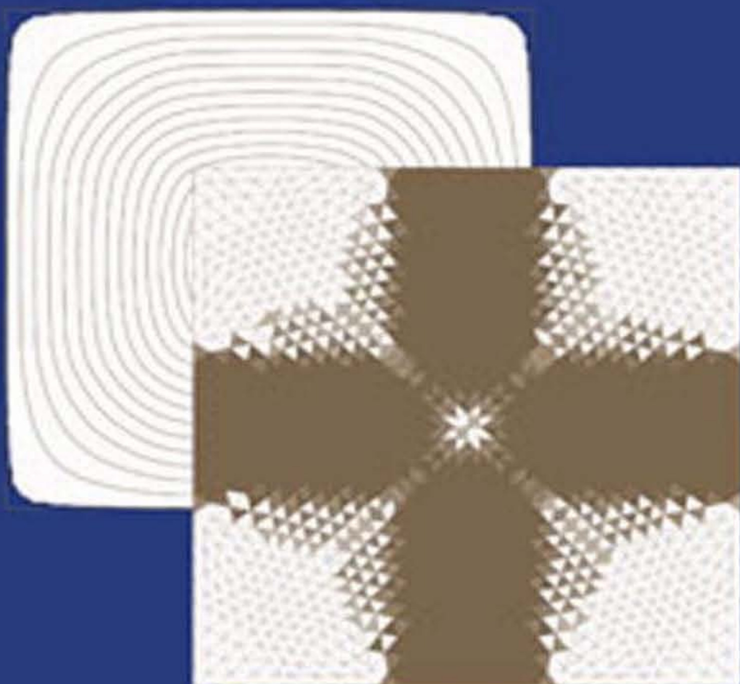
Shapes and Geometries

Metrics, Analysis, Differential Calculus, and Optimization

SECOND EDITION



M. C. Delfour
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Advances in Design and Control
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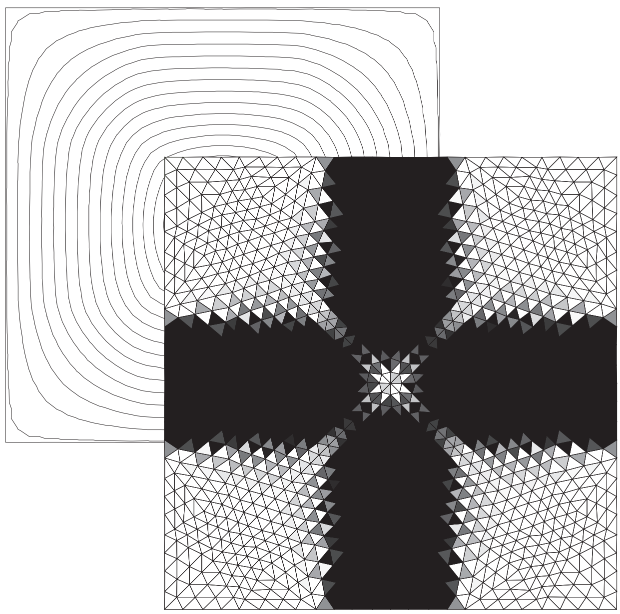
Shapes and Geometries



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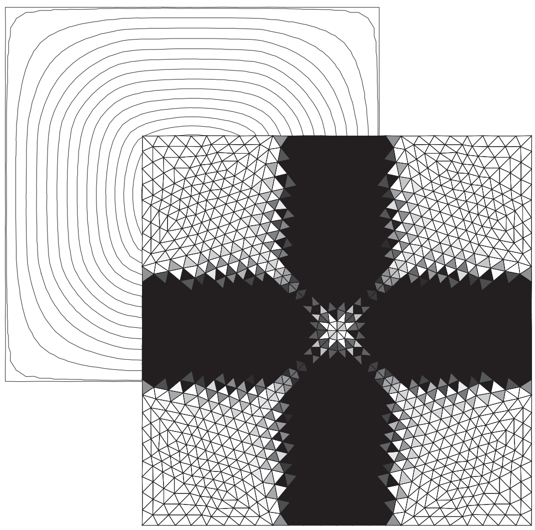

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This book is dedicated to
Alice, Jeanne, Jean, and Roger

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Preface

1 Objectives and Scope of the Book

The objective of this book is to give a comprehensive presentation of mathematical constructions and tools that can be used to study problems where the modeling, optimization, or control variable is no longer a set of parameters or functions but the shape or the structure of a geometric object. In that context, a good analytical framework and good modeling techniques must be able to handle the occurrence of singular behaviors whenever they are compatible with the mechanics or the physics of the problems at hand. In some optimization problems, the natural intuitive notion of a geometric domain undergoes *mutations* into relaxed entities such as microstructures. So the objects under consideration need not be smooth open domains, or even sets, as long as they still makes sense mathematically.

This book covers the basic mathematical ideas, constructions, and methods that come from different fields of mathematical activities and areas of applications that have often evolved in parallel directions. The scope of research is frighteningly broad because it touches on areas that include classical geometry, modern partial differential equations, geometric measure theory, topological groups, and constrained optimization, with applications to classical mechanics of continuous media such as fluid mechanics, elasticity theory, fracture theory, modern theories of optimal design, optimal location and shape of geometric objects, free and moving boundary problems, and image processing. Innovative modeling or new issues raised in some applications force a new look at the fundamentals of well-established mathematical areas such as geometry, to relax basic notions of volume, perimeter, and curvature or boundary value problems, and to find suitable relaxations of solutions. In that spirit, Henri Lebesgue was probably a pioneer when he relaxed the intuitive notion of volume to the one of *measure* on an equivalence class of *measurable* sets in 1907. He was followed in that endeavor in the early 1950s by the celebrated work of E. De Giorgi, who used the relaxed notion of *perimeter* defined on the class of Caccioppoli sets to solve Plateau's problem of minimal surfaces.

The material that is pertinent to the study of geometric objects and the entities and functions that are defined on them would necessitate an encyclopedic investment to bring together the basic theories and their fields of applications. This objective is obviously beyond the scope of a single book and two authors. The coverage of this book is more modest. Yet, it contains most of the important fundamentals at this stage of evolution of this expanding field.

Even if shape analysis and optimization have undergone considerable and important developments on the theoretical and numerical fronts, there are still cultural barriers between areas of applications and between theories. The whole field is extremely active, and the best is yet to come with fundamental structures and tools beginning to emerge. It is hoped that this book will help to build new bridges and stimulate cross-fertilization of ideas and methods.

2 Overview of the Second Edition

The second edition is almost a new book. All chapters from the first edition have been updated and, in most cases, considerably enriched with new material. Many chapters or parts of chapters have been completely rewritten following the developments in the field over the past 10 years. The book went from 9 to 10 chapters with a more elaborate sectioning of each chapter in order to produce a much more detailed table of contents. This makes it easier to find specific material.

A series of illustrative generic examples has been added right at the beginning of the introductory Chapter 1 to motivate the reader and illustrate the basic dilemma: parametrize geometries by functions or functions by geometries? This is followed by the big picture: a section on background and perspectives and a more detailed presentation of the second edition.

The former Chapter 2 has been split into Chapter 2 on the *classical descriptions and properties of domains and sets* and a new Chapter 3, where the important material on *Courant metrics* and the generic constructions of A. M. Micheletti have been reorganized and expanded. Basic definitions and material have been added and regrouped at the beginning of Chapter 2: Abelian group structure on subsets of a set, connected and path-connected spaces, function spaces, tangent and dual cones, and geodesic distance. The coverage of domains that verify some segment property and have a local epigraph representation has been considerably expanded, and Lipschitzian (graph) domains are now dealt with as a special case.

The new Chapter 3 on domains and submanifolds that are the image of a fixed set considerably expands the material of the first edition by bringing up the general assumptions behind the generic constructions of A. M. Micheletti that lead to the *Courant metrics* on the quotient space of families of transformations by subgroups of isometries such as identities, rotations, translations, or flips. The general results apply to a broad range of groups of transformations of the Euclidean space and to arbitrary closed subgroups. New complete metrics on the whole spaces of homeomorphisms and C^k-diffeomorphisms are also introduced to extend classical results for transformations of compact manifolds to general unbounded closed sets and open sets that are crack-free. This material is central in classical mechanics and physics and in modern applications such as imaging and detection.

The former Chapter 7 on *transformations versus flows of velocities* has been moved right after the Courant metrics as Chapter 4 and considerably expanded. It now specializes the results of Chapter 3 to spaces of transformations that are generated by the flow of a *velocity field* over a generic time interval. One important motivation is to introduce a notion of semiderivatives as well as a tractable criterion for continuity with respect to Courant metrics. Another motivation for the velocity point of view is the general framework of R. Azencott and A. Trouvé starting in 1994 with applications in imaging. They construct complete metrics in relation with *geodesic paths* in spaces of diffeomorphisms generated by a velocity field.

The former Chapter 3 on the *relaxation to measurable sets* and Chapters 4 and 5 on *distance and oriented distance functions* have become Chapters 5, 6, and 7. Those chapters have been renamed *Metrics Generated by ...* in order to emphasize one of the main thrusts of the book: the construction of complete metrics on shapes and geometries.[1] Those chapters emphasize the *function analytic* description of sets and domains: construction of metric topologies and characterization of compact families of sets or submanifolds in the Euclidean space. In that context, we are now dealing with equivalence classes of sets that may or may not have an invariant open or closed representative in the class. For instance, they include Lebesgue measurable sets and Federer's sets of positive reach. Many of the classical properties of sets can be recovered from the smoothness or function analytic properties of those functions.

The former Chapter 6 on optimization of shape functions has been completely rewritten and expanded as Chapter 8 on *shape continuity and optimization*. With meaningful metric topologies, we can now speak of continuity of a geometric objective functional such as the volume, the perimeter, the mean curvature, etc., compact families of sets, and existence of optimal geometries. The chapter concentrates on continuity issues related to shape optimization problems under state equation constraints. A special family of state constrained problems are the ones for which the objective function is defined as an infimum over a family of functions over a fixed domain or set such as the eigenvalue problems. We first characterize the continuity of the transmission problem and the upper semicontinuity of the first eigenvalue of the generalized Laplacian with respect to the domain. We then study the continuity of the solution of the homogeneous Dirichlet and Neumann boundary value problems with respect to their underlying domain of definition since they require different constructions and topologies that are generic of the two types of boundary conditions even for more complex nonlinear partial differential equations. An introduction is also given to the concepts and results from *capacity theory* from which very general families of sets stable with respect to boundary conditions can be constructed. Note that some material has been moved from one chapter to another. For instance, section 7 on the *continuity of the Dirichlet boundary problem* in the former Chapter 3 has been merged with the content of the former Chapter 4 in the new Chapter 8.

The former Chapters 8 and 9 have become Chapters 9 and 10. They are devoted to a modern version of the *shape calculus*, an introduction to the *tangential differential calculus*, and the shape derivatives under a state equation constraint. In Chapters 3, 5, 6, and 7, we have constructed complete metric spaces of geometries. Those spaces are nonlinear and nonconvex. However, several of them have a group

[1] This is in line with current trends in the literature such as in the work of the 2009 Abel Prize winner M. Gromov [**1**] and its applications in imaging by G. Sapiro [**1**] and F. Mémoli and G. Sapiro [**1**] to identify objects up to an isometry.

structure and, in some cases, it is possible to construct C^1-paths in the group from velocity fields. This leads to the notion of *Eulerian semiderivative* that is somehow the analogue of a derivative on a smooth manifold. In fact, two types of semiderivatives are of interest: the weaker *Gateaux style semiderivative* and the stronger *Hadamard style semiderivative*. In the latter case, the classical chain rule is still available even for nondifferentiable functions. In order to prepare the ground for shape derivatives, an enriched self-contained review of the pertinent material on semiderivatives and derivatives in topological vector spaces is provided.

The important Chapter 10 concentrates on two generic examples often encountered in shape optimization. The first one is associated with the so-called *compliance problems*, where the shape functional is itself the minimum of a domain-dependent energy functional. The special feature of such functionals is that the adjoint state coincides with the state. This obviously leads to considerable simplifications in the analysis. In that case, it will be shown that theorems on the differentiability of the minimum of a functional with respect to a real parameter readily give explicit expressions of the Eulerian semiderivative even when the minimizer is not unique. The second one will deal with shape functionals that can be expressed as the saddle point of some appropriate Lagrangian. As in the first example, theorems on the differentiability of the saddle point of a functional with respect to a real parameter readily give explicit expressions of the Eulerian semiderivative even when the solution of the saddle point equations is not unique. Avoiding the differentiation of the state equation with respect to the domain is particularly advantageous in shape problems.

3 Intended Audience

The targeted audience is applied mathematicians and advanced engineers and scientists, but the book is also suitable for a broader audience of mathematicians as a relatively well-structured initiation to shape analysis and calculus techniques. Some of the chapters are fairly self-contained and of independent interest. They can be used as lecture notes for a mini-course. The material at the beginning of each chapter is accessible to a broad audience, while the latter sections may sometimes require more mathematical maturity. Thus the book can be used as a graduate text as well as a reference book. It complements existing books that emphasize specific mechanical or engineering applications or numerical methods. It can be considered a *companion* to the book of J. Sokołowski and J.-P. Zolésio [**9**], *Introduction to Shape Optimization*, published in 1992.

Earlier versions of parts of this book have been used as lecture notes in graduate courses at the Université de Montréal in 1986–1987, 1993–1994, 1995–1996, and 1997–1998 and at international meetings, workshops, or schools: Séminaire de Mathématiques Supérieures on *Shape Optimization and Free Boundaries* (Montréal, Canada, June 25 to July 13, 1990), short course on *Shape Sensitivity Analysis* (Kénitra, Morocco, December 1993), course of the COMETT MATARI European Program on *Shape Optimization and Mutational Equations* (Sophia-Antipolis, France, September 27 to October 1, 1993), CRM Summer School on *Boundaries,* *Interfaces and Transitions* (Banff, Canada, August 6–18, 1995), and CIME course on *Optimal Design* (Troia, Portugal, June 1998).

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Michel Delfour
Jean-Paul Zolésio

August 13, 2009

Chapter 1

Introduction: Examples, Background, and Perspectives

1 Orientation

1.1 Geometry as a Variable

The central object of this book[1] is the *geometry* as a variable. As in the theory of functions of real variables, we need a differential calculus, spaces of geometries, evolution equations, and other familiar concepts in analysis when the variable is no longer a scalar, a vector, or a function, but is a geometric domain. This is motivated by many important problems in science and engineering that involve the geometry as a modeling, design, or control variable. In general the geometric objects we shall consider will not be parametrized or structured. Yet we are not starting from scratch, and several building blocks are already available from many fields: geometric measure theory, physics of continuous media, free boundary problems, the parametrization of geometries by functions, the set derivative as the inverse of the integral, the parametrization of functions by geometries, the Pompéiu–Hausdorff metric, and so on.

As is often the case in mathematics, spaces of geometries and notions of derivatives with respect to the geometry are built from well-established elements of functional analysis and differential calculus. There are many ways to structure families of geometries. For instance, a domain can be made variable by considering

[1]The *numbering* of equations, theorems, lemmas, corollaries, definitions, examples, and remarks is by chapter. When a reference to another chapter is necessary it is always followed by the words *in Chapter* and the *number of the chapter*. For instance, "equation (2.18) in Chapter 9." The text of theorems, lemmas, and corollaries is slanted; the text of definitions, examples, and remarks is normal shape and ended by a square □. This makes it possible to aesthetically emphasize certain words especially in definitions. The bibliography is by author in alphabetical order. For each author or group of coauthors, there is a numbering in square brackets starting with [**1**]. A reference to an item by a single author is of the form J. Dieudonné [**3**] and a reference to an item with several coauthors S. Agmon, A. Douglis, and L. Nirenberg [**2**]. *Boxed formulae* or *statements* are used in some chapters for two distinct purposes. First, they emphasize certain important definitions, results, or identities; second, in long proofs of some theorems, lemmas, or corollaries, they isolate key intermediary results which will be necessary to more easily follow the subsequent steps of the proof.

the images of a fixed domain by a family of diffeomorphisms that belong to some function space over a fixed domain. This naturally occurs in physics and mechanics, where the deformations of a continuous body or medium are smooth, or in the numerical analysis of optimal design problems when working on a fixed grid. This construction naturally leads to a group structure induced by the composition of the diffeomorphisms. The underlying spaces are no longer topological vector spaces but groups that can be endowed with a nice complete metric space structure by introducing the *Courant metric*. The practitioner might or might not want to use the underlying mathematical structure associated with his or her constructions, but it is there and it contains information that might guide the theory and influence the choice of the numerical methods used in the solution of the problem at hand.

The parametrization of a fixed domain by a fixed family of diffeomorphisms obviously limits the family of variable domains. The topology of the images is similar to the topology of the fixed domain. Singularities that were not already present there cannot be created in the images. Other constructions make it possible to considerably enlarge the family of variable geometries and possibly open the doors to pathological geometries that are no longer open sets with a nice boundary. Instead of parametrizing the domains by functions or diffeomorphisms, certain families of functions can be parametrized by sets. A single function completely specifies a set or at least an equivalence class of sets. This includes the distance functions and the characteristic function, but also the support function from convex analysis. Perhaps the best known example of that construction is the Pompéiu–Hausdorff metric topology. This is a very weak topology that does not preserve the volume of a set. When the volume, the perimeter, or the curvatures are important, such functions must be able to yield relaxed definitions of volume, perimeter, or curvatures. The characteristic function that preserves the volume has many applications. It played a fundamental role in the integration theory of Henri Lebesgue at the beginning of the 20th century. It was also used in the 1950s by E. De Giorgi to define a relaxed notion of perimeter in the *theory of minimal surfaces*.

Another technique that has been used successfully in *free* or *moving boundary problems*, such as motion by mean curvature, shock waves, or detonation theory, is the use of level sets of a function to describe a free or moving boundary. Such functions are often the solution of a system of partial differential equations. This is another way to build new tools from functional analysis. The choice of families of function parametrized sets or of families of set parametrized functions, or other appropriate constructions, is obviously problem dependent, much like the choice of function spaces of solutions in the theory of partial differential equations or optimization problems. This is one aspect of the geometry as a variable. Another aspect is to build the equivalent of a differential calculus and the computational and analytical tools that are essential in the characterization and computation of geometries. Again, we are not starting from scratch and many building blocks are already available, but many questions and issues remain open.

This book aims at covering a small but fundamental part of that program. We had to make difficult choices and refer the reader to appropriate books and references for *background material* such as geometric measure theory and *specialized topics* such as homogenization theory and microstructures which are available in excellent books in English. It was unfortunately not possible to include references to the considerable literature on numerical methods, free and moving boundary problems, and optimization.

1.2 Outline of the Introductory Chapter

We first give a series of generic examples where the shape or the geometry is the modeling, control, or optimization variable. They will be used in the subsequent chapters to illustrate the many ways such problems can be formulated. The first example is the celebrated problem of the optimal shape of a column formulated by Lagrange in 1770 to prevent buckling. The extremization of the eigenvalues has also received considerable attention in the engineering literature. The free interface between two regions with different physical or mechanical properties is another generic problem that can lead in some cases to a *mixing* or a *microstructure*. Two typical problems arising from applications to condition the thermal environment of satellites are described in sections 7 and 8. The first one is the design of a thermal diffuser of minimal weight subject to an inequality constraint on the output thermal power flux. The second one is the design of a thermal radiator to effectively radiate large amounts of thermal power to space. The geometry is a volume of revolution around an axis that is completely specified by its height and the function which specifies its lateral boundary. Finally, we give a glimpse at *image segmentation*, which is an example of shape/geometric identification problems. Many chapters of this book are of direct interest to *imaging sciences*.

Section 10 presents some background and perspectives. A fundamental issue is to find *tractable* and preferably analytical representations of a geometry as a variable that are compatible with the problems at hand. The generic examples suggest two types of representations: the ones where the *geometry is parametrized by functions* and the ones where a family of *functions is parametrized by the geometry.* As is always the case, the choice is very much problem dependent. In the first case, the topology of the variable sets is fixed; in the second case the families of sets are much larger and topological changes are included. The book presents the two points of view. Finally, section 11 sketches the material in the second edition of the book.

2 A Simple One-Dimensional Example

A general feature of minimization problems with respect to a shape or a geometry subject to a state equation constraint is that they are generally not convex and that, when they have a solution, it is generally not unique. This is illustrated in the following simple example from J. Céa [**2**]: minimize the *objective function*

$$J(a) \stackrel{\text{def}}{=} \int_0^a |y_a(x) - 1|^2 \, dx,$$

where $a \geq 0$ and y_a is the solution of the boundary value problem (*state equation*)

$$\frac{d^2 y_a}{dx^2}(x) = -2 \text{ in } \Omega_a \stackrel{\text{def}}{=} (0, a), \quad \frac{dy_a}{dx}(0) = 0, \quad y_a(a) = 0. \tag{2.1}$$

Here the one-dimensional geometric domain $\Omega_a =]0, a[$ is the minimizing variable. We recognize the classical structure of a control problem, except that the minimizing variable is no longer under the integral sign but in the limits of the integral sign. One consequence of this difference is that even the simplest problems will usually not be convex or convexifiables. They will require a special analysis.

In this example it is easy to check that the solution of the state equation is

$$y_a(x) = a^2 - x^2 \quad \text{and} \quad J(a) = \frac{8}{15}a^5 - \frac{4}{3}a^3 + a.$$

The graph of J, shown in Figure 1.1, is not the graph of a convex function. Its global minimum in $a_0 = 0$, local maximum in a_1, and local minimum in a_2,

$$a_1 \stackrel{\text{def}}{=} \sqrt{\frac{3}{4}\left(1 - \frac{1}{\sqrt{3}}\right)}, \quad a_2 \stackrel{\text{def}}{=} \sqrt{\frac{3}{4}\left(1 + \frac{1}{\sqrt{3}}\right)},$$

are all different.

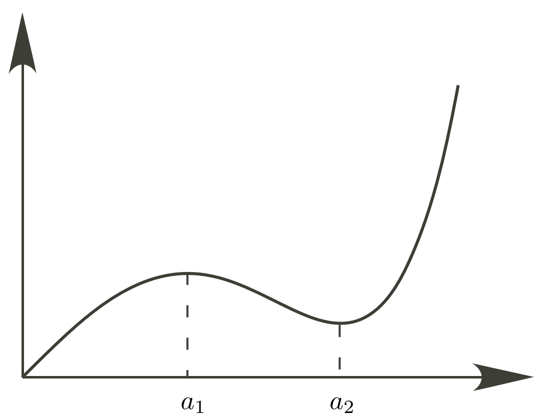


Figure 1.1. *Graph of $J(a)$.*

To avoid a trivial solution, a strictly positive lower bound must be put on a. A unique minimizing solution is obtained for $a \geq a_1$ where the gradient of J is zero. For $0 < a < a_2$, the minimum will occur at the preset lower or upper bound on a.

3 Buckling of Columns

The next example illustrates the fact that even simple problems can be nondifferentiable with respect to the geometry. This is generic of all eigenvalue problems when the eigenvalue is not simple.

One of the early optimal design problems was formulated by J. L. Lagrange [**1**] in 1770 (cf. I. Todhunter and K. Pearson [**1**]) and later studied by the Danish mathematician and astronomer T. Clausen [**1**] in 1849. It consists in finding the best profile of a vertical column of fixed volume to prevent buckling.

It turns out that this problem is in fact a hidden maximization of an eigenvalue. Many incorrect solutions had been published until 1992. This problem and other problems related to columns have been revisited in a series of papers by S. J. Cox [**1**], S. J. Cox and M. L. Overton [**1**], S. J. Cox [**2**], and S. J. Cox and C. M. McCarthy [**1**]. Since Lagrange many authors have proposed solutions, but a complete theoretical and numerical solution for the buckling of a column was given only in 1992 by S. J. Cox and M. L. Overton [**1**]. The difficulty was that the eigenvalue is not simple and hence not differentiable with respect to the geometry.

Consider a normalized column of unit height and unit volume (see Figure 1.2). Denote by ℓ the *magnitude of the normalized axial load* and by u the resulting transverse displacement. Assume that the potential energy is the sum of the bending and elongation energies

$$\int_0^1 EI\,|u''|^2\,dx - \ell \int_0^1 |u'|^2\,dx,$$

where I is the second moment of area of the column's cross section and E is its Young's modulus. For sufficiently small load ℓ the minimum of this potential energy with respect to all admissible u is zero. *Euler's buckling load* λ of the column is the largest ℓ for which this minimum is zero. This is equivalent to finding the following minimum:

$$\lambda \stackrel{\text{def}}{=} \inf_{0\neq u\in V} \frac{\int_0^1 EI\,|u''|^2\,dx}{\int_0^1 |u'|^2\,dx}, \tag{3.1}$$

where $V = H_0^2(0,1)$ corresponds to the clamped case, but other types of boundary conditions can be contemplated. This is an eigenvalue problem with a special Rayleigh quotient.

Assume that E is constant and that the second moment of area $I(x)$ of the column's cross section at the height x, $0 \le x \le 1$, is equal to a constant c times its

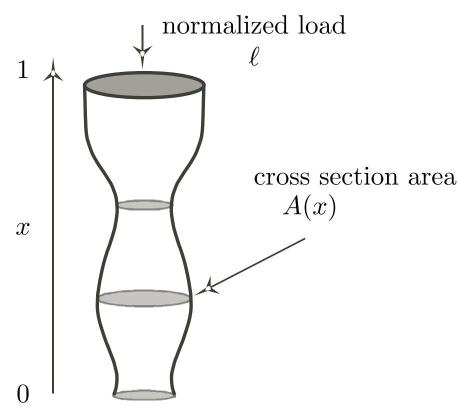


Figure 1.2. *Column of height one and cross section area A under the load ℓ.*

cross-sectional area $A(x)$,

$$I(x) = c\,A(x) \quad \text{and} \quad \int_0^1 A(x)\,dx = 1.$$

Normalizing λ by cE and taking into account the engineering constraints

$$\exists 0 < A_0 < A_1,\, \forall x \in [0,1], \quad 0 < A_0 \le A(x) \le A_1,$$

we finally get

$$\sup_{A\in\mathcal{A}} \lambda(A), \quad \lambda(A) \stackrel{\text{def}}{=} \inf_{0\ne u\in V} \frac{\int_0^1 A\,|u''|^2\,dx}{\int_0^1 |u'|^2\,dx}, \tag{3.2}$$

$$\mathcal{A} \stackrel{\text{def}}{=} \left\{ A \in L^2(0,1) \,:\, A_0 \le A \le A_1 \text{ and } \int_0^1 A(x)\,dx = 1 \right\}. \tag{3.3}$$

4 Eigenvalue Problems

Let D be a bounded open Lipschitzian domain in $\mathbf{R}^{\mathrm{N}}$ and $A \in L^\infty(D; \mathcal{L}(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}))$ be a matrix function defined on D such that

$$^*\!A = A \quad \text{and} \quad \alpha I \le A \le \beta I \tag{4.1}$$

for some coercivity and continuity constants $0 < \alpha \le \beta$ and $^*\!A$ is the transpose of A. Consider the minimization or the maximization of the first eigenvalue

$$\left.\begin{array}{l} \displaystyle\sup_{\Omega\in\mathcal{A}(D)} \lambda^A(\Omega) \\ \displaystyle\inf_{\Omega\in\mathcal{A}(D)} \lambda^A(\Omega) \end{array}\right| \quad \lambda^A(\Omega) \stackrel{\text{def}}{=} \inf_{0\ne\varphi\in H_0^1(\Omega)} \frac{\int_\Omega A\nabla\varphi\cdot\nabla\varphi\,dx}{\int_\Omega |\varphi|^2\,dx}, \tag{4.2}$$

where $\mathcal{A}(D)$ is a family of admissible open subsets of D (cf., for instance, sections 2, 7, and 9 of Chapter 8).

In the vectorial case, consider the following *linear elasticity* problem: find $U \in H_0^1(\Omega)^3$ such that

$$\forall W \in H_0^1(\Omega)^3, \quad \int_\Omega C\varepsilon(U)\cdot\cdot\,\varepsilon(W)\,dx = \int_\Omega F\cdot W\,dx \tag{4.3}$$

for some distributed loading $F \in L^2(\Omega)^3$ and a *constitutive law* C which is a bilinear symmetric transformation of

$$\mathrm{Sym}_3 \stackrel{\text{def}}{=} \left\{\tau \in \mathcal{L}(\mathbf{R}^3;\mathbf{R}^3) \colon {}^*\tau = \tau\right\}, \quad \sigma\cdot\cdot\,\tau \stackrel{\text{def}}{=} \sum_{1\le i,j\le 3} \sigma_{ij}\,\tau_{ij}$$

($\mathcal{L}(\mathbf{R}^3;\mathbf{R}^3)$ is the space of all linear transformations of $\mathbf{R}^3$ or 3×3-matrices) under the following assumption.

Assumption 4.1.
The *constitutive law* is a transformation $C \in \mathrm{Sym}_3$ for which there exists a constant $\alpha > 0$ such that $C\tau\cdot\cdot\,\tau \ge \alpha\,\tau\cdot\cdot\,\tau$ for all $\tau \in \mathrm{Sym}_3$. □

For instance, for the Lamé constants $\mu > 0$ and $\lambda \geq 0$, the special *constitutive law* $C\tau = 2\mu\,\tau + \lambda\,\mathrm{tr}\,\tau\, I$ satisfies Assumption 4.1 with $\alpha = 2\mu$.

The associated bilinear form is

$$a_\Omega(U, W) \stackrel{\text{def}}{=} \int_\Omega C\varepsilon(U) \cdot\cdot\, \varepsilon(W)\, dx,$$

where U is a vector function, $D(U)$ is the Jacobian matrix of U, and

$$\varepsilon(U) \stackrel{\text{def}}{=} \frac{1}{2}\left(D(U) + {}^*D(U)\right)$$

is the *strain tensor*. The first eigenvalue is the minimum of the *Rayleigh quotient*

$$\lambda(\Omega) = \inf\left\{ \frac{a_\Omega(U,U)}{\int_\Omega |U|^2\, dx} : \forall U \in H_0^1(\Omega)^3,\, U \neq 0 \right\}.$$

A typical problem is to find the sensitivity of the first eigenvalue with respect to the shape of the domain Ω. In 1907, J. Hadamard [**1**] used displacements along the normal to the boundary Γ of a C^∞-domain to compute the derivative of the first eigenvalue of the clamped plate. As in the case of the column, this problem is not differentiable with respect to the geometry when the eigenvalue is not simple.

5 Optimal Triangular Meshing

The *shape calculus* that will be developed in Chapters 9 and 10 for problems governed by partial differential equations (the *continuous model*) will be readily applicable to their *discrete model* as in the finite element discretization of elliptic boundary value problems. However, some care has to be exerted in the choice of the formula for the gradient, since the solution of a finite element discretization problem is usually less smooth than the solution of its continuous counterpart.

Most shape objective functionals will have two basic formulas for their shape gradient: a *boundary expression* and a *volume expression*. The boundary expression is always nicer and more compact but can be applied only when the solution of the underlying partial differential equation is smooth and in most cases *smoother* than the finite element solution. This leads to serious computational errors. The right formula to use is the less attractive volume expression that requires only the same smoothness as the finite element solution. Numerous computational experiments confirm that fact (cf., for instance, E. J. Haug and J. S. Arora [**1**] or E. J. Haug, K. K. Choi, and V. Komkov [**1**]). With the volume expression, the gradient of the objective function with respect to internal and boundary nodes can be readily obtained by plugging in the right velocity field.

A large class of linear elliptic boundary value problems can be expressed as the minimum of a quadratic function over some Hilbert space. For instance, let Ω be a bounded open domain in $\mathbf{R}^N$ with a smooth boundary Γ. The solution u of the boundary value problem

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma$$

is the minimizing element in the Sobolev space $H_0^1(\Omega)$ of the energy functional

$$E(v,\Omega) \stackrel{\text{def}}{=} \int_\Omega |\nabla v|^2 - 2f\,v\,dx,$$

$$J(\Omega) \stackrel{\text{def}}{=} \inf_{v\in H_0^1(\Omega)} E(v,\Omega) = E(u,\Omega) = -\int_\Omega |\nabla u|^2\,dx.$$

The elements of this problem are a Hilbert space V, a continuous symmetrical coercive bilinear form on V, and a continuous linear form ℓ on V. With this notation

$$\exists u \in V, \quad E(u) = \inf_{v\in V} E(v), \quad E(v) \stackrel{\text{def}}{=} a(v,v) - 2\,\ell(v)$$

and u is the unique solution of the variational equation

$$\exists u \in U,\, \forall v \in V, \quad a(u,v) = \ell(v).$$

In the finite element approximation of the solution u, a finite-dimensional subspace V_h of V is used for some small mesh parameter h. The solution of the approximate problem is given by

$$\exists u_h \in V_h, \quad E(u_h) = \inf_{v_h\in V_h} E(v_h), \quad \Rightarrow \exists u_h \in U_h,\, \forall v_h \in V_h, \quad a(u_h,v_h) = \ell(v_h).$$

It is easy to show that the error can be expressed as follows:

$$a(u-u_h, u-u_h) = \|u-u_h\|_V^2 = 2\,[E(u_h) - E(u)]\,.$$

Assume that Ω is a polygonal domain in $\mathbf{R}^N$. In the finite element method, the domain is partitioned into a set τ_h of small triangles by introducing nodes in $\bar{\Omega}$

$$\begin{aligned} M &\stackrel{\text{def}}{=} \{M_i \in \Omega \,:\, 1 \le i \le p\} \\ \partial M &\stackrel{\text{def}}{=} \{M_i \in \partial\Omega \,:\, p+1 \le i \le p+q\} \end{aligned} \quad \text{and} \quad \overline{M} \stackrel{\text{def}}{=} M \cup \partial M$$

for some integers $p \ge N+1$ and $q \ge 1$ (see Figure 1.3). Therefore the triangularization $\tau_h = \tau_h(\overline{M})$, the solution space $V_h = V_h(\overline{M})$, and the solution $u_h = u_h(\overline{M})$ are functions of the positions of the nodes of the set $\overline{M}$. Assuming that the total

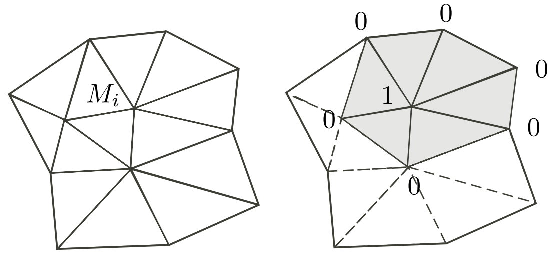


Figure 1.3. *Triangulation and basis function associated with node M_i.*

number of nodes is fixed, consider the following optimal triangularization problem:

$$\inf_{\overline{M}} j(\overline{M}), \quad j(\overline{M}) \overset{\text{def}}{=} E(u_h(\tau_h(\overline{M})), \Omega) = \inf_{v_h \in V_h(\overline{M})} E(v_h, \Omega),$$

$$\begin{aligned} \|u - u_h\|_V^2 = \int_\Omega |\nabla(u - u_h)|^2\, dx &= 2\, \left[E(u_h, \Omega(\tau_h(\overline{M}))) - E(u, \Omega)\right] \\ &= 2\, \left[J(\Omega(\tau_h(\overline{M}))) - J(\Omega)\right], \end{aligned}$$

$$J(\Omega(\tau_h(\overline{M}))) \overset{\text{def}}{=} \inf_{v_h \in V_h} E(v_h, \Omega(\tau_h(\overline{M}))) = E(u_h, \Omega(\tau_h(\overline{M}))) = -\int_\Omega |\nabla u_h|^2\, dx.$$

The objective is to compute the partial derivative of $j(\overline{M})$ with respect to the ℓth component $(M_i)^\ell$ of the node M_i:

$$\frac{\partial j}{\partial (M_i)^\ell}\,(\overline{M}).$$

This partial derivative can be computed by using the velocity method for the special velocity field (cf. M. C. Delfour, G. Payre, and J.-P. Zolésio [**3**])

$$V_{i\ell}(x) = b_{M_i}(x)\, \vec{e}_\ell,$$

where $b_{M_i} \in V_h$ is the (piecewise P^1) basis function associated with the node M_i: $b_{M_i}(M_j) = \delta_{ij}$ for all i, j. In that method each point X of the plane is moved according to the solution of the vector differential equation

$$\frac{dx}{dt}(t) = V(x(t)), \quad x(0) = X.$$

This yields a transformation $X \mapsto T_t(X) \overset{\text{def}}{=} x(t; X) : \mathbf{R}^2 \to \mathbf{R}^2$ of the plane, and it is natural to introduce the following notion of semiderivative:

$$dJ(\Omega; V) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{J(T_t(\Omega)) - J(\Omega)}{t}.$$

For $t \geq 0$ small, the velocity field must be chosen in such a way that triangles are moved onto triangles and the point M_i is moved in the direction $\vec{e}_\ell$:

$$M_i \to M_{it} = M_i + t\, \vec{e}_\ell.$$

This is achieved by choosing the following velocity field:

$$V_{i\ell}(t, x) = b_{M_{it}}(x)\, \vec{e}_\ell,$$

where $b_{M_{it}}$ is the piecewise P^1 basis function associated with node M_{it}: $b_{M_{it}}(M_j) = \delta_{ij}$ for all i, j. This yields the family of transformations

$$T_t(x) = x + t\, b_{M_i}(x)\, \vec{e}_\ell$$

which moves the node M_i to $M_i + t\, \vec{e}_\ell$ and hence

$$\frac{\partial j}{\partial (M_i)^\ell}(\overline{M}) = dJ(\Omega; V_{i\ell}).$$

Going back to our original example, introduce the shape functional

$$J(\Omega) \stackrel{\text{def}}{=} \inf_{v \in H_0^1(\Omega)} E(\Omega, v) = -\int_\Omega |\nabla u|^2 \, dx, \quad E(\Omega, v) = \int_\Omega |\nabla v|^2 - 2\, f\, v \, dx.$$

In Chapter 9, we shall show that we have the following boundary and volume expressions for the derivative of $J(\Omega)$:

$$dJ(\Omega; V) = -\int_\Gamma \left| \frac{\partial u}{\partial n} \right|^2 V \cdot n \, d\Gamma,$$

$$dJ(\Omega; V) = \int_\Omega A'(0) \nabla u \cdot \nabla u - 2 \left[\operatorname{div} V(0) f + \nabla f \cdot V(0) \right] u \, dx,$$

$$A'(0) = \operatorname{div} V(0)\, I - {}^*DV(0) - DV(0).$$

For a P^1-approximation

$$V_h \stackrel{\text{def}}{=} \left\{ v \in C^0(\bar{\Omega}) \, : \, v|_K \in P^1(K), \, \forall K \in \tau_h \right\}$$

and the trace of the normal derivative on Γ is not defined. Thus, it is necessary to use the volume expression. For the velocity field $V_{i\ell}$

$$DV_{i\ell} = \vec{e}_\ell \, {}^*\nabla b_{M_i}, \quad \operatorname{div} DV_{i\ell} = \vec{e}_\ell \cdot \nabla b_{M_i},$$

$$A'(0) = \vec{e}_\ell \cdot \nabla b_{M_i} \, I - \vec{e}_\ell \, {}^*\nabla b_{M_i} - \nabla b_{M_i} \, {}^*\vec{e}_\ell.$$

Since

$$\frac{\partial j}{\partial (M_i)^\ell} (\overline{M}) = dJ(\Omega; V_{i\ell}),$$

we finally obtain the formula for the derivative of the function $j(\overline{M})$ with respect to node M_i in the direction $\vec{e}_\ell$:

$$\begin{aligned} \frac{\partial j}{\partial (M_i)^\ell} (\overline{M}) = \int_\Omega & \left[\vec{e}_\ell \cdot \nabla b_{M_i} \, I - \vec{e}_\ell \, {}^*\nabla b_{M_i} - \nabla b_{M_i} \, {}^*\vec{e}_\ell \right] \nabla; u_h \cdot \nabla u_h \\ & - 2 \left[\vec{e}_\ell \cdot \nabla b_{M_i} \, f + \nabla f \cdot \vec{e}_\ell \, b_{M_i} \right] u_h \, dx. \end{aligned}$$

Since the support of b_{M_i} consists of the triangles having M_i as a vertex, the gradient with respect to the nodes can be constructed piece by piece by visiting each node.

6 Modeling Free Boundary Problems

The first step towards the solution of a shape optimization is the mathematical modeling of the problem. Physical phenomena are often modeled on relatively smooth or nice geometries. Adding an objective functional to the model will usually push the system towards rougher geometries or even microstructures. For instance, in the optimal design of plates the optimization of the profile of a plate led to highly oscillating profiles that looked like a comb with abrupt variations ranging from zero to maximum thickness. The phenomenon began to be understood in 1975 with the paper of N. Olhoff [**1**] for circular plates with the introduction of the mechanical notion of *stiffeners*. The optimal plate was a *virtual* plate, a microstructure, that is a homogenized geometry. Another example is the Plateau problem of minimal surfaces that experimentally exhibits surfaces with singularities. In both cases, it is mathematically natural to replace the geometry by a characteristic function, a function that is equal to 1 on the set and 0 outside the set. Instead of optimizing over a restricted family of geometries, the problem is relaxed to the optimization over a set of measurable characteristic functions that contains a much larger family of geometries, including the ones with boundary singularities and/or an arbitrary number of holes.

6.1 Free Interface between Two Materials

Consider the optimal design problem studied by J. Céa and K. Malanowski [**1**] in 1970, where the optimization variable is the distribution of two materials with different physical characteristics within a fixed domain D. It cannot a priori be assumed that the two regions are separated by a smooth interface and that each region is connected. This problem will be covered in more details in section 4 of Chapter 5.

Let $D \subset \mathbf{R}^N$ be a bounded open domain with Lipschitzian boundary ∂D. Assume for the moment that the domain D is partitioned into two subdomains Ω_1 and Ω_2 separated by a smooth interface $\partial\Omega_1 \cap \partial\Omega_2$ as illustrated in Figure 1.4. Domain Ω_1 (resp., Ω_2) is made up of a material characterized by a constant $k_1 > 0$ (resp., $k_2 > 0$). Let y be the solution of the *transmission problem*

$$\begin{cases} -k_1 \triangle y = f \text{ in } \Omega_1 \quad \text{and} \quad -k_2 \triangle y = f \text{ in } \Omega_2, \\ y = 0 \text{ on } \partial D \quad \text{and} \quad k_1 \dfrac{\partial y}{\partial n_1} + k_2 \dfrac{\partial y}{\partial n_2} = 0 \text{ on } \overline{\Omega}_1 \cap \overline{\Omega}_2, \end{cases} \tag{6.1}$$

where n_1 (resp., n_2) is the unit outward normal to Ω_1 (resp., Ω_2) and f is a given function in $L^2(D)$. Assume that $k_1 > k_2$. The objective is to maximize the equivalent of the *compliance*

$$J(\Omega_1) = -\int_D f y \, dx \tag{6.2}$$

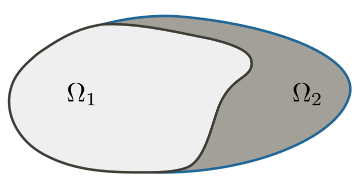


Figure 1.4. *Fixed domain D and its partition into Ω_1 and Ω_2.*

over all domains Ω_1 in D subject to the following constraint on the volume of material k_1 which occupies the part Ω_1 of D:

$$\boxed{\mathrm{m}(\Omega_1) \le \alpha, \quad 0 < \alpha < \mathrm{m}(D)} \tag{6.3}$$

for some constant α.

If χ denotes the characteristic function of the domain Ω_1,

$$\chi(x) = 1 \text{ if } x \in \Omega_1 \text{ and } 0 \text{ if } x \notin \Omega_1,$$

the compliance $J(\chi) = J(\Omega_1)$ can be expressed as the infimum over the Sobolev space $H_0^1(D)$ of an energy functional defined on the fixed set D:

$$J(\chi) = \min_{\varphi \in H_0^1(D)} E(\chi, \varphi), \tag{6.4}$$

$$E(\chi, \varphi) \overset{\text{def}}{=} \int_D (k_1\,\chi + k_2\,(1-\chi))\; |\nabla\varphi|^2 - 2\,\chi f \varphi\, dx. \tag{6.5}$$

$J(\chi)$ can be minimized or maximized over some appropriate family of characteristic functions or with respect to their relaxation to functions between 0 and 1 that would correspond to microstructures. As in the eigenvalue problem, the objective function is an infimum, but here the infimum is over a space that does not depend on the function χ that specifies the geometric domain. This will be handled by the special techniques of Chapter 10 for the differentiation of the minimum of a functional.

6.2 Minimal Surfaces

The celebrated Plateau's problem, named after the Belgian physicist and professor J. A. F. Plateau [**1**] (1801–1883), who did experimental observations on the geometry of soap films around 1873, also provides a nice example where the geometry is a variable. It consists in finding the surface of least area among those bounded by a given curve. One of the difficulties in studying the *minimal surface problem* is the description of such surfaces in the usual language of differential geometry. For instance, the set of possible singularities is not known.

Measure theoretic methods such as k-currents (k-dim surfaces) were used by E. R. Reifenberg [**1**, **2**, **3**, **4**] around 1960, H. Federer and W. H. Fleming [**1**] in 1960 (normals and integral currents), F. J. Almgren, Jr. [**1**] in 1965 (varifolds), and H. Federer [**5**] in 1969.

In the early 1950s, E. De Giorgi [**1**, **2**, **3**] and R. Caccioppoli [**1**] considered a hypersurface in the N-dimensional Euclidean space $\mathbf{R}^{\mathrm{N}}$ as the boundary of a set. In order to obtain a *boundary measure*, they restricted their attention to sets whose characteristic function is of bounded variation. Their key property is an associated natural notion of *perimeter* that extends the classical surface measure of the boundary of a smooth set to the larger family of *Caccioppoli sets* named after the celebrated Neapolitan mathematician Renato Caccioppoli.[2]

[2]In 1992 his tormented personality was remembered in a film directed by Mario Martone, *The Death of a Neapolitan Mathematician* (Morte di un matematico napoletano).

Caccioppoli sets occur in many shape optimization problems (or free boundary problems), where a surface tension is present on the (free) boundary, such as in the free interface water/soil in a dam (C. Baiocchi, V. Comincioli, E. Magenes, and G. A. Pozzi [**1**]) in 1973 and in the free boundary of a water wave (M. Souli and J.-P. Zolésio [**1**, **2**, **3**, **4**, **5**]) in 1988. More details will be given in Chapter 5.

7 Design of a Thermal Diffuser

Shape optimization problems are everywhere in engineering, physics, and medicine. We choose two illustrative examples that were proposed by the Canadian Space Program in the 1980s. The first one is the design of a thermal diffuser to condition the thermal environment of electronic devices in communication satellites; the second one is the design of a thermal radiator that will be described in the next section. There are more and more design and control problems coming from medicine. For instance, the design of endoprotheses such as valves, stents, and coils in blood vessels or left ventricular assistance devices (cardiac pumps) in interventional cardiology helps to improve the health of patients and minimize the consequences and costs of therapeutical interventions by going to mini-invasive procedures.

7.1 Description of the Physical Problem

This problem arises in connection with the use of high-power solid-state devices (HPSSD) in communication satellites (cf. M. C. Delfour, G. Payre, and J.-P. Zolésio [**1**]). An HPSSD dissipates a large amount of thermal power (typ. > 50 W) over a relatively small mounting surface (typ. 1.25 cm^2). Yet, its junction temperature is required to be kept moderately low (typ. 110°C). The thermal resistance from the junction to the mounting surface is known for any particular HPSSD (typ. 1°C/W), so that the mounting surface is required to be kept at a lower temperature than the junction (typ. 60°C). In a space application the thermal power must ultimately be dissipated to the environment by the mechanism of radiation. However, to radiate large amounts of thermal power at moderately low temperatures, correspondingly large radiating areas are required. Thus we have the requirement to efficiently spread the high thermal power flux (TPF) at the HPSSD source (typ. 40 W/cm^2) to a low TPF at the radiator (typ. 0.04 W/cm^2) so that the source temperature is maintained at an acceptably low level (typ. < 60°C) at the mounting surface. The efficient spreading task is best accomplished using heatpipes, but the snag in the scheme is that heatpipes can accept only a limited maximum TPF from a source (typ. max 4 W/cm^2).

Hence we are led to the requirement for a thermal diffuser. This device is inserted between the HPSSD and the heatpipes and reduces the TPF at the source (typ. > 40 W/cm^2) to a level acceptable to the heatpipes (typ. > 4 W/cm^2). The heatpipes then sufficiently spread the heat over large space radiators, reducing the TPF from a level at the diffuser (typ. 4 W/cm^2) to that at the radiator (typ. 0.04 W/cm^2). This scheme of heat spreading is depicted in Figure 1.5.

It is the design of the thermal diffuser which is the problem at hand. We may assume that the HPSSD presents a uniform thermal power flux to the diffuser

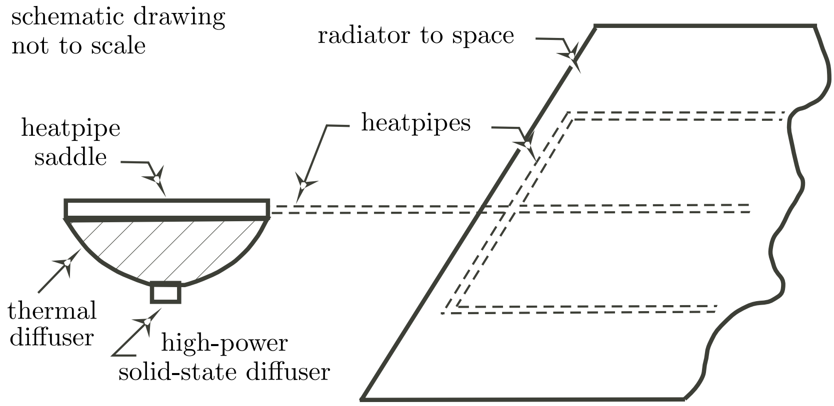


Figure 1.5. *Heat spreading scheme for high-power solid-state devices.*

at the HPSSD/diffuser interface. Heatpipes are essentially isothermalizing devices, and we may assume that the diffuser/heatpipe saddle interface is indeed isothermal. Any other surfaces of the diffuser may be treated as adiabatic.

7.2 Statement of the Problem

Assume that the thermal diffuser is a volume Ω symmetrical about the z-axis (cf. Figure 1.6 (A)) whose boundary surface is made up of three regular pieces: the mounting surface Σ_1 (a disk perpendicular to the z-axis with center in $(r, z) = (0, 0)$), the lateral adiabatic surface Σ_2, and the interface Σ_3 between the diffuser and the heatpipe saddle (a disk perpendicular to the z-axis with center in $(r, z) = (0, L)$).

The temperature distribution over this volume Ω is the solution of the stationary heat equation $k\Delta T = 0$ (ΔT, the Laplacian of T) with the following boundary conditions on the surface $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ (the boundary of Ω):

$$k\,\frac{\partial T}{\partial n} = q_{in} \text{ on } \Sigma_1, \quad k\,\frac{\partial T}{\partial n} = 0 \text{ on } \Sigma_2, \quad T = T_3 \text{ (constant) } \text{ on } \Sigma_3, \tag{7.1}$$

where n always denotes the outward unit normal to the boundary surface Σ and $\partial T/\partial n$ is the normal derivative to the boundary surface Σ,

$$\frac{\partial T}{\partial n} = \nabla T \cdot n \quad (\nabla T = \text{ the gradient of } T). \tag{7.2}$$

The parameters appearing in (7.1) are

k = thermal conductivity (typ. 1.8W/cm×°C),

q_{in} = uniform inward thermal power flux at the source (positive constant).

The radius R_0 of the mounting surface Σ_1 is fixed so that the boundary surface Σ_1 is already given in the design problem.

For practical considerations, we assume that the diffuser is solid without interior hollows or cutouts. The class of shapes for the diffuser is characterized by the design parameter $L > 0$ and the positive function $R(z)$, $0 < z \le L$, with $R(0) = R_0 > 0$. They are volumes of revolution Ω about the z-axis generated by the surface A between the z-axis and the function $R(z)$ (cf. Figure 1.6 (B)), that is,

$$\Omega \stackrel{\text{def}}{=} \left\{(x,y,z) \,:\, 0 < z < L,\, x^2 + y^2 < R(z)^2\right\}. \tag{7.3}$$

So the shape of Ω is completely specified by the length $L > 0$ and the function $R(z) > 0$ on the interval $[0, L]$.

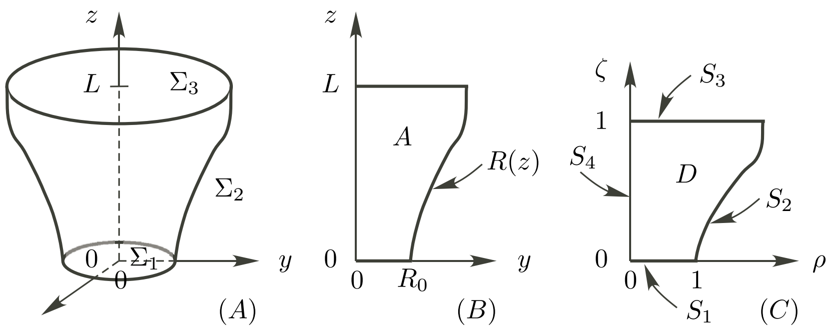


Figure 1.6. (A) Volume Ω and its boundary Σ; (B) Surface A generating Ω; (C) Surface D generating $\widetilde{\Omega}$.

Assuming that the diffuser is made up of a homogeneous material of uniform density (no hollow) the design objective is to minimize the volume

$$J(\Omega) \stackrel{\text{def}}{=} \int_\Omega dx = \pi \int_0^L R(z)^2\, dz \tag{7.4}$$

subject to a uniform constraint on the outward thermal power flux at the interface Σ_3 between the diffuser and the heatpipe saddle:

$$\sup_{p \in \Sigma_3} -k \frac{\partial T}{\partial z}(p) \le q_{out} \text{ or } k \frac{\partial T}{\partial n} + q_{out} \ge 0 \text{ on } \Sigma_3, \tag{7.5}$$

where q_{out} is a specified positive constant.

It is readily seen that the minimization problem (7.4) subject to the constraint (7.5) (where T is the solution of the heat equation with the boundary conditions (7.1)) is independent of the fixed temperature T_3 on the boundary Σ_3. In other words the optimal shape Ω, if it exists, is independent of T_3. As a result, from now on we set T_3 equal to 0.

7.3 Reformulation of the Problem

In a shape optimization problem the formulation is important from both the theoretical and the numerical viewpoints. In particular condition (7.5) is difficult to numerically handle since it involves the pointwise evaluation of the normal derivative on the piece of boundary Σ_3. This problem can be reformulated as the minimization of T on Σ_3, where T is now the solution of a variational inequality. Consider the following minimization problem over the subspace of functions that are positive or zero on Σ_3:

$$V^+(\Omega) \stackrel{\text{def}}{=} \left\{v \in H^1(\Omega) \,:\, v|_{\Sigma_3} \geq 0\right\}, \tag{7.6}$$

$$\inf_{v \in V^+(\Omega)} \int_\Omega \frac{1}{2}|\nabla v|^2\, dx - \int_{\Sigma_1} q_{in}\, v\, d\Sigma + \int_{\Sigma_3} q_{out}\, v\, d\Sigma. \tag{7.7}$$

$H^1(\Omega)$ is the usual Sobolev space on the domain Ω, and the inequality on Σ_3 has to be interpreted *quasi-everywhere* in the capacity sense. Leaving aside those technicalities, the minimizing solution of (7.7) is characterized by

$$\begin{aligned} &-k\,\Delta T = 0 \text{ in } \Omega, \quad k\frac{\partial T}{\partial n} = q_{in} \text{ on } \Sigma_1, \quad k\frac{\partial T}{\partial n} = 0 \text{ on } \Sigma_2, \\ &T \geq 0, \quad \left(k\frac{\partial T}{\partial n} + q_{out}\right) \geq 0, \quad T\left(k\frac{\partial T}{\partial n} + q_{out}\right) = 0 \text{ on } \Sigma_3. \end{aligned} \tag{7.8}$$

The former constraint (7.5) is verified and replaced by the new constraint

$$T = 0 \text{ on } \Sigma_3. \tag{7.9}$$

If there exists a nonempty domain Ω of the form (7.3) such that $T = 0$ on Σ_3, the problem is feasible.

In this formulation the pointwise constraint on the normal derivative of the temperature on Σ_3 has been replaced by a pointwise constraint on the less demanding trace of the temperature on Σ_3. Yet, we now have to solve a variational inequality instead of a variational equation for the temperature T.

7.4 Scaling of the Problem

In the above formulations the *shape parameter* L and the *shape function* R are not independent of each other since the function R is defined on the interval $[0, L]$. This motivates the following changes of variables and the introduction of the dimensionless temperature y:

$$x \mapsto \xi_1 = \frac{x}{R_0}, \quad y \mapsto \xi_2 = \frac{y}{R_0}, \quad z \mapsto \zeta = \frac{z}{L}, \quad 0 \leq \zeta \leq 1,$$

$$\tilde{L} = \frac{L}{R_0}, \quad \tilde{R}(\zeta) = \frac{R(L\zeta)}{R_0},$$

$$y(\xi_1, \xi_2, \zeta) = \frac{k}{Lq_{in}} T(R_0\xi_1, R_0\xi_2, L\zeta),$$

$$D \stackrel{\text{def}}{=} \left\{(\xi_1, \xi_2, \zeta) \,:\, 0 < \zeta < 1,\, \xi_1^2 + \xi_2^2 < \tilde{R}(z)^2\right\}.$$

The parameter $\tilde{L}$ now appears as a coefficient in the partial differential equation

$$\tilde{L}^2 \left(\frac{\partial^2 y}{\partial \xi_1^2} + \frac{\partial^2 y}{\partial \xi_2^2} \right) + \frac{\partial^2 y}{\partial \zeta^2} = 0 \text{ in } D$$

with the following boundary conditions on the boundary $S = S_1 \cup S_2 \cup S_3$ of D:

$$\frac{\partial y}{\partial \nu_A} = 1 \text{ on } S_1, \quad \frac{\partial y}{\partial \nu_A} = 0 \text{ on } S_2, \quad y = 0 \text{ on } S_3, \tag{7.10}$$

where ν denotes the outward normal to the boundary surface S and $\partial y/\partial \nu_A$ is the conormal derivative to the boundary surface S,

$$\frac{\partial y}{\partial \nu_A} = \tilde{L}^2 \left(\nu_1 \frac{\partial y}{\partial \xi_1} + \nu_2 \frac{\partial y}{\partial \xi_2} \right) + \nu_3 \frac{\partial y}{\partial \zeta}.$$

Finally, the optimal design problem depends only on the ratio $q = q_{out}/q_{in}$ through the constraint

$$\frac{\partial y}{\partial \nu_A} + \frac{q_{out}}{q_{in}} \geq 0 \text{ on } S_3.$$

The *design variables* are the parameter $\tilde{L} > 0$ and the function $\tilde{R} > 0$ now defined on the fixed interval $[0, 1]$.

7.5 Design Problem

The fact that this specific design problem can be reduced to finding a parameter and a function gives the false or unfounded impression that it can now be solved by standard mathematical programming and numerical methods. Early work on such problems revealed a different reality, such as oscillating boundaries and convergence towards nonphysical designs. Clearly, the geometry refused to be handled by standard methods without a better understanding of the underlying physics and its inception in the modeling of the geometric variable.

At the theoretical level, the existence of solution requires a concept of continuity with respect to the geometry of the solution of either the heat equation with an inequality constraint on the TPF or the variational inequality with an equality constraint on the temperature. The other element is the lower semicontinuity of the objective functional that is not too problematic for the volume functional as long as the chosen topology on the geometry preserves the continuity of the volume functional. For instance, the classical Hausdorff metric topology does not preserve the volume. In the context of fluid mechanics (cf., for instance, O. Pironneau [**1**]), it means that a drag minimizing sequence of sets with constant volume may converge to a set with twice the volume (cf. Example 4.1 in Chapter 6). A wine making industry exploiting the convergence in the Hausdorff metric topology could yield miraculous profits.

Other serious issues are, for instance, the lack of differentiability of the solution of a variational inequality at the continuous level that will inadvertently affect the differentiability or the evolution of a gradient method at the discrete level. We shall see that there is not only one topology for shapes but a whole range that selectively preserve some but not all of the geometrical features. Again the right choice is problem dependent, much like the choice of the right Sobolev space in the theory of partial differential equations.

8 Design of a Thermal Radiator

Current trends indicate that future communications satellites and spacecrafts will grow ever larger, consume ever more electrical power, and dissipate larger amounts of thermal energy. Various techniques and devices can be deployed to condition the thermal environment for payload boxes within a spacecraft, but it is desirable to employ those which offer good performance for low cost, low weight, and high reliability. A thermal radiator (or radiating fin) which accepts a given TPF from a payload box and radiates it directly to space can offer good performance and high reliability at low cost. However, without careful design, such a radiator can be unnecessarily bulky and heavy. It is the mass-optimized design of the thermal radiator which is the problem at hand (cf. M. C. Delfour, G. Payre, and J.-P. Zolésio [**2**]). We may assume that the payload box presents a uniform TPF (typ. 0.1 to 1.0 W/cm^2) into the radiator at the box/radiator interface. The radiating surface is a second surface mirror which consists of a sheet of glass whose inner surface has silver coating. We may assume that the TPF out of the radiator/space interface is governed by the T^4 radiation law, although we must account also for a constant TPF (typ. 0.01 W/cm^2) into this interface from the sun. Any other surfaces of the radiator may be treated as adiabatic. Two constraints restrict freedom in the design of the thermal radiator:

(i) the maximum temperature at the box/radiator interface is not to exceed some constant (typ. 50°); and

(ii) no part of the radiator is to be thinner than some constant (typ. 1 mm).

8.1 Statement of the Problem

Assume that the radiator is a volume Ω symmetrical about the z-axis (cf. Figure 1.7) whose boundary surface is made up of three regular pieces: the contact surface Σ_1 (a disk perpendicular to the z-axis with center at the point $(r, z) = (0, 0)$), the lateral adiabatic surface Σ_2, and the radiating surface Σ_3 (a disk perpendicular to the z-axis with center at $(r, z) = (0, L)$). More precisely

$$\begin{aligned} \Sigma_1 &= \left\{(x, y, z) \,:\, z = 0 \text{ and } x^2 + y^2 \le R_0^2\right\}, \\ \Sigma_2 &= \left\{(x, y, z) \,:\, x^2 + y^2 = R(z)^2,\ 0 \le z \le L\right\}, \\ \Sigma_3 &= \left\{(x, y, z) \,:\, z = L \text{ and } x^2 + y^2 \le R(L)^2\right\}, \end{aligned} \tag{8.1}$$

where the radius $R_0 > 0$ (typ. 10 cm), the length $L > 0$, and the function

$$R \,:\, [0, L] \to \mathbf{R}, \quad R(0) = R_0, \quad R(z) > 0,\ 0 \le z \le L, \tag{8.2}$$

are given ($\mathbf{R}$, the field of real numbers).

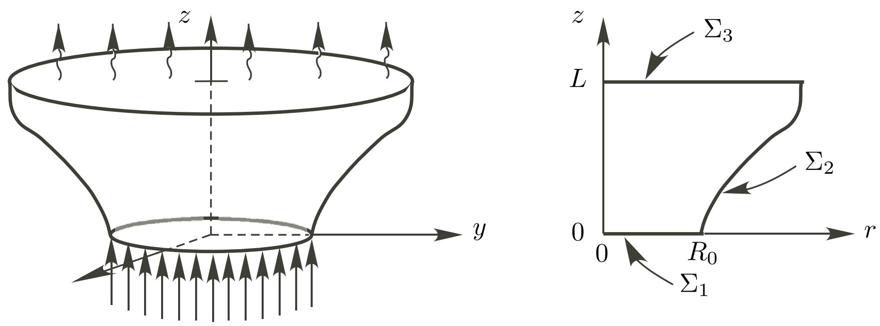


Figure 1.7. *Volume Ω and its cross section.*

The temperature distribution (in Kelvin degrees) over the volume Ω is the solution of the stationary heat equation

$$\Delta T = 0 \quad \text{(the Laplacian of } T\text{)} \tag{8.3}$$

with the following boundary conditions on the surface $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ (the boundary of Ω):

$$k\frac{\partial T}{\partial n} = q_{in} \text{ on } \Sigma_1, \quad k\frac{\partial T}{\partial n} = 0 \text{ on } \Sigma_2, \quad k\frac{\partial T}{\partial n} = -\sigma\varepsilon T\,|T|^3 + q_s \text{ on } \Sigma_3, \tag{8.4}$$

where n denotes the outward normal to the boundary surface Σ, $\partial T/\partial n$ is the normal derivative on the boundary surface Σ, and

$$\frac{\partial T}{\partial n} = \nabla T \cdot n \quad (\nabla T = \text{ the gradient of } T).$$

The parameters appearing in (8.1)–(8.4) are

$k =$ thermal conductivity (typ. 1.8 W/cm×°C),

$q_{in} =$ uniform inward thermal power flux at the source (typ. 0.1 to 1.0 W/cm^2),

$\sigma =$ Boltzmann's constant (5.67×10^{-8} W/m^{2}K^4),

$\varepsilon =$ surface emissivity (typ. 0.8),

$q_s =$ solar inward thermal power flux (0.01 W/cm^2).

The optimal design problem consists in minimizing the volume

$$J(\Omega) \stackrel{\text{def}}{=} \pi \int_0^L R(z)^2\, dz \tag{8.5}$$

over all length $L > 0$ and shape function R subject to the constraint

$$T(x, y, z) \le T_f \quad \text{(typ. } 50°\text{C)}, \quad \forall (x, y, z) \in \Sigma_1. \tag{8.6}$$

In this analysis we shall drop the requirement (ii) in the introduction.

8.2 Scaling of the Problem

As in the case of the diffuser, it is convenient to introduce the following dimensionless coordinates and temperature y (see Figure 1.8):

$$x \mapsto \xi_1 = \frac{x}{R_0}, \quad y \mapsto \xi_2 = \frac{y}{R_0}, \quad z \mapsto \zeta = \frac{z}{L}, \quad 0 \le \zeta \le 1,$$
$$\tilde{L} = \frac{L}{R_0}, \quad \tilde{R}(\zeta) = \frac{R(L\zeta)}{R_0},$$
$$y(\xi_1, \xi_2, \zeta) = \left(\frac{\sigma\varepsilon R_0}{k}\right)^{1/3} T(R_0\xi_1, R_0\xi_2, L\zeta),$$
$$D \overset{\text{def}}{=} \left\{(\xi_1, \xi_2, \zeta) \,:\, 0 < \zeta < 1,\, \xi_1^2 + \xi_2^2 < \tilde{R}(z)^2\right\}.$$

The parameter $\tilde{L}$ now appears as a coefficient in the partial differential equation

$$\tilde{L}^2 \left(\frac{\partial^2 y}{\partial \xi_1^2} + \frac{\partial^2 y}{\partial \xi_2^2}\right) + \frac{\partial^2 y}{\partial \zeta^2} = 0 \text{ in } D$$

with the following boundary conditions on the boundary $S = S_1 \cup S_2 \cup S_3$ of D:

$$\begin{aligned}
\frac{\partial y}{\partial \nu_A} &= \tilde{L}\left[\left(\frac{\sigma\varepsilon R_0}{k}\right)^{1/3} \frac{q_{in}}{k} R_0\right] && \text{on } S_1, \\
\frac{\partial y}{\partial \nu_A} &= 0 && \text{on } S_2, \\
\frac{\partial y}{\partial \nu_A} + \tilde{L} y |y|^3 &= \tilde{L}\left[\left(\frac{\sigma\varepsilon R_0}{k}\right)^{1/3} \frac{q_{in}}{k} R_0\right] \frac{q_s}{q_{in}} && \text{on } S_3,
\end{aligned} \tag{8.7}$$

where ν denotes the outward normal to the boundary surface S and $\partial y/\partial \nu_A$ is the conormal derivative to the boundary surface Σ,

$$\frac{\partial y}{\partial \nu_A} = \tilde{L}^2 \left(\nu_1 \frac{\partial y}{\partial \xi_1} + \nu_2 \frac{\partial y}{\partial \xi_2}\right) + \nu_3 \frac{\partial y}{\partial \zeta}.$$

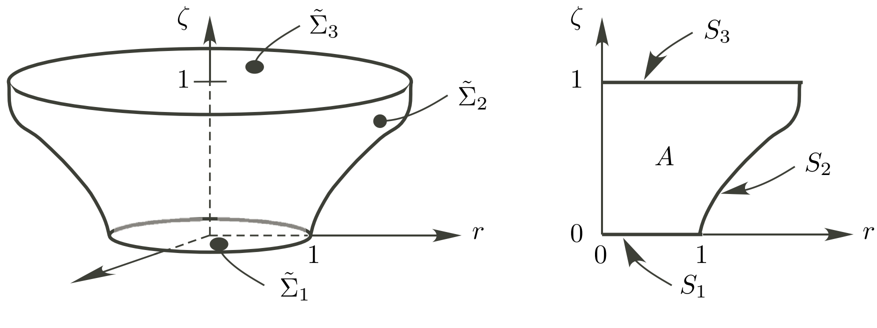


Figure 1.8. *Volume $\widetilde{\Omega}$ and its generating surface A.*

9 A Glimpse into Segmentation of Images

The study of the problem of linguistic or visual perceptions was initiated by several pioneering authors, such as H. Blum [**1**] in 1967, D. Marr and E. Hildreth [**1**] in 1980, and D. Marr [**1**] in 1982. It involves specialists of psychology, artificial intelligence, and experimentalists such as D. H. Hubel and T. N. Wiesel [**1**] in 1962 and F. W. Campbell and J. G. Robson [**1**] in 1968.

In the first part of this section, we revisit the pioneering work of D. Marr and E. Hildreth [**1**] in 1980 on the smoothing of the image by convolution with a sufficiently differentiable normalized function as a function of the *scaling parameter*. We extend the *space-frequency uncertainty principle* to N-dimensional images. It is the analogue of the *Heisenberg uncertainty principle* of quantum mechanics. We revisit the *Laplacian filter* and we generalize the *linearity assumption* of D. Marr and E. Hildreth [**1**] from linear to curved contours.

In the second part, we show how shape analysis methods and the shape and tangential calculuses can be applied to objective functionals defined on the whole contour of an image. We anticipate Chapters 9 and 10 on shape derivatives by the velocity method and show how they can be applied to snakes, active geodesic contours, and level sets (cf., for instance, the book of S. Osher and N. Paragios [**1**]). It shows that the Eulerian shape semiderivative is the basic ingredient behind those representations, including the case of the oriented distance function. In all cases, the evolution equation for the continuous gradient descent method is shown to have the same structure. For more material along the same lines, the reader is referred to M. Dehaes [**1**] and M. Dehaes and M. C. Delfour [**1**].

9.1 Automatic Image Processing

The first level of image processing is the detection of the contours or the boundaries of the objects in the image. For an ideal image $I : D \to \mathbf{R}$ defined in an open two-dimensional frame D with values in an interval of greys continuously ranging from white to black (see Figure 1.9), the edges of an object correspond to the loci of discontinuity of the image I (cf. D. Marr and E. Hildreth [**1**]), also called "step edges" by D. Marr [**1**]. As can be seen from Figure 1.9, the loci of discontinuity may only reveal part of an object hidden by another one and a subsequent and different level of processing is required. A more difficult case is the detection of black curves or cracks in a white frame (cf. Figure 1.10), where the function I becomes a measure supported by the curves rather than a function.

In practice, the frame D of the image is divided into periodically spaced cells P (square, hexagon, diamond) with a quantized value or pixel from 256 grey levels. For small squares a piecewise linear continuous interpolation or higher degree C^1-interpolation can be used to remove the discontinuities at the intercell boundaries. In addition, observations and measurements introduce noise or perturbations and the interpolated image I needs to be further *smoothed* or *filtered*. When the characteristics of the noise are known, an appropriate filter can do the job (see, for instance, the use of *low pass filters* in A. Rosenfeld and M. Thurston [**1**], A. P. Witkin [**1**], and A. L. Yuille and T. Poggio [**1**]).

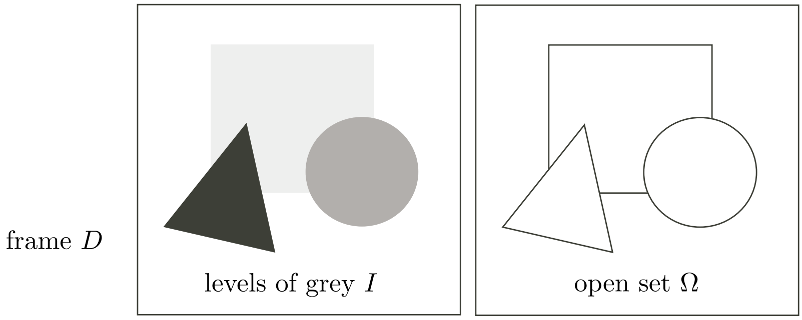


Figure 1.9. *Image I of objects and their segmentation in the frame D.*

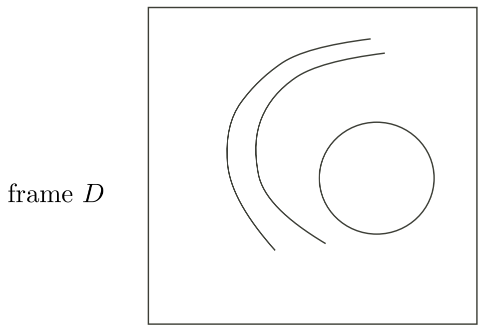


Figure 1.10. *Image I containing black curves or cracks in the frame D.*

Given an ideal *grey level image I* defined in a fixed bounded open *frame D* we want to identify the *edges* or boundaries of the objects contained in the image as shown in Figure 1.9. The *edge identification* or *segmentation* problem is a low-level processing of a *real* image which should also involve higher-level processings or tasks (cf. M. Kass, M. Witkin, and D. Terzopoulos [**1**]). Intuitively the edges coincide with the loci of discontinuities of the image function I. At such points the norm $|\nabla I|$ of the gradient ∇I is infinite. Roughly speaking the segmentation of an ideal image I consists in finding the loci of discontinuity of the function I. The idea is now to first smooth I by convolution with a sufficiently differentiable function. This operation will be followed and/or combined with the use of an *edge detector* as the zero crossings of the Laplacian. In the literature the term filter often applies to both the filter and the detector. In this section we shall make the distinction between the two operations.

9.2 Image Smoothing/Filtering by Convolution and Edge Detectors

In this section we revisit the work of D. Marr and E. Hildreth [**1**] in 1980 on the smoothing of the image I by convolution with a sufficiently differentiable normalized function ρ as a function of the *scaling parameter* $\varepsilon > 0$. In the second part of this section we revisit the *Laplacian filter* and we generalize the *linearity assumption* of D. Marr and E. Hildreth [**1**] from linear to curved contours.

9.2.1 Construction of the Convolution of I

Let $\rho : \mathbf{R}^{\mathrm{N}} \to \mathbf{R}$, $N \geq 1$, be a sufficiently smooth function such that

$$\rho \geq 0 \qquad \text{and} \qquad \int_{\mathbf{R}^{\mathrm{N}}} \rho(x)\, dx = 1.$$

Associate with the image I, ρ, and a *scaling parameter* $\varepsilon > 0$ the normalized convolution

$$I_\varepsilon(x) \stackrel{\text{def}}{=} (I * \rho_\varepsilon)(x) = \frac{1}{\varepsilon^N} \int_{\mathbf{R}^{\mathrm{N}}} I(y)\, \rho\left(\frac{x-y}{\varepsilon}\right) dy, \tag{9.1}$$

where for $x \in \mathbf{R}^{\mathrm{N}}$

$$\rho_\varepsilon(x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon^N} \rho\left(\frac{x}{\varepsilon}\right) \qquad \text{and} \qquad \int_{\mathbf{R}^{\mathrm{N}}} \rho_\varepsilon(x)\, dx = 1. \tag{9.2}$$

The function ρ_ε plays the role of a probability density and $\rho_\varepsilon(x)\, dx$ of a probability measure. Under appropriate conditions I_ε converges to the original image I as $\varepsilon \to 0$. For a sufficiently small $\varepsilon > 0$ the loci of discontinuity of the function I are transformed into loci of strong variation of the gradient of the convolution I_ε. When ρ has compact support, the convolution acts locally around each point in a neighborhood whose size is of order ε. A popular choice for ρ is the *Gaussian* with integral normalized to one in $\mathbf{R}^{\mathrm{N}}$ defined by

$$G^N(x) \stackrel{\text{def}}{=} \frac{1}{(\sqrt{2\pi})^N}\, e^{-\frac{1}{2}|x|^2} \qquad \text{and} \qquad \int_{\mathbf{R}^{\mathrm{N}}} G^N(x)\, dx = 1$$

and the *normalized Gaussian of variance* ε

$$G^N_\varepsilon(x) = \frac{1}{\varepsilon^N} \frac{1}{(\sqrt{2\pi})^N}\, e^{-\frac{1}{2}\left|\frac{x}{\varepsilon}\right|^2} \qquad \text{and} \qquad \int_{\mathbf{R}^{\mathrm{N}}} G^N_\varepsilon(x)\, dx = 1. \tag{9.3}$$

As noted in L. Alvarez, P.-L. Lions, and J.-M. Morel [**1**] in dimension $N = 2$, the function $u(t) = G^2_{\sqrt{2t}} * I$ is the solution of the parabolic equation

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x), \quad u(0,x) = I(x).$$

9.2.2 Space-Frequency Uncertainty Relationship

It is interesting to compute the Fourier transform $\mathcal{F}(G^N_\varepsilon)$ of G^N_ε to make explicit the relationship between the mean square deviations of G^N_ε and its Fourier transform $\mathcal{F}(G^N_\varepsilon)$. For $\omega \in \mathbf{R}^{\mathrm{N}}$, define the *Fourier transform* of a function f by

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{(\sqrt{2\pi})^N} \int_{\mathbf{R}^{\mathrm{N}}} f(x)\, e^{-i\,\omega\cdot x}. \tag{9.4}$$

In applications the integral of the square of f often corresponds to an energy. We shall refer to the L^2-norm of f as the *energy norm* and the function $f^2(x)$ as the *energy density*. The Fourier transform (9.4) of the normalized Gaussian (9.3) is

$$\mathcal{F}(G_\varepsilon^N)(\omega) \stackrel{\text{def}}{=} \frac{1}{(\sqrt{2\pi})^N} \int_{\mathbf{R}^N} G_\varepsilon^N(x)\, e^{-i\,\omega\cdot x} dx = \frac{1}{(\sqrt{2\pi})^N}\, e^{-\frac{1}{2}|\varepsilon\omega|^2}. \tag{9.5}$$

D. Marr and E. Hildreth [**1**] notice that there is an uncertainty relationship between the mean square deviation with respect to the energy density $f(x)^2$ and the mean square deviation with respect to the energy density $\mathcal{F}(f)(\omega)^2$ of its Fourier transform in dimension 1 using a result of R. Bracewell [**1**, pp. 160–161], where he uses the constant $1/(2\pi)$ instead of $1/\sqrt{2\pi}$ in the definition of the Fourier transform that yields $1/(4\pi)$ instead of $1/2$ as a lower bound to the product of the two mean square deviations.

This *uncertainty relationship* generalizes to dimension N. First define the notions of *centroïd* and *variance* with respect to the energy density $f(x)^2$. Given a function $f \in L^1(\mathbf{R}^N) \cap L^2(\mathbf{R}^N)$ and $x \in \mathbf{R}^N$, define the *centroïd* as

$$\overline{x} \stackrel{\text{def}}{=} \frac{\int_{\mathbf{R}^N} x\, f(x)^2\, dx}{\int_{\mathbf{R}^N} f(x)^2\, dx} \tag{9.6}$$

and the *variance* as

$$\langle \Delta x\rangle^2 \stackrel{\text{def}}{=} \langle x-\overline{x}\rangle^2 = \frac{\int_{\mathbf{R}^N} |x|^2\, f(x)^2\, dx}{\int_{\mathbf{R}^N} f(x)^2\, dx} - |\overline{x}|^2.$$

Theorem 9.1 (Uncertainty relationship)**.** *Given $N \geq 1$ and $f \in W^{1,1}(\mathbf{R}^N) \cap W^{1,2}(\mathbf{R}^N)$ such that $-ix\, f \in L^1(\mathbf{R}^N) \cap L^2(\mathbf{R}^N)$,*

$$\boxed{\langle \Delta x\rangle\, \langle \Delta\omega\rangle \;\geq\; \frac{N}{2}.} \tag{9.7}$$

For $f = G_\varepsilon^N$

$$f(x) = G_\varepsilon^N(x) = \frac{1}{\varepsilon^N}\frac{1}{(\sqrt{2\pi})^N}\, e^{-\frac{1}{2}\left|\frac{x}{\varepsilon}\right|^2} \quad \Rightarrow \langle \Delta x\rangle^2 \;=\; \frac{N\varepsilon^2}{2}, \tag{9.8}$$

$$\hat{f}(\omega) = \mathcal{F}(G_\varepsilon^N)(\omega) = \frac{1}{(\sqrt{2\pi})^N}\, e^{-\frac{1}{2}|\varepsilon\omega|^2} \quad \Rightarrow \langle \Delta \omega\rangle^2 \;=\; \frac{N}{2\,\varepsilon^2} \tag{9.9}$$

$$\Rightarrow \boxed{\langle \Delta x\rangle\langle \Delta\omega\rangle \;=\; \frac{N}{2} \qquad \text{for } G_\varepsilon^N \text{ and } \mathcal{F}(G_\varepsilon^N).} \tag{9.10}$$

So the *normalized Gaussian filter* is indeed an *optimal* filter since it achieves the lower bound in all dimensions as stated by D. Marr and E. Hildreth [**1**] in dimension 1 in 1980. In the context of quantum mechanics, this relationship is the analogue of the *Heisenberg uncertainty principle*.

9.2.3 Laplacian Detector

One way to detect the edges of a regular object is to start from the convolution-smoothed image I_ε. Given a direction v, $|v| = 1$, and a point $x \in \mathbf{R}^2$ an *edge point* will correspond to a local minimum or maximum of the directional derivative $f(t) \stackrel{\text{def}}{=} \nabla I_\varepsilon(x+tv)\cdot v$ with respect to t. Denote by $\hat{t}$ such a point. Then a necessary condition for an extremal is

$$D^2 I_\varepsilon(x+\hat{t}v)v\cdot v = 0.$$

The point $\hat{x} = x + \hat{t}v$ is a *zero crossing* following the terminology of D. Marr and E. Hildreth [**1**] of the second-order directional derivative in the direction v. Thus we are looking for the pairs $(\hat{x}, \hat{v})$ verifying the necessary condition

$$D^2 I_\varepsilon(\hat{x})\hat{v}\cdot\hat{v} = 0 \tag{9.11}$$

and, more precisely, lines or curves $\mathcal{C}$ such that

$$\forall x \in \mathcal{C},\ \exists v(x),\ |v(x)| = 1, \quad D^2 I_\varepsilon(x)v(x)\cdot v(x) = 0. \tag{9.12}$$

This condition is necessary in order that, in each point $x \in \mathcal{C}$, there exists a direction $v(x)$ such that $\nabla I_\varepsilon(x)\cdot v$ is extremal.

In order to limit the search to points x rather than to pairs (x, v), the Laplacian detector was introduced in D. Marr and E. Hildreth [**1**] under two assumptions: the linear variation of the intensity along the edges $\mathcal{C}$ of the object and the condition of zero crossing of the second-order derivative in the direction normal to $\mathcal{C}$.

Assumption 9.1 (Linear variation condition)**.**
The intensity I_ε in a neighborhood of lines parallel to $\mathcal{C}$ is locally linear (affine). □

Assumption 9.2.
The zero-crossing condition is verified in all points of $\mathcal{C}$ in the direction of the normal n at the point, that is,

$$\boxed{D^2 I_\varepsilon\, n\cdot n = 0 \text{ on } \mathcal{C}.} \tag{9.13}$$

□

Under those two assumptions and for a line $\mathcal{C}$, it is easy to show that the points of $\mathcal{C}$ verify the necessary condition

$$\Delta I_\varepsilon = 0 \text{ on } \mathcal{C}. \tag{9.14}$$

Such conditions can be investigated for edges $\mathcal{C}$ that are the boundary Γ of a smooth domain $\Omega \subset R^N$ of class C^2 by using the *tangential calculus* developed in section 5.2 of Chapter 9 and M. C. Delfour [**7**, **8**] for objects in $\mathbf{R}^{\mathrm{N}}$, $N \geq 1$, and not just in dimension $N = 2$. Indeed we want to find points $x \in \Gamma$ such that the function $\nabla I_\varepsilon(x)\cdot n(x)$ is an extremal of the function $f(t) = \nabla I_\varepsilon(x+t\,\nabla b_\Omega(x))\cdot\nabla b_\Omega(x)$ in $t = 0$. This yields the *local necessary condition* $D^2 I_\varepsilon \nabla b_\Omega\cdot\nabla b_\Omega = D^2 I_\varepsilon\, n\cdot n = 0$ on Γ of Assumption 9.2. As for Assumption 9.1, its analogue is the following.

Assumption 9.3.
The restriction to the curve $\mathcal{C}$ of the gradient of the intensity I_ε is a constant vector c, that is,

$$\nabla I_\varepsilon = c, \quad c \text{ is a constant vector on } \mathcal{C}. \tag{9.15}$$

□

Indeed the Laplacian of I_ε on $\mathcal{C}$ can be decomposed as follows (cf. section 5.2 of Chapter 9):

$$\Delta I_\varepsilon = \operatorname{div}_\Gamma \nabla I_\varepsilon + D^2 I_\varepsilon \, n \cdot n,$$

where $\operatorname{div}_\Gamma \nabla I_\varepsilon$ is the *tangential divergence* of ∇I_ε that can be defined as follows:

$$\operatorname{div}_\Gamma \nabla I_\varepsilon = \operatorname{div}(\nabla I_\varepsilon \circ p_\mathcal{C})|_\mathcal{C}$$

and $p_\mathcal{C}$ is the *projection* onto $\mathcal{C}$. Hence, under Assumption 9.3, $\operatorname{div}_\Gamma(\nabla I_\varepsilon) = 0$ on $\mathcal{C}$ since

$$\operatorname{div}_\Gamma(\nabla I_\varepsilon) = \operatorname{div}(\nabla I_\varepsilon \circ p_\Gamma)|_\Gamma = \operatorname{div}(c)|_\Gamma = 0,$$

and, in view of (9.13) of Assumption 9.2, $D^2 I_\varepsilon \, n \cdot n = 0$ on $\mathcal{C}$. Therefore

$$\Delta I_\varepsilon = \operatorname{div}_\Gamma(\nabla I_\varepsilon) + D^2 I_\varepsilon \, n \cdot n = 0 \text{ on } \mathcal{C},$$

and we obtain the necessary condition

$$\boxed{\Delta I_\varepsilon = 0 \text{ on } \mathcal{C}.} \tag{9.16}$$

9.3 Objective Functions Defined on the Whole Edge

With the pioneering work of M. Kass, M. Witkin, and D. Terzopoulos [**1**] in 1988 we go from a local necessary condition at a point of the edge to a global necessary condition by introducing objective functionals defined on the entire edge of an object. Here many computations and analytical studies can be simplified by adopting the point of view that a closed curve in the plane is the boundary of a set and using the whole machinery developed for shape and geometric analysis and the tangential and shape calculuses in Chapter 9.

9.3.1 Eulerian Shape Semiderivative

In this section we briefly summarize the main elements of the *velocity method*.

Definition 9.1.
Given D, $\varnothing \neq D \subset \mathbf{R}^N$, consider the set $\mathcal{P}(D) = \{\Omega : \Omega \subset D\}$ of subsets of D. The set D is the *holdall* or the *universe*. A *shape functional* is a map $J : \mathcal{A} \to E$ from an *admissible family* $\mathcal{A}$ in $\mathcal{P}(D)$ with values in a topological vector space E (usually $\mathbf{R}$). □

Given a *velocity field* $V : [0,\tau] \times \mathbf{R}^{\mathrm{N}} \to \mathbf{R}^{\mathrm{N}}$ (the notation $V(t)(x) = V(t,x)$ will often be used), consider the transformations

$$T(t,X) \stackrel{\text{def}}{=} x(t,X), \qquad t \geq 0,\ X \in \mathbf{R}^{\mathrm{N}}, \tag{9.17}$$

where $x(t,X) = x(t)$ is defined as the *flow* of the differential equation

$$\frac{dx}{dt}(t) = V(t,x(t)),\ t \geq 0, \quad x(0,X) = X \tag{9.18}$$

(here the notation $x \mapsto T_t(x) = T(t,x) : \mathbf{R}^{\mathrm{N}} \to \mathbf{R}^{\mathrm{N}}$ will be used). The *Eulerian shape semiderivative* of J in Ω in the direction V is defined as

$$dJ(\Omega;V) \stackrel{\text{def}}{=} \lim_{t \searrow 0} \frac{J(\Omega_t) - J(\Omega)}{t} \tag{9.19}$$

(when the limit exists in E), where $\Omega_t = T_t(\Omega) = \{T_t(x) : x \in \Omega\}$. Under appropriate assumptions on the family $\{V(t)\}$, the transformations $\{T_t\}$ are homeomorphisms that transport the boundary Γ of Ω onto the boundary Γ_t of Ω_t and the interior Ω onto the interior of Ω_t.

9.3.2 From Local to Global Conditions on the Edge

For simplicity, drop the subscript ε of the convolution-smoothed image I_ε and assume that the edge of the object is the boundary Γ of an open domain Ω of class C^2. Following V. Caselles, R. Kimmel, and G. Sapiro [**1**], it is important to choose objective functionals that are intrinsically defined and do not depend on an arbitrary parametrization of the boundary. For instance, given a *frame* $D =]0,a[\times]0,b[$ and a *smoothed image* $I : D \to \mathbf{R}$, find an extremum of the objective functional

$$E(\Omega) \stackrel{\text{def}}{=} \int_\Gamma \frac{\partial I}{\partial n}\, d\Gamma, \tag{9.20}$$

where the integrand is the normal derivative of I. Using the velocity method the *Eulerian shape directional semiderivative* is given by the expression (cf. Chapter 9)

$$dE(\Omega;V) = \int_\Gamma \left[H\,\frac{\partial I}{\partial n} + \frac{\partial}{\partial n}\left(\frac{\partial I}{\partial n}\right)\right] V \cdot n\, d\Gamma, \tag{9.21}$$

where $H = \Delta b_\Omega$ is the mean curvature and $n = \nabla b_\Omega$ is the outward unit normal. Proceeding in a formal way a necessary condition would be

$$\begin{aligned} & H\,\frac{\partial I}{\partial n} + \frac{\partial}{\partial n}\left(\frac{\partial I}{\partial n}\right) = 0 \text{ on } \Gamma \\ \Rightarrow\ & \Delta b_\Omega\, \nabla I \cdot \nabla b_\Omega + \nabla(\nabla I \cdot \nabla b_\Omega) \cdot \nabla b_\Omega = 0 \\ \Rightarrow\ & \Delta b_\Omega\, \nabla I \cdot \nabla b_\Omega + D^2 I \nabla b_\Omega \cdot \nabla b_\Omega + D^2 b_\Omega \nabla I \cdot \nabla b_\Omega = 0 \\ \Rightarrow\ & \boxed{D^2 I\, n \cdot n + H\,\frac{\partial I}{\partial n} = 0 \text{ on } \Gamma.} \end{aligned} \tag{9.22}$$

This *global condition* is to be compared with the *local condition* (9.13). It can also be expressed in terms of the Laplacian and the tangential Laplacian of I as

$$\boxed{\Delta I - \Delta_\Gamma I = 0 \text{ on } \Gamma,} \tag{9.23}$$

which can be compared with the local condition (9.16), $\Delta I = 0$. This arises from the decomposition of the Laplacian of I with respect to Γ using the following identity for a smooth vector function U:

$$\operatorname{div} U = \operatorname{div}_\Gamma U + DU\, n \cdot n, \tag{9.24}$$

where the *tangential divergence* of U is defined as

$$\operatorname{div}_\Gamma U = \operatorname{div}(U \circ p_\Gamma)|_\Gamma$$

and p_Γ is the projection onto Γ. Applying this to $U = \nabla I$ and recalling the definition of the tangential Laplacian

$$\Delta I = \operatorname{div}_\Gamma \nabla I + D^2 I\, n \cdot n = \Delta_\Gamma I + H \frac{\partial I}{\partial n} + D^2 I\, n \cdot n,$$

where the tangential gradient $\nabla_\Gamma I$ and the *Laplace–Beltrami* operator $\Delta_\Gamma I$ are defined as

$$\nabla_\Gamma I = \nabla(I \circ p_\Gamma)|_\Gamma \quad \text{and} \quad \Delta_\Gamma I = \operatorname{div}_\Gamma(\nabla_\Gamma I).$$

9.4 Snakes, Geodesic Active Contours, and Level Sets

9.4.1 Objective Functions Defined on the Contours

In the literature the *objective* or *energy functional* is generally made up of two terms: one (image energy) that depends on the image and one (internal energy) that specifies the smoothness of Γ. A general form of objective functional is

$$E(\Omega) \stackrel{\text{def}}{=} \int_\Gamma g(I)\, d\Gamma, \quad g(I) \text{ is a function of } I. \tag{9.25}$$

The directional semiderivative with respect to a velocity field V is given by

$$dE(\Omega; V) = \int_\Gamma \left[H g(I) + \frac{\partial}{\partial n} g(I) \right] n \cdot V\, d\Gamma. \tag{9.26}$$

This gradient will make the *snakes* move and will *activate* the contours.

9.4.2 Snakes and Geodesic Active Contours

If a *gradient descent method* is used to minimize (9.25) starting from an initial curve $\mathcal{C}_0 = \mathcal{C}$, the iterative process is equivalent to following the evolution $\mathcal{C}_t$ (boundary of the smooth domain Ω_t) of the closed curve $\mathcal{C}$ given by the equation

$$\frac{\partial \mathcal{C}_t}{\partial t} = -\left[H_t\, g_I + (\nabla g_I \cdot n_t) \right] n_t \text{ on } \mathcal{C}_t, \tag{9.27}$$

where H_t is the mean curvature, n_t is the unit exterior normal, and the right-hand side of the equation is formally the "derivative" of (9.25) given by (9.26). Equation (9.27) is referred to as the *geodesic flow.* For $g_I = 1$ it is the *motion by mean curvature*

$$\frac{\partial \mathcal{C}_t}{\partial t} = -H_t\, n_t \text{ on } \mathcal{C}_t. \tag{9.28}$$

9.4.3 Level Set Method

The idea to represent the contours $\mathcal{C}_t$ by the zero-level set of a function $\varphi_t(x) = \varphi(t,x)$ for a function $\varphi : [0,\tau] \times \mathbf{R}^N \to \mathbf{R}$ by setting

$$\mathcal{C}_t = \left\{x \in \mathbf{R}^N : \varphi(t,x) = 0\right\} = \varphi_t^{-1}\{0\}$$

and to replace (9.27) by an equation for φ is due to S. Osher and J. A. Sethian [**1**] in 1988. This approach seems to have been simultaneously introduced in image processing by V. Caselles, F. Catté, T. Coll, and F. Dibos [**1**] in 1993 under the name "geometric partial differential equations" with, in addition to the mean curvature term, a "transport" term and by R. Malladi, J. A. Sethian, and B. C. Vemuri [**1**] in 1995 under the name "level set approach" combined with the notion of "extension velocity."

Let $(t,x) \mapsto \varphi(t,x) : [0,\tau] \times \mathbf{R}^N \to \mathbf{R}$ be a smooth function and Ω be a subset of $\mathbf{R}^N$ of boundary $\Gamma = \overline{\Omega} \cap \overline{\complement\Omega}$ such that

$$\operatorname{int}\Omega = \left\{x \in \mathbf{R}^N : \varphi(0,x) < 0\right\} \quad \text{and} \quad \Gamma = \left\{x \in \mathbf{R}^N : \varphi(0,x) = 0\right\}. \tag{9.29}$$

Let $V : [0,\tau] \times \mathbf{R}^N \to \mathbf{R}^N$ be a sufficiently smooth velocity field so that the transformations $\{T_t\}$ are *diffeomorphisms.* Moreover, assume that the images $\Omega_t = T_t(\Omega)$ verify the following properties: for all $t \in [0,\tau]$

$$\operatorname{int}\Omega_t = \left\{x \in \mathbf{R}^N : \varphi(t,x) < 0\right\} \quad \text{and} \quad \Gamma_t = \left\{x \in \mathbf{R}^N : \varphi(t,x) = 0\right\}. \tag{9.30}$$

Assuming that the function $\varphi_t(x) = \varphi(t,x)$ is at least of class C^1 and that $\nabla\varphi_t \neq 0$ on $\varphi_t^{-1}\{0\}$, the *total derivative* with respect to t of $\varphi(t, T_t(x))$ for $x \in \Gamma$ yields

$$\frac{\partial}{\partial t}\varphi(t, T_t(x)) + \nabla\varphi(t, T_t(x)) \cdot \frac{d}{dt}T_t(x) = 0. \tag{9.31}$$

By substituting the velocity field in (9.18), we get

$$\begin{aligned} &\frac{\partial}{\partial t}\varphi(t, T_t(x)) + \nabla\varphi(t, T_t(x)) \cdot V(t, T_t(x)) = 0, \quad \forall\, T_t(x) \in \Gamma_t \\ &\Rightarrow \boxed{\frac{\partial}{\partial t}\varphi_t + \nabla\varphi_t \cdot V(t) = 0 \text{ on } \Gamma_t,\ t \in [0,\tau].} \end{aligned} \tag{9.32}$$

This last equation, the *level set evolution equation*, is verified *only* on the boundaries or *fronts* Γ_t, $0 \le t \le \tau$. Can we find a representative φ in the equivalence class

$$[\varphi]_{\Omega,V} = \left\{\varphi : \varphi_t^{-1}\{0\} = \Gamma_t \text{ and } \varphi_t^{-1}\{<0\} = \operatorname{int}\Omega_t, \forall t \in [0,\tau]\right\}$$

of functions φ that verify conditions (9.29) and (9.30) in order to extend (9.32) from Γ_t to the whole $\mathbf{R}^N$ or at least almost everywhere in $\mathbf{R}^N$? Another possibility is to consider the larger equivalence class

$$[\varphi]_{\Gamma,V} = \left\{\varphi : \varphi_t^{-1}\{0\} = \Gamma_t, \forall t \in [0,\tau]\right\}.$$

9.4.4 Velocity Carried by the Normal

In D. Adalsteinsson and J. A. Sethian [**1**], J. Gomes and O. Faugeras [**1**], and R. Malladi, J. A. Sethian, and B. C. Vemuri [**1**], the front moves under the effect of a velocity field carried by the normal with a scalar velocity that depends on the curvatures of the level set or the front. So we are led to consider velocity fields V of the form

$$V(t)|_{\Gamma_t} = v(t)\, n_t, \quad x \in \Gamma_t, \tag{9.33}$$

for a *scalar velocity* $(t,x) \mapsto v(t)(x) : [0,\tau] \times \mathbf{R}^N \to \mathbf{R}$. By using the expression $\nabla\varphi_t/|\nabla\varphi_t|$ of the exterior normal n_t as a function of $\nabla\varphi_t$, (9.32) now becomes

$$\boxed{\frac{\partial}{\partial t}\varphi_t + v(t)\,|\nabla\varphi_t| = 0 \text{ on } \Gamma_t.} \tag{9.34}$$

For instance, consider the example of the minimization of the total length of the curve $\mathcal{C}$ of (9.27) for the metric $g(I)\,d\mathcal{C}$ as introduced by V. Caselles, R. Kimmel, and G. Sapiro [**1**]. By using the computation (9.26) of the shape semiderivative with respect to the velocity field V of the objective functional (9.25), a natural direction of descent is given by

$$\boxed{v(t)n_t, \quad v(t) = -\left[H_t\, g(I) + \frac{\partial}{\partial n_t} g(I)\right] \text{ on } \Gamma_t.} \tag{9.35}$$

The normal n_t and the mean curvature H_t can be expressed as a function of $\nabla\varphi_t$:

$$n_t = \frac{\nabla\varphi_t}{|\nabla\varphi_t|} \quad \text{and} \quad H_t = \operatorname{div}_{\Gamma_t} n_t = \operatorname{div}_{\Gamma_t}\left(\frac{\nabla\varphi_t}{|\nabla\varphi_t|}\right) = \operatorname{div}\left(\frac{\nabla\varphi_t}{|\nabla\varphi_t|}\right)\Bigg|_{\Gamma_t}. \tag{9.36}$$

By substituting in (9.34), we get the following evolution equation:

$$\frac{\partial\varphi_t}{\partial t} - \left(H_t\, g(I) + \frac{\partial}{\partial n_t} g(I)\right) \nabla\varphi_t \cdot \frac{\nabla\varphi_t}{|\nabla\varphi_t|} = 0 \text{ on } \Gamma_t \tag{9.37}$$

and

$$\begin{cases} \dfrac{\partial\varphi_t}{\partial t} - \left(H_t\, g(I) + \dfrac{\partial}{\partial n_t} g(I)\right) |\nabla\varphi_t| = 0 & \text{on } \Gamma_t, \\ \varphi_0 = \varphi^0 & \text{on } \Gamma_0. \end{cases} \tag{9.38}$$

By substituting expressions (9.36) for n_t and H_t in terms of $\nabla\varphi_t$, we finally get

$$\boxed{\begin{cases} \dfrac{\partial\varphi_t}{\partial t} - \left[\operatorname{div}\left(\dfrac{\nabla\varphi_t}{|\nabla\varphi_t|}\right) g(I) + \nabla g(I)\cdot\dfrac{\nabla\varphi_t}{|\nabla\varphi_t|}\right]|\nabla\varphi_t| = 0 & \text{on } \Gamma_t, \\ \varphi_0 = \varphi^0 & \text{on } \Gamma_0. \end{cases}} \tag{9.39}$$

The negative sign arises from the fact that we have chosen the outward rather than the inward normal. The main references to the existence and uniqueness theorems related to (9.39) can be found in Y. G. Chen, Y. Giga, and S. Goto [**1**].

9.4.5 Extension of the Level Set Equations

Equation (9.39) on the fronts Γ_t is not convenient from both the theoretical and the numerical viewpoints. So it would be desirable to be able to extend (9.39) in a small *tubular neighborhood* of thickness h

$$U_h(\Gamma_t) \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : d_{\Gamma_t}(x) < h\right\} \tag{9.40}$$

of the front Γ_t for a small $h > 0$ (d_{Γ_t}, the distance function to Γ_t). In theory, the velocity field associated with Γ_t is given by

$$V(t) = -\left[\operatorname{div}\left(\frac{\nabla\varphi_t}{|\nabla\varphi_t|}\right) g(I) + \nabla g(I)\cdot\frac{\nabla\varphi_t}{|\nabla\varphi_t|}\right]\frac{\nabla\varphi_t}{|\nabla\varphi_t|} \text{ on } \Gamma_t. \tag{9.41}$$

Given the identities (9.36), the expressions of n_t and H_t extend to a neighborhood of Γ_t and possibly to $\mathbf{R}^{\mathrm{N}}$ if $\nabla\varphi_t$ is sufficiently smooth and $\nabla\varphi_t \neq 0$ on Γ_t.

Equation (9.39) can be extended to $\mathbf{R}^{\mathrm{N}}$ at the price of violating the assumptions (9.29)–(9.30), either by loss of smoothness of φ_t or by allowing its gradient to be zero:

$$\boxed{\begin{cases} \dfrac{\partial\varphi_t}{\partial t} - \left[\operatorname{div}\left(\dfrac{\nabla\varphi_t}{|\nabla\varphi_t|}\right) g(I) + \nabla g(I)\cdot\dfrac{\nabla\varphi_t}{|\nabla\varphi_t|}\right]|\nabla\varphi_t| = 0 & \text{in } \mathbf{R}^{\mathrm{N}}, \\ \varphi_0 = \varphi^0 & \text{in } \mathbf{R}^{\mathrm{N}}. \end{cases}} \tag{9.42}$$

In fact, the starting point of V. Caselles, F. Catté, T. Coll, and F. Dibos [**1**] was the following *geometric partial differential equation*:

$$\begin{cases} \dfrac{\partial\varphi_t}{\partial t} - \left[\operatorname{div}\left(\dfrac{\nabla\varphi_t}{|\nabla\varphi_t|}\right) g(I) + \nu\, g(I)\right]|\nabla\varphi_t| = 0 & \text{in } \mathbf{R}^{\mathrm{N}}, \\ \varphi_0 = \varphi^0 & \text{in } \mathbf{R}^{\mathrm{N}} \end{cases} \tag{9.43}$$

for a constant $\nu > 0$ and the function $g(I) = 1/(1+|\nabla I|^2)$. They prove the existence, in dimension 2, of a *viscosity solution* unique for initial data $\varphi^0 \in C([0,1]\times[0,1]) \cap W^{1,\infty}([0,1]\times[0,1])$ and $g \in W^{1,\infty}(\mathbf{R}^2)$.

9.5 Objective Function Defined on the Whole Image

9.5.1 Tikhonov Regularization/Smoothing

The convolution is only one approach to smoothing an image $I \in L^2(D)$ defined in a frame D. Another way is to use the *Tikhonov regularization*: given $\varepsilon > 0$, find a function $I_\varepsilon \in H^1(D)$ that minimizes the objective functional

$$E_\varepsilon(\varphi) \stackrel{\text{def}}{=} \frac{1}{2} \int_D \varepsilon |\nabla \varphi|^2 + |\varphi - I|^2 \, dx,$$

where the value of ε can be suitably adjusted to get the desired degree of smoothness. The minimizing function $I_\varepsilon \in H^1(D)$ is solution of the following variational equation:

$$\boxed{\forall \varphi \in H^1(D), \quad \int_D \varepsilon \, \nabla I_\varepsilon \cdot \nabla \varphi + (I_\varepsilon - I) \, \varphi \, dx = 0.} \tag{9.44}$$

The price to pay now is that the computation is made on the whole image.

This can be partly compensated by combining in a single operation or level of processing the smoothing of the image with the detection of the edges. To do that D. Mumford and J. Shah [**1**] in 1985 and [**2**] in 1989 introduced a new objective functional that received a lot of attention from the mathematical and engineering communities.

9.5.2 Objective Function of Mumford and Shah

Given a grey level *ideal* image in a fixed bounded open *frame* D we are looking for an open subset Ω of D such that its boundary reveals the edges of the two-dimensional objects contained in the image of Figure 1.9.

Definition 9.2.
Let D be a bounded open subset of $\mathbf{R}^{\mathrm{N}}$ with Lipschitzian boundary.

(i) An *image* in the *frame* D is specified by a function $I \in L^2(D)$.

(ii) We say that $\{\Omega_j\}_{j\in J}$ is an *open partition* of D if $\{\Omega_j\}_{j\in J}$ is a family of disjoint connected open subsets of D such that

$$\mathrm{m}_N(\cup_{j\in J}\Omega_j) = \mathrm{m}_N(D) \quad \text{and} \quad \mathrm{m}_N(\partial \cup_{j\in J} \Omega_j) = 0,$$

where m_N is the N-dimensional Lebesgue measure. Denote by $\mathcal{P}(D)$ the family of all such open partitions of D. □

The idea behind the formulation of D. Mumford and J. Shah [**1**] in 1985 is to do a Tikhonov regularization $I_{\varepsilon,j} \in H^1(\Omega_j)$ of I on each element Ω_j of the partition and then minimize the sum of the local minimizations over Ω_j over all open partitions of the image D: find an open partition $P = \{\Omega_j\}_{j\in J}$ in $\mathcal{P}(D)$ solution of the minimization problem

$$\inf_{P\in\mathcal{P}(D)} \sum_{j\in J} \inf_{\varphi_j\in H^1(\Omega_j)} \int_{\Omega_j} \varepsilon|\nabla\varphi_j|^2 + |\varphi_j - I|^2\, dx \tag{9.45}$$

for some fixed constant $\varepsilon > 0$. Observe that without the condition $\mathrm{m}_N(\cup_{j\in J}\Omega_j) = \mathrm{m}_N(D)$ the empty set would be a solution of the problem.

The question of existence requires a more specific family of open partitions or a penalization term that controls the "length" of the interfaces in some sense:

$$\inf_{P\in\mathcal{P}(D)} \sum_{j\in J} \inf_{\varphi_j\in H^1(\Omega_j)} \int_{\Omega_j} \varepsilon|\nabla\varphi_j|^2 + |\varphi_j - I|^2\, dx + c\, H_{N-1}(\partial \cup_{j\in J} \Omega_j) \tag{9.46}$$

for some $c > 0$, where H_{N-1} is the Hausdorff $(N-1)$-dimensional measure. Equivalently, we are looking for

$$\Omega = \cup_{j\in J}\Omega_j \text{ open in } D \text{ such that } \chi_\Omega = \chi_D \text{ a.e. in } D$$

(χ_Ω and χ_D, characteristic functions) solution of the following minimization problem:

$$\inf_{\substack{\Omega \text{ open } \subset D \\ \chi_\Omega=\chi_D \text{ a.e.}}} \inf_{\varphi\in H^1(\Omega)} \int_\Omega \varepsilon|\nabla\varphi|^2 + |\varphi - I|^2\, dx + c\, H_{N-1}(\partial\Omega). \tag{9.47}$$

In general, the Hausdorff measure is not a lower semicontinuous functional. So either the $(N-1)$-Hausdorff measure H_{N-1} is relaxed to a lower semicontinuous notion of perimeter or the minimization problem is reformulated with respect to a more suitable family of open domains and/or a space of functions larger than $H^1(\Omega)$.

Another way of looking at the problem would be to minimize the number J of connected subsets of the open partition, but this seems more difficult to formalize.

9.5.3 Relaxation of the $(N-1)$-Hausdorff Measure

The choice of a relaxation of the $(N-1)$-Hausdorff measure H_{N-1} is critical. Here the finite perimeter of Caccioppoli (cf. section 6 in Chapter 5) reduces to the perimeter of D since the characteristic function of Ω is almost everywhere equal to the characteristic function of D. However, the relaxation of $H_{N-1}(\partial\Omega)$ to the $(N-1)$-dimensional *upper Minkowski content* by D. Bucur and J.-P. Zolésio [**8**] is much more interesting and in view of its associated compactness theorem (cf. section 12 in Chapter 7) it yields a unique solution to the problem.

9.5.4 Relaxation to BV-, H^s-, and SBV-Functions

The other avenue to explore is to reformulate the minimization problem with respect to a space of functions larger than $H^1(\Omega)$.

One possibility is to use functions with possible jumps, such as functions of bounded variations. In dimension 1, the BV-functions can be decomposed into an absolutely continuous function in $W^{1,1}(0,1)$ plus a jump part at a countable number of points of discontinuity. Such functions have their analogue in dimension N. With this in mind one could consider the penalized objective function

$$J_{\mathrm{BV}}(\varphi) \stackrel{\text{def}}{=} \int_D |\varphi - I|^2\, dx + \varepsilon \|\varphi\|^2_{\mathrm{BV}(D)}, \tag{9.48}$$

which is similar to the Tikhonov regularization when rewritten in the form

$$J_{H^1}(\varphi) \stackrel{\text{def}}{=} \int_D |\varphi - I|^2\, dx + \varepsilon \|\varphi\|^2_{H^1(D)} = \int_D (1+\varepsilon)|\varphi - I|^2\, dx + \varepsilon |\nabla\varphi|^2\, dx,$$

where we square the penalization term to make it differentiable. Unfortunately, the square of the BV-norm is not differentiable.

Going back to Figure 1.9, assume that the triangle T has an intensity of 1, the circle C has an intensity of 2/3, and the remaining part of the square S has an intensity of 1/3. The remainder of the image is set equal to 0. Then the image functional is precisely

$$I(x) = \chi_T(x) + \frac{2}{3}\chi_C + \frac{1}{3}\chi_{S\setminus(T\cup C)}.$$

We know from Theorem 6.9 in section 6.3 of Chapter 5 that the characteristic function of a Lipschitzian set is a BV-function. Therefore, if we minimize the objective functional (9.48) over all $\varphi \in \mathrm{BV}(D)$, we get exactly $\hat{\varphi} = I$. If we insist on formulating the minimization problem over a Hilbert space as for the Tikhonov regularization, the norm of the penalization term can be chosen in the space $H^s(D)$ for some $s \in (0,1]$:

$$J_{H^s}(\varphi) \stackrel{\text{def}}{=} \int_D |\varphi - I|^2\, dx + \varepsilon\, \|\varphi\|^2_{H^s(D)}.$$

For $0 < s < 1/2$, the norm is given by the expression

$$\|\varphi\|_{H^s(D)} \stackrel{\text{def}}{=} \int_D dx \int_D dy\, \frac{|\varphi(y) - \varphi(x)|^2}{|y - x|^{N+2s}}$$

and the relaxation is very close to the one in $\mathrm{BV}(D)$ since $\mathrm{BV}(D) \cap L^\infty(D) \subset H^s(D)$, $0 < s < 1/2$ (cf. Theorem 6.9 (ii) in Chapter 5).

However, as seducing as those relaxations can be, it is not clear that the objectives of the original formulation are preserved. Take the BV-formulation. It is implicitly assumed that there are clean jumps across the interfaces. This is true in dimension 1, but not in dimension strictly greater than 1 as explained in L. Ambrosio [**1**], where he points out that the distributional gradient of a BV-function which is a vector measure has three parts: an absolutely continuous part, a jump part, and a nasty Cantor part. The space $\mathrm{BV}(D)$ contains pathological functions of Cantor–Vitali type that are continuous with an approximate differential 0 almost everywhere, and this class of functions is dense in $L^2(D)$, making the infimum of

J_{BV} over $\mathrm{BV}(D)$ equal to zero without giving information about the segmentation. To get around this difficulty, he introduces the smaller space $\mathrm{SBV}(D)$ of functions whose distributional gradient does not have a Cantor part. He considers the mathematical framework for the minimization of the objective functional

$$J(\varphi, K) \stackrel{\text{def}}{=} \int_{D\backslash K} |\varphi - I|^2 + \varepsilon|\nabla\varphi|^2 \, dx + c \, H_{N-1}(K)$$

with respect to all closed sets $K \subset \Omega$ and $\varphi \in H^1(\Omega\backslash K)$, where $I \in L^\infty(D)$. This problem makes sense and has a solution in $\mathrm{SBV}(D)$ when K is replaced by the set S_φ of points where the function $\varphi \in \mathrm{SBV}(D)$ has a jump, that is,

$$J(\varphi) \stackrel{\text{def}}{=} \int_D |\varphi - I|^2 + \varepsilon|\nabla\varphi|^2 \, dx + c \, H_{N-1}(S_\varphi),$$

that turns out to be well-defined on $\mathrm{SBV}(D)$.

9.5.5 Cracked Sets and Density Perimeter

The alternate avenue to explore is to reformulate the minimization problem with respect to a more suitable family of open domains (cf. section 15 of Chapter 7).

An additional reason to do that would be to remove the term on the length of the interface. When a penalization on the length of the segmentation is included, long slender objects are not “seen” by the numerical algorithms since they have too large a perimeter. In order to retain a segmentation of piecewise H^1-functions without a perimeter term, M. C. Delfour and J.-P. Zolésio [**38**] introduced the family of *cracked sets* (see Figure 1.11) that yields a compactness theorem in the $W^{1,p}$-topology associated with the *oriented distance function* (cf. Chapter 7).

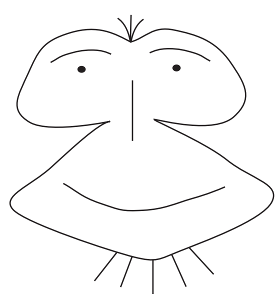


Figure 1.11. *Example of a two-dimensional strongly cracked set.*

The originality of this approach is that it does not require a penalization term on the *length of the segmentation* and that, within the set of solutions, there exists one with minimum *density perimeter* as defined by D. Bucur and J.-P. Zolésio [**8**].

Theorem 9.2. *Let D be a bounded open subset of $\mathbf{R}^N$ and $\alpha > 0$ and $h > 0$ be real numbers.*[3] *Consider the families*

$$\mathcal{F}(D,h,\alpha) \stackrel{\text{def}}{=} \left\{ \Omega \subset \overline{D} : \begin{array}{c} \Gamma \neq \varnothing \textit{ and } \forall x \in \Gamma,\, \exists d,\, |d| = 1, \\ \textit{such that } \displaystyle\inf_{0<t<h} \frac{d_\Gamma(x+td)}{t} \geq \alpha \end{array} \right\}, \tag{9.49}$$

$$C_b^{h,\alpha}(D) \stackrel{\text{def}}{=} \left\{ b_\Omega : \Omega \in \mathcal{F}(D,h,\alpha) \right\},$$

$$\mathcal{F}_s(D,h,\alpha) \stackrel{\text{def}}{=} \left\{ \Omega \subset \overline{D} : \begin{array}{c} \Gamma \neq \varnothing \textit{ and } \forall x \in \Gamma,\, \exists d,\, |d| = 1, \\ \textit{such that } \displaystyle\inf_{0<|t|<h} \frac{d_\Gamma(x+td)}{|t|} \geq \alpha \end{array} \right\}, \tag{9.50}$$

$$(C_b^{h,\alpha})_s(D) \stackrel{\text{def}}{=} \left\{ b_\Omega : \Omega \in \mathcal{F}_s(D,h,\alpha) \right\}.$$

Then $C_b^{h,\alpha}(D)$ and $(C_b^{h,\alpha})_s(D)$ are compact in $W^{1,p}(D)$, $1 \leq p < \infty$, where d_Γ is the distance function to the boundary of Ω.

In view of the above compactness, it can be shown that there exists a solution to the following minimization problem:

$$\inf_{\substack{\Omega \in \mathcal{F}(D,h,\alpha) \\ \Omega \text{ open } \subset D,\, \mathrm{m}_N(\Omega) = \mathrm{m}_N(D)}} \quad \inf_{\varphi \in H^1(\Omega)} \int_\Omega \varepsilon\, |\nabla \varphi|^2 + |\varphi - f|^2 \, dx. \tag{9.51}$$

Cracked sets form a very rich family of sets with a huge potential that is not yet fully exploited in the image segmentation problem. Indeed, they can be used not only to partition the frame of an image but also to detect isolated cracks and points provided an objective functional sharper than the one of Mumford and Shah is used. For instance, in view of the connection between image segmentation and *fracture theory* hinted at in J. Blat and J.-M. Morel [**1**], the theory may have potential applications in problems related to the detection of *fractures or cracks* or *fracture branching and segmentation* in geomaterials (cf. K. B. Broberg [**1**]), but this is way beyond the scope of this book. Some initial considerations about the numerical approximation of cracked sets can be found in M. C. Delfour and J.-P. Zolésio [**41**].

10 Shapes and Geometries: Background and Perspectives

10.1 Parametrize Geometries by Functions or Functions by Geometries?

In the buckling of the column and in the design of the thermal diffuser and the thermal radiator the geometry was a volume of revolution generated by rotation around the z-axis of the hypograph between the axis and the graph of the function

[3] In view of the fact that the distance function d_Γ is Lipschitzian with constant 1, we necessarily have $0 < \alpha \leq 1$.

defined in a variable interval $[0, L]$. They are examples of *sets parametrized by functions* or scalars. A well-documented weakness of this approach is that numerical computations of optimal shapes often yield boundary oscillations. This is typical of the parametrization of the boundary by a function. It provides a good control of the displacement in the direction normal to the graph but little control in the *tangential direction.*

Extending this approach to sets locally defined by several graphs becomes tricky. For instance, it is not possible to represent a sphere or a torus from a bi-Lipschitzian mapping from some domain in the plane unless we replace the plane by a surface with facets as illustrated in Figure 1.12 for a ball (cf. M. C. Delfour [**8**] for the construction of a surface with facets from a $C^{1,1}$-surface). Moreover, the surface with facets of Figure 1.12 is closely associated with the ball and would not be appropriate for a torus.

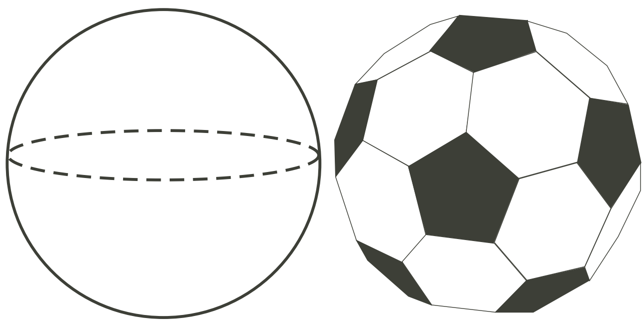


Figure 1.12. *Example of a surface with facets associated with a ball.*

Another family of function-parametrized sets is the family of images of a fixed set by a family of homeomorphisms or diffeomorphisms. This can be advantageous in problems where it is desirable to work with a fixed mesh of the original domain, thus avoiding remeshing the domain at each step of the optimization process. Function-parametrized sets are also used in the *identification of objects* (cf., for instance, the school of R. Azencott, A. Trouvé, and L. Younes). This approach can be traced back to R. Courant and D. Hilbert [**1**] in 1953 and the construction of Courant metrics and complete metric spaces of images of a fixed closed or crack-free open set by A. M. Micheletti [**1**] in 1972. This approach is less interesting in optimization problems where the topology of the set is part of the unknown features of the set. Indeed, the images of a fixed set by an homeomorphism cannot change the topology of the fixed set. For instance, it cannot create holes that are not present in the fixed set.

The example of the distribution of two materials in a fixed domain is generic of problems where the topology of the set is part of the unknowns. The variable chosen to represent the set is the characteristic function χ_A, a *function parametrized by the set A.* As such, arbitrary geometries that are only measurable can be used as independent variables. The topology (for instance, the number of holes) is not fixed a priori and extremely complex design can be obtained. Moreover, the underlying

L^p-spaces where the characteristic functions naturally live induce a metric

$$\rho([A_1],[A_2]) = \|\chi_{A_2} - \chi_{A_2}\|_{L^p(D)}$$

on the family of equivalence classes of Lebesgue measurable subsets of a bounded measurable holdall D. This type of topology has been used in the proof of the Maximum Principle by Pontryagin and later by Ekeland.

Other examples of set-parametrized functions include the uniformly Lipschitzian *distance function* d_A to a set A. The family of such functions for subsets of a holdall $\overline{D}$, D bounded open, can be considered as a family of functions in $C^0(\overline{D})$ or $W^{1,p}(D)$. For instance, the metric

$$\rho([A_1],[A_2]) = \|d_{A_2} - d_{A_2}\|_{C^0(\overline{D})}$$

($[A]$ is the equivalence class of all subsets of $\overline{D}$ with the same closure) coincides with the *écart mutuel* between two sets introduced by D. Pompéiu [**1**] in his thesis presented in Paris in March 1905 that was studied in more detail by F. Hausdorff [**2**, "Quellenangaben," p. 280, and Chap. VIII, sect. 6] in 1914. Unfortunately that metric does not preserve the volume functional, but the norm can be changed to obtain the new metric

$$\rho([A_1],[A_2]) = \|d_{A_2} - d_{A_2}\|_{W^{1,p}(D)}$$

that preserves the volume functional.

Another example of a set-parametrized function is the *support function*

$$\sigma_A(x) \stackrel{\text{def}}{=} \sup_{a\in A} a\cdot x$$

of convex analysis. Here the equivalence classes are

$$[A] \stackrel{\text{def}}{=} \left\{B \subset \overline{D} : \overline{\text{co}}\, B = \overline{\text{co}}\, A\right\},$$

that is, sets with the same closed convex hull. The associated metric is

$$\rho([A_1],[A_2]) = \|\sigma_{A_2} - \sigma_{A_2}\|_{C^0(\overline{B})},$$

where B is the unit open ball in $\mathbf{R}^{\mathrm{N}}$ (cf. M. C. Delfour and J.-P. Zolésio [**17**]).

The general pattern that emerges from the last three examples is the following one. Choose a fixed holdall D and let A denote a variable set in D. Select a family of set parametrized functions f_A and let

$$[A] \stackrel{\text{def}}{=} \left\{B \subset \mathbf{R}^{\mathrm{N}} : f_A = f_B\right\}$$

be the associate equivalence class. Finally, assume that the family of functions $\{f_A\}$ is contained in a Banach space $\mathcal{B}(D)$. Choose as a metric

$$\rho([A_1],[A_2]) = \|f_{A_2} - f_{A_2}\|_{\mathcal{B}(D)}.$$

This approach will be detailed in Chapters 5, 6, and 7 for the *characteristic function* χ_A, the *distance function* d_A, and the *oriented distance function* b_A, but many other set-parametrized functions can be used according to the requirements of the problem at hand.

10.2 Shape Analysis in Mechanics and Mathematics

The terminology *shape analysis* has been introduced independently in at least two different contexts: *continuum mechanics* and *mathematical theory of partial differential equations* (cf., for instance, E. J. Haug and J. Céa [**1**]).

In continuum mechanics, shape analysis encompasses contributions to structural mechanics of elastic bodies such as beams, plates, shells, arches, and trusses. In such problems, the objective is to optimize the *compliance*, e.g., the work of the applied loadings, by choosing the design parameters of the structure. Many of the early contributions are in dimension 2 and complete analytical solutions are often provided. Yet, it is not always easy to distinguish between a *shape optimization problem*, such as the shape of a two-dimensional plate, and a *distributed parameter problem*, such as the optimal thickness of that plate. The thickness that is often considered as a *shape parameter* is really a distributed parameter over the two-dimensional domain, which also specifies the plate. When the thickness goes to zero in some parts of the plate, holes are created and induce changes in the topology and the shape of the associated two-dimensional domain. This naturally leads to *topological optimization*, which deals with the connectivity of a domain, the number of holes, the fractal dimension of the boundary, and ultimately the appearance of a microstructure. This was exemplified by K.-T. Cheng and N. Olhoff [**2**]'s celebrated optimization of the compliance of the circular plate with respect to its thickness under prescribed loading and a constraint on the volume of material. Such questions have received a lot of attention in specific cases and have been analyzed by homogenization methods or Γ-convergence (cf., for instance, F. Murat and L. Tartar [**1**], the conference proceedings edited by M. Bendsøe and C. A. Mota Soares [**1**], the book by M. Bendsøe [**1**], and the book edited by A. Cherkaev and R. Kohn [**1**] that contains a selection of translations of key papers written in French or Russian). However, many fundamental questions still remain open. For instance, how does such an analysis affect the validity of the underlying mechanical or physical models?

For convenience we shall refer to this viewpoint as the *compliance analysis* that generally involves the extremum of the minimum of an energy or work functional with respect to some design parameters.

In the mathematical theory of partial differential equations, the analysis dealt with the sensitivity of the solution of boundary value problems with respect to the shape of the geometric domain on which the partial differential equation is defined. This was done for different applications, including free boundary problems, noncylindric problems,[4] and shape identification problems. This *shape sensitivity analysis* was simultaneously developed for the solution of the partial differential equation and for shape functionals depending on their solution.

In that context, the compliance analysis becomes a special case that, quite remarkably, does not require the shape sensitivity analysis of the solution of the associated partial differential equation (cf. section 2 of Chapter 10). This important

[4] A cylindric problem is a partial differential equation where the geometric domain is fixed and independent of the time variable. A noncylindric problem is a partial differential equation where the geometric domain changes with time.

simplification arises from the fact that the compliance is the minimum of an energy or work functional. An historical example that also benefits from this property is the shape derivative of the first eigenvalue of the plate studied at the beginning of the 20th century by J. HADAMARD [**1**]. As for the compliance, the shape sensitivity analysis of the first eigenfunction is not required even when the first eigenvalue is repeated. This follows from the fact that the first eigenvalue can be expressed as a minimum through Rayleigh's quotient or G. AUCHMUTY [**1**]'s dual variational principle.

Shape sensitivity analysis deals with a larger class of shape functionals (e.g., minimal drag, noise reduction) and partial differential equations (e.g., the wave equation, viscous or non-Newtonian fluids), where variational energy functionals are usually not available and for which the compliance analysis is no longer applicable. Yet, the shape sensitivity analysis of the solution of the partial differential equation can again be avoided by incorporating the partial differential equation into a Hamiltonian or Lagrangian formulation. As in *control theory*, this yields a partial differential equation for the *adjoint state* that is coupled with the initial partial differential equation or *state equation*. A precise mathematical justification of this approach can be given when the Lagrangian has saddle points and the shape derivative can be obtained from theorems on the differentiability of saddle points with respect to a parameter even when the saddle point solution is not unique (cf. section 5 in Chapter 10).

The shape sensitivity analysis through a family of diffeomorphisms which preserve the smoothness of the images of a fixed domain is primarily a local analysis. It is used to establish continuity, to define derivatives, or to optimize in a narrow class of domains with fixed regularities and topologies: it cannot create holes or singularities that were not present in the initial domain. Consider the problem of finding the best location and shape of a hole of given volume in a homogeneous elastic plate to optimize the compliance or some other criterion under a given loading. The expectation is that the presence of the hole would improve the compliance over a homogeneous plate without holes. This problem has its analogue in control theory, for instance the optimal placement of sensors and actuators for the control and stabilization of large flexible space structures or flexible arms of robots. Another important example is the localization of sensors and actuators to achieve noise reduction in structures. The optimal placement is usually an integral part of the control synthesis.

In the early 1970s J. CÉA, A. J. GIOAN, and J. MICHEL [**1**] proposed to introduce relaxed problems in which the optimal domain (here an optimal hole, the optimal location of the support of the optimal control) was systematically replaced by a density function ranging between zero and one and that would hopefully be a characteristic function (a bang bang control) in the optimal regime. Furthermore, in order to work on a fixed domain D without holes, they replaced the holes by a very weak elastic material. This changed the original topological optimization problem into the identification of distributed coefficients over the fixed domain D. Equivalently, this identification problem reduces to finding the optimal distribution of two materials for a transmission equation in D under a volume constraint on one of the two materials. Under appropriate conditions, the solution of this problem is a characteristic function, even when the space of distributed parameters is relaxed to the closed convex hull of the set of characteristic functions (cf. Chapter 5). This general technique can also be used to study the continuity of the solution of the homogeneous Dirichlet boundary value problems with respect to the domain Ω, by introducing a transmission problem over a fixed domain D and letting the distributed coefficient go to infinity in the complement of Ω with respect to D. Of course, the coefficient over the complement of Ω could be replaced by a Lagrange multiplier, thus providing a formulation over the fixed domain D. This is one of the many ways to make the domain *fictitious* and avoid dealing directly with the geometry (cf. R. Glowinski, T.-W. Pan, and J. Périaux [**1**]).

10.3 Characteristic Functions: Surface Measure and Geometric Measure Theory

The main advantage of the relaxation to a characteristic function in the formulation of problems involving a domain integral and/or a volume constraint on Lebesgue measurable sets is that the unknown set is completely described by a single function instead of a family of local diffeomorphisms.

Yet, such measurable sets can be quite unstructured. For example, in problems involving a surface tension on the free boundary of a fluid or on the interface between two fluids, the domain must have a locally finite boundary measure. Another example is when the objective functional is a function of the normal derivative of the state variable along the boundary (e.g., a flow or a thermal power flux through the boundary). In both cases the use of characteristic functions is limited by the fact that they are not differentiable on the boundary of the set and cannot be readily used to describe very smooth domains.

Fortunately, there is enough latitude to make sense of a locally finite boundary measure for sets for which the characteristic function is of bounded variation; that is, its gradient is a vector of bounded measures. Such sets are known as *Caccioppoli* or *finite perimeter sets*. They were a key ingredient in the contribution of E. De Giorgi to the *theory of minimal surfaces* in the 1950s, since the norm of that vector measure (which is the total variation of the characteristic function) turns out to be a relaxation of the boundary measure or "perimeter" of the set. To get a compactness result for a family of Caccioppoli sets it is sufficient to put a uniform bound on their perimeters. This field of activities is known as *geometric measure theory*, and its tools have been used very successfully in the theory of free and moving boundary problems. Even though it came rather late to *shape analysis*, this material is both important and fundamental.

10.4 Distance Functions: Smoothness, Normal, and Curvatures

Another set-parametrized function that plays a role similar to the characteristic function is the *distance function* between closed subsets of a fixed holdall[5] D of $\mathbf{R}^N$. The "*écart mutuel*" between two sets was introduced by D. Pompéiu [**1**] in

[5]This set D will play several roles in this book. It is the *universe* in which the variable subsets live. It will often be referred to as the underlying *holdall*. In other circumstances it will have a purely technical role, much like the control volume in fluid mechanics.

his thesis presented in Paris in March 1905. This is the first example of a metric between two sets in the literature. It was studied in more detail by F. Hausdorff [**2**, "Quellenangaben," p. 280, and Chap. VIII, sect. 6] in 1914. The Pompéiu–Hausdorff metric between two sets corresponds to the uniform norm of the difference of their respective distance functions. For most applications it is not a very interesting topology since the volume functional is not continuous in that topology. The continuity of the volume functional can be recovered by replacing the uniform norm by the $W^{1,p}$-norm since the characteristic function of the closure of the set can be expressed in terms of the gradient of the distance function.

In 1951, H. Federer [**1**] introduced the family of *sets of positive reach* and gave a first access to curvatures from the distance function in much the same spirit as the perimeter from the characteristic function. They are sets for which the projection onto the set is unique in a neighborhood of the set. Since the projection can be expressed in terms of the gradient of the square of the distance function, this is equivalent to requiring that the square of the distance function be $C^{1,1}$ in a neighborhood of the set. Since the gradient of the distance function has a jump discontinuity at the boundary of the set, he managed to recover from the distance function the *curvature measures* of the boundary and made sense of the classical Steiner formula for sets with positive reach.

The jump discontinuity of the characteristic function across the boundary can be bypassed by going to the *oriented distance function*. For smooth domains this function is quite remarkable since it inherits the same degree of smoothness in a neighborhood of the boundary as the boundary itself. Then the gradient, the Hessian matrix, and the higher-order derivatives in the neighborhood of the boundary can be used to characterize and compute normals, curvatures, and their derivatives along the boundary. This correspondence remains true for domains of class $C^{1,1}$. The gradient of the oriented distance function coincides with the outward unit normal on the boundary. This implicit orientation is at the origin of the terminology *oriented distance function* that was introduced by M. C. Delfour and J.-P. Zolésio [**17**] in 1994 to distinguish it from the *algebraic distance function* to a submanifold that can be defined only in terms of some discriminating criterion that distinguishes between what is *above* and what is *below* the submanifold. This is not always possible, while the oriented distance function to a set is always well-defined. The restriction of the Hessian matrix of the oriented distance function to the boundary coincides with the second fundamental form of differential geometry. Its eigenvalues are zero and the principal curvatures of the boundary.

A nice relaxation of the curvatures is obtained by considering sets for which the elements of the Hessian matrix of the oriented distance function are bounded measures. They are called *sets of locally bounded curvature* (cf. M. C. Delfour and J.-P. Zolésio [**17**, **32**]). To get a compactness result, it is sufficient to put a uniform bound on the total variation of the gradient (the elements of the Hessian matrix are bounded measures).

The oriented distance function also provides a framework to compare the smoothness of sets ranging from arbitrary sets with a nonempty boundary to sets of class C^∞ via the smoothness of the oriented distance function in a neighborhood of the boundary of the set. As a result Sobolev spaces can be used to introduce the notion of a *Sobolev domain*, which becomes intertwined with the classical notion of C^k-domains.

10.5 Shape Optimization: Compliance Analysis and Sensitivity Analysis

As in the vector space case, optimization and control problems with respect to geometry are of various degrees of difficulty. When the objective functional does not depend on the solution of a state equation or variational inequality defined on the variable domain, it is sufficient to invoke compactness and continuity arguments. In special cases such as the optimization of the compliance or of the first eigenvalue, the problem can be transformed into the optimization of an objective functional that is itself the minimum of some appropriate functional defined on a fixed function space. There a direct study of the dependence of the solution of the state equation with respect to the underlying domain can be bypassed. In the general case, a state equation constraint has to be handled very carefully from both the mathematical and the application viewpoints. When the analysis can be restricted to families of Lipschitzian or convex domains, it is usually possible to give a meaning and prove the continuity of the solution of the state equation with respect to the underlying varying domain.

When families of arbitrary bounded open domains are considered, new phenomena can occur. As the domains converge in some sense, the corresponding solutions may only converge in a weak sense to the solution of a different type of state equation over the limit domain: boundary conditions may no longer be satisfied, strange terms[6] may occur on the right-hand side of the equation, etc. In such cases, it is often a matter of modeling of the physical or technological phenomenon. Is it natural to accept a generalized solution or a relaxed formulation of the state equation or should the family of domains be sufficiently restricted to preserve the form of the original state equation and maintain the continuity of the solution with respect to the domain? The most suitable relaxation is not necessarily the most general mathematical relaxation. It must always be compatible with the physical or technological problem at hand. Too general a relaxation of the problem or too restrictive conditions on the varying domains can yield completely unsatisfactory solutions however nice the underlying mathematics are. A good balance of mathematical, physical, and engineering intuitions is essential.

The study of the shape continuity of the solution of the partial differential equation is also of independent mathematical interest. In the literature, this issue has been addressed in various ways. Some authors simply introduce a *stability assumption* that essentially says that the limiting domain is such that continuity occurs. Others introduce a set of more technical assumptions that correspond to a minimal set of conditions to make the crucial steps of the proof of the continuity work. On the constructive side the really challenging issue is to characterize the families of domains for which the continuity holds.

[6] Cf., for instance, D. Cioranescu and F. Murat [**1**, **2**].

Many authors have constructed compact families for that purpose. For instance the *Courant metric topology* was used by A. M. MICHELETTI [**1**] in 1972 for C^k-domains, the *uniform cone condition* by D. CHENAIS [**1**] in 1973 for uniformly Lipschitzian domains, and other metric topologies by F. MURAT and J. SIMON [**1**] in 1976 for Lipschitzian domains.

More general capacity conditions were introduced after 1994 by D. BUCUR and J.-P. ZOLÉSIO [**5**] in order to obtain compact subfamilies of domains with respect to the complementary Hausdorff topology and control the curvature of the boundaries of the domains. In dimension 2 they recover the nice result of V. ŠVERÁK [**2**] in 1993, which involves a uniform bound on the number of connected components of the complement of the sets. Intuitively the capacity conditions are such that, locally, the complement of the domains in the chosen family has "enough capacity" to preserve the homogeneous Dirichlet boundary condition in the limit. Yet, the capacity conditions might not be easy to use in a specific example. So D. BUCUR and J.-P. ZOLÉSIO [**1**, **5**] introduced a simpler geometric constraint, called the *flat cone condition*, under which the continuity and compactness results still hold. This generalizes the *uniform cone property* to a much larger class of open domains. All this will be further generalized in section 9 of Chapter 8.

10.6 Shape Derivatives

For functionals defined on a family of domains, it is important to distinguish between a function with values in a topological vector space over a fixed domain and a function such as the solution of a partial differential equation that lives in a Sobolev space defined on the varying domain. In the latter case special techniques have to be used to transport the solution onto a fixed domain or to first embed the varying domains into a fixed *holdall* D, extend the solutions to D, and enlarge the Sobolev space to a large enough space of functions defined over the fixed holdall D. In both cases the function is defined over a family of domains or sets belonging to some *shape space* which is generally nonlinear and nonconvex. Thus, defining derivatives on those spaces is more related to defining derivatives on differentiable manifolds than in vector spaces.

It is perhaps for groups of diffeomorphisms that the most complete theory of shape derivatives is available. This material will be covered in detail in Chapters 2, 3, and 9, but it is useful to briefly introduce the main ideas and definitions here. Given a Banach space Θ of mappings from a fixed open holdall $D \subset \mathbf{R}^{\mathrm{N}}$ into $\mathbf{R}^{\mathrm{N}}$, first consider the group of diffeomorphisms

$$\mathcal{F}(\Theta) \overset{\text{def}}{=} \left\{F : D \to D \,:\, F - I \in \Theta \text{ and } F^{-1} - I \in \Theta\right\}.$$

Then consider the images of a fixed domain Ω_0 in D by $\mathcal{F}(\Theta)$:

$$\mathcal{X}(\Omega_0) \overset{\text{def}}{=} \{F(\Omega_0) \,:\, \forall F \in \mathcal{F}(\Theta)\}.$$

They can be identified with the quotient group

$$\mathcal{F}(\Theta)/\mathcal{G}(\Omega_0), \quad \mathcal{G}(\Omega_0) \overset{\text{def}}{=} \{F \in \mathcal{F}(\Theta) \,:\, F(\Omega_0) = \Omega_0\}$$

of diffeomorphisms of D. For the specific choices of D, Θ, and Ω_0 that are of interest, this quotient group can be endowed with the so-called *Courant metric* to make it a complete metric space. This metric space is neither linear nor convex.

For unconstrained domains ($D = \mathbf{R}^N$) and "sufficiently small" elements $\theta \in \Theta$, transformations of the form $F = I + \theta$ belong to $\mathcal{F}(\Theta)$ and hence *perturbations of the identity* (i.e., of Ω_0) can be chosen in the vector space Θ. This makes it possible to define directional derivatives and speak of Gateaux and Fréchet differentiability with respect to Θ as in the classical case of functions defined on vector spaces. However, this approach does not extend to submanifolds D of $\mathbf{R}^N$ or to domains that are constrained in one way or another (e.g., constant volume, perimeter, etc.).

In the unconstrained case, we get an infinite-dimensional differentiable manifold structure on the quotient group $\mathcal{F}(\Theta)/\mathcal{G}(\Omega_0)$ and the tangent space to $\mathcal{F}(\Theta)$ is the whole linear space Θ. As a result ideas and techniques from *differential geometry* are readily applicable. For instance, the continuity in I (i.e., at Ω_0) can be characterized by the continuity along one-dimensional flows of velocity fields in $\mathcal{F}(\Theta)$ through the point I (that is, $I(\Omega_0) = \Omega_0$). This suggests defining a notion of directional derivative along one-dimensional flows associated with "velocity fields" V which keep the flows inside D. When D is a smooth submanifold of $\mathbf{R}^N$ such velocities are tangent to D and the set of all such velocities has a vector space structure. This key property generalizes to other types of holdalls. So it is possible to define directional derivatives and speak of shape gradient and shape Hessian with respect to the associated vector space of velocities. This second approach has been known in the literature as the *velocity method.* Another very nice property of that method is that perturbations of the identity, as previously defined in the unconstrained case, can be recovered by a special choice of velocity field, thus creating a certain unity in the methodology.

The concept of *topological derivative* was introduced by J. Sokołowski and A. Żochowski [**1**] for problems where the knowledge of the optimal topology of the domain is important. It gives some sensitivity of a shape functional to the presence of a small hole at a point of the domain as the size of that hole goes to zero. For a domain integral, that is, the integral of a locally Lebesgue integrable function, this derivative is the negative of the classical *set-derivative* which is equal to the function at almost every point. This is a direct consequence of the Lebesgue differentiation theorem. So, at least in its simplest form, this approach aims at extending the classical concept of *set-differentiation* as the inverse of integration over sets.

This type of derivative becomes more intricate as we look at the sensitivity of the solution of a boundary value problem or at shape functionals which are functions of that solution. For functions that are the integrals of an integrable function with respect to a Radon measure, the Lebesgue differentiation extends in the form of the Lebesgue–Besicovitch differentiation theorem, which says that the set-derivative of that integral is again equal to the function almost everywhere with respect to the Radon measure. So one could try to determine the class of shape functionals that can be expressed as an integral with respect to some Radon measure. Since this material requires extensive technical results to fully appreciate its impact, we did not include a chapter on this topic in the book, but the approach is an important addition to the global arsenal! Fortunately, the theoretical and numerical work is well documented in the literature and books by J. Sokołowski and his coauthors should be available in the near future.

10.7 Shape Calculus and Tangential Differential Calculus

The velocity method has been used in various applications and contexts, and a very complete *shape calculus* is now available (for instance, the reader is referred to the book of J. Sokołowski and J.-P. Zolésio [**9**]). In the computations of derivatives, and especially of second-order derivatives, an intensive use is made of the *tangential differential calculus*. In order to avoid parametrizations and local bases, tangential derivatives are defined through extensions of functions from the boundary to some small Euclidean neighborhood. Their importance should not be underestimated from both the theoretical and the computational points of view. The use of an intrinsic tangential gradient, divergence, or Laplacian can considerably simplify the computation and the final form of the expressions making more apparent their fine structure. Too many computations using local coordinates, Christoffel symbols, or intricate parametrizations are often difficult to decipher or to use effectively.

This book provides the latest and most important developments of that calculus. They arises from the systematic use of the oriented distance function in the theory of thin and asymptotic shells.[7] In that context, it has been realized that extending functions defined on the boundary Γ of a domain Ω by composition with the projection onto Γ results in sweeping simplifications in the tangential calculus. This is due to the fact that the projection can be expressed in terms of the gradient of the oriented distance function. Computing derivatives on Γ becomes as easy as computing derivatives in the Euclidean space. Curvature terms, when they occur, appear in the right place and in the right form through the Hessian matrix of the oriented distance function that coincides with the second fundamental form of Γ. Chapter 9 will provide a self-contained introduction to these techniques and show how the combined strengths of the shape calculus and the tangential differential calculus considerably simplify computations and expand our capability to tackle highly complex and challenging problems.

10.8 Shape Analysis in This Book

Problems in which the design, control, or optimization variable is no longer a vector of parameters or functions, but is the shape of a geometric domain, a set, or even a "fuzzy entity," cover a much broader range of applications than those for which the compliance or the shape sensitivity analysis have been used. Yet their analysis makes use of common mathematical techniques: partial differential equations, functional analysis, geometry, modern optimization and control theories, finite element analysis, large scale constrained numerical optimization, etc.

In this book the terminology *shape* will be used for domains ranging from unstructured sets to C^∞-domains. Relaxations of their geometric characteristics such as the volume, perimeter, connectivity, curvatures, and their derivatives will

[7] Cf. M. C. Delfour and J.-P. Zolésio [**19**, **20**, **25**] and M. C. Delfour [**3**, **4**, **6**, **7**].

be considered. Shape spaces (often metric and complete) corresponding to different levels of smoothness or degrees of relaxation of the geometry will be systematically constructed. For smooth domains we will emphasize the use of groups of diffeomorphisms endowed with the Courant metric. For more general domains we will use a generic construction based on the use of set-parametrized functions. The characteristic function will be associated with metric spaces of equivalence classes of Lebesgue measurable sets. The distance function will be associated not only with the uniform Hausdorff metric topology in the space of continuous functions but also with $W^{1,p}$-topologies for which the characteristic function and hence the volume function are continuous. The oriented distance function will be used in the same way to generate new metric topologies. In each case, the metric is constructed from the norm of one of the set-parametrized functions in an appropriate function space. The construction is generic and applies to other choices of set-parametrized functions and function spaces. For instance, the support function of convex analysis can be used to generate a complete metric topology on equivalence classes of sets with the same closed convex hull (cf. M. C. Delfour and J.-P. Zolésio [**17**]).

The nice property of the characteristic and distance functions over classical local diffeomorphisms is that the set is globally described in terms of the analytical properties of a single function. For instance, the gradient of the characteristic function yields a relaxed definition of the perimeter, and the Hessian of the distance function yields the boundary measure and curvature terms. But the characteristic function and the gradient of the distance function are both discontinuous at the boundary of the set. This seriously limits their use in the description of smooth domains.

In contrast, the oriented distance function can describe a broad spectrum of sets ranging from arbitrary sets with nonempty boundary to open C^∞-domains according to its degree of smoothness in a neighborhood of the boundary of the set. It readily combines the advantages of local diffeomorphisms and the characteristic and distance functions that are readily obtained from it. This provides, as in functional analysis, a common framework for the classification and comparison of domains according to their relative degree or lack of smoothness. As in geometric measure theory, compact families of sets will be introduced based on the degree of differentiability. One interesting family is made up of the sets with *locally bounded curvature*, which provide a sufficient degree of relaxation for most applications and for which nice compactness theorems are available (cf. sections 5 and 11 in Chapter 7). This family includes Federer's sets of positive reach and hence closed convex and semiconvex sets.

11 Shapes and Geometries: Second Edition

Except for the last two chapters, this second edition is almost a new book. Chapters 2, 3, and 4 give the classical description of sets and domains from the point of view of *differential geometry*. Special attention is paid to domains that verify some segment properties and have a local epigraph representation and to domains that are the image of a fixed set by a family of diffeomorphisms of the Euclidean space.

Chapters 5, 6, and 7 give a *function analytic* description of sets and domains via the set-parametrized characteristic functions, distance functions, and oriented distance functions. It emphasizes the fact that we are now dealing with equivalence classes of sets that may or may not have an invariant open or closed set representative in the class. In particular, they include Lebesgue measurable sets and Federer's sets of positive reach. Many of the classical properties of sets can be recovered from the regularity or function analytic properties of those functions. We concentrate on the basic properties, the construction of spaces of domains, metrics, and topologies, and the characterization of compact families.

Chapter 8 deals with problem formulations and *shape continuity and optimization* for some generic examples.

Chapters 9 and 10 are devoted to a modern version of the *shape calculus*, an introduction to the *tangential differential calculus*, and the shape derivatives under a state equation constraint.

11.1 Geometries Parametrized by Functions

Chapter 2 gives several classical characterizations and properties of sets or domains from the point of view of *differential geometry*: sets *locally described* in a neighborhood of each point of their boundary by an homeomorphism or a diffeomorphism (e.g., C^k or Hölderian diffeomorphisms), by the epigraph of a C^0 function (e.g., Lipschitzian or Hölderian domains), or by a geometric property (e.g., segment, cone, cusp), or sets *globally described* by the level sets of a C^1-function.

The sections on sets that are locally the epigraph of a C^0 function and sets having one of the segment properties have been completely reorganized, rewritten, and enriched with new results and older ones that are difficult to find in the literature other than in the form of *folk theorems*[8] or theorems without satisfying proofs. Special attention is given to the *uniform fat segment property* that was introduced in the first edition of the book under the name *uniform cusp property.* Several equivalent properties are given for domains that satisfy a *uniform segment property* and the stronger *uniform fat segment property* that can be expressed in terms of a *dominating function* and its modulus of continuity. All this is specialized to domains verifying a uniform cone or cusp property.

Chapter 3 adopts another point of view by considering families of sets that are the images of a fixed subset of $\mathbf{R}^{\mathrm{N}}$ by some family of transformations of $\mathbf{R}^{\mathrm{N}}$. The structure and the topology of the images can then be specified via the natural algebraic and topological structures of transformations or equivalence classes of transformations for which the full power of function analytic methods is available. In 1972, A. M. Micheletti [**1**] introduced the so-called *Courant metric* and gave what seems to be the first construction of a complete metric topology on the images

[8]As the term is understood by mathematicians, folk mathematics or mathematical folklore means theorems, definitions, proofs, or mathematical facts or techniques that are found by investigation and may circulate among mathematicians by word-of-mouth but have not appeared in print, either in books or in scholarly journals. Knowledge of folklore is the coin of the realm of academic mathematics, showing the relative insight of investigators (http://en.wikipedia.org/wiki/Mathematical_folklore).

of a fixed set. It is the nonlinear and nonconvex character of such *shape spaces* that will make the differential calculus and the analysis of shape optimization problems more challenging than their counterparts in topological vector spaces. The construction of A. M. MICHELETTI [**1**] is generic and readily extends to many families of transformations of $\mathbf{R}^N$.

The new version of Chapter 3 considerably expands the material and ideas of the first edition by extracting the fundamental assumptions behind the generic framework of A. M. Micheletti that leads to the *Courant metrics* on the quotient space of families of transformations by subgroup of isometries such as identities, rotations, translations, or flips. Constructions are given for a large spectrum of transformations of the Euclidean space and for arbitrary closed subgroups. New complete metrics on the whole spaces of homeomorphisms and C^k-diffeomorphisms are also introduced. They extend classical results for transformations of compact manifolds to general unbounded closed sets and open sets that are crack-free. This material is central in classical mechanics and physics and in modern applications such as imaging and detection.

The former Chapter 7 on *transformations versus flows of velocities* has been moved right after the Courant metrics as Chapter 4 and considerably expanded. It now specializes the results of Chapter 3 to spaces of transformations that are generated by the flow of a *velocity field* over a generic time interval. The main motivation is to introduce a notion of semiderivatives as well as a tractable criterion for continuity with respect to Courant metrics. The velocity point of view was also adopted by R. Azencott and A. Trouvé starting in 1994 to construct complete metrics and *geodesic paths* in spaces of diffeomorphisms generated by a velocity field with applications to imaging.

Chapter 4 also gives general equivalences between transformations and flows of velocity fields for unconstrained and constrained families of domains and further sharpens the results to the specific families of transformations associated with the Courant metrics studied in Chapter 2. Several examples of transformations and velocities associated with widely used families of domains are given: C^∞-domains, C^k-domains, Cartesian graphs, polar coordinates and star-shaped domains, and level sets. The chapter clarifies the long-standing issue of the equivalence of the continuity of shape functionals with respect to the Courant metrics and along the flow of velocity fields.

This chapter also prepares the ground for and motivates the definition of shape semiderivatives that will be given in Chapter 9. As in the case of the continuity the equivalent characterizations via transformations and flows of velocities are very much in the background and at the origin of the many seemingly different definitions which can be found in the literature. Preliminary considerations are first given to the definition of a shape functional and to two candidates for the definition of a directional shape semiderivative. They respectively correspond to *perturbations of the identity* associated with any one of the metric spaces constructed in Chapter 2 and to the *velocity method* associated with the flow of a generally nonautonomous vector field. The first one seems to be limited to domains in $\mathbf{R}^N$, while the second one naturally extends to domains living in a fixed smooth submanifold of $\mathbf{R}^N$. Moreover, the shape directional derivative obtained by perturbations of the identity can be recovered by a special choice of velocity field. Most definitions of shape derivatives which can be found in the literature can be brought down to one of the two approaches.

11.2 Functions Parametrized by Geometries

Almost all compactness theorems for families of sets are specified by function analytic conditions on special set-parametrized families of functions. In this book we single out the *characteristic function*, the *distance function*, and the *oriented distance function*. In the last case, a complete and convivial tangential differential calculus on the boundary of smooth sets will be obtained without local bases or Christoffel symbols and will be exploited in the computation of shape derivatives later in Chapters 9 and 10.

Chapter 5 relaxes the family of classical domains to the equivalence classes of Lebesgue measurable sets. Using the *characteristic function* associated with a set, complete *metric groups* of equivalence classes of characteristic functions are constructed. As for Courant metrics on groups, they are nonlinear and nonconvex. On one hand, this type of relaxation is desirable in optimization problems where the topology of the optimal set is not a priori specified; on the other hand, it necessitates the relaxation of the theory of partial differential equations on a smooth open domain to measurable sets that have no smoothness and may not even be open. Furthermore some optimization problems yield optimal solutions where the characteristic function is naturally relaxed to a function between zero and one. Such solutions can be interpreted as *microstructures*, *fuzzy sets*, probability measures, etc. As a first illustration of the use of characteristic functions in optimization, the solution of the original problem of J. Céa and K. Malanowski [**1**] for the optimization of the compliance with respect to the distribution of two materials is given with complete details. A second example deals with the buckling of columns, which is one of the very early optimal design problems formulated by Lagrange in 1770. This is followed by the construction of the *nice representative* of an equivalence class of measurable functions: the *measure theoretic representative*. The last section is devoted to the *Caccioppoli or finite perimeter sets*, which have been introduced to solve the Plateau problem of minimal surfaces (J. A. F. Plateau [**1**]). Even if, by nature, a characteristic function is discontinuous at the boundary of the associated set, the characteristic function of Caccioppoli sets has some smoothness: it belongs to $W^{\epsilon,p}(D)$, $1 \leq \epsilon < 1/p$, $p \geq 1$. This is sufficient to obtain compact families of sets by putting a uniform bound on the perimeter. One such family is the set of (locally) Lipschitzian (epigraph) domains contained in a fixed bounded *holdall* and satisfying a uniform cone property. This property puts a uniform bound on the perimeter of the sets. The use of the theory of finite perimeter sets is illustrated by an application to a free boundary problem in fluid mechanics: the modeling of the Bernoulli wave where the surface tension of the water enters via the perimeter of the free boundary. The chapter concludes with an approximation of the Dirichlet problem by transmission problems over a fixed larger space in order to study the continuity of its solution with respect to the underlying moving domains.

Chapter 6 moves on to the classical Pompéiu–Hausdorff metric topology which is associated with the space of equivalence classes of distance functions of sets with the same closure. As in Chapter 2, the construction of the metric on equivalence classes of sets with the same closure is generic. By going to the distance function of the complement of the set in the uniform topology of the continuous functions, we get the *complementary Hausdorff topology.* These uniform topologies are often too coarse for applications to physical or technological systems. But, since the distance function is uniformly Lipschitzian, it can also be embedded into $W^{1,p}$-Sobolev spaces, and finer metric topologies can be generated. They offer definite advantages over the uniform Hausdorff topology in the sense that they preserve the continuity of the volume of sets since the characteristic function is continuous with respect to $W^{1,p}$-topologies. Yet, we lose the compactness of the family of subsets of a fixed bounded holdall of $\mathbf{R}^N$. Compact families are recovered by imposing some smoothness on the Hessian matrix as in the case of the characteristic functions of Chapter 5. Sets for which the gradient of the distance function is a vector of functions of bounded variation are said to be of *bounded curvature* since their Hessian matrix is intimately connected with the curvatures of the boundary. This class of sets is sufficiently large for applications and at the same time sufficiently structured to obtain interesting theoretical results. For instance, such sets turn out to be Caccioppoli sets. Furthermore, the squared distance function is directly related to the projection of a point onto the set and is used to characterize *Federer's sets of positive reach* and construct compact families. Closed convex sets that are completely characterized by the convexity of their distance function are also of locally bounded curvature. To complete the list of families of sets that are associated with the distance function we introduce Federer's *sets of positive reach* and a first compactness theorem. The chapter is complemented with a general compactness theorem for families of sets of global or local bounded curvature in a tubular neighborhood of their boundary.

The use of the distance function of Chapter 6 to characterize the smoothness of sets is limited by the fact that its gradient presents a jump discontinuity at the boundary. This is similar to the jump discontinuity of the characteristic function. To get around this difficulty we introduce the *oriented distance function* that is obtained by subtracting from the distance function to a set the distance function to its complement, providing a level set description of the set. One remarkable property of this function is the fact that a set is of class $C^{1,1}$ (resp., C^k, $k \geq 2$) if and only if its oriented distance function is locally $C^{1,1}$ (resp., C^k) in a neighborhood of its boundary. It also provides an orientation of the boundary since its gradient coincides with the unit outward normal. This is why we use the terminology *oriented distance function.* As in Chapter 6, Hausdorff and $W^{1,p}$-topologies and sets of global or locally bounded curvature can be introduced. We now have a continuous classification of sets ranging from sets with a nonempty boundary to C^∞-sets, much as in the theory of functions. Closed convex sets are characterized by the convexity of their oriented distance function. Convex sets and semiconvex sets are of locally bounded curvature. This property extends to Federer's sets of positive reach.

Several compactness theorems in the $W^{1,p}$-topology of the oriented distance function are presented in this chapter for families of subsets of a bounded open holdall with either bounded curvature in tubular neighborhoods of their boundary, or with a bound on the *density perimeter* of D. Bucur and J.-P. Zolésio [**8**], or with a *uniform fat segment property* or equivalently the uniform boundedness and equicontinuity of all the local graphs of the sets in the family. The theorem on the compactness under the uniform fat segment property is specialized to the family of subsets of a bounded holdall satisfying a uniform cone or cusp property. It is more general than the one we obtained for Lipschitzian domains in Chapter 5, where the perimeter of each set was finite and uniformly bounded for all subsets of the holdall. Hölderian domains do not necessarily have a locally finite boundary measure. Yet, the fact that the boundary of a Hölderian domain may have cusp does not mean that all Hölderian domains do not have a locally finite boundary measure. In order to include applications where the perimeter is bounded, the general compactness theorem is specialized to families of subsets of a holdall that verify the uniform fat segment property with a uniform bound on either the De Giorgi perimeter or the density perimeter. A last section 15 deals with the family of *cracked sets.* They have been used in M. C. Delfour and J.-P. Zolésio [**38**] in the context of the image segmentation problem of D. Mumford and J. Shah [**2**] that will be detailed in this section. Cracked sets are more general than sets which are locally the epigraph of a continuous function in the sense that they include domains with cracks, and sets that can be made up of components of different codimensions. The Hausdorff $(N-1)$ measure of their boundary is not necessarily finite. Yet, compact families (in the $W^{1,p}$-topology) of such sets can be constructed.

11.3 Shape Continuity and Optimization

Chapter 8 deals with problem formulations, continuity, or semicontinuity of shape functionals or of the solutions of boundary value problems with respect to their underlying domain of definition for selected generic examples. Combined with compact families of sets studied in the previous chapters, they are all essential to getting the existence of optimal shapes. This is illustrated to some extent in Chapter 5 for the modeling of the transmission problem with the help of the characteristic function that occurs both in the model and in the specification of the metric on the equivalence classes of measurable sets.

This chapter first reviews the continuity of the transmission problem using characteristic functions. It characterizes the upper semicontinuity of the first eigenvalue of the generalized Laplacian with respect to the domain using the complementary Hausdorff topology. Then it studies the continuity of the solution of the homogeneous Dirichlet and Neumann boundary value problems with respect to their underlying domain of definition since they require different constructions and topologies that are generic of the two types of boundary conditions even for more complex nonlinear partial differential equations. In problems where the objective functional depends explicitly on the domain and the solution of an elliptic equation defined on the same domain, the strong continuity of the solution with respect to the underlying domain is the key element in the proof of the existence of optimal domains. To get that continuity, some extra conditions have to be imposed on the family of open domains, such as the uniform fat segment property.

The second part extends some of the results to a larger family of domains satisfying *capacity conditions* which turn out to be important to obtain the continuity of solutions of partial differential equations with homogeneous Dirichlet boundary conditions with respect to their underlying domain of definition. One special case of a capacity condition is the *flat cone condition* that generalizes the condition of V. Šverák [**2**] involving a bound on the number of connected components of the complement of the sets.

11.4 Derivatives, Shape and Tangential Differential Calculuses, and Derivatives under State Constraints

Chapter 9 is devoted to the essential *shape calculus* and the no less essential *tangential or boundary calculuses* since traces, normals, or tangential gradients will naturally occur in the final expressions.

After a self-contained review of differentiation in topological vector spaces that emphasizes Gateaux and Hadamard differentials, it introduces basic definitions of first- and second-order Eulerian shape semiderivatives and derivatives by the *velocity method*. General structure theorems are given for Eulerian semiderivatives of a shape functional. They arise from the fact that shape functionals are usually defined over equivalence classes of sets, and hence only the normal part of the velocity along the boundary really affects the shape functional. Bridges are provided with the *method of perturbations of the identity*. A section is devoted to a modern version of the shape calculus. It gives the general formulae for the shape derivative of domain and boundary integrals. From these formulae several examples are worked out, including the semiderivative of the boundary integral of the square of the normal derivative. For the computation of a broader range of shape derivatives the reader is referred to the book by J. Sokołowski and J.-P. Zolésio [**9**]. In most cases, both domain and boundary expressions are available for derivatives. The boundary expression usually contains more information on the structure of the derivative than its domain counterpart. Finally, to effectively deal with the differential calculus in boundary integrals, we provide the latest version of the *tangential calculus* on C^2-submanifolds of codimension 1 which has been developed in the context of the theory of shells (cf. M. C. Delfour and J.-P. Zolésio [**28**, **33**] and M. C. Delfour [**3**, **7**]). This calculus has been significantly simplified by using the projection associated with the oriented distance function studied in Chapter 7. This powerful tool combined with the shape calculus makes it possible to obtain clean explicit expressions of second-order shape derivatives of domain integrals along with a better understanding of their fine structure.

Chapter 10, the final chapter, completes the shape calculus by introducing the basic theoretical results and computational tools for the shape derivative of functionals that depend on a state variable that is usually the solution of a partial differential equation or inequality defined over the varying underlying domain. The first section concentrates on shape functionals that are of *compliance type*; that is, they are the minimum of an energy functional associated with the state equation or inequality. Such functionals are very nice in the sense that they do not generate an adjoint state equation, and their derivative can be obtained by theorems on the differentiability of a minimum with respect to a parameter even when the minimizers are not unique. A detailed generic example is provided to illustrate how to use the *function space parametrization* to transport the functions in Sobolev spaces over variable domains to a Sobolev space over the fixed larger holdall. These techniques extend to more complex situations. For instance, Sobolev spaces of vector functions with zero divergence can be transported by the so-called Piola transformation. Domain and boundary expressions are provided. The main theorem is applied to the example of the buckling of columns. An explicit expression of the semiderivative of *Euler's buckling load* with respect to the cross-sectional area is obtained from the main theorem, and a necessary and sufficient analytical condition is given to characterize the maximum Euler's buckling load with respect to a family of cross-sectional areas. The theory is further illustrated by providing the semiderivative of the first eigenvalue of several boundary value problems over a bounded open domain: Laplace equation, bi-Laplace equation, linear elasticity. In general, the first eigenvalue is not simple over an arbitrary bounded open domain and the eigenvalue is not differentiable; yet the main theorem provides explicit domain and boundary expressions of the semiderivatives.

For general shape functionals, a Lagrangian formulation is used to incorporate the state equation and to avoid the study of the derivative of the state equation with respect to the domain. The computation of the shape derivative of a state-constrained functional reduces to the computation of the derivative of a saddle point with respect to a parameter even when the saddle point solution is not unique. It yields an expression that depends on the associated *adjoint state equation*, much like in control theory, but here the domains play the role of the controls. The technique is illustrated on both the homogeneous Dirichlet and the Neumann boundary value problems by function space parametrization. An alternative to this method is the *function space embedding* combined with the use of Lagrange multipliers. It consists in extending solutions of the boundary value problems over the variable domains to a larger fixed holdall, rather than transporting them. This approach offers many technical advantages over the other one. The computations are easier and they apply to larger classes of problems. This is illustrated on the nonhomogeneous Dirichlet boundary value problem. Again domain and boundary expressions for the shape gradient are obtained. Yet the relative advantages of one method over the other are very much problem and objective dependent. Finally, it is important to acknowledge that the above techniques seem quite robust and are systematically used for nonlinear state equations and in contexts where optimization or saddle point formulations are not available.

Chapter 2

Classical Descriptions of Geometries and Their Properties

1 Introduction

This chapter is devoted to the classical descriptions of nonempty subsets of the finite-dimensional Euclidean space that are characterized by the smoothness or properties of their boundary.

A first approach is to assume that we can associate with each point of the boundary a diffeomorphism from a neighborhood of that point that *locally flattens* the boundary. Another way is to assume that the set is the union of the positive *level sets* of a continuous function and that the zero level is its boundary. A third way is to assume that in each point of the boundary the set is locally the *epigraph* of a function. The smoothness of the set is then characterized by the smoothness of the corresponding diffeomorphism, level function, or graph function. Those definitions are equivalent for sufficiently smooth sets. Domains that are locally the epigraph of a continuous function in the third category also belong to the first category, but the converse is generally not true.

The basic definitions and constructions for sets locally described by an isomorphism or a diffeomorphism are given in section 3, for sets locally described by the level sets of a function in section 4, and for sets locally described by the epigraph of a function in section 5. Section 6 deals with sets characterized by geometric *segment properties*. A set satisfying the basic *segment property* has the property that its boundary $\partial\Omega$ be locally a C^0 epigraph (cf. section 6.2). The stronger *uniform segment property* is further strengthened by introducing the *uniform fat segment property* and the concept of a *dominating function* that will be necessary to construct compact families of sets later in Chapter 7. Complete equivalences are provided between the three C^0 epigraph properties and the three geometric segment properties in sections 6.2 and 6.3. For sets with a compact boundary, the six properties are shown to be equivalent in section 6.2. Section 6.4 specializes the fat segment property to the geometric *uniform cusp* and *cone properties* that respectively characterize *Hölderian* and *Lipschitzian* domains. The uniform cone property has provided one of the early examples of a compact family of Lipschitzian domains (cf. section 6.4 in Chapter 5) in 1973.

2 Notation and Definitions

2.1 Basic Notation

$\mathbb{N}$ is the set of integers $\{1, 2, \dots\}$ and $\mathbf{R}$ the field of real numbers. The *interior* and the *closure* of a subset A in $\mathbf{R}^{\mathrm{N}}$ will be denoted, respectively, by $\operatorname{int} A$ and $\overline{A}$. The *relative complement* of A in B will be written as

$$\complement_B A \text{ (or } B\backslash A) \stackrel{\text{def}}{=} \{x \in B : x \notin A\}.$$

When $B = \mathbf{R}^{\mathrm{N}}$, we also write $\complement A$ or A^c. The *boundary* ∂A of A is defined as $\overline{A} \cap \overline{\complement A}$.

Since the superscript t will often appear in the book, the *transpose* of a vector v and a matrix A will be denoted, respectively, by ${}^*v$ and ${}^*A$. The inverse of A will be denoted by A^{-1}. The inner product and norm in $\mathbf{R}^{\mathrm{N}}$ will be written as

$$x \cdot y \stackrel{\text{def}}{=} \sum_{i=1}^{N} x_i\, y_i, \quad |x| = \sqrt{x \cdot x}.$$

For a linear transformation $A : \mathbf{R}^{\mathrm{N}} \to \mathbf{R}^K$, the *norm* of A is defined as

$$|A| \stackrel{\text{def}}{=} \max_{|x|_{\mathbf{R}^{\mathrm{N}}} \le 1} |Ax|_{\mathbf{R}^K},$$

and if A_{ij} and B_{ij} are the matrix representations of A and B with respect to some bases $\{a_1, \dots, a_N\}$ and $\{b_1, \dots, b_K\}$ of $\mathbf{R}^{\mathrm{N}}$ and $\mathbf{R}^K$, respectively, the *double inner product* is defined as

$$A \cdot\cdot B \stackrel{\text{def}}{=} \sum_{i=1}^{N} \sum_{j=1}^{K} A_{ij} B_{ij} \tag{2.1}$$

and the associated norm $\sqrt{A \cdot\cdot A}$ is equivalent to the norm $|A|$ of A.

2.2 Abelian Group Structures on Subsets of a Fixed Holdall D

Given a nonempty set D, consider the *power set* of D

$$\mathcal{P}(D) \stackrel{\text{def}}{=} \{A : A \subset D\}.$$

The power set is closed under the operation of union, intersection, and complement, but those operations do not have an inverse with respect to the neutral element $\varnothing$:

$$B \cup A = A \cup B, \quad A \cup \varnothing = A,$$
$$A \cup B = \varnothing \iff A = B = \varnothing.$$

Similarly, with the neutral element $\mathbf{R}^{\mathrm{N}}$

$$A \cap B = B \cap A, \quad A \cap \mathbf{R}^{\mathrm{N}} = A,$$
$$A \cap B = \mathbf{R}^{\mathrm{N}} \iff A = B = \mathbf{R}^{\mathrm{N}}.$$

2.2.1 First Abelian Group Structure on $(\mathcal{P}(D), \Delta)$

Denote by $\triangle$ the *symmetric difference* of two sets in $\mathcal{P}(D)$:

$$A \triangle B \stackrel{\text{def}}{=} \left[A \cap \complement B\right] \cup \left[B \cap \complement A\right] = [A \cup B] \cap \left[\complement A \cup \complement B\right] = [A \cup B] \cap \complement\,[A \cap B]\,.$$

By definition, $\triangle$ is commutative and associative:

$$A \triangle B = B \triangle A, \quad (A \triangle B) \triangle C = A \triangle (B \triangle C).$$

$\varnothing$ is a *neutral element*:

$$A \triangle \varnothing = A.$$

An inverse must verify

$$\begin{aligned} A \triangle B = \varnothing \iff & \left[A \cap \complement B\right] \cup \left[B \cap \complement A\right] = \varnothing \\ \iff & B \cap \complement A = \varnothing = A \cap \complement B \iff A = B. \end{aligned}$$

Therefore $A \triangle A = \varnothing$, every element A of $\mathcal{P}(D)$ is its own inverse, $A^{-1} = A$, and $(\mathcal{P}(D), \triangle)$ is an Abelian group. This yields a kind of *triangle inequality*:

$$(A \triangle B) \triangle (B \triangle C) = A \triangle C.$$

The symmetric difference of A and C is contained in the union of the symmetric difference of A and B and that of B and C. (But note that for the diameter of the symmetric difference the triangle inequality does not hold.)

Because every element in this group is its own inverse, $(\mathcal{P}(D), \triangle)$ is in fact a *vector space* over the field $\mathbb{Z}_2$ with two elements.

Finally, the *intersection* distributes over the *symmetric difference*:

$$A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C).$$

Hence the power set $\mathcal{P}(D)$ becomes a *ring* with *symmetric difference* as *addition* and *intersection* as *multiplication*. It is the prototypical example of a *Boolean ring*.

For operations involving the complement

$$\begin{gathered} \complement A \triangle \complement B = A \triangle B, \\ \complement(A \triangle B) = [A \cap B] \cup [\complement A \cap \complement B] = B \triangle \complement A = A \triangle \complement B \\ \Rightarrow A \triangle \mathbf{R}^{\mathrm{N}} = \complement A, \quad A \triangle \complement A = \mathbf{R}^{\mathrm{N}}. \end{gathered}$$

For operations involving the closure

$$\begin{gathered} \partial A = \complement\left(\overline{A} \triangle \overline{\complement A}\right) = \overline{A} \triangle \complement\overline{\complement A} = \complement\overline{A} \triangle \overline{\complement A} \\ \Rightarrow \partial A \triangle \complement\overline{\complement A} = \overline{A} \quad \text{and} \quad \partial A \triangle \complement\overline{A} = \overline{\complement A}. \end{gathered}$$

2.2.2 Second Abelian Group Structure on $(\mathcal{P}(D), \nabla)$

If for some reason $\varnothing$ is not acceptable as the *multiplicative neutral element*, $\mathbf{R}^{\mathrm{N}}$ can play that role with the following new multiplication:

$$A \triangledown B \overset{\text{def}}{=} \complement\left([A \cap \complement B] \cup [B \cap \complement A]\right) = \complement\left([A \cup B] \cap \complement[A \cap B]\right) = \complement[A \cup B] \cup [A \cap B].$$

By definition

$$A \triangledown B = B \triangledown A, \quad A \triangledown \mathbf{R}^{\mathrm{N}} = A.$$

A multiplicative inverse must verify

$$\begin{aligned} A \triangledown B = \mathbf{R}^{\mathrm{N}} \quad &\Leftrightarrow \quad [A \cap \complement B] \cup [B \cap \complement A] = \varnothing \\ \Leftrightarrow \quad B \cap \complement A = \varnothing = A \cap \complement B \quad &\Leftrightarrow \quad A = B. \end{aligned}$$

Therefore $A^{-1} = A$ and $(\mathcal{P}(D), \triangledown)$ are an Abelian group.

When $D = \mathbf{R}^{\mathrm{N}}$, some identities are interesting:

$$\begin{gathered} \complement A \triangledown \complement B = A \triangledown B, \\ A \triangledown \complement B = \complement\left([A \cap B] \cup [\complement B \cap \complement A]\right) = \complement(A \triangledown B) = \complement A \triangledown B, \\ \Rightarrow A \triangledown \varnothing = \complement A, \quad A \triangledown \complement A = \varnothing. \end{gathered}$$

2.3 Connected Space, Path-Connected Space, and Geodesic Distance

Definition 2.1. (i) A *connected space* is a topological space which cannot be represented as the disjoint union of two or more nonempty open subsets.

(ii) A topological space X is said to be *path-connected* (or *pathwise connected* or 0-*connected*) if for any two points x and y in X there exists a continuous function f from the unit interval $[0, 1]$ to X with $f(0) = x$ and $f(1) = y$. (This function is called a *path* in X from x to y.)

(iii) Given a path-connected subset S of $\mathbf{R}^{\mathrm{N}}$, the *geodesic distance* $\mathrm{dist}_S(x, y)$ between two points x and y of S is the infimum of the lengths of the paths in S joining x and y. □

The definitions of a connected space and a path-connected space are compatible with the *convention* that the empty set $\varnothing$ be both connected and path-connected.

Theorem 2.1.

(i) *The closure of a connected set is connected. But the closure of a path-connected set need not be path-connected.*

(ii) *Every path-connected space is connected. But a connected space need not be path-connected.*

(iii) *Y is path-connected if and only if it is connected, and each $y \in Y$ has a path-connected neighborhood. In particular, open subsets of $\mathbf{R}^{\mathrm{N}}$ are connected if and only if they are path-connected.*

Proof. Cf. J. DUGUNDJI [**1**]: (i) Thm. 1.4 and Ex. 4, p. 108. (ii) Thm. 5.3, p. 115. The topologist's sine curve $\{(x, y) : y = \sin 1/x,\ x > 0\} \cup \{(0, y) : |y| \leq 1\}$ in $\mathbf{R}^2$ is an example of a connected space that is not path-connected. (iii) Thm. 5.5, p. 116, Cor. 5.6, p. 116. □

Remark 2.1.
Subsets of the real line $\mathbf{R}$ are connected if and only if they are path-connected; these subsets are the intervals of $\mathbf{R}$. □

Definition 2.2.
Given a subset A of $\mathbf{R}^{\mathrm{N}}$,

$$\#(A) \stackrel{\text{def}}{=} \begin{cases} \text{number of connected components of } A, & \text{if } A \neq \varnothing, \\ 0, & \text{if } A = \varnothing. \end{cases}$$

We say that the set A is *hole-free* if $\complement A$ is connected or $A = \mathbf{R}^{\mathrm{N}}$. □

If we adopt the *convention* that the empty set $\varnothing$ is both connected and path-connected, then a set A is connected if and only if its complement $\complement A$ is hole-free.

Theorem 2.2. *Let A and Ω be subsets of $\mathbf{R}^{\mathrm{N}}$.*

(i) *A set $A \neq \varnothing$ is connected if and only if $\#(A) = 1$; a set A is hole-free if and only if $\#(\complement A) \leq 1$.*

(ii) *An open set $\Omega \neq \varnothing$ is path-connected if and only if $\#(\Omega) = 1$; an open set Ω is hole-free if and only if $\#(\complement\Omega) \leq 1$.*

Remark 2.2.
Observe that for a nonempty open subset Ω of an open holdall D in $\mathbf{R}^{\mathrm{N}}$, $\#(\complement\Omega \cap D)$ can be strictly greater than $\#(\complement\Omega)$. As an example, let D be the open square of the side equal to 2 centered in $(0, 0)$ and Ω be the open ball of center $(0, 0)$ and radius 1: $\#(\complement\Omega \cap D) = 4 > 1 = \#(\complement\Omega)$. □

2.4 Bouligand's Contingent Cone, Dual Cone, and Normal Cone

Definition 2.3.
Let $\varnothing \neq \Omega \subset \mathbf{R}^{\mathrm{N}}$ and let $x \in \overline{\Omega}$. The *dual cone* associated with Ω is defined as

$$\Omega^* \stackrel{\text{def}}{=} \left\{x^* \in \mathbf{R}^{\mathrm{N}} : \forall x \in \Omega, \quad x^* \cdot x \geq 0\right\}. \tag{2.2}$$

Ω^* is a closed convex cone in 0. □

Definition 2.4.
Let $\varnothing \neq \Omega \subset \mathbf{R}^{\mathrm{N}}$ and let $x \in \overline{\Omega}$.

(i) The vector $h \in \mathbf{R}^{\mathrm{N}}$ is an *admissible direction* for Ω in x if there exists a sequence $\{\varepsilon_n > 0\}$, $\varepsilon_n \searrow 0$ as $n \to \infty$, such that

$$\forall n, \quad \exists x_n \in \Omega \text{ and } \lim_{n\to\infty} \frac{x_n - x}{\varepsilon_n} = h. \tag{2.3}$$

$T_x\Omega$ will denote the set of all admissible directions for Ω in x and it will be referred to as *Bouligand's contingent cone*[1] to Ω in x.

(ii) The dual cone $(-T_x\Omega)^*$ associated with Ω and a point $x \in \overline{\Omega}$ will be referred to as the *normal cone* to Ω in $x \in \overline{\Omega}$. □

In general, $T_x\Omega$ is a closed cone in 0. If Ω is convex, then $T_x\Omega = \overline{\mathbf{R}(\Omega - x)}$ is a closed convex cone in 0. When $x \in \operatorname{int}\Omega$, $T_\Omega(x) = \mathbf{R}^{\mathrm{N}}$.

2.5 Sobolev Spaces

2.5.1 Definitions

For a detailed analysis of Sobolev spaces the reader is referred to R. A. Adams [**1**], J. Nečas [**1**], and Vo-Khac Khoan [**1**]. Let Ω be a bounded open subset of $\mathbf{R}^{\mathrm{N}}$. $\mathcal{D}(\Omega)$ is the space of infinitely continuously differentiable functions with compact support in Ω endowed with Schwartz's topology. Its topological dual $\mathcal{D}(\Omega)^*$ is called the *space of distributions*. Denote by $\mathbb{N}^N$ the set of all N-tuples $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^N$. An element of $\mathbb{N}^N$ will be called a *multi-index*. For each $\alpha \in \mathbb{N}^N$, define the *order* $|\alpha|$ of α and the *partial derivative* ∂^α as follows:

$$|\alpha| = \sum_{i=1}^{N} \alpha_i, \quad \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}}. \tag{2.4}$$

The Sobolev space $W^{m,p}(\Omega)$, $1 \le p \le \infty$, $m \in \mathbb{N}$, is the space of all distributions $T \in L^p(\Omega)^*$ with distributional partial derivatives $D^\alpha T \in L^p(\Omega)^*$ for all α, $|\alpha| \le m$. For $p = 2$ we shall use the notation $H^m(\Omega) = W^{m,2}(\Omega)$. By density of $\mathcal{D}(\Omega)$ in $L^p(\Omega)$ for $1 \le p < \infty$, we shall often identify a distribution $T \in L^p(\Omega)^*$ with a function $f \in L^p(\Omega)$, $p^{-1} + q^{-1} = 1$:

$$\forall \varphi \in \mathcal{D}(\Omega), \quad \langle T, \varphi \rangle = \int_\Omega f\, \varphi\, dx.$$

Endowed with the norm

$$\|v\|_{m,p,\Omega} \stackrel{\text{def}}{=} \left[\sum_{|\alpha| \le m} \|\partial^\alpha v\|^p_{L^p(\Omega)}\, dx \right]^{1/p}, \quad \|v\|_{m,\infty,\Omega} \stackrel{\text{def}}{=} \max_{|\alpha|\le m} \|\partial^\alpha v\|_{L^\infty(\Omega)}\, dx \tag{2.5}$$

[1] The *contingent cone* $T_\Omega(x)$ was introduced in the 1930s by G. Bouligand [**1**]. It naturally occurs in the *viability theory* of differential equations (cf. M. Nagumo [**1**] or J.-P. Aubin and A. Cellina [**1**, p. 174 and p. 180]).

the space $W^{m,p}(\Omega)$ is a Banach space. We shall also use the seminorm

$$|v|_{m,p,\Omega} = \left[\sum_{|\alpha|=m} \int_\Omega |\partial^\alpha v|^p_{L^p(\Omega)}\, dx\right]^{1/p}, \quad |v|_{m,\infty,\Omega} \overset{\text{def}}{=} \|\partial^m v\|_{L^\infty(\Omega)}\, dx. \tag{2.6}$$

When $p=2$ we drop the index p and write $\|v\|_{m,\Omega}$ and $|v|_{m,\Omega}$. By the Meyers and Serrin theorem $W^{m,p}(\Omega)$ coincides with the completion of $\{\varphi \in C^m(\Omega) : \|\varphi\|_{m,p} < \infty\}$ with respect to the norm $\|\varphi\|_{m,p}$ for $1 \le p < \infty$.[2]

The reader is referred to R. A. Adams [**1**] for details and extension of this definition to the case where m is not an integer. When f is a vector function from Ω to $\mathbf{R}^m$, the corresponding spaces will be denoted by $W^{s,p}(\Omega)^m$ or $W^{s,p}(\Omega, \mathbf{R}^m)$.

2.5.2 The Space $W_0^{m,p}(\Omega)$

Define the Sobolev space

$$W_0^{m,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{m,p}}, \tag{2.7}$$

where the closure is with respect to the norm $\|\cdot\|_{m,p,\Omega}$ of (2.5). When Ω is bounded and $1 \le p < \infty$, there exists a constant $C(\operatorname{diam}\Omega, m, p)$ such that

$$\forall\, v \in W_0^{m,p}(\Omega), \quad \|v\|_{m,p,\Omega} \le C(\operatorname{diam}\Omega, m, p)\ |v|_{m,p,\Omega} \tag{2.8}$$

and the seminorm $|\cdot|_{m,p,\Omega}$ is a norm on the space $W_0^{m,p}(\Omega)$ equivalent to the norm $\|\cdot\|_{m,p,\Omega}$ (cf. W. P. Ziemer [**1**, Thm. 4.4.1, p. 188]). We have the chain of continuous embeddings

$$W_0^{m,p}(\Omega) \to W^{m,p}(\Omega) \to L^p(\Omega).$$

For $p=2$, $m \ge 1$, and a bounded open Ω, the injection of $H_0^m(\Omega)$ into $H_0^{m-1}(\Omega)$ is compact (cf. R. A. Adams [**1**, Rellich–Kondrachov compactness Theorem 6.2, p. 144] for other embedding theorems).

Now assume that the set Ω is *Lipschitzian* in the sense of Definition 5.2 (ii) in section 5.2. There exists a constant $C(\Omega)$ such as

$$\forall\, v \in C^\infty(\overline{\Omega}), \quad \|v\|_{L^2(\Gamma)} \le C(\Omega)\, \|v\|_{1,\Omega}. \tag{2.9}$$

But, by Theorem 6.3 in section 6.1 $\overline{C^\infty(\overline{\Omega})} = H^1(\Omega)$, where the closure is with respect to the norm$\|\cdot\|_{1,\Omega}$. Hence there exists a continuous linear map

$$v \in H^1(\Omega) \mapsto v \in L^2(\Gamma)$$

that will be referred to as the *trace operator*. From this we get the following characterization:

$$H_0^1(\Omega) = \left\{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma\right\}. \tag{2.10}$$

[2] $C^m(\Omega)$ is defined in section 2.6.1.

Since the exterior unit normal $\nu = (\nu_1, \ldots, \nu_N)$ exists almost everywhere along Γ, the *normal derivative operator*

$$\frac{\partial}{\partial \nu} = \sum_{i=1}^{N} \nu_i \frac{\partial}{\partial x_i} \tag{2.11}$$

is also defined almost everywhere along Γ and the following characterization also holds:

$$H_0^2(\Omega) = \left\{ v \in H^2(\Omega) : v = 0 \text{ and } \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma \right\}. \tag{2.12}$$

Given two functions u, v in $H^1(\Omega)$, we get the following *Green formula*:

$$\int_\Omega u \frac{\partial v}{\partial x_i} dx = -\int_\Omega \frac{\partial u}{\partial x_i} v\, dx + \int_\Gamma uv\, \nu_i\, d\Gamma \tag{2.13}$$

is true for $1 \leq i \leq N$. From that formula, other Green formulae can be obtained. For instance, by replacing u by $\partial u / \partial x_i$ and summing from 1 to N, we get

$$\int_\Omega \nabla u \cdot \nabla v\, dx = \int_\Omega \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx = -\int_\Omega \Delta u\, v\, dx + \int_\Gamma \frac{\partial u}{\partial \nu} v\, d\Gamma \tag{2.14}$$

for all $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$.

2.5.3 Embedding of $H_0^1(\Omega)$ into $H_0^1(D)$

Let Ω and D be two open subsets of $\mathbf{R}^{\mathrm{N}}$ such that $\Omega \subset D$. Denote by $e_0(\varphi)$ the extension by zero of an element φ of $\mathcal{D}(\Omega)$ to D and consider the linear injection

$$\varphi \mapsto e_0(\varphi) : \mathcal{D}(\Omega) \to \mathcal{D}(D).$$

By definition, for $k \geq 1$,

$$\|\varphi\|_{H^k(\Omega)} = \|e_0(\varphi)\|_{H^k(D)}$$

and e_0 extends by continuity and density to a linear isometric map

$$\boxed{e_0 : H_0^k(\Omega) \overset{\text{def}}{=} \overline{\mathcal{D}(\Omega)}^{H^k} \to H_0^k(D) \overset{\text{def}}{=} \overline{\mathcal{D}(D)}^{H^k}.}$$

Denote by $H_0^k(\Omega; D)$ the image of $H_0^k(\Omega)$ by e_0.

Theorem 2.3. *Let $\Omega \subset D$ be two open subsets of $\mathbf{R}^{\mathrm{N}}$ such that D is bounded. The linear subspace $H_0^k(\Omega; D)$ of $H_0^k(D)$ is closed and isometrically isomorphic to $H_0^k(\Omega)$ with the following properties: for all $\psi \in H_0^k(\Omega; D)$*

$$\psi|_\Omega \in H_0^k(\Omega) \text{ and } \forall \alpha,\ |\alpha| \leq k, \quad \partial^\alpha \psi = 0 \text{ a.e. in } D \backslash \Omega. \tag{2.15}$$

Any convergent sequence in $H_0^k(\Omega)$-weak converges in $H_0^{k-1}(\Omega)$-strong.

Proof. (i) It is sufficient to prove that $H_0^k(\Omega; D)$ is closed. The other properties are easy to check. Pick a sequence $\{\varphi_n\}$ in $H_0^k(\Omega)$ such that $\{e_0(\varphi_n)\}$ is Cauchy in $H_0^k(D)$ and denote by Φ its limit in $H_0^k(D)$. Since e_0 is an isometry, then $\{\varphi_n\}$ is also Cauchy in $H_0^k(\Omega)$. Denote by φ its limit in $H_0^k(\Omega)$. Then

$$\begin{aligned}\|\Phi - e_0(\varphi)\|_{H^k(D)} &\leq \|\Phi - e_0(\varphi_n)\|_{H^k(D)} + \|e_0(\varphi_n - \varphi)\|_{H^k(D)} \\ &= \|\Phi - e_0(\varphi_n)\|_{H^k(D)} + \|\varphi_n - \varphi\|_{H^k(\Omega)} \to 0\end{aligned}$$

and there exists $\varphi \in H_0^k(\Omega)$ such that $\Phi = e_0(\varphi)$ in $H_0^k(D)$. Hence $\Phi \in H_0^k(\Omega; D)$.

(ii) Since Ω is bounded, there exists a sufficiently large open ball B such that $\Omega \subset B$. By the embedding of $H_0^k(\Omega)$ into $H_0^k(B)$, if a sequence φ_n converges to φ in $H_0^k(\Omega)$-weak, then $e_\Omega(\varphi_n)$ converges to $e_\Omega(\varphi)$ in $H_0^k(B)$-weak. Since the ball is sufficiently smooth, by Rellich's theorem, the sequence converges in $H_0^{k-1}(B)$-strong, and in view of the linear isometric isomorphism, φ_n converges to φ in $H_0^{k-1}(\Omega)$-strong. □

2.5.4 Projection Operator

Given $u \in H_0^1(D)$, let $y = y(\Omega) \in H_0^1(\Omega; D)$ be the solution of the variational problem

$$\forall \varphi \in H_0^1(\Omega; D), \quad \int_D \nabla y \cdot \nabla \varphi \, dx = \int_D \nabla u \cdot \nabla \varphi \, dx.$$

The projection operator P_Ω of $H_0^1(D)$ onto $H_0^1(\Omega)$ or $H_0^1(\Omega; D)$ is the mapping

$$u \mapsto P_\Omega u \stackrel{\text{def}}{=} y(\Omega) : H_0^1(D) \to H_0^1(\Omega; D)$$

that is linear and continuous since

$$\|\nabla P_\Omega u\|_{L^2(D)} \leq \|\nabla u\|_{L^2(D)}.$$

A fundamental issue will be the continuity of the projection P_Ω with respect to Ω.

2.6 Spaces of Continuous and Differentiable Functions

2.6.1 Continuous and C^k Functions

Let Ω be an open subset of $\mathbf{R}^N$. Denote by $C(\Omega)$ or $C^0(\Omega)$ the space of continuous functions from Ω to $\mathbf{R}$. For a multi-index $\alpha \in \mathbb{N}^N$, let $|\alpha|$ be the *order* of α and ∂^α be the *partial derivative* as defined in (2.4) of section 2.5.1. For an integer $k \geq 1$,

$$C^k(\Omega) \stackrel{\text{def}}{=} \left\{ f \in C^{k-1}(\Omega) : \partial^\alpha f \in C(\Omega), \forall \alpha, |\alpha| = k \right\}.$$

By convention $\partial^0 f$ will be the function f in order to make sense of the case $\alpha = 0$. When $|\alpha| = 1$, we also use the standard notation $\partial_i f$ or $\partial f / \partial x_i$. $\mathcal{D}^k(\Omega)$ or $C_c^k(\Omega)$ (resp., $\mathcal{D}(\Omega)$ or $C_c^\infty(\Omega)$) will denote the space of all k-times (resp., infinitely) continuously differentiable functions with compact support contained in the open set Ω.

Denote by $\mathcal{B}^0(\Omega)$ the space of bounded continuous functions from Ω to $\mathbf{R}$, and, for an integer $k \geq 1$, the space

$$\mathcal{B}^k(\Omega) \stackrel{\text{def}}{=} \left\{ f \in \mathcal{B}^{k-1}(\Omega) : \partial^\alpha f \in \mathcal{B}^0(\Omega), \forall \alpha, |\alpha| = k \right\},$$

that is, the space of all functions in $\mathcal{B}^0(\Omega)$ whose derivatives of order less than or equal to k are continuous and bounded in Ω. Endowed with the norm

$$\|f\|_{C^k(\Omega)} \stackrel{\text{def}}{=} \max_{0 \leq |\alpha| \leq k} \sup_{x \in \Omega} |\partial^\alpha f(x)|, \tag{2.16}$$

$\mathcal{B}^k(\Omega)$ is a Banach space.

When D is open but not necessarily bounded, the space $C^0(D)$ of continuous functions on D is endowed with the Fréchet topology of uniform convergence on compact subsets K of D, which is defined by the family of seminorms

$$\forall K \text{ compact} \subset D, \quad q_K(f) \stackrel{\text{def}}{=} \max_{x \in K} |f(x)|. \tag{2.17}$$

It is metrizable since the topology induced by the family of seminorms $\{q_K\}$ is equivalent to the one generated by the subfamily $\{q_{K_k}\}_{k \geq 1}$, where the compact sets $\{K_k\}_{k \geq 1}$ are chosen as follows:

$$K_k \stackrel{\text{def}}{=} \left\{ x \in D : d_{\complement D}(x) \geq \frac{1}{k} \text{ and } |x| \leq k \right\}, \quad k \geq 1 \tag{2.18}$$

(cf., for instance, J. Horváth [**1**, Ex. 3, p. 116]). Thus the Fréchet topology on $C^0(D)$ is equivalent to the topology defined by the metric

$$\delta(f, g) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{q_{K_k}(f-g)}{1 + q_{K_k}(f-g)}. \tag{2.19}$$

When $C^0(D)$ is endowed with that topology, it will be denoted by $C^0_{\text{loc}}(D)$.

If a function f is bounded and uniformly continuous[3] on Ω, it possesses a unique, continuous extension to the closure $\overline{\Omega}$ of Ω. Denote by $C^k(\overline{\Omega})$ the space of functions f in $C^k(\Omega)$ for which $\partial^\alpha f$ is bounded and uniformly continuous on Ω for all α, $0 \leq |\alpha| \leq k$. A function f in $C^k(\Omega)$ is said to *vanish at the boundary* of Ω if for every α, $0 \leq |\alpha| \leq k$, and $\varepsilon > 0$ there exists a compact subset K of Ω such that, for all $x \in \Omega \cap \complement K$, $|\partial^\alpha f(x)| \leq \varepsilon$. Denote by $C^k_0(\Omega)$ the space of all such functions. Clearly $C^k_0(\Omega) \subset C^k(\overline{\Omega}) \subset \mathcal{B}^k(\Omega) \subset C^k(\Omega)$. Endowed with the norm (2.16), $C^k_0(\Omega)$, $C^k(\overline{\Omega})$, and $\mathcal{B}^k(\Omega)$ are Banach spaces. Finally

$$C^\infty(\Omega) \stackrel{\text{def}}{=} \bigcap_{k \geq 0} C^k(\Omega), \quad C^\infty_0(\Omega) \stackrel{\text{def}}{=} \bigcap_{k \geq 0} C^k_0(\Omega), \quad \text{and } \mathcal{B}(\Omega) \stackrel{\text{def}}{=} \bigcap_{k \geq 0} \mathcal{B}^k(\Omega).$$

[3]A function $f : \Omega \to \mathbf{R}$ is uniformly continuous if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all x and y in Ω such that $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$.

When f is a vector function from Ω to $\mathbf{R}^m$, the corresponding spaces will be denoted by $C_0^k(\Omega)^m$ or $C_0^k(\Omega, \mathbf{R}^m)$, $C^k(\overline{\Omega})^m$ or $C^k(\overline{\Omega}, \mathbf{R}^m)$, $\mathcal{B}^k(\Omega)^m$ or $\mathcal{B}^k(\Omega, \mathbf{R}^m)$, $C^k(\Omega)^m$ or $C^k(\Omega, \mathbf{R}^m)$, etc.

We quote the following classical compactness theorem.

Theorem 2.4 (Ascoli–Arzelà theorem)**.** *Let Ω be a bounded open subset of $\mathbf{R}^{\mathrm{N}}$. A subset $\mathcal{K}$ of $C(\overline{\Omega})$ is precompact in $C(\overline{\Omega})$ provided the following two conditions hold:*

(i) *there exists a constant M such that for all $f \in \mathcal{K}$ and $x \in \Omega$, $|f(x)| \le M$;*

(ii) *for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $f \in \mathcal{K}$, x, y in Ω, and $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.*

2.6.2 Hölder ($C^{0,\ell}$) and Lipschitz ($C^{0,1}$) Continuous Functions

Given λ, $0 < \lambda \le 1$, a function f is $(0, \lambda)$-*Hölder continuous* in Ω if

$$\exists c > 0,\, \forall x,\, y \in \Omega, \quad |f(y) - f(x)| \le c\, |x - y|^\lambda.$$

When $\lambda = 1$, we also say that f is *Lipschitz* or *Lipschitz continuous*. Similarly for $k \ge 1$, f is (k, λ)-*Hölder continuous* in Ω if

$$\forall \alpha,\, 0 \le |\alpha| \le k,\, \exists c > 0,\, \forall x,\, y \in \Omega, \quad |\partial^\alpha f(y) - \partial^\alpha f(x)| \le c\, |x - y|^\lambda.$$

Denote by $C^{k,\lambda}(\Omega)$ the space of all $(k.\lambda)$-Hölder continuous functions on Ω. Define for $k \ge 0$ the subspaces[4]

$$C^{k,\lambda}(\overline{\Omega}) \stackrel{\text{def}}{=} \left\{ f \in C^k(\overline{\Omega}) \;:\; \begin{array}{c} \forall \alpha,\, 0 \le |\alpha| \le k,\, \exists c > 0,\, \forall x,\, y \in \Omega \\ |\partial^\alpha f(y) - \partial^\alpha f(x)| \le c\, |x - y|^\lambda \end{array} \right\} \tag{2.20}$$

of $C^k(\overline{\Omega})$. By definition for each α, $0 \le |\alpha| \le k$, $\partial^\alpha f$ has a unique, bounded, continuous extension to $\overline{\Omega}$. Endowed with the norm

$$\|f\|_{C^{k,\lambda}(\Omega)} \stackrel{\text{def}}{=} \max \left\{ \|f\|_{C^k(\Omega)},\, \max_{0 \le |\alpha| \le k} \sup_{\substack{x,y \in \Omega \\ x \ne y}} \frac{|\partial^\alpha f(y) - \partial^\alpha f(x)|}{|x - y|^\lambda} \right\} \tag{2.21}$$

$C^{k,\lambda}(\overline{\Omega})$ is a Banach space. Finally denote by $C_0^{k,\lambda}(\Omega)$ the space $C^{k,\lambda}(\overline{\Omega}) \cap C_0^k(\Omega)$.

2.6.3 Embedding Theorem

In general $C^{k+1}(\overline{\Omega}) \not\subset C^{k,1}(\overline{\Omega})$, but the inclusion is true for a large class of domains, including convex domains.[5] We first quote the following embedding theorem.

[4]The notation $C^{k,\lambda}(\overline{\Omega})$ should not be confused with the notation $C^{k,\lambda}(\Omega)$ for (k, λ)-*Hölder continuous* functions in Ω *without* the uniform boundedness assumption in Ω. In particular, $C^{k,\lambda}(\overline{\mathbf{R}^{\mathrm{N}}})$ is contained in but not equal to $C^{k,\lambda}(\mathbf{R}^{\mathrm{N}})$.

[5]For instance the convexity can be relaxed to the more general condition: there exists $M > 0$ such that for all x and y in Ω, there exists a path $\gamma_{x,y}$ in Ω (that is, a C^1 injective map from the interval $[0, 1]$ into Ω with $\gamma(0) = x$, $\gamma(1) = y$, and $\int_0^1 \|\gamma'(t)\|\, dt \le M\, |x - y|$).

Theorem 2.5 (R. A. ADAMS [**1**]). *Let $k \geq 0$ be an integer and $0 < \nu < \lambda \leq 1$ be real numbers. Then the following embeddings exist:*

$$C^{k+1}(\overline{\Omega}) \to C^{k}(\overline{\Omega}), \tag{2.22}$$

$$C^{k,\lambda}(\overline{\Omega}) \to C^{k}(\overline{\Omega}), \tag{2.23}$$

$$C^{k,\lambda}(\overline{\Omega}) \to C^{k,\nu}(\overline{\Omega}). \tag{2.24}$$

If Ω is bounded, then the embeddings (2.23) and (2.24) are compact. If Ω is convex, we have the further embeddings

$$C^{k+1}(\overline{\Omega}) \to C^{k,1}(\overline{\Omega}), \tag{2.25}$$

$$C^{k+1}(\overline{\Omega}) \to C^{k,\nu}(\overline{\Omega}). \tag{2.26}$$

If Ω is convex and bounded, then embeddings (2.22) and (2.26) are compact.

As a consequence of the second part of the theorem, the definition of $C^{k,\lambda}(\overline{\Omega})$ simplifies when Ω is convex:

$$\begin{aligned} C^{k,\lambda}(\overline{\Omega}) \overset{\text{def}}{=} \big\{ f \in \mathcal{B}^k(\Omega) \ : & \forall \alpha,\ |\alpha| = k,\ \exists c > 0,\ \forall x,\ y \in \Omega, \\ & |\partial^\alpha f(y) - \partial^\alpha f(x)| \leq c\,|x-y|^\lambda \big\}, \end{aligned} \tag{2.27}$$

and its norm is equivalent to the norm

$$\|f\|_{C^{k,\lambda}(\Omega)} \overset{\text{def}}{=} \|f\|_{C^k(\Omega)} + \max_{|\alpha|=k} \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|\partial^\alpha f(y) - \partial^\alpha f(x)|}{|x-y|^\lambda}. \tag{2.28}$$

When f is a vector function from Ω to $\mathbf{R}^m$, the corresponding spaces will be denoted by $C^{k,\lambda}(\overline{\Omega})^m$ or $C^{k,\lambda}(\overline{\Omega}, \mathbf{R}^m)$.

2.6.4 Identity $C^{k,1}(\overline{\Omega}) = W^{k+1,\infty}(\Omega)$: From Convex to Path-Connected Domains via the Geodesic Distance

By Rademacher's theorem (cf., for instance, L. C. EVANS and R. F. GARIEPY [**1**])

$$C^{k,1}(\overline{\Omega}) \subset W^{k+1,\infty}_{\text{loc}}(\Omega)$$

and, when Ω is convex,

$$C^{k,1}(\overline{\Omega}) = W^{k+1,\infty}(\Omega).$$

This last convexity assumption can be relaxed as follows. From Rademacher's theorem and the inequality (cf. H. BRÉZIS [**1**, p. 154])

$$\forall u \in W^{1,\infty}(\Omega) \text{ and } \forall x, y \in \Omega, \quad |u(x) - u(y)| \leq \|\nabla u\|_{L^\infty}(\Omega)\ \text{dist}_\Omega(x,y)$$

we have the following theorem.

Theorem 2.6. *Let Ω be a bounded, path-connected, open subset of $\mathbf{R}^{\mathrm{N}}$ such that*

$$\exists c, \forall x, y \in \overline{\Omega}, \quad \mathrm{dist}_\Omega(x, y) \leq c\,|x - y|. \tag{2.29}$$

Then we have $W^{1,\infty}(\Omega) = C^{0,1}(\overline{\Omega})$ algebraically and topologically, and there exists a constant c' such that

$$\forall u \in W^{1,\infty}(\Omega) \text{ and } \forall x, y \in \overline{\Omega}, \quad |u(x) - u(y)| \leq c' \, \|\nabla u\|_{L^\infty(\Omega)} |x - y|.$$

Hence for all integers $k \geq 0$, $W^{k+1,\infty}(\Omega) = C^{k,1}(\overline{\Omega})$ algebraically and topologically.

This is true for Lipschitzian domains (D. Gilbarg and N. S. Trudinger [**1**]).

Corollary 1. *Let Ω be a bounded, open, path-connected, and locally Lipschitzian*[6] *subset of $\mathbf{R}^{\mathrm{N}}$. Then property* (2.29) *and Theorem* 2.6 *are verified.*

Proof. By Theorem 5.8 later in section 5.4 of this chapter. □

3 Sets Locally Described by an Homeomorphism or a Diffeomorphism

Open domains in the Euclidean space $\mathbf{R}^{\mathrm{N}}$ are classically described by introducing at each point of their boundary a local diffeomorphism (that is, defined in a neighborhood of the point) that locally *flattens* the boundary. The smoothness of the boundary is determined by the smoothness of the local diffeomorphism. After introducing the main definition in section 3.1, we briefly recall the definition of an (embedded) *submanifold* of codimension greater than or equal to 1 in section 3.2 in order to make sense of the associated boundary integral and, more generally, the integral over smooth submanifolds of arbitrary dimension in $\mathbf{R}^{\mathrm{N}}$. The definition of the integral can be generalized by introducing the Hausdorff measures that extend the integration theory from smooth submanifolds to arbitrary Hausdorff measurable subsets of $\mathbf{R}^{\mathrm{N}}$, thus making the writing of the integral completely independent of the choice of the local diffeomorphisms. Section 3.3 completes the section by defining the fundamental forms and curvatures for a smooth submanifold of $\mathbf{R}^{\mathrm{N}}$ of codimension 1.

3.1 Sets of Classes C^k and $C^{k,\ell}$

Let $\{e_1, \dots, e_N\}$ be the standard unit orthonormal basis in $\mathbf{R}^{\mathrm{N}}$. We use the notation $\zeta = (\zeta', \zeta_N)$ for a point $\zeta = (\zeta_1, \dots, \zeta_N)$ in $\mathbf{R}^{\mathrm{N}}$, where $\zeta' = (\zeta_1, \dots, \zeta_{N-1})$. Denote by B the open unit ball in $\mathbf{R}^{\mathrm{N}}$ and define the sets

$$B_0 \stackrel{\text{def}}{=} \{\zeta \in B : \zeta_N = 0\}, \tag{3.1}$$

$$B_+ \stackrel{\text{def}}{=} \{\zeta \in B : \zeta_N > 0\}, \quad B_- \stackrel{\text{def}}{=} \{\zeta \in B : \zeta_N < 0\}. \tag{3.2}$$

The main elements entering Definition 3.1 are illustrated in Figure 2.1 for $N = 2$.

[6] Cf. Definition 5.2 of section 5 and Theorem 5.3, which says that a locally Lipschitzian domain is equi-Lipschitzian when $\partial\Omega$ is bounded.

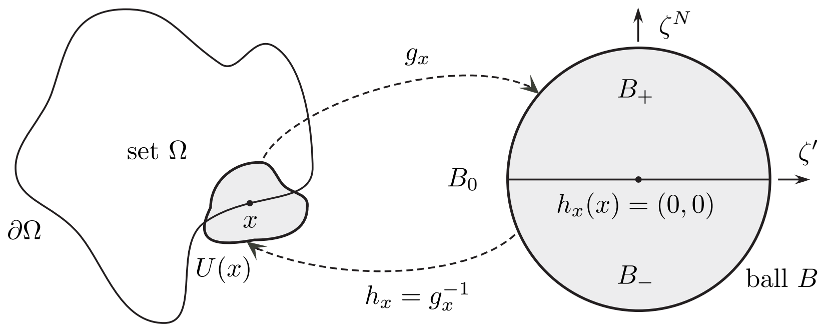


Figure 2.1. *Diffeomorphism g_x from $U(x)$ to B.*

Definition 3.1.
Let Ω be a subset of $\mathbf{R}^N$ such that $\partial\Omega \neq \varnothing$, $0 \leq k$ be an integer or $+\infty$, and $0 \leq \ell \leq 1$ be a real number.

(i) - Ω is said to be *locally* of class C^k at $x \in \partial\Omega$ if there exist

(a) a neighborhood $U(x)$ of x, and

(b) a bijective map $g_x : U(x) \to B$ with the following properties:

$$g_x \in C^k\big(U(x); B\big), \quad h_x \stackrel{\text{def}}{=} g_x^{-1} \in C^k\big(B; U(x)\big), \tag{3.3}$$

$$\operatorname{int}\Omega \cap U(x) = h_x(B_+), \tag{3.4}$$

$$\Gamma_x \stackrel{\text{def}}{=} \partial\Omega \cap U(x) = h_x(B_0), \quad B_0 = g_x(\Gamma_x). \tag{3.5}$$

- Given $0 < \ell < 1$, Ω is said to be *locally (k,ℓ)-Hölderian at $x \in \partial\Omega$* if conditions (a) and (b) are satisfied with a map $g_x \in C^{k,\ell}\big(U(x), B\big)$ with inverse $h_x = g_x^{-1} \in C^{k,\ell}\big(B, U(x)\big)$.

- Ω is said to be *locally k-Lipschitzian at $x \in \partial\Omega$* if conditions (a) and (b) are satisfied with a map $g_x \in C^{k,1}\big(U(x), B\big)$ with inverse $h_x = g_x^{-1} \in C^{k,1}\big(B, U(x)\big)$.

(ii) - Ω is said to be *locally of class C^k* if, for each $x \in \partial\Omega$, Ω is locally of *class* C^k at x.

- Given $0 < \ell < 1$, Ω is said to be *locally (k,ℓ)-Hölderian* if, for each $x \in \partial\Omega$, Ω is locally (k,ℓ)-Hölderian at x.

- Ω is said to be *locally k-Lipschitzian* if, for each $x \in \partial\Omega$, Ω is locally k-Lipschitzian at x. □

Remark 3.1.
By definition $\operatorname{int}\Omega \neq \varnothing$ and $\operatorname{int}\complement\Omega \neq \varnothing$. The above definitions are usually given for an open set Ω called a *domain*. This terminology naturally arises in partial differential equations, where this open set is indeed the *domain* in which the solution

of the partial differential equation is defined.[7] At this point it is not necessary to assume that the set Ω is open. The family of diffeomorphisms $\{g_x : x \in \partial\Omega\}$ characterizes the equivalence class of sets

$$[\Omega] = \{A \subset \mathbf{R}^{\mathrm{N}} : \operatorname{int} A = \operatorname{int} \Omega \text{ and } \partial A = \partial\Omega\}$$

with the same interior and boundary. The sets $\operatorname{int}\Omega$ and $\partial\Omega$ are invariants for the equivalence class $[\Omega]$ of sets of class $C^{k,\ell}$. The notation Γ for $\partial\Omega$ and the standard terminology *domain* for the unique (open) set $\operatorname{int}\Omega$ associated with the class $[\Omega]$ will be used. In what follows, a $C^{k,\ell}$-mapping g_x with $C^{k,\ell}$-inverse will be called a $C^{k,\ell}$-*diffeomorphism*. □

Consider the solution $y = y(\Omega)$ of the Dirichlet problem on the bounded open domain Ω:

$$-\Delta y = f \text{ in } \Omega, \quad y = g \text{ on } \partial\Omega. \tag{3.6}$$

When Ω is of class C^∞, the solution $y(\Omega)$ can be sought in any Sobolev space $H^m(\Omega)$, $m \geq 1$, by choosing sufficiently smooth data f and g and appropriate compatibility conditions. However, when Ω is only of class C^k, $1 \leq k < \infty$, m cannot be made arbitrary large by choosing smoother data f and g and appropriate compatibility conditions. The reader is referred to R. Dautray and J.-L. Lions [**1**, Chap. VII, sect. 3, pp. 1271–1304] for classical smoothness results of solutions to elliptic problems in domains of class C^k. It is important to understand that we shall consider shape problems involving the solution of a boundary value problem over domains with minimal smoothness. Since we know that, at least for elliptic problems, the smoothness of the solution depends not only on the smoothness of the data but also on the smoothness of the domain. This issue will be of paramount importance to make sure that the shape problems are well posed.

Remark 3.2.
We shall see in section 5 that domains that are locally the epigraph of a $C^{k,\ell}$, $k \geq 1$ (resp., Lipschitzian), function are of class $C^{k,\ell}$ (resp., $C^{0,1}$), but domains which are of class $C^{0,1}$ are not necessarily locally the epigraph of a Lipschitzian function (cf. Examples 5.1 and 5.2). Nevertheless globally $C^{0,1}$-mappings with a $C^{0,1}$ inverse are important since they transport L^p-functions onto L^p-functions and $W^{1,p}$-functions onto $W^{1,p}$-functions (cf. J. Nečas [**1**, Lems. 3.1 and 3.2, pp. 65–66]). □

For sets of class C^1, the unit exterior normal to the boundary $\Gamma = \partial\Omega$ can be characterized through the Jacobian matrices of g_x and h_x. By definition of B_0, $\{e_1, \ldots, e_{N-1}\} \subset B_0$ and the tangent space $T_y\Gamma$, $\Gamma = \partial\Omega$, at y to Γ_x is the vector space spanned by the $N-1$ vectors

$$\{Dh_x(\zeta', 0)e_i : 1 \leq i \leq N-1\}, \quad (\zeta', 0) = g_x(y) \in B_0, \tag{3.7}$$

[7] Classically for $k \geq 1$ a bounded open domain Ω in $\mathbf{R}^{\mathrm{N}}$ is said to be of class C^k if its boundary is a C^k-submanifold of $\mathbf{R}^{\mathrm{N}}$ of codimension 1 and Ω is located on one side of its boundary $\Gamma = \partial\Omega$ (cf. S. Agmon [**1**, Def. 9.2, p. 178]).

where $Dh_x(\zeta)$ is the Jacobian matrix of h_x at the point ζ:

$$(Dh_x)_{\ell m} \overset{\text{def}}{=} \partial_m (h_x)_\ell.$$

So from (3.7) a normal vector field to Γ_x at $y \in \Gamma_x$ is given by

$$m_x(y) = -\,{}^*(Dh_x)^{-1}(\zeta', 0)\, e_N = -\,{}^*Dg_x(y)\, e_N, \quad h_x(\zeta', 0) = y, \tag{3.8}$$

since

$$-m_x(y) \cdot Dh_x(\zeta', 0)\, e_i = e_N \cdot e_i = \delta_{iN}, \quad 1 \le i \le N.$$

Thus the *outward unit normal field* $n(y)$ at $y \in \Gamma_x$ is given by

$$\begin{aligned} \forall h_x(\zeta', 0) = y \in \Gamma_x, \quad n(y) &= -\frac{{}^*(Dh_x)^{-1}(\zeta', 0)\, e_N}{|\,{}^*(Dh_x)^{-1}(\zeta', 0)\, e_N|}, \\ \forall y \in \Gamma_x, \quad n(y) &= -\frac{{}^*(Dh_x)^{-1}(h_x^{-1}(y))\, e_N}{|\,{}^*(Dh_x)^{-1}(h_x^{-1}(y))\, e_N|}. \end{aligned} \tag{3.9}$$

It can be verified that n is uniquely defined on Γ by checking that for $y \in \Gamma_x \cap \Gamma_{x'}$, n is uniquely defined by (3.9).

3.2 Boundary Integral, Canonical Density, and Hausdorff Measures

3.2.1 Boundary Integral for Sets of Class C^1

The family of neighborhoods $U(x)$ associated with all the points x of Γ is an open cover of Γ. If $\Gamma = \partial\Omega$ is assumed to be compact, then there exists a finite open subcover: that is, there exists a finite sequence of points $\{x_j : 1 \le j \le m\}$ of Γ such that $\Gamma \subset U_1 \cup \dots \cup U_m$, where $U_j = U(x_j)$. For simplicity, index all the previous symbols by j instead of x_j.

The boundary integration on Γ is obtained by using a partition of unity $\{r_j : 1 \le j \le m\}$ for the family of open neighborhoods $\{U_j : 1 \le j \le m\}$ of Γ:

$$\begin{cases} r_j \in \mathcal{D}(U_j), \quad 0 \le r_j(x) \le 1, \\ \displaystyle\sum_{j=1}^m r_j(x) = 1 \text{ in a neighborhood } U \text{ of } \Gamma, \end{cases} \tag{3.10}$$

such that $\overline{U} \subset \cup_{j=1}^m U_j$, where $\mathcal{D}(U_j)$ is the set of all infinitely continuously differentiable functions with compact support in U_j. If $f \in C(\Gamma)$, then

$$(fr_j) \circ h_j \in C(B_0), \quad 1 \le j \le m. \tag{3.11}$$

Define the boundary integral of fr_j on Γ_j as

$$\int_{\Gamma_j} fr_j \, d\Gamma \overset{\text{def}}{=} \int_{B_0} (fr_j) \circ h_j(\zeta', 0)\, \omega_j(\zeta')\, d\zeta', \quad \Gamma_j = U(x_j) \cap \Gamma, \tag{3.12}$$

where $\omega_j = \omega_{x_j}$ and ω_x is the *density term*

$$\begin{aligned} \omega_x(\zeta') &= |m_x(h_x(\zeta',0))|\,|\det Dh_x(\zeta',0)|, \\ m_x(y) &= -\,{}^*(Dh_x)^{-1}(h_x^{-1}(y))\,e_N = -\,{}^*Dg_x(y)\,e_N. \end{aligned} \tag{3.13}$$

From this define the boundary integral of f on Γ as

$$\int_\Gamma f\,d\Gamma \stackrel{\text{def}}{=} \sum_{j=1}^m \int_{\Gamma_j} f r_j\,d\Gamma. \tag{3.14}$$

The expression on the right-hand side of (3.12) results from the parametrization of the boundary Γ_j by B_0 through the diffeomorphism h_j.

3.2.2 Integral on Submanifolds

In differential geometry there is a general procedure to define the canonical density for a d-dimensional submanifold V in $\mathbf{R}^N$ parametrized by a C^k-mapping, $k \geq 1$ (cf., for instance, M. Berger and B. Gostiaux [**1**, Def. 2.1.1, p. 48 (56) and Prop. 6.62, p. 214 (239)]).

Definition 3.2.
Let $\varnothing \neq S \subset \mathbf{R}^N$, let $k \geq 1$ and $1 \leq d \leq N$ be integers, and let $0 \leq \ell \leq 1$ be a real number.

(i) Given $x \in S$, S is said to be locally a C^k- (resp., $C^{k,\ell}$-) *submanifold* of dimension d at x in $\mathbf{R}^N$ if there exists an open subset $U(x)$ of $\mathbf{R}^N$ containing x and a C^k- (resp., $C^{k,\ell}$-) diffeomorphism g_x from $U(x)$ onto its open image $g_x\big(U(x)\big)$, such that

$$g_x\big(U(x) \cap S\big) = g_x\big(U(x)\big) \cap R^d,$$

where $R^d = \Big\{(x_1,\dots,x_d,0,\dots,0) \in \mathbf{R}^N : \forall (x_1,\dots,x_d) \in \mathbf{R}^d\Big\}$.

(ii) S is said to be a C^k- (resp., $C^{k,\ell}$-) *submanifold* of dimension d in $\mathbf{R}^N$ if, for each $x \in S$, it is locally a C^k- (resp., $C^{k,\ell}$-) *submanifold* of dimension d at x in $\mathbf{R}^N$. □

For $k \geq 1$ the canonical density ω_x on the submanifold S at a point $y \in U(x) \cap S$ is given by

$$\omega_x = \sqrt{|\det B|},$$

where the $d \times d$ matrix B is given by

$$(B)_{ij} = (Dh_x\,e_i) \cdot (Dh_x\,e_j),\ 1 \leq i,j \leq d, \quad h_x = g_x^{-1} \text{ on } g_x(U(x)).$$

In the special case $d = N-1$, it is easy to verify that

$${}^*Dh_x\,Dh_x = \begin{bmatrix} {}^*C\,C & {}^*C\,c \\ {}^*c\,C & {}^*c\,c \end{bmatrix}, \quad B = {}^*C\,C,$$

where C is the $(N \times (N-1))$-matrix and c is the N-vector defined by

$$C_{ij} = \{Dh_x\}_{ij}, \quad 1 \le i \le N, 1 \le j \le N-1, \quad c = Dh_x\, e_N.$$

Denote by $M(A)$ the matrix of cofactors associated with a matrix A: $M(A)_{ij}$ is equal to the determinant of the matrix obtained after deleting the ith row and the jth column times $(-1)^{i+j}$. Then $M(A) = (\det A)\, {}^*A^{-1}$, $M({}^*A) = {}^*M(A)$, and for two invertible matrices A_1 and A_2, $M(A_1A_2) = M(A_1)M(A_2)$. As a result

$$\det B = M({}^*Dh_x\, Dh_x)_{NN} = e_N \cdot M({}^*Dh_x\, Dh_x)e_N,$$

where $M({}^*Dh_x\, Dh_x)_{NN}$ is the NN-cofactor of the matrix ${}^*Dh_x\, Dh_x$. Then

$$\begin{aligned} M({}^*Dh_x\, Dh_x)_{NN} &= e_N \cdot M({}^*Dh_x\, Dh_x)e_N \\ &= e_N \cdot M({}^*Dh_x)\, M(Dh_x)e_N = |M(Dh_x)e_N|^2. \end{aligned}$$

In view of the previous considerations and (3.13)

$$\sqrt{\det B} = |M(Dh_x)e_N| = |\det Dh_x|\, |{}^*(Dh_x)^{-1}\, e_N| = \omega_x.$$

3.2.3 Hausdorff Measures

Definition 3.2 gives the classical construction of a d-dimensional *surface measure* on the boundary of a C^1-domain. In 1918, F. Hausdorff [**1**] introduced a d-dimensional measure in $\mathbf{R}^N$ which gives the same surface measure for smooth submanifolds but is defined on all measurable subsets of $\mathbf{R}^N$. When $d = N$, it is equal to the Lebesgue measure. To complete the discussion we quote the definition from F. Morgan [**1**, p. 8] or L. C. Evans and R. F. Gariepy [**1**, p. 60].

Definition 3.3.
For any subset S of $\mathbf{R}^N$, define the *diameter* of S as

$$\operatorname{diam}(S) \overset{\text{def}}{=} \sup\{|x-y| : x, y \in S\}.$$

Let $\alpha(d)$ denote the Lebesgue measure of the unit ball in $\mathbf{R}^d$. The d-dimensional Hausdorff measure $H_d(A)$ of a subset A of $\mathbf{R}^N$ is defined by the following process. For δ small, cover A efficiently by countably many sets S_j with $\operatorname{diam}(S_j) \le \delta$, add up all the terms

$$\alpha(d)\, (\operatorname{diam}(S_j)/2)^d,$$

and take the limit as $\delta \to 0$:

$$H_d(A) \overset{\text{def}}{=} \lim_{\delta \searrow 0} \inf_{\substack{A \subset \cup S_j \\ \operatorname{diam}(S_j) \le \delta}} \sum_j \alpha(d) \left(\frac{\operatorname{diam}(S_j)}{2} \right)^d,$$

where the infimum is taken over all countable covers $\{S_j\}$ of A whose members have diameter at most δ. $\square$

For $0 \le d < \infty$, H_d is a *Borel regular* measure, but not a *Radon* measure for $d < N$ since it is not necessarily finite on each compact subset of $\mathbf{R}^{\mathrm{N}}$. The *Hausdorff dimension* of a set $A \subset \mathbf{R}^{\mathrm{N}}$ is defined as

$$H_{dim}(A) \stackrel{\text{def}}{=} \inf\{0 \le s < \infty : H_s(A) = 0\}. \tag{3.15}$$

By definition $H_{dim}(A) \le N$ and

$$\forall k > H_{dim}(A), \quad H_k(A) = 0.$$

If a submanifold S of dimension d, $1 \le d < N$, of Definitions 3.2 and 3.3 is characterized by a single C^1-diffeomorphism g, that is,

$$g(S) = \mathbf{R}^d \text{ and } S = h(\mathbf{R}^d), \quad h = g^{-1},$$

then for any Lebesgue-measurable set $E \subset \mathbf{R}^d$

$$\int_E \omega\, dx = H_d(h(E)).$$

This is a generalization to submanifolds of codimension greater than 1 of formula (3.12).

The definition of the Hausdorff measure and the Hausdorff dimension extend from integers d to reals s, $0 \le s \le \infty$, by modifying Definition 3.3 as follows.

Definition 3.4.
For any real s, $0 \le s \le \infty$, the s-dimensional Hausdorff measure $H_s(A)$ of a subset A of $\mathbf{R}^{\mathrm{N}}$ is defined by the following process. For δ small, cover A efficiently by countably many sets S_j with $\operatorname{diam}(S_j) \le \delta$, and add all the terms

$$\alpha(s)\,(\operatorname{diam}(S_j)/2)^d\,,$$

where

$$\alpha(s) \stackrel{\text{def}}{=} \frac{\pi^{s/2}}{\Gamma(s/2+1)}, \quad \Gamma(t) \stackrel{\text{def}}{=} \int_0^\infty e^{-x}\, x^{t-1}\,. \tag{3.16}$$

Take the limit as $\delta \to 0$:

$$H_s(A) \stackrel{\text{def}}{=} \lim_{\delta \searrow 0} \inf_{\substack{A \subset \cup S_j \\ \operatorname{diam}(S_j) \le \delta}} \sum_j \alpha(s) \left(\frac{\operatorname{diam}(S_j)}{2}\right)^d,$$

where the infimum is taken over all countable covers $\{S_j\}$ of A whose members have diameter at most δ. The Hausdorff dimension is defined by the same formula (3.15). □

3.3 Fundamental Forms and Principal Curvatures

Consider a set Ω locally of class C^2 in $\mathbf{R}^{\mathrm{N}}$. Its boundary $\Gamma = \partial\Omega$ is an $(N-1)$-dimensional submanifold of $\mathbf{R}^{\mathrm{N}}$ of class C^2. At each point $x \in \Gamma$ there is a C^2-diffeomorphism g_x from a neighborhood $U(x)$ of x onto B. Denoting its inverse by

$h_x = g_x^{-1}$, the *covariant basis* at a point $y \in U(x) \cap \Gamma$ is defined as

$$a_\alpha(y) \overset{\text{def}}{=} \frac{\partial h_x}{\partial \zeta_\alpha}(\zeta', 0), \quad \alpha = 1, \dots, N-1, \quad h_x(\zeta', 0) = y,$$

and a_N is chosen as the *inward unit normal*:

$$a_N(y) \overset{\text{def}}{=} \frac{{}^*(Dh_x(h_x^{-1}(y)))^{-1} e_N}{|\,{}^*(Dh_x(h_x^{-1}(y)))^{-1} e_N|}.$$

The standard convention that the Greek indices range from 1 to $N-1$ and the Roman indices from 1 to N will be followed together with Einstein's rule of summation over repeated indices. The associated *contravariant basis* $\{a^i\} = \{a^i(y)\}$ is defined from the covariant one $\{a_i\} = \{a_i(y)\}$ as

$$a^i \cdot a_j = \delta_{ij},$$

where δ_{ij} is the Kronecker index function. The *first, second*, and *third fundamental forms* a, b, and c are defined as

$$a_{\alpha\beta} \overset{\text{def}}{=} a_\alpha \cdot a_\beta, \quad b_{\alpha\beta} \overset{\text{def}}{=} -a_\alpha \cdot a_{N,\beta}, \quad c_{\alpha\beta} \overset{\text{def}}{=} b_\alpha^\lambda \, b_{\lambda\beta},$$

where

$$a_{N,\beta} = \frac{\partial a_N}{\partial \zeta_\beta}, \quad b_\alpha^\lambda = a^\lambda \cdot a^\mu \, b_{\mu\alpha}.$$

The above definitions extend to sets of class $C^{1,1}$ for which $h_x \in C^{1,1}(B)$ and hence $h_x \in C^{1,1}(B_0)$. So by Rademacher's theorem in dimension $N-1$ (cf., for instance, L. C. Evans and R. F. Gariepy [**1**]), $h_x \in W^{2,\infty}(B_0)$ and the definitions of the second and third fundamental forms still make sense H_{N-1} almost everywhere on Γ. The eigenvalues of $b_{\alpha\beta}$ are the $(N-1)$ *principal curvatures* κ_i, $1 \le i \le N-1$, of the submanifold Γ. The *mean curvature* H and the *Gauss curvature* K are defined as

$$H \overset{\text{def}}{=} \frac{1}{N-1} \sum_{\alpha=1}^{N-1} \kappa_\alpha \quad \text{and} \quad K \overset{\text{def}}{=} \prod_{\alpha=1}^{N-1} \kappa_\alpha.$$

The choice of the *inner normal* for a_N is necessary to make the principal curvatures of the sphere (boundary of the ball) positive. The factor $1/(N-1)$ is used to make the mean curvature of the unit sphere equal to 1 in all dimensions. The reader should keep in mind the long-standing differences in usage between geometry and partial differential equations, where the outer unit normal is used in the integration-by-parts formulae for the Euclidean space $\mathbf{R}^N$. For integration by parts on submanifolds of $\mathbf{R}^N$ the sum of the principal curvatures will naturally occur rather than the mean curvature. It will be convenient to *redefine* H as the sum of the principal curvatures and introduce the notation $\bar{H}$ for the classical mean curvature:

$$H \overset{\text{def}}{=} \sum_{\alpha=1}^{N-1} \kappa_\alpha, \quad \bar{H} \overset{\text{def}}{=} \frac{1}{N-1} \sum_{\alpha=1}^{N-1} \kappa_\alpha. \tag{3.17}$$

4 Sets Globally Described by the Level Sets of a Function

From the definition of a set of class $C^{k,\ell}$, $k \geq 1$ and $0 \leq \ell \leq 1$, the set Ω can also be locally described by the level sets of the $C^{k,\ell}$-function

$$f_x(y) \stackrel{\text{def}}{=} g_x(y) \cdot e_N, \tag{4.1}$$

since by definition

$$\begin{aligned}
\operatorname{int} \Omega \cap U(x) &= \{y \in U(x) \, : \, f_x(y) > 0\}, \\
\partial\Omega \cap U(x) &= \{y \in U(x) \, : \, f_x(y) = 0\}.
\end{aligned}$$

The boundary $\partial\Omega$ is the zero level set of f_x and the gradient

$$\nabla f_x(y) = {}^*Dg_x(y)e_N \neq 0$$

is normal to that level set. Thus the *exterior normal* to Ω is given by

$$n(y) = -\frac{\nabla f_x(y)}{|\nabla f_x(y)|} = -\frac{{}^*Dg_x(y)e_N}{|{}^*Dg_x(y)e_N|} = -\frac{{}^*(Dh_x(g_x(y)))^{-1}e_N}{|{}^*(Dh_x(g_x(y)))^{-1}e_N|}.$$

From this local construction of the functions $\{f_x \, : \, x \in \partial\Omega\}$, a global function on $\mathbf{R}^N$ can be constructed to characterize Ω.

Theorem 4.1. *Given $k \geq 1$, $0 \leq \ell \leq 1$, and a set Ω in $\mathbf{R}^N$ of class $C^{k,\ell}$ with compact boundary, there exists a Lipschitz continuous $f : \mathbf{R}^N \to \mathbf{R}$ such that*

$$\operatorname{int} \Omega = \{y \in \mathbf{R}^N \, : \, f(y) > 0\} \quad \textit{and} \quad \partial\Omega = \{y \in \mathbf{R}^N \, : \, f(y) = 0\} \tag{4.2}$$

and a neighborhood W of $\partial\Omega$ such that

$$f \in C^{k,\ell}(W) \text{ and } \nabla f \neq 0 \text{ on } W \text{ and } n = -\frac{\nabla f}{|\nabla f|}, \tag{4.3}$$

where n is the outward unit normal to Ω on $\partial\Omega$.

Proof. Construction of the function f. Fix $x \in \partial\Omega$ and consider the function f_x defined by (4.1). By continuity of ∇f_x, let $V(x) \subset U(x)$ be a neighborhood of x such that

$$\forall y \in V(x), \quad |\nabla f_x(y) - \nabla f_x(x)| \leq \frac{1}{2}|\nabla f_x(x)|.$$

Furthermore, let $W(x)$ be a bounded open neighborhood of x such that $\overline{W(x)} \subset V(x)$. For $\partial\Omega$ compact there exists a finite subcover $\{W_j = W(x_j) \, : \, 1 \leq j \leq m\}$ of $\partial\Omega$ for a finite sequence $\{x_j \, : \, 1 \leq j \leq m\} \subset \partial\Omega$. Index all the previous symbols by j instead of x_j and define $W = \cup_{j=1}^m W_j$. Let $\{r_j : 1 \leq j \leq m\}$ be a partition of unity for $\{V_j = V(x_j)\}$ such that

$$\begin{cases}
r_j \in \mathcal{D}(V_j), \quad 0 \leq r_j(x) \leq 1, \quad 1 \leq j \leq m, \\
\displaystyle\sum_{j=1}^m r_j(x) = 1 \text{ in } \overline{W},
\end{cases} \tag{4.4}$$

where, by definition, W is an open neighborhood of $\partial\Omega$ such that $\overline{W} \subset V = \cup_{j=1}^m V_j$ and $\mathcal{D}(V_j)$ is the set of all infinitely continuously differentiable functions with compact support in V_j. Define the function $f : \mathbf{R}^{\mathrm{N}} \to \mathbf{R}$:

$$f(y) \stackrel{\text{def}}{=} d_{W\cup\Omega^c} - d_{W\cup\Omega} + \sum_{j=1}^m r_j(y) \frac{f_j(y)}{|\nabla f_j(x_j)|},$$

where $d_A(x) = \inf\{|y-x| : y \in A\}$ is the distance function from a point x to a nonempty set A of $\mathbf{R}^{\mathrm{N}}$. The function d_A is Lipschitz continuous with Lipschitz constant equal to 1. Since for all j, $f_j \in C^{k,\ell}(V_j)$, $r_j \in \mathcal{D}(V_j)$, and $\overline{W}_j \subset V_j$, $r_j f_j \in C^{k,\ell}(V_j)$ with compact support in V_j. Therefore since $k \geq 1$, $r_j f_j$ is Lipschitz continuous in $\mathbf{R}^{\mathrm{N}}$. So, by definition, f is Lipschitz continuous on $\mathbf{R}^{\mathrm{N}}$ as the finite sum of Lipschitz continuous functions on $\mathbf{R}^{\mathrm{N}}$.

Properties (4.2). Introduce the index set

$$J(y) \stackrel{\text{def}}{=} \{j : 1 \leq j \leq m, \, r_j(y) > 0\}$$

for $y \in V$. For all $y \in \overline{W}$, $J(y) \neq \varnothing$, $d_{W\cup\Omega^c} = 0 = d_{W\cup\Omega}$, and

$$f(y) = \sum_{j\in J(y)} r_j(y) \frac{f_j(y)}{|\nabla f_j(x_j)|}. \tag{4.5}$$

For $y \in \partial\Omega = \overline{W} \cap \partial\Omega$, $f_j(y) = 0$ for all $j \in J(y)$ and hence $f(y) = 0$; for $y \in \operatorname{int}\Omega \cap \overline{W}$, $f_j(y) > 0$ and $r_j(y) > 0$ for all $j \in J(y)$ and hence $f(y) > 0$; for $y \in \operatorname{int}\Omega^c \cap \overline{W}$, $f_j(y) < 0$ and $r_j(y) > 0$ for all $j \in J(y)$ and hence $f(y) < 0$. For $y \in \Omega\backslash\overline{W}$, $f_j \geq 0$, $r_j \geq 0$, $\sum_{j=1}^m r_j f_j \geq 0$, $d_{W\cup\Omega^c} > 0$, $d_{W\cup\Omega} = 0$, and $f > 0$; for $y \in \Omega^c\backslash\overline{W}$, $f_j \leq 0$, $r_j \geq 0$, $\sum_{j=1}^m r_j f_j \leq 0$, $d_{W\cup\Omega^c} = 0$, $d_{W\cup\Omega} > 0$, and $f < 0$. So we have proven that $\partial\Omega \subset f^{-1}(0)$, $\operatorname{int}\Omega \subset \{f > 0\}$, and $\operatorname{int}\Omega^c \subset \{f < 0\}$. Hence f has the properties (4.2).

Properties (4.3). Recall that on W the function f is given by expression (4.5). It belongs to $C^{k,\ell}(W)$, and, a fortiori, to $C^1(W)$, as the finite sum of $C^{k,\ell}$-functions on W since $k \geq 1$. The gradient of f in W is given by

$$\nabla f(y) = \sum_{j\in J(y)} \nabla r_j(y) \frac{f_j(y)}{|\nabla f_j(x_j)|} + r_j(y) \frac{\nabla f_j(y)}{|\nabla f_j(x_j)|}.$$

It remains to show that it is nonzero on $\partial\Omega$. On $\partial\Omega \cap V_j$, $f_j = 0$ and

$$\forall j \in J(y), \quad \frac{\nabla f_j(y)}{|\nabla f_j(y)|} = -n(y).$$

Therefore in W, ∇f can be rewritten in the form

$$\nabla f(y) = -n(y) \sum_{j\in J(y)} r_j(y) \frac{|\nabla f_j(y)|}{|\nabla f_j(x_j)|}$$

and

$$\nabla f(y) + n(y) = n(y) \sum_{j\in J(y)} r_j(y) \left[1 - \frac{|\nabla f_j(y)|}{|\nabla f_j(x_j)|}\right].$$

Finally, by construction of W,

$$|\nabla f(y)| \geq |n(y)| - |n(y)| \sum_{j \in J(y)} r_j(y) \left| 1 - \frac{|\nabla f_j(y)|}{|\nabla f_j(x_j)|} \right|$$
$$\geq 1 - \sum_{j \in J(y)} r_j(y) \frac{1}{2} = \frac{1}{2}$$

and $\nabla f \neq 0$ in W. This proves properties (4.3). □

This theorem has a converse.

Theorem 4.2. *Associate with a continuous function $f : \mathbf{R}^{\mathrm{N}} \to \mathbf{R}$ the set*

$$\Omega \stackrel{\text{def}}{=} \{y \in \mathbf{R}^{\mathrm{N}} \, : \, f(y) > 0\}. \tag{4.6}$$

Assume that

$$f^{-1}(0) \stackrel{\text{def}}{=} \{y \in \mathbf{R}^{\mathrm{N}} \, : \, f(y) = 0\} \neq \varnothing \tag{4.7}$$

and that there exists a neighborhood V of $f^{-1}(0)$ such that $f \in C^{k,\ell}(V)$ for some $k \geq 1$ and $0 \leq \ell \leq 1$ and that $\nabla f \neq 0$ in $f^{-1}(0)$. Then Ω is a set of class $C^{k,\ell}$,

$$\operatorname{int} \Omega = \Omega \quad \text{and} \quad \partial\Omega = f^{-1}(0). \tag{4.8}$$

Proof. By continuity of f, Ω is open, $\operatorname{int} \Omega = \Omega$, $\complement\Omega$ is closed,

$$\overline{\Omega} \subset \{y \in \mathbf{R}^{\mathrm{N}} \, : \, f(y) \geq 0\}, \quad \overline{\complement\Omega} = \complement\Omega = \{y \in \mathbf{R}^{\mathrm{N}} \, : \, f(y) \leq 0\},$$

and $\partial\Omega \subset f^{-1}(0)$. Conversely for each $x \in \partial\Omega$, define the function

$$g(t) \stackrel{\text{def}}{=} f(x + t\nabla f(x)).$$

There exists $\delta > 0$ such that for all $|t| < \delta$, $x + t\nabla f(x) \in V$ and the function g is C^1 in $(-\delta, \delta)$. Hence since $g(0) = 0$ and $g'(t) = \nabla f(x + t\nabla f(x)) \cdot \nabla f(x)$

$$f(x + t\nabla f(x)) = \int_0^t \nabla f(x + s\nabla f(x)) \cdot \nabla f(x) \, ds.$$

By continuity of ∇f in V and the fact that $\nabla f(x) \neq 0$, there exists δ', $0 < \delta' < \delta$, such that

$$\forall s, \, 0 \leq |s| \leq \delta', \quad \nabla f(x + s\nabla f(x)) \cdot \nabla f(x) \geq \frac{1}{2} |\nabla f(x)|^2 > 0.$$

Hence for all t, $0 < t \leq \delta'$, $f(x + t\nabla f(x)) > 0$. So any point in $f^{-1}(0)$ can be approximated by a sequence $\{x_n = x + t_n \nabla f(x) \, : \, n \geq 1, \, 0 < t_n \leq \delta'\}$ in Ω, $t_n \to 0$, and $f^{-1}(0) \subset \overline{\Omega}$. Similarly, using a sequence of negative t_n's, $f^{-1}(0) \subset \overline{\complement\Omega}$ and hence $f^{-1}(0) \subset \partial\Omega$. This proves (4.8).

Fix $x \in \partial\Omega = f^{-1}(0)$. Since $\nabla f(x) \neq 0$, define the unit vector $e_N(x) = \nabla f(x)/|\nabla f(x)|$. Associate with $e_N(x)$ unit vectors $e_1(x), \ldots, e_{N-1}(x)$, which form an orthonormal basis in $\mathbf{R}^{\mathrm{N}}$ with $e_N(x)$. Define the map $g_x : V \to \mathbf{R}^{\mathrm{N}}$ as

$$g_x(y) \stackrel{\text{def}}{=} \left(\{(y-x)\cdot e_\alpha(x)\}_{\alpha=1}^{N-1}, \frac{f(y)}{|\nabla f(x)|} \right) \quad \Rightarrow g_x \in C^{k,\ell}(V; \mathbf{R}^{\mathrm{N}}).$$

The transpose of the Jacobian matrix of g_x is given by

$${}^*Dg_x(y) = (e_1(x), \ldots, e_{N-1}(x), \nabla f(y)/|\nabla f(x)|)$$

and $Dg_x(x) = I$, the identity matrix in the $\{e_i(x)\}$ reference system. By the inverse mapping theorem g_x has a $C^{k,\ell}$ inverse h_x in some neighborhood $U(x)$ of x in V. Therefore,

$$\Omega \cap U(x) = \{y \in U(x) \,:\, f(x) > 0\} = \operatorname{int}\Omega \cap U(x),$$
$$\{y \in U(x) \,:\, f(x) = 0\} = \partial\Omega \cap U(x),$$

and the set Ω is of class $C^{k,\ell}$. □

We complete this section with the important theorem of Sard.

Theorem 4.3 (J. Dieudonné [**3**, Vol. III, sect. 16.23, p. 167])**.** *Let X and Y be two differential manifolds, $f : X \to Y$ be a C^∞-mapping, and E be the set of critical points of f. Then $f(E)$ is negligible in Y, and $Y - f(E)$ is dense in Y.*

Combining this theorem with Theorem 4.1, this means that, for almost all t in the range of a C^∞-function $f : \mathbf{R}^{\mathrm{N}} \to \mathbf{R}$, the set

$$\left\{ y \in \mathbf{R}^{\mathrm{N}} \,:\, f(x) > t \right\}$$

is of class C^∞ in the sense of Definition 3.1. The following theorem extends and completes Sard's theorem.

Theorem 4.4 (H. Federer [**5**, Thm. 3.4.3, p. 316])**.** *If $m > \nu \geq 0$ and $k \geq 1$ are integers, A is an open subset of $\mathbf{R}^m$, $B \subset A$, Y is a normed vector space, and*

$$f \,:\, A \to Y \text{ is a map of class } k, \quad \dim\ \operatorname{Im} Df(x) \leq \nu \text{ for } x \in B,$$

then

$$H_{\nu+(m-\nu)/k}(f(B)) = 0.$$

5 Sets Locally Described by the Epigraph of a Function

After diffeomorphisms and level sets, the local epigraph description provides a third point of view and another way to characterize the smoothness of a set or a domain. For instance, Lipschitzian domains, which are characterized by the property that their boundary is locally the epigraph of a Lipschitzian function, play a central role in the theory of Sobolev spaces and partial differential equations. They can equivalently be characterized by the geometric uniform cone property. We shall see that sets that are locally the epigraph of $C^{k,\ell}$-functions are equivalent to sets that are locally of class $C^{k,\ell}$ for $k \geq 1$.

5.1 Local C^0 Epigraphs, C^0 Epigraphs, and Equi-C^0 Epigraphs and the Space $\mathcal{H}$ of Dominating Functions

Let e_N in $\mathbf{R}^N$ be a *unit reference vector* and introduce the following notation:

$$H \stackrel{\text{def}}{=} \{e_N\}^\perp, \quad \forall \zeta \in \mathbf{R}^N, \quad \zeta' \stackrel{\text{def}}{=} P_H(\zeta), \quad \zeta_N \stackrel{\text{def}}{=} e_N \cdot \zeta \tag{5.1}$$

for the associated *reference hyperplane* H orthogonal to e_N, the orthogonal projection P_H onto H, and the *normal component* ζ_N of ζ. The vector ζ is equal to $\zeta' + \zeta_N e_N$ and will often be denoted by (ζ', ζ_N). In practice the vector e_N is chosen as $e_N = (0, \dots, 0, 1)$, but the actual form of the unit vector e_N is not important.

A *direction* in $\mathbf{R}^N$ is specified by a unit vector $d \in \mathbf{R}^N$, $|d| = 1$. Alternatively, it can be specified by Ae_N for some matrix A in the *orthogonal subgroup* of $N \times N$ matrices

$$\mathrm{O(N)} \stackrel{\text{def}}{=} \{A : {}^*\!A\, A = A\, {}^*\!A = I\}, \tag{5.2}$$

where ${}^*\!A$ is the transposed matrix of A. Conversely, for any unit vector d in $\mathbf{R}^N$, $|d| = 1$, there exists[8] $A \in \mathrm{O(N)}$ such that $d = Ae_N$. For all $x \in \mathbf{R}^N$ and $A \in \mathrm{O(N)}$, it is easy to check that $|Ax| = |x| = |A^{-1}x|$ for all $x \in \mathbf{R}^N$.

The main elements of Definition 5.1 are illustrated in Figure 2.2 for $N = 2$.

