

OPTIMAL VORTEX REDUCTION FOR INSTATIONARY FLOWS BASED ON TRANSLATION INVARIANT COST FUNCTIONALS*

K. KUNISCH[†] AND B. VEXLER[‡]

Abstract. We consider the problem of an appropriate choice of a cost functional for vortex reduction for unsteady flows described by the Navier–Stokes equations. This choice is directly related to a physically correct definition of a vortex. Therefore, we discuss different possibilities for the cost functional and analyze the resulting optimal control problems. Moreover, we present an efficient numerical realization of this concept based on space-time finite element discretization and demonstrate its behavior in some numerical experiments. It is demonstrated that the choice of cost functionals has a significant effect on the reduction of vortices.

Key words. optimal control, vortex reduction, Navier–Stokes equations

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1. Introduction. This work focuses on the choice of proper cost functionals in optimal control formulations for vortex reduction in incompressible fluids. The formalization of vorticity is still a major challenge and a subject of intense research within fluid mechanics research itself. In the context of optimal control the quantification must satisfy the additional requirement that it allows the description of vorticity as a scalar-valued functional in terms of observables of the fluid. Moreover, the mathematical properties of the functional have significant consequences for mathematical programming considerations and for the numerical realization of the resulting optimization problems.

Let us first summarize some of the cost functionals that were already used in the optimal control literature to formulate vortex reduction problems. We denote by $y(t, x)$ the velocity vector and by $p(t, x)$ the pressure of an incompressible fluid which extends over the time horizon $[0, T]$ and the spatial domain Ω . Further, let $\tilde{\Omega}$ be the subset of Ω over which vortex reduction is desired.

An intensively studied cost functional in the context of optimal control of vortex reduction is given by

$$(1.1) \quad \int_0^T \int_{\tilde{\Omega}} |\operatorname{curl} y(t, x)|^2 dx dt;$$

see, e.g., [AT, G]. One of the objections against this functional is that it is not Galilean invariant; i.e., it is not invariant under transformations of the form $Qx + dt$ of the flow field y , where Q is a time-independent matrix and d is a constant vector. Another frequently used functional is of the form

$$(1.2) \quad \int_0^T \int_{\tilde{\Omega}} |y(t, x) - y_{des}(t, x)|^2 dx dt,$$

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[†]Institute for Mathematics, University of Graz, Heinrichstraße 36, A-8010 Graz, Austria (karl.kunisch@uni-graz.at).

[‡]Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, Altenberger Straße 69, 4040 Linz, Austria (boris.vexler@oeaw.ac.at).

where y_{des} stands for a given desired flow field which contains some of the expected features of the controlled flow field without the undesired vortices. Typically y_{des} is chosen as the solution to the Stokes problem on the same flow geometry and with the same boundary conditions as those which are involved for the characterization of y . This functional is referred to as a tracking-type functional. Just like the functional in (1.1), the tracking-type functional is not Galilean invariant. From the mathematical programming point of view the functionals in (1.1), (1.2) behave quite differently, however. To explain this fact let us consider the following prototype boundary optimal control problem:

$$(1.3) \quad \begin{cases} \min J(y) + G(u) \\ \text{subject to} \\ y_t - \nu \Delta y + y \cdot \nabla y + \nabla p = f \text{ in } (0, T] \times \Omega, \\ -\operatorname{div} y = 0 \text{ in } (0, T] \times \Omega, \\ y(0, \cdot) = y_0 \text{ on } \Omega, \\ y = u \text{ on } (0, T] \times \Gamma_c, \quad y = 0 \text{ on } (0, T] \times (\partial\Omega \setminus \Gamma_c), \end{cases}$$

where $\nu > 0$, f and y_0 are given, and u denotes the control variable acting on $(0, T] \times \Gamma_c$ and satisfying $\int_{\Gamma_c} u(t) n \, dx = 0$, with n denoting the outer normal to $\partial\Omega$. Further J and G are real-valued functionals penalizing vorticity and control-action, respectively, with J as in (1.1) or (1.2), and $G(u) = \frac{1}{2}|u|^2$, where $|\cdot|$ denotes an appropriate norm on the control space. If u is an optimal solution to (1.3), then u , together with the associated velocity y and pressure p , satisfies the primal equations, which are the equations in (1.3), the adjoint equation

$$(1.4) \quad \begin{cases} -\lambda_t - \nu \Delta \lambda + (\nabla y)^t \lambda - (y \cdot \nabla) \lambda + \nabla \pi = J'(y) \text{ in } (0, T] \times \Omega, \\ -\operatorname{div} \lambda = 0 \text{ in } (0, T] \times \Omega, \\ \lambda(T, \cdot) = 0 \text{ on } \Omega, \\ \lambda = 0 \text{ on } (0, T] \times \partial\Omega, \end{cases}$$

with adjoint velocity λ and adjoint pressure π , and satisfies as well the so-called optimality condition formally given by

$$(1.5) \quad \nu \frac{\partial \lambda}{\partial n} + G'(u) - \pi n = 0 \text{ on } \Gamma_c.$$

We refer to [FGH] and [HK], for example, for rigorous frameworks for boundary control of the Navier–Stokes equations. Note that the adjoint equations related to (1.1) and (1.2) differ significantly with respect to the regularity of the right-hand sides: in the former case the right-hand side involves second order derivatives of the velocity field, whereas for the tracking-type cost functional only y without derivatives appears in (1.4). Moreover, in the case when the residue between y_{des} and y at the optimal control is sufficiently small, a second order optimality condition for (1.3) with J as in (1.2) holds [HK]. It appears to be difficult to obtain conditions which

lend themselves to an intuitive interpretation and which imply second order sufficient optimality for optimal control problems involving (1.1). For second order sufficient optimality conditions related to optimal control of the Navier–Stokes equations, we also refer to [TW].

To discuss candidates for Galilean invariant measures we decompose the velocity gradient tensor ∇y as

$$\nabla y = S + \Omega,$$

where $S = \frac{1}{2}(\nabla y + (\nabla y)^t)$ is the rate of strain tensor and $\Omega = \frac{1}{2}(\nabla y - (\nabla y)^t)$ is the vorticity tensor. The fact that Ω is used for both the spatial domain and the antisymmetric part of ∇y should not create confusion. We use this notation for both since they are quite standard in the literature.

The Δ -criterion (see [CPC] and [BMC]) is based on a local phase plane analysis of

$$(1.6) \quad \dot{\xi} = A\xi$$

with $A = \nabla y(t, x)$. For two-dimensional systems the geometry of the trajectories in terms of the eigenvalues of A can be found in many textbooks. For the case when A is a 3×3 matrix, a detailed analysis is given in [CPC], for example. In particular, if

$$(1.7) \quad \Delta = \frac{1}{2} \left(\frac{Q}{3} \right)^3 + \left(\frac{\det \nabla y}{2} \right)^2 > 0,$$

where

$$(1.8) \quad Q = \frac{1}{2}(|\Omega|^2 - |S|^2),$$

then the characteristic equation associated with A has one real and two complex eigenvalues. Thus, the regions in $(0, T) \times \Omega$, where Δ is positive, are candidates for local instantaneous stirring. In (1.8) we denote $|\Omega|^2 = \sum_{i,j} \Omega_{ij}^2$ and similarly for $|S|$. For incompressible fluids we have

$$Q = -\frac{1}{2} \text{trace}(A^2).$$

The research in [JH] contains an interesting discussion of some of the shortcomings of earlier characterizations of vortices, including (1.7), and it proposes to define vortices as regions where the second eigenvalue of the symmetric matrix $S^2 + \Omega^2$ satisfies

$$\lambda_2(S^2 + \Omega^2) < 0.$$

Under appropriate conditions this criterion guarantees an instantaneous local pressure minimum in a two-dimensional plane in a three-dimensional flow.

In the case when spatial domain Ω is two-dimensional, it can be easily verified by direct computation that the following criteria are equivalent:

- (i) The smaller eigenvalue of $S^2 - \Omega^2$ is negative;
- (ii) ∇y has complex eigenvalues;
- (iii) $Q > 0$;
- (iv) $\det \nabla y > 0$.

These considerations suggest the use of

$$(1.9) \quad \int_0^T \int_{\tilde{\Omega}} \max(0, \det \nabla y(t, x)) \, dx \, dt$$

as a cost functional in vortex-reduction formulations. Note that due to the max-operation the cost functional in (1.9) is nondifferentiable and therefore, for numerical optimization routines, regularization of the max-operator may be necessary. This cost functional was used for optimal vortex reduction in a driven-cavity problem in [HKS \bar{V}].

Let us return to the cost functional (1.1) involving the curl-operator and note that it can be equivalently expressed as

$$\int_0^T \int_{\tilde{\Omega}} |\Omega(t, x)|^2 \, dx \, dt.$$

Vorticity, together with thresholding, has been widely used for representing vortices; see [JH] and the references given there. However, it is now well accepted as an inadequate vorticity measure, for example, in the context of boundary layers. In particular, it was shown in [Lu] that maxima and minima of $|\Omega|$ in planar wall-bounded flows occur only at the wall.

A well-known Galilean invariant measure defines the vorticity region as the domain where the vorticity tensor dominates the rate of strain tensor, i.e.,

$$Q = \frac{1}{2}(|\Omega|^2 - |S|^2) = -\frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) > 0,$$

where λ_i are the eigenvalues of $S^2 + \Omega^2$.

This criterion was originally proposed in [O] and [W] for two-dimensional domains and investigated in [HWM] for three-dimensional domains. It is referred to as the Okubo–Weiss criterion, or Q-criterion, and readily lends itself to being used in a cost functional of the form

$$\int_0^T \int_{\tilde{\Omega}} \max(0, |\Omega(t, x)|^2 - |S(t, x)|^2) \, dx \, dt.$$

As mentioned above, the Okubo–Weiss criterion coincides with the $\det \nabla y > 0$ criterion in two dimensions.

Galilean invariant vortex criteria allow a classification which is invariant under frame changes that move at a constant speed relative to each other. In a variety of different theoretical and example-driven approaches (see, e.g., [H1], [H2], [LHK], [LKH], [TK]), it was established that Galilean invariance is not sufficient for reliable vortex identification. Rather, criteria must be invariant also under coordinate transformations of the form $\mathcal{Q}(t)x + d(t)$, where \mathcal{Q} is a time-dependent orthogonal matrix and d a time-dependent velocity vector. Such transformations are called *objective* in continuum mechanics and, in particular, they allow time-dependent rotations. An alternative way to point out the deficiencies of Galilean invariant criteria is based on taking the point of view of tracer dynamics. The gradient of the tracer q satisfies

$$\frac{D\nabla q}{Dt} = -(\nabla y)^t \nabla q,$$

where $\frac{D}{Dt}$ is the material derivative and $(\nabla y)^t$ denotes the transpose of the velocity gradient tensor. Studies have shown that the acceleration gradient tensor or second

derivatives of the pressure must be considered as well; see [LKH, LHK] and the references given there. In [LKH] an objective criterion is obtained in two dimensions which defines a rotation dominated region by means of

$$(1.10) \quad |r(y, p)| > 1,$$

where

$$(1.11) \quad r(y, p) = \frac{\omega}{\sigma} - \frac{\sigma_s(p_{x_1x_1} - p_{x_2x_2}) - 2\sigma_n p_{x_1x_2}}{\sigma^{\frac{3}{2}}},$$

and $\omega = (y_2)_{x_1} - (y_1)_{x_2}$, $\sigma_s = (y_2)_{x_1} + (y_1)_{x_2}$, $\sigma_n = (y_1)_{x_1} - (y_2)_{x_2}$, $\sigma = (\sigma_s^2 + \sigma_n^2)^{1/2}$. It can readily be used for vortex-reduction by introducing the cost functional

$$(1.12) \quad \int_0^T \int_{\bar{\Omega}} \max(r(t, x)^2 - 1, 0) dx dt,$$

for example.

