THE POROUS MEDIUM EQUATION

Mathematical theory

BY

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To my wife Mariluz

Preface

The Heat Equation is one of the three classical linear partial differential equations of second order that form the basis of any elementary introduction to the area of Partial Differential Equations. Its success in describing the process of thermal propagation has known a permanent popularity since Fourier's essay *Théorie Analytique de la Chaleur* was published in 1822, [237], and has motivated the continuous growth of mathematics in the form of Fourier analysis, spectral theory, set theory, operator theory, and so on. Later on, it contributed to the development of measure theory and probability, among other topics.

The high regard of the Heat Equation has not been isolated. A number of related equations have been proposed both by applied scientists and pure mathematicians as objects of study. In a first extension of the field, the theory of linear parabolic equations was developed, with constant and then variable coefficients. The linear theory enjoyed much progress, but it was soon observed that most of the equations modelling physical phenomena without excessive simplification are nonlinear. However, the mathematical difficulties of building theories for nonlinear versions of the three classical partial differential equations (Laplace's equation, heat equation and wave equation) made it impossible to make significant progress until the 20th century was well advanced. And this observation applies to other important nonlinear PDEs or systems of PDEs, like the Navier-Stokes equations.

The great development of Functional Analysis in the decades from the 1930's to the 1960's made it possible for the first time to start building theories for these nonlinear PDEs with full mathematical rigor. This happened in particular in the area of parabolic equations where the theory of Linear and Quasilinear Parabolic Equations in divergence form reached a degree of maturity reflected for instance in the classical books of Ladyzhenskaya et al. [357] and Friedman [239].

The aim of the present text is to provide a systematic presentation of the mathematical theory of the nonlinear heat equation

$$\text{(PME)} \qquad \partial_t u = \Delta(u^m), \quad m > 1\,,$$

usually called the Porous Medium Equation (shortly, PME), posed in the d-dimensional Euclidean space, with interest in the cases $d = 1, 2, 3$ for the applied scientist, with no dimension restriction for the mathematician. $\Delta = \Delta_x$ represents the Laplace operator acting on the space variables. We will also study the complete form, $u_t = \Delta(|u|^{m-1}u) + f$, but in a less systematic way. Other variants appear in the literature but will be given less attention, since we keep to the idea of presenting a rather complete account of the main results and methods for the basic PME.

The reader may wonder why such a simple-looking variation of the famous and well-known heat equation (HE): $u_t = \Delta u$, needs a book of its own. There are several answers to this question: the theory and properties of the PME depart strongly from the heat equation; it contains interesting and sometimes sophisticated developments of nonlinear analysis; there are a number of interesting applications where this theory, with all its differences, is necessary and useful; and, finally, similar treatises have been written for individual equations with a strong personality. As for the latter argument, we have the example of the heat equation itself, described in the monographs by Cannon [148] and Widder [525], and also the Stefan Problem, that is closely related to the HE and the PME, and was reported about in the books of Cannon [148], Rubinstein [454] and Meirmanov [388].

Let us now comment on the first aspects listed some lines above. The theory that has been developed and we present in this text not only settles the main problems of existence, uniqueness, stability, smoothness, dynamical properties and asymptotic behavior. In doing so, it contributes a wealth of new ideas with respect to the heat equation; great novelties occur also with respect to the standard nonlinear theories, represented by the theory of nonlinear parabolic equations in divergence form to which the Porous Medium Equation belongs. This is due to the fact that the equation is not parabolic at all points, but *only degenerate parabolic*, a fact that has deep mathematical consequences, both qualitative and quantitative. On the other hand, and as a sort of compensation, the equation enjoys a number of nice properties due to its simple form, like scaling invariance. This aspect makes the PME an interesting benchmark in the development of nonlinear analytical tools for the quite general classes of nonlinear, formally parabolic equations that continue to make their way into the pure and applied sciences, and then into the mainstream of mathematics.

There are a number of physical applications where the simple PME model appears in a natural way, mainly to describe processes involving fluid flow, heat transfer or diffusion. Other applications have been proposed in mathematical biology, lubrication, boundary layer theory, and other fields. All of these reasons support the interest of its study both for the mathematician and the scientist.

Context

In spite of the simplicity of the equation and of having some important applications, and due perhaps to its nonlinear and degenerate character, the mathematical theory of the PME has been only gradually developed in the last decades after the seminal paper of Oleinik et al. [408] in 1958; in the 1980's the theory was finally on firm ground and has been rounded up since then. The idea of the book arose out of the participation of the author in this progress in the last three decades. The immediate motivation for writing the text is the feeling that the time is ripe for a reasonably complete version of the mathematics of the PME, once the main mathematical issues have come to be fairly well understood, and every result receives a proof in the style of Analysis. We are also aware of the need for researchers to apply to more complex models the wealth of techniques that work so well here, hence the need for clear and balanced expositions to learn the material. Therefore, we aim at providing a description of the questions of existence, uniqueness and the main properties of the solutions, whereby everything is derived from basic estimates using standard Functional Analysis and well-known PDE results. And we have tried to provide sound physical foundations throughout.

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Contents

Chapter 1

Introduction

1.1 The subject

1.1.1 The porous medium equation

The aim of the text is to provide a systematic presentation of the mathematical theory of the nonlinear heat equation

$$\text{(PME)} \qquad \partial_t u = \Delta(u^m), \quad m > 1\,,$$

usually called the *Porous Medium Equation*, with due attention paid to its closest relatives. The default settings are: $u = u(x,t)$ is a nonnegative scalar function of space $x \in \mathbb{R}^d$ and time $t \in \mathbb{R}$, the space dimension is $d \geq 1$, and m is a constant larger than 1. $\Delta = \Delta_x$ represents the Laplace operator acting on the space variables. We will refer to the equation in short by the label PME. The equation can be posed for all $x \in \mathbb{R}^d$ and $0 < t < \infty$, and then initial conditions are needed to determine the solutions; but it is quite often posed, especially in practical problems, in a bounded subdomain $\Omega \subset \mathbb{R}^d$ for $0 < t < T$, and then determination of a unique solution asks for boundary conditions as well as initial conditions.

This equation is one of the simplest examples of nonlinear evolution equation of parabolic type. It appears in the description of different natural phenomena, and its theory and properties depart strongly from the heat equation, $u_t = \Delta u$, its most famous relative. Hence the interest of its study, both for the pure mathematician and the applied scientist. We will also discuss in less detail some important variants of the equation.

There are a number of physical applications where this simple model appears in a natural way, mainly to describe processes involving fluid flow, heat transfer or diffusion. Maybe the best known of them is the description of the flow of an isentropic gas through a porous medium, modelled independently by Leibenzon [367] and Muskat [394] around 1930. An earlier application is found in the study of groundwater infiltration by Boussisnesq in 1903, [123]. Another important application refers to heat radiation in plasmas, developed by Zel'dovich and coworkers around 1950, [533]. Indeed, this application was at the base of the rigorous mathematical development of the theory. Other applications have been proposed in mathematical biology, spread of viscous fluids, boundary layer theory, and other fields.

Most physical settings lead to the default restriction $u \geq 0$, which is mathematically convenient and currently followed. However, the restriction is not essential in developing a mathematical theory on the condition of properly defining the nonlinearity for negative values of u so that the equation is still (formally) parabolic. The most used choice is the antisymmetric extension of the nonlinearity, leading to the so-called *Signed PME*,

$$\text{(sPME)} \qquad \partial_t u = \Delta(|u|^{m-1}u).$$

We will also devote much attention to this equation. For brevity, we will often write u^m instead of $|u|^{m-1}u$ even if solutions have negative values in paragraphs where no confusion is to be feared. There is a second important extension, consisting of adding a *forcing term* in the right-hand side to get the complete form

$$\text{(cPME)} \qquad \partial_t u = \Delta(|u|^{m-1}u) + f,$$

where $f = f(x,t)$. The full form is the natural framework of the abstract functional theory for the PME, and has also received much attention when $f = f(u)$ and represents effects of reaction or absorption. Dependence of f on ∇u occurs when convection is taken into account. We will cover the complete form in the text, but the information on the qualitative and quantitative aspects is much less detailed in that generality, and we will not enter into the specific properties of reaction-diffusion models. Specially in the second part of the book, we want to concentrate on the plain equation PME, hence the simple label for that case. The *complete Porous Medium Equation* is also referred to as the PME with a source term, or the *forced PME*.

Equation (PME) for $m = 1$ is the famous *Heat Equation*, shortly HE, that has a well documented theory, cf. Widder [525]. The equation can also be considered for the range of exponents $m < 1$. Some of the properties in this range are similar to the case $m > 1$ studied here, but others are quite different, and it is called the *Fast Diffusion Equation* (FDE). Since it deserves a text of its own, the FDE will only be covered in passing in this book. Note: when $m < 0$ the FDE has to be written in the “modified form”

$$\partial_t u = \Delta(u^m/m) = \mathtt{div}(u^{m-1}\nabla u)$$

to keep the parabolic character of the equation. This form of the equation allows also to include the case $m = 0$ which reads $\partial_t u = \mathtt{div}(u^{-1}\nabla u) = \Delta \log(u)$, and is called logarithmic diffusion.

1.1.2 The PME as a nonlinear parabolic equation

The PME is an example of nonlinear evolution equation, formally of parabolic type. In a sense, it is the simplest possible nonlinear version of the classical heat equation, which can be considered as the limit $m \to 1$ of the PME. Written in its complete version and in divergence form,

$$\text{(1.1)} \qquad \partial_t u = \mathtt{div}(D(u)\nabla u) + f\,,$$

we see that the *diffusion coefficient* $D(u)$ of the PME equals mu^{m-1} assuming $u \ge 0$, and we have $D(u) = m|u|^{m-1}$ for signed solutions ($D(u) = |u|^{m-1}$ in the modified form). It is then clear that the equation is parabolic only at those points where $u \neq 0$, while the vanishing of $D(u)$ is recorded as saying that the PME *degenerates* wherever $u = 0$. In other words, the PME is a *degenerate parabolic equation*. The theory of nonlinear parabolic equations in divergence form deals with the class of nonlinear parabolic equations of the form

$$\partial_t u = \mathtt{div}\mathcal{A}(x,t,u,Du) + \mathcal{B}(x,t,u,Du)\,, \tag{1.2}$$

where the vector function $\mathcal{A} = (A_1,\cdots,A_d)$ and the scalar function $\mathcal{B}$ satisfy suitable structural assumptions and $\mathcal{A}$ satisfies moreover ellipticity conditions. This topic became a main area of research in PDEs in the second half of the last century, when the tools of Functional Analysis were ready for it. The theory extends to systems of the same form, in which $u = (u_1,\cdots,u_k)$ is a vector variable, $\mathcal{A}$ is an (m,d) matrix and $\mathcal{B}$ is an m-vector. Well-known areas, like Reaction-Diffusion are included in this generality. There is a large literature on this topic, cf. e.g. the books [239, 357, 482] that we take as reference works.

The change of character of the PME at the level $u = 0$ is most clearly demonstrated when we perform the calculation of the Laplacian of the power function in the case $m = 2$; assuming $u \geq 0$ for simplicity, we obtain the form

$$\partial_t u = 2u\,\Delta u + 2|\nabla u|^2. \tag{1.3}$$

It is immediately clear that in the regions where $u \neq 0$ the leading term in the right-hand side is the Laplacian modified by the variable coefficient $2u$; on the contrary, for $u \to 0$, the equation simplifies into $\partial_t u \sim 2|\nabla u|^2$, the *eikonal equation* (a first-order equation of Hamilton-Jacobi type, that propagates along characteristics). A similar calculation can be done for general $m \neq 1$ after introducing the so-called *pressure variable*, $v = cu^{m-1}$ for some $c \geq 0$. We then get

$$\partial_t v = av\,\Delta v + b\,|\nabla v|^2, \tag{1.4}$$

with $a = m/c$, $b = m/(c(m-1))$. This is a fundamental transformation in the theory of the PME that allows us to get similar conclusions about the behaviour of the equation for $u, v \sim 0$ when $m \neq 2$. The standard choice for c in the literature is $c = m/(m-1)$, because it simplifies the formulas ($a = m-1$, $b = 1$) and makes sense for dynamical considerations (to be discussed in Section 2.1), but $c = 1$ is also used. Mathematically, the choice of constant is not important.

Note that similar considerations apply to the FDE but then

$$D(u) = \frac{m}{|u|^{1-m}} \to \infty \quad \text{as} \quad u \to 0, \tag{1.5}$$

hence the name of fast diffusion which is well deserved when $u \sim 0$. The pressure can be introduced, but being an inverse power of u, its role is different than in the PME. All this shows the kinship and differences from the start between the two equations.

In spite of the simplicity of the equation and of having some important applications, a mathematical theory for the PME has been developed at a slow pace in several decades, due most probably to the fact that it is a nonlinear equation, and also a degenerate one. Though the techniques depart strongly from the linear methods used in treating the heat equation, it is interesting to remark that some of the basic techniques are not very difficult nor need a heavy machinery. What is even more interesting, they can be applied in, or adapted to, the study of many other nonlinear PDE's of parabolic type. The study of the PME can provide the reader with an introduction to, and practice of some interesting concepts and methods of nonlinear science, like the existence of free boundaries, the occurrence of limited regularity, and interesting asymptotic behaviour.

1.2 Peculiar features of the PME

When considering the linear and quasilinear parabolic theories, the main questions are asked in comparison to what happens for the heat equation, which is the model from which these theories take their inspiration. Thus, the three main questions of existence, uniqueness, and continuous dependence are posed in the literature, as well as the questions of regularity, the validity of maximum principles, the existence of Harnack inequalities, and so on; in some sense, these comparative questions receive positive answers, though the analogy breaks at some points, thus originating novelty and interest.

1.2.1 Finite propagation and free boundaries

The same golden rule of comparison with the HE is applied to the theory developed in this book for the PME. The main questions can be posed, but then we see that such questions, though important, do not convey the special flavor of the equation. Indeed, the PME offers a number of very peculiar traits that separate it from the core of the parabolic theory. Mathematically, the difficulties stem from the degenerate character, i. e., the fact that $D(u)$ is not always positive. Explaining the consequences implies changing the way the heat equation theory is developed. We will be led to introducing dynamical concepts to account for the main qualitative difference, which is the property called *Finite Propagation*, that will be precisely formulated and extensively explored in the text, especially in Chapters 14 and 15. This property is in strong contrast with one of the better known properties of the classical heat equation, the infinite speed of propagation, one of the most contested aspects of the HE on physical grounds. Let us express the contrast in simplest terms:

- HE: *"A nonnegative solution of the heat equation is automatically positive everywhere in its domain of definition"*; to be compared with
- PME: *"Disturbances from the level $u = 0$ propagate in time with finite speed for solutions of the porous medium equation"*.

In a sense, the property of finite propagation supports the physical soundness of the PME to model diffusion or heat propagation.

A first consequence of the finite propagation property for the theory of the PME is that the strong maximum principle cannot hold. On the positive side, it means that, whenever the initial data are zero in some open domain of the space, the property of finite propagation implies the appearance of a *free boundary* that separates the regions where the solution is positive (i. e. where "there is gas", according to the standard interpretation of u as a gas density, see Chapter 2), from the "empty region" where $u = 0$. Precisely, we define the free boundary as

$$\Gamma = \partial P_u \cap Q\,, \tag{1.6}$$

where Q is the domain of definition of the solution in space-time,

$$\mathcal{P}_u = \{(x,t) \in Q : u(x,t) > 0\} \tag{1.7}$$

is the *positivity set*, and ∂ denotes boundary. Since it moves as time passes, it is also called the *moving boundary*. In some cases, specially in one space dimension, the name *interface* is popular.

The theory of free boundaries, or propagation fronts, is an important and difficult subject of the mathematical investigation, covered for instance in the book by A. Friedman [240]. In principle, the free boundary of a nonlinear problem can be a quite complicated closed subset of Q. A main problem of the PME theory consists of proving that it is at least a Hölder continuous (C^α) hypersurface in $\mathbb{R}^{d+1}$, and then to investigate how smooth it really is. Let us advance that it is often C^∞ smooth, but not always.

Let us illustrate the two main situations that will be encountered. In the first of them, the space domain is $\mathbb{R}^d$, the initial data u_0 have compact support, i. e., there exists a bounded closed set $S_0 \subset \mathbb{R}^d$ such that $u_0(x) = 0$ for all $x \neq S_0$. In that case, we will prove that the solution $u(x,t)$ vanishes for all positive times $t > 0$ outside a compact set that changes with time. More precisely, if we define the *positivity set* at time t as $\mathcal{P}_u(t) = \{x \in \mathbb{R}^d : u(x,t) > 0\}$, and the *support* at time t as $\mathcal{S}_u(t)$ as the closure of $\mathcal{P}_u(t)$, then both families of bounded sets are shown to be expanding in time, or more precisely stated, non-contracting. Note that positivity sets and supports are not defined in the everywhere sense unless solutions are continuous; showing continuity of the solutions is a main issue in the PME theory, and it has been a hot topic in nonlinear elliptic and parabolic equations since the seminal papers of De Giorgi, Nash and Moser.

In the second scenario, the initial configuration 'has a hole in the support', i. e., there is a bounded subdomain $D_0 \neq \emptyset$ such that $u_0(x) = 0$ for every x in the closure of D_0, and $u_0(x) > 0$ otherwise. Then, the solution has a possibly smaller hole for $t > 0$. The fact that this hole does disappear in finite time (it is filled up), motivates one of the most beautiful mathematical developments of the PME theory, the so-called Focusing Problem, that we will study in Chapter 19.

1.2.2 The role of special solutions

Following a standard practice in Applied Nonlinear Analysis and Mechanics, before developing a fully-fledged theory, the question is posed whether there exist special solutions in explicit or quasi-explicit form that serve as representative examples of the typical or peculiar behaviour. The answer to that question is positive in our case; a reduced number of representative examples have been found and they give both insight and detailed information about the most relevant questions, like existence, finite propagation, optimal continuity, higher smoothness and so on.

A fundamental example of solution was obtained around 1950 in Moscow by Zel'dovich and Kompaneets [532] and Barenblatt [60], who found and analyzed a solution representing heat release from a point source. This solution has the explicit formula

$$\mathcal{U}(x,t) = t^{-\alpha} \left(C - k\,|x|^2 t^{-2\beta} \right)_+^{\frac{1}{m-1}}, \tag{1.8}$$

where $(s)_+ = \max\{s, 0\}$,

$$\alpha = \frac{d}{d(m-1)+2}, \quad \beta = \frac{\alpha}{d}, \quad k = \frac{\alpha(m-1)}{2\,m\,d} \tag{1.9}$$

and $C > 0$ is an arbitrary constant. The solution was subsequently found by Pattle [418] in 1959. The name *source-type solution* is due to the fact that it takes as initial data a Dirac mass: as $t \to 0$ we have $\mathcal{U}(x,t) \to M\,\delta(x)$, where M is a function of the free constant

C (and m and d). We will use the shorter term *source solution*, and very often the name *ZKB solution* that looks to us convenient. We recall that the names *Barenblatt solution* and *Barenblatt-Pattle solution* are found in the literature.

An analysis of this example shows many of the important features that we have been talking about. Thus, the source solution has compact support in space for every fixed time, since the free boundary is the surface given by the equation

$$t = c\,|x|^{d(m-1)+2}\,, \tag{1.10}$$

where $c = c(C, m, d)$. In physical terms, the disturbance propagates with a precise finite speed. This is to be compared with the properties of the Gaussian kernel,

$$E(x,t) = M\,(4\pi t)^{-n/2} \exp{(-x^2/4t)}, \tag{1.11}$$

which is the source solution for the HE.

There are many other special solutions that have been studied and shed light on different aspects of the theory. Some of the most important will be carefully examined in Chapter 4 and then used in the theory developed in this text. They take the main forms of separate-variables solutions, travelling waves and selfsimilar solutions. Chapter 16 is entirely devoted to constructing solutions. They play a prominent role in Chapters 18 and 19, where the focusing solutions have a key part in settling the regularity issue.

1.3 Nonlinear Diffusion. Related equations

The PME is but one example of partial differential equation in the realm of what is called Nonlinear Diffusion. Work in that wide area has frequent overlaps between the different models, both in phenomena to be described, results to be proved and techniques to be used. A quite general form of nonlinear diffusion equation, as it appears in the specialized literature, is

$$\partial_t H(x,t,u) = \sum_{i=1}^{d} \partial_{x_i}(A_i(x,t,u,Du)). \tag{1.12}$$

Suitable conditions should be imposed on the functions H and A_i. In particular, $\partial_u H(x,t,u) \ge 0$ and the matrix $(a_{ij}) = (\partial_{u_j} A_i(x,t,u,Du))$ should be positive semi-definite. If we want to consider reaction and convection effects, the term $\mathcal{B}(x,t,u,Du)$ is added to the right-hand side. A theory for equations in such a generality has been in the making during the last decades, but the richness of phenomena that are included in the different examples covered in the general formulation precludes a general theory with detailed enough information.

Progress has been quite remarkable on more specialized topics like ours. Let us mention next four natural extensions of the PME in that direction. Though they have some important traits in common with the PME, they are different territories and we think that the deep study deserves a separate text in each case.

(i) FAST DIFFUSION. Much of the theory can be and has been extended to the simplest generalization of the PME consisting of the same formal equation, but now in the range of exponents $m < 1$. Since the diffusion coefficient $D(u) = |u|^{m-1}$ goes now to infinity as $u \to 0$, the equation is called in this new range the *Fast Diffusion Equation*, FDE. In this terminology, the PME becomes a *Slow Diffusion Equation*.

There are strong analogies and also marked differences between the PME and the FDE. For instance, the free boundary theory of the PME disappears for the FDE. We will only make small incursions into it. We refer to the monograph [515] and its references as a source of further information.

(ii) FILTRATION EQUATIONS. A further extension is the *Generalized Porous Medium Equation,*

$$\text{(GPME)} \qquad \partial_t u = \Delta\Phi(u) + f\,,$$

also called the *Filtration Equation*, specially in the Russian literature; Φ is an increasing function: $\mathbb{R}_+ \mapsto \mathbb{R}_+$, and usually $f = 0$. The diffusion coefficient is now $D(u) = \Phi'(u)$, and the condition $\Phi'(u) \ge 0$ is needed to make the equation formally parabolic. Whenever $\Phi'(u) = 0$ for some $u \in \mathbb{R}$, we say that the equation degenerates at that u-level, since it ceases to be strictly parabolic. This is the cause for more or less serious departures from the standard quasilinear theory, as we have already explained in the PME case.

An important role in the development of the topic of the Filtration Equation has been played by the *Stefan Problem*, a simple but powerful model of phase transition, developed in the study of the evolution of a medium composed of water and ice. It can written as a filtration equation with

$$\text{(StE)} \qquad \Phi(u) = (u-1)_+ \quad \text{for } u \ge 0, \quad \Phi(u) = u \quad \text{for } u < 0\,.$$

More generally, we can put $\Phi(u) = c_1(u-L)_+$ for $u \ge 0$, and $\Phi(u) = c_2\, u$ for $u < 0$, where c_1, c_2 and L are positive constants. The Stefan Problem and the PME have had a somewhat parallel history.

Note: Due to the interest of other GPME models, we will develop a large part of the basic existence and uniqueness theory of this book for the GPME, and we will then specialize to the PME in the detailed analysis of the last part of the book.

(iii) p-LAPLACIAN EVOLUTIONS. There is another popular nonlinear degenerate parabolic equation:

$$\text{(PLE)} \qquad \partial_t u = \mathtt{div}(|\nabla u|^{p-2}\nabla u)\,,$$

called the *p-Laplacian evolution equation,* PLE, which has also attracted much attention from researchers. It is part of a general theory of diffusion with diffusivity depending on the gradient of the main unknown. It has a parallel, sometimes divergent, sometimes convergent theory. We can combine PME and PLE to get the so-called *Doubly nonlinear diffusion equation*

$$\text{(DNDE)} \qquad \partial_t u = \mathtt{div}(|\nabla u^m|^{p-2}\nabla u^m)\,.$$

Though these equations have many similarities with the PME, we will not deal with them in this book.

(iv) PME WITH LOWER ORDER TERMS. These are equations of the form

$$\partial_t u = \Delta\Phi(x,u) + B(x,t,u,\nabla u)\,. \tag{1.13}$$

We have written the general filtration diffusion, but $\Phi(s) = |s|^{m-1}s$ gives the PME. The lower order term takes several forms in the applications. The best known are:

1) the form $B = f(u)$ is a homogeneous reaction term, and the full equation is then a PME-based Reaction-Diffusion model; when $f \le 0$ we have the nonlinear diffusion-absorption model that has been studied extensively;

2) When $B = a \cdot \nabla u^q$ we have a convection term; a famous example is Burgers equation $u_t + uu_x = \mu\, u_{xx}$;

3) when $B = |\nabla u|^2$ we have a diffusive Hamilton-Jacobi equation.

We can see these latter equations as particular cases of the complete PME, but this could be misleading: their theory is quite rich. Of particular interest are the equations of the form

$$\partial_t u = \Delta(|u|^{m-1}u) + \nabla \cdot (\mathbf{a}(x)u)), \quad \mathbf{a}(x) = \nabla V(x), \tag{1.14}$$

called *Fokker-Planck equations*. The extra term stands for a confining effect due to a potential V. In the case $V(x) = c\,|x|^2$ these equations are closely connected to the study of the asymptotic behaviour of the plain PME/HE/FDE after a convenient rescaling (see details in Chapter 18).

1.4 Contents

In a classical mathematical style, the foundation of the book is the study of existence, uniqueness, stability and practical construction of suitably defined solutions of the equation plus appropriate initial and boundary data. This theory uses the machinery of Nonlinear Functional Analysis, as developed extensively in the last century. In the spirit of this theory, classical concepts of solution do not suffice, which leads to the introduction of suitable concepts of generalized solution, in the concrete form of *weak, limit, strong and mild solution*, among others.

1.4.1 The main problems and the classes of solutions

Three main problems that are posed in parabolic theories:

- Problem A is the initial-value problem in the whole space, $x \in \mathbb{R}^d$, $d \ge 1$, for a time $0 < t < T$ with T finite or infinite. It is usually called the Cauchy Problem, CP, and is considered the reference problem in the literature about the PME. It is usually posed for nonnegative solutions without a forcing term ($u \ge 0$ and $f = 0$), but we will also study it for signed solutions, and with a forcing term.

- Problem B is posed in a subdomain Ω or $\mathbb{R}^d$, and the additional data include initial conditions and boundary conditions of Dirichlet type, $u(x,t) = g(x,t)$ for $x \in \partial\Omega$ and $0 < t < T$. The same observations on the sign of u and on f apply. By default Ω is bounded, $u \ge 0$, $f = 0$, and $g = 0$.

- Problem C is similar to Problem B, but the data on the lateral boundary are Neumann data, $\partial_n u^m(x,t) = h(x,t)$. By default, Ω is bounded and $f = 0$, $h = 0$.

There is a number of other problems posed on spatial domains Ω with more general conditions of mixed or nonlinear type. In one space dimension a typical problem is posed in a semi-infinite domain $\Omega = (0, \infty)$. Typical data in that case are $u(0,t) = C$ or $(u^m)_x(0,t) = 0$.

Once the problems are shown to be well-posed in suitable functional settings, the next question is the study of the main qualitative properties. Prominent among them is the phenomenon of finite propagation and its consequences in the form of free boundaries. The emphasis shifts now into dynamical considerations and Differential Geometry.

A third important subject related to both previous ones is optimal regularity. Let us illustrate it on the source-type solution. We have seen that it is continuous in its domain of definition $Q = \mathbb{R}^d \times \mathbb{R}_+$. However, it is not smooth at the free boundary, again a consequence of the loss of the parabolic character of the equation when u vanishes. In fact, the function u^{m-1} is Lipschitz continuous in Q with jump discontinuities on Γ (i. e., there exists a regularity threshold). On the contrary, the solution is C^∞-smooth in $\mathcal{P}_u$. And we are interested in noting that though u is not smooth on Γ, nevertheless the free boundary is a C^∞ smooth surface given by the equation (1.10). However, not all free boundaries of solutions of the PME will be so smooth.

1.4.2 Chapter overview

The book is organized as follows: after this Introduction, we review the main applications in Chapter 2. This pays homage to the fundamental role played by these applications in motivating the mathematical research and supplying it with problems, intuitions, concepts and conjectures.

We continue with two preparatory chapters. In Chapter 3 we review the main facts and introduce the basic estimates we will need later in a classical framework. Chapter 4 examines the fundamental examples, and we use the opportunity to present in a simple and practical context some of the main topics of the theory, like the property of finite propagation, the appearance of free boundaries, the need for generalized solutions and the question of limited regularity. It even shows cases of blow-up and the evolution of signed solutions.

This gives way to the study of the classical problems of existence, uniqueness and regularity of a (generalized) solution for the tree main problems mentioned above. There have been two basic approaches to the existence theory for the PME in the literature: one of them is the so-called semigroup approach based on posing the problem in the setting of abstract ODEs in Banach spaces; the other one uses a priori estimates, approximation by related smooth problems (to which the estimates apply uniformly), and passage to the limit. Though both approaches have been fruitful, we have chosen to give priority to the latter, which uses as a cornerstone the preparatory work of Chapter 3. It is used in Chapters 5, 6 and 8 to study the Dirichlet boundary-value problem, and in Chapter 9 to treat the Cauchy Problem. An intermediate Chapter 7 establishes the continuity of the constructed solutions. Chapter 10 presents the semigroup approach which is very different in spirit and has had a fundamental importance in the historical development of the whole subject. The whole set of ideas is used Chapter 11 to treat the Neumann Problem as well as the problems posed on Riemannian manifolds. This completes the first half of the book. Three remarks are in order:

(i) at this general level, there is an interest in considering not only the PME but rather a wider class of equations to which most of methods apply. This is why a large part of the material is derived for the class of complete Generalized Porous Medium Equations,

$$\partial_t u = \Delta\Phi(u) + f\,.$$

(ii) For reasons of simplicity at this stage, most of the treatment is restricted to integrable data, a sound assumption on physical grounds, though not necessary from the point of view of mathematical analysis, as the sequel will show.

(iii) A main point of the study is the introduction of the different types of generalized solution that appear in the literature and are natural to the problem, and the careful analysis of their scope and mutual relationships.

With this foundation, Part Two of the book enters into more peculiar aspects of the theory of the PME; existence with optimal data, free boundaries, selfsimilar solutions, higher regularity, symmetrization and asymptotics; though relying on the previous foundation, the new material is not necessarily more difficult, and the aspects it covers can probably be more attractive for many active researchers, both for theoretical or practical purposes.

Let us examine the contents of the different chapters in this part. The existence and uniqueness theory is complemented with two beautiful chapters on solutions for general classes of data, i. e., data that are not assumed to be either integrable or bounded. Chapter 12 covers the theory of solutions with so-called growing data. Optimal growth conditions are found that allows for a theory of existence and uniqueness. Chapter 13 extends the analysis to solutions whose initial value (so-called trace) is a Radon measure.

We are now ready for the main topics of the qualitative theory, which are covered in the next block of four chapters:

The propagation properties, another fundamental topic in the PME theory, are discussed in detail in Chapter 14, including all questions related to finite propagation, free boundaries and evolution of the support.

The PME theory in several space dimensions presented many difficulties and was developed at a slow pace. Much of earlier progress focused on understanding the basic questions in a one dimensional setting. Actually, we have a much more detailed knowledge in that case, and we devote Chapter 15 to present the main features, like the 1D free boundary.

Chapter 16 contains the full analysis of selfsimilarity, which plays a big role in the theory of the PME.

Chapter 17 deals with the principles of symmetrization and concentration and their applications.

We devote the next three chapters to the questions of asymptotic behaviour as t goes to infinity and higher regularity. Chapter 18 does the asymptotics for the Cauchy Problem, and Chapter 20 for the homogeneous Dirichlet Problem. The former contains the famous result on stabilization of the integrable solutions of the PME towards the ZKB profile which is the analogous for $m > 1$ of the convergence towards the Gaussian profiles of the solutions of the heat equation. Since this convergence is a way of expressing the Central Limit Theorem of Probability Theory, the convergence of the PME flow towards the ZKB is a Nonlinear Central Limit Theorem.

Chapter 19 examines the actual regularity of the solutions of the Cauchy Problem; it concentrates on describing two of the main results for nonnegative and compactly supported solutions: the Lipschitz continuity of the pressure and the free boundary for large times and the lesser regularity for small times of the so-called focusing solutions (or hole-filling solutions). Partial C^∞ regularity is also shown according to Koch, and the concavity properties according to Daskalopoulos and Hamilton and Lee and Vazquez.

The last two chapters gather complements on the previous material. We devote Chapter 21 to collect further applications to the physical sciences.

We will use notations that are rather standard in PDE texts, like Evans [229], Gilbarg-Trudinger [261] or equivalent, which we assume known to the reader. A detailed summary of the main basic concepts and notations of Real and Functional Analysis is contained in the final Chapter 22. This chapter also contains a number of technical appendices on material that is used in the book and was considered not to have a place in the main flow of the text. One of these results is the proof of the lack of contractivity of the PME flow in L^p spaces with p large, which answers a question raised by some experts and posed some open problems.

1.4.3 What is not covered

This is a basic book on a very rich subject that keeps growing in many exciting directions. We list here some of the topics where much progress has been made and have been nevertheless left out of the presentation.

1. The theory of the so-called limit cases of the PME. First, the limit $m \to 1$, where we can get either the heat equation or the eikonal equation, $u_t = |\nabla u|^2$ depending on the scaling of the data, [50, 375]. We also have the limit as $m \to \infty$, leading to the famous Mesa Problem, [85, 141, 463, 242].

2. The detailed treatment of the Fast Diffusion Equation. The reader can find an expository account at a rather advanced level in the author's Lecture Notes [515]. A whole set of references is given.

3. The more detailed study of the behaviour for large times, using recent work on gradient flows, optimal transportation and the entropy-entropy dissipation method, [155, 413].

   Also, the question of asymptotic geometry, in particular the question of asymptotic concavity, cf. [196, 197, 365].

4. The theory of viscosity solutions for the PME developed by Caffarelli and Vazquez, [144], see also [125, 332].

5. More general boundary value problems: the general Dirichlet problem, and then Neumann and Mixed Problems.

6. The Lagrangian approach and particle trajectories, as developed in [279, 389]. See also [515].

7. Numerical computation of PME flows, see [232] .

8. Stochastic versions of the Porous Medium Equation, as in the work of da Prato *et al.* [192]

9. The Porous Medium Equation posed on a Riemannian Manifold, [413, 121]. See Section 11.5 below.

Of course, we have left out the developments for parallel equations and models, though their mathematical development has been closely connected to that of the PME, like

(i) The combination of nonlinear diffusion and reaction or absorption. This is a classical area where a wide literature exists.

(ii) The combined models involving nonlinear diffusion and convection, like $u_t = \Delta\Phi(u) + \nabla \cdot \mathbf{F}(u)$. This has also been a very active area of research in recent years.

(iii) Gradient flows and p-Laplacian equations, and their relation with the PME in 1D.

(iv) The detailed study of the so-called dual equation, $v_t = (\Delta u)^m$.

1.5 Reading the book

The whole book is aimed at providing a comprehensive coverage that hopes to be useful both to the beginning researcher as a text, and to the specialist as a reference. For that purpose, it is organized in blocks of different difficulty and scope.

While trying to present the most relevant basic results with whole proofs in each chapter, a parallel effort has been made to present an informative panorama of the relevant results known about the topics of the chapter. However, and especially in the second part of the book, many interesting results that can be easily traced and read in the sources were discussed more briefly by evident reasons of space. The more advanced sections have been marked with a star, $*$. On the other hand, we have included the proof of many new results that the author felt were needed to complete the presentation and were not reported in the literature. Chapters contain detailed introductions where the topics to be covered are announced and commented upon, and are supplied with a final section of Notes (comments, historical notes or recommended reading) and a list of problems. Problems contain many bits of proofs and some are used in later chapters. Solving them is recommended to the reader, since we believe that the best way of reading mathematics is active reading. We also include some advanced problems; they are marked with a star, $*$.

The first part of the book has been devised as an introductory course to Nonlinear Diffusion centered on the PME and the GPME. Selections of the text centered on the PME and versions of it have been taught as such to Ph. D. students having previously followed courses in Classical Analysis, Functional Analysis and PDEs. Knowing some Physics of continuous media or studying the subject in parallel is useful, but not required. Several selections are possible for one semester courses, the simplest one consisting of Chapters 2 to 11 plus 14, maybe jumping on most of 6 and 7. Relevant and elementary material is also contained in Chapters 15, 16, and 18. We will give extensive references when the material used is not standard.

This is a book in PDEs and Analysis at a theoretical level but covering the interests of what is usually called Applied Analysis. We will pay a serious attention to some, say, classical applications, but the reader need not be an expert in any physical or natural science or engineering, since all relevant concepts will be clearly defined.

The reader will notice that the subject is rich in methods and results, but also in concepts and denominations, many taken from different branches of the applied sciences, others from different areas of mathematics. We will underline all new concepts by writing them in italics the first time they are precisely defined and referencing the relevant ones in the index.

We hope that the material will make it easier for the interested reader to delve into deeper or more specific literature. We have already mentioned that, although we concentrate most of our effort in examining the nonnegative solutions of the PME, the natural functional framework leads the mathematician to work with the signed PME. A number of important issues are still open for signed solutions.

Notes

Some historical notes

We have seen the important contribution of Zel'dovich, Kompaneets [532], 1950, who found the source solutions in a particular case, and Barenblatt [60], who performed a complete study of these solutions in 1952. After the work in the decade by Barenblatt et al. on selfsimilar solutions and finite propagation, cf. [71] and the book [63], the systematic theory of the PME can be said to have begun with the fundamental work of Oleĭnik and her collaborators Kalashnikov and Czhou around 1958 [408], who introduced a suitable concept of generalized solution and analyzed both the Cauchy and the standard boundary value problems in one space dimension. The work was continued by Sabinina, [457], who extended the results to several space dimensions. The qualitative analysis was advanced by Kalashnikov and many authors followed. The survey of the last author contains a very complete reference list on the literature concerning different aspects of the PME and related equations at the time. For earlier history see the Notes of next chapter.

Since the 70's, the interest for the equation has touched many other scholars from different countries. Here are some important landmarks. Bénilan [79] and Crandall et al. [180, 178] constructed mild solutions, Brezis developed the theory of maximal monotone operators [128], Aronson studied the properties of the free boundary [35, 36, 37], Kamin began the analysis of the asymptotic behaviour [319, 320], and Peletier et al. studied selfsimilarity, [54]. In the 80's well-posedness in classes of general data was established in Aronson-Caffarelli [42] and Bénilan-Crandall-Pierre [91], and the study of solutions with measures as data was initiated in Brezis-Friedman [131] and advanced by Pierre [434] and Dahlberg-Kenig [187]. Basic continuity of solutions and free boundaries was proved by Caffarelli and Friedman [138, 139, 140] and refined by DiBenedetto [206, 207], Sacks [461] and a number of authors.

There exists today a relatively complete theory covering the subjects of existence and uniqueness of suitably defined generalized solutions, regularity, properties of the free boundary and asymptotic behaviour, for different initial and boundary-value problems. Their names will appear in the development.

Previous reports on the PME and related equations

The text has as a precedent the notes prepared on the basis of the course taught at the Université de Montréal in June-July of 1990, aimed at introducing the subject and its techniques to young researchers [508]. The material has been also used for graduate courses at the Universidad Autónoma de Madrid. It has several earlier precedents. A short survey was published by Peletier [425] in 1981 and has been much used. A much longer survey paper is due to Aronson [38], written in 1986. Another often cited contribution, more in the form of a summary but including a discussion of related nonlinear parabolic equations and a very extensive reference list is due to Kalashnikov [317] in 1987. These have been main references during these years. In his book on "Variational Principles and Free-Boundary Problems" [240], 1982, Friedman devotes a chapter to the PME because of its strong connection with free boundary problems. Recently, the book by four Chinese authors, Wu, Yin, Li, and Zhao, [527], 2001, about nonlinear diffusion equations is worth mentioning.

Both PME and p-Laplacian equations are tied together as degenerate diffusions in DiBenedetto's book [209]. The book [469] by Samarski, Galaktionov, Kurdyumov and Mikhailov is mainly devoted to reaction diffusion leading to blow-up but has wide information about PME, specially related to selfsimilarity. A similar observation applies to [255] by Galaktionov and the author which concentrates on asymptotic methods based on self-similarity and Dynamical Systems ideas. This book contains a chapter with the main facts about the PME that appear in the asymptotic studies.

A reference to the mathematics of Diffusion is Crank [182] which contains a bulk of basic information on the classical applied topics and results. Conduction of heat in solids is treated by Carslaw and Jaeger [159]. A general text on Reaction-Diffusion equations is Smoller's [482]. The Stefan problem is covered in the already mentioned books by Rubinstein [454] and Meirmanov [388].

Chapter 2

Main applications

The porous medium equation,

$$\partial_t u = \Delta_x u^m, \quad m > 1, \quad u = u(x,t), \tag{2.1}$$

is a prominent example of nonlinear partial differential equation. In the particular case $m = 2$ it is called Boussinesq's equation. We are going to describe a choice of the main applications found in the literature that have served as a motivation for the development of the mathematical theory. In Section 2.1 we describe the standard model of gas flow through a porous medium (Darcy-Leibenzon-Muskat), in Section 2.2 the model of nonlinear heat transfer (Zel'dovich-Raizer), in Section 2.3 Boussinesq's model of groundwater flow, in Section 2.4 a model of population dynamics (Gurtin-McCamy). Further applications will be found in Chapter 21.

An understanding of this chapter is recommended since we will be using some of the images and names suggested by these applications.

2.1 Gas flow through a porous medium

The Porous Medium Equation owes its name to its use in describing the flow of an ideal gas in a homogeneous porous medium. According to Leibenzon [367] and Muskat [394], this flow can be formulated from a macroscopic point of view in terms of the variables *density*, which we represent by ρ; *pressure*, represented by p; and *velocity*, represented by $\mathbf{V}$, which are functions of space x and time t (the former is a vector). These quantities are related by the following laws:

(i) *Mass balance*, also called continuity equation in fluid mechanics,

$$\varepsilon\, \rho_t + \nabla \cdot (\rho\, \mathbf{V}) = \mathbf{0}\,. \tag{2.2}$$

Here $\varepsilon \in (0,1)$ is the porosity of the medium, and $\nabla\cdot$ represents the divergence operator.

(ii) *Darcy's Law*, an empirical law formulated in 1856 by the French engineer H. Darcy, [193], which describes the dynamics of flows through porous media

$$\mu\, \mathbf{V} = -k\, \nabla p\,. \tag{2.3}$$

It replaces for that kind of media the usual Navier-Stokes law of standard fluid flows.

(iii) *State Equation*, which for perfect gases asserts that

$$p = p_0\,\rho^\gamma\,, \tag{2.4}$$

where γ, is called the so-called *polytropic exponent*. Its values in the two main cases covered by this state law when applied to gases are: $\gamma = 1$ for isothermal processes, and γ larger than 1 for adiabatic ones (for air at normal temperature, the value $\gamma = 1.405$ is derived from the experimental data). In any case $\gamma \ge 1$.

The parameters μ (the viscosity of the fluid), ε (the porosity of the medium), k (the permeability of the medium) and p_0 (the reference pressure) are assumed to be positive and constant, which constitutes an admissible simplification in many practical instances, but need not be the case in a more general situation. Accepting such hypothesis, an easy calculation allows to reduce (2.2)-(2.4) to the form

$$\rho_t = c\,\Delta(\rho^m)\,, \tag{2.5}$$

with exponent $m = 1 + \gamma$ and

$$c = \frac{\gamma k p_0}{(\gamma+1)\varepsilon\mu}. \tag{2.6}$$

The constant c can be easily scaled out (define for instance a new time, $t' = ct$), thus leaving us with the PME. Mathematically, we say that constants that can be scaled out play no role, though the engineer will need to take a look at them; this is an interesting philosophy that will be much used.

Observe that in the above applications the exponent m is always equal or larger than 2. The mathematical theory to be developed below does not find many differences between the exponents m as long as they are larger than 1, though the formulas look a bit simpler for $m = 2$. In all the formulas, the operators $\nabla\cdot = \texttt{div}$, $\nabla = \texttt{grad}$ and Δ, the Laplacian, are supposed to act on the space variables $x = (x_1, \cdots, x_d)$.

In order to adapt the notation to the mathematical taste and also adapt to current usage in the PME, we will use the letter u instead of ρ for the density; and the letter v is used for the pressure, which is exactly defined by the expression

$$v = \frac{m}{m-1} u^{m-1}, \tag{2.7}$$

so-called *mathematician's pressure*. This is an important definition that will be used frequently in the book. It allows to easily recover the above physical formulas with $\varepsilon = k = \mu = 1$, that is, forgetting about physical constants. Thus, Darcy's law for the velocity is written in the form

$$\mathbf{V} = -\nabla v = -m\,u^{m-2}\nabla u\,, \tag{2.8}$$

and the mass balance can be written in the form $\partial_t u + \nabla\cdot\mathbf{j} = 0$, where the quantity $\mathbf{j} = u\,\mathbf{V}$ in this formula is called the *mass flux*.

2.1.1 Extensions

Non-homogeneous media

The consideration of flows where ε, μ and k are not constant, but functions of space and maybe also time, provides us with a natural generalization of the PME. The equation is then written in the form suitable for inhomogeneous media, NHPME,

$$\varepsilon(x,t)\,\partial_t u = \nabla\cdot(c(x,t)\nabla u^m), \tag{2.9}$$

where ε and c are given positive functions (or even, nonnegative).

Filtration Equation

A quite different approach is assuming that the state law is not power-like, but has the form $p = p(\rho)$, as happens in general barotropic gases, and also that k and μ may depend on ρ. In that case we get a final equation for the density of the form

$$\rho_t = \Delta\Phi(\rho) + f, \tag{2.10}$$

where Φ is a given monotone increasing function of ρ, $\rho \geq 0$. This is called the Filtration Equation or Generalized Porous Medium Equation. In our application, $\Phi'(\rho) = \rho k(\rho)p'(\rho)/\mu(\rho)\varepsilon$. The last term $f = f(x,t)$ represents mass sources or sinks distributed in the medium.

We can also combine both types of extensions. We leave the detail to the reader. See also Section 5.11 for more general variants of the PME and the Filtration Equation.