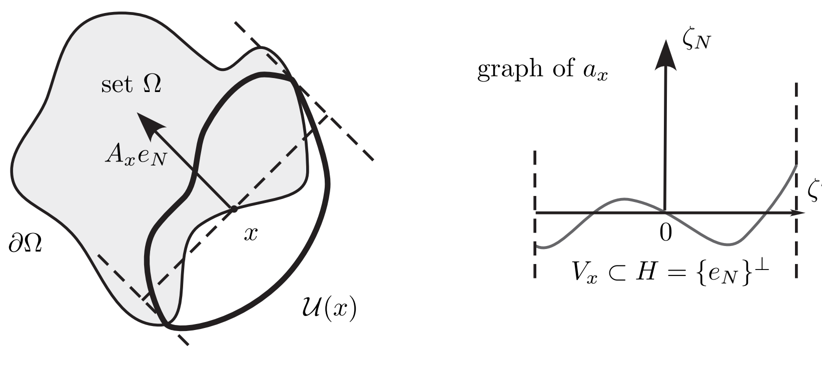


Figure 2.2. *Local epigraph representation* $(N = 2)$.

Definition 5.1.
Let e_N be a unit vector in $\mathbf{R}^N$, H be the hyperplane $\{e_N\}^\perp$, and Ω be a subset of $\mathbf{R}^N$ with nonempty boundary $\partial\Omega$.

(i) Ω is said to be *locally a C^0 epigraph* if for each $x \in \partial\Omega$ there exist

(a) an open neighborhood $\mathcal{U}(x)$ of x;

(b) a matrix $A_x \in \mathrm{O(N)}$;

[8] Complete the orthonormal basis with respect to d by adding $N-1$ unit vectors $d_1, d_2, \dots, d_{N-1}$ and construct the matrix whose columns are the vectors $A = [d_1\, d_2\, \dots d_{N-1}\, d]$ for which $Ae_N = d$.

(c) a bounded open neighborhood V_x of 0 in H such that

$$\mathcal{U}(x) \subset \{y \in \mathbf{R}^{\mathrm{N}} \,:\, P_H(A_x^{-1}(y-x)) \in V_x\}; \tag{5.3}$$

(d) and a function $a_x \in C^0(V_x)$ such that $a_x(0) = 0$ and

$$\mathcal{U}(x) \cap \partial\Omega = \mathcal{U}(x) \cap \left\{ x + A_x(\zeta' + \zeta_N e_N) : \begin{matrix} \zeta' \in V_x \\ \zeta_N = a_x(\zeta') \end{matrix} \right\}, \tag{5.4}$$

$$\mathcal{U}(x) \cap \operatorname{int}\Omega = \mathcal{U}(x) \cap \left\{ x + A_x(\zeta' + \zeta_N e_N) : \begin{matrix} \zeta' \in V_x \\ \zeta_N > a_x(\zeta') \end{matrix} \right\}. \tag{5.5}$$

(ii) Ω is said to be *a C^0 epigraph* if it is locally a C^0 epigraph and the neighborhoods $\mathcal{U}(x)$ and V_x can be chosen in such a way that V_x and $A_x^{-1}(\mathcal{U}(x) - x)$ are independent of x: there exist bounded open neighborhoods V of 0 in H and U of 0 in $\mathbf{R}^{\mathrm{N}}$ such that $P_H(U) \subset V$ and

$$\forall x \in \partial\Omega, \quad V_x = V \quad \text{and} \quad \exists A_x \in \mathrm{O}(\mathrm{N}) \text{ such that } \mathcal{U}(x) = x + A_x U.$$

(iii) Ω is said to be *an equi-C^0 epigraph* if it is a C^0 epigraph and the family of functions $\{a_x : x \in \partial\Omega\}$ is uniformly bounded and equicontinuous:

$$\exists c > 0 \text{ such that } \forall x \in \partial\Omega,\ \forall \xi' \in V, \quad |a_x(\xi')| \le c,$$

$$\begin{cases} \forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that} \\ \forall x \in \partial\Omega, \forall y, \forall z \in V \text{ such that } |z-y| < \delta, \quad |a_x(z) - a_x(y)| < \varepsilon. \end{cases} \tag{5.6}$$

□

Remark 5.1.
Conditions (5.3), (5.4), and (5.5) are equivalent to the following three conditions:

$$\mathcal{U}(x) \cap \partial\Omega \subset \left\{ x + A_x(\zeta' + \zeta_N e_N) : \begin{matrix} \zeta' \in V_x \\ \zeta_N = a_x(\zeta') \end{matrix} \right\}, \tag{5.7}$$

$$\mathcal{U}(x) \cap \operatorname{int}\Omega \subset \left\{ x + A_x(\zeta' + \zeta_N e_N) : \begin{matrix} \zeta' \in V_x \\ \zeta_N > a_x(\zeta') \end{matrix} \right\}, \tag{5.8}$$

$$\mathcal{U}(x) \cap \operatorname{int}\complement\Omega \subset \left\{ x + A_x(\zeta' + \zeta_N e_N) : \begin{matrix} \zeta' \in V_x \\ \zeta_N < a_x(\zeta') \end{matrix} \right\}. \tag{5.9}$$

In particular, Ω is locally a C^0 epigraph if and only if $\complement\Omega$ is locally a C^0 epigraph. As a result, conditions (5.3), (5.4), and (5.5) are also equivalent to conditions (5.3), (5.4), and the following condition:

$$\mathcal{U}(x) \cap \operatorname{int}\complement\Omega \subset \left\{ x + A_x(\zeta' + \zeta_N e_N) : \begin{matrix} \zeta' \in V_x \\ \zeta_N < a_x(\zeta') \end{matrix} \right\} \tag{5.10}$$

in place of condition (5.5). □

Remark 5.2.
Note that from (5.3) $P_H(A_x^{-1}(\mathcal{U}(x)-x)) \subset V_x$ and that this yields the condition $P_H(U) \subset V$ in part (iii) of the definition. □

Remark 5.3.
It is always possible to redefine $\mathcal{U}(x)-x$ or V_x to be open balls or open hypercubes centered in 0. For instance, in the case of balls, for $V'_x = B_H(0,\rho) \subset V_x$, choose the associated neighborhood $\mathcal{U}'(x) = \mathcal{U}(x) \cap \{y \in \mathbf{R}^{\mathrm{N}} : P_H(A_x^{-1}(y-x)) \in B_H(0,\rho)\}$. Similarly, for $\mathcal{U}'(x) = B(x,r) \subset \mathcal{U}(x)$ choose the associated neighborhood $V'_x = P_H(A_x^{-1}(B(x,r)-x) = B_H(0,r) \subset V_x$, since, from (5.3), $V_x \supset P_H(A_x^{-1}(\mathcal{U}(x)-x)) \supset P_H(A_x^{-1}B(x,r)-x)) = B_H(0,r)$. In both cases properties (5.3) to (5.5) are verified for the new neighborhoods. □

Remark 5.4.
Sets $\Omega \subset \mathbf{R}^{\mathrm{N}}$ that are locally C^0 epigraphs have a boundary $\partial\Omega$ with zero N-dimensional Lebesgue measure. □

The three cases considered in Definition 5.1 differ only when the boundary $\partial\Omega$ is unbounded. But we first introduce some notation.

Notation 5.1.
(i) Under condition (5.3), a point y in $\mathcal{U}(x) \subset \mathbf{R}^{\mathrm{N}}$ is represented by $(\zeta', \zeta_N) \in V_x \times \mathbf{R}$, where

$$\zeta' \overset{\text{def}}{=} P_H(A_x^{-1}(y-x)) \in V_x \subset H, \quad \zeta_N \overset{\text{def}}{=} A_x^{-1}(y-x) \cdot e_N(x).$$

We can also identify H with $R^{N-1} \overset{\text{def}}{=} \{(z',0) \in \mathbf{R}^{\mathrm{N}} : z' \in \mathbf{R}^{N-1}\}$.

(ii) The graph, epigraph, and hypograph of $a_x \colon V_x \to \mathbf{R}$ will be denoted as follows:

$$\begin{aligned} A_x^0 &\overset{\text{def}}{=} \{x + A_x(\zeta' + \zeta_N e_N) : \forall \zeta' \in V_x,\ \zeta_N = a_x(\zeta')\}, \\ A_x^+ &\overset{\text{def}}{=} \{x + A_x(\zeta' + \zeta_N e_N) : \forall \zeta' \in V_x,\ \forall \zeta_N > a_x(\zeta')\}, \\ A_x^- &\overset{\text{def}}{=} \{x + A_x(\zeta' + \zeta_N e_N) : \forall \zeta' \in V_x,\ \forall \zeta_N < a_x(\zeta')\}. \end{aligned}$$ □

The equi-C^0 epigraph case is important since we shall see in Theorem 5.2 that the three cases of Definition 5.1 are equivalent when the boundary $\partial\Omega$ is compact. We now show that, for an arbitrary boundary, the equicontinuity of the graph functions $\{a_x : x \in \partial\Omega\}$ can be expressed in terms of the continuity in 0 of a *dominating function* h. For this purpose, define the *space of dominating functions* as follows:

$$\mathcal{H} \overset{\text{def}}{=} \{h : [0,\infty[\to [0,\infty[\ : h(0) = 0 \text{ and } h \text{ is continuous in } 0\}. \tag{5.11}$$

By continuity in 0, h is locally bounded in 0:

$$\forall c > 0,\ \exists \rho > 0 \text{ such that } \forall \theta \in [0,\rho], \quad |h(\theta)| \le c.$$

The only part of the function h that will really matter is in a bounded neighborhood of zero. Its extension outside to $[0,\infty[$ can be relatively arbitrary.

Theorem 5.1. *Let Ω be a C^0-epigraph. The following conditions are equivalent:*

(i) *Ω is an equi-C^0-epigraph.*

(ii) *There exist $\rho > 0$ and $h \in \mathcal{H}$ such that $B_H(0,\rho) \subset V$ and for all $x \in \partial\Omega$*

$$\forall \zeta',\, \xi' \in B_H(0,\rho) \text{ such that } |\xi' - \zeta'| < \rho, \quad |a_x(\xi') - a_x(\zeta')| \leq h(|\xi' - \zeta'|). \tag{5.12}$$

(iii) *There exist $\rho > 0$ and $h \in \mathcal{H}$ such that $B_H(0,\rho) \subset V$ and for all $x \in \partial\Omega$*

$$\forall \xi' \in B_H(0,\rho), \quad a_x(\xi') \leq h(|\xi'|). \tag{5.13}$$

Inequalities (5.12) and (5.13) for $h \in \mathcal{H}$ are also verified with $\limsup h$ that also belongs to $\mathcal{H}$.

Proof. Since the a_x's are uniformly continuous, inequalities (5.12) and (5.13) with $h \in \mathcal{H}$ imply the same inequalities with $\limsup h$ that also belongs to $\mathcal{H}$ by taking the pointwise $\limsup$ of both sides of the inequalities.

(ii) $\Rightarrow$ (i). For $h \in \mathcal{H}$, there exists $\rho_1 > 0$ such that $0 < \rho_1 \leq \rho/2$ and $|h(\theta)| \leq 1$ for all $\theta \in [0, \rho_1]$. By construction $B_H(0,\rho_1) \subset V$ and from inequality (5.12)

$$\begin{aligned} \forall \zeta',\, \xi' \in B_H(0,\rho_1), \quad & |a_x(\xi') - a_x(\zeta')| \leq h(|\xi' - \zeta'|), \\ \forall\, \xi' \in B_H(0,\rho_1), \quad & |a_x(\xi')| \leq h(|\xi'|) \leq 1 \end{aligned}$$

and the family $\{a_x : B_H(0,\rho_1) \to \mathbf{R} \, : \, x \in \partial\Omega\}$ is equicontinuous and uniformly bounded.

(i) $\Rightarrow$ (ii). By Definition 5.1 (iii), there exist a neighborhood V of 0 in H and a neighborhood U of 0 in $\mathbf{R}^{\mathrm{N}}$ such that for all $x \in \partial\Omega$

$$V_x = V \quad \text{and} \quad \exists A_x \in \mathrm{O(N)} \text{ such that } \mathcal{U}(x) = x + A_x U,$$

and the family $\{a_x \in C^0(\overline{V}) : x \in \partial\Omega\}$ is uniformly bounded and equicontinuous. Choose $r > 0$ such that $B(0,3r) \subset U$. From property (5.3) in Definition 5.1

$$\mathcal{U}(x) \subset \left\{ y \in \mathbf{R}^{\mathrm{N}} : P_H A_x^{-1}(y - x) \in V \right\} \quad \Rightarrow B_H(0,3r) \subset V.$$

Define the *modulus of continuity*

$$\forall \theta \in [0,3r], \quad h(\theta) = \sup_{\substack{\zeta' \in H \\ |\zeta'|=1}} \sup_{x \in \partial\Omega} \sup_{\substack{\xi' \in V \\ \xi' + \theta\zeta' \in V}} |a_x(\theta\zeta' + \xi') - a_x(\xi')|. \tag{5.14}$$

The function $h(\theta)$, as the sup of the family $\theta \mapsto |a_x(\theta\zeta' + \xi') - a_x(\xi')|$ of continuous functions with respect to (ζ', ξ'), is *lower semicontinuous* with respect to θ and bounded in $[0,3r]$ since the a_x's are uniformly bounded. By construction $h(0) = 0$ and by equicontinuity of the a_x, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\theta \in [0,3r]$, $|\theta| < \delta$,

$$\forall x \in \partial\Omega,\, \forall \zeta' \in H,\, |\zeta'| = 1, \quad |a_x(\theta\zeta' + \xi') - a_x(\xi')| < \varepsilon \quad \Rightarrow |h(\theta)| < \varepsilon,$$

where h is continuous in 0. In view of Remark 5.3, choose $\rho = 3r$ and the new neighborhoods $V' = B_H(0,\rho)$ and $U' = \{\xi \in B(0,\rho) : P_H\xi \in V'\} = B(0,\rho)$ and extend h by the constant $h(\rho)$ to $[\rho, \infty[$. Therefore $h \in \mathcal{H}$ and, by definition of h, for all $x \in \partial\Omega$ and for all ξ' and ζ' in $B_H(0,\rho)$ such that $|\xi' - \zeta'| < \rho$

$$|a'_x(\xi') - a'_x(\zeta')| = \left| a'_x\left(\zeta' + |\xi' - \zeta'| \frac{\xi' - \zeta'}{|\xi' - \zeta'|}\right) - a'_x(\zeta') \right| \le h(|\xi' - \zeta'|)$$
$$\Rightarrow \forall x \in \partial\Omega,\ \forall \xi', \zeta' \in B_H(0,\rho),\ |\xi' - \zeta'| < \rho, \quad |a'_x(\xi') - a'_x(\zeta')| \le h(|\xi' - \zeta'|).$$

(ii) $\Rightarrow$ (iii) Choose $\zeta' = 0$ in inequality (5.12):

$$\forall \xi' \in B_H(0,\rho) \text{ such that } |\xi'| < \rho, \quad a_x(\xi') \le |a_x(\xi')| \le h(|\xi'|).$$

(iii) $\Rightarrow$ (ii). In view of Remark 5.3, choose the new neighborhoods $V' = B_H(0,\rho)$ and $U' = U \cap \{y \in \mathbf{R}^{\mathrm{N}} : P_H(y) \in B_H(0,\rho)\}$ and the restriction of the functions $\{a_x : V \to \mathbf{R} : \forall x \in \partial\Omega\}$ to V'. Choose $r > 0$ such that $B(0,4r) \subset U'$ (hence $4r \le \rho$) and consider the region

$$O \stackrel{\text{def}}{=} \{\zeta' + \zeta_N e_N : \zeta' \in B_H(0,2r) \text{ and } h(|\zeta'|) < \zeta_N < r\}.$$

For any $x \in \partial\Omega$ and $\zeta \in O$, consider the point $y_\zeta = x + A_x(\zeta' + \zeta_N e_N)$. It is readily seen that $|y_\zeta - x| = |\zeta| < \sqrt{(2r)^2 + r^2} < 3r$ and that $y_\zeta \in \mathcal{U}(x)$. In addition $y_\zeta \in A_x^+$. From (5.13) $a_x(\zeta') \le h(|\zeta'|)$ for all $\zeta' \in V = B_H(0,\rho)$ and hence $a_x(\zeta') \le h(|\zeta'|) < \zeta_N$ for all $\zeta \in O$ $(2r < \rho)$. As a result $y_\zeta \in \mathcal{U}(x) \cap A_x^+ = \mathcal{U}(x) \cap \operatorname{int}\Omega \subset \operatorname{int}\Omega$ and

$$\forall x \in \partial\Omega, \quad x + A_x O \subset \operatorname{int}\Omega. \tag{5.15}$$

Given ζ' and ξ' in $B_H(0,2r)$, consider the two points

$$y_{\zeta'} \stackrel{\text{def}}{=} x + A_x(\zeta' + a_x(\zeta')e_N) \in A_x^0 = \mathcal{U}(x) \cap \partial\Omega,$$
$$y_{\xi'} \stackrel{\text{def}}{=} x + A_x(\xi' + a_x(\xi')e_N) \in A_x^0 = \mathcal{U}(x) \cap \partial\Omega.$$

Since $y_{\xi'}$ and $y_{\zeta'}$ belong to $\mathcal{U}(x) \cap \partial\Omega$, $y_{\xi'}$ does not belong to $y_{\zeta'} + A_x O \subset \operatorname{int}\Omega$ or equivalently $y_{\xi'} - y_{\zeta'} \notin A_x O$. Hence

$$\xi' - \zeta' + (a_x(\xi') - a_x(\zeta'))e_N = A_x^{-1}(y_{\xi'} - y_{\zeta'}) \notin O.$$

For $|\xi' - \zeta'| < 2r$, this is equivalent to saying that one of the following inequalities is verified:

$$h(|\xi' - \zeta'|) \ge A_x e_N \cdot (y_{\xi'} - y_{\zeta'}) = a_x(\xi') - a_x(\zeta')$$
$$\text{or } r \le A_x e_N \cdot (y_{\xi'} - y_{\zeta'}) = a_x(\xi') - a_x(\zeta')$$

by definition of a_x and the fact that $\xi' - \zeta' \in B_H(0,2r) \subset B_H(0,\rho) = V'$. But the second inequality contradicts the continuity of a_x. Hence the first inequality holds and, by interchanging the roles of ξ' and ζ', we get inequality (5.12) with a new ρ equal to $2r$. □

Theorem 5.2. *When the boundary $\partial\Omega$ is compact, the three properties of Definition 5.1 are equivalent:*

$$\text{locally } C^0 \text{ epigraph} \iff C^0 \text{ epigraph} \iff \text{equi-}C^0 \text{ epigraph.}$$

Under such conditions, there exist neighborhoods V of 0 in H and U of 0 in $\mathbf{R}^{\mathrm{N}}$ such that $P_H U \subset V$ and the neighborhoods V_x of 0 in H and the neighborhoods $\mathcal{U}(x)$ of x in $\mathbf{R}^{\mathrm{N}}$ can be chosen of the form

$$V_x = V \quad \text{and} \quad \exists A_x \in \mathrm{O(N)} \text{ such that } \mathcal{U}(x) = x + A_x U. \tag{5.16}$$

Moreover, the family of functions $\{a_x : V \to \mathbf{R} : x \in \partial\Omega\}$ can be chosen uniformly bounded and equicontinuous with respect to $x \in \partial\Omega$. In addition, there exist $\rho > 0$ such that $B_H(0,\rho) \subset V$ and a function $h \in \mathcal{H}$ such that

$$\forall x \in \partial\Omega,\ \forall \xi', \zeta' \in B_H(0,\rho),\ |\xi' - \zeta'| < \rho, \quad |a_x(\xi') - a_x(\zeta')| \le h(|\xi' - \zeta'|). \tag{5.17}$$

Proof. It is sufficient to show that a locally C^0 epigraph is an equi-C^0 epigraph. For each $x \in \partial\Omega$, there exists $r_x > 0$ such that $B(x, r_x) \subset \mathcal{U}(x)$. Since $\partial\Omega$ is compact, there exist a finite sequence $\{x_i\}_{i=1}^m$ of points of $\partial\Omega$ and a finite subcover $\{B_i\}_{i=1}^m$, $B_i = B(x_i, R_i)$, $R_i = r_{x_i}$.

(i) We first claim that

$$\boxed{\exists R > 0, \forall x \in \partial\Omega, \exists i, 1 \le i \le m, \text{ such that } B(x,R) \subset B_i.}$$

We proceed by contradiction. If this is not true, then for each $n \ge 1$,

$$\begin{aligned} &\exists x_n \in \partial\Omega, \forall i, 1 \le i \le m, \quad B\big(x_n, 1/n\big) \not\subset B_i, \\ \Rightarrow\ &\forall i, 1 \le i \le m, \quad \exists y_{in} \in B\big(x_n, 1/n\big) \text{ such that } y_{in} \notin B_i. \end{aligned}$$

Since $\partial\Omega$ is compact, there exist a subsequence of $\{x_n\}$ and $x \in \partial\Omega$ such that $x_{n_k} \to x$ as k goes to infinity. Hence for each i, $1 \le i \le m$,

$$|y_{in_k} - x| \le |y_{in_k} - x_{n_k}| + |x_{n_k} - x| \le \frac{1}{n_k} + |x_{n_k} - x| \to 0$$

and $y_{in_k} \to x$ as k goes to infinity. For each i, the set $\complement B_i$ is closed. Thus $x \in \complement B_i$ and

$$y_{in_k} \in \complement B_i \to x \in \complement B_i \quad \Rightarrow \exists\, x \in \partial\Omega \text{ such that } x \notin \bigcup_{i=1}^m B_i.$$

But this contradicts the fact that $\{B_i\}_{i=1}^m$ is an open cover of $\partial\Omega$. This property is obviously also satisfied for all neighborhoods N of 0 such that $\varnothing \ne N \subset B(0,R)$.

(ii) *Construction of the neighborhoods V'_x and $\mathcal{U}'(x)$ and the maps A'_x and a'_x.* From part (i) for each $x \in \partial\Omega$, there exists i, $1 \le i \le m$, such that

$$\begin{aligned} &B(x,R) \subset B_i = B(x_i, R_i) \subset \mathcal{U}(x_i) \\ \Rightarrow\ &P_H(A_{x_i}^{-1}(B(x,R) - x_i)) \subset P_H(A_{x_i}^{-1}(\mathcal{U}(x_i) - x_i)) \subset V_{x_i} \\ \Rightarrow\ &P_H(A_{x_i}^{-1}(B(x - x_i, R))) \subset V_{x_i}. \end{aligned}$$

Since $x \in \partial\Omega \cap \mathcal{U}(x_i)$, it is of the form

$$\begin{aligned} x = x_i + A_{x_i}(\xi_i' + a_{x_i}(\xi_i')e_N), \quad \xi_i' = P_H(A_{x_i}^{-1}(x - x_i)) \\ \Rightarrow \; P_H(A_{x_i}^{-1}(B(x - x_i, R))) = P_H(A_{x_i}^{-1}(x - x_i)) + P_H(A_{x_i}^{-1}(B(0, R))) \\ = \xi_i' + B_H(0, R) \\ \Rightarrow \boxed{\xi_i' + B_H(0, R) \subset V_{x_i}.} \end{aligned} \tag{5.18}$$

Choose the neighborhoods

$$\boxed{U \overset{\text{def}}{=} B(0, R) \text{ and } V \overset{\text{def}}{=} B_H(0, R).}$$

Associate with $x \in \partial\Omega$ the following new neighborhoods and functions:

$$\begin{gathered} V_x' \overset{\text{def}}{=} V, \quad A_x' \overset{\text{def}}{=} A_{x_i}, \quad \mathcal{U}(x) \overset{\text{def}}{=} x + A_x' U, \\ \zeta' \mapsto a_x'(\zeta') \overset{\text{def}}{=} a_{x_i}(\zeta' + \xi_i') - a_{x_i}(\xi_i') : V \to \mathbf{R}\,. \end{gathered}$$

It is readily seen from identity (5.18) that a_x' is well-defined, $a_x'(0) = 0$, and $a_x' \in C^0(\overline{V})$. Since the family $\{a_{x_i} : 1 \le i \le m\}$ is finite,

$$\|a_x'\|_{C^0(\overline{V})} \le 2\, \|a_{x_i}\|_{C^0(\overline{V_{x_i}})} \le 2 \max_{1 \le i \le m} \|a_{x_i}\|_{C^0(\overline{V_{x_i}})} < \infty$$

and the family $\{a_x'; x \in \partial\Omega\}$ is uniformly bounded. Since each a_{x_i} is uniformly continuous, for all $\varepsilon > 0$ there exists $\delta_i > 0$ such that

$$\forall \zeta_1', \zeta_2' \in V_{x_i} \text{ such that } |\zeta_1' - \zeta_2'| < \delta_i, \quad |a_{x_i}(\zeta_1') - a_{x_i}(\zeta_2')| < \varepsilon.$$

And since the family $\{a_{x_i} : 1 \le i \le m\}$ is finite pick $\delta = \min_{1 \le i \le m} \delta_i$ to get the following: for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $i \in \{1, \ldots, m\}$,

$$\forall \zeta_1', \zeta_2' \in V_{x_i} \text{ such that } |\zeta_1' - \zeta_2'| < \delta, \quad |a_{x_i}(\zeta_1') - a_{x_i}(\zeta_2')| < \varepsilon.$$

Therefore, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall x \in \partial\Omega, \forall \zeta_1', \zeta_2' \in V \text{ such that } |\zeta_1' - \zeta_2'| < \delta, \quad |a_x'(\zeta_1') - a_x'(\zeta_2')| < \varepsilon \tag{5.19}$$

and the family $\{a_x'; x \in \partial\Omega\}$ is uniformly equicontinuous. By construction, properties (a) and (b) of Definition 5.1 are verified for $\mathcal{U}'(x)$ and A_x'.

By construction, properties (a) and (b) are verified for $\mathcal{U}'(x)$ and A_x'. Always by construction in $x \in \partial\Omega$

$$\begin{gathered} \mathcal{U}'(x) = x + A_x' B(0, R) = x + B(0, R) = B(x, R), \\ P_H((A_x')^{-1}(\mathcal{U}'(x) - x)) = P_H(B(0, R)) = B_H(0, R) = V = V_x' \\ \Rightarrow \mathcal{U}'(x) \subset \left\{ y \in \mathbf{R}^{\mathrm{N}} : P_H((A_x')^{-1}(y - x)) \in V_x' \right\} \end{gathered}$$

and condition (c) of Definition 5.1 is verified. For condition (d) of Definition 5.1 in $x \in \partial\Omega$, let x_i be the point associated with x such that $B(x, R) \subset \mathcal{U}(x_i)$. Since $\mathcal{U}'(x) = B(x, R)$,

$$\mathcal{U}'(x) \cap \partial\Omega \subset \mathcal{U}(x_i) \cap \partial\Omega = \{x_i + A_{x_i}(\zeta' + a_{x_i}(\zeta')e_N) : \forall \zeta' \in V_{x_i}\} \cap \partial\Omega$$

and for all $y \in \mathcal{U}'(x) \cap \partial\Omega$ there exists $\xi' \in V_{x_i}$ such that

$$y = x_i + A_{x_i}(\xi' + a_{x_i}(\xi')e_N) = x + A_{x_i}(\xi' + a_{x_i}(\xi')e_N) - (x - x_i).$$

But x also has a representation with respect to x_i

$$\begin{aligned} x &= x_i + A_{x_i}(\xi_i' + a_{x_i}(\xi_i')e_N), \quad \xi_i' \in V_{x_i} \\ \Rightarrow\ y &= x + A_{x_i}(\xi' - \xi_i' + (a_{x_i}(\xi') - a_{x_i}(\xi_i'))e_N) \end{aligned}$$

that can be rewritten as

$$\begin{aligned} y &= x + A_{x_i}(\xi' - \xi_i' + (a_{x_i}((\xi' - \xi_i') + \xi_i') - a_{x_i}(\xi_i'))e_N) \\ \Rightarrow\ \zeta' &\stackrel{\text{def}}{=} P_H(A_{x_i}^{-1}(y - x)) = \xi' - \xi_i' \in V \end{aligned}$$

since by (c) $P_H(A_{x_i}^{-1}(\mathcal{U}'(x) - x)) = P_H((A_x')^{-1}(\mathcal{U}'(x) - x)) \subset V_x = V$. Finally, there exists $\zeta' \in V$ such that

$$y = x + A_{x_i}(\zeta' + (a_{x_i}(\zeta' + \xi_i') - a_{x_i}(\xi_i'))e_N) = x + A_{x_i}(\zeta' + a_x'(\zeta')e_N)$$

and property (5.4) is verified.

The constructions and the proof of property (5.5) are similar:

$$\begin{aligned} \mathcal{U}'(x) \cap \operatorname{int}\Omega &\subset \mathcal{U}(x_i) \cap \operatorname{int}\Omega \\ &= \mathcal{U}(x_i) \cap \left\{x_i + A_{x_i'}(\zeta' + \zeta_N e_N) : \forall \zeta' \in V_{x_i},\ \forall \zeta_N > a_{x_i}(\zeta')\right\} \end{aligned}$$

and for all $y \in \mathcal{U}'(x) \cap \operatorname{int}\Omega$ there exists $(\xi', \xi_N) \in V_{x_i} \times \mathbf{R}$ such that $\xi_N > a_{x_i}(\xi')$ and

$$\begin{aligned} y &= x_i + A_{x_i}(\xi' + \xi_N e_N) = x + A_{x_i}(\xi' + \xi_N e_N) - (x - x_i) \\ &= x + A_{x_i}(\xi' - \xi_i' + (\xi_N - a_{x_i}(\xi_i'))e_N) \end{aligned}$$

$$\Rightarrow\ \zeta' \stackrel{\text{def}}{=} P_H(A_{x_i}^{-1}(y - x)) = \xi' - \xi_i' \in V, \quad \zeta_N \stackrel{\text{def}}{=} (A_{x_i}^{-1}(y - x)) \cdot e_N = \xi_N - a_{x_i}(\xi_i')$$

since by (c) $P_H(A_{x_i}^{-1}(\mathcal{U}'(x) - x)) = P_H((A_x')^{-1}(\mathcal{U}'(x) - x)) \subset V_x = V$. Finally, there exists $(\zeta', \zeta_N) \in V \times \mathbf{R}$ such that

$$\begin{aligned} y &= x + A_{x_i}(\zeta' + a_x'(\zeta')e_N), \\ \zeta_N = \xi_N - a_{x_i}(\xi_i') &= \xi_N - a_{x_i}(\xi') + a_{x_i}(\xi') - a_{x_i}(\xi_i') \\ &= \xi_N - a_{x_i}(\xi') + a_{x_i}(\zeta' + \xi_i') - a_{x_i}(\xi_i') \\ &= \xi_N - a_{x_i}(\xi') + a_x'(\zeta') > a_x'(\zeta') \end{aligned}$$

since $\xi_N > a_{x_i}(\xi')$ and property (5.5) in (d) is verified.

(iii) The other properties and (5.17) follow from Theorem 5.1. $\square$

5.2 Local $C^{k,\ell}$-Epigraphs and Hölderian/Lipschitzian Sets

We extend the three definitions of C^0 epigraphs to $C^{k,\ell}$ epigraphs. $C^{0,1}$ is the important family of *Lipschitzian sets* and $C^{0,\ell}$, $0 < \ell < 1$, of *Hölderian sets.*

Definition 5.2.
Let Ω be a subset of $\mathbf{R}^{\mathrm{N}}$ such that $\partial\Omega \neq \varnothing$. Let $k \geq 0$, $0 \leq \ell \leq 1$.

(i) Ω is said to be *locally a $C^{k,\ell}$ epigraph* if for each $x \in \partial\Omega$ there exist

(a) an open neighborhood $\mathcal{U}(x)$ of x;

(b) a matrix $A_x \in \mathrm{O(N)}$;

(c) a bounded open neighborhood V_x of 0 in H such that

$$\mathcal{U}(x) \subset \{y \in \mathbf{R}^{\mathrm{N}} \,:\, P_H(A_x^{-1}(y-x)) \in V_x\}; \tag{5.20}$$

(d) and a function $a_x \in C^{k,\ell}(V_x)$ such that $a_x(0) = 0$ and

$$\mathcal{U}(x) \cap \partial\Omega = \mathcal{U}(x) \cap \left\{ x + A_x(\zeta' + \zeta_N e_N) : \begin{array}{c} \zeta' \in V_x \\ \zeta_N = a_x(\zeta') \end{array} \right\}, \tag{5.21}$$

$$\mathcal{U}(x) \cap \operatorname{int}\Omega = \mathcal{U}(x) \cap \left\{ x + A_x(\zeta' + \zeta_N e_N) : \begin{array}{c} \zeta' \in V_x \\ \zeta_N > a_x(\zeta') \end{array} \right\}, \tag{5.22}$$

where $\zeta' = (\zeta_1, \ldots, \zeta_{N-1}) \in \mathbf{R}^{N-1}$.

We shall say that Ω is *locally Lipschitzian* in the $C^{0,1}$ case and *locally Hölderian of index ℓ* in the $C^{0,\ell}$, $0 < \ell < 1$, case.

(ii) Ω is said to be *a $C^{k,\ell}$ epigraph* if it is locally a $C^{k,\ell}$ epigraph and the neighborhoods $\mathcal{U}(x)$ and V_x can be chosen in such a way that V_x and $A_x^{-1}(\mathcal{U}(x)-x)$ are independent of x: there exist bounded open neighborhoods V of 0 in H and U of 0 in $\mathbf{R}^{\mathrm{N}}$ such that $P_H(U) \subset V$ and

$$\forall x \in \partial\Omega, \quad V_x \stackrel{\text{def}}{=} V \quad \text{and} \quad \exists A_x \in \mathrm{O(N)} \text{ such that } \mathcal{U}(x) \stackrel{\text{def}}{=} x + A_x U.$$

We shall say that Ω is *Lipschitzian* in the $C^{0,1}$ case and *Hölderian of index ℓ* in the $C^{0,\ell}$, $0 < \ell < 1$, case.

(iii) Given $(k,\ell) \neq (0,0)$, Ω is said to be *an equi-$C^{k.\ell}$ epigraph* if it is a $C^{k,\ell}$ epigraph and

$$\exists c > 0,\ \forall x \in \partial\Omega, \quad \|a_x\|_{C^{k,\ell}(V)} \leq c, \tag{5.23}$$

where $\|f\|_{C^{k,\ell}(V)}$ is the norm on the space $C^{k,\ell}(V)$ as defined in (2.21):

$$\|f\|_{C^{k,\ell}(V)} \stackrel{\text{def}}{=} \|f\|_{C^k(V)} + \max_{0 \leq |\alpha| \leq k} \sup_{\substack{x,y \in V \\ x \neq y}} \frac{|\partial^\alpha f(y) - \partial^\alpha f(x)|}{|y-x|^\ell}, \quad \text{if } 0 < \ell \leq 1,$$

$$C^{k,0}(V) \stackrel{\text{def}}{=} C^k(V) \quad \text{and} \quad \|f\|_{C^{k,0}(V)} \stackrel{\text{def}}{=} \|f\|_{C^k(V)}.$$

Ω is said to be *an equi-*$C^{0,0}$ *epigraph* if it is an equi-C^0 epigraph in the sense of Definition 5.1 (iii).

We shall say that Ω is *equi-Lipschitzian* in the $C^{0,1}$ case and *equi-Hölderian of index* ℓ in the $C^{0,\ell}$, $0 < \ell < 1$, case. □

Remark 5.5.
In Definition 5.1 (iii) for $(k,\ell) \neq (0,0)$ it follows from condition (5.23) that the functions $\{a_x : x \in \partial\Omega\}$ are uniformly equicontinuous on V. In the case $(k,\ell) = (0,0)$ this is no longer true. So we have to include it by using Definition 5.1 (iii). □

Theorem 5.2 readily extends to the $C^{k,\ell}$ case.

Theorem 5.3. *When the boundary $\partial\Omega$ is compact, the three types of sets of Definition* 5.2 *are equi-$C^{k,\ell}$ epigraphs in the sense of Definitions* 5.2 (iii) *and* 5.1 (iii).

Proof. The proof is the same as the one of Theorem 5.2 for the C^0 case. The equi-$C^{k,\ell}$ property is obtained by choosing in the proof of Theorem 5.2 the common constant as $c = \max_{1\leq i\leq m} c_{x_i}$, where c_{x_i} is the constant associated with a_{x_i} on V_{x_i}, $\|a_{x_i}\|_{C^k(V_{x_i})} \leq c_{x_i}$. □

Sets Ω that are *locally a $C^{k,\ell}$ epigraph* are Lebesgue measurable in $\mathbf{R}^{\mathrm{N}}$, their "volume" is locally finite, and their boundary has zero "volume."

Theorem 5.4. *Let $k \geq 0$, let $0 \leq \ell \leq 1$, and let Ω be* locally a $C^{k,\ell}$ epigraph.

(i) *The complement $\complement\Omega$ is also locally a $C^{k,\ell}$ epigraph, and*

$$\begin{aligned} \operatorname{int}\Omega \neq \varnothing, \quad \overline{\operatorname{int}\Omega} = \overline{\Omega}, \ \textit{and}\ \partial(\operatorname{int}\Omega) = \partial\Omega, \\ \operatorname{int}\complement\Omega \neq \varnothing, \quad \overline{\operatorname{int}\complement\Omega} = \overline{\complement\Omega}, \ \textit{and}\ \partial(\operatorname{int}\complement\Omega) = \partial\Omega. \end{aligned} \tag{5.24}$$

Moreover, for all $x \in \partial\Omega$

$$\mathrm{m}\big(\mathcal{U}(x) \cap \partial\Omega\big) = 0 \quad \Rightarrow \mathrm{m}\big(\partial\Omega\big) = 0, \tag{5.25}$$

where m *is the N-dimensional Lebesgue measure.*

(ii) *If $\Omega \neq \varnothing$ is open, then $\Omega = \operatorname{int}\overline{\Omega}$.*

Proof. (i) The fact that the complement of Ω is locally a $C^{k,\ell}$ epigraph follows from Remark 5.1. To complete the proof, it is sufficient to establish the first two properties of the first line of (5.24). The second line is the first line applied to $\complement\Omega$.

Since $\mathcal{U}(x)$ is a neighborhood of x, there exists $r > 0$ such that $B(x, 3r) \subset \mathcal{U}(x)$. Since $a_x \in C^0(\overline{V_x})$, $a_x(0) = 0$, and V_x is a neighborhood of 0, there exists ρ, $0 < \rho < r$, such that $B_H(0,\rho) \subset V_x$ and for all ζ', $|\zeta'| < \rho$, $|a_x(\zeta')| \leq r$. Given $\zeta' \in B_H(0,\rho)$, the point $y_{\zeta'} = x + A(\zeta' + a_x(\zeta')e_N) \in \partial\Omega \cap \mathcal{U}(x)$. Given a real α, $|\alpha| < r$, consider the point $z_\alpha = x + A(\zeta' + (a_x(\zeta') + \alpha)e_N)$. By construction $|z_\alpha - x| = |A(\zeta' + (a_x(\zeta') + \alpha)e_N)| = |\zeta' + (a_x(\zeta') + \alpha)e_N| < 3r$ and $z_\alpha \in \mathcal{U}(x)$. Hence the segment $(y_{\zeta'}, y_{\zeta'} + rA_x e_N) \subset A^+ \cap \mathcal{U}(x) = \operatorname{int}\Omega \cap \mathcal{U}(x)$ and $\operatorname{int}\Omega \neq \varnothing$. Also the segment $(y_{\zeta'}, y_{\zeta'} - rA_x e_N) \subset A^- \cap \mathcal{U}(x) = \operatorname{int}\complement\Omega \cap \mathcal{U}(x)$ and $\operatorname{int}\complement\Omega \neq \varnothing$.

Finally, associate with $y_{\zeta'} \in \partial\Omega$ and the sequence $\alpha_n = r/n$ the points $z_{\alpha_n} \in \operatorname{int}\Omega$. As n goes to infinity, $z_n \to y_{\zeta'}$ and $\partial\Omega \subset \overline{\operatorname{int}\Omega}$ and hence $\overline{\Omega} = \overline{\operatorname{int}\Omega}$. From the following Lemma 5.1 we get the third identity in the first line of (5.24).

(ii) From part (i) $\partial(\operatorname{int} \complement\Omega) = \partial\Omega$ and by Lemma 5.1 (ii) $\overline{\operatorname{int}\complement\Omega} = \overline{\complement\Omega}$. Since Ω is open, $\overline{\complement\overline{\Omega}} = \overline{\operatorname{int}\complement\Omega} = \complement\Omega$ and $\operatorname{int}\overline{\Omega} = \complement\overline{\complement\overline{\Omega}} = \Omega$. □

Lemma 5.1.

(i) $\partial(\operatorname{int}\Omega) = \partial\Omega \quad\iff\quad \overline{\operatorname{int}\Omega} = \overline{\Omega}$.

(ii) $\partial(\operatorname{int}\complement\Omega) = \partial\Omega \quad\iff\quad \overline{\operatorname{int}\complement\Omega} = \overline{\complement\Omega}$.

Proof. It is sufficient to prove (i). ($\Rightarrow$) By definition of the closure,

$$\overline{\Omega} = \operatorname{int}\Omega \cup \partial\Omega = \operatorname{int}\Omega \cup \partial(\operatorname{int}\Omega) = \overline{\operatorname{int}\Omega}.$$

Conversely ($\Leftarrow$) By definition $\partial\Omega = \overline{\Omega} \cap \overline{\complement\Omega}$,

$$\overline{\Omega} \cap \overline{\complement\Omega} = \overline{\operatorname{int}\Omega} \cap \overline{\complement\Omega} = \overline{\operatorname{int}\Omega} \cap \complement(\operatorname{int}\Omega) = \overline{\operatorname{int}\Omega} \cap \overline{\complement(\operatorname{int}\Omega)} = \partial(\operatorname{int}\Omega).$$ □

5.3 Local $C^{k,\ell}$-Epigraphs and Sets of Class $C^{k,\ell}$

Sets or domains which are locally the epigraph of a $C^{k,\ell}$-function, $k \geq 0$, $0 \leq \ell \leq 1$ (resp., locally Lipschitzian), are sets of class $C^{k,\ell}$ (resp., $C^{0,1}$). However, we shall see from Examples 5.1 and 5.2 that a domain of class $C^{0,1}$ is generally not locally the epigraph of a Lipschitzian function.

Theorem 5.5. *Let ℓ, $0 \leq \ell \leq 1$, be a real number.*

(i) *If Ω is locally a $C^{k,\ell}$ epigraph for $k \geq 0$, then it is locally of class $C^{k,\ell}$.*

(ii) *If Ω is locally of class $C^{k,\ell}$ for $k \geq 1$, then it is locally a $C^{k,\ell}$ epigraph.*

Proof. (i) For each point $x \in \partial\Omega$, let $\mathcal{U}(x)$, V_x, and a_x be the associated neighborhoods and the $C^{k,\ell}$-function. Define the $C^{k,\ell}$-mappings for $\xi = (\xi', \xi_N) \in V_x \times \mathbf{R}$

$$h_x(\xi) \overset{\text{def}}{=} x + A_x(\xi' + [\xi_N + a_x(\xi')]\, e_N), \tag{5.26}$$

$$g_x(y) \overset{\text{def}}{=} (P_H A_x^{-1}(y-x), e_N \cdot A_x^{-1}(y-x) - a_x(P_H A_x^{-1}(y-x))). \tag{5.27}$$

It is easy to verify that h_x is the inverse of g_x,

$$h_x(g_x(y)) = y \text{ in } \mathcal{U}(x) \text{ and } g_x(h_x(\xi)) = \xi \text{ in } g_x(\mathcal{U}(x)),$$

and that since for $y \in \mathcal{U}(x) \cap \partial\Omega$, $e_N \cdot A_x^{-1}(y-x) = a_x(P_H A_x^{-1}(y-x)$

$$g_x(\mathcal{U}(x) \cap \partial\Omega) \in H = \{(\xi', \xi_N) \in \mathbf{R}^{\mathrm{N}} : \xi_N = 0\},$$

and that since for $y \in \mathcal{U}(x) \cap \operatorname{int}\Omega$, $e_N \cdot A_x^{-1}(y-x) > a_x(P_H A_x^{-1}(y-x)$

$$g_x(\mathcal{U}(x) \cap \operatorname{int}\Omega) \subset \{(\xi', \xi_N) \in \mathbf{R}^{\mathrm{N}} : \xi_N > 0\}.$$

The set Ω is of class $C^{k,\ell}$.

(ii) In the discussion preceding Theorem 4.1, we showed that a set of class $C^{k,\ell}$ is locally the level set of a $C^{k,\ell}$-function. From the definition of a set of class $C^{k,\ell}$, $k \geq 1$ and $0 \leq \ell \leq 1$, the set Ω can be locally described by the level sets of the $C^{k,\ell}$-function

$$f_x(y) \stackrel{\text{def}}{=} g_x(y) \cdot e_N,$$

since by definition

$$\begin{aligned}\operatorname{int}\Omega \cap U(x) &= \{y \in U(x) : f_x(y) > 0\},\\ \partial\Omega \cap U(x) &= \{y \in U(x) : f_x(y) = 0\}.\end{aligned}$$

The boundary $\partial\Omega$ is the zero level set of f_x and the gradient

$$\nabla f_x(y) = {}^*Dg_x(y)e_N \neq 0$$

is normal to that level set. The exterior normal to Ω is given by

$$n(y) = -\frac{\nabla f_x(y)}{|\nabla f_x(y)|} = -\frac{{}^*Dg_x(y)e_N}{|{}^*Dg_x(y)e_N|} = -\frac{{}^*(Dh_x(g_x(y)))^{-1}e_N}{|{}^*(Dh_x(g_x(y)))^{-1}e_N|}.$$

Since f_x is C^1 and $\nabla f_x(x) \neq 0$, choose a smaller neighborhood of x, still denoted by $U(x)$, such that

$$\forall y \in U(x), \quad \nabla f_x(y) \cdot \nabla f_x(x) > 0.$$

To construct the graph function around the point $x \in \Gamma = \partial\Omega$, choose $A_x \in \mathrm{O(N)}$ such that $A_x e_N = -n(x)$ and $H = \{e_N\}^\perp$. Consider the $C^{k,\ell}$-function, $k \geq 1$,

$$\lambda(\zeta', \zeta_N) \stackrel{\text{def}}{=} f_x\left(x + A_x(\zeta' + \zeta_N e_N)\right).$$

By construction, $\lambda(0,0) = 0$ and

$$0 = \nabla\lambda(0,0) \cdot (0, \xi_N) = \nabla f_x(x) \cdot (0, \xi_N) = \xi_N\, |\nabla f_x(x)| \quad \Rightarrow \xi_N = 0.$$

Therefore, by the implicit function theorem, there exist a neighborhood $V_x \subset B_0$ of $(0,0)$ and a function $a_x \in C^{k.\ell}(V_x)$ such that $\lambda(0,0) = 0$ and

$$\forall \zeta' \in V_x, \quad \lambda(\zeta', a_x(\zeta')) = f_x(x + A_x(\zeta' + a_x(\zeta')e_N) = 0.$$

By construction, $x + A_x(\zeta' + a_x(\zeta')e_N) \in \Gamma \cap U(x)$. Choose

$$\mathcal{U}(x) \stackrel{\text{def}}{=} U(x) \cap \{x + A_x(\zeta' + \zeta_N e_N) : \zeta' \in V_x \text{ and } \zeta_N \in \mathbf{R}\}.$$

By construction, V_x and $\mathcal{U}(x)$ satisfy condition (5.3). Moreover, from (3.4)–(3.5)

$$\operatorname{int}\Omega \cap U(x) = h_x(B_+) \quad \text{and} \quad \partial\Omega \cap U(x) = h_x(B_0)$$
$$\Rightarrow \left|\begin{aligned}\operatorname{int}\Omega \cap U(x) &= \{y \in U(x) : f_x(y) > 0\},\\ \partial\Omega \cap U(x) &= \{y \in U(x) : f_x(y) = 0\},\end{aligned}\right.$$

and since $\mathcal{U}(x) \subset U(x)$ we get conditions (5.4) and (5.5):

$$\operatorname{int}\Omega \cap \mathcal{U}(x) = \{y \in \mathcal{U}(x) \,:\, f_x(y) > 0\} = A^+ \cap \mathcal{U}(x),$$
$$\partial\Omega \cap \mathcal{U}(x) = \{y \in \mathcal{U}(x) \,:\, f_x(y) = 0\} = A^0 \cap \mathcal{U}(x).$$

Recall that, by construction of $\mathcal{U}(x)$, for all $(\zeta', \zeta_N) \in A_x^{-1}(\mathcal{U}(x) - \{x\})$,

$$\zeta_N > a_x(\zeta') \iff \lambda(\zeta', \zeta_N) = f_x(x + A_x(\zeta' + \zeta_N e_N)) > 0,$$

since

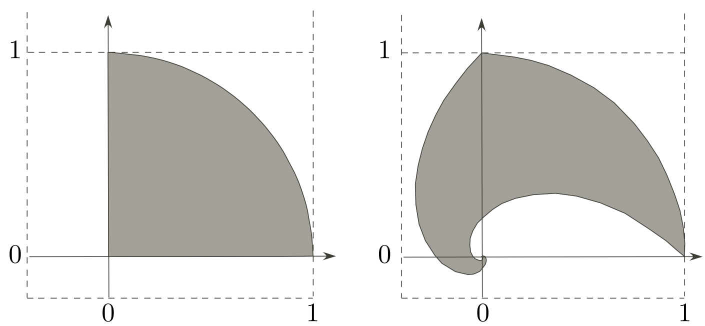
$$\frac{\partial \lambda}{\partial \zeta_N}(\zeta', \zeta_N) = \nabla f_x(x + A_x(\zeta' + \zeta_N e_n)) \cdot A_x e_N = \nabla f_x(x + A_x \zeta) \cdot \frac{\nabla f_x(x)}{|\nabla f_x(x)|} > 0$$

in the neighborhood $A_x^{-1}(\mathcal{U}(x) - \{x\}) \subset A_x^{-1}(U(x) - \{x\})$ of $(0,0)$. □

Example 5.1 (R. Adams, N. Aronszajn, and K. T. Smith [**1**])**.**
Consider the open convex (Lipschitzian) set $\Omega_0 = \{\rho\, e^{i\theta} \,:\, 0 < \rho < 1,\, 0 < \theta < \pi/2\}$ and its image $\Omega = T(\Omega_0)$ by the $C^{0,1}$-diffeomorphism (see Figure 2.3)

$$T(\rho\, e^{i\theta}) = \rho\, e^{i(\theta - \log \rho)}, \quad T^{-1}(\rho\, e^{i\theta}) = \rho\, e^{i(\theta + \log \rho)}.$$

It is readily seen that, as ρ goes to zero, the image of the two pieces of the boundary of Ω_0 corresponding to $\theta = 0$ and $\theta = \pi/2$ begin to spiral around the origin. As a result Ω is not locally the epigraph of a function at the origin. □



(a) quarter disk Ω_0 (b) $T(\Omega_0)$ spirals around 0

Figure 2.3. *Domain Ω_0 and its image $T(\Omega_0)$ spiraling around the origin.*

Example 5.2.
This second example can be found in F. Murat and J. Simon [**1**], where it is attributed to M. Zerner. Consider the Lipschitzian function λ defined on $[0,1]$ as follows: $\lambda(0) = 0$, and on each interval $[1/3^{n+1}, 1/3^n]$

$$\lambda(s) \stackrel{\text{def}}{=} \begin{cases} 2\left(s - \dfrac{1}{3^{n+1}}\right), & \dfrac{1}{3^{n+1}} \leq s \leq \dfrac{2}{3^{n+1}}, \\[2ex] -2\left(s - \dfrac{1}{3^n}\right), & \dfrac{2}{3^{n+1}} \leq s \leq \dfrac{1}{3^n}, \end{cases}$$

where n ranges over all integers $n \geq 0$. Associate with λ and a real $\delta > 0$ the set

$$\Omega \stackrel{\text{def}}{=} \{(x_1, x_2) \in \mathbf{R}^2 \,:\, 0 < x_1 < 1,\, |x_2 - \lambda(x_1)| < \delta\, x_1\}$$

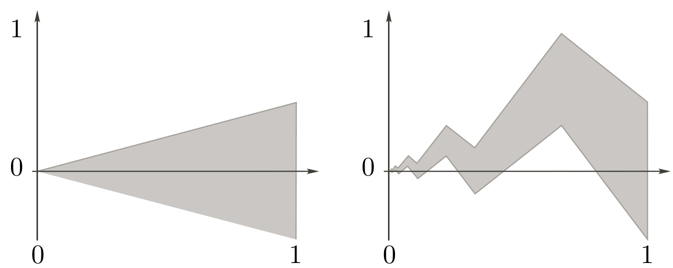
(see Figure 2.4). The set Ω is the image of the triangle

$$\Omega_0 \stackrel{\text{def}}{=} \{(x_1, x_2) \in \mathbf{R}^2 \,:\, 0 < x_1 < 1,\, |x_2| < \delta\, x_1\}$$

through the $C^{0,1}$-homeomorphism

$$T(x_1, x_2) \stackrel{\text{def}}{=} (x_1, x_2 + \lambda(x_1)), \quad T^{-1}(y_1, y_2) = (y_1, y_2 - \lambda(y_1)).$$

Since the triangle Ω_0 is Lipschitzian, its image is a set of class $C^{0,1}$. But Ω is not locally Lipschitzian in $(0, 0)$ since Ω zigzags like a lightning bolt as it gets closer to the origin. Thus, however small the neighborhood around $(0, 0)$, a direction $e_N(0, 0)$ cannot be found to make the domain locally the epigraph of a function. □



(a) triangle Ω_0 (b) $T(\Omega_0)$ zigzags to 0

Figure 2.4. *Domain Ω_0 and its image $T(\Omega_0)$ zigzagging towards the origin.*

5.4 Locally Lipschitzian Sets: Some Examples and Properties

The important subfamily of sets that are locally a Lipschitzian epigraph (that is, locally a $C^{0,1}$ epigraph) enjoys most of the properties of smooth domains. Convex sets and domains that are locally the epigraph of a C^1- or smoother function are locally Lipschitzian. In a bounded path-connected Lipschitzian set, the geodesic distance between any two points is uniformly bounded. Their boundary has zero *volume* and locally finite *boundary* measure (cf. Theorem 5.7 below). Sobolev spaces defined on Ω have linear extension to $\mathbf{R}^{\mathrm{N}}$.

5.4.1 Examples and Continuous Linear Extensions

According to this definition the whole space $\mathbf{R}^{\mathrm{N}}$ and the closed unit ball with its center or a small crack removed are not locally Lipschitzian since in the first case the boundary is empty and in the second case the conditions cannot be satisfied at the center or along the crack, which is a part of the boundary. Similarly the set

$$\Omega \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} \Omega_n, \quad \Omega_n \stackrel{\text{def}}{=} \left\{ y \in \mathbf{R}^{\mathrm{N}} : \left| y - \frac{1}{2^n} \right| < \frac{1}{2^{n+2}} \right\}$$

is not locally Lipschitzian since the conditions of Definition 5.2 (i) are not satisfied in $0 \in \partial\Omega$. However, the set

$$\Omega \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} \Omega_n, \quad \Omega_n \stackrel{\text{def}}{=} \left\{ y \in \mathbf{R}^{\mathrm{N}} : |y - n| < \frac{1}{2^{n+2}} \right\}$$

is locally Lipschitzian, but not Lipschitzian or equi-Lipschitzian in the sense of Definition 5.2.

One of the important properties of a bounded (open) Lipschitzian domain Ω is the existence of a *continuous linear extension* of functions of the Sobolev space $H^k(\Omega)$ from Ω to functions in $H^k(\mathbf{R}^{\mathrm{N}})$ defined on $\mathbf{R}^{\mathrm{N}}$, that is,

$$E\colon H^k(\Omega) \to H^k(\mathbf{R}^{\mathrm{N}}) \text{ and } \forall \varphi \in H^k(\Omega), \quad (E\varphi)\big|_{\Omega} = \varphi \tag{5.28}$$

(cf. the Calderón extension theorem in R. A. Adams [**1**, p. 83, Thm. 4.32, p. 91] and J. Nečas [**1**, Thm. 3.10, p. 80]). This property is also important in the existence of optimal domains when it is uniform for a given family. For instance, this will occur for the family of domains satisfying the uniform cone property of section 6.4.1.

5.4.2 Convex Sets

Theorem 5.6. *Any convex subset Ω of $\mathbf{R}^{\mathrm{N}}$ such that $\Omega \neq \mathbf{R}^{\mathrm{N}}$ and $\operatorname{int}\Omega \neq \varnothing$ is locally Lipschitzian, and for each $x \in \partial\Omega$ the neighborhood V_x and the function a_x can be chosen convex.*

Proof. (i) *Construction of the function a_x.* Start with the unit vector e_N and the hyperplane H orthogonal to e_N. Pick any point x_0 in the interior of Ω and choose $\varepsilon > 0$ such that $B(x_0, 2\varepsilon) \subset \operatorname{int}\Omega$. Since $\Omega \neq \mathbf{R}^{\mathrm{N}}$, $\partial\Omega \neq \varnothing$ and we can associate with each $x \in \partial\Omega$ the direction $d(x) = (x_0 - x)/|x_0 - x|$ and the matrix $A_x \in \mathrm{O(N)}$ such that $d(x) = A_x e_N$. Note that the hyperplane $A_x H$ is orthogonal to the vector $d(x) = A_x e_N$. By convexity, the point $x_0^- = x - (x_0 - x) \notin \overline{\Omega}$ and the minimum distance δ from x_0^- to $\overline{\Omega}$ is such that $0 < \delta < |x - x_0|$. As a result there exists ρ, $0 < \rho < \varepsilon$, such that for all $\zeta' \in H$, $|\zeta'| < \rho$, the line

$$L_{\zeta'} \stackrel{\text{def}}{=} \{x + A_x(\zeta' + \zeta_N e_N) : |\zeta_N| \leq |x - x_0|\}$$

from the point $x_0 + A_x\zeta'$ to $x_0^- + A_x\zeta'$ in the direction $A_x e_N$ contains a point of $\overline{\Omega}$ and a point of its complement. Hence there exists $\hat{\zeta}_N$, $|\hat{\zeta}_N| \leq |x - x_0|$, such that $\hat{y} = x + A_x(\zeta' + \hat{\zeta}_N e_N) \in \partial\Omega \cap L_{\zeta'}$ minimizes $\zeta_N = A_x e_N \cdot (y - x)$ over all $y \in \overline{\Omega} \cap L_{\zeta'}$. If $\hat{y}_1$ and $\hat{y}_2$ are two distinct minimizing points such that $\hat{\zeta}_{1N} = \hat{\zeta}_{2N}$, then $\hat{y}_1 = x + A_x(\zeta' + \hat{\zeta}_{1N} e_N) = x + A_x(\zeta' + \hat{\zeta}_{2N} e_N) = \hat{y}_2$. As a result, for all $\zeta' \in H$, $|\zeta'| < \rho$, the function

$$a_x(\zeta') \stackrel{\text{def}}{=} \inf_{y \in \overline{\Omega} \cap L_{\zeta'}} A_x e_N \cdot (y - x)$$

is well-defined and finite and there exists a unique $\hat{\zeta}_N$, $|\hat{\zeta}_N| < |x - x_0|$, such that $a_x(\zeta') = \hat{\zeta}_N$ and $\hat{y} = x + A_x(\zeta' + \hat{\zeta}_N e_N)$ is the unique minimizer.

Choose the neighborhoods

$$\boxed{\mathcal{U}(x) \stackrel{\text{def}}{=} \left\{ y \in \mathbf{R}^{\mathrm{N}} : \begin{array}{l} |P_H A_x^{-1}(y - x)| < \rho \\ |A_x^{-1}(y - x) \cdot e_N| < |x - x_0| \end{array} \right\}, \quad V_x \stackrel{\text{def}}{=} \{ z \in H : |z| < \rho \}.}$$

Then condition (5.20) of Definition 5.2 is satisfied and it is easy to verify that the function a_x satisfies conditions (5.21) and (5.22).

(ii) *Convexity of the function* a_x. By construction the neighborhoods V_x and $\mathcal{U}(x)$ are convex. The set $\mathcal{U}(x) \cap \overline{\Omega}$ is convex. Then recall from (5.21)–(5.22) that

$$\mathcal{U}(x) \cap \{ x + A_x(\zeta' + \zeta_N e_N) : \zeta' \in V_x,\, \zeta_N \geq a_x(\zeta') \} = \mathcal{U}(x) \cap \overline{\Omega}.$$

Thus the set on the left-hand side is convex. In particular,

$$\forall y^1, y^2 \in \overline{\Omega} \cap \mathcal{U}(x),\, \forall \alpha \in [0,1], \quad \alpha y^1 + (1-\alpha) y^2 \in \overline{\Omega} \cap \mathcal{U}(x).$$

In particular, for any two y^1 and y^2 in $\partial\Omega \cap \mathcal{U}(x)$,

$$y^j = x + A_x({\zeta^j}' + \zeta^j_N e_N), \quad \zeta^j_N = a_x({\zeta^j}'),\, j = 1, 2,$$

and

$$\begin{aligned} y^\alpha = \alpha y^1 + (1-\alpha) y^2 &= x + A_x[\alpha \zeta^1 + (1-\alpha)\zeta^2] \in \overline{\Omega} \cap \mathcal{U}(x) \\ &\Rightarrow \alpha \zeta^1_N + (1-\alpha)\zeta^2_N \geq a_x\Big((\alpha\zeta^1 + (1-\alpha)\zeta^2)' \Big) \\ &\Rightarrow \alpha a_x({\zeta^1}') + (1-\alpha) a_x({\zeta^2}') \geq a_x\big(\alpha {\zeta^1}' + (1-\alpha){\zeta^2}' \big) \end{aligned}$$

and a_x is convex.

(iii) *The function* a_x *is Lipschitzian.* It is sufficient to show that a_x is bounded in V_x. Then we conclude from I. Ekeland and R. Temam [**1**, Lem. 3.1, p. 11] that the function is continuous and convex and hence Lipschitz continuous in V_x. By construction the minimizing point $\hat{y}$ belongs to $\mathcal{U}(x)$, which is bounded. □

5.4.3 Boundary Measure and Integral for Lipschitzian Sets

The *boundary measure* on $\partial\Omega$, which was defined in section 3.2 from the local C^1-diffeomorphism, can also be defined from the local graph representation of the boundary and extended to the Lipschitzian case.

Assuming that $\partial\Omega$ is compact, there is a finite subcover of open neighborhoods $\mathcal{U}(x)$ for $\partial\Omega$ that can be represented by a finite family of Lipschitzian graphs. Let

$\{\mathcal{U}_j\}_{j=1}^m$, $\mathcal{U}_j \stackrel{\text{def}}{=} \mathcal{U}(x_j)$, be a finite open cover of $\partial\Omega$ corresponding to some finite sequence $\{x_j\}_{j=1}^m$ of points of $\partial\Omega$. Denote by e_N, H, $\{V_j\}_{j=1}^m$, $V_j = V_{x_j}$, and $\{a_j\}$, $a_j = a_{x_j}$, the associated elements of Definition 5.2. Introduce the notation

$$\Gamma = \partial\Omega, \qquad \Gamma_j = \Gamma \cap \mathcal{U}_j,\ 1 \le j \le m, \tag{5.29}$$

and the map $h_j = h_{x_j}$ and its inverse $g_j = g_{x_j}$ as defined in (5.26)–(5.27):

$$\begin{aligned} h_x(\xi) &\stackrel{\text{def}}{=} x + A_x(\xi' + [\xi_N + a_x(\xi')]\, e_N), \\ g_x(y) &\stackrel{\text{def}}{=} (P_H A_x^{-1}(y-x), e_N \cdot A_x^{-1}(y-x) - a_x(P_H A_x^{-1}(y-x))). \end{aligned}$$

By Rademacher's theorem the Lipschitzian function a_x is differentiable almost everywhere in V_x and belongs to $W^{1,\infty}(V_x)$ (cf., for instance, L. C. Evans and R. F. Gariepy [**1**]). Since h_j is defined from the Lipschitzian function a_j, Dh_j is defined almost everywhere in $V_x \times \mathbf{R}$, but also H_{N-1} almost everywhere on $V_x \times \{0\}$. Thus the canonical density for C^1-domains still makes sense and is given by the same formula (3.13):

$$\omega_j(\zeta') \stackrel{\text{def}}{=} \omega_{x_j}(\zeta') = |\det Dh_j(\zeta',0)|\,|\,{}^*(Dh_j)^{-1}(\zeta',0)\, e_N|. \tag{5.30}$$

It is easy to verify that for almost all $\xi' \in V_j$

$$Dh_j(\xi',\xi_N) = A_j \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & & & \ddots & 0 \\ \partial_1 a_j(\xi') & \partial_2 a_j(\xi') & \dots & \partial_{N-1} a_j(\xi') & 1 \end{bmatrix}, \ \det Dh_j(\xi',\xi_N) = 1,$$

where $\partial_i a_j$ is the partial derivative of a_j with respect to the ith component of $\zeta' = (\zeta_1, \dots, \zeta_{N-1}) \in V_j$. The matrix Dh_j is invertible and its coefficients belong to L^∞ and ${}^*(A_j)^{-1} = A_j$. By direct computation

$${}^*(Dh_j)^{-1}(\xi',\xi_N) = A_j \begin{bmatrix} 1 & 0 & \dots & 0 & -\partial_1 a_j(\xi') \\ 0 & \ddots & & \vdots & -\partial_2 a_j(\xi') \\ \vdots & & \ddots & 0 & \vdots \\ \vdots & & & \ddots & -\partial_{N-1} a_j(\xi') \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}, \tag{5.31}$$

and finally

$$\begin{aligned} \omega_j(\zeta') &= |\,{}^*(Dh_j)^{-1}(\zeta',0)\, e_N| \\ &= |(A_j)^{-1}\,{}^*(Dh_j)^{-1}(\zeta',0)\, e_N| = \sqrt{1 + |\nabla a_j(\zeta')|^2}. \end{aligned} \tag{5.32}$$

Let $\{r_1, \dots, r_m\}$ be a partition of unity for the $\mathcal{U}_j$'s, that is,

$$\begin{cases} r_j \in \mathcal{D}(\mathcal{U}_j), \quad 0 \le r_j(x) \le 1, \\ \displaystyle\sum_{j=1}^{m} r_j(x) = 1 \text{ in a neighborhood } \mathcal{U} \text{ of } \Gamma \end{cases} \tag{5.33}$$

such that $\overline{\mathcal{U}} \subset \cup_{j=1}^m \mathcal{U}_j$. For any function f in $C^0(\Gamma)$ the functions f_j and $f_j \circ h_j$ defined as

$$\begin{cases} f_j(y) = f(y) r_j(y), & y \in \Gamma_j, \\ f_j \circ h_j(\zeta', 0) = f_j(x + A_j(\xi' + a_j(\xi')\, e_N), & \zeta' \in V_j, \end{cases} \tag{5.34}$$

respectively, belong to $C^0(\Gamma_j)$ and $C^0(V_j)$. The integral of f on Γ is then defined as

$$\int_\Gamma f\, d\Gamma \stackrel{\text{def}}{=} \sum_{j=1}^{m} \int_{\Gamma_j} f_j\, d\Gamma_j, \quad \int_{\Gamma_j} f_j\, d\Gamma \stackrel{\text{def}}{=} \int_{V_j} f_j(h_j(\zeta', 0))\, \omega_j(\zeta')\, d\zeta'. \tag{5.35}$$

Since $\omega_j \in L^\infty(V_j)$, $1 \le j \le m$, this integral is also well-defined for all f in $L^1(\Gamma)$; that is, the function $f_j \circ h_j\, \omega_j$ belongs to $L^1(V_j)$ for all j, $1 \le j \le m$.

As in section 2, the tangent plane to Γ_j at $y = h_j(\zeta', 0)$ is defined by

$$Dh_j(\zeta', 0)\, A_j H,$$

where $A_j H$ is the hyperplane orthogonal to $A_j e_N$. An outward normal field to Γ_j is given by

$$m_x(\zeta) = -\,{}^*(Dh_j)^{-1}(\zeta', \zeta_N) e_N = A_j \begin{bmatrix} \partial_1 a_j(\zeta') \\ \vdots \\ \partial_{N-1} a_j(\zeta') \\ -1 \end{bmatrix}$$

and the unit outward normal to Γ_j at $y = h_j(\zeta', 0) \in \Gamma_j$ is given by

$$n(y) = n(h_j(\zeta', 0)) = \frac{1}{\sqrt{1 + |\nabla a_j(\zeta')|^2}}\, A_j \begin{bmatrix} \partial_1 a_j(\zeta') \\ \vdots \\ \partial_{N-1} a_j(\zeta') \\ -1 \end{bmatrix}. \tag{5.36}$$

The matrix A_j can be removed by redefining the functions a_x on the space $A_j H$ orthogonal to $A_j e_N$. The formulae also hold for $C^{k,\ell}$-domains, $k \ge 1$.

We have seen that sets that are locally a $C^{k,\ell}$ *epigraph*, $k \ge 0$, have nice properties. For instance, their boundary has zero "volume." However, they generally have no locally finite "surface measure" (cf. M. C. Delfour, N. Doyon, and J.-P. Zolésio [**1**, **2**] for specific examples) as we shall see in section 6.5. Fortunately, Lipschitzian sets have locally finite "surface measure."

Theorem 5.7. *Let $\Omega \subset \mathbf{R}^N$ be locally Lipschitzian. Then*

$$H_{N-1}\big(\mathcal{U}(x) \cap \partial\Omega\big) \le \sqrt{1+c_x^2}\; m_{N-1}\big(V_x\big), \tag{5.37}$$

where m_{N-1} and H_{N-1} are the respective $(N-1)$-dimensional Lebesgue measure and $(N-1)$-dimensional Hausdorff measure, and $c_x > 0$ is the Lipschitz constant of a_x in V_x. If, in addition, $\partial\Omega$ is compact, then

$$H_{N-1}(\partial\Omega) < \infty. \tag{5.38}$$

Proof. From Theorem 5.4 and from formula (5.32),

$$H_{N-1}\big(\mathcal{U}(x) \cap \partial\Omega\big) = \int_{V(x)} \big[1 + |\nabla a_x(\zeta')|^2\big]^{1/2} d\zeta' \le \big[1 + c_x^2\big]^{1/2} m_{N-1}\big(V_x\big).$$

When the boundary of Ω is compact, then we can find a finite subcover of open neighborhoods $\{\mathcal{U}(x_j)\}_{j=1}^m$. From the previous estimate, $H_{N-1}(\partial\Omega)$ is bounded by a finite sum of bounded terms. □

5.4.4 Geodesic Distance in a Domain and in Its Boundary

We can now justify property (2.29) in Theorem 2.6.

Theorem 5.8. *Let $\Omega \subset \mathbf{R}^N$ be bounded, open, and locally Lipschitzian.*

(i) *If Ω is path-connected, then*

$$\exists c_\Omega,\ \forall x, y \in \overline{\Omega}, \quad \mathrm{dist}_{\overline{\Omega}}(x, y) \le c_\Omega\, |x - y|. \tag{5.39}$$

(ii) *If $\partial\Omega$ is path-connected, then*

$$\exists c_{\partial\Omega},\ \forall x, y \in \partial\Omega, \quad \mathrm{dist}_{\partial\Omega}(x, y) \le c_{\partial\Omega}\, |x - y|. \tag{5.40}$$

Remark 5.6.
The boundary of a path-connected, bounded, open, and locally Lipschitzian set is generally not path-connected. As an example consider the set $\{x \in \mathbf{R}^2 : 1 < |x| < 2\}$ whose boundary is made up of the two disjoint circles of radii 1 and 2. □

Remark 5.7.
An open subset of $\mathbf{R}^N$ with compact path-connected boundary is generally not path-connected as can be seen from a domain in $\mathbf{R}^2$ made up of two tangent squares. □

Proof. For each $x \in \partial\Omega$, there exists $r_x > 0$ such that $B(x, r_x) \subset \mathcal{U}(x)$. Since $\partial\Omega$ is compact, there exist a finite sequence $\{x_i\}_{i=1}^m$ of points of $\partial\Omega$ and a finite subcover $\{B_i\}_{i=1}^m$, $B_i = B(x_i, R_i)$, $R_i = r_{x_i}$. Moreover, from part (i) of the proof of Theorem 5.2,

$$\exists R > 0, \forall x \in \partial\Omega, \exists i, 1 \le i \le m, \text{ such that } B(x, R) \subset B_i.$$

Always from part (i) of the proof of Theorem 5.2, the neighborhoods U and V can be chosen as

$$U \stackrel{\text{def}}{=} B(0,R) \quad \text{and} \quad V \stackrel{\text{def}}{=} B_H(0,R),$$

and to each point $x \in \partial\Omega$ we can associate the following new neighborhoods and functions:

$$V'_x \stackrel{\text{def}}{=} V, \quad A'_x \stackrel{\text{def}}{=} A_{x_i}, \quad \mathcal{U}(x) \stackrel{\text{def}}{=} x + A'_x U,$$
$$\zeta' \mapsto a'_x(\zeta') \stackrel{\text{def}}{=} a_{x_i}(\zeta' + \xi'_i) - a_{x_i}(\xi'_i) : V \to \mathbf{R}\,.$$

Since $V = B_H(0,R)$ is path-connected and for each $\zeta' \in V$ the map $\zeta_N \mapsto x + A_x[\zeta' + \zeta_N e_N]$ is continuous,

$$L_{\zeta'} \stackrel{\text{def}}{=} B(x,R) \cap \{x + A_x[\zeta' + \zeta_N e_N] : \zeta_N \in \mathbf{R}\}$$

is a (path-connected) line segment. This is nice, but we need to further reduce the size of the neighborhoods $\mathcal{U}'(x)$ and V'_x to make the piece of boundary $\partial\Omega \cap \mathcal{U}'(x)$ and hence $\overline{\Omega} \cap \mathcal{U}'(x)$ path-connected. Since the family of functions $\{a'_x(\zeta') : x \in \partial\Omega\}$ is uniformly bounded and equicontinuous, there exists $c > 0$ such that

$$\forall x \in \partial\Omega, \quad \forall \zeta'_1, \zeta'_2 \in V, \quad |a'_x(\zeta'_2) - a'_x(\zeta'_1)| \leq c\,|\zeta'_2 - \zeta'_1|.$$

Choose ρ, $0 < \rho \leq R/(4c+4)$. Then

$$\begin{aligned}\forall x \in \partial\Omega, \quad \forall \zeta' \in B_H(0,\rho), \quad |a'_x(\zeta')| \leq R/4 \quad &\Rightarrow |A_x(\zeta' + a_x(\zeta')e_N)| < R\\ \Rightarrow \{x + A_x[\zeta' + a_x(\zeta')e_N] : \zeta' \in B_H(0,\rho)\} &\subset \partial\Omega \cap B(x,R).\end{aligned}$$

Since $B_H(0,\rho)$ is path-connected and the map $\zeta' \mapsto x + A_x[\zeta' + a_x(\zeta')e_N]$ is continuous, the image of $B_H(0,\rho)$ is path-connected. Redefine V' as $B_H(0,\rho)$ and U' as $B(0,R) \cap \{y \in \mathbf{R}^{\mathrm{N}} : P_H(y) \in B_H(0,\rho)\}$ and take the restrictions of the functions a'_x to $B_H(0,\rho)$. From this construction, for all $x \in \partial\Omega$ the new sets $\mathcal{U}(x') \cap \partial\Omega$ and $\mathcal{U}(x') \cap \operatorname{int}\Omega$ are path-connected.

Using the same notation for the finite sequence $x_i \in \partial\Omega$, let $\{\mathcal{U}'(x_i) : 1 \leq i \leq k\}$ be the new finite subcovering of $\partial\Omega$. For each $x \in \operatorname{int}\Omega$ there exists $0 < r_x < R/2$ such that $B(x, r_x) \subset \operatorname{int}\Omega$. Consider the new set $\Omega_R = \Omega \backslash \cup_{i=1}^k \overline{\mathcal{U}'(x_i)}$. Since $\overline{\Omega_R}$ is compact, there exists a finite open covering of $\overline{\Omega_R}$ by the sets $\{B(x_i, r_i) : k+1 \leq i \leq m\}$, $x_i \in \Omega_R$, and $r_i = r_{x_i}$. By construction $B(x_i, r_i) \subset \operatorname{int}\Omega$.

(i) We first show that $\operatorname{dist}_{\overline{\Omega}}(x,y)$ is uniformly bounded for any two points of $\overline{\Omega}$. Since $\overline{\Omega}$ is covered by a finite number of balls, it is sufficient to show that the geodesic distance between any two points x and y in each ball is bounded. For $k+1 \leq i \leq m$ and $x, y \in B(x_i, r_i)$, $\operatorname{dist}_{\overline{\Omega}}(x,y) = |x-y| < R$.

For $1 \leq i \leq k$, and $x, y \in \mathcal{U}'(x_i)$, there is a graph representation $x = x_i + A_i(\zeta' + \zeta_N e_N)$ and $y = x_i + A_i(\xi' + \xi_N e_N)$. Choose as first path the line from $x = x_i + A_i(\zeta' + \zeta_N e_N)$ to $x_i + A_i(\zeta' + a_i(\zeta')e_N)$, follow the boundary along the second path $\{\lambda\zeta' + (1-\lambda)\xi', a_i(\lambda\zeta' + (1-\lambda)\xi') : \lambda \in [0,1]\}$ from $x_i + A_i(\zeta' + a_i(\zeta')e_N)$

to $x_i+A_i(\xi'+a_i(\xi')e_N)$, and finally as third path the line from $x_i+A_i(\xi'+a_i(\xi')e_N)$ to $y = x_i + A_i(\xi' + \xi_N e_N)$. Its length is bounded by

$$2R + \int_0^1 \sqrt{|\zeta' - \xi'|^2 + |\nabla a_i(\lambda\zeta' + (1-\lambda)\xi') \cdot (\zeta' - \xi')|^2}\, d\lambda + 2R,$$

which is bounded by $2R + \sqrt{1+c^2}\,2\rho + 2R$.

Finally we proceed by contradiction. By definition of the geodesic distance $\text{dist}_{\overline{\Omega}}(x,y)/|x-y| \geq 1$. If (5.39) is not true, there exist sequences $\{x_n\}$ and $\{y_n\}$ in $\overline{\Omega}$ such that $\text{dist}_{\overline{\Omega}}(x_n,y_n)/|x_n-y_n|$ goes to $+\infty$, and hence, by boundedness of $\text{dist}_{\overline{\Omega}}(x_n,y_n)$, $|x_n - y_n|$ goes to zero. Since $\overline{\Omega}$ is bounded there exist subsequences, still denoted by $\{x_n\}$ and $\{y_n\}$, and a point $x \in \overline{\Omega}$ such that $x_n \to x$ and $y_n \to x$. If $x \in \text{int}\,\Omega$, there exists $r' > 0$ such that $B(x,r') \subset \text{int}\,\Omega$ and N such that for all $n > N$, x_n, $y_n \in B(x,r')$, where $\text{dist}_{\overline{\Omega}}(x_n,y_n) = |x_n - y_n| \to 0$ and we get a contradiction.

If $x \in \partial\Omega$, $P_H(B(x,\rho)-x) = B_H(0,\rho) = V'$. There exists N such that, for all $n > N$, $x_n, y_n \in B(0,\rho)$ and there exist (ζ_n', ζ_{nN}), and (ξ_n', ξ_{nN}), $\zeta_n', \xi_n' \in B_H(0,\rho)$, $\xi_{nN} \geq a_x'(\xi_n)$, and $\zeta_{nN} \geq a_x'(\zeta_n)$ such that $x_n = x + A_x[\xi_n' + \xi_{nN}e_N]$ and $y_n = x + A_x[\zeta_n' + \zeta_{nN}e_N]$. The path $\{p(\lambda);\, 0 \leq \lambda \leq 1\}$ defined as

$$p(\lambda) = x + A_x \begin{bmatrix} \lambda\zeta_n' + (1-\lambda)\xi_n' \\ a_x\left(\lambda\zeta_n' + (1-\lambda)\xi_n'\right) + \lambda\left[\zeta_{nN} - a_x(\zeta_n')\right] + (1-\lambda)\left[\xi_{nN} - a_x(\xi_n')\right] \end{bmatrix}$$

stays in $B(x,R) \cap \overline{\Omega}$. Indeed $\lambda\zeta_n' + (1-\lambda)\xi_n' \in B_H(0,\rho)$,

$$\begin{aligned} &e_N \cdot A_x^{-1}(p(\lambda) - x) \\ &= a_x\left(\lambda\zeta_n' + (1-\lambda)\xi_n'\right) + \lambda\left[\zeta_{nN} - a_x(\zeta_n')\right] + (1-\lambda)\left[\xi_{nN} - a_x(\xi_n')\right] \\ &\geq a_x\left(\lambda\zeta_n' + (1-\lambda)\xi_n'\right), \end{aligned}$$

and $p(\lambda) \in B(x,R)$. This last property follows from the estimate

$$\begin{aligned} |p(\lambda) - x| \leq &\sqrt{|\lambda\zeta_n' + (1-\lambda)\xi_n'|^2 + |a_x\left(\lambda\zeta_n' + (1-\lambda)\xi_n'\right)|^2} \\ &+ |\lambda\left[\zeta_{nN} - a_x(\zeta_n')\right] + (1-\lambda)\left[\xi_{nN} - a_x(\xi_n')\right]| \\ \leq &\sqrt{1+c^2}\,|\lambda\zeta_n' + (1-\lambda)\xi_n'| + \lambda\,|\zeta_{nN} - a_x(\zeta_n')| + (1-\lambda)\,|\xi_{nN} - a_x(\xi_n')| \\ \leq &\sqrt{1+c^2}\,\rho + \lambda\,2\rho + (1-\lambda)\,2\rho = (2 + \sqrt{1+c^2})\,\rho < R. \end{aligned}$$

Since the path lies in $B(x,R) \cap \overline{\Omega}$, its length can be estimated by computing

$$p'(\lambda) = A_x \begin{bmatrix} \zeta_n' - \xi_n' \\ \nabla a_x(\lambda\zeta_n' + (1-\lambda)\xi_n') \cdot (\zeta_n' - \xi_n') + [\zeta_{nN} - \xi_{nN}] + a_x(\xi_n') - a_x(\zeta_n') \end{bmatrix}$$

$$\Rightarrow\ |p'(\lambda)| \leq |\zeta_n - \xi_n| + 2c|\zeta_n' - \xi_n'| \leq (1+2c)\,|y_n - x_n|$$

$$\Rightarrow\ \text{dist}_{\overline{\Omega}}(x_n,y_n) \leq \int_0^1 |p'(\lambda)|\,d\lambda \leq (1+2c)\,|y_n - x_n| \leq (1+2c)\,2\rho < R$$

and we again get a contradiction.

(ii) The case where $\partial\Omega$ is path-connected follows by the same constructions and proof as in part (i). $\square$

5.4.5 Nonhomogeneous Neumann and Dirichlet Problems

With the help of the previous definition of boundary measure in section 5.4.3, we can now make sense of the nonhomogeneous Neumann and Dirichlet problems (for the Laplace equation) in Lipschitzian domains. Assuming that the functions a_j belong to $W^{1,\infty}(V_j)$, it can be shown that the classical Stokes divergence theorem holds for such domains. Given a bounded smooth domain D in $\mathbf{R}^{\mathrm{N}}$ and a Lipschitzian domain Ω in D, then

$$\forall \vec{\varphi} \in C^1(\overline{D}, \mathbf{R}^{\mathrm{N}}), \quad \int_\Omega \operatorname{div} \vec{\varphi}\, dx = \int_\Gamma \vec{\varphi} \cdot n\, d\Gamma, \tag{5.41}$$

where the outward unit normal field n is defined by (5.36) for almost all $\in \Gamma_j$ (with $y = h_x(\zeta', 0)$, $\zeta' \in V_j$) as

$$n(y) = A_j \frac{{}^*(\partial_1 a_j(\zeta'), \dots, \partial_{N-1} a_j(\zeta'), -1)}{\sqrt{1 + |\nabla a_j(\zeta')|^2}}.$$

The trace of a function g in $W^{1,1}(D)$ on Γ is defined through a $W^{1,\infty}(D)^N$-extension N of the normal n as

$$\forall \varphi \in \mathcal{D}(\mathbf{R}^{\mathrm{N}}), \quad \int_\Gamma g\varphi\, d\Gamma \stackrel{\text{def}}{=} \int_\Omega \operatorname{div}(g\varphi N)\, dx \tag{5.42}$$

(cf. S. Agmon, A. Douglis, and L. Nirenberg [**1**, **2**]). The right-hand side is well-defined in the usual sense so that by the Stokes divergence theorem we obtain the trace $g|_\Gamma$ defined on Γ. This trace is uniquely defined, denoted by $\gamma_\Gamma g$, and

$$\gamma_\Gamma \in \mathcal{L}\big(W^{1,1}(\Omega), L^1(\Gamma)\big). \tag{5.43}$$

Let Ω be a Lipschitzian domain in D. Given $f \in L^2(D)$ and $g \in H^1(D)$, consider the following problem:

$$\begin{aligned} &\exists y \in H^1(\Omega) \text{ such that } \forall \varphi \in H^1(\Omega), \\ &\int_\Omega \nabla y \cdot \nabla \varphi\, dx = \int_\Omega f\varphi\, dx + \int_\Gamma g\, \varphi\, d\Gamma. \end{aligned} \tag{5.44}$$

If the condition

$$\int_\Omega f\, dx + \int_\Gamma g\, d\Gamma = 0 \tag{5.45}$$

is satisfied, then by the Lax–Milgram theorem, problem (5.44)–(5.45) has a unique solution in $H^1(\Omega)/\mathbf{R}$: any two solutions of problem (5.44) can differ only by a constant. For instance, we can associate with the solution $y = y(\Omega)$ in $H^1(\Omega)/\mathbf{R}$ of (5.44) the following objective function:

$$J(\Omega) \stackrel{\text{def}}{=} c\big(\Omega, y(\Omega)\big), \quad c(\Omega, \varphi) \stackrel{\text{def}}{=} \int_\Omega \left(|\nabla \varphi(x)| - G\right)^2\, dx \tag{5.46}$$

for some G in $L^2(D)$.

For the nonhomogeneous Dirichlet problem let Ω be a Lipschitzian domain in D and consider the weak form of the problem

$$\begin{aligned}&\exists y \in H^1(\Omega) \text{ such that } \forall \varphi \in H^2(\Omega) \cap H_0^1(\Omega),\\ &-\int_\Omega y \triangle \varphi \, dx = -\int_\Gamma g \frac{\partial \varphi}{\partial n} \, d\Gamma + \int_\Omega f \varphi \, dx\end{aligned} \tag{5.47}$$

for data (f, g) in $L^2(D) \times H^1(D)$. If this problem has a solution y, then by Green's theorem

$$\int_\Omega \nabla y \cdot \nabla \varphi \, dx - \int_\Gamma y \frac{\partial \varphi}{\partial n} \, d\Gamma = -\int_\Gamma g \frac{\partial \varphi}{\partial n} \, d\Gamma + \int_\Omega f \varphi \, dx \tag{5.48}$$

and

$$-\triangle y = f \in L^2(\Omega), \quad y|_\Gamma = g \in H^{\frac{1}{2}}(\Gamma). \tag{5.49}$$

Here, for instance, we can associate with y the objective function

$$J(\Omega) \stackrel{\text{def}}{=} c\big(\Omega, y(\Omega)\big), \quad c(\Omega, \varphi) \stackrel{\text{def}}{=} \frac{1}{2} \int_\Omega |\varphi - G|^2 \, dx, \quad G \in L^2(D). \tag{5.50}$$

6 Sets Locally Described by a Geometric Property

A large class of sets Ω can be characterized by geometric *segment properties*. The basic *segment property* is equivalent to the property that the set is locally a C^0 epigraph (cf. section 6.2). For instance, it is sufficient to get the density of $C^k(\overline{\Omega})$ in the Sobolev space $W^{m,p}(\Omega)$ for any $m \geq 1$ and $k \geq m$. In this section, we establish the equivalence of the *segment*, the *uniform segment*, and the more recent[9] *uniform fat segment* properties with the respective *locally* C^0, the C^0, and the *equi-*C^0 *epigraph* properties (cf. sections 6.2, 6.3, and 6.4). Under the uniform fat segment property the local functions all have the same modulus of continuity specified by the continuity of the dominating function at the origin (cf. section 6.3). For sets with a compact boundary the three segment properties are equivalent (cf. section 6.1). However, for sets with an unbounded boundary the uniform segment property is generally too *meager* to make the local epigraphs uniformly bounded and equicontinuous.

Section 6.4 specializes the uniform fat segment property to the uniform *cusp* and *cone properties*, which imply that $\partial\Omega$ is, respectively, an equi-Hölderian and equi-Lipschitzian epigraph.

Section 6.5 discusses the existence of a locally finite boundary measure. We have already seen in section 5.4.3 that Lipschitzian sets have a locally finite boundary or surface (that is Hausdorff H_{N-1}) measure. So we can make sense of boundary conditions associated with a partial differential equation on the domain. In general, this is not true for Hölderian domains. To illustrate that point, we construct examples of Hölderian sets for which the Hausdorff dimension of the boundary is strictly greater than $N - 1$ and hence $H_{N-1}(\partial\Omega) = +\infty$.

[9] This was introduced starting with the uniform cusp property in M. C. Delfour and J.-P. Zolésio [**37**] in 2001 and further refined and generalized in the sequence of papers by M. C. Delfour, N. Doyon, and J.-P. Zolésio [**1**, **2**] in 2005 and M. C. Delfour and J.-P. Zolésio [**43**] in 2007.

6.1 Definitions and Main Results

In this section we use the notation introduced at the beginning of section 5: e_N is a unit vector in $\mathbf{R}^{\mathrm{N}}$ and $H = \{e_N\}^{\perp}$. The *open segment* between two distinct points x and y of $\mathbf{R}^{\mathrm{N}}$ will be denoted by

$$(x, y) \stackrel{\text{def}}{=} \{x + t(y - x) \, : \, \forall t, \, 0 < t < 1\}.$$

Definition 6.1.
Let Ω be a subset of $\mathbf{R}^{\mathrm{N}}$ such that $\partial\Omega \neq \varnothing$.

(i) Ω is said to satisfy the *segment property* if

$$\begin{gathered}\forall x \in \partial\Omega, \, \exists r > 0, \, \exists \lambda > 0, \, \exists A_x \in \mathrm{O(N)}, \\ \text{such that} \quad \forall y \in B(x, r) \cap \overline{\Omega}, \quad (y, y + \lambda A_x e_N) \subset \operatorname{int}\Omega.\end{gathered}$$

(ii) Ω is said to satisfy the *uniform segment property* if

$$\exists r > 0, \, \exists \lambda > 0 \text{ such that } \forall x \in \partial\Omega, \, \exists A_x \in \mathrm{O(N)},$$

such that

$$\forall y \in B(x, r) \cap \overline{\Omega}, \quad (y, y + \lambda A_x e_N) \subset \operatorname{int}\Omega.$$

(iii) Ω is said to satisfy the *uniform fat segment property* if there exist $r > 0$, $\lambda > 0$, and an open region $\mathcal{O}$ of $\mathbf{R}^{\mathrm{N}}$ containing the segment $(0, \lambda e_N)$ and not 0 such that for all $x \in \partial\Omega$,

$$\exists A_x \in \mathrm{O(N)} \text{ such that } \forall y \in B(x, r) \cap \overline{\Omega}, \quad y + A_x \mathcal{O} \subset \operatorname{int}\Omega. \tag{6.1}$$

□

Of course, when Ω is an *open domain*, $\Omega = \operatorname{int}\Omega$, and we are back to the standard definition. Yet, as in Definition 3.1, those definitions involve only $\partial\Omega$, $\overline{\Omega}$, and $\operatorname{int}\Omega$ and the properties remain true for all sets in the class

$$[\Omega]_b = \left\{A \subset \mathbf{R}^{\mathrm{N}} \, : \, \overline{A} = \overline{\Omega} \text{ and } \partial A = \partial\Omega\right\}.$$

A set having the segment property must have an $(N-1)$-dimensional boundary and cannot simultaneously lie on both sides of any given part of its boundary. In fact, a domain satisfying the segment property is locally a C^0 epigraph in the sense of Definition 5.1, as we shall see in section 6.2. We first give a few general properties.

Theorem 6.1. *Let Ω be a subset of $\mathbf{R}^{\mathrm{N}}$ such that $\partial\Omega \neq \varnothing$.*

(i) *If*

$$\forall x \in \partial\Omega, \, \exists \lambda > 0, \, \exists A_x \in \mathrm{O(N)}, \quad (x, x + \lambda A_x e_N) \in \operatorname{int}\Omega, \tag{6.2}$$

then $\operatorname{int}\Omega \neq \varnothing$ *and*

$$\overline{\operatorname{int}\Omega} = \overline{\Omega} \quad \textit{and} \quad \partial(\operatorname{int}\Omega) = \partial\Omega.$$

(ii) Ω *satisfies the segment property if and only if* $\complement\Omega$ *satisfies the segment property. In particular*

$$\begin{aligned} \operatorname{int}\Omega \neq \varnothing, \quad \overline{\operatorname{int}\Omega} = \overline{\Omega}, \ \textit{and}\ \partial(\operatorname{int}\Omega) = \partial\Omega, \\ \operatorname{int}\complement\Omega \neq \varnothing, \quad \overline{\operatorname{int}\complement\Omega} = \overline{\complement\Omega}, \ \textit{and}\ \partial(\operatorname{int}\complement\Omega) = \partial\Omega. \end{aligned}$$

Proof. (i) Pick any point $x \in \partial\Omega$. By definition, for all $\mu \in]0,1[\,$, $x_\mu = x + \mu\,\lambda\,Ae_N \in \operatorname{int}\Omega$ and $\operatorname{int}\Omega \neq \varnothing$. As a result there exists a sequence $\mu_n > 0 \to 0$ such that

$$\operatorname{int}\Omega \ni x_{\mu_n} \to x \quad \Rightarrow \partial\Omega \subset \overline{\operatorname{int}\Omega} \subset \overline{\Omega} \quad \Rightarrow \overline{\Omega} \subset \overline{\operatorname{int}\Omega} \subset \overline{\Omega}.$$

The second identity follows from Lemma 5.1 (i).

(ii) Assume that Ω satisfies the segment property. For any $x \in \partial\Omega$, there exist $r > 0$, $\lambda > 0$, and $A \in \mathrm{O(N)}$ such that for all $y \in B(x,r) \cap \overline{\Omega}$, $(y, y+\lambda Ae_N) \subset \operatorname{int}\Omega$. We claim that

$$\forall y \in B(x,r) \cap \overline{\complement\Omega}, \quad (y, y + \lambda(-A)e_N) \subset \operatorname{int}\complement\Omega.$$

Indeed, if there exist $z \in B(x,r) \cap \overline{\complement\Omega}$ and $\alpha \in (0,\lambda)$ such that $y - \alpha Ae_N \in \overline{\Omega}$, then $y = (y - \alpha Ae_N) + \alpha Ae_N \in \operatorname{int}\Omega$, which leads to a contradiction. In the other direction we interchange Ω and $\complement\Omega$ and repeat the proof. Finally, the two sets of properties follow from part (i) applied to Ω and $\complement\Omega$. □

We summarize the equivalences between the epigraph and segment properties that will be detailed in Theorems 6.5, 6.6, and 6.7 with the associated constructions and additional motivation.

Theorem 6.2. *Let* Ω *be a subset of* $\mathbf{R}^\mathrm{N}$ *such that* $\partial\Omega \neq \varnothing$.

(i) (a) Ω *is locally a* C^0 *epigraph if and only if* Ω *has the segment property;*
 (b) Ω *is a* C^0 *epigraph if and only if* Ω *has the uniform segment property;*
 (c) Ω *is an equi-*C^0 *epigraph if and only if* Ω *has the uniform fat segment property.*

(ii) *If, in addition,* $\partial\Omega$ *is compact, the six properties are equivalent and there exists a* dominating function $h \in \mathcal{H}$ *that satisfies the properties of Theorems* 5.1 *and* 5.2.

Proof. (i) From Theorems 6.5, 6.6, and 6.7. (ii) When $\partial\Omega$ is compact, from part (i) and the equivalences in Theorem 5.2. □

We quote the following density results from R. A. ADAMS [**1**, Thm. 3.18, p. 54].

Theorem 6.3. *If* Ω *has the segment property, then the set*

$$\left\{ f|_{\operatorname{int}\Omega} : \forall f \in C_0^\infty(\mathbf{R}^\mathrm{N}) \right\}$$

of restrictions of functions of $C_0^\infty(\mathbf{R}^\mathrm{N})$ *to* $\operatorname{int}\Omega$ *is dense in* $W^{m,p}(\operatorname{int}\Omega)$ *for* $1 \leq p < \infty$ *and* $m \geq 1$. *In particular,* $C^k(\overline{\operatorname{int}\Omega})$ *is dense in* $W^{m,p}(\operatorname{int}\Omega)$ *for any* $m \geq 1$ *and* $k \geq m$.

6.2 Equivalence of Geometric Segment and C^0 Epigraph Properties

As for sets that are locally a C^0 epigraph in Definition 5.1, the three cases of Definition 6.1 differ only when $\partial\Omega$ is unbounded. We first deal with the first two and will come back to the third one in section 6.3.

Theorem 6.4. *Let $\partial\Omega \neq \varnothing$ be compact. Then the segment property and the uniform segment property of Definition* 6.1 *are equivalent.*

Proof. It is sufficient to show that the segment property implies the uniform segment property. Since $\partial\Omega$ is compact there exists a finite open subcover $\{B_i\}_{i=1}^m$, $B_i = B(x_i, r_{x_i})$, of $\partial\Omega$ for some finite sequence $\{x_i\}_{i=1}^m$ of points of $\partial\Omega$. From part (i) of the proof of Theorem 5.2

$$\exists r > 0, \forall x \in \partial\Omega, \quad \exists i, 1 \leq i \leq m, \text{ such that } B(x,r) \subset B_i.$$

Define $\lambda = \min_{1\leq i\leq m} \lambda_i > 0$. Therefore for each $x \in \partial\Omega$, there exists i, $1 \leq i \leq m$, such that $B(x,r) \subset B_i$ and hence, by choosing $A_x = A_{x_i} \in \mathrm{O(N)}$, we get

$$\forall y \in \overline{\Omega} \cap B_i, \quad (y, y + \lambda A_x e_N) \subset (y, y + \lambda_i A_{x_i} e_N) \subset \operatorname{int}\Omega.$$

By restricting the above property to $B(x,r) \subset B_i$,

$$\forall y \in \overline{\Omega} \cap B(x,r), \quad (y, y + \lambda A_x e_N) \subset (y, y + \lambda_i A_x e_N) \subset \operatorname{int}\Omega.$$

Since r and λ are independent of $x \in \partial\Omega$, Ω has the uniform segment property. $\square$

Theorem 6.5. *Let Ω be a subset of $\mathbf{R}^\mathrm{N}$ such that $\partial\Omega \neq \varnothing$.*

(i) *If Ω satisfies the segment property, then Ω is locally a C^0 epigraph. For $x \in \partial\Omega$, let $B(x, r_x)$, λ_x, and A_x be the associated open ball, the height, and the rotation matrix. Then there exists ρ_x,*

$$0 < \rho_x \leq r_{x\lambda} \stackrel{\text{def}}{=} \min\{r_x, \lambda_x/2\}, \tag{6.3}$$

which is the largest radius such that

$$B_H(0,\rho_x) \subset \left\{P_H(A_x^{-1}(y-x)) \,:\, \forall y \in B(x, r_{x\lambda}) \cap \partial\Omega\right\}.$$

The neighborhoods of Definition 5.1 *can be chosen as*

$$\begin{aligned} V_x &\stackrel{\text{def}}{=} B_H(0,\rho_x) \text{ and} \\ \mathcal{U}(x) &\stackrel{\text{def}}{=} B(x, r_{x\lambda}) \cap \left\{y \in \mathbf{R}^\mathrm{N} \,:\, P_H(A_x^{-1}(y-x)) \in V_x\right\}, \end{aligned} \tag{6.4}$$

where[10] *$B_H(0,\rho_x)$ denotes the open ball of radius ρ_x in the hyperplane H. For each $\zeta' \in V_x$, there exists a unique $y_{\zeta'} \in \partial\Omega\cap\mathcal{U}(x)$ such that $P_H A_x^{-1}(y_{\zeta'} - x) = \zeta'$ and the function*

$$\zeta' \mapsto a_x(\zeta') \stackrel{\text{def}}{=} (y_{\zeta'} - x) \cdot (A_x e_N) : V_x \to \mathbf{R}$$

[10] Note that $\mathcal{U}(x) - \{x\} = A_x\, [B(0, r_{x\lambda}) \cap \{\zeta : P_H\zeta \in V_x\}]$.

is well-defined, bounded,

$$\forall \zeta' \in V_x, \quad |a_x(\zeta')| < r_{x\lambda}, \tag{6.5}$$

and uniformly continuous in V_x*, that is,* $a_x \in C^0(\overline{V_x})$.

(ii) *Conversely, if* Ω *is locally a* C^0 *epigraph, both* Ω *and* $\complement\Omega$ *satisfy the segment property.*

Proof. (i) By assumption for each $x \in \partial\Omega$, $\exists r > 0$, $\exists \lambda > 0$, $\exists A \in \mathrm{O(N)}$ such that

$$\forall y \in B(x,r) \cap \overline{\Omega}, \quad (y, y + \lambda A e_N) \subset \operatorname{int}\Omega.$$

Consider the set

$$W \overset{\text{def}}{=} \left\{ P_H A^{-1}(y-x) : \forall y \in B(x, r_\lambda) \cap \partial\Omega \right\}, \quad r_\lambda = \min\{r, \lambda/2\}.$$

Therefore

$$\forall \zeta' \in W, \quad \exists y \in B(x,r) \cap \partial\Omega, \quad P_H A^{-1}(y-x) = \zeta'.$$

If there exist y_1, $y_2 \in B(x, r_\lambda) \cap \partial\Omega$ such that $P_H A^{-1}(y_1 - x) = P_H A^{-1}(y_2 - x) = \zeta'$, then $y_2 \in (y_1, y_1 + Ae_N \cdot (y_2 - y_1)\, Ae_N) \subset (y_1, y_1 + \lambda A e_N) \in \operatorname{int}\Omega$ since $|y_2 - y_1| < 2r_\lambda < \lambda$ and we get a contradiction. Therefore, the map

$$\boxed{\zeta' \mapsto a(\zeta') \overset{\text{def}}{=} Ae_N \cdot (y-x) = e_N \cdot A^{-1}(y-x) : W \to \mathbf{R}}$$

is well-defined and $a(0) = 0$.

(W *is open and* $0 \in W$). By definition, $0 \in W \subset B_H(0, r_\lambda)$. Given any point $\xi' \in W$, the segment property is satisfied at the point

$$y_{\xi'} \overset{\text{def}}{=} x + A(\xi' + a(\xi') e_N) \in \partial\Omega \cap B(x, r_\lambda),$$

and $(y_{\xi'}, y_{\xi'} + \lambda A e_N) \subset \operatorname{int}\Omega$ and $(y_{\xi'}, y_{\xi'} - \lambda A e_N) \subset \operatorname{int}\complement\Omega$ by Theorem 6.1 (ii):

$$\begin{aligned}(y_{\xi'}, y_{\xi'} + \lambda A e_N) \cap B(x, r_\lambda) &\subset \operatorname{int}\Omega \cap B(x, r_\lambda),\\ (y_{\xi'}, y_{\xi'} - \lambda A e_N) \cap B(x, r_\lambda) &\subset \operatorname{int}\complement\Omega \cap \cap B(x, r_\lambda).\end{aligned}$$

Therefore, there exists a neighborhood $B_H(\xi', R)$ of ξ' such that $B_H(\xi', R) \subset B_H(0, r_\lambda)$ and for all $\zeta' \in B_H(\xi', R)$, $L_{\zeta'} \cap \operatorname{int}\Omega \cap B(x, r_\lambda) \neq \varnothing$, where $L_{\zeta'} = x + A(\zeta' + \mathbf{R}\, e_N)$ is the line through $x + A\zeta'$ in the direction Ae_N.

If ξ' is not an interior point of W, there exists a sequence $\{\zeta'_n\} \subset B_H(\xi', R)$, $\zeta'_n \to \xi'$, such that for all n, $(L_{\zeta'_n} \cap B(x, r_\lambda)) \cap \partial\Omega = \varnothing$. Therefore, either $L_{\zeta'_n} \cap B(x, r_\lambda) \subset \operatorname{int}\Omega$ or $L_{\zeta'_n} \cap B(x, r_\lambda) \subset \operatorname{int}\complement\Omega$. Since for points $\zeta'_n \in B_H(\xi', R)$ we already know that $L_{\zeta'_n} \cap B(x, r_\lambda) \cap \operatorname{int}\Omega \neq \varnothing$, then $L_{\zeta'_n} \cap B(x, r_\lambda) \subset \operatorname{int}\Omega$. Choose

$$\alpha \overset{\text{def}}{=} \frac{1}{2}\left(a(\xi') - \sqrt{r_\lambda^2 - |\xi'|^2} \right)$$

and the sequence

$$z_n \stackrel{\text{def}}{=} x + A(\zeta'_n + \alpha e_N) \to z \stackrel{\text{def}}{=} y_{\xi'} - \frac{1}{2}\left(a(\xi') + \sqrt{r_\lambda^2 - |\xi'|^2}\right) Ae_N.$$

We get the following estimate:

$$\begin{aligned} |z - x| &< \frac{1}{2}\left(|y_{\xi'} - x| + r_\lambda\right) < r_\lambda, \\ |z_n - x| < |z_n - z| + \frac{1}{2}\left(|y_{\xi'} - x| + r_\lambda\right) &= |\zeta'_n - \xi'| + \frac{1}{2}\left(|y_{\xi'} - x| + r_\lambda\right). \end{aligned}$$

There exists $\bar{N}$ such that, for all $n \geq \bar{N}$,

$$|\zeta'_n - \xi'| = |z_n - z| < \frac{1}{2}\left(r_\lambda - |y_{\xi'} - x|\right) \quad \Rightarrow |z_n - x| < r_\lambda$$

and $z_n \in B(x, r_\lambda) \cap L_{\zeta'_n} \subset \operatorname{int} \Omega \cap B(x, r_\lambda)$. The limit point z belongs to $\overline{\Omega} \cap B(x, r_\lambda)$. However, $P_H A^{-1}(z - x) = \xi'$ and

$$Ae_N \cdot (z - y_{\xi'}) = -\frac{1}{2}\left(a(\xi') + \sqrt{r_\lambda^2 - |\xi'|^2}\right) < 0,$$

since, by construction of W and a, $|a(\xi')| < \sqrt{r_\lambda^2 - |\xi'|^2}$. This means that

$$z \in (y_{\xi'}, y_{\xi'} - \lambda Ae_N) \cap B(x, r_\lambda) \subset \operatorname{int} \complement\Omega \cap \cap B(x, r_\lambda)$$

and we get a contradiction with the fact that $z \in \overline{\Omega} \cap \cap B(x, r_\lambda)$. This proves that all points of W are interior points and, hence, that W is open. Since $0 \in W$, there exists ρ, $0 < \rho \leq r_\lambda$, such that ρ is the largest radius for which $B_H(0, \rho) \subset W$.

(*Choice of the neighborhoods and conditions of Definition* 5.1.) Finally the neighborhoods of Definition 5.1 can be chosen as

$$V \stackrel{\text{def}}{=} B_H(0, \rho) \text{ and } \mathcal{U} \stackrel{\text{def}}{=} B(x, r_\lambda) \cap \left\{y : P_H A^{-1}(y - x) \in B_H(0, \rho)\right\}.$$

By construction, condition (5.20) in Definition 5.2 is satisfied. It is then easy to check properties (5.21) and (5.22).

(*The function a is bounded and uniformly continuous in V.*) Consider a point $\xi' \in V$ and let $y_{\xi'} \in \partial\Omega \cap B(x, r_\lambda)$ be the associated unique point such that

$$\xi' = P_H A^{-1}(y_{\xi'} - x) \quad \text{and} \quad a(\xi') = Ae_N \cdot (y_{\xi'} - x).$$

By construction, since $\xi' \in \left\{P_H A^{-1}(y - x) : y \in B(x, r_\lambda) \cap \partial\Omega\right\}$, $|a(\xi')| \leq |y_{\xi'} - x| < r_\lambda$ and

$$\ell \stackrel{\text{def}}{=} \liminf_{\zeta' \to \xi'} a(\zeta') \quad \text{and} \quad L \stackrel{\text{def}}{=} \limsup_{\zeta' \to \xi'} a(\zeta')$$

are finite. Hence the points

$$\begin{aligned} y_\ell &\stackrel{\text{def}}{=} x + A(\zeta' + \ell e_N) \in \partial\Omega \cap B(x, r_\lambda), \\ y_L &\stackrel{\text{def}}{=} x + x + A(\zeta' + Le_N) \in \partial\Omega \cap B(x, r_\lambda) \end{aligned}$$

belong to $\partial\Omega$ as limit points of points of $\partial\Omega$. If $y_\ell \neq y_{\xi'}$, then $|y_\ell - y_{\xi'}| \leq |y_\ell - x| + |y_{\xi'} - x| < 2r_\lambda \leq \lambda$ and either $y_{\xi'} \in (y_\ell, y_\ell + \lambda d)$ or $y_\ell \in (y_{\xi'}, y_{\xi'} + \lambda d)$. By the segment property, this means that either $y_{\xi'}$ or y_ℓ belongs to $\operatorname{int}\Omega$, which contradicts the fact that both points belong to $\partial\Omega$. Therefore $y_\ell = y_{\xi'}$ and by the same argument $y_L = y_{\xi'}$. Hence

$$\liminf_{\zeta'\to\xi'} a(\zeta') = \ell = (y_\ell - x)\cdot Ae_N = (y_{\xi'} - x)\cdot Ae_N = a(\xi'),$$
$$a(\xi') = (y_{\xi'} - x)\cdot Ae_N = (y_L - x)\cdot Ae_N = L = \limsup_{\zeta'\to\xi'} a(\zeta'),$$

and a is continuous in $\xi' \in W$. In view of the fact that a is uniformly bounded, then $a \in C(\overline{W})$ and a is uniformly continuous in W.

(ii) Pick a point $x \in \partial\Omega$. By Definition 5.1, there exist neighborhoods $\mathcal{U}(x)$ and $V(x)$, $A_x \in \mathrm{O(N)}$, and a continuous function $a_x\colon V_x \to \mathbf{R}$ with the appropriate properties. Choose $r_x > 0$ such that $B(x, 2r_x) \subset \mathcal{U}(x)$. We want to show that for $\lambda_x = r_x$

$$\forall y \in B(x, r_x) \cap \overline{\Omega}, \quad (y, y + \lambda_x A_x e_N) \subset \operatorname{int}\Omega.$$

From property (5.3) in Definition 5.1

$$\mathcal{U}(x) \subset \left\{y \in \mathbf{R}^{\mathrm{N}} : P_H A_x^{-1}(y - x) \in V_x\right\} \quad \Rightarrow B_H(0, 2r_x) \subset V_x$$

and hence for all $y \in B(x, r_x) \cap \overline{\Omega}$ we have $\zeta' \stackrel{\text{def}}{=} P_H A^{-1}(y - x) \in V_x$. Moreover, $y \in B(x, r_x) \cap \overline{\Omega} \subset \mathcal{U}(x) \cap \overline{\Omega}$ implies that

$$(y - x)\cdot A_x e_N \geq a_x(\zeta').$$

Consider the segment $(y, y + \lambda_x A_x e_N)$. For each $0 < t < 1$

$$\begin{aligned}(y + t\lambda_x A_x e_N - x)\cdot A_x e_N &= (y - x)\cdot A_x e_N + t\lambda_x \geq a_x(\zeta') + t\lambda_x > a_x(\zeta')\\ &\Rightarrow (y, y + \lambda_x A_x e_N) \subset A^+,\\ |y + t\lambda_x A_x e_N - x| &\leq |y - x| + \lambda_x < r_x + \lambda_x < 2r_x\\ &\Rightarrow (y, y + \lambda_x A_x e_N) \subset B(x, 2r_x) \cap A^+ \subset \mathcal{U}(x) \cap \operatorname{int}\Omega\\ &\Rightarrow (y, y + \lambda_x A_x e_N) \subset \operatorname{int}\Omega\end{aligned}$$

and Ω satisfies the segment property.

Furthermore, since Ω is locally a C^0 epigraph, so is $\complement\Omega$ by changing $A_x e_N$ into $-A_x e_N$. Hence $\complement\Omega$ also satisfies the segment property. $\square$

Theorem 6.6. *Let Ω be a subset of $\mathbf{R}^{\mathrm{N}}$ such that $\partial\Omega \neq \varnothing$. Ω has the uniform segment property if and only if Ω is a C^0 epigraph. In particular, the neighborhoods $\mathcal{U}(x)$ at $x \in \partial\Omega$ in $\mathbf{R}^{\mathrm{N}}$ and V_x at 0 in H can be chosen in the following way: given $r_\lambda = \min\{r, \lambda/2\}$ and the largest radius $\rho > 0$ such that $B_H(0, \rho) \subset \{P_H z : \forall z \in B(x, r_\lambda) \cap \partial\Omega\}$, let*

$$V \stackrel{\text{def}}{=} B_H(0, \rho) \quad \textit{and} \quad U \stackrel{\text{def}}{=} B(0, r_\lambda) \cap \{y \,:\, P_H z \in B_H(0, \rho)\}$$

and, for each $x \in \partial\Omega$, define

$$V_x \stackrel{\text{def}}{=} V \quad \text{and} \quad \mathcal{U}(x) \stackrel{\text{def}}{=} x + A_x U.$$

Moreover, the functions $a_x : V \to \mathbf{R}$ can be chosen uniformly bounded,

$$\forall x \in \partial\Omega, \forall \zeta' \in V, \quad |a_x(\zeta')| \le r_\lambda,$$

and uniformly continuous in V, that is, $a_x \in C^0(\overline{V})$.

Remark 6.1.
It is important to notice that, when $\partial\Omega$ is unbounded, the *uniform segment property* does not generally imply that the family of functions $\{a_x : x \in \partial\Omega\}$ can be chosen uniformly bounded and equicontinuous. To restore that property the segment will have to be *fattened*, as we shall see in the next section. □

Remark 6.2.
The modulus of continuity in 0 of the function h of part (ii) determines the modulus of continuity of the family of functions $\{a_x : x \in \partial\Omega\}$. □

Proof. (i) By Theorem 6.5, Ω satisfies the segment property if and only if Ω is locally a C^0 epigraph. It remains to sharpen this result.

($\Rightarrow$) Ω satisfies a uniform segment property for some $r > 0$ and $\lambda > 0$, and we can now repeat the constructions in the proof of part (i) of Theorem 6.5. Let $r_\lambda = \min\{r, \lambda/2\}$. With each $x \in \partial\Omega$ can be associated

$$\left\{P_H A_x^{-1}(y - x) : \forall y \in B(x, r_\lambda) \cap \partial\Omega\right\} = \{P_H z : \forall z \in B(x, r_\lambda) \cap \partial\Omega\}$$

since $A_x^{-1} B(x, r_\lambda) = B(x, r_\lambda)$. The set

$$W \stackrel{\text{def}}{=} \{P_H z : \forall z \in B(x, r_\lambda) \cap \partial\Omega\}$$

is independent of x since r_λ is now independent of x. Hence the largest radius $\rho > 0$ such that $B_H(0, \rho) \subset W$ is also independent of x. The family of bounded uniformly continuous functions a_x defined on $V = B_H(0, \rho)$ such that $\partial\Omega$ is locally the graph function of a_x in the neighborhood

$$\begin{aligned} \mathcal{U}(x) &= B(x, r_\lambda) \cap \left\{y \,:\, P_H A_x^{-1}(y - x) \in B_H(0, \rho)\right\} \\ &= x + A_x U, \quad U \stackrel{\text{def}}{=} B(0, r_\lambda) \cap \{y \,:\, P_H z \in B_H(0, \rho)\} \end{aligned}$$

of x. The neighborhoods V_x and $\mathcal{U}(x)$ are now independent of x up to a rotation around the origin. By Definition 5.2 (ii), Ω is a C^0 epigraph. Moreover, by Theorem 6.5 (i) the functions $a_x : V \to \mathbf{R}$ can be chosen uniformly bounded by r_λ (cf. (6.5)), and uniformly continuous in V, that is, $a_x \in C^0(\overline{V})$.

($\Leftarrow$) The proof is the same as the proof of part (ii) of Theorem 6.5. The uniform segment property follows from the fact that the neighborhoods $\mathcal{U}(x) - x$ of 0 are, up to a rotation around 0, independent of $x \in \partial\Omega$. Therefore there exists $r > 0$ such that, for all $x \in \partial\Omega$, $B(0, 2r) \subset \mathcal{U}(x) - x$ and hence $B(x, 2r) \subset \mathcal{U}(x)$. As a result we can repeat the proof with r in place of r_x and $\lambda = r$, which are both independent of x. □

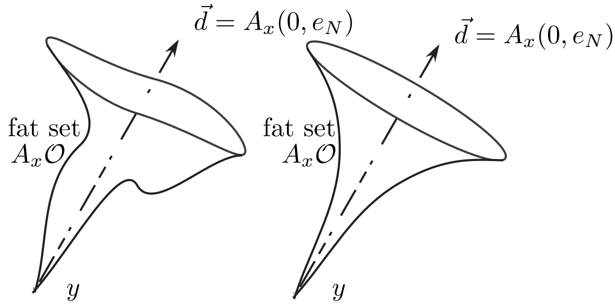


Figure 2.5. *Examples of arbitrary and axially symmetrical $\mathcal{O}$ around the direction $d = A_x(0, e_N)$.*

6.3 Equivalence of the Uniform Fat Segment and the Equi-C^0 Epigraph Properties

The *uniform fat segment property* strengthens the *uniform segment property* by fattening the segment of Definition 6.1 (ii). Before proving the equivalence with an equi-C^0 epigraph, we first show that the open set $\mathcal{O}$ of Definition 6.1 (iii) is equivalent to a parametrized axisymmetrical open region of the form

$$\mathcal{O}(h, \rho, \lambda) \stackrel{\text{def}}{=} \left\{\zeta' + \zeta_N e_N \in \mathbf{R}^{\mathrm{N}} : \zeta' \in B_H(0, \rho), \limsup h(|\zeta')|) < \zeta_N < \lambda\right\} \quad (6.6)$$

for some $\rho > 0$, $\lambda > 0$ and a *dominating function* h that belongs to the *space of dominating functions* introduced in (5.11) (see Figure 2.5):

$$\mathcal{H} = \{h : [0, \infty[\to [0, \infty[\, : \, h(0) = 0 \text{ and } h \text{ is continuous in } 0\} . \quad (6.7)$$

The use of the $\limsup h$ rather than h is necessary to make $\mathcal{O}(h, \rho, \lambda)$ open since we only assume that h is continuous in 0.

Lemma 6.1. (i) *Given $\rho > 0$, $\lambda > 0$, and $h \in \mathcal{H}$, the region $\mathcal{O}(h, \rho, \lambda)$ contains the segment $(0, \lambda e_N)$, does not contain 0, and is open.*

(ii) *Let $\lambda > 0$ be a real number and $\mathcal{O}$ be an open subset of $\mathbf{R}^{\mathrm{N}}$ containing $(0, \lambda e_N)$ and not 0. Then there exist $\rho > 0$ and a continuous function $h : [0, \rho] \to [0, \infty[$ which is monotone strictly increasing such that the set*

$$\mathcal{O}(h, \rho, \lambda/2) \stackrel{\text{def}}{=} \{\xi : \xi' \in B_H(0, \rho) \text{ and } h(|\zeta'|) < \xi_N < \lambda/2\}$$

is open and contains $(0, \lambda e_N/2)$ and not 0, and $\mathcal{O}(h, \rho, \lambda/2) \subset \mathcal{O}$.

Proof. (i) By definition

$$(0, \lambda e_N) = \left\{\zeta_N e_N : 0 = \lim_{\xi' \to 0} h(|\xi'|) < \zeta_N < \lambda\right\} \subset O(h, \rho, \lambda),$$

since $h(0) = 0$ and h is continuous at the origin. Also $0 = 0 + 0e_N \notin \mathcal{O}(h, \rho)$ since it would yield the contradiction $0 = \lim_{\xi' \to 0} h(|\xi'|) < \zeta_N = 0$. To show that $\mathcal{O}(h, \rho, \lambda)$

is open, it is sufficient to show that for each point $\xi = \xi' + \xi_N e_N \in \mathcal{O}(h, \rho, \lambda)$ there exists a neighborhood of ξ contained in $\mathcal{O}(h, \rho, \lambda)$. But this is true by definition of the lim sup: given $(\xi', \xi_N) \in \mathcal{O}(h, \rho, \lambda)$, $\limsup h(|\xi'|) < \xi_N < \lambda$ and there exists a neighborhood $V(\xi') \subset B_H(0, \rho)$ of ξ'. Hence

$$\xi \in \left\{\zeta' + \zeta_N e_N \in \mathbf{R}^{\mathrm{N}} \,:\, \zeta' \in V(\xi'),\, \limsup h(|\zeta')|) < \zeta_N < \lambda\right\} \subset \mathcal{O}(h, \rho, \lambda),$$

and $\mathcal{O}(h, \rho, \lambda)$ is open.

(ii) For each integer $n \geq 0$, let $\lambda_n = \lambda/2^{n+1}$. Since the segment $[\lambda_1 e_N, \lambda_0 e_N]$ is contained in $\mathcal{O}$, there exists a largest radius $r_0 > 0$ such that the cylinder

$$C_0 \stackrel{\text{def}}{=} \{\zeta : |\zeta'| < r_0 \text{ and } \lambda_1 < \zeta_N < \lambda_0\}$$

is contained in $\mathcal{O}$. Again, since the segment $[\lambda_2 e_N, \lambda_1 e_N]$ is contained in $\mathcal{O}$, there exists a largest radius $0 < r_1 \leq r_0/2$ such that the cylinder

$$C_1 \stackrel{\text{def}}{=} \{\zeta : |\zeta'| < r_1 \text{ and } \lambda_2 < \zeta_N < \lambda_1\}$$

is contained in $\mathcal{O}$. Similarly, for $n \geq 2$, there exists a largest radius $0 < r_n \leq r_{n-1}/2$ such that the cylinder

$$C_n \stackrel{\text{def}}{=} \{\zeta : |\zeta'| < r_1 \text{ and } \lambda_n < \zeta_N < \lambda_{n-1}\}$$

is contained in $\mathcal{O}$. By construction, $2^n r_n \leq r_0$, $r_n \searrow 0$, $\cup_{n\in\mathbb{N}} C_n \subset \mathcal{O}$ is open and contains $(0, \lambda e_N/2)$ and not 0. Since the boundary of that set is piecewise constant, we make a few adjustments to make it piecewise linear and continuous. First construct the set

$$O_1 \stackrel{\text{def}}{=} \left\{\zeta : \lambda_1 < \zeta_N < \lambda_0 \text{ and } |\zeta'| < r_0 \frac{\zeta_N - \lambda_1}{\lambda_0 - \lambda_1} + r_1 \frac{\lambda_0 - \zeta_N}{\lambda_0 - \lambda_1}\right\}.$$

For $n \geq 2$, choose

$$O_n \stackrel{\text{def}}{=} \left\{\zeta : \lambda_n < \zeta_N \leq \lambda_{n-1} \text{ and } |\zeta'| < r_{n-1} \frac{\zeta_N - \lambda_n}{\lambda_{n-1} - \lambda_n} + r_n \frac{\lambda_{n-1} - \zeta_N}{\lambda_{n-1} - \lambda_n}\right\}.$$

From the condition $0 < r_n \leq r_{n-1}/2$, the following union is open:

$$\mathcal{O}' \stackrel{\text{def}}{=} \cup_{n\geq 1} O_n, \quad (0, \lambda e_N/2) \subset \mathcal{O}' \subset \mathcal{O}, \text{ and } 0 \notin \mathcal{O}'.$$

Set $\rho = r_0$ and define the function $h : [0, \rho] \to [0, \infty[$ on each interval $[r_{n-1}, r_n[$ as follows:

$$h(\theta) \stackrel{\text{def}}{=} r_{n-1} \frac{\theta - \lambda_n}{\lambda_{n-1} - \lambda_n} + r_n \frac{\lambda_{n-1} - \theta}{\lambda_{n-1} - \lambda_n}, \quad r_{n-1} \leq \theta < r_n.$$

It is continuous and monotone strictly increasing since $r_{n-1} < r_n$ and $\lambda_{n-1} > \lambda_n$. Moreover, it is easy to check that $\mathcal{O}' = \mathcal{O}(h, \rho, \lambda/2)$. □

We now complete the equivalences between the epigraph and the segment properties of Theorems 6.5 and 6.6.

Theorem 6.7. *Let* Ω *be a subset of* $\mathbf{R}^{\mathrm{N}}$ *such that* $\partial\Omega \neq \varnothing$. Ω *is an equi-*C^0 *epigraph if and only if* Ω *has the uniform fat segment property.*

Proof. ($\Leftarrow$) Assume that the uniform fat segment property is verified for $r > 0$, $\lambda > 0$, and the open set $\mathcal{O}$. By Lemma 6.1 (ii), there exist $\rho > 0$ and a continuous function $h : [0, \rho] \to [0, \infty[$ which is monotone strictly increasing, the set

$$\mathcal{O}(h, \rho, \lambda/2) \overset{\text{def}}{=} \{\xi : \xi' \in B_H(0, \rho) \text{ and } h(|\zeta'|) < \xi_N < \lambda/2\}$$

is open and contains $(0, \lambda e_N/2)$ and not 0, and $\mathcal{O}(h, \rho, \lambda/2) \subset \mathcal{O}$. So the uniform fat segment property is verified for $r > 0$, $\lambda/2 > 0$, and the new open set $\mathcal{O}(h, \rho, \lambda/2)$. In particular, Ω satisfies the uniform segment property with $\lambda/2$ instead of λ. By Theorem 6.6, the neighborhoods at $x \in \partial\Omega$ can be chosen in the following way: given $r_\lambda = \min\{r, \lambda/4\}$ and the largest radius $\rho' > 0$ such that $B_H(0, \rho') \subset \{P_H z : \forall z \in B(x, r_\lambda) \cap \partial\Omega\}$, let

$$V \overset{\text{def}}{=} B_H(0, \rho') \quad \text{and} \quad U \overset{\text{def}}{=} B(0, r_\lambda) \cap \{y \,:\, P_H z \in B_H(0, \rho')\}$$

and, for each $x \in \partial\Omega$, define

$$V_x \overset{\text{def}}{=} V \quad \text{and} \quad \mathcal{U}(x) \overset{\text{def}}{=} x + A_x U.$$

By Theorem 6.5 (i) with $\lambda/2$ instead of λ, the function $a_x : V \to \mathbf{R}$ is uniformly continuous in V and

$$\forall \zeta' \in V, \quad |a_x(\zeta')| < r_\lambda.$$

Given ζ' and ξ' in $B_H(0, \rho')$ such that $\xi' - \zeta' \in B_H(0, \rho)$, consider the points

$$\begin{aligned} y_{\zeta'} &\overset{\text{def}}{=} x + A_x(\zeta' + a_x(\zeta')e_N) \in B(x, r_\lambda) \cap \partial\Omega, \\ y_{\xi'} &\overset{\text{def}}{=} x + A_x(\xi' + a_x(\xi')e_N) \in \mathcal{B}(x, r_\lambda) \cap \partial\Omega. \end{aligned}$$

By subtracting,

$$y_{\xi'} = y_{\zeta'} + A_x(\xi' - \zeta' + (a_x(\xi') - a_x(\zeta'))e_N).$$

If $a_x(\xi') - a_x(\zeta') > h(|\xi' - \zeta'|)$, then

$$\begin{aligned} &h(|\xi' - \zeta'|) < a_x(\xi') - a_x(\zeta') < \lambda/4 + \lambda/4 = \lambda/2 \\ \Rightarrow\ & y_{\xi'} = y_{\zeta'} + A_x(\xi' - \zeta' + (a_x(\xi') - a_x(\zeta'))e_N) \in y_{\zeta'} + A_x\mathcal{O}(h, \rho, \lambda/2) \subset \operatorname{int}\Omega. \end{aligned}$$

But this contradicts the fact that $y_{\xi'} \in \partial\Omega$.

Since the argument can be repeated with ξ' in place of ζ', for all $x \in \partial\Omega$,

$$\forall \zeta',\, \xi' \in B_H(0, \rho') \text{ such that } |\xi' - \zeta'| < \rho', \quad |a_x(\xi') - a_x(\zeta')| \leq h(|\xi' - \zeta'|). \tag{6.8}$$

Since h is continuous in 0, for all $\varepsilon > 0$, there exists δ, $0 < \delta < \rho$, such that $\theta < \delta$ implies $h(\theta) < \varepsilon$. Hence, for all $x \in \partial\Omega$,

$$\forall \zeta',\, \xi' \in B_H(0,\rho') \text{ such that } |\xi' - \zeta'| < \delta, \quad |a_x(\xi') - a_x(\zeta')| \leq h(|\xi' - \zeta'|) < \varepsilon$$

and the family $\{a_x\}$ is equicontinuous with respect to $x \in \partial\Omega$.

($\Rightarrow$) We go back to the proof of (i) $\Rightarrow$ (ii) in Theorem 5.1 that we copy below for the benefit of the reader in order to keep track of all the parameters. By Definition 5.1 (iii), Ω is *an equi-C^0 epigraph* and the family of functions $\{a_x \in C^0(\overline{V}) : x \in \partial\Omega\}$ is uniformly bounded and equicontinuous. In particular, $\mathcal{U}(x)$ and V_x can be chosen in such a way that V_x and $A_x^{-1}(\mathcal{U}(x) - x)$ are independent of x: there exist bounded open neighborhoods V of 0 in H and U of 0 in $\mathbf{R}^{\mathrm{N}}$ such that

$$\forall x \in \partial\Omega, \quad V_x \stackrel{\text{def}}{=} V \quad \text{and} \quad \exists A_x \in \mathrm{O(N)} \text{ such that } \mathcal{U}(x) \stackrel{\text{def}}{=} x + A_x U.$$

Choose $r > 0$ such that $B(0, 3r) \subset U$. From property (5.3) in Definition 5.1

$$\mathcal{U}(x) \subset \left\{ y \in \mathbf{R}^{\mathrm{N}} : P_H A_x^{-1}(y - x) \in V \right\} \quad \Rightarrow B_H(0, 3r) \subset V.$$

Use the *modulus of continuity* defined in (5.14):

$$\forall \theta \in [0, 3r], \quad h(\theta) = \sup_{\substack{\zeta' \in H \\ |\zeta'|=1}} \sup_{x \in \partial\Omega} \sup_{\substack{\xi' \in V \\ \xi' + \theta\zeta' \in V}} |a_x(\theta\zeta' + \xi') - a_x(\xi')|. \tag{6.9}$$

The function $h(\theta)$, as the sup of the family $\theta \mapsto |a_x(\theta\zeta' + \xi') - a_x(\xi')|$ of continuous functions with respect to (ζ', ξ'), is *lower semicontinuous* with respect to θ and bounded in $[0, 3r]$ since the a_x's are uniformly bounded. By construction $h(0) = 0$ and by equicontinuity of the a_x, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\theta \in [0, 3r]$, $|\theta| < \delta$,

$$\forall x \in \partial\Omega, \forall \zeta' \in H, |\zeta'| = 1, \quad |a_x(\theta\zeta' + \xi') - a_x(\xi')| < \varepsilon \quad \Rightarrow |h(\theta)| < \varepsilon,$$

where h is continuous in 0. In view of Remark 5.3, choose $\rho = 3r$ and the new neighborhoods $V' = B_H(0, \rho)$ and $U' = \{\xi \in B(0, \rho) : P_H\xi \in V'\} = B(0, \rho)$ and extend h by the constant $h(\rho)$ to $[\rho, \infty[$. Therefore $h \in \mathcal{H}$ and, by definition of h, for all $x \in \partial\Omega$ and for all ξ' and ζ' in $B_H(0, \rho)$ such that $|\xi' - \zeta'| < \rho$

$$\begin{aligned} |a'_x(\xi') - a'_x(\zeta')| &= \left| a'_x \left(\zeta' + |\xi' - \zeta'| \frac{\xi' - \zeta'}{|\xi' - \zeta'|} \right) - a'_x(\zeta') \right| \leq h(|\xi' - \zeta'|) \\ \Rightarrow \forall x \in \partial\Omega, \forall \xi', \zeta' \in B_H(0, \rho)\, |\xi' - \zeta'| < \rho, &\quad |a'_x(\xi') - a'_x(\zeta')| \leq h(|\xi' - \zeta'|). \end{aligned}$$

Now that we have recalled the proof of Theorem 5.1 we complete the proof by showing that the uniform fat segment property of Definition 6.1 is verified with $\lambda = r > 0$, $r > 0$, and

$$\mathcal{O}(h, r, r) \stackrel{\text{def}}{=} \left\{ \zeta' + \zeta_N e_N \in \mathbf{R}^{\mathrm{N}} \,:\, \zeta' \in B_H(0, r),\, \limsup h(|\zeta'|) < \zeta_N < r \right\}.$$

By Lemma 6.1 (i), the region $\mathcal{O}(h, r, r)$ contains the segment $(0, r\,e_N)$, does not contain 0, and is open. Given a point $x \in \partial\Omega$, we want to show that

$$\forall y \in B(x, r) \cap \overline{\Omega}, \quad y + A_x \mathcal{O}(h, r, r) \subset \operatorname{int} \Omega.$$

For $y \in B(x, r) \cap \overline{\Omega} \subset \mathcal{U}(x) \cap \overline{\Omega}$

$$\zeta' \stackrel{\text{def}}{=} P_H A_x^{-1}(y - x) \in V \quad \text{and} \quad \zeta_N \stackrel{\text{def}}{=} (y - x) \cdot A_x e_N \geq a_x(\zeta').$$

Associate with $\xi \in \mathcal{O}(h, r, r)$ the point

$$z \stackrel{\text{def}}{=} y + A_x \xi = x + A_x(\zeta' + \xi' + (\zeta_N + \xi_N) e_N).$$

By construction

$$\begin{aligned} P_H A_x^{-1}(z - x) = \zeta' + \xi' \text{ and } A_x e_N \cdot (z - x) = \zeta_N + \xi_N, \\ \zeta_N + \xi_N > a_x(\zeta') + \limsup h(|\xi'|) \geq a_x(\zeta') + \liminf h(|\xi'|) \geq a_x(\zeta') + h(|\xi'|) \\ \Rightarrow \zeta_N + \xi_N > a_x(\zeta') + h(|\xi'|) \\ > a_x(\zeta') + a_x(\zeta' + \xi') - a_x(\zeta') = a_x(\zeta' + \xi') \end{aligned}$$

and $z \in A_x^+$. In addition

$$|z - x| \leq |z - y| + |y - x| \leq \sqrt{2} r + r < 3r \quad \Rightarrow z \in \mathcal{U}(x) \cap A_x^+ = \mathcal{U}(x) \cap \operatorname{int} \Omega$$

and $y + A_x \mathcal{O}(h, r, r) \subset \operatorname{int} \Omega$. This proves that Ω satisfies the uniform fat segment property of Definition 6.1 with $\lambda = r > 0$, $r > 0$, and $\mathcal{O}(h, r, r)$. □

6.4 Uniform Cone/Cusp Properties and Hölderian/ Lipschitzian Sets

In this section we make the connection between the notions of Hölderian and Lipschitzian sets of Definition 5.2 and the dominating function h generating the set $\mathcal{O}(h, \rho, \lambda)$ of the uniform fat segment property. We first recall some elements of Definition 5.2.

Definition 6.2.
Let Ω be a subset of $\mathbf{R}^{\mathrm{N}}$ such that $\partial\Omega \neq \varnothing$.

(i) Ω is *locally Lipschitzian* if Ω is *locally a $C^{0,1}$ epigraph.*

Ω is *locally Hölderian* if Ω is *locally a $C^{0,\ell}$ epigraph* for some ℓ, $0 < \ell < 1$.

(ii) Ω is *Lipschitzian* if Ω is *a $C^{0,1}$ epigraph.*

Ω is *Hölderian* if Ω is *a $C^{0,\ell}$ epigraph* for some ℓ, $0 < \ell < 1$.

(iii) Ω is *equi-Lipschitzian* if Ω is *an equi-$C^{0,1}$ epigraph.*

Ω is *equi-Hölderian* if Ω is *an equi-$C^{0.\ell}$ epigraph* for some ℓ, $0 < \ell < 1$. □

It is natural to associate with Definition 6.2 the following family of dominating functions:

$$h_\ell(\theta) = \lambda\ (\theta/\rho)^\ell\,, \quad 0 < \ell \le 1,$$

in $\mathcal{H}$. For $\ell = 1$, the region $\mathcal{O}(h_1, \rho, \lambda)$ is the open *cone* of height λ and aperture ω given by $\tan\omega = \rho/\lambda$, $0 < \omega < \pi/2$,

$$h_1(\theta) = \lambda\,\frac{\theta}{\rho} = \frac{1}{\tan\omega}\,\theta.$$

For $0 < \ell < 1$, $\mathcal{O}(h_\ell, \rho, \lambda)$ defines a *cuspidal region* around λe_N of height λ.

6.4.1 Uniform Cone Property and Lipschitzian Sets

Lipschitzian sets have also been equivalently characterized by a purely geometric *uniform cone property* which seems to have been originally introduced by S. Agmon [**1**]. In this section we establish the equivalence of the two sets of definitions and express the previous properties and formulae of Definition 5.2 in terms of the parameters of the cone. One of the important properties of a family of open domains satisfying the uniform cone property is that the extension operator from the domains to $\mathbf{R}^N$ is uniformly continuous.

Notation 6.1.
Given $\lambda > 0$ and $0 < \omega < \pi/2$, denote by $C(\lambda, \omega)$ the open cone

$$C(\lambda,\omega) \stackrel{\text{def}}{=} \left\{ y \in \mathbf{R}^N : \frac{1}{\tan\omega}\big|P_H(y)\big| < y \cdot e_N < \lambda \right\},$$

where P_H is the orthogonal projection onto the hyperplane $H = \{e_N\}^\perp$ orthogonal to the direction e_N (cf. Figure 2.6). Further, associate with an arbitrary point

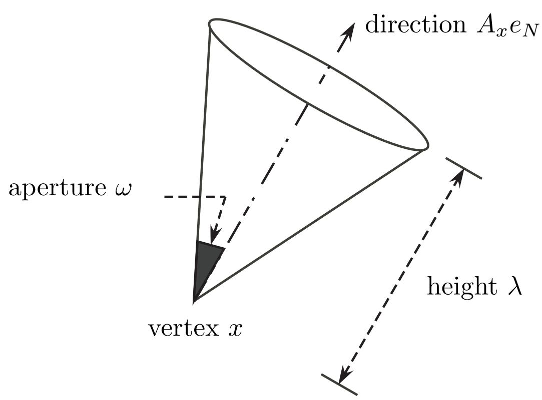


Figure 2.6. *The cone $x + A_x C(\lambda, \omega)$ in the direction $A_x e_N$.*

$x \in \mathbf{R}^N$, an $A_x \in O(N)$, a direction $A_x e_N$, and the translated and rotated cone $x + A_x C(\lambda, \omega)$. It is readily seen that

$$C(\lambda, \omega) = \mathcal{O}(h, \rho, \lambda) \text{ with } h(\theta) = \theta / \tan\omega \text{ and } \rho = \lambda \tan\omega. \qquad \square$$

Definition 6.3.
Let Ω be a subset of $\mathbf{R}^N$ such that $\partial\Omega \neq \varnothing$.

(i) Ω is said to satisfy the *local uniform cone property* if

$$\forall x \in \partial\Omega,\ \exists\lambda > 0,\ \exists\omega > 0,\ \exists r > 0,\ \exists A_x \in O(N),$$

such that

$$\forall y \in B(x,r) \cap \overline{\Omega}, \quad y + A_x C(\lambda, \omega) \subset \operatorname{int}\Omega.$$

(ii) Ω is said to satisfy the *uniform cone property* if

$$\exists\lambda > 0,\ \exists\omega > 0,\ \exists r > 0,\ \forall x \in \partial\Omega,\ \exists A_x \in O(N),$$

such that

$$\forall y \in B(x,r) \cap \overline{\Omega}, \quad y + A_x C(\lambda, \omega) \subset \operatorname{int}\Omega. \qquad \square$$

Theorem 6.8. *Let Ω be a subset of $\mathbf{R}^N$ such that $\partial\Omega \neq \varnothing$. Ω is equi-Lipschitzian if and only if Ω satisfies the uniform cone property.*

The proof will be the same as the one of Theorem 6.9 in the next section.

6.4.2 Uniform Cusp Property and Hölderian Sets

Definition 6.4.
Let Ω be a subset of $\mathbf{R}^N$ such that $\partial\Omega \neq \varnothing$, $0 < \ell < 1$, and $h_\ell(\theta) = \lambda(\theta/\rho)^\ell$.

(i) Ω is said to satisfy the *cusp property* of index ℓ, $0 < \ell < 1$, if

$$\forall x \in \partial\Omega,\ \exists h \in \mathcal{H},\ \exists\lambda > 0,\ \exists r > 0,\ \exists A_x \in O(N),$$

such that

$$\forall y \in B(x,r) \cap \overline{\Omega}, \quad y + A_x \mathcal{O}(h_\ell, \rho, \lambda) \subset \operatorname{int}\Omega.$$

(ii) Ω is said to satisfy the *uniform cusp property* of index ℓ, $0 < \ell < 1$, if

$$\exists\lambda > 0,\ \exists h \in \mathcal{H},\ \exists r > 0,\ \forall x \in \partial\Omega,\ \exists A_x \in O(N),$$

such that

$$\forall y \in B(x,r) \cap \overline{\Omega}, \quad y + A_x \mathcal{O}(h_\ell, \rho, \lambda) \subset \operatorname{int}\Omega. \qquad \square$$

The Lipschitzian case of Definitions 6.2 and 6.3 corresponds to $\ell = 1$ and $\rho = \lambda \tan\omega$ in the above definition, but we wanted to keep the two terminologies distinct.

Remark 6.3.
In some applications, it might be interesting to relax the uniform cusp property by permitting the axis of the cuspidal region to bend: this makes the region look

like a horn, and the corresponding property becomes a *horn condition* or *property.* Horn-shaped domains have been studied in several contexts in the literature. In particular conditions on domains have been introduced in the context of extension operators and embedding theorems: the domains of F. JOHN [**1**]; the (ε, δ)-domains of P. W. JOHN [**1**]; and the domains satisfying a *flexible horn condition* (which is a broader notion than the previous two) of O. V. BESOV [**1**, **2**]. □

Theorem 6.9. *Let Ω be a subset of $\mathbf{R}^N$ such that $\partial\Omega \neq \varnothing$ and let $0 < \ell < 1$. Ω is equi-Hölderian of index ℓ if and only if Ω has the uniform cusp property of index ℓ.*

Proof. By Theorem 6.7, Ω has the uniform fat segment property if and only if Ω is an equi-C^0 epigraph. It remains to sharpen the equivalence.

($\Leftarrow$) In addition, Ω has the uniform cusp property of index ℓ. From inequality (6.8) in the proof of Theorem 6.7 for all $x \in \partial\Omega$ and

$$\forall \zeta',\, \xi' \in B_H(0,\rho') \text{ such that } |\xi' - \zeta'| < \rho, \quad |a_x(\xi') - a_x(\zeta')| \leq h_\ell(|\xi' - \zeta'|)$$

and since $h_\ell(\theta) = \lambda(\theta/\rho)^\ell$, the functions a_x are equi-$C^{0,\ell}$ and Ω is equi-Hölderian of index ℓ.

($\Rightarrow$) In addition, Ω is equi-Hölderian of index ℓ. In the proof of Theorem 6.7, we have constructed the modulus of continuity $h \in \mathcal{H}$ in (6.9):

$$\forall \theta \in [0, 3r], \quad h(\theta) = \sup_{\substack{\zeta' \in H \\ |\zeta'|=1}} \sup_{x \in \partial\Omega} \sup_{\substack{\xi' \in V \\ \xi' + \theta\zeta' \in V}} |a_x(\theta\zeta' + \xi') - a_x(\xi')|.$$

Since the family $\{a_x : x \in \partial\Omega\}$ is equi-$C^{0,\ell}$, the modulus h in (6.9) is also $C^{0,\ell}$ and there exists c such that $|h(\theta)| \leq c\,|\theta|^\ell$. As a result for all $x \in \partial\Omega$ and all ζ', $\xi' \in V$ such that $|\xi' - \zeta'| < 3r$

$$|a_x(\xi') - a_x(\zeta')| \leq h(|\xi' - \zeta'|) \leq c|\xi' - \zeta'|^\theta.$$

Let $h_\ell(\theta) = c\,\theta^\ell$. Then $\mathcal{O}(h_\ell, r, r) \subset \mathcal{O}(h, r, r)$ and Ω satisfies the uniform fat segment property for $\mathcal{O}(h_\ell, r, r)$, but this is precisely the uniform cusp property of index ℓ. □

6.5 Hausdorff Measure and Dimension of the Boundary

We have already seen in section 5.4.3 that Lipschitzian sets have a locally finite boundary measure $H_{N-1}(\partial\Omega)$. In particular, when $\partial\Omega$ is compact, $H_{N-1}(\partial\Omega)$ is finite. We now construct examples of Hölderian sets of index α, $0 < \alpha < 1$, with compact boundary (verifying the uniform cusp property) for which the Hausdorff dimension of the boundary is strictly greater than $N - 1$ and hence their boundary measure $H_{N-1}(\partial\Omega) = +\infty$. We first give an upper bound.

Theorem 6.10. *Let Ω be a subset of $\mathbf{R}^N$ with compact boundary. If Ω satisfies the uniform cusp property of Definition* 6.4 (ii) *associated with the function $h(\theta) = \theta^\alpha$, $0 < \alpha < 1$, then the Hausdorff dimension of $\partial\Omega$ is less than or equal to $N - \alpha$.*

Proof. From Theorem 6.9, Ω is equi-Hölderian of index α, that is, an equi-$C^{0,\alpha}$ epigraph. Let $r > 0$, $\rho > 0$, and $\lambda > 0$ be the parameters of Theorem 6.6, and let $A_x e_N$, $A_x \in \mathrm{O(N)}$, be the direction associated with the point $x \in \partial\Omega$. By assumption the functions $a_x : V \to \mathbf{R}$ satisfy the condition

$$\forall \zeta_1', \zeta_2' \in V, \quad |a_x(\zeta_2') - a_x(\zeta_1')| \leq c\, |\zeta_2' - \zeta_1'|^\alpha . \tag{6.10}$$

Since $\partial\Omega$ is compact, there exists a finite number of points $\{x_i \in \partial\Omega : 1 \leq i \leq m\}$ such that $\partial\Omega \subset \cup_{i=1}^m \mathcal{U}(x_i)$.

Given ε, $0 < \varepsilon < \rho$, let $N_\Omega(\varepsilon)$ be the number of hypercubes of dimension N and side ε required to cover $\partial\Omega$ and let $N_{\Omega,i}(\varepsilon)$ be the number of hypercubes of dimension N and side ε required to cover $\partial\Omega \cap \mathcal{U}(x_i)$. We have the following estimate:

$$N_{\Omega,i}(\varepsilon) \leq \left(\frac{r_\lambda}{\varepsilon}\right)^{N-1} \frac{c(\sqrt{N-1}\,\varepsilon)^\alpha}{\varepsilon}.$$

Indeed the neighborhood

$$V = B_H(0,\rho) \subset B_H(0, r_\lambda)$$

can be covered by $[r_\lambda/\varepsilon]^{N-1}$ $(N-1)$-dimensional hypercubes of side ε. On each $(N-1)$-dimensional hypercube of side ε the variation between the minimum and the maximum of the function a_x is bounded by

$$c\left(\sqrt{(N-1)\varepsilon^2}\right)^\alpha = c\left(\sqrt{N-1}\,\varepsilon\right)^\alpha .$$

So the number of N-dimensional hypercubes of side ε necessary to cover the hypersurface above each $(N-1)$-dimensional hypercube of side ε is

$$\left[\frac{c}{\varepsilon}\left(\sqrt{N-1}\,\varepsilon\right)^\alpha\right].$$

Finally

$$\begin{aligned} N_{\Omega,i}(\varepsilon) &\leq \left(\frac{r_\lambda}{\varepsilon} + 1\right)^{N-1} \left(\frac{c}{\varepsilon}\left(\sqrt{N-1}\,\varepsilon\right)^\alpha + 1\right) \\ &\leq \frac{1}{\varepsilon^{N-1}} \frac{1}{\varepsilon^{1-\alpha}} \left(r_\lambda + \varepsilon\right)^{N-1} \left(c\left(\sqrt{N-1}\right)^\alpha + \varepsilon^{1-\alpha}\right) \\ &\leq \frac{1}{\varepsilon^{N-\alpha}} \left(r_\lambda + \varepsilon\right)^{N-1} \left(c\left(\sqrt{N-1}\right)^\alpha + \varepsilon^{1-\alpha}\right). \end{aligned}$$

As a result for all $\beta > N - \alpha$

$$\begin{gathered} N_{\Omega,i}(\varepsilon) \leq \sum_{i=1}^m N_{\Omega,i}(\varepsilon) \leq m\, \frac{1}{\varepsilon^{N-\alpha}} \left(r_\lambda + \varepsilon\right)^{N-1} \left(c\left(\sqrt{N-1}\right)^\alpha + \varepsilon^{1-\alpha}\right) \\ \Rightarrow N_\Omega(\varepsilon)\varepsilon^\beta \leq \varepsilon^{\beta-N+\alpha}\, m \left(r_\lambda + \varepsilon\right)^{N-1} \left(c\left(\sqrt{N-1}\right)^\alpha + \varepsilon^{1-\alpha}\right) \\ \Rightarrow \forall \beta > N - \alpha, \quad H_\beta(\partial\Omega) = 0. \end{gathered}$$

By definition, the Hausdorff dimension of $\partial\Omega$ is less than or equal to $N - \alpha$. $\square$

It is possible to construct examples of sets verifying the uniform cusp property for which the Hausdorff dimension of the boundary is strictly greater than $N-1$ and hence $H_{N-1}(\partial\Omega) = +\infty$.

Example 6.1.
This following two-dimensional example of an open domain with compact boundary satisfying the uniform cusp condition for the function $h(\theta) = \theta^\alpha$, $0 < \alpha < 1$, can easily be generalized to an N-dimensional example. Consider the open domain

$$\begin{aligned}\Omega \stackrel{\text{def}}{=} &\{(x,y) : -1 < x \le 0 \text{ and } 0 < y < 2\} \\ &\cap \{(x,y) : 0 < x < 1 \text{ and } f(x) < y < 2\} \\ &\cap \{(x,y) : 1 \le x < 2 \text{ and } 0 < y < 2\}\end{aligned}$$

in $\mathbf{R}^2$ (see Figure 2.7), where $f : [0,1] \to \mathbf{R}$ is defined as follows:

$$f(x) \stackrel{\text{def}}{=} d_C(x)^\alpha, \quad 0 \le x \le 1,$$

and C is the Cantor set on the interval $[0,1]$ and $d_C(x)$ is the distance function from the point x to the set C (d_C is uniformly Lipschitzian of constant 1). This function is equal to 0 on C. Any point in $[0,1]\backslash C$ belongs to one of the intervals of length 3^{-k}, $k \ge 1$, which has been deleted from $[0,1]$ in the sequential construction of the Cantor set. Therefore the distance function $d_C(x)$ is equal to the distance function to the two end points of that interval. In view of this special structure it can be shown that

$$\forall x, y \in [0,1], \quad |d_C(y)^\alpha - d_C(x)^\alpha| \le |y-x|^\alpha.$$

Denote by Γ the piece of the boundary $\partial\Omega$ specified by the function $f = d_C$. On Γ the uniform cusp condition is verified with $\rho = 1/6$, $\lambda = (1/6)^\alpha$, and $h(\theta) = \theta^\alpha$ (see

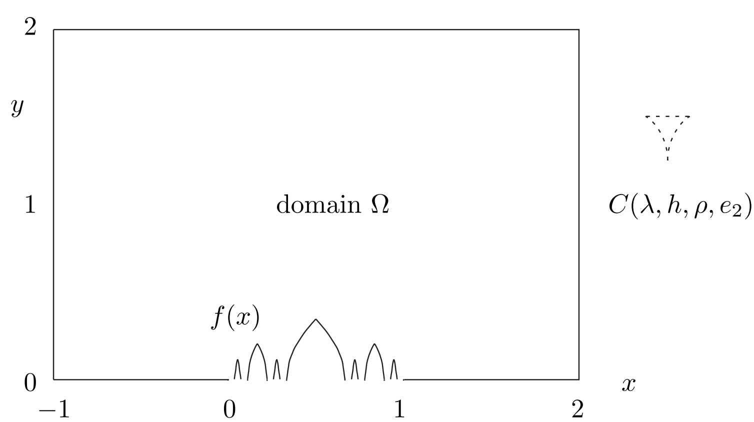


Figure 2.7. *Domain Ω for $N = 2$, $0 < \alpha < 1$, $e_2 = (0,1)$, $\rho = 1/6$, $\lambda = (1/6)^\alpha$, $h(\theta) = \theta^\alpha$.*

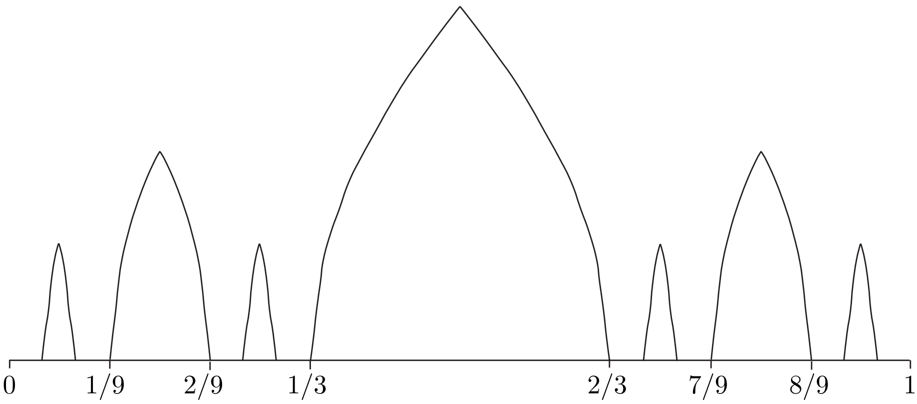


Figure 2.8. $f(x) = d_C(x)^{1/2}$ *constructed on the Cantor set* C *for* $2k+1=3$.

Figure 2.8). Clearly the number $N_\Omega(\varepsilon)$ of hypercubes of dimension N and side ε required to cover $\partial\Omega$ is greater than the number $N_\Gamma(\varepsilon)$ of hypercubes of dimension N and side ε required to cover Γ. The construction of the Cantor set is done by sequentially deleting intervals. At step $k=0$ the interval $(1/3, 2/3)$ of width 3^{-1} is removed. At step k a total of 2^k intervals of width $3^{-(k+1)}$ are removed. Thus if we pick $\varepsilon = 3^{-(k+1)}$, the interval $[0,1]$ can be covered with exactly $3^{(k+1)}$ intervals. Here we are interested in finding a lower bound to the total number of squares of side ε necessary to cover Γ. For this purpose we keep only the 2^k intervals removed at step k. Vertically it takes

$$\left\lceil \frac{\left(2^{-1}3^{-(k+1)}\right)^\alpha}{3^{-(k+1)}} \right\rceil \geq \frac{\left(2^{-1}3^{-(k+1)}\right)^\alpha}{3^{-(k+1)}} - 1.$$

Then we have for $\beta \geq 0$

$$\begin{aligned} N_\Omega(\varepsilon) \geq N_\Gamma(\varepsilon) &\geq 2^k \left(\frac{\left(2^{-1}3^{-(k+1)}\right)^\alpha}{3^{-(k+1)}} - 1 \right) \\ &\geq 2^{k-\alpha} 3^{(k+1)(1-\alpha)} - 2^k = \left(3^{(1-\alpha)}2\right)^k 2^{-\alpha}3^{(1-\alpha)} - 2^k \end{aligned}$$

and

$$\begin{aligned} N_\Omega(\varepsilon)(3^{-k})^{1+\beta} &\geq 3^{-k(1+\beta)} \left(\left(3^{(1-\alpha)}2\right)^k 2^{-\alpha}3^{(1-\alpha)} - 2^k \right) \\ &\geq \left(3^{-(\alpha+\beta)}2\right)^k 2^{-\alpha}3^{(1-\alpha)} - \left(\frac{2}{3^{(1+\beta)}}\right)^k. \end{aligned}$$

The second term goes to zero as k goes to infinity. The first term goes to infinity as k goes to infinity if $3^{-(\alpha+\beta)}2 > 1$, that is, $0 < \alpha+\beta < \ln 2/\ln 3$. Under this condition, $H_{1+\beta}(\partial\Omega) = H_{1+\beta}(\Gamma) = +\infty$ for all $0 < \alpha < \ln 2/\ln 3$ and all $0 \leq \beta < \ln 2/\ln 3 - \alpha$.

Therefore given $0 < \alpha < \ln 2/\ln 3$

$$\forall \beta,\ 0 \le \beta + \alpha < \ln 2/\ln 3, \quad H_{1+\beta}(\partial\Omega) = +\infty$$

and the Hausdorff dimension of $\partial\Omega$ is strictly greater than 1. □

Given $0 < \alpha < 1$, it is possible to construct an *optimal example* of a set verifying the uniform cusp property for which the Hausdorff dimension of the boundary is exactly $N - \alpha$ and hence $H_{N-1}(\partial\Omega) = +\infty$.

Example 6.2 (Optimal example of a set that verifies the uniform cusp property with $h(\theta) = |\theta|^\alpha$, $0 < \alpha < 1$, and whose boundary has Hausdorff dimension exactly equal to $N - \alpha$.)**.**

For that purpose, we need a generalization of the Cantor set. Denote by C_1 the Cantor set. Recall that each x, $0 \le x \le 1$, can be written uniquely (if we make a certain convention) as

$$x = \sum_{j=1}^{\infty} \frac{a_j(3,x)}{3^j},$$

where $a_j(3,x)$ can be regarded as the jth digit of x written in basis 3. From this the Cantor set is characterized as follows:

$$x \in C_1 \Longleftrightarrow \forall j, a_j(3,x) \neq 1.$$

Similarly for an arbitrary integer $k \ge 1$, each $x \in [0,1]$ can be uniquely written in the form

$$x = \sum_{j=1}^{\infty} \frac{a_j(2k+1,x)}{(2k+1)^j}$$

and we can define the set C_k as

$$x \in C_k \Longleftrightarrow \forall j, a_j(2k+1,x) \neq k.$$

In a certain sense, if $k_1 > k_2$, C_{k_1} contains more points than C_{k_2}. We now use these sets to construct the family of set D_k as follows:

$$x \in D_1 \Longleftrightarrow 2x \in C_1$$
$$\text{and for } k > 1, \quad x \in D_k \Longleftrightarrow 2^{k+1}(x - 2^k) \in C_k.$$

Note that if $k_1 \neq k_2$, $D_{k_1} \bigcap D_{k_2} = \varnothing$ since the D_k's contain only points from the interval $[1 - 2^{k-1}, 1 - 2^k]$. Consider now the set $D \stackrel{\text{def}}{=} \cup_{k=1}^{\infty} D_k$ and go back to Example 6.1 with the function f replaced by the function $f(x) \stackrel{\text{def}}{=} d_D(x)^\alpha$. Again it can be shown that for all $x, y \in [0,1]$, $|d_D(y)^\alpha - d_D(x)^\alpha| \le |y - x|^\alpha$. Note that on the interval $[1 - 2^{k-1}, 1 - 2^k]$ we have $d_D(x)^\alpha = d_{D_k}(x)^\alpha$.

Denote by Γ the piece of boundary $\partial\Omega$ specified by the function $f = d_D$ and by Γ_k the part of boundary $\partial\Omega$ specified by the function $f = d_D\,(= d_{D_k})$ on the interval $[1 - 2^{k-1}, 1 - 2^k]$. Once again on Γ the uniform cusp property is verified with $\rho = 1/6$, $\lambda = (1/6)^\alpha$, and $h(\theta) = \theta^\alpha$. Clearly the number $N_\Omega(\varepsilon)$ of hypercubes of dimension N and side ε required to cover $\partial\Omega$ is greater than the number $N_{\Gamma_k}(\varepsilon)$ of hypercubes of dimension N and side ε required to cover Γ_k. The construction of the set C_k is also done sequentially by deleting intervals. At step $j = 0$ the interval $]k/(2k+1), (k+1)/(2k+1)[$ of width $(2k+1)^{-1}$ is removed. At step j a total of 2^j intervals of width $(2j+1)^{-(j+1)}$ are removed. If we consider the intervals that remain at step j, a total of 2^{j+1} nonempty disjoint intervals of width $(k/(2k+1))^{j+1}$ remain in the set C_k. Each of these intervals contains a gap of length $(k/(2k+1))^{j+1}\, 1/(2k+1)$ created at step $j+1$.

If we construct the set D_k in the same way, at step j a total of 2^j nonempty disjoint intervals of width $(k/(2k+1))^{j+1}\, 1/2^k$ remain in the set D_k. Each of these intervals contains a gap of length $(k/(2k+1))^{j+1}\, 1/(2^k(2k+1))$. Pick

$$\varepsilon = \frac{1}{2^k}\left(\frac{k}{2k+1}\right)^{j+1}$$

and look for a lower bound on the number of squares of side ε necessary to cover Γ_k. For this purpose, consider only the 2^{j+1} nonempty disjoint intervals remaining at step j. As they each contain a gap of length

$$\left(\frac{k}{2k+1}\right)^{j+1} \frac{1}{2^k(2k+1)}$$

vertically it takes

$$\begin{aligned}
&\left[\left(\left(\frac{k}{2k+1}\right)^{j+1} \frac{1}{2^{k+1}(2k+1)}\right)^{\alpha} 2^k \left(\frac{2k+1}{k}\right)^{j+1}\right] \\
&\geq \left(\left(\frac{k}{2k+1}\right)^{j+1} \frac{1}{2^{k+1}(2k+1)}\right)^{\alpha} 2^k \left(\frac{2k+1}{k}\right)^{j+1} - 1
\end{aligned}$$

ε-cubes. Then we have for $\beta \geq 0$

$$\begin{aligned}
N_\Omega(\varepsilon) \geq N_\Gamma(\varepsilon) &\geq 2^{j+1}\left(\left(\left(\frac{k}{2k+1}\right)^{j+1} \frac{1}{2^{k+1}(2k+1)}\right)^{\alpha} 2^k \left(\frac{2k+1}{k}\right)^{j+1} - 1\right) \\
&\geq \left(\frac{2(2k+1)k^\alpha}{k(2k+1)^\alpha}\right)^{j+1} \left(\frac{2^k}{2^{\alpha(k+1)}(2k+1)^\alpha}\right) - 2^{j+1} \\
&= \left(2(2k+1)^{1-\alpha}k^{\alpha-1}\right)^{j+1} \left(\frac{2^{k(1-\alpha)-\alpha}}{(2k+1)^\alpha}\right) - 2^{j+1}
\end{aligned}$$

and hence

$$\begin{aligned}
&\varepsilon^{1+\beta}\, N_\Omega(\varepsilon) \\
&\geq \left(\frac{1}{2^k}\left(\frac{k}{2k+1}\right)^{j+1}\right)^{1+\beta}\left(\left(2(2k+1)^{1-\alpha}k^{\alpha-1}\right)^{j+1}\left(\frac{2^{k(1-\alpha)-\alpha}}{(2k+1)^\alpha}\right) - 2^{j+1}\right) \\
&\geq \left(\left(\frac{k}{2k+1}\right)^{\alpha+\beta} 2\right)^{j+1} \frac{2^{k(1-\alpha)-\alpha}}{2^{k(1+\beta)}(2k+1)^\alpha} - 2^{j+1}\left(\frac{1}{2^k}\left(\frac{k}{2k+1}\right)^{j+1}\right)^{1+\beta}.
\end{aligned}$$

The second term goes to zero as j goes to infinity. The first term goes to infinity as j goes to infinity if

$$\forall k, \quad \left(\frac{k}{2k+1}\right)^{\alpha+\beta} 2 > 1, \quad \text{that is } \forall k, \quad 0 < \alpha + \beta < \frac{\log 2}{\log\left((2k+1)/k\right)}.$$

As k can be chosen arbitrarily large, the former inequality reduces to $0 < \alpha+\beta < 1$.

Under this condition there exists an integer k for which

$$H_{1+\beta}(\partial\Omega) = H_{1+\beta}(\Gamma_k) = +\infty$$

for all $0 < \alpha < 1$ and all $0 \leq \beta < 1 - \alpha$. Therefore, given $0 < \alpha < 1$

$$\forall \beta,\ 0 \leq \beta < 1 - \alpha, \quad H_{1+\beta}(\partial\Omega) = +\infty.$$

This implies that the Hausdorff dimension of $\partial\Omega$ is greater than or equal to $2 - \alpha$, which is the upper bound we obtained in Theorem 6.10. $\square$

Chapter 3

Courant Metrics on Images of a Set

1 Introduction

A natural way to construct a family of variable domains is to consider the images of a fixed subset of $\mathbf{R}^N$ by some family of transformations of $\mathbf{R}^N$. The structure and the topology of the images can be specified via the natural algebraic and topological structures of the space of transformations or equivalence classes of transformations for which the full power of function analytic methods is available. There are many ways to do that, and specific constructions and choices are again very much problem dependent.

In 1972 A. M. Micheletti [**1**] introduced what may be one of the first complete metric topologies on a family of domains of class $C^k, k \geq 1$, that are the images of a fixed open domain through a family of C^k-diffeomorphisms of $\mathbf{R}^N$. There the natural underlying algebraic structure is the *group structure* of the composition of transformations with the identity transformation as the neutral element. Her analysis culminates with the construction of a complete metric on the quotient of the group by an appropriate closed subgroup of transformations leaving the fixed subset unaltered. She called it the *Courant metric*[1] because it is proved in the book of R. Courant and D. Hilbert [**1**, p. 420] that the nth eigenvalue of the Laplace operator depends continuously on the domain Ω, where $\Omega = (I + f)\Omega_0$ is the image of a fixed domain Ω_0 by $I + f$ and f is a smooth mapping. But there is no notion of a metric in that book. Her constructions naturally extend to other families of transformations of $\mathbf{R}^N$ or of fixed holdalls D.

In section 2 we first extend her generic constructions associated with the space $C_0^k(\mathbf{R}^N, \mathbf{R}^N)$ of mappings from $\mathbf{R}^N$ into $\mathbf{R}^N$ to a larger family of Banach spaces of mappings such as $C^k(\overline{\mathbf{R}^N}, \mathbf{R}^N)$, $C^{k,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)$, or $\mathcal{B}^k(\mathbf{R}^N, \mathbf{R}^N)$, and beyond to Fréchet spaces such as $\mathcal{B}(\mathbf{R}^N, \mathbf{R}^N)$ or $C_0^\infty(\mathbf{R}^N, \mathbf{R}^N)$ of infinitely continuously

[1] "Nello studiare la continuità dell'n-esimo autovalore dell'operatore di Laplace - Δ_Ω relativo ad un aperto limitato Ω con dati di Dirichlet nulli, considerato comme funzione dell'aperto Ω, Courant introduce una nozione di vicinanza tra due domini basata su un diffeomorphismo del tipo $I + \psi$ con $\psi \in C^1(\mathbf{R}^m)$, che transforma l'uno nell'altro." (cf. A. M. Micheletti [**1**]).

differentiables mappings. We emphasize the *geodesic character* of the construction of the metric and its interpretation as trajectories of bounded variation on the group.

The next step in the construction is the choice of the closed subgroup of transformations of $\mathbf{R}^N$ that is very much problem dependent. Originally, it was chosen as the set of transformations that leave the underlying set or pattern unaltered. However, in some applications, it could be unaltered up to a translation, a rotation, or a flip. The underlying set or pattern can be a closed set or an open crack-free set.[2] This includes closed submanifolds of $\mathbf{R}^N$. It is shown that, as long as the subgroup is closed, we get a complete Courant metric on the quotient group. In this section we also characterize the tangent space to the group of transformations of $\mathbf{R}^N$ that leads to the Courant metric. It is an example of an infinite-dimensional manifold (cf. Theorems 2.14 and 2.17 in sections 2.5 and 2.6).

In section 3, we free the constructions from the framework of bounded continuously differentiable transformations to reach the spaces of all homeomorphisms or C^k-diffeomorphisms of $\mathbf{R}^N$ or an open subset D of $\mathbf{R}^N$. Again it is shown that they are complete metric spaces. Hence, from section 2, their quotient by a closed subgroup yields a Courant metric and a complete metric topology. With such larger spaces, it now becomes possible to consider subgroups involving not only translations but also isometries, symmetries, or flips in $\mathbf{R}^N$ or D.

2 Generic Constructions of Micheletti

The original constructions of A. M. Micheletti [**1**] were carried out for the Banach space $C_0^k(\mathbf{R}^N, \mathbf{R}^N)$, $k \geq 0$.[3] In this section, we generalize them to Banach spaces Θ of mappings from $\mathbf{R}^N$ into $\mathbf{R}^N$ under fairly general assumptions.

2.1 Space $\mathcal{F}(\Theta)$ of Transformations of R^N

Associate with a real vector space Θ of mappings from $\mathbf{R}^N$ to $\mathbf{R}^N$ the following space of transformations of $\mathbf{R}^N$:

$$\boxed{\mathcal{F}(\Theta) \stackrel{\text{def}}{=} \left\{I + f : f \in \Theta,\, (I+f) \text{ bijective and } (I+f)^{-1} - I \in \Theta\right\},} \tag{2.1}$$

where $x \mapsto I(x) \stackrel{\text{def}}{=} x : \mathbf{R}^N \to \mathbf{R}^N$ is the identity mapping.[4] It will be shown that $\mathcal{F}(\Theta)$ is a group for the composition $(F \circ G)(x) \stackrel{\text{def}}{=} F(G(x))$ of transformations of $\mathbf{R}^N$. Equivalent right-invariant metrics on $\mathcal{F}(\Theta)$ will be introduced under Assumptions 2.1 and 2.2 and related to some notion of geodesics in the group $\mathcal{F}(\Theta)$. The completeness of $\mathcal{F}(\Theta)$ will require Assumption 2.3 and either

[2] Cf. Definition 7.1 in Chapter 8.

[3] $C_0^k(\mathbf{R}^N, \mathbf{R}^N)$ denotes the space of k times continuously differentiable functions from $\mathbf{R}^N$ to $\mathbf{R}^N$, whose derivatives vanish at infinity. It is also denoted by $\mathcal{B}_0^k(\mathbf{R}^N, \mathbf{R}^N)$ in the literature.

[4] For $\Theta = C_0^k(\mathbf{R}^N, \mathbf{R}^N)$, this definition is equivalent to the one of A. M. Micheletti [**1**]:

$$\mathcal{F}(C_0^k(\mathbf{R}^N, \mathbf{R}^N)) \stackrel{\text{def}}{=} \left\{F : \mathbf{R}^N \to \mathbf{R}^N \,:\, F - I \in C_0^k(\mathbf{R}^N, \mathbf{R}^N) \text{ and } F^{-1} \in C^k(\mathbf{R}^N, \mathbf{R}^N)\right\}.$$

(i) the assumption that $\Theta \subset C^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ and, for all $x \in \mathbf{R}^{\mathrm{N}}$, the mapping $f \mapsto f(x) : \Theta \to \mathbf{R}^{\mathrm{N}}$ is continuous that makes the group a complete metric space, or

(ii) Assumption 2.4 that makes the group a complete (metric) topological group.

Finally, the right-invariant *Courant metric* will be defined as the quotient metric

$$\boxed{\forall F,\, G \in \mathcal{F}(\Theta), \quad d_{\mathcal{G}}([F],[H]) \overset{\text{def}}{=} \inf_{G,\tilde{G}\in\mathcal{G}} d(F\circ G, H\circ\tilde{G}), \quad [F] \overset{\text{def}}{=} F\circ\mathcal{G}} \tag{2.2}$$

associated with the quotient group $\mathcal{F}(\Theta)/\mathcal{G}$ of $\mathcal{F}(\Theta)$ by a closed subgroup $\mathcal{G}$ of $\mathcal{F}(\Theta)$. The subgroup considered by A. M. Micheletti [**1**] was

$$\mathcal{G}(\Omega_0) \overset{\text{def}}{=} \{F \in \mathcal{F}(\Theta) : F(\Omega_0) = \Omega_0\}$$

to quotient out all F whose image of the fixed subset Ω_0 of $\mathbf{R}^{\mathrm{N}}$ is Ω_0 and make the quotient group $\mathcal{F}(\Theta)/\mathcal{G}(\Omega_0)$ isomorphic to the set of images

$$\mathcal{X}(\Omega_0) \overset{\text{def}}{=} \{F(\Omega_0) : \forall F \in \mathcal{F}(\Theta)\} \tag{2.3}$$

of Ω_0 by the elements of $\mathcal{F}(\Theta)$. The abstract results will be applied with Θ equal to Banach spaces such as $C_0^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}) \subset C^k(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}}) \subset \mathcal{B}^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ and $C^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$, $k \geq 0$, and, through special constructions, to the Fréchet spaces $C_0^\infty(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}) \subset \mathcal{B}(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}) = \cap_{k\geq 0}\mathcal{B}^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$.

We proceed by introducing the assumptions step by step.

Assumption 2.1.
Θ is a real vector space of mappings from $\mathbf{R}^{\mathrm{N}}$ into $\mathbf{R}^{\mathrm{N}}$ and

$$\forall g \in \Theta,\, \forall\, I + f \in \mathcal{F}(\Theta), \quad g\circ(I+f) \in \Theta. \qquad \square$$

Theorem 2.1. *Let Θ be a real vector space of mappings from $\mathbf{R}^{\mathrm{N}}$ into $\mathbf{R}^{\mathrm{N}}$. $\mathcal{F}(\Theta)$ is a group for the composition $\circ$ if and only if Assumption* 2.1 *is verified. In particular* $(I+f)^{-1} - I = -f\circ(I+f)^{-1} \in \Theta$.

Example 2.1.
(1) $\Theta = \mathcal{B}^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$, space of bounded continuous functions. For $g \in \mathcal{B}^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ and $I + f \in \mathcal{F}(\mathcal{B}^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}))$, the composition $g\circ(I+f)$ is continuous and $\|g\circ(I+f)\|_{C^0} = \|g\|_{C^0} < \infty$.

(2) $\Theta = C^0(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$, space of bounded uniformly continuous functions. For $g \in C^0(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$ and $I + f \in \mathcal{F}(C^0(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}}))$, $\|g\circ(I+f)\|_{C^0} = \|g\|_{C^0} < \infty$, and since g and $I+f$ are uniformly continuous, so is the composition $g\circ(I+f)$.

(3) $\Theta = C_0^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$, subspace of $C^0(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$. From (2) $\|g\circ(I+f)\|_{C^0} = \|g\|_{C^0} < \infty$ and $g\circ(I+f) \in C^0(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$. It remains to show that $g\circ(I+f) \in C_0^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$. Since $g \in C_0^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$, for all $\varepsilon > 0$, there exists $\rho_0 > 0$ such that for all x such that $|x| > \rho_0$, $|g(x)| < \varepsilon$. But f also belongs to $C_0^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ and

there exists $\rho \geq 2\rho_0$ such that for all x such that $|x| > \rho$, $|f(x)| < \rho_0$. In particular, $|x+f(x)| \geq |x|-|f(x)| > \rho-\rho_0 \geq \rho_0$ and hence $|g(x+f(x))| < \varepsilon$. In conclusion for all $\varepsilon > 0$, there exists $\rho > 0$ such that for all x such that $|x| > \rho$, $|g(x+f(x))| < \varepsilon$ and $g \circ (I+f) \in C_0^0(\mathbf{R}^N, \mathbf{R}^N)$.

(4) $\Theta = C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)$. This is the space of bounded Lipschitz continuous functions on $\mathbf{R}^N$ that is contained in $C^0(\overline{\mathbf{R}^N}, \mathbf{R}^N)$. For $f, g \in C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)$, $\|f \circ (I+g)\|_{C^0} = \|f\|_{C^0} < \infty$. Since g is Lipschitz with Lipschitz constant $c(g)$, $I+g$ is also Lipschitz with Lipschitz constant $1+c(g)$ and the composition is also Lipschitz with constant $c(f)\,(1+c(g))$. $\square$

Proof of Theorem 2.1. $I \in \mathcal{F}(\Theta)$ for $f=0$. If $I+f \in \mathcal{F}(\Theta)$, $(I+f)^{-1}$ is bijective, $(I+f)^{-1} - I \in \Theta$, $(F^{-1})^{-1} = F = I+f$, and $(F^{-1})^{-1} - I = f \in \Theta$. Hence $(I+f)^{-1} \in \mathcal{F}(\Theta)$ and

$$\begin{aligned} I = (I+f) \circ (I+f)^{-1} &= (I+f)^{-1} + f \circ (I+f)^{-1} \\ \Rightarrow\ (I+f)^{-1} - I &= -f \circ (I+f)^{-1}. \end{aligned}$$

The last property to check is the composition $(I+f) \circ (I+g)$. It is bijective as the composition of two bijective transformations. Moreover, by linearity of Θ and Assumption 2.1

$$(I+f) \circ (I+g) - I = g + f \circ (I+g) \in \Theta. \tag{2.4}$$

From (2.4) and the invertibility of the composition

$$[(I+f) \circ (I+g)]^{-1} - I = -[g + f \circ (I+g)] \circ (I+g)^{-1} \circ (I+f)^{-1} \in \Theta$$

by using Assumption 2.1 twice. Conversely, if (2.4) is true for all $f \in \Theta$ and $I+g \in \mathcal{F}(\Theta)$, by linearity of Θ, Assumption 2.1 is verified. $\square$

Associate with $F \in \mathcal{F}(\Theta)$ the following function of I and F:

$$\boxed{d(I,F) \stackrel{\text{def}}{=} \inf_{\substack{F = F_1 \circ \cdots \circ F_n \\ F_i \in \mathcal{F}(\Theta)}} \sum_{i=1}^{n} \|F_i - I\|_\Theta + \|F_i^{-1} - I\|_\Theta,} \tag{2.5}$$

where the infimum is taken over all *finite factorizations* of F in $\mathcal{F}(\Theta)$ of the form

$$F = F_1 \circ \cdots \circ F_n, \quad F_i \in \mathcal{F}(\Theta).$$

In particular $d(I,F) = d(I,F^{-1})$. Extend this function to all F and G in $\mathcal{F}(\Theta)$:

$$\boxed{d(F,G) \stackrel{\text{def}}{=} d(I, G \circ F^{-1}).} \tag{2.6}$$

By definition, d is right-invariant since for all F, G, and H in $\mathcal{F}(\Theta)$

$$d(F,G) = d(F \circ H, G \circ H).$$

To show that d is a metric[5] on $\mathcal{F}(\Theta)$ necessitates a second assumption.

Assumption 2.2.
$(\Theta, \|\cdot\|)$ is a normed vector space of mappings from $\mathbf{R}^{\mathrm{N}}$ into $\mathbf{R}^{\mathrm{N}}$ and for each $r > 0$ there exists a continuous function $c_0(r)$ such that

$$\forall \{f_i\}_{i=1}^n \subset \Theta \text{ such that } \sum_{i=1}^n \|f_i\| < \alpha < r; \tag{2.7}$$

then

$$\|(I + f_1) \circ \cdots \circ (I + f_n) - I\| < \alpha\, c_0(r). \tag{2.8}$$

□

Theorem 2.2. *Under Assumptions* 2.1 *and* 2.2, *d is a right-invariant metric on $\mathcal{F}(\Theta)$.*

Example 2.2.
(1), (2), (3) Since the norm on the three spaces $\mathcal{B}^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$, $C^0(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$, and $C_0^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ is the same and Assumption 2.2 involves only the norm, it is sufficient to check it for $\mathcal{B}^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$. Consider $\{f_i\}_{i=1}^n \subset \mathcal{B}^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ such that

$$\sum_{i=1}^n \|f_i\|_{C^0} < \alpha < r. \tag{2.9}$$

Using the convention $(I + f_n) \circ (I + f_n) = (I + f_n)$ in the summation

$$\begin{aligned}
&(I + f_1) \circ \cdots \circ (I + f_n) - I \\
&= (I + f_n) - I + \sum_{i=1}^{n-1} (I + f_i) \circ \cdots \circ (I + f_n) - (I + f_{i+1}) \circ \cdots \circ (I + f_n) \\
&= f_n + \sum_{i=1}^{n-1} f_i \circ (I + f_{i+1}) \circ \cdots \circ (I + f_n) \\
&\Rightarrow \|(I + f_1) \circ \cdots \circ (I + f_n) - I\|_{C^0} \le \sum_{i=1}^n \|f_i\|_{C^0} < \alpha < r \quad \Rightarrow c_0(r) = r.
\end{aligned}$$

(4) $\Theta = C^{0,1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$. Consider a sequence $\{f_i\}_{i=1}^n \subset C^{0,1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$ such that

$$\sum_{i=1}^n \max\{\|f_i\|_{C^0}, c(f_i)\} < \alpha < r. \tag{2.10}$$

[5] A function $d : X \times X \to \mathbf{R}$ is said to be a *metric* on X if (cf. J. Dugundji [**1**])

(i) $d(F.G) \ge 0$, for all F, G,

(ii) $d(F, G) = 0 \iff F = G$,

(iii) $d(F, G) = d(G, F)$, for all F, G,

(iv) $d(F, H) \le d(F, G) + d(G, H)$, for all F, G, H.

A metric d on a group $(\mathcal{F}, \circ)$ is said to be *right-invariant* if for all $H \in \mathcal{F}$, $d(F \circ H, G \circ H) = d(F, G)$.

From the previous example

$$\|(I+f_1)\circ\cdots\circ(I+f_n)-I\|_{C^0}\le\sum_{i=1}^{n}\|f_i\|_{C^0}<\alpha.$$

In addition,

$$\begin{aligned}(I+f_1)\circ\cdots\circ(I+f_n)-I&=f_n+\sum_{i=1}^{n-1}f_i\circ(I+f_{i+1})\circ\cdots\circ(I+f_n)\\ \Rightarrow\ c((I+f_1)\circ\cdots\circ(I+f_n)-I)&\le c(f_n)+\sum_{i=1}^{n-1}c(f_i)\,(1+c(f_{i+1}))\dots(1+c(f_1))\\ &\le c(f_n)+\sum_{i=1}^{n-1}c(f_i)\,e^{\sum_{k=1}^{i+1}c(f_k)}\\ &\le\left[\sum_{i=1}^{n}c(f_i)\right]\,e^{\left[\sum_{i=1}^{n}c(f_i)\right]}<\alpha\,e^{\alpha}\\ \Rightarrow\ \|(I+f_1)\circ\cdots\circ(I+f_n)-I\|_{C^{0,1}}&<\alpha\,\max\{e^{\alpha},1\}<\alpha\,e^{r}\end{aligned}$$

and we can choose $c_0(r)=e^r$. $\square$

Proof of Theorem 2.2. Properties (i) and (iii) are verified by definition. For the triangle inequality (iv), $F\circ H^{-1}=(F\circ G^{-1})\circ(G\circ H^{-1})$ is a finite factorization of $F\circ H^{-1}$. Consider arbitrary finite factorizations of $F\circ G^{-1}$ and $G\circ H^{-1}$:

$$\begin{aligned}F\circ G^{-1}&=(I+\alpha_1)\circ\cdots\circ(I+\alpha_m),\quad I+\alpha_i\in\mathcal{F}(\Theta),\\ G\circ H^{-1}&=(I+\beta_1)\circ\cdots\circ(I+\beta_n),\quad I+\beta_j\in\mathcal{F}(\Theta).\end{aligned}$$

This yields a new finite factorization of $F\circ H^{-1}$. By definition of d,

$$d(I,F\circ H^{-1})\le\sum_{i=1}^{m}\|\alpha_i\|+\|\alpha_i\circ(I+\alpha_i)^{-1}\|+\sum_{i=j}^{n}\|\beta_j\|+\|\beta_j\circ(I+\beta_j)^{-1}\|,$$

and by taking infima over each factorization,

$$d(F,H)=d(I,F\circ H^{-1})\le d(I,F\circ G^{-1})+d(I,G\circ H^{-1})=d(F,G)+d(G,H)$$

using the symmetry property (iii). To verify (ii) it is sufficient to show that $F=I\iff d(I,F)=0$. Clearly $I\circ I$ is a factorization of $F=I$ and $0\le d(I,F)\le 0$. Conversely, if $d(I,F)=0$, for ε, $0<\varepsilon<1$, there exists a finite factorization $F=F_1\circ\cdots\circ F_n$ such that

$$\sum_{i=1}^{n}\|F_i-I\|+\|F_i^{-1}-I\|<\varepsilon<1.$$

By Assumption 2.2, $\|F-I\|+\|F^{-1}-I\|<\varepsilon\,c_0(1)$, which implies that $F-I=0$ and $F^{-1}-I=0$. This completes the proof. $\square$

An important aspect of the previous construction was to *build in* the triangle inequality to make d a (right-invariant) metric. However, there are other ways to introduce a topology on $\mathcal{F}(\Theta)$. The interest of this specific choice will become apparent later on. For instance, the original *Courant metric* of A. M. MICHELETTI [**1**] in 1972 was constructed from the following slightly different right-invariant metric on $\mathcal{F}(\Theta)$: given $H \in \mathcal{F}(\Theta)$, consider finite factorizations $H_1 \circ \cdots \circ H_m$, $H_i \in \mathcal{F}(\Theta)$, $1 \le i \le m$, of H and finite factorizations $G_1 \circ \cdots \circ G_n$, $G_j \in \mathcal{F}(\Theta)$, $1 \le j \le n$, of H^{-1}, and define

$$d_1(I,H) \stackrel{\text{def}}{=} \inf_{\substack{H=H_1\circ\cdots\circ H_m \\ m\ge 1}} \sum_{i=1}^{m} \|H_i - I\| + \inf_{\substack{H^{-1}=G_1\circ\cdots\circ G_n \\ n\ge 1}} \sum_{i=1}^{n} \|G_i - I\|, \tag{2.11}$$

where the infima are taken over all *finite factorizations* of H and H^{-1} in $\mathcal{F}(\Theta)$ and

$$\forall F, G \in \mathcal{F}(\Theta), \quad d_1(G,F) \stackrel{\text{def}}{=} d_1(I, F \circ G^{-1}). \tag{2.12}$$

Another example is the topology generated by the right-invariant *semimetric*[6]

$$\begin{aligned} \forall H \in \mathcal{F}(\Theta), \quad d_0(I,H) &\stackrel{\text{def}}{=} \|H - I\| + \|H^{-1} - I\|, \\ \forall F, G \in \mathcal{F}(\Theta), \quad d_0(G,F) &\stackrel{\text{def}}{=} d_0(I, F \circ G^{-1}). \end{aligned} \tag{2.13}$$

Given $r > 0$ and $F \in \mathcal{F}(\Theta)$, let $B_r(F) = \{G \in \mathcal{F}(\Theta) : d_0(F,G) < r\}$. Let τ_0 be the weakest topology on $\mathcal{F}(\Theta)$ such that the family $\{B_r(F) : 0 \le r < \infty\}$ is the base for τ_0. By definition

$$d_1(F,G) \le d(F,G) \le d_0(F,G)$$

and the injections $(\mathcal{F}, d_1) \to (\mathcal{F}, d) \to (\mathcal{F}, d_0)$ are continuous.

Theorem 2.3. *Under Assumptions* 2.1 *and* 2.2*, the topologies generated by* d_1*,* d*, and* d_0 *are equivalent.*

Proof. Since d_0, d, and d_1 are right-invariant on the group $\mathcal{F}(\Theta)$, it is sufficient to prove the equivalence for d_0 and d_1 around I. For all $\varepsilon > 0$ with $\delta = \varepsilon$

$$\forall H \in \mathcal{F}(\Theta),\, d_0(I,H) < \delta, \quad d_1(I,H) < \varepsilon,$$

since $d_1(I,H) \le d_0(I,H)$. Conversely, by Assumption 2.2, for any δ, $0 < \delta < 1$,

$$d_1(I,H) < \delta < 1 \quad \Rightarrow d_0(I - H) = \|H - I\| + \|H^{-1} - I\| < 2\delta\, c(1).$$

So for any $\varepsilon > 0$, pick $\delta = \min\{1, \varepsilon/c(1)\}/2$. Hence

$$\forall \varepsilon > 0,\, \exists \delta > 0, \quad \forall H \in \mathcal{F}(\Theta),\, d_1(I,H) < \delta, \quad d_0(I,H) < \varepsilon.$$

This concludes the proof of the equivalences. $\square$

---

[6] Given a space X, a function $d : X \times X \to \mathbf{R}$ is said to be a *semimetric* if

(i) $d(F.G) \ge 0$, for all F, G,

(ii) $d(F,G) = 0 \iff F = G$,

(iii) $d(F,G) = d(G,F)$, for all F, G.

This notion goes back to Fréchet and Menger.

F. Murat and J. Simon [**1**] in 1976 considered as candidates for Θ the spaces

$$\begin{aligned}&W^{k+1,\bar{c}}(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})\\ &\overset{\text{def}}{=}\left\{f\in W^{k,\infty}(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})\,:\,\forall\, 0\le|\alpha|\le k+1,\,\partial^\alpha f\in C(\overline{\mathbf{R}^{\mathrm{N}}},\mathbf{R}^{\mathrm{N}})\right\}\end{aligned}$$

and $W^{k+1,\infty}(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})$, $k\ge 0$. The first space $W^{k+1,\bar{c}}(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})$ is equivalent to the space $C^{k+1}(\overline{\mathbf{R}^{\mathrm{N}}},\mathbf{R}^{\mathrm{N}})$, $k\ge 0$, algebraically and topologically. The second space $W^{k+1,\infty}(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})$ coincides with $C^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}},\mathbf{R}^{\mathrm{N}})$, $k\ge 0$. For the spaces $C^{k+1}(\overline{\mathbf{R}^{\mathrm{N}}},\mathbf{R}^{\mathrm{N}})$ and $C^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}},\mathbf{R}^{\mathrm{N}})$, $k\ge 0$, they obtained the following *pseudo triangle inequality* for the semimetric d_0:

$$\begin{aligned}d_0(F_1,&F_3)\\ &\le d_0(F_1,F_2)+d_0(F_2,F_3)\\ &\quad+d_0(F_1,F_2)\,d_0(F_2,F_3)\,P\left[d_0(F_1,F_2)+d_0(F_2,F_3)\right],\quad\forall F_1,F_2,F_3,\end{aligned}\tag{2.14}$$

where $P:\mathbf{R}^+\to\mathbf{R}^+$ is a continuous increasing function. With that additional property, they called d_0 a *pseudodistance*[7] and showed that they could construct a metric from it.

Lemma 2.1. *Assume that the semimetric d_0 satisfies the pseudo triangle inequality* (2.14)*. Then, for all α, $0<\alpha<1$, there exists a constant $\eta_\alpha>0$ such that the function $d_0^{(\alpha)}:\mathcal{F}(\Theta)\times\mathcal{F}(\Theta)\to\mathbf{R}^+$ defined as*

$$d_0^{(\alpha)}(F_1,F_2)\overset{\text{def}}{=}\inf\{d_0(F_1,F_2),\eta_\alpha\}^\alpha$$

is a metric on E.

The Micheletti construction with the metric d_1 was given in 2001 for the spaces $C^k(\overline{\mathbf{R}^{\mathrm{N}}},\mathbf{R}^{\mathrm{N}})$ and $C^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}},\mathbf{R}^{\mathrm{N}})$, $k\ge 0$, in M. C. Delfour and J.-P. Zolésio [**37**] so that pseudodistances can be completely bypassed. The metrics d and d_1 constructed from the semimetric d_0 under the general Assumptions 2.1 and 2.2 are both more general and interesting since they are of *geodesic type* by the use of infima over finite factorizations $(I+\theta_1)\circ\cdots\circ(I+\theta_n)$ of $I+\theta$ when the semimetric d_0 is interpreted as an *energy*

$$\sum_{i=1}^n d_0(I+\theta_i,I)=\sum_{i=1}^n\|\theta_i\|+\|-\theta_i\circ(I+\theta_i)^{-1}\|\tag{2.15}$$

[7] This terminology is not standard. In the literature a *pseudometric* or *pseudodistance* is usually reserved for a function $d:X\times X\to\mathbf{R}$ satisfying

(i) $d(F.G)\ge 0$, for all F,G,

(ii') $d(F,G)=0\Longleftarrow F=G$,

(iii) $d(F,G)=d(G,F)$, for all F,G.

(iv) $d(F,H)\le d(F,G)+d(G,H)$, for all F,G,H.

Condition (ii) for a metric is weakened to condition (ii') for a pseudodistance.

over families of piecewise constant trajectories $T : [0,1] \to \mathcal{F}(\Theta)$ starting from I at time 0 to $I+\theta$ at time 1 in the group $\mathcal{F}(\Theta)$ by assigning times t_i, $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$, at each jump:

$$T(t) \stackrel{\text{def}}{=} \begin{cases} I, & 0 \le t < t_1, \\ (I+\theta_1), & t_1 < t < t_2, \\ (I+\theta_2)\circ(I+\theta_1), & t_2 < t < t_3, \\ \dots & \\ (I+\theta_i)\circ\cdots\circ(I+\theta_1), & t_i < t < t_{i+1}, \\ \dots & \\ (I+\theta_n)\circ\cdots\circ(I+\theta_1), & t_n < t \le 1. \end{cases} \tag{2.16}$$

The *jump in the group* $\mathcal{F}(\Theta)$ at time t_i is given by

$$\begin{aligned} [T(t)]_{t_i} &\stackrel{\text{def}}{=} T(t_i^+)\circ T(t_i^-)^{-1} - I \\ &= [(I+\theta_i)\circ\cdots\circ(I+\theta_1)]\circ[(I+\theta_{i-1})\circ\cdots\circ(I+\theta_1)]^{-1} - I = \theta_i \end{aligned}$$

as an element of Θ that could be viewed as the tangent space to $\mathcal{F}(\Theta) \subset I+\Theta$.

The choice of d in a *geodesic context* is more interesting than the choice of d_1 that would involve a double infima in the energy (2.15). At this stage, in view of the equivalence Theorem 2.3, working with the metric d or d_1 is completely equivalent.

Assumption 2.3.
$(\Theta, \|\cdot\|)$ is a normed vector space of mappings from $\mathbf{R}^{\mathrm{N}}$ into $\mathbf{R}^{\mathrm{N}}$ and there exists a continuous function c_1 such that for all $I+f \in \mathcal{F}(\Theta)$ and $g \in \Theta$

$$\|g\circ(I+f)\| \le \|g\|\, c_1(\|f\|). \tag{2.17}$$

□

The next assumption is a kind of *uniform continuity.*

Assumption 2.4.
$(\Theta, \|\cdot\|)$ is a normed vector space of mappings from $\mathbf{R}^{\mathrm{N}}$ into $\mathbf{R}^{\mathrm{N}}$. For each $g \in \Theta$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall \gamma \in \Theta \text{ such that } \|\gamma\| < \delta, \quad \|g\circ(I+\gamma) - g\| < \varepsilon. \tag{2.18}$$

□

Theorem 2.4. *Under Assumptions* 2.1 *to* 2.4*,* $\mathcal{F}(\Theta)$ *is a topological (metric) group.*

Example 2.3 (Checking Assumption 2.3)**.**
(1), (2), (3) Since the norm on the three spaces $\mathcal{B}^0(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})$, $C^0(\overline{\mathbf{R}^{\mathrm{N}}},\mathbf{R}^{\mathrm{N}})$, and $C_0^0(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})$ is the same and Assumption 2.2 involves only the norm, it is sufficient to check it for $\mathcal{B}^0(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})$. Consider $g \in \mathcal{B}^0(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})$ and $I+f \in \mathcal{F}(\mathcal{B}^0(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}}))$. Since $I+f$ is a bijection

$$\|g\circ(I+f)\|_{C^0} \le \|g\|_{C^0} \tag{2.19}$$

and the constant function $c_1(r) = 1$ satisfies Assumption 2.3.

(4) $\Theta = C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)$. Consider $g \in C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)$ and $I + f \in \mathcal{F}(C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N))$. From (1)

$$\|g \circ (I + f)\|_{C^0} \leq \|g\|_{C^0}$$

and since all the functions are Lipschitz

$$\begin{aligned} c(g \circ (I + f)) &\leq c(g)\,(1 + c(f)) \\ \Rightarrow \|g \circ (I + f)\|_{C^{0,1}} &\leq \|g\|_{C^{0,1}}\,(1 + c(f)) \leq \|g\|_{C^{0,1}}\,(1 + \|f\|_{C^{0,1}}) \end{aligned}$$

and the continuous function $c_1(r) = 1 + r$ satisfies Assumption 2.3. □

Assumption 2.4 is a little trickier. It involves not only norms but also a "uniform continuity" of the function g that is not verified for $g \in \mathcal{B}^0(\mathbf{R}^N, \mathbf{R}^N)$ and for the Lipschitz part $c(g)$ of the norm of $C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)$.

Example 2.4 (Checking Assumption 2.4)**.**
(1) $\Theta = \mathcal{B}^0(\mathbf{R}^N, \mathbf{R}^N)$. In that space there exists some g that is not uniformly continuous. For that g there exists $\varepsilon > 0$ such that for all $n > 0$, there exists x_n, y_n, $|y_n - x_n| < 1/n$ such that $|g(y_n) - g(x_n)| \geq \varepsilon$. Define the function $\gamma_n(x) = y_n - x_n$. Therefore, there exists $\varepsilon > 0$ such that for all n with $\|\gamma_n\|_{C^0} < 1/n$,

$$\|g \circ (I + \gamma_n) - g\|_{C^0} \geq |g(x_n + \gamma_n(x_n)) - g(x_n)| = |g(x_n + y_n - x_n) - g(x_n)| \geq \varepsilon$$

and Assumption 2.4 is not verified. However, $\mathcal{B}^0(\mathbf{R}^N, \mathbf{R}^N) \subset C^0(\mathbf{R}^N, \mathbf{R}^N)$ and for all $x \in \mathbf{R}^N$, the mapping $f \mapsto f(x) : \mathcal{B}^0(\mathbf{R}^N, \mathbf{R}^N) \to \mathbf{R}$ is continuous.

(2) and (3) $\Theta = C^0(\overline{\mathbf{R}^N}, \mathbf{R}^N)$. By the uniform continuity of $g \in C^0(\overline{\mathbf{R}^N}, \mathbf{R}^N)$, for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall x, y \in \mathbf{R}^N, \quad |g(y) - g(x)| < \varepsilon/2.$$

For $\|\gamma\|_{C^0} < \delta$

$$\begin{aligned} \forall x \in \mathbf{R}^N, \quad & |(I + \gamma)(x) - x| = |\gamma(x)| \leq \|\gamma\|_{C^0} < \delta, \\ \forall x \in \mathbf{R}^N, \quad & |g(x + \gamma(x)) - g(x)| < \varepsilon/2 \quad \Rightarrow \|g \circ (I + \gamma) - g\|_{C^0} \leq \varepsilon/2 < \varepsilon \end{aligned}$$

and Assumption 2.4 is verified. In addition, $C^0(\overline{\mathbf{R}^N}, \mathbf{R}^N) \subset C^0(\mathbf{R}^N, \mathbf{R}^N)$ and for all $x \in \mathbf{R}^N$, the mapping $f \mapsto f(x) : C^0(\overline{\mathbf{R}^N}, \mathbf{R}^N) \to \mathbf{R}$ is continuous.

(4) $\Theta = C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)$. Since $g \in C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)$

$$\forall x, y \in \mathbf{R}^N, \quad |g(y) - g(x)| < c(g)\,|y - x|.$$

Therefore for

$$\|\gamma\|_{C^{0,1}} \stackrel{\text{def}}{=} \max\{\|\gamma\|_{C^0}, c(\gamma)\} < \delta,$$

$\|\gamma\|_{C^0} < \delta$ and we get

$$\|g \circ (I + \gamma) - g\|_{C^0} \leq c(g)\,\|\gamma\|_{C^0} < c(g)\,\delta.$$

Unfortunately, for the part $c(g \circ (I+\gamma))$ of the $C^{0,1}$-norm, for all $x \neq y \in \mathbf{R}^{\mathrm{N}}$

$$\frac{|g(y+\gamma(y)) - g(y) - (g(x+\gamma(x)) - g(x))|}{|y-x|} \leq c(g) + c(g)\, c(I+\gamma),$$
$$c(g \circ (I+\gamma) - g) \leq c(g)\,[1 + c(\gamma)] < c(g)\,[1+\delta]$$

and we cannot satisfy Assumption 2.4. However, $C^{0,1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}}) \subset C^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ and for all $x \in \mathbf{R}^{\mathrm{N}}$, the mapping $f \mapsto f(x) : C^{0,1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}}) \to \mathbf{R}$ is continuous. □

Remark 2.1.
Since the spaces $C_0^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}) \subset C^k(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}}) \subset \mathcal{B}^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ have the same norm and Assumptions 2.2 and 2.3 involve only the norms, they will always be verified for $C_0^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ and $C^k(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$ if they are verified for $\mathcal{B}^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$, $k \geq 0$. Solely Assumption 2.1 will require special attention. □

The fact that Assumption 2.4 is not always verified will not prevent us from proving that the space $(\mathcal{F}(\Theta), d)$ is complete. The completeness will be proved in Theorem 2.6 under Assumptions 2.1 to 2.3, where Assumption 2.4 will be replaced by the condition $\Theta \subset C^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ and, for all $x \in \mathbf{R}^{\mathrm{N}}$, the mapping $f \mapsto f(x) : \Theta \to \mathbf{R}^{\mathrm{N}}$ is continuous. The following theorem will be proved in Theorems 2.11, 2.12, and 2.13 of section 2.5 and Theorems 2.15 and 2.16 of section 2.6.

Theorem 2.5. *Let $k \geq 0$ be an integer.*

(i) $C_0^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ *and* $C^k(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$ *satisfy Assumptions* 2.1 *to* 2.4*,* $C_0^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}) \subset C^k(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}}) \subset C^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$*, and, for all* $x \in \mathbf{R}^{\mathrm{N}}$*, the mapping* $f \mapsto f(x) : C^k(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}}) \to \mathbf{R}^{\mathrm{N}}$ *is continuous.*

(ii) $\mathcal{B}^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ *and* $C^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$ *verify Assumptions* 2.1 *to* 2.3*,* $C^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}}) \subset \mathcal{B}^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}) \subset C^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$*, and, for all* $x \in \mathbf{R}^{\mathrm{N}}$*, the mapping* $f \mapsto f(x) : \mathcal{B}^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}) \to \mathbf{R}^{\mathrm{N}}$ *is continuous.*

Proof of Theorem 2.4. From J. L. Kelley [**1**], it is sufficient to prove that

$$\forall F \in \mathcal{F}(\Theta), \forall H \in \mathcal{F}(\Theta) \text{ such that } d(I,H) \to 0, \quad d(I, F^{-1} \circ H \circ F) \to 0.$$

Rewrite $F^{-1} \circ H \circ F$ as follows:

$$\begin{aligned} F^{-1} \circ H \circ F - I &= F^{-1} \circ [I + H - I] \circ F - F^{-1} \circ F \\ &= \left[F^{-1} \circ (I+\gamma) - F^{-1}\right] \circ F \\ &= \left[(F^{-1} - I) \circ (I+\gamma) - (F^{-1} - I)\right] \circ F + \gamma \circ F, \end{aligned}$$
$$\gamma \overset{\text{def}}{=} H - I \text{ and } g \overset{\text{def}}{=} (F^{-1} - I) \circ (I+\gamma) - (F^{-1} - I).$$

By Assumption 2.3,

$$\|\gamma \circ F\| \leq \|\gamma\|\, c_1(\|F - I\|), \quad \|g \circ F\| \leq \|g\|\, c_1(\|F - I\|);$$

by Assumption 2.4, for all $\varepsilon > 0$, there exists δ, $0 < \delta < \min\{1, \varepsilon/(4c_0(1)\, c_1(\|F - I\|))\}$, such that

$$\|\gamma\| = \|H - I\| < \delta\, c_0(1) \quad \Rightarrow \quad \begin{aligned} \|g\| =& \|(F^{-1} - I) \circ (I + \gamma) - (F^{-1} - I)\| \\ &< \varepsilon/(4\, c_1(\|F - I\|)). \end{aligned}$$

Combining the above properties

$$\begin{gathered} \forall H \text{ such that } d(I, H) < \delta, \quad \|H - I\| < \delta\, c_0(1) \\ \Rightarrow \|F^{-1} \circ H \circ F - I\| \le \|\gamma\|\, c_1(\|F - I\|) + \varepsilon/2 \le \delta\, c_0(1)\, c_1(\|F - I\|) + \varepsilon/4 \le \varepsilon/2. \end{gathered}$$

By Assumption 2.2, for all H such that $d(I, H) < \delta < 1$, we have $\|H - I\| < \delta\, c_0(1)$ and $\|H^{-1} - I\| < \delta\, c_0(1)$. Finally, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $d(I, H) < \delta$ implies $\|F^{-1} \circ H \circ F - I\| \le \varepsilon/2$. By a similar argument, for all $\varepsilon > 0$, there exists $\delta' > 0$ such that $d(I, H) < \delta'$ implies $\|F^{-1} \circ H^{-1} \circ F - I\| \le \varepsilon/2$. Therefore, for all $\varepsilon > 0$ there exists $\delta_0 = \min\{\delta, \delta'\} > 0$ such that for all H, $d(I, H) < \delta_0$,

$$d(I, F^{-1} \circ H \circ F) \le \|F^{-1} \circ H \circ F - I\| + \|F^{-1} \circ H^{-1} \circ F - I\| < \varepsilon$$

and $\mathcal{F}(\Theta)$ is a topological group for the metric d. □

Theorem 2.6. (i) *Let Assumptions* 2.1 *to* 2.3 *be verified for a Banach space* Θ *contained in* $C^0(\mathbf{R}^N, \mathbf{R}^N)$ *such that, for all* $x \in \mathbf{R}^N$*, the mapping* $f \mapsto f(x) : \Theta \to \mathbf{R}^N$ *is continuous. Then* $(\mathcal{F}(\Theta), d)$ *is a complete metric space.*

(ii) *Let Assumptions* 2.1 *to* 2.4 *be verified for a Banach space* Θ*. Then* $\mathcal{F}(\Theta)$ *is a complete topological (metric) group.*

Remark 2.2.
$\mathcal{F}(C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N))$ is a complete metric space since the assumptions of Theorem 2.6 are verified, but it is not a topological group since Assumption 2.4 is not verified. □

Proof of Theorem 2.6. (i) Let $\{F_n\}$ be a Cauchy sequence in $\mathcal{F}(\Theta)$.

(a) *Boundedness of* $\{F_n - I\}$ *and* $\{F_n^{-1} - I\}$ *in* Θ. For each ε, $0 < \varepsilon < 1$, there exists N such that, for all m, $n > N$, $d(F_m, F_n) < \varepsilon$. By the triangle inequality

$$|d(I, F_m) - d(I, F_n)| \le d(F_m, F_n) < \varepsilon,$$

$\{d(I, F_n\}$ is Cauchy, and, a posteriori, is bounded by some constant L. For each n, there exists a factorization $F_n = (I + f_1^n) \circ \cdots \circ (I + f_{\nu(n)}^n)$ such that

$$\begin{aligned} 0 \le& \sum_{i=1}^{\nu(n)} \|f_i^n\| + \|f_i^n \circ (I + f_1^n)^{-1}\| - d(I, F_n) < \varepsilon < 1 \\ \Rightarrow& \sum_{i=1}^{\nu(n)} \|f_i^n\| + \|f_i^n \circ (I + f_i^n)^{-1}\| < L + \varepsilon < L + 1. \end{aligned}$$

By Assumption 2.2

$$\|F_n - I\| + \|F_n^{-1} - I\| < 2(L+\varepsilon)\, c_0(L+1)$$

and the sequences $\{F_n - I\}$ and $\{F_n^{-1} - I\}$ are bounded in Θ.

(b) *Convergence of* $\{F_n - I\}$ *and* $\{F_n^{-1} - I\}$ *in* Θ. For any m, n

$$F_n - F_m = (F_n \circ F_m^{-1} - I) \circ F_m$$

and by Assumption 2.3

$$\begin{aligned} &\|F_m - F_n\| = \|(F_n \circ F_m^{-1} - I) \circ F_m\| \le \|F_n \circ F_m^{-1} - I\|\, c_1(\|F_m - I\|) \\ &\Rightarrow\ \|F_m - F_n\| \le c_1^+\, \|F_n \circ F_m^{-1} - I\|, \quad c_1^+ \overset{\text{def}}{=} \sup_m c_1(\|F_m - I\|) < \infty \end{aligned}$$

since $\{F_m - I\}$ is bounded. For all ε, $0 < \varepsilon < c_0(1)\, c_1^+$, there exists N such that

$$\forall n, m, > N, \quad d(I, F_n \circ F_m^{-1}) = d(F_m, F_n) < \varepsilon/(c_0(1)\, c_1^+) < 1.$$

By Assumption 2.2

$$\begin{aligned} &\|F_n \circ F_m^{-1} - I\| < c_0(1)\varepsilon/(c_0(1)\, c_1^+) = \varepsilon/c_1^+ \\ \Rightarrow\ &\forall n, m, > N, \quad \|F_m - I - (F_n - I)\| \le c_1^+\, \|F_n \circ F_m^{-1} - I\| < \varepsilon \end{aligned}$$

and $\{F_n - I\}$ is a Cauchy sequence in Θ that converges to some $F - I \in \Theta$ since Θ is a Banach space. Similarly, we get that $\{F_n^{-1} - I\}$ is a Cauchy sequence in Θ that converges to some $G - I \in \Theta$.

(c) *F is bijective, $G = F^{-1}$, and $F \in \mathcal{F}(\Theta)$.* For all n, F_n, F_n^{-1}, F, and G are continuous since $\Theta \subset C^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$. By continuity of $f \mapsto f(x) : \Theta \to \mathbf{R}^{\mathrm{N}}$, $f \mapsto x + f(x) : \Theta \to \mathbf{R}^{\mathrm{N}}$ is continuous and $[F_n^{-1}(x) - x| + x \to [G(x) - x] + x$. By the same argument for F_n, $x = F_n(F_n^{-1}(x)) \to F(G(x)) = (F \circ G)(x)$ and for all $x \in \mathbf{R}^{\mathrm{N}}$, $(F \circ G)(x) = x$. Similarly, starting from $F_n^{-1} \circ F_n = I$, we get $(G \circ F)(x) = x$. Hence, $G \circ F = I = F \circ G$. So F is bijective, $F - I, F^{-1} - I \in \Theta$, and $F \in \mathcal{F}(\Theta)$.

(d) *Convergence of $F_n \to F$ in $\mathcal{F}(\Theta)$.* To check that $F_n \to F$ in $\mathcal{F}(\Theta)$ recall that by Assumption 2.3

$$\begin{aligned} d(F_n, F) = d(I, F \circ F_n^{-1}) \le &\|I - F \circ F_n^{-1}\| + \|I - F_n \circ F^{-1}\| \\ &= \|(F_n - F) \circ F_n^{-1}\| + \|(F - F_n) \circ F^{-1}\| \\ &\le c_1(F_n^{-1} - I)\|F_n - F\| + c_1(F^{-1} - I)\|F - F_n\| \\ &\le \left(\max_n c_1(F_n^{-1} - I) + c_1(F^{-1} - I)\right) \|F - F_n\|, \end{aligned}$$

which goes to zero as $F_n - I$ goes to $F - I$ in Θ.

(ii) The proof (ii) differs from the proof of (i) only in part (c). Consider

$$G \circ F - I = G \circ F - G \circ F_n + (G - F_n^{-1}) \circ F_n.$$

Since $\{F_n - I\}$ is bounded in Θ and $F_n^{-1} \to G$, the term $(G - F_n^{-1}) \circ F_n$ goes to 0 as $n \to \infty$ by Assumption 2.3. The second terms can be rewritten as

$$G \circ F - G \circ F_n = F - F_n + (G - I) \circ F - (G - I) \circ F_n = F - F_n + g \circ F - g \circ F_n,$$

where $g = G - I \in \Theta$. The term $F - F_n$ goes to zero.

As for the second terms

$$g \circ F - g \circ F_n = \left\{ g \circ [I + (F - F_n) \circ F_n^{-1}] - g \right\} \circ F_n.$$

Let $g_n = g \circ [I + (F - F_n) \circ F_n^{-1}] - g$ and let $\gamma_n = (F - F_n) \circ F_n^{-1}$.

By boundedness of $\{F_n - I\}$ and $\{F_n^{-1} - I\}$ in Θ and Assumption 2.3,

$$\begin{gathered}
\|\gamma_n\| = \|(F - F_n) \circ F_n^{-1}\| \leq \|F - F_n\| \, c_1 \left(\|F_n^{-1} - I\|\right) \leq c_1^+ \, \|F - F_n\|, \\
\|g_n \circ F_n\| \leq \|g_n\| \, c_1 \left(\|F_n - I\|\right) \leq c_1^+ \, \|g_n\|, \\
c_1^+ \overset{\text{def}}{=} \sup_n \{c_1 \left(\|F_n - I\|\right), c_1 \left(\|F_n^{-1} - I\|\right)\} < \infty.
\end{gathered}$$

By Assumption 2.4, for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|\gamma_n\| < \delta \, c_1^+ \quad \Rightarrow \|g_n\| = \|g \circ (I + \gamma_n) - g\| < \varepsilon / c_1^+.$$

By combining for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned}
\|F - F_n\| < \delta \quad & \Rightarrow \|\gamma_n\| \leq c_1^+ \|F - F_n\| < c_1^+ \delta \\
& \Rightarrow \|g_n \circ F_n\| < c_1^+ \, \varepsilon / c_1^+ = \varepsilon \\
& \Rightarrow \|F - F_n\| < \delta \quad \Rightarrow \|g \circ F - g \circ F_n\| < \varepsilon.
\end{aligned}$$

As a result $\|G \circ F - I\|$ is bounded by an expression that goes to zero as $n \to \infty$ and $G \circ F = I$. By an analogous argument, we can prove that $F \circ G - I = 0$, that is, $F^{-1} = G$. So F is bijective, $F - I, F^{-1} - I \in \Theta$. Hence $F \in \mathcal{F}(\Theta)$. $\square$

2.2 Diffeomorphisms for $\mathcal{B}(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N})$ and $C_0^\infty(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N})$

Consider as other candidates for Θ the spaces

$$\mathcal{B}(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N}) \overset{\text{def}}{=} \cap_{k=0}^\infty \mathcal{B}^k(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N}) = \cap_{k=0}^\infty C^k(\overline{\mathbf{R}^\mathrm{N}}, \mathbf{R}^\mathrm{N}), \tag{2.20}$$

$$C_0^\infty(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N}) \overset{\text{def}}{=} \cap_{k=0}^\infty C_0^k(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N}) \tag{2.21}$$

and the resulting groups $\mathcal{F}(\mathcal{B}(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N}))$

$$\left\{ I + f : f \in \mathcal{B}(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N}),\ (I + f) \text{ bijective and } (I + f)^{-1} - I \in \mathcal{B}(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N}) \right\}$$

and $\mathcal{F}(C_0^\infty(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N}))$

$$\left\{ I + f : f \in C_0^\infty(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N}),\ (I + f) \text{ bijective and } (I + f)^{-1} - I \in C_0^\infty(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N}) \right\}$$

that satisfy Assumption 2.1 but not the other assumptions since $\mathcal{B}(\mathbf{R}^N, \mathbf{R}^N)$ and $C_0^\infty(\mathbf{R}^N, \mathbf{R}^N)$ are Fréchet but not Banach spaces. To get around this, observe that

$$\mathcal{F}(\mathcal{B}(\mathbf{R}^N, \mathbf{R}^N)) = \cap_{k=0}^\infty \mathcal{F}(\mathcal{B}^k(\mathbf{R}^N, \mathbf{R}^N)) = \cap_{k=0}^\infty \mathcal{F}(C^k(\overline{\mathbf{R}^N}, \mathbf{R}^N)),$$
$$\mathcal{F}(C_0^\infty(\mathbf{R}^N, \mathbf{R}^N)) = \cap_{k=0}^\infty \mathcal{F}(C_0^k(\mathbf{R}^N, \mathbf{R}^N)),$$

where $(\mathcal{F}(C^k(\overline{\mathbf{R}^N}, \mathbf{R}^N)), d_k)$ and $(\mathcal{F}(C_0^k(\mathbf{R}^N, \mathbf{R}^N)), d_k)$ are topological (metric) groups with the same metrics $\{d_k\}$ and the following monotony property:

$$d_k(F, G) \leq d_{k+1}(F, G)$$

resulting from the monotony of the norms

$$\|f\|_{C^k} = \max_{0 \leq i \leq k} \|f\|_{C^i} \leq \max_{0 \leq i \leq k+1} \|f\|_{C^i} = \|f\|_{C^{k+1}}.$$

Theorem 2.7. *The spaces $\mathcal{F}(\mathcal{B}(\mathbf{R}^N, \mathbf{R}^N))$ and $\mathcal{F}(C_0^\infty(\mathbf{R}^N, \mathbf{R}^N))$ are complete topological (metric) groups for the distance*

$$d_\infty(F, G) \stackrel{\text{def}}{=} \sum_{k=0}^\infty \frac{1}{2^k} \frac{d_k(F, G)}{1 + d_k(F, G)}, \tag{2.22}$$

where d_k is the metric associated with $C^k(\overline{\mathbf{R}^N}, \mathbf{R}^N)$ and $C_0^k(\mathbf{R}^N, \mathbf{R}^N)$.[8]

This theorem is a consequence of the following general lemma.

Lemma 2.2. *Assume that $(\mathcal{F}_k, d_k)$, $k \geq 0$, is a family of groups each complete with respect to their metric d_k and that*

$$\forall k \geq 0, \quad \mathcal{F}_{k+1} \subset \mathcal{F}_k \quad \textit{and} \quad d_k(F, G) \leq d_{k+1}(F, G), \; \forall F, G \in \cap_{k=0}^\infty \mathcal{F}_k. \tag{2.23}$$

Then the intersection $\mathcal{F}_\infty = \cap_{k=0}^\infty \mathcal{F}_k$ is a group that is complete with respect to the metric d_∞ defined in (2.22)*. If, in addition, each $(\mathcal{F}_k, d_k)$ is a topological (metric) group, then $\mathcal{F}_\infty$ is a topological (metric) group.*

Proof. By definition, $\mathcal{F}_\infty$ is clearly a group.

(i) *d_∞ is a metric on $\mathcal{F}_\infty$.* $d_\infty(F, G) = 0$ implies that $d_k(F, G) = 0$ for all k and $F = G$ and conversely. Also $d_\infty(F, G) = d_\infty(G, F)$. The function $x/(1 + x)$ is monotone strictly increasing with $x \geq 0$. Therefore, $d_k(F, H) \leq d_k(F, G) + d_k(G, H)$ implies

$$\begin{aligned}\frac{d_k(F, H)}{1 + d_k(F, H)} &\leq \frac{d_k(F, G)}{1 + d_k(F, G) + d_k(G, H)} + \frac{d_k(G, H)}{1 + d_k(F, G) + d_k(G, H)} \\ &\leq \frac{d_k(F, G)}{1 + d_k(F, G)} + \frac{d_k(G, H)}{1 + d_k(G, H)},\end{aligned}$$

$$\sum_{k=0}^\infty \frac{1}{2^k} \frac{d_k(F, H)}{1 + d_k(F, H)} \leq \sum_{k=0}^\infty \frac{1}{2^k} \frac{d_k(F, G)}{1 + d_k(F, G)} + \sum_{k=0}^\infty \frac{1}{2^k} \frac{d_k(G, H)}{1 + d_k(G, H)}$$

and $d_\infty(F, H) \leq d_\infty(F, G) + d_\infty(G, H)$.

[8]In view of (2.20), it is important to note that $\mathcal{F}(\mathcal{B}(\mathbf{R}^N, \mathbf{R}^N))$ is a topological group by using the family $(\mathcal{F}(C^k(\overline{\mathbf{R}^N}, \mathbf{R}^N)), d_k)$ in Lemma 2.2 and not the family $(\mathcal{F}(\mathcal{B}^k(\mathbf{R}^N, \mathbf{R}^N)), d_k)$.

(ii) $(\mathcal{F}_\infty.d_\infty)$ *is complete.* Let $\{F_n\}$ be a Cauchy sequence in $\mathcal{F}_\infty$; that is, for all ε, $0 < \varepsilon < 1/2$, there exists N_0 such that for $n, m > N_0$, $d_\infty(F_n, F_m) < \varepsilon$. In particular

$$\frac{d_0(F_n, F_m)}{1 + d_0(F_n, F_m)} \le \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{d_k(F_n, F_m)}{1 + d_k(F_n, F_m)} < \varepsilon \quad \Rightarrow d_0(F_n, F_m) < \varepsilon/(1-\varepsilon) < 2\varepsilon$$

and $\{F_n\}$ is a Cauchy sequence in $\mathcal{F}_0$ that converges to some F in $\mathcal{F}_0$. Choose a subsequence of $\{F_n\}$ such that there exists $N_1 \ge N_0$ such that for $n, m > N_1$, $d_\infty(F_n, F_m) < \varepsilon/2^2$. Then

$$\frac{1}{2} \frac{d_1(F_n, F_m)}{1 + d_1(F_n, F_m)} \le \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{d_k(F_n, F_m)}{1 + d_k(F_n, F_m)} < \frac{1}{2^2}\varepsilon \ \Rightarrow d_1(F_n, F_m) < \varepsilon/(1-\varepsilon) < 2\varepsilon$$

and $\{F_n\}$ is a Cauchy sequence in $\mathcal{F}_1$ that converges to some G_1 in $\mathcal{F}_1$. By continuous injection of $\mathcal{F}_1$ into $\mathcal{F}_0$, $G_1 = F \in \mathcal{F}_0 \cap \mathcal{F}_1$. Proceeding in that way, we get that there exists $F \in \mathcal{F}_\infty = \cap_{k=0}^\infty \mathcal{F}_k$ such that $F_n \to F$ in $\mathcal{F}_k$ for all $k \ge 0$. Therefore, $\mathcal{F}_\infty$ is complete. $\square$

2.3 Closed Subgroups $\mathcal{G}$

Given an arbitrary nonempty subset Ω_0 of $\mathbf{R}^{\mathrm{N}}$, consider the family

$$\mathcal{X}(\Omega_0) \stackrel{\text{def}}{=} \left\{F(\Omega_0) \subset \mathbf{R}^{\mathrm{N}} \ : \forall F \in \mathcal{F}(\Theta)\right\} \tag{2.24}$$

of images of Ω_0 by the elements of $\mathcal{F}(\Theta)$. By introducing the subgroup

$$\mathcal{G}(\Omega_0) \stackrel{\text{def}}{=} \{F \in \mathcal{F}(\Theta) \,:\, F(\Omega_0) = \Omega_0\}, \tag{2.25}$$

we obtain a bijection

$$[F] \mapsto F(\Omega_0) \,:\, \mathcal{F}(\Theta)/\mathcal{G}(\Omega_0) \to \mathcal{X}(\Omega_0) \tag{2.26}$$

between the set of images $\mathcal{X}(\Omega_0)$ and the quotient space $\mathcal{F}(\Theta)/\mathcal{G}(\Omega_0)$. Using this bijection, the topological structure of $\mathcal{X}(\Omega_0)$ will be identified with the topological structure of the quotient space.

Lemma 2.3. *Let Assumptions* 2.1 *to* 2.3 *hold with* $\Theta \subset C^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ *and, for all* $x \in \mathbf{R}^{\mathrm{N}}$*, let the mapping* $f \mapsto f(x) : \Theta \to \mathbf{R}^{\mathrm{N}}$ *be continuous.*[9] *If the nonempty subset* Ω_0 *of* $\mathbf{R}^{\mathrm{N}}$ *is closed or if* $\Omega_0 = \operatorname{int}\overline{\Omega_0}$*,*[10] *the family*

$$\mathcal{G}(\Omega_0) \stackrel{\text{def}}{=} \{F \in \mathcal{F}(\Theta) \,:\, F(\Omega_0) = \Omega_0\} \tag{2.27}$$

is a closed subgroup of $\mathcal{F}(\Theta)$.

[9] This means that $\mathcal{F}(\Theta) \subset I + \Theta \subset \operatorname{Hom}(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$.

[10] Such a set will be referred to as a *crack-free* set in Definition 7.1 (ii) of Chapter 8. Indeed, by definition Ω is crack-free if $\overline{\complement\Omega} = \overline{\complement\overline{\Omega}}$. If, in addition, Ω is open, then $\complement\Omega = \overline{\complement\Omega} = \overline{\complement\overline{\Omega}}$ and $\Omega = \complement\overline{\complement\overline{\Omega}} = \operatorname{int}\overline{\Omega}$.

Proof. $\mathcal{G}(\Omega_0)$ is clearly a subgroup of $\mathcal{F}(\Theta)$. It is sufficient to show that for any sequence $\{F_n\}$ in $\mathcal{G}(\Omega_0)$ that converges to F in $\mathcal{F}(\Theta)$, then $F \in \mathcal{G}(\Omega_0)$. The other properties are straightforward. If $\{F_n\}$ converges to F in $\mathcal{F}(\Theta)$, then by Assumption 2.3 we have

$$\|F_n \circ F^{-1} - I\| + \|F \circ F_n^{-1} - I\| \to 0.$$

From $\|F \circ F_n^{-1} - I\| \to 0$, for each $x \in \Omega_0$ the sequence $\{F(F_n^{-1}(x))\} \subset F(\Omega_0)$ converges to $x \in \Omega_0$ and hence $\Omega_0 \subset \overline{F(\Omega_0)}$. From the convergence $\|F_n \circ F^{-1} - I\| \to 0$ for each $x \in F(\Omega_0)$, $F^{-1}(x) \in \Omega_0$ and the sequence $\{F_n(F^{-1}(x))\} \subset \Omega_0$ converges to $x \in F(\Omega_0)$ and hence $F(\Omega_0) \subset \overline{\Omega_0}$. Therefore, $\overline{\Omega_0} \subset \overline{F(\Omega_0)} \subset \overline{\Omega_0}$ and $\overline{\Omega_0} = \overline{F(\Omega_0)}$. Since F is a homeomorphism, for all A, $\overline{F(A)} = F(\overline{A})$ and $F(\complement A) = \complement F(A)$. As a result $\operatorname{int} F(\overline{A}) = F(\operatorname{int} \overline{A})$. In view of the identity $\overline{\Omega_0} = \overline{F(\Omega_0)}$,

$$F(\overline{\Omega_0}) = \overline{F(\Omega_0)} = \overline{\Omega_0} \quad \text{and} \quad \operatorname{int} \overline{\Omega_0} = \operatorname{int} F(\overline{\Omega_0}) = F(\operatorname{int} \overline{\Omega_0}).$$

From this, if Ω_0 is closed, $\Omega_0 = \overline{\Omega_0}$ and $F(\Omega_0) = \Omega_0$. If $\Omega_0 = \operatorname{int} \overline{\Omega_0}$, we also get $F(\Omega_0) = \Omega_0$. In both cases $F \in \mathcal{G}(\Omega_0)$. □

Remark 2.3.
A. M. Micheletti [**1**] assumes that Ω_0 is a bounded connected open domain of class C^3 in order to make all the images $F(\Omega_0)$ bounded connected open domains of class C^3 with $\Theta = C_0^3(\mathbf{R}^N, \mathbf{R}^N)$. Open domains of class C^k, $k \geq 1$, are locally C^k epigraphs and domains that are locally C^0 epigraphs are crack-free[11] by Theorem 5.4 (ii) of Chapter 2. □

Remark 2.4.
F. Murat and J. Simon [**1**] were mainly interested in families of images of locally Lipschitzian (epigraph) domains. Recall that $\mathcal{F}(C^1(\overline{\mathbf{R}^N}, \mathbf{R}^N))$ transports locally Lipschitzian (epigraph) domains onto locally Lipschitzian (epigraph) domains (cf. A. Bendali [**1**] and A. Djadane [**1**]), but that $\mathcal{F}(C^{0,1}(\mathbf{R}^N, \mathbf{R}^N))$ does not, as shown in Examples 5.1 and 5.2 of section 5.3 in Chapter 2. □

Of special interest are the closed submanifolds of $\mathbf{R}^N$ of codimension greater than or equal to 1. For instance, we can choose $\Omega_0 = S^n$, $2 \leq n < N$, the n-sphere of radius 1 centered in 0. We get a family of closed curves in $\mathbf{R}^3$ for $n = 2$ and of closed surfaces in $\mathbf{R}^3$ for $n = 3$. Similarly, by choosing $\Omega_0 = \overline{B_1^n(0)}$, the closed unit ball of dimension n, we get a family of curves for $n = 1$ and a family of surfaces for $n = 2$. The choice of $[0,1]^n$ yields a family of surfaces with *corners* for $n = 2$.

Explicit expressions can be obtained for normals, curvatures, and density measures as in Chapter 2.

Example 2.5 (Closed surfaces in $\mathbf{R}^3$)**.**
For $\Omega_0 = S^2$ in $\mathbf{R}^3$, $f \in C^1(\overline{\mathbf{R}^3}, \mathbf{R}^3)$, and $F = I + f$, the tangent field at the point $\xi \in S^2$ is $T_\xi S^2 = \{\xi\}^\perp$ and the unit normal is ξ. At a point $x = F(\xi)$ of the image

[11] Cf. Definition 7.1 (ii) of Chapter 8.

$F(S^2)$, the tangent space and the unit normal are given by

$$T_{F(\xi)}F(S^2) = DF(\xi)\{\xi\}^\perp \text{ and } n(F(x)) = \frac{{}^*DF(\xi)^{-1}\xi}{|\,{}^*DF(\xi)^{-1}\xi|}. \qquad \square$$

Example 2.6 (Curves in $\mathbf{R}^N$)**.**
Let e_N be a unit vector in $\mathbf{R}^N$, let $H = \{e_N\}^\perp$, and let

$$\Omega_0 \overset{\text{def}}{=} \{\zeta_N e_N : \zeta_N \in [0,1]\} \subset \mathbf{R}^N.$$

For $f \in C^1(\overline{\mathbf{R}^N}, \mathbf{R}^N)$, and $F = I + f$, the tangent field at the point $\xi \in \Omega_0$ is $T_\xi \Omega_0 = \mathbf{R}\, e_N$ and the normal space $(T_\xi\Omega_0)^\perp$ is H. At a point $x = F(\xi)$ of the curve $F(\Omega_0)$, the tangent and normal spaces are given by

$$T_{F(\xi)}F(\Omega_0) = \mathbf{R}\, DF(\xi)e_N \quad \text{and} \quad T_{F(\xi)}F(\Omega_0)^\perp = \{DF(\xi)e_N\}^\perp. \qquad \square$$

In practice, one might want to use other subgroups, such as all the translations by some vector $a \in \mathbf{R}^N$:

$$F_a(x) \overset{\text{def}}{=} a + x, \quad x \in \mathbf{R}^N, \quad \Rightarrow F_a - I = a \in \Theta \tag{2.28}$$

$$\Rightarrow F_a^{-1}(x) - a + x = F_{-a}(x) \quad \Rightarrow F_a^{-1} - I = -a \in \Theta \tag{2.29}$$

for all Θ that contain the constant functions such as $\mathcal{B}^k(\mathbf{R}^N, \mathbf{R}^N)$, $C^k(\overline{\mathbf{R}^N}, \mathbf{R}^N)$, and $C^{k,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)$, but not $C^k_0(\overline{\mathbf{R}^N}, \mathbf{R}^N)$. Therefore

$$\mathcal{G}_\tau \overset{\text{def}}{=} \left\{F_a : \forall a \in \mathbf{R}^N\right\} \tag{2.30}$$

is a subgroup of $\mathcal{F}(\Theta)$. In that case, it is also a closed subgroup of $\mathcal{F}(\Theta)$:

$$F_a \circ F_b(x) = a + b + x = F_{a+b}(x), \quad F_0 = I$$

and $(\mathcal{G}_\tau, \circ)$ is isomorphic to the one-dimensional closed abelian group $(\mathbf{R}^N, +)$. This means that all the topologies on $\mathcal{G}_\tau$ are equivalent and $(\mathcal{G}_\tau, \circ)$ is closed in $\mathcal{F}(\Theta)$.

2.4 Courant Metric on the Quotient Group $\mathcal{F}(\Theta)/\mathcal{G}$

When $\mathcal{F}(\Theta)$ is a topological right-invariant metric group and $\mathcal{G}$ is a closed subgroup of $\mathcal{F}(\Theta)$, the *quotient metric*

$$d_{\mathcal{G}}(F \circ \mathcal{G}, H \circ \mathcal{G}) \overset{\text{def}}{=} \inf_{G, \tilde{G} \in \mathcal{G}} d(F \circ G, H \circ \tilde{G}) \tag{2.31}$$

induces a complete metric topology on the quotient group $\mathcal{F}(\Theta)/\mathcal{G}$.[12] The *quotient metric* $d_{\mathcal{G}(\Omega_0)}$ with $\Theta = C^k_0(\overline{\mathbf{R}^N}, \mathbf{R}^N)$ was referred to as the *Courant metric* by

[12]Cf., for instance, D. Montgomery and L. Zippin [**1** sect. 1.22, p. 34, sect. 1.23, p. 36])

(i) If $\mathcal{G}$ is a topological group whose open sets at e have a countable basis, then $\mathcal{G}$ is metrizable and, moreover, there exists a metric which is right-invariant.

(ii) If $\mathcal{G}$ is a closed subgroup of a metric group $\mathcal{F}$, then $\mathcal{F}/\mathcal{G}$ is metrizable.

A. M. Micheletti [**1**]. In that case $\mathcal{F}(\Theta)$ is a topological (metric) group. However, we have seen that for $\Theta = C^{0,1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$ and $\mathcal{B}^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$, $\mathcal{F}(\Theta)$ is not a topological group. Yet, the completeness of $(\mathcal{F}(\Theta), d)$ and the right-invariance of d are sufficient to recover the result for an arbitrary closed subgroup $\mathcal{G}$ of $\mathcal{F}$. It is important to have the weakest possible assumptions on $\mathcal{G}$ to secure the widest range of applications.

Theorem 2.8. *Assume that $(\mathcal{F}, d)$ is a group with right-invariant metric d that is complete for the topology generated by d. For any closed subgroup $\mathcal{G}$ of $\mathcal{F}$, the function $d_{\mathcal{G}} : \mathcal{F} \times \mathcal{F} \to \mathbf{R}$,*

$$d_{\mathcal{G}}(F \circ \mathcal{G}, H \circ \mathcal{G}) \stackrel{\text{def}}{=} \inf_{G \in \mathcal{G}} d(F, H \circ G), \tag{2.32}$$

is a right-invariant metric on $\mathcal{F}(\Theta)/\mathcal{G}$ and the space $(\mathcal{F}/\mathcal{G}, d_{\mathcal{G}})$ is complete. The topology induced by $d_{\mathcal{G}}$ coincides with the quotient topology of $\mathcal{F}/\mathcal{G}$.

Proof. (i) $\mathcal{F}/\mathcal{G}$ is clearly a group with unit element $\mathcal{G}$ under the composition law $[F \circ \mathcal{G}] \circ [H \circ \mathcal{G}] \stackrel{\text{def}}{=} [F \circ H] \circ \mathcal{G}$.

(ii) ($d_{\mathcal{G}}$ *is a metric*). Since d is right-invariant, the definition (2.31) is equivalent to

$$d_{\mathcal{G}}(F \circ \mathcal{G}, H \circ \mathcal{G}) \stackrel{\text{def}}{=} \inf_{G \in \mathcal{G}} d(F, H \circ G). \tag{2.33}$$

Since d is symmetrical, so is $d_{\mathcal{G}}$. For the triangle inequality let $g_1, g_2 \in \mathcal{G}$:

$$\begin{aligned} d_{\mathcal{G}}(F \circ \mathcal{G}, H \circ \mathcal{G}) &\le d(F, H \circ g_1 \circ g_2) \le d(F, G \circ g_2) + d(G \circ g_2, H \circ g_1 \circ g_2) \\ &\le d(F, G \circ g_2) + d(G, H \circ g_1) \end{aligned}$$

by right-invariance. The triangle inequality for $d_{\mathcal{G}}$ follows by taking the infima over $g_1, g_2 \in \mathcal{G}$. Since $\mathcal{F}/\mathcal{G}$ is a group, it remains to show that $d_{\mathcal{G}}(I \circ \mathcal{G}, H \circ \mathcal{G}) = 0$ if and only if $H \in \mathcal{G}$. Clearly $d_{\mathcal{G}}(I \circ \mathcal{G}, I \circ \mathcal{G}) \le d(I, I) = 0$. Conversely, let $d_{\mathcal{G}}(I \circ \mathcal{G}, H \circ \mathcal{G}) = 0$. For all $n > 0$, there exists $g_n \in G$ such that $d(g_n^{-1}, H) = d(I, H \circ g_n) < 1/n$. Therefore $g_n^{-1} \to H$ in $\mathcal{F}(\Theta)$. But since $\mathcal{G}$ is closed $\lim_{n \to \infty} g_n^{-1} \in \mathcal{G}$ and $H \in \mathcal{G}$. In particular, the canonical mapping $F \to \pi(F) = F \circ \mathcal{G} : \mathcal{F}(\Theta) \to \mathcal{F}(\Theta)/\mathcal{G}$ is continuous since $d_{\mathcal{G}}(I \circ \mathcal{G}, F \circ \mathcal{G}) \le d(I, F)$ for all $F \in \mathcal{F}(\Theta)$.

(iii) (*Completeness*). Consider a Cauchy sequence $\{F_n \circ \mathcal{G}\}$ in $\mathcal{F}/\mathcal{G}$. It is sufficient to show that there exists a subsequence which converges to some $F \circ \mathcal{G} \in \mathcal{F}(\Theta)/\mathcal{G}$. Construct the subsequence $\{F_\nu \circ \mathcal{G}\}$ such that

$$d_{\mathcal{G}}(F_\nu \circ \mathcal{G}, F_{\nu+1} \circ \mathcal{G}) < 1/(2^\nu).$$

Then there exists a sequence $\{H_\nu\}$ in $\mathcal{F}$ such that (i) $H_\nu \in F_\nu \circ \mathcal{G}$ and (ii) $d(H_\nu, H_{\nu+1}) < 1/(2^\nu)$. We proceed by induction. By definition of $d_{\mathcal{G}}$, if $d_{\mathcal{G}}(F_1 \circ \mathcal{G}, F_2 \circ \mathcal{G}) < 1/2$, there exist $H_1 \in F_1 \circ \mathcal{G}$ and $H_2 \in F_2 \circ \mathcal{G}$ such that $d(H_1, H_2) < 1/2$. Similarly, if $H_\nu \in F_\nu \circ \mathcal{G}$, it follows that there exists $H_{\nu+1}$ with the required property. Because $d_{\mathcal{G}}(F_\nu \circ \mathcal{G}, F_{\nu+1} \circ \mathcal{G}) < 1/2^\nu$, there exist G_1 and G_2 in $\mathcal{G}$ such that $d(F_\nu \circ G_1, F_{\nu+1} \circ G_2) < 1/2^\nu$. If G_3 is such that $F_\nu \circ G_1 = H_\nu \circ G_3$, then we can choose $H_{\nu+1} = F_{\nu+1} \circ G_2 \circ G_3^{-1}$ since

$$d(H_\nu, F_{\nu+1} \circ G_2 \circ G_3^{-1}) = d(H_\nu \circ G_3, F_{\nu+1} \circ G_2) = d(F_\nu \circ G_1, F_{\nu+1} \circ G_2).$$

It is easy to check that $\{H_\nu\}$ is a Cauchy sequence in $\mathcal{F}$. Indeed

$$\forall i < j, \quad d(H_i, H_j) \le \sum_{n=i}^{j-1} d(H_n, H_{n+1}) \le \frac{1}{2^i} + \cdots + \frac{1}{2^j} \le \frac{1}{2^{i-1}}.$$

By completeness of $\mathcal{F}$, $\{H_\nu\}$ converges to some H in $\mathcal{F}$. From (ii) the canonical map $\pi : \mathcal{F} \to \mathcal{F}/\mathcal{G}$ is continuous. So the sequence $\{\pi(H_\nu) = F_\nu \circ \mathcal{G}\}$ converges to $\pi(H) = H \circ \mathcal{G}$ in $\mathcal{F}/\mathcal{G}$.

(iv) Associate with $\mathcal{G}$ the equivalence relation $F \overset{R}{\sim} H$ if $F \circ H^{-1} \in \mathcal{G}$. By definition, the quotient topology on $\mathcal{F}/R$ is the finest topology $\mathcal{T}$ on $\mathcal{F}/R$ for which the canonical map $F \mapsto \pi(F) : \mathcal{F} \to \mathcal{F}/R$ is continuous. But, by definition, $\mathcal{F}/R = \mathcal{F}/\mathcal{G}$ and, from part (ii), π is also continuous for the right-invariant quotient metric. Therefore, the identity map $i : (\mathcal{F}/R, \mathcal{T}) \to (\mathcal{F}/\mathcal{G}, d_\mathcal{G})$ is continuous. The topology $\mathcal{T}$ in $[I]$ is defined by the fundamental system of neighborhoods $\{\pi(F) : d(F, I) < \varepsilon\}$. But $d_\mathcal{G}(\pi(F), \pi(I)) \le d(F, I) < \varepsilon$ and $i(\{\pi(F) : d(F, I) < \varepsilon\}) \subset \{\pi(F) : d_\mathcal{G}([F], [I]) < \varepsilon\}$ that is a neighborhood of $[I]$ in the $(\mathcal{F}/\mathcal{G}, d_\mathcal{G})$ topology. Therefore i^{-1} is continuous and the two topologies coincide. $\square$

We now have all the ingredients to conclude.

Theorem 2.9. *Let Θ be any of the spaces $\mathcal{B}^k(\mathbf{R}^N, \mathbf{R}^N)$, $C^k(\overline{\mathbf{R}^N}, \mathbf{R}^N)$, $C_0^k(\mathbf{R}^N, \mathbf{R}^N)$, $C^{k,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)$, $k \ge 0$, or $C^\infty(\overline{\mathbf{R}^N}, \mathbf{R}^N)$, $C_0^\infty(\mathbf{R}^N, \mathbf{R}^N)$.*

(i) *The group $(\mathcal{F}(\Theta), d)$ is a complete right-invariant metric space. For Θ equal to $C^k(\overline{\mathbf{R}^N}, \mathbf{R}^N)$ or $C_0^k(\mathbf{R}^N, \mathbf{R}^N)$, $0 \le k \le \infty$, $(\mathcal{F}(\Theta), d)$ is also a topological group.*

(ii) *For any closed subgroup $\mathcal{G}$ of $\mathcal{F}(\Theta)$, the function $d_\mathcal{G} : \mathcal{F}(\Theta) \times \mathcal{F}(\Theta) \to \mathbf{R}$,*

$$d_\mathcal{G}(F \circ \mathcal{G}, H \circ \mathcal{G}) \overset{\text{def}}{=} \inf_{G \in \mathcal{G}} d(F, H \circ G), \tag{2.34}$$

is a right-invariant metric on $\mathcal{F}(\Theta)/\mathcal{G}$ and the space $(\mathcal{F}(\Theta)/\mathcal{G}, d_\mathcal{G})$ is complete. The topology induced by $d_\mathcal{G}$ coincides with the quotient topology of $\mathcal{F}(\Theta)/\mathcal{G}$.

(iii) *Let Ω_0, $\varnothing \ne \Omega_0 \subset \mathbf{R}^N$, be closed or let $\Omega_0 = \operatorname{int} \overline{\Omega_0}$. Then*

$$\mathcal{G}(\Omega_0) \overset{\text{def}}{=} \{F \in \mathcal{F}(\Theta) : F(\Omega_0) = \Omega_0\} \tag{2.35}$$

is a closed subgroup of $\mathcal{F}(\Theta)$.

Finally, it is interesting to recall that A. M. Micheletti [**1**] used the quotient metric to prove the following theorem.

Theorem 2.10. *Fix $k = 3$, the Courant metric of $\mathcal{F}(C_0^3(\mathbf{R}^N, \mathbf{R}^N))/\mathcal{G}(\Omega_0)$, and a bounded open connected domain Ω_0 of class C^3 in $\mathbf{R}^N$. Let $\mathcal{X}(\Omega_0)$ be the family of images of Ω_0 by elements of $\mathcal{F}(C_0^3(\mathbf{R}^N, \mathbf{R}^N))/\mathcal{G}(\Omega_0)$.*

The subset of all bounded open domains Ω in $\mathcal{X}(\Omega_0)$ such that the spectrum of the Laplace operator $-\Delta$ on Ω with homogeneous Dirichlet conditions on the boundary $\partial\Omega$ does not have all its eigenvalues simple is of the first category.[13]

This theorem says that, up to an arbitrarily small perturbation of the domain, the eigenvalues of the Laplace operator of a C^3-domain can be made simple.

2.5 Assumptions for $\mathcal{B}^k(\mathrm{R}^N, \mathrm{R}^N)$, $C^k(\overline{\mathrm{R}^N}, \mathrm{R}^N)$, and $C_0^k(\mathrm{R}^N, \mathrm{R}^N)$

2.5.1 Checking the Assumptions

By definition, for all integers $k \geq 0$

$$C_0^k(\mathbf{R}^N, \mathbf{R}^N)) \subset C^k(\overline{\mathbf{R}^N}, \mathbf{R}^N) \subset \mathcal{B}^k(\mathbf{R}^N, \mathbf{R}^N) \subset \mathcal{B}^0(\mathbf{R}^N, \mathbf{R}^N) \subset C^0(\mathbf{R}^N, \mathbf{R}^N)$$
$$\forall x \in \mathbf{R}^N, \quad f \mapsto f(x) : \mathcal{B}^0(\mathbf{R}^N, \mathbf{R}^N) \to \mathbf{R}^N$$

with continuous injections between the spaces and continuity of the evaluation map. The spaces are all Banach spaces endowed with the same norm:[14]

$$\|f\|_{C^k} \stackrel{\text{def}}{=} \max_{0 \leq |\alpha| \leq k} \|\partial^\alpha f\|_C.$$

As a consequence, the evaluation map $f \mapsto f(x)$ is continuous for all the spaces.

The following results are quoted directly from A. M. Micheletti [**1**] and apply to all spaces $\Theta \subset C^k(\mathbf{R}^N, \mathbf{R}^N)$. Denote by

$$S(x)[y_1][y_2], \ldots, [y_k]$$

at the point $x \in \mathbf{R}^N$ a k-linear form with arguments $y_1, \ldots, y_k$.

Lemma 2.4. *Assume that F and G are two mappings from $\mathbf{R}^N$ to $\mathbf{R}^N$ such that F is k-times differentiable in an open neighborhood of x and G is k-times differentiable in an open neighborhood of $F(x)$. Then the kth derivative of $G \circ F$ in x is the sum of a finite number of k-linear applications on $\mathbf{R}^N$ of the form*

$$\begin{aligned}(h_1, h_2, \ldots, h_k) \mapsto G^{(\ell)}(F(x)) &\left[F^{(\lambda_1)}(x)[h_1][h_1] \ldots [h_{\lambda_1}]\right] \\ &\ldots \left[F^{(\lambda_\ell)}(x)[h_{k-\lambda_\ell+1}] \ldots [h_k]\right],\end{aligned} \tag{2.36}$$

where $\ell = 1, \ldots, k$ and $\lambda_1 + \cdots + \lambda_\ell = k$.