In [TK] the Okubo–Weiss criterion is reconsidered through study of the stability of fluid particles in the eigenbasis of the rate of strain tensor S . This results in the modified criterion

$$(1.13) \quad Q_s = \frac{1}{2}(|\Omega - \Omega_S|^2 - |S|^2) > 0,$$

where Ω_S is the matrix containing the time derivatives of the unit eigenvectors of S in the Lagrangian frame. This criterion is well defined and objective regardless of the spatial dimension; however, as noted in [TK], the physical principles used in deriving (1.13) are restricted to two dimensions. In the appendix of [TK] it is verified that in two dimensions the criteria (1.10) and (1.13) coincide.

An interesting new vorticity criterion [H1, H2] again departs from a stability consideration of $\xi = 0$ in (1.7). It utilizes the Lyapunov functional

$$V(t, \xi) = \frac{1}{2} \frac{d}{dt} |\xi|^2 = \xi^t S(t, x(t)) \xi.$$

By incompressibility of the velocity field, it can be argued that $Z = Z(t) = \{\xi : V(t, \xi) = 0\}$ defines a cone in $\mathbb{R}^3(\mathbb{R}^2)$ separating regions with different qualitative properties of the flow of (1.6) depending on the signs of the real eigenvalues of $\nabla y(t, x(t))$. To analyze the qualitative behavior of the flow, the strain acceleration tensor $M = S_t + (\nabla S)y + S\nabla y + (\nabla y)^t S$ is defined and its restriction M_Z to Z is considered. The elliptic region is defined as the set in (t, x) space, where M_Z is indefinite or $S(t, x)$ vanishes on Z . A vortex is a bounded connected set of fluid trajectories that remain in the elliptic region. This is an objective criterion valid for two and three dimensions. It appears that further considerations are necessary regarding how to design a practical cost functional based on this vortex definition which can be used in optimal control formulations.

In this paper we shall show the practical efficiency of the objective functional (1.12) for vortex reduction. We shall further conduct a comparison among the four functionals (1.1), (1.2), (1.9), and (1.12). Optimal control problems based on these four functionals can give surprisingly different results. A comparison among different cost functionals is, at first, impeded by the following difficulty: As indicated in the prototype problem (1.3), usually a term $G(u)$ representing control costs is utilized.

Mathematically it guarantees a priori bounds on minimizing sequences for (1.3) and, subsequently, existence of a minimizer for the optimal control problem (1.3). The optimal solution depends on G and therefore, if J is taken as one of the four functionals (1.1), (1.2), (1.9), (1.12), the question must be addressed of how to eliminate the effect of the control-cost term on the solutions of these optimal control problems. Here we take the approach of eliminating G altogether. As a consequence, we have to consider atypical existence problems for optimal control problems with the Navier–Stokes equations as constraints. In fact, there are no obvious a priori bounds for the control. We can only hope for a priori bounds for y due to J . Assuming that such bounds can be obtained it is, however, unfeasible to assume that boundedness of y implies boundedness of u for most practical norms for y and u , where y and u are linked through the Navier–Stokes equations. For this reason we consider finite dimensional control spaces only. This still leaves us with interesting existence problems for optimal control problems without control costs in the functional to be minimized.

We should also note the fact that some arbitrariness remains due to the fact that vorticity criteria of the type $c(t, x) > 0$ pointwise in the space-time cylinder must be converted to scalar-valued functionals; compare (1.10) and (1.12), for example.

Let us briefly outline the following sections. Section 2 is devoted to existence results for optimal control problems with the Navier–Stokes equations as constraints. Specifically we also consider the situation without control costs, where a priori bounds on the controls can result only from the differential equation which appears as a constraint. In section 3 we discuss optimality systems for the optimal control problems under consideration. Section 4 is devoted to algorithmic aspects concerning the optimization algorithm and the space-time finite element discretization. Numerical examples for a channel flow with an obstacle are given in section 5. In section 6 (appendix) the proofs for the theorems and propositions of sections 2 and 3 are provided.

2. Optimal control problem. In this section we formulate optimal control problems for vortex reduction and discuss the existence of solutions in some prototypical cases. In order to compare different vortex descriptions leading to different cost functionals, we choose a formulation where the control variable does not explicitly enter the cost functional; i.e., we consider optimal control problems without control costs. In general, existence of a solution for such problems cannot be guaranteed. Therefore we restrict ourselves to the consideration of finite dimensional control spaces. Even then, due to nonlinearity of the state equation and possible nonconvexity of the cost functional, existence of optimal controls does not follow from standard arguments. These arguments can be employed if the cost functional is radially unbounded with respect to the control, but this is not the case in our work; see, e.g., [AT, Li]. In practice, as well, the control variables are often restricted to a finite dimensional setting.

Throughout this section we consider the optimal control problem of vortex reduction on the spacial domain $\Omega \subset \mathbb{R}^2$, with boundary $\partial\Omega$ of C^1 -class, in the time interval $I = (0, T)$. The space-time cylinder is denoted by $Q = (0, T) \times \Omega$.

In order to formulate the optimal control problem we introduce the following spaces:

$$\mathcal{V} = \{v \in H^1(\Omega)^2 : \operatorname{div} v = 0\}, \quad \mathcal{V}_0 = \{v \in H_0^1(\Omega)^2 : \operatorname{div} v = 0\},$$

$$\mathcal{H} = \{v \in L^2(\Omega)^2 : \operatorname{div} v = 0\}, \quad H = \{v \in H_0^1(\Omega)^2 : \operatorname{div} v = 0\}^{-L^2(\Omega)^2},$$

where $-L^2(\Omega)^2$ denotes the closure in $L^2(\Omega)^2$, and \mathcal{V}^* is the dual space to \mathcal{V} . These

spaces build a Gelfand triple $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$; i.e., the imbeddings $\mathcal{V} \hookrightarrow \mathcal{H}$ and $\mathcal{H} \hookrightarrow \mathcal{V}^*$ are continuous and \mathcal{V} is dense in \mathcal{H} .

For an arbitrary space Y we use the abbreviations $L^p(Y) = L^p(0, T; Y)$, for $1 \leq p < \infty$, and $C(Y) = C([0, T]; Y)$. We further set

$$(2.1) \quad W = \{w \in L^2(\mathcal{V}) : w_t \in L^2(\mathcal{V}_0^*)\}, \quad W_0 = W \cap L^2(\mathcal{V}_0),$$

$$L^2(\Omega)/\mathbb{R} = \left\{v \in L^2(\Omega) : \int_{\Omega} v \, dx = 0\right\}.$$

The space W_{Σ} of admissible functions appearing in the Dirichlet boundary conditions is chosen as

$$W_{\Sigma} = \{\hat{g} = \tau g : g \in W\},$$

where $\tau : W \rightarrow L^2(H^{1/2}(\partial\Omega)^2)$ is the trace operator onto the lateral boundary $(0, T) \times \partial\Omega$ of the cylinder Q ; see [HK].

As motivated in the introduction, we choose a finite dimensional control space $U \cong \mathbb{R}^n$ ($n \in \mathbb{N}$) and consider a control operator $\hat{B} \in \mathcal{L}(U, W_{\Sigma})$. Then \hat{B} can be expressed as

$$\hat{B}u = \sum_{i=1}^n u_i \hat{\psi}_i, \quad \hat{\psi}_i = \tau \psi_i, \quad \text{with } \psi_i \in W.$$

Throughout we assume the operator \hat{B} to be injective, i.e., the functions $\{\hat{\psi}_i\}$ are linearly independent, and that

$$\psi_i(0) \in H, \quad i = 1, 2, \dots, n.$$

The latter condition implies that $\psi_i(0) \cdot n = 0$ on $\partial\Omega$ and also will be required for the initial condition y_0 below. Our results can easily be generalized to just requiring the compatibility condition

$$\left((\hat{B}u)(0) - y_0|_{\partial\Omega}\right) \cdot n = 0.$$

For later use, we introduce a prolongation operator $B \in \mathcal{L}(U, W)$ of the control operator \hat{B} with the property

$$\tau(Bu) = \hat{B}u \quad \text{for all } u \in U.$$

This prolongation may be defined by $Bu = \sum_i u_i \psi_i$ or as in Lemma 6.2, but each prolongation satisfying the above condition is admissible.

The state equation for the velocity field $y = y(t, x)$ and pressure $p = p(t, x)$ is formulated as follows:

$$(2.2) \quad \begin{cases} y_t - \nu \Delta y + y \cdot \nabla y + \nabla p = f & \text{in } (0, T] \times \Omega, \\ -\operatorname{div} y = 0 & \text{in } (0, T] \times \Omega, \\ y(0, \cdot) = y_0 & \text{on } \Omega, \\ y = \hat{B}u & \text{on } (0, T] \times \partial\Omega. \end{cases}$$

The data are assumed to satisfy $\nu > 0$, $f \in L^2(H^{-1}(\Omega)^2)$, and $y_0 \in H$, where $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$. The state equation (2.2) is understood in the distributional sense, allowing for a variational formulation for the velocity component y . The introduction of the pressure component and its regularity is discussed below in Proposition 2.1.

To introduce the weak formulation for the velocity component we define a semi-linear form $\bar{a}: W \times W_0 \rightarrow \mathbb{R}$ by

$$\bar{a}(y, \psi) = \int_0^T \{(y_t, \psi) + \nu(\nabla y, \nabla \psi) + (y \nabla y, \psi) - (f, \psi)\} dt + (y(0) - y_0, \psi(0)).$$

The velocity component y is called the variational solution of (2.2) if $y \in W$ satisfies

$$(2.3) \quad y \in Bu + W_0 : \bar{a}(y)(\psi) = 0 \quad \text{for all } \psi \in L^2(\mathcal{V}_0).$$

For the state equation formulated in this setting we have the following existence result.

PROPOSITION 2.1. *For every $u \in U$ there exists a unique variational solution $y \in W$ of the state equation (2.3) defining a continuous solution operator $S: U \rightarrow W$. Moreover, there exists a distribution p fulfilling (2.2) such that $p = \partial_t P$ with some $P \in C(L^2(\Omega)/\mathbb{R})$. The mapping $u \mapsto P$ is continuous from U to $C(L^2(\Omega)/\mathbb{R})$.*

The proof of this proposition is given in section 6 (appendix), where it is shown that the pair (y, P) satisfies

$$y(t) - y(0) - \nu \int_0^t \Delta y(s) ds + \int_0^t (y(s) \cdot \nabla) y(s) ds + \nabla P(t) = \int_0^t f(s) ds$$

in $C(H^{-1}(\Omega)^2)$.

The space of all pairs $x = (y, p)$ satisfying $y \in W$ and $p = \partial_t P$ (as distribution) with some $P \in C(L^2(\Omega)/\mathbb{R})$ is denoted by X , i.e.,

$$X = \{(y, p) : y \in W \text{ and } p = \partial_t P, P \in C(L^2(\Omega)/\mathbb{R})\}.$$

In the following proposition a regularity result for the solution $x = (y, p)$ of (2.2) is given. It will be shown that x lies in the space

$$\mathcal{X} = L^2(H^2(\Omega)^2) \cap H^1(L^2(\Omega)^2) \times L^2(H^1(\Omega)),$$

provided that additional assumptions are satisfied. This regularity in particular allows us to interpret the pressure component p as an almost everywhere defined function rather than as only a distribution.

PROPOSITION 2.2. *If $\partial\Omega$ is of C^2 -class, $f \in L^2(L^2(\Omega)^2)$, $y_0 \in \mathcal{V}_0$, $\psi_i \in W \cap L^2(H^2(\Omega)^2) \cap H^1(L^2(\Omega)^2)$, and $\psi_i(0) \in \mathcal{V}_0$, then $x \in \mathcal{X}$.*

The proof of this proposition is given in section 6 (appendix).

Under the assumptions of Proposition 2.2 a variational formulation incorporating the pressure component can be stated. To this end we introduce the space \mathcal{X}_0 by

$$\mathcal{X}_0 = \{(y, p) \in \mathcal{X} : \tau y = 0\}$$

and define for $x = (y, p) \in \mathcal{X}$ and $\zeta = (\psi, \xi) \in \mathcal{X}_0$ the semilinear form $a: \mathcal{X} \times \mathcal{X}_0 \rightarrow \mathbb{R}$ by

$$a(x)(\zeta) = \int_0^T \{ (y_t, \psi) + \nu(\nabla y, \nabla \psi) + (y \nabla y, \psi) - (p, \operatorname{div} \psi) - (f, \psi) + (\operatorname{div} y, \xi) \} dt \\ + (y(0) - y_0, \psi(0)).$$

We introduce a prolongation $\mathcal{B} \in \mathcal{L}(U, \mathcal{X})$ by $\mathcal{B}u = (Bu, 0)$ and state the corresponding variational formulation as follows:

$$(2.4) \quad x \in \mathcal{B}u + \mathcal{X}_0 : a(x, \zeta) = 0 \quad \text{for all } \zeta \in \mathcal{X}_0.$$

Under the assumptions of Proposition 2.2 the solution x satisfies the variational problem (2.4).

