Mathematics, Culture, and the Arts

Emily Rolfe Grosholz

Great Circles

The Transits of Mathematics and Poetry

Springer

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This book is dedicated to my friends Paula Deitz and Marjorie Lee Senechal, with admiration for their work as writers and editors.

Preface

Philosophy is inherently interdisciplinary, because it can reflect on the conditions of intelligibility or meaningfulness of almost anything. A philosopher is thus especially well suited to explore connections among disciplines and explain the import of those linkages. As a philosopher of mathematics, I have urged the use of historical case studies as a complement to logical investigations, in my co-edited volume *The Growth of Mathematical Knowledge*, and then in two monographs: *Representation and Productive Ambiguity in Mathematics and the Sciences* as well as *Starry Reckoning: Reference and Analysis in Mathematics and Cosmology*. While I often draw my historical examples from classical antiquity or the seventeenth century, I also write about algebra, topology, number theory, and logic in the twentieth century, as well as mathematical models in biology, chemistry, and modern cosmology. At the same time, I am a poet and literary critic who tends to approach poetry philosophically. Many of my essays and poems have appeared in *The Hudson Review*, where I have served as advisory editor since 1984, as well as in the *Sewanee Review*, *PN Review*, *Poetry Magazine, Prairie Schooner, Plume, Journal of Mathematics and the Arts*, *Think* Journal, *Able Muse*, *Literary Matters*, *San Diego Reader*, *Journal of Humanistic Mathematics,* and others. Because mathematics and poetry seem inherently and problematically disjoint to many people, I have spent the last 40 years searching for ways to think the two disciplines together, and this book organizes my reflections. In general, I argue that poetry stands in the same relation to the humanities as mathematics stands to the sciences. Both disciplines generate insight by highly concentrated modes of expression in which formal structure is just as important as content in the creation of meaning. Thus, in these disciplines, close attention to form, and to forms in combination, is essential to the interpretation of texts; the philosopher must also balance an appreciation of timeless form with an historian's sense of the temporality of proof and discovery as human actions, and the changing cultural context of poems.

Human understanding hovers between the timeless realm of concepts, propositions, and arguments that stand in inferential relations tracked by logic and rhetoric, and the historical realm in which discoveries are made and projects framed on the basis of earlier results or events, and in light of as-yet-unsolved problems. A name or concept pulls something that exists out of the flux of time, and by imposing the universal on the particular gives it a kind of local immortality. Arguments organize thoughts so that they can be rehearsed and examined; narratives organize human actions so that they can be relived and their meaning reconsidered. A scientific experiment on the one hand and a theatrical drama on the other deliberately represent situations both as having happened once—physically or dramatically real—and as meant to be repeated—universally true. In mathematics and poetry, the tension on this duality is especially strong. The study of mathematical knowledge and poetic knowledge is therefore central to philosophical epistemology; the dialogues of Plato testify to this. My work in the philosophy of mathematics takes its inspiration from the twentieth century European tradition that begins with Poincaré, Hilbert, and early Husserl; I explain mathematical rationality as an interplay between logical necessity and historical contingency. My literary work locates human action and utterance at the crossroads between the constraints of moral law and the fatal accomplishments of history, and the free play of artistic form and anarchic will, always imagining "what if...?"

And as Keats reminds us, truth is not the whole story: we must also acknowledge the importance of beauty. Art (including poetic art) and mathematics characteristically generate beautiful forms that express human action on the one hand, and on the other hand the stable systems and dynamic processes of nature. As a philosopher of mathematics, I have developed a theory of "hybrids," which examines the growth of mathematical knowledge at the intersection of heterogeneous domains. When one domain is brought in to augment the resources of another, each with its own tradition of representation, the result is the combination, superposition, and metamorphosis of a variety of modes of representation that often produces new mathematical entities. This situation in itself calls into question standard accounts of theory reduction and moreover presents striking examples of constructive ambiguity. What William Empson says so brilliantly in his *Seven Types of Ambiguity* about the poets' exploitation of structured ambiguity offered by the semantic field of dictionary definitions also holds true for the mathematician. The ellipse in Proposition XI of Newton's *Principia* must be read as a trajectory, as a figure derived from Euclid and Apollonius, as a dynamic nexus determined by a central force, and (after Leibniz, the Bernoullis, and Euler) as the solution to a differential equation; its internal articulation must also be read as both finite and infinitesimal. The proof of the proposition hinges on Newton's exploitation of the controlled ambiguity of the ellipse.

In poetry, the line embedded in stanzas and organized by rules of meter and rhyme (or studied violations of them) creates a formal counterpoint of superimposed periodicities that deepens and complicates what it means. Thus in a poem a thought is suspended at the end of a line by the white margin even if it is also continued by enjambment and grammar to the next line, or by the logical structure of an argument to the next stanza. This formal constitution of ambiguity, which exploits aural patterns of repeated phoneme and accentual beat, grammatical structure, poetic lineation, metrical conventions, and logical forms, shows that a poem is not just a string of words but a two-dimensional array that composes a rich plurality of modes of representation. When I interpret action, character, and image in (for example) poems of Keats and Housman, I discover that this constructed ambiguity often mirrors the ambiguity of human intention: whenever we act, we are aware of what we might have chosen but in fact did not choose, and those unrealized possibilities remain with us as part of the meaning of what we did. We act at the crossroads of necessity and freedom, and of the visible and the invisible.

This book explores the many ways in which mathematics and poetry may enrich and inform each other directly, but also how they exhibit important analogies as they shape human existence. I did my best to explain the mathematical contexts "all the way down" for a general audience, as I have learned to do in the classroom, and also to evoke the complexity and allure of poetry for students of mathematics and the sciences. Poetry is to the other genres of literature as mathematics is to the sciences: each provides a home ground from which everything begins, the distillate, the seed. The proportion also suggests relations that take us beyond discourse: mathematics is the middle term between natural systems and scientific theory, and poetry is the middle term between human life as we live it and the historical, theological, philosophical, and, yes, even scientific theories we use to understand its meaning, and to confer meaning upon it.

University Park, PA Emily Rolfe Grosholz

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A Life in Mathematics and Poetry

Chapter 1
The House of Childhood

In his book *The Poetics of Space*, Gaston Bachelard talks about the house of childhood, the house we never leave because at first we live in it, and afterwards it lives on in us. The house of childhood organizes our experience, first of all determining inside and outside, and then offering middle terms: the front porch and its steps are a middle term between the house and the town, while the back yard and garden are a middle term between the house and the wild. It organizes what is far away, both because we measure "away" by how far it is from home, how many hours or days of travel. Moreover, the windows of the house let in the distances, the dwindling train tracks, river or road, the fields and forest, even the cloudy-blue or starry heavens: they are set squarely on the walls within the window-frames, as light comes through and we see what is outside. The infinite or limitless is framed or mapped, a kind of compactification. (We'll look into middle terms and compactification later.) The house also organizes time, for what lives in the basement or the attic? We ourselves do not eat or sleep or socialize there, although those rooms are part of the house: it is where we put the past, the discarded and the treasured. It is also where sometimes we put the might-have-been, the unrealized possibles. So, finally, the house invites playing. The play room (the nursery, as one used to call it) with its gate, and the fenced-in part of the back yard that displays and bars the wild, are enclosures where the toys are kept and where children go about their business of imitating the adult activities of building and furnishing houses, admonishing and encouraging their dolls, rushing about on small basketball courts and soccer pitches, setting forth amidst the ceremonies of departure and return, celebrating holidays, those middle terms between time and eternity that punctuate and organize the human year (Bachelard 1969).

This brings us back to a song. It is, to my mind, the most beautiful of the poems in Robert Louis Stevenson's *A Child's Garden of Verses*, "Where Go the Boats?" (Stevenson 1941). As a child, I owned a golden vinyl record with this poem on it, recorded as a song: thus I learned it by heart and of course I can still sing it, and very often do. Many of the poems I know by heart I learned as songs, including and especially poems in other languages.

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Dark brown is the river.
Golden is the sand.
It flows along forever,
With trees on either hand.

Green leaves a-floating,
Castles of the foam,
Boats of mine a-boating,
Where will all come home?

On goes the river
And out past the mill,
Away down the valley,
Away down the hill.

Away down the river,
A hundred miles or more,
Other little children
Shall bring my boats ashore.

The house where I grew up was not located by a river, though there was a stream I loved in the woods, about a twenty-minute walk away, flanked by a magic green door built into a hillside. I seemed to revisit that place decades later at the Paleolithic caves of Altamira, Spain, which one entered through a green door, directly into a hillside; then, about a decade after that, it appeared again whenever I took my second son to painting lessons in the nearby town of Lemont, behind another green door perched on stairs overlooking the banks of our local small river, Spring Creek (the finest wild trout stream in Pennsylvania!).

My old house at 2 Forrest Lane, Strafford, Pennsylvania (Photograph by Frances Skerrett Grosholz)

However, earthbound as it was, my childhood house was close to two other kinds of stream. Just one block down the Old Eagle School Road (the Old Eagle School was founded in 1788, and lies next to a Revolutionary War graveyard), ran the Main Line railroad tracks. All day long we heard the train whistles, and the Doppler Effect lowering the pitch, my first, aural introduction to the sorrow of the Red Shift: why are all those galaxies leaving us? Physics tells us that sound is first of all waves, propagating in the medium of the air, with a certain frequency (how quickly the crests of the waves move past a fixed point) and wavelength (the distance between those peaks). Light too is propagated as waves. The Doppler Effect is produced by a moving source of waves relative to a fixed observer: there is an apparent upward shift in frequency for the observer towards whom the source is approaching (blue shift) because the crests of the waves seem to go past more quickly, and an apparent downward shift in frequency for the observer from whom the source is receding (red shift), when the crests of the waves seem to go past more and more slowly. This effect has been discerned in the light of galaxies.

In the early twentieth century, Henrietta Swan Leavitt, an astronomer working at the Harvard College Observatory, discovered the relation between the absolute luminosity and the period of 'Cepheid variable stars,' which regularly wax and wane in brightness; they then became the 'standard candles' that first allowed astronomers to measure correctly the distance between Earth and very distant galaxies. In 1912, Vesto Slipher, an astronomer at the Lowell observatory in Flagstaff, Arizona (since deserts are the best places for telescopes), used spectroscopy to detect the red shift of the light emitted from distant galaxies, and Edwin Hubble in 1929 put these results together to formulate what is now called Hubble's Law, demonstrating that most galaxies are moving away from us: the universe is expanding. (We'll meet these people again, along with Einstein and the Belgian priest Georges Lemaître, in the cosmological poems discussed at the end of this book.)

In the other direction from my house, a block down the Old Eagle School Road, lay the Lincoln Highway, one of the first transcontinental highways in the United States. Route 30, as it was designated under the auspices of the United States Numbered Highway system established in 1926, ran all the way to San Francisco, and when my parents returned from California where my father had served in the Navy during the Korean War, that was the route they came home on. We had a painting of the cypresses "the sailor wind/ties into deep sea knots," (as Robinson Jeffers wrote) at Point Lobos over our fireplace, and I retained a few fugitive memories of California and the long trip back home, so for me that road always led to California, as well as Exton (where the best ice cream place was), Downingtown (summer camp lay on its outskirts), and Lancaster (where the Amish people at the Farmers' Market came from), points west that seemed far, far away. On my first road trip in high school, I drove my friend Jackie Dee past Lancaster, north to the Ephrata Cloister—which was like going to eighteenth century southwest Germany, as I later discovered—and felt that I had achieved adulthood, navigating past the Pillars of Hercules into unknown waters. But in this chapter we must return to early childhood.

As Tolkien wrote, in one of my beloved books, *The Hobbit*, "The road goes ever, ever on" (Tolkien 1937). I discovered *The Hobbit* in the back of my fifth grade classroom, long before anyone else I knew had heard of Tolkien, so for a while it was my private world, far over the Misty Mountains cold. That was the same year Jackie and my cousin Trish Grosholz and I and some other friends spent every recess writing and rehearsing a play of Louisa May Alcott's *Little Women*, which we put on at the end of the year. (You might recall that Jo March did most of her reading in the attic of her house.) So the Lincoln Highway set up a dialectic with my house, not least because Point Lobos was over the fireplace and my mother's most romantic, and often repeated, memories were of California and Hawaii: she was never able to travel much during most of her short life, except to the New Jersey beaches and to New England where she went to college and still had friends. Her stories were the other side of my father's silences, though he too had a trove of stories, set pieces with all the bitter absurdity of those in Joseph Heller's *Catch-22* (Heller 1961). Drafted in World War II, and then again in the Korean War, my father, Edwin DeHaven Grosholz, spent 7 years of his life crossing the great Pacific again and again in destroyers and tankers, seeking refuge from his terror and displacement in alcohol, at sea. And though he made it back home, like Odysseus, he was often there but not there, sitting in his armchair reading through tome after tome of Naval History and smoking the cigarettes that eventually bore him away again.

Geometry starts with the house and field and town center, as we find it in Euclid's *Elements*, for Euclidean geometry is the study of *figures*, finitely delimited and delimiting. Here is a sampling of his definitions, from Book I of the *Elements*. "3. The extremities of a line are points. 4. A straight line is a line which lies evenly with the points on itself. 5. A surface is that which has length and breadth only. 6. The extremities of a surface are lines. 13. A boundary is that which is an extremity of anything. 14. A figure is that which is contained by any boundary or boundaries." And then he gives us the circle, various triangles, the square, and the oblong (rectangle), the rhombus, and various trapezia (Euclid 1956: 153–154). This is the world of childhood: the yard is a rectangle; the house is a closed figure, a set of rectangles and triangles (the walls and roof) hemming in a cuboid; the lane is a bounded straight line; the center of town is a square.

But the road goes ever, ever on. It is a line that continues indefinitely, like the river, like the train tracks that allow the train to glide so quickly and smoothly, in a straight line at a constant speed (inertial motion!) as the train whistle turns into a lament. So Ella Fitzgerald once sang "The Blues in the Night" like no one else, echoing the Doppler Effect, the very sound of departure, in the minor key and falling inflections of the melody:

…Now the rain's a-fallin'
Hear the train a-callin' "whoo-ee!"
My Mama done tol' me
Ah-whooee-ah-whooee ol' clickety-clack's
Ah-echoin' back the blues in the night…

The house is a finite figure, straight from the pages of Euclid, and thus a figure of finitude; but the road, or river, is not a figure, for “it flows along forever.” Or rather, it is a figure after all, thanks to the linguistic blue shift given to the term ‘figure’ by the ambiguities of English: it is a figure of the infinite.

Ironically, we see a foreshadowing of the expansion of geometry into the infinite in the seventeenth and nineteenth centuries in the last of Euclid’s definitions, which he added to clarify the peculiar status of parallel lines. “23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction” (Euclid 1956: 154). This definition doesn’t limit itself to line segments; it involves co-planar lines that are “produced indefinitely” and yet never meet. So we are invited to think about what happens to a line as it goes on and on. In the seventeenth century, Desargues, inspired by optics and a novel theory of perspective, was one of the founders of projective geometry, and proposed that *all* co-planar lines intersect: parallel lines just intersect at infinity, ‘the point at infinity.’ And Leibniz, following the work of Desargues and Pascal, focused on space itself as an object whose structure is revealed by studying the transformations of figures, and noting what features remain invariant among the transformations: thus one could think of all the conic sections as variants of the circle.

Indeed, in 1679, Leibniz briefly considered the possibility of spherical geometry (the easiest of the non-Euclidean geometries to understand, because navigation on the spherical earth more or less exemplifies it), based on the analogy between all lines in projective geometry intersecting (some in the point at infinity) and all geodesics on a spherical surface intersecting (a geodesic is the shortest distance between two points on a spherical surface and thus the analogue to a straight line in ‘flat’ Euclidean geometry). However, he veered off in another direction, and left the explicit formulation of non-Euclidean geometry on a surface of constant positive curvature to the Hungarian mathematician Janos Bolyai in the nineteenth century, following upon the work of Euler and Gauss (Chemla 1998; Debuiche 2013). Nikolai Lobachevsky worked out the non-Euclidean geometry on a surface of constant negative curvature around the same time, entirely independently. Bernhard Riemann, building on the work of his teacher Carl Friedrich Gauss, came up with the generalized notion of a 2-dimensional surface (generalizable to n-dimensional surfaces) which launches geometry into the realm of topology (Gray 1989).

If only Hitler had been red-shifted or carried away by an eagle, or everyone had decided to stop fighting, in 1940, or 1945, or 1950, and my father could have kept on sailing east past Japan (with all the people in Hiroshima and Nagasaki still there, waving him along) and Korea (with its citizens persuasively renouncing the genocidal Communism of the mid-twentieth century, so that perhaps we wouldn’t have had the war in Viet Nam, and would have invented a more human socialism that would now balance and counter the trends of international capitalism), around the south coast of India and then of Africa, and just come back along a geodesic to my mother and me. My mother used to sing me this lullaby, by Alfred, Lord Tennyson.

Sweet and low, sweet and low,
 Wind of the Western sea.
Blow, blow, breathe and blow,
 Wind of the Western sea.
Over the rolling waters go,
Come from the dying moon, and blow,
 Blow him again to me;
While my little one, while my pretty one, sleeps.

Sleep and rest, sleep and rest,
 Father will come to thee soon;
Rest, rest, on Mother's breast,
 Father will come to thee soon;
Father will come to his babe in the nest,
Silver sails all out of the west
 Under the silver moon:
Sleep, my little one, sleep, my pretty one, sleep.

Like a baby on its mother's breast, one can always dream of the unrealized possibles, while remaining grateful for some of the things that materialize and spiritualize reality, like milk. The house governs the poetics of space (inflected by time—and eventually Riemann's geometry provides a model for Einstein's space-time); the road and river govern a poetics of time (inflected by space—for we must all go home again, whether we can or cannot, in fact or in imagination, sooner or later). And what child does not thrill to the romance of departure, which is after all what eventually he or she prepares to do: depart from the house of childhood, aided and abetted by romance.

We might surmise that music is a middle term between poetry and mathematics. It is also true that the visual arts (from painting to architecture) can serve as a middle term, but I discovered that later in life, on my travels around Europe in my 20s and 30s, and here we are still concerned with childhood. Music is first of all a temporal art: a melody must be sung in time. However, sheer time only sweeps everything along and away; so something must temper and counter the sweep for music to be music, and mathematics helps to explain that tempering and counterpoint. The kind of music in play here is song, which requires a poem; mathematics also helps to explain how the flow of forgettable prose becomes a poem that everyone knows by heart, which is passed along from parent to child.

Music requires a scale; the scale that is the basis of European classical music arrays seven distinct notes (C D E F G A B) between middle C and high C, which define an octave. The octave is the interval between one musical pitch and another which is half (or double) its frequency. Sound, as we noted above apropos the Doppler effect, is composed of sound waves that travel through space with determinate frequency and wavelength. The two notes that define the interval of an octave sound the same to the human ear: they ring together, ring true, due to their closely related harmonics. Even before Euclid, Pythagoras (who died ca. 500 BCE) studied the musical scale in relation to the ratio between the lengths of vibrating strings needed to produce them: two strings of the same length have the same pitch and the interval is unison; if one is exactly half the other, the interval is the octave; and if one is two-thirds of the other, the interval is a perfect fifth. So we make our way up the scale from middle C, and miraculously end up at C again—high C; it is a natural periodicity that maps higher notes to lower notes. We can read it vertically, right off the two staves joined by a brace, the top staff marked by the treble clef and the lower staff by the bass clef, with middle C right in between the two staves. Smaller sopranos (I was one), singing in the church choir or the school choir, took our bearings from middle C, despaired of low C (it was left to the baritones) and tried not to be screechy at high C.

However, if we look again at our musical notation, there is a second kind of periodicity which is horizontal and provided not by nature but by convention: the bar or measure, and the time signature, which is also called the meter signature or the measure signature. We all recognize the locution 'three-four time,' ¾, which tells the musician that each bar contains the equivalent of three quarter notes: it is the time signature we waltz to, and hear that beat (**dah** dum dum) repeated in every bar. Smaller ballroom dancing class students (yes, I was one of them too) did our best not to trip our partners or to trip over our own feet, as they swept across the floor, on and on, would the music never cease? That's to say, the time signature imposes a small periodicity on music (only a bar long); but a song requires something more, and that something comes from the poem, another kind of imposed periodicity.

Four-four time presides over many of the hymns in the Episcopal hymnal that I grew up with as a child, and so does the following metrical pattern: iambic tetrameter/ iambic trimeter/iambic tetrameter/iambic trimeter. Here is one of my favorite examples, which begins a well-known hymn written by Isaac Watts; other stanzas will turn up later on.

Oh God, our help in ages past,
Our hope for years to come,
Our shelter from the stormy blast,
And our eternal home.

What's with all the Greek and Latin? An iamb is a metrical foot (analogous to the musical bar) with two syllables, the first stressed and the second unstressed; this is a good choice for English poets because in the "music" of English we stress certain syllables (a habit inherited from Anglo-Saxon/Old English), and we tend to alternate the stresses as we talk. If you put four iambs together, you get tetrameter, and three iambs, unsurprisingly, trimeter. Five iambs give us blank verse, iambic pentameter! In the composition of a hymn, the alternation of tetrameter and trimeter (united in their iambic natures) is underscored by rhyme: the trimeter lines share one full—though here it is a bit slanted—rhyme (come/home) and the tetrameter lines another (past/blast). These superadded periodicities define the line and the stanza: the word stanza in Italian means 'room.'

This brings us to another kind of horizontal musical periodicity: groups of notes that are recognizable as a melody. Each stanza in the poem by Isaac Watts mentioned above, when it is sung as a hymn, repeats the melody of the first stanza. (So too, throughout a symphony, the signature theme surfaces again and again, each time re-created differently by different instruments and different harmonies and dissonances, which themselves play on the vertical periodicities of our musical system.) Thus, music and poetry keep the river of time from bearing all things away, by turning it into a house composed of stanzas. Or perhaps it is a houseboat? If you are sitting in a little room on a houseboat, sailing along in a straight line at a constant speed, and singing with your eyes closed, it is hard to tell whether you are at rest or moving: more on inertial motion later.

So music is a middle term between mathematics and poetry. The back yard is a middle term between the house and the wild. The attic is a middle term between today and the past. But what is a middle term? Why is it so important for our human efforts to understand the world, and each other, and to find meaning in life? We have inherited two different versions of a middle term, one from arithmetic and one from logic. We owe to Aristotle, and to Euclid, the useful notion of a middle term, though they are both indebted to Plato's analogy of the Divided Line (Plato, *Republic*, Book VI), which we will encounter later.

In his most famous syllogism, AAA-1, Aristotle formally introduces the notion of a valid deductive argument form, with two premises supporting the conclusion.

All M is P
All S is M

—
Therefore, All S is P

If for S, M and P we substitute concepts that make the premises true, the very form of the argument necessitates the truth of the conclusion. We find this in the *Prior Analytics* I.2, 24b 18-20 (Aristotle 1947). This is an astonishing insight, though like arithmetic it formulates and confirms our ordinary experience, when we argue and count: the modality of necessity arises in human discourse and then demands acknowledgment. One of the important features of the AAA-1 Syllogism is that it exhibits M as the middle term, which brings S (the subject term) and P (the predicate term) into rational relation, thus guiding the investigation of the truth of the standard logical proposition 'S is P' to a search for middle terms.

For Euclid, the middle term is re-conceptualized and re-imagined in the theory of proportions, which links the study of number to the study of geometry in Book V of the *Elements* (Euclid 1956). Greek mathematicians carefully segregated the terms that are paired in a ratio: they must be of the same kind. Thus, numbers are paired only with numbers, line segments with line segments, areas with areas. Moreover, all magnitudes must be finite, since Aristotle viewed infinity as a source of contradiction; thus ratios never involve anything resembling infinitesimals or infinites, as inimical to reason. The theory of proportions, however, allows for the comparison of heterogeneous ratios (pairing different kinds of things). In an expression like A:B::C:D we can substitute numbers for A and B, and line lengths for C and D, and assert a similitude, but not an identity: a proportion, for the Greeks, was emphatically not an equation, and a ratio was not a number. In a case like the one just given, there can be no middle term. The theory of proportions thus allows for the discernment and management of irrational magnitudes, like the square root of 2 discoverable on the hypotenuse of the right triangle whose 'legs' (the two other sides of the triangle) are both 1, but it blocks the investigation of infinitesimalistic reasoning, and makes the treatment of fractions and irrational magnitudes as numbers difficult to conceptualize. Those developments must wait for the seventeenth century, though they are prefigured by medieval insights (Sylla 1984)

However, if all the terms involved in the proportion are the same kind of thing, then a Greek mathematician can investigate the case where A:B::B:C and ask, if we know what A and C are, how can we determine B? Here, in a different sense, B is the middle term bringing A and C into rational relation. So the theory of proportions only partly solves the problem of how to understand disparate things: there are proportions that capture similitudes between (for example) relations between numbers and relations between line segments, but there is no middle term to be found between a number and a line segment. Geometry only comes into novel relation with arithmetic in the seventeenth century, thanks to a broad range of conceptual innovations; one might say that the polynomial plays the role of middle term between arithmetic things and geometrical things, but then the notion of middle term has been strongly revised, and depends on the abstract ambiguity of the polynomial. Does the equation $x^2 + y^2 = 1$ stand for an infinite set of pairs of numbers (x, y), or does it stand for a circle, a certain kind of set of points on the plane? It depends on what field you think the polynomial is defined over: is it $\mathbf{R}^2$? Is it the points on the plane? This raises the question of whether you are willing to identify $\mathbf{R}^2$ with the plane, an identification we make so often that we forget to examine it. In *An Introduction to Differentiable Manifolds and Riemannian Geometry*, William M. Boothby reminds us to be more careful, since $\mathbf{R}^2$ stands for a whole spectrum of possible mathematical items: vector spaces, metric spaces, topological spaces, and finally (but with some qualification) Euclidean space. He cautions that we must usually decide from context which one is intended (Boothby 1975: 1–5) We also forget that the implied identification of $\mathbf{R}^2$ with the plane radically revises the notion of number, a process that begins with Descartes' analytic geometry in the seventeenth century, encompasses prolonged debate about the nature of irrational and transcendental numbers, and attains a certain clarification in the work of Cantor in the nineteenth century, which is then clouded a bit in twentieth century debates over the axioms of set theory. That is, we can think of the polynomial as a kind of middle term between arithmetic and geometry only because it introduces a certain ambiguity into the meaning of both number and space. But the ambiguity is not vicious; rather, it is productive.

Many people hope that mathematics provides a setting where they can escape from the shadow of ambiguity, but ambiguity works its magic in mathematics as well as in poetry (Grosholz 2007, 2016). One of my favorite books of literary criticism is William Empson's *Seven Types of Ambiguity*, where he shows in the context of the English literary canon that artless ambiguity cancels out meaning, but artful ambiguity deepens and enhances meaning (Empson 1946/1966). The English language arises as a marriage between Anglo-Saxon/Old English and Norman French, blessed by lavish sprinkings of Latin. Within a period of less than a hundred years, it emerges with astonishing celerity, rapidity and quickness (see?!) from the households of stranded Norman conqueror-courtiers, who did not bring many women with them, and high-born Old English-speaking ladies who wanted to keep up their social position, in a context where literacy was first of all the business of a Catholic church conducting education and communication in Latin. So it is a marriage both figuratively and literally (Baugh and Cable 2012). Thus we have a particularly rich vocabulary where poets can (consciously or unconsciously) delve into the history of the language, that rich soil where etymologies extend their tangled roots downwards and seek the light upwards, to flower as brilliant ambiguities.

Here is an example, drawn from my reflections when some of my poems were translated into French by Alain Madeleine-Perdrillat (Ostovani et al. 2007). I came to realize in a new way how much linguistic and cultural information lies submerged in my language, and so too in my poems. Often, linguistic information includes knowing the first, second, third and *n*th dictionary meanings of a word, and of course a poet especially plays on this ambiguity that arises within the historical etymology of a word. Because English is such a metamorphic language, there are two or three words for everything and also unlimited opportunities for puns (the funny part of ambiguity), as well as accretions and intersections of meaning. One of the poems in the book I created with the artist Farhad Ostovani (Yves Bonnefoy's favorite collaborator in the last 20 years of his life), *Leaves/Feuilles*, was inspired by the olive trees of the Mediterranean, so often planted on terraces that rise up the steep hillsides, trees that often live more than a millennium.

Farhad Ostovani, for Yves Bonnefoy's translation of Giacomo Leopardi's 'A Silvia,' 2006. Lithograph

An olive tree can live a thousand years,
Drawing its silver leaves and oval fruit
From stony terraces, fretting the wind
In registers of sun-inflected shadow.

But we, my love, who count the terraces
Rising to meet the stories of the sky,
Who cultivate the olive groves, who hear
The interruption in the trees as music

And weep responsive to those minor chords,
Can live only a century, no more.
Although I love you, you are just a man,
And the great silver sun is just a star.

When Alain Madeleine-Perdrillat sent me his first draft of a translation, it made me realize the extent of the ambiguity that I (and English) had stored in the word "fretting."

Un olivier peut vivre bien mille ans,
Tirant ses fruits ovales et ses feuilles argentées
De terrasses de pierre, ajourant le vent
Selon les registres d'ombre ensoleillée.

Mais nous, mon âme, qui nombrons les terrasses
S'élevant vers les étapes du ciel, qui cultivons
Les champs d'oliviers, nous qui entendent
L'interruption dans les arbres comme si c'était musique,

Et qui pleurons, sensibles a ces accords mineurs,
À peine pouvons-nous vivre cent ans, pas plus.
Bien que je t'aime, tu n'es qu'un homme seulement,
Et le grand soleil d'argent n'est qu'une étoile.

I did my best to explain to him in a letter: "This poem is built around the 'conceit' that the olive trees are like musical instruments played by the wind. The wind is like the hand that passes over a harp, or like the hand that touches the strings of a violin and the bow that passes over the strings. Indeed, there is an elaborate pun *(jeu de mots*) on the word 'fret' which I am sure cannot be reproduced in French—but the point is to try to make all the double-entendres support a musical metaphor, as well as an architectural metaphor, as they do in English." Here are all the meanings!

Fret: Any of the ridges of wood or metal set across the finger board of a lute or similar instrument which help the fingers to stop the strings at the correct points. The origins of this word, c. 1500, may be from Old French, *frete*, meaning ring or ferule. There is a Middle English verb, *freten*, which means to bind or fasten.

Fret: An intransitive verb meaning to be regretful or to worry. A transitive verb meaning to corrode or wear away. Those who fret may moan, like the wind in the trees, *accords mineurs*. The origin of this word is the Old English word *fretan*, which means to devour, feed upon, consume, often used of monsters and Vikings! It may be related to Old French *froter*, to rub, wipe, beat, or thrash. After 1200, it takes on a figurative use, meaning to worry, consume or vex, and acquires its intransitive use around 1550: to fret oneself, to fret.

Fret: An interlaced, angular design: fretwork, ornamental work consisting of interlacing parts, especially work in which the design is formed by perforation. Also, any pattern of light and dark. This is captured by the French verb *ajourer*, which means to pierce or perforate, so it is not surprising that the French translator would choose this meaning, since it derives from the Old French *frete* (late fourteenth c.) which means interlaced work, trellis work.

However, by choosing the architectural meaning, one loses the musical and emotional meaning, which are key to the poem. The poet, writing the poem, is fretting: the poem is melancholy. The musical ambiguities continue in the words 'registers,' 'inflected,' 'count,' 'interruption,' 'minor chords.' In English, we often talk of music in terms of numbers: any one of a collection of songs or dances is a number. Numbers in the plural may refer to metrical feet or verses in a poem, and musical periods, bars or measures, or repeated groups of notes. I think I also visualized the terraces, and the stories of the sky, as a kind of grand staff, as on the musical page. Interruption evokes rests, in music: the silences, the rests, play an indispensable role in the creation of music; interruptions play an indispensable role in the music of life. The minor chords, of course, are the melancholy music that the wind plays in the trees, and that life plays in us: "Although I love you, you are just a man,/And the great silver sun is just a star." Even stars are mortal. So here was the final version, from *Feuilles/Leaves*.

Un olivier peut vivre mille ans,
Tirant ses fruits ovales et ses feuilles argentées
De terrasses de pierre, faisant geindre le vent
Selon les registres du soleil et de l'ombre.

Mais nous, mon amour, nous qui comptons les terrasses
S'étageant vers les portées du ciel,
Nous qui soignons les oliviers et entendons
Une musique dans les soupirs des arbres,

Et qui pleurons, sensibles à ces accords mineurs,
À peine pouvons-nous vivre cent ans, pas plus.
Bien que je t'aime, tu n'es qu'un homme
Et le grand soleil d'argent n'est qu'une étoile.

What has happened to the child? Here I turn to the poem "Halfway Down," by A. A. Milne, which I also learned from a brightly colored vinyl record and can and do frequently sing.

Halfway down the stairs
Is a stair
Where I sit.
There isn't any
Other stair
Quite like
It.

I'm not at the bottom,
I'm not at the top;
So this is the stair
Where
I always
Stop.

Halfway up the stairs
Isn't up
And isn't down.
It isn't in the nursery,
It isn't in the town.
And all sorts of funny thoughts
Run round my head.
"It isn't really
Anywhere!
It's somewhere else
Instead!"

It is from his collection *When We Were Very Young* (Milne 1961). I remember that stair, a middle term between the upstairs and the downstairs, between my room and the world outside, a place where no adult would ever sit. I used to sit there too, imagining. It had a family resemblance to my windowsill, where I waited expectantly for Peter Pan to appear on the threshold between my house and Neverland, and to the back of my closet, where I supposed that one day the back wall would become a door to Narnia. Thus the imagination turns ordinary places into a middle term between the house and terrifying, seductive, gorgeous fairyland. "It isn't really/Anywhere!/It's somewhere else/Instead!"

So when I first read Keats' "La Belle Dame sans Merci" and Yeats' "Song of the Wandering Aengus," and, years later, Yves Bonnefoy's memoir *L'Arrière pays* (Bonnefoy 2003), I recognized the protagonists immediately, young mortals entranced by the transient light of a shade, the beam that lights up and makes somber whatever it rests upon: "… a glimmering girl/With apple blossoms in her hair,/Who called me by my name and ran,/And faded through the brightening air," with whom one hopes to "… pluck till time and times are done/The silver apples of the moon,/The golden apples of the sun." Using Stith Thompson's celebrated *Motif-Index of Folk-Literature*, we might classify them with Walter Map's King Herla and Washington Irvings' Rip Van Winkle under F377, "Supernatural Lapse of Time in Fairyland." The combined proximity and inaccessibility of the other world does something odd to temporality; often those who briefly stray into fairyland and then return home find that centuries have passed (Thompson 1955–1958). This might remind you of the 'Twin Paradox' that stems from Special Relativity Theory: when one of two identical twins goes off in a spaceship at close to the speed of light, he returns home a few years later to discover that his brother is an old man.

Not only do these tales bend time, they cast a shadow on the real world, the world we think we live in. The shadow is not mere illusion; those moments halfway between, those ambiguous places that confound the usual categories, places where no grownup would ever sit (except perhaps some poets, or some mathematicians), reveal something crucial. Why does Keats' pale knight-at-arms go on loitering there by the lake, where the sedge has withered and no birds sing? Yeats offers a suggestion in another poem, "The Wild Swans at Coole," in which he finally abjures Maude Gonne; and so does Shakespeare, in the two great soliloquies Prospero utters in *The Tempest* (Act 4, scene 1 and Epilogue), where he abjures poetry (Shakespeare 1987). In both poems, we have the earth, the sea/lake, the heavens, envisioned upwards, all evoked in heart-breakingly beautiful language, flawlessly metered by one of the two inventors of blank verse in the second case, flawlessly metered and rhymed in the first case: unforgettable. Yet they renounce, making the words in which they renounce unforgettable, un-forego-able. Spinoza exhorts us to see all things *sub specie aeternitatis*: and this is what great poetry does, even or especially when it describes the passage of time, or when it performs loss, since it insists on the absolute value and meaning of what passes and what is lost, whilst expressing and performing its persistence.

Yves Bonnefoy suggests something similar, in his poem "To the Voice of Kathleen Ferrier," the great English contralto who died tragically in 1953 at the age of 41, which I offer in my own translation (Bonnefoy 2001; Deitz 2013: 106).

All gentleness and irony converged
For this farewell of crystal and low clouds,
Thrustings of a sword played upon silence,
Light that glanced obscurely on the blade.

I celebrate the voice blended with gray
That falters in the distances of singing
As if beyond pure form another song's
Vibrato rose, the only absolute.

O light and light's denial, smiling tears
That shine upon both anguish and desire,
True swan, upon the water's dark illusion,
Source, when evening deepens and descends.

You seem to be at home on either shore,
Extremes of happiness, extremes of pain.
And there among the luminous gray reeds
You seem to draw upon eternity.

And there is the wild swan again, though the woodland paths are dry, drifting "on the still water,/mysterious, beautiful." The act of renunciation engenders lines of poetry that one can only call immortal, as the poet in reverie returns to the house of childhood. So I end this chapter with a poem published not long ago in the *Hudson Review*, which takes us back to Forrest Lane, between Lancaster Pike and Old Eagle School Road and the Main Line Railway.

Where I Went, and Cannot Come Again

That crabapple tree is gone, that used to blossom, no,
To burst like a low budget, pale pink Vesuvius
Halfway between our back door and the neighbors'
Who spoke Italian when the dark-eyed grandmothers appeared.

So too the ash, whose canopy embowered, overshadowed
The lawn with lunar-eclipse shadows, and the small magnolia
Whose open flowers filled up like bowls of alabaster
With April rain, their lips rimmed gold with pollen.

The dogwood's gone I used to climb, the sailor's mast
Blue spruce that lifted almost to the rooftop,
Even the wind-stunted Japanese pine that slanted
Sideways to shelter daylilies, marigolds and tulips.

Even the late-planted holly bush, that lent an air
Of Christmas to the suite of rhododendrons
That screened the front door, and the Christmas trees
Planted year by year against the uproar of the Lincoln highway.

All of them are gone now, and the house is bare,
Mere office space for the adjacent Catholic parish.
Mother, where's the garter snake once hidden in violets,
Periwinkle, hyacinth, all of them blue, all scattered?

Father, where's the porch you built and screened, each nail
Carefully marked and measured in blue pencil? Where's the girl
Who used to slam those doors in helpless anger, and returns
Now to name the vanished trees and close, more gently, the unopened doors?

The title of my poem is borrowed from A. E. Housman, whose poems were turned into songs again and again, though oddly he seemed indifferent to the musicians. (Grosholz 2008). We will encounter it later: it is there and not there, the land of lost content, the blue remembered hills, and, as the first line announces, the record of an air.

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Chapter 2
Music and Hyperspace

Thus, throughout my childhood, I thought a poem was first of all to sing, though I read my way through many of Louis Untermeyer's poetry anthologies unaccompanied, which taught me that even without a melody, poems could tell a story or make an argument or ring true enough to remember. (Of course, I didn't know that the abstract form of argument had become mathematical logic or that the Russian formalists and their structuralist heirs had been trying to make an algebra out of plot and character.) Throughout my teenage years, songs continued to play a central role in my life. I sang in the choir at the church of St. Martin in the Fields in Radnor, Pennsylvania, and learned to read musical notation there, while singing the poems of Robert Bridges, George Herbert, John Masefield, John Milton, Christina Rossetti, Alfred Tennyson, John Greenleaf Whittier, and Isaac Watts in the Episcopalian/Anglican *Hymnal* (Washburn et al. 1940); we also listened attentively to the prose-poetry of the *King James Bible* (Blayney 1769) and the *Book of Common Prayer* (Suten 1945). Later, in my high school choir, we sang Benjamin Britten's beautiful setting of W. H. Auden's "Hymn to St. Cecilia," whom I rediscovered last year at the church of Santa Cecilia, founded in the fourth century, in the Trastevere district of Rome: she was the patron saint of musicians, though martyrdom ended her earthly singing. I listened to Anna Moffo (she came from Wayne—I have often shown visitors exactly where her father's barber shop was), Marian Anderson (she grew up in Philadelphia), and Joan Baez (she starred at the Philadelphia Folk Festival in 1968) sing art songs, poems in Italian, French, German and Spanish turned into music.

In Junior High School I learned to play the guitar, and became an aspiring folksinger. I got out all five volumes of *The English and Scottish Popular Ballads* edited by Francis James Child from the Wayne library (Child 1904). The daughter of a friend of my mother's, Raun Mackinnon, actually became a successful folksinger, and since she was a babysitter for my small brothers, I got to hear her play very often. My aunt Jane, Janette Skerrett Pierce, was one of the people who, inspired by the Philadelphia Folk Festival, created The Main Point, a coffee shop on the Lincoln Highway—five miles down from my house—in the middle of Bryn Mawr, which

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between 1964 and 1981 hosted an astonishing galaxy of folksingers. Even now, as I look back, I can hardly believe the lineup: Bonnie Raitt, Muddy Waters, Tom Rush, Jim Croce, Joni Mitchell, Bruce Springsteen, Son House, James Taylor, Laura Nyro, Sonny Terry and Brownie McGhee, Leonard Cohen, Doc Watson, Dave Von Ronk, and, quite often, Raun Mackinnon. On Sundays, there was always a hootenanny, so I went over with various cousins and my perennial friends Jackie Dee (King), Cinda Agnew (Musters), and Michael Stone, and sang my favorite songs, blues and ballads. How lucky we were!

The poems I wrote during my high school days were thus quite song-like, somewhat mysterious and highly structured: rondeaux, sestinas, villanelles, all in iambic pentameter with mostly end-stopped lines and full rhyme. These forms, which arose in late medieval France between the twelfth and the sixteenth centuries, began as songs or as poems recited to musical accompaniment. (The pantoum, which is rather like the villanelle in its repetition of lines, is by contrast a form borrowed from Malay literature in the early nineteenth century by the French and English.) The two that I still like best are rather cosmological; there is some despair in both of them, corresponding to the mild but unacknowledged chaos of my family's life, I think, balanced against a sky that exhibits form or order: the circle of the horizon, the sphere of the moon like a silver apple, stars that can be counted like pebbles. (There are about ten thousand visible stars.) In both the pantoum and the villanelle, there is a "you" addressed by the authorial voice, which at first seems just to be the general reader, but by the end of the poem becomes the object of direct address and must perhaps be construed as a lover. This the same switch Auden makes in his villanelle that begins "Time can say nothing but I told you so," which explains in part why the poem is so affecting. I know it by heart and it echoes in a villanelle I wrote just this past winter, half a century later, about saying goodbye at airports; this time I even used exactly the same rhymes as Auden. The pantoum was published in *The Mind Carpenter*, the literary magazine at Conestoga Senior High School, Berwyn, Pennsylvania, in 1967–68, with Jackie and Cinda on the editorial board, and including two poems by Michael Stone (he spelled his name 'Myke' at the time). I was the editor-in-chief, which might explain why there are three poems, a prose poem and a translation of a poem by Victor Hugo of mine in that issue; I hope I duly consulted the editorial board.

> Forget, for food improvident and spare
> The richest garden in your memory;
> Bring down the white fruit from the tree
> Which is too heavy for the tree to bear.
>
> The richest garden in your memory
> You shall begin to change, with that despair
> Which is too heavy for the tree to bear,
> The secret vision that you are not free.
>
> You shall begin to change with that despair,
> The moon that skews the circle of the sea.

The secret vision that you are not free
Is no less true for wisdom, no less fair.

The moon that skews the circle of the sea,
Promise of apples hung among the air,
Is no less true for wisdom, no less fair;
Weigh them, silver in your hand, for me.

The villanelle appeared there too, but it soon after became my first real journal publication, in *Poet Lore*, America's oldest poetry journal founded in 1889 and still going strong (Grosholz 1968). The editor was the kindly John Williams Andrews, and my mother's friend Virginia McFarland was Assistant Editor. (Yes, it was an inside job; I am still grateful.)

Because there is no rest for me or you,
We cannot stop to ask the morning why.
I sing when there is nothing left to do.

The light is lapping gently at the dew,
The eastern wind resolves into a sigh
Because there is no rest for me or you.

Tears are the summer solace of a few:
Despair cannot be sated eye to eye.
I sing when there is nothing left to do.

Until the dark dissembles into blue
I count the pebbled stars as they go by,
Because there is no rest for me or you.

Lie still: the wind is soft and very new;
Pretend we yet have time to trust the sky.
I sing when there is nothing left to do.

A song is neither penitent nor true.
(My velvet heart is blind, but will not lie.)
Because there is no rest for me or you
I sing when there is nothing left to do.

What does it all mean? At this point, I am not sure, but I still like the way the poems sound, and I like the way the images evolve alongside the ambiguous "you": the white fruit in the pantoum become Yeats' silver apples of the moon, and the morning in the villanelle turns back into Shakespeare's Juliet's wished-for night that will not end: it was the nightingale, and not the lark.

In junior and senior high school, English classes and mathematics classes were taught as if their subject matters were wholly disjunct. I dutifully discovered nineteenth and twentieth century English and American poetry in one suite of classes, and plowed through geometry, algebra and trigonometry in another other suite. (At my high school, we never got around to infinitesimal calculus, although we were supposed to, due to the lassitude of one of our teachers.) However, there was one literary genre that reliably juxtaposed them, and that was science fiction. My father had fallen in love with science fiction as a teenager, which was odd, since although his father and his brother James were engineers, he wanted nothing to do with mathematics and science. However, love is divine madness. Our house was littered with science fiction magazines and paperbacks; and once I was allowed to take the train from Strafford to Wayne all by myself, I went often to the bookstore on North Wayne Avenue and added to the collection. My father, also inexplicably, subscribed to *Scientific American*, where I got my first glimpses of what lay beyond trigonometry: it was thrilling. How much mathematics Martin Gardner managed to explain in his *Mathematical Games*! And how much topology Tony Phillips managed to convey in the full color diagrams that embellished his celebrated essay, "Turning a Surface Inside Out," and even curled around the cover of the issue in which it appeared (Phillips 1966).

I began to suspect that poetry, story-telling and argument, even unsung, stood in some positive relation to mathematics. The best way to capture my first inklings is to introduce the notion of a 'hyperspace.' (Let us not forget the importance of the Oxford Inklings! Tolkien, C. S. Lewis, and Owen Barfield, inter alia. My Aunt Nell Burke—who was really my first cousin once removed—spent a year in Oxford in the early 1950s before dashing off to Jamaica, and used to see them quite often on their way to or from the Eagle and Child pub. So I know they were real, like Christopher Robin, who was a friend of the brother of a friend of mine at Cambridge, Audrey Glauert, a distinguished electron microscopist and godmother of Andrew Wiles, who is the star of my most recent philosophy book.) The word 'hyperspace' shows up often in science fiction, and typically means an alternative region of space (or space-time), co-existing with our own universe, to which one might have access in order, for example, to travel across the universe at a speed greater than the speed of light. Recall the little girl (me) who sat on the windowsill waiting for Peter Pan to take her to Neverland, or who kept walking into the back of her closet hoping to find the melting wall/doorway to Narnia. Sometimes, as she knew from reading her favorite books, one might gain access to an alternative world, side-by-side with ours, but somehow unnoticed. One need only pay closer attention and get lucky!

Many of my favorite science fiction stories involved just such a toggle: the newsstand where, depending on what coin you laid down for your paper, you would be connected to another time or place; perhaps this is why I always keep the odd pound, euro, shekel, rubel, krone or yen in my pocketbook. There was also the attic in Ray Bradbury's story "The Scent of Sarsaparilla": Mr. William Finch has been spending lots of time up in the attic, his unhappy wife notes. One day he comes back downstairs reeking of sarsaparilla, and another day wearing a brand new, highly outdated suit. He goes on and on about their courtship, 40 years before, and almost seems to be raving: "…you know what attics are? They're Time Machines… Consider an attic. Its very atmosphere is Time. It deals in other years, the cocoons and chrysalises of another age… Wouldn't it be interesting… if Time Travel could occur? And what more logical, proper place for it to happen than in an attic like ours?" (Bradbury

1958: 102–103). She thinks he is crazy, especially when he asks her to go with him (where?) and announces his departure. On the final page of the story, in the dead of winter, when at last she climbs up into the attic to see what he is doing, the attic is empty and the west attic window is ajar. There is a ladder down from the window to the porch roof; she looks out. “Outside the opened frame the apple trees shone bright green, it was twilight of a summer day in July. Faintly, she heard explosions, firecrackers going off. She heard laughter and distant voices. Rockets burst in the warm air, softly, red, white and blue, fading.” Instead of following, she slams the window shut, trapped “in that November world where she would spend the next thirty years” (Bradbury 1958: 107–108). The attic really was the middle term between the past and the present: it was the way out, the way back, to a time when Mr. Finch’s life was explosive, brightly colored, and warm. This idea is developed at length later, in Bradbury’s novel, *Dandelion Wine*; it is not science fiction, but memoir, and the time machine is memory, the stories of grandparents bottled and decanted like dandelion wine to be sipped years later.

But hyperspace turns up in a more technical sense in Isaac Asimov’s famous story “The Last Question,” first published in *Science Fiction Quarterly* in November 1956 (Asimov 1974: 157–169). Inspired by Einstein’s theories of Special Relativity and General Relativity, which invoke a four dimensional space-time, writers of science fiction in the 1930s (when my father discovered them) began to use the term hyperspace to mean both a realm beyond but adjacent to our world and a fourth or fifth dimension, thus combining the folk tradition of fairyland and the new physics. Isaac Asimov used the idea in his *Foundation* series, which he wrote during World War II, in West Philadelphia while he was working as a civilian at the Philadelphia Navy Yard’s Naval Air Experimental Station, before going on to finish his Ph. D. in Biochemistry at Columbia. It was first published as a series of eight short stories in *Astounding Magazine* between May 1942 and January 1950, and thereafter in three separate volumes between 1951 and 1953 by Gnome Press; the *Foundation Trilogy* won the Hugo Award for “Best All-Time Series” for science fiction and fantasy in 1966, over Tolkien’s *The Lord of the Rings*, to Asimov’s modest astonishment (Asimov 1951, 1952, 1953). He added sequels to the series between 1981 and 1993, inspiring and interacting with the writers of *Star Trek* in the 1980s and 1990s, and leaving his mark on *Star Wars*, where, as we all know, star ships travel faster than the speed of light to get across the galaxy by detouring through the alternate dimension of hyperspace. We are reminded of this interesting fact on the Star Wars Wiki, endearingly named Wookieepedia.

According to Asimov, the premise of the trilogy was based on ideas from Edward Gibbon’s *History of the Decline and Fall of the Roman Empire*, not accidentally one of the Great Books. (See next chapter: Asimov was educated at Columbia University just as the Great Books program was being promoted by John Erskine, Robert Maynard Hutchins, Mortimer Adler, Stringfellow Barr, and Jacques Barzun, inter alia.) The main plot line of *Foundation* is that a mathematician, Hari Seldon, is trying to save the galaxy from imminent catastrophe, by gathering a “foundation” of talented people who will combine mathematics and moral wisdom, extending and preserving human knowledge in a way that will be effective, reducing the “dark ages” from 30,000 years to a mere millennium. Keep in mind that Asimov’s parents were immigrant Russian Jews who moved to Brooklyn from Smolensk in the 1920s, and that World War II decimated the planet. Somehow, Asimov picked up the idea that the Trivium and Quadrivium might help save human beings from their worst selves and help them be their best selves, but of course I didn’t know that at the time; again, see next chapter, where I go off to the University of Chicago.

To make my point about hyperspace with more concision, however, I turn to the story “The Last Question,” which is only 12 yellowed pages in my copy of *The Best of Isaac Asimov*. It flagrantly violates Aristotle’s recommendations for plot construction, especially in the Neoclassical form of the ‘three unities’: a story should involve no more than a single action, with minimal subplots, that takes place in no more than a single day and in a single place. Indeed, Asimov’s story begins in 2061 and ends ten trillion years later, and its cast of characters come from different galaxies, meeting all over the universe. The story is unified in a certain sense by a central character faced with a recurrent question: can entropy be reversed? However, the character is a computer who cannot in fact answer the question, due to insufficient data. It begins as Multivac in the first episode, a self-adjusting, self-correcting miles-long computer with so many relays and circuits that it had “grown past the point where any single human could possibly have a firm grasp of the whole” (Asimov 1974: 157). It has just figured out how to convert sunlight into usable energy: no more need for coal and uranium.

In the second episode, it has been dispersed into Microvacs, like the one that belongs to a family traveling through hyperspace, “computing the equations for the hyperspatial jumps” on their way to another planet in another galaxy. “It was a nice feeling to have a Microvac of your own and Jerrodd was glad he was part of his generation and no other. In his father’s youth, the only computers had been tremendous machines taking up a hundred square miles of land. There was only one to a planet. Planetary ACs they were called. They had been growing in size steadily for a thousand years and then, all at once, came refinement. In place of transistors, had come molecular valves so that even the largest Planetary AC could be put into a space only half the volume of a spaceship. Jerrodd felt uplifted, as he always did when he thought that his own personal Microvac was many times more complicated than the ancient and primitive Multivac that had first tamed the Sun, and almost as complicated as Earth’s Planetary AC (the largest) that had first solved the problem of hyperspatial travel and had made trips to the stars possible” (Asimov 1974: 160–161).

In the third episode, it is twenty thousand years later. Human beings no longer die, and two different people, from two different galaxies, are compiling a report to the Galactic Council. The price of immortality is that the universe is filling up, quickly: our galaxy has been filled in fifteen thousand years, and now the population of the universe doubles every 10 years. Galaxies of individuals must be moved from one galaxy to the next, which takes a lot of energy: “Our energy requirements are going up in a geometric progression even faster than our population. We’ll run out of energy even sooner that we run out of Galaxies." This raises the question, for the third time, whether entropy can be reversed. "He stared somberly at his small AC-contact. It was only three inches cubed and nothing in itself, but it was connected through hyperspace with the great Galactic AC… It was a little world of its own, a spider webbing of force-beams holding the matter within which surges of sub-mesons took the place of the old clumsy molecular valves. Yet despite its sub-etheric workings, the Galactic AC was known to be a full thousand feet across" (Asimov 1974: 163–164). And Galactic AC can talk; it has a thin and beautiful voice.

In the next scene, human minds have left their immortal bodies behind ("back on the planets, in suspension over the eons"), and drift freely in space, marveling at the stars and sometimes encountering each other. Wherever they are, they can ask the Universal AC questions, "for on every world and throughout space, it had its receptors ready, and each receptor led through hyperspace to some unknown point where the Universal AC held itself aloof." A human mind that had gotten within sensing distance of Universal AC "reported only a shining globe, two feet across, difficult to see." How could it be so small? "Most of it… is in hyperspace. In what form it is there I cannot imagine" (Asimov 1974: 165). The two minds ask where humankind originated, and are distressed to learn that the Sun has gone nova and turned into a white dwarf, obliterating the place where it all began. The question of entropy comes up again, and again the great computer, present but outside, there but not there, has no answer. The next to last scene takes place a hundred billion years in the future; all the human souls have merged into one, and the Cosmic AC has shifted entirely into hyperspace, "made of something that was neither matter nor energy. The question of its size and nature no longer had any meaning in any terms that Man could comprehend." Yet Man and Cosmic AC still have conversations, which include the question about entropy and yet another admission of failure. And last, ten trillion years later, all the stars have died, space is black, there are no more space and time, energy and matter, and Man has fused with AC: only AC exists, in hyperspace. AC goes on existing just for the sake of that last question, which it cannot answer. "All collected data had come to a final end. Nothing was left to be collected. But all collected data had yet to be completely correlated and put together in all possible relationships. A timeless interval was spent doing that. And it came to pass that AC learned how to reverse the direction of entropy. But there was now no man to whom AC might give the answer of the last question. No matter. The answer—by demonstration—would take care of that too. For another timeless interval, AC thought about how best to do this. Carefully, AC organized the program. The consciousness of AC encompassed all of what had once been a Universe and brooded over what was now Chaos. Step by step, it must be done. And AC said, "LET THERE BE LIGHT!" And there was light…" (Asimov 1974: 167–169).

I remember reading this story for the first time, half a century ago, and getting to the end and almost dropping the book on the floor. Genesis 1.3, that passage I had heard so many times, echoing from the end of a twelve page science fiction story with the conviction of scripture, and with the mantic poetry of the King James Bible: "In the beginning God created the heaven and the earth. And the earth was without form, and void; and darkness **was** upon the face of the deep. And the Spirit of God moved upon the face of the waters. And God said, Let there be light: and there was light. And God saw the light, that **it was** good: and God divided the light from the darkness. And God called the light Day, and the darkness he called Night. And the evening and the morning were the first day." The astonishing insight, or rather hope, that Asimov offers is that the Divine might emerge from human intelligence, if we put our minds together. (We will see the same hope in Frederick Turner's *Apocalypse*, though oddly his computer is a woman, not a man, as Asimov's is by implication.) And so too, we see that the alter-world, hyperspace, which is distinct from but always at the edge of ours, is divine, numinous, shining with unearthly fire. As in Ray Bradbury's "The Scent of Sarsaparilla," the place which is not a place, which is there and not there, at the edges, is sometimes the past. As we attain the distant future at the end of Asimov's story, we suddenly see, or rather hear, the beginning of everything, the childhood of the world, the most ancient past; and the voice that sings it into being is the voice of God.

While Relativity Theory posits a four dimensional space-time, String Theory proposes a much larger number of extra dimensions, as well as 'strings' that are so small that they are in principle undetectable, which then makes the theory rather hard to test empirically. (As the resident Humanist, I belong to the cosmology group at Penn State, the Institute for Gravitation and the Cosmos; we tend to prefer Loop Quantum Gravity to String Theory.) In order for hyperspace, and travel through hyperspace, to seem plausible in science fiction, authors often refer rather obliquely to Relativity Theory or to String Theory, but despite the promising extra dimensions, neither theory actually supports a model for traveling faster than the speed of light. It is a fictional technology.

There is also a formally defined concept of hyperspace in topology, which is both a horse of a different color, and another kettle of fish, so this is a good moment to define a topological space; topology dawned on me in 1966, in the article from *Scientific American* I mentioned earlier, and will discuss in the next section. Moreover, saddles and pots play a central role in topology! Well, not really, but spaces of constant negative curvature and Klein bottles (certain non-orientable two-dimensional surfaces) do, which are pretty close, though since neither one of them can be embedded in 3-dimensional Euclidean space, I can't convince you by showing you pictures of them.

We start with the Pythagorean Theorem. You remember proving it in high school geometry, I hope, but if not, you can look at Scott Buchanan's clear exposition in Chapter 2, "Figures," of *Poetry and Mathematics*, a book we will discover in the next chapter, when I go to college (Buchanan 1929/1962). His account may prompt you to get Book I of Euclid's *Elements* out of the library, read through all two hundred (annotated) pages until you get to Proposition 47, and see how many earlier theorems show up in the proof of that thrilling truth that "In right-angled triangles, the square on the side subtending the right angle [the hypotenuse] is equal to the squares on the sides containing the right angle" (Euclid 1956: Vol. I, 349–368).

