Nonlinear Analysis, Theory, Methods & Applications, Vol. 20, No. 9, pp. 1043-1068, 1993.
Printed in Great Britain.

0362-546X/93 $6.00+ .00

THE STEADY NAVIER-STOKES EQUATIONS WITH MIXED BOUNDARY CONDITIONS: APPLICATION TO FREE BOUNDARY FLOWS

C. LE ROUX and B. D. REDDY
Department of Applied Mathematics, University of Cape Town, 7700 Rondebosch, South Africa

(*Received* 1 *December* 1991; *received for publication* 1 *August* 1992)

Key words and phrases: Navier-Stokes equations, mixed boundary conditions, existence theory, free boundary flows.

1. INTRODUCTION

THE NAVIER-Stokes equations, which constitute a model for the flow of viscous incompressible fluids, exhibit a wide range of complexities. This is not entirely unexpected, in view of the complex flow phenomena which these equations are assumed to model. Possibly for this reason, the vast majority of analyses relegate to a subsidiary role the question of which boundary conditions to include, and so one finds that the simplest, namely homogeneous Dirichlet boundary conditions are most often employed [1]. There exist many situations, however, in which other boundary conditions are to be found and, furthermore, in which the inclusion of these boundary conditions in the analysis is a less than trivial matter. For example, considerations of symmetry might lead to a formulation in which the boundary condition along the line of symmetry is one of zero normal velocity, and zero tangential stress. When considered together with boundary conditions of a different nature on other parts of the boundary, the overall effect is generally that great care has to be taken in carrying out the details of the analysis.

A specific class of problems in which the above considerations play a central role is that of free boundary problems (FBPs). The defining feature of these problems is that part of the boundary is unknown a priori, and has to be determined as part of the analysis. An important and reasonably successful approach to such problems has been to use the so-called *splitting method*: in the first stage of the method the free boundary, given by a function f, is fixed, that is, it is replaced by a rigid wall having perfect slip. The resulting problem, referred to as the *auxiliary problem*, is then solved, and the dependence of the solution on the function f is determined. In the second stage of the method this information is used, together with a remaining boundary condition on the free surface, to solve for the function f. The splitting method has been used successfully by Pukhnachov [2, 3], Jean [4] and Solonnikov [5] in the study of steady FBPs. For unsteady flows the situation is somewhat different and it is possible to develop an existence theory without recourse to the splitting method, as the works of Solonnikov [6], Beale [7, 8] and Allain [9] show.

The auxiliary problem in the steady case can include as many as four different boundary conditions on different parts of the boundary, depending on the physical problem and, in an analysis aimed at establishing existence of solutions to this problem, some care has to be taken in the manner in which these are treated. This has certainly been the case in the works cited above; however, the focus of these works has been on establishing the existence of *classical* solutions. As a consequence the results invariably require strong assumptions on the smoothness of data. A further consequence of the lack of results for weak or variational problems is found when approximate solution schemes are developed. Many numerical approaches to the problem make use of the finite element method (see, for example, the survey by Cuvelier and Schulkes [10]), and any complete analysis of this numerical approach would require an existence theory for the *variational* problem.

A similar problem arises in the study of contact problems involving elastic bodies, in which contact is accompanied by friction. The full problem is formulated as a variational inequality (see [11, Chapter 10]) but difficulties inherent in this problem, together with considerations related to algorithms for solving the problem numerically, have prompted a detailed study of a *reduced* or *auxiliary* problem, in which it is assumed that the contact surface and the pressure on this surface are known.

The purpose of this contribution is to carry out a detailed analysis, in a variational framework, of the steady Navier–Stokes equations, with particular attention paid to a variety of boundary conditions of both mathematical and practical interest. From the above remarks it is clear that one important application is to the auxiliary problem which occurs in FBPs. For convenience the examples to be studied are therefore formulated within the framework of FBPs and particular attention is given to the auxiliary problem. It will be clear enough, though, that this is not the only situation in which these problems would occur. In Section 2 we introduce four standard problems and formulate these problems in classical form. By considering these problems either in their own right or as auxiliary problems related to a FBP, weak or variational formulations are constructed in Section 3, and in Section 4 the existence theory for these variational formulations is presented.

In the existence theory a mixed or Lagrange multiplier approach is adopted in which both the hydrostatic pressure and the normal stress on the free boundary are treated as Lagrange multipliers. The treatment of the pressure as a multiplier is of course standard, but this is not the case for the normal stress. The reason for deliberately treating the normal stress as a multiplier is in order to obtain explicit information on this variable, since it is only this information which is carried forward from the auxiliary problem to the second stage, in the splitting method. The existence theory draws heavily on that established by Brezzi [12] for mixed variational problems; a detailed account of this theory appears in the text by Girault and Raviart [13].

We conclude by showing that a complete variational analysis of the FBP is not possible without additional regularity results on the velocity, pressure and hence the stress. Thus this contribution does not constitute a complete analysis for FBPs. Nevertheless, it does achieve two goals: first, a complete analysis of variational problems with a range of mixed boundary conditions and second, a clear indication of the status of steady FBPs within the variational framework.

2. MIXED BOUNDARY PROBLEMS

2.1. *Examples of free surface flows*

We shall restrict our attention to problems where variations in one direction are assumed to be negligible, so that the fluid motion is *two-dimensional*. Let Ω be a bounded domain in $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$ consisting of three mutually disjoint open manifolds Γ, Σ and Λ such that $\partial\Omega = \bar{\Gamma} \cup \bar{\Sigma} \cup \bar{\Lambda}$, meas$(\Gamma) > 0$, meas$(\Sigma) > 0$ and $\bar{\Gamma} \cap \bar{\Sigma} = \emptyset$. Greater smoothness will be assigned to portions of the boundary when the need arises. Here Ω represents the flow region, i.e. the space occupied by the liquid, Σ those parts of the boundary where the velocity is prescribed, Λ the parts of the boundary where, for example, mixed boundary conditions will be applied, and Γ is the a priori unknown (or free) part of the boundary. We shall focus on the following set of problems.

I. Consider the steady motion of a fluid in the annular space between a solid *cylindrical* surface Σ rotating with constant angular velocity ω and a free boundary Γ on which the pressure is prescribed as a function of the polar angle. The only body force assumed present is gravity, acting in a fixed direction perpendicular to the symmetry axis of the cylinder.

To be precise, we set $\Sigma = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| = 1\}$ (the unit circle), $\Lambda = \emptyset$ and we assume that Γ has the representation $\Gamma = \{\mathbf{x} \in \mathbb{R}^2 \mid r = f(\theta), 0 \leq \theta \leq 2\pi\}$, where $f\colon \mathbb{R} \to \mathbb{R}$ is an a priori unknown 2π-periodic function such that $0 < r_0 \leq f(\theta) \leq r_1 < 1 \; \forall\, \theta \in \mathbb{R}$, for chosen constants r_0 and r_1, as shown in Fig. 1.

II. Here we consider the motion of fluid filling an unbounded curvilinear *channel* Ω', the upper boundary Γ' of which is free, while the bottom Σ' represents a solid wall with *periodically* varying shape on which there are periodically distributed regions of fluid ingress and egress. It is assumed that these periods coincide (and equal unity), that the total intensity of the sinks and sources is zero, and that gravity is present.

For simplicity we assume that Σ' is of the form $\Sigma' = \{\mathbf{x} \in \mathbb{R}^2 \mid x_2 = b(x_1), x_1 \in \mathbb{R}\}$, with $b\colon \mathbb{R} \to \mathbb{R}$ a given function such that $b(x+1) = b(x) \; \forall\, x \in \mathbb{R}$, $\int_0^1 b(x)\,\mathrm{d}x = -d$ and $b(x) \leq \delta' - d \; \forall\, x \in \mathbb{R}$ (e.g. $b(x) \equiv -d$), for given constants $0 < \delta' < d$. Moreover, we assume that Γ' can be given the representation $\Gamma' = \{\mathbf{x} \in \mathbb{R}^2 \mid x_2 = f(x_1), x_1 \in \mathbb{R}\}$, where $f\colon \mathbb{R} \to \mathbb{R}$ is a (as yet

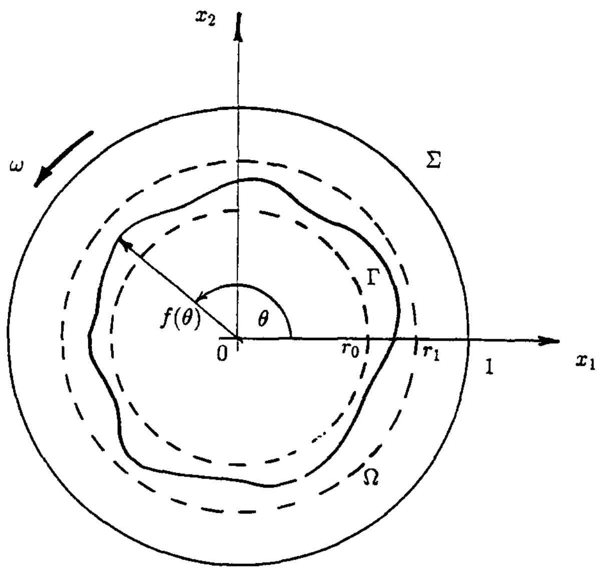

Fig. 1. Example I: the cylinder problem.

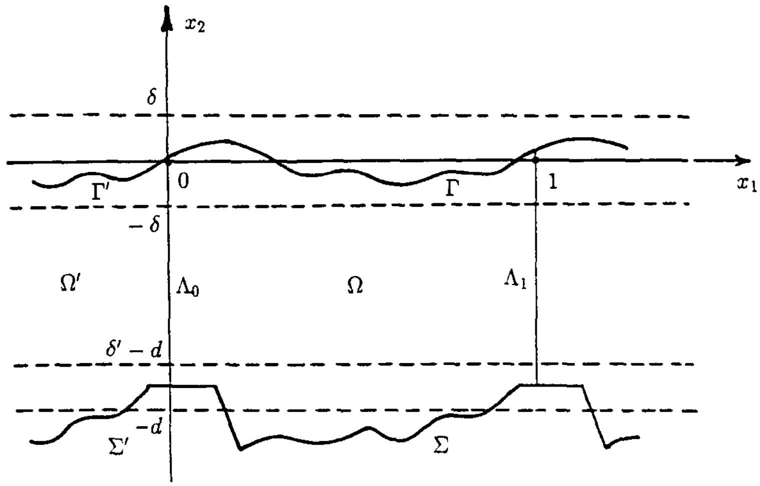

Fig. 2. Example II: the periodic channel.

unknown) function such that $|f(x)| \leq \delta \ \forall x \in \mathbb{R}$, for a given $0 < \delta < d - \delta'$. Since the data (Σ' and its distribution of sinks and sources) is one-periodic in x_1 we can assume that the flow (and hence f) is too. Therefore, the problem can be reduced to one on a bounded domain. Let $\Phi = \{\mathbf{x} \in \mathbb{R}^2 \mid 0 < x_1 < 1\}$ and define $\Gamma = \Phi \cap \Gamma'$, $\Sigma = \Phi \cap \Sigma'$, $\Omega = \Phi \cap \Omega'$ and $\Lambda = \Lambda_0 \cup \Lambda_1$ where $\Lambda_x = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 = x, b(x) < x_2 < f(x)\}$ (see Fig. 2).

III. The extrusion of a viscous incompressible jet from a die (or nozzle) into an inviscid fluid constitutes one of the best known free boundary problems. It is referred to as the *die-swell* problem since it is observed that far downstream the height of the extrudate can be different from that of the die. We shall assume the conventional geometry for the problem, i.e. we impose a priori the condition that the free surface should separate from the boundary at a specified point, such as a sharp lip (cf. [10, 14]). Moreover, it is assumed that there are no external forces.

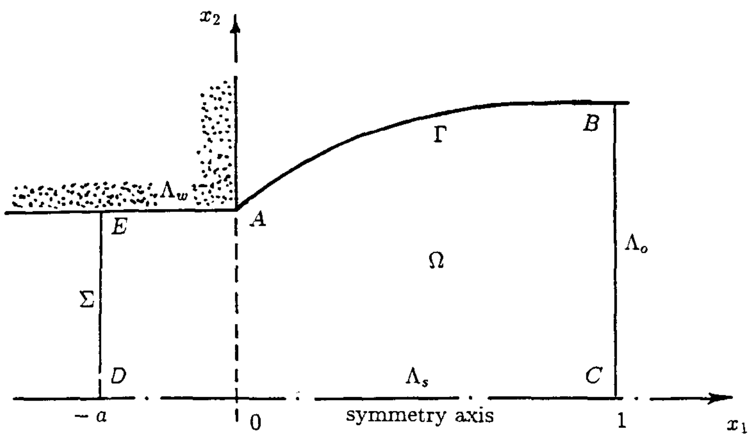

Fig. 3. Example III: the die-swell problem.

The situation is as in Fig. 3. In the absence of body forces we can assume that the flow is symmetric with respect to the x_1-axis and thus restrict the analysis to the half-plane $\{\mathbf{x} \in \mathbb{R}^2 \mid x_2 \geq 0\}$. The free surface is represented by $\Gamma = \{\mathbf{x} \in \mathbb{R}^2 \mid x_2 = f(x_1), 0 < x_1 < 1\}$, where the (unknown) function f must be such that $f(0) = b$ (separation at A) and $f'(1) = 0$ (vanishing slope at B). Furthermore, Σ is the inlet, Λ_w is the wet part of the wall, Λ_s represents the symmetry line, Λ_o is the outlet, and Ω is the bounded domain with boundary $\partial\Omega = \bar{\Gamma} \cup \bar{\Sigma} \cup \bar{\Lambda}$ where $\Lambda = \Lambda_w \cup \Lambda_s \cup \Lambda_o$.

IV. A problem of special interest concerns the behaviour of fluids partially filling an open container in a low-gravity environment, where capillary free boundaries must be taken into account. The steady motion is due to sinks and sources of total intensity zero distributed over a portion of the container wall. Consider a container Λ' consisting in part of two parallel rectilinear walls, say $\Lambda'_0 = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 = 0, x_2 > -h\}$ and $\Lambda'_1 = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 = 1, x_2 > -h\}$ for some $h > 0$, the lower endpoints of which are connected by a curve Λ'_l satisfying the condition $x_2 \leq -h \ \forall\, \mathbf{x} \in \Lambda'_l$. It is assumed that the free boundary $\bar{\Gamma}$ has a single point in common with each of Λ'_0 and Λ'_1 and does not intersect Λ' at any other point. Moreover, it is assumed that Γ is of the form $\Gamma = \{\mathbf{x} \in \mathbb{R}^2 \mid x_2 = f(x_1), 0 < x_1 < 1\}$, where the (unknown) function f: $[0, 1] \to \mathbb{R}$ is such that $|f(x)| \leq \delta < h \ \forall\, x \in [0, 1]$ for a given $\delta > 0$.