2.2 Nonlinear heat transfer

A quite important application, probably second in importance for the historic development of the field, happens in the theory of heat propagation with temperature-dependent thermal conductivity. The general equation describing such a process (in the absence of heat sources or sinks) takes the form

$$c\rho\,\frac{\partial T}{\partial t} = \mathtt{div}(\kappa\nabla T)\,, \tag{2.11}$$

where T is the temperature, c the specific heat (at constant pressure), ρ the density of the medium (which can be a solid, fluid or plasma) and κ the thermal conductivity. In principle all these quantities are functions of $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$. In the case where the variations of c, ρ and κ are negligible, we obtain the classical heat equation. However, when the range of variation of the temperatures is large, say hundreds or thousands of degrees, such an assumption is not very reasonable.

(i) The simplest case of variable coefficients corresponds to constant c and ρ and variable κ, a function of temperature, $\kappa = \phi(T)$. We then write (2.11) in the form

$$\partial_t T = \Delta\Phi(T)\,. \tag{2.12}$$

The *constitutive function* Φ is given by

$$\Phi(T) = \frac{1}{c\rho}\int_0^T \kappa(s)\,ds\,. \tag{2.13}$$

This is sometimes called Kirchhoff's transform. We find again the Filtration Equation (2.10), but now in a completely different applied context. If the dependence is given by a power function

$$\kappa(T) = a\,T^n\,, \tag{2.14}$$

with a and $n > 0$ constants, then we get

$$T_t = b\,\Delta(T^m) \quad \text{with} \quad m = n+1\,, \tag{2.15}$$

and $b = a/(c\rho\, m)$, thus the PME but for the constant b which is easily scaled out.

(ii) In case we also assume that $c\rho$ is variable, $c\rho = \psi(T)$, we still obtain a generalized PME, though we have to work a bit more. Thus, we introduce a new variable T' by the formula

$$T' = \Psi(T) \equiv \int_0^T \psi(s)\,ds\,. \tag{2.16}$$

We then obtain the following equation for T:

$$\partial_t \Psi(T) = \Delta\Phi(T), \tag{2.17}$$

which can also be written as a standard GPME in terms of the variable T' by inverting (2.16), i.e. $\partial_t T' = \Delta F(T')$, with $F = \Phi \circ \Psi^{-1}$. Again, if the dependences are given by power functions we obtain the PME with an appropriate exponent.

Zel'dovich and Raizer [533] propose model (i) to describe heat propagation by radiation occurring in plasmas (ionized gases) at very high temperatures. In that case energy is transferred mainly by electromagnetical radiation (as well as by conduction and convection, but these are of lesser importance). According to the mentioned reference, the radiation thermal conductivity is defined as

$$\kappa = \frac{lc}{3}\,c_{rad}, \quad c_{rad} = aT^3, \tag{2.18}$$

where c is speed of light, l is Rosseland's mean free path and the form of the radiation specific heat c_{rad} comes from the law of black body radiation law. This is an approximation valid under circumstances called the "optically thick" limit. If l is supposed to be constant we obtain the PME with $m = 4$. [1]

However, l is usually temperature dependent, $l \sim aT^n$, with different exponents depending on the type of high-energy approximations of the process. For multiply ionized gases the exponent n ranges the interval from 1.5 to 2.5, and then $m - 1 = n + 3 \sim 4.5 - 5.5$. This description is taken from Chapter X of [533] where further details can be found. Other references: Longmire [378], Ockendon et al. [404].

Remark. The fast diffusion equation is found in Plasma Physics in a different context. Plasma diffusion with the Okuda-Dawson scaling implies a diffusion coefficient ($D \sim u^{-1/2}$) in equation (1.1) where u is the particle density. This leads to the FDE with $m = 1/2$. See Berryman-Holland [106]. On the other hand, Berryman [105] reports that electron heat conduction in a plasma can be modelled with the PME with exponent $m = 3.5$.

[1] We will obtain the same exponent in the thin film example of Section 21.1.

2.3 Groundwater flow. Boussinesq's equation

We examine next another problem in fluid mechanics, this time related to liquids. It deals with the filtration of an incompressible fluid (typically, water) through a porous stratum, the main problem in groundwater infiltration. The model was developed first by Boussinesq in 1903 [123] and is related to the original motivation of Darcy [193]. See also Polubarinova-Kochina [439].

Modelling. We will impose the following simplifying assumptions:

1) the stratum has height H and lies on top of a horizontal impervious bed, which we label as $z = 0$,

2) we ignore the transversal variable y, and

3) the water mass which infiltrates the soil occupies a region described as

$$\Omega = \{(x, z) \in R : \ z \leq h(x, t)\}. \tag{2.19}$$

In practical terms, we are assuming that there is no region of partial saturation.

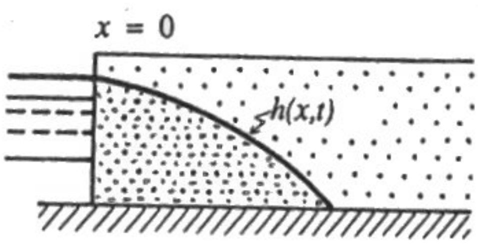

Figure 2.1: A schema of ground infiltration

This is an evolution model. Clearly, $0 \leq h(x,t) \leq H$ and the free boundary function h is also an unknown of the problem. In this situation, we arrive at a system of three equations with unknowns the two velocity components u, w and the pressure p in a variable domain: one equation of mass conservation for an incompressible fluid and two equations for the conservation of momentum of the Navier-Stokes type. Add initial and boundary conditions to the recipe. The resulting system is too complicated and can be simplified for the practical computation after introducing a suitable assumption, the hypothesis of *almost horizontal flow*, i. e., we assume that the flow has an almost horizontal speed $\mathbf{u} \sim (u, 0)$, so that h has small gradients. It follows that in the vertical component of the momentum equations

$$\rho\,(\frac{du_z}{dt} + \mathbf{u} \cdot \nabla u_z) = -\frac{\partial p}{\partial z} - \rho g,$$

we may neglect the inertial term (the left-hand side). Integration in z gives for this first approximation $p + \rho g z =$ constant. We now calculate the constant on the free surface $z = h(x,t)$. If we impose continuity of the pressure across the interface, we have $p = 0$ (assuming constant atmospheric pressure in the air that fills the pores of the dry region $z > h(x,t)$). We then get

$$p = \rho g(h - z). \tag{2.20}$$

In other words, the pressure is determined by means of the *hydrostatic approximation*.

We go now to the mass conservation law which will give us the equation. We proceed as follows: we take a section $S = (x, x+a) \times (0, C)$. Then,

$$\varepsilon \frac{\partial}{\partial t} \int_x^{x+a} \int_0^h dy dx = -\int_{\partial S} \mathbf{u} \cdot \mathbf{n} \, dl, \tag{2.21}$$

where ε is the porosity of the medium, i. e., the fraction of volume available for the flow circulation, and $\mathbf{u}$ is the velocity, which obeys Darcy's law in the form that includes gravity effects

$$\mathbf{u} = -\frac{k}{\mu} \nabla(p + \rho g z). \tag{2.22}$$

On the right-hand lateral surface we have $\mathbf{u} \cdot \mathbf{n} \approx (u, 0) \cdot (1, 0) = u$, i. e., $-(k/\mu) p_x$, while on the left-hand side we have $-u$. Using the formula for p and differentiating in x, we get

$$\varepsilon \frac{\partial h}{\partial t} = \frac{\rho g k}{\mu} \frac{\partial}{\partial x} \int_0^h \frac{\partial}{\partial x} h \, dz. \tag{2.23}$$

We thus obtain *Boussinesq's equation*

$$h_t = \kappa \, (h^2)_{xx} \tag{2.24}$$

with constant $\kappa = \rho g k / 2m\mu$. This is the PME with $m = 2$. It is a fundamental equation in groundwater infiltration. The system of nonlinear equations proposed in the initial model is reduced to solving a unique nonlinear heat equation that gives the height of the water mound. Once $h(x, t)$ is calculated, we may calculate the pressure via (2.20) and then the speed by means of Darcy's law.

We have made the final step of the derivation of Boussinesq's equation in one dimension for simplicity, but it generalizes immediately to several dimensions and gives

$$h_t = \kappa \, \Delta(h^2). \tag{2.25}$$

Extension

When there exists a water input into the porous stratum (by natural or artificial recharge), or an output (by sinks or pumping), the equation takes the complete form

$$h_t = \kappa \, \Delta(h^2) + f, \tag{2.26}$$

where function $f(x, z, t)$ reflects those effects. If we ideally assume that such effects take place at precise space locations, we are led to consider instead of a function f a sum of Dirac masses, which gives rise to interesting mathematical problems.

Remark. This is a fluid flow model and it involves a physical pressure, that is given by the hydrostatic law (2.20), a function of x and z. However, in average over z it amounts to $c\, h(x)$, which is in accordance with our assumption that $v \sim u^{m-1}$ of Section 2.1.

2.4 Population dynamics

A very interesting example concerns the spread of biological populations. The simplest law regarding a population consisting of a single species is

$$\partial_t u = \mathtt{div}(\kappa \, \nabla u) + f(u), \tag{2.27}$$

where u stands for the density or concentration of the species, and the reaction term $f(u)$ accounts for symbiotic interaction within the species; the medium is supposed to be homogeneous. According to Gurtin and McCamy [279], when populations behave so as to avoid crowding it is reasonable to assume that the *diffusivity* κ is an increasing function of the population density, hence

$$\kappa = \phi(u)\,, \quad \phi \quad \text{increasing.} \tag{2.28}$$

A realistic assumption in some particular cases is $\phi(u) = a\,u$. Disregarding the reaction term we obtain the PME with $m = 2$.

Of course, a complete study must take into account at least the reaction terms, and very often, the presence of several species. This leads to the consideration of nonlinear reaction-diffusion systems of equations of parabolic type containing lower order terms, whose diffusive terms are of PME type. Such equations and systems constitute therefore an interesting possibility of generalization of the theory of the PME. Similar equations appear in Chemistry in the study of diffusive and reacting media.

2.5 Other applications and equations

The previous applications show how naturally the PME appears to replace the classical heat equation in processes of heat transfer or diffusion of a substance or population dispersal, whenever the assumption of constancy of the thermal conductivity (resp. diffusivity) cannot be sustained, and, instead, it is reasonable to assume that it depends in a power-like fashion (or almost power-like fashion) on the temperature (resp. density or concentration).

Once the theory for the PME began to be known, a number of applications have been proposed. Some of them concern the Fast Diffusion Equation, the Generalized PME and the inhomogeneous versions already commented. There are numerous examples with lower order terms, in the areas of Reaction-Diffusion, where the PME is only responsible for one of the various mechanisms of the equation or system.

We do not want to break the flow of the presentation of the theory with more applications at this point. Therefore, we devote Chapter 21 to describe a number of interesting applications for the reader's benefit. For applications of the Fast Diffusion Equation we refer to the list of monograph [515].

2.6 Images, concepts and names taken from the applications

The presentation of the main applications of an equation or theory is a common practice in PDE's, and serves the purpose of justifying the attention paid to a particular topic, but also that of orienting the researcher in the difficult task of finding concepts and tools in the wild forest of applied nonlinear analysis.

It also serves another purpose that we want to stress here. It gives us the possibility of using a given application to put some flesh into the abstract thinking in the form of images, concepts and also a series of names that can be quite useful in coining a form of speech that allows for insight and communication.

Thus, starting with the name, it is quite common in the literature to talk about flows in porous media as the image behind the calculations. This brings us into talking about densities (u), pressures (v) and velocities ($-\nabla v$). Such a speech will be quite useful, specially when studying the propagation aspects, like the existence of free boundaries. In this point of view, the integral

$$M(\Omega, t) := \int_\Omega u(x,t)\,dx$$

is called the *mass of gas* contained in volume Ω at time t. If Ω is the whole domain of definition, we call it the *total mass* at time t. An important issue of the theory is the conservation of the total mass in time (mass conservation law), which holds for some problems and does not for others (e.g., if mass is allowed to flow through the boundary).

We must remind, however, that for the pure mathematician all this is a manner of speech, since our theories are model-independent; for the applied mathematician, we must recall that the theory aspires to serve the needs of different applied areas, and will at times use the images and denominations of those other areas.

In our case, a quite important area is thermal propagation. Changing the letter for the unknown in equation (2.15), this application gives us the possibility of seeing equation $\partial_t u = \Delta u^m$ in terms of heat transfer, thus allowing us to assign the meaning of temperature to u, and temperature-dependent diffusivity to $D(u) = mu^{m-1}$. We remind the interested reader that Fourier's law is now written as $\Phi = -\kappa(u)\nabla u = -\nabla u^m$, where Φ is the heat flux as defined in standard heat theory. The total mass becomes now a *total thermal energy* (but for the constant factor $c\rho$ that we imagine put to 1).

The areas of population dynamics and chemistry add the possibility of viewing u as a *concentration*, and now $D(u)$ is the concentration-dependent diffusivity. Concentration is the common concept in applications to Nonlinear Diffusion processes.

Notes

Some historical notes II. Let us review some of the early history, previous to the systematic theory, as far as we have discovered it. The French scientist J. Boussinesq seems to have been the first author to propose the porous medium equation as a mathematical model for a physical process, [123], precisely to calculate the height of the water mound in groundwater infiltration. He used as basic flow law the one proposed by H. Darcy [193] in 1856, and under the so-called Dupuit's assumption of small gradient, [223]. Note that the exponent is $m = 2$.

It is historically remarkable that, even if the PME looks like an innocent nonlinear version of the heat equation, it took many years for it to be correctly posed (in classes of weak solutions) and solved.

In the 30's the equation appeared again, this time for $m \geq 2$, in the study of gases in porous media, connected to oil extraction, in the works of two engineers, the Russian L. Leibenzon [367] and the American M. Muskat [394]. Polubarinova-Kochina [438] studied in 1948 the problem of groundwater infiltration into a porous stratum and proposed a selfsimilar solution that improved the knowledge of special solutions and their role in finite propagation.

A main progress was made in Moscow in the 1950's, when Ya. Zel'dovich and collaborators studied heat propagation in plasmas and landed again on the Porous Medium Equation, and its relative the Filtration Equation. Such simplified models are applicable for instance in the first stage after a nuclear explosion, when thermal waves are propagated in a gas that can still be considered stationary. Heat conduction happens mainly by radiation and the thermal conductivity is heavily dependent on temperature.

The mathematical study was seriously undertaken, attention to the presence of a front was duly paid, and the famous source-type solutions were found by Ya. Zel'dovich, A. Kompanyeets and B. Barenblatt, see Introduction or Chapter 4. Finally, the theory of well-posedness started with O. Oleĭnik and her group around 1958. Basic results were obtained in Moscow in the early 60's for the problem in several space dimensions (E. Sabinina, A. Kalashnikov, Yu. Dubinskii).

Reading notes. Earlier reference lists of applications of the PME can be found in: Berryman [105], Peletier [425], Lacey-Ockendon-Tayler [355], and Aronson [38], among other sources.

A general reference for the equations of fluid mechanics written with a mathematical audience in mind is Chorin and Marsden, [171]. Interesting further reading on flows in porous media: Bear's books [76, 77], Barenblatt-Entov-Rhyzhik [67]. See also the author's lecture notes on Flows in Porous Media [507]. A general reference for mathematical models in Biology are Murray's two volumes, [393].

Among the many works on nonlinear diffusion equations in population dynamics, let us mention the early papers of Aronson and Weinberger [53] and Aronson, Crandall and Peletier [46].

Problems

Problem 2.1 Scale out the constant c in Equation (2.5) by hiding it in the time variable, as indicated in the text. Do it also by hiding it in the space variable.

Problem 2.2 Derive the equation of filtration of a gas with a barotropic law in an inhomogenous medium.

Problem 2.3 Try to show formally that the mass conservation law should not hold for positive solutions of the PME defined in a domain $\Omega \subset \mathbb{R}^d$ and satisfying zero Dirichlet boundary conditions. How about zero Neumann conditions?

Problem 2.4 Derive the equation satisfied by the pressure v, defined by (2.7), when the density obeys the PME with exponent m.

Solution: $v_t = (m-1)v\,\Delta v + |\nabla v|^2$.

Problem 2.5 Pressure for the Filtration Equation. When we consider Equation (2.10), i. e., $\rho_t = \Delta\Phi(\rho)$, it can be written in the conservation form

$$\rho_t + \nabla\cdot(\rho\,\mathbf{V}) = \mathbf{0}\,.$$

(i) Show that this implies that $\mathbf{V} = -\nabla v$ if v is defined as a function of ρ by the formula

$$v = p(\rho) := \int_0^\rho \frac{\Phi'(s)}{s}\,ds,$$

whenever this integral is convergent.

(ii) Find the equation satisfied by v. [*Solution.* The equation is

$$v_t = a(v)\Delta v + |\nabla v|^2.$$

and $a(v) = \Phi'(\rho)$. See more in [343], [125].

(iii) Check that in the PME case this gives the usual formulas for the pressure and its equation.

Chapter 3

Preliminaries and Basic Estimates

This chapter covers preliminary material on parabolic equations needed to develop the main theories of the book. In this and following chapters we work on subdomains of the Euclidean space $\mathbb{R}^d$ or the whole such space. However, we will see in Chapter 11 that the main facts of the theory extend in a natural way to equations posed on a Riemannian manifold.

We start with a review of useful properties of quasilinear parabolic equations. Next, Section 3.2 is devoted to non-degenerate versions of the Generalized PME that will de used in approximating the degenerate cases. We derive for these better-behaved equations the basic estimates which will be used in developing the general theory for the class of possibly degenerate equations we have in mind.

We then specialize in Section 3.3 to properties that are formally satisfied by the PME; they will be justified in later chapters and used in the constructions of the different theories. Finally, Section 3.4 reviews the properties of the most popular alternative formulations of the PME.

In this chapter we consider solutions with changing sign. In most of the calculations Φ is not assumed to be a power. Sections 3.2 and 3.3 can be considered as basic material to be borrowed by later chapters.

3.1 Quasilinear equations and the PME

Let us review the properties of the solutions to quasilinear parabolic problems of the form

$$\partial_t u = \sum_{i=1}^{d} \frac{\partial}{\partial x_i} a_i(x, t, u, \nabla u) + b(x, t, u, \nabla u). \tag{3.1}$$

where $a_i(x, t, u, p_1, \cdots, p_d)$ and $b(x, t, u, p_1, \cdots, p_d)$ are called structural functions. They must satisfy certain conditions to ensure that a theory including existence, uniqueness and a certain regularity can be developed. The main condition is parabolicity to be explained presently. We will follow Ladyzhenskaya et al. [357], Friedman [239] or the more recent Lieberman [371] for reference to the classical theory of solutions of these equations.

3.1.1 Existence of classical solutions

In the classical theory, we assume that the structural functions $a_i(x,t,u,p_1,\cdots,p_d)$ and $b(x,t,u,p_1,\cdots,p_d)$ are bounded and C^∞ in their arguments. The *uniform parabolicity* condition is formulated as follows: there exist constants $0 < c_1 < c_2 < \infty$ such that for every vector $\xi = (\xi_1,\cdots,\xi_d)$, the following inequalities hold

$$c_1|\xi|^2 \le \sum_{i=1}^{d} \frac{\partial a_i}{\partial p_j}(x,t,u,u_{x_i})\,\xi_i\xi_j \le c_2|\xi|^2. \tag{3.2}$$

Here are some of the most basic results under the classical assumptions:

(i) Given bounded and continuous initial data, the Cauchy problem can be solved and the solution $u(x,t)$ is unique, C^∞ smooth in $Q = \mathbb{R}^d \times (0,\infty)$ and continuous down to $t=0$, i. e., $u \in C(\mathbb{R}^d \times [0,\infty))$ and $u(x,0) = u_0(x)$. If the initial data are only bounded, then the initial data are taken only in the sense of a. e. convergence (and more precisely, along time cones).

(ii) A main property in the theory of parabolic equations is the Maximum Principle, that is better termed the Comparison Principle in the nonlinear context. In the classical theory it takes a strong form that says:

Strong Maximum Principle. *Given two classical solutions $u(x,t)$ and $v(x,t)$ of the same equation of type* (3.1), *both defined and continuous in $S = \mathbb{R}^d \times [0,T]$, if we assume that $u(x,0) \le v(x,0)$, then either $u = v$ everywhere in S, or $u < v$ everywhere in S.*

(iii) The existence, uniqueness and regularity theory of classical solutions extends to the mixed problems posed in cylindrical domains of the form $Q = \Omega \times (0,T)$ where Ω is a bounded domain of $\mathbb{R}^d$ with smooth boundary. Then, we have to give information not only of the initial data but also of data on the lateral boundary $\Sigma = \partial\Omega \times [0,T)$, which takes the form of Dirichlet data, Neumann data or some other versions that are found in the literature. This is why the problems are usually called 'initial and boundary value problems', IBVPs. If the initial and boundary data are compatible for $x \in \partial\Omega$ and $t = 0$, these mixed problems also have existence, uniqueness and regularity and the Strong Maximum Principle holds: the same conclusion $u < v$ applies if $S = \Omega \times [0,T]$, Ω is a bounded open set with smooth boundary, and boundary data $u \le v$ are prescribed on $\Sigma = \partial\Omega \times [0,T]$..

3.1.2 Weak theories and the PME

In practice, the classical assumptions on a_i and b are not met in many problems of interest in the applied sciences. This is the origin of the weak theories, where relaxed conditions are accepted and then generalized solutions are obtained in Sobolev classes of weakly differentiable functions. The condition of uniform parabolicity is usually kept.

We will quote the results from the weak theory of non-degenerate quasilinear parabolic equations as the need arises. But let us mention that the strong maximum principle need not hold, and the typical comparison result states that if the data of a Cauchy problem are ordered by the relation $\le$, so are the solutions a. e. This applies also to the Dirichlet and Neumann problems with suitable ordering of the boundary data.

We turn now to the PME example. The assumptions of smoothness fail in our case since the PME is a particular case of equation (3.1) where

$$a(x,t,u,p) = |u|^{m-1}p$$

with $u \in \mathbb{R}$, $p \in \mathbb{R}^d$, and $b = 0$. The main problem is non-uniform parabolicity; indeed, even for bounded nonnegative solutions, condition (3.2) can only hold when $c_1 = 0$. This extension of the concept of parabolicity is called *degenerate parabolicity*. In physical speech, when thinking in terms of thermal propagation, it means that the thermal conductivity vanishes at zero temperature; in diffusion problems, we call it degenerate diffusivity. In any case and with any name, the study of the consequences of degenerate parabolicity is the reason of this book.

We still can save the classical theory as long as we consider 'non-degenerate' data u_0, i. e., data in the range $\varepsilon \le u_0(x) \le 1/\varepsilon$ with $\varepsilon > 0$. More generally, in the signed PME we may choose this option or $-1/\varepsilon \le u_0(x) \le -\varepsilon$. In order to solve the Cauchy problem for the PME with such data, we take $a(x,t,u,p) = m|u|^{m-1}p$ for $\varepsilon \le |u| \le 1/\varepsilon$, and extend the function as a linear function of u and p for u near 0 or infinity, making a smooth connection around the values $u = \pm\varepsilon$ and $u = \pm 1/\varepsilon$. With these modifications, we pose the problem of finding a solution of the perturbed equation

$$\partial_t u = \operatorname{div}(\phi(u)\nabla u) = \Delta\Phi(u).$$

Since the degeneracy has been eliminated, there is a unique classical solution, and it satisfies the same bounds $\varepsilon \le |u(x,t)| \le 1/\varepsilon$. But this means that u never takes values in the region of perturbed values, hence we get a classical solution of the PME. Let us state the result for the record

Theorem 3.1 Classical solutions of the PME. *Assume that u_0 is a continuous function in $\mathbb{R}$ with*

$$\varepsilon \le u_0(x) \le 1/\varepsilon$$

for some $\varepsilon > 0$ and all $x \in \mathbb{R}^d$. Then there exists a classical solution of the PME satisfying

$$\varepsilon < u(x,t) < 1/\varepsilon$$

for every $x \in \mathbb{R}^d$ and $0 < t < \infty$. If u_0 is C^k-smooth, so is u at $t = 0$; if u_0 is only bounded, then the convergence to the initial data takes place a. e. along time cones.

The same applies to the signed PME with values in the range $-1/\varepsilon \le u_0(x) \le -\varepsilon$; a classical solution exists and $-1/\varepsilon < u(x,t) < -\varepsilon$.

A similar argument applies to the Cauchy-Dirichlet problem when the boundary data satisfy the same condition $\varepsilon \le u(x,t) \le 1/\varepsilon$ for $x \in \partial\Omega$, $t \ge 0$, and are compatible with the initial data. It also applies to the problem with zero Neumann boundary conditions, and to other variants. The extensions to negative solutions also hold.

If, on the contrary, the data take zero values inside Ω the classical theories cannot apply, the strong maximum principle does not either, and there is no way to circumvent the weak theories. Moreover, a number of curious phenomena appear, like finite propagation and free boundaries, which are at the core of this book.

Let us finally recall that when data are unbounded we meet another problem, namely that $\partial a_i/\partial p_i = m|u|^{m-1}$ goes to infinity, so that the equation loses the upper bound on parabolicity. This is a different phenomenon, it will be less visible, but will also affect all calculations with large values of u, in the sense that the estimates will be different from the ones for the HE also in this case.

3.2 The GPME with good Φ. Main estimates

Our aim is to establish an existence and uniqueness theory of generalized solutions for the PME, and also the Generalized PME with quite general Φ. This will be done in Chapter 5 and following in the class of weak solutions, and we will also obtain the most important properties of that class of solutions. The program offers as a main difficulty the fact that the PME is a degenerate equation. Three other difficulties complicate the task: the generality of the nonlinearity Φ, the generality of the data, and the sign of the solutions.

A standard approach to the construction of solutions for the PME and other degenerate cases will be approximation with non-degenerate problems. A quite useful choice, though not the only one possible, is approximation with a GPME having a nonlinearity $\Phi : \mathbb{R} \mapsto \mathbb{R}$ which is C^2 smooth and with $\Phi'(u) > 0$ for all $s \in \mathbb{R}$. Under such assumptions, the equation is parabolic non-degenerate, and we may apply standard quasilinear theory to obtain the existence and uniqueness of *classical solutions*, i. e., solutions such all the derivatives appearing in the equation exist and are continuous and the equation is satisfied everywhere in the space-time domain where we are working. This is what we will do in this section as a preliminary for the full treatment. We will assume the normalization $\Phi(0) = 0$, since this implies no loss of generality (the equation is invariant under addition of a constant to Φ). We also ask the domain to have a smooth boundary, $\Gamma = \partial\Omega \in C^{2,\alpha}$. Actually, the consideration of inhomogeneous media recommends a bit more of generality and we will assume that $\Phi : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, it is smooth in both variables, it is strictly increasing in the second, and $\Phi(x, 0) = 0$ for all $x \in \Omega$.

There are two main problems: the homogeneous Dirichlet Problem is

$$\partial_t u = \Delta\Phi(x,u) + f \quad \text{in } Q_T\,, \tag{3.3}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega, \tag{3.4}$$

$$u(x,t) = 0 \quad \text{in } \Sigma_T, \tag{3.5}$$

where $Q_T = \Omega \times (0, T)$, and $\Sigma_T = \partial\Omega \times [0, T)$ is the lateral boundary. On the other hand, the homogeneous Neumann Problem consists of equations (3.3), (3.4) and

$$\frac{\partial}{\partial\nu}\Phi(x,u) = 0 \quad \text{in } \Sigma_T, \tag{3.6}$$

where ν is the outer normal to the boundary $\partial\Omega$ As for the data, we will assume that u_0 and f are bounded and C^α functions, and $u_0(x) = 0$ for $x \in \partial\Omega$. Under such assumptions, we apply the quasilinear theory to obtain the following existence result and, what is more important for later use, the main estimates on which the weak theory will be based.

Theorem 3.2 *Under the above regularity assumptions, the Dirichlet Problem* (3.3) – (3.5) *admits a classical solution u in the space $C^{2,1}(\overline{Q})$. If Φ, u_0 and f are C^∞, then so is u in Q. Similar results apply to the Neumann Problem.*

The above Dirichlet Problem usually serves to produce the approximate solutions that will be used to construct weak solutions of the PME and other cases of the Filtration Equation. Besides, it allows us to derive the main quantitative estimates on which the subsequent study is based. This is the content of the next subsections. The first two of them contain bounds for the solution. Next, we obtain the stability estimate in L^1 norm, one of most peculiar mathematical properties of these nonlinear diffusion processes. Three further estimates contain bounds for the derivatives that will be used to ensure compactness in the approximation processes.

3.2.1 Maximum Principle and Comparison

It applies to the solutions of both Dirichlet and Neumann problems. It has a simple form when Φ does not depend explicitly on x.

Lemma 3.3 *If $\Phi = \Phi(u)$, then the solutions of the homogeneous Dirichlet or Neumann problem for equation* (3.3) *satisfy*

$$\|u\|_{L^\infty(Q_T)} \le \|u_0\|_{L^\infty(\Omega)} + T\|f\|_{L^\infty(Q_T)} \tag{3.7}$$

Proof. Let $M = \sup(u_0)$ and $N = \sup_Q f$. As an immediate consequence of the classical Maximum Principle, we have

$$u(x,t) \le M + Nt \quad \text{in} \quad Q, \tag{3.8}$$

and a similar estimate applies as a lower bound. Hence, for bounded data u_0 and f we have a bound on the solution.

The Comparison Principle holds for smooth solutions: if $u, \widehat{u}$ are solutions with initial data such that $u_0 \le \widehat{u}_0$ a. e. in Ω and $f \le \widehat{f}$ a. e. in Q, then $u \le \widehat{u}$ a. e. in Q. In particular, if $u_0, f \ge 0$ in Ω, then $u \ge 0$ in Q. □

An inhomogeneous extension. In case Φ depends on x, things are not so simple. We have to make some assumption. Suppose to fix ideas that

$$\Delta_x \Phi(x,z) \le K_1 + K_2 z \tag{3.9}$$

for all $x \in \mathbb{R}$ and all $z \ge M_0$ and constants $K_1, K_2 \ge 0$. We argue on a point where $u(x,t)$ touches from below the function

$$U(x,t) = M + Ct + \varepsilon. \tag{3.10}$$

If $u < U$ we get an estimate. If not, there is a first contact point (x_0, t_0), and there the difference $f(x) = w(x,t_0) - \Phi(x, U(t_0))$, with $w = \Phi(x,u)$, attains a space maximum (equal 0). At that point $\Delta_x w \le \Delta_x \Phi(x, U(t_0))$, and

$$\partial_t u(x_0,t_0) = f(x_0,t_0) + \Delta_x w(x_0,t_0) \le N + \Delta_x \Phi(x, U(t_0)).$$

If moreover $U(x_0,t_0) > M_0$ we get

$$\partial_t u(x_0,t_0) \le N + K_1 + K_2(M+\varepsilon) + K_2 C t_0.$$

on the other hand, at the first touching point $u_t(x_0,t_0) \ge U_t(x_0,t_0) = C$. Therefore, we avoid the touching point if

$$N + K_1 + K_2(M+\varepsilon) + K_2 C t_0 < C.$$

Suppose now that $t_0 \in [0, 1/(2K_2)]$. Then we may take $C = 2(N + K_1 + K_2(M+\varepsilon))$. Letting $\varepsilon \to 0$ we get the result

Lemma 3.4 *Let us take the situation of Lemma* 3.3, *but now* $\Phi = \Phi(x,u)$. *If* (3.9) *holds, then*

$$u(x,t) \le \min\{M + Ct, M_0\} \quad C = 2(N + K_1 + K_2 M) \tag{3.11}$$

for all $x \in \Omega$ *and* $0 < t < 1/(2K_2)$.

If we want to extend the time interval when $K_2 \ne 0$, we argue in time steps of $1/(2K_2)$. We get in this way a possible exponential increase in time. On the other, a similar argument applies to the negative part by using the change of variables $\widetilde{u} = -u$. The necessary bound for $\Delta_x \Phi$ has now the form

$$-\Delta_x \Phi(x,z) \le K_1 - K_2 z \qquad \forall z \le -M_0.$$

3.2.2 Other boundedness estimates

We now start the typical technique of the weak theories consisting in multiplying the equation by suitable *multipliers*, integrating in space or in space-time and then performing a number of integrations by parts and other calculus tricks. In our first example we take a function $p \in C^1(\mathbb{R})$ such that $p'(s) \ge 0$ for all $s \in \mathbb{R}$, and let j be the primitive of p with $j(0) = 0$. Then, if Φ does not depend explicitly on x we have

$$\frac{d}{dt}\int j(u)\,dx = \int p(u)\partial_t u\,dx = -\int p'(u)\Phi'(u)|\nabla u|^2\,dx + \int f\,p(u)\,dx, \tag{3.12}$$

with integrals in Ω. The reader should check that this calculation applies to the solutions of both Dirichlet and Neumann problems. Since the term containing $|\nabla u|^2$ is negative, integrating in time from 0 to $t > 0$ we have

$$\int j(u(t))\,dx \le \int j(u_0)\,dx + \iint_{Q_t} f\,p(u)\,dxdt. \tag{3.13}$$

If $f = 0$ this means that $J(u)(t) = \int j(u(x,t))\,dx$ is a monotone nonincreasing function of time. Even if $f \ne 0$ we can get estimates. For instance, if f is bounded and $p(u)/j(u)$ bounded as $u \to \infty$, we get boundedness of $\int j(u(t))\,dx$ for bounded times, see Problem 3.3. An interesting particular case happens when $j(s) = |s|^r$ for some $r > 1$. When $f = 0$ we get monotonicity of the L^r norm

$$\frac{d}{dt}\int |u(t)|^r\,dx \le 0.$$

In case Φ depends on x, we above argument does not work because of the derivatives of $\Phi(x,z)$ with respect to x. We refrain from entering into the modifications which are not immediate.

3.2.3 The stability estimate. L^1-contraction

This is a very important estimate which has played a key role in the PME and the GPME theory. It will allow us to develop existence, uniqueness and stability theory in the space $L^1(\Omega)$. Actually, the concept of L^1-contraction turns out to be a very powerful tool in the theory of nonlinear diffusion equations. There is no problem in admitting explicit dependence of Φ on x.

Proposition 3.5 (L^1-Contraction Principle) *Let u and $\widehat{u}$ be two smooth solutions, possibly of changing sign, and with initial data u_0, $\widehat{u}_0$ and forcing terms f, $\widehat{f}$ respectively. We have for every $t > \tau \geq 0$*

$$\int_\Omega (u(x,t) - \widehat{u}(x,t))_+ dx \leq \int_\Omega (u(x,\tau) - \widehat{u}(x,\tau))_+ dx + \int_\tau^t \int_\Omega (f - \widehat{f})_+ \, dxdt. \tag{3.14}$$

As a consequence,

$$\|u(t) - \widehat{u}(t)\|_1 \leq \|u_0 - \widehat{u}_0\|_1 + \int_0^t \|f(s) - \widehat{f}(s)\|_1 ds. \tag{3.15}$$

Proof of the Proposition. This result applies to the solutions of both Dirichlet and Neumann problems. It is so important that we give two quite different proofs.

First Proof. This is a standard proof in the literature. The technique goes as follows: Let $p \in C^1(\mathbb{R})$ be such that $0 \leq p \leq 1$, $p(s) = 0$ for $s \leq 0$, $p'(s) > 0$ for $s > 0$. Let $w = \Phi(x,u) - \Phi(x,\widehat{u})$ - -. Subtracting the equations satisfied by u and $\widehat{u}$, multiplying by $p(w)$ and integrating in Ω, and observing that $p(w) = 0$ on Σ which vanishes on Σ in the Dirichlet problem, we have for $t > 0$

$$\begin{aligned}\int (u - \widehat{u})_t \, p(w) \, dx &= \int \Delta w \, p(w) \, dx + \int (f - \widehat{f}) \, p(w) \, dx \\ &\leq - \int |\nabla w|^2 p'(w) \, dx + \int (f - \widehat{f})_+ \, dx.\end{aligned}$$

Note the first term in the right-hand side is nonpositive. Therefore, letting p converge to the sign function sign_0^+, and observing that $\dfrac{\partial}{\partial t}(u - \widehat{u})_+ = (u - \widehat{u})_t \,\mathrm{sign}_0^+(u - \widehat{u})$, cf. [261], and also observing that

$$\mathrm{sign}_0^+(u - \widehat{u}) = \mathrm{sign}_0^+(\Phi(x,u) - \Phi(x,\widehat{u})),$$

(a crucial fact based on the strict monotonicity of Φ), we get

$$\frac{d}{dt} \int (u - \widehat{u})_+ dx \leq \int (f - \widehat{f})_+ \, dx,$$

which implies (3.14) for $u, \widehat{u}$. To obtain (3.15), combine (3.14) applied first to u and $\widehat{u}$ and then to $\widehat{u}$ and u. The Neumann problem is completely analogous . $\square$

Second Proof of the Proposition. It contains two arguments, one for ordered solutions, another one for the maximum of two solutions.

Lemma 3.6 *Assume that u and $\widehat{u}$ are two smooth solutions such that $u_0 \le \widehat{u}_0$ and $f \le \widehat{f}$. Then, for every $t > 0$ we have $u(t) \le \widehat{u}(t)$ and*

$$\int (\widehat{u}(x,t) - u(x,t))\, dx \le \int (\widehat{u}_0(x) - u_0(x))\, dx + \iint_{Q_t} (\widehat{f} - f)\, dxdt. \tag{3.16}$$

This result is immediate. Note that in the case of the Neumann Problem we have equality, for the Dirichlet Problem only inequality. The second lemma is also elementary.

Lemma 3.7 *Assume that u and $\widehat{u}$ are two smooth solutions, and let U be the solution with initial data $U(x,0) = \max\{u_0, \widehat{u}_0\}$ and forcing term $F = \max\{f, \widehat{f}\}$. Then, for every $t > 0$, $U(t) \ge \max\{u(t), \widehat{u}(t)\}$.*

In order to prove the contraction principle using these lemmas, we observe that for every $t > 0$ we have

$$U(t) - \widehat{u}(t) \ge \max\{u(t), \widehat{u}(t)\} - \widehat{u}(t) = (u(t) - \widehat{u}(t))_+$$

while equality holds at $t = 0$. Hence, by Lemma 3.6 we conclude that

$$\begin{aligned} \int_\Omega (u(t) - \widehat{u}(t))_+ dx &\le \int_\Omega (U(t) - \widehat{u}(t))\, dx \le \int_\Omega (U(0) - \widehat{u}_0)\, dx \\ &+ \iint_{Q_t} (F - \widehat{f})\, dxdt = \int_\Omega (u_0 - \widehat{u}_0)_+ dx + \iint_{Q_t} (f - \widehat{f})_+\, dxdt. \end{aligned}$$

This ends the proof. □

Taking $\hat{u} = 0$ we get an interesting consequence.

Corollary 3.8 *For every smooth solution and every $t > 0$*

$$\int_\Omega (u(x,t))_+ dx \le \int_\Omega (u_0(x))_+ dx + \iint_{Q_t} f_+(x,t)\, dxdt. \tag{3.17}$$

3.2.4 The energy identity

We want to control the derivatives of the solution or some function thereof in order to apply compactness arguments. With respect to spatial gradients, the natural function to control turns out to be $w = \Phi(x,u)$. In order to bound ∇w we need to introduce the function Ψ which is the primitive of Φ w.r. to u with $\Psi(x,0) = 0$, i. e.,

$$\Psi(x,s) = \int_0^s \Phi(x,\sigma)\, d\sigma\,. \tag{3.18}$$

Note that for the PME we have $\Psi(x,s) = |s|^{m+1}/(m+1)$. Generally, $\Psi(x,u) \ge 0$ and moreover, $\Psi(x,u) \ge O(|u|)$ for all large $|u|$. On the other hand, $\Psi(x,u) \le |\Phi(x,u)u|$.

Since we are assuming that Φ is smooth and the solution is classical, we can multiply equation (3.3) by $\Phi(x,u)$ and integrate in Q_T to obtain

$$\iint_{Q_T} |\nabla\Phi(x,u)|^2 dxdt + \int_\Omega \Psi(x,u(x,T))\, dx = \int_\Omega \Psi(x,u_0(x))\, dx + \iint_{Q_T} f\Phi(x,u)\, dxdt, \tag{3.19}$$

where we have integrated by parts in space in the term $\iint \Delta\Phi(x,u)\Phi(x,u)\,dxdt$ and integrated in time the term $\iint \Phi(x,u)u_t\,dxdt = \iint \Psi(x,u)_t\,dxdt$. This important formula will be used in Chapter 5 to provide key estimates in the existence theory for weak solutions. It is interesting therefore to supply some physical meaning to its terms. Thus, the estimate leads us to consider the expression

$$E_u(t) = \int_\Omega \Psi(x,u(t))\,dx \tag{3.20}$$

as a natural energy for the evolution, and then

$$DE(u) = \int_0^T \int_\Omega |\nabla\Phi(x,u)|^2\,dxdt \tag{3.21}$$

is the dissipated energy, while $\iint f\Phi(x,u)\,dxdt$, represents the work of the external forces. Formula (3.19) is known as the *energy identity*. If $f=0$, it takes the simple form

$$\iint\limits_{Q_T} |\nabla\Phi(x,u)|^2 dxdt + \int\limits_\Omega \Psi(x,u(x,T))\,dx = \int\limits_\Omega \Psi(x,u_0(x))\,dx. \tag{3.22}$$

In case $f \neq 0$, we use Hölder's inequality to split the last term into

$$(1/4c)\iint f^2\,dxdt + c\iint \Phi(x,u)^2\,dxdt.$$

In the case of the homogeneous Dirichlet Problem, the Poincaré inequality allows to control the last term by the first term in the right-hand side. In this way we get

$$\frac{1}{2}\iint\limits_{Q_T} |\nabla\Phi(x,u)|^2 dxdt + \int\limits_\Omega \Psi(x,u(x,T))\,dx \le \int\limits_\Omega \Psi(x,u_0(x))\,dx + C\iint\limits_{Q_T} f^2\,dxdt, \tag{3.23}$$

where C depends on Ω trough the constant in the Poincaré inequality. Since the right-hand side is bounded for every fixed $T>0$, it follows that $\nabla\Phi(x,u)$ is bounded in $L^2(Q)$.

In the Neumann Problem we still apply Hölder's inequality to the last term of (3.19); we can then bound $\iint \Phi(x,u)^2\,dxdt$ in terms of $\iint |\nabla\Phi(x,u)|^2\,dxdt$ and some L^p norm of Φ (or even of u is Φ behaves like a power). For that purpose we can use the boundedness estimates of Subsection 3.2.2. It then follows that it follows that $\nabla\Phi(x,u)$ is bounded in $L^2(Q)$.

Local version. An interesting version of this estimate proceeds by multiplying also by η^2, where η is a smooth cutoff function, $0 \le \eta \le 1$. If the rest of the process is the same we get

$$\begin{aligned}&\iint\limits_{Q_T} |\nabla\Phi(x,u)|^2\eta^2\,dxdt + \int\limits_\Omega \Psi(x,u(x,T))\eta^2\,dx = \int\limits_\Omega \Psi(x,u_0(x))\eta^2\,dx\\ &+\iint\limits_{Q_T} f\,\Phi(x,u)\eta^2\,dxdt - 2\iint\limits_{Q_T} \Phi(x,u)(\nabla\eta\cdot\nabla\Phi(x,u))\eta\,dxdt.\end{aligned}$$

Therefore,

$$\begin{aligned}&\iint\limits_{Q_T} |\nabla\Phi(x,u)|^2\eta^2\,dxdt + 2\int\limits_\Omega \Psi(x,u(x,T))\eta^2\,dx \le 2\int\limits_\Omega \Psi(x,u_0(x))\eta^2\,dx\\ &+\iint\limits_{Q_T} f^2\eta^2\,dxdt + \iint\limits_{Q_T} \Phi^2(u)|(4\nabla\eta|^2+\eta^2)\,dxdt.\end{aligned} \tag{3.24}$$

This allows to obtain local bounds in L^2 for $|\nabla\Phi(x,u)|$ when local bounds are available for $\Phi(x,u)$ and f, as well as for $\Psi(x,u_0)$ in $L^1_{loc}(\mathbb{R}^d)$.

3.2.5 Estimate of a time derivative

The function whose time derivative we control is $z(x,t) = \mathcal{Z}(x,u(x,t))$, where the new function $\mathcal{Z}$ is defined in terms of Φ by

$$\mathcal{Z}(x,s) = \int_0^s (\Phi_u(x,s))^{1/2} ds. \tag{3.25}$$

Hence, $\partial_t z = \Phi'(u)^{1/2}\partial_t u$. Note that since $2(\Phi_u(x,s))^{1/2} \le 1 + \Phi_u(x,s)$, we have $|\mathcal{Z}| \le (1/2)|s + \Phi(x,s)|$, hence $|z| \le (1/2)|u + \Phi(x,u)|$. For the PME we have $z = c(m)\,|u|^{(m+1)/2}$.

In order to estimate $\partial_t z$, we multiply the equation by $\partial_t w$, with $w = \Phi(x,u)$, and integrate by parts in space to obtain (both for Dirichlet and Neumann problems)

$$\begin{aligned}\int w_t\,\partial_t u\,dx &= \int w_t\,\Delta w\,dx + \int f w_t\,dx = -\int \nabla w\cdot\nabla w_t\,dx + \int f w_t\,dx\\ &= -\frac{1}{2}\frac{d}{dt}\int |\nabla w|^2\,dx + \int f w_t\,dx\,,\end{aligned}$$

where we have taken into account the fact that $w_t = 0$ (or $\nabla w = 0$) on Σ. Moreover, this estimate has a simple form if $f = 0$ upon integration in time

$$\iint_{Q_T^\tau} \Phi_u(x,u)|\partial_t u|^2\,dxdt + \frac{1}{2}\int_\Omega |\nabla w(x,T)|^2\,dx = \frac{1}{2}\int_\Omega |\nabla w(x,\tau)|^2\,dx. \tag{3.26}$$

If $f \ne 0$, there are several alternatives. Thus, multiplication by $\zeta(t)$, where ζ is a smooth function with $\zeta(0) = \zeta(T) = 1$, and integration in time can be used to obtain

$$\iint_{Q_T} \zeta\,\Phi_u(x,u)|\partial_t u|^2\,dxdt = \iint_{Q_T} \left\{\frac{\zeta_t}{2}|\nabla\Phi(x,u)|^2 - (f\zeta)_t\Phi(x,u)\right\}\,dxdt. \tag{3.27}$$

The middle term is bounded in view of the energy estimate, and the last one is also if f_t is bounded. Therefore, we get an estimate on the first, which is a nonnegative expression, another kind of energy.