[13] A set B is said to be *nowhere dense* if its closure has no interior or, alternatively, if $\complement B$ is dense. A set is said to be of the *first category* if it is the countable union of nowhere dense sets (cf. J. Dugundji [**1**, Def. 10.4, p. 250]).

[14] Here $\mathbf{R}^N$ is endowed with the norm $|x| = \left\{\sum_{i=1}^N x_i^2\right\}^{1/2}$.

Proof. We proceed by induction on k. The result is trivially true for $k = 1$. Assuming that it is true for $k-1$, we prove that it is also true for k. This is an obvious consequence of the observation that for a mapping from $\mathbf{R}^{\mathrm{N}}$ into $\mathcal{L}((\mathbf{R}^{\mathrm{N}})^{k-1}, \mathbf{R}^{\mathrm{N}})$ of the form

$$x \mapsto G^{(\ell)}(F(x))[F^{(\lambda_1)}(x)] \dots [F^{(\lambda_\ell)}(x)],$$

where $\ell = 1, \dots, k-1$ and $\lambda_1 + \dots + \lambda_\ell = k-1$, the differential in x is

$$\begin{aligned} G^{(\ell+1)}(F(x))[F^{(1)}][F^{(\lambda_1)}(x)] \dots [F^{(\lambda_\ell)}(x)] \\ + G^{(\ell)}(F(x))[F^{(\lambda_1+1)}(x)] \dots [F^{(\lambda_\ell)}(x)] \\ + \dots + G^{(\ell)}(F(x))[F^{(\lambda_1)}(x)] \dots [F^{(\lambda_\ell+1)}(x)], \end{aligned} \tag{2.37}$$

and the result is true for k. $\square$

Lemma 2.5. *Given g and f in $C^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$, let $\psi = g \circ (I + f)$. Then for each $x \in \mathbf{R}^{\mathrm{N}}$*

$$\begin{aligned} |\psi(x)| &= |g(x + f(x))|, \\ |\psi^{(1)}(x)| &\le |g^{(1)}(x + f(x))|\, [1 + |f^{(1)}(x)|], \\ |\psi^{(i)}(x)| &\le |g^{(1)}(x + f(x))|\, |f^{(i)}(x)| \\ &\quad + \sum_{j=2}^{i} |g^{(j)}(x + f(x))|\, a_j(|f^{(1)}(x)|, \dots, |f^{(i-1)}(x)|) \end{aligned}$$

for $i = 2, \dots, k$, where a_j is a polynomial with positive coefficients.

Proof. This is an obvious consequence of Lemma 2.4. $\square$

Theorem 2.11. *Assumptions* 2.1 *and* 2.3 *are verified for the spaces $C_0^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$, $C^k(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$, and $\mathcal{B}^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$, $k \ge 0$.*

Proof. (i) *Assumption* 2.3. From Lemma 2.5 it readily follows that $f \circ (I + g) \in \mathcal{B}^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ for all $f \in \mathcal{B}^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ and $I + g$ in $\mathcal{F}(\mathcal{B}^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}))$. Moreover,

$$\begin{aligned} \|\psi\|_C &= \|f\|_C, \\ \|\psi^{(1)}\|_C &\le \|f^{(1)}\|_C\, (1 + \|g^{(1)}\|_C), \\ \|\psi^{(i)}\|_C &\le \|f^{(1)}\|_C\, \|g^{(i)}\|_C + \sum_{j=2}^{i} \|f^{(j)}\|_C\, a_j(\|g^{(1)}\|_C, \dots, \|g^{(i-1)}\|_C) \end{aligned}$$

for $i = 2, \dots, k$, where a_j is a polynomial. Hence

$$\|\psi\|_{C^k} = \max_{i=0,\dots,k} \|\psi^i\|_C \le \|f\|_{C^k}\, c_1(\|g\|_{C^k})$$

for some polynomial function c_1 that depends only on k.

(ii) *Assumption* 2.1. The case of $\mathcal{B}^k(\mathbf{R}^N, \mathbf{R}^N)$ is a consequence of (i). For $C^k(\overline{\mathbf{R}^N}, \mathbf{R}^N)$ and each i, $\psi^{(i)}$ is uniformly continuous as the finite sum and product of compositions of uniformly continuous functions by Lemma 2.4. As for $C_0^k(\mathbf{R}^N, \mathbf{R}^N)$, for all $I + g \in \mathcal{F}(C_0^k(\mathbf{R}^N, \mathbf{R}^N))$ and $f \in C_0^k(\mathbf{R}^N, \mathbf{R}^N)$, $h = f \circ (I + g) \in C^k(\overline{\mathbf{R}^N}, \mathbf{R}^N)$ from the previous case. It remains to show that $h \in C_0^k(\mathbf{R}^N, \mathbf{R}^N)$. This amounts to showing that $|h(x)|$ and $|h^{(i)}(x)|$ go to zero as $|x| \to \infty$, $i = 1, \ldots, k$. We again proceed by induction on k. Since $|g(x)| \to 0$ as $|x| \to \infty$, $|x + g(x)| \geq |x| - |g(x)| \to \infty$ as $|x| \to \infty$ and $|h(x)| = |f(x + g(x))| \to 0$ as $|x| \to \infty$ and Assumption 2.1 is true for $k = 0$. Always, from Lemma 2.5 and the identity $h(x) = f(x + g(x))$,

$$|h^{(1)}(x)| \leq |f^{(1)}(x)| \, [1 + |g^{(1)}(x)|] \to 0$$

as $|x| \to 0$ and Assumption 2.1 is true for $k = 1$. For $k \geq 2$ and $2 \leq i \leq k$, again, from Lemma 2.5

$$\begin{aligned}|h^{(i)}(x)| &\leq |f^{(1)}(x + g(x))| \, \|g^{(i)}\|_C \\ &\quad + \sum_{j=2}^{i} |f^{(j)}(x + g(x))| \, a_j(\|g^{(1)}\|_C, \ldots, \|g^{(i-1)}\|_C)\end{aligned}$$

for $i = 2, \ldots, k$. Since for each j, $|f^{(j)}(x)| \to 0$ as $|x| \to \infty$, by the same argument, $|f^{(j)}(x + g(x))| \to 0$ as $|x| \to \infty$ and hence the whole right-hand side goes to 0. $\square$

Theorem 2.12. *Assumption* 2.2 *is verified for the spaces* $C_0^k(\mathbf{R}^N, \mathbf{R}^N)$, $C^k(\overline{\mathbf{R}^N}, \mathbf{R}^N)$, *and* $\mathcal{B}^k(\mathbf{R}^N, \mathbf{R}^N)$, $k \geq 0$.

It is a consequence of the following lemma of A. M. Micheletti [**1**].

Lemma 2.6. *Given integers $r \geq 0$ and $s > 0$ there exists a constant $c(r, s) > 0$ with the following property: if the sequence $f_1, \ldots, f_n$ in $C^r(\mathbf{R}^N, \mathbf{R}^N)$ is such that*

$$\sum_{i=1}^{n} \|f_i\|_{C^r} < \alpha, \quad 0 < \alpha < s, \tag{2.38}$$

then for the map $F = (I + f_n) \circ \cdots \circ (I + f_1)$,

$$\|F - I\|_{C^r} \leq \alpha \, c(r, s). \tag{2.39}$$

Proof. We again proceed by induction on r. For simplicity define

$$F_i \stackrel{\text{def}}{=} (I + f_i) \circ \cdots \circ (I + f_1), \quad F_n = F.$$

By the definition of F we have

$$\begin{aligned}F - I &= f_1 + f_2 \circ (I + f_1) + \cdots + f_n \circ (I + f_{n-1}) \circ \cdots \circ (I + f_1) \\ &= f_1 + f_2 \circ F_1 + \cdots + f_n \circ F_{n-1}.\end{aligned} \tag{2.40}$$

Then $\|F-I\|_{C^0} \le \sum_{i=1}^{n} \|f_i\|_{C^0} \le \alpha$ and $c(0,s)=1$ for all integers $s>0$. From (2.40) and Lemma 2.5 we further get

$$\begin{aligned}\sup_x |(F-I)^{(1)}(x)| &\le \|f_1\|_{C^1} + \|f_2\|_{C^1}\,[1+\|f_1\|_{C^1}] \\ &\qquad + \dots + \|f_n\|_{C^1}\,[1+\|f_{n-1}\|_{C^1}]\dots[1+\|f_1\|_{C^1}] \\ &\le \sum_{i=1}^{n} \|f_i\|_{C^1} \prod_{i=1}^{n}(1+\|f_i\|_{C^1}) \le \left(\sum_{i=1}^{n}\|f_i\|_{C^1}\right) e^{\left(\sum_{i=1}^{n}\|f_i\|_{C^1}\right)} \le \alpha\, e^{\alpha} \le \alpha\, e^{s}\end{aligned} \tag{2.41}$$

and $\|F-I\|_{C^1} \le \alpha\, e^{\alpha}$. Hence $c(1,s)=e^s$ for all integers $s>0$. We now show that if the result is true for $r-1$, it is true for r. We have to evaluate $|(F_n-I)^{(r)}(x)|$. It is obvious that $F_n - I = (I+f_n)\circ F_{n-1} - I = F_{n-1}-I+f_n\circ F_{n-1}$ and hence

$$(F_n-I)^{(r)}(x) = (F_{n-1}-I)^{(r)}(x) + (f_n\circ F_{n-1})^{(r)}(x). \tag{2.42}$$

For $r\ge 2$ from Lemma 2.5 we have

$$\begin{aligned}|(f_n\circ F_{n-1})^{(r)}(x)| &\le |f_n^{(1)}(F_{n-1}(x))|\,|(F_{n-1}-I)^{(r)}(x)| \\ &\quad + \sum_{j=2}^{r} |f_n^{(j)}(F_{n-1}(x))|\, a_j(|(F_{n-1}-I)^{(1)}(x)|,\dots,|(F_{n-1}-I)^{(r-1)}(x)|).\end{aligned}$$

By the induction assumption $|(F_{n-1}-I)^{(i)}(x)| \le \|(F_{n-1}-I)\|_{C^{r-1}} \le \alpha\, c(r-1,s)$, $i=1,\dots,r-1$, and the fact that a_j is a polynomial dependent on r for $j=2,\dots,r$, there exists a constant $L(r,s)$ such that

$$\begin{aligned}&|(f_n\circ F_{n-1})^{(r)}(x)| \\ &\le |f_n^{(1)}(F_{n-1}(x))|\,|(F_{n-1}-I)^{(r)}(x)| + L(r,s)\sum_{j=2}^{r}|f_n^{(j)}(F_{n-1}(x))| \\ &\le \|f_n\|_{C^r}|(F_{n-1}-I)^{(r)}(x)| + (r-1)L(r,s)\|f_n\|_{C^r}.\end{aligned} \tag{2.43}$$

Define $M(r,s)=(r-1)L(r,s)$. From (2.42) and (2.43), we get

$$|(F_n-I)^{(r)}(x)| \le [1+\|f_n\|_{C^r}]\,|(F_{n-1}-I)^{(r)}(x)| + M(r,s)\|f_n\|_{C^r}.$$

After repeating this procedure $n-1$ times we have

$$\begin{aligned}|(F_n-I)^{(r)}(x)| &\le [1+\|f_n\|_{C^r}]\dots[1+\|f_2\|_{C^r}]\|f_1\|_{C^r} \\ &\quad + [1+\|f_n\|_{C^r}]\dots[1+\|f_2\|_{C^r}]M(r,s)\|f_2\|_{C^r} + \dots + M(r,s)\|f_n\|_{C^r} \\ &\hspace{8cm} \le \max\{M(r,s),1\}\, e^{\alpha}\alpha.\end{aligned}$$

Henceforth $c(r,s)=\max\{M(r,s),1\}e^s$. □

Theorem 2.13. *Assumption* 2.4 *is verified for* $C^k(\overline{\mathbf{R}^{\mathrm{N}}},\mathbf{R}^{\mathrm{N}})$ *and* $C_0^k(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})$.

It is a consequence of the following slightly modified versions of lemmas of A. M. Micheletti [**1**].

Lemma 2.7. *Given $f \in C^k(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$ and $\gamma \in C^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\forall \gamma,\ \|\gamma\|_{C^0} < \delta, \quad \sup_{0 \le i \le k} \|f^{(i)} \circ (I + \gamma) - f^{(i)}\|_{C^0} < \varepsilon. \tag{2.44}$$

Proof. Since f and its derivatives $f^{(i)}$ all belong to $C^0(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$, they are uniformly continuous functions. Hence, for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall x, y \in \mathbf{R}^{\mathrm{N}},\ |x - y| < \delta, \quad \sup_{0 \le i \le k} |f^{(i)}(y) - f^{(i)}(x)| < \varepsilon.$$

In particular for $\gamma \in C^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ such that $\|\gamma\|_{C^0} < \delta$

$$|x + \gamma(x) - x| = |\gamma(x)| \le \|\gamma\|_{C^0} < \delta \quad \Rightarrow \sup_{0 \le i \le k} |f^{(i)}(x + \gamma(x)) - f^{(i)}(x)| < \varepsilon$$

and we get the result. □

Lemma 2.8. *For each $g \in C^k(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\forall \gamma \in C^k(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}}) \text{ such that } \|\gamma\|_{C^k} < \delta, \quad \|g \circ (I + \gamma) - g\|_{C^k} < \varepsilon. \tag{2.45}$$

Proof. Again proceed by induction on k. For $k = 0$ the result is a consequence of Lemma 2.7. Then check that if it is true for $k - 1$ it is true for k. From Lemma 2.4 the kth derivative of $g \circ (I + \gamma) - g$ is the sum of a finite number of k-linear mappings of the form

$$\begin{aligned} & g^{(r)}(x + \gamma(x))[(I + \gamma)^{(\lambda_1)}(x)] \dots [(I + \gamma)^{(\lambda_r)}(x)] - g^{(r)}(x)[I^{(\lambda_1)}(x)] \dots [I^{(\lambda_r)}(x)] \\ = & \left\{ g^{(r)}(x + \gamma(x)) - g^{(r)}(x) \right\} [(I + \gamma)^{(\lambda_1)}(x)] \dots [(I + \gamma)^{(\lambda_r)}(x)] \\ & + g^{(r)}(x)[(I + \gamma)^{(\lambda_1)}(x)] \dots [(I + \gamma)^{(\lambda_r)}(x)] - g^{(r)}(x)[I^{(\lambda_1)}(x)] \dots [I^{(\lambda_r)}(x)], \end{aligned}$$

where $r = 1, \dots, k$ and $\lambda_1 + \dots + \lambda_r = k$. Since

$$|(I + \gamma)^{(\lambda_1)}(x)|\, |(I + \gamma)^{(\lambda_2)}(x)| \dots |(I + \gamma)^{(\lambda_r)}(x)| \le (\|\gamma\|_{C^k} + 1)^r,$$

the norm of the mapping can be bounded above by the following expression:

$$\sup_x |g^{(r)}(x + \gamma(x)) - g^{(r)}(x)|\, (\|\gamma\|_{C^k} + 1)^r + \|\gamma\|_{C^k}\, p(\|\gamma\|_{C^k}, \|g\|_{C^k}), \tag{2.46}$$

where p is a polynomial. From this upper bound and Lemma 2.7 the result of the lemma is true for k. □

2.5.2 Perturbations of the Identity and Tangent Space

By construction

$$\mathcal{F}(\Theta) \subset I + \Theta$$

and the tangent space in each point of the affine space $I + \Theta$ is Θ. In general, $I + \Theta$ is not a subset of $\mathcal{F}(\Theta)$, but, for some spaces of transformations Θ, there is a sufficiently small ball B around 0 in Θ such that the so-called *perturbations of the identity*, $I + B$, are contained in $\mathcal{F}(\Theta)$. In particular, for $\theta \in B$, the family of transformations $T_t = I + t\theta \in \mathcal{F}(\Theta)$, $0 \le t \le 1$, is a C^1-path in $\mathcal{F}(\Theta)$ and the tangent space to each point of $\mathcal{F}(\Theta)$ can be shown to be exactly Θ.

Theorem 2.14. *Let Θ be equal to $C_0^k(\mathbf{R}^N, \mathbf{R}^N)$, $C^k(\overline{\mathbf{R}^N}, \mathbf{R}^N)$, or $\mathcal{B}^k(\mathbf{R}^N, \mathbf{R}^N)$, $k \ge 1$.*

(i) *The map $f \mapsto I + f : B(0,1) \subset \Theta \to \mathcal{F}(\Theta)$ is continuous.*

(ii) *For all $F \in \mathcal{F}(\Theta)$, the tangent space $T_F\mathcal{F}(\Theta)$ to $\mathcal{F}(\Theta)$ is Θ.*

Remark 2.5.
The generic framework of Micheletti is similar to an infinite-dimensional Riemannian manifold. □

Proof. (i) For $k \ge 1$ and f sufficiently small, F is bijective and has a unique inverse. Given $y \in \mathbf{R}^N$, consider the mapping

$$S(x) \stackrel{\text{def}}{=} y - f(x), \quad x \in \mathbf{R}^N .$$

Then for any x_1 and x_2,

$$\begin{aligned} S(x_2) - S(x_1) &= -[f(x_2) - f(x_1)] \\ \Rightarrow |S(x_2) - S(x_1)| &\le |f(x_2) - f(x_1)| \le \|f^{(1)}\|_{C^0}\, |x_2 - x_1|. \end{aligned}$$

For $\|f^{(1)}\|_{C^1} < 1$, S is a contraction and, for each y, there exists a unique x such that $y - f(x) = S(x) = x$, $[I + f](x) = y$. Therefore $I + f$ is bijective.

But in order to show that $F \in \mathcal{F}(\mathcal{B}^k(\mathbf{R}^N, \mathbf{R}^N))$ we also need to prove that $[I + f]^{-1} \in \mathcal{B}^k(\mathbf{R}^N, \mathbf{R}^N))$. The Jacobian matrix of $F(x)$ is equal to $I + f^{(1)}(x)$ and

$$\forall x \in \mathbf{R}^N, \quad |F^{(1)}(x) - I| \le \|F^{(1)} - I\|_{C^0} = \|f^{(1)}\|_{C^0}.$$

If there exists $x \in \mathbf{R}^N$ such that $F^{(1)}(x)$ is not invertible, then for some $0 \ne y \in \mathbf{R}^N$, $F^{(1)}(x)y = 0$ and

$$|y| = |F^{(1)}(x)y - y| \le |F^{(1)}(x) - I||y| \le \|f^{(1)}\|_{C^0}\, |y| \quad \Rightarrow 1 \le \|f^{(1)}\|_{C^0} < 1,$$

which is a contradiction. Therefore, for all $x \in \mathbf{R}^N$, $F^{(1)}(x)$ is invertible. As a result the conditions of the implicit function theorem are met and from L. Schwartz [**3**, Vol. 1, Thm. 29, p. 294, Thm. 31, p. 299] we have that $F^{-1} = [I + f]^{-1} \in C^k(\mathbf{R}^N, \mathbf{R}^N)$.

It remains to prove that $F^{-1} = [I+f]^{-1} \in \mathcal{B}^k(\mathbf{R}^N, \mathbf{R}^N)$ and the continuity. We again proceed by induction on k. Set $y = F(x)$. By definition $g(y) = F^{-1}(y) - y = x - F(x) = -f(x)$ and $\|g\|_{C^0} = \|f\|_{C^0}$. Always, from Lemma 2.5 and the identity $g(y) = -f(x) = -f(F^{-1}(y)) = -f(y + g(y))$,

$$|g^{(1)}(y)| \le |f^{(1)}(x)|\, [1 + |g^{(1)}(y)|] \quad \Rightarrow |g^{(1)}(y)|\, [1 - |f^{(1)}(x)|] \le |f^{(1)}(x)|.$$

Since $|f^{(1)}(x)| \le \|f\|_{C_0} < 1$

$$0 \le |g^{(1)}(y)| \le \frac{|f^{(1)}(x)|}{1-\|f\|_{C_0}} \quad \Rightarrow \|g^{(1)}\|_{C^0} \le \frac{|f^{(1)}(x)|}{1-\|f\|_{C_0}} \le \frac{\|f^{(1)}\|_{C_0}}{1-\|f\|_{C_0}}. \tag{2.47}$$

For $k \ge 2$, again from Lemma 2.5, we have

$$\begin{aligned} 0 \le |g^{(k)}(y)| &\le \frac{\sum_{j=2}^{k} |f^{(j)}(x)|\, a_j(|g^{(1)}(y)|, \dots, |g^{(k-1)}(y)|)}{1-|f^{(1)}(x)|} \\ &\le \frac{\sum_{j=2}^{k} |f^{(j)}(x)|\, a_j(|g^{(1)}(y)|, \dots, |g^{(k-1)}(y)|)}{1-\|f^{(1)}\|_{C^0}}. \end{aligned} \tag{2.48}$$

So there exists a constant c_k such that

$$\|F^{-1} - I\|_{\mathcal{C}^k} \le c_k \|F - I\|_{\mathcal{C}^k} = c_k \|f\|_{\mathcal{C}^k}.$$

Therefore, for $k \ge 1$,

$$\forall f \in \mathcal{B}^k(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N}) \text{ such that } \|f^{(1)}\|_{C^0} < 1, \quad F = I + f \in \mathcal{F}(\mathcal{B}^k(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N})).$$

Note that the condition is on $\|f^{(1)}\|_{C^0}$ and not $\|f\|_{C^k}$, but by definition $\|f^{(1)}\|_{C^0} \le \|f\|_{C^1} \le \|f\|_{C^k}$ and the condition $\|f\|_{C^k} < 1$ yields the same result. In particular, the map $f \mapsto I + f \,:\, B(0,1) \subset \mathcal{B}^k(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N}) \to \mathcal{F}^(\mathcal{B}^k(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N}))$ is continuous. All this can now be specialized to the spaces $C_0^k(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N})$ and $C^k(\overline{\mathbf{R}^\mathrm{N}}, \mathbf{R}^\mathrm{N})$ by using the identity $(I+f)^{-1} - I = -f \circ (I+f)^{-1}$.

(ii) Since $\mathcal{F}(\Theta)$ is contained in the affine space $I + \Theta$, any tangent element to $\mathcal{F}(\Theta)$ will be contained in Θ. From part (i) for $k \ge 1$ and for all $\theta \in \Theta$ there exists $\tau > 0$ such that for all t, $0 \le t \le \tau$, $I + t\Theta \in \mathcal{F}(\Theta)$. For any $F \in \mathcal{F}(\Theta)$, $t \mapsto (I + t\theta) \circ F : [0, \tau] \to \mathcal{F}(\Theta)$ is a continuous path in $\mathcal{F}(\Theta)$ and the limit of

$$\frac{[(I+t\theta)\circ F] \circ F^{-1} - I}{t} = \frac{(I+t\theta) - I}{t} = \theta$$

will be an element of the tangent space $T_F\mathcal{F}(\Theta)$ in F to $\mathcal{F}(\Theta)$. Therefore for all $F \in \mathcal{F}(\Theta)$, $T_F\mathcal{F}(\Theta) = \Theta$. □

2.6 Assumptions for $C^{k,1}(\overline{\mathbf{R}^\mathbf{N}}, \mathbf{R}^\mathbf{N})$ and $C_0^{k,1}(\mathbf{R}^\mathbf{N}, \mathbf{R}^\mathbf{N})$

2.6.1 Checking the Assumptions

By definition, for all integers $k \ge 0$

$$C^{k,1}(\overline{\mathbf{R}^\mathrm{N}}, \mathbf{R}^\mathrm{N}) \overset{\text{def}}{=} \left\{ f \in C^k(\overline{\mathbf{R}^\mathrm{N}}, \mathbf{R}^\mathrm{N}) \,:\, \begin{array}{l} \forall \alpha,\, 0 \le |\alpha| \le k,\, \exists c > 0,\, \forall x, \, y \in \Omega \\ |\partial^\alpha f(y) - \partial^\alpha f(x)| \le c\,|x-y| \end{array} \right\}$$

endowed with the same norm:[15]

$$\|f\|_{C^{k,1}} \stackrel{\text{def}}{=} \max\left\{\max_{0\le|\alpha|\le k} \|\partial^\alpha f\|_C,\, c_k(f)\right\} = \max\left\{\|f\|_{C^k},\, c_k(f)\right\}, \tag{2.49}$$

$$c(f) \stackrel{\text{def}}{=} \sup_{y\neq x} \frac{|f(y)-f(x)|}{|y-x|} \text{ and } \forall k\ge 1, \quad c_k(f) \stackrel{\text{def}}{=} \sum_{|\alpha|=k} c(\partial^\alpha f), \tag{2.50}$$

and the convention $c_0(f) = c(f)$. $C_0^{k,1}(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})$ is the space of functions $f \in C^{k,1}(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})$ such that f and all its derivatives go to zero at infinity. They are all Banach spaces. By definition

$$C_0^{k,1}(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}}) \subset C^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}},\mathbf{R}^{\mathrm{N}}) \subset \mathcal{B}^0(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}}) \subset C^0(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}}),$$
$$\forall x\in\mathbf{R}^{\mathrm{N}}, \quad f\mapsto f(x) : \mathcal{B}^0(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}}) \to \mathbf{R}^{\mathrm{N}}$$

with continuous injections between the spaces and continuity of the evaluation map. As a consequence, the evaluation map $f \mapsto f(x)$ is continuous for all the spaces. We deal only with the case $C^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}},\mathbf{R}^{\mathrm{N}})$. The case $C_0^{k,1}(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})$ is obtained from the case $C^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}},\mathbf{R}^{\mathrm{N}})$ as in the previous section.

Theorem 2.15. *Assumptions* 2.1 *and* 2.3 *are verified for the spaces* $C_0^{k,1}(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})$ *and* $C^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}},\mathbf{R}^{\mathrm{N}})$*,* $k\ge 0$*.*

Proof. The composition and multiplication of bounded and Lipschitz functions is bounded and Lipschitz. For $f \in C^{0,1}(\overline{\mathbf{R}^{\mathrm{N}}},\mathbf{R}^{\mathrm{N}})$ and $I+g \in \mathcal{F}(C^{0,1}(\overline{\mathbf{R}^{\mathrm{N}}},\mathbf{R}^{\mathrm{N}}))$, $\|f\circ(I+g)\|_{C^0} = \|f\|_{C^0} < \infty$. Since g is Lipschitz with Lipschitz constant $c(g)$, $I+g$ is also Lipschitz with Lipschitz constant $1+c(g)$ and the composition is also Lipschitz with constant $c(f\circ(I+g)) \le c(f)\,(1+c(g))$. Hence

$$\|f\circ(I+g)\|_{C^{0,1}} \le \|f\|_{C^{0,1}}\,(1+c(g)) \le \|f\|_{C^{0,1}}\,(1+\|g\|_{C^{0,1}})$$

and the continuous function $c_1(r) = 1+r$ satisfies Assumption 2.3. For $k=1$, $D(f\circ(I+g)) = Df\circ(I+g)\,(I+Dg)$ and

$$\begin{aligned}
\|D(f\circ(I+g))\|_C &\le \|Df\|_C\,(1+\|Dg\|_C),\\
c(D(f\circ(I+g))) &\le c(Df)\,(1+c(g))\,(1+\|Dg\|_C) + \|Df\|_C\,(1+c(Dg))\\
\Rightarrow\ \|D(f\circ(I+g))\|_{C^{0,1}} &\le \|Df\|_{C^{0,1}}\,(1+\|Dg\|_{C^{0,1}})\,(1+c(g))\\
&\le \|Df\|_{C^{0,1}}\,(1+\|Dg\|_{C^{0,1}})\,(1+\|g\|_{C^{0,1}})\\
\Rightarrow\ \|f\circ(I+g)\|_{C^{1,1}} &\le \|f\|_{C^{1,1}}\,(1+\|Dg\|_{C^{0,1}})\,(1+\|g\|_{C^{0,1}})\\
&\le \|f\|_{C^{1,1}}\,(1+\|g\|_{C^{1,1}})^2.
\end{aligned}$$

The general case is obtained by induction over k. □

Theorem 2.16. *Assumption* 2.2 *is verified for* $C_0^{k,1}(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})$ *and* $C^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}},\mathbf{R}^{\mathrm{N}})$*,* $k\ge 0$*.*

[15] Here $\mathbf{R}^{\mathrm{N}}$ is endowed with the norm $|x| = \left\{\sum_{i=1}^N x_i^2\right\}^{1/2}$.

Proof. Consider a sequence $\{f_i\}_{i=1}^n \subset C^{0,1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$ such that

$$\sum_{i=1}^n \max\{\|f_i\|_{C^0}, c(f_i)\} < \alpha < r. \tag{2.51}$$

From Theorem 2.12

$$\|(I+f_1)\circ\cdots\circ(I+f_n) - I\|_{C^0} \le \sum_{i=1}^n \|f_i\|_{C^0} < \alpha.$$

In addition,

$$\begin{aligned}
(I+f_1)\circ\cdots\circ(I+f_n) - I &= f_n + \sum_{i=1}^{n-1} f_i \circ (I+f_{i+1})\circ\cdots\circ(I+f_n) \\
\Rightarrow\; c((I+f_1)\circ\cdots\circ(I+f_n) - I) &\le c(f_n) + \sum_{i=1}^{n-1} c(f_i)\,(1+c(f_{i+1}))\dots(1+c(f_1)) \\
&\le c(f_n) + \sum_{i=1}^{n-1} c(f_i)\, e^{\sum_{k=1}^{i+1} c(f_k)} \\
&\le \left[\sum_{i=1}^n c(f_i)\right] e^{\left[\sum_{i=1}^n c(f_i)\right]} < \alpha\, e^\alpha \\
\Rightarrow\; \|(I+f_1)\circ\cdots\circ(I+f_n) - I\|_{C^{0,1}} &< \alpha \max\{e^\alpha, 1\} < \alpha\, e^r
\end{aligned}$$

and we can choose $c_0(r) = e^r$ for $k = 0$. We use the same technique for $k \ge 1$ but with heavier computations. □

2.6.2 Perturbations of the Identity and Tangent Space

By construction

$$\mathcal{F}(\Theta) \subset I + \Theta$$

and the tangent space in each point of the affine space $I + \Theta$ is Θ. In general, $I + \Theta$ is not a subset of $\mathcal{F}(\Theta)$, but, for some spaces of transformations Θ, there is a sufficiently small ball B around 0 in Θ such that the so-called *perturbations of the identity*, $I + B$, are contained in $\mathcal{F}(\Theta)$. In particular, for $\theta \in B$, the family of transformations $T_t = I + t\theta \in \mathcal{F}(\Theta)$, $0 \le t \le 1$, is a C^1-path in $\mathcal{F}(\Theta)$ and the tangent space to each point of $\mathcal{F}(\Theta)$ can be shown to be exactly Θ.

Theorem 2.17. *Let Θ be equal to $C_0^{k,1}(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ or $C^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$, $k \ge 0$.*

(i) *The map $f \mapsto I + f : B(0,1) \subset \Theta \to \mathcal{F}(\Theta)$ is continuous.*

(ii) *For all $F \in \mathcal{F}(\Theta)$, the tangent space $T_F\mathcal{F}(\Theta)$ to $\mathcal{F}(\Theta)$ is Θ.*

Remark 2.6.
The generic framework of Micheletti is similar to an infinite-dimensional Riemannian manifold. □

Proof. (i) Consider perturbations of the identity of the form $F = I + f$, $f \in C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)$. Consider for any $y \in \mathbf{R}^N$ the map

$$x \mapsto S(x) \overset{\text{def}}{=} y - f(x) : \mathbf{R}^N \to \mathbf{R}^N .$$

For all x_1 and x_2

$$\begin{gathered} S(x_2) - S(x_1) = -[f(x_2) - f(x_1)], \\ |S(x_2) - S(x_1)| = |f(x_2) - f(x_1)| \le c(f)\, |x_2 - x_1|, \end{gathered}$$

and f is contracting for $c(f) < 1$. Hence for each $y \in \mathbf{R}^N$ there exists a unique $x \in \mathbf{R}^N$ such that

$$S(x) = x \quad \Rightarrow y - f(x) = x \quad \Rightarrow F(x) = y$$

and F is bijective. It remains to prove that $g \overset{\text{def}}{=} F^{-1} - I \in C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)$ to conclude that $I + f \in \mathcal{F}(C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N))$.

For any $y \in \mathbf{R}^N$ and $x = F^{-1}(y)$

$$\begin{aligned} & F^{-1}(y) - y = x - F(x), \\ \|g\|_{C^0} = \|F^{-1} - I\|_{C^0} = \sup_{y \in \mathbf{R}^N} |F^{-1}(y) - y| &= \sup_{y \in \mathbf{R}^N} |(I - F)(F^{-1}(y))| \\ &= \sup_{x \in \mathbf{R}^N} |(I - F)(x)| = \|F - I\|_{C^0} < \infty \end{aligned}$$

since $F : \mathbf{R}^N \to \mathbf{R}^N$ is bijective and $\|g\|_{C^0} = \|f\|_{C^0}$.

For all y_1 and y_2 and $x_i = F^{-1}(y_i)$,

$$\begin{aligned} |g(y_2) - g(y_1)| &= |F^{-1}(y_2) - y_2 - (F^{-1}(y_1) - y_1)| \\ &\le |x_2 - F(x_2) - (x_1 - F(x_1)| = |f(x_2) - f(x_1)| \\ &\le c(f)|x_2 - x_1| \\ &\le c(f)|F^{-1}(y_2) - F^{-1}(y_1)| = c(f)|g(y_2) - g(y_1) + (y_2 - y_1)| \\ \Rightarrow\ |g(y_2) - g(y_1)| &\le \frac{c(f)}{1 - c(f)}\, |y_2 - y_1| \quad \Rightarrow c(g) \le \frac{c(f)}{1 - c(f)} < \infty. \end{aligned}$$

Therefore

$$\forall f \in C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N) \text{ such that } c(f) < 1, \quad F = I + f \in \mathcal{F}(C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)).$$

Moreover since $c(f) \le \|f\|_{C^{0,1}}$, the condition $\|f\|_{C^{0,1}} < 1$ yields $c(f) < 1$ and $I + f \in \mathcal{F}(C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N))$. Furthermore, the map $f \mapsto I + f : B(0,1) \subset C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N) \to \mathcal{F}(C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N))$ is continuous. We use the same technique for $k \ge 1$ plus the arguments of Theorem 2.14 (i).

(ii) Same proof as Theorem 2.14 (ii). $\square$

3 Generalization to All Homeomorphisms and C^k-Diffeomorphisms

With a variation of the generic constructions associated with the Banach space Θ of section 2, it is possible to construct a metric on the whole space of homeomorphisms $\text{Hom}(\mathbf{R}^N, \mathbf{R}^N)$ or on the whole space $\text{Diff}_k(\mathbf{R}^N, \mathbf{R}^N)$ of C^k-diffeomorphisms of $\mathbf{R}^N$, $k \geq 1$, or of an open subset D of $\mathbf{R}^N$.

First, recall that, for an open subset D of $\mathbf{R}^N$, the topology of the vector space $C^k(D, \mathbf{R}^N)$ can be specified by the family of seminorms

$$\forall K \text{ compact} \subset D, \quad n_K(F) \stackrel{\text{def}}{=} \|F\|_{C^k(K)}. \tag{3.1}$$

This topology is equivalent to the one generated by the monotone increasing subfamily $\{K_i\}_{i \geq 1}$ of compact sets

$$K_i \stackrel{\text{def}}{=} \left\{ x \in D : d_{\complement D}(x) \geq \frac{1}{i} \text{ and } |x| \leq i \right\}, \quad i \geq 1, \tag{3.2}$$

where $d_{\complement D}(x) = \inf_{y \in \complement D} |y - x|$. This leads to the construction of the metric

$$\delta_k(F, G) \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{n_{K_i}(F - G)}{1 + n_{K_i}(F - G)} \tag{3.3}$$

that generates the same topology as the initial family of seminorms and makes $C^k(D, \mathbf{R}^N)$ a Fréchet space with metric δ_k.

Given an open subset D of $\mathbf{R}^N$, consider the group

$$\begin{aligned} &\text{Hom}_k(D) \\ &\stackrel{\text{def}}{=} \left\{ F \in C^k(D, \mathbf{R}^N) : F : D \to D \text{ is bijective and } F^{-1} \in C^k(D, \mathbf{R}^N) \right\} \end{aligned} \tag{3.4}$$

of transformations of D. For $k = 0$, $\text{Hom}_0(D) = \text{Hom}(D)$ and for $k \geq 1$, $\text{Hom}_k(D) = \text{Diff}_k(D)$, where the condition $F^{-1} \in C^k(D, \mathbf{R}^N)$ is redundant. One possible choice of a topology on $\text{Hom}_k(D)$ is the topology induced by $C^k(D, \mathbf{R}^N)$. This is the so-called *weak topology* on $\text{Hom}_k(D)$ (cf., for instance, M. Hirsch [**1**, Chap. 2]). However, $\text{Hom}_k(D)$ is not necessarily complete for that topology.

In order to get completeness, we adapt the constructions of A. M. Micheletti [**1**]. Associate with $F \in \text{Hom}_k(D)$ and each compact subset $K \subset D$ the following function of I and F:

$$\boxed{q_K(I, F) \stackrel{\text{def}}{=} \inf_{\substack{F = F_1 \circ \cdots \circ F_n \\ F_\ell \in \text{Hom}_k(D)}} \sum_{\ell=1}^{n} n_K(I, F_\ell) + n_K(I, F_\ell^{-1}),} \tag{3.5}$$

where the infimum is taken over all *finite factorizations* $F = F_1 \circ \cdots \circ F_n$, $F_i \in \text{Hom}_k(D)$, of F in $\text{Hom}_k(D)$. By construction, $q_K(I, F^{-1}) = q_K(I, F)$. Extend this definition to all pairs F and G in $\text{Hom}_k(D)$:

$$\boxed{q_K(F, G) \stackrel{\text{def}}{=} q_K(I, G \circ F^{-1}).} \tag{3.6}$$

The function q_K is a *pseudometric*[16] and the family $\{q_K : K \text{ compact } \subset D\}$ defines a topology on $\mathrm{Hom}_k(D)$. This topology is metrizable for the metric

$$d_k(F,G) \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{q_{K_i}(F,G)}{1+q_{K_i}(F,G)}, \tag{3.7}$$

constructed from the monotone increasing subfamily $\{K_i\}_{i\geq 1}$ of compact sets defined in (3.2). By definition, $d_k(F,G) = d_k(G,F)$ and d_k is symmetrical. It is right-invariant since for all F, G, and H in $\mathrm{Hom}_k(D)$)

$$\forall F, G, \text{ and } H \in \mathrm{Hom}_k(D), \quad d_k(F,G) = d_k(F\circ H, G\circ H),$$
$$\forall F \in \mathrm{Hom}_k(D), \quad d_k(I,F) = d_k(I,F^{-1}).$$

Theorem 3.1. $\mathrm{Hom}_k(D)$*,* $k \geq 0$*, is a group under composition,* d_k *is a right-invariant metric on* $\mathrm{Hom}_k(D)$*, and* $(\mathrm{Hom}_k(D), d_k)$ *is a complete metric space.*

We need Lemmas 2.4 and 2.5 and the analogues of Lemma 2.6 and Theorem 2.11 for the spaces $C^k(K,\mathbf{R}^{\mathrm{N}})$, K a compact subset of D, whose elements are bounded and uniformly continuous.

Lemma 3.1. *Given integers* $r \geq 0$ *and* $s > 0$ *there exists a constant* $c(r,s) > 0$ *with the following property: for any compact* $K \subset D$ *and a sequence* $F_1, \dots, F_n$ *in* $C^r(K,\mathbf{R}^{\mathrm{N}})$ *is such that*

$$\sum_{\ell=1}^{n} \|F_\ell - I\|_{C^r(K)} < \alpha, \quad 0 < \alpha < s; \tag{3.8}$$

then for the map $F = F_n \circ \cdots \circ F_1$*,*

$$\|F - I\|_{C^r(K)} \leq \alpha\, c(r,s). \tag{3.9}$$

Theorem 3.2. *Given* $k \geq 0$*, there exists a continuous function* c_1 *such that for all* $F, G \in \mathrm{Hom}_k(D)$ *and any compact* $K \subset D$

$$\|F\circ G\|_{C^k(K)} \leq \|F\|_{C^k(K)}\, c_1(\|G\|_{C^k(K)}). \tag{3.10}$$

[16]A map $d : X \times X \to \mathbf{R}$ on a set X is called a *pseudometric* (or *écart*, or *gauge*) whenever

(i) $d(x,y) \geq 0$, for all x and y,

(ii) $x = y \Rightarrow d(x,y) = 0$,

(iii) $d(x,y) = d(y,x)$,

(iv) $d(x,z) \leq d(x,y) + d(y,z)$

(J. Dugundji [**1**, p. 198]). This notion is analogous to the one of seminorm for topological vector spaces, but for a group the terminology semimetric (cf. footnote 7) is used for a metric without the triangle inequality.

Proof. From Lemma 2.5 it readily follows that $\psi = F \circ G \in \mathrm{Hom}_k(D)$. Moreover,

$$\begin{aligned}
\|\psi\|_{C^0(K)} &= \|F\|_{C^0(K)}, \\
\|\psi^{(1)}\|_{C^0(K)} &\le \|F^{(1)}\|_{C^0(K)}\, \|G^{(1)}\|_{C^0(K)}, \\
\|\psi^{(i)}\|_{C^0(K)} &\le \|F^{(1)}\|_{C^0(K)}\, \|G^{(i)}\|_{C^0(K)} \\
&\quad + \sum_{j=2}^{i} \|F^{(j)}\|_{C^0(K)}\, a_j(\|G^{(1)}\|_{C^0(K)}, \dots, \|G^{(i-1)}\|_{C^0(K)})
\end{aligned}$$

for $i = 2, \dots, k$, where a_j is a polynomial. Hence

$$\|\psi\|_{C^k(K)} = \max_{i=0,\dots,k} \|\psi^i\|_{C(K)} \le \|F\|_{C^k(K)}\, c_1(\|G\|_{C^k(K)})$$

for some polynomial function c_1 that depends only on k. $\square$

Proof of Theorem 3.1. (i) $\mathrm{Hom}_k(D)$ *is a group.* The composition of C^k-diffeomorphisms is again a C^k-diffeomorphism. By definition, each element has an inverse and the neutral element is I.

(ii) d_k *is a metric.* $d_k(F,G) \ge 0$, $d_k(F,G) = d_k(G,F)$. For the triangle inequality $F \circ H^{-1} = (F \circ G^{-1}) \circ (G \circ H^{-1})$ is a finite factorization of $F \circ H^{-1}$. Consider arbitrary finite factorizations of $F \circ G^{-1}$ and $G \circ H^{-1}$:

$$\begin{aligned}
F \circ G^{-1} &= H_1 \circ \cdots \circ H_m, \quad H_i \in C^k(D,D), \\
G \circ H^{-1} &= L_1 \circ \cdots \circ L_n, \quad L_j \in C^k(D,D).
\end{aligned}$$

This yields a new finite factorization of $F \circ H^{-1}$. By definition of q_{K_i},

$$q_{K_i}(I, F \circ H^{-1}) \le \sum_{\ell=1}^{m} n_{K_i}(I, H_\ell) + n_{K_i}(I, H_\ell^{-1}) + \sum_{\ell'=1}^{n} n_{K_i}(I, L_{\ell'}) + n_{K_i}(I, L_{\ell'}^{-1})$$

and by taking infima over each factorization,

$$\begin{aligned}
q_{K_i}(F,H) = q_{K_i}(I, F \circ H^{-1}) &\le q_{K_i}(I, F \circ G^{-1}) + q_{K_i}(I, G \circ H^{-1}) \\
&= q_{K_i}(F,G) + q_{K_i}(G,H).
\end{aligned}$$

By definition of d_k, since $\{K_i\}$ is a monotonically increasing sequence of compact sets and $\{q_{K_i}\}$ is a monotonically increasing sequence of pseudometrics, we then get the triangle inequality

$$d_k(F,H) \le d_k(F,G) + d_k(G,H).$$

Since $\mathrm{Hom}_k(D)$ is a group and d_k is right-invariant, it remains to show that $d_k(I,F) = 0$ if and only if $F = I$. Clearly if $F = I$, $q_{K_i}(I,F) = 0$ since the q_{K_i}'s are pseudometrics and $d_k(I,F) = 0$. Conversely, if

$$d_k(I,F) \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{q_{K_i}(I,F)}{1 + q_{K_i}(I,F)} = 0,$$

then for all K_i, $q_{K_i}(I,F)=0$. For each K_i and all ε, $0<\varepsilon<1$, there exists a finite factorization $F=F_1\circ\cdots\circ F_n$ such that

$$\sum_{\ell=1}^{n} n_{K_i}(F_\ell - I) + n_{K_i}(F_\ell^{-1}-I) < \varepsilon < 1$$
$$\Rightarrow n_{K_i}(F-I) \le \sum_{\ell} n_{K_i}(F_\ell\circ\cdots\circ F_n - F_{\ell+1}\circ\cdots\circ F_n).$$

From Lemma 3.1

$$\forall\varepsilon>0,\quad n_{K_i}(F-I) < \varepsilon\, c(k,1)$$

and $(F-I)|_{K_i}=0$. Since this is true for all i, $F-I=0$ on D and d_k is a metric.

(iii) *Completeness.* Let $\{F_n\}$ be a Cauchy sequence in $\mathrm{Hom}_k(D)$.

(a) *Boundedness of* $\{F_n\}$ *and* $\{F_n^{-1}\}$ *in* $C^k(D,\mathbf{R}^{\mathrm{N}})$. For all ε, $0<\varepsilon<1$, there exists N such that, for all m, $n>N$, $d_k(F_m,F_n)<\varepsilon$. By the triangle inequality

$$|d_k(I,F_m)-d_k(I,F_n)| \le d_k(F_m,F_n) < \varepsilon.$$

For all i, $\{q_{K_i}(I,F_n\}$ is Cauchy and, a posteriori, bounded by some constant L. For each n, there exists a factorization $F_n=F_n^1\circ\cdots\circ F_n^{\nu(n)}$ such that

$$0 \le \sum_{i=1}^{\nu(n)} \|F_n^i - I\|_{C^k(K_i)} + \|(F_n^i)^{-1}-I\|_{C^k(K_i)} - d_k(I,F_n) < \varepsilon < 1$$
$$\Rightarrow \sum_{i=1}^{\nu(n)} \|F_n^i - I\|_{C^k(K_i)} + \|(F_n^i)^{-1}-I\|_{C^k(K_i)} < L+\varepsilon < L+1.$$

By Lemma 3.1

$$\|F_n-I\|_{C^k(K_i)} + \|F_n^{-1}-I\|_{C^k(K_i)} < 2(L+\varepsilon)\, c(k,L+1)$$

and the sequences $\{F_n-I\}$ and $\{F_n^{-1}-I\}$ and, a fortiori, $\{F_n\}$ and $\{F_n^{-1}\}$ are bounded in $C^k(K_i,\mathbf{R}^{\mathrm{N}})$.

(b) *Convergence of* $\{F_n\}$ *and* $\{F_n^{-1}\}$ *in* $C^k(D,\mathbf{R}^{\mathrm{N}})$. For any m, n

$$F_n - F_m = (F_n\circ F_m^{-1} - I)\circ F_m$$

and by Theorem 3.2

$$\begin{aligned}\|F_m-F_n\|_{C^k(K_i)} &= \|(F_n\circ F_m^{-1}-I)\circ F_m\|_{C^k(K_i)}\\ &\le \|F_n\circ F_m^{-1}-I\|_{C^k(K_i)}\, c_1(\|F_m\|_{C^k(K_i)})\\ \Rightarrow \|F_m-F_n\|_{C^k(K_i)} &\le c_1^+\, \|F_n\circ F_m^{-1}-I\|_{C^k(K_i)},\\ c_1^+ &\overset{\text{def}}{=} \sup_m c_1(\|F_m\|_{C^k(K_i)}) < \infty\end{aligned}$$

since $\{F_m\}$ is bounded in $C^k(K_i, \mathbf{R}^{\mathrm{N}})$. For all ε, $0 < \varepsilon < c(k,1)\, c_1^+$ (where $c(k,1)$ is the constant of Lemma 3.1),

$$\exists N,\, \forall n,m > N, \quad d_k(I, F_n \circ F_m^{-1}) = d_k(F_m, F_n) < \varepsilon/(c(k,1)\, c_1^+) < 1.$$

By Lemma 3.1

$$\begin{aligned} &\|F_n \circ F_m^{-1} - I\|_{C^k(K_i)} < c(k,1)\varepsilon/(c(k,1)\, c_1^+) = \varepsilon/c_1^+ \\ \Rightarrow\ \forall n,m, > N, \quad &\|F_m - F_n)\|_{C^k(K_i)} \le c_1^+\, \|F_n \circ F_m^{-1} - I\|_{C^k(K_i)} < \varepsilon \end{aligned}$$

and, since this is true for all K_i, $\{F_n\}$ is a Cauchy sequence in $C^k(D, \mathbf{R}^{\mathrm{N}})$ that converges to some $F \in C^k(D, \mathbf{R}^{\mathrm{N}})$ since $C^k(D, \mathbf{R}^{\mathrm{N}})$ is a Fréchet space for the family of seminorms $\{q_{K_i}\}$. Similarly, we get that $\{F_n^{-1}\}$ is a Cauchy sequence in $C^k(D, \mathbf{R}^{\mathrm{N}})$ that converges to some $G \in C^k(D, \mathbf{R}^{\mathrm{N}})$.

(c) *F is bijective, $G = F^{-1}$, and $F \in$* $\mathrm{Hom}_k(D)$. For all n, F_n, F_n^{-1}, F and G are continuous since $C^k(D, \mathbf{R}^{\mathrm{N}}) \subset C^0(D, \mathbf{R}^{\mathrm{N}})$. By continuity of $F \mapsto F(x) : C^k(D, \mathbf{R}^{\mathrm{N}}) \to \mathbf{R}^{\mathrm{N}}$, $F_n^{-1}(x) \to G(x)$. By the same argument for F_n, $x = F_n(F_n^{-1}(x)) \to F(G(x)) = (F \circ G)(x)$ and for all $x \in \mathbf{R}^{\mathrm{N}}$, $(F \circ G)(x) = x$. Similarly, starting from $F_n^{-1} \circ F_n = I$, we get $(G \circ F)(x) = x$. Hence, $G \circ F = I = F \circ G$. So F is bijective, $F, F^{-1} \in C^k(D, \mathbf{R}^{\mathrm{N}})$, and $F \in \mathrm{Hom}_k(D)$.

(d) *Convergence of $F_n \to F$ in* $\mathrm{Hom}_k(D)$. To check that $F_n \to F$ in $\mathrm{Hom}_k(D)$ recall that by Theorem 3.2 for each K_i

$$\begin{aligned} q_{K_i}(F_n, F) &= q_{K_i}(I, F \circ F_n^{-1}) \\ &\le \|I - F \circ F_n^{-1}\|_{C^k(K_i)} + \|I - F_n \circ F^{-1}\|_{C^k(K_i)} \\ &= \|(F_n - F) \circ F_n^{-1}\|_{C^k(K_i)} + \|(F - F_n) \circ F^{-1}\|_{C^k(K_i)} \\ &\le c_1(\|F_n^{-1}\|_{C^k(K_i)})\|F_n - F\|_{C^k(K_i)} + c_1(\|F^{-1}\|_{C^k(K_i)})\|F - F_n\|_{C^k(K_i)} \\ &\le \left(\max_n c_1(\|F_n^{-1}\|_{C^k(K_i)}) + c_1(\|F^{-1}\|_{C^k(K_i)})\right) \|F - F_n\|_{C^k(K_i)} \end{aligned}$$

that goes to zero as F_n goes to F in $C^k(D, \mathbf{R}^{\mathrm{N}})$. This completes the proof. □

Remark 3.1.
In section 2 it was possible to consider in $\mathcal{F}(\Theta)$ a subgroup $\mathcal{G}$ of translations since

$$F_a(x) = a + x \quad \Rightarrow\ (F_a - I)(x) = a \in \Theta$$

for spaces Θ that contain constants. Yet, since all of them were spaces of bounded functions in $\mathbf{R}^{\mathrm{N}}$, it was not possible to include rotations or flips. But this can be done now in $\mathrm{Hom}_k(\mathbf{R}^{\mathrm{N}})$, and we can quotient out not only by the subgroup of translations but also by the subgroups of isometries or rotation that are important in image processing. □

Chapter 4

Transformations Generated by Velocities

1 Introduction

In Chapter 3 we have constructed quotient groups of transformations $\mathcal{F}(\Theta)/\mathcal{G}$ and their associated complete Courant metrics. Such spaces are neither linear nor convex. In this chapter, we specialize the results of Chapter 3 to spaces of transformations that are generated at time $t = 1$ by the flow of a *velocity field* over a generic time interval $[0, 1]$ with values in the tangent space Θ. The main motivation is to introduce a notion of semiderivatives in the direction $\theta \in \Theta$ on such groups as well as a tractable criterion for continuity via C^1 or continuous paths in the quotient group endowed with the Courant metric. This point of view was adopted by J.-P. Zolésio [**2**, **7**] as early as 1973 and considerably expanded in his *thèse d'état* in 1979. One of his motivations was to solve a *shape differential equation* of the type $\mathcal{A}\,V(t) + G(\Omega_t(V)) = 0$, $t > 0$, where G is the *shape gradient* of a functional and $\mathcal{A}$ a duality operator.[1] At that time most people were using a simple perturbation of the identity to compute shape derivatives. The first comprehensive book systematically promoting the *velocity method* was published in 1992 by J. Sokołowski and J.-P. Zolésio [**9**]. Structural theorems for the Eulerian shape derivative of smooth domains were first given in 1979 by J.-P. Zolésio [**7**] and generalized to nonsmooth domains in 1992 by M. C. Delfour and J.-P. Zolésio [**14**]. The velocity point of view was also adopted in 1994 by R. Azencott [**1**], in 1995 by A. Trouvé [**2**], and in 1998 by A. Trouvé [**3**] and L. Younes [**2**] to construct complete metrics and *geodesic paths* in spaces of diffeomorphisms generated by a velocity field with a broad spectrum of applications to imaging.[2] The reader is referred to the forthcoming book of L. Younes [**6**] for a comprehensive exposition of this work and

[1] Cf. J.-P. Zolésio [**7**] and the recent book by M. Moubachir and J.-P. Zolésio [**1**].

[2] The first author would like to thank Robert Azencott, who pointed out his work and the contributions of his team during a visit in Houston early in 2010. Special thanks also to Alain Trouvé for an enlightening discussion shortly after in Paris.

to related papers such as the ones of P. W. Michor and D. Mumford [**1**, **2**, **3**] and L. Younes, P. W. Michor, I. Shah, and D. Mumford [**1**].

In view of the above motivations, this chapter begins with section 2, which specializes the results of Chapter 3 to transformations generated by velocity fields. It also explores the connections between the constructions of Azencott and Micheletti that implicitly use a notion of *geodesic path with discontinuities.*

Section 3 motivates and adapts the definitions of *Gateaux* and *Hadamard* semiderivatives in topological vector spaces (cf. Chapter 9) to shape functionals defined on shape spaces. The analogue of the Gateaux semiderivative for sets is obtained by the *method of perturbation of the identity operator*, while the analogue of the Hadamard semiderivative comes from the *velocity* (*speed*) *method.* The first notion does not extend to submanifolds and does not incorporate the *chain rule* for the semiderivative of the composition of functions (for instance, to get the semiderivative with respect to the parameters of an a priori parametrized geometry), while Hadamard does. In Chapter 9, flows of velocity fields will be adopted as the natural framework for defining shape semiderivatives. In section 3.3 the velocity and transformation viewpoints will be emphasized through a series of examples of commonly used families of transformations of sets. They include C^k-domains, Cartesian graphs, polar coordinates, and level sets.

The following two sections give technical results that will be used to characterize continuity and semidifferentiability along local paths without restricting the analysis to the subgroup G_Θ of $\mathcal{F}(\Theta)$ of section 2. In section 4 we establish the equivalence between deformations obtained from a family of C^1-paths and deformations obtained from the flow of a velocity field. Section 4.1 gives the equivalence under relatively general conditions. Section 4.2 shows that Lipschitzian perturbations of the identity operator can be generated by the flow of a nonautonomous velocity field. In section 4.3 the conditions of section 4.2 are sharpened for the special families of velocity fields in $C_0^k(\mathbf{R}^N,\mathbf{R}^N)$, $C^k(\overline{\mathbf{R}^N},\mathbf{R}^N)$, and $C^{k,1}(\overline{\mathbf{R}^N},\mathbf{R}^N)$. The *constrained case* where the family of domains are subsets of a fixed *holdall* is studied in section 5. In both sections 4 and 5 we show that, under appropriate conditions, starting from a family of transformations is locally equivalent to starting from a family of velocity fields. This key result bridges the two points of view.

Section 6 builds on the results of section 4 to establish that the continuity of a shape function with respect to the Courant metric on $\mathcal{F}(\Theta)/\mathcal{G}$ is equivalent to its continuity along the flows of all velocity fields V in $C^0([0,\tau];\Theta)$ for Θ equal to $C_0^{k+1}(\mathbf{R}^N,\mathbf{R}^N)$, $C^{k+1}(\overline{\mathbf{R}^N},\mathbf{R}^N)$, and $C^{k,1}(\overline{\mathbf{R}^N},\mathbf{R}^N)$, $k \geq 0$. This result is of both intrinsic and practical interest since it is generally easier to check the continuity along paths than directly with respect to the Courant metric. Finally, from the discussion in section 3, the technical results of section 4 will again be used in Chapter 9 to construct local C^1-paths generated by velocity fields V in $C^0([0,\tau];\Theta)$ to define the shape semiderivatives.

2 Metrics on Transformations Generated by Velocities

2.1 Subgroup G_Θ of Transformations Generated by Velocities

From Chapter 3 recall the definition of the group

$$\mathcal{F}(\Theta) \stackrel{\text{def}}{=} \left\{ I+\theta : \theta \in \Theta,\, (I+\theta)^{-1}\,\exists, \text{ and } (I+\theta)^{-1} - I \in \Theta \right\}$$

for the Banach space $\Theta = C^{0,1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$ with norm

$$\|\theta\|_\Theta \stackrel{\text{def}}{=} \max\left\{ \sup_{x\in\mathbf{R}^{\mathrm{N}}} |\theta(x)|, c(\theta) \right\}, \quad c(\theta) \stackrel{\text{def}}{=} \sup_{y\neq x} \frac{|\theta(y)-\theta(x)|}{|y-x|}, \tag{2.1}$$

and the associated complete metric d on $\mathcal{F}(\Theta)$. This section specializes to a subgroup of transformations $I+\theta$ of $\mathcal{F}(\Theta)$ that are specified via a vector field V on a *generic interval* $[0,1]$ with the following properties:

(i) for all $x \in \mathbf{R}^{\mathrm{N}}$, the function $t \mapsto V(t,x) : [0,1] \to \mathbf{R}^{\mathrm{N}}$ belongs to $L^1(0,1; \mathbf{R}^{\mathrm{N}})$ and

$$\int_0^1 \|V(t)\|_C \, dt = \int_0^1 \sup_{x\in\mathbf{R}^{\mathrm{N}}} |V(t,x)| \, dt < \infty; \tag{2.2}$$

(ii) for almost all $t \in [0,1]$, the function $x \mapsto V(t,x) : \mathbf{R}^{\mathrm{N}} \to \mathbf{R}^{\mathrm{N}}$ is Lipschitzian and

$$\int_0^1 c(V(t)) \, dt = \int_0^1 \sup_{x\neq y} \frac{|V(t,y)-V(t,x)|}{|y-x|} \, dt < \infty. \tag{2.3}$$

In view of the assumptions on V, it is natural to introduce the norm

$$\|V\|_{L^1(0,1;\Theta)} \stackrel{\text{def}}{=} \int_0^1 \|V(t)\|_\Theta \, dt. \tag{2.4}$$

Associate with $V \in L^1(0,1;\Theta)$ the flow

$$\frac{dT_t(V)}{dt} = V(t) \circ T_t(V), \quad T_0(V) = I. \tag{2.5}$$

In view of the assumptions on V, for each $X \in \mathbf{R}^{\mathrm{N}}$, the differential equation

$$x'(t;X) = V(t, x(t;X)), \quad x(0;X) = X, \tag{2.6}$$

has a unique solution in $W^{1,1}(0,1;\mathbf{R}^{\mathrm{N}}) \subset C([0,1],\mathbf{R}^{\mathrm{N}})$, the mapping $X \mapsto x(\cdot;X) : \mathbf{R}^{\mathrm{N}} \to C([0,1],\mathbf{R}^{\mathrm{N}})$ is continuous, and for all $t \in [0,1]$ the mapping $X \mapsto T_t(V)(X) \stackrel{\text{def}}{=} x(t;X) : \mathbf{R}^{\mathrm{N}} \to \mathbf{R}^{\mathrm{N}}$ is bijective. Define

$$G_\Theta \stackrel{\text{def}}{=} \left\{ T_1(V) : \forall V \in L^1([0,1];\Theta) \right\}. \tag{2.7}$$

Theorem 2.1. (i) *The set G_Θ with the composition $\circ$,*

$$\forall V_1,\, V_2 \in L^1(0,1;\Theta), \quad T_1(V_1) \circ T_1(V_2), \tag{2.8}$$

is a subgroup of $\mathcal{F}(\Theta)$ and for all $V \in L^1(0,1;\Theta)$,

$$\sup_{0\le t\le 1} \|T_t(V) - I\|_\Theta = \sup_{0\le t\le 1} \|\theta_V(t)\|_\Theta \le \|V\|_{L^1} \exp^{2(1+\|V\|_{L^1})}, \tag{2.9}$$

where $\theta_V(t) \overset{\text{def}}{=} T_t(V) - I$ and L^1 stands for $L^1(0,1;\Theta)$. In particular, for all t, $T_t(V) \in \mathcal{F}(\Theta)$ and $t \mapsto T_t(V) : [0,1] \to \mathcal{F}(\Theta)$ is a continuous path.

(ii) *For all $V \in L^1(0,1;\Theta)$, $(T_1(V))^{-1} = T_1(V^-)$ for $V^-(t,x) = -V(1-t,x)$,*

$$-\theta_V \circ (I+\theta_V)^{-1} = (T_1(V))^{-1} - I = T_1(V^-) - I = \theta_{V^-}, \tag{2.10}$$

$$\|V^-\|_{L^1(0,1;\Theta)} = \|V\|_{L^1(0,1;\Theta)}, \tag{2.11}$$

and

$$\sup_{0\le t\le 1} \|T_t(V)^{-1} - I\|_\Theta = \sup_{0\le t\le 1} \|\theta_V^-(t)\|_\Theta \le \|V\|_{L^1} \exp^{2(1+\|V\|_{L^1})}. \tag{2.12}$$

(iii) *For all $V \in L^1(0,1;\Theta)$,*

$$\|T_1(V) - I\|_\Theta + \|(T_1(V))^{-1} - I\|_\Theta \le 2\,\|V\|_{L^1(0,1;\Theta)} \exp^{2\,(1+\|V\|_{L^1(0,1;\Theta)})} \tag{2.13}$$

$$\Rightarrow\; d(T_1(V), I) \le 2\,\|V\|_{L^1(0,1;\Theta)} \exp^{2\,(1+\|V\|_{L^1(0,1;\Theta)})}. \tag{2.14}$$

(iv) *For all $V, W \in L^1(0,1;\Theta)$,*

$$\begin{aligned} &\max\left\{ \sup_{0\le t\le 1} \|\theta_W(t) - \theta_V(t)\|_\Theta,\, \sup_{0\le t\le 1} \|\theta_{W^-}(t) - \theta_{V^-}(t)\|_\Theta \right\} \\ &\le \|V\|_{L^1} \exp^{(1+\|V\|_{L^1})} \Big(1 + \|W\|_{L^1} \exp^{2(1+\|W\|_{L^1})}\Big) \|W(s) - V(s)\|_{L^1}, \end{aligned} \tag{2.15}$$

where L^1 stands for $L^1(0,1;\Theta)$.

Remark 2.1.
Note that the condition $(I-\theta)^{-1}$ exists and $(I-\theta)^{-1} \in \Theta$ in the definition of $\mathcal{F}(\Theta)$ is automatically verified for the transformations $I+\theta_V$ generated by a velocity field $V \in L^1(0,1;\Theta)$. □

Proof. (i) Choosing $V = 0$, the identity I is in G_Θ. To show that G_Θ is closed under the composition, it is sufficient to construct a $W \in L^1(0,1;\Theta)$ such that $T_1(W) = T_1(V_1) \circ T_1(V_2)$. Given $\tau \in (0,1)$ define the *concatenation*

$$(V_1 * V_2)(t,x) \overset{\text{def}}{=} \begin{cases} \dfrac{1}{\tau} V_2\left(\dfrac{t}{\tau}, x\right), & 0 \le t \le \tau, \\ \dfrac{1}{1-\tau} V_1\left(\dfrac{t-\tau}{1-\tau}, x\right), & \tau < t \le 1. \end{cases} \tag{2.16}$$

It is easy to check that $W = V_1 * V_2 \in L^1(0,1;\Theta)$ as the concatenation of the L^1-functions V_1 and V_2 so that the equation

$$\frac{dT_t(W)}{dt} = W(t) \circ T_t(W), \quad T_0(W) = I$$

has a unique solution and $T_1(W) \in G_\Theta$. On the interval $[0, \tau]$

$$\begin{aligned}
T_t(W)(X) - X = \int_0^t V_2\left(\frac{s}{\tau}, T_s(W)(X)\right) d\frac{s}{\tau} = \int_0^{t/\tau} V_2\left(s', T_{s'\tau}(W)(X)\right) ds' \\
\Rightarrow T_{t'\tau}(W)(X) - X = \int_0^{t'} V_2\left(s', T_{s'\tau}(W)(X)\right) ds', \quad 0 \le t' \le 1 \\
\Rightarrow T_{t'\tau}(W)(X) = T_{t'}(V_2)(X) \quad \text{and} \quad T_\tau(W) = T_1(V_2).
\end{aligned}$$

On the interval $[\tau, 1]$

$$\begin{aligned}
T_t(W)(X) - T_1(V_2) &= \int_\tau^t V_1\left(\frac{s-\tau}{1-\tau}, T_s(W)(X)\right) d\frac{s-\tau}{1-\tau} \\
&= \int_0^{\frac{t-\tau}{1-\tau}} V_2\left(s', T_{\tau+s'(1-\tau)}(W)(X)\right) ds' \\
\Rightarrow T_{\tau+t'(1-\tau)}t(W)(X) - T_1(V_2) &= \int_0^{t'} V_2\left(s', T_{\tau+s'(1-\tau)}(W)(X)\right) ds', \quad 0 \le t' \le 1, \\
T_{\tau+t'(1-\tau)}(W)(X) = T_{t'}(V_1) \circ T_1(V_2) &\text{ and } T_1(W)(X) = T_1(V_1) \circ T_1(V_2).
\end{aligned}$$

To show that each $T_1(V)$ has an inverse in G_Θ, consider the new function

$$y(t;X) \stackrel{\text{def}}{=} [T_{1-t}(V) \circ (T_1(V))^{-1}](X) = x(1-t; (T_1(V))^{-1}(X)), \quad 0 \le t \le 1,$$

for which $y(0;X) = [T_1(V) \circ (T_1(V))^{-1}](X) = X$ and $y(1;X) = (T_1(V))^{-1}(X)$. Its equation is given by

$$\frac{dy}{dt}(t;X) = -V(1-t, y(t;X)), \quad y(0;X) = X,$$

and the associated vector field in $V^-(t,x) \stackrel{\text{def}}{=} -V(1-t,x)$ that clearly belongs to $L^1(0,1;\Theta)$. Therefore G_Θ is a group of homeomorphisms of $\mathbf{R}^N$, that is, $G_\Theta \subset \text{Hom}(\mathbf{R}^N)$.

To show that $G_\Theta \subset \mathcal{F}(\Theta)$, it is necessary to show that for each V, $\theta = T_1(V) - I \in \Theta$. Given $X \in \mathbf{R}^N$, consider the functions $x(t) = T_t(V)(X)$ and

$\theta(t;X) = x(t) - X$. By definition

$$\theta(t;X) = x(t) - X = \int_0^t V(s, X + \theta(s;X))\, ds,$$

$$|\theta(t;X)| \leq \int_0^t |V(s, X + \theta(s;X)) - V(s,X)|\, ds + \int_0^t |V(s,X)|\, ds$$

$$\leq \int_0^t c(V(s))\, |\theta(s;X)|\, ds + \int_0^t |V(s,X)|\, ds,$$

$$\sup_X |\theta(t;X)| \leq \int_0^t c(V(s)) \sup_X |\theta(s;X)|\, ds + \int_0^t \sup_X |V(s,X)|\, ds.$$

Given $0 < \alpha < 1$ let $g_\alpha(t) = \exp(\int_0^t c(V(s))\, ds/\alpha)$. Then it is easy to show that

$$\sup_{t\in[0,1]} \frac{\sup_X |\theta(t;X)|}{g_\alpha(t)} \leq \frac{1}{1-\alpha} \int_0^1 \sup_X |V(s,X)|\, ds$$

$$\Rightarrow \sup_{t\in[0,1]} \|\theta(t)\|_C \leq \frac{g_\alpha(1)}{1-\alpha} \int_0^1 \|V(s)\|_C\, ds.$$

To prove that θ is Lipschitzian, associate with $X, Y \in \mathbf{R}^{\mathrm{N}}$, $X \neq Y$, the functions $x(t) = T_t(V)(X)$, $y(t) = T_t(V)(Y)$, $\theta(t;X) = x(t) - X$, and $\theta(t;Y) = y(t) - Y$. By definition

$$\theta(t;X) - \theta(t;Y) = \int_0^t V(s, Y + \theta(s;Y)) - V(s, X + \theta(s;X))\, ds,$$

$$|\theta(t;X) - \theta(t;Y)| \leq \int_0^t c(V(s))\, |\theta(s;Y)) - \theta(s;X)|\, ds + \int_0^t c(V(s))\, |Y - X|\, ds$$

$$\Rightarrow \sup_{X\neq Y} \frac{|\theta(t;X) - \theta(t;Y)|}{|Y-X|} \leq \int_0^t c(V(s)) \sup_{X\neq Y} \frac{|\theta(s;Y)) - \theta(s;X)|}{|Y-X|}\, ds + \int_0^t c(V(s))\, ds$$

$$\Rightarrow \sup_{t\in[0,1]} c(\theta(t)) \leq \frac{g_\alpha(1)}{1-\alpha} \int_0^1 c(V(s))\, ds.$$

Combining the last inequality with the one of the first part, for all α, $0 < \alpha < 1$,

$$\sup_{t\in[0,1]} \|\theta(t)\|_\Theta \leq \frac{g_\alpha(1)}{1-\alpha} \int_0^1 \|V(t)\|_\Theta\, dt.$$

For $V = 0$, $\theta = 0$; for $V \neq 0$, choose $\alpha = \int_0^1 \|V(t)\|_\Theta dt / (\int_0^1 \|V(t)\|_\Theta dt + 1)$. This yields (2.9). Since $(I+\theta)^{-1} = (T_1(V))^{-1} = T_1(V^-)$ is of the same form for the velocity field V^-, $(I+\theta)^{-1} - I = (T_1(V))^{-1} - I = T_1(V^-) - I \in \Theta$. Therefore G_Θ is a subgroup of $\mathcal{F}(\Theta)$.

(ii) From the construction of the inverse in part (i) and the fact that $(T_1(V))^{-1} - I = -\theta_V \circ (I + \theta_V)^{-1}$.

(iii) From inequality (2.9) applied to θ_V and θ_{V^-} and identity (2.11).

(iv) Let $\theta_V(t) = T_t(V) - I$ and let $\theta_W(t) = T_t(W) - I$:

$$\begin{aligned}
\theta_W(t) - \theta_V(t) &= \int_0^t W(s) \circ T_s(W) - V(s) \circ T_s(V)\, ds \\
&= \int_0^t (W(s) - V(s)) \circ T_s(W)\, ds \\
&\quad + \int_0^t V(s) \circ T_s(W) - V(s) \circ T_s(V)\, ds, \\
\|\theta_W(t) - \theta_V(t)\|_C &\leq \int_0^t \|W(s) - V(s)\|_C\, ds \\
&\quad + \int_0^t c(V(s))\, \|\theta_W(s) - \theta_V(s)\|_C\, ds \\
\Rightarrow \sup_{0\leq t\leq 1} \|\theta_W(t) - \theta_V(t)\|_C &\leq \frac{g_\alpha(1)}{1-\alpha} \int_0^1 \|W(s) - V(s)\|_C\, ds
\end{aligned}$$

for $0 < \alpha < 1$ and $g_\alpha(t) = \exp \int_0^t c(V(s)\, ds/\alpha$. Similarly,

$$\begin{aligned}
\theta_W(t) - \theta_V(t) &= \int_0^t (W(s) - V(s)) \circ T_s(W)\, ds \\
&\quad + \int_0^t V(s) \circ T_s(W) - V(s) \circ T_s(V)\, ds, \\
c(\theta_W(t) - \theta_V(t)) &\leq \int_0^t c(W(s) - V(s))\, (1 + c(\theta_W(s)))\, ds \\
&\quad + \int_0^t c(V(s))\, c(\theta_W(s) - \theta_V(s))\, ds \\
\Rightarrow \sup_{0\leq t\leq 1} c(\theta_W(t) - \theta_V(t)) &\leq \frac{g_\alpha(1)}{1-\alpha} \int_0^1 c(W(s) - V(s))\, (1 + c(\theta_W(s)))\, ds.
\end{aligned}$$

From part (i)

$$\sup_{0\leq s\leq 1} c(\theta_W(s)) \leq \|W\|_{L^1(0,1;\Theta)} \exp^{2(1+\|W\|_{L^1(0,1;\Theta)})}$$

and finally

$$\begin{aligned}
&\sup_{0\leq t\leq 1} \|\theta_W(t) - \theta_V(t)\|_\Theta \\
&\leq \frac{g_\alpha(1)}{1-\alpha} \left(1 + \|W\|_{L^1(0,1;\Theta)} \exp^{2(1+\|W\|_{L^1(0,1;\Theta)})}\right) \|W(s) - V(s)\|_{L^1(0,1;\Theta)}\, ds.
\end{aligned}$$

By choosing $\alpha = \|V\|_{L^1(0,1;\Theta)}/(1 + \|V\|_{L^1(0,1;\Theta)})$

$$\begin{aligned}
&\sup_{0\leq t\leq 1} \|\theta_W(t) - \theta_V(t)\|_\Theta \\
&\leq \|V\|_{L^1} \exp^{(1+\|V\|_{L^1})} \left(1 + \|W\|_{L^1(0,1;\Theta)} \exp^{2(1+\|W\|_{L^1})}\right) \|W(s) - V(s)\|_{L^1}\, ds. \qquad \square
\end{aligned}$$

2.2 Complete Metrics on G_Θ and Geodesics

Since G_Θ is a subgroup of the complete group $\mathcal{F}(\Theta)$, its closure $\overline{G}_\Theta$ with respect to the metric d of $\mathcal{F}(\Theta)$ is a closed subgroup whose elements can be "approximated" by elements constructed from the flow of a velocity field. Is (G_Θ, d) complete? Can a velocity field be associated with a limit element? It is very unlikely that (G_Θ, d) can be complete unless we strengthen the metric d to make the associated Cauchy sequence of velocities $\{V_n\}$ converge in $L^1(0,1;\Theta)$.

We have seen that for each $V \in L^1(0,1;\Theta)$, we have a continuous path $t \mapsto T_t(V)$ in G_Θ. Since the part of the metric on $I+\theta_V$ is defined as an infimum over all finite factorizations $(I+\theta_{V_1})\circ\cdots\circ(I+\theta_{V_n})$ of $I+\theta_V$ and $(I+\theta_{V_1^-})\circ\cdots\circ(I+\theta_{V_n^-})$ of $(I+\theta_V)^{-1}$,

$$T_1(V) = T_1(V_1)\circ T_1(V_2)\circ\cdots\circ T_1(V_n),$$
$$d(I, T_1(V)) = \inf_{\substack{T_1(V)=\\ T_1(V_1)\circ\cdots\circ T_1(V_n)}} \sum_{i=1}^{n} \|\theta_{V_i}\|_\Theta + \|\theta_{V_i^-}\|_\Theta,$$

is there a velocity $V^* \in L^1(0,1;\Theta)$ that achieves that infimum? Do we have a *geodesic path* between I and $T_1(V)$ that would be achieved by $V^* \in L^1(0,1;\Theta)$?

Given a velocity field we have constructed a continuous path $t \mapsto T_t(V) : [0,1] \to \mathcal{F}(\Theta)$. At each t, there exists $\delta > 0$ sufficiently small such that the path $t \mapsto T_{t+s}(V)\circ T_t^{-1}(V) : [0,\delta] \to \mathcal{F}(\Theta)$ is differentiable in t in the group $\mathcal{F}(\Theta)$:

$$\begin{aligned}\frac{T_{t+s}(V)\circ T_t^{-1}(V) - I}{s} &= \frac{1}{s}\int_t^{t+s} V(r)\circ T_r(V)\,dr\circ T_t^{-1}(V)\\ &\to V(t)\circ T_t(V)\circ T_t^{-1}(V) = V(t)\end{aligned} \tag{2.17}$$

$$\Rightarrow \frac{d_{\mathcal{F}}T_t}{dt} = V(t) \quad \text{a.e. in } \Theta. \tag{2.18}$$

With this definition a jump from $I+\theta_1$ to $[I+\theta_2]\circ[I+\theta_1]$ at time $1/2$ becomes

$$T_{1/2+s}(V)\circ T_{1/2}^{-1}(V) - I = [I+\theta_2]\circ[I+\theta_1]\circ[I+\theta_1]^{-1} - I = I+\theta_2 - I = \theta_2,$$

that is, a *Dirac delta function* θ_2 in the tangent space Θ to $\mathcal{F}(\Theta)$ at $1/2$. Now the norm of the velocity $V = \theta_2\,\delta_{1/2}$ in the space of measures becomes

$$\|V\|_{M_1((0,1);\Theta)} = \int_0^1 \|\theta_2\|_\Theta \delta_{1/2}\,dt = \|\theta_2\|_\Theta.$$

If we have n such jumps,

$$\|V\|_{M_1((0,1);\Theta)} = \sum_{1=1,\dots,n} \|\theta_i\|_\Theta.$$

The metric of Micheletti is the infimum of this norm over all finite factorizations.

A consequence of this analysis is that a factorization of an element $I+\theta$ of the form $(I+\theta_1)\circ(I+\theta_2)\circ\cdots\circ(I+\theta_n)$ is in fact a path $t \mapsto T_t : [0,1] \to \mathcal{F}(\Theta)$ of bounded variations in $\mathcal{F}(\Theta)$ between I and $I+\theta$ by assigning times $0 < t_1 < t_2 < \cdots < t_n < 1$ to each jump in the group $\mathcal{F}(\Theta)$:

$$T_t(V) \stackrel{\text{def}}{=} \begin{cases} I, & 0 \le t < t_1, \\ I+\theta_1, & t_1 \le t < t_2, \\ (I+\theta_1)\circ(I+\theta_2), & t_2 \le t < t_3, \\ \dots \\ (I+\theta_1)\circ(I+\theta_2)\circ\cdots\circ(I+\theta_n), & t_n \le t \le 1, \end{cases}$$

$$V(t) \stackrel{\text{def}}{=} \sum_{i=1}^n \theta_i\,\delta(t_i) \quad \Rightarrow \quad \frac{d_{\mathcal{F}}T_t}{dt} = V(t), \quad \|V\|_{M_1((0,1);\Theta)} = \sum_{1=1,\dots,n} \|\theta_i\|_\Theta,$$

where the time-derivative is taken in the group sense (2.17) and V is a bounded measure.

With this interpretation in mind, for $V \in L^1(0,1;\Theta)$ the metric of Micheletti *should* reduce to

$$d(I, T_1(V)) = \inf_{\substack{v\in L^1(0,1;\Theta)\\ T_1(v)=T_1(V)}} \int_0^1 \|v(t)\|_\Theta \, dt = \inf_{\substack{v\in L^1(0,1;\Theta)\\ T_1(v)=T_1(V)}} \int_0^1 \left\| \frac{d_{\mathcal{F}}T_t(v)}{dt} \right\|_\Theta dt,$$

where the derivative is in the group $\mathcal{F}(\Theta)$. As a result we could talk of a *geodesic path* in $\mathcal{F}(\Theta)$ between I and $T_1(V)$. If this could be justified, the next question is whether G_Θ is complete in $(\mathcal{F}(\Theta), d)$ or complete with respect to the metric constructed from the function

$$d_G(I, T_1(V)) \stackrel{\text{def}}{=} \inf_{\substack{v\in L^1(0,1;\Theta)\\ T_1(v)=T_1(V)}} \int_0^1 \|v(t)\|_\Theta \, dt, \tag{2.19}$$

$$d_G(T_1(V), T_1(W)) \stackrel{\text{def}}{=} d(T_1(V)\circ T_1(W)^{-1}, I). \tag{2.20}$$

By definition

$$d_G(T_1(V)^{-1}, I) = d_G(T_1(V), I), \quad d_G(T_1(V), T_1(W)) = d_G(T_1(W), T_1(V))$$

and d_G is right-invariant. The infimum is necessary since the velocity taking I to $T_1(V)$ is not unique.

Theorem 2.2. *Let Θ be equal to $C^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$, $C^{k+1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$, $C_0^{k+1}(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$, or $\mathcal{B}^{k+1}(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$, $k \ge 0$. Then d_G is a right-invariant metric on G_Θ.*

Remark 2.2.
The theorem is also true for the right-invariant metric

$$d'_G(T_1(V), I) \stackrel{\text{def}}{=} d(T_1(W), T_1(V)) + \inf_{\substack{W\in L^1(0,1;\Theta)\\ T_1(W)=T_1(V)}} \|W\|_{L^1(0,1;\Theta)}, \tag{2.21}$$

$$d'_G(T_1(V), T_1(W)) \stackrel{\text{def}}{=} d'_G(T_1(V)\circ T_1(W)^{-1}, I). \tag{2.22}$$

□

Proof. We give only the proof for $C^{0,1}(\overline{\mathbf{R}^{\rm N}}, \mathbf{R}^{\rm N})$. It remains to show that the triangle inequality holds and that $d_G(T_1(W), T_1(V)) = 0$ if and only if $T_1(W) = T_1(V)$. Given U, V, W in $L^1(0,1;\Theta)$, $T_1(W) \circ T_1(U)^{-1} = T_1(W) \circ T_1(V)^{-1} \circ T_1(V) \circ T_1(U)^{-1}$. For all $v \in L^1(0,1;\Theta)$ such that $T_1(v) = T_1(W) \circ T_1(V)^{-1}$ and for all $u \in L^1(0,1;\Theta)$ such that $T_1(v) = T_1(V) \circ T_1(U)^{-1}$, the concatenation $v * u$ is such that $T_1(v * u) = T_1(W) \circ T_1(U)^{-1}$. By definition

$$d_G(T_1(W) \circ T_1(U)^{-1}, I) \le \|v * u\|_{L^1} = \|v\|_{L^1} + \|u\|_{L^1}.$$

By taking infima with respect to v and u we get the triangle inequality

$$d_G(T_1(W) \circ T_1(U)^{-1}, I) \le d_G(T_1(W) \circ T_1(V)^{-1}, I) + d_G(T_1(V) \circ T_1(U)^{-1}, I).$$

If $T_1(V) = I$, then $\inf_{\substack{w \in L^1(0,1;\Theta) \\ T_1(w)=I}} \|w\|_{L^1(0,1;\Theta)} = 0$ and $d_G(T_1(V), I) = 0$. Conversely, if $d_G(T_1(V), I) = 0$, then $\inf_{\substack{v \in L^1(0,1;\Theta) \\ T_1(v)=T_1(V)}} \|v\|_{L^1(0,1;\Theta)} = 0$ and there exists a sequence $v_n \in L^1(0,1;\Theta)$ such that $v_n \to 0$ in $L^1(0,1;\Theta)$ and $T_1(v_n) = T_1(V)$. By continuity of $T_1(v_n) - I$ with respect to v_n from Theorem 2.1 (iv), $T_1(V) - I = T_1(v_n) - I \to T_1(0) - I = 0$ and $T_1(V) = I$. □

Getting the completeness with the metric d_G and even d'_G is not obvious. For a Cauchy sequence $\{T_1(V_n)\}$, the L^1-convergence of the velocities $\{V_n\}$ to some V would yield the convergence of $\{T_1(V_n)\}$ to $T_1(V)$ and even the convergence of the geodesics, but the metrics do not seem sufficiently strong to do that. To appreciate this point, we first establish an inequality that really follows from the triangle inequality for d_G. Associate with W a w such that $T_1(w) = T_1(W)$, with V a v such that $T_1(v) = T_1(V)$, and let u be such that $T_1(u) = T_1(W) \circ T_1(V)^{-1}$. Then the concatenation $w * v^-$ is such that $T_1(v * u) = T_1(W)$. From this we get the inequalities

$$\inf_{T_1(w)=T_1(W)} \|w\|_{L^1} \le \|v * u\|_{L^1} = \|v\|_{L^1} + \|u\|_{L^1}$$

and by taking infima we get the inequality

$$\begin{aligned} &\inf_{T_1(w)=T_1(W)} \|w\|_{L^1} \le \inf_{T_1(u)=T_1(W)\circ T_1(V)^{-1}} \|u\|_{L^1} + \inf_{T_1(v)=T_1(V)} \|v\|_{L^1} \\ \Rightarrow &\left| \inf_{T_1(w)=T_1(W)} \|w\|_{L^1} - \inf_{T_1(v)=T_1(V)} \|v\|_{L^1} \right| \le \inf_{T_1(u)=T_1(W)\circ T_1(V)^{-1}} \|u\|_{L^1}. \end{aligned}$$

Pick any Cauchy sequence $\{T_1(V_n)\}$. For any $\varepsilon > 0$, there exists N such that for all $n, m > N$

$$\left| \inf_{T_1(w)=T_1(V_m)} \|w\|_{L^1} - \inf_{T_1(v)=T_1(V_n)} \|v\|_{L^1} \right| \le \inf_{T_1(u)=T_1(V_m)\circ T_1(V_n)^{-1}} \|u\|_{L^1} < \varepsilon.$$

Therefore, there exists a sequence $\{v_n\}$ such that $T_1(v_n) = T_1(V_n)$ and

$$\begin{aligned} 0 \le \|v_n\|_{L^1} - \inf_{T_1(v)=T_1(V_n)} \|v\|_{L^1} < \varepsilon \\ \Rightarrow \big| \|v_m\|_{L^1} - \|v_n\|_{L^1} \big| < 3\varepsilon. \end{aligned}$$

The sequence $\{\|v_n\|_{L^1}\}$ is bounded in $L^1(0,1;\Theta)$, but this does not seem sufficient to get the convergence of the v_n's in $L^1(0,1;\Theta)$.

One could think of many ways to strengthen the metric. For instance, one could introduce the minimization of the whole $W^{1,1}(0,1;\Theta)$-norm of the path $t \mapsto T_t(v)$ with respect to v from which the minimizing velocity $v(t) = \partial T_t/\partial t \circ T_t^{-1}$ could be recovered, but the triangle inequality would be lost.

2.3 Constructions of Azencott and Trouvé

In 1994 R. Azencott [**1**] defined the following program. Given a smooth manifold M, consider the time continuous curve $T = (T_t)_{0\le t\le 1}$ in $Aut(M)$ solutions of

$$\frac{\partial T_t}{\partial t} = V(t) \circ T_t, \quad T_0 = I, \tag{2.23}$$

for a continuous time family $V(t)$ of vector fields in a Hilbert space $H \subset Aut(M)$ and define

$$d(I,\phi) = \inf_{\substack{V\in C([0,1];H)\\ T_1(V)=\phi}} l(\phi) \text{ with } l(\phi) \overset{\text{def}}{=} \int_0^1 |V(t)|_I\, dt. \tag{2.24}$$

A fully rigorous construction of this distance, in the context of infinite-dimensional Lie groups, was performed by A. Trouvé [**2**] in 1995 and [**3**] in 1998. He shows that the subset of $\phi \in Aut(M)$ for which $d(I,\phi)$ can be defined is a subgroup $\mathcal{A}$ of invertible mappings. He extends this distance between two arbitrary mappings ϕ and ψ by right-invariance: $d(\phi,\psi) \overset{\text{def}}{=} d(I, \psi\circ\phi^{-1})$. Hence, in this extended framework, the new problem should be to find in $\mathcal{A}$

$$\hat{\phi} = \text{ argmin } \frac{1}{2}\int_M |\tilde{f} - f\circ\phi|^2 dx + \frac{1}{2} d(Id,\phi)^2,$$

where M is a smooth manifold, the function $f : M \to \mathbf{R}$ is a *high dimensional representation template*, and $\tilde{f}$ is a new *observed image* belonging to this family (cf. A. Trouvé [**3**]). This approach was further developed by the group of Azencott (cf., for instance, L. Younes [**1**]).

Quoting from the introduction of A. Trouvé [**3**] in 1998:

> ...Hence, for a large deformation ϕ, one can consider ϕ as the concatenation of small deformations ϕ_{u_i}. More precisely, if $\phi = \Phi_n$, where the Φ_k are recursively defined by $\Phi_0 = Id$ and $\phi_{k+1} = \phi_{u_{k+1}} \circ \Phi_k$, the family $\Phi = (\Phi_k)_{0\le k\le n}$ defines a polygonal line in $Aut(M)$ whose length $l(\Phi)$ can be approximated by $l(\phi) = \sum_{k=1}^n |u_k|_e$. At a naive level, one could define the distance $d(Id,\phi)$ as the infimum of $l(\Phi_n)$ for all family Φ such that $\Phi_n = \phi$. For a more rigorous setting, one should consider the time continuous curve $\Phi = (\Phi_t)_{0\le t\le 1}$ in $Aut(M)$. ...

In fact what he describes as a naive approach is the very precise construction of A. M. Micheletti [**1**] in 1972. He is certainly not the first author who has overlooked that paper in Italian, where the central completeness result that is of interest to us was just a lemma in her analysis of the continuity of the first eigenvalue of the Laplacian. We brought it up only in the 2001 edition of this book.

By introducing a Hilbert space $H \subset \Theta = C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)$ and minimizing over $L^2(0,1;H)$ rather than $L^1(0,1;\Theta)$, the bounded sequence $\{v_n\}$ lives in $L^2(0,1;H)$, where there exist a $v \in L^2(0,1;H)$ and a weakly converging subsequence to v. By using the complete metric of Micheletti, we had the existence of a $F \in \mathcal{F}(\Theta)$ such that $T_1(v_n) \to F$, but now we have a candidate v for which $F = T_1(v)$. This is very much analogous to the *Hilbert uniqueness method* of J.-L. Lions [**1**] in the context of the controllability of partial differential equations (cf. also R. Glowinski and J.-L. Lions [**1**, **2**]).

Remark 2.3.
It is important to understand that finding a complete metric of *Courant type* for the various spaces of diffeomorphisms is a much less restrictive problem than constructing a metric from geodesics. The existence of a geodesic is not required for the Courant metrics. □

3 Semiderivatives via Transformations Generated by Velocities

3.1 Shape Function

All spaces $\mathcal{F}(\Theta)$ and $\mathcal{F}(\Theta)/\mathcal{G}(\Omega_0)$ in Chapter 3 associated with the family of sets

$$\mathcal{X}(\Omega_0) = \{F(\Omega_0) : \forall F \in \mathcal{F}(\Theta)\} \tag{3.1}$$

are nonlinear and nonconvex and the elements of $\mathcal{F}(\Theta)/\mathcal{G}(\Omega_0)$ are equivalence classes of transformations. Defining a differential with respect to such spaces is similar to defining a differential on an infinite-dimensional manifold. Fortunately, the tangent space is invariant and equal to Θ in every point of $\mathcal{F}(\Theta)$. This considerably simplifies the analysis.

Definition 3.1.
Given a nonempty subset D of $\mathbf{R}^N$, consider the set $\mathcal{P}(D) = \{\Omega : \Omega \subset D\}$ of subsets of D. The set D will be referred to as the underlying *holdall* or *universe*. A *shape functional* is a map

$$J : \mathcal{A} \to E \tag{3.2}$$

from some *admissible family* $\mathcal{A}$ of sets in $\mathcal{P}(D)$ into a topological space E. □

For instance, $\mathcal{A}$ could be the set $\mathcal{X}(\Omega)$. D can represent some physical or mechanical constraint, a submanifold of $\mathbf{R}^N$, or some mathematical constraint. In most cases, it can be chosen large and as smooth as necessary for the analysis. In the unconstrained case, D is equal to $\mathbf{R}^N$.

3.2 Gateaux and Hadamard Semiderivatives

Consider the solution (flow) of the differential equation

$$\frac{dx}{dt}(t,X) = V(t, x(t,X)),\ t \geq 0, \quad x(t,X) = X, \tag{3.3}$$

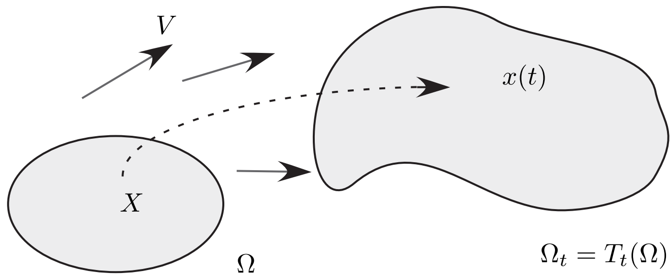


Figure 4.1. *Transport of Ω by the velocity field V.*

for velocity fields $V(t)(x) \stackrel{\text{def}}{=} V(t,x)$ (cf. Figure 4.1). It generates the family of transformations $\{T_t : \mathbf{R}^{\mathrm{N}} \to \mathbf{R}^{\mathrm{N}} : t \geq 0\}$ defined as follows:

$$X \mapsto T_t(X) \stackrel{\text{def}}{=} x(t;X) : \mathbf{R}^{\mathrm{N}} \to \mathbf{R}^{\mathrm{N}} . \tag{3.4}$$

Given an initial set $\Omega \subset \mathbf{R}^{\mathrm{N}}$, associate with each $t > 0$ the new set

$$\Omega_t \stackrel{\text{def}}{=} T_t(\Omega) = \{T_t(X) : \forall X \in \Omega\} . \tag{3.5}$$

This perturbation of the initial set is the basis of the *velocity* (*speed*) *method*.

The choice of the terminology "velocity" to describe this method is accurate but may become ambiguous in problems where the variables involved are themselves "physical velocities": this situation is commonly encountered in continuum mechanics. In such cases it may be useful to distinguish between the "artificial velocity" and the "physical velocity." This is at the origin of the terminology *speed method*, which has often been used in the literature. The latter terminology is convenient, but is not as accurate as the *velocity method*. We shall keep both terminologies and use the one that is most suitable in the context of the problem at hand.

It is important to work with the weakest notion of *semiderivative* that preserves the basic elements of the *classical differential calculus* such as the *chain rule* for the differentiation of the composition of functions. It should be able to handle the norm of a function or functions defined as an *upper* or *lower envelope* of a family of differentiable functions. Since the spaces of geometries are generally nonlinear and nonconvex, we also need a notion of semiderivative that yields differentials in the tangent space as for manifolds. We are looking for a very general notion of semidifferential *but not more*! In that context the most suitable candidate is the Hadamard semiderivative[3] that will be introduced later. Important nondifferentiable functions are Hadamard semidifferentiable and the chain rule is verified.

[3]In his 1937 paper, M. Fréchet [**1**] extends to function spaces the Hadamard derivative and promotes it as more general than his own Fréchet derivative since it does not require the space to have a metric in infinite-dimensional spaces. In the same paper, he also introduces a relaxation of the Hadamard derivative that is almost a semiderivative. A precise definition of the Hadamard semiderivative explicitly appears in J.-P. Penot [**1**] in 1978 and in A. Bastiani [**1**] in 1964 as a well-known notion.

We proceed step by step. A first candidate for the *shape semiderivative* of J at Ω in the direction V would be

$$dJ(\Omega;V) \stackrel{\text{def}}{=} \lim_{t\searrow 0} \frac{J(\Omega_t(V)) - J(\Omega)}{t} \quad \text{(when the limit exists).} \tag{3.6}$$

This very weak notion depends on the history of V for $t>0$ and will be too weak for most of our purposes. For instance, its composition with another function would not verify the chain rule. To make it compatible with the usual notion of semiderivative in a direction θ of the tangent space Θ to $\mathcal{F}(\Theta)$ in $F(\Omega)$, it must be strengthened as follows.

Definition 3.2.
Given $\theta \in \Theta$

$$dJ(\Omega;\theta) \stackrel{\text{def}}{=} \lim_{\substack{V\in C^0([0,\tau];\Theta)\\ V(0)=\theta\\ t\searrow 0}} \frac{J(\Omega_t(V)) - J(\Omega)}{t}, \tag{3.7}$$

where the limit depends only on θ and is independent of the choice of V for $t>0$. □

Equivalently, the limit $dJ(\Omega;\theta)$ is independent of the way we approach Ω for a fixed direction θ. This is precisely the adaptation of the Hadamard semiderivative.[4]

Definitions based on a *perturbation of the identity*

$$T_t(X) \stackrel{\text{def}}{=} x(t) = X + t\,\theta(X), \quad t \ge 0, \tag{3.8}$$

can be recast in the velocity framework by choosing the velocity field $V(t) = \theta\circ T_t^{-1}$, that is,

$$\forall x \in \mathbf{R}^{\mathrm{N}},\ \forall t\ge 0, \quad V(t,x) \stackrel{\text{def}}{=} \theta(T_t^{-1}(x)) = \theta\circ(I+t\theta)^{-1}(x), \tag{3.9}$$

which requires the existence of the inverse for sufficiently small t. With that choice for each X, $x(t) = T_t(X)$ is the solution of the differential equation

$$\frac{dx}{dt}(t) = V\big(t, x(t)\big), \quad x(0) = X. \tag{3.10}$$

Under appropriate continuity and differentiability assumptions,

$$\forall x \in \mathbf{R}^{\mathrm{N}}, \quad V(0)(x) \stackrel{\text{def}}{=} V(0,x) = \theta(x), \tag{3.11}$$

$$\forall x \in \mathbf{R}^{\mathrm{N}}, \quad \dot V(0)(x) \stackrel{\text{def}}{=} \frac{\partial V}{\partial t}(t,x)\bigg|_{t=0} = -\left[D\theta(x)\right]\theta(x), \tag{3.12}$$

where $D\theta(x)$ is the Jacobian matrix of θ at the point x. In compact notation,

$$V(0) = \theta \quad\text{and}\quad \dot V(0) = -\left[D\theta\right]\theta. \tag{3.13}$$

This last computation shows that at time 0, the points of the domain Ω are simultaneously affected by the velocity field $V(0)=\theta$ and the acceleration field

[4]Cf. footnote 2 in section 2 of Chapter 9 for the definitions and properties and the comparison of the various notions of derivatives and semiderivatives in Banach spaces.

$\dot{V}(0) = -[D\theta]\theta$. The same result will be obtained without *acceleration term* by replacing $I + t\theta$ by the transformation T_t generated by the solutions of the differential equation

$$\frac{dx}{dt}(t) = \theta(x(t)), \quad x(0) = X, \quad T_t(X) \stackrel{\text{def}}{=} x(t),$$

for which $V(0) = \theta$ and $\dot{V}(0) = 0$. Under suitable assumptions the two methods will produce the same first-order semiderivative.

The definition of a shape semiderivative

$$dJ(\Omega; \theta) \stackrel{\text{def}}{=} \lim_{t \searrow 0} \frac{J([I + t\theta]\Omega) - J(\Omega)}{t} \tag{3.14}$$

by perturbation of the identity is similar to a Gateaux semiderivative that will not verify the chain rule. Moreover, second-order semiderivatives will differ by an acceleration term that will appear in the expression obtained by the method of perturbation of the identity.

3.3 Examples of Families of Transformations of Domains

In section 3.2 we have formally introduced a notion of semiderivative of a shape functional via transformations generated by flows of velocity fields and perturbations of the identity operator as special cases of the velocity method.

Before proceeding with a more abstract treatment, we present several examples of definitions of shape semiderivatives that can be found in the literature. We consider special classes of domains (C^∞, C^k, Lipschitzian), Cartesian graphs, polar coordinates, and level sets that provide classical examples of parametrized and/or constrained deformations. In each case we construct the associated underlying family (not necessarily unique) of transformations $\{T_t : 0 \le t \le \tau\}$.

3.3.1 C^∞-Domains

Let Ω be an open domain of class C^∞ in $\mathbf{R}^{\mathrm{N}}$. Recall from section 5 of Chapter 2 that in any point $x \in \Gamma$ the unit outward normal is given by

$$\forall y \in \Gamma_x \stackrel{\text{def}}{=} \mathcal{U}(x) \cap \Gamma, \quad n(y) = \frac{m_x(y)}{|m_x(y)|}, \tag{3.15}$$

where

$$m_x(y) = -\,{}^*(Dh_x)^{-1}(h_x^{-1}(y))\, e_N \text{ in } \mathcal{U}(x) \tag{3.16}$$

$$\Rightarrow\ n = -\frac{{}^*(Dh_x)^{-1} e_N}{|\,{}^*(Dh_x)^{-1} e_N|} \circ h_x^{-1}. \tag{3.17}$$

When Γ is compact, it is possible to find a finite sequence of points $\{x_j : 1 \le j \le J\}$ in Γ such that

$$\Gamma \subset \mathcal{U} \stackrel{\text{def}}{=} \bigcup_{j=1}^{J} \mathcal{U}_j, \quad \mathcal{U}_j \stackrel{\text{def}}{=} \mathcal{U}(x_j).$$

As in the definition of the boundary integral, associate with $\{\mathcal{U}_j\}$ a partition of unity $\{r_j\}$:

$$r_j \in \mathcal{D}(\mathcal{U}_j), \quad 0 \leq r_j \leq 1, \quad \sum_{j=1}^{J} r_j = 1 \text{ in } \mathcal{U}_0$$

for some neighborhood $\mathcal{U}_0$ of Γ such that

$$\Gamma \subset \mathcal{U}_0 \subset \overline{\mathcal{U}_0} \subset \mathcal{U}.$$

For the C^∞-domain Ω the normal satisfies

$$n = \sum_{j=1}^{J} r_j \, n = \sum_{j=1}^{J} r_j \, \frac{m_j}{|m_j|} \in C^\infty(\Gamma; \mathbf{R}^{\mathrm{N}})$$

since

$$\forall j, \quad \frac{m_j}{|m_j|} \circ h_j \in C^\infty(B; \mathbf{R}^{\mathrm{N}}).$$

Given any $\rho \in C^\infty(\Gamma)$ and $t \geq 0$, consider the following perturbation Γ_t of Γ along the normal field n:

$$\Gamma_t \stackrel{\text{def}}{=} \{x \in \mathbf{R}^{\mathrm{N}} \, : \, x = X + t\rho(X)n(X), \, \forall X \in \Gamma\}. \tag{3.18}$$

We claim that for τ sufficiently small and all t, $0 \leq t \leq \tau$, the set Γ_t is the boundary of a C^∞-domain Ω_t by constructing a transformation T_t of $\mathbf{R}^{\mathrm{N}}$ which maps Ω onto Ω_t and Γ onto Γ_t. First construct an extension $N \in \mathcal{D}(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ of the normal field n on Γ. Define

$$m \stackrel{\text{def}}{=} \sum_{j=1}^{J} r_j m_j \in \mathcal{D}(\mathcal{U}, \mathbf{R}^{\mathrm{N}}). \tag{3.19}$$

By construction

$$m \in C^\infty(\Gamma; \mathbf{R}^{\mathrm{N}}), \tag{3.20}$$

$m \neq 0$ on Γ, and there exists a neighborhood $\mathcal{U}_1$ of Γ contained in $\mathcal{U}_0$ where $m \neq 0$ since m is at least C^1.

Now construct a function r_0 in $\mathcal{D}(\mathcal{U}_1)$, $0 \leq r_0(x) \leq 1$, and a neighborhood $\mathcal{V}$ of Γ such that

$$r_0 = 1 \text{ in } \mathcal{V} \quad \text{and} \quad \Gamma \subset \mathcal{V} \subset \overline{\mathcal{V}} \subset \mathcal{U}_1.$$

Define the vector field

$$\forall x \in \mathbf{R}^{\mathrm{N}}, \quad N(x) \stackrel{\text{def}}{=} r_0(x) \frac{m(x)}{|m(x)|}. \tag{3.21}$$

Hence N belongs to $\mathcal{D}(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ since $\operatorname{supp} N \subset \overline{\mathcal{V}}$ is compact. Moreover

$$N = \frac{m}{|m|} \quad \text{in } \mathcal{V} \quad \Rightarrow N(x) = n(x) \text{ on } \Gamma = \Gamma \cap \mathcal{V}.$$

For each j, $\rho \circ h_j \in C^\infty(\overline{B_0})$ and the extensions

$$\rho_j^h(\zeta) = \rho_j^h(\zeta', \zeta_N) \stackrel{\text{def}}{=} \rho(h_j(\zeta', 0)), \quad \forall \zeta \in B, \quad \tilde{\rho}_j \stackrel{\text{def}}{=} \rho_j^h \circ g_j,$$

belong, respectively, to $C^\infty(\overline{B})$ and $C^\infty(\overline{\mathcal{U}_j})$. Then

$$\tilde{\rho} \stackrel{\text{def}}{=} \sum_{j=1}^{J} r_j \tilde{\rho}_j \in \mathcal{D}(\mathbf{R}^{\mathrm{N}})$$

is an extension of ρ from Γ to $\mathbf{R}^{\mathrm{N}}$ with compact support since $\operatorname{supp} r_j \subset \mathcal{U}_j \subset \mathcal{U}$.

Define the following transformation of $\mathbf{R}^{\mathrm{N}}$:

$$T_t(X) \stackrel{\text{def}}{=} X + t\,\tilde{\rho}(X) N(X), \quad t \geq 0. \tag{3.22}$$

By construction, $\tilde{\rho}\,N$ is uniformly Lipschitzian in $\mathbf{R}^{\mathrm{N}}$ and, by Theorem 4.2 in section 4.2, there exists $0 < \tau$ such that T_t is bijective and bicontinuous from $\mathbf{R}^{\mathrm{N}}$ onto itself. As a result from J. Dugundji [**1**] for $0 \leq t \leq \tau$

$$\begin{aligned} \Omega_t &= T_t(\Omega) = [I + t\tilde{\rho}\,N](\Omega), \\ \partial\Omega_t &= T_t(\partial\Omega) = [I + t\tilde{\rho}\,N](\partial\Omega) = [I + t\rho\, n](\partial\Omega) = \Gamma_t. \end{aligned}$$

Since the domain Ω_t is specified by its boundary Γ_t, it depends only on ρ and not on its extension $\tilde{\rho}$. The special transformation T_t introduced here is of class C^∞, that is, $T_t \in C^\infty(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$, and $\frac{1}{t}\,[T_t - I]$ is proportional to the normal field n on Γ, but it is not proportional to the normal n_t on Γ_t for $t > 0$. In other words, at $t = 0$ the deformation is along n, but at $t > 0$ the deformation is generally not along n_t.

If $J(\Omega)$ is a real-valued shape functional defined on C^∞-domains in D, the semiderivative (if it exists) is defined as follows: for all $\rho \in C^\infty(\Gamma)$

$$d_n J(\Omega; \tilde{\rho}) \stackrel{\text{def}}{=} \lim_{t \searrow 0} \frac{J\big((I + t\tilde{\rho}N)\Omega\big) - J(\Omega)}{t}. \tag{3.23}$$

It turns out that this limit depends only on ρ and not on its extension $\tilde{\rho}$.

3.3.2 C^k-Domains

When Ω is a domain of class C^k with boundary Γ, the normal field n belongs to $C^{k-1}(\Gamma, \mathbf{R}^{\mathrm{N}})$. Therefore, choosing deformations along the normal would yield transformations $\{T_t\}$ mapping C^k-domains Ω onto C^{k-1}-domains $\Omega_t = T_t(\Omega)$. The obvious way to deal with C^k-domains is to relax the constraint that the perturbation $\tilde{\rho}N$ be carried by the normal. Choose vector fields θ in $\mathcal{D}^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ and consider the family of transformations

$$T_t = I + t\theta, \quad \Omega_t = T_t(\Omega), \quad t \geq 0. \tag{3.24}$$

This is a generalization of the family of transformations (3.22) in section 3.3.1 from $\tilde{\rho}N$ to θ. For $k \geq 1$, θ is again uniformly Lipschitzian in $\mathbf{R}^{\mathrm{N}}$ and, by Theorem 4.2 in section 4.2, there exists $\tau > 0$ such that T_t is bijective and bicontinuous from $\mathbf{R}^{\mathrm{N}}$ onto itself. Thus for $0 \leq t \leq \tau$,

$$\Omega_t = T_t(\Omega) = [I + t\theta](\Omega) \quad \text{and} \quad \partial\Omega_t = T_t(\partial\Omega) = [I + t\theta](\partial\Omega).$$

A more restrictive approach to get around the lack of sufficient smoothness of the normal n to Γ would be to introduce a transverse field p on Γ such that

$$p \in C^k(\Gamma; \mathbf{R}^{\mathrm{N}}), \quad \forall x \in \Gamma, \quad p(x) \cdot n(x) > 0. \tag{3.25}$$

Given p and $\rho \in C^k(\Gamma)$ define for $t \geq 0$

$$\Gamma_t \stackrel{\text{def}}{=} \left\{ x \in \mathbf{R}^{\mathrm{N}} : x = X + t\rho(X)p(X), \, \forall X \in \Gamma \right\}. \tag{3.26}$$

Choosing C^k-extensions $\tilde{\rho}$ and $\tilde{p}$ of ρ and p we can go back to the case where

$$\theta = \tilde{\rho}\tilde{p} \in \mathcal{D}^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}). \tag{3.27}$$

For any $\theta \in \mathcal{D}^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ the semiderivative is defined as

$$d_k J(\Omega; \theta) \stackrel{\text{def}}{=} \lim_{t \searrow 0} \frac{J\big((I + t\theta)(\Omega)\big) - J(\Omega)}{t}. \tag{3.28}$$

3.3.3 Cartesian Graphs

In many applications it is convenient to work with domains Ω which are the hypograph of some positive function γ in Cartesian coordinates. Such domains are typically of the form

$$\Omega \stackrel{\text{def}}{=} \Big\{ (x', x_N) \in \mathbf{R}^{\mathrm{N}} : x' \in U \subset \mathbf{R}^{N-1} \text{ and } 0 < x_N < \gamma(x') \Big\}, \tag{3.29}$$

where U is a connected open set in $\mathbf{R}^{N-1}$ and $\gamma \in C^0(\overline{U}; \mathbf{R}_+)$ is a positive function. Many free boundary and contact problems are formulated over such domains. Usually the domains Ω (and hence the functions γ) will be constrained. The function γ can be specified on $\overline{U} \backslash U$ or not. In some examples the derivative of γ could also be specified, that is, $\partial\gamma/\partial\nu = g$ on ∂U, where U is smooth and ν is the outward unit normal field along ∂U.

When γ (resp., $\partial\gamma/\partial\nu$) is specified along ∂U, the directions of deformation μ are chosen in $C^0(U; \mathbf{R}_+)$ such that

$$\mu = 0 \text{ (resp., } \partial\mu/\partial\nu = 0) \text{ on } \partial U. \tag{3.30}$$

For small $t \geq 0$ and each such μ, define the perturbed domain

$$\Omega_t = \left\{ (x', x_N) \in \mathbf{R}^{\mathrm{N}} : 0 < x_N < \gamma(x') + t\mu(x') \right\} \tag{3.31}$$

and the family of transformations

$$(x', x_N) \mapsto T_t(x', x_N) = \left(x', x_N + \frac{x_N}{\gamma(x')} t\mu(x') \right) : \Omega \to \Omega_t, \tag{3.32}$$

which can be extended to a neighborhood D of Ω containing the perturbed domains Ω_t, $0 \leq t \leq t_1$, for some small $t_1 > 0$. In general, D will be such that $\overline{D} = \overline{U} \times [0, L]$ for some $L > 0$.

This construction is also appropriate for domains that are Lipschitzian or of class C^k, $1 \leq k \leq \infty$, that is, when γ is a Lipschitzian or a C^k-function. Again the transformation T_t is equal to $I + t\theta$, where

$$\theta(x', x_N) \stackrel{\text{def}}{=} \left(0, x_N \frac{\mu(x')}{\gamma(x')}\right). \tag{3.33}$$

But of course this θ is not the only choice for that $\Omega_t = T_t(\Omega)$. For instance, let $\lambda : \mathbf{R}_+ \to \mathbf{R}_+$ be any smooth increasing function such that $\lambda(0) = 0$ and $\lambda(1) = 1$. Then we could consider the transformations

$$T_t(x', x_N) \stackrel{\text{def}}{=} \left(x', x_N + t\lambda\left(\frac{x_N}{\gamma(x')}\right)\mu(x')\right). \tag{3.34}$$

This example illustrates the general principle that *the transformation* T_t *of* $\overline{D}$ *such that* $\Omega_t = T_t(\Omega)$ *is not unique.*

This means that, at least for smooth domains, only the trace $T_t|_{\Gamma_t}$ on Γ_t is important, while the displacement of the inner points does not contribute to the definition of Ω_t. Nevertheless this statement is to be interpreted with caution. Here we implicitly assume that the objective function $J(\Omega)$ and the associated constraints are only a function of the shape of Ω. However, in some problems involving singularities at inner points of Ω (e.g., when the solution $y(\Omega)$ of the state equation has a singularity or when some constraints on the domain are active), the situation might require a finer analysis. One such example is the internal displacement of the interior nodes of a triangularization τ_h when the solution $y(\Omega)$ of a partial differential equation is approximated by a piecewise polynomial solution over the triangularized domain Ω_h in the finite element method. Such a displacement does not change the shape of Ω_h, but it does change the solution y_h of the problem. When the displacement of the interior nodes is a priori parametrized by the boundary nodes the solution y_h will depend only on the position of the boundary nodes, but the interior nodes will contribute through the choice of the specified parametrization.

3.3.4 Polar Coordinates and Star-Shaped Domains

In some examples domains are star-shaped with respect to a point. Since a domain can always be translated, there is no loss of generality in assuming that this point is the origin. Then such domains Ω can be parametrized as follows:

$$\Omega \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : x = \rho\zeta,\, \zeta \in S_{N-1},\, 0 \le \rho < f(\zeta)\right\}, \tag{3.35}$$

where S_{N-1} is the unit sphere in $\mathbf{R}^{\mathrm{N}}$,

$$S_{N-1} \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : |x| = 1\right\}, \tag{3.36}$$

and $f : S_{N-1} \to \mathbf{R}_+$ is a positive continuous mapping from S_{N-1} such that

$$m \stackrel{\text{def}}{=} \min\{f(\zeta) : \zeta \in S_{N-1}\} > 0. \tag{3.37}$$

Given any $g \in C^0(S_{N-1})$ and a sufficiently small $t \ge 0$ the perturbed domains are defined as

$$\Omega_t \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : x = \rho\zeta, \zeta \in S_{N-1}, 0 \le \rho < f(\zeta) + tg(\zeta)\right\}. \tag{3.38}$$

For example, choose t, $0 \le t \le t_1$, for some

$$t_1 = \frac{m}{\|g\|_{C^0(S_{N-1})}} > 0 \tag{3.39}$$

and define the transformation T_t as

$$\begin{cases} T_t(X) = 0, & \text{if } X = 0, \\ T_t(X) = \left[\rho + t\frac{\rho}{f(\zeta)} g(\zeta)\right] \zeta, & \text{if } X = \rho\zeta \neq 0. \end{cases} \tag{3.40}$$

As in the previous example, T_t is not unique and for any continuous increasing function $\lambda : \mathbf{R}_+ \to \mathbf{R}_+$ such that $\lambda(0) = 0$ and $\lambda(t) = 1$ the transformation

$$\begin{cases} T_t(X) = 0, & \text{if } X = 0, \\ T_t(X) = \left[\rho + t\lambda\left(\frac{\rho}{f(\zeta)}\right) g(\zeta)\right] \zeta, & \text{if } X = \rho\zeta \neq 0, \end{cases} \tag{3.41}$$

yields the same domain Ω_t.

3.3.5 Level Sets

In sections 3.3.1 to 3.3.4, the perturbed domain Ω_t always appears in the form $\Omega_t = T_t(\Omega)$, where T_t is a bijective transformation of $\mathbf{R}^{\mathrm{N}}$ and T_t is of the form $I + t\theta$. In some free boundary problems (e.g., plasma physics, propagation of fronts) the free boundary Γ is a level curve of a smooth function u defined over an open domain D. Assume that D is bounded open with smooth boundary ∂D. Let $u \in C^2(\overline{D})$ be a positive function on $\overline{D}$ such that

$$\begin{cases} u \ge 0 \text{ in } \overline{D}, \quad u = 0 \text{ on } \partial D, \\ \exists \text{ a unique } x_u \in D \text{ such that } \forall x \in \overline{D} - \{x_u\}, \quad |\nabla u(x)| > 0. \end{cases} \tag{3.42}$$

If $m = \max\left\{u(x) : x \in \overline{D}\right\}$, then for each t in $[0, m[$ the level set

$$\Gamma_t = u^{-1}(t) \tag{3.43}$$

is a C^2-submanifold of $\mathbf{R}^{\mathrm{N}}$ in D, which is the boundary of the open set

$$\Omega_t = \{x \in D : u(x) > t\}. \tag{3.44}$$

By definition $\Omega_0 = D$, for all $t_1 > t_2$, $\Omega_{t_1} \subset \Omega_{t_2}$, and the domains Ω_t converge in the Hausdorff complementary topology[5] to the point x_u. The outward unit normal field on Γ_t is given by

$$x \in \Gamma_t, \quad n_t(x) = -|\nabla u(x)|^{-1} \nabla u(x). \tag{3.45}$$

This suggests introducing the velocity field

$$\forall x \in D - \{x_u\}, \quad V(x) \stackrel{\text{def}}{=} |\nabla u(x)|^{-2} \nabla u(x), \tag{3.46}$$

which is continuous everywhere but at $x = x_u$. If V were continuous everywhere, then for each X, the path $x(t; X)$ generated by the differential equation

$$\frac{dx}{dt}(t) = V\big(x(t)\big), \quad x(0) = X \tag{3.47}$$

[5]Cf. section 2 of Chapter 6.

would have the property that

$$u\big(x(t)\big) = u(X) + t \tag{3.48}$$

since formally

$$\frac{d}{dt}u\big(x(t)\big) = \nabla u\big(x(t)\big) \cdot \frac{dx}{dt}(t) = 1. \tag{3.49}$$

This means that the map $X \mapsto T_t(X) = x(t; X)$ constructed from (3.47) would map the level set

$$\Gamma_0 = \big\{X \in \overline{D} : u(X) = 0\big\} \tag{3.50}$$

onto the level set

$$\Gamma_t = \big\{x \in \overline{D} : u(x) = t\big\} \tag{3.51}$$

and eventually Ω_0 onto Ω_t. Unfortunately, it is easy to see that this last property fails on the function $u(x) = 1 - x^2$ defined on the unit disk.

To get around this difficulty, introduce for some arbitrarily small $\varepsilon, 0 < \varepsilon < m/2$, an infinitely differentiable function $\rho_\varepsilon : \mathbf{R}^{\mathrm{N}} \to [0,1]$ such that

$$\rho_\varepsilon(x) = \begin{cases} 0, & \text{if } |\nabla u(x)| < \varepsilon, \\ 1, & \text{if } |\nabla u(x)| > 2\varepsilon, \end{cases} \tag{3.52}$$

and the velocity

$$V_\varepsilon(x) = \rho_\varepsilon(x) V(x), \quad x \in \overline{D}. \tag{3.53}$$

As above, define the transformation

$$X \mapsto T_t^\varepsilon(X) = x(t; X), \tag{3.54}$$

where $x(t; X)$ is the solution of the differential equation

$$\frac{dx}{dt}(t) = V_\varepsilon\big(x(t)\big), \quad t \geq 0, \quad x(0) = X \in \overline{D}. \tag{3.55}$$

For $0 \leq t < m - 2\varepsilon$, T_t maps Γ_0 onto Γ_t; for $0 \leq s < m - \varepsilon$ such that $s + t < m - \varepsilon$, T_t maps Γ_s onto Γ_{t+s}. However, for $s > m - \varepsilon$, T_t is the identity operator. As a result, for $0 \leq t < m - 2\varepsilon$,

$$T_t(\Omega_0) = \Omega_t \quad \text{and} \quad T_t(\Gamma_0) = \Gamma_t. \tag{3.56}$$

Of course $\varepsilon > 0$ is arbitrary and we can make the construction for t's arbitrary close to m. This is an example that can be handled by the velocity (speed) method and not by a perturbation of the identity. Here the domains Ω_t are implicitly constrained to stay within the larger domain D. We shall see in section 5 how to introduce and characterize such a constraint.

Another example of description by level sets is provided by the *oriented distance function* b_Ω for some open domain Ω of class C^2 with compact boundary Γ (cf. Chapter 7). We shall see that there exists $h > 0$ and a neighborhood

$$U_h(\Gamma) = \big\{x \in \mathbf{R}^{\mathrm{N}} : |b_\Omega(x)| < h\big\}$$

such that $b_\Omega \in C^2(U_h(\Gamma))$. Then for $0 \leq t < h$ the flow corresponding to the velocity field $V = \nabla b_\Omega$ maps Ω and its boundary Γ onto

$$T_t(\Omega) = \Omega_t = \big\{x \in \mathbf{R}^{\mathrm{N}} : b_\Omega(x)| < t\big\}, \quad T_t(\Gamma) = \Gamma_t = \big\{x \in \mathbf{R}^{\mathrm{N}} : b_\Omega(x)| = t\big\}.$$

4 Unconstrained Families of Domains

Here we study equivalences between the *velocity method* (cf. J.-P. Zolésio [**12**, **8**]) and methods using a family of transformations. In section 4.1, we give some general conditions to construct a family of transformations of $\mathbf{R}^N$ from a velocity field. Conversely, we show how to construct a velocity field from a family of transformations of $\mathbf{R}^N$. In section 4.2, this construction is applied to Lipschitzian perturbations of the identity. In section 4.3, the various equivalences of section 4.1 are specialized to velocities in $C_0^{k+1}(\mathbf{R}^N,\mathbf{R}^N)$, $C^{k+1}(\overline{\mathbf{R}^N},\mathbf{R}^N)$, and $C^{k,1}(\overline{\mathbf{R}^N},\mathbf{R}^N)$, $k \geq 0$.

4.1 Equivalence between Velocities and Transformations

Let the real number $\tau > 0$ and the map $V : [0,\tau]\times\mathbf{R}^N \to \mathbf{R}^N$ be given. If t is interpreted as an artificial time, the map V can be viewed as the (time-dependent) velocity field $\{V(t) : 0 \leq t \leq \tau\}$ defined on $\mathbf{R}^N$:

$$x \mapsto V(t)(x) \overset{\text{def}}{=} V(t,x) : \mathbf{R}^N \mapsto \mathbf{R}^N\,. \tag{4.1}$$

Assume that there exists $\tau = \tau(V) > 0$ such that

$$\text{(V)}\qquad \begin{aligned} &\forall x \in \mathbf{R}^N, \quad V(\cdot,x) \in C\big([0,\tau];\mathbf{R}^N\big),\\ &\exists c > 0,\ \forall x,y \in \mathbf{R}^N, \quad \|V(\cdot,y)-V(\cdot,x)\|_{C([0,\tau];\mathbf{R}^N)} \leq c|y-x|, \end{aligned} \tag{4.2}$$

where $V(\cdot,x)$ is the function $t \mapsto V(t,x)$. Note that V is continuous on $[0,\tau]\times\mathbf{R}^N$. Hence it is uniformly continuous on $[0,\tau]\times D$ for any bounded open subset D of $\mathbf{R}^N$ and

$$V(\cdot) \in C([0,\tau];C(\overline{D};\mathbf{R}^N)). \tag{4.3}$$

Associate with V the solution $x(t;V)$ of the vector ordinary differential equation

$$\frac{dx}{dt}(t) = V\big(t,x(t)\big), \quad t \in [0,\tau], \quad x(0) = X \in \mathbf{R}^N \tag{4.4}$$

and define the transformations

$$X \mapsto T_t(V)(X) \overset{\text{def}}{=} x_V(t;X) : \mathbf{R}^N \to \mathbf{R}^N \tag{4.5}$$

and the maps (whenever the inverse of T_t exists)

$$(t,X) \mapsto T_V(t,X) \overset{\text{def}}{=} T_t(V)(X) : [0,\tau]\times\mathbf{R}^N \to \mathbf{R}^N, \tag{4.6}$$

$$(t,x) \mapsto T_V^{-1}(t,x) \overset{\text{def}}{=} T_t^{-1}(V)(x) : [0,\tau]\times\mathbf{R}^N \to \mathbf{R}^N\,. \tag{4.7}$$

Notation 4.1.
In what follows we shall drop the V in $T_V(t,X)$, $T_V^{-1}(t,x)$, and $T_t(V)$ whenever no confusion arises. □

Theorem 4.1.

(i) *Under assumptions* (V) *the map* T *specified by* (4.4)–(4.6) *has the following properties:*

$$
\begin{aligned}
&\text{(T1)} \quad && \forall X \in \mathbf{R}^{\mathrm{N}}, \quad T(\cdot, X) \in C^1\big([0,\tau]; \mathbf{R}^{\mathrm{N}}\big) \text{ and } \exists c > 0, \\
& && \forall X, Y \in \mathbf{R}^{\mathrm{N}}, \quad \|T(\cdot, Y) - T(\cdot, X)\|_{C^1([0,\tau];\mathbf{R}^{\mathrm{N}})} \leq c|Y - X|, \\
&\text{(T2)} && \forall t \in [0,\tau],\ X \mapsto T_t(X) = T(t, X) : \mathbf{R}^{\mathrm{N}} \to \mathbf{R}^{\mathrm{N}} \text{ is bijective}, \\
&\text{(T3)} && \forall x \in \mathbf{R}^{\mathrm{N}}, \quad T^{-1}(\cdot, x) \in C\big([0,\tau]; \mathbf{R}^{\mathrm{N}}\big) \text{ and } \exists c > 0, \\
& && \forall x, y \in \mathbf{R}^{\mathrm{N}}, \quad \|T^{-1}(\cdot, y) - T^{-1}(\cdot, x)\|_{C([0,\tau];\mathbf{R}^{\mathrm{N}})} \leq c|y - x|.
\end{aligned}
\tag{4.8}
$$

(ii) *Given a real number* $\tau > 0$ *and a map* $T : [0,\tau] \times \mathbf{R}^{\mathrm{N}} \to \mathbf{R}^{\mathrm{N}}$ *satisfying assumptions* (T1) *to* (T3), *the map*

$$
(t, x) \mapsto V(t, x) \stackrel{\text{def}}{=} \frac{\partial T}{\partial t}\big(t, T_t^{-1}(x)\big) : [0,\tau] \times \mathbf{R}^{\mathrm{N}} \to \mathbf{R}^{\mathrm{N}} \tag{4.9}
$$

satisfies conditions (V), *where* T_t^{-1} *is the inverse of* $X \mapsto T_t(X) = T(t, X)$. *If, in addition,* $T(0, \cdot) = I$, *then* $T(\cdot, X)$ *is the solution of* (4.4) *for that* V.

(iii) *Given a real number* $\tau > 0$ *and a map* $T : [0,\tau] \times \mathbf{R}^{\mathrm{N}} \to \mathbf{R}^{\mathrm{N}}$ *satisfying assumptions* (T1) *and* (T2) *and* $T(0, \cdot) = I$, *then there exists* $\tau' > 0$ *such that the conclusions of part* (ii) *hold on* $[0, \tau']$.

A more general version of this theorem for constrained domains (Theorem 5.1) will be given and proved in section 5.1.

Proof. (i) *Conditions* (T1) follow by standard arguments.

Condition (T2). Associate with X in $\mathbf{R}^{\mathrm{N}}$ the function

$$
y(s) = T_{t-s}(X), \quad 0 \leq s \leq t.
$$

Then

$$
\frac{dy}{ds}(s) = -V(t - s, y(s)), \quad 0 \leq s \leq t, \quad y(0) = T_t(X). \tag{4.10}
$$

For each $x \in \mathbf{R}^{\mathrm{N}}$, the differential equation

$$
\frac{dy}{ds}(s) = -V(t - s, y(s)), \quad 0 \leq s \leq t, \quad y(0) = x \in \mathbf{R}^{\mathrm{N}} \tag{4.11}
$$

has a unique solution in $C^1([0,t]; \mathbf{R}^{\mathrm{N}})$. The solutions of (4.11) define the map

$$
x \mapsto S_t(x) \stackrel{\text{def}}{=} y(t) : \mathbf{R}^{\mathrm{N}} \to \mathbf{R}^{\mathrm{N}}
$$

such that

$$
\exists c > 0,\ \forall t \in [0,\tau],\ \forall x, y \in \mathbf{R}^{\mathrm{N}}, \quad |S_t(y) - S_t(x)| \leq c|y - x|. \tag{4.12}
$$

In view of (4.10) and (4.11)

$$S_t(T_t(X)) = y(t) = T_{t-t}(X) = X \quad \Rightarrow S_t \circ T_t = I \text{ on } \mathbf{R}^{\mathrm{N}}.$$

To obtain the other identity, consider the function

$$z(r) = y(t-r; x),$$

where $y(\cdot, x)$ is the solution of (4.11) for some arbitrary x in $\mathbf{R}^{\mathrm{N}}$. By definition

$$\frac{dz}{dr}(r) = V(r, z(r)), \quad z(0) = y(t, x),$$

and necessarily

$$\begin{aligned} x = y(0; x) = z(t) = T_t(y(t; x)) = T_t(S_t(x)) \\ \Rightarrow T_t \circ S_t = I \text{ on } \mathbf{R}^{\mathrm{N}} \quad \Rightarrow S_t = T_t^{-1} : \mathbf{R}^{\mathrm{N}} \to \mathbf{R}^{\mathrm{N}}. \end{aligned}$$

Conditions (T3). The uniform Lipschitz continuity in (T3) follows from (4.12), and we need only show that

$$\forall x \in \mathbf{R}^{\mathrm{N}}, \quad T^{-1}(\cdot, x) \in C([0, \tau]; \mathbf{R}^{\mathrm{N}}).$$

Given t in $[0, \tau]$ pick an arbitrary sequence $\{t_n\}$, $t_n \to t$. Then for each $x \in \mathbf{R}^{\mathrm{N}}$ there exists $X \in \mathbf{R}^{\mathrm{N}}$ such that

$$T_t(X) = x \quad \text{and} \quad T_{t_n}(X) \to T_t(X) = x$$

from the first condition (T1). But

$$\begin{aligned} T_{t_n}^{-1}(x) - T_t^{-1}(x) &= T_{t_n}^{-1}\left(T_t(X)\right) - T_t^{-1}\left(T_t(X)\right) \\ &= T_{t_n}^{-1}\left(T_t(X)\right) - T_{t_n}^{-1}\left(T_{t_n}(X)\right). \end{aligned}$$

By the uniform Lipschitz continuity of T_t^{-1}

$$|T_{t_n}^{-1}(x) - T_t^{-1}(x)| = |T_{t_n}^{-1}\left(T_t(X)\right) - T_{t_n}^{-1}\left(T_{t_n}(X)\right)| \leq c|T_t(X) - T_{t_n}(X)|,$$

and the last term converges to zero as t_n goes to t.

(ii) The first part of conditions (V) is satisfied since, for each $x \in \mathbf{R}^{\mathrm{N}}$ and t, s in $[0, \tau]$,

$$\begin{aligned} &|V(t, x) - V(s, x)| \\ &\leq \left| \frac{\partial T}{\partial t}(t, T_t^{-1}(x)) - \frac{\partial T}{\partial t}(t, T_s^{-1}(x)) \right| + \left| \frac{\partial T}{\partial t}(t, T_s^{-1}(x)) - \frac{\partial T}{\partial t}(s, T_s^{-1}(x)) \right| \\ &\leq c|T_t^{-1}(x) - T_s^{-1}(x)| + \left| \frac{\partial T}{\partial t}(t, T_s^{-1}(x)) - \frac{\partial T}{\partial t}(s, T_s^{-1}(x)) \right|. \end{aligned}$$

Thus from (T3) and (T1) $t \mapsto V(t,x)$ is continuous at $s = t$, and hence for all x in $\mathbf{R}^N$, $V(.,x) \in C([0,\tau];\mathbf{R}^N)$. The Lipschitzian property follows directly from the Lipschitzian properties (T1) and (T3): for all x and y in $\mathbf{R}^N$,

$$\begin{aligned}|V(t,y) - V(t,x)| &= \left|\frac{\partial T}{\partial t}(t, T_t^{-1}(y)) - \frac{\partial T}{\partial t}(t, T_t^{-1}(x))\right| \\ &\leq c\left|T_t^{-1}(y) - T_t^{-1}(x)\right| \leq cc'|y-x|.\end{aligned}$$

This proves that V satisfies condition (V).

(iii) From (T1) and (T2) for $f(t) = T_t - I$ and $t \geq s$,

$$\begin{aligned}f(t)(y) - f(t)(x) &= \int_0^t \frac{\partial T}{\partial t}(r, T_r(y)) - \frac{\partial T}{\partial t}(r, T_r(x))\, dr, \\ |f(t)(y) - f(t)(x)| &\leq \int_0^t c\,|T_r(y) - T_r(x)|\, dr \leq c^2\, t\, |y-x|.\end{aligned}$$

For $\tau' = \min\{\tau, 1/(2c^2)\}$ and $0 \leq t \leq \tau'$, $c(f(t)) \leq 1/2$,

$$\begin{aligned}g(t) = T_t^{-1} - I = (I - T_t) \circ T_t^{-1} &= -f(t) \circ [I + g(t)], \\ (1 - c(f(t)))\, c(g(t)) \leq c(f(t)) \quad &\Rightarrow c(g(t)) \leq 1 \text{ and } c(T_t^{-1}) \leq 2,\end{aligned}$$

and the second condition (T3) is satisfied on $[0,\tau']$. The first one follows by the same argument as in part (i). Therefore the conclusions of part (ii) are true on $[0,\tau']$. $\square$

This equivalence theorem says that we can start either from a family of velocity fields $\{V(t)\}$ on $\mathbf{R}^N$ or a family of transformations $\{T_t\}$ of $\mathbf{R}^N$ provided that the map V, $V(t,x) = V(t)(x)$, satisfies (V) or the map T, $T(t,X) = T_t(X)$, satisfies (T1) to (T3).

Starting from V, the family of homeomorphisms $\{T_t(V)\}$ generates the family

$$\Omega_t \stackrel{\text{def}}{=} T_t(V)(\Omega) = \{T_t(V)(X) : X \in \Omega\} \tag{4.13}$$

of perturbations of the initial domain Ω. Interior (resp., boundary) points of Ω are mapped onto interior (resp., boundary) points of Ω_t. This is the basis of the *velocity method* which will be used to define shape derivatives.

4.2 Perturbations of the Identity

In examples it is usually possible to show that the transformation T satisfies assumptions (T1) to (T3) and construct the corresponding velocity field V defined in (4.9). For instance, consider perturbations of the identity to the first ($A = 0$) or second order: for $t \geq 0$ and $X \in \mathbf{R}^N$,

$$T_t(X) \stackrel{\text{def}}{=} X + tU(X) + \frac{t^2}{2}A(X), \tag{4.14}$$

where U and A are transformations of $\mathbf{R}^N$. It turns out that for Lipschitzian transformations U and A, assumptions (T1) to (T3) are satisfied in some interval $[0,\tau]$.

Theorem 4.2. *Let U and A be two uniform Lipschitzian transformations of $\mathbf{R}^N$: there exists $c > 0$ such that for all $X, Y \in \mathbf{R}^N$,*

$$|U(Y) - U(X)| \le c|Y - X| \quad \textit{and} \quad |A(Y) - A(X)| \le c|Y - X|.$$

There exists $\tau > 0$ such that the map T given by (4.14) *satisfies conditions* (T1) *to* (T3) *on $[0, \tau]$. The associated velocity V given by*

$$(t, x) \mapsto V(t, x) = U\big(T_t^{-1}(x)\big) + tA\big(T_t^{-1}(x)\big) : [0, \tau] \times \mathbf{R}^N \to \mathbf{R}^N \tag{4.15}$$

satisfies conditions (V) *on $[0, \tau]$.*

Remark 4.1.
Observe that from (4.14) and (4.15)

$$V(0) = U, \quad \dot{V}(0)(x) = \frac{\partial V}{\partial t}(t, x)\big|_{t=0} = A - [DU]U, \tag{4.16}$$

where DU is the Jacobian matrix of U. The term $\dot{V}(0)$ is an *acceleration* at $t = 0$ which will always be present even when $A = 0$, but it can be eliminated by choosing $A = [DU]U$. □

Proof. (i) By the definition of T in (4.14), $t \mapsto T(t, X)$ and $t \mapsto \frac{\partial T}{\partial t}(t, X) = U(X) + tA(X)$ are continuous on $[0, \infty[$. Moreover, for all X and Y,

$$|T(t, Y) - T(t, X)| \le \left[1 + tc + \frac{t^2}{2}c\right] |Y - X|$$

and

$$\left|\frac{\partial T}{\partial t}(t, Y) - \frac{\partial T}{\partial t}(t, X)\right| \le [c + tc]\, |Y - X|.$$

Thus conditions (T1) are satisfied for any finite $\tau > 0$. To check condition (T2), consider for any $Y \in \mathbf{R}^N$ the mapping $h(X) \stackrel{\text{def}}{=} Y - [T_t(X) - X]$. For any X_1 and X_2

$$\begin{aligned} |h(X_2) - h(X_1)| &\le t|U(X_2) - U(X_1)| + \frac{t^2}{2}|A(X_2) - A(X_1)| \\ &\le t\,c\,|X_2 - X_1| + \frac{t^2}{2}\,c\,|X_2 - X_1| = t\,c\,[1 + t/2]\,|X_2 - X_1|. \end{aligned} \tag{4.17}$$

For $\tau = \min\{1, 1/(4c)\}$, and any t, $0 \le t \le \tau$, $t\,c\,[1 + t/2] < 1/2$ and h is a contraction. So for all $0 \le t \le \tau$ and $Y \in \mathbf{R}^N$, there exists a unique $X \in \mathbf{R}^N$ such that

$$Y - [T_t(X) - X] = h(X) = X \iff T_t(X) = Y$$

and T_t is bijective. Therefore (T2) is satisfied in $[0, \tau']$. The last part of the proof is the uniform Lipschitzian property of T_t^{-1}. In view of (4.17), for all t, $0 \le t \le \tau$,

$t\,c\,[1+t/2] < 1/2$ and

$$\begin{aligned} |T_t(X_2) - X_2 - (T_t(X_1) - X_1)| &= |h(X_2) - h(X_1)| \le \frac{1}{2}\,|X_2 - X_1| \\ \Rightarrow\ |X_2 - X_1| - |T_t(X_2) - T_t(X_1)| &\le \frac{1}{2}\,|X_2 - X_1| \\ \Rightarrow\ |X_2 - X_1| &\le 2\,|T_t(X_2) - T_t(X_1)|. \end{aligned}$$

In view of condition (T2) for all x and y,

$$|T_t^{-1}(y) - T_t^{-1}(x)| \ \le 2\,|T_t(T_t^{-1}(y)) - T_t(T_t^{-1}(x))| \ = 2\,|y - x|. \tag{4.18}$$

To complete our argument we prove the continuity with respect to t for each x. Let $X = T_t^{-1}(x)$. For any s in $[0,\tau]$

$$T_s^{-1}(x) - T_t^{-1}(x) = T_s^{-1}(T_t(X)) - T_t^{-1}(T_t(X)) = T_s^{-1}(T_t(X)) - T_s^{-1}(T_s(X)),$$

and in view of (4.18)

$$|T_s^{-1}(x) - T_t^{-1}(x)| \ \le\ 2|T_t(X) - T_s(X)|.$$

The continuity of $T_s^{-1}(x)$ at $s = t$ now follows from the continuity of $T_s(X)$ at $s = t$. Thus conditions (T3) are satisfied. □

4.3 Equivalence for Special Families of Velocities

In this section we specialize Theorem 4.1 to velocities in $C^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$, $C_0^{k+1}(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$, and $C^{k+1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$, $k \ge 0$. The following notation will be convenient:

$$f(t) \stackrel{\text{def}}{=} T_t - I, \quad f'(t) = \frac{dT_t}{dt}, \quad g(t) \stackrel{\text{def}}{=} T_t^{-1} - I,$$

whenever T_t^{-1} exists and the identities

$$\begin{aligned} g(t) &= -f(t) \circ T_t^{-1} = -f(t) \circ [I + g(t)], \\ V(t) &= \frac{dT_t}{dt} \circ T_t^{-1} = f'(t) \circ T_t^{-1} = f'(t) \circ [I + g(t)]. \end{aligned}$$

Recall also for a function $F : \mathbf{R}^{\mathrm{N}} \to \mathbf{R}^{\mathrm{N}}$ the notation

$$c(F) \stackrel{\text{def}}{=} \sup_{y \ne x} \frac{|F(y) - F(x)|}{|y - x|} \ \text{ and } \forall k \ge 1, \quad c_k(F) \stackrel{\text{def}}{=} \sum_{|\alpha| = k} c(\partial^\alpha F).$$

Theorem 4.3. *Let $k \ge 0$ be an integer.*

(i) *Given $\tau > 0$ and a velocity field V such that*

$$V \in C([0,\tau]; C^k(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})) \quad \textit{and} \quad c_k(V(t)) \le c \tag{4.19}$$

for some constant $c > 0$ *independent of* t*, the map* T *given by* (4.4)–(4.6) *satisfies conditions* (T1)*,* (T2)*, and*

$$f \in C^1([0,\tau]; C^k(\overline{\mathbf{R}^N}, \mathbf{R}^N)) \cap C([0,\tau]; C^{k,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)), \quad c_k(f'(t)) \le c, \tag{4.20}$$

for some constant $c > 0$ *independent of* t*. Moreover, conditions* (T3) *are satisfied and there exists* $\tau' > 0$ *such that*

$$g \in C([0,\tau']; C^k(\overline{\mathbf{R}^N}, \mathbf{R}^N)), \quad c_k(g(t)) \le ct, \tag{4.21}$$

for some constant c *independent of* t*.*

(ii) *Given* $\tau > 0$ *and* $T : [0,\tau] \times \mathbf{R}^N \to \mathbf{R}^N$ *satisfying conditions* (4.20) *and* $T(0,\cdot) = I$*, there exists* $\tau' > 0$ *such that the velocity field* $V(t) = f'(t) \circ T_t^{-1}$ *satisfies conditions* (V) *and* (4.19) *in* $[0,\tau']$*.*

Proof. We prove the theorem for $k = 0$. The general case is obtained by induction over k.

(i) By assumption on V, the conditions (V) given by (4.2) are satisfied and by Theorem 4.1 the corresponding family T satisfies conditions (T1) to (T3).

Conditions (4.20) *on* f. For any x and $s \le t$

$$\begin{aligned} T_t(x) - T_s(x) &= \int_s^t V(r) \circ T_r(x)\, dr, \\ |T_t(x) - T_s(x)| &\le \int_s^t c|T_r(x) - T_s(x)| + |V(r) \circ T_s(x)|\, dr, \\ |f(t)(x) - f(s)(x)| &\le \int_s^t c|f(r)(x) - f(s)(x)| + \|V(r)\|_C\, dr. \end{aligned}$$

By assumption on V and Gronwall's inequality,

$$\forall t, s \in [0,\tau], \quad \|f(t) - f(s)\|_C \le c\,|t-s|$$

for another constant c independent of t. Moreover,

$$\begin{aligned} &|(f(t) - f(s))(y) - (f(t) - f(s))(x)| = |(T_t - T_s)(y) - (T_t - T_s)(x)| \\ &\le \int_s^t |V(r) \circ T_r(y) - V(r) \circ T_r(x)|\, dr \\ &\le \int_s^t c|(T_r - T_s)(y) - (T_r - T_s)(x)| + c|T_s(y) - T_s(x)|\, dr \\ &\le \int_s^t c|(f(r) - f(s))(y) - (f(r) - f(s))(x)| + cc'|y-x|\, dr \end{aligned}$$

for some other constant c' by the second condition (T1). Again by Gronwall's inequality there exists another constant c such that

$$\begin{aligned} |(f(t) - f(s))(y) - (f(t) - f(s))(x)| &\le c|t-s|\,|y-x| \\ \Rightarrow c(f(t) - f(s)) &\le c\,|t-s| \end{aligned}$$

$$\Rightarrow \boxed{f \in C([0,\tau]; C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)) \text{ and } \|f(t) - f(s)\|_{C^{0,1}} \le c\,|t-s|.} \tag{4.22}$$

Moreover, $f'(t) = V(t) \circ T_t$ and

$$\begin{aligned}
|f'(t)(x) - f'(s)(x)| &\le |V(t)(T_t(x)) - V(s)(T_t(x))| \\
&\quad + |V(s)(T_t(x)) - V(s)(T_s(x))| \\
&\le \|V(t) - V(s)\|_C + c(V(s))\, \|T_t - T_s\|_C \\
&\le \|V(t) - V(s)\|_C + c\, \|f(t) - f(s)\|_C.
\end{aligned}$$

Finally,

$$\begin{aligned}
|f'(t)(y) - f'(t)(x)| &\le |V(t)(T_t(y)) - V(t)(T_t(x))| \\
&\le c(V(t))\, |T_t(y) - T_s(x)| \le c\, c(T_t)\, |y - x|
\end{aligned}$$

and $c(f'(t)) \le c$ for some new constant c independent of t. Therefore,

$$\boxed{f \in C^1([0, \tau]; C(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})) \text{ and } c(f'(t)) \le c.}$$

Conditions (4.21) *on* g. Since conditions (T1) and (T2) are satisfied there exists $\tau' > 0$ such that conditions (T3) are satisfied by Theorem 4.1 (iii). Moreover, from conditions (4.20)

$$\begin{aligned}
|g(t)(y) - g(t)(x)| &\le |f(t)(T_t^{-1}(y)) - f(t)(T_t^{-1}(x))| \\
&\le c(f(t))|T_t^{-1}(y) - T_t^{-1}(x)| \\
&\le c(f(t))\, (|g(t)(y) - g(t)(x)| + |y - x|) \\
\Rightarrow\ (1 - c(f(t)))\, |g(t)(y) - g(t)(x)| &\le c(f(t))\, |y - x| \le ct\, |y - x|.
\end{aligned}$$

Choose a new $\tau'' = \min\{\tau', 1/(2c)\}$. Then for $0 \le t \le \tau''$, $c(g(t)) \le 2ct$. Now

$$\begin{aligned}
g(t) - g(s) &= -f(t) \circ [I + g(t)] + f(s) \circ [I + g(s)], \\
\|g(t) - g(s)\|_C &\le \|f(t) \circ [I + g(t)] - f(t) \circ [I + g(s)]\|_C \\
&\quad + \|f(t) \circ [I + g(s)] - f(s) \circ [I + g(s)]\|_C \\
&\le c(f(t))\|g(t) - g(s)\|_C + \|f(t) - f(s)\|_C \\
&\le ct\, \|g(t) - g(s)\|_C + \|f(t) - f(s)\|_C.
\end{aligned}$$

For t in $[0, \tau'']$, $ct \le 1/2$, and

$$\begin{aligned}
&\|g(t) - g(s)\|_C \le 2\|f(t) - f(s)\|_C \\
\Rightarrow\ &\boxed{g \in C([0, \tau'']; C(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})) \text{ and } c(g(t)) \le 2ct.}
\end{aligned}$$

The conditions (4.20) on f are satisfied for $k = 0$. For $k = 1$ we start from the equation

$$DT_t - DT_s = \int_s^t DV(r) \circ T_r\, DT_r\, dr$$

and use the fact that $DT_t^{-1} = [DT_t]^{-1} \circ T_t^{-1}$ in connection with the identity

$$Dg(t) = -Df(t) \circ T_t^{-1}\, DT_t^{-1} = -(Df(t)[DT_t]^{-1}) \circ T_t^{-1}.$$

(ii) From conditions (4.20) on f, the transformation T satisfies conditions (T1). To check condition (T2) we consider two cases: $k \geq 1$ and $k = 0$. For $k \geq 1$ the function $t \mapsto Df(t) = DT_t - I : [0,\tau] \to C^{k-1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})^N$ is continuous. Hence $t \mapsto \det DT_t : [0,\tau] \to \mathbf{R}$ is continuous and $\det DT_0 = 1$. So there exists $\tau' > 0$ such that T_t is invertible for all t in $[0,\tau']$ and (T2) is satisfied in $[0,\tau']$. In the case $k = 0$ consider for any Y the map $h(X) = Y - f(t)(X)$. For any X_1 and X_2, $|h(X_2) - h(X_1)| \leq c(f(t))\,|X_2 - X_1|$. But by assumption $f \in C([0,\tau]; \mathcal{C}^{0,1}(\overline{\mathbf{R}^{\mathrm{N}}}))$ and $c(f(0)) = 0$ since $f(0) = 0$. Hence there exists $\tau' > 0$ such that $c(f(t)) \leq 1/2$ for all t in $[0,\tau']$ and h is a contraction. So for all Y in $\mathbf{R}^{\mathrm{N}}$ there exists a unique X such that

$$Y - [T_t(X) - X] = h(X) = X \iff T_t(X) = Y,$$

T_t is bijective, and condition (T2) is satisfied in $[0,\tau']$. By Theorem 4.1 (iii) from (T1) and (T2), there exists another $\tau' > 0$ for which conditions (T3) on g and (V) on $V(t) = f'(t) \circ T_t^{-1}$ are also satisfied. Moreover, we have seen in the proof of part (i) that conditions (4.21) on g follow from (T2) and (4.20). Using conditions (4.20) and (4.21),

$$\begin{aligned}|V(t)(y) - V(t)(x)| &\leq |f'(t)(T_t^{-1}(y)) - f'(t)(T_t^{-1}(x))| \\ &\leq c(f'(t))\,|T_t^{-1}(y) - T_t^{-1}(x)| \\ &\leq c(f'(t))\,[1 + c(g(t))]\,|y - x| \leq c'|y - x|\end{aligned}$$

and $c(V(t)) \leq c'$. Also

$$\begin{aligned}|V(t)(x) - V(s)(x)| &= |f'(t)(T_t^{-1}(x)) - f'(s)(T_s^{-1}(x))| \\ &\leq |f'(t)(T_t^{-1}(x)) - f'(t)(T_s^{-1}(x))| \\ &\quad + |f'(t)(T_s^{-1}(x)) - f'(s)(T_s^{-1}(x))| \\ &\leq c(f'(t))\,|T_t^{-1}(x) - T_s^{-1}(x)| + \|f'(t) - f'(s)\|_C \\ &\leq c\,\|g(t) - g(s)\|_C + \|f'(t) - f'(s)\|_C.\end{aligned}$$

Therefore, since both g and f' are continuous,

$$\boxed{V \in C([0,\tau']; C(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})) \text{ and } c(V(t)) \leq c}$$

for some constant c independent of t. This proves the result for $k = 0$. As in part (i), for $k = 1$ we use the identity

$$DV(t) = Df'(t) \circ T_t^{-1}\, DT_t^{-1} = (Df'(t)[DT_t]^{-1}) \circ T_t^{-1}$$

and proceed in the same way. The general case is obtained by induction over k. □

We now turn to the case of velocities in $C_0^k(\mathbf{R}^N, \mathbf{R}^N)$. As in Chapter 2, it will be convenient to use the notation $\mathcal{C}_0^k$ for the space $C_0^k(\mathbf{R}^N, \mathbf{R}^N)$, $\mathcal{C}^k(\overline{\mathbf{R}^N})$ for the space $C^k(\overline{\mathbf{R}^N}, \mathbf{R}^N)$, and $\mathcal{C}^{k,1}(\overline{\mathbf{R}^N})$ for the space $C^{k,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)$.

Theorem 4.4. *Let $k \geq 1$ be an integer.*

(i) *Given $\tau > 0$ and a velocity field V such that*

$$V \in C([0,\tau]; C_0^k(\mathbf{R}^N, \mathbf{R}^N)), \tag{4.23}$$

the map T given by (4.4)–(4.6) *satisfies conditions* (T1), (T2), *and*

$$f \in C^1([0,\tau]; C_0^k(\mathbf{R}^N, \mathbf{R}^N)). \tag{4.24}$$

Moreover, conditions (T3) *are satisfied and there exists $\tau' > 0$ such that*

$$g \in C([0,\tau']; C_0^k(\mathbf{R}^N, \mathbf{R}^N)). \tag{4.25}$$

(ii) *Given $\tau > 0$ and $T : [0,\tau] \times \mathbf{R}^N \to \mathbf{R}^N$ satisfying conditions* (4.24) *and $T(0,\cdot) = I$, there exists $\tau' > 0$ such that the velocity field $V(t) = f'(t) \circ T_t^{-1}$ satisfies conditions* (V) *and* (4.23) *on $[0,\tau']$.*

Proof. As in the proof of Theorem 4.3, we prove only the theorem for $k = 1$. The general case is obtained by induction on k, the various identities on f, g, f', and V, and the techniques of Lemma 2.4, Theorems 2.11 and 2.12, and Lemmas 2.5 and 2.6 of section 2.5 in Chapter 3.

(i) By the embedding $C_0^1(\mathbf{R}^N, \mathbf{R}^N) \subset C^1(\overline{\mathbf{R}^N}, \mathbf{R}^N) \subset C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)$, it follows from (4.23) that $V \in C([0,\tau]; C^{0,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N))$ and condition (4.19) of Theorem 4.3 are satisfied. Therefore, conditions (4.20) and (4.21) of Theorem 4.3 are also satisfied in some interval $[0,\tau']$, $\tau' > 0$.

Conditions (4.24) *on f.* It remains to show that $f(t)$ and $f'(t)$ belong to the subspace $C_0(\mathbf{R}^N, \mathbf{R}^N)$ of $C(\overline{\mathbf{R}^N}, \mathbf{R}^N)$ and to prove the appropriate properties for $Df(t)$ and $Df'(t)$. Recall from the proof of the previous theorems that there exists $c > 0$ such that

$$|f(t)(x)| \leq c \int_0^t |V(r)(x)|\, dr \leq c \int_0^t |(V(r) - V(0))(x)|\, dr + c\, t\, |V(0)(x)|.$$

By assumption on $V(0)$, for $\varepsilon > 0$ there exists a compact set K such that

$$\forall x \in \complement K, \quad |V(0)(x)| \leq \varepsilon/(2c)$$

and there exists δ, $0 < \delta < 1$, such that

$$\begin{gathered} \forall t,\, 0 \leq t \leq \delta, \quad \|V(r) - V(0)\|_C \leq \varepsilon/(2c) \\ \Rightarrow \forall t,\, 0 \leq t \leq \delta,\, \forall x \in \complement K, \quad |f(t)(x)| \leq \varepsilon \quad \Rightarrow f(t) \in \mathcal{C}_0. \end{gathered}$$

Proceeding in this fashion from the interval $[0,\delta]$ to the next interval $[\delta, 2\delta]$ using the inequality

$$|(f(t) - f(s))(x)| \leq c \int_s^t |(V(r) - V(\delta))(x)|\, dr + c\, |t - \delta|\, |V(\delta)(x)|,$$

the uniform continuity of V

$$\forall t, s, \quad |t-s| < \delta, \quad \|V(t) - V(s)\|_C \le \varepsilon/(2c),$$

and the fact that $V(\delta) \in \mathcal{C}_0$, that is, that there exists a compact set $K(\delta)$ such that

$$\forall x \in \complement K(\delta), \quad |V(\delta)(x)| \le \varepsilon/(2c),$$

we get $f(t) \in \mathcal{C}_0$, $\delta \le t \le 2\delta$, and hence $f \in C([0,\tau];\mathcal{C}_0)$. For $f'(t)$ we make use of the identity $f'(t) = V(t) \circ T_t$. Again by assumption for any $\varepsilon > 0$ there exists a compact set $K(t)$ such that $|V(t)(x)| \le \varepsilon$ on $\complement K(t)$. Thus by choosing the compact set $K'_t = T_t^{-1}(K(t))$, $|f'(t)(x)| \le \varepsilon$ on $\complement K'_t$, and $f' \in C([0,\tau];\mathcal{C}_0)$. In order to complete the proof, it remains to establish the same properties for $Df(t)$ and $Df'(t)$. The matrix $Df(t)$ is solution of the equations

$$\begin{aligned} &\frac{d}{dt} Df(t) = DV(t) \circ T_t \, DT_t, \quad Df(0) = 0 \\ \Rightarrow\ &\boxed{Df'(t) = DV(t) \circ T_t \, Df(t) + DV(t) \circ T_t.} \end{aligned} \tag{4.26}$$

From the proofs of Theorems 2.11 and 2.12 in section 2.5 of Chapter 3, for each t the elements of the matrix

$$A(t) \stackrel{\text{def}}{=} DV(t) \circ T_t = DV(t) \circ [I + f(t)]$$

belong to $\mathcal{C}_0$ since $DV(t)$ and $f(t)$ do. By assumption, $V \in C([0,\tau];\mathcal{C}_0^k)$ and V and all its derivatives $\partial^\alpha V$ are uniformly continuous in $[0,\tau] \times \mathbf{R}^{\mathrm{N}}$. Therefore, for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall\, |t-s| < \delta, \forall\, |y'-x'| < \delta, \quad |DV(t)(y') - DV(s)(x')| < \varepsilon.$$

Pick $0 < \delta' < \delta$ such that

$$\begin{aligned} &\forall\, |t-s| < \delta', \quad \|T_t - T_s\|_C = \|f(t) - f(s)\|_C < \delta \\ &\Rightarrow \forall x, \forall\, |t-s| < \delta', \quad |T_t(x) - T_s(x)| < \delta \\ &\Rightarrow |DV(t)(T_t(x)) - DV(s)(T_s(x))| < \varepsilon \\ &\Rightarrow \|A(t) - A(s)\|_C < \varepsilon \quad \Rightarrow A \in C([0,\tau];(\mathcal{C}_0)^N). \end{aligned}$$

For each x, $Df(t)(x)$ is the unique solution of the linear matrix equation (4.26). To show that $Df(t) \in (\mathcal{C}_0)^N$ we first show that $Df(t)(x)$ is uniformly continuous for x in $\mathbf{R}^{\mathrm{N}}$. For any x and y

$$\begin{aligned} |Df(t)(y) - Df(t)(x)| &\le \int_0^t |V(r, T_r(y)) - V(r, T_r(x))| \, dr \\ &\le \int_0^t c|T_r(y) - T_r(x)| \, dr \\ &\le c \int_0^t |f(r)(y) - f(r)(x)| + |y-x| \, dr. \end{aligned}$$

But $f \in C([0,\tau];\mathcal{C}_0)$ is uniformly continuous in (t,x): for each $\varepsilon > 0$ there exists δ, $0 < \delta < \varepsilon/(2c\tau)$, such that

$$\forall\, |t-s| < \delta,\, \forall\, |y-x| < \delta, \quad |f(t)(y) - f(s)(x)| < \varepsilon/(2c\tau).$$

Substituting in the previous inequality for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall\, t,\, \forall\, |y-x| < \delta, \quad |Df(t)(y) - Df(t)(x)| < \varepsilon.$$

Hence $Df(t)$ is uniformly continuous in $\mathbf{R}^{\mathrm{N}}$. Furthermore, from (4.26) we have the following inequality:

$$\begin{aligned} |Df(t)(x)| &\le \int_0^t |DV(r)(T_r(x))|\, |Df(r)(x)| + |DV(r)(T_r(x))|\, dr \\ &\le c \int_0^t (|Df(r)(x)| + 1)\, dr \end{aligned} \tag{4.27}$$

since $V \in C([0,\tau];\mathcal{C}_0^1)$. By Gronwall's inequality $|Df(t)(x)| \le c\,t$ for some other constant c independent of t. Hence $Df(t) \in C(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})^N$. Finally to show that $Df(t)$ vanishes at infinity we start from the integral form of (4.26):

$$\begin{aligned} Df(t)(x) &= \int_0^t DV(r)(T_r(x))\, DT_r(x)\, dr, \\ |Df(t)(x)| &\le c \int_0^t |DV(r)(T_r(x)) - DV(r)(x)| + |DV(r)(x)|\, dr \\ &\le c' \int_0^t |f(r)(x)| + |DV(r)(x)|\, dr. \end{aligned}$$

By the same technique as for $f(t)$, it follows that the elements of $Df(t)$ belong to $\mathcal{C}_0$ since both $f(s)$ and $DV(r)$ do. Finally for the continuity with respect to t

$$\begin{aligned} Df(t) - Df(s) &= \int_s^t A(r)Df(r) + A(r)\, dr, \\ \|Df(t) - Df(s)\|_C &\le \int_s^t \|A(r)\|_C\, \|Df(r) - Df(s)\|_C \\ &\qquad + \|A(r)\|_C\, (1 + \|Df(r)\|_C)\, dr. \end{aligned}$$

Again, by Gronwall's inequality, there exists another constant c such that

$$\|Df(t) - Df(s)\|_C \le c\,|t-s|.$$

Therefore, $Df \in C([0,\tau];(\mathcal{C}_0)^N)$ and $f \in C([0,\tau];\mathcal{C}_0^1)$. For Df' we repeat the proof for f' using the identity

$$Df'(t) = DV(t) \circ T_t\, Df(t) + DV(t) \circ T_t$$

to get

$$Df' \in C([0,\tau];(\mathcal{C}_0)^N) \quad \Rightarrow\; f' \in C([0,\tau];\mathcal{C}_0^1).$$

Conditions (4.25) *on* g. From the remark at the beginning of part (i) of the proof, the conclusions of Theorem 4.3 are true for g, and it remains to check the remaining properties for g and Dg using the identities

$$g(t) = -f(t) \circ [I + g(t)], \quad Dg(t) = -Df(t) \circ [I + g(t)]\,(I + Dg(t)).$$

From the proof of Theorems 2.11 and 2.12 in section 2.5 of Chapter 3, $g(t) \in \mathcal{C}_0$ since $Df(t)$ and $g(t)$ belong to $\mathcal{C}_0$. Therefore, $g(t) \in \mathcal{C}_0^1$. The continuity follows by the same argument as for f' and $g \in C([0,\tau];\mathcal{C}_0^1)$.

(ii) By assumption from conditions (4.24), conditions (T1) are satisfied. For (T2) observe that for $k \geq 1$ the function $t \mapsto Df(t) = DT_t - I : [0,\tau] \to C^{k-1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})^N$ is continuous. Hence $t \mapsto \det DT_t : [0,\tau] \to \mathbf{R}$ is continuous and $\det DT_0 = 1$. So there exists $\tau' > 0$ such that T_t is invertible for all t in $[0,\tau']$ and (T2) is satisfied in $[0,\tau']$. Furthermore, from the proof of part (i) conditions (T3) and (4.25) on g are also satisfied in some interval $[0,\tau']$, $\tau' > 0$. Therefore, the velocity field

$$V(t) = f'(t) \circ T_t^{-1} = f'(t) \circ [I + g(t)]$$

satisfies the conditions (V) specified by (4.2) in $[0,\tau']$. From the proof of Theorems 2.11 and 2.12 in section 2.5 of Chapter 3, $V(t) \in \mathcal{C}_0^k$ since $f'(t)$ and $g(t)$ belong to $\mathcal{C}_0^k$. By assumption, $f \in C^1([0,\tau];\mathcal{C}_0^k)$. Hence f' and all its derivatives $\partial^\alpha f'$, $|\alpha| \leq k$, are uniformly continuous on $[0,\tau] \times \mathbf{R}^{\mathrm{N}}$; that is, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall t, s,\ |t-s| < \delta,\ \forall y', x',\ |y'-x'| < \delta, \quad |\partial^\alpha f'(t)(y') - \partial^\alpha f'(s)(x')| < \varepsilon.$$

Similarly $g \in C([0,\tau'];\mathcal{C}_0^k)$ and there exists $0 < \delta' \leq \delta$ such that

$$\forall t, s,\ |t-s| < \delta',\ \forall y, x,\ |y-x| < \delta', \quad |\partial^\alpha g(t)(y) - \partial^\alpha g(s)(x)| < \delta.$$

Therefore, for $|t-s| < \delta'$

$$\|T_t^{-1} - T_s^{-1}\|_C = \|g(t) - g(s)\|_C < \delta,$$

and since $\delta' < \delta$

$$\begin{aligned} &\forall x, \quad |f'(t)(T_t^{-1}(x)) - f'(t)(T_s^{-1}(x))| < \varepsilon \\ \Rightarrow\ &\|V(t) - V(s)\|_C < \varepsilon \quad \Rightarrow V \in C([0,\tau'];\mathcal{C}_0). \end{aligned}$$

We then proceed to the first derivative of V,

$$DV(t) = Df'(t) \circ T_t^{-1}\, DT_t^{-1} = Df'(t) \circ [I + g(t)]\,[I + Dg(t)],$$

and by uniform continuity of the right-hand side $V \in C([0,\tau'];\mathcal{C}_0^1)$. By induction on k, we finally get $V \in C([0,\tau'];\mathcal{C}_0^k)$. □

The proof of the last theorem is based on the fact that the vector functions involved are uniformly continuous. The fact that they vanish at infinity is not an essential element of the proof. Therefore, the theorem is valid with $C^k(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$ in place of $C_0^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$.

Theorem 4.5. *Let $k \geq 1$ be an integer.*

(i) *Given $\tau > 0$ and a velocity field V such that*

$$V \in C([0,\tau]; C^k(\overline{\mathbf{R}^N}, \mathbf{R}^N)), \tag{4.28}$$

the map T given by (4.4)–(4.6) *satisfies conditions* (T1), (T2), *and*

$$f \in C^1([0,\tau]; C^k(\overline{\mathbf{R}^N}, \mathbf{R}^N)). \tag{4.29}$$

Moreover, conditions (T3) *are satisfied and there exists $\tau' > 0$ such that*

$$g \in C([0,\tau']; C^k(\overline{\mathbf{R}^N}, \mathbf{R}^N)). \tag{4.30}$$

(ii) *Given $\tau > 0$ and $T : [0,\tau] \times \mathbf{R}^N \to \mathbf{R}^N$ satisfying conditions* (4.29) *and $T(0,\cdot) = I$, there exists $\tau' > 0$ such that the velocity field $V(t) = f'(t) \circ T_t^{-1}$ satisfies conditions* (V) *and* (4.28) *on $[0,\tau']$.*

5 Constrained Families of Domains

We now turn to the case where the family of admissible domains Ω is constrained to lie in a fixed larger subset D of $\mathbf{R}^N$ or its closure. For instance, D can be an open set or a closed submanifold of $\mathbf{R}^N$.

5.1 Equivalence between Velocities and Transformations

Given a nonempty subset D of $\mathbf{R}^N$, consider a family of transformations

$$T : [0,\tau] \times \overline{D} \to \mathbf{R}^N \tag{5.1}$$

for some $\tau = \tau(T) > 0$ with the following properties:

$$\begin{aligned}
&(\mathrm{T1}_D) && \begin{array}{l} \forall X \in \overline{D}, \quad T(\cdot, X) \in C^1\big([0,\tau]; \mathbf{R}^N\big) \text{ and } \exists c > 0, \\ \forall X, Y \in \overline{D}, \quad \|T(\cdot, Y) - T(\cdot, X)\|_{C^1([0,\tau];\mathbf{R}^N)} \leq c|Y - X|, \end{array} \\
&(\mathrm{T2}_D) && \forall t \in [0,\tau], \quad X \mapsto T_t(X) = T(t,X) : \overline{D} \to \overline{D} \text{ is bijective,} \\
&(\mathrm{T3}_D) && \begin{array}{l} \forall x \in \overline{D}, \quad T^{-1}(\cdot, x) \in C\big([0,\tau]; \mathbf{R}^N\big) \text{ and } \exists c > 0, \\ \forall x, y \in \overline{D}, \quad \|T^{-1}(\cdot, y) - T^{-1}(\cdot, x)\|_{C([0,\tau];\mathbf{R}^N)} \leq c|y - x|, \end{array}
\end{aligned} \tag{5.2}$$

where under assumption $(\mathrm{T2}_D)$ T^{-1} is defined from the inverse of T_t as

$$(t,x) \mapsto T^{-1}(t,x) \overset{\text{def}}{=} T_t^{-1}(x) : [0,\tau] \times \overline{D} \to \mathbf{R}^N. \tag{5.3}$$

These three properties are the analogue for $\overline{D}$ of the same three properties obtained for $\mathbf{R}^N$. In fact, Theorem 4.1 extends from $\mathbf{R}^N$ to $\overline{D}$ by adding one assumption to (V). Specifically, consider a velocity field

$$V : [0,\tau] \times \overline{D} \to \mathbf{R}^N \tag{5.4}$$

for which there exists $\tau = \tau(V) > 0$ such that

$$\begin{aligned} &(\mathrm{V}1_D) \quad \begin{aligned} &\forall x \in \overline{D}, \quad V(\cdot, x) \in C\big([0,\tau]; \mathbf{R}^{\mathrm{N}}\big), \quad \exists c > 0, \\ &\forall x, y \in \overline{D}, \quad \|V(\cdot, y) - V(\cdot, x)\|_{C([0,\tau];\mathbf{R}^{\mathrm{N}})} \le c|y - x|, \end{aligned} \\ &(\mathrm{V}2_D) \quad \forall x \in \overline{D},\, \forall t \in [0,\tau], \quad \pm V(t,x) \in T_{\overline{D}}(x), \end{aligned} \tag{5.5}$$

where $T_{\overline{D}}(x)$ is Bouligand's contingent cone[6] to $\overline{D}$ at the point x in $\overline{D}$

$$T_{\overline{D}}(X) \overset{\text{def}}{=} \bigcap_{\varepsilon>0} \bigcap_{\alpha>0} \bigcup_{0<h<\alpha} \left[\frac{1}{h}(\overline{D} - X) + \varepsilon B\right] \tag{5.6}$$

and B is the unit disk in $\mathbf{R}^{\mathrm{N}}$ (cf. J.-P. Aubin and A. Cellina [**1**, p. 176]). This definition is equivalent to

$$T_{\overline{D}}(X) = \limsup_{t \searrow 0} \left\{\frac{\overline{D} - X}{t}\right\} = \left\{v \;\middle|\; \liminf_{t\searrow 0} \frac{1}{t} d_D(X + tv) = 0\right\} \tag{5.7}$$

(cf. J.-P. Aubin and H. Frankowska [**1**, pp. 121–122, 17, 21]). Note that when D is bounded in $\mathbf{R}^{\mathrm{N}}$,

$$V(\cdot) \in C([0,\tau]; C(\overline{D}; \mathbf{R}^{\mathrm{N}}) \cap \mathrm{Lip}\,(\overline{D}; \mathbf{R}^{\mathrm{N}})) = C([0,\tau]; C^{0,1}(\overline{D}; \mathbf{R}^{\mathrm{N}})).$$

When D is equal to $\mathbf{R}^{\mathrm{N}}$, $T_{\overline{D}}(x) = \mathbf{R}^{\mathrm{N}}$ for all x and condition $(\mathrm{V}2_D)$ can be dropped. When D is equal to the boundary ∂A of a set A of class $C^{1,1}$ in $\mathbf{R}^{\mathrm{N}}$, ∂A is a $C^{1,1}$-submanifold of $\mathbf{R}^{\mathrm{N}}$ and

$$\forall x \in \partial A, \quad \pm V(t,x) \in T_{\partial A}(x) \iff \forall x \in \partial A, \quad V(t,x) \cdot \nabla b_A(x) = 0;$$

that is, at each point of ∂A, the velocity field is tangent to ∂A: it belongs to the tangent linear space of ∂A.

The next theorem is a generalization of Theorem 4.1 from $\mathbf{R}^{\mathrm{N}}$ to an arbitrary set D which shows the equivalence between velocity and transformation viewpoints.

Theorem 5.1.

(i) *Let $\tau > 0$ and let V be a family of velocity fields satisfying conditions* $(\mathrm{V}1_D)$ *and* $(\mathrm{V}2_D)$ *and consider the family of transformations*

$$(t, X) \mapsto T(t, X) = x(t; X) : [0,\tau] \times \overline{D} \to \mathbf{R}^{\mathrm{N}}, \tag{5.8}$$

where $x(\cdot, X)$ is the solution of

$$\frac{dx}{dt}(t) = V\big(t, x(t)\big), \quad 0 \le t \le \tau,\, x(0) = X. \tag{5.9}$$

Then the family of transformations T satisfies conditions $(\mathrm{T}1_D)$ *to* $(\mathrm{T}3_D)$.

[6]This is an equivalent characterization of Bouligand's contingent cone of Definition 2.4 in section 2.4 of Chapter 2.

(ii) *Conversely, given a family of transformations* T *satisfying conditions* $(\mathrm{T}1_D)$ *to* $(\mathrm{T}3_D)$*, the family of velocity fields*

$$(t,x) \mapsto V(t,x) = \frac{\partial T}{\partial t}\big(t, T_t^{-1}(x)\big) : [0,\tau] \times \overline{D} \to \mathbf{R}^{\mathrm{N}} \tag{5.10}$$

satisfies conditions $(\mathrm{V}1_D)$ *and* $(\mathrm{V}2_D)$*. If, in addition,* $T(0,\cdot) = I$*, then* $T(\cdot, X)$ *is the solution of* (5.9) *for that* V*.*

(iii) *Given a real number* $\tau > 0$ *and a map* $T : [0,\tau] \times \overline{D} \to \overline{D}$ *satisfying assumptions* $(\mathrm{T}1_D)$ *and* $(\mathrm{T}2_D)$ *and* $T(0,\cdot) = I$*, then there exists* $\tau' > 0$ *such that the conclusions of part* (ii) *hold on* $[0,\tau']$*.*

Remark 5.1.
Assumption $(V2_D)$ is a *double viability condition*. M. Nagumo [**1**]'s usual viability condition

$$V(t,x) \in T_{\overline{D}}(x), \quad \forall t \in [0,\tau], \forall x \in \overline{D} \tag{5.11}$$

is a necessary and sufficient condition for a *viable solution* to (5.9):

$$\forall t \in [0,\tau], \forall X \in \overline{D}, \quad x(t;X) \in \overline{D} \, \big(\text{or } T_t(\overline{D}) \subset \overline{D}\big) \tag{5.12}$$

(cf. J.-P. Aubin and A. Cellina [**1**, p. 174 and p. 180]). Condition $(V2_D)$,

$$\forall t \in [0,\tau], \forall x \in \overline{D}, \quad \pm V(t,x) \in T_{\overline{D}}(x), \tag{5.13}$$

is a *strict viability condition* which says that T_t maps $\overline{D}$ into $\overline{D}$ and

$$\forall t \in [0,\tau], \quad T_t : \overline{D} \to \overline{D} \quad \text{is a homeomorphism.} \tag{5.14}$$

In particular it maps interior points onto interior points and boundary points onto boundary points (cf. J. Dugundji [**1**, pp. 87–88]). □

Remark 5.2.
Condition $(V2_D)$ is a generalization to an arbitrary set D of the following condition used by J.-P. Zolésio [**12**] in 1979: for all x in ∂D,

$$\begin{cases} V(t,x) \cdot n(x) = 0, & \text{if the outward normal } n(x) \text{ exists,} \\ 0, & \text{otherwise.} \end{cases}$$

□

Proof of Theorem 5.1. (i) *Existence and uniqueness of viable solutions to* (5.9). Apply M. Nagumo [**1**]'s theorem to the augmented system on $[0,\tau]$

$$\begin{cases} \dfrac{dx}{dt}(t) = V\big(t, x(t)\big), & x(0) = X \in \overline{D}, \\ \dfrac{dx_0}{dt}(t) = 1, & x_0(0) = 0, \end{cases} \tag{5.15}$$

that is,

$$\begin{cases} \dfrac{d\hat{x}}{dt}(t) = \hat{V}\big(\hat{x}(t)\big), \\ \hat{x}(0) = (0,X) \in \hat{D} \stackrel{\text{def}}{=} \mathbf{R}^+ \times \overline{D}, \end{cases} \tag{5.16}$$

where $\hat{x}(t) = \big(x_0(t), x(t)\big) \in \mathbf{R}^{N+1}$, $\hat{V}(\hat{x}) = \big(1, \tilde{V}(\hat{x})\big)$, and

$$\tilde{V}(x_0, x) = \left\{\begin{matrix} V(x_0, x), & 0 \le x_0 \le \tau \\ V(\tau, x), & \tau < x_0 \end{matrix}\right\}, \quad x \in \overline{D}. \tag{5.17}$$

It is easy to check that systems (5.15) and (5.16) are equivalent on $[0, \tau]$ and that $\hat{x}(t) = \big(t, x(t)\big)$. The new velocity field on $\hat{V} \subset \mathbf{R}^{N+1}$ is continuous at each point $\hat{x} \in \hat{D}$ by the first assumption $(V1_D)$ since

$$\hat{V}(\hat{y}) - \hat{V}(\hat{x}) = \left(0, \tilde{V}(y_0, y) - \tilde{V}(x_0, x)\right),$$

and for $0 \le x_0, y_0 \le \tau$

$$\begin{aligned} |V(y_0, y) - V(x_0, x)| &\le |V(y_0, y) - V(y_0, x)| + |V(y_0, x) - V(x_0, x)| \\ &\le n|y - x| + |V(y_0, x) - V(x_0, x)|. \end{aligned}$$

In addition,

$$\begin{aligned} & T_{\hat{D}}(\hat{x}) = T_{\mathbf{R}^+}(x_0) \times T_{\overline{D}}(x) \\ \Rightarrow\ & \hat{V}(\hat{x}) = \big(1, V(\hat{x})\big) \in T_{\mathbf{R}^+}(x_0) \times T_{\overline{D}}(x). \end{aligned}$$

Moreover $\hat{V}(\hat{D})$ is bounded and $\mathbf{R}^{N+1}$ is finite-dimensional. By using the version of M. Nagumo[**1**]'s theorem given in J.-P. Aubin and A. Cellina [**1**, Thm. 3, part b), pp. 182–183], there exists a viable solution $\hat{x}$ to (5.16) for all $t \ge 0$. In particular,

$$\forall t \in [0, \tau], \quad \hat{x}(t) \in \hat{D} = \mathbf{R}^+ \times \overline{D},$$

which is necessarily of the form $\hat{x}(t) = (t, x(t))$. Hence there exists a viable solution $x, x(t) \in \overline{D}$ on $[0, \tau]$, to (5.9). The uniqueness now follows from the Lipschitz condition $(V1_D)$. The Lipschitzian continuity $(T1_D)$ can be established by a standard argument.

Condition (T2_D). Associate with X in $\overline{D}$ the function

$$y(s) = T_{t-s}(X), \quad 0 \le s \le t.$$

Then

$$\frac{dy}{ds}(s) = -V\big(t - s, y(s)\big), \quad 0 \le s \le t, \quad y(0) = T_t(X). \tag{5.18}$$

For each $x \in \overline{D}$, the differential equation

$$\frac{dy}{ds}(s) = -V\big(t - s, y(s)\big), \quad 0 \le s \le t, \quad y(0) = x \in \overline{D} \tag{5.19}$$

has a unique viable solution in $C^1([0, t]; \mathbf{R}^{\mathrm{N}})$:

$$\forall s \in [0, t], \quad y(s) \in \overline{D}, \tag{5.20}$$

since by assumption $(V2_D)$

$$\forall t \in [0, \tau], \forall x \in \overline{D}, \quad -V(t, x) \in T_{\overline{D}}(x).$$

The proof is the same as above. The solutions of (5.19) define a Lipschitzian mapping

$$x \mapsto S_t(x) = y(t) : \overline{D} \to \overline{D}$$

such that

$$\exists c > 0, \forall t \in [0, \tau], \forall x, y \in \overline{D}, \quad |S_t(y) - S_t(x)| \le c|y - x|. \tag{5.21}$$

Now in view of (5.18) and (5.19)

$$S_t\big(T_t(X)\big) = y(t) = T_{t-t}(X) = X \quad \Rightarrow \quad S_t \circ T_t = I \text{ on } \overline{D}.$$

To obtain the other identity, consider the function

$$z(r) = y(t - r; x),$$

where $y(\cdot, x)$ is the solution of (5.19). By definition,

$$\frac{dz}{dr}(r) = V\big(r, z(r)\big), \quad z(0) = y(t, x)$$

and necessarily

$$\begin{aligned} x = y(0; x) = z(t) = T_t\big(y(t; x)\big) = T_t\big(S_t(x)\big) \\ \Rightarrow T_t \circ S_t = I \text{ on } \overline{D} \Rightarrow S_t = T_t^{-1} : \overline{D} \to \overline{D}. \end{aligned}$$

Condition (T3$_D$). The uniform Lipschitz continuity in (T3$_D$) follows from (5.21) and (T2$_D$), and we need only show that

$$\forall x \in \overline{D}, \quad T^{-1}(\cdot, x) \in C\big([0, \tau]; \mathbf{R}^{\mathrm{N}}\big).$$

Given t in $[0, \tau]$ choose an arbitrary sequence $\{t_n\}$, $t_n \to t$. Then for each $x \in \overline{D}$ there exists $X \in \overline{D}$ such that

$$T_t(X) = x \quad \text{and} \quad T_{t_n}(X) \to T_t(X) = x$$

from (T1$_D$). But

$$\begin{aligned} T_{t_n}^{-1}(x) - T_t^{-1}(x) &= T_{t_n}^{-1}\big(T_t(X)\big) - T_t^{-1}\big(T_t(X)\big) \\ &= T_{t_n}^{-1}\big(T_t(X)\big) - T_{t_n}^{-1}\big(T_{t_n}(X)\big). \end{aligned}$$

By the uniform Lipschitz continuity of T_t^{-1}

$$\begin{aligned} |T_{t_n}^{-1}(x) - T_t^{-1}(x)| &= |T_{t_n}^{-1}\big(T_t(X)\big) - T_{t_n}^{-1}\big(T_{t_n}(X)\big)| \\ &\le c|T_t(X) - T_{t_n}(X)|, \end{aligned}$$

and the last term converges to zero as t_n goes to t.

(ii) The first condition ($V1_D$) is satisfied since for each $x \in \bar{D}$ and t, s in $[0, \tau]$

$$\begin{aligned}|V(t,x) - V(s,x)| &\leq \left|\frac{\partial T}{\partial t}\big(t, T_t^{-1}(x)\big) - \frac{\partial T}{\partial t}\big(t, T_s^{-1}(x)\big)\right| \\ &\quad + \left|\frac{\partial T}{\partial t}\big(t, T_s^{-1}(x)\big) - \frac{\partial T}{\partial t}\big(s, T_s^{-1}(x)\big)\right| \\ &\leq c|T_t^{-1}(x) - T_s^{-1}(x)| \\ &\quad + \left|\frac{\partial T}{\partial t}\big(t, T_s^{-1}(x)\big) - \frac{\partial T}{\partial t}\big(s, T_s^{-1}(x)\big)\right|.\end{aligned}$$

The second condition ($V1_D$) follows from ($T1_D$) and ($T3_D$) and the following inequality: for all x and y in $\overline{D}$

$$\begin{aligned}|V(t,y) - V(t,x)| &= \left|\frac{\partial T}{\partial t}\big(t, T_t^{-1}(y)\big) - \frac{\partial T}{\partial t}\big(t, T_t^{-1}(x)\big)\right| \\ &\leq c\left|T_t^{-1}(y) - T_t^{-1}(x)\right| \leq cc'|y - x|.\end{aligned}$$

To check condition ($V2_D$), recall definition (5.6) of the Bouligand contingent cone:

$$T_{\overline{D}}(X) = \bigcap_{\varepsilon>0} \bigcap_{\alpha>0} \bigcup_{0<h<\alpha} \left[\frac{1}{h}(\overline{D} - X) + \varepsilon B\right],$$

where B is the unit disk in $\mathbf{R}^{\mathrm{N}}$. We first show that

$$\forall x \in \overline{D}, \quad V(t,x) = \frac{\partial T}{\partial t}\big(t, T_t^{-1}(x)\big) \in T_{\overline{D}}(x).$$

By ($T2_D$) T_t is bijective. So it is equivalent to show that

$$\forall X \in \overline{D}, \quad \frac{\partial T}{\partial t}(t, X) \in T_{\overline{D}}\big(T_t(X)\big).$$

For simplicity we use the notation

$$x(t) = T_t(X) = T(t,X) \quad \text{and} \quad x'(t) = \frac{\partial T}{\partial t}(t,X). \tag{5.22}$$

By the definition of $T_{\overline{D}}(x(t))$, we must prove that

$$\begin{gathered}\forall \varepsilon > 0, \forall \alpha > 0, \exists h \in\,]0, \alpha[\,, \exists u \\ \text{such that } x'(t) \in u + \varepsilon B \text{ and } x(t) + hu \in \overline{D}.\end{gathered}$$

Choose $\delta, 0 < \delta < \alpha$, such that

$$\forall s, \quad |s - t| < \delta \quad \Rightarrow \quad |x'(s) - s'(t)| < \varepsilon.$$

Then fix t'. For all t such that $0 < t' - t < \delta$,

$$\begin{aligned}x(t') - x(t) &= \int_t^{t'} \big[x'(s) - x'(t)\big]\, ds + (t' - t)x'(t) \\ &\Rightarrow \left|\frac{x(t') - x(t)}{t' - t} - x'(t)\right| < \varepsilon.\end{aligned}$$

Therefore choose $u = [x(t') - x(t)]/(t' - t)$. Now

$$x(t) + hu = x(t) + \frac{h}{t'-t}\big[x(t') - x(t)\big]$$

and choose $h = t' - t$ since $0 < t' - t < \delta < \alpha$:

$$T(t,X) + (t'-t)u = T(t,X) + \big[T(t',X) - T(t,X)\big] = T(t',X) \in \overline{D}$$

by assumption on T. This proves (5.22). The second part of ($\mathrm{V2}_D$) is

$$\forall x \in \overline{D}, \quad -V(t,x) = -\frac{\partial T}{\partial t}\big(t, T_t^{-1}(x)\big) \in T_{\overline{D}}(x),$$

which is equivalent to proving that

$$\forall X \in \overline{D}, \quad -\frac{\partial T}{\partial t}(t,X) \in T_{\overline{D}}\big(T_t(X)\big),$$

or with the simplified notation

$$-x'(t) \in T_{\overline{D}}\big(x(t)\big). \tag{5.23}$$

We proceed exactly as in the proof of (5.22) except that we choose t' such that $0 < t - t' < \delta, h = t - t'$, and $u = -[x(t) - x(t')]/(t - t')$. Then

$$|u + x'(t)| < \varepsilon$$
$$\text{and} \quad x(t) + hu = x(t) + (t - t')\left[-\left(\frac{x(t) - x(t')}{t - t'}\right)\right] = x(t') \in \overline{D}$$

and we get (5.23).

(iii) From ($\mathrm{T1}_D$) and ($\mathrm{T2}_D$)

$$f(t)(y) - f(t)(x) = \int_0^t \frac{\partial T}{\partial t}(r, T_r(y)) - \frac{\partial T}{\partial t}(r, T_r(x))\, dr,$$
$$|f(t)(y) - f(t)(x)| \le \int_0^t c\,|T_r(y) - T_r(x)|\, dr \le c^2\, t\, |y - x|.$$

For $\tau' = \min\{\tau, 1/(2c^2)\}$ and $0 \le t \le \tau'$, $c(f(t)) \le 1/2$,

$$g(t) = T_t^{-1} - I = (I - T_t) \circ T_t^{-1} = -f(t) \circ [I + g(t)],$$
$$(1 - c(f(t)))\, c(g(t)) \le c(f(t)) \quad \Rightarrow c(g(t)) \le 1 \text{ and } c(T_t^{-1}) \le 2,$$

and the second condition (T3) is satisfied on $[0, \tau']$. The first one follows by the same argument as in part (i). Therefore, the conclusions of part (ii) are true on $[0, \tau']$. This completes the proof of the theorem. $\square$

5.2 Transformation of Condition (V2$_D$) into a Linear Constraint

Condition (V2$_D$) is equivalent to

$$\boxed{\forall t \in [0,\tau], \forall x \in \overline{D}, \quad V(t,x) \in \{-T_D(x)\} \cap T_D(x)} \tag{5.24}$$

since $T_{\overline{D}}(x) = T_D(x)$. If $T_D(x)$ were convex, then the above intersection would be a closed linear subspace of $\mathbf{R}^{\mathrm{N}}$. This is true when D is convex. In that case $T_D(x) = C_D(x)$, where $C_D(x)$ is the Clarke tangent cone and

$$\boxed{L_D(x) = \{-C_D(x)\} \cap C_D(x)} \tag{5.25}$$

is a closed linear subspace of $\mathbf{R}^{\mathrm{N}}$. This means that (V2$_D$) reduces to

$$\boxed{\forall t \in [0,\tau], \forall x \in \overline{D}, \quad V(t,x) \in L_D(x).} \tag{5.26}$$

It turns out that for continuous vector fields $V(t,\cdot)$, the equivalence of (V2$_D$) and (5.26) extends to arbitrary domains D. This equivalence generally fails for discontinuous vector fields. Other equivalences might be possible between T_D and some intermediary convex cone between C_D and T_D, but there is no evidence so far of that fact. For smooth bounded open domains Ω, the two cones coincide and the condition reduces to $V \cdot n = 0$, n the normal to $\partial\Omega$, and $V(t,x)$ belongs to the tangent space to $\partial\Omega$ in each point of $\partial\Omega$.

Theorem 5.2.

(i) *Given a velocity field V satisfying* (V1$_D$), *condition* (V2$_D$) *is equivalent to*

$$(\mathrm{V}2_C) \quad \forall t \in [0,\tau],\ \forall x \in \overline{D}, \quad V(t,x) \in L_D(x) = \{-C_D(x)\} \cap C_D(x),$$

where $C_D(x)$ is the (closed convex) Clarke tangent cone to $\overline{D}$ at x,

$$C_D(x) = \left\{ v \in \mathbf{R}^{\mathrm{N}} : \lim_{\substack{t \searrow 0 \\ y \xrightarrow[\overline{D}]{} x}} \frac{d_D(y+tv)}{t} = 0 \right\}, \tag{5.27}$$

and $\xrightarrow[\overline{D}]{}$ denotes the convergence in $\overline{D}$.

(ii) *$L_D(x)$ is a closed linear subspace of $\mathbf{R}^{\mathrm{N}}$.*

Proof. (i) The equivalence of (V2$_D$) and (V2$_C$) is a direct consequence of the following lemma.

Lemma 5.1. *Given a vector field $W \in C(\overline{D};\mathbf{R}^{\mathrm{N}})$, the following two conditions are equivalent:*

$$\forall x \in \overline{D}, \quad W(x) \in T_D(x), \tag{5.28}$$

$$\forall x \in \overline{D}, \quad W(x) \in C_D(x). \tag{5.29}$$

(ii) The set $L_D(x)$ is closed as the intersection of two closed sets. To show that it is linear, we show that for all $\alpha \in \mathbf{R}$ and $V \in L_D(x)$, $\alpha V \in L_D(x)$, and for all V and W in $L_D(x)$, $V+W \in L_D(x)$. Since $\pm C_D(x)$ are cones,

$$\begin{gathered}\forall \alpha \in \mathbf{R},\ \forall V \in L_D(x), \quad \pm|\alpha| V \in C_D(x)\\ \Rightarrow \pm\alpha V \in C_D(x) \Rightarrow \alpha V \in L_D(x).\end{gathered}$$

By convexity of $\pm C_D(x)$

$$\forall V, W \in L_D(x), \quad \pm(V+W) \in C_D(x) \Rightarrow V+W \in L_D(x).$$

This completes the proof of the theorem. □

Proof of Lemma 5.1. Assume that (5.29) is verified. By definition $C_D(x) \subset T_D(x)$ and (5.29) $\Rightarrow$ (5.28). Conversely, either x is an isolated point and $T_D(x) = \{0\} = C_D(x)$ or there are points $x \neq y \in \overline{D}$ such that $y \underset{\overline{D}}{\longrightarrow} x$. In the latter case we know that

$$\liminf_{y \underset{\overline{D}}{\longrightarrow} x} T_D(y) = C_D(x)$$

(cf., for instance, J.-P. Aubin and H. Frankowska [**1**, Thm. 4.1.10, sect. 4.1.5, p. 130]). Since W is continuous in $\overline{D}$ and (5.28) is satisfied, then for each $x \in \overline{D}$

$$W(x) = \lim_{y \underset{\overline{D}}{\longrightarrow} x} W(y) \in \liminf_{y \underset{\overline{D}}{\longrightarrow} x} T_D(y) = C_D(x)$$

and (5.28) implies (5.29). □

Remark 5.3.
Lemma 5.1 essentially says that for continuous vector fields we can relax the condition of M. Nagumo [**1**]'s theorem from (V2$_D$) involving the Bouligand contingent cone to (V2$_C$) involving the smaller Clarke convex tangent cone. In dimension $N=3, L_D(x)$ is $\{0\}$, a line, a plane, or the whole space. □

Notation 5.1.
In what follows, it will be convenient to introduce the spaces and subspaces

$$\mathcal{L} = \left\{V : [0,\tau] \times \mathbf{R}^{\mathrm{N}} \to \mathbf{R}^{\mathrm{N}} : V \text{ satisfies (V) on } \mathbf{R}^{\mathrm{N}}\right\}, \tag{5.30}$$

and for an arbitrary domain D in $\mathbf{R}^{\mathrm{N}}$

$$\mathcal{L}_D = \{V : [0,\tau] \times \overline{D} \to \mathbf{R}^{\mathrm{N}} : V \text{ satisfies } (\mathrm{V1}_D) \text{ and } (\mathrm{V2}_C) \text{ on } \overline{D}\}. \tag{5.31}$$

For any integers $k \geq 0$ and $m \geq 0$ and any compact subset K of $\mathbf{R}^{\mathrm{N}}$ define the following subspaces of $\mathcal{L}$:

$$\mathcal{V}_K^{m,k} = C^m\Big([0,\tau], \mathcal{D}^k\big(K, \mathbf{R}^{\mathrm{N}}\big)\Big) \cap \mathcal{L}, \tag{5.32}$$

where $\mathcal{D}^k(K, \mathbf{R}^{\mathrm{N}})$ is the space of k-times continuously differentiable transformations of $\mathbf{R}^{\mathrm{N}}$ with compact support in K. In all cases $\mathcal{V}_K^{m,k} \subset \mathcal{L}_K$. As usual $\mathcal{D}^\infty(K, \mathbf{R}^{\mathrm{N}})$ will be written $\mathcal{D}(K, \mathbf{R}^{\mathrm{N}})$. □

6 Continuity of Shape Functions along Velocity Flows

Throughout this section, assume that Ω is a closed subset or an open crack-free[7] subset of $\mathbf{R}^{\mathrm{N}}$ and that B is a Banach space.

Since the construction of the Courant metric is relatively abstract, can the continuity of a shape functional be characterized in a more direct way? The answer is yes thanks to the sharp theorems giving the equivalence between transformations and velocities in the previous section without restricting the analysis to the subgroup G_Θ defined by (2.7) in section 2. This is due to the fact that continuity and semiderivatives are local properties and that the velocity character naturally pops out as a sequence of transformations goes to identity.

In this section, we give a characterization via velocities of the continuity of a shape functional of the form

$$\Omega' \mapsto J(\Omega') : \mathcal{X}(\Omega) \to B, \quad \mathcal{X}(\Omega) = \{F(\Omega) : \forall F \in \mathcal{F}(\Theta)\} \tag{6.1}$$

with respect to the Courant metric topology of the quotient $\mathcal{F}(\Theta)/\mathcal{G}(\Omega)$ using the equivalence Theorems 4.3, 4.4, and 4.5. Checking the continuity along flows of a velocity is easier and more natural. As in Definition 3.1, assume that J verifies the compatibility condition of Definition 3.1:

$$\forall H, F \in \mathcal{F}(\Theta) \text{ such that } H(F(\Omega)) = F(\Omega), \quad J(H(F(\Omega))) = J(F(\Omega)).$$

We specifically consider the continuity of shape functionals with respect to the Courant metric associated with Θ equal to $C_0^{k+1}(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$, $C^{k+1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$, and $C^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$, $k \geq 0$, but similar equivalences are true for the other spaces of Chapter 3.

Remark 6.1.
An obvious consequence of Theorems 6.1, 6.2, and 6.3 is that when a shape functional is semidifferentiable in the sense of Definition 3.2 (Hadamard), it is continuous at that point for the Courant metric topology in complete analogy with the classical Euclidean calculus. □

We begin with the space $\mathcal{C}_0^k(\mathbf{R}^{\mathrm{N}}) = C_0^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ of A. M. Micheletti [**1**].

Theorem 6.1. *Let $k \geq 1$ be an integer, B be a Banach space, and Ω be a nonempty open subset of $\mathbf{R}^{\mathrm{N}}$. Consider a shape functional $J : N_\Omega([I]) \to B$ defined in a neighborhood $N_\Omega([I])$ of $[I]$ in $\mathcal{F}(C_0^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}))/\mathcal{G}(\Omega)$. Then J is continuous at Ω for the Courant metric if and only if*

$$\lim_{t \searrow 0} J(T_t(\Omega)) = J(\Omega) \tag{6.2}$$

for all families of velocity fields $\{V(t) : 0 \leq t \leq \tau\}$ satisfying the condition

$$V \in C([0,\tau]; C_0^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})). \tag{6.3}$$

[7] Cf. Definition 7.1 (ii) of Chapter 8.

Proof. It is sufficient to prove the theorem for a real-valued function J. The Banach space case is readily obtained by considering the new real-valued function $j(T) = |J(T(\Omega)) - J(\Omega)|$.

(i) If J is d_G-continuous at Ω, then for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall T,\ [T] \in N_\Omega([I]), \quad d_G([T],[I]) < \delta, \quad |J(T(\Omega)) - J(\Omega)| < \varepsilon.$$

Condition (6.3) on V coincides with condition (4.23) of Theorem 4.4, which implies conditions (4.24) and (4.25):

$$\begin{aligned} &f \in C^1([0,\tau]; C_0^k(\mathbf{R}^N, \mathbf{R}^N)) \quad \text{and} \quad g \in C([0,\tau]; C_0^k(\mathbf{R}^N, \mathbf{R}^N)) \\ &\Rightarrow \|T_t - I\|_{\mathcal{C}^k(\mathbf{R}^N)} \to 0 \text{ and } \|T_t^{-1} - I\|_{\mathcal{C}^k(\mathbf{R}^N)} \to 0 \text{ as } t \to 0. \end{aligned}$$

But by definition of the metric d_G

$$d([T_t],[I]) \le \|T_t^{-1} - I\|_{\mathcal{C}^k} + \|T_t - I\|_{\mathcal{C}^k} \to 0 \text{ as } t \to 0,$$

and we get the convergence (6.2) of the function $J(T_t(\Omega))$ to $J(\Omega)$ as t goes to zero for all V satisfying (6.3).

(ii) Conversely, it is sufficient to prove that for any sequence $\{[T_n]\}$ such that $d_G([T_n],[I])$ goes to zero, there exists a subsequence such that

$$J(T_{n_k}(\Omega)) \to J(I(\Omega)) = J(\Omega) \text{ as } k \to \infty.$$

Indeed let

$$\ell = \liminf_{n\to\infty} J(T_n(\Omega)) \quad \text{and} \quad L = \limsup_{n\to\infty} J(T_n(\Omega)).$$

By definition of the liminf, there is a subsequence, still indexed by n, such that $\ell = \liminf_{n\to\infty} J(T_n(\Omega))$. But since there exists a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ such that $J(T_{n_k}(\Omega)) \to J(\Omega)$, then necessarily $\ell = J(\Omega)$. The same reasoning applies to the limsup, and hence the whole sequence $J(T_n(\Omega))$ converges to $J(\Omega)$ and we have the continuity of J at Ω.

We prove that we can construct a velocity V associated with a subsequence of $\{T_n\}$ verifying conditions (4.23) of Theorem 4.4 and hence conditions (6.3). By the same techniques as in Theorems 2.8 and 2.6 of Chapter 3, associate with a sequence $\{T_n\}$ such that $d_G([T_n],[I]) \to 0$ a subsequence, still denoted by $\{T_n\}$, such that

$$\|f_n\|_{\mathcal{C}^k} + \|g_n\|_{\mathcal{C}^k} = \|T_n^{-1} - I\|_{\mathcal{C}^k} + \|T_n - I\|_{\mathcal{C}^k} \le 2^{-2(n+2)}.$$

For $n \ge 1$ set $t_n = 2^{-n}$ and observe that $t_n - t_{n+1} = -2^{-(n+1)}$. Define the following C^1-interpolation in $(0,1/2]$: for t in $[t_{n+1}, t_n]$,

$$\boxed{T_t(X) \overset{\text{def}}{=} T_n(X) + p\left(\frac{t_{n+1} - t}{t_{n+1} - t_n}\right)(T_{n+1}(X) - T_n(X)), \quad T_0(X) \overset{\text{def}}{=} X,}$$

where $p \in P^3[0,1]$ is the polynomial of order 3 on $[0,1]$ such that $p(0) = 1$ and $p(1) = 0$ and $p^{(1)}(0) = 0 = p^{(1)}(1)$.

Conditions on f. By definition, for all t, $0 \le t \le 1/2$, $f(t) = T_t - I \in \mathcal{C}_0^k(\mathbf{R}^\mathrm{N})$. Moreover, for $0 < t \le 1/2$

$$T_{t_n}(X) = T_n(X),\ T_{t_{n+1}}(X) = T_{n+1}(X),\ \frac{\partial T}{\partial t}(t_n, X) = 0 = \frac{\partial T}{\partial t}(t_{n+1}, X),$$
$$\frac{\partial T}{\partial t}(t, X) = \frac{T_{n+1}(X) - T_n(X)}{|t_n - t_{n+1}|} p^{(1)}\!\left(\frac{t_{n+1} - t}{t_{n+1} - t_n}\right),$$

$f'(t) = \partial T/\partial t(t,\cdot) \in \mathcal{C}_0^k(\mathbf{R}^\mathrm{N})$, and $f(\cdot)(X) = T(\cdot, X) - I \in C^1((0,1/2];\mathbf{R}^\mathrm{N})$. By definition, $f(0) = 0$. For each $0 < t \le 1/2$ there exists $n \ge N$ such that $t_{n+1} \le t \le t_n$ and

$$\begin{aligned} \|f(t) - f(0)\|_{\mathcal{C}^k} &= \|f(t)\|_{\mathcal{C}^k} = \left\|f_n + p\left(\frac{t_{n+1} - t}{t_{n+1} - t_n}\right)(f_{n+1} - f_n)\right\|_{\mathcal{C}^k} \\ &\le 2\|f_n\|_{\mathcal{C}^k} + \|f_{n+1}\|_{\mathcal{C}^k} \le 2\,2^{-2(n+2)} + 2^{-2(n+3)} \le 2^{-(n+1)} \le t. \end{aligned}$$

Define at $t = 0$, $f'(t) = 0$. By the same technique, there exists a constant $c > 0$, and for each $0 < t \le 1/2$ there exists $n \ge N$ such that $t_{n+1} \le t \le t_n$ and

$$\begin{aligned} &\|f'(t) - f'(0)\|_{\mathcal{C}^k} = \|f'(t)\|_{\mathcal{C}^k} \\ &= \left\|\frac{\partial T}{\partial t}(t,\cdot)\right\|_{\mathcal{C}^k} \le c\,\frac{\|T_{n+1} - T_n\|_{\mathcal{C}^k}}{|t_{n+1} - t_n|} = c\,\frac{\|f_{n+1} - f_n\|_{\mathcal{C}^k}}{2^{-(n+1)}} \\ &\le c\,2\,2^{-2(n+2)}/2^{-(n+1)} \le c\,2^{-1}2^{-(n+1)} \le c\,2^{-(n+1)} \le c\,t \\ &\qquad \Rightarrow \boxed{\|f'(t)\|_{\mathcal{C}^k} \le ct.} \end{aligned}$$

So for each X the functions $t \mapsto f(t)(X)$ and $t \mapsto T_t(X)$ belong to $C^1([0,1/2];\mathbf{R}^\mathrm{N})$. By uniform C^k-continuity of the T_n's and the continuity with respect to t for each X, it follows that $f \in C^1([0,1/2];\mathcal{C}_0^k(\mathbf{R}^\mathrm{N}))$ and the condition (4.24) of Theorem 4.4 is satisfied. Hence the corresponding velocity V satisfies conditions (4.23). Finally V satisfies conditions (6.3) and by (6.2) for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall t,\ 0 \le t \le \delta, \quad |J(T_t(\Omega)) - J(\Omega)| < \varepsilon.$$

In particular there exists $N > 0$ such that for all $n \ge N$, $t_n \le \delta$, and

$$\forall n \ge N, \quad |J(T_n(\Omega)) - J(\Omega)| = |J(T_{t_n}(\Omega)) - J(\Omega)| < \varepsilon,$$

and this proves the d-continuity for the subsequence $\{T_n\}$. □

The case of the Courant metric associated with the space $\mathcal{C}^k(\overline{\mathbf{R}^\mathrm{N}}) = C^k(\overline{\mathbf{R}^\mathrm{N}}, \mathbf{R}^\mathrm{N})$ is a corollary to Theorem 6.1.

Theorem 6.2. *Let $k \ge 1$ be an integer, B be a Banach space, and Ω be a nonempty open subset of $\mathbf{R}^\mathrm{N}$. Consider a shape functional $J : N_\Omega([I]) \to B$ defined in a neighborhood $N_\Omega([I])$ of $[I]$ in $\mathcal{F}(C^k(\overline{\mathbf{R}^\mathrm{N}}, \mathbf{R}^\mathrm{N}))/\mathcal{G}(\Omega)$.*

Then J is continuous at Ω for the Courant metric if and only if

$$\lim_{t \searrow 0} J(T_t(\Omega)) = J(\Omega) \tag{6.4}$$

for all families of velocity fields $\{V(t) : 0 \leq t \leq \tau\}$ satisfying the condition

$$V \in C([0, \tau]; C^k(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})). \tag{6.5}$$

The proof of the theorem for the Courant metric topology associated with the space $\mathcal{C}^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}}) = C^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$ is similar to the proof of Theorem 6.1 with obvious changes.

Theorem 6.3. *Let $k \geq 0$ be an integer, Ω be a nonempty open subset of $\mathbf{R}^{\mathrm{N}}$, and B be a Banach space. Consider a shape functional $J : N_\Omega([I]) \to B$ defined in a neighborhood $N_\Omega([I])$ of $[I]$ in $\mathcal{F}(C^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}}))/\mathcal{G}(\Omega)$. Then J is continuous at Ω for the Courant metric if and only if*

$$\lim_{t \searrow 0} J(T_t(\Omega)) = J(\Omega) \tag{6.6}$$

for all families $\{V(t) : 0 \leq t \leq \tau\}$ of velocity fields in $C^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$ satisfying the conditions

$$V \in C([0, \tau]; C^k(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})) \quad \text{and} \quad c_k(V(t)) \leq c \tag{6.7}$$

for some constant c independent of t.

Proof. As in the proof of Theorem 6.1, it is sufficient to prove the theorem for a real-valued function J.

(i) If J is d_G-continuous at Ω, then for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall T,\ [T] \in N_\Omega([I]), \quad d_G([T], [I]) < \delta, \quad |J(T(\Omega)) - J(\Omega)| < \varepsilon.$$

Under condition (6.7) from Theorem 4.3

$$f = T - I \in C([0, \tau]; \mathcal{C}^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}})) \text{ and } \|T_t - I\|_{\mathcal{C}^{k,1}} \to 0 \text{ as } t \to 0,$$
$$g(t) = T_t^{-1} - I \in \mathcal{C}^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}}) \text{ and } \|T_t^{-1} - I\|_{\mathcal{C}^{k,1}} \leq ct \to 0 \text{ as } t \to 0.$$

But by definition of the metric d_G

$$d_G([T_t], [I]) \leq \|T_t^{-1} - I\|_{\mathcal{C}^{k,1}} + \|T_t - I\|_{\mathcal{C}^{k,1}} \to 0 \text{ as } t \to 0,$$

and we get the convergence (6.6) of the function $J(T_t(\Omega))$ to $J(\Omega)$ as t goes to zero for all V satisfying (6.7).

(ii) Conversely, as in the proof of Theorem 6.1, it is sufficient to prove that given any sequence $\{[T_n]\}$ such that $\delta([T_n], [I]) \to 0$ there exists a subsequence such that

$$J(T_{n_k}(\Omega)) \to J(I(\Omega)) = J(\Omega) \text{ as } k \to \infty.$$

By the same techniques as in Theorems 2.8 and 2.6 of Chapter 3, associate with a sequence $\{T_n\}$ such that $d_G([T_n],[I]) \to 0$ a subsequence, still denoted by $\{T_n\}$, such that

$$\|f_n\|_{\mathcal{C}^{k,1}} + \|g_n\|_{\mathcal{C}^{k,1}} = \|T_n^{-1} - I\|_{\mathcal{C}^{k,1}} + \|T_n - I\|_{\mathcal{C}^{k,1}} \le 2^{-2(n+2)}.$$

For $n \ge 1$ set $t_n = 2^{-n}$ and observe that $t_n - t_{n+1} = -2^{-(n+1)}$. Define the following C^1-interpolation in $(0, 1/2]$: for t in $[t_{n+1}, t_n]$,

$$\boxed{T_t(X) \overset{\text{def}}{=} T_n(X) + p\left(\frac{t_{n+1}-t}{t_{n+1}-t_n}\right)(T_{n+1}(X) - T_n(X)), \quad T_0(X) \overset{\text{def}}{=} X,}$$

where $p \in P^3[0,1]$ is the polynomial of order 3 on $[0,1]$ such that $p(0) = 1$ and $p(1) = 0$ and $p^{(1)}(0) = 0 = p^{(1)}(1)$.

Conditions on f. By definition for all t, $0 \le t \le 1/2$, $f(t) = T_t - I \in \mathcal{C}^{k,1}(\overline{\mathbf{R}^N})$. Moreover, for $0 < t \le 1/2$

$$T_{t_n}(X) = T_n(X),\ T_{t_{n+1}}(X) = T_{n+1}(X),\ \frac{\partial T}{\partial t}(t_n, X) = 0 = \frac{\partial T}{\partial t}(t_{n+1}, X),$$
$$\frac{\partial T}{\partial t}(t,X) = \frac{T_{n+1}(X) - T_n(X)}{|t_n - t_{n+1}|} p^{(1)}\left(\frac{t_{n+1}-t}{t_{n+1}-t_n}\right),$$

$f'(t) = \partial T/\partial t \in \mathcal{C}^{k,1}(\overline{\mathbf{R}^N})$, and $f(\cdot)(X) = T(\cdot, X) - I \in C^1((0,1/2]; \mathbf{R}^N)$. By definition, $f(0) = 0$. For each $0 < t \le 1/2$ there exists $n \ge N$ such that $t_{n+1} \le t \le t_n$ and

$$\|f(t) - f(0)\|_{\mathcal{C}^{k,1}} = \|f(t)\|_{\mathcal{C}^{k,1}} = \left\| f_n + p\left(\frac{t_{n+1}-t}{t_{n+1}-t_n}\right)(f_{n+1} - f_n)\right\|_{\mathcal{C}^{k,1}}$$
$$\le 2\|f_n\|_{\mathcal{C}^{k,1}} + \|f_{n+1}\|_{\mathcal{C}^{k,1}} \le 2\,2^{-2(n+2)} + 2^{-2(n+3)} \le 2^{-(n+1)} \le t.$$

Define at $t = 0$, $f'(t) = 0$. By the same technique there exists a constant $c > 0$, and for each $0 < t \le 1/2$ there exists $n \ge N$ such that $t_{n+1} \le t \le t_n$ and

$$\begin{aligned}
&\|f'(t) - f'(0)\|_{\mathcal{C}^{k,1}} = \|f'(t)\|_{\mathcal{C}^{k,1}} \\
&= \left\|\frac{\partial T}{\partial t}(t,\cdot)\right\|_{\mathcal{C}^{k,1}} \le c\,\frac{\|T_{n+1} - T_n\|_{\mathcal{C}^{k,1}}}{|t_{n+1} - t_n|} = c\,\frac{\|f_{n+1} - f_n\|_{\mathcal{C}^{k,1}}}{2^{-(n+1)}} \\
&\le c\,2\,2^{-2(n+2)}/2^{-(n+1)} \le c\,2^{-1}2^{-(n+1)} \le c\,2^{-(n+1)} \le c\,t \\
&\Rightarrow \boxed{\|f'(t)\|_{\mathcal{C}^{k,1}} \le ct \text{ and } c_k(f'(t)) \le ct}
\end{aligned}$$

and for each X the functions $t \mapsto f(t)(X)$ and $t \mapsto T_t(X)$ belong to $C^1([0,1/2]; \mathbf{R}^N)$. By uniform C^k-continuity of the T_n's and the continuity with respect to t for each X, it follows that $f \in C^1([0,1/2]; \mathcal{C}^k(\overline{\mathbf{R}^N}))$. Moreover, it can be shown that

$$c_k(f'(t)) \le ct \quad \Rightarrow \forall t, s \in [0,\tau], \quad c_k(f(t) - f(s)) \le c'|t - s|$$

for some $c' > 0$. The result is straightforward for $k = 0$ and then the general case follows by induction on k. As a result $f \in C([0,1/2]; \mathcal{C}^{k,1}(\overline{\mathbf{R}^N}))$ and the condition

(4.20) of Theorem 4.3 is satisfied. Hence the corresponding velocity V satisfies conditions (4.19). Finally the velocity field V satisfies conditions (6.7), and by (6.6) for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall t \leq \delta, \quad |J(T_t(\Omega)) - J(\Omega)| < \varepsilon.$$

In particular there exists $N > 0$ such that for all $n \geq N$, $t_n \leq \delta$ and

$$\forall n \geq N, \quad |J(T_n(\Omega)) - J(\Omega)| = |J(T_{t_n}(\Omega)) - J(\Omega)| < \varepsilon,$$

and we have the d_G-continuity for the subsequence $\{T_n\}$. □

Remark 6.2.
The conclusions of Theorems 6.1, 6.2, and 6.3 are generic. They also have their counterpart in the constrained case. The difficulty lies in the second part of the theorem, which requires a special construction to make sure that the family of transformations $\{T_t : 0 \leq t \leq \tau\}$ constructed from the sequence $\{T_n\}$ are homeomorphisms of $\overline{D}$. □

Chapter 5

Metrics via Characteristic Functions

1 Introduction

The constructions of the metric topologies of Chapter 3 are limited to families of sets which are the image of a fixed set by a family of homeomorphisms or diffeomorphisms. If it is connected, bounded, or C^k, $k \geq 1$, then the images will have the same properties under C^k-transformations of $\mathbf{R}^{\mathrm{N}}$. In this chapter we considerably enlarge the family of available sets by relaxing the smoothness assumption to the mere Lebesgue measurability and even just measurability to include *Hausdorff measures*. This is done by identifying $\Omega \subset \mathbf{R}^{\mathrm{N}}$ with its *characteristic function*

$$\chi_\Omega(x) \stackrel{\text{def}}{=} \left\{ \begin{matrix} 1, & \text{if } x \in \Omega \\ 0, & \text{if } x \notin \Omega \end{matrix} \right\} \text{ if } \Omega \neq \varnothing \quad \text{and} \quad \chi_\varnothing \stackrel{\text{def}}{=} 0. \tag{1.1}$$

We first introduce an Abelian group structure[1] on characteristic functions of *measurable*[2] subsets of a fixed holdall D in section 2 and then construct complete metric spaces of equivalence classes of measurable characteristic functions via the L^p-norms, $1 \leq p < \infty$, that turn out to be algebraically and topologically equivalent.

Starting with section 3, we specialize to the Lebesgue measure in $\mathbf{R}^{\mathrm{N}}$. The complete topology generated by the metric defined as the L^p-norm of the difference of two characteristic functions is called the *strong topology* in section 3.1. In section 3.2 we consider the *weak L^p-topology* on the family of characteristic functions. Weak limits of sequences of characteristic functions are functions with values in $[0, 1]$ which belong to the closed convex hull of the equivalence classes of measurable characteristic functions. This occurs in optimization problems where the function to be optimized depends on the solution of a partial differential equation on the variable domain, as was shown, for instance, by F. Murat [**1**] in 1971. It usually corresponds to the appearance of a *microstructure* or a *composite material* in mechanics.

[1] Cf. the group structure on subsets of a holdall D in section 2.2 of Chapter 2.

[2] Measurable in the sense of L. C. Evans and R. F. Gariepy [**1**]. It simultaneously covers Hausdorff and Lebesgue measures.

One important example in that category was the analysis of the optimal thickness of a circular plate by K.-T. CHENG and N. OLHOFF [**2**, **1**] in 1981. The reader is referred to the work of F. MURAT and L. TARTAR [**3**, **1**], initially published in 1985, for a comprehensive treatment of the calculus of variations and homogenization. Section 3.3 deals with the question of finding a *nice representative* in the equivalence class of sets. Can it be chosen open? We introduce the *measure theoretic representative* and characterize its interior, exterior, and boundary. It will be used later in section 6.4 to prove the compactness of the family of Lipschitzian domains verifying a uniform cone property. Section 3.4 shows that the family of convex subsets of a fixed bounded holdall is closed in the strong topology.

The use of the L^p-topologies is illustrated in section 4 by revisiting the optimal design problem studied by J. CÉA and K. MALANOWSKI [**1**] in 1970. By relaxing the family of characteristic functions to functions with values in the interval $[0, 1]$ a saddle point formulation is obtained. Functions with values in the interval $[0, 1]$ can also be found in the modeling of a dam by H. W. ALT [**1**] and H. W. ALT and G. GILARDI [**1**], where the "liquid saturation" in the soil is a function with values in $[0, 1]$. Another problem amenable to that formulation is the buckling of columns as will be illustrated in section 5 using the work of S. J. COX and M. L. OVERTON [**1**]. It is one of the very early optimal design problems, formulated by Lagrange in 1770.

The *Caccioppoli* or *finite perimeter sets* of the celebrated Plateau problem are revisited in section 6. Their characteristic function is a function of bounded variation. They provide the first example of compact families of characteristic functions in $L^p(D)$-strong, $1 \le p < \infty$. This was developed mainly by R. CACCIOPPOLI [**1**] and E. DE GIORGI [**1**] in the context of J. A. F. PLATEAU [**1**]'s problem of minimal surfaces. Section 6.3 exploits the embedding[3] of $\mathrm{BV}(D) \cap L^\infty(D)$ into the Sobolev spaces $W^{\varepsilon,p}(D)$, $0 < \varepsilon < 1/p$, $p \ge 1$, to introduce a cascade of complete metrics between $L^p(D)$ and $\mathrm{BV}(D)$ on characteristic functions. In section 6.4 we show that the family of Lipschitzian domains in a fixed bounded holdall verifying the uniform cone property of section 6.4.1 in Chapter 2 is also compact. This condition naturally yields a uniform bound on the perimeter of the sets in the family and hence can be viewed as a special case of the first compactness theorem. Section 7 gives an example of the use of the perimeter in the Bernoulli free boundary problem and in particular for the water wave. There the energy associated with the surface tension of the water is proportional to the perimeter, that is, the surface area of the free boundary.

2 Abelian Group Structure on Measurable Characteristic Functions

2.1 Group Structure on $\mathrm{X}_\mu(\mathbf{R}^\mathrm{N})$

From section 2.2 in Chapter 2, the Abelian group structure on the subsets of $\mathbf{R}^\mathrm{N}$ extends to the corresponding family of their characteristic functions.

[3] $\mathrm{BV}(D)$ is the space of functions of bounded variations in D.

Theorem 2.1. *Let $\varnothing \neq D \subset \mathbf{R}^{\mathrm{N}}$. The space of characteristic functions* $\mathrm{X}(D)$ *endowed with the multiplication* $\triangle$ *and the neutral multiplicative element* $\chi_\varnothing = 0$

$$\chi_A \triangle \chi_B \stackrel{\text{def}}{=} |\chi_A - \chi_B| = \chi_{(A\cap \complement B)\cup(B\cap\complement A)} = \chi_{A\triangle B} \tag{2.1}$$

is an Abelian group, that is,

$$\chi_A \triangle \chi_B = \chi_B \triangle \chi_A, \quad \chi_A \triangle \chi_\varnothing = \chi_A, \quad \chi_A \triangle \chi_A = \chi_\varnothing, \quad \chi_A^{-1} = \chi_A. \tag{2.2}$$

Proof. Note that the underlying group structure on subsets of D is the one of section 2.2 of Chapter 2. The product is well-defined in $\mathrm{X}(D)$ since it is the characteristic function of a subset of $A\cup B$ in D. The first two identities (2.2) are obvious. As for the inverse

$$\chi_A \triangle \chi_B = \chi_\varnothing \quad \Leftrightarrow \quad |\chi_A - \chi_B| = 0 \quad \Leftrightarrow \quad B = A,$$

and there is a unique multiplicative inverse $\chi_A^{-1} = \chi_A$ or $\chi_A \triangle \chi_A = \chi_\varnothing$. □

Remark 2.1.
We chose the product terminology $\triangle$, but we could have chosen to call that algebraic operation an addition $\oplus$. □

2.2 Measure Spaces

Definition 2.1.
Let X be a set and $\mathcal{P}(X)$ be the collection of subsets of X.

(i) A mapping $\mu : \mathcal{P}(X) \to [0,\infty]$ is called a *measure*[4] on X if

  (a) $\mu(\varnothing) = 0$, and
  (b) $\mu(A) \le \sum_{k=1}^\infty \mu(A_k)$, whenever $A \subset \cup_{k=1}^\infty A_k$.

(ii) A subset $A \subset X$ is *σ-finite with respect to* μ if it can be written $A = \cup_{k=1}^\infty B_k$, where B_k is μ-measurable and $\mu(B_k) < \infty$ for $k = 1, 2, \dots$.

(iii) A measure μ on $\mathbf{R}^{\mathrm{N}}$ is a *Radon* measure if μ is Borel regular and $\mu(K) < \infty$ for each compact set $K \subset \mathbf{R}^{\mathrm{N}}$. □

The *Lebesgue measure* m_N on $\mathbf{R}^{\mathrm{N}}$ is a Radon measure since $\mathbf{R}^{\mathrm{N}}$ is σ-finite with respect to m_N. The *s-dimensional Hausdorff measure* H_s, $0 \le s < N$, on $\mathbf{R}^{\mathrm{N}}$ (cf. section 3.2.3 in Chapter 2) is *Borel regular* but not necessarily a Radon measure since bounded H_s-measurable subsets of $\mathbf{R}^{\mathrm{N}}$ are not necessarily σ-finite with respect to H_s. Yet, a Borel regular measure μ can be made σ-finite by restricting it to a μ-measurable set $A \subset \mathbf{R}^{\mathrm{N}}$ such that $\mu(A) < \infty$ or, more generally, to a σ-finite μ-measurable set $A \subset \mathbf{R}^{\mathrm{N}}$. For smooth submanifolds of dimension s, s an integer, H_s gives the same area as the integral of the canonical density in section 3.2.2 of Chapter 2.

[4] A measure in the sense of L. C. Evans and R. F. Gariepy [**1**, Chap. 1]. It is called an *outer measure* in most texts.

Given a μ-measurable subset $D \subset \mathbf{R}^{\mathrm{N}}$, let $L^p(D,\mu)$ denote the Banach space of equivalence classes of μ-measurable functions such that $\int |f|^p \, d\mu < \infty$, $1 \le p < \infty$, and let $L^\infty(D,\mu)$ be the space of equivalence classes of μ-measurable functions such that $\operatorname{ess\,sup}_D |f| < \infty$. Denote by $[A]_\mu$ the equivalence class of μ-measurable subsets of D that are equal almost everywhere and

$$\mathrm{X}_\mu(D) \overset{\text{def}}{=} \{\chi_A : A \subset D \text{ is } \mu\text{-measurable}\}.$$

Since $A \mathbin{\triangle} B$ is μ-measurable for A and B μ-measurable, $\{[A]_\mu : A \subset D$ is μ-measurable$\}$ is a subgroup of $\mathcal{P}(D)$ and

$$\mathrm{X}_\mu(D) \cap L^1(D,\mu) = \{\chi_A : A \subset D \text{ is } \mu\text{-measurable and } \mu(A) < \infty\} \tag{2.3}$$

is a group in $L^p(D,\mu)$ with respect to $\triangle$. When $\mu = \mathrm{m}_N$, we drop the subscript μ.

2.3 Complete Metric for Characteristic Functions in L^p-Topologies

Now that we have revealed the underlying group structure of $\mathrm{X}_\mu(D) \cap L^1(D,\mu)$, we construct a metric on that group. We have a structure similar to the Courant metric of A. M. Micheletti [**1**] in Chapter 3 on groups of transformations of $\mathbf{R}^{\mathrm{N}}$. We identify equivalence classes of μ-measurable subsets of D and characteristic functions via the bijection $[A]_\mu \leftrightarrow \chi_A$.

Theorem 2.2. *Let $1 \le p < \infty$, let μ be a measure on $\mathbf{R}^{\mathrm{N}}$, and let $\varnothing \ne D \subset \mathbf{R}^{\mathrm{N}}$ be μ-measurable.*

(i) $\mathrm{X}_\mu(D) \cap L^1(D,\mu)$ *is closed in* $L^p(D,\mu)$*. The function*

$$\rho_D([A_2]_\mu, [A_1]_\mu) \overset{\text{def}}{=} \|\chi_{A_2} - \chi_{A_1}\|_{L^p(D,\mu)}$$

defines a complete metric structure on the Abelian group $\mathrm{X}_\mu(D) \cap L^1(D,\mu)$ *that makes it a topological group. If $\mu(D) < \infty$, then* $\mathrm{X}_\mu(D) \cap L^1(D,\mu) = \mathrm{X}_\mu(D)$.

(ii) *If, in addition, D is σ-finite with respect to μ for a family $\{D_k\}$ of μ-measurable subsets of D such that $\mu(D_k) < \infty$, for all $k \ge 1$, then* $\mathrm{X}_\mu(D)$ *is closed in* $L^p_{\mathrm{loc}}(D,\mu)$ *and*

$$\rho([A_2]_\mu, [A_1]_\mu) \overset{\text{def}}{=} \sum_{k=1}^{\infty} \frac{1}{2^n} \frac{\|\chi_{A_2} - \chi_{A_1}\|_{L^p(D_k,\mu)}}{1 + \|\chi_{A_2} - \chi_{A_1}\|_{L^p(D_k,\mu)}}$$

defines a complete metric structure on the Abelian group $\mathrm{X}_\mu(D)$ *that makes it a topological group. When μ is a Radon measure on $\mathbf{R}^{\mathrm{N}}$, the assumption that D is σ-finite with respect to μ can be dropped.*

Remark 2.2.
For the Lebesgue measure m_N and D measurable (resp., measurable and bounded), $\mathrm{X}(D)$ is a topological Abelian group in $L^p_{\mathrm{loc}}(D,\mathrm{m}_N)$ (resp., $L^p(D,\mathrm{m}_N)$). For Hausdorff measures H_s, $0 \le s < N$, the theorem does not say more. □

Proof. (i) Let $\{\chi_{A_n}\}$ be a Cauchy sequence of μ-measurable characteristic functions in $\mathrm{X}_\mu(D) \cap L^1(D,\mu) \subset L^p(D,\mu)$ converging to some $f \in L^p(D,\mu)$. There exists a subsequence, still denoted by $\{\chi_{A_n}\}$, such that $\chi_{A_n}(x) \to f(x)$ in D except for a subset Z of zero μ-measure. Hence $0 = \chi_{A_n}(x)\,(1-\chi_{A_n}(x)) \to f(x)\,(1-f(x))$. Define the set

$$A \stackrel{\text{def}}{=} \{x \in D\backslash Z \,:\, f(x) = 1\}.$$

Clearly A is μ-measurable and $\chi_A = f$ on $D\backslash Z$ since $f(x)\,(1-f(x)) = 0$ on $D\backslash Z$. Hence $f = \chi_A$ almost everywhere on D, $\chi_{A_n} \to \chi_A$ in $L^p(D,\mu)$, and $\chi_A \in \mathrm{X}_\mu(D) \cap L^1(D,\mu)$. Finally, the metric is compatible with the group structure

$$\rho_D([\Omega_2]_\mu, [\Omega_1]_\mu) \stackrel{\text{def}}{=} \|\chi_{\Omega_2} - \chi_{\Omega_1}\|_{L^p} = \|\chi_{\Omega_2} \,\triangle\, \chi_{\Omega_1}^{-1}\|_{L^p} = \|\chi_{\Omega_1} \,\triangle\, \chi_{\Omega_2}^{-1}\|_{L^p}$$

as in Chapter 3 and $\mathrm{X}_\mu(D) \cap L^1(D,\mu)$ is a topological group.

(ii) From part (i) using the Fréchet topology associated with the family of seminorms $q_k(f) = \|f\|_{L^p(D_k)}$, $k \geq 1$. □

For $p = 1$ and μ-measurable subsets A_1 and A_2 in D, the metric $\rho_D([A_2]_\mu, [A_1]_\mu)$ is the μ-measure of the symmetric difference

$$A_1 \Delta A_2 = (\complement A_1 \cap A_2) \cup (\complement A_2 \cap A_1)$$

and for $1 \leq p < \infty$ all the topologies are equivalent on $\mathrm{X}_\mu(D) \cap L^1(D,\mu)$.

Theorem 2.3. *Let $1 \leq p < \infty$, let μ be a measure on $\mathbf{R}^\mathrm{N}$, and let $\varnothing \neq D \subset \mathbf{R}^\mathrm{N}$ be μ-measurable.*

(i) *The topologies induced by $L^p(D,\mu)$ on $\mathrm{X}_\mu(D) \cap L^1(D,\mu)$ are all equivalent for $1 \leq p < \infty$.*

(ii) *If, in addition, D is σ-finite with respect to μ for a family $\{D_k\}$ of μ-measurable subsets of D such that $\mu(D_k) < \infty$, for all $k \geq 1$, then the topologies induced by $L^p_{\mathrm{loc}}(D,\mu)$ on $\mathrm{X}_\mu(D)$ are all equivalent for $1 \leq p < \infty$.*

Proof. For D μ-measurable and $1 \leq p < \infty$, $\mathrm{X}_\mu(D) \cap L^p(D,\mu) = \mathrm{X}_\mu(D) \cap L^1(D,\mu)$, and for any χ and $\overline{\chi}$ in $\mathrm{X}_\mu(D) \cap L^1(D,\mu)$

$$\|\overline{\chi} - \chi\|^p_{L^p(D,\mu)} = \int_D |\overline{\chi} - \chi(x)|^p \, d\mu = \int_D |\overline{\chi} - \chi(x)| \, d\mu = \|\overline{\chi} - \chi\|_{L^1(D,\mu)},$$

since $\chi(x)$ and $\overline{\chi}(x)$ are either 0 or 1 almost everywhere in D. Therefore the linear (identity) mapping between $\mathrm{X}_\mu(D) \cap L^p(D,\mu)$ and $\mathrm{X}_\mu(D) \cap L^1(D,\mu)$ is bicontinuous and the topologies are equivalent. □

Remark 2.3.
At this stage, it is not clear what the tangent space to this topological Abelian group is. If μ is a *Radon measure*, then for all $\varphi \in \mathcal{D}(\mathbf{R}^\mathrm{N})$

$$\frac{1}{\mu(B_\varepsilon(x))} \int_{\mathbf{R}^\mathrm{N}} \left[\chi_A \,\triangle\, \chi_{B_\varepsilon(x)}\right] \triangle\, \chi_A^{-1}\, \varphi \, d\mu = \frac{1}{\mu(B_\varepsilon(x))} \int_{\mathbf{R}^\mathrm{N}} \chi_{B_\varepsilon(x)}\, \varphi \, d\mu$$

and by the Lebesgue–Besicovitch differentiation theorem

$$\frac{1}{\mu(B_\varepsilon(x))}\int_{\mathbf{R}^\mathrm{N}} \left[\chi_A \bigtriangleup \chi_{B_\varepsilon(x)}\right] \bigtriangleup \chi_A^{-1}\, \varphi\, d\mu \to \varphi(x),\ \mu \text{ a.e. in } \mathbf{R}^\mathrm{N}. \qquad \square$$

3 Lebesgue Measurable Characteristic Functions

In this section we specialize to the Lebesgue measure and the corresponding family of characteristic functions

$$\mathrm{X}(\mathbf{R}^\mathrm{N}) \stackrel{\text{def}}{=} \left\{\chi_\Omega \,:\, \forall \Omega \text{ Lebesgue measurable in } \mathbf{R}^\mathrm{N}\right\}. \tag{3.1}$$

Clearly $\mathrm{X}(\mathbf{R}^\mathrm{N}) \subset L^\infty(\mathbf{R}^\mathrm{N})$, and for all $p \geq 1$, $\mathrm{X}(\mathbf{R}^\mathrm{N}) \subset L^p_{\mathrm{loc}}(\mathbf{R}^\mathrm{N})$. Also associate with $\varnothing \neq D \subset \mathbf{R}^\mathrm{N}$ the set

$$\mathrm{X}(D) \stackrel{\text{def}}{=} \{\chi_\Omega \,:\, \forall \Omega \text{ Lebesgue measurable in } D\}. \tag{3.2}$$

The above definitions are special cases of the definitions of section 2.3, and Theorems 2.2 and 2.3 apply to m_N as a Radon measure.

3.1 Strong Topologies and C^∞-Approximations

In view of the equivalence Theorem 2.3, we introduce the notion of strong convergence.

Definition 3.1. (i) If D is a bounded measurable subset of $\mathbf{R}^\mathrm{N}$, a sequence $\{\chi_n\}$ in $\mathrm{X}(D)$ is said to be *strongly convergent* in D if it converges in $L^p(D)$-strong for some p, $1 \leq p < \infty$.

(ii) A sequence $\{\chi_n\}$ in $\mathrm{X}(\mathbf{R}^\mathrm{N})$ is said to be *locally strongly convergent* if it converges in $L^p_{\mathrm{loc}}(\mathbf{R}^\mathrm{N})$-strong for some p, $1 \leq p < \infty$. $\square$

The following approximation theorem will also be useful

Theorem 3.1. *Let Ω be an arbitrary Lebesgue measurable subset of $\mathbf{R}^\mathrm{N}$. There exists a sequence $\{\Omega_n\}$ of open C^∞-domains in $\mathbf{R}^\mathrm{N}$ such that*

$$\chi_{\Omega_n} \to \chi_\Omega \quad \text{in } L^1_{\mathrm{loc}}(\mathbf{R}^\mathrm{N}).$$

Proof. The construction of the family of C^∞-domains $\{\Omega_n\}$ can be found in many places (cf., for instance, E. Giusti [**1**, sect. 1.14, p. 10 and Lem. 1.25, p. 23]). Associate with $\chi = \chi_\Omega$ the sequence of convolutions $f_n = \chi * \rho_n$ for a sequence $\{\rho_n\}$ of symmetric mollifiers. By construction, $0 \leq f_n \leq 1$. For t, $0 < t < 1$, define

$$F_{nt} \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^\mathrm{N} \,:\, f_n(x) > t\right\}.$$

By definition, $f_n - \chi > t$ in $F_{nt}\backslash\Omega$, $\chi - f_n > 1 - t$ in $\Omega\backslash F_{nt}$, and

$$|\chi - \chi_{F_{nt}}| \leq \frac{1}{\min\{t, 1-t\}}\, |\chi - f_n| \text{ a.e. in } \mathbf{R}^\mathrm{N}.$$

By Sard's Theorem 4.3 of Chapter 2, the set F_{nt} is a C^∞-domain for almost all t in $(0,1)$. Fix α, $0<\alpha<1/2$, and choose a sequence $\{t_n\}$ such that, for all n, $\alpha \le t_n \le 1-\alpha$, and define $\Omega_n \overset{\text{def}}{=} F_{nt_n}$. Therefore

$$|\chi_\Omega - \chi_{\Omega_n}| \le \frac{1}{\min\{t_n, 1-t_n\}} |\chi_\Omega - f_n| \le \frac{1}{\alpha} |\chi_\Omega - f_n| \text{ a.e. in } \mathbf{R}^{\mathrm{N}}$$

and for any bounded measurable subset D of $\mathbf{R}^{\mathrm{N}}$

$$\|\chi_\Omega - \chi_{\Omega_n}\|_{L^1(D)} \le \alpha^{-1} \|\chi_\Omega - f_n\|_{L^1(D)},$$

which goes to zero as n goes to infinity. □

3.2 Weak Topologies and Microstructures

Some *shape optimization* problems lead to apparent paradoxes. Their solution is no longer a geometric domain associated with a characteristic function, but a *fuzzy domain* associated with the relaxation of a characteristic function to a function with values ranging in $[0,1]$. The intuitive notion of a geometric domain is relaxed to the notion of a *probability distribution* of the presence of points of the set. When the underlying problem involves two different materials characterized by two constants $k_1 \neq k_2$, the occurrence of such a solution can be interpreted as the *mixing* or *homogenization* of the two materials at the microscale. This is also referred to as a *composite material* or a *microstructure*. Somehow this is related to the fact that the strong convergence needs to be relaxed to the weak L^p-convergence and the space $\mathrm{X}(D)$ needs to be suitably enlarged.

Even if $\mathrm{X}(D)$ is strongly closed and bounded in $L^p(D)$, it is not strongly compact. However, for $1<p<\infty$ its closed convex hull $\overline{\mathrm{co}}\,\mathrm{X}(D)$ is weakly compact in the reflexive Banach space $L^p(D)$. In fact,

$$\boxed{\overline{\mathrm{co}}\,\mathrm{X}(D) = \{\chi \in L^p(D) : \chi(x) \in [0,1] \text{ a.e. in } D\}.} \tag{3.3}$$

Indeed, by definition $\overline{\mathrm{co}}\,\mathrm{X}(D) \subset \{\chi \in L^p(D) : \chi(x) \in [0,1] \text{ a.e. in } D\}$. Conversely any χ that belongs to the right-hand side of (3.3) can be approximated by a sequence of convex combinations of elements of $\mathrm{X}(D)$. Choose

$$\chi_n = \sum_{m=1}^{n} \frac{1}{n} \chi_{B_{nm}}, \quad B_{nm} = \left\{x : \chi(x) \ge \frac{m}{n}\right\},$$

for which $|\chi_n(x) - \chi(x)| < 1/n$. The elements of $\overline{\mathrm{co}}\,\mathrm{X}(D)$ are not necessarily characteristic functions of a domain; that is, the identity

$$\chi(x)\,(1-\chi(x)) = 0 \text{ a.e. in } D$$

is not necessarily satisfied.

We first give a few basic results and then consider a classical example from the *theory of homogenization* of differential equations.

Lemma 3.1. *Let D be a bounded open subset of $\mathbf{R}^N$, let K be a bounded subset of $\mathbf{R}$, and let*

$$\mathcal{K} \stackrel{\text{def}}{=} \{k\colon D \to \mathbf{R} : k \text{ is measurable and } k(x) \in K \text{ a.e. in } D\}.$$

(i) *For any p, $1 \le p < \infty$, and any sequence $\{k_n\} \subset \mathcal{K}$ the following statements are equivalent:*

  (a) *$\{k_n\}$ converges in $L^\infty(D)$-weak$\star$;*

  (b) *$\{k_n\}$ converges in $L^p(D)$-weak;*

  (c) *$\{k_n\}$ converges in $\mathcal{D}(D)'$,*

  where $\mathcal{D}(D)'$ is the space of scalar distributions on D.

(ii) *If K is bounded, closed, and convex, then $\mathcal{K}$ is convex and compact in $L^\infty(D)$-weak$\star$, $L^p(D)$-weak, and $\mathcal{D}(D)'$.*

The above results remain true in the vectorial case when $\mathcal{K}$ is the set of mappings $k\colon D \to K$ for some bounded subset $K \subset \mathbf{R}^p$ and a finite integer $p \ge 1$.

Proof. (i) It is clear that (a) $\Rightarrow$ (b) $\Rightarrow$ (c). To prove that (c) $\Rightarrow$ (a) recall that since K is bounded, there exists a constant $c > 0$ such that $K \subset cB$, where B denotes the unit ball in $\mathbf{R}^N$. By density of $\mathcal{D}(D)$ in $L^1(D)$, any φ in $L^1(D)$ can be approximated by a sequence $\{\varphi_m\} \subset \mathcal{D}(D)$ such that $\varphi_m \to \varphi$ in $L^1(D)$. So for each $\varepsilon > 0$ there exists $M > 0$ such that

$$\forall m \ge M, \quad \|\varphi_m - \varphi\|_{L^1(D)} \le \frac{\varepsilon}{4c}.$$

Moreover, there exists $N > 0$ such that

$$\forall n \ge N, \forall \ell \ge N, \quad \left| \int_D \varphi_M (k_n - k_\ell)\, dx \right| \le \frac{\varepsilon}{2}.$$

Hence for each $\varepsilon > 0$, there exists N such that for all $n \ge N$ and $\ell \ge N$

$$\begin{aligned}\left| \int_D \varphi (k_n - k_\ell)\, dx \right| &\le \left| \int_D \varphi_M (k_n - k_\ell)\, dx \right| + \left| \int_D (\varphi - \varphi_M)(k_n - k_\ell)\, dx \right| \\ &\le \frac{\varepsilon}{2} + 2c\|\varphi - \varphi_M\|_{L^1(D)} \le \varepsilon.\end{aligned}$$

(ii) When K is bounded, closed, and convex, $\mathcal{K}$ is also bounded, closed, and convex. Since $L^2(D)$ is a Hilbert space, $\mathcal{K}$ is weakly compact. In view of the equivalences of part (i), $\mathcal{K}$ is also sequentially compact in all the other weak topologies.[5] □

[5] In a metric space the compactness is equivalent to the sequential compactness. For the weak topology we use the fact that if E is a separable normed space, then, in its topological dual E', any closed ball is a compact metrizable space for the weak topology. Since $\mathcal{K}$ is a bounded subset of the normed reflexive separable Banach space $L^p(D)$, $1 \le p < \infty$, the weak compactness of $\mathcal{K}$ coincides with its weak sequential compactness (cf. J. Dieudonné [**1**, Vol. 2, Chap. XII, sect. 12.15.9, p. 75]).

In view of this equivalence we adopt the following terminology.

Definition 3.2.
Let $\varnothing \neq D \subset \mathbf{R}^{\mathrm{N}}$ be bounded open. A sequence $\{\chi_n\}$ in $\mathrm{X}(D)$ is said to be *weakly convergent* if it converges for some topology between $L^\infty(D)$-weak$\star$ and $\mathcal{D}(D)'$. $\square$

It is interesting to observe that working with the weak convergence makes sense only when the limit element is not a characteristic function.

Theorem 3.2. *Let the assumptions of the theorem be satisfied. Let* $\{\chi_n\}$ *and* χ *be elements of* $\mathrm{X}(\mathbf{R}^{\mathrm{N}})$ (resp., $\mathrm{X}(D)$) *such that* $\chi_n \rightharpoonup \chi$ *weakly in* $L^2(D)$ (resp., $L^2_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$)*. Then for all* p*,* $1 \leq p < \infty$*,*

$$\chi_n \to \chi \text{ strongly in } L^p(D)\big(\text{resp., } L^p_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})\big).$$

Proof. It is sufficient to prove the result for D. The strong $L^2(D)$-convergence follows from the property

$$\begin{aligned}
&\int_D |\chi_n|^2\,dx = \int_D \chi_n 1\,dx \to \int_D \chi 1\,dx = \int_D |\chi|^2\,dx \\
\Rightarrow &\int_D |\chi_n - \chi|^2 dx = \int_D |\chi_n|^2 - 2\,\chi\chi_n + |\chi|^2 dx \to \int_D |\chi|^2 - 2\,\chi\chi + |\chi|^2 dx = 0.
\end{aligned}$$

The convergence for any $p \geq 1$ now follows from Theorem 2.2. $\square$

Working with the weak convergence creates new phenomena and difficulties. For instance when a characteristic function is present in the coefficient of the higher-order term of a differential equation the weak convergence of a sequence of characteristic functions $\{\chi_n\}$ to some element χ of $\overline{\mathrm{co}}\,\mathrm{X}(D)$,

$$\chi_n \rightharpoonup \chi \text{ in } L^2(D)\text{-weak} \quad \Rightarrow \chi_n \rightharpoonup \chi \text{ in } L^\infty(D)\text{-weak}\star,$$

does not imply the weak convergence in $H^1(D)$ of the sequence $\{y(\chi_n)\}$ of solutions to the solution of the differential equation corresponding to $y(\chi)$,

$$y(\chi_n) \not\rightharpoonup y(\chi) \text{ in } H^1(D)\text{-weak}.$$

By compactness[6] of the injection of $H^1(D)$ into $L^2(D)$ this would have implied convergence in $L^2(D)$-strong:

$$y(\chi_n) \to y(\chi) \text{ in } L^2(D)\text{-strong}.$$

This fact was pointed out in 1971 by F. Murat [**1**] in the following example, which will be rewritten to emphasize the role of the characteristic function.

[6] This is true since D is a bounded Lipschitzian domain. Examples of domains between two spirals can be constructed where the injection is not compact.