As before, Σ denotes those parts of the boundary where the flow is prescribed and Λ represents the remainder of the wet part of the container wall. We assume that $\Sigma \subset \Lambda'_l$ and that the vessel is tilted in such a way that the angle between the negative x_2-axis and the direction of gravity equals θ_0. Define $\Lambda_0 = \Lambda \cap \Lambda'_0$, $\Lambda_1 = \Lambda \cap \Lambda'_1$ and $\Lambda_l = \Lambda \cap \Lambda'_l$. The situation is illustrated in Fig. 4; a particular example of this situation is studied by Jean [4].

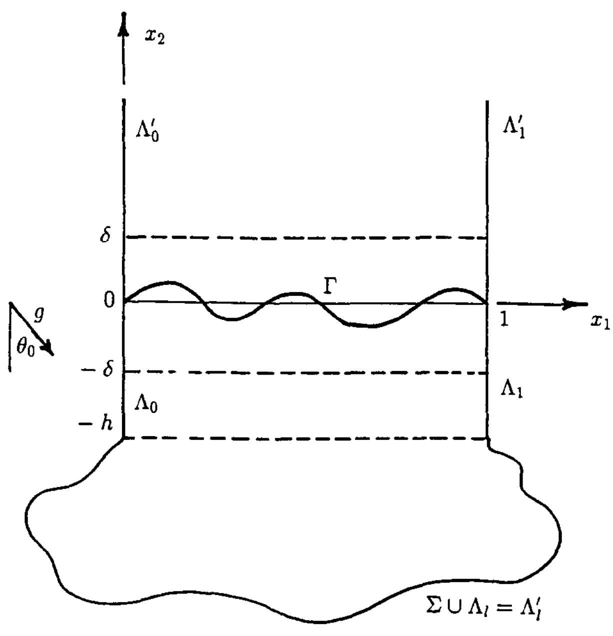

Fig. 4. Example IV: the container problem.

In each of the situations I–IV, the problem is to find the position of the free surface and corresponding velocity and pressure fields which satisfy the equations of motion and boundary conditions. Note, however, that these problems could equally be considered as mixed boundary problems in which Γ is assumed known; the only modification required would be to remove a subsidiary condition (see (2.9)–(2.11) later) on Γ.

It is instructive to compare the hydrodynamical free surface problems with the so-called *Signorini problem* in the theory of elasticity. Here one considers the unilateral contact of a body $\bar{\Omega}$ of linearly elastic material with a rigid foundation F. The body is subjected to surface tractions applied to a portion Λ of its boundary $\partial\Omega$ as well as body forces. The body is fixed along a portion Σ of its boundary and we denote by Γ a portion of the body which is a candidate contact surface, i.e. the actual surface which comes in contact with F is not known in advance but is contained in Γ [11].

2.2. *Governing equations and boundary conditions*

The steady Navier–Stokes equations for a viscous incompressible fluid are, in nondimensional form,

(momentum equations) $$u_j u_{i,j} - S_{ij,j} = f_i, \qquad i = 1, 2 \tag{2.1}$$

(incompressibility condition) $$u_{i,i} = 0, \tag{2.2}$$

where $S_{ij}(\mathbf{u}, p) = -p\delta_{ij} + (2/Re)D_{ij}(\mathbf{u})$ and $D_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$, $1 \le i, j \le 2$ (constitutive equation), and $Re = LU/\nu$ is the Reynolds number, where L is a characteristic (reference) length and U a characteristic velocity. The vector $\mathbf{u} = (u_1, u_2)$ denotes the velocity of the fluid, p is its pressure, $\mathbf{S} = [S_{ij}]$ is the stress tensor and $\mathbf{f} = (f_1, f_2)$ represents the external forces. Moreover, $(\cdot)_{,k}$ denotes $\partial(\cdot)/\partial k$, $k = 1, 2$, and the usual summation convention is used.

The following boundary condition applies to problems I–IV:

$$\mathbf{u} = \mathbf{u}_0 \qquad \text{on } \Sigma \text{ (prescribed velocity).} \tag{2.3}$$

The vector field $\mathbf{u}_0$ must satisfy the following conditions (listed with the numbers of the relevant problems):

I. $\mathbf{u}_0(\mathbf{x}) = \omega(-x_2, x_1)$, $\mathbf{x} \in \Sigma$ (velocity of rotating cylinder);
II. $\mathbf{u}_0(x_1 + 1, x_2) = \mathbf{u}_0(\mathbf{x})$, $\mathbf{x} \in \Sigma'$ (periodicity in x_1);
II, IV. $\int_\Sigma \mathbf{u}_0 \cdot \mathbf{n}\, ds = 0$ (zero total effect of sinks and sources);
III, IV. $\mathbf{u}_0$ must be compatible with the boundary conditions applied at the sections of $\partial\Omega$ adjoining Σ.

For the container problem it is usually assumed that

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Lambda \text{ (no-slip condition).} \tag{2.4}$$

However, it will be shown that well-posed boundary value problems can be formulated by using the boundary condition

$$\left.\begin{aligned} u_i n_i &= 0 \text{ (slip condition)} \\ S_{ij} n_j t_i &= 0 \text{ (tangential stress condition)} \end{aligned}\right\} \quad \text{on } \Lambda \tag{2.5}$$

or a combination of (2.4) and (2.5) applied to separate portions of Λ. Here, and elsewhere, $\mathbf{n} = (n_1, n_2)$ and $\mathbf{t} = (t_1, t_2)$ denote the outward unit normal and tangential vectors to $\partial\Omega$, respectively.

In the case of the die-swell problem, the condition of symmetry with respect to the x_1-axis is expressed by (2.5) on Λ_s. At Λ_w it is again possible to choose between (2.4) and (2.5). At Λ_o it is supposed that the flow is parallel to the x_1-axis and that there is no diffusive outflow of momentum:

$$\left.\begin{aligned} u_i t_i &= 0 \text{ (kinematic condition)} \\ S_{ij} n_j n_i &= 0 \text{ (normal stress condition).} \end{aligned}\right\} \quad \text{on } \Lambda_o. \tag{2.6}$$

Observe that the use of boundary conditions on Λ_0 and Λ_1 in the channel problem is avoided by requiring $\mathbf{u}$ to be periodic (in x_1) in Ω'.

We turn to Γ: in stationary situations the free boundary is a streamline

$$u_i n_i = 0 \qquad \text{on } \Gamma \text{ (kinematic condition).} \tag{2.7}$$

Moreover, a balance of forces must be fulfilled on the free boundary

$$S_{ij} n_j t_i = 0 \qquad \text{on } \Gamma \text{ (tangential stress condition),} \tag{2.8}$$

$$\frac{1}{R} = (Re.Oh)^2 (S_{ij} n_j n_i + p_a) \qquad \text{on } \Gamma \text{ (normal stress condition).} \tag{2.9}$$

In (2.9) $1/R$ denotes the curvature of the free surface, p_a is the pressure of the surrounding atmosphere (or inviscid fluid) and $Oh = \mu/\sqrt{(\rho\sigma L)}$ is the Ohnesorge number, with σ the coefficient of surface tension. The curvature assumes the form

$$\frac{1}{R} = \frac{f^2 + 2(f')^2 - ff''}{(f^2 + (f')^2)^{3/2}} \qquad \left(' \equiv \frac{\mathrm{d}}{\mathrm{d}\theta}\right)$$

in the cylinder problem, and in the other problems it becomes

$$\frac{1}{R} = \left[\frac{f'}{(1 + (f')^2)^{1/2}}\right]' = \frac{f''}{(1 + (f')^2)^{3/2}} \qquad \left(' \equiv \frac{\mathrm{d}}{\mathrm{d}x_1}\right).$$

Since the position of Γ (equivalently, f) is determined by the solution of a second-order ordinary differential equation (see (2.9)), two boundary conditions for the position of Γ are necessary. For problems I–IV these conditions are:

$$\begin{aligned} &\text{I.} \quad f(\theta + 2\pi) = f(\theta)\ \forall\, \theta \in \mathbb{R} \text{ (hence, } f(0) = f(2\pi) \text{ and } f'(0) = f'(2\pi)); \\ &\text{II.} \quad f(x + 1) = f(x)\ \forall\, x \in \mathbb{R} \text{ (periodicity condition);} \\ &\text{III.} \quad f(0) = b \text{ (fixed separation point), } f'(1) = 0 \text{ (vanishing slope at outflow);} \\ &\text{IV.} \quad f'(0) = -c,\ f'(1) = c \text{ (contact angle condition).} \end{aligned} \tag{2.10}$$

In the case of problems I, II and IV, f is determined only up to an additive constant by (2.9) and (2.10). To fix the position of Γ uniquely, the volume of the liquid in the container is prescribed

$$\operatorname{meas}(\Omega) = \int_\Omega \mathrm{d}x_1\, \mathrm{d}x_2 = \text{Vol (volume constraint).} \tag{2.11}$$

We can now formulate (for each of the settings I–IV) the *free boundary problem*:

(FBP) determine Γ (or f), $\mathbf{u}$ and p such that equations (2.1), (2.2) and (2.11) hold in Ω and the boundary conditions (2.5)–(2.10) are satisfied.

A well-established method for approaching this class of free boundary problems is the so-called splitting method:

(a) by treating the position of Γ as a given (but arbitrary) entity and ignoring one of the boundary conditions at Γ, namely (2.9), one obtains a well-posed (uniquely solvable) boundary value problem in $\mathbf{u}$ and p. This establishes a mapping of the form $\Gamma \to (\mathbf{u}, p)$;

(b) using the properties of this map, one proves that (2.9) has a unique solution. Calculation of the corresponding pair $(\mathbf{u}, p)$ then solves (FBP) (cf. [2–5, 15, 16]).

Our main aim is to carry out a detailed analysis of step (a) by using purely variational (weak) methods. For every fixed Γ (or f) satisfying the regularity requirements and conditions (2.10) we set the corresponding auxiliary problem:

(AUX) determine $\mathbf{u}$ and p such that equations (2.1) and (2.2) hold in Ω and boundary conditions (2.5)–(2.8) are satisfied.

3. VARIATIONAL FORM OF (AUX)

3.1. *Trace theorems*

Let Ω be a bounded open subset of $\mathbb{R}^2$ with a Lipschitz continuous boundary $\partial\Omega$. Let $L^2(\Omega)$ denote the space of (equivalence classes of) Lebesgue-square integrable functions with inner product $(\cdot,\ \cdot)_{0,\Omega}$ and norm $\|\cdot\|_{0,\Omega}$. For integer $m \geq 0$ let $H^m(\Omega)$ denote the Sobolev space with inner product $(\cdot,\ \cdot)_{m,\Omega}$ and norm $\|\cdot\|_{m,\Omega}$ defined in the usual manner. We shall also need the spaces $L^2(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$ which are Hilbert spaces with inner products and norms defined by

$$(u, v)_{0,\partial\Omega} = \int_{\partial\Omega} uv \, ds, \qquad \|u\|_{0,\partial\Omega} = (u, u)^{1/2}_{0,\partial\Omega}, \tag{3.1}$$

$$\begin{aligned}(u, v)_{1/2,\partial\Omega} &= (u, v)_{0,\partial\Omega} + \int_{\partial\Omega}\int_{\partial\Omega} \frac{[u(\mathbf{x}) - u(\mathbf{y})][v(\mathbf{x}) - v(\mathbf{y})]}{\|\mathbf{x} - \mathbf{y}\|^2} \, ds(x) \, ds(y), \\ \|u\|_{1/2,\partial\Omega} &= (u, u)^{1/2}_{1/2,\partial\Omega}.\end{aligned} \tag{3.2}$$

The importance of $H^{1/2}(\partial\Omega)$ is due to the classical trace theorem for $H^1(\Omega)$.

PROPOSITION 3.1 [17, p. 39]. Let Ω be a bounded open subset of $\mathbb{R}^2$ with a Lipschitz continuous boundary $\partial\Omega$. The operator γ defined by

$$\gamma(v) = v|_{\partial\Omega}, \qquad v \in C^1(\bar{\Omega}), \tag{3.3}$$

has a unique extension $\gamma \in \mathcal{L}(H^1(\Omega), H^{1/2}(\partial\Omega))$, which is surjective. This operator has a continuous linear right inverse. ■

In the above, $\partial\Omega$ may be replaced by any open subset Γ of $\partial\Omega$. We shall denote the corresponding trace operator by γ_Γ.

Let Ω be as above. Suppose that $\partial\Omega = \bar{\Gamma} \cup \bar{\Sigma}$ where Γ and Σ are nonempty disjoint open subsets of $\partial\Omega$. The spaces defined in (3.2) are sometimes inappropriate for dealing with problems with mixed boundary conditions (e.g. problems III and IV). For instance, if $u \in H^{1/2}(\Gamma)$ and $\tilde{u}$ is the extension by zero of u to Σ, then in general $\tilde{u}$ is not the trace of a function in $H^1(\Omega)$, i.e. $\tilde{u}$ is not a member of $H^{1/2}(\partial\Omega)$. This difficulty is overcome by restricting u to a special subspace of $H^{1/2}(\Gamma)$ which is defined as follows (cf. [17, pp. 57, 66]): let a function ρ be defined on $\bar{\Gamma}$ such that ρ is sufficiently smooth, positive on Γ and vanishes on the boundary $\partial\Gamma$ of Γ at a rate

$$r = \lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\rho(\mathbf{x})}{\operatorname{dist}(\mathbf{x}, \partial\Gamma)} \neq 0, \qquad \mathbf{x}_0 \in \partial\Gamma. \tag{3.4}$$

Then $H_{00}^{1/2}(\Gamma) = \{u \in H^{1/2}(\Gamma) \mid \rho^{-1/2} \cdot u \in L^2(\Gamma)\}$ is a Hilbert space with inner product and norm defined by

$$(u, v)_{00,\Gamma} = (u, v)_{1/2,\Gamma} + (\rho^{-1/2}u, \rho^{-1/2}v)_{0,\Gamma}, \qquad \|u\|_{00,\Gamma} = (u, u)_{00,\Gamma}^{1/2}. \tag{3.5}$$

Let U denote the closed subspace of $H^1(\Omega)$ given by $U = \{u \in H^1(\Omega) \mid \gamma_\Sigma(u) = 0\}$. Let γ_Γ^0 be the restriction of the trace operator $\gamma_\Gamma\colon H^1(\Omega) \to H^{1/2}(\Gamma)$ to U. Then we have the following trace theorem.