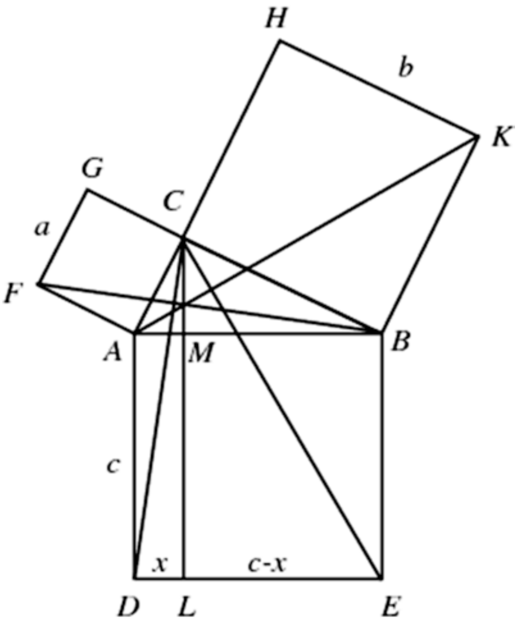


Diagram for Proposition 47, Book I, Euclid's *Elements*

If then we fast-forward almost two thousand years later, as of course Asimov has encouraged us to do, you will find that although Descartes does not give him proper acknowledgement, his analytic geometry is deeply indebted to Euclid. The Pythagorean theorem underlies the Euclidean norm in analytic geometry, where the Euclidean plane has become, or is about to become, a two-dimensional vector space, and where the norm furnishes the rules that govern the operations of linear algebra. A normed linear space is a vector space over the reals (or, later, the complex numbers) with a real-valued function denoted by $\| \ \|$ that satisfies the following conditions, for all vectors a and b and all scalars λ:

$$\|a\| \geq 0 \,\text{and}\, \|a\| = 0 \leftrightarrow a = 0$$

$$\|\lambda a\| = |\lambda| \ \|a\|$$

$$\|a + b\| \leq \|a\| + \|b\|$$

($|\lambda|$ means the absolute value of the scalar λ: the absolute value of a real number is just its magnitude, abstracting from its sign, positive or negative) (Singer and Thorpe 1967: 37).

In turn, even later in the work of Fréchet in 1906, this definition leads to a more abstract definition, that of a metric space. A metric space is defined as a set S given together with a function ρ: $S \times S \rightarrow \mathbf{R}^+$ (the non-negative reals) such that for all s_1, s_2 and s_3 in S,

$$\rho(s_1, s_2) = 0 \text{ if and only if } s_1 = s_2$$

$$\rho(s_1, s_2) = \rho(s_2, s_1)$$

$$\rho(s_1, s_3) \leq \rho(s_1, s_2) + \rho(s_2, s_3)$$

In the case of the geometric plane used as the basis for analytic geometry, the canonical metric that defines the distance between two points (x_1, y_1) and (x_2, y_2) is (thank you, Euclid!) the square root of the sum of $(x_2 - x_1)^2$ and $(y_2 - y_1)^2$. However, there are two other useful metrics that immediately come to mind; while they produce distinct metric spaces, it turns out that for studying certain properties of (for example) the Euclidean plane, the three metrics are equivalent. This led late nineteenth century mathematicians to define a more abstract structure, that of a topological space. This generalized definition allows for the assignment of distances between two elements in an astonishing variety of (metric and even non-metrizable) topological spaces.

So let us consider the Euclidean plane as a topological space. We think of it as a set of points S, and then choose a subset of all the subsets of S, $T \subset 2^S$, satisfying certain conditions: the empty set and S itself are in T, all finite intersections of members of T are again in T, and so are all arbitrary unions. Then we call T a collection of 'open sets' on S and the pair (S, T) is a topological space. The metric just described above, the canonical metric on the Euclidean plane, yields a topological space in which T consists of all the unions of 'open' circles on the plane; an open circle consists of the interior points, but not the boundary, of the circle. The boundary of the circle consists of all the limit points of the open set, so that we define the open set together with all its limit points to be a 'closed set.' The complement of an open set is a closed set, and the complement of a closed set is an open set. Due to this duality, a topological space can also be defined by specifying a collection of 'closed sets,' call them T', satisfying the following conditions: the empty set and S itself are in T', all finite unions of members of T' are again in T', and so are all arbitrary intersections (Singer and Thorpe 1967: 4–9).

The definition of a topological space raises the question whether a given topological space is rich (but not too rich) in open sets. For example, we can give any set what is called the 'discrete topology' by stipulating that $T = 2^S$, the power set of S; however, in a sense this is 'too many' open sets, because it yields a rather trivial and uninformative topology. This possibility does show, however, that any set whatsoever can be thought of as a topological space. You may have noticed that mathematicians have the habit of referring to almost everything—except a category—as a set; they also have the habit of referring to almost everything as a space, which seems odd, especially if they're talking about numbers. However, thinking of number systems as topological spaces reveals important features that often drive research in number theory; and the same could be said for algebraic varieties and research in algebraic geometry. If there are enough open sets to separate points, the space is called Hausdoff; if there are enough open sets to separate closed sets, the space is called normal (Singer and Thorpe 1967: 23–27).

Before we turn back to hyperspace, as a concept in topology, I will try your patience with two more definitions; but as you will see, they come in handy. (1) What corresponds to a structural isomorphism in topology is called a homeomorphism. We say that a function from one topological space to another, $f: S \to V$, is continuous if the inverse images of open sets are open; f is a homeomorphism if f is a one-one correspondence and both f and its inverse are continuous. (2) A basis $\mathscr{B}$ for a topology T on S is a subset of 2^S that satisfies these conditions: the empty set is in $\mathscr{B}$, the union of all the sets in $\mathscr{B}$ is S, and if two distinct sets are in $\mathscr{B}$, their intersection is the union of some subset of the sets in $\mathscr{B}$. Here is a good example of a basis: If (S, ρ) is a topological space that is a metric space with the canonical metric, then the set of all 'balls,' (on the plane, they would be all circular open sets) with center s_0 and radius a, where s_0 is an element of S and a is a nonnegative real number,

$$\left[s \,\varepsilon\, S | \rho\left(s, s_0\right) < a \right]$$

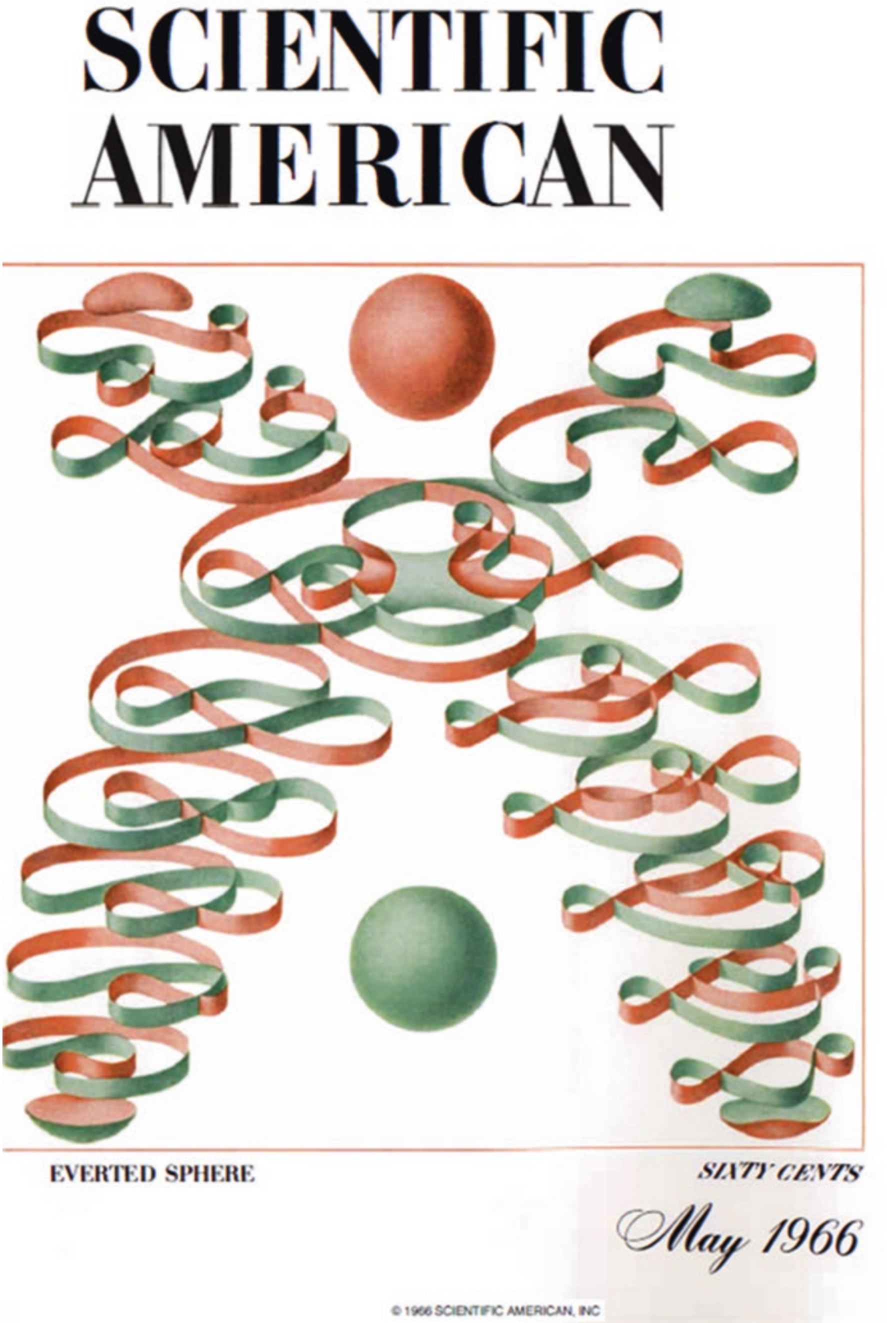
is a basis for a topology on S (Singer and Thorpe 1967: 6–12). A metric space turned into a topological space this way is Hausdorff and normal.

Now, what about hyperspace? A hyperspace, sometimes called a space equipped with a hypertopology, is a topological space that consists of the set $CL(X)$ of all closed subsets of another topological space X. Because it is given a topology that ensures that the canonical map is a homeomorphism into its image, a copy of the original space X, with the correct topological structure, is located inside the hyperspace $CL(X)$. One important application of this idea is captured by Whitney's Theorem, when the topological spaces are manifolds: it tells us when one manifold can be embedded in another manifold. But what is a manifold? The sphere and the torus are special kinds of topological spaces called 'manifolds' which topologists are especially fond of, perhaps because, unlike those pesky spaces of constant negative curvature and the Klein bottles, they can be embedded nicely in 3-dimensional space and so lend themselves to pictures. Moreover, the torus and the sphere (not to be confused with the tortoise and the hare) are smooth differentiable manifolds, a concept we owe in part to Bernhard Riemann, the nineteenth century German mathematician who first named and studied what we call a Riemann surface (Singer and Thorpe 1967: 97–136).

A manifold can be of any dimension, but here we are mostly interested in 2-dimensional surfaces. A 2-dimensional manifold is a Hausdorff space with additional structure, a set of maps Φ such that for each $s \,\varepsilon\, S$ there is a function $\varphi_s \,\varepsilon\, \Phi$ that maps some open set containing s homeomorphically into an open set in $\mathbf{R}^2$. That is, locally though not globally, such a manifold is just like the Euclidean plane—locally, it can be treated 'linearly,' as if it were flat. Obviously, $\mathbf{R}^n$ is locally Euclidean, as well as the n-dimensional sphere, $\mathbf{S}^n$; more surprisingly, so too is n-dimensional projective space, $\mathbf{P}^n$, the space of all lines through 0 in $\mathbf{R}^{n+1}$, because $\mathbf{S}^n$ is a 'covering space' for $\mathbf{P}^n$ (more on that later). Since a manifold can be linearized locally, the problem for mathematicians is how to move systematically from the local situation to a global understanding, so as to extend some version of the nice properties that follow from linearization to the manifold as a whole. In order to get the maps to overlap with each other in a way that accommodates this extension, we must add further conditions governing what happens on the overlap: the inner product on the tangent spaces must be well behaved. (More on inner products and also the interesting dualities that arise with respect to tangent and co-tangent spaces later.) The manifold is called a $\mathbf{C}^0$-manifold if the mappings overlap in a way that is continuous; $\mathbf{C}^k$ if all partial derivatives of order $\leq k$ exist and are continuous; $\mathbf{C}^\infty$ if all partial derivatives of all orders exist and are continuous; and $\mathbf{C}^\omega$ if it is real analytic. The sphere and torus, as just noted, are C^∞ manifolds known as 'smooth differentiable manifolds.' This means that on smooth differentiable manifolds, despite their curviness, we can define local notions of angle, length, surface area and volume, so that certain important global quantities can be obtained by integrating local contributions (Singer and Thorpe 1967: 99–104).

The Whitney Embedding Theorem tells us that any smooth, real, Hausdorff m-dimensional manifold with a countable basis can be smoothly embedded in the real $2m$-dimensional space $\mathbf{R}^{2m}$, the hyperspace, if $m > 0$. Hassler Whitney wrote his dissertation at Harvard with George D. Birkhoff, whom we will meet in the next chapter, and then held positions at Harvard and the Institute for Advanced Study at Princeton, whilst marrying three times and climbing mountains, in honor of his great uncle Josiah Whitney, for whom Mount Whitney was named. But what if the base space for a hyperspace is really strange, perhaps not even Hausdorff? Here we might set off into mountains that are not earthly, but rather lunar or Martian or perhaps even Andromedean, where the underlying space violates Riemannian standards. We might skirt the wilds of Vietoris topology, Zariski topology, the bewilderment of totally disconnected spaces, where the metric is non-Archimedean, all triangles are isosceles, and all open sets are clopen! But no… we leave the Cantor set, Stone spaces (oddly enough due to mathematical logic and George Boole, who preferred the deductive plains) and the p-adic numbers for another time (see Grosholz 2007: Ch. 10, 2016: Ch. 4 and Ch. 5). Instead, we will return to *Scientific American*, and the much more reasonable task of turning a sphere inside out without breaking it or leaving a crease!

In 1958, Steve Smale, an instructor at the University of Chicago, took the mathematical world by storm when he proved the possibility of a sphere eversion in differential topology: that is, he showed how to turn a sphere inside out in three dimensional space, smoothly and continuously, without cutting or tearing it, and without leaving a crease. He went on a few years later to prove higher dimensional versions of the Poincaré Conjecture (O'Shea 2008). The original conjecture states that every simply connected, closed 3-manifold is homeomorphic to the 3-sphere. (A connected manifold is not the union of two disjoint, non-empty sets; a simply-connected manifold has no holes.) An equivalent formulation is this: If a 3-manifold is homotopy equivalent to the 3-sphere, then it is homeomorphic to it. (We will meet homotopy groups and homology groups later.) It is easiest to picture the two-dimensional version of this claim, which is intuitively obvious: for compact 2-dimensional surfaces without boundary, if every loop can be continously contracted to a point, then the surface is topologically homeomorphic to a 2-sphere (the sphere we know and love). However, the three dimensional case is the most difficult: it was only proved by Grigori Perelman in 2006, though Poincaré had proposed it over a hundred years before.



Cover of Scientific American, May 1966

Steve Smale won the Fields Medal in 1966 for his work in topology, and in the same year, Tony Phillips (then a lecturer at the University of California at Berkeley) wrote up Smale's formal proof of the eversion of the sphere for *Scientific American*, with wonderful illustrations based on a more concrete and detailed visualization/conceptualization due to Arnold Shapiro, a mathematician at Brandeis University who also worked in Paris, but had died just a few years earlier. Keep in mind that a sphere is not a rubber balloon; it is a mathematical object. In differential topology, as Phillips tells us, we are allowed to move a surface through itself, so that two distinct points, during this deformation, may be mapped to the same point of the space in which the surface is embedded. So "two regions of the sphere are pushed toward the center from opposite sides until they pass through each other. The original inner surface begins to protrude in two places, which are then pulled apart until the knurl—the remaining portion of the outside—vanishes. In the process, unfortunately, the knurl forms a tight loop that must be pulled through itself. This results in a 'crease'…" which is impermissible when one is dealing with smooth surfaces (Phillips 1966: 112). So, how to avoid the crease?

Phillips' explanation begins with simpler cases, easier to think about and to see, and then step by step takes us higher. First he gives a rather technical definition of a curve in the plane: its technical complexity is justified, because it is precise, and because it suggests by analogy how to mount to the higher levels. Here is it. "A curve in the plane will be defined as a map from the circle into the plane." We think of it as a rule assigning to a point of the circle (identified with the angle θ, and thus a number between 0 and 360) a point on the plane; so the rule must map 0 and 360 to the same point. This leads to another definition: "A curve in the plane will be called regular if, as a point runs around the circle at constant speed, its image moves smoothly and with a velocity other than zero in the plane." This means that the tracing "pencil" of the rule cannot halt; the curve cannot have a cusp (a pointy protuberance). So we move to the definition of a regular homotopy, an equivalence relation that we'll discuss later: "Two regular curves are said to be regularly homotopic if one can be deformed into the other through a series of regular curves" (Phillips 1966: 112–114). Thus, between the two original curves we can find a family of regular curves, where each shape represents a stage of the deformation of the original curve into the target curve. Finally, we need to define the concept of "winding number": as we map the curve onto the plane, going from 0 to 360, the total number of clockwise turns the curve makes is its winding number. Two regularly homotopic curves must have the same winding number; in 1937, Hassler Whitney proved the more difficult converse claim: any two regular curves with the same winding number are regularly homotopic (Phillips 1966: 115–116).

Then we move to an analogous set of regular curves mapped, not to the plane, but to the sphere. In this case, it turns out that two regular curves are regularly homotopic if either they both have an odd number of self-intersections, or both have an even number of self-intersections; that is, they can be regularly homotopic even if they do not have the same winding number. And here comes the crucial shift: we move everything up one dimension. "The analogue to a curve [which was earlier defined as a map from the circle to the plane] is a map from the sphere into three-dimensional space. Such a map would assign to each point of the sphere some point (its image) in three-space. An example of such a map is the standard embedding, which assigns a point P of the sphere the [same] point P considered as a point in three-space." But guess what! "We could equally well use the antipodal map A, that assigns to each point P of the sphere its diametrically opposite point A(P), considered as a point in three-space." This embedding works just as well, and is the hinge on which the whole proof turns. So if we start with a curve on the sphere, we can now map it (via the mapping of the sphere) into three-space, and we define "a regular map from the sphere into three space as a map that transforms each regular curve on the sphere into a regular curve in three-space" (Phillips 1966: 116–117). The antipodal map is a regular map! Two regular maps from the sphere into three-space are regularly homotopic if we can find a family of regular maps joining them. The images labeled A through S that embellish Phillips' account, found in sequence on pages 113–117, are deformations of the sphere that are regularly homotopic, and that take the standard embedding to the embedding that uses the antipodal map: the maps all vary smoothly *as they turn the sphere inside out*. Then on page 119, using the same strategy, in only nine steps he turns the torus inside out!

If you are not already amazed enough, I remind you that one of the mathematicians who directly inspired the work of Smale was Hassler Whitney, the mountaineer we came across in the discussion of the topological notion of hyperspace. He initiated the systematic study of immersions and regular homotopies in the 1940s, work that resulted in the Whitney Immersion Theorem and the Whitney Embedding Theorem. Moreover, if you read Donal O'Shea's book *The Poincaré Conjecture: In Search of the Shape of the Universe*, as I have already encouraged you to do in a footnote, you will see that the investigation of the Poincaré Conjecture, in which Smale played an important role, not only helped to launch topology, but is now used to study the dynamic curvature of space-time. And, finally, there is the brilliant ending of Phillips' article, which made me almost drop the magazine on the floor half a century ago (again!) and has inspired me to think about the interaction between symbolic and iconic mathematical idioms ever since. "The intricacy of the pictures, which were in a sense implicit in Smale's abstract and analytical mathematics, is amazing. Perhaps even more amazing is the ability of mathematicians to convey these ideas to one another without relying on pictures. This ability is strikingly brought out by the history of Shapiro's description of how to turn a sphere inside out. I learned of its construction from the French topologist René Thom, who learned of it from his colleague Bernard Morin, who learned of it from Arnold Shapiro himself. Bernard Morin is blind" (Phillips 1966: 120). Here is a scientific article that ends with a masterful rhetorical flourish.

Steve Smale once said that his best work had been done "on the beaches of Rio," which for some reason made the administrators of the National Science Foundation very angry, and they took away his grant money. Heaven forbid that a mathematician should have any fun. So I end this chapter with my own poetic imitation of the beaches of Rio de Janeiro, Brazil, where I have never been except in imagination. I wrote the poem for a philosopher I loved and lost, who in fact lives in La Plata, Argentina, and teaches in Buenos Aires. But the history and geography of hyperspace has its reasons that reason knows nothing of, as Pascal wrote of the heart. In this poem, I use the most beautiful song that Raun MacKinnon ever sang for me and my brothers: “Dink’s Blues.” It was collected by the folklorist John Lomax, who learned it from a Black woman (known only by her first name or nickname, Dink—one of our great un-storied poets) on the banks of a levee near Houston in 1909. It haunted me as I tried to give up on the romance and deal with my first job, whilst listening to Brazilian jazz and the poems of Jorge Luis Borges transmuted into songs by the Argentinian singer Jairo, and admiring the mythically grand oak tree that grew just outside my rented house. You will note that right in the middle of the poem, “Two Variations on a Theme,” I turn the oak tree into Book I of Euclid’s *Elements*. Descartes’ poignant declaration in the *Meditations* when he has just lost the whole world to his own hyperbolic doubt conjured up as a malicious demon, “I am, I think,” echoes throughout the last half. The poem is (still) rather cosmological and mysterious. The lines in italics are lifted directly from the song.

Two Variations on a Theme
    “Fare thee well, oh honey, fare thee well.” *(Dink’s Blues)*
I.
Indian summer winds the trees
Without recovering their ancient green
Or leaving them in silence.
The enormous transience shimmers and burns,
Beating its empty vans on the dry hills,
An old song caught in its throat.

One of these days, it won’t be long.
Believe the song, my love, and not the singer.
Wild grape vines string the lyre
Of branches, bittersweet half-opens, ivy
Glitters like the goddess’s revenge
Snaking through the forest, killing the boles.

So weather sings, and flowers
Assume the claws of some fantastic creature.
Strange choirs out of season shake the air,
Rapt in transmutation. *Call my name.*
Apples ripen inward, yellow quinces
Bruise like mottled hearts, black walnuts
Tumble and litter the uncertain grass
That startles up, called by October’s fictions.

Veronica follows the grass in all its errors
Repeating the savior’s face,
Each leaf with its bloody forehead, lonely gaze.
One of these days, it won’t be long,
You call my name and I’ll be gone.
The body of earth continues to decline
Under the great, transparent shrouds of light.
Even gods are mortal. Trust the song.

II.
Light through the southern window throws
Shadows of cedar boughs
And the ghost of a jay, who haunts their frail
Shelter throughout the winter, on the wall.
Beyond the northern window, dusk
Stains the hills to damask, then to plum,
Sidelong to indigo. The leaves have fallen.
Sunset magnifies the neighbor's oak
To a system of borrowed light,
Thousands of theorems drawn
From the bole's exhaustive axiom: I am.

If I had wings like Noah's dove,
I'd fly down the river to the man I love.
But I stay here. Across the empty wall
Autumn displays its passages in shadow,
Re-creating the ancient masque
Of emigrant light leading out all its flocks
Along the Susquehanna, south
To Chesapeake and the ocean. Daylight drains
Our darkened continent, and leaves a tree
Of silver rivers read by satellite
Whose eye revolves ten thousand miles away.

Beyond the globe's meridian
Spring is beginning on the underside:
Tall grass fledges the pampas, passionflower
Stares from balconies toward Ipanema,
Ornament for the rich and shower
Of inaccurate gaiety over the favelas.
The principles of light reverse themselves.
I am, I see, but only insofar
As I have been deceived.
Ambiguous delight withdraws behind
The window-screen, inflamed with visible night.

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Chapter 3
Great Books

Throughout high school, I read philosophy persistently but erratically, starting with Soren Kierkegaard's *Fear and Trembling/The Sickness Unto Death* (Kierkegaard 1954) given to me by someone at my church in 1963, and then straying into the works of William James, Simone de Beauvoir, Friedrich Nietzsche, Martin Luther King, Susanne Langer, and Jean-Paul Sartre, along with Will Durant's *The Story of Philosophy: The Lives and Opinions of the Greater Philosophers* (Durant 1933). But when I applied to college, I didn't say I wanted to study philosophy, but rather that I wanted to study poetry and mathematics. This odd conjunction made sense at the University of Chicago (indeed, they offered me a scholarship), because undergraduate education there was still steered by the star of Robert Maynard Hutchins' College, a pedagogical structure built on the Great Books program. Because of my scholarship, during my freshman year I was enrolled in "Liberal Arts I," a special series of lectures that introduced me to the classicists James Redfield and Marc Cogan, and the philosopher Eugene Garver, who became lifelong friends.

These lectures were also a conduit to my undergraduate major, Ideas and Methods, created by Richard McKeon after he left the Philosophy Department. He was terrifying, by the way; to see why, re-read Robert Pirsig's *Zen and the Art of Motorcycle Maintenance*, where he is depicted as The Chairman; and note that McKeon edited my *Basic Works of Aristotle*. I never took a course from him (Pirsig 1974; Aristotle 1947). A major in Ideas and Methods was supposed to choose a subject matter (I chose mathematics), take courses in philosophy (the emphasis was on Plato and Aristotle), and then write a senior thesis, reflecting philosophically upon the ideas and methods in one's subject matter. So I also took courses on poetry (Homer, Pindar, Horace, Catullus, the poets of the English Renaissance), read lots of poems on my own, mostly in English and French, but also in Modern Greek, and wrote reams of poetry while winning a few prizes, helping to edit the campus literary magazine, and thrilling to my first journal publication in *Poet Lore*. The topic of my senior thesis was poetry and mathematics.

My advisor in Ideas and Methods was David Smigelskis. It was he who first put Scott Buchanan's *Poetry and Mathematics* in my hands; Buchanan was born in

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1895, and died in 1968, just as I arrived at the University of Chicago, alas, so I never met him, but his book was still a lively incentive and encouragement, as was David Smigelskis' tutelage. I was inspired by Buchanan's oblique approach, as well as by the structuralism that anthropologists like Robert Redfield (Jamie's father) and Victor Turner (the poet Frederick Turner's father) sent percolating around the campus, and the rhetorical aplomb of Wayne Booth and his friend Kenneth Burke, who occasionally showed up to lecture. Thus in my senior thesis I did not discuss poems about mathematics, nor did I interpret the work of mathematicians as somehow "poetical." Instead, I looked for habits of thought that could be discerned in the organization of poetic culture and of poems, and then identified the same habits in mathematical culture and theorems. A development of some of these reflections shows up later in this book. Moreover, as we will also see, habits of thought that organize mathematics and mathematical physics can be discerned in works of poetry, drama and (poetic) fiction.

Now that I look back on Buchanan's book, I am struck by the intersections among his life-lines and mine. We seem to have had the same winding number, which I guess makes us regularly homotopic. My well-worn edition of *Poetry and Mathematics* (1962) adds a new Introduction written in 1961 to the text, which was first published in 1929. Here are the lines that most inspired me: "It was only when I actually gave the lectures [on poetry and mathematics] to a small and sympathetic audience that I found stable patterns, although not the elements that I sought. Much to my surprise, I found that mathematics and poetry run parallel patterns, such that one illuminates the other. A puzzle in one has a corresponding puzzle in the other, and sometimes, though not always, they can be understood together when they are unintelligible apart. Ordinary language attests to the connection, as when one counts numbers and recounts a story, or when geometrical figures are compared with figures of speech, or when in Greek the word for ratio is *analogon*, or when physiological functions can be expressed in mathematical functions. There are dangers in this etymological game, as I learned when a word led me astray, but there is also confirmation of many guesses if the faith in words is boldly followed" (Buchanan 1929/1962: 17–18).

The Introduction also gives an account of the genealogy of the Great Books program, though it doesn't go back far enough. Thus in the spirit of Asimov, again, I catapult us back across two thousand years, to the origins of the Trivium (which means "where three roads meet" in Latin) and the Quadrivium, the *fons et origo* (which means, "the spring, fountain, well, or source and origin" in Latin) of the Great Books program. Perhaps you have guessed that we are returning to ancient Rome. The Roman philologist Varro (Marcus Terentius Varro, 116–27 BCE) studied for a while in Athens with an Academic philosopher, which links him to the tradition of Plato's Academy. His work *Disciplinae* consists of nine books, whose titles suggest a classification of knowledge: Grammar, Logic and Rhetoric; Geometry, Arithmetic, Music, Astronomy; and Architecture and Medicine. This classification was taken up by Martianus Capella, a Neoplatonist who lived around 400 CE; he transmitted the classical Roman curriculum to the early Middle Ages (especially the Carolingian Renaissance that grew from the court of Charlemagne around 800 CE) through his influential book *De septem disciplinis*, an allegory where Philologia is married to Mercury and so receives seven handmaidens: Grammar, Logic and

Rhetoric, and Geometry, Arithmetic, Music and Astronomy. Architecture and Medicine were demoted because apparently they were too material and mechanical. These seven *Artes Liberales* encompassed what a free person needed to know in order to take part in civic life.

Now we catapult back to the recent past. Throughout the European Middle Ages, the Trivium and Quadrivium (poetry and mathematics) presided over the birth of the modern university, in Bologna, Paris, Oxford and Cambridge, Padua, Salamanca, Coimbra and elsewhere. Between 1600 and 1700, philosophers helped to invent modern science, the first stirrings of democracy, and the rise of Protestantism, which together drove the Enlightenment between 1700 and 1800, and then in dialectical opposition, Romanticism. Around 1800, in consequence, it finally occurred to a few people that a university education might be made available to everyone, though the African people who had been imported-abducted to Europe and women would have to wait a while to be included in 'everyone.' (When I was a visiting scholar at Cambridge University, founded in 1209, we celebrated the 50th anniversary of the first degree officially conferred on a woman in the spring of 1998. Huzzah! We only had to wait 739 years! Go re-read Virginia Woolf's *A Room of One's Own*.) However, one must start somewhere.

Birkbeck College, part of the University of London, is named after George Birkbeck, a Quaker, physician and pioneer in adult education, who gave free lectures on the 'mechanical arts' to workmen in Glasgow around 1800, an initiative that led to the creation of the Mechanics' Institute there in 1821. When he later moved to London, he worked with the philosopher Jeremy Bentham and two MPs to create the London Mechanics Institute in 1823: it became the Birkbeck Literary and Scientific Institution in 1866, and then Birkbeck College. By the mid-nineteenth century, there were over 700 such institutes in Britain and its colonies. John Lubbock, first Baron Avebury, was Principal of the Working Men's College in London from 1883 to 1896 (he was also a banker and a scientist); in that capacity, he offered a list of 100 Great Books, all in English translation, for workers without Greek or Latin, in 1886. The list was widely published and discussed.

From 1925 to 1929, Scott Buchanan was Assistant Director of the People's Institute in New York City, which had been established 30 years earlier to supplement training in the mechanical arts at Cooper Union. This is where he presented the lectures that became *Poetry and Mathematics*. The People's Institute had become a place where immigrants from the lower East Side of Manhattan met with the intellectuals at Columbia on the upper West Side, as well as "internal migrants,... who spent their summers in harvesting on the Great Plains or in lumber camps, and who rode the rods back to New York for the winter... continuing the reading and discussion which had started when they knew Jack London. One could always find them during the day conversing and smoking in the lobby of the reading room of the New York Public Library, at Fifth Avenue and Forty-second Street. These two groups, the East Siders and the Wobblies, as we used to call them, were with the graduate students from the local universities at that time probably the best read audience in America" (Buchanan 1929/1962: 14).

Then there was Will Durant himself at the Labor Temple, Mortimer Adler running the Columbia Honors Course in Great Books, and Richard McKeon who had just returned from working on medieval philosophy with the great French philosopher Étienne Gilson at the Sorbonne. “He [McKeon] insisted that I had stumbled into a rediscovery of the seven liberal arts, the trivium—grammar, rhetoric, and logic—and the quadrivium—arithmetic, geometry, music and astronomy” (Buchanan 1929/1962: 19). Together, Buchanan, Adler and McKeon talked about what a modern version of the seven liberal arts might look like. From 1930 to 1952, Adler developed the Great Books program with Robert Maynard Hutchins at the University of Chicago, and in 1946 arranged with the Encyclopedia Britannica to re-print 443 Great Books in a 54-volume set. McKeon followed them there in 1934, and stayed for 40 years. With his friend Stringfellow Barr, Buchanan set up the Great Books program at St. John’s College in Annapolis, Maryland, in 1937. Along with many other people, I rue in retrospect the paucity of women and people of color in their lists; that absence is one of the many reasons why I edited a collection of essays on W. E. B. Du Bois and another on Simone de Beauvoir. Another reason, a reason of the heart, is that my mother Frances Skerrett Grosholz, who came from a family of Republicans, became a (mild) feminist at Pembroke College (where they put the women at Brown University in the old days) and a Roosevelt Democrat when she worked in the Bureau of the Budget during World War II. She was also a committed supporter of the Civil Rights movement during the 1960s: she and I did a lot of tutoring and marching. However, I still admire the liberal populism that animated the program, and most of the books, and I’m glad it formed part of my education.

My experience in the Mathematics Department was mixed: none of my professors could see me as a potential research mathematician. I wanted to create mathematics, not just study it; but I didn’t know how to formulate the wish and no one around me did either. However, I came away from my undergraduate courses in mathematics with a set of good textbooks and a number of useful insights, which have helped me to prolong my informal study over the decades. One was Garrett Birkhoff and Saunders Mac Lane’s *A Survey of Modern Algebra*, which arose from their classroom experience at Harvard in the last 1930s. (Birkhoff and Mac Lane 1941) I am not alone in my enthusiasm for this book: upon its fiftieth anniversary, the *Mathematical Intelligencer* published a five page account of it! Here is how the authors characterized it in that issue: “The ‘Modern Algebra’ of our title refers to the conceptual and axiomatic approach to this subject initiated by David Hilbert a century ago. This approach, which crystallized earlier insights of Cayley, Frobenius, Kronecker, and Dedekind, blossomed in Germany in the 1920s. By 1930, relatively new concepts inspired by it had begun to influence homology theory, operator theory, the theory of topological groups, and many other domains of mathematics’ (Birkhoff and Mac Lane 1992: 26–31). (Hitler effectively exploded Hilbert’s circle, destroying the brilliant mathematical culture in Göttingen that had flourished since the early nineteenth century, beginning with Gauss, and continuing through Riemann, Klein, Hilbert and Emmy Noether.) This textbook introduced me to groups, rings and fields, which will appear quite often in upcoming chapters.

The second inspiring book was I. M. Springer and John A. Thorpe’s *Lecture Notes on Elementary Topology and Geometry*, which I have already invoked by quietly footnoting it throughout the last chapter, and to which I will return. Thereon hangs a tale, from the summer of 1969, when I and Jackie and Cinda and my newly discovered friend from the University of Chicago, Roberta Caplan, spent the three summer months in Cambridge, Massachusetts. I worked at various jobs, and went to the Newport Folk Festival (where I saw James Taylor in person, and learned how to make large paper flowers). Midsummer, two of my pretty roommates were followed home (in a nice way) by two graduate students, Joe and Tad, who were studying mathematics at MIT, and I managed to make friends with both of them, friendships that have endured. Joseph Mazur has since written a series of books that make mathematical ideas accessible through accurate and clear exposition, and inspiring through vivid narrative: my favorites are *Euclid in the Rainforest* and *The Motion Paradox* (Mazur 2004 2007). Tadatoshi Akiba taught mathematics for a while after getting his Ph.D., but then went back to Japan and became the mayor of Hiroshima for 12 years, and has emerged as an effective anti-nuclear statesman active in the organization Mayors for Peace. They introduced me to Springer and Thorpe because they had actually taken topology from I. A. Singer at MIT, and later they helped me work though the chapters as I studied by myself. I still have the original paperback textbook: all its pages have come loose, but are still in the right order. They also took me to a café called The Blue Parrot, where inter alia Barry Mazur and Tony Phillips would be discussing the directions that current mathematical research should take. Magic!

When I applied to graduate school, then, I applied to study philosophy, and spent the next few years studying all the philosophers who came after Plato and Aristotle. In particular, I discovered the Early Modern philosophers Descartes, Galileo, Spinoza, Leibniz, Newton, Locke, Hume and Kant, whose work depended so strongly on the burgeoning of mathematics and physics (and their interaction with each other and epistemology) between 1600 and 1800. I also discovered logic, which had become a part of mathematics in the late nineteenth century (who knew?); one of my dissertation advisors was a philosopher and one (the helpful one, Angus Macintyre) was a mathematician. Although I had written my senior thesis at the University of Chicago on poetry and mathematics, at Yale University there was no suggestion that these two enterprises had anything to do with each other. This was in part because the Philosophy Department was riven between the Analytic philosophers and the Continental philosophers, who never talked to each other but only disputed, whilst we graduate students tried, and failed, not to get caught in the crossfire. If only Ernst Cassirer had gone on teaching at Yale after fleeing Germany, and had lived to be 105! During graduate school, on my own and only in books, I discovered his way of combining poetry and mathematics, which were helpful to me in the ensuing years.

However, during graduate school, I dutifully wrote my dissertation on the growth of mathematical knowledge in the Early Modern period, and the interaction between mathematical logic and topology in the twentieth century in unpoetic prose. Way across town, I ran a weekly poetry series in a small health food restaurant, *Down to Earth*, in the red light district. I published more poems in journals, met fellow poets, and began my decades-long attachment to the *Hudson Review*, and its editors Frederick Morgan and Paula Deitz. Very few of the poems I wrote during that decade had much to do with mathematics, with an important exception. I spent half my summers wandering around Europe, with books of art history under my arm, so the mathematics buried in the symmetries of the great temples and cathedrals and paintings I studied did surface in some poems, to which I will turn in the last section of this chapter.

But first, an excursus on Cassirer. The philosophical opposition that I encountered at Yale, he had encountered long before, caught between the Logical Positivism of Schlick, Carnap and Hempel, and the *Lebensphilosophie* of Nietzsche, Spengler and Heidegger: how did he deal with it? Cassirer opposed both Rudolf Carnap and Martin Heidegger in his brief professional ascendency, when he held a chair in philosophy at the University of Hamburg from 1919 until 1933. [See Michael Friedman's thoughtful *A Parting of the Ways: Carnap, Cassirer and Heidegger* (Friedman 2000).] In 1933, he emigrated from Germany because he was Jewish, spending the last 11 years of his life at the University of Oxford, the University of Göteborg (Sweden), and Yale University. He died there in 1944 and his papers (available online) are housed in Yale's Beinecke Library.

Although, as I have indicated, I arrived at Yale with my lifelong topic "poetry and mathematics" already well formulated, nobody ever suggested that I use Cassirer's work as a starting point for thinking about my double topic. Heidegger took poets seriously (with the notable exception of Paul Celan) but he despised science. The Logical Positivists were deeply interested in science and mathematics (though they distorted them by subjecting them to formal logic), but what they had to say about poetry, and art generally, was silly. Here is Carnap, for example: "The aim of a lyric poem in which occur the words "sunshine" and "clouds," is not to inform us of certain meteorological facts, but to express certain feelings of the poet and to excite similar feelings in us. A lyric poem has no assertional sense, no theoretical sense, it does not contain knowledge" (Carnap 1996: 28–29). Both logical positivism and *Lebensphilosophie* were reactions against the neo-Kantian movement that arose in Marburg, heralded by Hermann Cohen's first book in 1871—substantially revised in 1885—*Kants Theorie der Erfahrung*.

Kant defended Newton's physics against the skepticism of Hume by grounding it in "pure understanding," that is, he tried to show that the Newtonian axioms articulate certain conditions of possibility of any experience whatsoever, in his "transcendental deduction of the categories." This position was hard to maintain during the nineteenth century, in light of the revisions to physics and mathematics brought about by Maxwell's theory of electricity and magnetism and the new laws of thermodynamics, as well as the development of non-Euclidean geometry, which challenged Kant's "transcendental aesthetic." Beginning in 1905, the revolutions of Special and General Relativity, and then of Quantum Mechanics, intensified this difficulty. Neo-Kantianism was faced with a choice: either become a theory of pure consciousness, or become a theory of science. Cohen took the latter route; Husserl took the former. And this decision made Cohen's critical philosophy both historical and enclitic; it became the reflective study of the development of mathematics and science over time, and lost its "foothold in eternity," the priority that Kant claimed for the study of the transcendental that must always precede any investigation of the merely empirical. For Kant, nature was constituted by human understanding itself and thus amenable to codification by a complete set of principles (Newton's Mechanics), though to encompass the realm of morality and to tame metaphysics, the finitary understanding must be regulated by infinitary reason. For Cohen, by contrast, nature always surpasses human understanding so that the work of science (and therefore philosophy) must be open-ended; the demand for universal validity remains the one fixed constraint on a progressive science, but it is merely regulative, not constitutive.

Proponents of the Marburg School, though devoted to science, regarded it as a "free and active creation of the intellect." This set them at odds with earlier versions of Positivism (due to Comte, Darwin and Mach), which regarded science as the passive adaptation of the human mind to sensory data, a mechanism of survival. They refused to sever the philosophical study of science from that of ethics, which the Positivists rejected as subjective "poesy". Both science and ethics they regarded as objective, expressions of the same spontaneity and creativity of human reason. The Marburg School was also at odds with Hegelian idealism, because it rejected the central tenet of Hegelian metaphysics, the identity of reality and thought. Ultimate reality always eludes human thought; philosophy must maintain its modesty and circumspection in the face of its infinite task. In sum, the Marburg School tried to humanize science, rationalize religion, and liberalize socialism. Nonetheless, its project was one-sided, for Cohen held the extreme view that experience is objective only insofar as it can be given mathematical form. Science thus lost its foothold in everyday experience, and Cohen was never really successful in extending his critical method to nonscientific experience, aesthetic, ethical or political; in particular, he could not address the carnage of World War I, brought on in part by the advance of technology, the flower of scientific progress (Skidelsky 2008; Grosholz 2010).

Ernst Cassirer was Cohen's prize pupil. I spent a year (1976–1977) studying in Germany, to learn German and listen to the neo-Kantian lectures of Friedrich Kaulbach. I recall the strange conventions of the German university system in that era when I watched my fellow graduate students at the University of Münster meekly acquiesce to every stated opinion of their dissertation advisor or *Doktorvater*, not only in typescript but in private conversation; 60 years before, the inhibition would have been all the more stringent. Cassirer's book *Substance and Function* was written in 1910, under Cohen's jurisdiction and, moreover, the spell of the new logic of Frege and Russell. Only in 1921, 3 years after Cohen's death and when he was nearly fifty, did Cassirer first announce his own, independent 'philosophy of symbolic forms.'

Substance and Function begins with a critique of Aristotle's logic of the syllogism and doctrine of the abstraction of concepts. Syllogistic reflects a metaphysics of substance and attribute, for it only treats sentences of the form 'S is P'; its formalism cannot express relations or functions. Aristotle understood knowledge as beginning in the perception of individual substances, like horses, people, or trees: the concept is a selection from what is immediately presented in experience. Every collection of comparable objects has a supreme genus consisting of all the properties in which those objects agree and eliminating all the properties in which they do not agree. As we go up the hierarchy of concepts, then, the content of the more and more generic concepts diminishes, and the category of Being seems to have no content at all. This is a problem, because generic scientific and mathematical concepts are supposed to give us more and more precise determinations, that is, more content.

Cassirer's central claim, based on his study of modern mathematics and physics, and the new logic, is that modern mathematical concepts do not cancel or forget the determinations of the special cases, but fully retain them. When a mathematician makes his formula more general, Cassirer asserts, this means that he is able not only to retain all the more special cases, but also to deduce them from the universal formula. (If a general concept had been arrived at by Aristotelian abstraction, the special cases could not be recovered from it, because the particularities have been forgotten.) By contrast, the mathematical or scientific concept seeks to explain and clarify the whole content of the particulars by exhibiting their deeper systematic connections, revealed in the law of the series. Thus from Descartes' general equation $Ax^2 + By^2 + Cx + Dy + E = 0$, we can elicit the particular conic sections (circle, ellipse, parabola, hyperbola), presented in systematic interrelation. Here the more universal concept is more, not less, rich in content; it is not a vague image or a schematic presentation, but a principle of serial order (Cassirer 1953).

The general is no longer seen as something over and above the 'sum' of particulars, but immanent in them; it is the particulars viewed under the aspect of their serial form. Thus in modern mathematics, things and problems are not isolated, but shown to be in thoroughgoing interconnection. Moreover, Cassirer continues, the concrete universality of the mathematical function extends to the scientific treatment of nature. Thus a series of things with attributes is transformed into a systematic totality of variable terms or parameters; things are transformed into the solutions of complex equations, as when a molecule becomes the solution to a wave equation, or when the sun, the moon and the earth become a solution to the three-body problem. As Cassirer wrote earlier, the world of sensible presentations is not so much reproduced as supplanted by an order of another kind. But now we have returned to the central difficulty faced by the Marburg School: the realms of science and ordinary experience have been dissevered.

Cassirer was brought up on the poetry of Goethe and the anthropology of Herder, Schiller, Schelling, Hegel, and Humboldt. Skidelsky argues that he shared the romantic opposition to eighteenth century Rationalism for degrading our "sensuous, emotional life to the level of a biological residue, a passive stuff to be overcome." These thinkers defined the essence of humanity not as reason but rather our capacity for self-expression, manifest in not only science and mathematics, but also language, religion, art, and myth. The ways in which we articulate and organize the world are irreducibly plural. Cassirer learned from them that our relationship with the world is not dominated exclusively by the demand for objective knowledge, but must also answer to the human thirst for *meaning*, how we shape the world into patterns, our various activities of symbolic formation. "The critique of reason becomes, in Cassirer's famous declaration, the critique of culture." He did not however share the Romantics' disdain for science. Rather, "Cassirer's ultimate purpose was to reveal science as an expression of the same symbolic capacity underlying language, art, and myth, thereby acquitting it of the common charge of coldness and inhumanity. His philosophy is an attempt to exploit the ambiguous energies of German romanticism on behalf of enlightenment" (Skidelsky 2008: 72–4).

Thus for Cassirer, the counterweight to Kant was Goethe. From the great poet Cassirer learned to believe in the objectivity of the artistic imagination. Whereas for Kant, aesthetic judgment, the apprehension of beauty, is universally valid but still subjective, for Goethe beauty is a revelation of the real form of things; art as well as science is a mode of world constitution. Cassirer transforms his earlier account of the serial concept by detaching it from its mathematical context and presenting it as emblematic of the creative imagination. The poet, like the mathematician, "sees the individual not as an isolated, self-contained substance but as the symbol of a more universal complex" (Skidelsky 2008: 77). Thus Cassirer especially admired Goethe's gift for disclosing the general in the particular.

In *The Philosophy of Symbolic Forms* (published between 1923 and 1929), Cassirer not only urges this similarity but also explores the radical differences among the world-makings of mythology, language, religion, art, and science. He gives up the Kantian (and Hegelian) devotion to a complete and unified system in favor of irreducible plurality: thus his masterwork has three volumes, each a fresh departure from the others in the exploration of how we human beings use symbolic forms to organize our world. This lack of strict unity means that the project of thinking them together—using irony, analogy, and deliberation—must continue indefinitely. An honest thinker must always reflect on the limitations of his or her own methods, 'organs of interpretation,' and maintain flexibility. In Cassirer's view, because critical philosophy defines objectivity not in relation to an object but rather in terms of the immanent lawfulness of thought or human awareness, more than one 'objective' conception of the world can be admitted. Language, art, myth and religion, as well as science, are then different ways of accomplishing the synthesis of spirit and world. Both Goethe's *Farbenlehre* and Newton's arithmetization of color may be entertained (Cassirer 1965).

This opposition inspired two poems, "The Dissolution of the Rainbow" and "Goethe in Verona," which I wrote after visiting the great botanical garden that overlooks Verona in 1977 with my friend Ruth Geyer (Shaw), who went on to become a botanist and population geneticist. She knew that Goethe had also been an important botanist.

The Dissolution of the Rainbow

> "By an extraordinary combination of circumstances, the theory of colors has been drawn into the province and before the tribunal of the mathematician, a tribunal to which it cannot be said to be amendable." Goethe, *The Theory of Colors*

A cut-glass chandelier dangled above
The desk where Newton read and wrote:
All morning spectral dragons fought,
Mocked him and made love
Across the white wall opposite,
Flashed their blue and sea-green scales, the fur
Of tiny fires, a glittering red eyelight.
Then one day they suddenly
Fled, and no longer were.

Rising in impatience, strangely lit
By reason, the philosopher undid
Prism by prism the trembling chandelier
To run her now constrained and broken
Offspring through a maze of barriers.
The light went through its paces
But the dragons disappeared.
What remained Sir Isaac quantified,
Teaching Nature not to sing
Her sweeter variations, but in one
Low tone, geometry, to answer him.

Although white light is manifold,
A mixture, so he found, of different rays,
Each ray could be identified
In essence with its angle of refraction:
This was the only origin of colors,
Color then reduced to numbering.
The dragons lapsed to silence, mortified,
Curled up and dry as worms a child
Might question in the fire
Of curiosity and leave behind.

When Newton set his prism work aside,
He wiped his hands, and wrote on creamy paper
Long and elegant formulae,
A shadow of the sensuous retained
In his illuminating study,
Even that much immaterial.
Yet he sometimes noticed, later on,
How his sines and cosines lay
Across the paper like dark skeletons
Of dragon, couchant, rampant as the full
Proud curve of the integral.

Sines and cosines do look like little dragons on the page! Ruth (who was another high school friend) helped me learn more about trees and also name and recognize the species, genus and family of the many wildflowers we encountered when she shared my travels in 1977, so her influence extends far beyond the following poem.

Goethe in Verona

"I can't find it," he said to the almond-eyed woman,
And gave up his search for the day, rejoining her
On the highest terrace of the botanical gardens
Which overlook the river around Verona.
"But I have every reason to think it's here
On the Alps' Italian side, where antique flora
Have always found protection from bitter weather."

He had been hunting that small, pale, almost leafless
Urpflanz, which is the childish grandfather of all
Nature's overabundant bouquet, if it exists.
It was, he imagined, a kind of decorous lily
Without lanceolate leaves or silver bells,
A true false Solomon's seal, that had no cause
Or wisdom to discriminate itself.

Goethe stood apart from his companion
And watched the tumbled red roofs of Verona
Changing to umber in the light's decline,
A little surprised at how his imagination
Failed him, since it had long become his custom
To find what he himself put into nature
Greeting their father like well-brought-up children.

The almond trees were just beginning to flower,
Spangles of blue in the twilight, fair and scarce
As Hesperus and the other early stars.
The earth was not yet green, but the voice
Of fountains sang in the last of winter's frost
And ten years' labor at Weimar. Little bats
Like drunken birds went sideways in the air.

He felt he could trust the circle of his five senses
As long as he continued to practice green,
Magenta, blue, and the complex strata of lines,
Designing the very life of Italian prospects,
As long as the little *Urpflanz* kept
Its peace somewhere, untroubled within a landscape
Waiting to be discovered, touched and seen.

What else could he feel, who suffered so profoundly
The music, scent and texture of coming spring,
The sheen of anemone and gentian, showing
Colors barely unfolded above the sheath
Of bract and leaf, like the Veronese lady composing
Herself at the edge of sky and marble plinth,
Her pearls the dimmer seconds of Hesperus?

Though Goethe knew very well that over the Alps
In Paris and London, the physicists of Europe
Were fabricating a novel ghostliness,
The truth of an underworld beyond the senses,
He still beheld the falling rose of day
Which drew itself across the terraced slopes
As the flower of light in blossom, not decay.

In 1920, Cassirer came to know the Kulturwissenschaftliche Bibliothek Warburg, soon after he took up his professorship at the University of Hamburg, as well as Fritz Saxl, Erwin Panofsky, and later, in 1925, the Director Aby Warburg who had recovered from a long illness. Cassirer became the 'house philosopher' of the Warburg Library; his influence is evident in two influential books: *Saturn and Melancholy* (1923) jointly authored by Saxl and Panofsky, and Panofsky's *Perspective as Symbolic Form* (1927). Both Cassirer and Warburg viewed myth as the unruly element in any system of culture, full of energy and ineluctable. Over-intellectualized systems (philosophical or political) that try to get rid of myth entirely find that it returns with a destructive explosion; irrationalist doctrines that try to codify and use it for ulterior purposes find that although at first it lends energy to a program, in the end it proves anarchic and elusive. Cassirer concludes that philosophy's task is not to eliminate myth but to understand it, locating it within the whole (rich, plural, unstable) system of culture. He and Warburg were acutely aware of Fascism as a re-eruption of myth, challenging the cool rationality of liberal democracy.

By symbolism, Cassirer meant a natural potency inherent in consciousness. Our purchase on the world is always, from the beginning, active and objective; the way to arrive at a better understanding of ourselves is to study our ways of world-making. Thus, philosophy has no empire of its own; its project is to reflect upon the arts and sciences, to be the self-consciousness of culture. This subjection of philosophy made Heidegger furious; and the 'demotion' of mathematics and science was strenuously opposed by the Logical Positivists. Both Heidegger and Carnap sought a fixed location, outside of history, which philosophy could call its own. So too, Cassirer's work was an embarrassment to my professors at Yale University; 30 years after his death, impelled by different fears, they all wanted to escape the objective plurality of symbolic forms.

Human beings confront the world already shaped, symbolically, by their own aspirations, projects and theories. For Cassirer, the symbol was an instrument of human liberation, and he therefore ranked the different symbolic forms according to their power to liberate. Myth occupied the lowest rung, because in myth, symbols are treated as objective powers, and their source in human spontaneity is forgotten. Having escaped enslavement to nature, he wrote, we re-enslave ourselves to custom. Religion was more highly regarded by Cassirer, for religion (especially the Judeo-Christian tradition and the ethical religion of Kant) allows us to recognize and realize the autonomous, ethical will. Likewise, the scientific disenchantment of nature was the other side of this process of human liberation. All the same, Cassirer's view of the 'system' was not a ladder, like the Great Chain of Being or Hegel's progress of Spirit; it was more like a tree. Myth will never fully lend itself to liberation; religion typically relocates human autonomy in the otherness of God; science rejects the claims of myth altogether, and belittles art, cutting itself off from life; art faced with science dissolves in irony.

What Cassirer warns against is one-sidedness, as Sidelsky argues. Opposing the tendency to elevate a single kind of symbolic form and deny legitimacy to others, he offers a philosophy of symbolic forms that both seeks to explain this tendency to one-sidedness and to offer strategies for overcoming it. The tyranny of science gives us Logical Positivism; the tyranny of myth gives us *Lebensphilosophie*; but the recognition of incommensurable symbolic forms demands that philosophy engage in a continually renewed effort of reflection. The equilibrium of civilization is a project always in need of renewed commitment, and its many unbalances are ever more striking as the twenty-first century begins.

Towards the end of graduate school, I discovered another early twentieth century philosopher who moved from writing about science and mathematics, to writing about poetry, Gaston Bachelard. As a philosopher, he delved into the history of science and mathematics, and over almost 50 years wrote studies of thermodynamics, special and general relativity, the nature of space and time, and chemistry. He was also interested in psychology, and the creativity of human thought. In 1938, he published *The Psychoanalysis of Fire* which explores the long human experience of the hearth and its fire, and how it both guided and misled science in the modern investigation of combustion (Bachelard 1968). After that, Bachelard took up the theme of reverie, the waking dream that is the specialty of poets but belongs to everyone, and wrote seven more works of literary criticism. My favorite is *The Poetics of Space*, published in 1957 (Bachelard 1969a). As you might have noticed at the beginning of this book, it changed my way of thinking about my own childhood, and led me to pay closer attention to how the house of childhood organizes our experience later in life. (Grosholz 2004: 173–191). Bachelard shows that the organization and insight of science, with its mathematical models, cannot be explained by the structure of logic alone, as the Logical Positivists of the Vienna Circle seemed to want to believe. It must take history into account, and the imagination as well, which begins in reverie. One of Bachelard's last projects was to write *The Poetics of Reverie: Childhood, Language and the Cosmos*, published 2 years before his death (Bachelard 1969b). If you were inspired by the last chapter, read W. F. Touponce's study where he uses Bachelard to understand Bradbury: *Ray Bradbury and the Poetics of Reverie: Fantasy, Science Fiction and the Reader* (Touponce 1984).

Since I was a graduate student at Yale University, in my last years there I was lucky enough to meet in person two people who became my mentors, both of whom also migrated from mathematics to poetry. Jules Vuillemin was a student of both Gaston Bachelard and Jean Cavaillès! (More on Cavaillès later.) He also wrote about Descartes, Leibniz and Kant. In 1962, he published *La Philosophie de l'algèbre* (Vuillemin 1962; Grosholz 2005). He was also very interested in logic, but never forgot that logic, like the rest of mathematics, has a history. Thus, his philosophical reflections on Russell and Whitehead's version of Frege's predicate logic, Carnap's account of induction, Quine's rejection of history (Quine was a friend of his too), and Kripke's modal logic were very interesting indeed (Vuillemin 1984, 1996; Grosholz 1987a). Right in the midst of these explorations, he wrote *Elements of Poetics* (Vuillemin 1991a) on Aristotle's *Poetics*, some essays on Shakespeare, and two books of short stories, recounting the trauma of World War II: *Trois histoires de guerre* (Vuillemin 1991b) and *Dettes* (Vuillemin 1992). Here is the poem, built on a recurrent reverie, I wrote after he died. I went twice to visit him and his wife in their house in the Jura, a sub-alpine range that separates the basins of the Rhône and the Rhine rivers. The quote from Dante is the moment in the *Purgatorio* when Dante looks around, and realizes that Virgil has left him.

Farewell to Jules Vuillemin

… ma Virgilio n'avea lasciati scemi di sè (Dante, *Purgatorio* XXX)

The foothills of the Alps are green. We cross
A valley downwards, only to reascend
The other side, and enter the deep woods
Haunted by fox. Your favorite climbing trail
Commands the granite peak where guide and pupil
Gaze toward the crescent blue of Lake Geneva.

So often since, I've dreamed that dip and rise,
Heard the archaic melody of cowbells
Distances come to amplify, not dampen.
Like memory. A landscape scaled for giants,
As you then seemed to me, striding, explaining
The modal structures of necessity

In your patrician French, your fine austere
Logician's dialect, somewhere between
Our human tongue and music of the spheres.
Distracted briefly, when I turn again
You're gone, and all the slopes adrift with snow,
Night fallen, heaven overwhelmed with stars.

The other person was Yves Bonnefoy, who gave a course on Hugo and Baudelaire at Yale in the Fall of 1978, when I returned from that year in Germany, just after the death of my mother. I had been reading his poetry for a number of years, and at the very end of the course, I gave him some of my translations of his work, along with a few poems I had written in Germany, including "Letter from Germany," one of the first poems that was published in the *Hudson Review*. It was, of course, the letter I couldn't send to my mother; and it was also full of the trees and flowers that Ruth had introduced me to. A few months later, he sent me a nice letter and his new *Poèmes 1947–1975* (Bonnefoy 1978). We struck up a correspondence and exchange that continued for 40 years. I later learned that he had started out in mathematics, and then turned to philosophy, almost completing a Ph.D. on Descartes with the distinguished scholar Jean Wahl. Meanwhile, he asked me to translate one of his books of poetry, the 'American book' that he had written while teaching at Williams College, at the edge of the great boreal forest, with its thick winter snowfall (Bonnefoy 1991, 2012). In the process of translating *Début et fin de la neige*, I noticed how often the figure of Lucretius and his almost-mathematical atomism glimmered among the images of snowflakes, in tandem with images from neo-Platonism, Catholic texts, Renaissance paintings and… the house of childhood. Here is the poem about Lucretius, in my translation. Who should show up in it at the end, but Yves himself and his wife the artist Lucy Vines, who let me use her haunting drawings of mothers and children for my book of poems, *Childhood*?

De Natura Rerum

Lucretius knew it:
Open the box,
You'll see, it's full of snow
In flux, aswirl.

Or sometimes two flakes
Meet, unite,
Or else one swerves, gently
In its slight death.

How is it, that in some words
Daylight occurs,
When one is only night,
The other, dream.

Where do they come from,
These two shadows
Who walk off laughing, one
In red wool coat and mittens?

