Elliptic Problems in Nonsmooth Domains

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Philadelphia

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This SIAM edition is a republication of the work first published by Pitman Publishing Inc., Marshfield, Massachusetts, in 1985.

10 9 8 7 6 5 4 3 2 1

Library of Congress Cataloging-in-Publication Data

Grisvard, P. (Pierre)
Elliptic problems in nonsmooth domains / Pierre Grisvard. -- SIAM ed.
p. cm. -- (Classics in applied mathematics ; 69)
Originally published: Boston : Pitman Advanced Pub. Program, 1985.
Includes bibliographical references and index.
ISBN 978-1-611972-02-3 (pbk.)
1. Boundary value problems--Numerical solutions. 2. Differential equations, Elliptic--Numerical solutions. I. Title.
QA379.G74 2011
515'.782--dc23
2011028564

1985:
To Catherine, Olivier,
Béatrice, and Etienne

2011:
To the memory of our beloved father,
from Béatrice, Etienne, & Olivier Grisvard

Contents

Foreword

Since the publication of Pierre Grisvard's monograph in 1985, the theory of elliptic problems in nonsmooth domains has become increasingly important for research in partial differential equations and their numerical solutions. While significant advances have occurred during the last two decades, Grisvard's monograph remains an excellent introduction to the subject and a good source for the basic material.

I had the good fortune to obtain one of the last available copies of this monograph in the early 1990s just before it went out of print, and it has become a regular reference in my work ever since. I am, therefore, delighted to be able to play a role in bringing this monograph back into print in the SIAM Classics in Applied Mathematics series. I hope a new generation of mathematicians will also find it useful.

Susanne C. Brenner
Louisiana State University
June 2011

Preface

In this book, we focus our attention on elliptic boundary value problems in domains with nonsmooth boundaries and problems with mixed boundary conditions. So far this topic has been mainly ignored. Indeed most of the available mathematical theories about elliptic boundary value problems deal with domains with very smooth boundaries; few of them deal with mixed boundary conditions. However, the majority of the elliptic boundary value problems which arise in practice are naturally posed in domains whose geometry is simple but not smooth. These domains are very often three-dimensional polyhedra. For the purpose of solving them numerically these problems are usually reduced to two-dimensional domains. Thus the domains are plane polygons and the boundary conditions are mixed. Accordingly this book is primarily intended for mathematicians working in the field of elliptic partial differential equations as well as for numerical analysts and users of such elliptic equations.

Perhaps the main feature of elliptic boundary value problems in a domain with smooth boundary is the so-called 'shift theorem'. Let us describe it on the simplest example, the Dirichlet problem for the Laplace equation. This will be our model problem throughout this introduction. Accordingly we consider a function u which is a solution of the equation

$$\Delta u = f \tag{1}$$

in a bounded open subset Ω of the two-dimensional Euclidean space R^2. Here the function f is given and we assume that u coincides with some smooth given function g on the boundary Γ of Ω. The shift theorem can be phrased in the framework of either the Sobolev spaces or the Hölder spaces. Here, for simplicity, we describe only the Sobolev version.

We denote by $W_p^m(\Omega)$ the space of those functions defined in Ω whose derivatives up to the order m have their pth power integrable in Ω. We assume that p is strictly greater than 1 and is finite. For the time being, we also assume that the boundary of Ω is smooth, i.e. is a C^∞ manifold. Then when f is given in $W_p^m(\Omega)$, the corresponding solution u of the problem (1) belongs to $W_p^{m+2}(\Omega)$. In other words the order of the Sobolev space is shifted from m to $m+2$, by the inverse operator of Δ.

The particular case when $p=2$ has a simpler proof and is usually the only one needed by numerical analysts. However, the general case when p is allowed to differ from 2 (especially p large) is useful when one studies nonlinear boundary value problems by some kind of linearization or fixed point method. Most of the current error estimates for the numerical solution of an elliptic boundary value problem rely on this shift theorem. Therefore it is particularly important to know whether or not the same result holds for boundary value problems in a domain with a nonsmooth boundary.