Example 3.1.
Consider the following boundary value problem:

$$\begin{cases} -\dfrac{d}{dx}\left(k\dfrac{dy}{dx}\right) + ky = 0 \text{ in } D =]0,1[\ , \\ y(0) = 1 \quad \text{and} \quad y(1) = 2, \end{cases} \tag{3.4}$$

where

$$\mathcal{K} = \{k = k_1(x)\chi + k_2(x)(1-\chi) : \chi \in \mathrm{X}(D)\} \tag{3.5}$$

with

$$k_1(x) = 1 - \sqrt{\frac{1}{2} - \frac{x^2}{6}}, \quad k_2(x) = 1 + \sqrt{\frac{1}{2} - \frac{x^2}{6}}. \tag{3.6}$$

This is equivalent to

$$\mathcal{K} = \{k \in L^\infty(0,1) : k(x) \in \{k_1(x), k_2(x)\} \quad \text{a.e. in } [0,1]\}\,. \tag{3.7}$$

Associate with the integer $p \geq 1$ the function

$$k^p(x) = \begin{cases} 1 - \sqrt{\dfrac{1}{2} - \dfrac{x^2}{6}}, & \dfrac{m}{p} < x \leq \dfrac{2m+1}{2p}, \\ 1 + \sqrt{\dfrac{1}{2} - \dfrac{x^2}{6}}, & \dfrac{2m+1}{2p} < x \leq \dfrac{m+1}{p}, \end{cases} \qquad 0 \leq m \leq p-1, \tag{3.8}$$

and the characteristic function

$$\chi_p(x) = \begin{cases} 1, & \dfrac{m}{p} < x \leq \dfrac{2m+1}{2p}, \\ 0, & \dfrac{2m+1}{2p} < x \leq \dfrac{m+1}{p}, \end{cases} \qquad 0 \leq m \leq p-1. \tag{3.9}$$

It is readily seen that $k^p = k_1\, \chi_p + k_2\, (1-\chi_p)$. He shows that for each p, $k^p \in \mathcal{K}$ (resp., $\chi_p \in \mathrm{X}(D)$) and that

$$k^p \rightharpoonup k_\infty = 1 \quad \left(\text{resp., } \chi_p \rightharpoonup \frac{1}{2}\right) \text{ in } L^\infty(0,1)\text{-weak}\star, \tag{3.10}$$

$$\frac{1}{k^p} \rightharpoonup \frac{1}{2}\left[\frac{1}{k_1} + \frac{1}{k_2}\right] = \frac{1}{1/2 + x^2/6} \text{ in } L^\infty(0,1)\text{-weak}\,\star\,. \tag{3.11}$$

Moreover,

$$y_p \rightharpoonup y \text{ in } H^1(0,1)\text{-weak}, \tag{3.12}$$

where y_p denotes the solution of (3.4) corresponding to $k = k^p$ and y the solution of the boundary value problem

$$\begin{cases} -\dfrac{d}{dx}\left[\left(\dfrac{1}{2} + \dfrac{x^2}{6}\right)\dfrac{dy}{dx}\right] + y = 0 \text{ in }]0,1[\ , \\ y(0) = 1, \quad y(1) = 2. \end{cases} \tag{3.13}$$

Define the function

$$k_H(x) = \frac{1}{2} + \frac{x^2}{6}, \tag{3.14}$$

which corresponds to

$$\chi_H(x) = \frac{1}{2}\left[1 + \sqrt{\frac{1}{2} - \frac{x^2}{6}}\right] \in \overline{\text{co}}\,\text{X}(D). \tag{3.15}$$

Notice that k_H appears in the second-order term and k_∞ in the zeroth-order term in (3.13):

$$-\frac{d}{dx}\left[k_H \frac{dy}{dx}\right] + k_\infty y = 0 \text{ in }]0,1[\,. \tag{3.16}$$

It is easy to check that

$$y(x) = 1 + x^2 \text{ in } [0,1], \tag{3.17}$$

which is not equal to the solution y_∞ of (3.4) for the weak limit $k_\infty = 1$:

$$y_\infty(x) = \frac{2(e^x - e^{-x}) + e^{1-x} - e^{-(1-x)}}{e - e^{-1}}. \tag{3.18}$$

□

To our knowledge this was the beginning of the theory of homogenization in France. This is only one part of F. Murat [**1**]'s example. He also constructs an objective function for which the lower bound is not achieved by an element of $\mathcal{K}$. This is a nonexistence result.

The above example uses space varying coefficients $k_1(x)$ and $k_2(x)$. However, it is still valid for two positive constants $k_1 > 0$ and $k_2 > 0$. It is easy to show that

$$k_\infty = \frac{k_1 + k_2}{2}, \quad \chi_\infty = \frac{1}{2}, \quad \text{and} \quad \frac{1}{k_H} = \frac{1}{2}\left[\frac{1}{k_1} + \frac{1}{k_2}\right], \quad \chi_H = \frac{k_2}{k_1 + k_2},$$

and the solution y of the boundary value problem (3.16) is given by

$$y(x) = \frac{2\sinh cx + \sinh c(1-x)}{\sinh c}, \quad \text{where} \quad c = \frac{k_1 + k_2}{2\sqrt{k_1 k_2}} \geq 1,$$

and the solution y_∞ by

$$y_\infty(x) = \frac{2\sinh x + \sinh(1-x)}{\sinh 1}.$$

Thus for $k_1 \neq k_2$, or equivalently $c > 1$, $y \neq y_\infty$.

In fact, using the same sequence of χ_p's, the sequence $y_p = y(\chi_p)$ of solutions of (4.5) weakly converges to $y_H = y(\chi_H)$, which is different from the solution $y_\infty = y(\chi_\infty)$ for $k_1 \neq k_2$. This is readily seen by noticing that since χ_∞ and χ_H are constant

$$-\Delta(k_H\, y_H) = \chi_\infty\, f = -\Delta(k_\infty\, y_\infty)$$

$$\Rightarrow\ k_H\, y_H = k_\infty\, y_\infty \quad \Rightarrow\ y_H = \frac{(k_1 + k_2)^2}{4k_1\, k_2}\, y_\infty \neq y_\infty, \text{ for } k_1 \neq k_2.$$

Despite this example, we shall see in the next section that in some cases we obtain the existence (and uniqueness) of a maximizer in $\overline{\text{co}}\,\mathrm{X}(D)$ which belongs to $\mathrm{X}(D)$.

3.3 Nice or Measure Theoretic Representative

Since the strong and weak L^p-topologies are defined on equivalence classes $[\Omega]$ of (Lebesgue) measurable subsets Ω of $\mathbf{R}^{\mathrm{N}}$, it is natural to ask if there is a *nice representative* that is generic of the class $[\Omega]$. For instance we have seen in Chapter 2 that within the equivalence class of a set of class C^k there is a unique open and a unique closed representative and that all elements of the class have the same interior, boundary, and exterior. The same question will again arise for finite perimeter sets. As an illustration of what is meant by a nice representative, consider the smiling and the expressionless suns in Figure 5.1. The expressionless sun is obtained by adding missing points and lines "inside" Ω and removing the rays "outside" Ω. This "restoring/cleaning" operation can be formalized as follows.

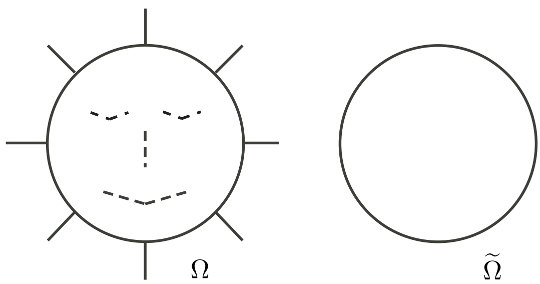


Figure 5.1. *Smiling sun Ω and expressionless sun $\widetilde{\Omega}$.*

Definition 3.3.
Associate with a Lebesgue measurable set Ω in $\mathbf{R}^{\mathrm{N}}$ the sets

$$\begin{aligned}
\Omega_0 &\overset{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : \exists \rho > 0 \text{ such that } \mathrm{m}\big(\Omega \cap B(x,\rho)\big) = 0\right\},\\
\Omega_1 &\overset{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : \exists \rho > 0 \text{ such that } \mathrm{m}\big(\Omega \cap B(x,\rho)\big) = \mathrm{m}\big(B(x,\rho)\big)\right\},\\
\Omega_{\bullet} &\overset{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : \forall \rho > 0 \text{ such that } 0 < \mathrm{m}\big(\Omega \cap B(x,\rho)\big) < \mathrm{m}\big(B(x,\rho)\big)\right\}
\end{aligned}$$

and the *measure theoretic exterior* O*, interior* I, and *boundary* $\partial_*\Omega$

$$\begin{aligned}
\mathrm{O} &\overset{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} \,:\, \lim_{r \searrow 0} \frac{\mathrm{m}(B(x,r) \cap \Omega)}{\mathrm{m}(B(x,r))} = 0\right\},\\
\mathrm{I} &\overset{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} \,:\, \lim_{r \searrow 0} \frac{\mathrm{m}(B(x,r) \cap \Omega)}{\mathrm{m}(B(x,r))} = 1\right\},\\
\partial_*\Omega &\overset{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} \,:\, 0 \le \liminf_{r \searrow 0} \frac{\mathrm{m}(B(x,r) \cap \Omega)}{\mathrm{m}(B(x,r))} < \limsup_{r \searrow 0} \frac{\mathrm{m}(B(x,r) \cap \Omega)}{\mathrm{m}(B(x,r))} \le 1\right\}.
\end{aligned}$$

We shall say that I is the *nice* or *measure theoretic representative* of Ω. $\square$

The six sets Ω_0, Ω_1, $\Omega_\bullet$, O, I, and $\partial_*\Omega$ are *invariant* for all sets in the equivalence class $[\Omega]$ of Ω. They are two different partitions of $\mathbf{R}^{\mathrm{N}}$

$$\Omega_0 \cup \Omega_1 \cup \Omega_\bullet = \mathbf{R}^{\mathrm{N}} \quad \text{and} \quad \mathrm{O} \cup \mathrm{I} \cup \partial_*\Omega = \mathbf{R}^{\mathrm{N}} \tag{3.19}$$

like $\operatorname{int} \complement\Omega \cup \operatorname{int}\Omega \cup \partial\Omega = \mathbf{R}^{\mathrm{N}}$. The next theorem links the six invariant sets and describes some of the interesting properties of the measure theoretic representative.

Theorem 3.3. *Let Ω be a Lebesgue measurable set in $\mathbf{R}^{\mathrm{N}}$ and let Ω_0, Ω_1, $\Omega_\bullet$,* O*,* I*, and $\partial_*\Omega$ be the sets constructed from Ω in Definition* 3.3.

(i) *The sets* O, I, *and $\partial_*\Omega$ are invariants of the equivalence class $[\Omega]$. Moreover, they are Borel measurable and*

$$\chi_{\mathrm{I}} = \chi_\Omega, \quad \chi_{\mathrm{O}} = \chi_{\complement\Omega}, \quad \textit{and} \quad \chi_{\partial_*\Omega} = 0 \quad \textit{a.e. in } \mathbf{R}^{\mathrm{N}}.$$

(ii) *The sets Ω_0, Ω_1, and $\Omega_\bullet$ are invariants of the equivalence class $[\Omega]$. The sets Ω_0 and Ω_1 are open, $\Omega_\bullet$ is closed, and*

$$\operatorname{int}\Omega \subset \Omega_1 = \operatorname{int}\mathrm{I},\ \operatorname{int}\complement\Omega \subset \Omega_0 = \operatorname{int}\mathrm{O},\ \partial\Omega \supset \overline{\complement\mathrm{I}} \cap \overline{\complement\mathrm{O}} = \Omega_\bullet \supset \partial_*\Omega, \tag{3.20}$$

$$\partial_*\Omega \cup \partial\mathrm{O} \cup \partial\mathrm{I} \subset \Omega_\bullet \subset \partial\Omega. \tag{3.21}$$

(iii) *The condition* $\mathrm{m}(\partial\Omega) = 0$ *implies* $\mathrm{m}(\partial\mathrm{I}) = 0$*. When* $\mathrm{m}(\partial\mathrm{I}) = 0$*, then* $\operatorname{int}\mathrm{I}$ *and* $\overline{\mathrm{I}}$ *can be chosen as the respective open and closed representatives of* Ω*. In particular, this is true for Lipschitzian sets and hence for convex sets.*

Proof. (i) All the sets involved depend only on χ_Ω almost everywhere. They are invariants of the equivalence class $[\Omega]$. From L. C. Evans and R. F. Gariepy [**1**, Lem. 2, sect. 5.11, p. 222, Cor. 3, sect. 1.7.1, p. 45], I, O, and $\partial_*\Omega$ are Borel measurable and $\mathrm{m}(\mathrm{I}\backslash\Omega \cup \Omega\backslash\mathrm{I}) = 0$, and

$$\lim_{r \searrow 0} \frac{\mathrm{m}(B(x,r) \cap \Omega)}{\mathrm{m}(B(x,r))} = 1 \text{ a.e. in } \Omega, \quad \lim_{r \searrow 0} \frac{\mathrm{m}(B(x,r) \cap \Omega)}{\mathrm{m}(B(x,r))} = 0 \text{ a.e. in } \complement\Omega.$$

Therefore, $\mathrm{m}(\partial_*\Omega) = 0$, $\chi_{\mathrm{I}} = \chi_\Omega$, and $\chi_{\mathrm{O}} = \chi_{\complement\Omega}$ almost everywhere in $\mathbf{R}^{\mathrm{N}}$.

(ii) We use the proof given by E. Giusti [**1**, Prop. 4.1, pp. 42–43] for Ω_0 and Ω_1. To show that Ω_0 is open, pick any point x in Ω_0. By construction, there exists $\rho > 0$ such that $m\big(\Omega \cap B(x,\rho)\big) = 0$. For any $y \in B(x,\rho)$ (that is, $|y - x| < \rho$), choose $\rho_0 = \rho - |x - y| > 0$. Then $B(y,\rho_0) \subset B(x,\rho)$ and $m\big(B(y,\rho_0) \cap \Omega\big) \le m\big(B(x,\rho) \cap \Omega\big) = 0$. So $B(x,\rho) \subset \Omega_0$ and Ω_0 is open. Similarly, by repeating the argument for $\complement\Omega$, we obtain that Ω_1 is open. By complementarity, $\Omega_\bullet$ is closed.

By definition, $\operatorname{int}\Omega \subset \Omega_1 \subset I$ and, since Ω_1 is open, $\Omega_1 \subset \operatorname{int}\mathrm{I}$. For all $x \in \operatorname{int}\mathrm{I}$, there exists $r > 0$ such that $B(x,r) \subset \mathrm{I}$ and

$$\mathrm{m}\big(B(x,r)\big) = \mathrm{m}\big(B(x,r) \cap \mathrm{I}\big) = \mathrm{m}\big(B(x,r) \cap \Omega\big).$$

Hence $\operatorname{int}\mathrm{I} \subset \Omega_1$ and necessarily $\operatorname{int}\mathrm{I} = \Omega_1$. Similarly, $\operatorname{int}\complement\Omega \subset \Omega_0 = \operatorname{int}\mathrm{O}$. By taking the union term by term of the two chains $\operatorname{int}\Omega \subset \operatorname{int}\mathrm{I} \subset \Omega_1 = \operatorname{int}\mathrm{I} \subset \mathrm{I}$

and $\operatorname{int} \complement\Omega \subset \Omega_0 = \operatorname{int} \mathrm{O} \subset \mathrm{O}$ and then the complement, we get $\partial\Omega \supset \overline{\complement \mathrm{I}} \cap \overline{\complement \mathrm{O}} = \Omega_\bullet \supset \partial_*\Omega$. From the first two relations (3.20), $\Omega_1 = \operatorname{int} \mathrm{I}$ implies $\complement\Omega_1 = \overline{\complement \mathrm{I}} \subset \overline{\mathrm{O}}$ and $\Omega_0 = \operatorname{int} \mathrm{O}$ implies $\complement\Omega_0 = \overline{\complement \mathrm{O}} \subset \overline{\mathrm{I}}$. Therefore, $\Omega_\bullet = \complement\Omega_1 \cap \complement\Omega_0 \supset \overline{\complement \mathrm{I}} \cap \overline{\mathrm{I}} = \partial \mathrm{I}$ and similarly $\Omega_\bullet = \complement\Omega_1 \cap \complement\Omega_0 \supset \overline{\complement \mathrm{O}} \cap \overline{\mathrm{O}} = \partial \mathrm{O}$.

(iii) From part (ii). □

The inclusions (3.20) indicate that the measure theoretic interior and exterior are enlarged and that the measure theoretic boundary is reduced. In general those operations do not commute with the set theoretic operations. For instance, they do not commute with the closure, as can be seen from the following simple example.

Example 3.2.
Consider the set Ω of all rational numbers in $[0,1]$. Then

$$\left.\begin{array}{r} \mathrm{O} = \Omega_0 = \mathbf{R} \\ \mathrm{I} = \Omega_1 = \varnothing \\ \partial_*\Omega = \Omega_\bullet = \varnothing \end{array}\right| \Rightarrow \mathrm{I}(\Omega) = \varnothing,$$

$$\left.\begin{array}{r} \mathrm{O}(\overline{\Omega}) = (\overline{\Omega})_0 = \mathbf{R} \backslash [0,1] \\ \mathrm{I}(\overline{\Omega}) = (\overline{\Omega})_1 =]0,1[\quad\quad \\ \partial_*(\overline{\Omega}) = (\overline{\Omega})_\bullet = \{0,1\} \quad \end{array}\right| \Rightarrow \mathrm{I}(\overline{\Omega}) =]0,1[\,. \qquad \square$$

However, the complement operation commutes with the other set of operations and determines the same sets

$$(\complement\Omega)_0 = \Omega_1, \quad (\complement\Omega)_1 = \Omega_0, \quad (\complement\Omega)_\bullet = \Omega_\bullet, \tag{3.22}$$

$$\mathrm{I}(\complement\Omega) = \mathrm{O}(\Omega), \quad \mathrm{I}(\Omega) = \mathrm{O}(\complement\Omega), \quad \partial_*(\Omega) = \partial_*(\complement\Omega). \tag{3.23}$$

The next theorem is a companion to Lemma 5.1 in Chapter 2.

Theorem 3.4. *The following conditions are equivalent:*

(i) $\overline{\operatorname{int}\Omega} = \overline{\Omega}$ *and* $\overline{\operatorname{int} \complement\Omega} = \overline{\complement\Omega}$.

(ii) $\operatorname{int}\Omega = \Omega_1, \quad \operatorname{int}\complement\Omega = \Omega_0, \quad \partial\Omega = \partial\Omega_0 = \partial\Omega_1\, (= \Omega_\bullet)$.

If either of the above two conditions is satisfied, then

$$\operatorname{int} \mathrm{I} = \operatorname{int}\Omega, \quad \operatorname{int} \mathrm{O} = \operatorname{int} \complement\Omega, \quad \overline{\complement \mathrm{I}} \cap \overline{\complement \mathrm{O}} = \partial\Omega = \partial\overline{\Omega} = \partial\overline{\complement\Omega}.$$

Proof. (i) $\Rightarrow$ (ii). First note that $\Omega_1 = \complement(\Omega_0 \cup \Omega_\bullet) = \complement\Omega_0 \cap \complement\Omega_\bullet \subset \complement\Omega_0$. Similarly, $\overline{\Omega_0} \subset \complement\Omega_1$. From the first two relations (3.20), $\operatorname{int}\Omega \subset \Omega_1$ and $\operatorname{int}\complement\Omega \subset \Omega_0$. Combining them with the two identities in (i), $\overline{\Omega} = \overline{\operatorname{int}\Omega} \subset \overline{\Omega}_1 \subset \complement\Omega_0$ and $\overline{\complement\Omega} = \overline{\operatorname{int}\complement\Omega} \subset \overline{\Omega}_0 \subset \complement\Omega_1$. By taking the intersection of the two chains, $\partial\Omega \subset \complement\Omega_0 \cap \complement\Omega_1 = \complement(\Omega_0 \cup \Omega_1) = \Omega_\bullet$. But $\operatorname{int}\Omega \cup \operatorname{int}\complement\Omega \cup \partial\Omega$ and $\Omega_1 \cup \Omega_0 \cup \Omega_\bullet$ are two disjoint partitions of $\mathbf{R}^{\mathrm{N}}$ such that $\operatorname{int}\Omega \subset \Omega_1$, $\operatorname{int}\complement\Omega \subset \Omega_0$, and $\partial\Omega \subset \Omega_\bullet$. We conclude that $\operatorname{int}\Omega = \Omega_1$, $\operatorname{int}\complement\Omega = \Omega_0$, $\partial\Omega = \Omega_\bullet$, and $\partial\Omega = \Omega_\bullet = \partial\Omega_0 = \partial\Omega_1$ since, by Lemma 5.1 in Chapter 2, $\overline{\operatorname{int}\Omega} = \overline{\Omega}$ and $\overline{\operatorname{int}\complement\Omega} = \overline{\complement\Omega}$ imply $\partial\operatorname{int}\Omega = \partial\Omega = \partial\operatorname{int}\complement\Omega$.

(ii) $\Rightarrow$ (i). By assumption,

$$\begin{aligned}
\overline{\Omega} &= \operatorname{int}\Omega \cup \partial\Omega = \Omega_1 \cup \partial\Omega_1 = \overline{\Omega_1} = \overline{\operatorname{int}\Omega},\\
\overline{\complement\Omega} &= \operatorname{int}\complement\Omega \cup \partial\Omega = \Omega_0 \cup \partial\Omega_0 = \overline{\Omega_0} = \overline{\operatorname{int}\complement\Omega}\\
\Rightarrow\ \partial\Omega &= \overline{\Omega} \cap \overline{\complement\Omega} = \overline{\operatorname{int}\Omega} \cap \overline{\complement\Omega} = \overline{\complement\overline{\complement\Omega}} \cap \overline{\complement\Omega} = \partial\,\overline{\complement\Omega}\\
\Rightarrow\ \partial\Omega &= \overline{\Omega} \cap \overline{\complement\Omega} = \overline{\Omega} \cap \overline{\operatorname{int}\complement\Omega} = \overline{\Omega} \cap \overline{\complement\overline{\Omega}} = \partial\,\overline{\Omega}.
\end{aligned}$$ $\square$

3.4 The Family of Convex Sets

Convex sets will play a special role here and in the subsequent chapters. We shall say that an equivalence class $[\Omega]$ of Lebesgue measurable subsets of D is *convex* if there exists a convex Lebesgue measurable subset Ω^* of D such that $[\Omega] = [\Omega^*]$. We also introduce the notation

$$\mathcal{C}(D) \stackrel{\text{def}}{=} \{\chi_\Omega \,:\, \Omega \text{ convex subset of } D\}. \tag{3.24}$$

Theorem 3.5. *Let $\Omega \neq \varnothing$ be a convex subset of $\mathbf{R}^N$.*

(i) $\overline{\Omega}$ *and* $\operatorname{int}\Omega$ *are convex.*[7]

(ii) $\complement(\operatorname{int}\Omega) = \overline{\complement\Omega} = \overline{\complement\overline{\Omega}} = \overline{\operatorname{int}\complement\Omega}$.
If $\operatorname{int}\Omega \neq \varnothing$*, then* $\overline{\operatorname{int}\Omega} = \overline{\Omega}$.
If $\operatorname{int}\Omega \neq \varnothing$ *and* $\Omega \neq \mathbf{R}^N$*, then* $\partial\Omega \neq \varnothing$ *and* $\operatorname{int}\complement\Omega \neq \varnothing$.

(iii) *Given a measurable* (resp., *bounded measurable*) *subset D in* $\mathbf{R}^N$

$$\forall 1 \le p < \infty, \quad \mathcal{C}(D) \text{ is closed in } L^p_{\text{loc}}(D) \text{ (resp., } L^p(D)).$$

Proof. (i) For all x and y in $\overline{\Omega}$, there exist $\{x_n\}$ and $\{y_n\}$ in Ω such that $x_n \to x$ and $y_n \to y$. Then for all $\lambda \in [0,1]$

$$\Omega \ni \lambda\, x_n + (1-\lambda)\, y_n \to \lambda\, x + (1-\lambda)\, y \in \overline{\Omega}.$$

If $\operatorname{int}\Omega \neq \varnothing$, then for all x and y in $\operatorname{int}\Omega$, there exist $r_x > 0$ and $r_y > 0$ such that $B(x, r_x) \subset \Omega$ and $B(y, r_y) \subset \Omega$. So for all $\lambda \in [0,1]$

$$\begin{aligned}
x_\lambda \stackrel{\text{def}}{=} \lambda x + (1-\lambda) y \in{} & B(x_\lambda, \lambda r_x + (1-\lambda) r_y)\\
& \subset \lambda B(x, r_x) + (1-\lambda) B(y, r_y) \subset \Omega
\end{aligned}$$

and $x_\lambda \in \operatorname{int}\Omega$.

(ii) By definition, $\operatorname{int}\Omega = \complement\overline{\complement\Omega}$ and $\complement\operatorname{int}\Omega = \overline{\complement\Omega}$. If $\Omega = \mathbf{R}^N$, $\complement\overline{\Omega} \subset \complement\Omega = \varnothing$. This implies $\overline{\operatorname{int}\complement\Omega} = \overline{\complement\overline{\Omega}} = \overline{\complement\Omega} = \varnothing$ and $\overline{\operatorname{int}\Omega} = \overline{\Omega}$. If $\Omega \neq \mathbf{R}^N$ and $\operatorname{int}\Omega \neq \varnothing$,

[7] By convention $\varnothing$ is convex.

from Theorem 5.6 in Chapter 2, Ω is locally Lipschitzian, and from Theorem 5.4 in Chapter 2, $\partial\Omega \neq \varnothing$, $\operatorname{int} \complement\Omega \neq \varnothing$, $\overline{\operatorname{int}\Omega} = \overline{\Omega}$, and $\overline{\complement\overline{\Omega}} = \overline{\complement\Omega}$. Finally, if $\operatorname{int}\Omega = \varnothing$, then $\overline{\complement\Omega} = \mathbf{R}^{\mathrm{N}}$ and Ω belongs to a linear subspace L of $\mathbf{R}^{\mathrm{N}}$ of dimension strictly less than N. Therefore $\complement\Omega \supset \complement\overline{\Omega} \supset \complement L$, $\mathbf{R}^{\mathrm{N}} = \overline{\complement\Omega} \supset \overline{\complement\overline{\Omega}} \supset \overline{\complement L} = \mathbf{R}^{\mathrm{N}}$, and $\overline{\complement\Omega} = \overline{\complement\overline{\Omega}}$.

(iii) It is sufficient to prove the result for D bounded. For any Cauchy sequence $\{\chi_n\}$ in $\mathcal{C}(D)$, there exist a sequence of convex sets $\{\Omega_n\}$ and χ_Ω in $\mathrm{X}(D)$ such that $\chi_{\Omega_n} \to \chi_\Omega$ in $L^p(D)$. In particular, there exists a subsequence $\chi_k = \chi_{\Omega_{n_k}}$ such that $\chi_k(x) \to \chi_\Omega(x)$ almost everywhere in D. Define

$$\Omega^* \stackrel{\text{def}}{=} \{x \in D : \chi_k(x) \to 1 \text{ as } k \to \infty\}.$$

To show that Ω^* is convex consider x and y in Ω^*. There exists $K \geq 1$ such that

$$\forall k \geq K, \quad |\chi_k(x) - 1| < 1/2, \quad |\chi_k(y) - 1| < 1/2,$$

and, since χ_k is either 0 or 1, for all $k \geq K$, $\chi_k(x) = 1 = \chi_k(y)$. By convexity for all $\lambda \in [0, 1]$, $x_\lambda = \lambda x + (1 - \lambda)y \in \Omega_{n_k}$ and

$$\forall k \geq K, \quad \chi_k(\lambda x + (1 - \lambda)y) = 1 \to 1 \quad \Rightarrow x_\lambda \in \Omega^*.$$

But $\chi_k(x) \to \chi_\Omega(x)$ almost everywhere and

$$\lim_{k\to\infty} \chi_k(x) \in \{0, 1\} \text{ a.e.}$$

So for $x \in D\backslash\Omega^*$, $\chi_k(x) \to 0 = \chi_{\Omega^*}(x)$ almost everywhere in D. By the Lebesgue-dominated convergence theorem, $\chi_k \to \chi_{\Omega^*}$ in $L^p(D)$ and necessarily $\chi_\Omega = \chi_{\Omega^*}$. This means that $[\Omega]$ is a convex equivalence class. □

We shall show in section 6.1 (Corollary 1) that for bounded domains D, $\mathcal{C}(D)$ is also compact for the $L^p(D)$ topology, $1 \leq p < \infty$.

3.5 Sobolev Spaces for Measurable Domains

The lack of a priori smoothness on Ω may introduce technical difficulties in the formulation of some boundary value problems. However, it is possible to relax such boundary value problems from smooth bounded open connected domains Ω to measurable domains (cf. J.-P. Zolésio [**7**]). For instance consider the homogeneous Dirichlet boundary value problem

$$-\triangle y = f \text{ in } \Omega, \quad y = 0 \text{ on } \Gamma \tag{3.25}$$

over a bounded open connected domain Ω with a boundary Γ of class C^1 and associate with its solution $y = y(\Omega)$ the objective function and volume constraint

$$J(\Omega) = \frac{1}{2}\int_\Omega |y - g|^2\, dx, \quad \int_\Omega dx = \pi. \tag{3.26}$$

There is a priori no reason to assume that an optimal (minimizing) domain Ω^* is of class C^1 or is connected. So the problem must be suitably relaxed to a large enough class of domains, which preserves the meaning of the underlying function spaces, the well-posedness of the original problem, and the volume.

To extend problem (3.25)–(3.26) to Lebesgue measurable sets, we first have to make sense of the Sobolev space for measurable subsets Ω of D.

Theorem 3.6. *Let D be an open domain in $\mathbf{R}^N$. For any Lebesgue measurable subset Ω of $\mathbf{R}^N$, the spaces*

$$H^1_\bullet(\Omega; D) \stackrel{\text{def}}{=} \left\{\varphi \in H^1_0(D) : (1-\chi_\Omega)\nabla\varphi = 0 \text{ a.e. in } D\right\}, \tag{3.27}$$

$$H^1_\diamond(\Omega; D) \stackrel{\text{def}}{=} \left\{\varphi \in H^1_0(D) : (1-\chi_\Omega)\varphi = 0 \text{ a.e. in } D\right\} \tag{3.28}$$

are closed subspaces of $H^1_0(D)$ and hence Hilbert spaces.[8] *Similarly, for any $\chi \in \overline{\text{co}}\, \mathrm{X}(D)$,*

$$H^1_\bullet(\chi; D) \stackrel{\text{def}}{=} \left\{\varphi \in H^1_0(D) : (1-\chi)\nabla\varphi = 0 \text{ a.e. in } D\right\}, \tag{3.29}$$

$$H^1_\diamond(\chi; D) \stackrel{\text{def}}{=} \left\{\varphi \in H^1_0(D) : (1-\chi)\varphi = 0 \text{ a.e. in } D\right\} \tag{3.30}$$

are also closed subspaces of $H^1_0(D)$ and hence Hilbert spaces. Furthermore[9]

$$H^1_\diamond(\Omega; D) \subset H^1_\bullet(\Omega; D), \quad H^1_\diamond(\chi; D) \subset H^1_\bullet(\chi; D).$$

Proof. We give only the proof for $H^1_\bullet(\Omega; D)$. Let $\{\varphi_n\}$ in $H^1_\bullet(\Omega; D)$ be a Cauchy sequence. It converges to an element φ in the $H^1_0(D)$-topology. Hence $\{\nabla\varphi_n\}$ converges to $\nabla\varphi$ in $L^2(D)$. But for all n

$$(1-\chi_\Omega)\nabla\varphi_n = 0 \quad \text{in} \quad L^2(D).$$

By Schwarz's inequality the map $\varphi \mapsto (1-\chi_\Omega)\nabla\varphi$ is continuous and

$$(1-\chi_\Omega)\nabla\varphi = 0 \quad \text{in} \quad L^2(D).$$

So finally $\varphi \in H^1_\bullet(\Omega; D)$ and $H^1_\bullet(\Omega; D)$ is a closed subspace of $H^1_0(D)$. □

Assuming that D is bounded, the variational problems

$$\begin{cases} \text{find } y = y(\Omega) \in H^1_\bullet(\Omega; D) \text{ such that} \\ \forall\varphi \in H^1_\bullet(\Omega; D), \quad \displaystyle\int_D \nabla y \cdot \nabla\varphi\, dx = \int_D \chi_\Omega f\varphi\, dx, \end{cases} \tag{3.31}$$

$$\begin{cases} \text{find } y = y(\Omega) \in H^1_\diamond(\Omega; D) \text{ such that} \\ \forall\varphi \in H^1_\diamond(\Omega; D), \quad \displaystyle\int_D \nabla y \cdot \nabla\varphi\, dx = \int_D \chi_\Omega f\varphi\, dx \end{cases} \tag{3.32}$$

[8] Observe that for two open domains in the same equivalence class, the spaces defined by (3.27)–(3.28) coincide. Therefore their functions do not see cracks in the underlying domain. The case of cracks will be handled in Chapter 8 by capacity methods.

[9] From L. C. Evans and R. F. Gariepy [**1**, Thm. 4, p. 130], $\nabla\varphi = 0$ a.e. on $\{\varphi = 0\}$.

now make sense and have unique solutions for measurable subsets Ω of D (or even $\chi \in \overline{\mathrm{co}}\,\mathrm{X}(D)$), and the associated objective function

$$J(\Omega) = h\big(\chi_\Omega, y(\Omega)\big), \quad h(\chi, \varphi) \stackrel{\text{def}}{=} \frac{1}{2}\int_D \chi|\varphi - g|^2\,dx \tag{3.33}$$

can be minimized over all measurable subsets Ω of D (or all $\chi \in \overline{\mathrm{co}}\,\mathrm{X}(D)$) with fixed measure $m(\Omega) = \pi$.

The above problems are now well-posed, and their restriction to smooth bounded open connected domains coincides with the initial problem (3.25)–(3.26). Indeed, if Ω is a connected open domain with a boundary Γ of class C^1, then Γ has a zero Lebesgue measure and the definition (3.28) of $H^1_\bullet(\Omega; D)$ coincides with $H^1_\diamond(\Omega; D)$. Therefore, the problem specified by (3.31)–(3.33) is a well-defined extension of problem (3.25)–(3.26). In general, when Ω contains holes, the elements of $H^1_\bullet(\Omega; D)$ are not necessarily equal to zero on the boundary of each hole and can be equal to different constants from hole to hole, as in physical problems involving a potential.

Example 3.3.
Let $D =]-2, 2[$, $\Omega =]-2, -1[\cup]1, 2[$, and $f = 1$ on D. The solution of (3.31) is given by

$$\begin{cases} -\dfrac{d^2y}{dx^2} = 1 \quad \text{on }]-2,-1[, \\ y(-2) = 0, \quad \dfrac{dy}{dx}(-1) = 0, \end{cases} \quad \begin{cases} -\dfrac{d^2y}{dx^2} = 1 \quad \text{on }]1,2[, \\ y(2) = 0, \quad \dfrac{dy}{dx}(1) = 0, \end{cases}$$

$$y = \frac{1}{2} \quad \text{on } [-1, 1],$$

$$\Rightarrow y(x) = \begin{cases} -(x+2)\frac{x}{2} & \text{in }]-2,-1[, \\ \frac{1}{2} & \text{in } [-1,1], \\ -(x-2)\frac{x}{2} & \text{in }]1,2[. \end{cases}$$

□

Example 3.4.
Let $D = B(0, 2)$, the open ball of radius 2 centered in 0 in $\mathbf{R}^2$, $H = \overline{B(0, 1)}$, $\Omega = \complement_D H$, and $f = 1$. In polar coordinates the solution of (3.31) is given by

$$\begin{cases} -\dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{dy}{dr}\right) = 1 \quad \text{in }]1,2[, \\ y(2) = 0, \quad \dfrac{dy}{dr}(1) = 0, \end{cases} \text{and} y(r) = \frac{1}{2}\ln\frac{1}{2} + \frac{3}{4} \quad \text{in } [0,1],$$

or explicitly

$$y(r) = \begin{cases} \frac{1}{2}\ln\frac{r}{2} + 1 - \frac{r^2}{4} & \text{in }]1,2[\,, \\ \frac{1}{2}\ln\frac{1}{2} + \frac{3}{4} & \text{in } [0,1]. \end{cases}$$

□

The last example is a special case that retains the characteristics of the one-dimensional example. In higher dimensions the normal derivative is not necessarily zero on the "internal boundary" of Ω.

Example 3.5.
Let $D = B(0,1)$ in $\mathbf{R}^2$, let H be a bounded open connected hole in D such that $\overline{H} \subset D$, and let $\Omega = \complement_D \overline{H}$. Then it can be checked that the solution of (3.31) is of the form

$$-\Delta y = f \text{ in } \Omega, \quad y = 0 \text{ on } \partial D,$$

where the constant c on H is determined by the condition

$$\forall \varphi \in H^1_\bullet(\Omega; D), \quad \int_{\partial\Omega} \frac{\partial y}{\partial n} \varphi \, d\gamma = 0,$$

or equivalently

$$\int_{\partial H} \frac{\partial y}{\partial n} \, d\gamma = 0. \qquad \square$$

Example 3.6.
Let $D =]-2,2[\times]-2,2[$ and let Ω_1 and Ω_2 be two open squares in $G =]-1,1[\times]-1,1[$ such that $\overline{\Omega_1} \subset G$, $\overline{\Omega_2} \subset G$, and $\overline{\Omega_1} \cap \overline{\Omega_2} = \varnothing$. Define the domain as

$$\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2, \quad \Omega_0 = D \backslash \overline{G} \quad \text{(cf. Figure 5.2).}$$

Then the solution of (3.31) is characterized by

$$-\Delta y = f \text{ in } \Omega, \quad y = 0 \text{ on } \partial D, \quad y = c \text{ on } G \backslash \left[\overline{\Omega}_1 \cup \overline{\Omega}_2 \right],$$

where the constant c is determined by the condition

$$\int_{\partial\Omega_0^{int}} \frac{\partial y}{\partial n} \, d\gamma + \int_{\partial\Omega_1} \frac{\partial y}{\partial n} \, d\gamma + \int_{\partial\Omega_2} \frac{\partial y}{\partial n} \, d\gamma = 0$$

and $\partial\Omega_0^{\text{int}} = \partial G$, the interior boundary of Ω_0. $\square$

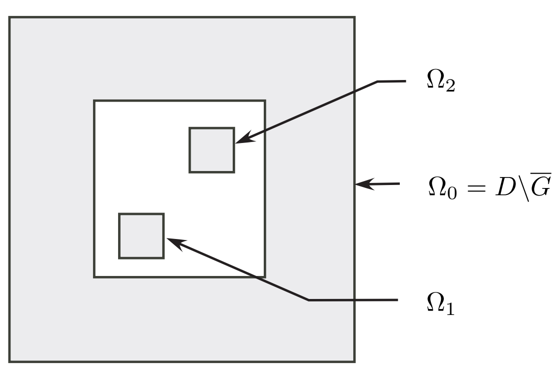


Figure 5.2. *Disconnected domain* $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$.

4 Some Compliance Problems with Two Materials

As a first example consider the optimal compliance problem where the optimization variable is the distribution of two materials with different physical characteristics within a fixed domain D. It cannot a priori be assumed that the two regions are separated by a smooth boundary and that each region is connected. The optimal solution may lead to a nonsmooth interface and even to the mixing of the two materials. This type of solution occurs in control or optimization problems over a bounded nonconvex subset of a function space. In general, their relaxed solution lies in the closed convex hull of that subset. In control theory, the phenomenon is known as *chattering control*. To illustrate this approach it is best to consider a generic example. In section 4.1, we consider a variation of the optimal design problem studied by J. Céa and K. Malanowski [**1**] in 1970 and then discuss its original version in section 4.2. The variation of the problem is constructed in such a way that the form of the associated variational equation is similar to the one of Example 3.1, which provided a counterexample to the weak continuity of the solution. Yet the maximization of the minimum energy yields as solution a characteristic function in both cases. However, this is no longer true for the minimization of the same minimum energy as was shown by F. Murat and L. Tartar [**1**, **3**].

4.1 Transmission Problem and Compliance

Let $D \subset \mathbf{R}^N$ be a bounded open domain with Lipschitzian boundary ∂D. Assume that the domain D is partitioned into two subdomains Ω_1 and Ω_2 separated by a smooth boundary $\partial\Omega_1 \cap \partial\Omega_2$ as illustrated in Figure 5.3. Domain Ω_1 (resp., Ω_2) is made up of a material characterized by a constant $k_1 > 0$ (resp., $k_2 > 0$). Let y be the solution of the *transmission problem*

$$\begin{cases} -k_1 \triangle y = f \text{ in } \Omega_1, \quad -k_2 \triangle y = 0 \text{ in } \Omega_2, \\ y = 0 \text{ on } \partial D, \quad k_1 \dfrac{\partial y}{\partial n_1} + k_2 \dfrac{\partial y}{\partial n_2} = 0 \text{ on } \partial\Omega_1 \cap \partial\Omega_2, \end{cases} \tag{4.1}$$

where n_1 (resp., n_2) is the unit outward normal to Ω_1 (resp., Ω_2) and f is a given function in $L^2(D)$. Our objective is to maximize the equivalent of the *compliance*

$$J(\Omega_1) = -\int_{\Omega_1} f y \, dx \tag{4.2}$$

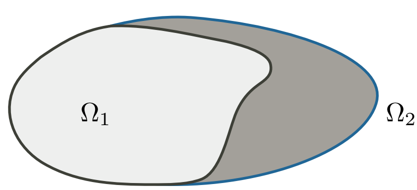


Figure 5.3. *Fixed domain D and its partition into Ω_1 and Ω_2.*

over all domains Ω_1 in D. In mechanics the compliance is associated with the total work of the body forces f.

Denote by Ω the domain Ω_1. By complementarity $\Omega_2 = \complement_D \overline{\Omega}$ and let $\chi = \chi_\Omega$. Problem (4.1) can be rewritten in the following variational form:

$$\boxed{\begin{aligned}&\text{find } y = y(\chi) \in H_0^1(D) \text{ such that } \forall \varphi \in H_0^1(D),\\ &\int_D [k_1\chi + k_2(1-\chi)]\,\nabla y \cdot \nabla\varphi\,dx = \int_D \chi f \varphi\,dx.\end{aligned}} \tag{4.3}$$

For $k_1 > 0$ and $k_2 > 0$ and all χ in $\mathrm{X}(D)$

$$\begin{gathered} k(x) \stackrel{\text{def}}{=} k_1\,\chi(x) + k_2\,(1-\chi(x)),\\ 0 < \min\{k_1, k_2\} \le k(x) \le \max\{k_1, k_2\} \text{ a.e. in } D. \end{gathered} \tag{4.4}$$

By the Lax–Milgram theorem, the variational equation (4.3) still makes sense and has a unique solution $y = y(\chi)$ in $H_0^1(D)$ that coincides with the solution of the boundary value problem

$$-\mathrm{div}\,(k\,\nabla y) = \chi f \text{ in } D, \quad y = 0 \text{ on } \partial D. \tag{4.5}$$

Notice that the high-order term has the same form as the term in Example 3.1. As for the objective function, it can be rewritten as

$$J(\chi) = -\int_D \chi f y(\chi)\,dx. \tag{4.6}$$

Thus the initial boundary value problem (4.1) has been transformed into the variational problem (4.3), and the initial objective function (4.2) into (4.6). Both make sense for χ in $\mathrm{X}(D)$ and even in $\overline{\mathrm{co}}\,\mathrm{X}(D)$. The family of characteristic functions $\mathrm{X}(D)$ and its closed convex hull $\overline{\mathrm{co}}\,\mathrm{X}(D)$ have been defined and characterized in (3.2) and (3.3).

The objective function (4.6) can be further rewritten as a minimum

$$J(\chi) = \min_{\varphi \in H_0^1(D)} E(\chi, \varphi) \tag{4.7}$$

for the *energy function*

$$E(\chi, \varphi) \stackrel{\text{def}}{=} \int_D (k_1\,\chi + k_2\,(1-\chi))\,|\nabla\varphi|^2 - 2\,\chi f \varphi\,dx \tag{4.8}$$

associated with the variational problem (4.3). The initial optimal design problem becomes a max-min problem

$$\boxed{\max_{\chi \in \mathrm{X}(D)} J(\chi) = \max_{\chi \in \mathrm{X}(D)} \min_{\varphi \in H_0^1(D)} E(\chi, \varphi),} \tag{4.9}$$

where $\mathrm{X}(D)$ can be considered as a subset of $L^p(D)$ for some p, $1 \le p < \infty$.

This problem can also be relaxed to functions χ with value in $[0,1]$

$$\boxed{\max_{\chi\in\overline{\mathrm{co}}\,\mathrm{X}(D)} J(\chi) = \max_{\chi\in\overline{\mathrm{co}}\,\mathrm{X}(D)} \min_{\varphi\in H_0^1(D)} E(\chi,\varphi).} \tag{4.10}$$

We shall refer to this problem as the *relaxed problem.*

These formulations have been introduced by J. Céa and K. Malanowski [**1**], who used the variable $k(x)$ and the following *equality constraint* on its integral:

$$\int_D k(x)\,dx = \gamma \quad \text{or} \quad \int_D \chi(x)\,dx = \frac{\gamma - k_2\, m(D)}{k_1 - k_2}$$

for some appropriate $\gamma > 0$. Moreover the force f in (4.1)–(4.3) was exerted everywhere in D and not only in Ω. So the objective function (4.2)–(4.6) was the integral of fy over all of D. We shall see in section 4.2 that the fact that the support of f is Ω or all of D does not affect the nature of the results.

In both maximization problems (4.1)–(4.2) and (4.3)–(4.6) it is necessary to introduce a *volume constraint* in order to avoid a trivial solution. Notice that by using (4.3) with $\varphi = y(\chi)$ the objective function (4.6) becomes

$$J(\chi) = -\int_D (k_1\chi + k_2(1-\chi))\, |\nabla y(\chi)|^2\, dx \le 0.$$

So maximizing $J(\chi)$ is equivalent to minimizing the integral of $k(x)\,|\nabla y|^2$. Therefore, $\chi = 0$ (only material k_2) is a maximizer since the corresponding solution of (4.3) is $y = 0$. In order to eliminate this situation we introduce the following constraint on the volume of material k_1:

$$\int_D \chi\, dx \ge \alpha > 0 \tag{4.11}$$

for some α, $0 < \alpha \le m(D)$. The case $\alpha = \mathrm{m}(D)$ yields the unique solution $\chi = 1$ (only material k_1). So we can further assume that $\alpha < \mathrm{m}(D)$.

For $0 < \alpha < \mathrm{m}(D)$, the optimal design problem becomes

$$\max_{\substack{\chi\in\mathrm{X}(D)\\ \int_D \chi\,dx\ge\alpha}} J(\chi) = \max_{\substack{\chi\in\mathrm{X}(D)\\ \int_D \chi\,dx\ge\alpha}} \min_{\varphi\in H_0^1(D)} E(\chi,\varphi) \tag{4.12}$$

and its relaxed version

$$\max_{\substack{\chi\in\overline{\mathrm{co}}\,\mathrm{X}(D)\\ \int_D \chi\,dx\ge\alpha}} J(\chi) = \max_{\substack{\chi\in\overline{\mathrm{co}}\,\mathrm{X}(D)\\ \int_D \chi\,dx\ge\alpha}} \min_{\varphi\in H_0^1(D)} E(\chi,\varphi). \tag{4.13}$$

We shall now show that problem (4.13) has a unique solution χ^* in $\overline{\mathrm{co}}\,\mathrm{X}(D)$ and that χ^* is a characteristic function, $\chi^* \in \mathrm{X}(D)$, for which the inequality constraint is saturated:

$$\chi^* \in \mathrm{X}(D) \quad \text{and} \quad \int_D \chi^*\, dx = \alpha. \tag{4.14}$$

At this juncture it is advantageous to incorporate the volume inequality constraint into the problem formulation by introducing a Lagrange multiplier $\lambda \geq 0$. The formulation of the relaxed problem becomes

$$\max_{\chi \in \overline{\mathrm{co}}\, \mathrm{X}(D)} \min_{\substack{\varphi \in H_0^1(D) \\ \lambda \geq 0}} G(\chi, \varphi, \lambda), \quad G(\chi, \varphi, \lambda) \stackrel{\text{def}}{=} E(\chi, \varphi) + \lambda \left(\int_D \chi \, dx - \alpha \right) \tag{4.15}$$

and $E(\chi, \varphi)$ is the energy function given in (4.7).

We first establish the existence of saddle point solutions to problem (4.15). We use a general result in I. Ekeland and R. Temam [**1**, Prop. 2.4, p. 164]. The set $\overline{\mathrm{co}}\, \mathrm{X}(D)$ is a nonempty bounded closed convex subset of $L^2(D)$ and the set $H_0^1(D) \times [\mathbf{R}^+ \cup \{0\}]$ is trivially closed and convex. The function G is concave-convex with the following properties:

$$\begin{cases} \forall \chi \in \overline{\mathrm{co}}\, \mathrm{X}(D), (\varphi, \lambda) \mapsto G(\chi, \varphi, \lambda) \text{ is convex, continuous, and} \\ \exists \chi_0 \in \overline{\mathrm{co}}\, \mathrm{X}(D) \text{ such that } \displaystyle\lim_{\|\varphi\|_{H_0^1} + |\lambda| \to \infty} G(\chi_0, \varphi, \lambda) = +\infty; \end{cases} \tag{4.16}$$

$$\begin{cases} \forall \varphi \in H_0^1(D),\, \forall \lambda \geq 0, \quad \text{the map } \chi \mapsto G(\chi, \varphi, \lambda) \text{ is affine and} \\ \text{continuous for } L^2(D)\text{-strong.} \end{cases} \tag{4.17}$$

For the first condition recall that $0 < \alpha < m(D)$ and pick $\chi_0 = 1$ on D. To check the second condition pick any sequence $\{\chi_n\}$ in $\overline{\mathrm{co}}\, \mathrm{X}(D)$ which converges to some χ in $L^2(D)$-strong. Then the sequence also converges in $L^2(D)$-weak and by Lemma 3.1 in $L^\infty(D)$-weak*. Hence $G(\chi_n, \varphi, \lambda)$ converges to $G(\chi, \varphi, \lambda)$.

The set of saddle points $(\hat{\chi}, y, \hat{\lambda})$ is of the form $X \times Y \subset \overline{\mathrm{co}}\, \mathrm{X}(D) \times \{H_0^1(D) \times [\mathbf{R}^+ \cup \{0\}]\}$ and is completely characterized by the following variational equation and inequalities (cf. I. Ekeland and R. Temam [**1**, Prop. 1.6, p. 157]):

$$\forall \varphi \in H0^1(D), \quad \int_D [k_2 + (k_1 - k_2)\hat{\chi}] \, \nabla y \cdot \nabla \varphi - \hat{\chi} f \varphi \, dx = 0, \tag{4.18}$$

$$\forall \chi \in \overline{\mathrm{co}}\, \mathrm{X}(D), \quad \int_D \left[(k_1 - k_2)|\nabla y|^2 - 2fy + \hat{\lambda} \right] (\chi - \hat{\chi}) \, dx \leq 0, \tag{4.19}$$

$$\left(\int_D \hat{\chi} \, dx - \alpha \right) \hat{\lambda} = 0, \quad \int_D \hat{\chi} \, dx - \alpha \geq 0, \quad \hat{\lambda} \geq 0. \tag{4.20}$$

But for each $\chi \in \overline{\mathrm{co}}\, \mathrm{X}(D)$ there exists a unique $y(\chi)$ solution of (4.18) and then

$$\forall \hat{\chi} \in X, \quad \{\hat{\chi}\} \times Y \subset \{\hat{\chi}\} \times \left\{ \{y(\hat{\chi})\} \times [\mathbf{R}^+ \cup \{0\}] \right\}.$$

Therefore, $y(\hat{\chi})$ is independent of $\hat{\chi} \in X$, that is, $Y = \{y\} \times \Lambda$ and

$$\forall \hat{\chi} \in X, \hat{\lambda} \in \Lambda, \quad y(\hat{\chi}) = y.$$

For each $\hat{\lambda}$, inequality (4.19) is completely equivalent to the following characterization of the maximizer $\hat{\chi} \in X$:

$$\hat{\chi}(x) = \begin{cases} 1, & \text{if } (k_1 - k_2)|\nabla y|^2 - 2fy + \hat{\lambda} > 0, \\ \in [0, 1], & \text{if } (k_1 - k_2)|\nabla y|^2 - 2fy + \hat{\lambda} = 0, \\ 0, & \text{if } (k_1 - k_2)|\nabla y|^2 - 2fy + \hat{\lambda} < 0. \end{cases} \tag{4.21}$$

Associate with an arbitrary $\lambda \geq 0$ the sets

$$\begin{aligned} D_+(\lambda) &= \{x \in D : (k_1 - k_2)|\nabla y|^2 - 2fy + \lambda > 0\}, \\ D_0(\lambda) &= \{x \in D : (k_1 - k_2)|\nabla y|^2 - 2fy + \lambda = 0\}. \end{aligned} \tag{4.22}$$

In order to complete the characterization of the optimal triplets, we need the following general result.

Lemma 4.1. *Consider a function*

$$G : A \times B \to \mathbf{R} \tag{4.23}$$

for some sets A and B. Define

$$g \stackrel{\text{def}}{=} \inf_{x \in A} \sup_{y \in B} G(x, y), \quad A_0 \stackrel{\text{def}}{=} \left\{ x \in A : \sup_{y \in B} G(x, y) = g \right\}, \tag{4.24}$$

$$h \stackrel{\text{def}}{=} \sup_{y \in B} \inf_{x \in A} G(x, y), \quad B_0 \stackrel{\text{def}}{=} \left\{ y \in B : \inf_{x \in A} G(x, y) = h \right\}. \tag{4.25}$$

When $g = h$ the set of saddle points *(possibly empty) will be denoted by*

$$S \stackrel{\text{def}}{=} \{(x, y) \in A \times B : g = G(x, y) = h\}. \tag{4.26}$$

Then the following hold.

(i) *In general $h \leq g$ and*

$$\forall (x_0, y_0) \in A_0 \times B_0, \quad h \leq G(x_0, y_0) \leq g. \tag{4.27}$$

(ii) *If $h = g$, then $S = A_0 \times B_0$.*

Proof. (i) If $A_0 \times B_0 = \varnothing$, there is nothing to prove. If there exist $x_0 \in A_0$ and $y_0 \in B_0$, then by definition

$$h = \inf_{x \in A} G(x, y_0) \leq G(x_0, y_0) \leq \sup_{y \in B} G(x_0, y) = g. \tag{4.28}$$

(ii) If $h = g$, then in view of (4.26)–(4.27), $A_0 \times B_0 \subset S$. Conversely if there exists $(x_0, y_0) \in S$, then $h = G(x_0, y_0) = g$ and by the definitions of A_0 and B_0, $(x_0, y_0) \in A_0 \times B_0$. □

Associate with an arbitrary solution $(\hat{\chi}, y, \hat{\lambda})$, the characteristic function

$$\chi_{\hat{\lambda}} = \begin{cases} \chi_{D_+(\hat{\lambda})}, & \text{if } D_+(\hat{\lambda}) \neq \varnothing, \\ 0, & \text{if } D_+(\hat{\lambda}) = \varnothing. \end{cases} \tag{4.29}$$

Then from (4.21)

$$\hat{\chi} \left\{ (k_1 - k_2)|\nabla y|^2 - 2fy + \hat{\lambda} \right\} = \chi_{\hat{\lambda}} \left\{ (k_1 - k_2)|\nabla y|^2 - 2fy + \hat{\lambda} \right\} \text{ a.e. in } D$$

and

$$\begin{aligned}
G(\hat{\chi}, y, \hat{\lambda}) &= \int_D [\hat{\chi} k_1 + (1-\hat{\chi}) k_2]\, |\nabla y|^2 - 2\hat{\chi} f y\, dx + \hat{\lambda}\Big(\int_D \hat{\chi}\, dx - \alpha \Big) \\
&= \int_D k_2 |\nabla y|^2 + \hat{\chi}\left[(k_1 - k_2)|\nabla y|^2 - 2fy + \hat{\lambda} \right] dx - \hat{\lambda}\alpha \\
&= \int_D k_2 |\nabla y|^2 + \chi_{\hat{\lambda}}\left[(k_1 - k_2)|\nabla y|^2 - 2fy + \hat{\lambda} \right] dx - \hat{\lambda}\alpha \\
&= G(\chi_{\hat{\lambda}}, y, \hat{\lambda}).
\end{aligned}$$

From Lemma 4.1, $(\chi_{\hat{\lambda}}, y, \hat{\lambda})$ is also a saddle point. So there exists a maximizer $\chi_{\hat{\lambda}} \in \mathrm{X}(D)$ that is a characteristic function and necessarily

$$\max_{\chi \in \mathrm{X}(D)} \min_{\substack{\varphi \in H_0^1(D) \\ \lambda \geq 0}} G(\chi, \varphi, \lambda) = \max_{\chi \in \overline{\mathrm{co}}\, \mathrm{X}(D)} \min_{\substack{\varphi \in H_0^1(D) \\ \lambda \geq 0}} G(\chi, \varphi, \lambda).$$

But we can show more than that. If there exists $\hat{\lambda} \in \Lambda$ such that $\hat{\lambda} > 0$, then by construction $\hat{\chi} \geq \chi_{\hat{\lambda}}$ and by (4.20)

$$\alpha = \int_D \hat{\chi}\, dx \geq \int_D \chi_{\hat{\lambda}}\, dx = \alpha.$$

Since $\hat{\chi} = \chi_{\hat{\lambda}} = 1$ almost everywhere in $D_+(\hat{\lambda})$, then $\hat{\chi} = \chi_{\hat{\lambda}}$. But, always by construction, $\chi_{\hat{\lambda}}$ is independent of $\hat{\chi}$. Thus the maximizer is unique, it is a characteristic function, and its integral is equal to α.

The case $\Lambda = \{0\}$ is a degenerate one. Set $\varphi = y$ in (4.18) and regroup the terms as follows:

$$\begin{aligned}
0 &= \int_D (k_2 + (k_1 - k_2)\hat{\chi})\, |\nabla y|^2 - f y \hat{\chi}\, dx \\
&= \int_D \left(k_2 + \frac{k_1 - k_2}{2} \hat{\chi} \right) |\nabla y|^2\, dx + \int_D \left\{ \frac{k_1 - k_2}{2}\, \hat{\chi}\, |\nabla y|^2 - f y \right\} \hat{\chi}\, dx.
\end{aligned}$$

The integrand of the first integral is positive. From the characterization of $\hat{\chi}$, the integrand of the second one is also positive. Hence they are both zero almost everywhere in D. As a result

$$m(D_+(0)) = 0 \quad \text{and} \quad \nabla y = 0.$$

Therefore $y = 0$ in D. Hence the saddle points are of the general form $(\hat{\chi}, y, \hat{\lambda}) = (\hat{\chi}, 0, 0)$, $\hat{\chi} \in X$. In particular, $D_+(0) = \varnothing$ and from our previous considerations $\chi_\varnothing$; that is, $\hat{\chi} = 0$ is a solution. But this is impossible since $\int_D \hat{\chi}\, dx \geq \alpha > 0$. Therefore $\hat{\lambda} = 0$ cannot occur.

In conclusion, there exists a (unique for $\alpha > 0$) maximizer χ^* in $\overline{\mathrm{co}}\, \mathrm{X}(D)$ that is in fact a characteristic function, and necessarily

$$\max_{\substack{\chi \in \mathrm{X}(D) \\ \int_D \chi\, dx \geq \alpha}} \min_{\varphi \in H_0^1(D)} E(\chi, \varphi) = \max_{\substack{\chi \in \overline{\mathrm{co}}\, \mathrm{X}(D) \\ \int_D \chi\, dx \geq \alpha}} \min_{\varphi \in H_0^1(D)} E(\chi, \varphi). \tag{4.30}$$

Moreover, for $0 < \alpha \le \mathrm{m}(D)$,

$$\int_D \chi^* \, dx = \alpha \tag{4.31}$$

and χ^* is the unique solution of the problem with an equality constraint:

$$\max_{\substack{\chi \in \mathrm{X}(D) \\ \int_D \chi\, dx = \alpha}} \; \min_{\varphi \in H_0^1(D)} E(\chi, \varphi) = \max_{\substack{\chi \in \overline{\mathrm{co}}\, \mathrm{X}(D) \\ \int_D \chi\, dx \ge \alpha}} \; \min_{\varphi \in H_0^1(D)} E(\chi, \varphi). \tag{4.32}$$

As a numerical illustration of the theory consider (4.1) over the diamond-shaped domain

$$D = \{(x, y) \, : \, |x| + |y| < 1\} \tag{4.33}$$

with the function f of Figure 5.4. This function has a sharp peak in $(0, 0)$ which has been scaled down in the picture. The variational form (4.3) of the boundary value problem was approximated by continuous piecewise linear finite elements on each triangle, and the function χ by a piecewise constant function on each triangle. The constant on each triangle was constrained to lie between 0 and 1 together with the global constraint on its integral over the whole domain D. Figure 5.5 shows the optimal partition (the domain has been rotated by 45 degrees to save space). The grey triangles correspond to the region $D_0(\hat{\lambda}_m)$, where $\hat{\chi} \in [0, 1]$. The presence of this grey zone in the approximated problem is due to the fact that equality for the total area where $\hat{\chi} = 1$ could not be exactly achieved with the chosen triangulation of the domain. Thus the problem had to adjust the value of $\hat{\chi}$ between 0 and 1 in

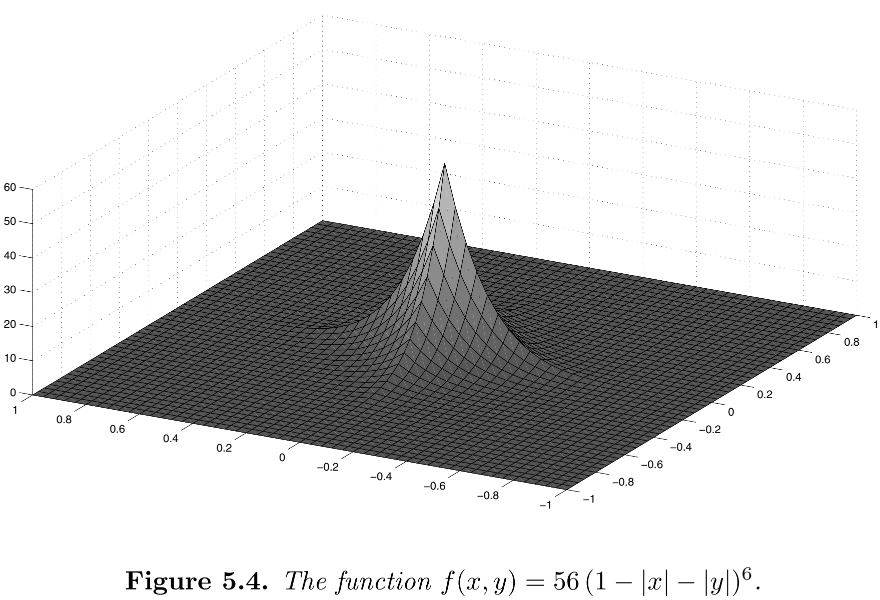


Figure 5.4. *The function* $f(x, y) = 56\,(1 - |x| - |y|)^6$.

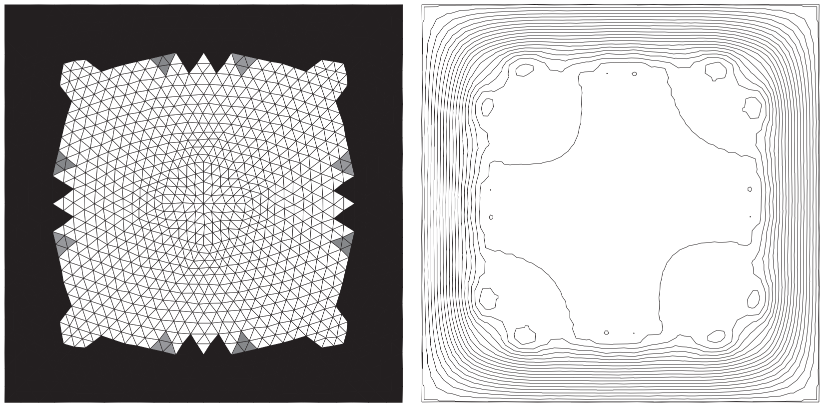


Figure 5.5. *Optimal distribution and isotherms with* $k_1 = 2$ *(black) and* $k_2 = 1$ *(white) for the problem of section* 4.1.

a few triangles in order to achieve equality for the integral of $\hat{\chi}$. For this example the Lagrange multiplier associated with the problem is strictly positive.

4.2 The Original Problem of Céa and Malanowski

We now put the force f everywhere in the fixed domain D. The same technique and the same conclusion can be drawn: there exists at least one maximizer of the compliance, which is a characteristic function. For completeness we give the main elements below.

Fix the bounded open Lipschitzian domain D in $\mathbf{R}^{\mathrm{N}}$ and let Ω be a smooth subset of D. Let y be the solution of the *transmission problem*

$$\begin{cases} -k_1\triangle y = f \text{ in } \Omega, \quad -k_2\triangle y = f \text{ in } D\backslash\overline{\Omega}, \\ y = 0 \text{ on } \partial D, \\ k_1\dfrac{\partial y}{\partial n_1} + k_2\dfrac{\partial y}{\partial n_2} = 0 \text{ on } \partial\Omega\cap D, \end{cases} \tag{4.34}$$

where n_1 (resp., n_2) is the unit outward normal to Ω (resp., $\complement_D\overline{\Omega}$) and f is a given function in $L^2(D)$. Again problem (4.34) can be reformulated in terms of the characteristic function $\chi = \chi_\Omega$:

$$\begin{aligned} &\text{find } y = y(\chi) \in H_0^1(D) \text{ such that } \forall\varphi \in H_0^1(D), \\ &\int_D [k_1\chi + k_2(1-\chi)]\ \nabla y\cdot\nabla\varphi\,dx = \int_D f\varphi\,dx, \end{aligned} \tag{4.35}$$

with the objective function

$$J(\chi) = -\int_D fy(\chi)\,dx \tag{4.36}$$

to be maximized over all $\chi \in \mathrm{X}(D)$. As in the previous case, the function $J(\chi)$ can be rewritten as the minimum of the energy function

$$E(\chi, \varphi) \stackrel{\text{def}}{=} \int_D [k_1\chi + k_2(1-\chi)]\, |\nabla\varphi|^2 - 2f\varphi\, dx \tag{4.37}$$

over $H_0^1(D)$,

$$J(\chi) = \min_{\varphi \in H_0^1(D)} E(\chi, \varphi), \tag{4.38}$$

and we have the relaxed max-min problem

$$\max_{\chi \in \overline{\text{co}}\, \mathrm{X}(D)} \min_{\varphi \in H_0^1(D)} E(\chi, \varphi). \tag{4.39}$$

Without constraint on the integral of χ, the problem is trivial and $\chi = 1$ (resp., $\chi = 0$) if $k_1 > k_2$ (resp., $k_2 > k_1$). In other words it is optimal to use only the strong material. In order to make the problem nontrivial assume that the strong material is k_1, that is, $k_1 > k_2$, and put an upper bound on the volume of material k_1 which occupies the part Ω of D:

$$\int_D \chi\, dx \le \alpha, \quad 0 < \alpha < m(D). \tag{4.40}$$

The case $\alpha = 0$ trivially yields $\chi = 0$. Under assumption (4.40) the case $\chi = 1$ with only the strong material k_1 is no longer admissible. Thus we consider for $0 < \alpha < \mathrm{m}(D)$ the problem

$$\max_{\substack{\chi \in \overline{\text{co}}\, \mathrm{X}(D) \\ \int_D \chi\, dx \le \alpha}} \min_{\varphi \in H_0^1(D)} E(\chi, \varphi). \tag{4.41}$$

We shall now show that problem (4.41) has a unique solution χ^* in $\overline{\text{co}}\, \mathrm{X}(D)$ and that in fact χ^* is a characteristic function for which the inequality constraint is saturated:

$$\int_D \chi^*\, dx = \alpha. \tag{4.42}$$

This solves the original problem of Céa and Malanowski with the equality constraint:

$$\max_{\substack{\chi \in \mathrm{X}(D) \\ \int_D \chi\, dx = \alpha}} \min_{\varphi \in H_0^1(D)} E(\chi, \varphi). \tag{4.43}$$

As in section 4.1 it is convenient to reformulate the problem with a Lagrange multiplier $\lambda \ge 0$ for the constraint inequality (4.40):

$$\max_{\chi \in \mathrm{X}(D)} \min_{\substack{\varphi \in H_0^1(D) \\ \lambda \ge 0}} G(\chi, \varphi, \lambda), \quad G(\chi, \varphi, \lambda) = E(\chi, \varphi) - \lambda \left[\int_D \chi\, dx - \alpha \right]. \tag{4.44}$$

We then relax the problem to $\overline{\text{co}}\,\text{X}(D)$:

$$\max_{\chi\in\overline{\text{co}}\,\text{X}(D)}\ \min_{\substack{\varphi\in H_0^1(D)\\ \lambda\geq 0}} G(\chi,\varphi,\lambda). \tag{4.45}$$

By the same arguments as the ones used in section 4.1 we have the existence of saddle points $(\hat{\chi}, y, \hat{\lambda})$ which are completely characterized by

$$\forall\varphi\in H_0^1(D),\quad \int_D [k_2+(k_1-k_2)\hat{\chi}]\nabla y\cdot\nabla\varphi - f\varphi\,dx = 0, \tag{4.46}$$

$$\forall\chi\in\overline{\text{co}}\,\text{X}(D),\quad \int_D \left[(k_1-k_2)|\nabla y|^2-\hat{\lambda}\right](\chi-\hat{\chi})\,dx\leq 0, \tag{4.47}$$

$$\left[\int_D \hat{\chi}\,dx-\alpha\right]\hat{\lambda}=0,\quad \int_D \hat{\chi}\,dx-\alpha\leq 0,\quad \hat{\lambda}\geq 0. \tag{4.48}$$

As before y is unique and the set of saddle points of G is the closed convex set

$$X\times\big\{\{y\}\times\Lambda\big\}\subset \text{X}(D)\times\big\{\{y\}\times\{\lambda:\lambda\geq 0\}\big\}.$$

The closed convex set Λ has a minimal element $\hat{\lambda}_m\geq 0$.

For each $\hat{\lambda}$, each maximizer $\hat{\chi}\in X$ is necessarily of the form

$$\hat{\chi}(x)=\begin{cases}1, & \text{if } (k_1-k_2)|\nabla y|^2-\hat{\lambda}>0,\\ \in[0,1], & \text{if } (k_1-k_2)|\nabla y|^2-\hat{\lambda}=0,\\ 0, & \text{if } (k_1-k_2)|\nabla y|^2-\hat{\lambda}<0.\end{cases} \tag{4.49}$$

Associate with each $\lambda>0$ the sets

$$D_+(\lambda)=\left\{x\in D:(k_1-k_2)|\nabla y|^2-\lambda>0\right\}, \tag{4.50}$$

$$D_0(\lambda)=\left\{x\in D:(k_1-k_2)|\nabla y|^2-\lambda=0\right\}, \tag{4.51}$$

$$D_-(\lambda)=\left\{x\in D:(k_1-k_2)|\nabla y|^2-\lambda<0\right\}. \tag{4.52}$$

Define the characteristic function $\chi_m=\chi_{D_+(\lambda_m)}$. By construction all $\hat{\chi}\in X$ are of the form

$$\chi_m\leq\hat{\chi},\quad \int_D\hat{\chi}\,dx\leq\alpha. \tag{4.53}$$

Again it is easy to show that $(\chi_m, y, \hat{\lambda}_m)$ is also a saddle point of G. Therefore, we have a maximizer $\chi_m\in\text{X}(D)$ over $\overline{\text{co}}\,\text{X}(D)$ which is a characteristic function and

$$\int_D\chi_m\,dx\leq\int_D\hat{\chi}\,dx\leq\alpha.$$

If there exists $\hat{\lambda}>0$ in Λ, then for all $\hat{\chi}\in X$

$$\int_D\chi_m\,dx=\alpha=\int_D\hat{\chi}\,dx.$$

As a result the maximizer χ_m is unique, it is a characteristic function, and its integral is equal to α.

The case $\Lambda = \{0\}$ cannot occur since the triplet $(1, y_1, 0)$ would be a saddle point, where y_1 is the solution of the variational equation (4.34) for $\chi = 1$. To see this, first observe that for $k_1 > k_2$

$$\forall \chi, \varphi, \quad G(\chi, \varphi, 0) = E((\chi, \varphi) \leq E((1, \varphi) = G(1, \varphi, 0).$$

As a result

$$\begin{aligned} \inf_{\varphi,\lambda} G(\chi, \varphi, \lambda) &\leq \inf_{\varphi} G(\chi, \varphi, 0) \leq \inf_{\varphi} G(1, \varphi, 0), \\ \inf_{\varphi} G(1, \varphi, 0) &= G(1, y_1, 0) \leq \sup_{\chi} \inf_{\varphi} G(\chi, \varphi, 0) \\ \Rightarrow \sup_{\chi} \inf_{\varphi,\lambda} G(\chi, \varphi, \lambda) &\leq \sup_{\chi} \inf_{\varphi} G(\chi, \varphi, 0) = G(1, y_1, 0). \end{aligned}$$

But we know that there exists a unique $y \in H_0^1(D)$ such that

$$\sup_{\chi} \inf_{\varphi} G(\chi, \varphi, 0) \leq \sup_{\chi} G(\chi, y, 0) = \inf_{\varphi,\lambda} \sup_{\chi} G(\chi, \varphi, \lambda).$$

So from the above two inequalities,

$$\sup_{\chi} \inf_{\varphi,\lambda} G(\chi, \varphi, \lambda) = G(1, y_1, 0) = \inf_{\varphi,\lambda} \sup_{\chi} G(\chi, \varphi, \lambda),$$

$(1, y_1, 0)$ is a saddle point of G, and

$$\alpha \geq \int_D \chi \, dx = \mathrm{m}(D).$$

This contradicts the fact that $\alpha < \mathrm{m}(D)$.

In conclusion in all cases there exists a (unique when $0 \leq \alpha < \mathrm{m}(D)$) maximizer χ^* in $\overline{\mathrm{co}}\, \mathrm{X}(D)$, which is in fact a characteristic function, and

$$\max_{\substack{\chi \in \mathrm{X}(D) \\ \int_D \chi \, dx \geq \alpha}} \min_{\varphi \in H_0^1(D)} E(\chi, \varphi) = \max_{\substack{\chi \in \overline{\mathrm{co}}\, \mathrm{X}(D) \\ \int_D \chi \, dx \geq \alpha}} \min_{\varphi \in H_0^1(D)} E(\chi, \varphi).$$

Moreover, for $0 \leq \alpha < \mathrm{m}(D)$

$$\int_D \chi^* \, dx = \alpha.$$

This is precisely *the* solution of the original problem of Céa and Malanowski and

$$\max_{\substack{\chi \in \mathrm{X}(D) \\ \int_D \chi \, dx = \alpha}} \min_{\varphi \in H_0^1(D)} E(\chi, \varphi) = \max_{\substack{\chi \in \overline{\mathrm{co}}\, \mathrm{X}(D) \\ \int_D \chi \, dx \geq \alpha}} \min_{\varphi \in H_0^1(D)} E(\chi, \varphi).$$

Again, as an illustration of the theoretical results, consider (4.34) over the diamond-shaped domain D defined in (4.33) with the function f of Figure 5.4 in section 4.1.

The variational form (4.35) of the boundary value problem was approximated in the same way as the variational form (4.3) of (4.1) in section 4.1. Figure 5.6 shows the optimal partition of the domain (rotated 45 degrees). The grey triangles correspond to the region $D_0(\hat{\lambda}_m)$, where $\hat{\chi} \in [0,1]$. The black region corresponds to the points where $|\nabla y|^2 > \hat{\lambda}_m/(k_1 - k_2)$ and the white region to the ones where $|\nabla y|^2 < \hat{\lambda}_m/(k_1 - k_2)$, as can be readily seen in Figure 5.6. For this example the Lagrange multiplier associated with the problem is strictly positive. It is interesting to compare this computation with the one of Figure 5.5 in section 4.1, where the support of the force was restricted to Ω.

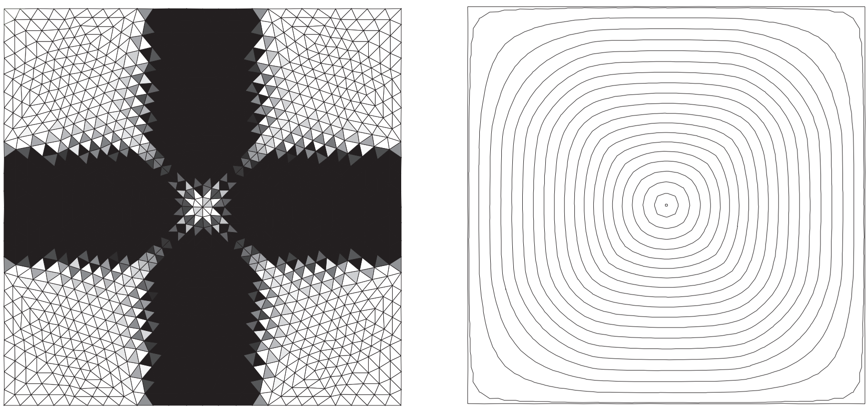


Figure 5.6. *Optimal distribution and isotherms with $k_1 = 2$ (black) and $k_2 = 1$ (white) for the problem of Céa and Malanowski.*

Remark 4.1.
The formulation and the results remain true if the generic variational equation is replaced by a variational inequality (unilateral problem)

$$\max_{\substack{\chi \in X(D) \\ \int_D \chi\, dx = \alpha}} \min_{\varphi \in \{\psi \in H_0^1(D) : \psi \geq 0 \text{ in } D\}} E(\chi, \varphi). \qquad \square$$

4.3 Relaxation and Homogenization

In sections 4.1 and 4.2 the possible homogenization phenomenon predicted in Example 3.1 of section 3.2 did not take place, and in both examples the solution was a characteristic function. Yet, it occurs when the maximization is changed to a minimization in the problem of J. Céa and K. Malanowski [**1**] of section 4.2:

$$\inf_{\substack{\chi \in X(D) \\ \int_D \chi\, dx = \alpha}} \min_{\varphi \in H_0^1(D)} E(\chi, \varphi).$$

In 1985 F. Murat and L. Tartar [**1**] gave a general framework to study this class of problems by relaxation. It is based on the use of L. C. Young [**2**]'s generalized functions (measures). They present a fairly complete analysis of the homogenization theory of second-order elliptic problems of the form

$$-\sum_{ij} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = f.$$

They give as examples the maximization and minimization versions of the problem of section 4.2. This material, which would have deserved a whole chapter in this book, is fortunately available in English (cf. F. Murat and L. Tartar [**3**]). More results on composite materials can be found in the book edited by A. Cherkaev and R. Kohn [**1**], which gathers a selection of translations of key papers originally written in French and Russian.

5 Buckling of Columns

One of the very early optimal design problem was formulated by Lagrange in 1770 (cf. I. Todhunter and K. Pearson [**1**]) and later studied by T. Clausen in 1849. It consists in finding the best profile of a vertical column to prevent buckling. This problem and other problems related to columns have been revisited in a series of papers by S. J. Cox [**1**], S. J. Cox and M. L. Overton [**1**], S. J. Cox [**2**], and S. J. Cox and C. M. McCarthy [**1**]. Since Lagrange many authors have proposed solutions, but a complete theoretical and numerical solution for the buckling of a column was given only in 1992 by S. J. Cox and M. L. Overton [**1**].

Consider a normalized column of unit height and unit volume. Denote by ℓ the *magnitude of the normalized axial load* and by u the resulting transverse displacement. Assume that the potential energy is the sum of the bending and elongation energies

$$\int_0^1 EI\,|u''|^2\,dx - \ell \int_0^1 |u'|^2\,dx,$$

where I is the second moment of area of the column's cross section and E is its Young's modulus. For sufficiently small load ℓ the minimum of this potential energy with respect to all admissible u is zero. *Euler's buckling load* λ of the column is the largest ℓ for which this minimum is zero. This is equivalent to finding the following minimum:

$$\lambda \overset{\text{def}}{=} \inf_{0\neq u\in V} \frac{\int_0^1 EI\,|u''|^2\,dx}{\int_0^1 |u'|^2\,dx}, \tag{5.1}$$

where $V = H_0^2(0,1)$ corresponds to the clamped case, but other types of boundary conditions can be contemplated. This is an eigenvalue problem with a special Rayleigh quotient.

Assume that E is constant and that the second moment of area $I(x)$ of the column's cross section at the height x, $0 \leq x \leq 1$, is equal to a constant c times its cross-sectional area $A(x)$,

$$I(x) = c\,A(x) \quad \Rightarrow \int_0^1 A(x)\,dx = 1.$$

Normalizing λ by cE and taking into account the engineering constraints

$$\exists 0 < A_0 < A_1,\, \forall x \in [0,1], \quad 0 < A_0 \le A(x) \le A_1,$$

we finally get

$$\sup_{A\in\mathcal{A}} \lambda(A), \quad \lambda(A) \stackrel{\text{def}}{=} \inf_{0\neq u\in V} \frac{\int_0^1 A\,|u''|^2\,dx}{\int_0^1 |u'|^2\,dx}, \tag{5.2}$$

$$\mathcal{A} \stackrel{\text{def}}{=} \left\{ A \in L^2(0,1) \,:\, A_0 \le A \le A_1 \text{ and } \int_0^1 A(x)\,dx = 1 \right\}. \tag{5.3}$$

This problem can also be reformulated by rewriting

$$A(x) = A_0 + \chi(x)\,(A_1 - A_0), \quad \int_0^1 \chi(x)\,dx = \alpha \stackrel{\text{def}}{=} \frac{1 - A_0}{A_1 - A_0}$$

for some $\chi \in \overline{\text{co}}\, X([0,1])$. Clearly the problem makes sense only for $0 < A_0 \le 1$. Then

$$\sup_{\substack{\chi\in\overline{\text{co}}\,X([0,1])\\ \int_0^1 \chi(x)\,dx=\alpha}} \tilde{\lambda}(\chi), \quad \tilde{\lambda}(\chi) \stackrel{\text{def}}{=} \inf_{0\neq u\in V} \frac{\int_0^1 [A_0 + (A_1 - A_0)\,\chi]\,|u''|^2\,dx}{\int_0^1 |u'|^2\,dx}. \tag{5.4}$$

Rayleigh's quotient is not a nice convex concave function with respect to (v, A), and its analysis necessitates tools different from the ones of section 4. One of the original elements of the paper of S. J. Cox and M. L. Overton [**1**] was to replace Rayleigh's quotient by G. Auchmuty [**1**]'s dual variational principle for the eigenvalue problem (5.1). We first recall the existence of solution to the minimization of Rayleigh's quotient. In what follows we shall use the norm $\|u'\|_{L^2}$ for the space $H_0^1(0,1)$ and $\|u''\|_{L^2}$ for the space $H_0^2(0,1)$.

Theorem 5.1. *There exists at least one nonzero solution $u \in V$ to the minimization problem*

$$\lambda(A) \stackrel{\text{def}}{=} \inf_{0\neq u\in V} \frac{\int_0^1 A\,|u''|^2\,dx}{\int_0^1 |u'|^2\,dx}. \tag{5.5}$$

Then $\lambda(A_0) > 0$ and for all $A \in \mathcal{A}$, $\lambda(A) \ge \lambda(A_0)$, and

$$\forall v \in V, \quad \int_0^1 |v'|^2\,dx \le \lambda(A_0)^{-1} \int_0^1 A\,|v''|^2\,dx. \tag{5.6}$$

The solutions are completely characterized by the variational equation:

$$\exists u \in V,\, \forall v \in V, \quad \int_0^1 A\,u''\,v''\,dx = \lambda(A) \int_0^1 u'\,v'\,dx. \tag{5.7}$$

Proof. The infimum is bounded below by 0 and is necessarily finite. Let $\{u_n\}$ be a minimizing sequence such that $\|u_n'\|_{L^2} = 1$. Then the sequence $\|u_n''\|_{L^2(D)}$ is bounded. Hence $\{u_n\}$ is a bounded sequence in $H^2(0,1)$ and there exist $u \in H^2(0,1)$ and a subsequence, still indexed by n, such that $u_n \rightharpoonup u$ in $H_0^2(0,1)$-weak. Therefore the subsequence strongly converges in $H_0^1(0,1)$ and

$$1 = \int_0^1 |u_n'|^2\,dx \to \int_0^1 |u'|^2\,dx \quad \text{and} \quad \int_0^1 A\,|u''|^2\,dx \le \liminf_{n\to\infty} \int_0^1 A\,|u_n''|^2\,dx$$
$$\Rightarrow \frac{\int_0^1 A\,|u''|^2\,dx}{\int_0^1 |u'|^2\,dx} \le \liminf_{n\to\infty} \frac{\int_0^1 A\,|u_n''|^2\,dx}{\int_0^1 |u_n'|^2\,dx} = \lambda(A),$$

and, by the definition of $\lambda(A)$, $u \in H_0^2(0,1)$ is a minimizing element. Notice that

$$\begin{aligned}\forall A \in \mathcal{A}, \quad \lambda(A_0) &= \inf\left\{\frac{\int_0^1 A_0\,|u''|^2\,dx}{\int_0^1 |u'|^2\,dx} : \forall u \in H_0^2(0,1),\ u \ne 0\right\} \\ &\le \inf\left\{\frac{\int_0^1 A\,|u''|^2\,dx}{\int_0^1 |u'|^2\,dx} : \forall u \in H_0^2(0,1),\ u \ne 0\right\} = \lambda(A)\end{aligned}$$

and similarly $\lambda(A) \le \lambda(A_1)$. If $\lambda(A_0) = 0$, we repeat the above construction and end up with an element $u \in H_0^2(0,1)$ such that

$$\int_0^1 |u'|^2\,dx = 1 \quad \text{and} \quad A_0 \int_0^1 |u''|^2\,dx = \int_0^1 A_0 |u''|^2\,dx = 0,$$

which is impossible in $H_0^2(0,1)$ since $A_0 \ne 0$ and $u'' = 0$ implies $u = 0$.

If $u \ne 0$, then Rayleigh's quotient is differentiable and its directional semiderivative in the direction v is given by

$$2\frac{\int_0^1 A\,u''\,v''\,dx}{\|u'\|_{L^2}^2} - 2\int_0^1 A\,|u''|^2\,dx \frac{\int_0^1 u'\,v'\,dx}{\|u'\|_{L^2}^4}.$$

A nonzero solution u of the minimization problem is necessarily a stationary point, and for all $v \in H_0^2(0,1)$

$$\int_0^1 A\,u''\,v''\,dx = \frac{\int_0^1 A_0\,|u''|^2\,dx}{\int_0^1 |u'|^2\,dx} \int_0^1 u'\,v'\,dx = \lambda(A) \int_D u'\,v'\,dx.$$

Conversely any nonzero solution of (5.7) is necessarily a minimizer of Rayleigh's quotient. □

The dual variational principle of G. Auchmuty [**1**] for the eigenvalue problem (5.5) can be chosen as

$$\mu(A) \stackrel{\text{def}}{=} \inf\left\{L(A,v) : v \in H_0^2(0,1)\right\}, \tag{5.8}$$

$$L(A,v) \stackrel{\text{def}}{=} \frac{1}{2}\int_0^1 A\,|v''|^2\,dx - \left[\int_0^1 |v'|^2\,dx\right]^{1/2}. \tag{5.9}$$

Theorem 5.2. *For each $A \in \mathcal{A}$, there exists at least one minimizer of $L(A, v)$,*

$$\mu(A) = -\frac{1}{2\lambda(A)}, \tag{5.10}$$

and the set of minimizers of (5.8) *is given by*

$$E(A) \stackrel{\text{def}}{=} \left\{ u \in H_0^2(0,1) : \begin{array}{c} u \text{ is solution of (5.7) and} \\ \left\{ \int_0^1 |u'|^2 \, dx \right\}^{1/2} = 1/\lambda(A) \end{array} \right\}. \tag{5.11}$$

Proof. The existence of the solution follows from the fact that $v \mapsto L(A, v)$ is weakly lower semicontinuous and coercive. From Theorem 5.1 the set $E(A)$ is not empty, and for any $u \in E(A)$

$$\mu(A) \leq L(A, u) = -\frac{1}{2\lambda(A)} < 0. \tag{5.12}$$

Therefore, the minimizers of (5.7) are different from the zero functions. For $u \neq 0$, the function $L(A, u)$ is differentiable and its directional semiderivative is given by

$$dL(A, u; v) = \int_0^1 A\, u''\, v''\, dx - \frac{1}{\|u'\|_{L^2}} \int_0^1 u'\, v'\, dx, \tag{5.13}$$

and any minimizer u of $L(A, v)$ is a stationary point of $dL(A, u; v)$, that is,

$$\forall v \in H_0^2(0,1), \quad \int_0^1 A\, u''\, v''\, dx - \frac{1}{\|u'\|_{L^2}} \int_0^1 u'\, v'\, dx = 0. \tag{5.14}$$

Therefore, u is a solution of the eigenvalue problem with

$$\lambda' = \frac{1}{\|u'\|_{L^2}} \quad \Rightarrow -\frac{1}{2\lambda'} = \mu(A) \leq -\frac{1}{2\lambda(A)}$$

from inequality (5.12). By the minimality of $\lambda(A)$, we necessarily have $\lambda' = \lambda(A)$, and this concludes the proof of the theorem. $\square$

Theorem 5.3.

(i) *The set $\mathcal{A}$ is compact in the $L^2(0,1)$*-weak *topology.*

(ii) *The function $A \mapsto \mu(A)$ is concave and upper semicontinuous with respect to the $L^2(0,1)$*-weak *topology.*

(iii) *There exists A in $\mathcal{A}$ which maximizes $\mu(A)$ over $\mathcal{A}$ and, a fortiori, which maximizes $\lambda(A)$ over $\mathcal{A}$.*

Proof. (i) $\mathcal{A}$ is convex, bounded, and closed in $L^2(0,1)$. Hence it is compact in $L^2(0,1)$ weak.

(ii) *Concavity.* For all λ in $[0,1]$ and A and A' in $\mathcal{A}$,

$$\begin{aligned} L(\lambda A+(1-\lambda)A',v) &= \lambda L(A,v)+(1-\lambda)L(A',v) \\ &\geq \lambda \inf_{v\in V} L(A,v)+(1-\lambda)\inf_{v\in V} L(A',v) \end{aligned}$$
$$\Rightarrow \mu(\lambda A+(1-\lambda)A') = \inf_{v\in V} L(\lambda A+(1-\lambda)A',v) \geq \lambda\mu(A)+(1-\lambda)\mu(A').$$

Upper semicontinuity. Let $u_A \in \mathcal{A}$ be a minimizer of $L(A,v)$ and $\{A_n\}$ be a sequence converging to A in the $L^2(0,1)$-weak topology. By Lemma 3.1, $\{A_n\}$ converges to A in the $L^\infty(0,1)$-weak$\star$ topology. Then

$$\begin{aligned} \mu(A_n)-\mu(A) &\leq L(A_n,u_A)-L(A,u_A) = \frac{1}{2}\int_0^1 (A_n-A)\,|u_A''|^2\,dx \\ \Rightarrow \limsup_{n\to\infty}\mu(A_n)-\mu(A) &\leq \lim_{n\to\infty}\frac{1}{2}\int_0^1 (A_n-A)\,|u_A''|^2\,dx = 0. \end{aligned}$$

(iii) The existence of solution follows from the concavity and upper semicontinuity of $\mu(A)$. Hence it is weakly upper semicontinuous over the weakly compact subset $\mathcal{A}$ of $L^2(0,1)$. The existence of a minimizer of $\lambda(A)$ follows from identity 5.10. $\square$

6 Caccioppoli or Finite Perimeter Sets

The notion of a *finite perimeter set* has been introduced and developed mainly by R. Caccioppoli [**1**] and E. De Giorgi [**1**] in the context of J. A. F. Plateau [**1**]'s problem, named after the Belgian physicist and professor (1801–1883), who did experimental observations on the geometry of soap films. A modern treatment of this subject can be found in the book of E. Giusti [**1**]. One of the difficulties in studying the *minimal surface problem* is the description of such surfaces in the usual language of differential geometry. For instance, the set of possible singularities is not known. Finite perimeter sets provide a geometrically significant solution to Plateau's problem without having to know ahead of time what all the possible singularities of the solution can be. The characterization of all the singularities of the solution is a difficult problem which can be considered separately.

This was a very fundamental contribution to the theory of variational problems, where the optimization variable is the geometry of a domain. This point of view has been expanded, and a variational calculus was developed by F. J. Almgren, Jr. [**1**]. This is the *theory of varifolds.*

> *Perhaps the one most important virtue of varifolds is that it is possible to obtain a geometrically significant solution to a number of variational problems, including Plateau's problem, without having to know ahead of time what all the possible singularities of the solution can be.*
> (cf. F. J. Almgren, Jr. [**1**, p. viii]).

The treatment of this topic is unfortunately much beyond the scope of this book, and the interested reader is referred to the above reference for more details.

6.1 Finite Perimeter Sets

Given an open subset D of $\mathbf{R}^\mathrm{N}$, consider L^1-functions f on D with distributional gradient ∇f in the space $M^1(D)^N$ of (vectorial) bounded measures; that is,

$$\vec{\varphi} \mapsto \langle \nabla f, \vec{\varphi}\rangle_{\mathcal{D}} \overset{\text{def}}{=} -\int_D f \operatorname{div} \vec{\varphi}\, dx \colon \mathcal{D}^1(D;\mathbf{R}^\mathrm{N}) \to \mathbf{R}$$

is continuous with respect to the topology of uniform convergence in D:

$$\|\nabla f\|_{M^1(D)^N} \overset{\text{def}}{=} \sup_{\substack{\vec{\varphi}\in\mathcal{D}^1(D;\mathbf{R}^\mathrm{N})\\ \|\vec{\varphi}\|_C\le 1}} \langle \nabla f, \vec{\varphi}\rangle_{\mathcal{D}} < \infty, \quad \|\vec{\varphi}\|_C \overset{\text{def}}{=} \sup_{x\in D} |\vec{\varphi}(x)|_{\mathbf{R}^\mathrm{N}}, \tag{6.1}$$

where $M^1(D) \overset{\text{def}}{=} \mathcal{D}^0(D)'$ is the topological dual of $\mathcal{D}^0(D)$, and

$$\nabla f \in \mathcal{L}\big(\mathcal{D}^0(D;\mathbf{R}^\mathrm{N}),\mathbf{R}\big) \equiv \mathcal{L}\big(\mathcal{D}^0(D),\mathbf{R}\big)^N = M^1(D)^N.$$

Such functions are known as *functions of bounded variation.* The space

$$\mathrm{BV}(D) \overset{\text{def}}{=} \big\{f \in L^1(D) : \nabla f \in M^1(D)^N\big\} \tag{6.2}$$

endowed with the norm

$$\|f\|_{\mathrm{BV}(D)} = \|f\|_{L^1(D)} + \|\nabla f\|_{M^1(D)^N} \tag{6.3}$$

is a Banach space.

Definition 6.1 (L. C. Evans and R. F. Gariepy [**1**]).
A function $f \in L^1_{\mathrm{loc}}(U)$ in an open subset U of $\mathbf{R}^\mathrm{N}$ has *locally bounded variation* if for each bounded open subset V of U such that $\overline{V} \subset U$, $f \in \mathrm{BV}(V)$. The set of all such functions will be denoted by $\mathrm{BV}_{\mathrm{loc}}(U)$. □

Theorem 6.1. *Given an open subset U of $\mathbf{R}^\mathrm{N}$, f belongs to $\mathrm{BV}_{\mathrm{loc}}(U)$ if and only if for each $x \in U$ there exists $\rho > 0$ such that $\overline{B(x,\rho)} \subset U$ and $f \in \mathrm{BV}(B(x,\rho))$, $B(x,\rho)$ the open ball of radius ρ in x.*

Proof. If $f \in \mathrm{BV}_{\mathrm{loc}}(U)$, then the result is true by specializing to balls in each point of U. Conversely, by assumption, the open set U is covered by a family $\mathcal{F}(U)$ of balls $B(x,\rho_x)$ such that $\overline{B(x,\rho_x)} \subset U$ and $f \in \mathrm{BV}(B(x,\rho_x))$. Given a bounded open subset V of U such that $\overline{V} \subset U$, the compact subset $\overline{V}$ can be covered by a finite number of balls $\{B(x_i,\rho_i)\}$, $1 \le i \le n$, in $\mathcal{F}(U)$ for points $x_i \in \overline{V}$, $1 \le i \le n$,

$$\overline{V} \subset \bigcup_{i=1}^n B_i, \quad B_i \overset{\text{def}}{=} B(x_i,\rho_{x_i}).$$

Denote by $\{\psi_i \in \mathcal{D}(B_i)\}_{i=1}^n$ a partition of unity for the family $\{B_i\}$ such that

$$\forall i,\ 0 \le \psi_i \le 1 \text{ in } B_i, \text{ and } \sum_{i=1}^n \psi_i = 1 \text{ on } \overline{V}.$$

Then for each $\vec{\varphi} \in \mathcal{V}^1(V)^N$

$$\vec{\varphi} = \sum_{i=1}^{n} \vec{\varphi}_i \text{ on } V, \quad \vec{\varphi}_i \stackrel{\text{def}}{=} \psi_i \, \vec{\varphi},$$

and by construction $\vec{\varphi}_i \in \mathcal{V}^1(B_i \cap V)^N$. Therefore, for all $\vec{\varphi} \in \mathcal{V}^1(V)^N$ such that $\|\vec{\varphi}\|_{C(V)} \leq 1$

$$\begin{aligned} -\int_V f \operatorname{div} \vec{\varphi} \, dx &= -\sum_{i=1}^{n} \int_V f \operatorname{div} \vec{\varphi}_i \, dx = -\sum_{i=1}^{n} \int_{V \cap B_i} f \operatorname{div} \vec{\varphi}_i \, dx \\ \Rightarrow \left| \int_V f \operatorname{div} \vec{\varphi} \, dx \right| &\leq \sum_{i=1}^{n} \left| \int_{B_i \cap V} f \operatorname{div} \vec{\varphi}_i \, dx \right| \\ \Rightarrow \left| \int_V f \operatorname{div} \vec{\varphi} \, dx \right| &\leq \sum_{i=1}^{n} \|\nabla f\|_{M^1(B_i \cap V)} \leq \sum_{i=1}^{n} \|\nabla f\|_{M^1(B_i)} < \infty, \end{aligned}$$

since $\mathcal{V}^1(B_i \cap V)^N \subset \mathcal{V}^1(B_i)^N$ and $\|\vec{\varphi}_i\|_{C(B_i)} \leq 1$. Therefore, $\|\nabla f\|_{M^1(V)}$ is finite and $f \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$. □

For more details and properties see also F. Morgan [**1**, p. 117], H. Federer [**5**, sect. 4.5.9], L. C. Evans and R. F. Gariepy [**1**], W. P. Ziemer [**1**], and R. Temam [**1**].

As in the previous section, consider measurable subsets Ω of a fixed bounded open subset D of $\mathbf{R}^{\mathrm{N}}$. Their characteristic functions $\chi_\Omega \in \mathrm{X}(D)$ are $L^1(D)$-functions

$$\|\chi_\Omega\|_{L^1(D)} = \int_D \chi_\Omega \, dx = \mathrm{m}(\Omega) \leq \mathrm{m}(D) < \infty \tag{6.4}$$

with distributional gradient

$$\forall \vec{\varphi} \in \big(\mathcal{D}(D)\big)^N, \quad \langle \nabla \chi_\Omega, \vec{\varphi} \rangle_{\mathcal{D}} \stackrel{\text{def}}{=} -\int_D \chi_\Omega \operatorname{div} \vec{\varphi} \, dx. \tag{6.5}$$

When Ω is an open domain with boundary Γ of class C^1, then by the Stokes divergence theorem

$$-\int_{\Omega \cap D} \operatorname{div} \vec{\varphi} \, dx = -\int_{\partial(\Omega \cap D)} \vec{\varphi} \cdot n \, d\Gamma = -\int_{\Gamma \cap D} \vec{\varphi} \cdot n \, d\Gamma,$$

where n is the outward normal field along $\partial(\Omega \cap D)$. Since Γ is of class C^1 the normal field n along Γ belongs to $C^0(\Gamma)$, and from the last identity the maximum is

$$P_D(\Omega) \stackrel{\text{def}}{=} \|\nabla \chi_\Omega\|_{M^1(D)} = \int_{\Gamma \cap D} |n|^2 \, d\Gamma = \int_{\Gamma \cap D} d\Gamma = H_{N-1}(\Gamma \cap D),$$

the $(N-1)$-dimensional Hausdorff measure of $\Gamma \cap D$. As a result

$$H_{N-1}(\Gamma) = P_D(\Omega) + H_{N-1}(\Gamma \cap \partial D). \tag{6.6}$$

Thus the norm of the gradient provides a natural relaxation of the notion of perimeter to the following larger class of domains.

Definition 6.2.
Let Ω be a Lebesgue measurable subset of $\mathbf{R}^N$.

(i) The *perimeter of* Ω *with respect to an open subset* D *of* $\mathbf{R}^N$ is defined as

$$P_D(\Omega) \stackrel{\text{def}}{=} \|\nabla\chi_\Omega\|_{M^1(D)^N}. \tag{6.7}$$

Ω is said to have *finite perimeter with respect to* D if $P_D(\Omega)$ is finite. The family of measurable characteristic functions with finite measure and finite relative perimeter in D will be denoted as

$$\mathrm{BX}(D) \stackrel{\text{def}}{=} \{\chi_\Omega \in \mathrm{X}(D) \,:\, \chi_\Omega \in \mathrm{BV}(D)\}.$$

(ii) Ω is said to have *locally finite perimeter* if for all bounded open subsets D of $\mathbf{R}^N$, $\chi_\Omega \in \mathrm{BV}(D)$, that is, $\chi_\Omega \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^N)$.

(iii) Ω is said to have *finite perimeter* if $\chi_\Omega \in \mathrm{BV}(\mathbf{R}^N)$. $\square$

Theorem 6.2. *Let* Ω *be a Lebesgue measurable subset of* $\mathbf{R}^N$. Ω *has* locally finite perimeter, *that is,* $\chi_\Omega \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^N)$, *if and only if for each* $x \in \partial\Omega$ *there exists* $\rho > 0$ *such that* $\chi_\Omega \in \mathrm{BV}(B(x,\rho))$, $B(x,\rho)$ *the open ball of radius* ρ *in* x.

Proof. We use Theorem 6.1. In one direction the result is obvious. Conversely, if $x \in \operatorname{int}\Omega$, then there exists $\rho > 0$ such that $B(x,\rho) \subset \operatorname{int}\Omega$, and for all $\vec{\varphi} \in \mathcal{D}^1(B(x,\rho))$

$$-\int_{B(x,\rho)} \chi_\Omega \operatorname{div}\vec{\varphi}\,dx = -\int_{B(x,\rho)} \operatorname{div}\vec{\varphi}\,dx = -\int_{\partial B(x,\rho)} \vec{\varphi}\cdot n\,dH_{N-1} = 0$$

since $\vec{\varphi} = 0$ on the boundary of $B(x,\rho)$. Thus $\chi_\Omega \in \mathrm{BV}(B(x,\rho))$. If $x \in \operatorname{int}\complement\Omega$, then there exists $\rho > 0$ such that $B(x,\rho) \subset \operatorname{int}\complement\Omega$, and for all $\vec{\varphi} \in \mathcal{D}^1(B(x,\rho))$

$$-\int_{B(x,\rho)} \chi_\Omega \operatorname{div}\vec{\varphi}\,dx = 0.$$

So again $\chi_\Omega \in \mathrm{BV}(B(x,\rho))$. Finally, if $x \in \partial\Omega$, then by assumption there exists $\rho > 0$ such that $\chi_\Omega \in \mathrm{BV}(B(x,\rho))$. Therefore, by Theorem 6.1, $\chi_\Omega \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^N)$. $\square$

The interest behind this construction is twofold. First, the notion of perimeter of a set is extended to measurable sets; second, this framework provides a first compactness theorem which will be useful in obtaining existence of optimal domains.

Theorem 6.3. *Assume that* D *is a bounded open domain in* $\mathbf{R}^N$ *with a Lipschitzian boundary* ∂D. *Let* $\{\Omega_n\}$ *be a sequence of measurable domains in* D *for which there exists a constant* $c > 0$ *such that*

$$\forall n, \quad P_D(\Omega_n) \le c. \tag{6.8}$$

Then there exist a measurable set Ω *in* D *and a subsequence* $\{\Omega_{n_k}\}$ *such that*

$$\chi_{\Omega_{n_k}} \to \chi_\Omega \text{ in } L^1(D) \text{ as } k \to \infty \quad \text{and} \quad P_D(\Omega) \le \liminf_{k\to\infty} P_D(\Omega_{n_k}) \le c. \tag{6.9}$$

Moreover $\nabla\chi_{\Omega_{n_k}}$ *"converges in measure" to* $\nabla\chi_\Omega$ *in* $M^1(D)^N$*; that is, for all* $\vec{\varphi}$ *in* $\mathcal{D}^0(D,\mathbf{R}^\mathrm{N})$,

$$\lim_{k\to\infty}\langle\nabla\chi_{\Omega_{n_k}},\vec{\varphi}\rangle_{M^1(D)^N}\to\langle\nabla\chi_\Omega,\vec{\varphi}\rangle_{M^1(D)^N}. \tag{6.10}$$

Proof. This follows from the fact that the injection of the space $\mathrm{BV}(D)$ endowed with the norm (6.3) into $L^1(D)$ is continuous and compact (cf. E. Giusti [**1**, Thm. 1.19, p. 17], V. G. Maz'ja [**1**, Thm. 6.1.4, p. 300, Lem. 1.4.6, p. 62], C. B. Morrey, Jr. [**1**, Thm. 4.4.4, p. 75], and L. C. Evans and R. F. Gariepy [**1**]). □

Example 6.1 (The staircase)**.**
In (6.9) the inequality can be strict, as can be seen from the following example (cf. Figure 5.7). For each $n\geq 1$, define the set

$$\Omega_n=\bigcup_{j=1}^{n}\left[\frac{j}{n},\frac{j+1}{n}\right]\times\left[0,1-\frac{j}{n}\right].$$

Its limit is the set

$$\Omega=\{(x,y):0\leq x\leq 1, 0\leq y\leq x\}.$$

It is easy to check that the sets Ω_n are contained in the holdall $D=]-1,2[\times]-1,2[$ and that

$$\forall n\geq 1,\quad P_D(\Omega_n)=4,\quad P_D(\Omega)=2+\sqrt{2},$$
$$\chi_{\Omega_n}\to\chi_\Omega\quad\text{in}\quad L^p(D),\ 1\leq p<\infty.$$

Each Ω_n is uniformly Lipschitzian, but the Lipschitz constant and the two neighborhoods of Definition 3.2 in Chapter 2 cannot be chosen independently of n. □

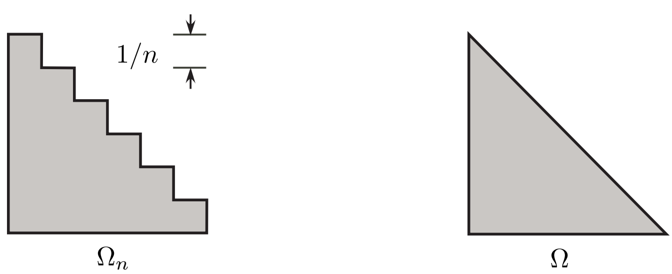


Figure 5.7. *The staircase.*

Corollary 1. *Let D be open bounded and Lipschitzian. There exists a constant $c>0$ such that for all convex domains Ω in D*

$$P_D(\Omega)\leq c,\quad m(\Omega)\leq c$$

and the set $\mathcal{C}(D)$ of convex subsets of D is compact in $L^p(D)$ for all p, $1\leq p<\infty$.

Proof. If Ω is a convex set with zero volume, $\mathrm{m}(\Omega) = 0$, then its perimeter $P_D(\Omega) = 0$ (cf. E. GIUSTI [**1**, Rem. 1.7 (iii), p. 6]). If $\mathrm{m}(\Omega) > 0$, then $\operatorname{int}\Omega \neq \varnothing$. For convex sets with a nonempty interior the perimeter and the volume enjoy a very nice monotonicity property. If $\mathcal{C}^\bullet$ denotes the set of nonempty closed convex subsets of $\mathbf{R}^{\mathrm{N}}$, then the map $\Omega \mapsto m(\Omega)\colon \mathcal{C}^\bullet \to \mathbf{R}$ is strictly increasing:

$$\forall A, B \in \mathcal{C}^\bullet, \quad A \subsetneq B \implies m(A) < m(B)$$

(cf. M. BERGER [**1**, Vol. 3, Prop. 12.9.4.3, p. 141]). Similarly if $P(\Omega)$ denotes the perimeter of Ω in $\mathbf{R}^{\mathrm{N}}$, the map $\Omega \mapsto P(\Omega)\colon \mathcal{C}^\bullet \to \mathbf{R}$ is strictly increasing (cf. M. BERGER [**1**, Vol. 3, Prop. 12.10.2, p. 144]). As a result

$$\forall n, \quad m(\Omega_n) \le m(D) < \infty \quad \text{and} \quad P_D(\Omega_n) \le P(\Omega_n) \le P(\operatorname{co} D) < \infty,$$

since the perimeter of a bounded convex set is finite. Then the conditions of the theorem are satisfied and the conclusions of the theorem follow. But we have seen in Theorem 3.5 (iii) that the set $\mathcal{C}(D)$ is closed in $L^1(D)$. Therefore, Ω can be chosen convex in the equivalence class. This completes the proof. $\square$

This theorem and the lower (resp., upper) semicontinuity of the shape function $\Omega \mapsto J(\Omega)$ will provide existence results for domains in the class of finite perimeter sets in D. One example is the transmission problem (4.1) to (4.3) of section 4.1 with the objective function

$$J(\Omega) = \frac{1}{2}\int_\Omega |y(\Omega) - g|^2\, dx + \alpha P_D(\Omega), \quad \alpha > 0, \tag{6.11}$$

for some $g \in L^2(D)$. We shall come back later to the homogeneous Dirichlet boundary value problem (3.25)–(3.26).

It is important to recall that even if a set Ω in D has a finite perimeter $P_D(\Omega)$, its *relative boundary* $\Gamma \cap D$ can have a nonzero N-dimensional Lebesgue measure. To illustrate this point consider the following adaptation of Example 1.10 in E. GIUSTI [**1**, p. 7].

Example 6.2.
Let $D = B(0,1)$ in $\mathbf{R}^2$ be the open ball in 0 of radius 1. For $i \ge 1$, let $\{x_i\}$ be an ordered sequence of all points in D with rational coordinates. Associate with each i the open ball

$$B_i = \{x \in D : |x - x_i| < \rho_i\}, \quad 0 < \rho_i \le \min\{2^{-i}, 1 - |x_i|\}.$$

Define the new sequence of open subsets of D,

$$\Omega_n = \bigcup_{i=1}^n B_i,$$

and notice that for all $n \ge 1$,

$$\mathrm{m}(\partial\Omega_n) = 0, \quad P_D(\Omega_n) \le 2\pi,$$

where $\mathrm{m} = \mathrm{m}_2$ is the Lebesgue measure in $\mathbf{R}^2$ and $\partial\Omega_n$ is the boundary of Ω_n. Moreover, since the sequence of sets $\{\Omega_n\}$ is increasing,

$$\chi_{\Omega_n} \to \chi_\Omega \text{ in } L^1(D), \quad \Omega = \bigcup_{i=1}^{\infty} B_i, \quad P_D(\Omega) \le \liminf_{n\to\infty} P_D(\Omega_n) \le 2\pi.$$

Observe that $\overline{\Omega} = \overline{D}$ and that $\partial\Omega = \overline{\Omega} \cap \overline{\complement\Omega} \supset \overline{D} \cap \complement\Omega$. Thus

$$\mathrm{m}(\partial\Omega) = \mathrm{m}(\overline{D} \cap \complement\Omega) \ge \mathrm{m}(\overline{D}) - \mathrm{m}(\Omega) \ge \frac{2\pi}{3}$$

since

$$\mathrm{m}(D) = \pi \quad \text{and} \quad \mathrm{m}(\Omega) \le \sum_{1=1}^{\infty} \pi 2^{-2i} = \frac{\pi}{3}.$$

Recall that

$$\mathrm{m}(\Omega_n) \le \sum_{i=1}^{n} \mathrm{m}(B_i) \le \sum_{i=1}^{n} \pi 2^{-2i} \quad \Rightarrow \mathrm{m}(\overline{D}) \le \sum_{i=1}^{\infty} \mathrm{m}(B_i) \le \sum_{i=1}^{\infty} \pi 2^{-2i}.$$

For p, $1 \le p < \infty$, the sequence of characteristic functions $\{\chi_{\Omega_n}\}$ converges to χ_Ω in $L^p(D)$-strong. However, for all n, $\mathrm{m}(\partial\Omega_n) = 0$, but $\mathrm{m}(\partial\Omega) > 0$. $\square$

In fact we can associate with the perimeter $P_D(\Omega)$ the *reduced boundary*, $\partial^*\Omega$, which is the set of all $x \in \partial\Omega$ for which the normal $n(x)$ exists. We quote the following interesting theorems from W. H. Fleming [**1**, p. 455] (cf. also E. De Giorgi [**2**], L. C. Evans and R. F. Gariepy [**1**, Thm. 1, sect. 5.1, p. 167, Notation, pp. 168–169, Lem. 1, p. 208], and H. Federer [**2**]).

Theorem 6.4. *Let $\chi_\Omega \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$. There exist a Radon measure $\|\partial\Omega\|$ on $\mathbf{R}^{\mathrm{N}}$ and a $\|\partial\Omega\|$-measurable function $\nu_\Omega : \mathbf{R}^{\mathrm{N}} \to \mathbf{R}^{\mathrm{N}}$ such that*

(i) $|\nu_\Omega(x)| = 1$, $\|\partial\Omega\|$ *almost everywhere and*

(ii) $\int_{\mathbf{R}^{\mathrm{N}}} \chi_\Omega \operatorname{div} \varphi \, dx = -\int_{\mathbf{R}^{\mathrm{N}}} \varphi \cdot \nu_\Omega \, d\|\partial\Omega\|$, *for all* $\varphi \in C^1_c(\mathbf{R}^{\mathrm{N}}; \mathbf{R}^{\mathrm{N}})$.

Definition 6.3.
Let $\chi_\Omega \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$. The point $x \in \mathbf{R}^{\mathrm{N}}$ belongs to the *reduced boundary* $\partial^*\Omega$ if

(i) $\|\partial\Omega\|(B_r(x)) > 0$, for all $r > 0$,

(ii) $\lim_{r\searrow 0} \frac{1}{\mathrm{m}(B_r(x))} \int_{B_r(x)} \nu_\Omega \, d\|\partial\Omega\| = \nu_\Omega(x)$, and

(iii) $|\nu_\Omega(x)| = 1$. $\square$

Theorem 6.5. *Let Ω have finite perimeter $P(\Omega)$. Let $\partial^*\Omega$ denote the reduced boundary of Ω. Then*

(i) $\partial^*\Omega \subset \partial_*\Omega \subset \Omega_\bullet \subset \partial\Omega$ *and* $\overline{\partial^*\Omega} = \partial\Omega$,

(ii) $P(\Omega) = H_{N-1}(\partial^*\Omega)$ (cf. E. Giusti [**1**, Chap. 4]),

(iii) *the Gauss–Green theorem holds with* $\partial^*\Omega$, *and*

(iv) $\mathrm{m}(\partial^*\Omega) = \mathrm{m}(\partial_*\Omega) = 0, \quad H_{N-1}(\partial_*\Omega\backslash\partial^*\Omega) = 0.$

We quote the following density theorem (cf. E. Giusti [**1**, Thm. 1.24 and Lem. 1.25, p. 23]), which complements Theorem 3.1.

Theorem 6.6. *Let* Ω *be a bounded measurable domain in* $\mathbf{R}^{\mathrm{N}}$ *with finite perimeter. Then there exists a sequence* $\{\Omega_j\}$ *of* C^∞*-domains such that as* j *goes to* ∞

$$\int_{\mathbf{R}^{\mathrm{N}}} |\chi_{\Omega_j} - \chi_\Omega|\, dx \to 0 \quad \text{and} \quad P(\Omega_j) \to P(\Omega).$$

6.2 Decomposition of the Integral along Level Sets

We complete this section by giving some useful theorems on the decomposition of the integral along the *level sets* of a function. The first one was used by J.-P. Zolésio [**6**] (Grad's model in plasma physics) in 1979 and [**9**, sect. 4.4, p. 95] in 1981, by R. Temam [**1**] (monotone rearrangements) in 1979, by J. M. Rakotoson and R. Temam [**1**] in 1987, and more recently in 1995 by M. C. Delfour and J.-P. Zolésio [**19**, **20**, **21**, **25**] (intrinsic formulation of models of shells). We quote the version given in L. C. Evans and R. F. Gariepy [**1**, Prop. 3, p. 118]. The original theorem can be found in H. Federer [**3**] and L. C. Young [**1**].

Theorem 6.7. *Let* $f : \mathbf{R}^{\mathrm{N}} \to \mathbf{R}$ *be Lipschitz continuous with*

$$|\nabla f| > 0 \quad a.e.$$

For any Lebesgue summable function $g : \mathbf{R}^{\mathrm{N}} \to \mathbf{R}$, *we have the following decomposition of the integral along the level sets of* f:

$$\int_{\{f>t\}} g\, dx = \int_t^\infty \left(\int_{\{f=s\}} \frac{g}{|\nabla f|}\, dH_{N-1} \right) ds.$$

The second theorem (W. H. Fleming and R. Rishel [**1**]) uses a BV function instead of a Lipschitz function.

Theorem 6.8. *Let* D *be an open subset of* $\mathbf{R}^{\mathrm{N}}$. *For any* $f \in \mathrm{BV}(D)$ *and real* t, *let* $E_t \stackrel{\text{def}}{=} \{x : f(x) < t\}$. *Then*

$$\|\nabla f\|_{M^1(D)^N} = \int_{-\infty}^{\infty} P(E_t)\, dt$$

(cf., for instance, H. Whitney [**1**, Chap. 11]).

This is known as the *co-area formula* (see also E. Giusti [**1**, Thm. 1.23, p. 20] and E. De Giorgi [**4**]).

6.3 Domains of Class $W^{\varepsilon,p}(D)$, $0 \le \varepsilon < 1/p$, $p \ge 1$, and a Cascade of Complete Metric Spaces

There is a general property enjoyed by functions in $\mathrm{BV}(D)$.

Theorem 6.9. *Let D be a bounded open Lipschitzian domain in $\mathbf{R}^{\mathrm{N}}$.*

(i) $\mathrm{BV}(D) \subset W^{\varepsilon,1}(D), \quad 0 \le \varepsilon < 1.$

(ii) $\mathrm{BV}(D) \cap L^\infty(D) \subset W^{\varepsilon,p}(D), \quad 0 \le \varepsilon < \frac{1}{p},\ 1 \le p < \infty.$

This theorem says that for a Caccioppoli set Ω in D,

$$\forall p \ge 1,\ 0 \le \varepsilon < \frac{1}{p}, \quad \chi_\Omega \in W^{\varepsilon,p}(D).$$

The special case $p = 2$ was proved by C. Baiocchi, V. Comincioli, E. Magenes, and G. A. Pozzi [**1**] in the context of the celebrated problem of the dam. They showed that the domain Ω is the hypograph of a continuous monotonically decreasing function on a closed interval is $H^\varepsilon(D) = W^{\varepsilon,2}(D)$, $0 \le \varepsilon < 1/2$, in $\mathbf{R}^2$.

Proof of Theorem 6.9. (i) Given f in $\mathrm{BV}(D)$ we want to show that $f \in W^{\varepsilon,1}(D)$, $0 < \varepsilon < 1$. This is equivalent to showing that the double integral

$$I = \int_D dx \int_D dy\, \frac{|f(y) - f(x)|}{|y-x|^{N+\varepsilon}}$$

is finite. Given $\alpha > 0$ small we break the above integral into two parts:

$$I_1 = \int_D dx \int_{D\cap B(x,\alpha)} dy\, \frac{|f(y) - f(x)|}{|y-x|^{N+\varepsilon}}, \quad I_2 \ = \int_D dx \int_{\complement_D B(x,\alpha)} dy\, \frac{|f(y) - f(x)|}{|y-x|^{N+\varepsilon}}.$$

The second integral I_2 is bounded since $|y-x| \ge \alpha$. For the first one we change the variable y to $t = y - x$,

$$I_1 = \int_D dx \int_{B(0,\alpha)} dt\, \frac{|(\chi_D f)(x+t) - (\chi_D f)(x)|}{|t|^{N+\varepsilon}},$$

and after a change in the order of integration,

$$I_1 = \int_{B(0,\alpha)} dt\, \frac{1}{|t|^{N+\varepsilon}} \int_D dx\, |(\chi_D f)(x+t) - (\chi_D f)(x)|.$$

But the integral over D is bounded by $|t|\, \|\nabla f\|_{M^1(D)}$. The result is true for smooth functions (cf. V. G. Maz'ja [**1**, Lem. 1.4.6, p. 62]). Pick a sequence of smooth functions such that

$$f_n \to f \text{ in } L^1(D), \quad \nabla f_n \to \nabla f \text{ in } M^1(D)\text{-weak;}$$

then

$$\|\nabla f\|_{M^1(D)} \le \limsup_{n\to\infty} \|\nabla f_n\|_{M^1(D)} = \limsup_{n\to\infty} \|\nabla f_n\|_{L^1(D)}$$

is bounded, and by going to the limit on both sides of the inequality,

$$\int_D dx\, |(\chi_D f_n)(x+t) - (\chi_D f_n)(x)| \leq |t| \|\nabla f_n\|_{L^1(D)},$$

we obtain the desired result. Now coming back to the estimate of I_1,

$$I_1 \leq \int_{B(x,\alpha)} |t|^{1-N-\varepsilon}\, dt\, \|\nabla f\|_{M^1(D)} \leq c(\alpha) \frac{\alpha^{1-\varepsilon}}{1-\varepsilon} \|\nabla f\|_{M^1(D)}, \quad 0 < \varepsilon < 1.$$

(ii) The function f belongs to $W^{\varepsilon,p}(D)$ if the integral

$$I = \int_D dx \int_D dy\, \frac{|f(y) - f(x)|^p}{|y-x|^{N+\varepsilon p}} < +\infty.$$

For each $f \in \mathrm{BV}(D) \cap L^\infty(D)$, there exists $M > 0$ such that

$$|f(x)| \leq M \quad \text{a.e. in } D.$$

Then

$$I = (2\,M)^p \int_D dx \int_D dy\, \frac{\left|\frac{f(y)-f(x)}{2\,M}\right|^p}{|y-x|^{N+\varepsilon p}},$$

and since

$$\left|\frac{f(y) - f(x)}{2\,M}\right| \leq 1,$$

then for each p, $1 \leq p < \infty$,

$$\left|\frac{f(y) - f(x)}{2\,M}\right|^p \leq \left|\frac{f(y) - f(x)}{2\,M}\right|.$$

As a result

$$I \leq (2\,M)^{p-1} \int_D dx \int_D dy\, \frac{|f(y) - f(x)|}{|y-x|^{N+\varepsilon p}}.$$

But we have seen in part (i) that for $f \in \mathrm{BV}(D)$, the double integral is finite if

$$0 \leq \varepsilon\, p < 1 \Rightarrow 0 \leq \varepsilon < \frac{1}{p}.$$

This complete the proof. □

For a characteristic function χ_Ω of a measurable set Ω and $\varepsilon > 0$, we have $0 \leq \varepsilon < 1/p$ for all $p \geq 1$ and

$$\|\chi_\Omega\|^p_{W^{\varepsilon,p}(D)} = \int_D \int_D \frac{|\chi_\Omega(x) - \chi_\Omega(y)|}{|x-y|^{N+p\varepsilon}}\, dx\, dy + \|\chi_\Omega\|^p_{L^p(D)}. \tag{6.12}$$

The $W^{\varepsilon,p}(D)$-norm is equivalent to the $W^{\varepsilon',p'}(D)$ for all pairs (p',ε'), $p' \geq 1$, $0 \geq \varepsilon' < 1/p'$, such that $p'\,\varepsilon' = p\,\varepsilon$. At this stage, it is not clear whether that norm is related to some Hausdorff measure of the relative boundary $\Gamma \cap D$ of the set.

For $p = 2$ and a characteristic function, χ_Ω, of a measurable set Ω, we get

$$\|\chi_\Omega\|^2_{H^\varepsilon(D)} = 2\int_\Omega \int_{\complement_D \Omega} |x-y|^{-(N+2\varepsilon)}\,dx\,dy + m(\Omega \cap D). \tag{6.13}$$

The space $\mathrm{X}(D) \cap W^{\varepsilon,2}(D)$ is a closed subspace of the Hilbert space $W^{\varepsilon,2}(D)$ and the square of its norm is differentiable. A direct consequence for optimization problems is that a penalization term of the form $\|\chi_\Omega\|^2_{W^{\varepsilon,2}(D)}$ is now differentiable and can be used in various minimization problems or to regularize the objective function to obtain existence of approximate solutions. For instance, to obtain existence results when minimizing a function $J(\Omega)$ (defined for all measurable sets Ω in D), we can consider a regularized problem in the following form:

$$J_\alpha(\Omega) = J(\Omega) + \alpha\|\chi_\Omega\|^2_{H^\varepsilon(D)}, \quad \alpha > 0. \tag{6.14}$$

Section 6.1 provided the important compactness Theorem 6.3 for Caccioppoli or finite perimeter sets. For a bounded open holdall D, this also provides a new complete metric topology on $\mathrm{X}(D) \cap \mathrm{BV}(D)$ when endowed with the metric

$$\rho_{\mathrm{BV}}(\chi_1, \chi_2) \overset{\text{def}}{=} \|\chi_2 - \chi_1\|_{\mathrm{BV}(D)} = \|\chi_2 - \chi_1\|_{L^1(D)} + \|\nabla\chi_2 - \nabla\chi_1\|_{M^1(D)}.$$

Similarly, the intermediate spaces of Theorem 6.9 between $\mathrm{BV}(D)$ and $L^p(D)$ provide little rougher metrics for $p \geq 1$ and $\varepsilon < 1/p$ on $\mathrm{X}(D) \cap W^{\varepsilon,p}(D)$ when endowed with the metric

$$\rho_{W^{\varepsilon,p}}(\chi_1, \chi_2) \overset{\text{def}}{=} \|\chi_2 - \chi_1\|_{W^{\varepsilon,p}(D)}.$$

Theorem 6.10. *Given a bounded open holdall D in $\mathbf{R}^{\mathrm{N}}$, $\mathrm{X}(D) \cap \mathrm{BV}(D)$ and $\mathrm{X}(D) \cap W^{\varepsilon,p}(D)$, $p \geq 1$, $\varepsilon < 1/p$, are complete metric spaces and*

$$\mathrm{X}(D) \cap \mathrm{BV}(D) \subset \mathrm{X}(D) \cap W^{\varepsilon,p}(D) \subset \mathrm{X}(D).$$

6.4 Compactness and Uniform Cone Property

In Theorem 6.3 of section 6.1 we have seen a first compactness theorem for a family of sets with a uniformly bounded perimeter. In this section we present a second compactness theorem for measurable domains satisfying a *uniform cone property* due to D. Chenais [**1**, **4**, **6**]. We provide a proof of this result that emphasizes the fact that the perimeter of sets in that family is uniformly bounded. It then becomes a special case of Theorem 6.3. This result will also be obtained as Corollary 2 to Theorem 13.1 in section 13 of Chapter 7. It will use completely different arguments and apply to families of sets that do not even have a finite boundary measure.

Theorem 6.11. *Let D be a bounded open holdall in $\mathbf{R}^{\mathrm{N}}$ with uniformly Lipschitzian boundary ∂D. For $r > 0$, $\omega > 0$, and $\lambda > 0$ consider the family*

$$L(D, r, \omega, \lambda) \overset{\text{def}}{=} \left\{\Omega \subset D : \begin{matrix}\Omega \textit{ is Lebesgue measurable and satisfies} \\ \textit{the uniform cone property for } (r,\omega,\lambda)\end{matrix}\right\}. \tag{6.15}$$

For p, $1 \le p < \infty$, the set

$$X(D, r, \omega, \lambda) \stackrel{\text{def}}{=} \{\chi_\Omega : \forall \Omega \in L(D, r, \omega, \lambda)\} \tag{6.16}$$

is compact in $L^p(D)$ and there exists a constant $p(D, r, \omega, \lambda) > 0$ such that

$$\forall \Omega \in L(D, r, \omega, \lambda), \quad H_{N-1}(\partial\Omega) \le p(D, r, \omega, \lambda). \tag{6.17}$$

We first show that the perimeter of the sets of the family $L(D, r, \omega, \lambda)$ is uniformly bounded and use Theorem 6.8 to show that any sequence of $X(D, r, \omega, \lambda)$ has a subsequence converging to the characteristic function of some finite perimeter set Ω. We use the full strength of the compactness of the injection of $\mathrm{BV}(D)$ into $L^1(D)$ rather than checking directly the conditions under which a subset of $L^p(D)$ is relatively compact. The proof will be completed by showing that the nice representative (Definition 3.3) I of Ω satisfies the same uniform cone property. The proof uses some elements of D. Chenais [**4**, **6**]'s original proof.

Proof of Theorem 6.11. Each $\Omega \in L(D, r, \omega, \lambda)$ satisfies the same uniform cone property (cf. Definition 6.3 (ii)) of Chapter 2 with parameters r, λ, and r_λ. By Theorem 6.8 of Chapter 2 it is uniformly Lipschitzian in the sense of Definition 5.2 (iii) of Chapter 2. As a result ρ, the largest radius such that $B_H(0, \rho) \subset \{P_H(y - x) : \forall y \in B(x, r_\lambda) \cap \partial\Omega\}$, and the neighborhoods $V = B_H(0, \rho)$ and $U = B(0, r_\lambda) \cap \{y : P_H y \in B_H(0, \rho)\}$ can be chosen as specified in Theorem 6.6 (i) of Chapter 2. They are the same for all $\Omega \in L(D, r, \omega, \lambda)$: for all $\Omega \in L(D, r, \omega, \lambda)$ and all $x \in \partial\Omega$, $V_x = V$, and there exists $A_x \in \mathrm{O(N)}$ such that $\mathcal{U}(x) = x + A_x U$. By construction

$$B(x, \rho) \subset B(x, r_\lambda) \cap \{y : P_H A_x^{-1}(y - x) \in B_H(0, \rho)\} = x + A_x U = \mathcal{U}(x).$$

The family $\left\{B\left(z, \rho/2\right) : z \in \overline{D}\right\}$ is an open cover of $\overline{D}$. Since $\overline{D}$ is compact, there exists a finite subcover $\{B_i\}_{i=1}^m$, $B_i = B\left(z_i, \rho/2\right)$, of $\overline{D}$. Now for any $\Omega \in L(D, r, \omega, \lambda)$, $\partial\Omega \subset \overline{D}$ and there is a subcover $\{B_{i_k}\}_{k=1}^K$ of $\{B_i\}_{i=1}^m$ such that

$$\partial\Omega \subset \bigcup_{k=1}^K B_{i_k} \text{ and } \forall k, \quad 1 \le k \le K, \quad \partial\Omega \cap B_{i_k} \ne \varnothing.$$

Pick a sequence $\{x_k\}_{k=1}^K$ such that $x_k \in \partial\Omega \cap B_{i_k}$, $1 \le k \le K$, and notice that

$$B_{i_k} = B\left(z_{i_k}, \frac{\rho}{2}\right) \subset B(x_k, \rho)$$

or

$$\exists K \le m, \quad \exists \{x_k\}_{k=1}^K \subset \partial\Omega, \quad \partial\Omega \subset \bigcup_{k=1}^K B(x_k, \rho).$$

Thus from the estimate (5.37) in Theorem 5.7 of Chapter 2 with $c_x = \tan\omega$,

$$\begin{aligned} H_{N-1}(\partial\Omega) &\le \sum_{k=1}^{K} H_{N-1}\big(\partial\Omega \cap B(x_k,\rho)\big) \le \sum_{k=1}^{K} H_{N-1}\big(\partial\Omega \cap \mathcal{U}(x_k)\big) \\ &\le \sum_{k=1}^{K} \rho^{N-1}\alpha_{N-1}\sqrt{1+(\tan\omega)^2} \le m\,\rho^{N-1}\alpha_{N-1}\sqrt{1+(\tan\omega)^2}, \end{aligned}$$

where α_{N-1} is the volume of the unit $(N-1)$-dimensional ball.

So the right-hand side of the above inequality is a constant that is equal to $p = p(D,r,\omega,\lambda) > 0$ and is independent of Ω in $L(D,r,\omega,\lambda)$:

$$\forall \Omega \in L(D,r,\omega,\lambda), \quad P_D(\Omega) = H_{N-1}(\partial\Omega) \le p.$$

Now from Theorem 6.3 for any sequence $\{\Omega_n\} \subset L(D,r,\omega,\lambda)$, there exists Ω such that $\chi_\Omega \in \mathrm{BV}(D)$ and a subsequence, still denoted by $\{\Omega_n\}$, such that

$$\chi_{\Omega_n} \to \chi_\Omega \text{ in } L^1(D) \text{ and } P_D(\Omega) \le p.$$

(ii) To complete the proof we consider the representative I of Ω (cf. Definition 3.3) and show that $\mathrm{I} \in L(D,r,\omega,\lambda)$. We need the following lemma.

Lemma 6.1. *Let $\chi_{\Omega_n} \to \chi_\Omega$ in $L^p(D)$, $1 \le p < \infty$, for $\Omega \subset D$, $\Omega_n \subset D$, and let I be the measure theoretic representative of Ω. Then*

$$\forall x \in \overline{\mathrm{I}},\ \forall R > 0,\ \exists N(x,R) > 0,\ \forall n \ge N(x,R), \quad \mathrm{m}\big(B(x,R) \cap \Omega_n\big) > 0.$$

Moreover,

$$\begin{gathered} \forall x \in \partial\mathrm{I},\ \forall R > 0,\ \exists N(x,R) > 0,\ \forall n \ge N(x,R), \\ \mathrm{m}\big(B(x,R) \cap \Omega_n\big) > 0 \quad \text{and} \quad \mathrm{m}\big(B(x,R) \cap \complement\Omega_n\big) > 0 \end{gathered}$$

and $B(x,R) \cap \partial\Omega_n \neq \varnothing$.

Proof. We proceed by contradiction. Assume that

$$\exists x \in \overline{\mathrm{I}}, \quad \exists R > 0, \quad \forall N > 0, \quad \exists n \ge N, \quad \mathrm{m}\big(B(x,R) \cap \Omega_n\big) = 0.$$

So there exists a subsequence $\{\Omega_{n_k}\}$, $n_k \to \infty$, such that

$$\begin{gathered} m\big(B(x,R) \cap \Omega\big) = \lim_{k\to\infty} m\big(B(x,R) \cap \Omega_{n_k}\big) = 0 \\ \Rightarrow \exists R > 0, \quad m\big(B(x,R) \cap \Omega\big) = 0 \implies x \in \Omega_0 = \operatorname{int} \complement\mathrm{I} = \complement\overline{\mathrm{I}}. \end{gathered}$$

But this is in contradiction with the fact that $x \in \overline{\mathrm{I}}$. For $x \in \partial\mathrm{I} = \overline{\mathrm{I}} \cap \overline{\complement\mathrm{I}}$, simultaneously apply the first statement to $\overline{\mathrm{I}}$ and $\chi_{\Omega_n} \to \chi_\Omega$ and to $\overline{\complement\mathrm{I}}$ and $\chi_{\complement\Omega_n} \to \chi_{\complement\Omega}$ since $\chi_{\complement\Omega} = \chi_{\complement\mathrm{I}}$ almost everywhere by Theorem 3.3 (i) and choose the larger of the two integers. As for the last property it follows from the fact that the open ball $B(x,R)$ cannot be partitioned into two nonempty disjoint open subsets. □

We wish to prove that

$$\forall x \in \partial \mathrm{I}, \quad \exists A_x \in \mathrm{O(N)}, \quad \forall y \in \bar{\mathrm{I}} \cap B(x,r), \quad y + A_x C(\lambda,\omega) \subset \operatorname{int} \mathrm{I}.$$

Since Ω_n is Lipschitzian, $\complement\Omega_n$ is also Lipschitzian and

$$\chi_{\overline{\Omega_n}} = \chi_{\Omega_n} \quad \text{and} \quad \chi_{\overline{\complement\Omega_n}} = \chi_{\complement\Omega_n}$$

almost everywhere. So we apply the second part of the lemma to $x \in \partial\mathrm{I}$, $\chi_{\overline{\Omega_n}} \to \chi_\Omega$ and $\chi_{\overline{\complement\Omega_n}} \to \chi_{\complement\Omega}$. So for each $x \in \partial\mathrm{I}$, for all $k \geq 1$, there exists $n_k \geq k$ such that

$$B\left(x, \frac{r}{2^k}\right) \cap \partial\Omega_{n_k} \neq \varnothing.$$

Denote by x_k an element of that intersection:

$$\forall k \geq 1, \quad x_k \in B\left(x, \frac{r}{2^k}\right) \cap \partial\Omega_{n_k}.$$

By construction, $x_k \to x$. Next consider $y \in B(x,r) \cap \bar{\mathrm{I}}$. By Lemma 6.1, there exists a subsequence of $\{\overline{\Omega}_{n_k}\}$, still denoted by $\{\overline{\Omega}_{n_k}\}$, such that

$$\forall k \geq 1, \quad B\left(y, \frac{r}{2^k}\right) \cap \overline{\Omega}_{n_k} \neq \varnothing.$$

For each $k \geq 1$ denote by y_k a point of that intersection. By construction,

$$y_k \in \overline{\Omega}_{n_k} \to y \in \bar{\mathrm{I}} \cap B(x,r).$$

There exists $K > 0$ large enough such that for all $k \geq K$, $y_k \in B(x_k, r)$. To see this note that $y \in B(x,r)$ and that

$$\exists \rho > 0, \quad B(y,\rho) \subset B(x,r) \quad \text{and} \quad |y-x| + \frac{\rho}{2} < r.$$

Now

$$\begin{aligned} |y_k - x_k| &\leq |y_k - y| + |y - x| + |x - x_k| \\ &\leq \frac{r}{2^k} + r - \frac{\rho}{2} + \frac{r}{2^k} \leq r + \left[\frac{r}{2^{k-1}} - \frac{\rho}{2}\right] < r \end{aligned}$$

for

$$\frac{r}{2^{k-1}} - \frac{\rho}{2} < 0 \implies k > 2 + \frac{\log(r/\rho)}{\log 2}.$$

So we have constructed a subsequence $\{\Omega_{n_k}\}$ such that for $k \geq K$,

$$x_k \in \partial\Omega_{n_k} \to x \in \partial\mathrm{I} \quad \text{and} \quad y_k \in \overline{\Omega}_{n_k} \cap B(x_k, r) \to y \in \bar{\mathrm{I}} \cap B(x,r).$$

For each k, there exists $A_k \in \mathrm{O(N)}$ such that

$$y_k + A_k C(\lambda,\omega) \subset \operatorname{int} \Omega_{n_k}.$$

Pick another subsequence of $\{\Omega_{n_k}\}$, still denoted by $\{\Omega_{n_k}\}$, such that

$$\exists A \in \mathrm{O(N)}, \quad A_k \to A,$$

since O(N) is compact. Now consider $z \in y + AC(\lambda, \omega)$. Since z is an interior point

$$\exists \rho > 0, \quad B(z, \rho) \subset y + AC(\lambda, \omega).$$

So there exists $K' \geq K$ such that

$$\forall k \geq K', \quad B\left(z, \frac{\rho}{2}\right) \subset y_k + A_k C(\lambda, \omega) \subset \operatorname{int} \Omega_{n_k}.$$

Therefore, for $k \geq K'$

$$\mathrm{m}\left(B\left(z, \frac{\rho}{2}\right)\right) = \mathrm{m}\left(B\left(z, \frac{\rho}{2}\right) \cap \Omega_{n_k}\right),$$

and, as $k \to \infty$,

$$\mathrm{m}\left(B\left(z, \frac{\rho}{2}\right)\right) = \mathrm{m}\left(B\left(z, \frac{\rho}{2}\right) \cap \Omega\right),$$

and by Definition 3.3, $z \in \Omega_1$ and $y + AC(\lambda, \omega) \subset \Omega_1 = \operatorname{int} \mathrm{I}$. This proves that $\mathrm{I} \subset \overline{D}$ satisfies the uniform cone property and $\mathrm{I} \in L(D, \lambda, \omega, r)$. □

7 Existence for the Bernoulli Free Boundary Problem

7.1 An Example: Elementary Modeling of the Water Wave

Consider a fluid in a domain Ω in $\mathbf{R}^3$ and assume that the velocity of the flow u satisfies the *Navier–Stokes equations*

$$u_t + Du\, u - \nu\, \Delta u + \nabla p = -\rho\, g \text{ in } \Omega, \quad \operatorname{div} u = 0 \text{ in } \Omega, \tag{7.1}$$

where $\nu > 0$ and $\rho > 0$ are the respective *viscosity* and *density* of the fluid, p is the pressure, and g is the gravity constant. The second equation characterizes the incompressibility of the fluid. A standard example considered by physicists is the water wave in a channel. The boundary conditions on $\partial\Omega$ are the *sliding conditions* at the bottom S and on the free boundary Γ, that is,

$$u \cdot n = 0 \text{ on } S \cup \Gamma. \tag{7.2}$$

Assume that a *stationary regime* has been reached so that the velocity of the fluid is no longer a function of the time. Furthermore assume that the motion of the fluid is *irrotational*. By the classical Hodge decomposition, the velocity can be written in the form

$$u = \nabla\varphi + \operatorname{curl} \xi. \tag{7.3}$$

As curl $u =$ curl curl $\xi = 0$ we conclude that $\xi = 0$, since curl curl is a *good* isomorphism. Then $u = \nabla\varphi$ and the incompressibility condition becomes

$$\Delta\varphi = 0 \text{ in } \Omega. \tag{7.4}$$

Therefore the Navier–Stokes equation reduces to

$$D(\nabla\varphi)\,\nabla\varphi + \nu\,\nabla(\Delta\varphi) + \nabla p = -\nabla(\rho g z), \tag{7.5}$$

but

$$D(\nabla\varphi)\,\nabla\varphi = \frac{1}{2}\nabla(|\nabla\varphi|^2), \tag{7.6}$$

so that

$$\nabla\left(\frac{1}{2}|\nabla\varphi|^2 + p + \rho g z\right) = 0 \text{ in } \Omega. \tag{7.7}$$

Then if Ω is connected, there exists a constant c such that

$$\frac{1}{2}|\nabla\varphi|^2 + p + \rho g z = c \text{ in } \Omega. \tag{7.8}$$

In the domain Ω, that *Bernoulli condition* yields explicitly the pressure p as a function of the velocity $|\nabla\varphi|$ and the height z in the fluid. The flow is now assumed to be independent of the transverse variable y so that the initial three-dimensional problem in a perfect channel reduces to a two-dimensional one. Since the problem has been reduced to a two-dimensional one, introduce the harmonic conjugate ψ, the so-called *stream function*, so that the boundary condition $\partial\varphi/\partial n = 0$ takes the form $\psi =$ constant on each connected component of the boundary $\partial\Omega$.

On the *free boundary* at the top of the wave, the pressure is related to the existing *atmospheric pressure* p_a through the surface tension $\sigma > 0$ and the mean curvature H of the free boundary. Actually

$$p - p_a = -\sigma\, H \tag{7.9}$$

on the free boundary of the stationary wave, where H is the mean curvature associated with the fluid domain. Also, from the Cauchy conditions, we have $|\nabla\psi| = |\nabla\varphi|$ so that the boundary condition (7.8) on the free boundary of the wave becomes

$$\frac{1}{2}|\nabla\psi|^2 + \rho g z + \sigma H = p_a. \tag{7.10}$$

In order to simplify the presentation we replace the equation $\Delta\psi = 0$ by $\Delta\psi = f$ in Ω to avoid a *forcing term* on a part of the boundary, and we show that the resulting free boundary problem has the following *shape variational formulation*. Let D be a fixed, sufficiently large, smooth, bounded, and open domain in $\mathbf{R}^2$. Let a be a real number such that $0 < a < \mathrm{m}(D)$. To find $\Omega \subset D$, $\mathrm{m}(\Omega) = a$ and $\psi \in H_0^1(D)$ such that

$$-\Delta\psi = f \text{ in } \Omega \tag{7.11}$$

and $\psi = constant$ and satisfies the boundary condition (7.10) on the free part $\partial\Omega \cap D$ of the boundary.

For a fixed Ω, the solution of this problem is a minimizing element of the following variational problem:

$$J(\Omega) \stackrel{\text{def}}{=} \inf_{\varphi \in H^1_\diamond(\Omega;D)} \int_\Omega \left(\frac{1}{2}|\nabla\varphi|^2 - f\varphi + \rho g z \right) dx + \sigma\, P_D(\Omega), \tag{7.12}$$

where

$$H^1_\diamond(\Omega; D) \stackrel{\text{def}}{=} \left\{ u \in H^1_0(D) : u(x) = 0 \text{ a.e. } x \text{ in } D \setminus \Omega \right\} \tag{7.13}$$

is the relaxation of the definition of the Sobolev space $H^1_0(\Omega)$[10] for any measurable subset Ω of D. Its properties were studied in Theorem 3.6.

With that formulation there exists a measurable $\Omega^* \subset D$, $|\Omega^*| = a$, such that

$$\forall \Omega \subset D,\ |\Omega| = a, \quad J(\Omega^*) \leq J(\Omega). \tag{7.14}$$

By using the methods of Chapter 10 it follows that if $\partial\Omega^*$ is sufficiently smooth, the *shape Euler condition* $dJ(\Omega^*; V) = 0$ yields the original free boundary problem and the free boundary condition (7.10). The existence of a solution will now follow from Theorem 6.3 in section 6.1. The case without surface tension is physically important. It occurs in phenomena with "evaporation " (cf. M. Souli and J.-P. Zolésio [**2**]). In that case the previous Bernoulli condition becomes

$$\left| \frac{\partial \xi}{\partial n} \right|^2 = g^2 \quad (g \geq 0)$$

on the free boundary, so that if we assume (in the channel setting) that there is no cavitation or recirculation in the fluid, then $\partial\xi/\partial n > 0$ on the free boundary and we get the Neumann-like condition $\partial\xi/\partial n = g$ together with the Dirichlet condition. In section 7.4 we shall consider the case with surface tension.

7.2 Existence for a Class of Free Boundary Problems

Consider the following free boundary problem, studied in J.-P. Zolésio [**25**, **26**, **29**]: find Ω in a fixed holdall D and a function y on Ω such that

$$-\Delta y = f \text{ in } \Omega, \quad y = 0 \text{ and } \frac{\partial y}{\partial n} = Q^2 \text{ on } \partial\Omega, \tag{7.15}$$

where f and Q are appropriate functions defined in D. To study this type of problem H. W. Alt and L. A. Caffarelli [**1**] have introduced the following function:

$$J(\varphi) \stackrel{\text{def}}{=} \int_D \frac{1}{2}|\nabla\varphi|^2 - f\varphi\, dx + \int_D Q^2 \chi_{\varphi>0}\, dx, \tag{7.16}$$

[10]In Chapter 8 the space $H^1_0(\Omega; D)$ of extensions by zero to D of elements of $H^1_0(\Omega)$ is defined for Ω open. It is then characterized in Lemma 6.1 of Chapter 8 by a capacity condition on the complement $D\backslash\Omega$. This characterization extends to quasi-open sets Ω as defined in section 6 of Chapter 8.

which is minimized over

$$K \stackrel{\text{def}}{=} \{u \in H_0^1(D) : u(x) \geq 0 \text{ a.e. in } D\}, \tag{7.17}$$

where $\chi_{\varphi>0}$ (resp., $\chi_{\varphi\neq 0}$) is the characteristic function of the set $\{x \in D : \varphi(x) > 0\}$[11] (resp., $\{x \in D : \varphi(x) \neq 0\}$). The existence of a solution is based on the following lemma.

Lemma 7.1. *Let $\{u_n\}$ and $\{\chi_n\}$ be two converging sequences such that $u_n \to u$ in $L^2(D)$-strong, and let the χ_n's be characteristic functions, $\chi_n(1-\chi_n) = 0$, which converge to some function λ in $L^2(D)$-weak. Then*

$$\forall n, \quad (1-\chi_n)u_n = 0 \quad \Rightarrow \quad \lambda \geq \chi_{u\neq 0}. \tag{7.18}$$

Proof. We have $(1-\chi_n)u_n = 0$, and in the limit $(1-\lambda)u = 0$. Thus on the set $\{x : u(x) \neq 0\}$ we have $\lambda = 1$; elsewhere λ lies between 0 and 1 as the weak limit of a sequence of characteristic functions. □

Proposition 7.1. *Let f and Q be two elements of $L^2(D)$ such that $f \geq 0$ almost everywhere. There exists u in K which minimizes the function J over the positive cone K of $H_0^1(D)$.*

Proof. Let $\{u_n\} \subset K$ be a minimizing sequence for the function J over the convex set K. Denote by χ_n the characteristic function of the set $\{x \in D : u_n(x) > 0\}$, which is in fact equal to the subset $\{x \in D : u_n(x) \neq 0\}$. It is easy to verify that the sequence $\{u_n\}$ remains bounded in $H_0^1(D)$. Still denote by $\{u_n\}$ a subsequence that weakly converges in $H_0^1(D)$ to an element u of K. That convergence holds in $L^2(D)$-strong so that Lemma 7.1 applies and we get $\lambda \geq \chi_{u\neq 0}$ for any weak limiting element λ of the sequence $\{\chi_n\}$ (which is bounded in $L^2(D)$). Denote by j the minimum of J over K. Then $J(u_n)$ weakly converges to j. We get

$$\int_D \frac{1}{2}|\nabla u|^2 - f u\, dx \leq \liminf_{n\to\infty} \int_D \frac{1}{2}|\nabla u_n|^2 - f u_n\, dx, \tag{7.19}$$

$$\int_D \chi_{u\neq 0} Q^2\, dx \leq \int_D \lambda Q^2 dx = \lim_{n\to\infty} \int_D \chi_n Q^2\, dx. \tag{7.20}$$

Finally, by adding these two estimates we get $J(u) \leq j$. □

Obviously in the upper bound in (7.20), Q^2 is a positive function. Moreover

(i) the set Ω is given by $\{x \in D : u(x) > 0\}$, and

(ii) the restriction $u|_\Omega$ of u to Ω is a weak solution of the free boundary problem

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ and } \frac{\partial u}{\partial n} = Q^2 \text{ on } \partial\Omega. \tag{7.21}$$

[11] This set defined up to a set of zero measure is a *quasi-open* set in the sense of section 6 in Chapter 8.

Effectively, the minimization problem (7.16)–(7.17) can be written as a shape optimization problem. First introduce for any measurable $\Omega \subset D$ the positive cone

$$H_0^1(\Omega)_+ \stackrel{\text{def}}{=} \left\{u \in H_\diamond^1(\Omega; D) : u(x) \geq 0 \text{ a.e. } x \in D\right\}$$

in the Hilbert space $H_\diamond^1(\Omega; D)$. Then consider the following shape optimization problem:

$$\inf \left\{E(\Omega) : \Omega \text{ is measurable subset in } D\right\}, \tag{7.22}$$

where the energy function E is given by

$$E(\Omega) \stackrel{\text{def}}{=} \min_{\varphi \in H_0^1(\Omega)_+} \left\{\int_\Omega \frac{1}{2}|\nabla\varphi|^2 - f\varphi\, dx\right\} + \int_\Omega Q^2\, dx. \tag{7.23}$$

The necessary condition associated with the minimum could be obtained by the techniques introduced in section 2.1 of Chapter 10. The important difference with the previous formulation (7.16)–(7.17) is that the independent variable in the objective function is no longer a function but a domain. The shape formulation (7.22)–(7.23) now makes it possible to handle constraints on the volume or the perimeter of the domain, which were difficult to incorporate in the first formulation.

As we have seen in Theorem 3.6, $H_\diamond^1(\Omega; D)$ endowed with the norm of $H_0^1(D)$ is a Hilbert space, and $H_0^1(\Omega)_+$ a closed convex cone in $H_\diamond^1(\Omega; D)$, so that for any measurable subset Ω in D, problem (7.23) has a unique solution y in $H_0^1(\Omega)_+$. Thus we have the following equivalence between problems (7.22)–(7.23) and the minimization (7.16)–(7.17) of J over K.

Proposition 7.2. *Let u be a minimizing element of J over K. Then*

$$\Omega \stackrel{\text{def}}{=} \{x \in D : u(x) > 0\}$$

is a solution of problem (7.22) *and $y = u_{|\Omega}$ is a solution of* (7.23)*. Conversely, if Ω is a measurable subset of D and y is a solution of* (7.22)–(7.23) *in $H_0^1(\Omega)_+$, then the element u defined by*

$$u(x) \stackrel{\text{def}}{=} \begin{cases} y(x), & \text{if } x \in \Omega, \\ 0, & \text{if } x \in D \setminus \Omega, \end{cases}$$

belongs to K and minimizes J over K.

7.3 Weak Solutions of Some Generic Free Boundary Problems

7.3.1 Problem without Constraint

Problem (7.22)–(7.23) can be relaxed as follows: given any f in $L^2(D)$, G in $L^1(D)$,

$$(\mathcal{P}_0) \quad \inf \{E(\Omega) : \Omega \subset D \text{ measurable}\}, \tag{7.24}$$

where for any measurable subset Ω of D the energy function is now defined by

$$E(\Omega) \stackrel{\text{def}}{=} \min_{\varphi \in H^1_\diamond(\Omega;D)} E_D(\varphi) + \int_\Omega G\,dx, \quad E_D(\varphi) \stackrel{\text{def}}{=} \int_D \frac{1}{2}|\nabla\varphi|^2 - f\varphi\,dx, \tag{7.25}$$

where $H^1_\diamond(\Omega;D)$ is defined in (7.13). By Theorem 3.6 $H^1_\diamond(\Omega;D)$ is contained in $H^1_\bullet(\Omega;D)$, and for any element u in $H^1_\diamond(\Omega;D)$ we have $\nabla u(x) = 0$ for almost all x in $D \setminus \Omega$ so that

$$\forall \varphi \in H^1_\diamond(\Omega;D), \quad E_D(\varphi) = \int_\Omega \frac{1}{2}|\nabla\varphi|^2 - f\varphi\,dx \tag{7.26}$$

$$\Rightarrow E(\Omega) = \min_{\varphi \in H^1_\diamond(\Omega;D)} \left\{ \int_\Omega \frac{1}{2}|\nabla\varphi|^2 - f\varphi\,dx + \int_\Omega G\,dx \right\}. \tag{7.27}$$

We have the following existence result for problem $(\mathcal{P}_0)$.

Theorem 7.1. *For any f in $L^2(D)$, $G = Q^2$ in $L^1(D)$, there exists at least one solution to problem $(\mathcal{P}_0)$.*

Proof. Let $\{\Omega_n\}$ be a minimizing sequence for problem $(\mathcal{P}_0)$, and for each n let u_n be the solution to problem (7.25) with $\Omega = \Omega_n$. If χ_n is the characteristic function of the measurable set Ω_n, we have u_n in $H^1_\diamond(\Omega_n;D)$, which implies that $(1-\chi_n)u_n = 0$. On the other hand, the sequence $\{u_n\}$ remains uniformly bounded in $H^1_0(D)$. Taking $\varphi = 0$ in (7.25) we get

$$\int_D \frac{1}{2}\,|\nabla u_n|^2 - f u_n\,dx \le 0,$$

and the conclusion follows from the equivalence of norms in $H^1_0(D)$. We can assume that the sequence $\{\chi_n\}$ weakly converges in $L^2(D)$ to an element λ and that the sequence $\{u_n\}$ weakly converges in $H^1_0(D)$ to an element u. From Lemma 7.1 we get $\lambda \ge \chi_{u\neq 0}$ almost everywhere in D. Define

$$\Omega(u) \stackrel{\text{def}}{=} \{x \in D : u(x) \neq 0\}.$$

Then u belongs to $H^1_0\big(\Omega(u)\big)$ and we have

$$\mathrm{m}(\Omega(u)) = \mathrm{m}(\{x \in D \,:\, \lambda(x) = 1\}) \le \alpha,$$

since we have

$$\alpha = \mathrm{m}(\{x : \lambda(x) = 1\}) + \mathrm{m}(\{x : 0 \le \lambda(x) < 1\}).$$

In the limit with $\Omega = \Omega(u)$ we get

$$\int_\Omega \frac{1}{2}|\nabla u|^2 - f u\,dx = \int_D \frac{1}{2}|\nabla u|^2 - f u\,dx \le \liminf \int_D \frac{1}{2}|\nabla u_n|^2 - f u_n\,dx, \tag{7.28}$$

$$\int_\Omega G\,dx \le \int_D \lambda G\,dx = \lim_{n\to\infty} \int_{\Omega_n} G\,dx. \tag{7.29}$$

By adding (7.28) and (7.29) we get that Ω minimizes E and u minimizes $E(\Omega)$. $\square$

7.3.2 Constraint on the Measure of the Domain Ω

Consider an important variation of problem (7.22)–(7.23). Given any f in $L^2(D)$, G in $L^1(D)$, and a real number α, $0 < \alpha < \mathrm{m}(D)$,

$$(\mathcal{P}_0^\alpha) \quad \inf \{E(\Omega) : \Omega \subset D \text{ measurable and } \mathrm{m}(\Omega) = \alpha\}. \tag{7.30}$$

We have the following existence result for problem $(\mathcal{P}_0^\alpha)$.

Theorem 7.2. *For any f in $L^2(D)$, $G = 0$, and any real number α, $0 < \alpha < \mathrm{m}(D)$, there exists at least one solution to problem $(\mathcal{P}_0^\alpha)$.*

Proof. Let $\{\Omega_n\}$ be a minimizing sequence for problem $(\mathcal{P}_0)$, and for each n let u_n be the unique solution to problem (7.25). If χ_n is the characteristic function of the measurable set Ω_n, we have u_n in $H^1_\diamond(\Omega_n; D)$, which implies that $(1 - \chi_n)u_n = 0$. On the other hand, by picking $\varphi = 0$ in (7.25), $\{u_n\}$ remains bounded in $H^1_0(D)$,

$$\int_D \frac{1}{2}|\nabla u_n|^2 - f u_n \, dx \le 0,$$

and the conclusion follows from the equivalence of norms in $H^1_0(D)$. We can assume that $\{\chi_n\}$ weakly converges in $L^2(D)$ to an element λ and that $\{u_n\}$ weakly converges in $H^1_0(D)$ to an element $u \in H^1_0(D)$. In the limit we get

$$\int_D \lambda(x)\, dx = \lim_{n\to\infty} \int_D \chi_n(x)\, dx = \alpha.$$

From Lemma 7.1 we get $\lambda \ge \chi_{u\neq 0}$ almost everywhere in D. Define

$$\Omega(u) \stackrel{\text{def}}{=} \{x \in D : u(x) \neq 0\}.$$

Then u belongs to $H^1_\diamond(\Omega(u); D)$ and we have

$$\mathrm{m}(\Omega(u)) = \mathrm{m}(\{x \in D \,:\, \lambda(x) = 1\}) \le \alpha$$

(as we have $\alpha = \mathrm{m}(\{x : \lambda(x) = 1\}) + \mathrm{m}(\{x : 0 \le \lambda(x) < 1\})$). In the limit we get for $\Omega = \Omega(u)$

$$\int_\Omega \frac{1}{2}|\nabla u|^2 - f\, u\, dx = \int_D \frac{1}{2}|\nabla u|^2 - f u\, dx \le \liminf_{n\to\infty} \int_D \frac{1}{2}|\nabla u_n|^2 - f u_n\, dx, \tag{7.31}$$

$$\int_D \lambda G\, dx = \lim_{n\to\infty} \int_{\Omega_n} G\, dx, \tag{7.32}$$

so that

$$\int_\Omega \frac{1}{2}|\nabla u|^2 - f u\, dx + \int_D \lambda G\, dx \le \inf_{\substack{\Omega' \subset D \\ \mathrm{m}(\Omega')=\alpha}} E(\Omega').$$

If $G \ge 0$ almost everywhere in D, we get

$$E(\Omega) \le \inf_{\substack{\Omega' \subset D \\ \mathrm{m}(\Omega')=\alpha}} \{E(\Omega')\},$$

but Ω does not necessarily satisfy the constraint $\mathrm{m}(\Omega) = \alpha$. Note that for any measurable set Ω' such that $\Omega \subset \Omega' \subset D$, we have

$$\int_{\Omega'} \frac{1}{2}|\nabla u|^2 - f\, u\, dx = \int_{\Omega} \frac{1}{2}|\nabla u|^2 - f u\, dx$$

so that in expression (7.31) Ω can be enlarged to any such Ω'. The inclusion of Ω in Ω' implies the inclusion of $H^1_\diamond(\Omega; D)$ in $H^1_\diamond(\Omega'; D)$. So if $G \le 0$ almost everywhere in D, then it is readily seen that from (7.25) $E(\Omega') \le E(\Omega)$. In view of the previous assumption on G, now $G = 0$ almost everywhere in D. To conclude the proof we just have to select Ω' with $\mathrm{m}(\Omega') = \alpha$. That measurable set Ω' is admissible and minimizes the objective function in (7.30), and we have

$$E(\Omega') = E(\Omega) = \inf_{\substack{\Omega'' \subset D \\ \mathrm{m}(\Omega'')=\alpha}} \{E(\Omega'')\}. \qquad \square$$

Corollary 1. *Assume that $G = 0$ and f in $L^2(D)$ and $f = Q^2$ in $L^1(D)$. Then the following problem has an optimal solution:*

$$(\mathcal{P}_0^{\alpha-}) \qquad \inf_{\substack{\Omega \subset D \\ \mathrm{m}(\Omega)\le\alpha}} E(\Omega). \tag{7.33}$$

Proof. The proof is similar to the proof of the theorem where the minimizing sequence is chosen such that $\mathrm{m}(\Omega_n) \le \alpha$, so that in the weak limit we get

$$\mathrm{m}(\Omega) \le \int_D \lambda(x)\, dx \le \alpha. \qquad \square$$

7.4 Weak Existence with Surface Tension

Problems $(\mathcal{P}_0)$, $(\mathcal{P}_0^\alpha)$, and $(\mathcal{P}_0^{\alpha-})$ have optimal solutions, but as they are associated with the homogeneous Dirichlet boundary condition, u in $H^1_\diamond(\Omega; D)$, the optimal domain Ω is in general not allowed to have holes, that is to say, roughly speaking, that the topology of Ω is a priori specified. In many examples it turns out that the solution u physically corresponds to a potential and the classical homogeneous Dirichlet boundary condition is not the appropriate one. The physical condition is that the potential u should be constant on each connected component of the boundary $\partial\Omega$ in D. When Ω is a simply connected domain in $\mathbf{R}^2$, then $\partial\Omega$ has a single connected component so the constant can be taken as zero. In general this constant can be fixed only in one connected component; in the others the constant is an unknown of the problem.

The minimization problems $(\mathcal{P}_0)$, $(\mathcal{P}_0^\alpha)$, and $(\mathcal{P}_0^{\alpha-})$ associated with the Hilbert space $H^1_\bullet(\Omega; D)$ fail (in the sense that the previous techniques for existence of an optimal Ω fail). The main reason is that Lemma 7.1 is no longer true when u_n is replaced by ∇u_n weakly converging in $L^2(D)^N$. The key idea is to recover the equivalent of Lemma 7.1 by imposing the strong $L^2(D)$-convergence of the sequence $\{u_n\}$. In practice $\{u_n\}$ corresponds to the sequence of characteristic functions

χ_{Ω_n} of a minimizing sequence $\{\Omega_n\}$. To obtain the strong $L^2(D)$-convergence of a subsequence we add a constraint on the perimeters. Consider the family of finite perimeter sets in D of Definition 6.2 (ii) in section 6.1,

$$\mathrm{BPS}(D) \stackrel{\text{def}}{=} \{\Omega \subset D : \chi_\Omega \in \mathrm{BV}(D)\},$$

where $\mathrm{BV}(D)$ is defined in (6.2). Introduce the following problem indexed by $\sigma > 0$:

$$(\mathcal{P}^\alpha_\sigma) \qquad \inf_{\substack{\Omega\subset D\\ \mathrm{m}(\Omega)=\alpha}} E_\sigma(\Omega), \quad E_\sigma(\Omega) \stackrel{\text{def}}{=} E(\Omega) + \sigma P_D(\Omega), \tag{7.34}$$

$$E(\Omega) \stackrel{\text{def}}{=} \min_{\varphi\in H^1_\bullet(\Omega;D)} \left\{ \int_\Omega \frac{1}{2}|\nabla\varphi|^2 - f\varphi\, dx + \int_\Omega G\, dx \right\}. \tag{7.35}$$

Theorem 7.3. *Let f in $L^2(D)$, G in $L^1(D)$, $\sigma > 0$, and $0 \le \alpha < \mathrm{m}(D)$. Then problem $(\mathcal{P}^\alpha_\sigma)$ has at least one optimal solution Ω in* BPS(D).

Proof. Let $\{\Omega_n\}$ be a minimizing sequence for $(\mathcal{P}^\alpha_\sigma)$ and let $\{\chi_n\}$ be the corresponding sequence of characteristic functions associated with $\{\Omega_n\}$. By picking $\varphi = 0$ in (7.35),

$$P_D(\Omega_n) \le \sigma^{-1}\left[c\|f\|^2_{L^2(D)} + \int_D |G|\, dx + \sigma P_D(D)\right].$$

By Theorem 6.3 there exist a subsequence of $\{\Omega_n\}$, still indexed by n, and $\Omega \subset D$ with finite perimeter such that $\chi_n \to \chi = \chi_\Omega$ in $L^1(D)$-strong and $\alpha = \mathrm{m}(\Omega_n) = \mathrm{m}(\Omega)$. For each n, let u_n in $H^1_\bullet(\Omega_n; D)$ be the unique minimizer of (7.35). That sequence remains bounded in $H^1_0(D)$. Pick a subsequence, still indexed by n, such that $\{u_n\}$ weakly converges to an element u in $H^1_0(D)$. From the identity $(1 - \chi_n)\nabla u_n = 0$ almost everywhere in D, we get in the limit $(1 - \chi_\Omega)\nabla u = 0$ almost everywhere in D, so that the limiting element u belongs to $H^1_\bullet(\Omega; D)$. Hence we have

$$\begin{aligned}\int_\Omega \frac{1}{2}|\nabla u|^2 - fu\, dx &= \int_D \frac{1}{2}|\nabla u|^2\, dx - \int_\Omega fu\, dx \\ &\le \liminf_{n\to\infty} \int_D \frac{1}{2}|\nabla u_n|^2\, dx - \int_{\Omega_n} fu_n\, dx,\end{aligned} \tag{7.36}$$

$$\int_\Omega fu\, dx = \lim_{n\to\infty}\int_{\Omega_n} fu_n\, dx, \qquad \int_\Omega G\, dx = \lim_{n\to\infty}\int_{\Omega_n} G\, dx, \tag{7.37}$$

so that

$$\int_\Omega \frac{1}{2}|\nabla u|^2 - fu\, dx + \int_\Omega G\, dx + \sigma P_D(\Omega) \le \inf_{\substack{\Omega\subset D\\ \mathrm{m}(\Omega)=\alpha}} E_\sigma(\Omega). \tag{7.38}$$

□

Chapter 6

Metrics via Distance Functions

1 Introduction

In Chapter 5 the characteristic function was used to embed the equivalence classes of measurable subsets of D into $L^p(D)$ or $L^p_{\text{loc}}(D)$, $1 \le p < \infty$, and induce a complete metric on the equivalence classes of measurable sets. This construction is generic and extends to other set-dependent functions embedded in an appropriate function space. The Hausdorff metric is the result of such a construction, where the distance function plays the role of the characteristic function. The distance function embeds equivalence classes of subsets A of a closed holdall D with the same closure $\overline{A}$ into the space $C(D)$ of continuous functions. When D is bounded the C^0-norm of the difference of the distance functions of two subsets of D is the Hausdorff metric. The Hausdorff topology has many much desired properties. In particular, for D bounded, the set of equivalence classes of nonempty subsets A of D is compact.

Yet, the volume and perimeter are not continuous with respect to the Hausdorff topology. This can be fixed by changing the space $C(D)$ to the space $W^{1,p}(D)$ since distance functions also belong to that space. With that metric the volume is now continuous. The price to pay is the loss of compactness when D is bounded. But other sequentially compact families can easily be constructed. By analogy with Caccioppoli sets we introduce the sets for which the elements of the Hessian matrix of second-order derivatives of the distance function are bounded measures. They are called sets of *bounded curvature*. Their closure is a Caccioppoli set, and they enjoy other interesting properties. Convex sets belong to that family. General compactness theorems are obtained for such families under global or local conditions. This chapter also deals with the family of open sets characterized by the distance function to their complement. They are discussed in parallel along with the distance functions to a set.

The properties of distance functions and Hausdorff and Hausdorff complementary metric topologies are studied in section 2. The projections, skeletons, cracks, and differentiability properties of distance functions are discussed in section 3. $W^{1,p}$-topologies are introduced in section 4 and related to characteristic functions. The compact families of *sets of bounded and locally bounded curvature* are characterized in section 5.

The notion of *reach* and Federer's sets of *positive reach* are studied in section 6. The smoothness of smooth closed submanifolds is related to the smoothness of the square of the distance function in a neighborhood. Approximation of distance functions by dilated sets/tubular neighborhoods and their critical points are presented in section 7. Convex sets are characterized in section 8 along with the special family of convex sets. Both sets of positive reach and convex sets will be further investigated in Chapter 7. Finally section 9 gives several compactness theorems under global and local conditions on the Hessian matrix of the distance function.

2 Uniform Metric Topologies

2.1 Family of Distance Functions $C_d(D)$

In this section we review some properties of distance functions and present the general approach that will be followed in this chapter.

Definition 2.1.
Given $A \subset \mathbf{R}^N$ the *distance function* from a point x to A is defined as

$$\boxed{d_A(x) \stackrel{\text{def}}{=} \begin{cases} \inf_{y \in A} |y-x|, & A \neq \varnothing, \\ +\infty, & A = \varnothing, \end{cases}} \tag{2.1}$$

and the family of all distance functions of nonempty subsets of D as

$$\boxed{C_d(D) \stackrel{\text{def}}{=} \left\{ d_A : \forall A,\, \varnothing \neq A \subset \overline{D} \right\}.} \tag{2.2}$$

When $D = \mathbf{R}^N$ the family $C_d(\mathbf{R}^N)$ is denoted by C_d. □

By definition d_A is finite in $\mathbf{R}^N$ if and only if $A \neq \varnothing$.
Recall the following properties of distance functions.

Theorem 2.1. *Assume that A and B are nonempty subsets of $\mathbf{R}^N$.*

(i) *The map $x \mapsto d_A(x)$ is uniformly Lipschitz continuous in $\mathbf{R}^N$:*

$$\forall x, y \in \mathbf{R}^N, \quad |d_A(y) - d_A(x)| \leq |y - x| \tag{2.3}$$

and $d_A \in C^{0,1}_{\text{loc}}(\overline{\mathbf{R}^N})$.[1]

(ii) *There exists $p \in \overline{A}$ such that $d_A(x) = |p - x|$ and $d_A = d_{\overline{A}}$ in $\mathbf{R}^N$.*

(iii) $\overline{A} = \{x \in \mathbf{R}^N : d_A(x) = 0\}$.

[1] A function f belongs to $C^{0,1}_{\text{loc}}(\overline{\mathbf{R}^N})$ if for all bounded open subsets D of $\mathbf{R}^N$ its restriction to D belongs to $C^{0,1}(\overline{D})$.

(iv) $d_A = 0$ *in* $\mathbf{R}^N \iff \overline{A} = \mathbf{R}^N$.

(v) $\overline{A} \subset \overline{B} \iff d_A \geq d_B$.

(vi) $d_{A\cup B} = \min\{d_A, d_B\}$.

(vii) d_A *is (Fréchet) differentiable almost everywhere and*

$$|\nabla d_A(x)| \leq 1 \quad a.e.\ in\ \mathbf{R}^N. \tag{2.4}$$

Proof. (i) For all $z \in A$ and $x, y \in \mathbf{R}^N$

$$|z-y| \leq |z-x| + |y-x|,$$
$$d_A(y) = \inf_{y\in A} |z-y| \leq \inf_{y\in A} |z-x| + |y-x| = d_A(x) + |y-x|$$

and hence the Lipschitz continuity.

(ii) As the function $y \mapsto |y-x|$ is continuous (hence upper semicontinuous),

$$d_A(x) = \inf_{y\in A} |y-x| = \inf_{y\in \overline{A}} |y-x| = d_{\overline{A}}(x).$$

Since the function $|y-x| \geq 0$ is bounded below, its infimum over $y \in \overline{A}$ is finite. Let $\{y_n\} \subset \overline{A}$ be a minimizing sequence

$$d_{\overline{A}}(x) = \inf_{y\in \overline{A}} |y-x| = \lim_{n\to\infty} |y_n - x|.$$

Then $\{y_n - x\}$ and hence $\{y_n\}$ are bounded sequences. Hence there exists a subsequence converging to some $y \in \overline{A}$ and by continuity $d_{\overline{A}}(x) = |y-x|$.

(iii) By the continuity of d_A, $d_A^{-1}\{0\}$ is closed and

$$A \subset d_A^{-1}\{0\} \quad \Rightarrow \overline{A} \subset d_A^{-1}\{0\}.$$

Conversely, if $d_A(x) = 0$, there exists y in $\overline{A}$ such that $0 = d_A(x) = |y-x|$ and necessarily $x = y \in \overline{A}$.

(iv) follows from (iii).

(v) From (ii) for $x \in \overline{A}$, $d_A(x) = 0$ and $d_B(x) \leq d_A(x) = 0$ necessarily implies $d_B(x) = 0$ and $x \in \overline{B}$. Conversely, if $\overline{A} \subset \overline{B}$, then

$$d_B(x) = d_{\overline{B}}(x) = \inf_{y\in \overline{B}} |y-x| \leq \inf_{y\in \overline{A}} |y-x| = d_{\overline{A}}(x) = d_A(x).$$

(vi) is obvious.

(vii) follows from Rademacher's theorem. □

2.2 Pompéiu–Hausdorff Metric on $C_d(D)$

Let D be a nonempty subset of $\mathbf{R}^N$ and associate with each nonempty subset A of $\overline{D}$ the equivalence class

$$[A]_d \overset{\text{def}}{=} \left\{B : \forall B, B \subset \overline{D} \text{ and } \overline{B} = \overline{A}\right\},$$

since from Theorem 2.1 (v) $d_A = d_B$ if and only if $\overline{A} = \overline{B}$. So $\overline{A}$ is the invariant *closed representative* of the class $[A]_d$. Consider the set

$$\boxed{\mathcal{F}_d(D) \stackrel{\text{def}}{=} \left\{[A]_d : \forall A,\, \varnothing \neq A \subset \overline{D}\right\}}$$

that can be identified with the set

$$\left\{A : \forall A,\, \varnothing \neq A \text{ closed } \subset \overline{D}\right\}.$$

In general, there is no *open representative* in the class $[A]_d$ since

$$d_{\overline{A}} = d_A \leq d_{\text{int}\, A},$$

where $\text{int}\, A = \complement\overline{\complement A}$ denotes the interior of A. By the definition of $[A]_d$ the map

$$[A]_d \mapsto d_A \colon \mathcal{F}_d(D) \to C_d(D) \subset C(\overline{D})$$

is injective. So $\mathcal{F}_d(D)$ can be identified with the subset of distance functions $C_d(D)$ in $C(\overline{D})$. The distance function plays the same role as the set $\mathrm{X}(D)$ in $L^p(D)$ of equivalence classes of characteristic functions of measurable sets.

When D is bounded, $C(\overline{D})$ is a Banach space when endowed with the norm

$$\|f\|_{C(D)} = \sup_{x \in D} |f(x)|. \tag{2.5}$$

As for the characteristic functions of Chapter 5, this induces a complete metric

$$\boxed{\rho([A]_d, [B]_d) \stackrel{\text{def}}{=} \|d_A - d_B\|_{C(D)} = \sup_{x \in D} |d_A(x) - d_B(x)|} \tag{2.6}$$

on $\mathcal{F}_d(D)$, which turns out to be equal to the classical Pompéiu–Hausdorff metric[2]

$$\boxed{\rho_H([A]_d, [B]_d) \stackrel{\text{def}}{=} \max\left\{\sup_{x \in B} d_A(x), \sup_{y \in A} d_B(y)\right\}}$$

(cf. J. Dugundji [**1**, Chap. IX, Prob. 4.8, p. 205] for the definition of ρ_H). Indeed for $x \in D$, $x_A \in A$, and $x_B \in B$

$$\begin{aligned}
&|x - x_A| \leq |x - x_B| + |x_B - x_A| \\
&\Rightarrow \quad d_A(x) \leq |x - x_B| + d_A(x_B) \leq |x - x_B| + \sup_{y \in B} d_A(y) \\
&\Rightarrow \quad d_A(x) \leq d_B(x) + \sup_{y \in B} d_A(y) \quad \Rightarrow \quad d_A(x) - d_B(x) \leq \sup_{y \in B} d_A(y).
\end{aligned}$$

[2]The "*écart mutuel*" between two sets was introduced by D. Pompéiu [**1**] in his thesis presented in Paris in March 1905. This is the first example of a metric between two sets in the literature. It was studied in more detail by F. Hausdorff [**2**, "Quellenangaben," p. 280, and Chap. VIII, sect. 6] in 1914.

By interchanging the roles of A and B

$$\forall x \in D, \quad |d_A(x) - d_B(x)| \leq \max\left\{\sup_{z\in B} d_A(z), \sup_{y\in A} d_B(y)\right\}.$$

The converse inequality follows from the fact that $A \subset \overline{D}$ and $B \subset \overline{D}$.

When D is open but not necessarily bounded, we use $C^0_{\mathrm{loc}}(D)$ (cf. (2.19) in section 2.6.1 of Chapter 2), the space $C(D)$ of continuous functions on D endowed with the Fréchet topology of uniform convergence on compact subsets K of D, which is defined by the family of seminorms

$$\forall K \text{ compact} \subset D, \quad q_K(f) \stackrel{\text{def}}{=} \max_{x\in K} |f(x)|. \tag{2.7}$$

The Fréchet topology on $C(D)$ is metrizable since the topology induced by the family of seminorms $\{q_K\}$ is equivalent to the one generated by the subfamily $\{q_{K_k}\}_{k\geq 1}$, where the compact sets $\{K_k\}_{k\geq 1}$ are chosen as follows:

$$K_k \stackrel{\text{def}}{=} \left\{x \in D : d_{\complement D}(x) \geq \frac{1}{k} \text{ and } |x| \leq k\right\}, \quad k \geq 1. \tag{2.8}$$

It is equivalent to the topology defined by the metric

$$\delta(f,g) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{q_{K_k}(f-g)}{1+q_{K_k}(f-g)}. \tag{2.9}$$

It will be shown below that $C_d(D)$ is a closed subset of $C_{\mathrm{loc}}(D)$ and that this will induce the following complete metric on $\mathcal{F}_d(D)$:

$$\boxed{\rho_\delta([A]_d, [B]_d) \stackrel{\text{def}}{=} \delta(d_A, d_B) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{q_{K_k}(d_A - d_B)}{1+q_{K_k}(d_A - d_B)},} \tag{2.10}$$

which is a natural extension of the Hausdorff metric to an unbounded domain D.

Theorem 2.2. *Let $D \neq \varnothing$ be an open* (resp., *bounded open*) *holdall in* $\mathbf{R}^N$.

(i) *The set $C_d(D)$ is closed in $C_{\mathrm{loc}}(D)$* (resp., $C(\overline{D})$) *and ρ_δ* (resp., ρ) *defines a complete metric topology on $\mathcal{F}_d(D)$.*

(ii) *When D is a bounded open subset of $\mathbf{R}^N$, the set $C_d(D)$ is compact in $C(\overline{D})$ and the metrics ρ and ρ_H are equal.*

Proof. (i) We give the proof of C. Dellacherie [**1**, Thm. 2, p. 42 and Rem. 1, p. 43]. The basic constructions will again be used in Chapter 7. Consider a sequence $\{A_n\}$ of nonempty subsets of $\overline{D}$ such that d_{A_n} converges to some element f of $C_{\mathrm{loc}}(D)$. We wish to prove that $f = d_A$ for

$$A \stackrel{\text{def}}{=} \{x \in \overline{D} : f(x) = 0\}$$

and that the closed subset A of $\overline{D}$ is nonempty. Fix x in $\overline{D}$; then

$$\begin{aligned}\forall n, \exists y_n \in \overline{A}_n, \quad |y_n - x| &= \inf_{z\in A_n} |z - x| = d_{A_n}(x)\\ \Rightarrow \lim_{n\to\infty} |y_n - x| &= \lim_{n\to\infty} d_{A_n}(x) = f(x).\end{aligned}$$

Hence $\{y_n - x\}$ and $\{y_n\} \subset \overline{D}$ are bounded and there exists a subsequence, still indexed by n, which converges to some $y \in \overline{D}$

$$y_n \to y, \text{ and } |y - x| = f(x).$$

In particular, $f(y) = 0$, since in the inequality

$$f(y) \le f(y) - d_{A_n}(y) + d_{A_n}(y) - d_{A_n}(y_n) + d_{A_n}(y_n),$$

the last term is zero and d_{A_n} is Lipschitz continuous with constant equal to 1:

$$|f(y)| \le |f(y) - d_{A_n}(y)| + |y - y_n|,$$

and both terms go to zero. By the definition of A, $y \in A$ and A is not empty. Therefore for each $x \in \overline{D}$, there exists $y \in A$ such that

$$f(x) = |y - x| \ge \inf_{z\in A} |z - x| = d_A(x).$$

Next we prove the inequality in the other direction. By construction for any $A_n \subset \overline{D}$

$$\forall x, y \in \overline{D}, \quad |d_{A_n}(x) - d_{A_n}(y)| \le |x - y|$$

and

$$|f(x) - f(y)| \le |f(x) - d_{A_n}(x)| + |d_{A_n}(x) - d_{A_n}(y)| + |d_{A_n}(y) - f(y)|.$$

By uniform convergence the first and last terms converge to zero, and by Lipschitz continuity of d_{A_n},

$$\forall x, y \in \overline{D}, \quad |f(x) - f(y)| \le |x - y|.$$

Hence for all $x \in \overline{D}$ and $y \in A$, $f(y) = 0$ and

$$f(x) \le f(y) + |x - y| = |x - y| \;\Rightarrow\; \forall x \in \overline{D}, \quad f(x) \le \inf_{y\in A} |x - y| = d_A(x).$$

This proves the reverse equality.

(ii) Observe that on a compact set $\overline{D}$ and for any $A \subset \overline{D}$

$$\begin{gathered}d_A(x) = \inf_{y\in A} |y - x| \le \sup_{y\in A} |y - x| \le c \stackrel{\text{def}}{=} \sup_{y,x\in\overline{D}} |y - x| < \infty,\\ \forall x, y \in \overline{D}, \quad |d_A(y) - d_A(x)| \le |y - x|.\end{gathered}$$

The compactness of $C_d(D)$ now follows by Ascoli–Arzelà Theorem 2.4 of Chapter 2 and part (i). By construction, ρ and ρ_δ are metrics on $\mathcal{F}_d(D)$. By definition, for A and B in the compact $\overline{D}$,

$$\begin{aligned}\rho(A,B) &= \max_{x\in D} |d_A(x) - d_B(x)| \\ &\geq \max\left\{\max_{x\in B} |d_A(x)-d_B(x)|, \max_{x\in A} |d_A(x)-d_B(x)|\right\} \\ &\geq \max\left\{\max_{x\in B} d_A(x),\ \max_{x\in A} d_B(x)\right\} = \rho_H(A,B).\end{aligned}$$

Conversely, for any $x \in \overline{D}$ and y in A,

$$d_A(x) - d_B(x) \leq |y-x| - d_B(x)$$

and there exists $x_B \in \overline{B}$ such that $d_B(x) = |x - x_B|$. Therefore,

$$\begin{aligned}&\forall y \in A, d_A(x) - d_B(x) \leq |y-x| - |x-x_B| \leq |y - x_B| \\ \Rightarrow\ & d_A(x) - d_B(x) \leq \inf_{y\in A} |y - x_B| = d_A(x_B) \leq \max_{x\in B} d_A(x).\end{aligned}$$

Similarly

$$d_B(x) - d_A(x) \leq \max_{x\in A} d_B(x),$$

and for all x in $\overline{D}$

$$|d_B(x) - d_A(x)| \leq \max\left\{\max_{x\in B} d_A(x), \max_{x\in A} d_B(x)\right\} \ \Rightarrow\ \rho(A,B) \leq \rho_H(A,B). \qquad \square$$

When D is bounded the family $\mathcal{F}_d(D)$ enjoys more interesting properties.

Theorem 2.3. *Let D be a nonempty open* (resp., *bounded open*) *subset of $\mathbf{R}^N$. Define for a subset S of $\mathbf{R}^N$ the sets*

$$\begin{aligned}H(S) &\stackrel{\text{def}}{=} \left\{d_A \in C_d(D) \,:\, S \subset \overline{A}\right\}, \\ I(S) &\stackrel{\text{def}}{=} \left\{d_A \in C_d(D) \,:\, \overline{A} \subset S\right\}, \\ J(S) &\stackrel{\text{def}}{=} \left\{d_A \in C_d(D) \,:\, \overline{A} \cap S \neq \varnothing\right\}.\end{aligned}$$

(i) *Let S be a subset of $\mathbf{R}^N$. Then $H(S)$ is closed in $C_{\text{loc}}(D)$* (resp., $C(\overline{D})$).

(ii) *Let S be a closed subset of $\mathbf{R}^N$. Then $I(S)$ is closed in $C_{\text{loc}}(D)$* (resp., $C(\overline{D})$). *If, in addition, $S \cap \overline{D}$ is compact, then $J(S)$ is closed in $C_{\text{loc}}(D)$* (resp., $C(\overline{D})$).

(iii) *Let S be an open subset of $\mathbf{R}^N$. Then $J(S)$ is open in $C_{\text{loc}}(D)$* (resp., $C(\overline{D})$). *If, in addition, $\complement S \cap \overline{D}$ is compact, then $I(S)$ is open in $C_{\text{loc}}(D)$* (resp., $C(\overline{D})$).

(iv) *For a nonempty bounded holdall D, the map*

$$[A]_d \mapsto \#(\overline{A}) \colon \mathcal{F}_\#(D) \stackrel{\text{def}}{=} \left\{[A]_d : \forall A,\ \varnothing \neq A \subset \overline{D} \text{ and } \#(\overline{A}) < \infty\right\} \to \mathbf{R}$$

is lower semicontinuous, where $\#(\overline{A})$ *is the number of connected components as defined in Definition* 2.2 *of Chapter* 2*. For a fixed number* $c > 0$ *the subset*

$$\{d_A \in C_d(D) : \#(\overline{A}) \leq c\}$$

of $C_d(D)$ *is compact in* $C(\overline{D})$*. In particular, the subset*

$$\{d_A \in C_d(D) : \overline{A} \text{ is connected}\}$$

of $C_d(D)$ *is compact in* $C(\overline{D})$*.*

Proof. (i) If $S \not\subset \overline{D}$, then $H(S) = \varnothing$ and there is nothing to prove. Assume that $S \subset \overline{D}$. From Theorem 2.1 (v), $S \subset \overline{A} \Rightarrow d_S \geq d_A$. So for any sequence $\{d_{A_n}\}$ in $H(S)$ converging to d_A in $C_{\text{loc}}(D)$,

$$\forall n, \quad d_S \geq d_{A_n} \quad \Rightarrow d_S \geq d_A \quad \Rightarrow S \subset \overline{S} \subset \overline{A} \quad \Rightarrow d_A \in H(S).$$

(ii) We shall use the same technique for $I(S)$ as for $H(S)$, but here we need $S = \overline{S}$ to conclude. For $\overline{D} \cap S = \varnothing$, $J(S) = \varnothing$ and there is nothing to prove. Assume that $\overline{D} \cap S \neq \varnothing$ and consider a sequence $\{d_{A_n}\}$ in $J(S)$ converging to d_A in $C_{\text{loc}}(D)$. Assume that $\overline{A} \cap S = \varnothing$. Then $\overline{A} \subset \overline{D}$ implies $\overline{A} \cap [S \cap \overline{D}] = \overline{A} \cap S = \varnothing$. By assumption, $S \cap \overline{D}$ is compact and

$$\exists \delta > 0, \forall x \in S \cap \overline{D}, \quad d_A(x) \geq \delta,$$
$$\exists N \geq 1, \forall n \geq N, \quad \|d_{A_n} - d_A\|_{C(D)} \leq \delta/2.$$

So for all $x \in \overline{D} \cap S$

$$d_{A_n}(x) \geq d_A(x) - |d_{A_n}(x) - d_A(x)| \geq \delta - \delta/2 > 0$$
$$\Rightarrow \overline{A}_n \cap S = \overline{A}_n \cap [\overline{D} \cap S] = \varnothing,$$

and this contradicts the fact that $A_n \in J(S)$.

(iii) By definition, for any S, $\complement J(S) = I(\complement S)$,

$$\complement J(S) = \left\{d_A \in C_d(D) : \overline{A} \cap S = \varnothing\right\} = \left\{d_A \in C_d(D) : \overline{A} \subset \complement S\right\} = I(\complement S).$$

Since $\complement S$ is closed, $\complement J(S)$ is closed from part (ii) and $J(S)$ is open. Similarly, replacing S by $\complement S$ in the previous identity, $I(S) = \complement J(\complement S)$. So from part (ii) if $\overline{D} \cap \complement S$ is compact, then $\complement I(S)$ is closed and $I(S)$ is open. From part (ii) if $\overline{D} \cap \complement S$ is compact, then $\complement I(S)$ is closed and $I(S)$ is open.

(iv) Let $\{A_n\}$ and A be nonempty subsets of $\overline{D}$ such that d_{A_n} converges to d_A in $C(\overline{D})$. Assume that $\#(\overline{A}) = k$ is finite. Then there exists a family of disjoint open sets $G_1, \ldots, G_k$ such that

$$\overline{A} \subset G = \cup_{i=1}^k G_i \quad \text{and} \quad \forall i, \overline{A} \cap G_i \neq \varnothing.$$

In view of the definitions of $I(S)$ and $J(S)$ in part (i)

$$\overline{A} \in \mathcal{U} = \cap_{i=1}^k J(G_i) \cap I(G).$$

But $\mathcal{U}$ is not empty and open as the finite intersection of $k+1$ open sets. As a result there exist $\varepsilon > 0$ and an open neighborhood of $[A]_d$:

$$N_\varepsilon([A]_d) = \{[B]_d : \|d_B - d_A\| < \varepsilon\} \subset \mathcal{U}.$$

Hence since d_{A_n} converges to d_A, there exists $\overline{n} > 0$ such that, for all $n \geq \overline{n}$, $[A_n]_d \in \mathcal{U}$, and necessarily

$$\forall n \geq \overline{n}, \quad \overline{A}_n \subset G, \quad \overline{A}_n \cap G_i \neq \varnothing, \forall i \quad \Rightarrow \#(\overline{A_n}) \geq \#(\overline{A}).$$

Therefore, $[A]_d \mapsto \#(\overline{A})$ is lower semicontinuous. □

This theorem has many interesting corollaries. For instance part (iii) says something about the function that gives the number of connected components of $\overline{A}$ (cf. T. J. Richardson [**1**] for an application to image segmentation).

2.3 Uniform Complementary Metric Topology and $C_d^c(D)$

In the previous section we dealt with a theory of closed sets since the equivalence class of the subsets of $\overline{D}$ was completely determined by their unique closure. In partial differential equations the underlying domain is usually open. To accommodate this point of view, consider the set of open subsets Ω of a fixed nonempty open holdall D in $\mathbf{R}^{\mathrm{N}}$, endowed with the Hausdorff topology generated by the distance functions $d_{\complement\Omega}$ to the complement of Ω. This approach has been used in several contexts, for instance, in J.-P. Zolésio [**6**, sect. 1.3, p. 405] in the context of free boundary problems. Also, V. Šverák [**2**] uses the family $C_d^c(D)$ for open sets Ω such that $\#([\complement\Omega]) \leq c$ for some fixed $c > 0$. His main result is that in dimension 2 the convergence of a sequence $\{\Omega_n\}$ to Ω of such sets implies the convergence of the corresponding projection operators $\{P_{\Omega_n} \colon H_0^1(D) \to H_0^1(\Omega_n)\}$ to $P_\Omega \colon H_0^1(D) \to H_0^1(\Omega)$, where the projection operators are directly related to homogeneous Dirichlet linear boundary value problems on the corresponding domains $\{\Omega_n\}$ and Ω. In dimension 1 the constraint on the number of components can be dropped. This result will be discussed in Theorem 8.2 of section 8 in Chapter 8.

By analogy with the constructions of the previous section, consider for a nonempty subset D of $\mathbf{R}^{\mathrm{N}}$ the family

$$\left\{d_{\complement A} : \varnothing \neq \complement A \text{ and } A \subset \overline{D}\right\}.$$

By definition,

$$\complement A \supset \complement \overline{D} \quad \Rightarrow d_{\complement A} = 0 \text{ in } \complement \overline{D},$$

and, by Lipschitz continuity, $d_{\complement A} = 0$ in $\overline{\complement \overline{D}}$ and $d_{\complement A} = 0$ on $\partial \overline{D} = \overline{D} \cap \overline{\complement \overline{D}}$. If $\operatorname{int} \overline{D} = \varnothing$, then $d_{\complement A} = 0$ in $\mathbf{R}^{\mathrm{N}}$ and for all sets A in the family $\operatorname{int} A = \varnothing$. If $\operatorname{int} \overline{D} \neq \varnothing$, associate with each A the open set

$$\Omega \overset{\text{def}}{=} \operatorname{int} A = \complement \overline{\complement A} \quad \Rightarrow \complement\Omega = \overline{\complement A} \text{ and } d_{\complement\Omega} = d_{\complement A}.$$

By definition and the previous considerations,

$$\complement A \supset \complement \overline{D} \quad \Rightarrow \Omega = \complement\overline{\complement A} \subset \complement\overline{\complement\overline{D}} = \operatorname{int}\overline{D}.$$

So finally, for $\operatorname{int}\overline{D} \neq \varnothing$,

$$\left\{d_{\complement A} : \varnothing \neq \complement A \text{ and } A \subset \overline{D}\right\} = \left\{d_{\complement\Omega} : \Omega \neq \mathbf{R}^{\mathrm{N}} \text{ and } \Omega \text{ open } \subset \operatorname{int}\overline{D}\right\}.$$

The invariant set $\operatorname{int} A$ is the *open representative* in the equivalence class

$$[A]_d^c \stackrel{\text{def}}{=} \left\{B : \overline{\complement B} = \overline{\complement A} \neq \varnothing\right\},$$

but, in general, there is no *closed representative* in that class since

$$d_{\complement \operatorname{int} A} = d_{\overline{\complement A}} = d_{\complement A} \leq d_{\complement \overline{A}},$$

as in the case of d_A, where $d_{\overline{A}} = d_A \leq d_{\operatorname{int} A}$.

From this analysis it will be sufficient to consider the family of open subsets of an open holdall D.

Definition 2.2.
Let D be a nonempty open subset of $\mathbf{R}^{\mathrm{N}}$. Define the family of functions

$$\boxed{C_d^c(D) \stackrel{\text{def}}{=} \left\{d_{\complement\Omega} : \forall\Omega \text{ open subset of } D \text{ and } \Omega \neq \mathbf{R}^{\mathrm{N}}\right\}} \tag{2.11}$$

corresponding to the family of open sets

$$\mathcal{G}(D) \stackrel{\text{def}}{=} \left\{\Omega \subset D : \forall\Omega \text{ open and } \Omega \neq \mathbf{R}^{\mathrm{N}}\right\}. \tag{2.12}$$

For D bounded define the *Hausdorff complementary metric* ρ_H^c on $\mathcal{G}(D)$:

$$\boxed{\rho_H^c(\Omega_2, \Omega_1) \stackrel{\text{def}}{=} \|d_{\complement\Omega_2} - d_{\complement\Omega_1}\|_{C(D)}.} \tag{2.13}$$

$\square$

Note that for $\Omega_1 \subset D$ and $\Omega_2 \subset D$, $d_{\complement\Omega_1} = d_{\complement\Omega_2} = 0$ in $\complement D$. Therefore,

$$d_{\complement\Omega_1} = d_{\complement\Omega_2} \text{ in } D \iff \complement\Omega_1 = \complement\Omega_2 \iff \Omega_1 = \Omega_2.$$

Theorem 2.4. *Let D be a nonempty open subset of $\mathbf{R}^{\mathrm{N}}$.*

(i) *The set $C_d^c(D)$ is closed in $C_{\mathrm{loc}}(D)$.*

(ii) *If, in addition, D is bounded, then $C_d^c(D)$ is compact in $C_0(D)$ and $(\mathcal{G}(D), \rho_H^c)$ is a compact metric space.*

(iii) (Compactivorous property)
Let $\{\Omega_n\}$ and Ω be sets in $\mathcal{G}(D)$ such as

$$d_{\complement\Omega_n} \to d_{\complement\Omega} \quad \text{in } C_{\mathrm{loc}}(D).$$

Then for any compact subset $K \subset \Omega$, there exists an integer $N(K) > 0$ such that

$$\forall n \geq N(K), \quad K \subset \Omega_n.$$

Proof. (i) Let $\{\Omega_n\}$ be a sequence of open subsets in D such that $\{d_{\complement\Omega_n}\}$ is a Cauchy sequence in $C_{\text{loc}}(D)$. For all n, $d_{\complement\Omega_n} = 0$ in $\complement D$ and $\{d_{\complement\Omega_n}\}$ is also a Cauchy sequence in $C_{\text{loc}}(\mathbf{R}^{\mathrm{N}})$. By Theorem 2.2 (i) the sequence $\{d_{\complement\Omega_n}\}$ converges in $C_{\text{loc}}(\mathbf{R}^{\mathrm{N}})$ to some distance function d_A. By construction

$$\begin{aligned}&\Omega_n \subset D \implies \complement\Omega_n \supset \complement D \implies d_{\complement\Omega_n} \le d_{\complement D}\\ &\implies d_A \le d_{\complement D} \implies \overline{A} \supset \complement D \implies \complement\overline{A} \subset D.\end{aligned}$$

By choosing the open set $\Omega = \complement\overline{A}$ in D we get

$$d_{\complement\Omega_n} \to d_{\complement\Omega} \quad \text{in } C_{\text{loc}}(\mathbf{R}^{\mathrm{N}}) \text{ and } C_{0,\text{loc}}(D).$$

(ii) To prove the compactness we use the compactness of $C_d(D)$ from Theorem 2.2 (ii) and the fact that $C_d^c(D)$ is closed in $C_0(D)$. Observe that since $\complement\Omega_n \supset \complement D$, $d_{\complement\Omega_n} = 0$ in $\complement D$, $d_{\complement\Omega_n} \in C_0(D)$, and

$$\begin{aligned}&\complement\Omega_n = [\complement\Omega_n \cap \overline{D}] \cup [\complement\Omega_n \cap \complement D] \subset [\complement\Omega_n \cap \overline{D}] \cup \complement D \subset \complement\Omega_n\\ &\Rightarrow \quad d_{\complement\Omega_n} = d_{[\complement\Omega_n \cap \overline{D}] \cup \complement D} = \min\{d_{\complement\Omega_n \cap \overline{D}}, d_{\complement D}\}.\end{aligned}$$

There exist a subsequence, still indexed by n, and a set A, $\varnothing \ne \overline{A} \subset \overline{D}$, such that

$$\begin{aligned}&d_{\complement\Omega_n \cap \overline{D}} \to d_{\overline{A}} \quad \text{in } C_0(D)\\ &\Rightarrow \quad d_{\complement\Omega_n} = \min\{d_{\complement\Omega_n \cap \overline{D}}, d_{\complement D}\} \to \min\{d_{\overline{A}}, d_{\complement D}\} = d_{\overline{A} \cup \complement D} \quad \text{in } C(\overline{D})\\ &\Rightarrow \quad d_{\complement\Omega_n} \to d_{\overline{A} \cup \complement D} \quad \text{in } C(\overline{\mathbf{R}^{\mathrm{N}}})\end{aligned}$$

since $d_{\complement\Omega_n} = 0 = d_{\overline{A} \cup \complement D}$ in $\complement D$. The result now follows by choosing the open set

$$\Omega \stackrel{\text{def}}{=} \complement[\overline{A} \cup \complement D] = \complement\overline{A} \cap D \subset D.$$

(iii) Define

$$m \stackrel{\text{def}}{=} \inf_{x \in K} \inf_{y \in \complement\Omega} |x - y| = \inf_{x \in K} d_{\complement\Omega}(x).$$

Since $K \subset \Omega \subset D$, there exist $\hat{x} \in K$, $\hat{y} \in \partial\Omega$ such that

$$m = \inf_{x \in K} \inf_{y \in \partial\Omega} |x - y| = |\hat{x} - \hat{y}|.$$

Necessarily, $m > 0$ since Ω is open, $\hat{x} \in K \subset \Omega$, and $\hat{y} \in \partial\Omega$. Now

$$\begin{aligned}&\exists N > 0, \forall n \ge N, \quad \|d_{\complement\Omega_n} - d_{\complement\Omega}\|_{C(K)} < m/2\\ &\Rightarrow \forall x \in K, \quad d_{\complement\Omega_n}(x) \ge d_{\complement\Omega}(x) - |d_{\complement\Omega_n}(x) - d_{\complement\Omega}(x)| \ge m - m/2 > 0\\ &\Rightarrow x \ne \overline{\complement\Omega_n} \quad \Rightarrow x \in \complement\overline{\complement\Omega_n} = \operatorname{int}\Omega_n = \Omega_n \quad \Rightarrow K \subset \Omega_n.\end{aligned}$$

□

Notation 2.1.
It will be convenient to write

$$\Omega_n \stackrel{H^c}{\to} \Omega$$

for the Hausdorff complementary convergence of open sets of $\mathcal{G}(D)$

$$d_{\complement\Omega_n} \to d_{\complement\Omega} \quad \text{in } C_{\text{loc}}(D).$$

□

We have the analogue of Theorem 2.3 for the distance function to the complement of an open subset of D.

Theorem 2.5. *Let D be a nonempty open* (resp., *bounded open*) *subset of* $\mathbf{R}^{\mathrm{N}}$. *Define for a subset S of $\mathbf{R}^{\mathrm{N}}$ and open subsets Ω of D the following families:*

$$H^c(S) \stackrel{\text{def}}{=} \left\{ d_{\complement\Omega} \in C_d^c(D) \,:\, S \subset \complement\Omega \right\},$$
$$I^c(S) \stackrel{\text{def}}{=} \left\{ d_{\complement\Omega} \in C_d^c(D) \,:\, \complement\Omega \subset S \right\},$$
$$J^c(S) \stackrel{\text{def}}{=} \left\{ d_{\complement\Omega} \in C_d^c(D) \,:\, \complement\Omega \cap S \neq \varnothing \right\}.$$

(i) *Let S be a subset of $\mathbf{R}^{\mathrm{N}}$. Then $H^c(S)$ is closed in $C_{\mathrm{loc}}(D)$* (resp., $C(\overline{D})$).

(ii) *Let S be a closed subset of $\mathbf{R}^{\mathrm{N}}$. Then $I^c(S)$ is closed in $C_{\mathrm{loc}}(D)$* (resp., $C(\overline{D})$). *If, in addition, $S \cap \overline{D}$ is compact, then $J^c(S)$ is closed in $C_{\mathrm{loc}}(D)$* (resp., $C(\overline{D})$).

(iii) *Let S be an open subset of $\mathbf{R}^{\mathrm{N}}$. Then $J^c(S)$ is open in $C_{\mathrm{loc}}(D)$* (resp., $C(\overline{D})$). *If, in addition, $\complement S \cap \overline{D}$ is compact, then $I^c(S)$ is open in $C_{\mathrm{loc}}(D)$* (resp., $C(\overline{D})$).

(iv) *For a nonempty bounded open holdall D, the map*

$$\complement\Omega \mapsto \#(\complement\Omega) \colon \mathcal{G}_\#(D) \stackrel{\text{def}}{=} \left\{ \complement\Omega : \Omega \textit{ open} \subset D \textit{ and } \#\left(\complement\Omega\right) < \infty \right\} \to \mathbf{R}$$

is lower semicontinuous, where $\#(\complement\Omega)$ is the number of connected components of $\complement\Omega$ as defined in Definition 2.2 *of Chapter* 2. *For a fixed number $c \geq 0$, the subset*

$$\{ d_{\complement\Omega} \in C_d^c(D) : \#\left(\complement\Omega\right) \leq c \}$$

is compact in $C(\overline{D})$. In particular, the subset

$$\{ d_{\complement\Omega} \in C_d^c(D) : \Omega \textit{ is hole-free} \}$$

is compact in $C(\overline{D})$, where hole-free *is in the sense of Definition* 2.2 *of Chapter* 2.

2.4 Families $C_d^c(E;D)$ and $C_{d,\mathrm{loc}}^c(E;D)$

Now and then, it is desirable to avoid having the empty set as a solution of a shape optimization problem. This can occur for the $C(\overline{D})$-topology on $C_d^c(D)$. One way to get around this is to assume that each set of the family contains a fixed nonempty set or, more generally, that each set of the family contains a translated and rotated image of a fixed nonempty set.

Theorem 2.6. *Let D, $\varnothing \neq D \subset \mathbf{R}^{\mathrm{N}}$, be bounded open, let E, $\varnothing \neq E \subset D$, and let*

$$C_d^c(E;D) \stackrel{\text{def}}{=} \{ d_{\complement\Omega} : E \subset \Omega \textit{ open} \subset D \}, \tag{2.14}$$

$$C_{d,\mathrm{loc}}^c(E;D) \stackrel{\text{def}}{=} \left\{ d_{\complement\Omega} : \begin{array}{l} \exists x \in \mathbf{R}^{\mathrm{N}},\, \exists A \in \mathrm{O}(\mathrm{N}) \textit{ such that} \\ x + AE \subset \Omega \textit{ open} \subset D \end{array} \right\}. \tag{2.15}$$

(i) *If E is open, $C_d^c(E;D)$ and $C_{d,\mathrm{loc}}^c(E;D)$ are compact in $C(\overline{D})$.*

(ii) *If E is closed, $C_d^c(E;D)$ and $C_{d,\mathrm{loc}}^c(E;D)$ are open in $C(\overline{D})$.*

Proof. (a) We first deal with $C_d^c(E;D)$ using the fact that

$$C_d^c(E;D) = \left\{ d_{\complement\Omega} : \complement\Omega \subset \complement E \text{ and } \Omega \text{ open} \subset D \right\} = I^c(\complement E)$$

and Theorem 2.5. If E is open, $\complement E$ is closed and $I^c(\complement E)$ is closed and hence compact since D is bounded. If E is closed, $\complement E$ is open, $E \cap \overline{D}$ is compact, and $I^c(\complement E)$ is open.

(b) Assume that E is open. Let $\{d_{\complement\Omega_n}\}$ be a Cauchy sequence in $C_D^c(D)$ converging to some $d_{\complement\Omega}$. Denote by $x_n \in \mathbf{R}^{\mathrm{N}}$ and $A_n \in \mathrm{O(N)}$ the translation and rotation of E such that $x_n + A_n E \subset \Omega_n$ and $d_{\complement x_n + A_n E} \le d_{\complement\Omega}$. Since, for all n, $x_n + A_n E \subset D$ and D is bounded, there exist $x \in \mathbf{R}^{\mathrm{N}}$ and $A \in \mathrm{O(N)}$ and subsequences, still indexed by n, such that $x_n \to x$ and $A_n \to A$. But

$$d_{\complement x_n + A_n E}(z) = d_{\complement E}(A_n(z - x_n))$$

since $A^{-1} = A$ in $\mathrm{O(N)}$ and $d_{\complement E}(A_n(z-x_n)) \to d_{\complement E}(A(z-x))$. Finally, $d_{\complement x + AE} = d_{\complement E}(A(z-x)) \le d_{\complement\Omega}$ and, since $\complement x + AE$ and $\complement\Omega$ are both closed, $\complement x + AE \subset \complement\Omega$, which implies $x + AE \supset \Omega$ and $d_{\complement\Omega} \in C_{d,\mathrm{loc}}^c(E;D)$. Therefore $C_{d,\mathrm{loc}}^c(E;D)$ as a closed subset of the compact set $C_d^c(D)$ is compact. We leave the proof that $C_{d,\mathrm{loc}}^c(E;D)$ is open when E is closed to the reader. □

3 Projection, Skeleton, Crack, and Differentiability

In this section we study the connection between the gradient of d_A, the *set of projections onto* $\overline{A}$, and the characteristic function of $\overline{A}$. We partition the set of singularities of the gradient of d_A into the *skeleton*[3] and *set of cracks* of A.

Definition 3.1.
Let $A \subset \mathbf{R}^{\mathrm{N}}$, $\varnothing \ne A$, and denote by

$$\operatorname{Sing}(\nabla d_A) \stackrel{\text{def}}{=} \left\{ x \in \mathbf{R}^{\mathrm{N}} \, : \, \nabla d_A(x) \nexists \right\} \tag{3.1}$$

the *set of singularities of the gradient of* d_A.

(i) Given $x \in \mathbf{R}^{\mathrm{N}}$, a point $p \in \overline{A}$ such that $|p - x| = d_A(x)$ is called a *projection onto* $\overline{A}$. The *set of all projections onto* $\overline{A}$ will be denoted by

$$\Pi_A(x) \stackrel{\text{def}}{=} \left\{ p \in \overline{A} : |p - x| = d_A(x) \right\}. \tag{3.2}$$

When $\Pi_A(x)$ is a singleton, its element will be denoted by $p_A(x)$.

[3]Our definition of a skeleton does not exactly coincide with the one used in *morphological mathematics*, where it is defined as the closure $\overline{\mathrm{Sk}(A)}$ of our skeleton $\mathrm{Sk}(A)$ (cf., for instance, G. Matheron [**1**] or A. Rivière [**1**], and J. Serra [**1**] for the pioneering applications in mining engineering in 1968 and later in image processing).

(ii) The *skeleton of* A is the set of all points of $\mathbf{R}^{\mathrm{N}}$ whose projection onto $\overline{A}$ is not unique. It will be denoted by

$$\operatorname{Sk}(A) \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : \Pi_A(x) \text{ is not a singleton}\right\}. \tag{3.3}$$

(iii) The *set of cracks* is defined as

$$\operatorname{Ck}(A) \stackrel{\text{def}}{=} \operatorname{Sing} \nabla d_A \backslash \operatorname{Sk}(A). \tag{3.4}$$

Note that $\operatorname{Sk}(\overline{A}) = \operatorname{Sk}(A)$, $\operatorname{Ck}(\overline{A}) = \operatorname{Ck}(A)$, and $\operatorname{Sing}(\overline{A}) = \operatorname{Sing}(A)$ since $d_A = d_{\overline{A}}$ by Theorem 2.1 (ii). □

Even for a set A with smooth boundary ∂A, the gradient $\nabla d_A(x)$ may not exist far from the boundary as shown in Figure 6.1, where $\nabla d_A(x)$ exists everywhere outside $\overline{A}$ except on $\operatorname{Sk}(A)$, a semi-infinite line. Yet, $\nabla d_A(x)$ exists close to the boundary. An example of the set of cracks is given by the eyes, nose, mouth, and rays of the smiling sun in Figure 5.1 of Chapter 5 that belong to the boundary of the set as opposed to the skeleton $\operatorname{Sk}(A)$ that lies outside of $\overline{A}$.

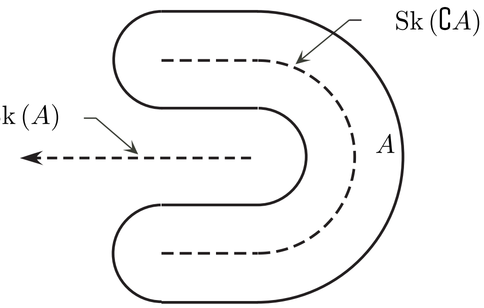


Figure 6.1. *Skeletons* $\operatorname{Sk}(A)$, $\operatorname{Sk}(\complement A)$, *and* $\operatorname{Sk}(\partial A) = \operatorname{Sk}(A) \cup \operatorname{Sk}(\complement A)$.

The level sets of d_A generate dilations/tubular neighborhoods of A.

Definition 3.2.
Let $\varnothing \neq A \subset \mathbf{R}^{\mathrm{N}}$ and let $h > 0$.

(i) The *open h-tubular neighborhood*, the *closed h-tubular neighborhood*, and the *h-boundary* of A are respectively defined as

$$U_h(A) \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : d_A(x)| < h\right\}, \; A_h \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : d_A(x)| \leq h\right\}, \text{ and } d_A^{-1}\{h\}.$$

We shall also use the terminology open or closed *dilated sets*.

(ii) For $0 < s < h < \infty$

$$U_{s,h}(A) \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : s < d_A(x)| < h\right\} \text{ and } A_{s,h} \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : s \leq d_A(x)| \leq h\right\}.$$

This terminology is justified by the fact that a dilated set is a relatively nice set.

Theorem 3.1. *Let $\varnothing \neq A \subset \mathbf{R}^N$. For $r > 0$,*

$$A_r = \overline{U_r(A)},\ \partial A_r = \partial U_r(A) = d_A^{-1}\{r\},\ \overline{\complement A_r} = \complement U_r(A),\ \operatorname{int} A_r = U_r(A). \tag{3.5}$$

In particular, for $0 < s < r$, $\overline{U_{s,r}(A)} = A_{s,r}$.

Proof. By definition $\overline{A} \subset U_r(A) \subset A_r$ and $\overline{U_r(A)} \subset A_r$. For any $x \in A_r$, $d_A(x) \leq r$. If $d_A(x) < r$, then $x \in U_r(A) \subset \overline{U_r(A)}$. If $d_A(x) = r > 0$, then $x \notin \overline{A}$ and there exists $p \in \partial\overline{A}$ such that $d_A(x) = |x - p| = r > 0$. Consider the points

$$x_n \stackrel{\text{def}}{=} p + \left(1 - \frac{1}{n}\right) r \, \frac{x - p}{|x - p|} \to p + r \, \frac{x - p}{|x - p|} = x.$$

Then $p \in \Pi_A(x_n)$, and $d_A(x_n) = r\,(1 - 1/n) < r$, $d_A(x_n) \to d_A(x) = r$, and $x \in \overline{U_r(A)}$. Therefore, since $U_r(A)$ is open, $A_r = \overline{U_r(A)} = U_r(A) \cup \partial U_r(A)$ implies that $\partial U_r(A) = d_A^{-1}\{r\}$. Moreover, since $\complement U_r(A)$ is closed, $\complement U_r(A) = \operatorname{int} \complement U_r(A) \cup \partial U_r(A) = \complement A_r \cup \partial U_r(A)$ implies that $\overline{\complement A_r} = \complement U_r(A)$. The proof is the same for $\overline{U_{s,r}(A)} = A_{s,r}$. □

We now give several important technical results that will be used later for sets of positive reach and in Chapter 7. We first give a few additional definitions.

Definition 3.3. (i) A function $f : U \to \mathbf{R}$ defined in a convex subset U of $\mathbf{R}^N$ is *convex* if for all x and y in U and all $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y).$$

It is *concave* if the function $-f$ is convex. A function $f : U \to \mathbf{R}$ defined in a convex subset U of $\mathbf{R}^N$ is *semiconvex* (resp., *semiconcave*) if

$$\exists c \geq 0, \quad f_c(x) = c\,|x|^2 + f(x) \quad (\text{resp., } f_c(x) = c\,|x|^2 - f(x))$$

is convex in U.

(ii) A function $f : U \to \mathbf{R}$ defined in a subset U of $\mathbf{R}^N$ is *locally convex* (resp., *locally concave*) if it is convex (resp., concave) in every convex subset of U. A function $f : U \to \mathbf{R}$ defined in a subset U of $\mathbf{R}^N$ is *locally semiconvex* (resp., locally semiconcave) if it is semiconvex (resp., semiconcave) in every convex subset of U.
When U is convex the local definitions coincide with the global ones. □

The next theorem is a slight extension of a result in J. H. G. Fu [**1**, Prop. 1.2].

Theorem 3.2. *Let $A \subset \mathbf{R}^N$ be such that $A \neq \varnothing$.*

(i) *Given $h > 0$, the function*

$$k_{A,h}(x) \stackrel{\text{def}}{=} \begin{cases} |x|^2 - 2\,h\,d_A(x), & \text{if } d_A(x) \geq h, \\ |x|^2 - d_A^2(x) - h^2, & \text{if } d_A(x) < h, \end{cases}$$

is convex (and continuous) in $\mathbf{R}^{\mathrm{N}}$ *and the function*

$$x \mapsto \frac{k_{A,h}(x)}{2h} = \frac{|x|^2}{2\,h} - d_A(x) : \mathbf{R}^{\mathrm{N}} \backslash U_h(A) \to \mathbf{R}$$

is locally convex (and continuous) in $\mathbf{R}^{\mathrm{N}} \backslash U_h(A)$.

(ii) *The function*

$$x \mapsto f_A(x) \stackrel{\text{def}}{=} \frac{1}{2} \left(|x|^2 - d_A^2(x)\right) : \mathbf{R}^{\mathrm{N}} \to \mathbf{R}$$

is convex and continuous, $d_A^2(x)$ *is the difference of two convex functions*

$$d_A^2(x) = |x|^2 - \left(|x|^2 - d_A^2(x)\right), \tag{3.6}$$

∇f_A *and* ∇d_A^2 *belong to* $\mathrm{BV_{loc}}(\mathbf{R}^{\mathrm{N}})^N$, *and* $\nabla d_A \in \mathrm{BV_{loc}}\left(\mathbf{R}^{\mathrm{N}} \backslash \partial \overline{A}\right)^N$.

Proof. (i) For all $p \in \overline{A}$ define the convex function

$$\ell_p(x) = \begin{cases} |x-p| - h, & |x-p| \geq h, \\ 0, & |x-p| < h. \end{cases}$$

Since ℓ_p is nonnegative, ℓ_p^2 is convex and

$$\ell_p^2(x) = \begin{cases} |x|^2 - 2\,h\,|x-p| + |p|^2 + h^2 - 2\,x \cdot p, & |x-p| \geq h, \\ 0, & |x-p| < h. \end{cases}$$

By subtracting the constant term $|p|^2 + h^2$ and the linear term $-2\,x \cdot p$ from ℓ_p^2, we get the new convex function

$$m_p(x) = \begin{cases} |x|^2 - 2\,h\,|x-p|, & |x-p| \geq h, \\ |x|^2 - |p-x|^2 - h^2, & |x-p| < h. \end{cases}$$

Then the function

$$k(x) \stackrel{\text{def}}{=} \sup_{p \in A} m_p(x) = \left\{ \begin{array}{ll} |x|^2 - 2\,h\,d_A(x), & d_A(x) \geq h, \\ |x|^2 - d_A^2(x) - h^2, & d_A(x) < h \end{array} \right\}$$

is convex in $\mathbf{R}^{\mathrm{N}}$. Moreover, it is finite for each $x \in \mathbf{R}^{\mathrm{N}}$ and hence continuous in x. Indeed, if x is such that $d_A(x) \geq h$, by Theorem 2.1 (ii) there exists $\bar{p} \in \overline{A}$ such that $|x - \bar{p}| = d_A(x) \geq h$ and

$$|x - \bar{p}| = d_A(x) = \inf_{\substack{p \in \overline{A} \\ |x-p| \geq h}} |x - p|$$

and $k(x) = |x|^2 - 2\,h\,d_A(x)$. If $d_A(x) < h$, then there exists $\bar{p} \in \overline{A}$ such that $|x - \bar{p}| = d_A(x) < h$ and

$$|x-\bar{p}| = d_A(x) = \inf_{\substack{p\in\overline{A}\\ |x-p|<h}} |x-p|$$

and $k(x) = |x|^2 - d_A^2(x) - h^2$. We recover the result of J. H. G. Fu [**1**, Prop. 1.2] by observing that the restriction of k to $\mathbf{R}^{\mathrm{N}} \backslash U_h(A)$ is locally convex. In addition the function $|x|^2 - d_A^2(x)$ is locally convex in $U_h(A)$.

(ii) Since for all $h > 0$, $|x|^2 - d_A^2(x)$ is locally convex in $U_h(A)$ it is locally convex and continuous in $\mathbf{R}^{\mathrm{N}} = \cup_{h>0} U_h(A)$ and hence convex in $\mathbf{R}^{\mathrm{N}}$. Therefore, from L. C. Evans and R. F. Gariepy [**1**, Thm. 3, p. 240], $\nabla f_A \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})^N$ and $\nabla d_A^2 \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})^N$. Finally, $\mathbf{R}^{\mathrm{N}} \backslash \partial\overline{A}$ is the union of the two disjoint open sets $\mathbf{R}^{\mathrm{N}} \backslash \overline{A}$ and $\operatorname{int} \overline{A}$. On $\operatorname{int}\overline{A}$, $\nabla d_A = 0$. For any point x in $\mathbf{R}^{\mathrm{N}} \backslash \overline{A}$, there exists ρ, $0 < 3\,\rho < d_A(x)$, such that the open ball $B(x, 2\,\rho)$ is contained in $\mathbf{R}^{\mathrm{N}} \backslash U_\rho(A)$, where $k_{A,\rho}$ is locally convex by part (i). Hence $k_{A,\rho}$ is convex in $B(x, 2\,\rho)$ and $\nabla k_{A,\rho}$, and ∇d_A belong to $\mathrm{BV}_{\mathrm{loc}}(B(x, 2\rho))^N$ and, by Definition 6.1 in Chapter 5, ∇d_A belong to $\mathrm{BV}(B(x,\rho))^N$. Finally, by Theorem 6.1 in Chapter 5, ∇d_A belongs to $\mathrm{BV}_{loc}(\mathbf{R}^{\mathrm{N}} \backslash \overline{A})^N$. □

We now give some basic results.

Theorem 3.3. *Let A, $\varnothing \neq A \subset \mathbf{R}^{\mathrm{N}}$, and let $x \in \mathbf{R}^{\mathrm{N}}$.*

(i) *The set $\Pi_A(x)$ is nonempty, compact, and*

$$\forall x \notin \overline{A} \quad \Pi_A(x) \subset \partial\overline{A} \quad \textit{and} \quad \forall x \in \overline{A} \quad \Pi_A(x) = \{x\}.$$

(ii) *For all x and v in $\mathbf{R}^{\mathrm{N}}$, the Hadamard semiderivative*[4] *of d_A^2 always exists,*

$$\begin{aligned} d_H d_A^2(x;v) = \lim_{t\searrow 0} \frac{d_A^2(x+tv) - d_A^2(x)}{t} &= \min_{p\in\Pi_A(x)} 2(x-p)\cdot v \\ &= 2(x\cdot v - \sigma_{\Pi_A(x)}(v)), \\ d_H f_A(x;v) = \lim_{t\searrow 0} \frac{f_A(x+tv) - f_A(x)}{t} &= \sigma_{\Pi_A(x)}(v) = \sigma_{\operatorname{co}\Pi_A(x)}(v), \end{aligned}$$

where σ_B is the support function of the set B,

$$\sigma_B(v) = \sup_{z\in B} z\cdot v,$$

and $\operatorname{co} B$ is the convex hull of B. In particular,

$$\operatorname{Sk}(A) = \left\{x \in \mathbf{R}^{\mathrm{N}} \,:\, \nabla d_A^2(x) \nexists\right\} \subset \mathbf{R}^{\mathrm{N}} \backslash \overline{A}. \tag{3.7}$$

[4] A function $f : \mathbf{R}^{\mathrm{N}} \to \mathbf{R}$ has a Hadamard semiderivative in x in the direction v if

$$d_H f(x;v) \overset{\text{def}}{=} \lim_{\substack{t\searrow 0\\ w\to v}} \frac{f(x+tw)-f(x)}{t} \text{ exists}$$

(cf. Chapter 9, Definition 2.1 (ii)).

Given $v \in \mathbf{R}^N$, *for all* $x \in \mathbf{R}^N \backslash \partial \overline{A}$ *the Hadamard semiderivative of* d_A *exists,*

$$d_H d_A(x;v) = \min_{p \in \Pi_A(x)} 2 \frac{x-p}{|x-p|} \cdot v, \tag{3.8}$$

and for $x \in \partial \overline{A}$, $d_H d_A(x;v)$ *exists if and only if*

$$\lim_{t \searrow 0} \frac{d_A(x+tv)}{t} \quad \textit{exists}. \tag{3.9}$$

(iii) *The following statements are equivalent:*

(a) $d_A^2(x)$ *is (Fréchet) differentiable at* x.

(b) $d_A^2(x)$ *is Gateaux differentiable at* x.

(c) $\Pi_A(x)$ *is a singleton.*

(iv) $\nabla f_A(x)$ *exists if and only if* $\Pi_A(x) = \{p_A(x)\}$ *is a singleton. In that case*

$$p_A(x) = \nabla f_A(x) = x - \frac{1}{2} \nabla d_A^2(x).$$

For all x *and* y *in* $\mathbf{R}^N$,

$$\forall p(x) \in \Pi_A(x), \quad \frac{1}{2}\left(|y|^2 - d_A^2(y)\right) \geq \frac{1}{2}\left(|x|^2 - d_A^2(x)\right) + p(x) \cdot (y-x), \tag{3.10}$$

or equivalently,

$$\forall p(x) \in \Pi_A(x), \quad d_A^2(y) - d_A^2(x) - 2\,(x - p(x)) \cdot (y-x) \leq |x-y|^2. \tag{3.11}$$

For all x *and* y *in* $\mathbf{R}^N$

$$\forall p(x) \in \Pi_A(x), \forall p(y) \in \Pi_A(y), \quad (p(y) - p(x)) \cdot (y-x) \geq 0. \tag{3.12}$$

(v) *The functions*

$$p_A : \mathbf{R}^N \backslash \mathrm{Sk}\,(A) \to \mathbf{R}^N \quad \textit{and} \quad \nabla d_A^2 : \mathbf{R}^N \backslash \mathrm{Sk}\,(A) \to \mathbf{R}^N$$

are continuous. For all $x \in \mathbf{R}^N \backslash \mathrm{Sk}\,(A)$

$$p_A(x) = x - \frac{1}{2} \nabla d_A^2(x) = \nabla f_A(x). \tag{3.13}$$

In particular, for all $x \in \overline{A}$, $\Pi_A(x) = \{x\}$ *and* $\nabla d_A^2(x) = 0$,

$$\begin{aligned} \mathrm{Sk}\,(A) &= \left\{x \in \mathbf{R}^N : \nabla d_A^2(x) \nexists \textit{ and } \nabla d_A(x) \nexists\right\} \subset \mathbf{R}^N \backslash \overline{A}, \\ \mathrm{Ck}\,(A) &= \left\{x \in \mathbf{R}^N : \nabla d_A^2(x) \exists \textit{ and } \nabla d_A(x) \nexists\right\} \subset \partial \overline{A}, \end{aligned} \tag{3.14}$$

and $\mathrm{Sing}\,(\nabla d_A) = \mathrm{Sk}\,(A) \cup \mathrm{Ck}\,(A)$.

(vi) *The function*

$$\nabla d_A : \mathbf{R}^{\mathrm{N}} \setminus \big(\mathrm{Sk}\,(A) \cup \partial \overline{A}\big) \to \mathbf{R}^N$$

is continuous. For all $x \in \operatorname{int} \overline{A}$, $\nabla d_A(x) = 0$ *and for all* $x \in \mathbf{R}^{\mathrm{N}} \setminus \big(\mathrm{Sk}\,(A) \cup \overline{A}\big)$

$$\nabla d_A(x) = \frac{\nabla d_A^2(x)}{2\, d_A(x)} = \frac{x - p_A(x)}{|x - p_A(x)|} \quad \text{and} \quad |\nabla d_A(x)| = 1. \tag{3.15}$$

If $\nabla d_A(x)$ *exists for* $x \in \overline{A}$, *then* $\nabla d_A(x) = 0$. *In particular,* $\nabla d_A = 0$ *almost everywhere in* $\overline{A}$.

(vii) *The functions* d_A^2 *and* d_A *are differentiable almost everywhere and*

$$\mathrm{m}(\mathrm{Sk}\,(A)) = \mathrm{m}(\mathrm{Ck}\,(A)) = \mathrm{m}(\mathrm{Sing}\,(\nabla d_A)) = 0.$$

For $\partial A \neq \varnothing$

$$\mathrm{Sk}\,(\partial A) = \mathrm{Sk}\,(A) \cup \mathrm{Sk}\,(\complement A) \subset \mathbf{R}^{\mathrm{N}} \backslash \partial A, \tag{3.16}$$

$$\mathrm{Ck}\,(\partial A) = \mathrm{Ck}\,(A) \cup \mathrm{Ck}\,(\complement A) \subset \partial(\partial A). \tag{3.17}$$

If $\partial A = \varnothing$, *then either* $A = \mathbf{R}^{\mathrm{N}}$ *or* $A = \varnothing$.

(viii) *Given* $A \subset \mathbf{R}^{\mathrm{N}}$, $\varnothing \neq A$ (resp., $\varnothing \neq \complement A$),

$$\chi_{\overline{A}}(x) = 1 - |\nabla d_A(x)|, \qquad \chi_{\operatorname{int} \complement A}(x) = |\nabla d_A(x)| \text{ in } \mathbf{R}^{\mathrm{N}} \backslash \mathrm{Sing}\,(\nabla d_A)$$

$$\text{(resp., } \chi_{\overline{\complement A}}(x) = 1 - |\nabla d_{\complement A}(x)|, \quad \chi_{\operatorname{int} A}(x) = |\nabla d_{\complement A}(x)| \text{ in } \mathbf{R}^{\mathrm{N}} \backslash \mathrm{Sing}\,(\nabla d_{\complement A}))$$

and the above identities hold for almost all x *in* $\mathbf{R}^{\mathrm{N}}$.

(ix) *Given* $x \in \mathbf{R}^{\mathrm{N}}$, $\alpha \in [0,1]$, $p \in \Pi_A(x)$, *and* $x_\alpha \stackrel{\text{def}}{=} p + \alpha\,(x - p)$, *then*

$$d_A(x_\alpha) = |x_\alpha - p| = \alpha\,|x - p| = \alpha\, d_A(x) \quad \text{and} \quad \forall \alpha \in [0,1],\ \Pi_A(x_\alpha) \subset \Pi_A(x).$$

In particular, if $\Pi_A(x)$ *is a singleton, then* $\Pi_A(x_\alpha)$ *is a singleton and* $\nabla d_A^2(x_\alpha)$ *exists for all* α, $0 \leq \alpha \leq 1$. *If, in addition,* $x \neq \overline{A}$, *then for all* $0 < \alpha \leq 1$ $\nabla d_A(x_\alpha)$ *exists and* $\nabla d_A(x_\alpha) = \nabla d_A(x)$.

Remark 3.1.
In general, for each $v \in \mathbf{R}^{\mathrm{N}}$

$$\exists \hat{p} = p(v) \in \Pi_A(x), \quad d_H d_A^2(x; v) = 2(x - \hat{p}) = \inf_{p \in \Pi_A(x)} 2(x - p) \cdot v,$$

since $\Pi_A(x)$ is compact. However, when $\Pi_A(x)$ is not a singleton, $\hat{p}$ is not necessarily unique since $2(x-p)$ depends on the direction v. For points x outside of $\overline{A}$ and $\mathrm{Sk}\,(A)$ the norm of $\nabla d_A(x)$ is equal to 1. When A is sufficiently smooth, $\nabla d_A(x)$ coincides with the outward unit normal to $\overline{A}$ at the point $p_A(x)$. When A is not smooth, this normal is not always unique as shown in Figure 6.2. □

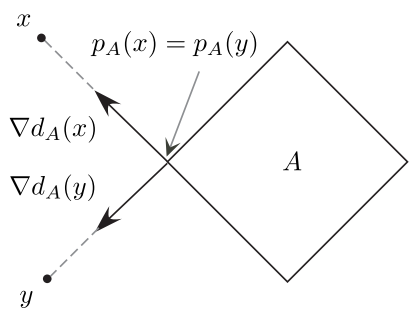


Figure 6.2. *Nonuniqueness of the exterior normal.*

It is useful to consider examples to illustrate the subtleties of the theorem. The set of cracks $\mathrm{Ck}\,(A)$ of a set A has zero measure, but its boundary ∂A can have a nonempty interior and a nonzero measure.

Example 3.1.
Let $B_1(0)$ be the open ball of radius 1 at the origin in $\mathbf{R}^{\mathrm{N}}$ and let

$$A = \{x \in B_1(0) \,:\, x \text{ has rational coordinates}\}.$$

Then $\overline{A} = \overline{B_1(0)}$, $\overline{\complement A} = \mathbf{R}^N$, $\partial A = \overline{B_1(0)}$, $\partial \overline{A} = \partial B_1(0)$,

$$\nabla d_A(x) = \nabla d_A^2(x) = 0 \text{ in } B_1(0) \quad \text{and} \quad \nabla d_A(x) = \frac{x}{|x|} = p_A(x) \text{ in } \complement \overline{B_1(0)},$$

$$\mathrm{Sk}\,(A) = \varnothing \quad \text{and} \quad \mathrm{Ck}\,(A) = \partial B_1(0).$$

In general $\partial \overline{A} \subset \partial A$ and this inclusion can be strict as illustrated by this example. Also note that the boundary ∂A has a nonempty interior $\mathrm{int}\, \partial A = B_1(0)$ and a nonzero Lebesgue measure (nonzero volume in dimension $N = 3$). Thus the boundary of two members of the equivalence class $[A]$ can be different. □

The boundary ∂A of a closed or an open set A has no interior; that is, ∂A is *nowhere dense*. Yet, its boundary ∂A still does not necessarily have a zero measure. If A is closed, then $A = \mathrm{int}\, A \cup \partial A$. If $\mathrm{int}\, A \neq \varnothing$, there exist $x \in \partial A$ and $r > 0$ such that $B_r(x) \subset \partial A \subset A$ and $x \in \mathrm{int}\, A$, a contradiction. If A is open, $\complement A$ is closed, $\complement A = \mathrm{int}\, \complement A \cup \partial A$, and ∂A has no interior.

Example 3.2 (Modified version of Example 1.10 in E. Giusti [**1**, p. 7])**.**
Let $B_1(0)$ in $\mathbf{R}^2$ be the open ball in 0 of radius 1. For $i \geq 1$, let $\{x_i \,:\, i \in \mathbb{N}\}$ be the sequence of all points in $B_1(0)$ with rational coordinates for some choice of ordering. Associate with each i the open ball

$$B_i = \{x \in \mathbf{R}^2 : |x - x_i| < \rho_i\}, \quad 0 < \rho_i \leq \min\{2^{-i}, 1 - |x_i|\} \quad \Rightarrow B_i \subset B_1(0).$$

Consider the increasing sequence of open subsets of $B_1(0)$

$$\Omega_n \stackrel{\text{def}}{=} \bigcup_{i=1}^{n} B_i \nearrow \Omega \stackrel{\text{def}}{=} \bigcup_{i=1}^{\infty} B_i$$

and the closed set $A \stackrel{\text{def}}{=} \complement\Omega$. We get

$$\overline{A} = A = \complement\Omega, \quad \complement A = \Omega, \quad \overline{\complement A} = \overline{\Omega} = \overline{B_1(0)}, \quad \partial A = \overline{B_1(0)} \cap \complement\Omega.$$

As a result, we have the following decreasing sequence of closed subsets of $\overline{B_1(0)}$:

$$\begin{aligned} D_n \stackrel{\text{def}}{=} \overline{B_1(0)} \cap \complement\Omega_n \searrow \partial A = \overline{B_1(0)} \cap \complement\Omega \\ \Rightarrow \mathrm{m}(D_n) \geq \mathrm{m}(\overline{B_1(0)}) - \sum_{i=1}^{n} \mathrm{m}(B_i) = \pi - \pi \sum_{i=1}^{n} \left(\frac{1}{2^i}\right)^2 \\ \Rightarrow \mathrm{m}(\partial A) \geq \pi - \pi \sum_{i=1}^{\infty} \left(\frac{1}{2^i}\right)^2 = \frac{2}{3}\pi. \end{aligned}$$

□

Proof of Theorem 3.3. (i) *Existence.* The function

$$z \mapsto |z - x|^2 \colon \mathbf{R}^{\mathrm{N}} \to \mathbf{R} \tag{3.18}$$

is continuous and, a fortiori, upper semicontinuous, and hence

$$\inf_{a \in A} |a - x|^2 = \inf_{a \in \overline{A}} |a - x|^2.$$

If A is bounded, there exists $p \in \overline{A}$ such that $|p - x|^2 = \inf_{a \in \overline{A}} |a - x|^2$. If A is unbounded, the function (3.18) is coercive with respect to $\overline{A}$,

$$\lim_{a \in \overline{A},\, |a| \to \infty} \frac{|a - x|^2}{|a|} = +\infty,$$

and we also have the existence of $p \in \overline{A}$ such that $|p - x|^2 = \inf_{a \in \overline{A}} |a - x|^2$. By definition, $\Pi_A(x) \subset \overline{A}$. For $x \in \mathbf{R}^{\mathrm{N}} \setminus \overline{A}$, $d_A(x) > 0$. Assume that there exists $p \in \Pi_A(x)$ such that $p \in \operatorname{int} \overline{A}$. Therefore, there exists r, $0 < r < d_A(x)/2$, such that $B_r(p) \subset \overline{A}$. Choose the point

$$\begin{aligned} y = p + \frac{r}{2} \frac{x - p}{|x - p|} \quad &\Rightarrow |y - p| = r/2 \quad \Rightarrow y \in B_r(p) \subset \overline{A} \\ \Rightarrow y - x = p - x + \frac{r}{2} \frac{x - p}{|x - p|} &= \left[1 - \frac{r}{2d_A(x)}\right] (p - x) \\ \Rightarrow \exists y \in \overline{A} \text{ such that } |y - x| &= \left[1 - \frac{r}{2d_A(x)}\right] |p - x| < d_A(x); \end{aligned}$$

this contradicts the minimality of $d_A(x)$. Hence $x \in \partial \overline{A}$.

(ii) *Semidifferentiability of* f_A. The semiderivative in a direction v can be readily computed by using a theorem on the differentiability of the min with respect to a parameter (cf. Chap. 10, sect. 2.3, Thm. 2.1). It is sufficient to prove the semidifferentiability of d_A^2 in any direction v, that is, for each x and v the existence of the limit

$$dd_A^2(x;v) \overset{\text{def}}{=} \lim_{t\searrow 0} \frac{d_A^2(x+tv)-d_A^2(x)}{t}.$$

Recall that for $t \geq 0$,

$$\Pi_A(x+t\,v) = \left\{p_t \in \bar{A} : |x+tv-p_t|^2 = d_A^2(x+tv)\right\}$$

and consider for $t>0$ the quotient

$$q_t = \frac{d_A^2(x+tv)-d_A^2(x)}{t}.$$

For all $p \in \Pi_A(x)$ and $p_t \in \Pi_A(x+t\,v)$

$$\begin{aligned} q_t &= \frac{|x+tv-p_t|^2-|x-p|^2}{t} \\ &\leq \frac{|x+tv-p|^2-|x-p|^2}{t} = t\,|v|^2 + 2\,v\cdot(x-p), \end{aligned}$$

and for all $p \in \Pi_A(x)$

$$\limsup_{t\searrow 0} q_t \leq 2\,v\cdot(x-p) \quad \Rightarrow \bar{q} = \limsup_{t\searrow 0} q_t \leq 2 \inf_{p\in\Pi_A(x)} v\cdot(x-p).$$

In the other direction choose a sequence $t_k > 0$ such that $t_k \to 0$ and $q_{t_k} \to \underline{q} = \liminf_{t\searrow 0}$. The corresponding sequence p_{t_k} in $\overline{A}$ is uniformly bounded since

$$\begin{aligned} |p_{t_k}| &\leq |x+t_kv-p_{t_k}| + |x+t_kv| \\ &\leq d_A(x+t_kv) - d_A(x) + d_A(x) + |x+t_kv| \\ &\leq t_k|v| + d_A(x) + |x+t_kv|, \end{aligned}$$

and we can find $p_0 \in \bar{A}$ and a subsequence of $\{p_{t_k}\}$, still denoted by $\{p_{t_k}\}$, such that $p_{t_k} \to p_0$. But by continuity

$$\begin{aligned} &d_A(x+t_kv) \to d_A(x) \quad \text{and} \quad |x+t_kv-p_{t_k}| \to |x-p_0| \\ &\Rightarrow \exists p_0 \in \bar{A} \text{ such that } d_A(x) = |x-p_0| \quad \Rightarrow p_0 \in \Pi_A(x). \end{aligned}$$

Therefore,

$$\begin{aligned} q_{t_k} &= \frac{|x+t_kv-p_{t_k}|^2-|x-p_0|^2}{t_k} \\ &\geq \frac{|x+t_kv-p_{t_k}|^2-|x-p_{t_k}|^2}{t_k} = t_k|v|^2 + 2\,v\cdot(x-p_{t_k}) \\ \Rightarrow &\underline{q} \geq 2\,v\cdot(x-p_0) \geq 2 \inf_{p\in\Pi_A(x)} v\cdot(x-p). \end{aligned}$$

Hence

$$dd_A^2(x;v) = 2 \inf_{p\in\Pi_A(x)} v\cdot(x-p).$$

Finally, by Theorem 2.4 (ii), $d_H d_A^2(x;v) = dd_A^2(x;v)$ since d_A^2 is locally Lipschitzian. The same remark applies to $dd_A(x;v)$ in $\mathbf{R}^{\mathrm{N}}\backslash\partial\overline{A}$.

(iii) From part (ii), $d_A^2(x)$ is Gateaux differentiable at x if and only if $\Pi_A(x)$ is a singleton. The (Fréchet) differentiability is a standard part of Rademacher's theorem, but we reproduce it here for completeness as a lemma.

Lemma 3.1. *Let $f : \mathbf{R}^{\mathrm{N}} \to \mathbf{R}$ be a locally Lipschitzian function. Then f is (Fréchet) differentiable at x, that is,*

$$\lim_{y\to x} \frac{f(y)-f(x)-\nabla f(x)\cdot(y-x)}{|y-x|} = 0$$

if and only if it is Gateaux differentiable at x, that is,

$$\forall v\in\mathbf{R}^{\mathrm{N}},\quad df(x;v) = \lim_{t\searrow 0}\frac{f(x+tv)-f(x)}{t} \text{ exists}$$

and the map $v\mapsto df(x;v)$ is linear and continuous.

Proof. It is sufficient to prove that the Gateaux differentiability implies that

$$\lim_{y\to x} \frac{f(y)-f(x)-\nabla f(x)\cdot(y-x)}{|y-x|} \to 0.$$

The function f is locally Lipschitzian and there exist $\delta_1>0$ and $c_1>0$ such that f is uniformly Lipschitzian in the open ball $B(x,\delta_1)$ with Lipschitz constant c_1. Therefore, $|\nabla f(x)|\le c_1$. The unit sphere $S(0,1)=\{v\in\mathbf{R}^{\mathrm{N}} : |v|=1\}$ is compact and can be covered by the family of open balls $\{B(v,\varepsilon/(4c_1)) : v\in S(0,1)\}$ for an arbitrary $\varepsilon>0$. Hence there exists a finite subcover $\{B(v_n,\varepsilon/(4c_1)) : 1\le n\le N\}$ of $S(0,1)$. As a result, given any $v\in S(0,1)$, there exists n, $1\le n\le N$, such that $|v-v_n|\le\varepsilon/(4c_1)$. Define the function

$$g(x,v,t) = \frac{f(x+tv)-f(x)}{t} - \nabla f(x)\cdot v$$

for $x\in\mathbf{R}^{\mathrm{N}}$, $v\in S(0,1)$, and $t>0$. Since $f(x)$ is Gateaux differentiable

$$\exists\delta,\ 0<\delta<\delta_1,\ \forall n,\ 1\le n\le N,\ \forall t<\delta,\quad |g(x,v_n,t)|<\varepsilon/2.$$

Hence for $|y-x|<\delta$

$$\begin{aligned}
&\left|g\left(x,\frac{y-x}{|y-x|},|y-x|\right)\right| \\
&\le |g(x,v_n,|y-x|)| + \left|g\left(x,\frac{y-x}{|y-x|},|y-x|\right) - g(x,v_n,|y-x|)\right| \\
&\le |g(x,v_n,|y-x|)| + 2\,c_1\left|\frac{y-x}{|y-x|}-v_n\right| \le \frac{\varepsilon}{2} + 2\,c_1\,\frac{\varepsilon}{(4c_1)} = \varepsilon,
\end{aligned}$$

and we conclude that $f(x)$ is differentiable at x. □

(iv) From part (ii), $\nabla f_A(x)$ exists if and only if $\Pi_A(x)$ is a singleton. Inequality (3.10) follows directly from the inequality

$$\forall x \text{ and } y \in \mathbf{R}^{\mathrm{N}},\ \forall\, p(x) \in \Pi_A(x), \quad d_A^2(y) \le |p(x) - y|^2$$

since

$$\begin{aligned}
&d_A^2(y) - |y|^2 \le |p(x) - x + x - y|^2 - |y|^2 \\
&\le |p(x) - x|^2 + |x - y|^2 + 2\,(p(x) - x) \cdot (x - y) - |y|^2 \\
&\le |p(x) - x|^2 - |x|^2 + 2\,(p(x) - x) \cdot (x - y) + |x - y|^2 + |x|^2 - |y|^2
\end{aligned}$$

and

$$\begin{aligned}
&-2 f_A(y) \\
&\le -2 f_A(x) + 2\,p(x) \cdot (x - y) - 2\,x \cdot (x - y) + |x - y|^2 + |x|^2 - |y|^2 \\
&\le -2 f_A(x) - 2\,p(x) \cdot (y - x) \ \Rightarrow\ f_A(y) \ge f_A(x) + 2\,p(x) \cdot (y - x).
\end{aligned}$$

Inequality (3.11) is (3.10) rewritten. Inequality (3.12) follows by adding inequality (3.10) to the same inequality with x and y permuted.

(v) From the equivalences in (ii), when d_A^2 is differentiable at x, then $\Pi_A(x) = \{p_A(x)\}$ is a singleton, and from part (i)

$$\nabla d_A^2(x) = 2(x - p_A(x)),$$

which yields the expression for $p_A(x)$. When $x \in \overline{A}$, $\Pi_A(x) = \{x\}$, and by substitution $\nabla d_A^2(x) = 0$. Finally, it is sufficient to prove the continuity of p_A. Given $x \in \mathbf{R}^{\mathrm{N}} \backslash \mathrm{Sk}\,(A)$, consider a sequence $\{x_n\} \subset \mathbf{R}^{\mathrm{N}} \backslash \mathrm{Sk}\,(A)$ such that $x_n \to x$ and the associated sequence $\{p_A(x_n)\} \subset \overline{A}$. For all n

$$\begin{aligned}
|p_A(x_n) - x| &\le |p_A(x_n) - x_n| + |x_n - x| \le |p_A(x) - x_n| + |x_n - x| \\
&\le |p_A(x) - x| + 2|x_n - x| < d_A(x) + 2r, \quad r \overset{\text{def}}{=} \sup_n |x_n - x| < \infty.
\end{aligned}$$

So the sequence $\{p_A(x_n)\} \subset \overline{A}$ is bounded and there exists $p \in \overline{A}$ and a subsequence, still denoted by $\{p_A(x_n)\}$, such that $p_A(x_n) \to p$. Therefore

$$\begin{aligned}
&d_A(x) \leftarrow d_A(x_n) = |x_n - p_A(x_n)| \to |x - p| \\
&\Rightarrow \exists p \in \overline{A}, \quad d_A(x) = |x - p| \quad \Rightarrow p_A(x) = p
\end{aligned}$$

and p_A is continuous at x since $p = p_A(x)$ is independent of the choice of the converging subsequence.

To complete the proof of identities (3.14), recall that the projection of each point of $\overline{A}$ onto $\overline{A}$ being a singleton, $\mathrm{Sk}\,(A) \subset \mathbf{R}^{\mathrm{N}} \backslash \overline{A}$. Therefore, for any $x \in \mathrm{Sk}\,(A)$, we have $d_A(x) > 0$ and if $\nabla d_A^2(x)$ does not exist, then $\nabla d_A(x)$ cannot exist. This sharpens the characterization (3.7) of $\mathrm{Sk}\,(A)$. From this we readily get the characterization of $\mathrm{Ck}\,(A) = \mathrm{Sing}\,(\nabla d_A) \backslash \mathrm{Sk}\,(A)$ and $\mathrm{Ck}\,(A) \subset \overline{A}$. This last property can be improved to $\mathrm{Ck}\,(A) \subset \partial\overline{A}$ by noticing that for $x \in \operatorname{int}\overline{A}$, $d_A(x) = 0$ and $\nabla d_A^2(x) = \nabla d_A(x) = 0$.