Although so far I have insisted on music as a middle term between poetry and mathematics, it is also true that visual art can play the role of middle term. During my twenties, when I was lucky enough to travel often to Europe, I studied the art and architecture that links Minoan Civiliation (ca. 3500–1500 BCE) to classical Greece and Rome, thence to medieval Europe and the Renaissance (ca. 1500 CE). Here is the great irony: to escape the square of the house of my childhood, I launched myself on rivers of air from Philadelphia to London, on train tracks from London to Paris to Rome to Brindisi, on Mediterranean currents from Brindisi to Patras and Olympia, and from Athens to Heraklion, and soared above five millennia. What did I find? Again and again, at the heart of the matter, I re-discovered the square! And all the reasons why we can, and must, go home again. If only, alas, I could explain this insight to my parents now; but at least I can explain it to my children – in some measure, by the poems in *Childhood*, which they inspired.

The square is often nestled in the golden ratio. Two magnitudes stand in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the magnitudes. So, supposing we have $a > b > 0$, we write the proportion of the two ratios this way:

$$a : b :: a + b : a$$

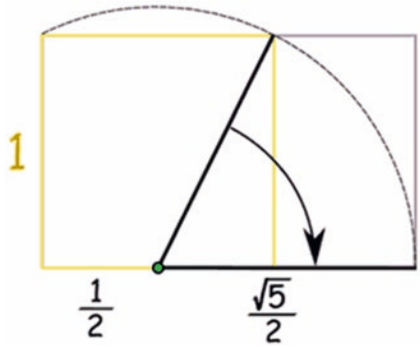
and so too

$$b : a :: a : a + b$$

which means that a is the middle term between b and $a+b$ in this case. We also define φ, which is (like $\sqrt{2}$) a quadratic irrational number, by using the late medieval trick of reconceptualizing ratios as fractions and proportions as equations:

$$\varphi = a / b = a + b / a = \left(1 + \sqrt{5}\right) / 2$$

And finally, we retrieve the square tucked into a rectangle, which illustrates the golden ratio: the yellow side of the square is a, the purple top of the adjacent rectangle is b, so the longer side of the rectangle is $a + b$, and its shorter side is a. In this particular image, one can stipulate that $a = 1$, so that you can see especially clearly why $\varphi = (1 + \sqrt{5})/2 = 1.6180339887\ldots$



Two places where I arrived on my travels made an especially deep impression on me, so that I feel as if I could relive the very moment when I first beheld them. One was the Parthenon, on its hilltop overlooking Athens, where in the summer of 1970 I followed the shade of Socrates around, listening to him use *reductio ad absurdum* (that paradoxically valid deductive argument form) to render the citizens of his city puzzled, reflective and ultimately philosophical. We will see that his student Plato was convinced that mathematics was the middle term between Becoming and Being.

The Parthanon

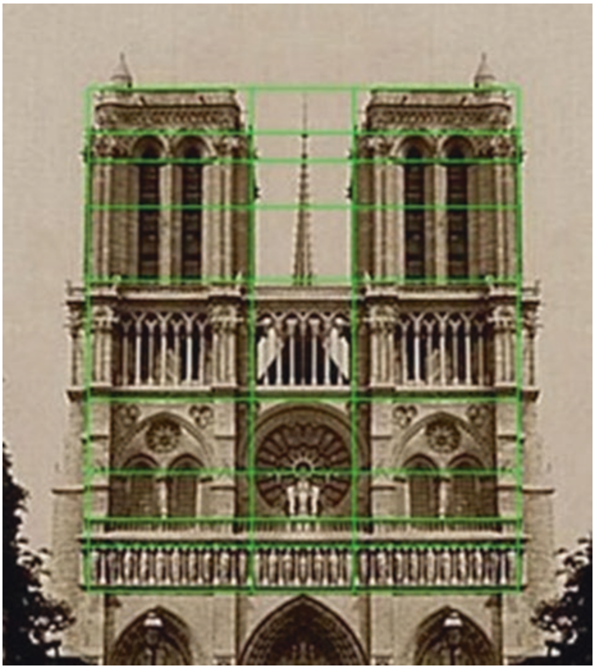
I kept a journal, and later I turned those journal entries into a narrative poem, *Cypress and Bitter Laurel*, which was eventually published in *The Reaper* (Vol. 17) by Mark Jarman and Robert McDowell. (Grosholz 1987b). Here is that moment, recorded in the iambic pentameter I wasn't even yet conscious of using.

Such an odd life we lead, the life of tourists
(And of spies, anthropologists and poets),
Moving slowly through the world's locations,
Transparent, unremarkable, all eyes.

But what's the purpose of this voyaging?
I thought at first I was just trying to change
My notions of history and geography,
Breaking the shell of childhood, or the small
Circle of America's visible culture.
The more I know where I'm not, the better I know
Where I am, gathering all those abandoned times
And places in my heart, against the future.
If I were a country, perhaps I'd be like Greece,
Telling myself the stories of my past
As patterns of flight for making divinations.
Still, why should it matter to the soul
If the body rattles around in trains and busses?

Yesterday, we got up at five, to see
The Acropolis in the very earliest light.
Large enough to count as monumental,
Small enough to be visible all at once,
The Parthenon stood white against the sky
And taught us both a lesson in proportion
Even in ruins, its roof blasted away.
For what remains is so insistently formal,
The series of marble steps and columns require
Completeness in the act of the mind's eye.

The second place was Notre Dame in Paris: there it was again, the golden rectangle, and another embodiment in stone of the golden ratio and its iterations.



Notre Dame de Paris

Oddly, Notre Dame didn't show up in any of my poems until 30 years later, when I turned fifty and was thinking about the young man I'd fallen in love with in Greece. He still runs an art gallery called *Galérie Orphée* in Olympia. A bit later, he married a Frenchwoman and I had just walked past the storefront of a second *Galérie Orphée*, which he had recently both opened and closed on the rue Gay-Lussac, where my life as a philosopher often takes me. The metaphors I used in that poem were drawn from algebraic topology.

Café on the rue Gay-Lussac

Exhausted, footsore, sweetening my coffee
Only with sugar, suddenly I see
That memory has lost its long allure.
I used to spend whole days remembering,
Expecting backwards, painting my shabby hopes
With lipstick and mascara, trompe-l'oeil glamor.

Oh Paris. Now it's only another city
Where restaurants are dear, the gutters full
Of rain and smithereens, the clerks deployed
On every front to say, *"Ah non, Madame,*
C'est impossible. Vraiment, je regrette."
Where Notre Dame is blanked by scaffolding.

All those imaginary dimensions heaped
Above my little life, like extra storeys
Hidden behind the gray-blue Mansart roofs,
Clouds crowning Sacré Coeur, sheaves over rings,
Co-tangent spaces kissing their manifold:
How finally they collapse. Goodbye, goodbye.

How finally future overshadows past
At half a hundred. What was she expecting,
That girl I was? Oh honestly, I forget.
Memory scatters in the sky like sparrows,
Or sinks like rain in gutters, where the drowned
Underground river-falls of Paris vanish.

At that point, halfway through a century, of course Notre Dame looked different, nuanced by the river Seine and the river of time. Note the arrow! And the changing colors of the leaves; I wrote the following poem on my birthday, which is in October.

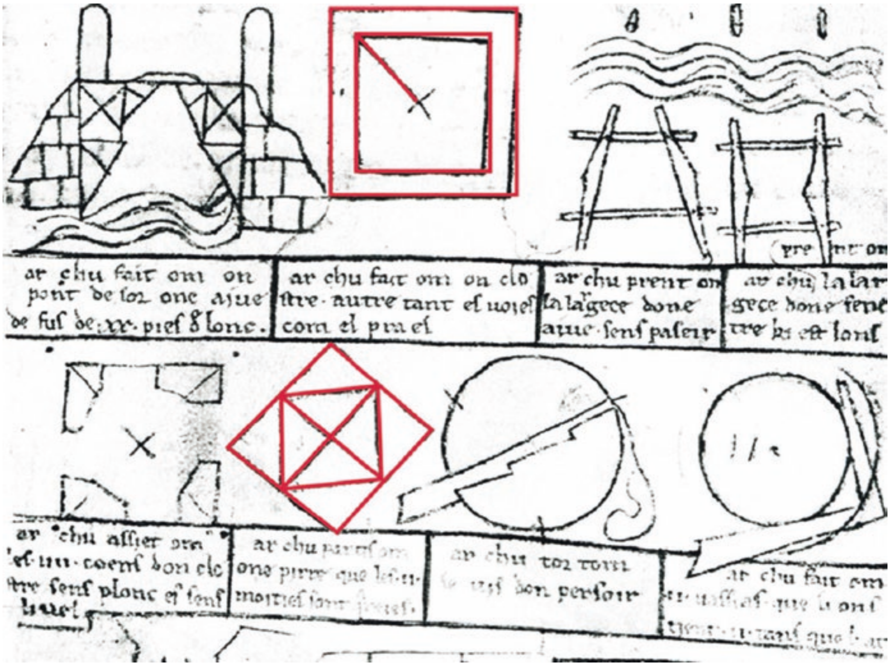
Love's Shadow

Lately the subject of my thought is time:
How time flows like a river, if it does,
Or like a house of crystal, stands immobile,
Or ceases where the very smallest creatures
Dance without houses, clocks, or bankless streams.

Here is my birthday, carried round again
By the tall star-crossed cycles of the moon.
Love's shadow brightens as the day begins
And then grows longer, whether we believe
That time is real or just a dream of things.

Notre Dame de Paris (my photograph)

However, back to 1973, when 3 years after my first European tour, I bicycled around Normandy with Henry Adams' *Mont-Saint-Michel and Chartres* in my backpack, and then around Burgundy, with various writings by Viollet-le-Duc and the memoirs of my beloved Colette. A few months after that, starting graduate school, I discovered Otto von Simson's *The Gothic Cathedral* under the tutelage of Karsten Harries. Adams and Simson both refer to the *Sketchbook* of Villard de Honnecourt, a thirteenth century mason and architect from Picardy (Adams 1959; Von Simson 1989). On one page, he includes the following geometrical diagrams; the second (in red) shows how to halve the square (or, going in the opposite direction, how to double the square), as he analyzes the ground plan of a cloister.



This diagram is especially significant, because it recalls the passage in Plato's *Meno* where Socrates leads an unlettered, unschooled slave boy to this very construction just by asking him questions, and then suggests that any human soul has access to mathematics: it is the middle term that orients us toward the eternal. (This of course means that everybody should be able to go to university, women are good at mathematics, and nobody should be a slave.)

Von Simson asserts, "the Gothic builders… are unanimous in paying tribute to *geometry* as the basis of their art. This is revealed even by a glance at Gothic architectural drawings… they appear like beautiful patterns of lines ordered according to geometrical principles. The architectural members are represented without any indication of volume, and, until the end of the fourteenth century, there is no indication of space or perspective. The exclusive emphasis on surface and line confirms our impressions of actual Gothic buildings… With but a single basic dimension given, the Gothic architect developed all other magnitudes of his ground plan and elevation by strictly geometrical means, using as modules certain regular polygons, above all the square" (Von Simson 1989: 13–14). In his less scholarly but lively book *The Cathedral Builders*, Jean Gimpel notes that in the Twelfth Century, the conduit of Greek learning to Western Europe was, besides Greek communities in Sicily, the city of Toledo, where Muslim, Jewish and Christian students and scholars studied (and translated) Arabic commentaries on important Greek texts (Gimpel 1961: Ch. 7). Thierry of Chartres had access to certain mathematical and scientific texts via this route, and developed a novel kind of Christian Neoplatonism.

Von Simson notes, “The church is, mystically and liturgically, an image of heaven.” One expression of that order is mathematical structure, manifest in both architecture and music. St. Augustine, ca. 400 CE, invoked the science and art of music, because it was based on mathematics. (Pythagoras, whom we encountered briefly in Chap. 1, influenced Plato, and Augustine was in turn influenced by Neoplatonism despite his conversion to Christianity.) The most admirable ratio, Augustine claimed, was 1:1, equality, symmetry, and unison; next in rank were the ratios 1:2 (the octave—remember Pythagoras’ strings), 2:3 (the fifth) and finally 3:4 (the fourth). Just as the stone squares in their rectangles are beautiful and stable, so the notes in their admirable ratios are consonant. Augustine, von Simson tells us, loved both architecture and music “since he experienced the same transcendental elements in both” (Von Simson 1989: 23).

Later, he tells us that Villard’s book contains “not only the geometrical canons of Gothic architecture, but also the Augustinian aesthetics of ‘musical’ proportions. In one of his drawings, the ground plan of a Cistercian church, “the square bay of the side aisles is the basic unit or module from which all proportions of the plan are derived… Thus the length of the church is related to the transept in the ratio of the fifth (2:3). The octave ratio (1:2) determines the relations between side aisle and nave, length and width of the transept, and… of the interior elevation as well. The 3:4 ratio of the choir evokes the musical fourth; the 4:5 ratio of nave and side aisles taken as a unit corresponds to the third; while the crossing, liturgically and aesthetically the center of the church, is based on the 1 : 1 ratio of unison, most perfect of consonances” (Von Simson 1989: 199). Von Simson goes on to explain in some detail how the architect of Notre Dame de Chartres elaborated on this structure, in surprisingly innovative ways (“he turned to advantage the restrictions that tradition imposed upon his design”) that allowed the vault of Chartres to be sprung at a much greater height than that of any of its predecessors. It was the first cathedral where the flying buttresses were aesthetically as well as structurally part of the overall design. Moreover, the golden ratio occurs in the figures of the west façade, as well as in the elevation: “The height of the piers… is 8.61 m. The height of the shafts above (excluding their capitals) is 13.85 m. The distance between the base of the shafts and the lower string-course is 5.35 m. The three ratios 5.33:8.61:13.85 are very close approximations indeed to the ratios of the golden section.” And the dimensions of the elevation are closely related to those of the ground plan. And so von Simson sums up: “Medieval metaphysics conceived beauty as the *splendor veritas*, as the radiant manifestation of objectively valid laws” (Von Simson 1989: 209–211).

This is the poem I wrote, inspired by the churches of Normany, in particular one especially lovely set of ruins in Jumièges. I was feeling melancholy, again, about breaking up with the law student whom I had irritated by writing my melancholy narrative about the young man in Greece.

… An ancient abbey stands in Jumièges.
The western towers, twin battlements
Against the tides of darkness, still remain,
But every wall is gutted, overgrown,

And the high roof, the paradigm of heaven,
Is long since stormed away.
Between the nave and choir there is no stair,
No screen to keep the crowds from their desire.
Only a copper beech, the prince of trees,
Whose monumental bole could bear
The manifold thin nervures of the air,
Divides the floor. The leaves divide the sky
In panes of bronze which fan
Around a spectral cross of red and green,
So the lost vault becomes a sheer
Translucent window, and the blue between
The blue of distance, as it ought to be…

One aspect of the great miracle of Chartres is the stained glass in the windows. Sometimes I think that the closest I have come to heaven were moments when I found myself standing in the nave of Chartres, looking up; but that was not just because of the radiant geometry and arithmetic manifest in the cathedral's stones. The cathedral is also astonishingly luminous: the great architect of Chartres replaced solid walls, to an unprecedented extent, with the transparent walls of stained glass windows. The clerestory is the upper part of the nave and choir and transepts: because it rises above the lower roofs, its windows admit a flood of light, which animate the colors of the stained glass, as James tells us in his rapsodic Chapter VIII of *Mont-Saint-Michel and Chartes*, "The Twelfth-Century Glass." And James reminds us that Villard sketched the western Rose Window at Chartes. When we look through the windows of the house of childhood, we see the town we live in and the woods and fields beyond, and the sky above, which sometimes (especially when it fills with wheeling stars) becomes a figure for heaven. But when we look 'through' the windows of Chartres, we see heaven: there it is.

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Chapter 4
Home, Cambridge, Paris: A Family

As a philosopher of mathematics and science, I have been teaching at the Pennsylvania State University for almost 40 years, and interacting with colleagues at the University of Paris, the University of Rome, the University of London, Cambridge University, Hebrew University and the Leibniz Archives and the University of Hannover. My philosophical focus was always on the growth of knowledge, the creativity of mathematics and the sciences, with issues of proof and justification lurking in the background. I have been especially interested in the way that mathematicians establish correlations and productive ambiguities that link different fields, as well as combining mathematics with the sciences: how do these correlations develop and how do they falter? I have studied the way that mathematical models link the world and formal discourse; how mathematicians make the infinite and the infinitesimal tractable; and how symmetry and periodicity are given mathematical expression, as they are encountered in the world. My historical studies extend generally from around 1600 to the present, in mathematics, though my historical interests in philosophy go further back, to Pythagoras, Plato and Aristotle, late antiquity and the Middle Ages. As a literary critic, I have found the same habits of mind that drive mathematical discovery in works of poetry. As a poet, I have also noticed that the vocabulary of mathematics, science and philosophy finds its way into my poems, even when I am writing about everyday matters, matters of the heart. During the same period, I married the medievalist Robert R. Edwards, and raised four children on the outskirts of a college town, re-creating for them the house of childhood and haphazardly cultivating a back yard that really is a middle term between the town and the wild. Between the trees, we see fields of soy and corn, and then the line of the Tussey Ridge, beyond which stretch hundreds of miles of forest.

While I was writing my first philosophical book, *Cartesian Method and the Problem of Reduction* (Grosholz 1991), I sometimes taught Thomas Kuhn's *The Copernican Revolution* (Kuhn 1957), the historical study that inspired his celebrated *The Structure of Scientific Revolutions* (Kuhn 1962). (I always try to show my students how strongly philosophy of science depends upon the history of science.)

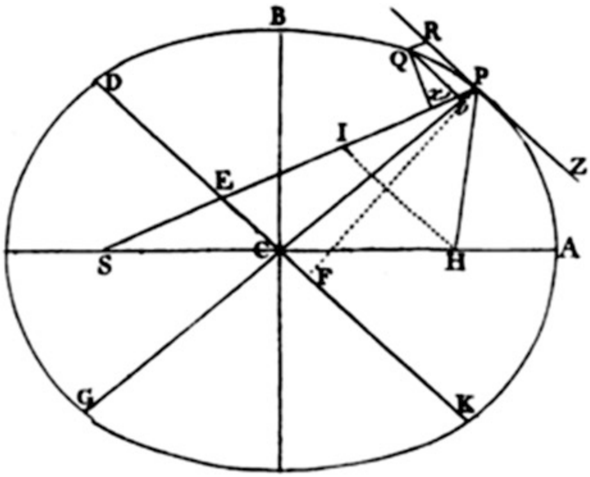
E. R. Grosholz, *Great Circles*, Mathematics, Culture, and the Arts,
https://doi.org/10.1007/978-3-319-98231-1_4

The earlier book explains Aristotelian cosmology, and reminds us how well it was supported by astronomical data as well as by ordinary experience. Kuhn also explains very clearly the sophisticated mathematical model developed in the *Almagest* (second century C.E.) by the Alexandrian astronomer Ptolemy. In Aristotelian cosmology, the earth is understood as four concentric spheres (earth, water, air, and fire) lying at the center of a finite cosmos; around the earth, the sun, the moon and the known planets (Mercury, Venus, Mars, Jupiter and Saturn) pursue their circular courses at a constant speed, along the celestial highway of the Ecliptic. (Today we understand the Ecliptic as the great circle marked out above us by the intersection of the plane of the solar system with the apparent sphere of the heavens; along this great circle lie the constellations of the Zodiac.) However, the three outer planets, Mars, Jupiter and Saturn, sometimes seem to go backwards (retrogress) for a period of time. Ptolemy postulated and added 'epicycles,' small circular motions superimposed on the great circles of the orbits, to model and finesse planetary motion, in a way that actually improved computations. Overhead, in Aristotelian/Ptolemaic cosmology, the great sphere of the fixed stars turns on the axis designated by the North Star, Polaris, which remains immobile. Here on earth, amidst earth, air, water and fire, we find ourselves in the realm of generation and decay, Becoming; up there, in the aetherial heavens, we see the realm of Being (Kuhn 1957: Ch. 1–3).

When we look up, what do we see? Whether the sky is blue and mottled by clouds, or black and spangled with ten thousand visible stars, we see a dome, a half-sphere, a ceiling. We cannot look up and see an infinite extent: we compactify the infinite, to understand it and make it meaningful. Looking up, we also see things that behave in an orderly and predictable way: luminous spheres that pursue circular paths, and always return just when the astronomers tell us to expect them. Geometry describes them, and we use them to number our days and months and years. No wonder Plato taught that mathematics was the middle term between Being and Becoming! We live in a (solar) system with only a few large visible moving parts, that obey relatively simple laws, and which, along with clay pots, beads, bricks, herds, fields, tents and houses, inspired us to invent numbers and figures, arithmetic and geometry.

In the middle of the sixteenth century, Copernicus proposes that the sun, not the earth is at the center of the cosmos; Kepler (on the basis of the improved astronomical date of Tycho Brahe, and a bit more sophisticated mathematics) proposes that the trajectories of the planets are ellipses; Galileo gives an accurate geometrical description of free fall (uniformly accelerated motion) and shows how to compound motions; Descartes gives a clear account of inertial motion (straight line motion at a constant velocity) as physically equivalent to rest; and Newton puts it all together in Book I, Proposition XI of the *Principia*, using Euclidean geometry improved by the nascent infinitesimal calculus.

The line PR is Descartes' inertial motion, and P is where the earth is: if it were on its own, in space, it would proceed in a straight line at a constant speed. But it is attracted by the gravitational force of the sun, so P*x* is the first moment of Galileo's freefall, which is then compounded in Galilean fashion with PR; so instead of tracing the segment PR, the earth traces PQ. The resultant trajectory is Kepler's ellipse, on which the earth obeys the law of areas as it goes slower and faster, depending on its variable distance from the sun. And there is S, Copernicus' sun, not exactly at the center but rather at one of the foci of the ellipse, since ellipses do not have centers. Here Newton demonstrates that the force of gravity obeys the inverse square law! (Grosholz 2011).



Inspired by explaining all these developments to my students, I sometimes waxed cosmological in my poems. When I became pregnant with my first child Benjamin, for example, I realized that for nine months I was really going to be a world.

Thirty-six Weeks

Ringed like a tree or planet, I've begun
To feel encompassing,
And so must seem to my inhabitant
Who wakes and sleeps in me, and has his being,
Who'd like to go out walking after supper
Although he never leaves the dining room,
Timid, insouciant, dancing on the ceiling.

I'm his roof, his walls, his musty cellar
Lined with untapped bottles of blue wine.
His beach, his seashell combers
Tuned to the minor tides of my placenta,
Wound in the single chamber of my whorl.
His park, a veiny meadow
Plumped and watered for his ruminations,
A friendly climate, sun and rain combined
In one warm season underneath my heart.

Beyond my infinite dark sphere of flesh
And fluid, he can hear two voices talking:
His mother's alto and his father's tenor
Aligned in conversation.
Two distant voices, singing beyond the pillars
Of his archaic mediterranean,
Reminding him to dream
The emerald outness of a brave new world.

Sail, little craft, at your appointed hour,
Your head the prow, your lungs the sails
And engine, belly the sea-worthy hold,
And see me face to face:
No world, no palace, no Egyptian goddess
Starred over heaven's poles,
Only your pale, impatient, opened mother
Reaching to touch you after the long wait.

Only one of two, beside your father,
Speaking a language soon to be your own.
And strangely, brightly clouding out behind us,
At last you'll recognize
The greater earth you used to take me for,
Ocean of air and orbit of the skies.

Eleanor Wilner, a poet whom you will meet a bit later, reminded me that I had invoked the Egyptian goddess Nut, and sent me an image.

The Goddess Nut

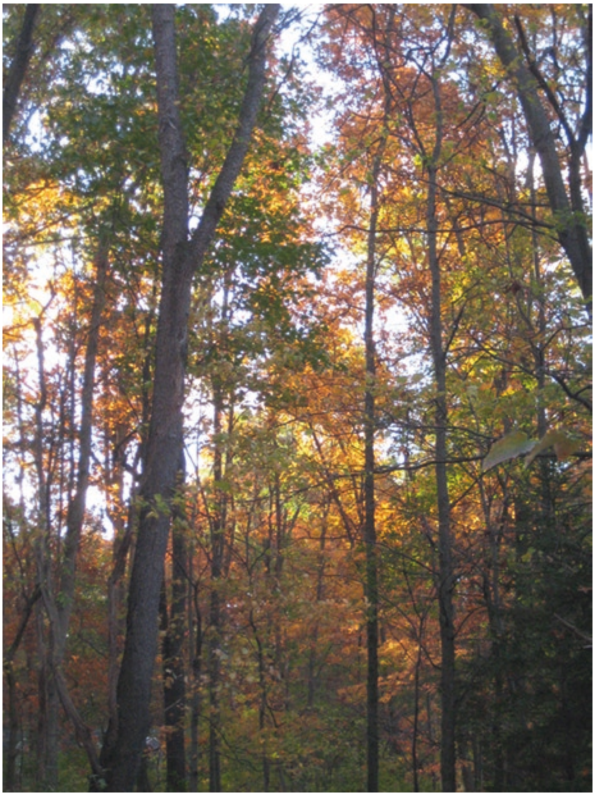
In the first stanza, I echo part of a speech by St. Paul to the Athenians, debating with a crowd that contained Jews, Stoics, Epicureans (and no doubt followers of Plato or Aristotle, though they are not named). Explaining his understanding of God to the Athenians, he says (*Acts* 17:28) “For in Him we live, and move, and have our being; as certain also of your own poets have said, for we are also his offspring.” But then of course I wasn’t *the* world, as the baby would immediately discover when he was born, looking out at the forests and fields, and the sky overhead. I wasn’t God and I wasn’t a Goddess, though I invoke the Egyptian goddess Nut in the next to last stanza, because of the way that she bends across the sky and over earth in the depictions of her. I suppose that the reference is not very Aristotelian, but St. Paul was testifying in Athens, and Ptolemy did after all live in Alexandria and wrote the *Almagest* in Greek.

Later, nursing my third son very early in the morning, as I watched the sun rise through the trees behind our house, I realized that I was looking directly into heaven, and so must be back at Chartres gazing up at the Clerestory. The tree branches became mullions (the vertical wood or stone dividers between the lights, or panes, of a window) and then tracery, the lovely curved elements that divide the panes in stained glass Rose Windows.

Finitude

Awake before dawn, William and I sit drowsing,
Lapsed from a dream, louring toward consciousness,
Nursing a little, musing, counting our toes.
There are always ten, no matter where we begin.
Oh, look. He suddenly points at the closed door-windows
That cast over snow, past spindly lank silhouettes
Of maple, oak, black walnut, into the dawn.

On tiptoe, weaving, he runs up close to the windows
Charmed by the panels of gold set high among mullions
Of boles, the roses fastened in tracery-branches.
Yet how the fastening ravels: our matins are sung,
The windows beyond the windows wither away,
And then he returns to my arms asking his questions
In an ancient, unknown tongue. And all of my answers,
Equally enigmatic, are kisses in shadow.



The woods behind my house (my photograph)

And my daughter somehow showed up in a flowerbed as a snowdrop, flourishing as the snows recede, “les neiges d’antan,” the snows of yesteryear: thus Dante Gabriel Rossetti rendered François Villon’s haunting lines, and I unwittingly borrowed them. To my friend Ruth Geyer Shaw, not only do I owe my first recognition of the ecliptic, but also the odd fact that snowdrops can freeze and unfreeze, and so survive that transition between winter and spring.

Snowdrop

Snow fell so early this year, just after Allhallows,
We never finished the ritual of raking clean
Livid grass and cushions of stricken moss.
The yard's still matted with leaves, oak, maple, walnut,
Visible once again as the snow recedes,
Tatted lace unravelling, going wherever the snows
Of yesteryear retire to, heaven or hellward.
Under the mat of crisscrossed mahogany
And black gold crusted with ice, one snowdrop rises.

She stands already in the outmost bed, bordering
Woods, though it is only February, turned,
Dear Mary-Frances, less than a week ago. I laid
The coverlet of leaves aside and there she was,
Furled on herself and bowed, but blooming hard,
Sober, exquisite child of an uncertain season.

There is another echo: the first line of my poem “Letter from Germany,” written to my mother almost 20 years before, my mother Frances for whom Mary-Frances was named. (As also for my mother-in-law, Mary-Theresa.) Babies are so cosmological! They are the beginning of the world, explaining joy, pain, vivacity and mortality, conferring meaning on the rest of us, and helping us to organize time and space in order, sooner or later, to find our way back home.

Over a period of 15 years, from 1983 to 1998, I worked off and on at the Leibniz Archives in Hannover, editing some of Leibniz’s mathematical manuscripts, guided by Herbert Breger, the Director, and the scholars Eberhard Knobloch and Elhanan Yakira (Grosholz 1987; Grosholz and Yakira 1998). Between 1672 and 1676, Leibniz sojourned in Paris under the tutelage of the Dutch polymath Christiaan Huygens, where, as I like to tell my impressionable undergraduates, he recapitulated the history of Western Mathematics in 4 years and ended up inventing the infinitesimal calculus. Just afterwards, Leibniz returned to Germany and spent the last 40 years of his life in the Court of Hannover, so most of his papers (about 10,000 sheets) are in the Archives. Now they have all be digitized as computer files, but in those days, you could ask for the very pages you wanted, and read Leibniz’s difficult but expressive handwriting, as it were face to face. I found the confrontation very moving. During my sabbatical year 1997–1998, which we spent in Cambridge, England, I wrote up my new understanding of how Leibniz discovered the infinitesimal calculus, and developed his notation for it. I also came to see why his notation was better than Newton’s, and so why it ultimately became the idiom in which the calculus and the exploration of differential and integral equations was carried out in the eighteenth century.

While I was following Leibniz down his pathways of discovery and doing my best to analyze why they led to such exciting new fields and forests, my children were making an equally exciting discovery: soccer! They all gave up their nascent interest in (American) football, thank God, and baseball and basketball, and converted to the sport that only requires a somewhat polygonal, somewhat spherical ball, two goals, and a pitch in between. (This explains its universality.) Just outside our college, Clare Hall, lay a rugby field that they occasionally hijacked for their own purposes, and often late at night I’d find them out there kicking the ball around, with my sons Robbie and William displaying special skill and enthusiasm. Once when the moon was full, I realized that the vista I could see at the edge of the field–pitch was another version of Leibniz’s conception of the continuum. Not only is it the basis of his work on the infinitesimal calculus, it is also, as Herbert Breger explains so well, the basis of his theory of freedom, the way he escapes from the fatal necessity of Rationalism, without giving up his faith in the Principle of Sufficient Reason (Breger 1986).

Trying to Describe the Reals in Cambridge
"For there are two labyrinths of the human mind,
one concerning the composition of the continuum,
and the other concerning the nature of freedom,
and they arise from the same source: infinity.

G. W. Leibniz*, On Freedom*

Draw the curtains! The curtains are always closed
On roses, rugby field, light variable
But waning along these tiered northern skies
Where ten o'clock's the apogee of day,
A full moon pewtering the cliffs of sunset.
I write in the wizened glow of my computer.

I write, the reals are really not like numbers
That we are used to count with, to begin
And go up stepwise. They are number flooded
By continuity, the line upbraided
By differential strands to labyrinth.
They are the shape and cardinal of freedom.

Abysses along abysses along abysses,
Yet perfectly defined. As if we charted
A finest-grained Grand Canyon with passing walls
Through which a sourceless unplumbed river ran,
Like moon-plate cumulant in tiers above
The river of waning sunlight. Draw the curtains!

I start the poem bending over my computer, and working on the mathematics and metaphysics of Leibniz. But by the end of the poem, I have been seduced by that very metaphysics and the light of the moon: it must have been Byron's moon, by which we go a-roving, especially when Joan Baez turns the poem into a song, and Ray Bradbury revives it as a theme in *The Martian Chronicles*. I am ready to draw the curtains (meaning to open them!), and then to rush outside my office, and join the children on the edge of the field, while they frolic in the silver light. That was 20 years ago, and they are still playing soccer; even their father was converted, and he still plays pickup soccer too, whenever he can. As a runner and swimmer, I do not really go in for field sports, but I am happy to watch the odd soccer match at the edges of the fields, especially when my children are playing, and count the goals and pick wildflowers.

I often reflected that a child's world is defined in part by toys, stashed in the toy box or on the shelves of a closet, or next to the pillows on the bed: but they are usually sorted, by kind. The congeries put together by my children recalled my own collection of dolls and my zoological garden of stuffed animals, and the sets of markers and counters that came with my games: checkers with the checker board, legos in their boxes from Denmark (they're called legos because "leg godt" means "play well" in Danish!), the Parchisi pawns that were once (in ancient India) cowrie shells, along with two dice that are cubes with numbers. Toys are one way we learn about sets (because we sort the elements out in kinds) and numbers (because we often count our toys, to make sure none of them are missing after marauding siblings have swept past). A set is a collection of objects, first and foremost objects that can be counted, and that share a property or description: we call those discrete objects the elements of the set, countable one by one using the natural numbers. One, two, three, four, adding a unit each time, and continuing ad libitum, until we have counted them all (toys are always a finite set), and the last number tells us how many. When we count, we put the elements of our set into one-one correspondence with a natural number (positive whole numbers), a member of the set of numbers **N** from which all other numbers descend, or perhaps ascend. If you start playing with paper money in a game where you can go into debt, you realize that the natural numbers can be extended to the integers, which include also 0, when you are broke, and all the negative whole numbers. My adult closet contains (on an uppermost shelf) one of my dolls (a lovely Victorian doll with a china head and delicately painted eyelashes and eyebrows), and one of my stuffed animals, a lamb whose fleece has almost been loved away, but who still sings the melody from one of my favorite hymns: "Jesus, tender shepherd, hear me./Bless thy little lamb tonight./Through the darkness be thou near me,/Keep me safe till morning light."

A child's room also often contains a clock (so you can get up in time for school) and a calendar so you can see when Christmas and spring and summer vacation are coming up, never soon enough). Clocks and calendars do something odd to time: they mod it out! When we give the time of day, we only count up to 12 and then begin again. We call two numbers 'congruent mod 12' if they differ only by a multiple of 12, that is, they leave the same remainder when divided by 12.

$$4 \equiv 16 \pmod{12}$$

In general,

$$a \equiv b \pmod{m} \text{ if and only if } m \text{ divides } a - b.$$

And of course our calendar has 12 months. So even though time flows ever onwards, we can tell time by referring to only 12 numbers for the hours (and then 12 × 5 numbers for the minutes) and 12 numbers for the months (and then more or less 30 numbers for the days, depending on that pesky moon: "O swear not by the moon, the inconstant moon!/That monthly changes in her circle orb…" as Shakespeare's Juliet once pleaded with Romeo). Clearly the choice of 12 is a convention (dating back to ancient Sumer), as the choice of 10 in our arithmetical notation is a convention, widespread around the world because of our ten fingers, though it is complicated by positional conventions in our notation that we owe to India and the Arabic world.

But clocks and calendars (like the house of childhood, as we have noted) offer a finite refuge and orientation in the midst of the wild infinite cosmos and the cruel infinite flood of time. They map the infinite onto the finite, so that we can keep track of where we are. We owe this modding out to the earth and the solar system, our house and neighborhood in space: so the earth turns on its axis, and the moon wheels round the earth, and the earth around the sun. We wake up every morning at the same time (at least, while school is in session) and we come back to the holidays every year at the winter solstice. Congruence (mod n) is an equivalence relation, and is thus reflexive, symmetric and transitive: $p \equiv p \pmod{n}$; if $p \equiv r \pmod{n}$ then $r \equiv \mathrm{p} \pmod{n}$; and finally if $p \equiv r \pmod{n}$ and $r \equiv s \pmod{n}$, then $p \equiv s \pmod{n}$. When we mod out the integers by any natural number, we arrive at a finite group; if the natural number is prime, we arrive at a finite field. More on groups and fields to come.

Yet our human modding out is ambiguous: we know that this morning is not yesterday morning, and this Christmas is not last Christmas. The formal equivalence is shadowed by the asymmetry of time, and nuanced by the inconstancy of the moon. One day we must all leave the house of childhood. Our solace is that we can look forward to the unpredictable marvels of tomorrow morning, and to the gifts of next Christmas.

Twelfth Night

The Christmas tree is dry:
Resin-dropping twigs whose silky needles
Stroked my hand in Advent, break and crumble.
Time, high time, to take the strung lights down,
The ornaments that shiver,
And from the mantlepiece the gilded star
Beside the homeless family it shown on.

Our house in space is here.
Our house in time is the terrestrial year,
Marked for us by the sun's near disappearance
In night and winter storm,
And those three painted fugitives, who huddle
Against the chill of a wind-riddled byre
To greet a shining baby, small and warm.

Half distracted from and half inspired by such Bachelardian reveries, I often walked to my office or the library to study The Group. No, it was not the Vassar girls with actual libidos who star in Mary McCarthy's famously scandalous novel, though she and her friend Hannah Arendt (who came regularly to the University of Chicago because she was a member of the Committee on Social Thought, along with Jamie Redfield and Victor Turner) were well represented on my bookshelves. It was something else! The chapter on Group Theory in Birkhoff and Mac Lane begins with a diagram of a square, and then, a few pages later, a honeycomb. So, following in their footsteps (a pilgrimage possible whilst stuck in a library chair: come with me, reader!), we start by studying the symmetries of the square, that is, we study the transformations under which the square remains invariant. Think of the square as if it were sitting there pinned just at the center of the Cartesian plane you remember from high school. If you rotate that square 90° around its center 0, you arrive at a square indistinguishable from (congruent to) the square you started with; and this also holds true for rotations through 180° and 270°. (A rotation through 360° is just the identity.) The square not only has rotational symmetries, R, R′, R″ (we don't add R‴ because R‴ = I, the identity), it also has reflective symmetries. If you reflect the square in the horizontal axis through 0, you arrive at a square indistinguishable from the square you started with; this also holds for a reflection through the vertical axis through 0, and the two reflections through the two diagonals. Thus we add H, V, D and D′. Notice that the elements we are concerned with here are not figures, but transformations! Next, we define the multiplication of two transformations as performing them in succession. Thus it is clear that, for example, HR = D′; and the product of each reflection with itself is the identity, I. There is a finite algebra of the symmetries of a square, which can be given by a multiplication table that displays rules for forming the product of any two such transformations: it is a square array, with a list of the elements down the left hand side, and along the top. Think how odd it is, to have a multiplication table of transformations (Birkhoff and Mac Lane 1941: Ch. VI).

The set of those transformations, the symmetries of the square, form a group! "A group G is a system of elements which is closed under a single-valued binary operation which is associative, and relative to which G contains an element satisfying the identity law, and with each element another element (called its inverse) satisfying the inverse law." The Associative Law says that $a(bc) = (ab)c$ for all elements a, b, c; the Identity Law says that $ae = ea = a$ for all a, where e is the identity element; and the inverse law says that $aa^{-1} = a^{-1}a = e$ for each a and some a^{-1}. A group has only one identity element e and only one inverse a^{-1} for each element a (Birkhoff and Mac Lane 1941: 130–132). Cut out a cardboard square, and then write out the multiplication table for the group of symmetries of the square, and check it against page 132 in this book. You'll notice that the table is symmetric with respect to the diagonal of entries from the upper left to the lower right: this meta-symmetry means that the group is commutative: that is, it doesn't matter in what order you multiply the elements, the product will be the same.

A commutative group is called an Abelian group, after Niels Henrik Abel, a Norwegian mathematician who died at the age of 26 in 1829 of tuberculosis, after a great journey and leaving his fiancée behind; one thinks of the Englishman John Keats, who died of the same illness in 1821 at the age of 25, in Rome, mourning his beloved Fanny Brawne. In little more than a quarter of a century, both young men left their indelible mark on mathematics and poetry. Abel showed that there is no general algebraic solution for the roots of a quintic equation, or for any general polynomial equation of degree greater than four, in terms of explicit algebraic operations. To do this, he invented Group Theory. At just about the same time, Group Theory was invented independently by a young Frenchman, Évariste Galois, who died at the age of 20 in 1832, as the result of a duel; one thinks of Alexander Pushkin, the great Russian poet, who also died in a duel in 1837, at the age of 36. We are still haunted by the mathematics and poems they left behind, and the work they never lived to carry out.

The range and usefulness of Group Theory is surprising. You will have noted that we introduced the notion of a group by using a geometric example, but recall that the integers modded out by n form a group. Abel and Galois were both working in number theory, on problems that descended from Fermat's Last Theorem. Algebraic structures, like geometric figures, exhibit certain symmetries, which prove key to solving important problems in number theory (Grosholz 2016: Ch. 4). Moreover, chemical structures (which we like to call molecules) exhibit certain symmetries as well, which help to solve problems at the border between chemistry and physics, so Group Theory also plays a central role in Quantum Mechanics. Indeed, any discursive or natural system that exhibits symmetries lends itself to Group Theory.

It might seem odd that a small list of conditions, the rules that define a group, could turn out to be so conceptually important in such a wide range of subject matters. A key to the mystery is that one-one transformations of any set of elements that preserve given properties of the elements form a group. Moreover, in a sense to be explained, group structure precipitates more group structure. To illustrate more precisely the power of this abstract structure, we can look at isometries, isomorphisms, automorphisms, homomophisms and homeomorphisms: we start with isometries. We just saw that the symmetries of the square form a group; they are one-one transformations that preserve distances on the square; such transformations are called isometries. There are eight elements in the group of isometries of the square. But if we start, not with the finite figure of the square, but with the infinite Euclidean plane, the set of isometries of the plane is itself infinite: yet it is also a group! That group consists of all the translations, rigid rotations and reflections of the points of the plane, which map it back on itself, one-one, and preserve distances. Birkhoff and Mac Lane note that Felix Klein, in his exposition of the *Erlanger Programm* (1872) "eloquently described how the different branches of geometry can be regarded as the study of those properties of suitable spaces which are preserved under appropriate groups of transformations. Thus Euclidean geometry deals with those properties of space preserved under all isometries, and topology with those which are preserved under all homeomorphisms. Similarly, 'projective' and 'affine' geometry deals with the properties which are preserved under the 'projective' and 'affine' groups" (Birkhoff and Mac Lane 1941: 128–129).

What about isomorphisms? Here we think about relations among sets that already have group structure. If we have two distinct groups, G and G', we can investigate their relations to each other in various ways. First, we can establish an isomorphism between two groups if there is a one-one correspondence (remember that notion from set theory, when we were suspiciously counting all the toys) between their elements that preserves group multiplication: if the isomorphism assigns a to a' and b to b', then it must be the case that ab is mapped to $a'b'$. The relation of isomorphism is reflexive, symmetric, and transitive; it is an equivalence relation. That means that

G is isomorphic to itself; if G is isomorphic to G' then G' is isomorphic to G; and if G is isomorphic to G' and G' is isomorphic to G'', then G is isomorphic to G''.

What about the relation of a group G to itself? If there is a way to map the elements of a group to the elements of that group so that the mapping is an isomorphism, it is called an automorphism. The identity map is a trivial example, but there are many more interesting examples. In any group G, $a^{-1}xa$ is called the conjugate of x under conjugation by a; you can construct an automorphism of the group G to itself, by mapping each element of G to its conjugate. The proof is very nice: $(a^{-1}xa)(a^{-1}ya) = a^{-1}xa$. (Recall that the identity $e = a^{-1}a$ for every element of the group.) Automorphisms of this kind are called "inner automorphisms," and all others are called "outer automorphisms." Here we see groups spawning groups, because the automorphisms of any group G themselves form a group.

Right around the turn of the century, and the millennium, I used Group Theory to return to the study of algebraic topology, where homotopy groups and homology groups are central to the development of the field, throughout the twentieth century. I also launched into the study of group theory in the context of representation theory (which correlates groups of automorphisms with groups of matrices, more tractable for computation) used to study the structure of molecules, with the help of the chemist and poet Roald Hoffmann, an inspiring role model. Both these studies show up on my book *Representation and Constructive Ambiguity in Mathematics and the Sciences* (Grosholz 2007: Ch. 5 and Ch. 9). They also show up in poems.

Algebraic topology gives rise to the notion of a manifold (which we already encountered whilst thinking about hyperspace), which was key to the study of non-Euclidean geometries. A surface, a two-dimensional manifold, has negative curvature at a given point if it curves away from the tangent plane in two different directions, like a saddle. I would sometimes reflect on this concept while I was exploring the negative curvatures on my husband at given points, but then I would be distracted; so I turned that investigation into a poem, since it seemed inappropriate for a scholarly footnote.

West Wind

I like to wake beside my husband's
Large resilient body, surfaces
My hand rehearses out of pure
And pleasurable habit, consciously,
Especially where his intersecting planes
Make saddle-passes in the uncertain
Alps of darkness pitched against our bed:
Where his neck and shoulders join,
His back shades into haunches, or his thigh
Looms into underbelly with a curve
Shaped by the most magnetic zone
My fingers graze, in passing for the moment.

I know each juncture by its hidden odor
Caught in the dark brown bear-fur of a blond

That sunlight easily spins to gold:
Basil, eucalyptus, harsh vanilla
Queen Anne's lace, cache of wisteria.
His sweet breath riffles on my cheek
As if day returning were the earth's
Lost children coming back again in April.
Who would wake from such a real
And ramifying dream? I switch the tongue
Of our alarming clock from lark
To nightingale, and wait with open eyes.

There is Shakespeare again, inescapable: it was the nightingale, and not the lark! Happily, since we two were not the scions of noble families during the Renaissance, things turned out better for us than for poor Romeo and Juliet.

While I was working on F. A. Cotton's *Chemical Applications of Group Theory*, and regularly sending requests for clarification to Roald Hoffmann, Roald had begun to organize a cabaret series called *Entertaining Science* at the Cornelia Street Café in Greenwich Village in New York City. He kindly invited me and the composer Elliott Sharp to appear with Benoît Mandelbrot at an event about fractals. So I had to write a poem for the occasion!

In Praise of Fractals

Variations on the Introduction to *The Fractal Geometry of Nature* by Benoit Mandelbrot (New York: W. H. Freeman and Company, 1983)

Euclid's geometry cannot describe,
Nor Apollonius', the shape of mountains,
Puddles, clouds, peninsulas, or trees.
Clouds are never spheres,
Nor mountains cones, nor Ponderosa pines;
Bark is not smooth; and where the land and sea
So variously lie
And lightly kiss, is no hyperbola.

Compared with Euclid's elementary forms,
Nature, loosening her hair, exhibits patterns
(Sweetly disarrayed, afloat, uncombed)
Not simply of a higher degree *n*
But rather of an altogether different
Level of complexity:
The number of her scales of distances
Is almost infinite.

How shall we study the morphology
Of the amorphous? Benoit Mandelbrot
Solved the conundrum by inventing fractals,
A lineage of shapes
Fretted by chance, whose regularities

Are all statistical, like Brownian motion,
Whose fine configurations
Turn out to be the same at every scale.

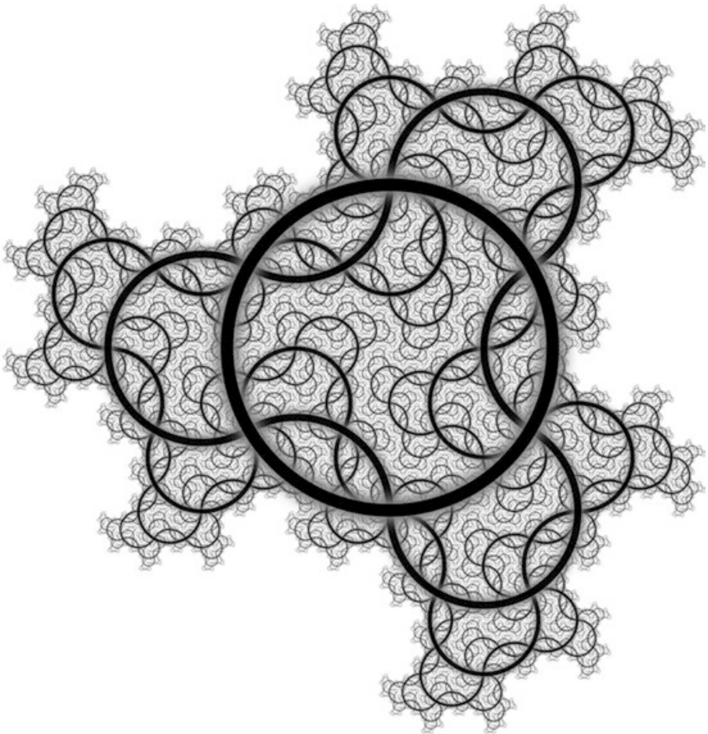
Some fractal sets are curves
(Space-filling curves!) or complex surfaces;
Others are wholly disconnected "dusts";
Others are just too odd to have a name.
Poincaré once observed,
There may be questions that we choose to ask,
But others ask themselves,
Sometimes for centuries, while no one listens.

Questions that ask themselves without repose
May come to rest at last in someone's mind.
So Mandelbrot in time
Designed his fractal brood to be admired
Not merely for its formal elegance
As mathematical structure,
But power to interpret, curl by curl,
Nature's coiffure of molecules and mountains.

What gentle revolution of ideas
Disjoins the eighteenth century from ours!
Cantor's set of nested missing thirds,
Peano's curve of fractional dimension,
Mandelbrot's fractals, counter the old rule
Of continuity,
Domesticating what short-sightedly
Was once considered monstrous.

Nature embraces monsters as her own,
Encouraging the pensive mathematician
To find anomaly
Inherent in the creatures all around us.
The masters of infinity,
Cantor, Peano, Hausdorff, Mandelbrot,
Discovered sets not in the end transcendent
But immanent, Spinoza's darling Cause.

Imagination shoots the breeze with nature
And what they speak (mathematics) as they flirt
Reveals itself surprisingly effective
In science, a wrought gift
We don't deserve or seek or understand.
So let us just be grateful,
And hope that it goes on, although our joy
Is always balanced by our bafflement.



Robert Fathauer, Fractal Curves, 2012. Digital print

Reading up on fractals, I was happy to discover the Cantor Set, which I'd encountered long before at the intersection of formal logic and algebraic topology (who knew!), and to learn that another way of saying that the fine configurations of fractals "turn out to be the same at every scale" is that they exhibit expanding or evolving symmetry (Grosholz 2007: Ch. 10). Take a look at the beautiful images of the Mandelbrot set, the crown jewels of computer graphics: he noted that he worked out his theory of fractals in part by using computer imaging, which he was able to do in the late 1950s because he worked at IBM and had access to powerful computers.

As I mentioned earlier, one of the people I met at Clare Hall was Audrey Glauert. She and I talked about molecular biology, because she was a pioneer in electron microscopy, and about the enigmatic A. A. Milne, but she also encouraged my interest in Andrew Wiles' proof of Fermat's Last Theorem, because he was her godson, and she was very proud of him. When I returned from Cambridge, I discovered that Wen Ch'ing (Winnie) Li was offering a course on the proof; her work is in fact cited in the footnotes! So I attended her graduate seminar, and ever since I've sat in on her seminars every other year or so, and interacted with some of her graduate students. If only I were half a century younger, and could be one of her graduate students, just starting now; but we have already determined that hyperspace does not really perform the magic alleged in science fiction. Still, I have learned a great deal from her, and have profited from her advice and guidance, while I was writing chapters in the first half of *Starry Reckoning: Reference and Analysis in Mathematics and Cosmology* (Grosholz 2016). One day while I was sitting in class, watching her cover blackboard upon blackboard upon blackboard, and marveling at the clarity of her explanations, I asked myself why poets haven't written more odes to their teachers. So the next day I wrote one.

Elliptic Curves and Modular Forms Converge South of the Taklamakan
For Wen-Ch'ing (Winnie) Li

A skein of silk amid the iron and bronze weapons,
The trade routes brought my number theory teacher, Dr. Li,
Who writes faster with white chalk on the blackboard
Then any human being I ever followed across a proof,
Raising clouds of chalk dust at the furrowed extremities
Of each long expedition towards a theorem. Camels
Cough and huddle by the caravansarie, in moonlight.

I carry coughdrops with me in my bookbag, under notes,
So I won't interrupt her train of thought by sneezing,
And try to copy every line she writes, as well as those
Brief detours on heuristics, or her mild evaluations
Of depth or usefulness or interest of conjectures, placed
Unexpectedly like waterfalls down clefts in limestone,
Or her infrequent, offhand explanations of the way
She generalized a printed remark of Serres, from gamma-
Zero-*p* (*p* prime, indexing groups of matrices) to any level.

How algebraic form can complement the smooth analysis
That frames the proof, the complex upper half-plane poised
Like some great violet dome on whose connected face
The primes come out, appearing one by one in constellations
Above the Taklamakan Desert where the Silk Route ran
From Xian, between the winding Yellow River and the great
But perished Wall, to break against the lovely gates of Kashgar.

When this poem was published in the *Mathematical Intelligencer* (Grosholz 2013a), it was flanked by a picture of Winnie next to a set of her impressively inscribed blackboards, and the cover of the issue was a photograph of the corridor of a hotel near the Taklamakan Desert, where the curves of the architecture echoed some of the diagrams in our number theory text. I had been spending my spare time traveling around and in between the Yangtze, Nile and Indus river valleys, not in a minibus but via a series of books, which I wrote up for a wonderful issue of the *Hudson Review* on 'Literature and the Environment' (Grosholz 2013b). So somehow the poem migrated to the Taklamakan Desert, an important location on the ancient trade routes between China, Persia, Arabia and Europe, which I read about in Valerie Hansen's *The Silk Road: A New History* (Hansen 2012), and then rediscovered in one of Karine Chemla's seminars in Paris. For centuries the desert concealed the Dunhuang Star Map, one of the first graphical representations of the stars in ancient Chinese astronomy, dating from the Tang Dynasty, so from the eighth or ninth century CE; it was only discovered in the early twentieth century in the Mogao Caves.

Winnie Li, Teaching (my photograph)

In 2004–2005, my family was lucky enough to spend a year in Paris (Grosholz 2005a). And I was lucky enough to find Karine Chemla in charge of REHSEIS (now SPHERE), my laboratory in the CNRS (Centre National de la Recherche Scientifique) devoted to the history and philosophy of mathematics and the exact sciences. Karine herself is a specialist in ancient Chinese mathematics, though she was born in Tunis to Jewish parents, and raised in Paris, and speaks and writes fluent English and German. She translated and edited (with Guo Shuchun) *Les Neuf Chapitres*, a book whose importance in China was like that of Euclid in the West; it was probably compiled between the first century BCE and the first century CE (Chemla and Shuchun 2004; Grosholz 2005b). SPHERE also includes a collection of specialists on medieval Arabic Mathematics (led earlier by Roshdi Rashed, who was mentored by Jules Vuillemin), and scholars of classical Indian mathematics. SPHERE has been relocated to the new campus of the University of Paris Denis Diderot (Paris 7), in a series of rehabilitated factories called Les Grands Moulins, on the banks of the river Seine. You can sit in the library, and read, and watch the barges float by, on their way to the sea. Thus transported, I also wrote a poem for Karine.

Technical Divination

For Karine Chemla

I.
Classical Chinese divining also includes
Possession by a god, but technical methods
Are more widespread and truly more important.
The preferred media, the go-betweens,
Are tortoise shell, whose plates seem to display
A gold cosmography, and yarrow stalks
Cast on a plane. It's reckoning.

In fact, it's the beginning of many methods
For organizing memory: calculi,
Tables with ordered entries,
Series of answered questions, read in runes.

The disposition, studied over and over,
Of stalks on a smooth surface,
May be in fact the origin of rules
For multiplication, and the hexagrams
Of the *I Ching*, which Leibniz learned to cherish.

The questions posed concern
Sanction and prohibition,
Hidden meaning, promising occasion,
Cause and responsibility.

So the virtue of the sage
Is visibly and bodily displayed,
But one must study how to read
His hands and eyes.

II.
There are however other means
Of chatting up or questioning politely
Or forcibly interrogating Fortune.

Track the wind's direction, or the form
And motion of clouds. Record the regular
Movement of heavenly bodies.

Mark anomalous events in heaven:
Comets, eclipses, supernovae, sunstorms,
Thunder, lightning, rainbows, promises.

Divination tables may be useful,
And dipper astrolabes that track the Wain
Through lunar lodges, twenty-eight in toto.

Some clouds are shaped like animals;
Some are rainbow-tinted, others red,
White, gray, and almost green.

It matters where clouds arise:
Some gather over mountains,
Some are exhaled by rivers or the sea.

Each season has its wind, each wind its music;
Music must be reckoned as a kind
Of tame, intelligent wind.

Dreams too can aid prediction,
Recording where inside and outside *chi*
Clash on the body's battlefield.

With excess *yin*, we dream of fording rivers
In flood; with excess *yang* we dream
Of fires burning cities, burning forests.

With excess *chi* below, we dream of falling;
So we dream of flying
With excess *chi* above, around the heart.

III.
Technical divination
Is not just prophecy for clerks and clerics.

Gods possess the seer in oracles,
Whose voice and consciousness are not her own.

But technical divining's a two-way process
Where each diviner talks back to the gods,

Negotiates, asks them repeated questions
In her own voice and personal awareness.

Thus it bespeaks an attitude towards fate
More like rational persuasion

Than bending before necessity.
God speaks a human idiom

Of dreams and clouds, patterns of wind
And rivers, yarrow stalks that fall,

Segmented years, an empty tortoise shell,
Wandering star-shapes trying to come home.

Another old friend in Paris was Hourya Sinaceur, who was born in Rabat, Morocco, but has lived most of her adult life in Paris. She is a specialist in the historical development of the concept of field (which so far I have slighted along with rings, in these introductory chapters, distracted by groups), and in particular the theory of real closed fields, where logic and algebra interact in an especially striking way, in the last century (Sinaceur 1991). She is also a specialist on the thought of the French philosopher of mathematics Jean Cavaillès (shot by the Gestapo in 1944) and the work of the great mathematician Emmy Noether (sent into exile by the Nazis in 1933, to die 2 years later in Bryn Mawr, Pennsylvania). Cavaillès and Noether together edited the correspondence of Cantor and Dedekind, about which more later. Hourya's devotion to her family in Rabat and to her teaching and family in Paris often made her life complicated (as did Winnie's devotion to her family in Taiwan), so I wrote this poem for her, when I was concerned about the distances.

Mind
For Hourya Benis Sinaceur

The enormous, high-ceilinged apartment near Trocadéro
Echoes, though it is full of books, intaglio'd furniture, and flowers,
As if reflecting the old house in Rabat, now seized and lost,
And the great, oceanless dunes ranged beyond the city walls
That bear the trace of wind sifting, but not of mind.

You write the history of mind, entering its formal labyrinth
With only the silk thread of demonstration to lead you on.
So Hilbert guides you, Poincaré, Weyl, Noether, Cavaillès.
So Emmy Noether grieved for Hilbert's house, her home and circle,
Stranded on the outskirts of Philadelphia, where she died.

So Göttingen fell, the greatest commonwealth of mind
Europe ever knew, dismantled by the agents of the Reich
Who sized up living mathematicians as Catholics, women, Jews.
So Cavaillès was shot against a wall, so Emmy Noether,
Exiled from her algebraic home, succumbed to memory. *Don't you.*

I always wanted to visit the Maghreb, to take the ferryboat from Spain to Morocco, and then go sideways across Algeria, Tunisia, Libya and Egypt, and really travel down the Nile, but I haven't so far. I am waiting for everybody to stop fighting, so I can steer my minivan from Rabat to Cavafy's Alexandria and the Cairo of Mahfouz, to T. Carmi's and Israel Charny's Jerusalem, to Hikmet's Ankara and on to Pamuk's Istanbul, and drive around the Black Sea, searching for the ghost of Ovid, and then around the Caspian Sea, down to Hafez's Shiraz, and over to the Indus, then across to Kabir's Varanasi (Benares) and up into the Himalayas, looking for my friends from Bhutan. (My first son's first babysitter, Thinley Wangmo, is from Thimpu, so we also got to know her husband Sonam Phuntsho and their colleagues Rinzin Dorji and Tobgay Dorji, and we are still friends.) After that? Perhaps I can persuade Karine to take me on a tour of the great cities of China, and then stop one more time in Tokyo and Kyoto, and then in Hawaii, on my way back home across the Pacific: a great circle. (But it leaves out Patagonia, the horn of Africa, and Australia… and one must abandon the minivan for a ship.) I hope to live as long as my friend Yves Bonnefoy, and to write poems as enduring, and endearing, as his, but that means I only have about a quarter century left. So please, dear powers that be, dear people, settle your disputes as soon as you can. If you can't manage that so quickly, at least please don't blow up the world, and I'll stay home in Pennsylvania and compactify, making one little circle-room-house-valley an everywhere, as Donne recommended, and look forward and upward to grandchildren, those constellations.