PROPOSITION 3.2. Let $\gamma_\Gamma^0\colon U \to H^{1/2}(\Gamma)$ be the operator defined above. Then γ_Γ^0 maps U onto $H_{00}^{1/2}(\Gamma)$ and $\gamma_\Gamma^0 \in \mathcal{L}(U, H_{00}^{1/2}(\Gamma))$. This map has a continuous linear right inverse. ■

Suppose that $\partial\Omega = \bar{\Gamma} \cup \bar{\Sigma} \cup \bar{\Lambda}$ with Γ, Σ and Λ nonempty, disjoint, open and connected, $\bar{\Gamma} \cap \bar{\Sigma} = \{A\}$ and $\bar{\Gamma} \cap \bar{\Lambda} = \{B\}$. Let $U' = \{u \in H^1(\Omega) \mid \gamma_\Sigma(u) = 0\}$. In this situation the spaces $H^{1/2}(\Gamma)$ and $H_{00}^{1/2}(\Gamma)$ are both inappropriate for characterizing the traces on Γ of functions in U'. We need a space "intermediate" to them: let ρ_A be a function defined on $\bar{\Gamma}$ with properties identical to those of the function ρ in (3.4), but with $\partial\Gamma$ replaced by $\{A\}$ and ρ_A approaching, say, 1 at B. Then $H_A^{1/2}(\Gamma) = \{u \in H^{1/2}(\Gamma) \mid \rho_A^{-1/2} \cdot u \in L^2(\Gamma)\}$ is a Hilbert space with inner product and norm defined as in (3.4) with ρ replaced by ρ_A. It can be proved that γ_Γ maps U' onto $H_A^{1/2}(\Gamma)$ and $\gamma_\Gamma^A \in \mathcal{L}(U', H_A^{1/2}(\Gamma))$ where γ_Γ^A is the restriction of γ_Γ to U'.

Definitions and results analogous to these apply to vector-valued functions $\mathbf{v}$ in $H^1(\Omega)^2$ and their boundary values. In order to deal with the boundary conditions of problems I–IV it is necessary to decompose the traces $\gamma(\mathbf{v})$ in $H^{1/2}(\partial\Omega)^2$ of such functions into well-defined normal and tangential components. The following proposition shows that this can be done when the boundary $\partial\Omega$ is sufficiently smooth.

PROPOSITION 3.3 [13]. Let Ω be a bounded open subset of $\mathbb{R}^2$ with a Lipschitz continuous boundary $\partial\Omega$. Let Γ be an open subset of $\partial\Omega$ of class $C^{1,\alpha}$, $\alpha > \frac{1}{2}$.

(a) Every $\mathbf{w} \in H^{1/2}(\Gamma)^2$ has a unique decomposition $\mathbf{w} = w_n\mathbf{n} + w_t\mathbf{t}$ with $w_n, w_t \in H^{1/2}(\Gamma)$. The map $(w_1, w_2) \to (w_n, w_t)$ is an isomorphism form $H^{1/2}(\Gamma)^2$ onto itself.

(b) There exist uniquely determined operators $\gamma_{\Gamma n}, \gamma_{\Gamma t} \in \mathcal{L}(H^1(\Omega)^2, H^{1/2}(\Gamma))$ such that

$$\gamma_\Gamma(\mathbf{v}) = \gamma_{\Gamma n}(\mathbf{v})\mathbf{n} + \gamma_{\Gamma t}(\mathbf{v})\mathbf{t} \qquad \forall\, \mathbf{v} \in H^1(\Omega)^2$$

and

$$\gamma_{\Gamma n}(\mathbf{v}) = \mathbf{v}|_\Gamma \cdot \mathbf{n}, \qquad \gamma_{\Gamma t}(\mathbf{v}) = \mathbf{v}|_\Gamma \cdot \mathbf{t} \qquad \forall\, \mathbf{v} \in C^1(\bar{\Omega})^2.$$

Furthermore, the map $(\gamma_{\Gamma n}, \gamma_{\Gamma t})\colon H^1(\Omega)^2 \to H^{1/2}(\Gamma)^2$ is surjective and has a continuous linear right inverse.

(c) Results analogous to those in (a) and (b) apply when $H^1(\Omega)$, $H^{1/2}(\Gamma)$ and γ_Γ are replaced by U, $H_{00}^{1/2}(\Gamma)$ and γ_Γ^0, or by U', $H_A^{1/2}(\Gamma)$ and γ_Γ^A. ■

For problems I–IV we shall henceforth assume that Γ and the other portions of $\partial\Omega$ on which Neumann boundary conditions are applied are (separately) of class $C^{1,1}$.

The space of admissible flows is defined as the subspace of $H^1(\Omega)^2$ consisting of the velocity fields $\mathbf{v}$ satisfying the Dirichlet boundary conditions specified at Λ and Σ (with $\mathbf{u}_0$ replaced by $\mathbf{0}$) for $\mathbf{u}$ in (AUX). Hence, for the respective problems we define:

$$
\begin{array}{ll}
\text{I.} & V_1 = \{\mathbf{v} \in H^1(\Omega)^2 \mid \gamma_\Sigma(\mathbf{v}) = \mathbf{0}\}; \\
\text{II.} & V_2 = \{\mathbf{v} \in H^1(\Omega)^2 \mid \mathbf{v} = \mathbf{v}'|_\Omega, \mathbf{v}' \in H^1(\Phi' \cap \Omega')^2, \mathbf{v}'(x_1 + 1, x_2) = \mathbf{v}'(\mathbf{x}) \\
 & \qquad \text{for a.e. } \mathbf{x} \in \Omega,\ \gamma_\Sigma(\mathbf{v}) = \mathbf{0}\} \text{ where } \Phi' = \{\mathbf{x} \in \mathbb{R}^2 \mid 0 < x_1 < 2\}; \\
\text{III.} & V_3 = \{\mathbf{v} \in H^1(\Omega)^2 \mid \gamma_{\Sigma \cup \Lambda_w}(\mathbf{v}) = \mathbf{0},\ \gamma_{\Lambda_s n}(\mathbf{v}) = 0,\ \gamma_{\Lambda_o t}(\mathbf{v}) = 0\}; \\
\text{IV.} & V_4 = \{\mathbf{v} \in H^1(\Omega)^2 \mid \gamma_{\Sigma \cup \Lambda}(\mathbf{v}) = \mathbf{0}\}.
\end{array}
\tag{3.6}
$$

In the case of a slip condition at Λ_w, the first constraint in V_3 is replaced by $\gamma_\Sigma(\mathbf{v}) = \mathbf{0}$, $\gamma_{\Lambda_w n}(\mathbf{v}) = 0$. The same idea applies to V_4. The periodicity condition on $\mathbf{v}$ in V_2 (i.e. $v'_{i,j} \in L^2(\Phi' \cap \Omega')$, $i, j = 1, 2$) is equivalent to the "intrinsic" condition $\gamma_{\Lambda_0}(\mathbf{v})(x_2) = \gamma_{\Lambda_1}(\mathbf{v})(x_2)$ for a.a. x_2. (This follows easily from the definition of the distribution $v'_{i,j} \in \mathfrak{D}(\Phi' \cap \Omega')$.)

For $i = 1, \ldots, 4$, V_i is a closed subspace of $H^1(\Omega)^2$ and therefore a Hilbert space with the norm $\|\cdot\|_{1,\Omega}$. For V_2 we replace $H^1(\Omega)^2$ and $H^{1/2}(\partial\Omega)^2$ by the corresponding subspaces of one-periodic elements, which are closed. Since $\text{meas}(\Sigma) > 0$, an equivalent norm on V_i is given by $|\mathbf{v}|_1 = (v_{k,j}, v_{k,j})_{0,\Omega}^{1/2}$ (cf. [13, lemma 3.1]). Moreover, for $i = 1, \ldots, 4$, the set $\gamma_0(V_i)$ of traces of admissible flows is a closed subspace of $H^{1/2}(\partial\Omega)^2$ and therefore a Hilbert space.

We shall now define, for $i = 1, \ldots, 4$, a Hilbert space Z_i, which is isomorphic to $\gamma_0(V_i)$ and represents the nonzero (or unprescribed) parts of the traces in $\gamma_0(V_i)$, and a corresponding surjective trace operator $\gamma_i \in \mathcal{L}(V_i, Z_i)$:

$$
\begin{array}{l}
\text{I. } Z_1 = H^{1/2}(\Gamma) \times H^{1/2}(\Gamma),\ \gamma_1 = (\gamma_{\Gamma n}, \gamma_{\Gamma t}); \\
\text{II. } Z_2 = H_1^{1/2}(\Gamma) \times H_1^{1/2}(\Gamma),\ \gamma_2 = (\gamma^1_{\Gamma n}, \gamma^1_{\Gamma t}) \text{ where}
\end{array}
\tag{3.7}
$$

$$H_1^{1/2}(\Gamma) = \{v \in H^{1/2}(\Gamma) \mid v = v'|_\Gamma,\ v' \in H^{1/2}(\Phi' \cap \Gamma'),\ v'(x_1 + 1, x_2) = v'(\mathbf{x}) \text{ for a.e. } \mathbf{x} \in \Gamma\}$$

and γ^1 denotes the restriction of γ to V_2;

III. $Z_3 = H_{00}^{1/2}(\Gamma) \times Y_3$, $\gamma_3 = (\gamma^0_{\Gamma n}, \gamma^A_{\Gamma t}, \gamma_{\Lambda_o n}, \gamma^D_{\Lambda_s t})$ where Y_3 is the subspace of

$$H_A^{1/2}(\Gamma) \times H^{1/2}(\Lambda_o) \times H_D^{1/2}(\Lambda_s)$$

consisting of elements (u, v, w) such that $\mathbf{a} \in H_{00}^{1/2}(\Gamma')^2$, $\Gamma' = \Gamma \cup \Lambda_o \cup \Lambda_s$, if $\mathbf{a}$ is defined by

$$
\mathbf{a} = \begin{cases} u\mathbf{t} & \text{on } \Gamma \\ v\mathbf{n} & \text{on } \Lambda_o \\ w\mathbf{t} & \text{on } \Lambda_s; \end{cases}
$$

IV. $Z_4 = H_{00}^{1/2}(\Gamma) \times H_{00}^{1/2}(\Gamma)$, $\gamma_4 = (\gamma^0_{\Gamma n}, \gamma^0_{\Gamma t})$.

The condition $\|\mathbf{a}\|_{00,\Gamma'} < +\infty$ in (3.7 III) is equivalent to $\|\mathbf{a}\|_{1/2,\Gamma'} < +\infty$ and can be expressed as compatibility conditions to the effect that "$u\mathbf{t} = v\mathbf{n}$ at A" and "$v\mathbf{n} = w\mathbf{t}$ at D". Similarly, the periodicity condition $\|v'\|_{1/2,\Phi' \cap \Gamma'} < +\infty$ in (3.7 II) is equivalent to a condition of the form "$v(0) = v(1)$". The precise formulation can be found in [18, theorem 1.5.2.3 (c)]. Finally, it is easily verified that the spaces Z_i and operators γ_i do have the stated properties; e.g. the completeness of Z_3 follows from that of $\gamma_0(V_3)$.

3.2. *Green's formulas*

For each fixed k with $1 \le k \le 4$, we define the space of symmetric stress fields on Ω by

$$\mathcal{S} = \{\tau = [\tau_{ij}] \mid \tau_{ij} \in L^2(\Omega),\ \tau_{ij} = \tau_{ji},\ i, j = 1, 2\} \tag{3.8}$$

where Ω is understood to denote Ω_k, the domain in problem k. For simplicity we shall denote quantities of which the definitions are formally identical in the four problems without the subscript k. It follows from the theory presented in [11] that the inner product space defined by

$$\mathfrak{I} = \{\tau \in \mathcal{S} \mid \tau_{ij,j} \in L^2(\Omega),\ i = 1, 2\}, \tag{3.9}$$

$$((\sigma, \tau))_{\mathfrak{I}} = (\sigma_{ij}, \tau_{ij})_{0,\Omega} + (\sigma_{ij,j}, \tau_{ij,j})_{0,\Omega}, \tag{3.10}$$

is a Hilbert space. Moreover, the following Green's formula for stresses can be derived.

THEOREM 3.1. Let the notation be as above. Then, for $k = 1, \dots, 4$, there exists a uniquely determined operator $\pi_k \in \mathcal{L}(\mathfrak{I}, Z_k')$ such that Green's formula

$$\int_\Omega \tau_{ij} v_{i,j}\, \mathrm{d}x + \int_\Omega \tau_{ij,j} v_i\, \mathrm{d}x = \langle\langle \pi_k(\tau), \gamma_k(\mathbf{v}) \rangle\rangle_k \tag{3.11}$$

holds $\forall\, \tau \in \hat{\mathfrak{I}}$ and $\forall\, \mathbf{v} \in V_k$, where

$$\langle\langle \pi_k(\tau), \gamma_k(\mathbf{v}) \rangle\rangle_k = \int_{\partial\Omega} \gamma_0(\tau_{ij}) n_j \gamma_0(v_i)\, \mathrm{d}s \qquad \forall\, \mathbf{v} \in V_k \tag{3.12}$$

when $\tau_{ij} \in H^1(\Omega)$, $i, j = 1, 2$, and where $\langle\langle\cdot, \cdot\rangle\rangle_k$ denotes duality pairing on $Z_k' \times Z_k$. ■

3.3. *Weak form of* (*AUX*)

We begin by posing (AUX) with the weakest possible regularity constraints on the data: given a domain Ω satisfying the regularity requirements set out earlier, $\mathbf{f} \in L^2(\Omega)^2$ and $\mathbf{u}_0 \in H^{1/2}(\Sigma)^2$ satisfying the relevant conditions given in Section 2, find $\mathbf{u} \in H^2(\Omega)^2$ and $p \in H^1(\Omega)$ such that

$$u_j u_{i,j} - \tau_{ij,j} = f_i \qquad \text{a.e. in } \Omega, \qquad i = 1, 2, \tag{3.13}$$

$$\text{where } \tau_{ij} = -p\delta_{ij} + \frac{2}{Re}(u_{i,j} + u_{j,i}),$$

$$u_{i,i} = 0 \qquad \text{a.e. in } \Omega, \tag{3.14}$$

$$\gamma_\Sigma(\mathbf{u}) = \mathbf{u}_0, \tag{3.15}$$

$$\gamma_{\Gamma n}(\mathbf{u}) = 0, \tag{3.16}$$

$$\gamma_\Gamma(\tau_{ij}) n_j t_i = 0, \tag{3.17}$$

II. $$\gamma_{\Lambda_0}(\mathbf{u})(x_2) = \gamma_{\Lambda_1}(\mathbf{u})(x_2) \qquad \text{for a.a. } x_2, \tag{3.18}$$

