Numer. Math. 50, 697–721 (1987)

Numerische Mathematik

Finite Element Approximation of Incompressible Navier-Stokes Equations with Slip Boundary Condition

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Summary. We consider a mixed finite element approximation of the stationary, incompressible Navier-Stokes equations with slip boundary condition, which plays an important rôle in the simulation of flows with free surfaces and incompressible viscous flows at high angles of attack and high Reynold's numbers. The central point is a saddle-point formulation of the boundary conditions which avoids the well-known Babuška paradox when approximating smooth domains by polyhedrons. We prove that for the new formulation one can use any stable mixed finite element for the Navier-Stokes equations with no-slip boundary condition provided suitable bubble functions on the boundary are added to the velocity space. We obtain optimal error estimates under minimal regularity assumptions for the solution of the continous problem. The techniques apply as well to the more general Navier boundary condition.

Subject Classifications: AMS(MOS): 65N30; CR: G1.8.

1. Introduction

This paper is concerned with mixed finite element approximations of the stationary, incompressible Navier-Stokes equations with slip boundary condition

$$-\nu \Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1.1a}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.1b}$$

$$\mathbf{u}\cdot\mathbf{n} = g \quad \text{on } \Gamma := \partial\Omega, \tag{1.1c}$$

$$\mathbf{n}\cdot\mathfrak{T}(\nu\mathbf{u}, p)\cdot\mathbf{t}_k = 0 \quad \text{on } \Gamma, \quad 1 \leqq k \leqq d-1, \tag{1.1d}$$

in a simply connected bounded domain $\Omega \subset \mathbb{R}^d$, $d=2,3$. Here,

$$\mathfrak{T}(\mathbf{v}, q)_{ij} := -q\,\delta_{ij} + \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}, \quad 1 \leqq i,j \leqq d,$$

denotes the stress tensor, ν is the viscosity, $\mathbf{n}$ is the unit exterior normal, and $\mathbf{t}_1, \ldots, \mathbf{t}_{d-1}$ form an orthonormal set of tangent vectors. In order to ensure that problem (1.1) is well-posed, the function g must have mean value 0 and the function $\mathbf{f}$ must be orthogonal to all rigid body rotations of Ω (cf. Eq. (2.1)). We assume that the boundary Γ is of class C^3, that $\mathbf{f}$ is a square integrable, and that g is the trace of an H^2-function. These conditions guarantee a certain amount of regularity for the solutions of problem (1.1) (cf. Eq. (3.2)). Note, that condition (1.1c) is an essential boundary condition, whereas condition (1.1d) is a natural one.

The slip boundary condition (1.1c, d) is the appropriate physical model for flow problems with free boundaries (e.g., the coating problem [14, 18, 19]), for flows past chemically reacting walls [6], and for flows at high angles of attack and high Mach and Reynold's numbers (e.g., re-entry of a space orbiter), where the classical no-slip condition of Stokes [20] $\mathbf{u}=0$ is no longer valid. Both slip and no-slip boundary conditions are special cases of Navier's boundary condition [15]

$$\alpha_k \mathbf{u}\cdot\mathbf{t}_k + \mathbf{n}\cdot\mathfrak{T}(\nu\mathbf{u}, p)\cdot\mathbf{t}_k(1-\alpha_k) = g_k, \qquad 0 \leqq \alpha_k \leqq 1, \quad 1 \leqq k \leqq d,$$

where, for ease of notation, we have used the convention that $\mathbf{t}_d := \mathbf{n}$. The methods developed in this paper can also be applied to Navier's boundary condition with different values of α_k on different parts of the boundary. However, special care must be taken of the singularities which may occur where the α_k change.

The slip and no-slip boundary conditions apparently describe different physical situations. This is also reflected in the mathematical properties. For example, the solutions of problem (1.1) are not stable with respect to perturbations of the boundary which are not sufficiently smooth. To see this, consider the corresponding linearized Stokes problem:

$$\begin{aligned} -\Delta\mathbf{u} + \nabla p = f, \qquad & \operatorname{div}\mathbf{u} = 0, \quad \text{in } \Omega \\ \mathbf{u}\cdot\mathbf{n} = g, \qquad & \mathbf{n}\mathfrak{T}(\mathbf{u}, p)\mathbf{t}_k = 0, \quad 1 \leqq k \leqq d-1, \text{ on } \Gamma. \end{aligned} \tag{1.2}$$

One easily checks that the stream function formulation of problem (1.2) in two dimensions is equivalent to the problem of a simply supported plate. The well-known Babuška-paradox [4] of plane elasticity, however, shows that this problem cannot be approximated by the corresponding problem on a polyhedral approximation of Ω. Moreover, the regularity results of [8] for plate problems in plane domains with corners imply that it is not legitimate to restrict the analysis of problems (1.1), (1.2) to the case of polyhedral domains.

In order to overcome these difficulties, we introduce in Sect. 3 a saddle-point formulation of the essential boundary condition (1.1c), which fits into the abstract framework of Babuška and Brezzi [5, 9] for mixed problems. The corresponding Lagrange multiplier can be interpreted as the normal stress component $\mathbf{n}\cdot\mathfrak{T}(\nu\mathbf{u}, p)\cdot\mathbf{n}$. Thus, the method yields additional information on an important physical parameter.

Section 4 is concerned with the mixed finite element approximation of the variational problem introduced in Sect. 3. To simplify the exposition, we only consider simplicial finite elements on polyhedral approximations Ω_h of Ω. The discussion of higher order approximations of the boundary is postponed to Sect. 7.

In Brezzi's abstract framework for saddle-point problems [9], the primal variable must balance the influence of the Lagrange multiplier. For the Stokes problem with no-slip condition, this means that the velocity-space has to weigh up the pressure-space (cf. condition (H1) in Sect. 4). Many finite element spaces satisfying this condition are known in the literature (cf. the overview in [13]). When applying Brezzi's theory to the saddle-point problem of Sect. 3, the velocity-space must balance two Lagrange multipliers: the pressure and the normal stress component. We prove that this can be achieved for any pair of finite element spaces, which satisfy the stability condition for the Stokes problem with no-slip condition, by adding "bubble" functions to the velocity-space. Exactly one "bubble" function corresponds to each boundary face S in the tesselation of Ω_h. This function is a polynomial, which vanishes on ∂S. For the practical computation, it can therefore be eliminated by static condensation techniques.

The abstract results of Brezzi [9], standard approximation properties of finite elements [11], and the upper bounds for the consistency error, which is introduced by the approximation of the domain, immediately yield error estimates for the mixed finite element approximation of problem (1.2). In particular, we obtain optimal error estimates for the velocity in the H^1-norm, for the pressure in the L^2-norm, and for the normal stress component in the $H^{-1/2}$- and L^2-norms. Due to the piecewise linear approximation of the boundary, the well-known duality argument of Aubin [3] and Nitsche [16] yields no improved error estimates for the velocity in the L^2-norm. This drawback will be overcome in Sect. 7.

In Sect. 6, the same error estimates are established for the mixed finite element approximation of branches of nonsingular solutions of problem (1.1). Our analysis closely follows that of Brezzi et al. [10]. Due to the approximation of the boundary, however, we have to consider an auxiliary problem.

In order to avoid technical difficulties, we assume throughout Sects. 5 and 6 that Ω is convex. This restriction is overcome in Sect. 7 by constructing continuations $\tilde{\mathbf{u}}, \tilde{p}$, and $\tilde{\mathbf{f}}$ of $\mathbf{u}$, p, and $\mathbf{f}$ such that $-\nu\Delta\tilde{\mathbf{u}}+\nabla\tilde{p}-\tilde{\mathbf{f}}$ and $\operatorname{div}\tilde{\mathbf{u}}$ are "small" in a "small" strip about the boundary Γ. These continuations also allow us to treat higher order approximations of the boundary. When suitably choosing the polynomial degree of the functions which approximate the velocity, the pressure, the normal stress component, and the boundary, the convergence order of the finite element approximation only depends on the regularity of the solutions of problem (1.1).

The methods of this paper are currently implemented in collaboration with Aviations Marcel Dassault/Breguet Aviation. The numerical results will be presented in a subsequent paper.

2. Some Function Spaces

We denote by $W^{k,p}(\Omega)$ and $L^p(\Omega)=W^{0,p}(\Omega)$, $k\in\mathbb{N}$, $1\leqq p\leqq\infty$, the usual Sobolev and Lebesgue spaces equipped with the norm (cf. [1])

$$\|\varphi\|_{k,p,\Omega}:=\begin{cases}\{\sum\limits_{|\alpha|\leqq k}\int\limits_\Omega |D^\alpha\varphi|^p\,dx\}^{1/p}, & 1\leqq p<\infty,\quad k\in\mathbb{N},\\ \max\limits_{|\alpha|\leqq k}\operatorname*{ess\,sup}\limits_{x\in\Omega}|D^\alpha\varphi(x)|, & p=\infty,\quad k\in\mathbb{N}.\end{cases}$$

In the case $p=2$, we simply write $H^k(\Omega)$ instead of $W^{k,2}(\Omega)$ and omit the index $p=2$ in the corresponding norm. The scalar product of $L^2(\Omega)$ is denoted by $(.,.)_{0,\Omega}$. Since no confusion can arise, we use the same notation for the corresponding norms and scalar products on $H^k(\Omega)^d$ and $L^2(\Omega)^d$. We denote by $H^{-1}(\Omega)$ the dual space of $H^1(\Omega)$ and by $\langle.,.\rangle_\Omega$ the corresponding duality pairing. Finally, $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ with zero boundary condition.

We will frequently refer to the spaces $H^{k-1/2}(\Gamma)$ and $W^{k-1/p,p}(\Gamma)$, $k=1,2$, $2<p<\infty$, of traces of functions in $H^k(\Omega)$ and $W^{k,p}(\Omega)$, respectively (cf. [1]).

Denote by $H^{-1/2}(\Gamma)$ the dual space of $H^{1/2}(\Gamma)$ and by $\langle.,.\rangle_\Gamma$ the corresponding duality pairing.

The rigid body rotations of Ω generate the space

$$\begin{aligned}\mathscr{S}:=\operatorname{span}\{\mathbf{u}(\mathbf{x})=\boldsymbol{\beta}\wedge\mathbf{x}:\ &\boldsymbol{\beta}\in\mathbb{R}^3,\ |\boldsymbol{\beta}|=1,\\ &\boldsymbol{\beta}\text{ is an axis of symmetry of }\Omega\},\end{aligned}\tag{2.1}$$

where $\wedge$ denotes the vector product. To simplify the notation, we introduce the following Hilbert and Banach spaces

$$X:=H^1(\Omega)/\mathscr{S},\tag{2.2a}$$

$$X_{n0}:=\{\mathbf{u}\in X:\mathbf{u}\cdot\mathbf{n}=0\ \text{ on }\ \Gamma\},\tag{2.2b}$$

$$Y:=L_0^2(\Omega):=\{p\in L^2(\Omega):\ \int\limits_\Omega p=0\},\tag{2.2c}$$

$$Z:=H^{-1/2}(\Gamma),\tag{2.2d}$$

$$V:=\{\mathbf{u}\in X_{n0}:\ \operatorname{div}\mathbf{u}=0\}.\tag{2.2e}$$

The division by $\mathscr{S}$ in Eq. (2.2a) takes into account that the solution of problem (1.2) is unique only up to a rigid body rotation of Ω.

Denote by

$$\mathfrak{D}(\mathbf{u})_{ij}:=\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i},\qquad 1\leqq i,\ j\leqq d,\tag{2.3}$$

the *deformation tensor*. One easily checks, that the *Green's formula*

$$\begin{aligned}&\int\limits_\Omega\{-\Delta\mathbf{u}+\nabla p\}\cdot\mathbf{v}\\ &\quad=\tfrac{1}{2}\int\limits_\Omega\mathfrak{D}(\mathbf{u})\cdot\mathfrak{D}(\mathbf{v})-\int\limits_\Omega p\operatorname{div}\mathbf{v}-\int\limits_\Gamma\mathbf{n}\cdot\mathfrak{T}(\mathbf{u},p)\cdot\mathbf{v}\end{aligned}\tag{2.4}$$

holds for all $\mathbf{v}\in H^1(\Omega)^d$, $p\in H^1(\Omega)$, and $\mathbf{u}\in H^2(\Omega)^d$ with $\operatorname{div}\mathbf{u}=0$.

In the sequel, we will often use *Korn's second inequality*

$$\tfrac{1}{2}\int_\Omega |\mathfrak{D}(\mathbf{u})|^2 \geqq c_1 \|\mathbf{u}\|_{1,\Omega}^2 - c_0 \|\mathbf{u}\|_{0,\Omega}^2 \qquad \forall \mathbf{u} \in H^1(\Omega)^d \tag{2.5}$$

and the following *Poincaré-Morrey inequality*

$$c_2 \|\mathbf{u}\|_{0,\Omega}^2 \leqq \int_\Omega |\mathfrak{D}(\mathbf{u})|^2 + \int_\Gamma |\mathbf{u}\cdot\mathbf{n}|^2 \qquad \forall \mathbf{u} \in X \tag{2.6}$$

where the constants c_0, c_1, c_2 only depend on Ω. Inequality (2.5) is proved in [17] for domains with Lipschitz boundary. The proof of Inequality (2.6) follows from Inequality (2.5) by standard arguments (cf. [21]).

