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Estimates Near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions II*

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Introduction

This paper is the second part of a study of estimates pertaining to solutions of boundary problems for linear elliptic equations. The first part [1], hereinafter referred to as ADN (and sometimes as Part I), dealt with a single equation for one dependent variable; here we treat systems of several equations for an equal number of dependent variables. The estimates, as in ADN, are of two kinds: (a) Schauder estimates, in which Hölder norms for the solutions and their derivatives are assessed from the values of similar norms for the given data; (b) L_p estimates for the solutions and their derivatives. The boundary conditions considered are of differential type and satisfy a condition of suitability to the system of differential equations which we call the Complementing Condition. This Complementing Condition is necessary and sufficient in order that it be possible to estimate all the derivatives occurring in the system, without loss of order, i.e., that inequalities of "coercive" type be valid. It is expressed algebraically in Section 2, but also can be formulated in the following non-algebraic way very useful in practice (see Lopatinskii [16]). If P is an arbitrary point on the boundary of the region $\mathscr{D}$, make a coordinate transformation flattening the boundary in a neighborhood Ω of P into a portion of a plane, which we represent by $t = 0$ with the half-space $t \geqq 0$ containing the image under the transformation of $\Omega \cap \mathscr{D}$. Then consider the problem to which the given problem reduces when all the "leading" coefficients in the differential equations and in the boundary conditions are fixed equal to their values at P and the lower order coefficients dropped out. For this system with constant coefficients, consider the homogeneous boundary value problem in the half-space $t \geqq 0$. The Complementing Condition requires that, in this new problem, the

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only solution u defined for $t \geqq 0$, tending to zero as $t \to \infty$, and consisting of a finite linear combination of exponential polynomial solutions is $u \equiv 0$.

The following is another way of formulating the Complementing Condition; the new formulation is again in terms of the homogeneous problem with constant coefficients introduced just above. Let x refer to coordinates in the hyperplane $t = 0$. For any real vector $\xi \neq 0$ in this hyperplane, consider solutions of the problem of the form $e^{ix \cdot \xi}$ times a function v of t; v is thus a solution of a system of ordinary differential equations with constant coefficients. The Complementing Condition requires that $v \equiv 0$ for every $\xi \neq 0$, if v is bounded on the semi-axis $t \geqq 0$.

This paper is concerned only with estimates. Existence theory under the same boundary conditions is developed with the aid of L_2 estimates in Hörmander's book [12]. Peetre's method, generalized from a single equation [21] to systems, also should lead to an adequate existence theory.

Since the publication of Part I of this paper, much further work has been done on elliptic boundary value problems. Some of this work is compiled in a supplementary list in the Russian translation of ADN; to make this list more accessible, we have added it to the end of our own bibliography below. We now mention more recent papers on this subject.* Fife [11] has shown that Schauder estimates still can be obtained when Hölder continuity assumptions with respect to some of the independent variables are dropped. Cordes [9] has derived L_2 estimates for systems of fixed order. Lions and Magenes [15] have extended previous results in various directions with the aid of interpolation theorems for mappings in Banach spaces. Schechter (see, for instance, [22], which contains additional references to other work) has obtained quite general results for single elliptic equations; see also Miranda [17]. Some authors have discussed boundary conditions not purely local in character, i.e., not purely of differential type, Bade [5] abstractly formulated boundary conditions (for a single equation) and Agranovich and Dynin [3] boundary conditions (for a system) expressed with the aid of singular integral operators. Respecting the latter type of boundary condition, we refer also to the authors listed in [3] and mention Atiyah's and Singer's application [4] to the problem of the index of elliptic operators on manifolds. Many of the works mentioned above overlap with the present paper, and a few in some respects are more general. None, however, outside of [12], [24] and 1** treats systems of such comprehensive type as are considered here*.

The systems of equations treated by us are those of [10], which are permitted to be of different orders in the different dependent variables: the maximum order allowed of the j-th dependent variable in the i-th equation, namely, is $s_i + t_j$, where s_i and t_j are integers. The boundary conditions, which, as noted, are of differential type, similarly may involve varying orders of the dependent variables;

* See also references 1** to 7** added in proof at the end of the Bibliography; the results of this paper have also been announced by Solonnikov in 7**.

the maximum order allowed of the j-th dependent variable in the h-th boundary condition is $r_h + t_j$, where the r_h are integers.

Our treatment follows closely that of Part I, many details as well as applications therefore being only summarily handled. Chapter I describes our formal hypotheses concerning the differential equations and the boundary conditions. In Chapter II, explicit Poisson kernels are developed for systems in a half-space with constant leading coefficients and zero lower terms. The constructions are based on a consideration of general initial value problems for systems of ordinary differential equations; believed to be of independent interest, this theory is given self-contained treatment, in Section 3. After the paper was completed we found that Volevich [24] has also recently constructed Poisson kernels for general systems. Chapters III and IV are devoted to Schauder and L_p estimates, including *local* L_p estimates which had been left out of ADN, but omitting equations of variational form such as had been treated in Chapter III. Chapter V consists of a few applications and comments; the last section contains remarks on singular integral representations.

A question was raised in ADN (p. 649) concerning the relation between two different ways of expressing the fact that a function has a fractional derivative of order $1 - 1/p$ belonging to L_p. More specifically, the question concerned the relation between two norms, called $|f|_{1-1/p,L_p}$ and $|f|^*_{1-1/p,L_p}$ (see $(3.13)^*_p$ in ADN). We had conjectured the two norms to be equivalent, but J. P. Kahane, in the summer of 1960, showed us a simple example of a function constructed with the aid of lacunary series for which, for $1 < p < 2$, the first norm is infinite and the second norm not. Eli Stein subsequently demonstrated that

$$|f|^*_{1-1/p,L_p} \leqq \text{constant} \cdot |f|_{1-1/p,L_p}$$

for $1 < p < 2$ and that the opposite inequality holds for $p > 2$; see also M. Taibleson [23].

It is natural to ask whether a given system of elliptic equations in some domain admits *any* boundary conditions of complementing type. In Section 2 we have considered only strongly elliptic equations and shown that Dirichlet conditions are of complementing type. A complete (unpublished) answer to this question has been given, however, by R. Bott. We mention also that recently P. D. Lax and R. Phillips, in connection with their work on hyperbolic symmetric systems, have shown that when $A_j, j = 1, \cdots, n$, denote real, symmetric N by N matrices, the first order operator $\sum_{j=1}^{n} A_j(\partial/\partial x_j)$ can be elliptic only for certain values of n.

CHAPTER I

FORMULATION OF BOUNDARY PROBLEM. BASIC NOTATION AND ASSUMPTIONS

1. The Differential Equations Considered

We shall be concerned with functions defined in domains of $(n + 1)$-dimensional Euclidean space, $n \geqq 1$, with coordinates $P = (x_1, \cdots, x_{n+1})$, and we use the notation

$$\partial_i = \partial/\partial x_i, \quad \partial = (\partial_1, \cdots, \partial_{n+1}).$$

Differentiations of order l, for instance $\partial_1^{\beta_1} \cdots \partial_{n+1}^{\beta_{n+1}}$, where $\sum_i \beta_i = l$, are symbolized in terms of the multi-index $\beta = (\beta_1, \cdots, \beta_{n+1})$ by ∂^β or, more schematically, by ∂^l. Monomials $\xi_1^{\beta_1} \cdots \xi_{n+1}^{\beta_{n+1}}$ of degree $l = |\beta| = \Sigma\beta_i$ in the components of any vector $\Xi = (\xi_1, \cdots, \xi_{n+1})$ are similarly abbreviated as Ξ^β or Ξ^l. By $|\Xi|$ we shall mean $\left(\sum_{i=1}^{n+1} \xi_i^2\right)^{1/2}$.

The systems of partial differential equations considered will be represented as

$$\sum_{j=1}^{N} l_{ij}(P, \partial)\, u_j(P) = F_i(P), \qquad i = 1, \cdots, N, \tag{1.1}$$

where the $l_{ij}(P, \partial)$, linear differential operators, are polynomials in ∂ with coefficients depending on P over some domain $\mathscr{D}$ in $x_1, \cdots, x_{n+1}$-space. The orders of these operators, as in [10], will be assumed to depend on two systems of integer weights, $s_1, \cdots, s_N$ and $t_1, \cdots, t_N$, attached to the equations and to the unknowns, respectively, s_i corresponding to the i-th equation and t_j to the j-th dependent variable u_j. The manner of the dependence is expressed by the inequality

$$\deg l_{ij}(P, \Xi) \leqq s_i + t_j, \qquad i, j = 1, \cdots, N, \tag{1.2}$$

"deg" referring of course to the degree in Ξ. It is to be understood that $l_{ij} = 0$ if $s_i + t_j < 0$.

Adding a suitable constant to one system of weights and subtracting the constant from the other, we readily achieve, as a normalization, the condition

$$s_i \leqq 0. \tag{1.3}$$

Since, for fixed j, not all l_{ij} vanish, and since $s_i + t_j \geqq 0$ for non-vanishing l_{ij}, we thus also have

$$0 \leqq t_j \leqq t', \tag{1.4}$$

t' being the maximum of the t_j.

Only *elliptic* systems of the form (1.1) will be considered. These are defined as systems for which

$$(1.5) \qquad L(P, \Xi) \equiv \det (l'_{ij}(P, \Xi)) \neq 0 \quad \text{for real} \quad \Xi \neq 0,$$

where $l'_{ij}(P, \Xi)$ consists of the terms in $l_{ij}(P, \Xi)$ which are just of the order $s_i + t_j$.

Remark. Our definition of ellipticity is a bit artificial for it is not obvious how to determine whether a given system (1.1) is elliptic, i.e., when weights s_i, t_j can be found so that (1.5) holds. Recently, however, Volevich [25] has given an equivalent formulation which is easily verifiable. To describe it consider a general system of the form (1.1), and let $\alpha_{ij} = \deg l_{ij}$. The determinant $A(P, \Xi) = \det |l_{ij}|$ consists of sums and differences of terms of the form l_{1i_1}, $l_{2i_2}, \cdots, l_{ni_n}$. Let R be the maximal degree of such terms and r the degree of A (so that in general $r \leq R$). Our definition of ellipticity is equivalent to the following: $r = R$ and $A(P, \Xi)$ is elliptic, i.e., for any P, $\Xi = 0$ is the only real root of $A'(P, \Xi) = 0$, where A' is the leading part, the part of order r, of A. That our definition of ellipticity implies this formulation is immediate; Volevich gives an elegant proof of the converse using a result in linear programming.

All elliptic systems in three or more independent variables are known (cf., for example, ADN, pp. 631–632) to satisfy a further condition described below. In the two variable case, however, we must additionally assume this.

SUPPLEMENTARY CONDITION ON L. *$L(P, \Xi)$ is of even degree $2m$ (with respect to Ξ). For every pair of linearly independent real vectors Ξ, Ξ', the polynomial $L(P, \Xi + \tau\Xi')$ in the complex variable τ has exactly m roots with positive imaginary part.*

This condition is actually used only at points P of the boundary $\dot{\mathscr{D}}$ of $\mathscr{D}$ with Ξ a tangent, and Ξ' the normal to $\dot{\mathscr{D}}$, at P.

When $m = 0$ we may explicitly solve (1.1) for the u_k in terms of the F_i and their derivatives as has been seen in [10, p. 506], We do not lose, therefore, in requiring here that

$$(1.6) \qquad m = \tfrac{1}{2} \deg (L(P, \Xi)) > 0.$$

Uniform ellipticity will be required in the sense that there is a positive constant A such that

$$(1.7) \qquad A^{-1} |\Xi|^{2m} \leqq |L(P, \Xi)| \leqq A\, |\Xi|^{2m}$$

for every real vector $\Xi = (\xi_1, \cdots, \xi_{n+1})$ and for every point P in the closure of the domain $\mathscr{D}$ considered.

To summarize, we shall always assume for our systems conditions (1.2), (1.3), (1.4), (1.6), (1.7), and also the above Supplementary Condition on L. Accompanying continuity requirements for different purposes will be stated in Sections 9 and 10.

Remarks on elliptic systems of first order. Any elliptic system satisfying (1.2) can be remodeled so that $s_i + t_j \leqq 1$: first order elliptic systems in this sense are the most general. This fact was first pointed out to us by Atiyah and Singer. It is not used in this paper, but, being of interest in itself, will be justified here.

The remodeling procedure consists in introducing as new dependent variables, which are added to the original dependent variables, appropriate derivatives of the latter, and as new equations, added to the original system, the equations which define the new variables. The procedure is well known, but it destroys ellipticity in the ordinary sense and hence has not heretofore seemed significant for elliptic systems. What Atiyah and Singer noticed is that this procedure does preserve ellipticity in our sense. Consider as a simple illustration Laplace's equation $\partial_1^2 u + \partial_2^2 u = 0$, $\partial_k = \partial/\partial x_k$, which in terms of the new variables $u_1 = \partial_1 u$ and $u_2 = \partial_2 u$ is reducible to the first order system

$$\begin{aligned} &\partial_1 u_1 + \partial_2 u_2 = 0, \\ &\partial_1 u - u_1 = 0, \\ &\partial_2 u - u_2 = 0. \end{aligned} \tag{1.8}$$

The characteristic determinant of (1.8) by the usual definition is

$$\begin{vmatrix} 0 & \xi_1 & \xi_2 \\ \xi_1 & 0 & 0 \\ \xi_2 & 0 & 0 \end{vmatrix} = 0;$$

the system thus is not elliptic in the ordinary sense. The characteristic determinant under our definition, when the weights

$$t_0 = 2, \quad t_1 = t_2 = 1$$

are assigned to u, u_1, u_2, respectively, and

$$s_0 = 0, \quad s_1 = s_2 = -1$$

to the first, second, and third equations, respectively, is, however,

$$\begin{vmatrix} 0 & \xi_1 & \xi_2 \\ \xi_1 & -1 & 0 \\ \xi_2 & 0 & -1 \end{vmatrix} = (\xi_1^2 + \xi_2^2),$$

so that the system is elliptic in our sense, as contended.

Let us now prove that the procedure described is applicable to an arbitrary system of equations of higher than first order which is elliptic in our sense. By inductive reasoning, it suffices to show that, when the *first* derivatives are added as new dependent variables, the resulting system has maximum order one less than that of the original system and is elliptic. Recalling that max $s_i = 0$, we first divide the functions u_k into two classes, $\{u_{k'}\}$ and $\{u_{k''}\}$, the indices written as k' taking the values of k for which $t_k > 1$, the k'' the remaining values of k; we may suppose $1 \leqq k' \leqq K$ and $K + 1 \leqq k'' \leqq N$. With $\partial_k = \partial/\partial x_k$, we introduce as new unknowns the first derivatives of the functions $u_{k'}$: $\partial_j u_{k'} = u_{k',j}$, and we introduce the additional equations

$$(1.9)_{k',j} \qquad \partial_j u_{k'} - u_{k',j} = 0, \qquad k' = 1, \cdots, K, \quad j = 1, \cdots, n.$$

We also rewrite each operator $l_{ik}u_k$ (summation not used) in the following way. The terms in which u_k is not differentiated are not changed. Any other term, which consists of a coefficient times $\partial^\rho \partial_j u_k$ for some j and some ρ (possibly $|\rho| = 0$), is rewritten as coefficient times $\partial^\rho u_{k,j}$. This way of rewriting is clearly not unique. In any case, the expression $\Sigma_k l_{ik} u_k$ is replaced by one of the form

$$(1.9)_i \qquad \sum_{k'} \lambda_{ik'} u_{k'} + \sum_{k''} \lambda_{ik''} u_{k''} + \sum_{k',j} \lambda_{ik',j} u_{k',j},$$

where the $\lambda_{ik'}$ and $\lambda_{ik''}$ are functions and $\lambda_{ik',j}$ a differential operator of order $\leqq s_i + t_k - 1$. Our new system of equations consists of $(1.9)_{k',j}$ and $(1.9)_i$ together. Weights, $\bar{s}_{k',j}$, $\bar{s}_i$, $\bar{t}_{k',j}$, and $\bar{t}_k$, for the new system are assigned as follows:

$$\bar{s}_i = s_i, \quad \bar{t}_k = t_k, \quad \bar{s}_{k',j} = 1 - t_{k'}, \quad \bar{t}_{k',j} = t_{k'} - 1.$$

We note that the new system is completely equivalent to the old and that its order is reduced, with the original variables differentiated, in fact, no more than once.

To see that the new system is elliptic, we have only to show that, if

$$\xi = (\xi_1, \cdots, \xi_{n+1}) \neq 0,$$

the corresponding characteristic determinant is not zero, i.e., the following system of linear equations for complex variables η_k, $\eta_{k',j}$ has only the null solution:

$$\xi_j \eta_{k'} - \eta_{k',j} = 0, \qquad k' = 1, \cdots, K, \quad j = 1, \cdots, n,$$

$$\textstyle\sum_1 \lambda_{ik}\eta_k + \sum \lambda'_{ik',j}(\xi)\eta_{k',j} = 0, \qquad i = 1, \cdots, N,$$

Σ indicating summation over all k' and j, Σ_1 summation over the values of k for which $s_i + t_k = 0$, and $\lambda'_{ik',j}(\xi)$ the characteristic polynomial of the leading part of $\lambda_{ik',j}$, the part of order $s_i + t_{k'} - 1$. By substituting the first set of equations into the second, we find

$$\textstyle\sum_1 \lambda_{ik}\eta_k + \sum \lambda'_{ik',j}(\xi)\xi_j\eta_{k'} = 0, \qquad i = 1, \cdots, N.$$

By the construction of the new system, these equations, however, are simply the equations

$$\sum_j l'_{ij}(\xi)\eta_j = 0, \qquad i = 1, \cdots, N.$$

Hence, $\eta_k = 0$ and, consequently, $\eta_{k',j} = 0$. It follows that the reduced system is elliptic, as stated.

Boundary conditions for the original system of differential equations go over, usually in several possible ways, into equivalent boundary conditions for the reduced system. If the boundary conditions were to complement the original system, any equivalent form of these conditions would also complement the new. This is because 1) the new and original boundary value problems are completely equivalent, and 2) the Complementing Conditions are both necessary and sufficient for our coercive inequalities to hold (Section 11).

2. Complementing Boundary Conditions

The boundary conditions we consider refer to a regular portion Γ of $\dot{\mathscr{D}}$. They are expressed as

$$\sum_{j=1}^{N} B_{hj}(P, \partial)\, u_j(P) = \phi_h(P) \quad \text{on} \quad \Gamma, \qquad h = 1, \cdots, m, \tag{2.1}$$

in terms of given polynomials in Ξ, $B_{hj}(P, \Xi)$, with complex coefficients depending on P. The orders of the boundary operators, like those of the operators in (1.1), depend on two systems of integer weights, in this case the system $t_1, \cdots, t_N$ already attached to the dependent variables and a new system $r_1, \cdots, r_m$ of which r_h pertains to the h-th condition in (2.1), $h = 1, \cdots, m$. The exact dependence is that expressed by the inequality

$$\deg B_{hj}(P, \Xi) \leqq r_h + t_j, \tag{2.2}$$

it being understood that $B_{hj} = 0$ when $r_h + t_j < 0$.

Let $B'_{hj}(P, \Xi)$ consist of the terms in $B_{hj}(P, \Xi)$ which are just of the order $r_h + t_j$.

The boundary conditions must "complement" the differential equations in presenting a well-posed problem for both. They will do so or not according to an algebraic criterion involving the $l'_{ij}(P, \Xi)$ and the $B'_{hj}(P, \Xi)$ which we shall describe below.

At any point P of Γ, let $\mathbf{n}$ denote the normal, and $\Xi \neq 0$ any tangent, to Γ (Ξ, in particular, is real). Denote by $\tau_k^+(P, \Xi)$, $k = 1, \cdots, m$, the m roots (in τ) with positive imaginary part of the characteristic equation $L(P, \Xi + \tau\mathbf{n}) = 0$. The existence of these roots is assured by the Supplementary Condition on L. Set

$$M^+(P, \Xi, \tau) = \prod_{h=1}^{m} (\tau - \tau_h^+(P, \Xi)).$$

Let $(L^{jk}(P, \Xi + \tau\mathbf{n}))$ denote the matrix adjoint to $(l'_{ij}(P, \Xi + \tau\mathbf{n}))$. The above mentioned criterion for the boundary problem (1.1), (1.2) to be coercive is that the following algebraic condition be satisfied.

Complementing Boundary Condition. *For any $P \in \Gamma$ and any real, non-zero vector Ξ tangent to Γ at P, let us regard $M^+(P, \Xi, \tau)$ and the elements of the matrix*

$$\sum_{j=1}^{N} B'_{hj}(P, \Xi + \tau\mathbf{n}) L^{jk}(P, \Xi + \tau\mathbf{n}) \tag{2.3}$$

as polynomials in the indeterminate τ. The rows of the latter matrix are required to be linearly independent modulo $M^+(P, \Xi, \tau)$, i.e.,

$$\sum_{h=1}^{m} C_h \sum_{j=1}^{N} B'_{hj} L^{jk} \equiv 0 \ (\text{mod } M^+),$$

only if the constants C_h are all zero.