We are now prepared to introduce the optimization problems which will further be investigated. In the following section we shall derive optimality systems under the mild regularity requirements of Proposition 2.1 as well as under the stronger ones of Proposition 2.2. The corresponding variational formulations are given in (2.3) and (2.4), respectively.

The optimization problems are of the form

$$(2.5) \quad \text{minimize } J(x) \quad \text{subject to (2.2), } \quad x \in X, u \in U,$$

where $J: X \rightarrow \mathbb{R}$ and the solutions to (2.2) are understood in the sense of Proposition 2.1.

We stress that due to the absence of a control cost term, one cannot use standard techniques to ensure the existence of a solution of (2.5). In the following, we first provide the existence of solutions for two choices of the functional J :

$$(2.6) \quad J_1(y) = \int_0^T \int_{\Omega} |y(t, x) - y_{des}(t, x)|^2 dx dt,$$

$$(2.7) \quad J_2(y) = \int_0^T \int_{\Omega} |\operatorname{curl} y(t, x)|^2 dx dt,$$

where $y_{des} \in L^2(L^2(\Omega))$ is a given desired velocity field.

THEOREM 2.3. *There exists a solution for the optimal control problem (2.5) for both choices of the cost functional $J = J_1$ and $J = J_2$.*

The proof of this theorem is given in section 6 (appendix).

As discussed in the previous section, the cost functionals J_1 and J_2 defined in (2.6) and (2.7) are not based on Galilean invariant or objective vortex definitions. Therefore we additionally consider the functional obtained from the Q-criterion:

$$(2.8) \quad J_3(y) = \int_0^T \int_{\Omega} g_3(\det \nabla y) dx dt,$$

which is Galilean invariant, and the functional based on the vortex criterion from [LKH]:

$$(2.9) \quad J_4(y, p) = \int_0^T \int_{\Omega} g_4(r(y, p)) dx dt,$$

which is even objective. Here, $r(y, p)$ is defined as in (1.11) and the functions $g_3, g_4 \in C^2(\mathbb{R})$ are chosen as follows:

$$g_3(t) = \begin{cases} 0, & t \leq 0, \\ l(t), & t > 0, \end{cases} \quad g_4(t) = \begin{cases} l(-t-1), & t < -1, \\ 0, & -1 \leq t \leq 1, \\ l(t-1), & t > 1, \end{cases} \quad l(t) = \frac{t^3}{t^2+1}.$$

The techniques presented in section 6 for ensuring the existence of optimal solutions for optimal control problem (2.5) with $J = J_1$ and $J = J_2$ (without the control cost term) cannot be directly applied for the cost functionals J_3 and J_4 . For J_3 we obtain the following result.

THEOREM 2.4. *The optimal control problem (2.5) with $J = J_3$ and additional control constraints $u_a \leq u \leq u_b$ ($u_a, u_b \in U$) possesses an optimal solution.*

The proof of this theorem is given in section 6 (appendix).

Remark 2.1. The discussion of the case $J = J_4$ requires more regularity of the state variable for this cost functional to be well defined. For the required regularity, including $p \in L^2(0, T; H^2(\Omega))$, strong compatibility assumptions on the data are necessary; see, e.g., [T]. A detailed analysis for existence and optimality conditions is not within the scope of this paper.

3. Optimality system. In this section we discuss necessary optimality conditions for (2.5). The derivation is rigorous for J_1, J_2, J_3 , but only formal for J_4 .

In order to set up the optimality system, we introduce the adjoint equation for $z = (\lambda, \pi)$:

$$(3.1) \quad \begin{cases} -\lambda_t - \nu \Delta \lambda + (\nabla y)^t \lambda - (y \cdot \nabla) \lambda + \nabla \pi = J'_y(y, p) \text{ in } (0, T] \times \Omega, \\ -\operatorname{div} \lambda = J'_p(y, p) \text{ in } (0, T] \times \Omega, \\ \lambda(T, \cdot) = 0 \text{ on } \Omega, \\ \lambda = 0 \text{ on } (0, T] \times \partial \Omega. \end{cases}$$

This equation is understood in the distributional sense, allowing for a variational formulation for the velocity component λ . The adjoint pressure π is introduced in Theorem 3.1 similarly to how the primal pressure p was introduced in Proposition 2.1.

We note that for $J = J_1, J_2, J_3$ the term J'_p vanishes and the adjoint velocity field λ is divergence-free. This is not the case for the choice $J = J_4$. For J_4 , moreover, regularity beyond $x \in X$ is required to make $J'_{4,y}$ rigorous; see Remark 2.1 above. In fact, the derivatives $J'_{4,y}$ and $J'_{4,p}$ are given by

$$J'_{4,y}(x)(\delta y) = \int_0^T \int_{\Omega} g'_4(r(y, p)) r'_y(y, p)(\delta y) \, dx \, dt,$$

$$J'_{4,p}(x)(\delta p) = \int_0^T \int_{\Omega} g'_4(r(y, p)) r'_p(y, p)(\delta p) \, dx \, dt,$$

where $r'_y(y, p)(\delta y)$ and $r'_p(y, p)(\delta p)$ are directional derivatives of $r(y, p)$ defined in (1.11).

The following theorem ensures the existence of the solution for this adjoint equation for the choices $J = J_1, J_2, J_3$, where the velocity component λ of the adjoint state z is given in the following variational sense:

$$(3.2) \quad \lambda \in L^2(\mathcal{V}_0) : \bar{a}'(y)(\psi, \lambda) = J'(y)(\psi) \quad \text{for all } \psi \in W_0.$$

THEOREM 3.1. *The functionals $J = J_1, J_2, J_3$ are Gateaux differentiable on $L^2(H^1(\Omega)^2)$, and for every $x = (y, p) \in X$ there exists a unique distributional solution $z = (\lambda, \pi)$ of the adjoint equation (3.1) with $\lambda \in L^2(\mathcal{V}_0)$, $\lambda_t \in L^{4/3}(\mathcal{V}_0^*)$, and $\pi = \partial_t \Pi$ with $\Pi \in C(L^2(\Omega)/\mathbb{R})$. If, in addition, the assumptions of Proposition 2.2 are fulfilled, then $z \in \mathcal{X}_0$.*

The proof is given in section 6 (appendix). It implies that the solution of Theorem 3.1 satisfies

$$\lambda(t) - \nu \int_t^T \Delta \lambda \, ds + \int_t^T ((\nabla y)^t \lambda - (y \cdot \nabla) \lambda) \, ds + \nabla \Pi(t) = \int_t^T J'_y(y) \, ds.$$

The existence of the adjoint state allows the formulation of first order optimality conditions for the problem (2.5). Due to the fact that the functionals J_i ($i = 1, 2, 3$) do not depend on p , we can formulate the optimality system using only the velocity components y of x and λ of z , respectively:

$$(3.3) \quad y \in Bu + W_0 : \bar{a}(y)(\psi) = 0 \quad \text{for all } \psi \in L^2(\mathcal{V}_0),$$

$$(3.4) \quad \lambda \in L^2(\mathcal{V}_0) : \bar{a}'(y)(\psi, \lambda) = J'(y)(\psi) \quad \text{for all } \psi \in W_0,$$

$$(3.5) \quad u \in U_{ad} : J'(y)(B(v - u)) - \bar{a}'(y)(B(v - u), \lambda) \geq 0 \quad \text{for all } v \in U_{ad},$$

where $U_{ad} = U$ in the case when $J = J_1$ or $J = J_2$, and $U_{ad} = \{u \in U : u_a \leq u \leq u_b\}$ in the case when $J = J_3$.

THEOREM 3.2. *Let $(u, x) \in U \times X$ be a local solution of the optimal control problem (2.5) for the choices $J = J_1, J_2, J_3$. Then the triple (u, x, z) fulfills the optimality system (3.3)–(3.5), where $z = (\lambda, \pi)$ is the adjoint state. In the case when $U = U_{ad}$ the inequality (3.5) can be replaced by an equality.*

The proof is given in section 6 (appendix).

If the assumptions of Proposition 2.2 are fulfilled, then we have $x, z \in \mathcal{X}$. In this case the optimality system can be equivalently rewritten using the semilinear form $a(\cdot)(\cdot)$ involving pressure components:

$$(3.6) \quad x \in \mathcal{B}u + \mathcal{X}_0 : a(x)(\zeta) = 0 \quad \text{for all } \zeta \in \mathcal{X}_0,$$

$$(3.7) \quad z = (\lambda, \pi) \in \mathcal{X}_0 : a'(x)(\zeta, z) = J'(x)(\zeta) \quad \text{for all } \zeta \in \mathcal{X}_0,$$

$$(3.8) \quad u \in U_{ad} : J'(x)(\mathcal{B}(v - u)) - a'(x)(\mathcal{B}(v - u), z) \geq 0 \quad \text{for all } v \in U_{ad}.$$

Moreover, integration by parts, Green's formula, and the fact that $Bv \in W$ for all $v \in U$ imply that

$$(3.9) \quad J'(x)(\mathcal{B}(v - u)) - a'(x)(\mathcal{B}(v - u), z) = \int_0^T \int_{\partial\Omega} (\nu \nabla \lambda \cdot n) \cdot \hat{B}(v - u) \, ds \, dt.$$

Here, the trace $\nu \nabla \lambda \cdot n$ in (3.9) can be understood in a usual $L^2(L^2(\partial\Omega))$ sense.

Remark 3.1. On the discrete level the equality (3.9) does not hold anymore due to the lack of the appropriate formulas for integration by parts of the discretized solutions. As suggested in [V], we use the integrated residual (3.8) of the adjoint equation on the discrete level, which allows a higher order of convergence, with respect to the maximal cell-size h , than the discretization based on the boundary integral of the flux $\nu \nabla \lambda \cdot n$. A similar technique for the approximation of boundary integrals using a representation as volume integrals in the context of finite element discretization is discussed, e.g., in [GLLS].

4. Algorithmic aspects. In this section we describe a solution algorithm for optimal control problems of vortex reduction. The problem is reformulated as an unconstrained optimization problem by eliminating the state equation. Based on this formulation we describe the Newton method for solving this problem on the continuous level. Subsequently the optimization problem is discretized by space-time finite element methods. This allows a natural translation of the optimality conditions from the continuous to the discrete level due to the fact that the approaches *optimize-then-discretize* and *discretize-then-optimize* coincide for Galerkin-type discretizations. For more details of the finite element discretization of nonstationary optimal control problems we refer to [BMV].

4.1. Optimization algorithm. Before describing the discretization, we discuss the solution algorithm based on Newton's method on the continuous level. Since a finite element discretization is used, the continuous algorithm can then be simply translated into a discrete one by projection. Throughout this section, we require the assumptions of Proposition 2.2, which ensures the existence of a solution operator $S: U \rightarrow \mathcal{X}$ for the state equation in formulation (2.4) such that

$$S(u) \in \mathcal{B}u + \mathcal{X}_0 : a(S(u))(\zeta) = 0 \quad \text{for all } \zeta \in \mathcal{X}_0 \quad \text{for all } u \in U.$$

This gives rise to the introduction of a reduced cost functional $j: U \rightarrow \mathbb{R}$ by

$$(4.1) \quad j(u) = J(S(u)),$$

and allows us to reformulate the optimization problem (2.5) as an unconstrained problem

$$(4.2) \quad \text{minimize } j(u), \quad u \in U.$$

For J_3 and J_4 we should, in principle, replace U by U_{ad} . Since, for the numerical examples we consider, the inequality constraints did not become active, we do not consider U_{ad} here.

