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An AZ-style identity and Bollobás deficiency

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Abstract

The powerful AZ identity is a sharpening of the famous LYM-inequality. More generally, Ahlswede and Zhang discovered a generalization in which the Bollobás inequality for two set families can be lifted to an identity.

In this paper, we show another generalization of the AZ identity. The new identity implies an identity which characterizes the deficiency of the Bollobás inequality for an intersecting Sperner family. We also give some consequences relating to Helly families and LYM-style inequalities.

Keywords: Antichain; LYM; Bollobás inequality; AZ identity

1. Introduction

Let $[n] = \{1, 2, \dots, n\}$, $2^{[n]}$ denote the family of all subsets of $[n]$, and $\emptyset$ be the empty set. If $\emptyset \neq \mathcal{F} \subseteq 2^{[n]}$ and $A \not\subset B$ for all $A, B \in \mathcal{F}$, $A \neq B$, then $\mathcal{F}$ is called a *Sperner family* or an *antichain*. The well-known LYM inequality (Lubell, Yamamoto, Meshalkin) is

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leqslant 1,$$

where $\mathcal{F}$ is an arbitrary antichain.

There are many contributions on application or extension of the LYM inequality [8]. Effort to sharpen or generalize the LYM is one of study directions: for example, adding to left-hand

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side of LYM suitable terms [5,6,11] or proposing an inequality stronger than the LYM (Bollobás inequality for two set systems [7]), unifying inequalities with a common generalization [4]. Recently, Katona has given a revised presentation of the LYM with linear or non-linear profile-vectors [5,15]. Another direction is to extend the LYM inequality to other types of posets, e.g. lattice of divisors [9], partition lattice or chain poset [4,12] in which there are certain adaptations to the original type of LYM. This study direction also gives LYM-style inequalities for set families with specific properties such as intersecting antichains [8].

Specifically, Ahlswede and Zhang [1] discovered an elegant identity (called *AZ identity*) which is a sharpening of the original LYM inequality. This work characterizes deficiency of the LYM, i.e. a non-trivial formula for the deficiency of the LYM inequality. Ahlswede and Cai [2] also found a generalization of the AZ identity: this second identity implies Bollobás inequality for two set systems [7]. The same authors established successfully AZ type identities of several other posets [3].

On the other hand, some important posets do not have inequalities or identities in the same type as the original LYM. Erdös and Székely gave AZ type identities which imply pseudo-LYM inequalities for these posets [12]. Especially, Patkós [16] applied the *AZ identity* to introduce a new proof of the uniqueness case in the theorem of Erdös for k-Sperner families (i.e. families which contain no chain of $k+1$ length). He also proposed an AZ type identity for k-chains in k-Sperner families. Radcliffe and Szaniszló [17] characterize extremal cases of the Ahlswede–Cai identity.

In this paper, we present our work on AZ identities in which two main identities are proposed. This work not only implies original AZ identities [1,2] but also characterizes deficiencies of inequalities such as: LYM inequality, Bollobás inequality for two set systems [7], Bollobás inequality for intersecting antichains [8], Tuza inequality for Helly families [20]. Firstly, we recall some notations and previous theorems which relate to our work.

Let $\mathcal{G}$ be the family of all $\mathcal{F}$ such that $\emptyset \neq \mathcal{F} \subseteq 2^{[n]}$. For every $\mathcal{F} \in \mathcal{G}$ we define the downset

$$\mathcal{D}(\mathcal{F}) = \{D \subset [n]\colon\ D \subset F \text{ for some } F \in \mathcal{F}\},$$

and the upset

$$\mathcal{U}(\mathcal{F}) = \{U \subset [n]\colon\ U \supset F \text{ for some } F \in \mathcal{F}\}.$$

We also write $c\mathcal{F}$ for the set $\{[n]\backslash F\colon\ F \in \mathcal{F}\}$. Then for $X \subseteq [n]$, we put

$$Z_{\mathcal{F}}(X) = \begin{cases} \emptyset & \text{if } X \notin \mathcal{U}(\mathcal{F}), \\ \bigcap_{X \supseteq F \in \mathcal{F}} F & \text{otherwise.} \end{cases}$$

The following important identity is found by Ahlswede and Zhang.

Theorem 1A. *(See Ahlswede–Zhang [1].) If $\emptyset \notin \mathcal{F} \in \mathcal{G}$ then*

$$\sum \frac{|Z_{\mathcal{F}}(X)|}{|X|\binom{n}{|X|}} = 1, \tag{AZ1}$$

where summation is over $X \in \mathcal{U}(\mathcal{F})$.

