stability of $e^{A_{P_{0,\infty}} t}$ by (6.3.1.14), combined with the regularity (6.3.1.1) of $u^0_{w,\infty}$ and $R y^0_{w,\infty}$ in $L_2(0, \infty; \cdot)$. □

Given $p_{w,T}(t; y_0)$ by (6.2.2.1), the legitimacy of calling by $p_{w,\infty}(t; y_0)$ the quantity at the right-hand side of (6.3.2.1) is established next.

Proposition 6.3.2.2 *Assume (H.1)–(H.6). With reference to $p_{w,T}$ in (6.2.2.1) and $p_{w,\infty}$ in (6.3.2.1), we have for $y_0 \in Y$:*

$$\|p_{w,T}(t;y_0) - p_{w,\infty}(t;y_0)\|_Y \to 0, \quad \text{as } T \uparrow \infty, \text{ for each } t \text{ fixed, indeed, uniformly on compact } t\text{-sets;} \tag{6.3.2.3a}$$

$$\|\tilde{p}_{w,T}(\,\cdot\,;y_0) - p_{w,\infty}(\,\cdot\,;y_0)\|_{L_2(0,\infty;Y)} \to 0;$$
$$\|\tilde{p}_{w,T}(\,\cdot\,;y_0) - p_{w,\infty}(\,\cdot\,;y_0)\|_{C([0,\infty];Y)} \to 0, \quad \text{as } T \uparrow \infty, \tag{6.3.2.3b}$$

where $\tilde{p}_{w,T}$ denotes the extension of $p_{w,T}(t;y_0)$ by zero for $t > T$.

Proof.

Step 1 With reference to (6.2.2.26), $p_{w,T}(t;y_0)$ is the unique solution of

$$\dot{p}_{w,T}(t;y_0) = -A^*_{P_{0,\infty}} p_{w,T}(t;y_0) + P_{0,\infty}Bu^0_{w,T}(t;y_0) - R^*Ry^0_{w,T}(t;y_0) \text{ in } [\mathcal{D}(A)]', \tag{6.3.2.4a}$$

$$p_{w,T}(T;y_0) = 0, \tag{6.3.2.4b}$$

as one sees by adding and subtracting $-P_{0,\infty}BB^*p_{w,T}(\,\cdot\,;y_0) = P_{0,\infty}Bu^0_{w,T}(\,\cdot\,;y_0)$ by (6.2.2.9a) and as is given explicitly by the more convenient formula

$$p_{w,T}(t;y_0) = \int_t^T e^{A^*_{P_{0,\infty}}(\tau-t)}\left[-P_{0,\infty}Bu^0_{w,T}(\tau;y_0) + R^*Ry^0_{w,T}(\tau;y_0)\right]d\tau \tag{6.3.2.5a}$$

$$\in C([0,T];Y) \tag{6.3.2.5b}$$

(compare with (6.2.2.1) and with (6.2.2.11)), where the continuity in (6.3.2.5b) again follows by convolution [Sadosky, 1979, p. 29], since (the extension of) $P_{0,\infty}B \in \mathcal{L}(U;Y)$ by (6.3.1.7).

Step 2 Using (6.3.2.1) and (6.3.2.5) with $y_0 \in Y$, we readily estimate for t fixed

$$\begin{aligned}
&\|p_{w,\infty}(t;y_0) - p_{w,T}(t;y_0)\|_Y \\
&\le \left\| \int_t^T e^{A^*_{P_{0,\infty}}(\tau-t)}\left\{-P_{0,\infty}B\left[u^0_{0,\infty}(\tau;y_0) - u^0_{w,T}(\tau;y_0)\right]\right\}d\tau \right\|_Y \\
&\quad + \left\| \int_t^T e^{A^*_{P_{0,\infty}}(\tau-t)}R^*R\left[y^0_{w,\infty}(\tau;y_0) - y^0_{w,T}(\tau;y_0)\right]d\tau \right\|_Y \\
&\quad + \left\| \int_T^\infty e^{A^*_{P_{0,\infty}}(\tau-t)}\left[-P_{0,\infty}Bu^0_{w,\infty}(\tau;y_0) + R^*Ry^0_{w,\infty}(\tau;y_0)\right]d\tau \right\|_Y .
\end{aligned} \tag{6.3.2.6}$$

We now move the norms inside the integrals, use the fact that $P_{0,\infty}B \in \mathcal{L}(U;Y)$ from (6.3.1.7), extend $u^0_{w,T}$ and $y^0_{w,T}$ by zero for $t > T$, and invoke the convergence in

(6.3.1.3) and (6.3.1.4), as well as the uniform stability in (6.3.1.14), to obtain the desired convergence in (6.3.2.3a) and (6.3.2.3b), the latter first in $L_2(0, \infty; Y)$ and then in $C([0, \infty]; Y)$. □

We now draw a few corollaries from the convergence of Proposition 6.3.2.2.

Corollary 6.3.2.3 *Assume (H.1)–(H.6). Then $p_{w,\infty}$ in (6.3.2.1) can be rewritten as*

$$p_{w,\infty}(t; y_0) = \int_t^{t_0} e^{A^*(\tau - t)} R^* R y^0_{w,\infty}(\tau; y_0)\, d\tau + e^{A^*(t_0 - t)} p_{w,\infty}(t_0; y_0), \quad (6.3.2.7)$$

where t_0 is an arbitrary point $t_0 > t$.

Proof. We return to identity (6.2.2.11) and take the limit as $T \uparrow \infty$ on both sides. Invoking the L_2-convergence of $(R^*R)^{\frac{1}{2}} y^0_{w,T}$ to $(R^*R)^{\frac{1}{2}} y^0_{w,\infty}$ in (6.3.1.4) as well as the convergence (6.3.2.3) for $p_{w,T}$ to $p_{w,\infty}$ we obtain (6.3.2.7). □

Corollary 6.3.2.4 *Assume (H.1)–(H.6). With reference to (6.3.2.1), the following ("decoupling") identity holds true for $y_0 \in Y$:*

$$p_{w,\infty}(t; y_0) = P_{0,\infty} y^0_{w,\infty}(t; y_0) + r_{w,\infty}(t) \in L_2(0, \infty; Y), \quad (6.3.2.8)$$

where

$$r_{w,\infty}(t) \equiv \lim_{T \uparrow \infty} r_{w,T}(t) = p_{w,\infty}(t; y_0 = 0) - P_{0,\infty} y^0_{w,\infty}(t; y_0 = 0) \in L_2(0, \infty; Y). \quad (6.3.2.9)$$

Proof. We return to identity (6.2.2.7) and take the $L_2(0, \infty; Y)$-limit as $T \uparrow \infty$ after extending $p_{w,T}$, $y^0_{w,T}$, $r_{w,T}(t)$ by zero for $t > T$. On the left, we use (6.3.2.3b) and on the right we use (6.3.1.6). Thus, (6.3.2.8) follows, with (6.3.2.9) as a consequence of (6.2.2.8), again by (6.3.2.3) and (6.3.1.6). □

We next provide the differential versions of (6.3.2.1) and of (6.3.2.7).

Corollary 6.3.2.5 *Assume (H.1)–(H.6). With reference to (6.3.2.1) we have for all $y_0 \in Y$:*

$$\begin{aligned} \dot{p}_{w,\infty}(t; y_0) &= -(A - BB^* P_{0,\infty})^* p_{w,\infty}(t; y_0) \\ &\quad + P_{0,\infty} B u^0_{w,\infty}(t; y_0) - R^* R y^0_{w,\infty}(t; y_0) \quad \text{in } [\mathcal{D}(A)]' \quad (6.3.2.10a) \\ &= -A^* p_{w,\infty}(t; y_0) - R^* R y^0_{w,\infty}(t; y_0) \quad \text{in } [\mathcal{D}(A)]'. \quad (6.3.2.10b) \end{aligned}$$

Proof. From (6.3.2.1) and (6.3.2.7), we take the inner product with $x \in \mathcal{D}(A)$, we differentiate in t, and we readily obtain the conclusions. □

6.3.3 The Equation for $r_{w,\infty}(t)$

Proposition 6.3.3.1 *Assume (H.1)–(H.6). The function $r_{w,\infty}(t)$ defined by (6.3.2.9) satisfies the equation*

$$\dot{r}_{w,\infty}(t) = -A^*_{P_{0,\infty}} r_{w,\infty}(t) - P_{0,\infty} G w(t) \tag{6.3.3.1}$$

and is thus given explicitly by

$$r_{w,\infty}(t) = \int_t^\infty e^{A^*_{P_{0,\infty}}(\tau - t)} P_{0,\infty} G w(\tau)\, d\tau \in L_2(0,\infty; Y) \cap C([0,\infty]; Y), \tag{6.3.3.2}$$

with terminal condition

$$r_{w,\infty}(\infty) = 0. \tag{6.3.3.3}$$

Proof. The strategy is to start from the defining relation (6.3.2.8); we then differentiate in t by using Eqn. (6.3.2.10) for $\dot{p}_{w,\infty}$, Eqn. (6.1.1.1) for $\dot{y}^0_{w,\infty}$, and the algebraic Riccati equation (ARE) (6.3.1.12) for $P_{0,\infty}$. Thus, we take $x \in \mathcal{D}(A_{P_{0,\infty}})$ and recall from (6.3.1.16) that $P_{0,\infty} A y^0_{w,\infty}(\cdot\,; y_0) \in L_2(0, T; Y)$, $\forall\, T < \infty$, $y_0 \in Y$. Then, we obtain, from (6.3.2.8) a.e. in t,

$$\begin{aligned}
\frac{d}{dt}(r_{w,\infty}(t), x)_Y &= \frac{d}{dt}(p_{w,\infty}(t; y_0), x)_Y - \frac{d}{dt}\big(P_{0,\infty} y^0_{w,\infty}(t; y_0), x\big)_Y \\
&= -(A^*_{P_{0,\infty}} p_{w,\infty}(t; y_0), x)_Y + \big(\cancel{P_{0,\infty} B u^0_{w,\infty}(t; y_0)}, x\big)_Y \\
&\quad - \big(R^* R y^0_{w,\infty}(t; y_0), x\big)_Y - \big(P_{0,\infty} A y^0_{w,\infty}(t; y_0), x\big)_Y \\
&\quad - \big(\cancel{P_{0,\infty} B u^0_{w,\infty}(t; y_0)}, x\big)_Y - \big(P_{0,\infty} G w(t), x\big)_Y,
\end{aligned} \tag{6.3.3.4}$$

where we note that all terms in (6.3.3.4) are well defined a.e. t with $x \in \mathcal{D}(A_{P_{0,\infty}})$, also because $P_{0,\infty} B \in \mathcal{L}(U; Y)$, $P_{0,\infty} G \in \mathcal{L}(V; Y)$ by (6.3.1.7). Next, with $x \in \mathcal{D}(\hat{A}^\epsilon)$, $0 < \epsilon \le 1 - \delta$, and $y^0_{w,\infty} \in \mathcal{D}(\hat{A}^\epsilon)$ a.e. in t by (6.3.1.15), we invoke the ARE (6.3.1.12) to obtain

$$\begin{aligned}
\big(P_{0,\infty} A y^0_{w,\infty} + R^* R y^0_{w,\infty}, x\big)_Y &= \big(y^0_{w,\infty}, A^* P_{0,\infty} x + R^* R x\big)_Y \\
&= -\big(y^0_{w,\infty}, P_{0,\infty} A_{P_{0,\infty}} x\big)_Y && (6.3.3.5\text{a}) \\
\text{(by (6.3.1.12))} \qquad &= -\big(A^*_{P_{0,\infty}} P_{0,\infty} y^0_{w,\infty}, x\big)_Y, \quad x \in \mathcal{D}(\hat{A}^\epsilon). && (6.3.3.5\text{b})
\end{aligned}$$

Validity of (6.3.3.5a), not only for $x \in \mathcal{D}(A_{P_{0,\infty}})$, but more generally for $x \in \mathcal{D}(\hat{A}^\epsilon) \supset \mathcal{D}(A_{P_{0,\infty}})$, $0 < \epsilon \le 1 - \delta$, (6.3.1.10b), follows from

$$P_{0,\infty} A_{P_{0,\infty}} x = P_{0,\infty}[A - BB^* P_{0,\infty}]x \in Y, \quad x \in \mathcal{D}(\hat{A}^\epsilon). \tag{6.3.3.6}$$

Indeed, with $x \in \mathcal{D}(\hat{A}^{\epsilon})$, replacing A by its translation $-\hat{A}$ we have that

$$\begin{aligned} &P_{0,\infty}\hat{A}[I - \hat{A}^{-1}BB^*P_{0,\infty}]x \\ &= P_{0,\infty}\hat{A}^{1-\epsilon}\hat{A}^{\epsilon}[I - \hat{A}^{-1}BB^*P_{0,\infty}]x \\ &= P_{0,\infty}\hat{A}^{1-\epsilon}[I - \hat{A}^{-(1-\epsilon)}BB^*P_{0,\infty}\hat{A}^{-\epsilon}]\hat{A}^{\epsilon}x \in Y \end{aligned} \tag{6.3.3.7}$$

is well defined $\forall \epsilon > 0$ such that $\delta < 1 - \epsilon < 1$, recalling (6.3.1.7) and (6.1.1.6a). Thus, using (6.3.3.6), we see that (6.3.3.5) holds $\forall x \in \mathcal{D}(\hat{A}^{\epsilon})$. Inserting (6.3.3.5b) into (6.3.3.4) we have a.e. in t:

$$\begin{aligned} \frac{d}{dt}(r_{w,\infty}(t), x)_Y &= -\left(A^*_{P_{0,\infty}}\left[p_{w,\infty}(t;y_0) - P_{0,\infty}y^0_{w,\infty}(t;y_0)\right], x\right)_Y \\ &\quad -(P_{0,\infty}Gw(t), x)_Y \\ \text{(by (6.3.2.8))} \quad &= -\left(A^*_{P_{0,\infty}}r_{w,\infty}(t), x\right)_Y - (P_{0,\infty}Gw(t), x)_Y, \end{aligned} \tag{6.3.3.8}$$

first for all $x \in \mathcal{D}(\hat{A}^{\epsilon})$, $\forall \epsilon < 1 - \delta$, and next $\forall x \in Y$ by extension. Then (6.3.3.8) yields (6.3.3.1) as desired. Equation (6.3.3.1) has, plainly, the unique solution given by (6.3.3.2), where $P_{0,\infty}G \in \mathcal{L}(V;Y)$ by (6.3.1.7). Moreover, (6.3.3.2), in turn, implies the terminal condition (6.3.3.3) at $t = \infty$ by virtue of the exponential decay (6.3.1.14). $\square$

6.3.4 Regularity of the Abstract Equation Driven by $A_{P_{0,\infty}} = A - BB^*P_{0,\infty}$

Recalling from (6.3.1.9) that $A_{P_{0,\infty}} = (A - BB^*P_{0,\infty})$, with maximal domain, in this section we consider the abstract equation

$$\dot{\zeta} = (A - BB^*P_{0,\infty})\zeta + Bg \quad \text{on } [\mathcal{D}(A^*)]'; \quad \zeta(0) = \zeta_0 = 0 \tag{6.3.4.1}$$

to be intended in the sense of the map

$$\zeta(t) = (\mathcal{L}_{P_{0,\infty}}g)(t) \equiv \int_0^t e^{A_{P_{0,\infty}}(t-\tau)}Bg(\tau)\,d\tau. \tag{6.3.4.2}$$

Its L_2-adjoint in the sense

$$(\mathcal{L}_{P_{0,\infty}}g, v)_{L_2(0,\infty;Y)} = (g, \mathcal{L}^*_{P_{0,\infty}}v)_{L_2(0,\infty;U)} \tag{6.3.4.3}$$

is given by

$$B^*\eta(t) \equiv (\mathcal{L}^*_{P_{0,\infty}}v)(t) \equiv B^*\int_t^\infty e^{A^*_{P_{0,\infty}}(\tau-t)}v(\tau)\,d\tau \tag{6.3.4.4}$$

and corresponds to

$$\dot{\eta} = (A - BB^*P_{0,\infty})^*\eta + v; \quad \eta(\infty) = 0. \tag{6.3.4.5}$$

The regularity properties of $\mathcal{L}_{P_{0,\infty}}$ and $\mathcal{L}^*_{P_{0,\infty}}$ are stated next.

Theorem 6.3.4.1 *Assume hypotheses (H.1)–(H.6). Then, with reference to (6.3.4.2) and (6.3.4.4), we have*

$$\mathcal{L}_{P_{0,\infty}} : \textit{continuous } L_2(0,\infty;U) \to L_2(0,\infty;Y); \tag{6.3.4.6}$$

$$\mathcal{L}^*_{P_{0,\infty}} : \textit{continuous } L_2(0,\infty;Y) \to L_2(0,\infty;U). \tag{6.3.4.7}$$

Similarly,

$$(\mathcal{W}_{P_{0,\infty}} w)(t) \equiv \int_0^t e^{A_{P_{0,\infty}}(t-\tau)} G w(\tau)\,d\tau$$
$$: \textit{continuous } L_2(0,\infty;V) \to L_2(0,\infty;Y); \tag{6.3.4.8}$$

$$(\mathcal{W}^*_{P_{0,\infty}} f)(t) \equiv G^* \int_t^\infty e^{A^*_{P_{0,\infty}}(\tau-t)} f(\tau)\,d\tau$$
$$: \textit{continuous } L_2(0,\infty;Y) \to L_2(0,\infty;V). \tag{6.3.4.9}$$

Proof. We prove (6.3.4.6) for $\mathcal{L}_{P_{0,\infty}}$ or that

$$(\mathcal{L}_{P_{0,\infty}} u)(t) = \int_0^t (\lambda_0 - A_{P_{0,\infty}}) e^{A_{P_{0,\infty}}(t-\tau)} (\lambda_0 - A_{P_{0,\infty}})^{-1} B u(\tau)\,d\tau \tag{6.3.4.10a}$$

$$: \text{ continuous } L_2(0,\infty;U) \to L_2(0,\infty;Y), \tag{6.3.4.10b}$$

for some fixed λ_0, say, $\lambda_0 > 0$. Since $e^{A_{P_{0,\infty}}t}$ is a s.c. analytic, uniformly stable semigroup by Theorem 6.1.3.1, it suffices to show that

$$(\lambda_0 - A_{P_{0,\infty}})^{-1} B \in \mathcal{L}(U;Y) \tag{6.3.4.11}$$

and to appeal to the standard regularity result of Eqn. (0.4) of Chapter 0.

Indeed, writing

$$\lambda - A_{P_{0,\infty}} = (\lambda - A)[I + R(\lambda, A) B B^* P_{0,\infty}], \tag{6.3.4.12}$$

we obtain

$$(\lambda - A_{P_{0,\infty}})^{-1} B = [I + R(\lambda, A) B B^* P_{0,\infty}]^{-1} R(\lambda, A) B, \tag{6.3.4.13}$$

where the inverse in the square brackets exists as a bounded operator on Y for all λ real sufficiently large, since from (6.1.1.3) and $\lambda > \omega$ we have by (6.1.1.6):

$$\|R(\lambda, A)B\| = \|R(\lambda-\omega, -\hat{A})B\| = \|R(\lambda-\omega, -\hat{A})\hat{A}^{\delta}\hat{A}^{-\delta}B\|$$
$$\le \frac{c}{(\lambda-\omega)^{1-\delta}} \|\hat{A}^{-\delta}B\| \to 0 \ \text{ as } \lambda \to \infty, \tag{6.3.4.14}$$

by the usual resolvent estimate on generators of analytic semigroups [Chapter 2, Eqn. (6.2.1.6)]. Then, (6.3.4.14) used in (6.3.4.13) proves (6.3.4.11) via (6.3.1.7) on $B^* P_{0,\infty} \in \mathcal{L}(Y;U)$ and (6.1.1.6) on $\hat{A}^{-\delta} B \in \mathcal{L}(U;Y)$, $\forall\, \lambda$ large. $\square$

6.3.5 Proof That $B^* r_{w,\infty} \in L_2(0, \infty; U)$ and $B^* P_{0,\infty} y^0_{w,\infty} \in L_2(0, \infty; U)$

We now state the following important result.

Corollary 6.3.5.1 *Assume hypotheses (H.1)–(H.6). Then, with reference to Eqn. (6.3.3.2) we have*

$$B^* r_{w,\infty}(t) \in L_2(0, \infty; Y). \tag{6.3.5.1}$$

Proof. By (6.3.3.2) and (6.3.4.4) we may write

$$B^* r_{w,\infty}(t) = B^* \int_t^\infty e^{A^*_{P_{0,\infty}}(\tau - t)} P_{0,\infty} G w(\tau)\, d\tau \tag{6.3.5.2}$$

$$= \{\mathcal{L}^*_{P_{0,\infty}} P_{0,\infty} G w\}(t) \in L_2(0, \infty; U), \tag{6.3.5.3}$$

where the conclusion follows from (6.3.4.7), since, by (6.3.1.17), $P_{0,\infty} G \in \mathcal{L}(V; Y)$. □

The proof of the next corollary is in the style of the proof of Theorem 2.3.3.4.

Corollary 6.3.5.2 *Assume hypotheses (H.1)–(H.6). With reference to (6.3.2.1) or (6.3.2.7) for $p_{w,\infty}$ and to $u^0_{w,\infty}$ guaranteed by (6.3.1.3), we have*

$$u^0_{w,\infty}(t; y_0) = -B^* p_{w,\infty}(t; y_0) \in L_2(0, \infty; U). \tag{6.3.5.4}$$

Proof.

Step 1 Returning to the operators $\mathcal{L}_{P_{0,\infty}}$, $\mathcal{L}^*_{P_{0,\infty}}$ in (6.3.4.2), (6.3.4.4), we introduce their restrictions on $0 \le t \le T$:

$$(\mathcal{L}_{P_{0,\infty},T}\, g)(t) = \int_0^t e^{A_{P_{0,\infty}}(t-\tau)} B g(\tau)\, d\tau$$
$$: \text{continuous } L_2(0, T; U) \to L_2(0, T; Y); \tag{6.3.5.5}$$

$$(\mathcal{L}^*_{P_{0,\infty},T}\, f)(t) = \int_t^T B^* e^{A^*_{P_{0,\infty}}(\tau - t)} f(\tau)\, d\tau$$
$$: \text{continuous } L_2(0, T; Y) \to L_2(0, T; U), \tag{6.3.5.6}$$

where the indicated regularity is, of course, a consequence of Theorem 6.3.4.1.

The following lemma is proved in Chapter 2, Lemma 2.3.3.2.

Lemma 6.3.5.3 *Let $f_T(t) \in L_2(0, T; Y)$ for all $T > 0$ and extend $f_T(t)$ by zero for $t > T$. Let $f \in L_2(0, \infty; Y)$, and assume that*

$$\|f_T(\cdot) - f(\cdot)\|_{L_2(0,\infty;Y)} \to 0 \quad \text{as } T \uparrow \infty. \tag{6.3.5.7}$$

Then, with reference to the operators $\mathcal{L}^*_{P_{0,\infty}}$, $\mathcal{L}^*_{P_{0,\infty},T}$ defined by (6.3.4.4) and (6.3.5.6) *we have*

$$\left\|\mathcal{L}^*_{P_{0,\infty}} f - \mathcal{L}^*_{P_{0,\infty},T} f_T\right\|_{L_2(0,\infty;U)} \to 0 \quad \text{as } T \uparrow \infty, \tag{6.3.5.8}$$

where $(\mathcal{L}^*_{P_{0,\infty},T} f_T)(t)$ *is extended by zero for* $t > T$.

Step 2 We return to (6.3.2.5) for $p_{w,T}$ and (6.3.2.1) for $p_{w,\infty}$, and we write, by use of (6.3.4.4) and (6.3.5.6),

$$\begin{aligned} &B^* p_{w,\infty}(\cdot\,; y_0) - B^* p_{w,T}(\cdot\,; y_0) \\ &= \mathcal{L}^*_{P_{0,\infty}}\left[-P_{0,\infty} B u^0_{w,\infty}(\cdot\,; y_0) + R^* R y^0_{w,\infty}(\cdot\,; y_0)\right] \\ &\quad - \mathcal{L}^*_{P_{0,\infty},T}\left[-P_{0,\infty} B u^0_{w,T}(\cdot\,; y_0) + R^* R y^0_{w,T}(\cdot\,; y_0)\right]. \end{aligned} \tag{6.3.5.9}$$

We then extend $u^0_{w,T}$ and $y^0_{w,T}$ by zero for $t > T$, and we recall (6.3.1.7) on $P_{0,\infty} B \in \mathcal{L}(U; Y)$, as well as the convergence results (6.3.1.3) and (6.3.1.4). All this then allows us to invoke Lemma 6.3.5.3 in (6.3.5.9) and conclude that

$$\|B^* p_{w,\infty}(\cdot\,; y_0) - B^* p_{w,T}(\cdot\,; y_0)\|_{L_2(0,\infty;U)} \to 0. \tag{6.3.5.10}$$

Step 3 We next return to (6.2.2.9a): $u^0_{w,T} = -B^* p_{w,T}$ and take the $L_2(0,\infty;U)$-limit as $T \uparrow \infty$. On the left-hand side we use (6.3.1.3), and on the right-hand side we use (3.5.10), thus obtaining (6.3.5.4). □

Corollary 6.3.5.4 *Assume (H.1)–(H.6). We have*

$$B^* P_{0,\infty} y^0_{w,\infty}(t; y_0) \in L_2(0,\infty;U). \tag{6.3.5.11}$$

Proof. We return to identity (6.3.2.8) and write

$$\begin{aligned} B^* P_{0,\infty} y^0_{w,\infty}(t; y_0) &= B^* p_{w,\infty}(t; y_0) - B^* r_{w,\infty}(t) \\ \text{(by (6.3.5.4))} \qquad &= -u^0_{w,\infty}(t; y_0) - B^* r_{w,\infty}(t) \in L_2(0,\infty;U), \end{aligned} \tag{6.3.5.12}$$

and the conclusion (6.3.5.11) follows from (6.3.5.12) via (6.3.1.3) and (6.3.5.1). □

6.3.6 The Equation for $y^0_{w,\infty}$ (Stable Form)

In addition to the equation for the optimal dynamics

$$\dot{y}^0_{w,\infty}(t; y_0) = A y^0_{w,\infty}(t; y_0) + B u^0_{w,\infty}(t; y_0) + G w(t) \quad \text{on } [\mathcal{D}(A^*)]', \tag{6.3.6.1}$$

or its mild form, (6.1.1.9), we shall now provide another representation, as a consequence of Corollary 6.3.5.4, which is more useful in describing the behavior at infinity.

Proposition 6.3.6.1 *Assume (H.1)–(H.6). The optimal solution $y^0_{w,\infty}(t; y_0)$ satisfies the following equation with stable generator:*

$$\dot{y}^0_{w,\infty}(t; y_0) = (A - BB^* P_{0,\infty}) y^0_{w,\infty}(t; y_0) - BB^* r_{w,\infty}(t) + Gw(t) \quad \text{on } [\mathcal{D}(A^*)]', \tag{6.3.6.2}$$

where, by (6.3.5.11) and (H.2) = (6.1.1.6),

$$(A - BB^* P_{0,\infty}) y^0_{w,\infty}(t; y_0) = A(I - A^{-1}BB^* P_{0,\infty}) y^0_{w,\infty}(t; y_0) \in [\mathcal{D}(A^*)]', \tag{6.3.6.3}$$

and by (6.3.5.1) and (H.2) = (6.1.1.6),

$$BB^* r_{w,\infty}(t) = AA^{-1}BB^* r_{w,\infty}(t) \in L_2(0, \infty; [\mathcal{D}(A^*)]'). \tag{6.3.6.4}$$

The solution of (6.3.6.2) may then be written in the following stable form:

$$y^0_{w,\infty}(t; y_0) = e^{A_{P_{0,\infty}} t} y_0 + \int_0^t e^{A_{P_{0,\infty}}(t-\tau)} [-BB^* r_{w,\infty}(\tau) + Gw(\tau)]\, d\tau \tag{6.3.6.5a}$$

$$= e^{A_{P_{0,\infty}} t} y_0 - \left\{\mathcal{L}_{P_{0,\infty}} B^* r_{w,\infty}\right\}(t) + \left\{\mathcal{W}_{P_{0,\infty}} w\right\}(t) \in L_2(0, \infty; Y). \tag{6.3.6.5b}$$

Proof. We add and subtract $BB^* P_{0,\infty} y^0_{w,\infty}(t; y_0) \in L_2(0, \infty; [\mathcal{D}(A^*)]')$ to the right-hand side of (6.3.6.1) and use (6.3.5.12) to obtain (6.3.6.2). Clearly, (6.3.6.5a) gives the unique solution of (6.3.6.2), and the regularity (6.3.6.5b) is a consequence of the formula (6.3.6.5a) by use of Theorem 6.3.4.1, Eqns. (6.3.4.6) and (6.3.4.8), via (6.3.5.1) for $B^* r_{w,\infty}$. Then, (6.3.6.3) follows from (6.3.6.5b) on $y^0_{w,\infty}$. □

6.4 Collection of Explicit Formulae for $p_{w,\infty}$, $r_{w,\infty}$, and $y^0_{w,\infty}$ in Stable Form

For convenience, we collect in this section the relevant formulae for $p_{w,\infty}$, $r_{w,\infty}$, and $y^0_{w,\infty}$ obtained in the preceding section that display a stable generator (stable form). In connection with future developments, it is expedient to rewrite them by introducing suitable operators. To this end, recalling the stable generator $A_{P_{0,\infty}}$ in (6.3.1.10), we let (see Young's Inequality [Sadosky, 1979, p. 29])

$$\left(\mathcal{K}_{P_{0,\infty}} f\right)(t) = \int_0^t e^{A_{P_{0,\infty}}(t-\tau)} f(\tau)\, d\tau \tag{6.4.1a}$$

$$: \text{ continuous } L_2(0, \infty; Y) \to L_q(0, \infty; Y) \cap C([0, \infty]; Y), \quad \forall q \geq 2; \tag{6.4.1b}$$

$$\left(\mathcal{K}^*_{P_{0,\infty}} v\right)(t) = \int_t^\infty e^{A^*_{P_{0,\infty}}(\tau-t)} v(\tau)\, d\tau \tag{6.4.2a}$$

$$: \text{ continuous } L_2(0, \infty; Y) \to L_q(0, \infty; Y) \cap C([0, \infty]; Y), \quad \forall q \geq 2. \tag{6.4.2b}$$

Recalling $\mathcal{L}_{P_{0,\infty}}$, $\mathcal{W}_{P_{0,\infty}}$, $\mathcal{L}^*_{P_{0,\infty}}$ and $\mathcal{W}^*_{P_{0,\infty}}$ introduced in (6.3.4.2), (6.3.4.8), (6.3.4.4), and (6.3.4.9) we have

$$\mathcal{L}_{P_{0,\infty}} = \mathcal{K}_{P_{0,\infty}} B; \qquad \mathcal{W}_{P_{0,\infty}} = \mathcal{K}_{P_{0,\infty}} G : \text{continuous } L_2(0,\infty;\ \cdot) \to L_2(0,\infty; Y); \tag{6.4.3}$$

$$\mathcal{L}^*_{P_{0,\infty}} = B^* \mathcal{K}^*_{P_{0,\infty}}; \qquad \mathcal{W}^*_{P_{0,\infty}} = G^* \mathcal{K}^*_{P_{0,\infty}} : \text{continuous } L_2(0,\infty; Y) \to L_2(0,\infty;\ \cdot), \tag{6.4.4}$$

where $\cdot$ stands for either U (if B is involved) or V (if G is involved).