(vi) For all $x \in \operatorname{int}\overline{A}$, $d_A(x) = 0$ and hence $\nabla d_A(x) = 0$ is constant and, a fortiori, continuous in $\operatorname{int}\overline{A}$. For $x \in \mathbf{R}^{\mathrm{N}} \backslash (\mathrm{Sk}\,(A) \cup \overline{A})$, $d_A(x) > 0$ and for all $v \in \mathbf{R}^{\mathrm{N}}$ and $t > 0$ sufficiently small

$$\frac{d_A(x+tv) - d_A(x)}{t} = \frac{d_A(x+tv)^2 - d_A(x)^2}{t} \frac{1}{d_A(x+tv) + d_A(x)}$$
$$\Rightarrow dd_A(x;v) = \frac{dd_A^2(x;v)}{2d_A(x)} = \frac{1}{2d_A(x)} \nabla d_A^2(x) \cdot v \quad \Rightarrow \nabla d_A(x) = \frac{1}{2d_A(x)} \nabla d_A^2(x).$$

By continuity of $\nabla d_A^2(x)$ from part (iv) and continuity of d_A, $\nabla d_A(x)$ is continuous since $d_A(x) > 0$. Moreover, from part (iv), $\nabla d_A(x) = 2(x - p_A(x))/2d_A(x)$ and $|\nabla d_A(x)| = 1$.

If $\nabla d_A(x)$ exists in some $x \in \overline{A}$, then $d_A(x) = 0$ and for all $v \in \mathbf{R}^{\mathrm{N}}$, the differential quotient converges:

$$0 \leq \lim_{t \searrow 0} \frac{d_A(x+tv)}{t} = \lim_{t \searrow 0} \frac{d_A(x+tv) - d_A(x)}{t} = \nabla d_A(x) \cdot v$$
$$\Rightarrow \forall v, \quad 0 \leq \pm \nabla d_A(x) \cdot v \quad \Rightarrow \nabla d_A(x) = 0.$$

Since ∇d_A exists almost everywhere, $\nabla d_A = 0$ almost everywhere in $\overline{A}$.

(vii) The function d_A is Lipschitzian and the function d_A^2 is locally Lipschitzian. By Rademacher's theorem they are both differentiable almost everywhere. The other properties follow from parts (iii), (iv), and (v).

By definition $\mathrm{Sk}\,(\partial A) = \{x \in \mathbf{R}^{\mathrm{N}} : \nabla d_{\partial A}^2(x) \nexists\} \subset \mathbf{R}^{\mathrm{N}} \backslash \partial A = \mathbf{R}^{\mathrm{N}} \backslash \overline{A} \cup \mathbf{R}^{\mathrm{N}} \backslash \overline{\complement A}$. If $x \in \mathrm{Sk}\,(\partial A) \cap \mathbf{R}^{\mathrm{N}} \backslash \overline{A}$, then $d_{\partial A}(x) = d_A(x)$, $\nabla d_A^2(x) \nexists$, and $x \in \mathrm{Sk}\,(A)$; if $x \in \mathrm{Sk}\,(\partial A) \cap \mathbf{R}^{\mathrm{N}} \backslash \overline{\complement A}$, then $d_{\partial A}(x) = d_{\complement A}(x)$, $\nabla d_{\complement A}^2(x) \nexists$, and $x \in \mathrm{Sk}\,(\complement A)$. Conversely, if $x \in \mathrm{Sk}\,(A) \subset \mathbf{R}^{\mathrm{N}} \backslash \overline{A}$, then $d_{\partial A}(x) = d_A(x)$, $\nabla d_{\partial A}^2(x) \nexists$, and $x \in \mathrm{Sk}\,(\partial A)$; if $x \in \mathrm{Sk}\,(\complement A) \subset \mathbf{R}^{\mathrm{N}} \backslash \overline{\complement A}$, then $d_{\partial A}(x) = d_{\complement A}(x)$, $\nabla d_{\partial A}^2(x) \nexists$, and $x \in \mathrm{Sk}\,(\partial A)$. Finally, $\mathrm{Sk}\,(\partial A) = \mathrm{Sk}\,(A) \cup \mathrm{Sk}\,(\complement A) \subset \partial\overline{A} \cup \partial\overline{\complement A} = \partial(\partial A)$.

The property that $\mathrm{Ck}\,(\partial A) = \mathrm{Ck}\,(A) \cup \mathrm{Ck}\,(\complement A)$ is a consequence of the following property for $x \in \partial A$: $\nabla d_{\partial A}(x)$ exists if and only if $\nabla d_A(x)$ and $\nabla d_{\complement A}(x)$ exist. Indeed, for $x \in \partial A$, $d_{\partial A}(x) = d_A(x) = d_{\complement A}(x) = 0$. From part (v), if $\nabla d_{\partial A}(x)$ exists, it is 0. But $\partial A \subset \overline{A}$ implies $d_{\partial A} \geq d_A$ and for all $v \in \mathbf{R}^{\mathrm{N}}$ and $t > 0$

$$0 \leq \frac{d_A(x+tv) - d_A(x)}{t} \leq \frac{d_{\partial A}(x+tv) - d_{\partial A}(x)}{t} \to 0 \quad \Rightarrow \nabla d_A(x) = 0$$

and similarly $\nabla d_{\complement A}(x) = 0$. Conversely, from part (v), if $\nabla d_A(x)$ and $\nabla d_{\complement A}(x)$ exist, they are 0. Moreover, $d_{\partial A}(x) = \max\{d_A(x), d_{\complement A}(x)\}$, where

$$0 \leq \frac{d_{\partial A}(x+tv) - d_{\partial A}(x)}{t} = \max\left\{\frac{d_A(x+tv) - d_A(x)}{t}, \frac{d_{\complement A}(x+tv) - d_{\complement A}(x)}{t}\right\}.$$

Since both $\nabla d_A(x) = 0$ and $\nabla d_{\complement A}(x) = 0$, then $\nabla d_{\partial A}(x) = 0$. Finally, it is easy to check that $\partial\overline{A} \cup \partial\overline{\complement A} = \partial(\partial A)$. The other properties for $\partial A = \varnothing$ are obvious.

(viii) From part (i), if $\nabla d_A(x)$ exists for $x \in \overline{A}$, it is equal to 0. Hence $\chi_{\overline{A}}(x) = 1 = 1 - |\nabla d_A(x)|$ on $\overline{A}\backslash \mathrm{Ck}\,(A)$. From the definition of $\mathrm{Sk}\,(A)$, $\nabla d_A(x)$ exists and $|\nabla d_A(x)| = 1$ for all points in $\mathrm{int}\,\complement\overline{A}\backslash \mathrm{Sk}\,(A)$. Hence from part (i), $\chi_{\overline{A}}(x) = 0 = 1 - |\nabla d_A(x)| = 0$ on $\complement\overline{A}\backslash \mathrm{Sk}\,(A)$. From this, we get the first line of identities in $\mathbf{R}^{\mathrm{N}} \backslash \mathrm{Sing}\,(\nabla d_A)$. Since $\mathrm{m}(\mathrm{Sing}\,(\nabla d_A)) = 0$, the above identities are satisfied almost everywhere in $\mathbf{R}^{\mathrm{N}}$. We get the same results with $\complement A$ in place of A.

(ix) If $x \in \overline{A}$, $\Pi_A(x) = \{x\}$ and there is nothing to prove. If $x \notin \overline{A}$, then $d_A(x) = |x - p| > 0$ and $p \neq x$ for all $p \in \Pi_A(x)$. First

$$d_A(x_\alpha) \leq |x_\alpha - p| = \alpha\, |x - p|.$$

If the inequality is strict, then there exists $p_\alpha \in \overline{A}$ such that $|x_\alpha - p_\alpha| = d_A(x_\alpha)$ and

$$|x - p_\alpha| \leq |x - x_\alpha| + |x_\alpha - p_\alpha| = (1-\alpha)|x - p| + d_A(x_\alpha) < |x - p| = d_A(x).$$

This contradicts the minimality of $d_A(x)$ with respect to A. Therefore,

$$d_A(x_\alpha) = \alpha\, |x - p| = |x_\alpha - p| \quad \Rightarrow p \in \Pi_A(x_\alpha).$$

Now for any $p_\alpha \in \Pi_A(x_\alpha)$, $p_\alpha \in \overline{A}$ and

$$|p_\alpha - x| \leq |p_\alpha - x_\alpha| + |x_\alpha - x| = \alpha\, |x - p| + (1-\alpha)\, |p - x| = |p - x| = d_A(x)$$

and $p_\alpha \in \Pi_A(x)$. Hence $\Pi_A(x_\alpha) \subset \Pi_A(x)$. □

4 $W^{1,p}$-Metric Topology and Characteristic Functions

4.1 Motivations and Main Properties

Since distance functions are locally Lipschitzian, they belong to $C^{0,1}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$ and hence to $W^{1,p}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$ for all $p \geq 1$. Thus the constructions of section 2.1 can be repeated with $W^{1,p}_{\mathrm{loc}}(D)$ in place of $C_{\mathrm{loc}}(D)$ to generate new $W^{1,p}$-metric topologies on the family $C_d(D)$. One big advantage is that the $W^{1,p}$-convergence of sequences will imply the L^p-convergence of the corresponding characteristic functions of the closure of the sets and hence the convergence of volumes (cf. Theorem 3.3 (viii)).

In the uniform Hausdorff topology, the convergence of volumes and perimeters is usually lost. In general, the measure of a closed set is only upper semicontinuous with respect to that topology. This includes Lebesgue and Hausdorff measures. The next example shows that the Hausdorff metric convergence is not sufficient to get the L^p-convergence of the characteristic functions of the closure of the corresponding sets in the sequence. The volume function is only upper semicontinuous with respect to the Hausdorff topology. The perimeters do not converge either as illustrated by the example of the staircase of Example 6.1 and Figure 5.7 in Chapter 5, where the volumes converge but not the perimeters.

Example 4.1.
Denote by $D =]-1,2[\times]-1,2[$ the open unit square in $\mathbf{R}^2$, $p \geq 1$ a real number, and for each $n \geq 1$ define the sequence of closed sets

$$A_n \stackrel{\text{def}}{=} \left\{(x_1,x_2) \in D : \frac{2k}{2n} \leq x_1 \leq \frac{2k+1}{2n}, 0 \leq k < n,\, 0 \leq x_2 \leq 1\right\}.$$

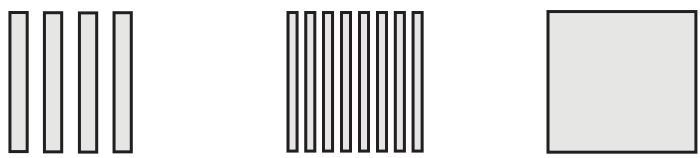


Figure 6.3. *Vertical stripes of Example* 4.1.

This defines n vertical stripes of equal width $1/2n$ each distant of $1/2n$ (cf. Figure 6.3). Let $S = [0,1]\times[0,1]$ be the closed unit square. Clearly, for all $n \geq 1$,

$$m(A_n) = \frac{1}{2}, \quad P_D(A_n) = 2n+1,$$
$$\forall x \in S, \quad d_{A_n}(x) \leq \frac{1}{4n}, \quad \|\nabla d_{A_n}\|_{L^p(D)} \geq 2^{-1/p},$$

where $m(A_n)$ is the surface and $P_D(A_n)$ the perimeter of A_n. Hence

$$d_{A_n} \to d_S \quad \text{in } C(\overline{D}), \quad m(S) = 1, \quad P_D(S) = 4.$$

But

$$m(\overline{A}_n) = m(A_n) = \frac{1}{2} \nrightarrow 1 = m(S) \quad \Rightarrow \chi_{A_n} \nrightarrow \chi_S \text{ in } L^p(D),$$
$$P_D(A_n) \nrightarrow P_D(S), \quad \chi_{A_n} \rightharpoonup \frac{1}{2}\chi_S \text{ in } L^p(D)\text{-weak}.$$

Since the characteristic functions do not converge, the sequence $\{\nabla d_{A_n}\}$ does not converge in $L^p(D)^N$ and $\{d_{A_n}\}$ does not converge in $W^{1,p}(D)$. $\square$

Theorem 4.1. *Let μ be a measure*[5] *and let $D \subset \mathbf{R}^N$ be bounded open such that $\mu(D) < \infty$.*

(i) *The map*

$$d_A \mapsto \mu(\overline{A}) : C_d(D) \to \mathbf{R} \tag{4.1}$$

is upper semicontinuous with respect to the topology of uniform convergence (Hausdorff topology on $C_d(D)$).

[5] A measure in the sense of L. C. Evans and R. F. Gariepy [**1**, Chap. 1]. It is called an *outer measure* in most texts.

(ii) *The map*

$$d_{\complement A} \mapsto \mu(\operatorname{int} A) : C_d^c(D) \to \mathbf{R} \tag{4.2}$$

is lower semicontinuous with respect to the topology of uniform convergence (Hausdorff complementary topology on $C_d^c(D)$).

Note that μ can be the Lebesgue measure but also any of the Hausdorff measures.

Proof. We prove (ii). The proof of (i) is similar and simpler. It is sufficient to work with open sets $\Omega = \operatorname{int} A = \complement\overline{\complement A}$. Let $\{\Omega_n\}$ be a sequence of open subsets of D such that $d_{\complement\Omega_n} \to d_{\complement\Omega}$ in $C(\overline{D})$ for some open $\Omega \subset D$. Given $\varepsilon > 0$, there exists N such that for all $n > N$

$$\|d_{\complement\Omega_n} - d_{\complement\Omega}\}_{C(\overline{D})} \le \varepsilon \quad \Rightarrow \complement\Omega_n \subset (\complement\Omega)_\varepsilon.$$

By definition of a measurable set

$$\begin{aligned} \mu(\Omega_n) = \mu(\Omega_n \cap D) = \mu(D) - \mu(D \cap \complement\Omega_n) &\ge \mu(D) - \mu(D \cap (\complement\Omega)_\varepsilon) \\ \Rightarrow \liminf_{n\to\infty} \mu(\Omega_n) &\ge \mu(D) - \mu(D \cap (\complement\Omega)_\varepsilon). \end{aligned}$$

But $D \cap (\complement\Omega)_\varepsilon$ is monotonically decreasing to $\cap_{\varepsilon>0} D \cap (\complement\Omega)_\varepsilon = D \cap \complement\Omega$ and hence

$$\liminf_{n\to\infty} \mu(\Omega_n) \ge \mu(D) - \lim_{\varepsilon\to 0} \mu(D \cap (\complement\Omega)_\varepsilon) = \mu(D) - \mu(D \cap \complement\Omega) = \mu(\Omega)$$

since $D \cap \Omega = \Omega$. □

Theorem 4.2. *Let D be an open* (resp., *bounded open*) *subset of $\mathbf{R}^N$.*

(i) *The topologies induced by $W^{1,p}_{\text{loc}}(D)$* (resp., *$W^{1,p}(D)$*) *on $C_d(D)$ and $C_d^c(D)$ are all equivalent for p, $1 \le p < \infty$.*

(ii) *$C_d(D)$ is closed in $W^{1,p}_{\text{loc}}(D)$* (resp., *$W^{1,p}(D)$*) *for p, $1 \le p < \infty$, and*

$$\rho_D([A_2],[A_1]) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|d_{A_2} - d_{A_1}\|_{W^{1,p}(B(0,n))}}{1 + \|d_{A_2} - d_{A_1}\|_{W^{1,p}(B(0,n))}}$$
$$(\text{resp., } \rho_D([A_2],[A_1]) \overset{\text{def}}{=} \|d_{A_2} - d_{A_1}\|_{W^{1,p}(D)})$$

defines a complete metric structure on $\mathcal{F}(D)$. For p, $1 \le p < \infty$, the map

$$d_A \mapsto \chi_{\overline{A}} = 1 - |\nabla d_A| \,:\, C_d(D) \subset W^{1,p}_{\text{loc}}(D) \to L^p_{\text{loc}}(D)$$

is "Lipschitz continuous": for all bounded open subsets K of D and nonempty subsets A_1 and A_2 of D

$$\|\chi_{\overline{A_2}} - \chi_{\overline{A_1}}\|_{L^p(K)} \le \|\nabla d_{A_2} - \nabla d_{A_1}\|_{L^p(K)} \le \|d_{A_2} - d_{A_1}\|_{W^{1,p}(K)}.$$

(iii) *$C_d^c(D)$ is closed in $W^{1,p}_{\mathrm{loc}}(D)$* (resp., $W_0^{1,p}(D)$) *for p, $1 \le p < \infty$, and*

$$\rho_D(\Omega_2, \Omega_1) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|d_{\complement\Omega_2} - d_{\complement\Omega_1}\|_{W^{1,p}(B(0,n))}}{1 + \|d_{\complement\Omega_2} - d_{\complement\Omega_1}\|_{W^{1,p}(B(0,n))}}$$

$$(\text{resp., } \rho_D(\Omega_2, \Omega_1) \stackrel{\text{def}}{=} \|d_{\complement\Omega_2} - d_{\complement\Omega_1}\|_{W^{1,p}(D)})$$

defines a complete metric structure on the family $\mathcal{G}(D)$ of open subsets of D. For p, $1 \le p < \infty$, the map

$$d_{\complement\Omega} \mapsto \chi_\Omega = |\nabla d_{\complement\Omega}| \,:\, C_d^c(D) \subset W^{1,p}_{\mathrm{loc}}(D) \to L^p_{\mathrm{loc}}(D)$$

is "Lipschitz continuous": for all bounded open subsets K of D and open subsets $\Omega_1 \neq \mathbf{R}^{\mathrm{N}}$ and $\Omega_2 \neq \mathbf{R}^{\mathrm{N}}$ of D

$$\|\chi_{\Omega_2} - \chi_{\Omega_1}\|_{L^p(K)} \le \|\nabla d_{\complement\Omega_2} - \nabla d_{\complement\Omega_1}\|_{L^p(K)} \le \|d_{\complement\Omega_2} - d_{\complement\Omega_1}\|_{W^{1,p}(K)}.$$

Proof. (i) The proof is similar to the proof of Theorem 2.3 in Chapter 5. It is sufficient to prove it for D bounded open. Since D is bounded it is contained in a sufficiently large ball of radius c. Therefore,

$$d_A(x) = \inf_{y \in A} |x - y| \le |x| + |y| \le 2c$$

since $y \in \overline{A} \subset \overline{D}$, and by Theorem 2.1 (vii)

$$|\nabla d_A(x)| \le 1 \text{ a.e. in } D.$$

For all $p > 1$ the injection of $W^{1,p}(D)$ into $W^{1,1}(D)$ is continuous since

$$\|d_A\|_{W^{1,1}(D)} \le \|d_A\|_{W^{1,p}(D)} \, \mathrm{m}(D)^{1/q}$$

with $1/p + 1/q = 1$. Conversely, we have the continuity in the other direction. For $p > 1$ and any d_A and d_B in $C_d(D)$,

$$\begin{aligned}
&\int_D |d_A - d_B|^p + |\nabla d_A - \nabla d_B|^p \, dx \\
&= \int_D |d_A - d_B| \, |d_A - d_B|^{p-1} + |\nabla d_A - \nabla d_B| \, |\nabla d_A - \nabla d_B|^{p-1} \, dx \\
&\le \max\{(2c)^{p-1}, 2^{p-1}\} \int_D |d_A - d_B| \, + |\nabla d_A - \nabla d_B| \, dx \\
&\quad \Rightarrow \; \|d_A - d_B\|^p_{W^{1,p}(D)} \le (2 \max\{c, 1\})^{p-1} \, \|d_A - d_B\|_{W^{1,1}(D)}.
\end{aligned}$$

Therefore, for any $\varepsilon > 0$, pick $\delta = \varepsilon^p / (2 \max\{c, 1\})^{p-1}$ and

$$\|d_A - d_B\|_{W^{1,1}(D)} \le \delta \quad \Rightarrow \|d_A - d_B\|_{W^{1,p}(D)} \le \varepsilon.$$

(ii) It is sufficient to prove it for D bounded open. Let $\{d_{A_n}\}$ be a Cauchy sequence in $C_d(D)$ which converges to some f in $W^{1,p}(D)$-strong. By Theorem 2.2 (ii),

$C_d(D)$ is compact in $C(\overline{D})$ and there exist a subsequence, still denoted by $\{d_{A_n}\}$, and $\varnothing \neq A \subset \overline{D}$ such that $d_{A_n} \to d_A$ in $C(\overline{D})$ and, a fortiori, in $L^p(D)$-strong since D is bounded. By uniqueness of the limit in $L^p(D)$-strong, $f = d_A$ and d_{A_n} converges to d_A in $W^{1,p}(D)$-strong. Therefore $C_d(D)$ is closed in $W^{1,p}(D)$. For the Lipschitz continuity, recall that the distance function d_A is differentiable almost everywhere in $\mathbf{R}^N$ for $A \neq \varnothing$. In view of Theorem 3.3 (viii) $\chi_{\overline{A}} = 1 - |\nabla d_A(x)|$ almost everywhere in $\mathbf{R}^N$. Given two nonempty subsets A_1 and A_2 of D

$$|\nabla d_{A_2}| \leq |\nabla d_{A_1}| + |\nabla d_{A_2} - \nabla d_{A_1}|$$
$$\Rightarrow \chi_{A_1} \leq \chi_{A_2} + |\nabla d_{A_2} - \nabla d_{A_1}| \quad \Rightarrow \int_D |\chi_{A_1} - \chi_{A_2}|^p \, dx \leq \|d_{A_2} - d_{A_1}\|^p_{W^{1,p}(D)}$$

for $1 \leq p < \infty$ and with the ess-sup norm for $p = \infty$.

(iii) Again it is sufficient to prove the result for D bounded. In that case $\complement\Omega \neq \varnothing$ for all open subsets Ω of D. Let $\{\Omega_n\}$ be a sequence of open subsets of D such that $\{d_{\complement\Omega_n}\}$ is Cauchy in $W^{1,p}(D)$. By assumption $\Omega_n \subset D$, $\complement\Omega_n \supset \complement D$,

$$\forall n \geq 1, \quad d_{\complement\Omega_n} = 0 \text{ in } \complement D \quad \Rightarrow d_{\complement\Omega_n} \in W_0^{1,p}(D),$$

and the Cauchy sequence converges to some $f \in W_0^{1,p}(D)$. By Theorem 2.4 (ii), $C_d^c(D)$ is compact in $C(\overline{D})$ and there exist a subsequence, still denoted by $\{d_{\complement\Omega_n}\}$, and an open set $\Omega \subset D$ such that $d_{\complement\Omega_n} \to d_{\complement\Omega}$ in $C(\overline{D})$ and hence in $L^p(D)$-strong since D is bounded. By uniqueness of the limit in $L^p(D)$, $f = d_{\complement\Omega}$ and the Cauchy sequence $d_{\complement\Omega_n}$ converges to $d_{\complement\Omega}$ in $W_0^{1,p}(D)$. The other part of the proof is similar to that of part (ii). □

4.2 Weak $W^{1,p}$-Topology

We have the following general result.

Theorem 4.3. *Let D be a bounded open domain in $\mathbf{R}^N$.*

(i) *If $\{d_{A_n}\}$ weakly converges in $W^{1,p}(D)$ for some p, $1 \leq p < \infty$, then it weakly converges in $W^{1,p}(D)$ for all p, $1 \leq p < \infty$.*

(ii) *If $\{d_{A_n}\}$ converges in $C(\overline{D})$, then it weakly converges in $W^{1,p}(D)$ for all p, $1 \leq p < \infty$. Conversely if $\{d_{A_n}\}$ weakly converges in $W^{1,p}(D)$ for some p, $1 \leq p < \infty$, it converges in $C(\overline{D})$.*

(iii) *$C_d(D)$ is compact in $W^{1,p}(D)$*-weak *for all p, $1 \leq p < \infty$.*[6]

(iv) *Parts* (i) to (iii) *also apply to $C_d^c(D)$.*

[6] In a metric space the compactness is equivalent to the sequential compactness. For the weak topology we use the fact that if E is a separable normed space, then, in its topological dual E', any closed ball is a compact metrizable space for the weak topology. Since $C_d(D)$ is a bounded subset of the normed reflexive separable Banach space $W^{1,p}(D)$, $1 \leq p < \infty$, the weak compactness of $C_d(D)$ coincides with the weak sequential compactness (cf. J. Dieudonné [**1**, Vol. II, Chap. XII, sect. 12.15.9, p. 75]).

Proof. (i) For D bounded there exists a constant $c > 0$ such that for all $d_A \in C_d(D)$

$$d_A(x) \le c \text{ and } |\nabla d_A(x)| \le 1 \text{ a.e. in } D.$$

If $\{d_{A_n}\}$ weakly converges in $W^{1,p}(D)$, then

$$\{d_{A_n}\} \text{ weakly converges in } L^p(D), \quad \{\nabla d_{A_n}\} \text{ weakly converges in } L^p(D)^N.$$

By Lemma 3.1 (iii) in Chapter 5 both sequences weakly converge for all $p \ge 1$, and hence $\{d_{A_n}\}$ weakly converges in $W^{1,p}(D)$ for all $p \ge 1$.

(ii) If $\{d_{A_n}\}$ converges in $C(\overline{D})$, then by Theorem 2.2 (i) there exists $d_A \in C_b(D)$ such that $d_{A_n} \to d_A$ in $C(\overline{D})$ and hence in $L^p(D)$. So for all $\varphi \in \mathcal{D}(D)^N$,

$$\int_D \nabla d_{A_n} \cdot \varphi \, dx = -\int_D d_{A_n} \operatorname{div} \varphi \, dx \to -\int_D d_A \operatorname{div} \varphi \, dx = \int_D \nabla d_A \cdot \varphi \, dx.$$

By density of $\mathcal{D}(D)$ in $L^2(D)$, $\nabla d_{A_n} \to \nabla d_A$ in $L^2(D)^N$-weak and hence $d_{A_n} \to d_A$ in $W^{1,2}(D)$-weak. From part (i) it converges in $W^{1,p}(D)$-weak for all p, $1 \le p < \infty$. Conversely, the weakly convergent sequence converges to some f in $W^{1,p}(D)$. By compactness of $C_b(D)$ there exist a subsequence, still indexed by n, and d_A such that $d_{A_n} \to d_A$ in $C(\overline{D})$ and hence in $W^{1,p}(D)$-weak. By uniqueness of the limit $d_A = f$. Therefore, all convergent subsequences in $C(\overline{D})$ converge to the same limit, so the whole sequence converges in $C(\overline{D})$.

(iii) Consider an arbitrary sequence $\{d_{A_n}\}$ in $C_d(D)$. From Theorem 2.2 (ii) $C_d(D)$ is compact and there exist a subsequence $\{d_{A_{n_k}}\}$ and $d_A \in C_d(D)$ such that $d_{A_{n_k}} \to d_A$ in $C(\overline{D})$. From part (ii) the subsequence weakly converges in $W^{1,p}(D)$ and hence $C_d(D)$ is compact in $W^{1,p}(D)$-weak. □

Theorem 4.4. *Let D be a closed subset of $\mathbf{R}^N$ and $\{A_n\}$, $\varnothing \ne A_n \subset D$, be a sequence such that $d_{A_n} \to d_A$ in $C_{\text{loc}}(D)$ for some A, $\varnothing \ne A \subset D$. Then*

$$\forall x \in \mathbf{R}^N \setminus \overline{A}, \; \lim_{n\to\infty} \chi_{\overline{A_n}}(x) = \chi_{\overline{A}}(x) = 0 \quad \textit{and} \quad \forall x \in \mathbf{R}^N, \; \limsup_{n\to\infty} \chi_{\overline{A_n}}(x) \le \chi_{\overline{A}}(x),$$

and for any compact subset K of D

$$\limsup_{n\to\infty} \int_K \chi_{\overline{A_n}} \, dx \le \int_K \chi_{\overline{A}} \, dx.$$

Corollary 1. *Let $D \ne \varnothing$ be a bounded open subset of $\mathbf{R}^N$ and $\{\Omega_n\}$, $\varnothing \ne \Omega_n \subset D$, be a sequence of open subsets of D converging to an open subset Ω, $\varnothing \ne \Omega \subset D$, of D in the Hausdorff complementary topology: $d_{\complement\Omega_n} \to d_{\complement\Omega}$ in $C(\bar{D})$. Then*

$$\forall x \in \Omega, \; \lim_{n\to\infty} \chi_{\Omega_n}(x) = \chi_\Omega(x) = 1, \quad \textit{and} \quad \forall x \in \mathbf{R}^N, \; \liminf_{n\to\infty} \chi_{\Omega_n}(x) \ge \chi_\Omega(x),$$

and

$$\liminf_{n\to\infty} \int_D \chi_{\Omega_n} \, dx \ge \int_D \chi_\Omega \, dx.$$

Proof of Theorem 4.4. For all $x \notin \overline{A}$, $d_A(x) > 0$ and

$$\begin{aligned}
&\exists N_x > 0, \forall n \geq N_x, \quad \|d_A - d_{A_n}\| \leq d_A(x)/2 \\
&\Rightarrow \exists N_x > 0, \forall n \geq N_x, \quad d_{A_n}(x) \geq d_A(x)/2 > 0 \\
&\Rightarrow \exists N_x > 0, \forall n \geq N_x, \quad x \notin \overline{A}_n \text{ and } \chi_{\overline{A}_n}(x) = 0.
\end{aligned}$$

So for all $x \notin \overline{A}$

$$\lim_{n\to\infty} \chi_{\overline{A}_n}(x) = \chi_{\overline{A}}(x) = 0.$$

Finally, for all $x \in \overline{A}$ and all $n \geq 1$

$$\chi_{\overline{A}_n}(x) \leq 1 = \chi_{\overline{A}}(x)$$

and the result follows trivially by taking the limsup of each term.

We conclude that

$$\limsup_{n\to\infty} \chi_{\overline{A}_n} \leq \chi_{\overline{A}},$$

and by using the analogue of Fatou's lemma for the limsup we get for all compact subsets K of D

$$\limsup_{n\to\infty} \int_K \chi_{\overline{A}_n}\, dx \leq \int_K \limsup_{n\to\infty} \chi_{\overline{A}_n}\, dx \leq \int_K \chi_{\overline{A}}\, dx. \qquad \square$$

In order to get the L^p-convergence of the characteristic functions of the closure of the sets in the sequence, we need the L^p-convergence of the gradients of the distance functions which are related to the characteristic functions of the closure of the sets (cf. Theorem 3.3 (viii)).

We have seen in Example 4.1 that the weak convergence of the characteristic functions is not sufficient to obtain the strong convergence of the sequence $\{d_{A_n}\}$ to d_A in $W^{1,2}(D)$. However, if we assume that $\{\chi_{\overline{A}_n}\}$ is strongly convergent, it converges to the characteristic function χ_B of some measurable subset B of D. Is this sufficient to conclude that $\chi_B = \chi_{\overline{A}}$? The answer is negative. The counterexample is provided by Example 6.2 in Chapter 5, where

$$d_{A_n} \rightharpoonup d_D \text{ in } W^{1,2}(D)\text{-weak} \quad \text{and} \quad \chi_{A_n} \to \chi_B \text{ in } L^2(D)\text{-strong}$$

for some $B \subset D$ such that

$$m(D) = \pi > \frac{\pi}{3} \geq \mathrm{m}(B) \quad \Rightarrow \chi_D \neq \chi_B.$$

Remark 4.1.

In view of part (ii) of Theorem 4.2 an optimization problem with respect to the characteristic functions χ_Ω of open sets Ω in D for which we have the continuity with respect to χ_Ω can be transformed into an optimization problem with respect to $d_{\complement\Omega}$ in $W^{1,1}(D)$ since $\chi_\Omega = |\nabla d_{\complement\Omega}|$. This would apply to the transmission problem (4.3)–(4.6) in section 4.1 of Chapter 5. $\square$

5 Sets of Bounded and Locally Bounded Curvature

From Theorem 6.1 in Chapter 5 and Theorem 3.2, it is readily seen that

$$\nabla d_A \in \mathrm{BV}_{\mathrm{loc}}\left(\mathbf{R}^{\mathrm{N}}\right)^N \quad \iff \quad \forall x \in \partial\overline{A},\, \exists \rho > 0 \text{ such that } \nabla d_A \in \mathrm{BV}(B(x,\rho))^N.$$

The global properties of ∇d_A depend only on its local properties around $\partial\overline{A}$. This local property is fairly general and is verified for sets with corners, that is, sets with discontinuities in the orientation of the normal along the boundary.

This suggests introducing a new family of sets and motivates the study of their general properties, We shall later see that convex sets have this property and later in Chapter 7 that the larger family of *sets with positive reach* (cf. section 6) also does. One remarkable property is that such sets are Caccioppoli sets (cf. section 6 in Chapter 5). We give the definitions for an arbitrary set A, but they can be specialized to $\complement A$ and ∂A (associated with the distance functions $d_{\complement A}$ and $d_{\partial A}$) that would require a finer terminology such as *exterior, interior*, or *boundary curvature* to distinguish them.

Definition 5.1. (i) Given a bounded open holdall $D \subset \mathbf{R}^{\mathrm{N}}$ and A, $\varnothing \neq A \subset \overline{D}$, the set A is said to be of *bounded curvature* with respect to D if

$$\nabla d_A \in \mathrm{BV}(D)^N. \tag{5.1}$$

The family of sets with bounded curvature will be denoted by

$$\mathrm{BC}_d(D) \stackrel{\text{def}}{=} \left\{ d_A \in C_d(D) \,:\, \nabla d_A \in \mathrm{BV}(D)^N \right\}.$$

(ii) A set A, $A \neq \varnothing$, in $\mathbf{R}^{\mathrm{N}}$ is said to be of *locally bounded curvature* if

$$\nabla d_A \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})^N.$$

The family of sets with locally bounded curvature will be denoted by

$$\mathrm{BC}_d \stackrel{\text{def}}{=} \left\{ d_A \in C_d(\mathbf{R}^{\mathrm{N}}) \,:\, \nabla d_A \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})^N \right\}. \qquad \square$$

As in Theorem 6.2 of Chapter 5 for Caccioppoli sets, it is sufficient to satisfy the BV property in a neighborhood of each point of the boundary $\partial\overline{A}$.

Theorem 5.1. *A, $\varnothing \neq A \subset \mathbf{R}^{\mathrm{N}}$, is of locally bounded curvature if and only if*

$$\forall x \in \partial\overline{A},\, \exists \rho > 0 \text{ such that } \nabla d_A \in \mathrm{BV}(B(x,\rho))^N, \tag{5.2}$$

where $B(x,\rho)$ is the open ball of radius $\rho > 0$ in x.

Proof. From Theorem 6.1 and Definition 6.1 in Chapter 5, a function belongs to $\mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$ if and only if for each $x \in \mathbf{R}^{\mathrm{N}}$ it belongs to $\mathrm{BV}(B(x,\rho))$ for some $\rho > 0$. The equivalence now follows from Theorem 3.2. $\square$

Recall that the relaxation of the perimeter of a set was obtained from the norm of the gradient of its characteristic function for Caccioppoli sets. For distance functions $\chi_{\overline{A}} = 1 - |\nabla d_A|$ almost everywhere and the gradient of d_A also has a jump discontinuity along the boundary $\partial\overline{A}$ of magnitude at most 1. So it is not too surprising that a set $\overline{A}$ with bounded curvature is a Caccioppoli set.

Theorem 5.2.

(i) *Let $D \subset \mathbf{R}^{\mathrm{N}}$ be a bounded open holdall and let A, $\varnothing \neq A \subset \overline{D}$. If $\nabla d_A \in \mathrm{BV}(D)^N$, then*

$$\|\nabla\chi_{\overline{A}}\|_{M^1(D)} \leq 2\,\|D^2 d_A\|_{M^1(D)}, \quad \chi_{\overline{A}} \in \mathrm{BV}(D),$$

and $\overline{A}$ has finite perimeter with respect to D.

(ii) *For any subset A of $\mathbf{R}^{\mathrm{N}}$, $A \neq \varnothing$,*

$$\nabla d_A \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})^N \quad \Rightarrow \quad \chi_{\overline{A}} \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$$

and $\overline{A}$ has locally finite perimeter.

Proof. Given ∇d_A in $\mathrm{BV}(D)^N$, there exists a sequence $\{u_k\}$ in $C^\infty(D)^N$ such that

$$u_k \to \nabla d_A \text{ in } L^1(D)^N \quad \text{and} \quad \|Du_k\|_{M^1(D)} \to \|D^2 d_A\|_{M^1(D)}$$

as k goes to infinity, and since $|\nabla d_A(x)| \leq 1$, this sequence can be chosen in such a way that $|u_k(x)| \leq 1$ for all $k \geq 1$. This follows from the use of mollifiers (cf. E. Giusti [**1**, Thm. 1.17, p. 15]). For all V in $\mathcal{D}(D)^N$

$$-\int_D \chi_{\overline{A}} \mathrm{div}\, V\, dx = \int_D (|\nabla d_A|^2 - 1)\mathrm{div}\, V\, dx = \int_D |\nabla d_A|^2 \mathrm{div}\, V\, dx.$$

For each u_k

$$\int_D |u_k|^2 \mathrm{div}\, V\, dx = -2\int_D {}^*[Du_k]u_k \cdot V\, dx = -2\int_D u_k \cdot [Du_k]V\, dx,$$

where ${}^*Du_k$ is the transpose of the Jacobian matrix Du_k and

$$\begin{aligned}\left|\int_D |u_k|^2 \mathrm{div}\, V\, dx\right| &\leq 2\int_D |u_k|\,|Du_k|\,|V|\, dx \\ &\leq 2\,\|Du_k\|_{L^1}\|V\|_{C(\overline{D})} \leq 2\,\|Du_k\|_{M^1}\|V\|_{C(\overline{D})}\end{aligned}$$

since for $W^{1,1}(D)$-functions $\|\nabla f\|_{L^1(D)^N} = \|\nabla f\|_{M^1(D)^N}$. Therefore,

$$\left|\int_D \chi_{\overline{A}} \mathrm{div}\, V\, dx\right| = \left|\int_D |\nabla d_A|^2 \mathrm{div}\, V\, dx\right| \leq 2\|D^2 d_A\|_{M^1}\|V\|_{C(\overline{D})}$$

as k goes to infinity, where $D^2 d_A$ is the Hessian matrix of second-order partial derivatives of d_A. Finally, $\nabla\chi_{\overline{A}} \in M^1(D)^N$. □

5.1 Examples

It is useful to consider the following three simple illustrative examples (cf. Figure 6.4). By convexity, they are all of locally bounded curvature.

Example 5.1 (Half-plane in $\mathbf{R}^2$)**.**
Consider the domain

$$A = \{(x_1, x_2) : x_1 \leq 0\}, \quad \partial A = \{(x_1, x_2) : x_1 = 0\}.$$

It is readily seen that

$$d_A(x_1, x_2) = \max\{x_1, 0\}, \quad \nabla d_A(x_1, x_2) = \begin{cases} (0,0), & x_1 < 0, \\ (1,0), & x_1 > 0, \end{cases}$$
$$\langle \partial_{11} d_A, \varphi \rangle = \int_{\partial A} \varphi \, dH_1, \quad \partial_{12} d_A = \partial_{21} d_A = \partial_{22} d_A = 0,$$
$$\langle \Delta d_A, \varphi \rangle = \int_{\partial A} \varphi \, dH_1 \quad \Rightarrow \quad \Delta d_A = H_1.$$

Thus Δd_A is the one-dimensional Hausdorff measure of ∂A. $\square$

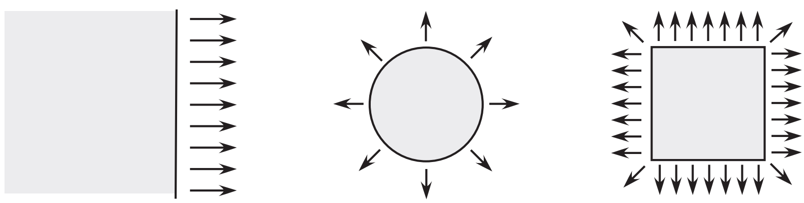


Figure 6.4. ∇d_A *for Examples* 5.1*,* 5.2*, and* 5.3*.*

Example 5.2 (Ball of radius $R > 0$ in $\mathbf{R}^2$)**.**
Consider the domain

$$A = \{x \in \mathbf{R}^2 : |x| \leq R\}, \quad \partial A = \{x \in \mathbf{R}^2 : |x| = R\}.$$

Clearly

$$d_A(x) = \max\{0, |x| - R\}, \quad \nabla d_A(x) = \begin{cases} x/|x|, & |x| > R, \\ (0,0), & |x| < R, \end{cases}$$
$$\langle \partial_{11} d_A, \varphi \rangle = \int_0^{2\pi} R \cos^2(\theta) \varphi \, d\theta + \int_{\complement A} \frac{x_2^2}{(x_1^2 + x_2^2)^{3/2}} \varphi \, dx,$$
$$\langle \partial_{22} d_A, \varphi \rangle = \int_0^{2\pi} R \sin^2(\theta) \varphi \, d\theta + \int_{\complement A} \frac{x_1^2}{(x_1^2 + x_2^2)^{3/2}} \varphi \, dx,$$
$$\langle \partial_{12} d_A, \varphi \rangle = \langle \partial_{21} d_A, \varphi \rangle = \int_0^{2\pi} R \cos(\theta) \sin(\theta) \varphi \, d\theta + \int_{\complement A} \frac{x_2 x_1}{(x_1^2 + x_2^2)^{3/2}} \varphi \, dx,$$
$$\langle \Delta d_A, \varphi \rangle = \int_{\partial A} \varphi \, dH_1 + \int_{\complement A} \frac{1}{(x_1^2 + x_2^2)^{1/2}} \varphi \, dx,$$

where H_1 is the one-dimensional Hausdorff measure. Δd_A contains the one-dimensional Hausdorff measure of the boundary ∂A plus a term that corresponds to the volume integral of the mean curvature over the level sets of d_A in $\complement A$. $\square$

Example 5.3 (Unit square in $\mathbf{R}^2$).
Consider the domain $A = \{x = (x_1, x_2) : |x_1| \le 1, |x_2| \le 1\}$. Since A is symmetrical with respect to both axes, it is sufficient to specify d_A in the first quadrant. We use the notation Q_1, Q_2, Q_3, and Q_4 for the four quadrants in the counterclockwise order and c_1, c_2, c_3, and c_4 for the four corners of the square in the same order. We also divide the plane into three regions:

$$\begin{aligned} D_1 &= \{(x_1, x_2) : |x_2| \le \min\{1, |x_1|\}\}, \\ D_2 &= \{(x_1, x_2) : |x_1| \le \min\{1, |x_2|\}\}, \\ D_3 &= \{(x_1, x_2) : |x_1| \ge 1 \text{ and } |x_2| \ge 1\}. \end{aligned}$$

Hence

$$d_A(x) = \begin{cases} \min\{x_2 - 1, 0\}, & x \in D_2 \cap Q_1, \\ |x - c_1|, & x \in D_3 \cap Q_1, \\ \min\{x_1 - 1, 0\}, & x \in D_1 \cap Q_1, \end{cases}$$

$$\nabla d_A(x) = \begin{cases} (0,1), & x \in D_2 \cap Q_1 \text{ and } x_2 > 1, \\ \frac{x - c_1}{|x - c_1|}, & x \in D_3 \cap Q_1, \\ (1,0), & x \in D_1 \cap Q_1 \text{ and } x_1 > 1, \\ (0,0), & x \in Q_1,\ x_1 < 1 \text{ and } x_2 < 1, \end{cases}$$

$$\begin{aligned} \langle \partial_{11} d_A, \varphi \rangle &= \sum_{i=1}^{4} \int_{D_3 \cap Q_i} \frac{(x_2 - c_{i,2})^2}{|x - c_i|^2} \varphi \, dx + \int_{\partial A \cap Q_i \cap D_1} \varphi \, dH_1, \\ \langle \partial_{22} d_A, \varphi \rangle & \sum_{i=1}^{4} \int_{D_3 \cap Q_i} \frac{(x_1 - c_{i,1})^2}{|x - c_i|^2} \varphi \, dx + \int_{\partial A \cap Q_i \cap D_2} \varphi \, dH_1, \\ \langle \partial_{12} d_A, \varphi \rangle = \langle \partial_{21} d_A, \varphi \rangle &= \sum_{i=1}^{4} \int_{D_3 \cap Q_i} \frac{(x_2 - c_{i,2})(x_1 - c_{i,1})}{|x - c_i|^2} \varphi \, dx, \\ \langle \Delta d_A, \varphi \rangle &= \sum_{i=1}^{4} \int_{D_3 \cap Q_i} \frac{1}{|x - c_i|} \varphi \, dx + \int_{\partial A} \varphi \, dH_1. \end{aligned}$$

Notice that the structure of the Laplacian is similar to that observed in the previous examples. $\square$

The norm $\|D^2 d_A\|_{M^1(U_h(A))}$ is decreasing as h goes to zero. The limit is particularly interesting since it singles out the behavior of the singular part of the Hessian matrix in a shrinking neighborhood of the boundary ∂A.

Example 5.4.
Let $N = 2$. For the finite square and the ball of finite radius,

$$\lim_{h \searrow 0} \|\Delta d_A\|_{M^1(U_h(\partial A))} = H_1(\partial A),$$

where H_1 is the one-dimensional Hausdorff measure. □

6 Reach and Federer's Sets of Positive Reach

The material in this section is mostly taken from the pioneering 1959 work of H. Federer [**3**, sect. 4], where he introduced the notion of *reach* and the *sets of positive reach*.

6.1 Definitions and Main Properties

By definition, $\overline{A} \cap \mathrm{Sk}\,(A) = \varnothing$. For $a \in \mathrm{int}\,\overline{A}$, there exists $r > 0$ such that $B_r(a) \subset \overline{A}$. Thus $\mathrm{Sk}\,(A) \cap B_r(a) = \varnothing$ and

$$\sup\{r > 0 : \mathrm{Sk}\,(A) \cap B_r(a) = \varnothing\} > 0. \tag{6.1}$$

If $a \in \partial\overline{A}$, then either $a \notin \overline{\mathrm{Sk}\,(A)}$ or $a \in \overline{\mathrm{Sk}\,(A)}$. If $a \notin \overline{\mathrm{Sk}\,(A)}$, there exists $r > 0$ such that $B_r(a) \cap \overline{\mathrm{Sk}\,(A)} = \varnothing$. Thus $B_r(a) \cap \mathrm{Sk}\,(A) = \varnothing$ and (6.1) is verified. If $a \in \overline{\mathrm{Sk}\,(A)}$, then for all $r > 0$, $B_r(a) \cap \mathrm{Sk}\,(A) \neq \varnothing$. This leads to the following definition.

Definition 6.1.
Given A, $\varnothing \neq A \subset \mathbf{R}^N$, and a point $a \in \overline{A}$

$$\mathrm{reach}\,(A, a) \overset{\text{def}}{=} \begin{cases} 0, & \text{if } a \in \partial\overline{A} \cap \overline{\mathrm{Sk}\,(A)}, \\ \sup\{r > 0 : \mathrm{Sk}\,(A) \cap B_r(a) = \varnothing\}, & \text{otherwise,} \end{cases}$$

$$\mathrm{reach}\,(A) \overset{\text{def}}{=} \inf\{\mathrm{reach}\,(A, a) : a \in A\}.$$

The set A is said to have *positive reach* if $\mathrm{reach}\,(A) > 0$. □

Remark 6.1.
(1) Since $\mathrm{Sk}\,(\overline{A}) = \mathrm{Sk}\,(A)$, $\mathrm{reach}\,(\overline{A}) = \mathrm{reach}\,(A)$.

(2) For any set A, $\mathrm{Sk}\,(A) \subset \mathrm{Sk}\,(\partial A)$ and

$$\forall a \in \partial A, \quad 0 \le \mathrm{reach}\,(\partial A, a) \le \mathrm{reach}\,(A, a) \le +\infty.$$

(3) For all $a \notin \partial\overline{A} \cap \overline{\mathrm{Sk}\,(A)}$, $\mathrm{reach}\,(A, a) > 0$. □

Lemma 6.1. *Let $\varnothing \neq A \subset \mathbf{R}^N$. The following statements are equivalent:*

(i) *There exists $a \in \overline{A}$ such that* $\mathrm{reach}\,(A, a) = +\infty$.

(ii) $\mathrm{Sk}\,(A) = \varnothing$.

(iii) *For all* $a \in \overline{A}$, $\operatorname{reach}(A, a) = +\infty$.

(iv) $\operatorname{reach}(A) = +\infty$.

Proof. (i) $\Rightarrow$ (ii) If there exists $a \in \overline{A}$ such that $\operatorname{reach}(A, a) = +\infty$, then $\mathbf{R}^{\mathrm{N}} = \cup_{r>0} B_r(a) \subset \mathbf{R}^{\mathrm{N}} \backslash \operatorname{Sk}(A)$ and $\operatorname{Sk}(A) = \varnothing$. (ii) $\Rightarrow$ (iii) If $\operatorname{Sk}(A) = \varnothing$, for all $a \in \overline{A}$ $\operatorname{reach}(A, a) = +\infty$. (iii) $\Rightarrow$ (iv) If for all $a \in \overline{A}$ $\operatorname{reach}(A, a) = +\infty$, then $\operatorname{reach}(A) = \inf_{a \in \overline{A}} \operatorname{reach}(A, a) = +\infty$. (iv) $\Rightarrow$ (i) is obvious. $\square$

Remark 6.2.
We shall see later that all nonempty convex sets A have positive reach and that $\operatorname{reach}(A) = +\infty$. Bounded domains and submanifolds of class C^2 also have positive reach. $\square$

Theorem 6.1. *Let* A, $\varnothing \neq A \subset \mathbf{R}^{\mathrm{N}}$, *be such that* $\operatorname{Sk}(A) \neq \varnothing$.

(i) *The function* $a \mapsto \operatorname{reach}(A, a) : \overline{A} \to \mathbf{R}$ *is continuous.*

(ii) *If* $\operatorname{reach}(A) > 0$, *then for all* h, $0 < h < \operatorname{reach}(A)$, $U_h(A) \cap \operatorname{Sk}(A) = \varnothing$. *Conversely, if there exists* $h > 0$ *such that* $U_h(A) \cap \operatorname{Sk}(A) = \varnothing$, $\operatorname{reach}(A) \geq h$, *and* A *has positive reach.*

(iii) *If* $\operatorname{reach}(A) > 0$, *then* $\overline{A} \cap \overline{\operatorname{Sk}(A)} = \varnothing$. *If* $\overline{A} \cap \overline{\operatorname{Sk}(A)} = \varnothing$ *and* A *is bounded, then* $\operatorname{reach}(A) > 0$.

Remark 6.3.
When A is unbounded and $\overline{A} \cap \overline{\operatorname{Sk}(A)} = \varnothing$, we can possibly have $\operatorname{reach}(A) = 0$ even if for all $a \in \overline{A}$, $\operatorname{reach}(A, a) > 0$ as readily seen from the following example:

$$A \overset{\text{def}}{=} \{(x_1, x_2) : |x_2| \geq e^{x_1}\}, \text{ where } \operatorname{Sk}(A) = \{(x_1, 0) : x_1 \in \mathbf{R}\}. \qquad \square$$

Proof. (i) We prove that the function is both lower and upper semicontinuous. For h, $\operatorname{reach}(A, a) > h$, there exists r, $h < r < \operatorname{reach}(A, a)$, such that $B_r(a) \cap \operatorname{Sk}(A) = \varnothing$. Let $a' \in B_\delta(a)$, $\delta = (r - h)/2 > 0$. Then $r - \delta = (r + h)/2 > h$ and for all $a' \in B_\delta(a) \cap \overline{A}$, $B_{r-\delta}(a') \subset B_r(a)$. Thus $B_{r-\delta}(a') \cap \operatorname{Sk}(A) \subset B_r(a) \cap \operatorname{Sk}(A) = \varnothing$, $\operatorname{reach}(A, a') \geq r - \delta > h$, and $\operatorname{reach}(A, a)$ is lower semicontinuous. Similarly, for k, $\operatorname{reach}(A, a) < k$, there exists r, $\operatorname{reach}(A, a) > r > k$, such that $B_r(a) \cap \operatorname{Sk}(A) \neq \varnothing$. Let $a' \in B_\delta(a)$, $\delta = (k - r)/2 > 0$. Then $r + \delta = (r + k)/2 < k$ and for all $a' \in B_\delta(a) \cap \overline{A}$, $B_r(a) \subset B_{r+\delta}(a')$. Thus $\varnothing \neq B_r(a) \cap \operatorname{Sk}(A) \subset B_{r+\delta}(a') \cap \operatorname{Sk}(A)$, $\operatorname{reach}(A, a') \leq r + \delta < k$, and $\operatorname{reach}(A, a)$ is upper semicontinuous.

(ii) By definition of $\operatorname{reach}(A)$ as an infimum. Conversely, for all $0 < r < h$ and all $x \in \overline{A}$, $B_r(x) \cap \operatorname{Sk}(x) = \varnothing$, and $\operatorname{reach}(A, x) \geq r$. Since this is true for all $0 < r < h$, $\operatorname{reach}(A, x) \geq h$ and $\operatorname{reach}(A) = \inf_{x \in \overline{A}} \operatorname{reach}(A, x) \geq h > 0$.

(iii) By separation of a closed set and a compact set. $\square$

Theorem 6.2. *Let* $\varnothing \neq A \subset \mathbf{R}^{\mathrm{N}}$.

(i) *Associate with* $a \in \overline{A}$ *the sets*

$$P(a) \overset{\text{def}}{=} \left\{v \in \mathbf{R}^{\mathrm{N}} : \Pi_A(a + v) = \{a\}\right\}, \quad Q(a) \overset{\text{def}}{=} \left\{v \in \mathbf{R}^{\mathrm{N}} : d_A(a + v) = |v|\right\}.$$

Then $P(a)$ *and* $Q(a)$ *are convex and* $P(a) \subset Q(a) \subset (-T_a A)^*$.

(ii) *Given* $a \in \overline{A}$ *and* $v \in \mathbf{R}^{\mathrm{N}}$, *assume that*

$$0 < r(a,v) \stackrel{\text{def}}{=} \sup\{t > 0 : \Pi_A(a+tv) = \{a\}\}.$$

Then for all t, $0 \le t < r(a,v)$, $\Pi_A(a+tv) = \{a\}$ *and* $d_A(a+tv) = t\,|v|$. *Moreover, if* $r(a,v) < +\infty$, *then* $a + r(a,v)\,v \in \overline{\mathrm{Sk}\,(A)}$.

(iii) *If there exists* h *such that* $0 < h \le \mathrm{reach}\,(A)$, *then for all* $x \in U_h(A)\backslash\overline{A}$ *and* $0 < t < h$, $\Pi_A(x(t)) = \{p_A(x)\}$ *and* $d_A(x(t)) = t$, *where*

$$x(t) \stackrel{\text{def}}{=} p_A(x) + t\,\nabla d_A(x) = p_A(x) + t\,\frac{x - p_A(x)}{d_A(x)}. \tag{6.2}$$

(iv) *Given* $b \in \overline{A}$, $x \in \mathbf{R}^{\mathrm{N}}\backslash\overline{\mathrm{Sk}\,(A)}$, $a = p_A(x)$, *such that* $\mathrm{reach}\,(A,a) > 0$, *then*

$$(x-a)\cdot(a-b) \ge -\frac{|a-b|^2|x-a|}{2\,\mathrm{reach}\,(A,a)}. \tag{6.3}$$

(v) *Given* $0 < r < q < \infty$ *and*

$$x \in \mathbf{R}^{\mathrm{N}}\backslash\overline{\mathrm{Sk}\,(A)} \textit{ such that } d_A(x) \le r \textit{ and } \mathrm{reach}\,(A, p_A(x)) \ge q,$$
$$y \in \mathbf{R}^{\mathrm{N}}\backslash\overline{\mathrm{Sk}\,(A)} \textit{ such that } d_A(y) \le r \textit{ and } \mathrm{reach}\,(A, p_A(y)) \ge q,$$

then

$$|p_A(y) - p_A(x)| \le \frac{q}{q-r}\,|y-x|. \tag{6.4}$$

(vi) *If* $0 < \mathrm{reach}\,(A) < +\infty$, *then for all* h, $0 < h < \mathrm{reach}\,(A)$,

$$|p_A(y) - p_A(x)| \le \frac{\mathrm{reach}\,(A)}{\mathrm{reach}\,(A) - h}\,|y-x|, \quad \forall x,y \in A_h = \{x \in \mathbf{R}^{\mathrm{N}} : d_A(x) \le h\},$$
$$\left|\nabla d_A^2(y) - \nabla d_A^2(x)\right| \le 2\left(1 + \frac{\mathrm{reach}\,(A)}{\mathrm{reach}\,(A)-h}\right)|y-x|, \quad \forall x,y \in A_h; \tag{6.5}$$

for all $0 < s < h < \mathrm{reach}\,(A)$ *and all* $x, y \in A_{s,h} = \big\{x \in \mathbf{R}^{\mathrm{N}} : s \le d_A(x) \le h\big\}$,

$$|\nabla d_A(y) - \nabla d_A(x)| \le \frac{1}{s}\left(2 + \frac{\mathrm{reach}\,(A)}{\mathrm{reach}\,(A)-h}\right)|y-x|.$$

If $\mathrm{reach}\,(A) = +\infty$,

$$\begin{aligned} \forall x,y \in \mathbf{R}^{\mathrm{N}}, \quad |p_A(y) - p_A(x)| &\le |y-x|, \\ \left|\nabla d_A^2(y) - \nabla d_A^2(x)\right| &\le 4\,|y-x|; \end{aligned} \tag{6.6}$$

for all $0 < s$ *and all* $x, y \in \{x \in \mathbf{R}^{\mathrm{N}} : s \le d_A(x)\}$,

$$|\nabla d_A(y) - \nabla d_A(x)| \le \frac{3}{s}\,|y-x|.$$

(vii) *If* $\operatorname{reach}(A) > 0$*, then, for all* $0 < r < \operatorname{reach}(A)$*,* $U_r(A)$ *is a set of class* $C^{1,1}$*,* $\overline{U_r(A)} = A_r$*,* $\partial U_r(A) = \{x \in \mathbf{R}^N : d_A(x) = r\}$*, and* $\complement A_r = \complement U_r(A) = \{x \in \mathbf{R}^N : d_A(x) \geq r\}$*.*

(viii) *For all* $a \in \overline{A}$*,*

$$T_a A = \left\{ h \in \mathbf{R}^N : \liminf_{t \searrow 0} \frac{d_A(a + th)}{t} = 0 \right\}.$$

In particular, if $a \in \operatorname{int} \overline{A}$*,* $T_a A = \mathbf{R}^N$*.*

Remark 6.4.
For part (viii), more properties of $T_a A$ and the normal cone $(-T_a A)^*$ (cf. Definitions 2.3 and 2.4 in Chapter 2) can be found in H. Federerer [**3**, sect. 4]. □

Proof. (i) By definition for all $a \in \overline{A}$,

$$\begin{aligned} a \in P(A) \quad &\iff \quad \forall b \in A \backslash \{a\}, \quad |a + v - b| > |v|, \\ a \in Q(A) \quad &\iff \quad \forall b \in A \backslash \{a\}, \quad |a + v - b| \geq |v| \end{aligned}$$

and $P(a) \subset Q(a)$. To prove the convexity, we need the identity

$$|a + v - b|^2 - |v|^2 = (a - b) \cdot (2v + a - b).$$

The convexity of $P(a)$ and $Q(a)$ now follow from the following identities: for all $\lambda \in [0, 1]$ and $v, w \in \mathbf{R}^N$,

$$\begin{aligned} &|a + (\lambda v + (1 - \lambda) w) - b|^2 - |(\lambda v + (1 - \lambda) w|^2 \\ &= (a - b) \cdot (2((\lambda v + (1 - \lambda) w) + a - b) \\ &= \lambda\, (a - b) \cdot (2v + a - b) + (1 - \lambda)\, (a - b) \cdot (2w + a - b) \\ &= \lambda\, \left(|a + v - b|^2 - |v|^2\right) + (1 - \lambda)\, \left(|a + w - b|^2 - |v|^2\right). \end{aligned}$$

Finally, for $h \in T_a A$, there exist sequences $\{b_n\} \subset \overline{A}$ and $\{\varepsilon_n > 0\}$, $\varepsilon_n \searrow 0$, such that $(b_n - a)/\varepsilon_n \to h$ and $b_n \to a$. For all $v \in Q(a)$,

$$2 \frac{b_n - a}{\varepsilon_n} \cdot v \leq \frac{|b_n - a|^2}{\varepsilon_n} \, |b_n - a| \leq 0 \quad \Rightarrow h \cdot v \leq 0 \quad \Rightarrow h \in [-T_a A]^*.$$

(ii) Let $S \stackrel{\text{def}}{=} \{t > 0 : \Pi_A(a + tv) = \{a\}\}$. Since, by assumption, $r(a, v) = \sup S > 0$, then $S \neq \varnothing$ and $v \neq 0$. For each $t \in S$, $\Pi_A(a + tv) = \{a\}$ and $d_A(a + tv) = t\,|v|$. In particular for all $0 < s < t$, $d_A(a + sv) \leq s\,|v|$. For any $p \in \Pi_A(a + sv) \subset \overline{A}$, $d_A(a + sv) = |a + sv - p| \leq s\,|v|$ and

$$\begin{aligned} d_A(a + tv) &\leq |a + tv - p| = |a + sv - p + (t - s) v| \\ &\leq |a + sv - p| + (t - s)|v| \leq s\,|v| + (t - s) v| = t\,|v| = d_A(a + tv). \end{aligned}$$

As a result $p \in \Pi_A(a + tv) = \{a\}$, $p = a$, and necessarily $\Pi_A(a + sv) = \{a\}$ and $d_A(a + sv) = s\,|v|$. This proves that for all $t \in S$, $\{s : 0 < s \leq t\} \subset S$ and hence for all $0 \leq t < r(a, v)$.

Assume that when $r = r(a, v) < \infty$, $a + rv \notin \overline{\text{Sk}\,(A)}$. Then there exists $\delta > 0$ such that $B_\delta(a+rv) \cap (\overline{\text{Sk}\,(A)} \cup \overline{A}) = \varnothing$. By continuity of the projection, $\Pi_A(a+rv) = \{a\}$ and $d_A(a+rv) = r\,|v| > 0$ so that $a+rv \notin \overline{A}$. By Theorem 3.3 (vi), the function $\nabla d_A : \mathbf{R}^N \setminus \big(\text{Sk}\,(A) \cup \partial\overline{A}\big) \to \mathbf{R}$ is continuous and for all $x \in \mathbf{R}^N \setminus \big(\text{Sk}\,(A) \cup \overline{A}\big)$, $\nabla d_A(x) = (x - p_A(x))/|x - p_A(x)|$. For $0 < t < r$, $p_A(a+tv) = a$, $d_A(a+tv) = t|v|$, and $\nabla d_A(a + tv) = v/|v|$. By continuity, $\nabla d_A(a + rv) = v/|v|$.

Since ∇d_A is continuous in $B_\delta(a+rv)$, the Caratheodory conditions are verified and there exists ε, $0 < \varepsilon < \delta$, such that the differential equation

$$x'(t) = \nabla d_A(x(t)), \quad x(0) = a + rv = a + r\,|v|\frac{v}{|v|}$$

has a solution in $(-\varepsilon, \varepsilon)$. This yields

$$\begin{aligned}\frac{d}{dt}d_A(x(t)) = \nabla d_A(x(t)) \cdot x'(t) &= \nabla d_A(x(t)) \cdot \nabla d_A(x(t)) = 1 \\ \Rightarrow\ d_A(x(t)) &= r|v| + t \text{ in } (-\varepsilon, \varepsilon).\end{aligned}$$

For $-\varepsilon < p < q < \varepsilon$, the length of the curve $x(t)$ is given by

$$\int_p^q |x'(t)|\,dt = \int_p^q \frac{d}{dt}d_A(x(t))\,dt = d_A(x(q)) - d_A(x(p)) \leq |x(q) - x(p)|$$

so that the length of the curve $x(t)$ between the points $x(p)$ and $x(q)$ is less than or equal to the distance $|x(q) - x(p)|$ between $x(p)$ and $x(q)$. Therefore $x(t)$ is a line between $x(p)$ and $x(q)$:

$$\begin{aligned}
&x(t) = x(p) + (t-p)\frac{x(q) - x(p)}{q - p}, \quad x'(t) = \frac{x(q) - x(p)}{q - p} \\
&\Rightarrow\ \frac{x(q) - x(p)}{q - p} = x'(0) = \nabla d_A(x(0)) = \nabla d_A(a + rv) = \frac{v}{|v|} \\
&\Rightarrow\ \forall t,\ 0 < t < \varepsilon, \quad x(t) = x(0) + (t - 0)\frac{v}{|v|} = a + rv + t\frac{v}{|v|} \\
\Rightarrow\ &|x(t) - a| = r|v| + t = d_A(x(t)) \quad \Rightarrow p_A(x(t)) = a \text{ and } \Pi_A(a + r|v| + t) = \{a\}
\end{aligned}$$

and we get the contradiction $\sup S \geq r + t/|v| > r = \sup S$. Therefore, $a + rv \notin \overline{\text{Sk}\,(A)}$.

(iii) If there exists $0 < h \leq \text{reach}\,(A)$, then for all $a \in \overline{A}$ and all $0 < s < h$, $B_s(a) \cap \text{Sk}\,(A) = \varnothing$. For $x \in U_h(A)\backslash\overline{A}$, $0 < d_A(x) < h$ and for all $p \in \Pi_A(x)$

$$(d_A(x) + h)/2 < h \leq \text{reach}\,(A) \quad \Rightarrow B_{(d_A(x)+h)/2}(p) \cap \text{Sk}\,(A) = \varnothing.$$

Since $x \in B_{(d_A(x)+h)/2}(p)$, $\Pi_A(x)$ is a singleton and $p = p_A(x)$. From part (ii), $S \stackrel{\text{def}}{=} \{t > 0 : \Pi_A(p_A(x) + t\,\nabla d_A(x)) = \{p_A(x)\}\} \neq \varnothing$ and $r_A \stackrel{\text{def}}{=} r(p_A(x), \nabla d_A(x)) \geq d_A(x) > 0$. If $r_A = +\infty$, then for all $t \geq 0$ we have $\Pi_A(x(t)) = \{p_A(x)\}$ and $d_A(x(t)) = t$. If $r_A < +\infty$, then $x(r_A) \in \overline{\text{Sk}\,(A)}$ and for all $\rho > 0$, $B_{r_A+\rho}(p_A(x)) \cap \text{Sk}\,(A) \neq \varnothing$. This implies that, for all $\rho > 0$, $\text{reach}\,(p_A(x), A) \leq r_A + \rho$ and

$\operatorname{reach}(A) \le \operatorname{reach}(p_A(x), A) \le r_A$. Thus, for all $0 < t < h$, $t < h \le \operatorname{reach}(A) \le r_A$ and we have $\Pi_A(x(t)) = \{p_A(x)\}$ and $d_A(x(t)) = t$.

(iv) If $b = a$, the inequality is verified. For $b \ne a$, let

$$v \stackrel{\text{def}}{=} \frac{x-a}{|x-a|}, \quad S \stackrel{\text{def}}{=} \{t > 0 : \Pi_A(a+tv) = \{a\}\}.$$

By assumption $p_A(x) = a$ and

$$x = a + |x-a|\, v \quad \Rightarrow |x-a| \in S, \quad \sup S \ge |x-a| > 0,$$

and, from part (ii), $a + \sup S\, v \in \overline{\operatorname{Sk}(A)}$.

If $\operatorname{reach}(A,a) > \sup S$, then for ρ, $\operatorname{reach}(A,a) > \rho > \sup S$,

$$B_\rho(a) \cap \operatorname{Sk}(A) = \varnothing \quad \Rightarrow \overline{B_{\sup S}(a)} \cap \overline{\operatorname{Sk}(A)} = \varnothing$$

and this contradicts the fact that $a + \sup S\, v \in \overline{\operatorname{Sk}(A)}$. Therefore, $\operatorname{reach}(A,a) \le \sup S$. Given t, $0 < t < \operatorname{reach}(A,a)$, define $x_t = a + tv$. Then $t \in S$ and $t < \sup S$. Thus $\Pi_A(x_t) = \{a\}$, $p_A(x_t) = a$, and $d_A(x_t) = |x_t - a| = t$. For $b \in \overline{A}$, $|x_t - b| \ge |x_t - a| = t > 0$ and

$$\left| a + t\frac{x-a}{|x-a|} - b \right|^2 \ge t^2 \quad \Rightarrow (a-b)\cdot(x-a) \ge -|a-b|^2 \frac{|x-a|}{2t}.$$

To conclude, choose a sequence $\{t_n\}$, $0 < t_n < \operatorname{reach}(A,a)$, such that $t_n \to \operatorname{reach}(A,a)$ in the above inequality.

(v) Apply the inequality of part (iii) twice to the triplets (a,x,b) and (b,y,a):

$$(a-b)\cdot(x-a) \ge -|a-b|^2 \frac{|x-a|}{2\operatorname{reach}(A,a)} \ge -|a-b|^2 \frac{r}{2\,q},$$
$$(b-a)\cdot(y-b) \ge -|a-b|^2 \frac{|y-b|}{2\operatorname{reach}(A,b)} \ge -|a-b|^2 \frac{r}{2\,q}.$$

By adding the two inequalities

$$(x-a+b-y)\cdot(a-b) \ge -\frac{r}{q}|a-b|^2 \quad \Rightarrow (x-y)\cdot(a-b) \ge \left(1 - \frac{r}{q}\right)|a-b|^2$$
$$\Rightarrow |p_A(x) - p_A(y)| = |a-b| \le \frac{q}{q-r}|x-y|.$$

(vi) Given $0 < h < r < \operatorname{reach}(A)$, for all $a \in \overline{A}$,

$$0 < h < r < \operatorname{reach}(A) \le \operatorname{reach}(A,a) \quad \text{and} \quad B_r(a) \cap \operatorname{Sk}(A) = \varnothing.$$

Therefore $A_h \subset U_r(A) = \cup_{a\in\overline{A}} B_r(a) \subset \mathbf{R}^{\mathrm{N}} \backslash \operatorname{Sk}(A)$ and from (iv)

$$\forall x, y \in A_h,\ \forall r,\ h < r < \operatorname{reach}(A), \quad |p_A(x) - p_A(y)| \le \frac{r}{r-h}|x-y|.$$

The function $r \mapsto r/(r-h) : (h, \infty) \to \mathbf{R}$ is strictly decreasing to 1. If $\text{reach}\,(A) = +\infty$, then for all $a \in \overline{A}$, $\text{reach}\,(A, a) = +\infty$ and by letting r go to infinity

$$\forall x, y \in A_h, \quad |p_A(x) - p_A(y)| \le |x - y|.$$

If $\text{reach}\,(A)$ is finite, then for all $a \in \overline{A}$, $\text{reach}\,(A, a) \ge \text{reach}\,(A) > r$ and by letting r go to $\text{reach}\,(A)$

$$\forall x, y \in A_h, \quad |p_A(x) - p_A(y)| \le \frac{\text{reach}\,(A)}{\text{reach}\,(A) - h}\,|x - y|.$$

The inequalities for ∇d_A^2 follow from identity (3.13) in Theorem 3.3 (v).

Finally, the Lipschitz continuity of ∇d_A follows from identity (3.15) in Theorem 3.3 (vi). For all pairs $0 < s < h < \text{reach}\,(A)$ and all $x, y \in \{x \in \mathbf{R}^{\mathrm{N}} : s \le d_A(x) \le h\}$

$$\begin{aligned}
&|\nabla d_A(y) - \nabla d_A(x)| = \left| \frac{y - p_A(y)}{d_A(y)} - \frac{x - p_A(x)}{d_A(x)} \right| \\
&= \frac{|(y - p_A(y))\, d_A(x) - (x - p_A(x))\, d_A(y)|}{d_A(x)\, d_A(y)} \\
&\le \frac{|(y - p_A(y))\,(d_A(x) - d_A(y))|}{d_A(x)\, d_A(y)} + \frac{|[(y - p_A(y)) - (x - p_A(x))]\, d_A(y)|}{d_A(x)\, d_A(y)} \\
&\le \frac{1}{d_A(x)}\,(|d_A(x) - d_A(y)| + |y - x| + |p_A(y) - p_A(x)|) \\
&\le \frac{1}{s}\left(2 + \frac{\text{reach}\,(A)}{\text{reach}\,(A) - h}\right)|y - x|\,.
\end{aligned}$$

When $\text{reach}\,(A) = +\infty$, for all $s > 0$ and all $x, y \in \{x \in \mathbf{R}^{\mathrm{N}} : s \le d_A(x)\}$

$$|\nabla d_A(y) - \nabla d_A(x)| \le \frac{3}{s}\,|y - x|\,.$$

(vii) For $0 < r < \text{reach}\,(A)$, there exists h such that $0 < r < h < \text{reach}\,(A)$. The open set $U_r(A)$ is defined via the r-level set $d_A^{-1}\{r\} = \{x \in \mathbf{R}^{\mathrm{N}} : d_A(x) = r\}$ of the function d_A. From part (vi), for each $x \in d_A^{-1}\{r\}$, there exists a neighborhood $V(x) = B_\rho(x)$, $\rho = \min\{r, h - r\}/2$, of x in $U_h(A)\backslash\overline{A}$ such that $d_A \in C^{1,1}(\overline{V(x)})$. Therefore, ∇d_A exists and is not zero on the level set $\partial A_r = d_A^{-1}\{r\}$ since $|\nabla d_A| = 1$. From Theorem 4.2 in Chapter 2, the set $U_r(A)$ is of class $C^{1,1}$, $\overline{U_r(A)} = A_r$, and $\partial U_r(A) = d_A^{-1}\{r\}$, and $\overline{\complement A_r} = \complement U_r(A)$.

(viii) Given $h \in T_a A$ (cf. Definition 2.3 in Chapter 2), there exist sequences $\{a_n\} \subset \overline{A}$ and $\{\varepsilon_n\}$, $\varepsilon_n \searrow 0$, such that $(a_n - a)/\varepsilon_n \to h$. For each n,

$$0 \le \frac{d_A(a + \varepsilon_n h)}{\varepsilon_n} \le \frac{|a + \varepsilon_n h - a_n|}{\varepsilon_n} = \left| h - \frac{a_n - a}{\varepsilon_n} \right| \to 0 \;\Rightarrow\; \liminf_{t \searrow 0} \frac{d_A(a + th)}{t} = 0.$$

Conversely, there exists $\{t_n > 0\}$, $t_n \searrow 0$, such that $d_A(a + t_n h)/t_n \to 0$. This means that there exists a sequence of projections $\{a_n\} \subset \overline{A}$ such that

$$\left| h - \frac{a_n - a}{t_n} \right| = \frac{|a + t_n h - a_n|}{t_n} = \frac{d_A(a + t_n h)}{t_n} \to 0$$

and $h \in T_a A$. $\square$

The next theorem summarizes several characterizations of sets of positive reach. Note that condition (ii) is a global condition on the smoothness of d_A^2 in the tubular neighborhood $U_h(A)$.

Theorem 6.3. *Given $\varnothing \neq A \subset \mathbf{R}^{\mathrm{N}}$, the following conditions are equivalent:*

(i) *A has positive reach, that is,* $\mathrm{reach}\,(A) > 0$.

(ii) *There exists $h > 0$ such that $d_A^2 \in C^{1,1}(\overline{U_h(A)})$.*

(iii) *There exists $h > 0$ such that for all s, $0 < s < h$, $d_A \in C^{1,1}(\overline{U_{s,h}(A)})$.*

(iv) *There exists $h > 0$ such that $d_A \in C^1(U_h(A) \backslash \overline{A})$.*

(v) *There exists $h > 0$ such that for all $x \in U_h(A)$, $\Pi_A(x)$ is a singleton.*

Proof. (i) $\Rightarrow$ (ii) For all h, $0 < h < \mathrm{reach}\,(A)$, and all $a \in \overline{A}$, $B_h(a) \cap \mathrm{Sk}\,(A) = \varnothing$. Hence for all $x \in U_h(A)$ $\Pi_A(x)$ is a singleton and $\nabla d_A^2(x)$ exists. The functions d_A^2 and ∇d_A^2 are bounded and uniformly continuous in $U_h(A)$:

$$\forall x \in U_h(A), \quad d_A^2(x) \leq h^2, \quad \nabla d_A^2(x) = |2(x - p_A(x)| = 2\, d_A(x) \leq 2h,$$
$$\forall x, y \in U_h(A), \quad |d_A^2(y) - d_A^2(x)| = (|d_A(y) - d_A(x))\, |d_A(y) - d_A(x)| \leq 2h\, |y - x|,$$
$$\forall x, y \in U_h(A), \quad \left|\nabla d_A^2(y) - \nabla d_A^2(x)\right| \leq 2 \left(1 + \frac{\mathrm{reach}\,(A)}{\mathrm{reach}\,(A) - h} \right) |y - x|$$

from inequality (6.5) of Theorem 6.2 (vi). By definition, $d_A^2 \in C^{1,1}(\overline{U_h(A)})$.

(ii) $\Rightarrow$ (iii) From the proof of Theorem 6.2 (vi).

(iii) $\Rightarrow$ (iv) By assumption, for all $0 < s < h$, $d_A \in C^1(U_{s,h}(A))$. Since $U_h(A) \backslash \overline{A} = \cup_{0<s<h} U_{s,h}(A)$, $d_A \in C^1(U_h(A) \backslash \overline{A})$.

(iv) $\Rightarrow$ (v) For all $x \in U_h(A) \backslash \overline{A}$, $d_A(x) > 0$ and $\nabla d_A^2(x) = 2\, d_A(x) \nabla d_A(x)$ exists; from Theorem 3.3 (v), for all $x \in \overline{A}$, $\nabla d_A^2(x) = 0$. This means that $\Pi_A(x)$ is a singleton in $U_h(A) = U_h(A) \backslash \overline{A} \cup \overline{A}$.

(v) $\Rightarrow$ (i) By assumption, for all $a \in \overline{A}$ and all $x \in B_h(a)$, $B_h(a) \cap \mathrm{Sk}\,(A) = \varnothing$. This implies that for all $a \in \overline{A}$, $\mathrm{reach}\,(A, a) \geq h$, which finally implies that $\mathrm{reach}\,(A) \geq h > 0$. $\square$

6.2 C^k-Submanifolds

For arbitrary closed submanifolds M of $\mathbf{R}^{\mathrm{N}}$ of codimension greater than or equal to one, ∇d_M has a discontinuity along M and generally does not exist. In that case the smoothness of M is related to the existence and the smoothness of ∇d_M^2 in a neighborhood of M that is, a fortiori, locally of positive reach. The following analysis of the smoothness of M was given by J.-B. Poly and G. Raby [**1**] in 1984.

Theorem 6.4. *Let A be a closed nonempty subset of $\mathbf{R}^{\mathrm{N}}$ and $k \geq 2$ be an integer* ($k = \infty$ and ω included[7]). *Then*

$$\operatorname{sing}_k A = A \cap \operatorname{sing}_k d_A^2,$$

where $\operatorname{sing}_k d_A^2 = \mathbf{R}^{\mathrm{N}} \backslash \operatorname{reg}_k d_A^2$, $\operatorname{sing}_k A = \mathbf{R}^{\mathrm{N}} \backslash \operatorname{reg}_k A$, *and*

$$\operatorname{reg}_k d_A^2 \stackrel{\text{def}}{=} \left\{ x \in \mathbf{R}^{\mathrm{N}} : d_A^2 \text{ is } C^k \text{ in a neighborhood of } x \right\},$$
$$\operatorname{reg}_k A \stackrel{\text{def}}{=} \left\{ x \in A : A \text{ is a } C^k\text{-submanifold of } \mathbf{R}^{\mathrm{N}} \text{ in a neighborhood of } x \right\}.$$

It is more interesting to rephrase their theorem in terms of regularities than singularities.

Theorem 6.5. *Let $\varnothing \neq A \subset \mathbf{R}^{\mathrm{N}}$ and let $k \geq 2$ be an integer* ($k = \infty$ and ω included), *and $x \in \overline{A}$.*[8]

(i) *d_A^2 is C^k in a neighborhood of a point $x \in \overline{A}$ if and only if $\overline{A}$ is a C^k-submanifold in a neighborhood of x.*

(ii) *Under conditions* (i), *the dimension of $\overline{A}$ in x is equal to the rank of $Dp_A(x)$ (dimension of the subspace $Dp_A(\mathbf{R}^{\mathrm{N}})$) and $Dp_A(x)$ is the* orthogonal projector onto the tangent space *to $\overline{A}$ in x.*

(iii) *If $\partial \overline{A} = \varnothing$, then $\overline{A} = \mathbf{R}^{\mathrm{N}}$. If $\partial \overline{A} \neq \varnothing$ and d_A^2 is C^2 in a neighborhood of $x \in \overline{A}$, then there exists an open neighborhood $V(x)$ of x such that*

  (a) *if $x \in \operatorname{int} \overline{A}$, then $d_A = d_{\mathbf{R}^{\mathrm{N}}} = 0$ in $V(x)$;*

  (b) *if $x \in \partial \overline{A}$, then $d_A = d_{\partial \overline{A}}$ in $V(x)$, $\operatorname{int} \overline{A} \cap V(x) = \varnothing$, $d_{\complement \overline{A}} = 0$ in $V(x)$, $\overline{A}$ is a C^2-submanifold of dimension less than or equal to $N-1$ in $V(x)$, and $\mathrm{m}(\partial \overline{A} \cap V(x)) = 0$.*

Proof. (i) This follows from Theorem 6.4.

(ii) For $y \in V(x)$

$$f_A(y) \stackrel{\text{def}}{=} \frac{1}{2}\left(|y|^2 - d_A^2(y)\right) \quad \Rightarrow \quad \left|\begin{array}{l} \nabla f_A(y) = y - \dfrac{1}{2}\nabla d_A^2(y) = p_A(y), \\ D^2 f_A(y) = I - \dfrac{1}{2} D^2 d_A^2(y) = Dp_A(y). \end{array}\right.$$

Since p_A is a projection, $p_A(p_A(y)) = p_A(y)$ implies that $Dp_A \circ p_A \, Dp_A = Dp_A$ and $(Dp_A)^2 = Dp_A$ on $\overline{A}$. Therefore, if $R(x)$ denotes the image of $Dp_A(x)$, $Dp_A(x)$ is an orthogonal projector onto $R(x)$ and $I - Dp_A(x)$ is an orthogonal projector onto $R(x)^\perp$ and

$$\forall v \in \mathbf{R}^{\mathrm{N}}, \quad v = Dp_A(x)v + [I - Dp_A(x)]v \in R(x) \oplus R(x)^\perp.$$

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[7] ω indicates the analytical case.

[8] Note that $\overline{A}$ can have several connected components with the same smoothness k but different dimension. When $\overline{A} = \mathbf{R}^{\mathrm{N}}$, d_A is identically zero. Another set of results for Hölderian sets was obtained by G. M. Lieberman [**1**] (see also V. I. Burenkov [**1**]) by introducing a regularized distance function.

Consider the C^1-mapping

$$T_x(y) \stackrel{\text{def}}{=} (Dp_A(x)(p_A(y)-x), [I - Dp_A(x)](y - p_A(y))) : W(x) \to R(x) \oplus R(x)^\perp.$$

T_x is a C^1-diffeomorphism from a neighborhood $U(x) \subset V(x)$ of x onto a neighborhood $W(0)$ of 0 since DT_x is continuous in $V(x)$ and

$$\begin{aligned} DT_x(y) &\stackrel{\text{def}}{=} (Dp_A(x)Dp_A(y), [I - Dp_A(x)](I - Dp_A(y))), \\ DT_x(x) &\stackrel{\text{def}}{=} (Dp_A(x)Dp_A(x), [I - Dp_A(x)](I - Dp_A(x))) \\ &= (Dp_A(x)Dp_A(x), [I - Dp_A(x)](I - Dp_A(x)) \\ &= (Dp_A(x), I - Dp_A(x)) = I. \end{aligned}$$

By construction

$$\begin{gathered} T_x(U(x) \cap \overline{A}) = V(0) \cap R(x), \\ V(0) \cap R(x)^\perp = \{(0, [I - Dp_A(x)](y - p_A(y))) \,:\, \forall y \in U(x)\}, \end{gathered}$$

and $U(x) \cap \overline{A}$ is a C^1-submanifold in $\mathbf{R}^N$ of codimension r, where $N - r$ is the rank of $Dp_A(x)$.

(iii) If $\partial\overline{A} = \varnothing$, $\overline{\complement\overline{A}} \subset \complement\overline{A}$ and $\complement\overline{A}$ is both open and closed. Since $A \neq \varnothing$, then $\overline{A} = \mathbf{R}^N$. Let $\partial\overline{A} \neq \varnothing$. If $x \in \partial\overline{A}$, then $\complement\overline{A} \neq \varnothing$. We claim that $\operatorname{int}\overline{A} = \varnothing$. If this is not true, there exists a neighborhood $V(x)$ of x where d_A^2 is C^2 and $\operatorname{int}\overline{A} \cap V(x) \neq \varnothing$ where $d_A = 0$ and $D^2 d_A^2 = 0$. This implies that $Dp_A = I$ in $\operatorname{int}\overline{A} \cap V(x)$. In $\complement\overline{A} \cap V(x)$, $D^2 d_A^2$ is continuous and $d_A > 0$. Therefore, ∇d_A and $D^2 d_A$ exist, $|\nabla d_A| = 1$, and

$$\begin{gathered} \frac{1}{2} D^2 d_A^2 = \nabla d_A\, {}^*\nabla d_A + d_A\, D^2 d_A \\ \Rightarrow\; Dp_A = I - \nabla d_A\, {}^*\nabla d_A - d_A\, D^2 d_A \quad \text{and} \quad Dp_A\, \nabla d_A = 0 \end{gathered}$$

since $|\nabla d_A|^2 = 1$ implies that $D^2 d_A\, \nabla d_A = 0$. As a result the rank of Dp_A is less than or equal to $N - 1$ in $\complement\overline{A} \cap V(x)$ and equal to N in $\operatorname{int}\overline{A} \cap V(x)$. This contradicts the continuity of $D^2 d_A^2$ and Dp_A in x. Therefore, $\overline{A} \cap V(x) = \partial\overline{A} \cap V(x)$, and $d_{\partial\overline{A}} = d_A$ in $V(x)$ and, by the previous argument in $\complement\overline{A} \cap V(x)$, $\overline{A}$ is a C^2-submanifold of dimension less than or equal to $N - 1$ in a neighborhood of x. $\square$

At this juncture, several interesting remarks can be made.

(a) The equivalence in (i) fails in the direction ($\Rightarrow$) for $d_A^2 \in C^{1,1}$ as seen from Example 6.1 of J.-B. Poly and G. Raby [**1**].

(b) The equivalence in (i) also fails in the direction ($\Leftarrow$) for a $C^{1,\ell}$ submanifold, $0 \leq \ell < 1$, as seen from Example 6.2 of S. G. Krantz and H. R. Parks [**1**].

(c) The square d_A^2 is really pertinent for C^k-submanifolds of codimension greater than or equal to one and not for sets of class C^k. Indeed, when $\partial\overline{A} \neq \varnothing$ and d_A^2 is C^k, $k \geq 2$, in a neighborhood of each point of $\overline{A}$, then $\overline{A}$ is necessarily a C^k-submanifold of *codimension greater than or equal to one*. In particular, $\overline{\complement\overline{A}} = \mathbf{R}^N$, $\overline{A} = \partial\overline{A}$, and $\mathrm{m}(\overline{A}) = 0$.

Example 6.1.
Consider the half-space in $\mathbf{R}^{\mathrm{N}}$

$$A \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : x \cdot e_N \le 0\right\},$$

$$d_A(x) = \begin{cases} 0, & x \cdot e_N \le 0, \\ x \cdot e_N, & x \cdot e_N > 0, \end{cases} \qquad \nabla d_A^2(x) = \begin{cases} 0, & x \cdot e_N \le 0, \\ 2\,(x \cdot e_N)\, e_N, & x \cdot e_N > 0, \end{cases}$$

and d_A^2 is locally $C^{1,1}$. For any point $x \in \partial A = \{x \in \mathbf{R}^{\mathrm{N}} : x \cdot e_N = 0\}$, $A \cap V(x)$ is not even a manifold in any neighborhood $V(x)$ of x. $\square$

The equivalence in (i) also fails in the direction ($\Leftarrow$) for $C^{1,\ell}$, $0 \le \ell < 1$, as seen from Example 6.2, that shows that for a set of class $C^{1,\ell}$, $0 \le \ell < 1$, we do not get $b_A \in C^{1,\lambda}$ in a neighborhood of the boundary ∂A.

Example 6.2.
Consider the two-dimensional domain Ω defined as the epigraph of the function f:

$$\Omega \stackrel{\text{def}}{=} \{(x, z) \,:\, f(x) > z,\, \forall x \in \mathbf{R}\}, \quad f(x) \stackrel{\text{def}}{=} |x|^{2-\frac{1}{n}}$$

for some arbitrary integer $n \ge 1$. Ω is a set of class $C^{1,1-1/n}$ and $\partial\Omega$ is a $C^{1,1-1/n}$-submanifold of dimension 1. In view of the presence of the absolute value in $(0,0)$, it is the point where the smoothness of $\partial\Omega$ is minimum. We claim that

$$\operatorname{Sk}(\partial\Omega) = \operatorname{Sk}(\Omega) = \{(0, y) \,:\, y > 0\} \quad \text{and} \quad \partial\Omega \cap \overline{\operatorname{Sk}(\partial\Omega)} = (0,0) \neq \varnothing.$$

Because the skeleton is a line touching $\partial\Omega$ in $(0,0)$, $d_{\partial\Omega}^2$ and d_Ω^2 cannot be C^1 in any neighborhood of $(0,0)$.

Indeed, for each $(0, y)$

$$d_{\partial\Omega}^2(0, y) = \inf_{x \in \mathbf{R}} \left\{x^2 + |f(x) - y|^2\right\}.$$

The point $(0, y)$, $y > 0$, belongs to $\operatorname{Sk}(\partial\Omega)$ since there exist two different points $\hat{x}$ that minimize the function

$$F(x, y) \stackrel{\text{def}}{=} x^2 + +|f(x) - y|^2. \tag{6.7}$$

Since f is symmetric with respect to the y-axis it is sufficient to show that there exists a strictly positive minimizer $\hat{x} > 0$. A locally minimizing point $\hat{x} \ge 0$ must satisfy the conditions

$$F_x'(\hat{x}, y) = 2\hat{x} + 2\left(|\hat{x}|^{2-\frac{1}{n}} - y\right)\left(2 - \frac{1}{n}\right)\hat{x}^{1-\frac{1}{n}} = 0, \tag{6.8}$$

$$\frac{1}{2}F_x''(\hat{x}, y) = 1 + \left(2 - \frac{1}{n}\right)\left(3 - \frac{2}{n}\right)\hat{x}^{2-\frac{2}{n}} - \left(2 - \frac{1}{n}\right)\left(1 - \frac{1}{n}\right) y\, \hat{x}^{-\frac{1}{n}} \ge 0. \tag{6.9}$$

Equation (6.8) can be rewritten as

$$2\hat{x}^{1-\frac{1}{n}}\left[\hat{x}^{\frac{1}{n}} + \left(\hat{x}^{2-\frac{1}{n}} - y\right)(2 - 1/n)\right] = 0,$$

and $\hat{x} = 0$ is a solution. The second factor can be written as an equation in the new variable $X \stackrel{\text{def}}{=} \hat{x}^{1/n}$ and

$$g(X) \stackrel{\text{def}}{=} \frac{n}{2n-1} X + X^{2n-1} - y = 0.$$

It has exactly one solution $\hat{X}$ since

$$\forall X, \quad \frac{dg}{dX}(X) = \frac{n}{2n-1} + (2n-1)X^{2n-2} > 0,$$

$g(0) = -y$, and $g(X)$ goes to infinity as X goes to infinity. In particular, $\hat{X} > 0$ for $y > 0$ and $\hat{X} = 0$ for $y = 0$. For $y > 0$ and $\hat{x} = \hat{X}^n$

$$\frac{1}{2} F'_x(\hat{x}, y) = 0 \quad \Rightarrow \boxed{y = \hat{x}^{2-\frac{1}{n}} + \frac{n}{2n-1} \hat{x}^{\frac{1}{n}}} \tag{6.10}$$

and

$$\begin{aligned}
\frac{1}{2} F''_x(\hat{x}, y) =& 1 + \left(2 - \frac{1}{n}\right)\left(3 - \frac{2}{n}\right) \hat{x}^{2-\frac{2}{n}} \\
&- \left(2 - \frac{1}{n}\right)\left(1 - \frac{1}{n}\right) \left[\hat{x}^{2-\frac{2}{n}} + \frac{n}{2n-1}\right] \\
=& 1 - \left(\frac{n-1}{n}\right) + \left(\frac{2n-1}{n}\right)^2 \hat{x}^{2-\frac{2}{n}} = \frac{1}{n} + \left(\frac{2n-1}{n} \hat{x}^{1-\frac{1}{n}}\right)^2 > 0.
\end{aligned}$$

Therefore, $\hat{x}$ is a *local minimum*. To complete the proof for $y > 0$ we must compare $F(0, y) = y^2$ for the solution corresponding to $\hat{x} = 0$ with

$$F(\hat{x}, y) = \hat{x}^2 + |\hat{x}^{2-\frac{1}{n}} - y|^2$$

for the solution $\hat{x} > 0$. Again using identity (6.10)

$$F(\hat{x}, y) = \hat{x}^2 + \left(\frac{n}{2n-1}\right)^2 \hat{x}^{\frac{2}{n}},$$

$$F(0, y) = \left|\hat{x}^{2-\frac{1}{n}} + \frac{n}{2n-1} \hat{x}^{\frac{1}{n}}\right|^2 = \left(\hat{x}^{2-\frac{1}{n}}\right)^2 + \left(\frac{n}{2n-1}\right)^2 \hat{x}^{\frac{2}{n}} + 2\frac{n}{2n-1} \hat{x}^2$$

$$\Rightarrow F(0, y) - F(\hat{x}, y) = \left(\frac{2n}{2n-1} - 1\right) \hat{x}^2 + \left(\hat{x}^{2-\frac{1}{n}}\right)^2 = \frac{1}{2n-1} \hat{x}^2 + \left(\hat{x}^{2-\frac{1}{n}}\right)^2 > 0.$$

This proves that for $y > 0$, $\hat{x} > 0$ is the minimizing positive solution. Therefore, for all $n \geq 1$, $f(x) = |x|^{2-1/n}$ yields a domain Ω for which the strictly positive part of the y-axis is the skeleton. It is interesting to note that, for each point x, the points $(x, f(x)) \in \partial\Omega$ are the projections of the point

$$(0, y) = \left(0, |x|^{2-\frac{1}{n}} + \frac{n}{2n-1} |x|^{\frac{1}{n}}\right)$$

and that the square of the distance function is equal to

$$b_\Omega^2(0,y) = x^2 + \left(\frac{n}{2n-1}\right)^2 x^{\frac{2}{n}},$$

$$\begin{aligned}\nabla b_\Omega(x, f(x)) &= \frac{1}{\sqrt{1+f'(x)^2}}(f'(x), -1)\\ &= \frac{1}{\sqrt{1+\left(\frac{2n-1}{n}\right)^2 |x|^{2-\frac{2}{n}}}}((2-1/n)\, x\, |x|^{-\frac{1}{n}}, -1).\end{aligned}$$

Finally for all $n \geq 1$, $f \notin C^{1,1}$. For $n = 1$, $f \in C^{0,1}$, and for $n > 1$, $f \in C^{1,1-1/n}$. This gives an example in dimension 2 of a domain Ω of class $C^{1,\lambda}$, $0 \leq \lambda < 1$, with skeleton converging to the boundary. □

6.3 A Compact Family of Sets with Uniform Positive Reach

The following family of sets of positive reach and the associated compactness theorem were first given in H. Federer [**3**, Thm. 4.13, Rem. 4.14].

Theorem 6.6. *Given an open* (resp., *bounded open*) *holdall* D *in* $\mathbf{R}^{\mathrm{N}}$ *and* $h > 0$, *the family*

$$C_{d,h}(D) \stackrel{\text{def}}{=} \left\{d_A : \varnothing \neq \overline{A} \subset \overline{D},\ \text{reach}\,(A) \geq h\right\} \tag{6.11}$$

is closed in $C_{\text{loc}}(D)$ (resp., *compact in* $C(\overline{D})$).

Proof. It is sufficient to prove the theorem for a bounded open D. Given any $C(\overline{D})$-Cauchy sequence $\{d_{A_n}\} \subset C_{d,h}(D)$, there exists $d_A \in C_d(D)$ such that $d_{A_n} \to d_A$ in $C(\overline{D})$. Choose W open such that $\overline{W} \subset U_h(A)$. Then $\sup_{x\in W} d_A(x) < h$. Choose r, $\sup_{x\in W} d_A(x) < r < h$. Then $\overline{W} \subset U_r(A) \subset A_r \subset U_h(A)$. Choose N such that

$$\forall n > N, \quad \sup_{x\in W} d_{A_n}(x) < r < h.$$

Then for all $n > N$, $\overline{W} \subset U_r(A_n)$.

From Theorem 6.2 (v), for all $n > N$ and

$$\forall x, y \in (A_n)_r, \quad \left|\nabla d_{A_n}^2(y) - \nabla d_{A_n}^2(x)\right| \leq 2\left(1 + \frac{h}{h-r}\right) |y - x|. \tag{6.12}$$

But $\overline{W} \subset U_r(A) \subset A_r \subset U_h(A)$, $d_{A_n}^2 \to d_A^2$ in $C(\overline{D})$, and

$$\forall n > N, \quad d_{A_n}^2 \in C^{1,1}(\overline{W}) \subset C^1(\overline{W}).$$

From Theorem 2.5 in Chapter 2, the embedding $C^{1,1}(\overline{W}) \subset C^1(\overline{W})$ is compact. Therefore, for all open subsets such that $\overline{W} \subset U_h(A)$, $d_{A_n}^2 \to d_A^2 \in C^1(\overline{W})$, $d_A^2 \in C^1(\overline{U_r(A)})$, and reach $(A) \geq r$. Since this is true for all $r < h$, we finally get reach $(A) \geq h$ and $d_A \in C_{d,h}(D)$. □

The compactness of Theorem 6.7 can also be expressed as follows.

Theorem 6.7. *Let D be a fixed bounded open subset of $\mathbf{R}^N$. Let $\{A_n\}$, $A_n \neq \varnothing$, be a sequence of subsets of $\overline{D}$. Assume that there exists $h > 0$ such that*

$$\forall n, \quad d^2_{A_n} \in C^{1,1}(\overline{U_h(A_n)}). \tag{6.13}$$

Then there exist a subsequence $\{A_{n_k}\}$ and $A \subset \overline{D}$, $A \neq \varnothing$, such that reach $(A) \geq h$ *and for all r, $0 < r < h$, $d^2_A \in C^{1,1}(\overline{U_r(A)})$,*

$$d^2_{A_{n_k}} \to d^2_A \text{ in } C^1(\overline{U_r(A)}) \quad \text{and} \quad d_{A_{n_k}} \to d_A \text{ in } C(\overline{D}). \tag{6.14}$$

This condition in an h-tubular neighborhood is generic of other such conditions that will be expressed in different function spaces.

7 Approximation by Dilated Sets/Tubular Neighborhoods and Critical Points

The $W^{1,p}$-topology turns out to be an appropriate setting for the approximation of the closure of a set by its dilated sets as defined in Definition 3.2.

Theorem 7.1. *Let $\varnothing \neq A \subset \mathbf{R}^N$. For $r > 0$,*

$$A_r = \overline{U_r(A)},\ \partial A_r = \partial U_r(A) = d_A^{-1}\{r\},\ \overline{\complement A_r} = \complement U_r(A),\ \operatorname{int} A_r = U_r(A), \tag{7.1}$$

$$d_{U_r(A)} = d_{A_r}(x) = \begin{cases} d_A(x) - r, & \text{if } d_A(x) \geq r, \\ 0, & \text{if } d_A(x) < r, \end{cases} \tag{7.2}$$

$$\forall x \in \mathbf{R}^N, \quad 0 \leq d_A(x) - d_{A_r}(x) \leq r, \tag{7.3}$$

$$d_{U_r(A)} = d_{A_r}(x) \to d_A(x) \text{ uniformly in } \mathbf{R}^N \quad \text{as } r \to 0,$$
$$d_{U_r(A)} = d_{A_r} \to d_A \text{ in } W^{1,p}_{\mathrm{loc}}(\mathbf{R}^N) \quad \text{and} \quad \chi_{A_r} \to \chi_{\overline{A}} \text{ in } L^p_{\mathrm{loc}}(\mathbf{R}^N) \tag{7.4}$$

for all p, $1 \leq p < \infty$.

Proof. The first line of properties follows from Theorem 3.1. For $r > 0$, $\overline{A} \subset U_r(A) \subset A_r$ and $d_A(x) \geq d_{A_r}(x)$ for all $x \in \mathbf{R}^N$. For $x \in \mathbf{R}^N$ such that $d_A(x) < r$, $d_{A_r}(x) = 0$ and $0 \leq d_A(x) - d_{A_r}(x) < r$. For $x \in \mathbf{R}^N$ such that $d_A(x) \geq r$ and all $a \in A$ and $a_r \in A_r$,

$$\begin{aligned} |x - a| \leq |x - a_r| + |x - a_r| \quad &\Rightarrow d_A(x) \leq |x - a_r| + d_A(a_r) \leq |x - a_r| + r \\ &\Rightarrow d_A(x) \leq d_{A_r}(x) + r \quad \Rightarrow 0 \leq d_A(x) - d_{A_r}(x) \leq r. \end{aligned}$$

We show that $d_A(x) - d_{A_r}(x) = r$. Assume that $d_A(x) - d_{A_r}(x) < r$. There exists a unique $p \in \overline{A}$ such that $d_A(x) = |x - p| \geq r > 0$. Consider the point

$$x_r \stackrel{\text{def}}{=} p + r\frac{x - p}{|x - p|} \quad \Rightarrow |x - x_r| = d_A(x) - r \quad \text{and} \quad |x_r - p| = r.$$

By definition, $d_A(x_r) \le |x_r - p| = r$ and $x_r \in A_r$. As a result,

$$d_A(x) < d_{A_r}(x) + r \le |x - x_r| + r = d_A(x) - r + r = d_A(x)$$

yields a contradiction. This establishes identity (7.2) and for all $x \in \mathbf{R}^{\mathrm{N}}$, $|d_{A_r}(x) - d_A(x)| \le r \to 0$ as r goes to zero.

When $\nabla d_{A_r}(x)$ exists at a point $\in A_r$, $\nabla d_{A_r}(x) = 0$ (cf. Theorem 3.3 (vi)). This means that $\nabla d_{A_r} = 0$ almost everywhere in A_r and that the gradient of d_{A_r} is almost everywhere equal to

$$\nabla d_{A_r}(x) = \begin{cases} \nabla d_A(x), & d_A(x) > r, \\ 0, & d_A(x) \le r. \end{cases}$$

For any compact $K \subset \mathbf{R}^{\mathrm{N}}$

$$\int_K |\nabla d_{A_r} - \nabla d_A|\, dx = \int_K |\nabla d_A|\, \chi_{A_r}\, dx = \int_K (1 - \chi_{\overline{A}})\, \chi_{A_r}\, dx = \int_K \left(\chi_{A_r} - \chi_{\overline{A}}\right)\, dx.$$

Since $\{K \cap A_r\}$ is a decreasing family of closed sets as $r \to 0$ and $\cap_{r>0} A_r = \overline{A}$, $\mathrm{m}(K \cap A_r) \to m(K \cap \overline{A})$ (cf. W. RUDIN [**1**, Thm. 1.1.9 (e), p. 16]). Finally, $\|\nabla d_{A_r} - \nabla d_A\|_{L^1} \to 0$ and $d_{A_r} \to d_A$ in $W^{1,1}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$. □

S. FERRY [**1**] proved in 1975 that if $N = 2$ or 3, then $d_A^{-1}\{r\}$ is an $(N-1)$-manifold for almost all r, and that if A is a finite polyhedron in $\mathbf{R}^{\mathrm{N}}$, then $d_A^{-1}\{r\}$ is an $(N-1)$-manifold for all sufficiently small r. On the other hand, there is a Cantor set K in $\mathbf{R}^4$ such that $d_K^{-1}\{r\}$ is not a 3-manifold for any r between 0 and 1. The set K generalizes to higher dimensions.

Generalizing those results, J. H. G. FU [**1**] investigated the regularity properties of ∂A_r and proved the following theorem involving the notion of *critical point of* d_A and of *Lipschitz manifold*. In order to make things simple we use a specific characterization of the critical points of d_A as a definition (cf. J. H. G. FU [**1**, **2**] and Figure 6.5).

Definition 7.1.
Given $\varnothing \ne A \subset \mathbf{R}^{\mathrm{N}}$, $x \in \mathbf{R}^{\mathrm{N}} \backslash \overline{A}$ is said to be a *critical point of* d_A if $x \in \mathrm{co}\, \Pi_A(x)$. The set of all critical points of d_A will be denoted by $\mathrm{crit}\,(d_A)$. □

Definition 7.2.
An m-dimensional *Lipschitz manifold* is a paracompact metric space M such that there is a system of open sets $\{U_\alpha\}$ covering M and, for each α, a bi-Lipschitzian homeomorphism φ_α of U_α onto an open subset of $\mathbf{R}^m$. □

Theorem 7.2. *Let $A \subset \mathbf{R}^{\mathrm{N}}$ be compact. The subset $C(A) = d_A(\mathrm{crit}\,(d_A)) \subset \mathbf{R}$ is compact:*

$$C(A) \subset \left[0, \left(\frac{N}{2N+2}\right)^{1/2} \mathrm{diam}\,(A)\right] \quad \text{and} \quad H_{(N-1)/2}(C(A)) = 0$$

(the assertion about the measure of $C(A)$ is nontrivial only for $N \le 3$). For all $r \notin C(A)$, $\overline{\complement A_r}$ has positive reach and ∂A_r is a Lipschitz manifold.

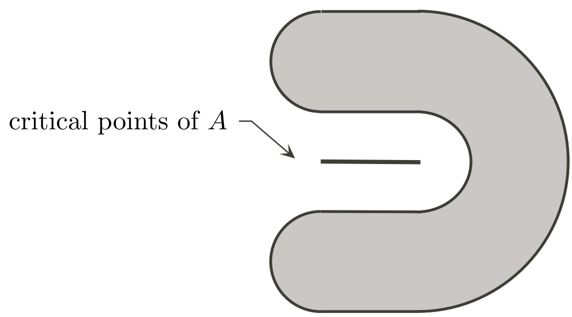


Figure 6.5. *Set of critical points of A.*

8 Characterization of Convex Sets

8.1 Convex Sets and Properties of d_A

In the convex case the squared distance function is differentiable everywhere and this property can be used to characterize the convexity of a set.

Theorem 8.1. *Let $\varnothing \neq A \subset \mathbf{R}^{\mathrm{N}}$. The following statements are equivalent:*

(i) *There exists $a \in \overline{A}$ such that* $\mathrm{reach}\,(A, a) = +\infty$.

(ii) $\mathrm{Sk}\,(A) = \varnothing$.

(iii) *For all $a \in \overline{A}$,* $\mathrm{reach}\,(A, a) = +\infty$.

(iv) $\mathrm{reach}\,(A) = +\infty$.

(v) *For all $x \in \mathbf{R}^{\mathrm{N}}$, $\Pi_A(x)$ is a singleton.*[9]

(vi) *$\overline{A}$ is convex.*

(vii) *d_A is convex.*

In particular,

$$\forall x, \forall y \in \mathbf{R}^{\mathrm{N}}, \quad |p_A(y) - p_A(x)| \leq |y - x| \tag{8.1}$$

and d_A^2 belongs to $C^{1,1}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$ (and, a fortiori, to $W^{2,\infty}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$). Moreover, if A is convex, then $\overline{\complement\overline{A}} = \overline{\complement A}$ and $d_{\complement\overline{A}} = d_{\complement A}$.

Remark 8.1.
The convexity of d_A implies the convexity of $\overline{A}$ but not of A. The set A of all points of the unit open ball with rational coordinates has a convex closure but is not convex. □

[9]This part of the theorem is related to deeper results on the convexity of *Chebyshev sets* in metric spaces. A subset A of a metric space X is called a Chebyshev set provided that every point x of X has a unique projection $p_A(x)$ in A. The reader is referred to V. L. Klee [**1**] for details and background material.

Proof. The equivalence of properties (i) to (v) follows from Lemma 6.1.

(v) $\Rightarrow$ (vi) Given $x, y \in \overline{A}$ and $\lambda \in [0,1]$, consider the point $x_\lambda = \lambda x + (1-\lambda) y$ and the continuous function $\lambda \mapsto d_A(x_\lambda) : [0,1] \to \mathbf{R}$ is continuous. There exists $\mu \in [0,1]$ such that $d_A(x_\mu) = \sup_{\lambda \in [0,1]} d_A(x_\lambda)$. If $\sup_{\lambda \in [0,1]} d_A(x_\lambda) = 0$, then $x_\lambda \in \overline{A}$. If $\sup_{\lambda \in [0,1]} d_A(x_\lambda) > 0$, then $x \neq y$, $d_A(x_\mu) > 0$, $\mu \in (0,1)$, and $x_\mu \notin \overline{A}$. Since $\mathrm{Sk}\,(A) = \varnothing$, $\nabla d_A(x_\mu)$ exists and the maximum is characterized by

$$\frac{d}{d\lambda} d_A(x_\lambda)\bigg|_{\lambda=\mu} = 0 \quad \Rightarrow \nabla d_A(x_\mu) \cdot (x-y) = 0.$$

By assumption there exists a unique projection $p_\mu = p_A(x_\mu) \in \overline{A}$ such that $d_A(x_\mu) = |x_\mu - p_\mu|$ and by Theorem 3.3 (vi)

$$\nabla d_A(x_\mu) = \frac{x_\mu - p_\mu}{|x_\mu - p_\mu|}.$$

From inequality (6.6) in Theorem 6.2 (vi)

$$\begin{aligned} |p_\mu - x| = |p_A(x_\mu) - p_A(x)| \leq |x_\mu - x| = (1-\mu)\,|x-y|, \\ |p_\mu - y| = |p_A(x_\mu) - p_A(y)| \leq |x_\mu - y| = \mu\,|x-y| \\ \Rightarrow |x-y| \leq |p_\mu - x| + |p_\mu - y| \leq |x-y|. \end{aligned}$$

Therefore the point p_μ lies on the segment between x and y: there exists $\mu' \in [0,1]$, $\mu' \neq \mu$, such that $p_\mu = x'_\mu$ and $x_\mu - p_\mu = x_\mu - x_{\mu'} = (\mu - \mu')\,(x-y)$. Finally,

$$\frac{\mu - \mu'}{|\mu - \mu'|}|x-y| = \frac{x_\mu - x_{\mu'}}{|x_\mu - x_{\mu'}|} \cdot (x-y) = \frac{x_\mu - p_\mu}{|x_\mu - p_\mu|} \cdot (x-y) = \nabla d_A(x_\mu) \cdot (x-y) = 0$$

and we get the contradiction $x = y$.

(vi) $\Rightarrow$ (v) By definition for each $x \in \mathbf{R}^{\mathrm{N}}$

$$d_A^2(x) = \inf_{z \in A} |z-x|^2 = \inf_{z \in \overline{A}} |z-x|^2,$$

and since $\overline{A}$ is closed and convex and $z \mapsto |z-x|^2$ is strictly convex and coercive, there exists a unique minimizing point $p_A(x)$ in $\overline{A}$ and $\Pi_A(x)$ is a singleton.

(vi) $\Rightarrow$ (vii) Given x and y in $\mathbf{R}^{\mathrm{N}}$, there exist $\overline{x}$ and $\overline{y}$ in $\overline{A}$ such that $d_A(x) = |x - \overline{x}|$ and $d_A(y) = |y - \overline{y}|$. By convexity of $\overline{A}$, for all λ, $0 \leq \lambda \leq 1$, $\lambda\,\overline{x} + (1-\lambda)\,\overline{y} \in \overline{A}$ and

$$\begin{aligned} d_A(\lambda\,x + (1-\lambda)\,y) \leq & |\lambda\,x + (1-\lambda)\,y - (\lambda\,\overline{x} + (1-\lambda)\,\overline{y})| \\ \leq & \lambda\,|x - \overline{x}| + (1-\lambda)\,|y - \overline{y}| = \lambda\,d_A(x) + (1-\lambda)\,d_A(y) \end{aligned}$$

and d_A is convex in $\mathbf{R}^{\mathrm{N}}$.

(vii) $\Rightarrow$ (vi) If d_A is convex, then

$$\forall \lambda \in [0,1], \forall x, y \in \overline{A}, \quad d_A(\lambda x + (1-\lambda) y) \leq \lambda d_A(x) + (1-\lambda) d_A(y).$$

But x and y in $\overline{A}$ imply that $d_A(x) = d_A(y) = 0$ and hence

$$\forall \lambda \in [0,1], d_A(\lambda x + (1-\lambda)y) = 0.$$

Thus $\lambda x + (1-\lambda)y \in \overline{A}$ and $\overline{A}$ is convex.

Properties (8.1) follow from Theorem 6.2 and the identity on the complements follows from Theorem 3.5 (ii) in Chapter 5. $\square$

The next theorem is a consequence of the fact that the gradient of a convex function is locally of bounded variations.

Theorem 8.2. *For all subsets A of $\mathbf{R}^N$ such that $\partial A \neq \varnothing$ and $\overline{A}$ is convex,*

(i) *∇d_A belongs to $\mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^N)^N$,*

(ii) *the Hessian matrix $D^2 d_A$ of second-order derivatives is a matrix of signed Radon measures that are nonnegative on the diagonal, and*

(iii) *d_A has a second-order derivative almost everywhere and for almost all x and y in $\mathbf{R}^N$,*

$$\left| d_A(y) - d_A(x) - \nabla d_A(x) \cdot (y-x) - \frac{1}{2}(y-x) \cdot D^2 d_A(x)\,(y-x) \right| = o(|y-x|^2)$$

as $y \to x$.

Proof. From the equivalence of (vi) and (vii) in Theorem 8.1 d_A is convex and continuous and the result follows from L. C. Evans and R. F. Gariepy [**1**, Thm. 3, p. 240, Thm. 2, p. 239, and Aleksandrov's Thm., p. 242]. $\square$

8.2 Semiconvexity and BV Character of d_A

Since the BV properties in Theorem 8.2 arise solely from the convexity of the distance function d_A, the BV property will extend to sets A such that d_A is semiconvex.

Theorem 8.3. *Let A be a nonempty subset of $\mathbf{R}^N$ such that*

$$\forall x \in \partial A,\ \exists \rho > 0 \text{ such that } d_A \text{ is semiconvex in } B(x,\rho).$$

Then $\nabla d_A \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^N)^N$.

Proof. For all $x \in \operatorname{int} A$, $d_A = 0$ and there exists $\rho > 0$ such that $B(x,\rho) \subset \operatorname{int} A$ and the result is trivial. For each $x \in \mathbf{R}^N \backslash \overline{A}$, $d_A(x) > 0$. Pick $h = d_A(x)/2$. Then for all $y \in B(x,h)$ and all $a \in A$,

$$|y-a| \geq |x-a| - |y-x| \geq d_A(x) - |y-x| > 2h - h = h$$

and $B(x,h) \subset \mathbf{R}^N \backslash A_h$. By Theorem 3.2 (i) the function $|x|^2/(2h) - d_A(x)$ is locally convex in $\mathbf{R}^N \backslash A_h$ and, a fortiori, convex in $B(x,h)$. Finally, by assumption for all $x \in \partial A$, there exists $\rho > 0$ such that d_A is semiconvex. Hence for each $x \in \mathbf{R}^N$, ∇d_A belongs to $\mathrm{BV}(B(x,\rho/2))$ for some $\rho > 0$. Therefore, ∇d_A belongs to $\mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^N)^N$. $\square$

However, it is never semiconcave in $\mathbf{R}^{\mathrm{N}}$ for nontrivial sets.

Theorem 8.4.

(i) *Given a subset* A *of* $\mathbf{R}^{\mathrm{N}}$, $A \neq \varnothing$,

$$\exists c \geq 0, \quad f_c(x) = c\,|x|^2 - d_A(x) \text{ is convex in } \mathbf{R}^{\mathrm{N}}$$

if and only if

$$\exists h > 0, \exists c \geq 0, \quad f_c(x) = c\,|x|^2 - d_A(x)$$

is locally convex in $U_h(A)$.

(ii) *Given a subset* A *of* $\mathbf{R}^{\mathrm{N}}$, $\varnothing \neq \overline{A} \neq \mathbf{R}^{\mathrm{N}}$,

$$\nexists c \geq 0, \quad f_c(x) = c\,|x|^2 - d_A(x) \text{ is convex in } \mathbf{R}^{\mathrm{N}}\,.$$

Proof. (i) If f_c is convex in $\mathbf{R}^{\mathrm{N}}$, it is locally convex in $U_h(A)$. Conversely, assume that there exist $h > 0$ and $c \geq 0$ such that f_c is locally convex in $U_h(A)$. From Theorem 3.2 (i),

$$\frac{1}{h}\,|x|^2 - d_A(x)$$

is locally convex in $\mathbf{R}^{\mathrm{N}} \backslash A_{h/2}$. Therefore, for $\overline{c} = \max\{c, 1/h\}$, the function $f_{\overline{c}}$ is locally convex in $\mathbf{R}^{\mathrm{N}}$ and hence convex on $\mathbf{R}^{\mathrm{N}}$.

(ii) Assume the existence of a $c \geq 0$ for which f_c is convex in $\mathbf{R}^{\mathrm{N}}$. For each $y \in \overline{A}$, the function

$$x \mapsto F_c(x, y) = c\,|x - y|^2 - d_A(x)$$

is also convex since it differs from $f_c(x)$ by the linear term

$$x \mapsto c\,(|y|^2 - 2\,x \cdot y).$$

Since $\varnothing \neq \overline{A} \neq \mathbf{R}^{\mathrm{N}}$, there exist $x \in \complement \overline{A}$ and $p \in \overline{A}$ such that

$$0 < d_A(x) = |x - p| \leq \frac{1}{2c}.$$

For any $t > 0$, define $x_t = p - t\,(x - p)$ and $\lambda = t/(1 + t) \in]0, 1[$ and observe that

$$x_\lambda \stackrel{\text{def}}{=} \lambda\, x + (1 - \lambda)\, x_t = p \quad \text{and} \quad F_c(x_\lambda, p) = 0.$$

But

$$\begin{aligned}
&\lambda\, F_c(x, p) + (1 - \lambda)\, F_c(x_t, p) \\
&= \frac{t}{1+t}\,[c\,|x - p|^2 - d_A(x)] + \frac{1}{1+t}\,[c\,|x_t - p|^2 - d_A(x_t)] \\
&\leq \frac{t}{1+t}\,[c\,d_A(x)^2 - d_A(x) + c\,t d_A(x)^2] \\
&\leq \frac{t}{1+t}\,d_A(x)\,[(1+t)\,c\,d_A(x) - 1] \\
&\leq \frac{t}{1+t}\,d_A(x)\,\left[(1+t)\,\frac{1}{2} - 1\right] = \frac{d_A(x)\,t}{1+t}\,\frac{t-1}{2},
\end{aligned}$$

since by construction $d_A(x) > 0$ and $c\, d_A(x) < 1/2$. Therefore for t, $0 < t < 1$, the above quantity is strictly negative and we have constructed two points x and x_t and a λ, $0 < \lambda < 1$, such that

$$F_c(\lambda\, x + (1-\lambda)\, x_t, p) = 0 > \lambda\, F_c(x,p) + (1-\lambda)\, F_c(x_t,p).$$

This contradicts the convexity of the function $x \mapsto F_c(x,p)$ and, a fortiori, of f_c. □

8.3 Closed Convex Hull of A and Fenchel Transform of d_A

Another interesting property is that the closed convex hull of A is completely characterized by the convex envelope of d_A that can be obtained from the double Fenchel transform of d_A.

Theorem 8.5. *Let A be a nonempty subset of $\mathbf{R}^{\mathrm{N}}$:*

(i) *The Fenchel transform*

$$d_A^*(x^*) = \sup_{x\in\mathbf{R}^{\mathrm{N}}} x^* \cdot x - d_A(x)$$

of d_A is given by

$$d_A^*(x^*) = \sigma_A(x^*) + I_{\overline{B(0,1)}}(x^*), \tag{8.2}$$

where $\sigma_A(x^*)$ is the support function of A and

$$I_{\overline{B(0,1)}}(x^*) = \begin{cases} +\infty, & x \notin \overline{B(0,1)}, \\ 0, & x \in \overline{B(0,1)}, \end{cases}$$

is the indicator function of the closed unit ball $\overline{B(0,1)}$ at the origin.

(ii) *The Fenchel transform of d_A^* is given by*

$$d_A^{**} = d_{\mathrm{co}\, A} = d_{\overline{\mathrm{co}}\, A}. \tag{8.3}$$

In particular, $d_{\mathrm{co}\, A}$ is the convex envelope of d_A.

The function d_A and the indicator function $I_{\overline{A}}$ are both zero on $\overline{A}$ and the Fenchel transform of $I_{\overline{A}}$,

$$I_{\overline{A}}^* = \sigma_{\overline{A}} = \sigma_A,$$

coincides with d_A^* on $\overline{B(0,1)}$.

Proof. (i) From the definition,

$$\begin{aligned}
d_A^*(x^*) &= \sup_{x\in\mathbf{R}^{\mathrm{N}}} x^* \cdot x - d_A(x) = \sup_{x\in\mathbf{R}^{\mathrm{N}}} \left[x^* \cdot x - \inf_{p\in A} |x-p| \right] \\
&= \sup_{x\in\mathbf{R}^{\mathrm{N}}} \sup_{p\in A} x^* \cdot x - |x-p| = \sup_{p\in A} \sup_{x\in\mathbf{R}^{\mathrm{N}}} x^* \cdot x - |x-p| \\
&= \sup_{p\in A} \sup_{x\in\mathbf{R}^{\mathrm{N}}} x^* \cdot p + x^* \cdot (x-p) - |x-p| \\
&= \sup_{p\in A} x^* \cdot p + \sup_{x\in\mathbf{R}^{\mathrm{N}}} \left[x^* \cdot (x-p) - |x-p| \right] \\
&= \sup_{p\in A} x^* \cdot p + \sup_{x\in\mathbf{R}^{\mathrm{N}}} \left[x^* \cdot x - |x| \right] = \sigma_A(x^*) + I_{\overline{B(0,1)}}(x^*).
\end{aligned}$$

(ii) We now use the property that $\sigma_A = \sigma_{\text{co}\,A}$:

$$\begin{aligned}
d_A^{**}(x^{**}) &= \sup_{x^*\in\mathbf{R}^N} x^{**}\cdot x^* - d_A^*(x^*) = \sup_{x^*\in\mathbf{R}^N} x^{**}\cdot x^* - \sigma_A(x^*) - I_{\overline{B(0,1)}}(x^*)\\
&= \sup_{|x^*|\le 1} x^{**}\cdot x^* - \sigma_A(x^*) = \sup_{|x^*|\le 1}\left[x^{**}\cdot x^* - \sup_{x\in \text{co}\,A} x^*\cdot x\right]\\
&= \sup_{|x^*|\le 1}\ \inf_{x\in\text{co}\,A} (x^{**}-x)\cdot x^*\\
&\le \sup_{|x^*|\le 1} (x^{**}-p(x^{**}))\cdot x^* \le |x^{**}-p(x^{**})| = d_{\text{co}\,A}(x^{**}).
\end{aligned}$$

But in fact we have a saddle point. If $x^{**}\in\overline{\text{co}}\,A$, pick $x^*=0$ and

$$d_A^{**}(x^{**}) = \sup_{|x^*|\le 1}\ \inf_{x\in\text{co}\,A}(x^{**}-x)\cdot x^* \ge 0 = d_{\text{co}\,A}(x^{**}).$$

For $x^{**}\notin\overline{\text{co}}\,A$ rewrite the function as

$$(x^{**}-x)\cdot x^* = (x^{**}-p(x^{**}))\cdot x^* + (p(x^{**})-x)\cdot x^*.$$

Since $p(x^{**})$ is the minimizing point of the function $|x-x^{**}|^2$ over the closed convex set $\overline{\text{co}}\,A$ it is completely characterized by the variational inequality

$$\forall x\in\overline{\text{co}}\,A,\quad (p(x^{**})-x^{**})\cdot(x-p(x^{**}))\ge 0. \tag{8.4}$$

By choosing the following special x^*,

$$x^* = \frac{x^{**}-p(x^{**})}{|x^{**}-p(x^{**})|},$$

we get

$$\begin{aligned}
d_A^{**}(x^{**}) &= \sup_{|x^*|\le 1}\ \inf_{x\in\text{co}\,A}(x^{**}-x)\cdot x^*\\
&\ge |x^{**}-p(x^{**})| + (p(x^{**})-x)\cdot\frac{x^{**}-p(x^{**})}{|x^{**}-p(x^{**})|}\\
&\ge |x^{**}-p(x^{**})| = d_{\text{co}\,A}(x^{**})
\end{aligned}$$

in view of (8.4). Therefore

$$\sup_{|x^*|\le 1}\ \inf_{x\in\text{co}\,A}(x^{**}-x)\cdot x^* = d_{\text{co}\,A}(x^{**}) = \inf_{x\in\text{co}\,A}\ \sup_{|x^*|\le 1}(x^{**}-x)\cdot x^*.$$

Finally from the previous theorem $d_{\text{co}\,A}$ is a convex function that coincides with the convex envelope since it is the bidual of d_A. □

8.4 Families of Convex Sets $\mathcal{C}_d(D)$, $\mathcal{C}_d^c(D)$, $\mathcal{C}_d^c(E;D)$, and $\mathcal{C}_{d,\text{loc}}^c(E;D)$

Theorem 8.6. *Let D be a nonempty open subset of $\mathbf{R}^N$. The subfamily*

$$\mathcal{C}_d(D) \overset{\text{def}}{=} \{d_A\in C_d(D) : A\neq\varnothing \ \textit{convex}\} \tag{8.5}$$

of $C_d(D)$ is closed in $C_{\text{loc}}(D)$. It is compact in $C(\overline{D})$ when D is bounded. The subfamily

$$\mathcal{C}_d^c(D) \overset{\text{def}}{=} \{d_{\complement\Omega} \in C_d^c(D) \, : \, \Omega \text{ open and convex}\} \tag{8.6}$$

and for E open, $\varnothing \neq E \subset D$, the subfamilies

$$\mathcal{C}_d^c(E;D) \overset{\text{def}}{=} \{d_{\complement\Omega} : E \subset \Omega \text{ open and convex } \subset D\}, \tag{8.7}$$

$$\mathcal{C}_{d,\text{loc}}^c(E;D) \overset{\text{def}}{=} \left\{ d_{\complement\Omega} : \begin{array}{l} \exists x \in \mathbf{R}^{\text{N}},\, \exists A \in \text{O(N)} \text{ such that} \\ x + AE \subset \Omega \text{ open and convex } \subset D \end{array} \right\} \tag{8.8}$$

of $C_d^c(D)$ are closed in $C_{\text{loc}}(D)$. If D is bounded, they are compact in $C(\overline{D})$.

Proof. The set of all convex functions in $C_{\text{loc}}(D)$ is a closed convex cone with vertex at 0. Hence its intersection with $C_d(D)$ is closed. Any Cauchy sequence in $\mathcal{C}_d(D)$ converges to some convex d_A. From part (i) $\overline{A}$ is convex. But $d_A = d_{\overline{A}}$ and the nonempty convex set $\overline{A}$ can be chosen as the limit set. For the complementary distance function, it is sufficient to prove the compactness for D bounded. Given any sequence $\{d_{\complement\Omega_n}\}$, there exist an open subset Ω of D and a subsequence, still indexed by n, such that $d_{\complement\Omega_n} \to d_{\complement\Omega}$ in $C(\overline{D})$. If Ω is empty, there is nothing to prove. If Ω is not empty, consider two points x and y in Ω and $\lambda \in [0,1]$. There exists $r > 0$ such that

$$B(x,r) \subset \Omega \text{ and } B(y,r) \subset \Omega \quad \Rightarrow d_{\complement\Omega}(x) \geq r \text{ and } d_{\complement\Omega}(y) \geq r.$$

There exists $N > 0$ such that for all $n > N$, $\|d_{\complement\Omega_n} - d_{\complement\Omega}\|_{C(D)} < r/2$ and

$$\begin{aligned} d_{\complement\Omega_n}(x) &\geq d_{\complement\Omega}(x) - r/2 = r/2 \Rightarrow B(x, r/2) \subset \Omega_n, \\ d_{\complement\Omega_n}(y) &\geq d_{\complement\Omega}(y) - r/2 = r/2 \Rightarrow B(y, r/2) \subset \Omega_n. \end{aligned}$$

By convexity of Ω_n, $x_\lambda = \lambda x + (1-\lambda)y \in \Omega_n$,

$$\begin{aligned} &B(x_\lambda, r/2) \subset \lambda\, B(x, r/2) + (1-\lambda) B(y, r/2) \subset \Omega_n \\ \Rightarrow \forall n > N, \quad &d_{\complement\Omega_n}(x_\lambda) \geq r/2 \quad \Rightarrow d_{\complement\Omega}(x_\lambda) \geq r/2 \quad \Rightarrow B(x_\lambda, r/2) \subset \Omega. \end{aligned}$$

Therefore $x_\lambda \in \Omega$, Ω is convex, and $\mathcal{C}_d^c(D)$ is compact. Finally, $\mathcal{C}_d^c(E;D) = C_d^c(E;D) \cap \mathcal{C}_d^c(D)$ and $\mathcal{C}_{d,\text{loc}}^c(E;D) = C_{d,\text{loc}}^c(E;D) \cap \mathcal{C}_d^c(D)$ are closed since $\mathcal{C}_d^c(D)$ is closed and $C_d^c(E;D)$ and $C_{d,\text{loc}}^c(E;D)$ are closed by Theorem 2.6. □

9 Compactness Theorems for Sets of Bounded Curvature

We have seen two compactness conditions in Theorems 6.6 and 6.7 of section 6.3 for sets of positive reach from a condition in a tubular neighborhood. In this section we give compactness theorems for families of sets of global and local bounded curvature.

For the family of sets with bounded curvature, the key result is the compactness of the embeddings

$$\mathrm{BC}_d(D) = \left\{d_A \in C_d(D) \,:\, \nabla d_A \in \mathrm{BV}(D)^N\right\} \to W^{1,p}(D), \tag{9.1}$$

$$\mathrm{BC}_d^c(D) = \left\{d_{\complement\Omega} \in C_d^c(D) \,:\, \nabla d_{\complement\Omega} \in \mathrm{BV}(D)^N\right\} \to W^{1,p}(D) \tag{9.2}$$

for bounded open Lipschitzian subsets D of $\mathbf{R}^N$ and p, $1 \le p < \infty$. It is the analogue of the compactness Theorem 6.3 of Chapter 5 for Caccioppoli sets

$$\mathrm{BX}(D) = \{\chi \in \mathrm{X}(D) \,:\, \chi \in \mathrm{BV}(D)\} \to L^p(D), \tag{9.3}$$

which is a consequence of the compactness of the embedding

$$\mathrm{BV}(D) \to L^1(D) \tag{9.4}$$

for bounded open Lipschitzian subsets D of $\mathbf{R}^N$ (cf. C. B. Morrey, Jr. [**1**, Def. 3.4.1, p. 72, Thm. 3.4.4, p. 75] and L. C. Evans and R. F. Gariepy [**1**, Thm. 4, p. 176]).

As for characteristic functions in Chapter 5, we give a first version involving global conditions on a fixed bounded open Lipschitzian holdall D. In the second version, the sets are contained in a bounded open holdall D with local conditions in their tubular neighborhood or the tubular neighborhood of their boundary.

9.1 Global Conditions in D

Theorem 9.1. *Let D be a nonempty bounded open Lipschitzian holdall in $\mathbf{R}^N$. The embedding* (9.1) *is compact. Thus for any sequence $\{A_n\}$, $\varnothing \ne A_n$, of subsets of $\overline{D}$ such that*

$$\exists c > 0, \forall n \ge 1, \quad \|D^2 d_{A_n}\|_{M^1(D)} \le c, \tag{9.5}$$

there exist a subsequence $\{A_{n_k}\}$ and $A \ne \varnothing$, such that $\nabla d_A \in \mathrm{BV}(D)^N$ and

$$d_{A_{n_k}} \to d_A \text{ in } W^{1,p}(D)\text{-strong}$$

for all p, $1 \le p < \infty$. Moreover, for all $\varphi \in \mathcal{D}^0(D)$,

$$\lim_{n\to\infty} \langle \partial_{ij} d_{A_{n_k}}, \varphi\rangle = \langle \partial_{ij} d_A, \varphi\rangle, \; 1 \le i,j \le N, \quad \text{and} \quad \|D^2 d_A\|_{M^1(D)} \le c. \tag{9.6}$$

Proof. Given $c > 0$ consider the set

$$S_c \stackrel{\text{def}}{=} \left\{d_A \in C_d(D) \,:\, \|D^2 d_A\|_{M^1(D)} \le c\right\}.$$

By compactness of the embedding (9.4), given any sequence $\{d_{A_n}\}$, there exist a subsequence, still denoted by $\{d_{A_n}\}$, and $f \in \mathrm{BV}(D)^N$ such that $\nabla d_{A_n} \to f$ in $L^1(D)^N$. But by Theorem 2.2 (ii), $C_d(D)$ is compact in $C(\overline{D})$ for bounded D and there exist another subsequence $\{d_{A_{n_k}}\}$ and $d_A \in C_d(D)$ such that $d_{A_{n_k}} \to d_A$ in $C(\overline{D})$ and, a fortiori, in $L^1(D)$. Therefore, $d_{A_{n_k}}$ converges in $W^{1,1}(D)$ and also in

$L^1(D)$. By uniqueness of the limit, $f = \nabla d_A$ and $d_{A_{n_k}}$ converges in $W^{1,1}(D)$ to d_A. For $\Phi \in \mathcal{D}^1(D)^{N\times N}$ as k goes to infinity

$$\int_D \nabla d_{A_{n_k}} \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx \to \int_D \nabla d_A \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx$$
$$\Rightarrow \left| \int_D \nabla d_A \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx \right| = \lim_{k\to\infty} \left| \int_D \nabla d_{A_{n_k}} \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx \right| \le c\|\Phi\|_{C(D)},$$

$\|D^2 d_A\|_{M^1(D)} \le c$, and $\nabla d_A \in \mathrm{BV}(D)^N$. This proves the compactness of the embedding for $p = 1$ and properties (9.6). The conclusions remain true for $p \ge 1$ by the equivalence of the $W^{1,p}$-topologies on $C_d(D)$ in Theorem 4.2 (i). $\square$

When D is bounded open, $C_d^c(D)$ is compact in $C(\overline{D})$ and closed in $W_0^{1,p}(D)$, $1 \le p < \infty$, and we have the analogue of the previous compactness theorem.

Theorem 9.2. *Let D, $\varnothing \ne D \subset \mathbf{R}^N$, be bounded open Lipschitzian. The embedding (9.2) is compact. Thus for any sequence $\{\Omega_n\}$ of open subsets of D such that*

$$\exists h > 0,\ \exists c > 0,\ \forall n, \quad \|D^2 d_{\complement\Omega_n}\|_{M^1(D)} \ \le\ c, \tag{9.7}$$

there exist a subsequence $\{\Omega_{n_k}\}$ and an open subset Ω of D such that $\nabla d_{\complement\Omega} \in \mathrm{BV}(D)^N$ and

$$d_{\complement\Omega_{n_k}} \to d_{\complement\Omega} \ \text{in}\ W_0^{1,p}(D) \tag{9.8}$$

for all p, $1 \le p < \infty$. Moreover, for all $\varphi \in \mathcal{D}^0(D)^{N\times N}$,

$$\langle D^2 d_{\complement\Omega_n}, \varphi\rangle \ \to\ \langle D^2 d_{\complement\Omega}, \varphi\rangle, \quad \|D^2 d_{\complement\Omega}\|_{M^1(D)} \ \le\ c, \quad \text{and} \quad \chi_\Omega \in \mathrm{BV}(D). \tag{9.9}$$

Proof. Given $c > 0$ consider the set

$$S_c^c \overset{\text{def}}{=} \left\{ d_{\complement\Omega} \in C_d^c(D) \,:\, \|D^2 d_{\complement\Omega}\|_{M^1(D)} \le c \right\}.$$

By compactness of the embedding (9.4), given any sequence $\{d_{\complement\Omega_n}\}$ there exist a subsequence, still denoted by $\{d_{\complement\Omega_n}\}$, and $f \in \mathrm{BV}(D)^N$ such that $\nabla d_{\complement\Omega_n} \to f$ in $L^1(D)^N$. But by Theorem 2.4 (ii), $C_d^c(D)$ is compact in $C_0(D)$ for bounded D and there exist another subsequence $\{d_{\complement\Omega_{n_k}}\}$ and $d_{\complement\Omega} \in C_d^c(D)$ such that $d_{\complement\Omega_{n_k}} \to d_{\complement\Omega}$ in $C_0(D)$ and, a fortiori, in $L^1(D)$. Therefore, $d_{\complement\Omega_{n_k}}$ converges in $W_0^{1,1}(D)$ and also in $L^1(D)$. By uniqueness of the limit, $f = \nabla d_{\complement\Omega}$ and $d_{\complement\Omega_{n_k}}$ converges in $W_0^{1,1}(D)$ to $d_{\complement\Omega}$. For all $\Phi \in \mathcal{D}^1(D)^{N\times N}$ as k goes to infinity

$$\int_D \nabla d_{\complement\Omega_{n_k}} \cdot \overrightarrow{\operatorname{div}}\Phi\, dx \to \int_D \nabla d_{\complement\Omega} \cdot \overrightarrow{\operatorname{div}}\Phi\, dx$$
$$\Rightarrow \left| \int_D \nabla d_{\complement\Omega} \cdot \overrightarrow{\operatorname{div}}\Phi\, dx \right| = \lim_{k\to\infty} \left| \int_D \nabla d_{\complement\Omega_{n_k}} \cdot \overrightarrow{\operatorname{div}}\Phi\, dx \right| \le c\|\Phi\|_{C(D)},$$

$\|D^2 d_{\complement\Omega}\|_{M^1(D)} \le c$, $\nabla d_{\complement\Omega} \in \mathrm{BV}(D)^N$, and $\chi_\Omega \in \mathrm{BV}(D)$. This proves the compactness of the embedding for $p = 1$ and properties (9.9). The conclusions remain true for $p \ge 1$ by the equivalence of the $W^{1,p}$-topologies on $C_d^c(D)$ in Theorem 4.2 (i). $\square$

9.2 Local Conditions in Tubular Neighborhoods

The global conditions (9.5) and (9.7) can be weakened to local ones in a neighborhood of each set of the sequence, and the Lipschitzian condition on D can be removed since only the uniform boundedness of the sets of the sequence is required.

Theorem 9.3. *Let D, $\varnothing \neq D \subset \mathbf{R}^N$, be a bounded open holdall and $\{A_n\}$, $\varnothing \neq A_n$, be a sequence of subsets of $\overline{D}$. Assume that there exist $h > 0$ and $c > 0$ such that*

$$\forall n, \quad \|D^2 d_{A_n}\|_{M^1(U_h(\partial \overline{A_n}))} \;\leq\; c. \tag{9.10}$$

Then there exist a subsequence $\{A_{n_k}\}$ and a subset A, $\varnothing \neq A$, of $\overline{D}$ such that $\nabla d_A \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^N)^N$, and for all p, $1 \leq p < \infty$,

$$d_{A_{n_k}} \to d_A \text{ in } W^{1,p}(U_h(D))\text{-strong.} \tag{9.11}$$

Moreover, for all $\varphi \in \mathcal{D}^0(U_h(A))$

$$\lim_{k\to\infty} \langle \partial_{ij} d_{A_{n_k}}, \varphi\rangle = \langle \partial_{ij} d_A, \varphi\rangle, \; 1 \leq i,j \leq N, \quad \|D^2 d_A\|_{M^1(U_h(A))} \leq c, \tag{9.12}$$

and $\chi_{\overline{A}}$ belongs to $\mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^N)$.

Proof. First notice that, since A_n is bounded and nonempty, $\partial \overline{A_n} \neq \varnothing$. Further $U_h(\partial \overline{A_n})$ can be replaced by $U_h(A_n)$ in condition (9.10) since $d_{A_n} = 0$ in $\overline{A_n}$. So it is sufficient to prove the theorem for that case.

Lemma 9.1. *For any A, $\partial \overline{A} \neq \varnothing$ and $h > 0$,*

$$\|D^2 d_A\|_{M^1(U_h(A))} = \|D^2 d_A\|_{M^1(U_h(\partial \overline{A}))}.$$

The proof of the lemma will be given after the proof of the theorem.

(i) The assumption $A_n \subset \overline{D}$ implies $U_h(A_n) \subset U_h(D)$. Since $U_h(D)$ is bounded, there exist a subsequence, still indexed by n, and a set A, $\varnothing \neq A \subset \overline{D}$, such that $d_{A_n} \to d_A$ in $C(\overline{U_h(D)})$ and another subsequence, still denoted by $\{d_{A_n}\}$, such that

$$d_{A_n} \to d_A \quad \text{in } H^1(U_h(D))\text{-weak.}$$

For all $\varepsilon > 0$, $0 < 3\varepsilon < h$, there exists $N > 0$ such that for all $n \geq N$ and x in $U_h(D)$

$$d_{A_n}(x) \leq d_A(x) + \varepsilon, \quad d_A(x) \leq d_{A_n}(x) + \varepsilon.$$

Therefore,

$$\overline{A}_n \subset U_{h-2\varepsilon}(A_n) \subset U_{h-\varepsilon}(A) \subset U_h(A_n), \tag{9.13}$$

$$\complement U_{h-\varepsilon}(A) \subset \complement U_{h-2\varepsilon}(A_n) \subset \complement \overline{A}_n. \tag{9.14}$$

From (9.10) and (9.13)

$$\forall n \geq N, \quad \|D^2 d_{A_n}\|_{M^1(U_{h-\varepsilon}(A))} \leq c.$$

In order to use the compactness of the embedding (9.4) as in the proof of Theorem 9.1, we would need $U_{h-\varepsilon}(A)$ to be Lipschitzian. To get around this, we construct a bounded Lipschitzian set between $U_{h-2\varepsilon}(A)$ and $U_{h-\varepsilon}(A)$. Indeed, by definition,

$$U_{h-\varepsilon}(A) = \cup_{x\in\overline{A}} B(x, h-\varepsilon) \quad \text{and} \quad \overline{U_{h-2\varepsilon}(A)} \subset U_{h-\varepsilon}(A),$$

and by compactness, there exists a finite sequence of points $\{x_i\}_{i=1}^n$ in $\overline{A}$ such that $|x_i - x_j| < (h-\varepsilon)/2$ for all $i \neq j$ so that no two balls are tangent, and

$$\overline{U_{h-2\varepsilon}(A)} \subset U_B \overset{\text{def}}{=} \cup_{i=1}^n B(x_i, h-\varepsilon) \subset U_h(D).$$

Since U_B is Lipschitzian as the union of a finite number of nontangent balls, it now follows by compactness of the embedding (9.4) for U_B that there exist a subsequence, still denoted by $\{d_{A_n}\}$, and $f \in \mathrm{BV}(U_B)^N$ such that $\nabla d_{A_n} \to f$ in $L^1(U_B)^N$. Since $U_h(D)$ is bounded, $C_d(U_h(D))$ is compact in $C(\overline{U_h(D)})$ and there exist another subsequence, still denoted by $\{d_{A_n}\}$, and $\varnothing \neq A \subset \overline{D}$ such that $d_{A_n} \to d_A$ in $C(\overline{U_h(D)})$ and, a fortiori, in $L^1(U_h(D))$. Therefore, d_{A_n} converges in $W^{1,1}(U_B)$ and also in $L^1(U_B)$. By uniqueness of the limit, $f = \nabla d_A$ on U_B and d_{A_n} converges to d_A in $W^{1,1}(U_B)$. By Definition 3.3 and Theorem 5.1, ∇d_A and ∇d_{A_n} all belong to $\mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^\mathrm{N})^N$ since they are BV in tubular neighborhoods of their respective boundaries. Moreover, by Theorem 5.2 (ii), $\chi_{\overline{A}} \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^\mathrm{N})$. The above conclusions also hold for the subset $U_{h-2\varepsilon}(A)$ of U_B.

(ii) *Convergence in* $W^{1,p}(U_h(D))$. Consider the integral

$$\begin{aligned}&\int_{U_h(D)} |\nabla d_{A_n} - \nabla d_A|^2\, dx \\ &= \int_{U_{h-2\varepsilon}(A)} |\nabla d_{A_n} - \nabla d_A|^2\, dx + \int_{U_h(D)\backslash U_{h-2\varepsilon}(A)} |\nabla d_{A_n} - \nabla d_A|^2\, dx.\end{aligned}$$

From part (i) the first integral on the right-hand side converges to zero as n goes to infinity. The second integral is on a subset of $\complement U_{h-2\varepsilon}(A)$. From (9.14) for all $n \geq N$,

$$\begin{gathered}|\nabla d_{A_n}(x)| = 1 \text{ a.e. in } \complement\overline{A}_n \supset \complement U_{h-3\varepsilon}(A_n) \supset \complement U_{h-2\varepsilon}(A), \\ |\nabla d_A(x)| = 1 \text{ a.e. in } \complement\overline{A} \supset \complement U_{h-2\varepsilon}(A).\end{gathered}$$

The second integral reduces to

$$\int_{U_h(D)\backslash U_{h-2\varepsilon}(A)} |\nabla d_{A_n} - \nabla d_A|^2\, dx = \int_{U_h(D)\backslash U_{h-2\varepsilon}(A)} 2\,(1 - \nabla d_{A_n} \cdot \nabla d_A)\, dx,$$

which converges to zero since $\nabla d_{A_n} \rightharpoonup \nabla d_A$ in $L^2(U_h(D))^N$-weak in part (i) and the fact that $|\nabla d_A| = 1$ almost everywhere in $U_h(D)\backslash U_{h-2\varepsilon}(A)$. Therefore, since $d_{A_n} \to d_A$ in $C(\overline{U_h(D)})$,

$$d_{A_n} \to d_A \text{ in } H^1(U_h(D))\text{-strong},$$

and by Theorem 4.2 (i) the convergence is true in $W^{1,p}(U_h(D))$ for all $p \geq 1$.

(iii) *Properties* (9.12). Consider the initial subsequence $\{d_{A_n}\}$ which converges to d_A in $H^1(U_h(D))$-weak constructed at the beginning of part (i). This sequence is independent of ε and the subsequent constructions of other subsequences. By convergence of d_{A_n} to d_A in $H^1(U_h(D))$-weak for each $\Phi \in \mathcal{D}^1(U_h(A))^{N\times N}$,

$$\lim_{n\to\infty} \int_{U_h(A)} \nabla d_{A_n} \cdot \overrightarrow{\operatorname{div}}\, \Phi \, dx = \int_{U_h(A)} \nabla d_A \cdot \overrightarrow{\operatorname{div}}\, \Phi \, dx.$$

Each such Φ has compact support in $U_h(A)$, and there exists $\varepsilon = \varepsilon(\Phi) > 0$, $0 < 3\varepsilon < h$, such that

$$\overline{\operatorname{supp} \Phi} \subset U_{h-2\varepsilon}(A).$$

From part (ii) there exists $N(\varepsilon) > 0$ such that

$$\forall n \geq N(\varepsilon), \quad U_{h-2\,\varepsilon}(A_n) \subset U_{h-\varepsilon}(A) \subset U_h(A_n).$$

For $n \geq N(\varepsilon)$ consider the integral

$$\begin{aligned}
\int_{U_h(A)} \nabla d_{A_n} \cdot \overrightarrow{\operatorname{div}}\, \Phi \, dx &= \int_{U_{h-2\varepsilon}(A)} \nabla d_{A_n} \cdot \overrightarrow{\operatorname{div}}\, \Phi \, dx = \int_{U_h(A_n)} \nabla d_{A_n} \cdot \overrightarrow{\operatorname{div}}\, \Phi \, dx \\
\Rightarrow \left| \int_{U_h(A)} \nabla d_{A_n} \cdot \overrightarrow{\operatorname{div}}\, \Phi \, dx \right| &\leq \|D^2 d_{A_n}\|_{M^1(U_h(A_n))} \, \|\Phi\|_{C(U_h(A_n))} \\
&\leq c \, \|\Phi\|_{C(U_{h-2\varepsilon}(A))} = c \, \|\Phi\|_{C(U_h(A))}.
\end{aligned}$$

By convergence of ∇d_{A_n} to ∇d_A in $L^2(D \cup U_h(A))$-weak, for all $\Phi \in \mathcal{D}^1(U_h(A))^{N\times N}$

$$\left| \int_{U_h(A)} \nabla d_A \cdot \overrightarrow{\operatorname{div}}\, \Phi \, dx \right| \leq c \, \|\Phi\|_{C(U_h(A))} \quad \Rightarrow \|D^2 d_A\|_{M^1(U_h(A))} \leq c.$$

Finally, the convergence remains true for all subsequences constructed in parts (i) and (ii). This completes the proof. □

Proof of Lemma 9.1. First check that

$$U_h(A) = U_h(\partial \overline{A}) \cup_{x \in A_{-h}} B(x,h), \quad A_{-h} \stackrel{\text{def}}{=} \left\{ x \in \mathbf{R}^{\mathrm{N}} \, : \, B(x,h) \subset A \right\}.$$

It is sufficient to show that all points x of $\operatorname{int} A$ are contained in the right-hand side of the above expression. If $d_{\partial \overline{A}}(x) < h$, then $x \in U_h(\partial \overline{A})$; if $d_{\partial \overline{A}}(x) \geq h$, then $B(x,h) \subset A$ and $x \in A_{-h}$. For all $\Phi \in \mathcal{D}^1(U_h(A))^{N\times N}$, Φ has compact support and there exists $\varepsilon > 0$, $0 < 2\varepsilon < h$, such that

$$\overline{\operatorname{supp} \Phi} \subset U_{h-2\varepsilon}(A).$$

But $\overline{U_{h-\varepsilon}(A)} \subset U_h(A)$ and there exists $\{x_j \in A_{-h} \, : \, 1 \leq j \leq m\}$ such that

$$\overline{U_{h-\varepsilon}(A)} \subset U_h(\partial \overline{A}) \cup_{j=1}^m B(x_j, h).$$

Let $\psi_0 \in \mathcal{D}(U_h(\partial \overline{A}))$ and let $\psi_j \in \mathcal{D}(B(x_j,h))$ be a partition of unity such that

$$0 \leq \psi_j \leq 1, \quad \sum_{j=0}^{m} \psi_j = 1 \text{ on } \overline{U_{h-2\varepsilon}(A)}.$$

Consider the integral

$$\int_{U_h(A)} \nabla d_A \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx = \int_{U_{h-2\varepsilon}(A)} \nabla d_A \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx = \int_{U_{h-2\varepsilon}(A)} \nabla d_A \cdot \overrightarrow{\operatorname{div}} \left(\sum_{j=0}^{m} \psi_j \Phi \right) dx$$

$$= \int_{U_{h-2\varepsilon}(A)} \nabla d_A \cdot \overrightarrow{\operatorname{div}} \left(\psi_0 \Phi\right) dx + \sum_{j=1}^{m} \int_{U_{h-2\varepsilon}(A)} \nabla d_A \cdot \overrightarrow{\operatorname{div}} \left(\psi_j \Phi\right) dx.$$

By construction for $\|\Phi\|_{C(U_h(A))} \leq 1$

$$\psi_0 \Phi \in \mathcal{D}^1(U_{h-2\varepsilon}(A) \cap U_h(\partial \overline{A}))^{N\times N}, \quad \|\psi_0 \Phi\|_{C(U_h(\partial\overline{A}))} \leq 1,$$
$$\psi_j \Phi \in \mathcal{D}^1(U_{h-2\varepsilon}(A) \cap B(x_j, h))^{N\times N}, \quad \|\psi_j \Phi\|_{C(B(x_j,h))} \leq 1, \quad 1 \leq j \leq m.$$

But for j, $1 \leq j \leq m$, $B(x_j, h) \subset \operatorname{int} A$, where both d_A and ∇d_A are identically zero. As a result the above integral reduces to

$$\int_{U_h(A)} \nabla d_A \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx = \int_{U_{h-2\varepsilon}(A)} \nabla d_A \cdot \overrightarrow{\operatorname{div}}\, (\psi_0\Phi)\; dx = \int_{U_h(\partial\overline{A})} \nabla d_A \cdot \overrightarrow{\operatorname{div}}\, (\psi_0\Phi)\; dx$$

$$\Rightarrow \left| \int_{U_h(A)} \nabla d_A \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx \right| \leq \|D^2 d_A\|_{M^1(U_h(\partial\overline{A}))} \|\psi_0 \Phi\|_{C(U_h(\partial\overline{A}))}$$

$$\Rightarrow \|D^2 d_A\|_{M^1(U_h(A))} \leq \|D^2 d_A\|_{M^1(U_h(\partial\overline{A}))},$$

and this completes the proof. □

Theorem 9.4. *Let D, $\varnothing \neq D \subset \mathbf{R}^{\mathrm{N}}$, be a bounded open holdall. Let $\{\Omega_n\}$, $\varnothing \neq \Omega_n \subset D$, be a sequence of open subsets and assume that*

$$\exists h > 0 \text{ and } \exists c > 0 \text{ such that } \quad \forall n, \quad \|D^2 d_{\complement\Omega_n}\|_{M^1(U_h(\partial\Omega_n))} \leq c. \tag{9.15}$$

Then there exist a subsequence $\{\Omega_{n_k}\}$ and an open subset Ω of D such that $\nabla d_{\complement\Omega} \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})^N$ and for all p, $1 \leq p < \infty$,

$$d_{\complement\Omega_{n_k}} \to d_{\complement\Omega} \text{ in } W^{1,p}_0(D)\text{-strong}. \tag{9.16}$$

Moreover for all $\varphi \in \mathcal{D}^0(U_h(\complement\Omega))$

$$\lim_{k\to\infty} \langle \partial_{ij} d_{\complement\Omega_{n_k}}, \varphi\rangle = \langle \partial_{ij} d_{\complement\Omega}, \varphi\rangle,\ 1 \leq i,j \leq N, \quad \|D^2 d_{\complement\Omega}\|_{M^1(U_h(\complement\Omega))} \leq c, \tag{9.17}$$

and χ_Ω belongs to $\mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$.

Proof. First note that $\partial\Omega_n \neq \varnothing$ since Ω_n is bounded and nonempty. By Lemma 9.1, $U_h(\partial\Omega_n)$ can be replaced by $U_h(\complement\Omega_n)$ in condition (9.15). Moreover since $\complement\Omega_n$ can be unbounded, we shall work with the bounded neighborhoods $U_h(\complement_{\overline{D}}\Omega_n)$ of $\complement_{\overline{D}}\Omega_n$ rather than with $U_h(\complement\Omega_n)$.

Lemma 9.2. *Let $h > 0$ and let $\varnothing \neq \Omega \subset D \subset \mathbf{R}^{\mathrm{N}}$ be bounded open. Then*

$$\|D^2 d_{\complement\Omega}\|_{M^1(U_h(\complement_{\overline{D}}\Omega))} = \|D^2 d_{\complement\Omega}\|_{M^1(U_h(\complement\Omega))}.$$

If $\partial\Omega \neq \varnothing$,

$$\|D^2 d_{\complement\Omega}\|_{M^1(U_h(\partial\Omega))} = \|D^2 d_{\complement\Omega}\|_{M^1(U_h(\complement\Omega))}.$$

The proof of this lemma will be given after the proof of the theorem.

(i) By Theorem 2.4 (ii) since D is bounded, there exist a subsequence, still denoted by $\{d_{\complement\Omega_n}\}$, and an open subset Ω of D such that

$$d_{\complement\Omega_n} \to d_{\complement\Omega} \quad \text{in } C_0(D) \text{ and } C_c(D \cup U_h(\complement\Omega))$$

and another subsequence, still denoted by $\{d_{\complement\Omega_n}\}$, such that

$$d_{\complement\Omega_n} \to d_{\complement\Omega} \quad \text{in } H_0^1(D)\text{-weak and } H_0^1(D \cup U_h(\complement\Omega))\text{-weak.}$$

For all $\varepsilon > 0$, $0 < 3\varepsilon < h$, there exists $N > 0$ such that for all $n \geq N$

$$d_{\complement\Omega_n}(x) \leq d_{\complement\Omega}(x) + \varepsilon, \quad d_{\complement\Omega}(x) \leq d_{\complement\Omega_n}(x) + \varepsilon.$$

Clearly, since $\Omega \subset D$ and $\Omega_n \subset D$ and D is open, $\complement_{\overline{D}}\Omega_n \neq \varnothing$ and $\complement_{\overline{D}}\Omega \neq \varnothing$. Furthermore,

$$\complement_{\overline{D}}\Omega_n \subset U_{h-2\,\varepsilon}(\complement_{\overline{D}}\Omega_n) \subset U_{h-\varepsilon}(\complement_{\overline{D}}\Omega) \subset U_h(\complement_{\overline{D}}\Omega_n), \tag{9.18}$$

$$\complement_{\overline{D}}U_{h-\varepsilon}(\complement_{\overline{D}}\Omega) \subset \complement_{\overline{D}}U_{h-2\,\varepsilon}(\complement_{\overline{D}}\Omega_n) \subset \Omega_n. \tag{9.19}$$

From (9.15) and (9.18)

$$\forall n \geq N, \quad \|D^2 d_{\complement\Omega_n}\|_{M^1(U_{h-\varepsilon}(\complement_{\overline{D}}\Omega))} \leq c. \tag{9.20}$$

As in part (i) of the proof of Theorem 9.3, we can construct a bounded open Lipschitzian set U_B between $U_{h-2\varepsilon}(\complement_{\overline{D}}\Omega)$ and $U_{h-\varepsilon}(\complement_{\overline{D}}\Omega)$ such that

$$\overline{U_{h-2\varepsilon}(\complement_{\overline{D}}\Omega)} \subset U_B \subset U_{h-\varepsilon}(\complement_{\overline{D}}\Omega).$$

Since U_B is bounded and Lipschitzian, it now follows by compactness of the embedding (9.4) for U_B that there exist a subsequence, still denoted by $\{d_{\complement\Omega_n}\}$, and $f \in \mathrm{BV}(U_B)^N$ such that $\nabla d_{\complement\Omega_n} \to f$ in $L^1(U_B)^N$. Since $D \cup U_B$ is bounded, $C_d^c(D \cup U_B)$ is compact in $C_0(D \cup U_B)$, and there exist another subsequence, still denoted by $\{d_{\complement\Omega_n}\}$, and an open subset Ω of D such that $d_{\complement\Omega_n} \to d_{\complement\Omega}$ in $C_0(D \cup U_B)$ and, a fortiori, in $L^1(D \cup U_B)$. Therefore $d_{\complement\Omega_n}$ converges in $W^{1,1}(U_B)$ and also in $L^1(U_B)$. By uniqueness of the limit, $f = \nabla d_{\complement\Omega}$ on B_D and $d_{\complement\Omega_n}$ converges to $d_{\complement\Omega}$ in $W^{1,1}(U_B)$. By Definition 3.3 and Theorem 5.1, $\nabla d_{\complement\Omega}$ and $\nabla d_{\complement\Omega_n}$ all belong to $\mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})^N$ since they are BV in tubular neighborhoods $U_h(\complement_{\overline{D}}\Omega)$ and $U_h(\complement_{\overline{D}}\Omega_n)$ of their respective boundaries $\partial\Omega$ and $\partial\Omega_n$. Moreover, by Theorem 5.2 (ii), $\chi_{\complement\Omega} \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$ and, a fortiori, χ_Ω since Ω is open. The above conclusions also hold for the subset $U_{h-2\varepsilon}(\complement_{\overline{D}}\Omega)$ of U_B.

(ii) *Convergence in* $W_0^{1,p}(D)$. Consider the integral

$$\begin{aligned}
&\int_D |\nabla d_{\complement\Omega_n} - \nabla d_{\complement\Omega}|^2\, dx \\
&= \int_{D\cap U_{h-2\varepsilon}(\complement_{\overline{D}}\Omega)} |\nabla d_{\complement\Omega_n} - \nabla d_{\complement\Omega}|^2\, dx + \int_{D\setminus U_{h-2\varepsilon}(\complement_{\overline{D}}\Omega)} |\nabla d_{\complement\Omega_n} - \nabla d_{\complement\Omega}|^2\, dx.
\end{aligned}$$

From part (i) the first integral on the right-hand side converges to zero as n goes to infinity. The second integral is on a subset of $\complement_{\overline{D}} U_{h-2\varepsilon}(\complement_{\overline{D}}\Omega)$. From the relations (9.19) for all $n \geq N$,

$$|\nabla d_{\complement\Omega_n}(x)| = 1 \text{ a.e. in } \Omega_n \supset \complement_{\overline{D}} U_{h-3\varepsilon}(\complement_{\overline{D}}\Omega_n) \supset \complement_{\overline{D}} U_{h-2\varepsilon}(\complement_{\overline{D}}\Omega),$$
$$|\nabla d_{\Omega}(x)| = 1 \text{ a.e. in } \Omega \supset \complement_{\overline{D}} U_{h-2\varepsilon}(\complement_{\overline{D}}\Omega).$$

The second integral reduces to

$$\int_{D\backslash U_{h-2\varepsilon}(\complement_{\overline{D}}\Omega)} |\nabla d_{\complement\Omega_n} - \nabla d_{\complement\Omega}|^2\, dx = \int_{D\backslash U_{h-2\varepsilon}(\complement_{\overline{D}}\Omega)} 2\,(1 - \nabla d_{\complement\Omega_n} \cdot \nabla d_{\complement\Omega})\, dx,$$

which converges to zero by weak convergence of $\nabla d_{\complement\Omega_n}$ to $\nabla d_{\complement\Omega}$ in $L^2(D)^N$ and the fact that $|\nabla d_{\complement\Omega}| = 1$ almost everywhere in $D\backslash U_{h-2\varepsilon}(\complement_{\overline{D}}\Omega)$. Therefore, since $d_{\complement\Omega_n} \to d_{\complement\Omega}$ in $C_0(D)$

$$d_{\complement\Omega_n} \to d_{\complement\Omega} \text{ in } H_0^1(D)\text{-strong},$$

and by Theorem 4.2 (i) the convergence is true in $W_0^{1,p}(D)$ for all $p \geq 1$.

(iii) *Properties* (9.17). Consider the initial subsequence $\{d_{\complement\Omega_n}\}$, which converges to $d_{\complement\Omega}$ in $H_0^1(D \cup U_h(\complement\Omega))$-weak constructed at the beginning of part (i). This sequence is independent of ε and the subsequent constructions of other subsequences. By convergence of $d_{\complement\Omega_n}$ to $d_{\complement\Omega}$ in $H_0^1(D \cup U_h(\complement\Omega))$-weak for each $\Phi \in \mathcal{D}^1(U_h(\complement\Omega))^{N\times N}$,

$$\lim_{n\to\infty} \int_{U_h(\complement\Omega)} \nabla d_{\complement\Omega_n} \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx = \int_{U_h(\complement\Omega)} \nabla d_{\complement\Omega} \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx.$$

Now each $\Phi \in \mathcal{D}^1(U_h(\complement\Omega))^{N\times N}$ has compact support in $U_h(\complement\Omega)$ and there exists $\varepsilon = \varepsilon(\Phi) > 0$, $0 < 3\varepsilon < h$, such that

$$\overline{\operatorname{supp}\Phi} \subset U_{h-2\varepsilon}(\complement\Omega).$$

From part (ii) there exists $N(\varepsilon) > 0$ such that

$$\forall n \geq N(\varepsilon), \quad U_{h-2\,\varepsilon}(\complement\Omega_n) \subset U_{h-\varepsilon}(\complement\Omega) \subset U_h(\complement\Omega_n).$$

For $n \geq N(\varepsilon)$ consider the integral

$$\begin{aligned}
&\int_{U_h(\complement\Omega)} \nabla d_{\complement\Omega_n} \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx = \int_{U_{h-2\varepsilon}(\complement\Omega)} \nabla d_{\complement\Omega_n} \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx = \int_{U_h(\complement\Omega_n)} \nabla d_{\Omega_n} \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx \\
&\Rightarrow \left| \int_{U_h(\complement\Omega)} \nabla d_{\complement\Omega_n} \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx \right| \leq \|D^2 d_{\complement\Omega_n}\|_{M^1(U_h(\complement_D\Omega_n))}\, \|\Phi\|_{C(U_h(\complement_D\Omega_n))} \\
&\qquad\qquad \leq c\, \|\Phi\|_{C(U_{h-2\varepsilon}(\complement_D\Omega))} = c\, \|\Phi\|_{C(U_h(\complement\Omega))}.
\end{aligned}$$

By convergence of $\nabla d_{\complement\Omega_n}$ to $\nabla d_{\complement\Omega}$ in $L^2(D \cup U_h(\complement\Omega))$-weak, for all $\Phi \in \mathcal{D}^1(U_h(\complement\Omega))^{N\times N}$

$$\left| \int_{U_h(\complement\Omega)} \nabla d_{\complement\Omega} \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx \right| \leq c\, \|\Phi\|_{C(U_h(\complement\Omega))} \quad \Rightarrow \|D^2 d_{\complement\Omega}\|_{M^1(U_h(\complement\Omega))} \leq c.$$

Finally the convergence remains true for all subsequences constructed in parts (i) and (ii). This completes the proof. $\square$

Proof of Lemma 9.2. The proof is similar to that of Lemma 9.1 with

$$U_h(\complement\Omega) = U_h(\complement_{\overline{D}}\Omega) \cup \operatorname{int}\complement D = U_h(\complement_{\overline{D}}\Omega) \cup_{x \in (\complement\overline{D})_{-h}} B(x,h),$$
$$(\complement\overline{D})_{-h} \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : B(x,h) \subset \complement\overline{D}\right\},$$

since both $d_{\complement\Omega}$ and $\nabla d_{\complement\Omega}$ are identically zero on $\overline{\complement D}$. □

Chapter 7

Metrics via Oriented Distance Functions

1 Introduction

In this chapter we study *oriented distance functions* and their role in the description of the geometric properties and smoothness of domains and their boundary. They constitute a special family of the so-called *algebraic* or *signed distance functions*. Signed distance functions make sense only when some appropriate *orientation* of the underlying set is available to assign a sign, while oriented distance functions are always well-defined. Our choice of terminology emphasizes the fact that, for a smooth open domain, the associated oriented distance function specifies the *orientation* of the normal to the boundary of the set. They enjoy many interesting properties. For instance, they retain the nice properties of the distance functions but also generate the classical geometric properties associated with sets and their boundary. The smoothness of the oriented distance function in a neighborhood of the boundary of the set is equivalent to the local smoothness of its boundary. Similarly, the convexity of the function is equivalent to the convexity of the closure of the set. In addition, their respective gradient and Hessian matrix respectively coincide with the *unit outward normal* and the *second fundamental form* on the boundary of the set. Finally, they provide a framework for the classification of domains and sets according to their degree of smoothness, much like Sobolev spaces and spaces of continuous and Hölderian functions do for functions.

The first part of the chapter deals with the basic definitions and constructions and the main results. The second part specializes to specific subfamilies of oriented distance functions. The last part concentrates on compact families of subsets of oriented distance functions. In the first part, section 2 presents the basic properties, introduces the uniform metric topology, and shows its connection with the Hausdorff and complementary Hausdorff topologies of Chapter 6. Section 3 is devoted to the differentiability properties and the associated set of projections onto the boundary, and it completes the treatment of skeletons and cracks. Section 8 gives the equivalence of the smoothness of a set and the smoothness of its oriented distance function in a neighborhood of its boundary for sets of class $C^{1,1}$ or better.

When the domain is sufficiently smooth the trace of the Hessian matrix of second-order partial derivatives on the boundary is the classical second fundamental form of geometry. Section 4 deals with the $W^{1,p}$-topology on the set of oriented distance functions and the closed subfamily of sets for which the volume of the boundary is zero.

In the second part of the chapter, section 5 studies the subfamily of sets for which the gradient of the oriented distance function is a vector of functions of bounded variation: the sets with *global* or *local bounded curvature*. There are large classes for which general compactness theorems will be proved in section 11. Examples are given to illustrate the behavior of the norms in tubular neighborhoods as the thickness of the neighborhood goes to zero in section 5.1. Section 9 introduces *Sobolev or $W^{s,p}$-domains* which provide a framework for the classification of sets according to their degree of smoothness. For smooth sets this classification intertwines with the classical classification of C^k- and Hölderian domains. Section 10 extends the characterization of closed convex sets by the distance function to the oriented distance function and introduces the notion of *semiconvex sets*. The set of equivalence classes of convex subsets of a compact holdall D is again compact, as it was for all the topologies considered in Chapters 5 and 6. Section 7 shows that sets of positive reach introduced in Chapter 6 are sets of locally bounded curvature and that the boundary of their closure has zero volume. A compactness theorem is also given. Section 12 introduces the notion of γ-density perimeter, which is a relaxation of the $(N-1)$-dimensional *upper Minkowski content*, and a compactness theorem for a family of subsets of a bounded holdall that have a uniformly bounded γ-density perimeter.

In the last part of the chapter, section 11 gives compactness theorems for sets of global and local bounded curvatures from a uniform bound in tubular neighborhoods of their boundary. They are the analogues of the theorems of Chapter 6. Section 13 gives the compactness theorem for the family of subsets of a bounded holdall that have the *uniform fat segment property*. Equivalent conditions are given on the local graph functions of the family in section 13.2. The results are specialized to the compactness under the *uniform cusp/cone* property. In particular we recover the compactness of Theorem 6.11 of section 6.4 in Chapter 5 for the associated family of characteristic functions in $L^p(D)$. Section 14 combines the compactness under the uniform fat segment property with a bound on the De Giorgi perimeter of Caccioppoli sets in section 14.1 and the γ-density perimeter in section 14.2.

Section 15 introduces the families of *cracked sets* and uses them in section 16 to provide an original solution to a variation of the image segmentation problem of D. MUMFORD and J. SHAH [**2**]. Cracked sets are more general than sets which are locally the epigraph of a continuous function in the sense that they include domains with cracks, and sets that can be made up of components of different co-dimensions. The Hausdorff $(N-1)$ measure of their boundary is not necessarily finite. Yet compact families (in the $W^{1,p}$-topology) of such sets can be constructed.

2 Uniform Metric Topology

2.1 The Family of Oriented Distance Functions $C_b(D)$

The distance function d_A provides a good description of a domain A from the "outside." However, its gradient undergoes a jump discontinuity at the boundary of A which prevents the extraction of information about the smoothness of A from the smoothness of d_A in a neighborhood of ∂A. To get around this, it is natural to take into account the negative of the distance function of its complement $\complement A$ that cancels the jump discontinuity at the boundary of smooth sets.

Definition 2.1.
Given $A \subset \mathbf{R}^{\mathrm{N}}$, the *oriented distance function* from $x \in \mathbf{R}^{\mathrm{N}}$ to A is defined as

$$\boxed{b_A(x) \stackrel{\text{def}}{=} d_A(x) - d_{\complement A}(x).} \tag{2.1}$$

This function gives a level set description of a set whose boundary coincides with the zero level set. From Chapter 6 the function b_A is finite in $\mathbf{R}^{\mathrm{N}}$ if and only if $\varnothing \neq A \neq \mathbf{R}^{\mathrm{N}}$, since $b_A = d_A = +\infty$ when $A = \varnothing$ and $b_A = -d_{\complement A} = -\infty$ when $\complement A = \varnothing$. This condition is completely equivalent to $\partial A \neq \varnothing$, in which case b_A coincides with the *algebraic distance function* to the boundary of A:

$$\boxed{b_A(x) = \begin{cases} d_A(x) = d_{\partial A}(x), & x \in \operatorname{int} \complement A, \\ 0, & x \in \partial A, \\ -d_{\complement A}(x) = -d_{\partial A}(x), & x \in \operatorname{int} A. \end{cases}} \tag{2.2}$$

Noting that $b_{\complement A} = -b_A$, it means that we have implicitly chosen the negative sign for the interior of A and the positive sign for the interior of its complement. As b_A is an increasing function from its interior to its exterior, we shall see later that for sets with a smooth boundary, the restriction of the gradient of b_A to ∂A coincides with the outward unit normal to ∂A. Changing the sign of b_A gives the inward orientation to the gradient and the normal.

Associate with a nonempty subset D of $\mathbf{R}^{\mathrm{N}}$ the family

$$\boxed{C_b(D) \stackrel{\text{def}}{=} \left\{ b_A : A \subset \overline{D} \text{ and } \partial A \neq \varnothing \right\}.} \tag{2.3}$$

Theorem 2.1. *Let A be a subset of $\mathbf{R}^{\mathrm{N}}$. Then the following hold:*

(i) $A \neq \varnothing$ *and* $\complement A \neq \varnothing \iff \partial A \neq \varnothing$.

(ii) *Given b_A and b_B in $C_b(D)$,*

$$A \supset B \Rightarrow b_A \leq b_B \quad \text{and} \quad A = B \Rightarrow b_A = b_B,$$

$$b_A \leq b_B \text{ in } D \iff \overline{B} \subset \overline{A} \text{ and } \overline{\complement A} \subset \overline{\complement B},$$

$$b_A = b_B \text{ in } D \iff \overline{B} = \overline{A} \text{ and } \overline{\complement A} = \overline{\complement B} \iff \overline{B} = \overline{A} \text{ and } \partial A = \partial B.$$

In particular, $b_{\mathrm{int}\,A} \le b_A \le b_{\overline{A}}$ *and*

$$b_{\overline{A}} = b_A \iff \partial\overline{A} = \partial A \quad \text{and} \quad b_{\mathrm{int}\,A} = b_A \iff \partial\,\mathrm{int}\,A = \partial A. \tag{2.4}$$

(iii) $|b_A| = d_A + d_{\complement A} = \max\{d_A, d_{\complement A}\} = d_{\partial A}$ *and* $\partial A = \{x \in \mathbf{R}^{\mathrm{N}} : b_A(x) = 0\}$.

(iv) $b_A \ge 0 \iff \overline{\complement A} \supset \partial A \supset \overline{A} \iff \partial A = \overline{A}$.

(v) $b_A = 0 \iff \overline{\complement A} = \partial A = \overline{A} \iff \partial A = \mathbf{R}^{\mathrm{N}}$.

(vi) *If* $\partial A \neq \varnothing$, *the function* b_A *is uniformly Lipschitz continuous in* $\mathbf{R}^{\mathrm{N}}$ *and*

$$\forall x, y \in \mathbf{R}^{\mathrm{N}}, \quad |b_A(y) - b_A(x)| \le |y - x|. \tag{2.5}$$

Moreover, b_A *is (Fréchet) differentiable almost everywhere and*

$$|\nabla b_A(x)| \le 1 \quad \text{a.e. in } \mathbf{R}^{\mathrm{N}}. \tag{2.6}$$

Proof. (i) The proof is obvious.

(ii) By assumption

$$d_A = b_A^+ \le b_B^+ = d_B \text{ and } d_{\complement A} = b_A^- \ge b_B^- = d_{\complement B} \text{ in } \overline{D}.$$

Since $A \subset \overline{D}$ and $B \subset \overline{D}$, then $\complement A \supset \complement\overline{D}$ and $\complement B \supset \complement\overline{D}$ and

$$d_{\complement A} = d_{\complement B} = 0 \text{ in } \complement\overline{D} \quad \Rightarrow d_{\complement A} \ge d_{\complement B} \text{ in } \mathbf{R}^{\mathrm{N}} \quad \Rightarrow \overline{\complement A} \subset \overline{\complement B}.$$

Also since $\overline{B} \subset \overline{D}$ for all $x \in \overline{B}$, $d_A(x) \le d_B(x) = 0$ and $x \in \overline{A}$. Therefore, $\overline{\complement A} \subset \overline{\complement B}$ and $\overline{B} \subset \overline{A}$. Conversely,

$$\overline{B} \subset \overline{A} \quad \Rightarrow d_A \le d_B \text{ in } \mathbf{R}^{\mathrm{N}} \quad \text{and} \quad \overline{\complement A} \subset \overline{\complement B} \quad \Rightarrow d_{\complement B} \le d_{\complement A} \text{ in } \mathbf{R}^{\mathrm{N}}$$

and, a fortiori, in D. The equality case follows from the fact that $b_A = b_B$ if and only if $b_A \ge b_B$ and $b_A \le b_B$. By definition $\mathrm{int}\,A = \complement\overline{\complement A} \subset A \subset \overline{A}$ and $b_{\mathrm{int}\,A} \le b_A \le b_{\overline{A}}$.

(iii) For x in $\overline{A}$

$$b_A(x) = -d_{\complement A}(x) \implies |b_A(x)| = d_{\complement A}(x) \le d_{\partial A}(x)$$

since $\overline{\complement A} \supset \partial A$ and the inf over $\overline{\complement A}$ is smaller than the inf over its subset ∂A. Similarly, for x in $\overline{\complement A}$

$$b_A(x) = d_A(x) \implies |b_A(x)| = d_A(x) \le d_{\partial A}(x),$$

and finally

$$|b_A(x)| \le \max\{d_{\complement A}(x), d_A(x)\} \le d_{\partial A}(x).$$

Conversely, for each x in $\overline{\complement A}$, the set of projections $\Pi_A(x) \subset \overline{A} \cap \overline{\complement A} = \partial A$ is not empty. Hence

$$|b_A(x)| = d_A(x) = \min_{y \in \Pi_A(x)} |x - y| \ge \inf_{y \in \partial A} |x - y| = d_{\partial A}(x),$$

and similarly for all x in $\overline{A}$,

$$|b_A(x)| = d_{\complement A}(x) \geq d_{\partial A}(x).$$

Therefore $|b_A(x)| \geq d_{\partial A}(x)$.

(iv) If $b_A = d_A - d_{\complement A} \geq 0$, then $d_A \geq d_{\complement A}$ and $\overline{A} \subset \overline{\complement A}$, and necessarily $\overline{A} \subset \partial A$ and $\overline{A} = \partial A$. Conversely, if $x \in \partial A$, then by definition $b_A(x) = 0$. If $x \notin \partial A$, then $x \in \complement \partial A = \complement \overline{A} = \operatorname{int} \complement A$ and $b_A(x) = d_A(x) \geq 0$.

(v) $b_A = 0$ is equivalent to $b_A \geq 0$ and $b_{\complement A} = -b_A \geq 0$. Then we apply (v) twice. But $\overline{\complement A} = \partial A = \overline{A} \iff \operatorname{int} \complement A = \varnothing = \operatorname{int} A \iff \partial A = \mathbf{R}^{\mathrm{N}}$.

(vi) Clearly,

$$\begin{aligned} &\forall x, y \in \overline{A}, \quad |b_A(y) - b_A(x)| = |d_{\complement A}(y) - d_{\complement A}(x)| \leq |y - x|, \\ &\forall x, y \in \overline{\complement A}, \quad |b_A(y) - b_A(x)| = |d_A(y) - d_A(x)| \leq |y - x|. \end{aligned}$$

For $x \in \overline{A}$ and $y \in \operatorname{int} \complement A = \complement \overline{A}$, $d_A(y) > 0$ and

$$b_A(y) - b_A(x) = d_A(y) + d_{\complement A}(x) > 0 \quad \Rightarrow |b_A(y) - b_A(x)| = d_A(y) + d_{\complement A}(x).$$

By assumption $B(y, d_A(y)) \subset \operatorname{int} \complement A$. Define the point

$$\begin{aligned} \overline{x} &= y + \frac{d_A(y)}{|x-y|}(x - y) \in \overline{\complement A} \\ \Rightarrow\ d_{\complement A}(x) \leq |x - \overline{x}| &= \left| \left(1 - \frac{d_A(y)}{|x-y|}\right)(y - x) \right| = |x - y| - d_A(y) \\ \Rightarrow\ |b_A(y) - b_A(x)| &= d_A(y) + d_{\complement A}(x) \leq |x - y|. \end{aligned}$$

The argument is similar for $x \in \operatorname{int} A$ and $y \in \overline{\complement A}$. The differentiability follows from Theorem 2.1 (vii) in Chapter 6. □

2.2 Uniform Metric Topology

From Theorem 2.1 (i) the function b_A is finite at each point when $\partial A \neq \varnothing$. This excludes $A = \varnothing$ and $A = \mathbf{R}^{\mathrm{N}}$. The zero function $b_A(x) = 0$ for all x in $\mathbf{R}^{\mathrm{N}}$ corresponds to the equivalence class of sets A such that

$$\overline{A} = \partial A = \overline{\complement A} \text{ or } \partial A = \mathbf{R}^{\mathrm{N}}.$$

This class of sets is not empty. For instance, choose the subset of points of $\mathbf{R}^{\mathrm{N}}$ with rational coordinates or the set of all lines parallel to one of the coordinate axes with rational coordinates.

Let D be a nonempty subset of $\mathbf{R}^{\mathrm{N}}$ and associate with each subset A of $\overline{D}$, $\partial A \neq \varnothing$, the equivalence class

$$[A]_b \stackrel{\text{def}}{=} \left\{ B : \forall B,\ B \subset \overline{D},\ \overline{B} = \overline{A} \text{ and } \partial A = \partial B \right\}$$

and the family of equivalence classes

$$\boxed{\mathcal{F}_b(D) \stackrel{\text{def}}{=} \left\{[A]_b \,:\, \forall A,\, A \subset \overline{D} \text{ and } \partial A \neq \varnothing\right\}.}$$

The equivalence classes induced by b_A are finer than those induced by d_A since both the closures and the boundaries of the respective sets must coincide.

For $C_d(D)$, we have an invariant *closed representative* $\overline{A}$ in the equivalence class $[A]_d$ and for $C_d^c(D)$ an invariant *open representative* $\operatorname{int} A$ for $[\complement A]_d$. For $C_b(D)$, the deficiencies of $C_d(D)$ and $C_d^c(D)$ combine

$$d_{\overline{A}} = d_A \leq d_{\operatorname{int} A} \quad \text{and} \quad d_{\complement \operatorname{int} A} = d_{\overline{\complement A}} = d_{\complement A} \leq d_{\complement \overline{A}},$$

and, in general, there is neither a closed nor an open representative, since

$$b_{\overline{A}} \leq b_A \leq b_{\operatorname{int} A}.$$

One invariant in the class is ∂A, but it does not completely describes the class without one of the other two $\overline{A}$ or $\overline{\complement A}$. We shall see that for sets verifying a uniform segment property $b_{\overline{A}} = b_A = b_{\operatorname{int} A}$ and for convex sets that $b_{\overline{A}} = b_A$.

As in the case of d_A we identify $\mathcal{F}_b(D)$ with the family $C_b(D)$ through the embedding

$$[A]_b \mapsto b_A \colon \mathcal{F}_b(D) \to C_b(D) \subset C(\overline{D}).$$

When D is bounded, the space $C(\overline{D})$ endowed with the norm $\|f\|_{C(D)}$ is a Banach space. Moreover, for each $A \subset \overline{D}$, b_A is bounded, uniformly continuous on D, and $b_A \in C(\overline{D})$. This will induce the following complete metric:

$$\boxed{\rho([A]_b, [B]_b) \stackrel{\text{def}}{=} \|b_A - b_B\|_{C(D)}} \tag{2.7}$$

on $\mathcal{F}_b(D)$.

When D is open but not necessarily bounded, we use the space $C_{\text{loc}}(D)$ defined in section 2.2 of Chapter 6 endowed with the complete metric ρ_δ defined in (2.9) for the family of seminorms $\{q_K\}$ defined in (2.8). It will be shown below that $C_b(D)$ is a closed subset of $C_{\text{loc}}(D)$ and that this induces the following complete metric on $\mathcal{F}_b(D)$:

$$\boxed{\rho_\delta([A]_b, [B]_b) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{q_{K_k}(b_A - b_B)}{1 + q_{K_k}(b_A - b_B)}.} \tag{2.8}$$

The subfamilies

$$\mathcal{F}_b^0(D) \stackrel{\text{def}}{=} \{[A]_b \,:\, \forall A,\, A \subset D,\, \partial A \neq \varnothing,\, \mathrm{m}(\partial A) = 0\},$$

$$\boxed{C_b^0(D) \stackrel{\text{def}}{=} \{b_A \in C_b(D) \,:\, \mathrm{m}(\partial A) = 0\}}$$

of $\mathcal{F}_b(D)$ and $C_b(D)$ will be important. We now have the equivalent of Theorem 2.2 in Chapter 6.

Theorem 2.2. *Let $D \neq \varnothing$ be an open* (resp., *bounded open*) *holdall in* $\mathbf{R}^{\mathrm{N}}$.

(i) *The set $C_b(D)$ is closed in $C_{\mathrm{loc}}(D)$* (resp., $C(\overline{D})$) *and ρ_δ* (resp., ρ) *defines a complete metric topology on $\mathcal{F}_b(D)$.*

(ii) *For a bounded open subset D of $\mathbf{R}^{\mathrm{N}}$, the set $C_b(D)$ is compact in $C(\overline{D})$.*

(iii) *For a bounded open subset D of $\mathbf{R}^{\mathrm{N}}$, the map*

$$b_A \mapsto (b_A^+, b_A^-, |b_A|) = (d_A, d_{\complement A}, d_{\partial A}) : C_b(D) \subset C(\overline{D}) \to C(\overline{D})^3$$

is continuous: for all b_A and b_B in $C_b(D)$,

$$\begin{aligned} &\max\left\{\|d_B - d_A\|_{C(D)}, \|d_{\complement B} - d_{\complement A}\|_{C(D)}, \|d_{\partial B} - d_{\partial A}\|_{C(D)}\right\} \\ &\leq \|b_B - b_A\|_{C(D)}. \end{aligned} \tag{2.9}$$

Proof. (i) It is sufficient to consider the case for which D is bounded. Consider a sequence $\{A_n\} \subset \overline{D}$, $\partial A_n \neq \varnothing$, such that $b_{A_n} \to f$ in $C(\overline{D})$ for some f in $C(\overline{D})$. Associate with each g in $C(D)$ its positive and negative parts

$$g^+(x) = \max\{g(x), 0\}, \quad g^-(x) = \max\{-g(x), 0\}.$$

Then by continuity of this operation

$$\begin{gathered} d_{A_n} = b_{A_n}^+ \to f^+ \text{ and } d_{\complement A_n} = b_{A_n}^- \to f^- \text{ in } C(\overline{D}), \\ d_{\partial A_n} = |b_{A_n}| \to |f| \text{ in } C(\overline{D}). \end{gathered}$$

By Theorem 2.2 (i) of Chapter 6, there exists a closed subset F, $\varnothing \neq F \subset \overline{D}$, such that

$$f^+ = d_F \text{ in } \mathbf{R}^{\mathrm{N}} \quad \text{and} \quad f^+ > 0 \text{ in } \complement\overline{D}.$$

Moreover $F \neq \mathbf{R}^{\mathrm{N}}$ since D is bounded. By Theorem 2.4 and the remark at the beginning of section 2.3 of Chapter 6, there exists an open subset $G \subset D$, $G \neq \mathbf{R}^{\mathrm{N}}$, such that

$$f^- = d_{\complement G} \text{ in } \mathbf{R}^{\mathrm{N}} \text{ and } d_{\complement G} \in C_0(D).$$

Therefore,

$$\begin{gathered} f = f^+ - f^- = d_F - d_{\complement G}, \\ f = f^+ - f^- = d_F \text{ in } \overline{\complement\overline{D}} \quad \text{and} \quad f > 0 \text{ in } \complement\overline{D}. \end{gathered}$$

Define the sets

$$\begin{aligned} A^+ &\stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : f(x) > 0\right\} = \left\{x \in \overline{D} : f(x) > 0\right\} \cup \complement\overline{D}, \\ A^- &\stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : f(x) < 0\right\} = \left\{x \in \overline{D} : f(x) < 0\right\}, \\ A^0 &\stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : f(x) = 0\right\} = \left\{x \in \overline{D} : f(x) = 0\right\}. \end{aligned}$$

They form a partition of $\mathbf{R}^N$, $\mathbf{R}^N = A^- \cup A^0 \cup A^+$, and

$$\begin{gathered}\mathbf{R}^N \neq A^0 \cup A^- = F \neq \varnothing, \quad A^0 \cup A^+ = \complement G \neq \varnothing \quad \Rightarrow A^0 = F \cap \complement G \neq \varnothing,\\ \complement \overline{D} \subset A^+ \quad \text{and} \quad F = A^0 \cup A^- \subset \overline{D},\\ f = d_{A^0 \cup A^-} - d_{A^0 \cup A^+}.\end{gathered}$$

If A^0 were empty, $\mathbf{R}^N$ could be partitioned into two disjoint nonempty closed subsets. Since A^0 is closed

$$\begin{aligned}\partial A^0 &= A^0 \cap \overline{\complement A^0} = A^0 \cap \overline{A^+ \cup A^-} = A^0 \cap [\overline{A^+} \cup \overline{A^-}]\\ &= A^0 \cap [A^+ \cup A^- \cup \partial A^- \cup \partial A^+] = A^0 \cap [\partial A^- \cup \partial A^+].\end{aligned}$$

Moreover since A^- and A^+ are open, $\overline{A^-} \subset A^- \cup A^0$, $\overline{A^+} \subset A^+ \cup A^0$,

$$\begin{gathered}\partial A^- = \complement A^- \cap \overline{A^-} \subset [A^0 \cup A^+] \cap [A^0 \cup A^-] = A^0,\\ \partial A^+ = \complement A^+ \cap \overline{A^+} \subset [A^0 \cup A^-] \cap [A^0 \cup A^+] = A^0\\ \Rightarrow \partial A^- \cup \partial A^+ = [\partial A^- \cup \partial A^+] \cap A^0 = \partial A^0\\ \Rightarrow \boxed{\partial A^- \cup \partial A^+ = \partial A^0} \quad \Rightarrow \boxed{\partial A^0 = \partial A^- \cup (\partial A^+ \backslash \partial A^-).}\end{gathered}$$

Let $\mathbb{Q}$ be the subset of points in $\mathbf{R}^N$ with rational coordinates. Define

$$B^0 \stackrel{\text{def}}{=} \operatorname{int} A^0, \quad B^0_+ \stackrel{\text{def}}{=} B^0 \cap \mathbb{Q}, \quad B^0_- \stackrel{\text{def}}{=} B^0 \cap \complement \mathbb{Q},$$

and notice that by density of $\mathbb{Q}$ and $\complement\mathbb{Q}$ in $\mathbf{R}^N$,

$$\overline{B^0_+} = \overline{B^0} = \overline{B^0_-}.$$

Consider the following new partition of $\mathbf{R}^N$:

$$\mathbf{R}^N = A^+ \cup (\partial A^+ \backslash \partial A^-) \cup A^- \cup \partial A^- \cup B^0_+ \cup B^0_-.$$

Define

$$\boxed{A \stackrel{\text{def}}{=} A^- \cup (\partial A^+ \backslash \partial A^-) \cup B^0_+} \quad \Rightarrow \complement A = A^+ \cup \partial A^- \cup B^0_-$$

and

$$\begin{aligned}\overline{A} &= \overline{A^-} \cup (\overline{\partial A^+ \backslash \partial A^-}) \cup \overline{B^0_+} = A^- \cup \partial A^- \cup (\overline{\partial A^+ \backslash \partial A^-}) \cup \overline{B^0},\\ \overline{\complement A} &= \overline{A^+} \cup \partial A^- \cup \overline{B^0_-} = A^+ \cup \partial A^+ \cup \partial A^- \cup \overline{B^0}.\end{aligned}$$

But

$$\partial A^- \cup \partial A^+ = \partial A^- \cup (\partial A^+ \backslash \partial A^-) \subset \partial A^- \cup (\overline{\partial A^+ \backslash \partial A^-}) \subset \partial A^- \cup \partial A^+,$$

and since $\partial A^- \cup \partial A^+ = \partial A^0$

$$\left.\begin{aligned}\overline{A} &= A^- \cup \partial A^0 \cup \overline{\operatorname{int} A^0} = A^- \cup A^0\\ \overline{\complement A} &= A^+ \cup \partial A^0 \cup \overline{\operatorname{int} A^0} = A^+ \cup A^0\end{aligned}\right| \quad \Rightarrow \partial A = A^0 \neq \varnothing$$
$$\Rightarrow\ b_A = d_A - d_{\complement A} = b_{\overline{A}} - d_{\overline{\complement A}} = d_F - d_{\complement G} = f.$$

(ii) For the compactness, consider any sequence $\{b_{A_n}\} \subset C_b(D) \subset C^{0,1}(\overline{D})$. Since D is bounded, b_{A_n} and ∇b_{A_n} are both pointwise uniformly bounded in $\overline{D}$. From Theorem 2.5 in Chapter 2, the injection of $C^{0,1}(\overline{D})$ into $C(\overline{D})$ is compact and there exist $f \in C(\overline{D})$ and a subsequence $\{b_{A_{n_k}}\}$ such that $b_{A_{n_k}} \to f$ in $C(\overline{D})$. From the proof of the closure in part (i), there exists $A \subset \overline{D}$, $\partial A \neq \varnothing$, such that $f = b_A$.

(iii) For all b_A and b_B in $C_b(D)$ and x in $\overline{D}$,

$$\begin{aligned}|b_B(x)| &\leq |b_A(x)| + |b_B(x) - b_A(x)|\\ \Rightarrow\ \big|\,|b_B(x)| - |b_A(x)|\,\big| &\leq |b_B(x) - b_A(x)|\\ \Rightarrow\ \|d_{\partial B} - d_{\partial A}\|_{C(D)} = \|\,|b_B| - |b_A|\,\|_{C(D)} &\leq \|b_B - b_A\|_{C(D)}.\end{aligned}$$

Moreover, $d_A = b_A^+ = (|b_A| + b_A)/2$ and $d_{\complement A} = b_A^- = (|b_A| - b_A)/2$, and necessarily

$$\|b_B^\pm - b_A^\pm\|_{C(D)} \leq \|b_B - b_A\|_{C(D)}.$$

By combining the above three inequalities we get (2.9). $\square$

We have the analogue of Theorem 2.3 of Chapter 6 and its corollary for A, $\complement A$, and ∂A. In the last case it takes the following form (to be compared with T. J. Richardson [**1**, Lem. 3.2, p. 44] and S. R. Kulkarni, S. K. Mitter, and T. J. Richardson [**1**] for an application to image segmentation).

Corollary 1. *Let D be a nonempty open* (resp., *bounded open*) *holdall in* $\mathbf{R}^N$. *Define for a subset S of* $\mathbf{R}^N$ *the sets*

$$\begin{aligned}H_b(S) &\overset{\text{def}}{=} \{b_A \in C_b(D) \,:\, S \subset \partial A\},\\ I_b(S) &\overset{\text{def}}{=} \{b_A \in C_b(D) \,:\, \partial A \subset S\},\\ J_b(S) &\overset{\text{def}}{=} \{b_A \in C_b(D) \,:\, \partial A \cap S \neq \varnothing\}.\end{aligned}$$

(i) *Let S be a subset of* $\mathbf{R}^N$. *Then $H_b(S)$ is closed in $C_{\text{loc}}(D)$* (resp., $C(\overline{D})$).

(ii) *Let S be a closed subset of* $\mathbf{R}^N$. *Then $I_b(S)$ is closed in $C_{\text{loc}}(D)$* (resp., $C(\overline{D})$). *If, in addition, $S \cap \overline{D}$ is compact, then $J_b(S)$ is closed in $C_{\text{loc}}(D)$* (resp., $C(\overline{D})$).

(iii) *Let S be an open subset of* $\mathbf{R}^N$. *Then $J_b(S)$ is open in $C_{\text{loc}}(D)$* (resp., $C(\overline{D})$). *If, in addition, $\complement S \cap \overline{D}$ is compact, then $I_b(S)$ is open in $C_{\text{loc}}(D)$* (resp., $C(\overline{D})$).

(iv) *For D bounded, associate with an equivalent class $[A]_b$ the number*

$$\#\,\partial A = \textit{number of connected components of } \partial A.$$

Then the map

$$[A]_b \mapsto \#\,\partial A \colon \mathcal{F}_{\#b}(D) \overset{\text{def}}{=} \left\{[A]_b : \forall A \subset \overline{D},\, \partial A \neq \varnothing,\ \textit{and}\ \#\,\partial A < \infty\right\} \to \mathbf{R}$$

is lower semicontinuous. For a fixed number $c > 0$*, the subset*

$$\{b_A \in C_b(D) : \#\,\partial A \leq c\}$$

of $C_b(D)$ *is compact in* $C(\overline{D})$*. In particular, the subset*

$$\{b_A \in C_b(D) : \partial A \textit{ is connected}\}$$

of $C_b(D)$ *is compact in* $C(\overline{D})$*.*

Proof. The proof is the same as for Theorem 2.3 of Chapter 6 using the Lipschitz continuity of the map $b_A \mapsto d_{\partial A} = |b_A|$ from $C(\overline{D})$ to $C(\overline{D})$. □

3 Projection, Skeleton, Crack, and Differentiability

This section is the analogue of section 3 in Chapter 6: connection between the gradient of b_A and the projection onto ∂A and the characteristic functions associated with ∂A, singularities of the gradients, and the notions of skeleton[1] and cracks.

Let $A \subset \mathbf{R}^{\mathrm{N}}$, $\varnothing \neq \partial A$, and recall the notation

$$\mathrm{Sing}\,(\nabla b_A) \overset{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}}\ :\ \nabla b_A(x) \nexists\right\} \tag{3.1}$$

for the *set of singularities of the gradient of* b_A,

$$\Pi_{\partial A}(x) \overset{\text{def}}{=} \{z \in \partial A : |z - x| = d_{\partial A}(x)\} \tag{3.2}$$

for the *set of projections of* x *onto* ∂A (when $\Pi_{\partial A}(x)$ is a singleton, the unique element is denoted by $p_{\partial A}(x)$), and

$$\mathrm{Sk}\,(\partial A) \overset{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}}\ :\ \Pi_{\partial A}(x) \text{ is not a singleton}\right\} = \left\{x \in \mathbf{R}^{\mathrm{N}} : \nabla d^2_{\partial A}(x) \nexists\right\} \tag{3.3}$$

for the *skeleton* of ∂A. Since $\Pi_{\partial A}(x)$ is a singleton for all $x \in \partial A$ we have $\mathrm{Sk}\,(\partial A) \subset \mathbf{R}^{\mathrm{N}} \backslash \partial A$.

In general, the functions b_A and $d_{\partial A}$ are different and $\mathrm{Sing}\,\nabla b_A$ is usually smaller than $\mathrm{Sing}\,(\nabla d_{\partial A}) = \mathrm{Sk}\,(\partial A) \cup \mathrm{Ck}\,(\partial A)$. As a result, using the oriented distance function b_A rather than $d_{\partial A}$ requires a new definition of the set of cracks with respect to b_A that will be justified in Theorem 3.1 (iv).

Definition 3.1.
Given $A \subset \mathbf{R}^{\mathrm{N}}$, $\varnothing \neq \partial A$, the *set of* b*-cracks* of A is defined as

$$\mathrm{Ck}_{\,b}(A) \overset{\text{def}}{=} \mathrm{Sing}\,\nabla b_A \backslash \mathrm{Sk}\,(\partial A). \tag{3.4}$$

$\mathrm{Sk}\,(\partial A)$, $\mathrm{Ck}_{\,b}(A)$, and $\mathrm{Sing}\,(\nabla b_A)$ have zero N-dimensional Lebesgue measure, since b_A is Lipschitz continuous and hence differentiable almost everywhere. □

[1] Our definition of a skeleton does not exactly coincide with the one used in *morphological mathematics* (cf., for instance, G. Matheron [**1**] or A. Rivière [**1**]).

Remark 3.1.
(1) In general $\mathrm{Ck}_b(A) \subset \mathrm{Ck}(\partial A) = \mathrm{Ck}(A) \cup \mathrm{Ck}(\complement A)$, but $\mathrm{Ck}_b(A)$ can be strictly smaller than $\mathrm{Ck}(\partial A)$. Consider the open ball of unit radius $A = B_1(0)$:

$$b_A(x) = |x| - 1, \quad \nabla b_A(x) = \frac{x}{|x|}, \quad D^2 b_A(x)_{ij} = \frac{1}{|x|}\left(\delta_{ij} - \frac{x_i}{|x|}\frac{x_j}{|x|}\right),$$
$$\mathrm{Sing}(\nabla b_A) = \{0\}, \quad \mathrm{Sk}(\partial A) = \{0\}, \quad \text{and} \quad \mathrm{Ck}_b(A) = \varnothing.$$

It is easy to check that for all $x \in \partial A$, $\nabla b_A(x)$ is the outward unit normal, and $D^2 b_A(x)$ is the second fundamental form with eigenvalues 0 for the space spanned by the normal x and 1 for the tangent space to ∂A in x orthogonal to x. But

$$d_{\partial A}(x) = ||x| - 1|,$$
$$\mathrm{Sing}(\nabla d_{\partial A}) = \{0\} \cup \partial B_1(0), \quad \mathrm{Sk}(\partial A) = \{0\}, \quad \text{and} \quad \mathrm{Ck}(\partial A) = \partial B_1(0).$$

Using b_A rather than $d_{\partial A}$ or d_A removes *artificial singularities*.

(2) In general $\mathrm{Sk}(\partial A) \subset \mathbf{R}^{\mathrm{N}} \backslash \partial A$, but if ∂A has positive reach $h > 0$ in the sense of Definition 6.1 in section 6 of Chapter 6 (there exists $h > 0$ such that $b_A^2 \in C^{1,1}(U_h(\partial A))$, the skeleton will remain at a distance h from ∂A. □

The function b_A enjoys properties similar to the ones of d_A and $d_{\complement A}$ since $|b_A| = d_{\partial A} = \max\{d_A, d_{\complement A}\}$ and we have the analogues of Theorems 3.2 and 3.3 in Chapter 6. Recall the definition of

$$f_{\partial A}(x) \stackrel{\text{def}}{=} \frac{1}{2}\left(|x|^2 - d_{\partial A}^2(x)\right).$$

Theorem 3.1. *Let A be a subset of $\mathbf{R}^{\mathrm{N}}$ such that $\varnothing \neq \partial A$, and let $x \in \mathbf{R}^{\mathrm{N}}$.*

(i) *For all $x \in \mathbf{R}^{\mathrm{N}}$, $d_{\partial A}^2(x) = b_A^2(x)$,*

$$f_{\partial A}(x) = \frac{1}{2}\left(|x|^2 - b_A^2(x)\right)$$

is convex and continuous, the function

$$x \mapsto \frac{k_{\partial A,h}(x)}{2h} = \frac{|x|^2}{2\,h} - |b_A|(x) : \mathbf{R}^{\mathrm{N}} \backslash U_h(\partial A) \to \mathbf{R}$$

is locally convex (and continuous) in $\mathbf{R}^{\mathrm{N}} \backslash U_h(\partial A)$, $\nabla f_{\partial A}$ and ∇b_A^2 belong to $\mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})^N$, and $\nabla d_{\partial A} \in \mathrm{BV}_{\mathrm{loc}}\left(\mathbf{R}^{\mathrm{N}} \backslash \partial(\partial A)\right)^N$.

(ii) *The set $\Pi_{\partial A}(x)$ is nonempty and compact, and*

$$\forall x \notin \partial A \quad \Pi_{\partial A}(x) \subset \partial(\partial A) \quad \text{and} \quad \forall x \in \partial A \quad \Pi_{\partial A}(x) = \{x\}.$$

For $x \notin \overline{A}$, $\Pi_{\partial A}(x) = \Pi_A(x) \subset \partial\overline{A}$ and for $x \notin \overline{\complement A}$, $\Pi_{\partial A}(x) = \Pi_{\complement A}(x) \subset \partial\overline{\complement A}$. Moreover, $\partial(\partial A) = \partial\overline{A} \cup \partial\overline{\complement A}$.

(iii) *For all x and v in $\mathbf{R}^N$, the Hadamard semiderivative*[2] *of d_A^2 always exists,*

$$d_H b_A^2(x;v) = \lim_{t \searrow 0} \frac{b_A^2(x+tv) - b_A^2(x)}{t} = \min_{z \in \Pi_{\partial A}(x)} 2(x-z)\cdot v,$$

$$d_H f_{\partial A}(x;v) = \lim_{t \searrow 0} \frac{f_{\partial A}(x+tv) - f_{\partial A}(x)}{t} = \sigma_{\Pi_{\partial A}(x)}(v) = \sigma_{\mathrm{co}\,\Pi_{\partial A}(x)}(v),$$

where σ_B is the support function of the set B,

$$\sigma_B(v) = \sup_{z\in B} z\cdot v,$$

and $\mathrm{co}\,B$ is the convex hull of B. In particular,

$$\mathrm{Sk}\,(\partial A) = \left\{x \in \mathbf{R}^N : \nabla b_A^2(x) \nexists\right\} \subset \mathbf{R}^N \backslash \partial A. \tag{3.5}$$

Given $v \in \mathbf{R}^N$, for all $x \in \mathbf{R}^N \backslash \partial(\partial A)$ the Hadamard semiderivative of b_A exists,

$$d_H b_A(x;v) = \frac{1}{b_A(x)} \min_{p\in\Pi_A(x)} (x-p)\cdot v, \tag{3.6}$$

and for $x \in \partial(\partial A)$, $d_H b_A(x;v)$ exists if and only if

$$\lim_{t\searrow 0} \frac{b_A(x+tv)}{t} \quad \text{exists.} \tag{3.7}$$

(iv) *The following statements are equivalent:*

(a) *$b_A^2(x)$ is (Fréchet) differentiable at x.*
(b) *$b_A^2(x)$ is Gateaux differentiable at x.*
(c) *$\Pi_{\partial A}(x)$ is a singleton.*

(v) *$\nabla f_{\partial A}(x)$ exists if and only if $\Pi_{\partial A}(x) = \{p_{\partial A}(x)\}$ is a singleton. In that case*

$$p_{\partial A}(x) = \nabla f_{\partial A}(x) = x - \frac{1}{2}\nabla b_A^2(x). \tag{3.8}$$

For all x and y in $\mathbf{R}^N$,

$$\forall p(x) \in \Pi_{\partial A}(x), \quad \frac{1}{2}\left(|y|^2 - b_A^2(y)\right) \geq \frac{1}{2}\left(|x|^2 - b_A^2(x)\right) + p(x)\cdot(y-x), \tag{3.9}$$

or equivalently,

$$\forall p(x) \in \Pi_{\partial A}(x), \quad b_A^2(y) - b_A^2(x) - 2\left(x - p(x)\right)\cdot(y-x) \leq |x-y|^2. \tag{3.10}$$

For all x and y in $\mathbf{R}^N$,

$$\forall p(x) \in \Pi_{\partial A}(x), \forall p(y) \in \Pi_{\partial A}(y), \quad (p(y)-p(x))\cdot(y-x) \geq 0. \tag{3.11}$$

[2] A function $f : \mathbf{R}^N \to \mathbf{R}$ has a Hadamard semiderivative in x in the direction v if

$$d_H f(x;v) \overset{\text{def}}{=} \lim_{\substack{t\searrow 0\\ w\to v}} \frac{f(x+tw)-f(x)}{t} \text{ exists}$$

(cf. Chapter 9, Definition 2.1 (ii)).

(vi) *The functions*

$$p_{\partial A} : \mathbf{R}^{\mathrm{N}} \backslash \mathrm{Sk}\,(\partial A) \to \mathbf{R}^N \quad \textit{and} \quad \nabla b_A^2 : \mathbf{R}^{\mathrm{N}} \backslash \mathrm{Sk}\,(\partial A) \to \mathbf{R}^N$$

are continuous. For all $x \in \mathbf{R}^{\mathrm{N}} \backslash \mathrm{Sk}\,(\partial A)$

$$p_{\partial A}(x) = x - \frac{1}{2}\nabla b_A^2(x) = \nabla f_{\partial A}(x). \tag{3.12}$$

In particular, for all $x \in \partial A$, $\Pi_{\partial A}(x) = \{x\}$, $\nabla b_A^2(x) = 0$,

$$\begin{aligned} \mathrm{Sk}\,(\partial A) &= \left\{x \in \mathbf{R}^{\mathrm{N}} : \nabla b_A^2(x) \nexists \textit{ and } \nabla b_A(x) \nexists\right\} \subset \mathbf{R}^{\mathrm{N}} \backslash \partial A, \\ \mathrm{Ck}_{\,b}(A) &= \left\{x \in \mathbf{R}^{\mathrm{N}} : \nabla b_A^2(x) \exists \textit{ and } \nabla b_A(x) \nexists\right\} \subset \partial(\partial A), \end{aligned} \tag{3.13}$$

and $\mathrm{Sing}\,(\nabla b_A) = \mathrm{Sk}\,(\partial A) \cup \mathrm{Ck}_{\,b}(A)$.

(vii) *The function*

$$\nabla b_A : \mathbf{R}^{\mathrm{N}} \setminus (\mathrm{Sk}\,(\partial A) \cup \partial(\partial A)) \to \mathbf{R}^N$$

is continuous. For all $x \in \mathrm{int}\,\partial A$, $\nabla b_A(x) = 0$*; for all* $x \in \mathbf{R}^{\mathrm{N}} \setminus (\mathrm{Sk}\,(\partial A) \cup \partial A)$,

$$\nabla b_A(x) = \frac{\nabla b_A^2(x)}{2\,b_A(x)} = \left\{ \begin{aligned} &\frac{x - p_{\partial A}(x)}{|x - p_{\partial A}(x)|}, && x \in \mathbf{R}^{\mathrm{N}} \backslash \overline{A}, \\ -&\frac{x - p_{\partial A}(x)}{|x - p_{\partial A}(x)|}, && x \in \mathbf{R}^{\mathrm{N}} \backslash \overline{\complement A}, \end{aligned} \right\}, \quad |\nabla b_A(x)| = 1. \tag{3.14}$$

If $\nabla d_{\partial A}(x)$ *exists for some* $x \in \partial A$, *then* $\nabla b_A(x) = \nabla d_{\partial A}(x) = 0$. *In particular,* $\nabla b_A(x) = 0$ *almost everywhere in* ∂A.

(viii) *The functions* b_A^2 *and* b_A *are differentiable almost everywhere and*

$$\mathrm{m}(\mathrm{Sk}\,(\partial A)) = \mathrm{m}(\mathrm{Ck}_{\,b}(A)) = \mathrm{m}(\mathrm{Sing}\,(\nabla b_A)) = 0.$$

If $\partial A \neq \varnothing$,

$$\begin{aligned} \mathrm{Sk}\,(\partial A) &= \mathrm{Sk}\,(A) \cup \mathrm{Sk}\,(\complement A) \subset \mathbf{R}^{\mathrm{N}} \backslash \partial A, \\ \mathrm{Ck}_{\,b}(A) \subset \mathrm{Ck}\,(\partial A) &= \mathrm{Ck}\,(A) \cup \mathrm{Ck}\,(\complement A) \subset \partial(\partial A). \end{aligned} \tag{3.15}$$

If $\partial A = \varnothing$, *then either* $A = \mathbf{R}^{\mathrm{N}}$ *or* $A = \varnothing$.

(ix) *Given* $A \subset \mathbf{R}^{\mathrm{N}}$, $\partial A \neq \varnothing$,

$$\chi_{\partial A}(x) = 1 - |\nabla b_A(x)| \textit{ in } \mathbf{R}^{\mathrm{N}} \backslash \mathrm{Sing}\,(\nabla b_A), \tag{3.16}$$

the identity holds for almost all x *in* $\mathbf{R}^{\mathrm{N}}$, *and the set*

$$\partial_b A \overset{\text{def}}{=} \{x \in \partial A : \nabla b_A(x) \textit{ exists and } \nabla b_A(x) \neq 0\} \subset \partial(\partial A) \tag{3.17}$$

has zero N*-dimensional Lebesgue measure.*

(x) *Let* μ *be a measure in the sense of* L. C. Evans and R. F. Gariepy [**1**, Chap. 1] *and* $D \subset \mathbf{R}^N$ *be bounded open such that* $\mu(D) < \infty$. *The map*

$$b_A \mapsto \mu(\partial A) : C_b(D) \to \mathbf{R} \tag{3.18}$$

is upper semicontinuous with respect to the topology of uniform convergence on $C_b(D)$. *Moreover, if* $b_{A_n} \to b_A$ *in* $C(\overline{D})$ *for* b_{A_n} *and* b_A *in* $C_b(D)$, *then*

$$\mu(\text{int}\, A) \leq \liminf \mu(\text{int}\, A_n) \leq \limsup \mu(\overline{A_n}) \leq \mu(\overline{A}). \tag{3.19}$$

If $\mu(\partial A) = 0$, *the map* $b_A \mapsto \mu(A) : C_b(D) \to \mathbf{R}$ *is continuous in* b_A.

(xi) *Given* $x \in \mathbf{R}^N$, $\alpha \in [0,1]$, $p \in \Pi_{\partial A}(x)$, *and* $x_\alpha \overset{\text{def}}{=} p + \alpha\,(x-p)$, *then*

$$b_A(x_\alpha) = \alpha\, b_A(x) \quad \textit{and} \quad \forall \alpha \in [0,1]\,, \quad \Pi_{\partial A}(x_\alpha) \subset \Pi_{\partial A}(x).$$

In particular, if $\Pi_{\partial A}(x)$ *is a singleton, then* $\Pi_{\partial A}(x_\alpha)$ *is a singleton and* $\nabla b_A^2(x_\alpha)$ *exists for all* α, $0 \leq \alpha \leq 1$. *If, in addition,* $x \neq \partial A$, *then for all* $0 < \alpha \leq 1$ $\nabla b_A(x_\alpha)$ *exists and* $\nabla b_A(x_\alpha) = \nabla b_A(x)$.

Proof. (i) to (v) follow from Theorems 3.2 and 3.3 of Chapter 6 using the identity $b_A^2 = d_{\partial A}^2$.

(v) To complete the proof of identities (3.13), recall that the projection of all points of ∂A onto ∂A being a singleton, $\text{Sk}\,(\partial A \subset \mathbf{R}^N \backslash \partial A$. Therefore, for any $x \in \text{Sk}\,(\partial A)$, we have $b_A(x) \neq 0$ and if $\nabla b_A^2(x)$ does not exist, then $\nabla b_A(x)$ cannot exist. This sharpens the characterization (3.5) of $\text{Sk}\,(\partial A)$. From this we readily get the characterization of $\text{Ck}_{\,b}(A) = \text{Sing}\,(\nabla b_A) \backslash \text{Sk}\,(\partial A)$ and $\text{Ck}_{\,b}(A) \subset \partial A$. This last property can be improved to $\text{Ck}_{\,b}(A) \subset \partial(\partial A)$ by noticing that for $x \in \text{int}\,\partial A$, $b_A(x) = 0$ and $\nabla b_A^2(x) = \nabla b_A(x) = 0$.

(vi) From Theorem 3.3 (iv) of Chapter 6.

(vii) In $\mathbf{R}^N \backslash \overline{A}$, $b_A = d_A$ and ∇b_A is continuous in $\mathbf{R}^N \backslash (\text{Sk}\,(A) \cup \overline{A})$; in $\mathbf{R}^N \backslash \overline{\complement A}$, $b_A = -d_{\complement A}$ and ∇b_A is continuous in $\mathbf{R}^N \backslash (\text{Sk}\,(\complement A) \cup \overline{\complement A})$; in $\text{int}\,\partial A$, $b_A = d_A = d_{\complement A} = 0$ and $\nabla b_A = 0$. So, in view of part (v) it is continuous in

$$\mathbf{R}^N \backslash (\text{Sk}\,(A) \cup \overline{A}) \cup \mathbf{R}^N \backslash (\text{Sk}\,(\complement A) \cup \overline{\complement A}) \cup \text{int}\,\partial A = \mathbf{R}^N \setminus (\text{Sk}\,(\partial A) \cup \partial(\partial A))\,.$$

If $x \in \mathbf{R}^N \backslash \overline{\complement A}$, then $b_A = -d_{\complement A}$ and repeat the proof of Theorem 3.3 (vi) in Chapter 6 to obtain

$$-\nabla b_A(x) = \nabla d_{\complement A}(x) = \frac{x - p_{\partial A}(x)}{|x - p_{\partial A}(x)|} = \frac{x - p_{\partial A}(x)}{-b_A(x)}$$

and $p_{\partial A}(x) \in \partial \overline{\complement A}$ is unique. Similarly, for $x \in \mathbf{R}^N \backslash \overline{A}$, $b_A = d_A$,

$$\nabla b_A(x) = \nabla d_A(x) = \frac{x - p_{\partial A}(x)}{|x - p_{\partial A}(x)|} = \frac{x - p_{\partial A}(x)}{b_A(x)}$$

and $p_{\partial A}(x) \in \partial \overline{A}$ is unique. In both cases $|\nabla b_A(x) = 1$.

Finally, from Theorem 3.3 (i) in Chapter 6, whenever $d_{\partial A}$ is differentiable at a point $x \in \partial A$, $\nabla d_{\partial A}(x) = 0$. But for all v in $\mathbf{R}^{\mathrm{N}}$ and $t > 0$

$$\left|\frac{b_A(x+tv) - b_A(x)}{t}\right| = \frac{|b_A(x+tv)|}{t} = \frac{d_{\partial A}(x+tv) - d_{\partial A}(x)}{t}$$

since $|b_A| = d_{\partial A}$. Therefore, if $\nabla d_{\partial A}(x)$ exists, $\nabla b_A(x)$ exists and $\nabla b_A(x) = \nabla d_{\partial A}(x) = 0$. Finally, since $\nabla d_{\partial A}(x)$ exists almost everywhere in $\mathbf{R}^{\mathrm{N}}$, then for almost all $x \in \partial A$, $\nabla b_A(x) = 0$.

(viii) From Theorem 3.3 (vii) in Chapter 6 and the fact that $\mathrm{Ck}_b(A) \subset \mathcal{C}(\partial A)$. Indeed, for $x \in \partial A$, if $\nabla d_A(x)$ and $\nabla d_{\complement A}(x)$ exist, they are 0 and for all $t > 0$ and $v \in \mathbf{R}^{\mathrm{N}}$

$$\left|\frac{b_A(x+tv) - b_A(x)}{t}\right| = \left|\frac{d_A(x+tv) - d_A(x)}{t} - \frac{d_{\complement A}(x+tv) - d_{\complement A}(x)}{t}\right| \to 0$$

as $t \searrow 0$. Hence $\nabla b_A(x) = 0$. As a consequence, if $\nabla b_A(x)$ does not exist, then either $\nabla d_A(x)$ or $\nabla d_{\complement A}(x)$ does not exist and $\mathrm{Ck}_b(A) \subset \mathcal{C}(A) \cup \mathcal{C}(\complement A) = \mathcal{C}(\partial A) \subset \partial(\partial A)$ from Theorem 3.3 (vii) in Chapter 6.

(ix) Since b_A is a Lipschitzian function, it is differentiable almost everywhere in $\mathbf{R}^{\mathrm{N}}$, and in view of the previous considerations, when it is differentiable

$$|\nabla b_A(x)| = \begin{cases} 1, & x \notin \partial A, \\ 0 & \text{a.e. in } \partial A \end{cases} \quad \Rightarrow \chi_{\partial A}(x) = 1 - |\nabla b_A(x)| \quad \text{a.e. in } \mathbf{R}^{\mathrm{N}}.$$

Therefore, the set $\partial_b A$ has at most a zero measure.

(x) From Theorem 4.1 in Chapter 6 applied to $\overline{A}$, $\overline{\complement A}$, and ∂A.

(xi) Same proof as the one of Theorem 3.3 (ix) in Chapter 6. □

Remark 3.2.
In general $\nabla d_A(x)$ and $\nabla d_{\complement A}(x)$ do not exist for $x \in \partial A$. This is readily seen by constructing the directional derivatives for the half-space

$$A = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \leq 0\} \tag{3.20}$$

at the point $(0,0)$. Nevertheless $\nabla b_A(0,0)$ exists and is equal to $(1,0)$, which is the outward unit normal at $(0,0) \in \partial A$ to A. Note also that for all $x \in \partial A$, $|\nabla b_A(x)| = 1$. This is possible since $\mathrm{m}_N(\partial A) = 0$. If $\nabla b_A(x)$ exists, then it is easy to check that $|\nabla b_A(x)| \leq 1$. For sufficiently smooth domains, the subset $\partial_b A$ of the boundary ∂A coincides with the reduced boundary of finite perimeter sets. □

4 $W^{1,p}(D)$-Metric Topology and the Family $C_b^0(D)$

4.1 Motivations and Main Properties

From Theorem 2.1 (vi) b_A is locally Lipschitzian and belongs to $W^{1,p}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$ for all $p \geq 1$. So the previous constructions for d_A and $d_{\complement A}$ can be repeated with the space

$W^{1,p}_{\text{loc}}(\mathbf{R}^N)$ in place of $C_{\text{loc}}(D)$ to generate new $W^{1,p}$ metric topologies on the family $C_b(D)$. Moreover the other distance functions can be recovered from the map

$$b_A \mapsto (b_A^+, b_A^-, |b_A|) = (d_A, d_{\complement A}, d_{\partial A})$$

and the characteristic functions from the maps

$$b_A \mapsto b_A^- = d_{\complement A} \mapsto \chi_{\text{int}\, A} = |\nabla d_{\complement A}|, \tag{4.1}$$

$$b_A \mapsto b_A^+ = d_A \mapsto \chi_{\text{int}\, \complement A} = |\nabla d_A|, \tag{4.2}$$

$$b_A \mapsto \chi_{\partial A} = 1 - |\nabla b_A|. \tag{4.3}$$

One of the advantages of the function b_A is that the $W^{1,p}$-convergence of sequences will imply the L^p-convergence of the corresponding characteristic functions of int A, int $\complement A$, and ∂A, that is, continuity of the volume of these sets.

It will be useful to introduce the following set and terminology.

Definition 4.1. (i) A set $A \subset \mathbf{R}^N$, $\partial A \neq \varnothing$, is said to have a *thin*[3] boundary if $m_N(\partial A) = 0$.

(ii) Denote by

$$C_b^0(D) \stackrel{\text{def}}{=} \{b_A : \forall A,\ A \subset \overline{D},\ \partial A \neq \varnothing \text{ and } m(\partial A) = 0\} \tag{4.4}$$

the subset of oriented distance functions with thin boundaries. □

In this section we give the analogues of Theorems 4.2 and 4.3 in Chapter 6.

Theorem 4.1. *Let D be an open* (resp., *bounded open*) *subset of* $\mathbf{R}^N$.

(i) *The topologies induced by $W^{1,p}_{\text{loc}}(D)$* (resp., *$W^{1,p}(D)$*) *on $C_b(D)$ are all equivalent for p, $1 \leq p < \infty$.*

(ii) *$C_b(D)$ is closed in $W^{1,p}_{\text{loc}}(D)$* (resp., *$W^{1,p}(D)$*) *for p, $1 \leq p < \infty$, and*

$$\rho_D([A_2]_b, [A_1]_b) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|b_{A_2} - b_{A_1}\|_{W^{1,p}(B(0,n))}}{1 + \|b_{A_2} - b_{A_1}\|_{W^{1,p}(B(0,n))}}$$

$$(\text{resp., } \rho_D([A_2]_b, [A_1]_b) \stackrel{\text{def}}{=} \|b_{A_2} - b_{A_1}\|_{W^{1,p}(D)})$$

defines a complete metric structure on $\mathcal{F}_b(D)$.

(iii) *For p, $1 \leq p < \infty$, the map*

$$b_A \mapsto \chi_{\partial A} = 1 - |\nabla b_A| : C_b(D) \subset W^{1,p}_{\text{loc}}(D) \to L^p_{\text{loc}}(D)$$

is "Lipschitz continuous": for all bounded open subsets K of D and b_{A_1} and b_{A_2} in $C_b(D)$,

$$\|\chi_{\partial A_2} - \chi_{\partial A_1}\|_{L^p(K)} \leq \|\nabla b_{A_2} - \nabla b_{A_1}\|_{L^p(K)} \leq \|b_{A_2} - b_{A_1}\|_{W^{1,p}(K)}.$$

In particular, $C_b^0(D)$ is a closed subset of $C_b(\overline{D})$ for the $W^{1,p}$-topology.

[3] This terminology should not be confused with the notion of *thin set* in the capacity sense.

(iv) *Let D be a bounded open subset of $\mathbf{R}^N$. For each $b_A \in C_b(D)$,*

$$\begin{aligned} \|b_A\|_{W^{1,p}(D)} = \|\,|b_A|\,\|_{W^{1,p}(D)} = \|d_{\partial A}\|_{W^{1,p}(D)}, \\ \|d_A\|_{W^{1,p}(D)} \le \|b_A\|_{W^{1,p}(D)}, \quad \|d_{\complement A}\|_{W^{1,p}(D)} \le \|b_A\|_{W^{1,p}(D)}. \end{aligned} \tag{4.5}$$

The map

$$b_A \mapsto (b_A^+, b_A^-, |b_A|) = (d_A, d_{\complement A}, d_{\partial A}) \,:\, C_b(D) \subset W^{1,p}(D) \to W^{1,p}(D)^3 \tag{4.6}$$

is continuous.

(v) *Let D be an open* (resp., *bounded open*) *subset of $\mathbf{R}^N$. For all p, $1 \le p < \infty$, the map*

$$\begin{aligned} & b_A \mapsto (\chi_{\partial A}, \chi_{\operatorname{int} A}, \chi_{\operatorname{int} \complement A}) \\ & : W^{1,p}_{\mathrm{loc}}(D) \to L^p_{\mathrm{loc}}(D)^3 (\text{resp.,} W^{1,p}(D) \to L^p(D)^3) \end{aligned} \tag{4.7}$$

is continuous.

Proof. The proofs of (i) and (ii) are essentially the same as the proof of Theorem 4.2 of Chapter 6 using the properties established for $C_b(D)$ in $C(\overline{D})$ of Theorem 2.2.

(iii) From Theorem 3.1 (ix) for any two subsets A_1 and A_2 of $\overline{D}$ with nonempty boundaries and for any open U in D

$$\begin{aligned} & |\nabla b_{A_2}| \le |\nabla b_{A_1}| + |\nabla b_{A_2} - \nabla b_{A_1}| \\ \Rightarrow\ & \chi_{\partial A_1} \le \chi_{\partial A_2} + |\nabla b_{A_2} - \nabla b_{A_1}| \quad \Rightarrow |\chi_{\partial A_1} - \chi_{\partial A_2}| \le |\nabla b_{A_2} - \nabla b_{A_1}| \\ \Rightarrow\ & \int_U |\chi_{\partial A_2} - \chi_{\partial A_1}|^p\, dx \le \|\nabla b_{A_2} - \nabla b_{A_1}\|^p_{L^p(U)} \le \|b_{A_2} - b_{A_1}\|^p_{W^{1,p}(U)} \end{aligned}$$

for p, $1 \le p < \infty$, and with the ess sup norm for $p = \infty$. The closure of $C_b^0(D)$ follows from the continuity of the maps

$$b_A \mapsto \mathrm{m}(\partial A \cap U) = \int_{\partial A \cap U} \chi_{\partial A}\, dx \,:\, W^{1,p}(U) \to \mathbf{R}$$

for all bounded open subsets U of D.

(iv) First observe that

$$\begin{aligned} |b_A(x)| = d_A(x) + d_{\complement A}(x) \quad & \text{and} \quad \nabla |b_A(x)| = \nabla d_A(x) + \nabla d_{\complement A}(x), \\ b_A(x) = d_A(x) - d_{\complement A}(x) \quad & \text{and} \quad \nabla b_A(x) = \nabla d_A(x) - \nabla d_{\complement A}(x) \end{aligned}$$

and since $\nabla b_A = \nabla d_A = \nabla d_{\complement A} = 0$ almost everywhere on $\partial A = \overline{A} \cap \overline{\complement A}$, then

$$|\nabla b_A(x)| = |\nabla d_A(x)| + |\nabla d_{\complement A}(x)| = |\nabla |b_A(x)|\,| \text{ for almost all } x \text{ in } \mathbf{R}^N.$$

From this we readily get (4.5). However, this is not sufficient to prove the continuity since the map (4.6) is not linear. To get around this consider a sequence $\{b_{A_n}\} \subset C_b(D)$ converging to b_A in $W^{1,p}(D)$. From Theorem 2.2 (iii), $|b_{A_n}| = d_{\partial A_n}$, $b_{A_n}^+ = d_{A_n}$, and $b_{A_n}^- = d_{\complement A_n}$ converge to $|b_A| = d_{\partial A}$, $b_A^+ = d_A$, and $b_A^- = d_{\complement A}$ in $C(D)$

and hence in $W^{1,p}(D)$-weak. To prove that the convergence is strong, consider the L^2-norm

$$\begin{aligned}
&\int_D |\nabla d_{A_n} - \nabla d_A|^2 + |\nabla d_{\complement A_n} - \nabla d_{\complement A}|^2\,dx \\
&= \int_D |\nabla d_{A_n}|^2 + |\nabla d_A|^2 + |\nabla d_{\complement A_n}|^2 + |\nabla d_{\complement A}|^2 \\
&\qquad\qquad - 2\,\nabla d_{A_n}\cdot\nabla d_A - 2\,\nabla d_{\complement A_n}\cdot\nabla d_{\complement A}\,dx \\
&= \int_D |\nabla b_{A_n}|^2 + |\nabla b_A|^2 - 2\,\nabla d_{A_n}\cdot\nabla d_A - 2\,\nabla d_{\complement A_n}\cdot\nabla d_{\complement A}\,dx \\
&\to \int_D 2\,|\nabla b_A|^2 - 2\,|\nabla d_A|^2 - 2\,|\nabla d_{\complement A}|^2\,dx = \int_D 2\,|\nabla b_A|^2 - 2\,|\nabla b_A|^2\,dx = 0
\end{aligned}$$

by weak L^2-convergence of ∇d_{A_n} and $\nabla d_{\complement A_n}$ to ∇d_A and $\nabla d_{\complement A}$. So both sequences d_{A_n} and $d_{\complement A_n}$ converge in $W^{1,2}(D)$-strong, and hence from part (i) in $W^{1,p}(D)$-strong for all p, $1 \le p < \infty$.

(v) It is sufficient to prove the result for D bounded open. From part (iii) it is true for $\chi_{\partial A}$, and from part (iv) and Theorem 4.2 (ii) and (iii) of Chapter 6 for the other two. □

4.2 Weak $W^{1,p}$-Topology

Theorem 4.2. *Let D be a bounded open domain in $\mathbf{R}^N$.*

(i) *If $\{b_{A_n}\}$ weakly converges in $W^{1,p}(D)$ for some p, $1 \le p < \infty$, then it weakly converges in $W^{1,p}(D)$ for all p, $1 \le p < \infty$.*

(ii) *If $\{b_{A_n}\}$ converges in $C(\overline{D})$, then it weakly converges in $W^{1,p}(D)$ for all p, $1 \le p < \infty$. Conversely, if $\{b_{A_n}\}$ weakly converges in $W^{1,p}(D)$ for some p, $1 \le p < \infty$, it converges in $C(\overline{D})$.*

(iii) *$C_b(D)$ is compact in $W^{1,p}(D)$-weak for all p, $1 \le p < \infty$.*

Proof. (i) Recall that for D bounded there exists a constant $c > 0$ such that for all $b_A \in C_b(D)$

$$|b_A(x)| \le c \text{ and } |\nabla b_A(x)| \le 1 \text{ a.e. in } D.$$

If $\{b_{A_n}\}$ weakly converges in $W^{1,p}(D)$ for some $p \ge 1$, then

$$\{b_{A_n}\} \text{ weakly converges in } L^p(D),$$
$$\{\nabla b_{A_n}\} \text{ weakly converges in } L^p(D)^N.$$

By Lemma 3.1 (i) in Chapter 5 both sequences weakly converge for all $p \ge 1$ and hence $\{b_{A_n}\}$ weakly converges in $W^{1,p}(D)$ for all $p \ge 1$.

(ii) If $\{b_{A_n}\}$ converges in $C(\overline{D})$, then by Theorem 2.2 (ii) there exists $b_A \in C_b(D)$ such that $b_{A_n} \to b_A$ in $C(\overline{D})$ and hence in $L^p(D)$. So for all $\varphi \in \mathcal{D}(D)^N$

$$\int_D \nabla b_{A_n}\cdot\varphi\,dx = -\int_D b_{A_n}\mathrm{div}\,\varphi\,dx \to -\int_D b_A\mathrm{div}\,\varphi\,dx = \int_D \nabla b_A\cdot\varphi\,dx.$$

By density of $\mathcal{D}(D)$ in $L^2(D)$, $\nabla b_{A_n} \to \nabla b_A$ in $L^2(D)^N$-weak and hence $b_{A_n} \to b_A$ in $W^{1,2}(D)$-weak. From part (i) it converges in $W^{1,p}(D)$-weak for all p, $1 \le p < \infty$. Conversely, the weakly convergent sequence converges to some f in $W^{1,p}(D)$. By compactness of $C_b(D)$, there exist a subsequence, still indexed by n, and b_A such that $b_{A_n} \to b_A$ in $C(\overline{D})$ and hence in $W^{1,p}(D)$-weak. By uniqueness of the limit, $b_A = f$. Therefore, all convergent subsequences in $C(\overline{D})$ converge to the same limit. So the whole sequence converges in $C(\overline{D})$. This concludes the proof.

(iii) Consider an arbitrary sequence $\{b_{A_n}\}$ in $C_d(D)$. From Theorem 2.2 (ii) $C_b(D)$ is compact and there exist a subsequence $\{b_{A_{n_k}}\}$ and $b_A \in C_b(D)$ such that $b_{A_{n_k}} \to b_A$ in $C(\overline{D})$. From part (ii) the subsequence weakly converges in $W^{1,p}(D)$ and hence $C_b(D)$ is compact in $W^{1,p}(D)$-weak. □

For sets with thin boundaries the strong and weak $W^{1,p}(D)$-convergences of elements of $C_b^0(D)$ to an element of $C_b^0(D)$ are equivalent as in the case of Theorem 3.2 in section 3.2 of Chapter 5 for characteristic functions in $L^p(D)$-strong.

Theorem 4.3. *Let $\varnothing \neq D \subset \mathbf{R}^{\mathrm{N}}$ be bounded open.*

(i) *Let $\{A_n\}$ be a sequence of subsets of $\overline{D}$ such that $\partial A_n \neq \varnothing$ and $\mathrm{m}(\partial A_n) = 0$ and $A \subset \bar{D}$ be such that $\partial A \neq \varnothing$ and $\mathrm{m}(\partial A) = 0$. Then*

$$b_{A_n} \rightharpoonup b_A \text{ in } W^{1,2}(D)\text{-weak} \quad \Rightarrow b_{A_n} \to b_A \text{ in } W^{1,2}(D)\text{-strong},$$

and hence in $W^{1,p}(D)$-strong for all p, $1 \le p < \infty$.

(ii) *The map*

$$b_A \mapsto \mathrm{m}_N(\operatorname{int} A) = \mathrm{m}_N(A) = \mathrm{m}_N(\overline{A}) : C_b^0(D) \to \mathbf{R} \tag{4.8}$$

is continuous with respect to the $W^{1,p}$-topology on $C_b^0(D)$ for all p, $1 \le p < \infty$.

Proof. (i) From Theorem 4.1 (i) on the equivalence of the $W^{1,p}$-topologies, it is sufficient to prove the result for $p = 2$. By Theorem 4.2 (ii), the weak L^2-convergence of $\{b_{A_n}\}$ to b_A implies the strong convergence in $C(\overline{D})$ and hence in $L^2(D)$-strong. Since, for all $n \ge 1$, $\mathrm{m}(\partial A_n) = 0 = \mathrm{m}(\partial A)$, $|\nabla b_A| = 1 = |\nabla b_{A_n}|$ almost everywhere in D by Theorem 3.1 (vii). As a result

$$\begin{aligned} \int_D |\nabla b_{A_n} - \nabla b_A|^2 \, dx &= \int_D |\nabla b_{A_n}|^2 + |\nabla b_A|^2 - 2\nabla b_{A_n} \cdot \nabla b_A \, dx \\ &= 2 \int_D (1 - \nabla b_{A_n} \cdot \nabla b_A) \, dx \to 2 \int_D (1 - |\nabla b_A|^2) \, dx = 2 \int_D \chi_{\partial A} \, dx = 0. \end{aligned}$$

Therefore $\nabla b_{A_n} \to \nabla b_A$ in $L^2(D)^N$-strong and $b_{A_n} \to b_A$ in $W^{1,2}(D)$-strong, since the convergence $b_{A_n} \to b_A$ in $L^2(D)$-strong follows from the weak convergence in $W^{1,2}(D)$. The convergence in $W^{1,p}(D)$-strong follows from the equivalence of the topologies on $C_b(D)$ (cf. Theorem 4.1 (i)).

(ii) From Theorem 3.1 (x) since $\mathrm{m}_N(\partial A) = 0$ for $b_A \in C_b^0(D)$ and $C_b^0(D)$ is closed in the $W^{1,p}$-topology. □

5 Boundary of Bounded and Locally Bounded Curvature

We introduce the family of sets with bounded or locally bounded curvature which are the analogues for b_A of the sets of exterior and interior bounded curvature associated with d_A and $d_{\complement A}$ in section 5 of Chapter 6. They include $C^{1,1}$-domains, convex sets, and the sets of positive reach of H. FEDERER [**3**]. They lead to compactness theorems for subfamilies of $C_b(D)$ in $W^{1,p}(D)$.

Definition 5.1. (i) Given a bounded open nonempty holdall D in $\mathbf{R}^{\mathrm{N}}$ and a subset A of $\overline{D}$, $\partial A \neq \varnothing$, its *boundary* ∂A is said to be of *bounded curvature* with respect to D if

$$\nabla b_A \in \mathrm{BV}(D)^N. \tag{5.1}$$

This family of sets will be denoted as follows:

$$\mathrm{BC}_b(D) \overset{\text{def}}{=} \left\{ b_A \in C_b(D) \,:\, \nabla b_A \in \mathrm{BV}(D)^N \right\}.$$

(ii) Given a subset A of $\mathbf{R}^{\mathrm{N}}$, $\partial A \neq \varnothing$, its *boundary* ∂A is said to be of *locally bounded curvature* if ∇b_A belongs to $\mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})^N$. The family of sets whose boundary is of locally bounded curvature will be denoted by

$$\mathrm{BC}_b \overset{\text{def}}{=} \left\{ b_A \in C_b(\mathbf{R}^{\mathrm{N}}) \,:\, \nabla b_A \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})^N \right\}. \qquad \square$$

As in Theorem 6.2 of Chapter 5 for Caccioppoli sets, and in Theorem 5.1 of Chapter 6 for sets with a locally bounded curvature, it is sufficient to satisfy the BV property in a neighborhood of each point of the boundary $\partial\overline{A}$.

Theorem 5.1. *Let A, $\partial A \neq \varnothing$, be a subset of $\mathbf{R}^{\mathrm{N}}$. Then ∂A is of locally bounded curvature if and only*

$$\forall x \in \partial(\partial A),\ \exists \rho > 0 \text{ such that } \nabla b_A \in \mathrm{BV}(B(x,\rho))^N, \tag{5.2}$$

where $B(x,\rho)$ is the open ball of radius $\rho > 0$ in x.

Proof. From Definition 5.1 and Theorem 3.1 (i). $\square$

Theorem 5.2.

(i) *Let D be a nonempty bounded open Lipschitzian holdall in $\mathbf{R}^{\mathrm{N}}$ and let $A \subset \overline{D}$, $\partial A \neq \varnothing$. If $\nabla b_A \in \mathrm{BV}(D)^N$, then*

$$\|\nabla \chi_{\partial A}\|_{M^1(D)} \leq 2\,\|D^2 b_A\|_{M^1(D)}, \quad \chi_{\partial A} \in \mathrm{BV}(D),$$

and ∂A has finite perimeter with respect to D.

(ii) *For any subset A of $\mathbf{R}^{\mathrm{N}}$, $\partial A \neq \varnothing$,*

$$\nabla b_A \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})^N \quad \Rightarrow \quad \chi_{\partial A} \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}}),$$

and ∂A has locally finite perimeter.

Proof. Given ∇b_A in $\mathrm{BV}(D)^N$, there exists a sequence $\{u_k\}$ in $C^\infty(D)^N$ such that

$$u_k \to \nabla b_A \text{ in } L^1(D)^N \quad \text{and} \quad \|Du_k\|_{M^1(D)} \to \|D^2 b_A\|_{M^1(D)}$$

as k goes to infinity, and since $|\nabla b_A(x)| \le 1$, this sequence can be chosen in such a way that

$$\forall k \ge 1, \quad |u_k(x)| \le 1.$$

This follows from the use of mollifiers (cf. E. Giusti [**1**, Thm. 1.17, p. 15]). For all V in $\mathcal{D}(D)^N$

$$-\int_D \chi_{\partial A} \operatorname{div} V\, dx = \int_D (|\nabla b_A|^2 - 1) \operatorname{div} V\, dx = \int_D |\nabla b_A|^2 \operatorname{div} V\, dx.$$

For each u_k

$$\int_D |u_k|^2 \operatorname{div} V\, dx = -2\int_D [\,{}^*Du_k] u_k \cdot V\, dx = -2\int_D u_k \cdot [Du_k] V\, dx,$$

where ${}^*Du_k$ is the transpose of the Jacobian matrix Du_k and

$$\begin{aligned}\left|\int_D |u_k|^2 \operatorname{div} V\, dx\right| &\le 2\int_D |u_k|\, |Du_k|\, |V|\, dx \\ &\le 2\, \|Du_k\|_{L^1} \|V\|_{C(D)} \le 2\, \|Du_k\|_{M^1} \|V\|_{C(D)}\end{aligned}$$

since for $W^{1,1}(D)$-functions $\|\nabla f\|_{L^1(D)^N} = \|\nabla f\|_{M^1(D)^N}$. Therefore, as k goes to infinity,

$$\left|\int_D \chi_{\partial A} \operatorname{div} V\, dx\right| = \left|\int_D |\nabla b_A|^2 \operatorname{div} V\, dx\right| \le 2\|D^2 b_A\|_{M^1} \|V\|_{C(D)}.$$

Therefore, $\nabla \chi_{\partial A} \in M^1(D)^N$. □

5.1 Examples and Limit of Tubular Norms as h Goes to Zero

It is informative to compute the Laplacian of b_A for a few examples. The first three examples are illustrated in Figure 7.1.

Example 5.1 (Half-plane in $\mathbf{R}^2$)**.**
(cf. Example 5.1 in Chapter 6). Consider the domain

$$A = \{(x_1, x_2) : x_1 \le 0\}, \quad \partial A = \{(x_1, x_2) : x_1 = 0\}.$$

It is readily seen that

$$b_A(x_1, x_2) = x_1, \quad \nabla b_A(x_1, x_2) = (1, 0), \quad \Delta b_A(x) = 0,$$

and

$$b_{\partial A}(x_1, x_2) = |x_1|, \quad \nabla b_{\partial A}(x_1, x_2) = \left(\frac{x_1}{|x_1|}, 0\right),$$
$$\langle \Delta b_{\partial A}, \varphi \rangle = 2 \int_{\partial A} \varphi \, dx. \qquad \square$$

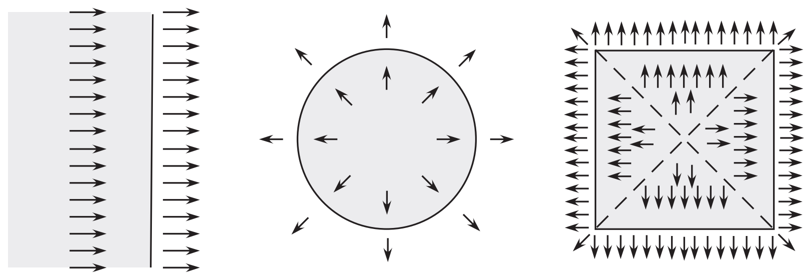


Figure 7.1. ∇b_A *for Examples* 5.1, 5.2, *and* 5.3.

Example 5.2 (Ball of radius $R > 0$ in $\mathbf{R}^2$)**.**
(cf. Example 5.2 in Chapter 6). Consider the domain

$$A = \{x \in \mathbf{R}^2 : |x| \le R\}, \quad \partial A = \{x \in \mathbf{R}^2 : |x| = R\}.$$

Clearly,

$$b_A(x) = |x| - R, \quad \nabla b_A(x) = \frac{x}{|x|},$$
$$\langle \Delta b_A, \varphi \rangle = \int_{\mathbf{R}^2} \frac{1}{|x|} \varphi \, dx.$$

Also

$$b_{\partial A}(x) = \big||x| - R\big|, \quad \nabla b_{\partial A}(x) = \begin{cases} \frac{x}{|x|}, & |x| > R, \\ -\frac{x}{|x|}, & |x| < R, \end{cases}$$
$$\langle \Delta b_{\partial A}, \varphi \rangle = 2 \int_{\partial A} \varphi \, ds - \int_A \frac{1}{|x|} \varphi \, dx + \int_{\complement A} \frac{1}{|x|} \varphi \, dx.$$

Again $\Delta b_{\partial A}$ contains twice the boundary measure on ∂A. $\square$

Example 5.3 (Unit square in $\mathbf{R}^2$)**.**
(cf. Example 5.3 in Chapter 6). Consider the domain

$$A = \{x = (x_1, x_2) : |x_1| \le 1, |x_2| \le 1\}.$$

Since A is symmetrical with respect to both axes, it is sufficient to specify b_A in the first quadrant. We use the notation Q_1, Q_2, Q_3, and Q_4 for the four quadrants in the counterclockwise order and c_1, c_2, c_3, and c_4 for the four corners of the square in the same order. We also divide the plane into three regions:

$$\begin{aligned}
D_1 &= \{(x_1, x_2) : |x_2| \le \min\{1, |x_1|\}\}\,, \\
D_2 &= \{(x_1, x_2) : |x_1| \le \min\{1, |x_2|\}\}\,, \\
D_3 &= \{(x_1, x_2) : |x_1| \ge 1 \text{ and } |x_2| \ge 1\}\,.
\end{aligned}$$

Hence for $1 \le i \le 4$

$$b_A(x) = \begin{cases} |x_2| - 1, & x \in D_2 \cap Q_i, \\ |x - c_i|, & x \in D_3 \cap Q_i, \\ |x_1| - 1, & x \in D_1 \cap Q_i, \end{cases} \quad \nabla b_A(x) = \begin{cases} (0,1), & x \in D_2 \cap Q_i, \\ \frac{x-c_i}{|x-c_i|}, & x \in D_3 \cap Q_i, \\ (1,0), & x \in D_1 \cap Q_i, \end{cases}$$

and for the whole plane

$$\langle \Delta b_A, \varphi \rangle = \sum_{i=1}^{4} \int_{D_3 \cap Q_i} \frac{1}{|x - c_i|} \varphi \, dx + \sqrt{2} \int_{D_1 \cap D_2} \varphi \, dx,$$

where $D_1 \cap D_2$ is made up of the two diagonals of the square where ∇b_A has a singularity (that is, the skeleton). Moreover, for $1 \le i \le 4$

$$b_{\partial A}(x) = \begin{cases} \left||x_2| - 1\right|, & x \in D_2 \cap Q_i, \\ |x - c_i|, & x \in D_3 \cap Q_i, \\ \left||x_1| - 1\right|, & x \in D_1 \cap Q_i, \end{cases}$$

and

$$\langle \Delta b_{\partial A}, \varphi \rangle = \sum_{i=1}^{4} \int_{D_3 \cap Q_i} \frac{1}{|x - c_i|} \varphi \, dx - \sqrt{2} \int_{D_1 \cap D_2} \varphi \, dx + 2 \int_{\partial A} \varphi \, dx.$$

Notice that the structures of the Laplacian are similar to the ones observed in the previous examples except for the presence of a singular term along the two diagonals of the square. □

All C^2-domains with a compact boundary belong to all the categories of Definition 5.1 and of Definition 5.1 of Chapter 6. The h-dependent norms $\|D^2 b_A\|_{M^1(U_h(\partial A))}$, $\|D^2 d_{\complement A}\|_{M^1(U_h(\complement A))}$, and $\|D^2 d_A\|_{M^1(U_h(A))}$ are all decreasing as h goes to zero. The limit is particularly interesting since it singles out the behavior of the singular part of the Hessian matrix in a shrinking neighborhood of the boundary ∂A.