3. A Saddle-Point Formulation of the Slip Boundary Condition

In this section we first recall some regularity results for problems (1.1), (1.2) which will be needed in the sequel. Then we introduce a saddle-point formulation of the slip boundary condition, which avoids the Babuška paradox and which fits into the abstract framework of [5, 9] for mixed problems.

The following weak formulation of problem (3.1) is considered in [14, 19]:

Find $\mathbf{u} \in X$ and $p \in Y$ such that $\mathbf{u}\cdot\mathbf{n} = g$ on Γ and

$$\begin{aligned} \tfrac{1}{2}\int_\Omega \mathfrak{D}(\mathbf{u})\cdot\mathfrak{D}(\mathbf{v}) - \int_\Omega p \operatorname{div}\mathbf{v} &= \int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall \mathbf{v} \in X_{n0} \\ -\int_\Omega q \operatorname{div}\mathbf{u} &= 0 \qquad \forall q \in Y. \end{aligned} \tag{3.1}$$

Inequalities (2.5) and (2.6) imply that problem (3.1) has a unique solution. Moreover, the regularity estimate

$$\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega} \leqq c\{\|\mathbf{f}\|_{0,\Omega} + \|g\|_{3/2,\Gamma}\} \tag{3.2}$$

holds [19]. If the boundary Γ is sufficiently smooth, the estimate

$$\|\mathbf{u}\|_{k+2,\Omega} + \|p\|_{k+1,\Omega} \leqq c_k\{\|\mathbf{f}\|_{k,\Omega} + \|g\|_{k+3/2,\Gamma}\}, \qquad k \geqq 1,$$

can be derived from (3.2) using the general results of Agmon et al. [2]. Applying a standard bootstrapping argument, the same regularity results can be established for the solutions of the Navier-Stokes equations (1.1).

In the sequel, we consider a saddle-point formulation of the slip boundary condition. That is, we seek the solution of problem (1.2) as the saddle-point of the Lagrange functional

$$\mathscr{L}(\mathbf{u}, p, \rho) := \tfrac{1}{4}\int_\Omega |\mathfrak{D}(\mathbf{u})|^2 - \int_\Omega \mathbf{f}\cdot\mathbf{u} - \int_\Omega p \operatorname{div}\mathbf{u} - \langle \mathbf{u}\cdot\mathbf{n} - g, \rho\rangle_\Gamma \tag{3.3}$$

in the space $H^1(\Omega)^d \times L_0^2(\Omega) \times H^{-1/2}(\Gamma)$.

In order to simplify the notation, we define two bilinear functionals on the spaces $X \times X$ and $X \times (Y \times Z)$:

$$a(\mathbf{u},\mathbf{v}):=\frac{1}{2}\int_\Omega \mathfrak{D}(\mathbf{u})\cdot\mathfrak{D}(\mathbf{v}) \tag{3.4a}$$

$$b(\mathbf{u};p,\rho):=-\int_\Omega p\operatorname{div}\mathbf{u}-\langle\rho,\mathbf{u}\cdot\mathbf{n}\rangle_\Gamma. \tag{3.4b}$$

The saddle-point problem (3.3) then leads to the following weak formulation of problem (1.2):

Find $\mathbf{u}\in X$, $p\in Y$, $\rho\in Z$ such that

$$\begin{aligned} a(\mathbf{u},\mathbf{v})+b(\mathbf{v};p,\rho)&=\langle\mathbf{f},\mathbf{v}\rangle \qquad &&\forall\,\mathbf{v}\in X\\ b(\mathbf{u};q,\sigma)&=-\langle g,\sigma\rangle_\Gamma &&\forall\, q\in Y,\quad \sigma\in Z. \end{aligned} \tag{3.5}$$

Dividing Eq. (1.1a, d) by v und replacing p/v by p, we obtain the corresponding weak formulation of problem (1.1):

Find $\mathbf{u}\in X$, $p\in Y$, $\rho\in Z$ such that

$$\begin{aligned} a(\mathbf{u},\mathbf{v})+b(\mathbf{v};p,\rho)&=\frac{1}{v}\langle\mathbf{f}-(\mathbf{u}\cdot\nabla)\,\mathbf{u},\mathbf{v}\rangle_\Omega \qquad &&\forall\,\mathbf{v}\in X\\ b(\mathbf{u};q,\sigma)&=-\langle g,\sigma\rangle_\Gamma &&\forall\, q\in Y,\quad \sigma\in Z. \end{aligned} \tag{3.6}$$

In order to get a physical interpretation of the Lagrange multiplier ρ, multiply Eq. (1.2) by a $\mathbf{v}\in H^1(\Omega)^d$ and integrate over Ω. Recalling Green's formula (2.4) and the boundary condition (1.1d), we then get

$$\begin{aligned}\int_\Omega \mathbf{f}\cdot\mathbf{v}&=\int_\Omega\{-\Delta\mathbf{u}+\nabla p\}\cdot\mathbf{v}=\frac{1}{2}\int_\Omega\mathfrak{D}(\mathbf{u})\cdot\mathfrak{D}(\mathbf{v})-\int_\Omega p\operatorname{div}\mathbf{v}-\int_\Gamma\mathbf{n}\cdot\mathfrak{T}(\mathbf{u},p)\cdot\mathbf{v}\\ &=\frac{1}{2}\int_\Omega\mathfrak{D}(\mathbf{u})\cdot\mathfrak{D}(\mathbf{v})-\int_\Omega p\operatorname{div}\mathbf{v}-\int_\Gamma\mathbf{n}\cdot\mathfrak{T}(\mathbf{u},p)\cdot\mathbf{n}\,\mathbf{v}\cdot\mathbf{n}.\end{aligned}$$

Hence, ρ can be interpreted as the normal stress component $\mathbf{n}\cdot\mathfrak{T}(\mathbf{u},p)\cdot\mathbf{n}$.

Problems (3.1) and (3.5) are weak formulations of the Stokes problem (1.2), which lead to different finite element discretizations. For the continuous problem, however, one easily checks that both weak formulations are equivalent.

The following lemma proves that problem (3.5) fits into the abstract framework of Babuška and Brezzi [5, 9] for mixed problems. In particular, it implies the unique solvability of problem (3.5).

Lemma 3.1. *The space*

$$W:=\{\mathbf{u}\in X:\ b(\mathbf{u};p,\rho)=0\ \ \forall\, p\in Y,\rho\in Z\}$$

is not empty. There are two constants $\alpha>0$ and $\beta>0$ such that

$$a(\mathbf{u},\mathbf{u})\geqq\alpha\,\|\mathbf{u}\|_{1,\Omega}^2\qquad\forall\,\mathbf{u}\in W \tag{3.7}$$

and

$$\inf_{\substack{0\neq p\in Y\\ 0\neq\rho\in Z}}\ \sup_{0\neq\mathbf{u}\in X}\frac{b(\mathbf{u};p,\rho)}{\|u\|_{1,\Omega}\{\|p\|_{0,\Omega}^2+\|\rho\|_{-1/2,\Gamma}^2\}^{1/2}}\geqq\beta. \tag{3.8}$$

Proof. Since $\mathbf{u} \in H^1(\Omega)^d$ implies that $\operatorname{div} \mathbf{u} \in L^2(\Omega)$ and $\mathbf{u} \cdot \mathbf{n} \in H^{1/2}(\Gamma)$, we conclude that W equals the space V defined in Eq. (2.2e). Hence, Inequality (3.7) follows from Inequality (2.5) and (2.6) with $\alpha = c_1 c_2/(c_2 + 2c_0)$.

In order to prove Inequality (3.8), take a $p \in Y$ and a $\rho \in Z$ with $\|p\|_{0,\Omega}^2 + \|\rho\|_{-1/2,\Gamma}^2 = 1$. Then, there are a $\hat{p} \in L^2(\Omega)$ and a $\hat{\rho} \in H^{1/2}(\Gamma)$ with

$$\|\hat{p}\|_{0,\Omega} = \|\hat{\rho}\|_{1/2,\Gamma} = 1, \quad (p, \hat{p})_{0,\Omega} \geqq \tfrac{1}{2} \|p\|_{0,\Omega}, \quad \langle \rho, \hat{\rho} \rangle_\Gamma \geqq \tfrac{1}{2} \|\rho\|_{-1/2,\Gamma}.$$

Put $\gamma := \{ -\int_\Omega \hat{p} + \int_\Gamma \hat{\rho} \}/\operatorname{meas}(\Omega)$. Then the Neumann problem

$$-\Delta \phi = \hat{p} + \gamma \quad \text{in } \Omega, \qquad \frac{\partial \phi}{\partial n} = -\hat{\rho} \quad \text{on} \quad \Gamma$$

has a unique weak solution ϕ which fulfills the regularity estimate

$$\|\phi\|_{2,\Omega} \leqq c_0 \{ \|\hat{p} + \gamma\|_{0,\Omega} + \|\hat{\rho}\|_{1/2,\Gamma} \} \leqq c_1.$$

Thus the vector field $\mathbf{v} := \nabla \phi$ satisfies

$$\|\mathbf{v}\|_{1,\Omega} \leqq c_2 \|\phi\|_{2,\Omega} \leqq c_3$$

and

$$\begin{aligned} b(\mathbf{v}; p, \rho) &= -\int_\Omega p \operatorname{div} \mathbf{v} - \langle \rho, \mathbf{v} \cdot \mathbf{n} \rangle_\Gamma = \int_\Omega p \hat{p} + \gamma \int_\Omega p + \langle \rho, \hat{\rho} \rangle_\Gamma \\ &\geqq \tfrac{1}{2} \{ \|p\|_{0,\Omega} + \|\rho\|_{-1/2,\Gamma} \} \geqq c_4 \{ \|p\|_{0,\Omega}^2 + \|\rho\|_{-1/2,\Gamma}^2 \}^{1/2} \|\mathbf{v}\|_{1,\Omega}, \end{aligned}$$

where we have used that $\int_\Omega p = 0$. Together with a homogeneity argument, this establishes Inequality (3.8). □

4. Mixed Finite Element Approximation

This section is concerned with the mixed finite element approximation of problems (3.5) and (3.6). To simplify the exposition, we only consider simplicial finite elements on polyhedral approximations Ω_h of Ω. The discussion of higher order approximations of the boundary is postponed to the end of Sect. 7.

The curved domain Ω is approximated by a family of polyhedrons Ω_h, $h > 0$, with vertices on Γ and sides of length less than h. Denote by Γ_h the boundary of Ω_h, by $\mathbf{n}_h$ the unit exterior normal to Ω_h, and by $\mathbf{t}_{hi}$, $1 \leqq i \leqq d-1$, an orthonormal set of tangent vectors. In the sequel, norms and scalar products referring to Ω_h or Γ_h will be denoted by an index Ω_h or Γ_h, respectively.

Each polyhedron Ω_h is subdivided into d-simplices with sides of length less than h. We assume that the corresponding family $\mathcal{T}_h$ of partitions satisfies the usual regularity assumptions (cf. e.g., [11]):

(i) each vertex of Ω_h is the vertex of a $T \in \mathcal{T}_h$,

(ii) each $T \in \mathcal{T}_h$ has at least one vertex in the interior of Ω_h,

(iii) any two d-simplices $T, T' \in \mathcal{T}_h$ may meet at most in a whole l-face, $0 \leqq l < d$.

(iv) each $T\in\mathcal{T}_h$ with $T\cap\Gamma_h\neq\emptyset$ contains a ball with radius $c_0 h$ and is contained in a ball with radius $c_1 h$.

Here and in the sequel, c_0, $c_1,\ldots$ denote various constants which may depend on the context, but which are independent of h.

Denote by $\mathcal{O}_h$ the partition of Γ_h which is induced by $\mathcal{T}_h$.

Let P_k, $k\geqq 0$, be the set of all polynomials in $x_1,\ldots,x_d$ of degree $\leqq k$ and put

$$S_h^k:=\{\phi\colon\Omega_h\to\mathbb{R}\colon \phi|_T\in P_k,\ \forall\, T\in\mathcal{T}_h\}. \tag{4.1}$$

The velocity $\mathbf{u}$, the pressure p, and the normal stress component ρ are approximated by functions from a subspace $\tilde{X}_h$ of $\{S_h^k\cap C(\bar{\Omega}_h)\}^d$, $k\geqq 1$, from a subspace Y_h of $S_h^l\cap L_0^2(\Omega_h)$, $l\geqq 0$, and from the space

$$Z_h:=\{\phi\colon\Gamma_h\to\mathbb{R}\colon \phi|_S\in P_m,\ \forall\, S\in\mathcal{O}_h\},\qquad m\geqq 0, \tag{4.2}$$

respectively.