For problems in which the Complementing Boundary Condition is satisfied, an important type of constant, called a "minor constant", pertaining to Γ or a suitable partition of Γ, will figure in the later estimates. Let us first describe such a constant in connection with a portion Γ' of Γ. For $P \in \Gamma'$, reduce the polynomials (in τ) (2.3) modulo $M^+(P, \Xi, \tau)$, as is described in the congruence

$$\sum_{j=1}^{N} B'_{hj}(P, \Xi + \tau\mathbf{n})\, L^{jk}(P, \Xi + \tau\mathbf{n}) \equiv \sum_{\beta=0}^{m-1} q_h^{k\beta}(P, \Xi)\tau^\beta \ (\text{mod } M^+(P, \Xi, \tau)). \tag{2.4}$$

Then construct the matrix $Q = (q_h^{k\beta})$ having m rows: $h = 1, \cdots, m$, and mN columns: $\beta = 0, \cdots, m-1$, $k = 1, \cdots, N$. Under the Complementing Boundary Condition, the rank of Q will be m. Hence, if

$$M^1(P, \Xi), \cdots, M^a(P, \Xi)$$

denote all the m-rowed minors of Q, not all the M^i will be zero, and, in particular,

$$\max_{j=1,\cdots,a} |M^j(P, \Xi)|$$

will not be zero. Also non-zero, at least when Γ' is compact, is the infimum $\Delta'_{\Gamma'}$ of these quantities for all real unit vectors Ξ which are tangent to Γ' at P, and for all $P \in \Gamma'$. When Γ' is plane, its minor constant $\Delta_{\Gamma'}$ is defined as $\Delta'_{\Gamma'}$. When Γ' is not plane, a certain change of variables will be known on occasion that, in particular, makes Γ' plane; $\Delta_{\Gamma'}$ in such a case will be defined as the value $\Delta'_{\Gamma'}$ has in the changed variables. When Γ itself, say, is covered by a union of subportions $\Gamma^{(i)}$, and each $\Gamma^{(i)}$ has a minor constant $\Delta_{\Gamma^{(i)}}$ as described above, the corresponding minor constant for Γ will be defined as

$$\Delta \equiv \Delta_\Gamma = \inf \Delta_{\Gamma^{(i)}}.$$

Except in the case in which $\Gamma = \Gamma'$ is given plane, these definitions, of course, are not complete; they will be made so in appropriate contexts below.

Strongly elliptic systems. As an example of an elliptic system with complementing boundary conditions, we mention a strongly elliptic system which has been treated for instance in [19]. This is a system

$$\sum_{1}^{N} l_{ij}(P, (1/i)\partial)u_j = F_i, \qquad i = 1, \cdots, N,$$

for which $\deg l_{ij} = t'_j + t'_i$, i.e., $s_i = t_i = t'_i \geqq 0$. Let l'_{ij} denote the principal part of l_{ij}, i.e., the terms of order precisely $t'_i + t'_j$. Strong ellipticity is the requirement that, at every point P of $\bar{\mathscr{D}}$,

$$\mathscr{Re}\, l'_{kj}(P, \Xi)\eta_k\bar{\eta}_j \geqq a\sum_1^N |\Xi|^{2t'_j}\,|\eta_j|^2 \tag{2.5}$$

for all complex numbers $\eta_1, \cdots, \eta_n$ and all real vectors $\Xi = (\xi_1, \cdots, \xi_{n+1})$, a being a positive constant.

In Dirichlet boundary conditions, the normal derivatives

$$(\partial/\partial\mathbf{n})^q u_j, \qquad q = 0, \cdots, t'_j - 1, \quad j = 1, \cdots, N, \tag{2.6}$$

are prescribed. (When $t'_j = 0$, u_j goes unprescribed.) The prescribed values are called the Dirichlet data in the problem. That the Dirichlet boundary conditions are complementing is easily seen. Suppose that ξ is a vector tangent to the boundary at a fixed point P_0 and $\mathbf{n}$ the inner normal at P_0. We must show that a solution of

$$l'_{jk}(P_0, (1/i)\partial)u_j = 0, \qquad k = 1, \cdots, N,$$

of the form $u = e^{ix\cdot\xi}v(n \cdot x)$, decaying as $n \cdot x \to \infty$ and having zero Dirichlet data for $n \cdot x = 0$, is identically zero. We observe that $v(t)$ is a solution of the system of ordinary differential equations

$$l'_{jk}\left(P_0, \xi + \frac{1}{i}\mathbf{n}\frac{d}{dt}\right)v_j = 0 \tag{2.7}$$

satisfying the boundary conditions $(d/dt)^q v_j = 0$ at $t = 0$ for $q \leqq t'_j - 1$. Multiply (2.7) by $\bar{v}_k(t)$, sum over k, and integrate with respect to t on the half-line $t \geqq 0$. The resulting integral, after suitable integrations by parts, may be expressed as the integral of a quadratic polynomial in the derivatives of the v_j of orders not greater than t'_j; because of the boundary conditions, no contribution occurs from the initial point $t = 0$. Now extend the domain of integration of this integral to the entire t-axis, for this purpose defining each $v_j(t)$ as zero for $t < 0$. In terms of the Fourier transform $\hat{v}_j(\tau) = \int e^{-it\tau}v_j(t)\,dt$, by Parseval's theorem, we have as a result

$$\int l'_{kj}(P_0, \xi + \tau\mathbf{n})\hat{v}_j(\tau)\overline{\hat{v}_k(\tau)}\,d\tau = 0,$$

which, in view of (2.5), proves that $v \equiv 0$. Dirichlet conditions are, therefore, complementing as claimed.

CHAPTER II

EQUATIONS WITH CONSTANT COEFFICIENTS IN A HALF-SPACE

Estimates in general problems (Sections 9 and 10) are based on those developed in this chapter for problems with constant coefficients in a half-space. Such problems arise already for ordinary differential equations, with which we find it advantageous to begin, see also [12], Chapter 10.

3. On Ordinary Differential Equations

Some questions in the constant coefficient case will be reduced by plane wave representation to problems for ordinary differential equations. These are treated here.

The problems we are led to consider are for systems of N linear, ordinary differential equations with constant (complex) coefficients for N functions $u = (u_1(t), \cdots, u_N(t))$ defined on the semi-axis $t \geqq 0$. Denoting $(1/i)\ (d/dt)$, by D, we write the system (here we use the summation convention) as

$$l_{ij}(D)u_j = 0, \qquad i = 1, \cdots, N, \tag{3.1}$$

where the l_{ij} are polynomials in D. Let $L(D)$ denote det $(l_{ij}(D))$. We assume that L is not identically zero; thus, L is a differential operator, say of order ρ, $\rho > 0$.

Let (L^{jk}) denote the adjoint to the matrix (l_{ij}), so that, in particular,

$$l_{ij}L^{jk} = \delta_i^k L. \tag{3.2}$$

Only the set of exponentially decaying solutions of (3.1) is pertinent to the applications intended below. However, the set of all solutions will be analogously discussed, if briefly, as being of interest in itself.

We shall first carry out a well known process for reducing the matrix $(l_{ij}(\tau))$ to triangular form with zeros below the diagonal, see Ince [13]. This reduction is carried out by two kinds of elementary operations: (a) the interchange of two rows, and (b) the addition to some row of another row multiplied by any polynomial in τ. The new (triangular) system of equations is equivalent to the original, since each such operation is reversible. The triangular matrix $(\tilde{l}_{ij}(\tau))$ is in fact obtainable from $(l_{ij}(\tau))$ by matrix multiplication on the left by a square matrix A whose entries are polynomials in τ and whose determinant equals ± 1. Thus, $\tilde{L}(\tau) = \det(\tilde{l}_{ij}) = \pm L(\tau)$. It follows, incidentally, that $\pm \tilde{L}^{ij} a_j^k = L^{ik}$, the a_j^k being the elements of the matrix A.

Let $\rho_i = \deg(\tilde{l}_{ii}(\tau))$. Obviously, $\rho = \sum_i \rho_i$.

What now follows, up to Theorem 3.1, will relate specifically to the full set of solutions of (3.1). We begin by proving that the dimension of the latter set is equal to ρ. Indeed, the system $\tilde{l}_{ij}u_j = 0$ being triangular, we have $\tilde{l}_{ij} = 0$ for $i > j$. Hence, the obvious procedure in constructing independent solutions of (3.1) is first to solve the last equation $\tilde{l}_{NN}u_N = 0$ of the triangularized system for u_N obtaining ρ_N independent solutions, then to solve the next to the last equation for u_{N-1} in terms of u_N, and so on. The ultimate result is that there are exactly ρ linearly independent solutions of (3.1), as contended.

By the above procedure, plainly a unique solution of (3.1) can be found with prescribed values of $D^k u_j$, $0 \leqq k < \rho_j$, $j = 1, \cdots, N$, at $t = 0$. The initial value problem thus described is an instance of

PROBLEM 1. To find a solution of (3.1) for which

$$B_{\alpha j}(D)u_j = a_\alpha \quad \text{at} \quad t = 0, \qquad \alpha = 1, \cdots, \rho, \tag{3.3}$$

where the a_α are given numbers and the $B_{\alpha j}(D)$, $\alpha = 1, \cdots, \rho$, $j = 1, \cdots, N$, differential operators with constant coefficients.

In the absence of any restrictions on the $B_{\alpha j}$, obviously some of the conditions listed in (3.3) might be redundant, or they might be incompatible with others of these conditions or with (3.1). Appropriate safeguards against these eventualities are described in

COMPLEMENTING CONDITION 1. *The rows of the ρ by N matrix, with polynomial entries,*

$$B_{\alpha j}(\tau)L^{jk}(\tau)$$

are linearly independent modulo the polynomial $L(\tau)$; i.e., the equations

$$\sum_\alpha C_\alpha B_{\alpha j}(\tau)L^{jk}(\tau) \equiv 0 \ (\mathrm{mod}\ L(\tau)) \quad \text{for} \quad k = 1, \cdots, N$$

imply that the constants C_α are all zero.

The pertinence of this condition is readily understood, in fact, from the following theorem, in which our main results concerning the full set of solutions of (3.1) are summarized.

THEOREM 3.1. *The following statements are all equivalent to one another:*

(a) *Complementing Condition* 1 *holds.*

(b) *Problem* 1 *has a solution for arbitrary a_α.*

(c) *Any solution of Problem* 1 *is unique: $u_j = 0$ is the only solution with all $a_\alpha = 0$.*

Theorem 3.1 can be proved by means analogous to those used below in demonstrating Theorem 3.2, the latter pertaining to the set of exponentially decaying solutions of (3.1). We now turn to this.

Assume that the polynomial $L(\tau)$ has exactly ρ^+ complex roots with $\mathscr{I}m\, \tau > 0$; then factor $L(\tau)$ as

$$L(\tau) = L^+(\tau) \cdot L^-(\tau),$$

where the roots of $L^+(\tau)$ are those of L for which $\mathscr{I}m\ \tau > 0$. Also factor the diagonal elements of the triangularized matrix above as $\tilde{l}_{ii}(\tau) = l_i^+(\tau)l_i^-(\tau)$, where the roots of $l_i^+(\tau)$ are just those of $\tilde{l}_{ii}$ for which $\mathscr{I}m\ \tau > 0$. Then

$$L^+(\tau) = \pm\prod_i l_i^+(\tau), \tag{3.4}$$

and $\rho^+ = \sum_i \rho_i^+$, ρ_i^+ being the degree of l_i^+.

Our first result concerning the set of exponentially decaying solutions of (3.1) is that its dimension is ρ^+. This is deduced with the help of the following

Lemma. *Let $P(D)$ be a differential operator with constant coefficients for a single variable u. If $f(t)$ dies down exponentially as $t \to \infty$, then the equation*

$$P(D)u = f$$

has a solution also dying down exponentially.

Postponing the proof of the lemma, we now show how to obtain ρ^+ independent solutions of (3.1) dying down exponentially at ∞. Consider the triangular system $\tilde{l}_{ij}u_j = 0$ equivalent to (3.1). The last equation has ρ_N^+ linearly independent, exponentially decaying solutions which are obtainable, namely, as the solutions of $l_N^+u = 0$. Choose one of these, and regard the next to the last equation

$$\tilde{l}_{N-1,N-1}u_{N-1} = -\tilde{l}_{N-1,N}u_N$$

as an equation for u_{N-1}. By the lemma, this has a solution dying down exponentially, and to it can be added any linear combination of the ρ_{N-1}^+ linearly independent solutions of $l_{N-1}^+u = 0$. There are thus, in all, $\rho_N^+ + \rho_{N-1}^+$ linearly independent vectors (u_{N-1}, u_N) satisfying the final two equations of the triangularized system. Continuing in this way, we ultimately find $\Sigma\rho_i^+ = \rho^+$ independent solutions of the whole triangularized system that die down exponentially. Furthermore, we may determine a unique solution with prescribed values of the derivatives

$$D^k u_j, \quad k = 0, 1, \cdots, \rho_j^+ - 1, \quad j = 1, \cdots, N, \tag{3.5}$$

at the origin. (No prescription is made when $\rho_j^+ = 0$.) This proves our contention as to dimensionality.

Proof of the Lemma: Factoring P, we easily reduce the lemma to the case of a first order operator

$$\frac{du}{dt} + au = f.$$

In this, a can be assumed to be real; for, when $a = \alpha + i\beta$ the transformation $v = e^{i\beta t}u$ reduces the equation to

$$\frac{dv}{dt} + \alpha v = e^{i\beta t}f.$$

Suppose $|f(t)| \leqq Ke^{-bt}$, $b > 0$. If $a \geqq b$, the solution

$$u = e^{-at} \int_0^t e^{a\tau} f(\tau)\, d\tau$$

decays exponentially, while if $a < b$, the solution

$$u = -e^{-at} \int_t^\infty e^{a\tau} f(\tau)\, d\tau$$

decays exponentially. Exhibiting these solutions proves the contention.

The problem of prescribing initial values of the derivatives (3.5) is an instance of

PROBLEM 2. To find an exponentially decaying solution of (3.1) for which

$$B_{\sigma j}(D) u_j = a_\sigma \quad \text{at} \quad t = 0, \qquad \sigma = 1, \cdots, \rho^+, \tag{3.6}$$

where the a_σ are given numbers and the $B_{\sigma j}(D)$, $\sigma = 1, \cdots, \rho^+$, $j = 1, \cdots, N$, differential operators with constant coefficients.

Appropriate safeguards against redundancy or incompatibility of conditions (3.6) are here furnished by

COMPLEMENTING CONDITION 2. *The rows of the ρ^+ by N matrix*

$$B_{\sigma j}(\tau) L^{jk}(\tau)$$

are linearly independent modulo the polynomial $L^+(\tau)$.

This condition is satisfied, as will be seen below, when the quantities prescribed for $t = 0$ are the derivatives (3.5). Its appropriateness is evident from

THEOREM 3.2. *The following statements are all equivalent to one another:*

(a) *Complementing Condition 2 holds.*

(b) *Problem 2 has a solution for arbitrary a_σ.*

(c) *Any solution of Problem 2 is unique: $u_j = 0$ is the only exponentially decaying solution with all $a_\sigma = 0$.*

Preliminary to proving Theorem 3.2 we observe that the Complementing Conditions are satisfied relative to equations (3.1) if and only if they also are satisfied relative to the triangularized equations $\tilde{l}_{ij} u_j = 0$. Indeed, if the rows of $B_{\sigma j}(\tau)\tilde{L}^{jk}(\tau)$ are linearly dependent mod $L^+(\tau)$, so also are the rows of $B_{\sigma j}(\tau) L^{jk}(\tau) = B_{\sigma j}(\tau) \tilde{L}^{ji}(\tau) a_i^k(\tau)$. Conversely, if the rows of $B_{\sigma j} L^{jk}$ are linearly dependent mod $L^+(\tau)$, so also are the rows of $B_{\sigma j}\tilde{L}^{jk}$, the elements of the matrix A^{-1} being again polynomials in τ.

Let us also verify that the particular boundary conditions mentioned above in connection with Problem 2, which consist in prescribing the values of the derivatives (3.5) at $t = 0$, satisfy Complementing Condition 2. (An analogous statement regarding Problem 1 is similarly proved.) By the preceding paragraph, it is sufficient to consider the triangular system $\tilde{l}_{ij} u_j = 0$. The matrix $(\tilde{L}^{ij})$ also is triangular, i.e., zero below the diagonal. Now suppose Complementing Condition

2 not to be satisfied. From the form of the present boundary conditions, it easily follows that there exist N polynomials $p_j(\tau)$, the degree of p_j being less than ρ_j^+ and not all the p_j being zero, such that

$$p_j(\tau)\tilde{L}^{jk}(\tau) = 0 \ (\mathrm{mod}\ L^+(\tau)) \quad \text{for} \quad k = 1, \cdots, N.$$

Since $(\tilde{L}^{jk})$ is triangular, we thus have, in particular,

$$p_1(\tau)\tilde{L}^{11}(\tau) \equiv 0 \ (\mathrm{mod}\ L^+(\tau)).$$

However, $\tilde{l}_{11}\tilde{L}^{11} = \prod_j \tilde{l}_{jj}$ so that $\tilde{L}^{11} = \prod_{j \neq 1} \tilde{l}_{jj}$. It follows from (3.4) that

$$p_1 \prod_{j \neq 1} \tilde{l}_{jj} \equiv 0 \ (\mathrm{mod} \prod_j l_j^+)$$

or

$$p_1 \prod_{j \neq 1} l_j^- \equiv 0 \ (\mathrm{mod}\ l_1^+).$$

Since each l_j^- has its roots in the lower half-plane, this is possible only if

$$p_1 \equiv 0 \ (\mathrm{mod}\ l_1^+),$$

and, since the degree of p_1 is less than $\rho_1^+ =$ degree of l_1^+, only if $p_1 \equiv 0$. Similarly we find, in turn, that $p_2, p_3, \cdots, p_N$ all vanish.

We shall now give the proof of Theorem 3.2; that of Theorem 3.1 is just the same, L^+ and ρ^+, wherever used, simply being replaced by L and ρ.

Proof of Theorem 3.2: (i) Equivalence of existence and uniqueness, i.e., of (b) and (c): The exponentially decaying solutions of (3.1) constitute, as noted, a ρ^+-dimensional space. We map this space into C^{ρ^+}, the space of vectors $a = (a_1, \cdots, a_{\rho^+})$ with ρ^+ (complex valued) components, by associating with any exponentially decaying solution $u = (u_1, \cdots, u_N)$ the vector $a(u)$ whose components are $B_{\sigma j}u_j\big|_{t=0}$, $\sigma = 1, \cdots, \rho^+$. This mapping being a homomorphism will be onto if and only if the statement $a(u) = 0$ implies $u = 0$. Thus, a u exists to any a if and only if such a u is necessarily unique. This is to say that (b) and (c) are equivalent, as asserted.

(ii) Sufficiency of the Complementing Conditions ((a) implies (b)): If L^+ has the form $L^+ = \tau^{\rho^+} + a_1\tau^{\rho^+-1} + \cdots$, we introduce ρ^+ polynomials

$$L_\beta^+(\tau) = \tau^\beta + a_1\tau^{\beta-1} + \cdots + a_\beta, \qquad \beta = 0, \cdots, \rho^+ - 1,$$

analogous to those of equations (1.5) in ADN. These polynomials have the property that

$$\frac{1}{2\pi i}\oint_\gamma \frac{L^+_{\rho^+-1-\beta}(\tau)}{L^+(\tau)}\,\tau^\alpha\,d\tau = \delta_\beta^\alpha, \qquad \alpha, \beta = 0, \cdots, \rho^+ - 1, \tag{3.7}$$

where γ is any rectifiable Jordan contour in the complex plane enclosing the roots of $L^+(\tau)$.

We also write

$$B_{\sigma j}(\tau)L^{jk}(\tau) = \sum_{\beta=0}^{\rho^+-1} q_\sigma^{k\beta}\tau^\beta \ (\mathrm{mod}\ L^+). \tag{3.8}$$

The matrix $Q = (q_\sigma^{k\beta})$ is to be regarded as having ρ^+ rows: $\sigma = 1, \cdots, \rho^+$, and $\rho^+ N$ columns: $\beta = 0, \cdots, \rho^+ - 1$, $k = 1, \cdots, N$. The Complementing Condition asserts that its rank is ρ^+. Given arbitrary numbers a_σ, we shall construct an exponentially decaying solution of (3.1) satisfying (3.6) at $t = 0$. Because Q has rank ρ^+, there exist numbers $c_{k\beta}$ (not necessarily uniquely determined) such that (the summation convention is used here)

$$c_{k\beta} q_\sigma^{k\beta} = a_\sigma. \tag{3.9}$$

We now set, for $j = 1, \cdots, N$,

$$u_j = \frac{1}{2\pi i} \int_\gamma c_{k\beta} \frac{L^{jk}(\tau)}{L^+(\tau)} L^+_{\rho^+ - \beta - 1}(\tau) e^{it\tau} \, d\tau, \tag{3.10}$$

where γ is a contour enclosing the roots of $L^+(\tau)$. Since these roots lie in the open upper half-plane, we may also choose γ to lie in that half-plane. The functions u_j therefore decay exponentially as $t \to \infty$; they also clearly satisfy (3.1). Now at $t = 0$,

$$\begin{aligned} B_{\sigma j} u_j &= \frac{1}{2\pi i} \int_\gamma c_{k\beta} B_{\sigma j}(\tau) \frac{L^{jk}(\tau)}{L^+(\tau)} L^+_{\rho^+ - \beta - 1}(\tau) \, d\tau \\ &= \frac{1}{2\pi i} \int_\gamma c_{k\beta} q_\sigma^{k\alpha} L^+_{\rho^+ - \beta - 1}(\tau) \frac{\tau^\alpha}{L^+(\tau)} \, d\tau \end{aligned}$$

by (3.8). Here summation is over j and k from 1 to N, and over α, β from 0 to $N - 1$. By (3.7) and (3.9) we thus have

$$B_{\sigma j} u_j = c_{k\beta} q_\sigma^{k\beta} = a_\sigma \quad \text{for} \quad t = 0.$$

Hence the functions u_j given in (3.10) have all the desired properties.