Without the extra condition on f, a typical approach consists of multiplication by t and integration in time from 0 to T to give

$$\begin{aligned}&\iint t\Phi_u(x,u)|\partial_t u|^2\,dxdt + \frac{T}{2}\int |\nabla w(x,T)|^2\,dx\\ &= \frac{1}{2}\iint |\nabla w|^2\,dxdt + \iint tf\Phi_u(x,u)\partial_t u\,dxdt\,.\end{aligned} \tag{3.28}$$

In order to obtain a uniform bound on the right-hand side, and since the term $\iint |\nabla w|^2\,dxdt$ is bounded, we only need to control the last term, that we may estimate as

$$\iint tf\Phi_u(x,u)\partial_t u\,dxdt \le \frac{1}{2}\iint t\Phi_u(x,u)f^2\,dxdt + \frac{1}{2}\iint t\Phi_u(x,u)|\partial_t u|^2\,dxdt.$$

The last term is absorbed by the first term of the previous expression. We get

(3.29)
$$\iint t\Phi_u(x,u)|\partial_t u|^2\,dxdt + T\int |\nabla w(x,T)|^2\,dx \le \iint |\nabla w|^2\,dxdt + \iint t\Phi_u(x,u)f^2\,dxdt\,.$$

and the last term is bounded for bounded f and bounded u. This estimate means that for every $T>\tau>0$ the integral $\int_\tau^T\int \Phi_u(x,u)|\partial_t u|^2\,dxdt = \iint z_t^2\,dxdt$ is bounded.

As a further alternative, we drop the multiplication by t and integrate in time from τ to T we get

(3.30)
$$\begin{aligned}&\iint_{Q^\tau}\Phi_u(x,u)|\partial_t u|^2\,dxdt + \frac{1}{2}\int_\Omega |\nabla w(x,T)|^2\,dx\\ &= \frac{1}{2}\int_\Omega |\nabla w(x,\tau)|^2\,dx + \iint_{Q_T^\tau} f\Phi_u(x,u)u_t\,dxdt\,.\end{aligned}$$

Local version. The same idea of multiplying also by η^2 allows to derive local versions of the time derivative estimates under some assumptions. We have

$$\int w_t\,u_t\eta^2\,dx = -\int \nabla w\cdot\nabla w_t\eta^2\,dx + \int f w_t\eta^2\,dx - 2\int \nabla w w_t\nabla\eta\eta\,dx,$$

so that

$$\int w_t\,u_t\eta^2\,dx + \frac{1}{2}\frac{d}{dt}\int |\nabla w|^2\eta^2\,dx = \int f w_t\eta^2\,dx - 2\int \nabla w w_t\nabla\eta\eta\,dx.$$

In order to proceed, assume that u is bounded, $|u|\le M$, and that $\Phi_u(s)\le c|s|$ for $|s|\le M$ (note: this happens for the PME). Then, $|w_t|\ge c|u_t|$ and, integrating in space in Ω and in time from $\tau>0$ to T we get

(3.31)
$$\begin{aligned}&\frac{1}{c}\iint (w_t)^2\eta^2\,dxdt + \int |\nabla w(T)|^2\eta^2\,dx\\ &\le \int |\nabla w(\tau)|^2\eta^2\,dx + C\iint f^2\eta^2\,dx + C\iint |\nabla w|^2|\nabla\eta|^2\,dxdt.\end{aligned}$$

The last term is supposed to be bounded by the first local estimate, (3.24).

3.2.6 The BV estimates

We consider the solutions of the Dirichlet Problem. We differentiate the equation with respect to t and put $v=\partial_t u$ to get the equation

(3.32)
$$\partial_t v = \Delta(\Phi_u(x,u)v).$$

We multiply by $p(\Phi_u(x,u)v)$ where p is an approximation of the sign function with the properties already mentioned above: $p\in C^1(\mathbb{R})$ be such that $0\le p\le 1$, $p(s)=0$ for $s\le 0$, $p'(s)>0$ for $s>0$. Then, putting $w=\Phi_u(x,u)v$ and integrating in Ω, we get

$$\int p(w)\partial_t v\,dx = \int \Delta(w)p(w)\,dx = -\int p'(w)|\nabla w|^2\,dx \le 0.$$

Therefore, $\int p(w)\partial_t v\,dx \le 0$. Now we let p tend to the sign function and observe that $\mathrm{sign}(w) = \mathrm{sign}(v)$ and $\partial_t v\,\mathrm{sign}(v) = |v|_t$ a. e. to conclude that

$$\frac{d}{dt}\int |\partial_t u|\,dx \le 0. \tag{3.33}$$

Actually, we can let p tend to the function sign^+ (resp. sign^-) to obtain the partial results

$$\frac{d}{dt}\int (\partial_t u)_+\,dx \le 0 \quad \left(\frac{d}{dt}\int (\partial_t u)_-\,dx \le 0\right). \tag{3.34}$$

Together, they imply (3.33).

Space estimates. When Φ does not depend explicitly on x, this trick can be repeated with any space derivative, putting $v = \partial u/\partial x_i$. We also get

$$\frac{d}{dt}\int |\partial_i u|\,dx \le 0. \tag{3.35}$$

and the corresponding estimates for the positive and negative signs.

With these estimates we can control u in the space $W^{1,1}(Q)$ if the initial data satisfy certain estimates. It is important to note that, when dealing with more general equations by approximation and passing to the limit in the approximations, the L^1 estimates obtained here may become estimates in the space of measures (since bounded sets in L^1 are not closed under weak convergence). Therefore, the estimates become estimates in the space of functions of bounded variation, $BV(Q)$.

3.3 Properties of the PME

The mathematical study of the PME and the GPME has a drawback common to all non-linear theories, the absence of good representation formulas for the solutions in terms of the data, think of the role of the Gaussian kernel in the heat equation and the Green function in the Laplace equation. On the other hand, the very simplicity of the PME implies a number of interesting properties of other types, like scaling invariance, conservation laws and dissipation laws, that play a big role as technical tools. This properties hold for the GPME as long as Φ does not depend on x.

3.3.1 Elementary invariance

We assume the restriction $\Phi(u)$ in this subsection.

Translations

The HE, the PME, the FDE and more generally, the GPME, are invariant under displacement of the coordinate axes, since their behaviour is homogeneous in space and time. To be specific, if $u(x,t)$ is a solution of the PME defined in a space-time domain Q, then, for every $h \in \mathbb{R}^d$ and $\tau \in \mathbb{R}$ the function

$$\widehat{u}(y,s) = u(y-h, s-\tau) \tag{3.36}$$

is also a solution, now defined in the translated domain

$$Q' = Q + (h, \tau) = \{(x + h, t + \tau) : (x, t) \in Q\}. \tag{3.37}$$

Space symmetries

The PME and its relatives mentioned above are invariant under the symmetry with respect to a coordinate space hyperplane. Thus, if $u(x,t)$ is a solution in a domain Q, so is

$$\widehat{u}(y,t) = u(-y_1, y_2, \cdots, y_d, t), \tag{3.38}$$

defined in Q' the domain of space-time that is symmetric of Q respect to the symmetry w.r.t. $x_1 = 0$. The same happens for any other space variable. By iteration, we may consider symmetries in a number of coordinates.

Space rotations

Indeed, we can perform any rigid motion in space since the Laplacian commutes with all the transformations in the orthogonal group. If A is the matrix of such a transformation and $u(x,t)$ is a solution in a domain Q, so is

$$\widehat{u}(y,t) = u(Ay, t), \tag{3.39}$$

defined in $Q' = \{(A^{-1}x, t) : (x,t) \in Q\}$. These arguments apply to any filtration equation $\partial_t u = \Delta\Phi(u)$.

Sign change

The filtration equation is invariant under the symmetry $u \mapsto -u$, if we change the nonlinearity Φ into $\widehat{\Phi}(s) = -\Phi(-s)$. To be precise, if a function u is a solution of Problem HDP with initial data u_0 and nonlinearity Φ, then $\widehat{u}(x,t) = -u(x,t)$ is a solution with data $\widehat{u}_0(x) = -u_0(x)$ and nonlinearity $\widehat{\Phi}(s) = -\Phi(-s)$.

3.3.2 Scaling

The HE, the PME and the FDE also share a powerful property inherited from the power-like form of the nonlinearity. This is the invariance under a transformation group of homotheties, usually known *the scaling group*. Indeed, whenever $u(x,t)$ is a classical solution of the equation $\partial_t u = \Delta(|u|^{m-1}u)$, the rescaled function

$$\tilde{u}(x,t) = K\ u(Lx, Tt) \tag{3.40}$$

is also a solution if the three real parameters $K, L, T > 0$ are tied by the relation

$$K^{m-1}L^2 = T. \tag{3.41}$$

We get in this way a two-parameter family of transformed solutions.

We can further restrict the family to a one-parameter family by imposing another condition. This happens for instance when the solutions are defined in the whole space and we impose the condition of preserving the L^p-norm of the data or the solution. In the first case, it reads

$$\int_{\mathbb{R}^d} |u|^p(x,0)\,dx = \int_{\mathbb{R}^d} K^p\,|u|^p(Lx,0)\,dx, \tag{3.42}$$

which implies the condition $K^p = L^n$. This allows to determine two parameters in terms of the third, and we can choose at will the free parameter but for some exceptional cases. See Problem 3.2.

As a first practical application, in Section 4.4 we will impose the conservation of the L^1-norm in time and we will find the Source Solutions, probably the most relevant example of the whole theory.

Later on we will introduce classes of generalized solutions (weak, strong, mild,...) and we will show that scaling applies to them.

3.3.3 Conservation and dissipation

These arguments apply to any equation $\partial_t u = \Delta\Phi(x,u)$ with $\Phi(x,0) = 0$ and Φ nondecreasing. The space domain is $\mathbb{R}^d$, $d \geq 1$.

Mass conservation

Given a classical solution $u(x,t)$ of the CP for the FE, we can multiply the equation by a cutoff function $\zeta(x)$ and integrate to obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d} u(x,t)\zeta(x)\,dx = \int \Delta\Phi(x,u)\zeta\,dx = \int \Phi(x,u)\Delta\zeta\,dx.$$

If u is integrable in space and goes to zero at infinity then we may let $\zeta \to 1$ and get in the limit

$$\int u(x,t)\,dx = \int u(x,0)\,dx. \tag{3.43}$$

This is called mass conservation.

The same argument holds for the IBVP posed in a bounded domain with zero Neumann data $\partial_n\Phi(x,u) = 0$, since $\zeta = 1$ is an admissible multiplier (i. e., it does not produce extra boundary terms when integrating by parts). In the case of Dirichlet data $u = 0$ on the boundary (and $u \geq 0$), we do not get conservation but decrease

$$\frac{d}{dt}\int u(x,t)\,dx \leq 0. \tag{3.44}$$

These formal computations will be carefully justified for the classes of weak solutions that make up the bulk of our theory.

Conservation of the first moment

Assume that the solution of the Cauchy Problem for the filtration equation is such that the integral $\int |x|\,u(x,t)\,dx$ is finite. Then, using that ζ vanishes for large $|x|$ and that $\Delta x_i = 0$, we formally get

$$\frac{d}{dt}\int_{\mathbb{R}^d} x_i u(x,t)\zeta(x)\,dx = \int \Delta\Phi(x,u)\,x_i\zeta\,dx = \int \Phi(x,u)x_i\Delta\zeta\,dx + 2\int \Phi(x,u)\partial_i\zeta\,dx.$$

Passing to the limit $\zeta = 1$ we get the result

$$\frac{d}{dt}\int x_i u(x,t)\,dx = 0. \tag{3.45}$$

This result is still true in one dimension for the problem posed in $\Omega = (0, \infty)$ with zero boundary data at $x = 0$ and suitable decay at infinity, since the boundary terms obtained when integrating by parts both vanish. But the result is not true for the Dirichlet or Neumann problems in general.

Conservation for the homogeneous Dirichlet problem

When such a problem is posed in a bounded smooth domain Ω we may use as a multiplier the solution ζ of the problem

$$\Delta\zeta = -1 \quad \text{in } \Omega, \qquad \zeta = 0 \quad \text{in } \partial\Omega, \tag{3.46}$$

to get the estimate

$$\frac{d}{dt}\int \zeta(x)\, u(x,t)\, dx = -\int \Phi(x, u(x,t))\, dx. \tag{3.47}$$

Dissipation and the L^p norms

The following formal computation works for the solutions of the Cauchy Problem that tend to zero at infinity, and also for the solutions of the homogeneous Dirichlet and Neumann problems:

$$\frac{d}{dt}\int |u|^p\, dx = p\int |u|^{p-2}u\Delta\Phi(u)\, dx = -p(p-1)\int \Phi_u(u)|u|^{p-2}|\nabla u|^2\, dx \le 0, \tag{3.48}$$

which shows that the L^p norm decays with time. Moreover, integration in time gives

$$\int |u(x,t)|^p\, dx + p(p-1)\int_0^t \Phi_u(u)|u|^{p-2}|\nabla u|^2\, dx = \int |u_0(x)|^p\, dx. \tag{3.49}$$

The second integral is therefore finite when $t \to \infty$ and measures the amount of dissipation of the L^p norm in time.

3.4 Alternative formulations of the PME and associated equations

There are some alternative formulations of the PME, where the lack of parabolicity is seen in a slightly different way. There are also some equations that can be derived from the PME $\partial_t u = \Delta(|u|^{m-1}u)$ through transformations.

3.4.1 Formulations

(A) One of alternative formulations consists in making the change of variables $|u|^{m-1}u = w$ (or simply, $w = u^m$ if $u \ge 0$). Formally, when $m > 1$ we arrive at the equation

$$\partial_t w = m\,|w|^n \Delta w, \tag{3.50}$$

with exponent $n = (m-1)/m \in (0,1)$. Now, we fall into the theory of nonlinear equations in non-divergence form, and it is immediately seen that the equation is parabolic for $|w| > 0$ and degenerates at $w = 0$.

(B) The second change is the *pressure* formulation introduced in Section 1.1.2, that is used for nonnegative solutions, mostly when $m > 1$, and uses the variable $v = cu^{m-1}$. If $c = m/(m-1)$ we get

$$\partial_t v = (m-1)\, v\Delta v + |\nabla v|^2, \tag{3.51}$$

which is again non-divergence. Cf. formulas (1.3), (1.4) and the physical interpretation of Section 2.1, see (2.7), hence the usual name of pressure variable that we will keep. It has an extra gradient term, but the nonlinearity of the right-hand side is homogeneous quadratic in u, and this is very useful for many calculations.

A technical detail: sometimes the equation is written as $\partial_t u = \Delta(u^m/m)$. Then we may define the pressure as $v = (1/(m-1))u^{m-1}$ and the equation for the pressure is still (3.51).

If the equation is the GPME the calculations are proposed as Problem 3.6.

3.4.2 Dual Equation

Suppose that we have a smooth solution the GPME defined in the whole space $\mathbb{R}^d$ for $0 < t < T$ and let us assume that $u(t) = u(\cdot, t) \in L^1(\mathbb{R}^d)$ for all t. Assume also that $d \geq 3$. We can take the Newtonian potential of u at every time $t > 0$ to get

$$v(t) = N(u(t)) = E_d \star u(t)$$

so that v is a uniquely defined function in the Marcinkiewicz space $M^{d/(d-2)}(\mathbb{R}^d)$ and $\Delta v(t) = -u(t)$ (see more on potentials in Section 22.6, on Marcinkiewicz spaces in Section 22.5). Then,

$$\partial_t v(x,t) = E_d \star (\Delta\Phi(x, u(t)) = -\Phi(x, u(t))\,.$$

In other words, v solves the nonlinear evolution equation $\partial_t v = -\Phi(x, u)$, i. e.,

$$\partial_t v = \widetilde{\Phi}(\Delta v) \tag{3.52}$$

where $\widetilde{\Phi}(x, u) = -\Phi(x, -u)$. This is called the *Dual Filtration Equation*. It is a formally parabolic, in principle degenerate, equation in non-divergence form. In the case of the PME the Dual Equation is just

$$\partial_t v = |\Delta v|^{m-1}\Delta v. \tag{3.53}$$

Some questions are better understood in terms of the dual equation satisfied by the potentials, see e.g. Chapter 13. The key point is that solutions of the dual equation have a better regularity since they are potentials. This is a fundamental calculation, obtained after differentiation and integration by parts

$$\frac{d}{dt}\int_{\mathbb{R}^d} |\nabla v|^2\, dx = -2\int_{\mathbb{R}^d} u\,\Phi(x,u)\, dx. \tag{3.54}$$

Another useful observation comes from integration of equation (3.52) in time. We get for $0 \leq s < t < T$,

$$v(x,t) + \int_s^t \Phi(u(x,\tau))\, d\tau = v(s,t).$$

so that bounds on v at time s and a condition like $v \geq 0$ imply bounds on the integrated function $U(x,t) = \int \Phi(u)\, dt$. This function is the function to be controlled in some theories.

In dimensions $d = 1, 2$ the potential approach in the whole space has some difficulties that we need not treat at this point. Let us give some details about the treatment in a bounded domain. Using the Green function with zero boundary conditions, $G = G_\Omega(x)$, as explained in Section 22.6 to define $\mathcal{G}f \in W_0^{1,1}(\Omega)$ by $v(x,t) = \mathcal{G}u(t)(x) := \int_{\mathbb{R}^d} u(y,t)G_\Omega(x,y)\,dy$. Then, $-\Delta v(t) = u(t)$ again, but now we have

$$\partial_t v = \mathcal{G}(\partial_t u) = \mathcal{G}\Delta\Phi(u).$$

Now, for general smooth functions it is not true that $\mathcal{G}\Delta = -I$ (because of the boundary conditions) and we can only conclude that

$$\partial_t v = h - \Phi(u), \tag{3.55}$$

where h is a harmonic function with boundary conditions $h|_{\partial\Omega} = \Phi(u)|_{\partial\Omega}$. Of course, if $\Phi(u)$ satisfies zero Dirichlet data we have $h = 0$ and the same type of dual equation holds.

We will return to potentials and dual equations in Chapter 13.

3.4.3 The p-Laplacian Equation in $d = 1$

When $d = 1$ and u is depends on two variables $(x,t) \in Q_T$, we can imagine that the filtration equation is just the condition that makes a differential form exact. A bit of reflection shows that such a form is $\omega = u\,dx + \Phi(x,u)_x dt$. Therefore, we can define a function of two variables

$$v(x,t) = v(x_0,t_0) + \int_\gamma (u\,dx + \Phi(x,u)_x dt) \tag{3.56}$$

along any piecewise continuous path that joins the fixed point (x_0,t_0) to any $(x,t) \in Q_T$. The integral does not depend on the path γ. We easily see that $v_x = u$ and that v satisfies the PDE

$$\partial_t v = (\Phi(x,v_x))_x. \tag{3.57}$$

If $\Phi(u) = |u|^{m-1}u$, then the equation for v is the standard p-Laplacian equation

$$\partial_t v = (|v_x|^{p-2}v_x)_x, \quad p = m+1. \tag{3.58}$$

This calculation performed for classical solutions has to be justified in the theory when dealing with generalized solutions.

Unfortunately, this relationship between the equations (PME and PLE) does not extend to higher space dimensions.

Notes

§3.1. References [357], [239] and [371] can be consulted as the need arises.

§**3.2.** The proof of the Energy bound for $\nabla\Phi(u)$ and z_t is adapted from Bénilan and Crandall [88].

§**3.3.** Scaling arguments are well-known and very successful in the applied literature, see Barenblatt's book [63] and our Chapter 16. In Chapter 17 we will use scaling and special solutions that are scaling-invariant, in combination with symmetrization and mass comparison, as basic tools in obtaining basic estimates for the solutions.

§**3.4.** The dual equation was used in [202] in the study of extinction in fast diffusion, in [191] in the study of uniqueness of general weak solutions, in [103] in the study of selfsimilarity and asymptotics.

The p-Laplacian equation has a very extensive literature, cf. the monograph [209].

Problems

Problem 3.1 Signed PME. Make the change of variables $w = |u|^{r-1}u$ for some $r > 0$. Obtain the equation for w

$$w_t = m\,|w|^n \Delta w + c|w|^{n-2} w |\nabla w|^2, \tag{3.59}$$

with $n = (m-1)/r$ and $c = m(m-r)/r$.

Problem 3.2 Scaling transformation. (i) Prove that the scaling transformation that preserves the PME and the L^p-norm of the data can be solved for K and L in terms of T unless $n(1-m) = 2p$ (which implies $m < 1$ since $p \geq 1$). Find the explicit expressions

$$K = T^{n/(n(m-1)+2p)}, \quad L = T^{p/(n(m-1)+2p)}.$$

(ii) Find out the admissible scaling if $n(1-m) = 2p$.

(iii) Explore the possibilities of taking K, L and T negative. Derive for $L = -1$ an invariance under symmetry. What happens when $K = -1$? Is $T = -1$ admissible?

Problem 3.3 Consider the homogeneous GPME. (i) Prove the boundedness of $\int j(u(t))\,dx$ in formula (3.13) for finite times when f is bounded and $|p(u)| \leq C_1 j(u) + C_2$.

(ii) Prove that for every $r \geq 1$

$$\int |u(t)|^r\,dx \leq \int |u_0|^r\,dx + r \iint f|u|^{r-1}\,dxdt.$$

Derive from this that whenever $\int_T^\infty \|f(t)\|_r\,dt$ is bounded then $\|u(t)\|_r$ is uniformly bounded for $0 \leq t < \infty$.

(iii) Put $f = 0$ and obtain the estimate

$$\int |u(t)|^r\,dx \leq \int |u_0|^r\,dx.$$

Problem 3.4 There is another useful conservation law for the GPME when the problem is posed in an exterior domain $\Omega = \mathbb{R}^d - K$, where K is a compact set with smooth boundary. Prove that in dimensions $d \geq 3$ there exist a solution $\zeta > 0$ of

$$\Delta\zeta = 0 \quad \text{in } \Omega, \quad \zeta = 0 \quad \text{in } \partial\Omega, \tag{3.60}$$

with the additional condition $\zeta \to 1$ as $|x| \to \infty$. Show that if u is a classical solution of the exterior problem with $u = 0$ on $\partial\Omega$, and u decays at infinity so that $u(\cdot, t)$ is integrable in space, then

$$\frac{d}{dt} \int \zeta(x)\, u(x,t)\, dx = 0.$$

This law is fundamental in the study of large time asymptotics done in [124].

Problem 3.5 Prove the following local energy estimate as a variant of estimate (3.24). We take $f = 0$ for simplicity. For every $\eta \in C_c^2(Q_T)$, we have

$$\iint_{Q_T} |\nabla\Phi(u)|^2 \eta\, dxdt = \frac{1}{2} \iint_{Q_T} (\Phi(u))^2 \Delta\eta\, dxdt + \iint_{Q_T} \Psi(x,u)\eta_t.$$

This means that there is a bound for $\nabla\Phi(u)$ in $L^2_{loc}(Q_T)$ in terms of the local norms of $\Phi(u)$ in $L^2_{loc}(Q_T)$ and $\Psi(x,u)$ in $L^1_{loc}(Q_T)$.

Problem 3.6 Take the GPME $\partial_t u = \Delta\Phi(u)$ and take as new variable $w = \Phi(u)$. Putting $\beta(\cdot) = \Phi(\cdot)^{-1}$, the equation becomes

$$\partial_t \beta(w) = \Delta w. \tag{3.61}$$

This generalizes (3.50). If β is differentiable the equation becomes

$$\beta'(w)\partial_t w = \Delta w. \tag{3.62}$$

which is convenient in the theory of fast diffusion. Write down the calculation for the so-called superslow diffusion equation where $\Phi(u) = e^{-1/u}$.

Problem 3.7 (i) In order to generalize the pressure change to the case $\partial_t u = \Delta\Phi(u)$, we write

$$v(x,t) = P(u(x,t)), \quad P(u) = \int_a^u \frac{\Phi'(u)}{u}\, du. \tag{3.63}$$

Writing $\nabla\Phi(u) = u\nabla v$, $\partial_t v = \Phi'(u)\partial_t u/u$, the equation for v is then

$$\partial_t v = \sigma(v)\Delta v + |\Delta v|^2. \tag{3.64}$$

where $\sigma(v) = \Phi'(u)$. Work out the details and compare with (3.51).

(ii) Assume that Φ is C^2 for $u > 0$. Prove that σ is C^1 at $v = 0$ if and only if there exists

$$\sigma'(0) = \lim_{u\to 0} \frac{u\Phi''(u)}{\Phi'(u)}.$$

(iii) Calculate the pressure in the superslow diffusion case $\Phi(u) = e^{-1/u}$, $u \geq 0$.

See further details in [125].

Problem 3.8 Derive carefully the associated equations (3.53), (3.55), and (3.58).

Problem 3.9 * Try to derive the a priori estimates of Section 3.2 for an inhomogeneous equation of the form

$$\partial_t u = \sum_{i=1}^{d} \partial_{x_i} \left(a_{ij}(x,t) \partial_{x_j} \Phi(u) \right). \tag{3.65}$$

where (a_{ij}) is a symmetric positive-definite matrix defending smoothly on x and t.

Chapter 4

Basic Examples

In this chapter we present five interesting types of solutions that will play a role in the development of the theory: separate-variables solutions, travelling waves, source-type solutions, blow-up solutions and constant-height solutions. Other solutions, like dipoles and general fronts, serve to complete the picture.

We will use the presentation to introduce and use important concepts for the sequel, like scaling, limit solutions, finite propagation, free boundaries, existence under optimal conditions, blow-up, limited regularity, and initial traces. These questions will receive a full rigorous treatment later on.

Solutions with changing sign will also be considered; therefore the equation by default is the signed PME, $u_t = \Delta(|u|^{m-1}u)$. The main emphasis should be laid however on nonnegative solutions of the standard PME. The last two sections can be skipped in a first reading.

4.1 Some very simple solutions

The PME admits a number of explicit solutions that play an role in developing the theory. Without any doubt, the simplest solutions are the ones that do not change in time, called *stationary solutions*. They satisfy the condition $u_t = 0$, hence u depends only on the space variable, $u = u(x)$, and $w = u^m$ has to satisfy the equation

$$\Delta w = 0. \tag{4.1}$$

Therefore, any harmonic function $w(x)$ provides a stationary solution of the PME putting $u(x,t) = w(x)^{1/m}$ if $w \geq 0$, $u(x,t) = |w(x)|^{1/m}\text{sign}(w)$ for signed solutions. If in particular we ask for solutions defined and nonnegative in the whole space, then such solutions must be constant. We call such solutions *trivial solutions*. They are the simplest solutions.

In one dimension the rest of the stationary solutions are linear functions, $u^m = Ax + B$, $A \neq 0$. If we insist on nonnegativity, then we must restrict the definition to the hyperspace where $u > 0$. Thus, the solutions defined for $x > 0$ and vanishing at the lateral boundary $x = 0$ are given by the formula $u = Cx^{1/m}$, $C \in \mathbb{R}$. Note that they are not C^1 functions on the boundary! The restriction $x > 0$ is not necessary if signed solutions are admitted but then we have to worry about the concept of solution at the transition point $x = 0$. This will be the task of the next chapter. The same applies to stationary "solutions" in two dimensions like

$$w(x, y, t) = x^2 - y^2 + c, \quad c \in \mathbb{R}$$

in $d = 2$ with $w = |u|^{m-1}u$. Here, the problematic locus is $x^2 = y^2 - c$.

4.2 Separation of variables

For our first model of non-trivial special solution, we follow the typical procedure of the Fourier approach for the linear Heat Equation (which is formally the case $m = 1$ of the PME), and we make the ansatz

$$u(x, t) = T(t)\, F(x). \tag{4.2}$$

This leads to separate equations for $T(t)$, the *time factor*, and $F(x)$, called the *space profile*:

$$T'(t) = -\lambda T(t)^m, \qquad \Delta F^m(x) + \lambda F(x) = 0. \tag{4.3}$$

The constant λ is in principle arbitrary, but it serves to couple both equations. When it is zero, the solutions are stationary in time, a case already discussed. Assuming in the sequel that $\lambda \neq 0$, the first equation is easy to solve and gives

$$T(t) = (C + (m-1)\lambda t)^{-1/(m-1)}.$$

Therefore, we have reduced finding these special solutions of the PME to solving the Nonlinear Elliptic equation for F, right-hand formula of (4.3). This is a nonlinear version of the eigenvalue problem to be solved in the Fourier analysis of the Heat Equation. The usual process is also the same: a domain Ω is chosen and boundary conditions are assigned on $\partial\Omega$; the boundary problem is then solved. But the results are remarkably different. As usual, the analysis depends on the sign of λ.

(i) Positive λ. Nonlinear Eigenvalue problem

The first curious feature of the nonlinear elliptic problem is that the general value $\lambda > 0$ can be reduced to $\lambda = 1$ by changing appropriately the value of F. In fact, if $F_1(x)$ is a solution of the equation with $\lambda = 1$,

$$\Delta(|F_1|^{m-1}F_1) + F_1 = 0. \tag{4.4}$$

the transformation

$$F(x) = \mu F_1(x), \quad \mu = \lambda^{1/(m-1)}, \tag{4.5}$$

is a solution of the original equation with $\lambda > 0$, $\lambda \neq 1$, $\Delta(|F|^{m-1}F) + \lambda F = 0$, and conversely. Equation (4.5) is the simplest case of what we call a scaling transformation[1].

[1] We have seen in Subsection 3.3.2 transformations in which the space and time variables are also changed by homothety.

It is convenient to further change the variable F into $G = |F|^{m-1}F$ and write the Nonlinear Elliptic Equation as

$$\Delta G(x) + |G(x)|^{p-1}G(x) = 0, \tag{4.6}$$

where $p = 1/m \in (0, 1)$. When this equation is posed in a bounded domain with regular boundary and we take zero boundary conditions, there exists precisely one positive solution of the problem, and not many as in the linear case. In case Ω has a special shape, like a ball or a cube, the solution is easy to find by standard ODE methods, see Problem 4.1 below. In the case of a general bounded domain, the existence and uniqueness of a positive solution of (4.6) can be obtained by variational methods, as in [511]. We will find an indirect proof in the Chapter 5.

Summing up, the problem is quite different from the linear eigenvalue problem: there is existence and uniqueness of a positive solution for all $\lambda > 0$. Let us note that the solution is continuous (actually, Hölder continuous) up to the boundary of Ω, and C^∞ smooth inside Ω.

Granted the existence of that solution, $F(x; \lambda, \Omega)$, the semi-explicit solution that we get for the PME has the form $u(x, t) = (C + (m-1)t)^{-1/(m-1)}F(x)$, that we can also write as

$$u(x, t) = ((m-1)(t - t_0))^{-1/(m-1)}F(x), \tag{4.7}$$

where t_0 is arbitrary. This form is called a separated variables solution. It is clear that the formula produces a classical solution of the PME in the space-time domain $\Omega \times (t_0, \infty)$ and takes on zero boundary data. There is no essential restriction in assuming that $t_0 = 0$ (since the difference is a time translation), but the family of solutions depends on Ω through the profile F in a nontrivial way. Let us point out a quite strange feature: The initial data at $t = t_0$ is $u(x, t_0) \equiv +\infty$, something unheard of in the linear case.

Note: the method does not produce any classical solution defined in the whole space $\mathbb{R}^d$ for the PME.

(ii) Negative $\lambda = -l < 0$. Blow-up.

We get solutions with time factor

$$T(t) = (C - (m-1)l\,t)^{-1/(m-1)} = ((m-1)l\,(t_0 - t))^{-1/(m-1)},$$

which blows up in finite time. We can again reduce us to the case $l = 1$ and solve the elliptic equation for the profile

$$\Delta F^m(x) = F(x) \tag{4.8}$$

after a scaling. We cannot solve that problem in the same setting as before and obtain nontrivial solutions. But it is easy to find radially symmetric solutions defined in the whole space by solving the corresponding ODE. A particular solution is well-known, and will be studied below in Section 4.5, devoted to presenting blow-up solutions, see formula (4.44).

Remark. We have concentrated on the values $m > 1$. Separable solutions can be constructed for $m < 1$, but they take the form $U(x, t) = (T - t)^{1/(1-m)}F(x)$. F still solves an elliptic equation, see Problem 5.14.

4.3 Planar travelling waves

In the second model, we look for solutions of the form

$$u = f(\eta), \quad \eta = x_1 - ct \in \mathbb{R}. \tag{4.9}$$

This type of solution represents a wave that moves in time along an axis (here, x_1), without changing its shape. The form does not depend on the variables $x_2, \ldots, x_d$, hence the name planar (that is usually omitted so that they are simply known as travelling waves, TWs for short). The parameter c is the *wave speed.* We may assume that $c \neq 0$, since for $c = 0$ we find again the stationary solutions, and the case $c < 0$ can be reduced to $c > 0$ by a reflection (changing $u(x,t)$ into $u(-x,t)$ we find another solution of the equation moving in opposite direction). Note that a wave with $c > 0$ travels in the positive direction of the axis. Note finally that TWs are one-dimensional in its space dependence.

We have taken as wave direction a coordinate axis, but the invariance under rotations explained in Section 3.3 above allows us to find a wave that travels along any straight direction $\mathbf{n}$ of space $\mathbb{R}^d$. The formula would then be (4.9) with $\eta = \mathbf{x} \cdot \mathbf{n} - ct$.

Taking thus $c > 0$ fixed, and substituting (4.9) into $u_t = \Delta u^m$ we arrive at the ODE

$$(f^m)'' + cf' = 0, \tag{4.10}$$

where prime indicates derivative respect to η. Integrating once we get

$$(f^m)' + cf = K, \tag{4.11}$$

with arbitrary integration constant $K \in \mathbb{R}$. In order to choose this constant we think of the situation where the wave advances against an 'empty region', i. e., we want $f(\eta) = f'(\eta) = 0$ for all $\eta \gg 0$. This condition leads to the conclusion that $K = 0$, so that (4.11) becomes

$$mf^{m-2}f' + c = 0, \tag{4.12}$$

which is easily integrated to give

$$\frac{m}{m-1} f^{m-1} = -c\,\eta + K_1 = c(\eta_0 - \eta). \tag{4.13}$$

This conclusion is very neat, since, according to our definition of the mathematical pressure, cf. formulas (1.4), (2.7), it means that the pressure is a linear function

$$v(x,t) = K_1 - c(x - ct) = c(x_0 + ct - x). \tag{4.14}$$

This is a perfectly valid classical solution of the PME in the expanding region $\{(x,t) : x < x_0 + ct\}$, where u is positive.

Analytical problems and ways of solution

However, this conclusion is not satisfactory, since formula (4.14) fails to provide a solution of the PME in the whole space, which is the natural framework for a TW. The problem is serious; actually, v becomes negative for $x > x_0 + ct$, a situation that goes squarely against the physics of many problems. One such problem was the heat transfer that the Moscow group was trying to solve around the 1950. The way out of this dilemma is a crucial moment in the history of the PME, and a powerful argument in favor of the influence of the applications on the theory. It consists of two parallel moves: *drastically modifying formula* (4.14), and *abandoning the concept of classical solutions*. Both are quite natural today, but we are talking about 1950.

Indeed, the solution of the difficulty is quite natural and relies on the *strategy of the limit problem:* to solve an approximate problem for which the difficulty is not present, to pass to the limit and to examine the obtained result. Specifically, we take as boundary condition for equation (4.11)

$$f(\infty) = \varepsilon, \quad f'(\infty) = 0, \tag{4.15}$$

so that the problem is non-degenerate. We obtain for K the value $K = \varepsilon\, c > 0$. We then write (4.11) as

$$f' = -c\frac{f-\varepsilon}{mf^{m-1}} \tag{4.16}$$

which is an ODE in separate-variable form, immediate to integrate, at least graphically and implicitly. We are interested in solutions $f \geq \varepsilon$.

Proposition 4.1 *For every $\varepsilon \in (0,1)$ there exists a unique solution $f_\varepsilon(\eta)$ of equation* (4.16) *satisfying the initial condition $f_\varepsilon(0) = 1$. It has end condition $f_\varepsilon(\infty) = \varepsilon$. Moreover $f_\varepsilon : \mathbb{R} \to (\varepsilon, \infty)$ is a monotone decreasing and C^∞ function such that $f_\varepsilon(-\infty) = \infty$. In the limit $\varepsilon \to 0$ we have*

$$\lim_{\varepsilon\to 0} \frac{m}{m-1} f_e^{m-1}(\eta) = c(\eta_0 - \eta)_+ \tag{4.17}$$

with $\eta_0 = m/c(m-1)$. The limit is uniform in sets of the form $[a, \infty)$.

We have taken the normalization value $f_\varepsilon(0) = 1$ without loss of generality. Since the equation is autonomous, we can get a one-parameter family of C^∞ solutions $f > \varepsilon$ with $f \to \varepsilon$ as $\eta \to \infty$ by horizontal translation of the one obtained in the proposition. Since the proof of the proposition is based on a simple phase plane analysis, and we assume that the reader is familiar with the elements of that technique, we assign the task as Problem 4.2. We are also asking him/her to perform the graphical integration to get a visual evidence. Thanking the reader in advance, we will devote the spared space to discuss the meaning of the result.

4.3.1 Limit solutions

Inserting formula (4.17) into the form (4.9) and passing to the pressure, we get the formula

$$v(x,t) = c(x_0 + ct - x)_+. \tag{4.18}$$

Since it is obtained as a limit of perfectly safe classical solutions, a quite strong intuition developed in the applied sciences tells us that this qualifies as a valid physical solution in some sense to be made precise. In the meantime, we will call it a *limit solution*.

The introduction of limit solutions solves some problems and poses a number of other problems. Thus, we now have a concept of travelling wave to describe the movement of

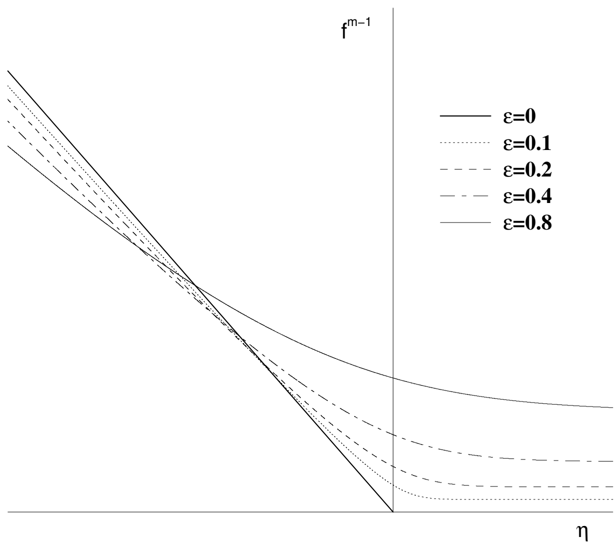

Figure 4.1: Travelling waves for $\varepsilon > 0$ and their limit

a mass of gas (or liquid, or heat) bordering on its right-hand side with empty space, a situation of enormous applied importance.

On the other hand, let us examine the problems. This limit solution is not a classical solution of the PME. Closer inspection shows that it is a broken version of the formula obtained by purely algebraic computations, (4.14), and it has a problem of differentiability at the line $x = x_0 + ct$, precisely, at the points where the equation passes from the classical state to the degenerate state. This set is called the *free boundary*, an important object of study as we have said.

Another problem of the concept of limit solution is the possibility of obtaining different solutions for the same problem, depending on the type of approximation used. We will introduce in subsequent chapters different concepts of *generalized solutions* that allow to (i) solve the problem in a unique way, (ii) include the classical solutions if they exist, as well as their limits, and (iii) show existence in cases where there is no classical solution. Such an effort will prove that the limit solution we have just accepted is indeed a ‘good solution’.

4.3.2 Finite propagation and Darcy’s law

Travelling waves consist of space functions (called profiles) that propagate with constant speed without changing shape. They are also called *constant-shape fronts*. Such fronts exist also for the heat equation, but they have different form and properties. Let us examine more closely the differences between the PME and the HE. For the latter, the TWs have the form

$$u(x,t) = Ce^{c(ct-x)}.$$

On one hand, they are classical solutions. On the other hand, they are always positive, and reach the level $u = 0$ at $x = \infty$ after developing an infinitely long *exponential tail*. It is precisely this property of the heat equation, namely that nonnegative solutions are actually positive everywhere, what is sometimes mentioned as an un-physical property of an otherwise quite effective model.

We have constructed an explicit (limit) solution of the PME that has a sharp and finite front separating the regions $\{u > 0\}$ and $\{u = 0\}$, and this front propagates in time with constant speed. This is the first appearance of the property of finite speed of propagation, usually called finite propagation, a fundamental property of the PME, that we will discuss at length later on.

Darcy's law on the Free Boundary. In trying to understand the lack of regularity of the TW at the free boundary, it is quite useful to go back to the modelling of a gas in a porous medium, Section 2.1. Putting $x_0 = 0$ without loss of generality, the pressure in the gas is given by

$$v(x,t) = c(ct - x)_+.$$

According to Darcy's law, the speed is $\mathbf{V} = -\nabla v = c\,\mathbf{e_1}$. In physical terms, every particle of the whole mass of gas moves with the same speed and the pressure must grow linearly near the free boundary to account for Darcy's law. On the other hand, $v = 0$ on the empty region according to our mathematical definition (2.7).

Summing up, we have concluded that Darcy's law forces the gradient of v to jump at the free boundary points $x = ct$.

4.4 Source-type solutions. Selfsimilarity

In the next example we look for the solution corresponding to PME flow starting from a finite mass concentrated at a single point of space, say, $x = 0$. A classical problem in the thermal propagation theory is to describe the evolution of a heat distribution after a point source release. In mathematical terms, we want to find a solution of the HE with initial data

$$u(x,0) = M\,\delta(x), \tag{4.19}$$

where $M > 0$ and δ is Dirac's delta function. This is called in engineering a point source, hence the name *source solution* widely used in the Russian literature. Such type of solution is well-known in the case of the Heat Equation (i. e., for $m = 1$) and is called the *fundamental solution*, with formula

$$E(x,t) = M\,(4\pi t)^{-n/2} \exp\left(-x^2/4t\right). \tag{4.20}$$

The *Gaussian kernel*, as it is also known, plays a fundamental role in developing the PDE theory of the Heat Equation, and is also of paramount importance in the probabilistic approach to diffusion (central limit theorems).

This motivates the interest in the similar question about existence of a source solution for our nonlinear diffusion equation, PME. Indeed, as we have indicated in Section 1.1, the source solution exists for $m > 1$ and, fortunately enough, it is explicitly given by a formula that we will now write as

$$\mathcal{U}(x,t;M) = t^{-\alpha} F(x\,t^{-\alpha/d}), \quad F(\xi) = (C - \kappa\,\xi^2)_+^{\frac{1}{m-1}}, \tag{4.21}$$

where

$$\alpha = \frac{d}{d(m-1)+2}, \quad \kappa = \frac{(m-1)\alpha}{2md}. \tag{4.22}$$

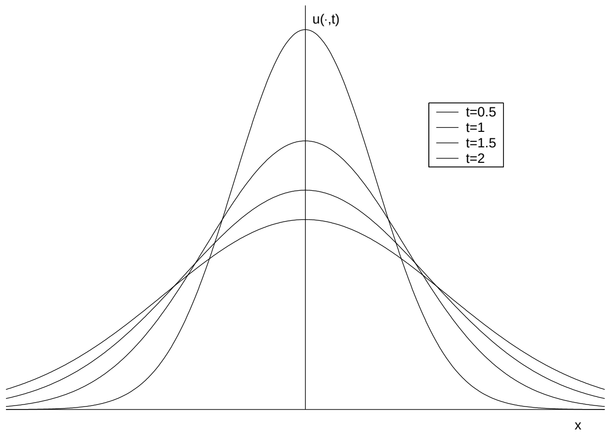

Figure 4.2: Fundamental solution of the heat equation

We ask the reader to check that indeed it takes on a Dirac delta as initial trace, i. e., that

$$\lim_{t\to 0}\mathcal{U}(x,t) = M\,\delta_0(x), \tag{4.23}$$

in the sense of measures. The free parameter $C > 0$ in formula (4.21) is in principle arbitrary; it can be uniquely determined by the condition of total mass, $\int U dx = M$, which gives the following relation between the 'mass' M and C:

$$M = a(m,d)\,C^{\gamma}, \quad \gamma = \frac{d}{2(m-1)\alpha}. \tag{4.24}$$

Note a and γ are functions of only m and d; the exact calculation of a is not needed at this point, cf. Section 17.5. We shall use quite often in the sequel the name ZKB solutions for the source solutions as explained in the Introduction; they are also quite widely known as Barenblatt solutions or Barenblatt-Pattle solutions. Notation: using the mass as parameter we denote it by $\mathcal{U}(x,t;M)$, or even $U_m(x,t;M)$ if the dependence on m is important.

According to the previous calculations, putting

$$C = \kappa\xi_0^2\,M^{2(m-1)\alpha/d},$$

then formula (4.21) is transformed into

$$\mathcal{U}_m(x,t;M) = M\,\mathcal{U}_m(x,M^{m-1}t;1) = \frac{M^{2\alpha/d}}{t^{\alpha}}F_{m,1}\left(\frac{x}{(M^{m-1}t)^{\alpha/d}}\right), \tag{4.25}$$

where $F_{m,1} = (\kappa(\xi_0^2-\xi^2))_+^{1/(m-1)}$ is the profile with exponent m and mass 1.

It is maybe a good idea to write the ZKB solution in terms of the pressure variable $v = u^{m-1}m/(m-1)$ and then we get the formula

$$\boxed{V_m(x,t;M) = \frac{(C\,t^{2\alpha/n} - b\,x^2)_+}{t}} \tag{4.26}$$

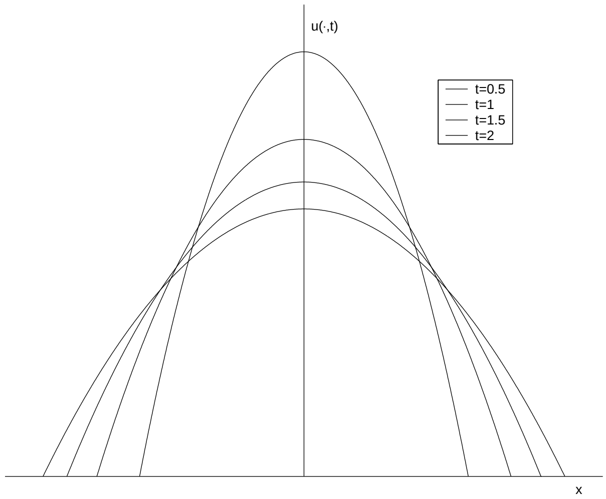

Figure 4.3: The ZKB solution of the PME

with $b = \alpha/2d$ and $C > 0$ is a free parameter. We see that in terms of the pressure, the ZKB has a simpler expression, a parabolic shape for all $m > 1$. This observation has strongly influenced a number of developments of the theory. About the shape see Problem 4.6.

We can also pass to the limit $m \to 1$ (with a fixed choice of the mass M) and obtain the fundamental solution of the heat equation,

$$\lim_{m\to 1} \mathcal{U}_m(x, t; M) = M\, E(x, t). \tag{4.27}$$

This is a relatively easy calculus result. We ask the reader to try his/her calculus abilities.