Note that $\tilde{X}_h\subset H^1(\Omega_h)^d$. Put

$$\tilde{X}_{h0}:=\{\mathbf{u}_h\in\tilde{X}_h\colon \mathbf{u}=0 \text{ on } \Gamma_h\}. \tag{4.3}$$

The spaces $\tilde{X}_h$ and Y_h have to satisfy the following compatibility condition and approximation properties:

(H1) There is a constant $\tilde{\beta}>0$, which does not depend on h, such that

$$\inf_{0\neq p_h\in Y_h}\ \sup_{0\neq \mathbf{u}_h\in\tilde{X}_{h0}} \frac{\int_{\Omega_h} p_h \operatorname{div}\mathbf{u}_h}{\|p_h\|_{0,\Omega_h}\,\|\mathbf{u}_h\|_{1,\Omega_h}}\geqq\tilde{\beta}.$$

(H2)
$$\inf_{p_h\in Y_h}\|p-p_h\|_{0,\Omega_h}\leqq ch\,\|p\|_{1,\Omega_h}\qquad \forall\, p\in H^1(\Omega_h).$$

(H3) There is a continuous linear operator $\Pi_h\colon H^1(\Omega_h)\to\tilde{X}_h$, which satisfies

$$\begin{aligned}
&\Pi_h(H_0^1(\Omega_h)^d)\subset\tilde{X}_{h0},\\
&\|\mathbf{u}-\Pi_h\mathbf{u}\|_{s,\Omega_h}\leqq ch^{t-s}\,\|\mathbf{u}\|_{t,\Omega_h}\qquad \forall\,\mathbf{u}\in H^t(\Omega_h)^d,\quad s=0,1,\ t=1,2,\\
&\|\mathbf{u}-\Pi_h\mathbf{u}\|_{0,\Gamma_h}\leqq ch^{1/2}\,\|\mathbf{u}\|_{1,\Omega_h}\qquad \forall\,\mathbf{u}\in H^1(\Omega_h)^d.
\end{aligned}$$

The approximation properties (H2), (H3) are fulfilled for the familiar finite element spaces (cf. [11]). The construction of the operator Π_h is due to P. Clément [12]. Assumption (H1) is the discrete version of the Babuška-Brezzi condition [5, 9] for the Stokes problem with no-slip boundary condition. An extensive list of finite element spaces satisfying condition (H1) is given in [13, 21] and the literature cited there.

When considering the Lagrange functional (3.3), the pressure p and the normal stress component ρ are Lagrange multipliers corresponding to the continuity equation $\operatorname{div}\mathbf{u}=0$ and to the boundary condition $\mathbf{u}\cdot\mathbf{n}=g$ respectively. In the framework of Babuška and Brezzi [5, 9], their influence must be balanced by the velocity-space. Concerning the pressure, this is expressed in condition (H1). In order to balance the influence of the parameter ρ, we have to enrich the velocity-space. To this end, consider a $T\in\mathcal{T}_h$ which has a face S

on Γ_h. Enumerate the vertices of T such that the vertices of S are numbered first and denote by $\lambda_{T1}, \ldots, \lambda_{Td+1}$ the barycentric co-ordinates of T. Put

$$\phi_s := \begin{cases} \mathbf{n}_h \prod_{i=1}^{d} \lambda_{Ti}, & \text{on } T, \\ 0, & \text{on } \Omega_h \setminus T, \end{cases} \tag{4.4}$$

and define

$$\mathscr{B}_h := \operatorname{span}\{\phi_s \sigma : \sigma \in P_m, S \in \mathcal{O}_h\}, \tag{4.5}$$

$$X_h := (\tilde{X}_h \oplus \mathscr{B}_h)/\mathscr{S}_h. \tag{4.6}$$

The space $\mathscr{S}_h$ is defined as in Eq. (2.1) with Ω replaced by Ω_h. The division by $\mathscr{S}_h$ takes into account that the solution of problem (3.5) is unique only up to rigid body rotations.

Note, that the functions ϕ_s of Eq. (4.4) are piecewise polynomials of degree d. Since they vanish on ∂S and outside T, we conclude that $\mathscr{B}_h \subset H^1(\Omega_h)^d$ and that $X_h \cap H_0^1(\Omega_h)^d = \tilde{X}_{h0}$.

Replacing Ω by Ω_h, we define approximations a_h and b_h to the bilinear forms a and b:

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) \quad &:= \tfrac{1}{2} \int_{\Omega_h} \mathfrak{D}(\mathbf{u}_h) \cdot \mathfrak{D}(\mathbf{v}_h) && \forall\, \mathbf{u}_h, \mathbf{v}_h \in X_h, \\ b_h(\mathbf{u}_h; p_h, \rho_h) &:= -\int_{\Omega_h} p_h \operatorname{div} \mathbf{u}_h - \int_{\Omega_h} \mathbf{u}_h \cdot \mathbf{n}_h \rho_h && \forall\, \mathbf{u}_h \in X_h, p_h \in Y_h,\ \rho_h \in Z_h. \end{aligned} \tag{4.7}$$

Denote by $I_h g$ the continuous, piecewise linear function corresponding to $\mathcal{O}_h$, which interpolates g in all vertices in $\mathcal{O}_h$, and put

$$g_h := I_h g - \int_{\Gamma_h} I_h g. \tag{4.8}$$

Since g is the trace of an $H^2(\Omega)$-function, the function g_h is well defined and satisfies the compatibility condition $\int_{\Gamma_h} g_h = 0$.

We consider the following mixed finite element approximation of the Stokes problem (3.5):

Find $\mathbf{u}_h \in X_h$, $p_h \in Y_h$, $\rho_h \in Z_h$ such that

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h; p_h, \rho_h) &= \int_{\Omega_h} \mathbf{f} \cdot \mathbf{v}_h && \forall\, \mathbf{v}_h \in X_h, \\ b_h(\mathbf{u}_h; q_h, \sigma_h) &= -\int_{\Gamma_h} g_h \sigma_h && \forall\, q_h \in Y_h,\ \sigma_h \in Z_h. \end{aligned} \tag{4.9}$$

Similarly, the mixed finite element approximation of the Navier-Stokes problem (3.6) is given by:

Find $\mathbf{u}_h \in X_h$, $p_h \in Y_h$, $\rho_h \in Z_h$ such that

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h; p_h, \rho_h) &= \frac{1}{\nu} \int_{\Omega_h} [\mathbf{f} - (\mathbf{u}_h \cdot \nabla) \mathbf{u}_h] \cdot \mathbf{v}_h && \forall\, \mathbf{v}_h \in X_h, \\ b_h(\mathbf{u}_h; q_h, \sigma_h) &= -\int_{\Gamma_h} g_h \sigma_h && \forall\, q_h \in Y_h,\ \sigma_h \in Z_h. \end{aligned} \tag{4.10}$$

In the remainder of this section, we prove that problem (4.9) fits into the abstract framework for mixed problems. To this end, define

$$V_h := \{\mathbf{u}_h \in X_h : b_h(\mathbf{u}_h; p_h, \rho_h) = 0 \ \forall p_h \in Y_h, \rho_h \in Z_h\}.$$

The space V_h is an approximation of the space V of Eq. (2.2). In general it is not a subspace of V. Thanks to condition (H1), the space V_h is not empty. Take a $\mathbf{u}_h \in V_h$. The definition of V_h and Z_h implies that $\mathbf{u}_h \cdot \mathbf{n}_h$ is orthogonal to all piecewise constant functions with respect to the $L^2(\Gamma_h)$ inner product. Standard approximation results and inverse estimates for finite elements therefore imply that

$$\|\mathbf{u}_h \cdot \mathbf{n}_h\|_{0,\Gamma_h} \leqq ch \, \|\mathbf{u}_h\|_{1,\Gamma_h} \leqq ch^{1/2} \, \|\mathbf{u}_h\|_{1,\Omega_h} \qquad \forall \mathbf{u}_h \in V_h. \tag{4.11}$$

Combining Inequalities (2.5), (2.6), and (4.11) we immediately obtain:

Proposition 4.1 *(V_h-ellipticity of a_h). There are two constants $h_0 > 0$ and $\alpha > 0$ such that the estimate*

$$a_h(\mathbf{u}_h, \mathbf{u}_h) \geqq \alpha \, \|\mathbf{u}_h\|_{1,\Omega_h}^2$$

holds for all $\mathbf{u}_h \in V_h$ and $0 < h \leqq h_0$. □

The next lemma shows that the space $\mathscr{B}_h$ of "bubble" functions balances the influence of the Lagrange multiplier ρ.

Lemma 4.2. *There is a constant $\gamma > 0$, which does not depend on h, such that*

$$\inf_{0 \neq \rho_h \in Z_h} \ \sup_{0 \neq \mathbf{u}_h \in X_h} \frac{-\int_{\Gamma_h} \rho_h \mathbf{u}_h \cdot \mathbf{n}_h}{\|\mathbf{u}_h\|_{1,\Omega_h} \, \|\rho_h\|_{-1/2,\Gamma_h}} \geqq \gamma.$$

Proof. Consider a $\rho_h \in Z_h$ with $\|\rho_h\|_{-1/2,\Gamma_h} = 1$ and define the function $\mathbf{u}_h \in \mathscr{B}_h$ as

$$\mathbf{u}_h := - \sum_{\substack{T \in \mathscr{T}_h \\ S := T \cap \Gamma_h \in \mathscr{O}_h}} \rho_h|_S \, \boldsymbol{\phi}_S. \tag{4.12}$$

The uniformity condition (iv) implies that

$$c_0 h^d \leqq \operatorname{meas}(T) \leqq c_1 h^d, \qquad c_2 h^{d-1} \leqq \operatorname{meas}(S) \leqq c_3 h^{d-1} \tag{4.13}$$

for all $T \in \mathscr{T}_h$ with $S := T \cap \Gamma_h \in \mathscr{O}_h$. Equations (4.12) and (4.13) yield

$$\begin{aligned} -\int_{\Gamma_h} \mathbf{u}_h \cdot \mathbf{n}_h &= \sum_{\substack{T \in \mathscr{T}_h \\ S := T \cap \Gamma_h \in \mathscr{O}_h}} |\rho_h|_S|^2 \int_S \boldsymbol{\phi}_s \cdot \mathbf{n}_s \\ &\geqq c_4 \, \|\rho_h\|_{0,\Gamma_h}^2 \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} \|\mathbf{u}_h\|_{1,\Omega_h}^2 &\leqq c_5 h^{d-2} \sum_{\substack{T \in \mathscr{T}_h \\ S := T \cap \Gamma_h \in \mathscr{O}_h}} |\rho_h|_S|^2 \\ &\leqq c_6 h^{-1} \, \|\rho_h\|_{0,\Gamma_h}^2. \end{aligned} \tag{4.15}$$

Combining Inequalities (4.14) and (4.15), we obtain an inf-sup condition with respect to a wrong norm:

$$\sup_{0 \neq \mathbf{v}_h \in X_h} \frac{-\int_{\Gamma_h} \rho_h \mathbf{v}_h \cdot \mathbf{n}_h}{\|\mathbf{v}_h\|_{1,\Omega_h}} \geqq c_7 h^{1/2} \|\rho_h\|_{0,\Gamma_h}, \tag{4.16}$$

where $c_7 = c_4/c_6$.