Formula for $\boldsymbol{p_{w,\infty}}$ In the new notation (6.4.1a) (left), we can rewrite formula (6.3.2.1) for $p_{w,\infty}$ as (recall $P_{0,\infty}B \in \mathcal{L}(U; Y)$ by (6.3.1.7))

$$p_{w,\infty}(t; y_0) = \int_t^\infty e^{A^*_{P_{0,\infty}}(\tau - t)} \big[-P_{0,\infty} B u^0_{w,\infty}(\tau; y_0) + R^* R y^0_{w,\infty}(\tau; y_0) \big]\, d\tau \tag{6.4.5a}$$

$$= \big\{ -\mathcal{K}^*_{P_{0,\infty}} P_{0,\infty} B u^0_{w,\infty}(\,\cdot\,; y_0) + \mathcal{K}^*_{P_{0,\infty}} R^* R y^0_{w,\infty}(\,\cdot\,; y_0) \big\}(t) \tag{6.4.5b}$$

$$\in C([0,\infty]; Y) \cap L_q(0,\infty; Y), \quad \forall q \ge 2.$$

Formula for $\boldsymbol{r_{w,\infty}(t)}$ By (6.4.1a) (left) formula (6.3.3.2) is rewritten as

$$r_{w,\infty}(t) = \int_t^\infty e^{A^*_{P_{0,\infty}}(\tau - t)} P_{0,\infty} G w(\tau)\, d\tau$$

$$= \big\{ K^*_{P_{0,\infty}} P_{0,\infty} G w \big\}(t) \in L_q(0,\infty; Y) \cap C([0,\infty]; Y), \quad \forall q \ge 2, \tag{6.4.6}$$

recalling that $P_{0,\infty}G \in \mathcal{L}(V; Y)$ by (6.3.1.7).

Formula for $\boldsymbol{y^0_{w,\infty}(t)}$ By (6.4.1a) (right) and (6.3.4.2) [or (6.4.3)], formula (6.3.6.5a) is rewritten now as

$$y^0_{w,\infty}(t; y_0) = e^{A_{P_{0,\infty}} t} y_0 + \int_0^t e^{A_{P_{0,\infty}}(t-\tau)} [-BB^* r_{w,\infty}(\tau) + G w(\tau)]\, d\tau \tag{6.4.7a}$$

$$= e^{A_{P_{0,\infty}} t} y_0 - \big\{ \mathcal{L}_{P_{0,\infty}} B^* r_{w,\infty}(\,\cdot\,) \big\}(t) + \big\{ \mathcal{W}_{P_{0,\infty}} w(\,\cdot\,) \big\}(t) \tag{6.4.7b}$$

$$\in L_2(0,\infty; Y).$$

Remark 6.4.1 These stable dynamics will be critically used below in Section 6.6 to define explicitly the critical value γ_c and in Section 6.7 to study the problem of maximization of $J^0_{w,\infty}(y_0)$ over $w \in L_2(0,\infty; Y)$ directly over $0 \le t \le \infty$ (unlike the minimization of J over u as in Sections 6.2 and 6.3). In this way, explicit formulae for all quantities involved will be obtained, which will involve $P_{0,\infty}$.

Using (6.4.6) in identity (6.3.2.8) for $p_{w,\infty}$, and in the expression (6.4.7b) for $y^0_{w,\infty}$, we obtain respectively, via (6.4.4),

$$p_{w,\infty}(\,\cdot\,; y_0) = P_{0,\infty} y^0_{w,\infty}(\,\cdot\,; y_0) + \mathcal{K}^*_{P_{0,\infty}} P_{0,\infty} G w \in C([0, \infty]; Y); \tag{6.4.8}$$

$$y^0_{w,\infty}(\,\cdot\,; y_0) = e^{A_{P_{0,\infty}}\cdot} y_0 - \mathcal{L}_{P_{0,\infty}} \mathcal{L}^*_{P_{0,\infty}} P_{0,\infty} G w + \mathcal{W}_{P_{0,\infty}} w \in L_2(0, \infty; Y). \tag{6.4.9}$$

Notice that (6.4.9) gives the optimal solution $y^0_{w,\infty}$ *explicitly in terms of the problem data* via the (unique) Riccati operator $P_{0,\infty}$ for $w \equiv 0$. Then, (6.4.9) inserted into (6.4.8) provides likewise an explicit expression for $p_{w,\infty}$, hence for the optimal control $u^0_{w,\infty} = -B^* p_{w,\infty}$ (see (6.3.5.4)), *directly in terms of the problem data*. The same conclusion will be reached in the next section for the optimal cost $J_{w,\infty}$ when $y_0 = 0$.

6.5 Explicit Expression for the Optimal Cost $J^0_{w,\infty}(y_0 = 0)$ as a Quadratic Term

We introduce the bounded, self-adjoint operators in $L_2(0, \infty; V)$ by

$$S \equiv G^* P_{0,\infty} \mathcal{L}_{P_{0,\infty}} \mathcal{L}^*_{P_{0,\infty}} P_{0,\infty} G - \left[\mathcal{W}^*_{P_{0,\infty}} P_{0,\infty} G + G^* P_{0,\infty} \mathcal{W}_{P_{0,\infty}}\right] \tag{6.5.1a}$$

$$\in \mathcal{L}(L_2(0, \infty; Y)); \tag{6.5.1b}$$

$$E_\gamma \equiv \gamma^2 I + S \in \mathcal{L}(L_2(0, \infty; V)). \tag{6.5.2}$$

By (6.3.4.6), (6.3.4.7), (6.4.1b), and (6.4.2b), we have that S is bounded on $L_2(0, \infty; Y)$ and is explicitly expressed in terms of the systems data. It will be shown below to be negative definite: $S \leq 0$. The goal of this section is to demonstrate

Theorem 6.5.1 *Assume (H.1)–(H.6). With reference to (6.5.2), the optimal cost $J^0_{w,\infty}(y_0 = 0)$ corresponding to a fixed $w \in L_2(0, \infty; V)$ and $y_0 = 0$ is*

$$J^0_{w,\infty}(y_0 = 0) = J\left(u^0_{w,\infty}(\,\cdot\,; y_0 = 0),\ y^0_{w,\infty}(\,\cdot\,; y_0 = 0)\right) = -(E_\gamma w, w)_{L_2(0,\infty; V)}. \tag{6.5.3}$$

Proof of Theorem 6.5.1 The proof is broken into two propositions.

Step 1. Proposition 6.5.2 Assume (H.1)–(H.6). Then, with reference to (6.5.1), we have:

(i)

$$\begin{aligned}(w, G^* P_{0,\infty} e^{A_{P_{0,\infty}}\cdot} y_0)_{L_2(0,\infty; V)} &- (Sw, w)_{L_2(0,\infty; V)} \\ &= (Gw, p_{w,\infty}(\,\cdot\,; y_0))_{L_2(0,\infty; Y)}.\end{aligned} \tag{6.5.4}$$

(ii) Specializing (6.5.4) to $y_0 = 0$:

$$-(Sw, w)_{L_2(0,\infty; V)} = (Gw, p_{w,\infty}(\,\cdot\,; y_0 = 0))_{L_2(0,\infty; Y)}. \tag{6.5.5}$$

Proof of Proposition 6.5.2 All inner products below are in $L_2(0,\infty)$. By recalling (6.4.8), we obtain

$$(Gw, p_{w,\infty}(\,\cdot\,;y_0)) = \big(Gw, P_{0,\infty}y^0_{w,\infty}(\,\cdot\,;y_0)\big) + \big(Gw, \mathcal{K}^*_{P_{0,\infty}}P_{0,\infty}Gw\big). \quad (6.5.6)$$

Next, recalling (6.4.9) for $y^0_{w,\infty}$ in (6.5.6), as well as (6.4.4), we obtain from (6.5.6)

$$\begin{aligned}
&(Gw, p_{w,\infty}(\,\cdot\,;y_0)) - (w, G^*P_{0,\infty}e^{A_{P_{0,\infty}}\cdot}y_0)\\
\text{(by (6.4.4))}\quad &= (Gw, -P_{0,\infty}\mathcal{L}_{P_{0,\infty}}\mathcal{L}^*_{P_{0,\infty}}P_{0,\infty}Gw)\\
&\quad + (Gw, P_{0,\infty}\mathcal{W}_{P_{0,\infty}}w) + (w, \mathcal{W}^*_{P_{0,\infty}}P_{0,\infty}Gw)\\
&= (w, [-G^*P_{0,\infty}\mathcal{L}_{P_{0,\infty}}\mathcal{L}^*_{P_{0,\infty}}P_{0,\infty}G]w)\\
&\quad + (w, [G^*P_{0,\infty}\mathcal{W}_{P_{0,\infty}} + \mathcal{W}^*_{P_{0,\infty}}P_{0,\infty}G]w)\\
\text{(by (6.5.1a))}\quad &= -(w, Sw)_{L_2(0,\infty;V)}, \qquad (6.5.7)
\end{aligned}$$

as desired, where in the last step we have invoked (6.5.1a). We note explicitly that all terms above are well defined. Equation (6.5.7) proves (6.5.4). □

Step 2. Proposition 6.5.3 Assume (H.1)–(H.6). Let $y_0 = 0$. Then

$$\begin{aligned}
&J^0_{w,\infty}(y_0 = 0) + \gamma^2(w, w)_{L_2(0,\infty;V)}\\
&= \big(y^0_{w,\infty}(\,\cdot\,;y_0 = 0), R^*Ry^0_{w,\infty}(\,\cdot\,;y_0 = 0)\big)_{L_2(0,\infty;Y)}\\
&\quad + \big(u^0_{w,\infty}(\,\cdot\,;y_0 = 0), u^0_{w,\infty}(\,\cdot\,;y_0 = 0)\big)_{L_2(0,\infty;U)} \qquad (6.5.8)\\
&= (Gw, p_{w,\infty}(\,\cdot\,;y_0 = 0))_{L_2(0,\infty;Y)} \qquad (6.5.9)\\
&= -(Sw, w)_{L_2(0,\infty;V)}. \qquad (6.5.10)
\end{aligned}$$

Thus, S is nonpositive definite: $S \le 0$.

Proof of Proposition 6.5.3 The equality in (6.5.8) is just the definition of $J^0_{w,\infty}$ $(y_0 = 0)$. To prove equality (6.5.9), we eliminate $R^*Ry^0_{w,\infty}$ from the identity (6.4.5b), that is, we use

$$p_{w,\infty} + \mathcal{K}^*_{P_{0,\infty}}P_{0,\infty}Bu^0_{w,\infty} = \mathcal{K}^*_{P_{0,\infty}}R^*Ry^0_{w,\infty}, \qquad (6.5.11)$$

and we recall expression (6.4.9) for $y^0_{w,\infty}(\,\cdot\,;y_0 = 0)$, here rewritten by (6.4.4) and (6.4.1a) as

$$y^0_{w,\infty}(\,\cdot\,;y_0 = 0) = \mathcal{K}_{P_{0,\infty}}\big[-B\mathcal{L}^*_{P_{0,\infty}}P_{0,\infty}Gw + Gw\big]. \qquad (6.5.12)$$

We then compute, by (6.5.12), with $y_0 = 0$ throughout (though this is not explicitly noted),

$$\begin{aligned}
\big(y^0_{w,\infty}, R^*Ry^0_{w,\infty}\big) &= \big([-B\mathcal{L}^*_{P_{0,\infty}}P_{0,\infty}Gw + Gw], \mathcal{K}^*_{P_{0,\infty}}R^*Ry^0_{w,\infty}\big)\\
\text{(by (6.5.11))}\quad &= \big(\big[-B\mathcal{L}^*_{P_{0,\infty}}P_{0,\infty}Gw + Gw\big], p_{w,\infty}\big)\\
&\quad + \big(\mathcal{K}_{P_{0,\infty}}[-B\mathcal{L}^*_{P_{0,\infty}}P_{0,\infty}Gw + Gw], P_{0,\infty}Bu^0_{w,\infty}\big)\\
\text{(by (6.5.12))}\quad &= -\big(\mathcal{L}^*_{P_{0,\infty}}P_{0,\infty}Gw, B^*p_{w,\infty}\big) + (Gw, p_{w,\infty})\\
&\quad + \big(y^0_{w,\infty}, P_{0,\infty}Bu^0_{w,\infty}\big). \qquad (6.5.13)
\end{aligned}$$

Notice that each term in (6.5.13) is well defined. Adding and subtracting $(u^0_{w,\infty}, u^0_{w,\infty}) = (B^* p_{w,\infty}, B^* p_{w,\infty})$ (by (6.3.5.4)) to both sides of (6.5.13), we obtain, by (6.4.4), still with $y_0 = 0$ throughout,

$$\begin{aligned}
&\left(y^0_{w,\infty}, R^*Ry^0_{w,\infty}\right) + \left(u^0_{w,\infty}, u^0_{w,\infty}\right) \\
&\quad = -\left(B^*\mathcal{K}^*_{P_{0,\infty}} P_{0,\infty} Gw, B^* p_{w,\infty}\right) + (Gw, p_{w,\infty}) \\
&\qquad + \left(y^0_{w,\infty}, P_{0,\infty} B u^0_{w,\infty}\right) + (B^* p_{w,\infty}, B^* p_{w,\infty})
\end{aligned} \tag{6.5.14}$$

(replacing $p_{w,\infty} = P_{0,\infty} y^0_{w,\infty} + \mathcal{K}^*_{P_{0,\infty}} P_{0,\infty} Gw$ by (6.4.8) in the left term in the last inner product of (6.5.14)), after a cancellation, and recalling (6.3.5.4))

$$= (Gw, p_{w,\infty}) + \left(y^0_{w,\infty}, P_{0,\infty} B u^0_{w,\infty}\right) + \left(B^* P_{0,\infty} y^0_{w,\infty}, B^* p_{w,\infty}\right) \tag{6.5.15}$$

(using now $u^0_{w,\infty} = -B^* p_{w,\infty}$ from (6.3.5.4) results in a cancellation)

$$= (Gw, p_{w,\infty}), \tag{6.5.16}$$

and (6.5.9) is proved, as desired.

Finally, (6.5.9) implies (6.5.10) by recalling (6.5.5) of Proposition 6.5.2. □

To complete the proof of Theorem 6.5.1, all we need to do is to recall the definition (6.5.2) of E_γ in (6.5.10). □

6.6 Definition of the Critical Value γ_c. Coercivity of E_γ for $\gamma > \gamma_c$

On the basis of Theorem 6.5.1, we now define the critical value, $\gamma_c \geq 0$, of the parameter γ in terms of the problem data by

$$\gamma_c^2 = \sup_{\|w\|=1} (-Sw, w)_{L_2(0,\infty;V)} = -\inf_{\|w\|=1} (Sw, w)_{L_2(0,\infty;V)}, \tag{6.6.1}$$

where $(-S) \geq 0$ is the nonnegative, self-adjoint operator (by Proposition 6.5.3) in $\mathcal{L}(L_2(0,\infty; V))$, defined explicitly in terms of the problem data by (6.5.1). Thus, the $\inf(-Sw, w)$ over all w with $L_2(0,\infty; V)$-norm $\|w\| = 1$ gives the lowest point (achieved) of the spectrum of the nonnegative, self-adjoint operator $(-S)$.

Proposition 6.6.1 *Assume (H.1)–(H.6). The bounded, self-adjoint operator E_γ in (6.5.2) satisfies*

$$(E_\gamma w, w)_{L_2(0,\infty;V)} \geq \left(\gamma^2 - \gamma_c^2\right) \|w\|^2_{L_2(0,\infty;V)} \tag{6.6.2}$$

and is strictly positive if and only if $\gamma > \gamma_c$, in which case $E_\gamma^{-1} \in \mathcal{L}(L_2(0,\infty; V))$.

Proof. The conclusion follows from (6.5.2) and the definition (6.6.1). □

6.7 Maximization of $J^0_{w,\infty}$ over w Directly on $[0,\infty]$ for $\gamma > \gamma_c$. Characterization of Optimal Quantities

Henceforth, we shall let $\gamma > \gamma_c$ unless otherwise stated. We return to the optimal $J^0_{w,\infty}(y_0)$ in (6.3.1.13) for fixed $w \in L_2(0,\infty; V)$. In this section, we consider the following optimal problem:

$$\text{maximize } J^0_{w,\infty}(y_0), \quad \text{equivalently minimize } -J^0_{w,\infty}(y_0), \qquad \text{over all } w \in L_2(0,\infty; V). \tag{6.7.1}$$

We shall first show, on the basis of the results of Sections 6.5 and 6.6, that a unique (optimal) solution $w^*(\cdot\,; y_0)$ exists for problem (6.7.1). Next, taking advantage that now $y^0_{w,\infty}$ is written in stable form as in (6.4.7), we shall study problem (6.7.1) directly over the infinite time interval ($T = \infty$) (unlike the inf problem in u in Section 6.3) to characterize the optimal solution w^* via, say, Lagrange multiplier theory and Liusternik's theorem [Luenberger, 1969, p. 243] (or "completing the square") as in Chapter 2, Eqns. (2.4.1.16)–(2.4.1.23) and Remark 2.4.1.1.

Theorem 6.7.1 *Assume (H.1)–(H.6). Let $\gamma > \gamma_c$ (see (6.6.1)).*

(i) For each $y_0 \in Y$, there exists a unique optimal solution of problem (6.7.1), denoted by $w^*(\cdot\,; y_0)$, which is characterized by

$$\gamma^2 w^*(\cdot\,; y_0) = G^* p^*(\cdot\,; y_0) \in L_2(0,\infty; V), \tag{6.7.2}$$

where we set henceforth

$$\begin{gathered} p^*(\cdot\,; y_0) = p_{w=w^*,\infty}(\cdot\,; y_0); \quad r^*(\cdot) = r_{w=w^*,\infty}(\cdot); \\ y^*(\cdot\,; y_0) = y_{w=w^*,\infty}(\cdot\,; y_0); \end{gathered} \tag{6.7.3}$$

$$u^*(\cdot\,; y_0) = u_{w=w^*,\infty}(\cdot\,; y_0) = -B^* p^*(\cdot\,; y_0) \in L_2(0,\infty; U). \tag{6.7.4}$$

(ii) Thus, from (6.4.5)–(6.4.9) specialized for $w = w^*$, we obtain

$$\begin{aligned} p^*(t; y_0) &= \int_t^\infty e^{A^*_{P_{0,\infty}}(\tau - t)}[-P_{0,\infty} B u^*(\tau; y_0) + R^* R y^*(\tau; y_0)]\, d\tau \\ &= \{\mathcal{K}^*_{P_{0,\infty}}[-P_{0,\infty} B u^*(\cdot\,; y_0) + R^* R y^*(\cdot\,; y_0)]\}(t) && \text{(6.7.5a)} \\ &\in C([0,\infty]; Y) \cap L_q(0,\infty; Y), \quad \forall q \geq 2; && \text{(6.7.5b)} \end{aligned}$$

$$\begin{aligned} r^*(t; y_0) &= \int_t^\infty e^{A^*_{P_{0,\infty}}(\tau - t)} P_{0,\infty} G w^*(\tau; y_0)\, d\tau \\ &= \left\{\mathcal{K}^*_{P_{0,\infty}} P_{0,\infty} G w^*(\cdot\,; y_0)\right\}(t) && \text{(6.7.6a)} \\ &\in L_q(0,\infty; Y) \cap C([0,\infty]; Y), \quad \forall q \geq 2; && \text{(6.7.6b)} \end{aligned}$$

$$y^*(t; y_0) = e^{A_{P_{0,\infty}} t} y_0 + \int_0^t e^{A_{P_{0,\infty}}(t-\tau)}[-BB^* r^*(\tau; y_0) + Gw^*(\tau; y_0)]\, d\tau \tag{6.7.7a}$$

$$= e^{A_{P_{0,\infty}} t} y_0 + \left\{-\mathcal{L}_{P_{0,\infty}} B^* r^*(\,\cdot\,; y_0) + \mathcal{K}_{P_{0,\infty}} Gw^*(\,\cdot\,; y_0)\right\}(t) \tag{6.7.7b}$$

(by (6.7.6), (6.4.1), and (6.4.3))

$$= e^{A_{P_{0,\infty}} t} y_0 - \left\{\mathcal{L}_{P_{0,\infty}} \mathcal{L}^*_{P_{0,\infty}} P_{0,\infty} Gw^*(\,\cdot\,; y_0) + \mathcal{W}_{P_{0,\infty}} w^*(\,\cdot\,; y_0)\right\}(t) \tag{6.7.7c}$$

$$\in L_2(0, \infty; Y); \tag{6.7.7d}$$

$$p^*(\,\cdot\,; y_0) = P_{0,\infty} y^*(\,\cdot\,; y_0) + r^*(\,\cdot\,; y_0) \tag{6.7.8a}$$

(by (6.7.6))
$$= P_{0,\infty} y^*(\,\cdot\,; y_0) + \mathcal{K}^*_{P_{0,\infty}} P_{0,\infty} Gw^*(\,\cdot\,; y_0) \tag{6.7.8b}$$

$$\in L_2(0, \infty; Y); \tag{6.7.8c}$$

$$J^*(y_0) = J^0_{w=w^*,\infty}(y_0) \geq J^0_{w=0,\infty}(y_0) = (P_{0,\infty} y_0, y_0) \geq 0; \tag{6.7.9}$$

$$u^*(t; y_0) = -B^* p^*(t; y_0)$$

$$= -B^* P_{0,\infty} y^*(t; y_0) - Br^*(t; y_0) \in L_2(0, \infty; U) \tag{6.7.10a}$$

$$= -B^* P_{0,\infty} e^{A_{P_{0,\infty}} \cdot} y_0 + \big[B^* P_{0,\infty} \mathcal{L}_{P_{0,\infty}} \mathcal{L}^*_{P_{0,\infty}} P_{0,\infty} G + B^* P_{0,\infty} \mathcal{W}_{P_{0,\infty}} - \mathcal{L}^*_{P_{0,\infty}} P_{0,\infty} G\big] w^*(\,\cdot\,; y_0)), \tag{6.7.10b}$$

where Eqn. (6.7.10a) follows from (6.7.4) and (6.7.8), while (6.7.10b) follows from (6.7.10a) via (6.7.6a) and (6.7.7c).

Proof. The optimal cost $J^0_{w,\infty}(y_0)$ is, as noted in (6.3.1.13), given by

$$J^0_{w,\infty}(y_0) = J^0_{w,\infty}(y_0 = 0) + J^0_{w=0,\infty}(y_0) + \mathcal{X}_{w,\infty}(y_0), \tag{6.7.11}$$

where, on the right of (6.7.11), the second term is constant in w, the third term is linear in w, so that

$$\|\mathcal{X}_{w,\infty}(y_0)\| \leq \epsilon \|w\|^2_{L_2(0,\infty; V)} + C_\epsilon \|y_0\|^2_Y, \quad \forall \epsilon > 0, \tag{6.7.12}$$

while the first term satisfies (6.5.3) with $(E_\gamma w, w)_{L_2(0,\infty; V)}$, a positive definite quadratic form if and only if $\gamma > \gamma_c$ by Proposition 6.6.1. Thus, all this leads to the lower bound

$$-J^0_{w,\infty}(y_0) \geq \left[\gamma^2 - \left(\gamma_c^2 + \epsilon\right)\right] \|w\|^2_{L_2(0,\infty; V)} - J^0_{w=0}(y_0) - c_\epsilon \|y_0\|^2_Y, \quad y_0 \in Y, \tag{6.7.13}$$

for the quadratic functional $-J^0_{w,\infty}(y_0)$. Thus, as a consequence, $-J^0_{w,\infty}(y_0)$ admits a unique minimum in w for $\gamma > \gamma_c$; call it $w^* = w^*(\,\cdot\,; y_0) \in L_2(0, \infty; V)$. To characterize it, we now take advantage of the stable dynamics (6.4.7b) for $y^0_{w,\infty}$, so that (unlike in Section 6.3) we can study the optimization problem (6.7.1) directly over the infinite time interval $[0, \infty]$; either by Lagrange multiplier (Liusternik's theorem [Luenberger, 1969, p. 243]) as in Chapter 2, Section 2.4.1 or else by "completing the square" [Chapter 2, Remark 2.4.1.1]. We follow the first approach. For $\{y^0_{w,\infty}, r_{w,\infty}, \lambda\} \in [L_2(0, \infty; Y)]^3$ considered all as free parameters, we introduce the Lagrangean after recalling (6.3.5.4), (6.3.2.8), for $u^0_{w,\infty}$ and (6.4.7b) for $y^0_{w,\infty}$:

$$\begin{aligned} L\big(y^0_{w,\infty}, r_{w,\infty}, \lambda\big) = \frac{1}{2}\big\{\big(R^*Ry^0_{w,\infty}, y^0_{w,\infty}\big)_{L_2(0,\infty;Y)} - \gamma^2(w, w)_{L_2(0,\infty;V)} \\ + \big(B^*\big[P_{0,\infty}y^0_{w,\infty} + r_{w,\infty}\big], B^*\big[P_{0,\infty}y^0_{w,\infty} + r_{w,\infty}\big]\big)_{L_2(0,\infty;U)}\big\} \\ + \big(\lambda, y^0_{w,\infty} - e^{A_{P_{0,\infty}}\cdot}y_0 + \mathcal{L}_{P_{0,\infty}}B^*r_{w,\infty} - \mathcal{W}_{P_{0,\infty}}w\big)_{L_2(0,\infty;Y)}, \end{aligned} \tag{6.7.14}$$

which is always finite for all $w \in L_2(0, \infty; V)$. Taking the variation with respect to $y^0_{w,\infty}$, we obtain

$$\begin{aligned} L_{y^0_{w,\infty}} = 0 : \big(R^*Ry^0_{w,\infty}, \delta y^0_{w,\infty}\big) \\ + \big(B^*\big[P_{0,\infty}y^0_{w,\infty} + r_{w,\infty}\big], B^*P_{0,\infty}\delta y^0_{w,\infty}\big) + \big(\lambda, \delta y^0_{w,\infty}\big) = 0, \end{aligned} \tag{6.7.15}$$

and recalling $-u^0_{w,\infty} = B^*[P_{0,\infty}y^0_{w,\infty} + r_{w,\infty}]$ from (6.3.5.4) and (6.3.2.8), we obtain, from (6.7.15),

$$\lambda^* = \big[P_{0,\infty}Bu^0_{w,\infty} - R^*Ry^0_{w,\infty}\big]_{w=w^*} = P_{0,\infty}Bu^* - R^*Ry^*. \tag{6.7.16}$$

Taking the variation with respect to w from (6.7.14), we get

$$L_w = 0 : -\gamma^2(w, \delta w) - (\lambda, \mathcal{W}_{P_{0,\infty}}\delta w) = 0; \tag{6.7.17}$$

hence, by (6.7.17), (6.7.16), and (6.4.4),

$$\gamma^2 w^* = -\mathcal{W}^*_{P_{0,\infty}}\lambda^* = -G^*\mathcal{K}^*_{P_{0,\infty}}[P_{0,\infty}Bu^* - R^*Ry^*] = G^*p^*, \tag{6.7.18}$$

where in the last step we have used (6.4.5b) for $w = w^*$. Thus, (6.7.2) follows from (6.7.18), where we have noted the dependence on y_0. □

It is instructive to compare the next two identities of the following corollary with, respectively, (6.21.8) for w^* and (6.21.9) for u^*, in the case where e^{At} is uniformly stable, or say (6.2.1.2a) for $T < \infty$.

Corollary 6.7.2 *Assume (H.1)–(H.6). Let $\gamma > \gamma_c$. Then*

$$\gamma^2 w^*(\cdot\,; y_0) = \mathcal{W}^*_{P_{0,\infty}}R^*Ry^*(\cdot\,; y_0) - \mathcal{W}^*_{P_{0,\infty}}P_{0,\infty}Bu^*(\cdot\,; y_0); \tag{6.7.19}$$

$$u^*(\cdot\,; y_0) = -\mathcal{L}^*_{P_{0,\infty}}R^*Ry^*(\cdot\,; y_0) + \mathcal{L}^*_{P_{0,\infty}}P_{0,\infty}Bu^*(\cdot\,; y_0). \tag{6.7.20}$$

Proof. To obtain (6.7.11), we insert (6.7.5b) into (6.7.2) and recall (6.4.4) (right). To obtain (6.7.12), we insert (6.7.5b) into (6.7.10) (left) and recall (6.4.4) (left). □

6.8 Explicit Expression of $w^*(\cdot\,; y_0)$ in Terms of the Data via E_γ^{-1} for $\gamma > \gamma_c$

The next result gives an explicit expression of the optimal $w^*(\cdot\,; y_0)$ for $\gamma > \gamma_c$ (defined by (6.6.1)), in terms of the data of the problem via the operator E_γ^{-1} defined by (6.5.2) and Proposition 6.6.1.

Theorem 6.8.1 *Assume (H.1)–(H.6). Let $\gamma > \gamma_c$. Then*

$$w^*(\cdot\,; y_0) = E_\gamma^{-1}[G^* P_{0,\infty} e^{A_{P_{0,\infty}} \cdot} y_0] \in L_2(0, \infty; V). \tag{6.8.1}$$

Proof. We return to the characterization (6.7.2) for w^*, with p^* given by (6.7.8), thus obtaining

$$\gamma^2 w^*(\cdot\,; y_0) = G^* p^*(\cdot\,; y_0) = G^*\big[P_{0,\infty} y^*(\cdot\,; y_0) + \mathcal{K}^*_{P_{0,\infty}} P_{0,\infty} G w^*(\cdot\,; y_0)\big]. \tag{6.8.2}$$

We now insert (6.7.7c) for y^* into (6.8.2) and recall (6.4.2a) to obtain

$$\begin{aligned}\gamma^2 w^*(\cdot\,; y_0) = G^* P_{0,\infty}\big[&e^{A_{P_{0,\infty}} \cdot} y_0 - \mathcal{L}_{P_{0,\infty}} \mathcal{L}^*_{P_{0,\infty}} P_{0,\infty} G w^*(\cdot\,; y_0)\\ &+ \mathcal{W}_{P_{0,\infty}} w^*(\cdot\,; y_0)\big] + \mathcal{W}^*_{P_{0,\infty}} P_{0,\infty} G w^*(\cdot\,; y_0).\end{aligned} \tag{6.8.3}$$

Recalling now (6.5.2) and (6.5.1), we rewrite (6.8.3) as

$$E_\gamma w^*(\cdot\,; y_0) = G^* P_{0,\infty}[e^{A_{P_{0,\infty}} \cdot} y_0]. \tag{6.8.4}$$

By Proposition 6.6.1, for $\gamma > \gamma_c$ we then obtain (6.8.1), as desired, from (6.8.4). □

6.9 Smoothing Properties of the Operators $\hat{L}$, $\hat{L}^*$, $\hat{W}$, $\hat{W}^*$: The Optimal u^*, y^*, w^* Are Continuous in Time

As we know from Chapter 1, Section 1.3 and Chapter 2, Section 2.3.5, as well as from the illustrations in Chapter 3, continuity in time $y \in C([0, T]; Y)$ for the solutions of Eqn. (6.1.1.1) is generally *false* for $\delta, \rho \geq 1/2$ in (6.1.1.6), (6.1.1.7) (where Y is, in PDE mixed problems, a correctly chosen, sharp space of parabolic regularity). However, in the mildly unbounded case, where constants δ and ρ in (6.1.1.6) and (6.1.1.7) are $< 1/2$, one obtains the desired regularity for the *optimal* solutions $y^0_w(t; y_0) \in C([0, \infty]; Y)$ (as opposed to a general solution), in particular, $y^*(t; y_0) \in C([0, \infty]; Y)$, and $w^*(t; y_0) \in C([0, \infty]; Y)$, as it follows directly from Theorem 6.9.1(i) below. Instead, in the general case where $1/2 < \delta, \rho < 1$, the continuity in time of the optimal solutions $y^*(t; y_0)$, $w^*(t; y_0)$, and $u^*(t; y_0)$ is still true – and this is a key achievement of the theory – but it requires a much more complicated bootstrap argument, as in the corresponding linear quadratic regulator problem where the disturbance $w \equiv 0$ [Chapter 2, Section 2.3.5]. This will be our next objective.