If $\mathcal{F}$ is an antichain, we have $Z_{\mathcal{F}}(F) = F$ for all $F \in \mathcal{F}$, then (AZ1) can be rewritten as follows

$$\sum_{F\in\mathcal{F}}\frac{1}{\binom{n}{|F|}}+\sum_{X\in\mathcal{U}(\mathcal{F})\setminus\mathcal{F}}\frac{|Z_{\mathcal{F}}(X)|}{|X|\binom{n}{|X|}}=1.$$

The second term of the left-hand side of the above identity is always non-negative, so the identity implies the famous LYM-inequality for any antichain $\mathcal{F}$:

$$\sum_{F\in\mathcal{F}}\frac{1}{\binom{n}{|F|}}\leqslant 1.$$

More generally, Ahlswede and Cai found the second identity for two set systems as follows.

Theorem 1B. *(See Ahlswede–Cai [2].) Let $\mathcal{A}=\{A_1,A_2,\dots,A_q\}\in\mathcal{G}$ and $\mathcal{B}=\{B_1,B_2,\dots,B_q\}\in\mathcal{G}$ such that $A_i\subset B_j$ if and only if $i=j$. Then*

$$\sum_{i=1}^{q}\frac{1}{\binom{n-|B_i|+|A_i|}{|A_i|}}+\sum_{X\in\mathcal{U}(\mathcal{A})\setminus\mathcal{D}(\mathcal{B})}\frac{|Z_{\mathcal{A}}(X)|}{|X|\binom{n}{|X|}}=1. \tag{AZ2}$$

For each antichain $\mathcal{A}$, (AZ2) *implies* (AZ1) *when setting $B_i=A_i$. Moreover, the second term of* (AZ2) *is non-negative, so* (AZ2) *is a generalization of the Bollobás inequality* [7]:

$$\sum_{i=1}^{q}\frac{1}{\binom{n-|B_i|+|A_i|}{|A_i|}}\leqslant 1.$$

Another well-known inequality of Bollobás relates to intersecting Sperner family.

Theorem 2. *(See Bollobás [8].) Let $\mathcal{A}=\{A_1,A_2,\dots,A_q\}$ be an intersecting antichain of subsets of $[n]$ such that $|A_i|\leqslant\frac{n}{2}$ for each $1\leqslant i\leqslant q$. Then*

$$\sum_{i=1}^{q}\frac{1}{\binom{n-1}{|A_i|-1}}\leqslant 1. \tag{BB}$$

The inequality (BB) *implies*

$$\sum_{i=1}^{q}\frac{1}{\binom{n-1}{|A_i|-1}}+\Delta(\mathrm{BB})=1,$$

in which Δ(BB) *is non-negative. We call Δ*(BB) *the Bollobás deficiency in this paper.*

In Section 2 of the paper, we present our main identity that can be considered as a new generalization of (AZ1) but different from (AZ2). In fact, this identity is parameterized by an integer m. Section 3 gives some consequences of the parameterized identity when m takes certain values. Then we introduce in Section 4 another parameterized identity which implies (AZ2). Finally, comments and dual identities are presented in Section 5.

2. Main theorem

Our main identity is given in the following theorem.

Theorem 3. *Let m be an integer, $\emptyset \notin \mathcal{A} \in \mathcal{G}$. If $|A| + m > 0$ for each $A \in \mathcal{A}$, then*

$$\sum_{X \in \mathcal{U}(\mathcal{A})} \frac{|Z_{\mathcal{A}}(X)| + m}{(|X| + m)\binom{n+m}{|X|+m}} = 1. \tag{T1}$$

We found (T1) when considering an elementary identity that is a generalization of the case $\mathcal{A} = \{A\}$ in the (AZ1) identity. Now we show an induction proof of (T1) in the style of [18,19].