We note that, although there was a great deal of arbitrariness in choosing the $c_{k\beta}$ to satisfy (3.9), the functions u_j are independent of the particular choice. This is due to Theorem 3.2, which guarantees uniqueness when solutions of Problem 2 exist for arbitrary a_σ. An easy consequence of this is the following

Corollary to Theorem 3.2. *If the Complementing Conditions are satisfied, and if $c_{k\beta}$ are constants satisfying $\sum_{k,\beta} c_{k\beta} q_\sigma^{k\beta} = 0$ for $\sigma = 1, \cdots, \rho^+$, then the polynomials*

$$\sum_{k=1}^{N} \sum_{\beta=0}^{\rho^+ - 1} c_{k\beta} L^{jk}(\tau) L^+_{\rho^+ - \beta - 1} \textit{ are } \equiv 0 \pmod{L^+(\tau)}$$

for $j = 1, \cdots, N$.

A direct algebraic proof of this was rather lengthy.

(iii) Necessity of the Complementing Condition ((c) implies (a)): Suppose that the rank of the matrix Q is less than ρ^+. We shall construct a non-zero, exponentially decaying solution of (3.1) with $B_{\sigma j} u_j = 0$ at $t = 0$, $\sigma = 1, \cdots, \rho^+$, contradicting uniqueness. We may suppose that (3.1) is in triangular form: $\tilde{l}_{ij}u_j = 0$. As we observed earlier there is a system of boundary operators $\tilde{B}_{\sigma j}u_j$ at $t = 0$, for which existence and uniqueness hold, and which satisfies the Complementing Condition (such a system might consist, namely, of the derivatives (3.5) at $t = 0$). The corresponding matrix $\tilde{Q}$ then has rank ρ^+ and, therefore, contains a row which is linearly independent (mod $L^+(\tau)$) of the rows of Q; suppose this is the σ_0-th row. Then, if we write

$$\tilde{B}_{\sigma_0 j}L^{jk} \equiv \sum_{\beta=0}^{\rho^+-1} \tilde{q}_{\sigma_0}^{k\beta}\tau^\beta \;(\text{mod } L^+(\tau)),$$

the row $\tilde{q}_{\sigma_0}^{k\beta}$ is independent of the rows $q_\sigma^{k\beta}$, $\sigma = 1, \cdots, \rho^+$, which correspond to Q according to (3.8). Hence there are constants $c_{k\beta}$ such that

$$c_{k\beta}q_\sigma^{k\beta} = 0, \quad c_{k\beta}\tilde{q}_{\sigma_0}^{k\beta} = 1$$

(the summation convention has been employed). If we substitute these constants into the expression (3.10), the resulting functions u_j decay exponentially, satisfy (3.1), and satisfy, at $t = 0$, the homogeneous boundary conditions

$$B_{\sigma j}u_j = 0.$$

These functions are not identically zero, however, because of the additional boundary condition $\tilde{B}_{\sigma_0 j}u_j = 1$. Thus, condition (c) of Theorem (3.2) is violated, this fact proving that (c) implies (a), completing our proof of Theorem 3.2.

4. Explicit Solution of Boundary Problems for Homogeneous Equations

In this section, we shall be concerned with functions defined in a half-space given, say, by $x_{n+1} > 0$. Usually, it will be convenient to write $t = x_{n+1}$ and to denote a point of the space by $P = (x, t)$, x here abbreviating $(x_1, \cdots, x_n)$. With $\xi = (\xi_1, \cdots, \xi_n)$, the symbol (ξ, τ) will denote a vector with $n + 1$ components; ∂_t will denote differentiation with respect to t, $\partial_x = (\partial_1, \cdots, \partial_n)$ differentiation with respect to $x_1, \cdots, x_n$, and ∂, as before, differentiation with respect to $x_1, \cdots, x_n, t$, while ∂_x^β or ∂_x^l, $l = |\beta|$, represents higher differentiations with respect to the x_i. If $P = (x, t)$ and $Q = (y, \tau)$, we represent the Euclidean distance between P and Q as

$$|P - Q| = (|x - y|^2 + (t - \tau)^2)^{1/2}, \qquad |x - y|^2 = \sum_1^n (x_i - y_i)^2.$$

The scalar product of two real vectors x and y will be denoted by $x \cdot y = \sum_1^n x_i y_i$.

This section is restricted to systems of partial differential equations of the form (1.1) with (complex) constant coefficients such that $l_{ij} = l'_{ij}$. Writing $l_{ij}(\xi, \tau)$ in place of $l_{ij}(P; \xi, \tau)$, we thus assume this polynomial, if not identically zero, to be of the degree

$$\deg (l_{ij}(\xi, \tau)) = s_i + t_j, \tag{4.1}$$

where s_i and t_j are integers. (If $s_i + t_j < 0$, we assume $l_{ij} \equiv 0$.) In accordance with (1.3), (1.4), and (1.6), we also assume

$$s_i \leqq 0, \tag{4.2}$$

$$t_j \geqq 0, \tag{4.3}$$

and

$$m = \tfrac{1}{2} \deg (L(\xi, \tau)) > 0, \tag{4.4}$$

where $L(\xi, \tau) = \det (l_{ij}(\xi, \tau))$.

According to the Condition on L, m is an integer, and corresponding to any real vector $\xi \neq 0$ there are m roots $\tau_k^+(\xi)$, $k = 1, \cdots, m$, of the equation $L(\xi, \tau) = 0$ with positive imaginary part. Defining

$$M^+(\xi, \tau) = \prod_{h=1}^{m} (\tau - \tau_h^+(\xi)) \equiv \sum_{p=0}^{m} a_p(\xi)\tau^{m-p}, \tag{4.5}$$

we note that the $a_p(\xi)$ are homogeneous functions of ξ_i of degree p which are analytic for real $\xi \neq 0$. $M^-(\xi, \tau)$ is analogously defined.

Let $(L^{jk}(\xi, \tau))$ designate the adjoint to the characteristic matrix $(l_{ij}(\xi, \tau))$. Thus, in particular,

$$\sum_{j=1}^{N} l_{ij}(\xi, \tau)L^{jk}(\xi, \tau) = \delta_i^k L(\xi, \tau), \qquad i, k = 1, \cdots, N, \tag{4.6}$$

δ_i^k being Kronecker's delta. If L^{jk} is not identically zero,

$$\deg (L^{jk}(\xi, \tau)) = 2m - s_k - t_j. \tag{4.7}$$

The boundary conditions considered in this section are assumed to have constant coefficients and to be such that $B_{hj} \equiv B'_{hj}$. Writing $B_{hj}(\xi, \tau)$ in place of $B_{hj}(P; \xi, \tau)$, we thus assume that the polynomial $B_{hj}(\xi, \tau)$, if not identically zero, is of the degree

$$\deg (B_{hj}(\xi, \tau)) = r_h + t_j, \tag{4.8}$$

r_h denoting an integer. ($B_{hj} \equiv 0$ if $r_h + t_j < 0$.)

For each fixed vector ξ, let us regard the elements of the matrix

$$\sum_{j=1}^{N} B_{hj}(\xi, \tau)L^{jk}(\xi, \tau)$$

as polynomials in the indeterminate τ. The Complementing Boundary Condition requires that, for each fixed $\xi \neq 0$, the rows of this matrix be linearly independent, modulo $M^+(\xi, \tau)$, over the scalar field.

In this section, we shall solve the following

BOUNDARY PROBLEM. *Let $\phi_j(x), j = 1, \cdots, m$, be C^∞ functions of compact support in $x_1, \cdots, x_n$-space. Under the foregoing assumptions, we seek C^∞ functions $u_j(x, t)$ in the half-space $t \geqq 0$ which satisfy the system of homogeneous partial differential equations*

$$\sum_{j=1}^{N} l_{ij}(\partial_x, \partial_t) u_j(x, t) = 0 \tag{4.10a}$$

and the boundary conditions

$$\sum_{j=1}^{N} B_{hj}(\partial_x, \partial_t) u_j(x, t) \Big|_{t=0} = \phi_h(x), \qquad h = 1, \cdots, m. \tag{4.10b}$$

The Complementing Boundary Conditions and the Condition on L will be seen to be both necessary and sufficient, when the other assumptions (4.1), (4.4), and (4.8) apply, for this boundary problem to have a unique solution decaying to zero as $t \to \infty$.

4.1. Explicit solution of the boundary problem. A solution of the above boundary problem will be obtained from convolutions of the ϕ_j with "Poisson kernels" (see also [24])

$$K_{jh}(x, t) \qquad j = 1, \cdots, N, \quad h = 1, \cdots, m.$$

The construction of the Poisson kernels, like that given in ADN, Section 2 for single equations, is based on Fritz John's resolution [14, Chapter 1]

$$\phi(x) = -\frac{1}{(2\pi i)^n q!} \Delta_x^{(n+q)/2} \int \phi(y) \left(\int_{|\xi|=1} [(x-y)\cdot\xi]^q \log \frac{(x-y)\cdot\xi}{i}\, d\omega_\xi \right) dy \tag{4.11}$$

of an arbitrary differentiable function $\phi(x)$ with compact support into plane waves, in which formula q is any positive integer of the same parity as n, $\Delta_x = \sum_i^n \partial_i^2$, and $d\omega_\xi$ denotes the element of area on the unit sphere $|\xi| = 1$. When $n = 1$, $\int_{|\xi|=1} \cdots d\omega_\xi$ is to be interpreted as $\sum_{\xi=\pm 1}$, the formula then reducing to

$$\phi(x) = \frac{1}{2(q!)} \left(\frac{d}{dx}\right)^{q+1} \int \phi(y)\, |x-y|^q\, dy. \tag{4.11$_1$}$$

An important role will be played by the functions

$$G(z; M) = \begin{cases} \dfrac{-1}{(2\pi i)^n M!} z^M \left(\log (z/i) - \sum_{k=1}^{M} 1/k \right) & \text{for } M > 0, \\ \dfrac{-1}{(2\pi i)^n} \log (z/i) & \text{for } M = 0, \\ \dfrac{(-1)^M (-M-1)!}{(2\pi i)^n} z^M & \text{for } M < 0, \end{cases}$$

in which M is an integer and z an ordinary complex variable and the principal branch of the logarithm is taken with the z-plane slit along the negative real axis. Obviously, $G((x-y)\cdot\xi;\ q)$ essentially enters (4.11). We note the formula

$$\left(\frac{d}{dz}\right)^{\alpha} G(z;\ M) = G(z;\ M-\alpha) \tag{4.12}$$

valid for any integer M and any positive integer α.

Let

$$\begin{aligned} s' &= -\min s_k, \quad t' = \max t_j, \quad r = \max r_h, \quad \rho = -\min(r_h, 0), \\ r' &= r + \rho, \quad r'_h = r_h + \rho, \quad P = r' + t' + s'. \end{aligned} \tag{4.13}$$

We set

$$g(z) = G(z;\ 2m + q + P).$$

In arriving at the Poisson kernels, an intermediate step, as in the related method of Fourier integrals, is to solve (4.10a) under certain plane wave boundary conditions. What is required, specifically, is m auxiliary solutions $(R_h^1(x, t;\ \xi),$ $R_h^2(x, t;\ \xi), \cdots, R_h^N(x, t;\ \xi))$, $h = 1, \cdots, m$, of the differential equations

$$\sum_{j=1}^{N} l_{ij}(\partial_x, \partial_t) R_h^j(x, t;\ \xi) = 0, \quad i = 1, \cdots, N, \quad h = 1, \cdots, m, \tag{4.14}$$

depending continuously on parameters $\xi = (\xi_1, \cdots, \xi_n)$ for $|\xi| = 1$, such that

$$\sum_{j=1}^{N} B_{hj}(\partial_x, \partial_t) R_{h'}^j(s, t;\ \xi)\Big|_{t=0} = g^{(2m+t'+s'+r'_h)}(x\cdot\xi)\delta_{hh'}, \quad h, h' = 1, \cdots, m. \tag{4.15}$$

These m auxiliary solutions are constructed as follows. First, we introduce

$$\begin{aligned} g^{jk}(x, t;\ \xi, \tau) &= L^{jk}(\partial_x, \partial_t) g^{(s_k+s'+t'+\rho)}(x\cdot\xi + \tau t) \\ &= L^{jk}(\xi, \tau) g^{(2m+s'+t'-t_j+\rho)}(x\cdot\xi + \tau t); \end{aligned}$$

these are of class C^{r+q+t_j} for $t > 0$, $\mathscr{I}m\,\tau > 0$. For them, we have by (4.6)

$$\sum_{j=1}^{N} l_{ij}(\partial_x, \partial_t) g^{jk} = \delta_i^k L(\xi, \tau) g^{(2m+s_k+s'+t'+\rho)}(x\cdot\xi + \tau t), \quad i, k = 1, \cdots, N, \tag{4.16}$$

and by (4.7) and (4.8)

$$\sum_{j=1}^{N} B_{hj}(\partial_x, \partial_t) g^{jk} = g^{(2m+s'+t'+r'_h)}(x\cdot\xi + t\tau) \sum_{j=1}^{N} B_{hj}(\xi, \tau) L^{jk}(\xi, \tau).$$

If, for each fixed vector ξ, we regard $M^+(\xi, \tau)$, $B_{hj}(\xi, \tau)$, and $L^{jk}(\xi, \tau)$ as polynomials in τ, and if we write

$$\sum_{j=1}^{N} B_{hj}(\xi, \tau) L^{jk}(\xi, \tau) \equiv \sum_{l=0}^{m-1} q_h^{kl}(\xi)\tau^l \ (\mathrm{mod}\ M^+(\xi, \tau)),$$
$$h = 1, \cdots, m, \quad k = 1, \cdots, N,$$

the last condition above leads to

$$(4.17)\qquad \sum_{j=1}^{N} B_{hj}(\partial_x, \partial_t) g^{jk}\Big|_{t=0} = g^{(2m+s'+t'+r_h')}(x\cdot\xi)\sum_{l=0}^{m-1} q_h^{kl}(\xi)\tau^l \pmod{M^+(\xi,\tau)}.$$

This observation can be regarded as the first step in the construction of the R_h^j.

According to the Complementing Boundary Condition,

$$\operatorname{rank}\,(q_h^{1l}(\xi), \cdots, q_h^{Nl}(\xi)) = m$$

for real $\xi \neq 0$. Hence, if ξ_0 is any fixed vector $\neq 0$, we can determine coefficients $c_{klp}(\xi)$, continuous in a suitable neighborhood of ξ_0, such that

$$(4.18)\qquad \sum_{k=1}^{N}\sum_{l=0}^{m-1} q_h^{kl}(\xi) c_{klh'}(\xi) = \delta_{hh'}, \qquad h, h' = 1, \cdots, m.$$

This is the second consideration that enters into the construction of the R_h^j.

As the third consideration, we introduce m polynomials (in τ) associated with M^+, analogous to those of equation (1.5) in ADN, as follows:

$$(4.19)\qquad M_l^+(\xi,\tau) = \tau^l + a_1(\xi)\tau^{l-1} + \cdots + a_l(\xi), \qquad l = 0, 1, \cdots, m-1.$$

These polynomials have the property that

$$(4.20)\qquad \frac{1}{2\pi i}\oint_\gamma \frac{M_{m-1-l}^+(\xi,\tau)}{M^+(\xi,\tau)}\tau^k\, d\tau = \delta_{lk}, \qquad k, l = 0, \cdots, m-1,$$

where γ is any rectifiable Jordan contour in the complex plane enclosing all the roots $\tau_k^+(\xi)$ for $|\xi| = 1$. (This is noted in ADN, Section 1.)

Let us now cover the unit n-dimensional sphere Ω: $|\xi| = 1$ with a finite number of domains, say S_β, $\beta = 1, \cdots, b$, within each of which equations (4.18) have a (non-unique) solution $c_{klh,\beta}(\xi)$ continuously dependent upon ξ. Let $(\zeta_1(\xi), \zeta_2(\xi), \cdots, \zeta_b(\xi))$ denote a continuous partition of unity for Ω subordinate to the indicated covering. Thus, each $\zeta_\beta(\xi)$ is continuous on Ω, $\zeta_\beta \geqq 0$, vanishes outside S_β, and $\sum_{\beta=1}^{b} \zeta_\beta = 1$.

The desired auxiliary solutions are defined as

$$R_h^j(x, t; \xi) = \sum_{\beta=1}^{b} \zeta_\beta(\xi) R_{h,\beta}^j(x, t; \xi),$$

where $R_{h,\beta}^j(x, t; \xi)$, defined just for $\xi \in S_\beta$, is given by

$$(4.21)\qquad R_{h,\beta}^j(x, t; \xi) = \frac{1}{2\pi i}\int_\gamma \sum_{k=1}^{N}\sum_{l=0}^{m-1} g^{jk}(x, t; \xi, \tau) c_{klh,\beta}(\xi)\frac{M_{m-l-1}^+(\xi,\tau)}{M^+(\xi,\tau)}\, d\tau,$$
$$j = 1, \cdots, N, \quad h = 1, \cdots, m, \quad \beta = 1, \cdots, b.$$

Here, γ is a simple closed curve in the upper complex τ-plane enclosing the zeros (in τ) of $M^+(\xi, \tau)$ for $|\xi| = 1$. The R_h^j satisfy conditions (4.14) by (4.16), and they satisfy (4.15) by (4.17), (4.20), and (4.18). By the Corollary to Theorem 3.2, $R^j_{h,\beta}(x, t; \xi)$ for given ξ, despite appearances, is independent of β; hence, R^j_h is unique.

In terms of the m auxiliary solutions (4.21), we can now give the solution of the problem formulated above.

THEOREM 4.1. *The boundary problem formulated on p. 53 can be solved if the Complementing Condition is satisfied by the formula*

$$u_j(x, t) = \sum_{h=1}^{m} K_{jh} * \phi_h = \sum_{h=1}^{m} \int K_{jh}(x - h, t)\phi_h(y)\, dy \tag{4.22}$$

in which the Poisson kernels $K_{jh}(x, t)$ are given by

$$K_{jh}(x, t) = \Delta_x^{(n+q)/2} \int_{|\xi|=1} \left(\sum_{i=1}^{n} \xi_i \partial_i \right)^{r-r_h} R_h^j(x, t; \xi)\, d\omega_\xi . \tag{4.23}$$

(A more convenient formula for calculation will be given below.) Here, q is a positive integer, entering the definition of $g(z)$, which is of the parity of n and for present purposes also satisfies the inequality $q > r + t'$.

Proof: By (4.14), equations (4.10a) hold for $t > 0$. It will be seen below that $u_j \in C^\infty$ for $t \geqq 0$. By (4.15), (4.13), and the resolution (4.11),

$$\begin{aligned} \sum_{j=1}^{N} B_{hj}(\partial_x, \partial_t) u_j(x, t)\big|_{t=0} &= \Delta_x^{(n+q)/2} \int \phi_h(y)\, dy \\ &\quad \times \int_{|\xi|=1} \left(\sum_{i=1}^{n} \xi_i \partial_i \right)^{r-r_h} g^{(2m+s'+t'+r_h')}((x - y)\cdot \xi)\, d\omega_\xi \\ &= \Delta_x^{(n+q)/2} \int \phi_h(y)\, dy \int_{|\xi|=1} G((x-y)\cdot\xi; q)\, d\omega_\xi \\ &= \phi_h(x). \end{aligned} \tag{4.24}$$

Except for remarks on the differentiability of u_j at the plane $t = 0$, this concludes our proof of Theorem 4.1.