III. $$\gamma_{\Lambda_s n}(\mathbf{u}) = 0, \qquad \gamma_{\Lambda_s}(\tau_{ij}) n_j t_i = 0,$$

$$\gamma_{\Lambda_o t} = 0, \qquad \gamma_{\Lambda_o}(\tau_{ij}) n_j n_i = 0,$$

$$\gamma_{\Lambda_w}(\mathbf{u}) = 0 \qquad (\text{or } \gamma_{\Lambda_w n}(\mathbf{u}) = 0,\ \gamma_{\Lambda_w}(\tau_{ij}) n_j t_i = 0),$$

IV. $$\gamma_\Lambda(\mathbf{u}) = 0 \qquad (\text{or as in III where slip is allowed}).$$

Let $1 \leq k \leq 4$ be fixed and suppose that $(\mathbf{u}, p)$ is a solution of the corresponding problem (3.13)–(3.18), denoted by $(\mathrm{AUX})_k$. Then (3.13) implies that

$$\int_\Omega u_j u_{i,j} v_i \, \mathrm{d}x - \int_\Omega \tau_{ij,j} v_i \, \mathrm{d}x = \int_\Omega f_i v_i \, \mathrm{d}x \qquad \forall\, \mathbf{v} \in L^2(\Omega)^2.$$

Since $p \in H^1(\Omega)$, $\mathbf{u} \in H^2(\Omega)^2$ (or since $\mathbf{u} \in H^1_L(\Omega)^2$ and $\mathbf{u}$ satisfies (3.14)) and $\tau = \tau(\mathbf{u}, p)$ is symmetric by definition, it follows that $\tau \in \mathfrak{T}$. Thus Green's formula (3.11) holds for τ and for every $\mathbf{v} \in V_k$. By the definition and symmetry of τ, this implies that

$$\begin{aligned}\int_\Omega u_j u_{i,j} v_i \, \mathrm{d}x + \frac{2}{Re}\int_\Omega D_{ij}(\mathbf{u}) D_{ij}(\mathbf{v}) \, \mathrm{d}x - \int_\Omega p v_{i,i} \, \mathrm{d}x \\ = \int_\Omega f_i v_i \, \mathrm{d}x + \langle\!\langle \pi_k(\tau(\mathbf{u}, p)), \gamma_k(\mathbf{v}) \rangle\!\rangle_k \qquad \forall\, \mathbf{v} \in V_k.\end{aligned} \tag{3.19}$$

Equations (3.14) and (3.16) are equivalent to

$$(q, u_{i,i})_{0,\Omega} + (\mu, \gamma_{\Gamma n}(\mathbf{u}))_{1/2,\Gamma} = 0 \qquad \forall\, (q, u) \in L^2(\Omega) \times H^{1/2}(\Gamma). \tag{3.20}$$

Moreover, (3.12) holds because $\tau_{ij} \in H^1(\Omega)$, $i, j = 1, 2$. Since $(A \times B)' = A' \times B'$ for all inner product spaces A and B, and (recall proposition 3.3) the traces of τ_{ij} and $\mathbf{v}$ satisfy

$$\begin{aligned}\tau_{ij} n_j v_i &= (\tau\mathbf{n}) \cdot \mathbf{v} \\ &= \{(\tau\mathbf{n})_n \mathbf{n} + (\tau\mathbf{n})_t \mathbf{t}\} \cdot \{v_n \mathbf{n} + v_t \mathbf{t}\} \\ &= v_n (\tau\mathbf{n}) \cdot \mathbf{n} + v_t (\tau\mathbf{n}) \cdot \mathbf{t} \\ &= \tau_{ij} n_j (n_i v_n + t_i v_t)\end{aligned}$$

on the relevant portions of $\partial\Omega$, it is easy to see that for the respective problems (3.12) takes the form

I.
$$\begin{aligned}\langle \pi_n(\tau), \gamma_{\Gamma n}(\mathbf{v}) \rangle_{1/2,\Gamma} + \langle \pi_t(\tau), \gamma_{\Gamma t}(\mathbf{v}) \rangle_{1/2,\Gamma} \\ = \int_\Gamma \gamma_\Gamma(\tau_{ij}) n_j (n_i \gamma_{\Gamma n}(\mathbf{v}) + t_i \gamma_{\Gamma t}(\mathbf{v})) \, \mathrm{d}s \qquad \forall\, \mathbf{v} \in V_1,\end{aligned}$$

where $\pi_1 = (\pi_n, \pi_t)$ and $\langle \cdot, \cdot \rangle_{1/2,\Gamma}$ denotes duality pairing on $H^{1/2}(\Gamma)' \times H^{1/2}(\Gamma)$;

II. as in I, with $H^{1/2}_1(\Gamma)$ in place of $H^{1/2}(\Gamma)$, etc.;

III.
$$\begin{aligned}\langle \pi^0_n(\tau), \gamma^0_{\Gamma n}(\mathbf{v}) \rangle_{00,\Gamma} + \langle \pi^3(\tau), (\gamma^A_{\Gamma t}, \gamma_{\Lambda_o n}, \gamma^D_{\Lambda_s t})\mathbf{v} \rangle_3 \\ = \int_\Gamma \gamma_\Gamma(\tau_{ij}) n_j (n_i \gamma^0_{\Gamma n}(\mathbf{v}) + t_i \gamma^A_{\Gamma t}(\mathbf{v})) \, \mathrm{d}s + \int_{\Lambda_o} \gamma_{\Lambda_o}(\tau_{ij}) n_j n_i \gamma_{\Lambda_o}(\mathbf{v}) \, \mathrm{d}s \\ + \int_{\Lambda_s} \gamma_{\Lambda_s}(\tau_{ij}) n_j t_i \gamma^D_{\Lambda_s t}(\mathbf{v}) \, \mathrm{d}s \qquad \forall\, \mathbf{v} \in V_3,\end{aligned}$$

where $\pi_3 = (\pi^0_n, \pi^3)$ and $\langle \cdot, \cdot \rangle_3$ denotes duality pairing on $Y_3' \times Y_3$;

IV. as in I, with $H^{1/2}_{00}(\Gamma)$ in place of $H^{1/2}(\Gamma)$, etc. (3.21)

Equations (3.17) and (3.18 III) imply that $\pi_t(\tau) = 0$ in I above, and similarly in II and IV, while $\pi^3(\tau) = 0$ in III. In the case of a slip condition at Λ_w in III, the operator π_n^0 cannot be "isolated" from π_3 as above in Green's formula. However, (3.17) and (3.18 III) imply that $\langle\langle \pi_3(\tau), \gamma_3(\mathbf{v})\rangle\rangle_3$ depends only on $\gamma_{\Gamma n}(\mathbf{v})$. The situation is similar when a slip condition is applied at Λ_1 or Λ_2 in IV. Hence, in every case the last term in (3.19) reduces to the first term of the corresponding relation in (3.21). For convenience we shall henceforth denote the relevant trace space, inner product, duality pairing and trace operator by N_k, $[\cdot, \cdot]_k$, $\langle \cdot, \cdot \rangle_k$ and γ_n^k, respectively. As before, we shall sometimes omit the subscript k. Thus, by Riesz' theorem, there exists a unique $\lambda = \lambda(\mathbf{u}, p) \in N_k$ such that the last term in (3.19) equals $[\lambda, v_n]_k$, where v_n denotes $\gamma_n^k(\mathbf{v})$, for every $\mathbf{v} \in V_k$.

We shall need the following result.

LEMMA 3.1. Let $\gamma \in \mathcal{L}(V, Z)$ be a surjective map from a Hilbert space V onto a Hilbert space Z. Denote $\ker \gamma$ $(=\{v \in V \mid \gamma v = 0\})$ by V_0. Then there exists a right inverse $\delta \in \mathcal{L}(Z, V)$ of γ (i.e. $\gamma \cdot \delta$ is the identity map from Z onto itself).

Sketch of proof. The existence of a right inverse δ can be deduced from the fact that, for any $v_0 \in V_0$, the map $\gamma\colon v_0 + V_0^\perp \to Z$ is a continuous linear bijection, where

$$V_0^\perp = \{v \in V \mid (u, v)_V = 0 \ \forall\, u \in V_0\}$$

is the orthogonal complement of V_0. The continuity of δ follows from the open mapping theorem. ∎

Let $\mathbf{u}^0 \in H^1(\Omega)^2$ be such that $\gamma(\mathbf{u}^0)$ satisfies conditions (3.15) and (3.18). The existence of $\mathbf{u}^0$ is assured if:

$$\begin{array}{ll}
\text{I.} & \mathbf{u}_0 \in H^{1/2}(\Sigma)^2; \\
\text{II.} & \mathbf{u}_0 \in H_1^{1/2}(\Sigma)^2; \\
\text{III.} & u_{0t} \in H_{00}^{1/2}(\Sigma) \text{ and } u_{0n} \in H_E^{1/2}(\Sigma) \ (u_{0n} \in H^{1/2}(\Sigma) \text{ if slip occurs at } \Lambda_w); \\
\text{IV.} & \mathbf{u}_0 \in H_{00}^{1/2}(\Sigma)^2 \text{ (with suitable changes when slip is allowed at } \Lambda).
\end{array} \tag{3.22}$$

In fact, if we denote the trace spaces listed above by U_0 and denote the corresponding closed subspaces $\{\mathbf{v} \in H^1(\Omega)^2 \mid \mathbf{v}$ satisfies (3.16) and (3.18)$\}$ by U^0, then it is not difficult to prove (by repeated application of the ideas and results earlier in this section and the fact that Γ is bounded away from Σ) that $\gamma_\Sigma \in \mathcal{L}(U^0, U_0)$ and that this operator is surjective. Thus, by lemma 3.1, it has a right inverse $\delta \in \mathcal{L}(U_0, U^0)$. Let $\mathbf{v} = \delta(\mathbf{u}_0)$. In the case of problems I, II and IV, it follows from the conditions on $\mathbf{u}_0$ below (2.5) that

$$\int_{\partial\Omega} \gamma_n(\mathbf{v})\, \mathrm{d}s = 0.$$

In the case of problem III, $\mathbf{v}$ can be adjusted to also have this property. (Set $q = 0$, $\tau = 0$ in the proof of lemma 4.4 (c) later.) This proves that the mapping $\gamma_\Sigma\colon \{\mathbf{v} \in U^0 \mid \operatorname{div} \mathbf{v} = 0\} \to U_0$ is surjective, so that it has a bounded linear right inverse. The important point is that $\|\mathbf{g}\|_{1/2,\partial\Omega} \le c(\Omega)\|\mathbf{u}_0\|_\Sigma$ where $\mathbf{g} = \gamma(\mathbf{v})$. It can be shown (cf. [13, p. 24]) that there exists a $\mathbf{u}^0 \in H^1(\Omega)^2$ such that $\operatorname{div} \mathbf{u}^0 = 0$, $\gamma(\mathbf{u}^0) = \mathbf{g}$ and $\|\mathbf{u}^0\|_1 \le k(\Omega)\|\mathbf{g}\|_{1/2}$. Hence, $\mathbf{u}^0$ satisfies (3.14)–(3.16), (3.18) and $\|\mathbf{u}^0\|_1 \le K(\Omega)\|\mathbf{u}_0\|_\Sigma$.

Let $\mathbf{w} = \mathbf{u} - \mathbf{u}^0$. Then $\mathbf{w} \in V_k$ since both $\mathbf{u}$ and $\mathbf{u}_0$ satisfy (3.15) and (3.18). Moreover, from (3.19) and the above remarks we get

$$\int_\Omega \left\{(w_j w_{i,j} + w_j u^0_{i,j} + u^0_j w_{i,j})v_i + \frac{2}{Re} D_{ij}(\mathbf{w}) D_{ij}(\mathbf{v})\right\} \mathrm{d}x - \int_\Omega p v_{i,i} \, \mathrm{d}x - [\lambda(\mathbf{u}, p), v_n]_k$$
$$= \int_\Omega \left\{(f_i - u^0_j u^0_{i,j})v_i - \frac{2}{Re} D_{ij}(\mathbf{u}^0) D_{ij}(\mathbf{v})\right\} \mathrm{d}x \qquad \forall\, \mathbf{v} \in V_k. \tag{3.23}$$

Since u_n, u^0_n and w_n belong to N_k ($\mathbf{u}$, $\mathbf{u}^0$ satisfy (3.18) and Γ is bounded away from Σ, so that it is always possible to construct functions in V_k that have the same boundary values on Γ), (3.20) is equivalent to

$$(q, w_{i,i})_{0,\Omega} + [\mu, w_n]_k = -(q, u^0_{i,i})_{0,\Omega} - [\mu, u^0_n]_k \qquad \forall\,(q, \mu) \in L^2(\Omega) \times N_k. \tag{3.24}$$

On the basis of the discussion above, we may assume that the right-hand side of (3.24) is identically zero. Hence, we have shown that to every solution $(\mathbf{u}, p)$ of (AUX) there corresponds a solution $(\mathbf{w}, p, \lambda)$ of the *variational problem* given by:

(VAR) find $\mathbf{w} \in V_k$, $p \in L^2(\Omega)$ and $\lambda \in N_k$ which satisfy equations (3.23) and (3.24).

The reason for including the normal stress λ in (VAR) as a multiplier becomes clear upon inspection of the supplementary equation (2.9), which is used in the second stage of the splitting method: we see from (2.9) that it is only the normal stress which is carried over from the first stage to the second, so that it helps to have this variable treated explicitly in the auxiliary problem, from both a mathematical as well as an algorithmic point of view.

4. EXISTENCE AND UNIQUENESS RESULTS

In this section we establish results on the existence and uniqueness of solutions to the weak problem (VAR). Throughout this section V and M will denote Hilbert spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_M$, respectively. The topological dual of a space W is denoted by W'. The following results are either proved in, or follow easily from the theory given in [13, pp. 57–61].

