NON-STATIONARY FLOW OF AN IDEAL INCOMPRESSIBLE LIQUID*

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Our aim in this paper is to prove that there is a single-valued solution "in the whole" of the two-dimensional problem with initial data for the equations of motion of an ideal incompressible liquid contained in some vessel. The existence and uniqueness of a solution on a sufficiently small time interval has been proved for sufficiently smooth data in the three-dimensional problem by Gyunter (see [1]) and Lichtenstein (see [2], 482-493).

Here we consider generalized solutions and give detailed proofs of the results given in [3]. A separate paper will be devoted to smooth solutions and to the problem of the flow of a liquid through a given region (see [4]).

Let us give a brief summary of the contents of the paper.

Section 1 is devoted to *a priori* estimates of the solutions of the Euler equations. The basic result is an *a priori* estimate of the maximum of the curl for a certain class of flow patterns which includes plane and axisymmetric cases. We do not exclude the case when the vessel containing the liquid is deformed with time in a given way.

In Section 2 we give a brief summary of certain properties of elliptic equations and integrals of potential type (see [3], [5] - [7]) which will be needed later. We also introduce certain space functionals and study their properties.

In Sections 3 - 4 we give the definition of a generalized solution

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and prove its uniqueness and existence for a simply connected flow region. Certain properties of the generalized solution, in particular its smoothness, are examined.

A similar examination is made in Section 5 for multiply connected regions.

It is shown in Section 6 that the resulting generalized solution enables us to determine the pressure and the trajectory of the particles of liquid, and the results of an examination of the external problem are given.

1. *A priori* estimates of the solutions of the problem with initial data for Euler's equations

We shall examine here the problem of determining the flow of an ideal incompressible liquid contained in some vessel for given initial data and operating forces. The problem reduces to the determination of the velocity vector $\mathbf{v}(x, t) = (v_1, v_2, v_3)$ and pressure $P(x, t)$ given the conditions*

$$\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} = -\frac{\partial P}{\partial x_i} + F_i(x, t), \tag{1.1}$$

$$\frac{\partial v_i}{\partial x_i} = 0, \tag{1.2}$$

$$v_n|_S = 0, \tag{1.3}$$

$$v_i|_{t=0} = a_i(x). \tag{1.4}$$

where F_i, a_i are given, S is the boundary of the flow region, Ω, $v_n = \mathbf{vn}$, where $\mathbf{n}$ is the external normal to S.

At the end of this section we consider the more general problem where S may depend on the time t and where condition (1.3) is replaced by the corresponding non-homogeneous condition. We shall assume there also that the region Ω is bounded.

The basic result of this section is the *a priori* estimate which we establish for the curl of the velocity for a certain class of flow patterns which includes plane and axisymmetric flow. From these estimates

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* With the usual convention of omitting the symbol Σ.

we then naturally go on to determine the generalized solution of problem (1.1)-(1.4). Logically this section is not connected with those which follow. However it is the *a priori* estimates given here which are the basis of all the subsequent analysis.

We shall assume that S and all the functions which come into the system (1.1)-(1.4) are sufficiently smooth.

Lemma 1.1. The solution of problem (1.1)-(1.4) satisfies the relation

$$\|\mathbf{v}\|_{L_2(\Omega)} \leqslant \|\mathbf{a}\|_{L_2(\Omega)} + \int_0^t \|F\|_{L_2(\Omega)}\, d\tau, \tag{1.5}$$

where

$$\|\mathbf{v}\|_{L_2(\Omega)} = \left[\int \mathbf{v}^2(x, t)\, dx\right]^{1/2}.$$

When $\mathbf{F} = 0$ the inequality is replaced by an equality.

To prove the lemma we multiply the i-th equation of (1.1) by v_i, integrate w.r.t. Ω and sum for i from 1 to 3. We have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \mathbf{v}^2(x, t)\, dx = \int_\Omega \mathbf{F}\mathbf{v}\, dx. \tag{1.6}$$

Using the Bunyakovskii inequality we estimate the right-hand side of this equation and obtain

$$\frac{d}{dt}\|\mathbf{v}\|_{L_2(\Omega)} \leqslant \|\mathbf{F}\|_{L_2(\Omega)},$$

from which (1.5) is obtained immediately by integrating w.r.t. t.

We then estimate the curl of the velocity, $\boldsymbol{\omega} = \operatorname{rot} \mathbf{v}$. If we apply the operation rot to (1.1) we obtain the well-known Helmholtz equation for the curl*:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{v}, \nabla)\, \boldsymbol{\omega} = (\boldsymbol{\omega}, \nabla)\, \mathbf{v} + \operatorname{rot} \mathbf{F}. \tag{1.7}$$

We give the following definition. The function $g(x, t)$ is called a point integral of equation (1.1) if it is constant on the trajectory of any liquid particle, i.e. satisfies the equation

* Here and below ∇ is the Hamiltonian operator: $\nabla = i_k(\partial/\partial x_k)$; i_k are unit coordinate vectors.

$$\frac{dg}{dt} \equiv \frac{\partial g}{\partial t} + \frac{\partial g}{\partial x_i} v_i = 0. \tag{1.8}$$

The following lemma is essentially equivalent to the well-known Helmholtz - Thomson theorems about curls and puts them in a form which is especially convenient later.

Lemma 1.2. Let rot $\mathbf{F} = 0$, and let the function $g(x, t)$ be a point integral of the problem (1.1)-(1.4). Then the quantity $\boldsymbol{\omega}\nabla g$ will also be a point integral:

$$\frac{d}{dt}(\boldsymbol{\omega}\nabla g) = 0. \tag{1.9}$$

Proof. Let us calculate the left-hand side of (1.9). We have, using (1.7)-(1.8),

$$\begin{aligned} \frac{d}{dt}(\boldsymbol{\omega}\ \nabla g) = \frac{d}{dt}\omega_s \frac{\partial g}{\partial x_s} = \frac{d\omega_s}{dt}\frac{\partial g}{\partial x_s} + \omega_s \frac{d}{dt}\frac{\partial g}{\partial x_s} = \\ = \omega_k \frac{\partial v_s}{\partial x_k}\frac{\partial g}{\partial x_s} + \omega_s \frac{d}{dt}\frac{\partial g}{\partial x_s}. \end{aligned} \tag{1.10}$$

We then have

$$\begin{aligned} \frac{d}{dt}\frac{\partial g}{\partial x_s} = \frac{\partial}{\partial t}\frac{\partial g}{\partial x_s} + v_k \frac{\partial}{\partial x_k}\frac{\partial g}{\partial x_s} = \frac{\partial}{\partial x_s}\left(\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x_k}\cdot v_k\right) - \\ - \frac{\partial g}{\partial x_k}\frac{\partial v_k}{\partial x_s} = -\frac{\partial g}{\partial x_k}\frac{\partial v_k}{\partial x_s}. \end{aligned} \tag{1.11}$$

Now (1.9) follows at once from (1.10)-(1.11).

If it is not necessary that the external forces $\mathbf{F}$ be potential forces, instead of (1.9) we have the relation

$$\frac{d}{dt}\boldsymbol{\omega}\ \mathrm{grad}\ g = \mathrm{rot}\ \mathbf{F}\ \mathrm{grad}\ g. \tag{1.12}$$

We note that the converse of Lemma 1.2 is also true.

Lemma 1.3. Given the operator L which sets the solenoidal vector $\mathbf{a}(x)$, $x \in \Omega$, which is smooth w.r.t. x, t in correspondence with every smooth solenoidal vector $\mathbf{v}(x, t)$. Suppose that Lemma 1.2 applies to this operator (i.e. that (1.8) implies (1.9)). Then the vector $\boldsymbol{\omega} = \mathrm{rot}\ \mathbf{v}$ satisfies equation (1.7) when rot $\mathbf{F} = 0$.

To prove the lemma we write

$$\boldsymbol{\omega} = \mathrm{rot}\ \mathbf{v}, \quad \frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\Phi} \tag{1.13}$$

and show that $\Phi_i = \omega_k \frac{\partial v_i}{\partial x_k}$. We introduce Lagrange coordinates $(\alpha_1, \alpha_2, \alpha_3)$, the Cartesian coordinates of a particle of liquid when $t = 0$. The coordinates of the particle (x_1, x_2, x_3) at any time t are found by solving the system of equations

$$\frac{dx_i}{dt} = v_i(x, t); \qquad x_i|_{t=0} = \alpha_i, \quad i = 1, 2, 3. \tag{1.14}$$

It is clear that any function $g(\alpha_1, \alpha_2, \alpha_3)$ satisfies equation (1.8). We have now to determine Φ_i from the condition that equation (1.9) holds for the function g. We have

$$\begin{aligned}\frac{d}{dt}\,\omega\nabla g = \Phi_s \frac{\partial g}{\partial x_s} - \omega_k \frac{\partial g}{\partial x_s}\frac{\partial v_s}{\partial x_k} = \frac{\partial g}{\partial x_s}\left(\Phi_s - \omega_k \frac{\partial v_s}{\partial x_k}\right) = \\ = \frac{\partial g}{\partial \alpha_r}\frac{\partial \alpha_r}{\partial x_s}\left(\Phi_s - \omega_k \frac{\partial v_s}{\partial x_k}\right) = 0.\end{aligned} \tag{1.15}$$

Inserting $g = \alpha_1, \alpha_2, \alpha_3$ successively in (1.15) we find

$$\frac{\partial \alpha_r}{\partial x_s}\left(\Phi_s - \omega_k \frac{\partial v_s}{\partial x_k}\right) = 0. \tag{1.16}$$

From the continuity equation (1.2) it follows, as we know, that $\det(\partial\alpha_r/\partial x_s) = 1$. Therefore from (1.16) it follows that

$$\Phi_s = \omega_k \frac{\partial v_s}{\partial x_k},$$

and this and (1.13) give (1.7).

Let us discuss the physical meaning of Lemma 1.2. Suppose that at the initial moment of time $t = 0$ the surface element $d\sigma_0$, consiting of liquid particles, is chosen. We shall trace its change over time due to the motion of the liquid (this motion is described by the system (1.14)). Suppose that at time t the element $d\sigma_0$ has moved to $d\sigma_t$. Then, with the conditions of Lemma 1.2, the flux of the curl across $d\sigma_t$ does not depend on the time*, i.e.

$$\omega(x, t)\cdot d\sigma_t = \omega(x, 0)\, d\sigma_0. \tag{1.17}$$

In certain cases Lemma 1.2 enables us to find an *a priori* estimate for $\max|\omega|$.

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* It is incorrectly stated in [8] that the curl itself is a point integral. The proof given is of course false.

Example 1. Let us consider plane flow. Then $v_3 = 0$, $\omega_1 = \omega_2 = 0$, v_1, v_2, ω_3 do not depend on x_3. It is easy to see that $g = x_3$ is, in this case, a point integral $(dx_3/dt = v_3 = 0)$. It then follows from (1.9) that

$$\frac{d\omega_3}{dt} = 0, \tag{1.18}$$

i.e. in this case the curl is preserved for each particle of liquid. It follows that $\omega_3(x, t)$ as a function of x is equimeasurable with $\omega_3(x, 0)$ and therefore

$$\max_x |\omega_3(x, t)| = \max_x |\omega_3(x, 0)|, \tag{1.19}$$

$$\int_\Omega f(\omega_3(x, t))\, dx = \int_\Omega f(\omega_3(x, 0))\, dx \tag{1.20}$$

for any smooth function f.

Example 2. Axisymmetric motion. We take a cylindrical coordinate system r, θ, z and consider the case when $v_\theta = 0$, v_r, v_z do not depend on θ. Then $\omega_r = \omega_z = 0$. Since the flow is in the planes θ = const. we have the trivial integral $g = \theta, d\theta/dt = v_\theta/r = 0$. It follows from (1.9) that the quantity

$$\omega \operatorname{grad} \theta = \frac{\omega_\theta}{r} \tag{1.21}$$

will also be an integral. Then reasoning as in the plane case we obtain the relations

$$\max_{r,z} \left| \frac{\omega_\theta(r, z, t)}{r} \right| = \max_{r,z} \left| \frac{\omega_\theta(r, z, 0)}{r} \right|, \tag{1.22}$$

$$\int_\Omega f\left(\frac{\omega_\theta}{r}\right) d\Omega = \int_\Omega f\left(\frac{\omega_\theta(r, z, 0)}{r}\right) d\Omega, \tag{1.23}$$

where f is an arbitrary smooth function.

The integral given by formula (1.9) may sometimes turn out to be trivial. For example, if v_r, v_θ, v_z do not depend on θ, then it is easy to prove that $g_1 = rv_\theta$ is an integral. But from formula (1.9) we obtain $\omega \operatorname{grad} g_1 = 0$.

We note further that the relations (1.19) - (1.20), (1.22) - (1.23) are not basically associated with any boundary condition, and refer to any mass of liquid.

We now state a lemma which put these lemmas in somewhat more general form.

Lemma 1.4. Let q_1, q_2, q_3 be orthogonal curvilinear coordinates such that the Lamé coefficients H_1, H_2, H_3 do not depend on q_3, and suppose that the flow of liquid is such that the velocity components v_1, v_2, v_3 satisfy the condition

$$v_3 = 0, \qquad v_1 = v_1(q_1, q_2, t), \qquad v_2 = v_2(q_1, q_2, t),$$

and the components of the external forces the condition

$$F_3 = 0, \qquad F_1 = F_1(q_1, q_2, t), \qquad F_2 = F_2(q_1, q_2, t).$$

Then we have the relation

$$\frac{d}{dt}\frac{\omega_3}{H_3} = \frac{1}{H_3}(\mathrm{rot}\ \mathbf{F})_3, \tag{1.24}$$

in particular if the external forces are potential forces it follows from (1.24) that ω_3/H_3 is a point integral.

In the conditions of the lemma $g = q_3$ is a point integral and so (1.24) follows from (1.12).

Let us write out the Euler equation for the conditions of Lemma 1.4:

$$\begin{aligned} \frac{\partial v_1}{\partial t} - v_2\omega_3 &= F \ - \frac{1}{H_1}\frac{\partial P}{\partial q_1}, \\ \frac{\partial v_2}{\partial t} + v_1\omega_3 &= F_2 - \frac{1}{H_2}\frac{\partial P}{\partial q_2} \\ v_3 &= 0. \end{aligned} \tag{1.25}$$

Here ω_3 is the component of the curl of the velocity along the coordinate q_3:

$$\omega_3 = \frac{1}{H_1H_2}\left[\frac{\partial(H_2v_2)}{\partial q_1} - \frac{\partial(H_1v_1)}{\partial q_2}\right], \tag{1.26}$$

and the other components ω_1, ω_2 are equal to zero.

The continuity equation has the form

$$\frac{\partial}{\partial q_1}(H_2H_3v_1) + \frac{\partial}{\partial q_2}(H_3H_1v_2) = 0. \tag{1.27}$$

As in the examples given above the estimate for ω_3/H_3 follows from (1.24).

Finally, let us consider the case when the flow region $\Omega = \Omega_t$ is deformed with time in a given way and the boundary condition (1.3) is replaced by the more general condition

$$v_n|_{S_t} = q_n(x, t), \tag{1.28}$$

where q_n is the projection on the external normal to the boundary S_t of the region Ω_t of the velocity of the point S_t in the course of its deformation. Condition (1.28) means that S_t is a deformable wall impenetrable to liquid. In view of the fact that the liquid is incompressible, the volume of the region Ω must remain constant.