The Poisson kernels defined by (4.23) can be represented as

$$K_{jh}(x, t) = \int_{|\xi|=1} d\omega_\xi \frac{1}{2\pi i} \int_\gamma G(x\cdot\xi + t\tau; r_h + t_j - n) \frac{N_{jh}(\xi, \tau)}{M^+(\xi, \tau)}\, d\tau \qquad \text{for } t > 0, \tag{4.25}$$

where

$$N_{jh}(\xi, \tau) = \sum_{k=1}^{N} \sum_{l=0}^{m-1} L^{jk}(\xi, \tau) c_{klh}(\xi) M^+_{m-l-1}(\xi, \tau).$$

(The dependence of the c_{klh} upon β is not displayed in this argument, because, as noted, the integral over γ in (4.25) is independent of β.) To justify (4.25), we note from (4.23) and (4.21) that

$$K_{jh}(x, t) = \Delta_x^{(n+q)/2} K_{jh,q}(x, t), \tag{4.26}$$

where

$$K_{jh,q}(x, t) = \int_{|\beta|=1} d\omega_\xi \left(\sum_{i=1}^n \xi_i \partial_i \right)^{r-r_h} \times \frac{1}{2\pi i} \int_\gamma \sum_{k=1}^N \sum_{l=0}^{m-1} g^{jk}(x, t; \xi, \tau) c_{klh}(\xi) \frac{M^+_{m-l-1}(\xi, \tau)}{M^+(\xi, \tau)} \, d\tau.$$

From the definition of g^{jk} we have

$$\left(\sum_{i=1}^n \xi_i \partial_i \right)^{r-r_h} g^{jk}(x, t; \xi, \tau) = L^{jk}(\xi, \tau) g^{(2m+P-r_h-t_j)}(x \cdot \xi + t\tau);$$

hence, from the definitions of $g(z)$ and N_{jh} and from (4.12),

$$K_{jh,q}(x, t) = \int_{|\xi|=1} d\omega_\xi \frac{1}{2\pi i} \int_\gamma G(x \cdot \xi + t\tau; q + r_h + t_j) \frac{N_{jh}(\xi, \tau)}{M^+(\xi, \tau)} \, d\tau \quad \text{for } t > 0. \tag{4.27}$$

Formula (4.25) is an immediate consequence of this and (4.26).

It is easily seen that $K_{jh,q}$ and all its derivatives up to those of order $q + r_h + t_j - 1$ are continuous in the closed half-space $t \geqq 0$ (the same is true of derivatives of order $q + r_h + t_j$ if $n \geqq 2$). From this observation, (4.22) and (4.26), u_j can be proved to be infinitely differentiable in the closed half-space $t \geqq 0$ by the same argument as given in ADN, p. 637, in the case of a single equation. This settles the last remaining contention of Theorem 4.1.

Remark 1. If q is of the parity of n, $0 \leqq s \leqq q - 1$, and $n + q - r_h \geqq 0$, and if the ϕ_h are compactly supported functions belonging to C^{n+q-r_h}, we can write

$$\partial^{s+t_j} \Delta_x^{(n+q)/2} \int K_{jh,q}(x - y, t) \phi_h(y) \, dy$$

as a combination of terms of the form

$$\int \partial^{s+t_j+r_h} K_{jh,q}(x - y, t) \cdot \partial_y^{n+q-r_h} \phi_h(y) \, dy,$$

these being continuous for $t \geqq 0$. Hence, $u_j \in C^{s+t_j}$ for $t \geqq 0$. This conclusion holds, in particular, if $q = 1 + s$ or $2 + s$, according to the parity of n, and $s \geqq \max(0, r - n - 1)$. In sum, therefore, $u_j \in C^{s+t_j}$ for $t \geqq 0$ if

$$s \geqq \max(0, r - n - 1) \text{ and } \phi_h \in C^{n+2+s-r_h}.$$

4.2. Estimates relating to the Poisson kernels. The above representations lead to useful estimates, relating to the Poisson kernels, which are expressible in terms of a "characteristic constant"

$$E = n + N + t' + \sum |r_h| + A + b + \Delta^{-1},$$

where b is a bound for the coefficients in equations (4.10a) and conditions (4.10b), Δ the minor constant for the boundary plane $t = 0$ (Section 2), and A the ellipticity constant (Section 1). The most basic of these estimates is that of

LEMMA 4.1. *The kernels* $K_{jh,q}(x, t)$ *are of class* C^∞ *in the half-space* $t \geqq 0$, *except at the origin, and satisfy*

$$|\partial^s K_{jh,q}| \leqq C(s, E)(|x|^2 + t^2)^{(r_h+t_j+q-s)/2}(1 + |\log (|x|^2 + t^2)|), \qquad s \geqq 0. \tag{4.28}$$

Furthermore, if $s \geqq r_h + t_j + q + 1$, *then* $\partial^s K_{jh,q}$ *is homogeneous of degree* $r_h + t_j + q - s$, *and the logarithmic term in the inequality may be omitted. Here,* $C(s, E)$ *is a constant depending only on* s, q, *and* E.

The proof is as that of ADN, Appendix 1.

From (4.26), we have

COROLLARY 4.1. *The Poisson kernels* $K_{jh}(x, t)$ *are of class* C^∞ *for* $t \geqq 0$, *except at the origin, and satisfy*

$$|\partial^s K_{jh}| \leqq C(s, E)(|x|^2 + t^2)^{(r_h+t_j-n-s)/2}(1 + |\log (|x|^2 + t^2)|), \qquad s \geqq 0. \tag{4.28'}$$

If $s > r_h + t_j - n$, *then* $\partial^s K_{jh}$ *is homogeneous of degree* $r_h + t_j - n - s$, *and the logarithm in* (4.28)′ *can be omitted.*

The remaining estimates refer to combinations of the kernels such as occur in the boundary conditions. They will be applied in Theorem 4.2 below to give a generalization of the maximum principle.

COROLLARY 4.2. *For* $h, h' = 1, \cdots, m$, *we have*

$$|\sum_j B_{hj}(\partial)K_{jh'}(x, t)| \leqq C(E)t(|x|^2 + t^2)^{(r_{h'}-r_h-n-1)/2}(1 + |\log (|x|^2 + t^2)|). \tag{4.29}$$

Hence, in particular,

$$|\sum_j B_{hj}(\partial)K_{jh}(x, t)| \leqq C(E)\frac{t}{(|x|^2 + t^2)^{(n+1)/2}}. \tag{4.29'}$$

For $s_0 = n + q + r_{h'} - r_h \geqq 0$, $x \neq 0$, $h \neq h'$,

$$|\partial_x^{s_0}\sum_j B_{hj}(\partial)K_{jh',q}(x, t)| \leqq C(s, E, q)\frac{t}{(|x|^2 + t^2)^{(n+1)/2}}. \tag{4.30}$$

Proof: The Poisson kernels possess the following "reproducing" property: for all C^∞ functions ϕ_h with compact support,

$$\sum_{h'}\sum_j\int[B_{hj}(\partial)K_{jh'}(x - y, t)]_{t=0}\phi_{h'}(y)\,dy = \phi_h(x).$$

Hence,

$$\sum_j [B_{hj}(\partial)K_{jh'}(x, t)]_{t=0} = 0 \quad \text{for} \quad x \neq 0, \quad h, h' = 1, \cdots, m.$$

This, (4.28)′, and the theorem of the mean suffice to prove (4.29). Inequality (4.30) follows in a similar way from the fact, a consequence of (4.15), that

$$[\sum_j B_{hj}(\partial)K_{jh',q}(x, t)]_{t=0} = 0 \quad \text{for} \quad x \neq 0, h \neq h'.$$

Remark 2. Let $l_0 = \max(0, r_h)$, and for $l \geqq l_0$ and $h = 1, \cdots, m$, let $\phi_h(x)$ be functions of class $C^{l+n-r_{h'}+2}$ for all $h' = 1, \cdots, m$ such that

$$\partial^k \phi_h(x) = O(|x|^{-1-r_h-k}(1 + |\log|x||)), \qquad k = 0, \cdots, \max_{h'}(l - r_{h'})$$

for large $|x|$. Let $\partial_x^{l-r_h}$ denote a particular differential operator with respect to x of order $l - r_h$, and for $t > 0$ define

$$v_h(x, t) = \sum_{h',i} \int \partial_x^{l-r_h} B_{hi}(\partial) K_{ih'}(x - y, t)\phi_{h'}(y)\, dy.$$

Then v_h may be continued as a continuous function into the closed half-space $t \geqq 0$, and $v_h(x, 0) = \partial^{l-r_h}\phi_h(x)$.

Proof: The integral converges absolutely, because of (4.28)′ and because of the conditions imposed on the ϕ_h; ∂_x can be replaced by ∂_y, and partial integrations can be carried out, without contributions from infinity, to transfer $\partial_y^{l-r_h}$ from $K_{ih'}$ to $\phi_{h'}$. The remainder of the proof is like that of ADN, p. 640, using Remark 1 and (4.29).

The next result estimates combinations of derivatives of the solution such as are to be found in the boundary conditions. It is a kind of generalization of the maximum principle. In it, we make use of the following norm defined for any non-negative integer j:

$$[f(x)]_j = \sum_{|\beta|=j} \text{l.u.b.}\, |D^\beta f|.$$

THEOREM 4.2. *Consider the solution $u_j(x, t)$ given by (4.22), and let l be an integer such that $l \geqq r_h$, $h = 1, \cdots, m$. Then for any $i = 1, \cdots, m$ and any differentiation $\partial_x^{l-r_h}$ with respect to x of order $l - r_h$, we have*

$$\sup_{t \geqq 0} |\partial_x^{l-r_h} \sum_i B_{hi}(\partial) u_i(x, t)| \leqq C(l, E) \sum_j [\phi_j]_{l-r_j}.$$

Proof: By (4.22), it suffices to show for any fixed h, j that

$$\sup_{t \geqq 0} |\partial_x^{l-r_h} \sum_i B_{hi}(\partial) K_{ih'}(x, t) * \phi_{h'}(x)| \leqq C(l, E)[\phi_h]_{l-r_h}.$$

In the case $h = h'$, we transfer $\partial_x^{l-r_h}$ to $\phi_h(x)$ and apply (4.29)′. In the case $h \neq h'$, we make the substitution (4.26) with $q + n - l + r_{h'} \geqq 0$, transfer a suitable number of Δ to $\phi_{h'}$ in whole or in part, and use (4.30). Details are as in ADN, p. 641.

4.3. Boundary problems not well posed. In this section, we shall show by examples that, if the Complementing Boundary Condition, or the Condition on L, is violated, then there are solutions $(u_1, \cdots, u_N)$ of the homogeneous system (4.10) ($\phi_h = 0$) such that for any positive integer σ the derivatives of u_j of orders up to $\sigma + t_j$, inclusive, are bounded, while for at least one index k, $\partial_t^{\sigma+t_k} u_k$ has Hölder coefficients (with any exponent $\alpha < 1$) as large as desired in any half neighborhood of the boundary $t = 0$.

Suppose first that the Condition on L holds, and consider homogeneous boundary conditions

$$\sum_{j=1}^{m} B_{hj}(\partial_x, \partial_t) u_j \Big|_{t=0} = 0 \tag{4.31}$$

such that the Complementing Condition fails for some value of ξ. For this value of ξ, which we keep fixed throughout the following reasoning, the discussion at the end of Section 3 assures us that there exist constants c_{kl} such that the functions

$$v_j(x, t) = \frac{1}{2\pi i} \int_\gamma \sum_{k=1}^{N} \sum_{l=0}^{m-1} L^{jk}(i\xi, i\tau) e^{i(x\cdot\xi + t\tau)} c_{kl} i^{s_k} \frac{M^+_{m-l-1}(\xi, \tau)}{M^+(\xi, \tau)} \, d\tau \tag{4.32}$$

are not all identically zero, where γ is a contour enclosing the zeros (in τ) of $M^+(\xi, \tau)$. These functions satisfy the differential equations (4.10a) by (4.1) and the boundary conditions (4.31) by (4.8), (4.20), and (4.25). Not all of these functions reduce for any fixed value of x to a polynomial in t. Hence, in particular, for any positive integer σ and for at least one value of k, $1 \leqq k \leqq N$, there is a constant a_k for which

$$c_k = \partial_t^{\sigma+t_k} v_k(0, a_k) - \partial_t^{\sigma+t_k} v_k(0, 0) \neq 0. \tag{4.33}$$

For $\lambda > 0$, define

$$u_j(x, t; \lambda) = \lambda^{-\sigma-t_j} v_j(\lambda x, \lambda t), \qquad j = 1, \cdots, N. \tag{4.34}$$

From (4.1) and (4.8), it is clear that the u_j satisfy both the differential equations (4.10a) and the boundary conditions (4.31). It also is clear that the derivatives of u_k with respect to x, t of orders up to $\sigma + t_k$ are bounded uniformly as $\lambda \to \infty$. Since

$$\begin{aligned} &\frac{\partial_t^{\sigma+t_k} u(0, a_k/\lambda; \lambda) - \partial_t^{\sigma+t_k} u(0, 0; \lambda)}{(a_k/\lambda)^\alpha} \\ &\qquad\qquad = (\lambda/a_k)^\alpha [\partial_t^{\sigma+t_k} v_k(0, a_k) - \partial_t^{\sigma+t_k} v_k(0, 0)] \\ &\qquad\qquad = (\lambda/a_k)^\alpha c_k, \end{aligned}$$

the Hölder coefficients for $\partial_t^{\sigma+t_k} u_k$, however, tend to infinity with λ. The example of the u_j thus shows that the Complementing Boundary Conditions are necessary, if the desired estimates are to hold, as contended.

Suppose now that the Condition on L is violated; this is possible only for $n = 1$. Define also in this case

$$M^+(\xi, \tau) = \prod_{1}^{r(\xi)} (\tau - \tau_k^+(\xi)),$$

where $\tau_1^+(\xi), \cdots, \tau_{r(\xi)}^+(\xi)$ are the $r(\xi)$ roots with positive imaginary part of $L(\xi, \tau)$. There exists in this case a unit vector $\xi = \pm 1$ such that $r(\xi) \geqq m + 1$. Since, for such ξ, $M^+(\xi, \tau)$ is of degree $\geqq m + 1$, whereas the number of boundary operators is m, it follows that there exists a solution v_j of (4.10a) of the form (4.33) which satisfies any given homogeneous boundary conditions (4.31). The functions $u_j(x, t; \lambda)$, defined from these v_j by (4.34), again violate the estimates in question. Thus, the Condition on L, like the Complementing Boundary Condition, is seen to be necessary as well as sufficient in order that the desired Hölder estimates hold for the boundary problems considered.

5. Estimates for Singular Integrals of Certain Types

In this section, we shall give estimates for singular integrals of the form

$$u(x, t) = \int K(x - y, t) f(y)\, dy, \qquad t > 0, \tag{5.1}$$

with kernels of the specific type

$$K(x, t) = \partial^{n+q+r_h+t_j} K_{jh,q}(x, t). \tag{5.2}$$

(Integration is over the entire n-dimensional x-space.) The estimates are immediate consequences of results obtained in ADN, Section 3, for convolutions (5.1) in which

$$K(x, t) = \frac{\Omega\left(\dfrac{x}{|P|}, \dfrac{t}{|P|}\right)}{|P|^n}, \tag{5.3}$$

where $P = (x, t)$ and $|P| = (|x|^2 + t^2)^{1/2}$. The function $\Omega(x, t)$ is supposed to be continuous on the half-sphere $t \geqq 0$, $|x|^2 + t^2 = 1$, and at least to satisfy a uniform Hölder condition at points of intersection of this half-sphere with the plane $t = 0$. In addition, it is required that

$$\int_{|x|=1} \Omega(x, 0)\, d\omega_x = 0, \tag{5.4}$$

where $d\omega_x$ denotes the element of surface area on the unit sphere $|x| = 1$. For $n = 1$, the latter assumption takes the form $\Omega(x, 0) = -\Omega(-x, 0)$.

It is easily seen that the kernels given by (5.2) possess the above properties. First, Lemma 4.1 shows that these kernels are of the requisite form (5.3). Secondly, their behavior on the unit sphere is adequately controlled by (4.28). It remains only to prove (5.4) for these kernels. Since by construction $\sum_{j'} l_{ij'}(\partial) K_{j'h,q} = 0$, we see by applying $L^{ji}(\partial)$ and summing over i that $L(\partial) K_{jh,q} = 0$. This fact enables us to deduce (5.4) from the corollary of ADN, p. 645.

We shall have to state estimates which are concerned with several norms or semi-norms of $f(x)$ and $u(P)$ already used in ADN. The first are those of Hölder type, namely

$$[f]_\alpha = \operatorname*{l.u.b.}_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

and

$$[u]_\alpha = \operatorname*{l.u.b.}_{P \neq Q} \frac{|u(P) - u(Q)|}{|P - Q|^\alpha}$$

with $0 < \alpha < 1$. The norm of $f(x)$ in L_p, $1 < p < \infty$, we denote by $|f|_{L_p}$, and for fixed $t > 0$, the L_p-norm of $u(x, t)$, regarded as a function of x, by $|u|_{L_p,t}$. For any non-negative integer j and again for $1 < p < \infty$, we also define

$$|u|_{j,L_p} = \left(\sum_{|\beta|=j} \int_{t>0} |\partial^\beta u|^p \, dx \, dt \right)^{1/p}.$$

Finally, if $f(x)$ represents the boundary values of a function $v(x, t)$ defined for $t \geqq 0$, i.e., $f(x) = v(x, 0)$, and if $|v|_{j,L_p}$ is finite, we define

$$|f|_{j-1/p,L_p} = \text{g.l.b.}\ |v|_{j,L_p},$$

the greatest lower bound being taken over all functions v as described.

The indicated character of the kernels (5.2) and Theorems 3.1, 3.2, and 3.3 of ADN enable us to infer fundamental inequalities which govern the corresponding singular integrals. These are stated below in a theorem analogous to Theorem 3.4 of ADN.

THEOREM 5.1. *Let $u(x, t)$ be the convolution* (5.1) *with kernel* (5.2).

(a) *If f is in L_p for some finite $p > 1$ and $[f]_\alpha < \infty$ for some positive $\alpha < 1$, then $[u]_\alpha$ is finite and, in fact,*

$$[u]_\alpha \leqq C(E, q, \alpha)[f]_\alpha.$$

(b) *If f is in L_p for some finite $p > 1$, then so is $u(x, t)$ for every t, and*

$$|u|_{L_p,t} \leqq C(E, q, p)\ |f|_{L_p}.$$

(c) *If f is in L_p for some finite $p > 1$ and $|f|_{1-1/p,L_p}$ is finite, then $|u|_{1,L_p}$ also is finite, and*

$$|u|_{1,L_p} \leqq C(E, q, p)\ |f|_{1-1/p,L_p}.$$

6. A Representation Formula in the Inhomogeneous Boundary Problem

In this section, we shall obtain an integral representation for a C^∞ solution $(u_j(x, t))$, $j = 1, \cdots, N$, with compact support in the half-space $t \geqq 0$, of an inhomogeneous boundary problem

$$(6.1)\qquad \begin{aligned} &\sum_{j=1}^N l_{ij}(\partial) u_j(x, t) = f_i(x, t), &&i = 1, \cdots, N, \quad t > 0, \\ &\sum_{j=1}^N B_{hj}(\partial) u_j(x, t)\Big|_{t=0} = \phi_h(x), &&h = 1, \cdots, m. \end{aligned}$$

The construction is based on the fundamental solution $\Gamma(P-\bar{P})$, $P=(x,t)$, $\bar{P}=(\bar{x},\bar{t})$, of the elliptic equation $L(\partial)u=0$ with singularity at $P=\bar{P}$. We note the estimates

$$(6.2)\qquad |\partial^a\Gamma(P)|\leqq \text{const.}\,|P|^{2m-n-1-a}\,(1+|\log|P||)\quad\text{for}\quad a\geqq 0,$$

in which the logarithmic term is omissible, if $a>2m-n-1$. (The constant depends on s, m, n, and the ellipticity constant for L.)