4.4.1 Comparison with Gaussian profiles. Anomalous diffusion

We begin by pointing out that the ZKB solutions start from a point source and spread in space like $O(t^\beta)$. Since $\beta < 1/2$ for $m > 1$, this is a slower rate (i.e., with a smaller power) than the average spread rate $O(t^{1/2})$ of the Heat Equation; such a spread rate was indicated by Einstein in his famous 1905 paper [538] as the characteristic average spread rate of Brownian motion. Observe furthermore than $\beta \to 0$ as $m \to \infty$. Such a deviation is not just a particular case, since we will show in Chapter 18 that the ZKB solutions represent the standard asymptotic behaviour of finite mass solutions both in size and spread rate (see Chapter 18); we conclude that the PME is an example of *anomalous diffusion* in the sense described for instance in [537].

Then, we notice the difference in the form of propagation. While the HE solution travels immediately to the whole space, the PME solution is supported in the region $|x| \le r(t)$ behind the free boundary

$$r(t) = (C/\kappa)^{1/2} t^\beta.$$

Use of the Maximum Principle will allow us to conclude in later chapters that all weak solutions $u \ge 0$ of the PME with bounded and compactly supported initial data are located for positive and bounded times $t > 0$ in an expanding but still bounded region.

Secondly, the fundamental solution of the HE is a C^∞ function, while the ZKB solutions are only Hölder continuous. Actually, the regularity depends on m but the regularity of the pressure does not. It is Lipschitz continuous and not C^1, just as in the TWs.

Besides, it is not difficult to check that Darcy's law holds on the free boundary in the sense that

$$\lim_{|x|\to r(t)-} \nabla v = -r'(t), \tag{4.28}$$

where the limit is taken as $|x| \to r(t)$ but only for $|x| < r(t)$, i. e., only in the gas region. This formula equates the speed of the particles with the speed of the moving surface.

Let us finally say that the Fundamental Solution of the HE allows to derive the whole theory of the equation using the representation formulas, a most powerful tool of linear analysis. This is not to be expected in the case of the PME where no valid equivalent of such formulas has been found. However, skillful use of the properties of the ZKB solution have allowed to obtain enormous progress in the theory of the PME. Therefore, carefully inspecting the properties of the ZKB is a most fruitful investment and we shall do it quite often. See in this direction Problem 4.5, part (vi). However, the development of the theory of the PME owes much to the other solutions mentioned in this chapter.

4.4.2 Selfsimilarity. Derivation of the ZKB solution

The most natural way of deriving formula (4.21) is using selfsimilarity, a most important concept in the theory that follows. It means that there is a scaling of the variables after which the ZKB become stationary solutions. Precisely, there holds that

$$u' = f(x'), \qquad \text{with} \qquad u' = ut^{\alpha}, \quad x' = xt^{-\beta}. \tag{4.29}$$

The selfsimilar form is then

$$\mathcal{U}(x,t) = t^{-\alpha} f(\eta), \quad \eta = xt^{-\beta}. \tag{4.30}$$

The exponents α and β are called *similarity exponents*, and function f is the *selfsimilar profile*. In particular, α is the density contraction rate, while β is the space expansion rate. We have to determine exponents and profile so that the resulting function $\mathcal{U}$ is a solution and has suitable additional data.

Selfsimilarity is a capital concept in Mechanics and, generally speaking in the applied sciences. Note that the fundamental solution of the heat equation (4.20) is selfsimilar with exponents $\alpha = d/2$, $\beta = 1/2$, and a Gaussian function as profile.

• In the case of the PME, we try the selfsimilar ansatz (4.30) in the PME. Since

$$\mathcal{U}_t = -\alpha t^{-\alpha-1} f(\eta) + t^{-\alpha}\nabla f(\eta)\cdot x t^{-\beta-1}(-\beta) = -t^{-\alpha-1}(\alpha f(\eta) + \beta\nabla f(\eta)\cdot\eta),$$

and

$$\Delta(\mathcal{U}^m) = t^{-\alpha m}\Delta_x(f^m(xt^{-\beta})) = t^{-\alpha m-2\beta}\Delta_\eta(f^m)(\eta),$$

equation $\mathcal{U}_t = \Delta\mathcal{U}^m$ becomes

$$t^{-\alpha-1}(-\alpha f(\eta) - \beta\eta\cdot\nabla f(\eta)) = t^{-\alpha m-2\beta}\Delta f^m(\eta). \tag{4.31}$$

• We now eliminate the time dependence (this is a kind of separated variables argument). This implies a first relation between the exponents:

$$\alpha(m-1)+2\beta=1, \tag{4.32}$$

and allows to express one exponent in terms of the other (e.g., α in terms of β). We then get the *profile equation*,

$$\Delta f^m+\beta\eta\cdot\nabla f+\alpha f=0, \tag{4.33}$$

which is a nonlinear elliptic equation with a free parameter (say, β). We only need to specify the boundary or other conditions to get a well-specified *Nonlinear Eigenvalue Problem.*

• We will see in Chapter 16 how to solve this problem for different values of β, and in another particular example in Section 4.6 below. In the present case, the 'eigenvalue' β is fixed by means of a physical law, conservation of mass: $\int \mathcal{U}(x,t)\,dx=$ constant. When applied to the selfsimilar formula, it gives

$$\int \mathcal{U}(x,t)dx=\int t^{-\alpha}f(xt^{-\beta})\,dx=t^{-\alpha}t^{\beta d}\int f(\eta)d\eta=\ \text{const}(t), \tag{4.34}$$

which implies the relation $\alpha=d\beta$. Summing up, we have

$$\alpha(m-1)+2\beta=1, \qquad \alpha=\beta d, \tag{4.35}$$

so that the exponents have the values:

$$\beta=\frac{1}{d(m-1)+2}\ , \quad \alpha=\frac{d}{d(m-1)+2}. \tag{4.36}$$

• We still have to solve Equation (4.33) in $\mathbb{R}^d$ for these values of α and β. We want nonnegative solutions. Since the problem is rotationally invariant we look for a radially symmetric solution, $f=f(r)$, $r=|x|$. We have

$$\frac{1}{r^{d-1}}(r^{d-1}(f^m)')'+\beta rf'+d\beta f=0,$$

that can be written as

$$(r^{d-1}(f^m)'+\beta r^n f)'=0.$$

This is a fortunate calculation, since we can integrate once to get

$$r^{d-1}(f^m)'+\beta(r^d f)=C. \tag{4.37}$$

Boundary conditions enter: since we want $f\to 0$ as $r\to\infty$, we take $C=0$, so that

$$(f^m)'+\beta rf=0, \quad mf^{m-2}f'=-\beta r, \tag{4.38}$$

hence

$$\frac{m}{m-1}f^{m-1}=-\frac{\beta}{2}r^2+C, \quad f^{m-1}=A-\frac{\beta(m-1)}{2m}r^2. \tag{4.39}$$

This is the end of the integration. We have obtained the announced *quadratic profile for the pressure* of the source solution.

Problems again

(i) We find that the formula produces a smooth solution of the PME whenever $\mathcal{U} > 0$, but we again face the problem of negative values if we want this formal solution to serve as a solution in the whole space. Since this is precisely the situation we have encountered in the study of TWs, we know how to proceed. Approximate the delta function by a positive function solve the classical problems and pass to the limit. Unfortunately, we are not technically strong enough to perform that feat. But Zeldovich *et al.* found numerically that the result is similar: taking the maximum between 0 and the formal solution. In other words, cutting off the unwanted part of the profile. We thus arrive at formula (4.21).

This way of waving hands at the proofs is quite unsatisfying (but see Problems 4.4 to 4.6). We will devote the next chapters to develop the theory of weak solutions. We will prove that (4.21) is a weak solution. We will prove that when initial data are taken in a suitable class of integrable functions, and weak solutions are suitably defined, the weak solution of the problem exists and is unique; moreover, it is shown that classical solutions are weak solution, and so are their limits.

(ii) Another problem appears. In Chapter 9 the class where weak solutions lie is $C([0,T) : L^1(\mathbb{R}^d))$. Now, $\mathcal{U}(\cdot,t) \in L^1(\mathbb{R}^d)$ for every $t > 0$, but not for $t = 0$, we have a problem with the initial data. We will have to enlarge the class of data to measures in order to have a well-posed generalized theory that includes our favorite special solution.

This discussion leads to an important conclusion: the abstract theory has been strongly influenced by underlying physical considerations and special solutions.

4.4.3 Extension to $m < 1$

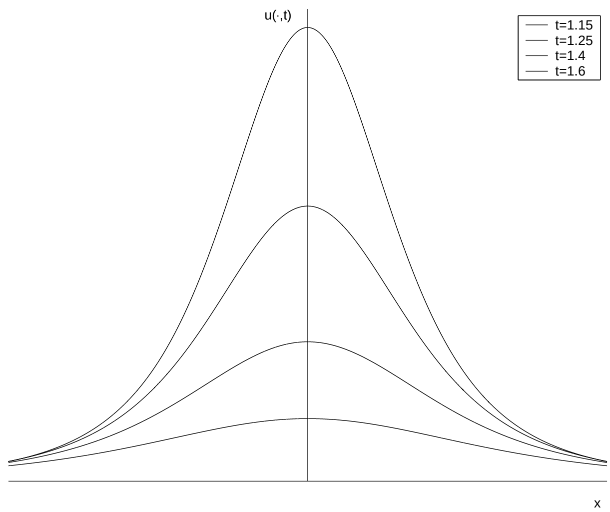

Figure 4.4: Source solution for FDE for $d = 3$, $m = 1/2$

It was soon realized that the source solution also exists with many similar properties as long as $\alpha > 0$, i. e., it can be extended to the Fast Diffusion Equation, $m < 1$, but only in the range $m_c < m < 1$, cf. [359], with

$$m_c = 0 \quad \text{for} \quad d = 1, 2, \qquad m_c = (d-2)/d \quad \text{for} \quad d \geq 3.$$

Formula (4.21) is basically the same, but now $m - 1$ and k are negative numbers, so that $\mathcal{U}_m$ is everywhere positive with power-like tails at infinity. More precisely,

$$\mathcal{U}_m(x, t; M) = t^{-\alpha} F(x/t^{\alpha/d}), \quad F(\xi) = (C + \kappa_1 \xi^2)_+^{-\frac{1}{1-m}}. \tag{4.40}$$

with same value of α and $\kappa_1 = -\kappa = (1-m)\alpha/(2d)$.

4.5 Blow-up. Limits for the existence theory

Let us start by an elementary, but interesting observation. If $u(x, t)$ is a classical solution of the PME and it is given as an smooth expression of x^2 and t in the form

$$u = F(x^2, t),$$

then

$$\widetilde{u} = F(-x^2, -t)$$

is again a classical solution of the equation. This trick is an extended form of the scaling transformations studied in Subsection 3.3.2. We may try the trick on the solutions of the last section, though they are not classical, and see what happens. Using it on equation (4.21) we get the formula

$$\widetilde{\mathcal{U}}(x, t) = (-t)^{-\alpha} \left(C + \kappa\, |x|^2 (-t)^{-2\beta} \right)_+^{\frac{1}{m-1}} \tag{4.41}$$

with α and κ given in (4.22). Let us examine this formula for the different values of the free constant C.

(i) When $C > 0$ formula (4.41) produces a function that is well-defined and positive in the domain where $x \in \mathbb{R}^d$ and $t < 0$. It is moreover a classical solution of the PME in that domain and tends to infinity as $t \to 0$ at every point x. This is what we call *blow-up*.

It is customary to change the origin of time to some $T > 0$, write the solution as

$$\widetilde{\mathcal{U}}(x, t; C) = (T-t)^{-\alpha} \left(C + \kappa\, |x|^2 (T-t)^{-2\beta} \right)^{\frac{1}{m-1}}, \tag{4.42}$$

and consider times $0 \leq t \leq T$ (or even $-\infty < t < T$). The formula is even easier in terms of the pressure

$$\widetilde{V}(x, t; C) = \frac{C(T-t)^{2\beta} + K\, |x|^2}{T-t} \tag{4.43}$$

where $K = \alpha/2d = m\kappa/(m-1)$, and $C > 0$ is arbitrary.

(ii) Case $C = 0$. We get an explicit solution whose pressure is a quadratic function that blows up in finite time

$$\widetilde{V}(x, t; 0) = \frac{K\, |x|^2}{T-t}. \tag{4.44}$$

This is a classical blow-up solution for the pressure equation, it has a separate variables form, and is defined in $\{(x,t) : t < T\}$.

(ii) Case $C = -D^2 < 0$. In this case the solution that we obtain is not classical:

$$\widetilde{U}(x,t;-D) = (T-t)^{-\alpha}\left(K\,|x|^2(T-t)^{-2\beta} - D^2\right)_+^{\frac{1}{m-1}} \tag{4.45}$$

with a free boundary given by the hypersurface

$$|x| = DK^{-1/2}\,(T-t)^{\beta}\,. \tag{4.46}$$

Notice that in this case the empty region (where $u = 0$) is a contracting hole located inside the support.

4.5.1 Optimal existence versus blow-up

The existence of solutions that blow up in finite time can be combined with the Maximum Principle to show that nonnegative solutions of the PME whose initial data grow as $|x| \to \infty$ not less than $O(|x|^{2/(m-1)})$ must necessarily cease to exist at a time T that can be estimated by using the above formulas. It is then shown that such a growth estimate is optimal, in the sense that solutions with data

$$u_0(x) = o((1+|x|^2)^{1/(m-1)}) \tag{4.47}$$

exist for all time (here, we use symbols O and o in the sense of Landau). This is no wonder, since similar transformations and blow-up solutions exist for the heat equation. But, whereas in the case of the HE the maximum admitted growth is square exponential, for the PME it is power-like with exponent $1/(m-1)$ (i. e., quadratic growth for the pressure). We conclude that as m grows the class of existence decreases.

The question of existence for optimal classes of data will be investigated carefully in Chapter 12, where all the statements will be proved.

4.5.2 Non-contractivity in uniform norm

One of the most important properties of the class of filtration equations studied in Chapter 3 is the property of (non-strict) contraction with respect to the L^1-norm. This property is one of the cornerstones on which the general theory of the PME if founded. One may wonder if the PME evolution is also contractive with respect to other L^p spaces. Actually, the Heat Equation is for all $p \in [1,\infty]$ and this is quite easy to prove and useful. We will show below that the PME is not contractive with respect to the L^p norms for any $p > 1$. The main idea is based on the following observation about the blow-up solutions (4.42), or in pressure terms (4.43). We note that for two different constants $0 < C_1 < C_2$ we have

$$\widetilde{V}(x,t;C_2) - \widetilde{V}(x,t;C_1) = \frac{C_2 - C_1}{(T-t)^{\alpha(m-1)}}. \tag{4.48}$$

If $m = 2$ this immediately implies that the L^∞ norm of the difference of two solutions $\widetilde{U}_1$ and $\widetilde{U}_2$ increases with time like an inverse power of $T-t$, so that it actually goes to infinity.

The reader may object that our solutions are not bounded themselves. The adaptation will have to wait for the theory to be developed. Moreover, examples of non-contractivity will be constructed when $m \neq 2$ or when $p \in (1,\infty)$, but we will have to know a bit more about the theory. The proofs are contained in Appendix 22.11.

4.6 Two solutions in groundwater infiltration

There are a number of selfsimilar solutions that play important roles in the theory. We present in this section two solutions for problems posed in a half-line of one-dimensional space. They are motivated by the model introduced in Section 2.3 for groundwater infiltration into a horizontal porous stratum, which leads to the PME with $m = 2$ as we have shown. In an idealized situation, typical of the selfsimilar analysis, we assume that the stratum is horizontal with an impervious lower bed at $z = 0$ and point the x-axis in the direction perpendicular to the border, which is supposed to be $x = 0$. Forgetting the y direction, the PME is posed for the variable $z = h(x, t)$ with $x > 0$ and boundary conditions

$$h(0, t) = H_0 > 0, \quad h(\infty, t) = 0 \tag{4.49}$$

The first condition represents border infiltration so that the groundwater level is kept constant at $x = 0$. There are then two main cases that have been studied.

4.6.1 The Polubarinova-Kochina solution

In case the height $H_0 > 0$ there exists a selfsimilar solution the form (4.30). The boundary condition is compatible with this form only if $\alpha = 0$. But then the compatibility with the equation (4.32) implies that $\beta = 1/2$. The solution is therefore written as

$$h(x, t = f(\eta), \quad \eta = x/t^{1/2}. \tag{4.50}$$

Corresponding initial data are

$$h(x, 0) = 0 \quad \text{for} \quad x > 0. \tag{4.51}$$

It represents infiltration into an empty stratum from a lateral source with constant height, and was studied by Mrs. Polubarinova-Kochina in 1948.

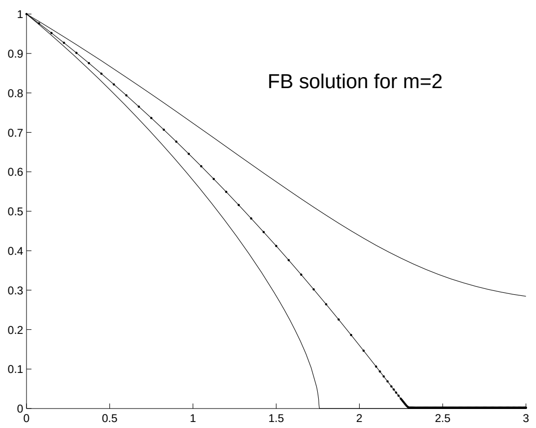

Figure 4.5: Groundwater solution and other orbits

The groundwater model problem is reduced to solve the ODE problem

$$\begin{aligned} &(f(\eta)^m)'' + \frac{1}{2}\eta\, f'(\eta) = 0, \quad 0 < \eta < \infty \\ &f(0) = H_0, \qquad f(\infty) = 0. \end{aligned} \tag{4.52}$$

The constant H_0 is inessential and can be replaced by 1 without loss of generality by rescaling. Paper [438] makes a numerical study of this ODE and concludes that there is a correct solution that lands on the x-axis at a finite distance η_*, even if the slope does not go to zero at this point. We get in this way finite propagation in the way we have seen above. Figure 4.5 shows the solutions as well as nearby orbits obtained by shooting in the ODE problem with different initial slopes. These calculations will be rigorously established in this book as the theory proceeds.

4.6.2 The Dipole Solution

There is a second solution that has the explicit form,

$$U_{dip}(x,t) = t^{-\frac{1}{m-1}}|x|^{1/m}\mathrm{sign}(x)\left(Ct^{\frac{m+1}{2m^2}} - \frac{m-1}{2m(m+1)}|x|^{\frac{m+1}{m}}\right)_+^{\frac{1}{m-1}}. \tag{4.53}$$

It corresponds to selfsimilarity with exponents $\alpha = 1/m$ and $\beta = 1/(2m)$, and profile

$$f_{dip}(\xi) = |\xi|^{1/m}\mathrm{sign}(\xi)\left(C - \kappa\,|\xi|^{\frac{m+1}{m}}\right)_+^{\frac{1}{m-1}}. \tag{4.54}$$

We usually consider this solution as defined in the quarter of plane, $Q_1 = (0,\infty)\times(0,\infty)$, and then $U_{dip}\ge 0$, and the mass $M(t) = \int_0^\infty u(x,t)\,dx$ decreases with time like $O(t^{-1/2m})$, while the momentum is conserved,

$$\int_0^\infty xu(x,t)\,dx = C_1, \tag{4.55}$$

cf. formula (3.45) of Section 3.3.3. A curious situation happens, namely that both the initial and boundary conditions vanish:

$$U_{dip}(x,0) = 0 \quad \text{for all } x>0, \quad U_{dip}(0,t) = 0 \quad \text{for all } t>0. \tag{4.56}$$

The question is then: should not the solution be trivial? an explanation of the negative answer comes from the realization that a singularity occurs (namely, u is unbounded) as we approach the corner point $(x=0, t=0)$. We conclude that uniqueness of unbounded solutions is not necessarily true (even if the solution is nonnegative and the divergence takes place only as $t\to 0$); this observation will affect the theory to be developed in this book. We will discuss in Section 13.5 a theory which allows for solutions with initial singularities like the dipole. Another illuminating observation consists of looking at the behaviour of the flux $(U_{dip}^m)_x$ near zero. Indeed, we have $(U_{dip}^m)_x \sim Ct^{-(2m+1)/m}$ which diverges as $t\to 0$.

The phenomenon is better understood if we consider the function as a signed solution of $u_t = (|u|^{m-1}u)_{xx}$ in $Q_2 = \mathbb{R}\times(0,\infty)$. Then, it is not difficult to see that for every test function $\zeta\in C_c^\infty(\mathbb{R})$

$$\lim_{t\to 0}\int_{\mathbb{R}} U_{dip}(x,t)\zeta(x)\,dx = M\,\zeta'(0) \tag{4.57}$$

with $M = M(m,C) > 0$. In other words, $U_{dip}(\cdot,t)\to M\,\delta'(x)$, where $\delta'(x)$ is the distributional derivative of the delta function. This is called in Physics the elementary dipole, hence the name Dipole Solution of the PME for U_{dip}. The data that are taken in the sense

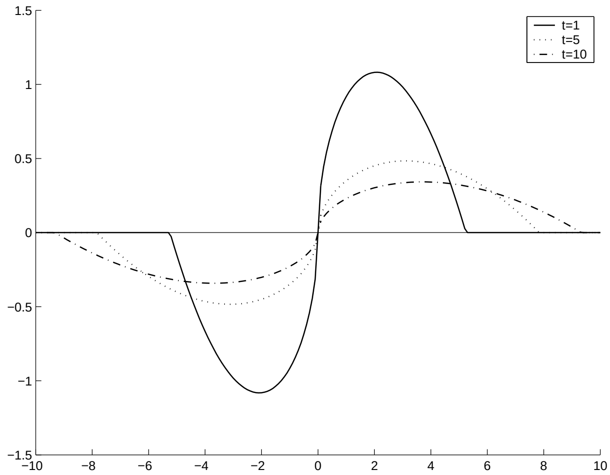

Figure 4.6: The dipole solution at different times, $m = 2$

of the weak limit, (4.57), are called *initial traces*. They are a very natural and general form of data and will be considered in detail when the so-called theory with optimal data is done in Chapter 12.

The dipole solution can be extended to the FDE with $0 < m < 1$ but then it reads

$$U_{dip}(x,t)^{1-m} = \frac{t\,|x|^{(1-m)/m}\mathrm{sign}(x)}{C\,t^{\frac{m+1}{2m^2}} + \frac{1-m}{2m(m+1)}\,|x|^{(m+1)/m}}. \tag{4.58}$$

Note the behaviour as $x \to \infty$, $u^{1-m} \sim ct/x^2$, which is the same as in the ZKB and is typical of the FDE, both in exponents and coefficients.

See Problem 4.12 for the derivation of the Dipole Solution from the source solution of the p-Laplacian equation and the limit $m \to 1$.

4.6.3 Signed selfsimilar solutions

The dipole solution opens the question of constructing other selfsimilar signed solutions of the PME in the whole space having compact support, maybe in a explicit or semi-explicit way. We will see in Chapter 16 how to construct selfsimilar solutions in a systematic way. The case of compactly supported solutions in $d = 1$ was analyzed by Hulshof in [296]. The form is

$$u(x,t) = t^{-\alpha}U(\eta), \quad \eta = xt^{-\beta},$$

with the standard constraint $(m-1)\alpha + 2\beta = 1$; then, U satisfies the differential equation

$$(|U|^{m-1}U)'' + \beta\eta U' + \alpha U = 0.$$

The theorem says that there exists a strictly decreasing sequence $\alpha_0 = 1/(m+1) < \alpha_1 = 1/m < \alpha_2 < \cdots \uparrow 1/(m-1)$ such that compactly supported similarity solutions of the type above exist if and only if $\alpha = \alpha_k$. The first exponent corresponds to the Barenblatt solution, the second to the dipole. The third was investigated in [103] where it was shown that the exponent is not derived from a conservation law (in other words, it is anomalous, and the simple extrapolation of the elementary algebraic conjecture breaks down).

It is also proved in [296] that k equals the number of times $U(\eta)$ changes its sign and $U(\eta)$ is symmetric if k is even (antisymmetric if k is odd). The reader should note that for the wrong α (i.e., not in the list) selfsimilar solutions exist but they are compactly supported. Actually, they are not integrable.

A similar analysis holds for compactly supported, radially symmetric solutions of the PME in $\mathbb{R}^d$, $d > 1$.

4.7 General planar front solutions

As an extension of the theory of travelling waves, we now examine the class of solutions that propagate with a certain speed $c(t)$ and keep their space shape constant in time but for a scale factor $A(t)$. We call them *fronts*. The general form is then

$$u(x,t) = A(t)\, U(x_1 - s(t), x_2, \cdots, x_d) \tag{4.59}$$

but for a possible rotation of the space direction. The PME implies then the differential equation

$$A'(t)U(\eta) - c(t)A(t)U'_{\eta_1} = A^m(t)\Delta U^m(\eta),$$

with $c(t) = s'(t)$. Since we want nontrivial solutions, we assume $A(t) \neq 0$. In that case, a simple separation of variables argument implies that the following two conditions must hold:

$$A'(t) = -\lambda A^m(t), \quad c(t)A(t) = \mu A^m(t). \tag{4.60}$$

The case $\lambda = 0$ allows us to recover the Travelling Waves of Section 4.3, and the case $\mu = 0$ gives $c(t) = 0$, hence, the solutions in separate-variables form.

New solutions are obtained when both parameters are nonzero. In that case, we get from the first equation

$$A(t) = \frac{1}{(C + \lambda(m-1)t)^{1/(m-1)}}, \quad c(t) = \frac{\mu}{(C + \lambda(m-1)t)}. \tag{4.61}$$

Case $\lambda > 0$. In this case the second equation integrates to give a speed of the form

$$s(t) = c_0 \log(C + \lambda(m-1)t),$$

which goes to infinity but slows down as $t \to \infty$, something that also happens for the ZKB solution (with a different rate though). The time factor also decreases in a power fashion, $A(t) = O(t^{-1/(m-1)})$, the same as the separate-variables solutions. The equation for the profile U becomes

$$\Delta U^m(\eta) + \lambda U(\eta) + \mu U'_{\eta_1}(\eta) = 0, \tag{4.62}$$

which is a variation of the basic nonlinear elliptic equation (4.4). Using scaling, there is no loss of generality in reducing the case $\mu > 0$ to $\mu = 1$. The case of negative μ can be reduced to positive μ by reflection (which only changes the direction of the wave).

4.7.1 Solutions with a blow-up interface

The case $\lambda = -l < 0$ is more interesting, because it leads to interesting solutions whose interface blows up in a finite time. Indeed, in this case we get

$$s(t) = -c_0 \log(C - l(m-1)t) = c_0 \log(1/(T-t)) + c_1. \tag{4.63}$$

which blows up as $t \to T = C/l(m-1)$. This means that the location of the interface, if there is one, reaches infinity in finite time with a logarithmic rate. Also, the scale factor $A(t)$ blows up as $t \to T$. Let us calculate the profile U in that case. The equation becomes

$$\Delta U^m(\eta) - lU(\eta) + \mu U'_{\eta_1}(\eta) = 0. \tag{4.64}$$

Existence and analysis of a special solution

Since this type of solution has a certain interest and seems not be described in the literature, we pay some attention to a particular solution in one space dimension. The exercise allows us to review some interesting ODE techniques.

Again, there is no loss of generality in fixing $\mu = 1$. In $d = 1$ we can write

$$(U^m)''(\eta) - lU(\eta) + U'(\eta) = 0. \tag{4.65}$$

Our next task is integrating this equation. We use a phase plane argument. Letting $V = -(U^m)'$, we have $V' = -lU + U'$. We get the system:

$$\begin{cases} \dfrac{dU}{d\eta} = -\dfrac{1}{m} V U^{1-m} \\ \dfrac{dV}{d\eta} = -lU - \dfrac{1}{m} V U^{1-m}, \end{cases} \tag{4.66}$$

so that

$$\frac{dV}{dU} = 1 + \frac{lmU^m}{V}. \tag{4.67}$$

We cannot integrate this ODE explicitly, but the usual qualitative techniques allow us to understand the existence and behaviour of the different solutions $(U(\eta), V(\eta))$. Let us recall that $dV/dU = 1$ is the equation for the TWs of Section 4.3, while $dV/dU = lmU^m/V$ describes the blow-up solution of Section 4.5 in 1 dimension. Therefore, we expect (4.67) to combine properties of both equations.

• Let us now examine the different orbits $V = V(U)$ in the first quadrant of the (U, V) plane. We see that they are increasing with $dV/dU > 1$. They can get started at any point of the vertical axis, i. e., $U_0 = 0$ and $V_0 > 0$, and then the initial slope is $dV/dU = 1$. We can also shoot from the horizontal axis, $U_0 > 0$ and $V_0 = 0$ with initial slope $dV/dU = \infty$. Finally, there exists one separatrix solution between the two families, which starts from $(0, 0)$. Such a separatrix is unique by monotonicity arguments.

• Local analysis of that orbit near the origin is as follows: since $dV/dU > 1$, we have $V > U$, hence the equation simplifies for $U \sim 0$ to $dV/dU \sim 1$ (the last term is smaller, $O(U^{m-1})$), so that $V/U \to 1$. We can now use the definition of V to conclude that there exists a constant η_0 such that

$$U(\eta)^{m-1} \sim \frac{m-1}{m}(\eta_0 - \eta)$$

as $\eta \to \eta_0$ with $\eta < \eta_0$; η_0 is an arbitrary constant, and we can take $\eta_0 = 0$ to normalize. We have obtained the correct behaviour near a free boundary.

• Behaviour for large values. As $U \to \infty$, we derive an estimate as follows: since $dV/dU > U^m/V$ we have $V^2 > C_1 U^{m+1}$; using the differential equation we conclude that $dV/dU < 1 + U^{(m-1)/2}$, hence $V < C_2 U^{(m+1)/2}$ for all large U. But that means that the equation can be simplified for all large U into $dV/DU \sim mlU^m/V$, which gives the exact estimate to first order

$$V = c_3 U^{(m+1)/2} + \cdots, \quad c_3 = (2lm/(m+1))^{1/2}.$$

Recalling that $V = -mu^{m-1}U'$ and integrating, this gives the estimate as $\eta \to -\infty$:

$$U^{m-1} = c_4 \eta^2 + \cdots, \quad c_4 = \left(\frac{m-1}{2}\right)^2 \frac{2l}{m(m+1)}.$$

It follows that

$$\frac{m}{m-1} u^{m-1}(x,t) = \frac{1}{2(m+1)} \frac{x^2}{T-t} + \cdots,$$

as $x \to -\infty$. This is precisely the behaviour of the standard blow-up solution (4.44) in $d = 1$.

Remark on the blow-up rate. As $t \to T$, and uniformly on bounded sets $|x| \le K$, the solution blows $U(x,t)$ up with the rate

$$U(x,t) \sim A(t)U(s(t)) \sim \frac{c}{(T-t)^{1/(m-1}} (\log(1/(T-t))^2 \tag{4.68}$$

This is faster than the blow-up rate of the standard blow-up solutions of Section 4.5. See Open Problem 4.OP1.

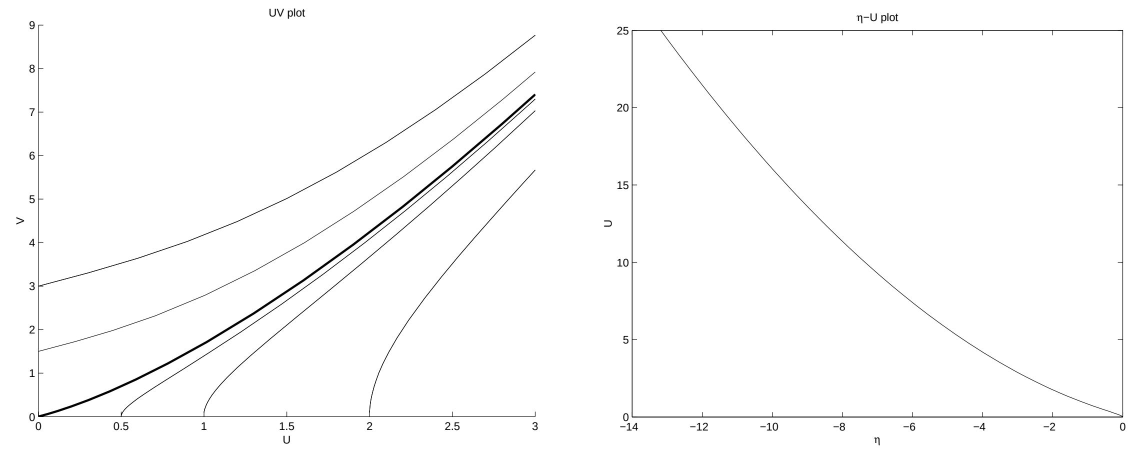

Figure 4.7: General front with blow-up. Left, the phase-plane with the solution in bold line. Right, the plot of u versus η. Parameters $m = 2$, $l = 1$

Notes

§4.2. Separate variables solution. The existence of this solution in a bounded domain was rigorously proved by Aronson and Peletier in [49]. If we eliminate the restriction of nonnegativity, we may obtain an infinite family of solutions of the elliptic problem with increasingly complicated sign-change patterns, cf. [511].

The method can be applied to the Fast Diffusion Equation, and then it produces solutions which vanish in finite time of the form

$$u(x,t) = ((1-m)(T-t))^{-1/(m-1)}F(x), \tag{4.69}$$

where T is the extinction time, a free parameter. Existence of a positive profile vanishing on the boundary of a bounded domain is proved for $m > (d-2)/(d+2)$ while for $0 < m \le (d-2)/(d+2)$ the profile is positive in the whole space, cf. [99].

§4.3. Travelling waves. They are constant-shape fronts moving with constant speed $|\mathbf{V}| = c$ wherever $u > 0$, while in the empty region (where $u = 0$) the speed, defined as the gradient of the pressure v, is zero. This gives rise to a discontinuous function to represent $\mathbf{V} = -\nabla v$. This reminds us of the discontinuous solutions of standard gas dynamics and their shock waves, cf. [171, 175, 359]. Indeed, this analogy has been used to develop the theory, cf. [501].

§**4.4. Source-type solutions.** The origin of these solutions has been explained in the Introduction and Chapter 2. The applied problem that motivated the studies was a problem in plasma Physics, nothing to do with porous media!

Continuation. The ZKB solutions can be continued algebraically for $m < (d-2)/d$, but they have different geometry, they do not solve the same initial value problem and they also lose their important role as asymptotic patterns for a large class of initial data. For a detailed analysis of this issue cf. our monograph [515].

§**4.5. Blow-up solutions.** They will play a big role in setting limits to the existence theory, and as handy comparison functions.

The example of expansion of the L^∞-norm of Subsection 4.5.2 seems to be due to the author.

§**4.6.** The first type of solutions of this section was studied by Polubarinova-Kochina in 1948, [438]. A study for the general equation $u_t = (D(u)u_x)_x$ is performed by [431] in 1955. This author finds cases with explicit solutions in [432]. These are interesting early references to mathematical work on the PME.

These solutions have appeared often in studies of groundwater infiltration. Solutions with changing sign in $d = 1$ were constructed by van Duijn et al. [221].

The dipole solution is due to Zel'dovich and Barenblatt [530] and has been used by Kamin and Vazquez [326] in describing the asymptotic behaviour of more general signed solutions. The extension to fast diffusion is new. For recent theory and experiments on dipole solutions see King-Woods [342].

A dipole solution for the signed PME in several space dimensions was constructed by Hulshof and Vazquez [299].

§**4.7. General planar front solutions.** They seem to be new in the literature. Our special solution shows that the free boundary may blow up in finite time.

Note on Selfsimilarity. This is a capital concept in Mechanics and, generally speaking, in the applied sciences. Thus, it has been pointed out in many papers and corroborated by numerical experiments that similarity solutions furnish the asymptotic representation for solutions of a wide range of problems in mathematical physics. The reader is referred to the book of G. Barenblatt [63, 64] for a detailed discussion of this subject. As these books say, it must be born in mind that, even if arisen to solve problems in the practical applications, selfsimilar solutions are ideal constructs that represent idealized situations and will only represent observed behaviour in a limit sense. However, it is discovered by the practical scientist that this sense has the deepest influence on the rest of the theory.

There are a number of other selfsimilar solutions that play a prominent role in the theory, like the hole-filling solutions of Graveleau and Aronson [48] that played a big role in the studies of optimal regularity. We will devote the whole Chapter 16 to the study of selfsimilarity. This set of ideas is better known in Theoretical Physics as Renormalization Group.

Eternal solutions. This is the name given to solutions that are defined for the whole time span, $-\infty < t < \infty$. The travelling wave solutions are eternal, and some of the planar fronts also are; the rest of the examples of this chapter exist either forward in time (separate-variables, source-type, dipole, constant-height solution) or backward in time (the blow-up solutions).

Problems

Problem 4.1 SEPARATED VARIABLES. (i) Take $\Omega = B_R(0)$ and solve the nonlinear elliptic problem (4.5) by writing $F_1(x) = f(r)$, $r = |x|$, and solving the ODE for $g(r) = f^m(r)$

$$g''(r) + \frac{d-1}{r}\, g'(r) + g(r)^p = 0, \quad p = \frac{1}{m},$$

with $g(0) = h$, $g'(0) = 0$. Find $h > 0$ so that $g(R) = 0$.

(ii) Check that taking $\lambda \neq 1$ still produces the same family of solutions (4.7) in separated variables.

Problem 4.2 TRAVELLING WAVES. Prove Proposition 4.1.

Hint: The function f_ε is obtained by integration of equation (4.16). It is better to think that it defines η in terms of f in the range $f \geq \varepsilon$. Here is a work plan:

(i) Get the formula for $\eta = \eta_\varepsilon(f)$:

$$\eta = -\frac{m}{c} \int_1^f \frac{f^{m-1}}{f-\varepsilon} df.$$

(ii) Show that η ranges from 0 to ∞ while f goes from ε to ∞, and the dependence is monotone decreasing. Note that $\eta_\varepsilon(1) = 0$ for every $\varepsilon \in (0, 1)$.

(iii) Show that as $\varepsilon \to 0$ we get uniform converge on sets of the form $[1/a, a]$ to the solution of the limit equation

$$\frac{d\eta}{df} = -\frac{m}{c}\, f^{m-2}.$$

(iv) Conclude the announced result.

(v) Perform the explicit computation for $m = 2$ and pass to the limit in the obtained formula.

Problem 4.3 SIGNED TRAVELLING WAVES. Construct a travelling wave with changing sign by considering the case $K < 0$ in formula (4.11). Show that $f \to K/c < 0$ as $\eta \to \infty$. Sketch the profile f and determine the optimal regularity in terms of m.

Solution: $f \in C^{1/m}(\mathbb{R})$, with minimal regularity at the sign transition, $f = 0$.

Conclude from the analysis that the transition from plus to minus sign of these signed solutions implies a blow-up for the gradient of the pressure.

Problem 4.4 TRAVELLING WAVES FOR THE GPME. Consider the existence of TWs for the equation $\partial_t u = \Delta\Phi(u)$ of the form (4.9), $u = f(x - ct)$. (i) Show that the equation of the TW such that $u(x,t) \to 0$ as $x \to \infty$ is given by the ODE

$$\Phi(f)' + cf = 0, \tag{4.70}$$

which leads to the implicit expression

$$\int_1^f \frac{d\Phi(s)}{s} = c(\eta_1 - \eta) \tag{4.71}$$

(ii) Show that the TW has a finite interface if and only if

$$\int_0^1 \frac{d\Phi(s)}{s} < \infty. \tag{4.72}$$

(iii) Check that this happens for $\Phi(u) = u^m$ iff $m > 1$.

See continuation in Problem 15.11.

Problem 4.5 THE ZKB SOLUTION. (i) Show that formula (4.21) is actually a classical solution of the PME in the region where it is positive.

(ii) Show that the initial data are taken in the sense of (4.23), i. e., that for every test function $\phi \in C_0(\mathbb{R}^d)$, $\phi \geq 0$ we have

$$\lim_{t\to 0} \int_{\mathbb{R}^d} \mathcal{U}(x,t)\phi(x)\, dx = M\,\phi(0).$$

(iii) Show that the pressure V of the ZKB solution is Lipschitz continuous but not C^1 on the free boundary.

(iv) Check Darcy's law for the ZKB solution.

(v) Prove formula (4.27) for the convergence of ZKB profiles to Gaussian profiles as $m \to 1$. Determine in what sense and where the limit is taken.

(vi) Write the formula for the pressure and prove that

$$\Delta V = -\frac{C}{t} \tag{4.73}$$

in the set $\{(x,t) : U > 0\}$. The solutions is even concave on that set. This is a much used property of the ZKB pressure inside the solution support.

Problem 4.6 SHAPE OF THE ZKB SOLUTION. Show that for $m = 2$ the shape of the density u of the source solution in terms of $|x|$ for fixed time is a parabola. Show that for $m = 3$ it is an ellipse with vertical slope at the front. Show that for $m = 3/2$ it is a fourth-order polynomial with flat contact with the x-axis. Write the explicit expression for $d = 1$, $t = 1$ and $C = 1/15$.

Problem 4.7 In later chapters we will be able to show that the ZKB solution is a limit solution by following this program: approximate the delta function by a sequence of positive functions $u_{0n}(x)$; solve the PME with these data and find classical solutions $u_n(x,t)$; pass to the limit as $n \to \infty$ and find the ZKB solution.

Study those chapters and perform this program.

Problem 4.8 Derive the formula for the velocity of the ZKB solutions and show that it is a discontinuous function. See more in Subsection 18.7.2.

Problem 4.9 Write the formula for the pressure of solution (4.45) and show that it is not a C^1 function. Calculate the velocity. How does it evolve with time on the free boundary?

Problem 4.10 BLOW-UP. Write the formulas of explicit blow-up solutions for the Heat Equation. The simplest has the form

$$U(x,t) = (T-t)^{-d/2} \exp\left(\frac{|x|^2}{4(T-t)}\right).$$

Find a whole family. Compare the rate of blow-up with the PME case.

Problem 4.11 THE DIPOLE SOLUTION. (i) Check that function U_{dip} defined in (4.53) is singular near $(0,0)$ by checking that $\|U_{dip}(\cdot,t)\|_\infty = c(m,C)t^{-1/m}$, and the maximum for fixed t is reached along a line of the form $x = c_1 t^{1/2m}$.

(ii) Calculate the relation between M and C in formula (4.57).

(iii) Calculate the constant $\kappa(m)$.

(iv) The free boundary of the dipole solution propagates like $|x| = O(t^{1/2m})$, while the ZKB propagates like $|x| = O(t^{1/(m+1)})$ in 1D. Find please a justification for the smaller rate.

Problem 4.12 THE BARENBLATT SOLUTION FOR THE p-LAPLACIAN EQUATION. (i) Show that the function

$$W(x,t) = t^{-\frac{1}{m}} \left(C - k(m)|\xi|^{\frac{m+1}{m}}\right)_+^{\frac{m}{m-1}}$$

with $C > 0$ arbitrary, $\xi = x\,t^{-1/2m}$ and

$$k = \frac{m-1}{2m}(2m)^{-1/m},$$

is a generalized solution of the p-Laplacian equation $w_t = (|w_x|^{m-1}w_x)_x$ for every $m > 1$, in the sense that the equation is satisfied in the classical sense whenever $w_x \neq 0$, and it is C^1 function for all (x,t). Cf. [68].

(ii) Prove that $W(\cdot,t) \to M\,\delta(x)$ for some $M = M(m,C) > 0$. This justifies the name of source-type solution.

(iii) Differentiate this solution with respect to x to find the Dipole Solution of the PME, $U_{dip} = -W_x$.

(iv) Show that in the limit $m \to 1$ with constant moment we obtain the dipole of the heat equation in $d = 1$

$$U_{dip}(x,t;m=1) = \frac{Mx}{t^{3/2}\exp\left(-x^2/4t\right)}. \tag{4.74}$$

Problem 4.13 (i) Use ODE techniques to obtain a description of the solutions of system (4.66) that enter the region $V < 0$. Draw the corresponding profiles $U(\eta)$.

(ii)) Use ODE techniques to obtain a description of the solutions of Section 4.7 when $\lambda > 0$. Draw the corresponding profiles $U(\eta)$.

Hint: The analysis at the origin is delicate. ODE techniques are intensely studied in Chapter 16.

Open problem 4.OP1. (i) The free boundary of the solution constructed in Section 4.7.1 blows up with a logarithmic rate, $s(t) = O(|\log(T-t)|)$. Are there any free boundary solutions of the PME with free boundaries which blow up in finite time with a power rate? The task is to construct one such solution in semi-explicit form.

(ii) The correction factor in the blow-up expression (4.68) is logarithmic. Is it the unique possible form? are there any solutions with faster rates?

Chapter 5

The Dirichlet Problem I. Weak solutions

In this long chapter we start the systematic study of the questions of existence, uniqueness and main properties of the solutions of the PME by concentrating on the first boundary-value problem posed in a spatial domain Ω, which is a bounded subdomain of $\mathbb{R}^d$, $d \geq 1$. We focus on homogeneous Dirichlet boundary conditions, $u = 0$ on $\partial\Omega$, in order to obtain a simple problem for which a fairly complete theory can be easily developed as a first stage in understanding the theory of the PME. This is called the homogeneous Cauchy-Dirichlet problem, or more simply, the homogeneous Dirichlet problem.

Even if the main goal of the text is to develop a theory for nonnegative solutions of the PME, the theory of this chapter can be safely done for the complete Generalized Porous Medium Equation, also called Filtration Equation, $u_t = \Delta\Phi(u) + f$, under conditions that include the whole range of exponents $0 < m < \infty$ of the PME, HE and FDE. We pursue this course for four reasons: it does not imply undue extra effort, the generality can be illustrative of the functional analysis involved, it will be of help in the future, and finally the Filtration Equation is an important subject of study in itself. We recall that the full form is important for its application to the study of reaction-diffusion processes where the forcing term depends on u, $f = f(x, t, u)$, while convection processes include a term of the form $f = \sum_i \partial_i F_i(x, t, u)$.

The problem is shown to be well-posed globally in time in particular classes of generalized solutions, specifically, in a class of weak energy solutions. The main points for future reference are the definition of solution, Definition 5.4, the uniqueness result, Theorem 5.3, and the existence result, Theorem 5.7.