Let us show that in this case also relations of the type (1.19) - (1.20) hold. We have (for any smooth function f)

$$\frac{d}{dt}\int\limits_{\Omega_t} f\left(\frac{\omega_3}{H_3}\right) dx = \int\limits_{\Omega_t} \frac{\partial}{\partial t} f\left(\frac{\omega_3}{H_3}\right) dx + \oint\limits_{S_t} f\left(\frac{\omega_3}{H_3}\right) q_n dS. \tag{1.29}$$

Using relations (1.24), (1.28) and transforming the second integral on the right-hand side of (1.29) using the Ostrogradskii-Gauss formula we find

$$\begin{aligned} \frac{d}{dt}\int\limits_{\Omega_t} f\left(\frac{\omega_3}{H_3}\right) dx &= \int\limits_{\Omega_t} \left[\frac{\partial}{\partial t} f\left(\frac{\omega_3}{H_3}\right) + \mathbf{v}\,\mathrm{grad}\, f\left(\frac{\omega_3}{H_3}\right)\right] dx = \\ &= \int\limits_{\Omega_t} f'\left(\frac{\omega_3}{H_3}\right) \frac{1}{H_3} (\mathrm{rot}\,\mathbf{F})_3\, dx. \end{aligned} \tag{1.30}$$

When rot $\mathbf{F} = 0$ it follows at once from (1.30) that

$$\int\limits_{\Omega_t} f\left(\frac{\omega_3}{H_3}\right) dx = \int\limits_{\Omega_0} f\left(\frac{\omega_3(x, 0)}{H_3}\right) dx, \tag{1.31}$$

and so, since f is arbitrary, (1.19) follows at once for this case (it is sufficient to put $f(z) = z^{2k}$ in (1.31), take the $2k$-th root of both sides of the equation and go to the limit as $k \to \infty$). When rot $\mathbf{F}$ is not identically equal to zero substituting $f(z) = z^{2k}$ in (1.30) and writing

$$\int\limits_{\Omega_t} \left(\frac{\omega_3}{H_3}\right)^{2k} dx = \gamma_k^{2k}(t), \tag{1.32}$$

we obtain

$$\frac{d\gamma_k}{dt} = \frac{1}{\gamma_k^{2k-1}} \int\limits_{\Omega_t} \left(\frac{\omega_3}{H_3}\right)^{2k-1} \frac{(\mathrm{rot}\,\mathbf{F})_3}{H_3}\, dx \leqslant \left\| \frac{(\mathrm{rot}\,\mathbf{F})_3}{H_3} \right\|_{L_{2k}(\Omega_t)}. \tag{1.33}$$

Integrating (1.33) w.r.t. t we find

$$\gamma_k(t) \leqslant \gamma_k(0) + \int_0^t \left\| \frac{(\text{rot } \mathbf{F}_3)}{H_3} \right\|_{L_{2k}(\Omega_t)} d\tau. \qquad (1.34)$$

Taking k to infinity in (1.34) we have

$$\max_x \left| \frac{\omega_3(x, t)}{H_3} \right| \leqslant \max_x \left| \frac{\omega_3(x, 0)}{H_3} \right| + \int_0^t \max_x \left| \frac{(\text{rot } \mathbf{F})_3}{H_3} \right| d\tau. \qquad (1.35)$$

We have thus proved the following lemma.

Lemma 1.5. Let the flow region Ω_t be deformed with time, leaving Ω_0 homeomorphic, where the volume of Ω_t does not depend on the time t. Let us suppose that for any fixed t Ω_t is such that its sections by coordinate surfaces q_3 = const. are congruent. Then any solution of the problem (1.25), (1.4), (1.28) satisfies the relations (1.34), (1.35), or when rot $\mathbf{F} = 0$ (1.31) and (1.19) (with ω_3/H_3 instead of ω_3).

2. Basic functional spaces

1. Let us consider the first boundary problem for a second order elliptic equation: to find a function $u(x)$, $x = (x_1, x_2, \ldots, x_n) \in \Omega$, with the conditions

$$Lu \equiv \sum_{i,k=1}^n a_{ik}(x) \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + c(x)\, u = f(x), \qquad (2.1)$$

$$u|_S = 0 \qquad (2.2)$$

where Ω is a bounded region of n-dimensional space E^n, S is its boundary; $a_{ik}(x) = a_{ki}(x)$, $b_i(x)$, $c(x)$, $f(x)$ are given functions. The ellipicity requirement means that the condition

$$\sum_{i,k=1}^n a_{ik}(x)\, \xi_i \xi_k \geqslant \lambda_1 \sum_{i=1}^n \xi_i^2, \qquad (2.3)$$

is satisfied, where λ_1 is a positive constant, $\xi_1, \ldots, \xi_n$ are arbitrary numbers.

We shall then assume that the region Ω possesses a certain regularity, sufficient for S.L. Sobolev's insertion theorem [9] to be applicable. We have the following theorem (see [3], [5] - [6]):

Theorem 2.1. Let $S \in C^{r+2}$, $r \geqslant 0$ *, the functions $b_i(x)$, $c(x)$ have bounded r-th generalized derivatives, $a_{ik}(x) \in B^{r,\mu}$ $0 < \mu \leqslant 1$. Then the solution of problem (2.1) - (2.2) of the class $W_p^{(r+2)}$ has the estimate ($p \geqslant p_0 > 1$)

$$\|u\|_{W_p^{(r+2)}(\Omega)} \leqslant Cp\|f\|_{W_p^{(r)}(\Omega)} + C_1 p^{\frac{1-\lambda}{\mu}}\|u\|_{B^{r+1,\lambda}(\Omega)}, \tag{2.4}$$

where C, C_1 depend on Ω and p_0 but not on f and p. If for any $u \in W_p^{(2)}(\Omega)$, $u|_S = 0$, we have the inequality

$$\int_\Omega uLudx \geqslant m\|u\|^2_{W_2^{(1)}(\Omega)},$$

then a solution of problem (2.1) - (2.2) exists and the estimate (2.4) can be put in the form

$$\|u\|_{W_p^{(r+2)}} \leqslant Cp\|f\|_{W_p^{(r)}}. \tag{2.5}$$

Note. Let us introduce the "norm" of the boundary S of the region Ω of the class $C^{(r)}$. Let $2\rho(x)$ be the maximum radius of a neighbourhood of the point $x \in S$, where the equation of the boundary $x_n = \varphi(x_1, \ldots, x_{n-1})$ with the function $\varphi \in C^{(r)}$, is still valid. We put

$$\|S\|_r = \sup_{x \in S} \sup_{|x-y| \leqslant \rho(x)} \sum_{0 < \alpha_1 + \ldots + \alpha_{n-1} \leqslant r} \left| \frac{\partial^k \varphi(y)}{\partial y_1^{\alpha_1} \ldots \partial y_{n-1}^{\alpha_{n-1}}} \right|.$$

Then it follows from the reasoning given in [5], [6] that the constants C and C_1 in the inequalities (2.4) - (2.5) depend only on $\|S\|_{r+2}$, the constant λ_1 from (2.3) and the norms of the coefficients of the equation (2.1) given in the theorem. This makes it possible to extend Theorem 2.1 to the case of an unbounded region Ω. In a definite sense Theorem 2.1 is true for $r = -1$ also. Consider, for example, the first boundary problem for the Poisson equation

$$\Delta u = \frac{\partial f(x)}{\partial x_k}, \qquad u|_S = 0. \tag{2.6}$$

We introduce the Hilbert space H_1, the closure of the set of functions which are smooth in Ω and equal to zero near the boundary S in the norm

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* i.e. The equation of the boundary S in the neighbourhood of any point of this boundary has the form $x_n = \varphi(x_1, \ldots, x_{n-1})$ in the local coordinate system with the $r+2$ times continuously differentiable function φ.

generated by the scalar product

$$(u, v)_{H_1} = \int_\Omega \nabla u \nabla v dx. \tag{2.7}$$

A generalized solution of problem (2.6) for $f \in L_2$ is a function $u(x) \in H_1$ satisfying the integral identity

$$(u, \phi)_{H_1} = \int_\Omega f \frac{\partial \phi}{\partial x_k} dx \tag{2.8}$$

for any $\varphi \in H_1$.

Theorem 2.2. (see [3], [7]). Let $f \in L_p$, $p > 2$; $S \in C^2$. Then a generalized solution of problem (1.6) in the sense (2.8) $u(x) \in W_p^{(1)}$ and we have the estimate

$$\| u \|_{W_p^{(1)}(\Omega)} \leqslant Cp \| f \|_{L_p(\Omega)}, \tag{2.9}$$

where C does not depend on f, p, but only on $\| S \|_2$.

We shall need yet another lemma, about integrals of potential type (see [5]).

Lemma 2.1. Let Ω be a bounded region of E^n and

$$u(x) = \int_\Omega \frac{f(y)}{|x - y|^{n-1}} dy, \tag{2.10}$$

where $|f| \leqslant M$. Then $u(x) \in B^{0,\mu}(\Omega)$ for any $0 \leqslant \mu < 1$ and

$$\| u(x) \|_{B^{0,\mu}(\Omega)} \leqslant \frac{C}{1 - \mu} M, \tag{2.11}$$

$$| u(x^1) - u(x^2) | \leqslant C_1 M \delta (1 + |\ln \delta|), \tag{2.12}$$

where $x^1, x^2 \in \Omega$; $|x^1 - x^2| = \delta$; C, C_1 are constants which depend only on the region Ω.

Proof. We have $(\Omega_{x,\delta} = \Omega \cap (|x - y| < \delta))$

$$\begin{aligned} u(x^1) - u(x^2) | \leqslant \int_{\Omega_{x^1,2\delta}} \frac{|f(y)|}{|x^1 - y|^{n-1}} dy + \int_{\Omega_{x^1,2\delta}} \frac{|f(y)|}{|x^2 - y|^{n-1}} dy + \\ + \int_{\Omega \setminus \Omega_{x^1,2\delta}} |f(y)| \left| \frac{1}{|x^1 - y|^{n-1}} - \frac{1}{|x^2 - y|^{n-1}} \right| dy = I_1 + I_2 + I_3. \end{aligned} \tag{2.13}$$

Let us estimate the integrals on the right-hand side in succession

For the first term I_1 we find easily

$$|I_1| \leqslant M \int\limits_{\Omega_{x^1,2\delta}} \frac{dy}{|x^1-y|^{n-1}} < M\omega_{n-1}\int\limits_0^{2\delta} dr = M\omega_{n-1}2\delta, \tag{2.14}$$

where ω_{n-1} is the area of the $n-1$ dimensional unit sphere.

The second term can be estimated as follows:

$$|I_2| \leqslant M \int\limits_{\Omega_{x^2,3\delta}} \frac{dy}{|x^2-y|^{n-1}} \leqslant M\omega_{n-1}3\delta. \tag{2.15}$$

Using the formula of finite increments we have for I_3

$$|I_3| \leqslant MC_2\delta \int\limits_{\Omega\setminus\Omega_{x^1,2\delta}} \frac{1}{|x^*-y|^n}\,dy,$$

where C_2 is an absolute constant, x^* some point of the segment x^1x^2.

Since, when $y \in \Omega\setminus\Omega_{x^1,2\delta}$, $|x^*-y| > \frac{1}{2}|x'-y|$, we obtain

$$\begin{aligned}|I_3| \leqslant MC_2\delta\, 2^n \int\limits_{\Omega\setminus\Omega_{x^1,2\delta}} \frac{dy}{|x'-y|^n} &\leqslant MC_2\delta 2^n \int\limits_{\Omega_{x^1,R}\setminus\Omega_{x^1,2\delta}} \frac{dy}{|x^1-y|^n} = \\ &= MC_2\delta 2^n\omega_{n-1}\ln\frac{R}{\delta}.\end{aligned} \tag{2.16}$$

(R is the diameter of the region Ω). Now (2.12) follows immediately from (2.14) - (2.16). The inequality (2.11) can be derived similarly (it can also be obtained from (2.12)). This proves the lemma.

We note that the estimates (2.5), (2.9), (2.11), (2.12) are exact with respect to the increase of their right-hand sides, as the example of the function $u(x) = x_1x_2\ln|x|$ in the unit circle shows.

2. We now introduce a few functional spaces and establish some of their properties.

Let V be the space of functions which are generalized solutions of the first boundary problem for the Poisson equation:

$$\Delta u = f(x); \quad u|_S = 0 \tag{2.17}$$

in the two-dimensional region Ω with boundary S, where f is an essentially bounded function*. The norm in the space V is defined by the equation**

$$\|\varphi\|_V = \max_{x \in \Omega} |\Delta\varphi(x)|. \tag{2.18}$$

Let V_1 be the space of functions of x, t defined in the cylinder $Q_T = \Omega \times [0, T]$ (where $T > 0$ is some number) for almost all t, belonging as a function of x to the space V with norm

$$\|\psi(x, t)\|_{V_1} = \max_{0 \leqslant t \leqslant T} \|\psi(x, t)\|_V, \tag{2.19}$$

and let C' be the Banach space of functions of x, t defined in Q_T for almost all t, equal to zero on S and belonging to $C^{(1)}$ in the region Ω with norm

$$\|\psi\|_{C'} = \max_{0 \leqslant t \leqslant T} \left(\left|\frac{\partial\psi}{\partial x_1}\right| + \left|\frac{\partial\psi}{\partial x_2}\right| \right). \tag{2.20}$$

Lemma 2.2. Let the boundary of the region $S \in C^2$. Then any function $\varphi \in V$ possesses second generalized derivatives w.r.t. x_1, x_2 in the region Ω which belong to any L_p, $p \geqslant 1$ and for sufficiently small p, $p \geqslant p_0 > 1$ we have the estimate

$$\left\|\frac{\partial^2\varphi}{\partial x_i \partial x_k}\right\|_{L_p(\Omega)} \leqslant Cp\|\varphi\|_V, \tag{2.21}$$

where C depends only on $\|S\|_2$ and not on φ or p. The first derivatives of the function φ satisfy the Hölder condition with any $0 \leqslant \lambda < 1$ and

$$\|\varphi\|_{B^{0,\lambda}} \leqslant \frac{C_1}{1-\lambda}\|\varphi\|_V, \tag{2.22}$$

the modulus of continuity $\omega_\delta(D\varphi)$ of any first derivative $D\varphi$ satisfies the condition

$$\omega_\delta(D\varphi) \leqslant C_2\delta(1 + |\ln\delta|). \tag{2.23}$$

The inequality (2.21) follows immediately from Theorem 2.1. Let us go on to prove inequalities (2.22) - (2.23). With the assumptions we have made with respect to the region Ω Green's function of the Dirichlet problem (see [10]) for which

---

* This definition differs slightly from that given in [3].

** By max we understand everywhere essential maximum.

$$|G(x, y)| \leqslant C_3\left(1 + \left|\ln|x-y|\right|\right);$$
$$\left|\frac{\partial G}{\partial x_i}\right| \leqslant \frac{C_1}{|x-y|}. \tag{2.24}$$

exists. Using Green's function, $\varphi \in V$ can be put in the form

$$\varphi(x) = \int_\Omega G(x, y)\, \Delta\varphi(y)\, dy. \tag{2.25}$$

Let us differentiate (2.25):

$$\frac{\partial\varphi}{\partial x_i} = \int_\Omega \frac{\partial G(x, y)}{\partial x_i} \Delta\varphi(y)\, dy. \tag{2.26}$$

Now the estimates (2.22)-(2.23) follow from (2.24), (2.26) and Lemma 2.1.

The Hilbert space H_1' is defined as the closure of the set of smooth functions of x, t defined in the cylinder $Q_T = \Omega \times [0, T]$ and equal to zero near the boundary S of the region Ω in the norm generated by the scalar product

$$(\varphi, \psi)_{H_1'} = \int_0^T (\varphi(x, t), \psi(x, t))_{H_1}\, dt, \tag{2.27}$$

where H_1 is the space defined in (2.7).

We then use another Banach space $L_{p,r}$, $r > p > 1$ of functions defined in the cylinder Q_T with norm

$$\| f(x, t)\|_{L_{p,r}} = \left(\int_0^T \|f\|_{L_p(\Omega)}^r dt\right)^{1/r}; \tag{2.28}$$

The space $L_{p,r}$ is separable from a uniformly convex sphere; the space conjugate to it is

$$L_{p'r'}, \text{ where } p' = \frac{p}{p-1}, \qquad r' = \frac{r}{r-1}.$$

Any bounded set in $L_{p,r}$ is weakly compact. The operators of the generalized differentiation are weakly closed in $L_{p,r}$. The proof of all these facts reproduces almost work for word the proof for the spaces L_p (see [9]) and so we shall not spend time on it.

We note one more well known property of the spaces L_p which is used below:

$$\lim_{p\to\infty} \| f \|_{L_p} = \max | f |, \tag{2.29}$$

and the property of the spaces $L_{p,r}$ which follows from it:

$$\lim_{r\to\infty} \| f (x, t) \|_{L_{p,r}} = \max_{0\leqslant t\leqslant T} \| f (x, t) \|_{L_p}{}^{(\Omega)}. \tag{2.30}$$

Of course (2.29) and (2.30) are only valid on the assumption that their right-hand sides exist.