The first step in finding the desired representation of u_j is to construct explicitly a vector $(v_j(x,t))$ satisfying the stipulated differential equations, if not the boundary conditions. For this purpose we assume the f_i extended in the manner of ADN, p. 652, to all $(n+1)$-space as functions with compact support of class C^{R-s_i} for R sufficiently large. Let $f_{R,i}(P)$, $i=1,\cdots,N$, denote the extended functions. We then define

$$(6.3)\qquad v_j(P)=\int\Gamma(P-\bar{P})\sum_{i=1}^{N}L^{ji}(\bar{\partial})f_{R,i}(\bar{P})\,d\bar{P},$$

$\bar{\partial}$ acting on $\bar{P}$, and the integration being over all $(n+1)$-space, see [10, p. 519]. We easily see that $v_j\in C^{R+t_j-2}$ and also that the v_j satisfy the stipulated differential equations

$$\sum_{j=1}^{N}l_{ij}(\partial)v_j(P)=f_i(P)\quad\text{for}\quad t>0.$$

Remark. If $0<\alpha<1$, $b\geqq 2m$, and $f_i\in C^{b-s_i+\alpha}$, then of course

$$L^{ji}(\partial)f_i\in C^{b-2m+t_j+\alpha},$$

and it is easily seen with the aid of a classical result (see Theorem 3.1, ADN) that $v_j\in C^{b+t_j+\alpha}$ and

$$[v_j]_{b+t_j+\alpha}\leqq\text{const}\cdot\sum_i[f_{R,i}]_{b-s_i+\alpha}\leqq\text{const}\cdot\sum_i[f_i]_{b-s_i+\alpha}.$$

Since the $f_{R,i}$ have compact support, for sufficiently large $|P|$ the integral defining v_j in (6.3) will not be singular, and integrations by parts, as well as differentiations with respect to P, will be possible on a scale only restricted by R. For sufficiently large $|P|$, we thus have, in particular,

$$\begin{aligned}\partial^{a+t_j}v_j&=\partial^{a+t_j}\int\Gamma(P-\bar{P})\bar{\partial}^{2m-s_i-t_j}f_{R,i}(\bar{P})\,d\bar{P}\\&=(-1)^{2m-t_j}\partial^{a+t_j}\int\bar{\partial}^{2m-t_j}\Gamma(P-\bar{P})\bar{\partial}^{-s_i}f_{R,i}(\bar{P})\,d\bar{P}\\&=\int\partial^{2m+a}\Gamma(P-\bar{P})\bar{\partial}^{-s_i}f_{R,i}(\bar{P})\,d\bar{P}\end{aligned}$$

for $a\geqq 0$; hence, by (6.2),

$$(6.4)\qquad |\partial^{a+t_j}v_j|\leqq\text{const.}\,|P|^{-n-1-a}\quad\text{for}\quad a\geqq 0.$$

Therefore,

$$|\partial^{a-r_h} B_{hj}(\partial) v_j(P)| \leqq \text{const. } |P|^{-n-1-a}$$

for $a \geqq 0$ and sufficiently large $|P|$, and for the boundary quantities

$$g_h(x) = \sum_{j=1}^{N} B_{hj}(\partial) v_j(x, t)\Big|_{t=0}, \tag{6.5}$$

we have for large $|x|$

$$|\partial^{a-r_h} g_h(x)| \leqq \text{const. } |x|^{-n-1-a} \quad \text{for} \quad a \geqq 0. \tag{6.6}$$

The above estimates will be of aid in justifying representations we shall shortly describe, not of the u_j, indeed, but of derivatives of the u_j of suitable orders. Again let

$$l_0 = \max(0, r_h).$$

THEOREM 6.1. *If the vector $(u_j(x, t))$ is a C^∞ solution of* (6.1) *with compact support in the half-space $t \geqq 0$, then the representation formulas*

$$\partial^{l_0+t_j} u_j(x, t) = \partial^{l_0+t_j} v_j(x, t) + \int \sum_{h=1}^{m} \partial^{l_0+t_j} K_{jh}(x-y, t)(\phi_h(y) - g_h(y))\, dy \tag{6.7}$$

are valid for $t > 0$. Here, v_j and g_h are defined by (6.3) *and* (6.5) *with $f_{R,i}$ a sufficiently smooth extension of f_i to the whole $(n + 1)$-dimensional x, t-space.*

The representation formula is obtained from the lemma and the uniqueness theorem (Theorem 6.2) below by the argument used for Theorem 4.1 of ADN.

LEMMA. *Let $h_p = f_p - g_p$, and let l be any integer $\geqq l_0$. Then the integrals*

$$I_{jp,l}(x, t) = \int \partial^{l+t_j} K_{jp}(x-y, t) h_p(y)\, dy, \qquad t > 0,$$

converge absolutely. Furthermore,

$$|I_{jp,l}(x, t)|_{L_2,t} \leqq \textit{constant independent of } t, \tag{6.8}$$

and

$$|I_{jp,l}(x, t)|_{1,L_2} < \infty. \tag{6.9}$$

The first derivatives $\partial I_{jp,l}$ are absolutely integrable on every plane $t = \text{const.} > 0$, their integrals converging uniformly in any interval $0 < \varepsilon \leqq t \leqq T$.

Proof: The integrals converge absolutely by (4.28)′, (6.6), and the fact that f_p has compact support. By (4.26),

$$I_{jp,l}(x, t) = \int \partial^{l+t_j} \Delta_x^{(n+q)/2} K_{jp,q}(x-y, t) h_p(y)\, dy.$$

If $l - r_p$ is even, choose q such that $n + q - l + r_p > 0$ (q also must be even).

Integrating by parts, we are then able to transform the right member above into

$$\int \partial^{l+t_j} \Delta_x^{(n+q-l+r_p)/2} K_{jp,q}(x-y,t) \cdot \Delta_y^{(l-r_p)/2} h_p(y)\, dy,$$

there being no contributions from infinity in view of (4.28) and (6.6). Since, in accordance with (6.6), $\partial^{l-r_p} h_p$ belongs to L_2, estimate (6.8) now follows from Theorem 5.1(b) in the case in which $l - r_p$ is even. Estimate (6.9) we similarly obtain in that case from Theorem 5.1(c), once we have verified that $|\partial^{l-r_p} h_p|_{1-1/2,L_2} < \infty$, or, in other words, that $\partial^{l-r_p} h_p(x)$ is the restriction to the plane $t = 0$ of a function $V_p(x, t)$ with square-integrable first derivatives in the half-space $t > 0$. Such a function is given, however, by the formula

$$V_p(x,t) = \zeta(t)\partial^{l-r_p} h_p(x),$$

where $\zeta(t)$ is in C^∞ for $t \geqq 0$, $\zeta(0) = 1$, and $\zeta(t) = 0$ for $t \geqq 1$.

The final statement of the lemma is easily proved, in the case in which $l - r_p$ is even, from the previous formula and the estimates (4.28) and (6.6) already used.

The case in which $l - r_p$ is odd, say $l - r_p = 2k + 1$, is treated by the same arguments as above applied to the formula

$$I_{jp,l}(x,t) = \sum_{i=1}^{n} \int \partial^{l+t_j} \partial_{x_i} \Delta_x^{(n+q)/2-k-1} K_{jp,q}(x-y,t)\partial_{y_i} \Delta_y^k h_p(y)\, dy.$$

This completes the proof of the lemma.

The uniqueness theorem referred to above remains to be justified. Its proof will depend on Fourier transformation and for this reason will be formulated in terms of $D_x = (1/i)\partial_x$, $D_t = (1/i)\partial_t$, and $v_j = i^{t_j} u_j$ rather than of ∂_x, ∂_t, and u_j. In these terms, the differential equations take the form

$$\sum_j l_{ij}(D_x, D_t) v_j = 0 \quad \text{for} \quad t > 0, \qquad i = 1, \cdots, N, \tag{6.14}$$

and the boundary conditions

$$\sum_j B_{hj}(D_x, D_t) v_j = 0 \quad \text{for} \quad t = 0, \qquad h = 1, \cdots, m. \tag{6.15}$$

THEOREM 6.2. *Let v_j, $j = 1, \cdots, N$, be functions of class C^{l_j} for $t \geqq 0$, where $l_j = t_j + \max_{i,h}(s_i, r_h)$, and satisfy the homogeneous differential equations* (6.14) *and the homogeneous boundary conditions* (6.15). *Assume that v_j and its derivatives up to order l_j are absolutely integrable on each plane $t =$ const. > 0, the integrals converging uniformly with respect to t in every finite interval $0 < \varepsilon \leqq t \leqq R$. Assume also that v_j and its derivatives of orders up to l_j possess uniformly bounded L_2 norms on the planes $t =$ const. $\geqq 0$. Finally, suppose that*

$$\int_{t>0} \sum_j |v_j(x,t)|^2\, dx\, dt < \infty. \tag{6.16}$$

Then $v_j \equiv 0$.

Proof: For $t > 0$, the Fourier transforms

$$\hat{v}_j(\xi, t) = (2\pi)^{-n/2} \int e^{-ix\cdot\xi} v_j(x, t)\, dx$$

are continuous in (ξ, t) and possess continuous derivatives with respect to t of orders up to l_j. From (6.14) we have

$$\sum_j l_{ij}(\xi, D_t)\hat{v}_j(\xi, t) = 0. \tag{6.17}$$

Intending to apply Theorem 3.2, we next prove that the $\hat{v}_k$ exponentially decay as $t \to \infty$. First, the $\hat{v}_k$ all satisfy the ordinary differential equation

$$L(\xi, D_t)\hat{v}_k(\xi, t) = 0$$

obtained by applying $L^{ki}(\xi, D_t)$ on the left of the members of (6.17) and summing over i. Any solution of the latter equation is a linear combination, with coefficients which are polynomials in t, of exponentials $e^{\mu t}$, where μ is a root of the algebraic equation $L(\xi, \mu/i) = 0$. Secondly, by condition (6.16), $\int |\hat{v}_k(\xi, t)|^2\, dt < \infty$ for almost all ξ, and for these ξ evidently $\mathscr{R}e\ \mu < 0$. Since, by ellipticity, $\mathscr{R}e\ \mu \neq 0$ for any ξ, we conclude by continuity that $\mathscr{R}e\ \mu < 0$ for every ξ, or, in other words, that the $\hat{v}_k(\xi, t)$ decay exponentially, as asserted.

Now we shall show that

$$\sum_j B_{hj}(\xi, D_t)\hat{v}_j(\xi, t) = 0 \quad \text{for} \quad t = 0. \tag{6.18}$$

Theorem 3.2 will then prove, in view of (6.17) and the exponential decay of the $\hat{v}_j$, that $\hat{v}_j(\xi, t) = 0$ and thus that $v_j(x, t) = 0$, as contended. To prove (6.18), it suffices to show that

$$\lim_{t\downarrow 0} \int \sum_j \phi(\xi) B_{hj}(\xi, D_t)\hat{v}_j(\xi, t)\, d\xi = 0$$

for every ϕ in C^∞ of compact support. The integral on the left, however, by Parseval's formula, is equal to

$$\int \sum_j \bar{\hat{\phi}}(x) B_{hj}(D_x, D_t) u_j(x, t)\, dx,$$

and by (6.15) the limit of this integral is zero, as $t \downarrow 0$, because of the assumed uniform boundedness of the L_2 norms involved. This completes the proof of the theorem.

CHAPTER III

THE SCHAUDER ESTIMATES

7. Notation

Let $\mathscr{D}$ be a k-dimensional domain (not necessarily bounded), and denote by $\dot{\mathscr{D}}$ and $\overline{\mathscr{D}}$ its boundary and closure, respectively. $C^l(\mathscr{D})$ will denote the space of functions possessing continuous derivatives up to order $l \geqq 0$ in $\mathscr{D}$; the meaning of $C^l(\overline{\mathscr{D}})$ will be similar. For f in $C^l(\mathscr{D})$, we define the pseudonorms

$$[f]_l = [f]_l^{\mathscr{D}} = \text{l.u.b. } |\partial^l f|,$$

where the least upper bound is taken over all derivatives of order l, as well as over the domain $\mathscr{D}$, and we define the norm

$$|f|_l = |f|_l^{\mathscr{D}} = \sum_{j=0}^{l} [f]_j.$$

The subclass of those functions in $C^l(\overline{\mathscr{D}})$ whose derivatives of order l satisfy a uniform Hölder condition of order α, $0 < \alpha < 1$, is denoted by $C^{l+\alpha}(\mathscr{D})$. For $f \in C^{l+\alpha}(\mathscr{D})$, we define the pseudonorm

$$[f]_{l+\alpha} = \text{l.u.b. } \frac{|\partial^l f(P) - \partial^l f(Q)|}{|P - Q|^\alpha},$$

where the l.u.b. is over $P \neq Q$ in $\overline{\mathscr{D}}$ and all derivatives of order l, and the norm

$$|f|_{l+\alpha} = |f|_l + [f]_{l+\alpha}.$$

Let $\Sigma = \Sigma_R$: $|x|^2 + t^2 < R^2$, $t \geqq 0$, be a half-sphere in x,t-space with $x = (x_1, \cdots, x_n)$. Let $\sigma = \sigma_R$ denote its planar boundary $|x|^2 < R^2$, $t = 0$. For the linear spaces $C^l(\Sigma)$, $C^l(\sigma)$, and $C^{l+\alpha}(\Sigma)$, $C^{l+\alpha}(\sigma)$, $0 < \alpha < 1$, we introduce seminorms such as

$$[f]_{a,l} = [f]_{a,l}^{\Sigma} = \text{l.u.b. } d_P^{a+l} |\partial^l f(P)| \leqq R^{a+l} |f|_l^{\Sigma}$$

for any integer a with $a + l \geqq 0$. If $a + l < 0$, we define $[f]_{a,l} = 0$. Here, the least upper bound is taken over all derivatives of order l and all points P in Σ, d_P denotes the distance from P to the spherical part of the boundary of Σ, $|x|^2 + t^2 = R^2$. Correspondingly, we also define

$$|f|_{a,l} = \sum_{j=0}^{l} [f]_{a,j},$$

$$[f]_{a,l+\alpha} = \underset{4|P-Q|<d_P,d_Q}{\text{l.u.b.}} \; d_P^{a+l+\alpha} \frac{|\partial^l f(P) - \partial^l f(Q)|}{|P - Q|^\alpha},$$

$$|f|_{a,l+\alpha} = |f|_{a,l} + [f]_{a,l+\alpha}.$$

Norms $|\phi|_{a,b}$ for functions in $C^a(\sigma)$ are defined in the same way.

It should be pointed out that Lemma 5.1 is stated incorrectly in ADN. The condition stated in the lemma in the words, "There is a $\delta > 0$ such that every point $P \in D$ is the extremity of a segment of length δ lying entirely in D", should be replaced by the cone condition, viz., "There is a finite right spherical half-cone such that every point $P \in D$ is the vertex of a congruent half-cone that lies wholly in D".

8. The Schauder Estimates for Systems of Equations with Constant Coefficients

In this section, we shall still consider boundary problems

$$\begin{aligned} &\sum_{j=1}^{N} l_{ij} u_j = f_i(x, t), && i = 1, \cdots, N, \quad t > 0, \\ &\sum_{j=1}^{N} B_{hj} u_j \Big|_{t=0} = \varphi_h(x), && h = 1, \cdots, m, \end{aligned} \tag{8.1}$$

for equations with constant coefficients in a half-space of the type introduced in Chapter II.

As before, let $l_0 = \max(0, r_h)$; let l be an integer $\geqq l_0$. Throughout this section, we shall assume that $u_j(x, t) \in C^{l_0+t_j+\alpha}$ and $f_i(x, t) \in C^{l-s_i+\alpha}$ for $t \geqq 0$ and that $\varphi_h(x) \in C^{l-r_h+\alpha}$ for all x, α being a positive number less than one.

The basic estimate is for functions of compact support. From it, the general inequalities will be deduced.

THEOREM 8.1. *In addition to the preceding hypotheses, assume each u_j has compact support. Then $u_j \in C^{l+t_j+\alpha}$, and*

$$[u_j]_{l+t_j+\alpha} \leqq \text{const.} \left\{ \sum_{i=1}^{N} [f_i]_{l-s_i+\alpha} + \sum_{h=1}^{m} [\varphi_h]_{l-r_h+\alpha} \right\}, \tag{8.2}$$

where the constant depends only on l, α, and E. (E was defined before Lemma 4.1.)

Proof: We shall consider only the case in which $u_i \in C^\infty$, the whole theorem being deducible from that case by a procedure of approximation. Formula (6.7), after $(l - l_0)$-fold differentiation, gives us

$$\partial^{l+t_j} u_j(x, t) = \partial^{l+t_j} v_j(x, t) + \sum_{p=1}^{m} I_{jp,l}(x, t), \tag{8.3}$$

the integrals $I_{jp,l}(x, t)$ being those defined in the Lemma of Section 6 and v_j the function (6.3). By the Remark made in Section 6,

$$[v_j]_{l+t_j+\alpha} \leqq \text{const.} \sum_{i=1}^{N} [f_i]_{l-s_i+\alpha}, \tag{8.4$'$}$$

the constant depending only on l, α, and E. From this and (6.5),

$$[g_p]_{l-r_p+\alpha} \leqq \text{const.} \sum_{j=1}^{N} [v_j]_{l+t_j+\alpha} \leqq \text{const.} \sum_{i=1}^{N} [f_i]_{l-s_i+\alpha}. \tag{8.4$''$}$$

From formulas given (in the two cases distinguished) in the proof of the Lemma in Section 6,

$$I_{jp,l} = \int \partial^{n+q+r_p+t_j} K_{jp,q}(x-y, t) \cdot \partial^{l-r_p}(\varphi_p(y) - g_p(y))\, dy.$$

Applying Theorem 5.1(a), we thus deduce

$$\begin{aligned}[I_{jp,l}]_\alpha &\leqq C(E, q, \alpha)[\varphi_p - g_p]_{l-r_p+\alpha} \\ &\leqq \text{const.} \left([\varphi_p]_{l-r_p+\alpha} + \sum_{i=1}^{N} [f_i]_{l-s_i+\alpha}\right),\end{aligned}$$

the latter inequality following because of (8.4)″. This, (8.3), and (8.4)′ suffice to prove (8.2).

Corollary. *The representation* (6.7) *of Theorem* 6.1 *is valid when* $u_j \in C^{l_0+t_j+\alpha}$, *provided the* u_j *have compact support in the half-space* $t \geqq 0$.

The above estimates for solutions with compact support in a half-space now are applied to the study of solutions defined just in a hemisphere. After this, we shall see how to modify estimate (8.2) to apply to solutions in a half-space that are not of compact support, but are regulated less severely at infinity.

Theorem 8.2. *Let* $u_j \in C^{l_0+t_j+\alpha}$, $0 < \alpha < 1$, *be a solution of* (8.1) *in the half-sphere* $\Sigma = \Sigma_R$. *Assume that for an integer* l *with* $l \geqq l_0$

$$K = \sum_{i=1}^{N} [f_i]_{l'+s_i, l-s_i+\alpha} + \sum_{h=1}^{m} [\varphi_h]_{l'+r_h, l-r_h+\alpha} < \infty.$$

Then $u_j \in C^{l+t_j+\alpha}$, *and*

$$[u_j]_{l'-t_j, l+t_j+\alpha} \leqq \text{const.} \left(K + \sum_{k=1}^{N} |u_k|_{l'-t_k, 0}\right), \tag{8.5}$$

where the constant depends only on l, α, *and* E.

Proof: A condition for deriving (8.5) is the prior knowledge that $u_j \in C^{l+t_j+\alpha}$, $j = 1, \cdots, N$. By assumption, $u_j \in C^{l_0+t_j+\alpha}$; hence, (8.5) is deducible, in particular, for $l = l_0$. We shall now show how, accepting inequality (8.5) for $l = l_0$, we can prove $u_j \in C^{l_0+1+t_j+\alpha}$. Inequality (8.5) can then be demonstrated for $l = l_0 + 1$ and the whole procedure repeated as many times as required.

Inequality (8.5) with $l = l_0$ is applicable to x-difference quotients for the u_j, as in ADN, p. 661, paragraph (b), with the result that $\partial_x u_j \in C^{l_0+t_j+\alpha}$. Knowing this, we shall seek to express the pure t-derivatives $v_j = \partial_t^{l_0+t_j} u_j$ in terms of derivatives of the u_k of the form $\partial^{l_0+t_k-1}\partial_x u_k$, to this end applying $\partial_t^{l_0-s_i}$ to the i-th differential equation in (8.1). Thereby we have

$$\sum_j \lambda_{ij}(\partial_x, \partial_t) u_j = \partial_t^{l_0-s_i} f_i,$$

where

$$\lambda_{ij}(\xi, \tau) = \tau^{l_0-s_i} l_{ij}(\xi, \tau).$$

(The degree of λ_{ij} is $l_0 + t_j$.) From the first sentence in this paragraph,

$$\sum_j \lambda_{ij}(0, \partial_t)u_j = \partial_t^{l_0-s_i}f_i + \sum_j(\lambda_{ij}(0, \partial_t) - \lambda_{ij}(\partial_x, \partial_t))u_j \in C^{1+\alpha},$$

the derivatives of each u_j entering the right member above being of order $l_0 + t_j$ and containing ∂_x. Moreover,

$$\lambda_{ij}(0, \tau) = c_{ij}\tau^{l_0+t_j},$$

and, incidentally,

$$l_{ij}(0, \tau) = c_{ij}\tau^{s_i+t_j},$$

where c_{ij} is a constant. Hence,

$$\sum_j c_{ij}v_j \in C^{1+\alpha},$$

while det $(c_{ij}) \neq 0$ because, for $\tau \neq 0$,

$$0 \neq \det(l_{ij}(0, \tau)) = \det(c_{ij}\tau^{s_i+t_j}) = \tau^{2m}\det(c_{ij}).$$

Hence, $v_j \in C^{1+\alpha}$, and $u_j \in C^{l_0+1+t_j+\alpha}$, as contended.

From the foregoing remarks, in justifying inequality (8.5) we can validly assume $u_j \in C^{l+t_j+\alpha}$, $j = 1, \cdots, N$, and proceed as in the proof of Theorem 6.2 in ADN, pp. 660–662, but omit the rather intricate details.

Now that estimates have been obtained for solutions of (8.1) defined just in a hemisphere, we are in a position to generalize Theorem 8.1, as noted above, to a class of solutions without compact support.

THEOREM 8.3. *Let $u_j(x, t) \in C^{l_0+t_j+\alpha}$, $0 < \alpha < 1$, satisfy* (8.1) *in the half-space $t \geqq 0$. Assume that for a fixed integer $l \geqq l_0$ both the quantities*

$$K_0 = \sum_{i=1}^{N} [f_i]_{l-s_i+\alpha} + \sum_{h=1}^{m} [\varphi_h]_{l-r_h+\alpha}$$

and

$$M_0 = \varlimsup_{R\to\infty} \sum_{j=1}^{N} R^{-(l+t_j+\alpha)} \max_{\Sigma_R} |u_j|$$

are finite. Then $u_j \in C^{l+t_j+\alpha}$, and

$$[u_j]_{l+t_j+\alpha} \leqq \text{const.}\ (K_0 + M_0).$$

The proof of this theorem is like that of Theorem 6.3 in ADN.