Lemma 1. *Let a, b, c be integers such that $a \geqslant 0$, $b > 0$, $c \geqslant a + b$. Then*

$$\sum_{k=0}^{a} \binom{a}{k} \frac{1}{(b+k)\binom{c}{b+k}} = \frac{1}{b\binom{c-a}{b}}. \tag{1}$$

Proof. Let $F(a, b, c)$ be the left-hand side of (1). If $a \geqslant 1$ by using

$$\binom{a}{k} = \binom{a-1}{k} + \binom{a-1}{k-1},$$

then arrange the terms of sum, we can rewrite $F(a, b, c)$ as follows

$$\sum_{k=0}^{a-1} \binom{a-1}{k} \frac{1}{(b+k)\binom{c}{b+k}} + \sum_{k=0}^{a-1} \binom{a-1}{k} \frac{1}{(b+k+1)\binom{c}{b+k+1}}$$

which is

$$F(a-1, b, c) + \sum_{k=0}^{a-1} \binom{a-1}{k} \frac{1}{(b+k+1)\binom{c}{b+k+1}}.$$

Then we use

$$\frac{1}{(b+k+1)\binom{c}{b+k+1}} = \frac{1}{(b+k)\binom{c-1}{b+k}} - \frac{1}{(b+k)\binom{c}{b+k}}$$

to obtain

$$F(a, b, c) = F(a-1, b, c) + F(a-1, b, c-1) - F(a-1, b, c),$$
$$F(a, b, c) = F(a-1, b, c-1). \tag{2}$$

Finally, we apply (2) many times to get

$$F(a, b, c) = F(0, b, c-a) = \frac{1}{b\binom{c-a}{b}}. \qquad \square$$

Lemma 2. *(See [10,18].) Let $\emptyset \notin \mathcal{A} \in \mathcal{G}$, $\emptyset \notin \mathcal{B} \in \mathcal{G}$ and put*

$$\mathcal{A} \vee \mathcal{B} = \{A \cup B\colon\ A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}.$$

Then for each $\emptyset \neq X \subset [n]$ we have

$$\left|Z_{\mathcal{A} \cup \mathcal{B}}(X)\right| = \left|Z_{\mathcal{A}}(X)\right| + \left|Z_{\mathcal{B}}(X)\right| - \left|Z_{\mathcal{A} \vee \mathcal{B}}(X)\right|.$$

The proof of Lemma 2 is given in detail in [10,18].

Induction proof of Theorem 3. Put

$$W(X) = \big(|X| + m\big)\binom{n+m}{|X|+m}.$$

Case 1. $|\mathcal{A}| = 1$ so $\mathcal{A} = \{A\}$, $A \neq \emptyset$, $|A| + m > 0$. For each $[n] \supseteq X \supseteq A$ we have $Z_{\mathcal{A}}(X) = A$. Put $q = |A| > 0$, then the left-hand side of (T1) is

$$\mathrm{LHS(T1)} = \sum_{k=0}^{n-q} \binom{n-q}{k} \frac{q+m}{(q+k+m)\binom{n+m}{q+k+m}} = b\sum_{k=0}^{a}\binom{a}{k}\frac{1}{(b+k)\binom{c}{b+k}},$$

in which $a = n - q \geqslant 0$, $b = q + m = |A| + m > 0$, $c = n + m = a + b$.

By Lemma 1, we have

$$\mathrm{LHS(T1)} = b\frac{1}{b\binom{c-a}{b}} = \frac{1}{\binom{b}{b}} = 1.$$

Case 2. We assume (T1) holds for $1 \leqslant |\mathcal{A}| < h$ and consider the case $\mathcal{A} = \{A_1, A_2, \dots, A_h\}$ with $h > 1$ and $|A_i| + m > 0$ for each $1 \leqslant i \leqslant h$. Let $\mathcal{B} = \{A_1, \dots, A_{h-1}\}$ and $\mathcal{D} = \{A_h\}$. We have

$$\mathcal{U}(\mathcal{A}) = \mathcal{U}(\mathcal{B}) \cup \mathcal{U}(\mathcal{D}), \qquad \mathcal{U}(\mathcal{B} \vee \mathcal{D}) = \mathcal{U}(\mathcal{B}) \cap \mathcal{U}(\mathcal{D}).$$