With reference to the translated operator $-\hat{A} = A - \omega I$ in (6.1.1.3), which is the generator of a s.c. analytic, stable semigroup $e^{-\hat{A}t}$ (see (6.1.1.4)), we introduce the following operators corresponding to $\hat{A}$:

$$(\hat{L}u)(t) \equiv \int_0^t e^{-\hat{A}(t-\tau)} Bu(\tau)\, d\tau \tag{6.9.1a}$$

$$: \text{continuous } L_2(0, \infty; U) \to L_2(0, \infty; Y), \tag{6.9.1b}$$

$$(\hat{W}w)(t) \equiv \int_0^t e^{-\hat{A}(t-\tau)} Gw(\tau)\, d\tau \tag{6.9.2a}$$

$$: \text{continuous } L_2(0, \infty; U) \to L_2(0, \infty; Y), \tag{6.9.2b}$$

with corresponding L^2-adjoints

$$(\hat{L}^* v)(t) \equiv B^* \int_t^\infty e^{-\hat{A}^*(\tau-t)} v(\tau)\, d\tau \tag{6.9.3a}$$

$$: \text{continuous } L_2(0, \infty; Y) \to L_2(0, \infty; U), \tag{6.9.3b}$$

$$(\hat{W}^* f)(t) \equiv G^* \int_t^\infty e^{-\hat{A}^*(\tau-t)} f(\tau)\, d\tau \tag{6.9.4a}$$

$$: \text{continuous } L_2(0, \infty; Y) \to L_2(0, \infty; V). \tag{6.9.4b}$$

The indicated regularity is conservative. Indeed, the key feature in the present analytic case is the following theorem [Chapter 2, Theorem 2.3.5.1; see also Chapter 1, Theorem 1.4.4.3, which we rewrite for convenience in the present notation for δ and ρ]. (We know that the proof uses, of course, assumptions (H.2) and (H.3) and Young's Inequality [Sadosky, 1979, p. 149].)

Theorem 6.9.1

(a) With reference to the operators $\hat{L}$ and $\hat{L}^*$ defined by (6.9.1) and (6.9.3) we have:

(i)

$$\hat{L} : \textit{continuous } L_2(0, \infty; U) \to L_{\ell_1}(0, \infty; Y), \tag{6.9.5}$$

where ℓ_1 is an arbitrary positive number satisfying $\ell_1 < \frac{2}{2\delta-1}$; here $\frac{2}{2\delta-1} > 2$ for $1/2 \le \delta < 1$, while for $0 \le \delta < 1/2$, we may take $\ell_1 = \infty$, and indeed replace $L_\infty(0, \infty;\ \cdot)$ with $C([0, \infty];\ \cdot)$.

(ii)

$$\hat{L}^* : \textit{continuous } L_{\ell_1}(0, \infty; Y) \to L_{\ell_2}(0, \infty; U), \tag{6.9.6}$$

where ℓ_1 is as in (i), and ℓ_2 is a positive number satisfying $\ell_2 < \frac{2}{4\delta-3}$; here $\frac{2}{4\delta-3} > \ell_1$ for $3/4 \le \delta < 1$; for $0 < \delta < 3/4$, we may take $\ell_2 = \infty$, and indeed replace $L_\infty(0, \infty;\ \cdot)$ with $C([0, \infty];\ \cdot)$.

(iii) *Generally, let $\ell_0 = 2$, and let ℓ_n, $n = 1, 2, \ldots$ be arbitrary positive numbers such that*

$$2 < \ell_1 < \cdots < \ell_n < \frac{2}{2n\delta - (2n-1)}, \quad n = 1, 2, \ldots \text{ for } \frac{2n-1}{2n} \le \delta < 1. \tag{6.9.7}$$

Then, for $n = 0, 2, 4, \ldots$ we have that

$$\hat{L} : \text{continuous } L_{\ell_n}(0, \infty; U) \to L_{\ell_{n+1}}(0, \infty; Y), \tag{6.9.8}$$

where for $0 \le \delta < \frac{2(n+1)-1}{2(n+1)}$ we may take $\ell_{n+1} = \infty$ and indeed replace $L_\infty(0, \infty; \cdot)$ with $C([0, \infty]; \cdot)$, and, moreover,

$$\hat{L}^* : \text{continuous } L_{\ell_{n+1}}(0, \infty; Y) \to L_{\ell_{n+2}}(0, \infty; U), \tag{6.9.9}$$

where ℓ_{n+1} in (6.9.9) is the same as in (6.9.8), and where for $0 \le \delta < \frac{2(n+1)-1}{2(n+2)}$ we may take $\ell_{n+2} = \infty$, and indeed replace $L_\infty(0, \infty; \cdot)$ with $C([0, \infty]; \cdot)$.

(iv) *For $p > \frac{1}{1-\delta}$,*

$$\hat{L} : \text{continuous } L_p(0, \infty; U) \to C([0, \infty]; Y), \tag{6.9.10}$$

$$\hat{L}^* : \text{ continuous } C([0, \infty]; Y) \to C([0, \infty]; U), \tag{6.9.11}$$

where $C([0, \infty]; \cdot)$ denotes the space of $\cdot$-valued continuous functions on $0 \le t \le \infty$, uniformly bounded on $[0, \infty]$.

(b) *Similar results apply for the operators $\hat{W}$ and $\hat{W}^*$ mutatis mutandis, that is, replacing δ in (6.1.1.6) with ρ in (6.1.1.7).*

Our next goal is to obtain – as a corollary – the desired time regularity property in $C([0, \infty]; \cdot)$ of the optimal quantities $u^*(t; y_0)$, $y^*(t; y_0)$, and $w^*(t; y_0)$. To this end, we need some preliminary results. As in Chapter 2, we introduce the quantities

$$\hat{u}^*(t; y_0) \equiv e^{-\omega t} u^*(t; y_0), \qquad \hat{y}^*(t; y_0) \equiv e^{-\omega t} y^*(t; y_0),$$
$$\hat{p}^*(t; y_0) \equiv e^{-\omega t} p^*(t; y_0), \qquad \hat{w}^*(t; y_0) \equiv e^{-\omega t} w^*(t; y_0). \tag{6.9.12}$$

We next collect some relations to be used later.

Lemma 6.9.2 *Assume (H.1)–(H.6). The following relations hold true:*

(i)

$$\hat{y}^*(t; y_0) = e^{-\hat{A}t} y_0 + \{\hat{L}\hat{u}^*(\,\cdot\,; y_0)\}(t) + \{\hat{W}\hat{w}^*(\,\cdot\,; y_0)\}(t); \tag{6.9.13}$$

(ii)

$$\hat{p}^*(t; y_0) \equiv \int_t^\infty e^{-\hat{A}^*(\tau - t)}[2\omega\hat{p}^*(\tau; y_0) + R^*R\hat{y}^*(\tau; y_0)]\,d\tau \tag{6.9.14a}$$

$$\in L_q(0, \infty; Y) \cap C([0, \infty]; Y), \quad \forall q \ge 2; \tag{6.9.14b}$$

(iii)

$$\hat{u}^*(t; y_0) = -\{\hat{L}^*[2\omega\hat{p}^*(\cdot\,; y_0) + R^*R\hat{y}^*(\cdot\,; y_0)]\}(t); \tag{6.9.15}$$

(iv)

$$\gamma^2\hat{w}^*(t; y_0) = \{\hat{W}^*[2\omega\hat{p}^*(\cdot\,; y_0) + R^*R\hat{y}^*(\cdot\,; y_0)]\}(t). \tag{6.9.16}$$

Proof. (i) The dynamics (6.1.1.1) at optimality becomes, via (6.9.12),

$$\dot{\hat{y}}^* = (A - \omega I)\hat{y}^* + B\hat{u}^* + G\hat{w}^*,$$

whose solution is given by (6.9.13). Alternatively, we multiply by $e^{-\omega t}$ across $y^* = e^{A\cdot}y_0 + Lu^* + Ww^*$.

(ii) From (6.3.2.10b) for $w = w^*$:

$$\frac{d}{dt}p^*(t; y_0) = -A^*p^*(t; y_0) - R^*Ry^*(t; y_0) \quad \text{in } [\mathcal{D}(A)]'.$$

We obtain, via (6.9.12),

$$\frac{d}{dt}\hat{p}^*(t; y_0) = -(A^* - \omega I)\hat{p}^*(t; y_0) - 2\omega\hat{p}^*(t; y_0) - R^*R\hat{y}^*(t; y_0), \tag{6.9.17}$$

whose solution is given by (6.9.14) by use of (6.1.1.3).

(iii) and (iv) Equations (6.9.15) and (6.9.16) are obtained from inserting (6.9.14) into $\hat{u}^*(t; y_0) = -B^*\hat{p}^*(t; y_0)$ (see (6.3.5.4)) and, respectively, into $\gamma^2\hat{w}^*(t; y_0) = G^*\hat{p}^*(t; y_0)$ (see (6.7.2)). □

We next obtain the desired regularity results as in Chapter 2, Corollary 2.3.5.2.

Corollary 6.9.3 *Assume (H.1)–(H.6). With reference to the optimal solutions $\{u^*, w^*, y^*\}$ guaranteed by Theorem 6.7.1, we have for any $y_0 \in Y$:*

$$\hat{u}^*(\cdot\,; y_0) \in C([0,\infty]; U), \quad \hat{y}^*(\cdot\,; y_0) \in C([0,\infty]; Y),$$
$$\hat{w}^*(\cdot\,; y_0) \in C([0,\infty]; U); \tag{6.9.18a}$$

$$u^*(\cdot\,; y_0) \in C([0,T_0]; U), y^*(\cdot\,; y_0) \in C([0,T_0]; Y), w^*(\cdot\,; y_0) \in C([0,T_0]; U),$$
$$\forall\, T_0 < \infty. \tag{6.9.18b}$$

Proof. The proof proceeds by a bootstrap argument playing alternatively between (6.9.13) on the one hand, and (6.9.15) and (6.9.16) on the other, using (6.9.14b) as well. To begin with, we already have from Theorem 6.7.1 that

$$u^*(\cdot\,; y_0) \in L_2(0,\infty; U), \quad w^*(\cdot\,; y_0) \in L_2(0,\infty; V), \quad y^*(\cdot\,; y_0) \in L_2(0,\infty; Y). \tag{6.9.19}$$

It is not restrictive to assume that $\delta = \rho$ for the purposes of this proof and thus use the same regularity results for $\hat{L}$ and $\hat{W}$ and for $\hat{L}^*$ and $\hat{W}^*$ from Theorem 6.9.1. Thus, invoking (6.9.5) for $\hat{L}$ and $\hat{W}$ we obtain that $\hat{L}\hat{u}^*, \hat{W}\hat{w}^* \in L_{\ell_1}(0,\infty; Y)$. Since $e^{-\hat{A}t}$ is exponentially stable as in (1.1.4), we then obtain from the optimal dynamics

(6.9.13) that $\hat{y}^* \in L_{\ell_1}(0, \infty; Y)$ as well. By (6.9.14b), since $q \geq 2$ is arbitrary there, we have that $(2\omega \hat{p}^* + R^* R y^*) \in L_{\ell_1}(0, \infty; Y)$. Next, we recall (6.9.15) and (6.9.16) and apply (6.9.6) for $\hat{L}^*$ and $\hat{W}^*$ to obtain that $\hat{w}^*, \hat{u}^* \in L_{\ell_2}(0, \infty; \cdot)$. We then repeat this bootstrap argument on $\hat{L}$ and $\hat{W}$ in (6.9.13), invoking (6.9.8), and on $\hat{L}^*$ and $\hat{W}^*$ in (6.9.15), (6.9.16), invoking (6.9.9), and we also use (6.9.14b) with $q \geq 2$ for $\hat{p}^*$. In this way, we obtain the desired time continuity for $\hat{u}^*(\cdot\,; y_0)$, $\hat{w}^*(\cdot\,; y_0)$, and $\hat{y}^*(\cdot\,; y_0)$, and hence for $u^*(\cdot\,; y_0)$, $w^*(\cdot\,; y_0)$, and $y^*(\cdot\,; y_0)$ by Theorem 6.9.1(iv), (v). □

Having obtained the desired time continuity for $u^*(\cdot\,; y_0)$, $w^*(\cdot\,; y_0)$, and $y^*(\cdot\,; y_0)$, we can next proceed to obtain the semigroup property of $y^*(\cdot\,; y_0)$.

6.10 A Transition Property for w^* for $\gamma > \gamma_c$

We have the following important property, whose proof follows from the technique used in Chapter 1, Proposition 1.4.3.1 or Chapter 2, Lemma 2.3.2.1.

Theorem 6.10.1 *Assume (H.1)–(H.6). Let $\gamma > \gamma_c$ (defined by (6.6.1)) and $y_0 \in Y$. Then*

$$w^*(t + \sigma; y_0) \underset{(\text{in } \sigma)}{=} w^*(\sigma; y^*(t; y_0)) \in C([0, \infty]; Y) \tag{6.10.1}$$

for each t fixed.

Proof. We rewrite (6.8.3), or (6.8.4), explicitly recalling the definitions of $\mathcal{L}_{P_{0,\infty}}$, $\mathcal{L}^*_{P_{0,\infty}}$, $\mathcal{W}_{P_{0,\infty}}$, and $\mathcal{W}^*_{P_{0,\infty}}$ in (6.3.4.2), (6.3.4.4), (6.3.4.8), and (6.3.4.9), thus obtaining

$$\begin{aligned}
&\gamma^2 w^*(t; y_0) + G^* P_{0,\infty} \int_0^t e^{A_{P_{0,\infty}}(t-\tau)} BB^* \left[\int_\tau^\infty e^{A^*_{P_{0,\infty}}(\alpha-\tau)} P_{0,\infty} G w^*(\alpha; y_0)\, d\alpha \right] d\tau \\
&\quad - \left[G^* P_{0,\infty} \int_0^t e^{A_{P_{0,\infty}}(t-\tau)} G w^*(\tau; y_0)\, d\tau \right. \\
&\quad \left. + G^* \int_t^\infty e^{A^*_{P_{0,\infty}}(\alpha-t)} P_{0,\infty} G w^*(\alpha; y_0)\, d\alpha \right] = G^* P_{0,\infty} e^{A_{P_{0,\infty}} t} y_0.
\end{aligned} \tag{6.10.2}$$

Replacing t and $t + \sigma$ in (6.10.2) we obtain

$$\begin{aligned}
&\gamma^2 w^*(t+\sigma; y_0) \\
&\quad + G^* P_{0,\infty} \int_0^{t+\sigma} e^{A_{P_{0,\infty}}(t+\sigma-\tau)} BB^* \left[\int_\tau^\infty e^{A^*_{P_{0,\infty}}(\alpha-\tau)} P_{0,\infty} G w^*(\alpha; y_0)\, d\alpha \right] d\tau \\
&\quad - \left[G^* P_{0,\infty} \int_0^{t+\sigma} e^{A_{P_{0,\infty}}(t+\sigma-\tau)} G w^*(\tau; y_0)\, d\tau \right. \\
&\quad \left. + G^* \int_{t+\sigma}^\infty e^{A^*_{P_{0,\infty}}(\alpha-(t+\sigma))} P_{0,\infty} G w^*(\alpha; y_0)\, d\alpha \right] \\
&= G^* P_{0,\infty} e^{A_{P_{0,\infty}}(t+\sigma)} y_0.
\end{aligned} \tag{6.10.3}$$

Next, returning to (6.10.2) and substituting here σ for t and $y^*(t; y_0)$ for y_0 yields

$$
\begin{aligned}
&\gamma^2 w^*(\sigma; y^*(t; y_0)) \\
&+ G^* P_{0,\infty} \int_0^\sigma e^{A_{P_{0,\infty}}(\sigma-\tau)} BB^* \left[\int_\tau^\infty e^{A^*_{P_{0,\infty}}(\alpha-\tau)} P_{0,\infty} G w^*(\alpha; y^*(t; y_0))\, d\alpha \right] d\tau \\
&- \left[G^* P_{0,\infty} \int_0^\sigma e^{A_{P_{0,\infty}}(\sigma-\tau)} G w^*(\tau; y^*(t; y_0))\, d\tau \right. \\
&\left. + G^* \int_\sigma^\infty e^{A^*_{P_{0,\infty}}(\alpha-\sigma)} P_{0,\infty} G w^*(\alpha; y^*(t; y_0))\, d\alpha \right] \\
&= G^* P_{0,\infty} e^{A_{P_{0,\infty}}(\sigma)} y^*(t; y_0). \qquad (6.10.4)
\end{aligned}
$$

We now invoke (6.7.7c), which gives $y^*(t; y_0)$ explicitly in terms of $w^*(\cdot\,; y_0)$, and insert it into the right-hand side (R.H.S.) of (6.10.4). We obtain

$$
\begin{aligned}
&\text{R.H.S. of (6.10.4)} = G^* P_{0,\infty} e^{A_{P_{0,\infty}}(t+\sigma)} y_0 \\
&+ G^* P_{0,\infty} \int_0^t e^{A_{P_{0,\infty}}(t+\sigma-\tau)} \left[-BB^* \int_\tau^\infty e^{A^*_{P_{0,\infty}}(\alpha-\tau)} P_{0,\infty} G w^*(\alpha; y_0)\, d\alpha \right. \\
&\left. + G w^*(\tau; y_0) \right] d\tau. \qquad (6.10.5)
\end{aligned}
$$

Now, on the right-hand side of (6.10.4) we insert (6.10.5), while on the left-hand side of (6.10.4) we add and subtract

$$
\begin{aligned}
&G^* P_{0,\infty} \int_t^{t+\sigma} e^{A_{P_{0,\infty}}(t+\sigma-\tau)} BB^* \int_\tau^\infty e^{A^*_{P_{0,\infty}}(\alpha-\tau)} P_{0,\infty} G w^*(\alpha; y_0)\, d\alpha\, d\tau \\
&= G^* P_{0,\infty} \int_0^\sigma e^{A_{P_{0,\infty}}(\sigma-\beta)} BB^* \int_\beta^\infty e^{A^*_{P_{0,\infty}}(s-\beta)} P_{0,\infty} G w^*(t+s; y_0)\, ds\, d\beta, \qquad (6.10.6)
\end{aligned}
$$

using first $\tau - t = \beta$ and then $\alpha - t = s$. We thus obtain

$$
\begin{aligned}
&\gamma^2 w^*(\sigma; y^*(t; y_0)) \\
&+ G^* P_{0,\infty} \int_0^{t+\sigma} e^{A_{P_{0,\infty}}(t+\sigma-\tau)} BB^* \int_\tau^\infty e^{A^*_{P_{0,\infty}}(\alpha-\tau)} P_{0,\infty} G w^*(\alpha; y_0)\, d\alpha\, d\tau \\
&- G^* P_{0,\infty} \int_0^\sigma e^{A_{P_{0,\infty}}(\sigma-\tau)} BB^* \int_\tau^\infty e^{A^*_{P_{0,\infty}}(\alpha-\tau)} P_{0,\infty} G[w^*(t+\alpha; y_0) \\
&- w^*(\alpha; y^*(\alpha; y^*(t; y_0)))]\, d\alpha\, d\tau \\
&- G^* P_{0,\infty} \int_0^{t+\sigma} e^{A_{P_{0,\infty}}(t+\sigma-\tau)} G w^*(\tau; y_0)\, d\tau \\
&+ G^* P_{0,\infty} \int_0^\sigma e^{A_{P_{0,\infty}}(\sigma-\alpha)} G[w^*(t+\alpha; y_0) - w^*(\alpha; y_0)]\, d\alpha \\
&+ G^* \int_\sigma^\infty e^{A^*_{P_{0,\infty}}(\alpha-\sigma)} P_{0,\infty} G w^*(\alpha; y^*(t; y_0))\, d\alpha = G^* P_{0,\infty} e^{A_{P_{0,\infty}}(t+\sigma)} y_0. \qquad (6.10.7)
\end{aligned}
$$

We now subtract (6.10.7) from (6.10.3), thereby obtaining, after a cancellation of six terms,

$$\begin{aligned}
&\gamma^2[w^*(t+\sigma; y_0) - w^*(\sigma; y^*(t; y_0))] \\
&\quad + G^* P_{0,\infty} \int_0^\sigma e^{A_{P_{0,\infty}}(\sigma-\tau)} BB^* \int_\tau^\infty e^{A^*_{P_{0,\infty}}(\alpha-\tau)} P_{0,\infty} G[w^*(t+\alpha; y_0) \\
&\quad - w^*(\alpha; y^*(t; y_0))]\, d\alpha\, d\tau - G^* P_{0,\infty} \int_0^\sigma e^{A_{P_{0,\infty}}(\sigma-\alpha)} G[w^*(\tau+\alpha; y_0) \\
&\quad - w^*(\alpha; y^*(t; y_0))]\, d\alpha + G^* \int_\sigma^\infty e^{A^*_{P_{0,\infty}}(s-\sigma)} P_{0,\infty} G[w^*(t+s; y_0) \\
&\quad - w^*(s; y^*(t; y_0))]\, ds = 0. \qquad (6.10.8)
\end{aligned}$$

The last term in (6.10.8) is obtained by the change of variable $\alpha - t = s$ in the last term on the left-hand side of (6.10.3). Recalling definitions (6.3.4.2), (6.3.4.4), (6.3.4.8), and (6.3.4.9), we rewrite (6.10.8) concisely as

$$\begin{aligned}
&\left\{\gamma^2 I + G^* P_{0,\infty} \mathcal{L}_{P_{0,\infty}} \mathcal{L}^*_{P_{0,\infty}} P_{0,\infty} G - [G^* P_{0,\infty} \mathcal{W}_{P_{0,\infty}} + \mathcal{W}^*_{P_{0,\infty}} P_{0,\infty} G]\right\} \\
&\quad \times [w^*(t + \cdot\,; y_0) - w^*(\cdot\,; y^*(t; y_0))] = 0, \qquad (6.10.9)
\end{aligned}$$

or, recalling (6.5.2) and (6.5.1),

$$E_\gamma[w^*(t + \cdot\,; y_0) - w^*(\cdot\,; y^*(t; y_0))] = 0. \qquad (6.10.10)$$

Then, for $\gamma > \gamma_c$, Proposition 6.6.1 applies and $E_\gamma^{-1} \in \mathcal{L}(L_2(0,\infty; V))$. Thus, (6.10.10) yields the desired equality (6.10.1), first in the $L_2(0,\infty; V)$-sense and next pointwise by the C-regularity of w^* in (6.9.18). □

6.11 A Transition Property for r^* for $\gamma > \gamma_c$

As a consequence of Theorem 6.10.1, we readily obtain

Theorem 6.11.1 *Assume (H.1)–(H.6). Let $\gamma > \gamma_c$ (defined by (6.6.1)) and $y_0 \in Y$. We then have*

$$r^*(t+\sigma; y_0) = r^*(\sigma; y^*(t; y_0)) \underset{(\text{in } \sigma)}{\in} C([0,\infty]; Y), \qquad (6.11.1)$$

for each t fixed.

Proof. We return to formula (6.7.6a) and compute

$$\begin{aligned}
r^*(t+\sigma; y_0) &= \int_{t+\sigma}^\infty e^{A^*_{P_{0,\infty}}(\tau-(t+\sigma))} P_{0,\infty} G w^*(\tau; y_0)\, d\tau, \\
(\tau - t = \alpha) \qquad &= \int_\sigma^\infty e^{A^*_{P_{0,\infty}}(\alpha-\sigma)} P_{0,\infty} G w^*(\alpha + t; y_0)\, d\alpha
\end{aligned}$$

$$\begin{aligned}
&\text{(by (6.10.1))} && = \int_\sigma^\infty e^{A^*_{P_{0,\infty}}(\alpha-\sigma)} P_{0,\infty} G w^*(\alpha; y^*(t; y_0))\, d\alpha \\
&\text{(by (6.7.6a))} && = r^*(\sigma; y^*(t; y_0)).
\end{aligned} \tag{6.11.2}$$

This last equality, (6.11.2), holds true for all t, σ and, in fact, in $C([0, \infty]; Y)$ as noted in (6.11.1), since $e^{A^*_{P_{0,\infty}} t}$ is uniformly stable; see (3.1.14). □

6.12 The Semigroup Property for y^* and a Transition Property for p^* for $\gamma > \gamma_c$

As a consequence of both Theorems 6.10.1 and 6.11.1, we have the semigroup property for $y^*(\cdot; y_0)$.

Theorem 6.12.1 *Assume (H.1)–(H.6). Let $\gamma > \gamma_c$ (defined by (6.6.1)) and $y_0 \in Y$. Then, for all t,*

(i)

$$y^*(t + \sigma; y_0) = y^*(\sigma; y^*(t; y_0)) \underset{(\text{in}\,\sigma)}{\in} C([0, \infty]; Y). \tag{6.12.1}$$

Thus, the operator $\Phi(t)$ (which depends on γ) defined by

$$y^*(t; x) = \Phi(t)x, \quad x \in Y \tag{6.12.2}$$

is a strongly continuous semigroup on Y.

(ii) Furthermore, $\Phi(t)$ is exponentially stable: There exist $c \ge 1$ and $k > 0$ such that

$$\|\Phi(t)\|_{\mathcal{L}(Y)} \le c e^{-kt}, \quad t \ge 0. \tag{6.12.3}$$

Proof. (i) The proof follows the same idea already employed in Theorem 6.10.1. We rewrite (6.7.7a) as

$$y^*(t; y_0) = e^{A_{P_{0,\infty}} t} y_0 + \int_0^t e^{A_{P_{0,\infty}}(t-\tau)}[-BB^* r^*(\tau; y_0) + G w^*(\tau; y_0)]\, d\tau. \tag{6.12.4}$$

Replacing t by $t + \sigma$ in (12.4), we obtain

$$\begin{aligned}
y^*(t + \sigma; y_0) &= e^{A_{P_{0,\infty}}(t+\sigma)} y_0 \\
&\quad + \int_0^{t+\sigma} e^{A_{P_{0,\infty}}(t+\sigma-\tau)}[-BB^* r^*(\tau; y_0) + G w^*(\tau; y_0)]\, d\tau \\
&= e^{A_{P_{0,\infty}}(t+\sigma)} y_0 \\
&\quad + \int_{-t}^{\sigma} e^{A_{P_{0,\infty}}(\sigma-\alpha)}[-BB^* r^*(t + \alpha; y_0) + G w^*(t + \alpha; y_0)]\, d\alpha,
\end{aligned} \tag{6.12.5}$$

after setting $\tau - t = \alpha$ in the last step. Using now the transition properties (6.10.1) for w^* and (6.11.1) for r^*, we obtain, from (6.12.5),

$$\begin{aligned} y^*(t+\sigma; y_0) &= e^{A_{P_{0,\infty}}(t+\sigma)} y_0 \\ &\quad + \int_0^\sigma e^{A_{P_{0,\infty}}(\sigma-\alpha)}[-BB^* r^*(\alpha; y^*(t; y_0)) + Gw^*(\alpha; y^*(t; y_0))]\, d\alpha \\ &\quad + \int_{-t}^0 e^{A_{P_{0,\infty}}(\sigma-\alpha)}[-BB^* r^*(\alpha; y^*(t; y_0)) + Gw^*(\alpha; y^*(t; y_0))]\, d\alpha. \end{aligned} \tag{6.12.6}$$

Next, returning to (6.12.4) and substituting here σ for t and $y^*(t; y_0)$ for y_0 yields

$$\begin{aligned} y^*(\sigma; y^*(t; y_0)) &= e^{A_{P_{0,\infty}}\sigma} y^*(t; y_0) \\ &\quad + \int_0^\sigma e^{A_{P_{0,\infty}}(\sigma-\alpha)}[-BB^* r^*(\alpha; y^*(t; y_0)) \\ &\quad + Gw^*(\alpha; y^*(t; y_0))]\, d\alpha. \end{aligned} \tag{6.12.7}$$

Using (6.12.4) for $y^*(t; y_0)$ in (6.12.7) gives

$$\begin{aligned} y^*(\sigma; y^*(t; y_0)) &= e^{A_{P_{0,\infty}}(\sigma+t)} y_0 \\ &\quad + \int_0^\sigma e^{A_{P_{0,\infty}}(\sigma-\alpha)}[-BB^* r^*(\alpha; y^*(t; y_0)) \\ &\quad + Gw^*(\alpha; y^*(t; y_0))]\, d\alpha + \int_0^t e^{A_{P_{0,\infty}}(\sigma+t-\tau)}[-BB^* r^*(\tau; y_0) \\ &\quad + Gw^*(\tau; y_0)]\, d\tau. \end{aligned} \tag{6.12.8}$$

Comparing (6.12.6) with (6.12.8), we see that identity (6.12.1) holds true, as desired, since we now show that the last integral term in (6.12.6) becomes precisely the last integral term of (6.12.8) after the change of variable $\tau - t = \alpha$ performed in the latter. Indeed,

$$\begin{aligned} &\int_0^t e^{A_{P_{0,\infty}}(\sigma+t-\tau)}[-BB^* r^*(\tau; y_0) + Gw^*(\tau; y_0)]\, d\tau \\ &= \int_{-t}^0 e^{A_{P_{0,\infty}}(\sigma-\alpha)}[-BB^* r^*(t+\alpha; y_0) + Gw^*(t+\alpha; y_0)]\, d\alpha \\ &= \int_{-t}^0 e^{A_{P_{0,\infty}}(\sigma-\alpha)}[-BB^* r^*(\alpha; y^*(t; y_0)) + Gw^*(\alpha; y^*(t; y_0))]\, d\alpha, \end{aligned} \tag{6.12.9}$$

after using once more (6.10.1) and (6.11.1). Thus, (6.12.9) shows that the last integrals in (6.12.6) and (6.12.8) are equal, as desired. Hence (6.12.1) is proved. This readily then implies that $\Phi(t)$ in (6.12.2) enjoys the semigroup property and is strongly continuous.

(ii) The proof of (ii) is then an immediate consequence of the semigroup property of (i), since $y^*(t; x) = \Phi(t)x \in L_2(0, \infty; Y)$ by (6.7.7d), $\forall\, x \in Y$, so that a well-known result [Datko, 1970] applies and yields (6.12.3). $\square$

Finally, as a consequence of Theorems 6.11.1 and 6.12.1, we obtain

Theorem 6.12.2 *Assume (H.1)–(H.6). Let $\gamma > \gamma_c$ (defined in (6.6.1)) and $y_0 \in Y$. Then, for all t,*

$$p^*(t+\sigma; y_0) = p^*(\sigma; y^*(t; y_0)) \underset{(\text{in}\,\sigma)}{\in} C([0,\infty]; Y). \tag{6.12.10}$$

Proof. We return to (6.7.8a) and compute

$$\begin{aligned} p^*(t+\sigma; y_0) &= P_{0,\infty} y^*(t+\sigma; y_0) + r^*(t+\sigma; y_0) \\ &= P_{0,\infty} y^*(\sigma; y^*(t; y_0)) + r^*(\sigma; y^*(t; y_0)) \\ \text{(by (6.7.8a))} \qquad &= p^*(\sigma; y^*(t; y_0)), \end{aligned} \tag{6.12.11}$$

after using (6.11.1) and (6.12.1) to obtain (6.12.11). □

6.13 Definition of P and Its Properties

With reference to $p^*(\cdot\,; y_0)$ in (6.7.3) and (6.7.5), we define an operator $P \in \mathcal{L}(Y)$ by setting

$$Px \equiv p^*(0; x), \quad x \in Y. \tag{6.13.1}$$

A few preliminary properties of P are collected below.