3. A simply-connected region. Definition of a generalized solution and its uniqueness

We now consider problem (1.1)-(1.4) for two space variables. We shall first take the region to be simply-connected. We introduce the current function $\psi(x, t)$ given by the equations

$$v_1 = \frac{\partial\psi}{\partial x_2}, \qquad v_2 = -\frac{\partial\psi}{\partial x_1}. \tag{3.1}$$

Eliminating the pressure P from (1.1) by cross differentiation we arrive at the following problem defining the function $\psi(x, t)$:

$$\frac{\partial\Delta\psi}{\partial t} + \frac{\partial\psi}{\partial x_2}\frac{\partial\Delta\psi}{\partial x_1} - \frac{\partial\psi}{\partial x_1}\frac{\partial\Delta\psi}{\partial x_2} = f (x, t), \tag{3.2}$$

$$\psi|_S = 0, \tag{3.3}$$

$$\psi|_{t=0} = \varphi (x), \tag{3.4}$$

where

$$f (x, t) = \frac{\partial F_1}{\partial x_2} - \frac{\partial F_2}{\partial x_1} \tag{3.5}$$

is the component of the curl of the external force $\mathbf{F}$ along the x_3-axis taken with the opposite sign; $\varphi(x)$ is the current function of the vector $\mathbf{a}(x)$ from (1.4). The condition (3.3) is obtained as follows: from (1.3) and (3.1) we deduce that $\partial\psi/\partial s = 0$ (where s is an arc of the contour S) and therefore $\psi(x, t)$ on S is a function of time t only. Since the region Ω is simply connected and the function $\psi(x, t)$ is determined from (3.1) to within an arbitrary function of time we can choose the latter so that condition (3.3) is satisfied. We note that we shall not use the fact that the region is simply connected again in our examination of the problem (3.2)-(3.4).

Now in order to solve the problem (1.1)-(1.4) we must find $\psi(x, t)$ from (3.2)-(3.4), calculate v_1, v_2 from (3.1) and find the pressure P from equation (1.1). We first consider the problem (3.2)-(3.4). We shall discuss the question of finding P below (see Section 6).

We shall assume that the following conditions are satisfied:

1) Ω is a bounded simply connected region of the plane $x = (x_1, x_2)$; its boundary $S \in C^{(3)}$;

2) $f(x, t)$ is a measurable bounded function;

3) $\varphi(x) \in V$.

Let us define the generalized solution of problem (3.2)-(3.4) in some cylinder $Q_T = \Omega \times [0, T]$ (where $T > 0$ is any number) as the function $\psi(x, t) \in V_1$, satisfying the integral identity

$$-\int_{\Omega} \Delta\varphi(x)\, \phi(x, 0)\, dx + \int_0^T \int_{\Omega} \left[-\Delta\psi \frac{\partial \phi}{\partial t} + \Delta\psi \left(\frac{\partial\psi}{\partial x_1}\frac{\partial\phi}{\partial x_2} - \frac{\partial\psi}{\partial x_2}\frac{\partial\phi}{\partial x_1} \right) \right] dx dt = \int_0^T \int_{\Omega} f\phi\, dx dt. \tag{3.6}$$

for any function $\varphi(x, t)$ smooth in Q_T and such that $\phi(x, T) = \phi|_S = 0$.

It is not difficult to see that if the generalized solution has all the derivatives occurring in (3.2) and these derivatives are continuous it will be a classical solution of the problem (3.2)-(3.4). To prove this it is sufficient to use integration by parts to transfer all the derivatives with φ in (3.6) into ψ and use the basic lemma of the variational calculus.

Let us give some of the properties of a generalized solution.

Lemma 3.1. The generalized solution $\psi(x, t)$ of problem (3.2)-(3.4) is, first, weakly continuous w.r.t. t on $[0, T]$ in the space H_1 (in particular it satisfies in this sense the initial condition (3.4)), and secondly, is a generalized solution in any cylinder $Q_{\tau_1, \tau_2} = \Omega \times [\tau_1, \tau_2]$ for $0 \leqslant \tau_1 < \tau_2 \leqslant T$.

In (3.6) we replace $\varphi(x, t)$ by the function $h(t)\varphi(x, t)$ where $h(t)$ is continuously differentiable on $[0, T]$. We have

$$-h(0)\int_{\Omega} \Delta\varphi(x)\, \phi(x, 0)\, dx + \int_0^T h'(t)\, K_t(\psi, \phi)\, dt + \int_0^T h(t) L_t(\psi, \phi) dt = 0, \tag{3.7}$$

where

$$K_t(\psi, \phi) = -\int_{\Omega} \Delta\psi(x, t)\, \phi(x, t)\, dx = (\psi, \phi)_{H_1}, \tag{3.8}$$

$$L_t(\psi, \phi) = \int_{\Omega} \left[-\Delta\psi \frac{\partial \phi}{\partial t} + \Delta\psi\left(\frac{\partial\psi}{\partial x_1}\frac{\partial \phi}{\partial x_2} - \frac{\partial\psi}{\partial x_2}\frac{\partial \phi}{\partial x_1}\right) - f\phi\right] dx; \tag{3.9}$$

K_t, L_t are bounded measurable functions of t. Choosing $h(t)$ so that $h(0) = 0$ and remembering the definition of a generalized derivative we conclude that

$$\frac{d}{dt} K_t = L_t \tag{3.10}$$

on $[0, T]$. Using the insertion theorem we find from (3.10) that K_t is a continuous function of time t on $[0, T]$.

Taking the limit it is easy to see that (3.7)-(3.10) are satisfied if instead of $\varphi(x, t)$ we take any function $\gamma(x) \in H_1$ and assume that $h(T) = 0$. It follows from the preceding argument that $(\psi, \gamma)_{H_1}$ is continuous w.r.t. t on $[0, T]$, and this means that $\psi(x, t)$ is by definition weakly continuous w.r.t. t in H_1 on $[0, T]$.

Further, from (3.7), (3.10), choosing h so that $h(T) = 0$, $h(0) = 1$ and integrating by parts, we have for any $\gamma(x) \in H_1$

$$\lim_{t\to 0} (\psi(x, t), \gamma)_{H_1} = \lim_{t\to 0} K_t\ (\psi, \gamma) = -\int_{\Omega} \Delta\varphi\gamma dx = (\varphi, \gamma)_{H_1}. \tag{3.11}$$

This means that $\psi(x, t)$ takes the initial value $\varphi(x)$ in the sense of weak convergence in H_1.

Now integrating (3.10) w.r.t. t from τ_1 to τ_2, $0 \leqslant \tau_1 < \tau_2 < T$ we have

$$\begin{gathered}(\psi(x, \tau_1), \phi(x, \tau_1))_{H_1} - (\psi(x, \tau_2), \phi(x, \tau_2))_{H_1} + \int_{\tau_1}^{\tau_2}\int_{\Omega}\left[-\Delta\psi\frac{\partial \phi}{\partial t} + \right. \\ \left. + \Delta\psi\left(\frac{\partial\psi}{?}\frac{\partial \phi}{} - \frac{\partial\psi}{\partial x_2}\frac{\partial \phi}{\partial x_1}\right)\right] dx\, dt = \int_{\tau_1}^{\tau_2}\int_{\Omega} f\,\phi\, dx dt.\end{gathered} \tag{3.12}$$

We note that (3.12) is satisfied for any function $\varphi(x, t)$ smooth in Q_{τ_1, τ_2} (and such that $\varphi|_S = 0$) since without losing the smoothness and the boundary condition we can continue the latter over the whole cylinder Q_T so that $\varphi(x, T)$ is equal to zero. In particular it follows from (3.12) that the second statement of the lemma is true.

Now let us show that when $\psi(x, T)$ is defined in the natural way the function $\psi(x, T)$ is weakly continuous in H_1 w.r.t. t for $t = T$. To do this we put $\varphi(x, t) = h(t)\gamma(x)$ in (3.12), where $h(t)$ is smooth in $[0, T]$ and $h(\tau_1) = 0$, and the function $\gamma(x) \in H_1$. Fixing τ_1 and making τ_2 tend to T using (3.12) it is easy to see that a weak limit of the function $\psi(x, t)$ exists as $t \to T$ and this gives what we were required to prove. We also note the relation

$$
\begin{aligned}
(\varphi(x), \phi(x, 0))_{H_1} - (\psi(x, T), \phi(x, T))_{H_1} + \int_0^T \int_\Omega \Big[-\Delta\psi \frac{\partial \phi}{\partial t} + \\
+ \Delta\psi \Big(\frac{\partial\psi}{\partial x_1} \frac{\partial \phi}{\partial x_2} - \frac{\partial\psi}{\partial x_2} \frac{\partial \phi}{\partial x_1} \Big) \Big] dx\, dt = \int_0^T \int_\Omega f \phi \, dx\, dt,
\end{aligned}
\tag{3.13}
$$

which can be used instead of (3.6) in the definition of a generalized solution*.

Lemma 3.2. Let $\psi(x, t)$ be a generalized solution of the problem (3.2)-(3.4) in the sense (3.6). Then there exist generalized derivatives $\partial^2\psi/\partial x_1 \partial t$, $\partial^2\psi/\partial x_2 \partial t$ and for any $p > 1$

$$
\max_{0 \leqslant t \leqslant T} \left\| \frac{\partial^2 \psi}{\partial x_i \partial t} \right\|_{L_p(\Omega)} \leqslant Cp \max_{0 \leqslant t \leqslant T} [\| \mathbf{F} \|_{L_p(\Omega)} + \| \psi \|_{V_1} \| \nabla\psi \|_{L_p(\Omega)}]. \tag{3.14}
$$

The proof is based on the fact that relations (3.2), (3.3) can be written in the form

$$
\Delta \frac{\partial\psi}{\partial t} = f - \frac{\partial}{\partial x_1} \Big(\Delta\psi \frac{\partial\psi}{\partial x_2} \Big) + \frac{\partial}{\partial x_2} \Big(\Delta\psi \frac{\partial\psi}{\partial x_1} \Big), \qquad \frac{\partial\psi}{\partial t} \Big|_S = 0 \tag{3.15}
$$

and Theorem 2.2 can be applied to (3.15).

Let us give the technical details of the proof. For any $r > 2$ we can find a sequence of functions $\{g_k(x, t)\}$, $k = 1, 2, \ldots$, such that Δg_k are continuously differentiable in Q_T, $g_k|_S = 0$ and $\Delta g_k \Rightarrow \Delta\psi$ in $L_{p, 2r}$:

---

* We note that if the derivatives with φ in the first two terms of (3.13) are transferred to ψ then the resulting identity will be true for any smooth φ.

$$\lim_{k\to\infty} \int_0^T \| \Delta g_k - \Delta\psi \|_{L_p}^{2r} \, dt = 0. \tag{3.16}$$

(For example we could take Δg_k as mean functions for $\Delta\psi$ and define g_k correspondingly.) Let us define the functions $q_k(x, t)$ as generalized solutions of the boundary problems

$$\Delta q_k = \frac{\partial}{\partial x_2}\left(\Delta g_k \frac{\partial g_k}{\partial x_1} + F_1\right) - \frac{\partial}{\partial x_1}\left(\Delta g_k \frac{\partial g_k}{\partial x_2} + F_2\right), \quad q_k|_S = 0; \tag{3.17}$$

F_1, F_2 have come in as a result of the substitution (3.5).

According to Theorem 2.2 these solutions exist and for them we have the relation

$$\| q_k \|_{W_p^{(1)}} \leqslant Cp\left(\| \mathbf{F} \|_{L_p} + \sum_{i=1}^{2} \left\| \Delta g_k \frac{\partial g_k}{\partial x_i} \right\|_{L_p}\right). \tag{3.18}$$

We introduce the functions $\psi_k(x, t)$ defined by the equations

$$\psi_k(x, t) = \varphi(x) + \int_0^t q_k(x, \tau)\, d\tau. \tag{3.19}$$

From (3.18), (3.19) using the insertion theorem and (3.16) we obtain the estimate

$$\sum_{i=1}^{2} \left\| \frac{\partial^2 \psi_k}{\partial x_i \partial t} \right\|_{L_{p,r}}^{r} \leqslant C_1^r p^r \left(\| \mathbf{F} \|_{L_{p,r}}^{r} + \| \, |\Delta\psi| \, |\nabla\psi| \, \|_{L_{p,2r}}^{2r}\right), \tag{3.20}$$

which is uniform w.r.t. k, $k = 1, 2, \ldots$, where C_1 depends only on the region Ω and not on k, p, r.

Then by the definition of a generalized solution of the problem (3.17) we have the integral identity

$$-\int_\Omega \nabla q_k \nabla \Phi \, dx = \int_\Omega \left[\left(-\Delta g_k \frac{\partial g_k}{\partial x_1} + F_1\right)\frac{\partial \Phi}{\partial x_2} + \left(\Delta g_k \frac{\partial g_k}{\partial x_2} + F_2\right)\frac{\partial \Phi}{\partial x_1}\right] dx \tag{3.21}$$

for any function $\Phi(x, t)$ smooth in Q_T, $\Phi(x, T) = \Phi|_S = 0$. We transform the left-hand side of relation (3.21):

$$-\int_\Omega \nabla q_k \nabla \Phi \, dx = -\int_\Omega \nabla \frac{\partial \psi_k}{\partial t} \nabla \Phi \, dx = -\frac{d}{dt}\int_\Omega \nabla\psi_k \nabla \Phi \, dx - \int_\Omega \Delta\psi_k \frac{\partial \Phi}{\partial t} \, dx. \tag{3.22}$$

Integrating equation (3.21) w.r.t. t and using (3.5), (3.22) we find

$$\int_{\Omega} \nabla\varphi\nabla\phi(x,0)dx+\int_0^T\int_{\Omega}\left[-\Delta\psi_k\frac{\partial\phi}{\partial t}+\Delta g_k\left(\frac{\partial g_k}{\partial x_1}\frac{\partial\phi}{\partial x_2}-\frac{\partial g_k}{\partial x_2}\frac{\partial\phi}{\partial x_1}\right)\right]dxdt = \\ =\int_0^T f\phi\,dxdt. \tag{3.23}$$

It follows from (3.16) that

$$\lim_{k\to\infty}\int_0^T\int_{\Omega}\Delta g_k\left(\frac{\partial g_k}{\partial x_1}\frac{\partial\phi}{\partial x_2}-\frac{\partial g_k}{\partial x_2}\frac{\partial\phi}{\partial x_1}\right)dxdt = \\ =\int_0^T\int_{\Omega}\Delta\psi\left(\frac{\partial\psi}{\partial x_1}\frac{\partial\phi}{\partial x_2}-\frac{\partial\psi}{\partial x_2}\frac{\partial\phi}{\partial x_1}\right)dx\,dt. \tag{3.24}$$

From (3.23), (3.24) and the definition of a generalized solution of (3.6) we find

$$\lim_{k\to\infty}\int_0^T\int_{\Omega}\Delta\psi_k\frac{\partial\phi}{\partial t}dx\,dt=\lim_{k\to\infty}\left(\psi_k,\ \frac{\partial\phi}{\partial t}\right)_{H_1'}=\left(\psi,\ \frac{\partial\phi}{\partial t}\right)_{H_1'}. \tag{3.25}$$

Now since the set of functions $\left\{\frac{\partial\phi}{\partial t}\right\}$ is dense in H_1' and the set of functions $\{\psi_k\}$, as it follows from (3.16), (3.18), (3.19), is uniformly bounded in H_1', we conclude that $\psi_k\to\psi$ weakly in H_1' and therefore strongly in $L_2(Q_T)$.

Using the property that generalized differentiation is weakly closed and the inequality (3.20) we see that the derivatives $\partial^2\psi/\partial x_i\partial t$ exist and that

$$\sum_{i=1}^{2}\left\|\frac{\partial^2\psi}{\partial x_i\partial t}\right\|_{L_{p,r}}\leqslant C_2p(\|\mathbf{F}\|_{L_{p,r}}+\|\psi\|_{V_1}\|\nabla\psi\|_{L_{p,r}}), \tag{3.26}$$

where C_2 is a constant which does not depend on ψ, $\mathbf{F}$, p, r. We have then then to make r tend to ∞ in (3.26) and use the fact that for an arbitrary function $\alpha(x,\ t)$

$$\overline{\lim_{r\to\infty}}\|\alpha\|_{L_{p,r}}=\max_t\|\alpha\|_{L_p(\Omega)} \tag{3.27}$$

(provided, of course, that the right-hand side of (3.27) exists), and we arrive at (3.14).

Now let us prove the uniqueness of the generalized solution..

Theorem 3.1. There cannot exist more than one generalized solution of the problem (3.2)-(3.4) in the sense (3.6).