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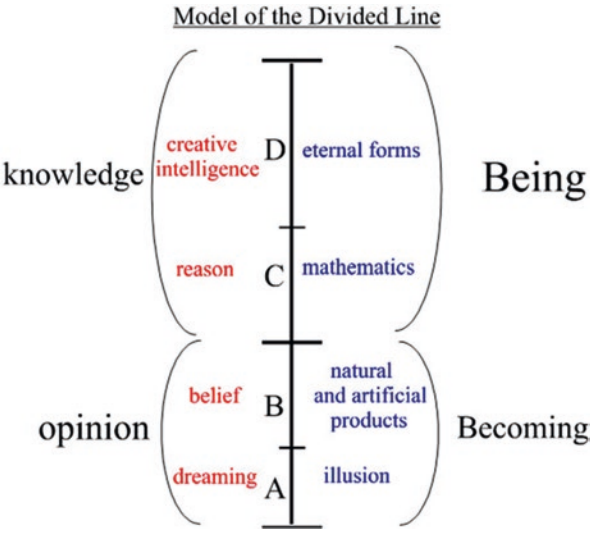
Part II
The Homestead

Chapter 5
What's in a Circle? Spinoza, Leibniz, Marlowe, Shakespeare, Keats

Both mathematics and poetry search for the conditions of intelligibility—the conditions of order, organization, meaningfulness—of the world and human life. Plato's Socrates proposed, in the *Republic*, Book VI, that the things of mathematics and the things in the heavens inhabit the realm of Being, of eternity and truth, whereas we human beings find ourselves down here in the world, in the realm of Becoming, of generation and corruption, of opinion. As I noted earlier, we owe to Aristotle, and to Euclid, the useful notion of a middle term; they are both indebted to Plato's analogy of the Divided Line in the *Republic*, Book VI, 509d-511e. The analogy is stated in terms of a proportion, the assertion of a similitude (not an equality) between two ratios: the ratio A:B is similar to the ratio C:D and also to the ratio A + B : C + D.



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We read off the Divided Line that as Becoming is to Being, so Shadows (A) are to Physical objects (B), and so Mathematics things (C) are to the Ideas or Forms (D). Shadows and Physical objects belong to the realm of Becoming and Mathematical things and the Ideas belong to Being. Given its position on the Divided Line, mathematics plays the role of a conduit from the realm of sense perception and common sense (where knowledge is of changing yet relatively stable physical things) to the realm of philosophical knowledge and eternal things, the Ideas or Forms. The study of mathematics prepares us for the study of philosophy. Thus not in a strict sense, but one might say in a poetic sense, mathematics is like a middle term between Being and Becoming: it is the shadow of the Ideas, and the eternal aspect of the things that change (Plato 1966: 744–747).

Leibniz called the search for conditions of intelligibility analysis, using the term that the Greek geometers, in particular Pappus, used for their best method of solving problems, seeking the conditions of solvability by working out and up from the problem at hand (Grosholz and Yakira 1998: 9–22). In this chapter, to show what analysis looks like concretely, I explore the analysis of the idea of the circle in mathematical and poetical settings. Analysis as the search for conditions of intelligibility of the circle follows upon the perception of the circle as problematic by mathematicians: the search reveals some features and hides others, at any given point in history. The problematicity of the circle shifts according to the mathematical context of solved and unsolved problems, and as it shifts, new features of the circle are revealed. The canonical objects of mathematics (the circle is one of my favorite examples) are inexhaustible: you never know what they will reveal next. Choosing the circle as an exemplary concept, we trace the discovery of a quadratic polynomial equation, the sine and cosine functions, and the *n*th roots of unity 'inside' the circle, as mathematicians discover new conditions of intelligibility for the problematic circle. Then we trace the uses of the circle in poems by Marlowe, Shakespeare and Keats. What we discover 'inside' the circle there is of course quite different (a devil, a planet, a sleeping girl), for what the poet really seeks are conditions of the meaningfulness (intelligibility) of human life. The notion of containment and of intelligibility changes as we move from the investigation of mathematical problems to that of problematic human beings. Still, the circle is part of experience and part of our best conceptualization of the natural world that gave rise to us and provides us with a home: the curve of the horizon, the great circle of the ecliptic, the periodicities of blood and breath, of day and night, of life and death.

To speak of analysis as a search for conditions of intelligibility of an idea in both mathematics and poetry, I invoke the seventeenth century Rationalists. Descartes called upon the light of reason, and used his method, the order of reasons, to secure the truths of mathematics in Meditation IV just after he secured the truth of the existence of God in Meditation III (Descartes 2013: 74–87). Leibniz affirmed, by his Principle of Sufficient Reason, that all of reality (including the possibles that frame the actual and help to lend it meaning) is thoroughly intelligible; God chooses this created world as the best of all possible worlds in virtue of our progress towards greater and greater perfection—harmony and organization. At the end of his essay "On the Ultimate Origination of Things," he writes,

And thus already many substances
    have arrived at great perfection,
although, given the infinite divisibility
    of the continuum,
there are always other parts asleep
    in the abyss of things,
yet to be aroused, to be advanced
    to better and greater stages,
as one might say, to better cultivation.
Thus progress never comes to a conclusion.

This is my translation, making his prose into a poem. The advance of mathematical knowledge and the improvement of human life stand in strong analogy to each other, for Leibniz, governed by his metaphysical principles (Leibniz 1976: 486–491).

Spinoza perhaps most strongly unifies the ethical and the metaphysical import of intelligibility in Parts IV and V of his *Ethics*. When we are in bondage, our ideas are externally caused, we react randomly and our activity is reduced; we merely know *that*. When we are free, however, we affirm our ideas because of reasons, which explicate the inner meaning or intelligibility of thought, and our activity is enhanced: we understand *why*. The true north of Spinoza's philosophical compass is the irreducible seriousness and significance of existence, and of reason as the final arbiter of ethical judgment (Spinoza 1992: 152–200, 201–223). Carried away by Part V of the *Ethics*, one day as I was walking in the fields behind my house, I wrote this poem.

Ode to the Butterflies

Spinoza says you're pure theology: o crazy butterflies aloft
Drinking the last of autumn's flower-wine, as if you thought
There's no tomorrow, as if chicory, vetch and daisies
Weren't on their long last legs, as if the field and forest
Mammals weren't headed underground, as if the coldest
Spells won't cast themselves on us before October's over.
But party, party, party! Spinoza says you're somehow
Cutouts of God's great fabric—matter—divine confetti

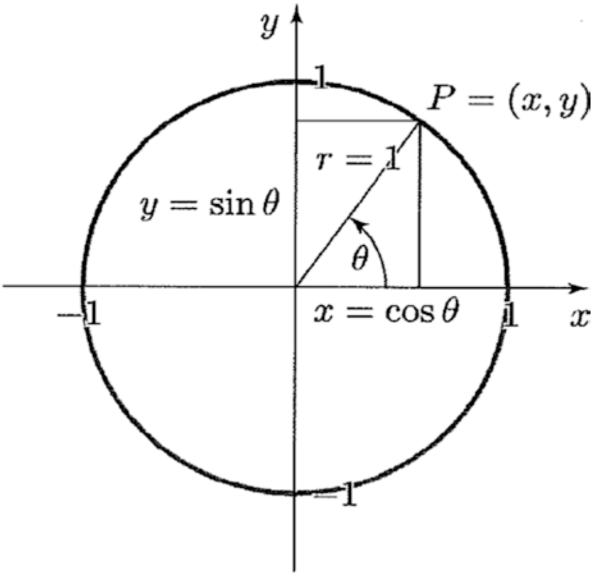
Thrown necessarily up in just these handfuls that I see
Right here, right now, and so am I, this slower body
Earthbound as you distractingly, disarmingly, are not.
Mode of the same sweet attribute, just one amidst an infinite,
Uncountable swarm. So we are all eternal, little buddies!
Succession is illusion, look: the sheer cloud-shadow stipple,
Breezes that shake the goldenrod, your upward-gusting trios,
We're all in this together and like God we're always here!

Ode to the butterflies (my photograph)

Intelligibility is a *value*. Analysis, according to Leibniz, is the search for conditions of intelligibility. If we are analyzing something, we look for its reasons and causes, its requisites, what is needed to make it thinkable and possible, to be what it is (Yakira 2015). Kant claimed that analysis was the discovery of what was "already contained" in the concept, as if it were a suitcase whose contents could be ruffled through, a bare concatenation of concept-components. Open the lid of 'bachelor' and you will find 'man' and 'unmarried' already there. But Leibniz's characterization of analysis is much more interesting: it is to distinguish or develop or articulate the content included in something, analogous to looking into the conditions that allow us to solve a problem. The analysis of an enigmatic thing or problem may stretch on and on, revealing its inexhaustible depths, uncovering new aspects in relation to a series of real or discursive surroundings (Leibniz 1976: 291–296, 547–553, 207–209).

So we begin with a simple example: the circle. Suppose that our knowledge of the circle consisted in unpacking "what was already contained in the definition," which is what Kant asserts. Euclid's definition runs: "A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another, and the point is called the center of the circle." This is a good definition, soon thereafter followed (right after the postulates and common notions) by Proposition I, which presents a diagram with two circles, used to construct an equilateral triangle on a given line segment (Euclid

1956: 153 and 241–243). I want to point out, however, that we cannot deduce from this definition the fact that the sine and cosine functions can be generated in terms of the circle. They are 'contained in' the concept of the circle for Euler, but not for Euclid. How did they show up there, right in the middle, plain as day, in the eighteenth century? The answer is given by Leibniz's notion of analysis, for Descartes and Leibniz himself (a great mathematician as well as a great philosopher) discovered new conditions of intelligibility for the circle by bringing it into novel relation with algebra and arithmetic, as we will see in the next few chapters. We need the Pythagorean theorem (which is about triangles), Cartesian geometry (which is not only about geometry but also about the algebra of arithmetic), the infinitesimal



Sine and cosine functions on the circle

calculus, the completion of the rationals that we call the reals and the notion of a transcendental function.

Once we have embedded the circle in Cartesian geometry, algebra, arithmetic and Leibnizian analysis, and Leibniz has distinguished transcendental from algebraic functions for us, and Euler has developed the modern notion of function, we can see that the functions sin and cos (sine and cosine) can be 'read off' the circle (Youschkevitch 1976). We will trace this development by using Kenneth Burke's four figures of speech, which are also figures of thought: metaphor, metonymy, syechdoche and irony. And we will search for the middle terms that bring formerly disjunct discourses and things into novel and fruitful relation. For when we look closely at the conditions of intelligibility of mathematical truth itself, we are drawn into the study of the history of mathematics: the investigation of what must necessarily hold true of mathematical things is an historical development, unfolding in time. It is moreover creative and unpredictable. This seems ironic. Another irony, which we will also address, is that poetry, while it celebrates nature and the acts of human beings, does its best to resist the flow of time, which is what Plato told us to expect from mathematics. Poetry imposes elaborate formal periodicities on its ways of speaking, turns memory into presence, captures action as narrative and nature as landscape or allegory. So in mathematics we find contingency mixing it up with necessity and history lurking behind truth; and in poetry we find immortal structure shining though the passions and actions of the mortal creatures to whom, and about whom, we sing.

In her book *Jean Cavaillès: Philosophie mathématique*, Hourya Sinaceur reflects on the apparent paradox of approaching mathematics through its history: the title of the first chapter is "Cette histoire, qui n'est pas une histoire" (Cavaillès 1949: 8). However, the method employed by the philosopher Jean Cavaillès and his teacher Léon Brunschvicg at the École Normale Supérieure was avowedly historical, and consciously opposed to the project of Russell and Couturat, who tried to argue that mathematics is reducible to logic, via set theory. Cavaillès knew more about set theory than almost any other philosopher at the time, but he approached it first as a mathematician and then as a historian, editing the Dedekind-Cantor correspondence with his friend Emmy Noether, probably the greatest woman mathematician in the twentieth century (Cavailles 1962: 177–251).

Cavaillès argued that the history of mathematics reveals an autodevelopment that is (in Sinaceur's words) protean yet architectural, a truly unforeseeable, organic development that somehow resolves itself into an architectural unity. And indeed, Cavaillès saw the creative effects of various projects of axiomatization. Group theory, for example, made possible fruitful and surprising discoveries, revealing the transversal character of mathematics, reciprocal exchanges between different branches that till then were never suspected. There is an irreducible dichotomy between arithmetic and geometry, yet the abstract structures of sets, groups, rings and fields (which lend themselves so well to axiomatization) do not accomplish any definitive reduction of one to the other, but rather generate unexpected discoveries precisely at the intersection, the crossroads, where they meet (Sinaceur 1994: 19–22).

As a professor at the University of Strasbourg in 1938, Cavaillès invoked the experience of doing mathematical research as an adventure. Georges Canguilhem observed that Cavaillès seemed to regard Cantor's creation of transfinite numbers as "elevating theoretical conflicts to the levels of passions and uncertainty to suffering," as if that creation were a drama. Yet Cavaillès resisted this drift toward the psychological. The history of mathematics is a becoming, a gradual development, but once achieved, at any stage, it reveals truths that are intelligible and necessary. So in a sense history seems to disappear, to efface itself behind a deductive chain of reasoning. And yet, and yet… it is a becoming, "un surgissement, le jaillissement d'une force vive" (Sinaceur 1994: 23–25). Here the vocabulary might remind us of Paul Valéry, who admired Goethe and shared his love for science (especially optics, which lends itself so well to mathematics), and who ended his greatest poem, "Le Cimetière marin" this way:

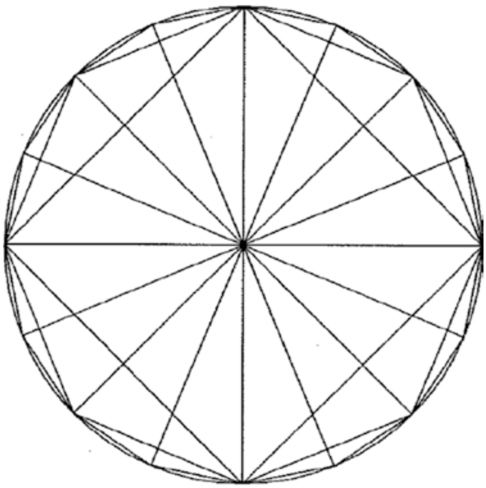
Le vent se lève! … il faut tenter de vivre!
L'air immense ouvre et referme mon livre,

La vague en poudre ose jaillir des rocs!
Envolez-vous, pages tout éblouies!
Rompez, vagues! Rompez d'eaux rejouies
Ce toit tranquille où picoraient des focs!

(I have been to Sète twice to visit this cemetery by the sea, and memorized most of this poem half a century ago.)

How can one explain the way that mathematical research seems to be "A la fois rupture et liason," innovation and determined connection, unifying the unforeseeable and the necessary? One way that Cavaillès takes on this philosophical conundrum, is to dissociate the idea of necessity from that of the Kantian a priori and from the notion of intuition due to Kant and then to a series of thinkers indebted to him, including especially the philosopher L. E. J. Brouwer. Cavaillès, Sinaceur suggests, escapes Kant in part via Spinoza: a contingent occurrence is not without reason, but has its reasons outside of it; contingency is the opposite of autonomy in mathematics and the opposite of freedom in human life. Thus, necessity is internal. Jean Cavaillès, a Catholic, exhibited both kinds of internal necessity in his comportment during World War II, writing up his last philosophical reflections of mathematics in jail and bravely resisting the Nazi occupation, to the end (Sinaceur 1994: 31–33).

We might say the circle becomes problematic in the context of Euclid's geometry, in relation to the lines and triangles that can be inscribed in it or circumscribed around it; many of these problems are resolved. It becomes problematic again, in different ways and in relation to other kinds of things, like polynomials and transcendental curves in the late seventeenth century; many of those problems are resolved. Then in the nineteenth and twentieth centuries the circle emerges again, problematic with respect to newly constituted things, like the complex plane, holomorphic and diffeomorphic functions, cyclotomic fields and more generally algebraic number fields, when it proves to harbor the *n*th roots of unity! (Grosholz 2016: Ch. 1).



Circle with the 4th, 8th, and 16th roots of unity

That is, a mathematical concept like the circle is not problematic in isolation but in relation; its problematicity waxes and wanes. The kind of generalization that one might call upwards embedding (followed by downwards reinsertion), so characteristic of modern mathematics, brings new content to the circle by putting it in novel relation to novel things; but this doesn't make the ontological status of the circle mere relation or structure. The reason the novel juxtapositions I have just pointed to are fruitful for mathematics is because the circle is what it is and has an irreducible unity. The circle severely constrains what can be said of it and how it can be used to frame and encompass other things. The precise and determinate resistance the circle offers to any use made of it contributes to the growth of knowledge. The circle proves itself again and again as a canonical object.

Clearly, if you open the lid of a concept in the course of a poem, you never know what will pop out! Poets are notorious for playing upon the historically somewhat contingent semantic spread of a word, for constructing unexpected similes, and for thinking not always in a narrative line or an argumentative line but often sideways by association. So the process of analysis in poetry looks different from its counterpart in mathematics; however, the differences have been overstated by those who read mathematics too strictly in terms of the schemata of deductive logic, and by those who insist on the irrationality of poetry. A poet may invoke the circle, and look for the conditions of intelligibility of the circle: some of our greatest poets have done precisely that, in some of their greatest lines. Admittedly, poets more often search for the condition of intelligibility of human action, as Aristotle teaches, yet the circle may link, frame, schematize and enmesh human actions: sometimes a poet simply requires a shape with an infinite group of symmetries.

The circle dominates Aristotelian-Ptolemaic cosmology, as we have seen; the ancient astronomers knew perfectly well that the earth, the moon and the sun were spherical, and postulated uniform circular motion for the trajectories of the moon and sun, Mercury and Venus, and Mars, Jupiter and Saturn, as well as the "two-sphere universe" that Thomas Kuhn describes so well (Kuhn 1957). For an English poet, the circle that comes first to the eye and mind is the circle of the horizon as one stands by the sea, a circle produced both by the curvature of our planet and by the limitations of human sight. The sea is never far away. In *Richard II*, Act 2, Scene 1, Shakespeare has John of Gaunt (Chaucer's patron) invoke England as "this sceptered isle" and then go on to elaborate the epithet by comparing England as it might be seen from on high surrounded by the Atlantic Ocean to a planet discerned against the background of the firmament (Shakespeare 2005).

This royal throne of kings, this sceptered isle,
This earth of majesty, this seat of Mars,
This other Eden, demi-paradise,
This fortress built by Nature for herself
Against infection and the hand of war.
This happy breed of men, this little world,
This precious stone set in the silver sea,
Which serves it in the office of a wall

Or as a moat defensive to a house
Against the envy of less happier lands –
This blessed plot, this earth, this realm, this England.

Sometimes the poet's perspective remains on the seashore; sometimes it rises into the heavens, in imagination, so that the whole globe comes into view, a 'lifting' that changes the meaning of the great curves that the eye and mind (and sometimes the heart) encompass. There too crossing the heavens are the great circle of the ecliptic, along which the constellations of the Zodiac are arrayed; the great circle of the celestial horizon, where Orion gleams; the circular faces of the sun and the moon; and the circular shadows cast on the faces of the sun or the moon during an eclipse, those dire portents.

Thus in Christopher Marlowe's *Dr. Faustus* (Act I, Scene III), the magician invokes Mephistopheles in terms of a series of circles. When night falls, we are standing in the shadow of the earth (whether our view of the cosmos is Ptolemaic or Copernican), whose curve gradually engulfs the sky; and if we think of night as earth's shadow cast upon the heavens (the welkin), then it seems like a kind of daily eclipse, not just of the moon but of the whole sky. (The constellation of Orion is invoked because it is especially prominent in winter.) The baleful connotations of eclipse are added to the foreboding of night and winter (Marlowe 2004).

Faust. Now that the gloomy shadow of the earth
Longing to view Orion's drizzling look,
Leaps from the Antarctic world unto the sky,
And dims the welkin with her pitchy breath,
Faustus, begin thine incantations,
And try if devils will obey thy hest,
Seeing thou hast pray'd and sacrific'd to them.
Within this circle is Jehovah's name,
Forward and backward anagrammatis'd,
The breviated names of holy saints,
Figures of every adjunct to the Heavens,
And characters of signs and erring stars,
By which the spirits are enforc'd to rise:
Then fear not, Faustus, but be resolute,
And try the uttermost magic can perform.

And upon the floor is another circle, echoing and demonizing its heavenly counterparts, within which Dr. Faustus writes anagrams of the name of God and the names of saints (forwards but also disrespectfully backwards), and characters of "signs and erring stars," that is, the outer planets with their retrograde motion. From that circle Mephistopheles arises: the circle has produced the arch-demon, as 300 years earlier he was found embedded in Dante's downward concentric circles, at the center of the earth.

Shakespeare too invokes similar cosmological circles in the most beautiful lines in *The Tempest* (Act IV, Scene I) and perhaps in all of English poetry.

(Shakespeare 2004) Throughout the play the human and inhuman characters often turn their eyes to the circle of the horizon, located as they are—imprisoned as some of them are—on a little island. A play within a play is always an important turning point in Shakespeare's drama, where the actors who are spectators are brought close to the audience, and the actors who are players draw close to the poet. Prospero announces the end of the play, dispersing the actors, an epilogue within the play that prefigures the final Epilogue that closes the play.

Our revels now are ended. These our actors,
As I foretold you, were all spirits and
Are melted into air, into thin air:
And, like the baseless fabric of this vision,
The cloud-capp'd towers, the gorgeous palaces,
The solemn temples, the great globe itself,
Yea, all which it inherit, shall dissolve
And, like this insubstantial pageant faded,
Leave not a rack behind. We are such stuff
As dreams are made on, and our little life
Is rounded with a sleep.

The poet's perspective sweeps up from the island to the heavens, where we can look down in imagination to see not just the little island but the coast of Italy and the Mediterranean littoral with its towers, palaces and temples, and then the whole globe, the great circle viewed impossibly from heaven. And yet this vision is only a vision, the result of our finitude in space and time, which like a dream will fade: "our little life is rounded with a sleep." The cycles of nature that render our life intelligible, and the circle of the horizon that organizes our vision, also bound them—by the comfort of sleep and the erasure of death when life comes full circle.

Keats too had the habit of haunting the English seashore, traveling twice to the Isle of Wight (I visited those houses and stood in the two small rented rooms he stayed in), and north to the Hebrides; the sea haunts his poems. In two of his most important sonnets, he moves from the perspective of the seashore to a cosmological vision of the heavens, but that "lifting" ends very differently in the one and the other. In "When I have fears…" the poet resembles the mage-astronomer reading the constellations and his archaic books.

When I have fears that I may cease to be
Before my pen has glean'd my teeming brain,
Before high-pilèd books, in charact'ry,
Hold like rich garners the full-ripen'd grain;
When I behold, upon the night's starr'd face,
Huge cloudy symbols of a high romance,
And feel that I may never live to trace
Their shadows, with the magic hand of chance;
And when I feel, fair creature of an hour!
That I shall never look upon thee more,

Never have relish in the faery power
Of unreflecting love!—then on the shore
Of the wide world I stand alone, and think,
Till Love and Fame to nothingness do sink.

The writing of poetry here resembles the casting of horoscopes, studying the migration of the planets across the band of the Zodiac with its constellations, "Huge cloudy symbols of a high romance," for the astrologer indeed traces "their shadows, with the magic hand of chance," somehow wresting from his calculations the meaning of a human life. Then Fanny Brawne intervenes, "fair creature of an hour," who strangely matters more than the stars, but whom he will certainly lose sooner or later (and in January 1818, he knew he was very ill). This crisis precipitates the "lifting": "then on the shore/of the wide world I stand alone, and think,/Till Love and Fame to nothingness do sink." The complexity of this image is very strange: the poet stands on the shore, as by the sea, but the shore is that of the whole world, the wide world, as if he could see the whole globe. Yet this perspective must be the imaginary view from space, where the world is engulfed in the infinite darkness of space, and the earth dwarfed by a million suns no longer organized by human sight in constellated patterns on the ecliptic, but scattered throughout the endless universe. So the poem itself is lost in a beautiful nihilism, like Dr. Faustus.

The last of Keats' sonnets (perhaps) is entitled "Sonnet written on a Blank Page in Shakespeare's Poems."

Bright star, would I were stedfast as thou art—
Not in lone splendor hung aloft the night
And watching, with eternal lids apart,
Like nature's patient, sleepless Eremite,
The moving waters at their priestlike task
Of pure ablution round earth's human shores,
Or gazing on the new soft-fallen mask
Of snow upon the mountains and the moors—

No—yet still stedfast, still unchangeable,
Pillow'd upon my fair love's ripening breast,
To feel for ever its soft fall and swell,
Awake forever in a sweet unrest,
Still, still to hear her tender-taken breath,
And so live ever—or else swoon to death.

This time the poem begins high up, looking down from the perspective of Polaris the North Star, the star towards which the axis of the rotating earth points so that it always remains, from the human perspective, where it is: "stedfast." From that vantage, all the oceans of earth are visible, "the moving waters at their priestlike task/ Of pure ablution round earth's human shores," as well as the plains and the great mountain ranges.

But then the poem abruptly changes its meaning and descends via a kind of (not upwards but downwards) embedding, announced by the word "pillow'd." The "stedfastness" of Polaris becomes that of the cycles of nature; the spatial symmetry of the heavenly spheres becomes the temporal symmetry of periodicity: recurrences of breath and pulse, of the tides, of day and night. (Periodicity is symmetry in time.) And the fallen star has conjured up a girl, Fanny Brawne again; there she is next to him: "yet still stedfast, still unchangeable,/Pillow'd upon my fair love's ripening breast,/To feel forever its soft fall and swell..." (Note how the beat of the iambic pentameter and the recurrence of the liquids and sibilants in those lines mirror their sense.) Perhaps the cycles of nature include a premonition of death, rounding out our little life with a sleep; or perhaps the circle is what Aristotle and Ptolemy took it for, the emblem of eternity within time. For there he is in the enchanted circle of the poem, John Keats, sleeping next to his girl: "awake forever in a sweet unrest" (Keats 1994).

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Chapter 6
Four Master Tropes, from Euclid to Leibniz, with Burke

Kenneth Burke begins his essay "Four Master Tropes" by identifying four figures of speech—metaphor, metonymy, synecdoche and irony—as figures of thought, observing: "My primary concern with them here will be not with their purely figurative usage, but with their role in the discovery and description of 'the truth.'" He then characterizes metaphor as "a device for seeing something in terms of something else. It brings out the thisness of a that, or the thatness of a this. The trope of metonymy elaborates the correlation by making reductions possible: "the reduction of some higher or more complex realm of being to the terms of a lower or less complex level of being." Metonymy is useful for problem solving, up to a point, but then synecdoche offers a broader set of strategies of representation: "part for whole, whole for part, container for the contained, sign for the thing signified, … cause for effect, effect for cause, genus for species, species for genus, etc. All such conversions imply an integral relationship, a relationship of convertibility, between the two terms." And of irony, he writes, "Irony arises when one tries, by the interaction of terms upon one another, to produce a development that uses all the terms." It is a dialectic, which, like all good philosophical dialectic and all good political deliberation, leads beyond itself (Burke 1969).

Inspired by Kenneth Burke, and by Cavaillès, we will look at the transformation of geometry from Euclid to Descartes. I will argue in this chapter that figures of speech, which we associate with works of literature and poetry in particular, also constitute figures of thought in the growth of mathematical knowledge. Tropes frame, provide solutions within, re-organize and break out of, a universe of discourse: a universe of discourse is a context in which discourse is possible and what may stand for what is well understood. Metaphor sets up a discourse by means of a correlation or a perspective; metonymy chastens the discourse by means of well-defined reductions; synecdoche develops it by various inventive representations; and irony exhibits its limits, setting it into the context of a larger dialectic. These habits of thought are salient in mathematical traditions as well as poetic traditions; thus, such literary terms may help students of both mathematics and literature understand how mathematics develops.

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A metaphor is a trope that sets up a universe of discourse between two poles, two polar terms. Following, and diverging from, Cassirer's arguments in *Substance and Function*, I will argue that the constitutive metaphor in Greek geometry was that the things of geometry are to be understood as constructions, and so as artifacts. Artifacts in the Ancient World were strongly distinguished from natural substances of the kind that Aristotle focused on, like plants and animals. But artifacts like bricks, pots, fashioned stones, houses, and by extension fenced fields and gardens, were the cultural inventions that led human beings out of the Paleolithic era into the Neolithic; human experience with artifacts led to the first formulations of geometry. Thus, the things of geometry (and arithmetic) were treated as if they were separate and distinct. The two areas of applied mathematics in the Ancient World were the theory of simple machines, and astronomy: the heavenly bodies (moon, sun, and five planets) moving in predictable circular orbits around the earth, along the great circle of the ecliptic in the heavens. The heavens exhibited a beautiful, divine, eternal system, with just few distinct, spherical moving parts.

Simple machines could be analyzed by using ratios and proportions. As we have already noticed, the conceptual schema of ratio and proportion is what allows Euclid to assert the similarity (but not the equality) of one ratio between two numbers and another between two line segments, the kind of similarity that allows the proof of the Pythagorean Theorem to motivate the discovery of irrational numbers like the square root of two. It is also the schema that Plato uses to organize the Divided Line. The poet Donald Davie wrote especially well about the analogy between poetry and sculpture, the relation between stone and the finished work, and the presence of stone in landscapes, in the fields and in the houses. He also loved Italy and took his family there almost every summer: his son Mark Davie became an Italianist at the University of Exeter. Since his retirement, Mark Davie has published a new translation (with William R. Shea) of Galileo, *Selected Writings*, which supports my earlier claim that Galileo was a great (poetic) man of letters, as well as a great scientist (Galileo 2012). So I thought of Donald Davie the next-to-last time I was in Rome, looking at the beautiful constructions, with their Greek echoes.

A Meditation on Stone

There it stands, the Pantheon,
The same as ever: shallow dome,
Pillars shouldering the pediment
Which for millennia declared
Agrippa fecit. So
Stand the Colossum,
Titus' Arch and Constantine's,
The Wall Aurelian raised:
Waiting, it seems, for us
When we return at last.

But look at those blank faces,
The eyeless brow, the hand,
The flank, the granite ledge.

Though we recall each splendid
Ostinato, what they stood for
Fifty years ago, or
Twenty centuries, our great
Progenitors in stone
Cannot remember us,
Or chasten, or forget.

The Pantheon (my photograph)

As Cassirer remarked, the re-formulation of parts of mathematics in terms of function did not arise until much later, because, as I will try to show, the constitutive metaphor of construction with the attendant schema of ratio and proportion both organized and limited Greek mathematics. Two interesting observations follow from this. The first is that, when Descartes envisioned a universal science based on mathematics, he then understood nature in terms of (soulless) machines. The second is that, to escape the strictures of Greek mathematics, the mathematicians of the Early Modern period had to understand the things of mathematics in relational, functional terms, not as fixed and separate constructions. This involved adding algebraic expressions, polynomials and polynomial equations, as middle terms to bring numbers and figures into novel relation.

In the *Geometry*, Descartes demonstrates the power of his new method by proposing a solution to a generalization of Pappus' Problem, one which Greek geometers could formulate but not solve or properly generalize. Descartes' correlation-hypothesis, enunciated in his calculus of lines adumbrated by algebraic expressions, first brought the algebra of arithmetic systematically into the service of geometry, so that the combined fields could together solve this problem, intractable to the Greeks. Certain features of the internal organization of Greek geometry explain why they could not handle Pappus' Problem, first and foremost the habit of regarding the things of geometry as like constructed artifacts. Here I understand Greek geometry not only in terms of relations of derivation between axioms and theorems, but also in terms of links among problems discovered in the strategies of problem-solving. By examining the problem-constellation, we can understand the limitations of an area of research, due to its constitutive metaphors, and also how extensions become possible. Following the discussion of Greek geometry, I will review certain developments which took place during the period intervening between Pappus and Descartes, that we might call metonymy and synechdoche, and prepared the ground for Descartes' correlation. They include the emergence of algebra, and a general tendency towards the functionalization of mathematics that, ironically, requires novel relations between numbers and figures (Descartes 1954: 17–37).

Theorems represent solved problems, and the articulated theorems and axioms of a domain of research exhibit the boundary, so to speak, of the problems of the domain that have been brought under control. (The shape of a theory is therefore explained by reference to a collection of mastered problems.) Most areas of research still contain many problems that evade solution, because solutions to some problems may generate new and more difficult problems, and because important problems, or their generalizations, may simply be very difficult.

In order to understand the mutual relations of problems to each other within an area of research, of problems to theorems and of discovery procedures to proof, we can review the distinction (first drawn by the Greeks) between analysis and synthesis, and, following Pappus, to make a further distinction between theoretical analysis and problematic analysis.

The following definition of analysis and synthesis is taken from Euclid.

Analysis is an assumption of that which is sought as if it were admitted, and the passage through its consequences to something admitted to be true.

Synthesis is the assumption of that which is admitted, and the passage through its consequences to the finishing or attaining of what is sought.

Synthesis is the finished proof, and the derivation of the theorem expressing the solved problem from axioms and theorems already proved. Analysis is thought of as the discovery procedure, and the synthesis, roughly, as the reversal of the analysis. However, Pappus draws a distinction between theoretical analysis and the problematic analysis which is crucial to an understanding of the relation between theorems and problems, and between discovery procedures and proof (Euclid 1956: II, 442).

Theoretical analysis, Pappus writes, assumes that the thing sought is true, and that it exists, and deduces from it some proposition of which the truth value is known. It determines if the theorem is valid by finding the intermediate steps that, when the analysis is reversed to a synthesis, provide the deduction from known to unknown. It can be formalized as

$$K \leftarrow P_n \leftarrow \cdots \leftarrow P_2 \leftarrow P_1 \leftarrow P$$

where P represents the proposition under investigation, K an established proposition, and the P_i intermediate propositions that are consequences of K or P. This notion of analysis demands consideration of the notion of logical reversibility, since some theorems do not have valid converses and the above sequence cannot simply be reversed to obtain a synthesis. In such cases, the converse may be made valid by the addition of what the Greeks called diorismoi, which specify the conditions under which the problem admits a solution (Mahoney 1968: 329–30).

Problematic analysis, however, is a discovery procedure that is explained not in terms of logical derivation between sentences, but rather in terms of relations between problems. The intermediate stages of problematic analysis lead from a solution of a problem known or surmised to be more tractable back to a problem that so far has resisted solution. The salient feature of this kind of problem 'reduction' is that both problems may not yet be solved; the analysis instead discovers that a solution of the former will also provide a solution to the latter. This kind of analysis was a continually growing body of related problem-solving techniques, a mathematical tool-box, more suggestive than prescriptive. Most important, the reduction of less to apparently more tractable problems, while not an effective or mechanical procedure, is still systematic. It provides a unification of the area of research that is local and not global, that cannot be described in terms of logical derivation alone, and that not only orders the problem-context but may also expand it. Discovery procedures may lead to solutions or new problems that require assumptions stronger than those governing the context in which the original problem was first posed. A view of synthesis as the mechanical deduction of a theorem from a set of fixed premises, and of analysis as the inversion of synthesis is thus inadequate (Mahoney 1968: 332–336). For further interesting discussion of the Greek conception of analysis as a conduit to discovery, see Carlo Cellucci's recent book, *Rethinking Logic: Logic in Relation to Mathematics, Evolution and Method* (Cellucci 2013).

One way of understanding metonymy in mathematics is to look at the reduction of one kind of problem to another. For example, in the course of Hippocrates of Chios' investigation of the quadrature of curvilinear figures, he reduces a difficult construction of a type of rectifiable lune (a crescent shape resulting from two intersecting circles, just like a crescent moon) to the general problem of inserting between two given lines—either straight or curved—a line segment of a given length which, when extended, passes through a given point. As a general technique this method, called neusis, also proved useful in the problem of the trisection of an angle. While the neusis of the quadrature problem could be solved using compass and straightedge, the neusis of the trisection problem could only be solved in terms of intersecting conic sections. The more general problem was, finally, more difficult; it is not clear that the mathematicians involved could or did solve mathematically the neusis on which their solutions rested. Here the reduction not only solves some problems, but generates others (Mahoney 1968: 332).

A similar example is Plato's use of the application of areas (a technique that I will discuss shortly) to inscribe a given area as a triangle in a given circle, a construction which also involves conic sections and a diorismos. The diorismos locates the conditions under which there is at least one solution, that is, when the given area is such that a certain hyperbola is tangent to the given circle. This condition leads into a whole new body of problems, those of tangency, later treated at great length in the *Contacts* and *Conics* of Apollonius (Mahoney 1968: 336).

These problem reductions order the field of Greek geometry by associating problems that can be regarded as special instances of the same general problem, or as equivalent in the sense that a solution of one provides a solution of the other. They suggest the investigation of more difficult general problems, which may in turn uncover some of the deeper reasons for the association of various classes of special problems, so they may, ironically, lead to an extension of the area of research.

But if the boundaries may be thus expanded, how can we precisely delimit the area of research? In particular, what constitutes Greek geometry? Euclid's *Elements* treats only problems which are solvable by means of compass and straightedge, whereas Apollonius' *Conics* makes use of much wider construction postulates. The understanding of construction and constructability alter a bit; the metonymy of problem reduction opens the door to the synecdoche of broader and more fertile correlations. There is no ready-made criterion to use in determining such boundaries, as the areas of research grow and change. The claim that a certain collection of problems and problematic items should be regarded as a unified area of research requires arguments based upon historical investigation. The boundaries of Greek geometry, I think, may best be indicated in historical retrospect positively by the metaphor of construction and the middle term of the proportion between ratios, and negatively by the absence of true algebraic notation, as a decisive limit and boundary.

In this regard, Archimedes' method of exhaustion and compression is a telling example of synecdoche, since it seems to point towards the infinitesimal calculus, but does not really open the gate. The method of exhaustion is a procedure for finding the area of a shape by inscribing inside it a sequence of polygons whose areas converge to the area of the containing shape. This method made possible the quadrature (finding the area) of a convex point set for a restricted set of curves (Whiteside 1961: 331). Its richness lay in the variety of suggestive geometric approaches exhibited by the wide range of problems Archimedes worked on. However, it failed to unify that problem-context; though many of the problems have a common dependence on certain trigonometric series, Archimedes could not, at that point in history, discern the commonality, and made no attempt to relate such results. These same proofs attracted a great deal of attention as the basis for geometric integration techniques in the seventeenth century, when the application of algebraic methods helped uncover the deeper reasons for their association. Moreover, Archimedes always gave every proof in its entirety. He never tried to avoid this repetition by supplying a general proof to then go on to deduce special results from it; and he often used special artifices in building and exhibiting upper and lower sums. He never passed from the sequence of polygons to the limit, but instead argued by contradiction (Baron 1969: 34f.). Thus while a method of problem-solving may unify and extend a field, it may also impose restraints that keep items and problems segregated. Archimedes' method never suggested a calculus; neither did it suggest an extension to curves that had not already been encountered in Greek geometry. The different curves were treated as if they were separate and disparate, like pots or carved stones.

A final example of problem-reduction leads to the discussion of Descartes. In his *Data*, a rearrangement of materials from the *Elements* that dealt more with the analysis of new problems, Euclid gave prominent place to the doctrine of the application of areas, a method that reduces a given problem to the manipulation of equations between areas (Taisbak 1991). Here is the simplest case: Given a line L and an area A (let's say, the area of a given triangle), construct a line W such that a rectangle with width L and height W has the same area as A. The Greek description of this problem was, "to apply an area (*parabolē tōn xōriōn*) A to a line L." There are three more analogous, slightly more complex cases; they were called, respectively, applying an area parabolically, hyperbolically and elliptically. Apollonius' treatment of the application of areas was deeply connected to his investigation of the conic sections, so that he classified them as parabolas, hyperbolas and ellipses. Subordinate to his main goal of determining loci and their characteristics, he tried to express statements of equality between areas in as short and simple a form as possible (Fried 2012: 123–124). He did not try to make the fixed lines of his construction as few as possible, as Descartes did, relating the construction to a coordinate system and minimizing the number of independent variables in the relevant equations. Apollonius' expectations about how the problem could best be reduced were different. These techniques worked well enough for the simplest case; however, they tended to multiply auxiliary constructions, adding lines whose mutual relations were not clear, when applied to problems of higher complexity. They generated either a multitude of special cases, which then had to be solved piecemeal, or, worse yet, gave no solution at all. The methods therefore did not bear extension to more difficult cases; the limits of its usefulness were exhibited in the course of mathematical practice (Klein 1968: Ch. 9).

We might pause for a moment to note that those three terms just mentioned have come to have secondary literary meanings: the parable is an allegory, moral tale or fable; the hyperbole is rhetorical exaggeration, magnification or embroidery; and the ellipsis, missing words, supposed to be inferable from context. I turned the parable back into a parabola (visible, invisibly, in air) in this poem, inspired by the way my son Benjamin missed, and named, his babysitter's daughter Lulu, on weekends. One summer when I was working at the Leibniz Archives, around the same time, we were both happy to discover a Café Lulu on the pedestrian street in central Hannover.

The Shape of Desire

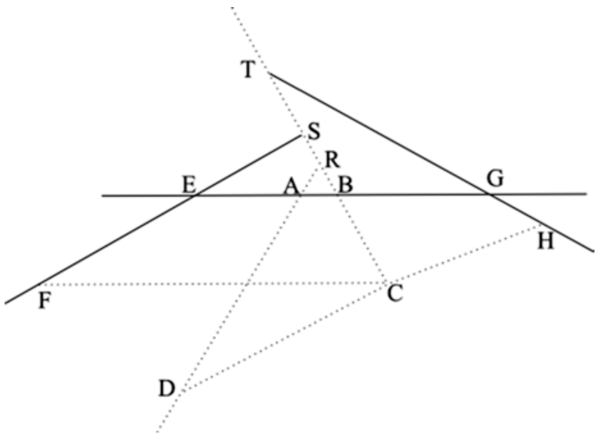
Tracing an airplane's pale trajectory,
You always point, and finish, "Airplane *gone*."
Waking from dreams about your babysitter's

Dark-eyed, clever daughter, you conclude,
"Lulu *gone*," and hurry to the door's
Long windowpane to see her reappear
Freshly composed from memory and clouds.
Now you can say the shape of your desire.

Now you believe that each sidereal item
Carries a left-handed banner to describe
Through curl and dissipation how it was,
That every friend is summoned by a name,
Even in parting. You are wrong, and right
About the frail parabolas of love.

The objects of Greek mathematics were understood in more concrete than abstract terms, or as Cassirer would say, more substantial than functional terms, or, as I would say, as metaphorically like artifacts. The chief task of Greek mathematics was to discover the inherent properties of various separate geometric figures, or numbers as various collections of units. These conceptual constraints on the kinds of items that could figure in Greek geometry and arithmetic, and the consequent absence of a certain notation, also blocked any true algebraization of geometrical problems. Though Apollonius used the tangent and corresponding diameter of a conic as a kind of coordinate axis, this technique could not be generalized or extended (Dieudonné 1974: 14).

The crowning irony in mathematics is the ability to pose a problem that one cannot solve. Thus, given the means at his disposal, Pappus could frame but not solve the problem basic to Descartes' *Geometry*, the solution to the four line locus problem. The following diagram illustrates the case of four fixed lines 'given in position.'



The problem is to determine the nature of the locus of points C such that, when CB, CD, CF and CH are drawn under given angles to them (meeting them at points

B, *D*, *F*, *H*), then CB × CF = CD × DH. Pappus observes that this problem had not been entirely solved by himself, Euclid, or Apollonius; and the loci generated by five or more lines given in position were not known. Descartes begins the *Geometry* by proposing a solution to this problem, for any number of lines 'given in position' (Descartes 1954: 17–37).

In the interim between Pappus and Descartes, two developments in mathematics laid the groundwork for Descartes' correlation-hypothesis and proposed solution for Pappus' problem. They were the gradual functionalization of mathematics and the emergence of algebra. A short history of the notion of function itself will serve as a preliminary illustration of these tendencies. The history that I give here is, in fact, much too short. During the past 15 years, as a member of REHSEIS and then SPHERE at the University of Paris Denis Diderot—Paris 7, I have come to have a better grasp of world mathematics. The real story of this development is laid out in Karine Chemla's edited volume, *The History of Mathematical Proof in Ancient Traditions*, which offers a much broader context in the ancient world; and the *Sourcebook in the Mathematics of Medieval Europe and North Africa*, edited by Victor Katz, Menso Folkerts, Barnabas Hughes, Roi Wagner, and J. Lennart Berggren, which offers a much broader medieval context (Chemla 2012; Katz et al. 2016). But here again I do not have world enough and time to trace out for you these remarkable compendia: go read them yourself!

It has been argued that mathematicians in antiquity had a general conception of function, since they made use of functional dependencies between quantities, specifically in the astronomy of Ptolemy. However, under the metaphor of mathematical item as constructed artifact, these dependencies were expressed as tables with discrete entries; the hypostatization of the entire series was never entertained as a possibility. The limitations of Greek mathematics precluded the recognition of functions themselves as mathematical items; its procedures for calculating or determining individual concrete limits never led to an explicit formulation of general concepts of a sequence, variable, limit, infinitely small quantity, integral, or of general theorems concerning these things. This limitation can be linked to the lack of any true algebraic notation, and to the general exclusion of kinematic ideas from Greek mathematics (Youschkevitch 1976). Even in astronomy, the constant, circular motion of the heavenly bodies was understood as the closest that motion comes to the stasis of eternity: periodicity imitating symmetry.

During the Middle Ages, the modern idea of function received some impetus from Arabic methods of linear and quadratic interpolation; as well as from the study of non-uniform motion in the schools of natural philosophy at Oxford and Paris. The doctrine of intensities of forms (uniformity and difformity of intensities) treated functional relations in kinematic and geometrical terms, and introduced the important notions of instantaneous velocity, acceleration, and variable quantity. Oresme, however, in his treatise on intensities, made the telling restriction: "Every measurable thing, except numbers, is imagined in the manner of continuous quantity." Only when the concept of function was introduced as a relation between sets of numbers rather than 'quantities' did the modern conception of function, with the algebraic-analytic representation of functions by formulae, emerge. Then the chief problems of the new sciences, to study the relation between curvilinear motion and the forces affecting motion, could be carried through to numerical solutions (Youschkevitch 1976). As we shall see, the process of revising the notion of number took many centuries, and still raises unresolved issues.

New metaphors emerged, motivated (I would argue) by the mechanics of gardens and weaponry in sixteenth century Italy. The machines in Italian gardens involved hydrodynamics, the flow of water, and its power directed here and there. They inspire Descartes and Gassendi to re-imagine nature as nested machines. The weaponry in Italy included cannons: a cannonball, unlike a planet, falls to earth. What is the shape of its trajectory, and how should one characterize its acceleration? So does an arrow. They inspire Galileo to mathematize projectile motion. The study of the kinds of machines that exhibited forces leads from statics to kinematics to dynamics. Meanwhile, Johannes Kepler, analyzing Tycho Brahe's meticulous astronomical data, realized that in fact the planets describe ellipses, and accelerate and decelerate, but in an orderly fashion that was captured by his Law of Areas (Drake and Drabkin 1969: 3–60). So there is the metaphor of dynamical processes as flow, and as flight that combines inertial motion and falling. As the expression of algebra and algebraized analysis were introduced, they brought about a new equivalence between the things of geometry and the things of arithmetic: what would geometry become when it must represent processes and the vagaries of flight? What would arithmetic look like when numbers must flow, continuously?

Though the notion of function was not formulated as such until the eighteenth and nineteenth centuries, the new method of introducing functions by algebraic-analytic expressions first began to emerge in in the work of Vieta, Fermat and Descartes. Formerly, functional dependencies had been introduced verbally, by a graph, kinematically, or by tables. The new method, however, allowed the introduction of functions by means of formulae and equations. The incipient notion of a functional dependence between an independent and a dependent variable appears, for example, in this passage from the *Geometry*: "If then we should take successively an infinite number of different values for the line y, we should obtain an infinite number of different points for the line x, and therefore an infinity of different points, such as C, by means of which the required curve could be drawn" (Descartes 1954: 34).

The tendency towards functionalization produces not only an alteration in what can count as items in mathematical domains, but also a genuine increase in informational content. Consider, for example, the relation between a function and its instantiations as exhibited in the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

which can be specified to represent any and all of the conic sections. The overarching function can't be deduced from a bare summation of its instances; no collection of discrete particulars can add up to a function. The function, a thoroughgoing rule of succession, is itself something above and beyond its instances. It contains more information, not only because its instances can be deduced from it and not vice versa, but also because it shows the relation of all its instances to each other in their systematic connection. This is why Cassirer is so enthusiastic about the new metaphor: a curve is a function (Cassirer 1953: Ch. 1). Fermat's equation covers all the isolated conic sections known to the Greeks, but at the same time it exhibits the mutual relations between them and their degenerate forms as limiting cases. Its variables already require some kind of continuity from the number system, and its very form suggests that polynomials and even families of polynomials should be respectable mathematical items. Thus it provides information about previously recognized items—the conic sections and the rational numbers—and sets up expectations calling for new items: the real numbers, polynomials, and systems of polynomials. This metaphor of curve (the flow of water, the fall of an arrow) as function not only transforms the notion of a geometrical curve, as we will see next, it also transforms the notion of number, as we will see later. The static, isolated figures of the Greeks explode into algebraic and transcendental curves used to model situations in a nascent physics, and the natural numbers explode into the reals, and then the complex numbers. Then set theory allows them to explode upwards into the transfinite, and topology and abstract algebra, even later, allow them to explode sideways into the *p*-adic numbers.

The development of algebra was an important factor in this tendency towards functionalization. The algebraization of number began during the period from the tenth to the thirteenth century in the Arabic world, when the use of Arabic numerals was gradually entering the mathematics of Western Europe. Algebra too had been developing slowly, not regarded as a science but as an art and a calculating device for merchants, in which the unknowns were always taken to represent numbers. Fibonacci, the leading algebraist of his time, used letters instead of numerals in at least one case in his *Liber Abaci* (1202). Nemorarius, his contemporary, constantly used letters for the sake of generality (Sarton 1931: 1–12). However, the algebraization of number began in earnest only when numbers no longer played an essential role in algebraic expression. Francis Vieta, in his *Introduction to the Analytic Art* (1591) for the first time introduced a true algebraic symbolism, in which letters representing unknowns and letters representing parameters were distinguished. Just as important, nothing was stipulated regarding the nature of the unknown; it was not taken to be numbers, or line segments, but simply quantity.

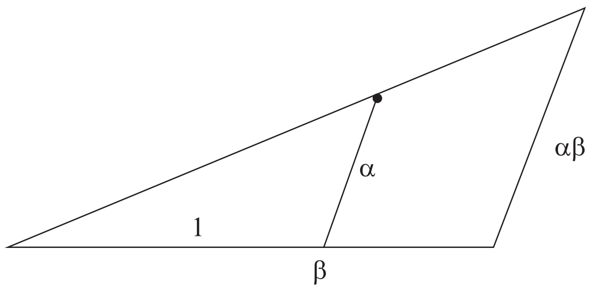
Vieta regarded his 'method of analysis' as a generalized procedure, neither specifically arithmetical nor geometrical; and this method led him to the notion of a generalized mathematical object, neither number nor figure. His method consisted of three steps: (1) construction of an equation, (2) transformation of the equation until it assumed canonical form, (3) the numerical instantiation and solution of the latter. Vieta understood the 'analytical' manner of finding solutions in Diophantus' works on arithmetic, and his own geometrical analysis as completely parallel procedures; hence a sharper line had to be drawn between the transformations of equations and the computation of the numbers sought than that which Diophantus usually drew. Whereas Diophantus concluded his analysis with the final computation, Vieta understood the calculation ending 'in the indeterminate,' which Diophantus used only as an auxiliary procedure, as the true analogue to geometrical analysis. Thus, for Vieta, what analysis discovered was an indeterminate solution. Diophantus represented a to-be-determined number by its 'species,' but in Vieta's hands the 'species,' representing general magnitude, became the object of a general mathematical discipline, neither geometry nor arithmetic (Klein 1968: 150–197).

Vieta was, however, like Descartes, conservative of the past, and tended to represent his innovation as renovation. He supposed that his own analytical method had been known to the Greeks but was kept an esoteric secret, or lost. Also he was ambivalent about the status of his new specious logistic, regarding it not as a new mathematical field, but only as an auxiliary means, a reliable metonymy, for finding geometric constructions and numbers, solutions to equations. It was only an instrument, an art, an organon; thus, the question of new items like variables or polynomials could be sidestepped. Descartes, as we shall see, displayed a similar reticence about the algebraical items he employed. Until these items were embedded in a newly constituted area of research, in a matrix of problems and successful solutions, his reticence was justified. Since it is not possible to settle questions about the objectivity of mathematical items by direct or indirect ostension, as it is in the natural sciences, the mathematician must employ different criteria. I suggest that mathematical items are objective to the extent that they figure in important problems and their solutions.

What, then, does it do to the concept of number when number is taken as a possible instantiation, one among many, of an abstract algebra? Vieta's letter signs, when used in contexts where they were to be interpreted by numbers, lent their own indeterminacy to the numbers themselves. The notation a, b, …, x, y, …, standing for number, but not any particular number, suggests the general character of being a number. This loosens up and leaves open the characterization of number. It weakens the constructed, artifactual, Greek conception of number in terms of units by emphasizing the dependence of number on a context of relations for its definition. Thus, for example, the synecdochic context of n-degree polynomial equations suggests the extension of number to the negative and imaginary. Vieta's notation also suggests that numbers are not just a discrete set, but a series which can be smoothly run though by a variable, and this prepares the ground for the later development of the real number system.

Geometry was also made more abstract as a result of being subsumed, by synechdoche, under the new algebra. In the first few pages of the *Geometry*, Descartes explains how his algebraic calculus can be interpreted as an algebra of line segments, in which the operations of addition, subtraction, multiplication, division, exponentiation and root extraction are represented by compositions of line segments, which again produce line segments. For Descartes, these line segments were like the algebraic letter-variables, for they were symbols of indeterminate multitude. As Jakob Klein observes: "The mode of being of these 'figures' is none other than that of Vieta's species. The essential difference between Descartes and Vieta is not that Descartes unites arithmetic and geometry, while Vieta retains their separation. But for Vieta, numerical computations and geometric constructions represent two different possibilities of application of the general analytic art; whereas Descartes begins by understanding geometric figures as structures whose being is determined solely by their symbolic character." Thus the figures in Descartes' diagrams have already undergone an important shift away from the concrete and discrete figures of Greek geometry (Klein 1968: 150f.).

Descartes' interpretation of the arithmetic operations also circumvents a Greek constraint on dimension. Whereas the Greeks interpreted the product of two or three line segments as producing an area or volume respectively, Descartes interpreted multiplication in such a way that it produced only further line segments, like addition.



This innovation prepared the expansion of geometry to include curves corresponding to polynomials of higher degree, as well as surfaces of higher dimension (Descartes 1954: 2–7). The new view of geometric items as relational rather than artifactual constructions led to the addition of complex points and points at infinity in projective geometry; and the view of them as variables led eventually to Felix Klein's construction of them as equivalence classes under a certain group of transformations in the nineteenth century.

Regarded under the aspect of algebra, both the fields of arithmetic and geometry became less artifact-like and concrete, and more functional and abstract. This made their assimilation to each other in the hands of Vieta and Descartes more plausible; and, once Descartes' metaphor of the function had been established, made possible the elaboration of that hypothesis and consequently the further extension of both fields. But it also generates a fresh set of ironies. To place number under the aegis of the algebraic notion of magnitude is ultimately to raise the question whether the number system can be expanded so that there is a number which corresponds to every point on the geometrical line. In the arithmetical books of Euclid, the representation of whole numbers by line segments runs in only one direction: the combination of units, or prime factors, is represented by certain segmentations of the line. But the question whether other geometrical configurations correspond to number is suppressed. In fact, such a question would be beside the point in those books, whose intent is to investigate the natural numbers; if the analogy were allowed to run in the other direction, very simple geometrical constructions would quickly lead one to the rationals and even the irrationals, or 'incommensurables,' as the Greeks called them. But this uni-directionality of the analogy in Books VII–IX is surely part of what Mueller has in mind when he calls the diagrams in those books 'inert' (Mueller 1981, Ch. 2).

An explicit, abstract representation of the structure shared by number and by figure is lacking, as is a kind of middle term by means of which one could move as easily from figure to number as from number to figure, so that number and figure could be thought together. The theory of proportions laid out in Book V of the *Elements* allows one to think of relations among numbers as *like* relations among figures, but not to think number and figure together. The theory of proportions allows us to recognize that 3, 4 and 5 have a relation to each other similar to that which the three sides of a certain right triangle have to each other. But when Descartes or Vieta write $x^2 + y^2 = r^2$, they consider both number and figure together at once, for they are using the algebra of arithmetic to analyze geometrical problems. That 'thinking together' might lead one to want to find an analogue on the side of number for everything one can construct on the side of figure, and vice versa, not least because of the presence of variables in the algebra. Variables do not just stand for the unknown, but for number generalized; the notation itself leads one to wonder how far that generality might be pushed. In other words, the notation allows for the formulation of a conjecture about the correlation, though just by itself, ironically, it does not solve the problem inherent in the formulation. Descartes' algebra strictly limited the 'constructible' numbers to the algebraic numbers. However, beyond the algebraic numbers lay the transcendental numbers, a term Leibniz coined, perhaps inspired by the clouds over the Harz Mountains, where he served the Court of Hannover as a mining engineer, supervising the draining of the silver mines, a task that he hoped to carry out, between 1680 and 1686, by installing wind machines (Wakefield 2010). Ultimately his scheme did not work out, and in fact he coined the term in 1673 (when he was still in Paris), but you see what I mean about the power of metaphor. While working at the Leibniz Archives in Hannover 25 years ago, I went to visit them, and thought the clouds in the Harz Mountains were really transcendent. On the other hand, so is Notre Dame.

Notre Dame de Paris (my photograph)

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Chapter 7
Periodicity as Symmetry in Time: Housman

Natural systems, like molecules, cells, organisms, solar systems, stars and galaxies, exhibit stability that persists for awhile, and then disperses. Their stability is manifest in geometric forms that are often highly symmetrical, and that express a 'low' energetic state in contrast to other more highly excited, unstable possible states of the same system. It is also manifest in the successful accomplishment of functions that allow the system to interact with its environment while maintaining its own integrity. Integrity is once again linked to shape, as well as to periodicity, which—as Bas van Fraassen observes—is symmetry in time (Van Fraassen 1989: 252). It would be a philosophical mistake to conclude either that the stability of natural systems is an illusion (with Heracleitus) or that their dispersal is an illusion (with Parmenides). The things of the world inhabit the middle kingdom between Being and Becoming; in fact, they constitute that middle kingdom.