In this chapter we will use the symbols $Q = \Omega \times \mathbb{R}_+$, $Q_T = \Omega \times (0, T)$, $Q^\tau = \Omega \times (\tau, \infty)$, and $Q_T^\tau = \Omega \times (\tau, T)$. We also use the sloppier notation $Q^* = \Omega \times (\tau, T)$. $\Sigma_T = \partial\Omega \times [0, T)$ is the lateral boundary, $\Sigma = \partial\Omega \times [0, \infty)$. We recall the fact that when Ω is a bounded domain with Lipschitz boundary, then $H_0^1(\Omega)$ coincides with the restriction of the functions $u \in L^2(\Omega)$ that belong to $H^1(\mathbb{R}^d)$ when extended by zero in $\mathbb{R}^d \setminus \Omega$.

5.1 Introducing generalized solutions

A consequence of the degeneracy of the PME is that we do not expect to have classical solutions of the problem when the initial data take on the value $u = 0$, say, in an open subset of Ω. Therefore, we need to introduce an appropriate concept of *generalized* solution of the equation. At the same time, we have to define in what sense the initial and boundary conditions are taken. In many cases, this latter information can be built into the definition of generalized solution.

There are different ways of defining generalized solutions, and we will explore in the book some natural choices, the most usual idea being that of multiplying the equation by suitable test functions, integrating by parts some or all of the terms, and asking for a regularity of the solution that allows this expression to make sense. Then, we say that the solution is a *weak solution*.

In any case, the concept of generalized solution changes the meaning of the term solution, so we have to be careful to ensure that the new definition makes good theoretical and practical sense. From the first point of view, we ask the theory to be well-posed. Then, the new solutions must be defined so that they include all classical solutions whenever the latter exist (compatibility). Moreover, a concept of generalized solution will be useful if the problem becomes *well-posed* for a reasonably wide class of data, i. e., if a unique such solution exists for each set of data in a given class and it depends continuously on the data in the appropriate topologies.

As we will see, it can happen that several concepts of generalized solution arise naturally. It is then important to check that they agree in their common domain of definition (i. e., for data which are compatible with two of them). Selecting one them as the preferred definition depends of several factors, the most important being in principle that of having the largest domain. However, one could consider a more restrictive definition which still covers the applications in mind if it involves simpler statements or more natural concepts, or when it leads to simpler proofs of its basic properties.

Let us review the contents of the chapter in some detail. The study starts in Section 5.2 by considering a rather general setting, where a natural concept of *weak solution* is introduced to solve the complete Filtration Equation with zero boundary data and integrable initial data and forcing term. The idea is to lay a firm foundation of subsequent existence and uniqueness theories. The alternative of defining so-called *very weak solutions* is introduced and briefly commented, but will not be further developed for the moment, since the chapter is focused on weak solutions. The proof of uniqueness is quite immediate in that setting, Section 5.3. The existence of data for which there can be no classical solution immediately follows, just justifying the need for a weak theory.

The class of weak solutions is rather general, and it is convenient in the development of the existence theory to restrict somewhat the generality in order to get simplicity and clarity, and to be able to make interesting calculations and approximations without undue effort in justifying them. In that sense, this chapter is based on the construction of the subclass of weak energy solutions, $\mathcal{WES}$, which are weak solutions that satisfy a version of the energy estimates which have been introduced in Section 3.2 in the classical setting. Such weak solutions form a class large enough for the purposes of the usual theory found in the literature. For instance, the class provides us with a unique solution when the initial data are bounded, a safe assumption for many purposes. They also serve as a foundation for the more advanced topics of next chapter, where we will strive for the largest generality of the data.

The construction of weak solutions in the case of general Φ and general data proceeds by approximation with smooth nonlinearities and data; the study of the question of existence under good assumptions on Φ and the data has been addressed in Section 3.2, and a number of important properties of the solutions have been obtained in that smooth setting. With this information at hand, the study of the problem with general Φ proceeds first for nonnegative data, § 5.4, and then for data with changing sign, § 5.5. The technique is based on a priori estimates that force some restrictions on the data that are characteristic of weak energy solutions; thus, the initial data are restricted to the space $L_\Psi(\Omega)$, a subspace of $L^1(\Omega)$ (for the PME, it equals $L^{m+1}(\Omega)$), and the forcing term must belong to some dual space, cf. Theorem 5.7 and Corollary 5.6. These restrictions are compensated by the fact that such solutions enjoy energy inequalities, see formulas (5.20) or (5.39), that would be lost for more general data.

Let us note that, though the solution of the approximate problems by classical methods is performed in spatial domains with a smooth boundary, the main facts of the theory are formulated for Lipschitz domains. Such a generality allows to consider domains with corners, typical in many applications.

Once existence and uniqueness of these weak solutions are settled, we establish some of the main properties in Section 5.6.

We consider in Section 5.7 the problem with non-homogeneous boundary data (on a smooth boundary). This is the most general setting that will appear sometimes in the sequel. It departs a bit from the rest of this chapter, centered on the problem with zero boundary data, but provides insight and is used as an auxiliary tool, e.g., to understand different super- and subsolutions.

We recall that it is to be expected in a parabolic problem that the solutions enjoy some extra regularity properties. We will not address at this point the question of continuity for the weak solutions of the GPME we have constructed in the context of several space dimensions, because this involves heavy work that will be tackled in Chapter 7.

We continue the basic theory with the topics of universal bounds and maximal solutions for the PME and also for filtration equations with strongly superlinear Φ. The existence of the universal bound is treated in Sections 5.8.

In Section 5.9 we establish the existence of a special solution with infinite initial data. This solution is unique and acts as an absolute upper bound for all solutions of the Dirichlet Problem. The existence of such a solution is a typical nonlinear effect, which is not possible in the linear theory. For the PME it takes the form $\tilde{U}(x,t) = f(x)\,t^{-\alpha}$ with decay rate $\alpha = 1/(m-1)$. Since it takes infinite initial data but it becomes bounded for positive times, this solution will be called *the Friendly Giant*.

We apply the same techniques in Section 5.10 to the fast diffusion equations with a different conclusion, extinction in finite time, and a comment on singular diffusion.

We end the chapter with a suggestion for advanced work: applying the techniques of this chapter to a number of more general equations of inhomogeneous media, Section 5.11.

The basic material is contained in Sections 5.2 to 5.7. The last section is an introduction to advanced reading.

5.2 Weak solutions for the complete GPME

We assume that Ω is a bounded domain in $\mathbb{R}^d$, $d \geq 1$, with Lipschitz continuous boundary $\Gamma = \partial\Omega$. We pose the Homogeneous Dirichlet Problem for the Filtration Equation in complete form, $u_t = \Delta\Phi(u) + f$. We make the following assumption on Φ, which we call the constitutive function, or, in more familiar terms, the 'nonlinearity'.

(H_Φ) *The function $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous, strictly increasing and $\Phi(\pm\infty) = \pm\infty$. We also admit the normalization $\Phi(0) = 0$.*

These assumptions will be kept throughout the chapter unless mention to the contrary. The PME and its signed counterpart are included as the special case $\Phi(s) = |s|^{m-1}s$ with $m > 1$. Note that the case $m = 1$ is also included. Relaxing the assumptions on Φ is possible with a small cost in complication that we have considered not convenient nor necessary. The possible dependence of Φ on x to account for the presence of so-called inhomogeneous media will be discussed in Section 5.11.

Problem HDP. *Given $u_0 \in L^1(\Omega)$, $f \in L^1(Q)$, find a locally integrable function $u = u(x,t)$ defined in Q_T, $T > 0$, that solves the set of equations*

$$u_t = \Delta\Phi(u) + f \quad \text{in } Q_T\,, \tag{5.1}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega, \tag{5.2}$$

$$u(x,t) = 0 \quad \text{in } \Sigma_T. \tag{5.3}$$

in a weak sense to be precisely defined. The time $T > 0$ can be finite or infinite. Moreover, we want to find u in a suitable functional class that guarantees uniqueness and continuous dependence on the data.

Though we will obtain solutions for all $T > 0$, i.e. with $T = \infty$, it is interesting for technical reasons to allow $T < \infty$.

We are going to introduce next precise definitions of what we understand by solution of Problem HDP. Since there are several options available for the concept of solution of the equation, and also for the sense in which the data are taken, it is important to carefully specify the choices we make at every instance.

5.2.1 Concepts of weak and very weak solution

First of all, we introduce a suitable concept of weak solution for the Filtration Equation in Q_T, avoiding at this moment any reference to initial or boundary data.

Definition 5.1 *A weak solution of Equation* (5.1) *in Q_T is a locally integrable function, $u \in L^1_{loc}(Q_T)$, such that*

(i) $w = \Phi(u) \in L^1_{loc}(0, T : W^{1,1}_{loc}(\Omega))$;

(ii) *the identity*

$$\iint_{Q_T} \{\nabla w \cdot \nabla \eta - u\eta_t\}\, dxdt = \iint_{Q_T} f\eta\, dxdt \tag{5.4}$$

holds for any test function $\eta \in C^1_c(Q_T)$.

Equation (5.4) is obtained by extrapolating a property of classical solutions. Indeed, if u is a smooth solution of the GPME in Q_T and we multiply the equation by η and integrate by parts we obtain (5.4). Observe that the equation is satisfied only in the sense that all these tests are true; this is called a weak sense. In particular, the definition does not require the derivatives appearing in equation (5.1) to be actual functions, they need merely exist in the sense of distributions.

Note that, in the PME the assumption $u \in L^1_{loc}(Q_T)$ is implied by the condition $\Phi(u) \in L^1_{loc}(Q_T)$, which is a part of (i). This implication is not necessarily true for more general Φ, like the FDE with $m < 1$.

The previous definition of weak solution of the GPME is not the only possibility at hand. Actually, there is a very natural alternative where the regularity assumptions are relaxed by integrating once again in space, so that no space derivatives appear in the statement.

Definition 5.2 *A very weak solution of Equation* (5.1) *in* Q_T *is a locally integrable function,* $u \in L^1_{loc}(Q_T)$*, such that* $w = \Phi(u) \in L^1_{loc}(Q_T)$*, and the identity*

$$\iint_{Q_T} \{w\, \Delta\eta + u\eta_t + f\eta\}\, dxdt = 0 \tag{5.5}$$

holds for any test function $\eta \in C^{2,1}_c(Q_T)$*.*

We can simply say that the equation is satisfied in the sense of distributions in Q_T or that it is a *distributional solution*. But note that we are asking u and $\Phi(u)$ to be integrable functions. We can also call these solutions *weak-0 solutions* to stress the fact that we do not use any derivatives of u or $\Phi(u)$ in defining them, and then the weak solutions become *weak-1 solutions*. It is clear that all weak solutions are very weak solutions according to these definitions.

There are advantages and disadvantages to both definitions. We will work in this chapter with the concept of weak solution which seems to us suitable to develop the basic theory and present the main techniques.

Remark. (1) In the definitions, we have chosen the space $L^1_{loc}(Q_T)$ as base space for the sake of generality. However, the simplicity of the most common existence and uniqueness proof recommends replacing such a space by other smaller $L^p_{loc}(Q_T)$ spaces, $1 < p \le \infty$. We recall that in the usual practice the weak solutions will be locally bounded, even continuous, either because we prove that a more general solution has this property, or because the author so assumes from the beginning.

(2) Since the weak theory is more restrictive than the very weak one, it allows for a simpler uniqueness proof. An existence theorem would be easier to prove in the more general context of very weak solutions, but we will obtain in this chapter a result on existence of weak solutions that is general enough for many purposes and allows us to develop interesting energy estimates.

On the other hand, very weak solutions allow for a more general theory that will be discussed in Section 6.2 as part of the more advanced topics. Quite strong uniqueness and comparison results will be proved.

(3) Note that we could go in the opposite direction of asking for more regularity than the theory provides; we will thus arrive at the quite useful concepts of *continuous weak solution* and *strong solution*. The former is discussed in Chapter 7 and used thereafter, while strong solutions are studied in Chapter 8; they are the preferred option in the study of the Cauchy Problem in Chapter 9. These are not the only options: *mild solutions* will appear as a consequence of the semigroup approach of Chapter 10. See related comment at the end of the chapter.

5.2.2 Definition of weak solutions for the HDP

The definition of weak solution we have proposed applies in the interior of the space-time domain (we usually say that it is a local weak solution), and does not take into account initial or boundary conditions, which are an essential part of Problem (5.1)–(5.3). Inserting the homogeneous boundary condition leads to the following standard definition.

Definition 5.3 *A measurable function u defined in Q_T is said to be a weak solution of Equation* (5.1) *with boundary condition* (5.3) *if*

(i) $u \in L^1(\Omega \times (\tau, T-\tau))$ *for all* $\tau > 0$ *and* $w = \Phi(u) \in L^1_{loc}(0, T : W_0^{1,1}(\Omega))$;

(ii) *the identity*

$$\iint_{Q_T} \{\nabla\Phi(u) \cdot \nabla\eta - u\eta_t\}\, dxdt = \iint_{Q_T} f\eta\, dxdt \tag{5.6}$$

holds for all test functions $\eta \in C^1(\overline{Q}_T)$ *which vanish on* Σ_T, *and also for* $0 \le t \le \tau$ *and for* $T - \tau \le t \le T$ *for some* $\tau > 0$.

We may wonder where is the boundary condition (5.3) included in this formulation. The answer is that it is hidden in the functional space $W_0^{1,1}(\Omega)$, a typical trick of weak theories. Caution: note the change in the class of test functions.

The final step consists of including the initial data. There are several ways of doing it. Following [408] and [457], we propose a definition of solution of the whole problem.

Definition 5.4 *A measurable function u defined in Q_T is said to be a weak solution of Problem* (5.1)–(5.3) *if*

(i) $u \in L^1(Q_T)$ *and* $w = \Phi(u) \in L^1(0, T : W_0^{1,1}(\Omega))$;

(ii) u *satisfies the identity*

$$\iint_{Q_T} \{\nabla\Phi(u) \cdot \nabla\eta - u\eta_t\}\, dxdt = \int_{\Omega} u_0(x)\eta(x,0)dx + \iint_{Q_T} f\eta\, dxdt \tag{5.7}$$

for any function $\eta \in C^1(\overline{Q}_T)$ *which vanishes on* Σ *and for* $t = T$.

We call $\mathcal{WS}$ the class of functions thus obtained when $u_0 \in L^1(\Omega)$ and $f \in L^1(Q_T)$. The initial function u_0 of condition (5.2) is built into the integral formulation (5.7), and is actually satisfied in a very weak sense. The function should be selected so that the integral involving u_0 in (5.7) be well defined. The natural option in the present setting is asking that $u_0 \in L^1(\Omega)$, which contains all spaces $L^p(\Omega)$ for $p > 1$.

Note that the space of test functions can be modified in several ways without modifying the defined class of weak functions. This remark will be of much use. Thus, we may replace the condition $\eta \in C^1(\overline{Q}_T)$ by $\eta \in C^\infty(\overline{Q}_T)$ which reduces in principle the amount of tests. But the full force of condition (ii) is recovered by approximation. In the other direction, we may enlarge to set of test functions the $\eta \in W^{1,\infty}(Q_T)$ as long as we may approximate it with functions η_ε in the class stated in the definition. This technically means that the trace of η on the parabolic boundary of Q_T has to be zero.

As in the case of weak solutions, the definition of very weak solution can be made precise to include initial and boundary data. See Section 6.2.

5.2.3 About the initial data

The inclusion of the initial data into the definition of weak solution is not the only natural option. As an indication of the scope of the above definition and its alternatives, we propose another natural way of defining a weak solution, and show that it is included in Definition 5.4.

Proposition 5.1 *Let $u \in L^1(Q_T)$ be such that*
(i) $\Phi(u) \in L^1(0, T : W_0^{1,1}(\Omega))$,
(ii) *For any function $\eta \in C_c^\infty(Q_T)$, u satisfies the identity*

$$\iint_{Q_T} \{\nabla\Phi(u)\cdot\nabla\eta - u\eta_t\}dxdt = \iint_{Q_T} f\eta\, dxdt; \tag{5.8}$$

(iii) *for every $t > 0$ we have $u(t) \in L^1(\Omega)$ and $u(t) \to u_0$ as $t \to 0$ in $L^1(\Omega)$.*
Then, u is a weak solution to Problem (5.1)–(5.3) *according to Definition* 5.4.

Proof. Suppose that u is as in the statement. We have to prove that (5.7) holds. Let η be as in (5.7) but we also assume that it vanishes in a neighbourhood of Σ. We take a cut-off function $\zeta \in C^\infty(\mathbb{R})$, $0 \le \zeta \le 1$, such that $\zeta(t) = 0$ for $t < 0$, $\zeta(t) = 1$ for $t \ge 1$ and $\zeta' \ge 0$, and let $\zeta_n(t) = \zeta(nt)$. Applying (5.8) with test function $\eta(x,t)\zeta_n(t)$ gives

$$\begin{aligned}&\iint_{Q} \{\nabla\Phi(u)\cdot\nabla\eta - u\eta_t\}\zeta_n - \iint_{Q_T} f\eta\zeta_n = \iint_{Q} u\eta\zeta_{n,t} = \\ &= \iint_{Q_{1/n}} u\eta\zeta_{n,t} = \iint_{Q_{1/n}} (u-u_0)\eta\zeta_{n,t} + \iint_{Q_{1/n}} u_0(x)\eta(x,t)\zeta_{n,t}(t)\,.\end{aligned} \tag{5.9}$$

Fix $\varepsilon > 0$ and let n be so large that $\|u - u_0\|_1 \le \varepsilon$ for $0 \le t \le 1/n$. Then the first integral in the end expression can be estimated as $\varepsilon\|\eta\|_\infty \int \zeta_{n,t}dt = \varepsilon\|\eta\|_\infty$, which vanishes as $n \to \infty$, $\varepsilon \to 0$. As for the last term, we get

$$\iint_{Q_{1/n}} u_0(x)\eta(x,t)\zeta_{n,t}(t)dxdt = \int_\Omega u_0(x)\eta(x,\frac{1}{n})dx - \iint_{Q_{1/n}} u_0\eta_t\zeta_n dxdt,$$

and this tends to $\int_\Omega u_0(x)\eta(x,0)dx$ as $n \to \infty$, which proves (5.7) in this case.

It is very easy to see that (5.7) continues to hold when $\eta \in C^1(Q_T)$ with $\eta = 0$ on the boundary of Q_T (Hint: approximate η with $\eta_\varepsilon \in C_c^\infty$ and pass to the limit). □

Actually, the definition of weak solution implies convergence to the initial data in a weaker sense. We ask the reader to prove the following statement (see Problem 5.2).

Proposition 5.2 *If u is a weak solution of* HDP *in the sense of Definition* 5.4 *then $u(t)$ converges to u_0 weakly in the sense that for every $\varphi \in C^1(\Omega)$ with $\varphi = 0$ on $\partial\Omega$ we have*

$$\lim_{t\to 0} \int_\Omega u(t)\varphi\, dx = \int_\Omega u_0 \varphi\, dx. \tag{5.10}$$

Such a poor convergence is immediate to obtain but not realistic. We will show below that the energy solutions constructed in this chapter, Sections 5.4 and 5.5, take the initial data in a much nicer way, indeed in the sense of strong convergence in $L^1(\Omega)$. Actually, this will apply to all limit solutions, as reflected in Theorem 6.2.

5.2.4 Examples of weak solutions for the PME

1. *Compatibility. Classical implies weak.* Every classical solution of Problem (5.1)–(5.3) is automatically a weak solution of the problem. This is the required property of agreement between classical and weak concepts, a must for every reasonable generalized solution.

2. We continue with less trivial examples for the PME with $f = 0$. One of them is the separate-variables solution

$$u(x,t;C) = T(t)F(x), \tag{5.11}$$

where $T(t) = (C + (1-m)t)^{1/(m-1)}$ and $F > 0$ is the solution of a certain nonlinear elliptic equation that vanishes on the boundary, cf. Section 4.2. Since F is $C^\alpha(\overline{\Omega})$ and C^∞ in Ω, and $F^m \in C^1(\overline{\Omega})$, it is clear that for every $C > 0$ this is a weak solution of the problem. Now, for $C = 0$ we obtain a limit solution that is perfect for $t > 0$, but takes on infinite values at $t = 0$ in the sense that

$$\lim_{t\to 0} u(x,t;0) = \infty \quad \forall x \in \Omega. \tag{5.12}$$

We thus find a kind of giant solution that is infinite everywhere at $t = 0$. But since it becomes bounded and smooth for $t > 0$, it is rather a friendly giant. We refer to Section 5.9 for a detailed construction of this special solution. Separate-variables solutions with changing sign can also be constructed, cf. [511].

3. Another nontrivial solution of the PME with $f = 0$ is the explicit source-type solution $\mathcal{U}(x,t) = \mathcal{U}(x,t;C)$ of Section 4.4. This is not a weak solution according to our definition because of two reasons: its initial data are singular, and the boundary data are not necessarily 0. However, we can obtain from it weak solutions in our setting by the following method: take $x_0 \in \Omega$, let $\tau > 0$ and let the constant C in $\mathcal{U}$ be small enough. Then the function

$$w(x,t) = \mathcal{U}(x - x_0, t+\tau; C) \tag{5.13}$$

is a weak solution of the Dirichlet Problem (5.1)–(5.3) in any time interval $(0, T)$ in which the free boundary lies inside of Ω, i. e., if

$$T + \tau \le c\,\mathrm{dist}(x_0, \partial\Omega)^{d(m-1)+2},$$

cf. (1.10). Observe that w is a weak solution but not a classical solution, which shows that the weak theory is a nontrivial extension of the classical theory. See Problem 5.3.

4. The Dipole solution $U_{dip}(x,t)$ given by formula (4.53) is a nonnegative solution of the PME in any cylinder of the form $Q_1 = (0,R) \times (0,T)$, hence $d = 1$, as long as the free boundary does not reach the fixed boundary $x = R$. If we want integrable initial data we have to insert a time delay and replace it by $U_{dip}(x, t+\tau)$.

When posed in the symmetric cylinder $Q_2 = (-R,R) \times (0,T)$, it is a signed solution of the signed PME under the same conditions. The change of sign takes place at $x = 0$, where we see that the solution is not C^1; to be precise, it is $C^{1/m}$ in x. This solution also shows that initial data more general than measures can occur in the theory with changing sign.

5.3 Uniqueness of weak solutions

The goal of the theory is to establish existence, uniqueness and other important properties of weak solutions of Problem (5.1)–(5.3). This will be done in the present chapter for the class of weak solutions under a small additional restriction. Moreover, the uniqueness of weak solutions is settled by means of an interesting and easy proof, based on using a quite specific test function.

Theorem 5.3 *Under the additional assumption that* $\Phi(u) \in L^2(0,T : H_0^1(\Omega))$ *and* $u \in L^2(Q_T)$*, Problem* (5.1)–(5.3) *has at most one weak solution.*

Proof. Suppose that we have two such solutions u_1 and u_2. We write $w_i = \nabla\Phi(u_i)$. By (5.7) we have

$$\iint_{Q_T} (\nabla(w_1 - w_2) \cdot \nabla\eta - (u_1 - u_2)\eta_t)\,dxdt = 0 \tag{5.14}$$

for all test functions η. We want to use as a test function the one introduced by Oleĭnik,

$$\eta(x,t) = \int_t^T (w_1(x,s) - w_2(x,s))\,ds \quad \text{if } 0 < t < T. \tag{5.15}$$

Even if η does not have the required smoothness, we may approximate it with smooth functions η_ε for which (5.7) will hold with these test functions. Since

$$\begin{cases} \eta_t &= -(w_1 - w_2) \in L^2(Q_T), \\ \nabla\eta &= \int_t^T (\nabla w_1 - \nabla w_2)\,ds \in L^2(Q_T), \end{cases} \tag{5.16}$$

and moreover $\eta(t) \in H_0^1(\Omega)$ and $\eta(T) = 0$, we may pass to the limit $\varepsilon \to 0$ and (5.7) will still hold for η. Hence,

$$\iint_{Q_T} (w_1 - w_2)(u_1 - u_2)\, dxdt +$$
$$\iint_{Q_T} (\nabla(w_1 - w_2)) \cdot (\int_t^T (\nabla w_1 - \nabla w_2)\, ds)\, dxdt = 0.$$

Integration of the last term gives

$$\iint_{Q_T} (w_1 - w_2)(u_1 - u_2)\, dx\, dt + \frac{1}{2} \int_\Omega \{\int_0^T (\nabla w_1 - \nabla w_2)\, ds\}^2\, dx = 0\,.$$

Since both terms are nonnegative, we conclude that $u_1 = u_2$ a.e. in Q. □

Remark on the approximation. Given a function $\eta \in L^2(0, T : H_0^1(\Omega))$, we first cut it at height n in the form

$$\eta_n = \max\{-n, \min\{n, \eta\}\}$$

to obtain a sequence of bounded functions $\eta_n \to \eta$ in $L^2(0, T : H_0^1(\Omega))$. In a second step, every η_n is approximated by functions $\eta_{\varepsilon,n}$ as in Definition 5.4.

5.3.1 Non-existence of classical solutions

As a consequence of the uniqueness of weak solutions and the constructed examples, we have the following:

Corollary 5.4 *There exist initial data for which Problem* HDP *for the PME does not admit a classical solution, even if u_0 is nonnegative and $f = 0$.*

Proof. This is a rather standard argument. Firstly, we note that a classical solution of Problem (5.1)–(5.3) is necessarily a weak solution in our sense. Secondly, we remark that the particular example of weak solution $w(x, t)$ defined in (5.13) has the regularity of Theorem 5.3 and is not a classical solution. By the uniqueness result, there cannot be any other weak solution of (5.1)–(5.3) with the same data. Therefore, no classical solution exists for those data. □

Remark. Such argument will apply to all the unique weak non-classical solutions that will be constructed in the sequel. Moreover, in later chapters we will have the opportunity of finding weak solutions of the PME corresponding to smooth initial data that cease to be classical after some time. One such example is presented in Problem 5.7. However, if the data are positive, then we will prove below that the solution stays classical.

5.3.2 The subclass of energy solutions

Uniqueness has been proved in a subclass of $\mathcal{WS}$ formed by the weak solutions such that $u \in L^2(Q_T)$ and $\Phi(u) \in L^2(0, T : H_0^1(\Omega))$. The extra regularity allows to define the dissipated energy

$$DE(u) = \int_0^T \int_\Omega |\nabla\Phi(u)|^2 \, dxdt \tag{5.17}$$

and try to reproduce in this general context the energy calculation of Subsection 3.2.4. That estimate leads us to consider the expression

$$E(u) = E_u(t) := \int_\Omega \Psi(u(t)) \, dx \tag{5.18}$$

(Ψ is defined in (5.19)) as a natural energy for the evolution. In this chapter, solutions will be constructed in the class of square integrable functions such that $DE(u)$ and E_u are finite. We will refer to this class as *weak energy solutions*, $\mathcal{WES}$.

5.4 Existence of weak energy solutions for general Φ. Case of nonnegative data

We address in this section the existence of nonnegative solutions u with nonnegative data, u_0 and f. This is the most typical problem that is solved for the PME. Though the results are superseded by the construction of Section 5.5 for data of any sign, the present construction uses less functional machinery and is the one currently found in the literature. The technique that we are going to use allows to cover the following generality:

- $\Phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous and strictly increasing in u with $\Phi(0+) = 0$.
- $\Phi(u)$ is smooth with $\Phi'(u) > 0$ for $u > 0$.

We want to construct solutions enjoying an energy estimate as discussed in Subsection 3.2.4. Such an estimate is essential in the existence proof that we give. We need to recall the function Ψ, the primitive of Φ defined as in (3.18):

$$\Psi(s) = \int_0^s \Phi(s) \, ds \, . \tag{5.19}$$

Concerning the initial data, such setting leads to assume that u_0 is a measurable function such that $\Psi(u_0(x)) \in L^1(\Omega)$. We call this space $X = L_\Psi(\Omega)$. It is a subspace of $L^1(\Omega)$. Note that for the PME, $X = L^{m+1}(\Omega)$. Concerning the forcing term f we need the expression $\iint f\Phi(u) \, dxdt$ to make sense. This leads us to ask f to belong to the dual space Y of the space $L^2(0, T : H_0^1(\Omega))$ where $\Phi(u)$ lies. Since our interest in f is minor (at least at this point), we will assume that f is bounded for simplicity.

This is the existence and comparison result for weak solutions that we prove at this stage.

Theorem 5.5 *Under the above assumptions on Φ, there exists a weak solution of Problem* (5.1)–(5.3) *with initial data $u_0 \in L^1(\Omega)$, $\Psi(u_0) \in L^1(\Omega)$, $u_0 \geq 0$, and forcing term $f \geq 0$, f bounded, where solution is understood in the weak sense of Definition* 5.4. *This solution is nonnegative and the time interval is unbounded ($T = \infty$).*

We have $\Psi(u) \in L^\infty(0, T : L^1(\Omega))$ for all $T > 0$ and $\Phi(u) \in L^2(0, T : H_0^1(\Omega))$. An energy inequality is satisfied

$$\iint_{Q_T} |\nabla\Phi(u)|^2 \, dxdt + \int_\Omega \Psi(u(x, T)) \, dx \leq \int_\Omega \Psi(u_0(x)) \, dx + \iint_{Q_T} f\Phi(u) \, dxdt. \tag{5.20}$$

It is therefore a weak energy solution. The Comparison Principle holds for these solutions: if $u, \widehat{u}$ are weak solutions with initial data such that $u_0 \le \widehat{u}_0$ a.e. in Ω and $f \le \widehat{f}$ a.e. in Q, then $u \le \widehat{u}$ a.e. in Q. In particular, if $u_0, f \ge 0$ in Ω, then $u \ge 0$ in Q.

Remark. The Comparison Principle is mentioned in the result because it is a basic property to be expected in a parabolic equation, linear or nonlinear, degenerate or not.

Proof. It will be divided into several steps. Firstly, we will consider the case of smooth functions u_0 and f and prove the existence result by approximation, compactness and monotone limit.

First step: We assume that $\Gamma = \partial\Omega \in C^{2+\alpha}$, that u_0 is a nonnegative and $C^2(\Omega)$ function with compact support in Ω, and $f \ge 0$ is continuous and bounded in $\overline{Q}$.

We begin by constructing a sequence of approximate initial data u_{0n} which do not take the value $u = 0$, so as to avoid the degeneracy of the equation. That allows us to use the results of Section 3.2. We may simply put

$$u_{0n}(x) = u_0(x) + \frac{1}{n}\,. \tag{5.21}$$

Let $M = \sup(u_0)$ and $N = \sup_Q f$. We also approximate f by a sequence of smooth functions f_n in a monotone decreasing way, keeping the bound $0 \le f_n \le N_n = N + 1/n$. We now solve the problem

$$(u_n)_t = \Delta\Phi(u_n) + f_n \quad \text{in} \quad Q, \tag{5.22}$$

$$u_n(x,0) = u_{0n}(x) \quad \text{in} \quad \overline{\Omega}, \tag{5.23}$$

$$u_n(x,t) = 1/n \quad \text{on} \quad \Sigma. \tag{5.24}$$

The Maximum Principle, which holds for classical solutions, implies that

$$\frac{1}{n} \le u_n(x,t) \le M + \frac{1}{n} + N_n t \quad \text{in} \quad \overline{Q}. \tag{5.25}$$

Therefore, we are dealing in practice with a uniformly parabolic problem. Actually, problem (5.22)–(5.24) has a unique solution $u_n \in C^{2,1}_{x,t}(\overline{Q})$. The rigorous justification uses the already mentioned trick consisting of replacing equation (5.22) by

$$(u_n)_t = \operatorname{div}(a_n(u_n)\,\nabla u_n) + f_n, \tag{5.26}$$

where $a_n(u)$ is a positive and smooth function, $a_n(u) \ge c > 0$, and $a_n(u) = \Phi'(u)$ in the interval $[1/n,\, M + 1/n + NT]$. This equation is not degenerate and a unique solution u_n of (5.26), (5.23), (5.24) exists in the space $C^{2,1}_{x,t}(\overline{Q})$ by the standard quasilinear theory of Chapter 3, and it satisfies (5.25). Moreover, by repeated differentiation and interior regularity results for parabolic equations, we are able to conclude that $u_n \in C^\infty(Q)$. Now, due to the definition of a_n, equations (5.22) and (5.26) coincide on the range of u_n. In this way, Problem (5.22)–(5.24) is solved in a classical sense and the degeneracy of the equation is avoided.

Moreover, again by the Maximum Principle

$$u_{n+1}(x,t) \le u_n(x,t) \quad \text{in} \quad \overline{Q} \tag{5.27}$$

for all $n \geq 1$. Hence, we may define the function

$$u(x,t) = \lim_{n\to\infty} u_n(x,t), \qquad (x,t) \in \overline{Q}. \tag{5.28}$$

as a monotone limit of bounded nonnegative functions. We see that u_n converges to u in $L^p(Q_T)$ for every $1 \leq p < \infty$. In order to show that this u is the weak solution of Problem (5.1)–(5.3), we need to estimate the spatial gradient of $\Phi(u_n)$. First of all, from (5.25) we get

$$0 \leq u(x,t) \leq M + Nt \quad \text{ in } \ \overline{Q}.$$

We control $\nabla\Phi(u_n)$ as in the energy identity of Subsection 3.2.4. Since $u_n = 1/n$ on the lateral boundary, we have to multiply equation (5.22) by $\eta_n = \Phi(u_n) - \Phi(1/n)$. Integrating by parts in Q_T, we obtain

$$\begin{aligned}
&\iint\limits_{Q_T} |\nabla\Phi(u_n)|^2 dxdt = \int\limits_\Omega \{\Psi(u_{0n}(x)) - \Phi(1/n)u_{0n}(x)\}\, dx \\
&- \int\limits_\Omega \{\Psi(u_n(x,T)) - \Phi(1/n)u_n(x,T)\}\, dx + \iint\limits_{Q_T} f_n(\Phi(u_n) - \Phi(1/n))\, dxdt \\
&\leq \int\limits_\Omega \Psi(u_0(x))\, dx + \int\limits_\Omega \Phi(1/n)\, u_{0n}(x)\, dx + \iint\limits_{Q_T} f_n\Phi(u_n)\, dxdt.
\end{aligned} \tag{5.29}$$

We may use the boundedness of f in L^2 and the Poincaré inequality on the last term to estimate it in the form:

$$\iint\limits_{Q_T} f_n\Phi(u_n)\, dxdt \leq C \iint\limits_{Q_T} f_n^2\, dxdt + C\iint\limits_\Sigma \Phi(1/n)^2\, dS + \frac{1}{2}\iint\limits_{Q_T} |\nabla\Phi(u_n)|^2\, dxdt.$$

We can absorb the influence of the last term into the first term of the left-hand side of (5.29), and the other two terms in this last formula are bounded. Then, since T is arbitrary, it follows that $\{\nabla\Phi(u_n)\}$ is uniformly bounded in $L^2(Q)$, and therefore a subsequence of it converges to some limit ψ weakly in $L^2(Q)$. Since also $\Phi(u_n) \to \Phi(u)$ everywhere, it follows that $\psi = \nabla\Phi(u)$ in the sense of distributions. The limit is uniquely defined so that the whole sequence must converge to it. Passing to the limit in (5.29), we get the energy identity transformed into the energy inequality (5.20).

On the other hand, since $u_n \in C(\overline{Q})$, $u_n(x,t) = 1/n$ on Σ and $0 \leq u \leq u_n$, we have

$$\lim_{(x,t)\to\Sigma} u(x,t) = 0$$

with uniform convergence. Hence $\Phi(u(\cdot,t)) \in H_0^1(\Omega)$ for a.e. $t > 0$.

Finally, since u_n is a classical solution of (5.1), it clearly satisfies (5.7) with u_0 replaced by u_{0n}. Letting $n \to \infty$ we obtain (5.7) for u. Therefore, u is a weak solution of (5.1)–(5.3).

Let us remark, to end this step, that if we have data (u_0, f) and $(\hat{u}_0, \widehat{f})$ such that $u_0 \leq \hat{u}_0$ and $f \leq \widehat{f}$, then the above approximation process produces ordered approximating sequences, $u_{0n} \leq \hat{u}_{0n}$. By the classical Maximum Principle, we have $u_n \leq \hat{u}_n$ for every $n \geq 1$. In the limit, $u \leq \hat{u}$.

Second step: We assume that u_0 is bounded and vanishes near the boundary, and f is bounded and nonnegative.

The method of the previous step can still be applied, but now f is approximated by a sequence of smooth functions f_n that converge to f a.e. According to the quasilinear theory, cf. [357], now the approximate solutions $u_n \in C^\infty(Q)\cap C^{2,1}(Q\cup\Sigma)$ are not continuous down to $t=0$ unless the data are; instead, they take the initial data in $L^p(\Omega)$ for every $p<\infty$. Passage to the limit in u_n is now based not on monotonicity, but on the L^1 dependence of the solutions on the data which is described in Subsection 3.2.3: it follows that $u_n \to u$ in $C([0,T) : L^1(\Omega))$. Since the functions are bounded, convergence also takes place in $C([0,T) : L^p(\Omega))$ for all $p<\infty$. The convergence of the gradients is unchanged and the proof ends as before. Comparison still applies.

Third step. General case.

For general Γ and general data, $\Psi(u_0)\in L^1(\Omega)$, $u_0\ge 0$, we first approximate the domain by an increasing family Ω_k of domains with $C^{2+\alpha}$-boundary Γ_k, we take an increasing sequence of cutoff functions $\zeta_k(x)$ which vanish near Γ_k, and consider the sequence of approximations of the initial data

$$u_{0k}(x) = \min\{u_0(x)\zeta_k(x), k\}\,. \tag{5.30}$$

Using Step 2 we solve Problem (5.1)–(5.3) with initial data u_{0k} and forcing term $f_k = f\zeta_k$ to obtain a unique weak solution u_k defined in $Q_k=\Omega_k\times(0,T)$. By the comparison remark, $u_{k+1}\ge u_k$ in Q_k (Note that $u_{k+1}\ge 0$ on Σ_k). On the other hand, by estimate (5.20) the family $\{\Psi(u_k)\}$ is uniformly bounded in $L^\infty(0,\infty : L^1(\Omega_k))$ and $\nabla\Phi(u_k)$ is likewise in $L^2(Q_k)$. Hence, extending u_k by 0 in $\Omega\setminus\Omega_k$, we have that $\{u_k\}$ converges a.e. to a function $u\in L^\infty(0,\infty : L_\Psi(\Omega))$. On the other hand, $\nabla\Phi(u_k)$ is uniformly bounded, hence it converges weakly in $L^2(Q)$ to $\nabla\Phi(u)$, and (5.20) holds for u. It follows that $\Phi(u)\in L^2(0,\infty : H_0^1(\Omega))$. Finally, equation (5.7) is satisfied, as the reader may easily check passing to the limit in the similar expressions for u_k. □

Motivation for the initial data. We see from the proof that the choice of space for the initial data depends essentially on the energy estimate (5.20), which is a cornerstone of this chapter. A priori estimates are one of the most powerful and widely used tools in the study of PDE. This approach will be stressed in our treatment of the existence, uniqueness and qualitative properties of solutions to the different problems.

5.4.1 Improvement of the assumption on f

The approximation proof used above can be performed under weaker assumptions on the forcing term:

Corollary 5.6 *The result of Theorem* 5.5 *on existence of weak energy solutions holds if* $f\ge 0$, $f\in L^p(Q)$, *with* $p=2d/(d+2)$ *if* $d\ge 3$; *for some* $p>1$ *if* $d=1,2$.

Actually, the technique of passage to the limit in the L^1 norm allows to obtain a limit $u=\lim u_n(x,t)$ even if f is not an L^2 function, as long as $f\in L^1(Q_T)$. However, if we want to obtain a weak solution in the sense of Definition 5.4 we need to keep a control for $\nabla\Phi(u)$ in $L^2(Q_T)$. In view of the energy estimate and Sobolev's Embedding Theorem, this is possible under the stated assumptions on f. These assumptions guarantee precisely that the energy estimate still holds with finite terms.

5.4.2 Nonpositive solutions

The results of this section not only show the existence of weak solutions with nonnegative data such that $\Psi(u_0) \in L^1(\Omega)$ and $f \geq 0$, but also the same problem with nonpositive data $u_0 \leq 0$ in the same integrable class, and $f \leq 0$, under a similar assumption of regularity of $\Phi(s)$ for $s < 0$. Indeed, the filtration equation is invariant under the symmetry $u \mapsto \widehat{u} = -u$, if we change the nonlinearity Φ into $\widehat{\Phi}(s) = -\Phi(-s)$. It is quite easy to check that, if a function u is a weak solution of Problem HDP with initial data u_0, f, and nonlinearity Φ, then $\widehat{u}(x,t) = -u(x,t)$ is a weak solution with data $\widehat{u}_0(x) = -u_0(x)$, $\widehat{f}(x,t) = -f(x,t)$ and nonlinearity $\widehat{\Phi}(s) = -\Phi(-s)$.

5.5 Existence of weak signed solutions

We address in this section the main problem of existence of signed solutions for the complete GPME. As in the previous section, our goal is to obtain weak energy solutions, which forces us to impose some conditions on the data. Such restrictions will eliminated in the following chapters by different techniques and with different concepts of solution. We assume that Φ satisfies the conditions (H_Φ) stated at the beginning of Section 5.2.

Theorem 5.7 *Assume that $u_0 \in L_\Psi(\Omega)$ and $f \in L^p(Q)$, with $p = 2d/(d+2)$ if $d \geq 3$ ($f \in L^p(Q)$ for some $p > 1$ if $d = 1, 2$). Then, Problem* (5.1)–(5.3) *has a weak solution defined in an infinite time interval, $T = \infty$. We have $u \in L^\infty(0, T : L_\Psi(\Omega))$ and $\Phi(u) \in L^2(0, T : H_0^1(\Omega))$, and the energy inequality* (5.20) *holds. Comparison holds as in Theorem* 5.5.

Proof. In our situation we cannot simply modify the initial data to obtain a problem with a classical solution, as we did in Theorem 5.5, since such approximations will necessarily be of changing sign, and the equation may degenerate (in the key case of the PME, it does so at the level $u = 0$, and that level cannot be avoided for solutions that change sign). Therefore, we also modify the equation into a non-degenerate parabolic equation by changing the nonlinearity Φ in the following form. We pick a sequence of functions Φ_n such that

(a) $\Phi_n \in C^\infty(\mathbb{R})$ and $\Phi_n'(u) > 0$ for every $n \geq 1$ and every $u \in \mathbb{R}$;

(b) $\Phi_n \to \Phi$ uniformly on compact sets.

(c) $\Phi_n(0) = 0$ for every $x \in \Omega$.

Lemma 5.8 *The result holds when u_0, f and f_t are bounded, $\Gamma = \partial\Omega \in C^{2+\alpha}$, and Φ is locally Lipschitz continuous in u.*

Proof. (i) We fix $T > 0$ and consider the approximate equations

$$u_t = \Delta\Phi_n(u) + f_n \quad \text{in} \quad Q_T\,, \tag{5.31}$$

where f_n is a smooth approximation converging to f in $L^p(Q_T)$. We solve the problem formed by (5.31) with initial data

$$u_n(x,0) = u_{0n}(x) \quad \text{ in } \overline{\Omega}, \tag{5.32}$$

where $u_{0n} \in C_c^\infty(\overline{\Omega})$ approximates u_0 in $L_\Psi(\Omega)$; it will be convenient to ask that $|u_{0n}(x)| \le n$. We also impose boundary data

$$u_n(x,t) = 0 \quad \text{on } \Sigma. \tag{5.33}$$

Since $\Gamma \in C^{2+\alpha}$, problem (5.31)–(5.33) has a unique solution $u_n \in C^{2,1}(\overline{Q}) \cap C^\infty(Q)$, cf. [357, Theorem 6, p. 452]. Moreover, if $M_1 = \sup(-u_0)$, $M_2 = \sup u_0$, $N_1 = \sup(-f)$, and $N_2 = \sup f$, we get by the standard maximum principle

$$-M_1 - N_1 t \le u_n(x,t) \le M_2 + N_2 t \quad \text{in } \overline{Q}. \tag{5.34}$$

(ii) Let $w_n = \Phi_n(u_n)$. We want to control the spatial derivative ∇w_n uniformly in n in terms of the initial data. As we know, the idea is to multiply the equation by w_n and integrate by parts in Q_τ to obtain for every $T > 0$ an energy estimate of the form

$$\int_\Omega \Psi_n(u_n(x,T))\,dx + \iint_{Q_T} |\nabla w_n|^2\,dxdt = \int_\Omega \Psi_n(u_{0n}(x))\,dx + \iint_{Q_T} f_n w_n\,dxdt\,, \tag{5.35}$$

where Ψ_n is the primitive of Φ_n with $\Psi_n(0) = 0$. Arguing as in (5.29), we conclude that the integral $\iint_Q |\nabla w_n|^2\,dxdt$ is bounded independently of n.

(iii) We produce next a compactness estimate in time as indicated in formula (3.27) for smooth solutions. The idea is to multiply the equation by $\zeta(t) w_{n,t}$, with $w_n = \Phi_n(u_n)$, $\zeta(t)$ a cutoff function, and integrate by parts in space to obtain the expression

$$\iint_{Q_T} \zeta\, \Phi_n'(u_n)|u_{n,t}|^2\,dxdt = \iint_{Q_T} \left\{ \frac{\zeta_t}{2}|\nabla \Phi_n(u_n)|^2 - (f\zeta)_t \Phi_n(u_n) \right\} dxdt.$$

We conclude from this, the previous energy estimate, and the assumption on f, that for every $\tau > 0$ the integral $\int_\tau^\infty \int \Phi_n'(u_n)|(u_n)_t|^2\,dxdt$ is uniformly bounded.

(iv) Under our assumptions, the u_n are uniformly bounded by some C_1 in Q_T, and Φ is locally Lipschitz continuous, so that $\Phi_n'(s) \le C$ for all n and for $|s| \le C_1$. Since $|(w_n)_t|^2 = (\Phi_n'(u_n)(u_n)_t)^2$, we conclude that $(w_n)_t \in L^2(\Omega \times (\tau,\infty))$ with bound independent of n.

The two previous estimates imply that the sequence $\{w_n\}$ is bounded in $H^1(Q^*)$, with $Q^* = \Omega \times (\tau, T)$. This allows us to pass to the limit $n \to \infty$ along a subsequence $\{n_j\}$ to obtain a function

$$w(x,t) = \lim_{j\to\infty} w_{n_j}(x,t), \tag{5.36}$$

and $w \in L^2(Q^*)$. Choosing a subsequence, the convergence $w_{n_j} \to w$ takes place almost everywhere. It is also clear that $w \in L^2(0,\infty : H_0^1(\Omega))$.