Let us suppose, on the contrary, that there exist two generalized solutions $\psi_1(x, t)$, $\psi_2(x, t)$ for the same φ and f. We form their difference $\alpha(x, t) = \psi_1(x, t) - \psi_2(x, t)$. We use Lemma 3.1. We write relations of the form (3.12) separately for ψ_1 and ψ_2 putting $\tau_1 = 0$, $\tau_2 = t$, $0 < t \leqslant T$. Subtracting one relation from the other we can see that

$$\begin{aligned} -\int_\Omega \nabla\alpha(x, t) \nabla\phi(x, t)\, dx + \int_0^t\int_\Omega \Big[-\Delta\alpha \frac{\partial\phi}{\partial t} + \Delta\alpha\Big(\frac{\partial\psi_1}{\partial x_1}\frac{\partial\phi}{\partial x_2} - \frac{\partial\psi_1}{\partial x_2}\frac{\partial\phi}{\partial x_1}\Big) + \\ + \Delta\psi_2\Big(\frac{\partial\alpha}{\partial x_1}\frac{\partial\phi}{\partial x_2} - \frac{\partial\alpha}{\partial x_2}\frac{\partial\phi}{\partial x_1}\Big)\Big] dx\, dt = 0 \end{aligned} \tag{3.28}$$

for any function $\varphi(x, t)$ which is smooth in Q_T and such that $\phi|_S = 0$.

Putting $\varphi = \alpha$ in (3.28) we obtain

$$-\int_\Omega |\nabla\alpha(x, t)|^2\, dx + \int_0^t\int_\Omega \Big[-\Delta\alpha\frac{\partial\alpha}{\partial t} + \Delta\alpha\Big(\frac{\partial\psi_1}{\partial x_1}\frac{\partial\alpha}{\partial x_2} - \frac{\partial\psi_1}{\partial x_2}\frac{\partial\alpha}{\partial x_1}\Big)\Big] dx\, dt = 0. \tag{3.29}$$

Integrating by parts (the legality of this operation is easily proved with the help of Lemma 3.2) we bring (3.29) to the form

$$\begin{aligned} \frac{1}{2}\int_\Omega |\nabla\alpha(x, t)|^2\, dx = \int_0^t\int_\Omega \Big\{\frac{\partial^2\psi_1}{\partial x_1\partial x_2}\Big[\Big(\frac{\partial\alpha}{\partial x_1}\Big)^2 - \Big(\frac{\partial\alpha}{\partial x_2}\Big)^2\Big] + \\ + \Big(\frac{\partial^2\psi_1}{\partial x_2^2} - \frac{\partial^2\psi_1}{\partial x_1^2}\Big)\frac{\partial\alpha}{\partial x_1}\frac{\partial\alpha}{\partial x_2}\Big\}\, dx\, dt. \end{aligned} \tag{3.30}$$

Since $\alpha \in V_1$ according to Lemma 2.2 $\partial\alpha/\partial x_1$, $\partial\alpha/\partial x_2$ are bounded

$$(\nabla\alpha)^2 = \Big(\frac{\partial\alpha}{\partial x_1}\Big)^2 + \Big(\frac{\partial\alpha}{\partial x_2}\Big)^2 \leqslant M^2 \tag{3.31}$$

in all Q_T. From (3.30), putting $z^2(t) = \int_\Omega (\nabla\alpha)^2\, dx$, we have

$$z\frac{dz}{dt} \leqslant \int_\Omega \Big(\Big|\frac{\partial^2\psi_1}{\partial x_1\partial x_2}\Big| + \Big|\frac{\partial^2\psi_1}{\partial x_2^2} - \frac{\partial^2\psi_1}{\partial x_1^2}\Big|\Big)(\nabla\alpha)^2\, dx. \tag{3.32}$$

Now let us estimate the right-hand side of (3.32), using the Hölder inequality, the inequality (3.31) and Lemma 2.2:

$$
\begin{aligned}
z\frac{dz}{dt} &\leqslant M^{\varepsilon}\int_{\Omega}\left(\left|\frac{\partial^2\psi_1}{\partial x_1\partial x_2}\right| + \left|\frac{\partial^2\psi_1}{\partial x_2^2} - \frac{\partial^2\psi_1}{\partial x_1^2}\right|\right)|\nabla\alpha|^{2-\varepsilon}\,dx \leqslant \\
&\leqslant M^{\varepsilon}\left(\left\|\frac{\partial^2\psi_1}{\partial x_1\partial x_2}\right\|_{L_{2/\varepsilon}} + \left\|\frac{\partial^2\psi_1}{\partial x_2^2} - \frac{\partial^2\psi_1}{\partial x_1^2}\right\|_{L_{2/\varepsilon}}\right) z^{2-\varepsilon} \leqslant M^{\varepsilon}C\frac{2}{\varepsilon}\|\psi_1\|_V z^{2-\varepsilon}
\end{aligned}
\tag{3.33}
$$

Since $\|\psi_1\|_V < M_1$ for $t \in [0, T]$ using (3.33) and integrating w.r.t. t we obtain

$$z^{\varepsilon}(t) \leqslant 2M^{\varepsilon}CM_1t, \tag{3.34}$$

or

$$z(t) \leqslant M\,(2CM_1t)^{1/\varepsilon}. \tag{3.35}$$

Now letting ε tend to zero in (3.35) for sufficiently small $t: 0 \leqslant t \leqslant 1/4CM_1 = \tau_0$ we obtain $z(t) = 0$ and therefore $\alpha(x, t) = 0$ also. Repeating the argument we show that $\alpha(x, t) = 0$ for values of t belonging to the segments $[\tau_0, 2\tau_0]$, $[2\tau_0, 3\tau_0]$ and so on. Thus we find that $\alpha(x, t) = 0$ in the whole cylinder Q_T. This contradiction proves the theorem.

4. A simply-connected region. Existence of a generalized solution

To prove the existence of a generalized solution of the problem (3.2)-(3.4) in the sense (3.6) we reduce the problem to a certain operator equation and show that the conditions of Schauder's principle of a fixed point [16] are satisfied.

We first consider the following linear problem:

$$\frac{\partial\Delta\psi'}{\partial t} + \frac{\partial\psi}{\partial x_2}\frac{\partial\Delta\psi'}{\partial x_1} - \frac{\partial\psi}{\partial x_1}\frac{\partial\Delta\psi'}{\partial x_2} = f(x, t), \tag{4.1}$$

$$\psi'|_S = 0, \tag{4.2}$$

$$\psi'|_{t=0} = \varphi(x). \tag{4.3}$$

We call the function $\psi' \in V_1$ satisfying the integral identity

$$(\varphi(x), \phi(x, 0))_{H_1} + \int_0^T\int_{\Omega}\left[-\Delta\psi'\frac{\partial\phi}{\partial t} + \Delta\psi'\left(\frac{\partial\psi}{\partial x_1}\frac{\partial\phi}{\partial x_2} - \frac{\partial\psi}{\partial x_2}\frac{\partial\phi}{\partial x_1}\right)\right]dxdt =$$

$$= \int_0^T \int_\Omega f \phi \, dx \, dt \tag{4.4}$$

for any function $\phi(x, t)$ which is smooth in the cylinder $Q_T = \Omega \times [0, T]$ and such that $\phi(x, T) = \phi|_S = 0$ a generalized solution of this problem in Q_T.

Lemma 4.1. Let $\psi(x, t) \in C'(Q_T)$, $f(x, t)$ be a bounded measurable function, $\varphi(x) \in V$, $S \in C^{(2)}$. Then there exists one and only one generalized solution of the problem (4.1)-(4.3) in the sense (4.4).

To prove uniqueness it is sufficient merely to repeat the argument given in the proof of Theorem 1, viz.: (1) to show that $\psi'(x, t)$ is a generalized solution in any cylinder Q_τ, $0 < \tau < T$; (2) to prove that there exist generalized derivatives $\partial^2\psi'/\partial x_i \partial t$ with

$$\max_t \left\| \frac{\partial^2 \psi'}{\partial x_i \partial t} \right\|_{L_p(\Omega)} \leqslant Cp \max_t (\|\mathbf{F}\|_{L_p(\Omega)} + \|\psi'\|_{V_1} \|\nabla\psi\|_{L_p(\Omega)}; \tag{4.5}$$

(3) to prove that ψ' is weakly continuous w.r.t. t in H_1 and putting $f = 0$, $\varphi = 0$, $\varphi = \psi'$ in (4.4) (where T is replaced by arbitrary $0 \leqslant t \leqslant T$) to obtain for $\psi'(x, t)$ a relation of the form (3.33) from which, as before, we obtain $\psi'(x, t) = 0$ if $f = \varphi = 0$.

To prove existence we consider the following mixed problem for a parabolic equations:

$$\frac{\partial u_\varepsilon}{\partial t} - \varepsilon^2 \Delta u_\varepsilon + \frac{\partial \psi}{\partial x_2} \frac{\partial u_\varepsilon}{\partial x_1} - \frac{\partial \psi}{\partial x_1} \frac{\partial v_\varepsilon}{\partial x_2} = f_\varepsilon(x, t), \tag{4.6}$$

$$u_\varepsilon|_S = 0, \qquad u_\varepsilon|_{t=0} = u_{0\varepsilon}(x), \tag{4.7}$$

where $u_{0\varepsilon}(x)$ is some function which is smooth in Ω and $u_{0\varepsilon}|_S = 0$; $f_\varepsilon(x, t)$ is a smooth function defined in Q_T. We know (see [11] - [13]) that there exists a function $u_\varepsilon(x, t)$ which solves the problem (4.6) (4.7) in the following sense:

1) $\partial u_\varepsilon/\partial t$, $\partial^2 u_\varepsilon/\partial x_i \partial x_k$ exist in Q_T as generalized derivatives and (4.6) is satisfied almost everywhere in Q_T,

2) the derivatives $\partial u_\varepsilon/\partial t$, $\partial^2 u_\varepsilon/\partial x_i \partial x_k \in L_2(Q_T)$,

3) $u_\varepsilon(x, t)$ is continuous in $\bar{Q}_T$ and conditions (4.7) are satisfied*.

The function $u_\varepsilon(x, t)$ satisfies the integral identity $(0 \leqslant \tau \leqslant T)$:

$$\int_\Omega u_\varepsilon(x, \tau)\, \phi(x, \tau)\, dx - \int_\Omega u_{0\varepsilon}(x)\, \phi(x, 0)\, dx + \int_0^\tau \int_\Omega \Big[-u_\varepsilon \frac{\partial \phi}{\partial t} + u_\varepsilon \Big(\frac{\partial \psi}{\partial x_1}\frac{\partial \phi}{\partial x_2} - \frac{\partial \psi}{\partial x_2}\frac{\partial \phi}{\partial x_1}\Big)\Big] dx dt + \varepsilon^2 \int_0^\tau \int_\Omega \nabla u_\varepsilon \nabla \phi\, dx dt = \int_0^\tau \int_\Omega f \phi\, dx dt, \tag{4.8}$$

where $\varphi(x, t)$ is an arbitrary function satisfying conditions (1) - (3).

Let us suppose that $u_{0\varepsilon}(x) \Rightarrow \Delta\varphi(x)$ in any $L_p(\Omega)$, $f_\varepsilon(x, t) \Rightarrow f(x, t)$ in any $L_p(Q_T)$, $p > 1$. This supposition is allowed since $\Delta\varphi$, f are bounded functions. We are now getting ready to state that as $\varepsilon \to 0$, u_ε in some sense converges to $\Delta\psi'$ where ψ' is a generalized solution of the problem (4.1)-(4.3). To do this we need a few estimates.

We put $\phi = u_\varepsilon^{2k-1}$ in (4.8). We see that

$$\int_0^\tau \int_\Omega u_\varepsilon \Big(\frac{\partial \psi}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial \psi}{\partial x_2}\frac{\partial}{\partial x_1}\Big) u_\varepsilon^{2k-1} dx dt = 0, \tag{4.9}$$

This can be established directly by integration by parts if ψ is twice continuously differentiable and for any $\psi \in C'$ by closure. Using (4.9) we have

$$\frac{1}{2k}\int_\Omega u_\varepsilon^{2k}(x, \tau)\, dx - \frac{1}{2k}\int_\Omega u_{0\varepsilon}^{2k}(x)\, dx + \varepsilon^2 (2k-1) \int_0^\tau \int_\Omega (u_\varepsilon^{k-1} \nabla u_\varepsilon)^2\, dx\, dt = \int_0^\tau \int_\Omega f_\varepsilon u_\varepsilon^{2k-1} dx\, dt. \tag{4.10}$$

From (4.10) we have the relation

$$\frac{1}{2k}\frac{d}{d\tau}\int_\Omega u_\varepsilon^{2k}(x, \tau)\, dx + (2k-1)\varepsilon^2 \int_\Omega (u_\varepsilon^{k-1}\nabla u_\varepsilon)^2\, dx = \int_\Omega f_\varepsilon u_\varepsilon^{2k-1} dx. \tag{4.11}$$

From (4.11), using the Hölder inequality for the estimate of the integral

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* In fact with the conditions given here $u_\varepsilon(x, t)$ possesses better differential properties. However this is sufficient for our purposes.

on the right-hand side we have

$$\|u_\varepsilon\|_{L_{2k}}^{2k-1}\frac{d}{d\tau}\|u_\varepsilon\|_{L_{2k}}\leqslant\|f_\varepsilon\|_{L_{2k}}\|u_\varepsilon\|_{L_{2k}}^{2k-1};$$
$$\|u_\varepsilon(x,\tau)\|_{L_{2k}(\Omega)}\leqslant\|u_{0\varepsilon}\|_{L_{2k}(\Omega)}+\int_0^\tau\|f_\varepsilon\|_{L_{2k}(\Omega)}\,d\tau. \tag{4.12}$$

From (4.10) with $k = 1$ and using (4.12) we find

$$\|u_\varepsilon\|_{H_1'}^2=\int_0^T\int_\Omega(\nabla u_\varepsilon)^2\,dx\,dt\leqslant\frac{M^2}{\varepsilon^2}, \tag{4.13}$$

where

$$M^2=\int_0^T\|f_\varepsilon\|_{L_2}\,dt\left(\|u_{0\varepsilon}\|_{L_2}+\int_0^T\|f_\varepsilon\|_{L_2}\,dt\right)+\frac{1}{2}\|u_{0\varepsilon}\|_{L_2}^2.$$

The quantity M^2, like the right-hand side of inequality (4.12) is bounded uniformly w.r.t. ε since $u_{0\varepsilon}, f_\varepsilon$ converge to $\Delta\varphi$, f in $L_p(\Omega)$ and $L_p(Q_T)$ respectively for any p.

From (4.12) there exists a sequence ε_r, $r = 1, 2, \ldots$, which tends to zero and is such that the corresponding sequence $u_{\varepsilon_r}(x, t)$ converges weakly in any $L_p(Q_T)$ to some function $u(x, t)$.

We note that as $\varepsilon \to 0$ for fixed $\phi \in H_1'$

$$I_\varepsilon=\varepsilon^2\int_0^T\int_\Omega\nabla u_\varepsilon\nabla\phi\,dx\,dt=\varepsilon^2(u_\varepsilon,\ \phi)_{H_1}\to 0. \tag{4.14}$$

This follows at once from (4.13) since

$$|I_\varepsilon|\leqslant\varepsilon^2\|u_\varepsilon\|_{H_1'}\|\phi\|_{H_1'}\leqslant M\|\phi\|_{H_1'}\varepsilon. \tag{4.15}$$

Now writing $\tau = T$, $\varepsilon = \varepsilon_r$ in (4.8) and taking r to infinity we find that $\Delta\psi' = u = \lim\limits_{r\to\infty} u_{\varepsilon_r}$ satisfies the integral identity (4.4) for smooth $\phi(x, t)$ such that $\phi|_S = 0$, $\phi(x, T) = 0$. It follows from (4.12) that

$$\max_{0\leqslant t\leqslant T}\|\Delta\psi'\|_{L_{2k}(\Omega)}\leqslant\|\Delta\varphi\|_{L_{2k}(\Omega)}+\int_0^T\|f\|_{L_{2k}(\Omega)}dt. \tag{4.16}$$

To prove (4.16) it is sufficient to note that the sequence u_{ε_r} is bounded uniformly in $L_{2k,l}$ for any $l > 1$ and therefore converges weakly to $\Delta\psi'$ (is weakly compact in $L_{2k,l}$ and converges weakly in $L_p(Q_T)$ to $\Delta\psi'$). The estimate (4.16) can be obtained by taking the norm in $L_l(0,T]$ of both sides of the inequality (4.12) and then letting $r \to \infty$ and $l \to \infty$ in succession.