Example 5.4.
If $A \subset \mathbf{R}^{\mathrm{N}}$ is of class C^2 with compact boundary, then

$$\lim_{h \searrow 0} \|D^2 b_A\|_{M^1(U_h(\partial A))} = 0. \qquad \square$$

Example 5.5.
Let $A = \{x_i\}_{i=1}^{I}$ be I distinct points in $\mathbf{R}^{\mathrm{N}}$. Then $\partial A = A$, and

$$\lim_{h \searrow 0} \|D^2 b_A\|_{M^1(U_h(\partial A))} = \begin{cases} 2\,I - 1, & N = 1, \\ 0, & N \ge 2. \end{cases} \qquad \square$$

Example 5.6.
Let A be a closed line in $\mathbf{R}^N$ of length $L > 0$. Then $\partial A = A$, and

$$\lim_{h \searrow 0} \|D^2 b_A\|_{M^1(U_h(\partial A))} = \begin{cases} 2\,L, & N = 2, \\ 0, & N \geq 3. \end{cases} \qquad \square$$

Example 5.7.
Let $N = 2$. For the finite square and the ball of finite radius

$$\lim_{h \searrow 0} \|\Delta d_A\|_{M^1(U_h(\partial A))} = H_1(\partial A),$$

where H_1 is the one-dimensional Hausdorff measure (cf. Examples 5.2 and 5.3 in Chapter 6). $\square$

Also, looking at Δb_A in $U_h(A)$ as h goes to zero provides information about $\mathrm{Sk}\,(A)$.

Example 5.8.
Let A be the unit square in R^2. Then

$$\lim_{h \searrow 0} \|\Delta b_A\|_{M^1(U_h(A))} = \frac{\sqrt{2}}{2}\, H_1(\mathrm{Sk}\,(A)),$$

where H_1 is the one-dimensional Hausdorff measure and $\mathrm{Sk}(A) = \mathrm{Sk}_{\mathrm{int}}(A)$ is the skeleton of A made up of the two interior diagonals (cf. Example 5.3). It would seem that in general

$$\lim_{h \searrow 0} \|\Delta b_A\|_{M^1(U_h(A))} = \int_{\mathrm{Sk}(A)} |\,[\nabla b_A] \cdot n|\, dH_1,$$

where $[\nabla b_A]$ is the jump in ∇b_A and n is the unit normal to $\mathrm{Sk}(A)$ (if it exists!). $\square$

6 Approximation by Dilated Sets/Tubular Neighborhoods

We have seen in Theorem 7.1 of Chapter 6 that d_A can be approximated by the distance function d_{A_r} of the dilated set $A_r = \{x \in \mathbf{R}^N : d_A(x) \leq r\}$, $r > 0$, of A in the $W^{1,p}$-topology. The approximation can also be achieved with the oriented distance function: $b_{A_r}(x) \to b_{\overline{A}}(x)$ uniformly in $\mathbf{R}^N$. But it requires that the measure of the boundary $\partial \overline{A}$ be zero to get the approximation in the $W^{1,p}$-topology.

Lemma 6.1. *Given $A \subset \mathbf{R}^N$, $\partial \overline{A} \neq \varnothing$ if and only if $A \neq \varnothing$ and there exists $h > 0$ such that $A_h \neq \mathbf{R}^N$.*

Proof. If $\partial \overline{A} \neq \varnothing$, then $\overline{A} \neq \varnothing$ and $\overline{\complement \overline{A}} \neq \varnothing$, $A \neq \varnothing$ and $\complement \overline{A} \neq \varnothing$, and

$$\mathbf{R}^N \neq \overline{A} = \cap_{h>0} A_h \quad \Rightarrow \exists h > 0 \text{ such that } A_h \neq \mathbf{R}^N\,.$$

Conversely, $\overline{A} \supset A \neq \varnothing$ and, for some $h > 0$, $\overline{A} = \cap_{k>0} A_k \subset A_h \neq \mathbf{R}^N$. Therefore $\overline{A} \neq \varnothing$ and $\overline{\complement \overline{A}} \neq \varnothing$ imply that $\partial \overline{A} \neq \varnothing$. $\square$

Theorem 6.1. *Given $A \subset \mathbf{R}^N$ such that $\partial \overline{A} \neq \varnothing$, there exists $h > 0$ such that for all r, $0 < r \leq h$, $\varnothing \neq A \subset \overline{A} \subset A_r \subset A_h \neq \mathbf{R}^N$.*

(i) *For all r, $0 < r \leq h$, $\complement \overline{A} \supset \complement A_r \neq \varnothing$, $\partial A_r \neq \varnothing$, $b_{U_r(A)} = b_{A_r}$, and $U_r(A)$ and A_r are invariant open and closed representatives in the equivalence class $[A_r]_b$. Moreover,*

$$\begin{aligned}
&d_{\complement U_r(A)}(x) = d_{\complement A_r}(x) = 0, && \text{if } d_A(x) \geq r, \\
&d_{\complement U_r(A)}(x) = d_{\complement A_r}(x) \geq r - d_A(x), && \text{if } 0 < d_A(x) < r, && (6.1)\\
&d_{\complement U_r(A)}(x) = d_{\complement A_r}(x) \geq d_{\complement \overline{A}}(x) + r, && \text{if } d_A(x) = 0, && (6.2)\\
&\quad \forall x \in \mathbf{R}^N, \quad 0 \leq b_{\overline{A}}(x) - b_{A_r}(x) \leq r, && && (6.3)
\end{aligned}$$

and, as $r \to 0$,

$$\begin{aligned}
&b_{U_r(A)}(x) = b_{A_r}(x) \to b_{\overline{A}}(x) \text{ uniformly in } \mathbf{R}^N,\\
&\chi_{A_r} \to \chi_{\overline{A}} \text{ in } L^p_{\mathrm{loc}}(\mathbf{R}^N), \quad \forall p,\ 1 \leq p < \infty.
\end{aligned}$$

(ii) *The following conditions are equivalent:*

(a) $\mathrm{m}(\partial \overline{A}) = 0$.

(b) *As $r \to 0$, $b_{U_r(A)} = b_{A_r} \to b_{\overline{A}}$ in $W^{1,p}_{\mathrm{loc}}(\mathbf{R}^N)$ for some p, $1 \leq p < \infty$.*

(c) *As $r \to 0$, $d_{\complement U_r(A)} = d_{\complement A_r} \to d_{\complement \overline{A}}$ in $W^{1,p}_{\mathrm{loc}}(\mathbf{R}^N)$ for some p, $1 \leq p < \infty$.*

If (a), (b), *or* (c) *is verified, then* (b) *and* (c) *are verified for all p, $1 \leq p < \infty$.*

In particular, (a) *is verified for all bounded sets $A \neq \varnothing$ in $\mathbf{R}^N$ such that* $\mathrm{m}(\partial A) = 0$.

Proof. The existence of the $h > 0$ such that $A_h \neq \mathbf{R}^N$ and $A \neq \varnothing$ is a consequence of Lemma 6.1. From this for all r, $0 < r \leq h$, $\varnothing \neq A \subset \overline{A} \subset A_r \subset A_h \neq \mathbf{R}^N$.

(i) From the initial remarks, $\mathbf{R}^N \neq \complement A \supset \complement \overline{A} \supset \complement A_r \supset \complement A_h \neq \varnothing$ and $\partial A_r \neq \varnothing$ and $\partial \overline{A} \neq \varnothing$, and the functions $d_{\complement \overline{A}}$, $d_{\complement A_r}$, $b_{\overline{A}}$, and b_{A_r} are well-defined. From Theorem 3.1 of Chapter 6, $A_r = \overline{U_r(A)}$, and $\overline{\complement A_r} = \complement U_r(A)$, and $\partial A_r = d_A^{-1}\{r\}$. Hence, $d_{A_r} = d_{U_r(A)}$. Hence, $d_{A_r} = d_{U_r(A)}$, $d_{\complement A_r} = d_{\complement U_r(A)}$, and $b_{A_r} = b_{U_r(A)}$.

For $x \in \mathbf{R}^N$ such that $d_A(x) \geq r$, $x \in \complement U_r(A)$ and $d_{\complement U_r(A)}(x) = 0$.

Inequality (6.1). For all $y \in \complement U_r(A)$ and all $p \in \overline{A}$,

$$\begin{aligned}
r &< d_A(y) \leq |y - p| \leq |y - x| + |x - p|,\\
r &\leq \inf_{y \in \complement U_r(A)} |y - x| + \inf_{p \in A} |x - p| = d_{\complement U_r(A)}(x) + d_A(x).
\end{aligned}$$

Inequality (6.2). Since $\overline{A} \cap \complement U_r(A) = \varnothing$, for $x \in \overline{A}$, there exists $p_r \in \partial \complement U_r(A) = d_A^{-1}\{r\}$ such that $d_{\complement U_r(A)}(x) = |p_r - x| > 0$. Therefore $d_A(p_r) = r$. Either $d_{\complement \overline{A}}(x) = 0$ or $d_{\complement \overline{A}}(x) > 0$. If $d_{\complement \overline{A}}(x) = 0$, then

$$d_{\complement U_r(A)}(x) = |p_r - x| \geq d_A(p_r) = r = d_{\complement \overline{A}}(x) + r.$$

If $d_{\complement\overline{A}}(x) > 0$, then $x \notin \overline{\complement\overline{A}}$ and $x \in \operatorname{int}\overline{A} = \overline{A}\backslash\partial\overline{A}$, $B_x \stackrel{\text{def}}{=} B(x, d_{\complement\overline{A}}(x)) \subset \operatorname{int} A$, and $\overline{B_x} \subset \overline{A}$. Define

$$p \stackrel{\text{def}}{=} x + d_{\complement\overline{A}}(x)\frac{p_r - x}{|p_r - x|} \in \overline{B_x} \quad \Rightarrow |p - x| = d_{\complement\overline{A}}(x) \text{ and } p \in \overline{B_x} \subset \overline{A}.$$

By construction

$$\begin{aligned} d_{\complement U_r(A)}(x) = |x - p_r| = |x - p| + |p - p_r| = d_{\complement\overline{A}}(x) + |p - p_r|, \\ |p - p_r| \geq \inf_{p' \in \overline{B_x}} |p' - p_r| \geq \inf_{p' \in \overline{A}} |p' - p_r| = d_A(p_r) = r \end{aligned}$$

and $d_{\complement U_r(A)}(x) \geq d_{\complement\overline{A}}(x) + r$.

Inequality (6.3). From Theorem 2.1 (ii), $\overline{A} \subset A_r$ implies that $b_{\overline{A}}(x) \geq b_{A_r}(x)$. As for the second part of inequality (6.3), we use identity (7.2) from Theorem 7.1 in Chapter 6 for $d_{U_r(A)} = d_{A_r}$ combined with (6.2) for $x \in \overline{A}$ and with (6.1) for $x \in \mathbf{R}^{\mathrm{N}} \backslash \overline{A}$:

$$\begin{aligned} r \leq d_A(x) \quad & \Rightarrow x \in \complement U_r(A) \subset \complement\overline{A} \quad \Rightarrow d_{\complement U_r(A)}(x) = d_{\complement\overline{A}}(x) = 0 \\ & \Rightarrow b_{U_r(A)}(x) = d_{U_r(A)}(x) = d_A(x) - r = b_{\overline{A}}(x) - r, \\ 0 < d_A(x) < r \quad & \Rightarrow d_{U_r(A)}(x) = 0 \text{ and } d_{\complement\overline{A}}(x) = 0 \\ & \Rightarrow b_{U_r(A)}(x) = -d_{\complement U_r(A)}(x) \leq d_A(x) - r = b_{\overline{A}}(x) - r, \\ d_A(x) = 0 \quad & \Rightarrow x \in \overline{A} \text{ and } d_{U_r(A)}(x) = 0 \\ & \Rightarrow b_{U_r(A)}(x) = -d_{\complement U_r(A)}(x) \leq -d_{\complement\overline{A}}(x) - r = b_{\overline{A}}(x) - r. \end{aligned}$$

(ii) If $\mathrm{m}(\partial\overline{A}) = 0$, from part (i), $d_{\complement U_r(A)}(x) \to d_{\complement\overline{A}}(x)$ implies that $\nabla d_{\complement U_r(A)} \rightharpoonup \nabla d_{\complement\overline{A}}$ in $L^2(D)^N$-weak for all bounded open sets. Consider the estimate

$$\begin{aligned} & \int_D |\nabla d_{\complement U_r(A)} - \nabla d_{\complement\overline{A}}|^2\, dx = \int_D \left(|\nabla d_{\complement U_r(A)}|^2 + |\nabla d_{\complement\overline{A}}|^2 - 2\,\nabla d_{\complement U_r(A)} \cdot \nabla d_{\complement\overline{A}}\right) dx \\ & = \int_D \left(1 - \chi_{\complement U_r(A)} + 1 - \chi_{\overline{\complement\overline{A}}} - 2\,\nabla d_{\complement U_r(A)} \cdot \nabla d_{\complement\overline{A}}\right) dx \end{aligned}$$

from the identities of Theorem 3.3 (viii) in Chapter 6. As $r \to 0$, $\{\complement U_r(A)\}$ is a family of increasing closed sets and

$$\complement U_r(A) \nearrow \cup_{r>0} \complement U_r(A) = \complement \cap_{r>0} U_r(A) = \complement\overline{A} \quad \Rightarrow \mathrm{m}(D \cap U_r(A)) \to \mathrm{m}(D \cap \complement\overline{A}).$$

Going to the limit

$$\begin{aligned} & \int_D \left(1 - \chi_{\complement U_r(A)} + 1 - \chi_{\overline{\complement\overline{A}}} - 2\,\nabla d_{\complement U_r(A)} \cdot \nabla d_{\complement\overline{A}}\right) dx \\ & \to \int_D \left(2 - \chi_{\complement\overline{A}} - \chi_{\overline{\complement\overline{A}}} - 2\,|\nabla d_{\complement\overline{A}}|^2\right) dx = \int_D \left(\chi_{\overline{\complement\overline{A}}} - \chi_{\complement\overline{A}}\right) dx = \int_D \chi_{\partial\overline{A}}\, dx = 0, \end{aligned}$$

since $|\nabla d_{\complement\overline{A}}| = 1 - \chi_{\overline{\complement\overline{A}}}$, $\partial\overline{A} = \overline{\complement\overline{A}}\backslash\complement\overline{A}$, and $\mathrm{m}(\partial\overline{A}) = 0$. Therefore $d_{\complement U_r(A)} \to d_{\complement\overline{A}}$ in $W^{1,2}(D)$ and hence in $W^{1,p}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$ for all p, $1 \leq p < \infty$. So, from Theorem 7.1 in Chapter 6, $b_{U_r(A)} = d_{U_r(A)} - d_{\complement U_r(A)} \to d_{\overline{A}} - d_{\complement\overline{A}} = b_{\overline{A}}$ in $W^{1,p}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$ for all p, $1 \leq p < \infty$. $\square$

7 Federer's Sets of Positive Reach

7.1 Approximation by Dilated Sets/Tubular Neighborhoods

We have seen in Theorem 6.1 that the oriented distance function $b_{\overline{A}}$ can be approximated uniformly in $\mathbf{R}^{\mathrm{N}}$ by the oriented distance function b_{A_r} of its dilated set A_r, $r > 0$, and that the $W^{1,p}$-convergence can be achieved if and only if $\mathrm{m}(\partial\overline{A}) = 0$. It turns out that the $W^{1,p}$-convergence can be obtained for sets with positive reach such that $\partial\overline{A} \neq \varnothing$ without assuming a priori that $\mathrm{m}(\partial\overline{A}) = 0$.

Theorem 7.1. *Let $A \subset \mathbf{R}^{\mathrm{N}}$ such that $\partial\overline{A} \neq \varnothing$ and* $\mathrm{reach}\,(A) > 0$.

(i) *$A \neq \varnothing$ and there exists h such that $0 < h \le \mathrm{reach}\,(A)$ and $A_h \neq \mathbf{R}^{\mathrm{N}}$. For all r, $0 < r < h$, the sets $U_r(A)$ are of class $C^{1,1}$, $b_{U_r(A)} = b_{A_r}$, and*

$$b_{A_r}(x) = b_{\overline{A}}(x) - r \text{ in } \mathbf{R}^{\mathrm{N}}. \tag{7.1}$$

As r goes to zero, $b_{A_r}(x) \to b_{\overline{A}}(x)$ uniformly in $\mathbf{R}^{\mathrm{N}}$,

$$b_{A_r} \to b_{\overline{A}} \text{ in } W^{1,p}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}}), \tag{7.2}$$

$$d_{A_r} \to d_{\overline{A}}, \quad d_{\complement A_r} \to d_{\complement \overline{A}}, \quad d_{\partial A_r} \to d_{\partial \overline{A}} \text{ in } W^{1,p}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}}), \tag{7.3}$$

$\mathrm{m}(\partial\overline{A}) = 0$, *and $\chi_{A_r} \to \chi_{\overline{A}} = \chi_{\mathrm{int}\,\overline{A}}$ in $L^p_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$ for all p, $1 \le p < \infty$.*

(ii) *Furthermore, $\partial\overline{A}$ is of locally bounded curvature and of locally finite perimeter:*

$$\nabla b_{\overline{A}} \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})^N \quad \text{and} \quad \chi_{\partial\overline{A}} \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}}). \tag{7.4}$$

Moreover, for all r, $0 < r < h$,

$$b_{\overline{A}}(x) = \frac{1}{r}\left[f_{\partial A_r}(x) - f_{\partial\overline{A}}(x)\right] + r = \frac{1}{r}\left[b^2_{\overline{A}}(x) - b^2_{A_r}(x)\right] + r, \tag{7.5}$$

$$\forall x \in \mathbf{R}^{\mathrm{N}}, \forall v \in \mathbf{R}^{\mathrm{N}}, \quad d_H b_{\overline{A}}(x; v) \text{ exists.} \tag{7.6}$$

Remark 7.1.
Recall from Theorem 3.1 (iii) that the Hadamard semiderivative of $b_{\overline{A}}$ always exists at points off the boundary. The last statement in part (ii) says that for a closed set with nonempty boundary and positive reach, the Hadamard semiderivative exists in all points of its boundary and there is an explicit expression for it. In particular, this is true for all closed convex sets. □

Proof. (i) The uniform convergence of b_{A_r} to $b_{\overline{A}}$ in $\mathbf{R}^{\mathrm{N}}$ was proved in Theorem 6.1 (i), and it remains to prove the $W^{1,p}$-convergence without the a priori assumption that $\mathrm{m}(\partial\overline{A}) = 0$ that will come as a consequence rather than as a hypothesis. To do that we first prove the convergence $d_{\complement U_r(A)} \to d_{\complement\overline{A}}$ in $W^{1,p}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$.

From Theorem 6.1 (i), it sufficient to show that inequalities (6.1) and (6.2) become equalities for $x \in U_h(A)$: for $0 < r < h$

$$d_{\complement U_r(A)}(x) = 0, \qquad \text{if } d_A(x) \ge r,$$

$$d_{\complement U_r(A)}(x) \ge r - d_A(x), \quad \text{if } 0 < d_A(x) < r, \tag{7.7}$$

$$d_{\complement U_r(A)}(x) \ge d_{\complement\overline{A}}(x) + r, \quad \text{if } d_A(x) = 0. \tag{7.8}$$

For $x \in \mathbf{R}^N$ such that $d_A(x) \geq r$, $d_{\complement \overline{A}}(x) = d_{\complement U_r(A)}(x) = 0$ and, from identity (7.2) of Chapter 6 in Theorem 6.1, $b_{A_r}(x) = d_{A_r}(x) = d_A(x) - r = b_{\overline{A}}(x) - r$. For $x \in \mathbf{R}^N$ such that $0 < d_A(x) < r$, $d_{\complement \overline{A}}(x) = 0 = d_{A_r}(x)$. From Theorem 6.2 (iii) of Chapter 6 for all r, $0 < d_A(x) < r < h \leq \text{reach}\,(A)$,

$$x(r) \stackrel{\text{def}}{=} p_A(x) + r\frac{x - p_A(x)}{d_A(x)} \quad \Rightarrow d_A(x(r)) = r \text{ and } p_A(x(r)) = p_A(x)$$

and $x(r) \in \partial A_r = \partial U_r(A)$. Since $x \notin \complement U_r(A)$, $d_{\complement U_r(A)}(x) = \inf_{p \in \partial A_r} |x - p|$ and

$$d_{\complement U_r(A)}(x) = d_{\complement A_r}(x) = d_{\partial A_r}(x) \leq |x(r) - x| = r - d_A(x).$$

Combining this with inequality (7.7), $b_{A_r}(x) = b_{\overline{A}}(x) - r$. Finally, for $x \in \mathbf{R}^N$ such that $0 = d_A(x)$, $x \in \overline{A}$, and since $\partial \overline{A} \neq \varnothing$, there exists $\bar{p} \in \partial \overline{A}$ such that $d_{\complement \overline{A}}(x) = |\bar{p} - x|$. There exists a sequence $\{y_n\} \subset U_r(A) \backslash \overline{A}$ such that $y_n \to p$ and c_r such that

$$|p_A(y_n) - p_A(\bar{p})| \leq c_r\,|y_n - \bar{p}| \quad \Rightarrow p_A(y_n) \to p_A(\bar{p}) = \bar{p}.$$

Associate with each y_n the points

$$q_n \stackrel{\text{def}}{=} p_A(y_n) + r\,\nabla d_A(y_n).$$

Since $0 < d_A(q_n) \leq r < h \leq \text{reach}\,(A)$, by Theorem 6.2 (iii) of Chapter 6

$$p_A(q_n) = p_A(y_n), \quad d_A(q_n) = r, \quad q_n \in \partial A_r.$$

The sequence $\{q_n\}$ is bounded since

$$\begin{aligned} |q_n - \bar{p}| \leq |q_n - p_A(q_n)| + |p_A(q_n) - \bar{p}| &= |q_n - p_A(q_n)| + |p_A(y_n) - p_A(\bar{p})| \\ &\leq r + c_r|y_n - \bar{p}|. \end{aligned}$$

There exist q and a subsequence, still indexed by n, such that

$$\begin{aligned} &q_n \to q \quad \Rightarrow r = d_A(q_n) \to d_A(q) \quad \Rightarrow p_A(y_n) = p_A(q_n) \to p_A(q) \\ &\Rightarrow d_A(q) = r \quad \text{and} \quad p_A(q) = \bar{p}, \quad r = d_A(q) = |q - p_A(q)| = |q - \bar{p}|. \end{aligned}$$

Finally, since $x \in \overline{A}$, $\overline{A} \cap \overline{\complement A_r} = \varnothing$, and $\partial \complement A_r = \partial A_r = d_A^{-1}\{r\}$

$$d_{\complement A_r}(x) = \inf_{d_A(p_r) = r} |x - p_r| \leq |x - q| \leq |x - \bar{p}| + |\bar{p} - q| = d_{\complement \overline{A}}(x) + r.$$

Hence, from inequality (7.8), $d_{\complement A_r}(x) = d_{\complement \overline{A}}(x) + r$, and since $d_A(x) = 0 = d_{A_r}(x)$, $b_{A_r}(x) = b_{\overline{A}}(x) - r$.

Since $\nabla b_{A_r} = \nabla b_{\overline{A}}$, for all bounded open subsets D of $\mathbf{R}^N$ as $r \to 0$,

$$\|b_{A_r} - b_{\overline{A}}\|_{W^{1,p}(D)} = \|b_{A_r} - b_{\overline{A}}\|_{L^p(D)} \leq r\,\mathrm{m}(D)^{1/p} \to 0.$$

From Theorem 7.1 of Chapter 6

$$d_{A_r} \to d_A \text{ in } W^{1,p}_{\text{loc}}(\mathbf{R}^{\mathrm{N}}) \quad \text{and} \quad \chi_{A_r} \to \chi_{\overline{A}} \text{ in } L^p_{\text{loc}}(\mathbf{R}^{\mathrm{N}})$$

for all p, $1 \le p < \infty$. For all $0 < r < h \le \text{reach}\,(A)$, $d_{\complement A_r} = d_{A_r} - b_{A_r} \to d_A - b_{\overline{A}} = d_{\complement \overline{A}}$ in $W^{1,p}_{\text{loc}}(\mathbf{R}^{\mathrm{N}})$ and $d_{\partial A_r} = d_{A_r} + d_{\complement A_r} \to d_{\overline{A}} + d_{\complement \overline{A}} = d_{\partial \overline{A}}$ in $W^{1,p}_{\text{loc}}(\mathbf{R}^{\mathrm{N}})$. Finally, since $b_{A_r} \to b_{\overline{A}}$ in $W^{1,p}_{\text{loc}}(\mathbf{R}^{\mathrm{N}})$, $0 = \chi_{\partial A_r} = 1 - |\nabla b_{A_r}| \to 1 - |\nabla b_{\overline{A}}| = \chi_{\partial \overline{A}}$ in $L^p_{\text{loc}}(\mathbf{R}^{\mathrm{N}})$ and $\chi_{\partial \overline{A}} = 0$ and $\mathrm{m}(\partial \overline{A}) = 0$.

(ii) For $0 < r < h < \text{reach}\,(A)$, the open set $U_r(A)$ is defined via the r-level set $d_A^{-1}\{r\} = \{x \in \mathbf{R}^{\mathrm{N}} : d_A(x) = r\}$ of the function d_A. From Theorem 6.2 (vi) in Chapter 6, for each $x \in d_A^{-1}\{r\} = \partial U_r(A)$ there exists a neighborhood $B_\rho(x)$, $\rho = \min\{r, h-r\}/2 > 0$, of x in $U_h(A)\backslash\overline{A}$ such that $d_A \in C^{1,1}(\overline{B_\rho(x)})$. From part (i) for all $0 < r < h$ and $y \in \overline{B_\rho(x)} \subset U_h(A)\backslash\overline{A}$, $d_{\complement \overline{A}}(y) = 0$, $b_{A_r}(y) = b_{\overline{A}}(y) - r = d_A(y) - r$. Therefore, $b_{A_r} \in C^{1,1}(\overline{B_\rho(x)})$, and, a fortiori, for all $x \in \partial A_r$, there exists $B_\rho(x)$ such that $\nabla b_{A_r} = \nabla d_A \in \mathrm{BV}(B_\rho(x))$. By Theorem 5.1, ∂A_r is of locally bounded curvature. By Definition 5.1, $\nabla b_{A_r} \in \mathrm{BV}_{\text{loc}}(\mathbf{R}^{\mathrm{N}})^N$ and, since $b_{A_r} = b_{\overline{A}} - r$ in $\mathbf{R}^{\mathrm{N}}$, we also have $\nabla b_{\overline{A}} = \nabla b_{A_r} \in \mathrm{BV}_{\text{loc}}(\mathbf{R}^{\mathrm{N}})^N$. By Theorem 5.2 (ii), $\chi_{\partial \overline{A}} \in \mathrm{BV}_{\text{loc}}(\mathbf{R}^{\mathrm{N}})$. Finally, since $b_{A_r}(x) = b_{\overline{A}}(x) - r$ in $\mathbf{R}^{\mathrm{N}}$ from part (i)

$$|x|^2 - b^2_{A_r}(x) = |x|^2 - b^2_{\overline{A}}(x) - r^2 + 2r\, b_{\overline{A}}(x),$$
$$f_{\partial A_r}(x) - f_{\partial \overline{A}}(x) + r^2 = r\, b_{\overline{A}}(x)$$

and $b_{\overline{A}}$ is the difference of two convex continuous functions for which the Hadamard semiderivative exists in all points $x \in \mathbf{R}^{\mathrm{N}}$ and for all directions $v \in \mathbf{R}^{\mathrm{N}}$. $\square$

7.2 Boundaries with Positive Reach

The next theorem is a specialization of Theorem 6.2 of Chapter 6 to the oriented distance function.

Theorem 7.2. *Let $A \subset \mathbf{R}^{\mathrm{N}}$, $\partial A \neq \varnothing$.*

(i) *Associate with $a \in \partial A$ the sets*

$$P(a) \overset{\text{def}}{=} \left\{v \in \mathbf{R}^{\mathrm{N}} : \Pi_{\partial A}(a+v) = \{a\}\right\}, \quad Q(a) \overset{\text{def}}{=} \left\{v \in \mathbf{R}^{\mathrm{N}} : d_{\partial A}(a+v) = |v|\right\}.$$

Then $P(a)$ and $Q(a)$ are convex and $P(a) \subset Q(a) \subset (-T_a \partial A)^*$.

(ii) *Given $a \in \partial A$ and $v \in \mathbf{R}^{\mathrm{N}}$, assume that*

$$0 < r(a,v) \overset{\text{def}}{=} \sup\{t > 0 : \Pi_{\partial A}(a+tv) = \{a\}\}.$$

Then for all t, $0 \le t < r(a,v)$, $\Pi_{\partial A}(a+tv) = \{a\}$ and $d_{\partial A}(a+tv) = t\,|v|$. Moreover, if $r(a,v) < +\infty$, then $a + r(a,v)\, v \in \overline{\mathrm{Sk}\,(\partial A)}$.

(iii) *If there exists h such that $0 < h \le \text{reach}\,(\partial A)$, then for all $x \in U_h(\partial A)\backslash\partial A$ and $0 < |t| < h$, $\Pi_{\partial A}(x(t)) = \{p_{\partial A}(x)\}$ and $b_A(x(t)) = t$, where*

$$x(t) \overset{\text{def}}{=} p_{\partial A}(x) + t\,\nabla b_A(x) = p_{\partial A}(x) + t\,\frac{x - p_{\partial A}(x)}{b_A(x)}. \tag{7.9}$$

(iv) *Given* $b \in \partial A$, $x \in \mathbf{R}^{\mathrm{N}} \setminus \overline{\mathrm{Sk}\,(\partial A)}$, $a = p_{\partial A}(x)$, *such that* $\mathrm{reach}\,(\partial A, a) > 0$, *then*

$$(x-a)\cdot(a-b) \geq -\frac{|a-b|^2|x-a|}{2\,\mathrm{reach}\,(\partial A, a)}. \tag{7.10}$$

(v) *Given* $0 < r < q < \infty$ *and*

$$x \in \mathbf{R}^{\mathrm{N}} \setminus \overline{\mathrm{Sk}\,(\partial A)} \text{ such that } |b_A(x)| \leq r \text{ and } \mathrm{reach}\,(\partial A, p_{\partial A}(x)) \geq q,$$
$$y \in \mathbf{R}^{\mathrm{N}} \setminus \overline{\mathrm{Sk}\,(\partial A)} \text{ such that } |b_A(y)| \leq r \text{ and } \mathrm{reach}\,(\partial A, p_{\partial A}(y)) \geq q,$$

then

$$|p_{\partial A}(y) - p_{\partial A}(x)| \leq \frac{q}{q-r}\,|y-x|. \tag{7.11}$$

(vi) *If* $0 < \mathrm{reach}\,(\partial A) < +\infty$, *then for all* h, $0 < h < \mathrm{reach}\,(\partial A)$, *and all* $x, y \in (\partial A)_h = \{x \in \mathbf{R}^{\mathrm{N}} : |b_A(x)| \leq h\}$

$$\begin{aligned} |p_{\partial A}(y) - p_{\partial A}(x)| &\leq \frac{\mathrm{reach}\,(\partial A)}{\mathrm{reach}\,(\partial A) - h}\,|y-x|, \\ \left|\nabla b_A^2(y) - \nabla b_A^2(x)\right| &\leq 2\left(1 + \frac{\mathrm{reach}\,(\partial A)}{\mathrm{reach}\,(\partial A) - h}\right)\,|y-x|; \end{aligned} \tag{7.12}$$

for all s, $0 < s < h < \mathrm{reach}\,(\partial A)$, $x, y \in \partial A_{s,h} = \left\{x \in \mathbf{R}^{\mathrm{N}} : s \leq |b_A(x)| \leq h\right\}$,

$$|\nabla b_A(y) - \nabla b_A(x)| \leq \frac{1}{s}\left(2 + \frac{\mathrm{reach}\,(\partial A)}{\mathrm{reach}\,(\partial A) - h}\right)|y-x|\,.$$

If $\mathrm{reach}\,(\partial A) = +\infty$,

$$\begin{aligned} \forall x, y \in \mathbf{R}^{\mathrm{N}}, \quad |p_{\partial A}(y) - p_{\partial A}(x)| &\leq |y-x|, \\ \left|\nabla b_A^2(y) - \nabla b_A^2(x)\right| &\leq 4\,|y-x|; \end{aligned} \tag{7.13}$$

for all $0 < s$ *and all* $x, y \in \{x \in \mathbf{R}^{\mathrm{N}} : s \leq d_{\partial A}(x)\}$

$$|\nabla b_A(y) - \nabla b_A(x)| \leq \frac{3}{s}\,|y-x|\,.$$

(vii) *If* $\mathrm{reach}\,(\partial A) > 0$, *then, for all* $0 < r < \mathrm{reach}\,(\partial A)$, $U_r(\partial A)$ *is a set of class* $C^{1,1}$, $\overline{U_r(\partial A)} = (\partial A)_r$, $\partial U_r(\partial A) = \{x \in \mathbf{R}^{\mathrm{N}} : |b_A(x)| = r\}$, *and* $\overline{\complement(\partial A)_r} = \complement U_r(\partial A) = \{x \in \mathbf{R}^{\mathrm{N}} : |b_A(x)| \geq r\}$.

(viii) *For all* $a \in \overline{\partial A}$,

$$T_a \partial A = \left\{h \in \mathbf{R}^{\mathrm{N}} : \liminf_{t \searrow 0} \frac{|b_A(a+th)|}{t} = 0\right\}.$$

In particular, if $a \in \mathrm{int}\,\overline{\partial A}$, $T_a \partial A = \mathbf{R}^{\mathrm{N}}$.

The next theorem is a specialization of Theorem 6.3 of Chapter 6 to the oriented distance function.

Theorem 7.3. *Given $A \subset \mathbf{R}^N$ such that $\partial A \neq \varnothing$, the following conditions are equivalent:*

(i) *∂A has positive reach, that is,* $\operatorname{reach}(\partial A) > 0$.

(ii) *There exists $h > 0$ such that $b_A^2 \in C^{1,1}(\overline{U_h(\partial A)})$.*

(iii) *There exists $h > 0$ such that for all s, $0 < s < h$, $b_A \in C^{1,1}(\overline{U_{s,h}(\partial A)})$.*

(iv) *There exists $h > 0$ such that $b_A \in C^1(U_h(\partial A)\backslash\partial A)$.*

(v) *There exists $h > 0$ such that for all $x \in U_h(\partial A)$, $\Pi_{\partial A}(x)$ is a singleton.*

Recall from Theorem 3.1 (viii) and (3.15) the following properties. If $\partial A \neq \varnothing$,

$$\begin{gathered}\operatorname{Sk}(\partial A) = \operatorname{Sk}(A) \cup \operatorname{Sk}(\complement A) \subset \mathbf{R}^N \backslash \partial A,\\ \operatorname{Ck}_b(A) \subset \operatorname{Ck}(\partial A) = \operatorname{Ck}(A) \cup \operatorname{Ck}(\complement A) \subset \partial(\partial A).\end{gathered}$$

If ∂A has positive reach, then both A and $\complement A$ have positive reach.

8 Boundary Smoothness and Smoothness of b_A

In this section the smoothness of the boundary ∂A is related to the smoothness of the function b_A in a neighborhood of ∂A. The next two theorems give a complete characterization of sets A such that $m(\partial A) = 0$ and $b_A \in C^{k,\ell}$ in a neighborhood of ∂A. We first present two equivalence theorems that will follow from the more detailed Theorems 8.3 and 8.4. It is useful to recall that for a set that is locally of class $C^{k,\ell}$ in a point of ∂A, ∂A is locally a $C^{k,\ell}$-submanifold of dimension $N-1$ in that point, but the converse is not true (cf. Definitions 3.1 and 3.2 in Chapter 2)[4].

Theorem 8.1 (Local version)**.** *Let $A \subset \mathbf{R}^N$, $\partial A \neq \varnothing$, let $k \geq 1$ be an integer, let $0 \leq \ell \leq 1$ be a real number, and let $x \in \partial A$.*

[4] It is important to note that Theorem 8.1 is not true when b_A is replaced by $d_{\partial A}$ since its gradient $\nabla d_{\partial A}$ is discontinuous across ∂A. Part (iii) of Theorem 8.1 in the direction ($\Rightarrow$) was asserted by J. Serrin [**1**] in 1969 for $N = 3$ and proved in 1977 by D. Gilbarg and N. S. Trudinger [**1**] for $k \geq 2$ (provided that $d_{\partial A}$ is replaced by b_A in Lemma 1, p. 382 in [**1**], and Lemma 14.16, p. 355 in [**2**]) with a different proof by S. G. Krantz and H. R. Parks [**1**] in 1981. Another proof with the function b_A was given in 1994 by M. C. Delfour and J.-P. Zolésio [**17**], who extended the result from (iii) to (ii) in the direction ($\Rightarrow$) down to the $C^{1,1}$ case and established the equivalence ($\Leftrightarrow$) in the whole range from C^∞ to $C^{1,1}$. The counterexample of Example 6.2 for domains of class $C^{1,1-\epsilon}$ is the same as the one provided earlier in S. G. Krantz and H. R. Parks [**1**], where they observe only that the domain is $C^{2-\varepsilon}$, leaving the reader under the misleading impression that part (ii) of Theorem 8.1 would not be true for domains ranging from class $C^{1,1}$ to C^2. Part (i) of Theorem 8.1 in the direction ($\Rightarrow$) was proved in the C^1 case by S. G. Krantz and H. R. Parks [**1**] in 1981. The equivalence ($\Leftrightarrow$) here for $C^{1,\lambda}$, $0 \leq \lambda < 1$, was given in 2004 by M. C. Delfour and J.-P. Zolésio [**40**].

(i) $(k = 1,\ 0 \le \ell < 1)$. *A is a set of class* $C^{1,\ell}$ *in a bounded open neighborhood* $V(x)$ *of* x *and* $\mathrm{Sk}\,(\partial A) \cap V(x) = \varnothing$ *if and only if*

$$\exists \rho > 0 \text{ such that } b_A \in C^{1,\ell}(\overline{B_\rho(x)}) \quad \text{and} \quad \mathrm{m}(\partial A \cap B_\rho(x)) = 0. \tag{8.1}$$

(ii) $(k = 1,\ \ell = 1)$. *A is locally a set of class* $C^{1,1}$ *at* x *if and only if*

$$\exists \rho > 0 \text{ such that } b_A \in C^{1,1}(\overline{B_\rho(x)}) \quad \text{and} \quad \mathrm{m}(\partial A \cap B_\rho(x)) = 0. \tag{8.2}$$

(iii) $(k \ge 2,\ 0 \le \ell \le 1)$. *A is locally a set of class* $C^{k,\ell}$ *at* x *if and only if*

$$\exists \rho > 0 \text{ such that } b_A \in C^{k,\ell}(\overline{B_\rho(x)}) \quad \text{and} \quad \mathrm{m}(\partial A \cap B_\rho(x)) = 0. \tag{8.3}$$

Moreover, in all cases, $\nabla b_A = n \circ p_{\partial A}$ *in* $\overline{B_\rho(x)}$, *where* n *is the unit exterior normal to* A *on* ∂A, *and* ∂A *is locally a* $C^{k,\ell}$*-submanifold of dimension* $N-1$ *at* x.

Remark 8.1.
The condition $\mathrm{Sk}\,(\partial A) \cap V(x) = \varnothing$ in part (i) is really necessary. Example 6.2 in Chapter 6 gives a family of two-dimensional sets of class $C^{1,\lambda}$, $0 \le \lambda < 1$, for which $b_A \in C^{1,\lambda}$ and $d^2_{\partial A} = b^2_A \notin C^1$ in any neighborhood of the point $(0,0)$ of its boundary ∂A. □

From Theorem 8.1 we get the global version that has had and still has an interesting history of inaccurate statements or incomplete proofs

Theorem 8.2 (Global version). *Let* $A \subset \mathbf{R}^{\mathrm{N}}$, $\partial A \ne \varnothing$, *let* $k \ge 1$ *be an integer, and let* $0 \le \ell \le 1$ *be a real number.*

(i) $(k = 1,\ 0 \le \ell < 1)$. *A is a set of class* $C^{1,\ell}$ *and* $\partial A \cap \overline{\mathrm{Sk}\,(\partial A)} = \varnothing$ *if and only if*

$$\mathrm{m}(\partial A) = 0 \quad \text{and} \quad \forall x \in \partial A,\ \exists \rho > 0 \text{ such that } b_A \in C^{1,\ell}(\overline{B_\rho(x)}). \tag{8.4}$$

(ii) $(k = 1,\ \ell = 1)$. *A is a set of class* $C^{1,1}$ *if and only if*

$$\mathrm{m}(\partial A) = 0 \quad \text{and} \quad \forall x \in \partial A,\ \exists \rho > 0 \text{ such that } b_A \in C^{1,1}(\overline{B_\rho(x)}). \tag{8.5}$$

(iii) $(k \ge 2,\ 0 \le \ell \le 1)$. *A is a set of class* $C^{k,\ell}$ *if and only if*

$$\mathrm{m}(\partial A) = 0 \quad \text{and} \quad \forall x \in \partial A,\ \exists \rho > 0 \text{ such that } b_A \in C^{k,\ell}(\overline{B_\rho(x)}). \tag{8.6}$$

Moreover, in all cases, $\nabla b_A = n \circ p_{\partial A}$ *in* $\overline{B_\rho(x)}$, *where* n *is the unit exterior normal to* A *on* ∂A, *and* ∂A *is a* $C^{k,\ell}$*-submanifold of dimension* $N-1$.

Remark 8.2.
Theorems 8.1 and 8.2 deal only with sets of class $C^{1,\ell}$ whose boundary is a $C^{1,\ell}$-submanifold of dimension $(N-1)$. For arbitrary closed submanifolds M of $\mathbf{R}^{\mathrm{N}}$ of codimension greater than or equal to 1, ∇b_M generally does not exist on M. In that case the smoothness of M is related to the existence and the smoothness of

∇d_M^2 in a neighborhood of M that implies that M is locally of positive reach. The reader is referred to Theorem 6.5 in section 6.2 of Chapter 6. $\square$

The above two equivalence theorems follow from the following two local theorems in each direction.

Theorem 8.3. *Let $A \subset \mathbf{R}^N$ be such that $\partial A \neq \varnothing$. Given $x \in \partial A$, assume that*

$$\exists \text{ a bounded open neighborhood } V(x) \text{ of } x \text{ such that } b_A \in C^{k,\ell}(\overline{V(x)}) \tag{8.7}$$

for some integer $k \geq 1$ and some real ℓ, $0 \leq \ell \leq 1$, and that $\mathrm{m}(\partial A \cap V(x)) = 0$.

(i) *Then A is locally a set of class $C^{k,\ell}$ in $V(x)$ and $b_A = b_{\mathrm{int}\, A} = b_{\overline{A}}$ in $V(x)$.*

(ii) $\Pi_{\partial A}(y) = \{p_{\partial A}(y)\}$ *is a singleton for all $y \in V(x)$ and*

$$b_A^2 \in C^{1,1}_{\mathrm{loc}}(V(x)) \cap C^{k,\ell}_{\mathrm{loc}}(V(x)) \quad \text{and} \quad p_{\partial A} \in C^{0,1}_{\mathrm{loc}}(V(x))^N \cap C^{k-1,\ell}_{\mathrm{loc}}(V(x))^N.$$

Remark 8.3.
If condition (8.7) is verified, the condition $\mathrm{m}(\partial A) = 0$ is equivalent to $\partial A \neq \mathbf{R}^N$. The condition $b_A \in C^0_b(\mathbf{R}^N)$ is necessary to rule out the case $\varnothing \neq \partial A = \mathbf{R}^N$ that yields $b_A = 0$ and $\overline{A} = \overline{\complement A} = \mathbf{R}^N$ that trivially satisfies condition (8.7). $\square$

Proof. (i) Since ∇b_A exists and is continuous everywhere in $V(x)$, $|\nabla b_A|$ is continuous in $V(x)$. But $|\nabla b_A(x)| = 1$ in $V(x)\backslash\partial A$ and $|\nabla b_A(x)| = 0$ almost everywhere in $V(x) \cap \partial A$. By continuity, either $|\nabla b_A| = 1$ or $|\nabla b_A| = 0$ in $V(x)$. If $\nabla b_A = 0$ in $V(x)$, then b_A is constant in $V(x)$ and equal to $b_A(x) = 0$. Therefore, $V(x) \subset \partial A$ and this yields the contradiction $0 < \mathrm{m}(V(x)) \leq \mathrm{m}(\partial A \cap V(x)) = 0$. Consider the set $\Omega \stackrel{\text{def}}{=} \{x \in \mathbf{R}^N : -b_A(x) > 0\}$. By Theorem 4.2 in Chapter 2, Ω is a set of class $C^{k,\ell}$ since $b_A \in C^{k,\ell}(\overline{V(x)})$ and $\nabla b_A \neq 0$ on $b_A^{-1}\{0\} \cap V(x)$. In particular, $\partial\Omega = b_A^{-1}\{0\} = \partial A$, $\mathrm{int}\, A = \Omega$, and $\mathrm{int}\, \complement A = \mathrm{int}\, \complement\Omega$ in $V(x)$. Finally $b_A = b_\Omega = b_{\mathrm{int}\, A} = b_{\overline{A}}$.

(ii) By assumption, $b_A \in C^{k,\ell}_{\mathrm{loc}}(V(x))$ and hence b_A^2 belongs to $C^{k,\ell}_{\mathrm{loc}}(V(x))$. By Theorem 3.1 (iv), $\Pi_{\partial A}(y) = \{p_{\partial A}(y)\}$ is a singleton in $V(x)$. By Theorem 7.3, $b_A^2 \in C^{1,1}_{\mathrm{loc}}(V(x))$ and, a fortiori, in $C^{k,\ell}_{\mathrm{loc}}(V(x))$. This yields $p_{\partial A} \in C^{0,1}_{\mathrm{loc}}(V(x)) \cap C^{k-1,\ell}_{\mathrm{loc}}(V(x))$. $\square$

In the other direction the smoothness of the domain implies the smoothness of b_A in a neighborhood of ∂A when the domain is at least of class $C^{1,1}$. A counterexample will be given in dimension 2.

Theorem 8.4. *Let $A \subset \mathbf{R}^N$, $\partial A \neq \varnothing$, let $k \geq 1$ be an integer, let $0 \leq \ell \leq 1$ be a real number, and let $x \in \partial A$ if it is not.*

(i) $(k = 1, 0 \leq \ell < 1)$. *If A is locally a set of class $C^{1,\ell}$ in a bounded open neighborhood $V(x)$ of x and* $\mathrm{Sk}\,(\partial A) \cap V(x) = \varnothing$, *then* $\mathrm{m}(\partial A \cap V(x)) = 0$ *and*

$$\exists \rho > 0 \text{ such that } b_A \in C^{1,\ell}(\overline{B_\rho(x)}) \quad \text{and} \quad p_{\partial A} \in C^{0,1}(\overline{B_\rho(x)}). \tag{8.8}$$

(ii) $(k = 1,\ \ell = 1)$. *If A is locally a set of class $C^{1,1}$ in a bounded open neighborhood $V(x)$ of x, then* $\mathrm{m}(\partial A \cap V(x)) = 0$ *and*

$$\exists \rho > 0 \text{ such that } b_A \in C^{1,1}(\overline{B_\rho(x)}) \quad \text{and} \quad p_{\partial A} \in C^{0,1}(\overline{B_\rho(x)}). \tag{8.9}$$

(iii) $(k \geq 2,\ 0 \leq \ell \leq 1)$. *If A is locally a set of class $C^{k,\ell}$ in a bounded open neighborhood $V(x)$ of x, then* $\mathrm{m}(\partial A \cap V(x)) = 0$ *and*

$$\exists \rho > 0 \text{ such that } b_A \in C^{k,\ell}(\overline{B_\rho(x)}) \quad \text{and} \quad p_{\partial A} \in C^{k-1,\ell}(\overline{B_\rho(x)}). \tag{8.10}$$

Moreover, in all cases, $\nabla b_A = n \circ p_{\partial A}$ in $\overline{B_\rho(x)}$, where n is the unit exterior normal to A on ∂A.

Proof. (i) From Definition 3.2 of Chapter 2 of a locally a $C^{1,\ell}$-submanifold of dimension $N-1$, $\mathrm{m}(\partial A \cap V(x)) = 0$ in a bounded open neighborhood $V(x)$ and the set A can be locally described by the level sets of the $C^{1,\ell}$-function

$$f(y) \stackrel{\text{def}}{=} g_x(y) \cdot e_N, \tag{8.11}$$

since by definition

$$\operatorname{int} A \cap V(x) = \{y \in V(x) : f(y) > 0\}, \quad \partial A \cap V(x) = \{y \in V(x) : f(y) = 0\}.$$

The boundary ∂A in $V(x)$ is the zero level set of f and the gradient

$$\nabla f(y) = Dg_x(y)^* e_N \neq 0 \text{ in } V(x)$$

is normal to that level set. Thus the *outward unit normal* n to A on $V(x) \cap \partial A$ is given by

$$n(y) = -\frac{\nabla f(y)}{|\nabla f(y)|} = -\frac{Dg_x(y)^* e_N}{|Dg_x(y)^* e_N|} = -\frac{Dh_x(g_x(y))^{-*} e_N}{|Dh_x(g_x(y))^{-*} e_N|}. \tag{8.12}$$

Since $V(x) \cap \mathrm{Sk}\,(\partial A) = \varnothing$, for all $y \in V(x)$,

$$\begin{aligned} &\exists \text{ unique } p_{\partial A}(y) \in \partial A, \quad d_{\partial A}(y) = \inf_{p \in \partial A} |y - p| = |y - p_{\partial A}(y)|, \\ &p_{\partial A}(y) = y - \frac{1}{2} \nabla d^2_{\partial A}(y) = y - \frac{1}{2} \nabla b^2_A(y). \end{aligned} \tag{8.13}$$

Choose $\rho > 0$ such that $\overline{B_{3\rho}(x)} \subset V(x)$. Then for all $y \in \overline{B_\rho(x)}$

$$\begin{aligned} &|p_{\partial A}(y) - x| \leq |p_{\partial A}(y) - y| + |y - x| \leq 2\,|y - x| \leq 2r \\ &\Rightarrow\ p_{\partial A}(y) \in \overline{B_{2\rho}(x)} \cap \partial A \subset V(x) \cap \partial A = V(x) \cap f^{-1}\{0\}. \end{aligned}$$

Therefore, for each $y \in \overline{B_\rho(x)}$, there exists a unique $p_{\partial A}(y) \in V(x) \cap f^{-1}\{0\}$ such that

$$d^2_{\partial A}(y) = \inf_{p \in V(x),\, f(p)=0} |p-y|^2 = |y - p_{\partial A}(y)|^2.$$

By using the Lagrange multiplier theorem for the Lagrangian

$$|z-y|^2 + \lambda\, f(z)$$

there exists a $\hat{\lambda} \in \mathbf{R}$ such that

$$\begin{aligned} 2(p_{\partial A}(y) - y) + \hat{\lambda}\, \nabla f(p_{\partial A}(y)) = 0 \quad &\text{and} \quad f(p_{\partial A}(y)) = 0 \\ \Rightarrow\ 2 d_{\partial A}(y) &= |\hat{\lambda}|\, |\nabla f(p_{\partial A}(y))| \\ \Rightarrow\ p_{\partial A}(y) = y + b_A(y) \frac{\nabla f(p_{\partial A}(y))}{|\nabla f(p_{\partial A}(y))|} &= y - b_A(y)\, n(p_{\partial A}(y)) \end{aligned}$$

since $\nabla f(p_{\partial A}(y)) \neq 0$ on $\partial A \cap V(x)$. Combining this with (8.13)

$$\begin{aligned} \forall y \in B_r(x), \quad y - \frac{1}{2} \nabla b_A^2(y) &= p_{\partial A}(y) = y - b_A(y)\, n(p_{\partial A}(y)) \\ \Rightarrow\ \forall y \in B_r(x) \backslash \partial A, \quad \nabla b_A(y) &= n(p_{\partial A}(y)) \end{aligned} \tag{8.14}$$

since $V(x) \cap \mathrm{Sk}\,(\partial A) = \varnothing$ implies that $\nabla b_A(y)$ exists in $V(x)\backslash\partial A$ by Theorem 3.1 (vii).

Consider the normal $n(z) = -\nabla f(z)/|\nabla f(z)|$ in $V(x)\backslash\partial A$. For z_2 and z_1 in $\partial A \cap \overline{B_{2\rho}(x)}$ the difference

$$\begin{aligned} |n(z_2) - n(z_1)| &= \left| \frac{\nabla f(z_2)}{|\nabla f(z_2)|} - \frac{\nabla f(z_1)}{|\nabla f(z_1)|} \right| \\ &= \left| \frac{1}{|\nabla f(z_2)|} (\nabla f(z_2) - \nabla f(z_1)) + \left(\frac{1}{|\nabla f(z_2)|} - \frac{1}{|\nabla f(z_1)|} \right) \nabla f(z_1) \right| \\ &= \frac{1}{|\nabla f(z_2)|} \left| \nabla f(z_2) - \nabla f(z_1) + (|\nabla f(z_1)| - |\nabla f(z_2)|) \frac{\nabla f(z_1)}{|\nabla f(z_1)|} \right| \\ &\leq 2 \frac{1}{|\nabla f(z_2)|} |\nabla f(z_2) - \nabla f(z_1)| \leq 2 \frac{1}{\inf_{z \in \partial A \cap \overline{B_{2\rho}(x)}} |\nabla f(z)|} |\nabla f(z_2) - \nabla f(z_1)|. \end{aligned}$$

But $f \in C^{1,\ell}(\overline{B_{2\rho}(x)})$ and since $|\nabla f| \neq 0$ is continuous on $V(x)$,

$$|\nabla f(z_2) - \nabla f(z_1)| \leq c_x\, |z_2 - z_1|^\ell \quad \text{and} \quad \alpha \overset{\text{def}}{=} \inf_{z \in \partial A \cap \overline{B_{2\rho}(x)}} |\nabla f(z)| > 0$$

and the outward unit normal is ℓ-Hölderian,

$$\forall z_2,\, z_1 \in \overline{B_{2\rho}(x)} \cap \partial A, \quad |n(z_2) - n(z_1)| \leq \frac{2}{\alpha}\, c_x\, |z_2 - z_1|^\ell,$$

where c_x is the Lipschitz constant of f in $\overline{B_{2\rho}(x)}$. By construction, $\overline{B_{3\rho}(x)} \cap \mathrm{Sk}\,(\partial A) \subset V(x) \cap \mathrm{Sk}\,(\partial A) = \varnothing$ and for all $z \in \overline{B_\rho(x)}$, $B_{2\rho}(z) \cap \mathrm{Sk}\,(\partial A) = \varnothing$

implies that reach $(\partial A, z) \geq 2r > 0$. By Theorem 6.2 (v) in Chapter 6 applied to ∂A with $0 < r = \rho < q = 2\rho$,

$$\forall z_1, z_2 \in \overline{B_\rho(x)}, \quad |p_{\partial A}(z_1) - p_{\partial A}(z_2)| \leq 2\,|z_1 - z_2| \tag{8.15}$$

and hence $p_{\partial A} \in C^{0,1}(\overline{B_\rho(x)})^N$. So the composition $n \circ p_{\partial A}$ belongs to $C^{0,\ell}(\overline{B_\rho(x)})^N$ since we have already shown that $n \in C^{0,\ell}(\overline{B_{2\rho}(x)} \cap \partial A)^N$. Finally, $\nabla b_A \in L^\infty(B_\rho(x))^N$, $n(p_{\partial A}) \in C^{0,\ell}(\overline{B_\rho(x)})^N$, $\mathrm{m}(\partial A) = 0$, and from (8.14)

$$\nabla b_A(y) = n(p_{\partial A}(y)) \quad \text{a.e. in } \overline{B_\rho(x)}$$

imply that $\nabla b_A \in C^{0,\ell}(\overline{B_\rho(x)})^N$, $b_A \in C^{1,\ell}(\overline{B_\rho(x)})$, and $\nabla b_A = n \circ p_{\partial A}$ in $\overline{B_\rho(x)}$.

(ii) The first steps are the same as in part (i) up to (8.12) with $C^{1,1}$ in place of $C^{1,\ell}$. For each $y \in V(x)$, $\Pi_{\partial A}(y)$ is compact and nonempty and

$$\forall p \in \Pi_{\partial A}(y), \quad d^2_{\partial A}(y) = \inf_{a \in \partial A} |a - y|^2 = |p - y|^2.$$

Choose $r > 0$ such that $\overline{B_{3r}(x)} \subset V(x)$. Then for all $y \in \overline{B_r(x)}$ and $p \in \Pi_{\partial A}(y)$

$$\begin{aligned} &|p - x| \leq |p - y| + |y - x| \leq 2\,|y - x| \leq 2r \\ \Rightarrow\ & p \in \overline{B_{2r}(x)} \cap \partial A \subset V(x) \cap \partial A = V(x) \cap f^{-1}\{0\}. \end{aligned}$$

Therefore, for each $y \in \overline{B_r(x)}$ and $p \in \Pi_{\partial A}(y)$, $p \in V(x) \cap f^{-1}\{0\}$ such that

$$d^2_{\partial A}(y) = \inf_{p \in V(x),\, f(p)=0} |p - y|^2 = |y - p_{\partial A}(y)|^2.$$

By using the Lagrange multiplier theorem for the Lagrangian

$$|a - y|^2 + \lambda\, f(a)$$

there exist $\hat{\lambda} \in \mathbf{R}$ and $p \in \Pi_{\partial A}(y)$ such that

$$\begin{aligned} 2(p - y) + \hat{\lambda}\, \nabla f(p) = 0 \text{ and } f(p) = 0 \quad &\Rightarrow\ 2 d_{\partial A}(y) = |\hat{\lambda}|\, |\nabla f(p)| \\ \Rightarrow\ p = y + b_A(y) \frac{\nabla f(p)}{|\nabla f(p)|} &= y - b_A(y)\, n(p) \\ \Rightarrow\ \forall y \in \overline{B_r(x)},\ \forall p \in \Pi_{\partial A}(y), \quad & p = y - b_A(y)\, n(p), \end{aligned} \tag{8.16}$$

since $\nabla f(p) \neq 0$ on $\partial A \cap V(x)$. By the same proof as in part (i) with $\ell = 1$

$$\forall z_2,\, z_1 \in \overline{B_{2r}(x)} \cap \partial A, \quad |n(z_2) - n(z_1)| \leq \frac{2}{\alpha}\, c_x\, |z_2 - z_1|,$$

where c_x is the Lipschitz constant associated with f in $\overline{B_{2r}(x)}$. Therefore, $n \in C^{0,1}(\overline{B_{2r}(x)} \cap \partial A)$. From (8.16) for $y \in \overline{B_r(x)}$ and for all $p_1, p_2 \in \Pi_{\partial A}(y)$,

$$|p_2 - p_1| \leq |b_A(y)|\, |n(p_2) - n(p_1)| \leq |b_A(y)| \frac{2\, c_x}{\alpha}\, |p_2 - p_1|.$$

By choosing h such that $0 < h < \min\{r, \alpha/(2\,c_x)\}$, $h\,2\,c_x/\alpha < 1$, for $y \in B_h(x) \subset \overline{B_r(x)} \cap \{z \in \mathbf{R}^{\mathrm{N}} : |b_A(z)| \le h\}$ and all $p_1, p_2 \in \Pi_{\partial A}(y)$, $p_2 = p_1$, $\Pi_{\partial A}(y)$ is a singleton, and

$$p_{\partial A}(y) = y - b_A(y)\, n(p_{\partial A}(y)).$$

By construction, $B_h(x) \cap \mathrm{Sk}\,(\partial A) = \varnothing$ and for all $z \in \overline{B_{h/3}(x)}$, $B_{2h/3}(z) \cap \mathrm{Sk}\,(\partial A) = \varnothing$ implies that $\mathrm{reach}\,(\partial A, z) \ge 2h/3 > 0$. By Theorem 6.2 (v) in Chapter 6 applied to ∂A with $0 < r = h/3 < q = 2h/3$

$$\forall z_1, z_2 \in \overline{B_{h/3}(x)}, \quad |p_{\partial A}(z_1) - p_{\partial A}(z_2)| \le 2\,|z_1 - z_2| \tag{8.17}$$

and hence $p_{\partial A} \in C^{0,1}(\overline{B_{h/3}(x)})^N$. Thus $n \circ p_{\partial A}$ belongs to $C^{0,1}(\overline{B_{h/3}(x)})^N$ since we have already shown that $n \in C^{0,1}(\overline{B_{2r}(x)} \cap \partial A)^N$. The remainder and the conclusion of the proof is the same as in part (i) with $\rho = h/3$.

(iii) When A is of class $C^{k,\ell}$ in $V(x)$ for an integer k, $k \ge 2$, and a real number ℓ, $0 \le \ell \le 1$, it is $C^{1,1}$ and the previous constructions and conclusions of (ii) are verified. There exists $\rho > 0$ such that for each $y \in \overline{B_\rho(x)}$ there exists a unique $p_{\partial A}(y) \in \overline{B_{2\rho}(x)}$ such that

$$p_{\partial A}(y) = y - b_A(y)\, \frac{\nabla f(p_{\partial A}(y))}{|\nabla f(p_{\partial A}(y))|}.$$

Since ∇f is continuous and $\nabla f \ne 0$ on $\partial A \cap V(x)$, there exists r, $0 < 2r < \rho$, such that

$$\inf_{z \in \overline{B_{2r}(x)}} |\nabla f(z)| > 0$$

and defining $N(z) = \nabla f(z)/|\nabla f(z)|$,

$$\nabla N(z) = \frac{1}{|\nabla f(z)|}\left[D^2 f(z) - \frac{\nabla f(z)}{|\nabla f(z)|}\ {}^*\!\left(D^2 f(z) \frac{\nabla f(z)}{|\nabla f(z)|}\right)\right]$$

belongs to $C^{k-1,\ell}(\overline{B_{2r}(x)})$, where ${}^*v$ denotes the transpose of a vector $v \in \mathbf{R}^{\mathrm{N}}$, so that $(u\ {}^*v)_{ij} = u_i v_j$ is the tensor product. Consider the map

$$(y, z) \mapsto G(y, z) \stackrel{\text{def}}{=} z - y - b_A(y) \frac{\nabla f(z)}{|\nabla f(z)|} \;:\; B_r(x) \times B_{2r}(x) \to \mathbf{R}^{\mathrm{N}}.$$

From part (ii), there exists a unique $p(y) = p_{\partial A}(y) \in B_{2r}(x) \cap \partial A$ such that $p_{\partial A} \in C^{0,1}(\overline{B_{2r}(x)})^N$, $b_A \in C^{1,1}(\overline{B_r(x)})$, and

$$G(y, p(y)) = 0 \quad \Rightarrow DG_y(y, p(y)) + DG_z(y, p(y))\, Dp(y) = 0,$$

where

$$DG_z(y, z) = I - \frac{b_A(y)}{|\nabla f(z)|}\left[D^2 f(z) - \frac{\nabla f(z)}{|\nabla f(z)|}\ {}^*\!\left(D^2 f(z) \frac{\nabla f(z)}{|\nabla f(z)|}\right)\right],$$
$$DG_y(y, z)) = -I - N(z)\, {}^*\nabla b_A(y).$$

$DG_z(y,z)$ is at least $C^{1,1}$ in (y,z) since $b_A \in C^{1,1}$ and N is at least $C^{1,1}$ for $k \geq 2$. Since f is at least C^2 in $V(x)$, f and its first and second derivatives are bounded in some smaller neighborhood $\overline{B_{r'}(x)}$ of x. By reducing the upper bound on $|b_A(y)|$, there exists a new smaller neighborhood $B_{r''}(x)$ of x such that $D_zG(y,z)$ is invertible for all $(y,z) \in B_{r''}(x) \times B_{2r''}(x)$. So the conditions of the implicit function theorem are met. There exist a neighborhood $W(x) \subset B_{r''}(x)$ of x and a unique C^1-mapping

$$p : W(x) \to \mathbf{R}^{\mathrm{N}} \text{ such that } \forall y \in W(x), \quad G(y, p(y)) = 0$$

that coincides with $p_{\partial A}$ (cf., for instance, L. Schwartz [**3**, Thm. 26, p. 283]). Therefore

$$Dp_{\partial A}(y) = (DG_z(y, p_{\partial A}(y)))^{-1} \left[I + N(p_{\partial A}(y))\, {}^*\nabla b_A(y)\right]$$

and $Dp_{\partial A}$ is C^0 and $p_{\partial A}$ is C^1. Thus $\nabla b_A = N \circ p_{\partial A} \in C^1$ and b_A is now C^2. Since $N \in C^{k-1,\ell}$ and $D^2 f \in C^{k-2,\ell}$, we can repeat this argument until we get $Dp_{\partial A}$ in $C^{k-2,\ell}$, and finally $p_{\partial A} \in C^{k-1,\ell}$, $\nabla b_A = N \circ p_{\partial A} \in C^{k-1,\ell}$, and $b_A \in C^{k,\ell}$. □

From Theorem 8.1 (ii) additional information is available on the flow of ∇b_Ω.

Theorem 8.5. *Let $\Omega \subset \mathbf{R}^{\mathrm{N}}$ be of class $C^{1,1}$. Denote its boundary by $\Gamma \stackrel{\text{def}}{=} \partial\Omega$.*

(i) *For each $x \in \Gamma$ there exists a bounded open neighborhood $W(x)$ of x such that*

$$b_\Omega \in C^{1,1}(\overline{W(x)}), \quad \nabla b_\Omega = n \circ p_\Gamma = \nabla b_\Omega \circ p_\Gamma, \quad |\nabla b_\Omega| = 1 \text{ in } W(x), \tag{8.18}$$

where n is the unit exterior normal to Ω on Γ.

(ii) *Associate with $x \in \Gamma$ and y in $W(x)$ the flow $T_t(y)$ of ∇b_Ω in $W(x)$, that is,*

$$\frac{dx}{dt}(t) = \nabla b_\Omega(x(t)), \quad x(0) = y, \quad T_t(y) \stackrel{\text{def}}{=} x(t). \tag{8.19}$$

Then for t in a neighborhood of 0 we have the following properties:

$$\boxed{p_\Gamma \circ T_t = p_\Gamma,} \quad \boxed{\nabla b_\Omega \circ T_t = \nabla b_\Omega,} \quad \boxed{T_t(y) = y + t\,\nabla b_\Omega(y).} \tag{8.20}$$

(iii) *Almost everywhere in $W(x)$*

$$DT_t = I + t\,D^2 b_\Omega, \quad \frac{d}{dt} DT_t = D^2 b_\Omega \tag{8.21}$$

$$\Rightarrow D^2 b_\Omega \circ T_t\; DT_t = D^2 b_\Omega, \quad D^2 b_\Omega \circ T_t = D^2 b_\Omega \left[I + tD^2 b_\Omega\right]^{-1}, \tag{8.22}$$

$$\boxed{(DT_t)^{-1} \nabla b_\Omega = \nabla b_\Omega = {}^*(DT_t)^{-1} \nabla b_\Omega.} \tag{8.23}$$

Proof. For simplicity, we use the notation $b = b_\Omega$ and $p = p_\Gamma$.

(i) The result follows from Theorem 8.4 (i).

(ii) By assumption from part (i) the function b belongs to $C^{1,1}(\overline{W(x)})$ and hence $p \in C^{0,1}(\overline{W(x)}; \mathbf{R}^{\mathrm{N}})$. Thus almost everywhere in $W(x)$,

$$Dp = I - \nabla b\, {}^*\nabla b - b D^2 b \quad \Rightarrow Dp \nabla b = 0.$$

Now

$$\frac{d}{dt}(p \circ T_t) = Dp \circ T_t \frac{dT_t}{dt} = Dp \circ T_t \nabla b \circ T_t = (Dp\, \nabla b) \circ T_t = 0$$
$$\Rightarrow\; p \circ T_t = p \text{ in } W(x).$$

But since p and T_t belong to $C^{0,1}(\overline{W(x)}; \mathbf{R}^{\mathrm{N}})$, the identity necessarily holds everywhere in $W(x)$. Moreover,

$$\nabla b \circ T_t = n \circ p \circ T_t = n \circ p = \nabla b \text{ in } W(x).$$

Finally

$$\frac{dT_t}{dt} = \nabla b \circ T_t = \nabla b \quad \Rightarrow T_t(y) = y + t\nabla b(y) \text{ in } W(x).$$

In particular, almost everywhere in $W(x)$,

$$DT_t = I + t\, D^2 b \quad \Rightarrow \frac{d}{dt} DT_t = D^2 b,$$
$$DT_t \nabla b = \nabla b \quad \Rightarrow (DT_t)^{-1} \nabla b = \nabla b = {}^*(DT_t)^{-1} \nabla b. \qquad \square$$

9 Sobolev or $W^{m,p}$ Domains

The N-dimensional Lebesgue measure of the boundary of a Lipschitzian subset of $\mathbf{R}^{\mathrm{N}}$ is zero. However, this is generally not true for sets of locally bounded curvature, as can be seen from the following example.

Example 9.1.
Let B be the open unit ball centered in 0 of $\mathbf{R}^2$ and define

$$A = \{x \in B \,:\, x \text{ with rational coordinates}\}.$$

Then $\partial A = \overline{B}$, $b_A = d_B$, and for all $h > 0$

$$\nabla b_A \in \mathrm{BV}(U_h(\partial A))^2,$$
$$\langle \Delta b_A, \varphi \rangle = \int_{\partial B} \varphi\, dx + \int_{\complement B} \frac{1}{|x|} \varphi\, dx. \qquad \square$$

The next natural question that comes to mind is whether the boundary or the skeleton has locally finite $(N-1)$-Hausdorff measure. Then come questions about the mean curvature of the boundary. For the function b_A, $\Delta b_A = \operatorname{tr} D^2 b_A$ is proportional to the mean curvature of the boundary, which is only a measure in $\mathbf{R}^{\mathrm{N}}$ for sets of bounded local curvature. This calls for the introduction of some classification of sets that would complement the classical Hölderian terminology introduced in Chapter 2, would fill the gap in between, and would possibly say something about sets whose boundary is not even continuous. The following definition seems to have been first introduced in M. C. Delfour and J.-P. Zolésio [**32**].

Definition 9.1 (Sobolev domains).
Given $m > 1$ and $p \geq 1$, a subset A of $\mathbf{R}^{\mathrm{N}}$ is said to be an (m,p)-*Sobolev domain* or simply a $W^{m,p}$-*domain* if $\partial A \neq \varnothing$ and there exists $h > 0$ such that

$$b_A \in W^{m,p}_{\mathrm{loc}}(U_h(\partial A)). \qquad \square$$

This classification gives an analytical description of the smoothness of boundaries in terms of the derivatives of b_A. The definition is obviously vacuous for $m = 1$ since $b_A \in W^{1,\infty}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$ for any set A such that $\partial A \neq \varnothing$. For $1 < m < 2$ we shall see that it encompasses sets that are less smooth than sets of locally bounded curvature. For $m = 2$ the $W^{2,p}$-domains are of class $C^{1,1-N/p}$ for $p > N$. For $m > 2$ they are intertwined with sets of class $C^{k,\ell}$.

Theorem 9.1. *Given any subset A of $\mathbf{R}^{\mathrm{N}}$, $\partial A \neq \varnothing$,*

$$\begin{aligned} &\nabla b_A \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})^N \\ &\Rightarrow \forall p,\, 1 \leq p < \infty, \quad \forall \eta,\, 0 \leq \eta < \frac{1}{p}, \quad b_A \in W^{1+\eta,p}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}}). \end{aligned}$$

Proof. Since $|\nabla b_A(x)| \leq 1$ almost everywhere, $\nabla b_A \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})^N \cap L^\infty_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})^N$ and the theorem follows directly from Theorem 6.9 in Chapter 5. $\square$

It is quite interesting that a domain of locally bounded curvature is a $W^{2-\varepsilon,1}$-domain for any arbitrary small $\varepsilon > 0$. So it is *almost* a $W^{2,1}$-domain, and domains of class $W^{2-\varepsilon,1}$ seem to be a larger class than domains of locally bounded curvature. In addition, their boundary does not generally have zero measure.

Now consider the "threshold case" $m = 2$.

Theorem 9.2. *Given an integer $N \geq 1$, let A be a subset of $\mathbf{R}^{\mathrm{N}}$ such that $\partial A \neq \varnothing$.*

(i) *If $\mathrm{m}(\partial A) = 0$ and there exist $p > N$ and $h > 0$ such that*

$$b_A \in W^{2,p}_{\mathrm{loc}}(U_h(\partial A)), \tag{9.1}$$

then

$$\operatorname{reach}(\partial A) \geq h > 0, \quad b_A \in C^{1,1-N/p}_{\mathrm{loc}}(U_h(\partial A)),$$

and A is a Hölderian set of class $C^{1,1-N/p}$.

(ii) *In dimension $N = 2$ the condition $b_A \in W^{2,p}_{\mathrm{loc}}(U_h(\partial A))$ is equivalent to $\Delta b_A \in L^p_{\mathrm{loc}}(U_h(\partial A))$.*

Proof. (i) Consider the function $|\nabla b_A|^2$. Since $|\nabla b_A| \leq 1$, then

$$b_A \in W^{2,p}_{\mathrm{loc}}(U_h(\partial A)) \cap W^{1,\infty}_{\mathrm{loc}}(U_h(\partial A)) \quad \Rightarrow b_A^2 \in W^{2,p}_{\mathrm{loc}}(U_h(\partial A)) \cap W^{1,\infty}_{\mathrm{loc}}(U_h(\partial A)).$$

In particular, for all $x \in \partial A$,

$$b_A,\, b_A^2 \in W^{2,p}(B(x,h)) \cap W^{1,\infty}(B(x,h)).$$

For $p > N$, from R. A. ADAMS [**1**, Thm. 5.4, Part III, and Rem. 5.5 (3), p. 98],

$$\begin{aligned} & b_A,\, b_A^2 \in C^{1,\lambda}(B(x,h)), \quad 0 < \lambda \le 1 - N/p \\ \Rightarrow\ & b_A,\, b_A^2 \in C^{1,\lambda}_{\mathrm{loc}}(U_h(A)), \quad 0 < \lambda \le 1 - N/p. \end{aligned}$$

From Theorem 8.2 (i), if $\mathrm{m}(\partial A) = 0$, the set A is a set of class $C^{1,1-N/p}$, $\overline{\partial A \cap \mathrm{Sk}\,(\partial A)} = \varnothing$, and $\mathrm{reach}\,(\partial A) \ge h$.

(ii) From part (i), $|\nabla b_A(x)|^2 = 1$ in $U_h(\partial A)$, and $D^2 b_A(x)\, \nabla b_A(x) = 0$ almost everywhere. Hence in dimension $N = 2$,

$$\begin{aligned} \partial^2_{12} b_A(x) = \partial^2_{21} b_A(x) &= -\partial_1 b_A(x)\, \partial_2 b_A(x)\, \Delta b_A(x), \\ \partial^2_{11} b_A(x) &= (\partial_2 b_A(x))^2\, \Delta b_A(x), \\ \partial^2_{22} b_A(x) &= (\partial_1 b_A(x))^2\, \Delta b_A(x). \end{aligned}$$ □

As can be seen, a certain amount of work would be necessary to characterize all the Sobolev domains and answer the many associated open questions.

10 Characterization of Convex and Semiconvex Sets

10.1 Convex Sets and Convexity of $b_{\overline{A}}$

We have seen in Chapter 6 that the convexity of d_A is equivalent to the convexity of $\overline{A}$. This characterization remains true with b_A in place of d_A. However, if in both cases the convexity of d_A or b_A is sufficient to get the convexity of $\overline{A}$, it is not the case for A as can be seen from the following example.

Example 10.1.
Let B be the open unit ball in $\mathbf{R}^{\mathrm{N}}$ and A be the set B minus all the points in B with rational coordinates. By definition,

$$\begin{gathered} \bar{A} = \overline{B}, \quad \overline{\complement A} = \mathbf{R}^{\mathrm{N}}, \quad \partial A = \overline{B}, \quad \partial \overline{A} = \partial \overline{B} \neq \partial A, \\ d_A = d_B = d_{\overline{A}}, \quad d_{\complement A} = d_{\overline{\complement A}} = d_{\mathbf{R}^{\mathrm{N}}} = 0 \\ \Rightarrow\ b_A = d_B \neq b_{\overline{B}} = b_{\overline{A}}. \end{gathered}$$

By Theorem 8.1 of Chapter 6 the functions d_B and, a fortiori, b_A are convex, but A is not convex and $\partial \overline{A} \neq \partial A$. □

Theorem 10.1.

(i) *Let A be a convex subset of $\mathbf{R}^{\mathrm{N}}$. Then*

$$b_A = b_{\overline{A}} \quad \textit{and} \quad [A]_b = [\overline{A}]_b. \tag{10.1}$$

Hence, the equivalence class $[A]_b$ can be identified with the unique closed set $\overline{A}$ in the class.

(ii) *Let A be a convex subset of* $\mathbf{R}^N$ *such that* $\operatorname{int} A \neq \varnothing$. *Then*

$$b_{\operatorname{int} A} = b_A = b_{\overline{A}} \quad \text{and} \quad [\operatorname{int} A]_b = [A]_b = [\overline{A}]_b. \tag{10.2}$$

Hence, the equivalence class $[A]_b$ can be identified with either the unique closed set $\overline{A}$ or the unique open set $\operatorname{int} A = \complement\overline{\complement A} \neq \varnothing$ in the class. If, in addition, $A \neq \mathbf{R}^N$, then $\partial A \neq \varnothing$ and $\operatorname{int} \complement A \neq \varnothing$.

(iii) *Let A be a subset of* $\mathbf{R}^N$ *such that* $\partial A \neq \varnothing$.[5] *Then*

$$b_A \text{ is convex} \iff \overline{A} \text{ is convex}. \tag{10.3}$$

Proof. (i) If $A = \varnothing$, then $\overline{A} = \varnothing$, $\complement A = \mathbf{R}^N$, and $\complement\overline{A} = \mathbf{R}^N$. Hence $b_A = b_{\overline{A}}$. If $A = \mathbf{R}^N$, we use the identity $b_A = -b_{\complement A}$. If $\partial A \neq \varnothing$, then $d_A = d_{\overline{A}}$ and, by Theorem 3.5 (ii) in Chapter 5, $\overline{\complement\overline{A}} = \overline{\complement A}$, and $b_A = b_{\overline{A}}$.

(ii) From Theorem 3.5 (ii) in Chapter 5, $\overline{\operatorname{int} A} = \overline{A}$ and $\complement \operatorname{int} A = \complement\complement\overline{\complement A} = \overline{\complement A}$. Hence $b_A = b_{\operatorname{int} A}$. From part (i), $b_A = b_{\overline{A}}$. If, in addition, $A \neq \mathbf{R}^N$, always from Theorem 3.5 (ii) in Chapter 5, $\partial A \neq \varnothing$ and $\operatorname{int} \complement A \neq \varnothing$.

(iii) ($\Rightarrow$) Denote by x_λ the convex combination $\lambda x + (1-\lambda)y$ of two points x and y in $\overline{A}$ for some $\lambda \in [0,1]$. By convexity of b_A,

$$\begin{aligned} b_A(x_\lambda) &\leq \lambda b_A(x) + (1-\lambda) b_A(y) = -\left[\lambda d_{\complement A}(x) + (1-\lambda) d_{\complement A}(y)\right] \leq 0 \\ &\Rightarrow\ d_A(x_\lambda) = b_A^+(x_\lambda) = \max\{b_A(x_\lambda), 0\} = 0 \quad \Rightarrow x_\lambda \in \overline{A} \end{aligned}$$

and $\overline{A}$ is convex.

($\Leftarrow$) We first prove that for A closed the function b_A is convex. From this we conclude that if $\overline{A}$ is convex, then $b_{\overline{A}}$ is convex and from part (i) $b_A = b_{\overline{A}}$ is convex. So for A closed, consider three cases. The first one deserves a lemma.

Lemma 10.1. *Let A be a subset of* $\mathbf{R}^N$ *such that $\partial \overline{A} \neq \varnothing$ and let $\overline{A}$ be convex. Then*

$$b_{\overline{A}} = -d_{\complement\overline{A}} \text{ is convex in } \overline{A}.$$

Proof. Again we can assume that A is closed. Associate with x and y in A the radii $r_x = d_{\complement A}(x)$, $r_y = d_{\complement A}(y)$, and $r_\lambda = \lambda r_x + (1-\lambda) r_y$ and the closed balls $\overline{B}_x$ of center x and radius r_x, $\overline{B}_y$ of center y and radius r_y, and $\overline{B}_\lambda$ of center x_λ and radius r_λ. By the definition of $d_{\complement A}$, $\overline{B}_x \subset A$ and $\overline{B}_y \subset A$ since A is closed. Associate with

[5]In 1985 D. H. Armitage and Ü. Kura [**1**] showed that a proper closed domain Ω is convex if and only if $-b_A$ is superharmonic and that, for $N = 2$, the result still holds if $-b_A$ is superharmonic only on A. They also showed that, for A compact, *$d_{\partial A}$ is subharmonic on $\complement A$ if and only if A is convex.* This work was pursued in 1987 by M. J. Parker [**1**] and in 1988 by M. J. Parker [**2**]. In 1994 it was shown in M. C. Delfour and J.-P. Zolésio [**17**] that the property that *$\overline{A}$ is convex if and only if d_A is convex* remains true with b_A in place of d_A.

each $z \in \overline{B}_\lambda$ the points $z_x = x$ and $z_y = y$ if $r_\lambda = 0$, and if $r_\lambda > 0$ the points

$$\begin{aligned} z_x &\overset{\text{def}}{=} x + \frac{r_x}{r_\lambda}(z - x_\lambda) \quad \Rightarrow z_x \in \overline{B}_x, \\ z_y &\overset{\text{def}}{=} y + \frac{r_y}{r_\lambda}(z - x_\lambda) \quad \Rightarrow z_y \in \overline{B}_y \\ \Rightarrow \lambda z_x + (1-\lambda) z_y &= x_\lambda + \frac{\lambda r_x + (1-\lambda) r_y}{r_\lambda}(z - x_\lambda) = z. \end{aligned}$$

Obviously $z = x_\lambda$ if $r_\lambda = 0$. Therefore,

$$\overline{B}_\lambda \subset \lambda \overline{B}_x + (1-\lambda)\overline{B}_y \subset A,$$

since A is closed and convex. In particular

$$d_{\complement A}(x_\lambda) \geq d_{\complement \overline{B}_\lambda}(x_\lambda) \geq r_\lambda = \lambda d_{\complement A}(x) + (1-\lambda) d_{\complement A}(y),$$

and this proves the lemma. □

For the second case consider x and y in $\overline{\complement A}$. By definition

$$b_A(x) = d_A(x) \quad \text{and} \quad b_A(y) = d_A(y).$$

By Theorem 8.1 of Chapter 6, d_A is convex when A is convex and

$$b_A(x_\lambda) \leq d_A(x_\lambda) \leq \lambda d_A(x) + (1-\lambda) d_A(y) = \lambda b_A(x) + (1-\lambda) b_A(y).$$

The third and last case is the mixed one for $x \in \overline{\complement A}$ and $y \in A$. Define

$$x_\lambda \overset{\text{def}}{=} \lambda x + (1-\lambda) y, \quad p_\lambda \overset{\text{def}}{=} p_{\partial A}(x_\lambda).$$

Since A is convex, denote by H the tangent hyperplane to A through p_λ, by H^+ the closed half-space containing A, and by H^- the other closed subspace associated with H. By the definition of H,

$$d_{\partial A}(x_\lambda) = d_H(x_\lambda), \quad H \subset H^\pm,$$

and by convexity of A, $A \subset H^+$ and $H^- \subset \overline{\complement A}$. The projection onto H is a linear operator and

$$\begin{aligned} p_\lambda = p_H(x_\lambda) &= \lambda p_H(x) + (1-\lambda) p_H(y), \\ p_\lambda - x_\lambda &= \lambda(p_H(x) - x) + (1-\lambda)(p_H(y) - y). \end{aligned}$$

If $y \in H^+$ and $x \in H^+$,

$$\boxed{d_H(x_\lambda) = \lambda d_H(x) + (1-\lambda) d_H(y),} \tag{10.4}$$

and if $y \in H^+$ and $x \in H^-$,

$$\boxed{d_H(x_\lambda) = \begin{cases} \lambda d_H(x) - (1-\lambda) d_H(y), & \text{if } x_\lambda \in H^-, \\ (1-\lambda) d_H(y) - \lambda d_H(x), & \text{if } x_\lambda \in H^+. \end{cases}} \tag{10.5}$$

By convexity any $y \in \overline{A}$ belongs to H^+ and since $H^- \subset \overline{\complement A}$, we readily have

$$d_H(y) = d_{H^-}(y) \geq d_{\complement A}(y) = -b_A(y).$$

First consider the case $x_\lambda \in \complement A$. Then $d_A(x_\lambda) > 0$, $x_\lambda \in H^-$, and necessarily $x \in H^-$. From (10.5)

$$b_A(x_\lambda) = d_A(x_\lambda) = d_H(x_\lambda) = \lambda d_H(x) - (1-\lambda) d_H(y).$$

But since $x \in H^-$ and $A \subset H^+$

$$d_H(x) = d_{H^+}(x) \leq d_A(x)$$

and

$$b_A(x_\lambda) = d_H(x_\lambda) \leq \lambda d_A(x) - (1-\lambda) d_{\complement A}(y) = \lambda b_A(x) + (1-\lambda) b_A(y).$$

Next consider the case $x_\lambda \in A$ for which $x_\lambda \in H^+$ and

$$-b_A(x_\lambda) = d_{\complement A}(x_\lambda) = d_H(x_\lambda).$$

If $x \in H^-$, then since $A \subset H^+$,

$$d_H(x) = d_{H^+}(x) \leq d_A(x),$$

and from (10.5)

$$\begin{aligned} -b_A(x_\lambda) &= d_H(x_\lambda) \\ &= (1-\lambda) d_H(y) - \lambda d_H(x) \\ &\geq (1-\lambda) d_{\complement A}(y) - \lambda d_A(x) = -[(1-\lambda) b_A(y) + \lambda b_A(x)]. \end{aligned}$$

If, on the other hand, $x \in H^+$, then $x, y \in H^+$ and $x_\lambda \in A \subset H^+$. So from (10.4)

$$\begin{aligned} -b_A(x_\lambda) &= d_H(x_\lambda) \\ &= (1-\lambda) d_H(y) + \lambda d_H(x) \\ &\geq (1-\lambda) d_{\complement A}(y) - \lambda d_A(x) = -[(1-\lambda) b_A(y) + \lambda b_A(x)]. \end{aligned}$$

This covers all cases and concludes the proof. $\square$

10.2 Families of Convex Sets $\mathcal{C}_b(D)$, $\mathcal{C}_b(E;D)$, and $\mathcal{C}_{b,\text{loc}}(E;D)$

We have seen in Theorem 10.1 (i) that for a convex subset A of $\overline{D}$, $b_A = b_{\overline{A}}$ and that $\overline{A}$ is the unique closed representative in the equivalence class. To work with open convex subsets is more delicate. Always from Theorem 10.1 (ii), $\operatorname{int} A$ becomes the unique open representative in the equivalence class when the convex set A has a nonempty interior. In order to work with families of open rather than closed convex subsets, it will be necessary to add a constraint on the interior of the subsets to prevent the occurrence of subsets with an empty interior.

Theorem 10.2. *Let D be a nonempty open* (resp., *bounded open*) *subset of* $\mathbf{R}^{\mathrm{N}}$.

(i) *The family*

$$\boxed{\mathcal{C}_b(D) \overset{\text{def}}{=} \{b_A : A \subset \overline{D},\, \partial A \neq \varnothing,\, A \textit{ convex}\}} \tag{10.6}$$

is closed in $C_{\text{loc}}(D)$ (resp., *compact in* $C(\overline{D})$). *It coincides with the family*

$$\{b_A : A \subset \overline{D},\, \partial A \neq \varnothing,\, A \textit{ closed and convex}\}.$$

(ii) *Let E be a nonempty open subset of D. Then the families*

$$\mathcal{C}_b(E;D) \overset{\text{def}}{=} \{b_\Omega : E \subset \Omega \subset D,\, \partial\Omega \neq \varnothing,\, \Omega \textit{ open and convex}\}, \tag{10.7}$$

$$\mathcal{C}_{b,\text{loc}}(E;D) \overset{\text{def}}{=} \left\{ b_\Omega : \begin{array}{l} \exists x \in \mathbf{R}^{\mathrm{N}},\, \exists A \in \mathrm{O(N)} \textit{ such that } x + AE \subset \Omega \\ \textit{and } \Omega \textit{ is open and convex } \subset D \end{array} \right\} \tag{10.8}$$

are closed in $C_{\text{loc}}(D)$ (resp., *compact in* $C(\overline{D})$).

(iii) *The conclusions of parts* (i) *and* (ii) *remain true with* $W^{1,p}_{\text{loc}}(D)$- (resp., $W^{1,p}(D)$-) strong *in place of* $C_{\text{loc}}(D)$ (resp., $C(\overline{D})$).

Proof. It is sufficient to prove the result for D bounded. Furthermore, by compactness of $C_b(D)$ in $C(\overline{D})$, it is sufficient to prove that $\mathcal{C}_b(D)$, $\mathcal{C}_b(E;D)$, and $\mathcal{C}_{b,\text{loc}}(E;D)$ are closed in $C(\overline{D})$.

(i) Let $\{b_{A_n}\}$ be a Cauchy sequence in $\mathcal{C}_b(D)$. It converges to some $b_A \in C_b(D)$. The function b_A is convex as the pointwise limit of a family of convex functions. By Theorem 10.1 (iii), $\overline{A}$ is convex, and by Theorem 10.1 (i) $b_{\overline{A}} = b_A$ is convex. Therefore the Cauchy sequence converges to $b_{\overline{A}}$ in $\mathcal{C}_b(D)$ and $\mathcal{C}_b(D)$ is closed in $C(\overline{D})$.

(ii) From part (i), given a Cauchy sequence $\{b_{\Omega_n}\}$ in $\mathcal{C}_b(E;D)$, there exists an open convex subset Ω of D such that $b_{\Omega_n} \to b_\Omega$. Moreover, since $d_{\complement\Omega_n} \to d_{\complement\Omega}$,

$$E \subset \Omega_n \Rightarrow \complement E \supset \complement\Omega_n \Rightarrow d_{\complement E} \leq d_{\complement\Omega_n} \Rightarrow d_{\complement E} \leq d_{\complement\Omega} \Rightarrow \complement E \supset \complement\Omega \Rightarrow E \subset \Omega.$$

The proof for $\mathcal{C}_{b,\text{loc}}(E;D)$ is similar to the proof for $\mathcal{C}^c_{d,\text{loc}}(E;D)$ in Theorem 2.6 of Chapter 6.

(iii) Let $\{b_{A_n}\} \subset \mathcal{C}_b(D)$ be a Cauchy sequence in $W^{1,p}(D)$. From Theorem 4.2 (ii), it is also Cauchy in $C(\overline{D})$. Hence from part (i) there exists a convex set

A, $\partial A \neq \varnothing$, such that $b_{A_n} \to b_A$ in $C(\overline{D})$ and, a fortiori, in $W^{1,p}(D)$. This shows that $\mathcal{C}_b(D)$ is closed in $W^{1,p}(D)$. To show that it is compact consider a sequence $\{b_{A_n}\} \subset \mathcal{C}_b(D)$. From part (i) there exist a subsequence, still indexed by n, and a convex subset A, $\partial A \neq \varnothing$, of $\overline{D}$ such that $b_{A_n} \to b_A$ in $C(\overline{D})$. Since D is bounded, it also converges in $L^p(D)$. Since $|\nabla b_{A_n}|$ is pointwise bounded by one, there exists another subsequence of $\{b_{A_n}\}$, still indexed by n, such that ∇b_{A_n} converges to ∇b_A in $W^{1,p}(D)^N$-weak. But A_n and A are convex. Thus $\mathrm{m}(\partial A) = 0 = \mathrm{m}(\partial A_n)$ and $|\nabla b_{A_n}| = 1 = |\nabla b_A|$ almost everywhere in $\mathbf{R}^{\mathrm{N}}$. As a result for $p = 2$

$$\begin{aligned}&\int_D |\nabla b_{A_n} - \nabla b_A|^2\, dx = \int_D |\nabla b_{A_n}|^2 + |\nabla b_A|^2\, dx - 2\nabla b_{A_n} \cdot \nabla b_A\, dx \\ &= \int_D 2 - 2\nabla b_{A_n} \cdot \nabla b_A\, dx \to 0.\end{aligned}$$

Hence we get the compactness in $W^{1,2}(D)$-strong and by Theorem 4.1 (i) in $W^{1,p}(D)$-strong, $1 \leq p < \infty$. □

10.3 BV Character of b_A and Semiconvex Sets

The next theorem is a consequence of the fact that the gradient of a convex function is locally of bounded variations.

Theorem 10.3. *For all convex sets A such that $\partial A \neq \varnothing$,*

(i) $\nabla b_A \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})^N$, $\nabla \chi_{\partial A} \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$, $\nabla d_A \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})^N$, $\nabla \chi_{\overline{A}} \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$, *and for all $x \in \mathbf{R}^{\mathrm{N}}$ and $v \in \mathbf{R}^{\mathrm{N}}$, $d_H b_A(x, v)$ exists.*

(ii) *The Hessian matrix $D^2 b_A$ of second-order derivatives is a matrix of signed Radon measures which are nonnegative on the diagonal.*

(iii) *b_A has a second-order derivative almost everywhere, and for almost all x and y in $\mathbf{R}^{\mathrm{N}}$,*

$$\left| b_A(y) - b_A(x) - \nabla b_A(x) \cdot (y - x) - \frac{1}{2}(y - x) \cdot D^2 b_A(x)\,(y - x) \right| = o(|y - x|^2)$$

as $y \to x$.

Proof. Same argument as in the proof of Theorem 8.2 of Chapter 6. □

When A is convex and of class C^2, then for any $x_0 \in \partial A$, there exists a strictly convex neighborhood $N(x_0)$ of x_0 such that

$$b_A \in C^2(N(x_0)) \text{ and } \forall x \in N(x_0), \forall \xi \in \mathbf{R}^{\mathrm{N}}, \quad D^2 b_A(x)\xi \cdot \xi \geq 0, \tag{10.9}$$

or, since $D^2 b_A(x)\, \nabla b_A(x) = 0$ in $N(x_0)$, then for all $x \in N(x_0)$,

$$\forall \xi \in \mathbf{R}^{\mathrm{N}} \text{ such that } \xi \cdot \nabla b_A(x) = 0, \quad D^2 b_A(x)\xi \cdot \xi \geq 0. \tag{10.10}$$

This is related to the notion of *strong elliptic midsurface* in the theory of shells: there exists $c > 0$

$$\forall x \in \partial A, \forall \xi \in \mathbf{R}^{\mathrm{N}} \text{ such that } \xi \cdot \nabla b_A(x) = 0, \quad D^2 b_A(x)\, \xi \cdot \xi \geq c|\xi|^2.$$

All this motivates the introduction of the following notions.

Definition 10.1.
Let A be a closed subset of $\mathbf{R}^N$ such that $\partial A \neq \varnothing$.

(i) The set A is *locally convex* (resp., *locally strictly convex*) if for each $x_0 \in \partial A$ there exists a convex neighborhood $N(x_0)$ of x_0 such that

$$b_A \text{ is convex (resp., strictly convex) in } N(x_0).$$

(ii) The set A is *semiconvex* if

$$\exists \alpha \geq 0, \quad b_A(x) + \alpha |x|^2 \text{ is convex in } \mathbf{R}^N.$$

(iii) The set A is *locally semiconvex* if for each $x_0 \in \partial A$ there exists a convex neighborhood $N(x_0)$ of x_0 and

$$\exists \alpha \geq 0, \quad b_A(x) + \alpha |x|^2 \text{ is convex in } N(x_0). \qquad \square$$

Remark 10.1.
When A is a C^2 domain with a compact boundary, $D^2 b_A$ is bounded in a bounded neighborhood of ∂A and A is necessarily locally semiconvex. When A is a locally semiconvex set, for each $x_0 \in \partial A$ there exists a convex neighborhood $N(x_0)$ of x_0 such that $\nabla b_A \in \mathrm{BV}(N(x_0))^N$. If, in addition, ∂A is compact, then there exists $h > 0$ such that $\nabla b_A \in \mathrm{BV}(U_h(\partial A))^N$. $\square$

Remark 10.2.
Given a fixed constant $\beta > 0$, consider all the subsets of D that are semiconvex with constant $0 \leq \alpha \leq \beta$. Then this set is closed for the uniform and the $W^{1,p}$-topologies, $1 \leq p < \infty$. $\square$

11 Compactness and Sets of Bounded Curvature

For the family of sets with bounded curvature, the key result is the compactness of the embedding

$$\mathrm{BC}_b(D) = \left\{ b_A \in C_b(D) : \nabla b_A \in \mathrm{BV}(D)^N \right\} \to W^{1,p}(D) \tag{11.1}$$

for bounded open Lipschitzian subsets of $\mathbf{R}^N$ and p, $1 \leq p < \infty$. It is the analogue of the compactness Theorem 6.3 of Chapter 5 for Caccioppoli sets

$$\mathrm{BX}(D) = \{\chi \in \mathrm{X}(D) : \chi \in \mathrm{BV}(D)\} \to L^p(D), \tag{11.2}$$

which is a consequence of the compactness of the embedding

$$\mathrm{BV}(D) \to L^1(D) \tag{11.3}$$

for bounded open Lipschitzian subsets of $\mathbf{R}^N$ (cf. C. B. Morrey, Jr. [**1**, Def. 3.4.1, p. 72, Thm. 3.4.4, p. 75] and L. C. Evans and R. F. Gariepy [**1**, Thm. 4, p. 176]).

As in the case of characteristic functions in Chapter 5, we give a first version involving global conditions on a fixed bounded open Lipschitzian holdall D. In the second version the sets are contained in a bounded open holdall D with local conditions in the tubular neighborhood of their boundary.

11.1 Global Conditions on D

Theorem 11.1. *Let D be a nonempty bounded open Lipschitzian subset of $\mathbf{R}^{\mathrm{N}}$. The embedding (11.1) is compact. Thus for any sequence $\{A_n\}$, $\partial A_n \neq \varnothing$, of subsets of $\overline{D}$ such that*

$$\exists c > 0, \forall n \geq 1, \quad \|D^2 b_{A_n}\|_{M^1(D)} \leq c, \tag{11.4}$$

there exist a subsequence $\{A_{n_k}\}$ and a set A, $\partial A \neq \varnothing$, such that $\nabla b_A \in \mathrm{BV}(D)^N$ and

$$b_{A_{n_k}} \to b_A \text{ in } W^{1,p}(D)\text{-strong}$$

for all p, $1 \leq p < \infty$. Moreover, for all $\varphi \in \mathcal{D}^0(D)$,

$$\lim_{n\to\infty} \langle \partial_{ij} b_{A_{n_k}}, \varphi\rangle = \langle \partial_{ij} b_A, \varphi\rangle, \quad 1 \leq i, j \leq N, \quad \|D^2 b_A\|_{M^1(D)} \leq c. \tag{11.5}$$

Proof. Given $c > 0$ consider the set $S_c \stackrel{\text{def}}{=} \left\{b_A \in C_b(D) : \|D^2 b_A\|_{M^1(D)} \leq c\right\}$. By compactness of the embedding (11.3), given any sequence $\{b_{A_n}\}$ there exist a subsequence, still denoted by $\{b_{A_n}\}$, and $f \in \mathrm{BV}(D)^N$ such that $\nabla b_{A_n} \to f$ in $L^1(D)^N$. But by Theorem 2.2 (ii), $C_b(D)$ is compact in $C(\overline{D})$ for bounded D and there exist another subsequence $\{b_{A_{n_k}}\}$ and $b_A \in C_b(D)$ such that $b_{A_{n_k}} \to b_A$ in $C(\overline{D})$ and, a fortiori, in $L^1(D)$. Therefore, $b_{A_{n_k}}$ converges in $W^{1,1}(D)$ and also in $L^1(D)$. By uniqueness of the limit, $f = \nabla b_A$ and $b_{A_{n_k}}$ converges in $W^{1,1}(D)$ to b_A. For $\Phi \in \mathcal{D}^1(D)^{N\times N}$ as k goes to infinity

$$\begin{aligned} &\int_D \nabla b_{A_{n_k}} \cdot \overrightarrow{\operatorname{div}}\, \Phi \, dx \to \int_D \nabla b_A \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx \\ \Rightarrow &\left| \int_D \nabla b_A \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx \right| = \lim_{k\to\infty} \left| \int_D \nabla b_{A_{n_k}} \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx\right| \leq c\|\Phi\|_{C(D)}, \end{aligned}$$

$\|D^2 b_A\|_{M^1(D)} \leq c$, and $\nabla b_A \in \mathrm{BV}(D)^N$. This proves the sequential compactness of the embedding (11.1) for $p = 1$ and properties (11.5). The conclusions remain true for $p \geq 1$ by the equivalence of the $W^{1,p}$-topologies on $C_b(D)$ in Theorem 4.1 (i). $\square$

11.2 Local Conditions in Tubular Neighborhoods

The global condition (11.4) is now weakened to a local one in a neighborhood of each set of the sequence. Simultaneously, the Lipschitzian condition on D is removed since only the uniform boundedness of the sets of the sequence is required.

Theorem 11.2. *Let $\varnothing \neq D \subset \mathbf{R}^{\mathrm{N}}$ be bounded open and $\{A_n\}$, $\partial A_n \neq \varnothing$, be a sequence of subsets of $\overline{D}$. Assume that there exist $h > 0$ and $c > 0$ such that*

$$\forall n, \quad \|D^2 b_{A_n}\|_{M^1(U_h(\partial A_n))} \leq c. \tag{11.6}$$

Then, there exist a subsequence $\{A_{n_k}\}$ and $A \subset \overline{D}$, $\varnothing \neq \partial A$, such that $\nabla b_A \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})^N$ and for all p, $1 \leq p < \infty$,

$$b_{A_{n_k}} \to b_A \text{ in } W^{1,p}(U_h(D))\text{-strong}. \tag{11.7}$$

Moreover, for all $\varphi \in \mathcal{D}^0(U_h(\partial A))$,

$$\lim_{k\to\infty} \langle \partial_{ij} b_{A_{n_k}}, \varphi\rangle = \langle \partial_{ij} b_A, \varphi\rangle, \quad 1 \le i,j \le N, \quad \|D^2 b_A\|_{M^1(U_h(\partial A))} \le c, \tag{11.8}$$

and $\chi_{\partial A}$ *belongs to* $\mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$.

Proof. (i) By assumption, $A_n \subset \overline{D}$ implies that $U_h(A_n) \subset U_h(D)$. Since $U_h(D)$ is bounded, there exist a subsequence, still indexed by n, and a subset A of $\overline{D}$, $\partial A \neq \varnothing$, such that

$$b_{A_n} \to b_A \quad \text{in } C(\overline{U_h(D)})\text{-strong}$$

and another subsequence, still indexed by n, such that

$$b_{A_n} \to b_A \quad \text{in } H^1(U_h(D))\text{-weak}.$$

For all $\varepsilon > 0$, $0 < 3\varepsilon < h$, there exists $N > 0$ such that for all $n \ge N$ and all $x \in \overline{U_h(D)}$,

$$d_{\partial A_n}(x) \le d_{\partial A}(x) + \varepsilon, \quad d_{\partial A}(x) \le d_{\partial A_n}(x) + \varepsilon. \tag{11.9}$$

Therefore,

$$\partial A_n \subset U_{h-2\,\varepsilon}(\partial A_n) \subset U_{h-\varepsilon}(\partial A) \subset U_h(\partial A_n), \tag{11.10}$$

$$\complement U_{h-\varepsilon}(\partial A) \subset \complement U_{h-2\,\varepsilon}(\partial A_n) \subset \complement \partial A_n. \tag{11.11}$$

From (11.6) and (11.10)

$$\forall n \ge N, \quad \|D^2 b_{A_n}\|_{M^1(U_{h-\varepsilon}(\partial A))} \le c.$$

In order to use the compactness of the embedding (11.3) as in the proof of Theorem 11.1, we would need $U_{h-\varepsilon}(\partial A)$ to be Lipschitzian. To get around this we construct a bounded Lipschitzian set between $U_{h-2\varepsilon}(\partial A)$ and $U_{h-\varepsilon}(\partial A)$. Indeed by definition,

$$U_{h-\varepsilon}(\partial A) = \cup_{x\in\partial A} B(x, h-\varepsilon) \quad \text{and} \quad \overline{U_{h-2\varepsilon}(\partial A)} \subset U_{h-\varepsilon}(\partial A),$$

and by compactness there exists a finite sequence of points $\{x_i\}_{i=1}^n$ in ∂A such that

$$\overline{U_{h-2\varepsilon}(\partial A)} \subset U_B \stackrel{\text{def}}{=} \cup_{i=1}^n B(x_i, h-\varepsilon) \subset U_h(D).$$

Since U_B is Lipschitzian as the union of a finite number of balls, it now follows by compactness of the embedding (11.3) for U_B that there exist a subsequence, still denoted by $\{b_{A_n}\}$, and $f \in \mathrm{BV}(U_B)^N$ such that $\nabla b_{A_n} \to f$ in $L^1(U_B)^N$. Since $U_h(D)$ is bounded, $C_b(U_h(D))$ is compact in $C(\overline{U_h(D)})$ and there exist another subsequence, still denoted by $\{b_{A_n}\}$, and $A \subset \overline{D}$, $\partial A \neq \varnothing$, such that $b_{A_n} \to b_A$ in $C(\overline{U_h(D)})$ and, a fortiori, in $L^1(U_h(D))$. Therefore, b_{A_n} converges in $W^{1,1}(U_B)$ and also in $L^1(U_B)$. By uniqueness of the limit, $f = \nabla b_A$ on U_B and b_{A_n} converges to b_A in $W^{1,1}(U_B)$. By Definition 5.1 and Theorem 3.1 (i), ∇b_A and ∇b_{A_n} all belong to $\mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})^N$ since they are of bounded variation in tubular neighborhoods of their respective boundaries. Moreover, by Theorem 5.2 (ii), $\chi_{\partial A} \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$. The above conclusions also hold for the subset $U_{h-2\varepsilon}(\partial A)$ of U_B.

(ii) *Convergence in* $W^{1,p}(U_h(D))$. Consider the integral

$$\begin{aligned}&\int_{U_h(D)} |\nabla b_{A_n} - \nabla b_A|^2\, dx\\ &= \int_{U_{h-2\varepsilon}(\partial A)} |\nabla b_{A_n} - \nabla b_A|^2\, dx + \int_{U_h(D)\backslash U_{h-2\varepsilon}(\partial A)} |\nabla b_{A_n} - \nabla b_A|^2\, dx.\end{aligned}$$

From part (i), the first integral on the right-hand side converges to zero as n goes to infinity. The second integral is on a subset of $\complement U_{h-2\varepsilon}(\partial A)$. From (11.11) for all $n \geq N$,

$$\begin{aligned}|\nabla b_{A_n}(x)| = 1 \text{ a.e. in } \complement\partial A_n \supset \complement U_{h-3\varepsilon}(\partial A_n) \supset \complement U_{h-2\varepsilon}(\partial A),\\ |\nabla b_A(x)| = 1 \text{ a.e. in } \complement\partial A \supset \complement U_{h-2\varepsilon}(\partial A).\end{aligned}$$

The second integral reduces to

$$\int_{U_h(D)\backslash U_{h-2\varepsilon}(\partial A)} |\nabla b_{A_n} - \nabla b_A|^2\, dx = \int_{U_h(D)\backslash U_{h-2\varepsilon}(\partial A)} 2\,(1 - \nabla b_{A_n} \cdot \nabla b_A)\, dx,$$

which converges to zero by weak convergence of ∇b_{A_n} to ∇b_A in the space $L^2(U_h(D))^N$ in part (i) and the fact that $|\nabla b_A| = 1$ almost everywhere in $U_h(D)\backslash U_{h-2\varepsilon}(\partial A)$. Therefore, since $b_{A_n} \to b_A$ in $C(\overline{U_h(D)})$

$$b_{A_n} \to b_A \text{ in } H^1(U_h(D))\text{-strong},$$

and by Theorem 4.1 (i) the convergence is true in $W^{1,p}(U_h(D))$ for all $p \geq 1$.

(iii) *Properties* (11.8). Consider the initial subsequence $\{b_{A_n}\}$ which converges to b_A in $H^1(U_h(D))$-weak constructed at the beginning of part (i). This sequence is independent of ε and the subsequent constructions of other subsequences. By convergence of b_{A_n} to b_A in $H^1(U_h(D))$-weak for each $\Phi \in \mathcal{D}^1(U_h(\partial A))^{N\times N}$,

$$\lim_{n\to\infty} \int_{U_h(\partial A)} \nabla b_{A_n} \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx = \int_{U_h(\partial A)} \nabla b_A \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx.$$

Each such Φ has compact support in $U_h(\partial A)$, and there exists $\varepsilon = \varepsilon(\Phi) > 0$, $0 < 3\varepsilon < h$, such that

$$\overline{\operatorname{supp}\Phi} \subset U_{h-2\varepsilon}(\partial A).$$

From part (ii) there exists $N(\varepsilon) > 0$ such that

$$\forall n \geq N(\varepsilon), \quad U_{h-2\,\varepsilon}(\partial A_n) \subset U_{h-\varepsilon}(\partial A) \subset U_h(\partial A_n).$$

For $n \geq N(\varepsilon)$ consider the integral

$$\int_{U_h(\partial A)} \nabla b_{A_n} \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx = \int_{U_{h-2\varepsilon}(\partial A)} \nabla b_{A_n} \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx = \int_{U_h(\partial A_n)} \nabla b_{A_n} \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx$$

$$\begin{aligned} \Rightarrow \left| \int_{U_h(\partial A)} \nabla b_{A_n} \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx \right| &\leq \|D^2 b_{A_n}\|_{M^1(U_h(\partial A_n))} \|\Phi\|_{C(U_h(\partial A_n))} \\ &\leq c\, \|\Phi\|_{C(U_{h-2\varepsilon}(\partial A))} = c\, \|\Phi\|_{C(U_h(\partial A))}. \end{aligned}$$

By convergence of ∇b_{A_n} to ∇b_A in the space $L^2(U_h(D))$-weak, then for all $\Phi \in \mathcal{D}^1(U_h(\partial A))^{N\times N}$

$$\left| \int_{U_h(\partial A)} \nabla b_A \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx \right| \leq c\, \|\Phi\|_{C(U_h(\partial A))} \quad \Rightarrow \|D^2 b_A\|_{M^1(U_h(\partial A))} \leq c.$$

Finally, the convergence remains true for all subsequences constructed in parts (i) and (ii). This completes the proof. □

12 Finite Density Perimeter and Compactness

The *density perimeter* introduced by D. Bucur and J.-P. Zolésio [**14**] is a relaxation of the $(N-1)$-dimensional *upper Minkowski content* which leads to the compactness Theorem 12.1.

Definition 12.1.
Let $h > 0$ be a fixed real number and let $\varnothing \neq A \subset \mathbf{R}^{\mathrm{N}}$, $\partial A \neq \varnothing$. Consider the quotient

$$P_h(\partial A) \overset{\text{def}}{=} \sup_{0<k<h} \frac{\mathrm{m}_N(U_k(\partial A))}{2k}, \tag{12.1}$$

where $U_k(\partial A) = \{x \in \mathbf{R}^{\mathrm{N}} : d_{\partial A}(x) < k\}$. We say that A has a finite *h-density perimeter* if $P_h(\partial A)$ is finite. □

Clearly for all k, $0 < k < h$,

$$\int_{\partial A} d\mathrm{m}_N \leq \int_{U_k(\partial A)} d\mathrm{m}_N \leq c\, 2k \quad \Rightarrow \mathrm{m}_N(\partial A) = 0.$$

The compactness result of D. Bucur and J.-P. Zolésio [**14**] has been revisited and established in the $W^{1,p}$-topology by M. C. Delfour and J.-P. Zolésio [**38**], from which convergence in all other topologies of Theorem 4.1 follows.

Theorem 12.1. *Let $D \neq \varnothing$ be a bounded open subset of $\mathbf{R}^{\mathrm{N}}$ and $\{A_n\}$, $\partial A_n \neq \varnothing$, be a sequence of subsets of $\overline{D}$. Assume that*

$$\exists h > 0 \text{ and } c > 0 \text{ such that } \forall n, \quad P_h(\partial A_n) \leq c. \tag{12.2}$$

Then there exist a subsequence $\{A_{n_k}\}$ and $A \subset \overline{D}$, $\partial A \neq \varnothing$, such that

$$P_h(\partial A) \leq \liminf_{n\to\infty} P_h(\partial A_n) \leq c, \tag{12.3}$$

$$\forall p,\ 1 \leq p < \infty, \quad b_{A_{n_k}} \to b_A \text{ in } W^{1,p}(U_h(D))\text{-strong}. \tag{12.4}$$

Proof. The proof essentially rests on Theorem 4.3 since $P_h(\partial A) \le c$ implies $\mathrm{m}_N(\partial A) = 0$. Since D is bounded, the family of oriented distance functions $C_b(D)$ is compact in $C(\overline{D})$ and $W^{1,p}(D)$-weak for all p, $1 \le p < \infty$ (cf. Theorems 2.2 (ii) and 4.2 (iii)). So there exist $b_A \in C_b(D)$ and a subsequence, still indexed by n, such that $b_{A_n} \to b_A$ in the above topologies. Moreover, for all k, $0 < k < h$, and all ε, $0 < \varepsilon < h - k$,

$$\exists N(\varepsilon) > 0 \text{ such that } \forall n \ge N(\varepsilon), \quad U_{k-\varepsilon}(\partial A_n) \subset U_k(\partial A) \subset U_{k+\varepsilon}(\partial A_n)$$

(cf. proof of part (i) of Theorem 11.2). As a result for all $n \ge N(\varepsilon)$,

$$\begin{aligned} \frac{\mathrm{m}_N(U_{k-\varepsilon}(\partial A_n))}{2(k-\varepsilon)} \frac{k-\varepsilon}{k} &\le \frac{\mathrm{m}_N(U_k(\partial A))}{2k} \le \frac{\mathrm{m}_N(U_{k+\varepsilon}(\partial A_n))}{2(k+\varepsilon)} \frac{k+\varepsilon}{k} \le c\,\frac{k+\varepsilon}{k} \\ \Rightarrow \frac{\mathrm{m}_N(U_k(\partial A))}{2k} &\le \frac{\mathrm{m}_N(U_{k+\varepsilon}(\partial A_n))}{2(k+\varepsilon)} \frac{k+\varepsilon}{k} \le P_h(\partial A_n) \frac{k+\varepsilon}{k} \\ \Rightarrow \frac{\mathrm{m}_N(U_k(\partial A))}{2k} &\le \liminf_{n\to\infty} P_h(\partial A_n) \frac{k+\varepsilon}{k}. \end{aligned} \tag{12.5}$$

Going to the limit as ε goes to zero in the second and fourth terms

$$\begin{aligned} \forall k,\ 0 < k < h, \quad \frac{\mathrm{m}_N(U_k(\partial A))}{2k} &\le \liminf_{n\to\infty} P_h(\partial A_n) \le c \\ \Rightarrow P_h(\partial A) \le \liminf_{n\to\infty} P_h(\partial A_n) &\le c \Rightarrow \mathrm{m}_N(\partial A) = 0. \end{aligned}$$

The theorem now follows from the fact that $\mathrm{m}_N(\partial A) = 0$ and Theorem 4.3. □

Corollary 1. *Let $D \neq \varnothing$ be a bounded open subset of $\mathbf{R}^{\mathrm{N}}$ and A, $\partial A \neq \varnothing$, be a subset of $\overline{D}$ such that*

$$\exists h > 0 \text{ and } c > 0 \text{ such that } P_h(\partial A) \le c. \tag{12.6}$$

Then the mapping $b_{A'} \to P_h(\partial A') : C_b(D) \to \mathbf{R} \cup \{+\infty\}$ is lower semicontinuous in A for the $W^{1,p}(D)$-topology.

Proof. Since we have a metric topology, it is sufficient to prove the property for $W^{1,p}(D)$-converging sequences $\{b_{A_n}\}$ to b_A. From that point on the argument is the same as the one used to get (12.5) in the proof of Theorem 12.1 after the extraction of the subsequence. □

Remark 12.1.

It is important to notice that even if $\{A_n\}$ is a $W^{1,p}$-convergent sequence of bounded open subsets of $\mathbf{R}^{\mathrm{N}}$ with a uniformly bounded perimeter, the limit set A need not be an open set or have a nonempty interior $\operatorname{int} A$ such that $b_A = b_{\operatorname{int} A}$. It would be tempting to say that $b_{A_n} \to b_A$ implies $d_{\complement A_n} \to d_{\complement A}$ and use the open set $\operatorname{int} A = \complement\overline{\complement A}$ for which $d_{\complement A} = d_{\complement \operatorname{int} A}$ to conclude that $b_A = b_{\operatorname{int} A}$. This is incorrect as can be seen in the following example shown in Figure 7.2. Consider a family $\{A_n\}$ of open rectangles in $\mathbf{R}^2$ of width equal to 1 and height $1/n$, $n \ge 1$, an integer. Their density perimeter is bounded by 4, and the b_{A_n}'s converge to b_A for A equal to the line of length 1 which has an empty interior. □

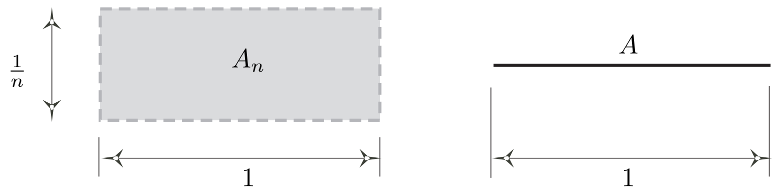


Figure 7.2. *$W^{1,p}$-convergence of a sequence of open subsets $\{A_n : n \geq 1\}$ of $\mathbf{R}^2$ with uniformly bounded density perimeter to a set with empty interior.*

13 Compactness and Uniform Fat Segment Property

13.1 Main Theorem

In Theorem 6.11 of section 6.4 of Chapter 5, we have seen a compactness theorem for the family of subsets of a bounded holdall D satisfying the uniform cone property. This compactness theorem is no longer true when the uniform cone property is replaced by a uniform segment property. This is readily seen by considering the following example.

Example 13.1.
Given an integer $n \geq 1$, consider the following sequence of open domains in $\mathbf{R}^2$:

$$\Omega_n \stackrel{\text{def}}{=} \left\{(x, y) \in \mathbf{R}^2 \,:\, |x| < 1 \text{ and } |x|^{1/n} < y < 2\right\}.$$

They satisfy the uniform segment property of Definition 6.1 (ii) of Chapter 2 by choosing $\lambda = r = 1/4$. The sequence $\{\overline{\Omega_n}\}$ converges to the closed set

$$A \stackrel{\text{def}}{=} \left\{(x, y) \in \mathbf{R}^2 \,:\, |x| \leq 1 \text{ and } 1 \leq y \leq 2\right\} \cup L,$$
$$L \stackrel{\text{def}}{=} \left\{(0, y) \in \mathbf{R}^2 \,:\, 0 \leq y \leq 1\right\}$$

in the uniform topologies associated with d_{Ω_n} and b_{Ω_n} or in the L^p-topologies associated with χ_{Ω_n} and $\chi_{\complement\Omega_n}$. However, the segment property is not satisfied along the line L and the corresponding family of subsets of the holdall $D = B(0, 4)$ satisfying the uniform segment property with $r = \lambda = 1/4$ is not closed and, a fortiori, not compact. $\square$

This example shows that a uniform segment property is too *meager* to make the corresponding family compact. Looking back at the proof of Theorem 6.11 of section 6.4 in Chapter 5, the fact that the cone is an open set around the segment is used critically. That was the motivation to introduce the more general uniform fat segment property (Definition 6.1 (iii) in section 6 of Chapter 2) that includes as special cases both the uniform cone and the uniform cusp properties (cf. Definitions 6.3 and 6.4 in section 6.4 of Chapter 2). However, the proof of Theorem 6.11 in Chapter 5 was based on the fact that the perimeter of all the sets in the family were uniformly bounded. There is no hope of getting that property for general Hölderian domains, and a completely different proof is necessary.

Given a bounded open subset D of $\mathbf{R}^{\mathrm{N}}$ consider the family

$$L(D, r, \mathcal{O}, \lambda) \stackrel{\text{def}}{=} \left\{ \Omega \subset \overline{D} : \begin{array}{l} \Omega \text{ satisfies the uniform fat} \\ \text{segment property for } (r, \mathcal{O}, \lambda) \end{array} \right\}. \tag{13.1}$$

We first give a general proof of the compactness of $L(D, r, \mathcal{O}, \lambda)$ in the $C(\overline{D})$- and $W^{1,p}(D)$-topologies associated with the oriented distance function b_Ω. As a consequence, we get the compactness for the $C(\overline{D})$- and $W^{1,p}(D)$-topologies associated with d_Ω and $d_{\complement\Omega}$ and the $L^p(D)$-topologies associated with χ_Ω and $\chi_{\complement\Omega}$. We recover the compactness Theorem 6.11 of Chapter 5 under the uniform cone property for χ_Ω in $L^p(D)$ as a special case. In the last part of this section, we give several other ways to specify the compact families by using the equivalent conditions of Chapter 2 of the uniform fat segment property in terms of conditions on the local epigraphs via the dominating function and equicontinuity of the local graph functions.

Theorem 13.1. *Let D be a nonempty bounded open subset of $\mathbf{R}^{\mathrm{N}}$ and let $1 \leq p < \infty$. Assume that $L(D, r, \mathcal{O}, \lambda)$ is not empty for $r > 0$, $\lambda > 0$, and an open subset $\mathcal{O}$ of $\mathbf{R}^{\mathrm{N}}$ such that $(0, e_N) \subset \mathcal{O}$ and $0 \notin \mathcal{O}$. Then the family*

$$B(D, r, \mathcal{O}, \lambda) \stackrel{\text{def}}{=} \{b_\Omega : \forall \Omega \in L(D, r, \mathcal{O}, \lambda)\}$$

is compact in $C(\overline{D})$ and $W^{1,p}(D)$. As a consequence the families

$$\begin{aligned} B_d(D, r, \mathcal{O}, \lambda) &\stackrel{\text{def}}{=} \{d_\Omega : \forall \Omega \in L(D, r, \mathcal{O}, \lambda)\}, \\ B_d^c(D, r, \mathcal{O}, \lambda) &\stackrel{\text{def}}{=} \{d_{\complement\Omega} : \forall \Omega \in L(D, r, \mathcal{O}, \lambda)\}, \\ B_d^\partial(D, r, \mathcal{O}, \lambda) &\stackrel{\text{def}}{=} \{d_{\partial\Omega} : \forall \Omega \in L(D, r, \mathcal{O}, \lambda)\} \end{aligned}$$

are compact in $C(\overline{D})$ and $W^{1,p}(D)$, and the families

$$\begin{aligned} X(D, r, \mathcal{O}, \lambda) &\stackrel{\text{def}}{=} \{\chi_\Omega : \forall \Omega \in L(D, r, \mathcal{O}, \lambda)\}, \\ X^c(D, r, \mathcal{O}, \lambda) &\stackrel{\text{def}}{=} \{\chi_{\complement\Omega} : \forall \Omega \in L(D, r, \mathcal{O}, \lambda)\} \end{aligned}$$

are compact in $L^p(D)$.

From Lemma 6.1 of Chapter 2, Theorem 13.1 can be specialized to structured open sets $\mathcal{O}$.

Corollary 1. *The conclusions of Theorem* 13.1 *hold when $\mathcal{O}$ is specialized to open regions of the form*

$$\mathcal{O}(h, \rho, \lambda) = \left\{ \xi \in \mathbf{R}^{\mathrm{N}} : \xi' \in B_H(0, \rho) \text{ and } h(|\xi'|) < \xi_N < \lambda \right\} \tag{13.2}$$

for dominating functions $h \in \mathcal{H}$.

Theorem 13.1 can be further specialized to Lipschitzian and Hölderian sets.

Corollary 2. *The conclusions of Theorem* 13.1 *hold under the* uniform cone property *of Definition* 6.3 (ii) *in section* 6.4 *of Chapter* 2 *with*

$$\mathcal{O} = \mathcal{O}(h, \rho, \lambda) \ \textit{ with } h(\theta) = \theta/\tan\omega \ \textit{ and } \rho = \lambda \tan\omega, \quad 0 < \omega < \pi/2, \tag{13.3}$$

and the uniform cusp property *of Definition* 6.4 (ii) *of Chapter* 2 *with*

$$\mathcal{O} = \mathcal{O}(h_\ell, \rho, \lambda) \ \textit{ with } h_\ell(\theta) = \lambda\,(\theta/\rho)^\ell, \quad 0 < \ell < 1.$$

The proof of Theorem 13.1 will require Theorem 4.3 and the following lemma.

Lemma 13.1. *Given a sequence* $\{b_{\Omega_n}\} \subset C_b(D)$ *such that* $b_{\Omega_n} \to b_\Omega$ *in* $C(\overline{D})$ *for some* $b_\Omega \in C_b(D)$*, we have the following properties:*

$$\forall x \in \overline{\Omega}, \ \forall R > 0, \quad \exists N(x,R) > 0, \ \forall n \geq N(x,R), \quad B(x,R) \cap \Omega_n \neq \varnothing,$$

and for all $x \in \overline{\complement\Omega}$,

$$\forall R > 0, \quad \exists N(x,R) > 0, \quad \forall n \geq N(x,R), \quad B(x,R) \cap \complement\Omega_n \neq \varnothing. \tag{13.4}$$

Moreover,

$$\begin{gathered}\forall x \in \partial\Omega, \quad \forall R > 0, \quad \exists N(x,R) > 0, \quad \forall n \geq N(x,R), \\ B(x,R) \cap \Omega_n \neq \varnothing \quad \textit{and} \quad B(x,R) \cap \complement\Omega_n) \neq \varnothing,\end{gathered}$$

and $B(x,R) \cap \partial\Omega_n \neq \varnothing$.

Proof. We proceed by contradiction. Assume that

$$\exists x \in \overline{\Omega}, \quad \exists R > 0, \quad \forall N > 0, \quad \exists n \geq N, \quad B(x,R) \cap \Omega_n = \varnothing.$$

So there exists a subsequence $\{\Omega_{n_k}\}$, $n_k \to \infty$, such that

$$b_{\Omega_{n_k}}(x) = d_{\Omega_{n_k}}(x) \geq R \neq 0 = d_\Omega(x) \geq b_\Omega(x),$$

which contradicts the fact that $b_{\Omega_{n_k}} \to b_\Omega$. Of course, the same assertion is true for the complements and for all $x \in \overline{\complement\Omega}$,

$$\forall R > 0, \quad \exists N(x,R) > 0, \quad \forall n \geq N(x,R), \quad B(x,R) \cap \complement\Omega_n \neq \varnothing.$$

When $x \in \partial\Omega$, $x \in \overline{\Omega} \cap \overline{\complement\Omega}$ and we combine the two results. For the last result use the fact that the open ball cannot be partitioned into two nonempty disjoint open subsets. □

Proof of Theorem 13.1. (i) *Compactness in* $C(\overline{D})$. Since for all Ω in $L(D, r, \mathcal{O}, \lambda)$, Ω is locally a C^0-epigraph, $\partial\Omega \neq \varnothing$, $b_\Omega \in C_b(D)$, and $m_N(\partial\Omega) = 0$. Consider an arbitrary sequence $\{\Omega_n\}$ in $L(D, r, \mathcal{O}, \lambda)$. For $\overline{D}$ compact, $C_b(D)$ is compact in $C(\overline{D})$ and there exist $\Omega \subset \overline{D}$, $\partial\Omega \neq \varnothing$, and a subsequence $\{\Omega_{n_k}\}$ such that $b_{\Omega_{n_k}} \to b_\Omega$ in $C(\overline{D})$. It remains to prove that $\Omega \in L(D, r, \mathcal{O}, \lambda)$, that is,

$$\forall x \in \partial\Omega, \ \exists A \in \mathrm{O(N)} \ \text{ such that } \forall y \in \overline{\Omega} \cap B(x,r), \quad y + \lambda A\mathcal{O} \subset \operatorname{int}\Omega.$$

From Lemma 13.1, for each $x \in \partial\Omega$, for all $k \geq 1$, there exists $n_k \geq k$ such that

$$B\left(x, r/2^k\right) \cap \partial\Omega_{n_k} \neq \varnothing.$$

Denote by x_k an element of that intersection:

$$\forall k \geq 1, \quad x_k \in B\left(x, r/2^k\right) \cap \partial\Omega_{n_k}.$$

By construction $x_k \to x$. Next consider $y \in B(x,r) \cap \overline{\Omega}$. From the first part of the lemma, there exists a subsequence of $\{\Omega_{n_k}\}$, still denoted by $\{\Omega_{n_k}\}$, such that for all $k \geq 1$, $B(y, r/2^k) \cap \Omega_{n_k} \neq \varnothing$. For each $k \geq 1$ denote by y_k a point of that intersection. By construction

$$y_k \in \overline{\Omega}_{n_k} \to y \in \overline{\Omega} \cap B(x,r).$$

There exists $K > 0$ large enough such that for all $k \geq K$, $y_k \in B(x_k, r)$. To see this, note that $y \in B(x,r)$ and that

$$\exists \rho > 0, \quad B(y,\rho) \subset B(x,r) \quad \text{and} \quad |y - x| + \rho/2 < r.$$

Now

$$\begin{aligned} |y_k - x_k| &\leq |y_k - y| + |y - x| + |x - x_k| \\ &\leq \frac{r}{2^k} + r - \frac{\rho}{2} + \frac{r}{2^k} \leq r + \left[\frac{r}{2^{k-1}} - \frac{\rho}{2}\right] < r. \end{aligned}$$

Since $r/\rho > 1$ the result is true for

$$\frac{r}{2^{k-1}} - \frac{\rho}{2} < 0 \quad \Rightarrow k > 2 + \log\left(\frac{r}{\rho}\right).$$

So we have constructed a subsequence $\{\Omega_{n_k}\}$ such that for $k \geq K$

$$x_k \in \partial\Omega_{n_k} \to x \in \partial\Omega \quad \text{and} \quad y_k \in \overline{\Omega}_{n_k} \cap B(x_k, r) \to y \in \overline{\Omega} \cap B(x,r).$$

For all k, there exists $A_k \in \mathrm{O(N)}$, $A_k\, {}^*\!A_k = {}^*\!A_k\, A_k = I$, such that $y_k + \lambda A_k \mathcal{O} \subset \operatorname{int}\Omega_{n_k}$. Pick another subsequence of $\{\Omega_{n_k}\}$, still denoted by $\{\Omega_{n_k}\}$, such that

$$\exists A \in \mathrm{O(N)},\ A\, {}^*\!A = {}^*\!A\, A = I, \quad A_k \to A.$$

Now consider $z \in y + \lambda A\mathcal{O}$. Since $y + \lambda A\mathcal{O}$ is open there exists $\rho > 0$ such that $B(z,\rho) \subset y + \lambda A\mathcal{O}$, and there exists $K' \geq K$ such that

$$\begin{gathered} \forall k \geq K', \quad B(z,\rho/2) \subset y_k + \lambda A_k\mathcal{O} \subset \operatorname{int}\Omega_{n_k} = \complement\overline{\complement\Omega_{n_k}} \\ \Rightarrow \complement B(z,\rho/2) \supset \overline{\complement\Omega_{n_k}} \Rightarrow 0 < \rho/2 = d_{\complement B(z,\rho/2)}(z) \leq d_{\complement\Omega_{n_k}}(z) \to d_{\complement\Omega}(z) \\ \Rightarrow 0 < \rho/2 \leq d_{\complement\Omega}(z) \quad \Rightarrow z \in \complement\overline{\complement\Omega} = \operatorname{int}\Omega \quad \Rightarrow y + \lambda A\mathcal{O} \subset \operatorname{int}\Omega. \end{gathered}$$

Hence $\Omega \subset \overline{D}$ satisfies the uniform fat segment property.

(ii) $W^{1,p}(D)$*-compactness.* From Theorem 4.1 (i), it is sufficient to prove the result for $p = 2$. Consider the subsequence $\{\Omega_{n_k}\} \subset L(D, \lambda, \mathcal{O}, r)$ and let $\Omega \in L(D, \lambda, \mathcal{O}, r)$ be the set previously constructed such as $b_{\Omega_{n_k}} \to b_\Omega$ in $C(\overline{D})$. Hence $b_{\Omega_{n_k}} \to b_\Omega$ in $L^2(D)$. Since $\overline{D}$ is compact

$$\forall \Omega \subset \overline{D}, \quad \int_D |b_{\Omega_{n_k}}|^2 \, dx \leq \int_D \operatorname{diam}(D)^2 \, dx \leq \operatorname{diam}(D)^2 \mathrm{m}_N(D),$$
$$\int_D |\nabla b_{\Omega_{n_k}}|^2 \, dx \leq \int_D dx = \mathrm{m}_N(D),$$

and there exists a subsequence, still denoted by $\{b_{\Omega_{n_k}}\}$, which converges weakly to b_Ω. Since all the sets in $L(D, r, \mathcal{O}, \lambda)$ verify a uniform segment property, they are locally C^0-epigraphs, $\mathrm{m}_N(\partial\Omega_{n_k}) = 0 = \mathrm{m}_N(\partial\Omega)$ and, by Theorem 4.3, $b_{\Omega_{n_k}} \to b_\Omega$ in $W^{1,p}(D)$-strong, $1 \leq p < \infty$.

(iii) The other compactness follows by continuity of the maps (4.6) and (4.7) in Theorem 4.1 and the fact that $\mathrm{m}_N(\partial\Omega) = 0$ implies $\chi_{\operatorname{int}\Omega} = \chi_\Omega$ and $\chi_{\operatorname{int}\complement\Omega} = \chi_{\complement\Omega}$ almost everywhere for $\Omega \in L(D, \lambda, \mathcal{O}, r)$. □

13.2 Equivalent Conditions on the Local Graph Functions

We have shown in Theorem 6.2 of Chapter 2 that the uniform fat segment property is equivalent to the equi-C^0 epigraph property. This means that we can equivalently characterize the compact family $L(D, r, \mathcal{O}, \lambda)$ in terms of conditions on the local graph functions such as the ones introduced in Theorem 5.1 of Chapter 2. Recall the following equivalent conditions:

(i) Ω verifies the fat segment property.

(ii) Ω is an equi-C^0-epigraph; that is, Ω is a C^0-epigraph and the local graph functions are uniformly bounded and equicontinuous.

(iii) Ω is a C^0-epigraph and there exist $\rho > 0$ and $h \in \mathcal{H}$ such that $B_H(0, \rho) \subset V$ and for all $x \in \partial\Omega$

$$\forall \zeta', \, \xi' \in B_H(0, \rho) \text{ such that } |\xi' - \zeta'| < \rho, \quad |a_x(\xi') - a_x(\zeta')| \leq h(|\xi' - \zeta'|). \tag{13.5}$$

(iv) Ω is a C^0-epigraph and there exist $\rho > 0$ and $h \in \mathcal{H}$ such that $B_H(0, \rho) \subset V$ and for all $x \in \partial\Omega$

$$\forall \xi' \in B_H(0, \rho), \quad a_x(\xi') \leq h(|\xi'|). \tag{13.6}$$

Recall also from Definition 5.1 (ii) in Chapter 2 that Ω is said to be *a C^0 epigraph* if it is locally a C^0-epigraph and the neighborhoods $\mathcal{U}(x)$ and V_x can be chosen in such a way that V_x and $A_x^{-1}(\mathcal{U}(x) - x)$ are independent of x: there exist bounded open neighborhoods V of 0 in H and U of 0 in $\mathbf{R}^{\mathrm{N}}$ such that $P_H(U) \subset V$ and

$$\forall x \in \partial\Omega, \quad V_x = V \quad \text{and} \quad \exists A_x \in \mathrm{O}(\mathrm{N}) \text{ such that } \mathcal{U}(x) = x + A_x U.$$

By applying each of the last three characterizations to all members of a family of subsets of a bounded holdall D, we get the analogue of the compactness Theorem 13.1 under the uniform fat segment property.

Theorem 13.2. *Let D be a bounded open nonempty subset of $\mathbf{R}^{\mathrm{N}}$ and consider the family $L(D,\rho,U)$ of all subsets Ω of $\overline{D}$ that verify the C^0-epigraph property: there exist $\rho>0$ and a bounded open neighborhood U of 0 such that*

$$U \subset \left\{\zeta \in \mathbf{R}^{\mathrm{N}} \,:\, P_H(\zeta) \in B_H(0,\rho)\right\}, \quad V \stackrel{\mathrm{def}}{=} B_H(0,\rho), \tag{13.7}$$

and for each $\Omega \in L(D,\rho,U)$ and each $x \in \partial\Omega$, there exists $A_x^\Omega \in \mathrm{O(N)}$ such that

$$\mathcal{U}^\Omega(x) \stackrel{\mathrm{def}}{=} x + A_x^\Omega U \tag{13.8}$$

and a C^0-graph function $a_x^\Omega : V \to \mathbf{R}$ such that $a_x^\Omega(0)=0$ and

$$\begin{aligned} \mathcal{U}^\Omega(x) \cap \partial\Omega &= \mathcal{U}^\Omega(x) \cap \left\{x + A_x(\zeta' + \zeta_N e_N) : \zeta' \in V,\, \zeta_N = a_x^\Omega(\zeta')\right\}, \\ \mathcal{U}^\Omega(x) \cap \operatorname{int}\Omega &= \mathcal{U}^\Omega(x) \cap \left\{x + A_x(\zeta' + \zeta_N e_N) : \zeta' \in V,\, \zeta_N > a_x^\Omega(\zeta')\right\}. \end{aligned} \tag{13.9}$$

Then, for all $h \in \mathcal{H}$, the subfamilies

$$L_0(D,\rho,U) \stackrel{\mathrm{def}}{=} \left\{ \Omega \in L(D,\rho,U) \,:\, \begin{array}{l} \{a_x^\Omega : \Omega \in L(D,\rho,U),\, x \in \partial\Omega\} \\ \textit{is uniformly bounded} \\ \textit{and equicontinuous} \end{array} \right\}, \tag{13.10}$$

$$L_1(D,\rho,U,h) \stackrel{\mathrm{def}}{=} \left\{\Omega \in L(D,\rho,U) \,:\, \forall x \in \partial\Omega,\, \forall \zeta' \in V,\; a_x^\Omega(\zeta') \le h(|\zeta'|)\right\}, \tag{13.11}$$

$$L_2(D,\rho,U,h) \stackrel{\mathrm{def}}{=} \left\{ \Omega \in L(D,\rho,U) \,:\, \begin{array}{l} \forall x \in \partial\Omega,\, \forall \zeta',\, \xi' \in V \\ \textit{such that } |\xi' - \zeta'| < \rho \\ |a_x(\xi') - a_x(\zeta')| \le h(|\xi' - \zeta'|) \end{array} \right\} \tag{13.12}$$

are compact (possibly empty) in $C(\overline{D})$ and $W^{1,p}(D)$, $1 \le p < \infty$, and the conclusions of Theorem 13.1 *hold.*

Remark 13.1.
The geometric fat segment property of Corollary 1 was introduced in the form (13.2) in the book of M. C. Delfour and J.-P. Zolésio [**37**] under the name *uniform cusp property* in 2001, where the condition was specialized to (13.3) for Lipschitzian and Hölderian domains as stated in Corollary 2.

Condition (13.10) is a streamlined version of the sufficient condition given in an unpublished report by D. Tiba [**2**], announced without proof in the note by W. Liu, P. Neittaanmäki, and D. Tiba [**1**] in 2000, and later documented in D. Tiba [**3**] in 2003. He gets a compactness result by introducing the following conditions on the local C^0-epigraphs: the boundaries of the domains belong (after a rotation and a translation) to a family $\mathcal{F}$ of uniformly equicontinuous functions defined in a fixed neighborhood V of 0 in $\mathbf{R}^{N-1}$. This means that there exists a modulus of continuity $\mu(\varepsilon) > 0$ such that

$$\forall \varepsilon > 0,\, \forall f \in \mathcal{F},\, \forall x, y \in V, \quad |x-y| < \mu(\varepsilon) \quad \Rightarrow |f(y) - f(x)| < \varepsilon.$$

Then the compactness follows by Ascoli–Arzelà Theorem 2.4 of Chapter 2.

The connection between that condition and our geometric condition was first established in M. C. DELFOUR, N. DOYON, and J.-P. ZOLÉSIO [**1**, **2**] in 2005. Condition (13.11) that relaxes the equicontinuity condition (13.10) to the simpler dominating function condition and condition (13.12) were both introduced in M. C. DELFOUR and J.-P. ZOLÉSIO [**43**] in 2007 along with the general form of the uniform fat segment property that involves only the unparametrized open set $\mathcal{O}$. □

14 Compactness under the Uniform Fat Segment Property and a Bound on a Perimeter

Domains that are locally Lipschitzian epigraphs or equivalently satisfy the local uniform cone property enjoy the additional property that for any bounded open set D the $(N-1)$-Hausdorff measure of $D \cap \partial\Omega$ is finite. In general this is no longer true for domains verifying a uniform cusp property for some function $h(\theta) = |\theta/\rho|^\alpha$, $0 < \alpha < 1$ (cf., for instance, Example 6.2 given[6] in section 6.5 of Chapter 2 of a bounded domain Ω in $\mathbf{R}^N$ for which $H_{N-1}(\partial\Omega) = +\infty$ and the Hausdorff dimension of $\partial\Omega$ is exactly $N - \alpha$). Yet, there are many domains with cusps whose boundary has finite H_{N-1} measure.

14.1 De Giorgi Perimeter of Caccioppoli Sets

One of the classical notions of perimeter is the one introduced in Definition 6.2 in section 6 of Chapter 5 in the context of the problem of minimal surfaces for Caccioppoli sets. Theorem 13.1 extends to the following subfamily of $L(D, r, \mathcal{O}, \lambda)$ for a bounded open subset D of $\mathbf{R}^N$ and a constant c':

$$L(D, r, \mathcal{O}, \lambda, c') \stackrel{\text{def}}{=} \left\{ \Omega \subset \overline{D} : \begin{array}{l} \Omega \text{ satisfies the uniform fat} \\ \text{segment property for } (r, \mathcal{O}, \lambda) \\ \text{and } P_D(\Omega) \le c' \end{array} \right\}. \tag{14.1}$$

Theorem 14.1. *Let D be a nonempty bounded open subset of $\mathbf{R}^N$ and let $1 \le p < \infty$. Assume that $L(D, r, \mathcal{O}, \lambda, c')$ is not empty for $r > 0$, $\lambda > 0$, $c' > 0$, and a bounded open subset $\mathcal{O}$ of $\mathbf{R}^N$ such that $(0, e_N) \subset \mathcal{O}$ and $0 \notin \mathcal{O}$. Then the family*

$$B(D, r, \mathcal{O}, \lambda, c') \stackrel{\text{def}}{=} \{b_\Omega : \forall \Omega \in L(D, r, \mathcal{O}, \lambda, c')\}$$

is compact in $C(\overline{D})$ and $W^{1,p}(D)$. As a consequence the families

$$\begin{aligned} B_d(D, r, \mathcal{O}, \lambda, c') &\stackrel{\text{def}}{=} \{d_\Omega : \forall \Omega \in L(D, r, \mathcal{O}, \lambda, c')\}, \\ B_d^c(D, r, \mathcal{O}, \lambda, c') &\stackrel{\text{def}}{=} \{d_{\complement\Omega} : \forall \Omega \in L(D, r, \mathcal{O}, \lambda, c')\}, \\ B_d^\partial(D, r, \mathcal{O}, \lambda, c') &\stackrel{\text{def}}{=} \{d_{\partial\Omega} : \forall \Omega \in L(D, r, \mathcal{O}, \lambda, c')\} \end{aligned}$$

[6] This was originally in M. C. DELFOUR, N. DOYON, and J.-P. ZOLÉSIO [**1**].

are compact in $C(\overline{D})$ *and* $W^{1,p}(D)$, *and the families*

$$X(D, r, \mathcal{O}, \lambda, c') \stackrel{\text{def}}{=} \{\chi_\Omega : \forall \Omega \in L(D, r, \mathcal{O}, \lambda, c')\},$$
$$X^c(D, r, \mathcal{O}, \lambda, c') \stackrel{\text{def}}{=} \{\chi_{\complement\Omega} : \forall \Omega \in L(D, r, \mathcal{O}, \lambda, c')\}$$

are compact in $L^p(D)$.

14.2 Finite Density Perimeter

It is also possible to use the density perimeter introduced in Definition 12.1 in place of the De Giorgi perimeter.

The proof of the next result combines Theorem 14.1, which says that the family $L(D, r, \mathcal{O}, \lambda, c')$ is compact, with Theorem 12.1, which says that the family of sets verifying (12.2) is compact in $W^{1,p}(D)$. The intersection of the two families of oriented distance functions is compact in $W^{1,p}(D)$.

Theorem 14.2. *For fixed* $h > 0$ *and an open subset* Ω *of* $\mathbf{R}^N$, *Theorem* 14.1 *remains true when* $P_D(\Omega)$ *is replaced by the density perimeter* $P_h(\partial\Omega)$.

15 The Families of Cracked Sets

In this section we introduce new families of thin sets, that is, sets A such that $m_N(\partial A) = 0$, by introducing conditions on the semiderivative of the distance function at points of the boundary. They have been used in M. C. Delfour and J.-P. Zolésio [**38**] in the context of the image segmentation problem of D. Mumford and J. Shah [**2**]. Cracked sets are more general than sets which are locally the epigraph of a continuous function in the sense that they include domains with cracks and sets that can be made up of components of different codimensions. The Hausdorff $(N-1)$ measure of their boundary is not necessarily finite. Yet, compact families (in the $W^{1,p}$-topology) of such sets can be constructed.

First recall lower and upper semiderivatives[7] of Dini type for the differential quotient of a function $f : V(x) \subset \mathbf{R}^N \to \mathbf{R}$ defined in a neighborhood $V(x)$ of a point $x \in \mathbf{R}^N$ in the direction $d \in \mathbf{R}^N$:

$$\liminf_{t \searrow 0} \frac{f(x+td) - f(x)}{t} \stackrel{\text{def}}{=} \lim_{\delta \searrow 0} \inf_{0<t<\delta} \frac{f(x+td) - f(x)}{t},$$
$$\limsup_{t \searrow 0} \frac{f(x+td) - f(x)}{t} \stackrel{\text{def}}{=} \lim_{\delta \searrow 0} \sup_{0<t<\delta} \frac{f(x+td) - f(x)}{t},$$
$$\liminf_{\substack{t \searrow 0 \\ w \to d}} \frac{f(x+tw) - f(x)}{t} \stackrel{\text{def}}{=} \lim_{\delta \searrow 0} \inf_{\substack{0<t<\delta \\ |w-d|_{\mathbf{R}^N}<\delta \\ (t,w)\neq(0,v)}} \frac{f(x+tw) - f(x)}{t},$$

[7] Note that the (t, d), $0 < t < \delta$, is allowed in the inf and the sup of the last two definitions and the constraint $(t, w) \neq (0, v)$ can be removed.

$$\limsup_{\substack{t\searrow 0\\ w\to d}} \frac{f(x+tw)-f(x)}{t} \overset{\text{def}}{=} \lim_{\delta\searrow 0} \sup_{\substack{0<t<\delta\\ |w-d|_{\mathbf{R}^{\mathrm{N}}}<\delta\\ (t,w)\neq(0,v)}} \frac{f(x+tw)-f(x)}{t}.$$

Definition 15.1.
Let $\Omega \subset \mathbf{R}^{\mathrm{N}}$ be such that $\Gamma \overset{\text{def}}{=} \partial\Omega \neq \varnothing$.

(i) Ω is said to be *weakly cracked* if

$$\forall x \in \Gamma,\ \exists d \in \mathbf{R}^{\mathrm{N}},\ |d| = 1,\ \text{such that } \limsup_{\substack{t\searrow 0\\ w\to d}} \frac{d_\Gamma(x+tw)}{t} > 0. \tag{15.1}$$

(ii) Ω is said to be *cracked* if

$$\forall x \in \Gamma,\ \exists d \in \mathbf{R}^{\mathrm{N}},\ |d| = 1,\ \text{such that } \liminf_{t\searrow 0} \frac{d_\Gamma(x+td)}{t} > 0. \tag{15.2}$$

(iii) Ω is said to be *strongly cracked*[8] if

$$\forall x \in \Gamma,\ \exists d \in \mathbf{R}^{\mathrm{N}},\ |d| = 1,\ \text{such that } \liminf_{t\to 0} \frac{d_\Gamma(x+td)}{|t|} > 0. \tag{15.3}$$

(iv) A set $\varnothing \neq A \subset \mathbf{R}^{\mathrm{N}}$ is said to be *thin* if $\mathrm{m}_N(A) = 0$. □

Strongly cracked implies cracked, and cracked implies weakly cracked since

$$\begin{aligned}\liminf_{\substack{t\searrow 0\\ w\to d}} \frac{d_\Gamma(x+tw)}{t} &\le \liminf_{t\searrow 0} \frac{d_\Gamma(x+td)}{t}\\ &\le \limsup_{t\searrow 0} \frac{d_\Gamma(x+td)}{t} \le \limsup_{\substack{t\searrow 0\\ w\to d}} \frac{d_\Gamma(x+tw)}{t}.\end{aligned}$$

The special terminology of Definition 15.1 is introduced to provide an intuitive description of the sets. It is motivated by the fact that the boundary of such a set has zero N-dimensional Lebesgue measure (cf. Lemma 15.1) and can be made up of internal cracks, external hairs, cusps, or points, as shown in Figure 7.3.

The weakly cracked property is verified in any point of the boundary where the gradient of d_Γ does not exist; in boundary points where the gradient exists it is not identically 0. This is a very large family of sets that includes domains which are locally the epigraph of a continuous function. There are obvious variations of the above definitions and the forthcoming compactness Theorem 15.1 by replacing d_Γ by d_Ω, $d_{\complement\Omega}$, or b_Ω.

[8]Here the definition is

$$\liminf_{t\to 0} \frac{f(x+td)-f(x)}{|t|} \overset{\text{def}}{=} \lim_{\delta\searrow 0} \inf_{0<|t|<\delta} \frac{f(x+td)-f(x)}{t}.$$

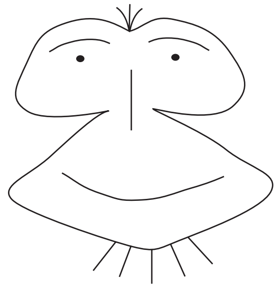


Figure 7.3. *Example of a two-dimensional strongly cracked set.*

Lemma 15.1. *Let $\Omega \subset \mathbf{R}^N$, let $\Gamma \neq \varnothing$, let $x \in \Gamma$, and let d, $|d| = 1$, be a direction in $\mathbf{R}^N$. If the semiderivative $dd_\Gamma(x; d)$ does not exist, then*

$$\limsup_{\substack{t \searrow 0 \\ w \to d}} \frac{d_\Gamma(x + tw)}{t} > 0.$$

Proof. Since the function d_Γ is Lipschitzian, the limit of the quotient

$$dd_\Gamma(x; d) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{d_\Gamma(x + td) - d_\Gamma(x)}{t}$$

exists if and only if the limit of the quotient

$$d_H d_\Gamma(x; d) \overset{\text{def}}{=} \lim_{\substack{t \searrow 0 \\ w \to d}} \frac{d_\Gamma(x + tw) - d_\Gamma(x)}{t}$$

exists (cf. Theorem 2.4 (i) in Chapter 9). Moreover, by the Lipschitz continuity,

$$\left| \frac{d_\Gamma(x + tw) - d_\Gamma(x)}{t} \right| \leq \left| \frac{tw}{t} \right| = |w| \to |d|$$

and hence the liminf and the limsup of the quotient exist and are finite.

For $x \in \Gamma$, $d_\Gamma(x) = 0$. Therefore $dd_\Gamma(x; d)$ does not exist if and only if

$$\limsup_{\substack{t \searrow 0 \\ w \to d}} \frac{d_\Gamma(x + tw)}{t} > \liminf_{\substack{t \searrow 0 \\ w \to d}} \frac{d_\Gamma(x + tw)}{t} \geq 0$$

since the last term is nonnegative. This completes the proof. $\square$

Theorem 15.1. *Let $\varnothing \neq \Omega \subset \mathbf{R}^N$.*

(i) *Given a weakly cracked set Ω,*

$$m_N(\Gamma) = 0 \quad \text{and} \quad \Gamma \subset \overline{\operatorname{int} \Omega} \cup \overline{\operatorname{int} \complement\Omega}.$$

Moreover in any point $x \in \Gamma$ either $\nabla b_\Omega(x)$ exists and is different from zero or $\nabla b_\Omega(x)$ does not exists (set of cracks).

(ii) *Given a cracked set* Ω*, for each* $x \in \Gamma$

$$\exists \text{ a direction } d \in \mathbf{R}^N,\ |d| = 1, \text{ such that } 1 \geq \underline{\ell}(x) \stackrel{\text{def}}{=} \liminf_{t \searrow 0} \frac{d_\Gamma(x + td)}{t} > 0$$

and for all ε*,* $0 < \varepsilon < \underline{\ell}(x)$*, there exists* $\delta > 0$ *such that*

$$\begin{aligned} &\forall t,\ 0 < t < \delta, \quad d_\Gamma(x + td) \geq (\underline{\ell}(x) - \varepsilon)t \\ \Rightarrow\ & x + C(\delta \cos\omega, \omega, d) \subset \mathbf{R}^N \backslash \Gamma, \quad \sin\omega \stackrel{\text{def}}{=} \underline{\ell}(x) - \varepsilon,\ 0 < \omega \leq \pi/2 \end{aligned}$$

($C(\lambda, \omega, d)$ *is the open cone in* 0 *of direction* d*, height* λ*, and aperture* ω*).*

(iii) *Given a strongly cracked set* Ω*, for each* $x \in \Gamma$ *there exists a direction* $d \in \mathbf{R}^N$*,* $|d| = 1$*, such that*

$$1 \geq \underline{\ell}(x) \stackrel{\text{def}}{=} \liminf_{t \to 0} \frac{d_\Gamma(x + td)}{|t|} > 0$$

and for all ε*,* $0 < \varepsilon < \underline{\ell}(x)$*, there exists* $\delta > 0$ *such that*

$$\begin{aligned} &\forall t,\ 0 < |t| < \delta, \quad d_\Gamma(x + td) \geq (\underline{\ell}(x) - \varepsilon)|t| \\ \Rightarrow\ & x \pm C(\delta \cos\omega, \omega, d) \subset \mathbf{R}^N \backslash \Gamma, \quad \sin\omega = \underline{\ell}(x) - \varepsilon,\ 0 < \omega \leq \pi/2. \end{aligned}$$

This means that in each point of the boundary of a cracked set, it is possible to place an open cone that does not intersect the boundary. If, in addition, the set is strongly cracked, this cone can be replaced by a *double cone*.

Proof. (i) We already know that ∇d_Γ exists almost everywhere in $\mathbf{R}^N$ and that, whenever it exists,

$$|\nabla d_\Gamma(x)| = \begin{cases} 0, & \text{if } x \in \Gamma, \\ 1, & \text{if } x \notin \Gamma \end{cases}$$

(cf. Theorem 3.3 (vi) in Chapter 6). Therefore, if $\nabla d_\Gamma(x)$ exists in a point $x \in \Gamma$, $\nabla d_\Gamma(x) = 0$ and for all d, $|d| = 1$,

$$\limsup_{\substack{t \searrow 0 \\ w \to d}} \frac{d_\Gamma(x + tw)}{t} = \limsup_{\substack{t \searrow 0 \\ w \to d}} \frac{d_\Gamma(x + tw) - d_\Gamma(x)}{t} = \nabla d_\Gamma(x) \cdot d = 0,$$

which contradicts the weakly cracked property. Hence the points of Γ are points where $\nabla d_\Gamma(x)$ does not exist, which is itself a set of zero measure. Moreover, if $\nabla b_\Omega(x)$ exists in a point $x \in \partial\Omega$, then $b_\Omega(x) = 0$ and for all d, $|d| = 1$,

$$\lim_{\substack{t \searrow 0 \\ w \to d}} \frac{b_\Omega(x + tw)}{t} = \lim_{\substack{t \searrow 0 \\ w \to d}} \frac{b_\Omega(x + tw) - b_\Omega(x)}{t} = \nabla b_\Omega(x) \cdot d.$$

If $\nabla b_\Omega(x) = 0$, then for all d

$$\lim_{\substack{t\searrow 0\\ w\to d}} \frac{b_\Omega(x+tw)}{t} = \lim_{\substack{t\searrow 0\\ w\to d}} \frac{b_\Omega(x+tw)-b_\Omega(x)}{t} = \nabla b_\Omega(x)\cdot d = 0$$
$$\Rightarrow \lim_{\substack{t\searrow 0\\ w\to d}} \frac{d_\Gamma(x+tw)}{t} = \lim_{\substack{t\searrow 0\\ w\to d}} \left|\frac{b_\Omega(x+tw)}{t}\right| = 0$$

and $\nabla d_{\partial\Omega}(x) = 0$, which contradicts our assumption. Therefore if $\nabla b_\Omega(x)$ exists in a point $x \in \partial\Omega$, $\nabla b_\Omega(x) \neq 0$. For any $x \in \Gamma$ introduce the notation

$$\bar{\ell}(x) \overset{\text{def}}{=} \limsup_{\substack{t\searrow 0\\ w\to d}} \frac{d_\Gamma(x+tw)}{t}.$$

By assumption $\bar{\ell}(x) > 0$ and for all $\delta > 0$

$$\bar{\ell}(x) \leq \sup_{\substack{0<t<\delta\\ |w-d|<\delta}} \frac{d_\Gamma(x+tw)}{t}.$$

Hence there exist sequences $\{t_n\}$, $t_n \to 0$, and $\{w_n\}$, $w_n \to d$, such that

$$0 < \bar{\ell}(x)/2 \leq \frac{d_\Gamma(x+t_n w_n)}{t} \quad \Rightarrow \mathbf{R}^{\mathrm{N}}\backslash\Gamma \ni x_n \overset{\text{def}}{=} x + t_n w_n \to x$$

and necessarily $\Gamma \subset \overline{\operatorname{int}\Omega \cup \operatorname{int}\complement\Omega} \subset \overline{\operatorname{int}\Omega} \cup \overline{\operatorname{int}\complement\Omega}$.

(ii) Given a cracked set Ω, for each $x \in \Gamma$ there exists a direction $d \in \mathbf{R}^{\mathrm{N}}$, $|d| = 1$, such that

$$\underline{\ell}(x) \overset{\text{def}}{=} \liminf_{t\searrow 0} \frac{d_\Gamma(x+td)}{t} > 0$$

and for all ε, $0 < \varepsilon < \underline{\ell}(x)$, there exists $\delta > 0$ such that

$$\forall t,\ 0 < t < \delta, \quad d_\Gamma(x+td) \geq (\underline{\ell}(x) - \varepsilon)t.$$

Recall that since d_Γ is Lipschitzian of constant 1, we necessarily have $1 \geq \underline{\ell}(x)$ for $|d| = 1$. Therefore

$$x + C(\delta\cos\omega, \omega, d) \subset \mathbf{R}^{\mathrm{N}}\backslash\Gamma, \quad \sin\omega = \underline{\ell}(x) - \varepsilon,\ 0 < \omega \leq \pi/2.$$

(iii) This is similar to the proof of part (ii). □

Theorem 15.2. *Let D be a bounded open subset of $\mathbf{R}^{\mathrm{N}}$ and α, $0 < \alpha \leq 1$, and $h > 0$ be real numbers.*[9] *Consider the families*

$$\mathcal{F}(D,h,\alpha) \overset{\text{def}}{=} \left\{ \Omega \subset \bar{D} : \begin{array}{c} \Gamma \neq \varnothing \textit{ and } \forall x \in \Gamma,\ \exists d,\ |d| = 1, \\ \textit{such that } \displaystyle\inf_{0<t<h} \frac{d_\Gamma(x+td)}{t} \geq \alpha \end{array} \right\}, \tag{15.4}$$

$$C_b^{h,\alpha}(D) \overset{\text{def}}{=} \{b_\Omega : \Omega \in \mathcal{F}(D,h,\alpha)\},$$

[9] In view of the fact that the distance function d_Γ is Lipschitzian with constant 1, we necessarily have $0 < \alpha \leq 1$.

$$\mathcal{F}_s(D,h,\alpha) \stackrel{\text{def}}{=} \left\{ \Omega \subset \bar{D} \; : \; \begin{array}{c} \Gamma \neq \varnothing \; \textit{and} \; \forall x \in \Gamma, \, \exists d, \, |d|=1, \\ \textit{such that} \; \inf\limits_{0<|t|<h} \dfrac{d_\Gamma(x+td)}{|t|} \geq \alpha \end{array} \right\}, \tag{15.5}$$

$$(C_b^{h,\alpha})_s(D) \stackrel{\text{def}}{=} \{b_\Omega \, : \, \Omega \in \mathcal{F}_s(D,h,\alpha)\}.$$

Then $C_b^{h,\alpha}(D)$ *and* $(C_b^{h,\alpha})_s(D)$ *are compact in* $W^{1,p}(D)$, $1 \leq p < \infty$.

Proof. (i) The family $C_b^{h,\alpha}(D)$ is contained in $C_b(D)$, which is compact in the uniform topology of $C(\overline{D})$ and in the weak topology of $W^{1,p}(D)$, $1 \leq p < \infty$ (cf. Theorem 2.2 (ii) and Theorem 4.2 (iii) in Chapter 6). Therefore, given a sequence $\{b_{\Omega_n}\}$ in $C_b^{h,\alpha}(D)$, there exist a subsequence, still denoted by $\{b_{\Omega_n}\}$, and $b_\Omega \in C_b(D)$ such that $b_{\Omega_n} \to b_\Omega$ in $W^{1,p}(D)$-weak and $C(\overline{D})$. In addition, by definition of the elements of $C_b^{h,\alpha}(D)$, condition (15.2) is verified and each Ω_n has thin boundary. We want to show that $b_\Omega \in C_b^{h,\alpha}(D)$. Once this is proved, from Theorem 15.1 (i), Ω has a thin boundary. Hence the weak $W^{1,p}(D)$ convergence implies the strong $W^{1,p}(D)$ convergence by Theorem 4.3. In view of the continuity of the map $b_\Omega \to d_\Gamma = |b_\Omega| : C(\overline{D}) \to C(\overline{D})$, $d_{\Gamma_n} \to d_\Gamma$ in $C(\overline{D})$. From Lemma 13.1, given $x \in \Gamma$, there exists a subsequence of $\{b_{\Omega_n}\}$, still denoted by $\{b_{\Omega_n}\}$, and for each $n \geq 1$ points $x_n \in \Gamma_n \cap B(x, 1/n)$. Hence $x_n \to x$. By assumption

$$\forall n \geq 1, \, \exists d_n \in \mathbf{R}^{\mathrm{N}}, \, |d_n| = 1, \text{ such that } \inf_{0<t<h} \frac{d_{\Gamma_n}(x_n+td_n)}{t} \geq \alpha.$$

Since the d_n's have norm 1, there exist a subsequence, still denoted by $\{d_n\}$, and d, $|d|=1$, such that $d_n \to d$. Fix t, $0<t<\delta$. Given $\varepsilon > 0$, there exists N such that for all $n \geq N$

$$|x_n - x| < \varepsilon t, \quad |d_n - d| < \delta_n < \varepsilon, \quad \|d_{\Gamma_n} - d_\Gamma\|_{C(\overline{D})} < \varepsilon t.$$

Fix $n = N$ and consider the following estimates:

$$\begin{aligned} &\frac{d_\Gamma(x+td)}{t} \\ &\geq \frac{d_{\Gamma_n}(x_n+td_n)}{t} - \frac{|d_{\Gamma_n}(x+td_n) - d_{\Gamma_n}(x_n+td)|}{t} \\ &\quad - \frac{|d_\Gamma(x+td) - d_\Gamma(x_n+td)|}{t} - \frac{|d_\Gamma(x_n+td) - d_{\Gamma_n}(x_n+td)|}{t} \\ &\geq \alpha - \varepsilon - |d-d_n| - \frac{|x-x_n|}{t} - \frac{\|d_\Gamma - d_{\Gamma_n}\|_{C\overline{D})}}{t} \geq \alpha - 4\varepsilon \\ &\Rightarrow \forall \varepsilon > 0, \inf_{0<t<h} \frac{d_\Gamma(x+td)}{t} \geq \alpha - 4\varepsilon \quad \Rightarrow \inf_{0<t<h} \frac{d_\Gamma(x+td)}{t} \geq \alpha. \end{aligned}$$

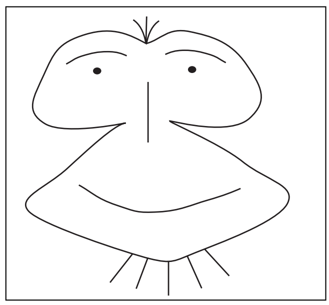
Therefore $b_\Omega \in C_b^{h,\alpha}(D)$ and this completes the proof of the compactness.

(ii) The proof for $(C_b^{h,\alpha})_s(D)$ is identical with obvious changes. □

16 A Variation of the Image Segmentation Problem of Mumford and Shah

16.1 Problem Formulation

In this section we specialize to the segmentation of (N-dimensional) images where the segmentation can be composed of objects of codimension greater than or equal to one. To represent the set of Figure 7.3 as the boundary of an open set one can use the unbounded plane $\mathbf{R}^2$ minus all the lines and the points in Figure 7.3. If it is important that the open set be bounded, a fixed open *frame* D is introduced. The open set Ω in Figure 7.4 is then defined as the interior of the bounded open frame D minus all the lines and points used to draw the picture.



open frame D

Figure 7.4. *The two-dimensional strongly cracked set of Figure* 7.3 *in an open frame* D.

Definition 16.1.
Let D be a bounded open subset of $\mathbf{R}^{\mathrm{N}}$ with Lipschitzian boundary.

(i) An *image* in the *frame* D is specified by a function $f \in L^2(D)$.

(ii) We say that $\{\Omega_i\}_{i\in I}$ is an *open partition* of D if $\{\Omega_i\}_{i\in I}$ is a family of disjoint connected open subsets of D such that

$$\mathrm{m}_N(\cup_{i\in I}\Omega_i) = \mathrm{m}_N(D) \text{ and } \mathrm{m}_N(\partial \cup_{i\in I} \Omega_i) = 0.$$

Denote by $\mathcal{P}(D)$ the family of all such open partitions of D. □

Given an open partition $\{\Omega_i\}_{i\in I}$ of D, associate with each $i \in I$ a function $\varphi_i \in H^1(\Omega_i)$. In its intuitive form the problem formulated by D. Mumford and J. Shah [**2**] aims at finding an open partition $P = \{\Omega_i\}_{i\in I}$ in $\mathcal{P}(D)$ solution of the following minimization problem:

$$\inf_{P\in\mathcal{P}(D)} \sum_{i\in I} \inf_{\varphi_i\in H^1(\Omega_i)} \int_{\Omega_i} \varepsilon\, |\nabla\varphi_i|^2 + |\varphi_i - f|^2\, dx \tag{16.1}$$

for some fixed constant $\varepsilon > 0$. The term in ε times the norm of the gradient can be seen as a Tikhonov regularization on each Ω_i. Observe that without the condition $\mathrm{m}_N(\cup_{i\in I}\Omega_i) = \mathrm{m}_N(D)$ the empty set would be a solution of the problem or a phenomenon of the type discussed in Remark 12.1 and the phenomenon of Figure 7.2 could occur.

The question of existence requires a more specific family of open partitions or a penalization term which preserves the "length" of the interfaces in some appropriate sense:

$$\inf_{P\in\mathcal{P}(D)} \sum_{i\in I} \inf_{\varphi_i\in H^1(\Omega_i)} \int_{\Omega_i} \varepsilon\,|\nabla\varphi_i|^2 + |\varphi_i - f|^2\,dx + c\,H_{N-1}(\partial\cup_{i\in I}\Omega_i) \tag{16.2}$$

for some $c > 0$. The choice of the relaxation of the $(N-1)$-Hausdorff measure H_{N-1} is critical as discussed in section 9 of Chapter 1. Another way of looking at the problem would be to minimize the number I of open subsets of the open partition, but this seems more difficult to formalize.

16.2 Cracked Sets without the Perimeter

In this section we specialize the compact family of Theorem 15.2 to get the existence of a solution to the following minimization problem:

$$\boxed{\inf_{\substack{\Omega\in\mathcal{F}(D,h,\alpha)\\ \Omega \text{ open } \subset D,\, \mathrm{m}_N(\Omega)=\mathrm{m}_N(D)}} \quad \inf_{\varphi\in H^1(\Omega)} \int_{\Omega} \varepsilon\,|\nabla\varphi|^2 + |\varphi - f|^2\,dx.} \tag{16.3}$$

This allows for open sets Ω with $H_{N-1}(\Gamma) = \infty$. The pair (h, α) are the control parameters of the segmentation. Recall the characterization of Theorem 15.1 (ii), which says that in each point of the boundary there exists a small open cone of uniform height and aperture which does not intersect the boundary.

Theorem 16.1. *Given a bounded open frame $D \subset \mathbf{R}^N$ with a Lipschitzian boundary and real numbers h, α, and $\varepsilon > 0$, there exist an open subset Ω^* of D in $\mathcal{F}(D, h, \alpha)$ such that $\mathrm{m}_N(\Omega^*) = \mathrm{m}_N(D)$ and a function $y \in H^1(\Omega^*)$ that are solutions of problem* (16.3).

16.2.1 Technical Lemmas

The proof of the existence theorems will require the following technical results.

Lemma 16.1. *Given a subset A of $\mathbf{R}^N$ with nonempty boundary ∂A,*

$$\exists \text{ an open subset } \Omega \subset \mathbf{R}^N \text{ such that } b_A = b_\Omega$$

if and only if $d_A = d_{\mathrm{int}\,A}$ or equivalently $\bar{A} = \overline{\mathrm{int}\,A}$.

Proof. By definition for Ω open

$$b_A = b_\Omega \iff d_A = d_\Omega \text{ and } d_{\complement\Omega} = d_{\complement A},$$

which is also equivalent to

$$\bar{A} = \bar{\Omega} \text{ and } \complement\Omega = \overline{\complement A} \iff \bar{A} = \bar{\Omega} \text{ and } \Omega = \operatorname{int} A.$$

Hence the necessary and sufficient condition finally reduces to $\overline{\operatorname{int} A} = \bar{A}$. $\square$

Lemma 16.2. *Let $D \subset \mathbf{R}^N$ be bounded open with Lipschitzian boundary. Then*

$$\Omega \subset D,\ \Gamma \neq \varnothing,\ \mathrm{m}_N(\Gamma) = 0, \text{ and } \mathrm{m}_N(\Omega) = \mathrm{m}_N(D) \quad \Rightarrow \overline{\operatorname{int}\Omega} = \overline{\Omega}.$$

Proof. By contradiction. If $\overline{\operatorname{int}\Omega} \subsetneq \overline{\Omega}$, there exists $x \in \overline{\Omega}$ such that $d_{\operatorname{int}\Omega}(x) = \rho > 0$ and hence $B(x,\rho) \subset \complement\Omega$. Therefore $\overline{\Omega} \cap B(x,\rho) \subset \Gamma$ and $\mathrm{m}_N(\overline{\Omega} \cap B(x,\rho)) \leq \mathrm{m}_N(\Gamma) = 0$. By assumption $\mathrm{m}_N(D) = \mathrm{m}_N(\Omega) = \mathrm{m}_N(\overline{\Omega})$ implies $\mathrm{m}_N(D \cap B(x,\rho)) = \mathrm{m}_N(\overline{\Omega} \cap B(x,\rho)) = 0$. Since $\Omega \subset D$, there exists $x \in \overline{D}$ such that $\mathrm{m}_N(D \cap B(x,\rho)) = 0$. But this is a contradiction since D is an open set with Lipschitzian boundary. $\square$

16.2.2 Another Compactness Theorem

The compactness of the following special family of cracked sets contained in a frame is a corollary to the compactness Theorem 15.2.

Theorem 16.2. *Let D be a bounded open subset of $\mathbf{R}^N$ with Lipschitzian boundary and $h > 0$ and $\alpha > 0$ be real numbers. Consider the family*

$$\mathcal{F}^c(D,h,\alpha) \stackrel{\text{def}}{=} \left\{ \Omega \subset \bar{D} \,:\, \begin{array}{l} \Omega \textit{ open } \subset D \textit{ and } \mathrm{m}_N(\Omega) = \mathrm{m}_N(D) \\ \textit{and } \forall x \in \Gamma,\ \exists d,\ |d| = 1, \\ \textit{such that } \displaystyle\inf_{0<t<h} \frac{d_\Gamma(x+td)}{t} \geq \alpha \end{array} \right\}, \tag{16.4}$$

$$C_b^{c,h,\alpha}(D) \stackrel{\text{def}}{=} \left\{ b_\Omega \,:\, \Omega \in \mathcal{F}^c(D,h,\alpha) \right\}.$$

Then $C_b^{c,h,\alpha}(D)$ is compact in $W^{1,p}(D)$, $1 \leq p < \infty$.

Proof. By standard arguments and Lemma 16.2. The conclusion follows from Theorem 15.2 by adding the constraint $m_N(\Omega_n) = \mathrm{m}_N(D)$ which will be verified for the limit set Ω for which a subsequence of $\{b_{\Omega_n}\}$ converges to b_Ω in $W^{1,1}(D)$ and hence $\{\chi_{\Omega_n}\}$ converges to χ_Ω in $L^1(D)$. $\square$

16.2.3 Proof of Theorem 16.1

Proof of Theorem 16.1. (i) For each open $\Omega \in \mathcal{F}^c(D,h,\alpha)$, the problem

$$\inf_{\varphi \in H^1(\Omega)} F(\Omega,\varphi), \quad F(\Omega,\varphi) \stackrel{\text{def}}{=} \int_\Omega \varepsilon|\nabla\varphi|^2 + |\varphi - f|^2\, dx$$

has a unique solution y in $H^1(\Omega)$ since the objective function $F(\Omega,\varphi)$ is continuous and coercive on $H^1(\Omega)$. Define

$$m \stackrel{\text{def}}{=} \inf_{\Omega \in \mathcal{F}^c(D,h,\alpha)} \inf_{\varphi \in H^1(\Omega)} \int_\Omega \varepsilon\, |\nabla\varphi|^2 + |\varphi - f|^2\, dx.$$

(ii) The minimum is finite since the objective function is positive and

$$\forall \text{ open } \Omega \in \mathcal{F}^c(D,h,\alpha), \qquad \inf_{\varphi \in H^1(\Omega)} \int_\Omega \varepsilon\, |\nabla\varphi|^2 + |\varphi - f|^2\, dx \le \int_\Omega |f|^2\, dx$$

by choosing $\varphi = 0$. Let $\{\Omega_n\}$ be a minimizing sequence of open subsets of D in $\mathcal{F}^c(D,h,\alpha)$ and for each n let $y_n \in H^1(\Omega_n)$ be the minimizing element of $F(\Omega_n, \varphi)$ over $H^1(\Omega_n)$. Therefore

$$\begin{aligned} &\int_{\Omega_n} \varepsilon\, |\nabla y_n|^2 + |y_n - f|^2\, dx \to m \\ \Rightarrow\ \exists c > 0 \text{ such that } \forall n, \quad &\int_{\Omega_n} \varepsilon\, |\nabla y_n|^2 + |y_n - f|^2\, dx \le c. \end{aligned}$$

By coercivity the sequence $\{y_n\}$ is uniformly bounded in $H^1(\Omega_n)$; that is, there exists a constant $c > 0$ such that

$$\forall n, \quad \|y_n\|_{L^2(\Omega_n)} \le c \quad \text{and} \quad \|\nabla y_n\|_{L^2(\Omega_n)} \le c.$$

By Theorem 16.2, there exist a subsequence of $\{\Omega_n\}$ and an open set $\Omega \in \mathcal{F}^c(D,h,\alpha)$ such that $b_{\Omega_n} \to b_\Omega$ in $H^1(D)$ and $C(\overline{D})$. In particular $d_{\complement\Omega_n} \to d_{\complement\Omega}$ in $C(\overline{D})$. By the compactivorous property of Theorem 2.4 (iii) in Chapter 6

$$\forall K \text{ compact} \subset \operatorname{int}\Omega, \quad \exists N \text{ such that } \forall n \ge N, \quad K \subset \Omega_n.$$

Moreover

$$\chi_{\Omega_n} \to \chi_\Omega \in L^2(D) \quad \Rightarrow \chi_{\Omega_n} \rightharpoonup \chi_\Omega \in L^\infty(D)\text{-weak}* \tag{16.5}$$

$$\Rightarrow\ f\chi_{\Omega_n} \rightharpoonup f\chi_\Omega \in L^2(D)\text{-weak.} \tag{16.6}$$

Define the distributions

$$\begin{aligned} \langle \tilde{y}_n, \varphi\rangle &\overset{\text{def}}{=} \int_{\Omega_n} y_n\, \varphi\, dx, \quad \forall \varphi \in \mathcal{D}(D), \\ \langle \widetilde{\nabla y_n}, \Phi\rangle &\overset{\text{def}}{=} \int_{\Omega_n} \nabla y_n \cdot \Phi\, dx, \quad \forall \Phi \in \mathcal{D}(D)^N, \\ \langle \nabla\tilde{y}_n, \Phi\rangle &= -\int_D \tilde{y}_n \operatorname{div}\Phi\, dx, \quad \forall \Phi \in \mathcal{D}(D)^N. \end{aligned}$$

It is readily seen that we can identify $\tilde{y}_n$ and $\widetilde{\nabla y_n}$ with the extensions of y_n and ∇y_n by zero from Ω_n to D. As a result there exist subsequences, still denoted by $\{\tilde{y}_n\}$ and $\{\widetilde{\nabla y_n}\}$, and $\tilde{y} \in L^2(D)$ and $Y \in L^2(D)^N$ such that $\tilde{y}_n \rightharpoonup \tilde{y}$ in $L^2(D)$-weak and $\widetilde{\nabla y_n} \rightharpoonup Y$ in $L^2(D)^N$-weak.

By the compactivorous property, for all $\Phi \in \mathcal{D}(\ \Omega)$, there exists N such that

$$\forall n > N, \quad \operatorname{supp}\Phi \subset \Omega_n \quad \Rightarrow \Phi \in \mathcal{D}(\Omega_n).$$

Therefore for all $n > N$

$$\langle \nabla \tilde{y}_n - \widetilde{\nabla y_n}, \Phi \rangle = -\int_{\Omega_n} y_n \operatorname{div} \Phi \, dx - \int_{\Omega_n} \nabla y_n \cdot \Phi \, dx = 0$$

since $y_n \in H^1(\Omega_n)$. But $\mathcal{D}(\Omega_n) \subset \mathcal{D}(D)$ and by letting n go to infinity

$$\begin{aligned} 0 = -\int_{\Omega_n} y_n \operatorname{div} \Phi \, dx - \int_{\Omega_n} \nabla y_n \cdot \Phi \, dx \to -\int_D \tilde{y} \operatorname{div} \Phi \, dx - \int_D Y \cdot \Phi \, dx \\ \Rightarrow \forall \Phi \in \mathcal{D}(\Omega), \quad \int_D \tilde{y} \operatorname{div} \Phi \, dx + \int_D Y \cdot \Phi \, dx = 0. \end{aligned}$$

Define the new distribution

$$\langle y, \varphi \rangle \stackrel{\text{def}}{=} \int_\Omega \tilde{y} \, \varphi \, dx, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

It is easy to check that $y \in L^2(\Omega)$ and hence

$$\begin{aligned} \forall \Phi \in \mathcal{D}(\Omega), \quad 0 = \int_D \tilde{y} \operatorname{div} \Phi \, dx + \int_D Y \cdot \Phi \, dx = \int_\Omega \tilde{y} \operatorname{div} \Phi \, dx + \int_\Omega Y \cdot \Phi \, dx \\ \Rightarrow \forall \Phi \in \mathcal{D}(\Omega), \quad \langle \nabla y, \Phi \rangle = -\int_\Omega \tilde{y} \operatorname{div} \Phi \, dx = \int_\Omega Y \cdot \Phi \, dx \\ \Rightarrow \nabla y = Y|_\Omega \in L^2(\Omega) \quad \Rightarrow y \in H^1(\Omega). \end{aligned}$$

(iv) Coming back to our objective function

$$\begin{aligned} \inf_{\varphi \in H^1(\Omega_n)} \int_{\Omega_n} \varepsilon \, |\nabla \varphi|^2 + |\varphi - f|^2 \, dx &= \int_{\Omega_n} \varepsilon \, |\nabla y_n|^2 + |y_n - f|^2 \, dx \\ &= \int_D \varepsilon \, |\widetilde{\nabla y_n}|^2 + |\tilde{y}_n - f\chi_{\Omega_n}|^2 \, dx. \end{aligned}$$

By convexity and continuity of the objective function with respect to the pair $(\tilde{y}_n - f\chi_{\Omega_n}, \widetilde{\nabla y_n})$ in $L^2(D) \times L^2(D)^N$ and the fact that, from (16.6), $(\tilde{y}_n - f\chi_{\Omega_n}, \widetilde{\nabla y_n}) \rightharpoonup (\tilde{y} - f\chi_\Omega, Y)$ in $L^2(D) \times L^2(D)^N$-weak,

$$\begin{aligned} \int_D \varepsilon \, |Y|^2 + |\tilde{y} - f\chi_\Omega|^2 \, dx &\le \liminf_{n \to \infty} \int_D \varepsilon \, |\widetilde{\nabla y_n}|^2 + |\tilde{y}_n - f\chi_{\Omega_n}|^2 \, dx \\ &= \liminf_{n \to \infty} \int_{\Omega_n} \varepsilon \, |\nabla y_n|^2 + |y_n - f|^2 \, dx = m \\ \Rightarrow \int_\Omega \varepsilon \, |\nabla y|^2 + |y - f|^2 \, dx &= \int_D \varepsilon \, |\nabla y|^2 + |y - f\chi_\Omega|^2 \, dx \le m. \end{aligned}$$

By definition of the minimum we have the equality and there exist an open set $\Omega \in \mathcal{F}^c(D, h, \alpha)$ and $y \in H^1(\Omega)$ solution of the segmentation problem. $\square$

16.3 Existence of a Cracked Set with Minimum Density Perimeter

Theorem 16.1 gives an existence result for the family $\mathcal{F}(D, h, \alpha)$ of open sets Ω such that $\mathrm{m}_N(\Omega) = \mathrm{m}_N(D)$ without constraint on the "perimeter" of Ω. Denote by $\mathcal{F}^*(D, h, \alpha)$ the set of solutions to problem (16.3). In general, the perimeter can be infinite as can be seen from the following example.

Example 16.1.
The function $f : D \to \mathbf{R}$ is defined as follows:

$$f(x) = \begin{cases} 1, & \text{if } x \in \Omega_1, \\ 0, & \text{if } x \in \Omega_2, \end{cases} \tag{16.7}$$

where $D = \{(x, y) : -2 < x < 3,\ -1 < y < 3\}$, $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_2 = D \backslash \overline{\Omega_1}$, and the open set Ω_1 is constructed below (see Figure 7.5). The set Ω with $y = f$ is a solution of problem (16.3) with infinite perimeter. The set Ω_1 is a two-dimensional example similar to Examples 6.1 and 6.2 of section 6.5 in Chapter 2 of an open domain satisfying the uniform cusp condition for the function $h(\theta) = \theta^\alpha$, $0 < \alpha < 1$. It can easily be generalized to an N-dimensional example. Consider the open domain Ω_1 in $\mathbf{R}^2$

$$\begin{aligned} \Omega_1 \stackrel{\text{def}}{=} & \{(x, y)\ :\ -1 < x \le 0 \text{ and } 0 < y < 2\} \\ & \cap \{(x, y)\ :\ 0 < x < 1 \text{ and } f(x) < y < 2\} \\ & \cap \{(x, y)\ :\ 1 \le x < 2 \text{ and } 0 < y < 2\}, \end{aligned}$$

where $f : [0, 1] \to \mathbf{R}$ is defined as follows:

$$f(x) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} f_k(x), \quad f_k : \left[1 - \frac{1}{2^k}, 1 - \frac{1}{2^{k+1}}\right] \to \mathbf{R}\,.$$

Associate with $0 < \alpha < 1$ and $k \ge 0$ the even integer $\eta_k = 2\,[(2^{k+1})^{\frac{\alpha}{1-\alpha}}]$, where $[\beta]$ is the smallest integer greater than or equal to β. Assume that for each $k \ge 0$

$$f_k \stackrel{\text{def}}{=} \sum_{j=1}^{\eta_k/2} g_{k,j} \quad g_{k,j} : [x_{k,j-1}, x_{k,j-1} + \delta_k] \to \mathbf{R},$$

$$x_{k,j} \stackrel{\text{def}}{=} 1 - \frac{1}{2^k} + (j-1)2\delta_k,\ 1 \le j \le \eta_k/2, \quad \delta_k \stackrel{\text{def}}{=} \frac{1}{\eta_k\, 2^{k+1}}$$

and that the function $g_{k,j}$ is given by the expression

$$g_{k,j}(x) \stackrel{\text{def}}{=} \begin{cases} 0, & 0 \le x < x_{k,j-1}, \\ (x - x_{k,j-1})^\alpha, & x_{k,j-1} \le x \le x_{k,j-1} + \delta_k, \\ (2x_{k,j-1} - x)^\alpha, & x_{k,j-1} + \delta_k \le x \le x_{k,j-1} + 2\delta_k, \\ 0, & x > x_{k,j-1} + 2\delta_k. \end{cases}$$

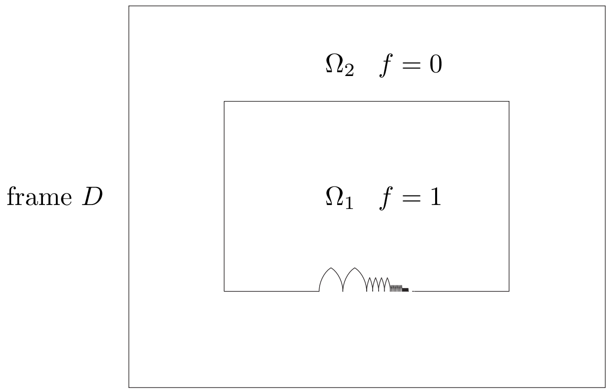


Figure 7.5. *The two open components Ω_1 and Ω_2 of the open domain Ω for $N = 2$.*

Note that

$$g_{k,j}(x_{k,j-1} + \delta_k) = (\delta_k)^\alpha$$

is independent of j and is the maximum of the function $g_{k,j}$.

The uniform cusp property is verified for $\rho = 1/8$, $\lambda = h(\rho)$, and $h(\theta) = \theta^\alpha$. The boundary of Ω_1 is made up of straight lines of total length 9 plus the length of the curve

$$C \stackrel{\text{def}}{=} \{(x, f(x)) \,:\, 0 \le x < 1\}, \quad C = \cup_{k=0}^{\infty} C_k, \quad C_k = \cup_{j=1}^{\eta_k/2} C_{k,j},$$
$$C_{k,j} \stackrel{\text{def}}{=} \{(x, f(x)) \,:\, x_{k,j-1} \le x < x_{k,j-1} + \delta_k\}.$$

The length of the curve $C_{k,j}$ is bounded below by

$$H_{N-1}(C_{k,j}) \ge 2\sqrt{(\delta_k)^2 + (\delta_k)^{2\alpha}} \ge 2\,(\delta_k)^\alpha \quad \Rightarrow H_{N-1}(C_k) \ge \sum_{j=1}^{\eta_k/2} H_{N-1}(C_{k,j}),$$

$$H_{N-1}(C_k) \ge \frac{\eta_k}{2}\, 2\,(\delta_k)^\alpha = \eta_k\,(\delta_k)^\alpha = \eta_k \left(\frac{1}{\eta_k\, 2^{k+1}}\right)^\alpha = {\eta_k}^{1-\alpha} \left(\frac{1}{2^{k+1}}\right)^\alpha,$$

$$\begin{aligned} H_{N-1}(C_k) \ge {\eta_k}^{1-\alpha} \left(\frac{1}{2^{k+1}}\right)^\alpha &= \left(2[(2^{k+1})^{\frac{\alpha}{1-\alpha}}]\right)^{1-\alpha} \left(\frac{1}{2^{k+1}}\right)^\alpha \\ &\ge 2^{1-\alpha}(2^{k+1})^\alpha \left(\frac{1}{2^{k+1}}\right)^\alpha = 2^{1-\alpha} \end{aligned}$$

$$\Rightarrow H_{N-1}(C) = \sum_{k=0}^{\infty} H_{N-1}(C_k) \ge +\infty\, 2^{1-\alpha} = +\infty. \qquad \square$$

When at least one solution has a bounded perimeter, it is possible to show that there is one that minimizes the h-density perimeter.

Theorem 16.3. *Assume that the assumptions of Theorem* 16.1 *are verified. There exists an* Ω^* *in* $\mathcal{F}^*(D, h, \alpha)$ *which minimizes the* h*-density perimeter.*

Proof. If for all Ω in $\mathcal{F}^*(D, h, \alpha)$ the h-density perimeter is $+\infty$, the theorem is true. If for some $\Omega \in \mathcal{F}^*(D, h, \alpha)$, $P_h(\Gamma) \leq c$, then there exists a sequence $\{\Omega_n\}$ in $\mathcal{F}^*(D, h, \alpha)$ such that

$$P_h(\Gamma_n) \to \inf_{\Omega \in \mathcal{F}^*(D,h,\alpha)} P_h(\Gamma).$$

By the compactness Theorems 12.1 and 16.2, there exist a subsequence and Ω^*, $\Gamma^* \neq \varnothing$, such that $b_{\Omega_n} \to b_{\Omega^*}$ in $W^{1,p}(D)$, $\Omega^* \in F(D, h, \alpha)$, $\mathrm{m}_N(\Omega) = \mathrm{m}_N(D)$, and

$$P_h(\Gamma^*) \leq \liminf_{n \to \infty} P_h(\Gamma_n) \leq c.$$

Finally by going back to the proof of Theorem 16.1 and using the fact that all the Ω_n's are already minimizers in $F^*(D, h, \alpha)$, it can be shown that Ω^* is indeed one of the minimizers in the set $F^*(D, h, \alpha)$. □

16.4 Uniform Bound or Penalization Term in the Objective Function on the Density Perimeter

To complete the results on the segmentation problem, we turn to the existence of a segmentation for a family of sets with a uniform bound or with a penalization term in the objective function on the h-density perimeter.

Theorem 16.4. *Given a bounded open frame* $\varnothing \neq D \subset \mathbf{R}^{\mathrm{N}}$ *with a Lipschitzian boundary and real numbers* $h > 0$ *and* $c > 0$,[10] *there exists an open subset* Ω^* *of* D, $\Gamma^* \neq \varnothing$, *with finite* h*-density perimeter such that* $\mathrm{m}_N(\Omega^*) = \mathrm{m}_N(D)$ ($P_h(\Gamma^*) \leq c$ *for* (16.8)), *and* $y \in H^1(\Omega^*)$ *solutions of the respective problems*

$$\inf_{\substack{\Omega \text{ open } \subset D,\, P_h(\Gamma) \leq c \\ \mathrm{m}_N(\Omega) = \mathrm{m}_N(D)}} \inf_{\varphi \in H^1(\Omega)} \int_\Omega \varepsilon\, |\nabla \varphi|^2 + |\varphi - f|^2 \, dx, \tag{16.8}$$

$$\inf_{\substack{\Omega \text{ open } \subset D \\ \mathrm{m}_N(\Omega) = \mathrm{m}_N(D)}} \inf_{\varphi \in H^1(\Omega)} \int_\Omega \varepsilon\, |\nabla \varphi|^2 + |\varphi - f|^2 \, dx + c\, P_h(\Gamma). \tag{16.9}$$

Proof. The proof for the objective function (16.8) is exactly the same as the one of Theorem 16.1. It uses Lemma 16.2 to show that the minimizing set has a thin boundary and the compactness Theorem 12.1. The proof for the objective function (16.9) uses the fact that there is a minimizing sequence for which the h-density perimeter is uniformly bounded and the lower semicontinuity of the density perimeter in the $W^{1,p}$-topology is given by Corollary 1. □

Problem (16.9) was originally considered in D. Bucur and J.-P. Zolésio [**8**]. The above two identification problems can be further specialized to the family of cracked sets $\mathcal{F}(D, h, \alpha)$.

[10] Note that the constant c must be large enough to take into account the contribution of the boundary of D.

Corollary 1. *Given a bounded open frame $D \subset \mathbf{R}^{\mathrm{N}}$ with a Lipschitzian boundary and real numbers $h > 0$, $\alpha > 0$, and $c > 0$, there exists an open subset Ω^* of D in $\mathcal{F}(D, h, \alpha)$ such that $\mathrm{m}_N(\Omega^*) = \mathrm{m}_N(D)$ ($P_h(\Gamma^*) \leq c$ for (16.10)), and $y \in H^1(\Omega^*)$ solutions of the problem*

$$\boxed{\inf_{\substack{\Omega \text{ open } \subset D, \Omega \in \mathcal{F}(D,h,\alpha) \\ P_h(\Gamma) \leq c, \, \mathrm{m}_N(\Omega) = \mathrm{m}_N(D)}} \; \inf_{\varphi \in H^1(\Omega)} \int_\Omega \varepsilon \, |\nabla \varphi|^2 + |\varphi - f|^2 \, dx,} \tag{16.10}$$

$$\boxed{\inf_{\substack{\Omega \text{ open } \subset D, \Omega \in \mathcal{F}(D,h,\alpha) \\ \mathrm{m}_N(\Omega) = \mathrm{m}_N(D)}} \; \inf_{\varphi \in H^1(\Omega)} \int_\Omega \varepsilon \, |\nabla \varphi|^2 + |\varphi - f|^2 \, dx + c \, P_h(\Gamma).} \tag{16.11}$$

Proof. Since the minimizing sequence $\{b_{\Omega_n}\}$ constructed in the proof of Theorem 16.1 strongly converges to b_{Ω^*} in $W^{1,p}(D)$ for all p, $1 \leq p < \infty$, from property (12.3) in Theorem 12.1, we have

$$P_h(\Gamma^*) \leq \liminf_{n \to \infty} P_h(\Gamma_n) \leq c$$

and the optimal Ω^* constructed in the proof of the theorem satisfies the additional constraint on the density perimeter. $\square$

Chapter 8

Shape Continuity and Optimization

1 Introduction and Generic Examples

The underlying philosophy behind Chapters 5, 6, and 7 was to introduce metric topologies on sets associated with families of functions parametrized by sets rather than parametrize sets by functions. In the case of the characteristic function we have seen several examples where it appears explicitly in the modeling of the problem. The distance functions provide other topologies and constructions. For instance the oriented distance function gives a direct analytic access to geometric entities such as the normal and the fundamental forms on the boundary of a geometric domain without ad hoc local bases or Christoffel symbols. Many of the advantages of working in a truly intrinsic framework for shape derivatives will be illustrated in Chapters 9 and 10.

Now that we have meaningful topologies on sets, we can consider the continuity of a geometric objective functional such as the volume, the perimeter, the mean curvature, etc. In this chapter we concentrate on continuity issues related to shape optimization problems under state equation constraints. A special family of state constrained problems are the ones for which the objective function is defined as an infimum over a family of functions over a fixed domain or set such as eigenvalue and compliance problems. What is nice about them is that no adjoint system is necessary to characterize the minimizing function. Other problems have the structure of *optimal control theory*, that is, a state equation depending on the control that describes the evolution of the state and an objective functional that depends on the state and the control. In that case, the characterization of the optimal control involves an adjoint state equation coupled with the state equation.

In the context of *shape and geometric optimization* the control will be the underlying geometry. The state equation will be a static or dynamical partial differential equation on the domain, and the objective functional will depend on the domain and the state that itself depends on the domain. As in *control theory*, we need continuity of the objective functional and the state with respect to the geometry and compactness of the family of domains over which the optimization takes place. Since it is not possible to give a complete general theory of shape optimization problems within this book, we choose to concentrate on a few simple generic examples that are illustrative of the underlying technicalities involved.

Several examples of optimization problems involving the L^p-topology on measurable sets via the characteristic function were given in Chapter 5. In this chapter, we first characterize the continuity of the transmission problem and the upper semicontinuity of the first eigenvalue of the generalized Laplacian with respect to the domain. We then study the continuity of the solution of the homogeneous Dirichlet and Neumann boundary value problems with respect to their underlying domain of definition since they require different constructions and topologies that are generic of the two types of boundary conditions even for more complex nonlinear partial differential equations. In the above problems the strong continuity of solutions of the elliptic equation or the eigenvector equation with respect to the underlying domain is the key element in the proof of the existence of optimal domains. To get that continuity, some extra conditions have to be imposed on the family of open domains.

Those issues have received a lot of attention for the Laplace equation with homogeneous Dirichlet boundary conditions. With a sequence of domains in a fixed holdall D associate a sequence of extensions by zero of the solutions in the fixed space $H_0^1(D)$. The Poincaré inequality is uniform for that sequence, as the first eigenvalue of the Laplace equation in each domain is dominated by the one associated with the larger holdall D. By a classical compactness argument, the sequence of extensions converges to some limiting element y in $H_0^1(D)$.

To complete the proof of the continuity, two more fundamental questions remain. Is y the solution of the Laplace equation for the limit domain? Does y satisfy the Dirichlet boundary condition on the boundary of the limit domain?

The first issue can be resolved by assuming that the Sobolev spaces associated with the moving domains converge in the Kuratowski sense, i.e., with the property that any element in the Sobolev space associated with the limit domain can be approached by a sequence of elements in the moving Sobolev spaces. That property is obtained in examples where the *compactivorous property*[1] follows from the choice of the definition of convergence of the domains. If the domains are open subsets of D, then the complementary Hausdorff topology has that property. The family of subsets of D satisfying a uniform fat segment property has that property.

For the second issue it is necessary to impose some constraints at least on the limit domain. The *stability condition* introduced by J. Rauch and M. Taylor [**1**] and used by E. N. Dancer [**1**] and D. Daners [**1**] precisely assumes that the limit domain is sufficiently smooth, just enough to have the solution in $H_0^1(\Omega)$. Of course, assuming such a regularity on the limit domain makes things easier but is a little bit artificial. The more fundamental issue is to identify the families of domains for which this stability property is preserved in the limit. The uniform cone property was used in the context of shape optimization by D. Chenais [**1**, **2**] in 1973. In 1994 D. Bucur and J.-P. Zolésio [**3**, **5**, **6**, **8**, **9**] showed that this is true under general *capacitary* conditions and have constructed new compact subfamilies

[1] A sequence of open sets converging to a limit open set in some topology has the compactivorous property if any compact subset of the limit set is contained in all sets of the sequence after a certain rank: the sequence *eats up* the compact set after a certain rank (cf. Theorem 2.4 (iii) in Chapter 6).

of domains with respect to the Hausdorff complementary topology. Furthermore D. Bucur [**1**] proved that the condition given in 1993 by V. Šverák [**1**, **2**] in dimension 2 involving a bound on the number of connected components of the complement of the domain can be recovered from the more general capacity conditions that are sufficient in the case of the Laplacian with homogeneous Dirichlet boundary conditions. In a more recent paper D. Bucur [**5**] proved that they are *almost necessary.* Intuitively, those capacity conditions are such that, locally, the complement of the domains in the family under consideration has *enough capacity* to preserve and retain the homogeneous Dirichlet boundary condition in the limit.

1.1 First Generic Example

The first example is the optimization of the first eigenvalue of an associated elliptic differential operator:

$$\left.\begin{array}{l}\displaystyle\sup_{\Omega\in\mathcal{A}(D)} \lambda^A(\Omega)\\[2ex] \displaystyle\inf_{\Omega\in\mathcal{A}(D)} \lambda^A(\Omega)\end{array}\right| \quad \lambda^A(\Omega) \stackrel{\text{def}}{=} \inf_{0\neq\varphi\in H_0^1(\Omega)} \frac{\int_\Omega A\nabla\varphi\cdot\nabla\varphi\,dx}{\int_\Omega |\varphi|^2\,dx}, \tag{1.1}$$

where $\mathcal{A}(D)$ is a family of admissible open subsets of D.

1.2 Second Generic Example

As a second generic example, consider the *transmission problem* over the fixed bounded open nonempty domain D associated with the Laplace equation and a measurable subset Ω of D: find $y = y(\chi_\Omega) \in H_0^1(D)$ such that

$$\forall\varphi \in H_0^1(D), \quad \int_D (k_1\chi_\Omega + k_2(1-\chi_\Omega))\,\nabla y\cdot\nabla\varphi\,dx = \langle f,\varphi\rangle \tag{1.2}$$

for some strictly positive constants k_1 and k_2, $\alpha = \max\{k_1, k_2\} > 0$, $f \in H^{-1}(D)$, $H^{-1}(D)$ the topological dual of $H_0^1(D)$, and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(D)$ and $H_0^1(D)$. This problem was studied in section 4 of Chapter 5 in the context of the compliance problem with two materials.

1.3 Third Generic Example

As a third example, consider the minimization of the objective function

$$J(\Omega) \stackrel{\text{def}}{=} \frac{1}{2}\int_\Omega |u_\Omega - g|^2\,dx \tag{1.3}$$

over the family of open subsets Ω of a bounded open holdall D of $\mathbf{R}^{\mathrm{N}}$, where g in $L^2(D)$, $u_\Omega \in H_0^1(\Omega)$ is the solution of the variational problem

$$\forall\varphi \in H_0^1(\Omega), \quad \int_\Omega A\nabla u_\Omega\cdot\nabla\varphi\,dx = \langle f|_\Omega,\varphi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}, \tag{1.4}$$

$A \in L^\infty(D;\mathcal{L}(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}}))$ is a matrix function on D such that ${}^*\!A = A$ and $\alpha I \leq A \leq \beta I$ for some coercivity and continuity constants $0 < \alpha \leq \beta$, and $f|_\Omega$ denotes the restriction of f in $H^{-1}(D)$ to $H^{-1}(\Omega)$.

1.4 Fourth Generic Example

As a fourth generic example, consider the previous example but with the homogeneous Dirichlet boundary value problem replaced by the *homogeneous Neumann boundary value problem of the mathematician* for the Laplacian

$$-\Delta y + y = f \text{ in } \Omega, \quad \frac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega. \tag{1.5}$$

The *classical* (physical) homogeneous Neumann boundary value problem

$$-\Delta y = f \text{ in } \Omega, \quad \frac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega \tag{1.6}$$

has a unique solution up to a constant if $\int_D f\, dx = 0$.

2 Upper Semicontinuity and Maximization of the First Eigenvalue

Given a bounded open nonempty domain $D \subset \mathbf{R}^N$, consider the minimization of the *Raleigh quotient* for an open nonempty subset Ω of D:

$$\lambda^A(\Omega) \overset{\text{def}}{=} \inf_{0\neq\varphi\in H_0^1(\Omega;D)} \frac{\int_D A\nabla\varphi\cdot\nabla\varphi\, dx}{\int_D |\varphi|^2\, dx} = \inf_{\substack{\varphi\in H_0^1(\Omega;D)\\ \|\varphi\|_{L^2}=1}} \int_D A\nabla\varphi\cdot\nabla\varphi\, dx, \tag{2.1}$$

where $A \in L^\infty(D;\mathcal{L}(\mathbf{R}^N,\mathbf{R}^N))$ is a matrix function on D such that

$${}^*A(x) = A(x) \quad \text{and} \quad \alpha I \leq A(x) \leq \beta I \tag{2.2}$$

for some coercivity and continuity constants $0 < \alpha \leq \beta$. We set $\lambda^A(\varnothing) = +\infty$. The minimizers are characterized as follows.

Theorem 2.1. (i) *Given a bounded open nonempty domain Ω in $\mathbf{R}^N$, there exist $u_\Omega \in H_0^1(\Omega; D)$, $u_\Omega \neq 0$, and $\lambda^A(\Omega) > 0$ such that*

$$\forall \varphi \in H_0^1(\Omega; D), \quad \int_\Omega A\nabla u_\Omega\cdot\nabla\varphi - \lambda^A(\Omega)\, u_\Omega\, \varphi\, dx = 0 \tag{2.3}$$

and

$$\forall \varphi \in H_0^1(\Omega; D), \quad \lambda^A(\Omega) = \frac{\int_D A\nabla u_\Omega\cdot\nabla u_\Omega\, dx}{\int_D |u_\Omega|^2\, dx} \leq \frac{\int_D A\nabla\varphi\cdot\nabla\varphi\, dx}{\int_D |\varphi|^2\, dx}. \tag{2.4}$$

Conversely, if there exist $u \in H_0^1(\Omega; D)$, $u \neq 0$, and λ such that

$$\forall \varphi \in H_0^1(\Omega; D), \quad \int_\Omega A\nabla u\cdot\nabla\varphi - \lambda\, u\, \varphi\, dx = 0 \tag{2.5}$$

and

$$\forall \varphi \in H_0^1(\Omega; D), \quad \lambda \leq \frac{\int_D A\nabla\varphi \cdot \nabla\varphi \, dx}{\int_D |\varphi|^2 \, dx}, \tag{2.6}$$

then

$$\lambda = \frac{\int_D A\nabla u \cdot \nabla u \, dx}{\int_D |u|^2 \, dx} = \inf_{\varphi \in H_0^1(\Omega;D)} \frac{\int_D A\nabla\varphi \cdot \nabla\varphi \, dx}{\int_D |\varphi|^2 \, dx},$$

u is a nonzero solution of the minimization problem (2.1)*, and λ is equal to the first eigenvalue $\lambda^A(\Omega)$ associated with Ω.*

(ii) *Given a bounded open nonempty domain D, there exists $\lambda_D^A > 0$ (that depends only on the diameter of D and A) such that for all open subsets Ω of D,*

$$\lambda^A(\Omega) \geq \lambda^A(D) \geq \lambda_D^A > 0,$$

where $\lambda^D(\Omega)$ is the first eigenvalue of the symmetrical linear operator $\varphi \mapsto -\mathrm{div}\,(A\,\nabla\varphi)$ on D.

Proof. (i) For any bounded open Ω, the infimum is bounded below by 0 and hence finite. Let $\{\varphi_n\}$ be a minimizing sequence such that $\|\varphi_n\|_{L^2(\Omega)} = 1$. By coercivity, $\alpha\,\|\nabla\varphi_n\|^2_{L^2(\Omega)}$ is bounded. Hence $\{\varphi_n\}$ is a bounded sequence in $H_0^1(\Omega)$ and there exist $\varphi \in H_0^1(\Omega)$ and a subsequence, still indexed by n, such that $\varphi_n \rightharpoonup \varphi$ in $H^1(\Omega)$-weak. By the Rellich–Kondrachov compactness theorem, the subsequence strongly converges in $L^2(\Omega)$. Then

$$1 = \int_\Omega \varphi_n^2 \, dx \to \int_\Omega \varphi^2 \, dx \quad \text{and} \quad \int_\Omega A\nabla\varphi \cdot \nabla\varphi \, dx \leq \liminf_{n\to\infty} \int_\Omega A\nabla\varphi_n \cdot \nabla\varphi_n \, dx$$
$$\Rightarrow \frac{\int_\Omega A\nabla\varphi \cdot \nabla\varphi \, dx}{\int_\Omega \varphi^2 \, dx} \leq \liminf_{n\to\infty} \frac{\int_\Omega A\nabla\varphi_n \cdot \nabla\varphi_n \, dx}{\int_\Omega \varphi_n^2 \, dx} = \lambda^A(\Omega).$$

By definition of $\lambda^A(\Omega)$, $0 \neq \varphi \in H_0^1(\Omega)$ is a minimizing element and inequality (2.4) is verified. Since $\varphi \neq 0$, the Rayleigh quotient is differentiable in φ and its directional derivative in the direction ψ is given by

$$2\frac{\int_\Omega A\nabla\varphi \cdot \nabla\psi \, dx}{\|\varphi\|^2_{L^2(\Omega)}} - 2\int_\Omega A\nabla\varphi \cdot \nabla\varphi \, dx \, \frac{\int_\Omega \varphi\,\psi \, dx}{\|\varphi\|^4_{L^2(\Omega)}}.$$

A nonzero solution φ of the minimization problem on Ω is necessarily a stationary point, and for all $\psi \in H_0^1(\Omega)$,

$$\int_\Omega A\nabla\varphi \cdot \nabla\psi \, dx = \frac{\int_\Omega A\nabla\varphi \cdot \nabla\varphi \, dx}{\|\varphi\|^2_{L^2(\Omega)}} \int_\Omega \varphi\,\psi \, dx = \lambda^A(\Omega) \int_\Omega \varphi\,\psi \, dx$$

and we get (2.3). Conversely, any λ and nonzero $u \in H_0^1(\Omega; D)$ that verify (2.5) yield $\lambda \geq \lambda^A(\Omega)$. But, from inequality (2.6), $\lambda \leq \lambda^A(\Omega)$ and $\lambda = \lambda^A(\Omega)$. Hence, u is a minimizer of the Rayleigh quotient over $H_0^1(\Omega; D)$.

(ii) Let $\operatorname{diam}(D)$ be the diameter of D and $x \in \mathbf{R}^N$ be a point such that $D \subset B_x = B(x, \operatorname{diam}(D))$. Therefore, from Theorem 2.3 in Chapter 2,

$$H_0^1(\Omega; B_x) \subset H_0^1(D; B_x) \subset H_0^1(B_x), \quad H_0^1(\Omega; D) \subset H_0^1(D)$$

with the associated isometries. Hence, by definition of the minimum of a functional over an increasing sequence of sets, $\lambda^A(\Omega) \geq \lambda^A(D) \geq \lambda^A(B_x)$. If $\lambda^A(B_x) = 0$, we repeat the above construction with $H_0^1(B_x)$ in place of $H_0^1(\Omega; B_x)$ and end up with an element $\varphi \in H_0^1(B_x)$ such that

$$\int_{B_x} \varphi^2 \, dx = 1 \quad \text{and} \quad \alpha \int_{B_x} |\nabla \varphi|^2 \, dx \leq \int_{B_x} A\nabla\varphi \cdot \varphi \, dx = 0,$$

which is impossible in $H_0^1(B_x)$. Finally $\lambda^A(B_x)$ is independent of the choice of x since the eigenvalue is invariant under a translation of the domain: it depends only on the diameter of D. The case $\lambda^A(\varnothing) = +\infty$ is compatible with the results. $\square$

Remark 2.1.
For A equal to the identity operator I, denote by $\lambda(\Omega)$, $\lambda(D)$, and λ_D the quantities $\lambda^A(\Omega)$, $\lambda^A(D)$, and λ_D^A of Theorem 2.1. For any open $\Omega \subset D$ and any $\varphi \in H_0^1(\Omega; D)$

$$\|\varphi\|_{L^2(D)} \leq \frac{1}{\sqrt{\lambda_D}} \|\nabla\varphi\|_{L^2(D)} \tag{2.7}$$

$$\Rightarrow \forall \Omega \text{ open } \subset D, \quad \|\varphi\|_{H^1(\Omega)} \leq \sqrt{1 + \frac{1}{\lambda_D}}\, \|\nabla\varphi\|_{L^2(\Omega)} \tag{2.8}$$

and the spaces $H_0^1(\Omega)$ and $H_0^1(D)$ will be endowed with the respective equivalent norms $\|\nabla\varphi\|_{L^2(\Omega)}$ and $\|\nabla\varphi\|_{L^2(D)}$. $\square$

Before considering the existence of minimizers/maximizers, we investigate the question of the continuity of the first eigenvalue $\lambda^A(\Omega)$ with respect to Ω for the uniform complementary Hausdorff topology on

$$C_d^c(D) \stackrel{\text{def}}{=} \left\{ d_{\complement\Omega} : \forall \Omega \text{ open subset in } D \text{ and } \Omega \neq \mathbf{R}^N \right\}$$

(cf. (2.11) of Definition 2.2 in Chapter 6).

Theorem 2.2. *Let D be a bounded open domain in $\mathbf{R}^N$ and A be a matrix function satisfying assumption* (2.2)*. The mapping*

$$d_{\complement\Omega} \mapsto \lambda^A(\Omega) : C_d^c(D) \subset C_0(D) \to \mathbf{R} \cup \{+\infty\} \tag{2.9}$$

is upper semicontinuous.

Proof. Since $\lambda^A(\Omega)$ is finite in $\Omega \neq \varnothing$ and $+\infty$ in $\Omega = \varnothing$, it is upper semicontinuous in $\varnothing$. For $\Omega \neq \varnothing$, consider a sequence $\{\Omega_n\}$ of open subsets of D such that $d_{\complement\Omega_n} \to d_{\complement\Omega}$ in $C(\overline{D})$ and the associated sequence $\{\lambda^A(\Omega_n)\}$. We want to show that

$$\limsup \lambda^A(\Omega_n) \geq \lambda^A(\Omega).$$

Consider the new sequence $\{\Omega_n \cap \Omega\}$. Since $\Omega \subset D$, for all n, $\complement(\Omega_n \cap \Omega) = \complement\Omega_n \cup \complement\Omega \supset \complement D \neq \varnothing$ and

$$d_{\complement(\Omega_n\cap\Omega)} = d_{\complement\Omega_n \cup \complement\Omega} = \min\{d_{\complement\Omega_n}, d_{\complement\Omega}\} \to d_{\complement\Omega} \text{ in } C(\overline{D}).$$

Let $u_n \in H_0^1(\Omega_n \cap \Omega)$, $\|u_n\|_{L^2(D)} = 1$, be an eigenvector for the eigenvalue $\lambda_n = \lambda^A(\Omega_n \cap \Omega)$. From Theorem 2.1 u_n is solution of the variational equation (2.3):

$$\forall \varphi_n \in H_0^1(\Omega_n; D), \quad \int_D A\nabla u_n \cdot \nabla \varphi_n \, dx = \lambda_n \int_D u_n \, \varphi_n \, dx. \tag{2.10}$$

Since $\Omega \neq \varnothing$, there exists a compact ball $K_\rho = \overline{B(x,\rho)} \subset \Omega$ for some $x \in \Omega$ and $\rho > 0$. By the *compactivorous property* of Theorem 2.4 in section 2.3 of Chapter 6, there exists an integer $N(K_\rho)$ such that for all $n \geq N(K_\rho)$, $K_\rho \subset \Omega_n \cap \Omega$ and by Theorem 2.1 (ii), $0 < \lambda_n \leq \lambda^A(B(x,\rho))$. Therefore, the sequence of eigenvalues $\{\lambda_n\}$ is bounded. By coercivity

$$\alpha \|\nabla u_n\}^2_{L^2(D)} \leq \lambda_n \, \|\nabla u_n\}^2_{L^2(D)} = \lambda_n$$

and $\{u_n\}$ is bounded in $H_0^1(\Omega; D)$ since $\Omega_n \cap \Omega \subset \Omega$. Letting $\lambda^* = \limsup \lambda_n$, there exist subsequences, still indexed by n, and $u \in H_0^1(\Omega; D)$ such that

$$\lambda_n \to \lambda^*, \quad \|u_n\|_{L^2(D)} = 1, \quad u_n \rightharpoonup u \text{ in } H_0^1(\Omega; D)\text{-weak} \quad \Rightarrow \|u\|_{L^2(D)} = 1.$$

Since for all n, $\lambda^A(\Omega_n) \leq \lambda^A(\Omega_n \cap \Omega)$, it is readily seen that

$$\limsup \lambda^A(\Omega_n) \leq \limsup \lambda^A(\Omega_n \cap \Omega) = \lambda^*$$

and that it is sufficient to show that $\lambda^* = \lambda^A(\Omega)$ to get the upper semicontinuity. We cannot directly go to the limit in (2.10), since the test functions φ_n depend on Ω_n. To get around this difficulty, recall the density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega; D)$ and use the *compactivorous property* a second time. For any $\varphi \in \mathcal{D}(\Omega)$, $K = \operatorname{supp} \varphi \subset \Omega$ is compact and there exists $N(K)$ such that for all $n \geq N(K)$, $\operatorname{supp} \varphi \subset \Omega_n \cap \Omega$. Therefore, for all $n \geq N(K)$

$$u_n \in H_0^1(\Omega_n \cap \Omega; D) \subset H_0^1(\Omega; D), \quad \int_D A\nabla u_n \cdot \nabla \varphi \, dx = \lambda_n \int_D u_n \, \varphi \, dx.$$

By letting $n \to \infty$

$$\forall \varphi \in \mathcal{D}(\Omega), \quad \int_D A\nabla u \cdot \nabla \varphi \, dx = \lambda^* \int_D u \, \varphi \, dx,$$

and, by density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega; D)$, the variational equation is verified in $H_0^1(\Omega; D)$. Hence, $u \in H_0^1(\Omega; D)$, $\|u\|_{L^2(\Omega)} = 1$, satisfies the variational equation

$$\forall \varphi \in H_0^1(\Omega; D), \quad \int_D A\nabla u \cdot \nabla \varphi \, dx = \lambda^* \int_D u \, \varphi \, dx.$$

Therefore, since $\lambda^A(\Omega)$ is the smallest eigenvalue with respect to $H_0^1(\Omega; D)$, $\lambda^A(\Omega) \leq \lambda^*$. To show that $\lambda^* = \lambda^A(\Omega)$, it remains to prove that $\lambda^* \leq \lambda^A(\Omega)$. For each n

$$\forall \varphi_n \in H_0^1(\Omega_n \cap \Omega; D), \quad \lambda^A(\Omega_n \cap \Omega) \leq \frac{\int_D A\nabla\varphi_n \cdot \nabla\varphi_n \, dx}{\|\varphi_n\|^2_{L^2(D)}}.$$

By the compactivorous property for each $\varphi \in \mathcal{D}(\Omega)$, $\varphi \neq 0$, there exists N such that $\operatorname{supp} \varphi \subset \Omega \cap \Omega_n$ for all $n \geq N$. Therefore for all $n \geq N$

$$\lambda^A(\Omega_n \cap \Omega) \leq \frac{\int_D A\nabla\varphi \cdot \nabla\varphi \, dx}{\|\varphi\|^2_{L^2(D)}}$$

and by taking the lim sup

$$\forall \varphi \in \mathcal{D}(\Omega), \quad \lambda^* \leq \frac{\int_D A\nabla\varphi \cdot \nabla\varphi \, dx}{\|\varphi\|^2_{L^2(D}}.$$

By density this is true for all $\varphi \in H_0^1(\Omega; D)$, $\varphi \neq 0$, and $\lambda^* \leq \lambda^A(\Omega)$. As a result $\lambda^* = \lambda^A(\Omega)$. Finally,

$$\lambda^A(\Omega_n) \leq \lambda^A(\Omega_n \cap \Omega) \quad \Rightarrow \limsup \lambda^A(\Omega_n) \leq \limsup \lambda^A(\Omega_n \cap \Omega) = \lambda^* = \lambda^A(\Omega),$$

and we have the upper semicontinuity of the map $d_{\complement\Omega} \mapsto \lambda^A(\Omega)$. □

In view of the previous theorem, we now deal with the maximization of the first eigenvalue $\lambda^A(\Omega)$ associated with the operator $\mathcal{A}$ with respect to a family of open subsets Ω of D.

Theorem 2.3. *Let D be a bounded open domain in $\mathbf{R}^{\mathrm{N}}$ and A be a matrix function satisfying assumption* (2.2).

(i) *Then*

$$\sup_{\Omega \in C_d^c(D)} \lambda^A(\Omega) = \lambda^A(\varnothing) = +\infty. \tag{2.11}$$

The conclusion remains true when $C_d^c(D)$ is replaced by

$$\mathcal{C}_d^c(D) = \{d_{\complement\Omega} : \Omega \textit{ convex and open } \subset D\} \tag{2.12}$$

(cf. Theorem 8.6 *in Chapter* 6*).*

(ii) *Given an open set E, $\varnothing \neq E \subset \mathbf{R}^{\mathrm{N}}$, consider the compact family*

$$C_{d,\mathrm{loc}}^c(E; D) = \left\{ d_{\complement\Omega} : \begin{array}{l} \exists x \in \mathbf{R}^{\mathrm{N}}, \exists A \in \mathrm{O(N)} \textit{ such that} \\ x + AE \subset \Omega \textit{ open} \subset D \end{array} \right\}$$

of Theorem 2.6 *in Chapter* 6*. Then, there exists an open Ω^* such that $d_{\complement\Omega^*} \in C_{d,\mathrm{loc}}^c(E; D)$ and*

$$\lambda^A(\Omega^*) = \sup_{\Omega \in C_{d,\mathrm{loc}}^c(E;D)} \lambda^A(\Omega) \leq \beta\, \lambda^I(E). \tag{2.13}$$

The conclusion remains true when $C^c_{d,\mathrm{loc}}(E;D)$ is replaced by $C^c_d(E;D)$ or the subfamilies of convex subsets $\mathcal{C}^c_d(E;D)$ and $\mathcal{C}^c_{d,\mathrm{loc}}(E;D)$ of Theorem 8.6 *in Chapter* 6.

(iii) *There exists $\Omega^* \in L(D,r,\mathcal{O},\lambda)$ such that*

$$\lambda^A(\Omega^*) = \sup_{\Omega \in L(D,r,\mathcal{O},\lambda)} \lambda(\Omega) \leq \beta \lambda^I(\mathcal{O}). \tag{2.14}$$

Proof. (i) From Theorem 2.2 and the compactness of $C^c_d(D)$ in $C(\overline{D})$.

(ii) All the families are compact by Theorems 2.6 and 8.6 in Chapter 6. So there is a maximizer $d_{\complement\Omega^*}$ of $\lambda^A(\Omega)$. The upper bound is obvious.

(iii) By compactness of $L(D,r,\mathcal{O},\lambda)$ in $W^{1,p}(D)$ and hence in $C^c_d(D)$ for the $C(\overline{D})$-topology. □

3 Continuity of the Transmission Problem

As a second generic example, consider the *transmission problem* over the fixed bounded open nonempty domain D associated with the Laplace equation and a measurable subset Ω of D: find $y = y(\chi_\Omega) \in H^1_0(D)$ such that

$$\forall \varphi \in H^1_0(D), \quad \int_D (k_1 \chi_\Omega + k_2(1-\chi_\Omega))\, \nabla y \cdot \nabla \varphi \, dx = \langle f, \varphi \rangle \tag{3.1}$$

for some strictly positive constants k_1 and k_2, $\alpha = \max\{k_1, k_2\} > 0$, $f \in H^{-1}(D)$, $H^{-1}(D)$ the topological dual of $H^1_0(D)$, and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(D)$ and $H^1_0(D)$. This problem was studied in section 4 of Chapter 5 in the context of the compliance problem with two materials.

Theorem 3.1. *Let D be a bounded open domain in $\mathbf{R}^N$ and k_1 and k_2 be some strictly positive constants such that $\alpha = \max\{k_1, k_2\} > 0$ and $f \in H^{-1}(D)$. Denote by $y(\chi)$ the solution of the transmission problem* (3.1) *associated with $\chi \in \mathrm{X}(D)$. The mapping*

$$\chi \mapsto y(\chi) : \mathrm{X}(D) \to H^1_0(D)$$

is continuous when $\mathrm{X}(D)$ is endowed with the strong $L^p(D)$-topology, $1 \leq p < \infty$. The same result is true with $\overline{\mathrm{co}}\, \mathrm{X}(D)$ in place of $\mathrm{X}(D)$.

Remark 3.1.
This theorem applies to the compact families of sets of Theorems 13.1, 14.1, and 14.2 in section 13 of Chapter 7 under the uniform fat segment property and for the conditions on the local graph functions of Theorem 13.2 (cf. Theorem 5.1 of Chapter 2) since the convergence of $b_{\Omega_n} \to b_\Omega$ in $W^{1,p}(D)$ implies the convergence of $\chi_{\Omega_n} \to \chi_\Omega$ in $L^p(D)$. It also applies to the compact family of convex sets $\mathcal{C}(D) = \{\chi_\Omega : \Omega \text{ convex subset of } D\}$ introduced in (3.24) in section 3.4 of Chapter 5 (cf. Theorem 3.5 (iii) and Corollary 1 to Theorem 6.3 in section 6.1 of Chapter 5). □

Proof of Theorem 3.1. The proof follows the arguments of J.-P. Zolésio [**2**], [**24**, sect. 4.2.1, pp. 446–448]. It is sufficient to prove that for any convergent sequence $\chi_n \to \chi$ in $L^2(D)$-strong, the corresponding solutions $y_n = y(\chi_n) \in H_0^1(D)$ converge to $y = y(\chi)$ in $H_0^1(D)$-strong. First, the sequence $\{y_n\}$ is bounded in $H_0^1(D)$:

$$\begin{aligned}\alpha\|\nabla y_n\|^2_{L^2(D)} &\le \int_D (k_1\chi_n + k_2(1-\chi_n))\,\nabla y_n\cdot\nabla y_n\,dx\\ &= \langle f, y_n\rangle \le \|f\|_{H^{-1}(D)}\,\|y_n\|_{H_0^1(D)}\end{aligned}$$

$$\Rightarrow\ \alpha\|\nabla y_n\|^2_{L^2(D)} \le \|f\|_{H^{-1}(D)}\,\|y_n\|_{H_0^1(D)} \le \|f\|_{H^{-1}(D)}\sqrt{1+\frac{1}{\lambda_D}}\,\|\nabla y_n\|_{L^2(\Omega)}$$

since we have the equivalence of norms from (2.7) in Remark 2.1. So there is a subsequence, still denoted by $\{y_n\}$, that weakly converges to some $y \in H_0^1(D)$. To show that y is solution of (3.1) corresponding to χ, we go to the limit in

$$\forall\varphi\in H_0^1(D),\quad \int_D (k_1\chi_n + k_2(1-\chi_n))\,\nabla\varphi\cdot\nabla y_n\,dx = \langle f,\varphi\rangle.$$

Since $(k_1\chi_n + k_2 1-\chi_n))\nabla\varphi \to (k_1\chi + k_2(1-\chi))\nabla\varphi$ in $L^2(D)$-strong and $\nabla y_n \to \nabla y$ in $L^2(D)$-weak, we get that $y \in H_0^1(D)$ is the unique solution of

$$\forall\varphi\in H_0^1(D),\quad \int_D (k_1\chi + k_2(1-\chi))\,\nabla y\cdot\nabla\varphi\,dx = \langle f,\varphi\rangle.$$

Finally, we prove the strong convergence. Indeed

$$\begin{aligned}\alpha\,\|\nabla(y_n-y)\|^2_{L^2(D)} \le& \int_D (k_1\chi_n + k_2(1-\chi_n))\,\nabla(y_n-y)\cdot\nabla(y_n-y)\,dx\\ \le& \int_D (k_1\chi_n + k_2(1-\chi_n))\,\nabla y_n\cdot\nabla y_n\,dx\\ &- 2\int_D (k_1\chi_n + k_2(1-\chi_n))\,\nabla y_n\cdot\nabla y\,dx\\ &+ \int_D (k_1\chi_n + k_2(1-\chi_n))\,\nabla y\cdot\nabla y\,dx\\ =&\langle f,y_n\rangle - 2\,\langle f,y\rangle + \int_D (k_1\chi_n + k_2(1-\chi_n))\,\nabla y\cdot\nabla y\,dx\\ &\to -\langle f,y\rangle + \int_D (k_1\chi + k_2(1-\chi))\,\nabla y\cdot\nabla y\,dx = 0.\end{aligned}$$

Since every strongly convergent subsequence converges to the same limit, the whole sequence strongly converges to y in $H_0^1(D)$. □

4 Continuity of the Homogeneous Dirichlet Boundary Value Problem

4.1 Classical, Relaxed, and Overrelaxed Problems

We have seen in section 3.5 of Chapter 5 two relaxations of the homogeneous Dirichlet boundary value problem for the Laplacian to measurable subsets Ω of a bounded open holdall D by introducing the following closed subspaces of $H_0^1(D)$:

$$H^1_\diamond(\Omega; D) \overset{\text{def}}{=} \left\{\varphi \in H_0^1(D) : (1-\chi_\Omega)\varphi = 0 \text{ a.e. in } D\right\}, \tag{4.1}$$

$$H^1_\bullet(\Omega; D) \overset{\text{def}}{=} \left\{\varphi \in H_0^1(D) : (1-\chi_\Omega)\nabla\varphi = 0 \text{ a.e. in } D\right\} \tag{4.2}$$

(cf. (3.27) and (3.28) in Theorem 3.6 of Chapter 5). It is readily seen that $H^1_\diamond(\Omega; D) \subset H^1_\bullet(\Omega; D)$. For Ω open, $H^1_0(\Omega; D) \subset H^1_\diamond(\Omega; D) \subset H^1_\bullet(\Omega; D)$ and the three closed subspaces of $H_0^1(D)$ are generally not equal as can be seen from Examples 3.3, 3.4, and 3.5 in section 3.5 of Chapter 5.

The choice of a relaxation is very much problem dependent and should be guided by the proper modeling of the underlying physical or technological phenomenon at hand. In general, there is a price to pay for the above relaxations. Both $H^1_\diamond(\Omega; D)$ and $H^1_\bullet(\Omega; D)$ depend on the equivalence class of measurable sets $[\Omega]$ and not specifically on Ω. In general, there is not even an *open representative* of Ω in the class. From section 3.3 in Chapter 5, $[\Omega] = [I]$, where I is the *measure theoretic interior* of Ω,

$$\mathrm{I} = \left\{x \in \mathbf{R}^{\mathrm{N}} : \lim_{r\searrow 0} \frac{\mathrm{m}(B(x,r)\cap\Omega)}{\mathrm{m}(B(x,r))} = 1\right\},$$

$\operatorname{int} I = \Omega_1$, where

$$\Omega_1 = \left\{x \in \mathbf{R}^{\mathrm{N}} : \exists \rho > 0 \text{ such that } \mathrm{m}\big(\Omega \cap B(x,\rho)\big) = \mathrm{m}\big(B(x,\rho)\big)\right\}$$

is open, and $\operatorname{int}\Omega \subset \operatorname{int} I$ and $\partial I \subset \partial\Omega$ (cf. Definition 3.3 and Theorem 3.3 in Chapter 5). In view of the *cleaning operation* in the definition of I, all cracks of zero measure will be deleted or not seen by the functions of the spaces $H^1_\diamond(\Omega; D)$ and $H^1_\bullet(\Omega; D)$. When $\mathrm{m}_N(\partial\Omega) = 0$, $[\operatorname{int}\mathrm{I}] = [\Omega]$ and $\operatorname{int}\mathrm{I} = \Omega_1$ is an open representative in the class $[\Omega]$.

Let D be a bounded open subset of $\mathbf{R}^{\mathrm{N}}$ and let $f \in H^{-1}(D)$, where $H^{-1}(D)$ is the topological dual of $H_0^1(D)$. For all open subsets Ω of D

$$\varphi \mapsto \langle f, \varphi\rangle_{H^{-1}(D)\times H_0^1(D)} : \mathcal{D}(\Omega) \to \mathbf{R} \tag{4.3}$$

is well-defined and continuous with respect to the $H^1(\Omega)$-topology and extends to an element $f|_\Omega \in H^{-1}(\Omega)$. The map

$$\varphi \mapsto \langle f, \varphi\rangle_{H^{-1}(D)\times H_0^1(D)} : H^1_\diamond(\Omega; D) \to \mathbf{R}$$

is also well-defined and continuous with respect to the closed subspace $H^1_\diamond(\Omega; D)$ of $H_0^1(D)$ and defines an element $f|_\Omega^\diamond$ of the dual $H^1_\diamond(\Omega; D)'$ of $H^1_\diamond(\Omega; D)$. Similarly, the map

$$\varphi \mapsto \langle f, \varphi\rangle_{H^{-1}(D)\times H_0^1(D)} : H^1_\bullet(\Omega; D) \to \mathbf{R}$$

is also well-defined and continuous with respect to the closed subspace $H^1_\bullet(\Omega; D)$ of $H_0^1(D)$ and defines an element $f|_\Omega^\bullet$ of the dual $H^1_\bullet(\Omega; D)'$ of $H^1_\bullet(\Omega; D)$.

Let $A \in L^\infty(D; \mathcal{L}(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}))$ be a symmetrical matrix function on D such that

$${}^*\!A = A \quad \text{and} \quad \alpha I \le A \le \beta I \tag{4.4}$$

for some coercivity and continuity constants $0 < \alpha \le \beta$. Consider the *classical* homogeneous Dirichlet boundary value problem for the generalized Laplacian

$$\begin{aligned}&\exists y(\Omega) \in H_0^1(\Omega; D) \text{ such that,}\\ &\forall \varphi \in H_0^1(\Omega; D), \quad \int_D A\nabla y(\Omega) \cdot \nabla \varphi \, dx = \langle f|_\Omega, \varphi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)},\end{aligned} \tag{4.5}$$

the *relaxed* homogeneous Dirichlet boundary value problem

$$\begin{aligned}&\exists y(\chi_\Omega) \in H_\diamond^1(\Omega; D) \text{ such that,}\\ &\forall \varphi \in H_\diamond^1(\Omega; D), \quad \int_D A\nabla y(\chi_\Omega) \cdot \nabla \varphi \, dx = \langle f|_\Omega^\diamond, \varphi\rangle_{H_\diamond^1(\Omega;D)'\times H_\diamond^1(\Omega;D)},\end{aligned} \tag{4.6}$$

and the *overrelaxed* homogeneous Dirichlet boundary value problem

$$\begin{aligned}&\exists y(\chi_\Omega) \in H_\bullet^1(\Omega; D) \text{ such that,}\\ &\forall \varphi \in H_\bullet^1(\Omega; D), \quad \int_D A\nabla y(\chi_\Omega) \cdot \nabla \varphi \, dx = \langle f|_\Omega^\bullet, \varphi\rangle_{H_\bullet^1(\Omega;D)'\times H_\bullet^1(\Omega;D)}.\end{aligned} \tag{4.7}$$

By convention, set $y(\varnothing) = 0$ and $y(\chi_\varnothing) = 0$ as elements of $H_0^1(D)$. In addition we shall keep the terminology *homogeneous Dirichlet boundary value problem* for what we just called the *classical* homogeneous Dirichlet boundary value problem for the generalized Laplacian.

Since, in general, $H_0^1(\Omega; D)$ is a distinct closed subspace of $H_\diamond^1(\Omega; D)$, a solution of (4.6) is not necessarily a solution of (4.5). In the other direction, a solution $y = y(\Omega) \in H_0^1(\Omega; D)$ of (4.5) belongs to $H_\diamond^1(\Omega; D)$ and is a solution of

$$\begin{aligned}&\exists y \in H_\diamond^1(\Omega; D) \text{ such that,}\\ &\forall \varphi \in H_0^1(\Omega; D), \quad \int_D A\nabla y \cdot \nabla \varphi \, dx = \langle f|_\Omega^\diamond, \varphi\rangle_{H_\diamond^1(\Omega;D)'\times H_\diamond^1(\Omega;D)},\end{aligned} \tag{4.8}$$

but y is not necessarily unique in $H_\diamond^1(\Omega; D)$ since this variational equation is verified only on the closed subspace $H_0^1(\Omega; D)$ of $H_\diamond^1(\Omega; D)$. However, the space $H_\diamond^1(\Omega; D)$ is very close to the space $H_0^1(\Omega; D)$ and the two spaces will coincide on smooth sets and more generally on *crack-free sets* that will be introduced in section 7 (cf. Definition 7.1 (ii) and Theorem 7.3 (ii)). The following terminology has been introduced by J. Rauch and M. Taylor [**1**].

Definition 4.1.
An open subset Ω of D is said to be *stable* with respect to D if $H_0^1(\Omega; D) = H_\diamond^1(\Omega; D)$. □

The same considerations apply to the overrelaxed problem (4.7) in $H_\bullet^1(\Omega; D)$ with the additional observation that even for sets with a very smooth boundary, the solution does not coincide with the one of the (classical) Dirichlet problem when the $\complement\Omega$ has more than one connected component as seen from Examples 3.3, 3.4, and 3.5 in section 3.5 of Chapter 5. The problem in $H_\bullet^1(\Omega; D)$ is associated with different physical phenomena.

4.2 Classical Dirichlet Boundary Value Problem

We have the following general results.

Theorem 4.1. *Let D be a bounded open nonempty subset in $\mathbf{R}^N$. Associate with each open subset Ω of D the solution $y(\Omega) \in H_0^1(\Omega; D)$ of the classical Dirichlet problem* (4.5) *on Ω.*

(i) *Let Ω and the sequence $\{\Omega_n\}$ be open subsets of D such that $d_{\complement\Omega_n} \to d_{\complement\Omega}$ in $C(\overline{D})$. Then*

$$y(\Omega_n \cap \Omega) \to y(\Omega) \textit{ in } H_0^1(D)\text{-strong} \tag{4.9}$$

and there exist a subsequence $\{\Omega_{n_k}\}$ and $y_0 \in H_\diamond^1(\Omega; D)$ such that

$$y(\Omega_{n_k}) \rightharpoonup y_0 \textit{ in } H_0^1(D)\text{-weak} \tag{4.10}$$

and y_0 is a (not necessarily unique) solution of (4.8).

(ii) *The function*

$$d_{\complement\Omega} \mapsto y(\Omega) : C_d^c(D) \to H_0^1(D)\text{-strong} \tag{4.11}$$

is continuous for each Ω such that $H_0^1(\Omega; D) = H_\diamond^1(\Omega; D)$.

Remark 4.1.
We have the additional information that $y_0 \in H_\diamond^1(\Omega; D) \subset H_0^1(\operatorname{int}\overline{\Omega}; D)$ in part (i) of the theorem (cf. Theorem 7.3 (i)). □

Remark 4.2.
The *projection operator*

$$P_\Omega : H_0^1(D) \to H_0^1(\Omega; D) \tag{4.12}$$

is defined for $u \in H_0^1(D)$ as the solution of the variational equation

$$\exists P_\Omega u \in H_0^1(\Omega; D),\ \forall \varphi \in H_0^1(\Omega; D), \qquad \int_D \nabla(P_\Omega u) \cdot \nabla\varphi\, dx = \int_D \nabla u \cdot \nabla\varphi\, dx.$$

Applying the theorem to a sequence of open subsets $\{\Omega_n\}$ of D converging to Ω in the complementary Hausdorff topology with

$$\langle f, \varphi \rangle \overset{\text{def}}{=} \int_D \nabla u \cdot \nabla\varphi\, dx,$$

there exist $u_0 \in H_\diamond^1(\Omega; D)$ and a subsequence $\{\Omega_{n_k}\}$ such that

$$d_{\complement\Omega_n} \to d_{\complement\Omega} \text{ in } C(\overline{D}), \quad P_{\Omega_{n_k}} u \rightharpoonup u_0 \text{ in } H_0^1(D)\text{-weak}.$$

The projection operator P_Ω is continuous at Ω if $H_0^1(\Omega; D) = H_\diamond^1(\Omega; D)$, but this is only a *sufficient condition.* □

Proof. (i) The proof of the continuity of (4.9) follows the same steps as the proof of the upper semicontinuity of the first eigenvalue of the Laplacian in Theorem 2.2. As for the convergence (4.10), it is readily seen that the sequence $\{y_n = y(\Omega_n)\}$ of elements of $H_0^1(\Omega_n; D)$ is bounded in the bigger space $H_0^1(D)$: by coercivity

$$\alpha \|\nabla y_n\|_{L^2(D)}^2 \le \|f\|_{H^{-1}(D)} \|y_n\|_{H_0^1(D)} \le \sqrt{1+\frac{1}{\lambda_D}} \|f\|_{H^{-1}(D)} \|\nabla y_n\|_{L^2(D)}$$

from Remark 2.1, where $\lambda_D > 0$ is the first eigenvalue of the Laplacian on D. There exist a subsequence, still indexed by n, and some y_0 in $H_0^1(D)$ such that $y_n \rightharpoonup y_0$ in $H_0^1(D)$-weak and hence $y_n \to y_0$ in $L^2(D)$-strong. From Corollary 1 to Theorem 4.4 in Chapter 6

$$\begin{aligned} &\forall x \in \mathbf{R}^{\mathrm{N}}, \quad \liminf_{n\to\infty} \chi_{\Omega_n}(x) \ge \chi_\Omega(x) \\ \Rightarrow \forall \text{ a.a. } x \in \mathbf{R}^{\mathrm{N}}, &\quad 0 = \liminf_{n\to\infty} (1-\chi_{\Omega_n}(x))|y_n(x)|^2 \ge (1-\chi_\Omega(x))|y_0(x)|^2 \end{aligned}$$

since $(1-\chi_{\Omega_n}(x)) \ge 0$, $|y_n|^2 \ge 0$, and $|y_n|^2 \to 0$ a.e. in D. By definition, $y_0 \in H_\diamond^1(\Omega; D)$. Finally, by the *compactivorous property* of Theorem 2.4 in section 2.3 of Chapter 6, for each $\varphi \in \mathcal{D}(\Omega)$ there exists N such that for all $n > N$, $\operatorname{supp} \varphi \subset \Omega_n$. Therefore, by substituting φ in (4.5) on Ω_n, for all $n > N$

$$\int_D A\nabla y_n \cdot \nabla \varphi \, dx = \langle f|_{\Omega_n}, \varphi \rangle_{H^{-1}(\Omega_n)\times H_0^1(\Omega_n)} = \langle f|_\Omega, \varphi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \tag{4.13}$$

and by going to the limit

$$\forall \varphi \in \mathcal{D}(\Omega), \quad \int_D A\nabla y_0 \cdot \nabla\varphi \, dx = \langle f|_\Omega, \varphi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \tag{4.14}$$

and, by density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega; D)$, $y_0 \in H_\diamond^1(\Omega; D)$ is a solution of (4.8).

(ii) If, in addition, $H_\diamond^1(\Omega; D) = H_0^1(\Omega; D)$, then $y_0 \in H_0^1(\Omega; D)$ is the unique solution of (4.5). Hence $y_0 = y(\Omega)$, solution of the classic problem (4.5), is independent of the choice of the converging subsequence. Therefore the whole sequence $\{y_n\}$ converges to $y_0 = y(\Omega)$ in $H_0^1(D)$-weak.

The strong continuity in $H_0^1(D)$ now follows from the convergence of $u_n \to u_0$ in $L^2(D)$-strong:

$$\begin{aligned} &\alpha \|\nabla(u_n - u_0)\|_{L^2(D)}^2 \\ &\le \int_D A\nabla(u_n-u_0)\cdot\nabla(u_n-u_0)\,dx \\ &= \int_D A\nabla u_n \cdot \nabla u_n \, dx + \int_D A\nabla u_0\cdot\nabla u_0\,dx - 2\int_D A\nabla u_n\cdot\nabla u_0\,dx \\ &= \langle f, u_n\rangle + \langle f, u_0\rangle - 2\int_D A\nabla u_n\cdot\nabla u_0\,dx \\ &\to 2\,\langle f, u_0\rangle - 2\int_D A\nabla u_0\cdot\nabla u_0\,dx = 0 \end{aligned}$$

and the whole sequence $\{y_n\}$ converges to $y_0 = y(\Omega)$ in $H_0^1(D)$-strong. Therefore, the function (4.11) is continuous in Ω. $\square$

4.3 Overrelaxed Dirichlet Boundary Value Problem

In this section we concentrate on the overrelaxation (4.7) of the homogeneous Dirichlet boundary value problem. To show the continuity with respect to the characteristic function of domains, we first approximate its solution by transmission problems.

4.3.1 Approximation by Transmission Problems

The overrelaxed Dirichlet problem (4.7) corresponding to $\chi = \chi_\Omega \in X(D)$ is first approximated by a transmission problem over D indexed by $\varepsilon > 0$ going to zero:

$$\begin{aligned}
&\text{find } y_\varepsilon = y_\varepsilon(\chi) \in H_0^1(D) \text{ such that } \forall \varphi \in H_0^1(D) \\
&\int_D \left[\chi + \frac{1}{\varepsilon}(1-\chi)\right] \nabla y_\varepsilon \cdot \nabla \varphi \, dx = \int_D \chi f \varphi \, dx.
\end{aligned} \tag{4.15}$$

They are special cases of the general transmission problem (3.1) over the fixed bounded open nonempty domain D associated with the Laplace equation and a measurable subset Ω of D for $k_1 = 1$ and $k_2 = 1/\varepsilon$ and $\chi = \chi_\Omega$.