There is a function $\hat{\rho} \in H^{1/2}(\Gamma_h)$ with

$$\|\hat{\rho}\|_{1/2,\Gamma_h} = 1 \quad \text{and} \quad \int_{\Gamma_h} \rho_h \hat{\rho} \geqq \tfrac{1}{2} \|\rho_h\|_{-1/2,\Gamma_h} = \tfrac{1}{2}. \tag{4.17}$$

Put $\delta := \int_{\Gamma_h} \hat{\rho}/\text{meas}(\Omega_h)$. Then the Stokes problem

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p = 0, \quad & \operatorname{div} \mathbf{u} = \delta \quad \text{in} \quad \Omega_h, \\ \mathbf{u} \cdot \mathbf{n}_h = \hat{\rho}, \quad & \mathbf{n}_h \cdot \mathfrak{T}(\mathbf{u}, p) \cdot \mathbf{t}_{hi} = 0 \quad \text{on} \quad \Gamma_h, \quad 1 \leqq i \leqq d-1 \end{aligned} \tag{4.18}$$

has a unique weak solution $\hat{\mathbf{u}} \in H^1(\Omega_h)^d$, $\hat{p} \in L_0^2(\Omega_h)$, which satisfies the stability estimate

$$\|\hat{\mathbf{u}}\|_{1,\Omega_h} \leqq c_8 \|\hat{\rho}\|_{1/2,\Gamma_h} = c_8. \tag{4.19}$$

Let $\hat{\mathbf{u}}_h := \Pi_h \hat{\mathbf{u}}$, where Π_h is the operator of condition (H3). Equations (4.17)-(4.19) and condition (H3) imply that

$$\|\hat{\mathbf{u}}_h\|_{1,\Omega_h} \leqq c_9 \|\hat{\mathbf{u}}\|_{1,\Omega_h} \leqq c_{10} \tag{4.20}$$

and

$$\begin{aligned} \int_{\Gamma_h} \rho_h \hat{\mathbf{u}}_h \cdot \mathbf{n}_h &= \int_{\Gamma_h} \rho_h \hat{\mathbf{u}} \cdot \mathbf{n}_h + \int_{\Gamma_h} \rho_h (\hat{\mathbf{u}}_h - \hat{\mathbf{u}}) \cdot \mathbf{n}_h \\ &\geqq \int_{\Gamma_h} \rho_h \hat{\rho} - \|\rho_h\|_{0,\Gamma_h} \|\hat{\mathbf{u}} - \hat{\mathbf{u}}_h\|_{0,\Gamma_h} \\ &\geqq \tfrac{1}{2} - c_{11} h^{1/2} \|\rho_h\|_{0,\Gamma_h}. \end{aligned} \tag{4.21}$$

Combining Inequalities (4.16), (4.20), and (4.21), we finally obtain

$$\begin{aligned} \sup_{0 \neq \mathbf{v}_h \in X_h} \frac{-\int_{\Gamma_h} \rho_h \mathbf{v}_h \cdot \mathbf{n}_h}{\|\mathbf{v}_h\|_{1,\Omega_h}} &\geqq \max \left\{ c_7 h^{1/2} \|\rho_h\|_{0,\Gamma_h}, \frac{\int_{\Gamma_h} \rho_h \hat{\mathbf{u}} \cdot \mathbf{n}_h}{\|\hat{\mathbf{u}}_h\|_{1,\Omega_h}} \right\} \\ &\geqq \min_{\eta \geqq 0} \max \{ c_7 \eta, c_{13} - c_{14} \eta \} \\ &\geqq \frac{c_7 c_{13}}{c_7 + c_{14}} =: \gamma. \end{aligned}$$

Together with a homogeneity argument, this completes the proof. □

Combining condition (H1) and Lemma 4.2, we are now able to establish an inf-sup condition for b_h:

Proposition 4.3 *(inf-sup condition for b_h). There is a constant $\beta > 0$, which does not depend on h, such that*

$$\inf_{\substack{0 \neq p_h \in Y_h \\ 0 \neq \rho_h \in Z_h}} \sup_{0 \neq \mathbf{u}_h \in X_h} \frac{b_h(\mathbf{u}_h; p_h, \rho_h)}{\|\mathbf{u}_h\|_{1,\Omega_h} \{\|p_h\|_{0,\Omega_h}^2 + \|\rho_h\|_{-1/2,\Gamma_h}^2\}^{1/2}} \geqq \beta.$$

Proof. Let $p_h \in Y_h$ and $p_h \in Z_h$ satisfy $\|p_h\|_{0,\Omega_h}^2 + \|\rho_h\|_{-1/2,\Gamma_h}^2 = 1$. Thanks to Lemma 4.2, there is a function $\hat{\mathbf{u}}_h \in X_h$ with

$$\|\hat{\mathbf{u}}_h\|_{1,\Omega_h} = \mathbf{1} \quad \text{and} \quad -\int_{\Gamma_h} \rho_h \hat{\mathbf{u}}_h \cdot \mathbf{n}_n \geqq \gamma \|\rho_h\|_{-1/2,\Gamma_h}. \tag{4.22}$$

Since the spaces $\tilde{X}_{h0}$ and Y_h satisfy the inf-sup condition (H1), the discrete Stokes problem

$$\begin{aligned} &\tfrac{1}{2} \int_{\Omega_h} \mathfrak{D}(\mathbf{u}_h^*) \mathfrak{D}(\mathbf{v}_h) - \int_{\Omega_h} p_h^* \operatorname{div} \mathbf{v}_h = 0 \qquad \forall \mathbf{v}_h \in \tilde{X}_{h0} \\ &-\int_{\Omega_h} q_h \operatorname{div} \mathbf{u}_h^* = \int_{\Omega_h} q_h \left\{ \frac{p_h}{\|p\|_{0,\Omega_h}} + \operatorname{div} \hat{\mathbf{u}}_h \right\} \qquad \forall q_h \in X_h \end{aligned} \tag{4.23}$$

has a unique solution $\mathbf{u}_h^* \in \tilde{X}_{h0}$, $p_h^* \in Y_h$. Put $\mathbf{u}_h := \hat{\mathbf{u}}_h + \mathbf{u}_h^*$. Equations (4.22) and (4.23) imply that

$$\begin{aligned} b_h(\mathbf{u}_h; p_h, \rho_h) &= -\int_{\Omega_h} p_h \operatorname{div} \mathbf{u}_h - \int_{\Gamma_h} \rho_h \mathbf{u}_h \cdot \mathbf{n}_h = \int_{\Omega_h} \frac{p_h^2}{\|p_h\|_{0,\Omega_h}} - \int_{\Gamma_h} \rho_h \hat{\mathbf{u}}_h \cdot \mathbf{n}_h \\ &) \geqq \|p_h\|_{0,\Omega_h} + \gamma \|\rho_h\|_{-1/2,\Gamma_h} \\ &\geqq \min\{1, \gamma\}. \end{aligned} \tag{4.24}$$

In order to prove the proposition, we have to bound $\|\mathbf{u}_h\|_{1,\Omega h}$ independently of h, p_h, and ρ_h. Inequalities (2.5), (2.6), (4.22), and (4.23) imply that

$$\begin{aligned} \frac{c_1}{1+2c_2} \|\mathbf{u}_h^*\|_{1,\Omega_h}^2 &\leqq \tfrac{1}{2} \int_{\Omega_h} |\mathfrak{D}(\mathbf{u}_h^*)|^2 = -\int_{\Omega_h} \frac{p_h^* p_h}{\|p_h\|_{0,\Omega_h}} - \int_{\Omega_h} p_h^* \operatorname{div} \hat{\mathbf{u}}_h \\ &\leqq c_3 \|p_h^*\|_{0,\Omega_h} \{1 + \|\hat{\mathbf{u}}_h\|_{1,\Omega_h}\} \leqq c_4 \|p_h^*\|_{0,\Omega_h}. \end{aligned} \tag{4.25}$$

From the inf-sup condition (H1) and Eq. (4.23), we conclude that

$$\tilde{\beta} \|p_h^*\|_{0,\Omega_h} \leqq \sup_{0 \neq \mathbf{v}_h \in X_h} \frac{\int_{\Omega_h} p_h^* \operatorname{div} \mathbf{v}_h}{\|\mathbf{v}_h\|_{1,\Omega_h}} \leqq c_5 \|\mathbf{u}_h^*\|_{1,\Omega_h}. \tag{4.26}$$

Inequalities (4.21), (4.25), and (4.26) immediately yield

$$\begin{aligned} \|\mathbf{u}_h\|_{1,\Omega_h} &\leqq \|\hat{\mathbf{u}}_h\|_{1,\Omega_h} + \|\mathbf{u}_h^*\|_{1,\Omega_h} \\ &\leqq 1 + c_4(1+2c_2)/(c_1 \tilde{\beta}) =: c_6. \end{aligned} \tag{4.27}$$

Combining Inequalities (4.24) and (4.27) and applying a homogeneity argument, we finally obtain the inf-sup condition for b_h. □

The restriction to simplicial finite elements is not essential. Similar results can easily be established for quadrilateral elements. One only has to modify the definition of $\mathscr{B}_h$ suitably. We refer to [21] for the technical details.

5. Error Estimates for the Linear Problem

To avoid additional technical difficulties, we assume in this and the following section that Ω is *convex*. This ensures $\bar{\Omega}_h \subset \bar{\Omega}$ for all $h>0$. We will comment in Sect. 7 on the treatment of nonconvex domains.

In order to estimate the consistency error, which is due to the approximation of Ω, we have to compare functions on Γ and Γ_h. To this end, denote by π the orthogonal projection of Γ onto Γ_h (see Fig. 1).

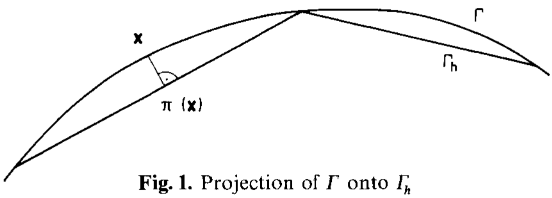

Fig. 1. Projection of Γ onto Γ_h

Since Ω is convex, the mapping π is bijective. The same holds for nonconvex domains provided $h \leqq c_\Gamma$, where the constant c_Γ only depends on the curvature of Γ. Let $1 \leqq s \leqq r < \infty$ and $\varphi \in W^{1,r}(\Omega)$. Using the mean value theorem and Hölder's inequality, one easily checks that the estimate

$$\|\varphi \circ \pi^{-1} - \varphi\|_{0,s,\Gamma_h} \leqq c h^{2\left(1-\frac{1}{r}\right)} \|\varphi\|_{1,r,\Omega} \tag{5.1}$$

holds with a constant c, which is independent of h (cf. Lemma 5.1 of [21]).

Theorem 5.1. *Let* $\mathbf{u} \in X$, $p \in Y$, $\rho \in Z$ *and* $\mathbf{u}_h \in X_h$, $p_h \in Y_h$, $\rho_h \in Z_h$ *be the unique solution of the Stokes problem* (3.5) *and of its mixed finite element approximation* (4.9), *respectively. Then, the error estimate*

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega_h} + \|p-p_h\|_{0,\Omega_h} + \|\rho\circ\pi^{-1}-\rho_h\|_{-1/2,\Gamma_h} + h^{1/2}\|\rho\circ\pi^{-1}-\rho_h\|_{0,\Gamma_h} \\ \leqq c h\{\|\mathbf{f}\|_{0,\Omega} + \|g\|_{3/2,\Gamma}\}\end{aligned} \tag{5.2}$$

holds.

Proof. Since we proved in Sects. 3 and 4 that problems (3.5) and (4.9) fit into the abstract framework for mixed problems, we know from [9] that each of them admits a unique solution. Recall that $\rho = \mathbf{n} \cdot \mathfrak{T}(\mathbf{u}, p) \cdot \mathbf{n}$. The regularity estimate (3.2) for the Stokes problem (1.2) then yields

$$\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega} + \|\rho\|_{1/2,\Gamma} \leqq c\{\|\mathbf{f}\|_{0,\Omega} + \|g\|_{3/2,\Gamma}\}. \tag{5.3}$$

Hence, we may multiply Eq. (1.2a) by a $\mathbf{v}_h \in X_h$ and integrate over Ω_h. Recalling Green's formula (2.4), we obtain

$$\begin{aligned}\int_{\Omega_h} \mathbf{f}\cdot\mathbf{v}_h &= \int_{\Omega_h} \{-\Delta\mathbf{u}+\nabla p\}\cdot\mathbf{v}_h \\ &= \tfrac{1}{2}\int_{\Omega_h} \mathfrak{D}(\mathbf{u})\cdot\mathfrak{D}(\mathbf{v}_h) - \int_{\Omega_h} p\operatorname{div}\mathbf{v}_h - \int_{\Gamma_h} \mathbf{n}_h\cdot\mathfrak{T}(\mathbf{u},p)\cdot\mathbf{v}_h \\ &= a_h(\mathbf{u},\mathbf{v}_h) + b_h(\mathbf{v}_h; p, \rho\circ\pi^{-1}) - \sum_{i=1}^{d-1}\int_{\Gamma_h} \mathbf{n}_h\cdot\mathfrak{T}(\mathbf{u},p)\cdot\mathbf{t}_{hi}\,\mathbf{v}_h\cdot\mathbf{t}_{hi} \\ &\quad - \int_{\Gamma_h} \{\mathbf{n}_h\cdot\mathfrak{T}(\mathbf{u},p)\cdot\mathbf{n}_h - \rho\circ\pi^{-1}\}\,\mathbf{v}_h\cdot\mathbf{n}_h, \qquad \forall\,\mathbf{v}_h\in X_h. \end{aligned} \tag{5.4a}$$

Similarly, we get

$$0 = \int_{\Omega_h} q_h \operatorname{div}\mathbf{u} \qquad \forall\, q_h\in Y_h \tag{5.4b}$$

and

$$\begin{aligned} -\int_{\Gamma_h} g\circ\pi^{-1}\sigma_h &= -\int_{\Gamma_h} (\mathbf{u}\cdot\mathbf{n})\circ\pi^{-1}\sigma_h \\ &= -\int_{\Gamma_h} \mathbf{u}\cdot\mathbf{n}_h\sigma_h - \int_{\Gamma_h} \{(\mathbf{u}\cdot\mathbf{n})\circ\pi^{-1} - \mathbf{u}\cdot\mathbf{n}_h\}\,\sigma_h, \qquad \forall\,\sigma_h\in Z_h \end{aligned} \tag{5.4c}$$