As $k \to \infty$ in (4.16)

$$\|\psi'(x,t)\|_{V_1} \leqslant \|\varphi\|_V + \int_0^T \max_{x\in\Omega} |f|\,dt \equiv R. \tag{4.17}$$

We have used the fact that $\lim_{k\to\infty} \|f\|_{L_{2k}} = \max_x |f|$ and applied Lebesgue's theorem about passage to the limit under the integral sign.

Thus we have shown that $\psi'(x, t)$ is a generalized solution of the problem (4.1)-(4.3) in the sense (4.4).

According to Lemma 4.1 the operator A which sets the element $\psi' = A\psi \in V_1$ in correspondence with any element $\psi \in C'$ is defined. Comparing (3.6) and (4.4) we conclude that the problem of finding a generalized solution of the problem (3.2)-(3.4) reduces to the solution of the equation

$$\psi = A\psi. \tag{4.18}$$

Lemma 4.2. The operator A is completely continuous in C' and translates the whole space C' into a certain sphere of radius R_1.

Suppose that the sequence $\{\psi_r(x,t)\}$ converges to $\psi'(x, t)$ in C'. From (4.17) we have for $\psi_r' = A\psi_r$

$$\|\psi_r'\|_{V_1} \leqslant \|\varphi\|_V + \int_0^T \max_x |f|\,d\tau = R. \tag{4.19}$$

Therefore the sequence $\{\Delta\psi_r'\}$ is weakly compact in $L_p(Q_T)$ for any $p > 1$. Let $\bar{\psi}$ be one of the limit points. Passing to the limit in the integral identities of the form (4.4) written for the functions ψ_r' it is easy to see that ψ is a generalized solution of the problem (4.1)-(4.3) corresponding to the function ψ. Because of the uniqueness of the generalized solution $\bar{\psi} = A\psi$. This also proves that the operator A is closed in C'.

The fact that the operator A translates C' into a sphere follows from the inequality ($\psi' = A\psi$)

$$\|\psi'\|_{C'} \leqslant C\,\|\psi'\|_{V'} \leqslant CR \equiv R_1, \tag{4.20}$$

which follows from the insertion theorem and (4.17).

It remains to show that the operator A is compact. Let $\{\psi_r\}$ be some sequence of C' and let $\|\psi_r\|_{C'} < M$ uniformly w.r.t. r. From (4.17) and the estimate for $\psi_r' = A\psi$

$$\sum_{i=1}^{2} \max_t \left\| \frac{\partial^2\psi_r'}{\partial x_i \partial t} \right\| \leqslant C_1, \tag{4.21}$$

which follows from the inequality (4.5) it follows that

$$\left\| \frac{\partial\psi_r'}{\partial x_i} \right\|_{W_k^{(1)}(Q_T)} < C_2 \tag{4.22}$$

uniformly w.r.t. r for any $k > 1$. If we take $k > 3$, then using the insertion theorem it will follow from (4.22) that

$$\left\| \frac{\partial\psi_r'}{\partial x_i} \right\|_{B^{0,\lambda}(Q_T)} < C_3 \tag{4.23}$$

for $\lambda = (k - 3)/k$. The equicontinuity of the sequences ψ_r', $\partial\psi_r'/\partial x_1$, $\partial\psi_r'/\partial x_2$ follows from (4.23). Applying the Arzelà criterion we can see that the sequence ψ_r' is compact in C'. This completes the proof that the operator A is fully continuous.

Theorem 4.1. Let $S \in C^{(2)}$, $\varphi(x) \in V$, $f(x, t)$ be bounded, $\mathbf{F}(x, t)$ be strongly continuous w.r.t. t on $[0, T]$ in any L_p, $p > 1$. Then there exists a generalized solution of the problem (3.2)-(3.4) in the sense (3.6). The limit conditions (3.3)-(3.4) are satisfied in the classical sense; $\Delta\psi$, $\partial^2\psi/\partial x_i\partial t$ are strongly continuous w.r.t. t on $[0, T]$ in any $L_p(\Omega)$.

According to Lemma 4.2 the operator A in the space C' satisfies the conditions of Schauder's principle of a fixed point. Therefore equation (4.18) has the solution $\psi(x, t)$ in C'. But the operator A operates from C' into V_1. Therefore $\psi \in V_1$. This proves the existence of a generalized solution.

Now let us prove that $\Delta\psi(x, t)$ is a function which is continuous w.r.t. t in any $L_p(\Omega)$. Since $\psi(x, t)$, from Lemma 3.1, is a generalized solution in any cylinder $Q_{\tau_1,\tau_2} = \Omega \times [\tau_1, \tau_2]$, $0 \leqslant \tau_1 < \tau_2 \leqslant T$ we have the identity

$$\int_{\Omega} \Delta\psi(x, \tau_2)\phi(x, \tau_2)\,dx - \int_{\Omega} \Delta\psi(x, \tau_1)\phi(x, \tau_1)\,dx + \int_{\tau_1}^{\tau_2}\int_{\Omega}\Big[-\Delta\psi\frac{\partial\phi}{\partial t} + \Delta\psi\Big(\frac{\partial\psi}{\partial x_1}\frac{\partial\phi}{\partial x_2} - \frac{\partial\psi}{\partial x_2}\frac{\partial\phi}{\partial x_1}\Big)\Big]\,dx\,dt = \int_{\tau_1}^{\tau_2}\int_{\Omega} f\phi\,dx\,dt \tag{4.24}$$

for any function $\varphi(x, t)$ smooth in Q_T. Using (4.24), as τ_2 tends to τ_1, we have

$$\begin{aligned}\lim_{\tau_2\to\tau_1}\int_{\Omega}[\Delta\psi(x, \tau_2) - \Delta\psi(x,\tau_1)]\phi(x, \tau_1)\,dx &= \lim_{\tau_2\to\tau_1}\Big\{\int_{\Omega}[\Delta\psi(x, \tau_2) - \Delta\psi(x, \tau_1)]\phi(x, \tau_1)\,dx + \\ &+ \int_{\Omega}\Delta\psi(x, \tau_2)[\phi(x, \tau_2) - \phi(x, \tau_1)]\,dx\Big\} = \\ &= \lim_{\tau_2\to\tau_1}\int_{\Omega}[\Delta\psi(x, \tau_2)\phi(x, \tau_2) - \Delta\psi(x, \tau_1)\phi(x, \tau_1)]\,dx = 0.\end{aligned} \tag{4.25}$$

We note that we can take any function which is smooth in Ω as the function $\varphi(x, \tau_1)$. Since the set of such functions is dense in any $L_p(\Omega)$ and $\|\Delta\psi(x, t)\|_{L_p(\Omega)}$ are uniformly bounded w.r.t. $t \in [0, T]$ we find that as $\tau_2 \to \tau_1$ the sequence $\Delta\psi(x, \tau_2)$ is weakly convergent to $\Delta\psi(x, \tau_1)$ in any $L_p(\Omega)$, $p > 1$.

Then

$$\|\Delta\psi(x, \tau_2)\|_{L_p(\Omega)} \leqslant \|\Delta\psi(x, \tau_1)\|_{L_p(\Omega)} + \int_{\tau_1}^{\tau_2}\|f(x, t)\|_{L_p(\Omega)}. \tag{4.26}$$

(This can be derived from (4.24) in the same way as (4.16) can be derived from (4.4).)

As τ_2 tends to τ_1 we find from (4.26) that

$$\overline{\lim_{\tau_2\to\tau_1}}\|\Delta\psi(x, \tau_2)\|_{L_p(\Omega)} \leqslant \|\Delta\psi(x, \tau_1)\|_{L_p(\Omega)}. \tag{4.27}$$

Thus as $\tau_2 \to \tau_1$, $\Delta\psi(x, \tau_2)$ converges to $\Delta\psi(x, \tau_1)$ and (4.27) holds. From the Radon-Riesz theorem [14] it follows that $\Delta\psi(x, \tau_2)$ converges strongly to $\Delta\psi(x, \tau_1)$ as $\tau_2 \to \tau_1$:

$$\lim_{\tau_2\to\tau_1}\|\Delta\psi(x, \tau_2) - \Delta\psi(x, \tau_1)\|_{L_p(\Omega)} = 0. \tag{4.28}$$

Since

$$\| \psi (x, \tau_2) - \psi (x, \tau_1) \|_{W_p^{(2)}(\Omega)} \leqslant C_p \| \Delta\psi (x, \tau_2) - \Delta\psi (x, \tau_1) \|_{L_p}, \quad (4.29)$$

we find that $\psi(x, t)$ is strongly continuous w.r.t. t in $W_p^{(2)}(\Omega)$ on $[0, T]$. Then using Lemma 3.3 we can show that $\partial\psi/\partial x_i \in W_k^{(1)}(Q_T)$ for any $k > 1$ and therefore, from the insertion theorem, $\partial\psi/\partial x_i \in B^{0,\lambda}(Q_T)$. From this it follows that $\psi(x, t)$ is continuous in Q_T and that the boundary condition (3.3) is satisfied, and also the initial condition (3.4) is satisfied in a stronger form:

$$\lim_{t\to 0} \| \psi (x, t) - \varphi (x) \|_{B^{1,\lambda}(\Omega)} = 0 \quad (4.30)$$

for any $0 \leqslant \lambda < 1$.

Finally, the strong continuity of the functions $\partial^2\psi/\partial x_i\partial t$ w.r.t. t on $[0, T]$ in $L_p(\Omega)$ (for any $p > 1$) follows from the relation

$$\frac{\partial^2\psi}{\partial x_i\partial t} = L_1^{(i)} \left(\Delta\psi \frac{\partial\psi}{\partial x_i} \right) + L_2^{(i)} \left(\Delta\psi \frac{\partial\psi}{\partial x_2} \right) + M_1^{(i)} F_1 + M_2^{(i)} F_2, \quad (4.31)$$

which is a consequence of (3.15) where $L_k^{(i)}$, $M_k^{(i)}$ are linear operators, continuous in $L_p(\Omega)$ and independent of t.

5. A multiply connected region

Now let Ω be an $(n + 1)$-times connected bounded region, the boundary of which consists of closed contours $S_0, S_1, \ldots, S_n$, the contour S_0 containing the others.

We assume that $S_k \in C^{(2)}$

Considering the problem (1.1) - (1.4) we again introduce the current function $\psi(x, t)$ with equation (3.1). As before the function $\psi(x, t)$ satisfies equation (3.2) and the initial condition (3.4). Instead of (3.3) we obtain the boundary condition

$$\psi |_{S_0} = 0, \qquad \psi |_{S_k} = \lambda_k (t), \quad k = 1, 2, \ldots, n, \quad (5.1)$$

where $\lambda_k(t)$ are known functions of time. To derive (5.1) we first note that (1.3), (3.1) give $(\partial\psi/\partial s)_S = 0$ ($\partial/\partial s$ is the derivative along the contour S). Further, since the relations (3.1) define the function ψ (for given $\mathbf{v}$) to within an arbitrary function of time t we can so choose the latter that the first of the relations (5.1) is satisfied.

Thus in this case there are new unknowns $\lambda_1(t), \ldots, \lambda_n(t)$. The missing n equations are given by the conditions that it is possible to determine the pressure $P(x, t)$ as a single-values function from the relations (1.1). For sufficiently smooth $\mathbf{v}(x, t)$, $\mathbf{F}(x, t)$ these conditions, as we know, have the form

$$\oint_{S_k} \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\,\mathbf{v} - \mathbf{F}\right] \mathbf{s}\, ds = 0, \qquad k = 1, 2, \ldots, n \tag{5.2}$$

where $\mathbf{s}$ is the unit tangent to S and $\mathbf{n}$, $\mathbf{s}$ (where $\mathbf{n}$ is the external normal) forms a right-handed pair. We now show that (5.2) can be transformed to a form which contains area integrals only.

We define the functions $\psi_k(x)$, $x \in \Omega$ by the conditions

$$\Delta\psi_k = 0, \qquad \psi_k|_{S_r} = \delta_{kr}, \quad k = 1, 2, \ldots, n, \quad r = 0, 1, \ldots, n. \tag{5.3}$$

It is not difficult to see that there exists a function $\tilde{\psi}_k$, as smooth as we please, which satisfies the same boundary conditions as ψ_k. To prove this it is sufficient to draw a sufficiently smooth contour $\tilde{S}_k \subset \Omega$ in a small neighbourhood of the contour S_k, to introduce curvilinear coordinates $\rho, \tilde{s}$ the distance between the point x and $\tilde{S}_k$ and the arc of the contour $\tilde{S}_k$, and then to put $\tilde{\psi}_k(x) = 1$ for x lying between S_k and $\tilde{S}_k$, $\tilde{\psi}_k(x) = 0$ if $\rho \geqslant \varepsilon$ (if the point lies inside the region bounded by the contour $\tilde{S}_k$ then $\rho > 0$) for sufficiently small fixed $\varepsilon > 0$ and at the other points of the region to find $\tilde{\psi}_k$ in the form of a polynomial in powers of ρ with coefficients which depend on $\tilde{s}$.

Now, using Theorem 2.1, we see that $\psi_k - \tilde{\psi}_k \in V$ and thus from Lemma 2.2 we have the estimate

$$\| D^2\psi_k \|_{L_p(\Omega)} \leqslant Cp, \tag{5.4}$$

where C depends only on the region Ω.

We introduce the vectors $\mathbf{u}_k(x)$, $k = 1, 2, \ldots, n$ with the equations

$$\mathbf{u}_k(x) = \left(\frac{\partial \psi_k}{\partial x_2}, \ -\frac{\partial \psi_k}{\partial x_1}\right). \tag{5.5}$$

Lemma 5.1. Let it be required to find the single-valued function $Q(x)$, $x = (x_1, x_2) \in \Omega$ from the equation

$$\nabla Q = \mathbf{b}, \tag{5.6}$$

where $\mathbf{b}$ is a given smooth vector. A necessary and sufficient condition for this problem to have a solution is that the relations

$$\frac{\partial b_2}{\partial x_1} - \frac{\partial b_1}{\partial x_2} = 0, \tag{5.7}$$

$$\int_{\Omega} \mathbf{b}\mathbf{u}_k \, dx = 0, \qquad k = 1, 2, \ldots, n \tag{5.8}$$

shall be satisfied.

To prove the lemma we transform (5.8) using (5.5), (5.7) and integrating by parts to the form

$$\begin{aligned}\int_{\Omega} \mathbf{b}\mathbf{u}_k \, dx = \int_{\Omega} \left(b_1 \frac{\partial \psi_k}{\partial x_2} - b_2 \frac{\partial \psi_k}{\partial x_1} \right) dx = \int_{\Omega} \left[\frac{\partial}{\partial x_1} (b_1 \psi_k) - \frac{\partial}{\partial x_2} (b_1 \psi_k) \right] dx = \\ = \sum_{r=0}^{n} \oint_{S_r} \psi_k \, (\mathbf{bs}) \, ds = 0.\end{aligned} \tag{5.9}$$

Using (5.3) and (5.9) we find

$$\oint_{S_k} \mathbf{bs} \, ds = 0, \qquad k = 1, 2, \ldots, n. \tag{5.10}$$

Thus (5.7), (5.8) are the same as the well-known conditions (5.7), (5.10) for the solvability of the problem (5.6), given in [15] for example. This proves the lemma.