Proposition 6.13.1 *Assume (H.1)–(H.6). Let $\gamma > \gamma_c$. For $y_0 \in Y$ we have*

(i)

$$p^*(t; y_0) = Py^*(t; y_0) = P\Phi(t)y_0 \in L_2(0,\infty; Y) \cap C([0,\infty]; Y); \tag{6.13.2}$$

(ii)

$$u^*(t; y_0) = -B^*p^*(t; y_0) = -B^*Py^*(t; y_0) \in L_2(0,\infty; U) \cap C([0,\infty]; U); \tag{6.13.3}$$

(iii)

$$\gamma^2 w^*(t; y_0) = G^*p^*(t; y_0) = G^*Py^*(t; y_0) \in L_2(0,\infty; Y) \cap C([0,\infty]; Y). \tag{6.13.4}$$

(iv) For $x \in Y$, the following identity holds true:

$$Px = \int_0^{t_0} e^{A^*\tau} R^*R\Phi(\tau)x\,d\tau + e^{A^*t_0} P\Phi(t_0)x, \tag{6.13.5}$$

where t_0 is an arbitrary point $0 < t_0 < \infty$.

(v) For $x \in Y$, the following formula holds true:

$$Px = P_{0,\infty}x + \int_0^{\infty} e^{A^*_{P_{0,\infty}}\tau} P_{0,\infty} G\{E_\gamma^{-1}[G^*P_{0,\infty}e^{A_{P_{0,\infty}}\cdot}]\}(\tau)\,d\tau, \tag{6.13.6}$$

which expresses P in terms of the problem data, via E_γ in (5.2).

(vi) For $x_1, x_2 \in Y$,

$$(Px_1, x_2)_Y = (P_{0,\infty}x_1, x_2)_Y + \left(E_\gamma^{-1}[G^*P_{0,\infty}e^{A_{P_{0,\infty}}\cdot}x_1], G^*P_{0,\infty}e^{A_{P_{0,\infty}}\cdot}x_2\right)_{L_2(0,\infty;V)}, \quad (6.13.7)$$

so that P is a positive, self-adjoint operator: $P = P^* > P_{0,\infty} \geq 0$.

Proof. (i) By Eqn. (6.12.10) of Theorem 6.12.2 with $\sigma = 0$, we obtain, as desired,

$$p^*(t; y_0) = p^*(0, y^*(t; y_0)) = Py^*(t; y_0), \quad (6.13.8)$$

where in the last step we have invoked the definition (6.13.1).

(ii), (iii) These follow from (6.7.10) and (6.7.2), respectively, via (6.13.2).

(iv) To obtain (6.13.5), we return to (3.2.7) for $t = 0$, and use (6.13.1), (6.12.2), and (6.13.2) for $t = t_0$.

(v) Returning to (6.7.8) for $t = 0$ we have, via (6.13.1),

$$\begin{aligned} Px = p^*(0; x) &= P_{0,\infty}y^*(0; x) + r^*(0; x) \\ &= P_{0,\infty}x + \int_0^\infty e^{A^*_{P_{0,\infty}}\tau}P_{0,\infty}Gw^*(\tau; x)\,d\tau, \end{aligned} \quad (6.13.9)$$

where in the last step we have recalled (6.7.6a) for $r^*(0; x)$. We next insert (6.8.1) for w^* in (6.13.9) and obtain (6.13.6).

(vi) Identity (6.13.7) is an immediate consequence of (6.13.6) and shows that P is positive, self-adjoint on Y, since $P_{0,\infty}$ is nonnegative, self-adjoint on Y by Theorem 6.3.1.1(ii) (xi), and E_γ^{-1} is positive, self-adjoint on $L_2(0, \infty; V)$ by Proposition 6.6.1 for $\gamma > \gamma_c$. □

Proposition 6.13.2 *Assume (H.1)–(H.6). Let $\gamma > \gamma_c$ and $x_1, x_2 \in Y$. Then*

(i) The following symmetric relation holds true:

$$\begin{aligned} (Px_1, x_2)_Y = \int_0^\infty &[(Ry^*(t; x_1), Ry^*(t; x_2))_Z + (u^*(t; x_1), u^*(t; x_2))_U \\ &- \gamma^2(w^*(t; x_1), w^*(t; x_2))_V]\,dt. \end{aligned} \quad (6.13.10)$$

(ii) Hence for $x \in Y$,

$$\begin{aligned} (Px, x)_Y &= J^*(x) = J(u^*(\cdot\,; x), w^*(\cdot\,; x)) \\ &= \int_0^\infty \left[\|Ry^*(t; x)\|_Z^2 + \|u^*(t; x)\|_U^2 - \gamma^2\|w^*(t; x)\|_V^2\right] dt, \end{aligned} \quad (6.13.11)$$

which reproves that P is positive, self-adjoint on Y by (6.7.9).

Proof. (i) By (6.13.1), recalling (6.7.5a), we obtain

$$Px_1 = p^*(0, x_1) = \int_0^\infty e^{A^*_{P_{0,\infty}}\tau}[-P_{0,\infty}Bu^*(\tau; x_1) + R^*Ry^*(\tau; x_1)]\,d\tau, \quad (6.13.12)$$

so that

$$(Px_1, x_2)_Y = \int_0^\infty ([-P_{0,\infty}Bu^*(\tau; x_1) + R^*Ry^*(\tau; x_1)], e^{A_{P_{0,\infty}}\tau}x_2)_Y. \quad (6.13.13)$$

We now recall (6.7.7b,c) and insert

$$e^{A_{P_{0,\infty}}\cdot}x_2 = y^*(\cdot\,; x_2) + \mathcal{L}_{P_{0,\infty}}B^*r^*(\cdot\,; x_2) - \mathcal{W}_{P_{0,\infty}}w^*(\cdot\,; x_2) \quad (6.13.14)$$

on the right of (6.13.13), thus obtaining with inner products in $L_2(0, \infty; \cdot)$:

$$\begin{aligned}(Px_1, x_2)_Y = &- (u^*(\cdot\,; x_1), B^*P_{0,\infty}y^*(\cdot\,; x_2)) \\ &- (\mathcal{L}^*_{P_{0,\infty}}P_{0,\infty}Bu^*(\cdot\,; x_1), B^*r^*(\cdot\,; x_2)) \\ &+ (\mathcal{W}^*_{P_{0,\infty}}P_{0,\infty}Bu^*(\cdot\,; x_1), w^*(\cdot\,; x_2)) + (Ry^*(\cdot\,; x_1), Ry^*(\cdot\,; x_2)) \\ &+ (\mathcal{L}^*_{P_{0,\infty}}R^*Ry^*(\cdot\,; x_1), B^*r^*(\cdot\,; x_2)) \\ &- (\mathcal{W}^*_{P_{0,\infty}}R^*Ry^*(\cdot\,; x_1), w^*(\cdot\,; x_2)).\end{aligned} \quad (6.13.15)$$

□

6.14 The Feedback Generator A_F and Its Preliminary Properties for $\gamma > \gamma_c$

For $\gamma > \gamma_c$, we return to the s.c. semigroup $\Phi(t)$ that defines the optimal solution $y^*(t; x) = \Phi(t)x$ by (6.12.2), and call A_F (F stands for "feedback") its infinitesimal generator, so that

$$\Phi(t)x = e^{A_F t}x, \quad x \in Y; \quad \frac{d\Phi(t)x}{dt} = A_F\Phi(t)x = \Phi(t)A_Fx, \ x \in \mathcal{D}(A_F). \quad (6.14.1)$$

We recall from (6.12.3) that $\Phi(t)$ is uniformly stable.

We next provide information about A_F essentially as a consequence of (6.13.3) and (6.13.4) being inserted into Eqn. (6.1.1.1) for y^* given by (6.12.2).

Theorem 6.14.1 *Assume (H.1)–(H.6). Let $\gamma > \gamma_c$. For $x \in Y$ and a.e. in $t \geq 0$,*

(i)

$$\frac{d\Phi(t)x}{dt} = [A - BB^*P + \gamma^{-2}GG^*P]\Phi(t)x \in [\mathcal{D}(A^*)]'. \quad (6.14.2)$$

(ii) Thus, by (6.14.1) and $t \geq 0$,

$$[A - BB^*P + \gamma^{-2}GG^*P]\Phi(t)x = A_F\Phi(t)x = \Phi(t)A_Fx \in Y, \quad x \in \mathcal{D}(A_F),\ t \geq 0; \quad (6.14.3a)$$

$$[A - BB^*P + \gamma^{-2}GG^*P]x = A_Fx \in Y, \quad x \in \mathcal{D}(A_F); \quad (6.14.3b)$$

$$\Phi(t)x = e^{(A-BB^*P+\gamma^{-2}GG^*P)t}x, \quad x \in Y. \quad (6.14.3c)$$

(iii) The resolvent $R(\lambda; A_F)$ of A_F satisfies the estimate

$$\|R(\lambda; A_F)\|_{\mathcal{L}(Y)} \leq \frac{C_{r_0}}{|\lambda - \omega_0|}, \quad \forall\, \lambda \textit{ with } \operatorname{Re}\lambda \geq \textit{ some } r_0 > 0, \quad (6.14.4)$$

and thus A_F generates a s.c. analytic semigroup on Y:

$$y^*(t; x) = \Phi(t)x = e^{A_F t}x = e^{(A-BB^*P+\gamma^{-2}GG^*P)t}x, \quad x \in Y. \quad (6.14.5)$$

(iv) The resolvents $R(\lambda, A - BB^*P)$ and $R(\lambda, A + \gamma^{-2}GG^*P)$ of the operators $A - BB^*P$ and $A + \gamma^{-2}GG^*P$ with maximal domains satisfy estimates like (6.14.4):

$$\begin{cases} \|R(\lambda; A - BB^*P)\|_{\mathcal{L}(Y)} \leq \frac{c_{r_0}}{|\lambda|}; & \\ & \forall\, \lambda \text{ with } \operatorname{Re}\lambda \geq \text{ some } r_0 > 0, \\ \|R(\lambda; A + \gamma^{-2}GG^*P)\|_{\mathcal{L}(Y)} \leq \frac{c'_{r_0}}{|\lambda|}; & \end{cases} \quad (6.14.6)$$

so that

$$e^{(A-BB^*P)t} \quad \text{and} \quad e^{(A+\gamma^{-2}GG^*P)t} \quad (6.14.7)$$

are both s.c. analytic semigroups on Y.

Proof. A proof may be given that closely follows the proof of the corresponding result [Chapter 2, Theorem 2.3.8.1] in the case $w \equiv 0$.

(i), (ii) We return to the min–max solution dynamics,

$$y^*(t; y_0) = e^{At}y_0 + \int_0^t e^{A(t-\tau)}Bu^*(\tau; y_0)\,d\tau + \int_0^t e^{A(t-\tau)}Gw^*(\tau; y_0)\,d\tau, \quad (6.14.8)$$

take the inner product of (6.14.8) with $y \in \mathcal{D}(A^*)$, and differentiate in t with $y_0 \in \mathcal{D}(A_F)$, thus obtaining

$$\begin{aligned} \left(\frac{dy^*(t; y_0)}{dt}, y\right)_Y = \left(\frac{d\Phi(t)y_0}{dt}, y\right)_Y &= (\Phi(t)A_F y_0, y)_Y \\ &= (\Phi(t)y_0, A^*y)_Y + (Bu^*(t; y_0), y)_Y \\ &\quad + (Gw^*(t; y_0), y)_Y \\ \text{(by (6.13.3) and (6.13.4))} \qquad &= ([A - BB^*P + \gamma^{-2}GG^*P]\Phi(t)y_0, y)_Y, \\ & \qquad y_0 \in \mathcal{D}(A_F), \; y \in \mathcal{D}(A^*), \end{aligned} \quad (6.14.9)$$

after using, in the last step, (6.13.3) for $u^*(t; y_0)$ and (6.13.4) for $w^*(t; y_0)$. Then, (6.14.9) shows readily (6.14.2) and (6.14.3).

(iii) First Proof. The usual perturbation formula is (see Chapter 2, Eqn. (2.3.8.5) for $w = 0$)

$$(\lambda I - A_F)^{-1} = \{I - R(\lambda, A)[-BB^*P + \gamma^{-2}GG^*P]\}^{-1}R(\lambda, A), \quad \operatorname{Re}\lambda > \omega_0, \quad (6.14.10)$$

recalling (6.1.1.2), where, by (6.1.1.6) and (6.13.20),

$$\begin{aligned} \|R(\lambda, A)BB^*P\|_{\mathcal{L}(Y)} &= \|R(\lambda, A)\hat{A}^{\delta}\hat{A}^{-\delta}BB^*P\|_{\mathcal{L}(Y)} \\ &\leq \|R(\lambda, A)\hat{A}^{\delta}\|_{\mathcal{L}(Y)} \to 0 \text{ as } \operatorname{Re}\lambda \to \infty, \end{aligned} \quad (6.14.11)$$

where the right-hand side goes to zero by virtue of the usual estimate for the generator $\hat{A}$ of a stable analytic semigroup:

$$\|\hat{A}^{\theta} R(\lambda, \hat{A})\|_{\mathcal{L}(Y)} \leq \frac{c}{|\lambda|^{1-\theta}}, \quad 0 \leq \theta < 1, \quad \operatorname{Re} \lambda > 0. \tag{6.14.12}$$

Thus, (6.14.11) used in (6.14.10) yields the desired (6.14.4):

$$\|R(\lambda, A_F)\|_{\mathcal{L}(Y)} \leq c_{r_0} \|R(\lambda, A)\|_{\mathcal{L}(Y)} \leq \frac{C_{r_0}}{|\lambda - \omega_0|}, \quad \operatorname{Re} \lambda > r_0 > 0, \tag{6.14.13}$$

since e^{At} is an analytic semigroup. Similar proofs apply for part (iv).

Second Proof. (See Chapter 2, Eqns. (2.3.8.9), (2.3.8.10).) The operators $(GG^*P)^*$ and $(BB^*P)^*$ are relatively bounded with respect to $(\hat{A}^*)^{\rho}$ and $(\hat{A}^*)^{\delta}$:

$$\begin{aligned} \|(GG^*P)^*x\|_Y &= \|(G^*P)^*G^*(\hat{A}^*)^{-\rho}(\hat{A}^*)^{\rho}x\|_Y \\ &\leq \|(G^*P)^*G^*(\hat{A}^*)^{-\rho}\| \|(\hat{A}^*)^{\rho}x\|_Y, \quad x \in \mathcal{D}((\hat{A}^*)^{\rho}); \end{aligned} \tag{6.14.14}$$

$$\begin{aligned} \|(BB^*P)^*x\|_Y &= \|(B^*P)^*B^*(\hat{A}^*)^{-\delta}(\hat{A}^*)^{\delta}x\|_Y \\ &\leq \|(B^*P)^*B^*(\hat{A}^*)^{-\delta}\| \|(\hat{A}^*)^{\delta}x\|_Y, \quad x \in \mathcal{D}((\hat{A}^*)^{\delta}), \end{aligned} \tag{6.14.15}$$

since $G^*P \in \mathcal{L}(Y; V)$, $B^*P \in \mathcal{L}(Y; U)$ by (6.13.20), and $G^*(\hat{A}^*)^{-\rho} \in \mathcal{L}(Y; V)$, $B^*(\hat{A}^*)^{-\delta} \in \mathcal{L}(Y; U)$ by (6.1.1.6) and (6.1.1.7). Since $\delta < 1$, and $\rho < 1$, estimates (6.14.14) and (6.14.15) permit one to invoke a well-known perturbation result [Pazy, 1983, p. 81] applied to the generator A^* of a s.c. analytic semigroup and conclude that: The operators

$$A^* - PBB^* \text{ with domain equal to } \mathcal{D}((\hat{A}^*)^{\delta}); \tag{6.14.16}$$

$$A^* + \gamma^{-2}PGG^* \text{ with domain equal to } \mathcal{D}((\hat{A}^*)^{\rho}); \tag{6.14.17}$$

$$A_F^* = A^* - PBB^* + \gamma^{-2}PGG^* \text{ with domain equal to } \mathcal{D}((\hat{A}^*)^{\sigma}),\ \sigma = \max\{\delta, \rho\}, \tag{6.14.18}$$

are all generators of s.c., analytic semigroups on Y. Taking adjoints we obtain a second proof of parts (iii) and (iv) of Lemma 6.14.1. □

Corollary 6.14.2 *Assume (H.1)–(H.6). Let $\gamma > \gamma_c$. With reference to (6.14.3), we have*

(i)

$$\begin{aligned} \mathcal{D}(A_F) &= \{x \in \mathcal{D}(\hat{A}^{1-\sigma}) : \hat{A}^{1-\sigma}x - \hat{A}^{-\sigma}BB^*Px + \gamma^{-2}\hat{A}^{-\sigma}GG^*Px \in \mathcal{D}(\hat{A}^{\sigma})\} \\ &\subset \mathcal{D}(\hat{A}^{1-\sigma}), \quad \sigma = \max\{\delta, \rho\}. \end{aligned} \tag{6.14.19}$$

(ii) For $x \in Y$ and $t > 0$,

$$e^{A_F t}x \subset \mathcal{D}(\hat{A}^{1-\sigma}). \tag{6.14.20}$$

Proof (as in Chapter 2, Corollary 2.3.8.2). With σ as in (6.14.19), we have, by (6.1.1.6) and (6.1.1.7):

$$\hat{A}^{-\sigma}B \in \mathcal{L}(U;Y); \qquad \hat{A}^{-\sigma}G \in \mathcal{L}(V;Y). \tag{6.14.21}$$

Let $x \in \mathcal{D}(A_F) = \mathcal{D}(\hat{A}_F)$, where $\hat{A}_F = -\hat{A} - BB^*P + \gamma^{-2}GG^*P$ is a translation of A_F; then

$$\begin{aligned}\hat{A}_F x &= [-\hat{A} - BB^*P + \gamma^{-2}GG^*P]x \\ &= \hat{A}^{\sigma}[-\hat{A}^{1-\sigma} - \hat{A}^{-\sigma}BB^*P + \gamma^{-2}\hat{A}^{-\sigma}GG^*P]x = z \in Y.\end{aligned} \tag{6.14.22}$$

By (6.14.21) and (6.13.20),

$$[\hat{A}^{-\sigma}BB^*P - \gamma^{-2}\hat{A}^{-\sigma}GG^*P]x \in Y. \tag{6.14.23}$$

Hence, (6.14.22) and (6.14.23) imply

$$-\hat{A}^{1-\sigma}x = \hat{A}^{-\sigma}z + [\hat{A}^{-\sigma}BB^*P - \gamma^{-2}\hat{A}^{-\sigma}GG^*P]x \in Y, \tag{6.14.24}$$

that is, $x \in \mathcal{D}(\hat{A}^{1-\sigma})$, and (6.14.19) is established. □

(ii) By analyticity of $e^{A_F t}$ established in Theorem 6.14.1, we have then, via (6.14.19),

$$e^{A_F t}x \in \mathcal{D}(A_F) \subset \mathcal{D}(\hat{A}^{1-\sigma}), \quad x \in Y, \quad t > 0, \tag{6.14.25}$$

and (6.14.20) is proved. □

6.15 The Operator P is a Solution of the Algebraic Riccati Equation, ARE$_\gamma$ for $\gamma > \gamma_c$

We finally obtain the ultimate goal of our analysis in the section.

Theorem 6.15.1 *(Existence) Assume (H.1)–(H.6). Let $\gamma > \gamma_c$. Then:*

The operator P defined by (6.13.1) satisfies the algebraic Riccati equation, ARE$_\gamma$, in (6.13.1), that is,

$$\begin{aligned}&(A^*Px, z)_Y + (PAx, z)_Y + (Rx, Rz)_Y \\ &\quad = (B^*Px, B^*Pz)_U - \gamma^{-2}(G^*Px, G^*Pz)_V,\end{aligned} \tag{6.15.1}$$

for all $x, z \in \mathcal{D}(\hat{A}^{\epsilon})$, in particular for all $x, z \in \mathcal{D}(A_F) \subset \mathcal{D}(\hat{A}^{1-\sigma})$, σ as in (6.14.18).

Proof (See Chapter 2, Theorem 2.3.9.1).

Step 1 We first show that (6.15.1) holds true $\forall\, x \in \mathcal{D}(A_F)$ and $\forall\, z \in \mathcal{D}(A)$. To this end we return to (6.13.5), which we rewrite as

$$(Px, z)_Y = \left(\int_0^{t_0} e^{A^*\tau}R^*R\Phi(\tau)x\,d\tau + e^{A^*t_0}P\Phi(t_0)x, z\right)_Y, \tag{6.15.2}$$

where we recall that t_0 is an arbitrary point $t_0 \geq 0$. We now specialize to $x \in \mathcal{D}(A_F)$, $z \in \mathcal{D}(A)$ and differentiate (6.15.2) with respect to t_0, thus obtaining, after invoking (6.14.1),

$$0 = (e^{A^* t_0} R^* R \Phi(t_0) x, z)_Y + (e^{A^* t_0} P \Phi(t_0) x, Az)_Y + (e^{A^* t_0} P \Phi(t_0) A_F x, z)_Y \tag{6.15.3}$$

for all $t_0 \geq 0$. Setting $t_0 = 0$ in (6.15.3) yields

$$(Px, Az)_Y + (PA_F x, z)_Y + (R^* R x, z)_Y = 0, \quad \forall\, x \in \mathcal{D}(A_F), \quad \forall\, z \in \mathcal{D}(A) \tag{6.15.4}$$

[Eqn. (6.15.4) corresponds to Eqn. (2.3.9.7) of Chapter 2.].

Step 2 (See Chapter 2, proof of Theorem 2.3.9.1, Step 2.) We extend the validity of (6.15.4) to all $x \in Y$ and all $z \in \mathcal{D}(\hat{A}^\epsilon)$. First, by (6.13.19),

$$(A^* P x, z)_Y = \text{well defined for all } x \in Y, \quad \text{all } z \in \mathcal{D}(\hat{A}^\epsilon). \tag{6.15.5}$$

Next, we can write the second term in (6.15.4) with A replaced by $-\hat{A}$ as

$$\begin{aligned}
&(P[\hat{A} - BB^* P + \gamma^{-2} GG^* P]x, z)_Y \\
&\quad = (P\hat{A}[I - \hat{A}^{-1} BB^* P + \gamma^{-2} \hat{A}^{-1} GG^* P]x, z)_Y \\
&\quad = (\hat{A}^\epsilon [I - \hat{A}^{-1} BB^* P + \gamma^{-2} \hat{A}^{-1} GG^* P]x, (\hat{A}^*)^{1-\epsilon} P z)_Y \\
&\quad = \left(\hat{A}^\epsilon x - \hat{A}^{-(1-\epsilon)} BB^* P x + \gamma^{-2} \hat{A}^{-(1-\epsilon)} GG^* P x, (\hat{A}^*)^{1-\epsilon} P z\right)_Y,
\end{aligned} \tag{6.15.6}$$

where the right-hand side of (6.15.6) holds true for $z \in Y$ (by (6.13.19)), and $x \in \mathcal{D}(\hat{A}^\epsilon)$ for $\rho, \delta \leq 1-\epsilon < 1$, by (6.1.1.6) and (6.1.1.7), and $B^* P \in \mathcal{L}(Y; U)$, $G^* P \in \mathcal{L}(Y; V)$; see (6.13.20). Thus, in conclusion, (6.15.4) can be extended to all $x, z \in \mathcal{D}(\hat{A}^\epsilon)$, as claimed, as elements of $\mathcal{D}(\hat{A}^\epsilon)$ can be approximated in the Y-norm by elements of $\mathcal{D}(A_F)$ and $\mathcal{D}(A)$. The same considerations above, centered on (6.15.6), permit us to split the second term in (6.15.4) via (6.14.3b) and obtain (6.15.1) $\forall\, x, z \in \mathcal{D}(\hat{A}^\epsilon)$.

We finally recall (6.14.19). □

6.16 The Semigroup Generated by $(A - BB^* P)$ Is Uniformly Stable

We return to the s.c. analytic semigroup generated by $A - BB^* P$ in Lemma 6.14.2 (iv). We now show that this semigroup shares with, indeed inherits from, $e^{A_F t}$ the property of being uniformly stable.

Proposition 6.16.1 *Assume (H.1)–(H.6). Let $\gamma > \gamma_c$. The s.c. semigroup $e^{(A-BB^* P)t}$ is uniformly stable on Y: There exist constants $C_1 \geq 1$, $a_1 > 0$ such that*

$$\left\| e^{(A-BB^* P)t} \right\|_{\mathcal{L}(Y)} \leq C_1 e^{-a_1 t}, \quad t \geq 0. \tag{6.16.1}$$

Proof.

Step 1 We write

$$z(t; z_0) = e^{(A-BB^*P)t}z_0 \in C([0, T]; Y); \quad \dot{z} = (A - BB^*P)z, \quad z(0) = z_0 \in Y. \tag{6.16.2}$$

Hence, recalling (6.14.5),

$$\dot{z} = (A - BB^*P + \gamma^{-2}GG^*P)z - \gamma^{-2}GG^*Pz = A_F z - \gamma^{-2}GG^*Pz;$$
$$z(t; z_0) = e^{A_F t}z_0 - \gamma^{-2}\int_0^t e^{A_F(t-\tau)}GG^*Pz(\tau; z_0)\,d\tau. \tag{6.16.3}$$

Step 2 With reference to (6.16.3), we shall show below in Steps 4 and 5 that

$$G^*Pz(\cdot\,; z_0) \in L_2(0, \infty; Y), \quad \forall\, z_0 \in Y. \tag{6.16.4}$$

Taking (6.16.4) for granted, we return to identiy (6.16.3), where $e^{A_F t}$ is uniformly stable by (6.12.3), so that $\|e^{A_F t}\|_{\mathcal{L}(Y)} \in L_1(0, \infty)$. The integral in (6.16.3), being a convolution between an L_1- and an L_2-function (see (6.16.4)) is also in $L_2(0, \infty; Y)$ (Young's inequality) [Sadosky, 1979, p. 28]. Thus, it follows that

$$z(t; y_0) = e^{(A-BB^*P)t}z_0 \in L_2(0, \infty; Y), \quad \forall\, z_0 \in Y. \tag{6.16.5}$$

Step 3 We apply a well-known result [Datko, 1970] to (6.16.5) and obtain the exponential decay (6.16.1) for the semigroup $e^{(A-BB^*P)t}$.

Step 4 It remains to show (6.16.4). To this end, we note first that:

$$z(t; z_0) \in \begin{cases} C((0, T]; \mathcal{D}(\hat{A}^\epsilon)) & \text{for all } \epsilon < 1 - \delta, \; z_0 \in Y; \qquad (6.16.6a) \\ C([0, T]; \mathcal{D}(\hat{A}^\epsilon)) & \text{for all } \epsilon < 1 - \delta, \; z_0 \in \mathcal{D}(\hat{A}^\epsilon). \qquad (6.16.6b) \end{cases}$$

In fact, from (6.16.2), we write

$$\hat{A}^\epsilon z(t; z_0) = \hat{A}^\epsilon e^{At}z_0 + \int_0^t \hat{A}^\epsilon e^{A(t-\tau)}\hat{A}^\delta \hat{A}^{-\delta}Bz(\tau; z_0)\,d\tau, \tag{6.16.7}$$

where we already know that $z(\tau; z_0) \in C([0, T]; Y)$. Thus, the analyticity estimate (6.1.1.5), combined wtih (6.1.1.6), permits us to obtain, via Young's inequality on the convolution integral [Sadosky, 1970, p. 29] between an L_1-function and a C-function [as, e.g., in Chapter 1, Eqn. (1.4.4.21) and ff.],

$$\int_0^t \hat{A}^\epsilon e^{A(t-\tau)}\hat{A}^\delta \hat{A}^{-\delta}Bz(\tau; z_0)\,d\tau \in C([0, T]; Y), \quad z_0 \in Y. \tag{6.16.8}$$

Then, (6.16.8) leads to (6.16.6a,b), according to the assumed smoothness of z_0 for the first term on the right-hand side of (6.16.7).

Step 5 We now take the inner product of the z-equation in (16.2) with $Pz = Pz(t; z_0)$ for $t \geq 0$, P self-adjoint:

$$\frac{1}{2}\frac{d}{dt}(z, Pz)_Y = ((A - BB^*P)z, Pz)_Y = (PAz, z)_Y - \|B^*Pz\|_U^2 \tag{6.16.9a}$$

$$= \mathrm{Re}(PAz, z)_Y - \|B^*Pz\|_U^2, \tag{6.16.9b}$$

since integrating (6.16.9a) on $[0, T]$ for all T yields that $(PAz(t; z_0), z(t; z_0))$ is real. We note that, in view of (6.13.19) and of the regularity (6.16.6), we have that Eqns. (6.16.9a,b) are well defined for $t \geq 0$ and $z_0 \in \mathcal{D}(\hat{A}^\epsilon)$. We now invoke the ARE$_\gamma$ (6.15.1) with $x = z = z(t; y_0) \in \mathcal{D}(\hat{A}^\epsilon)$ by (6.16.6), thus obtaining

$$2\,\mathrm{Re}(PAz, z)_Y + \|Rz\|_Z^2 = \|B^*Pz\|_U^2 - \gamma^{-2}\|G^*Pz\|_V^2, \quad t \geq 0,\ z_0 \in \mathcal{D}(\hat{A}^\epsilon). \tag{6.16.10}$$

Solving (6.16.10) for $\mathrm{Re}(PAz, z)$ and substituting in (6.16.9b) results in

$$\frac{d}{dt}(z, Pz)_Y = -\|B^*Pz\|_U^2 - \|Rz\|_Z^2 - \gamma^{-2}\|G^*Pz\|_V^2. \tag{6.16.11}$$

Integrating (6.16.11) over $[0, T]$ yields

$$\int_0^T \left[\|B^*Pz(t; z_0)\|_U^2 + \|Rz(t; z_0)\|_Z^2\right] dt + \gamma^{-2}\int_0^T \|G^*Pz(t; z_0)\|_Y^2$$
$$+ (z(T), Pz(T))_Y = (z_0, Pz_0)_Y, \quad z_0 \in \mathcal{D}(\hat{A}^\epsilon). \tag{6.16.12}$$

Since P is positive definite by Proposition 6.13.1, (6.13.7), we can drop the inner product on the left-hand side of (6.16.12). Then, (6.16.12) implies

$$\int_0^\infty \left[\|B^*Pz(t; z_0)\|_U^2 + \|Rz(t; z_0)\|_Z^2\right] dt + \gamma^{-2}\int_0^\infty \|G^*Pz(t; z_0)\|_Y^2\, dt$$
$$\leq (z_0, Pz_0)_Y, \tag{6.16.13}$$

at least for $z_0 \in \mathcal{D}(\hat{A}^\epsilon)$. But, $\mathcal{D}(\hat{A}^\epsilon)$ is dense in Y and hence (6.16.13) can be extended to all $z_0 \in Y$. This proves (6.16.4). The proof of (6.16.1) for $(A - BB^*P)$ is complete. □

Remark 6.16.1 The above proof applies to *any* operator $P = P^* \geq 0$ in $\mathcal{L}(Y)$ such that: (i) $A_F = A - BB^*P + \gamma^{-2}GG^*P$ is the generator of a s.c. uniformly stable semigroup on Y for some $\gamma > 0$; and (ii) P is a solution of the corresponding ARE$_\gamma$ in (6.1.3.1), $\forall\, x, z \in \mathcal{D}(\hat{A}^\epsilon)$, with the properties that $B^*P \in \mathcal{L}(Y; U)$, $G^*P \in \mathcal{L}(Y; V)$.

6.17 The Case $0 < \gamma < \gamma_c$: $\sup J^0_{w,\infty}(y_0) = +\infty$

We now consider the case where $\gamma_c > 0$ and $0 < \gamma < \gamma_c$, of Theorem 6.1.3.1(a).