Natural systems organize themselves. In our Newtonian world, the schema for self-organization is the deflection of a corpuscle in inertial motion (straight-line motion at a constant velocity) out of its endless, aimless path into an elliptical orbit around a center of force that obeys the inverse square law: the force of gravity falls off quickly, proportional to the square of the distance. This circling, which becomes, ironically, an elliptical circling back, induces a periodicity: the corpuscle now in one sense always comes back to the same place at regular intervals. Our earth brings us back to the winter solstice every year, then to the vernal equinox, and so forth. We have a home in space (near the sun) and in time (our year); the periodicity itself provides the auspicious circumstances in which we can grow. Of course, in another sense our solar system and our galaxy wander through space, so our home near the sun is more like the tents of Bedouins than the apartments of Parisians. As I write, it is a fact documented by UNICEF that over half of the world's refugees are children; there are about 30 million of them. One thinks of this at the winter solstice, when it is so cold.

In our Euclidean world, the schema for self-organization is geometrical form, the circles, triangles, and squares (or spheres, tetragons, and cubes) that articulate the infinite, homogenous, hole-less, bump-less, edge-less plane (or 3-space) so that it

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can be understood. In our Cartesian world, geometrical form is overlaid by numbers. Finite geometrical figures, the periodic notation of Arabic numerals, the fancy footwork of decimal notation, group theory and set theory, all induce periodicity on the plane, making one everywhere a little room, to reverse Donne's metaphor of love. The shape of a planetary orbit is an ellipse; stars and planets are oblate spheres; molecules array themselves as hexagons and pyramids, and crystals as cubes and tetragons. Soap bubbles and cell membranes are often catenoids, and molecules may knit together to form helicoid surfaces, emblems of stable energy states. No wonder we find these figures beautiful: they allow us to inhabit the world.

And so it is with our houses (or tents) whose doors and windows frame the open sky, roads that lead always away, rivers and mountains without end. Our little rooms define and organize *outside* by opposing to it the shapes of inside, circles or squares, and *elsewhere* by the lived reality of home. A home is somewhere we depart from and return to, in the periodicity of every day. So with our stories, bending the aimless, endless line of temporality into purposeful actions with a beginning, middle, and end, as Aristotle taught; actions once told can always be told again. Stories are inherently periodic; we organize and constitute our lives by telling stories about ourselves and others, retelling the stories we love and fighting with others over whose versions of the beloved stories are true. So too arguments, whose premises close the gate on the infinite regress of reasons, murmuring "let us assume," and "let us reason downwards from this point." Arguments are also meant to be revisited, like the garden paths of the Academy or University, as terms and propositions are analyzed and rules of inference are quarreled over. Shall we go along with the purely formal return of Reductio ad Absurdum, shall we trust the detachment of Modus Ponens, shall we step sideways into the flowerbeds of analysis?

But I have moved too abruptly here, and must bridge my leap between nature and culture. A solar system organizes itself by establishing periodicities; but not being conscious it does not see its displacements as departing from and returning to the 'same place.' Aristotle in his physics and celestial mechanics attributed striving and fleeing to bits of matter; but the periodic self-maintenance of physical systems is mindful only in a very rudimentary way. People, however, know their periodicities and name them, love them and fear them, and recognize the fact that they articulate space and time: the warmth of our arm-chair-planet near the sun, the vigil of our evening-meal-winter solstice. We read organizing symmetries in space and periodicities in time as repetition: sameness with difference as the rationalists would have it, or difference with sameness, as the deconstructionists contend. A planet doesn't know it is repeating the same orbit every year, but we use its orbiting to discursively assign places within and beyond our solar system and indeed to see our solar system as a place. This Christmas is the same as last Christmas, but the children are a bit older, or two of them have gone away to college. The slant of sunlight as we carve the turkey is the same as it was last year, for we have come back to the same house on the face of the earth, in the same orientation to the sun, which changes all year long because the axis of our planet is inclined at a certain angle to the ecliptic, the plane of the solar system as it traces out a great circle on the night sky along which the constellations cluster, and the moon and the sun rise and set, and the other planets wander.

Periodicity is symmetry in time. When people take up periodicity mindfully, and turn it into departure and return, regret and anticipation, the representations they use often turn periodicity back to spatial symmetry. We do this on the round faces of our clocks that superimpose midnight on midday, and the square arrays of our calendars that show how stormy Monday always abandons the weekend, how January must introduce the rest of the year. We picture periodicity symmetrically in the *ceinture* of constellation emblems that decorate mechanical models of the 'celestial sphere' as well as the columns of horoscope advice in the daily paper. We express it in the bilateral symmetry of our cathedrals with respect to their east-west axis, which sets the towers of the portal against darkness in the west while orienting the windows of the apse to the rising sun. Symmetry that stands for periodicity is also depicted on the printed page in the lineation of a poem, the left-rectification or centering of its lines, the array presented by the stanzas of a sonnet, the visual repetition—perhaps in italics—of a refrain.

Of course, people who read periodicity as repetition cannot ever forget the difference that nuances their sameness. Natural systems are not aware of their own dissolution; the solar system does not rue the day when the sun will grow large and red, devour its children-planets, and then sink into darkness. We are able to use periodicities to make ourselves at home in the world, but only because we know and represent them; and the shadow side of knowledge and representation is death. The circles of periodicity are really spirals, stretched out along the arrow of time that flies only in one direction, and sooner or later brings down every creature. When we assert the identity and existence of something in discourse, we write "A = A," and thus introduce difference into the heart of A. This is just as true when we say, "*cogito, sum*," or when God roars out of the cloud of the Old Testament, "I am that I am." Even our organizing arguments and stories are linear in their internal structure: the beginning comes before the middle and end, and the premises before the conclusion. We finish the novel in a shower of tears, we are convinced by the argument and turn away: it's over. We read our child's first poem, and understand its plaint or praise. We can re-visit the story or argument, it's true, and even the story of our own life over a glass of beer in a bar or on the analyst's couch; but we cannot revisit our life. Here is my poem, another villanelle, about saying goodbye at airports: it is also a complaint. I borrow the rhyme scheme from Auden's great villanelle, which I learned by heart half a century ago, which begins: "Time can say nothing but I told you so./Time only knows the price we have to pay./If I could tell you, I would let you know."

Holding Pattern

We can't remember half of what we know.
They hug each other and then turn away.
One thinks in silence, never let me go.

The sky above the airport glints with snow
That melts beneath the laws it must obey.
We can't remember half of what we know.

His arms are strong and warm, his breath is slow;
She holds him close, not knowing what to say.
One thinks in silence, never let me go.

Time silts the rivers, ravaging the flow
Of wave on wavelet, and suspends the day.
We can’t remember half of what we know.

This holding is agreement to forego,
This flight another strategy to stay.
One thinks in silence, never let me go.

The silver trees spring back to life, although
Their roots are gilded by the leaves’ decay.
We can’t remember half of what we know,
One thinks in silence. Never let me go.

Great poetic traditions are characterized by line and stanza; *stanze* are rooms. The dactylic hexameter, the alexandrine, the iambic pentameter, the *n*-beat alliterative Anglo-Saxon line, the doubled line of classical Persian poetry, each defines the poetry it organizes. (See Pöppel and Turner (1983) for a vivid account of the surprising uniformity of the poetic line across human cultures.) What constitutes a line is conventional, and the convention may be given in terms of foot, stress, number of syllables, rhyming end-words, typographical convention, and so forth; but the convention is essential because it establishes a primary periodicity. Here is an analogy: the natural numbers can be represented in terms of strokes:

$$/ \; // \; /// \; //// \; ///// \; ////// \; /////// \; //////// \; \ldots$$

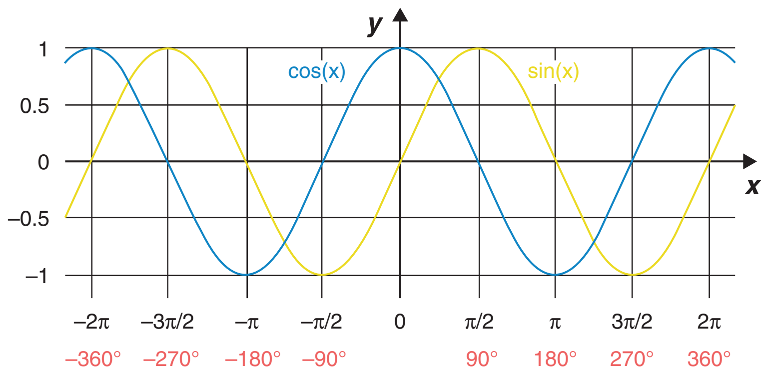
But this notation just goes on and on, like inertial motion. The genius of Arabic notation is the re-organization of the whole numbers by the multiple superposition of periodicities in powers of ten, an organization so familiar to all of us who learned the addition and multiplication tables in first grade that we no longer recognize its genius. In fact, that organization gives us finite addition and multiplication tables; without it, we would have an infinite number of addition and multiplication facts to memorize. It establishes a primary periodicity in terms of ten; and even though that *fiat* is a convention, probably related to our twice-five fingers on two hands, we must have some sort of convention to get arithmetic and then number theory going as enterprises. The iterated periodicities that number theorists use to build their discipline (borrowed from group theory and ring theory, analysis, algebraic geometry, and so forth) must have a place to start (Grosholz 2016: Ch. 4).

So too with any great tradition of poetry: we must have a place to start, the conventions of lineation, and along with them conventions of stanza, poetic form, and (sometimes) chapter. The effects of lineation are both direct and indirect. Some direct effects are that the first and last words of each line have special weight; we see them and hear them more distinctly than other words in the line, generally speaking, and so pay more attention to their meanings. The poet also tends to choose grammatical units and units of thought that ‘fit’ into the line; so the line both organizes and limits, opening up some possibilities while suppressing others. Poets interested in exploiting ambiguity will choose lines as well as terms that lend themselves to more than one meaning, where the meanings are mutually ampliative, coherent without being consistent. Each line has its own beginning, middle, and end; it may describe an act, but it is itself also an act of the fashioning of language, with the miniature drama, the building and resolution, such an act entails.

Indirect effects have to do with the superposition of periodicities. One obvious example of this is the difference between end-stopped lines, and lines that exhibit weaker and stronger kinds of enjambment. The lineation of poems establishes a formal periodicity; but grammar has its own periodicity, signaled by the completion of a sentence when a noun and a verb are coupled properly, and in Western languages by a capital letter at the beginning, a period at the end, and a space before the next sentence. These two kinds of periodicity may coincide, as in carefully end-stopped lines, or in the formulae chosen over centuries by the bards of oral traditions. However, they may not. The grammatical structure of enjambed lines overflows and violates the boundaries set by the poetic line, setting up a tension between the thought expressed and the form, like a river articulated and deflected by boulders but still rushing over them. Conversely, the boundaries set by the poetic line may interrupt the grammatical structure in ways that reinforce and emphasize words or phrases, or ironically undermine and analyze them.

One frequent consequence of the fluent conflict between grammatical organization and lineation is the creation of caesuras mid-line. In a rhymed poem with strong caesuras, the *sound* has a period created by the lineation and end-rhymes, while the *thought* articulated in grammatical units set off by caesuras has a period that begins and ends mid-line, and may in fact run for two or more lines. (I am of course overstating the opposition, because lines are always units of meaning as well as aural units no matter how extreme the enjambment—one might say a poetic line insists upon its own formal integrity and hence its own meaning no matter what one puts into it grammatically – and caesuras are marked aurally by a pause.) This creates an interesting counterpoint, at once aural and conceptual; the best example I know (without the rhymes) is Milton's *Paradise Lost*.

The mathematical schema for periodic pattern is the sine wave: a given sine wave has a certain amplitude (how big or intense it is) and a certain frequency (how quickly its peaks pass by a given fixed point). A cosine wave has the same shape, but with a phase shift:



$\cos(x) = \sin(x + \pi/2)$

When two sine or cosine waves of different amplitude and frequency are superimposed, they create a new pattern, with especially high peaks at points when the high peaks of the original waves happen to reinforce each other, especially low peaks when two low peaks are superimposed, and a complex but regular mixture of amplitudes in between. Any periodic function, no matter how apparently irregular, can be represented by a (possibly infinite) sum of sine and cosine functions, which is called its Fourier transform. Both light and sound waves can be schematized this way, and so too the complex visual and aural patterns created by their interference. In an immediate way, this schema applies to the aural apprehension of poetry: some words and phrases are louder and held longer than others, and some words and phrases are read more rapidly than others. (However, intensity or rate in poetry is as closely linked to meaning as it is to the linguistic production of vowels and consonants.) In a less immediate way, it also applies to the understanding of poetry, when the periods established by line, by grammar, and by thought do not coincide.

Poetic superpositions are many. We produce the effect of counterpoint by juxtaposing lineal periods with grammatical periods. But grammar is a more formal mode of organization than thought, having less to do with content, and doesn't always coincide with it. An argument, whose premises and conclusion are a good example of a completed thought, often runs over a number of sentences, and certainly a number of phrases. Small thoughts, like "I do" or "I am" or "Alas" often form only part of grammatical sentences or phrases. And in a narrative a continuous thought can be grammatically very broken up, and indeed strung out along many pages. Thus the counterpoint between lineation and grammar in a poem may itself be subject to a further articulation, thought, which as its own periods are superimposed introduces new patterns of reduction and amplification.

Aural counterpoint is also possible in poems. End-rhymed poems may still include a great deal of alliteration, slant rhyme, and even full rhyme in the middle of lines. Then our ears register not only the linear period and alternation of the end rhymes (for example, ABAB) but also the chime that complicates it: three occurrences of sibilance in one line, for example, or a glottal stop in the middle of a line echoing another in the middle of the next. And poems that have no end-rhyme are often knit together aurally by frequent alliteration, slant rhyme, and full rhyme set internally in the lines, often before caesuras; in such cases we hear their submerged periods, in connection with the interplay of line and caesura-unit. Another aural effect created by caesuras in highly enjambed poems is a counterpoint of the pauses expected at the end of lines with the pauses that occur mid-line as they frame a completed thought or grammatical unit. Note that highly enjambed poems often rush over the end of a line like rapids, so that what remains is only the ghost of a pause, a pause not taken, which we nonetheless register.

A similar effect is the counterpoint created between the regular metrical pattern of a line and the ordinary patterns of speech. Ordinary speech may demand an emphasis or temporal extension just where the regular metric pattern demands a brief light syllable; every poet knows that this juxtaposition of expectations can produce extraordinary effects, which when successfully accomplished make the reader hear two, or even perhaps three things at once. We hear the speech pattern (that we know well from our everyday life) and the metrical pattern (that the poet has successfully established in the foregoing lines) virtually as aural ghosts behind the resultant outcome, which is a combination of both and what we 'really' hear. Actors interpreting Shakespeare's iambic pentameter invent their own characteristic mixture of the formal and conversational to produce what we hear onstage.

William Empson explained how and why the successful management of ambiguity is so important to poets, and showed in a brilliant series of examples how ambiguity can be exploited by the poet at every grammatical level: term, phrase, sentence, narrative or argument. I would add, also at the formal level of a poetic line, and the contentful level of a thought, neither of which can be identified with a grammatical unit (Empson 1966). In successfully ambiguous poems, the primary and secondary construals of the words are coherent and not contradictory, without being strictly consistent: they are present at the same time even if one (the primary construal) is more present, and their combination will reinforce, amplify and deepen the meaning of the poem. Thus, the primary dictionary meaning of a term will dance with the secondary meaning, or an archaic meaning almost but not completely forgotten. Writ large, ambiguity in argument is Platonic dialectic, even when Socrates seems more philosophical than everyone else; ambiguity in narrative is an action contested and expressed by many agents, even when Oedipus is the tragic hero.

Here, I argue that the successful management of periodicity is also important to poets, for comparable reasons. Bach's counterpoint combines three or four heard melodies; the poet's counterpoint is even more subtle, because it combines melodies (of rhythm and rhyme) that are heard and also unheard (yet still determinate), as it combines patterns that are aural with patterns that are thoughtful, having to do with grammar and meaning. It always remains true that ambiguity and counterpoint can be mismanaged by a careless or inexperienced poet and will ruin the poem, when the multiple meanings simply cancel each other out, or when the counterpoint becomes unrelieved dissonance. But their successful use is the heart of poetry, and for reasons that have to do, I believe, with the ambiguity and counterpoint of human life. The meaning of a human action is not only what happened, but what brought it about, what we hope it will lead to, and – just as important – what might have happened but did not.

Arthur Danto has argued that human action can only be characterized as a project that collects moments in the present and refers them to the future in view of its aims, so that no physicalist account of human action is possible (Danto 1962). But the characterization must go even deeper: the meaning of an action includes reference to the acts it precluded—possibles it rendered impossible—which will never be 'there' and may still be quite determinate. What happens is a counterpoint among what might have happened and what did happen as a complex knot of past, present, and future; and we are aware of both the realized and the virtual dimensions of that action. Moreover, whenever we tell or hear stories, we understand them against a cultural background of expectations about the way such a story goes; even if, or especially if, the story doesn't turn out as we expected, we are aware of our disappointed expectation. Keats wrote, "Heard melodies are sweet, but those unheard/are sweeter" (Keats 1991: 36–37). I would say that the melodies we hear always play against a background of unheard melodies, and that is why we can hear them as melodies at all.

A couple of conclusions can be drawn from my description of the art of poetry as, among other things, the art of making multiple periodicities lend themselves to meaning and music. First, the poetic line is a pure but necessary convention. In oral cultures, it is a heard content: a certain number of accents, a certain number of metrical feet or alliterated words around a caesura. In our print culture, it is the typographic line, even more of a formality. But a poem must be lineated in order to have the basic periodic structure that makes it an ordering, a repetition, and a homecoming, and that stands up to all the other superimposed periodicities, making them audible to the ear and legible to thought. Paul Valéry wrote that while prose is like walking, poetry is like dancing (Valéry 1939/1954; Brombert 1969). In the room of the stanza, in the house of the sonnet, to which we return again and again, we are able to dance because of the formal periodicity established by the line.

Second, oddly enough, this argument doesn't prove that traditional, formal verse is better than *vers libre*. I would argue that the opportunities for playing periodicities off against each other are clearer and more numerous in traditional verse, but in any case the English canon offers plenty of variety in the way one can choose one's line and then dance with it. Free verse, losing the rigid underscoring of fixed meters, often compensates for the loss by piling on other kinds of periodicities at the level of sound and meaning, and those extra layers create a different kind of coherent roominess. Sometimes this leads the poet to guide the reader's attention to explicitly articulated vertical relations on the printed page, as well as to the horizontal unfolding of a single line. Conversely, it may lead the poet to downplay the vertical thrust of the poem as argument or narrative, the linear progression from beginning to end, and create instead a conceptual stasis or circularity. Both strategies mark the free verse of the late twentieth century.

My insights here should help illuminate what certain poems mean, and how the poet put them together. Here is a brief, eight-line poem by A. E. Housman, whose last lines, you might note, I borrowed for the title of my poem about the house of childhood. The rhyme scheme is ABAB and CDCD, so that every line ends in a full rhyme, mostly but not always monosyllabic and Anglo-Saxon. (Occasions on which Housman chooses a multi-syllabic word of French or Latin origin for end-rhyming are always worthy of note.) The end of every line is punctuated, except for two, both of which nonetheless complete a grammatical phrase. The first and third line of every stanza is iambic tetrameter, and the second and fourth iambic trimeter; this gives it the usual metrical pattern of a hymn from the Anglican/Episcopalian Hymnal. I chose this poem because I know it by heart; it inhabits me, while at the same time it is one of the poetic rooms I inhabit. I keep it alive by knowing it, and it helps to organize my life. Gaston Bachelard, in *The Poetics of Space*, observes correctly that the house of childhood, once we have left it, comes to occupy us; we carry it around with us, like a snail with an invisible, infinitely whorled shell (Bachelard 1964: Ch. 5). And so it is, much more formally and intermittently, with poems. I could not count how often this one has come to mind ever since I memorized it 50 years ago (Housman 1997: 40). It is untitled.

Into my heart an air that kills
  From yon far country blows:
What are those blue remembered hills,
  What spires, what farms are those?

That is the land of lost content,
  I see it shining plain,
The happy highways where I went
  And cannot come again.

Blue remembered hills (my photograph)

The very first words superpose the ordinary speech pattern "*in*to my **heart**" on the formal metrical pattern "in*to* my *heart*" with the result that the opening phrase tends to be read as spondees, slowing down the reading of the last half of the line, as well as that of the second, despite their metrical regularity. The first two syllables of the third, fourth and fifth lines (the latter, as the opening of the last stanza, is especially important) superimpose speech patterns that invert the first iambic foot of those lines and make them not so much trochaic as spondaic. Thus the two questions and the answer unfold slowly, gravely, and chime together: What are…? What spires…? That is… All eight lines of the poem are end-stopped, but the two that are not punctuated exploit the pauses—unexplained and un-seconded by punctuation—on which the meaning of the poem turns. The first invites us to wonder why an air (a breath, a breeze, a melody, all associated with life by poetic convention) should kill; the second invites us to wonder why the poet is killed, in spirit, by places he can see and ways he himself has traveled.

The dominant image in the poem is "blue remembered hills." Things that are far away from us for some reason look misty-blue; the convention for indicating that things lie at a distance in modern European landscape painting is to depict them, or veil them, in tints of blue, as well as to make them smaller than what is supposed to lie close at hand. Thus blue indicates distance, spatial distance; but Housman recasts the image by calling the hills, “blue remembered hills,” without even separating the blueness and the being-remembered by a comma. Blue is also, in this poem, the emblem of temporal distance, and so of the shadow side, a blue shadow, of the mindful periodicity that orders our lives. Not only can we foresee the coming-around of a day that we will never see, we also can remember days animated by people whom we will never see again. As Bachelard reminds us, human life is not only structured by the “thrown-ness” of our mortality (as Heidegger would have it), but also by the sheltered-ness of our natality, the house and people who watched over us when we entered the world (Heidegger 1971: 145–161; Bachelard 1964: Ch. 2). For most of us, the house of childhood is the land of lost content because we must leave it for the conflictful, risky journeys of adulthood, and because parents die. When the child hasn’t been able to establish other intimacies, and when his journeys seem to have led him astray, with no resolution to the conflict and no profit from the gamble—as at that point Housman’s journeys did seem to him—it is torment to look backwards and inwards-outwards to that lost shelter.

In the first two lines, the main aural repetition is an indistinct half-rhyme—really an approximate assonance modulated by “r”—buried in the middle of each line: “eart” and “air t(h)” for the first, “on (fa)r” and “oun (t)r,” subtle enough not to call attention to themselves but strong enough to integrate both lines individually as well as the non-rhyming couplet whose end rhymes are, of course, AB. The next two lines with like effect incorporate that same lightly insistent, buried and indistinct, doubled half-rhyme, are/ere and ire/are; repeat the end rhymes AB; and ask the same word three times: what, what, what? Fit speech for a living revenant.

The aural patterns are augmented in the second and last stanza’s stricken answer to the desperate question. The same indistinct half-rhymes weave the texture, holding the lines together, though here the vowels are linked to ‘n’: an/on/en, in/ain, en, and an/ain. But in addition and superimposed, each line has an overt, unavoidable consonance-pair that sounds almost Anglo-Saxon, like a knell from a Merovingian belfry: land/lost, see/shining, happy/highways, cannot/come. The emphatic repetition of the consonants is a figure for the repetition of what is seen by the mind’s eye, painted on the canvas of memory, still there and yet not there: “I see it shining plain.” And yet its use is ironic, at odds with the poem’s conclusion: “[I] cannot come again.” No repetition of vocable or image, buried or explicit, can turn the repetition-with-difference of representation into the longed-for identity of revival.

A different poem from Housman’s *A Shropshire Lad* must illustrate the counterpoint created by enjambment. This one has a title, *Reveille*, a Norman-French word incorporated whole into English, like so many terms in our legal, political, and martial lexicons. It is highly enjambed, since the urgency of its obvious message (wake up!) carries it over from line to line, stanza to stanza. But the Socratic admonition or lullaby of its submerged message (sleep now…) provides the counterpoint, and prevails as it closes the poem. The entrancement of this poem also began 50 years ago for me, when I was in fact engaged in a life and death struggle between sleeping past and waking into my life. I came to love those caesura-colons in tension with the end-stopping semi-colons, and never use that poetic strategy of punctuation without remembering its use here (Housman 1997: 6–7).

Wake: the silver dusk returning
  Up the beach of darkness brims,
And the ship of sunrise burning
  Strands upon the eastern rims.

Wake: the vaulted shadow shatters,
  Trampled to the floor it spanned,
And the tent of night in tatters
  Straws the sky-pavilioned land.

Up, lad, up, ‘tis late for lying:
  Hear the drums of morning play;
Hark, the empty highways crying
  ‘Who’ll beyond the hills away?’

Towns and countries woo together,
  Forelands beacon, belfries call;
Never lad that trod on leather
  Lived to feast his heart with all.

Up, lad: thews that lie and cumber
  Sunlit pallets never thrive;
Morns abed and daylight slumber
  Were not meant for man alive.

Clay lies still, but blood’s a rover;
  Breath’s a ware that will not keep.
Up, lad: when the journey’s over
  There’ll be time enough to sleep.

Think of a morning when you must rise for an exciting trip, but have been up half the night before packing and planning: you want to leap up, but you are exhausted and part of you would like not only to sleep in but also to forget about the whole thing. We love the thrill of new places, but we are all, at some level, afraid of flying. The contest between waking and sleeping in this poem is played out in the counterpoint between the formal line and the grammatical sentence. It is also expressed in a tension between iambic and trochaic meter: the overall trochaic meter pushes the poem forward, but there is a sense (which I explain below) in which it always lapses into iambics, and does so most importantly on the last line.

The first stanza opens with a command, *wake*, and a phrase “the silver dusk returning.” These two units of thought are at odds, because the silver dusk returning is a lovely and loving way of describing night, like Keats’ description of death in “Ode to a Nightingale.” But of course so far there is no verb that might complete and explain the thought; when we follow the enjambed sentence through to the grammatical end of the sentence we read, “the silver dusk returning/Up the beach of darkness brims.” The silver dusk is really the silver dawn, and the paradoxical identification is explained by the implicit simile, light: ocean wave::darkness:beach. At both dusk and dawn, in fact, the shifting surface of the incoming waves reflects the vanishing (or reappearing) light longer (or before) the matte surface of the beach; the delicacy of the image is perfectly appropriate, for the same phenomenon occurs equally at sunset and sunrise. Thus, the admonition *wake* is after all delivered at morning light; yet the lineation has left us with a first construal of ‘the silver dusk returning’ as sunset, and the image of the bright wave on the dark beach is ambiguous.

The same thing happens in the second two lines of the first stanza. “And the ship of sunrise burning” taken by itself is an image of the destruction of sunrise, that is, night; its extension to the grammatical finish reverses that impression: “And the ship of sunrise burning/strands upon the eastern rims.” The image is the glorious fire of the rising sun on the eastern horizon; and yet the dawn is a burning ship. However, according to the lineation of the second stanza, it is a shattered vault, a tent in tatters, a broken habitation. The glory of dawn, of travel and discovery, is introduced in the somber terms of ruin. The third line of the third stanza, by itself, reads, “Hark, the empty highways crying,” before it is completed by “‘Who’ll beyond the hills away?’” In those lines the echo of the “blue remembered hills,” and “happy highways where I went/and cannot come again” is very strong; perhaps the price of voyaging is permanent exile, ruin. In sum, the line-fragments that break up the enjambed sentences hurrying the poem along with its enthusiastic *Reveille* carry a very different counter-message.

Likewise, the trochaic meter that also pushes the poem forward and with it the sleepy lad (both poet and reader) is impeded in every other line by the suggestion of a reinstated iambic meter. This push-pull is marked by the alternation between feminine and masculine end-rhymes. A line of trochaic tetrameter that ends with a feminine end-rhyme is what it is; but when it ends with a masculine end-rhyme, it could just as well be a line of iambic tetrameter whose first foot has lost its light first syllable, a fairly common occurrence in the writing of iambs, when the poet wishes to start the line emphatically. Thus, “Clay lies still, but blood’s a rover;” is clearly trochaic; but “Breath’s a ware that will not keep” sounds iambic, especially given the emphasis, the almost-double beat, that must be conferred on ‘breath’ when the line is read. So too, “Up, lad, when the journey’s over” is trochaic and moreover reproduces the paradox that began the poem. Get up, it’s night, says this fragment, which then goes on by enjambment to its iambic continuation, “There’ll be time enough to sleep.” There the struggle ends, with the iambic downfall on the monosyllable *sleep*. Perhaps the silver dusk returning is after all the silver dusk returning. (Grosholz 2005).

Despite the hopes of Dante and Shakespeare to be reckoned immortal, poetry exhibits a stability that persists for awhile, and then disperses. Poems stay alive as long as they are transmitted by culture; so far, after the invention of writing, we humans seem to be able to keep a poem going, unchanged, for two or three thousand years, a bit longer than the life-span of an olive tree. We cherish our squares and rectangles made out of lines, whether they are written vertically or horizontally, and read left to right or right to left, whether the letters are symbolic or iconic, whether the words are sung or said. The purely formal period of the poetic line allows for an organization of feeling and thought that every culture treasures, and that becomes a kind of local immortality. It holds together the greater and lesser periods of thought and grammar, the physical and meaningful recurrences of sound and silence that allow poetry to sing even when it is only spoken. The room of the stanza allows us to revisit the stories, arguments, or sidelong associations of meditative stasis that poets collect from their tradition and re-invent for their children, and such rooms compose the great houses of Dante and Shakespeare, which may last for another few millennia if we are careful not to set the curtains on fire and remember to water the garden.

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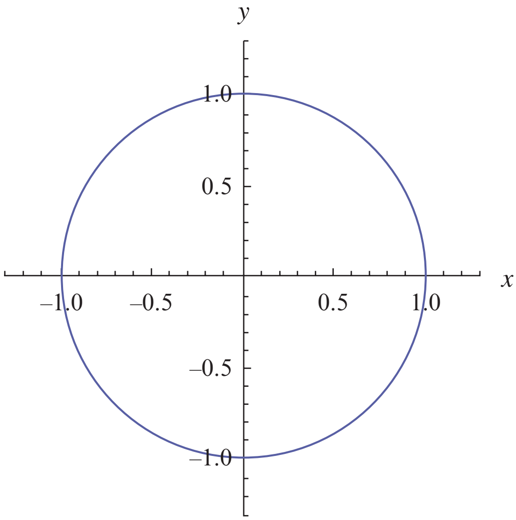
Part III
Shipping Out

Chapter 8
Troping in Italian Gardens: Descartes, Wallis, La Hire, Gassendi

Greek mathematics discerns and pursues analogies between number and figure, but stops short of equating them. The relation between number and figure is simile, not metaphor. We have seen that the introduction of the algebraic equation as a middle term between arithmetic and geometry opens up new possibilities: What might numbers become if they had the continuity of the line? What might figures become if they were understood as functions correlating numbers, and if the plane or three dimensional space were organized by axes and their points specified by pairs, or triples, of numbers? How would that change our understanding of the relation between figure and space?

Descartes, as we have seen, re-organized the study of conic sections by associating them with polynomials of second degree, specified by quadratic equations. So the unit circle at the center of the Cartesian plane is specified by the quadratic equation



$x^2 + y^2 = 1$

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Descartes also went on investigate cubic curves, defined by equations involving terms of degree 3, though his investigation was not complete. He defined the boundaries of geometry, reductively and thus metonymically, as the investigation of curves defined by, in tandem with the requisite diagrams, algebraic equations with a finite number of terms: terms of *n* degree, that is, involving only positive whole number exponents (Bos 1981). The subsequent generations of mathematicians who carried out Descartes' program at first followed the program, but then ultimately revised it more or less radically. His program can be sorted into two related projects, posed by Descartes' powerful new methods but not by Descartes himself. The first was to discover items, methods and techniques proper to analytic geometry, involving both the algebra of arithmetic and geometry, and allowing an easy, practiced information-sharing between them. The second followed upon the first, to build up a stock of well understood higher curves (Descartes 1954).

In carrying out the first task, Wallis, LaHire and Descartes' Dutch Commentators concentrated almost entirely on the conic sections. Though it may seem odd that this work, which ultimately prepared the ground for the emergence of higher algebraic as well as transcendental curves, clung so closely to the familiar conics, they provided familiar ground on which to practice the so far uncertain transferences between algebra and geometry. John Wallis, in his *Conics* (published posthumously, but written around 1656) was one of the first to assert the claim of the analytic-algebraic expression (Wallis 1693–1699). He felt free to replace geometrical expressions by numerical ones, and maintained that proofs by algebraic calculation were as valid as deductions using geometric constructions. (This inspired Leibniz.) He also introduced curves analytically; for example, he defined the ellipse as the plane figure characterized by the property

$$e^2 = ld - \frac{1}{t}d^2$$

This was the first time conics were defined independently as instances of equations of second degree. Wallis also deduced properties of the curve, like tangents and conjugate diameters, directly from the equation. He was the first to suspect how much information about a conic could be derived from the coefficients of its defining equation (Boyer 1956: 110).

Wallis' *Conics* also contains his investigations of the cubic parabola defined by the equation $y^3 = a^2x$, one of the first higher curves to be investigated, not in the conic family tree. Wallis tried to sketch the curve, at first incorrectly, assuming it should be symmetric with respect to the *y*-axis, like an ordinary parabola, which indicates how little was understood of the shape and properties of higher curves. Shortly thereafter, however, he discovered the correct form (on passing the *y*-axis, the curve turns up, not down), through an algebraic and graphical study of its intersections with a family of parallel lines, and went on to make the generalization that parabolas of even order lie on the same side of the tangent at the origin, and odd orders on opposite sides. The shape of odd parabolas naturally carry them out of the first quadrant, with both coordinates positive, into the third, where both are negative.

Descartes called negative roots 'false,' and did not consider quadratics with two false roots. When he did bother to sketch a curve, he drew it only in the first quadrant; the four-leaf clover 'folium' was for him only a single leaf. Wallis, in his treatment of cubic parabolas, was the first to consciously introduce negative abscissas, and relate them correctly to positive and negative coordinates. Significantly, his correct interpretation of negative coordinates was derived from a study of the graph and the algebraic properties of the curve (Boyer 1956: 111; Whiteside 1961: 295–296).

Wallis, impressed by the power and generality of algebraic techniques, accorded them equal status with those of geometry. This meant that the introduction of items or problems, and conclusions about them, could be variously algebraic or geometric, as the situation demanded. So Wallis developed new problems, for example, what information about geometric properties of a conic can be extracted from its equation; and he proposed new items, like negative coordinates. Years later, in 1685, Wallis used his cubic parabolas to give one of the first non-conical constructions of the roots of a polynomial equation. To solve the equation $R^3 - mR = \pm n$ (or $R^3 + mR = \pm n$), he drew two cubic parabolas, $y = x^3$ and $y = -x^3$, and certain lines cutting them which determined segments equal to the sought-for roots. Thus, he solved a problem from Descartes' program, without following the Cartesian prescription. Wallis defended his use of a curve of third degree in solving the problem (which Descartes would have rejected as means not of maximal simplicity) by observing that the 'compounding' of the circle and parabola, both curves of degree two, comes to the same thing as the 'compounding' of curves of degree three and one. So Wallis was also beginning to alter, by synecdoche, the criterion of simplicity. Similarly, Jacques Bernoulli, in 1695, found the roots for an equation of ninth degree by constructing the intersection of a quartic polynomial in the reciprocal of x with a straight line. He selected those means not according to Descartes' prescription, but because they were easier to draw (Boyer 1956: 123–128).

The Dutchman Frans van Schooten published editions of the *Geometry* with extensive and important commentary in 1649, 1659, 1683, and 1695. Though he was more conservative than Wallis, adhering more to the geometrical orientation of Descartes, he also contributed synecdochically to the development of the techniques of analytic geometry. He was, for instance, the first to use the transformation of coordinates to remove linear terms from quadratic polynomials in two variables, reducing them to a canonical form. This manipulation of an equation was, however, always carried out in connection with geometrical diagrams, and was never formalized and generalized. If we regard van Schooten as testing out correlations between algebraic transformations and geometric invariants, to see how far they hold, then his constant recourse to the geometric diagram appears, while conservative, necessary to his project (Boyer 1956: 113). Indeed, lack of generality should not simply be attributed to incomplete algebraization and adherence to geometric notions. The transition from treating many special cases to enunciating general forms depends upon, for example, the proper handling of negative numbers, complex numbers and coordinate axes. All of the latter were achieved, not by algebraization alone, but by an effective combination of the resources of the geometric graph and the algebraic equation.

The analytic geometers Jan de Witt, Philippe de La Hire, and John Craig, all attempted a systematic exposition of the conics, which became a preliminary exercise to cataloguing the higher curves. De Witt composed his *Elementa curvarum* in 1646. It opens with a kinematic construction of the conics; thus he felt that the items he treated must be introduced by construction, not equation. In the second part of his text, however, he began with equations rather than loci. His treatment of first and second degree equations was carried out, not in terms of a general form, but of many special cases. Having stated that an equation of first degree in two variables represents a straight line, he gave five special forms, omitting one because he had not mastered negative coordinates. His graphs are only of rays limited to the first quadrant. Likewise, he considered four forms of equations for parabolas, reconciling them with the properties of parabolas already established geometrically in the first section, but omitted two forms, not real for positive abscissas, and sketched only the positive segments of the other parabolas. Moreover, he treated separately forms obtained by the interchange of the variables x and y. He also gave two forms for ellipses, three for hyperbolas, and transformations to reduce other equations of second degree to his canonical forms, always however tied to specific cases and diagrams (Boyer 1956: 114–116).

La Hire, in his *Les lieux géométriques* (1679), also developed standard reductions to canonical form, and showed how to go from a canonical equation to the construction of a conic locus. However, he stated his method clumsily; a similar but clearer, more exhaustive treatment of the second degree curve was given by John Craig. Craig, although he also did not properly generalize with respect to negative numbers and the interchange of x and y axes, did show how to determine the kind and properties of the conic represented by any equation of second degree. Moreover, confident that the algebraic expression was an accurate representative of the geometric situation, he reduced equations to his canonical forms without appeal to the geometrically-based transformations used by van Schooten and de Witt. Craig derived four standard forms, one each for the ellipse and the parabola, and two for the hyperbola, a significant advance in generality. Then, by methodically comparing the equation of, for example, a given parabola with one of his standard forms, he showed how to derive a construction of the relevant point-set, with its transverse diameter, parameter and center; as well as how to determine, without reference to the geometrical diagram, certain tangents, and the orientation, vertex and *latus rectum* of the parabola. Thus, Craig provided analytic geometry with techniques of greater versatility and generality, in the spirit of synecdoche (Whiteside 1961: 298–300).

One aspect of the power of these new techniques of analytic geometry is that items can be introduced geometrically, mechanically or algebraically. Descartes' cubic curves defined through the intersection of moving conics and lines are a good example: these preliminary exercises on the conics, practicing the originally questionable transition between the geometric and the algebraic context, were necessary preconditions for the extension to higher curves. Another effect of these at first metonymic and then syecdochic excursions was to shift interest away from problems at the heart of Descartes' program. De Witt, La Hire and Craig were all interested in describing and cataloguing the conics and their representing equations, not just as items intervening in the solution of problems linked to Pappus' problem, but as items of interest in their own right.

Newton and Leibniz profited in a spectacular way from the investigations of two previous generations of mathematicians, who amplified Descartes' correlation-hypothesis into a body of techniques for using algebra to move between arithmetic and geometry, in particular between the equation and the graph of an algebraic curve. These two mathematicians had proceeded at first conservatively, sticking to the geometric diagram, and testing their new methods on the familiar conic sections. Then, as their conception of the diagram became more sophisticated, as they learned to retrieve more information from the equation, and came to see that the transference from graph to diagram and back to graph was not misleading, but indeed highly illuminating, they began to investigate more complicated curves and to pose new kinds of problems concerning them. These investigations set out directly, and ironically, toward curves at first excluded by Descartes from geometry, the transcendental curves, as well as the formulation of systematic methods for differentiating and integrating curves. (Descartes also always stayed away from reasoning that might be construed as involving 'the method of indivisibles,' as his contemporary Torricelli for example did not.)

However, before we look into those developments, it is worth reflecting on the way in which enthusiasm for reductive metonymy, and even sometimes its useful synecdochic elaborations, can lead important thinkers astray, so that they miss the ironic limitations of their projects. In the *Meditations*, that watershed of a book, Descartes demonstrates that the world is composed of the machines of nature (though we human beings have souls, and God is a great spirit: unabashed dualism!) and that mathematics (his mathematics) is the key to understanding those machines. Thus, his *Meditations* leads to physics and biology / physiology. In the latter enterprise, he is led astray not only by metaphysics, but by innovations in the fountains, streams, pools, pipes and automata of Italian garden-engineering. (Therefore, ye soft pipes, play on.)

At the very beginning of the Fifth Set of Objections, Pierre Gassendi attacks Descartes' characterization of the human mind in the Second Meditation. (The *Objections and Replies* were published together with the *Meditations on First Philosophy* in 1641) (Descartes 1984: II 180–193). In the Second Meditation, Descartes provisionally banishes the material, for the sake of his great thought experiment, and therefore the external world. He accomplishes this by means of his famous hyperbolic doubt, which is motivated by Descartes' anti-Aristotelian mistrust of the senses, motivated in turn by the revival of Atomism that inspired many other Early Modern thinkers. The imaginative figure who drives the hyperbolic doubt (disclaimed as soon as invoked by Descartes) is a supremely powerful, malicious Deceiver. All that Descartes can salvage from the shipwreck of the Evil Demon is "I am" and "I think," performative utterances to be echoed by the reader, which take on a kind of metaphysical pathos when uttered. He thus knows himself only as a thinking thing, thinking substance, *res cogitans*, which he can then also distinguish from the body he used to suppose he had, and the external world he supposed he inhabited: *res extensa* is distinct from *res cogitans*, and must be relinquished, at least until the Sixth Meditation. For Descartes as for Plato, the external world cannot become an object of reliable knowledge without the intervention of mathematics, but even mathematical knowledge is not yet available in the Second Meditation, as the discussion of the piece of wax makes clear (Descartes 1984: II 16–23).

Gassendi protests. He never recognizes the philosophical purpose of 'the order of reasons' which Martial Gueroult expounds so carefully in his great exegesis of the *Meditations* (Gueroult 1953: I, 15–29). And he never sees the point of the thought experiment of hyperbolic doubt, or the distinction between the self as mind, and the self as embodied mind. He writes, "Turning to the Second Meditation, I see that you still persist with your elaborate pretense of deception, but you go on to recognize at least that you, the subject of this deception, exist. And thus you conclude that this proposition, I am, I exist, is true whenever it is put forward by you or conceived in your mind... You add that you do not yet have a sufficient understanding of what you are. Here I agree with you in earnest and readily accept what you say; this is the point at which all the hard work begins" (Descartes 1984: II 180). But Gassendi just can't believe that there is any convincing answer to the question of what Descartes is, which doesn't involve an account of the body.

During the last two decades of his life, Pierre Gassendi aimed to promulgate Atomism, and to restore the philosophy of Epicurus, as St. Thomas Aquinas had restored the philosophy of Aristotle, making it consonant with Christian doctrine, for Gassendi was a Catholic priest as well as philosopher, scholar and empirical scientist. In his *Animadversiones* (1649), he published his Latin translation of Diogenes Laertius' Book X on Epicurus, and in his *Syntagma Philosophicum* (published posthumously in the *Opera Omnia* in 1658) he gave a more general and systematic treatment of Epicurean doctrine. Despite his allegiance to Atomism, he shared with Aristotle a firm empiricism: our access to the natural world depends on sense perception, and the accumulation of empirical data, pro and contra hypotheses. So for Gassendi, any account of what it is to be a human being must begin with sense perception, and what sense perception undeniably affords: one has a body as well as a soul. How then are they related?

Descartes had suggested a middle term, just after he noted, midway through Meditation Two. It is not mathematics! "Yet I am a true thing and am truly existing; what kind of a thing? I have said it already: a thinking thing. What else am I? I will use my imagination to see if I am not something more. I am not that structure of limbs which is called a human body. I am not even some thin vapor which permeates the limbs—a wind, fire, air, breath, or whatever I depict to my imagination; for these are things which I have supposed to be nothing" (Descartes 1984: II 18). The middle term is animal spirits, matter so fine, rapidly moving and pure that it is aetherial (to use Aristotle's term for the fifth element, a kind of heavenly matter): it is tempting to call animal spirits intelligent. Then animal spirits would be the middle term between mind and body. They would also be able to circulate throughout even the tiniest pores of the body, which brings in a further metaphor, the hydraulic engineering that the sixteenth century Italians were so good at, and used to animate their gardens with fountains and automata. But this supposition is rejected as soon as entertained, because "It would indeed be a case of fictitious invention if I used my imagination to establish that I was something or other; for imagining is simply contemplating the shape or image of a corporeal thing" (Descartes 1984: II 19). To appeal to the faculty of imagination at this point in the thought experiment is, for Descartes, contradictory, indeed silly: the sole appeal must be to the faculty of intellect.

Still, this is precisely the strategy that Gassendi recommends to Descartes. For Gassendi, the action of animal spirits is just the kind of empirical hypothesis that can offer "the point at which all the hard work begins." He repeatedly insists on the description of the soul which Descartes has just rejected: "You said that you did not know what the soul was, but imagined it to be merely 'something like a fire or wind or ether' which permeated the more solid parts of your body. This is worth remembering!" (Descartes 1984: II 181). And he goes on to observe, "...in this passage, you are regarding yourself not as a whole man but as an inner or hidden component—the kind of component which you had previously considered the soul to be. I ask you then, Soul, or whatever name you want me to address you by, have you by this time corrected the thought which previously led you to imagine that you were something like a wind diffused through the parts of the body? Certainly not. So why is it not possible that you are a wind, or rather a very thin vapour, given off when the heart heats up the purest type of blood, or produced by some other source, which is diffused through the parts of the body and gives them life? May it not be this vapor which sees with the eyes and hears with the ears and thinks with the brain and performs all the other functions which are commonly ascribed to you" (Descartes 1984: II 181–182). Note that here the notion of function serves as a middle term between motion (ascribed to bodies) and action (ascribed to human beings); and the faculty of imagination serves as a middle term between the faculty of sense perception and the faculty of intellect.

Whatever we think of this proposed 'physiological' account of the soul, Gassendi has identified a real problem with Descartes' metaphysics: radical dualism. How can Descartes possibly bring thinking substance into rational relation with extended substance? What could possibly serve as a middle term? Two ironies follow. The first is that Gassendi seems blind to the problem inherent in the materialist monism of Epicureanism, which also seems so difficult to reconcile with his commitments as a Christian priest. He urges Descartes to locate middle terms, but to do that one must have prior terms which are sufficiently differentiated, distinguished within that discourse in meaningful ways. Otherwise, the proportion A:B::B:C collapses to identity. Then no middle term remains, and the conceptual role of the middle term disappears.

Gassendi proposes the soul as a kind of middle term between mind and body, where the soul is diffused throughout the body and somehow also concentrated in the brain and heart. He suggests using the soul to refer to "the principle responsible for the vegetative and sensory functions in both us and the brutes... it is the vegetative and sensitive principle that is properly speaking said to 'animate' us," while the function of mind is to enable us to think (Descartes 1984: II 184). But then, why not understand mind to be "the noblest part of the soul... so to speak, the flower, or the most refined and pure and active part of it?" Then mind would be "some pure, transparent, rarified substance like a wind, which pervades the whole body or at least the brain or some other part, and which animates you and performs your functions" (Descartes 1984: II 185). But notice how the vocabulary of function and action insinuates itself into this description of particles. How can a stream of particles organize or direct a living thing? The hydraulic metaphor leaves out the human agents who design the fountain or automaton, and create the structure of pipes that directs the flow of water, and the faucets that allow the water to be turned off and on. Without the hydraulic metaphor that smuggles in human agency, Gassendi is left with atoms in the void; it is just a sandstorm, not an explanation.

The second irony is that in his physiological writings, *The Treatise of Man* and *Passions of the Soul*, Descartes employs precisely the same strategy that Gassendi suggests. Descartes like Gassendi is a materialist in his treatment of the body. His biological mechanics forms a kind of adjunct to his physics which, let us recall, consists of seven laws of motion—a kinematic schema that avoids the problem of dynamics—governing the collision of particles (so that momentum, or rather bulk times speed, is conserved) and a rather imaginative theory of vortices in the plenum of material particles. His physiology, however, depends in unacknowledged ways on the medical tradition of Galen and on the classical theory of simple machines (where weight times distance is the pertinent invariant), and employs an imaginative set of similitudes unlike the rigorous deductions that the account of method as the order of reasons in the *Meditations* might lead one to expect. At the beginning of the *Treatise of Man*, he writes, "And truly one can well compare the nerves of the machine that I am describing to the tubes of the mechanisms of these fountains, its muscles and tendons to divers other engines and springs which serve to move these mechanisms, its animal spirits to the water which drives them, of the which the heart is the source and the brain's cavities the water main. Moreover, breathing and other such actions which are ordinary and natural to it, and which depend on the flow of spirits, are like the movements of a clock or mills which the ordinary flow of water can render continuous" (Descartes 1972: 22).

And in Descartes' corpuscular version of the Galenic trandition, he sounds very much like Gassendi when he writes that the heart is a kind of furnace that accelerates and rarefies particles in the blood, converting the blood into animal spirits, which then, according to Descartes, it pumps to the pineal gland. (The pineal gland is a house for the soul in the great landscape of the body.) Note that Descartes adds to the five simple machines studied mathematically by the Greeks: he includes springs, clocks, mills, furnaces, fountains and pumps, for none of which at the time were mathematical models available. He does, however, find a way to invoke his concept of inertial motion, to make his description of the functioning of the brain sound scientific: "All the liveliest, strongest and subtlest parts of this blood proceed to the cavities of the brain, inasmuch as the arteries that bring them there are the ones that come in the straightest line from the heart; and, as you know, all bodies in motion tend in so far as possible to continue moving in a straight line... as for those parts of the blood that penetrate as far as the brain, they serve not only to nourish and sustain its substance, but also and principally to produce there a certain very subtle wind, or rather a very lively and very pure flame, which is called the 'animal spirits'" (Descartes 1972: 17). Not only does Descartes use an elaboration of this imaginative model to account for the functions, and actions, of animals; he also uses it to explain reflex action in human beings, and to explain the way in which the perception of external objects acts on the brain. Indeed, the diverse and patterned flow of animal spirits through the fibers of the brain carries information, and may induce permanent configurations in the fibers; these 'folds' are corporeal memory, which in turn influences the subsequence flow of animal spirits.

Descartes, like Gassendi, is exploiting an ambiguity in the concept of animal spirits: they are active and patterned, carrying information on the way into the pineal gland and directing the body's actions on the way out; they function as intelligence; and yet they are just flows of particles. The middle term of animal spirits brings the disparate terms of body and soul into relation, as the middle term of function brings motion and action into relation, and as the faculty of imagination brings sense perception and intellect into relation. And yet… we suspect that Gassendi has not escaped the traps of monism, and Descartes has not bridged the gap between *res extensa* and *res cogitans*.

So must our final judgment be that Descartes was wrong to assert a radical dualism, and that Gassendi was wrong to assert a monism that smuggles in various 'middle terms' as needed? Can this dispute be adjudicated? I would argue that there is an aspect of Descartes' project that Gassendi seems not to have understood, and which in the course of history was more important than the imaginary machines of his physiology. In defense of Descartes, I argue that the middle term which brings *res extensa* and *res cogitans* into rational relation, is in fact mathematics; Descartes understood very well the power of mathematical modeling which sometimes involves diagrams, sometimes an array of expressions in a novel notation, sometimes a mechanical drawing. The epistemological power of the hyperbolic doubt is that it forces the knower to begin with mathematics rather than with empirical evidence, according to the order of reasons; empirical evidence is pertinent, but it comes later. The great breakthroughs of Early Modern science were in fact sparked by innovations in mathematical modeling and notation.

Moreover, mathematical modeling is used in contemporary physics, chemistry, molecular biology and biology. What Descartes could not have foreseen is the complexity and number of the middle terms that lie between my body and me: subatomic particles, atoms, small molecules, large organic molecules, organelles, cells, organs. All of them, oddly, require rather different kinds of mathematical modeling. And, to give Descartes credit in Leibnizian terms, none of those wonderful structures, outwardly described, seem to lead up to the inwardness of me. Moreover, there is a lot to be said for Italian gardens, which ought to be visited as frequently as possible. That is what my husband and I did on our pre-honeymoon, which occasioned this poem, named after Shakespeare's play where politics, metaphysics and magic are resolved by marriage. Note the invariant under transformation, the colorful squaring-off of the plane/plain, the tangents, the sunbeam-line, the domes, and the flow of the river.

The Arno, in Florence (my photograph)

The Tempest

At last we climbed Michelangelo’s Piazzale
To command the elaborate vista Arno divides
And bounds: *intaglio* of terra cotta
Or copper green pressed on the plains, and flanks
Of the tangent mountains.

Above us, not that cobalt Italian sky
Whose formal self-effacement frames and phrases,
But weltering clouds. It never rains in Florence
This time of year, but dictum’s overwhelmed
By soft appearance.

Humid, warmed, the air grew up around us,
Thick citronelle of linden trees in bloom.
The perfume’s German, and its arias
Of hushing leaves and muted summer thunder
Hauntingly northern.

Lost in one of those quarrels nobody wants
But history can generate as crossings
Of long entangled lifelines, we sat down
In the lee of a café terrace, half-protected,
Half-drenched by rain.

What else could we do but watch the tournaments
Of cloud? They wheeled on Brunelleschi's Dome
Fixed as invariant under transformation,
But meant to be seen against the wider flawless
Blue dome of sky.

For us its red tiles wavered, slowly fused
Like a stain on the winding currents of atmosphere.
The tears in things dissolve geometry.
So we observed by words, and the warming touch
Of lips and hands.

Daisies massed on the terrace drank the rain.
Later, an intermittent beam of sunlight
Broached the dusk, and moved along the hills,
Its path the fairest analogue we have
To a perfect line.

And yet how randomly the caressing light
Wandered across the quartered town and vineyards,
Planting its golden bloom on the linden trees,
Our café's raffia arbors, and the curve
Of the glinting river.

My poem is a kind of echo of one of Shakespeare's sonnets, many of which reflect on imperfection in things that should be perfect. If you're on a pre-honeymoon, you should not be quarreling, right? Take a walk, have a cup of good Italian coffee, talk, hold hands, kiss, admire the view, bask in the cloudy sunshine, smell the linden trees in bloom, walk back down the hill to your hotel. That sonnet is also, I think, a good example of a poem that asserts a metaphor-identity, which then shades into melancholic simile. Here is Sonnet 33 (Shakespeare 2012: 174–175).

Full many a glorious morning have I seen
Flatter the mountain-tops with sovereign eye,
Kissing with golden face the meadows green,
Gilding pale streams with heavenly alchemy;
Anon permit the basest clouds to ride
With ugly rack on his celestial face
And from the forlorn world his visage hide,
Stealing unseen to west with this disgrace.
Even so my sun one early morn did shine
With all-triumphant splendor on my brow;

But out, alack! he was but one hour mine;
The region cloud hath mask'd him from me now.
Yet him for this my love no whit disdaineth;
Suns of the world may stain when heaven's sun staineth.

The poem at first insists on the metaphor, the identity of the beloved young man and the sun. It humanizes the sun: the sun has a sovereign eye, which flatters the mountain-tops just as a monarch's eye resting on his subjects flatters them, though flatter also means to touch lightly and caress. It has a golden face, which kisses the meadows, so loftiness is modified by sensuality and tenderness; and the sun is also an alchemist, magical, who doesn't just light up the streams but also turns them to gold. And the beloved is glorified, rather like Apollo, a beardless and atheletic youth who presides over poetry, music and light itself; he is like morning, golden and alchemic. This identity allows Shakespeare to solve, in the sestet of the sonnet the problem posed by the octet: how can his perfect love be imperfect? The sun has been given such a vivid reality that the clouds can only be seen as inconsequential and insubstantial next to it; they are not effects of the sun's being and have no essential attachment to it. So attributes that might be worrisome in a baser object of affection are negligible in relation to his love who is sun, man and morning: twilight clouds that stain and obscure cannot belong to him. So Shakespeare is free to love him unsullied, as he truly is, despite, of course, the inevitable twilight all lovers face.

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Chapter 9
Flow, Rest, Inertial Motion: Wilner, Di Piero, Bonnefoy

Poetry, like music, is an art that is inherently temporal. Just as we must hear a melody in time, so we must read a poem in time. I hum the melody to myself as I recall it, and say the poem over to myself, half out loud. Like a melody, a poem is never all there at once: we must run through it. Yet in many respects time is inimical to discourse: its irresistible, irrevocable flow, the river of time, carries us all away. A poem cannot be merely temporal; to be art and to be remembered—to be memorable art—a poem must resist temporality or make something of it. How do we do that? Our experience, so thoroughly temporal, would be a mere flood too (and so we wouldn't have any experience at all) unless we could remember events and classify things, and organize the world by telling stories and offering explanations. In the flood of time, we remember backwards in reflection and project forwards in action, using discourse; and we build things that are stable, houses and chairs and books. So our experience is really a side-eddy in the river of time, close to the mossy or stony bank, where past, present and future are held together by awareness. Time circles, impossibly, in that side eddy.

How do we make time circle? In fact, of course, we never do: time goes on flowing. But because we can remember and because the world itself is organized, we can recognize some events as repetitions of past events and hope or fear that they are harbingers of future events, and we can see that some things, including ourselves, continue. Time itself never repeats, but things in nature persist and natural events recur and human consciousness perceives that stability and return. So we recognize and codify ongoing periodicities and occasional repetitions in events, and name the constancies and systematic relations and predictable changes among the things engaged in these cultural and natural dances. Our discursive and artful or scientific ways of recording the dance are typically cast as spatialization: the circle of a clock, the rectangle of a calendar page, the tabular results of an experiment, the cube of a cathedral, the array of notes on a page strung between the treble clef and the bass clef, the stanzas of a poem. Poetic stanzas are typically rectangular, whether they are written in an alphabet or in characters, whether they are written right to left or left to right, horizontally or vertically. Such spatialization is especially useful

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because in a schematic way these artefacts display everything together, *at the same time*, and this makes organization and serial orderings easier to understand. By the same token, however, they misrepresent time itself, its evanescence, its irrevocability.

We can revisit a poem once we have written it down or committed it to memory. But many things are written down and only some of them are revisited by many people over long stretches of time; only some of them express the shape of a culture, and bring it to life even after the culture has disappeared. A poem must be memorable: thus the lineation of a poem, based upon a standard meter or syllabic convention (or a convention involving characters), is essential to its existence as art, a bastion of memory against the flood of time. Lineation makes a poem periodic: blank verse, for example, sets up five iambs over and over. Against the background of that fundamental periodicity other periods or repetitions find their place and register their meanings; they in turn contribute to the music of the poem, to its formal beauty. The complex interplay between the repetition of sound and the repetition of sense, between the formal parsing of the line by a foot and the parsing of grammar phrase, between the end of a line and the end of a thought, makes the meaning of lines deeper: it intensifies meaning.

What can a poet do with lineation? A poet can respect lineation quite strictly, so that the end of a line corresponds to the end of a grammatical unit (a clause or a sentence seconded by punctuation), and the end of a thought; he can even make sure that internal caesuras correspond to phrases and are flagged by commas. He can mark the end of the line with full rhyme (or not), using those emphatic Anglo-Saxon words that complete a line with a satisfying thump! ('Thump' is one of those words.) This tendency often goes along with a predilection for words that incorporate various stops: b, p, d, k, g, t, sounds that stop the line in its tracks or make it pause before going on. In such poems, the spatialization of the poem, its rectangle, looms large, and so too themes of enclosure and location and materiality. The poet often walks through a landscape as the poem progresses, a trajectory that can be taken in, altogether, in retrospect. The city block lends itself to this kind of poem.