For the application of Newton's method to this optimization problem, we have to compute the derivatives of the reduced cost functional j . This is addressed in the following proposition.

PROPOSITION 4.1. *Let j be the reduced cost functional defined in (4.1). Its derivatives can be expressed as follows:*

(a) *For an arbitrary direction $\delta u \in U$ we have*

$$j'(u)(\delta u) = J'(x)(\mathcal{B}\delta u) - a'(x)(\mathcal{B}\delta u, z),$$

where $x = S(u)$ is the solution of the state equation (2.4) and $z \in \mathcal{X}_0$ is the solution of the adjoint equation (3.7).

(b) *For arbitrary directions $\delta u, \tau u \in U$ we have*

$$j''(u)(\delta u, \tau u) = J''(x)(\delta x, \mathcal{B}\tau u) - a''(x)(\delta x, \mathcal{B}\tau u, z) - a'(x)(\mathcal{B}\tau u, \delta z),$$

where $z \in \mathcal{X}_0$ is the solution of the adjoint equation (3.7), $\delta x \in \mathcal{X}$ is determined by the tangent equation

$$(4.3) \quad \delta x \in \mathcal{B}\delta u + \mathcal{X}_0 : a'(x)(\delta x, \zeta) = 0 \quad \text{for all } \zeta \in \mathcal{X}_0,$$

and $\delta z \in \mathcal{X}_0$ is the solution of the dual Hessian equation

$$(4.4) \quad \delta z \in \mathcal{X}_0 : a'(x)(\zeta, \delta z) = J''(x)(\delta x, \zeta) - a''(x)(\delta x, \zeta, z) \quad \text{for all } \zeta \in \mathcal{X}_0.$$

The proof is similar to [HK] and [BMV].

In the following we describe the solution of the optimization problem (4.2) by Newton's method on the continuous level. Starting with an initial guess $u^0 \in U$, the next iterate u^{n+1} is computed by an update step

$$u^{n+1} = u^n + \delta u^n,$$

where δu^n solves

$$(4.5) \quad j''(u^n)(\delta u^n, v) = -j'(u^n)(v) \quad \text{for all } v \in U.$$

To solve (4.5) we use the conjugate gradient (cg) method, which requires only the evaluation of the right-hand side and of matrix-vector products. Thus we have to evaluate $j'(u^n)(v)$ and $j''(u^n)(\delta u^n, v)$ for fixed v . This can be done efficiently based on Proposition 4.1. Note that the second derivative $a''(x)$ involved in the representation of $j''(u)$ does not depend on the state x due to the quadratic structure of the Navier–Stokes equations.

Remark 4.1. For one step of the cg method, we have to solve one tangent equation (4.3) and one dual-Hessian equation (4.4). In some cases, if the dimension of U is small, it might be more efficient to build up the Hessian $\nabla^2 j(u^n)$; see [BMV] for a detailed discussion and a comparison.

4.2. Finite element discretization. In order to apply Newton's method described before, we consider a space-time finite element discretization of the optimal control problem under consideration. For the time discretization we use the dG (discontinuous Galerkin) or the cG (continuous Galerkin) method; see, e.g., [EJT].

For the time grid

$$0 = t_0 < \dots < t_l < \dots < t_M = T, \quad k_l = t_l - t_{l-1},$$

and a space mesh \mathcal{T}_h consisting of quadrilaterals, we consider a space of spatially continuous and cellwise bilinear (biquadratic) and discontinuous in time piecewise polynomial functions of order r , X_{hk}^r . A similar space with continuous and piecewise polynomial functions in time of order s is denoted by Y_{hk}^s . The Galerkin method using X_{kh}^r as the trial and the test spaces leads to dG discretization. If the continuous in time space Y_{hk}^s is used as a trial space, this results in a cG discretization. For the detailed description of discrete equations to be solved within one step of the Newton method we refer to [BMV]. In our practical realization we use the dG(0) method, which results in a variant of the backward Euler and the cG(1) method, which is very similar to the Crank–Nicolson method. We emphasize that the space-time finite element discretization leads to the exact representation of the first and second derivatives of the discrete reduced cost functional, which is important for the convergence of the optimization algorithms. The derivation of these representations follows along the same lines as in the continuous case; cf. Proposition 4.1. The first directional derivatives of the reduced cost functional are given by

$$j'_{kh}(u)(\delta u) = J'(x_{kh})(B_h \delta u) - a'_k(x)(B_h \delta u, z_{kh}),$$

where x_{kh} and z_{kh} are the solutions of the discretized state and adjoint equations, respectively, and $a_k(\cdot, \cdot)(\cdot)$ is the discrete analogue of the semilinear form $a(\cdot, \cdot)(\cdot)$. The operator B_h is the extension of the control operator \hat{B} in the discrete state space, with the property that $(B_h \delta u)(t, x_i) = 0$ for all interior nodes x_i of the mesh

\mathcal{T}_h . This choice leads to the fact that the integration in the above representation is done only over the cells adjacent to the boundary. We refer to [KV] for a more detailed discussion of this construction.

Remark 4.2. The use of (at least) quadratic elements for the pressure is essential for the practical realization for $J = J_4$. This is due to the fact that the second derivatives of the pressure are involved in the definition of J_4 and they must be accurately approximated by the numerical scheme in order to reliably compare the results of the four cost functionals.

Remark 4.3. The solution of the underlying state equation is required in the whole time interval for the computation of the dual, tangent, and dual-Hessian equations. If all data are stored, the storage grows linearly with respect to the number of time intervals in the time grid and also linearly with respect to the number of degrees of freedom in the space discretization. This makes the optimization procedure prohibitive for fine discretizations. This difficulty can be overcome by using storage reduction techniques known as “check-pointing” or “windowing”; see, e.g., [Gr], [BGL], and [BMV] for an application to optimization problems governed by parabolic equations.

Remark 4.4. For the examples that will be presented in the following section we use isoparametric biquadratic finite elements for the space discretization of both pressure and velocities. We add further terms to the semilinear form a in order to obtain a stable formulation with respect to both the pressure-velocity coupling and convection dominated flows. This type of stabilization technique is based on local projections of the pressure gradients (LPS method) first introduced in [BB]. In the context of optimal control problems this type of stabilization is analyzed in [RV, BV].

5. Numerical examples. In this section we discuss some numerical examples illustrating the effect of different choices of the cost functional in the context of optimal vortex reduction. For these examples we chose the dG(0) method for time and biquadratic elements for space discretization, as described in the previous section. In the context of the optimization, we use trust region techniques for globalization of the convergence; see, e.g., [NW]. The use of such techniques in the examples described below is necessary, particularly for the optimization of the cost functional J_4 .

We use two different configurations, both based on the computational domain Ω ; see Figure 5.1.

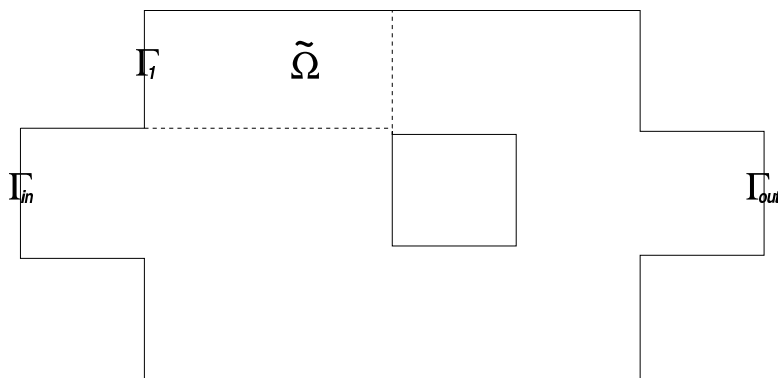


FIG. 5.1. Computational domain.

In both configurations we start with the following uncontrolled situation: We have constant parabolic inflow on Γ_{in} , “no-slip” boundary conditions on $\partial\Omega \setminus (\Gamma_{in} \cup \Gamma_{out})$, and “do nothing” boundary conditions on Γ_{out} (see [HRT]), i.e.,

$$\nu \nabla y \cdot n - p \cdot n = 0 \text{ on } \Gamma_{out}.$$

The flow with Reynolds number $Re \approx 10^3$ is considered on the time horizon $(0, T)$ with $T = 3$. The initial velocity field y_0 is chosen as the solution of the nonstationary Stokes equation on the same configuration after several time steps.

The solution of the uncontrolled state equation for $t = 2.4$ is shown in Figure 5.2. In this figure we observe two primary “vortex regions.”

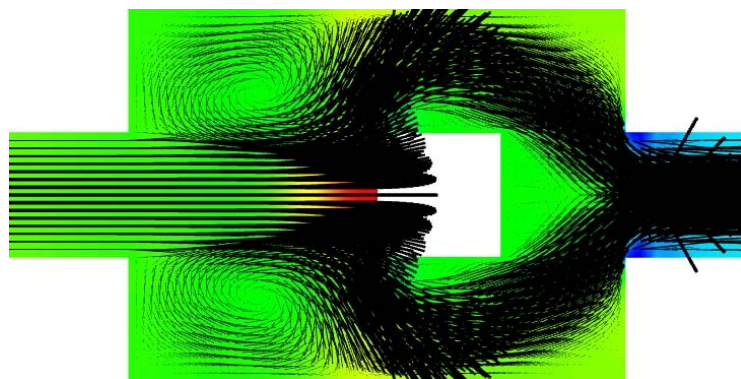


FIG. 5.2. *Uncontrolled flow, $t = 2.4$.*

In our first test we consider Dirichlet control on the part of boundary Γ_1 given as follows: $y = u\hat{y}_1$ on Γ_1 , with a parabolic profile \hat{y}_1 . The control space U is one-dimensional here, i.e., $U = \mathbb{R}$. In the following, we study the dependency of four different cost functionals J_1 , J_2 , J_3 , and J_4 on $u \in [-8; 8]$ with observation region $\tilde{\Omega}$ (see Figure 5.1) and the whole Ω (see Figures 5.3–5.6).

Figure 5.9 We conclude that for the present situation the vortex reduction with the help of these four cost functions leads to very different results. The reduced cost functional seems to be convex for J_1 and J_3 and to have several local extrema for the functional J_4 . In our second configuration we compare the optimal solutions in more detail.

For the second configuration we set the following Dirichlet boundary conditions:

$$y = 0 \quad \text{on } \partial\Omega \setminus (\Gamma_{in} \cup \Gamma_{out}),$$

$$y = g(u) \hat{y}_{in} \text{ on } \Gamma_{in},$$

where \hat{y}_{in} is a fixed parabolic profile and

$$g(u)(t) = (c_0/T) + \sum_{k=1}^n (u_{2k-1} \sin(2\pi kt/T) + u_{2k} \cos(2\pi kt/T)).$$

The control variable u is searched for in the space $U = \mathbb{R}^{2n}$. For this setting we have for all $u \in U$

$$\int_0^T \int_{\Gamma_{in}} y \cdot n \, ds \, dt = c_0 \int_{\Gamma_{in}} \hat{y}_{in} \cdot n \, ds$$

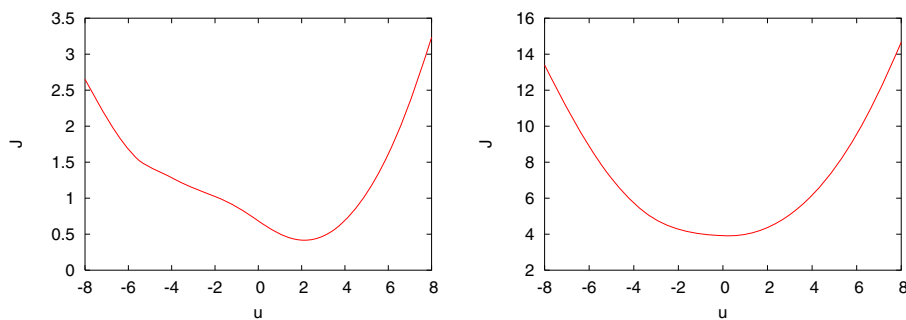


FIG. 5.3. Cost functional J_1 (tracking) for $u \in [-8, 8]$, observation domain $\tilde{\Omega}$ (left) and Ω (right).