It is straightforward to check that, under these circumstances, the uniformly bounded sequence $\{u_{n_j}\}$ also converges to a bounded function u a.e. and that $w = \Phi(u)$ a.e. Moreover, we have

$$\iint_{Q_T} |\nabla w|^2\,dxdt \le \int_\Omega \Psi(u_0(x))\,dx + \iint f\,w\,dxdt\,. \tag{5.37}$$

By virtue of Estimate (3.29) applied to the approximations, after passing to the limit we also have $t^{1/2}w \in L^\infty(0,\infty : H_0^1(\Omega))$ and

$$\frac{T}{2}\int_\Omega |\nabla w(x,T)|^2\,dx \le \iint_{Q_T} |\nabla w|^2\,dxdt + C(T)\,. \tag{5.38}$$

No major difficulty arises in checking that u satisfies the conditions of Definition 5.4, so that it is a weak solution of the problem. By uniqueness, we conclude that the whole sequence u_n converges to u, the unique solution of the problem. Estimate (5.35) becomes in the limit

$$\int_\Omega \Psi(u(T))\,dx + \iint_{Q_T} |\nabla\Phi(u)|^2\,dxdt \le \int_\Omega \Psi(u_0)\,dx + \iint_{Q_T} f\Phi(u)\,dxdt\,, \tag{5.39}$$

while the time derivative estimate can be written as

$$\iint_{Q_T} \zeta\,(\mathcal{Z}(u)_t)^2\,dxdt = \iint_{Q_T} \left\{\frac{\zeta_t}{2}|\nabla\Phi(u)|^2 - (f\zeta)_t\Phi(u)\right\}\,dxdt, \tag{5.40}$$

with $\mathcal{Z}$ as in (3.25), or in the alternative form

$$\begin{aligned}\iint_{Q_T} t\Phi'(u)|u_t|^2\,dxdt &+ \frac{T}{2}\int_\Omega |\nabla w(T)|^2\,dx \\ &\le \frac{1}{2}\iint_{Q_T} |\nabla w|^2\,dxdt + \iint_{Q_T} tfw_t\,dxdt.\end{aligned} \tag{5.41}$$

We also observe that in the limit of the approximations,

$$-M_1 - N_1 t \le u(x,t) \le \sup M_2 + N_2 t\,. \tag{5.42}$$

Let us recall at this stage that the Maximum Principle holds for the approximate problems. If we have two initial data $u_0, \hat{u}_0$ such that $u_0 \le \hat{u}_0$, $f \le \widehat{f}$, then the above approximation process can be performed so as to produce ordered approximating sequences, $u_n \le \hat{u}_n$. In the limit, $u \le \hat{u}$. Therefore, the proof is complete in this case. □

Lemma 5.9 *The result also holds when the previous condition on Φ is eliminated.*

Proof. We have to tackle now the case where Φ is not Lipschitz continuous, for instance in the case of the FDE ($0 < m < 1$). In that case we cannot conclude that w_t is bounded in some space and we need a slight modification in the passage to the limit of the previous step to arrive at the desired conclusion also in this case. Here is a way: we introduce the nondecreasing function $Z(s)$ defined by the differential rule, $dZ = \min\{ds, d\Phi(s)\}$, and its approximations

$$Z_n(s) = \int_0^s \min\{1, \Phi_n'(s)\}\,ds.$$

Clearly, the Z_n are strictly increasing functions, uniformly Lipschitz continuous, we have $|Z_n(s)| \le |s|$, $|Z_n(s)| \le |\Phi_n(s)|$, and finally $Z_n(s) \to Z(s)$ locally uniformly in $\mathbb{R}$.

We then define $z_n(x,t) = Z_n(u_n(x,t))$, and immediately see that the sequence $z_n(x,t)$ is uniformly bounded in Q_T. Moreover, from steps (ii) and (iii) we conclude that $(z_n)_t$

and ∇z_n are uniformly bounded in $L^2(Q^*)$. Therefore, after passing to a subsequence, z_n converges in $L^2(Q^*)$ and a.e. to a bounded function z. It is then easy to conclude that also $w_n \to w$, $u_n \to u$ weakly and a.e. and that $w = \Phi(u)$, since both are related to z by continuous and increasing functions. See Problem 5.4. The rest is similar to Step (iv). □

Lemma 5.10 *The result also holds when the initial data u_0 is not bounded and/or f and f_t are not bounded.*

Proof. We use approximations of these functions by functions u_{0n}, f_n as in the lemma, and such that: u_{0n} is uniformly bounded in $L_\Psi(\Omega)$ and $u_{0n} \to u_0$ in $L^1(\Omega)$; f_n is uniformly bounded in $L^p(Q_T)$ and $f_n \to f$ in $L^1(Q_T)$. Using the L^1 stability result, Proposition 3.5, we conclude that u_n converges in $L^\infty(0,T : L^1(\Omega))$ towards a function u. By the a priori estimates, $u \in L^\infty(0,T : L_\Psi(\Omega))$. Moreover, the energy argument used above implies that w_n converges weakly to some $w \in L^2(Q_T)$ with ∇w_n converging in the same way to ∇w. We also have $w = \Phi(u)$ a. e. In the limit of the weak formulation satisfied by u_n, we conclude that u is a weak solution of the problem. □

End of Proof of the Theorem. We still need to consider the case where Γ is not $C^{2+\alpha}$ smooth. As in the end of proof of Theorem 5.5, we approximate Ω by an increasing sequence Ω_k of domains strictly contained in Ω and having $C^{2+\alpha}$-boundary Γ_k, we take an increasing sequence of cutoff functions $\zeta_k(x)$ supported in Ω_k, and define $u_{0k} = u_0\zeta_k$, $f_k = f\zeta_k$. Then, solving the problems with these data in $Q_k = \Omega_k \times (0,T)$, and extending u_k by 0 in $\Omega \setminus \Omega_k$, the uniform estimates give boundedness of $\Psi(u_n)$ and compactness of the corresponding functions z_k, so that in the end $u_k \to u$, which is a solution of the desired problem in Q_T. □

Remarks. (1) Every solution with changing sign obtained as a limit of this process is bounded above by the nonnegative solution with data $u_0^+(x) = \sup\{u_0(x), 0\}$, $f_+(x,t) = \sup\{f(x,t), 0\}$, and below by the nonpositive solution with initial data $u_0^-(x) = \inf\{u_0(x), 0\}$, and forcing term $f_-(x,t) = \min\{f(x,t), 0\}$.

(2) In the PME case, it is convenient to organize the approximation of Φ as follows: we first pick a function $\Phi_1 \in C^\infty(\mathbb{R})$ such that: (i) $\Phi_1(s) = \Phi(s)$ for $|s| \ge 1$; (ii) $\Phi_1(-s) = -\Phi(s)$; (iii) Φ_1 is linear in the interval $(-1/2, 1/2)$, $\Phi = cs$; (iv) Φ_1 is convex for $s \ge 0$.

We then define for every integer $n \ge 1$ the function

(5.43) $$\Phi_n(s) = n^{-m}\Phi_1(ns)\,.$$

Observe that $\Phi_n(s)$ is just $\Phi(s)$ if $|s| \ge 1/n$, while $\Phi = n^{1-m}s$ for $|s| \le 1/(2n)$.

(3) Let us recall that the convergence of u_n to u in the approximations takes place in $L^1(Q)$ without having to introduce any time delay.

5.5.1 Constant boundary data

We can also modify Theorem 5.7 to solve some problems with non-zero boundary data. We note that solving those problems implies introducing a suitable concept of solution, a task that will be performed in detail in Section 5.7. For the moment, we can use the recently proved theorem to solve the question with constant boundary data as follows. We observe that in the smooth case, the vertical displacement of a solution u of the GPME

$u_t = \Delta\Phi(u) + f$ produces another solution $\widetilde{u} = u + C$ of the GPME with a new nonlinearity, $\widetilde{\Phi}(s) = \Phi(s + C) - \Phi(C)$, that is also in the same class as Φ. Namely, $\widetilde{u}_t = \Delta\widetilde{\Phi}(\widetilde{u}) + f$. Besides, if $u = 0$ on Σ, then $\widetilde{u} = C$ on Σ. We take this transformation as the definition of solution for the new boundary value problem. We immediately have:

Corollary 5.11 *We can uniquely solve the Dirichlet problem for the GPME under the same assumptions on u_0 and f but with constant non-zero boundary data $u = C$ on Σ. If $C > 0$ the solutions are larger than in the standard HDP, if $C < 0$ they are smaller.*

The comparison principle holds; it is easy in the smooth case, it is justified in the general case by approximation.

5.6 Some properties of weak solutions

Now that we know that weak solutions exist and are unique, we may proceed with the qualitative analysis. Though weak solutions with data which are not strictly positive need not be classical solutions, they enjoy some interesting regularity properties, some of them a consequence of the estimates satisfied by smooth solutions that we have presented in Section 3.2, and some others that will be derived as a consequence of new estimates. In the first type, let us mention:

• The energy inequality given by formulas (5.20) or (5.39), that asserts that $\Phi(u) \in L^2(0, T : H_0^1(\Omega))$ with a bound that depends only on the norm of u_0 in $L_\Psi(\Omega)$ and the norm of f in $L^p(Q_T)$. It also asserts that $u(t)$ is estimated in $L^\infty(0, T : L_\Psi(\Omega))$ in the same way.

• Time derivative control is given by (5.40), showing that $Z(u)_t$ is bounded in $L^2(Q_T^\tau)$ if $f, f_t \in L^p(Q_T)$. In the case of the PME, this means that $\partial_t(u^{(m+1)/2}) \in L^2(Q_T^\tau)$. When Φ' is bounded, also $\Phi(u)_t \in L^2(Q_T^\tau)$. The same happens when u is bounded for $t \ge \tau$ and Φ is locally Lipschitz. We will find in Section 5.8 a priori bounds for the sup norm of $|u|$ when Φ is superlinear at infinity and f is bounded.

• When $\Phi'(u)$ is bounded, inequality (5.41) implies that $\nabla\Phi(u)$ is actually bounded in $L^\infty(\tau, T : H_0^1(\Omega))$. Same comment for bounded u as before.

These estimates take on a much nicer form when applied to the incomplete equation, i. e., for $f = 0$, which is the case usually considered in the PME theory. Thus, the energy estimate becomes

$$\int_\Omega \Psi(u(x,T))\,dx + \iint\limits_{Q_T} |\nabla w|^2\,dxdt \le \int_\Omega \Psi(u_0(x))\,dx, \tag{5.44}$$

which means that $\int_\Omega \Psi(u(x,t))\,dx$ is a nonincreasing function of time, and that $\nabla\Phi(u)$ is square integrable in the whole $Q = \Omega \times (0, \infty)$. For future reference, we write this estimate in the case of the PME:

$$\frac{1}{m+1}\int_\Omega |u(x,T)|^{m+1}\,dx + \iint\limits_{Q_T} |\nabla(|u|^{m-1}u)|^2\,dxdt \le \frac{1}{m+1}\int_\Omega |u_0(x)|^{m+1}\,dx. \tag{5.45}$$

On the other hand, the time derivative estimate reads

$$\iint_{Q_T} t\Phi'(u)|u_t|^2\,dxdt + \frac{T}{2}\int_\Omega |\nabla w(x,T)|^2\,dx \le \frac{1}{2}\iint_{Q_T} |\nabla w|^2\,dxdt, \tag{5.46}$$

or in the alternative forms, like

$$\iint\limits_{Q_T^\tau} \Phi'(u)|u_t|^2\,dxdt + \frac{1}{2}\int_\Omega |\nabla w(x,T)|^2\,dx \le \frac{1}{2}\int_\Omega |\nabla w(x,\tau)|^2\,dx. \tag{5.47}$$

For the PME this estimate reads

$$2m\iint\limits_{Q_T^\tau} |u|^{m-1}|u_t|^2\,dxdt + \int_\Omega |\nabla(|u|^{m-1}u)(x,T)|^2\,dx \le \int_\Omega |\nabla(|u|^{m-1}u)(x,\tau)|^2\,dx. \tag{5.48}$$

These estimates are satisfied with equality for classical solutions. The question will be discussed for weak solutions in Subsection 8.2.1.

• A very important property of the approximate equations is the contractivity with respect to the $L^1(\Omega)$ norm. This property passes to the limit and gives for two weak energy solutions u and $\widehat{u}$ as in Theorem 5.7 the estimate

$$\|u(t)-\widehat{u}(t)\|_1 \le \|u_0-\widehat{u}_0\|_1 + \int_0^t \|f(s)-\widehat{f}(s)\|_1 ds. \tag{5.49}$$

This implies the stability of such solutions. We will develop this issue in depth in the next chapter.

• We can also obtain a priori bounds in the norm $L^\infty(0,T:L^p(\Omega))$ when both u_0 and f are L^p functions by passing to the limit the estimates of Subsection 3.2.2. When $f=0$, we can also obtain monotonicity in all the L^p norms, $1\le p<\infty$.

Proposition 5.12 *In the situation of Theorem* 5.7, *if moreover the initial data belong to the space* $L^p(\Omega)$, $p\ge 1$, *and* $f=0$, *then* $u(\cdot,t)\in L^p(\Omega)$ *for any* $t>0$ *and*

$$\|u(\cdot,t)\|_p \le \|u_0\|_p\,. \tag{5.50}$$

Proof. It is based on passing to the limit the estimate obtained for smooth solutions, (3.13). In the case of the PME the complete calculation reads

$$\frac{4q(q+1)m}{(q+m)^2}\iint\limits_{Q_T} \left|\nabla\left(u^{\frac{q+m}{2}}\right)\right|^2 dxdt + \int_\Omega u^{q+1}(x,T)dx \le \int_\Omega u_0^{q+1}(x)dx\,, \tag{5.51}$$

valid for $q>0$. To get the case $p=1$ we pass to the limit as $q\to 0$. The proof is justified by approximation. □

5.7 Weak solutions with non-zero boundary data

We consider here the extension of the theory developed thus far in this chapter to the case where the boundary data are not homogeneous. Let us assume that Ω is a bounded domain in $\mathbb{R}^d$, $d\ge 1$, with regular boundary $\Gamma=\partial\Omega\in C^{2+\alpha}$; as in Section 5.2 we assume that $\Phi:\mathbb{R}\mapsto\mathbb{R}$ is a continuous increasing function with $\Phi(\pm\infty)=\pm\infty$. We pose the General Dirichlet Problem for the Filtration Equation:

Problem GDP. *Given measurable functions u_0 in Ω, g in Σ_T, and f in Q_T, find a locally integrable function $u = u(x,t)$ defined in Q_T that solves the set of equations*

$$u_t = \Delta\Phi(u) + f \qquad \text{in } Q_T\,, \tag{5.52}$$

$$u(x,0) = u_0(x) \qquad \text{in } \Omega, \tag{5.53}$$

$$\Phi(u(x,t)) = g(x,t) \qquad \text{in } \Sigma_T. \tag{5.54}$$

in a weak sense to be precisely defined. The time $T > 0$ can be finite or infinite.

Functional setting. Traces

We want to find u in a suitable functional class that guarantees uniqueness and continuous dependence on the data. Depending on that functional choice, suitable functional spaces are chosen for the data u_0, f and g. Definition 5.1 is still good enough as a local weak solution. We need some changes to define a suitable concept of solution for the new problem that accounts for non-zero boundary data. We will ask $\Phi(u) \in L^2(0,T : H^1(\Omega))$, forgetting about the zero boundary conditions. Next, we need to recall some facts about the theory of *boundary traces*:

(i) Functions $f \in H^1(\Omega)$ have boundary values called *traces*, $T_{\partial\Omega} f$, on the boundary $\partial\Omega$; moreover, the linear *trace map* $T_{\partial\Omega}$ maps $H^1(\Omega)$ onto the space $H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$. [1]

(ii) In the time dependent context, the trace operator can be naturally extended into a continuous linear map

$$T_\Sigma : L^2(0,T : H^1(\Omega)) \to L^2(0,T : H^{1/2}(\partial\Omega)) \subset L^2(\Sigma_T). \tag{5.55}$$

(iii) We will also need a further result. The trace operator admits a continuous lifting map, $j : H^{1/2}(\partial\Omega) \mapsto H^1(\Omega)$ such that $T_\Sigma(j(g)) = g$ for every $g \in H^{1/2}(\partial\Omega)$; we say that j is a right inverse of T_Σ. This extends to a lifting map

$$J : L^2(0,T : H^{1/2}(\partial\Omega)) \mapsto L^2(0,T : H^1(\Omega)).$$

After these considerations, we propose the following definition.

Definition 5.5 *Given $u_0 \in L^1(\Omega)$, $g \in L^2(0,T : H^{1/2}(\partial\Omega))$, and $f \in L^1(Q_T)$, a locally integrable function u defined in Q_T is said to be a weak solution of Problem* (5.52)–(5.54) *if*

(i) $\Phi(u) \in L^2(0,T : H^1(\Omega))$, *and* $T_\Sigma(\Phi(u)) = g$;

(ii) $u \in L^2(\Omega \times (0,T))$ *for all* $\tau > 0$;

(iii) u *satisfies the identity*

$$\iint_{Q_T} \{\nabla\Phi(u)\cdot\nabla\eta - u\eta_t\}\,dxdt = \int_\Omega u_0(x)\eta(x,0)dx + \iint_{Q_T} f\,\eta\,dxdt \tag{5.56}$$

for any function $\eta \in C^1(\overline{Q}_T)$ which vanishes on Σ and for $t = T$.

Clearly, the weak solutions we have constructed for the Homogeneous Dirichlet Problem HDP are the particular case of this definition which assumes zero boundary trace, $g = 0$.

[1] References for this topic are e. g. Adams [4] or Dautray-Lions [198].

This theory covers also the existence for constant boundary data advanced in Corollary 5.11. Note however that the have restricted the generality of the discussion of the HDP to the case of weak energy solutions.

As in the homogeneous problem, the goal of the theory is to establish existence, uniqueness, continuous dependence and other important properties of weak solutions of the General Problem (5.52)–(5.54). The uniqueness of weak solutions as defined above is settled by exactly the same result as in Theorem 5.3, and even the proof is the same.

Theorem 5.13 *Problem* (5.52)–(5.54) *has at most one weak solution.*

The reader should only notice that the test function η defined in (5.15) is still acceptable because it continues to have zero boundary trace, and also that the data u_0 and f disappear from the weak formulation when subtracting the expressions satisfied by the two solutions.

Concerning the existence theory, we repeat the assumptions on the initial data and forcing term made in Sections 5.4 and 5.5 and repeat the outline of the existence proofs with a suitable choice of boundary data. The choice is somewhat stricter:

(HG) We assume that there is a function $G \in L^2(0,T : H^1(\Omega))$ such that $g = T_\Sigma(G)$, and we assume further that $G, G_t, G_{tt} \in L^\infty(\Omega)$.

Theorem 5.14 *Under the above assumptions on G, for every $u_0 \in L_\Psi(\Omega)$ and $f \in L^2(Q_T)$, there exists a weak solution of Problem GDP with $u \in L^\infty(0,\infty : L_\Psi(\Omega))$. The Comparison Principle applies to these solutions: if $u, \widehat{u}$ are weak solutions and $u_0 \le \widehat{u}_0$ a. e. in Ω, $f \le \widehat{f}$ a. e. in Q_T, and $g \le \widehat{g}$ a. e. in Σ, then $u \le \widehat{u}$ a. e. in Q_T. In particular, If $u_0, f, g \ge 0$, then $u \ge 0$.*

Proof. The proof we give follows the outline of Theorem 5.7. Therefore, we need only to stress the differences. We perform an approximation process where the data are bounded; we take as boundary value for the approximate solutions, $\Phi_n(u_n) = g_n(x,t)$, where g_n is the trace on Σ of a smooth and positive function G_n that approximates G in its space, $L^2(0,T : H^1(\Omega)) \cap L^\infty(Q_T)$. We call the solutions u_n and put $w_n = \Phi_n(u_n)$. Multiplying the equation satisfied by the smooth solution u_n by $\Phi_n(u_n) - G_n \in L^2(0,T : H_0^1(\Omega))$, we get in the usual way

$$\iint_{Q_T} \nabla\Phi_n(u_n)\cdot(\nabla\Phi_n(u_n) - \nabla G_n)\,dxdt + \iint_{Q_T} (\Phi_n(u_n) - G_n)u_{n,t}\,dxdt$$
$$= \iint_{Q_T} f_n\,(\Phi_n(u_n) - G_n)\,dxdt.$$

Hence,

$$\iint_{Q_T} |\nabla\Phi_n(u_n)|^2\,dxdt + \int_\Omega \Psi_n(u_n(T))\,dx + \iint_{Q_T} G_{n,t}u_n\,dxdt + \iint_\Omega u_n(0)G_n(0)\,dx$$
$$= \int_\Omega \Psi(u_{n,0})\,dx + \iint_{Q_T} f_n\,(\Phi_n(u_n) - G_n)\,dxdt + \iint_{Q_T} (\nabla\Phi_n(u_n)\cdot\nabla G_n)\,dxdt$$
$$+ \int_\Omega u_n(T)G_n(T)\,dx.$$

After some easy computations, and using the regularity of the data and Sobolev's embeddings, we may derive estimates of the form

$$\iint_{Q_T} |\nabla \Phi_n(u_n)|^2 \, dxdt \le C, \qquad \int_\Omega \Psi_n(u_n)\, dx \le C,$$

which are uniform in n and in $t \in (0,T)$.

At least in the signed case, we also need an estimate on a time derivative just as in Theorem 5.7. We multiply the equation satisfied by u_n by $\partial_t(w_n - G_n)$ and integrate by parts in space to obtain

$$\int_\Omega (w_n - G_n)_t \, (u_n)_t \, dx = -\int_\Omega \nabla w_n \cdot \nabla (w_n - G_n)_t \, dx + \int_\Omega f_n (w_n - G_n)_t \, dx.$$

Multiplying now by a smooth function $\zeta(t) \ge 0$ that vanishes for $t = 0$ and $t = T$, and integrating in time and rearranging, we get (integrals in Q_T)

$$\begin{aligned}\iint \zeta \Phi_n'(x, u_n)|(u_n)_t|^2 \, dxdt = \frac{1}{2}\iint \zeta' |\nabla w_n|^2 \, dx - \iint (\zeta (G_n)_t)_t \, u_n \, dxdt \\ + \iint \nabla \zeta w_n \cdot \nabla (G_n)_t \, dxdt - \iint (\zeta f_n)_t \, (w_n - G_n) \, dxdt.\end{aligned}$$

In this way, a uniform estimate is obtained for $\iint \Phi_n'(u_n)|(u_n)_t|^2 \, dxdt$.

The rest of the proof offers few novelties and is left to the reader as a long review exercise. Finally, the Maximum Principle applies to the approximate problems, and this property is conserved in the limit. □

Remarks. (1) The regularity of G_{tt} is not needed when treating nonnegative solutions under the assumptions on Φ made in Theorem 5.5 using monotonicity. Also the assumption on G_t may be relaxed.

(2) The condition on the forcing term for the result to be true can be weakened into $f \in L^p(Q)$, with $p = 2d/(d+2)$ if $d \ge 3$, and some $p > 1$ if $d = 1, 2$.

Examples. All the examples of "naive" solutions considered in Chapter 4 are all of them weak solutions of the GDP for the PME when restricted to a proper cylinder of the form $Q = \Omega \times (0,T)$. This applies for instance to the stationary solutions of the form $u = |w|^{1/m}\mathrm{sign}(w)$ with $\Delta w = 0$ in $\mathbb{R}^d$; when restricted to $x \in \Omega$ they are acceptable weak solutions with sign change.

The ZKB and the TW solutions show that weak nonnegative solutions of the GDP need not be differentiable functions. But, since we also see that the lack of differentiability concerns only the free boundary, we may propose a compromise in the form of the concept of *classical free boundary solution*. We refer the reader to Problem 5.13 for this topic.

5.7.1 Properties of radial solutions

Weak solutions for the complete problem have properties that extend the ones derived in Section 5.6 for the homogeneous case. Instead of revising them, we will devote some space to consider the special properties of solutions in the so-called radially symmetric case in a homogeneous medium. We will use them in the study of initial continuity in Section 7.5.1.

We assume that the domain is a ball $\Omega = B_R(0)$, the data are radially symmetric, $u_0(x) = \phi(r)$, and also $f(x,t) = \psi(r,t)$ and the boundary data are constant in space, $g(x,t) = g(t)$.

Proposition 5.15 (Property of radial symmetry) *Under those assumptions, the weak solution of Theorem* 5.14 *is also radially symmetric in the space variable,* $u(x,t) = \widehat{u}(r,t)$.

This follows from the invariance of the equation under orthogonal transformations plus the uniqueness for weak solutions that we have already proved. By abuse of language, we simply say that the solution is 'radial' and write $u = u(r,t)$, as well as $u_0(r)$, $f(r,t)$.

Proposition 5.16 (Property of radial monotonicity) *Assume moreover that the radial profile is nondecreasing in* r, *i. e.,* $(u_0(r))' \ge 0$, *that* $\partial_r f(r,t) \ge 0$, *and finally that* $g'(t) \ge 0$ *and* $g(0) \ge u_0(R)$. *Then, the solution satisfies* $\partial_r u(r,t) \ge 0$.

Proof. The result is first proved for smooth solutions of filtration equations with smooth Φ such that $\Phi'(u) > 0$ and $\Phi''(u) \ne 0$, and smooth data u_0, f and g with $u_0'(r) \ge 0$ and $g(0) = u_0(R)$. In this case, the result is a consequence of the Maximum Principle applied to the equation for $v = u_r := \partial_r u(r,t)$, which is:

$$v_t = \Delta_r(\Phi'(u)v) - \frac{d-1}{r^2}\Phi'(u)v + f_r(r,t),$$

where Δ_r is the radial version of the Laplacian. As boundary conditions we take $v = 0$ at $r = 0$ due to smoothness and symmetry. At $r = R$ we have $u = g$, which implies $u_t = g' \le 0$ so that $\Delta\Phi(u) \ge 0$, i. e., $(r^{d-1}\Phi'(u)v)_r \ge 0$. In view of the values of $r = R$ and $\Phi'(0), \Phi''(0)$, we get an expression of the form

$$a(t)v_r + b(t)v^2 \ge 0,$$

with $a(t) > 0$. Since $v(r,0) \ge 0$, the Maximum Principle implies that $v \ge 0$.

For general Φ and general radial data, the result follows by approximation. □

Of course, the same result holds if we replace the condition of radially nondecreasing by radially nonincreasing and change the signs of g; then we would get $\partial_r u(r,t) \le 0$.

5.8 Universal Bound in Sup Norm

We investigate here a very well-known property of the PME, the boundedness for positive times of the solutions of Problem HDP. This bound holds also for the GPME with a strongly superlinear nonlinearity. It is a useful tool what will give us a convenient control on the solution used in many calculations.

Proposition 5.17 *Every weak energy solution* u *of Problem* HDP *for the complete PME* $(m > 1)$ *with bounded* f *is bounded above in* $Q_T^\tau = \Omega \times (\tau, T)$ *for every* $T > \tau > 0$. *Moreover, we have a universal decay estimate of the form*

$$u(x,t) \le c(m,d)\left(R^{\frac{2}{m-1}} + (N/R^2)^{1/m} T^{\frac{1}{m-1}}\right) t^{-\frac{1}{m-1}}\,, \tag{5.57}$$

where $c(m,d)>0$, R is the radius of a ball containing Ω, and $N=\sup_{Q_T} f$.

The same result holds for the GPME under the following growth condition on Φ: for all large $u \ge c_0$, Φ is C^1-smooth and

$$\Phi'(u) \ge a\Phi^{(m-1)/m} \quad \text{for some } a>0, m>1. \tag{5.58}$$

Then, R must be large and c depends also on c_0.

By *universal* we mean that the bound does not depend in any way on the size of the initial data we are considering, neither in the form of the expression nor in the constants that appear.

Proof. We will use the fact that the solutions are constructed by approximation with smooth functions. We state and do the proof first for the PME, where the estimate is quite explicit and accurate.

(i) Let us first consider the case where u_0 is continuous and vanishes on $\partial\Omega$. We will construct an explicit super-solution $z(x,t)$ with which to compare the approximate solutions u_n of (5.22)–(5.24).

In fact, we fix $T>0$ and take a ball $B_R=B_R(0)$ of radius R strictly containing Ω, i. e., with $\Gamma=\partial\Omega\subset B_R$, and consider the function $z(x,t)$ defined in $B_R\times(0,T)$ by

$$z^m(x,t)=A(t+\tau)^{-\alpha}(R^2-x^2) \tag{5.59}$$

for suitable constants A,τ and $\alpha>0$ to be chosen presently. To begin with, we put $\alpha=m/(m-1)$. We want to prove that $u_n(x,t)\le z(x,t)$ in Q_T. This implies checking on the parabolic boundary: since function z is positive in $B_R\times(0,\infty)$, for all large n we have

$$u_n(x,t)=\frac{1}{n}<z(x,t) \quad \text{in } \Sigma,$$

if A,τ are kept fixed. Moreover, we choose τ small enough so that

$$u_{0n}(x)\le z(x,0).$$

Finally, we will obtain the inequality $z_t-\Delta(z^m)\ge f_n$ whenever

$$2dA\ge(t+\tau)^{m/(m-1)}f_n(x,t)+\frac{1}{m-1}A^{1/m}(R^2-x^2)^{1/m} \tag{5.60}$$

for $|x|\le R$ and $0\le t\le T$. This happens for instance if

$$A\ge c_1R^{2/(m-1)}, \quad A\ge c_2NT^{m/(m-1)}.$$

With these choices, and since $u_{n,t}-\Delta\Phi_n(u_n)-f_n=0$, and $\Phi_n(z)=z^m$ due to the fact that $z(x,t)\ge 1/n$, the classical Maximum Principle implies that $u_n(x,t)\le z(x,t)$ in Q_T. Passing to the limit $n\to\infty$ and $\tau\to 0$, we finally get

$$u(x,t)\le A^{1/m}t^{-\frac{1}{m-1}}\left(R^2-x^2\right)^{1/m}\le A^{1/m}R^{2/m}t^{-\frac{1}{m-1}}. \tag{5.61}$$

By approximation, (5.61) holds for every weak solution obtained as a limit.

(ii) Let us now consider the GPME. We assume that $\Phi'(u) \ge cu^{m-1}$ for $u \ge n_0$, and we also assume that the approximate constitutive functions Φ_n satisfy the same condition $\Phi_n'(u) \ge cu^{m-1}$ for $u \ge c_0$. We repeat the outline of the previous proof, taking

$$w = \Phi_n(z_n(x,t)) := A(t+\tau)^{-\alpha}(R^2 - x^2) \tag{5.62}$$

with R large enough so that z_n will be larger than c_0 on $\partial\Omega$ for $0 < t < T$. We have to pay attention to the supersolution condition for the equation that now reads $z_{n,t} - \Delta w - f_n \ge 0$ in Q_T, or, in another form,

$$w_t \ge \Phi_n'(z_n)(\Delta w + f_n).$$

Since $w_t, \Delta w < 0$ and $\Phi_n'(z) \ge cw^{(m-1)/m}$, this means that

$$|\Delta w| \ge f_n + \frac{1}{a}|w_t|w^{-(m-1)/m},$$

and we arrive at (5.60) but for a factor $1/c$ in the last term, that is not important. □

Remarks. (1) The existence of a universal upper bound is **not** true for the heat equation, $u_t = \Delta u$, simply because it is linear, so that given any solution $u(x,t) \ge 0$, we can also consider all multiples $cu(x,t)$, and this fact makes a universal bound impossible. The main requirement in order to obtain a universal upper bound is superlinearity of Φ at infinity.

(2) There are however estimates that imply boundedness for positive times when the nonlinearity has only linear growth, but then the L^∞-norm of $u(t)$ must depend on the L^1-norm of the u_0 (or other convenient measure of the size of initial data). Symmetrization techniques are very useful in establishing such results. See Chapter 17.

(3) Since the bound is universal in its form, it will still be true when we extend the solutions to deal with $L^1(\Omega)$ initial data in the next chapter. Indeed, it is a universal bound.

(4) The estimate is accurate. Indeed, for $f = 0$ we will construct in the next section an actual exact solution that has the predicted decay for the PME, $O(t^{-1/(m-1)})$.

(5) A convenient condition of superlinearity that appears in the literature is

$$\frac{s\Phi'(s)}{\Phi(s)} \ge c > 1 \quad \text{for all} \quad s \ge c_0. \tag{5.63}$$

It is easy to prove that this implies $\Phi(s) \ge Cs^c$ for all large s, hence $\Phi'(s) \ge K\Phi^{1-1/c}$, the condition used in the proof with $c = m$.

(6) The growth assumption on Φ can also be weakened.

(7) On the other hand, the assumption on f can be weakened; for instance the universal bound that we have obtained depends only on the L^∞ norm of $F(x,t) = t^{m/(m-1)}f(x,t)$. However, improving f is not a priority for us. We may also take f in an L^p space with large p.

We can also get a universal bound for the problem with boundary data.

Proposition 5.18 *Let u be a weak solution [u] of* (5.52)–(5.54) *constructed by approximation with smooth functions, and assume that $f^+ \in L^\infty(Q_T)$, $g^+ \in L^\infty(\Sigma_T)$. If Φ is* *superlinear in the sense of Proposition* 5.17*, then u is bounded above in Q^τ for every $\tau > 0$, and we have a universal decay estimate of the form*

$$u(x,t) \le F(t), \tag{5.64}$$

where F is a decreasing function of t that depends on $\|f^+\|_\infty$, $\|g^+\|_\infty$, and the radius R of a ball strictly containing Ω. Moreover, for small $t > 0$ the estimate has the form

$$u(x,t) \le C(m,d)\, R^{\frac{2}{m-1}}\, t^{-\frac{1}{m-1}}\,. \tag{5.65}$$

Proof. We only need to consider nonnegative data and solutions. We still try a supersolution of the form (5.59),

$$z^m(x,t) = A(t+\tau)^{-m/(m-1)}(1-bx^2).$$

We need to satisfy the conditions: $u_{0n}(x) \le z(x,0)$, which offers no novelty; $u_n(x,t) < z(x,t)$ on Σ, which is satisfied if

$$A(1-bR^2) \ge \|g^+\|_\infty (t+\tau)^{m/(m-1)};$$

and $z_t \ge \Delta(z^m) + f^+(x,t)$, which is implied by the two conditions

$$dbA \ge (t+\tau)^{m/(m-1)}\|f^+\|_\infty, \quad db(m-1)A^{(m-1)/m} \ge 1.$$

The result follows. □

5.9 Construction of the Friendly Giant

We want to explore now the question of how precise is the universal bound of the previous section. We investigate that issue by constructing a suitable solution that will later play a role in the theory. Considering for simplicity the case where $f = 0$, we show that there exists a special solution $\widetilde{U}$ which is the largest element in the class of functions which are weak solutions of the Dirichlet Problem in Q in the sense of Definition 5.3 with $f = 0$. This solution is the *maximal solution* of the Cauchy-Dirichlet Problem. It takes infinite initial data everywhere in Ω. Following Dahlberg and Kenig, we call this solution the Friendly Giant. Moreover, when the equation is the PME, $\widetilde{U}$ is a solution in separated-variables form; actually, it is the special solution discussed in Section 4.2, that is obtained here as a nice consequence of the general theory.

Theorem 5.19 *Let us assume that Φ satisfies the growth condition* (5.58)*. Then there exists a unique weak solution of the Dirichlet Problem for the GPME with $f = g = 0$ that takes initial values $u_0(x,0) = +\infty$ and the divergence is uniform away from the boundary. This solution is an upper bound for all weak energy solutions of Problem* (5.1)–(5.3) *with $f = 0$. It is a decreasing function of time for all $x \in \Omega$.*

Proof. (i) The concept of weak solution with infinite data is meant in the sense of Definition 5.3 and such that $u \in C((0,\infty) : L^1(\Omega))$ and for all $\tau > 0$ $v(x,t) = u(x,t+\tau)$ is a weak energy solution (this is assumed to simplify matters at this stage, cf. Section 6.5 below). For every integer $n \ge 1$ we solve the problem

$$(P_n) \quad \begin{cases} \partial_t u_n & = \Delta\Phi(u_n) & \text{in } Q\,, \\ u_n(x,0) & = n & \text{in } \Omega\,, \\ u_n(x,t) & = 0 & \text{on } \Sigma\,. \end{cases}$$

Let u_n be the weak solution to this problem. Clearly, the sequence $\{u_n\}$ is monotone: $u_{n+1} \ge u_n$. We also know from Proposition 5.17 that for every n

$$u_n(x,t) \le F(t) \quad \text{in} \quad Q\,, \tag{5.66}$$

where F is a decreasing function of t that does not depend on n. Therefore, we may pass to the limit and find a function

$$\widetilde{U}(x,t) = \lim_{n\to\infty} u_n(x,t),$$

also satisfying estimate (5.66). Let us examine the properties of $\widetilde{U}$:

As a monotone limit of bounded solutions u_n in Q^τ such that the functions $\Phi(u_n)$ are bounded above by a function in $L^2(\tau,\infty : H_0^1(\Omega))$, it is straightforward to conclude that $\widetilde{U}$ is a weak solution of the Cauchy-Dirichlet problem for the GPME in any time interval (τ,∞).

It is also clear that it takes on the value $\widetilde{U}(x,0) = +\infty$ everywhere in Ω. The divergence is uniform thanks to a simple barrier argument: since the solutions $u_n(x,t)$ are continuous down to $t=0$ at all interior points (see Proposition 7.13 for a proof), for every $\varepsilon > 0$ there exists $\tau > 0$ such that $u_n(x,t) \ge n - \varepsilon$ if $d(x,\partial\Omega) \ge \varepsilon$ and $0 < t < \tau$. Now, recall that $u_n \le \widetilde{U}$ to conclude.

(ii) Let us now prove that $\widetilde{U}$ is larger than any weak solution of the Cauchy-Dirichlet Problem in Q with $f = 0$. By Proposition 5.17 we know that every such solution satisfies

$$u(x,\tau) \le F(\tau) < \infty\,.$$

Taking $n \ge F(\tau)$, it follows from the Maximum Principle that

$$u(x,t+\tau) \le u_n(x,t) \le \widetilde{U}(x,t) \quad \text{in} \quad Q\,.$$

Using the fact that $u \in C([0,\infty) : L^1(\Omega))$ (see Section 6.1 for more details on this issue) and letting now $\tau \to 0$ we get $u(x,t) \le \widetilde{U}(x,t)$ in Q as desired.

(iii) Next, we prove the uniqueness of the solution with $u(x,0) = +\infty$. Assume that v is another such solution. Since we assume that $v(x,t+\tau)$ is a weak solution of problem (5.1)–(5.3), $v(x,\tau)$ must be an element in $H_0^1(\Omega)$, hence $v(x,2\tau)$ is bounded by Proposition 5.17. By comparison with the sequence u_n we conclude that $v(x,t+2\tau) \le u_n(x,t)$ in Q for some n large enough. Letting $\tau \to 0$ we get

$$v(x,t) \le \widetilde{U}(x,t)\,. \tag{5.67}$$

On the other hand, a function v which has infinite initial values is larger than the solutions u_n, hence $v \ge \widetilde{U}$. The precise argument is as follows: the uniform divergence of v at $t=0$ and the contraction property imply that for any n there is a small $\tau = \tau(n)$ such that

$$\int_\Omega (u_n(0) - v(\tau))_+\,dx \le \varepsilon,$$

since $u_n(0) = n$. Therefore, $\int_\Omega (u_n(t) - v(t+\tau))_+\,dx \le \varepsilon$ for every $t \ge 0$. In the limit, $\widetilde{U} \le v$. Putting both inequalities together, we get $v = \widetilde{U}$.

(iv) To prove the monotonicity in time, we fix $\tau > 0$ and observe that, by the a priori estimate, there exists $n_1 = n_1(\tau)$ such that for every $n \ge 1$

$$u_n(x,\tau) \le n_1 = u_{n_1}(x,0)\,.$$

By the Maximum Principle, we conclude that $u_n(x,t+\tau) \le u_{n_1}(x,t)$ in Q. In the limit we have $\widetilde{U}(x,t+\tau) \le u_{n_1}(x,t)$ for every $t \ge 0$, hence

$$\widetilde{U}(x,t+\tau) \le \widetilde{U}(x,t) \quad \text{in} \quad Q\,. \tag{5.68}$$

This proves the monotonicity. $\square$

Theorem 5.20 *For the PME this special function has the separate-variables form*

$$\widetilde{U}(x,t) = t^{\frac{1}{m-1}}\, F(x)\,. \tag{5.69}$$

$\widetilde{U}$ *can be characterized as the maximal solution of the PME in* Q *with zero Dirichlet conditions. Besides,* $g = F^m$ *is the unique positive solution of the nonlinear eigenvalue problem*

$$\Delta g + \frac{1}{m-1} g^{\frac{1}{m}} = 0\,, \quad g \in H_0^1(\Omega)\,. \tag{5.70}$$

Proof. To show that $\widetilde{U}$ has the form (5.69), we introduce the scaling transformation

$$(\mathcal{T}u)(x,t) = \lambda u(x, \lambda^{m-1}t)\,, \quad \lambda > 0\,. \tag{5.71}$$

This transformation leaves the equation invariant (see Subsection 3.3.2 for more details on scaling). It is interesting to see what happens when it is applied to our latter sequence $\{u_n\}$: checking the initial and boundary values, we see that

$$(\mathcal{T}u_n)(x,t) = u_{\lambda n}(x,t) \quad \text{in} \quad Q. \tag{5.72}$$

Passing to the limit $n \to \infty$ in (5.72) we get

$$(\mathcal{T}\widetilde{U})(x,t) = \widetilde{U}(x,t), \tag{5.73}$$

which holds for every $(x,t) \in Q$ and every $\lambda > 0$. Fixing (x,t) and setting $\lambda = t^{-1/(m-1)}$ we get (5.69) with $F(x) = \widetilde{U}(x,1)$.

The fact that $g = F^m$ satisfies (5.70) is also obvious. $\square$

Remarks. (1) The reader should compare this function with the similar situation for the linear case $m = 1$ for $d = 1$. Then, the solution of the equation equivalent to (5.70), i. e., $\Delta F + cF = 0$, is the sine,

$$F(x) = A\,\sin(\omega\, x), \quad \text{with} \quad \omega = \pi/|\Omega|, \tag{5.74}$$

$|\Omega|$ being the length of the interval Ω, and $c = \omega^2$. Thus, we may say that for $m > 1$ the profile of the giant is a kind of *nonlinear sine function*. In the linear case we have a free parameter $A > 0$ which does *not* exist in the nonlinear case.

Moreover, $U = e^{-\lambda_1 t}F(x)$, $\lambda_1 = \omega^2$, is the asymptotic first approximation for nonnegative solutions, but not an universal upper bound.

(2) The maximal solution shows that more general data are possible than the ones covered in this chapter. We will pursue this issue later in the book, starting with the next chapter. It is immediate to see that the present solution is also maximal with respect to the limit solutions defined there.

(3) There is no essential reason to consider maximal solutions only for forcing term $f = 0$. In fact, the proof of Theorem 5.19 goes through under the restriction $f \le C$ or even $f \le Ct^{-m/(m-1)}$. See Problem 5.11.

(4) The friendly giant will play a prominent role in the study of asymptotic behaviour of Section 20.1, where a new proof of existence will be given.

5.10 Properties of Fast Diffusion

We will be mainly interested in the PME equation where $\Phi(u) = |u|^{m-1}u$ with $m > 1$, and the associated properties like finite propagation and free boundaries. But the concepts and construction of solutions of this chapter apply equally well to the fast diffusion range $0 < m < 1$. There are, however, marked qualitative differences like *extinction* that we comment next.

5.10.1 Extinction in finite time

The techniques of Section 5.8 can be applied to the Fast Diffusion Equation, $0 < m < 1$, but they lead to very different conclusions. In that case we have:

Proposition 5.21 *Every weak energy solution u of Problem* HDP *for the signed FDE with bounded initial data (and $f = 0$) vanishes identically after a finite time $T > 0$ with a bound that depends on $\|u_0\|_\infty$. Moreover, we have the upper estimate*

$$u(x,t) \le c(m,d)\, R^{-\frac{2}{1-m}}\, (T-t)^{\frac{1}{1-m}}\,, \tag{5.75}$$

where $c(m,d) > 0$, R is the radius of a ball containing Ω and $T \ge c_1(m,d)M^{1-m}R^2$ where $M = \sup_\Omega u_0$. Similar estimates apply to the negative part.

Proof. The construction is similar to the PME case of Proposition 5.17. We assume that the ball $B_R(0)$ of radius R contains Ω, and consider the function $z(x,t)$ defined in $B_{2R} \times (0,T)$ by

$$z^m(x,t) = A\,(T-t)^\alpha(4R^2 - x^2) \tag{5.76}$$

for suitable constants A, T, and $\alpha = m/(1-m)$; note the sign changes with respect to the PME case. We want to prove that there exist approximations as in the PME case such that $u_n(x,t) \le z(x,t)$ in Q_T. This implies checking on the parabolic boundary: since function z is positive in $B_{2R} \times (0,\infty)$, for all large n we have

$$u_n(x,t) = \frac{1}{n} < z(x,t) \quad \text{in } \Sigma,$$

if A, T are kept fixed. Moreover, we choose T large enough so that $u_{0n}(x) \le z(x,0)$. This happens if

$$M^m \le 4A\,T^\alpha R^2. \tag{5.77}$$

Next, we obtain the inequality $z_t - \Delta(z^m) \geq 0$ whenever

$$2dA \geq \frac{1}{1-m} A^{1/m}(4R^2 - x^2)^{1/m} \tag{5.78}$$

for $|x| \leq 2R$ and $0 \leq t \leq T$. This happens if

$$A \leq c_2 R^{-2/(1-m)}$$

with $c_2 = c_2(m, d) > 0$. Our value for c_2 is $(2d(1-m))^{m/(1-m)} 4^{-1/(1-m)}$. Note that when we choose the best value for A according to this restriction, we get from (5.77) the condition $T \geq c_1(m, d) M^{1-m} R^2$. With these choices, and since $u_{n,t} - \Delta\Phi_n(u_n) = 0$, and $\Phi_n(z) = z^m$ due to the fact that $z(x,t) \geq 1/n$, the classical Maximum Principle implies that $u_n(x,t) \leq z(x,t)$ in Q_T. Passing to the limit, we get $u(x,t) \leq z(x,t)$. □

Corollary 5.22 *All weak energy solutions of the FDE with bounded initial data vanish in finite time.*

Remark. Finding that the weak solution with nontrivial data becomes identically zero for a degenerate parabolic equation was in its day a big surprise. It is tied to the fact that the exponent is less than one. The simplest case of an evolution equation where the phenomenon of extinction happens is the ODE

$$\frac{du}{dt} = -u^p \quad \text{for} \quad 0 < p < 1,$$

with initial data $u(0) > 0$. Actually, there is a proof of the phenomenon of extinction based on energy inequalities that leads to an ODE like this one. Suppose that $d \geq 3$ and the solution is nonnegative, and smooth so that the calculations are justified. Then, for every $q > 1$ we have by the usual methods of integration by parts applied to the FDE

$$\frac{d}{dt}\int_\Omega u^q\,dx = -q(q-1)m\int_\Omega u^{m+q-3}|\nabla u|^2\,dx \leq -C(m,d,q,\Omega)\left(\int_\Omega u^p\right)^r$$

with $p = (m+q-1)d/(d-2)$ and $r = (d-2)/d$. We have used the Sobolev embedding in the last inequality. We now choose $q \geq d(1-m)/2$ so that $p \geq q$, and put $I_q = \int_\Omega u^q\,dx$ to get from the comparison of L^p and L^q norms

$$\frac{dI_q}{dt} \leq -C\,I_q^\gamma, \qquad \gamma = 1 - \frac{1-m}{q} \in (0,1) \tag{5.79}$$

$(\gamma = rp/q)$. The ODE for I_q leads to extinction in a finite time T depending only on m, q, Ω and $I_q(0)$. This estimate is conserved when we make an approximation process. We leave to the reader to prove the cases $d = 1, 2$ with similar conclusion that we state next.