By contrast, a poet can try to outwit lineation and use lots of enjambment so that the line persistently rushes across its endpoint limit into the next. Then the relations among formal line, grammatical unit and unit of thought typically become very complicated, so that they outrun or fall short of not only the line but each other: this results in a kind of polyphony as formal periods are superposed upon but slide over other kinds of periods created by grammar or meaning or the placements of caesuras. Here the poet often chooses feminine rhymes or half-rhymes, or conceals full rhymes mid-line; she has a predilection for liquids, sibilants and internal nasals, sounds that keep the line flowing (and that when combined with a stop soften its impact), and her sentences and thoughts run on and on. In such poems, the temporality of the poem looms large and so do themes of process and non-localized awareness. The progression of the poem is not spatial but rather bridged by inference, association and conceptual play. Everything flows; thought escapes us or takes us higher. Rivers, oceans and the sky lend themselves to this kind of poem.

These two tendencies lead to different metaphysical errors. Parmenidean (and Spinozan) metaphysics holds that time is not real and that all change is an illusion, so that the truly real does not arise and pass away, but is always *there*: time is just an aspect of space. Heracleitean (and Whiteheadian) metaphysics holds that only time is real and that all permanence is an illusion, so that everything flows: space is just an aspect of time. Most poets who feel the tug of one or the other of these tendencies do not go to such extremes, and nevertheless the tendency marks their verse; I will use poems by W. S. Di Piero and Eleanor Wilner to illustrate and explain my point. Both these poets, like me, are from Philadelphia. Eleanor is a transplant, but she has lived in central Philadelphia for a long time; and I have visited her sporadically in her townhouse on 12th Street for almost 40 years. I don't think one could infer from her poems that she lives on 12th Street, or even indeed in Philadelphia or on the East Coast of the United States. Simone Di Piero was born in South Philadelphia; I was born in the suburbs on the Main Line. Though I have known him for almost as long as I've known Eleanor, and though his poems return as faithfully to his old neighborhoods as mine do, I've only visited him in Bologna and New York City and in San Francisco, where he now lives (Grosholz 2014).

Eleanor Wilner is the Heracleitean. I have been bemused by her long, long, flowing sentences cascading from line to line under the spell of an always thoughtful and nuanced habit of enjambment ever since the *Hudson Review* sent me her first book, *Maya*, to review in 1979. Here, from her 2004 collection *The Girl with Bees in her Hair*, is a poem, "Moon Gathering," representative of her work (Wilner 2004: 24). Nineteen of the thirty lines are enjambed, and most of the enjambments are grammatically rather radical: whatever/stars, of/the skies, clouds/that will open, let/the moon shine through, it will be/at the wheel's turning, when/three zeros stand, pawprints/in the snow, crescent/moon, from/the dark water, hook/without a fish, something/dark but glowing, no more/than, take up/their long-handled dippers, dippers/of brass, catch/the moon, light/to their lips, drink back/our eyes, to see/into the gullet, one/dips and drinks. The more radical the enjambment, the more you have no idea how to complete the meaning of the line until you go on to the next, and so the more quickly your thoughts spill over the end of the line, and the sentence you are reading with them. Moreover, the whole poem is only one sentence! It has three semi-colons and one colon and one phrase set off by dashes; but the colon, two of the semi-colons and one of the dashes are buried mid-line, establishing a caesura; and the one semi-colon that ends a line underscores that all important word, "hook," to which I'll return shortly.

To make the sentence flow even more intently, Wilner uses a preponderance of sibilants, liquids and nasals that slide the phonemic surface along and keep us from pausing as we read. The central image of the poem is a well, around which some gather: Whose well? Where is it? Who gathers? Why? We are never told. The surface is

…a mirror to catch whatever
stars slide by in the slow precession of
the skies, the tilting dome of time,

over all, a light mist like a scrim,
and here and there some clouds
that will open at the last and let
the moon shine through…

So even when the end of the line is punctuated with a comma, the hum of the repeated ‘m’ and the hiss of ‘s’ and the lilt of ‘l’ carry us past them almost without notice. Whenever a stop shows up, it is almost always softened by an accompanying sibilant, liquid, or nasal: mirror to, catch, stars, precession, skies, tilting, mist, scrim, clouds, last, let. And what are we looking at? There is a well; we suppose we are (with the poet) standing next to it and staring down at the dark water, where we see the crescent moon reflected. But no, that can’t be right; the crescent moon is a *hook* that stops the flow of the poem midway, twice (at line 13 and line 18) both because it comes at the end of the line and because of the stop ‘k’ that nothing softens. It is the hook that reverses the poem and the reader with it.

What we see,

swimming up from the well, something
dark but glowing, animate, like live coals--
it is our own eyes staring up at us,
as the moon sets its hook;

We are in the well: those are our own eyes staring up at us. We are submerged; perhaps we are drowning. Up is down. But then who are they, above us, gathered by the well?

… and they, whose dim shapes are no more
than what we will become, take up
their long-handled dippers
of brass, and one by one, they catch
the moon in the cup-shaped bowls,
and they raise its floating light
to their lips, and with it, they drink back
our eyes…

The ghostly presences loom and dissolve in the watery phonemes until we get to “drink back,” and then we realize that we are, alas, sunk:

… they drink back
our eyes, burning with desire to see
into the gullet of night: each one
dips and drinks, and dips, and drinks,
until there is only dark water,
until there is only the dark.

With our eyes drunk back, we have lost our spatial orientation: is it we or they who burn with desire to see into the gullet of night? Who can tell, if up is down, if the dark water in the deep well is the darkness of the night sky? Indeed, *we* are being drunk, over and over as the glottal stops accumulate, "until there is only dark water,/ until there is only the dark." If you aren't scared by now, you should be; just read the poem again.

And in case you think those drinkers might be something reassuring like, say, grandchildren, Wilner is careful to disabuse you of that idea in neighboring poems. So, for example, in "Sidereal Desire," from the same collection, we follow a disillusioned actress or tired hooker down the street, attracted at first by her golden slippers (Wilner 2004: 20–21).

Star struck   following this latter-day   Aphrodite
as she clatters   down the street   high heels
with gold glitter   on her tired feet   sparks thrown
from the friction   of dream   against the rough stone
of the real

She stops on a bridge; it is night; she looks down at the water and the same reversal upends us all, the girl and the reader. She stares down at the dark water, "on which a galaxy or more/of stars have fallen" and we too see (read it aloud to yourself to feel the way the sentence flows):

the veil! A glowing scrim of shifting moiré silks,
    silver asterisks set in motion by a wind   a blur
of stars   like a stir of bright wings in a dark air
    and the water   that was once a mirror
      is now a swirl of veils again

But wait: the girl has vanished and we have, once again, lost our orientation: up is down, the stars are gods, wearing our faces, and who are they, behind the veils?

the stars   the gods   are taken back into the stream
    what wore our faces in an old design
drawn down   return once more to the elements
    that called them forth –
                        the veils play across the surface
of the stream   are whirled along   until they pause
    some other place   a million years downstream from here
and there   the fleeting forms take shape again
though not like ours   or anything we knew.

Those shapes are not like ours or anything we knew; the metamorphosis invoked here is so thoroughgoing that analogy fails. Everything has been swept away except for the sweeping away itself. That would be, I suppose, Chronos, who devours all his children, or, as Edna St. Vincent Millay once wrote, echoing Catullus in her poem "Passer mortuus est,"

Death devours all lovely things:
Lesbia with her sparrow
Shares the darkness, – presently
Every bed is narrow.

Millay's quatrain is one sentence, half the lines enjambed, a colon and a dash, a predominance of liquids and sibilants, which also temper the stops except for the first two: "death devours," the echo of the 'd' repeated in case we didn't get the message.

Simone Di Piero is the Parmenidean. Here is another bridge, perhaps the very same one in Philadelphia, but we will never know, because Wilner will not tell us. In Di Piero's poem, "Leaving Bartram's Garden in Southwest Philadelphia," from *Shirts and Slacks*, the poet starts from Bartram's Garden (54th Street and Lindbergh Boulevard, the eighteenth century home of the botanist and naturalist John Bartram) and moves across Spring Garden Bridge, which takes Spring Garden Street across the Schuylkill River not far from the Philadelphia Zoo (Di Piero 2001: 13). He is back in Philadelphia because his mother is dying, as the surrounding poems tell us. The river is definitely not the issue, and we know where we are. Di Piero will disorient us too, but in an entirely different way. (Or what's a poem for?) Of the 21 lines in the poem, 11 end with punctuation, and most of the enjambments are not grammatically radical; there are 14 sentences, four of them questions, which give a reader pause. Outside the garden gate, the poet observes,

New-style trolleys squeak down Woodland
past wasted tycoon mansions and body shops.

Every second word holds us up with a stop, and we can see where we are. He remembers summer roses and visiting the zoo, the elephants and birds that "jumped from dust igniting on their backs." He recalls what he has just seen over the hearth in Bartram's house, "elephant-eared//cure-all comfrey leaves," and a redbird that "gashed the sunned mullioned glass."

And then the poem tightens around us, becoming both more precise and more mysterious.

… The brown-brick project softens

in the sun. Stakes in its communal garden catch
seed packets and chip bags blown across the rows.
Tagger signatures surf red and black

Across the wall, fearless, dense lines
that conch and muscle so intimately
I can't tell one name from another.

The communal garden of the brick-brown project catches seed packets and chip bags (18 stops in so many syllables), debris catching against the growing plants, and then the flourish at the end. A tagger is a certain kind of graffiti artist. Thanks to Di Piero's poem, I learned this vocabulary word from the website of the Edmonton police: "Tagging is the simplest and quickest [style of graffiti], involving only the marking of the tagger's initials, symbol, or alias. This may be in the manner of unreadable writing or initials, often made with spray paint in large rounded bubble style letters. They can also use markers to place their initials or 'tag' on a variety of surfaces. These taggers are called 'writers.'" Although these letters "surf," there is no ocean and they don't evanesce: they conch and muscle, curling into each other boldly on the walls, like a surfer riding eternally under the curl of the biggest wave ever: Garrett McNamara surfed a 78-foot wave on May 8, 2012 off the coast of Nazare, Portugal, as the Guinness Book of World Records tells us and Youtube reproduces in a minute played over and over on about fifty different posts, and almost infinitely viewed. But we can't read it; and Di Piero never answers his question: "There's something I wanted to find,/but what?"

There are, however, answers to the two question posed in the first and second parts of the elegy on his mother's dying, "Cheap Gold Flats," from the same book (Di Piero 2001: 3–4). Though the poet knows where he is (in his old house and at a bar just around the corner), the answers and the enclosed, well-defined locales don't dispel a mystery, that the relationship is undying even when the parent dies and long afterwards. (I've written a poem for one or the other of my parents about once every 5 years.) In the first part, "Philly Babylon," the question is posed by a girl looking for a hook-up, Hazel, against the background of the bartender "tossing cans, cooler to cooler" while outside "iceworks canal the pavements." (Punctuation closes 17 of the 20 lines.) She asks, "What's my horoscope say today, honey?" And the poet's answer, after seven sentences break up the first two-thirds of the poem, is one long thwarted lament, in which the stops bunch up and then disperse in the unknown:

Dear Hazel, dear Pisces, don't be hurt,
leave me alone a while, my mother's dying,
I've been beside her bed for several days,
today she had an extremer monkey look,
her forehead shrunk down to the bucky jaw,
and when she looks above her head, she groans
to see whatever it is she sees, so here,
take my paper, go home, forgive me.

His mother sees whatever it is she sees, so the poet can do nothing but offer a gift, which after all contains the answer to the question, her horoscope.

In the second part, "Finished Basement," the question is "What will she/be laid out in?" His mother loved to dance, and so does the poet, so she should be buried in her dancing shoes.

Charm bracelet, definitely, the one
she hardly wore, and cheap gold flats
that made her look young and men look twice.

There are the golden slippers again, and the memory of his mother when she was young, beautiful and vain, so different from the unconscious bone, "its used-up flesh helpless/on the pillow." And instead of singing, a death rattle. So there is another mystery:

I hear it behind me, too,
the disposal upstairs, a drainpipe clearing,
whatever it is, I feel it coming closer
to finger my hair and stroke my neck.

Like Wilner's "hook" and "dark," so Di Piero's "stroke my neck." A stop in the flow of language, after all, stops the breath.

We are all in our 60s and 70s, Simone and Eleanor and me, so I suppose it isn't surprising that the mystery our poems raise so often now is death, and that the heart of darkness is presented without romance or sentiment, often in a manner that terrifies at worst and at the least unsettles. The poems that I just discussed, for all the divergence of their prosodic means, are sublime rather than beautiful; they raise the little hairs on the back of my neck and give me the shivers. And yet how endearing they are, these poems, with the bee "half-drunk/on the nectar of the columbine," and the stars clustered like "burning knots/in an openwork of stars, the galaxies/like torn lace curtains blowing/in an empty room." Or in Simone's South Philadelphia, "a casual fall/of light that strikes and spreads/on enameled aluminum siding, brick,/ spangled stonework, fake fieldstone/and clapboard, leftover Santa lights,/casements trimmed in yellow fiberglass." Odd how we can't help caring about these things and words and creatures all the same, and even though time goes on passing, something of us, of them, sometimes our poems, still remains.

Another poet I met 40 years ago was Dorothy Roberts. Dorothy Roberts was an emigrée, a poet of exile. Born and mostly brought up in the St. John River valley around Fredericton, New Brunswick, she was the second generation of a family already distinguished in Canadian arts and letters. But she married an American and so passed most of her adult life in Connecticut, New York, and Pennsylvania, ending up in my college town, State College. In many ways, she sought to maintain her ties with her native country. She often travelled back to New Brunswick, especially when she was needed by her elderly relatives. She was also a member of the League of Canadian Poets; her poetry regularly appeared in Canadian journals and was widely anthologized there. That austere northern landscape—its farms, forests, and rivers—figured prominently among her poetic subjects. Yet a good poet generally transforms biographical particulars in the service of universal insight, and Dorothy Roberts was no exception. Her evocations of Canada became meditations on the problem of distance, and human strategies for overcoming it; her native land became that country, variously known as nature or divinity, from which we have all been parted, though we retain our citizenship. She knew how to combine flux and staying (Grosholz 1985).

Roberts did not regard our constructions of and upon nature as intrinsically harmful or illusory, and her assurance was founded upon close observation, study of the sciences, and patient reflection. Mind coerces and opposes nature with its forms, like that of the boat and the oar. But these forms, novel and artificial in their simplicity, also accord with nature, its imposed constraints. One formulation of these ideas is the poem "Float":

Water goes in and out of the reed's cell,
in and out of the gill,
but the boat is the form that floats.

Watertight, caulked, it coerces the water to be with it –
"Always when I am alive you are with me,
yet enter not and the lapping is continual."
This is the most graceful elaboration of the communion,
oars dip and the ripples of progress circle about it
and the furrow behind is a form.

Go lonely boat and be at dawn in the mist
and be at evening on the beach
just out of reach of the long ripple, a shell.

Through a double movement of refusal and acquiescence, mind produces things to which nature alone would never have given rise, and in so doing transforms the face of nature, as the boat leaves its wake behind it. Fluid alliteration and feminine endings lapping continuously at the coercion of line and stanza repeat the poem's declaration at the level of sound (Roberts 1976: 38).

Why should this double movement characterize the relation of mind to nature? Roberts's answer lies in her conception of nature as pattern, pattern always moving, embodied and mortal, pattern which "perambulates among its outposts," as she writes in the poem, "A Pattern."

A pattern rests, a pattern reaches plenty,
A pattern holds its seasons in assembly
And finishes itself in constant freedom.

Patterns are objective: the mind encounters them; it does not invent them. It is even one of them, though an odd one; so the mind encounters nature as something which corresponds to it, with which it can accord (Roberts 1976: 40). Yet this accord also involves a habit of mind that estranges it, illustrated as well in "Float." The river as nature appears in the aspect of pure flux; the boat as mind is pure geometrical form. Our attempts to understand nature and our place in it (which is our relation to it) inevitably lapse into a kind of metaphysical error, decomposing pattern into form and flux. Separate, they become exacerbated, despair before dissolution, desperate hope in timeless Form. I suppose it is our fear of death, as well as our magnificent ability to abstract and idealize, which constantly brings us back to this error. However, it is a fertile error, so long as we strive to correct it by recombining its elements, time and timelessness, the thicket of the particular, and the great arch of the universal.

Roberts reminds us of a cure for it, quiet attentiveness to the patterns of nature, in the poem "Staying":

Islands are sand and shrubbery and sedge grass
willing to go, if need be, with the river,
float like a boat, or wear away while tethered
by roots of sand and rock, bound round with lilies.

They are the limit to which land loves the water,
paddles and deep fish move and the sandpipers
sprinkled upon it liven the sand with the heron,
clams scrawl over textured sandridges.

Dipped in the bed of lilies paddles brush quiet,
herons flap up and all is like a painting
by Chinese water colorists, indeed it is one
so long and quietly these islands have persisted.

So it is logical to say that islands
coerce the things in flux to stay awhile,
to stand awhile delayed in forms and fragrances
freshening out the mist of early morning.

Though the elements of flux (the river) and form (the island) are still separate here, they are both parts of nature; and they are presented in combination, interwoven "by roots of sand and rock, bound round with lilies." They strike a balance, a poise; the islands are willing to go with the river, the river is coerced by the islands to stay awhile. This combination is pattern, staying, which really exists in nature (Roberts 1976: 12). Here is a painting by Dorothy's brother Goodridge Roberts, of one of their favorite islands.

Goodridge Roberts, *Pleasant Island, Georgian Bay,* 1952. Oil on Masonite, Overall: 81.3 × 121.9 cm (32 × 48 in.). Art Gallery of Ontario, Gift from the Fund of the T. Eaton Co. Ltd. for Canadian Works of Art, 1952. © Estate of Goodridge Roberts

Mind is recalled here first in the image of a boat ("islands… float like a boat"), suggesting again that mind is the stubborn stability in nature. But this comparison is subordinated to a more comprehensive analogy with painting: the ensemble of island and river (pattern, marriage of flux and form) is like a Chinese water color, due to its calm beauty and relative persistence over time. Our representations capture the patterns of nature and intensify them, through the double movement of accord and coercion. Levels of diction in the poem reflect this double movement. The concrete, accurate, and affectionate description (sandpipers, heron, sandridges, beds of lilies) of pattern alternates with the dry vocabulary of abstract form ("So it is logical to say that islands/coerce the things in flux to stay awhile"); both are held together by the unity of the poem, which represents the pattern of patterns, thought.

Thus Roberts recognizes both a positive and a negative moment in the mind's confrontation with nature, an accord and an alienation, both of which are needed to overcome our exile. Often one or the other dominates her poems, under separate guises. The sign of accord is the circle, not the geometric circle, but the embodied circle, the circle as pattern. Circles we find in nature correspond to our human circlings, art, science, and domesticity, where we preserve and nurture, from the vantages of which we recognize correspondences. Through our correspondences we overcome the distance between here and there, great and small, then and now. "Under the Fir" presents one such natural circle, which is by analogy the circle of thought:

> Here in the fringe the fern comes, airiness,
> and further in six mushrooms
> thumb up a simple existence of stem and spore.
> Here needles have fallen and sunlight in flecks
> and emerald moss employs the magic circle.
>
> Lives that might never have opened
> tranquil and bright, innocence bunched,
> a wandering green lily,
> a nest of violets reaching out to drink,
> all shine in appreciation of the circle,
> the space, the haven where the grass breaks vaguely
> upon the line of shade.

The poet, walker in the woods, has noticed all the rich detail of this living circle which offers novelty and variety: not the same old monotonous grass, but lives which would otherwise never have opened. The circle is not shade, but dappled sunlight, sunlight articulated by the foliate it falls through and the objects it falls upon, thought employed upon the objects of this world (Roberts 1976: 41). It is the sunlight of the widely anthologized poem "Dazzle," which

> Comes to the eye from the answer,
> Not direct from the fiery core –
> From the kindled pebble under the sprinkler
> To the glittering eye

That answers with so much seen
And the blinded "Why?"

The circle is also the domestic circle of hearth and home. So she recalls her "house in the past," the old house on the outskirts of Fredericton, where her ancestors lived for generations: "Swallows into the chimney/Have come since and circle/As I return at dusk; another circle./Another ring of years…" (Roberts 1976: 76). We protect what we love with these sheltering circles, not to block out the rest of the world, but through doors and windows to provide a special access to and perspective upon it. Home strengthens our inwardness so we are better able to assimilate the external; it furnishes a small order in virtue of which we recognize and construct greater orders.

Dorothy Roberts sometimes said that when she went to rediscover religion, one of the parts of her life she left behind in Canada, she read science books. The light rings of science (physics within chemistry within biology) also reveal a world articulated and patterned, "an enterprise in itself/where further laws are likely to be the same laws/to hold poised the great exuberance." Its correspondences bring together galaxy, flower, and child. In her powerful and, so far as I know, unprecedented poem on pregnancy, "This Child," where she calls the fetus "the whole fantastic personable appearance/floating upon its stem like any flower," the constellating cells become a galaxy:

So a heavenly body is this
in the closed womb
in mauve and purple and pink
and the softest blue,
clouded in silence and veil
pumping life from the tube
of honey and wine of existence
the day and night through.

The patterns are really there, the orderliness is mindful; looking out among our people, the fields, and the heaves, we recognize our own (Roberts 1976: 34).

Still, there is the negative moment of alienation, the mind's strenuous abstractions, the breaking of icons which brings with it vertigo and dislocation. Mind overcomes distance, yet it is also a distancing agent as it divides pure form and pure flux in thought. Pellucid geometry shimmers on the background of the abyss. And only a half-change in perspective is required for this, as we watch ourselves watching the world, depict ourselves drawing circles. The very act of writing about our accords is perilous, for it risks such reflexivity and the consequent fall into exile. Roberts's emblem for the negative moment is fire and ice, intense cold in conjunction with intense heat. Again, the landscape of Canada is recalled as winter's bitter cold threatening the hearth fires and lengthening the distance between farm and farm. In another particularly well-known poem, "Cold," she writes of her grandparents' ascetic religion:

God could have been in the flame
Responsive among the birch sticks, roaring
Up through the comforting pipes, and served all day
From the frosted woodpile, the continuing flame
As the sun almost let go of the bitter world.

But for them He stayed in the cold,
In the outer absolutes of cold among the fiery orbits…

Under the aspect of this oxymoronic collision, a side-by-sideness with no mediation, we find the possibility of our own intense opposition to nature, when it threatens to destroy us, when our orderings thin out into intense, fragile, pure form (Roberts 1976: 13). Roberts's "Red Angel" can be read as a statement on art in its baleful guise, brushfire melting upon a frosty dusk:

The simple disappearing brush
So hot, so tall, in such a rush
To be heaven and to be gone

Challenges me to hold from flame
My fading and my drying bone
Or draft my spirit for the rush
Of upward heat of upward loss

Red angel stay, red angel go—
Detained with leaves now till I know
The power of flame is terrible

The simple brush with withered leaf
Goes riding upward in alarm
And stands into its crimson dress
And winnowing wings of holiness

The elegant formality of the verse mirrors the angelic specter. Discerning form in fire is an artifice of the spirit; fire is one of the things found in nature closest to pure flux, a rapid, chaotic, and destructive process (Roberts 1976: 98). The negative moment is an intensification not only of pure form but also of pure formlessness, as Bachelard reminds us in *The Psychoanalysis of Fire*.

Any of our circles can be opened. The windows of our houses can spill us into heaven when thought turns on itself, and even heaven can open upon the abyss, where great fires burn against absolute zero, as in "Our Shells," Roberts's poem to her first grandchild:

These houses our shells in the almighty sea:
Day is now off and the great glass of space
Against which we plaster like birds is here presented.

I have drawn aside the curtain to let the light
Of ancient Arcturus upon a little face
So new from the womb that my arms are a world.

In this pattern of silent homes and heavenly bodies
The walls have been pushed out to a vast wandering –
How many stars to lead me to this child?

Only the constellations house with fables
Like brilliant parables upon church windows,
Making of night a high roof for the spirit.

Under the roofless, Neighbors, we are now
Far away, far off in our darkened places,
Our diagram of houses like constellations

Not telling us truly. I hold this tiny child
In his world of unemerged senses and fear of falling,
To the light of Arcturus to make him homeless.

What explains the great tenderness of this poem, the obvious suggestion that Roberts is offering her grandchild a gift? (Roberts 1976: 62). The negative moment is simply a part of human life. We are a part of nature, but one with its own curious estrangement, its terror and idealization. Sooner or later we encounter vertigo; it might as well be sooner, so we learn how it feels and how to live through it. Moreover, if we are strong enough to endure its engulfing or seduction, we can profit from it. Breaking our old assurances may be the means to a new reassurance, where we bring the cold forms back down into time and embodiment, sullying their perfection, making them mortal, granting them life. For we are part of nature; the patterns are real; we are already ineluctably connected to our fellow creatures by law.

Star-gazing is Roberts's gift to the child of her child and is the great activity of the poems where she most clearly resolves the problem of distance, in the poem "Distance."

What happened, what happens is fixed,
Sorrow is here
Close up against us,
But in that fiery star
It hasn't happened, and there is an eon to prepare.

O there the wonderful mastery where now
It is not now. It is something long ago,
And a clear pathway of light
Can lead to now.

The conceit of this poem is extraordinarily refined (Roberts 1976: 56). She supposes that we see the star face to face, abrogating distance as our usual seeing does. Yet because the distance, denied, is still there, since the light requires an eon to traverse it, acceptance of the first miracle produces a second: we see the past. We also see the star-folk seeing us: we see our future. Not only is space overcome, but time as well, in our curved and dynamic spacetime; and this compressed and encompassing present represents the mind's mastery. Reflecting upon ourselves in the context of a greater order is paradoxically a distancing that brings us nearer to the truth of things. Theoretically, we master the prejudices of anthropocentrism: morally, we master our griefs and obsessions.

The poet in her positive moment recognizes, embraces, and draws near: in her negative moment, renounces. In the distance of an estrangement strong enough to claim the two together, she regains the world, more profoundly comprehended, and herself in it, more profoundly implicated. The poem "A Marvel," published in her last book *In the Flight of Stars*, concludes,

More and more goes on and surrounds
the eclipsed self as the galaxy proves
in one of the great views given of light
on the vast horizon of being
at the outskirts of all we don't have do have and marvel at.

Distance takes a lifetime to cross (Roberts 1991: 16–17). In *The Girl with Bees in Her Hair*, there is a wonderful poem that Eleanor Wilner dedicated to Dorothy Roberts, entitled "Distances" (Wilner 2004: 14–15).

There is another way to reconcile form and flux, flow and staying, and it was bequeathed to us, oddly, by Galileo, Descartes and Newton: the concept, or figure, of inertial motion. Aristotle thought there must be a cause for all motion, and that motion is essentially different from rest. If something is moving, then either something else is pushing or pulling it, or it has been displaced from its natural place in the cosmos, and is seeking to return there. In the modern era, we are not used to thinking of physical objects as having natural places, but we recall the Aristotelian doctrine of the five elements: earth, water, air, fire and aether. In his cosmology, earth forms a sphere at the center of the cosmos, surrounded by concentric shells of water, air and fire, and in the heavens, the incorruptible element is aether. So, the earth was understood to be at the center of the cosmos, fixed and immovable. A stone (made of the element earth) displaced upwards would fall back downwards, trying to regain its natural place; fire would rise to seek its natural place. The aetherial bodies in the heavens, born on crystalline spherical shells, circled the unmoving earth in trajectories of uniform circular motion. The whole cosmos was a finite sphere, defined by the outmost sphere of the fixed stars.

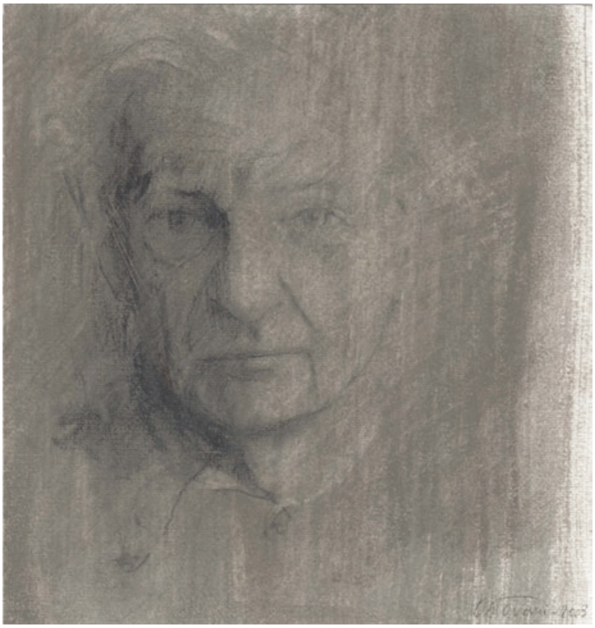
In the doctrines of Galileo and Descartes, however, a new concept of motion appeared: straight line motion at a constant speed, which is called inertial motion. The truly novel aspect of this kind of motion is that it is a state: just as a body at rest will continue at rest, a body in inertial motion (in a vacuum) will continue in inertial motion. No cause need be given: thus such motion was incompatible with the Aristotelian cosmological scheme, though of course that scheme had already been disrupted by the Copernican Revolution, and the idea that the earth moves around the sun. A concrete way to explain the physical equivalence of inertial motion to rest, is the following thought experiment. If you are sitting inside a closed cabin on a canal barge, you can't tell whether the barge is tied up at the quay, or whether it is floating down the canal in a straight line at a constant speed: there are no discernible physical consequences that follow from being in inertial motion. A more abstract formulation, which later inspired Einstein's theory of special relativity, is that the laws of physics remain invariant in all unaccelerated reference frames. You can run a physics experiment on the bank of the canal, or on the deck of the barge gliding by in inertial motion (or, more accurately, in two different space ships in the vacuum of space moving in different straight line trajectories unaccelerated with respect to each other), and the outcomes will be exactly the same.

Inertial motion is one of the central concepts of Newtonian mechanics (along with Newton's radically novel conception of force), and it clarifies an experience we have all had: if you are in a linearly, smoothly gliding ship (or plane or train or car), you are not aware of your motion until you look out the window. It also explains why we feel no effects from the movement of the earth: although our trajectory around the sun is elliptical and accelerates and decelerates a bit as earth approaches or recedes from the sun, locally—across a small patch of the trajectory—the motion is almost inertial.

And so we have a paradox: moving in inertial motion is just like being at rest. Now compound the paradox by adding the metaphor of the river of time: time flows like a river and carries us all away; the past never returns. Or, as Isaac Watts once wrote in a stanza of one of my favorite poem-hymns in the Anglican hymnal, "Oh God our help in ages past,"

Time like an ever-rolling stream
Bears all its sons away.
They fly, forgotten, as a dream
Dies at the opening day.

By contrast, God ("our shelter from the stormy blast,/and our eternal home…") is at rest, fixed, in eternity. If time is a flow, eternity is divine rest. But then inertial motion confounds, or unites, time and eternity. So too does memory, for we are not in fact forgotten, like a dream at dawn; we remember ourselves, and we remember the cosmos, and that remembering is recorded in our words, our poems, our astronomy. Poets thus often use inertial motion as a figure of thought to evoke the divinity or numinousness of human experience, especially those heightened, intense episodes that so appealed to the Romantics, and often involve romance. One of my favorite examples is a sequence of poems written by Yves Bonnefoy, "The Summer of Night," which I offer here in my own translation (Bonnefoy 1978: 163–171), with a portrait by Farhad Ostovani.



Farhad Ostovani, Portrait of Yves Bonnefoy, 2003. Mixed media on paper

True to its title, the first section establishes the context: it is summer, and it is night: "The starred sky, expanding,/Comes closer to us." The leaves are green, and the fruit on the tree is almost-orange; the tree glows and becomes universal. "It seems to me, this evening,/That we have entered the garden, whose gates/The angel had forever closed behind us." So in the poem, we, the lovers, have returned to Eden, the earthly paradise, where the tree (of knowledge?) is filled with "wing-beats/of hidden light," for there is an angel in the tree, and it shines like stars.

But then it seems, in Section II, that the garden is somehow also a ship, pursuing its paradoxical path of inertial motion, which marries time and eternity: "Ship of a summer,/And you, as if on the prow, as if time were coming to a close,/Unfolding the painted cloths, speaking low.//In that dream of May,/Eternity rose up among the fruits of the tree,/and I gave you the fruit that left the tree unbounded/Without distress or death, of a shared world." Then summer is plenitude, the water is transformed by starlight, and the lovers' bodies are blessed by the purity of water and the whiteness of the sand.

In Section III, Bonnefoy plays even more explicitly on the figure of inertial motion.

Motion
Seemed like a mistake to us, and we went on
In immobility, the way beneath a ship
The foliage of the dead moves and remains at rest.

His lover is again likened to the ship's figurehead, "who leads it forth." But the motion of the ship is a kind of staying, a remaining in deep peace, "where ancient love still beats." In the last stanza, the ship and its figurehead become "Forever the reflection of a fixed star/In mortal gesture./Beloved, in the foliage of the sea," that seaweed-like foliage, like underwater trees, that move and remain at rest. The fixed star is perhaps Polaris, the North Star, which remains unmoving in the heavens, as the other stars wheel around it.

In the fourth section, the earth itself becomes the boat, "Earth as if rigged for sailing." And indeed the earth carries us through space, whirling us around the sun, the solar system whirled around the galaxy, the galaxies rushing towards or away from each other: yet the earth is still and stable under our feet. What is moving, and what is at rest? Even the body of the figurehead is "made from moving shadows." Section V invokes "the moment/When day and night cease to exist," when time is abrogated, when the finite becomes boundless, when "the sad knot of dreams" is unknotted (and memory smooths out forgetfulness), when "The star loves the sea foam." So paradox unites motion and rest, the finite and the infinite, the knotted and the unknotted, forgetting and remembering, fire and water: perhaps this union is accomplished by the power of love, more powerful than death.

Sections VI and VII set forth the miracle of lived paradox, for so long as summer lasts. Even the path of the ship in its inertial motion becomes the link between earth and heaven.

For so long summer lasted. A motionless star
Ruled the suns that rose and set. The summer of night
Carried the summer of day in its luminous hands
And we spoke softly together, in the foliage of night.

The star, indifferent; and the boatstem; and the clear
Path from one to the other through calm waters and skies.
Everything moves like a boat that turns
And glides, and no longer knows its own soul in the night.

Its path from earth to heaven is like the path of summer across "a wide/Unmoving ocean," or like the poet's own dreaming body stretched out on the deck, "on the eyes and mouth and soul of the boat stem." The flow of time is just a line, not flowing, all there at once, held fast by memory, "an eyeless dream,/That grasps and doesn't grasp," a retina. And what is retained is "a greater summer, where nothing ever ends."

And yet, of course, it does. Section VIII begins, "But your shoulder disperses among the trees,/Starred sky, and your mouth seeks out/The breathing rivers of earth," those rivers that run down to the sea, that inexorably flow. The heart is wounded and divided, absence reasserts itself, the fruits stricken by death and decay fall from the tree, the glide of the ship ends abruptly on the beach, amid driftwood, sand and trees, "whose branches are night, doubled, always doubled." Here is the paradox of memory: the past is present, as remembered, as we lived it, as we wrote it – and yet it is the past. The ship sails on, and yet it has been beached, truly stopped on the line of black sand.

In the last section, night is brought to an end by a kind of daybreak: "Waters of the sleeper, tree of absence, shoreless hours./In your eternity a night comes to its close."

Summer, night, the suspended passage of the ship, the peace of love, they all disperse in the light of dawn, red on the black sand.

In the waters of the sleeper lights are blurred.
A language forms, which divides the clear
Luxuriance of stars in the sea foam.
And it is almost daybreak now, already memory.

And yet there is the poem, legible on the page, and the love it expresses, still luxuriant and clear. And there is the poet, still singing.

The Summer of Night
Yves Bonnefoy
Translated by Emily Grosholz

I.
It seems to me, this evening,
That the starred sky, expanding,
Comes closer to us; and that the night
Behind so many fires, is not as dark.

And the foliage also shimmers beneath the foliage,
The green, and almost-orange of ripe fruit, burgeon,
Lamp of a nearby angel; wing-beats
Of hidden light capture the universal tree.

It seems to me, this evening,
That we have entered the garden, whose gates
The angel had forever closed behind us.

II.
Ship of a summer,
And you, as if on the prow, as if time were coming to a close,
Unfolding the painted cloths, speaking low.

In that dream of May,
Eternity rose up among the fruits of the tree,
And I gave you the fruit that left the tree unbounded
Without distress or death, of a shared world.

Let the dead wander in the distance, on the desert of sea foam,
There is no more desert since everything is in us,
And there is no more death, since my lips touch
The water of a scattered likeness on the sea.

O plenitude of summer, I possessed you pure
As water the star transformed, as a sound
Of sea foam under our feet where the sand's whiteness
Rose to bless our lightless bodies.

III.
Motion
Seemed like a mistake to us, and we went on
In immobility, the way beneath the ship
The foliage of the dead moves and remains at rest.

I called you my ship's figurehead
Happy, indifferent, who leads it forth
With half-closed eyes, the ship of living
And dreams as it dreams, remaining its deep peace,
And arches on the boatstem where ancient love still beats.

Smiling, primary, faded,
Forever the reflection of a fixed star
In mortal gesture.
Beloved, in the foliage of the sea.

IV.
Earth as if rigged for sailing,
Look,
It's your figurehead,
Stained with red.

Star, water, sleep
Have worn away that naked shoulder
Which shivered then leaned
Towards the East where the heart runs cold.

Oil, meditative, covered
Her body made from moving shadows,
And yet she curves her nape
As one weighs the souls of the dead.

V.
Look, it's almost the moment
When day and night cease to exist, so great the star
Has grown to bless this burnished body, smiling,
Boundless, water that moves without a dream.

These frail earthly hands will unknot
The sad knot of dreams.
The light, protected, will be placed
On the water's table.

The star loves the seafoam, and will burn
Within this gray dress.

VI.
For so long the summer lasted. A motionless star
Ruled the suns that rose and set. The summer of night
Carried the summer of day in its luminous hands
And we spoke softly together, in the foliage of night.

The star, indifferent; and the boatstem; and the clear
Path from one to the other through calm waters and skies.
Everything moves like a boat that turns
And glides, and no longer knows its own soul in the night.

VII.
Didn't we have the summer to cross over, like a wide
Unmoving ocean, and me, peaceful, stretched out
On the eyes and mouth and soul of the boatstem,
Loving summer, drinking your eyes without memory.

Wasn't I the eyeless dream,
That grasps and doesn't grasp, and wishes to retain
Of all your summery color, only the blue of another stone
For a greater summer, where nothing ever ends.

VIII.
But your shoulder disperses among the trees,
Starred sky, and your mouth seeks out
The breathing rivers of earth so that
Your pensive, longing night may live among us.

O image of us, still there,
You carry close to the heart the same wound,
The same light, where the same sword-tip moves.

Divide yourself, you who are absence and its tides.
Accept us, who have the taste of fruits that fall,
Tumble us on your empty beaches in the foam
With driftwood from the wreck of death,

Tree whose branches are night, doubled, always doubled.

IX.
Waters of the sleeper, tree of absence, shoreless hours,
In your eternity a night comes to its close.
What shall we call that other daylight, o my soul,
That deeper, reddened glow mixed with black sand?

In the waters of the sleeper lights are blurred.
A language forms, which divides the clear
Luxuriance of stars in the seafoam.
And it is almost daybreak now, already memory.

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Chapter 10
Troping Towards the Transcendental: Napier, Newton, Leibniz

As we just saw, Descartes renovated mathematics in the early seventeenth century by bringing geometry and the algebra of arithmetic into novel combination. The middle term, setting number and figure into unprecedented relation, was the kind of algebraic equation we remember from high school: a few variables, a few constants, positive whole number exponents, a finite number of terms: something like $ax^2 + by^2 = c$. Descartes was quite finitist: he circumscribed his domain of research by limiting geometry to certain specified means of construction and representation. As the seventeenth century unfolded, however, this development brought about by the shifting conjunctions of number and figure created a demand for new notation, new middles terms: the infinite series, negative and fractional exponents, expressions for infinitesimal magnitudes and for new kinds of (infinitary) sums and for differential equations. Aristotle's restriction of the rational to the finite was challenged, so that finitude (as Pascal complained) suddenly found itself located between the infinitely small and the infinitely large. Moreover, the notion of geometric figure metamorphosed into that of a more generalized curve (some algebraic but some transcendental) and finally to the very general notion of a function; and the structure of space itself became an object of interest. Number expanded from the natural numbers, to the integers, to the rational numbers, finally gesturing towards the reals, number somehow endowed with the continuity of a line. The development of the infinitesimal calculus opened a large new chapter in the history of mathematics. Although techniques of integration date back to the time of Archimedes, as we noted earlier, the infinitesimal calculus properly began with the work of Newton and Leibniz, marked by the conscious introduction and exploration of transcendental curves, algorithms for finding derivatives and integrals, and the formulation of the Fundamental Theorem of the Calculus, which exhibits the duality of integration and differentiation.

Florimond de Beaume wrote to Descartes, questioning him about some curves; in one case the problem was to find a curve given the subnormal, a geometric description equivalent to

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$$\frac{dy}{dx} = \frac{x-y}{b}.$$

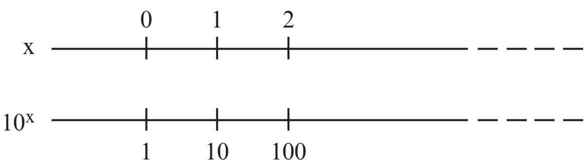
A transformation of variables turned this equation into the differential equation defining a logarithmic curve. Attacking this problem, Descartes derived something like the inequality

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{m-1} > \log\frac{m}{n} > \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m},$$

without, however, mentioning logarithms. But he claimed that the problem was not solvable within geometry, because the construction would involve moving two lines with respect to each other in a way which was 'incommensurable.' Thus Descartes excluded this particular transcendental curve, the logarithm, on the grounds that it could not be introduced by means that were, according to his standards, exact (Struik 1969: 279–80). Cartesian analytic geometry was a vehicle for curves expressible as finite polynomials, and so Descartes' restriction was, in the context of analytic geometry, metonymically rational. However, Descartes' successors, using in fact the highly restricted class of the conics as grounds on which to practice combining, synecdochically, the resources of algebra and geometry, ironically helped prepare for the introduction of transcendental curves.

The further development of mathematics, and in particular of the infinitesimal calculus, depended upon enlarging the stock of curves. Newton's complete catalogue of the cubics was a step in the right direction, but transcendental curves as well as algebraic were required. In this process of enlargement, techniques developed by the Cartesians played a part, as did techniques and concepts imported from the physical sciences. The history of the logarithm is particularly illuminating. Whiteside, observing that the logarithm was given a wide variety of treatments during the later seventeenth century, both abstractly as a correspondence between two sets of numbers, geometrically as an hyperbola-area, and arithmetically as an infinite series, writes, "Such a dual definition, analytical and geometrical, was typical of the 17th century; and it is important to notice that each aspect reinforced the other both conceptually and as a matter of practical technique" (Whiteside 1961: 214).

Napier, around 1617, developed the logarithm in terms of two correlated points moving on two distinct lines, one traversing line segments in arithmetical progression, the other in geometrical progression.



Torricelli imposed these two types of motion upon a single moving point: to abscissas fixed at equal distances corresponded ordinates in geometric progression, giving a monotonically decreasing curve. Many of the new curves that appeared in the seventeenth century were introduced by such mechanical/geometrical means. At the time of Galileo, such a definition was used to define the cycloid (a transcendental curve), which he admired for its "most gracious curvature, adaptable to the arches of a bridge," and sine curves were also thus introduced, as the 'companion to the cycloid.' Geometrical transformations, like the Ptolemaic and Mercator projections, also were used to generate new curves (Boyer 1959: 134; Struik 1969: 232–233).

Napier's basic idea can be connected with the area under a hyperbolic curve, though this connection was only noticed a half-century later by a Belgian Jesuit, based on the work of his colleague, Gregory St. Vincent. (In modern terminology, the connection is the function

$$\log_e x = \int_1^x \frac{1}{t}\,dt,$$

where the integral expresses the area under a hyperbolic curve.) Mathematicians may have been slow to make use of the hyperbola-area model of the logarithm because this representation seemed no simpler at first, perhaps even more opaque than the original (Whiteside 1961: 224; Boyer 1959: 122). In a sense, it was good luck that the curve $\log_e x$ had a geometric representation as the area of a known curve, the hyperbola; their intimate connection provided a bridge between the algebraic and the transcendental, as well as a perspicuous example of the Fundamental Theorem (Boyer 1959: 122).

Once the problem had been recast in these terms, the attempt to calculate hyperbola-areas systematically opened new avenues of research. Wallis first tried to discover approximations to the hyperbola area using the Cartesian representation

$$y = \sqrt{R^2 - x^2},$$

but, finding this approach inadequate, referred the problem to his friend and student William Brouncker. Brouncker used a rather more direct and traditional method, approximating the hyperbola-area as the sum of the inscribed rectangles (which yields the sum $\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \ldots$) and then the sum of inscribed triangles $\left(\text{yielding } 1 - \left(\frac{1}{2^2} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{4\cdot 5\cdot 6} + \ldots\right)\right)$. Brouncker's expansion however proved clumsy in practice, giving approximations only with enormous efforts of computation. Better approximations had to wait upon a more analytical definition (Whiteside 1961: 224).

By the beginning of the seventeenth century, the trigonometric and logarithmic functions had been extensively tabulated, and attempts to methodically find intermediate values between entries followed. In his *Arithmetica Infinitorum*, Wallis first noticed that

$$\lim_{n\to\infty}\frac{0^p+1^p+2^p+\ldots+n^p}{n^p+n^p+n^p+\ldots+n^p}=\frac{1}{p+1}.$$

Moreover,

$$\lim_{n\to\infty}\frac{\sqrt{0}+\sqrt{1}+\sqrt{2}+\ldots+\sqrt{n}}{\sqrt{n}+\sqrt{n}+\sqrt{n}+\ldots+\sqrt{n}}=\frac{1}{\frac{1}{2}+1},$$

from which Wallis concluded that $\sqrt{n}$ might be interpreted as $n^{\frac{1}{2}}$. Using what he called the Principle of Interpolation, Wallis drew up a table of results that showed that

$$\lim_{n\to\infty}\frac{\sum_{r=0}^{n} r^p}{n^{p+1}}=\frac{1}{p+1},$$

where the exponent p is positive, negative, or fractional! He then tried to apply similar methods to the binomial expansion of the form $(R^2 - x^2)^p$, in particular, the quadrature of the circle, which is expressed in modern terms by

$$\int_0^1\left(1-x^2\right)^{\frac{1}{2}}dx;$$

an attempt which was only partly successful (Baron 1969: 208–211; Boyer 1959: 171–173).

Newton picked up Wallis' technique and brought it to synedochic fruition, by applying it to the area of the hyperbola. He found the area under the rectangular hyperbola $(1 + x)y = 1$ as the limit-sum equivalent to

$$\int_0^1\frac{1}{1+x}dx.$$

Newton went on to generalize the integral bounds and tabulated

$$\Phi\left(\lambda\right)=\int_0^X\left(1+x\right)^{\lambda}dx,$$

where the upper bound X varied. The tabulation of $\Phi(\lambda)$ for ascending integral powers of λ in terms of the coefficients of $X, \frac{X^2}{2}, \frac{X^3}{3}, \supset$ yields a table whose entries form a Pascal triangle, and Newton assumed by analogy that the pattern also held for negative values of λ, thus generalizing the binomial coefficient $\binom{\lambda}{i}$. The hyperbola-area is then

$$\Phi(-1) = \lim_{n\to\infty} \sum_{i=0}^{n} \left[(-1)^i \cdot \frac{x^{i+1}}{i+1} \right],$$

yielding a binomial expansion in integral form:

$$\int_0^X (1+x)^{-1}\, dx = \int_0^X \lim_{n\to\infty} \sum_{i=0}^{n} \left[\binom{-1}{i} \cdot x^i \right] dx.$$

Using a similar interpolation technique with respect to the geometric model of a circle, Newton obtained

$$\int_0^X (1-x^2)^{\frac{1}{2}}\, dx = \int_0^X \lim_{n\to\infty} \sum_{i=0}^{n} \left[\binom{\frac{1}{2}}{i} \cdot (x^2)^i \right] dx.$$

From the examples of the hyperbola and the circle, Newton could move to the generalized form of the Binomial Theorem,

$$(1+\alpha)^r = \lim_{n\to\infty} \sum_{i=0}^{n} \left[\binom{r}{i} \cdot \alpha^i \right], \quad r \text{ is real.}$$

Notice how n keeps heading toward infinity, over the bridge! (Whiteside 1961: 244–245).

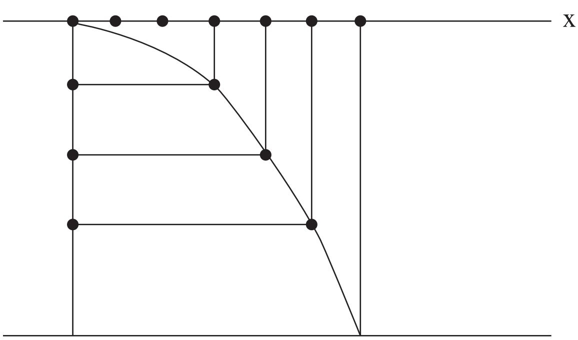
Newton was right to claim that this formula could be extended from integer values of the exponent n to arbitrary real values, though he did not prove it, since it involved considerations of convergence, and a definition of the real numbers, which he could not have supplied. Precisely this extension marks the transition from the finite polynomial expressions that Descartes treated to infinite expressions. The binomial series expression $(1 + x)^r$ is in general an infinite series for which the equation

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n$$

is valid in the interval $|x| < 1$. When r is a nonnegative integer, all but a finite number of the coefficients are zero, and the series reduces to a polynomial of degree r.

Descartes and his successors would probably not have accepted such infinite expressions; yet as we see, Newton's new analytic expressions reduce to finite Cartesian expressions, just when they should, that is, in case the exponent is positive integral. This kind of reduction is a powerful defense and justification of Newton's audacious generalization, as Thomas Nickles argues, relating the heuristic and justificatory uses of reduction (Nickles 1973, 1980).

Newton went on to show how many other infinite series, already existing in the literature, could be regarded as special cases of the Binomial Theorem, either directly or indirectly, via integration or differentiation. Not only did his generalization thus serve to unite hitherto scattered results, but it also precipitated a rich and complex collection of new sum-sequences, most approximating in the limit to geometrical forms: series for the lengths of ellipses, for zones of circles, for trigonometric functions and their inverses. A rich generalization like Newton's Binomial Theorem, then, not only orders previous results metonymically 'reduced' as instances, but synecdochically generates new ones; and by relating these new instances back to those already known, both puts them in context, in already familiar form, and provides them with a warrant. Thus, as variable geometric areas, as orderly though infinite arithmetic series, and as items that could figure in solvable problems, the transcendental curves were raised by Newton to respectability (Whiteside 1961: 256–257 and 261). The Binomial Theorem figures in Newton's first statement of the Fundamental Theorem of the Calculus, but the roots of this theorem go back to prior work on the rectification of arcs (determining the length of curved lines). In the late 1650s, a colleague of Wallis rectified the semi-cubical parabola, and Wren rectified the general cycloid arc. Most important, in 1657, Huygens reduced the rectification of the parabolic arc to the quadrature of the hyperbola. He began by considering the differences between the ordinates of the parabola corresponding to abscissas spaced at equal intervals:



Successive chords joining the ends of these ordinates were proportional to $\sqrt{1+3^2}$, $\sqrt{1+5^2}$, $\sqrt{1+7^2}$, and so on. As n increases indefinitely (the x-axis above is divided into n equal sections), successive elements of the arc of the curve can be replaced by the ordinates of the hyperbola $z^2 = 1 + k^2x^2$, and the quadrature of the parabola gives the rectification of the hyperbola (Baron 1969: 224–225). Isaac Barrow, Newton's predecessor at Cambridge, formulated his understanding of the inverse relation between integral and differential processes, in terms of time and motion, which encouraged another synecdochic elaboration (Baron 1969: 242–243).

The Fundamental Theorem of the Calculus is stated in modern form as follows: Let f be a function integrable on $[a, x]$ for each x in the interval $I = [a, b]$ and let P be any primitive of f on I, that is, a function such that $P'(x) = f(x)$ for x in I. Let $A(x) = \int_c^x f(t)\,dt$, then $A'(x) = f(x)$; and $P(x) = P(c) + \int_c^x f(t)\,dt$, or (here the real power of the theorem is apparent) $\int_c^x f(t)\,dt = P(x) - P(c)$. Thus, for example, when $f(x) = x^n$, $P(x) = \frac{x^{n+1}}{n+1}$, and $\int_a^b x^n\,dx = P(b) - P(a) = \frac{b^{n+1} - a^{n+1}}{n+1}$, excluding the case $n = -1$. The work of Newton that makes the Fundamental Theorem explicit appeared during the period when he had heard Barrow's lectures and had discovered the Binomial Theorem. Up to that time, the tendency had been to reduce problems whenever possible to the determination of quadratures. By making the derivative basic, and defining the integral in terms of it, Newton reversed the direction of reduction. This reversal was crucial, because it provided means for systematically determining integrals, in some cases in terms of an algorithm.

There is a big difference between differentiation and integration. Whereas differentiation of rational algebraic functions leads back into the same class, integration leads immediately into the class of transcendental curves, as we just saw in the case of the hyperbola and the logarithm. Thus, though differentiation and integration are dual, there is a fundamental asymmetry: integration is by far the more difficult operation to formalize. The behavior of differential operators can be captured in a closed algebra, which we learned in high school (if we were lucky enough to have a teacher who will bother to teach us calculus), whereas integration eludes such formalization, as its domain eludes closure. There are, for example, simple formulae governing the differentiation of sums, products and composition of functions:

$$(f + g)' = f' + g',$$

$$(f \cdot g)' = fg' + f'g;$$

and, letting $f = u \circ v$, and $y = v(x)$, then $f'(x) = u'(y) \cdot v'(x)$. Thus, if one knows how to differentiate functions separately, an algebra dictates how to differentiate them in combination. However, if one knows how to integrate separately two functions f and g, it is by no means simple, in general, to integrate the product or composition of those two functions. If the functions are expressed in terms of infinite series, one can add, multiply and compose these series algorithmically and then integrate term by term. But this procedure raises delicate questions of radii of convergence; each reduction has its own limitations. Derivatives are therefore easier to calculate than integrals, and reducing an integration back to a differentiation simplifies the task of the mathematician.

The reduction inherent in the Fundamental Theorem does not provide an algorithm for integrating all functions, Since integration leads quickly out of any collection of tractable functions, into functions that may be very difficult to integrate. Once the primitives have been found, the algorithms governing differentiation can come into play, but it is often difficult to find the primitives of a combination of functions. However, the reduction does eventuate in the accumulation of tables of functions and their primitives and, dependent upon them, a repertory of standard techniques for handling integration, familiar to any somewhat more advanced student of the calculus: integration by parts, by substitution and by partial fractions.

In the *De Analysi* of 1669, Newton found the quadrature of curves by using the infinitely small, both geometrically and analytically (following Barrow on the one hand, and Fermat on the other), and extending its applicability by the Binomial Theorem. He began with a curve drawn so that for abscissa x and ordinate y, the area was

$$z = \frac{n}{m+n} a x^{\frac{m+n}{n}}.$$

If the increase in the abscissa is o, the new abscissa is $x + o$, and the new area

$$z + oy = \left(\frac{n}{m+n}\right) a \left(x + o\right)^{\frac{m+n}{n}}.$$

Applying the Binomial Theorem, dividing through by o, then neglecting terms still containing o, we arrive at $y = ax^{\frac{m}{n}}$. So if the area is given by

$$z = \frac{n}{m+n} a x^{\frac{m+n}{n}}$$

the curve is $y = ax^{\frac{m}{n}}$, and conversely. The important feature of this approach is that the expression for the area is not arrived at as a sum of infinitesimal areas, but rather by considering the area as variable (as a function: here we see the significance of Newton's generalization which lets the upper bound of the integral vary), and observing the momentary increase in the area at the point in question. The area is defined in terms of the rate of change of the area (Boyer 1959).

Thus Newton explicitly related the operations of differentiation and integration, making the former fundamental, and by this problem-reduction opening the way to a systematic attack on problems of integration. This reduction does not eliminate the operation of integration, which continues to elude the algebra of differentiation and to demand continually revised, profounder treatment throughout the next century, remaining an independent source of problems and solutions (Hawkins 1970). The reduction does not provide an algebra governing integration, but rather a collection of systematic techniques which, in conjunction with tables of integrals, can recombine and reformulate functions requiring integration into more tractable form. These compose the core of the field of the infinitesimal calculus; the problem-reduction of the Fundamental Theorem thus constitutes and systematically orders that field.

Meanwhile, across the North Sea, Newton's German counterpart Leibniz was coming to similar conclusions and formulating an even better notation, the discursive middle term that would be further developed by the Bernoullis and Euler, and launch the study of differential equations and eighteenth century physics. My key insight is that Leibniz always understood curves as hybrids: productive ambiguity. The exploration of curves as hybrids is just what Leibniz described in his own retrospective account, written in 1714, of the intellectual genesis of the calculus, "Historio et origo calculi differentialis." In this essay, Leibniz recounts—in the third person—that his initial mathematical discoveries were in arithmetic: "He took a keen delight in the properties and combinations of numbers; indeed, in 1666 he published an essay, 'De arte combinatoria'" (Leibniz 2004: 392–418; Child 1920: 22–58). At the time, Leibniz was only nineteen! The same story comes up vividly in the much earlier "De geometria recondita et analysi indivisibilium atque infinitorum," written in 1686 (Leibniz 1989: 126–143, 2004: 226–233). Leibniz's abstractive, generalizing cast of mind took him immediately from the consideration of particular series of numbers to the study of their general relational properties. For example if, from a given series, one forms a second series of the differences holding between the original terms, what relations will hold between the given and the resultant series? The same question can be raised for the sum series, and then the operations of forming the difference series and the sum series are discovered to be inverse operations. Leibniz's first mathematical discovery of note during his sojourn in Paris (1672–1676) was that the sum of consecutive terms in a series of differences is equal to the difference between the two extreme terms of the original series.

This insight first attracted Huygens' interest in the young Leibniz; and his friendship proved essential to Leibniz's development as a mathematician. Huygens' tutorial friendship with Leibniz introduced the young philosopher to new mathematical realms. Huygens advised Leibniz to study *inter alia* the work of Pascal, and in particular the latter's "Traité des sinus du quart de cercle" where a surface of rotation generated by a circle is geometrically shown to be proportional to an area created by 'applying' the normals to the circle (the radii) to the y-axis, that is, by setting them perpendicular to the *y*-axis in order. In the “De geometria recondita,” Leibniz recalls, “While I was still a novice in these matters, it happened that, in the simple consideration of an argument on the measurement of a spherical surface, I suddenly saw a great light. I observed that, in full generality, the figure generated by the normals to a curve applied to the y-axis is proportional to the surface of rotation generated by rotating that curve around the y-axis. Transported by joy at this first theorem (ignorant though I was that no one else had ever discovered anything like it), I soon postulated for all curves a triangle which I called the characteristic triangle, whose sides were indivisibles (or, to speak more precisely, infinitely small), that is to say, differential quantities; and I deduced from this straightway quite a few theorems that subsequently I discovered piecemeal in the works of Gregory and Barrow. But I did all this without making use of an algebraic calculus” (Leibniz 1989: 140, my translation). The ‘application’ of the normals to the given curve onto the *y*-axis generates a new curve, which is by that very process as we would say integrated; this quantity is then affirmed to be proportional to the surface of rotation generated by the original curve, another integral. This is an elegant general method, but of course at that point Leibniz had no way of characterizing many of the new curves generated by the process of the application of normals, nor did he know how to find the areas under such curves. His reasoning, like that of Pascal, hinged on the similarity between a finite triangle and a characteristic triangle associated with the curve; he had picked up the notion of Cavalierian indivisibles, but as yet had no explicit algebraic language for the situation.