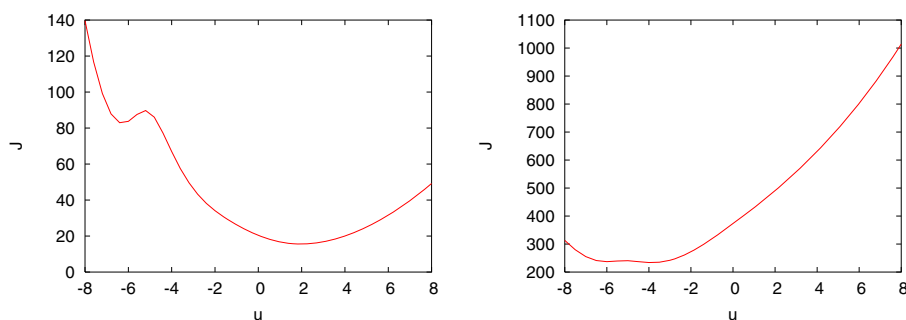


FIG. 5.4. Cost functional J_2 (curl) for $u \in [-8, 8]$, observation domain $\tilde{\Omega}$ (left) and Ω (right).

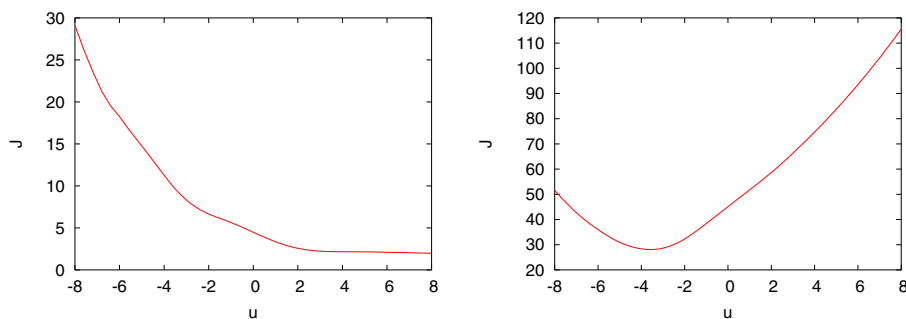


FIG. 5.5. Cost functional J_3 (det) for $u \in [-8, 8]$, observation domain $\tilde{\Omega}$ (left) and Ω (right).

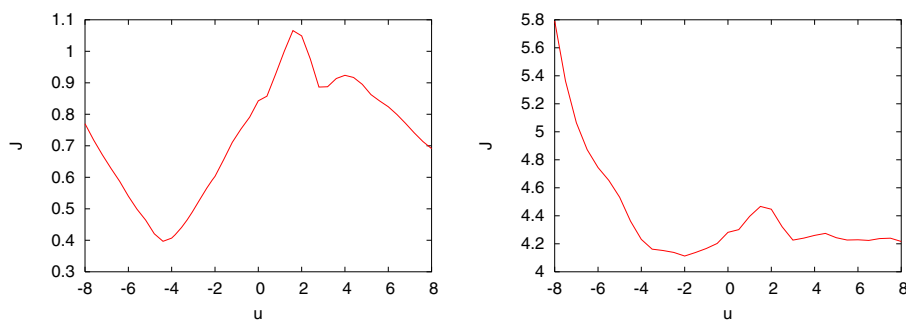


FIG. 5.6. Cost functional J_4 (LKH) for $u \in [-8, 8]$, observation domain $\tilde{\Omega}$ (left) and Ω (right).

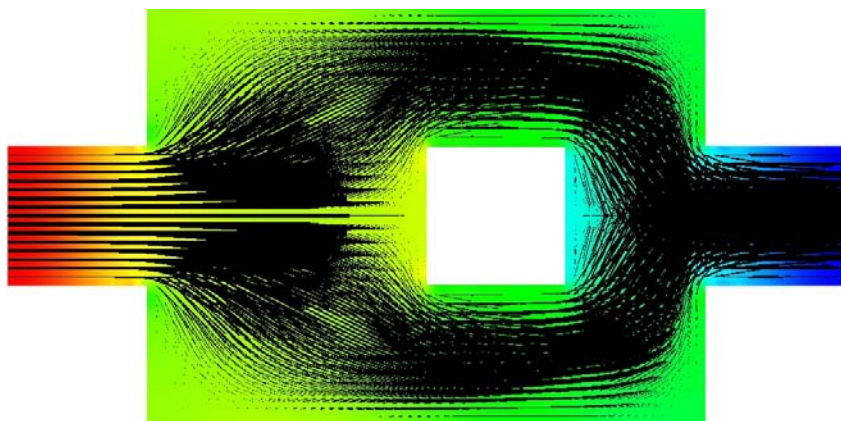


FIG. 5.7. Stokes flow, used as the desired state for the tracking-type functional.

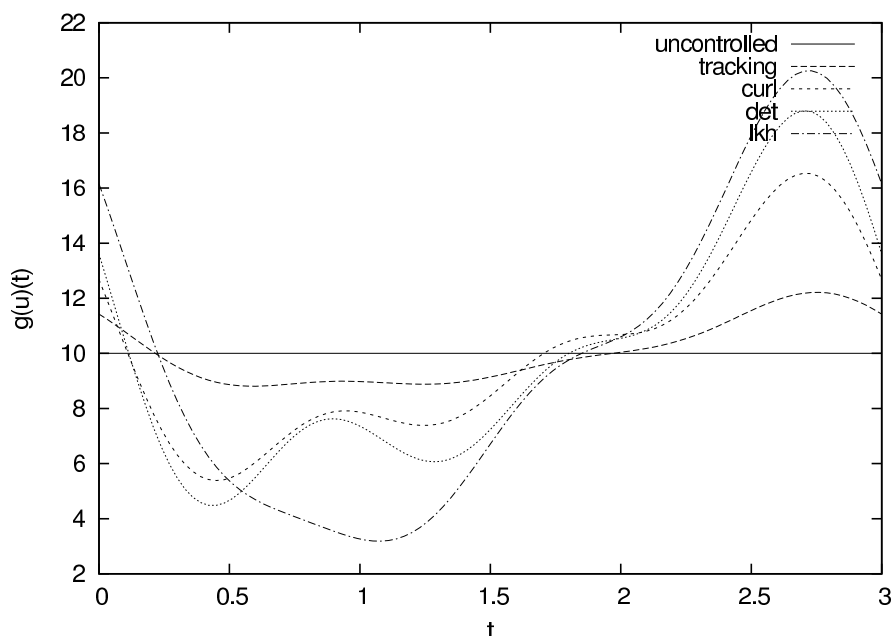


FIG. 5.8. Optimal controls $g(u)(t)$ for four different cost functionals.

independently of u . This condition has the following physical interpretation: The total flux through the inflow boundary in the time horizon $(0, T)$ does not depend on the control action. Thus we aim for the vortex reduction under the constraint that the total flux remains unchanged.

In Figures 5.8 and 5.9 we collect the results for the four cost functionals in the specified configuration. For the tracking-type functional we use the solution of the Stokes equation (see Figure 5.7) as the desired state y_{des} . In Figure 5.8 we show the optimal trajectories $g(u)(t)$ of the controls for the four cost functionals under consideration. In Figure 5.9 we collect the solutions of the state equation for the optimal control u with respect to the four different cost functionals J_1, J_2, J_3, J_4 . It can be noted that from the point of view of graphical vortex representation there is a significant reduction of “vorticity” as we move from J_1 to J_4 .

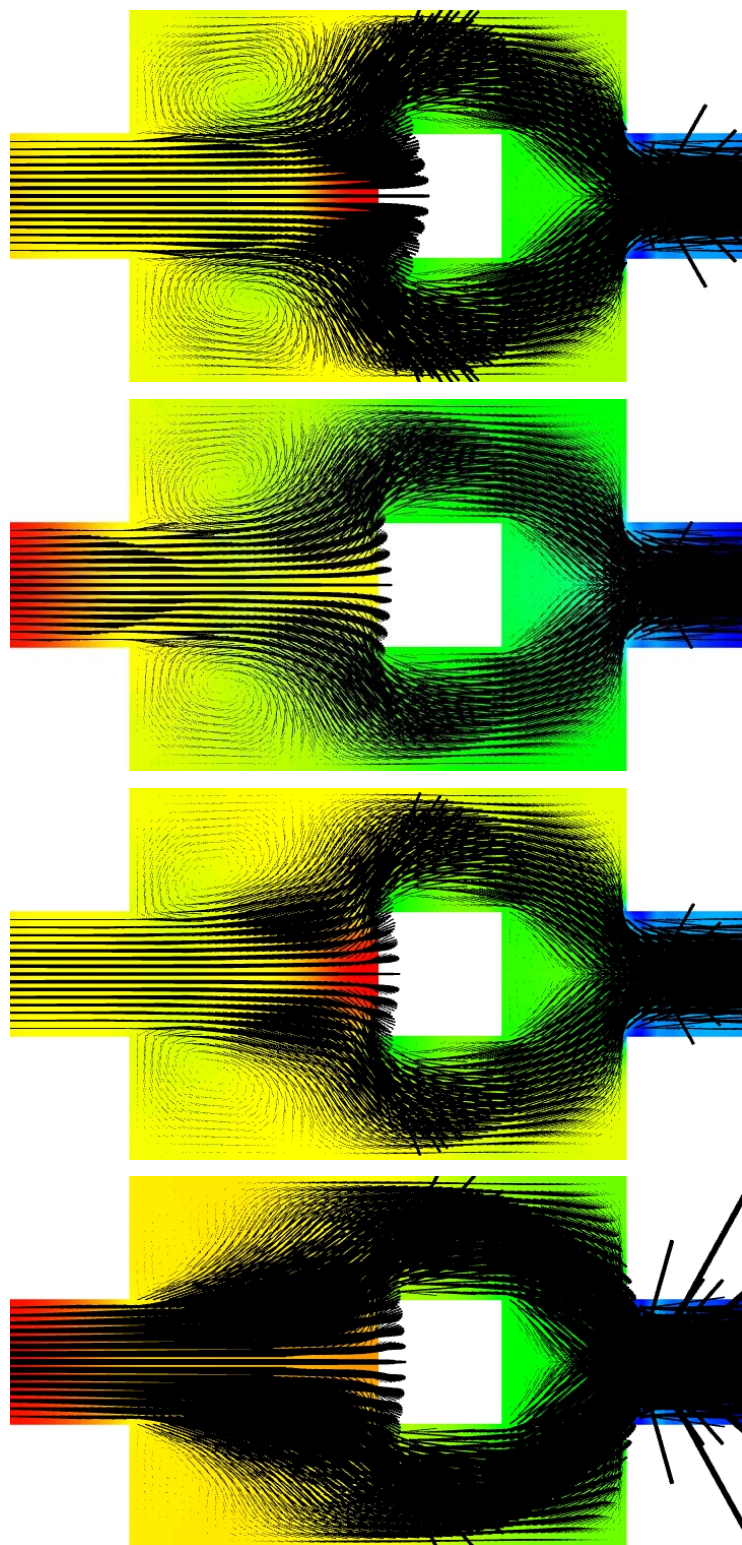


FIG. 5.9. Optimal flow with respect to four different cost functionals (from top to bottom): J_1 (tracking), J_2 (curl), J_3 (det), J_4 (LKH).

TABLE 5.1

Values of the four cost functionals for no control (u^0) and optimal control u^I, u^{II}, u^{III} , and u^{IV} .

	J_1	J_2	J_3	J_4
u^0	0.29608	5.60368	1.76442	0.525471
u^I	0.14346	3.72201	0.77298	0.350834
u^{II}	0.17125	3.52171	0.65593	0.390835
u^{III}	0.35245	3.79186	0.46048	0.422569
u^{IV}	0.18587	4.41063	0.93945	0.181102

In the first configuration, we have observed that the cost functionals J_2, J_3, J_4 may have local minima. Although it is impossible to check numerically, if the computed local minimum is a global one, we can provide the following results confirming our belief that we have found global minima. We denote by u^I, u^{II}, u^{III} , and u^{IV} the optimal solutions for the optimal control problems with the functionals J_1, J_2, J_3 , and J_4 , respectively. Moreover, we denote by $u^0 = 0$ the control which corresponds to the uncontrolled situation. In Table 5.1 we present the values of the four cost functionals J_1, J_2, J_3 , and J_4 for these controls. As expected the smallest value in the first column corresponds to the optimal solution with the cost functional J_1 , i.e., u^I , etc.