If $X \in \mathcal{U}(\mathcal{A})$ then $Z_{\mathcal{A}}(X)$ can be evaluated as follows:

$$\begin{aligned}
&X \in \mathcal{U}(\mathcal{B}) \quad \text{and} \quad X \notin \mathcal{U}(\mathcal{D}): \quad Z_{\mathcal{A}}(X) = Z_{\mathcal{B}}(X),\\
&X \notin \mathcal{U}(\mathcal{B}) \quad \text{and} \quad X \in \mathcal{U}(\mathcal{D}): \quad Z_{\mathcal{A}}(X) = Z_{\mathcal{D}}(X),\\
&X \in \mathcal{U}(\mathcal{B}) \quad \text{and} \quad X \in \mathcal{U}(\mathcal{D}) \quad \big(\text{i.e. } X \in \mathcal{U}(\mathcal{B} \vee \mathcal{D})\big) \quad \text{by Lemma 2:}\\
&\big|Z_{\mathcal{A}}(X)\big| = \big|Z_{\mathcal{B}}(X)\big| + \big|Z_{\mathcal{D}}(X)\big| - \big|Z_{\mathcal{B}\vee\mathcal{D}}(X)\big|.
\end{aligned}$$

Hence the LHS of (T1) is

$$\sum_{\substack{X \in \mathcal{U}(\mathcal{B})\\ X \notin \mathcal{U}(\mathcal{D})}} \frac{|Z_{\mathcal{B}}(X)| + m}{W(X)} + \sum_{\substack{X \notin \mathcal{U}(\mathcal{B})\\ X \in \mathcal{U}(\mathcal{D})}} \frac{|Z_{\mathcal{D}}(X)| + m}{W(X)} + \sum_{\substack{X \in \mathcal{U}(\mathcal{B})\\ X \in \mathcal{U}(\mathcal{D})}} \frac{|Z_{\mathcal{B}}(X)| + |Z_{\mathcal{D}}(X)| - |Z_{\mathcal{B}\vee\mathcal{D}}(X)| + m}{W(X)}.$$

By writing $|Z_{\mathcal{B}}(X)| + |Z_{\mathcal{D}}(X)| - |Z_{\mathcal{B}\vee\mathcal{D}}(X)| + m$ as

$$\big(\big|Z_{\mathcal{B}}(X)\big| + m\big) + \big(\big|Z_{\mathcal{D}}(X)\big| + m\big) - \big(\big|Z_{\mathcal{B}\vee\mathcal{D}}(X)\big| + m\big).$$

Then arranging terms of the sums, we can obtain

$$\sum_{X \in \mathcal{U}(\mathcal{B})} \frac{|Z_{\mathcal{B}}(X)| + m}{W(X)} + \sum_{X \in \mathcal{U}(\mathcal{D})} \frac{|Z_{\mathcal{D}}(X)| + m}{W(X)} - \sum_{X \in \mathcal{U}(\mathcal{B}\vee\mathcal{D})} \frac{|Z_{\mathcal{B}\vee\mathcal{D}}(X)| + m}{W(X)}.$$

Now $\mathcal{B}, \mathcal{D}, \mathcal{B} \vee \mathcal{D}$ satisfy the condition $|A| + m > 0$ for each set A belonging to them, and have cardinalities less than h. We apply induction hypothesis to have $\mathrm{LHS(T1)} = 1 + 1 - 1 = 1$. □

3. Some consequences

3.1. Special cases

Corollary 1. *For each $\emptyset \notin \mathcal{A} \in \mathcal{G}$, the AZ identity*

$$\sum_{X \in \mathcal{U}(\mathcal{A})} \frac{|Z_{\mathcal{A}}(X)|}{|X|\binom{n}{|X|}} = 1$$

holds.

Proof. Because of $\emptyset \notin \mathcal{A}$, we have $|A| + 0 > 0$ for each $A \in \mathcal{A}$. Then put $m = 0$ in the identity (T1), the AZ identity obviously holds. □

Corollary 2. *Let $\mathcal{A} \in \mathcal{G}$ be an antichain and $|A| > 1$ for each $A \in \mathcal{A}$. Then*

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n-1}{|A|-1}} + \sum_{X \in \mathcal{U}(\mathcal{A}) \setminus \mathcal{A}} \frac{|Z_{\mathcal{A}}(X)| - 1}{(|X| - 1)\binom{n-1}{|X|-1}} = 1. \tag{T2}$$

Proof. Because $|A| - 1 > 0$ for each $A \in \mathcal{A}$, we can apply the identity (T1) with $m = -1$ to obtain

$$\sum_{X \in \mathcal{U}(\mathcal{A})} \frac{|Z_{\mathcal{A}}(X)| - 1}{(|X| - 1)\binom{n-1}{|X|-1}} = 1.$$

If $\mathcal{A}$ is an antichain, $Z_{\mathcal{A}}(A) = A$ for each $A \in \mathcal{A}$, then the identity (T2) holds. □