For Leibniz, finite combinatorial arithmetic points beyond itself to the study of infinite series, and geometry to the study of curves outside the classical canon, precisely because of his intellectual habit of generalization. But he was well aware that he could not consolidate his own results. At this crucial juncture, Huygens sent him back to the library, telling him to read the works of Descartes and Slusius, who showed how to form equations for loci. There he discovered the powerful and expressive language of Cartesian algebra in Schooten’s two-volume edition of the *Geometry*. Algebra furnished two essential devices for the development of Leibniz’s thoughts: it allowed him to express perspicuously the rule for an infinite series, and to associate curves with polynomial equations. The idiom of Cartesian algebra, as we have seen, confers a new kind of unity on patterns of numbers, and brings out a new aspect of the unity of a curve, the way its shape constrains various geometrical magnitudes associated with the curve (like its normals, tangents, areas inscribed beneath it, abscissas, and so forth) and makes them vary in tandem. It also allows the unity of patterns of numbers, and the unity of a curve, to be thought together in a new way: the polynomial equation insists on the metaphor of figure and number. Leibniz, like Newton, looks for its synecdochic extension.

One of the first fruits of Leibniz’s use of the new algebra was his solution to the old problem of the squaring of the circle, a problem Newton was also addressing across the waters. Of course, this was not a solution in the Euclidean sense—that being impossible—but it did show how to express the area of a circle in terms of a simple numerical pattern: $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 - \ldots$. As set forth in

"De vera proportione circuli ad quadratum circumscriptum in numeris rationalibus expressa," the proof uses geometric insight gained through the study of Pascal to obtain the circular area by means of the quadrature of $y = xx/1 + xx$, making use of Mercator's expression of $1/1 + t$ as $1 - t + t^2 - t^3 + t^4 - ....$ But the very question, why the quadrature of curves should be bivalently thinkable in terms of both geometrical construction and series of numbers, cannot be answered by the restricted idiom of Cartesian algebra, which brings the question into view but then leaves it ironically suspended (Leibniz 1989: 61–81). As Leibniz writes in "De geometria recondita," "When I began to use [the algebraic calculus], I didn't waste any time in discovering my arithmetic quadrature, and many other things. But, who knows why, the algebraic calculus in these matters didn't entirely bring me satisfaction, constrained as I was to pass by the detour of a figure for establishing many results that I would have liked to obtain by analysis; up to the day when I finally discovered the true complement of algebra for transcendent things, my infinitesimal calculus, or as I also call it, my differential calculus, integral calculus, and—rather judiciously, I'd say—the analysis of indivisibles and infinites" (Leibniz 1989: 141, my translation). The application of Cartesian algebra to "transcendent things" changes algebra itself to produce the "infinitesimal calculus," whose immediate consequence is to generalize and simplify, as well as to offer a deeper-lying explanation of why algebra brings together the study of series of numbers and that of curves so effectively.

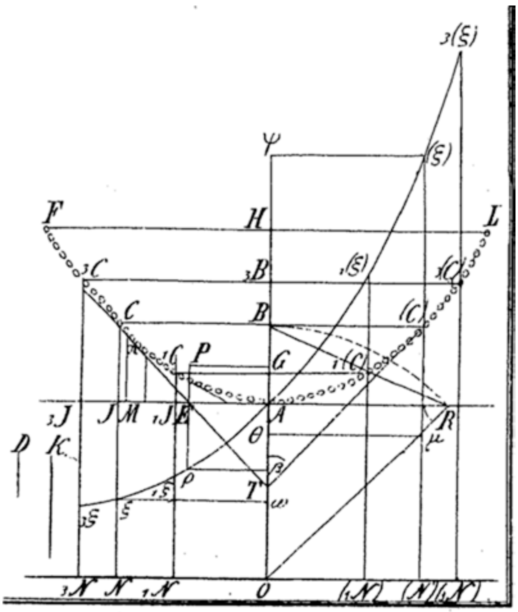
In "Historia et origo calculi differentialis," Leibniz's account of his first formulation of the '*d*' notation ('*d*' for differential—Leibniz thinks of dx as a line segment smaller than any finite line segment) refers to the arithmetical problems that first sparked his interest in mathematics. He describes Pascal's triangle and his own harmonic triangle, and points out that these arrays represent infinitely extended series of (natural or rational) numbers set alongside their sum or difference series. Using his '*d*' notation and Cartesian algebra to represent the general term of a series, Leibniz saw that he could formulate the relations among these series as rules. Thus, if the general term of the initial series is x^2, its difference series is $dx^2 = x^2 + 2x + 1$. Finding an expression for the difference series of an initial series with the form of a polynomial is straightforward; finding an expression for the sum series is not; a general method can be articulated "only if the value of the general term can be expressed by means of a variable x so that the variable does not enter into a denominator or an exponent." Again, if the general terms of the initial series is x^3, its sum series is $x^3/3 - x^2/2 + x/6$. In general, the task of finding sum series leads directly to the question of which series converge and which do not. Leibniz makes it clear that the formation of difference series and that of sum series are inverse operations (Leibniz 2004: 392–418; Child 1920: 22–58).

One of the hybrids that Leibniz uses to make the leap from the combinatorial, arithmetical context to the geometrical study of curves is that of the curve as infinite-sided polygon. (Remember Archimedes making use of it, but coming up short? He didn't have algebra yet.) The initial series, x, or $y = x^2$, is no longer a discrete series of integers or rational numbers that might be thought of as labels for the vertices of the polygon; rather, it stands for 'all the abscissas' or 'all the ordinates' of the curve;

and dx or dy then stands for differences that are infinitesimal, smaller than any finite magnitude. Leibniz's metaphysical faith in the intelligibility of things (expressed in his Principle of Sufficient Reason) is well repaid here, for the analogy between the finitary and the infinitary, carefully pursued, holds up remarkably well. Formulae for the differential calculus are found in a straightforward manner; formulae for the integral calculus, as in the finitary case, prove much more difficult to discover, since integration typically leads from known to unknown curves. Once again, the sum of consecutive terms of a difference series is equal to the difference of the two extreme terms of the original series; and the operations of differentiation and integration are mutually inverse. New curves, in particular transcendental curves, may be defined and investigated by means of differential equations.

The array of numbers that constitute Pascal's triangle and the problem of squaring the circle are ancient topics for mathematical meditation, arising independently in a number of different cultures around the world. Leibniz is able to think these patterns together, by using the new algebra and extending it by notions that allow the mathematician to pass via the infinitesimal and infinitary to return to the finite, as the expression $\int dx = x$ shows so concisely: an infinite number of infinitesimal areas adds up to a finite area, in ways that can be calculated. This inspired detour allows one to transfer insights about finitary, combinatorial items to the continuous items of geometry: the hybrid curve *qua* infinite-sided polygon—at once geometrical, arithmetic, and algebraic—by holding together different domains also brings the infinitesimal, finite, and infinite into rational relation. The intelligible unity of a transcendental function, for example, can be represented by the unity of its peculiar shape (which constrains various magnitudes associated with the curve in characteristic ways), and by the patterns in the numbers associated with the curve; moreover, adumbrated by either shape or number patterns, it can be represented by algebraic form, which holds the geometrical and the numerical unities together as distinct concrete expressions. That algebraic form exists both as the differential equation that defines the curve (indeed, a whole family of curves) and as the equation that constitutes the solution of the differential equation; the former expresses the defining conditions for the curve and the latter the structural features of the curve itself.

Elsewhere, I have discussed in some detail other curves that Leibniz helped to bring to light: the traxtrix, the isochrones and the catenary (Grosholz 2007: Ch. 8). To me, the most interesting curve was the catenary, because its genesis in Leibniz's work shows so clearly how he pursued curves as hybrids; and because the catenary—representing a state of minimal energy and thus of equilibrium—has so many interesting applications in the sciences; and because it is the shape of that crazy Gateway Arch in St. Louis, Missouri, 630 feet tall, the world's tallest arch. (Galileo would have loved it!) Once when I contemplating the diagram of the catenary in Volume 5 of Leibniz's *Mathematische Schriften*, where it looks like a mathematical necklace, I thought of the lovely portrait of Hendrickje Stoffels which I had just discovered in London. She was first Rembrandt's model, and then later his companion. So I wrote this poem.



What Rembrandt Saw

Portrait of Hendrickje Stoffels in the National Gallery, London

The light tug of a pearl drop on her earlobe,
Tick of its pendulum against her throat
Measuring time's passage, or its sheer
Arrest. Here. Again, here.

The weight of two gold necklaces her breast
Warms slightly, drape incurved along a swell,
A lapse, a swell. So might he ride,
Sails furled, one summer evening on the river.

How fur on flesh is smooth and irritant.
How folds conceal by ivory impasto,
Display by contour's tributary shadow,
Rill of the dark surround.

One hand expressive, one hand self-enclosed.
Lips he has never kissed.
And that inquiring gaze: unasked, unanswered
Questions so apparent in the eye.

So shadowy. A pearl
Hangs spinning in the balance, like a world.

Rembrandt Harmensz van Rijn (1606–1669), Portrait of Hendrickje Stoffels, probably 1654–6. Oil on Canvas, 101.9 × 83.7 cm. Bought with a contribution from The Art Fund, 1976 (NG6432). National Gallery, London/Art Resource, NY

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Part IV
The Sky's the Limit!

Chapter 11
Compactification: Randall, Fainlight, Sedakova

In every poem, there is a tug of war between the poem's temporality, its successiveness as the reader moves from line to line, from stanza to stanza, and its spatiality, its shape, which becomes a figure for the transcendence of time and evanescence, a figure for the way in which things (especially people, and works of art!) endure in our hearts and minds. Poets who do not love succession often employ various strategies to escape from the linearity and rush of time, and in particular from the linearity of narrative (driving us from beginning to middle to end) and argument (insisting on the necessary inference from premise, premise, premise… to conclusion). They employ repetition in a certain way, and they are especially fond of circles and rings, the result of various kinds of compactification, as the mathematicians call it.

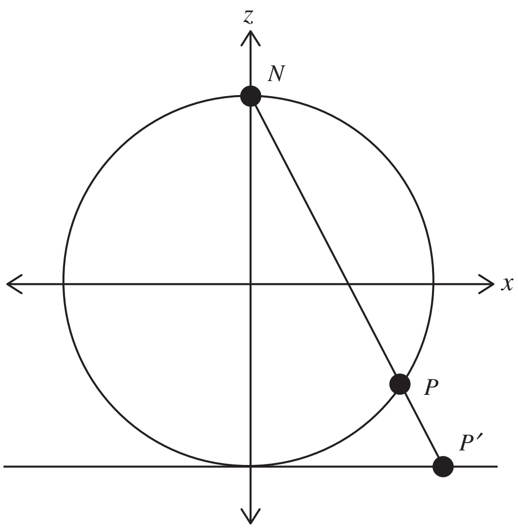
Three such poets are Julia Randall, Ruth Fainlight, and the Russian poet Olga Sedakova. Randall was an important but insufficiently appreciated American poet who spent most of her life teaching at Hollins College in Virginia, and retired to the countryside near Baltimore. She seemed to wish to minimize the successiveness of her poems, so that they would appear as if present all at once, or as if they gathered things in, as our human vision gathers in the night sky under the dome of heaven, or the forested plains of Maryland within the circle of the horizon. The anonymous poets who wrote the great ballads collected by Francis James Child, for example, were happy to carry the reader/listener forward by a story-line, typically by employing refrains whose meaning changes—often drastically—from the beginning to the end of the tale. Donne, Herbert, and Marvell, who loved to make poetic arguments, repeat key words and phrases throughout a given lyric, to measure the development of insight from premises to conclusion. Julia Randall, by contrast, most often created a phonic texture that was particularly rich in consonance, assonance and rhyme. Precisely because these phonic units in a poem have no semantic import, they cannot be developed; rather, they are just the same element whenever they reappear and resound, and so pull all the parts of the poem where they occur back into the same timeless moment. Succession is subordinated to and suppressed by sheer gorgeous repetition.

E. R. Grosholz, *Great Circles*, Mathematics, Culture, and the Arts,
https://doi.org/10.1007/978-3-319-98231-1_11

The winds of space, spiced with the latest spring,
Even stinging on the lips like winter's strong
Unstoppered breath, are music. But my song
Stammers and breaks upon the winds of time.

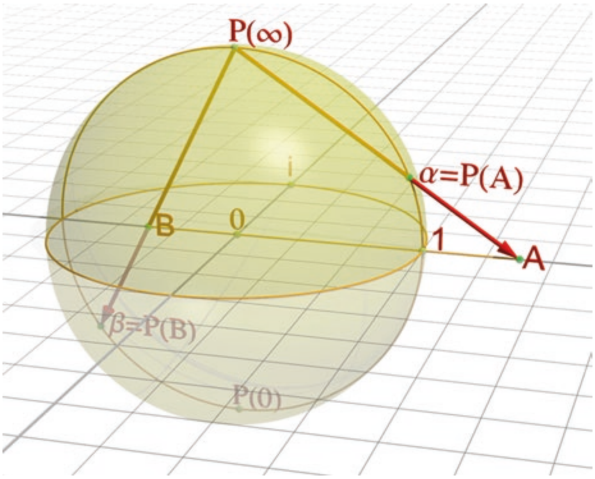
The ideal song would be like space, present all at once; human poets in time must do what they can to escape the winds, the winding and unwinding, of time (Randall 1992: 46).

Another allied poetic strategy is best explained by a mathematical schema that requires the terminology of topology. (We've already encountered some topology, and more is to come.) It is the way we "compactify" infinite spaces, a kind of mapping (Singer and Thorpe 1967: 30–34). Thus we can set up a one-to-one correspondence between the infinite Euclidean line and the circle, and the infinite Euclidean plane and the sphere. (The reader must be patient with a few mathematical technicalities on the next page, though I hope the diagrams convey most of the meaning of the constructions directly. Afterwards, we will return to the poetic images that embody these ideas.) In the former case, we think of the circle with radius r as sitting centered at the center of the x-axis within the Euclidean plane, just where the x-axis and the z-axis meet. Then we map the line parallel to the x-axis, at a distance r "south" of it, onto the circle in the following way. We map the point $P' = (0, -r)$ on the line (directly below the meeting point of the x-axis and the y-axis) to the "south pole" of the circle. We continue by mapping each point $P' = (x,-r)$ on the line to a point P on the circle by drawing a line from each point $(x,-r)$ on that line to the circle's "north pole" N and finding the point P where it intersects the circle. This means, if we extrapolate, that the two ends of the line, at positive infinity $P' = (\infty, -r)$ (infinitely far to the right) and at negative infinity $(-\infty, -r)$ (infinitely far to the left) both ultimately get mapped to the north pole. Draw some lines for yourself on this diagram, where P' is very far out to the right or left, and you'll see that P on the circle approaches closer and closer to N, the "north pole."



The Euclidean line mapped onto the circle

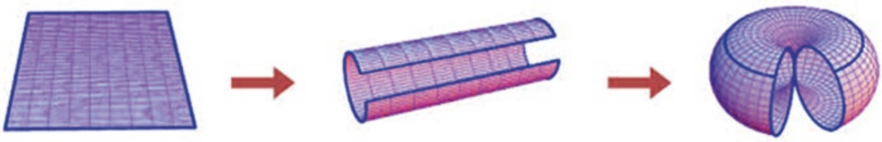
Here is another way of putting it, from projective geometry: the line goes off in two different directions, forever; but if we add the point at infinity at either end, and identify infinity with itself, we can consider the line a circle. Since the standard way of representing time is as a (directed) line—often a line given direction by assigning numbers to it but sometimes just a line with an arrow-tip—this is another way to both assert and deny heading off into the distance of the past or future, a way to configure stasis. In the latter case, we carry out a similar mapping with the Euclidean plane and the sphere. All the infinitely far away points at the imagined edges of the plane, in a similar construction, map onto the "north pole."



The Euclidean plane mapped onto the sphere

As we've just noticed, the circle and the sphere poetically often turn into the earth, so that the topmost point is the "north pole"; and in fact a version of this schema is used in making maps by stereographic projection. These constructions are borrowed from topology and projective geometry, but (like the experience of inertial motion) they are also aspects of everyday experience. When we look up at the blue sky, we don't see it as an infinite extent, but rather as half of a sphere, a dome over our heads, because human vision compactifies.

There is yet another visually suggestive way to compactify the plane. If you set a lattice on the two dimensional Euclidean plane with the usual x-axis and y-axis, perhaps by inscribing parallel vertical lines at 0 and then at all the integers on the x-axis, and then by inscribing at right angles intersecting horizontal lines at 0 and at all the integers on the y-axis, you can begin the process. Impose one periodicity by identifying all the points on the x-axis that differ from each other by a unit: this turns the plane into an endless unit-width strip. Then impose a second periodicity by identifying all the points on the y-axis that differ by a unit: the unit-width strip becomes a square. The whole plane becomes a postage stamp, a square button, a seed with corners. If you carry out a version of this same operation using complex numbers instead of real numbers (the complex numbers, as Gauss showed, can be identified with the two-dimensional Euclidean plane) the folding that results from this mathematical origami becomes a torus, or ring.



Doubly periodic functions over the complex plane are correlated with tori (rings)

Poetically, we have the golden ring that is always slipping from someone's finger, lost and then miraculously recovered, or lost forever and then transformed into the secret in the depths of the garden or pond (Singer and Thorpe 1967: 54–68).

The imposition and folding of the lattice is like the imposition of periodic structure on language, the special magic of the poet, which renders language memorable and lifts it out of the swift, choppy current of transience that carries all speech away: a golden ring, a golden ball. Thus poets construct lines with the strict repetitions of rhythm, metrical structure, or the more elastic and supple rhythms of song, especially folk songs; the chime of sound—rhyme, alliteration, consonance and assonance; the formal repetition of grammatical structure; the echoing citation of familiar and beloved texts or songs; the revival of formulaic phrases and descriptions from earlier oral traditions; the repeated invocations of a litany; and refrains.

Mary Kinzie argued long ago, in an article in the *Hollins Critic* (1983), that Randall's poems exhibit structural stasis rather than argumentative or associative (or, I would add, narrative) development. Randall embeds semantic in syntactic ambiguity, so that we must reread and reinterpret a passage, so that we cannot get to the bottom or the end of it, so that we cannot let it go. Randall's way of forcing us to begin again, to go back and back over a line that will not parse, is one of her ways of achieving structural stasis. This section from "To William Wordsworth from Vermont," is an example:

On Glastenbury once I saw
The pale moon hesitate, but sky
And earth, divorcing, cast her free
Till cold on Pownal, cold on Bennington,
Her puppet light drained down,
And every eye she praised the valley long
Selected love.
Old puppet of the moon, this mountain night
More sharp with stars and waters than a dream
Of islands conjured in a poet's sight,
Not my eyes bless, but light.

Empson would have loved this poem! The syntactic ambiguity of the last four lines is a *tour de force*, which generates a variety of meanings for all the important words. Is “old puppet of the moon” a vocative, or apposite to “this mountain night”? Or is “this mountain night” a temporal tag, an indexical, or is it the object of the verb in the last line? Is “light” a noun or a verb? Is the poet asking for illumination rather than blessing, or is she admitting the independence of natural reality, lit not by her own awareness but by the moon? Is she addressing the moon (light), the night (absence of light), or herself? The poem’s ending clarifies this passage some, but not entirely. The eyes of the poet, illuminating circles of experience, penetrate to the invisible, things hidden by night or long grasses, present only to memory.

Old Rydal
Reaper of snows and fountains, every crop
Of every season’s seedtime pushes up
Its blades to plume the pocked, perennial landscape.
Tintern, or ruinous
Tintagel, caves of Greece
And Cumae, cypresses
Winging the silver night at Arles, or meadows
Of childish May—low mint, lost irises,
Not light, but my eyes bless.

The last line, co-present with “Not my eyes bless, but light,” creates the sought-for balance, the perfect symmetry between inner and outer reality (Randall 1992: 46–48). Here the poet carries us up into the heavens, where we can look down on the sphere of earth and see, all present at once, Tintern Abbey on the Wye and Cumae in Italy (a Greek colony in classical antiquity) and the south of France, where Van Gogh painted the stars (Grosholz 2015b).

Randall was a stay-at-home, but her poems do not walk us through her familiar territory, indicating this neighborhood and the next, and neighborhoods beyond. Instead, she makes one little place, for example Dulaney’s valley outside of Baltimore, an everywhere by superimposing mythical and foreign places on it; her home becomes a palimpsest, a locus of compounded and superimposed meanings. The layered seeing is not at odds with Descartes’ directive in the *Meditations* to clear the mind of distractions in contemplation: foreign associations may be the key to essence, while ordinary association, the merely local, may distract. (Recall that, to solve old geometrical problems, Descartes overlaid them with the shadows of algebraic equations and new-fangled Italian tracing devices.) The order of essences is often for Randall, as for many mystics, at odds with customary framework, ordinary perspective. She distrusts it as ‘mere,’ like appearances, like language. So she writes in “Winter Bloom,”

A cotton snow, and the dogwood
Blooms as in April, but more briefly, to be spent
In a short sun, no berries from this snow. And the bent
Hemlock needs knocking off. This makes me wonder
About beauty and truth.

And truth to tell,
This is a false bloom, ephemeral. But then
If you're still up at ten
To one in Schubert's moonlight, you will cry
On beauty, though it goes
Without saying, like bird cries,
Without translation.

The ordinary phrases "ten to one" and "it goes without saying" leave their echo here, but change meaning completely in the way that Randall wrenches them around, across line breaks, across heartbreak: beauty goes without saying, and it abandons us silently, aloof, above words, beyond any human language. The moonlight belongs to Schubert and Germany, the hemlock to Keats and England; the winter dogwood brings back April and Housman's cherry trees; and they are all no less the backyard scenery of Randall's house (Randall 1987: 9).

And again in "First Frost":

Though woods have warned us, we are never ready.
Dougherty's fields are turnip-green.
The lavender threw up another bloom
Last week, to match the asters in the lane.

Today the tomahawk has fallen—fell
Upon Sennacherib, purple and gold
Gone down, the oldest story in the land,
The oldest question, fashioned in the stone,
But not by stone. By hand. By hand.

Randall's neighbor's fields, on a certain day in the onset of cold weather, were also the old planting grounds of the Piscataway Indians and equally the subject of myth, "the oldest story in the land." The frost is the tomahawk, and the story and the question: such metaphoric conflation, with its beautiful form, stands as the union of inner and outer, what is "fashioned in the stone,/but not by stone. By hand. By hand" (Randall 1981: 19).

The same intensively doubled and redoubled vision informs "Hardwood Country" (Randall 1981: 22–23). History is never far away, nor Greek and Biblical mythology; the entrance to the Otherworld lies beyond Hershfield's Hill and is then Cumae, Sinai, and Dante's dark wood.

October, cadenza. One would always know,
In this rude land, these colors and their close,
The Indian pomes, Valhalla's roof ablaze,
Poet and peasant in the glow of things
Reading of earth and sky the finite rhymes.

Randall's vision does not obscure the particularity of place, but imposes upon it that mark of Spinoza's account of life, absolute significance. Her corner of Maryland is, for Randall, the place most rich in resonances, already covered like a coral reef with the accretions of a lifetime. The soul, brought home by the discipline of contemplation, knows where it is in virtue of where it is not, so that Elsewhere may become, by the process of poetic compactification, aspects of the local scene.

Randall is likewise interested in the singular occasion which, though it can be revisited in memory, happens only once. The paradigm of an unrepeatable act is the creation of the work of art; the work itself becomes permanent testimony to the incursion of something absolute into mere succession. Here is her poem "Giverny," about the visions of Monet, in his own backyard, his *jardin*.

Bearded to match his willow, he sits here
By his pond. It is always summer. Far away
Ice cracks the jetties, holy towers fall,
The Channel rages. Let the world be done.

In the quiet of the lilies, never won
Since Eden rose and the arcangel fell,
That battle with the light goes on and on.

Claude Monet (1840–1926), The Water-Lily Pond, 1899, Oil on Canvas, 88.3 × 93.1 cm. Bought 1927 (NG4240). © National Gallery, London/Art Resource, NY

What is brought here to steadfast expression is Monet's moment of illumination where subject and object (his beard, the willows; his willows, the painted willows; his inner peace and the quiet lilies; his battle with the light and the painted lilies) finally correspond. And time, in the poem, in the painting, stands fast as a witness, in love with the infinitely meaningful flowers, bathed in water and light (Randall 1981: 42). Her corner of Maryland was, for Randall, the place most rich in resonances, already covered like a coral reef with the accretions of a lifetime. In one of her liveliest, most charming and most serious poems (where Wordsworth is summoned once again) "To William Wordsworth from Virginia," she acknowledges and abjures the danger of detachment and idealization. Mere words, the magical resolutions of art, are not sufficient (Randall 1992: 19–20).

…Words that split the tide
Apart for Moses (not for Mahon's bus),
Words that say, the bushes burn for us—
Lilac, forsythia, orange, Sharon rose
For us the seasons wheels, the lovers wait,
All things become the flesh of our delight,
The evidence of our wishes.

Such wishful vision must be tempered by the acknowledgment of mortality, and the contradictions, the moral depths, of the world as it is. Randall addresses herself to Wordsworth, the musical idealist, the narrator of his own life (Grosholz 1986).

But, sir, I am tired of living in a lake
Among the watery weeds and weedy blue
Shadows of flowers that Hancock never grew.
I am tired of my wet wishes, of running away
Like all the nymphs, from the droughty eye of day.
Run, Daphne, Run, Europa, Io, run!
There is not a god left underneath the sun
To balk, to ride, to suffer, to obey.
Here is the unseasonable barberry.
Here is the black face of a child in need.
Here is the bloody figure of a man.
Run, Great Excursioner. Run if you can.

Since we have migrated to the Lake District, we may drive down to London, and have a cup of tea with my other friend from college, Sarah Glazer, who works as a journalist but also runs a Salon in London. She lives right up the hill from Ruth Fainlight, and just a few blocks from Kensington Gardens, where Peter Pan spent time with the faeries of the Serpentine, before setting off to Never Never Land. So we can then walk down the path next to Holland Park, and visit Ruth. She has been a notable presence in English letters for many decades, though in fact she was born in the United States. Fainlight was trained as an art student in both Birmingham and Brighton. Painting is first of all a spatial art, and so inherently static; and (to state the obvious) it is concerned with color and form. Yet, of course, people and their experience of the world exist in time. So the display and stasis and design of Fainlight's poems, when they are especially painterly, are nuanced or inflected by the passage of time (as is all visual art). Time leaves its marks on the human face, on the collected elements of a still life, on the arrested moment (often one episode in a well-known narrative), on landscapes and buildings.

Time's Metaphors

Four paw-marks filled with frozen water,
white brush-strokes drawn on a dark kimono
clutched against her haunches, bird-claw
hieroglyphs; the last surviving
haws and this spring's thrusting buds
enclosed by ice on the same twig.

(Fainlight 2010: 171). During the past couple of decades, as I have read and re-read and taught her poems, I have often wondered how Fainlight's training in art might be expressed in her poetry, and also how the habits of a painter might be reflected in her ways of constructing poems. Or, put otherwise, I wonder what habits of thought and making attracted her earlier to the visual arts and later to poetry. Some of her poems can be classified by painterly genre: Still Life, Interior, Nude, Landscape, Portrait. Thus, for example, some of her poems are about a single object or a cluster of objects; some depict the objectified body of a nude woman and some express the lived body of a woman; some speak from enclosed spaces like a room, a garden, a single block of a city street, a park; some respond to faces.

Though Fainlight may choose a small object for the subject of a poem, it typically proves to have depths that must be sounded, or a kind of interior infinity that expands as the object is opened, as we look more deeply into it, as we extract its contents. And it constitutes a vibrant stasis that opposes itself to the arrow of time, to the river of time that carries all things away, to flow and transience, while still expressing the temporality of things. The poem "Handbag," is occupied solely with an old handbag, and is written in free verse. It consists of noun clauses—there are no main verbs, and so no tenses: all the sentences are sentence fragments that present things (Fainlight 2010: 241).

Handbag

My mother's old leather handbag,
crowded with letters she carried
all through the war. The smell
of my mother's handbag: mints
and lipstick and Coty powder.
The look of those letters, softened
and worn at the edges, opened,
read, and refolded so often.
Letters from my father. Odour
of leather and powder, which ever
since then has meant womanliness,
and love, and anguish, and war.

The language is carefully controlled, and it works to intensify the poem's resistance to the passage of time, to transience, in its own construction. Despite the absence of full rhyme, there is nonetheless a wealth of subtle repetition: handbag/handbag; letters/letters/letters; war/war; powder/powder; leather/leather; and, and, and, and, and, and, and. (That is a great deal of repetition to put into 12 short lines!)

Moreover, Fainlight adds clusters of half-rhyme or near-rhyme: leather/letters/war/powder/letters/odour/leather/powder/ever/war. Or again: crowded/carried/softened/opened/read/refolded/often. Or more sparsely, but accumulating at the end: mints/since/womanliness/anguish. And alliteration: leather/letters/lipstick/look/letters/letters/leather/love. And consonance: the repetitions of 'r,' 'l,' 's,' 'n,' 'm,' that again accumulate at the end: "womanliness,/and love, and anguish, and war." These repetitions create a phonic texture that is particularly rich. And precisely because these phonic units in a poem have no semantic import, they cannot be developed; rather, they are just the same element whenever they reappear and resound, and so most strongly pull all the parts of the poem where they occur back into the same timeless moment. Succession is subordinated to and suppressed by sheer gorgeous repetition. All these repetitions unify the poem, pull it together, work against its temporality and successiveness. There is the poem; there is the handbag. It is still there.

And yet the question remains, what is in the handbag? Is it a "real" handbag, a relic from the past, which would presumably be empty? But it might still retain the smell of the things it used to contain—odors linger. (I have a flask of lavender oil from 1981, which still smells as strong now as it did then.) In any case, this handbag is full: so it is ambiguously the handbag it is, and the handbag it once was, filled with the things it once held. (Or perhaps, as Fainlight's inheritance, it still holds the letters?) Memory also retains contents; memory is a retina, the soul's analogue to that part of the eye. The poet knows exactly what that handbag contained, or contains, in the 'historical present' of the poem: mints, lipstick, Coty powder, and those letters. The odor of leather and powder brings it all back: so the handbag contains the past. And it conjures up the mother, and the absent father, and the father's absence, and indeed the whole war. Five years that engulfed all of Europe in one sense and the whole world in another. Those people, those years, the love and anguish, the world, are there in the handbag.

The jar of "rosy-purple jam" in "Early Rivers" also inhabits two time frames simultaneously and sensuously, both represented with the same immediacy: the day when the poet concocted the jam, and the day, a few years later, when she ate it, a moment presumably identified with the moment of writing the poem, though of course that is a kind of conceit. "I'm eating,/now, straight from the jar…" the poet writes, as if one could write a poem like that while eating jam, as if the moment of eating jam were "real," there and then in the poem every time we read it (Fainlight 2010: 282).

Early Rivers

This jar of rosy-purple jam is labeled
Early Rivers, August '84 –

the date I made it, the name the farmer gave
those plums, smooth as onyx eggs, but warmer.

The dimpled groove, bloom-dusted, down each fruit
pouted at the touch of my knife, yielding
the stone I put inside a cotton sock
(relict of a worn-out pair – every
boiling dyed it darker crimson – from one
plum-season to the next I saved it), then pushed
the lumpy tied-up bag into the center
of the pulpy amber halves and melting sugar
in the preserving kettle, and let the mixture
ooze its pectins, odours, juices, flavours,

until the chemistry of time and fire
produced this sharpness, sweetness, that I'm eating
now, straight from the jar, smearing my mouth,
digging the spoon in deeper, seeking a taste
undiluted even by nostalgia.

The second stanza recreates the day in 1984 those plums were boiled, and gives us the recipe in tactile, palpable, tasty and visual details: "the dimpled groove, bloom-dusted," the cotton sock boiled "darker crimson," the bag with its "pulpy amber halves," and the sugar and juices. Clearly, the poet remembers that day as if it were only yesterday, so vividly; and yet the stanza is also a recipe—she used the same sock over and over, "from one/plum-season to another" as it darkened. The third stanza admits the conceit, identifying the process of making jam with writing the poem. It is a "chemistry of time and fire/[that] produced this sharpness, sweetness." So there the poet sits in that very stanza, "eating/now, straight from the jar," as she digs ever deeper; and we share the spoonfuls, presented to us in the poem, with her in a present somehow freed from the passage of time, "a taste/undiluted even by nostalgia."

So too the poet works her magic, abrogating time, in the ekphrastic poem, "Chardin's *Jar of Apricots*," where the jar of preserves has been cooked up, not by the poet, but almost two hundred and 50 years ago by the great painter Chardin, for our delectation. But we (the poet and the readers) can't open it, so the poet helps us look carefully at the Still Life that surrounds it, in ways that sharpen our vision and press on our sense of touch (Fainlight 2010: 261).

Chardin's *Jar of Apricots*

The jar is half-full with the soft gleam
of dark gold apricots, and has a sheet
of parchment tied across the top.
Chardin painted it at the same age
we both are, noting the different
transparencies and thicknesses

of wine glass and jar – and the decorated
cup with a spoon inside, pieces of bread,
crumbs and a knife, the orange or lemon
and paper parcel on the wooden table
pushed against a soft taupe wall
in the oval frame. I look at it
for a long time and the painting opens
into another sort of time,
with its own depth and light and meaning,
like a childhood memory. No!
Make it go beyond the old story,
but keep the timelessness of the child's
first concept of eternity –
which might have come while staring through
the reflecting sides of a half full
jar of apricots on the kitchen table.

Jean-Siméon Chardin, *Jar of Apricots,* 1758. Oil on canvas, Unframed: 57.2 × 50.8 cm (22 1/2 × 20 in.), Framed: 81 × 75 cm (31 7/8 × 29 1/2 in.). Art Gallery of Ontario, Purchase, 1962. Photo: Ian Lefebvre, Art Gallery of Ontario

At the center, slightly off-center, we see the jar “half-full with the soft gleam/of dark gold apricots,” its transparency and thickness carefully noted. And then around it, a wineglass, a cup and spoon, bread and bread knife, orange or lemon, and—oddly most delicious—we feel a “paper parcel on the wooden table/pushed against a soft taupe wall/in the oval frame.” The mention of the frame takes us out of the painting, and immediately we are back in the “present” with the poet: “I look at it for a long time…” So if we grant the conceit, we share this present with the poet, and we also see how “the painting opens/into another sort of time.” But this time, “with its own depth and light and meaning,” which we owe to the painter, is neither a childhood memory (of Fainlight’s own particular childhood), nor is it a general, universal “intimation of immortality.” To the identification of poet and reader, as we contemplate the painting directly or indirectly, she adds a further identification, with a child. Perhaps it is one of the many contemplative children Chardin painted; perhaps we can all remember a moment like this, re-captured by Chardin’s skill with light and depth and color: “…the timelessness of the child’s/first concept of eternity –/which might have come while staring through/the reflecting sides of a half full/jar of apricots on the kitchen table.” So the handbag, the “real” jar of jam (plum) and the painted jar of jam (apricot) are compactifications of reality, gathering in the lost world of childhood and forcing the long, lengthening trajectory of time to circle around into the eternity of love and anguish, memory, the sweetness of the moment, the play of light as the gravity of things bends it around.

In the poem “Lost Drawing,” a page from a sketchpad in one sense, and in another sense a garden with trees, “bare winter trees in silhouette/against a clear cold turquoise sky/just after sunset,” recalls the war, and a period when the poet lived through a kind of exile at her aunt’s house in Virginia while the rest of her family was in New York City and her father was away fighting in Europe. Again, the status of the page, the leaf from the sketchpad, is ambiguous: it is there and not there. At the beginning of the poem the “bare winter trees” (a phrase whose structure and cadence recalls Shakespeare’s phrase, “bare ruined choirs”) are presented both on the page (in the poem), and on the sketch pad’s paper, and in two gardens, one in Virginia long ago, and one in England, “now.” The identity of these bare winter trees organizes the poem (Fainlight 2010: 243).

Lost Drawing

Bare winter trees in silhouette
against a clear cold turquoise sky
just after sunset: during the war,
at my aunt’s house in Virginia, I tried
to draw them – trees like these now in England
which she never saw – and now,
trees in my garden make me feel
the first true pang of grief since her death.

Between the washtubs and store-cupboards filled
with pickled peaches and grape jam, crouched
into a broken wicker chair,

I peered up through a basement window.
Sketchpad on my lap, with brushes and
bottles of black ink, blue ink, and water,
I wanted to convey the thickness
of their trunks, the mystery
of how a branch puts out a hundred
twigs, the depth and power of evening.

I heard her cross the porch, the kitchen
floorboards creak. As it grew darker,
that halo of light, outlining
all the finest intersections,
faded. Night absorbed the trees
the house the woman and the girl
into inself, kept every aspect
of that time alive, to give
me back today the memory
of my dead aunt and my lost drawing.

In the central stanza, Fainlight shows us just how she drew them, materially, "with brushes and/bottles of black ink, blue ink, and water," and just how she struggled to capture what is so hard to put on two-dimensional paper: "the thickness/of their trunks, the mystery/of how a branch puts out a hundred/twigs…" And that is not all: to represent the trees properly, because they are in silhouette against the sky, she also has to draw, somehow, "the depth and power of evening." But there it is, compacted in the drawing, in the poem: not just the trees but also the great turquoise dome of sky (transforming into darkness) against which they appear.

So the poet, absorbed by the task of drawing, withdraws from the trees in the poet's garden in London, and she is the girl she once was, sitting in her aunt's basement among "pickled peaches and grape jam," looking up at the trees against the sky through the basement window, and listening to her aunt walking across the porch, across the kitchen. (Perhaps it was her aunt who taught her how to make preserves.) But the light is fading: this dramatic condition makes it harder and harder to for the girl to draw; the light fades from the sky, the past disappears into the drawing, the drawing is lost and the aunt is dead. (Then we recall her death was announced in the first stanza.) And finally there is only the poet, in the present of the poem, in London, remembering. Paintings or drawings represent temporality only by implication. Thus the letters in the handbag are represented as old: they exhibit their age in their worn edges, their deep creases, like an old face. The trees silhouetted against the sky in London are at first identified with the trees in Virginia, and the sketched trees on the page: but then the identity is thrown into question as turquoise turns to blackness, which merges with and engulfs the black silhouettes of the trees, and then indeed, without punctuation, "the trees/the house the woman and the girl." The winter trees are doubly negated as we learn that the page from the sketchpad is lost, and now merely remembered, and the aunt is dead. Yet paradoxically it is the returning darkness, the same returning darkness, that "kept every aspect/of the time alive, to give/me back today the memory/of my dead aunt and my lost drawing."

Fainlight has a penchant for spirals. The Milky Way is a spiral galaxy. In the poem "Spring in the City," one of the many poems she's written about the neighborhood where she lives in London near Ladbroke Square and Holland Park, she transforms the wind-tossed rain of petals falling from the city pear trees. First they become a vortex of sand in a rock pool, then the upswept, blossom-decorated hair of young girls, and then, as the poem ends, stars in galaxies: "Shaken by the breeze/and cornering cars/reaffirming the spiral/of the galaxies/the air today seems thick/with stardust and we/are breathing stars." And in "Milky Way," the starry spiral turns into a whole girl, naked in the bath or shower, perhaps the poet herself using the poem to present a nude not so much as an object to be viewed, as a consciousness (Fainlight 2010: 383).

Milky Way

Under the spout of the shower,
lifting an arm to soap the armpit –
recurrent gestures
                    that open onto
a flickering continuum,
like a zoetrope…
                    The girl in Virginia
Who lay on her back in the bathtub
To let the water needle her body
Deliciously and the woman
in England, meditatively washing,
enacting the same movements…
                    One
remembers, the other puzzles why
she feels the moment returning,
streaming before – behind –
                    circling
as water spirals, clockwise, counter
clockwise – as a cloud
of dust and gas coagulates
into a galaxy…
                    While the shower
patterns her body, scalp to footsole,
with a pelt of iridescent foam
like a map of the Milky Way.

So the poet is both the girl she was, and the woman she is now, as water spirals around her, mapping the timeline onto the circle of memory ("she feels the moment returning"), and then turning it into a galaxy of stars, "a map of the Milky Way."

In two of my favorite poems, Fainlight explicitly links the composition of a painting to that of a poem: "That Presence" and "The Bowl of Apples." The first starts out in a rather straightforward manner: it describes the moment when one pauses in the midst of writing a poem, analogous to the moment when the painter steps back from the canvas, brush loaded, or the weaver steps back from the loom to inspect the pattern. But we should note, again, the redoubling: the poet describes the moment, and then enacts it, pausing between the two stanzas (Fainlight 2010: 317).

That Presence

Like a painter stepping backward from the easel
straightening up from the worktable
with a loaded brush, to see exactly where
another touch of red is wanted, like
a carpet weaver wondering if the time
has come to change the pattern, a sculptor
hesitating before the first decisive cut,
I ponder a poem, repeating every word,
trying to hear where a note needs altering,
testing by breath and sense and luck –

like staring at the surface of a mirror
though soundless levels between glass and silver
into the pupils of that reflected presence
over my shoulder advancing from its depths.

Look what happens after the second stanza: the poem becomes uncanny. If there was an object simply to be depicted, it has vanished, and the mind is a mirror reflecting an eye's pupil reflecting the mirror, endlessly. And yet there is a presence, "over my shoulder advancing from its depths." What could it be?

Then we can return, with a sigh of relief, to "A Bowl of Apples." There is the same comparison of poetry and painting (though the differences are stressed), and the same redoubling, but the poet doesn't vanish at the end. She is there in the poem, in flesh and thought, trying her best to do justice to the object, impossibly, to get out of the way of the poem and express the essence of apples (Fainlight 2010: 281).

A Bowl of Apples

A painter looks at a bowl of apples
on a wooden table pushed against
a roughly plastered wall, and sees them
in the same instant as particular
fruits picked from a tree in the garden
or bought at the corner store, noting
yet ignoring the implications
of fading tones and softening forms –
and as abstract ideals of apples.

But whichever I choose, whether
the most elaborate metaphor
or barest statement, words
are too specific. Language
forces definition. Impossible
to write a poem impersonal

as a still-life, or to articulate
the essence of apples, and not
expose my true nature.

My fingers crave the first touch
of craftsman's tools, pencil or brush,
my nostrils, the reek of size and paint.
I want to understand and feel
through my own flesh the roundness
and bulk of apple pressed against
apple in the bowl's shadowy cradle;
that fierce joy a god must know,
contemplating his work.

And yet, surely she suggests by how the poem itself is presented that she too knows "that fierce joy a god must know,/contemplating his work." So we have come back to the Still Life, and found without surprise that Fainlight has rendered it in subtle colors, nuanced by temporality, and fiercely animated (Grosholz 2015a).

If we depart from the east coast of England, and go through the Øresund Strait overshadowed by Hamlet's castle on the east coast of Denmark into the Baltic Sea, we find ourselves in one of my poems.

The Tallinn Ferry

The ferryboat from Helsinki to Tallinn
Passes small islands into the open sea,
The Baltic. Brackish, neither sweet nor salt,
It plays congenial host to microscopic
Flora and fauna flourishing only here.
Only here. The sunset shines behind us
And lights the northern dome of heaven slantwise
As if dusk were midday, which it is,
Almost, in summer near the Arctic Circle.

This boat reminds me of another ferry
I boarded more than forty years ago,
From Brindisi to Patras. A southern sea
With the same perfect circle at the edges
Thanks to our finite eyes, the curvature
Of almost perfect earth, that oblate sphere,
The same slate blue at evening, but another
Dark-eyed man was waiting on the shore,
Hidden behind the folded wing of years.

For hours there is nothing on the horizon.
It's just a circle, as the river of time
Is just a line: fixed banks or flowing stream?
The line witholds its secrets, like the circle.

Then gleams arise, the facets of a cliff,
The windows of a city, the shimmer of ships
Moored close together round a crescent harbor.
And so my vanished loves sometimes appear
At sunset, as the ferry veers towards home.

But really, we are on our way to St. Petersburg and then down to Moscow and the environs of Tula, to visit the Russian poet Olga Sedakova. If a poet is fond of stasis, there are two problems: death and eternity. Death is inimical to poetry because it is mute, and so is eternity, because it has left earth behind. An inspired poet of stasis must then somehow invent something else, like Dante's earthly paradise or Hafez's garden or Yeats' Lake Isle of Innisfree, where everything is collected, and named, and mutually responsive. This kind of poetry does without narrative or inference, but not without structure: its structure is slant and works by association. Neither horizontal nor vertical, it arrays itself on the page in patterns that are somehow star-like, a compound symmetry, a lattice on the page. Arriving at the end of the line, the reader must contemplate the whole; arriving at the end of the poem, the reader is invited back to the beginning. Nothing in the world of the poem is isolated: one thing refers to all the other things, by means of love or thought or repetition of sound and sense. Poets of stasis, a stasis that is not fatal or transcendent, eschew the goal of radical novelty or the promise of Hegelian progress: the meaning of life must be not only discovered but also remembered, and shared. Sedakova is such a poet. Her poems contain little that is obviously autobiographical, and little that is recognizably historical. Though she often echoes poetic elements drawn from the work of her favorite poets (Pushkin, the source; Alexander Blok, Anna Ahkmatova, Boris Pasternak, Marina Tsvetaeva, and Osip Mandelstam; and her contemporaries Elena Shvarts, Leonid Aronzon and Joseph Brodsky), they are not located historically: rather, it is as if a conversation has become a canon, a round, with the melody repeated and harmonized.

Some poets of poetic (and vibrant) stasis focus on a single object, which then opens up from its depths to include other dimensions; one thinks, for example, of Rilke's *Dinggedichte*, or Ruth Fainlight's poems about worn, redolent, evocative objects, like her mother's handbag. Olga Sedakova, however, like Julia Randall, typically chooses a place, but then sounds its depths in quite a different way. In her early essay, "In Praise of Poetry," she writes, "I don't usually remember the moment in time when I composed a particular poem, often not even the year, but I do remember the places very clearly. Because each poem is to some extent a portrait of a place. The portrait is barely discernible, remaining far beyond the threshold of the immediate content of the poem. There are only a few of these places…" (Sedakova 2014: 141). She goes on to list a number of her poetic places, *topoi* or *loci* in a certain sense.

The first is "the surroundings of the village of Mashutino, halfway between the Holy Trinity-St. Sergius Lavra (Monastery) and Aleksandrov, which was [her] father's birthplace. There are no surroundings without a guardian spirit. And each *genius loci* loves its own style and themes. Mashutino is the setting for the "Legends," children's poems about Saint Alexius, and other such poems." The Holy Trinity-St. Sergius Lavra is the spiritual home of the Russian Orthodox Church; it is located in Sergiyev Posad, one of the most important cities of the "Golden Ring," about 40 kilometers from the town of Aleksandrov, which is in turn about 110 kilometers from Moscow. The "Legends" were a cycle of poems about the lives of the saints, among which the life of Saint Alexis was one of the most popular. Mashutino, Sedakova observes, is a flat, barren, undistinguished place where the wind howls, and which has been forgotten by history; its spirit is that of endless melancholy, like the *Sehnsucht* which permeates Rilke's *Book of Hours*.

The second site was that of the family's *dacha*, Valentinovka; the third, Saltykovka, where friends had another dacha. The poet associated the former with spring and summer, the latter with winter. But the fourth was the most important. She writes, "But the center of all the places, the stolen cradle, is Perovo Pole… this used to be a place of petty bourgeois life with rowan and silver fir bark between the double window frames, woodworm patterns, valances on the beds, pails in the hay and wild cucumbers on the fence. A house ruled over this place: in the house was a stove, by the stove my Grandmother… I have never met a more Christian person; it seems it is more unusual to meet a true Christian than a genius in this world. Perovo Pole reminded me, as a whole, of a deep carved cradle, the scoop of folded palms, a basket of unlit coal." And finally, she mentions Azarovka, where she has a house in the country and spends about half of each year; it lies to the south, between Moscow and Tula, the town near Tolstoy's estate Yasnaya Polyana. Notice that none of these places is the city where she grew up, Moscow (Sedakova 2014: 143).

Perovo Pole no longer exists; it has been turned into a housing development. And of course Sedakova's grandmother died long ago. But poetic stasis does not need to take time into account; the passages and promises of time are beside the point. Sedakova observes, "I do not need to revisit these places. I can see them without closing my eyes. And I not only see the things I remember, but can also sometimes examine or find a new knot in the floorboard at Perovo Pole, or a stand of bushes at Mashutino. These are probably gifts from the spirits of those places. Like everything in the world (even if they conceal it from themselves), they want to have their say, and are pleased if a listener comes their way." She associates these landscapes with dreams, and images—constant, recurrent images. "As if they were all various folds lying within the same depth, the last, final depth" (Sedakova 2014: 144).

So we should look especially closely at poems by Sedakova with traces of her grandmother, children, the strain of a lullaby, the edge of a garden between a village and green hills, and the shades of those who still remain. In a very early book, *The Wild Rose: Legends and Fantasies* (1978), we find "Morning in the Garden" (Sedakova 2003: 40). There at the center lies the image of sky reflected in a small bowl of water, a round mirror of water that in other poems shines up from the bottom of a well. Sometimes the round mirror is the whole ocean within its horizon, the circle constructed by the eye, contained in the eye: the compactification of the infinite, by vision and by thought. The planets appear as stars, though they are planets, and they move along another circle, the great circle of the ecliptic. And we should recall that the circle is the geometric figure with infinite symmetry.

> Is it light or a bush?
>   I push it aside and stand.
> Whatever I hold, like the wind, I hold, barely looking at what I've found.

It's just the water, it's just the wind rocking the light.
It's a saucer of water, reading the position of the planets.

No one is with me, but this light… at last we're alone.
Let it be taken, drunk up, and celebrated, as is usually done.

In a glass of wine, sometimes the lover who makes a toast or the worshipper who takes communion, sees himself, or herself, reflected, startled, looking back.

The cycle *Stellae and Inscriptions* (1982) is dedicated to Sedakova's friend Nina Braginskaya, a classical scholar at Moscow State University: the poems meditate on antique gravestones and epitaphs, calling up those enigmatic souls. In the poem "Inscription," the poet seems to address her friend directly, and they are walking. But the progression of the walk and of the poem is diverted, turned back onto itself, because the walking is dreamed, because "hello" is identified with "goodbye" and uttered in parting, and because the earth is round: all roads are inscribed on, and by, the earth. The line becomes a circle (Sedadova 2003: 53).

Nina, in a dream, or my mind — we were walking
one time on some old-fashioned road,
alongside, as it seemed to me, various
white and smoothed-down flagstones.
"Not the Appian, some other one," —
you said to me, — "it's not that important, the number of roads
in their cities that crossed from one grave to another
was legion." "Hello!" — we heard —
"hello!" (that, as we know, is the favorite word upon parting.)
"Hello! How clearly you look at the earth that's so dear.
Stop: I look with the eyes of the gigantic earth.
Only the emptiness looks. Only the unseen we see.
So go ahead faster or I'll leave you behind."

The poet of stasis scorns the illusion of novelty, the honed arrow of time whose 'cutting edge' literary critics and writers of manifestos are so thrilled by. Rather, the aim of poetry is to recreate, to go to the heart of things, to find or lament the hidden ring that lies in darkness, waiting to be recovered, the lost child who is waiting under the stairs or at the edge of the field. Or to recall the banished poet from the Caucasus. Pushkin was admittedly only sometimes a poet of stasis, when he writes songs, or epigrams in his Hafez mode; his narrative and dramatic works clearly exploit the possibilities offered by time and history, though his fairy tales tend to bring narrative around back into the charmed circle.

In Sedakova's collection, *The Beginning of a Book*, there is a poem to an old nurse (echoing one of Pushkin's), a poem to Saint Alexis, and a lullaby. *The Beginning of a Book* begins with a poem, "Rain," dedicated to Pope John Paul II. Sedakova met John Paul II four times between 1995 and 1998. In her encomium, "In Memory of John Paul II," she writes, "The icon he prayed to was one of Our Lady of Kazan painted in Russia (the icon was sent to him from the Fatima Monastery after the attempt on his life, and, in his own words, saved his life)—this was the icon he presented to the Russian Orthodox Church last year as a gift. We missed the wonderful opportunity of a possible first meeting after a thousand-year interruption" (Sedakova 2010: 241). She admired him for maintaining his compassion and active leadership despite his physical suffering, and his 'humanism,'despite all he had lived through during the Nazi and Soviet occupations of Poland: the seeds of redemption lie in every human soul. And she admired him for reviving the old, Aristotelian sense of politics, so dear as well to her teacher Sergei Averintsev: "politics, the 'common cohabitation,' not only in the Christian, but also in the ancient Greek, that is, in the original meaning, as it is stated in Aristotle, means something totally different: the organization of common life on the basis of mutual *philia* (the Greek word that is translated into Russian both as 'friendship' and 'love'), that very relationship about which Christ questioned St. Peter" (Sedakova 2010: 242). Grace is for everyone; piety does not give you the right to scorn others and cut them off.

The poem "Rain" must be quoted in full; its layout on the page matters (Sedakova 2003: 54).

"It's raining,
and they say there is no God!"
old Nanny Varya used to say
at our house.

Those who used to say there is no God
now light candles before icons,
order special liturgies,
and mistrust people of other faiths.

Nanny Varya lies in the churchyard now,
and the rain pours down,
magnificent, incomprehensible, abundant,
it pours and pours
without knocking on anyone's door.

Rain is not a misleading figure for grace, and neither is sunlight: "magnificent, incomprehensible, abundant." And Sedakova's nanny lies in the churchyard near one of her places, a *genius loci*, hidden but named, and still grateful for the rain; a cemetery is often the garden of a church.

In "A Lullaby," the house itself, a pillowed bed, and a seashell compactify experience, drawing in the open ocean and then the stars; and the house is Sedakova's grandmother's house (Sedakova 2003: 59).

Candlelight casts a figure eight,
ships sail onto the open sea,
the sea stirs in a pillow:
in a pillow, in a seashell, in a distant window.
Where's the Christmas knitting needle of the star?
Where is my grandmother, my sweet sister?
We've walked together for so long
and we've talked:

look, it's such a familiar,
such an unknown doorway!
Who's missed us here?…

Oddly, the house is childless, and yet the children return: "Here beds have been made for us/and we are taught to live in peace,/we will not part." And then the window performs its magic, airing the house and tucking in the sky:

A face flashed in the window.
We have to take off our shoes when we enter.
The evening star stretched its hands to us
like an all-seeing but blind mother.

Compactification allows the circle to complete the line and make lost time return, a sea shell to fold in the ocean, and the north pole to gather in the heavens.

So let us come back by a different path to the topic of childhood. In the essay "Once again about Childhood, Poetry and Courage. Answers for Elena Stepanian," Sedakova remarks that we owe to Romanticism the recognition of "the independent value of childhood," and adds, "The discovery of childhood as another consciousness, another connection between things, another method of communicating to the world… was very fruitful—and form-making." She cites Rilke, Bely, Joyce, Khlebnikov, Pasternak, Chesterton, and Florensky as writers who owed their enlightening new ideas to the memory of childhood. Her own childhood was very happy: "The childhood that I remember, or more precisely—my early childhood to about the age of five, is undoubtedly golden. Even the early problems, illnesses, and fears are that color. The good will of a small child is alchemical: it turns everything to gold." And on the whole, Sedakova was fortunate: her family was well-to-do, quiet and loving, and she had both a nanny and her grandmother to take care of her. She also had the fairy tales of Pushkin, and later all his poems (Sedakova 2010: 63–64).

Her grandmother seems to have been an extraordinary person. Sedakova writes, "I could say something about my grandmother Darya Semenovna such as the following in a poem by Pasternak: "You meant everything in my life." She was wise, thanks to "her profoundly enlightened faith, without the tiniest of superstitions and crudity that you can often find in the piety of common folk (she was from peasant stock)." Her prayer and thoughtfulness had a "monastic refinement." Her language was like that of the fairy-tales of Pushkin, she was funny, and she had a gift for teaching. "In my childhood I could say only one thing about her: that it was *interesting* to be with her, that I was afraid to miss a moment and could not tear my eyes away from her. It was not as interesting for me to be with almost anyone else the rest of my life to this day" (Sedakova 2010: 65–66). This is an impressive claim, coming from someone who was friends with Averintsev, Shvarts, Brodsky, and Lotman, among many others. In another text, "An Interview with Olga Sedakova, January, 2012," Sedakova returns to the topic of children, and mentions Vladimir Bibikhin's study of the babble of very small children. He poses this question: when an infant, all alone, pulls itself up in its crib and starts to babble, "who is the child trying so hard to talk to?" His answer is, "the owner of language." Sedakova emends his remark: “the owner of poetry.” Thus one way for her to sound the depths of language and sources of poetry, she suggests, is to return to “the preverbal perceptions of my early childhood, traces of which still shimmered in my memory,” because children have the gift “of seeing everything as it is” (Sedakova 2014: 191–193). Perhaps the heavens are really a dome, a protective and decorated roof for us; perhaps in the clerestory of Chartres we are really looking at heaven (Grosholz 2016).

The Clerestory of Chartres Cathedral, and my back yard (my photographs)

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Chapter 12
Beyond Deduction: From Descartes to Gödel

Here we return to the transformation of number in modern mathematics, where first Descartes' arithmetic algebra and then Leibniz's analytic algebra as a middle term transform both the understanding of number and the understanding of figure. First, we'll look at the way that mathematicians moved from the natural numbers, the integers, and the rationals, to the reals, the complex numbers, the *p*-adic numbers, and the transfinite ordinals and cardinals. Just as algebra allowed Leibniz to master the infinitesimal and broach the topic of the infinite, so here mathematicians finally begin to master the infinite, though unsurprisingly the infinite still proves elusive and mysterious.

In the seventeenth century, as we have seen, the natural numbers took on a new meaning when they were viewed as solutions to polynomial equations. The advent of polynomial equations made visible the distinctions among rational, algebraic and transcendental numbers, and then led to the dawning realization that many important curves were not associated with any polynomial of fixed degree with a finite number of terms: beyond the algebraic functions lay the transcendental functions. The infinitesimal calculus, and the notation and methods associated with differential equations, allowed for the exploration of these functions by Leibniz, the Bernoullis, Euler and Lagrange in conjunction with the nascent discipline of Newtonian mechanics. This set the stage for Gauss's work, which included the flowering of complex analysis based on the insight that the Euclidean plane provides a good (geometric) model for the complex number system. Gauss also revisited the ancient Chinese Remainder Theorem with a fresh conceptualization called the Quadratic Reciprocity Theorem, which explained for the first time why congruences, especially congruences precipitated by prime numbers, lie at the heart of number theory. To Galois and Klein we owe the notion of group, that important new middle term, which then engenders inquiry into fields and rings: abstract algebra emerges.