Proposition 6.17.1 *Assume (H.1)–(H.6). Let $0 < \gamma < \gamma_c$. Then, there exists a sequence $\{w_k\}_{k=1}^\infty$, $w_k \in L_2(0, \infty; V)$, such that for all $y_0 \in Y$ we have*

$$J^0_{w_k,\infty}(y_0) \to +\infty, \quad \text{as } k \to +\infty, \tag{6.17.1}$$

so that for all $y_0 \in Y$,

$$\sup_{w \in L_2(0,\infty;V)} J^0_{w,\infty}(y_0) = +\infty. \tag{6.17.2}$$

Proof. Let $\gamma_c > 0$ so that $-\gamma_c^2 = \inf_{\|w\|=1}(Sw, w)$ by (6.6.1), with norm and inner product of $L_2(0, \infty; V)$. Thus, given $\epsilon > 0$, there exists $w_\epsilon \in L_2(0, \infty; V)$, $\|w_\epsilon\| = 1$ such that

$$(Sw_\epsilon, w_\epsilon) < -\gamma_c^2 + \epsilon. \tag{6.17.3}$$

Recalling E_γ from (6.5.2), we then obtain by (6.17.3), under the present assumption $0 < \gamma < \gamma_c$:

$$(E_\gamma w_\epsilon, w_\epsilon) = \gamma^2 + (Sw_\epsilon, w_\epsilon) < \gamma^2 - \gamma_c^2 + \epsilon < 0, \tag{6.17.4}$$

after choosing ϵ sufficiently small. Recalling (6.3.1.13a–c), we have via (6.5.3):

$$\begin{aligned} -J^0_{w_\epsilon,\infty}(y_0) &= -J^0_{w_\epsilon,\infty}(y_0 = 0) - (P_{0,\infty}y_0, y_0)_Y - \mathcal{X}_{w_\epsilon,\infty}(y_0) \\ \text{(by (6.5.3))} \qquad &= (E_\gamma w_\epsilon, w_\epsilon)_{L_2(0,\infty;Y)} - (P_{0,\infty}y_0, y_0)_Y + (w_\epsilon, a_{y_0})_{L_2(0,\infty;V)}, \end{aligned} \tag{6.17.5}$$

where a_{y_0} is a suitable vector of $L_2(0, \infty; V)$, which depends on y_0. We now define the sequence w_k by setting $w_k = kw_\epsilon \in L_2(0, \infty; V)$. Then (6.17.5), evaluated for w_k, becomes

$$-J^0_{w_k,\infty}(y_0) = k^2(E_\gamma w_\epsilon, w_\epsilon) - (P_{0,\infty}y_0, y_0) + k(w_\epsilon, a_{y_0}) \to -\infty \quad \text{as } k \to +\infty, \tag{6.17.6}$$

where the limit $-\infty$ follows from (6.17.4) used on the first term on the right of (6.17.6). $\square$

Thus, Theorem 6.1.3.1 is fully proved.

6.18 Proof of Theorem 6.1.3.2

We now prove Theorem 6.1.3.2, which is the converse of Theorem 6.1.3.1.

Theorem 6.18.1 *Assume that $P = P^* \geq 0$ is an operator in $\mathcal{L}(Y)$ such that*

(i) the operator $A_F = A - BB^*P + \gamma^{-2}GG^*P$ is the generator of a s.c. uniformly stable semigroup $e^{A_F t}$ on Y for some $\gamma > 0$; and
(ii) P is a solution of the corresponding ARE_γ in (6.1.3.2), $\forall\, x, z \in \mathcal{D}(\hat{A}^\epsilon)$ with the properties that $B^*P \in \mathcal{L}(Y; U)$, $G^*P \in \mathcal{L}(Y; V)$.

Then, the cost functional $J^*(y_0)$ of the min–max problem game in (6.1.2.2) is finite for all $y_0 \in Y$, so that then $\gamma \geq \gamma_c$.

Proof. With such P, we define

$$\bar{p}(t; y_0) = Pe^{A_F t} y_0 \in L_2(0, \infty; Y) \cap C([0, \infty]; Y), \tag{6.18.1}$$

$$\gamma^2 \bar{w}(t; y_0) = G^* Pe^{A_F t} y_0 \in L_2(0, \infty; V) \cap C([0, \infty]; V), \tag{6.18.2}$$

$$\bar{u}(t; y_0) = -B^* Pe^{A_F t} y_0 \in C([0, \infty]; U), \quad y_0 \in \mathcal{D}(A_F). \tag{6.18.3}$$

Again, since P is a solution of the ARE$_\gamma$, with $B^*P \in \mathcal{L}(Y; U)$, $G^*P \in \mathcal{L}(Y; V)$, it follows, via (6.14.3),

$$\begin{aligned} -\frac{d}{dt}(Pe^{A_F t} y_0, e^{A_F t} y_0)_Y &= -2 \operatorname{Re}(P[A - BB^*P + \gamma^{-2} GG^*P]e^{A_F t} y_0, e^{A_F t} y_0)_Y \\ &= -2 \operatorname{Re}(PAe^{A_F t} y_0, e^{A_F t} y_0)_Y + 2\left\| B^* Pe^{A_F t} y_0 \right\|_U^2 \\ &\quad - 2\gamma^{-2} \left\| G^* Pe^{A_F t} y_0 \right\|_V^2. \end{aligned} \tag{6.18.4}$$

In view of (6.14.20), we may apply the ARE$_\gamma$ in (6.1.3.2) with $x = z = e^{A_F t} y_0 \in \mathcal{D}(\hat{A}^{1-\sigma})$, $t > 0$, $y_0 \in Y$ to the first term on the right-hand side of (6.18.4). We thus obtain

$$-\frac{d}{dt}(Pe^{A_F t} y_0, e^{A_F t} y_0) = \|B^* Pe^{A_F t} y_0\|_U^2 + \|Re^{A_F t} y_0\|_Z^2 - \gamma^{-2} \|G^* Pe^{A_F t} y_0\|_V^2, \quad y_0 \in Y. \tag{6.18.5}$$

Thus, since $e^{A_F t}$ is uniformly stable, integration in t over $[0, \infty]$ of (6.18.5) yields

$$(Py_0, y_0) = \int_0^\infty [\|Re^{A_F t} y_0\|_Z^2 + \|B^* Pe^{A_F t} y_0\|_U^2 - \gamma^{-2} \|G^* Pe^{A_F t} y_0\|_V^2]\, dt, \quad y_0 \in Y. \tag{6.18.6}$$

Moreover, $\bar{p}$, $\bar{w}$, and $\bar{u}$ as defined in (6.18.1)–(6.18.3), as well as $\bar{y}(t; y_0) = e^{A_F t} y_0$, satisfy the optimality conditions, so that (6.18.6) provides the achieved min–max cost $J^*(y_0)$. Since $J^*(y_0) < \infty$, then Proposition 6.17.1 implies that $\gamma \geq \gamma_c$. □

Part II: The Case Where e^{At} is Stable

6.19 Motivation, Statement of Main Results

Motivation for a Treatment of Stable Parabolic Problems Many free canonical parabolic, or parabolic-like, PDEs are intrinsically stable: That is, the constant ω_0 in (6.1.12a) is negative, $\omega_0 < 0$. Thus, via (6.1.12b), the s.c. semigroup e^{At} is (exponentially) stable in the uniform operator topology of $\mathcal{L}(Y)$ and hence satisfies condition (6.19.1) below. Examples include: (i) the pure heat equation with homogeneous Dirichlet BC, and also with Neumann B.C. if one factors out the one-dimensional null eigenspace of constant functions of the generator A [Chapter 3, Sections 3.1 and 3.3]; (ii) all structurally damped parabolic examples in Chapter 3, Sections 3.4 through 3.10; (iii) the thermo-elastic plates in Chapter 3, Sections 3.11–3.14]; etc. For the physically significant class of stable parabolic-like PDEs, it is desirable and possible to provide a more direct, less technical, streamlined version of the min–max theory (6.1.2.1), (6.1.2.2) of Part I, Sections 6.1 through 6.18. This is done in Sections 6.19-6.20. This simplified version of the min–max theory will also serve as a simplified version of the optimal quadratic cost problem over an infinite time horizon treated in Chapter 2, upon setting the disturbance $w \equiv 0$.

Assumptions of Part II. With reference to the dynamics (6.1.1) and the cost functional (6.1.2.1), the assumptions of the present Part II are: (H.1) on A; (H.2) = (6.1.1.6) on B; (H.3) = (6.1.1.7) on G; (H.4) = (6.1.1.8) on R, as well as the new stability assumption:

(S.1) There exist constants $M \geq 1$, $\omega > 0$ such that

$$\|e^{At}\|_{\mathcal{L}(Y)} \leq Me^{-\omega t}, \quad t \geq 0. \tag{6.19.1}$$

Accordingly, in view of (6.19.1), the control-theoretic assumptions (H.5) on the Finite Cost Condition and (H.6) on Detectability are automatically satisfied with control $u \equiv 0$ and operator $K = 0$, respectively. Moreover, it will be freely used below, that as a consequence of analyticity in (H.1) and of (S.1) = (6.19.1), the fractional powers $(-A)^{\theta}$ of $(-A)$ are well defined for $0 < \theta < 1$.

Simplifications and Benefits of the Stable Case Treatment In the present setting under (H.1)–(H.4) and (S.1), the overall line of argument of Part I can be markedly simplified at both the conceptual and technical level, with the additional benefit that all relevant quantities can be characterized directly in terms of the data $\{A, B, G, R, \gamma\}$. More specifically, one key advantage of the stable case is that its treatment eliminates altogether the need of introducing and using the Riccati operator $P_{0,\infty}$ corresponding to the optimal quadratic cost problem with no disturbance, $w \equiv 0$. Indeed, the purpose of $P_{0,\infty}$ in Part I was to introduce a stable form of the dynamics for $y^0_{w,\infty}$, $P_{w,\infty}$, and $r_{w,\infty}$ [see Section 6.4] in order to perform the maximization of $J^0_{w,\infty}(y_0)$ over w directly on the infinite time interval $[0, \infty]$, for $\gamma > \gamma_c$, and not as a limit process in T of the maximization on $[0, T]$. As a consequence, all relevant game-theoretic quantities can be expressed by explicit formulas directly in terms of the original data $\{A, B, G, R, \gamma\}$, without passing through $P_{0,\infty}$. This is illustrated in Table 6.1. In short, the stable case with $T = \infty$ is comporable to the general case with $T < \infty$, in terms of explicit formulas for the relevant quantities.

Thus, in this Part II we return to the abstract dynamics (6.1.1.1) subject to the assumptions (H.1) on the free dynamics operator A; (H.2) = (6.1.1.6) on the control operator B; (H.3) = (6.1.1.7) on the disturbance operator G; (H.4) = (6.1.1.8) on the observation operator R; along with the uniform stability assumption (S.1) = (6.19.1) on the s.c. analytic semigroup e^{At} on Y. We then reconsider the min–max game theory problem (6.1.2.1), (6.1.2.2), rewritten here for convenience: For a fixed $\gamma > 0$, we

Table 6.1. *Relevant quantities in Part I (unstable case) and Part II (stable case)*

Part I		Part II	
$\underbrace{\underbrace{A, B, R,}_{\downarrow} \; G, \gamma}$			
$\underbrace{P_{0,\infty} \quad\quad\quad}_{\downarrow}$		$\underbrace{A, B, G, R, \gamma}_{\downarrow}$	
γ_c	in (6.6.1)	γ_c	in (6.20.2.1)
$w^*(\cdot\,; y_0)$	in (6.8.1)	$w^*(\cdot\,; y_0)$	in (6.22.1)
$r^*(\cdot\,; y_0)$	in (6.7.6a)	$u^*(\cdot\,; y_0)$	in (6.22.5)
$y^*(\cdot\,; y_0)$	in (6.7.7a–c)	$Ry^*(\cdot\,; y_0), y^*(\cdot\,; y_0)$	in (6.22.6), (6.22.7)
$p^*(\cdot\,; y_0)$	in (6.7.8a–b)	$J^*(y_0), P$	in (6.22.8)
$u^*(\cdot\,; y_0)$	in (6.7.10a–b)		

associate with (6.1.1.1) the cost functional

$$J(u, w) = J(u, w, y(u, w)) = \int_0^\infty \left[\|Ry(t)\|_Z^2 + \|u(t)\|_U^2 - \gamma^2\|w(t)\|_V^2\right] dt, \tag{6.19.2}$$

and we study the following game-theory problem:

$$\sup_w \inf_u J(u, w), \tag{6.19.3}$$

where the infimum is taken over all $u \in L_2(0, \infty; U)$, for w fixed, and the supremum is taken over all $w \in L_2(0, \infty; Y)$. In (6.19.2), $y(t) = y(t; y_0)$ is the solution of (6.1.1.1) given explicitly by

$$y(t) = y(t; y_0) = e^{At}y_0 + (Lu)(t) + (Ww)(t), \tag{6.19.4}$$

where the operators L and W in (6.19.4) are given by

$$(Lu)(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \tag{6.19.5a}$$

$$: \text{continuous } L_2(0, \infty; U) \to L_2(0, \infty; Y), \tag{6.19.5b}$$

$$(Wu)(t) = \int_0^t e^{A(t-\tau)}Gw(\tau)\,d\tau \tag{6.19.6a}$$

$$: \text{continuous } L_2(0, \infty; V) \to L_2(0, \infty; Y). \tag{6.19.6b}$$

The dual versions of (6.19.5)–(6.19.6) are

$$(L^*f)(t) = B^*\int_t^\infty e^{A^*(\tau-t)}f(\tau)\,d\tau \tag{6.19.7a}$$

$$: \text{continuous } L_2(0, \infty; Y) \to L_2(0, \infty; U), \tag{6.19.7b}$$

$$(W^*v)(t) = G^* \int_t^\infty e^{A^*(\tau - t)} v(\tau)\, d\tau \tag{6.19.8a}$$

$$: \text{continuous } L_2(0,\infty; Y) \to L_2(0,\infty; V). \tag{6.19.8b}$$

Remark 6.19.1 In (6.19.5b) and (6.19.6b) – and, as a consequence, in (6.19.7b) and (6.19.8b) – in writing the regularity properties of L, W, L^*, and W^* over the infinite time interval, we have already taken advantage of the key stability assumption (S.1) = (6.19.1): Compare with Eqns. (6.2.3)–(6.2.6) in the general case. As we know by now, the above regularity results are conservative. Indeed, a key feature of the present analytic semigroup case is that L, W, L^*, and W^* are *smoothing* or *regularizing* operators. This will be noted and used at the appropriate spot in our analysis below, see Section 6.2.3.

We state the main result of Part II.

Theorem 6.19.1 *Assume (H.1), (H.2) = (6.1.1.6), (H.3) = (6.1.1.7), (H.4) = (6.1.1.8), and (S.1) = (6.19.1). Then, there exists a (critical) value $\gamma_c \geq 0$ defined explicitly in terms of the problem data by Eqn. (6.20.2.1) below such that:*

(a) *If $\gamma_c > 0$ and $0 < \gamma < \gamma_c$, then taking the supremum in w as in (6.19.3) leads to $+\infty$ for all initial conditions $y_0 \in Y$; that is, there is no finite solution of the game theory problem (6.19.3) (see Theorem 6.21.1(iii)).*

(b) *If $\gamma > \gamma_c$, then:*

(i) There exists a unique solution $\{u^*(\cdot\,; y_0); w^*(\cdot\,; y_0); y^*(\cdot\,; y_0)\}$ of the game-theory problem (6.19.3) (see Theorem 6.21.1(ii)).

(ii) There exists a bounded, nonnegative, self-adjoint operator, $P = P^* \in \mathcal{L}(Y)$, that satisfies the following algebraic Riccati equation ARE_γ, for all $x, z \in \mathcal{D}((-A)^\epsilon)$, $\forall \epsilon > 0$:

$$\begin{aligned}(PAx, z)_Y &+ (Px, Az)_Y + (Rx, Rz)_Y \\ &= (B^*Px, B^*Pz)_U - \gamma^{-2}(G^*Px, G^*Pz)_U \end{aligned}\tag{6.19.9}$$

(see Theorem 6.26.2.1 below), with the properties (see (6.26.1.2)–(6.26.1.3) of Proposition 6.26.1.1 below)

$$(-A)^\theta P \in \mathcal{L}(Y), \quad 0 \leq \theta < 1, \tag{6.19.10}$$

$$B^*P \in \mathcal{L}(Y; U); \quad G^*P \in \mathcal{L}(Y; U). \tag{6.19.11}$$

(iii) The following pointwise feedback relations hold:

$$u^*(t; y_0) = -B^*Py^*(t; y_0) \in L_2(0,\infty; U) \cap C([0,\infty]; U), \tag{6.19.12}$$

$$\gamma^2 w^*(t; y_0) = G^*Py^*(t; y_0) \in L_2(0,\infty; V) \cap C([0,\infty]; V) \tag{6.19.13}$$

(see Corollary 6.26.1.2 below).

(iv) The operator (F stands for "feedback") with maximal domain

$$A_F = A - BB^*P + \gamma^{-2}GG^*P; \tag{6.19.14}$$

$$\mathcal{D}(A_F) = \{x \in \mathcal{D}((-A)^{1-\sigma}) : (-A)^{1-\sigma}x - (-A)^{-\sigma}BB^*Px$$
$$+\gamma^{-2}(-A)^{-\sigma}GG^*Px \in \mathcal{D}((-A)^{\sigma})\} \tag{6.19.15a}$$
$$\in \mathcal{D}((-A)^{1-\sigma}), \tag{6.19.15b}$$

$\sigma = \max\{\rho, \delta\}$ is the generator of a s.c. semigroup e^{A_Ft} on Y, which is, moreover, analytic for $t > 0$ (see Corollary 6.26.2.2 below) and, in fact, for $y_0 \in Y$ (see (6.26.2.6) of Theorem 6.26.2.1 below):

$$y^*(t; y_0) = e^{A_Ft}y_0 = e^{(A-BB^*P+\gamma^{-2}GG^*P)t}y_0 \in L_2(0, \infty; Y) \cap C([0, \infty]; Y). \tag{6.19.16}$$

Moreover, the semigroup e^{A_Ft} is uniformly stable in Y (see Eqn. (6.25.7) of Corollary 6.25.3).

(v) The operators $(A - BB^*P)$ and $(A + \gamma^{-2}GG^*P)$, with maximal domains, generate s.c. analytic semigroups (Theorem 6.26.2.1 below), the first of which, $e^{(A-BB^*P)t}$, is, moreover, stable (as in Proposition 6.16.1).

Theorem 6.19.2 *Conversely, suppose that $P = P^* \geq 0$ is an operator in $\mathcal{L}(Y)$ such that:*

(a) The operator $A_F = A - BB^*P + \gamma^{-2}GG^*P$ is the generator of a s.c. uniformly stable semigroup on Y for some $\gamma > 0$;

(b) P is a solution of the corresponding ARE$_\gamma$ in (6.19.9), $\forall\, x, z \in \mathcal{D}((-A)^{\epsilon})$, $\forall\, \epsilon > 0$, with the properties that $B^*P \in \mathcal{L}(Y; U)$ and $G^*P \in \mathcal{L}(Y; V)$.

Then, the operators $(A - BB^*P)$ and $(A - \gamma^{-2}GG^*P)$ are likewise the generators of s.c. semigroups on Y, the first of which is uniformly stable, and, moreover, the game problem (6.19.3) has a finite optimal cost functional for all $y_0 \in Y$, so that then $\gamma \geq \gamma_c$ (as in Theorem 6.18.1).

Additional results are given in the treatment below.

6.20 Minimization of J over u for w Fixed

6.20.1 Existence of a Unique Optimal Pair and Optimality Conditions

We return to the cost functional J in (6.19.2). In this section we study the following minimization problem, given a fixed but arbitrary $w \in L_2(0, \infty; Y)$:

$$\inf_{u \in L_2(0,\infty;U)} J(u, w; y_0) \text{ holding } w \in L_2(0, \infty; V) \text{ fixed.} \tag{6.20.1.1}$$

The advantages of assumption (S.1) = (6.19.1) are reaped off at the very outset of the analysis, by obtaining all relevant quantities expressed directly in terms of the problem data.

Theorem 6.20.1.1 *Assume (H.1), (H.2) = (6.1.1.6), (H.3) = (6.1.1.7), and (H.4) = (6.1.1.8).*

(i) With reference to the minimization problem (6.20.1.1) for the dynamics (6.1.1.1), there exists a unique optimal pair denoted by $\{u_w^0(\cdot\,;y_0), y_w^0(\cdot\,;y_0)\}$, *with the corresponding optimal cost denoted by*

$$J_w^0(y_0) = J\big(u_w^0(\cdot\,;y_0), y_w^0(\cdot\,;y_0)\big)$$
$$= \int_0^\infty \big[\|Ry_w^0(t;y_0)\|_Z^2 + \|u_w^0(t;y_0)\|_U^2 - \gamma^2\|w(t)\|_V^2\big]\,dt. \tag{6.20.1.2}$$

Assume further (S.1) = (6.19.1). Then:

(ii) The optimal pair is related by

$$u_w^0(\cdot\,;y_0) = -L^*R^*Ry_w^0(\cdot\,;y_0), \tag{6.20.1.3}$$

and is explicitly given in terms of the problem data by the following formulas:

$$-u_w^0(\cdot\,;y_0) = [I + L^*R^*RL]^{-1}L^*R^*R[e^{A\cdot}y_0 + Ww] \in L_2(0,\infty;U) \tag{6.20.1.4a}$$

$$= -u_{w=0}^0(\cdot\,;y_0) - u_w^0(\cdot\,;y_0=0), \tag{6.20.1.4b}$$

$$y_w^0(\cdot\,;y_0) = [I + LL^*R^*R]^{-1}[e^{A\cdot}y_0 + Ww] \in L_2(0,\infty;Y) \tag{6.20.1.5a}$$

$$= y_{w=0}^0(\cdot\,;y_0) + y_w^0(\cdot\,;y_0=0), \tag{6.20.1.5b}$$

$$Ry_w^0(\cdot\,;y_0) = [I + RLL^*R^*]^{-1}[Re^{A\cdot}y_0 + RWw] \in L_2(0,\infty;Z), \tag{6.20.1.6}$$

where the inverse operators in (6.20.1.4a), (6.20.1.5a), and (6.20.1.6) are well defined as bounded operators on all of $L_2(0,\infty;U)$, $L_2(0,\infty;Y)$, *and* $L_2(0,\infty;Z)$, *respectively (for* $[I + LL^*R^*R]^{-1}$ *see Chapter 2, Appendix 2A). Moreover, the optimal dynamics is*

$$y_w^0(t;y_0) = e^{At}y_0 + \big\{Lu_w^0(\cdot\,;y_0)\big\}(t) + \{Ww(\cdot)\}(t) \in L_2(0,\infty;Y). \tag{6.20.1.7}$$

(iii) The optimal cost $J_w^0(y_0)$ *in (6.20.1.2) is given explicitly in terms of the data by the following formulas:*

$$J_w^0(y_0) = (Re^{A\cdot}y_0 + RWw, [I + RLL^*R^*]^{-1}[Re^{A\cdot}y_0 + RWw])_{L_2(0,\infty;Z)}$$
$$- \gamma^2(w,w)_{L_2(0,\infty;V)} \tag{6.20.1.8a}$$

$$= J_{w=0}^0(y_0) + J_w^0(y_0=0) + \chi_{y_{0,w}}, \tag{6.20.1.8b}$$

$$J_{w=0}^0(y_0) = (P_{0,\infty}y_0, y_0)_Y$$
$$= (Re^{A\cdot}y_0, [I + RLL^*R^*]^{-1}Re^{A\cdot}y_0)_{L_2(0,\infty;Z)}, \tag{6.20.1.9}$$

$$J_w^0(y_0 = 0) = (RWw, [I + RLL^*R^*]^{-1}RWw)_{L_2(0,\infty;Z)} \\ - \gamma^2(w, w)_{L_2(0,\infty;V)} \tag{6.20.1.10}$$

$$\equiv -(w, E_\gamma w)_{L_2(0,\infty;V)} = \text{quadratic in } w, \tag{6.20.1.11}$$

$$E_\gamma = \gamma^2 I - W^*R^*[I + RLL^*R^*]^{-1}RW = \gamma^2 I - S \in L_2(0, \infty; V), \tag{6.20.1.12}$$

$$S = W^*R^*[I + RLL^*R^*]^{-1}RW \in \mathcal{L}(L_2(0, \infty; V)) \tag{6.20.1.13a}$$

$$= \text{nonnegative, self-adjoint operator in } \mathcal{L}(L_2(0, \infty; V)). \tag{6.20.1.13b}$$

The cross term in (6.20.1.8b) is linear in w:

$$\chi_{y_0,w} = 2(Re^{A\cdot}y_0, [I + RLL^*R^*]^{-1}RWw)_{L_2(0,\infty;Z)} \\ = \text{linear in } w. \tag{6.20.1.14}$$

Proof. (i) For fixed $w \in L_2(0, \infty; V)$, the optimal problem is a standard quadratic (strictly convex) problem in u, which has a unique optimal solution.

(ii) Under the stability assumption (6.19.1), whereby properties (6.19.5)–(6.19.8) hold true, we consider the Lagrangean

$$L(u, y, \lambda) = \frac{1}{2}\left[(R^*Ry, y)_{L_2(0,\infty;Y)} + (u, u)_{L_2(0,\infty;U)}\right] \\ + (\lambda, y - e^{A\cdot}y_0 - Lu - Ww)_{L_2(0,\infty;Y)}, \tag{6.20.1.15}$$

to which we apply Liusternik's general Lagrange Multiplier Theorem [Luenberger, 1969, Theorem 1, p. 243] (note that for any $g \in L_2(0, \infty; Y)$, we can take $u = w = 0$ and $y = e^{A\cdot}y_0 + g$ to satisfy the required surjectivity condition, because of (6.19.1)): There exist $u_w^0 \in L_2(0, \infty; U)$, $y_w^0, \lambda_w^0 \in L_2(0, \infty; Y)$ such that $L_u = L_y = L_\lambda = 0$ at $(u_w^0, y_w^0, \lambda_w^0)$. From (6.20.1.15) we obtain in the appropriate inner products:

$$L_y = 0 : (R^*Ry, \delta y) + (\lambda, \delta y) = 0, \qquad \forall \delta y \in L_2(0, \infty; Y), \tag{6.20.1.16}$$

$$\lambda_w^0 = -R^*Ry_w^0; \tag{6.20.1.17}$$

$$L_u = 0 : (u, \delta u) - (L^*\lambda, \delta u) = 0, \qquad \forall \delta u \in L_2(0, \infty; U), \tag{6.20.1.18}$$

$$u_w^0 = L^*\lambda_w^0. \tag{6.20.1.19}$$

Then (6.20.1.17) inserted into (6.20.1.19) yields the basic relationship (6.20.1.3) between the optimal y_w^0 and the optimal u_w^0, as desired, where we have explicitly indicated the dependence on y_0.

Inserting the optimal dynamics (6.20.1.7) into (6.20.1.3) yields readily u_w^0 in (6.20.1.4). Moreover, applying L to (6.20.1.3) and inserting the resulting Lu_w^0 in the optimal dynamics (6.20.1.7) yields readily y_w^0 in (6.20.1.5). [We recall from Chapter 2, Appendix 2A that $[I + LL^*R^*R]^{-1} \in \mathcal{L}(L_2(0, \infty; Y))$.]

(iii) By (6.20.1.3), inserting $(u_w^0, u_w^0) = -(u_w^0, L^*R^*Ry_w^0) = -(Lu_w^0, R^*Ry_w^0)$ into (6.20.1.2) yields

$$
\begin{aligned}
J_w^0(y_0) &= \left(y_w^0, R^*Ry_w^0\right) + \left(u_w^0, u_w^0\right) - \gamma^2(w, w) \\
&= \left(y_w^0 - Lu_w^0, R^*Ry_w^0\right) - \gamma^2(w, w) \\
\text{(by (6.20.1.7))} \quad &= \left(e^{A\cdot} y_0 + Ww, R^*Ry_w^0\right) - \gamma^2(w, w) \\
\text{(by (6.20.1.6))} \quad &= (R[e^{A\cdot} y_0 + Ww], [I + RLL^*R^*]^{-1}R[e^{A\cdot} y_0 + Ww]) \\
&\quad - \gamma^2(w, w),
\end{aligned} \tag{6.20.1.20}
$$

and (6.20.1.8) is established. □

Remark 6.20.1.1 One could give better regularity results for the intermediate quantities $\{u_w^0, y_w^0\}$ using the subsequent Theorem 6.23.1; however, we shall do this for the final quantities$\{u^*, y^*, w^*\}$ of interest in Section 6.23.

Remark 6.20.1.2 In the above formula, one may note the identity

$$
[I + RLL^*R^*]^{-1}R = R[I + LL^*R^*R]^{-1} \in \mathcal{L}(L_2(0, \infty; Y); L_2(0, \infty; Z)), \tag{6.20.1.21}
$$

which is easily verified, once the inverse on the right-hand side is established by Chapter 2, Appendix 2A. Thus, in particular, we may rewrite S in (6.20.1.13) as

$$
S = W^*R^*R[I + LL^*R^*R]^{-1}W \in \mathcal{L}(L_2(0, \infty; V)). \tag{6.20.1.22}
$$

6.20.2 Strict Positive-Definiteness of the Operator E_γ for $\gamma > \gamma_c$

We return to the self-adjoint operator E_γ in $\mathcal{L}(L_2(0, \infty; Y))$ defined by (6.2.1.12). (We note that we defined $S \geq 0$ in (6.20.1.13) so that $E_\gamma = \gamma^2 I - S$ in (6.20.1.12). Instead, in Part I of the chapter, we defined $S \leq 0$ in (6.5.1), so that $E_\gamma = \gamma^2 I + S$ in (6.5.2).) Define the critical value γ_c of γ by:

$$
\gamma_c^2 \equiv \|S\|_{\mathcal{L}(L_2(0,\infty;V))} = \|W^*R^*[I + RLL^*R^*]^{-1}RW\|_{\mathcal{L}(L_2(0,\infty;V))} \tag{6.20.2.2a}
$$

$$
= \sup_{\|w\|=1} (W^*R^*[I + RLL^*R^*]^{-1}RWw, w)_{L_2(0,\infty;V)}, \tag{6.20.2.2b}
$$

where the norm of w is in $L_2(0, \infty; V)$. As a consequence of (6.20.2.1), we obtain:

Corollary 6.20.2.1 *Assume (H.1)–(H.4) and (S.1) = (6.19.1). The self-adjoint operator $E_\gamma \in \mathcal{L}(L_2(0, \infty; V))$ in (6.20.1.12) is (strictly) positive if and only if $\gamma > \gamma_c$ (defined by (6.20.2.1)), in which case:*

$$
(E_\gamma w, w)_{L_2(0,\infty;V)} \geq \left(\gamma^2 - \gamma_c^2\right)\|w\|^2_{L_2(0,\infty;V)}, \tag{6.20.2.3}
$$

in which case $E_\gamma^{-1} \in \mathcal{L}(L_2(0, \infty; V))$.