To simplify the notation, define

$$\begin{aligned} R_1(\mathbf{u},p\,\rho;\mathbf{v}_h) &:= \sum_{i=1}^{d-1}\int_{\Gamma_h} \mathbf{n}_h\cdot\mathfrak{T}(\mathbf{u},p)\cdot\mathbf{t}_{hi}\,\mathbf{v}_h\cdot\mathbf{t}_{hi} \\ &\quad + \int_{\Gamma_h} \{\mathbf{n}_h\cdot\mathfrak{T}(\mathbf{u},p)\cdot\mathbf{n}_h - \rho\circ\pi^{-1}\}\,\mathbf{v}_h\cdot\mathbf{n}_h, \end{aligned} \tag{5.5a}$$

$$\begin{aligned} R_2(\mathbf{u},g;\sigma_h) &:= -\int_{\Gamma_h} \{g\circ\pi^{-1} - g_h\}\,\sigma_h \\ &\quad + \int_{\Gamma_h} \{(\mathbf{u}\cdot\mathbf{n})\circ\pi^{-1} - \mathbf{u}\cdot\mathbf{n}_h\}\,\sigma_h, \end{aligned} \tag{5.5b}$$

and

$$R_1 := \sup_{0\neq\mathbf{v}_h\in X_h} \frac{|R_1(\mathbf{u},p,\rho;\mathbf{v}_h)|}{\|\mathbf{v}_h\|_{1,\Omega_h}}, \tag{5.6a}$$

$$R_2 := \sup_{0\neq\sigma_h\in Z_h} \frac{|R_2(\mathbf{u},g;\sigma_h)|}{\|\sigma_h\|_{-1/2,\Gamma_h}}. \tag{5.6b}$$

Equations (4.9), (5.4), and (5.5) yield

$$\begin{aligned} &a_h(\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h) + b_h(\mathbf{v}_h; p-p_h, \rho\circ\pi^{-1}-\rho_h) + b_h(\mathbf{u}-\mathbf{u}_h; q_h,\sigma_h) \\ &= R_1(\mathbf{u},p,\rho;\mathbf{v}_h) + R_2(\mathbf{u},g;\sigma_h) \qquad \forall\,\mathbf{v}_h\in X_h,\ q_h\in Y_h,\ \sigma_h\in Z_h. \end{aligned} \tag{5.7}$$

Using a standard argument for mixed problems (cf. the proof of Theorem 2.1 in [9]), we obtain from Propositions 4.1, 4.3 and Eqs. (5.6), (5.7) the error estimate

$$\begin{aligned} &\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega_h} + \|p-p_h\|_{0,\Omega_h} + \|\rho\circ\pi^{-1}-\rho_h\|_{-1/2,\Gamma_h} \\ &\leqq c\{\inf_{\mathbf{v}_h\in X_h}\|\mathbf{u}-\mathbf{v}_h\|_{1,\Omega_h} + \inf_{q_h\in Y_h}\|p-q_h\|_{0,\Omega_h} \\ &\quad + \inf_{\sigma_h\in Z_h}\|\rho\circ\pi^{-1}-\sigma_h\|_{-1/2,\Gamma_h} + R_1 + R_2\}. \end{aligned} \tag{5.8}$$

Assumptions (H2), (H3) and standard approximation results for finite elements imply that

$$\inf_{\mathbf{v}_h\in X_h}\|\mathbf{u}-\mathbf{v}_h\|_{1,\Omega_h}+\inf_{p_h\in Y_h}\|p-p_h\|_{0,\Omega_h}+\inf_{\sigma_h\in Z_h}\|\rho\circ\pi^{-1}-\sigma_h\|_{-1/2,\Gamma_h}$$
$$\leqq ch\{\|f\|_{0,\Omega}+\|g\|_{3/2,\Gamma}\}. \tag{5.9}$$

Next, we establish an upper bound for the consistency error R_1+R_2. Since Γ_h is a piecewise linear interpolation of Γ, we have

$$\sup_{x\in\Gamma_h}\left\{|\mathbf{n}_h(x)-\mathbf{n}\circ\pi^{-1}(x)|+\sum_{i=1}^{d-1}|\mathbf{t}_{hi}(x)-\mathbf{t}_i\circ\pi^{-1}(x)|\right\}\leqq ch. \tag{5.10}$$

Recalling the boundary condition (1.1d), applying the estimate (5.1) with $r=s=2$ to each component of the stress tensor, and using Inequalities (5.3) and (5.10), we conclude that

$$|R_1(\mathbf{u},p,\rho;\mathbf{v}_h)|$$
$$\leqq\|\mathbf{v}_h\|_{0,\Gamma_h}\left\{\sum_{i=1}^{d-1}\|\mathbf{n}_h\cdot\mathfrak{T}(\mathbf{u},p)\cdot\mathbf{t}_{hi}-(\mathbf{n}\cdot\mathfrak{T}(\mathbf{u},p)\cdot\mathbf{t}_i)\circ\pi^{-1}\|_{0,\Gamma_h}\right.$$
$$\left.+\|\mathbf{n}_h\cdot\mathfrak{T}(\mathbf{u},p)\cdot\mathbf{n}_h-(\mathbf{n}\cdot\mathfrak{T}(\mathbf{u},p)\cdot\mathbf{n})\circ\pi^{-1}\|_{0,\Gamma_h}\right\}$$
$$\leqq c_1\|\mathbf{v}_h\|_{1,\Omega_h}\{h\|\mathfrak{T}(\mathbf{u},p)\|_{0,\Gamma_h}+\|\mathfrak{T}(\mathbf{u},p)\circ\pi^{-1}-\mathfrak{T}(\mathbf{u},p)\|_{0,\Gamma_h}\}$$
$$\leqq c_3\|\mathbf{v}_h\|_{1,\Omega_h}h\{\|\mathbf{f}\|_{0,\Omega_h}+\|g\|_{3/2,\Gamma}\},\qquad\forall\,\mathbf{v}_h\in X_h.$$

Hence, we have

$$R_1\leqq c_3 h\{\|\mathbf{f}\|_{0,\Omega}+\|g\|_{3/2,\Gamma}\}. \tag{5.11}$$

Recall that g is the trace of a function $\tilde{g}\in H^2(\Omega)$. The definition of g_h implies that $g_h=I_h\tilde{g}-\int_{\Gamma_h}I_h\tilde{g}$, where I_h: $H^2(\Omega)\to S_h^1$ denotes the standard pointwise interpolation operator. Recalling the continuity of the imbedding $H^2(\Omega)\to W^{1,4}(\Omega)$, and applying the estimate (5.1) with $r=4$, $s=2$ to $\tilde{g}$ and each component of $\mathbf{u}$, we get

$$|R_2(\mathbf{u},g;\sigma_h)|$$
$$\leqq\left|\int_{\Gamma_h}\{\tilde{g}\circ\pi^{-1}-\tilde{g}\}\,\sigma_h\right|+\left|\int_{\Gamma_h}\{\tilde{g}-g_h\}\,\sigma_h\right|$$
$$+\left|\int_{\Gamma_h}\{(\mathbf{u}\cdot\mathbf{n})\circ\pi^{-1}-\mathbf{u}\cdot(\mathbf{n}\circ\pi^{-1})\}\,\sigma_h\right|+\left|\int_{\Gamma_h}\mathbf{u}\cdot(\mathbf{n}_h-\mathbf{n}\circ\pi^{-1})\,\sigma_h\right|$$
$$\leqq\|\sigma_h\|_{0,\Gamma_h}\{\|\tilde{g}\circ\pi^{-1}-\tilde{g}\|_{0,\Gamma_h}+\|\mathbf{u}\circ\pi^{-1}-\mathbf{u}\|_{0,\Gamma_h}\}$$
$$+\|\sigma_h\|_{-1/2,\Gamma_h}\{c_0 h\|\mathbf{u}\|_{1/2,\Gamma_h}+\|\tilde{g}-g_h\|_{1/2,\Gamma_h}\}$$
$$\leqq c_1 h^{3/2}\|\sigma_h\|_{0,\Gamma_h}\{\|\mathbf{u}\|_{2,\Omega}+\|\tilde{g}\|_{2,\Omega}\}$$
$$+\|\sigma_h\|_{-1/2,\Gamma_h}\{c_0 h\|\mathbf{u}\|_{1,\Omega}+\|\tilde{g}-I_h\tilde{g}\|_{1,\Omega_h}+\int_{\Gamma_h}|I_h\tilde{g}-\tilde{g}|+\int_{\Gamma_h}|\tilde{g}-\tilde{g}\circ\pi^{-1}|\}$$
$$\leqq c_2 h\{\|\sigma_h\|_{-1/2,\Gamma_h}+h^{1/2}\|\sigma_h\|_{0,\Gamma_h}\}\{\|\mathbf{f}\|_{0,\Omega}+\|g\|_{3/2,\Gamma}\}.$$

Together with the inf-sup condition (4.16), this yields

$$R_2 \leqq c_2 h\{\|\mathbf{f}\|_{0,\Omega} + \|g\|_{3/2,\Gamma}\}. \tag{5.12}$$

Combining Inequalities (5.8), (5.9), (5.11), and (5.12), and applying the inf-sup condition (4.16) to $\rho_h - \hat{\rho}_h$, we finally obtain the desired error estimate. □

The well-known duality argument of Aubin [3] and Nitsche [16] only yields the error estimate $\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega_h} \leqq ch$ for the velocity. This is due to the fact, that the consistency error $R_1 + R_2$ cannot be smaller than $\sup\{|\mathbf{n}_h(x) - \mathbf{n}\circ\pi^{-1}(x)|: x \in \Gamma_h\}$, and that the bound (5.10) is optimal for a polyhedral approximation of Ω. We will show in Sect. 7 that this drawback can be overcome, at the expense of a greater computational complexity, by considering higher order approximations of the boundary Γ.

6. Error Estimates for the Nonlinear Problem

In this section, we establish an error estimate similar to Theorem 5.1 for the finite element approximation (4.10) of the nonlinear problem (3.6). In order to avoid additional technical difficulties, we assume that Ω is *convex*. We refer to the next section for the general case.

Theorem 6.1. *Let $\Lambda \subset (0, \infty)$ be a compact interval. Consider a continuous branch $\lambda \to \mathbf{u}_\lambda$, $\lambda = \frac{1}{\nu}$, of solutions of the Navier-Stokes problem* (3.6), *which are non-singular in the sense of* [10]. *Denote by $\hat{\mathbf{u}}_{\lambda,h} \in X_h$ the best approximation to $\mathbf{u}_\lambda$ with respect to the $H^1(\Omega_h)^d$-norm. Then there are an $h_0 > 0$ and an $\alpha = \alpha(\lambda, h) > 0$ such that the discrete problem* (4.10) *has, for all $0 < h \leqq h_0$ and $\lambda \in \Lambda$, a unique solution $\mathbf{u}_{\lambda,h}$ in the ball $B_{\lambda,h}(\alpha) := \{\mathbf{v} \in H^1(\Omega_h)^d: \|\mathbf{v} - \hat{\mathbf{u}}_{\lambda,h}\|_{1,\Omega_h} \leqq \alpha\}$. Denote by $p_{\lambda,h}$, $\rho_{\lambda,h}$ and p_λ, ρ_λ the pressure and normal stress component corresponding to $\mathbf{u}_{\lambda,h}$ and $\mathbf{u}_\lambda$ respectively. Then the error estimate*

$$\begin{aligned}\|\mathbf{u}_\lambda - \mathbf{u}_{\lambda,h}\|_{1,\Omega_h} + \|p_\lambda - p_{\lambda,h}\|_{0,\Omega_h} + \|\rho_\lambda \circ \pi^{-1} - \rho_{\lambda,h}\|_{-1/2,\Gamma_h} \\ + h^{1/2}\|\rho_\lambda \circ \pi^{-1} - \rho_{\lambda,h}\|_{0,\Gamma_h} \leqq ch \quad \forall\, 0 < h \leqq h_0, \quad \lambda \in \Lambda\end{aligned}$$

holds with a constant c which is independent of h and λ. □

The proof of Theorem 6.1 closely follows the approach of Brezzi et al. [10] to nonlinear problems. However, we cannot apply their abstract results directly. Instead, we have to consider an auxiliary problem, which takes into account the approximation of the domain Ω. Since the technical arguments of the proof are rather lengthy but straightforward, we only sketch the essential steps of the proof and refer to [21] for the technical details.