We shall look for the current function $\psi(x, t)$ in the form

$$\psi = \psi_0 (x, t) + \sum_{k=1}^{n} \lambda_k (t) \psi_k (x). \tag{5.11}$$

Correspondingly we write the current function $\varphi(x)$ of the initial velocity vector $\mathbf{a}(x)$ as follows:

$$\varphi (x) = \varphi_0 (x) + \sum_{k=1}^{n} \lambda_{k0} \psi_k (x), \qquad \lambda_{k0} = \varphi |_{S_k}. \tag{5.12}$$

It is clear that $\psi_0 |_S = \varphi_0 |_S = 0$. Inserting (5.11) in (3.2), transforming condition (5.2) in accordance with Lemma 5.1 and using (5.1), (5.12) we obtain the problem

$$\frac{\partial \Delta \psi_0}{\partial t} + \frac{\partial}{\partial x_2} \left(\psi_0 + \sum_{k=1}^{n} \lambda_k \psi_k \right) \frac{\partial \Delta \psi_0}{\partial x_1} - \frac{\partial}{\partial x_1} \left(\psi_0 + \sum_{k=1}^{n} \lambda_k \psi_k \right) \frac{\partial \Delta \psi_0}{\partial x_2} = f (x, t), \tag{5.13}$$

$$\psi_0|_S = 0, \tag{5.14}$$

$$\psi_0|_{t=0} = \varphi_0(x), \tag{5.15}$$

$$\sum_{k=1}^{n} \lambda_k'(t) \int_\Omega \mathbf{u}_k \mathbf{u}_r \, dx - \sum_{k=1}^{n} \lambda_k(t) \int_\Omega \mathbf{u}_k \times \operatorname{rot} \mathbf{v}_0 \mathbf{u}_r \, dx + \\ + \int_\Omega \left(\frac{\partial \mathbf{v}_0}{\partial t} - \mathbf{v}_0 \times \operatorname{rot} \mathbf{v}_0 - \mathbf{F} \right) \mathbf{u}_r \, dx = 0, \qquad \mathbf{v}_0 = \left(\frac{\partial \psi_0}{\partial x_2}, \quad -\frac{\partial \psi_0}{\partial x_1} \right), \tag{5.16}$$

$$\lambda_r(0) = \lambda_{r0}, \qquad r = 1, 2, \ldots, n. \tag{5.17}$$

for $\psi_0(x, t)$, $\lambda_1(t)$, ..., $\lambda_n(t)$.

When ψ_0, λ_1, ..., λ_n are sufficiently smooth it is easy to prove, using Lemma 5.1, that the vector $\mathbf{v}$ defined by the relations (3.1),(5.11), together with the pressure $P(x, t)$ found from (1.1) gives a solution of the problem (1.1)-(1.4)..

Let us now find the generalized solution of the problem (5.13)-(5.17). We first prove a few lemmas.

Lemma 5.2. The vectors $\mathbf{u}_k(x)$ defined by the relation (5.5), are linearly independent and

$$\det \left(\int_\Omega \mathbf{u}_k \mathbf{u}_r \, dx \right) \neq 0. \tag{5.18}$$

Since this determinant is the Gramian of the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ (5.18) follows from the fact that these vectors are linearly independent.

Suppose that the constants $\gamma_1, \gamma_2, \ldots, \gamma_n$ are such that the equation

$$\sum_{k=1}^{n} \gamma_k \mathbf{u}_k = 0. \tag{5.19}$$

is satisfied. Let S_{0r} be some curve forming a diaphragm (i.e. a section along S_{0r} lowers the order of connectivity of the region Ω by 1) with ends on S_0 and S_r. Let us calculate the flux of a vector equal to the left-hand side of (5.19) in terms of S_{0r}. Let $\mathbf{s}_r$ the unit tangent to S_{0r} in a direction from S_0 to S_r, $\mathbf{n}$, the unit normal such that $\mathbf{n}$, $\mathbf{s}_r$ is a right-handed pair. Using (5.3), (5.5) we have

$$0 = \sum_{k=1}^{n} \gamma_k \int_{S_{0r}} \mathbf{u}_k \, \mathbf{n} \, ds = \sum_{k=1}^{n} \gamma_k \int_{S_{0r}} d\psi_k = \sum_{k=1}^{n} \gamma_k \delta_{kr} = \gamma_r. \tag{5.20}$$

Thus $\gamma_1 = \cdots = \gamma_n = 0$ and the lemma is proved.

Using Lemma 5.2 we can solve the relations (5.16) w.r.t. the derivatives $d\lambda_i/dt$. Again integrating w.r.t. t and using (5.17) we see that

$$\boldsymbol{\lambda}(t) = \boldsymbol{\lambda}_0 + B(\psi_0, \boldsymbol{\lambda}), \tag{5.21}$$

where we have put $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, $\boldsymbol{\lambda}_0 = (\lambda_{10}, \ldots, \lambda_{n0})$ and

$$[B(\psi_0, \boldsymbol{\lambda})]_s = \int_0^t \Lambda_s(\tau)\, d\tau,$$

$$\Lambda_s = \sum_{k,\, r=1}^{n} \xi_{rs}\lambda_k(t) \int_\Omega \mathbf{u}_k \times \operatorname{rot} \mathbf{v}_0 \mathbf{u}_r\, dx + \tag{5.22}$$
$$+ \sum_{k,\, r=1}^{n} \xi_{rs} \int_\Omega \left(\frac{\partial \mathbf{v}_0}{\partial t} - \mathbf{v}_0 \times \operatorname{rot} \mathbf{v}_0 - \mathbf{F}\right) \mathbf{u}_r\, dx;$$

ξ_{rs} are known numbers which depend only on the region Ω. Now let us generalize the definition of the operator A (4.18) to the case of a multiply-connected region. We introduce the space C'' of functions of x, t defined in the cylinder Q_T for almost all t, belonging to $C^{(1)}(\Omega)$ and equal to zero on S_0 and to certain functions of t only on $S_1, \ldots, S_n$ with the norm

$$\| \psi \|_{C''} = \max_{x,\, t} \left(\left| \frac{\partial \psi}{\partial x_1} \right| + \left| \frac{\partial \psi}{\partial x_2} \right| \right). \tag{5.23}$$

Lemma 5.3. To every function $\psi \in C''$ we can set in correspondence the function $A\psi = \psi'(x, t) \in V_1$, the generalized solution of a boundary problem of the form (4.1) - (4.3) (where φ is replaced by φ_0) in the sense (4.4). The operator A translates the whole space C'' into a sphere of the space C' and is completely continuous.

The proof is almost a word-for-word repetition of the proofs of Lemmas 4.1 and 4.2, since in the proofs of the latter we do not in fact use the condition $\psi|_S = 0$ but instead used the weaker condition $(\partial\psi/\partial s)|_S = 0$.

Lemma 5.3 enables us (at least for sufficiently smooth ψ_0) to derive the relation

$$\psi_0 = A\left(\psi_0 + \sum_{k=1}^{n} \lambda_k \psi_k\right). \tag{5.24}$$

from (5.13)-(5.15). We now naturally arrive at the following definition. We call the pair $(\psi_0(x, t), \boldsymbol{\lambda}(t))$ (where $\psi_0 \in V_1$, $\boldsymbol{\lambda}(t)$ is a vector-function which is continuous on $[0, T]$) satisfying the system of equations (5.21) and (5.24) a generalized solution of the problem (5.13) -

(5.17).

Theorem 5.1. Let $S \in C^{(2)}$, $f(x, t)$ be a measurable bounded function, $\varphi_0(x) \in V$. Then a generalized solution of the problem (5.13) - (5.17) exists and is unique. If in addition we suppose that $F(x, t)$ is strongly continuous w.r.t. t on $[0, T]$ in any $L_p(\Omega)$ then $\Delta\psi$, $\partial^2\psi/\partial x_i\partial t$ will possess the same property.

Let us first consider the proof of uniqueness. From the definition of the operator A and equation (5.24) it follows that

$$(\varphi_0(x), \phi(x, 0))|_{H_1} + \int_0^T \int_\Omega \left[- \Delta\psi_0 \frac{\partial \phi}{\partial t} + \Delta\psi_0 \left(\frac{\partial\psi}{\partial x_1} \frac{\partial \phi}{\partial x_2} - \frac{\partial\psi}{\partial x_2} \frac{\partial \phi}{\partial x_1} \right) \right] dx dt = \int_0^T \int_\Omega f\phi \, dx\, dt \tag{5.25}$$

for any function $\varphi(x, t)$ which is smooth in Q_T and such that

$$\phi(x, T) | = \phi|_S = 0, \qquad \psi = \psi_0(x, t) + \sum_{k=1}^n \lambda_k(t)\psi_k(x).$$

It follows from (5.3) that $\Delta\psi = \Delta\psi_0$. We transform the first term of equation (5.25) with the help of integration by parts and relations (5.3), (5.12):

$$\begin{aligned}(\varphi_0, \phi(x, 0))|_{H_1} &= \int_\Omega \nabla\varphi_0\nabla\phi(x, 0)\, dx = \int_\Omega \nabla\varphi\nabla\phi(x, 0)\, dx - \\ &- \sum_{k=1}^n \lambda_{k0} \int_\Omega \nabla\psi_k\nabla\phi(x, 0)\, dx = \int_\Omega \nabla\varphi\nabla\phi(x, 0)\, dx + \\ &+ \sum_{k=1}^n \lambda_{k0} \left(\int_\Omega \phi\Delta\psi_k dx + \oint_S \phi \frac{\partial\psi_k}{\partial n} ds \right) = \int_\Omega \nabla\varphi\nabla\phi(x, 0)\, dx.\end{aligned} \tag{5.26}$$

Now (5.25) can be put in the form

$$\begin{aligned}\int_\Omega \nabla\varphi\nabla\phi(x, 0)\, dx &+ \int_0^T \int_\Omega \left[\nabla\psi\nabla \frac{\partial \phi}{\partial t} + \Delta\psi \left(\frac{\partial\psi}{\partial x_1} \frac{\partial \phi}{\partial x_2} - \frac{\partial\psi}{\partial x_2} \frac{\partial \phi}{\partial x_2} \right) \right] dx dt - \\ &- \int_0^T \int_\Omega \left(F_1 \frac{\partial \phi}{\partial x_2} - F_2 \frac{\partial \phi}{\partial x_1} \right) dx\, dt = 0.\end{aligned} \tag{5.27}$$

We show that (5.27) is still satisfied if $\varphi(x, t)$ satisfies the condition

$\phi|_{S_0} = 0, \phi|_{S_k} = \eta_k(t)$, $k = 1, 2, \ldots, n$ instead of the stronger condition $\phi|_S = 0$. It is obviously sufficient to show that the identity (5.27) holds for $\phi = \eta_r(t)\psi_r(x)$. But this equation is obtained from (5.16) by multiplying it by $\eta_r(t)$ and integrating w.r.t. t from 0 to T and so is a consequence of (5.21).

The fact that the function $\psi(x, t)$ is defined uniquely by relation (5.27) can now be established with exactly the same reasoning as we used to prove Theorem 1. We have only to remember that, as before, the estimate

$$\|D^2\psi\|_{L_p(\Omega)} \leqslant Cp, \qquad p \geqslant p_0 > 1, \tag{5.28}$$

which follows from (5.11), (5.4) is satisfied, that ψ_0 belongs to the space V_1 and that $\lambda(t)$ is bounded on $[0, T]$. It then just remains to remember that $\psi_0(x, t)$, $\lambda(t)$ are uniquely determined by the function $\psi(x, t)$:

$$\lambda_k(t) = \psi|_{S_k}, \quad k = 1, 2, \ldots, n; \qquad \psi_0(x, t) = \psi(x, t) - \sum_{k=1}^{n} \lambda_k(t)\psi_k(x), \tag{5.29}$$

and the uniqueness is proved.

Going on to the proof of existence, we note first of all that, as in the case of a simply-connected region, we have the estimates

$$\|\psi'\|_{V_1} \leqslant \|\varphi_0\|_V + \int_0^T \max_x |f(x, t)|\, dt, \tag{5.30}$$

$$\left\| \frac{\partial^2\psi'}{\partial x_i \partial t} \right\|_{L_p(\Omega)} \leqslant Cp \left(\|\mathbf{F}\|_{L_p(\Omega)} + \|\psi_0\|_V \left\| \nabla\psi_0 + \sum_{k=1}^{n} \lambda_k \nabla\psi_k \right\|_{L_p(\Omega)} \right), \quad p \geqslant p_0 > 1 \tag{5.31}$$

for $\psi' = A\psi = A(\psi_0 + \Sigma\lambda_k\psi_k)$. We introduce the Banach space E of pairs $(\psi_0, \lambda) = W$ where $\psi_0 \in C'$, $\lambda(t) = (\lambda_1, \ldots, \lambda_n)$ is an n-dimensional vector-function, continuous on $[0, T]$ with norm

$$\|W\|_E = \|\psi_0\|_{C'} + \max_{0<t<T} |\lambda(t)| . \tag{5.32}$$

We replace the system of equations (5.21), (5.24) with the equivalent operator equation in the space E

$$W = LW, \tag{5.33}$$

where

$$LW = L(\psi_0, \lambda) = \{A\psi, \lambda_0 + B(A\psi, \lambda)\}, \qquad \psi = \psi_0 + \sum_{k=1}^{n} \lambda_k \psi_k. \tag{5.34}$$

To prove the existence of a solution of equation (5.33) we use the general Lere-Schauder principle of a fixed point [16]. The complete continuity of the operator L in the space E follows easily from inequalities (5.30), (5.31) and the insertion theorem. We now show that on spheres of sufficiently large radius the vector field $\Phi_1 W = W - LW$ has a curl equal to 1. To do this we show that it is homotopic to the field $\Phi_0 W = W$ with unit curl. On the sphere $S^{(R)}$ of the space E of radius R for any $\sigma \in [0, 1]$ we define the vector field

$$\begin{gathered} \Phi_\sigma W = \{\psi_0 - \sigma A\psi, \lambda - \sigma\lambda_0 - B(\sigma A\psi, \lambda)\}; \\ W = (\psi_0, \lambda) \in S^{(R)}, \qquad \sigma \in [0, 1]. \end{gathered} \tag{5.35}$$

We have to show, first, that the operator $W - \Phi_\sigma W$ is completely continuous on $S^{(R)} \times [0, 1]$ and, secondly, that $\Phi_\sigma W$ is not equal to zero on $S^{(R)}$ (for sufficiently large R). The first statement is quite obvious. To prove the second we show that the zeros of the vector field $\Phi_\sigma W$ cannot lie outside the spheres of the space E of a certain (known) radius.

In addition to (5.30) and (5.31) we need another energy estimate of the solution of equation (5.33). To derive it we note that, as in the case of a simply-connected region (see Lemma 3.1) from (5.27) we have the integral identity

$$\begin{gathered} -\int_\Omega \nabla\psi(x, t)\, \nabla\phi(x, t)\, dx + \int_\Omega \nabla\varphi(x)\, \nabla\phi(x, 0)\, dx + \int_0^t\int_\Omega \Big[\nabla\psi\nabla\frac{\partial\phi}{\partial t} + \\ + \Delta\psi\Big(\frac{\partial\psi}{\partial x_1}\frac{\partial\phi}{\partial x_2} - \frac{\partial\psi}{\partial x_2}\frac{\partial\phi}{\partial x_1}\Big)\Big]\, dx\, dt - \int_0^t\int_\Omega \Big(F_1\frac{\partial\phi}{\partial x_2} - F_2\frac{\partial\phi}{\partial x_1}\Big)\, dx\, dt = 0, \end{gathered} \tag{5.36}$$

which is valid for any function $\varphi(x, t)$ which is smooth in Q_T and such that $\phi|_{S_0} = 0$, $\phi|_{S_k} = \eta_k(t)$, $k = 1, 2, \ldots, n$, and for any $0 \leqslant t \leqslant T$. Writing $\varphi = \psi$ in (5.36) we find

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\nabla\psi)^2\, dx = \int_\Omega \Big(F_1\frac{\partial\psi}{\partial x_2} - F_2\frac{\partial\psi}{\partial x_1}\Big)\, dx. \tag{5.37}$$

From (5.37) we have an estimate for $\psi = \psi_0 + \sum_{k=1}^{n} \lambda_k \psi_k$:

$$\int_\Omega |\nabla\psi(x,t)|^2\,dx \leqslant \Big(\|\nabla\varphi\|_{L_2} + \int_0^T \|\mathbf{F}\|_{L_2}\,d\tau\Big)^2 \equiv C_1. \tag{5.38}$$

Using the notation (5.5) we can rewrite (5.38) in the form

$$\int_\Omega (\nabla\psi_0)^2\,dx + \Big(\sum_{k=1}^{n} \lambda_k u_k\Big)^2 \leqslant C_1, \tag{5.39}$$

since

$$\int_\Omega \nabla\psi_k \nabla\psi_0\,dx = -\int_\Omega \psi_0 \Delta\psi_k\,dx + \oint_S \psi_0 \frac{\partial\psi_k}{\partial n}\,ds = 0.$$