Corollary 3. *Let $\mathcal{A} \in \mathcal{G}$ be an antichain and a star. Then*

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n-1}{|A|-1}} \leqslant 1.$$

Proof. Because $\mathcal{A}$ is a star, we can assume $1 \in A$ for each $A \in \mathcal{A}$. So $1 \in Z_{\mathcal{A}}(X)$ and $|Z_{\mathcal{A}}(X)| \geqslant 1$ whenever $X \in \mathcal{U}(\mathcal{A})$. If there is some $A \in \mathcal{A}$ such that $|A| = 1$, then $A = \{1\}$ (since $1 \in A$) and antichain $\mathcal{A}$ must be $\{A\}$: the corollary obviously holds. Otherwise, $|A| > 1$ for each $A \in \mathcal{A}$, using (T2) to get the inequality of Corollary 3. □

3.2. Formula for deficiency of two inequalities

If $\mathcal{A} \subseteq 2^{[n]}$ is an intersecting antichain family and $|A| \leqslant \frac{n}{2}$, then by using Bollobás inequality (Theorem 2) and Corollary 2, we obtain the following formula for the Bollobás deficiency,

$$\Delta(\mathrm{BB}) = \sum_{X \in \mathcal{U}(\mathcal{A}) \setminus \mathcal{A}} \frac{|Z_{\mathcal{A}}(X)| - 1}{(|X| - 1)\binom{n-1}{|X|-1}} \geqslant 0.$$

Now we recall a result of Tuza for Helly families.

Theorem 4. *(See Tuza [20].) Let $\mathcal{A}$ be an H_k-family which is also an antichain. If $|A| \geqslant k + 1$ for each $A \in \mathcal{A}$, then*

$$\sum_{A\in\mathcal{A}}\frac{1}{\binom{n-1}{|A|-1}}\leqslant 1.$$

We can see easily that the deficiency of Tuza inequality is also $\Delta(\mathrm{BB})$.

3.3. An inequality for upsets

Lemma 3. *Let* $\emptyset\neq\mathcal{A}\in\mathcal{G}$. *Then*

(a) $$\sum_{X\in\mathcal{U}(\mathcal{A})}\frac{1}{\binom{n}{|X|}}+\sum_{X\in\mathcal{U}(\mathcal{A})}\frac{|Z_{\mathcal{A}}(X)|}{\binom{n}{|X|}}=n+1,$$

(b) $$\left|\mathcal{U}(\mathcal{A})\right|+\sum_{X\in\mathcal{U}(\mathcal{A})}\left|Z_{\mathcal{A}}(X)\right|\leqslant(n+1)\binom{n}{\lfloor\frac{n}{2}\rfloor}.$$

Proof. We have obviously $|A|+1>0$ for each $A\in\mathcal{A}$. So using (T1) with $m=1$ and then replacing $(|X|+1)\binom{n+1}{|X|+1}$ by $(n+1)\binom{n}{|X|}$ to obtain the identity (a). To prove (b) we use (a) and apply the inequality $\binom{n}{k}\leqslant\binom{n}{\lfloor\frac{n}{2}\rfloor}$. □

4. Identity for two set systems

The following theorem shows a natural extension of (T1) which is also a generalization of (AZ2).

Theorem 5. *Let* $\mathcal{A}=\{A_1,\dots,A_q\}\in\mathcal{G}$ *and* $\mathcal{B}=\{B_1,\dots,B_q\}\in\mathcal{G}$ *such that* $A_i\subset B_j$ *if and only if* $i=j$, *and an integer* m, $m+|A|>0$ *for every* $A\in\mathcal{A}$. *Then*

$$\sum_{i=1}^{q}\frac{1}{\binom{m+n-|B_i|+|A_i|}{m+|A_i|}}+\sum_{X\in\mathcal{U}(\mathcal{A})\backslash\mathcal{D}(\mathcal{B})}\frac{m+|Z_{\mathcal{A}}(X)|}{(m+|X|)\binom{m+n}{m+|X|}}=1.$$

Theorem 5 is proved in using Theorem 3 (the main identity, Section 2) and the following lemma.