Let $a_0(\cdot, \cdot)\colon V \times V \to \mathcal{R}$ and $b(\cdot, \cdot)\colon V \times M \to \mathcal{R}$ be two continuous bilinear forms with norms $\|a_0\|$ and $\|b\|$ defined in the usual manner. We also introduce the form

$$a(\cdot; \cdot, \cdot)\colon V \times V \times V \to \mathcal{R} \tag{4.1}$$

which is such that, for every $w \in V$, the mapping $a(w; \cdot, \cdot)\colon V \times V \to \mathcal{R}$ is a continuous bilinear form. We shall consider the following nonlinear variational problem:

(NVP) given $l \in V'$, find $w \in V$ and $n \in M$ which satisfy

$$a(w; w, v) + b(v, n) = \langle l, v\rangle \qquad \forall\, v \in V, \tag{4.2}$$

$$b(w, m) = 0 \qquad \forall\, m \in M. \tag{4.3}$$

Define the operator $\mathcal{B} \in \mathcal{L}(V, M')$ and its dual $\mathcal{B}' \in \mathcal{L}(M, V')$ by

$$\langle \mathcal{B}v, m\rangle = \langle \mathcal{B}'m, v\rangle = b(v, m) \qquad \forall\, v \in V,\ \forall\, m \in M. \tag{4.4}$$

For every $w \in V$, the operator $\mathcal{A}(w) \in \mathcal{L}(V, V')$ is defined by

$$\langle \mathcal{A}(w)u, v\rangle = a(w; u, v) \qquad \forall\, u, v \in V. \tag{4.5}$$

Then equations (4.2) and (4.3) can be written as

$$\mathcal{A}(w)w + \mathcal{B}'n = l \qquad \text{in } V', \tag{4.6}$$

$$\mathcal{B}w = 0 \qquad \text{in } M'. \tag{4.7}$$

Let

$$K = \ker \mathcal{B} = \{v \in V \mid \mathcal{B}v = 0\},$$

$$K^{\perp} = \{v \in V \mid (u, v)_V = 0\ \forall\, u \in K\}$$

and

$$K^{a} = \{g \in V' \mid \langle g, v\rangle = 0\ \forall\, v \in K\}.$$

Then K, $K^{\perp}$ and K^{a} are closed subspaces of, respectively, V, V and V' and are therefore Hilbert spaces.

We say that the bilinear form $b(\cdot, \cdot)$ satisfies the *inf-sup condition* if there exists a constant $\beta > 0$ such that

$$\inf_{m \in M} \sup_{v \in V} \frac{b(v, m)}{\|v\|_V \|m\|_M} \geq \beta. \tag{4.8}$$

For every $g \in V'$, let Πg be defined as the restriction of g to K: $\langle \Pi g, v\rangle = \langle g, v\rangle\ \forall\, v \in K$. Then $\|\Pi g\|_{K'} \leq \|g\|_{V'}$ and $\Pi \in \mathcal{L}(V', K')$.

For dealing with specific problems it is convenient to introduce the operator

$$B = j_M^{-1} \cdot \mathcal{B} \in \mathcal{L}(V, M),$$

where j_M is the Riesz isometry. Then

$$(Bv, m)_M = b(v, m) \qquad \forall\, v \in V,\ m \in M. \tag{4.9}$$

It is clear that the algebraic and topological properties of B are identical to those of $\mathcal{B}$. Note also that $\ker B = \ker \mathcal{B} = K$. We also have the following lemma.

Lemma 4.1 [12]. The inf-sup condition holds if and only if B is surjective, i.e. $RgB = M$. ■

We will require conditions on $a(\cdot; \cdot, \cdot)$ sufficient for solving the following problem:

(N) given $l \in V'$, find $w \in K$ such that

$$a(w; w, v) = \langle l, v\rangle \qquad \forall\, v \in K, \tag{4.10}$$

or equivalently, such that

$$\Pi \mathcal{A}(w)w = \Pi l \qquad \text{in } K'. \tag{4.11}$$

THEOREM 4.1. (a) Assume that the following conditions hold:

(i) the space K is separable, i.e. K has a countable dense subset;

(ii) there exists a constant $\alpha > 0$ such that

$$a(v; v, v) \geq \alpha \|v\|_V^2 \qquad \forall\, v \in K; \tag{4.12}$$

(iii) for each $v \in K$, the mapping $u \to a(u; u, v)$ is sequentially weakly continuous on K. Then problem (N) has at least one solution $w \in K$. Moreover, every solution satisfies the estimate

$$\|w\|_V \leq \|\Pi l\|_{K'}/\alpha \leq \|l\|_{V'}/\alpha. \tag{4.13}$$

(b) Assume that conditions (i) and (iii) are satisfied and that (4.12) holds $\forall\, v \in S_d = \{v \in K \mid \|v\|_K \leq d\}$ for some fixed $d > 0$. Then, for every $l \in V'$ with

$$\|\Pi l\|_{K'} \leq \alpha d, \tag{4.14}$$

problem (N) has at least one solution $w \in S_d$, for which (4.13) holds.

Proof. A proof of part (a) is given in [13, pp. 279–281]. The proof of (b) follows by a simple adaptation of the proof of (a). ■

THEOREM 4.2. (a) Assume that:

(i) the bilinear form $a(u; \cdot, \cdot)$ is uniformly K-elliptic with respect to u, i.e. there exists a constant $\alpha > 0$ such that

$$a(u; v, v) \geq \alpha \|v\|_K^2 \qquad \forall\, u, v \in K; \tag{4.15}$$

(ii) the mapping $u \to \Pi\mathcal{Q}(u): K \to \mathcal{L}(K, K')$ is uniformly Lipschitz continuous, with Lipschitz constant L.

Then, for every $l \in V'$ satisfying

$$L\|\Pi l\|_{K'}/\alpha^2 < 1, \tag{4.16}$$

problem (N) has a *unique* solution, for which the estimate (4.13) holds.

(b) Assume that condition (ii) is satisfied and that (4.15) holds for all $\mathbf{u} \in S_d$ for some fixed $d > 0$. Then, for every $l \in V'$ satisfying (4.14) and (4.16), problem (N) has a solution in S_d, which is *unique* in S_d and satisfies (4.13).

Proof. As in the proof of theorem 4.1, the proof of (a) given in [13, p. 282] can be adapted to establish (b). ■

It is easily verified that if (w, n) is a solution of problem (NVP), then w is a solution of problem (N). The converse statement holds when the inf–sup condition is satisfied.

THEOREM 4.3. (a) Assume that $b(\cdot, \cdot)$ satisfies the inf–sup condition. Then, for each solution w of problem (N), there exists a unique $n \in M$ such that (w, n) is a solution of problem (NVP).

(b) Let $B \in \mathcal{L}(V, M)$ be defined as in (4.9) and assume that $Rg(B)$ is closed. Then, for each solution w of problem (N), there exists a unique $n \in Rg(B)$ such that $(w, n + m)$ is a solution of problem (NVP) for every $m \in Rg(B)^\perp$.

Proof. (a) For a given l, assume that w is a solution of problem (N). Then $Bw = 0$ since $w \in K$. Furthermore, $\Pi(l - \mathcal{A}(w)w) = 0$ and thus $l - \mathcal{A}(w)w \in K^a$. By (4.8) and (4.9), $\mathcal{B}'$ is an isomorphism from M onto K^a, so that there exists a unique $n \in M$ such that $B'n = l - \mathcal{A}(w)w$.

(b) It is easily verified that $Rg(B) = \{n \in M \mid \text{there exists a } v \in V \text{ such that } (n, m)_M = b(v, m)\ \forall\, m \in M\}$. Set $M_1 = Rg(B)$ and let $b_1(\cdot, \cdot)$ be the restriction of $b(\cdot, \cdot)$ to $V \times M_1$. Then the corresponding operator $B_1 \in \mathcal{L}(V, M_1)$ is surjective since $Rg(B_1) = \{n \in M_1 \mid \text{ for some } v \in V,\ (n, m)_M = b(v, m)\ \forall\, m \in M_1\} = M_1$. Hence, by lemma 4.1 the inf–sup condition for $b_1(\cdot, \cdot)$ is satisfied.

Let $l \in V'$ be given. Then it follows from (a) that there exists a unique pair $(w, n) \in V \times M_1$ such that

$$\text{(i)} \qquad a(w; w, v) + b(v, n) = \langle l, v\rangle \qquad \forall\, v \in V,$$

$$\text{(ii)} \qquad b(w, m) = \langle k, m\rangle \qquad \forall\, m \in M_1.$$

Furthermore, $b(v, n_0) = (Bv, n_0)_M = 0\ \forall\, v \in V$ and $\forall\, n_0 \in Rg(B)^\perp$, so that n may be replaced by $n + n_0$ in (i). Finally, $b(w, n_0) = (Bw, n_0)_M = 0\ \forall\, n_0 \in M_1^\perp$, so that M_1 may be replaced by M. ■

It remains to record general a priori estimates for the solution(s) of problem (NVP). Let (w, n) be any solution: then

(i) if $a(\cdot; \cdot, \cdot)$ satisfies (4.12), then w satisfies (4.13) since

$$\alpha \|w\|_V^2 \leq a(w; w, w) = \langle l, w\rangle \leq \|\Pi l\|_{K'} \|w\|_V;$$

(ii) if $b(\cdot, \cdot)$ satisfies the inf–sup condition, or $Rg(B)$ is closed and $n \in Rg(B)$, then it follows [12] that $\mathcal{B}'$ is bounded below, and in fact

$$\|n\|_M \leq \|\mathcal{B}'n\|_{V'}/\beta \leq (\|l\|_{V'} + \|\mathcal{A}(w)\|_{\mathcal{L}(V, V')} \|w\|_V)/\beta. \tag{4.17}$$

4.1. *The Navier–Stokes problem*

With the notation as in Section 3, let $1 \leq k \leq 4$ be fixed. To put the Navier–Stokes problem into the framework of Section 4.1 we set $V = V_k$ and $M = L^2(\Omega) \times N_k$ with inner products $(\cdot, \cdot)_V = (\cdot, \cdot)_{1,\Omega}$, $((p, \lambda), (q, \tau))_M = (p, q)_{0,\Omega} + [\lambda, \tau]_k$. We also set

$$a(\mathbf{w}; \mathbf{u}, \mathbf{v}) = a_0(\mathbf{u}, \mathbf{v}) + a_1(\mathbf{w}, \mathbf{u}, \mathbf{v}) + a_1(\mathbf{u}^0, \mathbf{u}, \mathbf{v}) + a_1(\mathbf{u}, \mathbf{u}^0, \mathbf{v})$$

with

$$a_0(\mathbf{u}, \mathbf{v}) = \frac{2}{Re}(D_{ij}(\mathbf{u}), D_{ij}(\mathbf{v}))_0$$

and

$$a_1(\mathbf{w}, \mathbf{u}, \mathbf{v}) = \int_\Omega w_j u_{i,j} v_i \, \mathrm{d}x,$$

$$b(\mathbf{v}, (q, \tau)) = -((q, \tau), (\operatorname{div} \mathbf{v}, v_n)),$$

and

$$\langle l, \mathbf{v}\rangle = (\mathbf{f}, \mathbf{v})_0 - a_0(\mathbf{u}^0, \mathbf{v}) - a_1(\mathbf{u}^0, \mathbf{u}^0, \mathbf{v}) \tag{4.18}$$

We know from Section 3 that V_k, $L^2(\Omega)$ and N_k, and therefore V and M, are Hilbert spaces. It is also easy to see that $a(\cdot;\cdot,\cdot)$, $b(\cdot,\cdot)$, $\langle l,\cdot\rangle$ and $\langle k,\cdot\rangle$ are well-defined, linear in every argument and bounded. The operator B defined in (4.9) is given by

$$B\mathbf{v} = -(\operatorname{div}\mathbf{v}, v_n) \qquad \forall\, \mathbf{v} \in V_k. \tag{4.19}$$

Thus

$$K = \ker B = \{\mathbf{v} \in V_k \mid \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega,\ \gamma_n^k(\mathbf{v}) = 0 \text{ on } \Gamma\} \subset V_k \subset V_0 = \{\mathbf{v} \in H^1(\Omega)^2 \mid \gamma_\Sigma(\mathbf{v}) = \mathbf{0}\}.$$

Since there is no danger of ambiguity, in the rest of this Section we continue to refer to the problems (4.2), (4.3) and (4.10) with $a(\cdot;\cdot,\cdot)$, etc. defined in (4.18), as problems (NVP) and (N), respectively.

The K-ellipticity of $a_0(\cdot,\cdot)$ follows from the following lemma, the proof of which relies on one of Korn's inequalities (cf. [11, pp. 104–109, 115, 116; 19, pp. 110–116]).

LEMMA 4.2. Let Ω be a bounded domain in $\mathbb{R}^2$ with a Lipschitz continuous boundary. Then there exists a constant $\alpha' > 0$ such that

$$a_0(\mathbf{v},\mathbf{v}) \geq \alpha\|\mathbf{v}\|_1^2 \qquad \forall\, \mathbf{v} \in V_k \text{ with } \alpha = 2\alpha'/Re = A(\Omega)/Re. \quad \blacksquare \tag{4.20}$$

We need to characterize $Rg(B)$ (to check if $b(\cdot,\cdot)$ satisfies the LBB condition).

LEMMA 4.3. (a) The operator div maps $H_0^1(\Omega)^2$ onto the space

$$L_0^2(\Omega) = \left\{q \in L^2(\Omega) \,\middle|\, \int_\Omega q\,\mathrm{d}x = 0\right\}.$$

(b) For problems I, II and IV we have

$$Rg(B) = S = \left\{(q,\tau) \in M \,\middle|\, \int_\Omega q\,\mathrm{d}x = \int_\Gamma \tau\,\mathrm{d}s\right\}. \tag{4.21}$$

This is a closed subspace of M and its orthogonal complement is given by

$$Rg(B)^\perp = \operatorname{span}\{(1,\tau^*)\} = \{k(1,\tau^*) \mid k \in \mathbb{R}\} \tag{4.22}$$

where $\tau^* \in N_k$ is such that $[\tau^*,\tau]_k = -\int_\Gamma \tau\,\mathrm{d}s\ \forall\, \tau \in N_k$.

(c) In the case of problem III, B is surjective.

Proof. (a) is standard (cf. [1, pp. 15, 32; 13, p. 24]).

(b) (i) By definition of V_k, $\gamma_n(\mathbf{v}) = 0$ on $\partial\Omega\backslash\Gamma$ $\forall\, \mathbf{v} \in V_1$ and $\forall\, \mathbf{v} \in V_4$. Moreover, $\int_{\Lambda_0\cup\Lambda_1} \gamma_n(\mathbf{v})\,\mathrm{d}s = 0$ $\forall\, \mathbf{v} \in V_2$ due to the periodicity condition. Thus if $(q,\tau) = B\mathbf{v}$ for some $\mathbf{v} \in V_k$, then it follows via the standard Green's formula for functions in $H^1(\Omega)$ that

$$\int_\Omega q\,\mathrm{d}x = \int_\Omega 1\cdot v_{i,i}\,\mathrm{d}x = \int_{\partial\Omega} \gamma(1\cdot v_i)n_i\,\mathrm{d}s = \int_\Gamma v_n\,\mathrm{d}s = \int_\Gamma \tau\,\mathrm{d}s.$$

Conversely, suppose that $(q, \tau) \in M$ satisfies the compatibility condition $\int_\Omega q\,dx = \int_\Gamma \tau\,ds$. Since γ_n^k is surjective, there exists a $\mathbf{v} \in V_k$ such that $\gamma_n^k(\mathbf{v}) = \tau$. As above, it follows that

$$\int_\Omega \operatorname{div}\mathbf{v}\,dx = \int_\Gamma v_n\,ds = \int_\Gamma \tau\,ds = \int_\Omega q\,dx.$$

Hence $q - \operatorname{div}\mathbf{v} \in L_0^2(\Omega)$. By (a), there exists a $\mathbf{w} \in H_0^1(\Omega)^2 \subset V_k$ such that $\operatorname{div}\mathbf{w} = q - \operatorname{div}\mathbf{v}$. Set $\mathbf{u} = \mathbf{v} + \mathbf{w}$. Then $\mathbf{u} \in V_k$ and $B\mathbf{u} = (q, \tau)$. Thus $Rg(B) = S$.