We note that the second term in (5.39) is a positive definite quadratic form w.r.t. $\lambda_1, \ldots, \lambda_n$ since the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are linearly independent, according to Lemma 5.2. Therefore from (5.39) we have the estimates:

$$\int_\Omega (\nabla\psi_0)^2\,dx \leqslant C_1, \tag{5.40}$$

$$|\lambda|^2 = \sum_{k=1}^{2} \lambda_k^2 \leqslant C_2. \tag{5.41}$$

It is then not difficult to see that the fixed point $W_\sigma = (\psi_{0\sigma}; \lambda_\sigma)$ of the vector field $\Phi_\sigma W$ is the solution of an operator equation of the form (5.33) where f, φ are replaced, respectively, by σf, $\sigma\varphi$ and so estimates of the form (5.30), (5.31), (5.40), (5.41) hold for $\psi_{0\sigma}$, λ_σ Of course there is a constant C_3, depending only on the region Ω and the norms of f, φ given in the conditions of the theorem such that

$$\|W_\sigma\|_E < C_3 \tag{5.42}$$

uniformly w.r.t. $\sigma \in [0, 1]$. Thus the vector fields Φ_0 and Φ_1 are homotopic. Hence the curl of the field Φ_1 is equal to 1 and inside a sphere of radius C_3 it has not less than one fixed point. We have still to note that, from (5.24), $\psi_0 \in V_1$ since $A\psi \in V_1$. The strong continuity of $\Delta\psi$, $\partial^2\psi/\partial x_i\partial t$ w.r.t. t in $L_p(\Omega)$ is proved in the same way as in Theorem 4.1. In order to prove the continuity of ψ w.r.t. t on $[0, T]$ in $W_p^{(2)}$ it is sufficient to show that $\psi(x,t)$, $\nabla\psi(x,t)$ are continuous w.r.t. t in $L_2(\Omega)$ on $[0, T]$. We introduce the Hilbert space H_2, the closure of the set G of functions $g(x)$, smooth in Ω, and such that $g|_{S_0} = 0$, $g|_{S_k} = \eta_k = \text{const.}$, $k = 1, \ldots, n$, in the norm generated by the scalar product

$$(g_1, g_2)_{H_2} = \int_\Omega \nabla g_1 \nabla g_2 \, dx. \tag{5.43}$$

Let $0 \leqslant \tau_1 < \tau_2 \leqslant T$. We write (5.36) for $t = \tau_1$ and $t = \tau_2$ and subtract these equations from one another. We have

$$\begin{gathered} -(\psi(x, \tau_2), \phi(x, \tau_2))_{H_2} + (\psi(x, \tau_1), \phi(x, \tau_1))_{H_2} + \int_{\tau_1}^{\tau_2}\int_\Omega \Big[\nabla\psi\nabla\frac{\partial\phi}{\partial t} + \\ + \Delta\psi\Big(\frac{\partial\psi}{\partial x_1}\frac{\partial\phi}{\partial x_2} - \frac{\partial\psi}{\partial x_2}\frac{\partial\phi}{\partial x_1}\Big)\Big] dx\, dt - \int_{\tau_1}^{\tau_2}\int_\Omega \Big(F_1\frac{\partial\phi}{\partial x_2} - F_2\frac{\partial\phi}{\partial x_1}\Big) dx\, dt = 0. \end{gathered} \tag{5.44}$$

Making τ_2 tend to τ_1 we easily find that for any smooth $\phi(x, \tau_1) \in H_2$

$$\lim_{\tau_2 \to \tau_1} (\psi(x, \tau_2) - \psi(x, \tau_1), \phi(x, \tau_1))_{H_2} = 0. \tag{5.45}$$

Since, first, we can continue any sufficiently smooth function $\phi(x, \tau_1) \in H_2$ over the whole cylinder without losing its smoothness property or the boundary conditions, secondly the set of smooth functions is dense in H_2 and, thirdly, $\|\psi(x, t)\|_{H_2}$ is bounded uniformly w.r.t. $t \in [0, T]$, we can deduce from (5.45) that $\psi(x, t)$ is weakly continuous in H_2 w.r.t. t on $[0, T]$. Then integrating (5.37) w.r.t. t from τ_1 to τ_2 we find

$$\|\psi(x, \tau_2)\|^2_{H_2} - \|\psi(x, \tau_1)\|^2_{H_2} = 2\int_{\tau_1}^{\tau_2}\int_\Omega \Big(F_1\frac{\partial\psi}{\partial x_2} - F_2\frac{\partial\psi}{\partial x_1}\Big) dx\, dt. \tag{5.46}$$

From (5.46) we have the relation

$$\overline{\lim_{\tau_2 \to \tau_1}} \|\psi(x, \tau_2)\| = \|\psi(x, \tau_1)\|, \tag{5.47}$$

from which it also follows that ψ is strongly continuous w.r.t. t on $[0, T]$ in H_2, i.e. $\nabla\psi$ in $L_2(\Omega)$; $\psi(x, t)$ also is strongly continuous w.r.t. t in L_2 since for any function $g(x) \in H_2$ we have the Friedrich inequality

$$\|g\|_{L_2(\Omega)} \leqslant C\|g\|_{H_2}. \tag{5.48}$$

6. Determination of the pressure and trajectories of liquid particles. Generalized solution of the problem (1.1) - (1.4). The external problem

1. Let us first discuss the problem of finding the pressure $P(x, t)$. We shall examine the case of an $(n + 1)$-connected region Ω $(n \geqslant 0)$ and assume that the generalized solution of problem (5.13)-(5.17) has already been found and so, according to (3.1), (5.11) the velocity vector $\mathbf{v}(x,t)$ is known.

Lemma 6.1. Any smooth vector, solenoidal in Ω, $\mathbf{b}(x) = (b_1, b_2)$, $\operatorname{div} \mathbf{b} = 0$, with a normal component which is equal to zero on S can be put in the form

$$b_1 = \frac{\partial \phi}{\partial x_2}, \qquad b_2 = -\frac{\partial \phi}{\partial x_1}, \tag{6.1}$$

where φ is a smooth function and $\phi|_{S_0} = 0$, $\phi|_{S_k} = C_k$, $k = 1, 2, \ldots, n$, $C_1, \ldots, C_n$ are constants.

For the proof we introduce the vector $\mathbf{b}^* = (-b_2, b_1)$. It is clear that $\operatorname{rot} \mathbf{b}^* = 0$. The problem has reduced to one of finding the function φ from the equation

$$\nabla \phi = \mathbf{b}^*. \tag{6.2}$$

In order to prove that the problem (6.2) is solvable it is sufficient, according to Lemma 5.1, to verify that a condition of the form (5.8):

$$\int_\Omega \mathbf{b}^* \mathbf{u}_k \, dx = 0, \qquad k = 1, \ldots, n \tag{6.3}$$

is satisfied. Let us show that (6.3) is satisfied:

$$\begin{aligned}\int_\Omega \mathbf{b}^* \mathbf{u}_k \, dx &= -\int_\Omega \left(b_2 \frac{\partial \psi_k}{\partial x_2} + b_1 \frac{\partial \psi_k}{\partial x_1}\right) dx = -\int_\Omega \mathbf{b} \nabla \psi_k \, dx = \\ &= -\int_\Omega \operatorname{div}(\psi_k \mathbf{b}) \, dx = -\oint_S \psi_k \mathbf{b}\mathbf{n} \, ds = 0.\end{aligned} \tag{6.4}$$

This proves the lemma.

Lemma 6.2. There exists a function $P(x, t)$ which is strongly continuous w.r.t. t in any $W_p^{(1)}(\Omega)$, $p > 1$ and satisfies equations (1.1). $P(x, t)$ is continuous in Q_T and

$$\max_t \| \nabla P \|_{L_p(\Omega)} \leqslant Cp, \qquad p \geqslant p_0 > 1. \tag{6.5}$$

Replacing ψ in the integral identity for the current function (5.27) by $\mathbf{v}$, introducing the vector $\mathbf{b}'(x, t)$ by means of formulae of the form (6.1) and integrating by parts we obtain the relation

$$\int_0^T \int_\Omega \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\, \mathbf{v} - \mathbf{F}\right] \mathbf{b}'\, dx = 0, \tag{6.6}$$

which according to Lemma 6.1 holds for any smooth vector $\mathbf{b}'(x, t)$ such that $\operatorname{div} b' = 0$ and $\mathbf{b}' \cdot \mathbf{n}|_S = 0$, $\mathbf{b}'(x, T) = 0$. Putting $\mathbf{b}' = h(t)\, \mathbf{b}$ in (6.6), where $h(t)$ is a smooth function and $h(T) = 0$ and is otherwise arbitrary we have the identity

$$\int_\Omega \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\, \mathbf{v} - \mathbf{F}\right] \mathbf{b} dx = 0 \tag{6.7}$$

for any vector $b(x)$ such that $\operatorname{div} \mathbf{b} = 0$, $\mathbf{bn}|_S = 0$ for almost all t in $[0, T]$ and using the fact that the integral in (6.7) is continuous w.r.t. t (from Theorem 5.1), also for all $t \in [0, T]$. In accordance with the well known results of S.L. Sobolev [17] concerning the orthogonal expansion of a space of vector functions of the class L_2 (see also [18], [19]) this is sufficient to be able to state that there exists a function $P(x, t)$ such that

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\, \mathbf{v} - \mathbf{F} = \nabla P. \tag{6.8}$$

The other parts of the lemma follow from (6.8), the strong continuity of the functions $\partial v_i/\partial t$, $\partial v_i/\partial x_k$ w.r.t. t on $[0, T]$ in $L_p(\Omega)$ established in Theorem 5.1 and the insertion theorem.

2. The problem of finding the trajectory of a liquid particle consists of the integration of the system of ordinary differential equations

$$\frac{dx}{dt} = \mathbf{v}\,(x, t), \qquad i = 1, 2,$$

or

$$\frac{dx_1}{dt} = \frac{\partial \psi(x, t)}{\partial x_2}, \qquad \frac{dx_2}{dt} = -\frac{\partial \psi(x, t)}{\partial x_1}, \qquad x = (x_1, x_2) \tag{6.9}$$

with the initial conditions

$$x_i|_{t=0} = \alpha_i, \tag{6.10}$$

where $(\alpha_1, \alpha_2) = \alpha$ is the initial position of the particle.

Lemma 6.3. The problem (6.9), (6.10) for any $\alpha \in \Omega$ has a solution, which is unique, defined on any segment $[0, T]$ and continuously (even Hölder continuously) dependent on α.

The problem (6.9)-(6.10) has a solution defined on some segment $[0, \tau_0]$ since the right-hand sides in (6.9) are continuous. Let us prove that this solution is unique. Suppose, on the contrary, that there exist two solutions, x and x'. The difference $x - x' = y$ satisfies the equation

$$\frac{dy}{dt} = \mathbf{v}(x, t) - \mathbf{v}(x', t) \tag{6.11}$$

and the initial condition

$$y|_{t=0} = 0. \tag{6.12}$$

Now we recall that

$$\psi = \psi_0 + \sum_{k=1}^{n} \lambda_k(t)\, \psi_k(x),$$

where $\psi_0 \in V_1$; $\lambda_k(t)$ are continuous and $\psi_k(x)$ can be put in the form $\psi_k = \widetilde{\psi}_k + \psi'_k$, where $\widetilde{\psi}_k$ is a function as smooth as we please and $\psi'_k \in V$. Applying Lemma 2.2 we obtain the estimate

$$\| \mathbf{v}(x, t) \|_{B^{0,\lambda}(\Omega)} \leqslant C_1 \| \psi(x, t) \|_{B^{1,\lambda}(\Omega)} \leqslant \frac{C}{1-\lambda} \tag{6.13}$$

for any $0 \leqslant \lambda < 1$. Multiplying (6.11) by y and using (6.13) we obtain

$$\frac{d}{dt} \frac{|y|^2}{2} \leqslant \frac{C}{1-\lambda} |y|^{1+\lambda}. \tag{6.14}$$

From (6.14), dividing by $|y|^{1+\lambda}$ and integrating w.r.t. t, we have

$$|y(t)| \leqslant (Ct)^{1/(1-\lambda)}. \tag{6.15}$$

Assuming that $0 < t < 1/C$ and letting λ tend to 1 in (6.15) we find that $y(t) = 0$. Repeating the same argument we can show that $y(t) = 0$ for all t for which this vector function is defined.

In order to prove that the solution $x(t)$ of the problem (6.9), (6.10) can be continued on any segment $[0, T]$, we show that the point $x(t)$ does not go outside the region Ω. (Mechanically this fact is quite obvious, since the boundary S of the region Ω is, by hypothesis, an impenetrable wall.) If we assume the contrary we arrive at the conclusion that at some moment t_0 the point $x(t_0) = x^0 \in S$. Let us show that if a particle is on the boundary S at some time t_0 it was on S at all preceding moments of time and will remain on S at all subsequent moments of time. For since $(\partial\psi/\partial s)|_S = \mathbf{vn}|_S = 0$ one of the possible trajectories lies on S and the equation of motion of a point along it has the form

$$\frac{ds}{dt} = \frac{\partial\psi}{\partial n}, \qquad s|_{t=t_0} = 0 \tag{6.16}$$

(s is the arc of the contour on which the point lies at the initial moment), while uniqueness has already been proved. Thus the solution of the problem (6.9), (6.10) $x(t) \in \Omega$ and

$$\max|x(t)| \leqslant \max_{x\in\Omega}|x|. \tag{6.17}$$

The presence of the *a priori* estimate (6.17) also enables us to state that the solution $x(t)$ is defined for all t.

Now let x' and x'' be solutions of a problem of the form (6.9), (6.10) corresponding to initial data a' and a''. Putting $z = x' - x''$, $z_0 = a' - a''$ we shall have

$$\frac{dz}{dt} = \mathbf{v}(x', t) - \mathbf{v}(x'', t), \tag{6.18}$$

$$z|_{t=0} = z_0. \tag{6.19}$$

Making a scalar multiplication of (6.18) by z and using Lemma 2.2 we obtain

$$\frac{d}{dt}|z| \leqslant C|z|(1 + |\ln|z||). \tag{6.20}$$

Let $|z_0| \leqslant \delta < 1$. Then for sufficiently small values of t, $|z| < 1$ and $|\ln|z|| = -\ln|z|$. We divide (6.20) by $|z|$. We have

$$\frac{d}{dt}\ln|z| + C\ln|z| \leqslant C \tag{6.21}$$

and then

$$\ln|z| \leqslant e^{-Ct}\ln|z_0| + 1 - e^{-Ct}. \tag{6.22}$$

If, in particular, we take $|z_0| \leqslant \exp[1 - e^{CT}]$ then $|z|$ remains less than 1 and (6.22) is satisfied for $0 < t \leqslant T$. From (6.22) we can derive the estimate $(0 \leqslant t \leqslant T)$

$$| z (t) | \leqslant | z_0 |^{e^{-Ct}} e^{1-e^{-Ct}}, \tag{6.23}$$

from which the last part of the lemma follows.

3. Now let us find the generalized solution of problem (1.1)-(1.4) in the form of the pair $\mathbf{v}(x, t)$, $P(x, t)$ where $\mathbf{v}$ is the vector with current function $\psi(x, t)$ found in accordance with Theorem 5.1 and $P(x,t)$ is the pressure found in accordance with Lemma 6.2. Let us summarize the results obtained above for the problem (1.1)-(1.4) for two space variables.

Theorem 6.1. Let $S \in C^{(2)}$, $f(x, t) = -(\operatorname{rot} \mathbf{F})_3$ be a function bounded in the cylinder $Q_T = \Omega \times [0, T]$, $\mathbf{a}(x)$ a vector solenoidal in Ω, $\mathbf{a}n|_S = 0$, $|\operatorname{rot} \mathbf{a}|$ bounded and $\mathbf{F}(x, t)$ strongly continuous w.r.t. t in $L_p(\Omega)$. Then there exists a vector $\mathbf{v}(x, t) \in B^{0,\lambda}(Q_T)$ for any $0 < \lambda < 1$, $\mathbf{v} \in W_p^{(1)}(Q_T)$ for any $p > 1$, strongly continuous w.r.t. t on $[0, T]$ in $W_p^{(1)}(\Omega)$, and a function $P(x, t)$, continuous in Q_T and strongly continuous w.r.t. t on $[0, T]$ in $W_p^{(1)}$ such that (1) the equations (1.1), (1.2) are satisfied for any $t \in [0, T]$ almost everywhere in the region Ω; (2) the condition (1.3) is satisfied in the classical sense; (3) the condition (1.4) is satisfied in a stronger sense: as $t \to 0$

$$\| \mathbf{v}(x, t) - \mathbf{a}(x) \|_{B^{0,\lambda}(\Omega)} \to 0;$$

(4) $(\operatorname{rot} \mathbf{v})_3 = -\Delta\psi$ is a function bounded in Q_T; (5) when $p > 1$, $\| \nabla P \|_{L_p(\Omega)} + \| \mathbf{v} \|_{W_p^{(1)}(\Omega)} \leqslant Cp$; (6) the generalized solution is such that it is possible to find particle trajectories which are Hölder continuously dependent on the initial position of the particle.