As an immediate corollary, we have the following (see also Miranda [18]):

THEOREM OF LIOUVILLE TYPE. *Let $u_j(x, t) \in C^{l_0+t_j+\alpha}$, $0 < \alpha < 1$, satisfy* (8.1) *in the half-space $t \geqq 0$. Assume that the functions f_i and φ_h are polynomials of degrees not exceeding $k - s_i$ and $k - r_h$, respectively, $i = 1, \cdots, N$, $h = 1, \cdots, m$, where $k \geqq l_0$, and assume also that*

$$\varlimsup_{R\to\infty} \sum_{j=1}^{N} R^{-(k+t_j+\alpha)} \max_{\Sigma_R} |u_j| = 0.$$

Then u_j is a polynomial of degree at most $k + t_j$.

9. Equations with Variable Coefficients

The results just obtained for constant coefficient problems in a half-space or a hemisphere will be applied here to the study of solutions of a system of differential equations (1.1) with variable coefficients in a general domain $\mathscr{D}$, the solutions (u_j) satisfying Complementing Boundary Conditions, also with variable coefficients, on a portion Γ of $\dot{\mathscr{D}}$. Our goal, achieved in Theorem 9.3, is to estimate norms of the type $|u_j|_{l+t_j+\alpha}$ over arbitrary subdomains $\mathscr{A}$ of $\mathscr{D}$ abutting Γ. An important step towards this goal is to be able, as in Theorem 9.2, to bound norms of the type $|u_j|_{l'-t_j,l+t_j+\alpha}$ on a hemisphere, the flat face for the hemisphere being Γ. This step, in its turn, depends on a prior result, Theorem 9.1, which concerns the solutions of problems with variable coefficients in a half-space with Γ the entire boundary plane.

In all these results, whatsoever $\mathscr{D}$ and Γ, the hypotheses of Sections 1 and 2 always are presupposed. Suitable assumptions also are made concerning the behavior of F_i, ϕ_h, and the coefficients in

$$l_{ij}(P, \partial) = \sum_{|\rho|=0}^{s_i+t_j} a_{ij,\rho}(P)\partial^\rho$$

and

$$B_{hj}(P, \partial) = \sum_{|\sigma|=0}^{r_p+t_j} b_{pj,\sigma}(P)\partial^\sigma,$$

where ρ and σ denote multi-indices indicative of the precise differentiations involved. For a fixed integer $l \geqq l_0 = \max(0, r_h)$ and a fixed positive number α less than one, F_i and the coefficients $a_{ij,\rho}$, $|\rho| = 0, \cdots, s_i + t_j$, $j = 1, \cdots, N$, are assumed in particular to belong to $C^{l-s_i+\alpha}$ in $\bar{\mathscr{D}}$, and ϕ_h and the coefficients $b_{hj,\sigma}$, $|\sigma| = 0, \cdots, r_h + t_j$, $j = 1, \cdots, N$, to $C^{l-r_h+\alpha}$ on Γ. All these quantities furthermore are assumed to have finite norms of the pertinent type; let k denote an upper bound, in particular, for these norms of the coefficients. The resulting estimates of the u_j will involve constants, denoted C_1, C_2, or C_3, which depend on the latter coefficient norm bound k, the uniform ellipticity constant A (Section 1), the minor constant Δ (Section 2), the number of dimensions n, the number of dependent variables N, the Hölder exponent α, and the selected integer l. (The third constant C_3, which appears in Theorem 9.3 regarding a general domain $\mathscr{D}$, also will depend on the shape of the domain.) The constants do not depend, however, on the solution $(u_1, \cdots, u_N)$.

THEOREM 9.1. *Let $u_1, \cdots, u_N$ satisfy (1.1) in the half-space D^+: $t > 0$ and (2.1) on its boundary $t = 0$, u_j belonging to $C^{l+t_j+\alpha}$ in D^+ with $|u_j|_{l+t_j+\alpha} < \infty$, $j = 1, \cdots, N$. Assume the semi-norms*

$$[F_i]_{l-s_i+\alpha}, \qquad [\phi_h]_{l-r_h+\alpha}$$

and the norms

$$|a_{ij,\rho}|_{l-s_i+\alpha}, \qquad |b_{hj,\sigma}|_{l-r_h+\alpha}$$

also finite for all possible values of their indices. Then

$$|u_j|_{l+t_j+\alpha} \leqq C_1\left(\sum_{i=1}^{N}[F_i]_{l-s_i+\alpha} + \sum_{h=1}^{m}[\phi_h]_{l-r_h+\alpha} + \sum_{k=1}^{N}|u_k|_0\right), \qquad j = 1, \cdots, N.$$

THEOREM 9.2. *Let $u_1, \cdots, u_N$ satisfy* (1.1) *in the interior of the hemisphere $\Sigma = \Sigma_R$, $R \leqq 1$, and* (2.1) *on the planar portion of the boundary, u_j belonging to $C^{l_0+t_j+\alpha}$ in Σ, $j = 1, \cdots, N$. Assume that, for all values of their indices, the norms*

$$|F_i|_{l'+s_i, l-s_i+\alpha}, \qquad |\phi_h|_{l'+r_h, l-r_h+\alpha},$$

and

$$|a_{ij,\rho}|_{s_i+t_j-|\rho|, l-s_i+\alpha}, \qquad |b_{hj,\sigma}|_{r_h+t_j-|\sigma|, l-r_h+\alpha}$$

are finite. Then $u_j \in C^{l+t_j+\alpha}$ in Σ, and

$$|u_j|_{l'-t_j, l+t_j+\alpha} \leqq C_2\left(\sum_{i=1}^{N}|F_i|_{l'+s_i, l-s_i+\alpha} + \sum_{h=1}^{m}|\phi_h|_{l'+r_h, l-r_h+\alpha} + \sum_{k=1}^{N}|u_k|_{l'-t_k, 0}\right),$$
$$j = 1, \cdots, N.$$

The proofs of these theorems, the first relying on Theorem 8.2 and the interior estimates of [10], are so close to those of Theorems 7.1 and 7.2 in ADN, pp. 664–666, as not to require extended discussion. In demonstrating Theorem 9.1, it is well, noting the index J for which $[u_j]_{l+t_j+\alpha}$ is greatest, to consider, first, the possibility $[u_J]_{l+t_J+\alpha} \leqq \frac{1}{2}|u_J|_{l+t_J+\alpha}$, which is trivial, and, secondly, its opposite. In connection with the second possibility, a quantity analogous to $\bar{A}$ in ADN, equation (7.6), p. 665, is useful, defined here as

$$\hat{A} = \frac{|\partial^l u_J(P_0) - \partial^l u_J(Q)|}{|P_0 - Q|^\alpha},$$

where $P_0 = (0, t_0)$ and $Q = (y, \tau)$, $0 < t_0 \leqq \tau$, are such points, and ∂^l such a differentiation, that

$$\hat{A} > \tfrac{1}{2}|u_J|_{l+t_J+\alpha}.$$

A similar remark applies to the proof of Theorem 9.2.

The third theorem is concerned with a general $(n + 1)$-dimensional domain $\mathscr{D}$, which may be infinite, a portion Γ of its n-dimensional boundary $\dot{\mathscr{D}}$ (Γ may be the entire boundary), and a possibly infinite subdomain $\mathscr{A}$ of $\mathscr{D}$ such that, in the n-dimensional sense, $\dot{\mathscr{A}} \cap \dot{\mathscr{D}}$ is in the interior of Γ. With α and l fixed as described at the beginning of this section, it is assumed that there is a positive number d such that each point P in $\mathscr{A}$ within a distance d of $\dot{\mathscr{D}}$ has a neighborhood U_P with the following properties: (a) $\bar{U}_P \cap \dot{\mathscr{D}} \subset \Gamma$; (b) U_P contains the sphere about P of radius $d/2$; (c) the set $\bar{U}_P \cap \bar{\mathscr{D}}$ is the homeomorphic image of the closure of an $(n + 1)$-dimensional hemisphere $\Sigma_{R(P)}$, $R(P) \leqq 1$—the boundary subset $\bar{U}_P \cap \dot{\mathscr{D}}$ corresponding to the flat face F_P of the hemisphere—under a mapping T_P which, together with its inverse, is of class $C^{l+\lambda+\alpha}$, where $\lambda = \max(-s_i, -r_h, t_j)$. Over their respective domains, both T_P and its inverse are assumed to have finite $|\quad|_{l+\lambda+\alpha}$ norms bounded by a constant K independent of P.

Let Δ_P denote the minor constant for $\bar{U}_P \cap \dot{\mathscr{D}}$ (Section 2) that pertains to the transformation T_P, and define Δ here as

$$\Delta = \Delta_\Gamma = \inf_P \Delta_P;$$

it is assumed that

$$\Delta > 0.$$

It is important to see how the coefficients in (1.1) and (2.1) and the uniform ellipticity constant behave after the mapping T_P: $\hat{x} = \hat{x}(x)$. An arbitrary linear combination of differentiation operators, say $c(\partial) = c_{i_1,i_2,\cdots} \partial_1^{i_1}\partial_2^{i_2} \cdots$ (summation signs will be omitted in this argument), is transformed into

$$\hat{c}(\hat{\partial}) = \hat{c}_{j_1,j_2,\cdots} \hat{\partial}_1^{j_1}\hat{\partial}_2^{j_2} \cdots + \text{lower order differentiations,}$$

where $\hat{\partial}_j = \partial/\partial\hat{x}_j = \dfrac{\partial x_i}{\partial \hat{x}_j}\partial_i$, and each $\hat{c}_{j_1,j_2,\cdots}$ is a linear combination of the $c_{i_1,i_2,\cdots}$ with coefficients that are products of the $\partial\hat{x}_j/\partial x_i$. Correspondingly, $c(\Xi) = \hat{c}(\hat{\Xi})$, where $\hat{\Xi} = (\partial x_i/\partial\hat{x}_j)\,\Xi_i$. Hence, the coefficients in the transformed differential equations and transformed boundary conditions are of the respective continuity classes required above, and, moreover, the norms

$$(9.1\text{a}) \qquad |\hat{f}_i|_{l-s_i+\alpha}^{\Sigma_{R(P)}}, \qquad |\hat{\phi}_h|_{l-r_h+\alpha}^{F_P},$$

and

$$(9.1\text{b}) \qquad |\hat{a}_{ij,\rho}|_{l-s_i+\alpha}^{\Sigma_{R(P)}}, \qquad |\hat{b}_{hj,\sigma}|_{l-r_h+\alpha}^{F_P},$$

for $|\rho| = 0, \cdots, s_i + t_j$, $|\sigma| = 0, \cdots, r + t_j$, $i, j = 1, \cdots, N$, $h = 1, \cdots, m$, are bounded uniformly with respect to P, the accent ^ always referring to the pertinent transformed quantity. In conformity with previous notation, uniform bounds for the norms of types (9.1b) will be denoted by k and for the norms (9.1a) by $|f_i|_{l-s_i+\alpha}^{\mathscr{D}}$ and $|\phi_h|_{l-r+\alpha}^{\Gamma}$, respectively. (These bounds of course, will depend on κ.) Secondly, the characteristic determinant for the system (1.1), which we may denote by $L(Q, \Xi)$ at an arbitrary point Q in the domain of T_P, is invariant under T_P in the sense that $\hat{L}(\hat{Q}, \hat{\Xi}) = L(Q, \Xi)$. On the other hand, since both T_P and its inverse possess first derivatives which are subject to a uniform bound independent of P, there is a constant μ such that $\mu^{-1}|\hat{\Xi}| \leqq |\Xi| \leqq \mu\,|\hat{\Xi}|$, and the constant of uniform ellipticity for the transformed system of equations is seen to be $A\mu^{2m}$.

The above remarks enable us to reduce the present problem for a general domain to that for a hemisphere treated in Theorem 9.2 above. Using, in fact, the same methods as in ADN, p. 668, we thus obtain

THEOREM 9.3. *Under the assumptions above, let* $u_1, \cdots, u_N$ *satisfy* (1.1) *in* $\mathscr{D}$ *and* (2.1) *on* Γ. *If* $u_j \in C^{l_0+t_j+\alpha}$ *in* $\mathscr{D} \cup \Gamma$, *we can conclude that* $u_j \in C^{l+t_j+\alpha}$ *in* $\bar{\mathscr{A}}$, *and*

$$|u_j|^{\mathscr{A}}_{l+t_j+\alpha} \leqq C_3\left(\sum_{i=1}^{N} |F_i|^{\mathscr{D}}_{l-s_i+\alpha} + \sum_{h=1}^{m} |\phi_h|^{\Gamma}_{l-r_h+\alpha} + \sum_{k=1}^{N} |u_k|^{\mathscr{D}}_0\right), \qquad j = 1, \cdots, N,$$

the constant C_3 *depending on* k, A, Δ, n, N, α, l, κ, d.

Remark 1. In case $\mathscr{D}$ is bounded and $\Gamma = \dot{\mathscr{D}}$, we may take $\mathscr{A} = \mathscr{D}$ and replace each term $|u_k|^{\mathscr{D}}_0$ in the above estimate by

$$\int_{\mathscr{D}} |u_k| \, dV.$$

Remark 2. In case $\mathscr{D}$ is bounded and $\Gamma = \dot{\mathscr{D}}$, and in case the solution $(u_1, \cdots, u_N)$ is unique when $u_j \in C^{l_0+t_j+\alpha}$, we may take $\mathscr{A} = \mathscr{D}$ and also omit $\Sigma\, |u_k|^{\mathscr{D}}_0$ on the right of the above estimate. However, a new constant not explicitly known then replaces C_3.

These remarks follow just as in ADN, pp. 668–669.

CHAPTER IV

ESTIMATES OF INTEGRAL NORMS

10. L_p Estimates

Global, local interior, and local boundary estimates in L_p will be given under appropriate hypotheses, the results for general problems (Subsection 10.2) ensuing from those (Subsection 10.1) for equations with constant coefficients and solutions with compact support. In addition (Subsection 10.3), L_p theory will be applied to generalize previously stated estimates concerned with Hölder continuity. We follow ADN except in deriving local boundary estimates.

As in ADN, Section 14, we use the following notation:

$$|[f]|_j = |[f]|_{j,L_p} = \left(\sum_{|\beta|=j} \int |\partial^\beta f|^p \, dx\right)^{1/p},$$

$$\|f\|_j = \|f\|_{j,L_p} = \left(\sum_{i\leq j} (|[f]|_i)^p\right)^{1/p}.$$

By H_{j,L_p} is meant the Banach space obtained from the C^∞ functions in $\mathscr{D}$ by completing with respect to the norm $\| \quad \|_{j,L_p}$, and $H_{j-1/p,L_p}$ denotes the class of functions ϕ which serve as boundary values of functions v belonging to H_{j,L_p} in $\mathscr{D}$. This class is normed by

$$|[\phi]|_{j-1/p,L_p} = \text{g.l.b. } |[v]|_{j,L_p}$$

and

$$\|\phi\|_{j-1/p,L_p} = \text{g.l.b. } \|v\|_{j,L_p},$$

g.l.b. in both cases being taken over all functions v in H_{j,L_p} that equal ϕ on the boundary.

Norm symbols frequently will be abbreviated as $|[\phi]|_{j-1/p}$, $\|u\|_j$, etc., the index L_p being omitted.

10.1. Equations with constant coefficients, solutions with compact support. The two theorems of this subsection are concerned with systems of equations of the form shown in (8.1) and, as indicated, only with solutions of compact support.

THEOREM 10.1. (*Interior estimates.*) *If the u_j have compact support, and the derivatives $\partial^{t_j} u_j$ belong to L_p, $p > 1$, then*

$$\int |\partial^{t_j} u_j|^p \, dx \leqq \text{const.} \sum_i \int |\partial^{-s_i} \sum_k l_{ik}(\partial) u_k|^p \, dx. \tag{10.1}$$

Furthermore, if l is a positive integer, and if $|[\sum_k l_{ik}(\partial)u_k]|_{l-s_i,L_p}$ *is finite, so is* $|[u_j]|_{l+t_j,L_p}$, *and*

$$|[u_j]|_{l+t_j,L_p} \leqq \text{const.} \sum_i |[\sum_k l_{ik}(\partial)u_k]|_{l-s_i,L_p}. \tag{10.2}$$

The indicated constants depend only on p, n, and A (Section 1).

Inequality (10.1) is proved from Theorem 3.2, ADN, applied to the representation

$$\begin{aligned}\partial^{t_j}u_j &= \partial^{t_j}(\Gamma * L(\partial)u_j) = \partial^{t_j}(\Gamma * \sum_{i,k} L^{ji}l_{ik}u_k)\\ &= \partial^{t_j}(\partial^{2m-t_j}\Gamma * \sum_{i,k}\partial^{-s_i}l_{ik}u_k)\\ &= \partial^{2m}\Gamma * \sum_{i,k}\partial^{-s_i}l_{ik}u_k + \text{const.}\sum_{i,k}\partial^{-s_i}l_{ik}u_k,\end{aligned}$$

where Γ again denotes the fundamental solution of the elliptic equation $L(\partial)u = 0$. Inequality (10.2) then follows by a difference quotient procedure.

The next theorem is concerned with boundary problems in a half-space of the form (8.1), or

$$\begin{aligned}&\sum_{j=1}^{N} l_{ij}u_j = f_i(x, t), && i = 1, \cdots, N, \quad t > 0,\\ &\sum_{j=1}^{N} B_{hj}u_j\Big|_{t=0} = \phi_h(x), && h = 1, \cdots, m,\end{aligned} \tag{10.3}$$

with constant coefficients and with finite characteristic constant E (in particular, $\Delta > 0$, see Sections 2 and 4). Let $l_1 = \max(0, r_h + 1)$, and let l be an integer $\geq l_1$.

THEOREM 10.2. *Assume the u_j vanish for $|P| \geq 1$ and belong to H_{l+t_j,L_p} in the half-space $t \geq 0$. Then*

$$\|u_j\|_{l+t_j} \leqq \text{const.} \sum_i \|f_i\|_{l-s_i} + \sum_h \|\phi_h\|_{l-r_h-1/p}, \tag{10.4}$$

with constant depending on l, p, and E.

In proving this theorem, it suffices to consider solutions u_j which are infinitely differentiable. The representation formula (6.7) applies to such solutions, and then an argument can be given like that for Theorem 14.1, ADN, p. 701.

10.2. Equations with variable coefficients. The above results for compactly supported solutions of constant coefficient problems in a half-space will be applied here to solutions of general systems of differential equations (1.1) in arbitrary, bounded, sufficiently regular domains $\mathscr{D}$. Local interior, global, and local boundary estimates, in this order, will be given—the boundary estimates being for solutions u_j that satisfy Complementing Boundary Conditions (2.1) on the boundary $\dot{\mathscr{D}}$. Preliminary estimates (Theorems 10.3 and 10.4) still deal with solutions having compact support.

Our first result is an interior estimate for a solution u_j of (1.1) that vanishes outside a sufficiently small sphere of radius r. For a given integer $l \geqq 0$, it is assumed that, in the notation of Section 9,

$$F_i \in C^{l-s_i}, \qquad a_{ij,\rho} \in C^{l-s_i};$$

k will denote an upper bound for $|a_{ij,\rho}|_{l-s_i}$. It also is assumed that $\|F_i\|_{l-s_i} < \infty$.

THEOREM 10.3. *Positive constants r_1 and K_1 exist such that, if $r \leqq r_1$ and the $\|u_j\|_{t_j}$, $j = 1, \cdots, N$, are finite, then $\|u_j\|_{l+t_j}$ also is finite for $j = 1, \cdots, N$, and*

$$\|u_j\|_{l+t_j} \leqq K_1\Big(\sum_i \|F_i\|_{l-s_i} + \sum_j \|u_j\|_0\Big). \tag{10.5}$$

The constants r_1 and K_1 depend on n, N, t' (Section 4), A (Section 1), b (just before Lemma 4.1), p, k, and l, and also on the modulus of continuity of the leading coefficients in the l_{ij}.

The next result has to do with equations (1.1) defined in a hemisphere Σ_R, Complementing Boundary Conditions (2.1) being satisfied on the flat part of the boundary of the hemisphere. A solution u_j is considered that is zero outside a smaller hemisphere Σ_r, $0 < r < R$. With $l_1 = \max(0, r_h + 1)$ it is assumed that, for a given integer $l \geqq l_1$,

$$a_{ij,\rho} \in C^{l-s_i}, \qquad b_{hj,\sigma} \in C^{l-r_h};$$

k in this case denotes an upper bound for $|a_{ij,\rho}|^{\Sigma_R}_{l-s_i}$, $|b_{h,j\sigma}|^{t=0}_{l-r_h}$, the indices taking all values. It also is assumed that $\|F_i\|_{l-s_i}$ and $\|\phi_h\|_{l-r_h-1/p}$ are all finite.