6. Appendix.

Proof of Proposition 2.1. We abbreviate $\|\cdot\| = \|\cdot\|_{L^2(\Omega)^2}$, $(\cdot, \cdot) = (\cdot, \cdot)_{L^2(L^2(\Omega))}$, and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L^2(\mathcal{V}_0^*), L^2(\mathcal{V}_0)}$. We shall use the trilinear form

$$c(u, v, w) = \sum_{i,j=1}^2 \int_{\Omega} u_j \frac{\partial u_i}{\partial x_j} w_i dx \quad \text{for } u, v, w \in H^1(\Omega)^2,$$

and the following properties of c :

- (i) $c(u, v, w) = -c(u, w, v)$ for all $u \in \mathcal{V}_0, v, w \in H^1(\Omega)^2$;
 - (ii) $c(u, v, w) = -c(u, w, v)$ for all $u \in \mathcal{V}, v \in H^1(\Omega)^2, w \in H_0^1(\Omega)^2$;
 - (iii) $|c(u, v, w)| \leq \sqrt{2} \|u\|^{1/2} \|\nabla u\|^{1/2} \|\nabla v\| \|w\|^{1/2} \|\nabla w\|^{1/2}$ for all $u, v, w \in H^1(\Omega)$;
- see, e.g., [T].

First, we recall that $H_0^1(\Omega)^2$ can be expressed as

$$H_0^1(\Omega)^2 = \mathcal{V}_0 \oplus \mathcal{V}_0^\perp,$$

where

$$\mathcal{V}_0^\perp = \{(-\Delta_0)^{-1} \nabla q : q \in L^2(\Omega)/\mathbb{R}\},$$

and Δ_0 denotes the Laplacian with homogeneous Dirichlet boundary conditions; see [GR]. The forcing function $f \in L^2(H^{-1}(\Omega)^2)$ can therefore be decomposed as $f = (f_1, f_2) \in L^2(\mathcal{V}_0^*) \times L^2((\mathcal{V}_0^\perp)^*)$, and there exists $q_f \in L^2(L^2(\Omega)/\mathbb{R})$ such that

$$f_2(v) = \langle \nabla q_f, v \rangle_{L^2(H^{-1}(\Omega)^2), L^2(H_0^1(\Omega)^2)} \quad \text{for all } v \in L^2(\mathcal{V}_0^\perp).$$

We abbreviate $\hat{g} = \hat{B}u$ and note that $\hat{g}(0) \in H$ since $\psi_i(0) \in H$ for $i = 1, \dots, n$. The results in [HK], specifically Theorems 1.1 and 2.1, imply the existence of a unique velocity component $y \in W$ of (2.2) in the form

$$y = y_L + y_N,$$

where

$$(6.1) \quad \begin{cases} \langle y_{L,t}, v \rangle + \nu (\nabla y_L, \nabla v) = \langle f_1, v \rangle & \text{for all } v \in L^2(\mathcal{V}_0), \\ \tau y_L = \hat{g} & \text{in } W_\Sigma, \\ y_L(0) = y_0 & \text{in } H, \end{cases}$$

and

$$(6.2) \quad \begin{cases} \langle y_{N,t}, v \rangle + \nu (\nabla y_N, \nabla v) + \int_0^T c(y_N + y_L, y_N + y_L, v) dt = 0 \\ \text{for all } v \in L^2(\mathcal{V}_0), \\ \tau y_N = 0 & \text{in } W_\Sigma, \\ y_N(0, \cdot) = 0 & \text{in } H. \end{cases}$$

Moreover, there exists a constant K_1 , and a continuous function $K_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^+$, both independent of y_0 , f_1 , and \hat{g} , such that

$$(6.3) \quad \|y_L\|_{L^2(\mathcal{V})} + \|y_{L,t}\|_{L^2(\mathcal{V}_0^*)} + \|y_L\|_{C(\mathcal{H})} \leq K_1 (\|y_0\|_H + \|f_1\|_{L^2(\mathcal{V}_0^*)} + \|\hat{g}\|_{W_\Sigma})$$

and

$$(6.4) \quad \|y\|_{L^2(\mathcal{V})} + \|y_t\|_{L^2(\mathcal{V}_0^*)} + \|y\|_{C(\mathcal{H})} \leq K_2 (\|y_0\|_H, \|f_1\|_{L^2(\mathcal{V}_0^*)}, \|\hat{g}\|_{W_\Sigma}).$$

The results in [HK] are for $f = 0$ but can easily be extended to an arbitrary $f \in L^2(H^{-1}(\Omega)^2)$.

To argue continuity of $\hat{g} \mapsto y(\hat{g})$ from W_Σ to W , let $\hat{g}_n \rightarrow \hat{g}$ in W_Σ . Then $\{\hat{g}_n\}_{n=1}^\infty$ is bounded in W_Σ , and by (6.2) there exists a constant K_3 such that

$$(6.5) \quad \|y(\hat{g}_n)\|_W + \|y(\hat{g}_n)\|_{C(\mathcal{H})} \leq K_3 \quad \text{for all } n = 1, 2, \dots,$$

and this estimate holds for $y(\hat{g}_n)$ replaced by $y = y(\hat{g})$ as well. Let us henceforth denote $y^n = y(\hat{g}_n)$ with decomposition according to (6.1) and (6.2) as $y^n = y_L^n + y_N^n$. Further we write $y = y_L + y_N$ for $y = y(\hat{g})$. From (6.3) we have

$$(6.6) \quad \|y_L^n - y_L\|_{L^2(\mathcal{V})} + \|y_{L,t}^n - y_{L,t}\|_{L^2(\mathcal{V}_0^*)} + \|y_L^n - y_L\|_{C(\mathcal{H})} \leq K_1 \|\hat{g}^n - \hat{g}\|_{W_\Sigma}.$$

For $y_N^n - y_N$ we have the equation

$$(6.7) \quad \begin{cases} \langle y_{N,t}^n - y_{N,t}, v \rangle + \nu (\nabla(y_N^n - y_N), \nabla v) \\ + \int_0^T (c(y_N^n + y_L^n, y_N^n + y_L^n, v) - c(y_N + y_L, y_N + y_L, v)) dt = 0 \text{ for all } v \in L^2(\mathcal{V}_0), \\ \tau(y_N^n - y_N) = 0, \\ y_N^n(0) - y_N(0) = 0. \end{cases}$$

Abbreviating $w = y_N^n - y_N \in W_0$ and setting $v = w\chi_{(0,t)}$, where $\chi_{(0,t)} : (0, T) \rightarrow \mathbb{R}$ denotes the characteristic function of $(0, t)$ for $t > 0$, using $w(0) = 0$ we find

$$\int_0^t \left(\frac{1}{2} \frac{d}{ds} \|w(s)\|^2 + \nu \|\nabla w\|^2 + c(y_N^n + y_L^n, y_N^n - y_N, w) + c(y_N^n + y_L^n, y_L^n - y_L, w) \right. \\ \left. + c(y_N^n - y_N, y_N + y_L, w) + c(y_L^n - y_L, y_N + y_L, w) \right) ds = 0.$$

By (ii) this implies that

$$\int_0^t \left(\frac{1}{2} \frac{d}{ds} \|w(s)\|^2 + \nu \|\nabla w\|^2 + c(y^n, y_L^n - y_L, w) + c(w, y, w) + c(y_L^n - y_L, y, w) \right) ds = 0,$$

and hence by (i),

$$\frac{1}{2} \|w(t)\|^2 + \nu \int_0^t \|\nabla w\|^2 \leq \int_0^t (|c(y^n, w, y_L^n - y_L)| + |c(w, y, w)| + |c(y_L^n - y_L, w, y)|) ds.$$

By (iii), (6.5), and (6.6) there exists a constant K_4 independent of n such that

$$\frac{1}{2} \|w(t)\|^2 + \nu \int_0^t \|\nabla w(s)\|^2 ds \\ \leq \sqrt{2} \int_0^t \left(\|\nabla w(s)\| \|\nabla(y_L^n(s) - y_L(s))\|^{\frac{1}{2}} \|y_L^n(s) - y_L(s)\|^{\frac{1}{2}} (\|y_n(s)\|^{\frac{1}{2}} \|\nabla y_n(s)\|^{\frac{1}{2}} \right. \\ \left. + \|y(s)\|^{\frac{1}{2}} \|\nabla y(s)\|^{\frac{1}{2}}) + \|\nabla w(s)\| \|w(s)\| \|\nabla y(s)\| \right) ds \\ \leq K_4 \int_0^t \left(\|\nabla w(s)\| \|\nabla(y_L^n(s) - y_L(s))\|^{\frac{1}{2}} (\|\nabla y_n(s)\|^{\frac{1}{2}} + \|\nabla y(s)\|^{\frac{1}{2}}) \right. \\ \left. + \|\nabla w(s)\| \|w(s)\| \|\nabla y(s)\| \right) ds.$$

Using Young's inequality and absorbing terms we obtain

$$(6.8) \quad \|w(t)\|^2 + \nu \int_0^t \|\nabla w(s)\|^2 \leq \frac{4K_4^2}{\nu} \int_0^t \|\nabla(y_L^n(s) - y_L(s))\| (\|\nabla y_n(s)\| + \|\nabla y(s)\|) ds \\ + \frac{2K_4^2}{\nu} \int_0^t \|w(s)\|^2 \|\nabla y(s)\|^2 ds \leq \frac{8K_4^2 K_3}{\nu} \|y_L^n - y_L\|_{L^2(\mathcal{V})} \\ + \frac{2K_4^2}{\nu} \int_0^t \|w(s)\|^2 \|\nabla y(s)\|^2 ds.$$

By Gronwall's inequality we have with $K_5 = \frac{8K_4^2 K_3}{\nu}$

$$(6.9) \quad \|w(t)\|^2 \leq K_5 \exp \left(\frac{2K_4^2}{\nu} \|y\|_{L^2(\mathcal{V})}^2 \right) \|y_L^n - y_L\|_{L^2(\mathcal{V})}.$$

By (6.6) the right-hand side of (6.9) converges to 0 as $n \rightarrow \infty$ and from (6.8) we deduce that $w = y_N^n - y_N \rightarrow 0$ in W_0 as $n \rightarrow \infty$. Together with (6.6) this implies that $y^n \rightarrow y$ in W which establishes the desired continuity of $g \mapsto y(g)$ from W_Σ to W . Consequently $u \mapsto y(\hat{B}u)$ is continuous from U to W as well.

So far we worked in solenoidal spaces. To introduce the pressure component we first set

$$Y(t) = \int_0^t y(s) \, ds, \quad \hat{b}(t) = y_0 - y(t) - \int_0^t ((y(s) \cdot \nabla)y(s) - f_1(s)) \, ds.$$

We have, using (iii), that $Y \in H^1(H^1(\Omega)^2) \hookrightarrow C(H^1(\Omega)^2)$ and $\hat{b} \in C(H^{-1}(\Omega)^2)$, and

$$\nu(\nabla Y(t), \nabla v) - \langle \hat{b}(t), v \rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2} = 0 \quad \text{for all } v \in \mathcal{V}_0 \text{ and each } t \in (0, T).$$

Then by de Rham's theorem there exists a unique $\hat{P}(t) \in L^2(\Omega)/\mathbb{R}$ such that

$$\nu(\nabla Y(t), \nabla v) - \langle \nabla \hat{P}(t) - \hat{b}(t), v \rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2} = 0$$

$$\text{for all } v \in H_0^1(\Omega)^2 \text{ and each } t \in (0, T);$$

see, e.g., [DL, T]. Since the gradient operator is an isomorphism from $L^2(\Omega)/\mathbb{R}$ into $H^{-1}(\Omega)^2$ and $\hat{b}(t) + \nu \Delta Y(t) \in H^{-1}(\Omega)^2$ we conclude that $\hat{P} \in C(L^2(\Omega)/\mathbb{R})$.