6.21 Maximization of $J_w^0(y_0)$ over w: Existence of a Unique Optimal w^*

In this section we return to the optimal $J_w^0(y_0)$ in (6.20.1.9) for $w \in L_2(0, \infty; V)$ fixed and consider the problem:

$$\text{maximize } J_{w,\infty}^0, \text{ equivalently, minimize } -J_{w,\infty}^0(y_0), \text{ over all } w \in L_2(0, \infty; V). \tag{6.21.1}$$

Theorem 6.21.1 *Assume (H.1)–(H.4) and (S.1) = (6.19.1).*

(i) For $\gamma > \gamma_c$ (defined in (6.20.2.1)), and with reference to (6.20.1.8), the following estimate holds true for any $\epsilon > 0$ and every $w \in L_2(0, \infty; V)$:

$$-J_w^0(y_0) \geq \left[\gamma^2 - \left(\gamma_c^2 + \epsilon\right)\right] \|w\|^2_{L_2(0,\infty;V)} - J_{w=0}^0(y_0) - C_\epsilon \|y_0\|_Y^2. \tag{6.21.2}$$

(ii) For $\gamma > \gamma_c$ (defined in (6.20.2.1)), there exists a unique optimal solution $w^*(\,\cdot\,; y_0) \in L_2(0, \infty; V)$ for the optimal problem (6.21.1):

$$\max_{w \in L_2(0,\infty;V)} J_w^0(y_0) \equiv J_{w=w^*}^0(y_0) \equiv J^*(y_0). \tag{6.21.3}$$

(iii) Let $\gamma_c > 0$. If $0 < \gamma < \gamma_c$, then $\sup_w J_{w,\infty}^0(y_0) = +\infty$ for all initial conditions $y_0 \in Y$.

Proof. (i) We return to (6.20.1.8b), which gives $-J_w^0(y_0)$ as the sum of three contributions: a quadratic term in w, given by $-J_w^0(y_0 = 0) = (w, E_\gamma w)$ in (6.20.1.11), which satisfies (6.20.2.2); a linear term in w, given by $-\chi_{y_0,w}$ in (6.20.1.14), and satisfying, with $\mathcal{X} = W^* R^* [I + RLL^* R^*]^{-1} Re^{A\,\cdot}$,

$$|\chi_{y_0,w}| \leq 2\|w\|_{L_2(0,\infty;V)} \|\mathcal{X} y_0\|_{L_2(0,\infty;Y)} = \epsilon \|w\|^2_{L_2(0,\infty;V)} + \epsilon^{-1} \|\mathcal{X}\|^2 \, \|y_0\|_Y^2, \tag{6.21.4}$$

where the norm of $\mathcal{X}$ is in $\mathcal{L}(Y; L_2(0, \infty; Y))$; and finally a constant term in w given by $-J_{w=0}^0(y_0)$ in (6.20.1.9). Thus, (6.21.2) follows with $C_\epsilon = \epsilon^{-1}\|\mathcal{X}\|^2$.

(ii) The expression of $-J_w^0(y_0)$ given by (6.20.1.8b) as a quadratic functional, bounded below by part (i), guarantees that there exists a unique optimal solution w^* in $L_2(0, \infty; V)$.

(iii) If $0 < \gamma < \gamma_c$, then, by (6.20.2.1) and (6.20.1.12),

$$\inf_{\|w\|=1} (E_\gamma w, w)_{L_2(0,\infty;V)} = m < 0. \tag{6.21.5}$$

Hence, for $\epsilon > 0$ sufficiently small, there exists w_ϵ, $\|w_\epsilon\| = 1$ such that $(E_\gamma w_\epsilon, w_\epsilon)_{L_2(0,\infty;V)} < m + \epsilon < 0$. Then, define $w_k = k w_\epsilon \in L_2(0, \infty; V)$, for a real constant k. From (6.20.1.8b), (6.20.1.14) and (6.20.1.11), we have

$$-J_{w_k}^0(y_0) = k^2 (E_\gamma w_\epsilon, w_\epsilon)_{L_2(0,\infty;V)} - 2k(w_\epsilon, \mathcal{X} y_0) - J_{w=0}^0(y_0) \to -\infty \quad \text{as } k \to \infty, \tag{6.21.6}$$

as desired, since $(E_\gamma w_\epsilon, w_\epsilon) < 0$ by (6.21.5). (This proof coincides with that of Proposition 6.17.1.) $\square$

With the optimal w^* provided by Theorem 6.21.1(ii), we return to the optimal pair $\{u_w^0, y_w^0\}$ over u of Theorem 6.20.1.1, and set, along with (6.21.3):

$$u^*(\cdot\,; y_0) \equiv u^0_{w=w^*}(\cdot\,; y_0) \in L_2(0, \infty; U); \tag{6.21.7}$$
$$y^*(\cdot\,; y_0) \equiv y^0_{w=w^*}(\cdot\,; y_0) \in L_2(0, \infty; Y).$$

Theorem 6.21.2 *Assume (H.1)–(H.4) and (S.1) = (6.19.1).*

(i) The unique optimal $w^*(\cdot\,; y_0)$ provided by Theorem 6.3.1(ii) is given explicitly in terms of the problem data by (see (6.19.8) and (6.3.7)):

$$\gamma^2 w^*(\cdot\,; y_0) = W^* R^* R y^*(\cdot\,; y_0) \in L_2(0, \infty; V), \quad \gamma > \gamma_c. \tag{6.21.8}$$

(ii) Thus, for $\gamma > \gamma_c$ (defined by (6.20.2.1)), the original min–max problem (6.19.3) has a unique solution $\{u^*(\cdot\,; y_0), y^*(\cdot\,; y_0), w^*(\cdot\,; y_0)\}$ satisfying (6.21.8) and given by

$$-u^*(\cdot\,; y_0) = [I + L^*R^*RL]^{-1} L^*R^*R[e^{A\cdot} y_0 + Ww^*(\cdot\,; y_0)] \in L_2(0, \infty; U); \tag{6.21.9}$$

$$y^*(\cdot\,; y_0) = [I + LL^*R^*R]^{-1}[e^{A\cdot} y_0 + Ww^*(\cdot\,; y_0)] \in L_2(0, \infty; Y); \tag{6.21.10}$$

$$u^*(\cdot\,; y_0) = -L^*R^*Ry^*(\cdot\,; y_0); \tag{6.21.11}$$

$$Ry^*(\cdot\,; y_0) = [I + RLL^*R^*]^{-1}[Re^{A\cdot} y_0 + RWw^*(\cdot\,; y_0)] \in L_2(0, \infty; Z), \tag{6.21.12}$$

with optimal dynamics

$$y^*(t; y_0) = e^{At} y_0 + \{Lu^*(\cdot\,; y_0)\}(t) + \{Ww^*(\cdot\,; y_0)\}(t), \tag{6.21.13}$$

which therefore satisfies

$$\{[I + LL^*R^*R - \gamma^{-2}WW^*R^*R]y^*(\cdot\,; y_0)\}(t) = e^{At} y_0; \tag{6.21.14}$$
$$\{[I + RLL^*R^* - \gamma^{-2}RWW^*R^*]Ry^*(\cdot\,; y_0)\}(t) = Re^{At} y_0. \tag{6.21.15}$$

Proof. A first Lagrange multiplier proof, in the style of Chapter 1, Theorem 1.2.1.1 is as follows. Let

$$\left\{u_w^0, y_w^0, \lambda, w\right\} \in L_2(0, \infty; U) \times [L_2(0, \infty; Y)]^2 \times L_2(0, \infty; V)$$

be free parameters and consider the Lagrangean

$$L\left(u_w^0, y_w^0, \lambda, w\right) = \tfrac{1}{2}\left[\left(R^*Ry_w^0, y_w^0\right) + \left(u_w^0, u_w^0\right)\right] + \left(\lambda, y_w^0 - Lu_w^0 - e^{A\cdot} y_0 - Ww\right) \tag{6.21.16}$$

in the appropriate inner products, to which we apply Liusternik's theorem [Luenberger, 1966, Theorem 1, p. 243]: There exists a unique optimal solution $\{u^*, y^*, \lambda^*, w^*\}$ such that:

$$L_{u_w^0} = 0 : \left(u_w^0, \delta u_w^0\right) - \left(L^*\lambda, \delta u_w^0\right) = 0, \qquad \forall\, \delta u_w^0 \in L_2(0,\infty; U), \tag{6.21.17}$$

$$u^* \equiv u^0_{w=w^*} = L^*\lambda^*; \tag{6.21.18}$$

$$L_{y_w^0} = 0 : \left(R^*Ry_w^0, \delta y_w^0\right) + \left(\lambda, \delta y_w^0\right) = 0, \qquad \forall\, \delta y_w^0 \in L_2(0,\infty; Y), \tag{6.21.19}$$

$$\lambda^* = -R^*Ry^0_{w=w^*} \equiv -R^*Ry^*, \tag{6.21.20}$$

$$u^* = -L^*R^*Ry^*; \tag{6.21.21}$$

$$L_w = 0 : -\gamma^2(w, \delta w) - (W^*\lambda, \delta w) = 0, \qquad \forall\, \delta w \in L_2(0,\infty; V), \tag{6.21.22}$$

$$-\gamma^2 w^* = W^*\lambda^* = -W^*R^*Ry^*. \tag{6.21.23}$$

Thus, (6.21.8)–(6.21.11) are proved.

As a second proof, we set to zero the variation with respect to w of

$$J_w^0(y_0) = \left(R^*Ry_w^0, y_w^0\right) + \left(u_w^0, u_w^0\right) - \gamma^2(w, w), \tag{6.21.24}$$

that is, we set

$$\left(R^*Ry_w^0, \delta y_w^0\right) + \left(u_w^0, \delta u_w^0\right) - \gamma^2(w, \delta w) = 0, \tag{6.21.25}$$

where from (6.19.4), we insert $\delta y_w^0 = L\delta u_w^0 + W\delta w$ into (6.21.25) and recall (6.20.1.3) for $w = w^*$, to reobtain (6.21.21) and (6.21.23), from (6.21.25). Finally, inserting (6.21.8), (6.21.11), and (6.21.12) into (6.21.13) yields (6.21.14) and (6.21.15). □

Remark 6.21.1 We are not authorized to boundedly invert on $L_2(0,\infty; V)$ the operator $[I + LL^*R^*R - \gamma^{-2}WW^*R^*R]$ in (6.21.14) for all $\gamma > \gamma_c$; we can only do so for γ sufficiently large, in fact, $\gamma^2 > \gamma_1^2 = \|I + LL^*R^*R\| / \|WW^*R^*R\|$, in $L_2(0,\infty; Y)$-norms. A similar situation holds for the operator $[I + RLL^*R^* - \gamma^{-2}RWW^*R^*]$ in (6.21.15). Thus, a modified argument will be indicated in Proposition 6.4.1 below; see also Remark 6.4.1.

6.22 Explicit Expressions of $\{u^*, y^*, w^*\}$ and P for $\gamma > \gamma_c$ in Terms of the Data via E_γ^{-1}

We begin with an explicit expression of $w^*(\,\cdot\,; y_0)$ for $\gamma > \gamma_c$ in terms of the data via E_γ^{-1}.

Proposition 6.22.1 *Assume (H.1)–(H.4) and (S.1) = (6.19.1). For $\gamma > \gamma_c$ (defined by (6.20.2.1)), we have*

$$w^*(\,\cdot\,; y_0) = E_\gamma^{-1}W^*R^*R[I + LL^*R^*R]^{-1}(e^{A\,\cdot}y_0) \in L_2(0,\infty; V) \tag{6.22.1a}$$

$$(\text{by (6.20.1.21)}) \qquad = E_\gamma^{-1}W^*R^*[I + RLL^*R^*]^{-1}Re^{A\,\cdot}y_0. \tag{6.22.1b}$$

Proof. We insert (6.21.10), respectively (6.21.12), into (6.21.8), thereby obtaining

$$[\gamma^2 I - W^*R^*R[I + LL^*R^*R]^{-1}W]w^*(\cdot\,; y_0) = W^*R^*R[I + LL^*R^*R]^{-1}(e^{A\cdot}y_0), \tag{6.22.2a}$$

$$[\gamma^2 I - W^*R[I + RLL^*R^*]^{-1}RW]w^*(\cdot\,; y_0) = W^*R^*[I + RLL^*R^*]^{-1}Re^{A\cdot}y_0, \tag{6.22.2b}$$

respectively, or else we recall identity (6.20.1.21) to pass from (6.4.2a) to (6.4.2b). Recalling the definition (6.20.1.12) of E_γ (see also (6.20.1.22)), we rewrite (6.22.2) as

$$E_\gamma w^*(\cdot\,; y_0) = W^*R^*R[I + LL^*R^*R]^{-1}(e^{A\cdot}y_0) \tag{6.22.3a}$$

$$= W^*R^*[I + RLL^*R^*]^{-1}Re^{A\cdot}y_0, \tag{6.22.3b}$$

respectively, from which (6.22.1) follows for $\gamma > \gamma_c$ by Corollary 6.20.2.1. $\square$

Remark 6.22.1 If we apply $W^*R^*R[I + LL^*R^*R]^{-1}$ to Eqn. (6.21.14), we get

$$W^*R^*R[I + LL^*R^*R]^{-1}\{[I + LL^*R^*R]y^*(\cdot\,; y_0) - \gamma^{-2}WW^*R^*Ry^*(\cdot\,; y_0)\} = W^*R^*R[I + LL^*R^*R]^{-1}(e^{A\cdot}y_0). \tag{6.22.4}$$

We then use (6.21.8) to obtain Eqn. (6.22.2a). But, Eqn. (6.22.2a) is solvable for w^* for $\gamma > \gamma_c$, whereas Eqn. (6.21.14) is solvable for y^* for γ sufficiently large; see Remark 6.21.1.

However, inserting w^* given by (6.22.1) into the right-hand sides of (6.21.9), (6.21.10), and (6.20.1.8a) produces explicit expressions for u^*, y^*, and P for all $\gamma > \gamma_c$ in terms of the problem data. We obtain:

Corollary 6.22.2 *Assume (H.1)–(H.4) and (S.1) = (6.19.1). The following formulas complement (6.22.1) by giving explicit expressions of $u^*(\cdot\,; y_0)$, $y^*(\cdot\,; y_0)$, $Ry^*(\cdot\,; y_0)$, $J^*(y_0)$, and P, in terms of the data:*

$$-u^*(\cdot\,; y_0) = [I + L^*R^*RL]^{-1}L^*R^*\big\{I + RWE_\gamma^{-1}W^*R^*[I + RLL^*R^*]^{-1}\big\}Re^{A\cdot}y_0, \tag{6.22.5}$$

$$Ry^*(\cdot\,; y_0) = [I + RLL^*R^*]^{-1}\big\{I + RWE_\gamma^{-1}W^*R^*[I + RLL^*R^*]^{-1}\big\}Re^{A\cdot}y_0, \tag{6.22.6}$$

$$y^*(\cdot\,; y_0) = [I + LL^*R^*R]^{-1}\big\{I + WE_\gamma^{-1}W^*R^*R[I + LL^*R^*R]^{-1}\big\}e^{A\cdot}y_0, \tag{6.22.7}$$

$$J^*(y_0) = J^0_{w=w^*}(y_0) = (Py_0, y_0)_Y \tag{6.22.8}$$

$$= (Re^{A\cdot}y_0 + RWw^*, [I + RLL^*R^*]^{-1}[Re^{A\cdot}y_0 + RWw^*])_{L_2(0,\infty;Z)} \tag{6.22.9}$$

$$
\begin{aligned}
&= (Re^{A\cdot} y_0, [I + RLL^*R^*]^{-1} Re^{A\cdot} y_0)_{L_2(0,\infty;Z)} \\
&\quad + (RWE_\gamma^{-1} W^*R^*[I + RLL^*R^*]^{-1} Re^{A\cdot} y_0, [I + RLL^*R^*]^{-1} \\
&\quad \times RWE_\gamma^{-1} W^*R^*[I + RLL^*R^*]^{-1} Re^{A\cdot} y_0)_{L_2(0,\infty;Z)} \\
&\quad + 2(Re^{A\cdot} y_0, [I + RLL^*R^*]^{-1} RWE_\gamma^{-1} W^*R^* \\
&\quad \times [I + RLL^*R^*]^{-1} Re^{A\cdot} y_0)_{L_2(0,\infty;Z)}. \qquad (6.22.10)
\end{aligned}
$$

Remark 6.22.2 Formula (6.22.10) shows explicitly the quantitative relationship between the Riccati operator P of the min–max problem (6.19.3) and the Riccati operator $P_{w=0}$ of the optimal control problem (with no disturbance) of Chapter 2, that is, recalling (6.20.1.9),

$$
(P_{w=0}y_0, y_0) = J^0_{w=0}(y_0) = (Re^{A\cdot} y_0, [I + RLL^*R^*]^{-1} Re^{A\cdot} y_0)_{L_2(0,\infty;Z)}, \qquad (6.22.11)
$$

in particular,

$$
J^*(y_0) = (Py_0, y_0)_Y \geq J^0_{w=0}(y_0) = (P_{w=0}y_0, y_0)_Y, \qquad (6.22.12)
$$

as expected from the definition of the min–max value in (6.19.3).

6.23 Smoothing Properties of the Operators L, L^*, W, W^*: The Optimal u^*, y^*, w^* Are Continuous in Time

The present section is the counterpart of Section 6.9 in the general case, to whose first paragraph we refer for motivation and enlightenment. We are facing the same issue as in Chapter 1, Section 1.3 and Chapter 2, Section 2.3.5. It is the counterpart of Theorem 6.9.1 of Part I.

Theorem 6.23.1 *Assume (H.1), (H.2), (H.3), and (S.1) = (6.19.1).*

(a) With reference to the operators L and L^* defined by (6.19.5) and (6.19.7) we have:

(i)

$$
L : \textit{continuous } L_2(0,\infty;U) \to L_{\ell_1}(0,\infty;Y), \qquad (6.23.1)
$$

where ℓ_1 is an arbitrary positive number satisfying $\ell_1 < \frac{2}{2\delta-1}$; here $\frac{2}{2\delta-1} > 2$ for $1/2 < \delta < 1$, while for $0 \leq \delta < 1/2$, we may take $\ell_1 = \infty$ and indeed replace $L_\infty(0,\infty;\ \cdot)$ with $C([0,\infty];\ \cdot)$.

(ii)

$$
L^* : \textit{continuous } L_{\ell_1}(0,\infty;Y) \to L_{\ell_2}(0,\infty;U), \qquad (6.23.2)
$$

where ℓ_1 is as in (i), and ℓ_2 is positive number satisfying $\ell_2 < \frac{2}{4\delta-3}$; here $\frac{2}{4\delta-3} > \ell_1$ for $3/4 < \delta < 1$; for $0 < \delta < 3/4$, we may take $\ell_2 = \infty$ and indeed replace $L_\infty(0,\infty;\ \cdot\ )$ with $C([0,\infty];\ \cdot\ )$.

(iii) Generally, let $\ell_0 = 2$, *and let* ℓ_n, $n = 1, 2, \ldots$ *be arbitrary positive numbers such that*

$$\ell_n < \frac{2}{2n\delta - (2n-1)}, \quad n = 1, 2, \ldots. \tag{6.23.3}$$

Then, for $n = 0, 2, 4, \ldots$ *we have that*

$$L : \text{continuous } L_{\ell_n}(0, \infty; U) \to L_{\ell_{n+1}}(0, \infty; Y), \tag{6.23.4}$$

where for $0 \le \delta < \frac{2(n+1)-1}{2(n+1)}$ *we may take* $\ell_{n+1} = \infty$ *and indeed replace* $L_\infty(0, \infty; \,\cdot\,)$ *with* $C([0, \infty]; \,\cdot\,)$, *and, moreover,*

$$L^* : \text{continuous } L_{\ell_{n+1}}(0, \infty; Y) \to L_{\ell_{n+2}}(0, \infty; U), \tag{6.23.5}$$

where ℓ_{n+1} *in (6.23.5) is the same as in (6.23.4), and where for* $0 \le \delta < \frac{2(n+1)-1}{2(n+2)}$ *we may take* $\ell_{n+2} = \infty$ *and indeed replace* $L_\infty(0, \infty; \,\cdot\,)$ *with* $C([0, \infty]; \,\cdot\,)$.

(iv) For $p > \frac{1}{1-\delta}$,

$$L : \text{continuous } L_p(0, \infty; U) \to C([0, \infty]; Y), \tag{6.23.6}$$

(v)

$$L^* : \text{continuous } C([0, \infty]; Y) \to C([0, \infty]; U). \tag{6.23.7}$$

(b) Similar results apply for the operators W and W defined by (6.19.6) and (6.19.8),* mutatis mutandis, *that is, replacing* δ *in (6.1.1.2) with* ρ *in (6.1.1.3).*

Proof. See Chapter 1, Section 1.3 or Chapter 2, Section 2.3.5. □

As a corollary, we then obtain the desired time regularity property in $C([0, \infty]; \,\cdot\,)$ of the optimal quantities.

Corollary 6.23.2 *Assume (H.1)–(H.4) and (S.1) = (6.19.1). With reference to the optimal solutions* $\{u^*, w^*, y^*\}$, *we have for any* $y_0 \in Y$:

$$u^*(\,\cdot\,; y_0) \in C([0, \infty]; U); \quad y^*(\,\cdot\,; y_0) \in C([0, \infty]; Y);$$
$$w^*(\,\cdot\,; y_0) \in C([0, \infty]; U). \tag{6.23.8}$$

Proof (Counterpart of Corollary 6.9.3). The proof proceeds by a bootstrap argument playing alternatively between (6.21.13) on the one hand and (6.21.8) and (6.21.11) on the other. To begin with, we already have from Theorem 6.21.2 that

$$u^*(\,\cdot\,; y_0) \in L_2(0, \infty; U); \quad w^*(\,\cdot\,; y_0) \in L_2(0, \infty; V); \quad y^*(\,\cdot\,; y_0) \in L_2(0, \infty; Y). \tag{6.23.9}$$

It is not restrictive to assume that $\delta = \rho$ for the purposes of this proof and thus use the same regularity results for L and W and for L^* and W^* from Theorem 6.23.1. Thus, invoking (6.23.1) for L and W we obtain that $Lu^*, Ww^* \in L_{\ell_1}(0, \infty; Y)$. Since

e^{At} is exponentially stable as in (6.19.1), we then obtain from the optimal dynamics (6.21.13) that $y^* \in L_{\ell_1}(0,\infty;Y)$ as well. Next, recall (6.21.8) and (6.21.11) and apply (6.23.2) for L^* and W^* to obtain that $w^*,\ u^* \in L_{\ell_2}(0,\infty;\,\cdot)$. We then repeat this bootstrap argument on L and W in (6.21.13), invoking (6.23.4), and on L^* and W^* in (6.21.8) and (6.21.11), invoking (6.23.5), and we obtain the desired time continuity for $u^*(\,\cdot\,;y_0)$, $w^*(\,\cdot\,;y_0)$, and $y^*(\,\cdot\,;y_0)$, by Theorem 6.23.1(iv), (v). $\square$

Having obtained the desired time continuity for $u^*(\,\cdot\,;y_0)$, $w^*(\,\cdot\,;y_0)$, *and* $y^*(\,\cdot\,;y_0)$, we can next proceed, as in Sections 6.10 through 6.12 in the general case, to obtain the semigroup property of $y^*(\,\cdot\,;y_0)$.

6.24 A Transition Property for w^* for $\gamma > \gamma_c$

We have the following important property:

Theorem 6.24.1 *Assume (H.1)–(H.4) and (S.1) = (6.19.1). For $\gamma > \gamma_c$ (defined in (6.20.2.1), we have:*

$$w^*(t+\sigma;y_0) = w^*(\sigma;y^*(t;y_0)), \quad \forall\, t,\sigma > 0, \tag{6.24.1}$$

for t fixed, the equality being intended in $C([0,\infty];Y)$ in σ.

Proof.

Step 1. Lemma 6.24.2 *For $\gamma > \gamma_c$ (defined in (6.20.2.1), we have:*

$$[I + LL^*R^*R - \gamma^{-2}WW^*R^*R][y^*(t+\,\cdot\,;y_0) - y^*(\,\cdot\,;y^*(t;y_0))] = 0. \tag{6.24.2}$$

Remark 6.24.1 Recall Remark 6.4.1: For γ sufficiently large, not necessarily for all $\gamma > \gamma_c$, the operator in (6.24.2) is boundedly invertible on $L_2(0,\infty;Y)$, so that we obtain from (6.24.2): $y^*(t+\,\cdot\,;y_0) = y^*(\,\cdot\,;y^*(t;y_0))$, first in $L_2(0,\infty;Y)$ and then in $C([0,\infty];Y)$, by (6.23.8), which is the sought-after semigroup property, at least for γ large. The point is to show that this conclusion holds true for all $\gamma > \gamma_c$.

Proof of Lemma 6.24.2 (in the style of Chapter 1, Proposition 1.4.3.1). For $\gamma > \gamma_c$, we return to (6.21.14), which we rewrite as

$$y^*(t;y_0) + \{LL^*R^*Ry^*(\,\cdot\,;y_0)\}(t) - \gamma^{-2}\{WW^*R^*Ry^*(\,\cdot\,;y_0)\}(t) = e^{At}y_0, \tag{6.24.3}$$

that is, explicitly via (6.19.5)–(6.19.8),

$$\{LL^*R^*Ry^*(\,\cdot\,;y_0)\}(t) = \int_0^t e^{A(t-\tau)}BB^* \int_\tau^\infty e^{A^*(\alpha-\tau)}R^*Ry^*(\alpha;y_0)\,d\alpha\,d\tau, \tag{6.24.4}$$

$$\{WW^*R^*Ry^*(\,\cdot\,;y_0)\}(t) = \int_0^t e^{A(t-\tau)}GG^* \int_\tau^\infty e^{A^*(\alpha-\tau)}R^*Ry^*(\alpha;y_0)\,d\alpha\,d\tau. \tag{6.24.5}$$

Thus, (6.24.3) written for t replaced by $t + \sigma$ becomes explicitly

$$y^*(t+\sigma; y_0) + \int_0^{t+\sigma} e^{A(t+\sigma-\tau)} BB^* \int_\tau^\infty e^{A^*(\alpha-\tau)} R^*Ry^*(\alpha; y_0)\, d\alpha\, d\tau$$
$$-\gamma^{-2} \int_0^{t+\sigma} e^{A(t+\sigma-\tau)} GG^* \int_\tau^\infty e^{A^*(\alpha-\tau)} R^*Ry^*(\alpha; y_0)\, d\alpha\, d\tau = e^{A(t+\sigma)} y_0. \tag{6.24.6}$$

However, by (6.24.3) with t replaced by σ and y_0 replaced by $y^*(t; y_0)$, we obtain

$$y^*(\sigma; y^*(t; y_0)) + \int_0^{\sigma} e^{A(\sigma-\beta)} BB^* \int_\beta^\infty e^{A^*(r-\beta)} R^*Ry^*(r; y^*(t; y_0))\, dr d\beta$$
$$-\gamma^{-2} \int_0^{\sigma} e^{A(\sigma-\beta)} GG^* \int_\beta^\infty e^{A^*(r-\beta)} R^*Ry^*(r; y^*(t; y_0))\, dr\, d\beta = e^{A\sigma} y^*(t; y_0), \tag{6.24.7}$$

where, by (6.24.3)–(6.24.5),

$$e^{A\sigma} y^*(t; y_0) = e^{A(\sigma+t)} y_0 - \int_0^{t} e^{A(t+\sigma-\tau)} BB^* \int_\tau^\infty e^{A^*(\alpha-\tau)} R^*Ry^*(\alpha; y_0)\, d\alpha\, d\tau$$
$$- \int_0^{t} e^{A(t+\sigma-\tau)} GG^* \int_\tau^\infty e^{A^*(\alpha-\tau)} R^*Ry^*(\alpha; y_0)\, d\alpha\, d\tau. \tag{6.24.8}$$

Next, subtracting and adding on both sides of (6.24.7), the following quantity:

$$\int_t^{t+\sigma} e^{A(t+\sigma-\tau)} BB^* \int_\tau^\infty e^{A^*(\alpha-\tau)} R^*Ry^*(\alpha; y_0)\, d\alpha\, d\tau$$
$$= \int_0^{\sigma} e^{A(\sigma-\beta)} BB^* \int_{t+\beta}^\infty e^{A^*(\alpha-(t+\beta))} R^*Ry^*(\alpha; y_0)\, d\alpha\, d\beta$$
$$(\alpha - t = r) \qquad = \int_0^{\sigma} e^{A(\sigma-\beta)} BB^* \int_\beta^\infty e^{A^*(r-\beta)} R^*Ry^*(t+r; y_0))\, dr\, d\beta, \tag{6.24.9}$$

as well as a similar term with B replaced by $-G/\gamma$, and using (6.24.8), we finally rewrite (6.24.7) as

$$y^*(\sigma; y^*(t; y_0)) + \int_0^{\sigma} e^{A(\sigma-\beta)} BB^* \int_\beta^\infty e^{A^*(r-\beta)} R^*Ry^*(r; y^*(t; y_0))\, dr\, d\beta$$
$$-\gamma^{-2} \int_0^{\sigma} e^{A(\sigma-\beta)} GG^* \int_\beta^\infty e^{A^*(r-\beta)} R^*Ry^*(r; y^*(t; y_0))\, dr\, d\beta$$
$$- \int_0^{\sigma} e^{A(\sigma-\beta)} BB^* \int_\beta^\infty e^{A^*(r-\beta)} R^*Ry^*(r+t; y_0)\, dr\, d\beta$$

$$+ \gamma^{-2} \int_0^{\sigma} e^{A(\sigma-\beta)} GG^* \int_{\beta}^{\infty} e^{A^*(r-\beta)} R^* R y^*(r+t; y_0)\, dr\, d\beta \tag{6.24.10}$$

$$= - \int_0^{t+\sigma} e^{A(t+\sigma-\tau)} BB^* \int_{\tau}^{\infty} e^{A^*(\alpha-\tau)} R^* R y^*(\alpha; y_0)\, d\alpha\, d\tau + e^{A(t+\sigma)} y_0. \tag{6.24.11}$$

We finally subtract (6.24.11) from (6.24.6), and after a cancellation of six terms we obtain

$$\begin{aligned} &[y^*(t+\sigma; y_0) - y^*(\sigma; y^*(t; y_0))] \\ &+ \int_0^{\sigma} e^{A(\sigma-\beta)} BB^* \int_{\beta}^{\infty} e^{A^*(r-\beta)} R^* R[y^*(t+r; y_0) - y^*(r; y^*(t; y_0))]\, dr\, d\beta \\ &- \gamma^{-2} \int_0^{\sigma} e^{A(\sigma-\beta)} GG^* \int_{\beta}^{\infty} e^{A^*(r-\beta)} R^* R[y^*(t+r; y_0) - y^*(r; y^*(t; y_0))]\, dr\, d\beta = 0. \end{aligned} \tag{6.24.12}$$

Recalling (6.19.5)–(6.19.8), we rewrite (6.24.12) precisely as in (6.24.2) and Lemma 6.24.2 is proved. □

Step 2 Starting from (6.24.2) and applying to it the operator $W^* R^* R[I + LL^* R^* R]^{-1}$ as in Remark 6.22.1, we obtain

$$\begin{aligned} &W^* R^* R[y^*(t+\cdot\,; y_0) - y^*(\cdot\,; y^*(t; y_0))] \\ &\quad - \gamma^{-2} W^* R^* R[I + LL^* R^* R]^{-1} W W^* R^* R[y^*(t+\cdot\,; y_0) \\ &\quad - y^*(\cdot\,; y^*(t; y_0)] = 0. \end{aligned} \tag{6.24.13}$$

Step 3. Lemma 6.24.3 *For $\gamma > \gamma_c$ (defined in (6.20.2.1)) and with reference to (6.21.8), we have*

$$\gamma^2 w^*(t+\sigma; y_0) = \{W^* R^* R y^*(t+\ \cdot\,; y_0)\}(\sigma). \tag{6.24.14}$$

Proof of Lemma 6.24.3. The proof is by direct verification. By (6.21.8), rewritten explicitly via (6.19.8), we have after a change of variable:

$$\begin{aligned} \gamma^2 w^*(t+\sigma; y_0) &= G^* \int_{t+\sigma}^{\infty} e^{A^*(\beta-(t+\sigma))} R^* R y^*(\beta; y_0)\, d\beta \\ (\beta - t = \tau) \qquad &= G^* \int_{\sigma}^{\infty} e^{A^*(\tau-\sigma)} R^* R y^*(t+\tau; y_0)\, d\tau, \end{aligned} \tag{6.24.15}$$

which is precisely (6.24.14). □

Step 4 Using (6.24.14) and (6.21.8) in (6.24.13), we rewrite (6.24.13) as

$$\gamma^2[w^*(t+\sigma; y_0) - w^*(\sigma; y^*(t; y_0))] \\ - W^*R^*R[I + LL^*R^*R]^{-1}W[w^*(t + \cdot\,; y_0) - w^*(\cdot\,; y^*(t; y_0))] = 0, \tag{6.24.16}$$

or, recalling the definition of E_γ in (6.20.1.12), (6.20.1.22), or in (6.20.1.22),

$$E_\gamma[w^*(t + \cdot\,; y_0) - w^*(\cdot\,; y^*(t; y_0))] = 0. \tag{6.24.17}$$

Thus, by Corollary 6.20.2.1, if $\gamma > \gamma_c$, then $E_\gamma^{-1} \in \mathcal{L}(L_2(0, \infty; V))$, and so by (6.24.17) we obtain

$$w^*(t + \cdot\,; y_0) - w^*(\cdot\,; y^*(t; y_0)) = 0, \tag{6.24.18}$$

first in $L_2(0, \infty; V)$ and next in $C([0, \infty]; V)$, as desired, by the regularity of w^* in Corollary 6.23.2. Theorem 6.24.1 is proved. □

The implication on Ww^* of the property (6.24.1) of w^* is examined next.