THEOREM 10.4. *Positive constants r_2 and K_2 exist such that, if $r \leqq r_2$ and the quantities $\|u_j\|_{l_1+t_j}$ are all finite, then $\|u_j\|_{l+t_j}$ also is finite for $j = 1, \cdots, N$, and*

$$\|u_j\|_{l+t_j} \leqq K_2\Big(\sum_i \|F_i\|_{l-s_i} + \sum_h \|\phi_h\|_{l-r_h-1/p} + \sum_k \|u_k\|_0\Big). \tag{10.6}$$

K_2 and r_2 depend on p, k, l, E (Section 4), and the modulus of continuity of the leading coefficients of the l_{ij}.

The proof of this theorem is parallel to that of Theorem 15.1, ADN, pp. 702–703.

The preceding results enable us to derive, in boundary problems over arbitrary domains, L_p estimates for solutions not of compact support. We first consider global estimates in, for simplicity, a finite domain $\mathscr{D}$. Let u_j, $j = 1, \cdots, N$, denote a solution of equations (1.1) in $\mathscr{D}$ satisfying Complementing Boundary Conditions (2.1) on the boundary $\dot{\mathscr{D}}$. With $l_1 = \max(0, r_h + 1)$, let l denote a fixed integer $\geqq l_1$. We assume the boundary $\dot{\mathscr{D}}$ to be of class C^l in the following sense: $\dot{\mathscr{D}}$ is coverable by a finite number of ($(n + 1)$-dimensional) open sets U_β such that each intersection $\bar{U}_\beta \cap \overline{\mathscr{D}}$ is the image, under a 1-1 mapping T_β, of the closure of an $(n + 1)$-dimensional hemisphere Σ_{R_β}, $R_\beta \leqq 1$, the flat face of the hemisphere corresponding to $\bar{U}_\beta \cap \dot{\mathscr{D}}$; each T_β and its inverse have continuous derivatives of orders up to $l + t'$ bounded by a constant κ.

In the coordinate system appropriate to each hemisphere Σ_{R_β}, a minor constant Δ_β is determined by the definition of Section 2. As minor constant for $\dot{\mathscr{D}}$, we define

$$\Delta = \Delta_{\dot{\mathscr{D}}} = \min_\beta \Delta_\beta;$$

according to the remarks in Section 2,

$$\Delta > 0.$$

Respecting the coefficients in the differential equations and the boundary conditions, we assume

$$a_{ij,\rho} \in C^{l-s_i}(\overline{\mathscr{D}}), \qquad b_{hj,\sigma} \in C^{l-r_h}(\dot{\mathscr{D}});$$

k will denote a bound for the norms $|a_{ij,\rho}|_{l-s_i}^{\mathscr{D}}$ and $|b_{hj,\sigma}|_{l-r_h}^{\dot{\mathscr{D}}}$. For each i, we assume $\|F_i\|_{l-s_i} \equiv \|F_i\|_{l-s_i}^{\mathscr{D}}$ to be finite. Defining $\|\phi_h\|_{l-r_h-1/p} \equiv \|\phi_h\|_{l-r_h-1/p}^{\dot{\mathscr{D}}}$ as the maximum of the corresponding norms over the flat faces of the Σ_{R_β}, we assume these norms, for all h, too to be finite.

THEOREM 10.5. *A constant K exists such that, if $\|u_j\|_{l_1+t_j}$ is finite for $j = 1, \cdots, N$, then $\|u_j\|_{l+t_j}$ also is finite, and*

$$\|u_j\|_{l+t_j} \leqq K\Big(\sum_i \|F_i\|_{l-s_i} + \sum_h \|\phi_h\|_{l-r_h-1/p} + \sum_j \|u_j\|_0\Big). \tag{10.7}$$

K is dependent on p, l, κ, k, E (defined before Lemma 4.1), the domain $\mathscr{D}$, and the modulus of continuity of the leading coefficients in the l_{ij}.

Remark. The term $\Sigma \|u_j\|_0$ on the right may be replaced by $\Sigma \|u_j\|_{0,L_1}$ or may be omitted altogether when the solution in question is the only one for which u_j has L_p-derivatives of orders up to $l_i + t_j$, $j = 1, \cdots, N$. In either case, K may have to be replaced by another constant.

The proof of this theorem, depending on a partition of unity, is like that of Theorem 15.2, ADN, pp. 704–705, and thus need not be given.

COROLLARY TO THEOREM 10.5. *Under the conditions of Theorem 10.5, if p exceeds the dimension $n + 1$, so that $\alpha = 1 - (n + 1)/p > 0$, then $|u_j|_{l-1+t_j+\alpha}$ is majorized by $\|u_j\|_{l+t_j,L_p}$, and hence is majorized by the right side of (10.6).*

Now we shall derive "local" boundary estimates applicable to subregions of $\mathscr{D}$ abutting $\dot{\mathscr{D}}$. The estimates are formulated here for a hemisphere Σ_R whose flat face F_R is part of $\dot{\mathscr{D}}$. New norms are used, which we now introduce. Let $\alpha = (n + 1)(1 - 1/p)$, and let q be a non-negative integer. For $v \in H_{j,L_p}(\Sigma_R)$, define

$$\widetilde{\|\partial^j v\|}_q = \underset{0<d<R}{\text{l.u.b.}}\, d^{q+j+\alpha}\, \|\partial^j v\|_{L_p(\Sigma_{R-d})}.$$

For a function ϕ defined on F_R, which is the restriction to F_R of some $v \in H_{j,L_p}(\Sigma_R)$,

define
$$\tilde{\|}\phi\|_{j-1/p,q} = \underset{w}{\text{g.l.b.}} \underset{0<d<R}{\text{l.u.b.}} \sum_{k=0}^{j} d^{q+\alpha+k} \|\partial^k w\|_{L_p(\Sigma_{R-d})},$$

taking the g.l.b. for all $w \in H_{j,L_p}(\Sigma_R)$ that equal ϕ on F_R.

It is worth noting that

$$\tilde{\|}\phi\|_{j-1/p,q} \leqq \max\,(1, R^{j+q+\alpha})\, \tilde{\|}\phi\|_{j-1/p},$$

where

$$\tilde{\|}\phi\|_{j-1/p} = \underset{w}{\text{g.l.b.}} \|w\|_{j,L_p},$$

an alternative norm for ϕ. The g.l.b. in the last definition again is taken for all $w \in H_{j,L_p}(\Sigma_R)$ that equal ϕ on F_R.

Norms of the first kind above are subject to a basic inequality stated as follows:

LEMMA. *If $h > 0$ and $q > 0$, there is a constant K depending only on h, n, p, q such that, for $0 \leqq j < h$,*

$$\tilde{\|}\partial^j v\|_q \leqq K(\tilde{\|}\partial^h v\|_q)^a (R^q \|v\|_{L_1(\Sigma_R)})^{1-a} + KR^q \|v\|_{L_1(\Sigma_R)}, \tag{10.8}$$

where

$$a = (j+\alpha)/(h+\alpha).$$

COROLLARY. *For any $\varepsilon > 0$ and $j < h$, a constant K_1 of the same description as K exists such that*

$$\|\partial^j v\|_q \leqq K_1 \varepsilon\, \tilde{\|}\partial^h v\|_q + K_1(1 + \varepsilon^{-a/(1-a)}) R^q \|v\|_{L_1(\Sigma_R)}. \tag{10.8$'$}$$

Proof of the Lemma: The inequality in question is based on the existence of a constant k such that, in any hemisphere Ω of radius ρ,

$$\|\partial^j v\|_{L_p(\Omega)} \leqq k(\|\partial^h v\|_{L_p(\Omega)})^a \|v\|_{L_1(\Omega)}^{1-a} + k\, \|v\|_{L_1(\Omega)} \rho^{-j-\alpha}. \tag{10.9}$$

This fact is deducible by a dimensional argument from the case $\rho = 1$, for which reference is made to Remark 5 in Nirenberg [20], p. 126.

To deduce (10.8) from (10.9), let d be fixed such that $0 < d < R$ and

$$d^{q+j+\alpha} \|\partial^j v\|_{L_p(\Sigma_{R-d})} \geqq \tfrac{1}{2}\, \tilde{\|}\partial^j v\|_q. \tag{10.10}$$

Cover Σ_{R-d} by a finite number of hemispheres Ω_i of radius $d/2$ and contained in $\Sigma_{R-d/2}$; no more than a fixed number, independent of d and depending only on the dimension $n + 1$, of the Ω_i are permitted to have non-empty intersection. Applying (10.9) in each Ω_i and raising to the p-th power gives us

$$\int_{\Omega_i} |\partial^j v|^p\, dx \leqq \text{const.} \left(\int_{\Omega_i} |\partial^h v|^p\, dx\right)^a \left(\int_{\Omega_i} |v|\, dx\right)^{p(1-a)} + \text{const.}\, d^{-p(j+\alpha)} \left(\int_{\Omega_i} |v|\, dx\right)^p.$$

Next sum over i and apply Hölder's inequality, the existence of a limitation on the overlapping of the Ω_i, and the inequality $\Sigma\, c_i^p \leqq (\Sigma\, c_i)^p$ for $c_i \geqq 0$ and $p \geqq 1$, to obtain

$$\int_{\Sigma_{R-d}} |\partial^j v|^p \, dx \leqq \text{const.}\left(\sum_i \int_{\Omega_i} |\partial^h v|^p \, dx\right)^a \left(\sum_i \int_{\Omega_i} |v| \, dx\right)^{p(1-a)}$$

$$+ \text{const. } d^{-p(j+a)} \left(\sum_i \int_{\Omega_i} |v| \, dx\right)^p$$

$$\leqq \text{const.}\left(\int_{\Sigma_{R-d/2}} |\partial^h v|^p \, dx\right)^a \left(\int_{\Omega_R} |v| \, dx\right)^{p(1-a)}$$

$$+ \text{const. } d^{-p(j+a)} \left(\int_{\Omega_R} |v| \, dx\right)^p.$$

Taking the p-th roots of both sides and multiplying by d^{q+j+a} now gives us

$$d^{q+j+a} \, \|\partial^j v\|_{L_p(\Sigma_{R-d})} \leqq \text{const. } (\tilde{\|} \partial^h v \|_q)^a (R^q \, \|v\|_{L_1(\Sigma_R)})^{1-a} + \text{const. } R^q \, \|v\|_{L_1(\Sigma_R)},$$

the desired inequality (10.8) following from this and (10.10).

We now formulate an L_p estimate over Σ_R. As before, let $l \geqq \max(0, r_h + 1)$, and with this l assume the coefficients in the differential equations and the boundary conditions to satisfy the same continuity requirements as in Theorem 10.5, but now only in Σ_R or F_R, respectively. For each i and h, assume the norms

$$\tilde{\|} \partial^k F_i \|_{t'+s_i}, \qquad \tilde{\|} \phi_h \|_{l-r_h-1/p, t'+r_h}, \qquad k = 0, \cdots, l - s_i,$$

to be finite. We then shall prove the following result:

THEOREM 10.6 (*Local boundary estimate*). *Under the foregoing assumptions, a constant C exists such that, for $j = 1, \cdots, N$,*

$$\tilde{\|} \partial^{l+t_j} u_j \|_{t'-t_j} \leqq C \sum_i \sum_{k=0}^{l-s_i} \tilde{\|} \partial^k F_i \|_{s_i+t'} + C \sum_h \tilde{\|} \phi_h \|_{l-r_h-1/p, t'+r_h} \tag{10.11}$$

$$+ C \sum_j \|u_j\|_{L_1(\Sigma_R)},$$

where C is a constant independent of u_j, but involving R, p, l, E, the modulus of continuity of the leading coefficients in the l_{ij}, and bounds for certain derivatives of the other coefficients in the l_{ij} and the B_{hj}, in Σ_R.

Proof: In what follows, for any function a norm of integral index will refer to the restriction of the function to Σ_R and a norm of fractional index to the restriction of the function to F_R. C_k, $k = 1, 2, \cdots$, will denote constants of the same description as was given C in the statement of the theorem above.

Let $\mathscr{D}$ denote any C^∞ region containing Σ_R such that the boundary of $\mathscr{D}$ contains F_R, and consider any function v belonging to $H_{l+t_j}(\mathscr{D})$ with support in $\Sigma_{R-\varepsilon}$ for some positive ε. Theorem 10.5 applied to $\mathscr{D}$ provides the following estimate:

$$\sum_j \|\partial^{l+t_j} v_j\|_{L_p(\Sigma_R)} \leqq C_1 \sum_i \|\sum_j l_{ij} v_j\|_{l+s_i, L_p} \tag{10.12}$$

$$+ C_1 \sum_h \|\sum_j B_{hj} v_j\|_{l-r_h-1/p, L_p} + C_1 \sum_j \|v_j\|_{0, L_p}.$$

Let the largest of the quantities $\tilde{\|}\partial^{l+t_j}u_j\|_{t'-t_j}$ be that with the index $j = 1$, and let d be such that $0 < d < R$ and

$$d^{l+t'+\alpha}\,\|\partial^{l+t_1}u_1\|_{L_p(\Omega_{R-d})} \geqq \tfrac{1}{2}\,\tilde{\|}\partial^{l+t_1}u_1\|_{t'-t_1}. \tag{10.13}$$

Let ζ be a C^∞ function in Σ_{R-d} such that $\zeta = 1$ for $|x| \leqq R-d$, $\zeta = 0$ for $|x| \geqq R - d/2$, and $|\partial^j\zeta| \leqq \text{const.}/d^j$, the latter constant depending of course on j. We shall make the substitution

$$v_j = \zeta u_j$$

in (10.12). Note that, the b_{hjk} denoting known functions,

$$\begin{aligned}
&\|\sum_j B_{hj}(\zeta u_j)\|_{l-r_h-1/p,L_p} \\
&\quad = \|\zeta\phi_h + \sum_j \sum_{k=0}^{r_h+t_j-1} b_{hjk}(\partial^{r_h+t_j-k}\zeta)(\partial^k u_j)\|_{l-r_h-1/p,L_p} \\
&\quad \leqq \|\zeta\phi_h\|_{l-r_h-1/p,L_p} + \|\sum_j \sum_{k=0}^{r_h+t_j-1} b_{hjk}(\partial^{r_h+t_j-k}\zeta)(\partial^k u_j)\|_{l-r_h-1/p,L_p} \\
&\quad \leqq \|\zeta\phi_h\|_{l-r_h-1/p,L_p} + \|\sum_j \sum_{k=0}^{r_h+t_j-1} b_{hjk}(\partial^{r_h+t_j-k}\zeta)(\partial^k u_j)\|_{l-r_h,L_p} \\
&\quad \leqq \|\zeta\phi_h\|_{l-r_h-1/p,L_p} + C_2 \sum_j \sum_{k=0}^{l+t_j-1} d^{k-l-t_j}\,\|\partial^k u_j\|_{L_p(\Sigma_{R-d/2})}.
\end{aligned}$$

(C_2 might be replaced by $C_h'(1 + R + \cdots + R^{l-r_h})$, where C_h' is independent of R.) If w_h is any function in $H_{j,L({}_p\Sigma_R)}$ equalling ϕ_h on F_R, we have furthermore

$$\|\zeta\phi_h\|_{l-r_h-1/p,L_p} \leqq \|\zeta w_h\|_{l-r_h,L_p} \leqq C_3 \sum_{k=0}^{l-r_h} d^{k+r_h-l}\,\|\partial^k w_h\|_{L_p(\Sigma_{R-d/2})}.$$

From the foregoing, therefore,

$$\|\sum_j B_{hj}(\zeta u_j)\|_{l-r_h-1/p,L_p} \leqq C_4 d^{-l-t'-\alpha}\left(\tilde{\|}\phi_h\|_{l-r_h-1/p,t'+r_h} + \sum_j \sum_{k=0}^{l+t_j-1} \tilde{\|}\partial^k u_j\|_{t'-t_j}\right).$$

By a similar process of estimation,

$$\|\sum_j l_{ij}(\zeta u_j)\|_{l-s_i,L_p} \leqq C_5 d^{-l-t'-\alpha}\left(\sum_{k=0}^{l-s_i} \tilde{\|}\partial^k F_i\|_{t'+s_i} + \sum_j \sum_{k=0}^{l+t_j-1} \tilde{\|}\partial^k u_j\|_{t'-t_j}\right).$$

Both these results are used in (10.12) (with $v_j = \zeta u_j$), and then (10.8)′ is applied. Multiplying the result by $d^{l+t'+\alpha}$ we are thus led to inequalities of the form

$$\begin{aligned}
d^{l+t'+\alpha}\,\|\partial^{l+t_1}u_1\|_{L_p(\Sigma_{R-d})} &\leqq d^{l+t'+\alpha}\,\|\partial^{l+t_1}v_1\|_{L_p(\Sigma_R)} \\
&\leqq C_6 \sum_h \tilde{\|}\phi_h\|_{l-r_h-1/p,t'+r_h} + C_7 \sum_i \sum_{k=0}^{l-s_i} \tilde{\|}\partial^k F_i\|_{t'+s_i} \\
&\quad + C_8 \sum_j \sum_{k=1}^{l+t_j-1} \varepsilon^{-(k+\alpha)/(l+t_j-k)} R^{t'-t_j}\,\|u_j\|_{L_1(\Sigma_R)} \\
&\quad + C_9\varepsilon \sum_j \tilde{\|}\partial^{l+t_j}u_j\|_{t'-t_j}.
\end{aligned}$$

The first member of the latter inequality is estimated by (10.13) and the last summation in the third member by $N \tilde{\|} \partial^{l+t_1} u_1 \|_{t'-t_1}$. Choosing $\varepsilon = 1/4\, C_9$, we now deduce the desired inequality (10.11) for $j = 1$; by the maximizing property of this value of the index, the inequality immediately follows for all values of j.

10.3. Interior Schauder estimates improved. In an arbitrary domain $\mathscr{D}$, consider a system of differential equations of the form (1.1) with $F_i \in C^{\alpha - s_i}$ and $a_{ij,p} \in C^{\alpha - s_i}$ (Section 9), where $0 < \alpha < 1$; k will denote a bound for the norms $|a_{ij,p}|_{\alpha - s_i}$ over $\mathscr{D}$. The burden of the following theorem is that generalized solutions of these equations are strict solutions.

THEOREM 10.7. *Under the above hypotheses, suppose u_j has L_p-derivatives in $\mathscr{D}$ of orders up to t_j, $j = 1, \cdots, N$, and $u_1, \cdots, u_N$ satisfy equations* (1.1) *in $\mathscr{D}$ almost everywhere. Then $u_j \in C^{t_j + \alpha}$, and for every compact subdomain $\mathscr{A}$ of $\mathscr{D}$,*

$$|u_j|_{t_j+\alpha}^{\mathscr{A}} \leqq \text{const.} \Big(\sum_i |F_i|_{\alpha - s_i}^{\mathscr{D}} + \sum_j |u_j|_0^{\mathscr{D}} \Big). \tag{10.14}$$

The constant depends on n, N, α, t' (Section 4), A (Section 1), k, and on the minimum distance between $\mathscr{A}$ and $\dot{\mathscr{D}}$.

The facts stated in this theorem already were indicated briefly in a footnote in ADN, p. 722. The theorem is proved by applying results of Sobolev, Calderon and Zygmund, and E. Hopf to integral representations of u_j and its derivatives in a step by step procedure analogous to that of Appendix 5, ADN, pp. 719–722.

CHAPTER V

APPLICATIONS OF ESTIMATES AND COMMENTS

Using the Schauder or L_p estimates one may derive many properties of solutions of our boundary value problem as in ADN, Chapter IV. For instance one proves that the solutions are of class C^∞ in the closure of the domain provided this is true of the coefficients and the given data. Furthermore one sees that the solutions are compact, that the space of solutions of the homogeneous problems are finite dimensional, and that the differential operator has closed range. The argument in Browder ([6] Theorem 4) may also be extended to show that the range of the operator has finite codimension. Many of the results of Section 12 of ADN can also be extended; we mention that a simpler proof of Theorem 12.8 there (in fact of a more general result) has been given by Agmon, Nirenberg [2], Theorem 5.2. We shall merely present a few analogous results.

11. Necessity of Complementing Boundary Conditions and of Supplementary Condition on L

In deriving the estimates of Chapters III and IV, we assumed the Complementing Boundary Conditions (Section 2) and also the Supplementary Condition on L (Section 1). These conditions are essential, at least in a sense we now describe.

Consider equations (1.1) in a hemisphere Σ_R with boundary conditions (2.1) relating to the flat face of Σ_R, smoothness assumptions being made, say, as in Theorem 9.2. When the Complementing Boundary Conditions and the Supplementary Condition on L are valid, according to Theorem 9.2 a constant C exists such that

$$\sum_j |u_j|_{l_0+t_j+\alpha} \leqq C\{\sum_i |\sum_j l_{ij}u_j|_{l_0-s_i+\alpha} + \sum_h |\sum_j B_{hj}u_j|_{l_0-r_h+\alpha} + \sum_j |u_j|_0\}$$

for any set $u_1, \cdots, u_N$ of infinitely differentiable, compactly supported functions in Σ_R. Our contention is that no such constant C exists when any one of the Complementing Boundary Conditions, or the Supplementary Condition on L, is violated. A similar statement can be made relative to the L_p estimates of Section 10. These facts are easily proved by the method of ADN, p. 681, using the functions (4.34) with $\sigma = l_0$.