Lemma 4. *Let* $\emptyset\notin\mathcal{A}\in\mathcal{G}$ *and* $\emptyset\notin\mathcal{B}\in\mathcal{G}$, $\mathcal{A}=\mathcal{A}_1\cup\mathcal{A}_2$, $\mathcal{B}=\mathcal{B}_1\cup\mathcal{B}_2$ *such that* $\emptyset\notin\mathcal{A}_1,\mathcal{A}_2,\mathcal{B}_1,\mathcal{B}_2\in\mathcal{G}$ *and* $\mathcal{U}(\mathcal{A}_1)\cap\mathcal{D}(\mathcal{B}_2)=\emptyset=\mathcal{U}(\mathcal{A}_2)\cap\mathcal{D}(\mathcal{B}_1)$. *If* F *is a non-zero function defined on* $\mathcal{U}(\mathcal{A})$, *then*

$$\sum_{X\in\mathcal{U}(\mathcal{A})\backslash\mathcal{D}(\mathcal{B})}\frac{m+|Z_{\mathcal{A}}(X)|}{F(X)}=\sum_{X\in\mathcal{U}(\mathcal{A}_1)\backslash\mathcal{D}(\mathcal{B}_1)}\frac{m+|Z_{\mathcal{A}_1}(X)|}{F(X)}+\sum_{X\in\mathcal{U}(\mathcal{A}_2)\backslash\mathcal{D}(\mathcal{B}_2)}\frac{m+|Z_{\mathcal{A}_2}(X)|}{F(X)}-\sum_{X\in\mathcal{U}(\mathcal{A}_1\vee\mathcal{A}_2)}\frac{m+|Z_{\mathcal{A}_1\vee\mathcal{A}_2}(X)|}{F(X)}.\tag{4}$$

Proof. If $X\in\mathcal{U}(\mathcal{A})\backslash\mathcal{D}(\mathcal{B})$, i.e. $X\in\mathcal{U}(\mathcal{A})$ and $X\notin\mathcal{D}(\mathcal{B})$, then we can evaluate $Z_{\mathcal{A}}(X)$ in three cases:

- $X\notin\mathcal{U}(\mathcal{A}_1)$ which implies $X\in\mathcal{U}(\mathcal{A}_2)$: $Z_{\mathcal{A}}(X)=Z_{\mathcal{A}_2}(X)$,

- $X \in \mathcal{U}(\mathcal{A}_1)$ and $X \notin \mathcal{U}(\mathcal{A}_2)$: $Z_{\mathcal{A}}(X) = Z_{\mathcal{A}_1}(X)$,
- $X \in \mathcal{U}(\mathcal{A}_1)$ and $X \in \mathcal{U}(\mathcal{A}_2)$, by Lemma 2, $m + |Z_{\mathcal{A}}(X)|$ is

$$\bigl(m + \bigl|Z_{\mathcal{A}_1}(X)\bigr|\bigr) + \bigl(m + \bigl|Z_{\mathcal{A}_2}(X)\bigr|\bigr) - \bigl(m + \bigl|Z_{\mathcal{A}_1 \vee \mathcal{A}_2}(X)\bigr|\bigr).$$

Hence the left-hand side of (4) is

$$\begin{aligned}
&\sum_{\substack{X \notin \mathcal{U}(\mathcal{A}_1), X \in \mathcal{U}(\mathcal{A}_2) \\ X \notin \mathcal{D}(\mathcal{B})}} \frac{m + |Z_{\mathcal{A}_2}(X)|}{F(X)} + \sum_{\substack{X \in \mathcal{U}(\mathcal{A}_1), X \notin \mathcal{U}(\mathcal{A}_2) \\ X \notin \mathcal{D}(\mathcal{B})}} \frac{m + |Z_{\mathcal{A}_1}(X)|}{F(X)} \\
&\quad + \sum_{\substack{X \in \mathcal{U}(\mathcal{A}_1), X \in \mathcal{U}(\mathcal{A}_2) \\ X \notin \mathcal{D}(\mathcal{B})}} \frac{(m + |Z_{\mathcal{A}_1}(X)|) + (m + |Z_{\mathcal{A}_2}(X)|) - (m + |Z_{\mathcal{A}_1 \vee \mathcal{A}_2}(X)|)}{F(X)} \\
&= \sum_{\substack{X \in \mathcal{U}(\mathcal{A}_2) \\ X \notin \mathcal{D}(\mathcal{B})}} \frac{m + |Z_{\mathcal{A}_2}(X)|}{F(X)} + \sum_{\substack{X \in \mathcal{U}(\mathcal{A}_1) \\ X \notin \mathcal{D}(\mathcal{B})}} \frac{m + |Z_{\mathcal{A}_1}(X)|}{F(X)} \\
&\quad - \sum_{\substack{X \in \mathcal{U}(\mathcal{A}_1), X \in \mathcal{U}(\mathcal{A}_2) \\ X \notin \mathcal{D}(\mathcal{B})}} \frac{m + |Z_{\mathcal{A}_1 \vee \mathcal{A}_2}(X)|}{F(X)}.
\end{aligned}$$