From now on let us assume that Ω has one corner. For convenience we assume that this corner is at the origin of $\mathbb{R}^2$ and that, in some neighbourhood of the corner, Ω coincides with the sector

$$G=\{(r\cos\theta,\ r\sin\theta);\ r>0, 0<\theta<\omega\}$$

in the usual polar coordinates, where ω is the size of the angle at the origin. Otherwise we assume that Γ is smooth. For each positive integer k, we define a function u_k in the following way:

$$u_k=r^{k\pi/\omega}\sin(k\pi\theta/\omega)$$

when $k\pi/\omega$ is not an integer and

$$u_k=r^{k\pi/\omega}\{\ln r\sin(k\pi\theta/\omega)+\theta\cos(k\pi\theta/\omega)\}$$

when $k\pi/\omega$ is an integer. It is readily seen that u_k is harmonic in Ω (thus $f_k=\Delta u_k=0$) and that u_k coincides with a smooth function g_k on Γ. Indeed u_k vanishes on Γ near the origin when $k\pi/\omega$ is not an integer, while it vanishes on one side of G (for $\theta=0$) and coincides with the polynomial $(-1)^k\omega r^{k\pi/\omega}$ on the other side of G (for $\theta=\omega$) when $k\pi/\omega$ is an integer. Consequently if the shift theorem were valid on Ω, u_k ought to belong to the intersection of all the Sobolev spaces on Ω. This would imply that u_k has all its derivatives of all orders continuous in the closure of Ω by the well-known Sobolev imbedding theorem. However, it is easy to check from the explicit formula above for u_k, that u_k is l times continuously differentiable if and only if l is strictly smaller that $k\pi/\omega$. A little extra work shows that u_k belongs to the Sobolev space $W_p^l(\Omega)$ if and only if its Sobolev exponent $l-2/p$ is strictly smaller than $k\pi/\omega$, again.

So much for the shift theorem when Ω has a corner. Surprisingly enough, the functions u_k are all we need to formulate an alternative statement. Indeed, when f is given in $W_p^m(\Omega)$, the corresponding solution u of the problem (1) has the following property: there exist numbers c_k such that

$$u-\sum c_k u_k\in W_p^{m+2}(\Omega)$$

where the k in the summation ranges over all integers such that

$$\pi/\omega \leq k\pi/\omega < m+2-2/p,$$

provided the Sobolev exponent $m+2-2/p$ is not an integer itself. The limitation on k in the summation means that we exclude the u_k which belong to the space $W_p^{m+2}(\Omega)$. This result demonstrates that the solution has the usual regularity far from the corner while it describes accurately the behaviour near the corner of that part of the solution which does not belong to the required space.

The terms in the expansion of u above coincide with the terms in the formal power series derived by Lehman (1959).

The above modified version of the shift theorem does not express a regularity result in the whole of Ω. Thus the following question remains open: under which assumptions of f does the solution u belong to $W_p^{m+2}(\Omega)$? In other words when do the coefficients c_k vanish? These are continuous linear functionals of the data f and g. It turns out that they are local functionals if and only if $k\pi/\omega$ is an integer. This means that they only depend on the restriction of the data f and g to any neighbourhood of the corner. For instance we have

$$c_1 = f(0,0)/\pi$$

when $\omega = \pi/2$. On the other hand when $k\pi/\omega$ is not an integer the functional c_k is global; this means that c_k may not vanish even when the data f and g are zero near the corner. As a consequence the functional c_k depends on the geometry of Ω far from the corner and it is not possible to make it explicit in such a general case.

Deriving similar modified shift theorems for various boundary value problems is what this book is about. Let us now proceed with a detailed description of the various chapters.

Chapter 1

The properties of the Sobolev spaces have been thoroughly investigated even when they are defined on very rough domains. We review the only properties we need without proofs and rely on the well-known book by Nečas (1967) for the proofs and references. In dealing with boundary value problems, one cannot skip a preliminary study of the boundary values of the functions belonging to Sobolev spaces. Very little is available about this question when the boundary is a polygon, although a complete answer has been given by Nikol'skii (1956, 1958), in the framework of slightly different spaces more suitable in the approximation theory. Accordingly we describe completely the boundary properties of functions belonging to Sobolev spaces on domains with polygonal boundaries. We include the proofs which turn out to be very similar to Nikol'skii's proofs. Some extensions of the classical Green formula are also worked out in the spirit of Lions and Magenes (1963) in the more general case of nonsmooth domains. This is why Chapter 1 is surprisingly long.