Proof of Theorem 6.1. Denote by T: $H^{-1}(\Omega)^d \to H^1(\Omega)^d$ the Stokes operator which associates with each right-hand side $\mathbf{w} \in H^{-1}(\Omega)^d$ the unique solution $\mathbf{u} \in X$ of the Stokes problem

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}; p, \rho) &= \langle \mathbf{w}, \mathbf{v}\rangle_\Omega, \quad \forall\, \mathbf{v} \in X, \\ b(\mathbf{u}; q, \sigma) &= -\langle g, \sigma\rangle_\Gamma, \quad \forall\, q \in Y, \quad \sigma \in Z. \end{aligned}$$

The operator T is affine. Moreover, T is continuous, since by Lemma 3.1 we have

$$\|T(\mathbf{u})-T(\mathbf{v})\|_{1,\Omega} \leqq c_T \|\mathbf{u}-\mathbf{v}\|_{-1,\Omega} \qquad \forall\, \mathbf{u}, \mathbf{v} \in H^{-1}(\Omega)^d. \tag{6.1}$$

The nonlinearity of the Navier-Stokes problem is taken into account by the mapping G: $H^1(\Omega)^d \to L^{3/2}(\Omega)^d$,

$$G(\mathbf{u}) := (\mathbf{u} \cdot \nabla)\, \mathbf{u} - \mathbf{f}.$$

Put

$$F(\lambda, \mathbf{u}) := \mathbf{u} + \lambda\, T(G(\mathbf{u})).$$

Since, $\lambda \to \mathbf{u}_\lambda$ is a nonsingular branch of solutions of the Navier-Stokes equations (3.6), we know that

$$F(\lambda, \mathbf{u}_\lambda) = 0, \qquad \lambda = \frac{1}{\nu}, \tag{6.2}$$

and that $D_u F(\lambda, \mathbf{u}_\lambda)$ is a homeomorphism of $H^1(\Omega)^d$ onto itself.

Put

$$\gamma(\lambda) := \sup_{0 \neq \mathbf{v} \in H^1(\Omega)^d} \frac{1}{\|\mathbf{v}\|_{1,\Omega}} \|D_u F(\lambda, \mathbf{u}_\lambda)^{-1} \mathbf{v}\|_{1,\Omega}. \tag{6.3}$$

The continuity of the mapping $\lambda \to \mathbf{u}_\lambda$ and the compactness of Λ imply

$$\hat{\gamma} := \sup_{\lambda \in \Lambda} \gamma(\lambda) < \infty. \tag{6.4}$$

From the regularity estimate (3.2) for the Stokes problem and a well-known bootstrapping argument, we conclude that $\mathbf{u}_\lambda \in H^2(\Omega)^d$ and that

$$K := \max\{\|\mathbf{f}\|_{0,\Omega}, \|g\|_{3/2,\Gamma}, \sup_{\lambda \in \Lambda} \|\mathbf{u}_\lambda\|_{2,\Omega}\} < \infty. \tag{6.5}$$

In the sequel, $c_0, c_1, \ldots$ will denote constants which are independent of h and $\lambda \in \Lambda$.

In order to take into account the approximation of the domain Ω, we define the mapping G_h: $H^1(\Omega)^d \cup H^1(\Omega_h)^d \to L^{3/2}(\Omega)^d \cup L^{3/2}(\Omega_h)^d$ by

$$G_h(\mathbf{u}) := \chi_{\Omega_h} \cdot (\mathbf{u} \cdot \nabla)\, \mathbf{u} - \mathbf{f}.$$

Here, χ_{Ω_h} denotes the characteristic function of Ω_h. Let R_h be the operator which associates with each function ϕ: $\Omega \to \mathbb{R}$ its restriction to Ω_h. Put

$$\tilde{F}_h(\lambda, \mathbf{u}) := R_h[u + \lambda\, T(G_h(\mathbf{u}))].$$

The continuity of the imbeddings $H^1(\Omega) \to L^6(\Omega)$ and $H^2(\Omega) \to L^\infty(\Omega)$, together with Inequality (6.5) and Banach's fixed point theorem imply that there is an $h_1 > 0$ such that, for all $0 < h \leqq h_1$ and $\lambda \in \Lambda$, $D_u \tilde{F}_h(\lambda, \mathbf{u}_\lambda)$ is a homeomorphism of $H^1(\Omega_h)^d$ onto itself which satisfies the following estimates:

$$\|\tilde{F}_h(\lambda, \mathbf{u}_\lambda)\|_{1,\Omega_h} \leqq c_0 h^{4/3} K^2; \tag{6.6}$$

$$\sup_{0 \neq \mathbf{v} \in H^1(\Omega_h)^d} \frac{1}{\|\mathbf{v}\|_{1,\Omega_h}} \|D_u \tilde{F}_h(\lambda, \mathbf{u}_\lambda)^{-1} \mathbf{v}\|_{1,\Omega_h} \leqq 2\gamma(\lambda). \tag{6.7}$$

Denote by T_h: $H^{-1}(\Omega)^d \cup H^{-1}(\Omega_h)^d \to X_h$ the discrete Stokes operator which associates with each right-hand side $\mathbf{w} \in H^{-1}(\Omega)^d \cup H^{-1}(\Omega_h)^d$ the unique solution $\mathbf{u}_h \in X_h$ of the discrete Stokes problem

$$a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h; p_h, \rho_h) = \langle \mathbf{w}, \mathbf{v}_h \rangle_{\Omega_h} \qquad \forall\, \mathbf{v}_h \in X_h$$

$$b_h(\mathbf{u}_h; q_h, \sigma_h) = -\int_{\Omega_h} g_h \sigma_h. \qquad \forall\, q_h \in Y_h,\ \sigma_h \in Z_h$$

Here, the function g_h is defined in Eq. (4.8). The operator T_h is affine. It is continuous, since Propositions 4.1 and 4.3 imply that

$$\|T_h(\mathbf{u}) - T_h(\mathbf{v})\|_{1,\Omega_h} \leqq c'_T \|\mathbf{u} - \mathbf{v}\|_{-1,\Omega_h} \tag{6.8}$$

holds for all $\mathbf{u}, \mathbf{v} \in H^{-1}(\Omega)^d \cup H^{-1}(\Omega_h)^d$ and all $h > 0$. Note, that the constant c'_T is independent of h. Theorem 5.1 yields the error estimate

$$\|T(\mathbf{w}) - T_h(\mathbf{w})\|_{1,\Omega_h} \leqq c h \{\|\mathbf{w}\|_{0,\Omega} + \|g\|_{3/2,\Gamma}\} \qquad \forall\, \mathbf{w} \in L^2(\Omega)^d. \tag{6.9}$$

Put

$$F_h(\lambda, \mathbf{u}) := \mathbf{u} + \lambda\, T_h(G_h(\mathbf{u})).$$

Then, $\mathbf{u}_h \in X_h$ is a solution of the Navier-Stokes problem (4.10) if and only if it is a solution of

$$F_h(\lambda, \mathbf{u}_\lambda) = 0, \qquad \lambda = \frac{1}{\nu}. \tag{6.10}$$

Since $\hat{\mathbf{u}}_{\lambda,h}$ is the best approximation to $\mathbf{u}_\lambda$ in X_h with respect to the $H^1(\Omega_h)^d$-norm, condition (H3) yields the estimate

$$\|\mathbf{u}_\lambda - \hat{\mathbf{u}}_{\lambda,h}\|_{1,\Omega_h} \leqq c h K \qquad \forall\, h > 0. \tag{6.11}$$

Inequalities (6.6), (6.7), (6.8), (6.9), and (6.11) and Banach's fixed point theorem imply that there is an $0 < h_2 \leqq h_1$ such that, for all $0 < h \leqq h_2$ and $\lambda \in \Lambda$, $D_u F_h(\lambda, \hat{\mathbf{u}}_{\lambda,h})$ is a homeomorphism of $H^1(\Omega_h)^d$ onto itself which satisfies the following estimates:

$$\varepsilon_h(\lambda) := \|F_h(\lambda, \hat{\mathbf{u}}_{\lambda,h})\|_{1,\Omega_h} \leqq c_1 h K^2, \tag{6.12}$$

$$\sup_{0 \neq \mathbf{w} \in H^1(\Omega_h)^d} \frac{1}{\|\mathbf{w}\|_{1,\Omega_h}} \|D_u F_h(\lambda, \hat{\mathbf{u}}_{\lambda,h})\mathbf{w} - D_u \tilde{F}_h(\lambda, \mathbf{u}_\lambda)\mathbf{w}\|_{1,\Omega_h}$$
$$\leqq c_2 h K, \tag{6.13}$$

$$\sup_{0 \neq \mathbf{w} \in H^1(\Omega_h)^d} \frac{1}{\|\mathbf{w}\|_{1,\Omega_h}} \|D_u F_h(\lambda, \mathbf{v}_1)\mathbf{w} - D_u F_h(\lambda, \mathbf{v}_2)\mathbf{w}\|_{1,\Omega_h}$$
$$\leqq c_3 \|\mathbf{v}_1 - \mathbf{v}_2\|_{1,\Omega_h} \qquad \forall\, \mathbf{v}_1, \mathbf{v}_2 \in H^1(\Omega_h)^d, \tag{6.14}$$

$$\sup_{0 \neq \mathbf{w} \in H^1(\Omega_h)^d} \frac{1}{\|\mathbf{w}\|_{1,\Omega_h}} \|D_u F_h(\lambda, \hat{\mathbf{u}}_{\lambda,h})^{-1}\mathbf{w}\|_{1,\Omega_h} \leqq 4\gamma(\lambda). \tag{6.15}$$

Hence, we can define a mapping Φ: $H^1(\Omega_h)^d \to H^1(\Omega_h)^d$ by

$$\Phi(\mathbf{v}) := \mathbf{v} - D_u F_h(\lambda, \hat{\mathbf{u}}_{\lambda,h})^{-1} F_h(\lambda, \mathbf{v}).$$

Every fixed point of Φ is a solution of problem (6.10) and, therefore, of the discrete Navier-Stokes problem (4.10). Inequalities (6.12)–(6.15) imply that there is an $0<h_3\leqq h_2$ such that, for all $0<h\leqq h_3$ and $\lambda\in\Lambda$, Φ is a contraction of the ball

$$B_{\lambda,h}(8\gamma(\lambda)\varepsilon_h(\lambda)) := \{\mathbf{v}\in H^1(\Omega_h)^d \colon \|\mathbf{v}-\hat{\mathbf{u}}_{\lambda,h}\|_{1,\Omega_h}\leqq 8\gamma(\lambda)\varepsilon_h(\lambda)\}$$

into itself. Hence, Φ has a unique fixed point $\mathbf{u}_{\lambda,h}$ in the ball $B_{\lambda,h}(8\gamma(\lambda)\varepsilon_h(\lambda))$ for all $0<h\leqq h_3$ and $\lambda\in\Lambda$. The error estimate for the velocity follows from Inequalities (6.11) and (6.12).

In order to prove the error estimate for the pressure and the normal stress component, denote by $\tilde{\mathbf{u}}_{\lambda,h}\in X_h$, $\tilde{p}_{\lambda,h}\in Y_h$, $\tilde{\rho}_{\lambda,h}\in Z_h$ the unique solution of the discrete Stokes problem

$$\begin{aligned} a_h(\tilde{\mathbf{u}}_{\lambda,h},\mathbf{v}_h)+b_h(\mathbf{v}_h;\tilde{p}_{\lambda,h},\tilde{\rho}_{\lambda,h}) &= \lambda\langle \mathbf{f}-(\mathbf{u}_\lambda\cdot\nabla)\mathbf{u}_\lambda,\mathbf{v}_h\rangle_{\Omega_h} && \forall\,\mathbf{v}_h\in X_h \\ b_h(\tilde{\mathbf{u}}_{\lambda,h};q_h,\sigma_h) &= -\int_{\Gamma_h} g_h\sigma_h. && \forall\, q_h\in Y_h,\ \sigma_h\in Z_h \end{aligned}$$

Propositions 4.1, 4.3 and Inequality (4.16) yield the stability estimate

$$\begin{aligned} &\|\mathbf{u}_{\lambda,h}-\tilde{\mathbf{u}}_{\lambda,h}\|_{1,\Omega_h}+\|p_{\lambda,h}-\tilde{p}_{\lambda,h}\|_{0,\Omega_h}+\|\rho_{\lambda,h}-\tilde{\rho}_{\lambda,h}\|_{-1/2,\Gamma_h} \\ &\quad +h^{1/2}\|\rho_{\lambda,h}-\tilde{\rho}_{\lambda,h}\|_{0,\Gamma_h} \\ &\quad \leqq c\lambda\{\|\mathbf{u}_\lambda\|_{1,\Omega}+\|\mathbf{u}_{\lambda,h}\|_{1,\Omega_h}\}\,\|\mathbf{u}_\lambda-\mathbf{u}_{\lambda,h}\|_{1,\Omega_h} \\ &\quad \leqq c' hK^2. \end{aligned} \tag{6.16}$$