Note 1. Very similar methods can be used to study the axisymmetric case as well as the more general case mentioned in Lemma 1.4.

Note 2. For continuous rot $\mathbf{a}(x)$ and $f(x, t)$ we can show by changing to Lagrange variables and using Lemma 6.3 that rot $\mathbf{v}$ also turns out to be continuous.

Note 3. The generalized solution can be defined and analysed, as above, when the region is deformed with time.

4. In conclusion, let us say a few words about the external problem

(as above for the case of two space variables). We add the condition at infinity

$$\mathbf{v}\big|_{|x|\to\infty} = 0. \tag{6.24}$$

to conditions (1.1)-(1.4). The generalized solution of the problem (1.1)-(1.4), (6.24) is the pair $\mathbf{v}(x, t)$, $P(x, t)$ where the vector $\mathbf{v}(x, t)$ is strongly continuous w.r.t. t in $L_2(\Omega)$ and in $W_p^{(1)}(\Omega)$ for any $p > 1$ and has the curl $\operatorname{rot} \mathbf{v}(x, t)$ bounded in $Q_T = \Omega \times [0, T]$ and the function $P(x, t)$ is strongly continuous w.r.t. t in $W_p^{(1)}(\Omega)$ for any $p > 1$ and conditions (1.1)-(1.4), (6.24) are satisfied.

The existence of a generalized solution is easily proved given the previous conditions w.r.t. the boundary, the external forces and the initial velocity (of course now, for instance, the conditions $\mathbf{a} \in L_2$ and $\mathbf{a} \in W_p^{(1)}$ are independent) by taking the limit in finite regions, for example in the regions Ω_R obtained by adding the neighbourhood $S^{(R)}$ of large radius R to the boundary S. Estimates of the corresponding solutions $\mathbf{v}_R$ which are uniform w.r.t. R are used:

$$\| \mathbf{v}_R \|_{L_2(\Omega_R)} \leqslant \| \mathbf{a} \|_{L_2(\Omega)} + \int_0^T \| \mathbf{F} \|_{L_2(\Omega)}\, d\tau, \tag{6.25}$$

$$\| \operatorname{rot} \mathbf{v}_R \|_{L_p(\Omega_R)} \leqslant \| \operatorname{rot} \mathbf{a} \|_{L_p(\Omega)} + \int_0^t \| \operatorname{rot} \mathbf{F} \|_{L_p(\Omega)}\, d\tau. \tag{6.26}$$

The uniqueness also can be proved in a similar way to that used in the case of a finite region. Therefore we shall consider only those questions which apply specifically to the external problem: (a) for which conditions in (6.24) satisfied in the classical sense and (b) for which conditions is the pressure $P(x, t)$ a bounded function.

We derive one new estimate for the solution of problems (1.1)-(1.4) for a bounded region Ω to answer the first of these questions.

Lemma 6.4. Let all the conditions of Theorem 5.1 be satisfied. Then $(0 \leqslant t \leqslant T,\ 0 \overline{\in} \Omega)$

$$I_{rl}(t) \equiv \int_\Omega |x|^r\, |\Delta\psi|^l\, dx \leqslant M, \qquad 1 \leqslant r \leqslant 2,\ l > 1, \tag{6.27}$$

where the constant M depends only on r, l, T, the right-hand sides of the inequality (6.25) and the inequality (6.26) for $p = 2l/(2 - r)$, and also on the quantities

$$\max_t \int_\Omega | x|^r | f(x, t)|^l dx, \qquad I_{rl}(0).$$

Let us first derive (6.27) for sufficiently smooth solutions. We multiply equation (3.2) (which is valid for the current function ψ in in the case of a multiply connected region also) by $|x|^r | \Delta\psi|^{l-2}\Delta\psi$ and integrate over the region Ω. After integration by parts we obtain

$$\frac{1}{l}\frac{dI_{rl}}{dt} + \frac{r}{l}\int_\Omega | \Delta\psi|^l | x|^{r-1}\left(\frac{\partial\psi}{\partial x_1}\frac{x_2}{|x|} - \frac{\partial\psi}{\partial x_2}\frac{x_1}{|x|}\right)dx = \int_\Omega f|x|^r|\Delta\psi|^{l-2}\Delta\psi\, dx. \quad (6.28)$$

We factorize the integrand in the second term of (6.28):

$$(|x|^r|\Delta\psi|^l)^{(r-1)/r}|\Delta\psi|^{l/r}\left(\frac{\partial\psi}{\partial x_1}\frac{x_2}{|x|} - \frac{\partial\psi}{\partial x_2}\frac{x_1}{|x|}\right).$$

To estimate it we use the Hölder inequality with indices $p_1 = r/(r-1)$, $p_2 = 2r/(2-r)$, $p_3 = 2$, and to estimate the right-hand side of (6.28) we use the Hölder inequality with indices l, $l/(l-1)$, and find

$$\frac{dI_{rl}}{dt} \leqslant r\|\nabla\psi\|_{L_2(\Omega)}\|\Delta\psi\|_{L_{2l/(2-l)}(\Omega)} I_{rl}^{(r-1)/r} + lI_{rl}^{(l-1)/l}\left(\int_\Omega |f|^l|x|^r dx\right)^{1/l}. \quad (6.29)$$

Integrating (6.29) w.r.t. t and estimating the right-hand side of the resulting relation using inequalities of the type (6.25), (6.26) we obtain

$$K_{rl} \leqslant I_{rl}(0) + T\,(C_1K_{rl}^{(r-1)/r} + C_2K_{rl}^{(l-1)/l}) \quad (6.30)$$

for the quantity $K_{rl} = \max_t I_{rl}(t)$ from which it is easy to derive the required estimate (for example, by arguing the contrary) if we note that K_{rl} occurs in the right-hand side of (6.30) in powers less than 1. In order to do without the assumption that the solution is smooth we can repeat the argument of Lemma 4.1, the only difference being that we must put $\phi = |u_2|^{l-2}u_2|x|^r$ in (4.8).

Returning to the case of an unbounded region Ω we note that estimate (6.27) still applies here. It can be established, together with estimates of the type (6.25), (6.26) by making a limiting transition from finite regions. Let us show that the existence of this estimate is sufficient for the condition at infinity (6.25) to be satisfied in the classical sense.

Lemma 6.5. Let the function $\varphi(x)$ be defined in the unbounded region Ω with boundary S lying in a finite part of the plane E_2. Further, let $\nabla\varphi \in L_2(\Omega)$ and let an estimate of the form (6.27) be satisfied for certain fixed r, l such that $l > 2$, $(r-l+2)/l = \alpha_{rl} > 0$. Then as $|x| \to \infty$ the quantity $|\nabla\varphi| = \sqrt{(\partial\varphi/\partial x_1)^2 + (\partial\varphi/\partial x_2)^2}$ tends to zero as $|x|^{-\beta_{rl}}$; $\beta_{rl} = \min(2, \alpha_{rl})$.

With our conditions we can put the function $\varphi(x)$ in the region Ω_R outside the neighbourhood S_R of radius R in the form $\varphi = \varphi_1 + \varphi_2$, where

$$\varphi_1(x) = \int\limits_{\Omega_R} G(x, y)\, \Delta\varphi(y)\, dy, \tag{6.31}$$

and $G(x, y)$ is Green's function of the first boundary problem for the Poisson equation in the region Ω_R, $\varphi_2(x)$ is a harmonic function in the region Ω_R. From (6.31) we have

$$|\nabla\varphi_1(x)| \leqslant C \int\limits_{\Omega_R} \frac{|\Delta\varphi(y)|}{|x-y|}\, dy. \tag{6.32}$$

Continuing the estimate, using the Hölder inequality, we obtain

$$|\nabla\varphi_1(x)| \leqslant C \Big(\int\limits_{\Omega_R} |\Delta\varphi|^l\, |y|^r\, dy\Big)^{1/l} \Big(\int\limits_{\Omega_R} \frac{dy}{|x-y|^{l/(l-1)}\, |y|^{r/(l-1)}}\Big)^{(l-1)/l}. \tag{6.33}$$

The last integral in (6.33) does not exceed the quantity

$$K(x) = \int\limits_{E_2} \frac{dy}{|x-y|^{l/(l-1)}\, |y|^{r/(l-1)}}. \tag{6.34}$$

We make the substitution of variables $y_i = |x|\, y_i'$, $x_i = |x|\, x_i'$ in (6.34). Noting that $K(x)$ depends only on $|x|$ we obtain

$$K(x) = \frac{1}{|x|^{(r-l+2)/(l-1)}} \int\limits_{E_2} \frac{dy'}{|x'-y'|^{l/(l-1)}\, |y'|^{r/(l-1)}} = \frac{K}{|x|^{(r-l+2)/(l-1)}}, \tag{6.35}$$

where K is constant. From (6.33)-(6.35) we have

$$|\nabla\varphi_1| \leqslant \frac{C_1}{|x|^{\alpha_{rl}}}. \tag{6.36}$$

Let us consider the second term $\varphi_2(x)$. We have, for example,

$$\frac{\partial \varphi_2(x)}{\partial x_1} = \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{\beta_n}{z^n}\right), \quad z = x_1 + ix_2, \tag{6.37}$$

for the derivative w.r.t. x_1 where β_n are certain complex constants. When $\alpha_{rl} \leqslant 1$ from (6.37) φ decreases as φ_1 and the lemma is proved for this case. If $\alpha_{rl} > 1$ then it follows from (6.36) that $\nabla\varphi_1 \in L_2(\Omega)$. But then also $\nabla\varphi_2 = \nabla\varphi - \nabla\varphi_1 \in L_2(\Omega)$. Hence in this case $\beta_1 = 0$ and $|\nabla\varphi_2|$ decreases as $|x|^{-2}$. This proves the lemma for the case when $\alpha_{rl} > 1$.

Let us give some examples: (1) if $r \geqslant 2l-2$ then $|\nabla\varphi|$ does not decrease slower than $|x|^{-1}$; (2) if $r > 3l-2$, then $|\nabla\varphi|$ decreases like $|x|^{-2}$.

Lemma 6.6. Let all the conditions of Lemma 6.4 be satisfied, with the exception of the condition $r \leqslant 2$. Then as before we have the estimate (6.27).

The proof is basically the same as that of Lemma 6.4, but we use an estimate of the second term in equation (6.28):

$$\left|\int_{\Omega} |\Delta\psi|^l |x|^{r-1} \left(\frac{\partial\psi}{\partial x_1}\frac{x_2}{|x|} - \frac{\partial\psi}{\partial x_2}\frac{x_1}{|x|}\right) dx\right| \leqslant \max_{x\in\Omega}\left(\frac{|\nabla\psi|}{|x|}\right) I_{rl} \leqslant CI_{rl}, \tag{6.38}$$

where the second inequality follows from the considerations of Lemma 6.5.

Now let us estimate the pressure $P(x, t)$. In order to determine the pressure $P(x, t)$ uniquely we lay down the supplementary condition

$$P(x, t)\big|_{|x|\to\infty} \to 0. \tag{6.39}$$

Lemma 6.7. Let $\mathbf{a}(x)$ and $\mathbf{F}(x, t)$ be such that the right-hand sides of inequalities (6.25), (6.26) and, in addition, the quantities $\max_t \|\mathbf{F}(x, t)\|_{L_{p_0}}$ for some $1 < p_0 < 2$

$$\max_t \int_{\Omega} |x|^r |f(x, t)|^l dx, \qquad \int_{\Omega} |x|^r |\operatorname{rot} \mathbf{a}|^l dx, \quad r \geqslant 1, \qquad l > 2$$

are finite. Then we can define the pressure $P(x, t)$ so that condition (6.39) is satisfied, and $P(x, t)$ decreases, when $|x| \to \infty$, as $|x|^{-\delta_{rl}}$, $\delta_{rl} = \min(2\beta_{rl}, 2)$.

It follows from (1.1) that $P(x, t)$ for any $0 \leqslant t \leqslant T$ in the region $\Omega_R\ (|x| > R)$ can be put in the form*

$$P = P_1 + P_2 + \frac{\mathbf{v}^2}{2}, \tag{6.40}$$

where P_1 is a generalized solution, in the sense of Section 5, of the boundary problem

$$\Delta P_1 = \frac{\partial}{\partial x_i}[F_i + (\mathbf{v} \times \operatorname{rot} \mathbf{v})_i], \qquad P_1\big|_{|x|=R} = 0, \tag{6.40'}$$

and P_2 is harmonic in Ω_R.

Solving (6.40') w.r.t. P_1 using Green's function $G(x, y)$ we obtain

$$P_1(x, t) = -\int_{\Omega} \frac{\partial G(x, y)}{\partial y_i}[F_i + (\mathbf{v} \times \operatorname{rot} \mathbf{v})_i]\, dy \tag{6.41}$$

and further,

$$|P_1(x, t)| \leqslant C\Big(\int_{\Omega_R} \frac{|\mathbf{F}(y, t)|}{|x-y|}\, dy + \int_{\Omega_R} \frac{|\mathbf{v}(y, t)|\,|\operatorname{rot} \mathbf{v}(y, t)|}{|x-y|}\, dy\Big). \tag{6.42}$$

Let us estimate the second integral I in (6.42). Using the Hölder inequality we find

$$I \,| \leqslant \max_{y \in \Omega_R} |\mathbf{v}(y)|\, |y|^{\beta_{rl}} \Big|\Big(\int_{\Omega_R} |\operatorname{rot} \mathbf{v}|^l\, |y|^r\, dy\Big)^{1/l}\Big(\int_{\Omega_R} \frac{dy}{|x-y|^{l'}\,|y|^{\gamma_{rl}}}\Big)^{(l-1)/l}, \tag{6.43}$$

where

$$\beta_{rl} = \min\Big(2, \frac{r-l+2}{l}\Big), \qquad l' = \frac{l}{l-1}, \qquad \gamma_{rl} = \Big(\beta_{rl} + \frac{r}{l}\Big)\, l'.$$

From Lemmas 6.4 - 6.6 we know that the first two factors in (6.43) are bounded and the latter can be estimated as in Lemma 6.5. We have

$$|I| \leqslant \frac{C}{|x|^{1+r/l+\beta r_l - 2/l'}}. \tag{6.44}$$

From (6.42), (6.44), using Lemma 6.5 we find

$$|P_1(x, t)| \leqslant \frac{C}{|x|^{1+r/l+\beta_{rl}-2/l'}}. \tag{6.45}$$

Moreover using the theorem about singular integrals in L_p it is easy to

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* We omit an arbitrary function of time.

deduce from (6.41) that $\nabla P_1 \in L_{p_0}(\Omega)$. Then also $\nabla P_2 = \nabla P - \nabla P_1 \in L_{p_0}$ and since P_2 is harmonic we have the estimate

$$|P_2| < \frac{C}{|x|^2}. \tag{6.46}$$

The statement of the lemma now follows at once from (6.40), (6.45), (6.46) and from the fact noted above that $|\mathbf{v}|$ decreases like $|x|^{-\beta_{rl}}$. In addition we use the fact that $2\beta_{rl} \leqslant 1 + r/l + \beta_{rl} - 2/l'$. This leads to the following theorem.

Theorem 6.2. Let the vectors $\mathbf{F}(x, t)$ and $\mathbf{a}(x)$ be such that the right-hand sides of (6.25), (6.26) and (1) $\max\limits_t \|\mathbf{F}(x,t)\|_{L_{p_0}}$ for some $1 < p_0 < 2$; (2) $\max\limits_t \int\limits_\Omega |x|^r\, |\operatorname{rot} \mathbf{F}|^l\, dx$; (3) $\int\limits_\Omega |x|^r\, |\operatorname{rot} \mathbf{a}|^l\, dx$, $r - l + 2 > 0$ are finite. Moreover, let $\mathbf{F}(x, t)$ be continuous w.r.t. t on $[0, T]$ in any $L_p(\Omega)$, $(p > p_0)$. Then there exists a generalized solution, which is unique, of the problem (1.1)-(1.4), (6.24) such that conditions (6.24), (6.39) are satisfied in the classical sense.

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