Proposition 5.23 *Extinction in finite time happens in the FDE for all $0 < m < 1$ if the initial data u_0 belong to the space $L^q(\Omega)$ with $q > 1$, $q \geq d(1-m)/2$.*

We will not pursue more the study of extinction for Fast Diffusion since our aim is the study of the PME. But see Notes and Problem 5.14. The problem in the whole space is treated in full detail in the monograph [515].

5.10.2 Singular Fast Diffusion

The equation $u_t = \Delta u^m$ cannot be continued for $m \le 0$ because it is trivial for $m = 0$ and becomes inverse parabolic for $m < 0$. But the rescaled form

$$\partial_t u = \nabla \cdot (|u|^{m-1} \nabla u) \tag{5.80}$$

makes perfect sense as a *singular parabolic equation* (called singular because of the limit $D(u) = |u|^{m-1} \to \infty$ as $u \to 0$). It has appeared in several applications that have motivated the mathematical study in classes of nonnegative solutions. The theory has some surprising features in the form of non-existence and non-uniqueness of solutions for bounded data. We refer the reader to the detailed study contained in the monograph [515]. We point out that in order to use the notation of this chapter we should consider for $m = -n < 0$ a nonlinearity of the form

$$\Phi(u) = c - \frac{1}{n} u^{-n}, \quad u > 0.$$

It falls out of our assumption (H_Φ) because of the limits $\Phi(0+) = -\infty$, $\Phi(+\infty) = c < \infty$. In the case $m = 0$ we have $\Phi(u) = \log(u)$ with same conclusion.

5.11 Equations of inhomogeneous media. A short review

There are a number of extensions of the PME and its generalization the GPME that appear in the literature in the study of mass diffusion, heat propagation of gas flow in nonhomogeneous media. Here are some of the options.

• A natural generalization of the GPME in view of the existing theory of parabolic equations is the equation

$$\partial_t u = \sum_{i=1}^{d} \partial_{x_i} \left(a_{ij}(x,t)\, \partial_{x_j} \Phi(u) \right), \tag{5.81}$$

where Φ is as above and (a_{ij}) is a symmetric matrix of bounded measurable functions which is positive definite or at least non-negative. The equation is for instance suggested as a mathematical model for the flow of a gas in a nonhomogeneous porous medium according to the model of Section 2.1 when the permeability or the viscosity depend on x and/or t, see Subsection 2.1.1. We can think for instance of periodic media.

In order to re-do the theory of this chapter, the reader is advised to review the estimates of Section 3.2 and impose conditions on the derivatives of a_{ij} on x, t and u. The main item, the parabolicity conditions may read

$$\Lambda^{-1}\xi^2 \le a_{ij}(x,t)\xi_i\xi_j \le \Lambda \xi^2 \qquad \text{for a. e.} \quad (x,t) \in Q, \tag{5.82}$$

for all $\xi \in \mathbb{R}^d$, for some $\Lambda > 1$. We may also ask the coefficients a_{ij} to be continuous or differentiable. As an example, Bertsch and Kamin study in [112] the one-dimensional version of this problem under the assumptions: (i) $\Phi(u) = u^m$, (ii) $a(x,t)$ is a $C^{2,2}$ function and satisfies (5.82); (iii) $u_0 \ge 0$ is bounded and continuous; (iv) the space domain is $\mathbb{R}$.

• More generally, we may consider an equation of the form

$$\partial_t u = \sum_{1}^{d} \partial_{x_i} (A_i(x,t,u,\nabla u)) \tag{5.83}$$

and derive an existence and uniqueness theory of weak solutions under convenient assumptions on the functions A_i. We may write $a_{ij}(x,t) = \partial_{p_j} A_i(x,t,u,\mathbf{p})\big|_{\mathbf{p}=\nabla u}$, and impose parabolicity conditions as before. The very influential paper of Alt and Luckhaus [11], 1983, treats the initial-boundary value problems for quasilinear systems of the form

$$\partial_t b^j(u) - \nabla \cdot [a^j(b(u), \nabla_x u)] = f^j(b(u)), \tag{5.84}$$

$j = 1, \cdots, m$. General structure conditions (ellipticity of a and subdifferentiability of b) allow for elliptic-parabolic equations, nonsteady filtration problems and even Stefan problems. Existence, uniqueness and regularity results are established. Many subsequent papers have used and extended those results. This generality will be found below in extending the continuity results of Chapter 7, see e.g. DiBenedetto [207], and in extending the work on propagation, e.g. Antontsev [31] and Diaz-Véron [203].

The study of the so-called parabolic-elliptic boundary value problems has originated an extensive literature.

• The previous models concern equations in divergence form, an important feature in developing the mathematical theory. The consideration of the gas flow model in porous media with variable porosity leads to the equation of the form (2.9): $\rho(x,t)\,\partial_t u = \nabla\cdot((c(x,t)\nabla u^m)$, or more generally

$$\rho(x,t)\,\partial_t u = \nabla \cdot (c(x,t)\nabla\Phi(u)), \tag{5.85}$$

which have non-divergent form (note that $\rho > 0$ is given). A particular instance of this mathematical model was proposed by Kamin and Rosenau [322], [452], in the study of thermal propagation in an unbounded medium. The equation has the form

$$\rho(x)\,\partial_t u = \Delta\Phi(u)\,, \tag{5.86}$$

where u stands for the temperature and ρ is the mass density. In the last case we fix the total mass

$$m = \int_\Omega \rho(x)\,dx,$$

which may be finite or infinite; Ω is a bounded domain or $\mathbb{R}^d$. The thermal energy is then

$$E(t) = \int_\Omega u(x,t)\rho(x)\,dx.$$

The authors pose the problem in $d = 1$, $\Omega = \mathbb{R}$, with finite total mass and finite initial energy. The assumptions on the equation structure are: $\rho(x)$ is smooth, and Φ satisfies $\Phi(0) = 0$, $\Phi'(0) \geq 0$, $\Phi'(u) > 0$ for $u > 0$; the initial data satisfy $0 \leq u_0(x) \leq M$. Existence and uniqueness of solutions for this problem can be obtained by methods that are variations of the ones of this chapter and have been developed by a number of authors for $d = 1$ and $d > 1$ both in the case of a bounded domain or in the case of the whole space (to be treated in Chapter 9). In the latter case, the behaviour of the density at infinity is a matter of concern. The typical assumption is power decay as $|x| \to \infty$:

$$\rho(x) \sim |x|^{-a}, \quad a > 0.$$

There is a great difference between the case $a \leq d$ (infinite mass) and $a > d$ (finite mass). The value $a = 2$ is critical. Let us mention that the problem in the whole space leads to interesting non-uniqueness results even for bounded initial data, cf. [225], [280], [448].

• A simple inhomogeneous model that appears in the literature consists of the equation

$$u_t = \Delta\Phi(x,u) + f. \tag{5.87}$$

This version already appears in the pioneering work of Oleĭnik et al. [408] (with $f = 0$). A convenient assumption on Φ is

(H_Φ) $\Phi : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *is continuous in both variables and strictly increasing in the second. We also have* $\Phi(x,0) = 0$ *for all* $x \in \Omega$.

As indicated in Chapter 3, many of the basic estimates on which the theory relies can be easily adapted to this case, so that the whole theory of this chapter can be generalized. Additional assumptions on the dependence of Φ on x will be needed to round up the existence theorems.

• We have mentioned one famous case in which the GPME involves a function Φ that is not strictly increasing, namely the Stefan Problem, described in the Introduction, Section 1.3, to which many of the developments of this chapter apply. The combination of degenerate diffusion and the Stefan problem is treated by Bertsch et al. in [108].

On the other end, there is an interest in graphs Φ which have vertical parts, in other words, the inverse graph $c = \Phi^{-1}$ has a flat part. The corresponding equation

$$c(w)_t = \Delta w + f, \tag{5.88}$$

represents the so-called elliptic-parabolic problems, which also develop interesting free boundaries. Again, much of this chapter applies to such models. We refer for this topic to the work of J. Hulshof and coworkers [294, 109, 295].

• A different question is the solution of forward-backward nonlinear heat equations of the form

$$\partial_t u = \Delta\,\Phi(u), \tag{5.89}$$

where Φ is a non-monotone function, typically with a cubic type structure: it is increasing for large and small values of u but decreasing in an intermediate u-interval. The standard Dirichlet and Cauchy problems for this equation are ill-posed with the usual function spaces and topologies. Novick-Cohen and Pego [402] study the problem by means of a regularization of the form

$$\partial_t u = \Delta(\Phi(u) + \nu u_t), \qquad \nu > 0, \tag{5.90}$$

(Sobolev regularization), with Neumann boundary conditions $\mathbf{n}\cdot\nabla(\Phi(u) + \nu u_t) = 0$ on $\partial\Omega \times \mathbb{R}_+$ as a model for isothermal phase separation of a binary mixture. Padrón [416] finds this problem as a model of aggregating populations and uses the same regularization to find existence and uniqueness of global in time solutions of the HDP and certain regularity properties when Φ is coercive in some sense. The fine analysis of the weak limits and the hysteresis effects is done in Plotnikov [437] and Evans-Portilheiro [231].

These ill-posed problems can be regularized by a number of other methods with possibly different limits.

• As a curiosity, Antontsev and Shmarev [33] have recently studied a model of porous medium equation with variable exponent of nonlinearity:

$$u_t(x,t) - \mathrm{div}(|u|^{\gamma(x,t)}\nabla u(x,t)) = f(x), \tag{5.91}$$

for $(x,t) \in Q_T = \Omega \times (0,T]$ with initial data $u(0,x) = u_0(x)$, $x \in \Omega$, and Dirichlet boundary conditions $u(x,t) = 0$, $(x,t) \in \Gamma_T = \partial\Omega \times (0,T]$. They assume that $-1 < \gamma^- \le \gamma(x,t) \le \gamma^+ < +\infty$, for some given constants γ^-, γ^+. It is proved that the above-stated problem admits a unique weak solution if $\gamma(x,t) > 0$. Qualitative properties of the solution are derived in terms of the values of γ.

Notes

§5.2. As we have explained above, solutions for the Cauchy, Dirichlet and Neumann Problems were first announced by Oleĭnik [406], published in 1957, and explained in detail in [408], 1958. The case of one space dimension was considered, $f = 0$, and a class of so-called generalized solutions was introduced. Actually, a slightly more general equation was considered, $u_t = \Phi(x,u)_{xx}$ under convenient regularity assumptions on Φ and u_0. The uniqueness result, Theorem 5.3, follows the proof in [408]. Dubinskii [219] proves existence theorems for generalized solutions of the Dirchlet and the Cauchy problem for the PME and other much more general degenerating higher-order parabolic equations.

The semigroup approach to existence and uniqueness will be explained in Chapter 10. It has the advantage of allowing quite naturally for a greater generality for Φ which can then be a maximal monotone graph; this allows for instance to have graphs Φ with horizontal parts, like in the Stefan Problem. Such problems are very important in theory and applications but they are not our concern.

A study of the properties of weak solutions to the Dirichlet Problem was done by Aronson and Peletier in [49], who use a definition similar to our Definition 5.4. These works refer to nonnegative solutions, but the semigroup approach applies to both signs.

The change to L^2 instead of L^1 as the basic space behind the functional setting is done for convenience in the uniqueness proof, and is then supported by the existence result, but our larger goal is to work in L^1, a space that has a prominent role in the complete theory. This aspect will be explored in the next chapters.

§5.4, **§5.5.** Usually, proofs of the existence of solutions with changing sign were done in the framework of semigroups, thus obtaining mild solutions. We have chosen to offer a comparative presentation for nonnegative solutions and solutions of both signs, so that the reader can feel from the beginning the problems of extending the theory to the case of changing sign.

§5.7. A complete theory for the Nonhomogeneous Problem can be developed on this foundation. We will not pursue such a line of work, since the present text must address other more urgent issues. The interested reader is offered a continuation of the investigation in Problem 5.9.

The boundary conditions are taken in the sense of traces. However, in many practical applications the weak solutions will also be continuous inside the domain and up to the boundary, so that the concept of trace is simple.

§5.8. The universal bound in sup norm is a very strong regularity result. It is used to propose a new definition of weak solution with finite energy in the next chapter.

§5.9. The existence of the special solution (5.69) is established in [49] by a different method, consisting in studying the elliptic equation (5.66). A more general result can be found in Dahlberg and Kenig [190] who introduced the term Friendly Giant in 1988.

The uniqueness of the Friendly Giant by elliptic methods is discussed in [511]. This survey paper reviews the Dirichlet problem for the PME with special attention to the asymptotic behaviour as $t \to \infty$. See Chapter 20.

§5.10. The theory of the family of Fast Diffusion Equations with $1 > m > -\infty$ offers many theoretical surprises like instantaneous extinction, non-uniqueness, and lack of regularity. These aspects are studied in detail in the monograph [515].

The theory of solutions of the HDP for the GPME was studied by Evans in [227]. Under the assumption that Φ^{-1} is globally Lipschitz continuous, a unique solution is produced in the class of strong solutions (improved properties with respect to weak solutions, see Chapter 8).

The property of extinction in finite time was first proved by Sabinina [458, 459] for a class of one-dimensional parabolic equations of fast diffusion type in bounded intervals.

The extinction phenomenon for the GPME with general nonlinearities was studied by Diaz and Diaz who obtain in [202] the necessary and sufficient conditions on Φ for the existence of finite extinction time for solutions of the GPME in bounded domain. It reads

$$\int_0^u \frac{ds}{\Phi(s)} < \infty. \tag{5.92}$$

The study is generalized by a number of authors to the GPME with zero-order terms [315], [331], and with nonlinear boundary conditions in [369].

§5.11. Here are some additional observations:

(1) The study of reaction-diffusion equations with porous medium diffusion term has a very extensive literature that falls completely out of the scope of our text. We refer to the book of Samarski et al. [469] which specializes in blow-up problems. For early references we can mention [53, 46]. The presence of convection terms has also been studied by a number of authors, cf. [446] and its references.

(2) The theory of equations in non-smooth domains is important in the applications but not often treated in the theory. We refer for recent work to [1, 2].

Summary and perspective

Let us recapitulate our progress thus far. We have posed the problem, introduced a concept of weak solution, and proved existence and uniqueness results in that framework for a suitable class of data that includes all bounded functions u_0 and f. The solutions belong to the energy class. Moreover, for Φ similar to the PME case the solutions for nonnegative (or nonpositive) data can be constructed as limits of classical solutions of the same equation after approximating the data, while for data of both signs the equation has to be approximated too. The solutions also satisfy the expected comparison theorem.

Though we have shown the order properties of the constructed solutions, the proof of continuous dependence will be left to the next chapter where it will be addressed by the L^1 technique, an important tool that deserves some attention.

We have started the qualitative analysis by showing that solutions are uniformly bounded for $t \geq \tau > 0$ for the kind of equations we want to study. A number of other properties have been established. This fits into the picture we had in mind.

The chapter covers the basic existence and uniqueness theory and some of the main properties. A large number of more advanced questions are left open and will be tackled in the next chapters. Note finally that for most of the results of this chapter, the restriction of superlinear growth on Φ is not needed and the PME with $m > 0$ is acceptable.

Future chapters will introduce new definitions of generalized solution, like L^1-limit solutions, very weak solutions, continuous weak solutions, strong solutions and mild solutions, needed to account for more generality in the data, more general equations or different approach. And there are further options like entropy solutions, renormalized entropy solutions, viscosity solutions, kinetic solutions and dissipative solutions to be used in more general contexts, not needed at the basic level. We should not forget singular solutions which is a different direction. This variety is one of the aspects that makes the theory of nonlinear diffusion an active research field.

Problems

Problem 5.1 (i) In the context of Problem HDP, check that a classical solution of problem (5.1)–(5.3) is automatically a weak solution of the problem. (ii) Prove that a weak solution in Q_T is also a weak solution in Q_{T_1} if $0 < T_1 < T$.

Problem 5.2 The concept of initial data implicit in Definition 5.4 implies a weak form of convergence to the initial data as stated in Proposition 5.2. Prove it.

Problem 5.3 Prove that function w given by the ZKB formula (5.13) is a weak solution of the equation under the conditions stated below the formula.

Hint. In order to check the integral equalities (5.7) we may proceed as follows: First, we note that the function w is C^∞ away from the free boundary $|x| = r(t)$. We then divide Q_T into two regions $Q_1 = \{(x,t) \in Q_T : \ |x| < r(t)\}$, where $u > 0$, and $Q_2 = Q_T \setminus Q_1$ where $u = 0$. The integrals are then reduced to Q_1. Use now the fact that w is a classical solution of the equation inside Q_1 and also that w^m is C^1 up to $|x| = r(t)$ to eliminate the boundary terms in the integrations by parts.

Problem 5.4 Complete the convergence parts of Lemma 5.9. In particular, (i) show that $Z(s)$ and $Z_n(s)$ are strictly increasing continuous functions and that $Z_n(s) \to Z(s)$ uniformly on compacts; show that if $\Lambda = Z^{-1}$, $\Lambda_n = Z_n^{-1}$, they are also increasing continuous functions and that $\Lambda_n(s) \to \Lambda(s)$ uniformly on compacts; (ii) show that $u_n = \Lambda_n^{-1} z_n$ converges uniformly to u, the weak limit of u_n: show that $w_n = \Phi_n(\Lambda_n(z_n))$ also converges uniformly to w.

Problem 5.5 Using (5.51), obtain a decay rate for the PME of the form

$$\iint_{Q^\tau} |(u^q)_t|^2 \, dxdt = O(\tau^{-\frac{2(q-1)}{m-1}}) \iint_{Q^\tau} |\nabla u^m|^2 dxdt \,. \tag{5.93}$$

Hint. We only need to observe that

$$(u^q)_t = (2q/(m+1))u^{q-(m+1)/2}(u^{(m+1)/2})_t$$

and recall that u is bounded in Q^τ by Proposition 5.17. Combining inequalities (5.47) and (5.44) in (τ, T), with $T \to \infty$, with the L^∞-estimate (5.57), we get (5.93).

Problem 5.6 Prove that we have the following result for weak energy solutions of the PME:

$$\frac{8m}{(m+1)^2}\iint\limits_{Q_{12}} \left|\frac{d}{dt}(u^{(m+1)/2})\right|^2 + \int\limits_{\Omega} |\nabla u^m(x,t_2)|^2 dx \le \int\limits_{\Omega} |\nabla u^m(x,t_1)|^2 dx, \tag{5.94}$$

where $0 < t_1 \le t_2$ and $Q_{12} = \Omega \times (t_1, t_2)$.

Problem 5.7 Aronson's non-smoothness example. Take as domain a ball $\Omega = B_R(0)$, take smooth initial data $u_0(x)$, that are radially symmetric, and assume that $u_0(x) = c|x|^2$ for $0 \le x \le r_1 < R$, and is positive and integrable outside with $u_0(R) = 0$. Prove that in a finite time the solution cannot have a smooth pressure.

Hint: Write the equation for the pressure

$$p_t = (m-1)p\Delta p + |\nabla p|^2.$$

The solution is radially symmetric by uniqueness. As long as p is smooth, it must be zero at $x = 0$. Derive now the equation for θ, the Laplacian of the pressure:

$$\theta_t = (m-1)p\Delta\theta + 2m\nabla p\nabla\theta + (m-1)|\theta|^2 + 2\sum_{i,j}(\partial^2_{ij}p)^2.$$

At $x = 0$ we get, as long as $p = 0$,

$$\theta_t = (m-1)|\theta|^2 + 2\sum_{ij}(\partial^2_{ij}p)^2 \ge (m-1)|\theta|^2.$$

Since $\theta(0,0) = 2dc > 0$, integrating the inequality means that $\theta(0,t)$ blows up in finite time. At this time, p cannot be C^2 in space nor C^1 in time.

Problem 5.8 The Heat Equation. Adapt the theory of this chapter to the Heat Equation $u_t = \Delta u$. In particular,

(i) Use the methods of this section to prove existence, uniqueness and continuous dependence.

(ii) Show that the solutions are bounded for positive times but there cannot be a universal bound like (5.57).

(iii) Show that solutions are $C^\infty(Q)$ and not only continuous. A continuity higher than Hölder continuity is false for the PME due to the example of the ZKB solutions.

Problem 5.9 The Non-homogeneous Boundary Problem.

(i) Prove the boundedness of solutions of Proposition 5.18 under the assumptions

$$t^{m/(m-1)}f, \; t^{m/(m-1)}g \quad \text{bounded.}$$

(ii) Prove an L^1-contraction result for fixed boundary data: If u and $\widehat{u}$ are two weak solutions with data (u_0, f, g) and $(\widehat{u}_0, \widehat{f}, g)$ resp., then

$$\|(u(t) - \widehat{u}(t))_+\|_1 \le \|(u_0 - \widehat{u}_0)_+\|_1 + \int_0^t \|(f(s) - \widehat{f}(s))_+\|_1 ds. \tag{5.95}$$

(iii)* Use this estimate to construct a theory of weak solutions with L^1 initial data and forcing term, and bounded and regular data on Σ. See next chapter

(iv)* Consider the inhomogeneous boundary problem with less regular boundary.

Problem 5.10 * Prove a universal L^∞ bound as in Proposition 5.17 under weaker growth assumptions on Φ.

Problem 5.11 FRIENDLY GIANTS.

(i) Check that for every $\tau > 0$, $\widetilde{U}(x, t + \tau)$ is a weak solution of problem HDP.

(ii) Construct the special solution with initial data $U(x, 0) = +\infty$ and forcing term $f = C > 0$ and show that it is the maximal solution for a certain class of weak solutions.

(iii) Do the same for the PME with $f = t^{-m/(m+1)}C$. Find the associated nonlinear elliptic problem and solve it.

Problem 5.12 Continues the previous problem in $d = 1$.

(i) Compute numerically the *nonlinear sine function* $f_m(x)$ and discuss its shape as a function of m. Consider theoretically and numerically the limit situation $m \to \infty$.

[*Hint:* use an appropriate variable in order not to lose the detail of the asymptotic information. See [511].]

(ii) Study the convergence as $m \to 1$ of the Friendly Giant to the linear approximant in the generality of bounded domains in several space dimensions. Note that a convenient scaling is needed.

Problem 5.13 CLASSICAL FREE BOUNDARY SOLUTIONS. We assume for simplicity that $\Phi(s)$ is smooth for $s > 0$. We propose the following definition:

Definition 5.6 A function $u \ge 0$ defined in a closed cylinder $Q = \overline{\Omega} \times [0, T]$, Ω as before, is called a *classical free boundary solution* if there is a C^1-hypersurface $\Gamma \subset Q$ with normal not oriented along the t-axis, and such that

(i) $\Gamma = \partial\{u > 0\} \cap \{u = 0\}$,

(ii) $u \in C(Q)$, $u \in C^\infty(\{u > 0\})$ and

(iii) $\nabla_x \Phi(u)$ is continuous up to the free boundary Γ and $\nabla_x \Phi(u) = 0$ on Γ.

(Other variants are possible but need not bother us now; the condition on the normal means that there is always a well-defined space normal.)

(i) Prove that the (delayed) ZKB and the TWs are classical free boundary solutions.

(ii) Prove that a classical free boundary solution is a weak solution and satisfies the energy estimates.

Problem 5.14 Construct a separable solution of the FDE in the range $0 < m < 1$ of the form $U(x,t) = (T-t)^{1/(1-m)}F(x)$ by solving the elliptic equation for F. See Section 22.9.1. Or solve the ODE for F in case Ω is a ball and F is radially symmetric.

Problem 5.15 * Study project. Construct a Friendly Giant for the GPME in inhomogeneous media, $\Phi = \Phi(x,u)$. Derive a universal a priori bound.

Problem 5.16 * Establish the main existence and uniqueness results of this chapter without the assumption that Φ is strictly increasing.

Chapter 6

The Dirichlet Problem II. Limit solutions, very weak solutions and some other variants

We continue in this and the next chapter with the analysis of the initial-and-boundary value problem. In Chapter 5 the GPME was considered, the Dirichlet problem was posed in a spatial bounded domain Ω, and the problem was shown to be uniquely solvable in a class of weak solutions. It was also shown that these weak solutions are not always classical solutions. Some important questions were left open and are worth exploring, like: how general can the data be? Are there any natural and useful alternatives to the proposed definition of weak solution? Here, we address these questions and present extensions of the already developed theory. We recall that the central issue is to construct an existence theory as wide as possible and complement it with uniqueness and stability. Now, it is not automatic that the most natural class of data for existence purposes coincides with the class where uniqueness and stability can be proved. This is a standard source of complication in the theories, namely combining well-posedness with having the widest possible (or at least wide enough) class of data.

We first discuss stability and limit solutions. A main property of the classical solutions examined in Chapter 3 is the continuous dependence with respect to the data, that is shown to take place in L^1 norm according to Proposition 3.5. This idea can be extended to prove continuous dependence of the weak solutions constructed in Chapter 5 with respect to the data. In this way, well-posedness is established. But once this is done, it is quite easy to perform an extension of the class of solutions to encompass merely integrable data. This is done however at the price of resorting to a new solution concept, *limit solution*. See Section 6.1. We solve in this way the Homogeneous Dirichlet Problem for the GPME with general L^1 data. Limit solutions will appear again in Chapter 10 in a slightly different guise associated to time discretizations, and that version will be called mild solutions. The equivalence of both approaches must be proved!

Limit solutions are a real extension of the concept of weak solution, but lack an intrinsic functional characterization other than the indirect statement that they are limits of weak solutions. Section 6.2 addresses this inconvenience by resorting to the concept of *very weak solution*. Uniqueness results are proved in that setting, cf. Theorem 6.5 and Corollary 6.7, which improve in a substantial way the uniqueness of weak solutions of Theorem 5.3. A key point of this section is the technique of duality used in the uniqueness proof, which is presented here in a simple setting. The section also includes the definition of trace of a solution at a given time.

In Section 6.3 we briefly explore the dependence of the solutions on the variation of the domain, a question of practical interest.

In Section 6.4 we specialize to the case $f = g = 0$ and present the main ideas of semigroups applied to the GPME in the context of limit solutions.

We then revisit the basic theory of weak solutions to address the issue of solutions with L^1 initial data. Such extension can be obtained at a low cost if Φ is superlinear and f is assumed to be bounded. This is done in Section 6.5 and needs a modification of the old concept to accommodate the new data. The relation of both concepts of weak solution is carefully analyzed.

Finally, we return to the question of possible generality of the data and present two further extensions of the already developed theory. In Section 6.6 we consider the existence of weak and limit solutions with more general initial data, taken in weighted spaces. Section 6.7 contains another extension: we now allow for data in the space $H^{-1}(\Omega)$; this is not a space of functions, but a space of distributions.

We can consider the material of this chapter as advanced reading, except for Sections 6.1 and 6.4 that are recommended at the basic level.

6.1 L^1 theory. Stability. Limit solutions

This section takes into account the L^1-Contraction Principle, that we have proved in Section 3.2.3 for smooth solutions of the Filtration Equation, and that has appeared at some stages of the constructions of Chapter 5. We use this property to establish the stability of the constructed solutions and also to make an extension of the existence result.

6.1.1 Stability of weak solutions

It is easily seen that the L^1-Contraction Principle continues to hold in the limit for the weak solutions constructed from classical solutions by approximation. Let us explicitly state the property in our present setting.

Proposition 6.1 *The statement of Proposition* 3.5 *holds for the weak solutions constructed in Theorem* 5.7. *In other words, for two weak energy solutions u and $\widehat{u}$ with initial data u_0, $\widehat{u}_0$ and forcing terms f, $\widehat{f}$ respectively, we have for every $t > \tau \geq 0$*

$$\|(u(t) - \widehat{u}(t))_+\|_1 \leq \|(u(\tau) - \widehat{u}(\tau))_+\|_1 + \int_\tau^t \|(f(s) - \widehat{f}(s))_+\|_1 \, ds. \tag{6.1}$$

Remarks (1) This is a fundamental property that will allow us to develop existence, uniqueness and stability theory in the space $L^1(\Omega)$ below. For the moment it serves the purpose of providing us with a stability result for the just constructed weak energy solutions. We recall that, as pointed out in Section 3.2.3, formula (6.1) implies the plain contraction:

$$\|u(t) - \widehat{u}(t)\|_1 \leq \|u_0 - \widehat{u}_0\|_1 + \int_0^t \|f(s) - \widehat{f}(s)\|_1 ds. \tag{6.2}$$

(2) This result implies the uniqueness of solutions of problem (5.1)–(5.3) by a new technique (the L^1-technique) which is completely different from that of Theorem 5.3. Indeed, estimate (6.1) not only implies L^1-dependence of solutions on data, but also the Comparison Principle, as stated in the end of Theorem 5.5: if $u_0 \le \widehat{u}_0$ a.e. and $f \le \widehat{f}$ a.e. in Q, then $(u_0 - \widehat{u}_0)_+ = 0$ a.e., then by estimate (6.1) it follows that $(u(t) - \widehat{u}(t))_+ = 0$ a.e., hence, $u(t) \le \widehat{u}(t)$ a.e.

(3) The following is an important observation for the theory of the PME and related equations: the proof of the L^1-Contraction Principle does not depend on any particular properties of the nonlinearity $\Phi(u)$. It works in the same way whenever Φ is a monotone function. This has made the L^1 estimate a key item in the theory of the Filtration Equation $u_t = \Delta\Phi(u)$. On the contrary, similar estimates for L^p norms with $p > 1$ do not exist if the filtration equation is not linear (i. e., unless we deal with the Heat Equation).

6.1.2 Limit solutions in the L^1 setting

The L^1 techniques are quite different in spirit from the energy estimates that form the core of the previous chapter. We pursue here the exploitation of such L^1 estimates to construct generalized solutions of a new type for more general data.

Indeed, the continuous dependence in L^1 norm stated in Proposition 6.1 allows us to introduce a concept of solution of Problem HDP for data $u_0 \in L^1(\Omega)$ and $f \in L^1(Q_T)$. This is done by approximation with a sequence of data $(u_{0n}, f_n) \in L_\psi(\Omega) \times L^\infty(Q_T)$ such that $u_{0n} \to u_0$ in $L^1(\Omega)$ and $f_n \to f$ in $L^1(Q_T)$. We may even take as data for the approximations bounded or continuous functions, since these subspaces are dense in L^1. The limit is well-defined by virtue of estimate (6.1).

Definition 6.1 *We call every such function a limit solution of Problem HDP for the GPME. The class is denoted as $\mathcal{LS}$.*

We obtain the following result for limit solutions.

Theorem 6.2 *Let Φ be a monotone function as in Section* 5.2*. Then, for any $(u_0, f) \in L^1(\Omega) \times L^1(Q_T)$ there exists a unique $u \in C([0,\infty) : L^1(\Omega))$ that solves problem* HDP *in the sense of limit solutions. The weak solutions of Theorem* 5.7 *are limit solutions. The map: $(u_0, f) \mapsto u$ is an ordered contraction from $L^1(\Omega) \times L^1(Q_T)$ into $C([0,\infty) : L^1(\Omega))$ in the sense that* (6.1) *holds.*

Proof. Note that the last statement implies continuous dependence in L^1-norms and means that the problem is well-posed in those spaces. We have to prove that the limit is independent of the approximating sequence, and also that it is continuous from $[0, \infty)$ into $L^1(\Omega)$.

(i) The independence of the approximating sequence is an easy consequence of the L^1 dependence estimate.

(ii) For the proof of continuity, assume first that u_0 is continuous in $\overline{\Omega}$ and f bounded. Then, the method of initial barriers presented in detail in Section 7.5.1 proves that u is continuous at $t = 0$. Hence, for every $\varepsilon > 0$ there is a $\tau > 0$ such that $\|u(h) - u(0)\|_1 \le \varepsilon$ if $0 < h < \tau$. By the L^1 stability estimate

$$\|u(t + h) - u(t)\|_1 \le \|u(h) - u(0)\|_1 \le \varepsilon$$

for every $t > 0$ and $0 < h < \tau$. It follows that $u \in C([0,T] : L^1(\Omega))$.

(iii) For any $u_0 \ge 0$, we approximate with functions $\widehat{u}_0$, $\widehat{f}$ as above and write, using Proposition 6.1,

$$\begin{aligned}\|u(\tau) - u_0\|_1 &\le \|u(\tau) - \widehat{u}(\tau)\|_1 + \|u_0 - \widehat{u}_0\|_1 + \|\widehat{u}(\tau) - \widehat{u}_0\|_1 \le \\ &\le 2\|u_0 - \widehat{u}_0\|_1 + \int_0^\tau \|f(s) - \widehat{f}(s)\|_1\, ds + \|\widehat{u}(\tau) - \widehat{u}_0\|_1.\end{aligned}$$

Therefore, as $\widehat{u}_0 \to u_0$ and $\tau \downarrow 0$ we get $u(\tau) \to u_0$. This settles the continuity at $t = 0$. To settle it at any other time $t > 0$, we may displace the origin of time and argue as before at the times t and $t + \tau$. $\square$

Abstract dynamics. We have arrived at an interesting concept, seeing solutions as continuous curves moving around in an infinite-dimensional metric space X (here, the function space $L^1(\Omega)$). Viewing solutions as continuous curves in a general space is the starting point of the abstract theory of differential equations, a way that we will travel quite often. In the so-called Abstract Dynamics it is typical to forget the variable x in the notation and look at the map $t \mapsto u(t) \in X$, where $u(t)$ is the abbreviated form for $u(\cdot, t)$.

Remarks. (1) Note that the theorem allows to define the value $u(t)$ of a limit solution (in particular, of a weak solution) u at any time $t > 0$ as a well-defined element of $L^1(\Omega)$. Actually, in many cases, as when Φ is superlinear and f is bounded, it is an element of $L^\infty(\Omega)$.

(2) If u_0 and f are bounded the initial regularity is better. In that case the initial data are taken in the L^p sense: $\tilde{u}(t) \to \tilde{u}(0)$ in $L^p(\Omega)$, for every $p < \infty$. We will see later that the solution $u(x,t)$ is Hölder continuous for all $t > 0$; if u_0 is continuous, then the convergence takes place uniformly in x as $t \to 0$, see Section 7.5.1.

(3) Unfortunately, there are no equivalent L^1 estimates for the Dirichlet Problem with nonhomogeneous data $g \ne 0$.

We end this subsection with a simple but very useful consequence.

Corollary 6.3 *Let u be a limit solution with data $u_0 \in L^1(\Omega)$ and $f \in L^1(Q)$. If $t_1 > 0$, then $\tilde{u}(x,t) = u(x, t + t_1)$ is the limit solution with data $\tilde{u}_0(x) = u(x, t_1)$ and forcing term $\tilde{f}(x,t) = f(x, t + t_1)$.*

This important result is immediate for the approximations. We leave the details to the reader.

Remark. Let us note that any concept of limit solution depends on the type of admissible approximations and on the functional setting in which limits are taken. The definition we propose applies in the L^1 setting. If needed, these solutions will be called L^1-limit solutions. For an extension see Section 6.6.

6.2 Theory of very weak solutions

The continuous dependence with respect to the L^1-norm is a powerful property. It has allowed us to extend the existence result for weak solutions of the preceding section and consider as data any nonnegative function $u_0 \in L^1(\Omega)$ at the price of introducing the concept of limit solution, a function $u \in C([0,\infty) : L^1(\Omega))$ with $u(0) = u_0$ that is obtained as limit of weak energy solutions.

However, an important question remains: *Is the limit solution itself a weak solution according to Definition 5.4?* It turns out that in general we lose the control on $\nabla\Phi(u)$, which is important in giving a sense to identity (5.7). So, we are left with the problem of relating limit solutions to some weaker theory of solutions. Uniquely identifying the limit solutions as weak solutions in a certain sense is not an easy task. Though the text is not primarily intended to discuss the full theory of the GPME, we will explore in the sequel some aspects of the use of alternative theories of weak solutions to describe limit solutions.

We consider here the concept of very weak solution that was introduced in Definition 5.2 as a possible alternative to build a theory of generalized solutions. We recall that a very weak solution is a distribution solutions with certain integrable derivatives. We apply the definition of very weak solution to the general Dirichlet Problem with boundary data $\Phi(u) = g$ on Σ_T as follows.[1] We assume that u_0, f and g are integrable functions in their respective domains.

Definition 6.2 *An integrable function u defined in Q_T is said to be a very weak solution of Problem* (5.52)–(5.54) *if* (i) $u, \Phi(u) \in L^1(Q_T)$;

(ii) *the identity*

$$\iint_{Q_T} \{\Phi(u)\,\Delta\eta + u\eta_t + f\eta\}\,dxdt + \int_{\Omega} u_0(x)\eta(x,0)dx = \int_{\Sigma_T} g(x,t)\partial_\nu\eta(x,t)dSdt \tag{6.3}$$

holds for any function $\eta \in C^{2,1}(\overline{Q}_T)$ which vanishes on Σ_T and for $t = T$.

As an extension of the definition, if u satisfies a modified condition (ii) with inequality $\le$ (instead of equality) for every test function $\eta \ge 0$, then we call it a *very weak supersolution*; if the same happens with inequalities ≥ 0, then u is a *very weak subsolution* of the GPME.

We see that the present concept generalizes the work done so far.

Example 6.1 Let Φ be a good nonlinearity in the sense of Section 3.2, let us assume that the data f, g, u_0 are smooth, and let us define a classical supersolution as a $C^{2,1}$ smooth function u such that

$$\begin{cases} u_t \ge \Delta\Phi(u) + f & \text{in } Q_T, \\ u \ge g & \text{on } \Sigma_T. \end{cases} \tag{6.4}$$

Then u is a supersolution in the present sense. The proof only needs a convenient integration by parts justified by the regularity we have. The same applies to classical subsolutions.

Proposition 6.4 *The weak solution in the sense of Definitions* 5.4 *and* 5.5 *is a very weak solution in the present sense. All limit solutions of the Homogeneous Dirichlet Problem constructed in Subsection* 6.1.2 *are also very weak solutions.*

[1] As before, $\Sigma_T = \partial\Omega \times [0,T)$ is the lateral boundary with measure $dSdt$; ν is the outer normal vector field.

Proof. The two first statements are clear by integration by parts. For the limit solutions, assume first the situation applied to a classical solution. Then, equation (6.3) holds. For a general limit solution we perform a the passage to the limit. The control of the $L^1(Q)$-norm of u is guaranteed by the L^1 stability estimate. As for the control of the approximations $\Phi(u_n)$ in $L^1(Q)$, we need a further estimate that we will develop in Section 6.6. It is as follows: according to Formula (6.27), for any pair of approximating solutions

$$\begin{aligned} &\int |u_n - u_m|\,\zeta(x)\,dx + \iint |\Phi(u_n) - \Phi(u_m)|\,dxdt \\ &\le \int |u_{0n}(x) - u_{0m}(x)|\,\zeta(x)\,dx + \int_0^t \int |f_n(t) - f_m(t)|\,\zeta(x)\,dxdt. \end{aligned} \tag{6.5}$$

where ζ is the unique solution of the problem

$$\Delta\zeta = -1 \quad \text{in } \Omega, \qquad \zeta = 0 \quad \text{on } \partial\Omega.$$

This means that $\Phi(u_n)$ converges in $L^1(Q_T)$. By the monotonicity of Φ, the limit is $\Phi(u)$ a.e. □

Alternative definitions. There are equivalent definitions of weak and very weak solution where integration in time is done in an interval $[t_1, t_2]$ with $0 < t_1 < t_2 < T$ and the values at the end-times t_1 and t_2 enter the definition. These versions appear often in the literature. We refer to Problems 6.3 and 6.4 for that interesting issue.

6.2.1 Uniqueness of very weak solutions

As commented above, the introduction of generalized solutions poses two related problems, first the problem of recognizing them as such when a candidate is given, then the problem of uniqueness of such objects. While the first problem leads naturally to the desire to relax the conditions in the definition of solution, the second is obviously easier if the definition of solution is stricter. Therefore, very weak solutions are likely to have a problem with uniqueness.

We present next a quite general uniqueness result for very weak solutions that imposes however some mild assumption on the integrability of the solutions. The main idea is solving a dual problem.

Theorem 6.5 *Let Ω be a bounded domain with smooth boundary. Let u_1 be a very weak subsolution of the GPME defined in Q_T for data u_{01}, f_1, g_1, and let u_2 be a very weak supersolution for data u_{02}, f_2, g_2. Assume moreover that both satisfy $u_i, \Phi(u_i) \in L^2(Q_T)$. If the data are ordered, $u_{01} \le u_{02}$ a.e., $f_1 \le f_2$ a.e., and $g_1 \le g_2$, then $u_1 \le u_2$ in Q_T.*

Proof. (i) We write the weak inequalities satisfied by u_1 and u_2 with respect to a test function $\varphi \in C_0^{1,2}(\overline{Q_T})$. We subtract to get

$$0 \le \iint_{S_T} \{(u_1 - u_2)\varphi_t + (\Phi(u_1) - \Phi(u_2))\Delta\varphi\}\,dxdt.$$

We now write $u = u_1 - u_2$. Defining

$$a(x,t) = \frac{\Phi(u_1) - \Phi(u_2)}{u_1 - u_2}$$

where $u_1 \neq u_2$ and $a(x,t) = 0$ if $u_1 = u_2$, we may write $\Phi(u_1) - \Phi(u_2) = a(x,t)u(x,t)$ for a measurable function $a \geq 0$.

(ii) The next step is choosing a smooth test function $\theta(x,t) \geq 0$ compactly supported in Q_T and solving the inverse-time problem

$$\left\{ \begin{array}{ll} \varphi_t + a_\varepsilon \Delta\varphi + \theta = 0 & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(x,T) = 0 & \text{for } x \in \Omega, \end{array} \right. \tag{6.6}$$

where a_ε is a smooth approximation of a such that $\varepsilon \leq a_\varepsilon \leq K$. Note that this is a correct parabolic problem if we define a new time as $t' = T - t$ (i. e., inverse time). Therefore, it has a smooth solution $\varphi \geq 0$. We then get for the difference $u = u_1 - u_2$ the estimate:

$$\iint_{Q_T} u\,\theta\,dxdt \leq \iint_{Q_T} |u||a - a_\varepsilon||\Delta\varphi|\,dxdt = J. \tag{6.7}$$

In view of the estimates that will follow, we write the last term as

$$J \leq \left(\iint a_\varepsilon (\Delta\varphi)^2\,dxdt \right)^{1/2} \left(\iint \frac{|a - a_\varepsilon|^2}{a_\varepsilon} |u|^2\,dxdt \right)^{1/2}. \tag{6.8}$$

(iii) We need an a priori estimate for the term with $\Delta\varphi$. We multiply the equation satisfied by φ by $\zeta\Delta\varphi$ where $1/2 \leq \zeta(t) \leq 1$ is a smooth and positive function for $0 \leq t \leq T$ with $\zeta_t \geq c > 0$. Integrating gives

$$\iint \varphi_t \zeta \Delta\varphi\,dxdt + \iint \zeta a_\varepsilon (\Delta\varphi)^2\,dxdt + \iint \zeta\theta\Delta\varphi\,dxdt = 0.$$

Integration by parts of the first term, using that $\varphi(x,T) = 0$, gives

$$\iint \zeta\varphi_t \Delta\varphi\,dxdt = -\iint \zeta\nabla\varphi \cdot \nabla\varphi_t\,dxdt \geq \frac{1}{2} \iint |\nabla\varphi|^2\,\zeta_t dxdt.$$

It follows that

$$\frac{1}{2} \iint |\nabla\varphi|^2 \zeta_t\,dxdt + \iint \zeta a_\varepsilon (\Delta\varphi)^2\,dxdt \leq \iint \zeta(\nabla\theta \cdot \nabla\varphi)\,dxdt.$$

In view of the assumptions on ζ, a very easy application of Hölder's inequality gives the desired estimate in the form

$$\iint_{Q_T} a_\varepsilon |\Delta\varphi|^2\,dxdt + \iint |\nabla\varphi|^2\,dxdt \leq C \iint_{Q_T} |\nabla\theta|^2\,dxdt.$$

This estimate allows to return to (6.7), (6.8) and get

$$\iint_{Q_T} u\,\theta\,dxdt \leq C\|\nabla\theta\|_2 \left(\iint \frac{|a - a_\varepsilon|^2}{a_\varepsilon} |u|^2\,dxdt \right)^{1/2}. \tag{6.9}$$

(iv) At this stage we have to examine the way we construct the approximation so that the latter quantity goes to zero as $\varepsilon \to 0$, and the process is independent of θ. We do it like this: given $\varepsilon > 0$ we select two height $K > \varepsilon > 0$ and define $a_{K,\varepsilon} = \min\{K, \max\{\varepsilon, a\}\}$ (we will be taking K very large and ε very small). We take smooth approximations $a_n \to a_{K,\varepsilon}$ in L^p for all $p < \infty$. Then, we have

$$\iint |a-a_n|^2|u|^2\,dxdt \le 2\iint |a_{K,\varepsilon}-a_n|^2|u|^2\,dxdt + 2\iint ((a-K)_+ + \varepsilon)^2|u|^2\,dxdt.$$

Call the last integrals I_1 and I_2. The latter integrand is pointwise bounded by

$$2|u|^2(a^2+\varepsilon^2) = 2(\Phi(u_1)-\Phi(u_2))^2\chi(a>K) + 2\varepsilon^2|u|^2,$$

where $\chi(a>K)$ is the characteristic function of the indicated set. Therefore, using the square integrability of $\Phi(u_i)$ and u_i, we may take K large enough so that $I_2 \le (1/2)C\varepsilon^2$. Choosing now $n = n(\varepsilon, K)$ large enough we also get $I_1 \le C\varepsilon^2/2$. Then,

$$\iint |a-a_n|^2|u|^2\,dxdt \le C\varepsilon^2.$$

Since $a_n \ge \varepsilon$, we get in the end from (6.9) an estimate of the form

$$\iint_{Q_T} u\,\theta\,dxdt \le C\varepsilon^{1/2}\|\nabla\theta\|_2.$$

Finally, since $\varepsilon > 0$ was independent of θ, we conclude that

$$\iint_{Q_T} u\,\theta\,dxdt \le 0\,.$$

By the arbitrary choice of the smooth test function $\theta \ge 0$, we get $u \le 0$ a.e. in Q_T. □

The same line of proof can be used to treat the cases where the data u_0 and f are not ordered. We get the L^1 dependence in another way.