(ii) Assume that $(q, \tau) \in M$ and that there exists a sequence (q_n, τ_n) in S such that $(q_n, \tau_n) \to (q, \tau)$ in M. Then $q_n \to q$ in $L^2(\Omega)$ and $\tau_n \to \tau$ in N_k, so that

$$\begin{aligned}\left|\int_\Omega q\,dx - \int_\Gamma \tau\,ds\right| &= \left|\int_\Omega (q - q_n)\,dx + \int_\Gamma (\tau_n - \tau)\,ds\right| \qquad (\text{since } (q_n, \tau_n) \in S)\\ &\le |(1, q - q_n)_0| + |(1, \tau_n - \tau)_{0,\Gamma}|\\ &\le \|1\|_0\|q - q_n\|_0 + \|1\|_{0,\Gamma}\|\tau_n - \tau\|_k \to 0 \qquad \text{as } n \to \infty.\end{aligned}$$

Thus $\int_\Omega q\,dx = \int_\Gamma \tau\,ds$, i.e. $(q, \tau) \in S$. Hence S is closed in M.

(iii) By definition,

$$S^\perp = \left\{(p, \lambda) \in M \,\middle|\, (p, q)_0 + [\lambda, \tau]_k = 0 \ \forall\, (q, \tau) \in M \text{ satisfying } \int_\Omega q\,dx = \int_\Gamma \tau\,ds\right\}.$$

Let $(p, \lambda) \in S^\perp$. Then $(p, q)_0 = (p, q)_0 + [\lambda, 0]_k = 0 \ \forall\, q \in L_0^2(\Omega)$, i.e. $p \in L_0^2(\Omega)^\perp$. But

$$L_0^2(\Omega) = \{q \in L^2(\Omega) \mid (k, q)_0 = 0 \ \forall\, k \in \mathbb{R}\} = \mathbb{R}^\perp$$

(where $\mathbb{R}$ is understood to represent the subspace of a.e. constant functions in $L^2(\Omega)$) and $\mathbb{R}^{\perp\perp} = \mathbb{R}$ since $\mathbb{R}$ is a finite-dimensional, and therefore a closed, subspace of $L^2(\Omega)$. Thus $p \in \mathbb{R}$.

Suppose that $p = 0$. Then $[\lambda, \tau]_k = (0, q_\tau)_0 + [\lambda, \tau]_k = 0 \ \forall\, \tau \in N_k$ (with q_τ chosen so that $(q_\tau, \tau) \in S$), i.e. $\lambda = 0$. Since $S^\perp \neq \{(0, 0)\}$ (from (4.22)), it follows that there exists a pair $(p, \tau) \in S^\perp$ with $p \neq 0$.

Suppose that $Q = \operatorname{span}\{(p_i, \tau_i) \mid i \in J\} \subset S^\perp$, with J some index set. By the arguments above, we may assume that $0 \neq p_i \in \mathbb{R} \ \ \forall\, i \in J$. Thus $Q = \operatorname{span}\{(1, \tau_i/p_i) \mid i \in J\}$. Now $(1, \tau^*) \in S^\perp$ iff $(1, q)_0 + [\tau^*, \tau]_k = 0 \ \forall\, (q, \tau) \in M$ such that $(1, q)_0 = (1, \tau)_{0,\Gamma}$, i.e. iff $[\tau^*, \tau]_k = -(1, \tau)_{0,\Gamma} \ \forall\, \tau \in N_k$. By the Riesz representation theorem, this equation has a unique solution $\tau^* \in N_k$ since the right-hand side defines an element of N_k'. Hence $Q = \operatorname{span}\{(1, \tau^*)\}$. Since $S^\perp$ is a linear space, it follows that $S^\perp \subset \bar{Q} = Q$.

Conversely, let $k \in \mathbb{R}$ and set $(p, \lambda) = k(1, \tau^*)$. Then $(p, q)_0 + [\lambda, \tau]_k = k(1, q)_0 - k(1, \tau)_{0,\Gamma} = 0 \ \forall\, (q, \tau) \in S$. Thus $S^\perp = \operatorname{span}\{(1, \tau^*)\}$.

(c) Let $(q, \tau) \in L^2(\Omega) \times N_3$ be given. Since γ_n^3 is surjective, there exists a $\mathbf{v}^0 \in V_3$ such that $\gamma_n^3(\mathbf{v}^0) = \tau$. If $\int_{\Lambda_o} v_n^0\,ds = 0$, set $\mathbf{v} = \mathbf{v}^0$. If $\int_{\Lambda_o} v_n^0\,ds \neq 0$, then we can define $\Lambda_p \subset \Lambda_o$, with $\operatorname{meas}(\Lambda_p) > 0$ and $\bar{\Lambda}_p \subset \Lambda_o$, such that $\int_{\Lambda_p} v_n^0\,ds \neq 0$. Thus $\int_{\Lambda_p} \phi v_n^0\,ds \neq 0$ for some smooth function ϕ on Λ_o with $\phi = 0$ on $\Lambda_o \setminus \Lambda_p$ (else $v_n^0 = 0$ on Λ_p, since $\mathfrak{D}(\Lambda_p)$ is dense in $L^2(\Lambda_p)$). Now define

$$\boldsymbol{\Phi} = \begin{cases} \gamma(\mathbf{v}^0) & \text{on } \partial\Omega \setminus \Lambda_o \\ (1 + k\phi)\gamma(\mathbf{v}^0) & \text{on } \Lambda_o \end{cases}$$

where

$$k = \left(\int_\Omega q\,\mathrm{d}x - \int_\Gamma \tau\,\mathrm{d}s - \int_{\Lambda_o} v_n^0\,\mathrm{d}s\right)\Big/\int_{\Lambda_o} \phi v_n^0\,\mathrm{d}s.$$

Then $\boldsymbol{\phi} \in H^{1/2}(\partial\Omega)^2$ and therefore $\gamma(\mathbf{v}) = \boldsymbol{\phi}$ for some $\mathbf{v} \in H^1(\Omega)^2$. From the definition of $\boldsymbol{\phi}$ and V_3 it follows that $\mathbf{v} \in V_3$. Thus we have constructed a $\mathbf{v} \in V_3$ such that

$$\int_\Omega \operatorname{div} \mathbf{v}\,\mathrm{d}x = \int_{\Gamma\cup\Lambda_o} \gamma_n(\mathbf{v})\,\mathrm{d}s = \int_\Omega q\,\mathrm{d}x.$$

As in the last part of (b) (i), it now follows that $(q, \tau) \in Rg(B)$. ∎

LEMMA 4.4. (a) Let Ω be a bounded domain in $\mathbb{R}^2$ with a Lipschitz continuous boundary. Then the form $a_1(\cdot, \cdot, \cdot)$ is trilinear and continuous on $(H^1(\Omega)^2)^3$.

(b) Let Ω be as above and let Λ be a (possibly empty) portion of $\partial\Omega$. Let $\mathbf{v}, \mathbf{w} \in H^1(\Omega)^2$ with $\operatorname{div} \mathbf{w} = 0$ and $\gamma(\mathbf{w}) \cdot \mathbf{n}|_{\partial\Omega\setminus\Lambda} = 0$. Then

$$a_1(\mathbf{w}, \mathbf{v}, \mathbf{v}) = \tfrac{1}{2}\int_\Lambda \gamma(w_j)n_j\gamma(v_i)\gamma(v_i)\,\mathrm{d}s. \tag{4.23}$$

(c) Let $\mathbf{w} \in K$. Then for problems I–IV, we have

$$a_1(\mathbf{w}, \mathbf{v}, \mathbf{v}) = b \;\forall\, \mathbf{v} \in H, \tag{4.24}$$

$$a_1(\mathbf{w}, \mathbf{u}, \mathbf{v}) + a_1(\mathbf{w}, \mathbf{v}, \mathbf{u}) = c \;\forall\, \mathbf{u}, \mathbf{v} \in H, \tag{4.25}$$

where b, c and H are defined respectively by

I, IV. $H = H^1(\Omega)^2$, $b = c = 0$;

II. $H = \{\mathbf{v} \in H^1(\Omega)^2 \mid \gamma_{\Lambda_0}(\mathbf{v})(x_2) = \gamma_{\Lambda_1}(\mathbf{v})(x_2)$ for a.a. $x_2\}$, $b = c = 0$;

III. $H = \{\mathbf{v} \in H^1(\Omega)^2 \mid \gamma_{\Lambda_o t}(\mathbf{v}) = 0\}$, $b = \frac{1}{2}\int_{\Lambda_o} \gamma(w_1)\gamma(v_1)^2\,\mathrm{d}s$, $c = \int_{\Lambda_o} \gamma(w_1)\gamma(u_1)\gamma(v_1)\,\mathrm{d}s$.

(d) For every $\mathbf{v} \in H$ (with H as in (c)), the mapping $\mathbf{u} \to a_1(\mathbf{u}, \mathbf{u}, \mathbf{v})$ is sequentially weakly continuous on K.

Proof. (a) This is given in [20, p. 96; 13, p. 284].

(b) From the density of $\mathcal{D}(\bar{\Omega})$ in $H^1(\Omega)$ it follows from the standard Green's formula that, for $\mathbf{w}$ as given in the lemma and for every $\mathbf{v} \in \mathcal{D}(\bar{\Omega})^2$,

$$\begin{aligned} a_1(\mathbf{w}, \mathbf{v}, \mathbf{v}) &= \tfrac{1}{2}\int_\Omega w_j(v_iv_i)_{,j}\,\mathrm{d}x \\ &= \tfrac{1}{2}\int_{\partial\Omega} \gamma(w_j)v_iv_in_j\,\mathrm{d}s - \tfrac{1}{2}\int_\Omega w_{j,j}v_iv_i\,\mathrm{d}x \\ &= \tfrac{1}{2}\int_\Lambda \gamma(w_j)n_jv_iv_i\,\mathrm{d}s = a_2(\mathbf{w}, \mathbf{v}), \text{ say.} \end{aligned}$$

Let $\mathbf{v} \in H^1(\Omega)^2$. Then there exists a sequence $(\mathbf{v}^n)$ in $\mathcal{D}(\bar{\Omega})^2$ such that $\mathbf{v}^n \to \mathbf{v}$ in $H^1(\Omega)^2$. Moreover, every $\mathbf{v}^n$ satisfies the equation above. From (a) and the boundedness of $\|\mathbf{v}^n\|_1$ (since every (weakly or strongly) convergent sequence is bounded), it follows that

$$|a_1(\mathbf{w}, \mathbf{v}, \mathbf{v}) - a_1(\mathbf{w}, \mathbf{v}^n, \mathbf{v}^n)| \le \|a_1\| \cdot \|\mathbf{w}\|_1(\|\mathbf{v}\|_1 + \|\mathbf{v}^n\|_1)\|\mathbf{v} - \mathbf{v}^n\|_1 \to 0 \qquad \text{as } n \to \infty.$$

According to the Sobolev imbedding theorem, $H^{1/2}(\Lambda)$ is continuously imbedded in $L^4(\Lambda)$ (cf. [18, pp. 25, 27, 37]). Let $K = K(\Lambda)$ be the imbedding constant. Then

$$|a_2(\mathbf{w}, \mathbf{v})| \leq m\|\gamma(v_i)\gamma(v_i)\|_{0,\Lambda} = m(\|\gamma(v_1)\|_{0,4,\Lambda}^2 + \|\gamma(v_2)\|_{0,4,\Lambda}^2) \leq K^2 m\|\gamma(\mathbf{v})\|_{1/2,\Lambda}^2 < \infty$$

where $m = \|\gamma(w_j)n_j\|_{0,\Lambda}/2$. Furthermore,

$$\begin{aligned}
|a_2(\mathbf{w}, \mathbf{v}) - a_2(\mathbf{w}, \mathbf{v}^n)| &\leq \sum_{i=1}^{2} m\|\gamma(v_i)^2 - \gamma(v_i^n)^2\|_{0,\Lambda} \\
&= \sum_{i=1}^{2} m(\gamma(v_i - v_i^n)^2, \gamma(v_i + v_i^n)^2)_{0,\Lambda}^{1/2} \\
&\leq \sum_{i=1}^{2} m\|\gamma(v_i - v_i^n)\|_{0,4\Lambda}\|\gamma(v_i + v_i^n)\|_{0,4,\Lambda} \\
&\leq \sum_{i=1}^{2} K^2 m\|\gamma(v_i - v_i^n)\|_{1/2,\Lambda}\|\gamma(v_i + v_i^n)\|_{1/2,\Lambda} \\
&\leq \sum_{i=1}^{2} C\|v_i - v_i^n\|_1\|v_i + v_i^n\|_1 \\
&\leq 2C\|\mathbf{v} - \mathbf{v}^n\|_1(\|\mathbf{v}\|_1 + \|\mathbf{v}^n\|_1) \to 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

Thus we have proved that $a_1(\mathbf{w}, \mathbf{v}, \mathbf{v}) = \lim_{n\to\infty} a_1(\mathbf{w}, \mathbf{v}^n, \mathbf{v}^n) = \lim_{n\to\infty} a_2(\mathbf{w}, \mathbf{v}^n) = a_2(\mathbf{w}, \mathbf{v})$.

(c) By the definition of K (and V_k), if $\mathbf{w} \in K$ then div $\mathbf{w} = 0$ and (4.24) follows from (b) since

I, IV. $\gamma(\mathbf{w}) \cdot \mathbf{n} = 0$ on $\partial\Omega$ (with Λ an empty set);

II. $\gamma(\mathbf{w}) \cdot \mathbf{n} = 0$ on $\Gamma \cup \Sigma$ (with $\Lambda = \Lambda_0 \cap \Lambda_1$) and $\mathbf{v}$ and $\mathbf{w}$ are periodic;

III. $\gamma(\mathbf{w}) \cdot \mathbf{n} = 0$ on $\partial\Omega\backslash\Lambda_o$ (with $\Lambda = \Lambda_o$).

To obtain (4.25), we use (4.24) to evaluate $a_1(\mathbf{w}, \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v})$, expand the left-hand side via the multilinear properties of $a_1(\cdot, \cdot, \cdot)$ and then apply (4.24) to $a_1(\mathbf{w}, \mathbf{u}, \mathbf{u})$ and $a_1(\mathbf{w}, \mathbf{v}, \mathbf{v})$.