Theorem 4.2. *Let D be a bounded open nonempty subset in $\mathbf{R}^N$ and Ω be a measurable subset of D. Let $y(\chi_\Omega) \in H^1_\bullet(\Omega; D)$ be the solution of the overrelaxed Dirichlet problem (4.7). For ε, $0 < \varepsilon < 1$, let y_ε be the solution of problem (4.15) in $H_0^1(D)$. Then*

$$y(\chi_\Omega) = \lim_{\varepsilon \to 0} y_\varepsilon \ \textit{in } H_0^1(D)\text{-strong.}$$

Proof. For ε, $0 < \varepsilon < 1$, substitute $\varphi = y_\varepsilon$ in (4.15) to get the bounds

$$\|\nabla y_\varepsilon\|_{L^2(D)} \le c(D)\, \|f\|_{L^2(D)}, \quad \|(1-\chi_\Omega)\, \nabla y_\varepsilon\|_{L^2(D)} \le \sqrt{\varepsilon}\, c(D)\, \|f\|_{L^2(D)}.$$

There exist a sequence $\varepsilon_n \searrow 0$ and $y_0 \in H_0^1(D)$ such that $y_n = y_{\varepsilon_n} \rightharpoonup y_0$ in $H_0^1(D)$-weak and $(1-\chi_\Omega)\, \nabla y_n \to 0$ in $L^2(D)$-strong. Therefore, $(1-\chi_\Omega)\, \nabla y_0 = 0$ and $y_0 \in H^1_\bullet(\Omega; D)$. For $\varphi \in H^1_\bullet(\Omega; D)$, (4.15) reduces to

$$\forall \varphi \in H^1_\bullet(\Omega; D), \quad \int_D \nabla y_\varepsilon \cdot \nabla \varphi \, dx = \int_D \chi_\Omega f \varphi \, dx$$

and as $n \to \infty$

$$\forall \varphi \in H^1_\bullet(\Omega; D), \quad \int_D \nabla y_0 \cdot \nabla \varphi \, dx = \int_D \chi_\Omega f \varphi \, dx.$$

Hence $y_0 \in H^1_\bullet(\Omega; D)$ is the solution of (4.7). Since the limit is the same for all convergent sequences, $y_\varepsilon \to y_0$ in $H_0^1(D)$-weak. By subtracting the last two equations

$$\forall \varphi \in H^1_\bullet(\Omega; D), \quad \int_D [\nabla y_\varepsilon - \nabla y_0] \cdot \nabla \varphi \, dx = 0 \quad \Rightarrow \int_D [\nabla y_\varepsilon - \nabla y_0] \cdot \nabla y_0 \, dx = 0.$$

Finally, we estimate the following term:

$$\begin{aligned}
\int_D |\nabla(y_\varepsilon - y_0)|^2 dx &= \int_D [\nabla y_\varepsilon - \nabla y_0] \cdot \nabla y_\varepsilon \, dx - \int_D [\nabla y_\varepsilon - \nabla y_0] \cdot \nabla y_0 \, dx \\
&= \int_D [\nabla y_\varepsilon - \nabla y_0] \cdot \nabla y_\varepsilon \, dx \\
&= \int_D \nabla y_\varepsilon \cdot \nabla y_\varepsilon \, dx - \int_D \nabla y_0 \cdot \nabla y_\varepsilon \, dx \\
&= \int_D \chi_\Omega \, f y_\varepsilon \, dx - \int_D \nabla y_0 \cdot \nabla y_\varepsilon \, dx \\
&\to \int_D \chi_\Omega \, f y_0 \, dx - \int_D \nabla y_0 \cdot \nabla y_0 \, dx = 0.
\end{aligned}$$

As a result $y_\varepsilon \to y_0$ in $H_0^1(D)$-strong. □

4.3.2 Continuity with Respect to $\mathrm{X}(D)$ in the $L^p(D)$-Topology

Theorem 4.3. *Let D be a bounded open nonempty subset in $\mathbf{R}^{\mathrm{N}}$ and Ω be a measurable subset of D. Let $y(\chi_\Omega) \in H^1_\bullet(\Omega; D)$ be the solution of the overrelaxed Dirichlet problem (4.7). Then the function*

$$\chi_\Omega \mapsto y(\chi_\Omega) : \mathrm{X}(D) \to H_0^1(D)\text{-strong} \tag{4.16}$$

is continuous for all $L^p(D)$-topologies, $1 \le p < \infty$, on $\mathrm{X}(D)$.

Proof. Given $\chi = \chi_\Omega \in \mathrm{X}(D)$, let $\{\chi_{\Omega_n}\}$ in $\mathrm{X}(D)$ be a converging sequence to $\chi = \chi_\Omega$. In view of Theorem 4.2, let $\varepsilon_n > 0$ be such that the approximating solution $y_n \in H_0^1(D)$ of (4.15),

$$\forall \varphi \in H_0^1(D), \quad \int_D \left[\chi_{\Omega_n} + \frac{1}{\varepsilon_n}(1 - \chi_{\Omega_n})\right] \nabla y_n \cdot \nabla \varphi \, dx = \int_D \chi_{\Omega_n} f \varphi \, dx,$$

is such that

$$\|y_n - y(\chi_{\Omega_n})\|_{H^1(D)} < 1/n$$

and recall the bounds

$$\|\nabla y_n\|_{L^2(D)} \le c(D) \, \|f\|_{L^2(D)}, \tag{4.17}$$

$$\|(1 - \chi_{\Omega_n}) \nabla y_n\|_{L^2(D)} \le \sqrt{\varepsilon_n} \, c(D) \, \|f\|_{L^2(D)} \le c(D) \, \|f\|_{L^2(D)}. \tag{4.18}$$

Therefore the sequence $\{y(\chi_{\Omega_n}\}$ is also bounded in $H_0^1(D)$ and there exist subsequences, still indexed by n, and $y_0 \in H_0^1(D)$ such that

$$y_n \rightharpoonup y_0 \text{ and } y(\chi_{\Omega_n}) \rightharpoonup y_0 \text{ in } H_0^1(D)\text{-weak.}$$

Moreover, $(1 - \chi_{\Omega_n}) \nabla y(\chi_{\Omega_n}) = 0$. Hence $(1 - \chi_\Omega) \nabla y_0 = 0$ and $y_0 \in H^1_\bullet(\Omega; D)$.

For $\varphi \in H^1_\bullet(\Omega; D)$

$$\forall \varphi \in H_0^1(D), \quad \int_D \left[\chi_{\Omega_n} + \frac{1}{\varepsilon_n}(1 - \chi_{\Omega_n})\right] \nabla y_n \cdot (\chi_\Omega \nabla \varphi) \, dx = \int_D \chi_{\Omega_n} f \varphi \, dx.$$

The right-hand side converges to $\int_D \chi_\Omega f \varphi \, dx$. The left-hand side can be rewritten as

$$\begin{aligned}
&\int_D \chi_\Omega \, \chi_{\Omega_n} \, \nabla y_n \cdot \nabla \varphi + \chi_\Omega \, (1 - \chi_{\Omega_n}) \left[\frac{1}{\varepsilon_n} (1 - \chi_{\Omega_n}) \right] \nabla y_n \cdot \nabla \varphi \, dx \\
&\to \int_D \chi_\Omega \, \nabla y_0 \cdot \nabla \varphi \, dx = \int_D \nabla y_0 \cdot \nabla \varphi \, dx
\end{aligned}$$

since $\chi_\Omega \, \chi_{\Omega_n} \to \chi_\Omega$ and $\chi_\Omega \, (1 - \chi_{\Omega_n}) \to 0$ in $L^2(D)$-strong and the bound (4.18). Finally, $y_0 \in H^1_\bullet(\Omega; D)$ is solution of

$$\forall \varphi \in H^1_\bullet(\Omega; D), \quad \int_D \nabla y_0 \cdot \nabla \varphi \, dx = \int_D \chi_\Omega f \varphi \, dx.$$

Since the limit point $y_0 = y(\chi_\Omega)$ is independent of the choice of the sequence, $y(\chi_{\Omega_n}) \rightharpoonup y(\chi_\Omega)$ in $H^1_0(D)$-weak. Finally,

$$\begin{aligned}
&\int_D |\nabla \left(y(\chi_{\Omega_n}) - y(\chi_\Omega) \right)|^2 \, dx \\
&= \int_D |\nabla y(\chi_{\Omega_n})|^2 \, dx + \int_D |\nabla y(\chi_\Omega)|^2 \, dx - 2 \int_D \nabla y(\chi_{\Omega_n}) \cdot \nabla y(\chi_\Omega) \, dx \\
&= \int_D \chi_{\Omega_n} \, f \, y(\chi_{\Omega_n}) \, dx + \int_D \chi_\Omega \, f \, y(\chi_\Omega) \, dx - 2 \int_D \nabla y(\chi_{\Omega_n}) \cdot \nabla y(\chi_\Omega) \, dx \\
&\to 2 \int_D \chi_\Omega \, f \, y(\chi_\Omega) \, dx - 2 \int_D \nabla y(\chi_\Omega) \cdot \nabla y(\chi_\Omega) \, dx = 0
\end{aligned}$$

and $y(\chi_{\Omega_n}) \to y(\chi_\Omega)$ in $H^1_0(D)$-strong. □

Remark 4.3.
The penalization technique used in this section can be applied to higher-order elliptic, and even parabolic, problems and the Navier–Stokes equation (cf. R. Dziri and J.-P. Zolésio [**6**]) provided that the relaxation is acceptable in the modeling of the physical phenomenon. The continuity of the *classical* homogeneous Dirichlet problem for the Laplacian will be given in section 6 under capacity conditions. □

4.4 Relaxed Dirichlet Boundary Value Problem

In this section we consider the relaxation (4.6) of the homogeneous Dirichlet boundary value problem for the Laplacian to measurable subsets of a bounded open holdall D for the space $H^1_\diamond(\Omega; D)$.

Theorem 4.4. *Let D be a bounded open nonempty subset in $\mathbf{R}^N$ and Ω be a measurable subset of D. Let $y(\chi_\Omega) \in H^1_\diamond(\Omega; D)$ be the solution of the relaxed Dirichlet problem* (4.1)*. Then the function*

$$\chi_\Omega \mapsto y(\chi_\Omega) : \mathrm{X}(D) \to H^1_0(D)\text{-strong} \tag{4.19}$$

is continuous for all $L^p(D)$-topology, $1 \le p < \infty$, on $\mathrm{X}(D)$.

Proof. Given $\chi_\Omega \in \mathrm{X}(D)$, let $\{\chi_{\Omega_n}\}$ in $\mathrm{X}(D)$ be a converging sequence to χ_Ω and let $y(\chi_\Omega) \in H^1_\diamond(\Omega; D)$ and $y(\chi_{\Omega_n}) \in H^1_\diamond(\Omega_n; D)$ be the corresponding solutions of (4.1). Since $H^1_\diamond(\Omega; D) \subset H^1_\bullet(\Omega; D)$, we have the continuity $\chi_\Omega \mapsto y(\chi_\Omega)$ from Theorem 4.3. To complete the proof we need to show that the limiting element $y(\chi_\Omega)$ in the proof of Theorem 4.3 is solution of the variational equation (4.1) in the smaller space $H^1_\diamond(\Omega; D)$. Indeed, for each n, $(1-\chi_{\Omega_n})y(\chi_{\Omega_n}) = 0$ implies $(1-\chi_\Omega)y(\chi_\Omega) = 0$ and $y(\chi_\Omega) \in H^1_\diamond(\Omega; D)$. Finally, since $y(\chi_\Omega)$ is solution of the variational equation (4.7) for all $\varphi \in H^1_\bullet(\Omega; D)$, it still is solution for φ in the smaller subspace $H^1_\diamond(\Omega; D)$. Hence $y(\chi_\Omega) \in H^1_\diamond(\Omega; D)$ is the solution of (4.1). □

5 Continuity of the Homogeneous Neumann Boundary Value Problem

As a fourth generic example, consider the *homogeneous Neumann boundary value problem of mathematicians* for the Laplacian

$$-\Delta y + y = f \text{ in } \Omega, \quad \frac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega, \tag{5.1}$$

$$\exists y \in H^1(\Omega),\, \forall \varphi \in H^1(\Omega), \quad \int_\Omega \nabla y \cdot \nabla \varphi + y\,\varphi\, dx = \int_\Omega f\,\varphi\, dx. \tag{5.2}$$

The *classical* homogeneous Neumann boundary value problem

$$-\Delta y = f \text{ in } \Omega, \quad \frac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega, \tag{5.3}$$

$$\exists y \in H^1(\Omega),\, \forall \varphi \in H^1(\Omega), \quad \int_\Omega \nabla y \cdot \nabla \varphi\, dx = \int_\Omega f\,\varphi\, dx \tag{5.4}$$

has a unique solution up to a constant if $\int_D f\, dx = 0$. One way to get around the nonuniqueness is to introduce the closed subspace

$$H^1_a(\Omega) \overset{\text{def}}{=} \left\{ \varphi \in H^1(\Omega) : \int_\Omega \varphi\, dx = 0 \right\}$$

of functions with zero average. Then there exists a unique $y \in H^1_a(\Omega)$ such that

$$\forall \varphi \in H^1_a(\Omega), \quad \int_\Omega \nabla y \cdot \nabla \varphi\, dx = \int_\Omega f\,\varphi\, dx. \tag{5.5}$$

Moreover, the solution of (5.5) can be approximated by the ε-problems

$$\exists y_\varepsilon \in H^1(\Omega),\, \forall \varphi \in H^1(\Omega), \quad \int_\Omega \nabla y_\varepsilon \cdot \nabla \varphi + \varepsilon\, y_\varepsilon\, \varphi\, dx = \int_\Omega f\,\varphi\, dx \tag{5.6}$$

and $y_\varepsilon \to y$ in $H^1_a(\Omega)$-strong. Note that, by assumption on f, $y_\varepsilon \in H^1_a(\Omega)$.

One of the key conditions to get the continuity with respect to the domain Ω is the following density property.

Definition 5.1.
An open set Ω is said to have the H^1-*density property* if the set

$$\{\varphi|_\Omega \,:\, \varphi \in C_0^1(\mathbf{R}^\mathrm{N})\} \tag{5.7}$$

is dense in $H^1(\Omega)$. □

This property is not verified for a domain Ω with a crack in $\mathbf{R}^2$, since elements of $H^1(\Omega)$ have different traces on each side of the crack. Recall from section 6.1 in Chapter 2 that this condition is verified for a large class of sets Ω that can be characterized by the geometric *segment property* that is equivalent to the property that the set be locally a C^0-epigraph (cf. section 6.2 of Chapter 2). We recall Theorem 6.3 of Chapter 5.

Theorem 5.1. *If the open set Ω has the segment property, then the set*

$$\left\{f|_\Omega \,:\, \forall f \in C_0^\infty(\mathbf{R}^\mathrm{N})\right\}$$

of restrictions of functions of $C_0^\infty(\mathbf{R}^\mathrm{N})$ to Ω is dense in $W^{m,p}(\Omega)$ for $1 \le p < \infty$ and $m \ge 1$. In particular $C^k(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$ for any $m \ge 1$ and $k \ge m$.

Lemma 5.1. *Let D be a bounded open subset of $\mathbf{R}^\mathrm{N}$, let $f \in L^2(D)$, and let Ω and the sequence $\{\Omega_n\}$ be open subsets of D. Denote by $y_n \in H^1(\Omega_n)$ the solution of* (5.1) *in Ω_n and by $y \in H^1(\Omega)$ the solution of* (5.1) *in Ω. Assume the following:*

(i) $\chi_{\Omega_n} \to \chi_\Omega$ *in $L^2(D)$;*

(ii) $d_{\complement\Omega_n} \to d_{\complement\Omega}$ *in $C(\overline{D})$;*

(iii) Ω *has the* H^1-density property.

Then

$$(\tilde{y}_n, \widetilde{\nabla y_n}) \to (\tilde{y}, \widetilde{\nabla y}) \text{ in } (L^2(D) \times L^2(D))\text{-weak},$$

where $\tilde{z}$ denotes the extension by zero of a function z from Ω or Ω_n to D.

Proof. The proof of the lemma follows the approach of W. Liu and J. E. Rubio [**1**] in 1992. Since $f \in L^2(D)$, we readily have the following estimates:

$$\|\tilde{y}_n\|^2_{L^2(D)} + \|\widetilde{\nabla y_n}\|^2_{L^2(D)} = \|y_n\|^2_{L^2(\Omega_n)} + \|\nabla y_n\|^2_{L^2(\Omega_n)} \le \|f\|^2_{L^2(D)}.$$

There exist subsequences, still indexed by n, $z \in L^2(D)$, and $Z \in L^2(D)^N$ such that

$$\tilde{y}_n \rightharpoonup z \text{ weakly in } L^2(\mathbf{R}^\mathrm{N}) \quad\text{and}\quad \widetilde{\nabla y_n} \rightharpoonup Z \text{ weakly in } L^2(\mathbf{R}^\mathrm{N})^N.$$

By the H^1-density property of Ω, the variational equation (5.2) on Ω_n can be rewritten in the form

$$\forall \varphi \in C_0^1(\mathbf{R}^\mathrm{N}), \quad \int_{\mathbf{R}^\mathrm{N}} \chi_{\Omega_n} \left[\widetilde{\nabla y_n} \cdot \nabla\varphi + \tilde{y}_n\, \varphi\right] dx = \int_{RN} \chi_{\Omega_n} f\, \varphi\, dx \tag{5.8}$$

and we can go to the limit as $n \to \infty$:

$$\forall \varphi \in C_0^1(\mathbf{R}^{\mathrm{N}}), \quad \int_{\mathbf{R}^{\mathrm{N}}} \chi_\Omega \, [Z \cdot \nabla \varphi + z\,\varphi] \, dx = \int_{RN} \chi_\Omega f\, \varphi \, dx. \tag{5.9}$$

By definition of the distributional derivative, for all n

$$\int_{\Omega_n} \partial_i y_n \, \varphi_n \, dx = -\int_{\Omega_n} y_n \, \partial_i \varphi_n \, dx, \quad \forall \varphi_n \in \mathcal{D}(\Omega_n).$$

By the compactivorous property (cf. Theorem 2.4 (iii) in Chapter 6) and the convergence $d_{\complement\Omega_n} \to d_{\complement\Omega}$ in $C(\overline{D})$, for each $\varphi \in \mathcal{D}(\Omega)$, there exists N such that for all $n > N$, $\operatorname{supp}\varphi \subset \Omega_n$. Hence for all $n > N$

$$\int_\Omega \widetilde{\partial_i y_n} \, \varphi \, dx = -\int_\Omega \tilde{y}_n \, \partial_i \varphi \, dx, \quad \forall \varphi \in \mathcal{D}(\Omega)$$

and by going to the limit

$$\int_\Omega Z_i \, \varphi \, dx = -\int_\Omega z \, \partial_i \varphi \, dx, \quad \forall \varphi \in \mathcal{D}(\Omega), \quad \Rightarrow \nabla z = Z \in L^2(\Omega)$$

and from (5.9), $z \in H^1(\Omega)$ is solution of the variational equation

$$\forall \varphi \in C_0^1(\mathbf{R}^{\mathrm{N}}), \quad \int_\Omega \nabla z \cdot \nabla \varphi + z\, \varphi \, dx = \int_\Omega f\, \varphi \, dx.$$

By the H^1-density property of Ω, this extends to all $\varphi \in H^1(\Omega)$, which implies that $z = y$. Since all weakly convergent subsequences converge to the same limit, the whole sequence weakly converges and this completes the proof. □

Remark 5.1.
We know that the H^1-density condition holds for the family of domains satisfying the *segment property* of Definition 6.1 in Chapter 2 and not only for H^1-spaces but also $W^{m,p}$-spaces (cf. Theorem 6.3 in Chapter 2), thus extending the property and the results to a broader class of partial differential equations. We have shown in Theorem 6.5 of Chapter 2 that a domain satisfying the segment property is locally a C^0-epigraph in the sense of Definition 5.1 of Chapter 2. □

Theorem 5.2. *Let D be a bounded open subset of $\mathbf{R}^{\mathrm{N}}$, let $f \in L^2(D)$, and let $L(D, r, \mathcal{O}, \lambda)$ be the compact family of sets in $W^{1,p}(D)$, $1 \le p < \infty$, verifying the uniform fat segment property. Then the function*

$$\Omega \mapsto (\tilde{y}(\Omega), \widetilde{\nabla y(\Omega)}) : L(D, r, \mathcal{O}, \lambda) \to (L^2(D) \times L^2(D))\text{-weak},$$

where $\tilde{z}$ denotes the extension by zero of a function z from Ω to D, is continuous. It is also true when $L(D, r, \mathcal{O}, \lambda)$ is replaced by the family of convex sets

$$\mathcal{C}_b(D) \stackrel{\text{def}}{=} \left\{ b_\Omega : \Omega \subset \overline{D},\, \partial\Omega \neq \varnothing,\, \Omega \text{ convex} \right\} \tag{5.10}$$

(cf. Theorem 10.2 in Chapter 7).

Proof. In $L(D, r, \mathcal{O}, \lambda)$ endowed with the $W^{1,p}(D)$-topology, $1 \le p < \infty$, the three assumptions of the lemma are verified. $\square$

Remark 5.2.
Therefore the previous arguments are verified for the compact families of sets of Theorems 13.1, 14.1, and 14.2 in section 13 of Chapter 7 under the uniform fat segment property and for the conditions on the local graph functions of Theorem 13.2 (cf. Theorem 5.1 of Chapter 2). However, the subsets of a bounded holdall verifying a uniform segment property do not include the very important domains with cracks for which some specific results have been obtained by D. Bucur and J.-P. Zolésio [**2**, **4**] (and D. Bucur and N. Varchon [**1**] in dimension 2). $\square$

An issue that was not tackled is the nonhomogeneous boundary condition that involves a boundary integral in the variational formulation:

$$-\Delta y + y = f \text{ in } \Omega, \quad \frac{\partial y}{\partial n} = g \text{ on } \partial\Omega, \tag{5.11}$$

$$\exists y \in H^1(\Omega),\, \forall \varphi \in H^1(\Omega), \quad \int_\Omega \nabla y \cdot \nabla \varphi + y\, \varphi\, dx = \int_\Omega f\, \varphi\, dx + \int_\Gamma g\, \varphi\, d\Gamma. \tag{5.12}$$

6 Elements of Capacity Theory

Recall the relaxation of the homogeneous Dirichlet boundary value problem for the Laplacian (4.6) in section 4.4 by using the space

$$H^1_\diamond(\Omega; D) \stackrel{\text{def}}{=} \left\{\varphi \in H^1_0(D) : (1 - \chi_\Omega)\varphi = 0 \text{ a.e. in } D\right\}. \tag{6.1}$$

The classical solution lies in the closed subspace $H^1_0(\Omega; D)$ of $H^1_\diamond(\Omega; D)$, and we know that they are not equal when Ω is not stable with respect to D. The difference between the two spaces is that a function in $H^1_0(\Omega; D)$ is zero in $D\backslash\Omega$ as an $H^1_0(D)$-function and not just *almost everywhere* as an $L^2(D)$-function. Some information is lost in the relaxation, and this requires the finer notion of *quasi-everywhere* that will characterize the set over which an $H^1_0(D)$-function is zero.

This section introduces the basic elements and results from *capacity theory* that will play an essential role in establishing the continuity of the classical homogeneous Dirichlet problem (4.5) with respect to its underlying domain. The characterization of the space $H^1_0(\Omega; D)$ in terms of capacity will be given in Theorem 6.2. By introducing capacity constraints on sets, it will be possible to construct larger compact families that include domains with cracks.

6.1 Definition and Basic Properties

For $p > N$ the elements of $W^{1,p}(\mathbf{R}^N)$ can be redefined on sets of measure zero to be continuous functions (cf. R. A. Adams [**1**, Thm. 5.4, p. 97]):

$$W^{1,p}(\mathbf{R}^N) \to C^{1,\lambda}(\mathbf{R}^N), \quad 0 < \lambda \le 1 - N/p. \tag{6.2}$$

But for $p \le N$ this is no longer the case. We first recall the definition of $(1,p)$-*capacity* with respect to $\mathbf{R}^N$ and with respect to an open subset G of $\mathbf{R}^N$ along with a number of basic technical results.

Definition 6.1.
Let p, $1 < p < \infty$, be a real number and $N \ge 1$ be an integer.

(i) (L. I. Hedberg [**2**, p. 239]) The *capacity* is defined as follows: for a compact subset $K \subset \mathbf{R}^N$

$$\mathrm{Cap}_{1,p}(K) \stackrel{\text{def}}{=} \inf\left\{ \int_{\mathbf{R}^N} |\nabla\varphi|^p\, dx \,:\, \varphi \in C_0^\infty(\mathbf{R}^N),\, \varphi > 1 \text{ on } K\right\};$$

for an open subset $G \subset \mathbf{R}^N$

$$\mathrm{Cap}_{1,p}(G) \stackrel{\text{def}}{=} \sup\left\{\mathrm{Cap}_{1,p}(K) \,:\, \forall K \subset G,\, K \text{ compact }\right\};$$

for an arbitrary subset $E \subset D$

$$\mathrm{Cap}_{1,p}(E) \stackrel{\text{def}}{=} \inf\left\{\mathrm{Cap}_{1,p}(G) \,:\, \forall G \supset E,\, G \text{ open }\right\}.$$

In what follows, we shall use the simpler notation Cap_p as in L. C. Evans and R. F. Gariepy [**1**, pp. 147] instead of $\mathrm{Cap}_{1,p}$.

(ii) (L. C. Evans and R. F. Gariepy [**1**, pp. 160] and J. Heinonen, T. Kilpelainen, and O. Martio [**1**, p. 87]) A function f is said to be *quasi-continuous* on $D \subset \mathbf{R}^N$ if, for each $\varepsilon > 0$, there exists an open set G_ε such that $\mathrm{Cap}_{1,p}(G_\varepsilon) < \varepsilon$ and f is continuous on $D\backslash G_\varepsilon$.

(iii) A set E in $\mathbf{R}^N$ is said to be *quasi-open* if, for all $\varepsilon > 0$, there exists an open set G_ε such that $\mathrm{Cap}_{1,p}(G_\varepsilon) < \varepsilon$ and $E \cup G_\varepsilon$ is open. □

In the above definitions the capacity is defined with respect to $\mathbf{R}^N$. In J. Heinonen, T. Kilpelainen, and O. Martio [**1**, Chap. 2, p. 27], the capacity is defined relative to an open subset D of $\mathbf{R}^N$ instead of the whole space $\mathbf{R}^N$.

Definition 6.2.
Let $N \ge 1$ be an integer, let p, $1 < p < \infty$, and let D be a fixed open subset of $\mathbf{R}^N$.

(i) The $(1,p)$-*capacity* is defined as follows: for a compact subset $K \subset D$

$$\mathrm{Cap}_{1,p}(K,D) \stackrel{\text{def}}{=} \inf\left\{ \int_D |\nabla\varphi|^p\, dx \,:\, \varphi \in C_0^\infty(D),\, \varphi > 1 \text{ on } K\right\};$$

for an open subset $G \subset D$

$$\mathrm{Cap}_{1,p}(G,D) \stackrel{\text{def}}{=} \sup\left\{\mathrm{Cap}_{1,p}(K,D) \,:\, \forall K \subset G,\, K \text{ compact }\right\};$$

for an arbitrary subset $E \subset D$

$$\mathrm{Cap}_{1,p}(E,D) \stackrel{\text{def}}{=} \inf\left\{\mathrm{Cap}_{1,p}(G,D) \,:\, \forall G \text{ open such that } E \subset G \subset D\right\}.$$

(ii) A function f is said to be *quasi-continuous* in D if, for all $\varepsilon > 0$, there exists an open set $G_\varepsilon \subset D$ such that $\text{cap}_D(G_\varepsilon) < \varepsilon$ and f is continuous on $D \backslash G_\varepsilon$.

(iii) A set E in D is said to be *quasi-open* if, for all $\varepsilon > 0$, there exists an open set $G_\varepsilon \subset D$ such that $\text{cap}_D(G_\varepsilon) < \varepsilon$ and $E \cup G_\varepsilon$ is open.

Clearly, by definition,

$$\forall E \subset D, \quad \text{Cap}_{1,p}(E) \leq \text{Cap}_{1,p}(E, D). \tag{6.3}$$

The number $\text{Cap}_{1,p}(E, D) \in [0, \infty]$ is called the (*variational*) (p, m_N)-capacity of the *condenser* (E, G), where m_N is the N-dimensional Lebesgue measure. □

It can easily be shown that for a quasi-open set there exists a decreasing sequence $\{\Omega_n\}$ of open sets such that $\Omega_n \supset E$ and $\text{cap}_D(\Omega_n \backslash E)$ goes to zero as n goes to infinity.

We say that a property holds *quasi-everywhere* (q.e.) in D if it holds in the complement $D \backslash E$ of a set E of zero capacity. A set of zero capacity has zero measure, but the converse is not true. The capacity is a countably subadditive set function, but it is not additive even for disjoint sets. Hence the union of a countable number of sets of zero capacity has zero capacity.

6.2 Quasi-continuous Representative and H^1-Functions

The following theorem is from L. I. Hedberg [**2**] (sect. 2, p. 241 and Thm. 1.1, p. 237, and footnote, p. 238 referring to T. Wolff and the paper by L. I. Hedberg and Th. H. Wolff [**1**]). See also J. Heinonen, T. Kilpelainen, and O. Martio [**1**, Thm. 4.4, p. 89]

Theorem 6.1. *Let p, $1 < p < \infty$, be a real number, $N \geq 1$ be an integer, and $D \subset \mathbf{R}^{\mathrm{N}}$ be open.*

(i) *Every f in $W^{1,p}(D)$ has a quasi-continuous representative: there exists a quasi-continuous function f_1 defined on D such that $f_1 = f$ almost everywhere in D (hence f_1 is a representative of f in $W^{1,p}(D)$). Any two quasi-continuous representatives f_1 and f_2 of $f \in W^{1,p}(D)$ such that $f_1 = f_2$ almost everywhere are equal quasi-everywhere in D.*

(ii) *Let $f \in W^{1,p}(\mathbf{R}^{\mathrm{N}})$. Let $K \subset \mathbf{R}^{\mathrm{N}}$ be closed, and suppose that $f|_K = 0$ (trace defined quasi-everywhere). Then $f|_{\complement K} \in W_0^{1,p}(\complement K)$.*

The following key lemma completes the picture (cf. J. Heinonen, T. Kilpelainen, and O. Martio [**1**, Thm. 4.5, p. 90]). See also L. I. Hedberg [**2**, p. 241].

Lemma 6.1. (i) *Let Ω be an open subset of $\mathbf{R}^{\mathrm{N}}$, let p, $1 \leq p < \infty$, and consider an element u of $W^{1,p}(\Omega)$. Then $u \in W_0^{1,p}(\Omega)$ if and only if there exists a quasi-continuous representative u_1 of u in $\mathbf{R}^{\mathrm{N}}$ such that $u_1 = 0$ quasi-everywhere in $\complement\Omega$ and $u_1 = u$ almost everywhere in Ω.*

(ii) *Let* Ω *and* D *be two bounded open subsets of* $\mathbf{R}^{\mathrm{N}}$ *such that* $\Omega \subset D$ *and consider an element* u *of* $W_0^{1,p}(D)$. *Then* $u|_\Omega \in W_0^{1,p}(\Omega)$ *if and only if there exists a quasi-continuous representative* u_1 *of* u *such that* $u_1 = 0$ *quasi-everywhere in* $D \backslash \Omega$ *and* $u_1 = u$ *almost everywhere in* Ω.

(iii) *Let* Ω *and* D *be two bounded open subsets of* $\mathbf{R}^{\mathrm{N}}$ *such that* $\overline{\Omega} \subset D$ *and consider an element* u *of* $W^{1,p}(D)$. *Then* $u|_\Omega \in W_0^{1,p}(\Omega)$ *if and only if there exists a quasi-continuous representative* u_1 *of* u *such that* $u_1 = 0$ *quasi-everywhere in* $D \backslash \Omega$ *and* $u_1 = u$ *almost everywhere in* Ω.

A function φ in $W^{1,p}(D)$ is said to be *zero quasi-everywhere* in a subset E of D if there exists a quasi-continuous representative of φ which is zero quasi-everywhere in E. This makes sense since any two quasi-continuous representatives of an element φ of $W^{1,p}(D)$ are equal quasi-everywhere. For any $\varphi \in W_0^{1,p}(D)$ and $t \in \mathbf{R}$, the set $\{x \in D : \varphi(x) > t\}$ is quasi-open. Moreover, the subspace $W_0^{1,p}(\Omega; D)$ introduced in section 2.5.3 in Chapter 2 can now be characterized by the capacity.

Theorem 6.2. *Given two bounded open subsets* Ω *and* D *of* $\mathbf{R}^{\mathrm{N}}$,

$$W_0^{1,p}(\Omega; D) = \left\{\varphi \in W_0^{1,p}(D) : (1-\chi_\Omega)\varphi = 0 \text{ q.e. in } D\right\}, \quad \text{if } \Omega \subset D, \tag{6.4}$$

$$W_0^{1,p}(\Omega; D) = \left\{\varphi \in W^{1,p}(D) : (1-\chi_\Omega)\varphi = 0 \text{ q.e. in } D\right\}, \quad \text{if } \overline{\Omega} \subset D. \tag{6.5}$$

The characterization of $W_0^{1,p}(\Omega; D)$ requires the notion of *capacity* in a very essential way. It cannot be obtained by saying that the function and its derivatives are zero almost everywhere in $D \backslash \Omega$.

Recall the definition of the other extension $H^1_\diamond(\Omega; D)$ of $H^1(\Omega)$ to a measurable set D containing Ω (cf. Chapter 5, section 3.5, Theorem 3.6, identity (3.27)). Its definition readily extends from H^1 to $W^{1,p}$ as follows:

$$W_\diamond^{1,p}(\Omega; D) \overset{\text{def}}{=} \left\{\psi \in W_0^{1,p}(D) \,:\, (1-\chi_\Omega)\psi = 0 \text{ a.e. in } D\right\}.$$

By definition $W_0^{1,p}(\Omega; D) \subset W_\diamond^{1,p}(\Omega; D)$, but the two spaces are generally not equal, as can be seen from the following example. Denote by B_r, $r > 0$, the open ball of radius $r > 0$ in $\mathbf{R}^2$. Define $\Omega = B_2 \backslash \partial B_1$ and $D = B_3$. The circular crack ∂B_1 in Ω has zero measure but nonzero capacity. Since ∂B_1 has zero measure, $W_\diamond^{1,p}(\Omega; D)$ contains functions $\psi \in W_0^{1,p}(B_2)$ whose restrictions to B_2 are not zero on the circle ∂B_1 and hence do not belong to $W_0^{1,p}(\Omega)$. Yet for Lipschitzian domains and, more generally, for domains Ω that are stable with respect to D in the sense of Definition 4.1, the two spaces are indeed equal.

6.3 Transport of Sets of Zero Capacity

Any Lipschitz continuous transformation of $\mathbf{R}^{\mathrm{N}}$ which has a Lipschitz continuous inverse transports sets of zero capacity onto sets of zero capacity.

Lemma 6.2. *Let* D *be an open subset of* $\mathbf{R}^{\mathrm{N}}$ *and let* T *be an invertible transformation of* $\overline{D}$ *such that both* T *and* T^{-1} *are Lipschitz continuous. For any* $E \subset D$

$$\mathrm{Cap}_{1,2}(E, D) = 0 \iff \mathrm{Cap}_{1,2}(T(E), D) = 0.$$

Proof. (a) $E = K$, *K compact in D.* Observe that, in the definition of the capacity, it is not necessary to choose the functions φ in $C_0^\infty(D)$. They can be chosen in a larger space as long as

$$\int_D |\nabla\varphi|^2\,dx < \infty \text{ and } \varphi > 1 \text{ on } K.$$

So the capacity is also given by

$$\mathrm{Cap}_{1,2}(K,D) \stackrel{\mathrm{def}}{=} \inf\left\{\int_D |\nabla\varphi|^2\,dx \,:\, \varphi \in H_0^1(D)\cap C(\overline{D}),\ \varphi > 1 \text{ on } K\right\}.$$

But the space $H_0^1(D)\cap C(\overline{D})$ is stable under the action of T:

$$\varphi \in H_0^1(D)\cap C(\overline{D}) \iff T\circ\varphi \in H_0^1(D)\cap C(\overline{D}).$$

Consider the matrix

$$A(x) \stackrel{\mathrm{def}}{=} |\det(DT(x))|\ {}^*DT(x)({}^*DT(x))^{-1}.$$

By assumption on T, the elements of the matrix A belong to $L^\infty(D)$ and

$$\exists\alpha > 0, \quad \alpha I \le A(x) \le \alpha^{-1} I \text{ a.e. in } D.$$

Define $E(K) = \{\varphi \in H_0^1(D)\cap C(\overline{D}) \,:\, \varphi > 1 \text{ on } K\}$. For any $\varphi \in E(T(K))$, $\varphi\circ T \in E(K)$ and

$$\int_D |\nabla\varphi|^2\,dx = \int_D A\,\nabla(\varphi\circ T)\cdot\nabla(\varphi\circ T)\,dx \ge \alpha\int_D |\nabla(\varphi\circ T)|^2\,dx.$$

Let K be a compact subset such that $\mathrm{Cap}_{1,2}(K,D) = 0$. For each $\varepsilon > 0$ there exists $\varphi \in E(K)$ such that

$$\begin{aligned}\int_D |\nabla\varphi|^2\,dx \le \varepsilon \quad &\Rightarrow \forall\varepsilon, \quad \int_D |\nabla(\varphi\circ T)|^2\,dx \le \varepsilon/\alpha \text{ or } \varphi\circ T \in E(T^{-1}(K))\\ &\Rightarrow \forall\varepsilon, \quad \mathrm{Cap}_{1,2}(T^{-1}(K),D) \le \varepsilon/\alpha \quad \Rightarrow \mathrm{Cap}_{1,2}(T^{-1}(K),D) = 0\end{aligned}$$

and we can repeat the proof with T^{-1} in place of T.

(b) $E = G$, *G open.* Let

$$\mathrm{Cap}_{1,2}(G,D) = \sup\left\{\mathrm{Cap}_{1,2}(K,D) \,:\, \forall K \subset G,\ K \text{ compact}\right\} = 0.$$

Therefore $\mathrm{cap}_D(K) = 0$, which implies that $\mathrm{Cap}_{1,2}(T(K),D) = 0$ and hence

$$\begin{aligned}\mathrm{Cap}_{1,2}(T(G),D) &= \sup\left\{\mathrm{Cap}_{1,2}(K',D) \,:\, \forall K' \subset T(G),\ K' \text{ compact}\right\}\\ &= \sup\left\{\mathrm{Cap}_{1,2}(T(K),D) \,:\, \forall K \subset G,\ K \text{ compact}\right\} = 0.\end{aligned}$$

(c) *General case.* Let

$$\mathrm{Cap}_{1,2}(E,D) \stackrel{\mathrm{def}}{=} \inf\left\{\mathrm{cap}_D(G) \,:\, \forall G \supset E,\ G \text{ open}\right\}.$$

Therefore, for all $\varepsilon > 0$, there exists $G \supset E$ open such that

$$\mathrm{Cap}_{1,2}(E, D) \le \mathrm{Cap}_{1,2}(G, D) \le \varepsilon \quad \Rightarrow \mathrm{Cap}_{1,2}(T(E), D) \le \mathrm{Cap}_{1,2}(T(G), D)$$

since $T(E) \subset T(G)$ is open. By definition of $\mathrm{Cap}_{1,2}(G, D)$ we have

$$\begin{aligned}\forall K \subset G, \quad \mathrm{Cap}_{1,2}(K, D) \le \varepsilon \quad &\Rightarrow \mathrm{Cap}_{1,2}(T(K), D) \le \alpha\varepsilon \\ \Rightarrow \mathrm{Cap}_{1,2}(T(G), D) = \sup\{\mathrm{Cap}_{1,2}(T(K), D) \,:\, K \subset G\} &\le \alpha\varepsilon.\end{aligned}$$

Finally, for all $\varepsilon > 0$, $\mathrm{Cap}_{1,2}(T(E), D) \le \alpha\varepsilon$ and hence $\mathrm{Cap}_{1,2}(T(E), D) = 0$. □

7 Crack-Free Sets and Some Applications

In this section we show that the result for the second question raised in the introduction to this chapter is true for measurable sets in the following large family.

7.1 Definitions and Properties

Definition 7.1. (i) The *interior boundary* of a set Ω is defined as

$$\partial_i\Omega \overset{\text{def}}{=} \partial\Omega\backslash\partial\overline{\Omega}. \tag{7.1}$$

The *exterior boundary* of a set Ω is defined as

$$\partial_e\Omega \overset{\text{def}}{=} \partial\Omega\backslash\partial\overline{\complement\Omega}. \tag{7.2}$$

(ii) A subset Ω of $\mathbf{R}^N$ is said to be *crack-free* if $\partial_i\Omega = \varnothing$; its complement $\complement\Omega$ is said to be *crack-free* if $\partial_e\Omega = \varnothing$. □

The intuitive terminology *crack-free* arises from the following identities:

$$\partial_i\Omega = \partial\Omega\backslash\partial\overline{\Omega} = \mathrm{int}\,\overline{\Omega}\backslash\mathrm{int}\,\Omega, \quad \partial_e\Omega = \partial\Omega\backslash\partial\overline{\complement\Omega} = \mathrm{int}\,\overline{\complement\Omega}\backslash\mathrm{int}\,\complement\Omega. \tag{7.3}$$

A *crack-free* set Ω does not have *internal cracks* that would disappear after the *cleaning operation* with respect to the closure of the set[2]. It is verified for sets that

[2]Since $\partial\overline{\Omega} = \overline{\Omega} \cap \overline{\complement\overline{\Omega}} \subset \overline{\Omega} \cap \overline{\complement\Omega} = \partial\Omega$, the definition of a crack-free set Ω is equivalent to the property $\partial\overline{\Omega} = \partial\Omega$. In 1994 A. Henrot [**2**] introduced the terminology *Carathéodory set* for such a set that was later adopted by D. Tiba [**3**], P. Neittaanmäki, J. Sprekels, and D. Tiba [**1**], and A. Henrot and M. Pierre [**4**]. However, this terminology does not seem to be standard. For instance, in the literature on polynomial approximations in the complex plane $\mathbb{C}$, a Carathéodory set is defined as follows.

Definition 7.2 (O. Dovgoshey [**1**] or D. Gaier [**1**])**.**
A bounded subset Ω of $\mathbb{C}$ is said to be a *Carathéodory set* if the boundary of Ω coincides with the boundary of the unbounded component of the complement of $\overline{\Omega}$. A *Carathéodory domain* is a Carathéodory set if, in addition, Ω is simply connected. □

In V. A. Martirosian and S. E. Mkrtchyan [**1**], Ω is further assumed to be measurable. This definition excludes not only interior cracks but also bounded holes inside the set Ω as can be seen from the example of the annulus $\Omega = \{x \in \mathbf{R}^2 : 1 < |x| < 2\}$ in $\mathbf{R}^2$. So, it is more restrictive than $\partial\Omega = \partial\overline{\Omega}$. In order to avoid any ambiguity, we choose to keep the intuitive terminology *crack-free*.

are locally the epigraph of a C^0-function. Yet, it is also verified for sets that are not locally the epigraph of a C^0-function.

The following equivalent characterization of a crack-free set in terms of the oriented distance function was used as its definition in M. C. Delfour and J.-P. Zolésio [**43**].

Theorem 7.1. *A subset* Ω *of* $\mathbf{R}^N$ *is* crack-free *if and only if* $b_\Omega = b_{\overline{\Omega}}$. *Its complement* $\complement\Omega$ *is* crack-free *if and only if* $b_{\complement\Omega} = b_{\overline{\complement\Omega}}$ *(or, equivalently,* $b_\Omega = b_{\operatorname{int}\Omega}$*).*

Proof. If $\partial\Omega = \partial\overline{\Omega}$, then $|b_\Omega| = d_{\partial\Omega} = d_{\partial\overline{\Omega}} = |b_{\overline{\Omega}}|$. Hence, $d_\Omega + d_{\complement\Omega} = |b_\Omega| = |b_{\overline{\Omega}}| = d_{\overline{\Omega}} + d_{\complement\overline{\Omega}}$ implies $d_{\complement\Omega} = d_{\complement\overline{\Omega}}$ and $b_\Omega = b_{\overline{\Omega}}$. Conversely, if $b_\Omega = b_{\overline{\Omega}}$, then $d_{\partial\Omega} = |b_\Omega| = |b_{\overline{\Omega}}| = d_{\partial\overline{\Omega}}$ and $\partial\Omega = \partial\overline{\Omega}$ since both sets are closed. □

We also have the following equivalences:

$$\begin{aligned} b_{\complement\Omega} = b_{\operatorname{int}\complement\Omega} &\iff b_\Omega = b_{\overline{\Omega}} \iff \overline{\complement\overline{\Omega}} = \overline{\complement\Omega} \iff \partial\overline{\Omega} = \partial\Omega \\ &\iff \overline{\operatorname{int}\complement\Omega} = \overline{\complement\Omega} \iff \partial\operatorname{int}\complement\Omega = \partial\Omega \iff \operatorname{int}\Omega = \operatorname{int}\overline{\Omega} \end{aligned} \tag{7.4}$$

and

$$\begin{aligned} b_\Omega = b_{\operatorname{int}\Omega} &\iff b_{\complement\Omega} = b_{\overline{\complement\Omega}} \iff \overline{\complement\overline{\complement\Omega}} = \overline{\Omega} \iff \partial\overline{\complement\Omega} = \partial\Omega \\ &\iff \overline{\operatorname{int}\Omega} = \overline{\Omega} \iff \partial\operatorname{int}\Omega = \partial\Omega \iff \operatorname{int}\complement\Omega = \operatorname{int}\overline{\complement\Omega}. \end{aligned} \tag{7.5}$$

Remark 7.1.
An open set Ω is crack-free if and only if $\Omega = \operatorname{int}\overline{\Omega}$. Indeed, by definition, Ω is crack-free if and only if $\overline{\complement\overline{\Omega}} = \overline{\complement\Omega}$. But since Ω is open $\overline{\complement\overline{\Omega}} = \complement\Omega$ and $\Omega = \complement\overline{\complement\overline{\Omega}} = \operatorname{int}\overline{\Omega}$. □

We give the following general technical theorem and its corollary.

Theorem 7.2 (M. C. Delfour and J.-P. Zolésio [**43**])**.** *Let* K *be a compact subset of* $\mathbf{R}^N$ *and let* $v \in W^{1,p}(\mathbf{R}^N)$. *Then* $v = 0$ *almost everywhere on* $\complement K$ *implies that* v *has a quasi-continuous representative* v^* *such that* $v^*|_{\operatorname{int}K} \in W_0^{1,p}(\operatorname{int}K)$.

Corollary 1. *Let* Ω *be a bounded set in* $\mathbf{R}^N$ *and let* $v \in W^{1,p}(\mathbf{R}^N)$. *Then* $v = 0$ *almost everywhere on* $\complement\overline{\Omega}$ *implies that* v *has a quasi-continuous representative* v^* *such that* $v^*|_{\operatorname{int}\overline{\Omega}} \in W_0^{1,p}(\operatorname{int}\overline{\Omega})$.

Proof. Choose the quasi-continuous representative

$$v^*(x) \overset{\text{def}}{=} \lim_{\delta\searrow 0} \frac{1}{|B(x,\delta)|}\int_{B(x,\delta)} v\,dy \tag{7.6}$$

of v and let $\{G_\varepsilon\}$ be the associated family of open subsets of $(1,p)$-capacity less than ε of Definition 6.1 (ii).

(a) If there exists $\varepsilon > 0$ such that $G_\varepsilon \subset \operatorname{int}K$, then $\overline{\complement K} \subset \complement G_\varepsilon$ and v^* is continuous in $\overline{\complement K}$. But, by assumption, $v = 0$ almost everywhere in the open set $\complement K$. Hence, by definition of v^*, $v^* = 0$ everywhere in the open subset $\complement K \subset \overline{\complement K}$, and, by continuity, $v^* = 0$ everywhere in $\overline{\complement K}$.

(b) If for all $\varepsilon > 0$, $G_\varepsilon \cap \overline{\complement K} \neq \varnothing$, then either $x \in \cup_{\varepsilon>0} G_\varepsilon \cap \complement K$ or $x \in \cap_{\varepsilon>0} G_\varepsilon \cap \partial K$, since $\overline{\complement K} = \operatorname{int} \complement K \cup \partial K = \complement \overline{K} \cup \partial K$. In the first case, $v^*(x) = 0$ since $x \in \complement K$. In the second case, $\operatorname{Cap}_p(\cap_{\varepsilon>0} G_\varepsilon \cap \partial K) \leq \operatorname{Cap}_p(G_\varepsilon) < \varepsilon$ and $\cap_{\varepsilon>0} G_\varepsilon \cap \partial K$ has zero $(1,p)$-capacity. From (a) and (b), $v^* = 0$ quasi-everywhere in $\overline{\complement K}$. Therefore by Theorem 5 in L. I. Hedberg and Th. H. Wolff [**1**], which generalizes Theorem 1.1 in L. I. Hedberg [**2**], $v^*|_{\operatorname{int} K} \in W_0^{1,p}(\operatorname{int} K)$. □

Remark 7.2.
In L. I. Hedberg [**2**, **3**] a compact set K is said to be $(1,p)$-*stable* if $v = 0$ quasi-everywhere on $\complement K$ implies that $v|_{\operatorname{int} K} \in W_0^{1,p}(\operatorname{int} K)$. A sufficient condition (cf. Theorem 6.4 in L. I. Hedberg [**2**]) for the $(1,p)$-*stability* of K is that $\complement K$ be $(1,p)$-thick $(1,p)$ quasi-everywhere on ∂K. In the proof of Theorem 7.2 we have shown that, under the weaker sufficient condition $v = 0$ almost everywhere on $\complement K$, $v = 0$ quasi-everywhere on $\overline{\complement K}$. However, this condition would not be sufficient for the $(2,p)$-stability as shown in the example of L. I. Hedberg [**3**] (cf. Thm. 3.14, p. 100). □

Remark 7.3.
There is also a result in dimension $N = 2$ in Theorem 7 of L. I. Hedberg and Th. H. Wolff [**1**]: for p, $2 \leq p < \infty$, and q, $q^{-1} + p^{-1} = 1$, K is $(1,q)$-stable if and only if $\operatorname{Cap}_q(\partial K \cap e_{1,q}(\complement K)) = 0$, where $e_{1,q}(\complement K) = \{x \in \mathbf{R}^N : \complement K \text{ is } (1,q)\text{-thin at } x\}$. It says that the part of ∂K where $\complement K$ is thin has zero $(1,q)$-capacity. However, in the equivalence of parts (b) and (d) of Theorem 7, there seems to be a typo: the $(1,q)$-stability is defined as follows: "K is $(1,q)$-*stable* if $v = 0$ q.e. on $\complement K$ implies that $v \in W_0^{1,q}(\complement K)$." It should probably read "$v \in W_0^{1,q}(\operatorname{int} K)$." □

We now turn to our specific problem. Given a measurable subset Ω of a bounded open holdall $D \subset \mathbf{R}^N$, define the closed subspace

$$W_\diamond^{1,p}(\Omega; D) \overset{\text{def}}{=} \left\{ \psi \in W_0^{1,p}(D) \,:\, (1 - \chi_\Omega)\psi = 0 \text{ a.e.} \right\} \tag{7.7}$$

of $W_0^{1,p}(D)$. This space provides a relaxation of the Dirichlet boundary value problem to domains that are only measurable. The next theorem will show that in fact $W_\diamond^{1,p}(\Omega; D)$ can be embedded in $W_0^{1,p}(\operatorname{int} \overline{\Omega})$. Furthermore, for crack-free measurable subsets Ω of D, $W_\diamond^{1,p}(\Omega; D)$ coincides with $W_0^{1,p}(\operatorname{int} \Omega)$. This will yield the continuity of the homogeneous Dirichlet problem with respect to the open domains Ω in the family $L(D, r, \mathcal{O}, \lambda)$ of subsets of D verifying a uniform fat segment property.

Theorem 7.3. *Let D be bounded open in $\mathbf{R}^N$.*

(i) *For any measurable subset $\Omega \subset D$ such that* $\operatorname{int} \Omega \neq \varnothing$,

$$v \in W_\diamond^{1,p}(\Omega; D) \quad \Longrightarrow \quad \left| \begin{array}{l} \exists \textit{ a quasi-continuous representative } v^* \\ \textit{such that } v^*|_{\operatorname{int} \overline{\Omega}} \in W_0^{1,p}(\operatorname{int} \overline{\Omega}) \end{array} \right. \tag{7.8}$$

and $W_0^{1,p}(\operatorname{int} \Omega; D) \subset W_\diamond^{1,p}(\Omega; D) \subset W_0^{1,p}(\operatorname{int} \overline{\Omega}; D)$.

(ii) *For any measurable subset* $\Omega \subset D$ *such that* $\operatorname{int}\Omega \neq \varnothing$,

$$\left| \begin{array}{l} v \in W^{1,p}_{\diamond}(\Omega; D) \\ \text{and } \Omega \text{ crack-free} \end{array} \right. \quad \Longrightarrow \quad \left| \begin{array}{l} \exists \text{ a quasi-continuous representative } v^* \\ \text{such that } v^*|_{\operatorname{int}\Omega} \in W^{1,p}_0(\operatorname{int}\Omega) \end{array} \right. \tag{7.9}$$

and $W^{1,p}_0(\operatorname{int}\Omega; D) = W^{1,p}_{\diamond}(\Omega; D) = W^{1,p}_0(\operatorname{int}\overline{\Omega}; D)$.

Proof. (i) From Theorem 7.2 with $K = \overline{\Omega}$ and $\tilde{v}$, the extension by zero of v from D to $\mathbf{R}^N$. Given $\varphi \in W^{1,p}_0(\operatorname{int}\Omega)$, the zero extension $e_0\varphi \in W^{1,p}_0(\operatorname{int}\Omega; D)$ and $(1-\chi_{\operatorname{int}\Omega})\varphi = 0$ almost everywhere in D. This implies that $(1-\chi_\Omega)\varphi = 0$ almost everywhere in D and $\varphi \in W^{1,p}_{\diamond}(\Omega; D)$. (ii) From (i) and the equivalences (7.4): $\operatorname{int}\Omega = \operatorname{int}\overline{\Omega}$. □

Remark 7.4.
The condition that $v \in W^{1,p}_{\diamond}(\Omega; D)$ is weaker than the capacitary condition of D. Bucur and J.-P. Zolésio [**12**] and hence its conclusion that $v^* \in W^{1,p}_0(\operatorname{int}\overline{\Omega})$ is also weaker. It gives information only about the function on $\partial\overline{\Omega}$ rather than on the whole $\partial\Omega$. The "crack-free" property of Definition 7.1 is purely geometric and even set theoretic. It is different from and probably stronger than the capacitary condition of D. Bucur and J.-P. Zolésio [**12**]. Instead of forcing $\complement\Omega$ to be "uniformly thick" at quasi all points of the boundary $\partial\Omega$ in order to preserve the zero Dirichlet condition on the whole boundary, it erases the "interior boundary" $\partial\Omega\backslash\partial\overline{\Omega}$. Its advantage is that it does not require an a priori knowledge of the differential operator and does not necessitate the Maximum Principle or harmonic functions; its disadvantage is that it cannot handle problems where the trace of the function on $\partial\Omega\backslash\partial\overline{\Omega}$ is an important feature of the problem at hand. Yet, it should readily be extendable to Sobolev spaces on $C^{1,1}$-submanifolds of codimension 1 to deal with the Laplace–Beltrami or shell equations. □

7.2 Continuity and Optimization over $L(D, r, \mathcal{O}, \lambda)$

7.2.1 Continuity of the Classical Homogeneous Dirichlet Boundary Condition

In light of Theorem 4.1, it remains to find families of sets with properties that would enable us to sharpen part (i) or that would verify the stability condition $H^1_0(\Omega; D) = H^1_{\diamond}(\Omega; D)$. In the second case, the family of subsets of a bounded holdall D that satisfy the uniform fat segment property meets all the requirements.

Theorem 7.4. *Let D be a bounded open nonempty subset in $\mathbf{R}^N$ and Ω be open subsets of D. Denote by $y(\Omega) \in H^1_0(\Omega; D)$ the solution of the Dirichlet problem* (4.5) *on Ω. The map*

$$\Omega \mapsto y(\Omega) : L(D, r, \mathcal{O}, \lambda) \to H^1_0(D)\text{-strong} \tag{7.10}$$

is continuous with respect to any of the topologies on $L(D, r, \mathcal{O}, \lambda)$ of Theorem 13.1 *in Chapter* 7*, that is, d_Ω in $C(\overline{D})$ and $W^{1,p}(D)$, $d_{\complement\Omega}$ in $C(\overline{D})$ and $W^{1,p}(D)$, $b_\Omega, C(\overline{D})$ in $C(\overline{D})$ and $W^{1,p}(D)$, χ_Ω in $L^p(D)$, and $\chi_{\complement\Omega}$ in $L^p(D)$.*

Proof. Since all topologies are equivalent on $L(D,r,\mathcal{O},\lambda)$, consider a domain Ω and a sequence $\{\Omega_n\}$ such that $d_{\complement\Omega_n} \to d_{\complement\Omega}$ in $W^{1,p}(D)$ for some p, $1 \le p < \infty$. By Theorem 6.1 (ii) of Chapter 2, all sets in $L(D,r,\mathcal{O},\lambda)$ are crack-free since

$$\operatorname{int}\complement\Omega \ne \varnothing, \quad \overline{\complement\overline{\Omega}} = \overline{\operatorname{int}\complement\Omega} = \overline{\complement\Omega} \quad \Rightarrow b_\Omega = b_{\overline{\Omega}}.$$

By Theorem 7.3 (ii), $H^1_\diamond(\Omega;D) = H^1_0(\Omega;D)$. Finally, from part (ii) of Theorem 4.1, we get the continuity of the solutions of the classical Dirichlet problem. $\square$

7.2.2 Minimization/Maximization of the First Eigenvalue

Consider the first eigenvalue $\lambda^A(\Omega)$ introduced in (2.1)

$$\lambda^A(\Omega) \overset{\text{def}}{=} \inf_{0\ne\varphi\in H^1_0(\Omega;D)} \frac{\int_D A\nabla\varphi\cdot\nabla\varphi\,dx}{\int_D |\varphi|^2\,dx} \tag{7.11}$$

for the Laplace equation on the open Ω of D. It was shown in Theorem 2.2 that $\lambda^A(\Omega)$ is an upper semicontinuous function of Ω and that there is a solution to the maximization problem with respect to Ω. We now turn to the minimization of $\lambda^A(\Omega)$ with respect to Ω that requires the lower semicontinuity and hence the continuity of $\lambda^A(\Omega)$ with respect to Ω. To do that we choose the smaller family $L(D,r,\mathcal{O},\lambda)$ of sets verifying a uniform fat segment property.

Theorem 7.5. *Let D be a bounded open domain in $\mathbf{R}^N$ and A be a matrix function satisfying assumption* (2.2).

(i) *The mapping*

$$d_{\complement\Omega} \mapsto \lambda^A(\Omega) : L(D,r,\mathcal{O},\lambda) \to \mathbf{R}\cup\{+\infty\} \tag{7.12}$$

is continuous with respect to any of the topologies on $L(D,r,\mathcal{O},\lambda)$ of Theorem 13.1 *in Chapter 7, that is, d_Ω in $C(\overline{D})$ and $W^{1,p}(D)$, $d_{\complement\Omega}$ in $C(\overline{D})$ and $W^{1,p}(D)$, $b_\Omega, C(\overline{D})$ in $C(\overline{D})$ and $W^{1,p}(D)$, χ_Ω in $L^p(D)$, and $\chi_{\complement\Omega}$ in $L^p(D)$.*

(ii) *The minimization and maximization problems*

$$\inf_{\Omega\in L(D,r,\mathcal{O},\lambda)} \lambda(\Omega) \text{ and } \sup_{\Omega\in L(D,r,\mathcal{O},\lambda)} \lambda(\Omega) \tag{7.13}$$

have solutions in $L(D,r,\mathcal{O},\lambda)$. Moreover,

$$0 < \lambda^A(D) \le \inf_{\Omega\in L(D,r,\mathcal{O},\lambda)} \lambda(\Omega) \le \sup_{\Omega\in L(D,r,\mathcal{O},\lambda)} \lambda(\Omega) \le \beta\,\lambda^I(\mathcal{O}). \tag{7.14}$$

Proof. (i) *Continuity.* Go back to the proof of Theorem 2.2. Given $\Omega \in L(D,r,\mathcal{O},\lambda)$, consider a sequence $\{\Omega_n\}$ in $L(D,r,\mathcal{O},\lambda)$ such that $d_{\complement\Omega_n} \to d_{\complement\Omega}$ in $C(\overline{D})$. Let $\lambda^A(\Omega_n)$ and $u_n \in H^1_0(\Omega_n;D)$, $\|u_n\|_{L^2(D)} = 1$, be the eigenvalue and an eigenvector associated with Ω_n.

By definition of $L(D,r,\mathcal{O},\lambda)$, for all $x\in\partial\Omega$, there exists $A_x \in \mathrm{O(N)}$ such that for all $y \in B(x,r)\cap\overline{\Omega}$, $y + A_x\mathcal{O} \subset \Omega$. Fix $x_0 \in \partial\Omega$. There exist $z \in B(x_0,r)\cap\Omega$

and $\varepsilon > 0$ such that $\overline{B(z,\varepsilon)} \subset B(x_0, r) \cap \Omega$. As a result $\overline{B(z,\varepsilon)} + A_x\mathcal{O} \subset \Omega$ and $\overline{z + A_x\mathcal{O}} \subset \Omega$. By the compactivorous property, there exists N such that, for all $n > N$, $\overline{z + A_x\mathcal{O}} \subset \Omega_n$ and

$$\forall n > N, \quad \lambda_n = \lambda^A(\Omega_n) \leq \lambda^A(z + A_x\mathcal{O}) \leq \beta\lambda^I(z + A_x\mathcal{O}) = \beta\lambda^I(\mathcal{O})$$

since the eigenvalue is independent of a translation and a rotation of the set. This means that the sequence $\{\lambda_n\}$ of eigenvalues is bounded. By coercivity

$$\alpha\|\nabla u_n\}^2_{L^2(D)} \leq \lambda_n \, \|u_n\|^2_{L^2(D)} = \lambda_n$$

and $\{u_n\}$ is bounded in $H^1_0(D)$. There exist λ_0 and $u_0 \in H^1_\diamond(\Omega; D)$ and subsequences, still indexed by n, such that

$$\lambda_n \to \lambda_0, \quad \|u_n\|_{L^2(D)} = 1, \quad u_n \rightharpoonup u_0 \text{ in } H^1_0(D)\text{-weak} \quad \Rightarrow \|u_0\|_{L^2(D)} = 1.$$

But for all $\Omega \in L(D, r, \mathcal{O}, \lambda)$ we have $H^1_\diamond(\Omega; D) = H^1_0(\Omega; D)$ and $u_0 \in H^1_0(\Omega; D)$.

For any $\varphi \in \mathcal{D}(\Omega)$, $K = \operatorname{supp}\varphi \subset \Omega$ is compact and there exists $N(K)$ such that for all $n \geq N(K)$, $\operatorname{supp}\varphi \subset \Omega_n$. Therefore for all $n \geq N(K)$

$$u_n \in H^1_0(\Omega_n; D), \quad \int_D A\nabla u_n \cdot \nabla\varphi \, dx = \lambda_n \int_D u_n \, \varphi \, dx.$$

By letting $n \to \infty$

$$\forall \varphi \in \mathcal{D}(\Omega), \quad \int_D A\nabla u \cdot \nabla\varphi \, dx = \lambda_0 \int_D u \, \varphi \, dx,$$

and, by density of $\mathcal{D}(\Omega)$ in $H^1_0(\Omega; D)$, the variational equation is verified in $H^1_0(\Omega; D)$. Hence, $u_0 \in H^1_0(\Omega; D)$, $\|u\|_{L^2(\Omega)} = 1$, satisfies the variational equation

$$\forall \varphi \in H^1_0(\Omega; D), \quad \int_D A\nabla u_0 \cdot \nabla\varphi \, dx = \lambda_0 \int_D u \, \varphi \, dx.$$

Since $\lambda^A(\Omega)$ is the smallest eigenvalue with respect to $H^1_0(\Omega; D)$, $\lambda^A(\Omega) \leq \lambda_0$. To show that $\lambda_0 = \lambda^A(\Omega)$, it remains to prove that $\lambda_0 \leq \lambda^A(\Omega)$. For each n

$$\forall \varphi_n \in H^1_0(\Omega_n; D), \quad \lambda^A(\Omega_n) \leq \frac{\int_D A\nabla\varphi_n \cdot \nabla\varphi_n \, dx}{\|\varphi_n\|^2_{L^2(D)}}.$$

By the compactivorous property for each $\varphi \in \mathcal{D}(\Omega)$, $\varphi \neq 0$, there exists N such that $\operatorname{supp}\varphi \subset \Omega_n$ and $\varphi \in \mathcal{D}(\Omega_n)$. Therefore for all $n > N$

$$\lambda^A(\Omega_n) \leq \frac{\int_D A\nabla\varphi \cdot \nabla\varphi \, dx}{\|\varphi\|^2_{L^2(D)}}$$

and by going to the limit

$$\forall \varphi \in \mathcal{D}(\Omega), \quad \lambda_0 \leq \frac{\int_D A\nabla\varphi \cdot \nabla\varphi \, dx}{\|\varphi\|^2_{L^2(D}}.$$

By density this is true for all $\varphi \in H_0^1(\Omega; D)$, $\varphi \neq 0$, and $\lambda_0 \leq \lambda^A(\Omega)$. As a result $\lambda_0 = \lambda^A(\Omega)$. Therefore the whole sequence $\{\lambda^A(\Omega_n)\}$ converges to $\lambda^A(\Omega)$ and we have the continuity of the map $d_{\complement\Omega} \mapsto \lambda^A(\Omega)$. The strong continuity in $H_0^1(D)$ now follows from the convergence of $u_n \to u_0$ in $L^2(D)$-strong:

$$\begin{aligned}
&\alpha\|\nabla(u_n - u_0)\|^2_{L^2(D)} \\
&\leq \int_D A\nabla(u_n - u_0) \cdot \nabla(u_n - u_0)\, dx \\
&\quad = \int_D A\nabla u_n \cdot \nabla u_n \, dx + \int_D A\nabla u_0 \cdot \nabla u_0 \, dx - 2\int_D A\nabla u_n \cdot \nabla u_0 \, dx \\
&\quad = \lambda_n \int_D u_n \cdot u_n \, dx + \lambda_0 \int_D u_0 \cdot u_0 \, dx - 2\int_D A\nabla u_n \cdot \nabla u_0 \, dx \\
&\quad \to 2\lambda_0 \int_D u_0 \cdot u_0 \, dx - 2\int_D A\nabla u_0 \cdot \nabla u_0 \, dx = 0.
\end{aligned}$$

(ii) This follows from the continuity and the compactness of $L(D, r, \mathcal{O}, \lambda)$ in all the mentioned topologies. The inequalities have been proven in part (i). $\square$

Corollary 1. *Under the assumptions of Theorem* 7.5*, the maximization and minimization problems* (7.13) *also have solutions in* $L(D, r, \mathcal{O}, \lambda)$ *under an equality or an inequality constraint of the form*

$$\mathrm{m}(\Omega) = \alpha,\ \mathrm{m}(\Omega) \leq \alpha, \ \textit{or}\ \mathrm{m}(\Omega) \geq \alpha$$

provided that there exists $\Omega' \in L(D, r, \mathcal{O}, \lambda)$ *such that*

$$\mathrm{m}(\Omega') = \alpha,\ \mathrm{m}(\Omega') \leq \alpha, \ \textit{or}\ \mathrm{m}(\Omega') \geq \alpha.$$

Proof. This follows from Theorem 13.1 and its Corollaries 1 and 2 of section 13 in Chapter 7, where it is shown that the convergence is strong not only in the Hausdorff topologies but also for the associated characteristic functions in $L^1(D)$, thus making the volume function continuous with respect to open domains in $L(D, r, \omega, \lambda)$. For the spaces of convex subsets recall from Theorem 8.6 of Chapter 6 and Theorem 10.2 of Chapter 7 that the compactness remains true in the $W^{1,1}(D)$-topology. $\square$

8 Continuity under Capacity Constraints

We have established the continuity of the homogeneous Dirichlet problem, that is, the continuity of the function $d_{\complement\Omega} \mapsto u(\Omega) : C_d^c(D) \to H_0^1(D)$-strong for the family of domains verifying a uniform fat segment property. Domains with cracks lie between families of $L(D, r, \mathcal{O}, \lambda)$ type and $C_d^c(D)$. Topologies using the characteristic function χ_Ω are excluded since they do not see interior boundaries of zero measure. Families that contains sets with cracks must be chosen in such a way that the homogeneous Dirichlet boundary condition on the sequence of solutions $u(\Omega_n)$ in the converging sets Ω_n is preserved by the solution $u(\Omega)$ on the boundary of the limit domain Ω in section 4. We have to be careful since the relative capacity of the complement of the domains near the boundary must not vanish. In order to handle this point, we introduce the following concepts and terminology related to the local capacity of the complement near the boundary points. The following definition is due to J. Heinonen, T. Kilpelainen, and O. Martio [**1**, pp. 114–115].

Definition 8.1. (i) Given $r > 0$, a set $E \subset \mathbf{R}^{\mathrm{N}}$, and a point x, the function

$$\operatorname{cap}_{x,r}(E) \stackrel{\text{def}}{=} \frac{\operatorname{Cap}_p(E \cap B(x,r), B(x,2r))}{\operatorname{Cap}_p(B(x,r), B(x,2r))} \tag{8.1}$$

is called the *capacity density function.*

(ii) We say that a set E is *thick* at a point x if

$$\int_0^1 \left[\frac{\operatorname{Cap}_p(E \cap B(x,r), B(x,2r))}{\operatorname{Cap}_p(B(x,r), B(x,2r))}\right]^{1/(p-1)} \frac{dr}{r} = \infty. \tag{8.2}$$

This is called the *Wiener criterion.*

(iii) We say that an open set Ω satisfies the *Wiener density condition* if

$$\forall x \in \partial\Omega, \quad \int_0^1 \left[\frac{\operatorname{Cap}_p(\complement\Omega \cap B(x,r), B(x,2r))}{\operatorname{Cap}_p(B(x,r), B(x,2r))}\right]^{1/(p-1)} \frac{dr}{r} = \infty. \tag{8.3}$$

This means that the set $\complement\Omega$ is *thick* at every boundary point $x \in \partial\Omega$ (that is, $\complement\Omega$ has "sufficient" capacity around each boundary point). □

Remark 8.1.
If Ω satisfies the $(r,\ c)$-capacity density condition, the complement is thick in any point of the boundary. □

The Wiener criterion can be strengthened by introducing the following condition (J. Heinonen, T. Kilpelainen, and O. Martio [**1**, p. 127]).

Definition 8.2. (i) Given $r > 0$, $c > 0$, and an open set Ω, $\complement\Omega$ is said to satisfy the (r,c)-*capacity density condition* if

$$\forall x \in \partial\Omega,\ \forall r',\ 0 < r' < r, \quad \frac{\operatorname{Cap}_p(\complement\Omega \cap \overline{B(x,r')}, B(x,2r'))}{\operatorname{Cap}_p(\overline{B(x,r')}, B(x,2r'))} \geq c. \tag{8.4}$$

We also say that $\complement\Omega$ is *uniformly* (r,c)-*thick.*

(ii) Given $r > 0$, $c > 0$, and an open set Ω, $\complement\Omega$ is said to satisfy the *strong* (r,c)-*capacity density condition* if

$$\forall x \in \partial\Omega,\ \forall r',\ 0 < r' < r, \quad \frac{\operatorname{Cap}_p(\complement\Omega \cap B(x,r'), B(x,2r'))}{\operatorname{Cap}_p(B(x,r'), B(x,2r'))} \geq c. \tag{8.5}$$

We also say that $\complement\Omega$ is *strongly uniformly* (r,c)-*thick.*

(iii) For r, $0 < r < 1$, and $c > 0$ define the following family of open subsets of D:

$$\mathcal{O}_{c,r}(D) \stackrel{\text{def}}{=} \left\{ \Omega \subset D : \begin{array}{c} \Omega \text{ satisfies the strong} \\ (r,c)\text{-capacity density condition} \end{array} \right\}. \tag{8.6}$$

□

Example 8.1.
Only a few capacities can be computed explicitly. The computation of the *spherical condensers* $\text{Cap}_p(\overline{B(x,r)}, B(x,R))$ and $\text{Cap}_p(B(x,r), B(x,R))$ leads to some simplifications in the denominator of the above expression (cf. J. Heinonen, T. Kilpeläinen, and O. Martio [**1**, p. 35]). For $0 < r < R < \infty$

$$\text{Cap}_p(\overline{B(x,r)}, B(x,R)) = \text{Cap}_p(B(x,r), B(x,R) \tag{8.7}$$

and for $p > 1$

$$\text{Cap}_p(\overline{B(x,r)}, B(x,R)) = \begin{cases} \omega^{N-1} \left(\dfrac{|N-p|}{p-1} \right)^{p-1} \left| R^{\frac{p-N}{p-1}} - r^{\frac{p-N}{p-1}} \right|^{1-p}, & p \neq N, \\ \omega^{N-1} \left(\log \dfrac{R}{r} \right)^{1-N}, & p = N, \end{cases}$$

where ω^{N-1} is the volume (Lebesgue measure) of the ball in $\mathbf{R}^{N-1}$. In particular

$$\begin{aligned} &\text{Cap}_p(\overline{B(x,r)}, B(x,2r)) = c_1(N,p)\, r^{N-p}, && 1 < p, \\ &\text{Cap}_p(\overline{B(x,r)}) = \text{Cap}_p(\overline{B(x,r)}, \mathbf{R}^{\mathrm{N}}) = c_2(N,p)\, r^{N-p}, && 1 < p < N, \\ &\text{Cap}_p(\{x_0\}, \mathbf{R}^{\mathrm{N}}) = c_3(N,p)\, r^{N-p}, && p < N, \end{aligned} \tag{8.8}$$

for constants $c_i(N,p)$ that depend only on p and N. □

Remark 8.2.
Definition 8.2 (i) uses a slightly different definition of the previous *capacity density function* of Definition 8.1:

$$\overline{\text{cap}}_{x,r}(E) \stackrel{\text{def}}{=} \frac{\text{Cap}_p(E \cap \overline{B(x,r)}, B(x,2r))}{\text{Cap}_p(\overline{B(x,r)}, B(x,2r))}$$

instead of

$$\text{cap}_{x,r}(E) \stackrel{\text{def}}{=} \frac{\text{Cap}_p(E \cap B(x,r), B(x,2r))}{\text{Cap}_p(\overline{B(x,r)}, B(x,2r))}.$$

Note that

$$\text{Cap}_p(B(x,r), B(x,2r)) = \text{Cap}_p(\overline{B(x,r)}, B(x,2r)) = c_1(N,p)\, r^{N-p}, \quad 1 < p,$$

and $\text{Cap}_p(E \cap B(x,r), B(x,2r)) \leq \text{Cap}_p(E \cap \overline{B(x,r)}, B(x,2r))$ imply that

$$\forall r > 0, \forall x, \quad \overline{\text{cap}}_{x,r}(E) \geq \text{cap}_{x,r}(E)$$

and the *strong* (r,c)*-capacity density condition* implies the (r,c)*-capacity density condition*. But for an open subset Ω of the bounded open subset D of $\mathbf{R}^N$, the following two conditions are equivalent:

$$\forall x \in \partial\Omega,\ \forall\, 0 < r' < r, \quad \frac{\mathrm{Cap}_p(\complement\Omega \cap B(x,r'), B(x,2r'))}{\mathrm{Cap}_p(B(x,r'), B(x,2r'))} \geq c \tag{8.9}$$

and

$$\forall x \in \partial\Omega,\ \forall 0 < r' < r, \quad \frac{\mathrm{Cap}_p(\complement\Omega \cap \overline{B(x,r')}, B(x,2r'))}{\mathrm{Cap}_p(\overline{B(x,r')}, B(x,2r'))} \geq c, \tag{8.10}$$

and also from the $\mathcal{O}_{c,r}(D)$ family viewpoint the two definitions are equivalent. It is clear that (8.9) implies (8.10), but the converse is not obvious. □

Recall Theorem 4.1 and the notation u_n for the solution of (4.5) in $H_0^1(\Omega_n; D)$ and $u \in H_\diamond^1(\Omega; D)$ for the weak limit in $H_0^1(D)$ which satisfies (4.8). The following theorem gives the main continuity result.

Theorem 8.1. *Let D be a bounded open nonempty subset of $\mathbf{R}^N$ of class C^2 for $N \geq 3$ and Ω be an open subset of D. Assume that A is a matrix function that satisfies the conditions* (4.4) *and that the elements of A belong to $C^1(\overline{D})$. Denote by $u(\Omega) \in H_0^1(\Omega; D)$ the solution of the Dirichlet problem* (4.5) *on Ω. The map*

$$\Omega \mapsto u(\Omega) : \mathcal{O}_{c,r}(D) \to H_0^1(D)\text{-strong} \tag{8.11}$$

is continuous with respect to the uniform topology $C(\overline{D})$ on $C_d^c(D)$.

Proof. When the dimension N of the space is equal to 1, $H^1(D) \subset C(\overline{D})$ and no approximation of f is necessary. When $N \geq 2$, we first prove the result for $f \in H^s(D)$, $s > N/2 - 2$ (since for D of class C^2 the corresponding solution will belong to $H^{s+2}(D) \subset C(\overline{D})$), and then, by approximation of f, we prove it for $f \in H^{-1}(D)$.

(i) First consider $f \in H^s(D)$, $s > N/2 - 2$, for $N \geq 2$ ($s = -1$ for $N = 1$). The proof will make use of the following results from J. Heinonen, T. Kilpelainen, and O. Martio [**1**, Chap. 3, Thm. 3.70, p. 80].

Lemma 8.1. *Let the assumptions of Theorem* 8.1 *on the open domain D and the matrix function A be satisfied. Let v be an $\mathcal{A}$-harmonic function; that is, $v \in W_{\mathrm{loc}}^{1,p}(\Omega)$ is a weak solution of equation*

$$\forall \varphi \in \mathcal{D}(\Omega), \quad \int_D A\nabla v(\Omega) \cdot \nabla\varphi \, dx = 0 \tag{8.12}$$

in the open set Ω. Then there exists a continuous function u on Ω such that $v(\Omega) = u$ almost everywhere.

Note that the continuous representative from Lemma 8.1 is in fact a quasi-continuous $H_0^1(\Omega)$ representative. Indeed, let $v_1 = v$ almost everywhere and let v_1 be continuous on Ω. We want to show that v_1 is a quasi-continuous representative of v. There exists a quasi-continuous representative v_2 of v, which is equal to v almost everywhere. So v_1 is continuous, v_2 is quasi-continuous, and $v_1 = v_2$ almost everywhere. Using J. Heinonen, T. Kilpelainen, and O. Martio [**1**, Thm. 4.12], we get $v_1 = v_2$ quasi-everywhere. Always from J. Heinonen, T. Kilpelainen, and O. Martio [**1**, Thm. 6.27, p. 122] we get the following lemma.

Lemma 8.2. *Let the assumptions of Theorem* 8.1 *on the open domain D and the matrix function A be satisfied. Let Ω belong to $\mathcal{O}_{c,r}(D)$. If $\theta \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ and if h is an $\mathcal{A}$-harmonic function in Ω such that $h - \theta \in W_0^{1,p}(\Omega)$, then*

$$\forall x_0 \in \partial\Omega, \quad \lim_{x \to x_0} h(x) = \theta(x_0).$$

Note that the fact that Ω belongs to $\mathcal{O}_{c,r}(D)$ involves the notion of *thickness in any point* of its boundary, which necessarily occurs in the proof of the lemma (cf. Definition 8.1). Returning to Theorem 8.1, it will be sufficient to prove the continuity for a subsequence of $\{\Omega_n\}$. By Theorem 4.1 there exists a subsequence of $\{\Omega_n\}$, still indexed by n, such that u_n weakly converges to u in $H_0^1(D)$, and $u \in H^1_\diamond(\Omega; D)$ satisfies (4.8) in Ω. We now prove that under our assumptions $u_\Omega = u|_\Omega \in H_0^1(\Omega)$, which implies that $u = e_\Omega(u_\Omega)$ and $u \in H_0^1(\Omega; D)$ since $(1 - \chi_\Omega)u = (1 - \chi_\Omega)e_\Omega(u_\Omega) = 0$ almost everywhere in D. For that purpose we use Lemma 6.1, which says that it is sufficient to prove that $u = 0$ quasi-everywhere in $D \backslash \Omega$ for some quasi-continuous representative u. From Theorem 7.3 (i) $H_0^1(\operatorname{int}\Omega; D) \subset H^1_\diamond(\Omega; D) \subset H_0^1(\operatorname{int}\overline{\Omega}; D)$ and hence $u = 0$ quasi-everywhere in $D \backslash \overline{\Omega}$. So it remains to show that $u = 0$ quasi-everywhere in $\partial\Omega \cap D$. From the Banach–Saks theorem (cf. I. Ekeland and R. Temam [**1**]) there exists a sequence of averages

$$\psi_n \stackrel{\text{def}}{=} \sum_{k=n}^{N_n} \alpha_k^n u_n, \quad 0 \le \alpha_k^n \le 1 \;, \;\; \sum_{k=n}^{N_n} \alpha_k^n = 1$$

such that $\psi_n \to u$ in $H_0^1(D)$. Because of the strong convergence of $\{\psi_n\}$ to u in $H_0^1(D)$, we have

$$\psi_n(x) \to u(x) \quad \text{q.e. in } D$$

for a subsequence of $\{\psi_n\}$, still indexed by n. Let G_0 be the set of zero capacity on which $\{\psi_n(x)\}$ does not converge to $u(x)$. Given $x \in D \backslash (\Omega \cup G_0)$, we prove that, for all $\varepsilon > 0$, $|u(x)| < \varepsilon$. We have

$$|u(x)| \le |u(x) - \psi_n(x)| + |\psi_n(x)|.$$

There exists $N_{\varepsilon,x} > 0$ such that for all $n > N_{\varepsilon,x}$,

$$|u(x) - \psi_n(x)| < \varepsilon/2.$$

It remains to show that there exists N' such that for all $n \ge N'$, $|u_n(x)| < \varepsilon/2$ implies $\psi_n(x)| < \varepsilon/2$. Denote by u_D the solution of (4.5) in D. By assumption on D and f, the solution u_D is continuous in $\overline{D}$. Subtracting the corresponding equations, we obtain

$$\forall \varphi \in H_0^1(\Omega_n; D), \quad \int_D A\nabla(u_D - u_n) \cdot \nabla\varphi \, dx = 0,$$

$$\forall \varphi \in H_0^1(\Omega_n), \quad \int_{\Omega_n} A\nabla(u_D - u_{\Omega_n}) \cdot \nabla\varphi \, dx = 0.$$

Define

$$\tilde{h}_n \stackrel{\text{def}}{=} u_D - u_n, \quad h_n \stackrel{\text{def}}{=} u_D|_{\Omega_n} - u_{\Omega_n}, \quad \theta_n \stackrel{\text{def}}{=} u_D|_{\Omega_n}.$$

Therefore the restriction h_n of $\tilde{h}_n$ to Ω_n is $\mathcal{A}$-harmonic in Ω_n and, from Lemma 8.1, continuous in Ω_n. Moreover we have the continuity of u_{Ω_n} in the closure $\overline{\Omega}_n$, and u_{Ω_n} is zero on the boundary. To show that, we use Lemma 8.2 with h_n and θ_n. By definition, $h_n - \theta_n = -u_{\Omega_n}$ belongs to $H_0^1(\Omega_n)$. From the continuity of u_D we obtain that the continuous extension $\tilde{h}_n$ of h_n is equal to u_D on $\partial\Omega_n$. Hence the extension u_n of u_{Ω_n} to the boundary is zero. Using J. Heinonen, T. Kilpelainen, and O. Martio [**1**, Thm. 6.44], we obtain that if h_n is δ-Hölderian on $\partial\Omega_n$, then there exists δ_1, $\delta \le \delta_1 < 1$, such that it is δ_1-Hölderian on all Ω_n. We have that u_D is $(s+2-N/2)$-Hölderian on D (with $s+2-N/2 > 0$), with a constant M, because of the assumption on f and D. Finally, we get

$$\forall x, y \in \partial\Omega_n, \quad |h_n(x) - h_n(y)| = |u_D(x) - u_D(y)| \le M|x-y|^{s-N/2+2}.$$

So there exist $\delta_1 = \delta_1(N, \beta/\alpha, c)$ and

$$M_{1,n} = 80Mr^{-2}\max\{1, (\operatorname{diam}(\Omega_n))2\} \le 80Mr^{-2}\max\{1, (\operatorname{diam}(B))2\} = M_1$$

such that

$$\forall x, y \in \Omega_n, \quad |h_n(x) - h_n(y)| \le M_{1,n}|x-y|^{\delta_1} \le M_1|x-y|^{\delta_1}.$$

By a simple argument we obtain that this inequality holds in D, and hence there exists δ_2, $\delta_1 \le \delta_2 < 1$, such that for all $x, y \in D$ we have

$$\begin{aligned}|u_n(x) - u_n(y)| &\le |\tilde{h}_n(x) - \tilde{h}_n(y)| + |u_D(x) - u_D(y)| \\ &\le M_1|x-y|^{\delta_1} + M|x-y| \le M_2|x-y|^{\delta_2}.\end{aligned}$$

Choose $R > 0$ such that $M_2 R^{\delta_2} < \varepsilon/2$. Because of the H^c-convergence of Ω_n to Ω there exists an integer $n_R > 0$ such that for all $n \ge n_R$ we have $(D\backslash\Omega_n) \cap B(x,R) \ne \varnothing$. For $x_n \in (D\backslash\Omega_n) \cap B(x,R)$,

$$|u_n(x)| = |u_n(x) - u_n(x_n)| \le M_2|x-x_n|^{\delta_2} \le M_2 R^{\delta_2} \le \varepsilon/2$$

because $u_n(x_n) = 0$. So

$$\forall n > n_R, \quad |\psi_n(x)| = \left|\sum_{k=n}^{N_n} \alpha_k^n u_n(x)\right| \le \sum_{k=n}^{N_n} \alpha_k^n \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

Finally, we obtain $|u(x)| \le \varepsilon$. Because ε was arbitrary, we have $u(x) = 0$ quasi-everywhere on $D\backslash\Omega$, which implies that $u \in H_0^1(\Omega; D)$ and $u_\Omega = u|_\Omega \in H_0^1(\Omega)$. The strong convergence of $\{u_n\}$ to u now follows from Theorem 4.1 and $u = e_\Omega(u_\Omega)$.

(ii) In the next step we prove that the continuity is preserved for $f \in H^{-1}(D)$. The main idea is to use the continuous dependence of the solution $u_{\Omega,f} \in H_0^1(\Omega; D)$

with respect to f, which is uniform in Ω. Indeed, let $\Omega \subset D$, and let $f, g \in H^{-1}(D)$. Then, by a simple subtraction of the equations, we get

$$\int_D |\nabla(u_{\Omega,f} - u_{\Omega,g})|^2\,dx = \langle f-g, u_{\Omega,f} - u_{\Omega,g}\rangle$$
$$\Rightarrow \|u_{\Omega,f} - u_{\Omega,g}\|_{H_0^1(D)} \le \|f-g\|_{H^{-1}(D)}.$$

So, let $f \in H^{-1}(D)$ and $f_\varepsilon = f * \rho_\varepsilon$ for some mollifier ρ_ε. Letting $\varepsilon \to 0$ we have $\|f_\varepsilon - f\|_{H^{-1}(D)} \to 0$. Let $\{\Omega_n\} \subset \mathcal{O}_{c,r}(D)$ be a sequence that converges in the H^c-topology to an open set Ω ($d_{\complement\Omega_n} \to d_{\complement\Omega}$). Then we have

$$u_{\Omega_n,f_\varepsilon} \xrightarrow{H_0^1(D)} u_{\Omega,f_\varepsilon}$$

because $f_\varepsilon \in H^s(D)$ and, from the previous considerations,

$$u_{\Omega_n,f_\varepsilon} \xrightarrow{H_0^1(D)} u_{\Omega_n,f}$$

uniformly in Ω_n. Given $\delta > 0$, we get

$$\|u_{\Omega_n,f} - u_{\Omega,f}\|_{H_0^1(D)} \le \|u_{\Omega_n,f} - u_{\Omega_n,f_\varepsilon}\|_{H_0^1(D)}$$
$$+ \|u_{\Omega_n,f_\varepsilon} - u_{\Omega,f_\varepsilon}\|_{H_0^1(D)} + \|u_{\Omega,f_\varepsilon} - u_{\Omega,f}\|_{H_0^1(D)}.$$

Choose ε sufficiently small such that $\|f_\varepsilon - f\|_{H^{-1}(D)} < \delta/4$, and for each ε choose $n_{\varepsilon,\delta} > 0$ such that

$$\forall n > n_{\varepsilon,\delta}, \quad \|u_{\Omega_n,f_\varepsilon} - u_{\Omega,f_\varepsilon}\|_{H_0^1(D)} < \delta/2$$
$$\Rightarrow \forall n > n_{\varepsilon,\delta}, \quad \|u_{\Omega_n,f} - u_{\Omega,f}\|_{H_0^1(D)} \le \delta.$$

As δ was arbitrary, the proof is complete. □

We can now recover the result of V. Šverák [**2**] in dimension $N = 2$ using the fact that for a fixed number $c > 0$ the family

$$\{d_{\complement\Omega} : \Omega \text{ open } \subset D \text{ and } \#(\complement\Omega) \le c\}$$

is compact in $C(\overline{D})$, where $\#(\complement\Omega)$ denotes the number of connected components of $\complement\Omega$ (cf. Theorem 2.5 (iv) in Chapter 6 and Definition 2.2 in Chapter 2).

Theorem 8.2. *Let the assumptions of Theorem* 8.1 *on the matrix function A be satisfied. Let $N = 2$ and let $c > 0$ be a positive integer. Define the set*

$$\mathcal{O}_c \stackrel{\text{def}}{=} \{\Omega : \Omega \textit{ open } \subset D \textit{ and } \#(\complement\Omega) \le c\}.$$

Then the set $\mathcal{O}_c$ is compact and the map

$$\Omega \mapsto u(\Omega) : \mathcal{O}_c \to H_0^1(D)\text{-strong}$$

is continuous for the $C(\overline{D})$-topology on $C_d^c(D)$.

A proof of the result of V. Šverák [**2**] is given in D. Bucur [**1**] as a consequence of Theorem 8.1. The main idea of the proof is to consider $f \geq 0$ (because of the decomposition of $f = f^+ - f^-$) and a sequence $\{\Omega_n\} \subset \mathcal{O}_c$ such that Ω_n converges in the H^c-topology to an open set Ω, and $u_{\Omega_n} \rightharpoonup u$ in $H_0^1(D)$. He constructs extensions Ω_n^+ of the domains Ω_n with the following properties:

$$d_{\complement\Omega_n^+} \longrightarrow d_{\complement\Omega^+}, \quad \operatorname{cap}(\Omega^+\backslash\Omega) = 0, \quad \{\Omega_n^+\} \subset \mathcal{O}_{c,r}(D),$$

where c and r are suitable constants. In fact it can be proved that Ω_n^+ satisfy this capacity density condition, because in an open connected set any two points are linked by a continuous curve that lies in the set, and in the bidimensional case a curve has a positive capacity. From the previous theorem we get $u_{\Omega_n^+} \rightharpoonup u_{\Omega^+}$. We easily obtain that $u_{\Omega^+} = u_\Omega \geq u \geq 0$, which will imply that $u = 0$ quasi-everywhere on $\complement\Omega$, and this concludes the proof. For details, see D. Bucur [**1**].

9 Compact Families $\mathcal{O}_{c,r}(D)$ and $L_{c,r}(\mathcal{O}, D)$

We first prove the compactness of the family $\mathcal{O}_{c,r}(D)$ for the topology of uniform convergence of the distance function of the complement in $\overline{D}$. Then, by analogy with the "fat segment property," we specialize to a "thick set property" that generalizes the *flat cone property* of D. Bucur and J.-P. Zolésio [**10**].

9.1 Compact Family $\mathcal{O}_{c,r}(D)$

Theorem 9.1. *Let $D \subset \mathbf{R}^N$ be bounded open, let $c > 0$, and let $r > 0$.*

(i) *The subset*

$$\{d_{\complement\Omega} : \Omega \in \mathcal{O}_{c,r}(D)\} \tag{9.1}$$

of $C_d^c(D)$ is compact for the uniform topology of $C(\overline{D})$.

(ii) *Given a sequence of open subsets $\{\Omega_n\}$ and Ω in $\mathcal{O}_{c,r}(D)$, and the solutions $u(\Omega_n) \in H_0^1(\Omega_n; D)$ and $u(\Omega) \in H_0^1(\Omega; D)$ of the Dirichlet problem (4.5) on Ω_n and Ω, then*

$$d_{\complement\Omega_n} \to d_{\complement\Omega} \text{ in } C(\overline{D}) \quad \Rightarrow u(\Omega_n) \to u(\Omega) \text{ in } H^1(D)\text{-strong}. \tag{9.2}$$

We shall need the following technical lemma.

Lemma 9.1.[3] *Let D, $D \neq \varnothing$, be a bounded open subset of $\mathbf{R}^N$. Consider a sequence $\{\Omega_n\}$, $\complement\Omega_n \neq \varnothing$, of open subsets of D. Assume that*

$$d_{\complement\Omega_n} \to d_{\complement\Omega_0} \text{ in } C^0(\overline{D})$$

for some open $\Omega_0 \subset D$, $\partial\Omega_0 \neq \varnothing$.

[3] Part (ii) was proved by D. Bucur and J.-P. Zolésio [**10**, Lem. 3.1, p. 686] by a contradiction argument. We give a direct proof using the function b_Ω.

(i) *There exists N such that*

$$\forall n > N, \quad \partial\Omega_n \neq \varnothing.$$

(ii) *There exists a subsequence $\{\Omega_{n_k}\}$, $\partial\Omega_{n_k} \neq \varnothing$, of $\{\Omega_n : n > N\}$ such that for all $\varepsilon > 0$, there exists K such that*

$$\forall k > K, \quad \partial\Omega_0 \subset U_\varepsilon(\partial\Omega_{n_k}) \stackrel{\text{def}}{=} \{x \in \mathbf{R}^{\mathrm{N}} : d_{\partial\Omega_{n_k}}(x) < \varepsilon\}$$

and

$$\forall x_0 \in \partial\Omega_0,\ \exists \{x_{n_k}\},\ x_{n_k} \in \partial\Omega_{n_k} \text{ such that } x_{n_k} \to x_0.$$

Proof. (i) By the boundedness assumption on D, $\complement\Omega_n \neq \varnothing$ and $\complement\Omega_0 \neq \varnothing$. Since $\partial\Omega_0 \neq \varnothing$, $\overline{\Omega_0} \neq \varnothing$, and $\Omega_0 \neq \varnothing$,

$$\exists x_0 \in \Omega_0 \text{ such that } d \stackrel{\text{def}}{=} d_{\complement\Omega_0}(x_0) > 0.$$

There exists N such that

$$\begin{gathered}\forall n > N, \quad \|d_{\complement\Omega_n} - d_{\complement\Omega_0}\|_{C^0(\overline{D})} < d/2 \\ \Rightarrow d_{\complement\Omega_n}(x_0) \geq d_{\complement\Omega_0}(x_0) - |d_{\complement\Omega_n}(x_0) - d_{\complement\Omega_0}(x_0)| > d - d/2 > 0 \\ \forall n > N, \quad \Omega_n \neq \varnothing \quad \Rightarrow \partial\Omega_n \neq \varnothing.\end{gathered}$$

(ii) Since, for $n > N$, $\partial\Omega_n \neq \varnothing$, the subsequence $\{b_{\Omega_n} : n > N\}$ belongs to the compact family $C_b(D)$. Therefore there exist a subsequence, denoted by $\{b_{\Omega_n}\}$, and $\Omega \subset \overline{D}$, $\partial\Omega \neq \varnothing$, such that $b_{\Omega_n} \to b_\Omega$. Hence

$$\begin{gathered}d_{\complement\Omega_n} \to d_{\complement\Omega} \text{ and } d_{\partial\Omega_n} \to d_{\partial\Omega} \\ \Rightarrow \complement\Omega_0 = \overline{\complement\Omega} \quad \Rightarrow \Omega_0 = \complement\overline{\complement\Omega} = \operatorname{int}\Omega \quad \Rightarrow \partial\Omega_0 \subset \partial\Omega.\end{gathered}$$

Finally, for all $\varepsilon > 0$, there exists $N' > N$ such that

$$\begin{gathered}\forall n > N', \quad \|d_{\partial\Omega_n} - d_{\partial\Omega}\|_{C^0(\overline{D})} < \varepsilon \quad \Rightarrow \forall x \in \partial\Omega_0, \quad d_{\partial\Omega_n}(x) < \varepsilon \\ \Rightarrow \forall n > N', \quad \partial\Omega_0 \subset U_\varepsilon(_{\partial\Omega_n}) \stackrel{\text{def}}{=} \{x \in \mathbf{R}^{\mathrm{N}} : d_{\partial\Omega_n}(x) < \varepsilon\}.\end{gathered}$$

As a result, for all $x \in \partial\Omega_0$, there exists $\{x_n \in \partial\Omega_{x_n}\}$ such that

$$|x_n - x| = d_{\partial\Omega_n}(x) \to d_{\partial\Omega}(x) = 0. \qquad \square$$

Proof of Theorem 9.1. (i) It is sufficient to prove that for all $c > 0$ and all r, $0 < r < 1$, $\mathcal{O}_{c,r}(D)$ is closed in $C(\overline{D})$ for the uniform convergence topology. Consider a Cauchy sequence $\{d_{\complement\Omega_n}\}$ in $\mathcal{O}_{c,r}(D)$. There exists an open set $\Omega \subset D$ such that $d_{\complement\Omega_n} \to d_{\complement\Omega}$ in $C(\overline{D})$. If $\partial\Omega = \varnothing$, then $\complement\Omega \cap \overline{\Omega} = \varnothing$, $\complement\overline{\Omega} \subset \complement\Omega \subset \complement\overline{\Omega}$, and $\Omega = \varnothing \in \mathcal{O}_{c,r}(D)$. If $\partial\Omega \neq \varnothing$, by Lemma 9.1, there exist $K' \geq K$ and a subsequence $\{\complement\Omega_{n_k}\}_{k > K'}$ such that

$$\forall k > K', \quad \|d_{\complement\Omega_{n_k}} - d_{\complement\Omega}\|_{C(\overline{D})} \leq \varepsilon \quad (\Rightarrow \complement\Omega_{n_k} \subset (\complement\Omega)_\varepsilon = \{x \in \mathbf{R}^{\mathrm{N}} : d_{\complement\Omega}(x) \leq \varepsilon\})$$

and, for each $x \in \partial\Omega$, there exists $x_{n_k} \in \partial\Omega_{n_k}$ such that

$$\forall k > K', \quad |x_{n_k} - x| \le \varepsilon.$$

Define the translations

$$\tau_k(y) \stackrel{\text{def}}{=} y + x_{n_k} - x, \quad k > K'.$$

Then, for all $\rho > 0$, $\tau_k(B(x_{n_k}, \rho)) = B(x, \rho)$ and

$$\begin{aligned} &\|d_{\tau_k(\complement\Omega_{n_k})} - d_{\complement\Omega_{n_k}}\|_{C(\overline{D})} \le |x_{n_k} - x| \le \varepsilon \\ \Rightarrow\ &\tau_k(\complement\Omega_{n_k}) \subset (\complement\Omega_{n_k})_\varepsilon \subset ((\complement\Omega)_\varepsilon)_\varepsilon \subset (\complement\Omega)_{2\varepsilon}. \end{aligned}$$

The implication $((\complement\Omega)_\varepsilon)_\varepsilon \subset (\complement\Omega)_{2\varepsilon}$ follows from

$$\begin{aligned} &\forall y \in (\complement\Omega)_\varepsilon,\ d_{\complement\Omega}(z) \le d_{\complement\Omega}(y) + |z-y| \le \varepsilon + |z-y| \quad \Rightarrow d_{\complement\Omega}(z) \le \varepsilon + d_{(\complement\Omega)_\varepsilon}(z) \\ &\qquad \Rightarrow \forall z \in ((\complement\Omega)_\varepsilon)_\varepsilon, \quad d_{\complement\Omega}(z) \le 2\varepsilon \quad \Rightarrow ((\complement\Omega)_\varepsilon)_\varepsilon \subset (\complement\Omega)_{2\varepsilon}. \end{aligned}$$

From the identities $\tau_k(\complement\Omega_{n_k}) \subset (\complement\Omega)_{2\varepsilon}$ and $\tau_k(B(x_{n_k}, \rho)) = B(x, \rho)$, for all r', $0 < r' < r$,

$$\begin{aligned} c &\le \frac{\mathrm{Cap}_p(\complement\Omega_{n_k} \cap B(x_{n_k}, r'), B(x_{n_k}, 2r'))}{\mathrm{Cap}_p(B(x_{n_k}, r'), B(x_{n_k}, 2r'))} \\ &= \frac{\mathrm{Cap}_p(\tau_k(\complement\Omega_{n_k} \cap B(x_{n_k}, r')), \tau_k(B(x_{n_k}, 2r')))}{\mathrm{Cap}_p(\tau_k(B(x_{n_k}, r')), \tau_k(B(x_{n_k}, 2r')))} \\ &= \frac{\mathrm{Cap}_p(\tau_k(\complement\Omega_{n_k}) \cap B(x, r'), B(x, 2r'))}{\mathrm{Cap}_p(B(x, r'), B(x, 2r'))} \le \frac{\mathrm{Cap}_p((\complement\Omega)_{2\varepsilon} \cap B(x, r'), B(x, 2r'))}{\mathrm{Cap}_p(B(x, r'), B(x, 2r'))}, \end{aligned}$$

since Cap_p is invariant with respect to the translation that is a special case of an affine isometry. Now $(\complement\Omega)_{2\varepsilon}$ is monotone decreasing as $\varepsilon \to 0$ and $(\complement\Omega)_{2\varepsilon} \searrow \complement\Omega = \cap_{\varepsilon > 0}(\complement\Omega)_{2\varepsilon}$. From L. C. Evans and R. F. Gariepy [**1**, Thm. 2 (ix), p. 151],

$$\begin{aligned} &\lim_{\varepsilon \to 0} \mathrm{Cap}_p((\complement\Omega)_{2\varepsilon} \cap B(x, r'), B(x, 2r')) = \mathrm{Cap}_p(\complement\Omega \cap B(x, r'), B(x, 2r')) \\ &\quad \Rightarrow \forall x \in \partial\Omega,\ \forall r',\ 0 < r' < r, \quad c \le \frac{\mathrm{Cap}_p(\complement\Omega \cap B(x, r'), B(x, 2r'))}{\mathrm{Cap}_p(B(x, r'), B(x, 2r'))} \end{aligned}$$

and $\Omega \in \mathcal{O}_{c,r}(D)$. We conclude that $\mathcal{O}_{c,r}(D)$ is compact as a closed subset of the compact set $C_d^c(D)$ in the topology of uniform convergence on $\overline{D}$.

(ii) From part (i) and Theorem 8.1. $\square$

9.2 Compact Family $L_{c,r}(\mathcal{O}, D)$ and Thick Set Property

By analogy with the "fat segment property," we introduce a "thickness property."

Definition 9.1.
Given $r > 0$ and $c > 0$, let $\mathcal{O}$ be a bounded subset of $\mathbf{R}^{\mathrm{N}}$ such that $0 \in \overline{\mathcal{O}} \backslash \mathcal{O}$ and $\mathcal{O}$ satisfies the *strong* (r, c)*-capacity density condition* in 0:

$$\forall r',\ 0 < r' < r, \quad \frac{\mathrm{Cap}_p(\mathcal{O} \cap B(0, r'), B(0, 2r'))}{\mathrm{Cap}_p(B(0, r'), B(0, 2r'))} \geq c. \tag{9.3}$$

(i) An open set Ω is said to satisfy the $(r, c, \mathcal{O})$*-capacity density condition* if

$$\forall x \in \partial\Omega,\ \exists A_x \in \mathrm{O(N)} \text{ such that } x + A_x\mathcal{O} \subset \complement\Omega. \tag{9.4}$$

(ii) For r, $0 < r < 1$, and $c > 0$, define the following family of open subsets of D:

$$L_{c,r}(\mathcal{O}, D) \overset{\text{def}}{=} \{\Omega \subset D : \Omega \text{ satisfies the } (r, c, \mathcal{O})\text{-capacity density condition.}\} \tag{9.5}$$

□

As a special case of $\mathcal{O}$ we have the *flat cone property* introduced by D. Bucur and J.-P. Zolésio [**1**, **5**] in 1993: given an aperture ω, $0 < \omega < \pi/2$, a height $\lambda > 0$, and a unit normal ν,

$$\mathcal{O}_{\omega,\lambda,\nu} \overset{\text{def}}{=} \left\{ y \in \mathbf{R}^{\mathrm{N}} : \frac{1}{\tan\omega} |P_H(y)| < y \cdot e_N < \lambda \right\} \cap \{\nu\}^{\perp}, \tag{9.6}$$

where P_H is the orthogonal projection onto the hyperplane $H = \{e_N\}^{\perp}$ orthogonal to the direction e_N (cf. Figure 2.6 in Chapter 2). The intersection with the hyperplane $\{\nu\}^{\perp}$ makes the cone *flat.* This cone satisfies the *strong* (r, c)*-capacity density condition* in 0 of Definition 9.1 from the scaling property of this cone.

Theorem 9.2. *Given* $r > 0$, $\mathcal{O}_{\omega,\lambda,\nu}$ *associated with an aperture* ω, $0 < \omega < \pi/2$, *a height* $\lambda > 0$, *and a unit normal* ν *through identity* (9.6), *and*

$$c \overset{\text{def}}{=} \frac{\mathrm{Cap}_p(\mathcal{O}_{\omega,\lambda,\nu} \cap B(0, r), B(0, 2r))}{\mathrm{Cap}_p(B(0, r), B(0, 2r))}, \tag{9.7}$$

then $0 < c < \infty$, *for all* r', $0 < r' \leq r$,

$$\forall r',\ 0 < r' < r, \quad \frac{\mathrm{Cap}_p(\mathcal{O}_{\omega,\lambda,\nu} \cap B(0, r'), B(0, 2r'))}{\mathrm{Cap}_p(B(0, r'), B(0, 2r'))} \geq c, \tag{9.8}$$

and $\mathcal{O}_{\omega,\lambda,\nu}$ *satisfies the* strong (r, c)-capacity density condition *in* 0.

Proof. Since $\mathcal{O}_{\omega,\lambda,\nu}$ is a flat cone in the hyperplane $\{\nu\}^{\perp}$, the capacity of its intersection with $B(0, r)$, $\mathrm{Cap}_p(\mathcal{O}_{\omega,\lambda,\nu} \cap B(0, r), B(0, 2r))$, is strictly positive and finite.

From L. C. Evans and R. F. Gariepy [**1**, Thm. 2, p. 151],

$$\mathrm{Cap}_p(B(0,r'),B(0,2r')) = \left(\frac{r'}{r}\right)^{N-p} \mathrm{Cap}_p(B(0,r),B(0,2r)),$$

$$\mathrm{Cap}_p(\mathcal{O}_{\omega,\lambda,\nu}\cap B(0,r'),B(0,2r')) = \left(\frac{r'}{r}\right)^{N-p} \mathrm{Cap}_p((r/r'\mathcal{O}_{\omega,\lambda,\nu})\cap B(0,r),B(0,2r)),$$

$$\frac{\mathrm{Cap}_p(\mathcal{O}_{\omega,\lambda,\nu}\cap B(0,r'),B(0,2r'))}{\mathrm{Cap}_p(B(0,r'),B(0,2r'))} = \frac{\mathrm{Cap}_p((r/r'\mathcal{O}_{\omega,\lambda,\nu})\cap B(0,r),B(0,2r))}{\mathrm{Cap}_p(B(0,r),B(0,2r))}. \quad (9.9)$$

Now,

$$\begin{aligned}
\frac{r}{r'}\mathcal{O}_{\omega,\lambda,\nu} &= \frac{r}{r'}\left\{y\in\mathbf{R}^{\mathrm{N}} : \frac{1}{\tan\omega}\big|P_H(y)\big| < y\cdot e_N < \lambda\right\}\cap\{\nu\}^{\perp} \\
&= \left\{\frac{r}{r'}y\in\mathbf{R}^{\mathrm{N}} : \frac{1}{\tan\omega}\left|P_H\left(\frac{r}{r'}y\right)\right| < \frac{r}{r'}y\cdot e_N < \lambda\frac{r}{r'}\right\}\cap\{\nu\}^{\perp} \\
&= \left\{z\in\mathbf{R}^{\mathrm{N}} : \frac{1}{\tan\omega}\big|P_H(z)\big| < z\cdot e_N < \lambda\frac{r}{r'}\right\}\cap\{\nu\}^{\perp} \\
&\supset \left\{y\in\mathbf{R}^{\mathrm{N}} : \frac{1}{\tan\omega}\big|P_H(y)\big| < y\cdot e_N < \lambda\right\}\cap\{\nu\}^{\perp} = \mathcal{O}_{\omega,\lambda,\nu}
\end{aligned}$$

since $r/r'\geq 1$. Finally from (9.9)

$$\begin{aligned}
\frac{\mathrm{Cap}_p(\mathcal{O}_{\omega,\lambda,\nu}\cap B(0,r'),B(0,2r'))}{\mathrm{Cap}_p(B(0,r'),B(0,2r'))} &= \frac{\mathrm{Cap}_p(r/r'\mathcal{O}_{\omega,\lambda,\nu}\cap B(0,r),B(0,2r))}{\mathrm{Cap}_p(B(0,r),B(0,2r))} \\
&\geq \frac{\mathrm{Cap}_p(\mathcal{O}_{\omega,\lambda,\nu}\cap B(0,r),B(0,2r))}{\mathrm{Cap}_p(B(0,r),B(0,2r))} = c.
\end{aligned}$$

Therefore, $\mathcal{O}_{\omega,\lambda,\nu}$ satisfies the *strong (r,c)-capacity density condition* in 0. □

Lemma 9.2. (i) *For all $c>0$ and $r>0$,*

$$L_{c,r}(\mathcal{O},D)\subset\mathcal{O}_{c,r}(D),$$

where $\mathcal{O}_{c,r}(D)$ is the family of open subsets Ω of D satisfying the strong (r,c)-capacity density condition of Definition 8.2 (iii).

(ii) *Given a sequence of open subsets $\{\Omega_n\}$ and Ω in $L_{c,r}(\mathcal{O},D)$, and the solutions $u(\Omega_n)\in H_0^1(\Omega_n;D)$ and $u(\Omega)\in H_0^1(\Omega;D)$ of the Dirichlet problem* (4.5) *on Ω_n and Ω, then*

$$d_{\complement\Omega_n}\to d_{\complement\Omega} \text{ in } C(\overline{D}) \quad\Rightarrow u(\Omega_n)\to u(\Omega) \text{ in } H^1(D)\text{-strong}. \quad (9.10)$$

Proof. (i) By Definition 9.1 (i) for an open set Ω satisfying the following $(r,c,\mathcal{O})$-*capacity density condition*: for all $x\in\partial\Omega$ and all r', $0<r'<r$, $x+A_x\mathcal{O}\subset\complement\Omega$ and

$$\frac{\mathrm{Cap}_p(\complement\Omega\cap B(x,r'),B(x,2r'))}{\mathrm{Cap}_p(B(x,r'),B(x,2r'))} \geq \frac{\mathrm{Cap}_p([x+A_x\mathcal{O}]\cap B(x,r'),B(x,2r'))}{\mathrm{Cap}_p(B(x,r'),B(x,2r'))}. \quad (9.11)$$

From L. C. EVANS and R. F. GARIEPY [**1**, Thm. 2, p. 151] since $y \mapsto x + A_x y$ is an affine isometry and for all $\rho > 0$, $A_x B(x,\rho) = B(x,\rho)$,

$$\frac{\mathrm{Cap}_p([x + A_x\mathcal{O}] \cap B(x,r'), B(x,2r'))}{\mathrm{Cap}_p(B(x,r'), B(x,2r'))} = \frac{\mathrm{Cap}_p(\mathcal{O} \cap B(0,r'), B(0,2r'))}{\mathrm{Cap}_p(B(0,r'), B(0,2r'))} \geq c \quad (9.12)$$

$$\Rightarrow \forall x \in \partial\Omega,\ \forall r',\ 0 < r' < r, \quad \frac{\mathrm{Cap}_p(\complement\Omega \cap B(x,r'), B(x,2r'))}{\mathrm{Cap}_p(B(x,r'), B(x,2r'))} \geq c, \tag{9.13}$$

and Ω satisfies the *strong* (r,c)*-capacity density condition.*

(ii) From part (i) and Theorem 8.1. □

Theorem 9.3. *Let D be a bounded open subset of $\mathbf{R}^{\mathrm{N}}$ and $c > 0$ and $r > 0$ be constants. The subset*

$$\{d_{\complement\Omega} : \Omega \in L_{c,r}(\mathcal{O}, D)\} \tag{9.14}$$

of $C_d^c(D)$ is compact for the uniform topology of $C(\overline{D})$.

Proof. Consider an arbitrary sequence $\{d_{\complement\Omega_n}\}$, $\Omega_n \in L_{c,r}(\mathcal{O}, D)$. By compactness of $C_d^c(D)$, there exist an open subset Ω_0 of D and a subsequence of $\{d_{\complement\Omega_n}\}$, still denoted by $\{d_{\complement\Omega_n}\}$, that converges to $d_{\complement\Omega_0}$ in $C_d^c(D)$. If $\partial\Omega_0 = \varnothing$, then $\Omega \in L_{c,r}(\mathcal{O}, D)$. If $\partial\Omega_0 \neq \varnothing$, then, by Lemma 9.1 (ii), for each $x \in \partial\Omega_0$, there exist a subsequence $\{\Omega_{n_k}\}$ of $\{\Omega_n\}$ and a sequence $\{x_k\}$, $x_k \in \partial\Omega_{n_k}$, that converges to x. Since $\{\Omega_{n_k}\} \subset L_{c,r}(\mathcal{O}, D)$, by Definition 9.1 of $L_{c,r}(\mathcal{O}, D)$, for each k

$$\exists A_k \in \mathrm{O(N)} \quad \text{such that} \quad x_k + A_k\mathcal{O} \subset \complement\Omega_{n_k} \quad \Rightarrow d_{x_k + A_k\mathcal{O}} \geq d_{\complement\Omega_{n_k}}.$$

But since $A_k \in \mathrm{O(N)}$,

$$d_{x_k + A_k\mathcal{O}}(y) = d_{\mathcal{O}}(A_k^{-1}(y - x_k)) = d_{\mathcal{O}}({}^*A_k(y - x_k)).$$

There exist a subsequence of $\{A_k\}$, still denoted by $\{A_k\}$, and $A \in \mathrm{O(N)}$ such that $A_k \to A$ and since $x_k \to x$, $d_{\mathcal{O}}(A_k^{-1}(y - x_k)) \to d_{\mathcal{O}}(A^{-1}(y - x)) = d_{x + A\mathcal{O}}(y)$. Finally, since $d_{\complement\Omega_{n_k}} \to d_{\complement\Omega}$,

$$d_{x_k + A_k\mathcal{O}} \geq d_{\complement\Omega_{n_k}} \quad \Rightarrow d_{x + A\mathcal{O}} \geq d_{\complement\Omega} \quad \Rightarrow \forall x \in \partial\Omega,\ x + A\mathcal{O} \subset \complement\Omega$$

and $\Omega \in L_{c,r}(\mathcal{O}, D)$. □

9.3 Maximizing the Eigenvalue $\lambda^A(\Omega)$

A first example of extremal domain follows directly from Theorem 9.1.

Theorem 9.4. *Let D be a bounded open domain in $\mathbf{R}^{\mathrm{N}}$ and A be a matrix function satisfying assumption* (2.2)*. Given an open E, $\varnothing \neq E \subset D$, consider the compact families $C_d^c(E; D) \cap \mathcal{O}_{c,r}(D)$ and $C_{d,\mathrm{loc}}^c(E; D) \cap \mathcal{O}_{c,r}(D)$, where*

$$C_d^c(E; D) \overset{\mathrm{def}}{=} \{d_{\complement\Omega} : E \subset \Omega \text{ open} \subset D\}, \tag{9.15}$$

$$C_{d,\mathrm{loc}}^c(E; D) \overset{\mathrm{def}}{=} \left\{ d_{\complement\Omega} : \begin{array}{l} \exists x \in \mathbf{R}^{\mathrm{N}},\ \exists A \in \mathrm{O(N)} \text{ such that} \\ x + AE \subset \Omega \text{ open} \subset D \end{array} \right\} \tag{9.16}$$

are the compact families defined in Theorem 2.6 (i) *of Chapter* 6*. Then the maximization problems*

$$\sup \left\{\lambda^A(\Omega) \,:\, \forall \Omega \in \mathcal{O}_{c,r}(D) \text{ such that } E \subset \Omega \subset D\right\}, \tag{9.17}$$

$$\sup \left\{\lambda^A(\Omega) \,:\, \forall \Omega \in \mathcal{O}_{c,r}(D) \text{ such that } \begin{array}{l} \exists x \in \mathbf{R}^{\mathrm{N}},\, \exists A \in \mathrm{O(N)} \text{ such that} \\ x + AE \subset \Omega \text{ open} \subset D \end{array}\right\} \tag{9.18}$$

have solutions.[4]

Proof. From Theorem 2.6 (i) in Chapter 6, $C_d^c(E; D)$ and $C_{d,\mathrm{loc}}^c(E; D)$ are compact in $C(\overline{D})$. Since sets of that family are nonempty, $\lambda^A(\Omega) < +\infty$ and, from Theorem 2.2, the function $d_{\complement\Omega} \mapsto \lambda^A(\Omega) : C_d^c(D) \subset C_0(D) \to \mathbf{R} \cup \{+\infty\}$ is upper semicontinuous and the function $d_{\complement\Omega} \mapsto \lambda^A(\Omega) : C_d^c(E; D) \cap \mathcal{O}_{c,r}(D) \to \mathbf{R}$ is upper semicontinuous. So there exist maximizers in the compact set $C_d^c(E; D) \cap \mathcal{O}_{c,r}(D)$. The same applies for $C_{d,\mathrm{loc}}^c(E; D) \cap \mathcal{O}_{c,r}(D)$. □

9.4 State Constrained Minimization Problems

We now give the following general existence theorem.

Theorem 9.5. *Let the assumptions of Theorem* 8.1 *on the open domain D and the matrix function A be satisfied. Let u_Ω be the solution of* (4.5) *for Ω in $\mathcal{O}_{c,r}(D)$. If h is continuously defined from $H_0^1(D)$ into $\mathbf{R}$, then $J(\Omega) = h(e_\Omega(u_\Omega))$ is continuously defined from $\mathcal{O}_{c,r}(D)$ into $\mathbf{R}$ and reaches its extremal values on that set.*

Example 9.1.
Consider the following shape function for which we can get the existence of minimizing domains even if the assumptions of Theorem 9.5 on h are not satisfied. Given $\alpha > 0$,

$$J(\Omega) \stackrel{\text{def}}{=} \frac{1}{2}\int_\Omega |\nabla u_\Omega - \bar{z}|^2 dx + \alpha\, \frac{1}{\int_\Omega |u_\Omega|^2 dx}, \tag{9.19}$$

where u_Ω is the solution of (4.5) and $\bar{z} \in L^2(D; \mathbf{R}^{\mathrm{N}})$. From Theorem 2.3 in Chapter 2, $u \stackrel{\text{def}}{=} e_\Omega(u_\Omega)$ and ∇u_Ω are zero almost everywhere in $D \backslash \Omega$, and the function can be rewritten as

$$J(\Omega) = \frac{1}{2}\int_D |\nabla u - \bar{z}|^2\, dx + \alpha\, \frac{1}{\int_D |u|^2\, dx} - \frac{1}{2}\int_{D\backslash\Omega} |\bar{z}|^2\, dx. \tag{9.20}$$

The last term is not of the form $h(u_\Omega)$ for some h, but is simply of the form $h(\Omega)$. It turns out that J is not continuous but only lower semicontinuous for the H^c-topology from Theorem 4.1 of Chapter 6 applied to the last term. Hence it can be minimized. □

The minimization problem of Example 9.1 can now be formulated as follows on D. Let D be a bounded open nonempty domain in $\mathbf{R}^{\mathrm{N}}$, f be an element of

[4] In some sense the maximizing solution is an $\mathcal{O}_{c,r}(D)$-approximation of the set E.

$H^{-1}(D)$, and B be a ball containing D. Let μ be a positive measure on D and a be a positive constant such that $0 < a < \mu(D) < +\infty$. Let u_Ω be the solution of (4.5), and let $u = e_\Omega(u_\Omega)$, its extension by zero in $H_0^1(D)$. For $J(\Omega)$ defined in (9.19) consider the following problem:

$$\min\{J(\Omega) \,:\, \Omega \in \mathcal{O}_{c,r}(B),\ \ \Omega \subset D,\ \mu(\Omega) \le a\}. \tag{9.21}$$

Theorem 9.6. *Let the assumptions of Theorem* 8.1 *on the open domain D and the matrix function A be satisfied. For any constants $\alpha > 0$, $c > 0$ and r, $0 < r < 1$, problem* (9.21) *has at least one solution.*

Proof. It is sufficient to notice that the first term in (9.20) is continuous under the assumptions of Theorem 9.5. The second term is lower semicontinuous from Theorem 4.1 of Chapter 6 (with the measure of density $|\bar{z}|^2$, that is, $d\mu = \bar{z}|^2\, dz$). To complete the proof, recall that if $\Omega_n \subset D$ and $\Omega_n \xrightarrow{H^c} \Omega$, then $\Omega \subset D$. As $\alpha > 0$, a minimizing sequence cannot converge to $\varnothing$. In fact, for any admissible domain Ω_0 and any optimal solution u_{Ω_0},

$$\int_\Omega |u_{\Omega_0}|^2 dx \ge \frac{\alpha}{J(\Omega_0)}. \qquad \square$$

9.5 Examples with a Constraint on the Gradient

Let D be a fixed bounded smooth open domain in $\mathbf{R}^{\mathrm{N}}$, let f in $H^{-1}(D)$, and let g in $L^2(D)$. For any open subset Ω of D, let u_Ω be the solution of the Dirichlet problem (4.5) in $H_0^1(\Omega)$. Given $\alpha > 0$, $M > 0$, and an open subset E of D, consider the following minimization problem:

$$\inf\left\{J(\Omega) \,:\, \Omega \text{ open },\ E \subset \Omega \subset D, \quad \operatorname{ess\,sup} |\nabla u_\Omega| \le M\right\}, \tag{9.22}$$

$$J(\Omega) \overset{\text{def}}{=} \mathrm{m}(\Omega) + \alpha \int_\Omega |u_\Omega - g|^2\, dx.$$

It is understood that if $|\nabla u|$ is not in $L^\infty(D)$, then the $\operatorname{ess\,sup} |\nabla u|$ is $+\infty$. In a first step it is easy to check that the problem

$$\begin{aligned}\inf\Bigg\{&\mathrm{m}(\Omega) + \alpha \int_\Omega |u_\Omega - g|^2\, dx \\ &\qquad : \operatorname{ess\,sup} |\nabla u_\Omega| \le M,\ E \subset \Omega,\ \Omega \in \mathcal{O}_{c,r}(D)\Bigg\}\end{aligned} \tag{9.23}$$

has minimizing solutions. Let $\{\Omega_n\}$ be a minimizing sequence. By assumption there exist Ω and a subsequence, still indexed by n, such that $\Omega_n \to \Omega$ in the H^c-topology ($d_{\complement\Omega_n} \to d_{\complement\Omega}$ in $C(\overline{D})$) and

$$\operatorname{ess\,sup} |\nabla u_{\Omega_n}| \le M.$$

Then $|\nabla u_n|$ converges in $L^2(D)$ to $|\nabla u|$ from the previous sections. For any $\varphi \in L^2_+(D)$ we have

$$\int_D (|\nabla u_n| - M)\, \varphi\, dx \le 0,$$

and then, in the limit, we get the same inequality with u. Then, the function being lower semicontinuous with respect to that topology, we get the existence.

Lemma 9.3. *For any $f \in H^{-1}(D)$ and any $r > 0$ and $c > 0$, problem* (9.23) *has solutions in $\mathcal{O}_{c,r}(D)$*

In fact, the presence of the constraint on the gradient of u is helpful for getting the continuity of the application $\Omega \mapsto u = e_\Omega(u_\Omega)$, and we directly get the existence of solutions to the original problem (9.22). In the proof of Theorem 8.1 we only needed the equicontinuity of the family of solutions $\{u_n = e_{\Omega_n}(u_{\Omega_n})\}$ and the fact that if $x \in \complement\Omega$, then $u(x) = 0$ for a quasi-continuous representative. If $\Omega \in \mathcal{O}_{c,r}(D)$, those two assumptions are readily satisfied. Notice that the boundedness of the gradients in (9.22) implies the equicontinuity of any minimizing sequence $\{u_n\}$. So, in order to obtain a continuity result with constraints on the boundedness of the gradient, we only have to notice that if $u \in H_0^1(\Omega)$, $\operatorname{ess\,sup} |\nabla u| \leq M$. Then $u \in W^{1,\infty}(\Omega)$ and u is Lipschitzian with the constant M, or more exactly, there exists a Lipschitzian function almost everywhere equal to u, which is also a quasi-continuous representative of u. To obtain that $u(x) = 0$ in any point of the complement of Ω, we have to introduce the following capacity constraints on Ω: we require that Ω be capacity extended (see D. Bucur [**1**]), i.e., that $\Omega = \Omega^*$, where

$$\Omega^* \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : \exists \varepsilon_x > 0, \text{ such that } \operatorname{cap}_D(B(x, \varepsilon_x) \cap \complement\Omega) = 0\right\}. \tag{9.24}$$

Indeed, let u be continuous on D, $u = 0$ quasi-everywhere on $\complement\Omega$ and $\Omega = \Omega^*$. Then for all $x \in \complement\Omega$, for all $\varepsilon > 0$ we have $\operatorname{cap}_D(B(x, \varepsilon_x) \cap \complement\Omega) > 0$ and there exists a point x_ε in $B(x, \varepsilon_x) \cap \complement\Omega$, where $u(x_\varepsilon) = 0$. By continuity we get $u(x) = 0$. We recall from D. Bucur [**1**] the main properties of the capacity extension.

Proposition 9.1. *For any open $\Omega \subset D$, the set Ω^* is open,*

$$\Omega \subset \Omega^*, \quad \operatorname{cap}_D(\Omega^* \backslash \Omega) = 0, \quad \textit{and} \quad (\Omega^*)^* = \Omega^*.$$

As $\operatorname{cap}_D(\Omega^* \backslash \Omega) = 0$ we get $e_\Omega(u_\Omega) = e^*_\Omega(u_{\Omega^*})$ and any minimizing sequence can be made up only of capacity-extended domains.

Theorem 9.7. *Given $f, g \in L^2(D)$, $\alpha > 0$, and $M > 0$, problem* (9.22) *has minimizing solutions.*

Proof. From the boundedness of the gradient we obtain that $\{u_{\Omega_n}\}$ are uniformly Lipschitz continuous, and we can apply the same arguments as in Theorem 8.1, avoiding the capacity conditions. □

Consider the penalized version of problem (9.22). Given $M > 0$ and $\beta > 0$, define

$$J_\beta(\Omega) \stackrel{\text{def}}{=} J(\Omega) + \beta \sup_D \operatorname{ess} (|\nabla u_\Omega| - M)^+ \tag{9.25}$$

and consider the existence of solutions to the following problem:

$$\inf \{J_\beta(\Omega) : E \subset \Omega \text{ open } \subset D\}. \tag{9.26}$$

Theorem 9.8. *Given f and g in $L^2(D)$, $\alpha > 0$, $\beta > 0$, and $M \geq 0$, problem* (9.26) *has minimizing solutions.*

Proof. Let $\{\Omega_n\}$ be a minimizing sequence for the function J_β, $\Omega_n = \Omega_n^*$. We have

$$\operatorname{ess\,sup} |\nabla u_{\Omega_n}| \leq \max\left\{M, \beta^{-1} J(\Omega_1)\right\} = M'$$

for all $n \geq 1$. Then, from the previous considerations we have the strong convergence in $H_0^1(D)$ of $u_n = e_{\Omega_n}(u_{\Omega_n})$ to $u = e_\Omega(u_\Omega)$ for some subsequence $\{\Omega_n\}$ that converges to Ω in the H^c-topology. Hence Ω is a minimizing domain. To complete the proof note that the map

$$\Omega \mapsto \sup_D \operatorname{ess} (|\nabla u| - M)^+$$

is not lower semicontinuous from the H^c-topology to $\mathbf{R}$. Nevertheless, if $\{\Omega_n\}$ is a minimizing sequence for problem (9.26), the required property is satisfied. We can assume that the sequence is chosen such that

$$\sup_D \operatorname{ess} |\nabla u_n| \to c = \liminf_{n\to\infty} \left(\sup_D \operatorname{ess} |\nabla u_n|\right).$$

For any $\varepsilon > 0$ there exists $n_\varepsilon > 0$ such that for $n \geq n_\varepsilon$, $\sup_D \operatorname{ess} |\nabla u_{\Omega_n}| \leq c + \varepsilon$. Because of the $H_0^1(D)$-strong convergence of u_n to u, we get

$$\begin{aligned} &\forall \varepsilon > 0, \quad \sup_D \operatorname{ess} |\nabla u| \leq c + \varepsilon \\ \Rightarrow\ &\sup_D \operatorname{ess} |\nabla u| \leq \liminf \left(\sup_D \operatorname{ess} |\nabla u|\right). \end{aligned}$$ □

Remark 9.1.
We can change the L^∞-norm for a differentiable one; that is, we need

$$\|\nabla u\|_{1,p} \leq M, \quad \forall p > N.$$

By the continuous inclusion $W^{1,p}(D) \subset W^{\varepsilon,\infty}(D)$, ε small, the result still holds. □

Chapter 9

Shape and Tangential Differential Calculuses

1 Introduction

In Chapters 3, 5, 6, and 7, we have constructed nonlinear and nonconvex complete metric spaces of geometries. The spaces $\mathcal{F}(\Theta)$ and $\mathrm{X}(D)$ have group structures. For the groups $\mathcal{F}(\Theta)$, $\Theta \subset C^{0,1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$, it is possible to construct C^1-paths in $\mathcal{F}(\Theta)$ and we have shown that the tangent space is Θ in all points of $\mathcal{F}(\Theta)$ in Chapter 4. It is more difficult to fully characterize tangent spaces for the spaces of characteristic functions or distance functions. Yet, paths constructed from velocity fields can also be used to obtain C^1-paths in those spaces.

In the absence of sharper results, we shall concentrate on the notion of semiderivatives or derivatives of a shape functional (cf. section 3 of Chapter 4) with respect to a velocity field. This point of view was used in Chapter 4 to give an equivalent characterization of the continuity of a shape functional with respect to Courant metrics in terms of the continuity along continuous paths generated by velocities. Moreover the velocity approach readily extends to the constrained case.

As we pointed out in section 3.2 of Chapter 4, two types of semiderivatives are of interest: the weaker Gateaux style semiderivative (3.6) and the stronger Hadamard style semiderivative of Definition 3.2 for which the chain rule is available. This chapter has been structured along those general directions.

In section 2 we give a self-contained review of semiderivatives and derivatives in topological vector spaces in order to prepare the ground for shape derivatives.

Section 3 gives the definitions and the main properties of first-order shape semiderivatives and derivatives of shape functionals (Eulerian, Hadamard, Gateaux, and Fréchet). For the Hadamard semidifferentiability we get the Courant metric continuity of the shape functional as for Banach spaces. A complete structure theorem is given for the shape gradient in section 3.4.

Before going to second-order derivatives, the main elements of the *shape calculus* are introduced in section 4 and the basic formulae for domain and boundary integrals are given in section 4.1 and section 4.2. Their application is illustrated in a series of examples in section 4.3.

The final expressions of the gradients in the examples of section 4 always lead to a domain and a boundary expression. The boundary expression contains fundamental properties of the gradients, and the natural way to untangle some of the resulting terms is to use the *tangential calculus*. Section 5 gives the main elements of that calculus for a C^2-submanifold of $\mathbf{R}^N$ of codimension 1, including Stokes's and Green's formulae in section 5.5 and the relationship between tangential and covariant derivatives in section 5.6. This is applied to the derivative of the integral of the square of the normal derivative of section 4.3.3.

Section 6 extends definitions and structure theorems to second-order derivatives. In order to develop a better feeling for the abstract definitions the second-order derivative of the domain integral is computed in section 6.1 using the combined strengths of the shape and tangential calculuses. A basic formula for the second-order semiderivative of the domain integral is given in section 6.2. This is completed with structure theorems in the nonautonomous case in section 6.3 and the autonomous case in section 6.4. The shape Hessian is decomposed into a symmetrical term plus the gradient acting on the first half of the Lie bracket in section 6.5. This symmetrical part is itself decomposed into a symmetrical part that depends only on the normal component of the velocity field and a symmetrical term made up of the gradient acting on a generic group of terms that occurs in all examples considered in section 6.

2 Review of Differentiation in Topological Vector Spaces

In this section we review some elements of semiderivatives and derivatives in topological vector spaces that will be useful for defining shape semiderivatives and derivatives. In that context we begin with the weaker notion of *Gateaux semiderivative*. Its simplicity makes it useful for computing a semiderivative. Yet, the *chain rule* for the derivative of the composition of such functions does not hold and the function itself need not be continuous. The more interesting notion is the stronger one of *Hadamard semiderivative* that makes the function continuous and builds the chain rule in its definition. It also readily extends to the tangential semiderivative on smooth submanifolds. In the Euclidean space the *Hadamard derivative* coincides with the *total differential* or *Fréchet derivative*, but it is both weaker and more general in infinite-dimensional vector spaces where the existence of a metric on the space is not required. The chain rule is the central ingredient of a good differential or semidifferentiable calculus.

2.1 Definitions of Semiderivatives and Derivatives

The notion of semiderivative corresponds to the intuitive notion of *differential* that was formalized by Newton and Leibniz, who also introduced the notation in 1684.

Definition 2.1.
Let f be a real-valued function defined in a neighborhood of a point x of a topological vector space E.

(i) We say that f has a *Gateaux semiderivative*[1] at a point $x \in U$ in the direction $v \in E$ if the following limit exists:

$$\lim_{\varepsilon \searrow 0} \frac{f(x+\varepsilon v) - f(x)}{\varepsilon}; \tag{2.1}$$

when it exists it will be denoted by $df(x;v)$.

(ii) We say that f has a *Hadamard semiderivative*[2] at $x \in U$ in the direction $v \in E$ if the following limit exists:

$$\lim_{\substack{\varepsilon \searrow 0 \\ w \to v}} \frac{f(x+\varepsilon w) - f(x)}{\varepsilon}; \tag{2.2}$$

when it exists it will be denoted by $d_H f(x;v)$.

The above definitions extend from a real-valued function to a function f into another topological vector space F. □

Remark 2.1.
Our definition of Hadamard semiderivative is simple and general, but it somewhat hides the original definition of J. Hadamard [**2**] that builds into the definition the chain rule for the derivative of the composition of functions and the continuity of the function, as we shall see in Theorem 2.5. It is readily seen that Definition 2.1 (i) is equivalent to

$$\lim_{\substack{\varepsilon \searrow 0 \\ \forall \varphi:[0,\tau]\to E \text{ such that} \\ \varphi(0^+)=x,\, \varphi'(0^+)=v}} \frac{f(\varphi(\varepsilon)) - f(x)}{\varepsilon}, \tag{2.3}$$

where the limit is independent of the choice of path $\varphi : [0,\tau] \to E$ such that $\lim_{\varepsilon \searrow 0} \varphi(\varepsilon) = x$ and $\lim_{\varepsilon \searrow 0} (\varphi(\varepsilon) - \varphi(0))/\varepsilon = v$. Since we are using paths, this definition extends to paths on manifolds M with v in the tangent space to M in x. □

It is clear that if $d_H f(x;v)$ exists, the semiderivative $df(x;v)$ exists and is equal to $d_H f(x;v)$, but the converse is not true without additional assumptions, as

[1] In his original work (published after his death in 1914) R. Gateaux [**2**] introduces as *differential* the *directional derivative* (ε goes to 0 not only by positive values but also by negative values yielding the property $df(x-v) = -df(x;v)$), but he does not a priori assume that this differential is linear with respect to the direction. So he cannot get the differential of the norm $|x|$, but his definition of differential is very close to our version.

[2] In 1923 J. Hadamard [**2**] introduces a notion of differential that builds in the chain rule for the composition of functions. M. Fréchet [**1**] shows in 1937 that "the definition of the *total differential* of Stolz-Young is equivalent to the definition of J. Hadamard [**2**] (in finite dimension). However, when the latter is extended to functionals (functions of functions - infinite dimensional spaces), it is more general than the one of the author (M. Fréchet) and necessarily verifies the chain rule for the differentiation of the composition of functions." In the same paper M. Fréchet [**1**, p. 239] weakens the notion of differential by dropping the linearity with respect to the direction. Again this amounts to letting ε go to zero by positive and negative values in our definition. As in the case of Gateaux his definition is very close to our semiderivative version of the derivative of Hadamard that can be found in J.-P. Penot [**1**, p. 250] in 1978 and earlier in A. Bastiani [**1**, Def. 3.1, p. 17] in 1964 as a seemingly well-known notion.

can be seen from the following example. The Gateaux semiderivative is generally neither linear nor continuous with respect to the direction.

Example 2.1.
Consider the following function $f : \mathbf{R}^2 \to \mathbf{R}$:

$$f(x,y) = \begin{cases} \dfrac{x^5}{(y-x^2)^2 + x^8}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases} \tag{2.4}$$

By definition, $df((0,0);(0,0))$ is always 0, but here $d_H f((0,0);(0,0))$ does not exist. Choose the sequences

$$t_n = \frac{1}{n} \searrow 0 \text{ and } w_n = \left(\frac{1}{n}, \frac{1}{n^3}\right) \to (0,0) \text{ as } n \to +\infty$$

and consider the quotient

$$q_n \overset{\text{def}}{=} \frac{f((0,0) + t_n w_n) - f(0,0)}{t_n}.$$

As n goes to infinity,

$$q_n = \frac{(\frac{1}{n^2})^5}{\frac{1}{n}(\frac{1}{n^2})^8} = n^4 \to +\infty$$

and $d_H f((0,0);(0,0))$ does not exist. The function f is Gateaux semidifferentiable at $(0,0)$ in all directions $v = (v_1, v_2)$ in $\mathbf{R}^2$ and

$$df(0,0);(v_1,v_2)) = \begin{cases} v_1, & \text{if } v_2 = 0, \\ 0, & \text{if } v_2 \neq 0, \end{cases} \tag{2.5}$$

but it is neither linear nor continuous with respect to v ($v_n = (1, 1/n) \to (1,0)$). □

The Hadamard semiderivative is also generally not linear in v.

Example 2.2.
Let E be a normed vector space. The Hadamard semiderivative of the norm $f(x) = |x|_E$ at $x = 0$ is given by

$$\forall v \in E, \quad d_H f(0;v) = |v|_E, \tag{2.6}$$

which is continuous but not linear in v. □

We shall also need the following notions of (full) derivatives.

Definition 2.2.
Let f be a real-valued function defined in a neighborhood of a point x of a topological vector space E.

(i) The function f has a *Gateaux derivative* at x if

$$\begin{gathered} \forall v \in E, \quad df(x;v) \text{ exists and} \\ v \mapsto df(x;v) : E \to \mathbf{R} \text{ is linear and continuous.} \end{gathered} \tag{2.7}$$

Whenever it exists the linear map (2.7) will be denoted by $\nabla f : E \to E'$.

(ii) The function f has a *Hadamard derivative* at x if

$$\forall v \in E, \quad d_H f(x; v) \stackrel{\text{def}}{=} \lim_{\substack{t \searrow 0 \\ w \to v}} \frac{f(x+tw) - f(x)}{t} \text{ exists and}$$
$$v \mapsto d_H f(x; v) : E \to \mathbf{R} \text{ is linear and continuous.}$$

The above definitions extend from a real-valued function to a function f into a normed vector space F. □

The next example is a Gateaux differentiable function in 0 that is not continuous in 0 and not Hadamard differentiable in 0.

Example 2.3.
Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$:

$$f(x,y) \stackrel{\text{def}}{=} \begin{cases} \dfrac{x^6}{(y-x^2)^2 + x^8}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases} \tag{2.8}$$

It is readily seen that f has a Gateaux semiderivative at $(0,0)$ in all directions,

$$\forall v \in \mathbf{R}^2, \quad df\big((0,0); v\big) = 0, \tag{2.9}$$

and that $v \mapsto df\big((0,0); v\big) = 0$ is trivially linear and continuous with respect to v. But it is not Hadamard semidifferentiable in $(0,0)$ in the direction $(0,0)$ by the same argument as in Example 2.1. Similarly, for $\varepsilon > 0$ by choosing the directions

$$w(\varepsilon) = (1, \varepsilon) \to v = (1,0) \text{ as } \varepsilon \to 0,$$

then

$$\frac{f\big(\varepsilon w(\varepsilon)\big) - f\big((0,0)\big)}{\varepsilon} = \frac{1}{\varepsilon^3} \to +\infty$$

and $d_H f((0,0); (1,0))$ does not exist. Finally, f is not continuous in $(0,0)$. To see this follow the path $(\varepsilon, \varepsilon^2)$ as ε goes to 0:

$$\big|f(\varepsilon, \varepsilon^2) - f(0,0)\big| = \left|\frac{\varepsilon^6}{\varepsilon^8}\right| = \frac{1}{\varepsilon^2} \to +\infty \text{ as } \varepsilon \to 0. \qquad \square$$

2.2 Derivatives in Normed Vector Spaces

We now specialize to normed vector spaces. In a normed vector space E, we have the strong (norm) topology and the weak topology. As a result we have two notions of Hadamard semiderivatives,

$$d_H^s f(x; v) \stackrel{\text{def}}{=} \lim_{\substack{t \searrow 0 \\ w \to v \text{ in } E\text{-strong}}} \frac{f(x+tw) - f(x)}{t}, \tag{2.10}$$

$$d_H^w f(x; v) \stackrel{\text{def}}{=} \lim_{\substack{t \searrow 0 \\ w \rightharpoonup v \text{ in } E\text{-weak}}} \frac{f(x+tw) - f(x)}{t}, \tag{2.11}$$

and two notions of derivatives, where $w \to v$ denotes the strong convergence and $w \rightharpoonup v$ denotes the weak convergence. By definition, the weak Hadamard semiderivative is a stronger condition than the strong Hadamard semiderivative:

$$d_H^w f(x;v) \text{ exists} \quad \Rightarrow \quad d_H^s f(x;v) \text{ exists.} \tag{2.12}$$

The two notions coincide when E is finite-dimensional.

Definition 2.3.
Let f be a real-valued function defined in a neighborhood of a point x of a normed vector space E.

(i) The function f has a *strong Hadamard derivative* in x if

$$\forall v \in E, \quad d_H^s f(x;v) \overset{\text{def}}{=} \lim_{\substack{t \searrow 0 \\ w \to v \text{ in } E\text{-strong}}} \frac{f(x+tw)-f(x)}{t} \text{ exists and}$$
$$v \mapsto d_H^s f(x;v) \,:\, E \to \mathbf{R} \text{ is linear and continuous.}$$

(ii) The function f has a *weak Hadamard derivative* in x if

$$\forall v \in E, \quad d_H^w f(x;v) \overset{\text{def}}{=} \lim_{\substack{t \searrow 0 \\ w \rightharpoonup v \text{ in } E\text{-weak}}} \frac{f(x+tw)-f(x)}{t} \text{ exists and}$$
$$v \mapsto d_H^w f(x;v) \,:\, E \to \mathbf{R} \text{ is linear and continuous.}$$

(iii) The function f has a *Fréchet derivative* in x if there exists a continuous linear mapping $L(x) : E \to \mathbf{R}$ such that

$$\lim_{|h|_E \to 0} \frac{f(x+h)-f(x)-L(x)h}{|h|_E} \to 0.$$

The above definitions extend from a real-valued function to a function f into another normed vector space F. $\square$

If f has a Fréchet derivative in x, then it has a Gateaux derivative at x. Indeed for all $v \neq 0$ and $t \searrow 0$, $h = tv \to 0$ in norm, and

$$\frac{f(x+tv)-f(x)}{t} - L(x)v = |v| \frac{f(x+tv)-f(x)-L(x)tv}{|tv|_E} \to 0$$
$$\Rightarrow df(x;v) = L(x)v \text{ and } v \mapsto df(x;v) : E \to \mathbf{R} \text{ is linear and continuous.}$$

By definition $\langle \nabla f(x), v\rangle_E = L(x)v$, where $\langle \cdot,\cdot\rangle_E$ denotes the duality pairing between E' and E.

The following theorem makes explicit the connections and equivalences between the three notions of derivatives.

Theorem 2.1. *Let E be a normed vector space and $f : V(x) \to \mathbf{R}$ be a function defined in a neighborhood of a point $x \in E$.*

(i) *If f has a Fréchet derivative in x, then f has a weak Hadamard derivative in x and $d^w_H f(x;v) = \langle \nabla f(x), v\rangle_E = L(x)v$. If f has a weak Hadamard derivative in x, then f has a strong Hadamard derivative in x. If f has a strong Hadamard derivative in x, then f has a Gateaux derivative in x.*

(ii) *If E is a reflexive Banach space, f has a weak Hadamard derivative in x if and only if f has a Fréchet derivative in x.*

For a function f, the four notions are well ordered:

Fréchet derivative $\Rightarrow$ weak Hadamard derivative $\Rightarrow$ strong Hadamard derivative

and the last one implies that f has a Gateaux derivative in x. In view of Theorem 2.1, the first three notions are equivalent in finite dimension.

Theorem 2.2. *When E is finite-dimensional, the strong Hadamard derivative, the weak Hadamard derivative, and the Fréchet derivative of Definition* 2.3 *are equivalent and equal.*

Proof of Theorem 2.1. (i) Given $t > 0$ and $w \in E$, consider the quotient

$$q(t,w) \stackrel{\text{def}}{=} \frac{f(x+tw) - f(x)}{t}$$

and for $h \in E$ the quotient

$$Q(h) \stackrel{\text{def}}{=} \begin{cases} \dfrac{f(x+h) - f(x) - L(x)h}{|h|_E}, & h \neq 0, \\ 0, & h = 0. \end{cases}$$

Then

$$q(t,w) = Q(h(t,w))\, |w|_E + L(x)w.$$

Given $v \in E$, as $w \rightharpoonup v$ and $t \searrow 0$, $h(t,w) \stackrel{\text{def}}{=} tw \to 0$ in the E-norm. Hence $Q(h(t,w)) \to 0$ since f has a Fréchet derivative in x and $L(x)w \to L(x)v$ as $w \rightharpoonup v$ by continuity. Therefore

$$\lim_{\substack{w \rightharpoonup v \\ t \searrow 0}} q(t,w) = L(x)v$$

and $d^w_H f(x;v)$ exists and $v \mapsto d^w_H f(x;v) = L(x)v : E \to \mathbf{R}$ is linear and continuous with respect to v. The second statement follows from (2.12) and the last one by definition of the Gateaux and Hadamard derivatives.

(ii) Define the element $L(x)$ of E' as

$$\forall v \in E, \quad L(x)v \stackrel{\text{def}}{=} \langle \nabla f(x), v\rangle_E = d^w_H f(x;v).$$

Denote by $\overline{Q}$ the limsup of $|Q(h)|$ as $|h| \to 0$, which can possibly be infinite. Let $\{h_n\}$ be a sequence in E such that $|h_n| \to 0$ and $|Q(h_n)| \to \overline{Q}$. For $h \neq 0$

$$Q(h) = Q\left(|h| \, \frac{h}{|h|}\right).$$

Since E is a reflexive Banach space there exist a subsequence of $\{h_n\}$, still denoted by $\{h_n\}$, and $v \in E$ such that

$$w_n \stackrel{\text{def}}{=} \frac{h_n}{|h_n|} \rightharpoonup v \in E \quad \text{and} \quad |Q(h_n)| \to \overline{Q}.$$

By letting $t_n = |h_n|$,

$$\begin{aligned} Q(h_n) &= \frac{f(x + t_n w_n) - f(x)}{t_n} - \langle \nabla f(x), w_n \rangle_E \\ &= q(t_n, w_n) - \langle \nabla f(x), w_n \rangle_E \to d_H^w f(x; v) - \langle \nabla f(x), v \rangle_E = 0. \end{aligned}$$

Therefore $\overline{Q} = 0$ and f is Fréchet differentiable in x. □

We complete this section with a classical sufficient condition for the Fréchet differentiability.

Theorem 2.3. *Given a normed vector space E and a map $f : V(x) \to \mathbf{R}$, defined in a neighborhood $V(x)$ of $x \in E$, assume that*

(i) *for all $y \in V(x)$, f has a Gateaux derivative $\nabla f(y)$, and*

(ii) *the map $\nabla f : V(x) \to E'$-strong is continuous in x.*

Then f has a Fréchet derivative in x.

Proof. There exists $\rho > 0$ such that the open ball $B(x, \rho)$ is contained in $V(x)$. For $h \in E$, $0 < |h| < \rho$, consider the quotient

$$Q(h) \stackrel{\text{def}}{=} \frac{f(x + h) - f(x) - \langle \nabla f(x), h \rangle_E}{|h|_E}.$$

There exists $\alpha(h)$, $0 < \alpha(h) < 1$, such that

$$\begin{aligned} & f(x + h) - f(x) = \langle \nabla f(x + \alpha(h)h), h \rangle_E \\ \Rightarrow \; & Q(h) = \left\langle \nabla f(x + \alpha(h)h) - \nabla f(x), \frac{h}{|h|_E} \right\rangle_E \\ \Rightarrow \; & |Q(h)| \leq |\nabla f(x + \alpha(h)h) - \nabla f(x)|_{E'} \to 0 \text{ as } h \to 0. \end{aligned}$$

This shows that f is Fréchet differentiable in x. □

2.3 Locally Lipschitz Functions

The following theorem gives a sufficient condition for the equivalence of Gateaux and Hadamard semiderivatives (resp., Gateaux and Hadamard derivatives).

Theorem 2.4. *Let E be a normed vector space and $f : E \to \mathbf{R}$ be a function that is uniformly Lipschitzian in a neighborhood U of x in E, that is,*

$$\exists c(x) > 0,\ \forall y, z \in U, \quad |f(y) - f(z)| \le c(x)\,|y - z|_E. \tag{2.13}$$

(i) *If $df(x; v)$ exists, then $d^s_H f(x; v)$ exists and $d^s_H f(x; v) = df(x; v)$. Moreover, if $df(x; v)$ exists for all $v \in E$, then*

$$\forall v_2, v_1 \in E, \quad |d^s_H f(x; v_2) - d^s_H f(x; v_1)| \le c\,|v_2 - v_1|_E.$$

(ii) *If f has a Gateaux derivative at x, then*

$$\begin{gathered} \forall v,\ d^s_H f(x; v) \text{ exists and} \\ v \mapsto d^s_H f(x; v) : E \to \mathbf{R} \text{ is linear and continuous} \end{gathered}$$

and f has a strong Hadamard derivative in x.

Proof. (i) There exist $\bar{\varepsilon} > 0$ and a neighborhood W of v in E such that

$$\forall w \in W,\ \forall \varepsilon,\ 0 < \varepsilon \le \bar{\varepsilon}, \quad x + \varepsilon w \in U \text{ and } x + \varepsilon v \in U.$$

Then

$$\frac{1}{\varepsilon}\big[f(x + \varepsilon w) - f(x)\big] = \frac{1}{\varepsilon}\big[f(x + \varepsilon w) - f(x + \varepsilon v)\big] + \frac{1}{\varepsilon}\big[f(x + \varepsilon v) - f(x)\big]$$

and

$$\begin{aligned} &\left|\frac{1}{\varepsilon}\big[f(x + \varepsilon w) - f(x)\big] - df(x; v)\right| \\ &\le \left|\frac{1}{\varepsilon}\big[f(x + \varepsilon v) - f(x)\big] - df(x; v)\right| + c(x)|w - v|_E. \end{aligned}$$

So as $\varepsilon \to 0$ and $w \to v$, $d^s_H f(x; v) = df(x; v)$.

(ii) By definition, from part (i). □

2.4 Chain Rule for Semiderivatives

The family of Hadamard semidifferentiable functions is extremely interesting since this property is stable under composition. They form a sufficiently large family of nondifferentiable functions that include continuous convex functions as we shall see in section 2.5.

Theorem 2.5. *Let E and F be two topological vector spaces and let h be the composition of two mappings f and g*

$$h(x) = f\big(g(x)\big) \tag{2.14}$$

in a neighborhood U *of a point* x *in* E*, where*

$$g : U \subset E \to F \quad \textit{and} \quad f : g(U) \to \mathbf{R}. \tag{2.15}$$

Assume that

(i) g *has a Gateaux* (resp., *Hadamard*) *semiderivative at* x *in the direction* v*, and*

(ii) $d_H f(g(x); dg(x; v))$ *exists,*

where the Hadamard semiderivative is taken in the general sense of Definition 2.1 (ii) *with respect to the topologies of* E *and* F.[3] *Then*

$$\begin{gathered} dh(x; v) = d_H f(g(x); dg(x; v)) \\ \big(\text{resp., } d_H h(x; v) = d_H f(g(x); d_H g(x; v))\big). \end{gathered} \tag{2.16}$$

Proof. (a) For $\varepsilon > 0$ small enough let

$$m(\varepsilon) = \frac{g(x + \varepsilon v) - g(x)}{\varepsilon} - dg(x; v) \text{ in } F.$$

By assumption $m(\varepsilon) \to 0$ as $\varepsilon \to 0$. By the definition of $dh(x; v)$ we want to find the limit of the differential quotient

$$d(\varepsilon) = \frac{f\big(g(x + \varepsilon v)\big) - f\big(g(x)\big)}{\varepsilon},$$

which can be rewritten as

$$d(\varepsilon) = \frac{f\big(g(x) + \varepsilon\big(dg(x; v) + m(\varepsilon)\big)\big) - f\big(g(x)\big)}{\varepsilon},$$

where

$$dg(x; v) + m(\varepsilon) \to dg(x; v) \text{ as } \varepsilon \to 0.$$

So by the definition of $d_H f$,

$$\lim_{\varepsilon \searrow 0} d(\varepsilon) = d_H f\big(g(x); dg(x; v)\big).$$

(b) When g is Hadamard semidifferentiable we replace $m(\varepsilon)$ and $d(\varepsilon)$ by

$$m(\varepsilon, w) = \frac{g(x + \varepsilon w) - g(x)}{\varepsilon} - dg(x; v)$$

and

$$d(\varepsilon, w) = \frac{1}{\varepsilon}\Big[f\big(g(x) + \varepsilon\big(dg(x; v) + m(\varepsilon, w)\big)\big) - f\big(g(x)\big)\Big]$$

and proceed as in part (a). □

[3]For normed vector spaces this theorem gives two sets of results: one for the strong topology and one for the weak topology.

In general we cannot improve the semiderivative of h by improving the semiderivative of g when f is not Hadamard semidifferentiable.

Example 2.4.
Consider the composition of the function $f : \mathbf{R}^2 \to \mathbf{R}$ in Example 2.3 and the map

$$g : \mathbf{R} \to \mathbf{R}^2, \quad g(x) = (x, x^2). \tag{2.17}$$

The map f is Gateaux, but not Hadamard semidifferentiable, and the map g is infinitely differentiable. However, the composition

$$h(x) = f\big(g(x)\big) = 1/x^2 \tag{2.18}$$

is not even Gateaux semidifferentiable in 0. □

We reiterate that the Hadamard semidifferentiability is a key property for the "chain rule." In general the composition $h = f \circ g$ of two maps will fail to have a semiderivative unless f has a Hadamard semiderivative, even if the map $g : E \to F$ is Fréchet differentiable at the point x.

2.5 Semiderivatives of Convex Functions

Finally the class of functions that are Hadamard semidifferentiable is quite large since it contains the classical continuously differentiable functions and the convex continuous functions.

Theorem 2.6. *Let $f : U \subset E \to \mathbf{R}$ be a function defined in an open convex subset of a topological vector space E.*

(i) *f is convex in U if and only if*

$$\forall x \in U, \forall v \in E, \quad df(x;v) \text{ exists}, \tag{2.19}$$

$$\forall x \in U, \forall v \in E, \quad df(x;v) + df(x;-v) \geq 0, \tag{2.20}$$

$$\forall x, y \in U, \quad f(y) \geq f(x) + df(x; y-x). \tag{2.21}$$

(ii) *If E is a normed vector space and if f is convex in U and continuous in $x \in U$, there exists a neighborhood $V(x)$ of x such that*

$$\forall y \in V(x), \forall v \in E, \quad d^s_H f(y;v) \text{ exists}. \tag{2.22}$$

Remark 2.2.
In a normed vector space E, the norm $n_E(x) \stackrel{\text{def}}{=} |x|_E$ is a convex and Lipschitzian function in E. Hence for all $x, v \in E$, $d_H n_E(x;v)$ exists and $d_H n_E(0;v) = |v|_E$. □

Proof. Part (ii) is a consequence of part (i) and the fact that a continuous convex function at x is locally Lipschitzian in a neighborhood of x (cf. I. Ekeland and R. Temam, [**1**, pp. 11–12]) by direct application of Theorem 2.4. We give the proof of part (i) for completeness. Assume that f is convex in U. Let $\theta \in]0,1]$. Given $x \in U$ and $v \in E$

$$\exists \alpha, 0 < \alpha < 1, \text{ such that } x - \alpha v \in U,$$
$$\exists \theta_0, 0 < \theta_0 < 1, \text{ such that } \forall \theta \in]0, \theta_0], \quad x + \theta v \in U.$$

For this fixed α, we show that

$$\forall \theta \in]0, \theta_0[, \quad \frac{f(x) - f(x - \alpha v)}{\alpha} \leq \frac{f(x + \theta v) - f(x)}{\theta}. \tag{2.23}$$

This follows from the identity

$$x = \frac{\alpha}{\alpha + \theta}(x + \theta v) + \frac{\theta}{\alpha + \theta}(x - \alpha v)$$

and the convexity of f,

$$f(x) \leq \frac{\alpha}{\alpha + \theta} f(x + \theta v) + \frac{\theta}{\alpha + \theta} f(x - \alpha v).$$

This can be rewritten as

$$\frac{\theta}{\theta + \alpha}\left[f(x) - f(x - \alpha v)\right] \leq \frac{\alpha}{\theta + \alpha}\left[f(x + \theta v) - f(x)\right]$$

and yields (2.23). Define

$$\varphi(\theta) = \frac{f(x + \theta v) - f(x)}{\theta}, \quad 0 < \theta < \theta_0.$$

Now we show that φ is a *monotone increasing* function of $\theta > 0$. For all θ_1 and θ_2, $0 < \theta_1 < \theta_2 < \theta_0$:

$$\begin{aligned} f(x + \theta_1 v) - f(x) &= f\left(\frac{\theta_1}{\theta_2}(x + \theta_2 v) + \left(1 - \frac{\theta_1}{\theta_2}\right) x\right) - f(x) \\ &\leq \frac{\theta_1}{\theta_2} f(x + \theta_2 v) + \left(1 - \frac{\theta_1}{\theta_2}\right) f(x) - f(x), \end{aligned}$$

which implies that $\varphi(\theta_1) \leq \varphi(\theta_2)$. As φ is a monotone increasing function which is bounded below, its limit exists as θ goes to 0. By definition it is equal to $df(x;v)$. Now that we know that $df(x;v)$ exists in all directions v, go back to inequality (2.23) and let α go to zero to get (2.20).

Conversely, apply inequality (2.19) twice to x and $x + \theta(y - x)$ and to y and $x + \theta(y - x)$ with $\theta \in [0,1]$:

$$\begin{aligned} f(x) &\geq f(x + \theta(y - x)) + df(x + \theta(y - x); -\theta(y - x)), \\ f(y) &\geq f(x + \theta(y - x)) + df(x + \theta(y - x); (1 - \theta)(y - x)). \end{aligned}$$

Multiply the first inequality by $1 - \theta$ and the second by θ and add up each side:

$$\begin{aligned} (1 - \theta) f(x) + \theta f(y) \geq f(x + \theta(y - x)) &+ (1 - \theta)\theta df(x + \theta(y - x); -(y - x)) \\ &+ \theta(1 - \theta) df(x + \theta(y - x); y - x). \end{aligned}$$

By using property (2.20), we get

$$f(\theta y + (1 - \theta)x) \leq \theta f(y) + (1 - \theta) f(x)$$

and the convexity of f in U. □

2.6 Hadamard Semiderivative and Velocity Method

In section 3 of Chapter 4 we have drawn an analogy between Gateaux and Hadamard semiderivatives on one hand and shape semiderivatives obtained by a perturbation of the identity and the velocity method on the other hand. The next theorem relates the Hadamard semiderivative and the semiderivative obtained by the velocity method for real functions defined on $\mathbf{R}^{\mathrm{N}}$.

Theorem 2.7. *Let $f : N_X \to \mathbf{R}$ be a real function defined in a neighborhood N_X of a point X in $\mathbf{R}^{\mathrm{N}}$. Then f is Hadamard semidifferentiable at (X, v) if and only if there exists $\tau > 0$ such that for all velocity fields $V : [0, \tau] \to \mathbf{R}$ satisfying the following assumptions:*

(a) (V_1) $\forall x \in \mathbf{R}^{\mathrm{N}}, V(\cdot, x) \in C\big([0, \tau]; \mathbf{R}^{\mathrm{N}}\big)$*;*

(b) (V_2) $\exists c > 0,\ \forall x, y \in \mathbf{R}^{\mathrm{N}}, \quad \|V(\cdot, y) - V(\cdot, x)\|_{C([0,\tau];\mathbf{R}^{\mathrm{N}})} \le c|y - x|$*;*

(c) *the limit*

$$df(X; v) \stackrel{\text{def}}{=} \lim_{\substack{\forall V,\, V(0)=v \\ t\searrow 0}} \frac{f\big(T_t(V)(X)\big) - f(X)}{t} \tag{2.24}$$

exists and is independent of the choice of V satisfying (a) *and* (b)*, where*

$$T_t(V)(X) \stackrel{\text{def}}{=} x(t; X),$$
$$\frac{dx}{dt}(t; X) = V\big(t, x(t; X)\big), \quad 0 < t < \tau, \quad x(0; X) = X.$$

Proof. $(\Rightarrow)$ Let V be a vector field satisfying conditions (a), (b), and (c). Define

$$w(t) \stackrel{\text{def}}{=} \frac{1}{t} \int_0^t V\big(s, x(s)\big)\, ds, \quad 0 < t \le \tau.$$

It is continuous on $]0, \tau]$ and

$$w(t) - v = \frac{1}{t} \int_0^t \Big[V\big(s, x(s)\big) - V(0, X)\Big]\, ds.$$

Therefore

$$|w(t) - v| \le c \max_{[0,t]} |x(s) - X| + \max_{[0,t]} |V(s, X) - V(0, X)|$$
$$\Rightarrow \lim_{t\searrow 0} w(t) = v.$$

So

$$\lim_{t\searrow 0} \frac{f\big(x(t; X)\big) - f(X)}{t} = \lim_{\substack{w(t)\to v \\ t\searrow 0}} \frac{f\big(X + tw(t)\big) - f(X)}{t} = d_H f(X; v),$$

since f is Hadamard semidifferentiable at (X, v) and the limit depends only on $V(0) = v$.

($\Leftarrow$) Conversely saying that the $d_H f(x; v)$ exists is equivalent to saying that for the sequence $t_n = 2^{-n}$ and any sequence $w_n \to v$ the limit

$$\frac{f(x + t_n w_n) - f(x)}{t_n}$$

exists and depends only on v. We now associate with the sequence $\{w_n\}$ a velocity field V satisfying (a) to (c) such that $V(t_n, X) = w_n$ and $T_{t_n}(V)(X) = X + t_n w_n$. Then from properties (2.24) and $V(o) = v$ we conclude that $d_H f(x; v)$ exists. Set $t_0 = 1$, $t_n = 2^{-n}$, and $x_n = X + t_n w_n$ and observe that $t_n - t_{n+1} = -2^{-(n+1)}$. Define for $m \geq 1$ the following C^m-interpolation in $(0, 1]$: for t in $[t_{n+1}, t_n]$

$$\boxed{\begin{aligned} T_t(X) \overset{\text{def}}{=} x_n &+ p\left(\frac{t - t_n}{t_{n+1} - t_n}\right)(x_{n+1} - x_n) \\ &+ q_0\left(\frac{t - t_n}{t_{n+1} - t_n}\right)(t_{n+1} - t_n)w_n \\ &+ q_1\left(\frac{t - t_n}{t_{n+1} - t_n}\right)(t_{n+1} - t_n)w_{n+1}, \quad T_0(X) \overset{\text{def}}{=} X, \end{aligned}}$$

where p, q_0, $q_1 \in P^{2m+1}[0, 1]$ are polynomials of order $2m + 1$ on $[0, 1]$ such that $p(0) = 0$, $p(1) = 1$, and $p^{(\ell)}(0) = 0 = p^{(\ell)}(1)$, $1 \leq \ell \leq m$, $q_0(0) = 0 = q_0(1)$, $q_0'(0) = 1$, $q_0'(1) = 0$, and $q_0^{(\ell)}(0) = 0 = q_0^{(\ell)}(1)$, $2 \leq \ell \leq m$, and $q_1(0) = 0 = q_1(1)$, $q_1'(0) = 0$, $q_1'(1) = 1$, and $q_1^{(\ell)}(0) = 0 = q_1^{(\ell)}(1)$, $2 \leq \ell \leq m$. By definition $T(\cdot, X) \in C^m((0, 1]; \mathbf{R}^{\mathrm{N}})$. Moreover, for $1 \leq \ell \leq m$,

$$\begin{aligned} T_{t_n}(X) = x_n,\ T_{t_{n+1}}(X) = x_{n+1}, \quad &\frac{\partial T}{\partial t}(t_n, X) = w_n,\ \frac{\partial T}{\partial t}(t_{n+1}, X) = w_{n+1}, \\ \frac{\partial T}{\partial t}(t, X) = p'\left(\frac{t - t_n}{t_{n+1} - t_n}\right)&\frac{x_{n+1} - x_n}{t_{n+1} - t_n} + q_0'\left(\frac{t - t_n}{t_{n+1} - t_n}\right)w_n \\ &+ q_1'\left(\frac{t - t_n}{t_{n+1} - t_n}\right)w_{n+1}. \end{aligned}$$

We now show that T satisfies conditions (a) and (b), which are equivalent to condition (4.2) in Chapter 4 on V. First observe that $T_t(X) - X$ and $\partial T/\partial t$ are both independent of X and $(T_t - I)(X) \in C^m((0, 1]; \mathbf{R}^{\mathrm{N}})$. Define

$$f(t) \overset{\text{def}}{=} T_t(X) - X \quad \Rightarrow \frac{\partial T}{\partial t}(t, X) = \frac{\partial f}{\partial t}.$$

Hence T and $\partial T/\partial t$ clearly satisfy conditions (T1) in (4.8) in Chapter 4. It satisfies condition (T2) since $T_t^{-1}(Y) = Y - f(t)$, and, a fortiori, T_t^{-1} satisfies conditions (T3). Therefore, the velocity field

$$V(t, x) \overset{\text{def}}{=} \frac{\partial T}{\partial t}(t, T_t^{-1}(x)) = \frac{\partial f}{\partial t}(t)$$

satisfies conditions (V) given by (4.2) of Chapter 4. Thus properties (a) and (b) are satisfied. It remains to show that $\partial f/\partial t(t) \to v$ as $t \to 0$. It is easy to check that by construction[4] $-p'(\tau) + q_0'(\tau) + q_1'(\tau) = 1$ in $[0, 1]$ and that on each interval $[t_{n+1}, t_n]$

$$\frac{x_{n+1} - x_n}{t_{n+1} - t_n} = w_{n+1} - 2w_n,$$

$$\begin{aligned}\frac{\partial f}{\partial t}(t) = \; & p'\left(\frac{t - t_n}{t_{n+1} - t_n}\right)(w_{n+1} - 2w_n) + q_0'\left(\frac{t - t_n}{t_{n+1} - t_n}\right) w_n \\ & + q_1'\left(\frac{t - t_n}{t_{n+1} - t_n}\right) w_{n+1},\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial t}(t) - v = \; & p'\left(\frac{t - t_n}{t_{n+1} - t_n}\right)[(w_{n+1} - v) - 2(w_n - v)] \\ & + q_0'\left(\frac{t - t_n}{t_{n+1} - t_n}\right)(w_n - v) + q_1'\left(\frac{t - t_n}{t_{n+1} - t_n}\right)(w_{n+1} - v).\end{aligned}$$

The left-hand side now clearly converges since the three polynomials are bounded by a constant independent of n. This is sufficient to prove the theorem. □

3 First-Order Shape Semiderivatives and Derivatives

3.1 Eulerian and Hadamard Semiderivatives

Go back to Definition 3.1 of a *shape functional* in section 3.1 of Chapter 4 and the discussion of *Gateaux* and *Hadamard* semiderivatives in section 3.2 of the same chapter, where we have shown that results based on perturbations of the identity or asymptotic developments can be readily recovered by direct application of the velocity method with *nonautonomous velocity fields* $V(t, x)$. In view of this, we give all the basic definitions, constructions, and theorems within the context of the velocity method and emphasize the Hadamard semiderivative, which is important in situations where the shape is parametrized and the chain rule is necessary.

We give the definitions in the constrained case under assumptions (5.5) in Chapter 4 that can be specialized to the unconstrained case ($D = \mathbf{R}^{\mathrm{N}}$).[5] Consider a velocity field

$$V : [0, \tau] \times \overline{D} \to \mathbf{R}^{\mathrm{N}} \tag{3.2}$$

[4]By interpolation of the function $g(\tau) = \tau - 1$, $g(\tau) = g(0)p(\tau) + g'(0)q_0(\tau) + g'(1)q_1(\tau) = -p(\tau) + q_0(\tau) + q_1(\tau)$, and necessarily $1 = g'(\tau) = -p'(\tau) + q_0'(\tau) + q_1'(\tau)$.

[5]In the unconstrained case, everything reduces to the conditions (V) on the velocity field given by (4.2) in Chapter 4: there exists $\tau = \tau(V) > 0$ such that

$$\text{(V)} \quad \begin{aligned} & \forall x \in \mathbf{R}^{\mathrm{N}}, \quad V(\cdot, x) \in C\big([0, \tau]; \mathbf{R}^{\mathrm{N}}\big), \\ & \exists c > 0, \forall x, y \in \mathbf{R}^{\mathrm{N}}, \quad \|V(\cdot, y) - V(\cdot, x)\|_{C([0,\tau];\mathbf{R}^{\mathrm{N}})} \le c|y - x|. \end{aligned} \tag{3.1}$$

verifying condition $(\mathrm{V1}_D)$ and the equivalent form $(\mathrm{V2}_C)$ of condition $(\mathrm{V2}_D)$[6] for some $\tau = \tau(V) > 0$:

$$\begin{aligned} &(\mathrm{V1}_D) \quad \begin{array}{l} \forall x \in \overline{D}, \quad V(\cdot,x) \in C\big([0,\tau];\mathbf{R}^{\mathrm{N}}\big), \quad \exists c > 0, \\ \forall x, y \in \overline{D}, \quad \|V(\cdot,y) - V(\cdot,x)\|_{C([0,\tau];\mathbf{R}^{\mathrm{N}})} \leq c|y-x|, \end{array} \\ &(\mathrm{V2}_C) \quad \forall t \in [0,\tau], \forall x \in \overline{D}, \quad V(t,x) \in L_D(x) = \{-C_D(x)\} \cap C_D(x), \end{aligned} \tag{3.3}$$

where $L_D(x) = C_D(x) \cap \{-C_D(x)\}$ is the *linear tangent space* to D at the point $x \in \partial D$ and $C_D(x)$ is Clarke's tangent cone to $\overline{D}$ at x (Theorem 5.2 in Chapter 4). We shall refer to the set of conditions (3.2)–(3.3) as condition (V_D).

Introduce the following linear subspace of $\mathrm{Lip}\,(D,\mathbf{R}^{\mathrm{N}})$:

$$\boxed{\mathrm{Lip}_{\,L}(D,\mathbf{R}^{\mathrm{N}}) \stackrel{\mathrm{def}}{=} \big\{\theta \in \mathrm{Lip}\,(D,\mathbf{R}^{\mathrm{N}}) \,:\, \forall X \in \partial D,\, \theta(X) \in L_D(X)\big\}\,.} \tag{3.4}$$

Under the action of a velocity field V satisfying conditions (3.2)–(3.3), a set Ω in $\overline{D}$ is transformed into a new domain,

$$\boxed{\Omega_t(V) \stackrel{\mathrm{def}}{=} T_t(V)(\Omega) = \{T_t(V)(X) \,:\, \forall X \in \Omega\}\,,} \tag{3.5}$$

also contained in $\overline{D}$. The transformation $T_t : \overline{D} \to \overline{D}$ associated with the velocity field $x \mapsto V(t)(x) \stackrel{\mathrm{def}}{=} V(t,x) : \overline{D} \to \mathbf{R}^{\mathrm{N}}$ is given by

$$T_t(X) \stackrel{\mathrm{def}}{=} x(t,X), \quad t \geq 0,\, X \in \overline{D}, \tag{3.6}$$

where $x(\cdot,X)$ is solution of the differential equation

$$\frac{dx}{dt}(t,X) = V(t,x(t,X)),\, t \geq 0, \quad x(t,X) = X. \tag{3.7}$$

Recall Definition 3.1 of section 3 in Chapter 4 of a shape functional.

Definition 3.1.
Given a nonempty subset D of $\mathbf{R}^{\mathrm{N}}$, consider the set $\mathcal{P}(D) = \{\Omega : \Omega \subset D\}$ of subsets of D. The set D will be referred to as the underlying *holdall* or *universe*. A *shape functional* is a map

$$J : \mathcal{A} \to E \tag{3.8}$$

from some *admissible family* $\mathcal{A}$ of sets in $\mathcal{P}(D)$ into a topological space E. □

For instance, $\mathcal{A}$ could be the set $\mathcal{X}(\Omega) = \{F(\Omega) : \forall F \in \mathcal{F}(\Theta)\}$.

Definition 3.2.
Let J be a real-valued shape functional.

[6]We replace condition $(\mathrm{V2}_D)$ in (5.5) of Chapter 4

$$(\mathrm{V2}_D) \quad \forall x \in \overline{D}, \forall t \in [0,\tau], \quad \pm V(t,x) \in T_{\overline{D}}(x)$$

by its equivalent form $(\mathrm{V2}_C)$ by using Theorem 5.2 in Chapter 4.

(i) Let V be a velocity field satisfying conditions (3.2)–(3.3). J has an *Eulerian semiderivative* at Ω in the direction V if the limit

$$\lim_{t \searrow 0} \frac{J\big(\Omega_t(V)\big) - J(\Omega)}{t} \tag{3.9}$$

exists. It will be denoted by $dJ(\Omega; V)$. For simplicity, when $V(t,x) = \theta(x)$, $\theta \in \operatorname{Lip}_L(D, \mathbf{R}^N)$ an autonomous velocity field, we shall write $dJ(\Omega;\theta)$.

(ii) Let Θ be a topological vector subspace of $\operatorname{Lip}_L(D, \mathbf{R}^N)$ satisfying conditions (3.2)–(3.3). J has a *Hadamard semiderivative* at Ω in the direction $\theta \in \Theta$ if

$$\lim_{\substack{V \in C^0([0,\tau];\Theta)\\ V(0)=\theta \\ t \searrow 0}} \frac{J(\Omega_t(V)) - J(\Omega)}{t} \tag{3.10}$$

exists, depends only on θ, and is independent of the choice of V satisfying conditions (3.2)–(3.3). It will be denoted by $d_H J(\Omega;\theta)$. If $d_H J(\Omega;\theta)$ exists, J has an Eulerian semiderivative at Ω in the direction $V(t,x) = \theta(x)$ and

$$d_H J(\Omega;\theta) = dJ(\Omega; V(0)) = dJ(\Omega; V).$$

(iii) Let Θ be a topological vector subspace of $\operatorname{Lip}_L(D, \mathbf{R}^N)$ satisfying conditions (3.2)–(3.3). J has a *Hadamard derivative* at Ω with respect to Θ if it has a Hadamard semiderivative at Ω in all directions $\theta \in \Theta$ and the map

$$\theta \mapsto dJ(\Omega;\theta) \, : \, \Theta \to \mathbf{R} \tag{3.11}$$

is linear and continuous. The map (3.11) will be denoted $G(\Omega)$ and referred to as the *gradient* of J in the topological dual Θ' of Θ. □

The definition of an Eulerian semiderivative is quite general. For instance, it readily applies to shape functionals defined on closed submanifolds D of $\mathbf{R}^N$. It includes cases where $dJ(\Omega;V)$ is dependent not only on $V(0)$ but also on $V(t)$ in a neighborhood of $t = 0$. We shall see that this will not occur under some continuity assumption on the map $V \mapsto dJ(\Omega;V)$. When $dJ(\Omega;V)$ depends only on $V(0)$, the analysis can be specialized to autonomous vector fields V and the semiderivative can be related to the gradient of J.

Example 3.1.
For any measurable subset Ω of $\mathbf{R}^N$, consider the volume function

$$J(\Omega) = \int_\Omega dx. \tag{3.12}$$

For Ω with finite volume and V in $C([0,\tau]; C_0^1(\mathbf{R}^N, \mathbf{R}^N))$, consider the transformations $T_t(\Omega)$ of Ω:

$$J(T_t(\Omega)) = \int_{T_t(\Omega)} dx = \int_\Omega |\det(DT_t)|\, dx = \int_\Omega \det(DT_t)\, dx$$

for t small since $\det(DT_0) = \det I = 1$. This yields the Eulerian semiderivative

$$dJ(\Omega; V) = \int_\Omega \operatorname{div} V(0)\, dx. \tag{3.13}$$

By definition it depends only on $V(0)$ and hence J has a Hadamard semiderivative

$$d_H J(\Omega; V(0)) = \int_\Omega \operatorname{div} V(0)\, dx \tag{3.14}$$

and even a gradient $G(\Omega)$ for $\Theta = C_0^1(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N})$. □

The following simple continuity condition can also be used to obtain the Hadamard semidifferentiability.

Theorem 3.1. *Let Θ be a Banach subspace of $\operatorname{Lip}_L(D, \mathbf{R}^\mathrm{N})$ satisfying conditions (3.2)–(3.3), J be a real-valued shape functional, and Ω be a subset of $\mathbf{R}^\mathrm{N}$.*

(i) *Given $\theta \in \Theta$, if*

$$\forall V \in C([0,\tau]; \Theta) \text{ such that } V(0) = \theta, \quad dJ(\Omega; V) \text{ exists}, \tag{3.15}$$

and if the map

$$V \mapsto dJ(\Omega; V) : C([0,\tau]; \Theta) \to \mathbf{R} \tag{3.16}$$

is continuous for the subspace of V's such that $V(0) = \theta$, then J is Hadamard semidifferentiable at Ω in the direction θ with respect to Θ and

$$\forall V \in C([0,\tau]; \Theta),\ V(0) = \theta, \quad dJ(\Omega; V) = d_H J\big(\Omega; \theta\big). \tag{3.17}$$

(ii) *If for all V in $C([0,\tau]; \Theta)$, $dJ(\Omega; V)$ exists and the map*

$$V \mapsto dJ(\Omega; V) : C([0,\tau]; \Theta) \to \mathbf{R} \tag{3.18}$$

is continuous, then J is Hadamard semidifferentiable at Ω in the direction $V(0)$ with respect to Θ and

$$\forall V \in C([0,\tau]; \Theta), \quad dJ(\Omega; V) = dJ\big(\Omega; V(0)\big) = d_H J\big(\Omega; V(0)\big). \tag{3.19}$$

Proof. Given V in $C([0,\tau]; \Theta)$ such that $V(0) = \theta$, construct the sequence

$$V_n(t) \overset{\text{def}}{=} \begin{cases} V(t), & 0 \le t \le \tau/n, \\ V\left(\frac{\tau}{n}\right), & \tau/n < t \le \tau, \end{cases} \quad \text{for all integers } n \ge 1.$$

Note that for each $n \ge 1$, $dJ(\Omega; V_n) = dJ(\Omega; V)$ since for $t < \tau/n$, $T_t(V_n) = T_t(V)$. By continuity of V, $\{V_n\}$ converges in $C([0,\tau]; \Theta)$ to the autonomous field $\widetilde{V}(t) = V(0)$:

$$\begin{gathered} \forall n,\ \forall V \in C([0,\tau]; \Theta), \\ \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall t',\ |t'| < \delta, \quad \|V(t') - V(0)\|_\Theta < \varepsilon. \end{gathered}$$

Hence

$$\forall \varepsilon > 0,\ \exists N > 0,\ \forall n \geq N,\ \frac{\tau}{n} < \delta$$
$$\Rightarrow \quad \sup_{t \in [0, \frac{\tau}{n}]} \|V(t) - V(0)\|_\Theta < \varepsilon \quad \Rightarrow V_n \to \widetilde{V}.$$

Hence by continuity of the map (3.18)

$$dJ(\Omega; V_n) \to dJ\big(\Omega; \widetilde{V}\big) \quad \Rightarrow dJ(\Omega; V) = dJ\big(\Omega; \widetilde{V}\big).$$

But since $dJ\big(\Omega; \widetilde{V}\big)$ is dependent only on $V(0)$, for all $W \in \Theta$ such that $W(0) = V(0)$ we have the identity $dJ(\Omega; W) = dJ\big(\Omega; \widetilde{V}\big)$. Therefore, J has a Hadamard semiderivative at Ω in the direction $V(0)$ and

$$d_H J(\Omega; V(0)) = dJ\big(\Omega; \widetilde{V}\big) = dJ(\Omega; V).$$

This concludes the proof of the theorem. □

For simplicity, the last theorem was given only for a Banach space Θ. This includes the spaces $\mathcal{C}_0^k(\mathbf{R}^\mathrm{N})$, $\mathcal{C}^k(\overline{\mathbf{R}^\mathrm{N}})$, $\mathcal{C}^{0,1}(\overline{\mathbf{R}^\mathrm{N}})$, and $\mathcal{B}^k(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N})$ considered in Chapters 3 and 4. However, its conclusions are not limited to Banach spaces. Other constructions can be used as illustrated below in the unconstrained case. Consider velocity fields in

$$\boxed{\overrightarrow{\mathcal{V}}^{m,k} \stackrel{\text{def}}{=} \varinjlim_{K} \left\{ V_K^{m,k} : \forall K \text{ compact in } \mathbf{R}^\mathrm{N} \right\},} \tag{3.20}$$

where for $m \geq 0$,

$$\boxed{V_K^{m,k} \stackrel{\text{def}}{=} C^m\big([0,\tau]; \mathcal{D}^k(K, \mathbf{R}^\mathrm{N}) \cap \mathrm{Lip}(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N})\big)} \tag{3.21}$$

and $\varinjlim$ denotes the inductive limit set with respect to K endowed with its natural inductive limit topology. For autonomous fields, this construction reduces to

$$\overrightarrow{\mathcal{V}}^{k} \stackrel{\text{def}}{=} \varinjlim_{K} \left\{ V_K^{k} : \forall K \text{ compact in } \mathbf{R}^\mathrm{N} \right\}, \tag{3.22}$$

$$V_K^k \stackrel{\text{def}}{=} \begin{cases} \mathcal{D}^0(K, \mathbf{R}^\mathrm{N}) \cap \mathrm{Lip}\,(K, \mathbf{R}^\mathrm{N}), & k = 0, \\ \mathcal{D}^k(K, \mathbf{R}^\mathrm{N}), & 1 \leq k \leq \infty. \end{cases} \tag{3.23}$$

For $k \geq 1$, $\overrightarrow{\mathcal{V}}^k = \mathcal{D}^k(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N})$. In all cases the conditions (3.1) of the unconstrained case $D = \mathbf{R}^\mathrm{N}$ are verified.

Theorem 3.2. *Let J be a real-valued shape functional, Ω be a subset of $\mathbf{R}^\mathrm{N}$, and $k \geq 0$ be an integer.*

(i) *Given $\theta \in \overrightarrow{\mathcal{V}}^k$, assume that*

$$\forall V \in \overrightarrow{\mathcal{V}}^{0,k},\ V(0) = \theta, \quad dJ(\Omega; V) \text{ exists} \tag{3.24}$$

and that the map

$$V \mapsto dJ(\Omega; V) : \overrightarrow{\mathcal{V}}^{0,k} \to \mathbf{R} \tag{3.25}$$

is continuous for all V's such that $V(0) = \theta$. Then J is Hadamard semidifferentiable in Ω in the direction θ with respect to $\mathcal{V}^k$ and

$$\forall V \in \overrightarrow{\mathcal{V}}^{0,k},\ V(0) = \theta, \quad d_H J\big(\Omega; \theta\big) = dJ(\Omega; V) = dJ\big(\Omega; V(0)\big). \tag{3.26}$$

(ii) *Assume that for all $V \in \overrightarrow{\mathcal{V}}^{0,k}$, $dJ(\Omega; V)$ exists and that the map*

$$V \mapsto dJ(\Omega; V) : \overrightarrow{\mathcal{V}}^{0,k} \to \mathbf{R} \tag{3.27}$$

is continuous. Then J is Hadamard semidifferentiable in Ω in the direction $V(0)$ with respect to $\mathcal{V}^k$ and

$$\forall V \in \overrightarrow{\mathcal{V}}^{0,k}, \quad d_H J\big(\Omega; V(0)\big) = dJ(\Omega; V) = dJ\big(\Omega; V(0)\big). \tag{3.28}$$

Proof. It is sufficient to prove the theorem for any compact subset K of $\mathbf{R}^{\mathrm{N}}$ and hence only for velocities in $\mathcal{V}_K^{0,k} = C([0,\tau]; \mathcal{V}_K^k)$, where $\mathcal{V}_K^k$ is a Banach space contained in $C_0^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$. So the theorem follows from Theorem 3.1. □

3.2 Hadamard Semidifferentiability and Courant Metric Continuity

We now relate the Hadamard semidifferentiability of a shape functional with the Courant metric continuity studied in section 6 of Chapter 4. Recall the generic metric spaces $\mathcal{F}(\Theta)$ of Micheletti with metric d and the quotient group $\mathcal{F}(\Theta)/\mathcal{G}$ with the Courant metric $d_{\mathcal{G}}$ in (2.31) of Chapter 3.

Theorem 3.3. *Let Θ be equal to $C_0^{k+1}(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$, $C^{k+1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$, or $C^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$, $k \geq 0$. Assume that $\mathcal{G} = \mathcal{G}(\Omega)$ for Ω closed or open and crack-free. If a shape functional J is Hadamard semidifferentiable for all $\theta \in \Theta$, then it is continuous with respect to the Courant metric $d_{\mathcal{G}(\Omega)}$.*

Proof. By definition of the Hadamard semiderivative and Theorem 6.1 for $\Theta = C_0^{k+1}(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$, Theorem 6.2 for $\Theta = C^{k+1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$, and Theorem 6.3 for $\Theta = C^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$ of Chapter 4. □

3.3 Perturbations of the Identity and Gateaux and Fréchet Derivatives

In the unconstrained case ($D = \mathbf{R}^{\mathrm{N}}$), we have introduced in sections 3 and 4.2 of Chapter 4 a notion of directional semiderivative associated with first- and second-order perturbations of the identity. Even if this approach does not naturally extend to the constrained case, it is interesting to compare the definitions and results with those associated with the velocity method of Definition 3.2.

Recall the generic metric spaces $\mathcal{F}(\Theta)$ of Micheletti with metric d and the quotient group $\mathcal{F}(\Theta)/\mathcal{G}$ with the Courant metric $d_{\mathcal{G}}$ in (2.31) in Chapter 3. Assume that $\mathcal{G} = \mathcal{G}(\Omega)$ for Ω closed or open and crack-free and that there exist a ball B_ε of radius $\varepsilon > 0$ in Θ and a constant $c > 0$ such that

$$\boxed{\forall f \in B_\varepsilon, \quad \|[I+f]^{-1} - I\|_\Theta \le c\,\|f\|_\Theta.} \tag{3.29}$$

Except for $C^0(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$ and $\mathcal{B}^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$, this is true in all Banach spaces $\Theta \subset C^{0,1}(\overline{\mathbf{R}^{\mathrm{N}}}, \mathbf{R}^{\mathrm{N}})$ considered in Chapters 3 and 4 (cf. Theorem 2.14 in section 2.5.2 and Theorem 2.17 in section 2.6.2 of Chapter 3). Hence the maps

$$f \mapsto [I+f] \mapsto [I+f] \,:\, B_\varepsilon \subset \Theta \to \mathcal{F}(\Theta) \to \mathcal{F}(\Theta)/\mathcal{G}(\Omega)$$

are well-defined and continuous in $f = 0$ since

$$d_{\mathcal{G}}(I, I+f) \le d(I, I+f) \le \|f\|_\Theta + \|[I+f]^{-1} - I\|_\Theta \le (1+c)\,\|f\|_\Theta.$$

For a shape functional J, the map

$$[I+f] \mapsto J_\Omega(f) \stackrel{\text{def}}{=} J([I+f](\Omega)) \,:\, \mathcal{F}(\Theta)/\mathcal{G}(\Omega) \to \mathbf{R}$$

is well-defined since J is invariant on $\mathcal{G}(\Omega)$ and the map

$$f \mapsto J_\Omega(f) \stackrel{\text{def}}{=} J([I+f](\Omega)) \,:\, \Theta \to \mathbf{R}$$

is continuous in $f = 0$ if J is continuous in Ω for the Courant metric on $\mathcal{F}(\Theta)/\mathcal{G}(\Omega)$.

The following definitions are now the extension of the standard definitions of section 2 for a function $J_\Omega(f)$ defined on the ball B_ε in the normed vector space Θ to topological vector spaces. They parallel the ones of Definition 3.2.

Definition 3.3.
Let J be a real-valued shape functional and Θ be a topological vector subspace of $\mathrm{Lip}\,(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$. For $f \in \Theta$, denote $[I+f](\Omega)$ by Ω_f.

(i) J_Ω is said to have a *Gateaux semiderivative* at f in the direction $\theta \in \Theta$ if the following limit exists and is finite:

$$dJ_\Omega(f;\theta) \stackrel{\text{def}}{=} \lim_{t \searrow 0} \frac{J\big([I+f+t\theta](\Omega)\big) - J([I+f](\Omega))}{t}. \tag{3.30}$$

(ii) J_Ω is said to be *Gateaux differentiable* at f if it has a *Gateaux semiderivative* at f in all directions $\theta \in \Theta$ and the map

$$\theta \mapsto dJ_\Omega(f;\theta) \,:\, \Theta \to \mathbf{R} \tag{3.31}$$

is linear and continuous. The map (3.31) is denoted $\nabla J_\Omega(f)$ and referred to as the gradient of J_Ω in the topological dual Θ' of Θ.

(iii) If, in addition, Θ is a normed vector space, we say that J is *Fréchet differentiable* at f if J is Gateaux differentiable at f and

$$\lim_{\|\theta\|_\Theta \to 0} \frac{|J([I+f+\theta](\Omega)) - J([I+f](\Omega)) - \langle \nabla J_\Omega(f), \theta\rangle_\Theta|}{\|\theta\|_\Theta} = 0. \tag{3.32}$$

□

The semiderivatives of J and J_Ω are related.

Theorem 3.4. *Let J be a real-valued shape functional, let $f \in \Theta$, and let $I + f \in \mathcal{F}(\Theta)$.*

(i) *Assume that J_Ω has a Gateaux semiderivative at f in the direction $\theta \in \Theta$; then J has an Eulerian semiderivative at Ω_f in the direction V_θ^f and*

$$\boxed{dJ_\Omega(f;\theta) = dJ(\Omega_f; V_\theta^f), \quad V_\theta^f(t) \overset{\text{def}}{=} \theta \circ [I + f + t\theta]^{-1}} \tag{3.33}$$

for t sufficiently small.

(ii) *If J has a Hadamard semiderivative at Ω_f in the direction $\theta \circ [I+f]^{-1}$, then J_Ω has a Gateaux semiderivative at f in the direction θ and*

$$\boxed{dJ_\Omega(f;\theta) = d_H J(\Omega_f; \theta \circ [I+f]^{-1}).} \tag{3.34}$$

Conversely, if J_Ω has a Gateaux semiderivative at f in the direction $\theta \circ [I+f]$, then J has a Hadamard semiderivative at Ω_f in the direction θ. If either $dJ_\Omega(f;\theta)$ or $d_H J(\Omega_f;\theta)$ is linear and continuous with respect to all θ in Θ, so is the other and

$$\forall \theta \in \Theta, \quad \boxed{\begin{aligned} \langle \nabla J_\Omega(f), \theta\rangle_\Theta &= \langle G(\Omega_f), \theta \circ [I+f]^{-1}\rangle_\Theta, \\ \langle G(\Omega_f), \theta\rangle_\Theta &= \langle \nabla J_\Omega(f), \theta \circ [I+f]\rangle_\Theta. \end{aligned}} \tag{3.35}$$

Proof. By definition

$$\begin{aligned} dJ_\Omega(f;\theta) &= \lim_{t \searrow 0} \frac{J\big([I+f+t\theta](\Omega)\big) - J([I+f](\Omega))}{t} \\ &= \lim_{t \searrow 0} \frac{J\big(T_t([I+f](\Omega))\big) - J([I+f](\Omega))}{t} = dJ([I+f](\Omega); V_\theta^f) \end{aligned}$$

for the family of transformations

$$T_t^f \overset{\text{def}}{=} [I+f+t\theta] \circ [I+f]^{-1},$$

and from Theorem 4.1 in section 4.1 of Chapter 4, T_t^f corresponds to the velocity field

$$V_\theta^f(t) \overset{\text{def}}{=} \frac{\partial T_t^f}{\partial t} \circ (T_t^f)^{-1} = \theta \circ [I+f+t\theta]^{-1}.$$

Identity (3.34) now follows from the fact that J has a Hadamard semiderivative at Ω_f. The other properties readily follow from the definitions. □

We have standard sufficient conditions for the Fréchet differentiability from Theorem 2.3.

Theorem 3.5. *Let J be a real-valued shape functional. Let Ω be a subset of $\mathbf{R}^{\mathrm{N}}$ and Θ be $C_0^{k+1}(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})$, $C^{k+1}(\overline{\mathbf{R}^{\mathrm{N}}},\mathbf{R}^{\mathrm{N}})$, $\mathcal{C}^{k,1}(\overline{\mathbf{R}^{\mathrm{N}}})$, or $\mathcal{B}^{k+1}(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})$, $k \geq 0$. If J_Ω is Gateaux differentiable for all f in B_ε and the map*

$$f \mapsto \nabla J_\Omega(f) : B_\varepsilon \to \Theta'$$

is continuous in $f = 0$, then J_Ω is Fréchet differentiable in $f = 0$.

3.4 Shape Gradient and Structure Theorem

In view of the previous discussion we now specialize to autonomous vector fields V to further study the properties and the structure of $dJ(\Omega;V)$. For simplicity we also specialize to the unconstrained case in $\mathbf{R}^{\mathrm{N}}$. The constrained case yields similar results but is technically more involved (cf. M. C. Delfour and J.-P. Zolésio [**14**] for D open in $\mathbf{R}^{\mathrm{N}}$).

The choice of a *shape gradient* depends on the choice of the topological vector subspace Θ of Lip$(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})$. We choose to work in the classical framework of the Theory of Distributions (cf. L. Schwartz [**3**]) with $\Theta = \mathcal{D}(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})$, the space of all infinitely differentiable transformations θ of $\mathbf{R}^{\mathrm{N}}$ with compact support. For these velocity fields V, conditions (3.1) are satisfied.

Definition 3.4.
Let J be a real-valued shape functional. Let Ω be a subset of $\mathbf{R}^{\mathrm{N}}$.

(i) The function J is said to be *shape differentiable* at Ω if it is differentiable at Ω for all θ in $\mathcal{D}(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})$.

(ii) The map (3.11) defines a vector distribution $G(\Omega)$ in $\mathcal{D}(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})'$, which will be referred to as the *shape gradient* of J at Ω.

(iii) When, for some finite $k \geq 0$, $G(\Omega)$ is continuous for the $\mathcal{D}^k(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})$-topology, we say that the shape gradient $G(\Omega)$ is of order k. □

The next theorem gives additional properties of shape differentiable functions.

Notation 3.1.
Associate with a subset A of $\mathbf{R}^{\mathrm{N}}$ and an integer $k \geq 0$ the set

$$L_A^k \stackrel{\text{def}}{=} \left\{V \in \mathcal{D}^k(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}}) : \forall x \in A, V(x) \in L_A(x)\right\},$$

where $L_A(x) = \{-C_A(x)\} \cap C_A(x)$ and $C_A(x)$ is given by (5.27) in Chapter 4. □

Theorem 3.6 (Structure theorem)**.** *Let J be a real-valued shape functional. Assume that J has a shape gradient $G(\Omega)$ for some $\Omega \subset \mathbf{R}^{\mathrm{N}}$ with boundary Γ.*

(i) *The support of the shape gradient $G(\Omega)$ is contained in Γ.*

(ii) *If Ω is open or closed in $\mathbf{R}^{\mathrm{N}}$ and the shape gradient is of order k for some $k \geq 0$, then there exists $[G(\Omega)]$ in $(\mathcal{D}^k/L^k_\Omega)'$ such that for all V in $\mathcal{D}^k \stackrel{\text{def}}{=} \mathcal{D}^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$*

$$dJ(\Omega; V) = \langle [G(\Omega)], q_L V \rangle_{\mathcal{D}^k/L^k_\Omega}, \tag{3.36}$$

where $q_L : \mathcal{D}^k \to \mathcal{D}^k/L^k_\Omega$ is the canonical quotient surjection. Moreover

$$G(\Omega) = (q_L)^*[G(\Omega)], \tag{3.37}$$

where $(q_L)^*$ denotes the transpose of the linear map q_L.

Proof. (i) Any V in $\mathcal{D}$ such that $V = 0$ on Γ satisfies assumptions $(V1_\Omega)$ and $(V2_\Omega)$ in (5.5) (with Ω in place of D) and $V \in L^\infty_\Omega$. Since $V = 0$ on Γ, $T_s(\Gamma) = \Gamma$. Moreover, by Theorem 5.1 (i) of Chapter 4, $T_s : \bar{\Omega} \to \bar{\Omega}$ is a homeomorphism and $T_s(\bar{\Omega}) = \bar{\Omega}$ and $(\Omega)_s = T_s(\Omega) = \Omega$ since $\bar{\Omega} = \Omega \cup \Gamma$ and $\Omega \cap \Gamma = \varnothing$. Thus when Ω is closed ($\Omega = \bar{\Omega}$) or open ($\Omega = \operatorname{int} \Omega$),

$$\forall s \geq 0, \quad T_s(\Omega) = \Omega \quad \Rightarrow \quad J(\Omega_s) = J(\Omega) \quad \Rightarrow \quad dJ(\Omega; V) = 0.$$

(ii) It is sufficient to prove that $dJ(\Omega; V) = 0$ for all V in L^k_Ω. The other statements follow by standard arguments and the fact that L^k_Ω is a closed linear subspace of $\mathcal{D}^k$. From part (i) we know that the result is true for all V in L^∞_Ω and hence by a density argument for all V in L^k_Ω. □

Remark 3.1.
When the boundary Γ of Ω is compact and J is shape differentiable at Ω, the distribution $G(\Omega)$ is of finite order. Once this is known, the conclusions of Theorem 3.6 (ii) apply with k equal to the order of $G(\Omega)$. Hence $G(\Omega)$ will belong to a Hilbert space $H^{-s}(\mathbf{R}^{\mathrm{N}})$ for some $s \geq 0$. □

The quotient space is very much related to a trace on the boundary Γ, and when the boundary Γ is sufficiently smooth we can indeed make that identification.

Corollary 1. *Assume that the assumptions of Theorem 3.6 are satisfied for an open domain Ω, that the order of $G(\Omega)$ is $k \geq 0$, and that the boundary Γ of Ω is C^{k+1}. Then for all x in Γ, $L_\Omega(x)$ is an $(N-1)$-dimensional hyperplane to Ω at x and there exists a unique outward unit normal $n(x)$ which belongs to $C^k(\Gamma; \mathbf{R}^{\mathrm{N}})$. As a result, the kernel of the map*

$$V \mapsto \gamma_\Gamma(V) \cdot n : \mathcal{D}^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}) \to C^k(\Gamma) \tag{3.38}$$

coincides with L^k_Ω, where $\gamma_\Gamma : \mathcal{D}^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}) \to C^k(\Gamma, \mathbf{R}^{\mathrm{N}})$ is the trace of V on Γ. Moreover, the map $p_L(V)$

$$q_L(V) \mapsto p_L\big(q_L(V)\big) \stackrel{\text{def}}{=} \gamma_\Gamma(V) \cdot n : \mathcal{D}^k/L^k_\Omega \to C^k(\Gamma) \tag{3.39}$$

is a well-defined isomorphism. In particular, there exists a scalar distribution $g(\Gamma)$ in $\mathbf{R}^{\mathrm{N}}$ with support in Γ such that $g(\Gamma) \in C^k(\Gamma)'$ and for all V in $\mathcal{D}^k(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$

$$\boxed{dJ(\Omega; V) = \langle g(\Gamma), \gamma_\Gamma(V) \cdot n \rangle_{C^k(\Gamma)}} \tag{3.40}$$

and

$$G(\Omega) = {}^*(q_L)[G(\Omega)], \quad [G(\Omega)] = {}^*(p_L)g(\Gamma). \tag{3.41}$$

When $g(\Gamma) \in L^1(\Gamma)$

$$dJ(\Omega; V) = \int_\Gamma g\, V \cdot n\, d\Gamma \text{ and } G = \gamma_\Gamma^*(g\, n), \tag{3.42}$$

where γ_Γ *is the trace operator on* Γ.

Proof. The surjectivity of (3.39) is a consequence of the fact that Γ is compact and that for a C^{k+1} boundary, $k \geq 0$, it is always possible to construct an extension N of the unit normal n on Γ which belongs to $\mathcal{D}^k(\mathbf{R}^N, \mathbf{R}^N)$ with support in a neighborhood of Γ. Then for any v in $C^k(\Gamma)$, there exists also an extension $\tilde{v}$ in $\mathcal{D}^k(\mathbf{R}^N)$ with support in a neighborhood of Γ, and the vector $V = \tilde{v}N$ belongs to $\mathcal{D}^k(\mathbf{R}^N, \mathbf{R}^N)$ and coincides with vn on Γ. □

Remark 3.2.
In 1907, J. Hadamard [**1**] used displacements along the normal to the boundary Γ of a C^∞-domain (as in section 3.3.1 of Chapter 4) to compute the derivative of the first eigenvalue of the clamped plate. Theorem 3.6 and its corollary are generalizations to arbitrary shape functionals of that property to open or closed domains with an arbitrary boundary. The structure theorem for shape functionals on open domains with a C^{k+1}-boundary is due to J.-P. Zolésio [**12**] in 1979 and not to Hadamard even if the formula (3.42) is often called the *Hadamard formula.* □

Example 3.2.
For any measurable subset Ω of $\mathbf{R}^N$, consider the volume shape function (3.12) of Example 3.1:

$$J(\Omega) = \int_\Omega dx.$$

For Ω with finite volume and V in $\mathcal{D}^1(\mathbf{R}^N, \mathbf{R}^N)$, we have seen in Example 3.1 that

$$dJ(\Omega; V) = \int_\Omega \operatorname{div} V\, dx = \int_{\mathbf{R}^N} \chi_\Omega \operatorname{div} V\, dx, \tag{3.43}$$

and this is formula (3.40) with $k = 1$. For an open domain Ω with a C^1 compact boundary Γ,

$$dJ(\Omega; V) = \int_\Gamma V \cdot n\, d\Gamma, \tag{3.44}$$

which is also continuous with respect to V in $\mathcal{D}^0(\mathbf{R}^N, \mathbf{R}^N)$. Here the smoothness of the boundary decreases the order of the distribution $G(\Omega)$. This raises the question of the characterization of the family of all subsets Ω of $\mathbf{R}^N$ for which the map

$$V \mapsto \int_\Omega \operatorname{div} V\, dx : \mathcal{D}^1(\mathbf{R}^N, \mathbf{R}^N) \to \mathbf{R} \tag{3.45}$$

can be continuously extended to $\mathcal{D}^0(\mathbf{R}^N, \mathbf{R}^N)$. But this is the family of *locally finite perimeter sets*: sets Ω whose characteristic function belongs to $\mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^N)$. □

4 Elements of Shape Calculus

In this section we recall a number of basic formulae from J.-P. ZOLÉSIO [**7**, **8**] for the derivative of *domain and boundary integrals*. The reader is also referred to the *companion book* of J. SOKOŁOWSKI and J.-P. ZOLÉSIO [**9**] for the computation of shape derivatives associated with a wide range of partial differential equations.

In this section, a more modern treatment of the derivative of boundary integrals is given that emphasizes a new approach to the *tangential calculus* on a C^2 submanifold of $\mathbf{R}^{\mathrm{N}}$ of codimension 1. This development took place in the context of the theory of shells, where a simple and self-contained intrinsic differential calculus has been developed by using the oriented distance function of Chapter 7. It completely avoids the use of local maps and bases and Christoffel's symbols. In this section a new, considerably simplified proof of the formula for the derivative of boundary integrals is presented by using the oriented distance function.

4.1 Basic Formula for Domain Integrals

The simplest examples of domain functions are given by *volume integrals* over a bounded open domain Ω in $\mathbf{R}^{\mathrm{N}}$. They use a basic formula in connection with the family of transformations $\{T_t : 0 \leq t \leq \tau\}$. Assume that condition (3.1) is satisfied by the velocity field $\{V(t) : 0 \leq t \leq \tau\}$. Further assume that $V \in C^0([0,\tau]; C^1_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}))$ and that $\tau > 0$ is such that the *Jacobian* J_t is strictly positive:

$$\forall t \in [0,\tau], \quad J_t(X) \stackrel{\text{def}}{=} \det DT_t(X) > 0, \quad (DT_t)_{ij} = \partial_j T_i,$$

where $DT_t(X)$ is the *Jacobian matrix* of the transformation $T_t = T_t(V)$ associated with the velocity vector field V. Given a function φ in $W^{1,1}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$, consider for $0 \leq t \leq \tau$ the volume integral

$$J(\Omega_t(V)) \stackrel{\text{def}}{=} \int_{\Omega_t(V)} \varphi \, dx, \tag{4.1}$$

where $\Omega_t(V) \stackrel{\text{def}}{=} T_t(V)(\Omega)$. By the change of variables formula

$$J(\Omega_t(V)) = \int_{\Omega_t(V)} \varphi \, dx = \int_{\Omega} \varphi \circ T_t \;\; J_t \;\; dx, \tag{4.2}$$

and the following formulae and results are easy to check.

Theorem 4.1. *Let φ be a function in $W^{1,1}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$. Assume that the vector field $V = \{V(t) : 0 \leq t \leq \tau\}$ satisfies condition (V).*

(i) *For each $t \in [0,\tau]$ the map*

$$\varphi \mapsto \varphi \circ T_t \, : \, W^{1,1}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}}) \to W^{1,1}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$$

and its inverse are both locally Lipschitzian and

$$\nabla(\varphi \circ T_t) = {}^*DT_t \, \nabla\varphi \circ T_t.$$

(ii) *If* $V \in C^0([0,\tau]; C^1_{\mathrm{loc}}(\mathbf{R}^N, \mathbf{R}^N))$, *then the map*

$$t \mapsto \varphi \circ T_t \;:\; [0,\tau] \to W^{1,1}_{\mathrm{loc}}(\mathbf{R}^N)$$

is well-defined and for each t

$$\frac{d}{dt}\varphi \circ T_t = (\nabla\varphi \cdot V(t)) \circ T_t \in L^1_{\mathrm{loc}}(\mathbf{R}^N). \tag{4.3}$$

Hence the function

$$t \mapsto \varphi \circ T_t \text{ belongs to } C^1([0,\tau]; L^1_{\mathrm{loc}}(\mathbf{R}^N)) \cap C^0([0,\tau]; W^{1,1}_{\mathrm{loc}}(\mathbf{R}^N)).$$

(iii) *If* $V \in C^0([0,\tau]; C^1_{\mathrm{loc}}(\mathbf{R}^N, \mathbf{R}^N))$, *then the map*

$$t \mapsto J_t \;:\; [0,\tau] \to C^0_{\mathrm{loc}}(\mathbf{R}^N)$$

is differentiable and

$$\frac{dJ_t}{dt} = [\,\mathrm{div}\, V(t)] \circ T_t \, J_t \in C^0_{\mathrm{loc}}(\mathbf{R}^N). \tag{4.4}$$

Hence the map $t \mapsto J_t$ *belongs to* $C^1([0,\tau]; C^0_{\mathrm{loc}}(\mathbf{R}^N))$.

Indeed it is easy to check that

$$\begin{aligned}
\frac{d}{dt}DT_t(X) &= DV(t, T_t(X))\, DT_t(X), \quad DT_0(X) = I,\\
\frac{d}{dt}\,\det\, DT_t(X) &= \mathrm{tr}\; DV(t, T_t(X))\,\det\, DT_t(X)\\
\Rightarrow\; \frac{d}{dt}\,\det\, DT_t(X) &= \mathrm{div}\; V(t, T_t(X))\,\det\, DT_t(X), \quad \det\, DT_0(X) = 1,
\end{aligned}$$

and (4.4) follows directly by definition of $J_t(X)$.

From (4.2), (4.3), and (4.4)

$$\begin{aligned}
dJ(\Omega; V) = \frac{d}{dt}\, J(\Omega_t(V))\Big|_{t=0} &= \int_\Omega \nabla\varphi \cdot V(0) + \varphi\, \mathrm{div}\, V(0)\, dx\\
\Rightarrow\; dJ(\Omega; V) &= \int_\Omega \mathrm{div}\,(\varphi\; V(0))\; dx.
\end{aligned}$$

If Ω has a Lipschitzian boundary, then by Stokes's theorem

$$dJ(\Omega; V) = \int_\Gamma \varphi\, V(0) \cdot n\; d\Gamma.$$

Theorem 4.2. *Assume that there exists* $\tau > 0$ *such that the velocity field* $V(t)$ *satisfies conditions* (V) *and* $V \in C^0([0,\tau]; C^1_{\mathrm{loc}}(\mathbf{R}^N, \mathbf{R}^N))$. *Given a function* $\varphi \in$

$C(0,\tau;W^{1,1}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}}))\cap C^1(0,\tau;L^1_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}}))$ and a bounded measurable domain Ω with boundary Γ, the semiderivative of the function

$$\boxed{J_V(t)\overset{\mathrm{def}}{=}\int_{\Omega_t(V)}\varphi(t)\,dx} \tag{4.5}$$

at $t=0$ is given by

$$\boxed{dJ_V(0)=\int_\Omega \varphi'(0)+\operatorname{div}\,(\varphi(0)\,V(0))\;dx,} \tag{4.6}$$

where $\varphi(0)(x)\overset{\mathrm{def}}{=}\varphi(0,x)$ and $\varphi'(0)(x)\overset{\mathrm{def}}{=}\partial\varphi/dt(0,x)$. If, in addition, Ω is an open domain with a Lipschitzian boundary Γ, then

$$\boxed{dJ_V(0)=\int_\Omega \varphi'(0)\,dx+\int_\Gamma \varphi(0)\,V(0)\cdot n\,dx.} \tag{4.7}$$

4.2 Basic Formula for Boundary Integrals

Given ψ in $H^2_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$, consider for some bounded open Lipschitzian domain Ω in $\mathbf{R}^{\mathrm{N}}$ the shape functional

$$J(\Omega)\overset{\mathrm{def}}{=}\int_\Gamma \psi\,d\Gamma. \tag{4.8}$$

This integral is invariant with respect to a homeomorphism which maps Ω onto itself (and hence Γ onto itself). Given the velocity field V and $t\geq 0$, consider the expression

$$J(\Omega_t(V))\overset{\mathrm{def}}{=}\int_{\Gamma_t(V)}\psi\,d\Gamma_t.$$

Using the change of variables $T_t(V)$ and the material introduced in (3.13) of section 3.2 and (5.30) of section 5 in Chapter 2, this integral can be brought back from Γ_t to Γ:

$$J(\Omega_t(V))\overset{\mathrm{def}}{=}\int_{\Gamma_t}\psi\,d\Gamma_t=\int_\Gamma \psi\circ T_t\,\omega_t\,d\Gamma, \tag{4.9}$$

where the density ω_t is given as

$$\omega_t=|M(DT_t)n|, \tag{4.10}$$

n is the outward normal field on Γ, and $M(DT_t)$ is the cofactor matrix of DT_t, that is,

$$M(DT_t)=J_t\;{}^*(DT_t)^{-1}\quad\Rightarrow\boxed{\omega_t=J_t\,|\,{}^*(DT_t)^{-1}n|.} \tag{4.11}$$

It can easily be checked from (4.10) and (4.11) that $t \mapsto \omega_t$ is differentiable in $C^0(\Gamma)$ and that the limit

$$\boxed{\omega' = \lim_{t \searrow 0} \frac{1}{t}(\omega_t - \omega) = \operatorname{div} V(0) - DV(0)n \cdot n} \tag{4.12}$$

in the $C^0(\Gamma)$-norm is linear and continuous with respect to $V(0)$ in the $C^1_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ Fréchet topology. Hence

$$dJ(\Omega; V) = \int_\Gamma \nabla\psi \cdot V(0) + \psi \, (\operatorname{div} V(0) - DV(0)n \cdot n) \, d\Gamma. \tag{4.13}$$

From Corollary 1 to the structure Theorem 3.6, $dJ(\Omega; V)$ depends only on the normal component v_n of the velocity field $V(0)$ on Γ:

$$v_n \stackrel{\text{def}}{=} v \cdot n, \quad v \stackrel{\text{def}}{=} V(0)|_\Gamma \tag{4.14}$$

through Hadamard's formula (3.40). In view of this property any other velocity field with the same smoothness and normal component on Γ will yield the same limit. Given $k > 0$ consider the *tubular neighborhood*

$$\boxed{S_k(\Gamma) \stackrel{\text{def}}{=} \left\{ x \in \mathbf{R}^{\mathrm{N}} \, : \, |b(x)| < k \right\}} \tag{4.15}$$

of Γ in $\mathbf{R}^{\mathrm{N}}$ for the oriented distance function $b = b_\Omega$ associated with Ω. Assuming that Γ is compact and of class C^2, there exists $h > 0$ such that $b \in C^2(S_{2h}(\Gamma))$. Let $\varphi \in \mathcal{D}(\mathbf{R}^{\mathrm{N}})$ be such that $\varphi = 1$ in $S_h(\Gamma)$ and $\varphi = 0$ outside of $S_{2h}(\Gamma)$. Consider the velocity field

$$W(t) \stackrel{\text{def}}{=} (V(0) \cdot \nabla b) \, \nabla b \, \varphi.$$

Clearly, the normal component of $W(0)$ on Γ coincides with v_n. Moreover, in $S_h(\Gamma)$

$$\nabla\psi \cdot W = \nabla\psi \cdot \nabla b \, V(0) \cdot \nabla b \quad \Rightarrow \boxed{\nabla\psi \cdot W|_\Gamma = \nabla\psi \cdot n \, V(0) \cdot n = \frac{\partial\psi}{\partial n} \, v_n,}$$

$$\begin{aligned}
DW &= V(0) \cdot \nabla b \, D^2 b + \nabla b \; {}^*\nabla(V(0) \cdot \nabla b), \\
\operatorname{div} W &= V(0) \cdot \nabla b \, \Delta b + \nabla b \cdot \nabla(V(0) \cdot \nabla b), \\
DW \nabla b \cdot \nabla b &= V(0) \cdot \nabla b \, D^2 b \nabla b \cdot \nabla b + \nabla(V(0) \cdot \nabla b) \cdot \nabla b \\
&= \nabla(V(0) \cdot \nabla b) \cdot \nabla b, \\
\operatorname{div} W - DW \nabla b \cdot \nabla b &= V(0) \cdot \nabla b \, \Delta b \\
\Rightarrow \; &\boxed{\operatorname{div} W - DW \nabla b \cdot \nabla b|_\Gamma = V(0) \cdot n \, H = H \, v_n}
\end{aligned}$$

since $\nabla b|_\Gamma = n$, $D^2 b \nabla b = 0$, and $H = \Delta b$ is the *additive curvature*, that is, the sum of the $N - 1$ curvatures of Γ or $N - 1$ times the mean curvature $\overline{H}$. Finally,

$$dJ(\Omega; V) = \int_\Gamma \left(\frac{\partial\psi}{\partial n} + \psi \, H \right) v_n \, d\Gamma. \tag{4.16}$$

We have proven the following result.

Theorem 4.3. *Let Γ be the boundary of a bounded open subset Ω of $\mathbf{R}^N$ of class C^2 and ψ be an element of $C^1([0,\tau]; H^2_{\mathrm{loc}}(\mathbf{R}^N))$. Assume that $V \in C^0([0,\tau]; C^1_{\mathrm{loc}}(\mathbf{R}^N, \mathbf{R}^N))$. Consider the function*

$$\boxed{J_V(t) \overset{\text{def}}{=} \int_{\Gamma_t(V)} \psi(t)\, d\Gamma_t.}$$

Then the derivative of $J_V(t)$ with respect to t in $t = 0$ is given by the expression

$$\boxed{\begin{aligned} dJ_V(0) &= \int_\Gamma \psi'(0) + \left(\frac{\partial \psi}{\partial n} + H\psi\right) V(0)\cdot n\, d\Gamma \\ &= \int_\Gamma \psi'(0) + \nabla\psi \cdot V(0) + \psi\, (\operatorname{div} V(0) - DV(0)n\cdot n)\, d\Gamma, \end{aligned}} \tag{4.17}$$

where $\psi'(0)(x) \overset{\text{def}}{=} \partial\psi/\partial t(0,x)$.

Note that, as in the case of the volume integral, we have two formulae. Hence we have the following identity:

$$\boxed{\begin{aligned} &\int_\Gamma \left(\frac{\partial \psi}{\partial n} + H\psi\right) V(0)\cdot n\, d\Gamma \\ &\quad = \int_\Gamma \nabla\psi \cdot V(0) + \psi\, (\operatorname{div} V(0) - DV(0)n\cdot n)\, d\Gamma. \end{aligned}} \tag{4.18}$$

4.3 Examples of Shape Derivatives

4.3.1 Volume of Ω and Surface Area of Γ

Consider the volume shape functional

$$J(\Omega) = \int_\Omega dx.$$

This shape functional is used as a constraint on the domain in several examples of shape optimization problems. We get

$$dJ(\Omega; V) = \int_\Omega \operatorname{div} V(0)\, dx \tag{4.19}$$

and, if Γ is Lipschitzian,

$$dJ(\Omega; V) = \int_\Gamma V(0)\cdot n\, d\Gamma. \tag{4.20}$$

A sufficient condition on the field $V(0)$ to preserve the volume is $\operatorname{div} V(0) = 0$ in Ω and, if Γ is Lipschitzian, $V(0)\cdot n = 0$ on Γ.

Consider the (shape) area function

$$J(\Omega) = \int_\Gamma d\Gamma.$$

Assuming that Γ is of class C^2 we get from (4.17)

$$dJ(\Omega; V) = \int_\Gamma H\, V(0) \cdot n\, d\Gamma, \tag{4.21}$$

where $H = \Delta b_\Omega$ is the additive curvature. The condition to keep the surface of Γ constant is that $V(0) \cdot n$ be orthogonal (in $L^2(\Gamma)$) to H.

4.3.2 $H^1(\Omega)$-Norm

Given ϕ and ψ in $H^2_{\text{loc}}(\mathbf{R}^{\text{N}})$, consider the shape functional

$$J(\Omega) = \int_\Omega \nabla\phi \cdot \nabla\psi\, dx.$$

By using the change of variables $T_t(V)$, $\Omega_t = \Omega_t(V) = T_t(V)(\Omega)$ and

$$\int_{\Omega_t} \nabla\phi \cdot \nabla\psi\, dx = \int_\Omega [A(V)(t)\nabla(\phi \circ T_t)] \cdot \nabla(\psi \circ T_t)\, dx, \tag{4.22}$$

where $A(V)$ is the following matrix associated with the field V:

$$A(V)(t) = J(t)\, (DT_t)^{-1}\ {}^*(DT_t)^{-1} \tag{4.23}$$

and $J(t) = \det(DT_t)$. Expression (4.23) is easily obtained from the identity

$$(\nabla\phi) \circ T_t = {}^*(DT_t)^{-1}\nabla(\phi \circ T_t). \tag{4.24}$$

If $V \in C^0([0,\tau]; C^k_{\text{loc}}(\mathbf{R}^{\text{N}}; \mathbf{R}^{\text{N}}))$, $k \geq 1$, T and T^{-1} belong to $C^1([0,\tau]; C^k_{\text{loc}}(\mathbf{R}^{\text{N}}, \mathbf{R}^{\text{N}}))$ and $A(V)$ to $C^1([0,\tau], C^{k-1}_{\text{loc}}(\mathbf{R}^{\text{N}}; \mathbf{R}^{N^2}))$. Then if ϕ, ψ belongs to $H^2_{\text{loc}}(\mathbf{R}^{\text{N}})$, we get $t \mapsto \phi \circ T_t$, which is differentiable in $H^1_{\text{loc}}(\mathbf{R}^{\text{N}})$, with $\partial\phi/\partial t \circ T_t|_{t=0} = \nabla\phi \cdot V(0)$, which belongs to $H^1_{\text{loc}}(\mathbf{R}^{\text{N}})$. Finally, we obtain

$$\begin{aligned} dJ(\Omega; V) = &\int_\Omega [A'(V)\nabla\phi] \cdot \nabla\psi\, dx \\ &+ \int_\Omega \{\nabla(\nabla\phi \cdot V(0)) \cdot \nabla\psi + \nabla\phi \cdot \nabla(\nabla\psi \cdot V(0))\}\, dx, \end{aligned} \tag{4.25}$$

where $A'(V)$ is the derivative in the $C^{k-1}_{\text{loc}}(\mathbf{R}^{\text{N}}, \mathbf{R}^{N^2})$-norm

$$A'(V) \stackrel{\text{def}}{=} \frac{\partial}{\partial t} A(V)(t)|_{t=0} = \operatorname{div} V(0)\, I - 2\varepsilon(V(0)) \tag{4.26}$$

and $\varepsilon(V(0))$ is the symmetrized Jacobian matrix (the strain tensor associated with the field $V(0)$ in elasticity)

$$\varepsilon(V(0)) = \frac{1}{2}[{}^*DV(0) + DV(0)]. \tag{4.27}$$

We finally obtain the *volume expression*

$$\boxed{\begin{aligned} dJ(\Omega; V) = \int_\Omega &[\operatorname{div} V(0)\, I - 2\varepsilon(V(0))]\, \nabla\phi \cdot \nabla\psi \\ &+ [\nabla(\nabla\phi \cdot V(0)) \cdot \nabla\psi + \nabla\phi \cdot \nabla(\nabla\psi \cdot V(0))]\, dx. \end{aligned}} \tag{4.28}$$

When Γ is a C^1-submanifold, $\phi, \psi \in H^2_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$, we obtain the simpler *boundary expression* by directly using formula (4.17):

$$\boxed{dJ(\Omega; V) = \int_\Gamma \nabla\phi \cdot \nabla\psi \, V(0) \cdot n \, d\Gamma.} \tag{4.29}$$

This simple example nicely illustrates the notion of density gradient g. From expression (4.28) it was obvious that the mapping $V \mapsto dJ(\Omega; V)$ was well-defined, linear, and continuous on $C^1([0,\tau]; C^1_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}}; \mathbf{R}^{\mathrm{N}}))$. By the structure theorem and Hadamard's formula we knew that dJ could be written in the form

$$\int_\Gamma g \, V(0) \cdot n \, d\Gamma.$$

But from (4.29) we know that g, which is an element of $\mathcal{D}^1(\Gamma)'$, is an element of $W^{1/2,1}(\Gamma)$ given by

$$g = \nabla\phi \cdot \nabla\psi \text{ (traces on } \Gamma). \tag{4.30}$$

The direct calculation of g from expression (4.24) would have been very fastidious.

4.3.3 Normal Derivative

Let Γ be of class C^2 and $\phi \in H^2_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$ be given. Consider the following shape function:

$$J(\Omega) \stackrel{\text{def}}{=} \int_\Gamma \left|\frac{\partial\phi}{\partial n}\right|^2 d\Gamma = \int_\Gamma |\nabla\phi \cdot n|^2 \; d\Gamma.$$

By the change of variables formula we get, with $\Omega_t = T_t(V)(\Omega)$ and $\Gamma_t = T_t(V)(\Gamma)$,

$$J(\Omega_t) \stackrel{\text{def}}{=} \int_{\Gamma_t} |\nabla\phi \cdot n_t|^2 \, d\Gamma_t = \int_\Gamma \left\{ {}^*(DT_t)^{-1} \nabla(\phi \circ T_t) \cdot (n_t \circ T_t) \right\}^2 \omega_t \, d\Gamma, \tag{4.31}$$

where $n_t \circ T_t$ is the transported normal field n_t from Γ_t onto Γ. The derivative can be obtained by using formula (4.17) of Theorem 4.3 and one of the above two expressions. However, the first expression first requires the construction of an extension N_t of the normal n_t in a neighborhood of Γ. In both cases the following result will be useful.

Theorem 4.4. *Let $k \geq 1$ be an integer. Given a velocity field $V(t)$ satisfying condition (V) such that $V \in C([0,\tau]; C^k_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}))$, then*

$$\boxed{n_t \circ T_t = \frac{{}^*(DT_t)^{-1} n}{|\,{}^*(DT_t)^{-1} n|} = \frac{M(DT_t) n}{|M(DT_t) n|},} \tag{4.32}$$

where n and n_t are the respective outward normals to Ω and Ω_t on Γ and Γ_t and $M(DT_t)$ is the cofactor's matrix of DT_t.

Proof. Go back to Definition 3.1 in section 3.1 of Chapter 2. We have shown in that section that for a domain Ω of class C^k the *unit outward normal* at any point $y \in \Gamma_x = \Gamma \cap U(x)$ is given by expression (3.9)

$$\forall y \in \Gamma_x, \quad n(y) = -\frac{{}^*(Dh_x)^{-1}\, e_N}{|\,{}^*(Dh_x)^{-1}\, e_N|}(h_x^{-1}(y)),$$

where h_x is the local diffeomorphism specified by (3.3):

$$g_x \in C^k\big(U(x), B\big), \quad h_x = g_x^{-1} \in C^k\big(B, U(x)\big).$$

For $\Omega_t = \Omega_t(V)$ and $x_t \overset{\text{def}}{=} T_t(x)$, choose the following new neighborhood and local diffeomorphism:

$$U_t \overset{\text{def}}{=} T_t(U(x)), \quad h_t \overset{\text{def}}{=} T_t \circ h_x : B \to U_t, \quad g_t = h_t^{-1} \overset{\text{def}}{=} g_x \circ T_t^{-1} : U_t \to B.$$

On $\Gamma_t(V) \cap U_t$ the normal in given by the same expression but with h_t in place of h_x:

$$n_t = -\frac{{}^*(Dh_t)^{-1} \circ h_t^{-1}\, e_N}{|\,{}^*(Dh_t)^{-1} \circ h_t^{-1}\, e_N|}.$$

However,

$$\begin{gathered} D(T_t \circ h_x) = DT_t \circ h_x\, Dh_x, \\ D(T_t \circ h_x) \circ (T_t \circ h_x)^{-1} = \big[DT_t\, Dh_x \circ h_x^{-1}\big] \circ T_t^{-1}, \\ {}^*(DT_t \circ h_x)^{-1} \circ (T_t \circ h_x)^{-1} e_N = \big[\,{}^*(DT_t)^{-1}\ {}^*(Dh_x)^{-1} \circ h_x^{-1} e_N\big] \circ T_t^{-1}. \end{gathered}$$

Therefore, by using the previous expressions for n and n_t,

$$n_t = \frac{{}^*(DT_t)^{-1} n}{|\,{}^*(DT_t)^{-1} n|} \circ T_t^{-1} \quad \Rightarrow n_t \circ T_t = \frac{{}^*(DT_t)^{-1} n}{|\,{}^*(DT_t)^{-1} n|}. \tag{4.33}$$

The other expression for $n_t \circ T_t$ in terms of the cofactor matrix $M(DT_t)$ of DT_t readily follows from the identity $M(DT_t) = J(t)\, {}^*(DT_t)^{-1}$. □

Recalling the expression for ω_t

$$\omega_t = |M(DT_t) n| = J(t)\, |\,{}^*(DT_t)^{-1} n|,$$

our boundary integral becomes

$$\int_{\Gamma_t} |\nabla \phi \cdot n_t|^2 \, d\Gamma_t = \int_\Gamma |[A(t) \nabla (\phi \circ T_t)] \cdot n\,|^2\, \omega_t^{-1}\, d\Gamma. \tag{4.34}$$

Using expression (4.26) of $A'(V)$ and expression (4.12) of ω' we get

$$
\begin{aligned}
&dJ(\Omega;V)\\
&= 2\int_\Gamma \frac{\partial\phi}{\partial n}\,\left[A'(V)\nabla\phi\cdot n + \nabla(\nabla\phi\cdot V(0))\cdot n\right] - \left|\frac{\partial\phi}{\partial n}\right|^2 \omega'\,d\Gamma\\
&= 2\int_\Gamma \frac{\partial\phi}{\partial n}\,\left\{\left[\operatorname{div} V(0)\,I - DV(0) - {}^*DV(0)\right]\,\nabla\phi\cdot n + \nabla(\nabla\phi\cdot V(0))\cdot n\right\}\\
&\qquad\qquad - \left|\frac{\partial\phi}{\partial n}\right|^2 (\operatorname{div} V(0) - DV(0)n\cdot n)\;d\Gamma\\
&= \int_\Gamma 2\frac{\partial\phi}{\partial n}\,\left\{\operatorname{div} V(0)\,\frac{\partial\phi}{\partial n} - DV(0)\,\nabla\phi\cdot n + D^2\phi\, V(0)\cdot n\right\}\\
&\qquad - \left|\frac{\partial\phi}{\partial n}\right|^2 (\operatorname{div} V(0) - DV(0)n\cdot n)\;d\Gamma\\
&= \int_\Gamma \left|\frac{\partial\phi}{\partial n}\right|^2 (\operatorname{div} V(0) - DV(0)n\cdot n)\\
&\qquad + 2\frac{\partial\phi}{\partial n}\,\left\{DV(0)n\cdot n\,\frac{\partial\phi}{\partial n} - DV(0)\,\nabla\phi\cdot n + D^2\phi\, V(0)\cdot n\right\}\,d\Gamma.
\end{aligned}
$$

This formula can be somewhat simplified by using identity (4.18) with

$$
\begin{aligned}
\psi = |\nabla\phi\cdot\nabla b|^2 \quad &\Rightarrow\; \psi|_\Gamma = \left|\frac{\partial\phi}{\partial n}\right|^2,\\
\nabla\psi = 2\,\nabla\phi\cdot\nabla b\;\nabla(\nabla\phi\cdot\nabla b) &= 2\,\nabla\phi\cdot\nabla b\,\left[D^2\phi\nabla b + D^2 b\nabla\psi\right]\\
\Rightarrow\;\nabla\psi|_\Gamma &= 2\frac{\partial\phi}{\partial n}\left[D^2\phi\, n + D^2 b\,\nabla\phi\right]\\
\Rightarrow\;\nabla\psi\cdot V(0)|_\Gamma &= 2\frac{\partial\phi}{\partial n}\left[D^2\phi\, n + D^2 b\,\nabla\phi\right]\cdot V(0)\\
\Rightarrow\;\frac{\partial\psi}{\partial n} = \nabla\psi\cdot\nabla b|_\Gamma &= 2\frac{\partial\phi}{\partial n}\left[D^2\phi\, n + D^2 b\,\nabla\phi\right]\cdot\nabla b = 2\frac{\partial\phi}{\partial n}\,D^2\phi\, n\cdot n.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
&\int_\Gamma \left(2\frac{\partial\phi}{\partial n}\,D^2\phi\, n\cdot n + H\left|\frac{\partial\phi}{\partial n}\right|^2\right) V(0)\cdot n\,d\Gamma\\
&= \int_\Gamma 2\frac{\partial\phi}{\partial n}\left[D^2\phi\, n + D^2 b\,\nabla\phi\right]\cdot V(0) + \left|\frac{\partial\phi}{\partial n}\right|^2 (\operatorname{div} V(0) - DV(0)n\cdot n)\,d\Gamma,
\end{aligned}
\tag{4.35}
$$

and hence

$$
\boxed{
\begin{aligned}
&dJ(\Omega;V)\\
&= \int_\Gamma \left(2\frac{\partial\phi}{\partial n}\,D^2\phi n\cdot n + H\left|\frac{\partial\phi}{\partial n}\right|^2\right) V(0)\cdot n\\
&\qquad + 2\frac{\partial\phi}{\partial n}\left\{\frac{\partial\phi}{\partial n}\,DV(0)n\cdot n - \nabla\phi\cdot\left({}^*DV(0)\,n + D^2 b\,V(0)\right)\right\}\,d\Gamma.
\end{aligned}
}
\tag{4.36}
$$

This formula can be more readily obtained from the first expression (4.31) and the extension

$$\boxed{N_t = \frac{{}^*(DT_t)^{-1}\nabla b}{|\,{}^*(DT_t)^{-1}\nabla b|} \circ T_t^{-1}} \tag{4.37}$$

of the normal n_t. To compute N', decompose N_t as follows:

$$\begin{aligned}
N_t &= f(t) \circ T_t^{-1}, \quad f(t) = \frac{g(t)}{\sqrt{g(t)\cdot g(t)}}, \quad g(t) = {}^*(DT_t)^{-1}\nabla b,\\
N' &= f' - Df(0)V(0), \quad g' = -\,{}^*DV(0)\nabla b, \quad f(0) = \nabla b,\\
f' &= \frac{g'\,|g(0)| - g(0)\cdot g'\,g(0)/|g(0)|}{|g(0)|^2}\\
&= g' - g'\cdot\nabla b\,\nabla b = DV(0)\nabla b\cdot\nabla b\,\nabla b - {}^*DV(0)\nabla b.
\end{aligned}$$

So finally

$$\boxed{N'|_\Gamma = (DV(0)n\cdot n)\,n - {}^*DV(0)n - D^2b\,V(0)} \tag{4.38}$$

and

$$\begin{aligned}
\frac{\partial}{dt}|\nabla\phi\cdot N_t|^2\Big|_{t=0} &= 2\frac{\partial\phi}{\partial n}\,\nabla\phi\cdot N'\\
&= 2\frac{\partial\phi}{\partial n}\,\nabla\phi\cdot\left\{(DV(0)n\cdot n)n - {}^*DV(0)\,n - D^2b\,V(0)\right\}.
\end{aligned}$$

The first part of the integral (4.36) depends explicitly on the normal component of $V(0)$. Yet we know from the structure theorem that for this function, the shape derivative depends only on the normal component of $V(0)$. To make this explicit, it is necessary to introduce some elements of tangential calculus.

5 Elements of Tangential Calculus

In this section some basic elements of the differential calculus on a C^2-submanifold of codimension 1 are introduced. The approach avoids local bases and coordinates by using the intrinsic tangential derivatives.

In the classical theory of shells (see, for instance, Ph. D. Ciarlet [**1**]), the midsurface ω is defined as the image of a flat smooth bounded connected domain U in $\mathbf{R}^2$ via a C^2-immersion $\varphi : U \to \mathbf{R}^3$. When U is sufficiently smooth and the thickness sufficiently small, the associated tubular neighborhood $S_h(\omega)$ of thickness $2h$ is a Lipschitzian domain that is identified with a thin shell of thickness $2h$ around ω. Local bases are generated by the derivatives of the immersion φ and tangential derivatives can be defined in the surface ω.

Such hypersurfaces ω in $\mathbf{R}^{\mathrm{N}}$ can be viewed as a subset of the boundary Γ of an open subset Ω of $\mathbf{R}^{\mathrm{N}}$. In that case, the gradient and Hessian matrix of the associated *oriented distance function* b_Ω to the underlying set Ω completely describes the normal and the N fundamental forms of ω, and a fairly complete intrinsic theory of Sobolev spaces on $C^{1,1}$-hypersurfaces is available in M. C. Delfour [**7**]. M. C. Delfour [**8**] showed that all smooth hypersurfaces and the hypersurfaces with little regularity of S. Anicic, H. Le Dret, and A. Raoult [**1**] are $C^{1,1}$ hypersurfaces that can be described by the oriented distance function b_Ω or the signed distance function b_ω. Such an approach has been used successfully in the theory of thin and asymptotic shells with a $C^{1,1}$-midsurface.

Let Ω be an open domain of class C^2 in $\mathbf{R}^{\mathrm{N}}$ with compact boundary Γ. Therefore, there exists $h > 0$ such that $b = b_\Omega \in C^2(S_{2h}(\Gamma))$. The *projection* of a point x onto Γ is given by

$$p(x) \stackrel{\text{def}}{=} x - b(x)\, \nabla b(x),$$

and the *orthogonal projection operator* of a vector onto the tangent plane $T_{p(x)}\Gamma$ is given by

$$P(x) \stackrel{\text{def}}{=} I - \nabla b(x)\, {}^*\nabla b(x).$$

Notice that, as a transformation of $T_{p(x)}\Gamma$,

$$P(x) \,:\, T_{p(x)}\Gamma \to T_{p(x)}\Gamma, \quad P(x) = I - \nabla b(x)\, {}^*\nabla b(x)$$

is the identity transformation on $T_{p(x)}\Gamma$. In fact $P(x)$ coincides with the *first fundamental form.* Similarly $D^2 b$ can be considered as a transformation of $T_{p(x)}$ since $D^2b(x)n(x) = D^2b(x)\nabla b(x) = 0$. We shall show in section 5.6 that

$$D^2b(x) \,:\, T_{p(x)} \to T_{p(x)}$$

coincides with the *second fundamental form* of Γ. Similarly $D^2b(x)^2$ coincides with the *third fundamental form.* Finally

$$Dp(x) = I - \nabla b(x)\, {}^*\nabla b(x) - b\, D^2b(x) \quad \text{and} \quad Dp|_\Gamma = P.$$

5.1 Intrinsic Definition of the Tangential Gradient

The classical way to define the tangential gradient of a scalar function $f \,:\, \Gamma \to \mathbf{R}$ is through an appropriately smooth extension F of f in a neighborhood of Γ using the fact that the resulting expression on Γ is independent of the choice of the extension F. In this section an equivalent direct intrinsic definition is given in terms of the extension $f \circ p$ of the function f. This is the basis of a simple differential calculus on Γ which uses the Euclidean differential calculus in the ambient neighborhood of Γ.

Given $f \in C^1(\Gamma)$, let $F \in C^1(S_{2h}(\Gamma))$ be a C^1-extension of f. Define

$$g(F) \stackrel{\text{def}}{=} \nabla F|_\Gamma - \frac{\partial F}{\partial n}\, n \quad \text{on } \Gamma.$$

This is the orthogonal projection $P(x)\nabla F(x)$ of $\nabla F(x)$ onto the tangent plane $T_x\Gamma$ to Γ at x. To get something intrinsic, $g(F)$ must be independent of the choice of F. It is sufficient to show that $g(F) = 0$ for $f = 0$. But $F = f = 0$ on Γ and the tangential component of ∇F is 0 on Γ, and

$$\nabla F|_\Gamma = \frac{\partial F}{\partial n}\, n \quad \Rightarrow g(F) = \nabla F|_\Gamma - \frac{\partial F}{\partial n} n = 0.$$

Definition 5.1 (General extension).
Assume that Γ is compact and that there exists $h > 0$ such that $b_\Omega \in C^2(S_{2h}(\Gamma))$. Given an extension $F \in C^1(S_{2h}(\Gamma))$ of $f \in C^1(\Gamma)$, the *tangential gradient* of f in a point of Γ is defined as

$$\nabla_\Gamma f \overset{\text{def}}{=} \nabla F|_\Gamma - \frac{\partial F}{\partial n}\, n. \qquad \square$$

The notation is quite natural. The subscript Γ of $\nabla_\Gamma f$ indicates that the gradient is with respect to the variable x in the submanifold Γ.

Theorem 5.1. *Assume that Γ is compact and that there exists $h > 0$ such that $b_\Omega \in C^2(S_{2h}(\Gamma))$ and that $f \in C^1(\Gamma)$. Then*

(i) $\nabla_\Gamma f = (P\,\nabla F)|_\Gamma$ *and* $n \cdot \nabla_\Gamma f = \nabla b \cdot \nabla_\Gamma f = 0$*;*

(ii) $\nabla(f \circ p) = [I - b\, D^2 b]\, \nabla_\Gamma f \circ p$ *and* $\nabla(f \circ p)|_\Gamma = \nabla_\Gamma f$*.*

Proof. (i) By definition

$$P\,\nabla F = (I - \nabla b\, {}^*\nabla b)\, \nabla F = \nabla F - \nabla F \cdot \nabla b\, \nabla b,$$
$$(P\,\nabla F)|_\Gamma = (\nabla F)|_\Gamma - \frac{\partial F}{\partial n}\, n = \nabla_\Gamma f.$$

Moreover

$$\nabla b \cdot P\,\nabla F = (I - \nabla b\, {}^*\nabla b)\, \nabla b \cdot \nabla F = 0$$
$$\Rightarrow\ (\nabla b)|_\Gamma \cdot (P\,\nabla F)|_\Gamma = 0 \quad \Rightarrow\ n \cdot \nabla_\Gamma f = 0.$$

(ii) $F = f \circ p$ is a C^1-extension of f and

$$\begin{aligned}\nabla(f \circ p) = \nabla(f \circ p \circ p) &= Dp\, \nabla(f \circ p) \circ p \\ &= [I - \nabla b\, {}^*\nabla b - b\, D^2 b]\, \nabla(f \circ p)|_\Gamma \circ p.\end{aligned}$$

But by definition of $\nabla_\Gamma f$

$$\begin{aligned}&\nabla(f \circ p)|_\Gamma = \nabla_\Gamma f + \frac{\partial (f \circ p)}{\partial n}\, n \\ &\Rightarrow\ \nabla(f \circ p) = [I - \nabla b\, {}^*\nabla b - b\, D^2 b] \left(\nabla_\Gamma f + \frac{\partial (f \circ p)}{\partial n}\, n \right) \circ p.\end{aligned}$$

Recall, from Theorem 8.4 (i) in Chapter 7, that $n \circ p = \nabla b$ so that

$$\begin{aligned}\nabla(f \circ p) &= [I - \nabla b\, {}^*\nabla b - b\, D^2 b] \left(\nabla_\Gamma f \circ p + \frac{\partial (f \circ p)}{\partial n} \nabla b \right) \\ &= [I - \nabla b\, {}^*\nabla b - b\, D^2 b]\, \nabla_\Gamma f \circ p.\end{aligned}$$

Again

$$\nabla b \cdot \nabla_\Gamma f \circ p = n \circ p \cdot \nabla_\Gamma f \circ p = (n \cdot \nabla_\Gamma f) \circ p = 0$$

from part (i) and

$$\nabla(f \circ p) = [I - b D^2 b]\, \nabla_\Gamma f \circ p \quad \Rightarrow\ \nabla(f \circ p)|_\Gamma = \nabla_\Gamma f. \qquad \square$$

In view of part (ii) of the theorem $f \circ p$ plays the role of a *canonical extension* of a map $f : \Gamma \to \mathbf{R}$ to a neighborhood $S_{2h}(\Gamma)$ of Γ and its gradient is tangent to the level sets of b. This suggests the use of the following definition of tangential gradient, which will be the clue to the tangential differential calculus.

Definition 5.2 (Canonical extension)**.**
Under the assumptions of Definition 5.1 on b_Ω, associate with $f \in C^1(\Gamma)$

$$\nabla_\Gamma f \stackrel{\text{def}}{=} \nabla(f \circ p)|_\Gamma . \tag{5.1}$$

□

Remark 5.1.
This definition naturally extends to nonempty sets A such that d_A^2 belongs to $C^{1,1}(S_{2h}(A))$, since the projection onto A,

$$p_A(x) = x - \frac{1}{2}\nabla d_A^2(x),$$

is $C^{0,1}$. They are the sets of positive reach introduced by Federer. They include convex sets and submanifolds of codimension larger than or equal to 1. □

Theorem 5.2. *Under the assumption of Definition* 5.1 *on* b_Ω *for* $f \in C^1(\Gamma)$ *the following hold:*

(i) $\nabla b \cdot \nabla(f \circ p) = 0$ *in* $S_h(\Gamma)$ *and* $n \cdot \nabla_\Gamma f = 0$ *on* Γ.

(ii) $\nabla F|_\Gamma - \frac{\partial F}{\partial n} n = (P\,\nabla F)|_\Gamma = \nabla_\Gamma f$ *and* $\nabla(f \circ p) = [I - b\, D^2 b]\, \nabla_\Gamma f \circ p$ *in* Γ.