Chapter 2

As a first step toward the generalization of the classical shift theorem, we attempt to find the minimal assumptions under which one of the classical methods of proof can be worked out. Our technique is to look at the problem locally, flatten the boundary by a change of variables, freeze the coefficients and use partial Fourier transforms. Basically this is the method followed in Agmon *et al.* (1959). It turns out that the minimal assumption under which one obtains solutions in the Sobolev space $W_p^m(\Omega)$ is that the boundary Γ is of class C^m. This means that Γ can be locally represented as the graph of a C^m function. Actually one can allow a boundary of class $C^{m-1,1}$. Consequently a variational solution to a second-order boundary value problem is shown to belong to $W_p^2(\Omega)$ provided the boundary is at least of class $C^{1,1}$ This assumption does not allow a polygonal boundary. We recall that $C^{1,1}$ denotes the class of the functions with Lipschitz first derivatives.

Chapter 3

The classical method outlined above includes the proof of an *a priori* estimate which looks roughly like this:

$$\int_\Omega \left|\frac{\partial u}{\partial x_i\, \partial x_j}\right|^p \mathrm{d}x \leqslant C\int_\Omega |\Delta u|^p \,\mathrm{d}x + \text{lower-order terms.} \tag{2}$$

Usually we have very poor control of the constant C involved in this inequality. This is due to the local character of the method of proof. However in the case when $p=2$, an alternative proof based on integration by parts leads to a very accurate evaluation of the constant C. This is achieved under very general (possibly nonlinear) boundary conditions on u, in any n-dimensional domain. Such a proof (for the Dirichlet boundary condition) goes back to Caccioppoli (1950–51). It turns out that the constant C depends only on the negative part of the curvature of Γ (roughly speaking). This allows one to take limits with respect to the domain Ω and to prove some regularity results in general convex domains as well as in domains with turning points. Such a technique has been used for the first time by Kadlec (1964).

Chapters 4 and 5

These chapters are devoted to the proof of a modified shift theorem similar to the one outlined at the beginning of this introduction for general boundary value problems for the Laplace equation in a plane polygon. On each side of the polygon, the condition is either a Dirichlet or a Neumann or an oblique boundary condition. In Chapter 4 we prove the regularity of the second derivatives of the solution, while Chapter 5 focuses on the higher derivatives. In addition, some boundary value problems involving operators with variable coefficients as well as nonhomogeneous operators are investigated.

Chapter 6

The same boundary value problems as in Chapters 4 and 5 are investigated in the framework of the spaces $C^{m,\sigma}(\bar{\Omega})$, i.e. the space of the functions which are m times continuously differentiable and whose derivatives of order m fulfil a uniform Hölder condition of order σ throughout Ω $(0<\sigma<1)$.

Chapter 7

This chapter is focused on the Dirichlet problem for the biharmonic equation in a plane polygon. We have chosen this particular problem as our model fourth-order problem because of its importance in several physical questions (bending of plates, elasticity, fluid dynamics). Again we prove a suitably modified shift theorem in the Sobolev spaces $W_p^m(\Omega)$. We follow very closely the method of Kondratiev (1967a) when $p=2$. The shift theorem is also reformulated for the linear Stokes system and for the stationary Navier–Stokes equations in a plane polygon.

Chapter 8

This chapter includes miscellaneous topics all closely related to the content of the previous chapters.

First, the Dirichlet problem for a strongly nonlinear elliptic equation in a convex plane polygon is solved by applying a classical global inversion theorem following a work by Najmi (1978). The method relies strongly on the results of Chapters 4 and 5.

The method of Chapter 3 is adapted to the heat equation for a domain which is not time-like (with only one space variable for simplicity). Here we follow a work by Sadallah (1976, 1977, 1978).

The third section of Chapter 8 describes without complete proofs the few results about the behaviour of the solution of a second-order boundary value problem in a three-dimensional polyhedron.

Finally the fourth section is devoted to the consequences of the results of the previous chapters for the numerical analysis of boundary value problems.

Singular solutions like the u_k above have a strong polluting effect on the classical finite element methods. This difficulty is usually overcome in two main ways which are described in this section. The first consists (in a few words) in augmenting the usual spaces of trial functions by the addition of some of the singular solutions which have been explicitly calculated here.

The second relies on mesh refinements near the corners. Again the way the mesh has to be refined is governed by the behaviour of the singular solutions near the corners. We give here an analysis of the related error estimates.

In conclusion, let me acknowledge that this book has been strongly influenced by the outstanding paper by Kondratiev about general elliptic boundary value problems in domains with conical points.

I wish to express my gratitude to the many mathematical colleagues in the Universities of Algiers, Maryland and Nice, with whom I have had so many fruitful talks.

Finally I wish to express my sincere appreciation to Pitman Publishing for their most efficient handling of the publication of this book.

Nice P.G.
August. 1984