Now from the hypothesis on $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_1$, $\mathcal{B}_2$ of Lemma 4, we have:

- $X \in \mathcal{U}(\mathcal{A}_2)$, $X \notin \mathcal{D}(\mathcal{B}) \Leftrightarrow X \in \mathcal{U}(\mathcal{A}_2)$, $X \notin \mathcal{D}(\mathcal{B}_2)$,
- $X \in \mathcal{U}(\mathcal{A}_1)$, $X \notin \mathcal{D}(\mathcal{B}) \Leftrightarrow X \in \mathcal{U}(\mathcal{A}_1)$, $X \notin \mathcal{D}(\mathcal{B}_1)$,
- $X \in \mathcal{U}(\mathcal{A}_1)$, $X \in \mathcal{U}(\mathcal{A}_2)$, $X \notin \mathcal{D}(\mathcal{B}) \Leftrightarrow X \in \mathcal{U}(\mathcal{A}_1)$, $X \in \mathcal{U}(\mathcal{A}_2) \Leftrightarrow X \in \mathcal{U}(\mathcal{A}_1 \vee \mathcal{A}_2)$.

Then we continue to evaluate the left-hand side of (4) to obtain its right-hand side. □

Induction proof of Theorem 5. Put $W_m(X) = (m + |X|)\binom{m+n}{m+|X|}$.

Case 1. $q = 1$ so $\mathcal{A} = \{A\}$, $\mathcal{B} = \{B\}$, $\emptyset \neq A \subset B$, $m + |A| > 0$. For every $X \in \mathcal{U}(\mathcal{A})$: $Z_{\mathcal{A}}(X) = A$ and $W_m(X) \neq 0$. We apply Theorem 3 to obtain $\sum_{X \in \mathcal{U}(\mathcal{A})} \frac{m+|Z_{\mathcal{A}}(X)|}{W_m(X)} = 1$ which is equivalent with

$$\sum_{X \in \mathcal{U}(\mathcal{A}) \setminus \mathcal{D}(\mathcal{B})} \frac{m + |Z_{\mathcal{A}}(X)|}{W_m(X)} + \sum_{X \in \mathcal{U}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})} \frac{m + |Z_{\mathcal{A}}(X)|}{W_m(X)} = 1.$$

The second term of LHS is $\sum_{A \subset X \subset B} \frac{m+|A|}{W_m(X)}$ which can be written as

$$\Delta = \sum_{k=0}^{|B|-|A|} \binom{|B| - |A|}{k} \frac{m + |A|}{(m + |A| + k)\binom{m+n}{m+|A|+k}}.$$

Then using Lemma 1 to get

$$\Delta = \bigl(m + |A|\bigr) \frac{1}{(m + |A|)\binom{m+n-(|B|-|A|)}{m+|A|}} = \frac{1}{\binom{m+n-|B|+|A|}{m+|A|}}.$$

Hence the identity of Theorem 5 holds in this case.

Case 2. We assume the identity holds for $1 \leqslant q < h$ and consider the case $\mathcal{A} = \{A_1, A_2, \dots, A_h\}$, $\mathcal{B} = \{B_1, B_2, \dots, B_h\}$ with $h > 1$ and $m + |A_i| > 0$ for each $1 \leqslant i \leqslant h$. Let

$$\mathcal{A}_1 = \{A_1, A_2, \dots, A_{h-1}\}, \qquad \mathcal{A}_2 = \{A_h\},$$
$$\mathcal{B}_1 = \{B_1, B_2, \dots, B_{h-1}\}, \qquad \mathcal{B}_2 = \{B_h\}.$$