(d) Let $\mathbf{u} \in K$ and let $(\mathbf{u}^n)$ be a sequence in K such that $\mathbf{u}^n \to \mathbf{u}$ weakly in K. Then $\mathbf{u}^n \to \mathbf{u}$ weakly in $H^1(\Omega)^2$ because $(H^1(\Omega)^2)' \subset K'$. By the Sobolev imbedding theorem, $H^1(\Omega)$ is compactly imbedded into $L^2(\Omega)$, which implies that $\mathbf{u}^n \to \mathbf{u}$ strongly in $L^2(\Omega)^2$.

For each of the problems I–IV, the space $\mathcal{H} = H \cap D(\bar{\Omega})^2$ is dense in H. Let $\mathbf{v} \in \mathcal{H}$. Since $K \subset H$, it follows from (4.25) that

$$a_1(\mathbf{u}, \mathbf{u}, \mathbf{v}) = c(\mathbf{u}, \mathbf{v}) - a_1(\mathbf{u}, \mathbf{v}, \mathbf{u}), \qquad a_1(\mathbf{u}^n, \mathbf{u}^n, \mathbf{v}) = c(\mathbf{u}^n, \mathbf{v}) - a_1(\mathbf{u}^n, \mathbf{v}, \mathbf{u}^n) \qquad \forall\, n,$$

where $c(\mathbf{u}, \mathbf{v}) = \int_{\Lambda_o} \gamma(u_1)^2\gamma(v_1)\,\mathrm{d}s$ in problem III and $c(\cdot, \cdot) = 0$ for the other problems. Since $\mathbf{v}$ is smooth, $|v_i|$ and $|v_{i,j}|$ are bounded on $\bar{\Omega}$ by some constant, say d. Hence,

$$\begin{aligned}
|a_1(\mathbf{u}, \mathbf{v}, \mathbf{u}) - a_1(\mathbf{u}^n, \mathbf{v}, u^n)| &\leq \int_\Omega |v_{i,j}| \cdot |u_i u_j - u_i^n u_j^n|\,\mathrm{d}x \\
&\leq \sum_{i,j=1}^{2} d\int_\Omega (|u_i - u_i^n| \cdot |u_j| + |u_i^n| \cdot |u_j - u_j^n|)\,\mathrm{d}x \\
&\leq 4d(\|\mathbf{u}\|_0 + \|\mathbf{u}^n\|_0)\|\mathbf{u} - \mathbf{u}^n\|_0 \to 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

If $T \in H^{1/2}(\Lambda_o)'$, then $T \cdot \gamma_{\Lambda_o} \in K'$ and thus $T(\gamma_{\Lambda_o}(\mathbf{u}^n)) \to T(\gamma_{\Lambda_o}(\mathbf{u}))$. Hence $\gamma_{\Lambda_o}(\mathbf{u}^n) \to \gamma_{\Lambda_o}(\mathbf{u})$ weakly in $H^{1/2}(\Lambda_o)$, and therefore strongly in $L^2(\Lambda_o)$, since $H^{1/2}(\Lambda_o)$ is compactly imbedded in $L^2(\Lambda_o)$. It now follows (for problem III) in formally the same manner as above that

$$|c(\mathbf{u}, \mathbf{v}) - c(\mathbf{u}^n, \mathbf{v})| \leq 4d(\|\gamma(\mathbf{u})\|_{0,\Lambda} + \|\gamma(\mathbf{u}^n)\|_{0,\Lambda})\|\gamma(\mathbf{u}) - \gamma(\mathbf{u}^n)\|_{0,\Lambda} \to 0 \qquad \text{as } n \to \infty.$$

Thus $\lim_{n\to\infty} a_1(\mathbf{u}^n, \mathbf{u}^n, \mathbf{v}) = a_1(\mathbf{u}, \mathbf{u}, \mathbf{v})\ \forall\, \mathbf{v} \in \mathcal{K}$.

Let $\mathbf{v} \in H$ and let $(\mathbf{v}^m)$ be a sequence in $\mathcal{K}$ such that $\mathbf{v}^n \to \mathbf{v}$ in $H^1(\Omega)^2$. Let

$$s = \sup_n \{\|\mathbf{u}^n\|_1, \|\mathbf{u}\|_1\} < \infty.$$

Then for every m and n,

$$|a_1(\mathbf{u}, \mathbf{u}, \mathbf{v}) - a_1(\mathbf{u}^n, \mathbf{u}^n, \mathbf{v})| \leq 2s^2\|a_1\| \cdot \|\mathbf{v} - \mathbf{v}^m\|_1 + A_{mn}$$

where $A_{mn} = |a_1(\mathbf{u}, \mathbf{u}, \mathbf{v}^m) - a_1(\mathbf{u}^n, \mathbf{u}^n, \mathbf{v}^m)|$.

Let $\varepsilon > 0$ be given. Choose M so that $\|\mathbf{v} - \mathbf{v}^M\|_1 < \varepsilon/(4s^2\|a_1\|)$. Since $\mathbf{v}^M \in \mathcal{K}$, we can choose N so that $A_{Mn} < \varepsilon/2\ \forall\, n \geq N$. Hence, $|a_1(\mathbf{u}, \mathbf{u}, \mathbf{v}) - a_1(\mathbf{u}^n, \mathbf{u}^n, \mathbf{v})| < \varepsilon\ \forall\, n \geq N$. Thus the equation above holds for every $\mathbf{v} \in H$. ■

It follows from (a) and the properties of $a_0(\cdot, \cdot)$ that $a(\cdot; \cdot, \cdot)$ is well-defined on $V_k \times V_k \times V_k$ and that the mapping $(\mathbf{u}, \mathbf{v}) \to a(\mathbf{w}; \mathbf{u}, \mathbf{v})$ is a continuous bilinear form on $V_k \times V_k$ for every $\mathbf{w} \in V_k$. In fact, for every $\mathbf{w} \in V_k$ we have

$$|a(\mathbf{w}; \mathbf{u}, \mathbf{v})| \leq (\|a_0\| + \|a_1\| \cdot \|\mathbf{w}\|_1 + 2\|a_1\| \cdot \|\mathbf{u}^0\|_1)\|\mathbf{u}\|_1\|\mathbf{v}\|_1 \qquad \forall\, \mathbf{u}, \mathbf{v} \in V_k,$$

so that

$$\|\mathcal{A}(\mathbf{w})\| \leq \|a_0\| + \|a_1\|(2\|\mathbf{u}^0\|_1 + \|\mathbf{w}\|_1).$$

Similarly, $l \in V_k'$ with $\|l\|_{V_k'} \leq \|\mathbf{f}\|_0 + (\|a_0\| + \|a_1\| \cdot \|\mathbf{u}^0\|_1)\|\mathbf{u}^0\|_1$.

It is clear from (a) that, for every $\mathbf{v}$ in K, the mapping $\mathbf{u} \to a_0(\mathbf{u}, \mathbf{v}) + a_1(\mathbf{u}^0, \mathbf{u}, \mathbf{v}) + a_1(\mathbf{u}, \mathbf{u}^0, \mathbf{v})$ belongs to K'. Using (d), this implies that the mapping $\mathbf{u} \to a(\mathbf{u}, \mathbf{u}, \mathbf{v})$ is sequentially weakly continuous on K for every $\mathbf{v}$ in K. Furthermore, for all $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{v}$ and $\mathbf{w}$ in V_k,

$$|a(\mathbf{u}_1, \mathbf{v}, \mathbf{w}) - a(\mathbf{u}_2, \mathbf{v}, \mathbf{w})| \leq \|a_1\| \cdot \|\mathbf{u}_1 - \mathbf{u}_2\|_1\|\mathbf{v}\|_1\|\mathbf{w}\|_1.$$

Hence $\Pi\mathcal{A}$ is uniformly Lipschitz continuous in K with Lipschitz constant $L = \|a_1\|$.

For problems I, II and IV, it follows from (c) that $a_1(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0\ \forall\, \mathbf{u} \in K,\ \forall\, \mathbf{v} \in V_k$. Moreover, from the properties of $\mathbf{u}^0$ and the proof of (c) it is clear that $a_1(\mathbf{u}^0, \mathbf{v}, \mathbf{v}) = 0\ \forall\, \mathbf{v} \in V_k$ (since $\gamma_\Sigma(\mathbf{v}) = \mathbf{0}$). Therefore

$$a(\mathbf{u}, \mathbf{v}, \mathbf{v}) = a_0(\mathbf{v}, \mathbf{v}) + a_1(\mathbf{v}, \mathbf{u}^0, \mathbf{v}) \geq (a_0 - \|a_1\| \cdot \|\mathbf{u}^0\|_1)\|\mathbf{v}\|_1^2 \qquad \forall\, \mathbf{u}, \mathbf{v} \in K.$$

Since $\|\mathbf{u}^0\|_1 \leq K(\Omega)\|\mathbf{u}_0\|_\Sigma$, it follows that the hypotheses (4.12) and (4.15) are satisfied, with $\alpha = \alpha_0 - \|a_1\| \cdot \|\mathbf{u}^0\|_1$, when $\|\mathbf{u}_0\|_\Sigma$ is sufficiently small. In the case of problem III,

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}, \mathbf{v}) &= a_0(\mathbf{v}, \mathbf{v}) + a_1(\mathbf{u}, \mathbf{v}, \mathbf{v}) + a_1(\mathbf{u}^0, \mathbf{v}, \mathbf{v}) + a_1(\mathbf{v}, \mathbf{u}^0, \mathbf{v}) \\ &\geq (a_0 - \|a_1\|(\|\mathbf{u}\|_1 + 2\|\mathbf{u}^0\|_1))\|\mathbf{v}\|_1^2 \qquad \forall\, \mathbf{u}, \mathbf{v} \in V_k. \end{aligned}$$

Let c and d be arbitrary positive constants such that $0 < 2c + d < \alpha_0/\|a_1\|$. Then (4.12) holds for all $\mathbf{v} \in S_d$ and (4.15) holds for all $\mathbf{u} \in S_d$ and $\mathbf{v} \in K$, with $\alpha = \alpha_0 - (2c + d)\|a_1\|$, if $\|\mathbf{u}^0\|_1 \leq c$.

Finally, since the Sobolev spaces are separable, K is too.

It was proved in Section 3 that it is always possible to define $\mathbf{u}^0$ such that $\|\mathbf{u}^0\|_1 \le \kappa\|\mathbf{u}_0\|_\Sigma$, where κ is a constant which depends only on Ω. Choose $U = \nu/L$. Then it follows from the estimates derived earlier that $\|a_0\| \le 2\sqrt{3} < 4$, $\|a_1\| \le C(\Omega)$, $\|\mathbf{f}\|_0 \le gm/\nu^2$ with $m(\Omega) = \sqrt{(\text{meas}(\Omega))/L^3}$, and therefore that $\|l\|_{V'} \le gm/\nu^2 + (4 + C\kappa u)\kappa u$, where u denotes $\|\mathbf{u}_0\|_\Sigma$. Furthermore, we may take $a_0 = A(\Omega)$.

For problems I, II and IV, a sufficient condition for (4.12) and (4.15) to hold is therefore that $u < A/C\kappa$. Then we may set $\alpha = A - C\kappa u > 0$, so that a sufficient condition for (4.16) to hold is that $gm/\nu^2 + (4 + C\kappa u)\kappa u < (A - C\kappa u)^2/C$ or, equivalently,

$$gm/\nu^2 + (4 + 2A)\kappa u < A^2/C. \tag{4.26}$$

It is easy to see that (4.26) implies $u < A/C\kappa$. For (4.12) and (4.15) to be satisfied with respect to S_d in the case of problem III, it is sufficient that $d + 2\kappa u < A/C$. Then

$$\alpha = A - (d + 2\kappa u)C > 0,$$

so that for (4.16) to hold it is sufficient that

$$(4 + C\kappa u)\kappa u < (A - (d + 2\kappa u)C)^2/C, \tag{4.27}$$

and for (4.14) to hold it is sufficient that

$$(4 + C\kappa u)\kappa u < d(A - (d + 2\kappa u)C). \tag{4.28}$$

From the results of this section and theorems 4.1–4.3 we can obtain the following theorem.

THEOREM 4.4. Assume that Ω and $\mathbf{u}_0$ satisfy the regularity requirements, etc. specified in Sections 2 and 3.

(a) Then, for each of problems I, II and IV, there exist constants C_1, C_2 and C_3 (with $C_3 < C_1$) which depend only on Ω such that problem (N) has at least one solution $\mathbf{w} \in K$ when $\|\mathbf{u}_0\|_\Sigma < C_1$. Moreover, $\mathbf{w}$ is unique if $C_2 g/\nu^2 + \|\mathbf{u}_0\|_\Sigma < C_3$.

For every solution $\mathbf{w}$ of problem (N), there exists a unique pair $(p, \lambda) \in S$ such that $(\mathbf{w}, p + c, \lambda + c\tau^*)$ is a solution of problem (NVP) for every $c \in \mathbb{R}$, with S and τ^* as in (4.21) and (4.22). Every solution of problem (VAR) is generated in this manner.

(b) In the case of problem III, there exist constants C_4 and C_5 which depend only on Ω such that problem (N) has at least one solution $\mathbf{w} \in S_d$ when $d + C_4\|\mathbf{u}_0\|_\Sigma < C_5$. Furthermore, for every $d < C_5$ there exists a constant $C_6 = C_6(\Omega, d)$ such that $\mathbf{w}$ is unique in S_d when $\|\mathbf{u}_0\|_\Sigma < C_6$.

For every solution $\mathbf{w}$ of problem (N) there exists a unique pair (p, λ) such that $(\mathbf{w}, p, \lambda)$ is a solution of problem (NVP). ■

Let $(\mathbf{w}, p, \lambda)$ be as above. For problems I, II and IV, it follows from (4.13) that when $u < A/C\kappa$, then

$$\begin{aligned}\|\mathbf{w}\|_1 &\le (gm/\nu^2 + (4 + C\kappa u)\kappa u)/(A - C\kappa u)\\ &\le C_7(\Omega)(g/\nu^2 + u) \text{ for sufficiently small } u.\end{aligned} \tag{4.29}$$

Since $\|\mathfrak{A}(\mathbf{w})\mathbf{u}\|_{V_k'} \leq (\|a_0\| + \|a_1\|(\|\mathbf{w}\|_1 + 2\|\mathbf{u}^0\|_1))\|\mathbf{u}\|_1\ \forall\, \mathbf{u}, \mathbf{w} \in V_k$, it follows from (4.17) and (4.29) that

$$\|p\|_0, \|\lambda\|_k \leq (gm/v^2 + (\|\mathbf{w}\|_1 + \kappa u)[4 + C(\|\mathbf{w}\|_1 + \kappa u)])/\beta$$
$$\leq C_8(\Omega)(g/v^2 + u) \text{ for sufficiently small } g/v^2, u. \tag{4.30}$$

In the case of problem III, similar estimates hold (with $g = 0$ and $A - C\kappa u$ replaced by $A - (d + 2\kappa u)C$ in (4.29)) when $d + 2\kappa u < A/C$.