Corollary 6.24.4 *For all $\gamma > \gamma_c$ (defined in (6.20.2.1)), we have for all $t, \sigma > 0$,*

$$\{Ww^*(\cdot\,; y_0)\}(t+\sigma) = \{Ww^*(\cdot\,; y_0)\}(\sigma) - e^{A\sigma}\{Ww^*(\cdot\,; y_0)\}(t) \equiv 0. \tag{6.24.19}$$

Proof. By (6.19.6), we compute

$$\{Ww^*(\cdot\,; y_0)\}(t+\sigma) - e^{A\sigma}\{Ww^*(\cdot\,; y_0)\}(t) - \{Ww^*(\cdot\,; y_0)\}(\sigma) \\ = \int_0^{t+\sigma} e^{A(t+\sigma-\tau)}Gw^*(\tau; y_0)\,d\tau - \int_0^t e^{A(t+\sigma-\tau)}Gw^*(\tau; y_0)\,d\tau \\ - \int_0^\sigma e^{A(\sigma-\beta)}Gw^*(\beta; y_0)\,d\beta. \tag{6.24.20}$$

We now add and subtract (use $\tau - t = \beta$) the quantity

$$\int_t^{t+\sigma} e^{A(t+\sigma-\tau)}Gw^*(\tau; y_0)\,d\tau = \int_0^\sigma e^{A(\sigma-\beta)}Gw^*(t+\beta; y_0)\,d\beta \tag{6.24.21}$$

to the right-hand side of (6.24.20) to obtain, after a cancellation,

$$\{Ww^*(\cdot\,; y_0)\}(t+\sigma) - \{Ww^*(\cdot\,; y_0)\}(\sigma) - e^{A\sigma}\{Ww^*(\cdot\,; y_0)\}(t) \\ = \int_0^\sigma e^{A(\sigma-\beta)}G[w^*(t+\beta; y_0) - w^*(\beta; y_0)]\,d\beta \equiv 0, \tag{6.24.22}$$

where, in the last step, we have used (6.24.1). □

6.25 The Semigroup Property for y^* for $\gamma > \gamma_c$ and Its Stability

Defining the operator $\Phi(t)$ (which depends on γ) by

$$y^*(t; x) = \Phi(t)x \in C([0, \infty]; Y), \quad \forall\, x \in Y, \tag{6.25.1}$$

we obtain the semigroup property:

Theorem 6.25.1 *Assume (H.1)–(H.4) and (S.1) = (6.19.1). For $\gamma > \gamma_c$ (see (20.2.1)), $y_0 \in Y$, and $t,\ \sigma > 0$ we have*

$$y^*(t + \sigma; y_0) = y^*(\sigma; y^*(t; y_0)) \in C([0, \infty]; Y), \tag{6.25.2}$$

so that $\Phi(t)$ is a s.c. semigroup on Y.

Proof.

Step 1. Lemma 6.25.2 *For $\gamma > \gamma_c$, $y_0 \in Y$, and $t, \sigma > 0$ we have*

$$\begin{aligned}[y^*(t + \sigma; y_0) - y^*(\sigma; y^*(t; y_0)] + \{LL^*R^*R[y^*(t + \,\cdot\,; y_0) - y^*(\,\cdot\,; y^*(t; y_0)]\}(\sigma) \\ = \{Ww^*(\,\cdot\,; y_0)\}(t + \sigma) - \{Ww^*(\,\cdot\,; y_0)\}(\sigma) - e^{A\sigma}\{Ww^*(\,\cdot\,; y_0)\}(t).\end{aligned} \tag{6.25.3}$$

Proof. We return to the optimal dynamics (6.21.13) rewritten via (6.21.11) as

$$y^*(t; y_0) + \{LL^*R^*Ry^*(\,\cdot\,; y_0)\}(t) = e^{At}y_0 + \{Ww^*(\,\cdot\,; y_0)\}(t). \tag{6.25.4}$$

From here on, the proof proceeds as the proof of Lemma 6.24.2 below (6.24.3). Details may be omitted.

Step 2 We now apply Corollary 6.6.4, Eqn. (6.6.19), to (6.7.3) and obtain

$$[I + LL^*R^*R][y^*(t + \,\cdot\,; y_0) - y^*(\,\cdot\,; y^*(t; y_0))] = 0. \tag{6.25.5}$$

Since $[I + LL^*R^*R]^{-1} \in \mathcal{L}(L_2(0, \infty; Y))$ [see Chapter 2, Appendix 2A], we obtain, from (2.25.5),

$$y^*(t + \sigma; y_0) - y^*(\sigma; y^*(t; y_0)) = 0, \tag{6.25.6}$$

first in $L_2(0, \infty; Y)$, and then, by the regularity of y^* in (6.23.8), in $C([0, \infty]; Y)$. □

Corollary 6.25.3 *Assume (H.1)–(H.4) and (S.1) = (6.19.1). The s.c. semigroup $\Phi(t)$ in (6.25.1), guaranteed by Theorem 6.25.1, is exponentially stable: There exist constants $C \geq 1$, $k > 0$ such that*

$$\|\Phi(t)\|_{\mathcal{L}(Y)} \leq Ce^{-kt}, \quad t \geq 0. \tag{6.25.7}$$

Proof. The s.c. semigroup $\Phi(t)$ satisfies $y^*(t; x) = \Phi(t)x \in L_2(0, \infty; Y)$ for all $x \in Y$ by optimality (6.21.7), and then conclusion (6.25.7) follows via a known theorem in Datko [1970]. □

6.26 The Riccati Operator, P, for $\gamma > \gamma_c$

Using the property of $\Phi(t)$ in (6.25.1) as a s.c. uniformly stable semigroup on Y (as guaranteed by Section 6.25) we shall first define (in terms of the problem data) an operator $P \in \mathcal{L}(Y)$ for $\gamma > \gamma_c$, and we shall next show that P is, in fact, a solution of an algebraic Riccati (operator) equation.

6.26.1 Definition of P and Its Preliminary Properties

For $\gamma > \gamma_c$, $x \in Y$, and recalling (H.4) $=$ (6.1.1.8), we define the operator $P \in \mathcal{L}(Y)$ by (see (6.25.1)):

$$Px = \int_0^\infty e^{A^*\sigma} R^*Ry^*(\sigma; x)\, d\sigma = \int_0^\infty e^{A^*\sigma} R^*R\Phi(\sigma)x\, d\sigma \tag{6.26.1.1a}$$

$$= \int_t^\infty e^{A^*(\tau-t)} R^*Ry^*(\tau - t; x)\, d\tau = \int_t^\infty e^{A^*(\tau-t)} R^*R\Phi(\tau - t)x\, d\sigma. \tag{6.26.1.1b}$$

We now collect preliminary properties of P.

Proposition 6.26.1.1 *Assume (H.1)–(H.4) and (S.1) $=$ (6.19.1). With reference to (6.26.1.1) we have for $\gamma > \gamma_c$:*

(i)

$$\text{range of } P = PY \subset \mathcal{D}((-A^*)^\theta), \quad \forall\, 0 \le \theta < 1; \tag{6.26.1.2a}$$

$$(-A^*)^\theta P \in \mathcal{L}(Y); \tag{6.26.1.2b}$$

(ii)

$$B^*P \in \mathcal{L}(Y); \qquad G^*P \in \mathcal{L}(Y). \tag{6.26.1.3}$$

Proof. As usual, applying $(-A^*)^\theta$, $0 \le \theta < 1$ to (6.26.1.1) and using the analytic semigroup bound (see (6.19.1))

$$\|(-A^*)^\theta e^{A^*t}\|_{\mathcal{L}(Y)} \le \frac{Ce^{-\omega t}}{t^\theta}, \quad t > 0, \tag{6.26.1.4}$$

yields readily (6.26.1.2) via, say, (6.25.7). Then (6.26.1.2) with $\theta = \delta < 1$ implies (6.26.1.3) by writing $B^*P = B^*(-A^*)^{-\delta}(-A^*)^\delta P$ and using (6.1.1.6). Similarly, we can prove the desired result for G^*P, recalling (6.1.1.7) with $\rho < 1$. $\square$

Corollary 6.26.1.2 *Assume (H.1)–(H.4) and (S.1) $=$ (6.19.1). With reference to (6.26.1.1), we have for $\gamma > \gamma_c$:*

(i)

$$u^*(t; y_0) = -B^*Py^*(t; y_0) = -B^*P\Phi(t)y_0 \ \text{a.e. in } t; \quad y_0 \in Y; \tag{6.26.1.5a}$$

$$B^*P\Phi(t) : \text{continuous } Y \to L_2(0, \infty; U) \cap C([0, \infty]; U); \tag{6.26.1.5b}$$

(ii)

$$\gamma^2 w^*(t; y_0) = G^* P y^*(t; y_0) = G^* P \Phi(t) y_0; \qquad (6.26.1.6a)$$

$$G^* P \Phi(t) : \textit{continuous } Y \to L_2(0, \infty; V) \cap C([0, \infty]; V); \qquad (6.26.1.6b)$$

(iii) the operator $P \in \mathcal{L}(Y)$ satisfies the symmetric relation for $x_1, x_2 \in Y$:

$$(Px_1, x_2) = \int_0^\infty [(Ry^*(t; x_1), Ry^*(t; x_2))_Y + (u^*(t; x_1), u^*(t; x_2))_U - \gamma^2 (w^*(t; x_1), w^*(t; x_2))_Y]\, dt, \qquad (6.26.1.7)$$

from which it follows that P is a nonnegative, self-adjoint operator: $P = P^* \geq 0$ on Y and that the optimal cost of problem (6.21.1) is

$$\begin{aligned}(Py_0, y_0)_Y &= J^*(y_0) \quad \textit{[optimal cost in (6.21.3)]} \\ &= J(u^*(\cdot\,; y_0), y^*(\cdot\,; y_0), w^*(\cdot\,; y_0)). \end{aligned} \qquad (6.26.1.8)$$

Proof. (i) One applies B^* to (6.26.1.1b) with x replaced by $y^*(t; y_0)$ and obtains (6.26.1.5) via the semigroup property $\Phi(\tau - t)\Phi(t) = \Phi(\tau)$, as well as (6.19.7) and (6.21.11). The proof for (ii) is similar, this time by use of (6.19.8) and (6.21.8). The indicated regularity follows from (6.26.1.3).

(ii) For $x_1, x_2 \in Y$ we write from (6.26.1.1) using $e^{A\cdot} x_2$ from (6.21.13):

$$\begin{aligned}(Px_1, x_2)_Y &= (R^* R y^*(\cdot\,; x_1), e^{A\cdot} x_2)_{L_2(0,\infty;Y)} \\ &= (Ry^*(\cdot\,; x_1), Ry^*(\cdot\,; x_2))_{L_2(0,\infty;Z)} \\ &\quad - (L^* R^* R y^*(\cdot\,; x_1), u^*(\cdot\,; x_2))_{L_2(0,\infty;U)} \\ &\quad - (W^* R^* R y^*(\cdot\,; x_1), w^*(\cdot\,; x_0))_{L_2(0,\infty;V)}, \end{aligned} \qquad (6.26.1.9)$$

and (6.26.1.7) follows from (6.26.1.9), recalling (6.21.11) and (6.21.8). □

6.26.2 The Feedback Generator A_F and Its Preliminary Properties for $\gamma > \gamma_4$. Merger with Section 6.14

Next, for $\gamma > \gamma_c$, we call A_F the infinitesimal generator of the s.c. uniformly stable semigroup, $\Phi(t)$ in (6.25.1):

$$\Phi(t)x = e^{A_F t} x, \quad x \in Y; \qquad \frac{d\Phi(t)x}{dt} - A_F \Phi(t)x = \Phi(t) A_F x, \quad x \in \mathcal{D}(A_F). \qquad (6.26.2.1)$$

We now identify A_F as in Section 6.14 in the present case.

Theorem 6.26.2.1 *Assume (H.1)–(H.4) and (S.1) = (6.19.1).*

(i) For $x \in Y$ and $t > 0$ we have that

$$\frac{d\Phi(t)}{dt} = [A - BB^* P + \gamma^{-2} GG^* P]\Phi(t)x \in [\mathcal{D}(A^*)]'. \qquad (6.26.2.2)$$

(ii) Moreover,

$$[A - BB^*P + \gamma^{-2}GG^*P]x = A_F x, \quad x \in \mathcal{D}(A_F). \tag{6.26.2.3}$$

(iii) The resolvent $R(\lambda; A_F)$ of A_F satisfies the estimate

$$\|R(\lambda; A_F)\|_{\mathcal{L}(Y)} \le \frac{C_{r_0}}{|\lambda|}, \quad \forall\, \lambda \text{ with } \operatorname{Re}\lambda \ge \text{some } r_0 > 0, \tag{6.26.2.4}$$

and thus A_F generates a s.c. analytic semigroup on Y:

$$\Phi(t) = e^{A_F t} = e^{(A - BB^*P + \gamma^{-2}GG^*P)t}. \tag{6.26.2.5}$$

(iv) The resolvents $R(\lambda, A - BB^*P)$ and $R(\lambda, A + \gamma^{-2}GG^*P)$ of the operators $(A - BB^*P)$ and $(A + \gamma^{-2}GG^*P)$ with maximal domains satisfy estimates like (6.26.2.4),

$$\|R(\lambda, A - BB^*P)\|_{\mathcal{L}(Y)} + \|R(\lambda, A + \gamma^{-2}GG^*P)\|_{\mathcal{L}(Y)} \le \frac{C_{r_0}}{|\lambda|},$$
$$\forall\, \lambda \text{ with } \operatorname{Re}\lambda \ge \text{some } r_0 > 0, \tag{6.26.2.6}$$

so that

$$e^{(A - BB^*P)t} \quad \text{and} \quad e^{(A + \gamma^{-2}GG^*P)t}$$

are both s.c. analytic semigroups on Y.

Proof. The proof is the same as the proof of Theorem 6.14.1. □

Corollary 6.26.2.2 *Assume (H.1)–(H.4) and (S.1) = (6.19.1). Let $\gamma > \gamma_c$. With reference to (6.26.2.3) we have*

(i)

$$\mathcal{D}(A_F) = \{x \in \mathcal{D}((-A)^{1-\sigma}) : (-A)^{1-\sigma}x - (-A)^{-\sigma}BB^*P \tag{6.26.2.7}$$
$$+\gamma^{-2}(-A)^{-\sigma}GG^*Px \in \mathcal{D}((-A)^{\sigma})\} \tag{6.26.2.8}$$
$$\subset \mathcal{D}((-A)^{1-\sigma}), \quad \sigma = \max\{\delta, \rho\}. \tag{6.26.2.9}$$

(ii) For $x \in Y$, and $t > 0$,

$$e^{A_F t}x \subset \mathcal{D}((-A)^{1-\sigma}). \tag{6.26.2.10}$$

Proof. This is the same as the proof of Corollary 6.14.2. □

6.26.3 The Operator P Satisfies the Algebraic Riccati Equation for $\gamma > \gamma_c$

We finally obtain the ultimate goal of our analysis.

Theorem 6.26.3.1 *Assume (H.1)–(H.4) and (S.1) = (6.19.1). For $\gamma > \gamma_c$, the operator P defined by (6.26.1.1) satisfies the algebraic Riccati equation, ARE_γ,*

$$(Px, Az)_Y + (Ax, Pz)_Y + (Rx, Rz)_Y = (B^*Px, B^*Pz)_Y - \gamma^{-2}(G^*Px, G^*Pz)_U \tag{6.26.3.1}$$

for all $x, z \in \mathcal{D}((-A)^\epsilon)$, $\forall \epsilon > 0$, in particular for all $x, z \in \mathcal{D}(A_F) \subset \mathcal{D}((-A)^{1-\sigma})$, σ as in (6.26.2.9).

Proof.

Step 1 We first show that (6.26.3.1) holds true $\forall x \in \mathcal{D}(A_F)$ and $\forall z \in \mathcal{D}(A)$. To this end, we return (6.26.1.1) with $\sigma = \tau - t$, which we rewrite via (6.26.2.1), and obtain

$$(Px, z)_Y = \int_t^\infty \left(re^{A_F(\tau-t)}x, Re^{A(\tau-t)}z\right) d\tau, \quad x, z \in Y. \tag{6.26.3.2}$$

We now specialize to $x \in \mathcal{D}(A_F)$, $z \in \mathcal{D}(A)$ and differentiate (6.26.3.2) in t, thus obtaining

$$(Rx, Rz)_Z + (PA_Fx, z)_Y + (Px, Az)_Y = 0, \quad \forall\, x \in \mathcal{D}(A_F), \quad \forall\, z \in \mathcal{D}(A), \tag{6.26.3.3}$$

which coincides with Eqn. (6.15.4).

Step 2 One then extends the validity of (6.26.3.3) to all $x, z \in \mathcal{D}((-A)^\epsilon)$, for $\rho, \delta \leq 1 - \epsilon < 1$, by use of (6.26.1.2) and (6.26.1.3) of Proposition 6.26.1.1 and (6.26.2.9) of Corollary 6.26.2.2, as in Step 2 of Theorem 6.15.1, recalling also (6.26.2.3). □

6A Optimal Control Problem with Nondefinite Quadratic Cost. The Stable, Analytic Case. A Brief Sketch

After the detailed treatment of the present chapter on the min–max game problem with (a specific) indefinite quadratic cost, it seems appropriate to conclude this volume with a brief mention of a related problem: the optimal control problem with nondefinite quadratic cost which is (strictly) coercive in the control function. There is no space in this volume for a detailed discussion of this natural generalization of the classical quadratic problem treated in it. Only a brief sketch will be given, and only in the stable case. The main result, which assumes uniqueness of the Optimal Control Problem, is given in the theorem at the end of this appendix. The point we wish to stress here is that most, if not all, of the ingredients necessary to cover this more general optimal control problem are already in place in the present volume (say, Chapter 2, and the present Chapter 6 for $T = \infty$). Mainly, it is the result on the pointwise feedback synthesis of the *assumed* unique optimal pair that requires a different approach in the present, nondefinite cost case. This is so due to the term $F_2 \neq 0$ in the optimality condition (6A.B.10): and it is done in Proposition 6A.F.1 below. Otherwise, for the most part, what remains to be done is to compile results of preceding chapters and rearrange them appropriately. We are pleased to note that contemporary work [Li, Yong, 1995], [Li, 1998] in this area by X. Li and his collaborators at Fudan University, Shanghai, China, as well as that of C. McMillan [1997; 1999] follows the thrust and the variational approach of the original work of the present authors, reported in the present volumes. McMillan's papers, like those of F. Bucci and L. Pandolfi, quoted in the references to Chapter 6, include also the singular case, where reference [Louis, Wexler, 1991] is relevant, along with the original work of Yakubovich quoted therein. We close these brief comments by mentioning paper [Lasiecka et al., 1997], where a 'pathological' optimal control problem is treated via *singular control theory* (in particular, via the 'dissipation operator inequality'). This pathological optimal control problem arises in the context of a distinctive class of parabolic PDE with control u acting on the boundary, whose corresponding abstract model includes also the time derivative u_t, a novelty, and a source of difficulty, to the problem. Previous different treatment, in the style of this Volume 1, but with additional technical difficulties in common, are given in Lasiecka et al. [1995] and Triggiani [1994b, c].

6A.A Mathematical Setting. Assumptions

Dynamics We consider the usual dynamics of the present volume

$$\dot{y} = Ay + Bu \in [\mathcal{D}(A^*)]', \quad y(0) = y_0; \qquad \text{or } y(t) = e^{At}y_0 + (Lu)(t), \tag{6A.A.1}$$

under the standard assumptions (H.1) = (6.1.1.2) and (H.2) = (61.1.6) that the s.c. semigroup e^{At} be analytic on Y, and that $A^{-\gamma}B \in \mathcal{L}(U; Y)$, $\gamma < 1$. In addition, however, we assume the stability hypothesis (6.19.1). The operator L has the usual meaning defined by (6.19.5), with its adjoint L^* defined by (6.19.7).

Cost Functional This time, however, we consider a more general quadratic cost

$$F(x, u) = (F_1x, x) + 2\,\mathrm{Re}(F_2x, u) + (u, u), \tag{6A.A.2}$$

which is a continuous Hermitian form on $Y \times U$, under the assumptions

$$F_1 = F_1^* \in \mathcal{L}(Y); \qquad F_2 \in \mathcal{L}(Y; U), \tag{6A.A.3}$$

with F_1 possibly indefinite. In (6A.A.2) we omit, for simplicity, specification of the three inner products on Y, U, and U, respectively. [The case where the quadratic control term (u, u) is replaced by (F_3u, u) with $F_3 = F_3^* \in \mathcal{L}(U)$ strictly positive: $\|F_3u\| \geq c\|u\|$, $c > 0$ is, without loss of generality, reduced to the case $F_3 =$ identity considered in (6A.A.2).] The quadratic cost is then

$$J(y_0; u) = \int_0^\infty [(F_1y, y) + 2\,\mathrm{Re}(F_2y, u) + \|u\|^2]\,dt, \tag{6A.A.4}$$

with y the solution of the dynamics (6A.A.1) due to $y_0 \in Y$ and $u \in L_2(0, \infty; U) \equiv \mathcal{U}$. We likewise set $L_2(0, \infty; Y) = \mathcal{Y}$, for brevity.

Optimal Control Problem The corresponding optimal control problem (OCP) is as follows: given $y_0 \in Y$, seek $u^0 = u^0(\cdot\,; y_0) \in \mathcal{U}$ such that

$$J^0(y_0) \equiv J(y_0; u^0) = \inf_{u \in \mathcal{U}} J(y_0; u) > -\infty. \tag{6A.A.5}$$

Any such u^0 is then called 'an optimal control,' and its corresponding trajectory y^0 obtained via (6A.A.1) 'an optimal solution.'

6A.B Existence, Uniqueness, Optimality Conditions. Explicit Formulas

The assumptions stated in Section A are in force and will not be repeated. First, we insert (6A.A.1) into (6A.A.4) and obtain, after straightforward algebraic manipulations:

Proposition 6A.B.1 *For $y_0 \in Y$ and $u \in \mathcal{U}$, the cost $J(y_0; u)$ in (6A.A.4) may be rewritten as a continuous form on $Y \times \mathcal{U}$ as:*

$$J(y_0; u) = (\Lambda u, u)_{\mathcal{U}} + 2\,\mathrm{Re}(K y_0, u)_{\mathcal{U}} + (S y_0, y_0)_Y, \tag{6A.B.1}$$

where the bounded operators Λ (self-adjoint), K, and S (self-adjoint) are given by

$$\Lambda = I + L^* F_1 L + F_2 L + L^* F_2^* \in \mathcal{L}(\mathcal{U}); \tag{6A.B.2}$$

$$K = [L^* F_1 + F_2] e^{A\,\cdot} \in \mathcal{L}(Y; \mathcal{U}); \tag{6A.B.3}$$

$$S = \int_0^\infty e^{A^* t} F_1 e^{At} dt \in \mathcal{L}(Y). \tag{6A.B.4}$$

Proposition 6A.B.2 (Existence) *Let $y_0 \in Y$. The OCP has a solution $u^0 = u^0(\cdot\,; y_0)$ such that (6A.A.5) holds true if and only if the following two conditions are satisfied:*

$$(\Lambda u, u)_{\mathcal{U}} \geq 0, \quad \forall u \in \mathcal{U}; \quad \text{and} \quad \Lambda u^0 + K y_0 = 0. \tag{6A.B.5}$$

Proof (As in Section 6.17). By contradiction on the first condition in (6A.B.5), suppose that there exists $0 \neq v \in \mathcal{U}$ such that $(\Lambda v, v) < 0$. Then, the control sequence $u_k = kv \in \mathcal{U}$, $k = 1, 2, 3, \ldots$ yields costs

$$J(y_0; u_k) = k^2(\Lambda v, v) + 2k\,\mathrm{Re}(K y_0, v) + (S y_0, y_0) \to -\infty \quad \text{as } k \to +\infty, \tag{6A.B.6}$$

and (6A.B.6) contradicts assumption (6A.A.5). The second condition in (6A.B.5) follows, for instance, by setting equal to zero the Frechet derivative of $J(y_0; u)$ in (6A.B.1) at the extreme point u^0.

Conversely, let $(\Lambda u, u)_{\mathcal{U}} \geq 0$, $\forall\, u \in \mathcal{U}$, and assume that there exists u^0 solution of $\Lambda u^0 + K y_0 = 0$. We want to show that such $u^0 = u^0(\cdot; y_0)$ is a solution of the O.C.P. (6A.A.5). In fact, recalling (6A.B.1), we obtain after adding and subtracting:

$$\begin{aligned} J(y_0; u) &= \|\Lambda^{\frac{1}{2}} u\|^2 + 2\mathrm{Re}(K y_0 + \Lambda u^0, u) - 2\mathrm{Re}(\Lambda u^0, u) + (S y_0, y_0) \\ &= \|\Lambda^{\frac{1}{2}} u\|^2 - 2\mathrm{Re}(\Lambda^{\frac{1}{2}} u^0, \Lambda^{\frac{1}{2}} u) + \|\Lambda^{\frac{1}{2}} u^0\|^2 + (S y_0, y_0) - \|\Lambda^{\frac{1}{2}} u^0\|^2 \\ &= \|\Lambda^{\frac{1}{2}} (u - u^0)\|^2 + (S y_0, y_0) - (\Lambda u^0, u^0). \end{aligned}$$

Thus, as desired,

$$\min J(y_0; u) = J(y_0, u^0) = (Sy_0, y_0) - (\Lambda u^0, u^0). \quad \square \tag{6A.B.7}$$

Proposition 6A.B.3 (Uniqueness) *Let $y_0 \in Y$. Assume that the corresponding OCP in (6A.A.5) has a unique solution $u^0 = u^0(\,\cdot\,; y_0) \in \mathcal{U}$. Then, the self-adjoint operator Λ in (6A.B.2) is positive definite:*

$$(\Lambda u, u)_{\mathcal{U}} > 0, \quad \forall\, 0 \neq u \in \mathcal{U}. \tag{6A.B.8}$$

Thus, Λ is injective and its inverse Λ^{-1} exists as an operator from the range $\mathcal{R}(\Lambda)$ of Λ onto $\mathcal{U}$, where $\overline{\mathcal{R}(\Lambda)} = Y$.

Proof. Let the optimal control u^0 be unique. To prove (6A.B.8), assume by contradiction, via Proposition 6A.B.2, that there exists $0 \neq v \in \mathcal{U}$ such that $(\Lambda v, v) = 0$; so that, $\mathrm{Re}(Ky_0, v) = 0$ as well, again by Proposition 6A.B.2. Then, the control sequence $u_k = u^0 + kv$, $k = \pm 1, \pm 2, \ldots$ yields, via (6A.B.1), the costs

$$J(y_0; u_k) = J(y_0; u^0) + k^2(\Lambda v, v) + 2k\, \mathrm{Re}(Ky_0, v) + 2k\, \mathrm{Re}(\Lambda u^0, v), \tag{6A.B.9}$$

after using the properties of v. Since, by optimality of u^0, $J(y_0; u_k) \geq J(y_0; u^0)$ for all $k = \pm 1, \pm 2, \ldots$, it follows from (6A.B.9) that we must have $\mathrm{Re}(\Lambda u^0, v) = 0$, and thus $J(y_0; u^0 + kv) = J(y_0; u^0)$, $k = \pm 1, \pm 2, \ldots$. This contradicts uniqueness of the optimal control. Finally, by (6A.B.8) the null space $\mathcal{N}(\Lambda^*) = \mathcal{N}(\Lambda) = 0$, and so $\mathcal{R}(\Lambda)$ is dense in Y. $\square$

Proposition 6A.B.4 (Optimality condition, explicit formulas) *Assume that the OCP in (6A.A.5) admits a unique optimal control $u^0 = u^0(\,\cdot\,; y_0) \in \mathcal{U}$, for all $y_0 \in Y$, with corresponding optimal trajectory $y^0 = y^0(\,\cdot\,; y_0) \in \mathcal{Y}$ obtained via (6A.A.1). Then*

(i)

$$[I + L^* F_2^*]u^0 = -[L^* F_1 + F_2]y^0 \tag{6A.B.10}$$

[this is the extension of [Chapter 1, (1.2.1.4)], [Chapter 6, (6.20.1.3), etc.];

(ii)

$$\Lambda u^0(\,\cdot\,; y_0) = -[L^* F_1 + F_2]e^{A\,\cdot}\, y_0 = -Ky_0 \in \mathcal{Y} \tag{6A.B.11}$$

(iii)

$$\Lambda^{-1} K = \Lambda^{-1}[L^* F_1 + F_2]e^{A\,\cdot} \in \mathcal{L}(Y; \mathcal{U}); \tag{6A.B.12}$$

(iv)

$$u^0(\,\cdot\,; y_0) = -\Lambda^{-1} K y_0 = -\Lambda^{-1}[L^* F_1 + F_2]e^{A\,\cdot}\, y_0 \in \mathcal{U}; \tag{6A.B.13}$$

$$y^0(\,\cdot\,; y_0) = e^{A\,\cdot}\, y_0 - L\Lambda^{-1} K y_0 = \{I - L\Lambda^{-1}[L^* F_1 + F_2]\}e^{A\,\cdot}\, y_0 \in \mathcal{Y}. \tag{6A.B.14}$$

$$J^0(y_0) = (Py_0, y_0)_Y; \quad P = S - K^*\Lambda^{-1}K \in \mathcal{L}(Y); \tag{6A.B.15a}$$

$$J(y_0; u) = (\Lambda u, u) - 2\,\mathrm{Re}(\Lambda u, u^0) + (Sy_0, y_0); \tag{6A.B.15b}$$

$$J(y_0; u) - J(y_0; u^0) = (\Lambda[u - u^0], [u - u^0]); \tag{6A.B.15c}$$

$$J^0(y_0) = -(\Lambda u^0, u^0) + (Sy_0, y_0) \tag{6A.B.15d}$$

[these are extensions of [Chapter 1, (1.2.1.1), (1.2.1.28)–(1.2.1.30)], [Chapter 6, (6.20.1.4a), (6.20.1.9), etc.].