Proof. (i) Consider

$$\begin{aligned}\nabla(f \circ p) \cdot \nabla b &= \nabla(f \circ p \circ p) \cdot \nabla b = Dp\, \nabla(f \circ p) \circ p \cdot \nabla b \\ &= \nabla(f \circ p) \circ p \cdot Dp\, \nabla b.\end{aligned}$$

But

$$Dp\, \nabla b = [I - \nabla b\, {}^*\nabla b - b\, D^2 b]\, \nabla b = 0$$

and

$$\nabla(f \circ p) \cdot \nabla b = 0, \quad \nabla_\Gamma f \cdot n = \nabla(f \circ p)|_\Gamma \cdot \nabla b|_\Gamma = 0.$$

(ii) By definition

$$P\,\nabla F = (I - \nabla b\ {}^*\nabla b)\, \nabla F.$$

However, since $F \circ p = f \circ p$ on Γ

$$\nabla(F \circ p) = \nabla(F \circ p \circ p) = Dp\, \nabla(F \circ p) \circ p = Dp\, \nabla(f \circ p) \circ p = Dp\, \nabla_\Gamma f \circ p$$

and

$$\nabla(F \circ p) = Dp\, \nabla F \circ p \quad \Rightarrow \boxed{Dp\, \nabla F \circ p = Dp\, \nabla_\Gamma f \circ p.}$$

By restricting to Γ

$$Dp|_\Gamma\, \nabla F|_\Gamma = Dp|_\Gamma\, \nabla_\Gamma f \quad \Rightarrow P\, \nabla F|_\Gamma = P\, \nabla_\Gamma f = \nabla_\Gamma f$$

□

5.2 First-Order Derivatives

The *tangential Jacobian matrix* of a vector function $v \in C^1(\Gamma)^M$, $M \geq 1$, is defined in the same way as the gradient:

$$\boxed{D_\Gamma v \stackrel{\text{def}}{=} D(v \circ p)|_\Gamma \text{ or } (D_\Gamma v)_{ij} = (\nabla_\Gamma v_i)_j.} \tag{5.2}$$

If ${}^*v = (v_1, \ldots, v_M)$, then

$${}^*D_\Gamma v = (\nabla_\Gamma v_1, \ldots, \nabla_\Gamma v_M),$$

where $\nabla_\Gamma v_i$ is a column vector. From the previous theorems about the tangential gradient we can recover the definition from an extension $V \in C^1(S_{2h}(\Gamma))^M$ of v:

$$\begin{aligned} {}^*D_\Gamma v &= (P\,\nabla V_1, \ldots, P\,\nabla V_M)|_\Gamma = (I - \nabla b\,{}^*\nabla b)\,{}^*DV|_\Gamma \\ &= {}^*DV|_\Gamma - \nabla b\,{}^*(DV\,\nabla b)|_\Gamma \end{aligned}$$

and

$$\boxed{D_\Gamma v = DV|_\Gamma - DV n\,{}^*n = (DV\,P)|_\Gamma.} \tag{5.3}$$

Also,

$${}^*D(v \circ p) = [I - b\,D^2 b]\,(\nabla_\Gamma v_1, \ldots, \nabla_\Gamma v_M) \circ p = [I - b\,D^2 b]\,{}^*D_\Gamma v \circ p,$$

and we have for the extension

$$\boxed{D(v \circ p) = D_\Gamma v \circ p\,[I - b\,D^2 b].} \tag{5.4}$$

Note that

$$\boxed{D(v \circ p)\,\nabla b = 0 \text{ and } D_\Gamma v\,n = 0.} \tag{5.5}$$

For a vector function $v \in C^1(\Gamma)^N$ define the *tangential divergence* as

$$\boxed{\operatorname{div}_\Gamma v \stackrel{\text{def}}{=} \operatorname{div}(v \circ p)|_\Gamma,} \tag{5.6}$$

and it is easy to show that

$$\operatorname{div}_\Gamma v = \operatorname{div}(v \circ p)|_\Gamma = \operatorname{tr} D(v \circ p)|_\Gamma = \operatorname{tr} D_\Gamma v,$$

$$\boxed{\operatorname{div}_\Gamma v = \operatorname{tr}[DV|_\Gamma - DV\,n\,{}^*n] = \operatorname{div} V|_\Gamma - DV\,n \cdot n.}$$

The *tangential linear strain tensor* of linear elasticity is given by

$$\boxed{\varepsilon_\Gamma(v) \stackrel{\text{def}}{=} \frac{1}{2}(D_\Gamma v + {}^*D_\Gamma v),} \tag{5.7}$$

$$\begin{aligned} \varepsilon(v \circ p) &= \frac{1}{2}(D(v \circ p) + {}^*D(v \circ p)) \\ &= \varepsilon_\Gamma(v) \circ p - \frac{b}{2}\,[D_\Gamma v \circ p\,D^2 b + D^2 b\,{}^*D_\Gamma v \circ p] \end{aligned} \tag{5.8}$$

$$\Rightarrow \boxed{\varepsilon_\Gamma(v) = \varepsilon(v \circ p)_{|\Gamma}.} \tag{5.9}$$

The tangential Jacobian matrix of the normal n is especially interesting since, from Theorem 8.4 (i) in Chapter 7, $n \circ p = \nabla b = \nabla b \circ p$. As a result

$$\boxed{D_\Gamma(n) = D^2 b|_\Gamma = {}^*D_\Gamma(n)} \quad \Rightarrow \boxed{\varepsilon_\Gamma(n) = D^2 b|_\Gamma.} \tag{5.10}$$

The *tangential vectorial divergence* of a matrix or tensor function A is defined as

$$(\tilde{\operatorname{div}}_\Gamma A)_i \stackrel{\text{def}}{=} \operatorname{div}_\Gamma (A_{i,\cdot}). \tag{5.11}$$

5.3 Second-Order Derivatives

Assume that Ω is of class C^3. The simplest second-order derivative is the *Laplace–Beltrami operator* of a function $f \in C^2(\Gamma)$, which is defined as

$$\boxed{\Delta_\Gamma f \stackrel{\text{def}}{=} \operatorname{div}_\Gamma(\nabla_\Gamma f).} \tag{5.12}$$

Recall from Theorem 5.2 the identity

$$\nabla(f \circ p) = [I - b\, D^2 b]\, \nabla_\Gamma f \circ p.$$

Then

$$\begin{aligned}
\operatorname{div}(\nabla_\Gamma f \circ p) &= \operatorname{div}(\nabla(f \circ p)) + \operatorname{div}(b\, D^2 b\, \nabla_\Gamma f \circ p),\\
\operatorname{div}(\nabla_\Gamma f \circ p) &= \Delta(f \circ p) + b \operatorname{div}(D^2 b\, \nabla_\Gamma f \circ p) + (D^2 b\, \nabla_\Gamma f \circ p) \cdot \nabla b,\\
\operatorname{div}(\nabla_\Gamma f \circ p) &= \Delta(f \circ p) + b \operatorname{div}(D^2 b\, \nabla_\Gamma f \circ p),
\end{aligned}$$

and by taking restrictions to Γ,

$$\boxed{\Delta_\Gamma f = \Delta(f \circ p)|_\Gamma.}$$

The *tangential Hessian matrix* of second-order derivatives is defined as

$$\boxed{D^2_\Gamma f \stackrel{\text{def}}{=} D_\Gamma(\nabla_\Gamma f).} \tag{5.13}$$

Here the curvatures of the submanifold begin to appear. The Hessian matrix is not symmetrical and does not coincide with the restriction of the Hessian matrix of the canonical extension. Specifically,

$$\begin{aligned}
D^2(f \circ p) &= D(\nabla(f \circ p)) = D([I - b\, D^2 b]\, \nabla_\Gamma f \circ p)\\
&= D(\nabla_\Gamma f \circ p) - D(b\, D^2 b\, \nabla_\Gamma f \circ p)\\
&= D(\nabla_\Gamma f \circ p) - b\, D(D^2 b\, \nabla_\Gamma f \circ p) - D^2 b\, \nabla_\Gamma f \circ p\, {}^*\nabla b,\\
D^2(f \circ p)|_\Gamma &= D_\Gamma(\nabla_\Gamma f) - D^2 b\, \nabla_\Gamma f\, {}^*\nabla b = D^2_\Gamma f - D^2 b\, \nabla_\Gamma f\, {}^*n.
\end{aligned}$$

Of course, since $D^2(f\circ p)$ is symmetrical we also have

$$\begin{aligned}D^2(f\circ p) &= {}^*D(\nabla(f\circ p)) = {}^*D([I-b\,D^2b]\,\nabla_\Gamma f\circ p)\\ &= {}^*D(\nabla_\Gamma f\circ p) - {}^*D(b\,D^2b\,\nabla_\Gamma f\circ p)\\ &= {}^*D(\nabla_\Gamma f\circ p) - b\,{}^*D(D^2b\,\nabla_\Gamma f\circ p) - \nabla b\,{}^*(D^2b\,\nabla_\Gamma f\circ p),\\ D^2(f\circ p)|_\Gamma &= {}^*D_\Gamma(\nabla_\Gamma f) - \nabla b\,{}^*(D^2b\,\nabla_\Gamma f) = {}^*D^2_\Gamma f - n\,{}^*(D^2b\,\nabla_\Gamma f).\end{aligned}$$

As a final result we have the following identity:

$$\boxed{D^2_\Gamma f - (D^2b\,\nabla_\Gamma f)\,{}^*n = D^2(f\circ p)|_\Gamma = {}^*D^2_\Gamma f - n\,{}^*(D^2b\,\nabla_\Gamma f),} \tag{5.14}$$

since by definition ${}^*D^2_\Gamma f = {}^*\,(D^2_\Gamma f)$. So the Hessian and its transpose differ by terms that contain a first-order derivative as is well known in differential geometry. Note that $D^2b\,\nabla_\Gamma f = -{}^*D_\Gamma(\nabla_\Gamma f)n = -{}^*D^2_\Gamma f\,n$ and that we also can write

$$\boxed{\begin{gathered}D^2_\Gamma f + {}^*D^2_\Gamma f\,[n\,{}^*n] = D^2(f\circ p)|_\Gamma = {}^*D^2_\Gamma f + [n\,{}^*n]\,D^2_\Gamma f\\ \Rightarrow\ PD^2_\Gamma f = {}^*(PD^2_\Gamma f).\end{gathered}} \tag{5.15}$$

5.4 A Few Useful Formulae and the Chain Rule

Associate with $F\in C^1(S_{2h}(\Gamma))$ and $V\in C^1(S_{2h}(\Gamma))^N$

$$f \overset{\text{def}}{=} F|_\Gamma,\quad v \overset{\text{def}}{=} V|_\Gamma,\quad v_n \overset{\text{def}}{=} v\cdot n,\quad v_\Gamma \overset{\text{def}}{=} v - v_n\,n, \tag{5.16}$$

where v_Γ and v_n are the respective tangential part and the normal component of v. In view of the previous definitions the following identities are easy to check:

$$\nabla F|_\Gamma = \nabla_\Gamma f + \frac{\partial F}{\partial n}\,n, \tag{5.17}$$

$$DV|_\Gamma = D_\Gamma v + DVn\;{}^*n, \tag{5.18}$$

$$\operatorname{div} V|_\Gamma = \operatorname{div}_\Gamma v + DVn\cdot n, \tag{5.19}$$

$$D_\Gamma v\,n = 0,\quad D^2b\,n = 0. \tag{5.20}$$

Decomposing v into its tangential part and its normal component,

$$D_\Gamma v = D_\Gamma v_\Gamma + v_n D^2b + n\,{}^*\nabla_\Gamma v_n, \tag{5.21}$$

$$\operatorname{div}_\Gamma v = \operatorname{div}_\Gamma v_\Gamma + \Delta b\,v_n = \operatorname{div}_\Gamma v_\Gamma + H\,v_n, \tag{5.22}$$

$$\nabla_\Gamma v_n = {}^*D_\Gamma v\,n + D^2b\,v_\Gamma. \tag{5.23}$$

Given $f\in C^1(\Gamma)$ and $g\in C^1(\Gamma;\Gamma)$, consider the canonical extensions $f\circ p\in C^1(S_{2h}(\Gamma))$ and $g\circ p\in C^1(S_{2h}(\Gamma);\mathbf{R}^{\mathrm{N}})$ and the gradient of the composition

$$\begin{aligned}\nabla(f\circ p\circ g\circ p) &= {}^*D(g\circ p)\,\nabla(f\circ p)\circ g\circ p\\ \Rightarrow &\ \boxed{\nabla_\Gamma(f\circ g) = {}^*D_\Gamma g\,\nabla_\Gamma f\circ g,}\end{aligned} \tag{5.24}$$

and for a vector-valued function $v\in C^1(\Gamma;\mathbf{R}^{\mathrm{N}})$,

$$\boxed{D_\Gamma(v\circ g) = D_\Gamma v\circ g\,D_\Gamma g.} \tag{5.25}$$

5.5 The Stokes and Green Formulae

One interesting application of the shape calculus in connection with the tangential calculus is the tangential Stokes formula. Given $v \in C^1(\Gamma)^N$, consider Stokes formula in $\mathbf{R}^{\mathrm{N}}$ for the vector function $v \circ p$:

$$\int_\Omega \operatorname{div}(v \circ p)\, dx = \int_\Gamma v \cdot n\, d\Gamma.$$

Given an autonomous velocity field V, differentiate both sides of Stokes's formula with respect to t,

$$\int_{\Omega_t(V)} \operatorname{div}(v \circ p)\, dx = \int_{\Gamma_t(V)} (v \circ p) \cdot n_t = \int_{\Gamma_t(V)} (v \circ p) \cdot N_t\, d\Gamma_t,$$

where N_t is the extension (4.37) of n_t. This gives the new identity

$$\int_\Gamma \operatorname{div}(v \circ p)\, V \cdot n\, d\Gamma = \int_\Gamma v \cdot N' + \left\{ \frac{\partial}{\partial n}[(v \circ p) \cdot \nabla b] + H\, v \cdot n \right\} V \cdot n\, d\Gamma.$$

Now choose the velocity field $V = \nabla b\, \psi$, where $\psi \in \mathcal{D}(S_{2h}(\Gamma))$ is chosen in such a way that $\psi = 1$ on $S_h(\Gamma)$. Using expression (4.38) for N',

$$\begin{aligned}
V \cdot n = n \cdot n = 1, &\quad \operatorname{div}(v \circ p)|_\Gamma = \operatorname{div}_\Gamma v,\\
N'|_\Gamma &= DVn \cdot n\, n - {}^*DVn - D^2 b\, V\\
&= D^2 b \nabla b \cdot \nabla b\, \nabla b - D^2 b \nabla b - D^2 b \nabla b = 0,\\
\nabla((v \circ p) \cdot \nabla b) &= D^2 b\, v \circ p + {}^*D(v \circ p)\nabla b,\\
\frac{\partial}{\partial n}[(v \circ p) \cdot \nabla b] &= \left[D^2 b\, v + {}^*D_\Gamma v\, n\right] \cdot n = v \cdot D^2 b \nabla b + n \cdot D_\Gamma v\, n = 0.
\end{aligned}$$

Finally we get the *tangential Stokes formula* with $H = \Delta b$:

$$\boxed{\int_\Gamma \operatorname{div}_\Gamma v\, d\Gamma = \int_\Gamma H\, v \cdot n\, d\Gamma.} \tag{5.26}$$

For a function $f \in C^1(\Gamma)$ and a vector $v \in C^1(\Gamma)^N$, the above formula also yields the *tangential Green's formula*

$$\boxed{\int_\Gamma f \operatorname{div}_\Gamma v + \nabla_\Gamma f \cdot v\, d\Gamma = \int_\Gamma H\, f\, v \cdot n\, d\Gamma.} \tag{5.27}$$

5.6 Relation between Tangential and Covariant Derivatives

One of the simplest examples of a tangential derivative is when Ω is the half-space

$$H_+ \stackrel{\text{def}}{=} \left\{\zeta \in \mathbf{R}^{\mathrm{N}} : \zeta \cdot e_N > 0\right\}$$

for some orthonormal basis $\{e_1, \dots, e_N\}$ in $\mathbf{R}^N$ and Γ is the boundary of H_+ denoted by

$$H \stackrel{\text{def}}{=} \left\{\zeta \in \mathbf{R}^N : \zeta \cdot e_N = 0\right\}.$$

If p_H denotes the projection onto H, then $p_H(\zeta) = p_H(\zeta', \zeta_N) = \zeta' \stackrel{\text{def}}{=} (\zeta_1, \dots, \zeta_{N-1}) \in H$, and for any function $\varphi \in C^1(H)$ the tangential gradient

$$\nabla_H \varphi = \nabla(\varphi \circ p_H)|_H$$

coincides with the gradient of φ in H:

$$\boxed{\nabla_H \varphi \cdot e_\alpha = \frac{\partial \varphi}{\partial \zeta_\alpha}, \quad 1 \le \alpha \le N-1.}$$

With the notation of section 3.1 of Chapter 2, consider a set Ω locally of class C^2 in $\mathbf{R}^N$. Its boundary $\Gamma = \partial\Omega$ is an $(N-1)$-dimensional submanifold of $\mathbf{R}^N$ of class C^2. At each point $x \in \Gamma$, there is a C^2-diffeomorphism h_x from the open unit ball B onto a neighborhood $U(x)$ of x such that

$$h_x(B_0) = \Gamma_x \stackrel{\text{def}}{=} \Gamma \cap U(x), \quad h_x(B_+) = \Omega \cap U(x),$$

where $B_+ = H_+ \cap B$ and $B_0 = H \cap B$ and H_+ and H are as defined above for some appropriate orthonormal basis.

For a function $f \in C^1(\Gamma)$, $f \circ \Phi \in C^1(B_0)$ with $\Phi \stackrel{\text{def}}{=} h_x|_{B_0} : B_0 \to \Gamma$. Observing that $f \circ \Phi \circ p_H = f \circ p \circ \Phi \circ p_H$, we have

$$\begin{gathered} \nabla(f \circ \Phi \circ p_H) = \nabla(f \circ p \circ \Phi \circ p_H), \\ \nabla(f \circ \Phi \circ p_H) = {}^*D(\Phi \circ p_H) \nabla(f \circ p) \circ \Phi \circ p_H, \\ \boxed{\nabla_H(f \circ \Phi) = {}^*D_H \Phi \, \nabla_\Gamma f \circ \Phi.} \end{gathered}$$

The covariant basis associated with Φ is defined as

$$\boxed{a_\alpha \stackrel{\text{def}}{=} \frac{\partial \Phi}{\partial \zeta_\alpha}, \quad 1 \le \alpha \le N-1, \quad a_N \stackrel{\text{def}}{=} \left.\frac{{}^*(Dh_x)^{-1} e_N}{|{}^*(Dh_x)^{-1} e_N|}\right|_{B_0},}$$

where from identity (3.9) of section 3.1 in Chapter 2 we know that $a_N \circ \Phi^{-1}$ is the inward unit normal to Ω, that is,

$$\boxed{n = -a_N \circ \Phi^{-1}.} \tag{5.28}$$

The *covariant partial derivatives* of f are defined as

$$\boxed{f_\alpha \stackrel{\text{def}}{=} \frac{\partial (f \circ \Phi)}{\partial \zeta_\alpha}, \quad 1 \le \alpha \le N-1.}$$

But from the previous observations

$$a_\alpha = \frac{\partial \Phi}{\partial \zeta_\alpha} = D_H \Phi e_\alpha, \quad f_\alpha = \frac{\partial (f \circ \Phi)}{\partial \zeta_\alpha} = e_\alpha \cdot \nabla_H (f \circ \Phi),$$

$f \circ \Phi \circ p_H = f \circ p \circ \Phi \circ p_H$, and

$$\begin{aligned}
\nabla(f \circ \Phi \circ p_H) = \nabla(f \circ p \circ \Phi \circ p_H) &= {}^*D(\Phi \circ p_H) \nabla(f \circ p) \circ \Phi \circ p_H, \\
&\boxed{\nabla_H (f \circ \Phi) = {}^*D_H(\Phi)\, \nabla_\Gamma f \circ \Phi,} \\
f_\alpha = e_\alpha \cdot \nabla_H (f \circ \Phi) = e_\alpha \cdot {}^*D_H \Phi\, \nabla_\Gamma f \circ \Phi &= D_H \Phi\, e_\alpha \cdot \nabla_\Gamma f \circ \Phi \\
&= a_\alpha \cdot \nabla_\Gamma f \circ \Phi \\
\Rightarrow &\boxed{f_\alpha = a_\alpha \cdot \nabla_\Gamma f \circ \Phi = e_\alpha \cdot \nabla_H (f \circ \Phi).}
\end{aligned}$$

For a vector function $v \in C^1(\Gamma; \mathbf{R}^N)$

$$\boxed{v_{,\alpha} \overset{\text{def}}{=} \frac{\partial (v \circ \Phi)}{\partial \zeta_\alpha}, \quad 1 \le \alpha \le N-1.}$$

Again, $v \circ \Phi \circ p_H = v \circ p \circ \Phi \circ p_H$ and

$$D(v \circ \Phi \circ p_H) = D(v \circ p \circ \Phi \circ p_H) = D(v \circ p) \circ \Phi \circ p_H\, D(\Phi \circ p_H),$$

$$\boxed{D_H (v \circ \Phi) = D_\Gamma v \circ \Phi\, D_H \Phi} \quad \Rightarrow \boxed{v_{,\alpha} = D_H (v \circ \Phi) e_\alpha = D_\Gamma v \circ \Phi\, a_\alpha.}$$

The *second fundamental form* of Γ is defined from the inward normal a_N,

$$\boxed{b_{\alpha\beta} \overset{\text{def}}{=} -a_\alpha \cdot a_{N,\beta},}$$

where from (5.28) $a_N = -n \circ \Phi$, and from (5.10) $D_\Gamma n = D^2 b|_\Gamma$,

$$\begin{aligned}
D_H a_N &= -D_\Gamma n \circ \Phi\, D_H \Phi = -D^2 b \circ \Phi\, D_H \Phi, \\
a_{N,\beta} &= -D_H a_N\, e_\beta = -D^2 b \circ \Phi\, D_H \Phi\, e_\beta = -D^2 b \circ \Phi\, a_\beta, \\
&\boxed{b_{\alpha\beta} = a_\alpha \cdot D^2 b \circ \Phi\, a_\beta.}
\end{aligned}$$

Recall that since $|\nabla b| = 1$, $D^2 b \nabla b = 0$ and ∇b is an eigenvector for the eigenvalue 0. As a result the eigenvalues of $D^2 b$ at a point of Γ are the eigenvalues of the second fundamental form $b_{\alpha\beta}$ (the $N-1$ *principal curvatures*) plus $\{0\}$. In particular,

$$\Delta b = \operatorname{tr} D^2 b = H = (N-1)\bar{H},$$

where $\bar{H}$ is the *mean curvature* of Γ and H the *additive curvature* (cf. section 3.3 of Chapter 2).

5.7 Back to the Example of Section 4.3.3

Coming back to formula (4.36) for $dJ(\Omega; V(0))$ of the boundary integral of the square of the normal derivative in section 4.3.3, the tangential calculus is now used on the term

$$\int_\Gamma 2\frac{\partial\phi}{\partial n}\left\{\frac{\partial\phi}{\partial n}DV(0)n\cdot n - \nabla\phi\cdot\left({}^*DV(0)\,n + D^2b\,V(0)\right)\right\}d\Gamma,$$

$$T \stackrel{\text{def}}{=} \frac{\partial\phi}{\partial n}DV(0)n\cdot n - \nabla\phi\cdot\left({}^*DV(0)\,n + D^2b\,V(0)\right).$$

From identity (5.18) with $v = V(0)|_\Gamma$ and identity (5.23),

$$\begin{aligned}
{}^*DV(0)n &= {}^*D_\Gamma v\,n + DV(0)n\cdot n\,n,\\
\nabla\phi\cdot\,{}^*DV(0)n &= \nabla\phi\cdot\,{}^*D_\Gamma v\,n + \frac{\partial\phi}{\partial n}DV(0)n\cdot n\\
\Rightarrow T = -\nabla\phi\cdot\left[{}^*D_\Gamma v\,n + D^2b\,v_\Gamma\right] &= -\nabla\phi\cdot\nabla_\Gamma v_n = -\nabla_\Gamma\phi\cdot\nabla_\Gamma v_n.
\end{aligned}$$

Therefore by using the tangential Stokes formula (5.26),

$$\begin{aligned}
&\int_\Gamma 2\frac{\partial\phi}{\partial n}T\,d\Gamma = -\int_\Gamma 2\frac{\partial\phi}{\partial n}\nabla_\Gamma\phi\cdot\nabla_\Gamma v_n\,d\Gamma\\
&= \int_\Gamma -2\,\mathrm{div}_\Gamma\left\{\frac{\partial\phi}{\partial n}v_n\,\nabla_\Gamma\phi\right\} + 2\,\mathrm{div}_\Gamma\left\{\frac{\partial\phi}{\partial n}\nabla_\Gamma\phi\right\}v_n\,d\Gamma\\
&= \int_\Gamma -2H\frac{\partial\phi}{\partial n}v_n\,\nabla_\Gamma\phi\cdot n + 2\,\mathrm{div}_\Gamma\left\{\frac{\partial\phi}{\partial n}\nabla_\Gamma\phi\right\}v_n\,d\Gamma\\
&= \int_\Gamma 2\,\mathrm{div}_\Gamma\left\{\frac{\partial\phi}{\partial n}\nabla_\Gamma\phi\right\}v_n\,d\Gamma = \int_\Gamma 2\,\mathrm{div}_\Gamma\left\{\frac{\partial\phi}{\partial n}\nabla_\Gamma\phi\right\}V(0)\cdot n\,d\Gamma.
\end{aligned}$$

Substituting into expression (4.36), we finally get the explicit formula in terms of $V(0)\cdot n$ as predicted by the structure theorem

$$\boxed{\begin{aligned}&dJ(\Omega;V)\\ &= \int_\Gamma\left\{2\frac{\partial\phi}{\partial n}D^2\phi\,n\cdot n + H\left|\frac{\partial\phi}{\partial n}\right|^2 + 2\,\mathrm{div}_\Gamma\left(\frac{\partial\phi}{\partial n}\nabla_\Gamma\phi\right)\right\}V(0)\cdot n\,d\Gamma.\end{aligned}} \tag{5.29}$$

6 Second-Order Semiderivative and Shape Hessian

The object of this section is to study second-order derivatives and semiderivatives by the *velocity method* for smooth and nonsmooth domains and discuss their relationship with the *method of perturbations of the identity* in the unconstrained case. The analysis of this section is motivated by first computing the second-order derivative of a domain integral by using the combined strengths of the shape and tangential calculuses in section 6.1. A basic formula for the second-order semiderivative of the domain integral is given in section 6.2, which also reveals the general structure of this derivative. Structure theorems for the second-order Eulerian semiderivative $d^2J(\Omega;V;W)$ of a function $J(\Omega)$ for two vector fields V and W are given in section 6.3 and section 6.4. A first theorem shows that under some natural continuity assumptions,

$$d^2J(\Omega;V;W) = d^2J\big(\Omega;V(0);W(0)\big) + dJ\big(\Omega;V'(0)\big),$$

where $V'(0)$ is the time-partial derivative $\partial_t V(t,x)$ at $t=0$. As in the study of first-order Eulerian semiderivatives, this first theorem reduces the study of second-order Eulerian semiderivatives to the autonomous case. So we then specialize to fields in $\mathcal{D}^k(D,\mathbf{R}^{\mathrm{N}})$ and give the equivalent of Hadamard's structure theorem for the term $d^2J\big(\Omega;V(0);W(0)\big)$.

This bilinear term is decomposed into a symmetrical term plus the gradient acting on the first half of the Lie bracket $[V(0),W(0)]$ in section 6.5. The symmetrical part is itself decomposed into a symmetrical part that depends only on the normal component of the velocity fields and a symmetrical term made up of the gradient acting on a generic group of terms, which occurs in all examples considered in this section.

6.1 Second-Order Derivative of the Domain Integral

Given $f \in C^2_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$ and a domain Ω of class C^2, consider the function

$$J(\Omega_{t,s}(V,W)) \stackrel{\text{def}}{=} \int_{\Omega_{t,s}(V,W)} f\,dx, \tag{6.1}$$

where

$$\Omega_{t,s}(V,W) \stackrel{\text{def}}{=} T_s(W)(\Omega_t(V)) = T_s(W)(T_t(V)(\Omega)),$$
$$J_{V,W}(t,s) \stackrel{\text{def}}{=} J(\Omega_{t,s}(V,W))$$

for some pair of autonomous velocity fields V and W satisfying condition (3.1) and additional smoothness conditions as necessary. The objective is to compute

$$\boxed{d^2J_{V,W} \stackrel{\text{def}}{=} \frac{\partial}{\partial s}\left\{\frac{\partial}{\partial t}J_{V,W}(t,s)\bigg|_{t=0}\right\}\bigg|_{s=0}.}$$

From formula (4.7) in Theorem 4.2, we already know that

$$\frac{\partial}{\partial t}J_{V,W}(t,s)\bigg|_{t=0} = dJ(\Omega_s(W);V) = \int_{\Gamma_s(W)} f\,V\cdot n_s\,d\Gamma_s = \int_{\Gamma_s(W)} f\,V\cdot N_s\,d\Gamma_s,$$

where $N_s = N_s(W)$ is the extension (4.37) of the normal n_s. So we can readily use formula (4.17) from Theorem 4.3:

$$\boxed{d^2J_{V,W} = \int_\Gamma f\,V\cdot N'(W) + \left\{\frac{\partial}{\partial n}(f\,V\cdot\nabla b) + H\,f\,V\cdot n\right\} W\cdot n\,d\Gamma,} \tag{6.2}$$

where $N'(W)$ is the derivative of the extension $N_s = N_s(W)$ given by expression (4.38). This yields

$$
\begin{aligned}
&d^2 J_{V,W} \\
&= \int_\Gamma f\, V \cdot \left\{ (DWn\cdot n)\, n - {}^*DW - D^2 b\, W \right\} \\
&\qquad + \left(\frac{\partial}{\partial n} (f\, V \cdot \nabla b) + H\, f\, V \cdot n \right) W \cdot n \, d\Gamma \\
&= \int_\Gamma f \left\{ V \cdot \left\{ (DWn\cdot n)\, n - {}^*DWn - D^2 b\, W \right\} + \frac{\partial}{\partial n} (V\cdot \nabla b)\; W \cdot n \right\} \\
&\qquad + \left(\frac{\partial f}{\partial n} + H\, f \right) V \cdot n\; W \cdot n \; d\Gamma.
\end{aligned}
$$

It remains to untangle the following term in the first part of the integral:

$$
T \stackrel{\text{def}}{=} V \cdot \left\{ (DWn\cdot n)\, n - {}^*DWn - D^2 b\, W \right\} + \frac{\partial}{\partial n} (V\cdot \nabla b)\; W \cdot n.
$$

Using the notation $v = V|_\Gamma$ and $w = W|_\Gamma$,

$$
\begin{aligned}
\nabla(V\cdot\nabla b) &= {}^*DV\nabla b + D^2 b\, V, \\
\nabla(V\cdot\nabla b)\cdot \nabla b &= {}^*DV\nabla b \cdot \nabla b + D^2 b\, V\cdot \nabla b = DV\nabla b\cdot \nabla b \\
\Rightarrow\; \frac{\partial}{\partial n} (V\cdot \nabla b) &= DVn\cdot n, \\
V\cdot {}^*DWn &= V\cdot \left[{}^*D_\Gamma w + n\; {}^*(DWn) \right]\, n \\
&= V\cdot \left[\nabla_\Gamma w_n - D^2 b\, w \right] + DWn\cdot n\; V\cdot n, \\
V\cdot \left[{}^*DWn + D^2 b\, W \right] &= \nabla_\Gamma w_n \cdot v_\Gamma + DWn\cdot n\; v_n.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
T &= DWn\cdot n\, v_n - (v_\Gamma \cdot \nabla_\Gamma w_n + DWn\cdot n\, v_n) + DVn\cdot n\, w_n \\
&= DVn\cdot n\, w_n - v_\Gamma \cdot \nabla_\Gamma w_n,
\end{aligned}
$$

and by using the tangential Stokes formula (5.26)

$$
\begin{aligned}
\int_\Gamma f\, T\, d\Gamma &= \int_\Gamma f\; \{ DVn\cdot n\, w_n - v_\Gamma \cdot \nabla_\Gamma w_n \}\; d\Gamma \\
&= \int_\Gamma f\, DVn\cdot n\, w_n - \operatorname{div}_\Gamma (f w_m v_\Gamma) + \operatorname{div}_\Gamma (f v_\Gamma)\; w_n \, d\Gamma \\
&= \int_\Gamma \{ f\, DVn\cdot n + \operatorname{div}_\Gamma (f v_\Gamma) \}\; w_n - f w_n\, v_\Gamma \cdot n\, d\Gamma \\
&= \int_\Gamma \{ f\; (DVn\cdot n + \operatorname{div}_\Gamma v_\Gamma) + \nabla_\Gamma f \cdot v_\Gamma \}\, w_n \, d\Gamma.
\end{aligned}
$$

Finally, we get two equivalent expressions:

$$\boxed{\begin{aligned} &d^2 J_{V,W} \\ &= \int_\Gamma \left(\frac{\partial f}{\partial n} + H\, f\right) v_n\, w_n + f\, \left(DVn \cdot n\, w_n - v_\Gamma \cdot \nabla_\Gamma w_n\right)\, d\Gamma \\ &= \int_\Gamma \left\{ \left(\frac{\partial f}{\partial n} + H\, f\right) v_n + f\, \left(DVn \cdot n + \operatorname{div}_\Gamma v_\Gamma\right) + \nabla_\Gamma f \cdot v_\Gamma \right\} w_n\, d\Gamma. \end{aligned}} \tag{6.3}$$

Since the above expressions involved the composition $T_s(W) \circ T_t(V)$, it is expected that the condition for the symmetry of expression (6.3) will involve the *Lie bracket* $[V, W] = DVW - DWV$. Indeed, by using identity (5.23)

$$\begin{aligned} DVW \cdot n &= (D_\Gamma v + DVn\, {}^*n\}\ W \cdot n \\ &= w \cdot {}^*D_\Gamma v\, n + DVn \cdot n\, w_n \\ &= w_\Gamma \cdot \left(\nabla_\Gamma v_n - D^2 b\, v_\Gamma\right) + DVn \cdot n\, w_n \end{aligned}$$

and substituting in the first expression, we get a symmetrical term plus the first half of the Lie bracket:

$$\boxed{\begin{aligned} &d^2 J_{V,W} \\ &= \int_\Gamma \left(\frac{\partial f}{\partial n} + H\, f\right) v_n\, w_n + f\, \left(D^2 b\, v_\Gamma \cdot w_\Gamma - v_\Gamma \cdot \nabla_\Gamma w_n - w_\Gamma \cdot \nabla_\Gamma v_n\right) \\ &\qquad\qquad + f\, DVW \cdot n\, d\Gamma. \end{aligned}} \tag{6.4}$$

Thus

$$\boxed{d^2_{V,W} = d^2_{W,V} \quad \Longleftrightarrow \quad \int_\Gamma f\, [V, W] \cdot n\, d\Gamma = 0,} \tag{6.5}$$

from which either $f\, [V, W] \cdot n = 0$ on Γ or $\operatorname{div}(f\, [V, W]) = 0$ on Ω can be used as sufficient conditions.

Example 6.1.
Let $\Omega = \{(x, y) : x^2 + y^2 < 1\}$ be the unit ball in $\mathbf{R}^2$ with boundary $\Gamma = \{(x, y) : x^2 + y^2 = 1\}$. Choose

$$\begin{aligned} &V(x, y) = (1, 0), \quad W(x, y) = (x^2/2, 0), \quad n = (x, y)/\sqrt{x^2 + y^2} \\ &\Rightarrow DVW - DWV = (x, 0) \ \Rightarrow [DVW - DWV] \cdot n = x^2/\sqrt{x^2 + y^2} = x^2. \qquad \square \end{aligned}$$

6.2 Basic Formula for Domain Integrals

We have proved the following result.

Theorem 6.1. *Let $f \in C^2([0, \tau] \times [0, \tau]; H^2_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}}))$ and let Γ be the boundary of a bounded open subset Ω of $\mathbf{R}^{\mathrm{N}}$ of class C^2. Assume that $V \in C^0([0, \tau]; C^2_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}))$*

and that $W \in C^0([0,\tau]; C^1_{\text{loc}}(\mathbf{R}^{\rm N}, \mathbf{R}^{\rm N}))$. *Consider the function*

$$\boxed{J_{V,W}(t,s) \stackrel{\text{def}}{=} \int_{\Omega_{t,s}(V,W)} f(t,s)\, dx.} \tag{6.6}$$

Then the partial derivative of $J_{V,W}(t,s)$ *with respect to* t *in* $t=0$ *is given by*

$$\begin{aligned} \frac{\partial}{\partial t} J_{V,W}(t,s)\Big|_{t=0} &= \int_{\Omega_s(W)} \frac{\partial f}{\partial t}(0,s) + \operatorname{div}\left(f(0,s)V(0)\right) dx \\ &= \int_{\Omega_s(W)} \frac{\partial f}{\partial t}(0,s)\, dx + \int_{\Gamma_s(W)} f(0,s)\, V(0)\cdot n_s \, d\Gamma_s \end{aligned} \tag{6.7}$$

and the second-order mixed derivative of $J_{V,W}(t,s)$ *in* $(t,s) = (0,0)$,

$$d^2 J_{V,W} \stackrel{\text{def}}{=} \frac{\partial}{\partial s}\left\{ \frac{\partial}{\partial t} J_{V,W}(t,s)\Big|_{t=0} \right\}\Big|_{s=0}, \tag{6.8}$$

is given by the expression

$$\boxed{\begin{aligned} d^2 J_{V,W} =& \int_\Omega \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial t}\right) + \operatorname{div}\left(\frac{\partial f}{\partial s} V(0) + \frac{\partial f}{\partial t} W(0)\right) \\ &\qquad + \operatorname{div}\left[\operatorname{div}\left(f\, V(0)\right)\, W(0)\right] dx \\ =& \int_\Omega \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial t}\right) dx + \int_\Gamma \left(\frac{\partial f}{\partial s} V(0) + \frac{\partial f}{\partial t} W(0)\right)\cdot n \\ &\qquad + \operatorname{div}\left(f\, V(0)\right)\, W(0)\cdot n \, d\Gamma. \end{aligned}} \tag{6.9}$$

The last term in the second integral can be expressed in terms of $v = V(0)|_\Gamma$ *and* $w = W(0)|_\Gamma$ *as follows:*

$$\boxed{\begin{aligned} &\int_\Gamma \operatorname{div}\left(f\, V(0)\right)\, W(0)\cdot n\, d\Gamma \\ &= \int_\Gamma \left\{ \left(\frac{\partial f}{\partial n} + H\, f\right) v_n + f\, \left(DVn\cdot n + \operatorname{div}_\Gamma v_\Gamma\right) + \nabla_\Gamma f\cdot v_\Gamma \right\} w_n \, d\Gamma \\ &= \int_\Gamma \left(\frac{\partial f}{\partial n} + H\, f\right) v_n\, w_n + f\, \left(D^2 b v_\Gamma\cdot w_\Gamma - v_\Gamma\cdot\nabla_\Gamma w_n - w_\Gamma\cdot\nabla_\Gamma v_n\right) \\ &\qquad + f\, DVW\cdot n\, d\Gamma. \end{aligned}} \tag{6.10}$$

6.3 Nonautonomous Case

The framework introduced in sections 4 and 5 of Chapter 4 has reduced the computation of the Eulerian semiderivative of $J(\Omega)$ to the computation of the derivative of the function

$$j(t) \stackrel{\text{def}}{=} J\big(\Omega_t(V)\big) \tag{6.11}$$

for a velocity field $V \in C([0,\tau]; C^k_{\mathrm{loc}}(\mathbf{R}^N;\mathbf{R}^N))$. In $t \geq 0$

$$j'(t) = dJ\big(\Omega_t(V); V_t\big) \tag{6.12}$$

since $T_{s+t}(V) = T_s(V_t) \circ T_t(V)$, where $V_t(s) \overset{\mathrm{def}}{=} V(t+s)$ and $V_t(0) = V(t)$.

This suggests the following definition.

Definition 6.1.
Let J be a real-valued shape functional. Let V and W satisfy conditions (V) (conditions (4.2) of Chapter 4) and assume that for all $t \in [0,\tau]$, $dJ\big(\Omega_t(W);\, V_t\big)$ exists for $\Omega_t(W) = T_t(W)(\Omega)$. The function J is said to have a *second-order Eulerian semiderivative* at Ω in the directions (V, W) if the following limit exists:

$$\lim_{t \searrow 0} \frac{dJ(\Omega_t(W); V_t) - dJ(\Omega; V)}{t}. \tag{6.13}$$

When it exists, it is denoted $d^2J(\Omega; V; W)$. □

If, for all t, J has a Hadamard semiderivative at $\Omega_t(W)$, recall that

$$\begin{aligned} dJ(\Omega_t(W); V_t) =& d_H J(\Omega_t(W); V_t(0)) \\ =& d_H J(\Omega_t(W); V(t)) = dJ(\Omega_t(W); V(t)), \end{aligned}$$

and the above definition reduces to

$$d^2J(\Omega; V; W) = \lim_{t \searrow 0} \frac{dJ\big(\Omega_t(W); V(t)\big) - dJ\big(\Omega; V(0)\big)}{t}.$$

Remark 6.1.
This last definition is compatible with the second-order expansion of $j(t)$ with respect to t around $t = 0$:

$$j(t) \cong j(0) + t j'(0) + \frac{t^2}{2} j''(0), \tag{6.14}$$

where

$$j(0) = J(\Omega), \quad j'(0) = dJ(\Omega; V), \quad j''(0) = d^2J(\Omega; V; V). \tag{6.15}$$

□

The next theorem is the analogue of Theorem 3.2 and provides the canonical structure of the second-order Eulerian semiderivative (cf. (3.20) to (3.22) in section 3.1 for the definitions of $\overrightarrow{\mathcal{V}}^{m,\ell}$ and $\mathcal{V}^\ell$).

Theorem 6.2. *Let J be a real-valued shape functional, Ω be a subset of $\mathbf{R}^N$, and $m \geq 0$ and $\ell \geq 0$ be two integers. Assume that*

(i) $\forall V \in \overrightarrow{\mathcal{V}}^{m+1,\ell}, \forall W \in \overrightarrow{\mathcal{V}}^{m,\ell}, \quad d^2J(\Omega; V; W)$ *exists;*

(ii) $\forall W \in \overrightarrow{\mathcal{V}}^{m,\ell}$, $\forall t \in [0,\tau]$, *J has a shape gradient of order* ℓ *and is Hadamard semidifferentiable at* $\Omega_t(W)$*;*

(iii) $\forall U \in \mathcal{V}^\ell$, *the map*

$$W \mapsto d^2 J(\Omega; U; W) : \overrightarrow{\mathcal{V}}^{m,\ell} \to \mathbf{R} \tag{6.16}$$

is continuous.

Then for all V *in* $\overrightarrow{\mathcal{V}}^{m+1,\ell}$ *and all* W *in* $\overrightarrow{\mathcal{V}}^{m,\ell}$,

$$d^2 J(\Omega; V; W) = d^2 J\big(\Omega; V(0); W(0)\big) + dJ\big(\Omega; V'(0)\big), \tag{6.17}$$

where

$$V'(0)(x) \stackrel{\text{def}}{=} \lim_{t \searrow 0} \frac{V(t,x) - V(0,x)}{t}. \tag{6.18}$$

Proof. The differential quotient (6.13) can be split into the sum of two terms:

$$\frac{dJ\big(\Omega_t(W); V(0)\big) - dJ\big(\Omega; V(0)\big)}{t} + \frac{dJ\big(\Omega_t(W); V(t)\big) - dJ\big(\Omega_t(W); V(0)\big)}{t}. \tag{6.19}$$

In view of (i) and (iii), for all U in $\mathcal{V}^\ell$,

$$d^2 J(\Omega; U; W) = d^2 J\big(\Omega; U; W(0)\big)$$

by the same argument as in the proof of Theorem 3.2 for the gradient. Hence the first term converges to

$$d^2 J\big(\Omega; V(0); W\big) = d^2 J\big(\Omega; V(0); W(0)\big).$$

For the second term recall that V belongs to $\overrightarrow{\mathcal{V}}^{m+1,\ell}$ and observe that the vector field

$$\widehat{V}(t) = \frac{V(t) - V(0)}{t}$$

belongs to $\overrightarrow{\mathcal{V}}^{m,\ell}$ and that $\widehat{V}(0) = V'(0)$. Thus by linearity of $dJ(\Omega; V)$, the second term in (6.19) can be written as

$$dJ\left(\Omega_t(W); \frac{V(t) - V(0)}{t}\right) = dJ\big(\Omega_t(W); \widehat{V}(t)\big),$$

$$dJ\big(\Omega_t(W); \widehat{V}(t)\big) = t\,\frac{1}{t}\left[dJ\big(\Omega_t(W); \widehat{V}(t)\big) - dJ\big(\Omega; \widehat{V}(0)\big)\right] + dJ\big(\Omega; \widehat{V}(0)\big).$$

But for any V in $\overrightarrow{\mathcal{V}}^{m+2,\ell}$, $\widehat{V}$ belongs to $\overrightarrow{\mathcal{V}}^{m+1,\ell}$. Then by assumption (i),

$$\lim_{t\searrow 0} \frac{dJ\big(\Omega_t(W);\widehat{V}(t)\big) - dJ\big(\Omega;\widehat{V}(0)\big)}{t} = d^2J(\Omega;\widehat{V};W),$$

which implies that

$$\lim_{t\searrow 0} dJ\big(\Omega_t(W);\widehat{V}(t)\big) = dJ\big(\Omega;\widehat{V}(0)\big) = dJ\big(\Omega;V'(0)\big).$$

Now by assumption (ii), the map $U \mapsto dJ(\Omega;U)$ is linear and continuous on $\mathcal{D}^\ell(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})$, and the map

$$V \mapsto V'(0) \mapsto dJ\big(\Omega;V'(0)\big) : \overrightarrow{\mathcal{V}}^{m+2,\ell} \to \mathcal{V}^\ell \to \mathbf{R}$$

is linear and continuous (hence uniformly continuous) for the topology $\overrightarrow{\mathcal{V}}^{m+1,\ell}$ for all V in the dense subspace $\overrightarrow{\mathcal{V}}^{m+2,\ell}$. Hence it uniquely and continuously extends to all elements of $\overrightarrow{\mathcal{V}}^{m+1,\ell}$. This completes the proof of the theorem. □

This important theorem gives the canonical structure of the second-order Eulerian semiderivative: a first term that depends on $V(0)$ and $W(0)$ and a second term that is equal to $dJ\big(\Omega;V'(0)\big)$. When V is autonomous the second term disappears and the semiderivative coincides with $d^2J\big(\Omega;V;W(0)\big)$, which can be separately studied for autonomous vector fields in $\mathcal{V}^\ell$.

We conclude this section with the explicit computation of the second-order Eulerian semiderivative for a shape function $J(\Omega)$ with respect to two velocity fields V and W satisfying the conditions of Theorem 6.2 and such that the shape gradient at any t is of the form

$$\boxed{dJ(\Omega_t(W);V(t)) = \int_{\Gamma_t(W)} g(t)\, V(t)\cdot n_t \, d\Gamma_t} \tag{6.20}$$

for some function $g(t) \in C(\Gamma_t(W))$. Further assume that the family of functions $g(t)$ has an extension $Q \in C^1([0,\tau];C^k_{\mathrm{loc}}(N(\Gamma);\mathbf{R}^{\mathrm{N}}))$ to an open neighborhood $N(\Gamma)$ of Γ such that $\cup\{\Gamma_t(W) : 0 \le t \le \tau\} \subset N(\Gamma)$. Therefore using the extension $N_t(W)$ of the normal n_t on $\Gamma_t(W)$, it amounts to differentiating the expression

$$j(t) = \int_{\Gamma_t(W)} Q(t)\, V(t)\cdot N_t(W)\, d\Gamma_t.$$

Apply the first formula (4.17) of Theorem 4.3 to get

$$\begin{aligned} j'(0) = \int_\Gamma & (Q'_W(0)\, V(0) + Q(0)\, V'(0))\cdot n + Q(0)\, V(0)\cdot N'(W) \\ & + \left(\frac{\partial}{\partial n}\, (Q(0)\, V(0)\cdot \nabla b) + H\, Q(0)\, V(0)\cdot n\right) W(0)\cdot n\, d\Gamma, \end{aligned}$$

where $Q'_W(0)$ depends only on W. But the last three terms have already been computed in several forms. They constitute expression (6.2) in section 6.1, which yields (6.3) and (6.4) with $f = Q(0)$, $V = V(0)$, and $W = W(0)$. This yields with the notation $v = V(0)|_\Gamma$ and $w = W(0)|_\Gamma$

$$\boxed{\begin{aligned} d^2J(V;W) = &\int_\Gamma Q'_W(0)\, v_n + Q(0)\, V'(0)\cdot n + \left(\frac{\partial Q(0)}{\partial n} + H\, Q(0)\right) v_n\, w_n \\ &\quad + Q(0)\, \left(D^2 b\, v_\Gamma \cdot w_\Gamma - v_\Gamma \cdot \nabla_\Gamma w_n - w_\Gamma \cdot \nabla_\Gamma v_n\right) \\ &\quad + Q(0)\, DVW \cdot n\, d\Gamma \\ = &\int_\Gamma Q'_W(0)\, v_n + Q(0)\, V'(0)\cdot n + \left(\frac{\partial Q(0)}{\partial n} + H\, Q(0)\right) v_n\, w_n \\ &\quad + \left\{Q(0)\, \left(DVn\cdot n + \operatorname{div}_\Gamma v_\Gamma\right) + \nabla_\Gamma Q(0)\cdot v_\Gamma\right\}\, w_n\, d\Gamma. \end{aligned}} \tag{6.21}$$

Remark 6.2.
When V is autonomous the term in $V'(0)$ disappears. In that case the first half of the Lie bracket can be eliminated by restarting the computation with $\mathcal{V}(t) = V \circ T_t^{-1}(W)$ in place of V since $\mathcal{V}(0) = V$ and

$$\begin{aligned} \mathcal{V}'(0) = -DV\, W \quad &\Rightarrow \int_\Gamma Q(0)\, \mathcal{V}'(0)\cdot n + Q(0)\, DVW\cdot n\, d\Gamma = 0 \\ &\Rightarrow\ d^2J(V,W) = d^2J(\mathcal{V},W) + dJ(\Omega; DVW). \end{aligned}$$

□

Remark 6.3.
Except for the terms that contain the first half of the Lie bracket DVW and $V'(0)$, the only term that might not be symmetrical in the first expression (6.21) is the one in $Q'_W(0)$. In fact, according to the second expression and our theorem, $Q'_W(0) = Q'_{W(0)}$ depends only on $W(0)$. Now choose autonomous velocity fields V and W. Furthermore, assume that W is of the form $W = w_\Gamma \circ p$. Since $W\cdot n = 0$ on Γ, $dJ(\Omega_t(W); V(0)) = dJ(\Omega; V(0))$ and necessarily $d^2J(\Omega; V; w_\Gamma \circ p) = 0$. Therefore, $d^2J(\Omega; V; W)$ depends only on w_n and hence $Q'_W(0) = Q'_{w_n}(0)$ and the integral

$$\int_\Gamma Q'_{w_n}(0)\, v_n\, d\Gamma$$

depends only on w_n and v_n. □

The above expressions give valuable information on the structure of the second-order Eulerian derivative. Other expressions can also be obtained. For instance, if $j(t)$ is transformed into the volume integral

$$j(t) = dJ(\Omega_t(W); V(t)) = \int_{\Gamma_t(W)} Q(t)\, V(t)\cdot n_t\, d\Gamma_t = \int_{\Omega_t(W)} \operatorname{div}\, \left(Q(t)\, V(t)\right)\, dx,$$

we get from formula (4.6) in Theorem 4.2 the equivalent volume expression

$$\begin{aligned} &d^2J(\Omega; V; W) \\ &= \int_\Omega \operatorname{div}\, \left\{Q'_{W(0)}(0)\, V(0) + Q(0)\, V'(0) + \operatorname{div}\, \left(Q(0)\, V(0)\right)\, W(0)\right\}\, dx, \end{aligned}$$

which can obviously be transformed into a boundary expression.

6.4 Autonomous Case

Definition 6.2.
Let J be a real-valued shape functional. Let Ω be a subset of $\mathbf{R}^N$.

(i) The function $J(\Omega)$ is said to be *twice shape differentiable* at Ω if

$$\forall V, \forall W \in \mathcal{D}(\mathbf{R}^N, \mathbf{R}^N), \quad d^2 J(\Omega; V; W) \text{ exists} \tag{6.22}$$

and the map

$$(V, W) \mapsto d^2 J(\Omega; V; W) : \mathcal{D}(\mathbf{R}^N, \mathbf{R}^N) \times \mathcal{D}(\mathbf{R}^N, \mathbf{R}^N) \to \mathbf{R} \tag{6.23}$$

is bilinear and continuous. We denote by h the map (6.23).

(ii) Denote by $H(\Omega)$ the vector distribution in $(\mathcal{D}(\mathbf{R}^N, \mathbf{R}^N) \otimes \mathcal{D}(\mathbf{R}^N, \mathbf{R}^N))'$ associated with h:

$$d^2 J(\Omega; V; W) = \langle H(\Omega), V \otimes W \rangle = h(V, W), \tag{6.24}$$

where $V \otimes W$ is the tensor product of V and W defined as

$$(V \otimes W)_{ij}(x, y) = V_i(x) W_j(y), \quad 1 \le i, j \le N, \tag{6.25}$$

and $V_i(x)$ (resp., $W_j(y)$) is the ith (resp., jth) component of the vector V (resp., W) (cf. L. Schwartz's [**1**] kernel theorem and M. Gelfand and N. Y. Vilenkin [**1**]). $H(\Omega)$ will be called the *shape Hessian* of J at Ω.

(iii) When there exists a finite integer $\ell \ge 0$ such that $H(\Omega)$ is continuous for the $\mathcal{D}^\ell(\mathbf{R}^N, \mathbf{R}^N) \otimes \mathcal{D}^\ell(\mathbf{R}^N, \mathbf{R}^N)$-topology, we say that $H(\Omega)$ is of order ℓ.

In what follows, the compact notation $\mathcal{D}^\ell$ will be used in place of $\mathcal{D}^\ell(\mathbf{R}^N, \mathbf{R}^N)$. □

Theorem 6.3. *Let J be a real-valued shape functional and Ω be a subset of $\mathbf{R}^N$ with boundary Γ. Assume that J is twice shape differentiable.*

(i) *The vector distribution $H(\Omega)$ has support in $\Gamma \times \Gamma$.*

(ii) *If Ω is an open or closed domain in $\mathbf{R}^N$ and $H(\Omega)$ is of order $\ell \ge 0$, then there exists a continuous bilinear form*

$$[h] : \left(\mathcal{D}^\ell / D^\ell_\Gamma\right) \times \left(\mathcal{D}^\ell / L^\ell_\Omega\right) \to \mathbf{R} \tag{6.26}$$

such that for all $[V]$ in $\mathcal{D}^\ell / D^\ell_\Gamma$ and $[W]$ in $\mathcal{D}^\ell / L^\ell_\Omega$,

$$\boxed{d^2 J(\Omega; V; W) = [h]\big(q_D(V), q_L(W)\big),} \tag{6.27}$$

where $q_D : \mathcal{D}^\ell \to \mathcal{D}^\ell / D^\ell_\Gamma$ and $q_L : \mathcal{D}^\ell \to \mathcal{D}^\ell / L^\ell_\Omega$ are the canonical quotient surjections and

$$\boxed{D^\ell_\Gamma = \left\{V \in \mathcal{D}^\ell\big(\mathbf{R}^N, \mathbf{R}^N\big) : \partial^\alpha V = 0 \text{ on } \Gamma, \forall \alpha, \ |\alpha| \le \ell\right\}.} \tag{6.28}$$

Proof. (i) It is sufficient to prove the following two properties:

(a) For all $V, W \in \mathcal{D}$ such that $W = 0$ in a neighborhood of Γ, $d^2J(\Omega; V; W) = 0$.

(b) For all $V, W \in \mathcal{D}$ such that $V = 0$ in a neighborhood of Γ, $d^2J(\Omega; V; W) = 0$.

In case (a) the proof is similar to the one in Theorem 3.6 for the gradient and we prove the stronger result that for W such that $W = 0$ on Γ,

$$\begin{aligned}\Omega_t(W) = \Omega, \forall t \geq 0 &\implies dJ\big(\Omega_t(W); V\big) = dJ(\Omega; V) \\ &\implies d^2J(\Omega; V; W) = 0.\end{aligned}$$

In case (b) $V = 0$ in a neighborhood N of Γ and in $\complement K$, the complement of the compact support K of V. So $U = \complement K$ is a neighborhood of Γ where $V = 0$. By construction $U \cap K = \varnothing$ and there exists a bounded neighborhood $\mathcal{U}$ of K such that $\overline{\mathcal{U}} \cap \Gamma = \varnothing$. Since $\overline{\mathcal{U}}$ is compact and Γ is closed, the minimum distance d from $\overline{\mathcal{U}}$ to Γ is finite and nonzero. Let

$$N(\Gamma) = \left\{ y \in \mathbf{R}^{\mathrm{N}} : d_\Gamma(y) < d/2 \right\},$$

where

$$d_\Gamma(y) = \inf\{|y - x| : x \in \Gamma\}.$$

For all X in Γ

$$T_t(X) - X = \int_0^t W\big(T_s(X)\big)\, ds = tW(X) + \int_0^t \Big[W\big(T_s(X)\big) - W(X) \Big]\, ds,$$

and by condition (3.1) on W,

$$|T_t(X) - X| \leq t|W(X)| + ct \max_{[0,t]} |T_s(X) - X|,$$

and it can easily be shown that for $t < 1/c$

$$\max_{[0,t]} |T_s(X) - X| < \frac{t}{1 - ct} |W(X)|.$$

Thus

$$\sup_{X \in \Gamma} \max_{[0,t]} |T_s(X) - X| \leq \frac{t}{1 - ct} \sup_{X \in \Gamma} |W(X)|.$$

But W is continuous with compact support. Therefore,

$$\sup_{X \in \Gamma} |W(X)| \leq \sup_{X \in \operatorname{supp} W} |W(X)| = \|W\|_{C(\mathbf{R}^{\mathrm{N}}; \mathbf{R}^{\mathrm{N}})} < \infty$$

and there exists $\tau > 0$ such that

$$\forall s \in [0, \tau], \quad \frac{s}{1 - cs} \|W\|_C < \frac{d}{2}.$$

By definition and the previous inequalities

$$d_\Gamma\big(T_s(X)\big) = \inf_{Y \in \Gamma} |T_s(X) - Y| \leq |T_s(X) - X| < \frac{d}{2}$$

for all s in $[0,\tau]$ and all $X \in \Gamma$. This implies that

$$\forall s \in [0,\tau], \forall X \in \Gamma, \quad \Gamma_s(W) = T_s(W)(\Gamma) \subset N(\Gamma).$$

By construction, $V = 0$ in $N(\Gamma)$ since the distance from K to Γ is greater than or equal to d. Therefore,

$$\forall s \in [0,\tau], \quad V \in L^\infty_{\Omega_s(W)}$$

and as in the proof of Theorem 3.6, $dJ(\Omega_s(W);V) = 0$ and necessarily $d^2J(\Omega; V;W) = 0$.

(ii) We have already established in (i) that the bilinear form

$$(V,W) \mapsto h(V,W) : \mathcal{D} \times \mathcal{D} \to \mathbf{R}$$

is zero for all $V \in \mathcal{D}$ and $W \in \mathcal{D}$ such that $W = 0$ on Γ and also zero for all $W \in \mathcal{D}$ and $V \in \mathcal{D}$ for which $V = 0$ in a neighborhood of Γ. By density all this is still true in $\mathcal{D}^\ell$, and now by the same argument as in the proof of Theorem 3.6 for all V in $\mathcal{D}^\ell$,

$$[W] \mapsto h(V,W) : \mathcal{D}^\ell / L^\ell \to \mathbf{R}$$

is well-defined, linear, and continuous. For the first component it is necessary to show that for all W in

$$D^\ell_\Gamma = \left\{ V \in \mathcal{D}^\ell(\mathbf{R}^N, \mathbf{R}^N) : \partial^\alpha V = 0 \text{ on } \Gamma,\ \forall \alpha, |\alpha| \le \ell \right\}$$

the bilinear form $h(V,W) = 0$. We first prove the result for the subspace

$$A = \mathcal{D}(\Omega; \mathbf{R}^N) \oplus \mathcal{D}(\complement\bar{\Omega}; \mathbf{R}^N).$$

Then by density and continuity the result holds for the $\mathcal{D}^\ell(\mathbf{R}^N, \mathbf{R}^N)$-closure $\overline{A}$ of A. Finally, we prove that $\overline{A} = D^\ell_\Gamma$. For any V in A, there exist $V_1 \in \mathcal{D}(\Omega; \mathbf{R}^N)$ and $V_2 \in \mathcal{D}(\complement\bar{\Omega}; \mathbf{R}^N)$ such that $V = V_1 + V_2$. Moreover,

$$K_1 = \operatorname{supp} V_1 \subset \Omega \quad \text{and} \quad K_2 = \operatorname{supp} V_2 \subset \complement\bar{\Omega}$$

are compact subsets of the open sets Ω and $\complement\bar{\Omega}$, respectively. Hence $V_1 = 0$ (resp., $V_2 = 0$) in the open neighborhood $\complement K_1$ (resp., $\complement K_2$) of Γ and necessarily $V = V_1 + V_2 = 0$ in the neighborhood $U = \complement(K_1 \cup K_2)$ of Γ. Hence from part (i) $h(V,W) = 0$. By definition of D^ℓ_Γ, $D^\ell_\Gamma \subset \mathcal{D}^\ell(\bar{\Omega}; \mathbf{R}^N) \oplus \mathcal{D}^\ell(\complement\Omega; \mathbf{R}^N)$. Now $A \subset D^\ell_\Gamma$,

$$\overline{A} = \overline{\mathcal{D}(\Omega; \mathbf{R}^N)} \oplus \overline{\mathcal{D}(\complement\bar{\Omega}; \mathbf{R}^N)},$$

and

$$\overline{\mathcal{D}(\Omega; \mathbf{R}^N)} = \mathcal{D}^\ell(\bar{\Omega}; \mathbf{R}^N), \quad \overline{\mathcal{D}(\complement\bar{\Omega}; \mathbf{R}^N)} = \mathcal{D}^\ell(\complement\Omega; \mathbf{R}^N).$$

By construction each V in $\bar{A}$ is of the form $V = V_1 + V_2$ for

$$\begin{aligned} V_1 \in \mathcal{D}^\ell(\bar{\Omega}; \mathbf{R}^N), \quad & K_1 = \operatorname{supp} V_1 \text{ compact in } \bar{\Omega}, \\ & \forall |\alpha| \le \ell, \quad \partial^\alpha V_1 = 0 \text{ on } \Gamma, \\ V_2 \in \mathcal{D}^\ell(\complement\Omega; \mathbf{R}^N), \quad & K_2 = \operatorname{supp} V_2 \text{ compact in } \complement\Omega, \\ & \forall |\alpha| \le \ell, \quad \partial^\alpha V_2 = 0 \text{ on } \Gamma. \end{aligned}$$

Hence

$$\operatorname{supp} V = K_1 \cup K_2 \text{ compact in } \left[\bar{\Omega}\right] \cup \left[\mathbf{R}^N \backslash \Omega\right] = \mathbf{R}^N$$

and $V \in \mathcal{D}^\ell(\mathbf{R}^N, \mathbf{R}^N)$. Moreover,

$$\forall \alpha,\ |\alpha| \leq \ell, \quad \partial^\alpha V = \partial^\alpha V_1 + \partial^\alpha V_2 = 0 \text{ on } \Gamma.$$

This proves that $\overline{A} \subset D^\ell_\Gamma$ and hence $\overline{A} = D^\ell_\Gamma$. To complete the proof notice that by continuity of $V \mapsto h(V, W)$, for all W in $\mathcal{D}^\ell$ the map

$$[V] \mapsto h(V, W,) : \mathcal{D}^\ell / D^\ell_\Gamma \to \mathbf{R}$$

is well-defined, linear, and continuous. Finally the map

$$\big([V], [W]\big) \mapsto h(V, , W) : (\mathcal{D}^\ell / L^\ell) \times (\mathcal{D}^\ell / D^\ell_\Gamma) \to \mathbf{R}$$

is well-defined, bilinear, and continuous. □

The next and last result is the extension of the structure Theorem 3.6 to second-order Eulerian semiderivatives. We need the result established in the corollary to Theorem 3.6. For a domain Ω with a boundary Γ which is $C^{\ell+1}$, $\ell \geq 0$, the map

$$q_L(W) \mapsto p_L\big(q_L(W)\big) = \gamma_\Gamma(W) \cdot n : \mathcal{D}^\ell_\Omega / L^\ell_\Omega \to C^\ell(\Gamma) \tag{6.29}$$

is a well-defined isomorphism. This will be used for the V-component. For the W-component we need the following lemma.

Lemma 6.1. *Assume that the boundary Γ of Ω is $C^{\ell+1}$, $\ell \geq 0$. Then the map*

$$q_D(V) \mapsto p_D\big(q_D(V)\big) = \gamma_\Gamma(V) : \mathcal{D}^\ell / D^\ell_\Gamma \to C^\ell(\Gamma, \mathbf{R}^N) \tag{6.30}$$

is a well-defined isomorphism, where

$$p_D : \mathcal{D}^\ell \to \mathcal{D}^\ell / D^\ell_\Gamma \tag{6.31}$$

is the canonical surjection and D^ℓ_Γ is given by (6.28).

Proof. The proof follows by standard arguments. □

Theorem 6.4. *Let J be a real-valued shape functional. Assume that the conditions of Theorem* 6.3 (ii) *are satisfied and that the boundary Γ of the open domain Ω is $C^{\ell+1}$ for $\ell \geq 0$.*

(i) *The map*

$$\begin{cases} (v, w) \mapsto h_{D\times L}(v, w) = [h](p_D^{-1} v, p_L^{-1} w) \\ : C^\ell(\Gamma, \mathbf{R}^N) \times C^\ell(\Gamma) \to \mathbf{R} \end{cases} \tag{6.32}$$

is bilinear and continuous, and for all V and W in $\mathcal{D}^\ell(\mathbf{R}^N, \mathbf{R}^N)$

$$d^2 J(\Omega; V; W) = h_{D\times L}\Big(\gamma_\Gamma P(V), \big((\gamma_\Gamma W) \cdot n\big)\Big), \tag{6.33}$$

where $P(V)$ is a linear combination of derivatives of V up to order ℓ.

(ii) *This induces a vector distribution* $h(\Gamma \otimes \Gamma)$ *on* $C^\ell(\Gamma, \mathbf{R}^N) \otimes C^\ell(\Gamma)$ *of order* ℓ,

$$h(\Gamma \otimes \Gamma) : C^\ell(\Gamma, \mathbf{R}^N) \otimes C^\ell(\Gamma) \to \mathbf{R}, \tag{6.34}$$

such that for all V *and* W *in* $\mathcal{D}^\ell(\mathbf{R}^N, \mathbf{R}^N)$

$$\langle h(\Gamma \otimes \Gamma), (\gamma_\Gamma V) \otimes \big((\gamma_\Gamma W) \cdot n\big)\rangle = d^2 J(\Omega; V; W), \tag{6.35}$$

where $(\gamma_\Gamma V) \otimes \big((\gamma_\Gamma W) \cdot n\big)$ *is defined as the tensor product*

$$\Big((\gamma_\Gamma V) \otimes \big((\gamma_\Gamma W) \cdot n\big)\Big)_i (x, y) = (\gamma_\Gamma V_i)(x)\big((\gamma_\Gamma W) \cdot n\big)(y), \quad x, y \in \Gamma, \tag{6.36}$$

$V_i(x)$ *is the* i*th component of* $V(x)$, *and*

$$\forall y \in \Gamma, \quad \big(\gamma_\Gamma(W) \cdot n\big)(y) = (\gamma_\Gamma W)(y) \cdot n(y). \tag{6.37}$$

In the regular case the reader is referred to the early papers of J.-P. Zolésio [**24**, p. 434] and D. Bucur and J.-P. Zolésio [**12**].

Remark 6.4.
Finally, under the assumptions of Theorems 6.3 and 6.4

$$\begin{aligned} &d^2 J(\Omega; V; W) \\ &= \langle h(\Gamma \otimes \Gamma), (\gamma_\Gamma P(V(0))) \otimes ((\gamma_\Gamma W(0)) \cdot n)\rangle + \langle (g(\Gamma), (\gamma_\Gamma V'(0)) \cdot n)\rangle \end{aligned} \tag{6.38}$$

for all V in $\overrightarrow{V}^{m+1,\ell}$ and W in $\overrightarrow{V}^{m,\ell}$. □

Example 6.2.
Go back to the example of the domain integral in section 6.1,

$$J(\Omega) \stackrel{\text{def}}{=} \int_\Omega f \, dx.$$

For $V \in \mathcal{D}^1(\mathbf{R}^N, \mathbf{R}^N)$

$$dJ(\Omega; V) = \int_\Gamma f \, V \cdot n \, d\Gamma = \int_\Omega \operatorname{div}(fV) \, dx.$$

Using the domain expression with V in $\mathcal{D}^2(\mathbf{R}^N, \mathbf{R}^N)$ and W in $\mathcal{D}^1(\mathbf{R}^N, \mathbf{R}^N)$

$$dJ(\Omega_s(W); V) = \int_{\Omega_s(W)} \operatorname{div}(fV) \, dx$$

we readily get

$$d^2 J(\Omega; V; W) = \int_\Gamma \operatorname{div}(fV) \, W \cdot n \, d\Gamma = \int_\Omega \operatorname{div}\big[\operatorname{div}(fV) \, W\big] \, dx \tag{6.39}$$

and if Γ is C^1,

$$d^2 J(\Omega; V; W) = \int_\Gamma \operatorname{div} V \, W \cdot n \, d\Gamma \tag{6.40}$$

is continuous for pairs

$$(V, W) \in \mathcal{D}^1(\mathbf{R}^N, \mathbf{R}^N) \times \mathcal{D}^0(\mathbf{R}^N, \mathbf{R}^N) \text{ or } C^1(\Gamma, \mathbf{R}^N) \times C^0(\Gamma, \mathbf{R}^N).$$ □

6.5 Decomposition of $d^2J(\Omega; V(0), W(0))$

One important observation in the explicit computation of the second-order Eulerian semiderivative of the domain integral (6.1) in section 6.1 was the lack of symmetry and the appearance of the first half of the Lie bracket in (6.4). The same phenomenon was observed in the final form of the basic formula (6.10) for domain integrals (6.6) in Theorem 6.1, and also in section 6.3 for the derivative of the shape gradient (6.21) when it can be represented in the integral form (6.20). In this section perturbations of the identity will be used to show that $d^2J(\Omega; V(0), W(0))$ can be further decomposed into a symmetric term plus the gradient applied to the velocity $DV(0)W(0)$:

$$d^2J(\Omega; V(0), W(0)) = \langle d^2J_\Omega(0)V(0), W(0)\rangle + dJ(\Omega; DV(0)W(0)).$$

Furthermore the symmetrical term can be obtained by the velocity method

$$d^2J_\Omega(0; V(0), W(0)) = \frac{d}{dt} J\left(\Omega_t(W(0); V(0)\circ T_t^{-1}(W(0))\right)\Big|_{t=0}.$$

Theorem 6.5. *Let J be a real-valued shape functional and Θ be a Banach subspace of* $\mathrm{Lip}\,(\mathbf{R}^N; \mathbf{R}^N)$.

(i) *Given f, θ, and ξ in B_ε, assume that there exists $\tau > 0$ such that*

$$\forall t \in [0,\tau], \quad d^2J_\Omega(f + t\xi; \theta) \text{ exists.}$$

Then

$$\boxed{d^2J_\Omega(f;\theta;\xi) \text{ exists} \iff d^2J(\Omega_f; \mathcal{V}; W_\xi) \text{ exists}} \tag{6.41}$$

for $\Omega_f = [I+f](\Omega)$ and the velocity fields

$$\boxed{W_\xi(t) \overset{\text{def}}{=} \xi\circ[I+f+t\xi]^{-1}} \text{ and } \boxed{\mathcal{V}(t) \overset{\text{def}}{=} \theta\circ[I+f+t\xi]^{-1}.} \tag{6.42}$$

(ii) *If f, θ belong to $\mathcal{V}^{\ell+1}$ and ξ to $\mathcal{V}^\ell$, and $\mathcal{V}$ and W_ξ satisfy the conditions of Theorem* 6.2, *then*

$$\boxed{\begin{aligned} d^2J_\Omega(f;\theta;\xi) =\, & d^2J(\Omega_f; \theta\circ[I+f]^{-1}; \xi\circ[I+f]^{-1}) \\ & - dJ(\Omega_f; D(\theta\circ[I+f]^{-1})\,\xi\circ[I+f]^{-1}), \end{aligned}} \tag{6.43}$$

$$d^2J(\Omega_f;\theta;\xi) = d^2J_\Omega(f;\theta\circ[I+f];\xi\circ[I+f]) + dJ(\Omega_f; D\theta\,\xi). \tag{6.44}$$

(iii) *If V and W satisfy the conditions of Theorem* 6.2 *and V belongs to $\overrightarrow{\mathcal{V}}^{m+1,\ell+1}$, then*

$$\begin{aligned} & d^2J(\Omega; V(0); W(0)) \\ & = d^2J_\Omega(0; V(0); W(0)) + dJ(\Omega; DV(0)\,W(0)), \end{aligned} \tag{6.45}$$

$$\boxed{\begin{aligned} & d^2J(\Omega; V; W) \\ & = d^2J_\Omega(0; V(0); W(0)) + dJ(\Omega; V'(0) + DV(0)\,W(0)), \end{aligned}} \tag{6.46}$$

and
$$\boxed{\begin{aligned}&d^2 J_\Omega(0; V(0); W(0))\\ &= \frac{d}{dt} dJ(\Omega_t(W(0)); V(0)\circ T_t^{-1}(W(0)))\Big|_{t=0}.\end{aligned}} \tag{6.47}$$

Proof. (i) Assume that $d^2 J_\Omega(f;\theta;\xi)$ exists and consider the differential quotient
$$q(t) \overset{\text{def}}{=} \frac{1}{t}\left[dJ_\Omega(f+t\xi;\theta) - dJ_\Omega(f;\theta)\right] \to d^2 J_\Omega(f;\theta;\xi).$$
From Theorem 3.4 (ii),
$$dJ_\Omega(f+t\xi;\theta) = dJ([I+f+t\xi](\Omega); \theta\circ[I+f+t\xi]^{-1}).$$
Define
$$T_t \overset{\text{def}}{=} [I+f+t\xi]\circ[I+f]^{-1}, \quad \mathcal{V}(t) \overset{\text{def}}{=} \theta\circ[I+f+t\xi]^{-1}.$$
From Theorem 4.1 in Chapter 4, $T_t = T_t(W_\xi)$ for the velocity field
$$W_\xi(t) \overset{\text{def}}{=} \frac{\partial T_t}{\partial t}\circ T_t^{-1} = \xi\circ[I+f+t\xi]^{-1}.$$
Therefore
$$\begin{aligned}dJ_\Omega(f+t\xi;\theta) &= dJ(T_t(W_\xi)(\Omega_f); \mathcal{V}(t)),\\ q(t) &= \frac{1}{t}\left[dJ(T_t(W_\xi)(\Omega_f);\mathcal{V}(t)) - dJ(\Omega_f;\mathcal{V}(0))\right] \to d^2(\Omega_f;\mathcal{V};W_\xi)\end{aligned}$$
since $q(t)$ converges as t goes to 0. The converse is obvious.

(ii) This is now a direct consequence of Theorem 6.2 and the fact that
$$\begin{aligned}&\mathcal{V}(t) = \theta\circ[I+f]^{-1}\circ T_t^{-1}(W_\xi)\\ \Rightarrow\ &\mathcal{V}'(0) = -D(\theta\circ[I+f]^{-1})\,\xi\circ[I+f]^{-1}.\end{aligned}$$

(iii) From Theorem 6.2 and part (ii) with $f=0$, $\theta = V(0)$, and $\xi = W(0)$. □

If $D^2 J_\Omega(f)$ exists in a neighborhood of $f=0$ and if it is continuous at $f=0$, then $D^2 J_\Omega(0)$ is symmetrical and this completes the decomposition of the shape gradient into a symmetrical operator and the gradient applied to the first half of the Lie bracket $[V(0), W(0)]$:
$$\begin{aligned}&d^2 J(\Omega; V(0); W(0))\\ &= \langle D^2 J_\Omega(0)\,V(0), W(0)\rangle + \langle G(\Omega), DV(0)\,W(0)\rangle.\end{aligned}$$

But this is not the end of the story. From the computation of the Hessian of the domain integral (6.4) in section 6.1,
$$\begin{aligned}&d^2 J_{V,W}\\ &= \int_\Gamma \left(\frac{\partial f}{\partial n} + H\,f\right) v_n\,w_n + \boxed{f\ \left(D^2 b\,v_\Gamma\cdot w_\Gamma - v_\Gamma\cdot\nabla_\Gamma w_n - w_\Gamma\cdot\nabla_\Gamma v_n\right)}\\ &\qquad + f\,DVW\cdot n\,d\Gamma,\end{aligned}$$

the result of the computation (6.10) in section 6.2,

$$\begin{aligned}&\int_\Gamma \operatorname{div}\,(f\,V(0))\;W(0)\cdot n\,d\Gamma\\&=\int_\Gamma\left(\frac{\partial f}{\partial n}+H\,f\right)v_n\,w_n+\boxed{f\,\left(D^2bv_\Gamma\cdot w_\Gamma-v_\Gamma\cdot\nabla_\Gamma w_n-w_\Gamma\cdot\nabla_\Gamma v_n\right)}\\&\qquad\qquad+f\,DVW\cdot n\,d\Gamma,\end{aligned}$$

and expression (6.21) of the derivative of (6.20) in section 6.3,

$$\begin{aligned}d^2J(V;W)=\int_\Gamma &Q'_{w_n}(0)\,v_n+Q(0)\,V'(0)\cdot n+\left(\frac{\partial Q(0)}{\partial n}+H\,Q(0)\right)v_n\,w_n\\&+\boxed{Q(0)\,\left(D^2b\,v_\Gamma\cdot w_\Gamma-v_\Gamma\cdot\nabla_\Gamma w_n-w_\Gamma\cdot\nabla_\Gamma v_n\right)}\\&+Q(0)\,DVW\cdot n\,d\Gamma,\end{aligned}$$

it is readily seen that the symmetrical term $\langle D^2J_\Omega(0)\,V(0),W(0)\rangle$ further decomposes into a symmetrical term which depends only on the normal components v_n and w_n of $V(0)$ and $W(0)$ and another symmetrical term, which is boxed in the above expressions. This last term is the same in all expressions and depends only on the trace of $G(0)$ (here of f and $Q(0)$) on Γ and the group of terms

$$\boxed{D^2b\,v_\Gamma\cdot w_\Gamma-v_\Gamma\cdot\nabla_\Gamma w_n-w_\Gamma\cdot\nabla_\Gamma v_n}\tag{6.48}$$

involving v_Γ, w_Γ, and tangential derivatives of v_n and w_n on Γ.

It would be interesting to further investigate this structure. In the context of a domain optimization problem, if the shape gradient is zero, both the term (6.48) and the first half of the Lie bracket will be multiplied by zero. Thus they will not contribute to the Hessian which will reduce to the symmetrical part that depends only on v_n and w_n; that is, for autonomous velocity fields V and W,

$$\int_\Gamma\left(\frac{\partial f}{\partial n}+H\,f\right)v_n\,w_n\,d\Gamma,$$

$$\int_\Gamma Q'_{w_n}(0)\,v_n+\left(\frac{\partial Q(0)}{\partial n}+H\,Q(0)\right)v_n\,w_n\,d\Gamma,$$

and the term

$$\int_\Gamma Q'_{w_n}(0)\,v_n\,d\Gamma=\int_\Gamma Q'_{v_n}(0)\,w_n\,d\Gamma$$

is symmetrical. For earlier results on the structure of the second-order shape derivative the reader is referred to J.-P. ZOLÉSIO [**24**, p. 434] and D. BUCUR and J.-P. ZOLÉSIO [**12**]. Of course, the task of establishing similar results in the constrained case ($D\neq\mathbf{R}^{\mathrm{N}}$) remains to be done.

Chapter 10

Shape Gradients under a State Equation Constraint

1 Introduction

When a shape functional depends on the solution of a boundary value problem defined on the underlying domain, it is said to be *constrained*. This special type of constraint is to be distinguished from constraints on the geometry such as the volume, perimeter, and curvatures. Using the generic terminology of *control theory* the solution of the boundary value problem will be called the *state* and its corresponding equation or inequality the *state equation* or *inequality*. The domains will be identified with the *controls* and the constraints on the domains with the *control constraints*. Additional constraints on the solution of the state equation are usually called *state constraints*. Such problems have received a considerable amount of attention over the last century, and a rich and abundant literature can be found in mechanics, control theory, and optimization. Treating the state equation as an equality constraint and using *Lagrange multipliers* naturally yields a *dual variable* which is solution of the *adjoint equation* of the linearized state equation. This dual variable is called the *adjoint state*.

Of course under appropriate differentiability assumptions with respect to the state and the control variables, necessary conditions can be obtained within the classical framework of the *calculus of variations*. Probably one of the most influential contributions of the last half of the 20th century to *optimal control theory* was made by L. Pontryagin, V. Boltyanskii, R. Gamkrelidze, and E. Mishchenko [**1**] in the 1960s when they showed that the differentiability with respect to the control could be relaxed and replaced by the pointwise maximization with respect to the control variable of the *Hamiltonian* constructed from the objective function and the coupled state and adjoint state equations, the so-called *Maximum Principle*.

One of the important technical advantages of the control theory approach is to avoid the differentiation of the state with respect to the control. This is true not only for the characterization of optimal controls but also to obtain explicit expressions of the derivative of an objective function constrained by a *state equation* with respect to the control. Using a Lagrangian or Hamiltonian formulation, the derivative of the constrained objective function with respect to the control is typically equal to the "partial" derivative of the Lagrangian or the Hamiltonian with respect to the control, where the adjoint variable is a solution of an appropriate adjoint state equation that is coupled with the state equation.

This chapter will concentrate on two generic examples often encountered in shape optimization. The first one is associated with the so-called *compliance problems*, where the shape functional is equal to the minimum of a domain-dependent energy functional. The special feature of such functionals is that the adjoint state coincides with the state. This obviously leads to considerable simplifications in the analysis. In that case it will be shown that theorems on the differentiability of the minimum of a functional with respect to a real parameter readily give explicit expressions of the Eulerian semiderivative even when the minimizer is not unique. The second one will deal with shape functionals that can be expressed as the saddle point of some appropriate Lagrangian. As in the first example, theorems on the differentiability of the saddle point of a functional with respect to a real parameter readily give explicit expressions of the Eulerian semiderivative even when the solution of the saddle point equations is not unique.

Avoiding the differentiation of the state equation with respect to the domain is particularly advantageous in shape problems. Here the state variable (the solution of the boundary value problem) lives in a function space (Banach, Hilbert, Sobolev) which depends on the control (*underlying variable domain*)! Thus the notion of derivative of the state with respect to the domain is more delicate. In this chapter two techniques will be presented to get around this difficulty: *function space parametrization* and *function space embedding*.

Function space parametrization consists in transporting the functions on variable domains back onto the initial domain, where they can be compared. It is related to the notion of *material derivative* in continuum and structural mechanics. This will be illustrated by computing the volume and boundary integral expressions of the Eulerian semiderivative of a simple shape functional for the homogeneous Dirichlet and Neumann boundary value problems by the theorems on the differentiation of a minimum and a saddle point.

Function space embedding consists in constructing extensions of the domain-dependent boundary value problems to a larger fixed domain. For homogeneous Dirichlet boundary conditions, it is sufficient to consider extensions by zero to $\mathbf{R}^{\mathrm{N}}$. The nonhomogeneous case requires the introduction of a multiplier to take into account the boundary condition. This technique will be illustrated by computing the boundary integral expression of the Eulerian semiderivative of a simple shape functional for the nonhomogeneous Dirichlet boundary value problem by the theorems on the differentiation of a saddle point.

In addition to the generic example, the theorem on the differentiation of an infimum with respect to a parameter will be applied to the example of the buckling of columns considered in section 5 of Chapter 5. In section 3 an explicit expression of the semiderivative of *Euler's buckling load* with respect to the cross-sectional area will be given. A necessary and sufficient condition will also be given to characterize the maximum Euler's buckling load with respect to a family of cross-sectional areas. The theory is further illustrated in section 4 by providing the semiderivative of the first eigenvalue of several boundary value problems over a bounded open domain: Laplace equation, bi-Laplace equation, linear elasticity. In general the first eigenvalue is not simple over an arbitrary bounded open domain and the eigenvalue is not differentiable; yet the main theorem provides explicit domain expressions for a bounded open domain and boundary expressions of the semiderivatives for a sufficiently smooth bounded open domain.

2 Min Formulation

2.1 An Illustrative Example and a Shape Variational Principle

Let Ω be a bounded open domain in $\mathbf{R}^{\mathrm{N}}$ with a smooth boundary Γ. Let $y = y(\Omega)$ be the solution of the Dirichlet problem

$$-\triangle y = f \text{ in } \Omega, \quad y = 0 \text{ on } \Gamma, \tag{2.1}$$

where f is a fixed function in $H^1(\mathbf{R}^{\mathrm{N}})$. The solution of (2.1) is the minimizing element in $H_0^1(\Omega)$ of the energy functional

$$E(\Omega, \varphi) = \int_\Omega \left[\frac{1}{2}|\nabla\varphi|^2 - f\varphi\right] dx. \tag{2.2}$$

Introduce the shape function

$$J(\Omega) \stackrel{\text{def}}{=} \inf_{\varphi\in H_0^1(\Omega)} E(\Omega, \varphi) = -\int_\Omega \frac{1}{2}|\nabla y|^2 \, dx. \tag{2.3}$$

We want to show that

$$dJ(\Omega; V) = -\frac{1}{2}\int_\Gamma \left|\frac{\partial y}{\partial n}\right|^2 V \cdot n \, d\Gamma. \tag{2.4}$$

This example is the prototype of a *free boundary* problem[1] which can be obtained from the following *shape variational* principle:

$$\forall V, \quad dJ(\Omega; V) = 0. \tag{2.5}$$

It yields the extra boundary condition[2]

$$\frac{\partial y}{\partial n} = 0 \text{ on } \Gamma. \tag{2.6}$$

Equations (2.1) and (2.6) characterize a free boundary problem. It is the simplest example of a large family of problems where the shape function is an extremal of a natural internal energy (cf. section 7 of Chapter 5). They correspond to the so-called *compliance problems* in elasticity theory. They also occur in fracture theory and image segmentation. The first-order variation of this shape functional yields the extra boundary condition which is characteristic of a free boundary problem.

[1]This usually comes with other constraints, such as a volume constraint on Ω and/or an area constraint on Γ.

[2]With a volume equality constraint we would get that $|\partial y/\partial n|^2$ is equal to a constant on Γ.

2.2 Function Space Parametrization

To compute the first-order derivative of $J(\Omega)$ we perturb the bounded open domain Ω by a velocity field V, which generates the family of transformations $\{T_t : 0 \leq t \leq \tau\}$ of $\mathbf{R}^{\mathrm{N}}$ and the family of domains $\{\Omega_t = T_t(\Omega) : 0 \leq t \leq \tau\}$. At t,

$$J(\Omega_t) = \inf_{\varphi \in H_0^1(\Omega_t)} E(\Omega_t, \varphi) \tag{2.7}$$

and the minimizing element $y_t = y(\Omega_t)$ is the solution of the Dirichlet problem

$$-\triangle y_t = f \text{ in } \Omega_t, \quad y_t = 0 \text{ on } \Gamma_t, \tag{2.8}$$

where Γ_t is the boundary of Ω_t. We want to compute the derivative

$$dj(0) = \lim_{t \searrow 0} \frac{j(t) - j(0)}{t} \tag{2.9}$$

of the function

$$j(t) \stackrel{\text{def}}{=} J(\Omega_t). \tag{2.10}$$

We need a theorem that would give the derivative of an infimum with respect to a parameter $t \geq 0$ at $t = 0$. The difficulty here is that the function space $H^1(\Omega_t)$ depends on the parameter t. To get around this and obtain an infimum with respect to a function space that is independent of t, we introduce the following parametrization:

$$H_0^1(\Omega_t) = \left\{\varphi \circ T_t^{-1} : \varphi \in H_0^1(\Omega)\right\}. \tag{2.11}$$

Notice that since T_t is a homeomorphism, it transforms the open domain Ω into the open domain Ω_t and sends the boundary Γ of Ω onto the boundary Γ_t of Ω_t. In particular, when V is sufficiently smooth for all φ in $H_0^1(\Omega)$, $\varphi \circ T_t^{-1} \in H_0^1(\Omega_t)$, and conversely, for all ψ in $H_0^1(\Omega_t)$, $\psi \circ T_t \in H_0^1(\Omega)$. This parametrization does not affect the value of the minimum $J(\Omega_t)$ but changes the functional E

$$J(\Omega_t) = \inf_{\varphi \in H_0^1(\Omega)} E(T_t(\Omega), \varphi \circ T_t^{-1}). \tag{2.12}$$

This parametrization is typical in *shape analysis*. It amounts to introducing the new energy functional

$$\widetilde{E}(t, \varphi) = E(T_t(\Omega), \varphi \circ T_t^{-1}), \quad \varphi \in H_0^1(\Omega). \tag{2.13}$$

Our objective in the next section is to compute the limit (2.9) for

$$j(t) = \inf_{\varphi \in H_0^1(\Omega)} \widetilde{E}(t, \varphi). \tag{2.14}$$

This will be done in section 2.3. Before closing, it is interesting to characterize the minimizing element y^t in $H_0^1(\Omega)$ of

$$\widetilde{E}(t, \varphi) = \int_{\Omega_t} \left[\frac{1}{2}|\nabla(\varphi \circ T_t^{-1})|^2 - f(\varphi \circ T_t^{-1})\right] dx, \tag{2.15}$$

which is the solution of the following variational equation: find $y^t \in H_0^1(\Omega)$ such that for all $\varphi \in H_0^1(\Omega)$

$$\int_{\Omega_t} \left\{\nabla(y^t \circ T_t^{-1}) \cdot \nabla(\varphi \circ T_t^{-1}) - f(\varphi \circ T_t^{-1})\right\} dx = 0. \tag{2.16}$$

Compare this expression with the characterization of the minimizing element y_t of $E(\Omega_t, \varphi)$ on $H_0^1(\Omega_t)$: find $y_t \in H_0^1(\Omega_t)$ such that for all $\varphi \in H_0^1(\Omega_t)$

$$\int_{\Omega_t} \{\nabla y_t \cdot \nabla \varphi - f\varphi\} \, dx = 0. \tag{2.17}$$

It is easy to verify that

$$y_t = y^t \circ T_t^{-1} \quad \text{and} \quad y^t = y_t \circ T_t. \tag{2.18}$$

So y^t is the solution y_t of (2.8) transported back onto the fixed domain Ω by the change of variables induced by T_t.

In view of this expression, (2.15) can be rewritten on the fixed domain Ω as

$$\widetilde{E}(t,\varphi) = \int_\Omega \frac{1}{2}\,(A(t)\nabla\varphi)\cdot\nabla\varphi - (f\circ T_t)\,\varphi\, J_t\, dx, \tag{2.19}$$

where for t in $[0,\tau]$ small

$$DT_t = \text{Jacobian matrix of } T_t, \tag{2.20}$$

$$J_t = |\det DT_t| = \det DT_t \text{ for } t \geq 0 \text{ small}, \tag{2.21}$$

$$A(t) = J_t[DT_t]^{-1}\ {}^*[DT_t]^{-1}. \tag{2.22}$$

With this change of variables y^t is now characterized by the variational equation

$$\begin{cases} y^t \in H_0^1(\Omega) \text{ and } \forall \varphi \in H_0^1(\Omega), \\ \displaystyle\int_\Omega A(t)\nabla y^t \cdot \nabla\varphi - J_t(f\circ T_t)\varphi\, dx = 0. \end{cases} \tag{2.23}$$

2.3 Differentiability of a Minimum with Respect to a Parameter

Consider a functional

$$G\colon [0,\tau] \times X \to \mathbf{R} \tag{2.24}$$

for some $\tau > 0$ and some set X. For each t in $[0,\tau]$ define

$$g(t) \overset{\text{def}}{=} \inf\{G(t,x) : x \in X\}, \tag{2.25}$$

$$X(t) \overset{\text{def}}{=} \{x \in X : G(t,x) = g(t)\}. \tag{2.26}$$

The objective is to characterize the limit

$$dg(0) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{g(t) - g(0)}{t} \tag{2.27}$$

when $X(t)$ is not empty for $0 \leq t \leq \tau$.

When $X(t) = \{x^t\}$ is a singleton, $0 \le t \le \tau$, and the derivative

$$\dot{x} = \lim_{t \searrow 0} \frac{x^t - x^0}{t} \tag{2.28}$$

of x is known, then it is easy to obtain $dg(0)$ under appropriate differentiability of the functional G with respect to t and x. When $\dot{x}$ is not readily available or when the sets $X(t)$ are not singletons, this direct approach fails or becomes very intricate. In this section we present a theorem that gives an explicit expression for $dg(0)$, the derivative of the min of the functional G with respect to t at $t = 0$. Its originality is that the differentiability of x^t is replaced by a continuity assumption on the set-valued function and the existence of the partial derivative of the functional G with respect to the parameter t. In other words this technique does not require a priori knowledge of the derivative $\dot{x}$ of the minimizing elements x^t with respect to t.

Theorem 2.1. *Let X be an arbitrary set, let $\tau > 0$, and let $G\colon [0,\tau] \times X \to \mathbf{R}$ be a well-defined functional. Assume that the following conditions are satisfied:*

(H1) *for all $t \in [0,\tau]$, $X(t) \neq \varnothing$;*

(H2) *for all x in $\bigcup_{t\in[0,\tau]} X(t)$, $\partial_t G(t,x)$ exists everywhere in $[0,\tau]$;*

(H3) *there exists a topology $\mathcal{T}_X$ on X such that for any sequence $\{t_n\} \subset\,]0,\tau]$, $t_n \to t_0 = 0$, there exist $x_0 \in X(0)$ and a subsequence $\{t_{n_k}\}$ of $\{t_n\}$, and for each $k \ge 1$, there exists $x_{n_k} \in X(t_{n_k})$ such that*

(i) *$x_{n_k} \to x^0$ in the $\mathcal{T}_X$-topology and*

(ii)

$$\liminf_{\substack{k\to\infty \\ t\searrow 0}} \partial_t G(t, x_{n_k}) \ge \partial_t G(0, x^0);$$

(H4) *for all x in $X(0)$, the map $t \mapsto \partial_t G(t,x)$ is upper semicontinuous at $t = 0$.*

Then there exists $x^0 \in X(0)$ such that

$$dg(0) = \lim_{t\searrow 0} \frac{g(t) - g(0)}{t} = \inf_{x\in X(0)} \partial_t G(0,x) = \partial_t G(0,x^0). \tag{2.29}$$

Remark 2.1.
In the literature condition (H3) (i) is known as *sequential semicontinuity for set-valued functions.* When $X(0)$ is a singleton $\{x_0\}$ we readily get $dg(0) = \partial_t G(0, x_0)$. □

Remark 2.2.
This theorem, and in particular the last part of property (2.29), extends a former result by B. Lemaire [**1**, Thm. 2.1, p. 38], where sequential compactness of the set X was assumed. It also completes and extends Theorem 1 in J.-P. Zolésio [**7**] and M. C. Delfour and J.-P. Zolésio [**3**]. □

Proof of Theorem 2.1. (i) We first establish upper and lower bounds to the differential quotient

$$\frac{\Delta(t)}{t}, \quad \Delta(t) \stackrel{\text{def}}{=} g(t) - g(0).$$

Choose arbitrary x_0 in $X(0)$ and x_t in $X(t)$. Then, by definition,

$$\begin{aligned} G(t, x_t) = \quad & g(t) \leq G(t, x_0), \\ -G(0, x_t) \leq & -g(0) = -G(0, x_0). \end{aligned}$$

Add the above two inequalities to obtain

$$G(t, x_t) - G(0, x_t) \leq \Delta(t) \leq G(t, x_0) - G(0, x_0).$$

By assumption (H2), there exist θ_t, $0 < \theta_t < 1$, and α_t, $0 < \alpha_t < 1$, such that

$$\begin{aligned} G(t, x_t) - G(0, x_t) &= t \partial_t G(\theta_t t, x_t), \\ G(t, x_0) - G(0, x_0) &= t \partial_t G(\alpha_t t, x_0), \end{aligned}$$

and by dividing by $t > 0$

$$\partial_t G(\theta_t t, x_t) \leq \frac{\Delta(t)}{t} \leq \partial_t G(\alpha_t t, x_0). \tag{2.30}$$

(ii) Define

$$\underline{d}g(0) = \liminf_{t \searrow 0} \frac{\Delta(t)}{t}, \quad \overline{d}g(0) = \limsup_{t \searrow 0} \frac{\Delta(t)}{t}.$$

There exists a sequence $\{t_n : 0 < t_n \leq \tau\}$, $t_n \to 0$, such that

$$\lim_{n \to \infty} \frac{\Delta(t_n)}{t_n} = \underline{d}g(0).$$

By assumption (H3), there exist $x^0 \in X(0)$ and a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ and for each $k \geq 1$, there exists $x_{n_k} \in X(t_{n_k})$ such that $x_{n_k} \to x^0$ in $\mathcal{T}_X$ and

$$\liminf_{\substack{t \searrow 0 \\ k \to \infty}} \partial_t G(t, x_{n_k}) \geq \partial_t G(0, x^0).$$

So, from the first part of the estimate (2.30) for $t = t_{n_k}$,

$$\partial_t G(\theta_{t_{n_k}} t_{n_k}, x_{n_k}) \leq \frac{\Delta(t_{n_k})}{t_{n_k}}$$

and

$$\partial_t G(0, x^0) \leq \liminf_{k \to \infty} \partial_t G(\theta_{t_{n_k}} t_{n_k}, x_{n_k}) \leq \lim_{k \to \infty} \frac{\Delta(t_{n_k})}{t_{n_k}} = \underline{d}g(0).$$

Therefore,

$$\exists x^0 \in X(0), \quad \partial_t G(0, x^0) \leq \underline{d}g(0),$$

and

$$\inf_{x\in X(0)} \partial_t G(0,x) \le \partial_t G(0,x^0) \le \underline{d}g(0).$$

From the second part of (2.30) and assumption (H4) we also obtain

$$\begin{aligned} \forall x \in X(0), \quad \partial_t G(0,x) \ge \overline{d}g(0),\\ \overline{d}g(0) \le \inf_{x\in X(0)} \partial_t G(0,x), \end{aligned} \tag{2.31}$$

and necessarily

$$\inf_{x\in X(0)} \partial_t G(0,x) = \underline{d}g(0) = \overline{d}g(0) = \inf_{x\in X(0)} \partial_t G(0,x).$$

In particular, from (2.3) and (2.31)

$$\partial_t G(0,x^0) = dg(0) = \inf_{x\in X(0)} \partial_t G(0,x)$$

and x^0 is a minimizing point of $\partial_t G(0,\cdot)$. □

2.4 Application of the Theorem

Our example has a unique minimizing point y^t for $t \ge 0$ small. Here $X = H_0^1(\Omega)$, $X(t) = \{y^t\}$, and it is sufficient to establish the continuity of the map $t \mapsto y^t$ at $t = 0$ for an appropriate topology on $H_0^1(\Omega)$.

We now check assumptions (H1) to (H4). Assume that $V \in C^0\big([0,\tau]; \mathcal{D}^2(\mathbf{R}^N, \mathbf{R}^N)\big)$ and that $f \in H^1(\mathbf{R}^N)$. Choose $\tau > 0$ small enough such that

$$J_t = |J_t|, \quad 0 \le t \le \tau, \tag{2.32}$$

and that there exist constants $0 < \alpha < \beta$ such that

$$\forall \xi \in \mathbf{R}^N, \quad \alpha|\xi|^2 \le A(t)\,\xi\cdot\xi \le \beta|\xi|^2, \text{ and } \alpha \le J_t \le \beta. \tag{2.33}$$

Since the bilinear form associated with (2.23) is coercive, there exists a unique solution y^t to (2.23) and

$$\forall t \in [0,\tau], \quad X(t) = \{y^t\} \ne \varnothing. \tag{2.34}$$

So assumption (H1) is satisfied. To check (H2) use expression (2.19) and compute

$$\partial_t \widetilde{E}(t,\varphi) = \int_\Omega \left\{ \frac{1}{2}\,[A'(t)\nabla\varphi]\cdot\nabla\varphi - [\operatorname{div} V_t\,(f\circ T_t) + J_t \nabla f\cdot V_t]\,\varphi \right\} dx, \tag{2.35}$$

where

$$V_t(X) = V(t, T_t(X)), \quad DV_t(X) = DV(t, T_t(X)), \tag{2.36}$$

$$A'(t) = (\operatorname{div} V_t)I - (DV_t)^* - DV_t. \tag{2.37}$$

By assumptions on V and f, $\partial_t \widetilde{E}(t,\varphi)$ exists everywhere in $[0,\tau]$ for all φ in $H_0^1(\Omega)$ and assumption (H2) is satisfied.

To check assumption (H3)(i) we first show that $\{y^t\}$ is bounded in $H_0^1(\Omega)$. From (2.33)

$$\begin{aligned}\alpha\|\nabla y^t\|^2_{L^2(\Omega)} &\le \int_\Omega A(t)\nabla y^t\cdot\nabla y^t\,dx\\ &= \int_\Omega J_t(f\circ T_t)\,y^t\,dx \le \|J_t\,f\circ T_t\|_{L^2(\Omega)}\,\|y^t\|_{L^2(\Omega)}.\end{aligned}\tag{2.38}$$

By using the norm

$$\|\varphi\|_{H_0^1(\Omega)} = \|\nabla\varphi\|_{L^2(\Omega)}$$

and the continuous injection of $H_0^1(\Omega)$ into $L^2(\Omega)$,

$$\exists c>0,\quad \|\varphi\|_{L^2(\Omega)} \le c\|\varphi\|_{H_0^1(\Omega)}.$$

So from (2.38)

$$\|y^t\|_{H_0^1(\Omega)} \le \frac{c}{\alpha}\|J_t f\circ T_t\|_{L^2(\Omega)}.\tag{2.39}$$

But $J_t\to 1$ as $t\to 0$ and $f\circ T_t\to f$ in $L^2(\Omega)$ by the following lemma.

Lemma 2.1. *Assume that $V\in C^0\big([0,\tau];\mathcal{D}^1(\mathbf{R}^{\mathrm{N}},\mathbf{R}^{\mathrm{N}})\big)$ satisfies assumptions* (V) *of Chapter* 4 *and that $f\in L^2(\mathbf{R}^{\mathrm{N}})$. Then*

$$\lim_{t\searrow 0} f\circ T_t = f \text{ and } \lim_{t\searrow 0} f\circ T_t^{-1} = f \text{ in } L^2(\mathbf{R}^{\mathrm{N}}).\tag{2.40}$$

So by (2.39) y^t is bounded:

$$\exists c>0,\quad \sup_{t\in[0,\tau]}\|y^t\|_{H_0^1(\Omega)} \le c.\tag{2.41}$$

The next step is to prove the continuity by subtracting (2.23) at $t>0$ from (2.23) at $t=0$:

$$\begin{aligned}\int_\Omega \nabla y^t\cdot\nabla\varphi\,dx + \int_\Omega (A(t)-I)\nabla y^t\cdot\nabla\varphi\,dx\\ = \int_\Omega f\varphi\,dx + \int_\Omega [J_t(f\circ T_t)-f]\varphi\,dx,\\ \int_\Omega \nabla y\cdot\nabla\varphi\,dx = \int_\Omega f\varphi\,dx.\end{aligned}$$

Subtract and set $\varphi = y^t - y$:

$$\begin{aligned}&\int_\Omega |\nabla(y^t-y)|^2\,dx\\ &= -\int_\Omega (A(t)-I)\nabla y^t\cdot\nabla(y^t-y) + [J_t(f\circ T_t)-f](y^t-y)\,dx\\ &\le |A(t)-I|\|\nabla y^t\|_{L^2(\Omega)}\|\nabla(y^t-y)\|_{L^2(\Omega)}\\ &\quad + \|J_t\,f\circ T_t - f\|_{L^2(\Omega)}\|y^t-y\|_{L^2(\Omega)}\end{aligned}$$

and

$$\|y^t - y\|_{H_0^1(\Omega)} \le c\,\{|A(t) - I| + \|J_t\, f \circ T_t - f\|_{L^2(\Omega)}\}.$$

But $A(t) - I \to 0$ and $J_t\, f \circ T_t \to f$ in $L^2(\Omega)$, and finally $y^t \to y$ in $H_0^1(\Omega)$. So assumption (H3)(i) is satisfied for $H_0^1(\Omega)$-strong.

For assumption (H3)(ii) we have

$$\partial_t G(t, \varphi) = \int_\Omega \left\{ \frac{1}{2} \left[A'(t)\nabla\varphi\right] \cdot \nabla\varphi - \left[\operatorname{div} V_t (f \circ T_t) + J_t \nabla f \circ V_t\right] \varphi \right\} dx$$

and for φ in $H_0^1(\Omega)$, f in $H^1(\mathbf{R}^{\mathrm{N}})$, and V in $C^0([0, \tau]; \mathcal{D}^2(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}))$

$$\begin{aligned} &\partial_t G(t, \varphi) - \partial_t G(0, y) \\ &= \int_\Omega \left\{ \frac{1}{2} (A'(t) - A'(0)) \nabla\varphi \cdot \nabla\varphi \right. \\ &\qquad \left. - \left[\operatorname{div} V_t (f \circ T_t) - J_t \nabla f \cdot V_t - \operatorname{div} V(0) f + \nabla f \cdot V(0)\right] \varphi \right\} dx \\ &\quad + \int_\Omega \frac{1}{2} \left\{ A'(0) \nabla\varphi \cdot \nabla\varphi - A'(0) \nabla y \cdot \nabla y \right\} dx. \end{aligned}$$

As φ goes to y in $H_0^1(\Omega)$ and $t \to 0$, the first term converges to zero since φ is bounded and

$$\begin{gathered} A'(t) \to A'(0), \\ \operatorname{div} V_t (f \circ T_t) \to \operatorname{div} V(0) f \text{ in } L^2(\Omega), \\ J_t \nabla f \cdot V_t \to \nabla f \cdot V(0) \text{ in } L^2(\Omega). \end{gathered}$$

The second term is continuous with respect to φ in $H_0^1(\Omega)$ and goes to zero as $\varphi \to y$ in $H_0^1(\Omega)$. So assumption (H3)(ii) is satisfied.

Finally, (H4) is satisfied since

$$t \mapsto \partial_t \widetilde{E}(t, y) = \int_\Omega \left\{ \frac{1}{2} A'(t) \nabla y \cdot \nabla y - \left[\operatorname{div} V_t f + \nabla f \cdot V_t\right] y \right\} dx \tag{2.42}$$

is continuous in $[0, \tau]$. So all the assumptions of Theorem 2.1 are satisfied and for V in $C^0([0, \tau]; \mathcal{D}^2(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}))$ and f in $H^1(\mathbf{R}^{\mathrm{N}})$,

$$dJ(\Omega; V) = \int_\Omega \left\{ \frac{1}{2} A'(0) \nabla y \cdot \nabla y - \left[\operatorname{div} V(0) f + \nabla f \cdot V(0)\right] y \right\} dx. \tag{2.43}$$

For V autonomous, (2.43) is continuous with respect to the space $\mathcal{D}^1(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$ and the shape gradient is of order 1. We know by the structure Theorem 3.6 of Chapter 9 that for Ω open and a C^2-boundary Γ,

$$\exists g(\Gamma) \in \mathcal{D}^1(\Gamma)', \quad dJ(\Omega; V) = \langle g(\Gamma), V \rangle_{\mathcal{D}^1(\Gamma)}.$$

The characterization of $g(\Gamma)$ will be given in the next section.

We complete this section with the proof of Lemma 2.1.

Proof of Lemma 2.1. (i) By density of $\mathcal{D}^1(\mathbf{R}^\mathrm{N})$ in $L^2(\mathbf{R}^\mathrm{N})$ for all $\varepsilon > 0$, there exists f_ε such that

$$\|f - f_\varepsilon\|_{L^2} < \frac{\varepsilon}{\max\{J_t^{-1} : 0 \le t \le \tau\}} \quad (\le \varepsilon\alpha \le \varepsilon).$$

Hence

$$\|f \circ T_t - f\| \le \|f_\varepsilon \circ T_t - f_\varepsilon\| + \|f \circ T_t - f_\varepsilon \circ T_t\| + \|f - f_\varepsilon\|.$$

The last term is less than ε and the middle term can be rewritten after a change of variables as

$$\int_{\mathbf{R}^\mathrm{N}} |f \circ T_t - f_\varepsilon \circ T_t|^2 dx = \int_{\mathbf{R}^\mathrm{N}} |f - f_\varepsilon|^2 J_t^{-1}\, dx \le \varepsilon^2.$$

A function f_ε in $\mathcal{D}^1(\mathbf{R}^\mathrm{N})$ has a compact support and is uniformly Lipschitz continuous, that is,

$$\exists c > 0, \forall x, y \in \mathbf{R}^\mathrm{N}, \quad |f_\varepsilon(y) - f_\varepsilon(x)| \le c|y - x|.$$

Thus for all X in $\mathbf{R}^\mathrm{N}$

$$|f_\varepsilon(T_t(X)) - f_\varepsilon(X)| \le c|T_t(X) - X|. \tag{2.44}$$

Now

$$T_t(X) = X + \int_0^t V(s, X)\, ds + \int_0^t [V(s, T_s(X)) - V(s, X)]\, ds.$$

Since $V \in C^0([0, \tau]; \mathcal{D}^1(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N}))$ is also uniformly Lipschitz continuous by assumption (V)

$$\exists c > 0, \forall (X, Y), \forall s \in [0, \tau], \quad |V(s, Y) - V(s, X)| \le c\, |Y - X|,$$

and for all t in $[0, \tau]$

$$|T_t(X) - X| \le t\|V(\cdot, X)\|_{C^0([0,\tau];\mathbf{R}^\mathrm{N})} + \int_0^t c|T_s(X) - X|\, ds.$$

It is now easy to verify that there exists a constant $c > 0$ such that

$$\max_{s\in[0,t]} |T_s(X) - X| \le ct\|V(\cdot, X)\|_{C^0([0,\tau];\mathbf{R}^\mathrm{N})}. \tag{2.45}$$

Finally, in view of (2.44) and (2.45) the term

$$\int_{\mathbf{R}^\mathrm{N}} |f_\varepsilon(T_t(X)) - f_\varepsilon(X)|^2 dX \le ct^2 \int_{\mathbf{R}^\mathrm{N}} \|V(\cdot, X)\|_{C^0}^2\, dX,$$

$$\begin{aligned}\int_{\mathbf{R}^\mathrm{N}} |f_\varepsilon(T_t(X)) - f_\varepsilon(X)|^2 dX &= \int_{K_\varepsilon} |f_\varepsilon(T_t(X)) - f_\varepsilon(X)|^2\, dX \\ &\le c\, t^2 \int_{K_\varepsilon} \|V(\cdot, X)\|_{C^0}^2\, dX \le c' t^2.\end{aligned}$$

So for t small enough the right-hand side of (2.4) is less than 3ε and this completes the proof of the first part of (2.41).

(ii) The second part of (2.41) can be obtained by the change of variables

$$\int_{\mathbf{R}^N} |f \circ T_t^{-1} - f|^2 \, dx = \int_{\mathbf{R}^N} |f - f \circ T_t|^2 J_t^{-1} \, dX$$

and the fact that $\beta^{-1} < J_t^{-1} < \alpha^{-1}$. This completes the proof of the lemma. □

2.5 Domain and Boundary Integral Expressions of the Shape Gradient

Expression (2.43) for the shape gradient is a volume (or domain) integral, and it is easy to check that the map

$$V \mapsto dJ(\Omega; V) : \overrightarrow{\mathcal{V}}^{0,1} \to \mathbf{R} \tag{2.46}$$

is linear and continuous (cf. section 3.4 in Chapter 9). So by Corollary 1 to the structure Theorem 3.6 in Chapter 9, we know that for a domain Ω with a C^2-boundary Γ there exists a scalar distribution $g(\Gamma)$ in $\mathcal{D}(\Gamma)'$ such that

$$dJ(\Omega; V) = \langle g(\Gamma), V(0) \cdot n \rangle_{\mathcal{D}^1(\Gamma)}. \tag{2.47}$$

The next objective is to further characterize the boundary expression. Recall that we have assumed that $f \in H^1(\mathbf{R}^N)$. So for a C^2 boundary Γ the solution $y = y(\Omega)$ of the problem

$$-\triangle y = f \text{ in } \Omega, \quad y = 0 \text{ on } \Gamma$$

belongs to $H^2(\Omega)$. For velocity fields V in $C^0([0,\tau]; \mathcal{D}^1(\mathbf{R}^N, \mathbf{R}^N))$ satisfying assumption (V) the transported solution y^t in $H^2(\Omega)$ is the solution of the system

$$-\mathrm{div}\,(A(t)\nabla y^t) = J_t f \circ T_t \text{ in } \Omega, \quad y^t = 0 \text{ on } \Gamma. \tag{2.48}$$

Knowing that for all t in $[0,\tau]$, $y^t \in H^2(\Omega) \cap H_0^1(\Omega)$, we can repeat the computation of $\partial_t \widetilde{E}(t,\varphi)$ for φ in $H^2(\Omega) \bigcap H_0^1(\Omega)$ instead of $H_0^1(\Omega)$. With this extra smoothness we can use the formula

$$\boxed{\frac{d}{dt} \int_{\Omega_t} F(t,x)\, dx \bigg|_{t=0} = \int_\Gamma F(0,x) V(0) \cdot n \, d\Gamma + \int_\Omega \frac{\partial F}{\partial t}(0,x)\, dx} \tag{2.49}$$

for a sufficiently smooth function $F\colon [0,\tau] \times \mathbf{R}^N \to \mathbf{R}$. Prior to applying this formula to expression (2.15) in section 2.2, notice that for φ in $H^2(\Omega)$,

$$\boxed{\dot{\varphi} = \frac{d}{dt} \varphi \circ T_t^{-1}\big|_{t=0} = -\nabla\varphi \cdot V(0) \in H^1(\Omega),} \tag{2.50}$$

but it generally does not belong to $H_0^1(\Omega)$. Then

$$\partial_t \widetilde{E}(t,\varphi)\big|_{t=0} = \int_\Gamma \left\{\frac{1}{2}|\nabla\varphi|^2 - f\varphi\right\} V(0)\cdot n\, d\Gamma + \int_\Omega \{\nabla\varphi\cdot\nabla\dot{\varphi} - f\dot{\varphi}\}\, dx. \tag{2.51}$$

Substitute $\varphi = y$ in (2.51):

$$\begin{aligned}\partial_t \widetilde{E}(t,y)\big|_{t=0} = & \int_\Gamma \left\{\frac{1}{2}|\nabla y|^2 - fy\right\} V(0)\cdot n\, d\Gamma \\ & - \int_\Omega \{\nabla y\cdot\nabla(\nabla y\cdot V(0)) - f(\nabla y\cdot V(0))\}\, dx.\end{aligned} \tag{2.52}$$

But $y \in H^2(\Omega)\bigcap H_0^1(\Omega)$ and

$$\int_\Omega \nabla y\cdot\nabla(\nabla y\cdot V(0))\, dx = -\int_\Omega \triangle y\, \nabla y\cdot V(0)\, dx + \int_\Gamma \frac{\partial y}{\partial n}\nabla y\cdot V(0)\, d\Gamma$$

and

$$\partial_t \widetilde{E}(t,y)\big|_{t=0} = \int_\Gamma \left\{\left[\frac{1}{2}|\nabla y|^2 - fy\right] V(0)\cdot n - \frac{\partial y}{\partial n}\nabla y\cdot V(0)\right\} d\Gamma. \tag{2.53}$$

But

$$y = 0 \text{ on } \Gamma \;\Rightarrow\; \nabla y = \frac{\partial y}{\partial n} n \text{ on } \Gamma,$$

and finally

$$\partial_t \widetilde{E}(t,y)\big|_{t=0} = -\int_\Gamma \frac{1}{2}\left|\frac{\partial y}{\partial n}\right|^2 V(0)\cdot n\, d\Gamma. \tag{2.54}$$

This is the boundary expression that is continuous for $V(0)$ in the space $\mathcal{D}^0(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$. It has been obtained via a parametrization of the function space appearing in the min formulation. Thus

$$dJ(\Omega; V) = -\int_\Omega \frac{1}{2}\left|\frac{\partial y}{\partial n}\right|^2 V(0)\cdot n\, d\Gamma, \tag{2.55}$$

as predicted in section 2.1.

Remark 2.3.
An original application of the computations made for the generic example can be found in M. C. Delfour, G. Payre, and J.-P. Zolésio [**3**]. In that paper the derivative of the energy function with respect to the nodes in a P^1 finite element approximation is obtained from the volume expression of the shape semiderivative of the continuous problem. This technique also applies to a broad class of boundary value problems and to mixed hybrid finite element approximations. As observed by several authors the boundary integral expression is not suitable since the finite element solution does not have the appropriate smoothness under which the boundary integral formula is obtained. This technique is used to obtain a triangularization which minimizes the approximation error of the solution. Other formulae of the same type have been obtained for mixed finite element approximations in

M. C. Delfour, Z. Mghzali, and J.-P. Zolésio [**1**] and several other boundary value problems. The reader is also referred to M. C. Delfour, G. Payre, and J.-P. Zolésio [**1**, **2**, **4**, **5**] for application of those techniques to thermal problems such as diffusers and space radiators. The last paper combines the above techniques with the systematic handling of parametrized geometries. □

3 Buckling of Columns

Auchmuty's dual principle was used for the functional associated with the optimal design of a column against buckling in section 5 of Chapter 5. In this section we first use this construction to compute the directional semiderivative with respect to the cross-sectional area and then use it to give a necessary and sufficient condition that characterizes the maximizers. Indeed, recall the identity

$$\mu(A) = -\frac{1}{2\lambda(A)}$$

and notice that A is a maximizer of λ over $\mathcal{A}$ if and only if A is a maximizer of μ over $\mathcal{A}$.

We have seen in Theorem 5.3 of section 5 of Chapter 5 that the concave upper semicontinuous functional

$$\mu(A) \overset{\text{def}}{=} \inf \left\{ L(A, v) \,:\, v \in H_0^2(0,1) \right\}, \tag{3.1}$$

$$L(A, v) \overset{\text{def}}{=} \frac{1}{2} \int_0^1 A\, |v''|^2\, dx - \left[\int_0^1 |v'|^2\, dx \right]^{1/2} \tag{3.2}$$

has maximizers over the weakly compact convex set $\mathcal{A}$. Therefore, if $\mu(A)$ has a directional semiderivative $d\mu(A; B)$, a maximizer is completely characterized by

$$\exists A \in \mathcal{A},\ \forall B \in \mathcal{A}, \quad d\mu(A; B - A) \le 0.$$

This directional semiderivative always exists for concave continuous functions, and Theorem 2.1 can be used to get an explicit expression of $d\mu(A; B)$.

Given $A \in \mathcal{A}$, $B \in L^\infty(0,1)$, and $t \ge 0$, let

$$A_t \overset{\text{def}}{=} A + tB.$$

In view of the fact that $A \ge A_0 > 0$, there exists $\tau > 0$ such that for all $0 \le t \le \tau$

$$\forall t,\ 0 \le t \le \tau, \quad A_t \ge A_0/2.$$

Define

$$\tilde{L}(t, v) \overset{\text{def}}{=} \frac{1}{2} \int_0^1 A_t\, |v''|^2\, dx - \left[\int_0^1 |v'|^2\, dx \right]^{1/2}$$

$$\Rightarrow\ \mu(A_t) = \inf \left\{ \tilde{L}(t, v) \,:\, v \in H_0^2(0,1) \right\}, \quad X(t) \overset{\text{def}}{=} E(A_t).$$

Theorem 3.1. *Let B be an arbitrary function in $L^\infty(0,1)$ and A be an element of $\mathcal{A}$.*

(i) *The directional semiderivative at A in the direction B is given by the following expression:*

$$d\mu(A;B) = \inf_{v \in E(A)} \frac{1}{2} \int_0^1 B\,|v''|^2\,dx, \tag{3.3}$$

and since μ is concave and continuous, $d_H\mu(A;B)$ also exists.

(ii) *The maximizing elements of $\mu(A)$ or $\lambda(A)$ in $\mathcal{A}$ are completely characterized by the variational inequality*

$$\exists A \in \mathcal{A},\ \forall B \in \mathcal{A}, \quad \inf_{v \in E(A)} \frac{1}{2} \int_0^1 (B-A)\,|v''|^2\,dx \le 0 \tag{3.4}$$

or equivalently

$$\exists A \in \mathcal{A},\ \forall B \in \mathcal{A}, \quad \inf_{v \in E(A)} \frac{1}{2} \int_0^1 B\,|v''|^2\,dx \le \frac{1}{2\lambda(A)} = -\mu(A). \tag{3.5}$$

If we replace $E(A)$ by

$$\tilde{E}(A) \overset{\text{def}}{=} \left\{ u \in H_0^2(0,1) \ : \ \begin{matrix} \displaystyle\int_0^1 |u'|^2\,dx = 1 \ \text{and}\ \forall v \in H_0^2(0,1) \\ \displaystyle\int_0^1 A\,u''\,v''\,dx = \lambda(A) \int_0^1 u'\,v'\,dx \end{matrix} \right\},$$

then the maximizing elements of $\mu(A)$ or $\lambda(A)$ in $\mathcal{A}$ are completely characterized by the variational inequality

$$\exists A \in \mathcal{A},\ \forall B \in \mathcal{A}, \quad \inf_{v \in \tilde{E}(A)} \int_0^1 B\,|v''|^2\,dx \le \lambda(A). \tag{3.6}$$

Proof. (i) From Theorem 5.2 in Chapter 5, $X(t) \neq \varnothing$ and assumption (H1) is satisfied. The partial derivative of $\tilde{L}(t,v)$ with respect to t is given by the expression

$$\partial_t \tilde{L}(t,v) = \frac{1}{2} \int_0^1 B\,|v''|^2\,dx, \tag{3.7}$$

which is independent of t and exists for all $B \in L^\infty(0,1)$. Hence assumptions (H2) and (H4) are trivially satisfied. From (5.7) in Theorem 5.1 of Chapter 5

$$\exists u_t \in H_0^2(0,1),\ \forall v \in H_0^2(0,1), \quad \int_0^1 A_t\,u_t''\,v''\,dx = \lambda(A_t) \int_0^1 u_t'\,v'\,dx \tag{3.8}$$

$$\Rightarrow\ \frac{A_0}{2} \int_0^1 |u_t''|^2\,dx \le \int_0^1 A_t\,|u_t''|^2\,dx = \lambda(A_t) \int_0^1 |u_t'|^2\,dx = \frac{1}{\lambda(A_t)}. \tag{3.9}$$

But

$$0 < \frac{A_0}{2} \le A_t \le A_B \overset{\text{def}}{=} A_1 + \tau \|B\|_{L^\infty}$$
$$\Rightarrow \frac{1}{2}\mu(A_0) \le \mu(A_t) \le \mu(A_B) \quad \text{and} \quad 2\lambda(A_0) \le \lambda(A_t) \le \lambda(A_B).$$

Therefore for any sequence $t_n \searrow 0$, there exist a subsequence $\{t_{n_k}\}$, μ, and $u \in H_0^2(0,1)$ such that

$$\mu_k \overset{\text{def}}{=} \mu(A_{t_{n_k}}) \to \mu < 0, \quad \lambda_k \overset{\text{def}}{=} \lambda(A_{t_{n_k}}) = -\frac{1}{2\mu_k} \to \lambda = -\frac{1}{2\mu} > 0,$$
$$u_k \rightharpoonup u \text{ in } H_0^2(0,1)\text{-weak} \quad \Rightarrow u_k \to u \text{ in } H_0^1(0,1).$$

Going to the limit in (3.8)

$$\forall v \in H_0^2(0,1), \quad \int_0^1 A_{t_{n_k}}\, u_k''\, v''\, dx = \lambda(A_{t_{n_k}}) \int_0^1 u_k'\, v'\, dx, \quad \|u_k'\|_{L^2} = \frac{1}{\lambda_k},$$

we get the following variational equation for $u \in H_0^2(0,1)$:

$$\forall v \in H_0^2(0,1), \quad \int_0^1 A\, u''\, v''\, dx = \lambda \int_0^1 u'\, v'\, dx,$$
$$\|u'\|_{L^2} = \lim_{k\to\infty} \|u_k'\|_{L^2} = \lim_{k\to\infty} \frac{1}{\lambda_k} = \frac{1}{\lambda} > 0.$$

To show that $u \in X(0) = E(A)$, let u_0 be any element of $E(A)$. By definition,

$$\frac{1}{2}\int_0^1 A_{t_{n_k}}\, |u_k''|^2\, dx - \|u_k'\|_{L^2} \le \frac{1}{2}\int_0^1 A_{t_{n_k}}\, |u_0''|^2\, dx - \|u_0'\|_{L^2}. \tag{3.10}$$

But $A_{t_{n_k}} \to A$ in $L^\infty(0,1)$-strong and

$$\begin{aligned}
\liminf_{k\to\infty} \int_0^1 A_{t_{n_k}}\, |u_k''|^2\, dx &= \liminf_{k\to\infty} \int_0^1 (A_{t_{n_k}} - A)\, |u_k''|^2\, dx + \int_0^1 A\, |u_k''|^2\, dx \\
&\ge \liminf_{k\to\infty} -\|A_{t_{n_k}} - A\|_{L^\infty} \int_0^1 |u_k''|^2\, dx + \int_0^1 A\, |u_k''|^2\, dx \\
&\ge \int_0^1 A\, |u''|^2\, dx
\end{aligned}$$

since $\|u_k''\|_{L^2}$ is bounded. Since $\|u_k'\|_{L^2} \to \|u'\|_{L^2}$, by going to the limit in (3.10), we get

$$\begin{aligned}
L(A,u) &= \frac{1}{2}\int_0^1 A\, |u''|^2\, dx - \|u'\|_{L^2} \\
&\le \liminf_{k\to\infty} \frac{1}{2}\int_0^1 A_{t_{n_k}}\, |u_k''|^2\, dx - \|u_k'\|_{L^2} \\
&\le \liminf_{k\to\infty} \frac{1}{2}\int_0^1 A_{t_{n_k}}\, |u_0''|^2\, dx - \|u_0'\|_{L^2} = \frac{1}{2}\int_0^1 A\, |u_0''|^2\, dx - \|u_0'\|_{L^2} \\
&\qquad = \mu(A) = \inf_{v\in V} L(A,v).
\end{aligned}$$

Therefore, by definition of the minimum $u \in E(A) = X(0)$ and assumption (H3) (i) is satisfied.

To check the second part of (H3) we first show that $\{u_k\}$ converges not only in $H_0^2(0,1)$-weak but also in $H_0^2(0,1)$-strong. Then assumption (H4) directly follows from that property

$$\frac{1}{2}\partial\tilde{L}(t,u_k) = \int_0^1 B\,|u_k''|^2\,dx \to \int_0^1 B\,|u''|^2\,dx = \partial\tilde{L}(t,u).$$

The strong convergence follows from the following chain of inequalities:

$$\begin{aligned}
\frac{A_0}{2}\int_0^1 |u_k''-u''|^2\,dx &\le \int_0^1 \frac{A_0}{2}|u_k''|^2\,dx + \int_0^1 \frac{A_0}{2}|u''|^2\,dx - \int_0^1 A_0 u_k''\,u''\,dx \\
&\le \int_0^1 A_k|u_k''|^2\,dx + \int_0^1 A_0|u''|^2\,dx - 2\int_0^1 A_0 u_k''\,u''\,dx \\
&= \lambda_k \int_0^1 |u_k'|^2\,dx + \int_0^1 A_0|u''|^2\,dx - 2\int_0^1 A_0 u_k''\,u''\,dx \\
&\to \lambda \int_0^1 |u'|^2\,dx - \int_0^1 A_0|u''|^2\,dx = 0
\end{aligned}$$

since $\lambda = \lambda(A)$. This complete the proof of part (i).

(ii) The first characterization now follows directly from the fact that μ is concave and continuous and $\mathcal{A}$ is closed and convex. The second characterization uses the following identity, which comes from (3.8) for the eigenvectors and their normalization (5.11) in Theorem 5.2 of Chapter 5 as elements of $E(A)$: for all $v \in E(A)$

$$\frac{1}{2}\int_0^1 A\,|v''|^2\,dx = \lambda(A)\,\frac{1}{2}\int_0^1 |v'|^2\,dx = \frac{1}{2\lambda(A)} = -\mu(A). \qquad \square$$

4 Eigenvalue Problems

The first eigenvalue $\lambda(\Omega)$ of a linear boundary value problem defined on a bounded open subset Ω of $\mathbf{R}^{\mathrm{N}}$ is a classical example of a shape functional which comes in the form of an infimum. It is generally not differentiable since the eigenvalue can be repeated, but Theorem 2.1 of section 2.3 can be used to compute its Eulerian semiderivative.

In this section we give the *volume expression* of the Eulerian semiderivative for the Laplacian and the bi-Laplacian with a homogeneous Dirichlet boundary condition for a general bounded open domain and its *boundary expression* when the domain is sufficiently smooth. In the case of the Laplacian over a smooth domain, the first eigenvalue is simple and differentiable. As in section 3 it is technically advantageous to work with G. Auchmuty [**1**]'s dual principle rather than with the Rayleigh quotient. It is also technically advantageous to embed the problem of the computation of the Eulerian semiderivative for the domain Ω into a larger and sufficiently smooth holdall D which will contain all the perturbations $\Omega_t = T_t(V)(\Omega)$ of Ω for t sufficiently small. The smoothness conditions that would normally occur on Ω will then occur on D. Once the expression for the semiderivative is obtained, D can be thrown away, leaving a final expression that does not require any smoothness assumption on Ω.

4.1 Transport of $H_0^k(\Omega)$ by $W^{k,\infty}$-Transformations of $\mathbf{R}^N$

Recall Theorem 2.3 in Chapter 8 on the embedding of $H_0^k(\Omega)$ into $H_0^k(D)$. As a consequence, the homogeneous Dirichlet boundary value problem in Ω is the following: find $y \in H_0^1(\Omega)$ such that

$$\forall \varphi \in H_0^1(\Omega), \quad \int_\Omega \nabla y \cdot \nabla \varphi \, dx = \int_\Omega f \, \varphi \, dx$$

is completely equivalent to the variational problem, find $Y \in H_0^1(\Omega; D)$ such that

$$\forall \Phi \in H_0^1(\Omega; D), \quad \int_D \nabla Y \cdot \nabla \Phi \, dx = \int_D f \, \Phi \, dx$$

and $Y|_\Omega = y$.

Let $k \geq 1$ and let T be a transformation of $\mathbf{R}^N$ such that

$$T, T^{-1} \in W^{k,\infty}(\mathbf{R}^N, \mathbf{R}^N).$$

Associate with T the map

$$\varphi \mapsto \mathcal{T}(\varphi) \stackrel{\text{def}}{=} \varphi \circ T^{-1} \, : \, \mathcal{D}(\mathbf{R}^N) \to H_0^k(\mathbf{R}^N).$$

By assumption on T, the norms $\|\varphi\|_{H^k}$ and $\|\mathcal{T}(\varphi)\|_{H^k} = \|\varphi \circ T^{-1}\|_{H^k}$ are equivalent and by density $\mathcal{T}$ extends to a linear bijection

$$\varphi \mapsto \mathcal{T}(\varphi) \stackrel{\text{def}}{=} \varphi \circ T^{-1} \, : \, H_0^k(\mathbf{R}^N) \to H_0^k(\mathbf{R}^N)$$

such that both $\mathcal{T}$ and $\mathcal{T}^{-1}$ are uniformly Lipschitzian. Given any bounded open subset Ω of $\mathbf{R}^N$, the map

$$\varphi \mapsto \mathcal{T}_\Omega(\varphi) \stackrel{\text{def}}{=} \mathcal{T}(\varphi)|_\Omega \, : \, H_0^k(\Omega) \to H_0^k(T(\Omega))$$

is again a linear bijection such that both $\mathcal{T}_\Omega$ and $\mathcal{T}_\Omega^{-1}$ are uniformly Lipschitzian.

Given a bounded open subset D of $\mathbf{R}^N$, further assume that T is such that

$$T(D) = D.$$

As a result $T(\mathbf{R}^N \backslash D) = \mathbf{R}^N \backslash D$ and $T(\partial D) = \partial D$ and

$$\mathcal{T}_D \, : \, H_0^k(D) \to H_0^k(D).$$

Let Ω be an open subset of $\mathbf{R}^{\mathrm{N}}$ such that $\Omega \subset D$. Then $T(\Omega) \subset D$, $T^{-1}(\Omega) \subset D$, and $H_0^1(\Omega)$ and $H_0^1(T(\Omega))$ can be identified with the subspaces $H_0^1(\Omega; D)$ and $H_0^1(T(\Omega); D)$ of $H_0^1(D)$. Moreover,

$$\forall \varphi \in \mathcal{D}(\Omega), \quad e_0(\varphi) \circ T^{-1} = e_0(\varphi \circ T^{-1})$$

and the following diagram commutes:

$$\begin{array}{ccc} H_0^k(\Omega) & \xrightarrow{e_0} & H_0^k(D) \\ \downarrow \mathcal{T}_\Omega & & \downarrow \mathcal{T}_D \\ H_0^k(T(\Omega)) & \xrightarrow{e_0} & H_0^k(D) \end{array}$$

$$\Rightarrow \boxed{\mathcal{T}_D(H_0^k(\Omega; D)) = H_0^k(T(\Omega); D).}$$

4.2 Laplacian and Bi-Laplacian

For an arbitrary bounded open subset Ω of $\mathbf{R}^{\mathrm{N}}$ the bounded open holdall D can be arbitrarily chosen such that $\overline{\Omega} \subset D$. For instance, D can be a large enough open ball. This will make it possible to throw on D any smoothness assumption that would normally occur on Ω and work with an arbitrary Ω. In this setup, D is used as an intermediate step and can be thrown away once the semiderivative has been computed.

Given a velocity field

$$V \in C([0, \tau]; W_0^{k,\infty}(D; \mathbf{R}^{\mathrm{N}})), \tag{4.1}$$

the flow mapping $T_t = T_t(V)$ maps D onto itself and ∂D onto itself, and is equal to the identity on $\mathbf{R}^{\mathrm{N}} \backslash D$. So it transports $H_0^1(D)$ onto itself.

Theorem 4.1. *For $k = 1, 2$ and $0 \le t < \tau$*

$$\varphi \in H_0^k(D) \iff \varphi \circ T_t(V) \in H_0^k(D),$$
$$\varphi \in H_0^k(\Omega; D) \iff \varphi \circ T_t(V) \in H_0^k(\Omega; D).$$

Hence for $\Omega_t = T_t(\Omega)$

$$H_0^k(\Omega_t; D) = \left\{ \varphi \circ T_t^{-1}(V) \, : \, \forall \varphi \in H_0^k(\Omega; D) \right\}. \tag{4.2}$$

Proof. The second statement follows from the fact that T_t and its inverse T_t^{-1} are both Lipschitz continuous and that, by the previous considerations or Lemma 6.2 in Chapter 8, they transport sets of zero capacity onto sets of zero capacity: $T_t(D \backslash \Omega) = D \backslash T_t(\Omega) = D \backslash \Omega_t$. □

As $\overline{\Omega}$ is compact and $\overline{\Omega} \subset D$, there exists $\tau_V > 0$ such that for all t, $0 \le t < \tau_V$, $\overline{\Omega_t(V)} \subset D$. In both cases the first eigenvalue is given by the Rayleigh quotient:

$$\lambda(\Omega_t(V)) = \inf \left\{ \frac{a_{\Omega_t}^k(\varphi, \varphi)}{\int_{\Omega_t} \varphi^2 \, dx} \, : \, \forall \varphi \in H_0^k(\Omega_t), \, \varphi \neq 0 \right\}, \quad k = 1, 2,$$

where

$$a^1_{\Omega_t}(\varphi,\psi) \overset{\text{def}}{=} \int_{\Omega_t} \nabla\varphi\cdot\nabla\psi\,dx \quad\text{and}\quad a^2_{\Omega_t}(\varphi,\psi) \overset{\text{def}}{=} \int_{\Omega_t} \Delta\varphi\,\Delta\psi\,dx.$$

In view of the previous constructions $H^k_0(\Omega_t)$ can be replaced by $H^k_0(\Omega_t;D)$:

$$\lambda(\Omega_t(V)) = \inf\left\{\frac{a^k_D(\varphi,\varphi)}{\int_D \varphi^2\,dx} : \forall\varphi\in H^k_0(\Omega_t;D),\ \varphi\neq 0\right\},\quad k=1,2. \tag{4.3}$$

Note that if $\varphi\neq 0$ is a minimizer, then for any real number $\alpha\neq 0$, $\alpha\varphi$ is also a minimizer.

Theorem 4.2. *Given $k=1,2$, let Ω be a bounded open subset of $\mathbf{R}^N$. Assume that*

$$V\in C([0,\tau];W^{k,\infty}_0(D;\mathbf{R}^N))$$

for some bounded open domain D in $\mathbf{R}^N$ such that $\Omega\subset D$. There exists at least one nonzero solution $\varphi\in H^k_0(\Omega_t;D)$ to the minimization problem (4.3), $\lambda(\Omega_t(V))\geq\lambda(D)>0$, and

$$\begin{aligned}
\forall\varphi\in H^1_0(D),&\quad \int_D\varphi^2\,dx \leq \lambda(D)^{-1}\int_D |\nabla\varphi|^2\,dx,\\
\forall\varphi\in H^2_0(D),&\quad \int_D\varphi^2\,dx \leq \lambda(D)^{-1}\int_D |\Delta\varphi|^2\,dx.
\end{aligned}$$

The solutions are completely characterized by the following variational equation: there exists $\varphi\in H^k_0(\Omega_t;D)$ such that

$$\forall\psi\in H^k_0(\Omega_t;D),\quad a^k_D(\varphi,\psi) = \lambda(\Omega_t(V))\int_D \varphi\,\psi\,dx \tag{4.4}$$

or equivalently

$$\forall\psi\in H^k_0(\Omega_t),\quad a^k_{\Omega_t}(\varphi,\psi) = \lambda(\Omega_t(V))\int_{\Omega_t} \varphi\,\psi\,dx. \tag{4.5}$$

Proof. Same technique as in the proof of Theorem 2.1 in Chapter 8. □

The minimization problem can be rewritten on the unit sphere in $L^2(D)$ by normalization:

$$\lambda(\Omega_t(V)) = \inf\left\{\int_D |\nabla\varphi|^2\,dx : \forall\varphi\in H^1_0(\Omega_t;D),\ \|\varphi\|_{L^2(D)}=1\right\}. \tag{4.6}$$

The minimizing elements of (4.6) cannot be zero since the injection of $H^1_0(D)$ into $L^2(D)$ is compact (cf. Theorem 2.3 in Chapter 8).

Remark 4.1.
One of the consequences of the use of the embedding of $H^1_0(\Omega)$ into $H^1_0(D)$ is that the characterization of the first eigenvalue on Ω requires only that Ω be a bounded open subset of $\mathbf{R}^N$. It is not necessary to assume that Ω is Lipschitzian in order to use Rellich's theorem, since D can be chosen sufficiently large ($\overline{\Omega}\subset D$) and smooth in order to contain all variations $\overline{\Omega}_t\subset D$ as t goes to zero. This technique will be exploited in the computation of the Eulerian semiderivative of $\lambda(\Omega)$. □

G. Auchmuty [**1**]'s dual variational for this eigenvalue problem can be chosen as

$$\mu(\Omega_t) \stackrel{\text{def}}{=} \inf \left\{ L^k(\Omega_t, \varphi) \,:\, \varphi \in H_0^k(\Omega_t) \right\}, \tag{4.7}$$

$$L^k(\Omega_t, \varphi) \stackrel{\text{def}}{=} \frac{1}{2} a^k_{\Omega_t}(\varphi, \varphi) - \left[\int_{\Omega_t} \varphi^2 \, dx \right]^{1/2}. \tag{4.8}$$

By using the embedding of $H_0^1(\Omega_t)$ into $H_0^1(D)$, this problem can be rewritten as

$$\mu(\Omega_t) \stackrel{\text{def}}{=} \inf \left\{ L^k(D, \varphi) \,:\, \varphi \in H_0^k(\Omega_t; D) \right\}, \tag{4.9}$$

$$L^k(D, \varphi) \stackrel{\text{def}}{=} \frac{1}{2} a^k_{D}(\varphi, \varphi) - \left[\int_{D} \varphi^2 \, dx \right]^{1/2}. \tag{4.10}$$

Theorem 4.3. *Given $k = 1, 2$, let Ω be a bounded open subset of $\mathbf{R}^{\mathrm{N}}$. Assume that*

$$V \in C^0([0, \tau[\,; W_0^{k,\infty}(D; \mathbf{R}^{\mathrm{N}}))$$

for some bounded open domain D in $\mathbf{R}^{\mathrm{N}}$ such that $\Omega \subset D$. Then for $0 \le t < \tau$

$$\mu(\Omega_t) = -\frac{1}{2\lambda(\Omega_t)} \tag{4.11}$$

and the set of minimizers of (4.7) *is given by*

$$E^k(\Omega_t) \stackrel{\text{def}}{=} \left\{ \varphi \in H_0^k(\Omega_t; D) \,:\, \begin{array}{c} \varphi \text{ is solution of (4.4) and} \\ \left\{ \displaystyle\int_D |\varphi|^2 \, dx \right\}^{1/2} = 1/\lambda(\Omega_t) \end{array} \right\}. \tag{4.12}$$

Proof. From the previous theorem the set $E(\Omega_t)$ is not empty, and for any $\varphi \in E(\Omega_t)$

$$\mu(\Omega_t) \le L(t, \varphi) = -\frac{1}{2\lambda(\Omega_t)} < 0.$$

Therefore the minimizers of (4.8) are different from the zero functions. For $\varphi \neq 0$, the functional $L(t, \varphi)$ is differentiable and its directional derivative is given by

$$dL(t, \varphi; \psi) = a^k_D(\varphi, \psi) - \frac{1}{\|\varphi\|_{L^2(D)}} \int_D \varphi \, \psi \, dx, \tag{4.13}$$

and any minimizer of $L(t, \varphi)$ is a stationary point of $dL(t, \varphi; \psi)$; that is,

$$\forall \psi \in H_0^k(\Omega_t; D), \quad a^k_D(\varphi, \psi) - \frac{1}{\|\varphi\|_{L^2(D)}} \int_D \varphi \, \psi \, dx = 0. \tag{4.14}$$

Therefore, φ is a solution of the eigenvalue problem with

$$\lambda' = \frac{1}{\|\varphi\|_{L^2(D)}} \quad \Rightarrow -\frac{1}{2\lambda'} = \mu(\Omega_t) \le -\frac{1}{2\lambda(\Omega_t)}.$$

By minimality of $\lambda(\Omega_t)$, we necessarily have $\lambda' = \lambda(\Omega_t)$ and this concludes the proof of the theorem. □

We now turn to the computation of the Eulerian semiderivative $d\lambda(\Omega; V)$ via $d\mu(\Omega; V)$ from the identity

$$\boxed{\lambda(\Omega_t) = -\frac{1}{2\mu(\Omega_t)},}$$

since whenever $d\mu(\Omega; V)$ exists

$$\boxed{d\lambda(\Omega; V) = \frac{1}{2\mu(\Omega)^2}\, d\mu(\Omega; V) = 2\,\lambda(\Omega)^2\, d\mu(\Omega; V).}$$

Assuming that V satisfies assumption (4.1), we use the function space parametrization of section 2.4 in conjunction with Theorem 2.1. From the characterization (4.2) of $H_0^k(\Omega_t; D)$, define the following new functional: for each $\varphi \in H_0^1(D)$,

$$\begin{aligned}\tilde{L}(t,\varphi) &\overset{\text{def}}{=} L(D, \varphi \circ T_t^{-1}(V)) \\ &= \frac{1}{2} a_D^k(\varphi \circ T_t^{-1}(V), \varphi \circ T_t^{-1}(V)) - \left\{ \int_D |\varphi \circ T_t^{-1}(V)|^2\, dx \right\}^{1/2}.\end{aligned}$$

After a change of variables for $k = 1$ and $\varphi \in H_0^1(D)$,

$$\begin{gathered}\tilde{L}(t,\varphi) = \frac{1}{2}\int_D A(t)\nabla\varphi \cdot \nabla\varphi\, dx - \left\{ \int_D J_t\, |\varphi|^2\, dx \right\}^{1/2}, \\ A(t) = J_t\, DT_t^{-1}\, {}^*DT_t^{-1}, \quad J_t = \det(DT_t),\end{gathered}$$

and for $k = 2$ and $\varphi \in H_0^2(D)$

$$\begin{gathered}\tilde{L}(t,\varphi) = \frac{1}{2}\int_D |\operatorname{div}(B(t)\nabla\varphi)|^2\, J_t\, dx - \left\{ \int_D J_t\, |\varphi|^2\, dx \right\}^{1/2}, \\ B(t) = DT_t^{-1}\, {}^*DT_t^{-1}.\end{gathered}$$

To apply Theorem 2.1, choose

$$X(t) \overset{\text{def}}{=} \left\{ \varphi^t \overset{\text{def}}{=} \varphi_t \circ T_t \,:\, \forall \varphi_t \in E^k(\Omega_t) \right\},$$

endowed with the weak topology of $H_0^1(D)$. Assumption (H1) is clearly satisfied. For all $\varphi \neq 0$, $\tilde{L}(t,\varphi)$ is differentiable. For $k = 1$ and $0 \neq \varphi \in H_0^1(D)$,

$$\partial_t \tilde{L}(t,\varphi) = \frac{1}{2}\int_D A'(t)\nabla\varphi \cdot \nabla\varphi\, dx - \frac{1}{\left\{ \int_D J_t\, |\varphi|^2\, dx \right\}^{1/2}} \int_D |\varphi|^2\, J_t'\, dx,$$

and for $k = 2$ and $0 \neq \varphi \in H_0^2(D)$

$$\begin{aligned}\partial_t \tilde{L}(t,\varphi) = &\int_D \operatorname{div}(B'(t)\nabla\varphi)\operatorname{div}(B(t)\nabla\varphi)\, J_t\, dx \\ &+ \frac{1}{2}\int_D |\operatorname{div}(B(t)\nabla\varphi)|^2\, J_t'\, dx \\ &- \frac{1}{\left\{ \int_D J_t\, |\varphi|^2\, dx \right\}^{1/2}} \int_D |\varphi|^2\, J_t'\, dx.\end{aligned}$$

Hence assumptions (H2) and (H4) are satisfied. In $t = 0$ the above expressions simplify. For $k = 1$ and $\varphi \in E^1(\Omega)$,

$$\begin{aligned}\partial_t \tilde{L}(0,\varphi) &= \frac{1}{2}\int_D A'(0)\nabla\varphi\cdot\nabla\varphi\,dx - \lambda(\Omega)\int_D |\varphi|^2 \operatorname{div} V(0)\,dx \\ &= -\int_D \varepsilon(V(0))\nabla\varphi\cdot\nabla\varphi\,dx + \int_D \left[\frac{1}{2}|\nabla\varphi|^2 - \lambda(\Omega)|\varphi|^2\right]\operatorname{div} V(0)\,dx,\\ A'(0) &= \operatorname{div} V(0) I - DV(0) - {}^*DV(0), \quad \varepsilon(U) \stackrel{\text{def}}{=} \frac{1}{2}\left[DU + {}^*DU\right],\end{aligned}$$

and for $k = 2$ and $\varphi \in E^2(\omega)$,

$$\begin{aligned}\partial_t \tilde{L}(0,\varphi) =&2\int_D \operatorname{div}\left(\varepsilon(V(0))\nabla\varphi\right)\Delta\varphi\,dx \\ &+ \int_D \left[\frac{1}{2}|\Delta\varphi|^2 - \lambda(\Omega)|\varphi|^2\right]\operatorname{div} V(0)\,dx,\\ B'(0) = &-DV(0) - {}^*DV(0) = -2\varepsilon(V(0)).\end{aligned}$$

For volume-preserving velocities, $\operatorname{div} V(0) = 0$, and the above expressions reduce to the first integral term.

In order to apply Theorem 2.1 it remains to check assumption (H3). This is the first example where the set of minimizers is not necessarily unique and for which we have a complete description of the Eulerian semiderivative.

Theorem 4.4. *Given $k = 1, 2$, let Ω be a bounded open subset of $\mathbf{R}^{\mathrm{N}}$. Assume that*

$$V \in C^0([0,\tau[\,; W_0^{k,\infty}(D;\mathbf{R}^{\mathrm{N}}))$$

for some bounded open domain D in $\mathbf{R}^{\mathrm{N}}$ such that $\overline{\Omega} \subset D$. Then for $k = 1$

$$\begin{aligned}\frac{1}{2} d\lambda(\Omega;V) = \inf_{\varphi\in\tilde{E}^1(\Omega)} &-\int_\Omega \varepsilon(V(0))\nabla\varphi\cdot\nabla\varphi\,dx \\ &+\int_\Omega \left[\frac{1}{2}|\nabla\varphi|^2 - \lambda(\Omega)|\varphi|^2\right]\operatorname{div} V(0)\,dx,\\ \varepsilon(U) &\stackrel{\text{def}}{=} \frac{1}{2}\left[DU + {}^*DU\right],\\ \tilde{E}^1(\Omega) \stackrel{\text{def}}{=} &\left\{\varphi \in H_0^1(\Omega) : \begin{array}{c} -\Delta\varphi = \lambda(\Omega)\varphi \textit{ in } \mathcal{D}(\Omega)' \\ \textit{and } \displaystyle\int_\Omega |\varphi|^2\,dx = 1\end{array}\right\};\end{aligned}$$

for $k = 2$

$$\begin{aligned}\frac{1}{2} d\lambda(\Omega;V) = \inf_{\varphi\in\tilde{E}^2(\Omega)} &\int_\Omega 2\operatorname{div}\left(\varepsilon(V(0))\nabla\varphi\right)\Delta\varphi\,dx \\ &+\int_\Omega \left[\frac{1}{2}|\Delta\varphi)|^2 - \lambda(\Omega)|\varphi|^2\right]\operatorname{div} V(0)\,dx,\\ \tilde{E}^2(\Omega_t) \stackrel{\text{def}}{=} &\left\{\varphi \in H_0^2(\Omega) : \begin{array}{c} \Delta(\Delta\varphi) = \lambda(\Omega)\varphi \textit{ in } \mathcal{D}(\Omega)' \\ \textit{and } \displaystyle\int_\Omega |\varphi|^2\,dx = 1\end{array}\right\}.\end{aligned}$$

In both cases $\lambda(\Omega)$ *has a Hadamard semiderivative in the sense of Definition* 3.2 (iii) *in Chapter* 9*, which is continuous with respect to* $V(0) \in W_0^{k,\infty}(D;\mathbf{R}^N)$*:*

$$d\lambda(\Omega;V) = d_H\lambda(\Omega;V(0)).$$

Proof. As $\overline{\Omega}$ is compact and $\overline{\Omega} \subset D$, there exists $\tau_V > 0$ such that for all t, $0 \le t < \tau_V$, $\overline{\Omega_t(V)} \subset D$. Observe that

$$\mu(D) = \inf_{\varphi \in H_0^1(D)} L(\varphi) \le \inf_{\varphi \in H_0^1(\Omega_t;D)} L(\varphi) = \mu(\Omega_t).$$

By continuity of $t \mapsto \tilde{L}(t,\varphi)$, for any $\varphi^t \in X(t)$ and $\varphi^0 \in X(0)$,

$$\begin{aligned}\tilde{L}(t,\varphi^t) \le \tilde{L}(t,\varphi^0) \;&\Rightarrow\; \limsup_{t\searrow 0} \tilde{L}(t,\varphi^t) \le \tilde{L}(0,\varphi^0) = \mu(\Omega)\\ &\Rightarrow\; \mu(D) \le \limsup_{t\searrow 0} \mu(\Omega_t) \le \mu(\Omega).\end{aligned}$$

Therefore there exists τ', $0 < \tau' \le \tau_V$, such that for all t, $0 \le t \le \tau'$,

$$\mu(D) \le \mu(\Omega_t) \le \frac{1}{2}\mu(\Omega) < 0 \quad \Rightarrow 0 < \lambda(D) \le \lambda(\Omega_t) \le 2\,\lambda(\Omega).$$

Since $A(t) \to I$ and $J_t \to 1$ as t goes to zero, there exist $c > 0$ and $0 < \tau'' \le \tau$ such that for all $0 \le t \le \tau''$,

$$\begin{aligned}\tilde{L}(t,\varphi) =& \frac{1}{2}\int_D A(t)\nabla\varphi\cdot\nabla\varphi\,dx - \left\{\int_D J_t\,|\varphi|^2\,dx\right\}^{1/2}\\ \ge& c\left\{\|\nabla\varphi\|^2_{L^2(D)} - \|\varphi\|_{L^2(D)}\right\} \ge c\|\nabla\varphi\|^2_{L^2(D)} - c'\|\nabla\varphi\|_{L^2(D)}\end{aligned}$$

for some $c' > 0$ by Poincaré's inequality. Therefore for all $\varphi^t \in X(t)$,

$$c\,\|\nabla\varphi^t\|^2_{L^2(D)} - c'\|\nabla\varphi^t\|_{L^2(D)} \le \tilde{L}(t,\varphi^t) \le 0 \quad \Rightarrow \|\nabla\varphi^t\|_{L^2(D)} \le c'/c.$$

For any sequence $t_n \searrow 0$, there exist subsequences μ, and $\varphi_0 \in H_0^1(\Omega;D)$ such that

$$\begin{aligned}\mu_n = \mu(\Omega_{t_n}) \to \mu \text{ and } \varphi_n = \varphi^{t_n} \rightharpoonup \varphi_0 \text{ in } H_0^1(\Omega;D)\text{-weak},\\ \lambda_n = \lambda(\Omega_{t_n}) \to \lambda = -\frac{1}{2\mu} > 0.\end{aligned}$$

Therefore, since $A(t_n) \to I$, $J_{t_n} \to 1$, and $\{\varphi_n\}$ converges in H^1-weak,

$$\begin{aligned}\forall\psi \in H_0^1(\Omega;D),\quad \frac{1}{2}\int_D A(t_n)\nabla\varphi_n\cdot\nabla\psi\;dx = \lambda_n\int_D \varphi_n\,\psi\,dx\\ \Rightarrow \forall\psi \in H_0^1(\Omega;D),\quad \frac{1}{2}\int_D A(0)\nabla\varphi_0\cdot\nabla\psi\;dx = \lambda\int_D \varphi_0\,\psi\,dx,\\ \lambda_n = \int_D J_{t_n}\,|\varphi_n|^2\,dx \to \int_D |\varphi_0|^2\,dx \quad \Rightarrow \int_D |\varphi_0|^2\,dx = \lambda > 0,\end{aligned}$$

and λ is an eigenvalue for the problem on Ω. But we know that

$$\mu(D) \leq \limsup_{n\to\infty} \mu(\Omega_{t_n}) = -\frac{1}{2\lambda}, \quad \mu(D) \leq -\frac{1}{2\lambda} \leq -\frac{1}{2\lambda(\Omega)},$$
$$\lambda(D) \leq \lambda \leq \lambda(\Omega) \quad \Rightarrow \quad \lambda = \lambda(\Omega),$$

since $\lambda(\Omega)$ is minimal and hence $\varphi_0 \in X(0)$. This proves condition (H3)(i). To prove condition (H3)(ii) and complete the proof we first prove that the weakly convergent subsequence strongly converges in $H_0^1(\Omega; D)$. By compactness of the injection of $H_0^1(D)$ into $L^2(D)$ (cf. Theorem 2.3 in Chapter 8)

$$\varphi_n \rightharpoonup \varphi_0 \text{ in } H_0^1(D)\text{-weak} \quad \Rightarrow \varphi_n \to \varphi_0 \text{ in } L^2(D)\text{-strong}, \tag{4.15}$$

and for some $\alpha > 0$ independent of n,

$$\begin{aligned}
\alpha \int_D |\nabla(\varphi_n - \varphi_0)|^2\, dx \leq & \int_D A(t_n)\nabla(\varphi_n - \varphi_0) \cdot \nabla(\varphi_n - \varphi_0)\, dx \\
= & \int_D A(t_n)\nabla\varphi_n \cdot \nabla\varphi_n - 2A(t_n)\nabla\varphi_0 \cdot \nabla\varphi_n \\
& \qquad + A(t_n)\nabla\varphi_0 \cdot \nabla\varphi_0\, dx \\
= & \int_D \lambda_n J_{t_n}\, \varphi_n\, \varphi_n - 2A(t_n)\nabla\varphi_0 \cdot \nabla\varphi_n \\
& \qquad + A(t_n)\nabla\varphi_0 \cdot \nabla\varphi_0\, dx \\
\to & \int_D \lambda\, \varphi_0\, \varphi_0 - 2\nabla\varphi_0 \cdot \nabla\varphi_0 \\
& \qquad + \nabla\varphi_0 \cdot \nabla\varphi_0\, dx = 0.
\end{aligned}$$

By the same technique as in section 2.4 when $n \to \infty$ and $t \searrow 0$

$$\begin{aligned}
\partial_t \tilde{L}(t, \varphi_n) = & \frac{1}{2} \int_D A'(t)\nabla\varphi_n \cdot \nabla\varphi_n\, dx \\
& - \frac{1}{\left\{ \int_D J_t\, |\varphi_n|^2\, dx \right\}^{1/2}} \int_D |\varphi_n|^2\, J_t'\, dx \\
\to & \frac{1}{2} \int_D A'(0)\nabla\varphi_0 \cdot \nabla\varphi_0\, dx \\
& - \frac{1}{\left\{ \int_D |\varphi_0|^2\, dx \right\}^{1/2}} \int_D |\varphi_0|^2 \operatorname{div} V(0)\, dx = \partial_t \tilde{L}(0, \varphi_0).
\end{aligned}$$

The case $k = 2$ is analogous. □

If Ω is assumed to be of class C^{2k}, the eigenvector functions belong to $H_0^k(\Omega) \cap H^{2k}(\Omega)$ and the volume expressions of the previous theorem can be expressed as boundary integrals as in section 2.5. In that case we can use formulae (2.49) and (2.50) to compute the partial derivative of $L_t = L(\Omega_t, \varphi \circ T_t^{-1})$. For $k = 1$,

$$L_t = \frac{1}{2} \int_{\Omega_t} |\nabla(\varphi \circ T_t^{-1})|^2\, dx - \left\{ \int_{\Omega_t} |\varphi \circ T_t^{-1}|^2\, dx \right\}^{1/2}.$$

From formula (2.49)

$$L' \overset{\text{def}}{=} \partial_t L_t|_{t=0} = \frac{1}{2}\int_\Gamma |\nabla\varphi|^2\, V(0)\cdot n\, d\Gamma - \frac{1}{2}\frac{\int_\Gamma |\varphi|^2\, V(0)\cdot n\, d\Gamma}{\left\{\int_\Omega |\varphi|^2\, dx\right\}^{1/2}}$$
$$+ \int_\Omega \nabla\varphi\cdot\nabla\dot\varphi\, dx - \frac{\int_\Omega \varphi\,\dot\varphi\, dx}{\left\{\int_\Omega |\varphi|^2\, dx\right\}^{1/2}}.$$

For $\varphi \in E^1(\Omega)$, $\varphi = 0$ on Ω, $-\Delta\varphi = \lambda(\Omega)\varphi$ in Ω, and $\lambda(\Omega)\|\varphi\|_{L^2(\Omega)} = 1$,

$$\begin{aligned} L' &= \int_\Gamma \frac{1}{2}|\nabla\varphi|^2\, V(0)\cdot n\, d\Gamma + \int_\Omega \nabla\varphi\cdot\nabla\dot\varphi\, dx - \lambda(\Omega)\int_\Omega \varphi\,\dot\varphi\, dx \\ &= \int_\Gamma \frac{1}{2}|\nabla\varphi|^2\, V(0)\cdot n\, d\Gamma + \int_\Gamma \frac{\partial\varphi}{\partial n}\,\dot\varphi\, d\Gamma. \end{aligned}$$

But $\varphi = 0$ on Γ implies that $\nabla\varphi = \partial\varphi/\partial n\, n$ on Γ, and from identity (2.50)

$$\begin{aligned} \dot\varphi = -\nabla\varphi\cdot V(0) \quad &\Rightarrow \dot\varphi = -\frac{\partial\varphi}{\partial n}\, V(0)\cdot n \text{ on } \Gamma \\ &\Rightarrow L' = -\int_\Gamma \frac{1}{2}\left|\frac{\partial\varphi}{\partial n}\right|^2 V(0)\cdot n\, d\Gamma. \end{aligned}$$

For $k = 2$ the computation of the partial derivative of $L_t = L(\Omega_t, \varphi\circ T_t^{-1})$ is similar, with obvious changes:

$$L_t = \frac{1}{2}\int_{\Omega_t} |\Delta(\varphi\circ T_t^{-1})|^2\, dx - \left\{\int_{\Omega_t} |\varphi\circ T_t^{-1}|^2\, dx\right\}^{1/2}.$$

From formula (2.49)

$$L' \overset{\text{def}}{=} \partial_t L_t|_{t=0} = \frac{1}{2}\int_\Gamma |\Delta\varphi|^2\, V(0)\cdot n\, d\Gamma - \frac{1}{2}\frac{\int_\Gamma |\varphi|^2\, V(0)\cdot n\, d\Gamma}{\left\{\int_\Omega |\varphi|^2\, dx\right\}^{1/2}}$$
$$+ \int_\Omega \Delta\varphi\,\Delta\dot\varphi\, dx - \frac{\int_\Omega \varphi\,\dot\varphi\, dx}{\left\{\int_\Omega |\varphi|^2\, dx\right\}^{1/2}}.$$

For $\varphi \in E^2(\Omega)$, $\varphi = 0$ on Ω, $\Delta(\Delta\varphi) = \lambda(\Omega)\varphi$ in Ω, and $\lambda(\Omega)\|\varphi\|_{L^2(\Omega)} = 1$,

$$\begin{aligned} L' &= \int_\Gamma \frac{1}{2}|\Delta\varphi|^2\, V(0)\cdot n\, d\Gamma + \int_\Omega \Delta\varphi\,\Delta\dot\varphi\, dx - \lambda(\Omega)\int_\Omega \varphi\,\dot\varphi\, dx \\ &= \int_\Gamma \frac{1}{2}|\Delta\varphi|^2\, V(0)\cdot n\, d\Gamma + \int_\Gamma \frac{\partial\Delta\varphi}{\partial n}\,\dot\varphi - \Delta\varphi\,\frac{\partial\dot\varphi}{\partial n}\, d\Gamma. \end{aligned}$$

But $\varphi = 0$ and $\partial\varphi/\partial n = 0$ on Γ imply that $\nabla\varphi = 0$ on Γ and

$$D^2\varphi = (D^2\varphi\, n)\, {}^*n \text{ on } \Gamma \quad \Rightarrow \Delta\varphi = (D^2\varphi\, n)\cdot n \text{ on } \Gamma.$$

From identity (2.50) we have on Γ

$$\begin{aligned}
\dot\varphi &= -\nabla\varphi\cdot V(0) \quad \Rightarrow \dot\varphi = 0 \text{ on } \Gamma\\
\Rightarrow\ \nabla\dot\varphi &= -\nabla(\nabla\varphi\cdot V(0)) = -D^2\varphi\, V(0) - {}^*DV(0)\nabla\varphi = -D^2\varphi\, V(0)\\
\Rightarrow\ \frac{\partial\dot\varphi}{\partial n} &= -D^2\varphi\, V(0)\cdot n = -D^2\varphi\, n\cdot n\, V(0)\cdot n = -\Delta\varphi\, V(0)\cdot n,\\
L' &= -\int_\Gamma \frac{1}{2}\,|\Delta\varphi|^2\; V(0)\cdot n\, d\Gamma.
\end{aligned}$$

The quantity $D^2\varphi\, n\cdot n$ is equal to $\partial^2\varphi/\partial n^2$. Let b_Ω be the oriented function associated with the domain Ω of class C^4. It is C^4 in a neighborhood of Γ. Therefore, in that neighborhood

$$\psi \overset{\text{def}}{=} \nabla\varphi\cdot\nabla b_\Omega, \quad \psi|_\Gamma = \frac{\partial\varphi}{\partial n}, \quad \frac{\partial}{\partial n}\left(\frac{\partial\varphi}{\partial n}\right) = \frac{\partial\psi}{\partial n} = \nabla\psi\cdot\nabla b_\Omega|_\Gamma\,,$$

$$\nabla\psi = \nabla(\nabla\varphi\cdot\nabla b_\Omega) = D^2 b_\Omega\nabla\varphi + D^2\varphi\nabla b_\Omega,$$

$$\nabla\psi\cdot\nabla b_\Omega = D^2 b_\Omega\nabla\varphi\cdot\nabla b_\Omega + D^2\varphi\nabla b_\Omega\cdot\nabla b_\Omega = D^2\varphi\nabla b_\Omega\cdot\nabla b_\Omega,$$

$$\boxed{\frac{\partial^2\varphi}{\partial n^2} = \frac{\partial}{\partial n}\left(\frac{\partial\varphi}{\partial n}\right) = D^2\varphi\nabla b_\Omega\cdot\nabla b_\Omega\big|_\Gamma = D^2\varphi\big|_\Gamma\; n\cdot n = \Delta\varphi|_\Gamma.}$$

We summarize the results in the next theorem.

Theorem 4.5. *Given $k=1,2$, let Ω be a bounded open subset of $\mathbf{R}^N$ of class C^{2k}. Assume that*

$$V \in C^0([0,\tau[\,;W_0^{k,\infty}(D;\mathbf{R}^N))$$

for some bounded open domain D in $\mathbf{R}^N$ such that $\overline{\Omega}\subset D$. Then for $k=1$

$$d\lambda(\Omega;V) = \inf_{\varphi\in\tilde E^1(\Omega)} -\int_\Gamma \left|\frac{\partial\varphi}{\partial n}\right|^2 V(0)\cdot n\, d\Gamma,$$

$$\tilde E^1(\Omega) \overset{\text{def}}{=} \left\{\varphi\in H_0^1(\Omega)\cap H^2(\Omega)\ :\ \begin{array}{c} -\Delta\varphi = \lambda(\Omega)\varphi \ \text{in } \Omega\\ \text{and } \displaystyle\int_\Omega |\varphi|^2\,dx = 1\end{array}\right\};$$

for $k=2$

$$d\lambda(\Omega;V) = \inf_{\varphi\in\tilde E^2(\Omega)} -\int_\Gamma \left|\frac{\partial^2\varphi}{\partial n^2}\right|^2 V(0)\cdot n\, d\Gamma,$$

$$\tilde E^2(\Omega_t) \overset{\text{def}}{=} \left\{\varphi\in H_0^2(\Omega)\cap H^4(\Omega)\ :\ \begin{array}{c} \Delta(\Delta\varphi) = \lambda(\Omega)\varphi \ \text{in } \Omega\\ \text{and } \displaystyle\int_\Omega |\varphi|^2\,dx = 1\end{array}\right\}.$$

In both cases $\lambda(\Omega)$ has a Hadamard semiderivative in the sense of Definition 3.2 (iii) *in Chapter* 9 *which is continuous with respect to $V(0)\in C_0(D;\mathbf{R}^N)$:*

$$d\lambda(\Omega;V) = d_H\lambda(\Omega;V(0)).$$

4.3 Linear Elasticity

The constructions and results of the previous section readily extend to the vectorial case of linear elasticity: find $U \in H_0^1(\Omega)^3$ such that

$$\forall W \in H_0^1(\Omega)^3, \quad \int_\Omega C\varepsilon(U) \cdot\cdot\, \varepsilon(W)\, dx = \int_\Omega F \cdot W\, dx \tag{4.16}$$

for some distributed loading $F \in L^2(\Omega)^3$ and a *constitutive law* C which is a bilinear symmetric transformation of

$$\mathrm{Sym} \stackrel{\text{def}}{=} \left\{\tau \in \mathcal{L}(\mathbf{R}^3; \mathbf{R}^3) : {}^*\tau = \tau\right\}$$

($\mathcal{L}(\mathbf{R}^3; \mathbf{R}^3)$ is the space of all linear transformations of $\mathbf{R}^3$ or 3×3-matrices) under the following assumption.

Assumption 4.1.
The *constitutive law* is a linear bijective and symmetric transformation $C\colon \mathrm{Sym} \to \mathrm{Sym}$ for which there exists a constant $\alpha > 0$ such that $C\tau \cdot\cdot\, \tau \geq \alpha\, \tau \cdot\cdot\, \tau$ for all $\tau \in \mathrm{Sym}$. □

For instance, for the Lamé constants $\mu > 0$ and $\lambda \geq 0$, the special constitutive law $C\tau = 2\mu\,\tau + \lambda\, \mathrm{tr}\,\tau\, I$ satisfies Assumption 4.1 with $\alpha = 2\mu$.

The associated bilinear form is

$$a_\Omega(U, W) \stackrel{\text{def}}{=} \int_\Omega C\varepsilon(U) \cdot\cdot\, \varepsilon(W)\, dx,$$

where the unknown is now a vector function. To make sense of (4.16) we shall use Korn's inequality on a larger bounded open Lipschitzian domain D, $\Omega \subset D$:

$$\exists c_D > 0,\ \forall W \in H_0^1(D)^3, \quad \int_D |W|^2\, dx \leq c_D \int_D |\varepsilon(W)|^2\, dx.$$

In view of the considerations of the previous section, for a bounded open domain Ω and the velocity fields $V \in C^0([0, \tau[\,; W_0^{1,\infty}(D; \mathbf{R}^3))$, for all $0 \leq t < \tau$

$$\exists c_D > 0,\ \forall W \in H_0^1(\Omega_t(V))^3, \quad \int_{\Omega_t(V)} |W|^2\, dx \leq c_D \int_{\Omega_t(V)} |\varepsilon(W)|^2\, dx.$$

So for any $F \in L^2(D)^3$, the variational equation (4.16) with Ω_t in place of Ω has a unique solution U_t in $H_0^1(\Omega_t(V))^3$.

With the same assumptions the first eigenvalue is given by the Rayleigh quotient

$$\lambda(\Omega_t(V)) = \inf\left\{ \frac{a_{\Omega_t}(U, U)}{\int_{\Omega_t} |U|^2\, dx} : \forall U \in H_0^1(\Omega_t)^3,\ U \neq 0 \right\}.$$

In view of the previous constructions $H_0^1(\Omega_t)$ can be replaced by $H_0^1(\Omega_t; D)$:

$$\lambda(\Omega_t(V)) = \inf\left\{ \frac{a_{\Omega_t}(U, U)}{\int_{\Omega_t} |U|^2\, dx} : \forall U \in H_0^1(\Omega_t; D)^3,\ U \neq 0 \right\}. \tag{4.17}$$

Theorem 4.6. *Let Ω be a bounded open subset of $\mathbf{R}^3$. Assume that*

$$V \in C^0([0,\tau[\,; W_0^{1,\infty}(D;\mathbf{R}^3))$$

for some bounded open Lipschitzian domain D in $\mathbf{R}^3$ such that $\Omega \subset D$. There exists at least one nonzero solution $U \in H_0^1(\Omega_t; D)^3$ to the minimization problem (4.17), $\lambda(\Omega_t(V)) \geq \lambda(D) > 0$, and

$$\forall W \in H_0^1(D)^3, \quad \int_D |W|^2\, dx \leq \lambda(D)^{-1} \int_D \|\varepsilon(W)\|^2\, dx.$$

The solutions are completely characterized by the following variational equation: there exists $U \in H_0^1(\Omega_t; D)^3$ such that

$$\forall W \in H_0^1(\Omega_t; D)^3, \quad a_D(U,W) = \lambda(\Omega_t(V)) \int_D U \cdot W\, dx \tag{4.18}$$

or equivalently

$$\forall W \in H_0^1(\Omega_t)^3, \quad a_{\Omega_t}(U,W) = \lambda(\Omega_t(V)) \int_{\Omega_t} U \cdot W\, dx. \tag{4.19}$$

G. Auchmuty [**1**]'s dual variational principle for this eigenvalue problem can be chosen as

$$\mu(\Omega_t) \stackrel{\text{def}}{=} \inf \left\{ L(\Omega_t, U) \,:\, U \in H_0^1(\Omega_t)^3 \right\}, \tag{4.20}$$

$$L(\Omega_t, U) \stackrel{\text{def}}{=} \frac{1}{2} a_{\Omega_t}(U,U) - \left[\int_{\Omega_t} |U|^2\, dx \right]^{1/2}. \tag{4.21}$$

By using the embedding of $H_0^1(\Omega_t)$ into $H_0^1(D)$, this problem can be rewritten as

$$\mu(\Omega_t) \stackrel{\text{def}}{=} \inf \left\{ L(D, U) \,:\, U \in H_0^1(\Omega_t; D)^3 \right\}, \tag{4.22}$$

$$L(D, U) \stackrel{\text{def}}{=} \frac{1}{2} a_D(U,U) - \left[\int_D |U|^2\, dx \right]^{1/2}. \tag{4.23}$$

Theorem 4.7. *Let Ω be a bounded open subset of $\mathbf{R}^3$. Assume that*

$$V \in C^0([0,\tau[\,; W_0^{1,\infty}(D;\mathbf{R}^3))$$

for some bounded open Lipschitzian domain D in $\mathbf{R}^3$ such that $\Omega \subset D$. Then for $0 \leq t < \tau$,

$$\mu(\Omega_t) = -\frac{1}{2\lambda(\Omega_t)} \tag{4.24}$$

and the set of minimizers of (4.20) is given by

$$E(\Omega_t) \stackrel{\text{def}}{=} \left\{ U \in H_0^1(\Omega_t; D)^3 \,:\, \begin{array}{c} U \text{ is solution of (4.18) and} \\ \left\{ \int_D |U|^2\, dx \right\}^{1/2} = 1/\lambda(\Omega_t) \end{array} \right\}. \tag{4.25}$$

From the identity

$$\boxed{\lambda(\Omega_t) = -\frac{1}{2\mu(\Omega_t)},}$$

if $d\mu(\Omega; V)$ exists, then $d\lambda(\Omega; V)$ exists and

$$\boxed{d\lambda(\Omega; V) = \frac{1}{2\mu(\Omega)^2}\, d\mu(\Omega; V) = 2\,\lambda(\Omega)^2\, d\mu(\Omega; V).}$$

We again use the function space parametrization of section 2.4 in conjunction with Theorem 2.1. From the characterization (4.2) of $H_0^1(\Omega_t; D)$, define the new functional: for each $U \in H_0^1(D)^3$

$$\begin{aligned}\tilde{L}(t, U) &\overset{\text{def}}{=} L(D, U \circ T_t^{-1}(V)) \\ &= \frac{1}{2} a_D(U \circ T_t^{-1}(V), U \circ T_t^{-1}(V)) - \left[\int_D |U \circ T_t^{-1}(V)|^2\, dx\right]^{1/2}.\end{aligned}$$

After a change of variables for $U \in H_0^1(D)^3$,

$$\tilde{L}(t, U) = \frac{1}{2}\int_D C\varepsilon(U \circ T_t^{-1}) \circ T_t \cdot\cdot\, \varepsilon(U \circ T_t^{-1}) \circ T_t\, J_t\, dx - \left[\int_D J_t\, |U|^2\, dx\right]^{1/2},$$

where $J_t = \det(DT_t)$ and

$$\begin{aligned} D(U \circ T_t^{-1}) \circ T_t &= D(U)(DT_t)^{-1}, \\ 2\,\varepsilon(U \circ T_t^{-1}) \circ T_t &= D(U)(DT_t)^{-1} + {}^*(DT_t)^{-1}\, {}^*D(U). \end{aligned}$$

Defining the transformation

$$4\,\widetilde{C}(t)\,\tau \cdot\cdot\, \sigma \overset{\text{def}}{=} C\, \left\{\tau(DT_t)^{-1} + {}^*(DT_t)^{-1}\, {}^*\tau\right\} \cdot\cdot\, \left\{\sigma(DT_t)^{-1} + {}^*(DT_t)^{-1}\, {}^*\sigma\right\},$$

the previous expression can be written in the following more compact form:

$$\tilde{L}(t, U) = \frac{1}{2}\int_D \widetilde{C}(t) D(U) \cdot\cdot\, D(U)\, J_t\, dx - \left[\int_D J_t\, |U|^2\, dx\right]^{1/2}.$$

To apply Theorem 2.1, choose

$$X(t) \overset{\text{def}}{=} \left\{U^t \overset{\text{def}}{=} U_t \circ T_t \,:\, \forall U_t \in E(\Omega_t)\right\}$$

endowed with the weak topology of $H_0^1(D)^3$. Assumption (H1) is clearly satisfied. For all $U \neq 0$, $\tilde{L}(t, U)$ is differentiable. For $0 \neq U \in H_0^1(D)^3$,

$$\begin{aligned}\partial_t \tilde{L}(t, U) = \frac{1}{2}\int_D &\widetilde{C}'(t) D(U) \cdot\cdot\, D(U)\, J_t + \widetilde{C}(t) D(U) \cdot\cdot\, D(U)\, J_t'\, dx \\ &- \frac{1}{\left\{\int_D J_t\, |U|^2\, dx\right\}^{1/2}} \int_D |U|^2\, J_t'\, dx,\end{aligned}$$

where

$$
\begin{aligned}
4\,\widetilde{C}'(t)\,\tau\cdot\cdot\,\sigma = & \,C\,\{\tau\mathcal{T}'(t) + {}^*\mathcal{T}'(t)\,{}^*\tau\}\cdot\cdot\,\left\{\sigma(DT_t)^{-1} + {}^*(DT_t)^{-1}\,{}^*\sigma\right\} \\
& + C\,\left\{\tau(DT_t)^{-1} + {}^*(DT_t)^{-1}\,{}^*\tau\right\}\cdot\cdot\,\{\sigma\mathcal{T}'(t) + {}^*\mathcal{T}'(t)\,{}^*\sigma\}\,, \\
\mathcal{T}(t) \stackrel{\text{def}}{=} & \,(DT_t)^{-1}, \quad \mathcal{T}'(t) = -(DT_t)^{-1}DV(t)\circ T_t.
\end{aligned}
$$

Hence assumptions (H2) and (H4) are satisfied. In $t = 0$ the above expressions simplify. For $U \in E(\Omega)$

$$
\begin{aligned}
\partial_t \tilde{L}(0, U) = & \,\frac{1}{2}\int_D \widetilde{C}'(0)D(U)\cdot\cdot\, D(U) + C\,\varepsilon(U)\cdot\cdot\,\varepsilon(U)\operatorname{div}V(0)\,dx \\
& - \frac{1}{\left\{\int_D |U|^2\,dx\right\}^{1/2}}\int_D |U|^2 \operatorname{div}V(0)\,dx
\end{aligned}
$$

and for all U and W

$$
\begin{aligned}
2\,\widetilde{C}'(0)\,D(U)\cdot\cdot\,D(W) = & - C\,\{D(U)\,DV(0) + {}^*DV(0)\,{}^*D(U)\}\cdot\cdot\,\varepsilon(W) \\
& - C\,\varepsilon(U)\cdot\cdot\,\{D(W)\,DV(0) + {}^*DV(0)\,{}^*D(W)\}\,.
\end{aligned}
$$

This is another example of the application of Theorem 2.1 to a case where the set of minimizers is not a singleton and for which we have a complete description of the Eulerian semiderivative.

Theorem 4.8. *Let Ω be a bounded open subset of $\mathbf{R}^3$. Assume that*

$$V \in C^0([0,\tau[\,; W_0^{1,\infty}(D;\mathbf{R}^3))$$

for some bounded open Lipschitzian domain D in $\mathbf{R}^3$ such that $\overline{\Omega} \subset D$. Then

$$
\begin{aligned}
\frac{1}{2}d\lambda(\Omega;V) = & \inf_{U\in\tilde{E}(\Omega)} -\int_\Omega \widetilde{C}'(0)D(U)\cdot\cdot\, D(U)\,dx \\
& \qquad + \int_\Omega \left[\frac{1}{2}C\,\varepsilon(U)\cdot\cdot\,\varepsilon(U) - \lambda(\Omega)|U|^2\right]\operatorname{div}V(0)\,dx \\
= & \inf_{U\in\tilde{E}(\Omega)} -\int_\Omega C\,\varepsilon(U)\cdot\cdot\,\{D(U)\,DV(0) + {}^*DV(0)\,{}^*D(U)\}\,dx \\
& \qquad + \int_\Omega \left[\frac{1}{2}C\,\varepsilon(U)\cdot\cdot\,\varepsilon(U) - \lambda(\Omega)|U|^2\right]\operatorname{div}V(0)\,dx, \\
\tilde{E}(\Omega) \stackrel{\text{def}}{=} & \left\{ U \in H_0^1(\Omega)^3 \;:\; \begin{array}{l} -\overrightarrow{\operatorname{div}}\,(\varepsilon(U)) = \lambda(\Omega)U \text{ in } \mathcal{D}(\Omega)' \\ \text{and } \displaystyle\int_\Omega |U|^2\,dx = 1 \end{array} \right\}.
\end{aligned}
$$

$\lambda(\Omega)$ has a Hadamard semiderivative in the sense of Definition 3.2 (iii) *in Chapter* 9 *which is continuous with respect to $V(0) \in W_0^{1,\infty}(D;\mathbf{R}^3)$:*

$$d\lambda(\Omega;V) = d_H\lambda(\Omega;V(0)).$$

If Ω is assumed to be of class C^2, the eigenvector functions belong to $H_0^1(\Omega)^3 \cap H^2(\Omega)^3$ and the volume expressions of the previous theorem can be expressed as boundary integrals as in section 2.5. In that case we can use formulae (2.49) and (2.50) to compute the partial derivative of $L_t = L(\Omega_t, U \circ T_t^{-1})$:

$$L_t = \frac{1}{2}\int_{\Omega_t} C\,\varepsilon(U \circ T_t^{-1})\cdot\cdot\,\varepsilon(U \circ T_t^{-1})\,dx - \left\{\int_{\Omega_t} |U \circ T_t^{-1}|^2\,dx\right\}^{1/2}.$$

From formula (2.49)

$$L' \stackrel{\text{def}}{=} \partial_t L_t|_{t=0} = \frac{1}{2}\int_\Gamma C\,\varepsilon(U)\cdot\cdot\,\varepsilon(U)\,V(0)\cdot n\,d\Gamma - \frac{1}{2}\frac{\int_\Gamma |U|^2\,V(0)\cdot n\,d\Gamma}{\left\{\int_\Omega |U|^2\,dx\right\}^{1/2}}$$
$$+ \int_\Omega C\,\varepsilon(U)\cdot\cdot\,\varepsilon(\dot U)\,dx - \frac{\int_\Omega U\cdot \dot U\,dx}{\left\{\int_\Omega |U|^2\,dx\right\}^{1/2}}.$$

For $U \in E(\Omega)$, $U = 0$ on Γ, $\lambda(\Omega)\|U\|_{L^2(\Omega)} = 1$, and $-\overrightarrow{\operatorname{div}}\,(\varepsilon(U)) = \lambda(\Omega)U$ in Ω,

$$L' = \frac{1}{2}\int_\Gamma C\,\varepsilon(U)\cdot\cdot\,\varepsilon(U)\,V(0)\cdot n\,d\Gamma + \int_\Omega C\,\varepsilon(U)\cdot\cdot\,\varepsilon(\dot U) - \lambda(\Omega)\,U\cdot\dot U\,dx$$
$$= \int_\Gamma \frac{1}{2}C\,\varepsilon(U)\cdot\cdot\,\varepsilon(U)\,V(0)\cdot n + [C\,\varepsilon(U)]n\cdot\dot U\,d\Gamma.$$

But $U = 0$ on Γ implies that $D_\Gamma(U) = 0$, $D(U) = [D(U)]n\ {}^*n$ on Γ, and from identity (2.50)

$$\dot U = -D(U)V(0) \in H_0^1(\Omega)^3 \quad \Rightarrow \dot U = -[D(U)]n\,V(0)\cdot n \text{ on } \Gamma$$
$$\Rightarrow\ L' = -\int_\Gamma [C\,\varepsilon(U)]n\cdot[D(U)]n\,V(0)\cdot n\,d\Gamma.$$

This expression can be rewritten only in terms of $\varepsilon(U)$ as follows:

$$2\,\varepsilon(U) = D(U)n\,{}^*n + n\,{}^*(D(U)n), \quad 2\,\varepsilon(U)n = D(U)n + (D(U)n)\cdot n\,n,$$
$$2\,\varepsilon(U)n\cdot n = 2\,D(U)n\cdot n,$$
$$[C\,\varepsilon(U)]n\cdot D(U)n = 2\,[C\,\varepsilon(U)]n\cdot\varepsilon(U)n - [C\,\varepsilon(U)]n\cdot n\ (\varepsilon(U)n)\cdot n.$$

We summarize the results in the next theorem.

Theorem 4.9. *Let Ω be a bounded open subset of $\mathbf{R}^3$ of class C^2. Assume that*

$$V \in C^0([0,\tau[;W_0^{1,\infty}(D;\mathbf{R}^3))$$

for some bounded open Lipschitzian domain D in $\mathbf{R}^3$ such that $\overline{\Omega} \subset D$. Then

$$d\lambda(\Omega;V) = \inf_{U\in\tilde E(\Omega)} -\int_\Gamma [C\,\varepsilon(U)]n\cdot[D(U)]n\,V(0)\cdot n\,d\Gamma,$$
$$d\lambda(\Omega;V) = \inf_{U\in\tilde E(\Omega)} -\int_\Gamma 2\,[C\,\varepsilon(U)]n\cdot\varepsilon(U)n - [C\,\varepsilon(U)]n\cdot n\ (\varepsilon(U)n)\cdot n\,V(0)\cdot n\,d\Gamma,$$
$$\tilde E(\Omega) \stackrel{\text{def}}{=} \left\{U \in H_0^1(\Omega)^3\cap H^2(\Omega)^3 :\ \begin{array}{l} -\overrightarrow{\operatorname{div}}\,(\varepsilon(U)) = \lambda(\Omega)\,U \text{ in } \Omega \\ \text{and } \int_\Omega |U|^2\,dx = 1\end{array}\right\}.$$

For the special constitutive law $C\tau = 2\mu\tau + \lambda \mathrm{tr}\,\tau\, I$*,*

$$d\lambda(\Omega; V) = \inf_{U\in\tilde{E}(\Omega)} -\int_\Gamma \left\{4\,\mu|[\varepsilon(U)]n|^2 + (\lambda - 2\mu)\,|\mathrm{tr}\,\varepsilon(U)|^2\right\}\, V(0)\cdot n\, d\Gamma.$$

$\lambda(\Omega)$ *has a Hadamard semiderivative in the sense of Definition* 3.2 (iii) *in Chapter* 9 *which is continuous with respect to* $V(0) \in C_0(D; \mathbf{R}^3)$*:*

$$d\lambda(\Omega; V) = d_H\lambda(\Omega; V(0)).$$

5 Saddle Point Formulation and Function Space Parametrization

5.1 An Illustrative Example

Let Ω be a bounded open domain in $\mathbf{R}^{\mathrm{N}}$ with a smooth boundary Γ. Let $y = y(\Omega)$ be the solution of the Neumann problem

$$-\triangle y + y = f \text{ in } \Omega, \quad \frac{\partial y}{\partial n} = 0 \text{ on } \Gamma, \tag{5.1}$$

where f is a fixed function in $H^1(\mathbf{R}^{\mathrm{N}})$. Associate with $y(\Omega)$ the objective function

$$J(\Omega) = \frac{1}{2}\int_\Omega |y(\Omega) - y_d|^2\, dx, \tag{5.2}$$

where y_d is a fixed function in $H^1(\mathbf{R}^{\mathrm{N}})$.

The solution of (5.1) coincides with the minimizing element of the following variational problem:

$$\inf\left\{E(\Omega, \varphi) : \varphi \in H^1(\Omega)\right\}, \tag{5.3}$$

$$E(\Omega, \varphi) \stackrel{\text{def}}{=} \frac{1}{2}\int_\Omega \left\{|\nabla\varphi|^2 + \varphi^2 - 2\,f\varphi\right\}\, dx. \tag{5.4}$$

The minimizing element y of (5.4) is the solution in $H^1(\Omega)$ of Euler's equation

$$dE(\Omega, y; \varphi) = 0, \quad \forall\varphi \in H^1(\Omega), \tag{5.5}$$

$$dE(\Omega, y; \varphi) = \int_\Omega \left[\nabla y \cdot \nabla\varphi + y\varphi - f\varphi\right]\, dx, \tag{5.6}$$

which is the *variational equation* for y.

The *objective function* $J(\Omega)$ is a shape functional, and the solution of (5.1) will be called the *state*. It is convenient to introduce the *objective function*

$$F(\Omega, \varphi) \stackrel{\text{def}}{=} \frac{1}{2}\int_\Omega |\varphi - y_d|^2\, dx, \tag{5.7}$$

which clearly expresses the dependence on Ω and φ. To sum up, we consider the objective function

$$J(\Omega) = F\big(\Omega, y(\Omega)\big), \tag{5.8}$$

where $y = y(\Omega)$ is the solution of

$$y \in H^1(\Omega), \quad \forall \varphi \in H^1(\Omega), \quad dE(\Omega, y; \varphi) = 0. \tag{5.9}$$

We wish to find an expression for the shape derivative $dJ(\Omega; V)$.

5.2 Saddle Point Formulation

The basic approach is the one of control theory. Equation (5.1) (or in its variational form (5.9)) is considered as a state constraint in the minimization problem. We construct a Lagrangian functional by introducing a *Lagrange multiplier* function or the so-called *adjoint state* ψ:

$$G(\Omega, \varphi, \psi) = F(\Omega, \varphi) + dE(\Omega, \varphi; \psi). \tag{5.10}$$

Then the objective function is given by

$$J(\Omega) = \min_{\varphi \in H^1(\Omega)} \sup_{\psi \in H^1(\Omega)} G(\Omega, \varphi, \psi) \tag{5.11}$$

since

$$\sup_{\psi \in H^1(\Omega)} G(\Omega, \varphi, \psi) = \begin{cases} F(\Omega, y(\Omega)), & \text{if } \varphi = y(\Omega), \\ +\infty, & \text{if } \varphi \neq y(\Omega). \end{cases} \tag{5.12}$$

In our example the Lagrangian G is convex and continuous with respect to the variable φ and concave and continuous with respect to the variable ψ. Moreover, the space $H^1(\Omega)$ is convex and closed. So the functional G has a saddle point if and only if the saddle point equations have a solution (y, p) (cf. I. Ekeland and R. Temam [**1**]):

$$p \in H^1(\Omega), \quad dG(\Omega, y, p; 0, \psi) = 0, \; \forall \psi \in H^1(\Omega), \tag{5.13}$$
$$y \in H^1(\Omega), \quad dG(\Omega, y, p; \varphi, 0) = 0, \; \forall \varphi \in H^1(\Omega). \tag{5.14}$$

They are completely equivalent to

$$y \in H^1(\Omega), \quad dE(\Omega, y; \psi) = 0, \; \forall \psi \in H^1(\Omega), \tag{5.15}$$
$$p \in H^1(\Omega), \quad dF(\Omega, y; \varphi) + d^2E(\Omega, y; p; \varphi) = 0, \; \forall \varphi \in H^1(\Omega), \tag{5.16}$$

or

$$-\triangle y + y = f \text{ in } \Omega, \quad \frac{\partial y}{\partial n} = 0 \text{ on } \Gamma, \tag{5.17}$$
$$-\triangle p + p + y - y_d = 0 \text{ in } \Omega, \quad \frac{\partial p}{\partial n} = 0 \text{ on } \Gamma. \tag{5.18}$$

System (5.15)–(5.16) has a unique solution in $H^1(\Omega) \times H^1(\Omega)$ which coincides with the unique saddle point of $G(\Omega, \varphi, \psi)$ in $H^1(\Omega) \times H^1(\Omega)$.

5.3 Function Space Parametrization

We have shown that the objective function $J(\Omega)$ can be expressed as a min max of a functional G with a unique saddle point (y, p) which is completely characterized by the variational equations (5.15)–(5.16). The same result holds when Ω is transformed into a domain $\Omega_t = T_t(\Omega)$ under the action of the velocity field V for $t \geq 0$:

$$J(\Omega_t) = \min_{\varphi \in H^1(\Omega_t)} \sup_{\psi \in H^1(\Omega_t)} G(\Omega_t, \varphi, \psi), \tag{5.19}$$

where the saddle point (y_t, p_t) is completely characterized by

$$y_t \in H^1(\Omega_t), \quad dE(\Omega_t, y_t; \psi) = 0, \quad \forall \psi \in H^1(\Omega_t), \tag{5.20}$$

$$p_t \in H^1(\Omega_t), \quad dF(\Omega_t, y_t; \varphi) + d^2E(\Omega_t, y_t; p_t, \varphi) = 0, \quad \forall \varphi \in H^1(\Omega_t). \tag{5.21}$$

We are looking for a theorem that will give an expression for the derivative of a min sup with respect to a parameter $t \geq 0$. However, in (5.19) the space $H^1(\Omega_t)$ depends on the parameter t. To get around this difficulty and obtain a min sup expression for $J(\Omega_t)$ over spaces that are independent of $t \geq 0$, we introduce the following parametrization:

$$H^1(\Omega_t) = \left\{\varphi \circ T_t^{-1} : \varphi \in H^1(\Omega)\right\} \tag{5.22}$$

since T_t and T_t^{-1} are diffeomorphisms. This parametrization does not affect the value of the saddle point $J(\Omega_t)$ but will change the parametrization of the functional G:

$$J(\Omega_t) = \inf_{\varphi \in H^1(\Omega)} \sup_{\psi \in H^1(\Omega)} G(\Omega_t, \varphi \circ T_t^{-1}, \psi \circ T_t^{-1}). \tag{5.23}$$

This parametrization is apparently unique to *shape analysis*. It amounts to introducing the new Lagrangian functional for all φ and ψ in $H^1(\Omega)$:

$$\widetilde{G}(t, \varphi, \psi) = G\left(T_t(\Omega), \varphi \circ T_t^{-1}, \psi \circ T_t^{-1}\right). \tag{5.24}$$

Our next objective is to find an expression for the limit

$$dg(0) = \lim \frac{g(t) - g(0)}{t}, \tag{5.25}$$

where

$$g(t) = J(\Omega_t) = \inf_{\varphi \in H^1(\Omega)} \sup_{\psi \in H^1(\Omega)} \widetilde{G}(t, \varphi, \psi). \tag{5.26}$$

This will be done in the next section.

Before closing, it is useful to look at the expression for $\widetilde{G}$ and the resulting saddle point (y^t, p^t) in $H^1(\Omega) \times H^1(\Omega)$. By definition, $\widetilde{G}$ is given by the expression

$$\begin{aligned}\widetilde{G}(t, \varphi, \psi) =& \frac{1}{2} \int_{\Omega_t} |\varphi \circ T_t^{-1} - y_d|^2 \, dx \\ &+ \int_{\Omega_t} \left[\nabla(\varphi \circ T_t^{-1}) \cdot \nabla(\psi \circ T_t^{-1}) \right. \\ &\qquad \left. + (\varphi \circ T_t^{-1}) - f\,(\psi \circ T_t^{-1})\right] \, dx,\end{aligned} \tag{5.27}$$

and its saddle point is the solution of the variational equations

$$\begin{aligned} & y^t \in H^1(\Omega) \text{ and } \forall \psi \text{ in } H^1(\Omega), \\ & \int_{\Omega_t} \left[\nabla(y^t \circ T_t^{-1}) \cdot \nabla(\psi \circ T_t^{-1}) \right. \\ & \qquad \left. + (y^t \circ T_t^{-1})(\psi \circ T_t^{-1}) - f(\psi \circ T_t^{-1}) \right] dx = 0, \end{aligned} \tag{5.28}$$

$$\begin{aligned} & p^t \in H^1(\Omega) \text{ and } \forall \phi \text{ in } H^1(\Omega), \\ & \int_{\Omega_t} [(y^t \circ T_t^{-1} - y_d)(\varphi \circ T_t^{-1}) + \nabla(y^t \circ T_t^{-1}) \cdot \nabla(\varphi \circ T_t^{-1}) \\ & \qquad + (y^t \circ T_t^{-1})(\varphi \circ T_t^{-1})] \, dx = 0. \end{aligned} \tag{5.29}$$

It is readily seen that $(y^t \circ T_t^{-1}, p^t \circ T_t^{-1})$ coincide with the saddle point (y_t, p_t) in $H^1(\Omega_t) \times H^1(\Omega_t)$:

$$y_t = y^t \circ T_t^{-1}, \quad p_t = p_t \circ T_t^{-1},$$

or equivalently

$$y^t = y_t \circ T_t, \quad p^t = p_t \circ T_t. \tag{5.30}$$

The solutions (y^t, p^t) can easily be interpreted as the solutions (y_t, p_t) on Ω_t transported back to the fixed domain Ω by the transformation T_t.

In view of this observation, we can rewrite expressions (5.27) to (5.29) on the fixed domain Ω by using the coordinate transformation T_t. Expression (5.27) becomes

$$\begin{aligned} \widetilde{G}(t, \varphi, \psi) = & \frac{1}{2} \int_\Omega |\varphi - y_d \circ T_t|^2 J_t dx \\ & + \int_\Omega A(t) \, \nabla\varphi \cdot \nabla\psi + J_t \left[\varphi\psi - (f \circ T_t)\psi \right] \, dx, \end{aligned} \tag{5.31}$$

where for $t > 0$ small,

$$DT_t = \text{Jacobian matrix of } T_t, \tag{5.32}$$

$$J_t = \det DT_t \quad (\text{since } \det DT_t = |\det DT_t| \text{ for } t \geq 0 \text{ small}), \tag{5.33}$$

$$A(t) = J_t [DT_t]^{-1} \; {}^*[DT_t]^{-1}. \tag{5.34}$$

Similarly the variational equations (5.28)–(5.29) reduce to

$$\begin{aligned} & y_t \in H^1(\Omega) \text{ and } \forall \psi \in H^1(\Omega), \\ & \int_\Omega \left\{ A(t) \nabla y^t \cdot \nabla\psi + J_t \left[y^t \psi - (f \circ T_t), \psi \right] \right\} \, dx = 0, \end{aligned} \tag{5.35}$$

$$\begin{aligned} & p^t \in H^1(\Omega) \text{ and } \forall \varphi \in H^1(\Omega), \\ & \int_\Omega \left\{ A(t) \nabla p^t \cdot \nabla\varphi + J_t \left[p^t \varphi + (y^t - y_d \circ T_t)\varphi \right] \right\} \, dx = 0. \end{aligned} \tag{5.36}$$

5.4 Differentiability of a Saddle Point with Respect to a Parameter

Consider a functional

$$G\colon [0,\tau] \times X \times Y \to \mathbf{R} \tag{5.37}$$

for some $\tau > 0$ and sets X and Y. For each t in $[0,\tau]$ define

$$g(t) = \inf_{x \in X} \sup_{y \in Y} G(t,x,y) \tag{5.38}$$

and the sets

$$X(t) = \left\{ x^t \in X : \sup_{y \in Y} G(t,x^t,y) = g(t) \right\}, \tag{5.39}$$

$$Y(t,x) = \left\{ y^t \in Y : G(t,x,y^t) = \sup_{y \in Y} \; G(t,x,y) \right\}. \tag{5.40}$$

Similarly define

$$h(t) = \sup_{y \in Y} \inf_{x \in X} G(t,x,y) \tag{5.41}$$

and the sets

$$Y(t) = \left\{ y^t \in Y : \inf_{x \in X} G(t,x,y^t) = h(t) \right\}, \tag{5.42}$$

$$X(t,y) = \left\{ x^t \in X : G(t,x^t,y) = \inf_{x \in X} G(t,x,y) \right\}. \tag{5.43}$$

In general we always have the inequality

$$h(t) \le g(t). \tag{5.44}$$

To complete the set of notations, we introduce the set of *saddle points*

$$S(t) = \left\{ (x,y) \in X \times Y : g(t) = G(t,x,y) = h(t) \right\}, \tag{5.45}$$

which may be empty.

Our objective is to find realistic conditions under which the limit

$$dg(0) = \lim_{t \searrow 0} \frac{g(t) - g(0)}{t} \tag{5.46}$$

exists. A case of special interest is when G has a saddle point for all t in $[0,\tau]$. It can be viewed as an extension of Theorem 2.1 in section 2.3 on the differentiability of a min with respect to a parameter. It is used when the functional to be minimized is a function of the state, which is itself a function of the domain through the boundary value problem. In that case the saddle point equations coincide with the "state equation" and the "adjoint state equation" as illustrated in the previous section. The main advantage of this approach is to avoid the problem of the existence and characterization of the derivative of the state x^t with respect to t. In a control problem this would be the directional derivative of the state with respect to the control variable. In particular, it is not necessary to invoke any implicit function theorem with possibly restrictive differentiability conditions. It will be sufficient to check two continuity conditions for the set-valued maps $X(\cdot)$ and $Y(\cdot)$. To complete this discussion we recall the following.

Lemma 5.1. *Fix t in $[0,\tau]$. Then*

$$\forall (x^t, y^t) \in X(t) \times Y(t), \quad h(t) \le G(t, x^t, y^t) \le g(t), \tag{5.47}$$

and if $h(t) = g(t)$,

$$X(t) \times Y(t) = S(t). \tag{5.48}$$

Proof. (i) If $X(t) \times Y(t) = \varnothing$, there is nothing to prove. If there exist $x^t \in X(t)$ and $y^t \in Y(t)$, then by definition

$$h(t) = \inf_{x \in X} G(t, x, y^t) \le G(t, x^t, y^t) \le \sup_{y \in Y} G(t, x^t, y^t) = g(t). \tag{5.49}$$

(ii) If $h(t) = g(t)$, then in view of (5.49), $X(t) \times Y(t) \subset S(t)$. Conversely, if there exists $(x^t, y^t) \in S(t)$, then $h(t) = G(t, x^t, y^t) = g(t)$ and by the definition of $X(t)$ and $Y(t)$, $(x^t, y^t) \in X(t) \times Y(t)$. □

It is important to keep in mind that identity (5.48) is always true when

$$h(t) = \sup_{y \in Y} \inf_{x \in X} G(t, x, y) = \inf_{x \in X} \sup_{y \in Y} G(t, x, y) = g(t)$$

but that $S(t)$ may be empty.

Theorem 5.1 (R. Correa and A. Seeger [**1**]). *Let the sets X and Y, the real number $\tau > 0$, and the functional*

$$G \colon [0, \tau] \times X \times Y \to \mathbf{R}$$

be given. Assume that the following assumptions hold:

(H1) *$S(t) \neq \varnothing$, $0 \le t \le \tau$;*

(H2) *for all (x, y) in $[\cup\{X(t) : 0 \le t \le \tau\} \times Y(0)] \cup [X(0) \times \cup\{Y(t) : 0 \le t \le \tau\}]$ the partial derivative $\partial_t G(t, x, y)$ exists everywhere in $[0, \tau]$;*

(H3) *there exists a topology $\mathcal{T}_X$ on X such that for any sequence $\{t_n : 0 < t_n \le \tau\}$, $t_n \to t_0 = 0$, there exist $x^0 \in X(0)$ and a subsequence $\{t_{n_k}\}$ of $\{t_n\}$, and for each $k \ge 1$, there exists $x_{n_k} \in X(t_{n_k})$ such that*

(i) *$x_{n_k} \to x^0$ in the $\mathcal{T}_X$-topology, and*

(ii) *for all y in $Y(0)$,*

$$\liminf_{\substack{t \searrow 0 \\ k \to \infty}} \partial_t G(t, x_{n_k}, y) \ge \partial_t G(0, x^0, y); \tag{5.50}$$

(H4) *there exists a topology* $\mathcal{T}_Y$ *on* Y *such that for any sequence* $\{t_n : 0 < t_n \le \tau\}$, $t_n \to t_0 = 0$, *there exist* $y^0 \in Y(0)$ *and a subsequence* $\{t_{n_k}\}$ *of* $\{t_n\}$, *and for each* $k \ge 1$, *there exists* $y_{n_k} \in Y(t_{n_k})$ *such that*

(i) $y_{n_k} \to y^0$ *in the* $\mathcal{T}_Y$*-topology, and*

(ii) *for all* x *in* $X(0)$,

$$\limsup_{\substack{t \searrow 0 \\ k \to \infty}} \partial_t G(t, x, y_{n_k}) \le \partial_t G(0, x, y^0). \tag{5.51}$$

Then there exists $(x^0, y^0) \in X(0) \times Y(0)$ *such that*

$$\begin{aligned} dg(0) = \inf_{x \in X(0)} \sup_{y \in Y(0)} \partial_t G(0, x, y) &= \partial_t G(0, x^0, y^0) \\ &= \sup_{y \in Y(0)} \inf_{x \in X(0)} \partial_t G(0, x, y). \end{aligned} \tag{5.52}$$

Thus (x^0, y^0) *is a saddle point of* $\partial_t G(0, x, y)$ *on* $X(0) \times Y(0)$.

Proof. (i) We first establish upper and lower bounds to the differential quotient

$$\frac{\Delta(t)}{t}, \quad \Delta(t) \stackrel{\text{def}}{=} g(t) - g(0).$$

Choose arbitrary x_0 in $X(0)$, x_t in $X(t)$, y_0 in $Y(0)$, and y_t in $Y(t)$. Then by definition

$$\begin{aligned} G(t, x_t, y_0) \le G(t, x_t, y_t) \le G(t, x_0, y_t), \\ -G(0, x_t, y_0) \le -G(0, x_0, y_0) \le -G(0, x_0, y_t). \end{aligned}$$

Add the above two chains of inequalities to obtain

$$G(t, x_t, y_0) - G(0, x_t, y_0) \le \Delta(t) \le G(t, x_0, y_t) - G(0, x_0, y_t).$$

By assumption (H2), there exist θ_t, $0 < \theta_t < 1$, and α_t, $0 < \alpha_t < 1$, such that

$$\begin{aligned} G(t, x_t, y_0) - G(0, x_t, y_0) = t \partial_t G(\theta_t t, x_t, y_0), \\ G(t, x_0, y_t) - G(0, x_0, y_t) = t \partial_t G(\alpha_t t, x_0, y_t), \end{aligned}$$

and by dividing by $t > 0$,

$$\partial_t G(\theta_t t, x_t, y_0) \le \frac{\Delta(t)}{t} \le \partial_t G(\alpha_t t, x_0, y_t). \tag{5.53}$$

(ii) Define

$$\underline{d}g(0) = \liminf_{t \searrow 0} \frac{\Delta(t)}{t}, \quad \overline{d}g(0) = \limsup_{t \searrow 0} \frac{\Delta(t)}{t}.$$

There exists a sequence $\{t_n : 0 < t_n \le \tau\}$, $t_n \to 0$, such that

$$\lim_{n \to \infty} \frac{\Delta(t_n)}{t_n} = \underline{d}g(0).$$

By assumption (H3), there exist $x^0 \in X(0)$ and a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ for each $k \geq 1$, there exists $x_{n_k} \in X(t_{n_k})$ such that $x_{n_k} \to x^0$ in $\mathcal{T}_X$, and

$$\forall y \in Y(0), \quad \liminf_{\substack{t \searrow 0 \\ k \to \infty}} \partial_t G(t, x_{n_k}, y) \geq \partial_t G(0, x^0, y).$$

Thus, from the first part of the estimate (5.53) for any $y \in Y(0)$ and $t = t_{n_k}$,

$$\partial_t G(\theta_{t_{n_k}} t_{n_k}, x_{n_k}, y) \leq \frac{\Delta(t_{n_k})}{t_{n_k}}$$

and

$$\partial_t G(0, x^0, y) \leq \liminf_{k \to \infty} \partial_t G(\theta_{t_{n_k}} t_{n_k}, x_{n_k}, y) \leq \lim_{k \to \infty} \frac{\Delta(t_{n_k})}{t_{n_k}} = \underline{d}g(0).$$

Therefore,

$$\exists x^0 \in X(0), \forall y \in Y(0), \quad \partial_t G(0, x^0, y) \leq \underline{d}g(0)$$

and

$$\inf_{x \in X(0)} \sup_{y \in Y(0)} \partial_t G(0, x, y) \leq \sup_{y \in Y(0)} \partial_t G(0, x^0, y) \leq \underline{d}g(0). \tag{5.54}$$

By a dual argument and assumption (H4) we also obtain

$$\begin{aligned} &\exists y^0 \in Y(0), \forall x \in X(0), \quad \partial_t G(0, x, y^0) \geq \overline{d}g(0), \\ \overline{d}g(0) \leq &\inf_{x \in X(0)} \partial_t G(0, x, y^0) \leq \sup_{y \in Y(0)} \inf_{x \in X(0)} \partial_t G(0, x, y), \end{aligned} \tag{5.55}$$

and necessarily

$$\begin{aligned} &\inf x \in X(0) \sup_{y \in Y(0)} \partial_t G(0, x, y) \\ &= \underline{d}g(0) = \overline{d}g(0) = \sup_{y \in Y(0)} \inf_{x \in X(0)} \partial_t G(0, x, y). \end{aligned}$$

In particular, from (5.54) and (5.55),

$$\sup_{y \in Y(0)} \partial_t G(0, x^0, y) = dg(0) = \inf_{x \in X(0)} \inf \partial_t G(0, x, y^0)$$

and (x^0, y^0) is a saddle point of $\partial_t G(0, \cdot, \cdot)$. □

Remark 5.1.
In the applications this formulation of the theorem presents some definite technical advantages over its original version.

(i) From identity (5.52), $\partial_t G(0, \cdot, \cdot)$ has a saddle point with respect to $X(0) \times Y(0)$.

(ii) Another important feature is the use of subsequences in assumptions (H3) and (H4). This makes it possible to work with weak topologies in reflexive Banach spaces and use the eventual boundedness of the sets of saddle points.

(iii) Finally, assumption (H2) and conditions (5.50) and (5.51) in (H3) and (H4) need only be checked on the family of saddle points at $t = 0$. For instance, the first part of assumptions (H3) and (H4) could be satisfied in $H^1(\Omega) \times H^1(\Omega)$. Yet, if the saddle points are smoother, say in $H^2(\Omega) \times H^2(\Omega)$, this extra smoothness can be used to satisfy (H2) and (5.50) and (5.51) in (H3) and (H4).

5.5 Application of the Theorem

Our example has a unique saddle point (y^t, p^t) for $t \geq 0$ small, and we can use the corollary to Theorem 5.1. The set-valued maps X and Y reduce to ordinary functions

$$t \mapsto X(t) = y^t, \quad t \mapsto Y(t) = p^t, \tag{5.56}$$

and it is sufficient to show their continuity at $t = 0$ in $H^1(\Omega)$. So we now check assumptions (H1) to (H4).

Assume that V belongs to $\mathcal{V}^1$, that is, $\mathcal{D}^1(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$, and that f and y belong to $H^1(\mathbf{R}^{\mathrm{N}})$. Choose $\tau > 0$ small enough such that

$$J_t = \det DT_t = |\det DT_t| = |J_t|, \quad 0 \leq t \leq \tau, \tag{5.57}$$

and that there exist constants $0 < \alpha < \beta$ such that

$$\forall \xi \in \mathbf{R}^{\mathrm{N}}, \quad \alpha|\xi|^2 \leq A(t)\,\xi \cdot \xi \leq \beta|\xi|^2 \text{ and } \alpha \leq J_t \leq \beta. \tag{5.58}$$

Since the bilinear forms associated with (5.35) and (3.34) are coercive, there exists a unique pair (y^t, p^t) solution of the system (5.35)–(5.36). Hence

$$\forall t \in [0, \tau], \quad X(t) = \{y^t\} \neq \varnothing, \; Y(t) = \{p^t\} \neq \varnothing. \tag{5.59}$$

So assumption (H1) is satisfied. To check (H2) we use expression (5.31) and compute for φ and ψ in $H^1(\Omega)$:

$$\begin{aligned} &\partial_t \widetilde{G}(t, \varphi, \psi) \\ &= \int_\Omega \left\{ \frac{1}{2}(\varphi - y_d \circ T_t)^2 \operatorname{div} V_t - (\varphi - y_d \circ T_t) \nabla y_d \cdot V_t J_t \right\} dx \\ &\qquad + \int_\Omega \{A'(t)\nabla\varphi \cdot \nabla\psi + \operatorname{div} V_t(\varphi\psi - f \circ T_t \psi) - J_t \nabla f \cdot V_t \psi\}\, dx, \end{aligned} \tag{5.60}$$

where

$$V_t(X) \overset{\text{def}}{=} V(T_t(X)), \; A'(t) = (\operatorname{div} V_t)I - {}^*DV_t - DV_t, \tag{5.61}$$

I is the identity matrix on $\mathbf{R}^{\mathrm{N}}$, and DV_t is the Jacobian matrix of V_t. By the choice of V in $\mathcal{D}^1(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}})$, $t \mapsto V_t$ and $t \mapsto DV_t$ are continuous on $[0, \tau]$. Moreover, f and y_d belong to $H^1(\mathbf{R}^{\mathrm{N}})$. As a result expression (5.60) is well-defined and $\partial_t \widetilde{G}(t, \varphi, \psi)$ exists everywhere in $[0, \tau]$ for all φ and ψ in $H^1(\Omega)$. This can be proved in many ways. For instance, we establish (5.60) for f and y_d in $\mathcal{D}(\mathbf{R}^{\mathrm{N}})$. Then we show that the affine map $(f, y_d) \mapsto \partial_t \widetilde{G}(\cdot, \varphi, \psi)$ is continuous from $H^1(\mathbf{R}^{\mathrm{N}}) \times H^1(\mathbf{R}^{\mathrm{N}})$ to

$C^1([0,\tau])$. So it extends by uniform continuity to all (f, y_d) in $H^1(\mathbf{R}^{\mathrm{N}}) \times H^1(\mathbf{R}^{\mathrm{N}})$ and density of $\mathcal{D}(\mathbf{R}^{\mathrm{N}})$ in $H^1(\mathbf{R}^{\mathrm{N}})$. Assumption (H2) is satisfied.

To check assumptions (H3)(i) and (H4)(i), we first show that for any sequence $\{t_n\} \subset [0,\tau]$, $t_n \to 0$, there exists a subsequence of $\{y^{t_n}\}$, still denoted by $\{y^{t_n}\}$, such that

$$\begin{aligned} y^{t_n} &\rightharpoonup y^0 = y \text{ in } H^1(\Omega)\text{-weak}, \\ p^{t_n} &\rightharpoonup p^0 = p \text{ in } H^1(\Omega)\text{-weak}, \end{aligned}$$

where (y,p) is the solution of system (5.15)–(5.16) or (5.17)–(5.18). By the choice of τ satisfying condition (5.58), there exists a constant $c > 0$ such that

$$\begin{aligned} \alpha \|y^t\|_{H^1(\Omega)} &\le \beta c \|f\|_{L^2(\mathbf{R}^{\mathrm{N}})}, \\ \alpha \|p^t\|_{H^1(\Omega)} &\le \beta c \|y^t - y_d\|_{L^2(\Omega)}. \end{aligned}$$

So the pair $\{y^t, p^t\}$ is bounded in $H^1(\Omega) \times H^1(\Omega)$ and there exist a subsequence $\{y^{t_n}, p^{t_n}\}$ and a pair (z,q) in $H^1(\Omega) \times H^1(\Omega)$ such that

$$y^{t_n} \rightharpoonup z \text{ in } H^1(\Omega)\text{-weak} \quad \text{and} \quad p^{t_n} \rightharpoonup q \text{ in } H^1(\Omega)\text{-weak}.$$

The pair (z,q) can be characterized by going to the limit in the variational equations (5.35)–(5.36):

$$\int_\Omega \left\{ A(t_n) \nabla y^{t_n} \cdot \nabla \psi + J_{t_n} \left[y^{t_n} \psi - (f \circ T_{t_n}) \psi \right] \right\} dx = 0,$$
$$\int_\Omega \left\{ A(t_n) \nabla p^{t_n} \cdot \nabla \varphi + J_{t_n} \left[p^{t_n} \varphi + (y^{t_n} - y_d \circ T_{t_n}) \varphi \right] \right\} dx = 0.$$

So we proceed as in section 2.4 and use Lemma 2.1 to obtain

$$\forall \psi, \quad \int_\Omega \{ \nabla z \cdot \nabla \psi + z \psi - f \psi \} \, dx = 0,$$
$$\forall \varphi, \quad \int_\Omega \{ \nabla q \cdot \nabla \varphi + q \varphi + (z - y_d) \varphi \} \, dx = 0.$$

By uniqueness $(z,q) = (y,p)$. We now proceed as in section 2.4 and prove that

$$y^{t_n} \to y \text{ in } H^1(\Omega)\text{-strong}, \quad p^{t_n} \to p \text{ in } H^1(\Omega)\text{-strong}$$

by the same argument. So assumptions (H3)(i) and (H4)(i) are satisfied for the strong topology of $H^1(\Omega)$. Finally, assumptions (H3)(ii) and (H4)(ii) are readily satisfied in view of the strong continuity of $(t,\varphi) \mapsto \partial_t \widetilde{E}(t,\varphi,\psi)$ and $(t,\psi) \mapsto \partial_t \widetilde{E}(t,\varphi,\psi)$. In fact, it would have been sufficient to check assumption (H3) with $H^1(\Omega)$-strong and (H4) with $H^1(\Omega)$-weak.

So all assumptions of Theorem 5.1 are satisfied and

$$\begin{aligned} dJ(\Omega;V) = &\int_\Omega \left\{ \frac{1}{2}(y - y_d)^2 \operatorname{div} V - (y - y_d) \nabla y_d \cdot V \right\} dx \\ &+ \int_\Omega \{ A'(0) \nabla y \cdot \nabla p + \operatorname{div} V(0), (yp - fp) - \nabla f \cdot V(0) p \} \, dx, \quad (5.62) \end{aligned}$$

where (y, p) is the solution of (5.17)–(5.18) or, in variational form,

$$y \in H^1(\Omega), \quad \forall \psi \in H^1(\Omega), \quad \int_\Omega \{\nabla y \cdot \nabla \psi + y\psi - f\psi\}\, dx = 0, \tag{5.63}$$

$$p \in H^1(\Omega), \quad \forall \varphi \in H^1(\Omega), \quad \int_\Omega \{\nabla p \cdot \nabla \varphi + p\varphi + (y - y_d)\varphi\}\, dx = 0. \tag{5.64}$$

5.6 Domain and Boundary Expressions for the Shape Gradient

Expression (5.62) for the shape gradient is a volume or domain integral. For y_d and f in $H^1(\mathbf{R}^\mathrm{N})$ it is readily seen that the map

$$V \mapsto dJ(\Omega; V) \colon \mathcal{D}^1(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N}) \to \mathbf{R} \tag{5.65}$$

is linear and continuous. So by Corollary 1 to the structure Theorem 3.6 in section 3.4 of Chapter 9, we know that for a domain Ω with a C^2-boundary Γ, there exists a scalar distribution $g(\Gamma)$ in $\mathcal{D}^1(\Gamma)'$ such that

$$dJ(\Omega; V) = \langle g(\Gamma), V \cdot n \rangle. \tag{5.66}$$

We now further characterize this boundary expression. In view of the assumptions on f, y_d, and Ω, the pair (y, p) is the solution in $H^2(\Omega) \times H^2(\Omega)$ of the system

$$-\triangle y + y = f \text{ in } \Omega, \quad \frac{\partial y}{\partial n} = 0 \text{ on } \Gamma, \tag{5.67}$$

$$-\triangle p + p + (y - y_d) = 0 \text{ in } \Omega, \quad \frac{\partial p}{\partial n} = 0 \text{ on } \Gamma. \tag{5.68}$$

Similarly, for V in $\mathcal{D}^1(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N})$ the system

$$-\mathrm{div}\, \left[A(t)\nabla y^t\right] + J_t y^t = J_t f \circ T_t \text{ in } \Omega, \quad \frac{\partial y^t}{\partial n} = 0 \text{ on } \Gamma, \tag{5.69}$$

$$-\mathrm{div}\, \left[A(t)\nabla p^t\right] + J_t p^t + (y^t - y_d \circ T_t)J_t = 0 \text{ in } \Omega, \quad \frac{\partial p^t}{\partial n} = 0 \text{ on } \Gamma \tag{5.70}$$

has a unique solution in $H^2(\Omega) \times H^2(\Omega)$ instead of $H^1(\Omega) \times H^1(\Omega)$. With this extra smoothness we can use the formula

$$\left.\frac{d}{dt} \int_{\Omega_t} F(t, x)\, dx\right|_{t=0} = \int_\Gamma F(0, x)V(0) \cdot n\, d\Gamma + \int_\Omega \frac{\partial F}{\partial t}(0, x)\, dx \tag{5.71}$$

for a sufficiently smooth function $F \colon [0, \tau] \times \mathbf{R}^\mathrm{N} \to \mathbf{R}$. We easily obtain

$$\begin{aligned}\partial_t \widetilde{G}(0, \varphi, \psi) = \int_\Gamma &\left\{\frac{1}{2}(\varphi - y_d)^2 + \nabla\varphi \cdot \nabla\psi + \varphi\psi - f\psi\right\}\, d\Gamma \\ &+ \int_\Omega \{(\varphi - y_d)\dot{\varphi} + \nabla\psi \cdot \nabla\dot{\varphi} + \psi\dot{\varphi}\}\, dx \\ &\qquad + \int_\Omega \left\{\nabla\varphi \cdot \nabla\dot{\psi} + \varphi\dot{\psi} - f\dot{\psi}\right\}\, dx,\end{aligned} \tag{5.72}$$

where

$$\dot{\varphi} = \frac{d}{dt}\varphi \circ T_t^{-1}\bigg|_{t=0} = -\nabla\varphi \cdot V(0) \tag{5.73}$$

and

$$\dot{\psi} = \frac{d}{dt}\ \psi \circ T_t^{-1}\bigg|_{t=0} = -\nabla\psi \cdot V(0). \tag{5.74}$$

Now substitute for (φ, ψ) the solution (y, p) of (5.63)–(5.64):

$$\begin{aligned}
&\partial_t \widetilde{G}(0, y, p) \\
&= \int_\Gamma \left\{ \frac{1}{2}(y - y_d)^2 + \nabla y \cdot \nabla p + yp - fp \right\} V \cdot n \, d\Gamma \\
&\qquad + \int_\Omega \ \{(y - y_d)(-\nabla y \cdot V) + \nabla p \cdot \nabla(-\nabla y \cdot V) + p(-\nabla p \cdot V)\} \, dx \\
&\qquad\qquad + \int_\Omega \{\nabla y \cdot \nabla(-\nabla p \cdot V) + y(-\nabla p \cdot V) - f(-\nabla p \cdot V)\} \, dx.
\end{aligned} \tag{5.75}$$

We recognize that the second term is (5.64) with $\varphi = -\nabla y \cdot V$ and that the third term is (3.63) with $\psi = -\nabla p \cdot V$. So they are both zero, and finally

$$dJ(\Omega; V) = \int_\Gamma \left\{ \frac{1}{2}(y - y_d)^2 + \nabla y \cdot \nabla p + yp - fp \right\} V \cdot n \, d\Gamma. \tag{5.76}$$

It must be emphasized that this last expression has been obtained under the assumption that both y and p belong to $H^2(\Omega)$. We shall see later that shape gradient can also be obtained by our technique for the finite element approximations of y and p. However, for piecewise linear elements, formula (5.76) fails since the finite element solutions y_h and p_h belong to $H^1(\Omega_h)$ but not to $H^2(\Omega_h)$. However, the domain formula (5.62) will remain true. The crucial point is that for the continuous problem, a smooth boundary plus f and y_d in $H^1(\mathbf{R}^\mathrm{N})$ put the solution (y, p) in $H^2(\Omega) \times H^2(\Omega)$. However, the smoothness of the finite element solution (y_h, p_h) cannot be improved.

6 Multipliers and Function Space Embedding

6.1 The Nonhomogeneous Dirichlet Problem

Let Ω be a bounded open domain in $\mathbf{R}^\mathrm{N}$ with a sufficiently smooth boundary Γ. Let $y = y(\Omega)$ be the solution of the nonhomogeneous Dirichlet problem

$$-\Delta y = f \text{ in } \Omega, \quad y = g \text{ on } \Gamma, \tag{6.1}$$

where f and g are fixed functions in $H^{1/2+\varepsilon}(\mathbf{R}^\mathrm{N})$ and $H^{2+\varepsilon}(\mathbf{R}^\mathrm{N})$, respectively, for some arbitrary fixed $\varepsilon > 0$. Associate with the solution of (6.1) the objective function

$$J(\Omega) = \frac{1}{2} \int_\Omega |y(\Omega) - y_d|^2 \, dx \tag{6.2}$$

for some fixed function y_d in $H^{1/2+\varepsilon}(\mathbf{R}^\mathrm{N})$ and some arbitrary fixed $\varepsilon > 0$. We want to compute the derivative of $J(\Omega)$ with respect to Ω subject to the state equation system (6.1). Our objective is to transform this problem into finding the saddle point of a volume Lagrangian functional. This technique can be applied to other boundary value problems with Dirichlet conditions.

6.2 A Saddle Point Formulation of the State Equation

When $g = 0$ problem (6.1) is equivalent to a variational problem on $H_0^1(\Omega)$. When $g \neq 0$ the extra constraint $\phi = g$ makes the Sobolev space dependent on g. To get around this difficulty, we introduce a Lagrange multiplier and the new functional

$$L(\phi, \psi, \mu) = \int_\Omega (\Delta\phi + f)\psi \, dx + \int_\Gamma (\phi - g)\mu \, d\Gamma \tag{6.3}$$

for all $\psi \in H^2(\Omega)$ and $\mu \in H^{1/2}(\Gamma)$. This is a convex-concave functional with a unique saddle point $(\hat{\phi}, \hat{\psi}, \hat{\mu})$ which is completely characterized by the equations

$$\Delta\hat{\phi} + f = 0 \text{ in } \Omega, \quad \hat{\phi} - g = 0 \text{ in } \Gamma, \tag{6.4}$$

$$\forall \phi \in H^2(\Omega), \quad \int_\Omega \Delta\phi \, \hat{\psi} \, dx + \int_\Gamma \phi \hat{\mu} \, d\Gamma = 0. \tag{6.5}$$

The last equation characterizes $\hat{\psi}$ and $\hat{\mu}$:

$$\Delta\hat{\psi} = 0 \text{ in } \Omega, \quad \hat{\psi} = 0 \text{ on } \Gamma, \tag{6.6}$$

$$\hat{\mu} = \frac{\partial\hat{\psi}}{\partial n} \text{ on } \Gamma \tag{6.7}$$

(cf., for instance, I. Ekeland and R. Temam [**1**, Prop. 1.6]). Of course, this implies that the saddle point is unique and given by

$$(\hat{\phi}, \hat{\psi}, \hat{\mu}) = (y, 0, 0). \tag{6.8}$$

The purpose of the above computation was to find out the form of the multiplier $\hat{\mu}$,

$$\hat{\mu} = \frac{\partial\hat{\psi}}{\partial n} \text{ on } \Gamma, \tag{6.9}$$

in order to rewrite the previous functional as a function of two variables instead of three:

$$L(\Omega, \phi, \psi) = \int_\Omega (\Delta\phi + f)\psi \, dx + \int_\Gamma (\phi - g)\frac{\partial\psi}{\partial n} \, d\Gamma \tag{6.10}$$

for (ϕ, ψ) in $H^2(\Omega) \times H^2(\Omega)$. It is also advantageous for shape problems to get rid of boundary integrals whenever possible. So noting that

$$\int_\Gamma (\phi - g)\frac{\partial\psi}{\partial n} \, d\Gamma = \int_\Omega \operatorname{div}\left[(\phi - g)\nabla\psi\right] dx, \tag{6.11}$$

we finally use the functional

$$L(\Omega, \phi, \psi) = \int_\Omega \{(\Delta\phi + f)\psi + (\phi - g)\Delta\psi + \nabla(\phi - g) \cdot \nabla\psi\} \, dx \tag{6.12}$$

on $H^2(\Omega) \times H^2(\Omega)$. It is readily seen that it has a unique saddle point $(\hat{\phi}, \hat{\psi})$ in $H^2(\Omega) \times H^2(\Omega)$ which is completely characterized by the saddle point equations

$$\Delta\hat{\phi} + f = 0 \text{ in } \Omega, \quad \hat{\phi} = g \text{ on } \Gamma, \tag{6.13}$$

$$\Delta\hat{\psi} = 0 \text{ in } \Omega, \quad \hat{\psi} = 0 \text{ on } \Gamma. \tag{6.14}$$

6.3 Saddle Point Expression of the Objective Function

Now repeat the above constructions taking into account the objective function. First introduce the objective function

$$F(\Omega, \phi) = \frac{1}{2} \int_\Omega |\phi - y_d|^2 \, dx \tag{6.15}$$

and the new Lagrangian functional

$$G(\Omega, \phi, \psi) = F(\Omega, \phi) + L(\Omega, \phi, \psi).$$

Then it is easy to verify that

$$J(\Omega) = \min_{\phi \in H^2(\Omega)} \max_{\psi \in H^2(\Omega)} G(\Omega, \phi, \psi). \tag{6.16}$$

The Lagrangian $G(\Omega, \phi, \psi)$ is given by the expression

$$\begin{aligned} G(\Omega, \phi, \psi) = \frac{1}{2} \int_\Omega & |\phi - y_d|^2 \, dx \\ & + \int_\Omega \{(\Delta\phi + f)\psi + (\phi - g)\Delta\psi + \nabla(\phi - g) \cdot \nabla\psi\} \, dx \end{aligned} \tag{6.17}$$

on $H^2(\Omega) \times H^2(\Omega)$. It is readily seen that it has a unique saddle point$(\hat{\phi}, \hat{\psi})$ which is completely characterized by the following saddle point equations:

$$\Delta\hat{\phi} + f = 0 \text{ in } \Omega, \quad \hat{\phi} = g \text{ on } \Gamma, \tag{6.18}$$

$$\forall \phi \in H^2(\Omega), \quad \int_\Omega \{(\hat{\phi} - y_d)\phi + \Delta\phi\hat{\psi} + \phi\Delta\hat{\psi} + \nabla\phi \cdot \nabla\hat{\psi}\} \, dx = 0. \tag{6.19}$$

But the last equation is equivalent to

$$\forall \phi \in H^2(\Omega), \quad \int_\Omega [(\hat{\phi} - y_d) + \Delta\hat{\psi}] \, \phi \, dx + \int_\Gamma \frac{\partial\phi}{\partial n} \, \hat{\psi} \, d\Gamma = 0 \tag{6.20}$$

or

$$\Delta\hat{\psi} + (\hat{\phi} - y_d) = 0 \text{ in } \Omega, \quad \hat{\psi} = 0 \text{ on } \Gamma \tag{6.21}$$

by using the theorem on the surjectivity of the trace. In what follows, we shall use the notation (y, p) for the saddle point $(\hat{\phi}, \hat{\psi})$. As a result, we have

$$J(\Omega) = \min_{\phi \in H^2(\Omega)} \max_{\psi \in H^2(\Omega)} G(\Omega, \phi, \psi). \tag{6.22}$$

We shall now use the above Lagrangian formulation combined with the velocity method to compute the shape gradient of $J(\Omega)$. Given a velocity field V in $\mathcal{D}^1(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N})$ and the parametrized domains $\Omega_t = T_t(\Omega)$,

$$J(\Omega_t) = \min_{\phi \in H^2(\Omega)} \max_{\psi \in H^2(\Omega)} G(\Omega_t, \phi, \psi). \tag{6.23}$$

There are two methods for getting rid of the time dependence in the underlying function spaces:

- the *function space parametrization* and
- the *function space embedding.*

In the first case, we parametrize the functions in $H^2(\Omega_t)$ by elements of $H^2(\Omega)$ through the transformation

$$\phi \mapsto \phi \circ T_t^{-1} = H^2(\Omega) \to H^2(\Omega_t), \tag{6.24}$$

where "$\circ$" denotes the composition of the two maps, and we introduce the *parametrized Lagrangian*,

$$\widetilde{G}(t, \phi, \psi) = G(T_t(\Omega), \phi \circ T_t^{-1}, \psi \circ T_t^{-1}), \tag{6.25}$$

on $H^2(\Omega) \times H^2(\Omega)$. In the function space embedding method, we introduce a large enough domain, D, which contains all the transformations $\{\Omega_t : 0 \le t \le \bar{t}\ \}$ of Ω for some small $\bar{t} > 0$.

In this section, we use the function space embedding method with $D = \mathbf{R}^\mathrm{N}$ and

$$J(\Omega_t) = \min_{\Phi \in H^2(\mathbf{R}^\mathrm{N})} \max_{\Psi \in H^2(\mathbf{R}^\mathrm{N})} G(\Omega_t, \Phi, \Psi). \tag{6.26}$$

As can be expected, the price to pay for the use of this method is the fact that the set of saddle points

$$S(t) = X(t) \times Y(t) \ \subset H^2(\mathbf{R}^\mathrm{N}) \times H^2(\mathbf{R}^\mathrm{N}) \tag{6.27}$$

is not a singleton anymore since

$$X(t) = \{\Phi \in H^2(\mathbf{R}^\mathrm{N}) : \Phi|_{\Omega_t} = y_t\}, \tag{6.28}$$

$$Y(t) = \{\Psi \in H^2(\mathbf{R}^\mathrm{N}) : \Psi|_{\Omega_t} = p_t\}, \tag{6.29}$$

where (y_t, p_t) is the unique solution in $H^2(\Omega_t) \times H^2(\Omega_t)$ to the previous saddle point equations on Ω_t

$$\Delta y_t + f = 0 \text{ in } \Omega_t, \quad y_t = g \text{ on } \Gamma_t, \tag{6.30}$$

$$\Delta p_t + (y_t - y_d) = 0 \text{ in } \Omega_t, \quad p_t = 0 \text{ on } \Gamma_t. \tag{6.31}$$

We can now apply the theorem of R. Correa and A. Seeger [**1**], which says that under appropriate assumptions (to be checked in the next section)

$$dJ(\Omega; V) = \min_{\Phi \in X(0)} \max_{\Psi \in Y(0)} \partial_t G(\Omega_t, \Phi, \Psi)\big|_{t=0}. \tag{6.32}$$

Since we have already characterized $X(0)$ and $Y(0)$, we need only compute the partial derivative of

$$G(\Omega_t, \Phi, \Psi) = \int_{\Omega_t} \left\{ \frac{1}{2}|\Phi - y_d|^2 + (\Delta\Phi + f)\Psi + (\Phi - g)\Delta\Psi \right. \\ \left. + \nabla(\Phi - g)\cdot\nabla\Psi \right\} dx. \tag{6.33}$$

If we assume that Ω_t is sufficiently smooth, then

$$f, y_d \in H^{1/2+\varepsilon}(\mathbf{R}^{\mathrm{N}}) \text{ and } g \in H^{2+\varepsilon}(\mathbf{R}^{\mathrm{N}}) \quad \Rightarrow p \in H^{5/2+\varepsilon}(\Omega), \tag{6.34}$$

and we can choose to consider our saddle points $S(t)$ in $H^{5/2+\varepsilon}(\mathbf{R}^{\mathrm{N}}) \times H^{5/2+\varepsilon}(\mathbf{R}^{\mathrm{N}})$ rather than $H^2(\mathbf{R}^{\mathrm{N}}) \times H^2(\mathbf{R}^{\mathrm{N}})$. If the functions Φ and Ψ belong to $H^{5/2+\varepsilon}(\mathbf{R}^{\mathrm{N}})$, then

$$\partial_t G(\Omega_t, \Phi, \Psi) = \int_{\Gamma_t} \left\{ \frac{1}{2}|\Phi - y_d|^2 + (\Delta\Phi + f)\Psi + (\Phi - g)\Delta\Psi \right. \\ \left. + \nabla(\Phi - g)\cdot\nabla\Psi \right\} V\cdot n_t \, d\Gamma_t. \tag{6.35}$$

This expression is an integral over the boundary Γ_t which will not depend on Φ and Ψ outside of $\overline{\Omega}_t$. As a result, the min and the max can be dropped in expression (6.32), which reduces to

$$dJ(\Omega; V) = \int_{\Gamma} \left\{ \frac{1}{2}(y - y_d)^2 + (\Delta y + f)p + (y - g)\Delta p \right. \\ \left. + \nabla(y - g)\cdot\nabla p \right\} V\cdot n \, d\Gamma. \tag{6.36}$$

However, $p = 0$ and $y - g = 0$ imply

$$\nabla p = \frac{\partial p}{\partial n} n \text{ and } \nabla(y - g) = \frac{\partial}{\partial n}(y - g)n \text{ on } \Gamma \tag{6.37}$$

and, finally,

$$dJ(\Omega; V) = \int_{\Gamma} \left\{ \frac{1}{2}|g - y_d|^2 + \frac{\partial}{\partial n}(y - g)\frac{\partial p}{\partial n} \right\} V\cdot n \, d\Gamma, \tag{6.38}$$

where

$$-\Delta y = f \text{ in } \Omega, \quad y = g \text{ on } \Gamma,$$
$$\Delta p + (y - y_d) = 0 \text{ in } \Omega, \quad p = 0 \text{ on } \Gamma.$$

6.4 Verification of the Assumptions of Theorem 5.1

As we have seen the computations of the shape gradient are both quick and easy. We now turn to the step by step verification of the assumptions of Theorem 5.1. Many of the constructions given below are "canonical" and can be repeated for different problems in different contexts.

Let y_d and $f \in H^1(\mathbf{R}^\mathrm{N})$ and $g \in H^{5/2}(\mathbf{R}^\mathrm{N})$ so that

$$X = Y = H^3(\mathbf{R}^\mathrm{N}). \tag{6.39}$$

The saddle points $S(t) = X(t) \times Y(t)$ are given by

$$X(t) = \{\Phi \in X : \Phi|_{\Omega_t} = y_t\}, \tag{6.40}$$

$$Y(t) = \{\Psi \in Y : \Psi|_{\Omega_t} = p_t\}. \tag{6.41}$$

The sets $X(t)$ and $Y(t)$ are not empty since it is always possible to construct a continuous linear extension

$$\Pi^m \colon H^m(\Omega) \to H^m(\mathbf{R}^\mathrm{N}) \tag{6.42}$$

for each $m \geq 1$. For instance with $m = 1$ and a boundary Γ which is $W^{1,\infty}$, see S. Agmon, A. Douglis, and L. Nirenberg [**1**, **2**], and for $m > 1$, see V. M. Babič [**1**] and J. Nečas [**1**]. Using this Π^m, we define the following extension:

$$\Pi^m_t \colon H^m(\Omega_t) \to H^m(\mathbf{R}^\mathrm{N}), \tag{6.43}$$

$$\Pi^m_t(\phi) = [\Pi^m(\phi \circ T_t)] \circ T_t^{-1}. \tag{6.44}$$

In what follows, m is fixed and equal to 3, so we shall drop the superscript m and define the extensions

$$Y_t = \Pi_t y_t, \quad P_t = \Pi_t p_t \tag{6.45}$$

of y_t and p_t, respectively. Hence,

$$Y_t \in X(t) \text{ and } P_t \in Y(t) \Rightarrow S(t) \neq \varnothing. \tag{6.46}$$

So condition (H1) is satisfied. Condition (H2) follows from the assumptions on f, y_d, and g. To check conditions (H3) and (H4), we need two general theorems, which can be used in various contexts and problems.

Theorem 6.1. *For $V \in \mathcal{D}^1(\mathbf{R}^\mathrm{N}, \mathbf{R}^\mathrm{N})$ and $\Phi \in L^2(\mathbf{R}^\mathrm{N})$,*

$$\lim_{t \searrow 0} \Phi \circ T_t = \Phi \text{ and } \lim_{t \searrow 0} \Phi \circ T_t^{-1} = \Phi \text{ in } L^2(\mathbf{R}^\mathrm{N}). \tag{6.47}$$

Proof. (i) The space $\mathcal{D}(\mathbf{R}^\mathrm{N})$ of continuous functions with compact support in $\mathbf{R}^\mathrm{N}$ is dense in $L^2(\mathbf{R}^\mathrm{N})$. So given $\varepsilon > 0$, there exists Φ_ε in $\mathcal{D}(\mathbf{R}^\mathrm{N})$ such that

$$\|\Phi - \Phi_\varepsilon\|^2_{L^2} < \frac{\varepsilon^2}{\max}\{J_t^{-1} : 0 \leq t \leq \tau\}.$$

Hence,

$$\|\Phi \circ T_t - \Phi\| \leq \|\Phi_\varepsilon \circ T_t - \Phi_\varepsilon\| + \|\Phi \circ T_t - \Phi_\varepsilon \circ T_t\| + \|\Phi - \Phi_\varepsilon\|. \tag{6.48}$$

But

$$\forall t \in [0, \tau], \quad \int_{\mathbf{R}^\mathrm{N}} |\Phi \circ T_t - \Phi_\varepsilon \circ T_t|^2 \, dx = \int_{\mathbf{R}^\mathrm{N}} |\Phi - \Phi_\varepsilon|^2 J_t^{-1} \, dx \leq \varepsilon^2.$$

So the last two terms in (6.48) are less than 2ε. It remains to evaluate the first term for a fixed function Φ_ε with compact support K in $\mathbf{R}^N$. Recall that, since $\Phi_\varepsilon = 0$ on the boundary ∂K of K, $T_t(K) = K$ for all t in $[0, \tau]$ (use M. NAGUMO [**1**]'s theorem twice as in the proof of Theorem 5.1 (i) in Chapter 4). Moreover, by the compactness of K, Φ_ε is uniformly continuous on $\mathbf{R}^N$ and

$$\exists \delta > 0, \forall x, y \in \mathbf{R}^N, \quad |x - y| < \delta \implies |\Phi_\varepsilon(y) - \Phi_\varepsilon(x)| < \frac{\varepsilon}{m(K)^{1/2}}.$$

However, T_t is also uniformly continuous on K and

$$\exists \eta > 0, \forall t, 0 \le t < \eta, \forall x \in K, \quad |T_t x - x| < \delta.$$

By construction,

$$\operatorname{supp}(\Phi_\varepsilon \circ T_t) = T_t(\operatorname{supp} \Phi_\varepsilon) \subset K,$$

and

$$\Phi_\varepsilon = 0 \text{ and } \Phi_\varepsilon \circ T_t = 0 \text{ outside of } K.$$

Finally,

$$\int_{\mathbf{R}^N} |\Phi_\varepsilon(T_t x) - \Phi_\varepsilon(x)|^2 \, dx = \int_K |\Phi_\varepsilon(T_t x) - \Phi_\varepsilon(x)|^2 \, dx \le \varepsilon^2,$$

and this implies that

$$\forall \varepsilon > 0, \exists \eta > 0, \forall 0 \le t \le \eta, \quad \|\Phi \circ T_t - \Phi\|_{L^2(\mathbf{R}^N)} \le 3\varepsilon.$$

(ii) For the second part of (6.47), we make a change of variables and use the result of part (i)

$$\int_{\mathbf{R}^N} |\Phi \circ T_t^{-1} - \Phi|^2 \, dx = \int_{\mathbf{R}^N} |\Phi - \Phi \circ T_t|^2 J_t \, dx \le \varepsilon^2.$$

This completes the proof. □

Corollary 1. *Under the assumptions of Theorem* 6.1 *for* $m \ge 1$, V *in* $\mathcal{D}^m(\mathbf{R}^N, \mathbf{R}^N)$, *and* $\Phi \in H^m(\mathbf{R}^N)$,

$$\lim_{t \searrow 0} \Phi \circ T_t = \Phi \textit{ and } \lim_{t \searrow 0} \Phi \circ T_t^{-1} = \Phi \textit{ in } H^m(\mathbf{R}^N). \tag{6.49}$$

Theorem 6.2. *Under the assumptions of Corollary* 1 *to Theorem* 6.1,

$$y^t \to y^0 \textit{ in } H^m(\Omega)\text{-strong} \,(\textit{resp.}, \text{-weak}) \tag{6.50}$$

implies that

$$Y_t \to Y_0 \textit{ in } H^m(\mathbf{R}^N)\text{-strong} \,(\textit{resp.}, \text{-weak}).$$

Proof. The strong case is obvious. We prove the weak case for $m = 0$. By definition,

$$Y_t = (\Pi y^t) \circ T_t^{-1},$$

and for all Φ in $L^2(\mathbf{R}^{\mathrm{N}})$, we consider

$$\int_{\mathbf{R}^{\mathrm{N}}} Y_t \Phi \, dx = \int_{\mathbf{R}^{\mathrm{N}}} (\Pi y^t) \circ T_t^{-1} \Phi \, dx = \int_{\mathbf{R}^{\mathrm{N}}} \Pi y^t \Phi \circ T_t J_t \, dx.$$

We have shown in Theorem 6.1 that

$$\Phi \circ T_t \to \Phi \text{ in } L^2(\mathbf{R}^{\mathrm{N}})\text{-strong}.$$

In addition, $J_t \to 1$, and by linearity and continuity of Π,

$$\Pi y^t \to \Pi y \text{ in } L^2(\mathbf{R}^{\mathrm{N}})\text{-weak}.$$

Hence,

$$\forall \Phi \in L^2(\mathbf{R}^{\mathrm{N}}), \quad \int_{\mathbf{R}^{\mathrm{N}}} Y_t \Phi \, dx \to \int_{\mathbf{R}^{\mathrm{N}}} \Pi y \Phi \, dx =\!\!\to \int_{\mathbf{R}^{\mathrm{N}}} Y_0 \Phi \, dx.$$

This proves the weak convergence. □

To satisfy condition (H3), we transform (y_t, p_t) on Ω_t to $(y^t, p^t) = (y_t \circ T_t, p_t \circ T_t)$ on Ω. The pair (y^t, p^t) is the transported pair of solutions from Ω_t to Ω. It is the unique solution in $H^1(\Omega) \times H^1(\Omega)$ of the system

$$-\mathrm{div}\,[A(t)\nabla y^t] = J_t f \circ T_t \text{ in } \Omega, \quad y^t = g \circ T_t \text{ on } \Gamma, \tag{6.51}$$

$$-\mathrm{div}\,[A(t)\nabla p^t] = J_t (y^t - y_d \circ T_t) \text{ in } \Omega, \quad p^t = 0 \text{ on } \Gamma, \tag{6.52}$$

where

$$A(t) = J_t [DT_t]^{-1\,*}[DT_t]^{-1}, \quad J_t = |\det DT_t|, \tag{6.53}$$

DT_t is the Jacobian matrix of T_t, and ${}^*[DT_t]^{-1}$ is the transpose of $[DT_t]^{-1}$.

For sufficiently smooth domains Ω and vector fields V, the pair $\{y^t, p^t\}$ is bounded in $H^1(\Omega) \times H^1(\Omega)$ as t goes to zero. Since $H^1(\Omega)$ is a Hilbert space, we can extract weakly convergent subsequences to some $(\bar{y}, \bar{p})$ in $H^1(\Omega) \times H^1(\Omega)$. However, by linearity of the equation with respect to (y^t, p^t) and continuity of the coefficients with respect to t, the limit point $(\bar{y}, \bar{p})$ will coincide with (y^0, p^0), since the system has a unique solution at $t = 0$. Then we go back to the equation for y^t and y and show that the convergence is strong in $H^1(\Omega)$. Finally, by using the regularity of the data and the classical regularity theorems, we show that $(y^t, p^t) \to (y, p)$ in $H^3(\Omega) \times H^3(\Omega)$.

For the verification of condition (H4), we go back to expression (6.35), which can be rewritten as a volume integral:

$$\begin{aligned} \partial_t G(\Omega_t, \Phi, \Psi) = \int_{\Omega_t} \mathrm{div} \bigg\{ \bigg[& \frac{1}{2}(\Phi - y_d)^2 + (\Delta\Phi + f)\Psi \\ & + (\Phi - g)\Delta\Psi + \nabla(\Phi - g) \cdot \nabla\Psi \bigg] V \bigg\} \, dx \end{aligned} \tag{6.54}$$

for $(\Phi, \Psi) \in H^3(\mathbf{R}^{\mathrm{N}}) \times H^3(\mathbf{R}^{\mathrm{N}})$. Now introduce the map

$$\begin{aligned}(\Phi, \Psi) \mapsto & F(\Phi, \Psi) \\ & = \left[\frac{1}{2}(\Phi - y_d)^2 + (\Delta\Phi + f)\Psi + (\Phi - g)\Delta\Psi + \nabla(\Phi - g) \cdot \nabla\Psi\right] V \\ : H^3(\mathbf{R}^{\mathrm{N}}) \times H^3(\mathbf{R}^{\mathrm{N}}) & \to (H^1(\mathbf{R}^{\mathrm{N}}))^N.\end{aligned}$$

It is bilinear and continuous. Finally, the map

$$(t, F) \mapsto \int_{\Gamma_t} F \circ n_t \, d\Gamma = \int_{\Omega_t} F \, dx = \int_{\Omega} (\operatorname{div} F) \circ T_t J_t^{-1} \, dx \tag{6.55}$$

from $[0, \tau] \times H^1(\mathbf{R}^{\mathrm{N}})$ to $\mathbf{R}$ is continuous. Then

$$(t, \Phi, \Psi) \mapsto \partial_t G(\Omega_t, \Phi, \Psi) = \int_{\Gamma_t} F(\Psi, \Psi) \cdot n_t \, d\Gamma_t \tag{6.56}$$

is continuous and condition (H4) is satisfied. This completes the verification of the four conditions of Theorem 5.1.

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