If the existence and uniqueness results for the problems (VAR) or (NVP) associated with problems I–IV are compared to the well-known theory of the corresponding Dirichlet problems, then it is clear that for the Dirichlet problem, solutions $(\mathbf{u}, p)$ exist for data of any magnitude. For sufficiently small data, $\mathbf{u}$ is unique and p is unique up to an additive constant. For problems I–IV, existence of solutions can apparently only be established for small data. Moreover, for problem III the solution is possibly nonunique regardless of how small the data is (since the theorem above only shows that $\mathbf{u}$ is unique in some neighbourhood of $\mathbf{0}$).

The reason why the theory for the Dirichlet problem fails to carry over to problem (VAR) in the nonlinear case is as follows: in the case of the Dirichlet problem, the form $a(\cdot\,; \cdot, \cdot)$ is exactly as in the present section, but $V = H_0^1(\Omega)^2$ and $B\mathbf{v} = \operatorname{div} \mathbf{v}$ so that $K = \{\mathbf{v} \in H_0^1(\Omega)^2 \mid \operatorname{div} \mathbf{v} = 0\} = K_0$, say. Consequently, the following result can be used to prove that $a(\cdot\,; \cdot, \cdot)$ satisfies (4.15).

LEMMA 4.5. Let Ω be a bounded domain in $\mathbb{R}^2$ with a Lipschitz continuous boundary $\partial\Omega$. Let $\Gamma_1, \ldots, \Gamma_m$ denote the connected components on $\partial\Omega$. Then, given a function $\mathbf{u}_0 \in H^{1/2}(\partial\Omega)^2$ with $\int_{\Gamma_i} \mathbf{g} \cdot \mathbf{n}\, ds = 0$, $i = 1, \ldots, m$, there exists for any $\varepsilon > 0$ a function $\mathbf{u}^0 = \mathbf{u}^0(\varepsilon) \in H^1(\Omega)^2$ such that $\operatorname{div} \mathbf{u}^0 = 0$, $\gamma(\mathbf{u}^0) = \mathbf{u}_0$, $|a_1(\mathbf{v}, \mathbf{u}^0, \mathbf{v})| \leq \varepsilon|\mathbf{v}|_1^2\ \forall\, \mathbf{v} \in K_0$. ■

Together with Korn's inequalities, this lemma represents the main technical step in solving the nonhomogeneous Dirichlet problem. The proof (due to E. Hopf) can be found in [13, pp. 287–291] and (for a C^2 domain) in [1, pp. 175–178]. Unfortunately, this proof fails to work when K_0 is replaced by the spaces K associated with problems I–IV.

5. CONCLUDING REMARKS: THE PROBLEM OF REGULARITY

In Section 4 it was proved under minimal conditions on the smoothness of the data (namely that $\partial\Omega$ is Lipschitz continuous, f is of class $C^{1,\alpha}$ with $\alpha > \frac{1}{2}$, $\mathbf{f} \in L^2(\Omega)^2$, $p_a \in L^\infty(\Gamma)$, and $\mathbf{u}_0$ belongs to (some subspace of) $H^{1/2}(\Sigma)^2$) that for every choice of the function f there exists a pair of functions $(\mathbf{u}(f), p(f))$ in $H^1(\Omega)^2 \times L^2(\Omega)$ which is a weak solution of problem (AUX). If in fact we assume that $f \in C^{1,1}[0, 1]$, then this implies that f'' is defined a.e. in $(0, 1)$ and belongs to the space $L^\infty(0, 1)$. It follows that the curvature operator $(1/R)(f)$ belongs to $L^\infty(0, 1)$, so that for the curvature/normal stress boundary condition (2.9) to be meaningful, it is necessary to prove that the quantity $\tau_{ij} = S_{ij}(\mathbf{u}(f), p(f))$ (or $\tau_n = \tau_{ij} n_j n_i$) is well-defined on Γ and belongs to $L^\infty(\Gamma)$. This is equivalent to proving that p and $u_{i,j}$, $i, j = 1, 2$, belong to $H^1(O)$ with their traces in $L^\infty(0, 1)$, where O is some subdomain of Ω that contains a neighbourhood of Γ.

It is apparent from the above that the need to establish regularity results for the solutions of problem (VAR) is unavoidable in any attempt to solve problem (FBP). This is confirmed by the literature discussed in Section 1, in which much effort is devoted to the analysis of the differentiability properties of the weak solution of the auxiliary problem and the derivation of suitable a priori estimates. In general the regularity results established in these papers rely on the results of Agmon *et al.* [21] or Solonnikov and Scadilov [22] for general elliptic systems. However, these theorems invariably require greater regularity from the data than we have assumed thus far. Typically, the regularity theorems for problem (AUX) in situations where there is no contact between Γ and $\partial\Omega\backslash\Gamma$, as in problems I and II, are of the form: if $\mathbf{u}_0 \in C^{2,h}(\Sigma)^2$, $p_a \in C^{1,h}(\Gamma)$ and $\mathbf{f} \in C^{0,h}(\bar{\Omega})$, then the weak solution $(\mathbf{u}, p)$ of the auxiliary problem belongs to $C^{2,h}(\bar{\Omega})^2 \times C^{1,h}(\bar{\Omega})$. (The free boundary is then sought in $C^{3,h}$.) Moreover, when there is contact between Γ and $\partial\Omega\backslash\Gamma$, as in problems III and IV, the problem is considerably complicated by the presence of corner points on the boundary of the flow domain, as can be seen in the analysis of Solonnikov [23].

The question of regularity of solutions is not pursued further here, but we remark that if $\mathbf{f} \in C^{\infty}(\bar{\Omega})^2$ in problems I–IV, the results given in [20] for the case of Dirichlet boundary conditions (cf. [20, theorem 3 in Section 1 of Chapter 2; theorem 6 in Section 5 of Chapter 5; pp. 115, 116 in the 1963 edition of this text]) suggest that the solution of problem (VAR) is smooth in every strictly interior domain and continuous on Γ, excluding neighbourhoods of the corner points, under slightly stronger assumptions than those used here.

We conclude by indicating how (2.9) can be formulated as a fixed point equation in f if it is assumed that the regularity problem has been resolved. We shall only consider problem IV, but similar ideas apply in the case of the other problems and in smoother settings.

It is easily shown that the solution f_0 corresponding to the static problem with zero gravity belongs to $C^{1,1}(0, 1)$. We assume that the domain has the following simple geometry: for every

$$f \in \mathfrak{F}(f_0, \varepsilon) = \left\{ f \in C^{1,1}(0,1) \,\middle|\, \int_0^1 f \, \mathrm{d}x = 0,\ -f'(0) = c = f'(1),\ \|f - f_0\|_{1,1} \le \varepsilon \right\},$$

set

$$\Omega(f) = \{\mathbf{x} \in \mathbb{R}^2 \mid 0 < x_1 < 1, -h < x_2 < f(x_1)\}, \qquad \Sigma = \{\mathbf{x} \in \mathbb{R}^2 \mid 0 < x_1 < 1, x_2 = -h\}$$

and define $\Gamma(f)$, Λ_0 and Λ_1 as before. (By taking p_a close to a constant and c small enough, it can be ensured that $\|f_0\|_{1,1} \le \delta/2$, say. Then $\|f\|_{1,1} \le \delta$ when $\varepsilon \le \delta/2$.)

Assume that for each $f \in \mathfrak{F}(f_0, \varepsilon)$ the solution(s) $(\mathbf{u}, p)$ of problem (AUX) is such that τ_n, as a function of x_1, belongs to $L^{\infty}(0, 1)$, where we define $\tau_n = E - q$, with $E = (2/Re)D_{ij}(\mathbf{u})n_j n_i$ on Γ and $p = q + r$ on $\bar{\Omega}$, with $\int_{\Omega} q \, \mathrm{d}x = 0$ and $r \in \mathbb{R}$ a constant that will be fixed later. Then (2.9) becomes

$$(1/R)(f)(x) = (Re \cdot Oh)^2(\tau_n(x) - r + p_a(x)), \qquad x \in [0, 1].$$

Assume furthermore that there exist constants U and V, which depend only on h and δ, such that $\|\tau_n(f_1) - \tau_n(f_2)\|_{\infty} \le UD\|f_1 - f_2\|\ \forall f_1, f_2 \in \mathfrak{F}$, $\|\tau_n(f) - \overline{\tau_n(f)}\|_{\infty} \le VD\ \forall f \in \mathfrak{F}$, where $\overline{(\cdot)}$ denotes $\int_0^1 (\cdot) \, \mathrm{d}x$.

Under these assumptions it follows from elementary but lengthy arguments, which we shall omit, that when $\|\mathbf{u}_0\|_{\Sigma} + g/v^2$ and $\|p_a - \overline{p_a}\|$ are sufficiently small, (2.9) is equivalent to the equation $f = F(f)$, where the operator F is a contraction on the complete metric space $\mathfrak{F}$ and is defined by

$$F(f)(x) = \int_0^x Q(f)(s) \, \mathrm{d}s - \int_0^1 \int_0^t Q(f)(s) \, \mathrm{d}s \, \mathrm{d}t$$

where $Q(f) = P(f)/\sqrt{(1 - P(f)^2)}$ and

$$P(f)(x) = k\int_0^x (\tau_n(f) + p_a)\,ds - krx - C$$
$$= k\left(\int_0^x (\tau_n(f) + p_a)\,ds - (\overline{\tau_n(f)} + \overline{p_a})x\right) + (2x - 1)C$$

with $r = \overline{\tau_n} + \overline{p_a} - 2C/k$, $C = c/\sqrt{(1 + c^2)}$. Hence, under the assumptions above, F has a unique fixed point f in B, so that problem (FBP) has a solution $(\mathbf{u}(f), p(f), f)$ which is locally unique.

Acknowledgement—The authors are grateful to the Foundation for Research Development for a postgraduate bursary (CleR) and a research grant (BDR).

REFERENCES

1. TEMAM R., *Navier-Stokes Equations*. North-Holland, Amsterdam (1979).
2. PUKHNACHOV V. V., A plane steady-state free-boundary problem for the Navier–Stokes equations, *J. Appl. Mech. Tech. Phys.* **3**, 91–102 (1972).
3. PUKHNACHOV V. V., Two methods of approximate description of steady-state motions of a viscous incompressible liquid with a free boundary, *J. Appl. Mech. Tech. Phys.* **5**, 126–134 (1972).
4. JEAN M., Free surface of the steady flow of a Newtonian fluid in a finite channel, *Archs ration. Mech. Analysis* **74**, 197–217 (1980).
5. SOLONNIKOV V. A., Solvability of a problem on the plane motion of a heavy viscous incompressible capillary liquid partially filling a container, *Math. USSR Izvestija* **14**, 193–221 (1980).
6. SOLONNIKOV V. A., Solvability of a problem on the motion of a viscous incompressible fluid bounded by a free surface, *Math. USSR Izvestija* **11**(6), 1323–1358 (1977).
7. BEALE J. T., The initial value problem for the Navier–Stokes equations with a free surface, *Communs pure appl. Math.* **34**, 359–392 (1981).
8. BEALE J. T., Large-time regularity of viscous surface waves, *Archs ration. Mech. Analysis* **84**, 307–352 (1984).
9. ALLAIN G., Small-time existence for the Navier–Stokes equations with a free surface, *Appl. Math. Optim.* **16**(1), 37–50 (1987).
10. CUVELIER C. & SCHULKES R. M. S. M., Some numerical methods for the computation of capillary free boundaries governed by the Navier–Stokes equations, *SIAM Rev.* **32**(3), 355–423 (1990).
11. KIKUCHI N. & ODEN J. T., *Contact Problems in Elasticity*. SIAM, Philadelphia (1988).
12. BREZZI F., On the existence, uniqueness and approximation of saddle-point problems arising from Lagrange multipliers, *RAIRO Anal. Numér.* **R2**, 129–151 (1974).
13. GIRAULT V. & RAVIART P. A., *Finite Element Methods for Navier–Stokes Equations*. Springer, Berlin (1986).
14. KRUYT N. P., CUVELIER C., SEGAL A. & VAN DER ZANDEN J., A total linearization method for solving viscous free boundary flow problems by the finite element method, *Int. J. numer. Meth. Fluid.* **8**, 351–363 (1988).
15. SOCOLESCU D., On some new boundary value problems for the stationary Navier–Stokes equations, *Methoden und Verfahren der mathematischen Physik 18, Free and Mixed Boundary Value Problems*, pp. 117–140. B.I.-Wissenschaftsverlag, Mannheim (1978).
16. SOCOLESCU D., On one free boundary problem for the stationary Navier–Stokes equations, *Methoden und Verfahren der mathematischen Physik 18, Free and Mixed Boundary Value Problems*, pp. 41–60. B.I.-Wissenschaftsverlag, Mannheim (1978).
17. LIONS J. L. & MAGENES E., *Non-Homogeneous Boundary Value Problems and Applications*, Vol. I. Springer, Berlin (1972).
18. GRISVARD P., *Elliptic Problems in Nonsmooth Domains*. Pitman, Boston (1985).
19. DUVAUT & LIONS, *Inequalities in Mechanics and Physics*. Springer, Berlin (1976).
20. LADYZHENSKAYA O. A., *The Mathematical Theory of Viscous Incompressible Flow*. Gordon and Breach, New York (1969).
21. AGMON S., DOUGLIS A. & NIRENBERG L., Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions: II, *Communs pure appl. Math.* **17**(1), 35–92 (1964).
22. SOLONNIKOV V. A. & SCADILOV V. E., On a boundary value problem for a stationary system of Navier–Stokes equations, *Proc. Steklov Inst. Math.* **125**, 186–199 (1973).
23. SOLONNIKOV V. A., On the Stokes equations in domains with nonsmooth boundaries and on viscous incompressible flow with a free surface, *Research Notes in Mathematics* **70**, *Nonlinear Partial Differential Equations and their Applications*, Vol. III, pp. 340–423. Pitman, Boston (1982).