12. Differentiability for Non-Linear Systems

Let $u_1, \cdots, u_N$ satisfy a system of non-linear differential equations

$$(12.1)\quad F_i(P;\ u_1, \cdots, u_N;\ \cdots;\ \partial^{s_i+t_1} u_1, \cdots, \partial^{s_i+t_N}u_N) = 0, \qquad i = 1, \cdots, N,$$

in an $(n + 1)$-dimensional domain $\mathscr{D}$. We assume this system, for the given solution, to be elliptic: the system of equations for first variations δu_j of the u_j is to be elliptic and, indeed, to fulfill all the requirements of Section 1. The latter system of "variational" equations is, of course, linear with coefficients that depend on the u_j and their derivatives of orders up to $s_i + t_j$.

Our first result is an interior estimate.

THEOREM 12.1. *Let $F_i \in C^{1-s_i}$ with respect to all its arguments. If $u_j \in C^{t_j}$ in $\mathscr{D}$, then, for any positive $\alpha' < 1$, $u_j \in C^{t_j+\alpha'}$ in $\mathscr{D}$. If $F_i \in C^{l-s_i+\alpha}$ with $0 < \alpha < 1$ and with l an integer $\geqq 1$, then $u_j \in C^{l+t_j+\alpha}$ in $\mathscr{D}$.*

The second part of the theorem follows from the first part and Theorem 5 of [10], p. 532. To prove the first part of the theorem, consider subdomains $\mathscr{A}$ and $\mathscr{B}$ of $\mathscr{D}$ such that $\bar{\mathscr{A}} \subset \mathscr{B}$, $\bar{\mathscr{B}} \subset \mathscr{D}$, and let ζ denote a non-negative C^∞ function $= 1$ in $\mathscr{A}$ and $= 0$ outside $\mathscr{B}$. We begin by deriving, from (12.1), equations for difference quotients

$$\delta u_j = \frac{u_j(x_1, \cdots, x_k + \Delta x_k, \cdots, x_{n+1}) - u_j(x_1, \cdots, x_k, \cdots, x_{n+1})}{\Delta x_k},$$

then obtaining from these analogous relations for $\zeta \delta u_j$, functions with compact support. If $\mathscr{B}$ is of sufficiently small diameter, Theorem 10.3 will furnish a bound for $\|\zeta \delta u_j\|^{\mathscr{B}}_{t_j, L_p}$ with any $p > 1$, and, in particular, with $p = (n+1)/(1-\alpha')$. This bound will apply to $\|\delta u_j\|^{\mathscr{A}}_{t_j, L_p}$. The bound being independent of Δx_k (assumed sufficiently small) and of k (and j), we easily conclude that $u_j \in H_{t_j+1, L_p}(\mathscr{A})$ with $\|u_j\|^{\mathscr{A}}_{t_j+1, L_p}$ also subject to the aforementioned bound. By a theorem of Sobolev, it now follows that $u_j \in C^{t_j+\alpha'}$ in $\mathscr{A}$. Since the position of $\mathscr{A}$ is arbitrary, the same statement holds in $\mathscr{D}$, as contended.

The remaining results are concerned with differentiability of the u_j near a portion Γ of the boundary of $\mathscr{D}$ on which m (possibly non-linear) conditions

$$G_h(x;\ u_1, \cdots, u_N;\ \cdots;\ \partial^{r_h+t_1} u_1, \cdots, \partial^{r_h+t_N} u_N) = 0 \tag{12.2}$$

are satisfied. Not losing generality, we take $\bar{\mathscr{D}}$ to include a hemisphere Σ_R and take Γ as the hemisphere's flat face. As in Section 4, coordinates $x_1, \cdots, x_n, t$ are assumed chosen in which Γ has the equation $t = 0$; then an arbitrary point of Σ_R is denoted by $P = (x, t)$, x representing the vector $(x_1, \cdots, x_n)$.

On Γ, the δu_j are subject to linear boundary conditions (the "variational" boundary conditions) consistent with (12.2). These will be assumed to satisfy the Complementing Boundary Conditions and other restrictions of Section 2. Two statements then are made which, combined, yield a general theorem concerning differentiability at the boundary in non-linear problems. The first statement we formulate as

THEOREM 12.2. *Let $0 < \alpha < 1$ and $l_0 = \max(0, r_h)$, and assume $u_j \in C^{l+t_j+\alpha^0}$ in Σ_R. Let $l \geqq l_0$ be a fixed integer, and assume $F_i \in C^{l-s_i+\alpha}$ and $G_h \in C^{l-r_h+\alpha}$ with respect to all arguments. Then $u_j \in C^{l+t_j+\alpha}$ in Σ_R.*

The proof of this result is parallel to that of Theorem 11.1, ADN; only the case $l > l_0$ is of interest. Theorem 9.2 first is applied to difference quotients tangent to Γ, these difference quotients satisfying a linear elliptic system in Σ_R, and Complementing Boundary Conditions on Γ, that fulfill all the required assumptions. By a limiting procedure, the derivatives of the form $\partial_x \partial^{l_0+t_j} u_j$ are found to belong to C^α. From the equations $\partial_t{}^{1-s_i+l_0} F_i = 0$, as in an analogous argument in the proof of Theorem 8.2, the pure t-derivatives $\partial_t{}^{l_0+t_j+1} u_j$ are expressible, however, in terms of mixed and lower order derivatives known already to belong to C^α. Hence, all the derivatives of u_j of order $l_0 + t_j + 1$ are in C^α, and the theorem is seen to hold for $l = l_0 + 1$. Repeating the above argument proves it for any given l.

Our second statement as to differentiability near the boundary we formulate in

THEOREM 12.3. *Let* $l_1 = \max(0, r_h + 1)$, *and assume* $u_j \in C^{l_1+t_j}$ *in* Σ_R. *Assume* $F_i \in C^{l_1-s_i+1}$ *and* $G_h \in C^{l_1-r_h+1}$ *with respect to all arguments. Then for every positive* $\alpha' < 1$, $u_j \in C^{l_1+t_j+\alpha'}$ *in* Σ_R.

To prove this, we first consider any set of difference quotients $\delta u_1, \cdots, \delta u_N$ tangent to the flat face Γ of Σ_R. For any sufficiently small, positive $r < R$ and any $p > 1$, reasoning as in the proof of Theorem 12.1 shows the norms $\|\delta u_j\|^{\Sigma_r}_{l_1+t_j,L_p}$ to be bounded; therefore, $\partial_x u_j \in H_{l_1+t_j,L_p}$ in Σ_r. Choosing $p = (n+1)/(1-\alpha')$, we thus have from Sobolev's lemma that $\partial_x \partial^{l_1+t_j-1} u_j \in C^{\alpha'}$ in Σ_r, and all that remains is to show

$$\partial_t^{l_1+t_j} u_j \in C^{\alpha'}$$

in Σ_r. This is done, as at the end of the proof of Theorem 12.2, by expressing the pure t-derivatives in question in terms of derivatives of the u_k, and other functions, now known to be in $C^{\alpha'}$.

We see, in particular, that if F_i and G_h are of class C^∞ in $\overline{\mathscr{D}}$, then so is the solution.

13. Perturbation of Non-Linear Problems

It is convenient to symbolize, say, $f(x; y_{11}, \cdots, y_{1s}; \cdots; y_{r1}, \cdots, y_{rs})$ by $f(x, y_{j,\beta})_{j=1,\cdots,r;\ \beta=1,\cdots,s}$ or, more briefly, just by $f(x, y_{j,\beta})$. Analogous notation is used when β, for instance, denotes a multi-index.

Let j be an index, and ρ and σ multi-indices, as in previous usage. For a parameter τ in a neighborhood of zero, let

$$F_i(\tau, P, u_{j,\rho})_{|\rho|=0,\cdots,s_i+t_j;\ j=1,\cdots,N}, \qquad i = 1, \cdots, N,$$

and

$$G_h(\tau, P, u_{j,\sigma})_{|\sigma|=0,\cdots,r_h+t_j;\ j=1,\cdots,N}, \qquad h = 1, \cdots, m,$$

be functions with continuous first and second derivatives with respect to τ belonging, with these derivatives, to $C^{l_0-s_i+\alpha}$ and $C^{l_0-r_h+\alpha}$, respectively, with respect to the other arguments. (Again, $m = \frac{1}{2}(\Sigma s_i + \Sigma t_j)$ is assumed to be an integer.) We shall consider the problem of solving the system of differential equations

$$F_i(\tau, P, \partial^\rho u_j) = 0, \qquad i = 1, \cdots, N, \tag{13.1}$$

in an $(n + 1)$-dimensional region $\mathscr{D}$ under the boundary conditions

$$G_h(\tau, P, \partial^\sigma u_j) = 0 \quad \text{on} \quad \dot{\mathscr{D}}, \qquad h = 1, \cdots, m. \tag{13.2}$$

For $\tau = 0$ a solution $u_j^{(0)} \in C^{l_0+t_j+\alpha}$ will be assumed known; it will be taken as $u_j^{(0)} \equiv 0$. The question as to whether a solution exists for any τ in a neighborhood of 0 is made to depend on the properties of the linear problem

$$\begin{aligned} &\sum_{j=1}^{N} \sum_{|\rho|=0}^{s_i+t_j} \frac{\partial F_i}{\partial u_{j,\rho}} (\tau, P, 0)\, \partial^\rho w_j = f_i \quad \text{in} \quad \mathscr{D}, \qquad i = 1, \cdots, N, \\ &\sum_{j=1}^{N} \sum_{|\sigma|=0}^{r_h+t_j} \frac{\partial G_h}{\partial u_{j,\sigma}} (\tau, P, 0)\, \partial^\sigma w_j = \phi_h \quad \text{on} \quad \dot{\mathscr{D}}, \qquad h = 1, \cdots, m. \end{aligned} \tag{13.3}$$

Theorem 13.1. *For $\tau = 0$, let $u_j \equiv 0$ solve* (13.1), (13.2). *Assume the linear problem* (13.3) *to satisfy all the conditions of Theorem* 9.3 *and to possess a unique solution* $(w_1, \cdots, w_N)$, *w_j belonging to $C^{l_0+t_j+\alpha}$, for arbitrary $f_i \in C^{l_0-s_i+\alpha}$ and $\phi_h \in C^{l_0-r_h+\alpha}$. Then for $|\tau|$ sufficiently small, problem* (13.1), (13.2) *has a unique solution* $(u_1, \cdots, u_N)$ *with $u_j \in C^{l_0+t_j+\alpha}$.*

The proof is based on Schauder estimates and is similar to that of Theorem 12.6, ADN.

Remark. It should be noted that one can use L_p estimates in place of the Schauder estimates. This is because the product of any two functions belonging to H_{j,L_p} is again in H_{j,L_p} provided $pj >$ dimension. Thus one may prove an analogue of Theorem 13.1 for equations (13.1), (13.2), obtaining a solution u with $u_j \in H_{l+t_j,L_p}$ provided $\max\,(l, l - r_h) > (n + 1)/p$.

14. Schauder Estimates for Systems of Semi-Linear Equations

Let l_{ij} and B_{hj} denote linear differential operators defined, respectively, in, and on the boundary of, a finite domain $\mathscr{D}$ and satisfying the conditions of Theorem 9.3 with $\mathscr{A} = \mathscr{D}$, $\Gamma = \dot{\mathscr{D}}$, $l = l_0$. Here we shall consider semi-linear problems

$$\begin{aligned} &\sum_j l_{ij} u_j = F_i \quad \text{in} \quad \mathscr{D}, \\ &\sum_j B_{hj} u_j = \phi_h \quad \text{on} \quad \dot{\mathscr{D}}, \end{aligned} \tag{14.1}$$

in which the functions

$$F_i = F_i(P, \partial^\rho u_j)_{|\rho|=0,\cdots,s_i+t_j-1;\, j=1,\ldots,N}$$

and

$$\phi_h = \phi_h(P, \partial^\sigma u_j)_{|\sigma|=0,\cdots,r_h+t_j-1;\, j=1,\ldots,N}$$

are permitted to depend non-linearly on the unknown functions and their derivatives of the indicated orders. When the non-linearities are not severe, the solutions of these problems are subject to estimates of Schauder type generalizing those for a single semi-linear equation of order $2m$ (ADN, Section 13). These estimates are to be obtained again by the method of Nagumo, who studied such matters in connection with second order equations.

We suppose that, when any set of functions $u_j(P) \in C^{l_0+t_j+\alpha}$, $j = 1, \cdots, N$, are inserted into F_i and ϕ_h, the resulting functions of P, namely

$$\tilde{F}_i(P) \equiv F_i(P, \partial^\rho u_j(P)), \qquad \tilde{\phi}_h(P) \equiv \phi_h(P, \partial^\sigma u_j(P)),$$

satisfy conditions of the form

$$|\tilde{F}_i|_{l_0-s_i+\alpha} + |\tilde{\phi}_h|_{l_0-r_h+\alpha} \leqq KI + \rho(J),$$

where (a) I is a finite sum of terms each of which is a finite product

$$\prod_{j,k} [u_j]_{a_{jk}}^{p_{jk}}$$

with $a_{jk} < l_0 + t_j + \alpha$, $p_{jk} \geqq 0$, and $\sum_j (l_0 + t_j + \alpha)^{-1} \sum_k a_{jk}p_{jk} < 1$, and also (b) J is a finite sum of similar products with $a_{jk} < l_0 + t_j + \alpha$, $p_{jk} \geqq 0$, and with $\sum_j (l_0 + t_j + \alpha)^{-1} \sum_k a_{jk}p_{jk} = 1$; K is a constant, and $\rho(S)$ is a fixed function defined for positive S which is $o(S)$ as $S \to \infty$.

Under these hypotheses, we state

THEOREM 14.1. *If bounds are known for* $|u_1|_0, \cdots, |u_N|_0$, *it is possible to estimate* $|u_j|_{l_0+t_j+\alpha}$ *for* $j = 1, \cdots, N$.

15. On the Representation of Solutions by Singular Integrals

The Poisson kernels of Section 4, which give integral representations in terms of boundary data for the solutions of equations with constant coefficients in a half-space, have been of obvious utility; this paper has been based upon them. Analogous Poisson kernels for equations with variable coefficients also would be useful, for instance in obtaining expressions for derivatives of a solution at the boundary as linear combinations of derivatives of singular integrals involving the boundary data. Such purposes are served almost as well, however, by "approximate" Poisson kernels that give, as singular integrals involving the prescribed boundary data, not the actual solution u, but a function v which approximates u in the sense that $u - v$ is more regular than u or v individually might be. We now indicate briefly how such "approximate" Poisson kernels might be obtained in the case of equations with sufficiently differentiable leading coefficients and a sufficiently smooth boundary. For brevity, we confine our discussion to a local problem and to homogeneous equations.

Consider a solution of (1.1) and (2.1) with $F_i = 0$, and suppose that near a point on the boundary, which we take to be the origin, the boundary has been flattened and represented by the equation $x_{n+1} = t = 0$, the domain lying on the side $t > 0$. As before, we use the coordinates (x, t), where x represents coordinates in the plane $t = 0$. In some neighborhood Σ_R: $|x|^2 + t^2 < R^2$, $t \geqq 0$, our system has the form (the summation convention is used)

$$
\begin{aligned}
&l_{ij}(x, t;\, \partial_x, \partial_t)u_j = 0, && i = 1, \cdots, N,\\
&B_{hj}(x;\, \partial_x, \partial_t)u_j|_{t=0} = \phi_h(x), && h = 1, \cdots, m.
\end{aligned} \tag{15.1}
$$

For $(x^0, t^0) \in \Sigma_R$, let $K_{jh}(x^0, t^0; x, t)$ be the Poisson kernels, as constructed in Section 4, for problems with constant coefficients of the associated type

$$
\begin{aligned}
&l'_{ij}(x^0, t^0;\, \partial_x, \partial_t)V_j = 0 \quad \text{for} \quad t > 0,\\
&B'_{hj}(x^0;\, \partial_x, \partial_t)V_j = \psi_h(x) \quad \text{for} \quad t = 0,
\end{aligned} \tag{15.2}
$$

with sufficiently differentiable $\psi_h(x)$ having compact support. These Poisson kernels and their derivatives $\partial^s K_{hj}$ with respect to x, t are governed by inequalities of the form (4.28)′. Moreover, if the coefficients in the operators $l'_{ij}(x, t;\, \partial_x, \partial_t)$ and $B'_{hj}(x;\, \partial_x, \partial_t)$ have derivatives of a sufficiently high order, the derivatives of a corresponding order of $\partial^s K_{hj}(x^0, t^0; x, t)$ with respect to x^0, t^0 exist and also are subject to inequalities of this form. In what follows, we shall suppose the number of such derivatives to be sufficient for our needs.

The solution of (15.2) is expressed in terms of the Poisson kernels as

$$
V_j(x^0, t^0; x, t) = \int K_{jh}(x^0, t^0; x - y, t)\psi_h(y)\, dy. \tag{15.2$'$}
$$

We apply this formula with $\psi_h(x) = \zeta(x, 0)\phi_h(x)$, where $\zeta(x, t)$ is a non-negative, infinitely differentiable function defined for $t \geqq 0$, vanishing outside the hemisphere Σ_R, and equalling one in a concentric hemisphere Σ_r, $r < R$. Then we define

$$
v_j(x, t) = V_j(x, t; x, t), \qquad j = 1, \cdots, N. \tag{15.2$''$}
$$

These v_j are approximations to the u_j of the desired kind; for, the differences $w_j = v_j - u_j$ are subject, for instance, to inequalities of the form

$$
\|\partial^{l+t_j} w_j\|_{L_p(\Sigma_{r/2})} \leqq C \sum_h \sum_{k=0}^{l-r_h-1} \|\partial^k \phi_h\|_{L_c(\Sigma_R)} \tag{15.3}
$$

with constant C and with $1/c = (1 + 1/n)/p$, l being an integer not less than zero or $1 + \max r_h$. (For the u_j and the v_j individually, derivatives of ϕ_h of orders up to $l - r_h$ would be expected to appear.)

We prove (15.3) from Theorem 10.6, or rather from its consequence,

$$
\begin{aligned}
(\tfrac{1}{2}r)^{l+t'+\alpha} \|\partial^{l+t_j} w_j\|_{L_p(\Sigma_{r/2})} &\leqq C_1 \sum_i \sum_{k=0}^{l-s_i} \|\partial^k l_{ij'} w_{j'}\|_{L_p(\Sigma_r)}\\
&+ C_1 \sum_h \|\tilde{B}_{hj'} w_{j'}\|^{\Sigma_r}_{l-r_h-1/p,\, t'+r_h} + C_1 \sum_{j'} \|w_{j'}\|_{L_1(\Sigma_r)},
\end{aligned} \tag{15.4}
$$

in which a superscript attached to a norm indicates the region to which the norm refers, and C_1 represents a constant expressible as a polynomial in r. C_2, C_3, and

C_4 below will denote constants of a similar description. Since $B_{hj}w_j = 0$ for $|x| \leqq r$, $t = 0$, the second summation on the right-hand side of (15.4) is zero. To estimate the terms in the first summation, we note that $l_{ij}w_j = l_{ij}v_j$ is expressible in terms of the integral operators on the ϕ_h with kernels that, by (15.2), are less singular than, say, the kernels occurring in the expressions for derivatives of v_j of order $s_i + t_j$. For any possible differentiation ∂^k, $\partial^k l_{ij}w_j$ is similarly representable in terms of singular integrals, and, moreover, the strength of the singularities can be controlled to some extent by integrating by parts. From these remarks, and in view of (5.28)′, we readily have, for instance, that in Σ_r

$$|\partial^k l_{ij}w_j| \leqq C_2 \sum_h \int (|x-y| + t)^{-n} \sum_{k'=0}^{k+s_i-r_h-1} |\partial^{k'}\psi_h(y)|\, dy, \tag{15.5}$$

the inner summation extending over all derivatives of the indicated orders unless $k + s_i - r_h - 1 < 0$, in which case only the term for $k' = 0$ appears. To (15.5) applies the general inequality

$$\left| \int (|x-y| + t)^{-n} f(y)\, dy \right|_{L_p} \leqq \text{const.}\, |f|_{L_c} \quad \text{for} \quad \frac{1}{c} = \frac{1}{p}\left(1 + \frac{1}{n}\right),$$

the L_p norm on the left, with $1 < p < \infty$, extending over the half of the x, t-space for which $t > 0$. Hence, for $l \geqq 0$, $l \geqq \max r_h + 1$,

$$\begin{aligned} \|\partial^{l-s_i} l_{ij}w_j\|_{L_p(\Sigma_r)} &\leqq C_3 \sum_{k=0}^{l-r_h-1} \|\partial^k \psi_h\|_{L_c} \\ &\leqq C_4 \sum_{k=0}^{l-r_h-1} \|\partial^k \phi_h\|_{L_c(\Sigma_R)}. \end{aligned}$$

These inequalities together with the preceding considerations imply inequality (15.3).

Inequalities comparable to (15.3) appraising lower order derivatives of the w_j in terms of lower order derivatives of the ϕ_h can be obtained by applying L_p estimates for equations written in variational form such as have been discussed in ADN.

When restricted to the boundary $t = 0$ the formulas (15.2)′, (15.2)″ yield expressions for the derivatives of the solution in terms of derivatives of the ϕ_h via singular integral operators in the sense of Calderon and Zygmund [8].

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