Now we insert the f_2 component of f and define

$$b(t) = y_0 - y(t) - \int_0^t ((y(s) \cdot \nabla)y(s) - f(s)) \, ds \quad \text{and} \quad P(t) = \hat{P}(t) + q_f(t).$$

Then $P \in C(L^2(\Omega)/\mathbb{R})$ and

$$\nu(\nabla Y(t), \nabla v) - \langle \nabla P(t) - b(t), v \rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2} = 0$$

$$\text{for all } v \in H_0^1(\Omega)^2 \text{ and each } t \in (0, T),$$

which is equivalent to the following equality in $H^{-1}(\Omega)^2$:

$$(6.10) \quad y(t) - y(0) - \nu \int_0^t \Delta y(s) \, ds + \int_0^t (y(s) \cdot \nabla)y(s) \, ds + \nabla P(t) = \int_0^t f(s) \, ds.$$

This allows us to introduce pressure p as the (distributional) derivative $p = \partial_t P$.

To argue continuity of $\hat{g} \mapsto P(\hat{g})$ from W_Σ to $C(L^2(\Omega)/\mathbb{R})$, we again consider a sequence $\hat{g}^n \rightarrow \hat{g}$, $P^n = P(\hat{g}^n)$, $y^n = y(\hat{g}^n)$, and $P = P(\hat{g})$, $y = y(\hat{g})$. For a constant K_6 independent of t and n we have from (6.10)

$$\begin{aligned} \|\nabla P^n(t) - \nabla P(t)\| &\leq K_6 \left(\|y^n - y\|_{C(\mathcal{H})} + \|y^n - y\|_{L^2(\mathcal{V})} \right. \\ &\quad \left. + \sup_{\|v\|_{L^2(H_0^1(\Omega)^2)}=1} \int_0^T |c(y^n(t), y^n(t), v(t)) - c(y(t), y(t), v(t))| \, dt \right). \end{aligned}$$

For the last term we have, using (ii) and (iii) and the fact that the sequence $\{\|y^n\|_{L^2(\mathcal{H})}\}$ is bounded, for a constant K_7 independent of n ,

$$\begin{aligned} & \int_0^T |c(y^n(t), y^n(t), v(t)) - c(y(t), y(t), v(t))| \, dt \\ & \leq \int_0^T (|c(y(t), v(t), y(t) - y^n(t))| + |c(y(t) - y^n(t), v(t), y^n(t))|) \, dt \\ & \leq K_7 \int_0^T \|\nabla(y(t) - y^n(t))\|^{\frac{1}{2}} \|\nabla v(t)\| \left(\|\nabla y(t)\|^{\frac{1}{2}} + \|\nabla y^n(t)\|^{\frac{1}{2}} \right) \, dt \\ & \leq \sqrt{2} K_7 \left(\int_0^T \|\nabla(y(t) - y^n(t))\| (\|\nabla y(t)\| + \|\nabla y^n(t)\|) \, dt \right)^{\frac{1}{2}} \\ & \leq \sqrt{2} K_7 \|y - y^n\|_{L^2(\mathcal{V})}^{\frac{1}{2}} \left(\|y\|_{L^2(\mathcal{V})}^{\frac{1}{2}} + \|y^n\|_{L^2(\mathcal{V})}^{\frac{1}{2}} \right) \rightarrow 0 \text{ for } n \rightarrow \infty. \end{aligned}$$

This proves that $\nabla P^n(t) \rightarrow \nabla P(t)$ in $H^{-1}(\Omega)^2$ uniformly in $t \in [0, T]$, and therefore $P^n \rightarrow P$ in $C(L^2(\Omega)/\mathbb{R})$. \square

Proof of Proposition 2.2. We recall the following two additional properties of $c(\cdot, \cdot, \cdot)$.

(iv) For all $u \in H^1(\Omega)^2, v \in H^2(\Omega)^2, w \in L^2(\Omega)^2$ there holds

$$|c(u, v, w)| \leq c_4 \|u\|^{\frac{1}{2}} \|u\|_{H^1(\Omega)^2}^{\frac{1}{2}} \|v\|_{H^1(\Omega)^2}^{\frac{1}{2}} \|v\|_{H^2(\Omega)^2}^{\frac{1}{2}} \|w\|.$$

(v) For all $u \in H^2(\Omega)^2, v \in H^1(\Omega)^2, w \in L^2(\Omega)^2$,

$$|c(u, v, w)| \leq c_5 \|u\|^{\frac{1}{2}} \|u\|_{H^2(\Omega)^2}^{\frac{1}{2}} \|v\|_{H^1(\Omega)^2} \|w\|;$$

see, e.g., [T].

We recall the decomposition

$$L^2(\Omega)^2 = H \oplus H^\perp, \quad H^\perp = \{\nabla q : q \in H^1(\Omega)\};$$

see, e.g., [GR]. The forcing function $f \in L^2(L^2(\Omega)^2)$ can therefore be decomposed as $f = (f_1, f_2) \in L^2(H) \times L^2(H^\perp)$.

We consider the decomposition $y = y_L + y_N$, where y_L fulfills (6.1) and y_N fulfills (6.2), respectively. Due to the fact that $\psi_i(0) \in V_0$ and $\psi_i \in W \cap L^2(H^2(\Omega)^2) \cap H^1(L^2(\Omega)^2)$ for all i , we have

$$\hat{g} = \tau g, \quad \text{with } g \in W \cap L^2(H^2(\Omega)^2) \cap H^1(L^2(\Omega)^2), \quad g(0) \in \mathcal{V}.$$

We consider $w = y_L - g$ fulfilling

$$\begin{cases} \langle w_t, v \rangle + \nu (\nabla w, \nabla v) = \langle f_1, v \rangle + \langle g_t, v \rangle + \nu (\nabla g, \nabla v) & \text{for all } v \in L^2(\mathcal{V}_0), \\ \tau w = 0 & \text{in } W_\Sigma, \\ w(0) = y_0 - g(0) & \text{in } H. \end{cases}$$

Using the regularity of g we obtain that

$$\langle f_1, v \rangle + \langle g_t, v \rangle + \nu (\nabla g, \nabla v)$$

is a linear continuous functional on $L^2(L^2(\Omega)^2)$ and $w(0) \in \mathcal{V}_0$. Therefore using a regularity result for Stokes equations with homogeneous Dirichlet boundary conditions (see, e.g., [DL]), we conclude $w \in L^2(H^2(\Omega)^2) \cap H^1(L^2(\Omega)^2) \cap C(H^1(\Omega)^2)$, $y_L \in L^2(H^2(\Omega)^2) \cap H^1(L^2(\Omega)^2) \cap C(H^1(\Omega)^2)$, and

$$(6.11) \quad \|y_L\|_{L^2(H^2(\Omega)^2)} + \|y_L\|_{H^1(L^2(\Omega)^2)} + \|y_L\|_{C(H^1(\Omega)^2)} \leq C_1 (\|f_1\|_{L^2(L^2(\Omega)^2)} + \|g\|_{H^1(L^2(\Omega)^2)} + \|g\|_{L^2(H^2(\Omega)^2)}),$$

with a constant C_1 dependent on f and g . To argue the corresponding result for y_N , we derive an a priori estimate for y_N in $L^2(H^2(\Omega)^2) \cap H^1(L^2(\Omega)^2) \cap C(H^1(\Omega)^2)$ using the fact that y_L satisfies (6.11). Then, the existence of a solution with asserted regularity can be obtained using a standard Galerkin procedure; see, e.g., [T].

We use $v = \chi_{(0,t)} \Delta y_N$ as a test function in (6.2), where $\chi_{(0,t)}$ is the characteristic function of $(0, t)$, for $t > 0$, and obtain

$$\begin{aligned} \frac{1}{2} \|\nabla y_N(t)\|^2 + \nu \int_0^t \|\Delta y_N(s)\|^2 ds \\ \leq \int_0^t (|c(y_N(s), y_N(s), \Delta y_N(s))| + |c(y(s), y_L(s), \Delta y_N(s))| \\ + |c(y_L(s), y_N(s), \Delta y_N(s))|) ds. \end{aligned}$$

For the first term we obtain, using (ii) and (iv),

$$\begin{aligned} \int_0^t |c(y_N(s), y_N(s), \Delta y_N(s))| ds &\leq c_4 \int_0^t \|y_N(s)\|^{\frac{1}{2}} \|\nabla y_N(s)\| \|\Delta y_N(s)\|^{\frac{3}{2}} ds \\ &\leq \frac{27c_4^4}{4\nu^3} \int_0^t \|y_N(s)\|^2 \|\nabla y_N(s)\|^4 ds + \frac{\nu}{4} \int_0^t \|\Delta y_N(s)\|^2 ds \\ &\leq C_2 \int_0^t \|\nabla y_N(s)\|^2 \|\nabla y_N(s)\|^2 ds + \frac{\nu}{4} \int_0^t \|\Delta y_N(s)\|^2 ds \end{aligned}$$

for a constant C_2 , where we used the fact that $\|y_N\|_{C(H)}$ is bounded according to (6.4). For the second and third terms we have, using (iv), (v), and (6.11),

$$\begin{aligned} \int_0^t (|c(y(s), y_L(s), \Delta y_N(s))| + |c(y_L(s), y_N(s), \Delta y_N(s))|) ds \\ \leq C_3 \int_0^t \left(\|y(s)\|^{\frac{1}{2}} \|y(s)\|_{\mathcal{V}}^{\frac{1}{2}} \|y_L(s)\|_{\mathcal{V}}^{\frac{1}{2}} + \|y_L(s)\|^{\frac{1}{2}} \|\nabla y_N(s)\| \right) \\ \times \|y_L(s)\|_{H^2(\Omega)^2}^{\frac{1}{2}} \|\Delta y_N(s)\| ds \leq C_4 \int_0^t \|y(s)\|_{\mathcal{V}} \|y_L(s)\|_{H^2(\Omega)^2} ds \\ + C_4 \int_0^t \|y_L(s)\|_{H^2(\Omega)^2} \|\nabla y_N(s)\|^2 ds + \frac{\nu}{4} \int_0^t \|\Delta y_N(s)\|^2 ds, \end{aligned}$$

with some constants C_3, C_4 . Absorbing terms we obtain

$$\begin{aligned} \|\nabla y_N(t)\|^2 + \nu \int_0^t \|\Delta y_N(s)\|^2 ds &\leq 2C_4 \|y\|_{L^2(V)} \|y_L\|_{L^2(H^2(\Omega)^2)} \\ &\quad + \int_0^t (2C_2 \|\nabla y_N(s)\|^2 + 2C_4 \|y_L\|_{L^2(H^2(\Omega)^2)}) \|\nabla y_N(s)\|^2 ds. \end{aligned}$$

Using Gronwall's inequality we first infer that y_N is bounded in $C(H^1(\Omega)^2) \cap L^2(H^2(\Omega)^2)$. The boundedness of $y_{N,t}$ in $L^2(L^2(\Omega)^2)$ is then obtained using (6.2). Using arguments similar to those for the introduction of the pressure in the proof of Proposition 2.1, we obtain $p \in L^2(H^1(\Omega))$. In fact $\nabla: H^1(\Omega)/\mathbb{R} \rightarrow H^\perp$ is a homeomorphism,

$$t \mapsto y_t - \nu \Delta y + (y \cdot \nabla)y - f_1 \in L^2(L^2(\Omega)),$$

and

$$(y_t(t) - \nu \Delta y(t) + (y \cdot \nabla)y(t) - f_1(t), v) = 0 \quad \text{for all } v \in H \text{ and a.e. } t \in (0, T).$$

Hence, there is $p_1 \in L^2(H^1(\Omega)/\mathbb{R})$ fulfilling the following equality in $L^2(\Omega)^2$:

$$y_t(t) - \nu \Delta y(t) + (y \cdot \nabla)y(t) + \nabla p_1 = f_1.$$

The second component of the pressure is given through the definition of H^\perp , i.e., $p = p_1 + p_2$, $\nabla p_2 = f_2$. This completes the proof. \square

In order to prove Theorem 2.3, we start with a regularity result for the Stokes equation that we need in what follows.