Since $(\mathbf{u}_\lambda\cdot\nabla)\mathbf{u}_\lambda\in L^2(\Omega)^d$, we conclude from Theorem 5.1 that

$$\begin{aligned} &\|\mathbf{u}_\lambda-\tilde{\mathbf{u}}_{\lambda,h}\|_{1,\Omega_h}+\|p_\lambda-\tilde{p}_{\lambda,h}\|_{0,\Omega_h}+\|\rho_\lambda\circ\pi^{-1}-\tilde{\rho}_{\lambda,h}\|_{-1/2,\Gamma_h} \\ &\quad +h^{1/2}\|\rho_\lambda\circ\pi^{-1}-\tilde{\rho}_{\lambda,h}\|_{0,\Gamma_h} \\ &\quad \leqq ch\{\|(\mathbf{u}_\lambda\cdot\nabla)\mathbf{u}_\lambda\|_{0,\Omega}+\|g\|_{3/2,\Gamma}\} \\ &\quad \leqq c' hK^2. \end{aligned} \tag{6.17}$$

Combining Inequalities (6.16) and (6.17), we immediately obtain the desired error estimate. □

7. Treatment of Nonconvex Domains and of Higher Order Approximations of the Boundary

If Ω is nonconvex or if we approximate Γ by functions which are piecewise polynomials of degree s, $s\geqq 2$, the set $\Omega_h\setminus\Omega$ is in general not empty. Hence, we cannot integrate Eq. (1.2) over Ω_h. Therefore, Eq. (5.7) is no longer valid. To overcome this difficulty, we consider for $\varepsilon>0$ the larger domain

$$\Omega_\varepsilon := \Omega\cup\{x\in\mathbb{R}^d \colon \inf_{y\in\Gamma}|x-y|<\varepsilon\}. \tag{7.1}$$

Then, there is an $\varepsilon_0>0$ such that continuations $\tilde{\mathbf{u}}\in H^2(\Omega_{\varepsilon_0})^d$, $\tilde{p}\in H^1(\Omega_{\varepsilon_0})$ and $\tilde{\mathbf{f}}\in L^2(\Omega_{\varepsilon_0})^d$ exist which satisfy (cf. [1])

$$\|\tilde{\mathbf{u}}\|_{2,\Omega_{\varepsilon_0}} \leq c\,\|\mathbf{u}\|_{2,\Omega}, \qquad \|\tilde{p}\|_{1,\Omega_{\varepsilon_0}} \leq c\,\|p\|_{1,\Omega},$$
$$\|\tilde{\mathbf{f}}\|_{0,\Omega_{\varepsilon_0}} \leq c\,\|f\|_{0,\Omega}. \tag{7.2}$$

For sufficiently small h, we have

$$\Omega_h \subset \Omega_{\varepsilon_0}. \tag{7.3}$$

Replace $\mathbf{f}$ by $\tilde{\mathbf{f}}$ in problems (4.9) and (4.10). Equation (5.7) then remains valid, if we replace $\mathbf{u}$ and p by $\tilde{\mathbf{u}}$ and $\tilde{p}$ respectively and add the term

$$R_3(\tilde{\mathbf{u}},\tilde{p},\tilde{\mathbf{f}};\mathbf{v}_h,q_h) := \int_{\Omega_h\setminus\Omega} \{-\Delta\tilde{\mathbf{u}} + \nabla\tilde{p} - \tilde{\mathbf{f}}\}\cdot\mathbf{v}_h - \int_{\Omega_h\setminus\Omega} q_h \operatorname{div}\tilde{\mathbf{u}} \tag{7.4}$$

to the right-hand side. Recalling the continuity of the imbedding $H^1(\Omega)\to L^6(\Omega)$, we conclude that

$$\begin{aligned} R_3 &:= \sup_{\mathbf{v}_h\in X_h,\, q_h\in Y_h} |R_3(\tilde{\mathbf{u}},\tilde{p},\tilde{\mathbf{f}};\mathbf{v}_h,q_h)|/\{\|\mathbf{v}_h\|_{1,\Omega_h} + \|q_h\|_{0,\Omega_h}\} \\ &\leq c\operatorname{meas}(\Omega_h\setminus\Omega)^{1/3}\{\|-\Delta\tilde{\mathbf{u}}+\nabla\tilde{p}-\tilde{\mathbf{f}}\|_{0,\Omega_h\setminus\Omega} + \|\operatorname{div}\tilde{\mathbf{u}}\|_{1,\Omega_h\setminus\Omega}\}. \end{aligned} \tag{7.5}$$

The arguments used in the proof of Theorem 5.1 and Inequalities (7.2), (7.5) yield the error estimate

$$\begin{aligned} &\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega_h\cap\Omega} + \|p-p_h\|_{0,\Omega_h\cap\Omega} + \|\rho\circ\pi^{-1}-\rho_h\|_{-1/2,\Gamma_h} \\ &\quad + h^{1/2}\|\rho\circ\pi^{-1}-\rho_h\|_{0,\Gamma_h} \\ &\leq ch\{\|\mathbf{f}\|_{0,\Omega} + \|g\|_{3/2,\Gamma}\} \\ &\quad + c\operatorname{meas}(\Omega_h\setminus\Omega)^{1/3}\{\|-\Delta\tilde{\mathbf{u}}+\nabla\tilde{p}-\tilde{\mathbf{f}}\|_{0,\Omega_h\setminus\Omega} + \|\operatorname{div}\tilde{\mathbf{u}}\|_{1,\Omega_h\setminus\Omega}\}. \end{aligned} \tag{7.6}$$

For polyhedral approximations of Ω and arbitrary continuations of $\mathbf{u}$, p and $\mathbf{f}$, satisfying Eq. (7.2), the second term on the right-hand side of Inequality (7.6) can at best be bounded by

$$\begin{aligned} &c\operatorname{meas}(\Omega_h\setminus\Omega)^{1/3}\{\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega} + \|\mathbf{f}\|_{0,\Omega}\} \\ &\leq c'\,h^{2/3}\{\|\mathbf{f}\|_{0,\Omega} + \|g\|_{3/2,\Gamma}\}. \end{aligned}$$

Hence, we lose a factor $h^{1/3}$ when comparing with the error estimate of Theorem 5.1 for convex domains.

The following proposition shows, that – taking into account the boundary conditions of problem (1.2) – we can choose the continuations such that Eq. (7.2) and the estimate

$$\begin{aligned} &\|-\Delta\tilde{\mathbf{u}}+\nabla\tilde{p}-\tilde{\mathbf{f}}\|_{0,\Omega_h\setminus\Omega} + \|\operatorname{div}\tilde{\mathbf{u}}\|_{1,\Omega_h\setminus\Omega} \\ &\leq c\operatorname{meas}(\Omega_h\setminus\Omega)^{1/3}\{\|\mathbf{f}\|_{0,\Omega} + \|g\|_{3/2,\Gamma}\} \end{aligned} \tag{7.7}$$

hold. The idea is to transform Ω locally to the upper half-plane and to use a suitable reflection principle there. For simplicity we assume that $g=0$. It will be evident from the arguments how to treat the inhomogeneous problem.

Proposition 7.1. *Let* $\mathbf{u}\in H^2(\Omega)^d$, $p\in H^1(\Omega)\cap L_0^2(\Omega)$ *be the weak solution of the Stokes problem* (1.2) *with* $\mathbf{f}\in L^2(\Omega)^d$ *and* $g=0$. *Then, there are an* $\varepsilon_0>0$, *which* *only depends on the curvature of* Γ, *and continuations* $\tilde{\mathbf{u}} \in H^2(\Omega_{\varepsilon_0})^d$, $\tilde{p} \in H^1(\Omega_{\varepsilon_0})$; *and* $\tilde{\mathbf{f}} \in L^2(\Omega_{\varepsilon_0})^d$ *of* $\mathbf{u}$, p *and* $\mathbf{f}$ *such that Eqs.* (7.2), (7.3), *and* (7.7) *hold for sufficiently small* $h > 0$.

Proof. Using a partition of unity, it is sufficient to consider a neighbourhood of a point on Γ. There, the boundary Γ can be represented as $\{\psi(s)\colon s \in \omega \subset \mathbb{R}^{d-1}\}$ with a sufficiently smooth function ψ. It is possible to choose the parametrization such that $\frac{\partial \psi}{\partial s_k}$ is the k-th tangent vector $\mathbf{t}_k$, $1 \leqq k \leqq d-1$. Denote by $\mathbf{n}(s)$ the unit exterior normal to Ω at the point $\psi(s)$. Define the function Φ: $\omega \times \mathbb{R} \to \mathbb{R}^d$ by $\Phi(s,r) := \psi(s) - r\mathbf{n}(s)$. Note, that $\Phi(s,r) \subset \Omega$ for all sufficiently small positive r. Since $D\Phi = (\mathbf{t}_1, \dots, \mathbf{t}_{d-1}, -\mathbf{n}) - r\left(\frac{\partial n}{\partial s_1}, \dots, \frac{\partial n}{\partial s_{d-1}}, 0\right)$, there is an $\varepsilon_0 > 0$ such that Φ is a diffeomorphism on $\omega \times (-\varepsilon_0, \varepsilon_0)$. Moreover, ε_0 can be chosen such that $\Phi(\omega \times (0, \varepsilon_0)) \subset \Omega$. Note, that ε_0 only depends on the curvature of Γ. The matrices $J_0 := (\mathbf{t}_1, \dots, \mathbf{t}_{d-1}, -\mathbf{n})$ and $A := (\mathbf{t}_1, \dots, \mathbf{t}_{d-1}, \mathbf{n})$ obviously are orthogonal. Moreover, we have $D\Phi^{-1} = J_0^T + rJ_1$ with a suitable $d \times d$ matrix J_1. Define the function $\mathbf{v} \in H^1(\omega \times (0, \varepsilon_0))^d$ by $\mathbf{v} := A^T \mathbf{u} \circ \Phi$. An elementary calculation shows that, for $g = 0$, the boundary condition (1.2c, d) is equivalent to

$$v_d = \frac{\partial v_k}{\partial r} - \sum_{j=1}^{d-1} \kappa_{kj} v_j = 0, \ 1 \leqq k \leqq d-1, \qquad \text{for } r = 0, \tag{7.8}$$

where $\kappa_{kj} := \mathbf{n} \cdot \frac{\partial \mathbf{t}_j}{\partial s_k}$, $1 \leqq k, j \leqq d-1$. Put

$$D := \begin{pmatrix} 1 & \dots & & 0 \\ & \ddots & & \\ \vdots & & 1 & \vdots \\ 0 & \dots & & -1 \end{pmatrix}, \qquad \hat{T}_1 := \begin{pmatrix} \kappa_{11} & \dots & \kappa_{1d-1} & 0 \\ \cdot & & & \cdot \\ \kappa_{d-11} & \dots & \kappa_{d-1d-1} & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix},$$

$$\hat{T} := D + 2r\hat{T}_1, \qquad T := A\hat{T}A^T, \qquad T_0 := ADA^T, \qquad T_1 := A\hat{T}_1A^T$$

and define the function $\tilde{\mathbf{v}}$ on $\omega \times (-\varepsilon_0, \varepsilon_0)$ by

$$\tilde{\mathbf{v}}(s,r) := \begin{cases} \mathbf{v}(s,r), & \text{if } r \geqq 0, \\ \hat{T}\mathbf{v}(s,-r), & \text{if } r < 0. \end{cases}$$

From the boundary condition (7.8), we conclude that $\mathbf{v} \in H^2(\omega \times (-\varepsilon_0, \varepsilon_0))^d$. Transforming back to the original co-ordinate system, we obtain the continuation of $\mathbf{u}$. Thus, we define the continuations $\tilde{\mathbf{u}}$, $\tilde{p}$, and $\tilde{\mathbf{f}}$ of $\mathbf{u}$, p and $\mathbf{f}$ by

$$\tilde{\mathbf{u}}(x) := \begin{cases} \mathbf{u}(x), & \text{if } x \in \bar{\Omega}, \\ T\mathbf{u} \circ \Phi(s,-r), & \text{if } x = \Phi(s,r) \in \Omega_{\varepsilon_0} \setminus \Omega, \ r < 0, \end{cases} \tag{7.9a}$$

$$\tilde{p}(x) := \begin{cases} p(x), & \text{if } x \in \bar{\Omega}, \\ p \circ \Phi(s,-r), & \text{if } x = \Phi(s,r) \in \Omega_{\varepsilon_0} \setminus \Omega, \ r < 0 \end{cases} \tag{7.9b}$$

$$\tilde{\mathbf{f}}(x) := \begin{cases} \mathbf{f}(x), & \text{if } x \in \bar{\Omega}, \\ T_0 \mathbf{f} \circ \Phi(s,-r), & \text{if } x = \Phi(s,r) \in \Omega_{\varepsilon_0} \setminus \Omega, \ r < 0. \end{cases} \tag{7.9c}$$