Because of $A_i \not\subset B_h$ and $A_h \not\subset B_i$ for each $1 \leqslant i < h$, we have

$$\mathcal{U}(\mathcal{A}_1) \cap \mathcal{D}(\mathcal{B}_2) = \emptyset = \mathcal{U}(\mathcal{A}_2) \cap \mathcal{D}(\mathcal{B}_1),$$

then by using Lemma 4 the sum

$$\sum_{X \in \mathcal{U}(\mathcal{A}) \setminus \mathcal{D}(\mathcal{B})} \frac{m + |Z_{\mathcal{A}}(X)|}{W_m(X)}$$

can be written as

$$\sum_{X \in \mathcal{U}(\mathcal{A}_1) \setminus \mathcal{D}(\mathcal{B}_1)} \frac{m + |Z_{\mathcal{A}_1}(X)|}{W_m(X)} + \sum_{X \in \mathcal{U}(\mathcal{A}_2) \setminus \mathcal{D}(\mathcal{B}_2)} \frac{m + |Z_{\mathcal{A}_2}(X)|}{W_m(X)}$$
$$- \sum_{X \in \mathcal{U}(\mathcal{A}_1 \vee \mathcal{A}_2)} \frac{m + |Z_{\mathcal{A}_1 \vee \mathcal{A}_2}(X)|}{W_m(X)}.$$

The third term of the above is 1 due to Theorem 3, applying induction hypothesis for $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_1$, $\mathcal{B}_2$, the sum

$$\sum_{X \in \mathcal{U}(\mathcal{A}) \setminus \mathcal{D}(\mathcal{B})} \frac{m + |Z_{\mathcal{A}}(X)|}{W_m(X)}$$

now is

$$\left(1 - \sum_{i=1}^{h-1} \frac{1}{\binom{m+n-|B_i|+|A_i|}{m+|A_i|}}\right) + \left(1 - \frac{1}{\binom{m+n-|B_h|+|A_h|}{m+|A_h|}}\right) - 1 = 1 - \sum_{i=1}^{h} \frac{1}{\binom{m+n-|B_i|+|A_i|}{m+|A_i|}}.$$

So the identity holds for the case of $q = h$. $\square$

5. Notes

In the preprint version of [14], authors of the paper [14] presented an inequality for *t-intersecting antichains* (with certain hypotheses and positive integer t) of the form

$$\sum_{A \in \mathcal{F}} \frac{1}{\binom{n-t}{|A|-t}} \leqslant 1.$$

By using Theorem 3 with $m = -t$ and $m + |A| > 0$ for all $A \in \mathcal{F}$, we can get a formula for the deficiency of the identity of these authors.

In the same way as in [10], we can obtain the dual identity of (T1) which can be presented in the following theorem.

Theorem 3D. *Let $\mathcal{A} \in \mathcal{G}$ and q be a positive integer. If $|A| < q$ for each $A \in \mathcal{A}$, then*

$$\sum_{X \in \mathcal{D}(\mathcal{A})} \frac{q - |Z^*_{\mathcal{A}}(X)|}{(q - |X|)\binom{q}{|X|}} = 1,$$

where the Z^* is

$$Z^*_{\mathcal{A}}(X) = \begin{cases} [n] & \text{if } X \notin \mathcal{D}(\mathcal{A}), \\ \bigcup_{X \subseteq A \in \mathcal{A}} A & \text{otherwise.} \end{cases}$$

We notice that the above dual identity does not contain a term depending on n. From this identity, we can get another formula of the Bollobás deficiency (also the deficiency of Tuza identity in Theorem 4) as follows (put $q = n-1$, $\mathcal{B} = c\mathcal{A}$):

$$\Delta(\mathrm{BB}) = \sum_{X \in \mathcal{D}(\mathcal{B}) \setminus \mathcal{B}} \frac{n-1-|Z^*_{\mathcal{B}}(X)|}{(n-1-|X|)\binom{n-1}{|X|}}.$$

Similarly, we learn from Hilton–Stirling [13] to establish the dual identity of the one in Theorem 5, which is described in Theorem 5D as follows.

Theorem 5D. *Let $\mathcal{A} = \{A_1, \dots, A_q\} \in \mathcal{G}$ and $\mathcal{B} = \{B_1, \dots, B_q\} \in \mathcal{G}$ such that $A_i \subset B_j$ if and only if $i = j$, and m be an integer, $m - |B_i| > 0$ for each $1 \leqslant i \leqslant q$. Then*

$$\sum_{i=1}^{q} \frac{1}{\binom{m-|B_i|+|A_i|}{|A_i|}} + \sum_{X \in \mathcal{D}(\mathcal{B}) \setminus \mathcal{U}(\mathcal{A})} \frac{m-|Z^*_{\mathcal{B}}(X)|}{(m-|X|)\binom{m}{|X|}} = 1.$$

This dual identity does not contain a term depending on n, either.

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