LEMMA 6.1. *Let $(v, s) \in X$ be the solution of the Stokes equation:*

$$(6.12) \quad \begin{cases} v_t - \nu \Delta v + \nabla s = f & \text{in } (0, T] \times \Omega, \\ -\operatorname{div} v = 0 & \text{in } (0, T] \times \Omega, \\ v(0, \cdot) = 0 & \text{on } \Omega, \\ v = 0 & \text{on } (0, T] \times \partial\Omega, \end{cases}$$

with $f \in L^q(Q)$, $q > d + 2$. Then the following estimate holds:

$$\|\nabla v\|_{L^\infty(Q)} + \|v(T)\|_{L^2(\Omega)} \leq c \|f\|_{L^q(Q)}$$

with a constant c independent of $f \in L^q(Q)$.

Proof. For the proof we introduce the Sobolev space $W_q^{k,l}(Q)$ consisting of functions whose derivatives of order $\leq k$ with respect to x and of order $\leq l$ with respect to t are in $L^q(Q)$. From [S2] we have the following result:

$$\|v\|_{W_q^{2,1}(Q)} \leq c \|f\|_{L^q(Q)};$$

see also [DL]. Using an embedding theorem from [S1] we obtain for $q > d + 2$,

$$\|\nabla v\|_{L^\infty(Q)} \leq c \|v\|_{W_q^{2,1}(Q)}.$$

Moreover, the following estimate is well known (see, e.g., [DL]):

$$\|v(T)\|_{L^2(\Omega)} \leq c \|v\|_W \leq c \|v\|_{W_2^{2,1}(Q)}.$$

This completes the proof. \square

In the following, we formulate two core lemmas for functionals J_1 and J_2 that will be used for proving the existence of solutions to (2.5).

LEMMA 6.2. *For a sequence $\{u_k\} \subset U$, let $(y_k, p_k) \in X$ denote the solutions of the state equation (2.2), and assume that $J_1(y_k) \leq C$ for a constant $C > 0$. Then the sequence $\{u_k\}$ is bounded in U .*

Proof. We introduce prolongation $\psi_i \in W$ of the functions $\hat{\psi}_i \in W_\Sigma$, which define \hat{B} , by means of the Stokes equations:

$$(6.13) \quad \begin{cases} (\psi_i)_t - \nu \Delta \psi_i + \nabla \zeta_i = 0 & \text{in } (0, T] \times \Omega, \\ -\operatorname{div} \psi_i = 0 & \text{in } (0, T] \times \Omega, \\ \psi_i(0, \cdot) = 0 & \text{on } \Omega, \\ \psi_i = \hat{\psi}_i & \text{on } (0, T] \times \partial\Omega. \end{cases}$$

This allows us to define a prolongation $B: U \rightarrow W$ of the control operator \hat{B} by means of

$$Bu = \sum_{i=1}^n u_i \psi_i$$

and the corresponding operator for the pressure $R: U \rightarrow L^2(L^2(\Omega)/\mathbb{R})$ by

$$Ru = \sum_{i=1}^n u_i \zeta_i.$$

Note that the family $\{\psi_i\}$ is linearly independent in W . Next, we set

$$z_k = y_k - Bu_k, \quad r_k = p_k - Ru_k.$$

These variables satisfy the following equations:

$$(6.14) \quad \begin{cases} (z_k)_t - \nu \Delta z_k + \nabla r_k = f - y_k \cdot \nabla y_k & \text{in } (0, T] \times \Omega, \\ -\operatorname{div} z_k = 0 & \text{in } (0, T] \times \Omega, \\ z_k(0, \cdot) = y_0 & \text{on } \Omega, \\ z_k = 0 & \text{on } (0, T] \times \partial\Omega. \end{cases}$$

We proceed by showing that $\{z_k\}$ is bounded in $L^{q'}(Q)$, where

$$\frac{1}{q'} + \frac{1}{q} = 1, \quad q > d + 2.$$

For this purpose we consider the following “dual” equation for an arbitrary function $\xi \in L^q(Q)$: Find $(v, s) \in X$ such that

$$(6.15) \quad \begin{cases} -v_t - \nu \Delta v - \nabla s = \xi & \text{in } (0, T] \times \Omega, \\ -\operatorname{div} v = 0 & \text{in } [0, T] \times \Omega, \\ v(T, \cdot) = 0 & \text{on } \Omega, \\ v = 0 & \text{on } [0, T] \times \partial\Omega. \end{cases}$$

From Lemma 6.1 we obtain

$$(6.16) \quad \|\nabla v\|_{L^\infty(Q)} + \|v(0)\|_{L^2(\Omega)} \leq c \|\xi\|_{L^q(Q)}.$$

Using (6.14) for z_k and (6.15) for v we obtain

$$\begin{aligned} \int_0^T (z_k, \xi) dt &= \int_0^T \{(-v_t, z_k) + \nu(\nabla v, \nabla z_k)\} dt \\ &= \int_0^T \{(v, (z_k)_t) + \nu(\nabla v, \nabla z_k)\} dt + (v(0), z_k(0)) - (v(T), z_k(T)) \\ &= \int_0^T (f, v) dt - \int_0^T (y_k \cdot \nabla y_k, v) dt + (v(0), y_0). \end{aligned}$$

For the second term in the last expression we have

$$\left| - \int_0^T (y_k \cdot \nabla y_k, v) dt \right| = \left| \int_0^T (y_k \cdot \nabla v, y_k) \right| \leq \|\nabla v\|_{L^\infty(Q)} \|y_k\|_{L^2(Q)}^2.$$

Using (6.16) we obtain

$$\left| \int_0^T (z_k, \xi) dt \right| \leq c (\|f\|_{L^2(V^*)} + \|y_k\|_{L^2(Q)}^2 + \|y_0\|_{L^2(\Omega)}) \|\xi\|_{L^q(Q)}.$$

Due to the fact that $J_1(y_k)$ is bounded, we have

$$\|z_k\|_{L^{q'}(Q)} = \sup_{\xi \in L^q(Q)} \frac{\int_0^T (z_k, \xi) dt}{\|\xi\|_{L^q(Q)}} \leq C,$$

with a generic positive constant C . Due to $1 < q' < 2$, this implies that

$$\|Bu_k\|_{L^{q'}(Q)} \leq \|z_k\|_{L^{q'}(Q)} + \|y_k\|_{L^{q'}(Q)} \leq C + \|y_k\|_{L^2(Q)} \leq C.$$

Since B is an injective mapping to $L^{q'}(Q)$, it follows that $\{u_k\}$ is bounded in U . This completes the proof. \square

LEMMA 6.3. *For a sequence $\{u_k\} \subset U$, let $(y_k, p_k) \in X$ denote the solutions of the state equation (2.2), and assume that $J_2(y_k) \leq C$, with a constant $C \in \mathbb{R}_+$. Then the sequence $\{u_k\}$ is bounded in U .*

The proof uses the same techniques as those for Lemma 6.2.

Using Proposition 2.1, Lemma 6.2, and Lemma 6.3 we are now able to prove Theorem 2.3.

Proof of Theorem 2.3. Let $\mathcal{A} \subset X \times U$ be a set of admissible pairs:

$$\mathcal{A} = \{(y, p), u\} \in X \times U : (y, p) \in X \text{ fulfills the state equation (2.2)}\}.$$

From Proposition 2.1 we have that \mathcal{A} is not empty, and due to boundedness from below the functionals J_1 and J_2 , we obtain in both cases the existence of a nonnegative real number J^* with

$$J^* = \inf_{((y,p),u) \in \mathcal{A}} J(y)$$

and a sequence $\{(y_k, p_k), u_k\} \subset \mathcal{A}$ with

$$\lim_{k \rightarrow \infty} J(y_k) = J^*.$$

Therefore $J(y_k)$ is bounded, and using Lemma 6.2 (respectively, Lemma 6.3) we obtain that $\{u_k\}$ is bounded as well. Choosing a subsequence u_{k_l} we have

$$u_{k_l} \rightarrow u^* \in U.$$

We set $(y^*, p^*) = S(u^*)$, and due to Proposition 2.1 we obtain

$$J^* = \lim_{l \rightarrow \infty} J(y_{k_l}) = J(y^*).$$

This completes the proof. \square

Proof of Theorem 2.4. The functional J_3 is well defined and continuous on $L^2(\mathcal{V})$. The reduced cost functional $j_3: U \rightarrow \mathbb{R}$ is defined as $j_3(u) = J_3(S(u))$, where S is the (continuous) solution operator of the state equation (2.2); see Proposition 2.1. Thus, j_3 is continuous as well.

Let $\{u_n\} \subset U$ be a minimizing sequence, i.e.,

$$J^* = \inf_{u \in U, u_a \leq u \leq u_b} j_3(u), \quad j_3(u_n) \rightarrow J^*.$$

Due to the facts that $\{u_n\}$ is bounded and U is finite dimensional, there exists a subsequence converging to $u^* \in U$. Continuity of j_3 completes the proof. \square

Proof of Theorem 3.1. The functionals $J_i: L^2(H^1(\Omega)^2) \rightarrow \mathbb{R}$ ($i = 1, 2, 3$) are continuous. Moreover, for any $y \in L^2(H^1(\Omega)^2)$ and $\psi \in L^2(H^1(\Omega)^2)$, there exist the directional derivatives $J'_i(y)(\psi)$ given by

$$J'_1(y)(\psi) = 2 \int_0^T \int_{\Omega} (y - y_{des}) \psi \, dx \, dt,$$

$$J'_2(y)(\psi) = 2 \int_0^T \int_{\Omega} \operatorname{curl} y \cdot \operatorname{curl} \psi \, dx \, dt,$$

$$J'_3(y)(\psi) = \int_0^T \int_{\Omega} g'_3(\det(\nabla y)) \cdot (y_{x_1}^1 \psi_{x_2}^2 + \psi_{x_1}^1 y_{x_2}^2 - y_{x_1}^2 \psi_{x_2}^1 - \psi_{x_1}^2 y_{x_2}^1) \, dx \, dt.$$

Due to the fact that $g'_3(t) \in [0, 3]$ for all $t \in \mathbb{R}$, we obtain that $g'_3(\det(\nabla y)) \in L^\infty(Q)$ and that

$$J'_i(y) \in L^2(H^{-1}(\Omega)^2), \quad i = 1, 2, 3.$$

The existence of the adjoint velocity $\lambda \in L^2(\mathcal{V}_0)$ with $\lambda_t \in L^{4/3}(\mathcal{V}_0^*)$ follows, e.g., from [HK]. The introduction of dual pressure π can be done as in the proof of Proposition 2.1. Under the assumptions from Proposition 2.2 additional regularity $z \in \mathcal{X}_0$ can be argued as in [HK]. \square

Proof of Theorem 3.2. The reduced cost functional $j: U \rightarrow \mathbb{R}$ is defined as $j(u) = J(S(u))$, where $S: U \rightarrow W$ is the (continuous) solution operator for the velocity component of the state equation (2.2); see Proposition 2.1. The solution operator S is directionally differentiable, and the directional derivative $\delta y = S'(u)(\delta u)$ fulfills the following linearized equation (see, e.g., [HK]):

$$\delta y \in B\delta u + W_0 : \bar{a}'(y)(\delta y, \psi) = 0 \quad \text{for all } \psi \in L^2(\mathcal{V}_0).$$

In the proof of Theorem 3.1 it is shown that the functionals $J = J_1, J_2$, and J_3 are directionally differentiable. Therefore, the reduced cost functional j is directionally differentiable too. A necessary optimality condition for the reduced cost functional is given by

$$j'(u)(\delta u - u) \geq 0 \quad \text{for all } \delta u \in U_{ad}.$$

To complete the proof it remains to show the representation of the directional derivative of j . For this purpose we recall the definition of the adjoint equation, and using the fact that $\delta y - B\delta u \in W_0$ we find

$$\begin{aligned} j'(u)(\delta u) &= J'(y)(\delta y) = J'(y)(\delta y - B\delta u) + J'(y)(B\delta u) \\ &= \bar{a}'(y)(\delta y - B\delta u, \lambda) + J'(y)(B\delta u) \\ &= -\bar{a}'(y)(B\delta u, \lambda) + J'(y)(B\delta u). \quad \square \end{aligned}$$

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