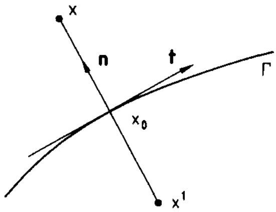

Fig. 2. Reflection of x

Condition (7.2) is obviously satisfied. Note, that the calculation of $\tilde{\mathbf{f}}$ is very simple. Consider an $x \in \Omega_{\varepsilon_0} \backslash \Omega$ and denote by x_0 its projection onto Γ and by x' its reflection across Γ (see Fig. 2). Then we have

$$\tilde{\mathbf{f}}(x) = \sum_{k=1}^{d-1} (\mathbf{f}(x') \cdot \mathbf{t}_k(x_0))\, \mathbf{t}_k(x_0) - (\mathbf{f}(x') \cdot \mathbf{n}(x_0))\, \mathbf{n}(x_0). \tag{7.10}$$

In order to establish Inequality (7.7), we denote in the sequel by R all functions that are of the form $r\psi_0 + \psi_1$ with an L^2-function ψ_0 and an H^1-function ψ_1, which can be bounded by

$$\begin{aligned} &\|\psi_0\|_{L^2(\omega \times (-\varepsilon_0, \varepsilon_0))} + \|\psi_1\|_{H^1(\omega \times (-\varepsilon_0, \varepsilon_0))} \\ &\leqq c\{\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega} + \|\mathbf{f}\|_{0,\Omega}\} \leqq c'\{\|\mathbf{f}\|_{0,\Omega} + \|g\|_{3/2,\Gamma}\}. \end{aligned} \tag{7.11}$$

We will collect in R all lower order derivatives of $\mathbf{u}$ and p and all terms that are proportional to r. From Inequality (7.11) and the imbedding theorem we obtain, for sufficiently small h, the estimate

$$\|R\|_{0, \Omega_h \backslash \Omega} \leqq c \operatorname{meas}(\Omega_h \backslash \Omega)^{1/3} \{\|\mathbf{f}\|_{0,\Omega} + \|g\|_{3/2,\Gamma}\}. \tag{7.12}$$

An elementary calculation yields for all $r < 0$:

$$\begin{aligned} (\Delta \tilde{\mathbf{u}}) \circ \boldsymbol{\Phi}(s,r) &= (\Delta_{s,r}(\tilde{\mathbf{u}} \circ \boldsymbol{\Phi}))(s,r) + R = T_0(\Delta_{s,r}(\mathbf{u} \circ \boldsymbol{\Phi}))(s,-r) + R \\ &= T_0(\Delta \mathbf{u}) \circ \boldsymbol{\Phi}(s,-r) + R \end{aligned}$$

and

$$\begin{aligned} (\nabla \tilde{p}) \circ \boldsymbol{\Phi}(s,r) &= J_0(\nabla_{s,r}(\tilde{p} \circ \boldsymbol{\Phi}))(s,r) + R = J_0 D(\nabla_{s,r}(p \circ \boldsymbol{\Phi}))(s,-r) + R \\ &= J_0 D J_0^T (\nabla p) \circ \boldsymbol{\Phi}(s,-r) + R = A D^3 A^T (\nabla p) \circ \boldsymbol{\Phi}(s,-r) + R \\ &= T_0(\nabla p) \circ \boldsymbol{\Phi}(s,-r) + R. \end{aligned}$$

Hence, we have for all $r < 0$:

$$(-\Delta \tilde{\mathbf{u}} + \nabla \tilde{p} - \tilde{\mathbf{f}}) \circ \boldsymbol{\Phi}(s,r) = T_0(-\Delta \mathbf{u} + \nabla p - \mathbf{f}) \circ \boldsymbol{\Phi}(s,-r) + R = R.$$

Together with Inequality (7.12), this establishes the upper bound (7.7) for $\|-\Delta \tilde{\mathbf{u}} + \nabla \tilde{p} - \tilde{\mathbf{f}}\|_{0, \Omega_h \backslash \Omega}$. Similarly, we get for $r < 0$:

$$(\operatorname{div} \tilde{\mathbf{u}}) \circ \boldsymbol{\Phi}(s,r) = (\operatorname{div} \mathbf{u}) \circ \boldsymbol{\Phi}(s,-r) + \bar{R} = \bar{R}.$$

Here, $\bar{R}$ is of the form $r\bar{\psi}_0+\bar{\psi}_1$ with an H^1-function $\bar{\psi}_0$ and an H^2-function $\bar{\psi}_1$ which can be bounded by

$$\|\bar{\psi}_0\|_{H^1(\omega\times(-\varepsilon_0,\varepsilon_0))}+\|\bar{\psi}_1\|_{H^2(\omega\times(-\varepsilon_0,\varepsilon_0))}\leqq c\{\|\mathbf{f}\|_{0,\Omega}+\|g\|_{3/2,\Gamma}\}.$$

Recalling the imbedding theorem, this establishes the upper bound (7.7) for $\|\operatorname{div}\tilde{\mathbf{u}}\|_{1,\Omega_h\Omega}$. □

Since $\operatorname{meas}(\Omega_h\backslash\Omega)\leqq ch^2$ for a piecewise linear interpolation of the boundary, Inequalities (7.6), (7.7), and Proposition 7.1, together with the arguments used in Sects. 5 and 6 immediately yield:

Corollary 7.2. *Replace in problems* (4.9) *and* (4.10) *the function* $\mathbf{f}$ *by its continuation* $\tilde{\mathbf{f}}$ *defined in Eq.* (7.10). *Let* $\mathbf{u}\in X$, $p\in Y$, $\rho\in Z$ *and* $\mathbf{u}_h\in X_h$, $p_h\in Y_h$, $\rho_h\in Z_h$ *be the unique solution of the Stokes problem* (3.5) *and of its mixed finite element approximation* (4.9), *respectively. Then, the error estimate*

$$\begin{aligned}&\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega_h\cap\Omega}+\|p-p_h\|_{0,\Omega_h\cap\Omega}+\|\rho\circ\pi^{-1}-\rho_h\|_{-1/2,\Gamma_h}\\&\quad+h^{1/2}\|\rho\circ\pi^{-1}-\rho_h\|_{0,\Gamma_h}\\&\quad\leqq ch\{\|\mathbf{f}\|_{0,\Omega}+\|g\|_{3/2,\Gamma}\}\end{aligned}$$

holds for sufficiently small h. The same error estimate is valid for the mixed finite element approximation (4.10) *of branches of nonsingular solutions of the Navier-Stokes equations* (3.6). □

The results of Sect. 5 show that we have to consider higher order approximations of the boundary Γ in order to obtain improved error estimates for the velocity in the L^2-norm. On the other hand we are interested in keeping the finite element spaces as simple as possible. We therefore consider tessalations $\mathscr{T}_n$ of Ω into *curved d-simplices*, such that each $T\in\mathscr{T}_h$ is the image of the reference d-simplex $T^*:=\left\{x\in\mathbb{R}^d\colon \sum_{i=1}^{d}x_i\leqq 1,\ x_i\geqq 0,\ 1\leqq i\leqq d\right\}$ under a mapping F_T, which can be decomposed as $F_T=\hat{F}_T+\Phi_T$ with an invertible affine mapping $\hat{F}_T$ and a sufficiently small polynomial Φ_T of degree $\leqq s$, $s\geqq 2$ (cf. Definition II.1 in [7]). Moreover, we assume that simplices in the interior of Ω are straight, i.e., $\Phi_T=0$ for all $T\in\mathscr{T}_h$ with $T\cap\Gamma=\emptyset$. Put $\Omega_h:=\bigcup_{T\in\mathscr{T}_h}T$. Since the boundary Γ_h of Ω_h is an interpolation of Γ by functions which are piecewise polynomials of degree $\leqq s$, we have

$$\begin{aligned}&\sup_{x\in\Gamma}\inf_{y\in\Gamma_h}|x-y|\leqq ch^{s+1},\qquad \operatorname{meas}(\Omega_h\backslash\Omega)\leqq ch^{s+1},\\&\qquad|\mathbf{n}-\mathbf{n}_h|\leqq ch^s\ ,\qquad|\mathbf{t}_k-\mathbf{t}_{hk}|\leqq ch^s,\quad 1\leqq k\leqq d-1.\end{aligned}\tag{7.13}$$

The finite element spaces are defined as "perturbations" of classical finite element spaces corresponding to families of straight d-simplices "close" to the curved d-simplices. Put $\hat{\mathscr{T}}_h:=\{\hat{F}_T\cdot F_T^{-1}(T)\colon T\in\mathscr{T}_h\}$ and $\hat{\Omega}_h:=\bigcup_{\hat{T}\in\hat{\mathscr{T}}_h}\hat{T}$. $\hat{\Omega}_h$, $h>0$, is a family of polyhedrons with vertices on Γ. Denote by $\hat{\Gamma}_h$ and $\hat{n}_h$ the boundary and unit exterior normal of $\hat{\Omega}_h$. Let $\hat{X}_h$, $\hat{Y}_h$, $\hat{Z}_h$ be finite element spaces as defined in Sect. 4 with Ω_h, Γ_h, $\mathbf{n}_h$, and $\mathcal{T}_h$ replaced by $\hat{\Omega}_h$, $\hat{\Gamma}_h$, $\hat{\mathbf{n}}_h$, and $\hat{\mathcal{T}}_h$ respectively. In particular, we assume that $\hat{X}_h$, $\hat{Y}_h$, $\hat{Z}_h$ satisfy conditions (H1)-(H3) and the inf-sup condition of Proposition 4.3. Define the space X_h by

$$X_h := \{\mathbf{u}_h : \Omega_h \to \mathbb{R}^d : \exists\, \hat{\mathbf{u}}_h \in \hat{X}_h \text{ such that } \mathbf{u}_h|_T = \hat{\mathbf{u}}_h \circ \hat{F}_T \circ F_T^{-1}\ \forall\, T \in \mathcal{T}_h\}.$$

Similarly, the spaces Y_h and Z_h are defined. One easily checks (cf. [21]) that the spaces X_h, Y_h, Z_h satisfy conditions (H1)-(H3) and the inf-sup condition of Proposition 4.3.

Hence, the approximation by curved finite elements fits into the abstract framework for mixed problems. We can therefore apply the techniques of Sects. 5 and 6 to establish the corresponding error estimates. Because of Inequalities (4.11), (5.1), and (7.13), the consistency errors R_1, R_2, R_3 of Eqs. (5.5), (5.8), (7.5) can now be bounded by

$$\begin{aligned} R_1 + R_2 + R_3 &\leqq c\{h^s + h^{(s+1)/2} + h^{2(s+1)/3}\}\{\|\mathbf{f}\|_{0,\Omega} + \|g\|_{3/2,\Gamma}\} \\ &\leqq c h^{(s+1)/2}\{\|\mathbf{f}\|_{0,\Omega} + \|g\|_{3/2,\Gamma}\}. \end{aligned} \tag{7.14}$$

Note, that the factor $h^{(s+1)/2}$ in Inequality (7.14) is due to the term $\int_{\Omega_h} \mathbf{n}_h \cdot \mathfrak{T}(\mathbf{u}, p) \cdot \mathbf{t}_{hk}\, \mathbf{v}_h \cdot \mathbf{t}_{hk}$ in Eq. (5.7). Hence, this factor can only be improved under regularity assumptions which are more restrictive than those of Inequality (3.2).

Using the techniques of Sect. 5 and applying a standard duality argument [3, 16], we conclude from Inequality (7.14) that the error estimate

$$\begin{aligned} h^{-t}\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_h \cap \Omega} &+ \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega_h \cap \Omega} + \|p - p_h\|_{0,\Omega_h \cap \Omega} \\ &+ \|\rho \circ \pi^{-1} - \rho_h\|_{-1/2,\Gamma_h} + h^{1/2}\|\rho \circ \pi^{-1} - \rho_h\|_{0,\Gamma_h} \\ &\leqq c h\{\|\mathbf{f}\|_{0,\Omega} + \|g\|_{3/2,\Gamma}\}, \qquad t = \min\{1, (s-1)/2\}, \end{aligned} \tag{7.15}$$

holds for the mixed finite element approximation of the Stokes problem (1.2). The same error estimate is valid for the mixed finite element approximation of branches of nonsingular solutions of the Navier-Stokes equations (1.1). Since we only assume H^2-regularity for the solution of the Stokes problem (1.2), the error estimate of Inequality (7.15) is optimal for $s \geqq 3$.

Since there are finite element spaces X_h, Y_h of arbitrary polynomial degree which satisfy the stability condition (H1) (cf. [13, 21]), we can always obtain optimal error estimates by choosing appropriately the polynomial degree of the finite element spaces X_h, Y_h, Z_h, and of the approximation of the boundary. The convergence rate of the finite element approximation then only depends on the regularity of the solution of the Stokes problem (1.2).

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Received October 15, 1986