Theorem 6.6 *Let Ω be a bounded domain with smooth boundary. Let u_1 be a very weak subsolution of the GPME defined in Q_T for data u_{01}, f_1, g_1, and let u_2 be a very weak supersolution for data u_{02}, f_2, g_2. Assume that both satisfy $u_i, \Phi(u_i) \in L^2(Q_T)$. Then, if $g_1 \le g_2$, we have for every $t_0 \in (0,T)$:*

$$\int_\Omega (u_1(x,t_0)-u_2(x,t_0))_+\,dx \le \int_\Omega (u_{01}(x)-u_{02}(x))_+\,dx + \int_0^{t_0}\int_\Omega (f_1-f_2)_+ dxdt. \tag{6.10}$$

Proof. We repeat the proof, but taking now into account the differences $u_{01}-u_{02}$ and $f_1 - f_2$ that now do not disappear since they do not have a definite sign. In the end we get the inequality

$$\iint_{Q_T} (u_1-u_2)\,\theta\,dxdt \le M\int_\Omega (u_{01}(x)-u_{02}(x))_+\,dx + M\iint_{Q_T} (f_1-f_2)_+\,dxdt, \tag{6.11}$$

where M is a uniform upper bound for the functions $\varphi(x,t)$ used in the preceding proof. Now, we choose $\theta = \psi(x)\rho_\varepsilon(t-t_0)$ with $0 \le \psi \le 1$ and ρ_ε a standard smoothing kernel in one variable. By the Maximum Principle, we find that φ is bounded by a function $C(t)$ with $C(T) = 0$, $C'(t) = -\rho_\varepsilon(t-t_0)$, which tends to the characteristic function $\chi_{[0,t_0]}(t)$. Hence, writing $u_{i\varepsilon}(x,t_0) = \int u_i(x,t)\rho_\varepsilon(t-t_0)\,dt$ we have

$$\int_\Omega (u_{1\varepsilon}(x,t_0)-u_{2\varepsilon}(x,t_0))\,\psi(x)\,dx \le \int_\Omega (u_{01}(x)-u_{02}(x))_+\,dx + \int_0^{t_0+\varepsilon}\int_\Omega (f_1-f_2)_+ dxdt.$$

Passing to the limit $\varepsilon \to 0$ we get the inequality. □

Corollary 6.7 *Very weak solutions of Problem* HDP *for the GPME defined in Q_T and such that $u, \Phi(u) \in L^2(Q_T)$ are uniquely determined by their data. They coincide therefore with the limit solutions. They are weak solutions if they also meet the conditions of Theorem* 5.7 *on the data.*

6.2.2 Traces of very weak solutions

The definition of very weak solution allows to identify the value of the solution u at almost every time $t \in (0,T)$ as a function $u(t) \in L^1(\Omega)$. But we would like to have a definite value at all times. This is possible with some extra work thanks to the theory of traces, that we start here. The result holds for a local very weak solution of the GPME $u_t = \Delta\Phi(u) + f$ in $Q_T = \Omega \times (0,T)$, in the sense that $u \in L^1(0,T : L^1_{loc}(\Omega))$ and Definition 6.2 holds for every $\eta \in C^{2,1}(\overline{Q}_T)$ which vanishes for $t = T$ and near Σ_T; also, $f \in L^1(0,T : L^1_{loc}(\Omega))$.

Theorem 6.8 *Let u be a local very weak solution of the GPME in the above sense. Then, for every $t \geq 0$ there exists a distribution $\mu(t)$ such that*

$$\lim_{s\to t} \int_\Omega u(x,s)\,\eta(x)\,dx = \langle \mu(t), \eta\rangle \tag{6.12}$$

holds for all test functions $\eta \in C_0^2(\Omega)$. Moreover, for a. e. t $\mu(t)$ is a measure with density u: $d\mu(t) = u(x,t)\,dx$. If $u \geq 0$, then $\mu(t)$ is a Radon measure.

Proof. (i) Take a test function $\varphi(x) \in C_c^\infty(\Omega)$ and define the function

$$L_\varphi(t) = \int_\Omega u(x,t)\varphi(x)\,dx, \tag{6.13}$$

which is a locally integrable function of $t \in (0,T)$, well-defined for a.e. t. We want to define $L_\varphi(t)$ for all t. In order to do that, we use a test function of the form $\eta(x,t) = \varphi(x)\theta(t)$ in the definition of very weak solution to get

$$-\int_0^T L_\varphi(t)\partial_t\theta\,dt = \iint_{Q_T} \{\Phi(u)\,\Delta\varphi(x) + f\varphi(x)\}\theta(t)\,dxdt. \tag{6.14}$$

Now take $0 \leq t_1 < t < t_2 < T$, take a test function $\theta(t) \geq 0$ such that $\theta(t) = 0$ for $0 < t \leq t_1$ and $t_2 \leq t < T$, but $\theta(t) = 1$ for $t_1 + h < t < t_2 - k$. Then, pass to the limit as $h, k \to 0$ to obtain the function $\widetilde{\theta}(t) = 1$ for $t_1 \leq t \leq t_2$, and zero otherwise. Due to the local integrability of f and $\Phi(u)$, the limit in the right-hand side exists for every $0 \leq t_1 < t < t_2 < T$. Taking limits in the left-hand side we have

$$L_\varphi(t_2) - L_\varphi(t_1) = \int_{t_1}^{t_2}\int_\Omega \{\Phi(u)\,\Delta\varphi + f\varphi\}\,dxdt \tag{6.15}$$

for a.e. $t \in [0,T)$; let us call this set of times $\mathcal{T}_\varphi$, the Lebesgue points of L_φ. But the right-hand side makes sense for all $0 < t_1, t_2 < T$, and it is in fact a continuous function of t_1, t_2. Taking t_1 fixed in $\mathcal{T}_\varphi$ and $t_{2j} \to t$ with $t_{2j} \in \mathcal{T}_\varphi$, we may use the formula

$$X(t) = L_\varphi(t_1) + \int_{t_1}^{t}\int_\Omega \{\Phi(u)\,\Delta\varphi(x) + f\varphi(x)\}\,dxdt \tag{6.16}$$

as definition of $L_\varphi(t)$ for all $0<t<T$. It is easy to see that the limit is independent of t_1. Since it is finite for all $\varphi \in C_0^2(\Omega)$, we conclude that there is a linear functional on the set of functions $C_0^2(\Omega)$, a distribution $\mu(t)$, such that

$$\lim_{t_1\to t, t_1\in \mathcal{T}} \int_\Omega u(x,t_1)\varphi(x)\,dx = \langle \mu(t),\varphi\rangle. \tag{6.17}$$

The limit has been taken as $t_1 \uparrow t$ but it is easy to see that the limit as $t_1 \downarrow t$ gives the same value. This formula is the definition of the trace of u at time t. We recall that for a.e. time t the trace is the value of $u(t)$; usually, we simply write $u(t)$ for the trace by abuse of notation. In that notation we can write the definition of very weak solution in the equivalent form

$$\int\limits_\Omega \{u(x,t_2)\eta(x,t_2) - u(x,t_1)\eta(x,t_1)\}\,dx = \iint\limits_{\Omega\times(t_1,t_2)} \{\Phi(u)\,\Delta\eta + u\eta_t + f\eta\}\,dxdt \tag{6.18}$$

for all $0<t_1<t_2<T$ and all test functions $\eta \in C^{2,1}(Q_T)$ which are compactly supported in the space variable (uniformly in time).

(ii) Some properties of the family $\mu(t)$ are immediate. Thus, equation (6.16) implies that

$$\mu(t_2) - \mu(t_1) = \Delta \int_{t_1}^{t_2} \Phi(u)\,dt + \int_{t_1}^{t_2} f\,dt \tag{6.19}$$

in the sense of distributions, $\mathcal{D}'(\Omega)$. Usually, μ is a function, but it need not be in general. A sufficient condition is: if $u \in L^\infty_{loc}(0,T : L^p_{loc}(\Omega))$ with $1<p<\infty$, then $\mu(t)\in L^p_{loc}(\Omega)$ for all t. □

We will make much use of traces in Chapter 13. We point out that the set of test functions that enter into formula (6.12) is $C_0^2(\Omega)$.

6.3 Problems in different domains

An interesting application of the preceding ideas happens when we consider the solutions of Problem HDP in two different domains $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^d$. On the one hand, we can compare the solution of the problem posed in $Q_2 = \Omega_2 \times (0,T)$ with data u_{02}, f_2, with the solution u_1 of the problem posed in $Q_1 = \Omega_1 \times (0,T)$ with initial data u_{01}, f_1, if we know that $u_2 \ge 0$.

Proposition 6.9 *Let u_1 be the energy weak (or limit) solution of the HDP posed in Q_1 with data u_{01}, f_1 and let u_2 be the solution of the HDP posed in Q_2 with data u_{02}, f_2. If $u_2 \ge 0$ in Q_2, $u_{01}(x) \le u_{02}(x)$ for $x\in\Omega_1$, and $f_1(x,t)\le f_2(x,t)$ in Q_1, then*

$$u_1(x,t) \le u_2(x,t) \quad \textit{for every} \quad (x,t)\in Q_1. \tag{6.20}$$

The proof relies on noting that we can easily take the approximations $u_{2,n}$ to solution u_2 in such a way that $u_{2,n}(x,t) \ge u_{1,n}$ on the parabolic boundary of Q_1, where $u_{1,n}$ are the approximations to solution u_1. Since the equation satisfied in Q_1 is the same but for the forcing term, the Maximum Principle implies that $u_{2,n}(x,t) \ge u_{1,n}(x,t)$ in Q_1. Note that a similar result holds if $u_2 \le 0$ if we change all the inequalities.

On the other hand, we have a continuity result with respect to the domain.

Proposition 6.10 *Let Ω_n a nondecreasing (resp. nonincreasing) family of bounded domains with Lipschitz continuous boundary and let Ω be a domain with the same regularity. We assume that $\Omega = \bigcup_n \Omega_n$ (resp., $\overline{\Omega} = \bigcap_n \overline{\Omega}_n$). Let u_n be the weak (or limit) solution of the HDP in Ω_n with data $(u_{0,n}, f_n)$, and let u be the weak (or limit) solution of the HDP in Ω_n with data (u_0, f). Under the assumption that $u_{0,n} \to u_0$ and $f_n \to f$ in L^1, we have $u_n \to u$ in the same norm.*

The convergence of the data is understood in the sense that we extend the data and solutions by 0 for $x \notin \Omega_n$, resp. $x \notin \Omega$, and then we assume that $u_{0,n} \to u_0$ in $L^1(\mathbb{R}^d)$ and $f_n \to f$ in $L^1(\mathbb{R}^d \times (0,T))$.

Proof. (i) Assume first that the family Ω_n is increasing. We will need a metric fact: the boundary of Ω_n tends to $\partial\Omega$ in the sense that

$$d_n = \max\{d(x, \partial\Omega) : x \in \partial\Omega_n\}$$

tends to zero as $n \to \infty$. We leave to the reader to check that fact.

Assume to begin with that all the data are uniformly bounded, so that the solutions are too. Then, with the notation of Theorem 5.7, the estimates on $\nabla_x \Phi(u_n)$ and $\partial_t Z(u_n)$ are uniform locally in $\mathbb{R}^d \times (0,T)$, so that we can pass to the limit and obtain a bounded function $u(x,t)$ with convergence a.e. in $\Omega \times (0,T)$. The energy estimate passes to the limit and we obtain $\Phi(u_n) \to \Phi(u)$ weakly in $L^2(0,T : H^1(\mathbb{R}^d))$. We now observe that since the support of all functions $\Phi(u_n)$ is contained in $\overline{\Omega}$, the limit takes place in $L^2(0,T : H_0^1(\Omega))$.

In order to check that the equation is satisfied we try to pass to the limit in the weak formulation of the solution u_n (given in formula (5.4)) for a test function η as in the Definition and with compact support in space. Since $d_n \to 0$, such a function is also an admissible test function for u_n when n is large enough. The weak convergences allow us to pass to the limit and show that u is a solution in $\Omega \times (0,T)$ with the correct data.

(ii) We consider now the case where the family Ω_n is decreasing under the same boundedness assumptions on the data. The same argument shows that $u_n \to u$ and $\Phi(u_n) \to \Phi(u)$ weakly and a.e. in $\Omega \times (0,T)$. The weak formulation of the equation is now immediately satisfied. We have to justify that $\Phi(u(t)) \in H_0^1(\Omega)$ for a.e. t and this follows from the fact that the support of $\Phi(u_n)$ is contained in $\overline{\Omega}_n \times [0,T]$ and the relation between Ω_n and Ω appearing in the statement.

(iii) Under any of the monotonicity assumptions on Ω_n, if the data are general and converge in $L^1(\mathbb{R}^d)$, $L^1(\mathbb{R}^d \times (0,T))$ resp., we use the L^1 stability to conclude the result for limit solutions. $\square$

We continue the study of the relation between the concept of solution in different nested domains in Problem 6.9.

6.4 Limit solutions build a Semigroup

Let us now pay attention to the functional properties of the class of solutions generated by the GPME with $f = g = 0$. In that situation, and as we have pointed out, if $u(x,t)$ is a limit solution with data $u_0(x)$ and $\tau > 0$, then $v(x,t) = u(x, t+\tau)$ is the solution corresponding to data $v_0(x) = u(x,\tau)$. This allows to show that the definition generates a very interesting functional object, a *semigroup of contractions*.

Definition 6.3 (Semigroup) *Let S_t, $t \geq 0$, be a family of maps of a metric space (E, d) into itself. It is called a semigroup if the following conditions hold*

(i) S_0 *is the identity map;*

(ii) *for every $t, s \geq 0$ we have*

$$S_{s+t} = S_t \circ S_s.$$

In case we have (iii)

$$\lim_{t \to 0} S_t x = x$$

for every $x \in E$, we say that S_t is a strongly continuous semigroup, also known as a C_0-semigroup.

Notice that in the usual notation S_t, the subscript t does not indicate partial derivative. In order to avoid confusions we favor the notation $S(t)x$, but the standard notation is as it is.

There are different classes of semigroups considered in the literature. Thus, in the quite developed linear theory, the metric space is a normed space, or better a Banach space, and the maps: $u_0 \mapsto u(t) = S_t u_0$ are linear transformations in the linear space. It is called a *linear* semigroup. But the theory of nonlinear operators can dispense with that requirement, and E is quite often a closed convex subset of a Banach space of functions. The denomination *nonlinear semigroup* refers to the general theory and includes in practice all semigroups, linear or not.

Semigroups as a language. Types of semigroups

This is not a book about semigroups. We rather think of Semigroup Theory as a convenient and motivating language in which our problems can be seen from a global point of view, which may also add some intuitions. Thus, a theory of existence and uniqueness is called existence of a semigroup, construction by approximation is seen as convergence of semigroups, a theory with comparison is termed an ordered semigroup, and the universal bound gives rise to a universally bounded semigroup.

Some questions are easier to understand in this new language. Thus, we will be interested in knowing whether our weak solutions are indeed bounded, or classical solutions, or at least continuous functions, or belong to a compact class. This translates in terms of classes of semigroups.

Definition 6.4 *A semigroup acting on a metric space E is called bounded if it maps bounded sets $K \subset E$ into bounded sets for every $t > 0$. If the bound of $S_t(K)$ is uniform for all $t > 0$, then we say that it is uniformly bounded.*

It is called contractive if S_t satisfies

$$d(S_t x, S_t y) \leq d(x, y).$$

We also say that it is a semigroup of contractions. Actually, the more accurate term should be non-expansive*, and contractive should be reserved for the case $d(S_t x, S_t y) < d(x, y)$, but the usual language in PDEs is as described.*

A semigroup is called regularizing, or smoothing, if it maps the space into a subspace F of smoother functions.

A semigroup is called compact if it maps bounded subsets of E into compact subsets for every $t > 0$.

The reference semigroups in Diffusion Theory are the ones generated by the Heat Equation. As is well-known, all the above properties apply in the case of the HDP in a bounded domain. See Problem 6.4.

The GPME Semigroup

Let us go back to the GPME with zero forcing term $f = 0$. We consider as linear space $X = L^1(\Omega)$, and as a special convex set

$$E = L^1(\Omega)_+ = \{g \in L^1(\Omega) : g \geq 0 \ a.e.\}.$$

We define the maps $S_t : X \to X$ or $S_t : E \to E$ by

$$S_t(u_0) = u(t), \tag{6.21}$$

where $u_0 \in E$ and $u(t)$ is the limit solution of the HDP for the PME. We have proved the following result.

Theorem 6.11 *The maps S_t define a continuous semigroup of contractions in $X = L^1(\Omega)$, and S_t preserves E. The semigroup is uniformly bounded. If Φ is superlinear, it is regularizing into $L^\infty(\Omega)$.*

Note that the semigroup property is equivalent to checking that, given a solution $u = u(t)$ with initial data u_0 and given a time $s > 0$, the solution with initial data $u(s)$ is

$$v(t) = u(s + t).$$

In other words, it is an existence and uniqueness theorem. At times we will refer to the contractions as L^1-contractions to make clear what is the norm used in the statement.

We will prove in the next chapter that bounded solutions are indeed C^α functions for some $\alpha \in (0, 1)$. In other words, we will prove that our semigroup regularizes from $L^1(\Omega)_+$ into $C^\alpha(\Omega)$. The same idea proves that the semigroup is compact. This property is important for many applications, for instance in the study of asymptotic behaviour.

6.5 Weak solutions with bounded forcing

We have extended the class of weak energy solutions into a larger class, the limit solutions, and we have mentioned that this new class enjoys the properties of well-posedness but lacks a good characterization as solutions of the equation. The characterization of limit solutions can be done by slightly modifying the concept of weak solution under some restriction on the forcing data. Recall that $f = 0$ is a current assumption anyway in the applications. Here, we admit bounded f. We also assume that Φ is superlinear as in Theorem 5.17.

It happens that, thanks to the universal bound, in passing to the limit $n \to \infty$ in the sequence u_n considered in §6.1.2 and checking that u is a weak solution, we encounter difficulties near $t = 0$. In general, u does not satisfy the condition $\Phi(u) \in L^2(0, \infty : H_0^1(\Omega))$, which is important in giving a sense to identity (5.7), therefore we must change our definition of weak solution.

A convenient modification of the definition of weak solution to circumvent that difficulty and deal with solutions with L^1 data is as follows.

Definition 6.5 *A nonnegative function $u \in C([0,\infty) : L^1(\Omega))$ is said to be a weak solution of problem* (5.1)–(5.3) *if*

(i) $\Phi(u) \in L^2_{loc}(0,\infty : H^1_0(\Omega))$,

(ii) *u satisfies the identity*

$$\iint_Q \{\nabla\Phi(u)\cdot\nabla\eta - u\eta_t - f\eta\}dxdt = 0 \tag{6.22}$$

for any function $\eta \in C^1_0(\overline{Q})$ which vanishes everywhere for $0 < t < \tau$ for some $\tau > 0$,

(iii) $u(0) = u_0$.

Note that $\Phi(u) \in L^2_{loc}(0,\infty : H^1_0(\Omega))$ means that $\Phi(u) \in L^2(\tau, T : H^1_0(\Omega))$ for every $0 < \tau < T < \infty$, but not necessarily for $\tau = 0$. We immediately see that a weak solution in the sense of Definition 5.4 is also a weak solution in the present sense if we can ensure that it belongs to the class $C([0,\infty) : L^1(\Omega))$. We will come back to the relation between both definitions. Let us for the moment denote both concepts of solution, old and new, as weak-1 and weak-2.

Theorem 6.12 *Let us assume that Φ is superlinear and f is bounded. Then, there exists a unique weak-2 solution of Problem* (5.1)–(5.3) *with given initial data $u_0 \in L^1(\Omega)$. The Comparison Principle, the Contraction Principle, and the Universal Sup Bound hold for this class of weak solutions.*

Proof. (i) *Existence.* We construct approximations u_n as indicated in § 6.1.2 and pass to the limit using the L^∞ estimate derived in Proposition 5.17, and the L^1 estimates of Propositions 3.5 and 6.1, plus the energy estimate (5.20), (5.39). The limit solution is a weak-2 solution, and the reader is asked to carefully verify the details.

(ii) *Uniqueness.* It relies on a rather tricky way of reducing the problem to the old uniqueness proof plus stability estimates. Let u_1, u_2 be weak-2 solutions of the problem with same initial data u_0. By the continuity assumption, given $\varepsilon > 0$, there exists $\tau > 0$ such that $\|u_1(t) - u_0\|_1$, $\|u_2(t) - u_0\|_1 < \varepsilon$ for $0 \le t \le \tau$.

Consider now the functions $\widetilde{u}_i(x,t) = u_i(x, t+\tau)$, $i = 1,2$. Function $\widetilde{u}_i$ satisfies the assumptions of Proposition 5.1, hence it is a weak-1 solution of the same problem with initial data $u_i(x,\tau)$ (see also Problem 6.3). On the other hand, the assumption $\Phi(u) \in L^2_{loc}(0,\infty : H^1_0(\Omega))$ implies that for a.e. $\tau > 0$, $\Phi(u_i(\tau)) \in L^2(\Omega)$; since $\Psi(u) \le |\Phi(u)u| \le C|\Phi(u)|^2$, for such a τ the weak-1 solution $\widetilde{u}_i(t)$ satisfies all the conclusions of Theorem 5.7, and also the L^1 dependence of Proposition 6.1. We thus get for $t > \tau$,

$$\begin{aligned}\|u_1(t) - u_2(t)\|_1 &= \|\widetilde{u}_1(t-\tau) - \widetilde{u}_2(t-\tau)\|_1 \\ &\le \|\widetilde{u}_1(0) - \widetilde{u}_2(0)\|_1 = \|u_1(\tau) - u_2(\tau)\|_1 < \varepsilon\,.\end{aligned}$$

We may now let $\varepsilon, \tau \to 0$ to get $u_1(t) = u_2(t)$ a.e. for every $t > 0$.

(iii) The validity of the Sup Bound (Proposition 5.17), the Contraction Principle (Proposition 3.5), and the Comparison Principle are just a consequence of the limit process. □

Eliminating the restriction of superlinearity

The assumption of superlinearity of Φ is used to ensure the existence of a universal L^∞ bound for the weak-1 solutions, which is used to prove that $u_\tau(x,t) = u(x, t+\tau)$ is a weak-1 solution for $\tau > 0$, hence $\nabla\Phi(u) \in L^2(\tau, T : L^2(\Omega))$. It is to be noted that any L^∞ bound depending on the initial L^1 or L^p norm will do the job:

(i) We show in Section 7.7 that in one space dimension weak solutions are automatically bounded for $t \geq \tau > 0$. Hence, the assumption of superlinearity on Φ is not needed in that case.

(ii) Bounds can be obtained under much less stringent conditions on Φ, like the one found by Bénilan and Berger [84] for $d \geq 3$:

$$\int_1^\infty \Phi(s)^{-d/(d-2)} ds < \infty,$$

and a similar growth condition as $s \to -\infty$. This condition is always implied by our standing assumption $|\Phi(u)| \geq c|u|$. Their bound for $|u(t)|$ depends on Φ and $\|u_0\|_1$. They put $f = 0$ and the proof is based on symmetrization techniques.

Therefore, Theorem 6.12 is true under such assumptions.

6.5.1 Relating the concepts of solution

We have been led to introduce two concepts of weak solution for the same initial and boundary value problem in Definitions 5.4 and 6.5. This is a bad situation, so we need to establish the relationship between both definitions and make a choice if possible. Fortunately for us, the relationship turns out to be clear and easy.

Theorem 6.13 *Under the above assumptions on Φ and f, if $u_0 \in L_\Psi(\Omega)$, the concepts of weak-1 energy solution and weak-2 solution are equivalent. If $u_0 \in L^1(\Omega)$, the limit solution is a weak-2 solution.*

Proof. (i) If u is a weak-1 solution and $u_0 \in L_\Psi(\Omega)$, then it is also a weak-2 solution. Indeed, we have proved the continuity of the solution curve in Theorem 6.2. This part does not need any assumption on Φ.

(ii) Suppose on the converse that u is a weak-2 solution and $u_0 \in L_\Psi(\Omega)$. By uniqueness, it must be the weak-1 solution constructed in Theorem 5.5. □

Both definitions have advantages, though Definition 6.5 seems to have the upper hand since it is an extension. It also has the advantage over Definition 5.4 that the comparison, boundedness and stability results proved in the last and this chapter are immediately seen to hold for all solutions with data in the larger class. We have started with the historical Definition 5.4 essentially because it has an easy uniqueness proof.

In comparison with the concept of limit solution, that has the advantage of a wider application, weak-2 solutions are easier to recognize by means of their characterization.

We can also prove that very weak solutions are weak solutions in some cases. Here is a first result in that direction. We use the notation $Q^* = \Omega \times (\tau, T)$.

Proposition 6.14 *Let u be a very weak solution of Problem* HDP *for the GPME and assume that $u \in C([0,T] : L^1(\Omega))$, $\Phi(u) \in L^2(Q^*)$ and $f \in L^p(Q^*)$ with p large as before (see Theorem* 5.7*). Then, u is the weak solution for positive times $t \ge \tau > 0$.*

This type of conditions will be met quite often in the future.

6.6 More general initial data. The case L^1_δ

In this section we extend the existence theory to data in the class of locally integrable functions that are allowed to diverge mildly at the boundary, since this more general setting fits nicely with the basic concept of L^1-stability. In order to develop such results, we have to introduce new estimates that are of interest in themselves. For simplicity we assume here that Ω has a $C^{2+\alpha}$ regular boundary.

We need some notation: we denote by $L^p_\delta(\Omega) = L^p(\Omega; \delta(x)dx)$ the class of functions $f \in L^p_{loc}(\Omega)$ such that

$$\int |f(x)|\, \delta(x) dx < \infty, \tag{6.23}$$

where $\delta(x) = d(x, \partial\Omega)$ is the distance from a point $x \in \Omega$ to the boundary $\partial\Omega$. Besides, let ζ be the unique solution of the problem

$$\Delta\zeta = -1 \quad \text{in } \Omega, \qquad \zeta = 0 \quad \text{on } \partial\Omega. \tag{6.24}$$

It is known that $\zeta \in C^\infty(\Omega)$, $\zeta > 0$ in Ω, and whenever $\partial\Omega \in C^2$, then $\zeta(x)$ is C^2 up to the boundary and behaves like $\delta(x)$ in the sense that there exist constants $c_1, c_2 > 0$ such that

$$c_1\delta(x) \le \zeta(x) \le c_2\delta(x)$$

in a neighbourhood of the boundary.

Theorem 6.15 *For every $u_0 \in L^1_\delta(\Omega)$ and $f \in L^1(0,T : L^1_\delta(\Omega))$ there exists a unique function $u \in C([0,\infty) : L^1_\delta(\Omega))$ which is a limit solution of the HDP for the GPME in the sense that it is obtained by approximation with weak solutions. We also have*

$$\int u(x,t)\zeta(x)\, dx + \iint \Phi(u)\, dxdt = \int u_0(x)\zeta(x)\, dx + \iint f\zeta\, dxdt, \tag{6.25}$$

and

$$\int |u(x,t)|\zeta(x)\, dx + \iint |\Phi(u)|\, dxdt \le \int |u_0(x)|\zeta(x)\, dx + \iint |f|\zeta\, dxdt. \tag{6.26}$$

Moreover, the Comparison Principle holds for these solutions: if $u, \widehat{u}$ are two such solutions with initial data $u_0, \widehat{u}_0$, and $u_0 \le \widehat{u}_0$ a. e. in Ω, $f \le \widehat{f}$ a. e. in Q_T, then $u \le \widehat{u}$ a.e. in Q. More precisely, for any two solutions we have

$$\begin{aligned} &\int (u-\widehat{u})_+\zeta(x)\, dx + \iint (\Phi(u) - \Phi(\widehat{u}))_+\, dxdt \\ &\le \int (u_0(x) - \widehat{u}_0(x))_+\zeta(x)\, dx + \int_0^t \int (f(x,t) - \widehat{f}(x,t))_+\zeta(x)\, dxdt. \end{aligned} \tag{6.27}$$

Proof. The proof should be easy after the developments of Chapter 5 and Section 6.1.2 once the new contraction inequality given by formula (6.27) is proved. We call such inequality the *Weighted Contraction Principle.*

(i) Let us indicate the calculations to obtain formula (6.27) for smooth solutions. Let $p \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be such that $p(s) = 0$ for $s \le 0$, $p'(s) > 0$ for $s > 0$ and $0 \le p \le 1$, and let $j(r) = \int_0^r p(s)ds$ be a primitive of p. We subtract the equation for both solutions, multiply by $p(w)\zeta$ with $w = \Phi(u_1) - \Phi(u_2)$, and integrate by parts to get

$$\begin{aligned}\int (u(x,t) - \widehat{u}(x,t))_t p(w)\zeta(x)\, dx &= -\int p'(w)|\nabla w|^2 \zeta\, dx \\ &- \int p(w)\nabla w\, \nabla\zeta\, dx + \int (f - \widehat{f})p(w)\zeta\, dx.\end{aligned}$$

Dropping the negative term and integrating the next one, we get

$$\int (u(x,t) - \widehat{u}(x,t))_t p(w)\zeta(x)\, dx \le \int j(w)\, \Delta\zeta\, dx + \int (f - \widehat{f})p(w)\zeta\, dx.$$

We now let p tend to the function sign-plus and integrate in time. See a more detailed similar proof in Lemma 9.1.

(ii) We can now construct the limit solution by first approximating u_0 and f with sequences of bounded functions u_{0n} and f_n that converge to u_0 in $L^1_\delta(\Omega)$ and f in $L^1(0,T : L^1_\delta(\Omega))$ respectively. If u_n is the sequence of solutions of the approximate problems, estimate (6.27) implies that

$$\begin{aligned}&u_n \to u \quad \text{in } L^\infty(0,T : L^1_\delta(\Omega)),\\ &\Phi(u_n) \to v \quad \text{in } L^1(0,T : L^1(\Omega)).\end{aligned}$$

By taking subsequences we may assume that the convergence takes place almost everywhere. It is then clear that $v(x,t) = \Phi(u(x,t))$ a.e.

(iii) The weighted contraction principle implies that all smooth approximations of this kind produce the same limit. It is also easy to prove that the weak energy solutions of the preceding chapter and the limit solutions of previous section are particular cases of these solutions. Finally, we can use the data of such cases as approximations in the construction and still get the same limit. We leave these details to the reader as a training exercise.

(iv) The proof of the fact that $u \in C([0,\infty) : L^1_\delta(\Omega))$ copies the proof done in the previous section for the L^1 case. We can be interested in the way the equation is satisfied since the energy inequality does not necessarily make sense as a relation between finite quantities. This topic will be further investigated in the next section. □

Note that for a.e. $\tau > 0$ we have $\Phi(u(\tau)) \in L^1(\Omega)$; when $\Phi(s)$ has linear or superlinear growth as $|s| \to \infty$ we then have $u(\tau) \in L^1(\Omega)$ and the standard theory of L^1 limit solutions applies for $t \ge \tau$. Moreover, if Φ is superlinear and f is bounded the solutions also enjoy the universal bound (5.57). It is also clear that when $f = 0$ we have

Theorem 6.16 *The HDP for the PME generates an ordered contraction semigroup in the space $L^1(\Omega; \zeta\, dx)$.*

Remark. Very weak solutions can be considered with data in the weighted spaces L^1_δ as in the previous section. We leave details as a problem.

6.7 More general initial data. The case H^{-1}

This section is devoted to a still different extension of the class of data, namely taking initial data in the space $H^{-1}(\Omega)$, dual of $H_0^1(\Omega)$. The difficulty does not lie now with the size but with the regularity: the data are not necessarily locally integrable functions. We take as Ω a bounded subset of $\mathbb{R}^d$ with $\Gamma = \partial\Omega \in C^{2+\alpha}$.

6.7.1. Review of Functional Analysis

The space $H = H^{-1}(\Omega)$ is defined as the dual of the Hilbert space $H_0^1(\Omega)$. It can be identified as the space of distributions that can be written in the form

$$f = f_0 + \sum_1^d \frac{\partial f_i}{\partial x_i}$$

for functions $f_0, f_1, \cdots, f_d \in L^2(\Omega)$. A key fact in the theory is the following: the map $A = -\Delta$ is an isomorphism from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$. Let us call its inverse G. For every $f \in H^{-1}(\Omega)$, $F = Gf$ is the weak solution of equation $\Delta F = -f$ with data $F = 0$ on Γ. We define a dot product in H by means of the formula

$$\langle f_1, f_2\rangle_H = \langle G(f_1), G(f_2)\rangle_{H_0^1} = \int \nabla F_1 \cdot \nabla F_2 \, dx, \tag{6.28}$$

where we write $F_i = G(f_i)$. In this way, H becomes a Hilbert space and $\|f\|_H = \|G(f)\|_{H_0^1}$. For more information, see [4], [373].

6.7.2. Basic Identities

The basic calculations are better performed under the assumptions of Section 3.2: $\Phi : \mathbb{R} \mapsto \mathbb{R}$ is C^2 smooth, $\Phi(0) = 0$, and $\Phi'(u) > 0$ for all $s \in \mathbb{R}$; u_0 and f are bounded and continuous functions, and $u_0(x) = 0$ for $x \in \partial\Omega$. Then u is smooth and we have the following computations:

(i) We apply to all terms of the GPME the operator G acting on the space functions for every fixed time t to obtain the equation

$$U_t = -\Phi(u) + F, \tag{6.29}$$

where $U(\cdot, t) = G(u(\cdot, t))$, $F(\cdot, t) = G(f(\cdot, t))$.

(ii) The important new computation concerns the H^{-1} norms:

$$\begin{aligned} \frac{d}{dt}\|u\|_H^2 = \frac{d}{dt}\|U\|_{H_0^1}^2 &= 2\int \nabla U \cdot \nabla U_t \, dx \\ = -2\int \Delta U\, U_t \, dx &= -2\int u\,\Phi(u)\, dx + 2\int uF\, dx. \end{aligned}$$

Since $\int uF\,dx = -\int (\Delta U)F\,dx = \int \nabla U \cdot \nabla F\,dx = \langle u, f\rangle_H$, we get

$$\frac{1}{2}\|u(t)\|_H^2 + \iint u\,\Phi(u)\, dxdt = \frac{1}{2}\|u_0\|_H^2 + \int_0^t \langle u(s), f(s)\rangle\, ds. \tag{6.30}$$

Therefore, the norm $\|u(t)\|_H$ stays bounded in any time interval. Dropping the term containing $u\Phi(u)$, we can get from $d\|u\|^2/dt \le 2\|u\|\|f(t)\|$ the following precise bound

$$\|u(t)\|_H \le \|u_0\|_H + \int_0^t \|f(s)\|_H \, ds.$$

(iii) This computation can be improved into a computation for the difference of two solutions u_1, u_2 with data (u_{01}, f_1) and (u_{02}, f_2) resp. We get

$$\frac{1}{2}\frac{d}{dt}\|u_1 - u_2\|_H^2 + 2\int (u_1 - u_2)(\Phi(u_1) - \Phi(u_2))\,dx = \langle u_1 - u_2, f_1 - f_2\rangle. \tag{6.31}$$

Note that the second term has a nonnegative integrand precisely because of the assumption that Φ is monotone nondecreasing. This implies the estimate

$$\|u_1(t) - u_2(t)\|_H \le \|u_{01} - u_{02}\|_H + 2\int_0^t \|f_1(s) - f_2\|_H\,ds.$$

(iv) We now make use of the estimate on $\iint u\Phi(u)\,dxdt$. Indeed, since

$$d(s\Phi(s)) = sd\Phi(s) + \Phi(s)ds \ge \Phi(s)ds = d\Psi(s),$$

we have $\Psi(s) \le s\Phi(s)$ for every s. Therefore, we have

$$\iint \Psi(u)\,dxdt \le \iint u\Phi(u)\,dxdt \le C\|u_0\|_H^2 + C(\int_0^t \|f(s)\|_H\,ds)^2. \tag{6.32}$$

This means that for a.e. $\tau > 0$ we have $\int u(\tau)\Phi(u(\tau))dx \in L^1(\Omega)$ and we enter into the energy calculations of weak energy solutions.

6.7.3. General setting. Existence of H^{-1} solutions

Assume now that Φ is a monotone function as introduced in Section 5.2. All of the preceding estimates can be used together with a process of approximation and passage to the limit in order to obtain the following result.

Theorem 6.17 *For any $u_0 \in H^{-1}(\Omega)$ and $f \in L^2(0,T : H^{-1}(\Omega))$ there exists a unique $u \in C([0,\infty) : H^{-1}(\Omega))$ obtained as limit of weak solutions of the HDP with data (u_{0n}, f_n) that approximate (u_0, f) in the indicated spaces. Moreover,*

(i) for a. e. $t > 0$, $\Phi(u(t)) \in L^1(\Omega)$, and (6.32) holds;

(ii) $u(t) \to u_0$ as $t \to 0$ in the sense of $H^{-1}(\Omega)$;

(iii) The map: $(u_0, f) \mapsto u$ is a contraction from $H^{-1}(\Omega) \times L^2(0,T : H^{-1}(\Omega))$ into $C([0,\infty) : H^{-1}(\Omega))$.

Note that the weak solutions of Theorem 5.7 are particular cases of H^{-1} solutions.

Case with no forcing. Time decay. It is interesting to discuss the special properties of these solutions when $f = 0$. We immediately see that the norm $\|u(t)\|_H$ decreases in time. If we combine this with the already known fact that $J(t) = \int \Psi(u(t))\,dx$ is nonincreasing in time, we then get the estimate

$$\int \Psi(u(t))\,dt \le \frac{C}{t}\|u_0\|_H^2. \tag{6.33}$$

We also know from estimate (5.47) that $\int |\nabla\Phi(u(x,T))|^2\,dx$ is nonincreasing in time so that

$$\int |\nabla\Phi(u(T))|^2\,dx \le (C/T)\iint |\nabla\Phi(u)|^2\,dxdt.$$

In this way, and using estimate (5.39) a second decay estimate is obtained:

$$\int |\nabla \Phi(u(t))|^2\,dx \le \frac{C}{t}\int \Psi(u(t/2))\,dx \le \frac{C}{t^2}\|u_0\|_H^2. \tag{6.34}$$

We have the following result

Theorem 6.18 *When $f=0$ the solutions of Theorem 6.17 are weak solutions in any interval $t\in(\tau,T)$, with $\tau>0$ and the decay estimates (6.33) and (6.34) hold. The GPME generates a semigroup of contractions in $H^{-1}(\Omega)$. This semigroup is compact.*

It is interesting to compare these decay estimates with the actual decay of the Friendly Giant that we have constructed in the previous chapter. In the case of the PME, the explicit formula (5.69) implies that

$$\int_\Omega U(t)^{m+1}\,dx = O(t^{-m/(m-1)}), \qquad \int_\Omega |\nabla U^m|^2\,dx = O(t^{-2m/(m-1)}),$$

which improve the exponents of the above a priori decay estimates for large t, but are worse for small t. We ask the reader to think about this fact.

Remark. We will prove in the next section that in the case of the PME, $u_t=\Delta(|u|^{m-1}u)$, the solutions have better regularity; they are actually strong solutions.

Notes

§**6.1.** In dealing with limit solutions we must bear in mind that since the weak energy solutions are also constructed by an approximation method, they can be justly called limit solutions. The point to be stressed in the new class is that we lack at this moment a functional characterization of the set $\mathcal{LS}$ as solutions in some weak or similar sense.

Stability is proved with respect to a different norm, L^1. This reflects the different type of estimate involved. The mixture of norms is typical of nonlinear problems. Actually, the new technique produces a new solution concept, the limit solution.

§**6.2.** The duality proof of Theorem 6.5 is inspired in the proof by Kamin for the Stefan Problem [318] and by Kalashnikov [313] who studied the case $d=1$ of the GPME; the idea was used for the PME and $d>1$ by Bénilan-Crandall-Pierre [91]. See more uses in Chapters 12 and 13. Traces are treated in [191].

§**6.4.** The generation of semigroups by abstract nonlinear differential equations was a main subject of research in the late 1960's and early 1970's. A main reference is Crandall and Liggett's [180]. We will study that aspect in greater detail in Chapter 10.

§**6.6.** We have basically followed [88].

§**6.7.** The problem in H^{-1} was investigated in the framework of the theory of contractive semigroups in Hilbert spaces by Brezis [127, 128]. We will study that theory in Chapter 10.

Problems

Problem 6.1 Show that the construction of limit solutions can be performed for boundary data g as in Theorem 5.14. Show that we can prove L^1-continuous dependence on u_0 and f, but not on g.

Problem 6.2 Prove the statements of Theorem 6.8 in detail. In particular:

(i) Show that the definition of very weak solution can be written in the equivalent form

$$\int_\Omega \{u(x,t_2)\eta(x,t_2) - u(x,t_1)\eta(x,t_1)\}\,dx = \iint_{\Omega\times(t_1,t_2)} \{\Phi(u)\,\Delta\eta + u\eta_t + f\eta\}\,dxdt \tag{6.35}$$

for all $0 < t_1 < t_2 < T$ and all test functions $\eta \in C^{2,1}(Q_T)$ which are compactly supported in the space variable (uniformly in time)

(ii) Show that we have

$$u(t_2) - u(t_1) = \Delta \int_{t_1}^{t_2} \Phi(u)\,dt + \int_{t_1}^{t_2} f\,dt \tag{6.36}$$

in the sense of distributions, $\mathcal{D}'(\Omega)$; the values of $u(t)$ are the traces.

Problem 6.3 Prove that the following definition of weak solution is equivalent to Definition 6.5 for the PME. The difference lies in the explicit occurrence of the initial and end-values of the solution.

Definition 6.6 *A nonnegative function $u \in C([0,\infty) : L^1(\Omega))$ is said to be a weak solution of problem* (5.1)–(5.3) *if*

(i) $\Phi(u) \in L^2_{loc}(0,\infty : H^1_0(\Omega))$,

(ii) *for every $0 < t_1 < t_2$, u satisfies the identity*

$$\iint_{Q_{12}} \{\nabla(\Phi(u))\cdot\nabla\eta - u\eta_t\}dxdt = \int_\Omega u(x,t_1)\eta(x,t_1)\,dx - \int_\Omega u(x,t_2)\eta(x,t_2)\,dx \tag{6.37}$$

for any function $\eta \in C^1(Q)$ that vanishes on the lateral boundary $\Sigma = \partial\Omega \times (0,\infty)$. Here, $Q_{12} = \Omega \times (t_1,t_2)$,

(iii) $u(0) = u_0$.

Problem 6.4 Show that for the HE a semigroup of contractions is generated in all spaces $L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, and not only in $L^1(\mathbb{R}^d)$.

Problem 6.5 CONTINUOUS DEPENDENCE ON Φ. Prove that when Φ_ε is smooth and approximates Φ then the solutions of the Dirichlet Problem converge as $\varepsilon \to 0$.

Problem 6.6 (i) Develop the theory for general initial data of Section 6.6 for the GPME.

(ii) Construct very weak solutions with data in weighted spaces (i. e., L^1_δ).

Problem 6.7 Repeat the theory of weak-2 solutions of Section 6.5 for the GPME $u_t = \Delta\Phi(u) + f$ when Φ satisfies the assumptions of [84] and f is bounded.

Problem 6.8 Repeat the theory of weak-2 solutions for H^{-1} initial data when $f \neq 0$ but is still regular.

Problem 6.9 RESTRICTION OF NONNEGATIVE SUPERSOLUTIONS. (i) Let u be a nonnegative very weak supersolution to the GPME posed in a domain Ω_1 with zero boundary data (in the sense of Definition 6.2 and subsequent comment). Let Ω be a domain strictly contained in Ω_1. Show that u is still a supersolution of the GPME posed in Ω with zero boundary data.

(ii) Show that the result is not true if we replace supersolution by solution or subsolution.

Hint. Part (ii) is easy by construction of examples. In part (i) we take the definition of supersolution

$$\iint_{Q_T} \{\Phi(u)\,\Delta\eta + u\eta_t + f\eta\}\,dxdt + \int_{\Omega} u_0(x)\eta(x,0)dx \leq 0 \tag{6.38}$$

for any nonnegative function $\eta \in C^{2,1}(\overline{Q}_T)$ which vanishes on Σ and for $t = T$. In order to prove this result we proceed as follows. First, we extend η to $\Omega_1 \times (0,T)$ by putting $u(x,t) = 0$ when $x \neq \Omega$. We make convolution with a smooth kernel $\rho_\varepsilon \geq 0$ to obtain a smooth function η_ε that is acceptable as a test function for u as a supersolution in $Q_1 = \Omega_1 \times (0,T)$. Therefore, we have

$$I := \iint_{Q_1} \{\Phi(u)\,\Delta\eta_\varepsilon + u\eta_{\varepsilon,t} + f\eta_\varepsilon\}\,dxdt + \int_{\Omega_1} u_0(x)\eta_\varepsilon(x,0)dx \leq 0.$$

We now observe that $\eta_\varepsilon, |\eta_{\varepsilon,t}|$ are uniformly bounded for all $\varepsilon > 0$ small, and $\Delta\eta_v$ is uniformly bounded below (though not above near the boundary of Ω where $\Delta\eta$ has a Dirac delta). We separate the integral in tree regions: the interior region Q_i where $x \in \Omega$ and $d(x,\partial\Omega) \geq 1/n$, the exterior Q_e where $x \in \Omega_1\backslash\Omega$ and $d(x,\partial\Omega) \geq 1/n$ and the neighbourhood of the boundary Q_b where $d(x,\partial\Omega) < 1/n$. Write integral I as $I_i + I_e + I_b$. Prove that $I_e = 0$ if ε is small. Prove also that $I_b \geq -\delta$ if ε and $1/n$ are small (use the integrability of u and $\Phi(u)$). Conclude that $I_i \leq \delta$. Take the limit to get the desired result.

Problem 6.10 * PROJECT. Extend as much as possible of the theory of this chapter to equations of the form $u_t = \Delta\Phi(x,u)$.

Open Problem 6.OP. Prove that any weak-1 solution in the sense of Definition 5.4 is a very weak solution in the sense of Definition 6.2.

Are weak-1 solutions always limit solutions? Same question for weak-2 solutions.

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