Proof. (i), (ii) Formula (6A.B.10) giving the optimality condition can be obtained from $\Lambda u^0 + Ky_0 = 0$ in (6A.B.5) via (6A.B.2), (6A.B.3) and the optimal dynamics y^0 by (6A.A.1); or else by Lagrange multiplier argument as in [Chapter 1, Section 1.4.1], [Chapter 6, Sections 6.7.1 and 6.20.1]. (6A.B.10) and (6A.B.11) are equivalent.

(iii) By Proposition 6A.B.3, we have that Λ^{-1} exists on the range $\mathcal{R}(\Lambda)$ of Λ. By (6A.B.11), $[L^*F_1 + F_2]e^{A\,\cdot}y_0 \in \mathcal{R}(\Lambda)$, for all $y_0 \in Y$. Thus, the operator $\Lambda^{-1}[L^*F_1 + F_2]e^{A\,\cdot}$ is well-defined on all of Y; moreover, it is closed, being the composition of Λ^{-1} with inverse Λ closed (bounded) and of a bounded operator $[L^*F_1 + F_2]e^{A\,\cdot}$ [Kato, 1966, p. 164]. Thus, the closed graph theorem yields (6A.B.12).

(iv) (6A.B.11) yields (6A.B.13) via (6A.B.12) and this, in turn, yields the optimal dynamics y^0 in (6A.B.14). Next, inserting (6A.B.13) into (6A.B.1) readily yields (6A.B.15a–b). The latter was also obtained in (6A.B.7). Specializing (6A.B.15b) for $u = u^0$ and subtracting the resulting expression from (6A.B.15b) for u yields (6A.B.15c). □

6A.C Continuity of Optimal Pair u^0, y^0. Feedback Semigroup

Henceforth, we assume that the OCP has a unique optimal control $u^0 = u^0(\,\cdot\,; y_0) \in L_2(0, \infty; U)$ for *all* $y_0 \in Y$, with corresponding optimal trajectory $y^0 = y^0(\,\cdot\,; y_0) \in L_2(0, \infty; Y)$ via (6A.A.1) as in Proposition 6A.B.4.

Proposition 6A.C.1 *The following regularity properties hold true for the optimal pair, with $y_0 \in Y$:*

$$u^0(\,\cdot\,; y_0) \in C_{ub}([0, \infty]; U); \qquad y^0(\,\cdot\,; y_0) \in C_{ub}([0, \infty]; Y), \tag{6A.C.1}$$

(as in [Chapter 1, several times], [Chapter 2, Section 6.3.5], [Chapter 6, Section 6.9]).

Proof. We, of course, use a boot-strap argument on (6A.B.10), (6A.A.1)

$$u^0 = -L^*F_2^*u^0 - L^*F_1y^0 - F_2y^0; \tag{6A.C.2}$$

$$y^0 = e^{A\,\cdot}y_0 + Lu^0, \tag{6A.C.3}$$

based on Theorem 6.9.1 in this chapter, or [Chapter 2, Theorem 2.3.5.1], under the stability hypothesis (6.19.1). At the start, $u^0 \in L_2(0, \infty; U)$. Thus, by (6A.C.3) and (6.9.5), $y^0 \in L_{\ell_1}(0, \infty; Y)$. Then, $u^0 \in L_{\ell_1}(0, \infty; U)$ by (6A.C.2). A new cycle then gives $y^0 \in L_{\ell_2}(0, \infty; U)$ by (6A.C.3) and (6.9.8), and so $u^0 \in L_{\ell_2}(0, \infty; U)$ by (6A.C.2). After a finite number of steps we obtain (6A.C.1). □

6A.D Transitivity Properties of the Optimal Pair $u^0(\cdot\,; y_0)$, $y^0(\cdot\,; y_0)$. The Feedback Semigroup Φ

The following properties are the counterpart of [Chapter 1, Proposition 1.4.3.1], [Chapter 6, Sections 6.10, 6.11, 6.12], etc.

Proposition 6A.D.1 *The following transitivity properties hold true for $u^0(\cdot\,; y_0)$, $y^0(\cdot\,; y_0)$, $y_0 \in Y$,*

$$u^0(t+\tau; y_0) = u^0(\tau; y^0(t; y_0)); \qquad y^0(t+\tau; y_0) = y^0(\tau; y^0(t; y_0)). \quad (6A.D.1)$$

Thus, setting

$$y^0(t; y_0) = \Phi(t) y_0 \in L_2(0, \infty; Y) \cap C_{ub}([0, \infty]; Y), \tag{6A.D.2}$$

we have that $\Phi(t)$ is a s.c. semigroup, which moreover is exponentially stable: there exist constant $C \geq 1$, $\delta > 0$ such that

$$\|\Phi(t)\|_{\mathcal{L}(Y)} \leq C e^{-\delta t}, \quad t \geq 0. \tag{6A.D.3}$$

Proof. The transitivity property for u^0 may be proved from the explicit formula (6A.B.13), in the style of previous chapters or [Chapter 6, Sections 6.10–6.12], and then leads to the same transitivity property for y^0 via the optimal dynamics (6A.A.1). The latter yields the semigroup property for $\Phi(t)$ defined in (6A.D.2).

Strong continuity of $\Phi(t)$ is a restatement of (6A.C.1). The regularity property $y^0 \in L_2(0, \infty; Y)$ in (6A.D.2) implies exponential decay (6A.D.3), as usual, by Datko's theorem. □

6A.E The Operator P: Explicit Expression and Properties

Proposition 6A.E.1 *Regarding the operator $P = P^* \in \mathcal{L}(Y)$ defined in (6A.B.15a), the following properties hold true:*

(i) for any $x \in Y$, and recalling $y^0(t; x) = \Phi(t)x$ by (6A.D.2), we have

$$Px = \int_0^\infty e^{A^*t}[F_1 \Phi(t)x + F_2^* u^0(t; x)]\, dt \tag{6A.E.1}$$

where $u^0(t; x)$ is given explicitly in terms of the problem's data by (6A.B.13) [counterpart of, say, [Chapter 2, Eqn. (2.3.7.1) with $\omega = 0$, stable case]];

(ii)

$$(-A^*)^\theta P \in \mathcal{L}(Y), \quad \text{for any } \theta < 1; \tag{6A.E.2}$$

(iii)

$$B^* P \in \mathcal{L}(U; Y) \tag{6A.E.3}$$

[the latter two properties are the counterpart of [Chapter 2, Eqn. (2.3.7.4), (2.3.7.5)].

Proof. (i) We return to (6A.B.15d) for $x \in Y$,

$$J^0(x) = (Px, x)_Y = -(\Lambda u^0(\cdot\,; x), u^0(\cdot\,; x))_{\mathcal{U}} + (Sx, x)_Y, \tag{6A.E.4}$$

where by (6A.B.4),

$$(Sx, x)_Y = \left(\int_0^\infty e^{A^* t} F_1 e^{At} x \, dt, x \right)_Y, \tag{6A.E.5}$$

and where we shall show below that

$$\begin{aligned} -(\Lambda u^0(\cdot\,; x), u^0(\cdot\,; x))_{\mathcal{U}} &= -(K^* \Lambda^{-1} K x, x)_{\mathcal{U}} = (u^0(\cdot; x), Kx)_{\mathcal{U}} \\ &= \left(\int_0^\infty e^{A^* t} [F_1 \{ L u^0(\cdot\,; x)\}(t) + F_2^* u^0(t; x)] \, dt, x \right)_Y . \end{aligned} \tag{6A.E.6}$$

Assuming for the moment the validity of identity (6A.E.6), we substitute (6A.E.5) and (6A.E.6) into (6A.E.4), to obtain for $x \in Y$:

$$\begin{aligned} J^0(x) &= (Px, x)_Y \\ &= \left(\int_0^\infty e^{A^* t} [F_1 (e^{At} x + \{ L u^0(\cdot\,; x)\}(t)) + F_2^* u^0(t; x)] \, dt, x \right)_Y \qquad (6A.E.7) \\ &= \left(\int_0^\infty e^{A^* t} [F_1 \Phi(t) x + F_2^* u^0(t; x)] \, dt, x \right)_Y, \qquad (6A.E.8) \end{aligned}$$

after recalling, in the last step, the optimal dynamics from (6A.D.2), (6A.A.1),

$$y^0(t; x) = \Phi(t) x = e^{At} x + \{ L u^0(\cdot\,; x)\}(t), \quad x \in Y. \tag{6A.E.9}$$

Then, (6A.E.8) yields (6A.E.1) since P is self-adjoint.

We now prove (6A.E.6). By (6A.B.15a), or (6A.B.13), followed by (6A.B.3), we compute:

$$\begin{aligned} -(\Lambda u^0(\cdot; x), u^0(\cdot; x))_{\mathcal{U}} &= -(K^* \Lambda^{-1} K x, x)_{\mathcal{U}} = (u^0(\cdot; x), Kx)_{\mathcal{U}} \qquad (6A.E.10) \\ &= (u^0(\cdot; x), [L^* F_1 + F_2] e^{A\cdot} x)_{\mathcal{U}} \qquad (6A.E.11) \\ &= (L u^0(\cdot; x), F_1 e^{A\cdot} x)_{\mathcal{Y}} + (u^0(\cdot; x), F_2 e^{A\cdot} x)_{\mathcal{U}} \qquad (6A.E.12) \\ &= \left(\int_0^\infty e^{A^* t} [F_1^* \{ L u^0(\cdot; x)\}(t) + F_2^* u^0(t; x)] \, dt, x \right)_Y . \qquad (6A.E.13) \end{aligned}$$

Then, (6A.E.13) yields (6A.E.6), as desired, since $F_1 = F_1^*$ by assumption (6A.A.3). The proof of (6A.E.1) is complete.

(ii), (iii) The regularity properties in (6A.C.1) of the optimal pair, combined with the analyticity of the exponentially stable semigroup e^{At}, see (6.19.1) and estimates (6.1.1.5), allow one to obtain (6A.E.2) from (6A.E.1) via the closed graph theorem. Then (6A.E.2) with $\theta = \gamma < 1$ yields $B^*P = B^*(-A^*)^{-\gamma}(-A^*)^{\gamma}P \in \mathcal{L}(Y; U)$, as desired, as usual. □

6A.F Optimal Synthesis. Feedback Generator A_P. Analyticity of $\Phi(t)$

The next result gives the optimal control $u^0(t; y_0)$ in a pointwise feedback form of the optimal solution $y^0(t; x) = \Phi(t)x$.

Proposition 6A.F.1 (Feedback synthesis of optimal pair) *For $y_0 \in Y$ we have*

$$u^0(t; y_0) = -[F_2 + B^*P]\Phi(t)y_0 \in L_2(0, \infty; U) \cap C_{ub}([0, \infty]; U). \tag{6A.F.1}$$

Proof. For $x \in Y$, we obtain from (6A.E.1), (6A.E.3),

$$B^*Px = \int_0^\infty B^*e^{A^*\sigma}[F_1\Phi(\sigma)x + F_2^*u^0(\sigma; x)]\, d\sigma. \tag{6A.F.2}$$

Setting $x = \Phi(t)y_0$, t fixed positive and $y_0 \in Y$, we then obtain from (6A.F.2), by virtue of the semigroup/transitivity properties in (6A.D.1):

$$B^*P\Phi(t)x = \int_0^\infty B^*e^{A^*\sigma}[F_1\Phi(\sigma)\Phi(t)y_0 + F_2^*u^0(\sigma; \Phi(t)y_0)]\, d\sigma \tag{6A.F.3}$$

$$(\text{by (6A.D.1)}) \qquad = \int_0^\infty B^*e^{A^*\sigma}[F_1\Phi(\sigma + t)y_0 + F_2^*u^0(\sigma + t; y_0)]\, d\sigma \tag{6A.F.4}$$

$$(\sigma = \tau - t) \qquad = \int_t^\infty B^*e^{A^*(\tau - t)}[F_1\Phi(\tau)y_0 + F_2^*u^0(\tau; y_0)]\, d\tau \tag{6A.F.5}$$

$$(\text{by (6.19.7)}) \qquad = \{L^*[F_1\Phi(\cdot)y_0 + F_2^*u^0(\cdot\,; y_0)\}(t), \tag{6A.F.6}$$

recalling, in the last step, the adjoint L^* defined by (6.19.7).

Finally, we recall the optimality condition (6A.B.10), conveniently rewritten here as

$$u^0(t; y_0) + F_2y^0(t; y_0) = -L^*[F_1y^0(\cdot\,; y_0) + F_2^*u^0(\cdot\,; y_0)]\}(t). \tag{6A.F.7}$$

Substituting (6A.F.7) in (6A.F.6) yields (6A.F.1), as desired, via (6A.D.2). □

According to (6A.F.1) we can re-write formula (6A.E.1) for P.

Corollary 6A.F.2 *For $x \in Y$,*

$$Px = \int_0^\infty e^{A^*t}[F_1 - F_2^*(F_2 + B^*P)]\Phi(t)x\, dt, \tag{6A.F.8}$$

where $\Phi(t)x$ is expressed explicitly from the data via (6A.B.14).

Proof. We insert (6A.F.1) in (6A.E.1). □

After the optimal synthesis (6A.F.1), we can obtain the counterpart of [Chapter 2, Section 2.3.8].

Corollary 6A.F.3 *For $x \in \mathcal{D}(A_P)$ and $t > 0$:*

$$\frac{d\Phi(t)x}{dt} = A_P\Phi(t)x = \Phi(t)A_Px, \tag{6A.F.9}$$

where the infinitesimal generator A_P of the s.c. semigroup $\Phi(t)$ is given by

$$A_P = A - B[F_2 + B^*P], \text{ with maximal domain}; \tag{6A.F.10a}$$

$$\mathcal{D}(A_P) = \{x \in \mathcal{D}((-A)^{1-\gamma}) : (-A)^{1-\gamma}BB^*Px \in \mathcal{D}((-A)^{\gamma})\} \subset \mathcal{D}((-A)^{1-\gamma}). \tag{6A.F.10b}$$

Thus, the s.c. semigroup $\Phi(t)$ is, moreover, analytic on Y.

Proof. See proof of [Chapter 2, Theorem 2.3.8.1]. □

6A.G The Operator P Satisfies the Algebraic Riccati Equation

The counterpart of [Chapter 2, Theorem 2.3.9.1], with an analogous proof, is now

Theorem 6A.G.1 *(i) The operator P in (6A.B.15a) satisfies the following Algebraic Riccati Equation:*

$$(A^*Px, y)_Y + (PAx, y)_Y + (F_1x, y)_Y = ([F_2 + B^*P]x, [F_2 + B^*P]y)$$
$$\forall\, x, y \in \mathcal{D}((-A)^{\epsilon}); \quad \textit{in particular, } x, y \in \mathcal{D}(A_P) \subset \mathcal{D}((-A)^{1-\gamma}). \tag{6A.G.1}$$

Proof. By (6A.F.8), after a change of variable,

$$(Px, y)_Y = \int_t^{\infty} [F_1 - F_2^*(F_2 + B^*P)]\Phi(\tau - t)x, e^{A(\tau - t)}y)_Y\, d\tau. \tag{6A.G.2}$$

We differentiate (6A.G.2) initially for $x \in \mathcal{D}(A_P)$ and $y \in \mathcal{D}(A)$ and we use Corollary 6A.F.3. Details are as in the proof of [Chapter 2, Theorem 2.3.9.1]. □

We summarize the results of Sections 6A.C–6A.G.

Theorem *Assume the hypotheses of Section 6A.A [analyticity and stability of the semigroup e^{At}, and $A^{-\gamma}B \in \mathcal{L}(U; Y)$, $\gamma < 1$]. Assume, further, that the OCP (A.5) has a unique solution for all $y_0 \in Y$. Then, the results of Sections 6A.C through 6A.G hold true.*

Notes on Chapter 6

Min–Max Theory

The general treatment of Sections 6.1–6.18 follows closely McMillan and Triggiani [1994(b)] with the exception of the new Section 6.2.3; while the stable case of Sections 6.19–6.26 is based upon McMillan and Triggiani [1994(c)], complemented by McMillan and Triggiani [1991]. (For reasons beyond the authors' control, Subsection 6.2.2 is missing in McMillan and Triggiani [1994(c)].)

Section 6.2 (including the deliberate detour in Section 6.2.3) contains, in passing, also a solution of the following minimization problem:

$$\inf_u \int_0^T \left[\|Ry(t)\|_Z^2 + \|u(t)\|_U^2\right] dt,$$

for all $u \in L_2(0, T; U)$, for the dynamics (6.1.1.1), subject to a fixed disturbance $w \in L_2(0, T; V)$ [this problem is clearly equivalent to (6.2.2)]. A finite-dimensional version of this problem may be found in [Lee, Markus, 1968, pp. 190–1]; see also Mitter [1966]. The parabolic version of this problem with B mildly unbounded ($\delta < 1/2$) and $G =$ Identity is in [J. L. Lions, 1971, Chapter 3]. All these references contain the function called r in the present chapter.

PDE Illustrations

For lack of space, we do not provide explicit PDE examples, several of which may be found in McMillan's Ph.D. thesis [Mitter, 1966] and in Triggiani's review paper [Triggiani, 1994(a)]. It is an easy matter, however, to use the applications in Chapter 3 to construct numerous PDE examples with point/boundary control and disturbance. One may also include the case of the observation R unbounded, with a controlled degree of unboundedness, as in Chapter 1, Section 1.8, or Chapter 2, Section 2.5.

H^∞-Problem

The min–max problem of this chapter is related to the so-called H^∞-robust, state feedback, stabilization problem, and variations thereof: More precisely, the min–max problem may be viewed as a state space version of the frequency domain H^∞-problem. A textbook presentation, which emphasizes the min–max state space formulation in the finite-dimensional case, is given in Basar and Bernhard [1995], where additional frequency domain oriented references are given. It was the appearance, in 1991, of the first edition of this finite-dimensional book, that has sparked interest on this topic in "infinite-dimensional spaces," soon afterward. The original frequency domain formulation of the H^∞-problem is beyond the scope of this Chapter 6 (and of this book). For the purposes of these Notes, it suffices to say that the original H^∞-optimal control problem, in its equivalent time domain formulation, is a min–max game problem. Here we can only focus, and briefly, on min–max game works, and related algebraic Riccati equations with indefinite quadratic term, which are pertinent to the abstract parabolic treatment of the present Chapter 6.

Finite Dimensional Theory on ARE with Indefinite Quadratic Term

An enlightening source of the finite-dimensional theory on ARE with indefinite quadratic term, updated as of 1991–1992, is the paper of S. Chen [1992]. This paper has several features for those interested in this subject:

(a) It summarizes diverse control problems leading to ARE with nondefinite quadratic term, such as: (a_1) the Linear-Quadratic problem with conflicting objectives; (a_2) the two-person zero sum differential game (as in this Chapter 6); and (a_3) the state feedback for the H^∞-control problem.

(b) It points out [Chen, 1992, p. 96] the revealing and mathematically critical fact that *most of the existing results provide only a statement of equivalence between the existence of specific solutions to ARE (with indefinite quadratic term) and the control problems mentioned above.*

(c) It then furnishes, for the first time, necessary and sufficient conditions for the existence of positive solutions to a ARE with indefinite quadratic term (along with an iterative procedure to compute an extremal solution of it).

In accordance to the second point (b) above made by S. Chen, the statement in [Bensoussan et al., 1992, Vol. II, Theorem 4.1, p. 33] also provides only an *equivalence* between the existence of a stabilizing matrix, whose transfer function (from the disturbance to the output) has H^∞-norm $< \gamma$, and the existence of a positive definite solution of an ARE with nondefinite quadratic term, as in this Chapter 6, with the matrix $A - (BB^* - \gamma^{-2}GG^*)$ stable.

Infinite-Dimensional Case

It appears that, under the stimulus of the just published (1991) first edition of the book by Basar and Bernhard [1995], the years 1991–1992 were a period of intense, and independent, min–max activities in infinite dimensions in various circles. Most of them concerned, however, the case of control operator B and disturbance operator G both bounded (Bensoussan and Bernhard [1992], Ichikawa [1991a, b], Curtain [1991], and references therein), or only slightly unbounded as in van Keulen [1993], as to be of only modest interest to applications to partial differential equations. In particular, the critical case of boundary/point control and boundary/point disturbance is plainly excluded from all these aforementioned references. By contrast, the contemporaneous papers of McMillan and Triggiani [1991; 1994(a)] and Barbu [1995] on *hyperbolic* PDE and McMillan and Triggiani [1994b, c] on *parabolic* PDE focus precisely on the boundary/point action case. The paper of Barbu [1995] also provides an *equivalence* *statement*, as in [Bensoussan et al., 1992, Vol. II, Theorem 4.1, p. 33] between the H^∞-problem and the existence of a positive solution of a ARE with nondefinite quadratic term. By contrast, the work of McMillan and Triggiani quoted above pursues the following strategy (detailed in this Chapter 6 in the abstract parabolic case):

(i) It first defines a critical value $\gamma_c > 0$ of the parameter γ, explicitly in terms of the problem data (see Eqn. (6.6.1) in the general case and Eqn. (6.20.2.1) in the stable case, both in the present parabolic treatment).
(ii) Next, if $\gamma > \gamma_c$, it constructs a candidate Riccati operator, explicitly in terms of the problem data (in the style of Lasiecka and Triggiani's work on the linear quadratic problem seen in Chapters 1 and 2), and finally shows that such P is, in fact, *a solution of the ARE with indefinite quadratic term.*
(iii) Finally, if $\gamma < \gamma_c$, it shows that the value of the $\sup_w$ is $+\infty$, for all initial conditions.

This strategy, which is apparently new even in the finite-dimensional case, answers in the affirmative the open question raised by the aforementioned equivalence statements, and formulated explicitly in [Curtain, 1991, p. 253]: "In Section 3 we consider the standard state-feedback H^∞-control problem which depends on the solution of a non-standard Riccati equation. This has been solved only recently in infinite dimensions, but the existence and uniqueness of solutions of these Riccati equations is still an open problem."

The approach of McMillan and Triggiani provides existence for the relevant case of all $\gamma > \gamma_c$. We recall that uniqueness generally fails; see Remark 6.1.3.2.

The H^∞-Robust Stabilization Problem with Partial Observation

We refer to Bensoussan and Bernhard [1992] for a detailed description of this problem, and for a solution when the control operator B and the disturbance operator G are both bounded. The following solution in the case of fully unbounded B and G as in (6.1.1.6) and (6.1.1.7), based on the theory of the analytic case of the present Chapter 6, is taken from Section 7 of the review paper by R. Triggiani [1994(a)].

Consider the dynamics (6.1.1.1) with cost (6.1.2.1), in short, the quadruplet $\{A, B, G, R\}$ subject to the assumptions (H.1) through (H.6) of the present Chapter 6. In addition, we associate with it the partial observation

$$z = Cy + \eta, \tag{6N.1}$$

where $C \in \mathcal{L}(Y; Z)$, and Z is the Hilbert space of observations and the measurement error is η. It is well known (e.g., Bensoussan and Bernhard [1992]) that the solution of the corresponding H^∞-robust stabilization problem with partial observation (6N.1) (described by Section 4.1 of Bensoussan and Bernhard [1992]) relies on the solution of the following three problems via duality arguments:

(i) *Problem 1:* This is the min–max problem for the quadruplet $\{A, B, G, R\}$, which is considered in the present chapter, and which culminates with the theory of Theorem 6.1.3.1.

(ii) *Problem 2:* This is a similar min–max problem, this time, however, for the quadruplet $\{\bar{A}, \bar{B}, \bar{G}, \bar{R}\}$ where

$$\bar{A} = A^* + \gamma^{-2} PGG^*; \qquad \bar{B} = C^*; \qquad \bar{G} = PB; \qquad \bar{R} = G^*. \quad \text{(6N.2)}$$

Here, P is the algebraic Riccati operator provided by Theorem 6.1.3.1 for $\gamma > \gamma_c$ in the theory of Problem 1 for $\{A, B, G, R\}$, whereas C is given by (6.N.1).

(iii) *Problem 3:* This is a similar min–max problem, this time, however, for the quadruple $\{\tilde{A}, \tilde{B}, \tilde{G}, \tilde{R}\}$ where

$$\tilde{A} = A^*; \qquad \tilde{B} = C^*; \qquad \tilde{G} = R^*; \qquad \tilde{R} = G. \quad \text{(6N.3)}$$

As to *Problem 3*, we note that the control operator $\tilde{B}$ and the disturbance operator $\tilde{G}$ are bounded, whereas it is the observation $\tilde{R}$ that is now unbounded. It is clear, and well known [Lasiecka, Triggiani, 1991, p. 48] that this problem is definitely easier than the one considered in this Chapter 6 and solved by Theorem 6.1.3.1. After all, the quadratic term of the ARE involves $\tilde{B}$ and $\tilde{G}$. As to *Problem 2*, that $\{\bar{A}, \bar{B}, \bar{G}, \bar{R}\}$ fits also into the setting of the present Chapter 6 (where B^*P and G^*P are bounded by (6.1.3.4)) is obvious.

References and Bibliography

V. Barbu, H^∞-boundary control with state feedback: the hyperbolic case, *SIAM J. Control Optim.* **33** (1995), 684–701.

T. Basar and P. Bernhard, *H^∞-Optimal Control and Related Mimimax Design Problems A Dynamic Game Approach*, 2nd ed., Birkhäuser, 1995.

A. Bensousson and P. Bernhard, Remarks on the theory of robust control, *International Series of Numerical Mathematics*, **107** (1992), 149–166, Birkhäuser.

F. Bucci, Frequency domain stability of nonlinear feedback systems with unbounded input operators, Discrete and continuous dynamical systems, in print.

F. Bucci and L. Pandolfi, The value function of the singular quadratic regulator problem with distributed control action, *SIAM J. Control Optim.* **36** (1998), 115–136.

S. Chen, Necessary and sufficient conditions for the existence of positive solutions to algebraic Riccati equations with indefinite quadratic term, *Appl. Math. Optim.* **26** (July 1992), 95–110.

R. Curtain, State-space approaches to H-infinity control for infinite-dimensional linear systems, *Trans. Inst. MC,* **13** (1991), 253–261.

R. Datko. Extending a theorem of Liapunov to Hilbert space, *J. Math. Anal. Appl.* **32** (1970), 610–616.

A. Ichikawa, H^∞-control with state feedback and quadratic games in Hilbert space, preprint 1991(a).

A. Ichikawa, H_∞-control and min–max problems in Hilbert spaces, preprint, 1991(b).

T. Kato, *Perturbation theory of linear operators*, Springer-Verlag 1966.

B. van Keulen, *H_∞-Control for Distributed Parameter Systems: A State-Space Approach*, Birkhäuser, 1993.

I. Lasiecka, D. Lukes, and L. Pandolfi, Input dynamics and nonstandard Riccati equations, *J. Opt. Theory & Appl.* **84**, n. 3 (1995), 549–574.

I. Lasiecka, L. Pandolfi, and R. Triggiani, A singular control approach to highly damped second-order abstract equations and applications, *Appl. Math. Optim.* **36** (1997), 67–107. Preliminary announcement appeared in "Control of Partial Differential Equations," Marcel Dekker, **174** (1995), 157–171.

I. Lasiecka and R. Triggiani, *Differential and Algebraic Riccati Equations with Applications to Boundary/Point Control Problems: Continuous Theory and Approximation Theory* Springer-Verlag Lectures Notes LNCIS Vol. 164, 1991, p. 160.

E. B. Lee and L. Markus, *Foundation of Optimal Control Theory*, Wiley, 1968.

X. Li, talk presented at IFIP Conference, Zhejiang University, Hangzhou, PRC, June 1998.

X. Li and J. Yong, *Optimal control theory for infinite dimensional systems*, Birkhäuser 1995.

J. L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, 1971.

J. Louis and D. Wexler, The Hilbert space regulator problem and operator Riccati equations under stabilizability, *Annales de la Societé Scientifique de Bruxelles*, T. 105, 4, (1991) pp. 137–165.

D. G. Luenberger, *Optimization by Vector Space Methods*, Wiley, 1969.

C. McMillan, *Min–max game theory problems for partial differential equations and applications to H^∞-robust stabilization*, May 1993, Ph.D. dissertation, Department of Applied Mathematics, University of Virginia.

C. McMillan, Equivalent conditions for the solvability of non-standard and singular LQ-problems with applications to PDE with continuous input-solution map, *J. of Systems, Estimation and Control*, Vol. 7 (1977), electronic manuscript.

C. McMillan, Equivalent conditions for the solvability of non-standard and singular LQ-problems with applications to Parabolic equations *J. Opt. Th. & Appl.*, to appear in 1999.

C. McMillan and R. Triggiani, Min–max game theory and algebraic Riccati equations for boundary control problems with continuous input-solution map. Part I: The stable case, *Marcel Dekker Lecture Notes in Pure and Applied Mathematics*, Proceedings of International Conference/Workshop in Evolution Equations, held at Han-sur-Lesse, Belgium, October 1991, P. Clément and G. Lumer, eds.

C. McMillan and R. Triggiani, Min–max game theory and algebraic Riccati Equations for boundary control problems with continuous input-solution map. Part II: The general case, *Appl. Math. Optim.* **29** (1994(a)), 1–65. A summary appears in Min–Max Game Theory for a Class of Boundary Control Problems, Springer-Verlag Lecture Notes in Control & Information Sciences, vol. 185, pp. 459–466, Proceedings of the 10th International Conference, Sophia-Antipolis, France, June 9–12, 1992; and also in algebraic Riccati equations arising in game theory and in H^∞-control problems for a class of abstract systems, in *Differential Equations with Applications to Mathematical Physics*, W. F. Ames, E. M. Harrell, and J. V. Herod, eds. Academic Press, 1993.

C. McMillan and R. Triggiani, Min–max game theory and algebraic Riccati equations for boundary control problems with analytic semigroups. Part II: The general case, *J. Nonlinear Anal.* **22** (1994(b)), 431–465.

C. McMillan and R. Triggiani, Min–max game theory and algebraic Riccati equations for boundary control problems with analytic semigroups: Part I: The stable case, *Marcel Dekker Lecture Notes in Pure and Applied Mathematics* **152** (1994(c)), 757–780, invited paper for special volume (Festschrift), in honor of L. Markus on occasion of his 70th birthday, K. D. Elworthy, W. N. Everitt, and E. B. Lee, eds.

S. K. Mitter, Successive approximation methods for the solution of optimal control problems, *Automatica* **3** (1966), 135–149.

L. Pandolfi, The standard regulator problem for systems with input delays: an approach through singular control theory, *Appl. Math. Optim.* **31** (1995), 119–136.

L. Pandolfi, The Kalman–Yakubovich–Popov Theorem: An overview and new results for hyperbolic control systems, *Nonlinear Anal.* **30** (1997), 735–745.

L. Pandolfi, Dissipativity and the Lur'e problem for parabolic boundary control, *SIAM J. Control Optim.* **36** (1998), 2061–2081.

A. Pazy, *Semigroups of Operators and Applications to Partial Differential Equations, Appl. Math. Sci.* **44**, Springer-Verlag, 1983.

C. Sadosky, *Interpolation of Operators and Singular Integrals*, Marcel Dekker, 1979.

R. Triggiani, Min–max game theory and H^∞-problems for partial differential equations with boundary/point control and disturbance, INRIA-IFIP Conference, held at Universite de Technologie de Compiegne, France, July 1993, *Springer-Verlag Lecture Notes*, **197** (1994(a)), 70–89.

R. Triggani, Optimal boundary control and new Riccati equations for highly damped second order equations, *Diff. & Integral Eqns.* Vol. 7 (1994(b)), 1109–1144 (Invited paper for special volume in memory of P. Hess.)

R. Triggiani, An optimal boundary control problem for wave and platelike equations with high internal damping: an abstract approach, *Marcel Dekker Lecture Notes in Pure & Appl. Math.* **165** (1994(c)), 215–271.

Index