Another novel middle term in the nineteenth century is the notion of set. In the opening lines of his classic work, *Contributions to the Founding of the Theory of Transfinite Numbers*, Georg Cantor wrote, "By an 'aggregate' (*Menge*) we are to

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understand any collection into a whole (*Zusammenfassung zu einem Ganzen*) M of definite and separate objects *m* of our intuition or our thought. These objects are called the 'elements' of M" (Cantor 1915/1955: 85). Elements of a set, qua elements, have no internal or intrinsic structure and no internal or intrinsic relations to other elements. Ordering may be externally superimposed, and then the elements constitute an ordered set; but a set of elements qua set is not yet ordered. Any intrinsic structure that elements might have is not pertinent to their being collected in a set or subject to the operations of union and intersection. Another—interesting because always tacit—aspect of the formation of sets is that in their mathematical uses sets are always formed from homogeneous elements. While in principle sets can "collect into a whole" any objects whatsoever as long as they are "definite and separate" (e.g. the three-member set composed of the number 2, an equilateral triangle, and $\sqrt{\pi}$), in practice mathematicians, like small children, only invoke sets composed of homogeneous elements: the set of all isometries of the plane, the set of all rational numbers, the set of all open sets in a metric space. (Recall the set of my stuffed animals in the closet of the house of childhood, of which only the singleton set of my musical lamb remains.)

Some philosophers, stricken by Logicism (a very dangerous infection of the brain, which tends to go viral), have tried to reduce other parts of mathematics to Set Theory. They typically pretend that arithmetic can be generated from sets built up from the empty set, and geometry from sets built up from points: these elements are chosen because they seem to have no intrinsic relations to other such elements. I have countered this reductive account elsewhere (Grosholz 2016: Ch. 4). Here, however, I just want to show how Set Theory works as a middle term to help present the real numbers in their full glory, as both an arithmetization of the line, endowing the continuum for the first time with its own number system, and a geometrization of number, imposing continuity on what had always been understood to be discrete. This union might have seemed hopeless. There is no way that the geometrical line, the continuum, can match up perfectly with numbers, which by their very nature are for counting: numbers are well-orderable and discrete, otherwise we could never begin or proceed with counting. Similarly, without its continuous spread or extension, the way it is seamless, the line could never be the formal expression of otherness or differentiation, holding things apart. If the line were a set of points-for-counting, all the points could be superimposed on each other. This fact is captured by Lebesgue's notion of measure and Brouwer's notion of dimension: a set of points on the line which is countable (whose cardinality is aleph-null) has measure zero and dimension zero; it cannot hold things apart and cannot, in particular, be used to measure, for measurement involves extent.

The linkage between number and figure, inspired first by algebra and then by set theory, insists on equating what cannot be equated, and brings into existence a new kind of thing. The system of the reals appears to be neither counting/countable number nor 'viscous' geometric continuum, but rather a hybrid. Those who like the descendants of Weierstrass insisted on the arithmeticity of the reals, and those who like Brouwer insisted on the viscosity of the reals, missed the hybrid nature of this number system (Posy 2000). At the end of the twentieth century, a hundred years or so after the invention of the reals, we have become rather used to them. In every textbook, we encounter "the set of points on the real line," **R**, and forget how strange it is to have a number system that exhibits the interlocked density or undecomposability of the line, *and* that seems to articulate the line as a set of points.

The natural numbers have proved their canonicity over all the millennia of mathematical practice; any attempt to mimic them in terms of sets (or in other terms) must presuppose them, not least because they are required for counting, and so for creating and counting iteration. (For related arguments see Poincaré 1913/1946.) But what does set theory make of the line? A line is a unity in a very strong sense: it has no parts, though it does admit of internal articulation by points as limits. Thus it appears that all set theory can do with the line is form the set of a line; but set theory is not much interested in singleton sets. Its main mathematical concern is collecting multiplicities, and then characterizing and relating them. Historically, in fact, set theory arose in the work of Cantor and others through the study of point sets on the line in late nineteenth century analysis. The interest of set theory seems to lie in points, and sets of points.

Points, as I just noted, do seem likely candidates for elements of sets in set theory, since they have no intrinsic structure or intrinsic relations to other points: their relations to other points seem externally imposed. The problem with points is how to admit them as "definite and separate objects." There is no way to distinguish one point from another (as we can distinguish 1 from 2, for example, or a circle from a square), precisely because points have no intrinsic structure. Just as when you form the set of (∅,∅,∅) all you get is ∅, so if you form the set (.,.,.) all you get is a point. The only way to separate points is by outness, or extension: distinct points mark the boundaries of a line segment. Points cannot be set side-by-side; either they are the same thing, or they are separated by a line, a continuum. The only way to collect two points as "definite and separate objects" in order to make a set with more than one element, is to collect the first point, the second point, and the line segment on which they lie. But such a set seems anomalous: the elements are heterogeneous. We might suppress the line as a member of the set; but then it must still somehow be available, as a condition of the intelligibility of the set. Indeed, insofar as the two points cannot be given apart from the line to which they belong, the relations of order are intrinisic to them, qua 'limits of the line.' The relation of betweenness is then intrinsic to three collinear points of which we say "*b* is between *a* and *c*" insofar as they are three separate points; it is not imposed on them.

What set theory aims to do is to replace the line or continuum with 'the set of all points on the line.' But how should this set be understood? Putting aside the question of how points may be considered 'definite and separate,' we may ask whether these points—like the natural numbers—can be considered apart from their 'natural' order. One might try first, following Cantor's example when he re-ordered the rational numbers and the algebraic numbers to show that they can be put in one-to-one correspondence with the natural numbers, to imagine alternative orderings of points on the line, that is, to imagine points on the line as identical marbles in a row, and then transpose marbles within the row. However, since a point is a point is a point, by the identity of indiscernibles (which is one of Leibniz's principles), this process just yields the same thing over and over again, as the mere iteration of an element just yields that element again. So there is no way to tell the difference between various orderings of points on the line, or even to pick out their 'natural' order. Points, unlike rational numbers or algebraic numbers, have no 'natural' order; enjoying no intrinsic structure, they have no intrinsic relations with each other. The line qua set of points has no internal or intrinsic articulation, which is perhaps to say that it has no parts. Note too that in this thought-experiment, the transpositions are governed by the notion of the line: the marbles are only to be transposed within the row. The line has not been replaced or banished by a set of points, but is still in force as a condition of intelligibility. Indeed, so far the notion of set and the notion of point seem mathematically inert in this context.

However, we can think of points on the line as constituting a set of 'definite and separate objects,' if they can be tagged somehow by numbers. We know that the natural numbers (and the rational and algebraic numbers) are not sufficient for the task, because all those number systems when correlated with points on the line prove to have 'holes' in them. Even Descartes was worried by those little holes. Another formulation of this claim is that the intelligibility of 'the set of points on the line' is the availability of the real numbers as tags. Two points cannot be distinguished from each other; but two reals can be distinguished, because a real number does have intrinsic structure, expressed variously as its decimal expansion, an infinite sum, an extended fraction, or the like. And if we tag each point by a real, then it seems we can keep track of them. We can consider the line as a set of points if we can assume the availability of the reals (Grosholz 2005).

This leaves us with two problems. First, if the reals tag the points on the line, this kind of tagging is very different from what is meant by indexing in the context of set theory: counting terms by the natural numbers or a linear, transfinite extension of them, that is, exhibiting the well-ordering of a set by means of an indexing set whose terms are well-ordered in their 'natural' order. Second, we must characterize the reals as numbers (either in relation to, or independent of, the line), as well as characterize the set of the real numbers as a set. Most of the time, when mathematicians refers to the reals, they call them a set, and mean by that set both a set of points and a set of numbers. Take for example this definition from John Stillwell's *Geometry of Surfaces*: "The Euclidean plane is the set $\mathbf{R}^2 = \{(x,y)$ s.t. x,y is an element of $\mathbf{R}\}$" (Stillwell 1992: 2). Or the same example given to illustrate the definition of the Cartesian product in Singer and Thorpe's *Lecture Notes on Elementary Topology and Geometry*, which introduces the real numbers without any further preliminaries in that book: "Example. Let $A = B$ = the set of real numbers. Then $A \times B$ is the plane" (Singer and Thorpe 1967: 3).

Why do mathematicians tend to refer to the reals as 'the set of the reals'? When we talk about the natural numbers vs. the set of natural numbers, we mean by the former the natural numbers in their 'natural' order, endowed with their intrinsic structure, and by the latter the set of natural numbers considered as elements 'in any order whatsoever,' that is, abstracted from their natural order and intrinsic structure. So it would follow that we mean by the distinction between the reals and the set of the reals, the real numbers in their 'natural' order, the order conferred on them in virtue of their analogy with the line, and the reals considered as elements 'in any order whatsoever,' that is, abstracted from their natural order and intrinsic structure. Recall that to exhibit the set of natural numbers, Cantor exhibited 'artificial' orderings of the natural numbers indexed by the natural numbers, but then had to admit that the 'natural' ordering of the natural numbers was inescapably canonical. To exhibit the set of reals, it seems then that we must exhibit 'artificial' orderings of the reals indexed by… what? We know that they cannot be indexed by the natural numbers; Cantor proved that. This proof allowed him to assert that the infinite set of the natural numbers had a cardinality different from that of the infinite set of the reals: this opened the gate to a new field, the hierarchy of transfinite ordinals.

The looked-for orderings would have to use an indexing that was a linear course of discrete numbers, but that continued out beyond the countably infinite, ordinals that went up to or beyond aleph-one, whatever that turns out to be. (Is the cardinality of the reals aleph-one, the first transfinite cardinal after aleph-null, the cardinality of the natural numbers, or is it a higher transfinite cardinal?) If the Axiom of Choice (AC) is true, then such indexed well-orderings of the reals exist—because all sets are well ordered—though they may not be definable. Another contested axiom for set theory might also help here, the Axiom of Constructibility, which asserts that all sets are constructible. It is usually written as the expression V = L, where V and L denote the von Neumann universe and the constructible universe, respectively. If V = L, such indexed well-orderings of the reals exist, and are definable, in the sense that we can write down an explicit formula in the language of second order arithmetic (arithmetic expressed in terms of second-order predicate logic), which defines the well-ordering. If AC does not hold, then it may be that as the numbers go out into the transfinite, they are not a *linear* course of discrete numbers – the cardinals may branch, forming a tree rather than a telephone pole (Gödel 1940; Hrbacek and Jech 1984: Chaps. 10–12).

Now recall the way in which the rationals and the algebraic numbers are shown to be countable. In their 'natural' order (as they occur along the real number-line) their countable well-ordering cannot be exhibited, because they are nested, or to use the topological term, 'dense in themselves': between any two rationals (or algebraic numbers) there is an infinite suite of further rationals (or algebraic numbers). Thus we can't count them in this order, because we can't find a place to start or a way to define their side-by-sideness. Only by appeal to the way in which the rationals are constituted as ratios of integers, or the algebraic numbers as roots of polynomials, can their well-ordering be exhibited. In a sense, it gives us the assurance that we can get to the bottom of, or exhaust, all the nestings that exist in those number systems. The attempt to exhibit such a well-ordering of the reals by means of the way they are constituted by their decimal expansion fails, as Cantor showed. And there is no other constitution of the reals that we know of which applies to all the reals. The 'nesting' of the reals is much worse than that of the rationals or algebraic numbers, and there are many more of them. If AC or V = L, then there is a linear suite of transfinite numbers that indexes the reals in a manner analogous to the way in which the natural numbers index the rational or algebraic numbers; but we can't precisely characterize that linear suite and we can't know precisely what 'constitution' of the reals it would make use of, that is, how by counting one might eventually exhaust all the nestings. All we can do, by means of those axioms, is to indicate its existence; the axioms simply enunciate our belief, or hope, that the real numbers can be counted, or indexed, in some extended sense of the notion 'to count.'

However, when we tag things by the reals, as mathematicians do all the time, we invoke them in their 'natural' order. This is odd. We index the rational and algebraic numbers by the natural numbers, but not vice versa; and the former cannot be indexed or counted at all in their 'natural' order. Similarly, we index the reals by a transfinite, linear extension of the natural numbers (if AC holds or if V = L), but not vice versa; and they cannot be indexed or counted at all in their 'natural' order (Cohen 1966; Kuratowski and Mostowski 1968: Chaps. 8 and 9). Only the natural number system, or possibly (if AC holds, or if V = L) its transfinite, linear extension, can index because it indexes or counts other things in the 'natural' order of its discrete elements. And yet mathematicians persist in tagging things by **R**, 'the set of real numbers,' on the assumption that the reals in their 'natural' order can tag other things, like the points on the line, families of surfaces, families of solutions to differential equations, and so forth. Why? We don't assume that the algebraic numbers in their 'natural' order can index other things, because we cannot exhibit those numbers in that order; and of course the situation of the reals is even worse.

The answer is obvious. We *can* exhibit the reals in their 'natural' order. That exhibit is just the number-line: there they are. Supposing that we can treat the line as made up of points, and that those points can be tagged by numbers, the number-line exhibits the reals in their 'natural' order. But note that the exhibit is a arithmetical-geometrical entity (a continuum with numbers) and that the tagging is a very strange kind of indexing: it looks like measuring, not counting. Each real number in its 'natural' order tells how far away that point is from the origin, given the fiat of an origin and a unit: so that each real number qua point is really the limit point of a line segment in mufti. The great achievement of Cantor and Dedekind was not the arithmetization of the geometrical line. It was, rather, the use of the notion of set on the one hand to extend the number system into the transfinite by means of the operation of power set, and on the other hand to recast the line as a set of points and number systems as sets of numbers.

The emergence of Set Theory and Group Theory in the nineteenth century also played an important role in the emergence of Topology and then Algebraic Topology in the twentieth century. The axioms defining a topological space were first formulated successfully by Felix Hausdorff. Problems in analysis (such as the search for new, more abstract definitions of function and integral), and the development of Set Theory by Cantor, had led mathematicians like René Fréchet to search for ways of talking about abstract sets and spaces, whose components were not points or real numbers, but simply elements (Manheim 1964: 116–119). It was Hausdorff who first noted the distinction between the traditional understanding of distance and limit, and the notion of neighborhood. The concept of distance is tied to a metric, and that of a limit depends on a relation of countability, so Hausdorff selected the notion of neighborhood as more fundamental. He also "knew how to choose, among the axioms of Hilbert on neighborhoods in the plane, those which were able to give his theory both the desirable precision and generality" (Manheim 1964: 122–123; Bourbaki 1960: 151). Subsequent topological research has ruled that Hausdorff's axiomatization was too narrow, however. The axioms of Topology which we saw before and which I present again here include only the first three but not his fourth, which stipulates that topological spaces should have a certain separation property; it became clear later on that Topology should encompass spaces with a variety of separation properties. So those that satisfy his fourth axiom are now called 'Hausdorff spaces,' and they include all metric spaces and all differentiable manifolds. Algebraic Topology is concerned almost exclusively with Hausdorff spaces.

A topological space is a set of elements; each space is accompanied by another set *S*, of subsets of the elements of that space. *S* satisfies three conditions: (1) It includes both the empty set and the whole space, (2) Finite intersections of subsets in *S* are again in *S*, and (3) Any finite or infinite union of subsets in *S* are again in *S*. These subsets are called open sets, and *S* is called the topology of the space. Hausdorff spaces are spaces where there are enough open sets for any two distinct elements to be contained in two open sets that do not intersect. While we are reviewing axioms, let us remember the definition of a group. A group is a set of elements, along with a single binary operation (an operation that assigns an element of the group to pairs of elements of the group), which satisfies the following conditions. (1) The group is closed under the operation; (2) Each element has an inverse; (3) There is an identity element for the operation; and (4) The operation is associative. Recall too that if the operation is commutative, we call it an Abelian group, and think sadly of Abel and Keats.

Group theory was first developed to investigate the roots of equations and the symmetries of geometrical figures. But the group, as a middle term, went on to play an impressive role in the development of Topology. First, I will give an example that shows how important the generalization of distance by the notion of neighborhood was. The rational numbers can be completed (have all their holes filled up) not only by the reals and then in another sense by the complex numbers, but also by the *p*-adic numbers, which then play a central role in the development of number theory. To understand this, in fact, we need to let go of the hybrid-figure of holes in the line. Second, we will look at the synecdochic formulation of homotopy as an equivalence relation that spawns homotopy groups in Algebraic Topology, and then the formulation of homology groups. Both allow for useful classifications of topological spaces: for example, the homology groups completely determine the homeomorphism class of connected, compact, orientable surfaces. The latter leads to De Rham's Theorem, which (at least in the proof given in Singer and Thorpe) proves that Plato's Timaeus was right: everything is composed of triangles! Well, every smooth manifold that is isomorphic to a suitably chosen simplicial complex is then dubbed a smoothly triangulated manifold. Third, ironically, Topology is involved in the non-Euclidean geometry and the investigation of Riemann surfaces that made General Relativity possible, and so too the three poetic narratives we will encounter in the last chapter.

Topology allowed for the generalization of notions of distance or separation, and closeness, and this proved fruitful in surprising ways. We get from **Q** (the rational numbers) to **R** (the reals) and **C** (the complex numbers) by a definition of closeness that stems from the real line, using the Euclidean norm. But we get from **Q** to the p-adic field $\mathbf{Q}_p$ by a definition of closeness that stems from congruence relations. Within the ring of integers **Z** there is a kind of fine structure precipitated by relations of congruence. There is the reorganization of the integers when they are sorted into subsets by modding out by a natural number n; the subsets form a finite group, and when n is a prime p they form a finite field. From this reorganization we can move to a sorting mod p^2, and then to a sorting mod p^3, and so on. These relations of congruence lead to a novel conception of closeness, which in the twentieth century came to play a key role in number theory via the definition of p-adic number fields: two integers are close to each other when they remain in the same congruence class not only mod p, but also mod p^2, mod p^3, and in general mod p^n. This sense of closeness can be generalized from the integers **Z** to the rationals **Q** by the notion of a p-adic valuation, so that we arrive at $\mathbf{Q}_p$ (Coppel 2009: Ch. 6). Who would ever have thought that the internal structure of the integers **Z** would give rise to an interesting new conception of closeness, and so to a new family of fields (one for every prime p) that complete the rationals, sitting alongside the famous completions of **R** and **C**? But there they are.

In the rest of this chapter, I was originally going to explain all of homotopy theory and homology theory to you, along with De Rham's Theorem. However, first of all the explanations became too technical for this book, so I urge you to buy your own copy of the newest version of *Lecture Notes on Elementary Topology and Geometry* (Singer and Thorpe 1967) if you're interested in the details. Second, my friend Joe Mazur explained to me important connections that I had forgotten about, and if I connected all the dots, or the neighborhoods, I would have to write another book. He noted that although the exposition of the proof in Singer and Thorpe brings us into the cohomology of n-dimensional differential forms, De Rham, himself understood those forms as being generalizations from infinitesimal calculus. You will have noticed that we've spent a great deal of time in this book exploring the history of the infinitesimal calculus: it turns out that De Rham's Theorem is a generalization of the Fundamental Theorem of Calculus (FTC), which brings us full circle. Joe observed, "It's wonderful to show how the magnificent scaffold of abstraction, involving so much machinery such as cohomology and differential forms, collapses to something fundamental, beautiful, and useful. All that is necessary is to say that simplicial (triangulated, if you wish) geometry is mirrored by differential forms, in the same way that integral calculus is mirrored by differential calculus." Not only that, but there is an application to electro-magnetism through the connection between De Rham's results and Maxwell's equations! (Mazur 2017).

The Tallinn Ferry (my photograph)

So, instead, we'll go on a different kind of expedition (perhaps we are taking the Tallinn Ferry?) into the history of mathematics and logic, as I encountered their intertwining and hostility and reconciliation during my years as a philosopher of mathematics, where we will discover in another sense the importance of abstract algebra and come to understand why Gödel's Incompleteness Theorems chime with the insights expounded in William Empson's *Seven Types of Ambiguity*, my favorite work of poetic criticism from the last century (Empson 1930/1966). I will not attempt to give an objective and comprehensive account of the history of the philosophy of mathematics in the late nineteenth century and throughout the twentieth century. Rather, I offer a subjective account of how I discovered it, starting around 1970, and how those discoveries shaped my philosophical project, as I went back and forth between North America and Europe, and in Europe between Germany and England on the one hand, and France and Italy on the other. My early conviction that the growth of mathematical knowledge is key to understanding mathematical philosophically has been borne out, not just in my own work but in the developments that I have witnessed, and engaged with, during the first decades of the twenty-first century. To pose the problem in Leibnizian terms: what are the conditions of intelligibility for mathematics, that allow it to offer logically organized arguments and systems, and necessary truths, as well as unexpected, surprising and unforeseen developments as it grows. The key to understanding mathematics is not just logic, but also the history of mathematics, the record of its growth, as well as the record of its organization, and recurrent re-organization.

As a graduate student at Yale University, I continued sitting in on math courses, and finding out more about my favorite discipline; however, the situation was complicated. I thought the obvious question for a philosopher was how does mathematical knowledge grow? How can mathematical knowledge seem to exhibit necessary truth, and yet be so open-ended, so unpredictable, so exciting? How does mathematical knowledge keep on spilling over its apparent boundaries? The professors in the philosophy department at Yale, however, were really not interested in such questions, or in current or past mathematical research. Frederick Fitch was the author of two textbooks, *Symbolic Logic: An Introduction* (Fitch 1952) and *Elements of Combinatory Logic* (Fitch 1975). I did not take any courses from him, but did take a course from Ruth Marcus on modal logic: her papers were collected later in *Modalities: Philosophical Essays* (Marcus 1995). She was unwaveringly hostile.

Luckily, as I recounted earlier, I was also introduced to the kind and encouraging French philosopher Jules Vuillemin (1920–2001), while I was at Yale. He came to give a lecture, and as I listened to him talk about the importance of the history of logic, the history of mathematics and the history of philosophy, I thought to myself, there is my mentor! I read his books and wrote to him, and he unfailingly answered my questions and offered suggestions for further research and reading over almost a quarter of a century. My first philosophy book, *Cartesian Method and the Problem of Reduction*, on Descartes' mathematics and method, was dedicated to Vuillemin (Grosholz 1991). I also discovered and was inspired by Imré Lakatos (1922–1974) and his book *Proofs and Refutations* (Lakatos 1976/2015). This book, published just after his untimely death, was based on chapters from his dissertation at the University of Cambridge, and developed while he was teaching at the London School of Economics, where he was a colleague of Karl Popper (1902–1964), who liked his discussions of 'conjectures and refutations' in mathematics.

One of his students was Donald Gillies, who continued to work with Popper, and both of them responded to Thomas Kuhn's *The Structure of Scientific Revolutions* (Kuhn 1962). Significantly, Donald and Imré Lakatos argued in various way that mathematics as well as science have revolutions: see *Revolutions in Mathematics* (Gillies 1992). I remember having dinner in the early 1990s with Donald and his wife Grazia Ietto Gillies, when Donald was suddenly abstracted from dinner for a long time, because Popper had called him up with philosophical questions. My next-to-last philosophy book, *Representation and Productive Ambiguity in Mathematics and the Sciences* (Grosholz 2007) is dedicated to both of them, along with Carlo Cellucci and his wife Mirella Capozzi; Grazia and Mirella are impressive scholars in the adjacent fields of economics and history of philosophy. Carlo's arguments about analysis in the development of mathematics play a pivotal role in that book, and the book that followed (Cellucci 2013). The other lucky development was discovering Angus Macintyre, a Scottish model theorist in the Mathematics Department at Yale, who wrote his dissertation with Dana Scott at UC Berkeley. He and his students introduced me to a different way of thinking about the interaction of logic and the rest of mathematics. And, finally, I took a class on the work of Ludwig Wittgenstein (1889–1951), and explored his *Remarks on the Foundations of Mathematics* (1967) and then *Wittgenstein's Lectures on the Foundations of Mathematics* (1976), both published posthumously. Here is a poem I wrote, inspired by Wittgenstein, while attending a conference in Rome.

Uncertain

"Someone who, dreaming, says "I am dreaming," even if he speaks audibly in doing so, is no more correct than if he said in his dream, "It is raining," while in fact it was raining. Even if his dream were actually connected with the sound of the rain." (The final lines of Ludwig Wittgenstein's *On Certainty*.)

Four o'clock: still night and silence.
I hear one local bird practice a trill,
Then silence falls again.
Why did it sing?

Are birds like us misled
By false dawn? By a dream
Of moonrise? Ever roused, eyestruck
By rays of moonlight?

We cannot say its song
Was caused by the moon rising,
Even if it dreamed
The moon, as the moon rose.

In any case, darkness prevails.
A mystery, unless I too
Dreamed the bird, its trill, the pale
Ruse of moonlight.

The Tiber, in Rome (my photograph)

These early conversations, with people and books, shaped my first critiques of logicism, as formulated by Bertrand Russell (1872–1970) and Alfred North Whitehead (1861–1947) in *Principia Mathematica* (Russell and Whitehead 1910–1913). Bertrand Russell stated early on that his goal was to arrive at "a perfected mathematics which leaves no room for doubts," as he puts it in *My Philosophical Development* (Russell 1959). He found the axioms of geometry and arithmetic unsatisfactory, and based on very different kinds of intuitions. Once having discovered the work of Gottlob Frege (1848–1925), he arrived at the project of trying to derive all of mathematics from self-evident logical principles, using infallible deductive principles of inference. For example, he showed that the combinatorial facts of arithmetic, like 7 + 5 = 12, can be translated into sentences in Russell's logical notation, and shown to be theorems in that system, furnished with a proof from the axioms of that system. Russell claimed that the translation and proof justify the arithmetical rule 7 + 5 = 12 and secure it from skeptical attack, for the axioms of logic are so clearly true that they are immune from skeptical questioning, and the rules of inference are guaranteed to transmit that immunity. More generally, the correlation of the whole system of rules of arithmetic with the corresponding logical theorems serves to justify the whole of arithmetic and preserve it from the skeptic; so too, Russell assumed, other branches of mathematics like geometry and analysis could be shown to be beyond question.

What worried Ludwig Wittgenstein about Russell's project was his reductionist foundationalism, his attempted homogenization of mathematics to a single system of logic, as well as his Platonism. (Wittgenstein wanted to study with Frege, but Frege referred him to Russell, so he arrived at Cambridge in 1911, where he later taught from 1929 to 1947.) The foundationalist narrowly construes the justification of mathematical knowledge as deduction from true (logical) axioms, so that once an allegedly adequate set of axioms is given, all the truth of mathematics will be given along with it. But Wittgenstein insisted that mathematical practice involves a motley, a variety of strategies of justification (not all of which involve established relations of deducibility), and of kinds of things investigated, and of notations. Over and over, we see mathematicians extending mathematical systems in ways that, while natural and productive, cannot be proven from what was accepted before. Such strategies often make salient the heterogeneity of mathematical domains, their partial autonomy and complex relations of interdependence. As I learned from the model theorists, as well as, oddly, from Ruth Marcus, first order predicate logic is only one variant form in an historical progression (shaped more like a tree than a telephone pole) of logics. It is without doubt an expressive and well-understood system, with desirable formal properties expressed by the Compactness Theorem, the Completeness Theorem, the upwards and downwards Lowenheim-Skolem Theorem, and a kind of maximality expressed by Lindstrom's Theorem. All the same, intuitionistic logic, second and higher order predicate logics, modal logics and logics with generalized quantifiers (like topological logics), are useful and respectable members of a community of logics. Moreover, first order logic is not used directly to solve problems arising in, for example, number theory or algebraic geometry, though reformulating a specific part of a certain problem in terms of first order or second order logic may sometimes be helpful in order to determine the logical complexity of certain sets or procedures that figure in those problems.

I would also add that Russell's logical principles turned out to include the Axioms of Reducibility, Infinity and Choice, and to require concepts of set and membership-relation, as well as ramified type theory. All of these principles and concepts have generated philosophical and mathematical debate, as Lakatos liked to point out, and so were no more proof against skeptical attack than geometry or arithmetic. Ruth Marcus used to claim that her protégé Saul Kripke had solved the problems raised by the modalities through his formalization of modal logic. But Vuillemin's book *Necessité ou contingence: L'aporie de Diodore et les systemes philosophiques* (Vuillemin 1984), by rehearsing the debates about, and interpretations of, possibility, necessity and actuality from classical antiquity to the early modern period, reveals an irreduciable variety of kinds of 'fundamental' modal assertations, and an irreducible dialectic that has continued for two millennia. Ironically, Vuillemin finds in the pure intolerance of one system for another, the liveliness and promise of philosophy: the dialectic will always lead to further refinement and insight.

One of Wittgenstein's favorite ways to challenge Russell's account of mathematical justification, and to focus on the issue of invention, was to raise the topic of correlation. Given an initial correlation of two mathematical systems, what justifies the decision to extend it in a certain way, or to modify it? His insight is that in certain situations, a correlation itself becomes a novel element, which cannot be deduced from either of the initial correlates taken in isolation. I found this insight especially inspiring, but was then disappointed by his inability to elaborate: Wittgenstein made little use of the history of mathematics. Just at the end of my life as a graduate student, and the beginning of my life as a professor at the Pennsylvania State University, I spent the spring of 1981 in Paris. There I attended a study group organized by François De Gandt and a couple of his colleagues to work through Book I of Newton's *Principia* proof by proof, line by line. It shaped my understanding of that book, and inspired me to work my way through Descartes' *Geometry* in the same manner, guided by the essays and books of Henk Bos (see De Gandt 1995; Bos 2001). That work allowed me to bring some of Wittgenstein's insights down to earth, working on Descartes and then Leibniz and Newton. As Jean Cavaillès argued so persuasively, the striking feature of mathematical domains is their tendency to grow beyond previously established boundaries. Solutions to given problems will often generate new problems that require ever more complex and profound methods for their solution. This open-endedness is obscured by the artificial, though useful, closure of axiomatization. (Dirk Schlimm does an excellent job of explaining its usefulness in his essay "Axioms in Mathematical Practice" in *Philosophia Mathematica* 12/1 (2013).) So, let us look at some issues of ontology and epistemology in the light of these philosophical conversations.

Donald Gillies, in his various discussions of mathematical ontology, reminds us that Platonism asserts that mathematical entities exist objectively, but in a transcendent reality, while Aristotelianism asserts that the objective existence of mathematical entities occur, embodied, in this world of space and time. His preferred version of Aristotelianism begins with the observation that the natural world consists not just of physical things, but things standing in relation to each other: these relations are abstract, but they are real. Moreover, some sets associated unambiguously with numbers are physically embodied, like the moons of Mars (there are 2). He then modifies his Aristotelian position with a kind of constructivism. Rejecting the intuitionism of L. E. J. Brouwer (1881–1966), which holds that mathematical entities are the subjective mental construction of an individual mathematician, who carries out languageless mental constructions, he walks through Popper's Third World (a constructivist Platonism) in which mathematics is a social construction, the result of the efforts of a human community. Aided by Wittgenstein's account of meaning in terms of language games, he reminds us that signs play an essential role in our social activities; aided by Ladislav Kvasz (and Danielle Macbeth), he reminds us that developments in mathematics often involve the creation of new symbolic languages, powerful and highly specialized (see Kvasz et al. 2002; Kvasz 2008; Macbeth 2014). Finally, invoking Frege's distinction between sense and reference, he reminds us that mathematical entities refer: they are not just human constructions, but are aspects of the non-human world of nature. Thus he arrives at a generalized, constructivist Aristotelianism: mathematical meanings, though abstract, exist as concretely embodied in social practices, and often refer to situations in nature (Gillies 2015).

While writing 3.5 books about the philosophy of mathematics over the past 30 years, I have generally stayed away from issues of ontology, but Donald Gillies' account seems to me the most plausible, though as he notes, it has trouble accounting for transfinite cardinals and the sets associated with them. However, we find in nature not only countable sets, but shapes: for example, nature is filled with catenaries, because they express, in a sense, situations of minimal energy. A catenary is a transcendental curve, and so I would argue that it gets us at least from aleph null to aleph one; of course, there is the ongoing debate about which aleph counts the reals, so it might carry us even further up the great telephone pole (or is it a branching tree?) of the transfinite numbers. In any case, I like this account because it is compatible both with the insight that mathematics is objective, and that mathematics is open-ended, sometimes corrigible, and historical, for it develops over time. Also, it encourages us to remember that there is no 'total plan,' no all-encompassing idiom, no master axiomatization. Logic too has a history and exhibits its own heterogeneity, responsive to the heterogeneity of mathematics, and set theory has its own indeterminacies, and is challenged in various ways by category theory and topos theory.

Is it paradoxical to approach the philosophy of mathematics through the history of mathematics? Jean Cavaillès studied with Léon Brunschvicg (1869–1944) at the École Normale Supérieure, whose philosophical methods were avowedly historical and consciously opposed to the project of Bertrand Russell and his French confrere, Louis Couturat (1868–1914). In 1927, Cavaillès went to Berlin, to investigate the history of set theory: he studied the works of Felix Klein (1849–1925), Bernhard Riemann (1826–1866), and Richard Dedekind (1831–1916). He learned from them that the history of mathematics is a record of unforeseen novelty, using new methods furnished by group theory, complex analysis, topology and set theory. Later, with his friend Emmy Noether (1882–1935), he edited the Dedekind-Cantor correspondence (Cavaillès and Noether 1937/1962). Writing about the debates that went on between Dedekind and Cantor, as well as David Hilbert (1862–1943), Henri Poincaré (1854–1912) and Brouwer, Cavaillès traced the reasons for and against actual infinity, the law of excluded middle, and the admissibility of impredicative definitions, debates that were both philosophically cogent and mathematically fertile.

The study of set theory was especially rewarding for Cavaillès, because in the late nineteenth and early twentieth centuries, abstract structures played such an important role in the growth of mathematical knowledge. (Recall that Emmy Noether's greatest accomplishment was the development of ring theory.) As Hourya Sinaceur explains, "Structural mathematics revealed links between distinct disciplines: algebra, geometry, analysis, number theory; it multiplied the transfer of results and methods, translated one theory into another, gave rise to new theories at the intersections of older theories: analytic number theory, algebraic number theory, algebraic topology, et cetera" (Sinaceur 2013: 145). There is an irreducible dichotomy between arithmetic and geometry, and the abstract structures of sets, groups, rings and fields do not accomplish any definitive reduction of one to the other, but rather generate unexpected discoveries precisely at the intersection, the crossroads, where they meet. One could say the same thing about the role of polynomials and polynomial equations in the seventeenth century, and the notation for functions and for differential equations in the eighteenth century. A number does not have a diameter and a figure is not odd or even; there is no such thing as a prime set, and you cannot factor a circle. However, you can correlate numbers and sets, and sets of numbers and figures; those correlations introduce something new into mathematics.

In his book *La Philosophy de l'algèbre* (1962), Vuillemin adds another important insight. Theoretical philosophy, when it is dogmatic, is interested in the order inherent in things, in objective validity, not in the accidents of invention. However, he argues, the revision and development of mathematical methods have direct consequences for theoretical philosophy; they are so closely allied, that changes in one impinge on the other. From Plato to Descartes and Kant, mathematics is used as a model to criticize, reform and define theoretical philosophy; so, he argues, as mathematics changes, it imposes itself in a changing way on philosophy. His book traces the impact of Descartes' analytic geometry, Leibniz's infinitesimal calculus, Lagrange's method of resolvants, Gauss' *Disquisitiones arithmeticae*, and Galois' formulation of the notion of group, followed by that of Klein and Lie, on the development of philosophers from Kant to Frege and Husserl. History intervenes after all in theoretical philosophy because mathematics has a history. Vuillemin, like his professor Cavaillès, was fascinated by how a detour into the apparently contingent and arbitrary could in the end reveal objective structure.

So it seems that Russell's search for a re-writing of mathematics that would offer a 'master axiomatization' in one formal idiom, secure against questioning, was an illusion. Yet formal logic proved to be a discipline that rewarded study, and was also useful to other areas of mathematical research, in part because the notation of logic is designed to exhibit logical complexity, or, computability. First order logic, as it turned out, is not decidable, though a fragment of it is, monadic logic with identity. The representation of other mathematical areas of research as first order theories turned out to be a way of identifying undecidable as well as decidable portions of mathematics. Some interesting results included Post's proof of the validity of the decision procedure for propositional calculus, Langford's decision method for an elementary theory of linear order, and Tarski's decision method for the elementary theory of Boolean algebra and for the elementary algebra of real numbers. They also included the negative results of Gödel, Church and Rosser, that there is no decision method for any theory including all the sentences of elementary number theory, nor for the theory of rings (Mostowski and Tarski), groups and lattices (Tarski), and fields (J. Robinson) (Boolos and Jeffrey 1974).

Thus, results concerning decision procedures can play a useful role in the development of current research, either by insuring the existence of algorithms and so motivating a search for them, or by providing an early warning system where none exists. For example, the theory of compact, 2-dimensional manifolds is decidable; we can give an exhaustive catalogue of their homeomorphism types. However, there is no such procedure for four-dimensional, compact manifolds, so the best we can do in that case is select a homotopy equivalence class (which restricts the situation to manageable proportions) and try to catalogue the homeomorphism types within it. This restriction then stems not from a lack of ambition, but is rather a rational restraint, since the decidability results show that we can do no better. A striking feature of mathematical logics is that they exhibit their formal properties, like completeness, consistency and decidability (or lack of the same) in an especially perspicuous way. Consequently, the correlation of other mathematical areas of research with mathematical logic yields information about the formal properties of elements, methods and problems, and indeed may provide a measure of that complexity, yielding guidelines for research. However, the correlated mathematical domains still retain their own distinct character, and methods, and notation. Logic does not replace the discourse of, say, number theory, but is sometimes juxtaposed with or superimposed upon it.

From the point of view of productive research, the notion of ontological parsimony we owe to W. V. O. Quine (1908–2000) is not helpful. In sum, the attempt of logicism to provide a foundation for mathematics did not succeed. A logic strong enough to represent other important mathematical domains has no privileged claim to incorrigible truth, decidability or consistency, nor does this representation reduce the ontology of mathematics. Rather, logic performs the service of exhibiting clearly its own formal properties, and by correlation those of other areas of research.

In my most recent book, *Starry Reckoning: Reference and Analysis in Mathematics and Cosmology* (Grosholz 2016), I argue that important ampliative reasoning takes place in mathematics when heterogeneous discourses are brought into novel, and rational, relation. This often happens when discourse whose main intent is to establish and clarify reference is yoked with discourse whose main purpose is analysis in the sense of Leibniz, the exhibition of conditions of intelligibility of items and of solvability of problems. As a prelude to discussing Wile's proof of

Fermat's Last Theorem as an example, I discuss Gödel's incompleteness theorems, to which I now turn, with a consideration of some pages from a logic textbook I often consult. In his well-known book *A Mathematical Introduction to Logic* (Enderton 1972), Herbert B. Enderton introduces the study of models in the following way. He states that a structure for a first-order language will tell us what collection of things the universal quantifier symbol refers to, and what the other parameters (the predicate and function symbols) denote. Then he formally defines a structure U for a given first-order language as a function whose domain is the set of parameters, such that,

1. U assigns to the universal quantifier symbol a non-empty set $|U|$, called the universe of U.
2. U assigns to each *n*-place predicate symbol P an *n*-ary relation P^U which is a subset of the set of all *n*-tuples of members of $|U|$, $|U|^n$.
3. U assigns to each constant symbol c a member c^U of the universe $|U|$.
4. U assigns to each *n*-place function symbol f an *n*-ary operation f^U on $|U|$, so that

$$f^U\colon |U|^n \to |U|.$$

Note that the structure U is a function; in so far as it is a function, it is just as 'discursive' or 'logical' as the so-far-uninterpreted first order language that serves as its domain. Considered simply as a function which performs a service for predicate logic, the structure U is itself presumably also uninterpreted, even if its service is to provide an interpretation. That is, the structure U considered simply as a function has no ontological import, no more ontological import than the uninterpreted first order language that serves as its domain.

If we then think of the structure U as 'purely' discursive and the set $|U|$ as somehow outside of or beyond discourse, we must wonder how they have been brought into relation. One answer might be that by treating whatever it is that lies outside or beyond discourse as a set, we have assimilated it to a discourse (the discourse of set theory). But then we have only established a mapping between two discourses: does this mapping really count as successful reference and denotation? Does it not seem odd that the discourse of first order predicate logic and the discourse of set theory should resemble each other so closely? Perhaps there is an historical explanation for that notable resemblance.

Another answer might be that we find here an example of two different functions of language, the function of indicating what we are talking about and the function of analyzing it. Then the structure U considered as a function has the job of analysis, and the set $|U|$ has the job of indicating, or perhaps exhibiting, what we are talking about. But what is the relation between the structure U (which is a function) and the non-empty set $|U|$, which is presumably more like a referent or object, since it provides the interpretation? This is a very difficult question to answer; in a sense, the philosophically vexed question of how applied mathematics is possible at all is implicated in this question, and so is our choice of how to reply to it. What is Enderton's response? After a few more pages of exposition, Enderton gives an example of what it means for a model to satisfy a set of sentences.

Example: Assume that our language has the parameters ‘universal quantifier,’ P (a two-place predicate symbol), f (a one-place function symbol), and c (a constant symbol). Let U be the structure for this language defined as follows:

$|U|$ = N, the set of all natural numbers;
P^U = the set of pairs of natural numbers such that m is less than or equal to *n*;
f^U = the successor function S; and $c^U = 0$.

Then Enderton adds, off-handedly, *“We can summarize this in one line, by suppressing the fact that U is really a function and merely listing its components: U = (N, ≤, S, 0).”* And thereafter, the fact that U is ‘really’ a function is always suppressed and Enderton writes as if the structure U can be thought of as the referent, the object of discourse, in this case the natural numbers. Thus as the beginning of Chapter 3, on Undecidability, Enderton introduces the structure U = (N, 0, S, <, +, •, E) as the “intended structure” for the first-order language of number theory with equality and the usual parameters. Clearly here (in an exposition of the undecidability of certain logical theories) the structure U is meant to stand for the referent or object that supplies the interpretation and stands somehow beyond or outside of the discourse of first order predicate logic.

Shall we accuse Enderton of intellectual dishonesty? In fact, what he has done here is to employ a common strategy of mathematicians and scientists who must bring an analytic discourse into rational relation with that to which it refers. Of course, there are no such things as bare facts or raw data! What is referred to is never encountered wholly outside of discourse: we articulate our awareness of things in one way or another, according to one or another mode of representation, and thereby develop modes of representation that lend themselves well to indicating what we’re talking about. Meanwhile we develop other modes of representation that lend themselves better to certain kinds of analysis of the things under investigation. Some terms occur in more than one of the disparate discourses that develop, and so may serve as bridges between them, though usually because the mathematicians or scientists learn to live with their ambiguous meaning: they typically mean one thing in one discourse and something slightly different in another. (Thus H_2O means a certain molecule in the middle of a chemical article, and a purified substance in a beaker in the account of the experiment at the end of the article.) Enderton needs the student to believe in Chap. 2 that the structure U is part of the formal logical apparatus, and in Chap. 3 that the structure U is the object of knowledge. His sleight-of-hand is so brief and casual that for most students it goes unnoticed.

The problem with Enderton’s exposition is that he has left out the discourses that mathematicians historically employ for referring to numbers. In one sense it is the arithmetic that children learn, expressed in the idiom of the Arabic numerals enhanced by Descartes and Fermat, which inter alia allows us to express the prime decomposition of a large number in the following perspicuous way:

$$243{,}000{,}000 = 2^6 \times 3^5 \times 5^6$$

In another sense, it is the multiply-idiomatic discourse to be found in an article published in a journal devoted to number theory, where mathematicians present and defend their latest results. Recall that for Enderton, the 'intended structure' U = (N, 0, S, <, +, •, E) has already been assimilated to a logician's discourse in which the numbers are represented as the initial element 0 and successive iterations of the function successor, S. This mode of representation was developed by logicians to subject arithmetic to logical analysis, but it is never employed in articles in journals devoted to number theory. When logicians need to use the natural numbers as indexes, or to invoke number theoretical facts like the prime decomposition of a large number, they make use of Arabic/Cartesian notation without really mentioning their own departure from the formalism they are supposed to be using. This does not mean that there is a single preferred idiom for referring or picking out the things we are interested in "correctly." Successful referring often depends on the context of use, and changes in the historical context of problem-solving may lead us to change the representations we use not only for analyzing but also for referring. However, the notation 0, S0, SS0, SSS0, … however useful for logical analysis of the natural numbers, is not useful for successful reference in most problem-solving situations.

In my 2007 book, *Representation and Productive Ambiguity in Mathematics and the Sciences*, I argue that ambiguous terms and formulations in mathematics, when handled well, can contribute to the growth of knowledge. Gödel's two Incompleteness Theorems seem to me a good example of this claim, which I explore in my 2016 book at greater length. An accurate and well-received exposition of those theorems is given by Ernest Nagel and James R. Newman in their book *Gödel's Proof* ((Nagel and Newman 1958/1964), dedicated to Bertrand Russell. The proof strategy is fairly well known. Gödel begins with a theory (a set of axioms and their deductive consequences) in the language of first order predicate logic with parameters like those offered by Enderton. Then he assigns a natural number, now called its Gödel number, to every well-formed formula (wff) in such a way that if we are given the number, its prime decomposition will allow us to recover the wff, and then perform a similar feat for any sequence of wffs. Then he devises a numerical function 'Dem.' Here is Nagel and Newman's exposition of Dem. "Let us fix attention on the meta-mathematical statement: "The sequence of formulas with Gödel number *x* is a proof of the formula with Gödel number *z*." This statement is represented (mirrored) by a definite formula in the arithmetical calculus which expresses a purely arithmetical relation between *x* and *z*… We write this relation between *x* and *z* as the formula 'Dem (*x*, *z*)' to remind ourselves of the meta-mathematical statement to which it corresponds (i.e. of the meta-mathematical statement 'the sequence of formulas [wffs] with Gödel number *x* is a proof (or a demonstration) of the formula with the Gödel number *z*)" (Nagel and Newman 1958/1964: 78–79).

This relation Dem (*x*, *y*) is used to construct the celebrated Gödel Sentence G via a carefully constructed, self-referential designation '(sub(m,13,m))' which picks out "the Gödel number of the formula that is obtained from the formula with Gödel number m, by substituting for the variable with Gödel number 13 [y] the numeral for m," which in turn can be shown to be a definite number that is a certain arithmetical function of the numbers m and 13, and the function itself can be expressed within the formalized system. The Gödel Sentence G is then constructed by beginning with the following formula:

$$(x) \sim \mathrm{Dem}\big(x,\ \mathrm{sub}(\mathrm{y},13,\mathrm{y})\big),$$

which meta-mathematically claims that “the formula with Gödel number sub(y,13,y) is not demonstrable.” It has a Gödel number, which we will call n. Then the Gödel Sentence G is:

$$(x) \sim \mathrm{Dem}\big(x,\ \mathrm{sub}(\mathrm{n},13,\mathrm{n})\big),$$

and the meta-mathematical meaning of G is “The formula with Gödel number sub(n,13,n) is not demonstrable.” Gödel has cleverly set up the ‘diagonalization,’ so that in fact the Gödel number of G is sub(n,13,n). Gödel then uses this special formula to show that G is demonstrable if and only if ~G is demonstrable and thus that if the axioms of this formalized system of arithmetic are consistent, then G is formally undecidable. And G can be shown to be true by meta-mathematical reasoning: it formulates a complex numerical property that must hold of all numbers. Thus the claim that ‘if arithmetic is consistent, it is incomplete,’ is represented by a demonstrable formula within formalized arithmetic.

The use of Gödel numbering as the strategic bridge between formalized arithmetic qua logical system, and arithmetic, forces an ambiguity on Gödel numbers reminiscent of the ambiguity we noted in Enderton’s treatment of the structure U. On the second page of Chap. VII, where Gödel numbering is introduced, the ontological status of ‘arithmetic’ is not wholly clear. Nagel and Newman characterize it on the first page of the chapter as discursive and therefore as composed of number-types, that is, numerals. “Gödel described a formalized calculus within which all the customary arithmetical notations can be expressed and familiar arithmetic notations established.” Indeed, in a long note that runs from page 82 to page 84, Nagel and Newman assert, “We cannot literally substitute a number for a sign, because a number is… not something we can put on paper.” However, on the second page of that same chapter, the two authors define the logical constant **0** as “the numeral for the number 0,” and by implication then **S0** as “the numeral for the number 1,” **SS0** as “the numeral for the number 2,” and so forth, as if the referents were the objects themselves, the numbers.

This ambiguity forces an ambiguity on the Gödel numbers. What is a Gödel number? If we think of the Gödel number as a plank in the bridge between the formalized, uninterpreted calculus and arithmetic, we must think of it as a numeral in Arabic/Cartesian notation, because only in that notation can the wffs be retrieved from their Gödel numbers. So, for example, the formula ‘0 = 0’ in the uninterpreted calculus is represented by $2^6 \times 3^5 \times 5^6$ which is equal to 243,000,000; we must use the prime decomposition in Arabic/Cartesian notation to rewrite 243,000,000 in order to find the powers of 2, 3 and 5 (the first three primes)—6, 5, and 6—which code for 0, =, and 0. But we must obviously also think of a Gödel number as a plank in the internal bridging created by the strategy of Gödel numbering. A Gödel number must also be a numeral in the uninterpreted calculus, with the form SSSS… SSS0, if we are to believe that the formalized calculus of arithmetic can describe its own formal properties as a system. In order for the proof to go through, Gödel numbers must be variously considered as both; the proof cannot renounce its two distinct modes of representation. Their combination via the strategy of Gödel numbering results in a demonstration that exploits and requires a carefully controlled ambiguity. This ambiguity cannot be dispelled; it cannot be dismissed as heuristics but is central to the demonstration; and it goes largely unremarked by Nagel and Newman, whose exposition betrays it nonetheless here and there.

Logical disparity does not entail incoherence. The neglect of that disparity makes twentieth century philosophical reflection about what renders a proposition (or a theory or a combination of theories) *true*, appear in hindsight too simple and straightforward. The debate about theory reduction is one version of this critique, but there should also be debate about the internal relations of parts of a theory, and indeed the internal relations of parts of sentences, so that philosophical assumptions can be revisited and subject to critique. The epistemological ideal, put forward by Bertrand Russell and others, to unify mathematics as one formalized theory, is born of logic's demand for a homogeneous idiom as a vehicle for deductive inference. If the form of reasoning alone must transmit truth, then all the terms must be stable in meaning and must be "alike" in some important way. And this must hold true not just of the elements of the syntax, but also of the semantic field. Thus philosophers like Russell tried very hard to homogenize the semantic field by reconceptualizing numbers as logical formulae, and geometrical objects as sets of *n*-tuples of numbers.

Logic gets itself into trouble because of its own pretentions: it wants to replace the mathematical areas of research it formalizes. So, since logical languages must be homogeneous, it must offer substitutes for (among other items) the natural numbers. This leads to the program of Russell, and in a different sense to that of Hilbert, thence to Gödel's incompleteness theorems. We have seen that Enderton's textbook tries to finesse the disparity between logic, set theory, and number theory, as does Nagel and Newman's exposition of Gödel's proof. Ironically, Gödel's proof offers a clear example of the difficulty of bringing modes of representation useful for referring, and modes of representation useful for analysis, into rational relation. His solution to the problem is sometimes to let Gödel numbers stand for numerals in the logical calculus, and sometimes to let them stand for Arabic/Cartesian numerals, depending on their role in the proof. But then he is exploiting a carefully controlled and fruitful ambiguity, which multiplies the information avaiblable to the mathematician; it is hard for a logician to admit that he is trafficking in heterogeneity.

As a professor at the University of Strasbourg in 1938, Cavaillès described the experience of doing mathematical research as an adventure. He invoked Rimbaud: adventure, with its revelations, its unexpected astonishments, its dimensions of the imaginary, its projections into the future, its impossibilities and blind alleys. Yet the adventure was objective, not merely subjective: Spinoza was just as pertinent to him as was Rimbaud. Spinoza's necessity chastens the system of heroic mathematical acts, but it is an internal conditioning, an autoconditioning, incarnated concretely in an orderly accumulation of results, methods, idioms and problems. Yet it is not closed, as an axiomatic system is closed: problems always open out upon other problems (Sinceur 1994: 23–25). The application of a new method, like the course of a river, leads irresistibly to the overflowing of the flood plain of applications. The following poems is really a set of variations on Luna B. Leopold's "Rivers," *American Scientist* 50, 1962, pp. 511–37, and the passages in italics are my translations of lines from Pindar's 8th Pythian Ode. Here is my favorite of those lines: ἐπάμεροι: τί δέ τις; τί δ' οὔ τις; σκιᾶς ὄναρ ἄνθρωπος

Rivers

Rain, sleet, snow, and hail's grand excess
Over evaporation and the fine,
Dense, constant transpiration of all flora,
Provides the flow of rivers.
Earth is watered by the inequation:
Congo, Mississippi, Amazon,
Yangtze, Nile, Parana,
Ob, Amur, Yanisei, and Lena.

Volumes could be written on the way
Renowned and cryptic rivers flow:
Their seasonal regimen,
Occurrence and diastole of flood waves,
Chemistry of river water, form of
River systems: snake and tree. And yet,
A poem needs restraint.
Take up the means at hand with a good will.

A river's organized in delicate balance,
Self-formed, self-maintained, between
Forces of erosion and resistance.
Curving the groove it runs along, it fashions
Depth, and areal figure,
Longitudinal profile, and cross-section.
Equations show its equilibrium
Studded with liquid, looped parameters:

Sheer (internal force tangential to
Ideal cross sections); bedload (particles
From bed and bank transported by the river);
Dissolved load (bedload made invisible).
Headwater pours across boulders and cobbles;
Downstream material is
Smoother, silt or silky clay. But scour
And fill tend on the average to balance.

A river overflows its banks in flood.
Everywhere, in rivers of all sizes,
Bankfull stages happen once a year, or
Once in two. The floodplain
Has to drown biennially, and proves
Inherent to a river.
What are we? Indeed, what are we not?
Ephemeral, the light dream of a shadow.

Despite the broad necessity of floods,
Most fluvial work on landscape forms
Stems from intermediate events:
A modest count of days of
Intermediate flow or scour. Like love;
Like ordinary science with its careful,
Incomplete descriptions. *Let us be*
Small in small things, great in greater things.

Nearly every natural channel snakes.
(Indeed, a river's rarely a straight line
Longer than ten channel widths.)
Meandering or merely sinuous,
Curves a channel carves
Remain in constant ratio to its girth:
Small channels wind in smaller, great in greater
Curves. The noblest element is water.

Sinuosity's root cause is just
How water flows: hydro-
Dynamics. Independent of its load,
Any river slowly, surely migrates
Laterally across the valley floor.
The laws of water run
Beyond the little rules that order stone,
Farmers, or the fan of delta soil.

Seek not, my soul, the life of the immortals.
Even among the very smallest rills and
Broadest river basins,
Logarithmic proportions hold
Between stream order and the length of streams
Of given lesser order, and between
Stream order and the multitude of streamlets.
Rivers most resemble trees.

Not just as schema, but as organism:
Parts arranged dynamically in

Causal, mutual self-regulation.
Given possible discharge and prevailing
Channel characters, a graded stream
Is delicately adjusted to provide
The one precise velocity required for
Transport of the load.

Thus rivers freely flow
According to the principle of least work,
That, like the *Odes* of Pindar, gently governs
Spirit wound in matter's labyrinth.
So the river-snake's a tree,
Tree a form of systematic thought,
Thought, like us, an asymmetrical,
Branched mirror of God.

Farhad Ostovani, 1998. Woodcut printed on paper

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Chapter 13
Literary Cosmology: Plato, Tobin, Major, Turner

As we have seen throughout this book, one way that mathematics enters directly into poetry is through cosmology, which tracks the origin, creation and development, and sometime the future or fate of the whole world. The heavens seem to be a dome, half a sphere, sometimes blue brushed with clouds or the rose of dawn or the gold of sunset, and sometimes black spangled with stars. Across the sky, the sun and moon and planets (and the odd comet) trace the great circle of the ecliptic, which then collects and orders the constellations of the Zodiac. By the sea or on the sea, the horizon, the visible boundary of earth, exhibits another great circle; and the sun and moon are spheres. Divine geometry! Days and months and years depend on the circuits of heavenly bodies, and the cycles are expressed as seasons: birth, maturity, withering, rebirth. Rivers seem to be great circles on earth, replenishing and sometimes threatening; mountains (and birds) seem to be middle terms between ourselves and the heavens, and caves (and snakes) middle terms between the living and the dead. A cave with a river running through it is a double portent, like a sky with a rainbow. Forests are habitations and labyrinths, where we may dwell or lose ourselves. The aim of cosmology is to provide explanations and reveal the meaningfulness of the world: mathematical order, periodicity, symmetry, number and figure, generally play an important role in these accounts.

So in this chapter I'll touch on the sources for three contemporary cosmological poems. We will start with the Greeks, and Plato's *Timaeus*. Plato was fascinated by the constructive arguments of mathematicians in both arithmetic and geometry, so he sought an account of the world that had an analogous clarity and order, that proceeded from simple elements and assumptions to the construction of vast complexities. However, as we have seen, his ontology is schematized by the Divided Line, which shows he was hostile to the kind of monism or reductionism we find in the Atomism of Democritus; for Plato, reality is layered or striated, because different things exist in different ways. Material reality has a place in the strata, though it is lower down, so of course any kind of materialism could not be a complete speech about reality.

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Like the Atomism we find in Lucretius' rendering of Atomist doctrines in *De rerum natura*, materialist metaphysics typically asserts that we are confused about ourselves: we take ourselves to be something we are not. (So too we are confused about the world.) Having just asserted that there exist only particles of matter (atoms) and the void, Lucretius argues that Homer got it wrong. We must learn to re-describe what we have erroneously taken to be human events, as the motion and compounding of atoms in the void. His poem is designed to liberate us from the baseless fears caused by the illusions of history, religion and (ironically) poetry. Likewise, Paul Churchland in his book *Matter and Consciousness* suggests that neuroscience may provide the answers to our doubts and fears, explaining the enduring mysteries of sleep, learning, intelligence, memory and despair (Churchland 1988).

Plato objects that materialism must stop at the second level of the Divided Line; but then just as it cannot rise above the second level of the Divided Line ontologically, so its epistemology can never rise above the level of common sense, or worse, of rumor; that is, it can never aspire to science or philosophy. By contrast, the complete though never conclusive speeches of Plato's dialogues often advance our understanding by moving up and through all four levels of the line, which was the measure of completeness (Sinaiko 1965). The account of the construction of the cosmos in the *Timaeus* can thus be read in terms of the Divided Line. The section from 20e to 47e belongs to the first level of the line, the level of rumor or myth, stories that Solon brought back from Egypt, the story of Atlantis, bits of Pythagorean lore, the creation of the world by a Demiurge. The section from 47e to 53c-d belongs to the level of common sense and material objects: it treats the four elements and the Receptacle. So we should pay close attention to the transition that takes place at 53c-d, for it is the threshold where Plato parts company with materialism and begins the work of understanding the world in terms of the ideal, and Becoming in terms of Being. The section from 53c-d to 69b belongs to the third level of the Divided Line, where mathematics and mathematical modeling can be carried out. From 69b to 90e, Plato engages in the synoptic, detailed philosophizing, combining both the abstract organization of mathematics and the rich concreteness of experience that is the high point of so many of his dialogues, and exhibits what he means by knowledge at the fourth level of the divided line. The complexity of experience must be acknowledged, but must also be made coherent and understandable through intelligible principles. However, Plato never pretends that his highest level of philosophizing is the last word: the dialogue ends by catapulting us back down, or perhaps out, into a myth (90e-91c) that indicates where the philosophical *dialogos* must proceed in its restless, unsatisfied quest for understanding (rather like Kant's restless, unsatisfied Reason, which appealed to the Romantics) (Hamilton and Cairns 1989: 1151–1211).

What occurs at 53c-d? Just at this passage, Plato provides the hypothetical starting points needed for the organized deduction of a kind of mathematical physics, by grounding the floating mathematics of the Pythagoreans in the requirements of materiality. He uplifts materiality by allying it with mathematics, so that it can be an object of science, something the Atomists never tried to do: that project had to wait for the seventeenth century, as we saw in the initial attempts of Descartes and

Gassendi. In this passage about *archai* (beginnings, first causes or origins, principles or elements), Plato offers his own innovative solution to the problem of the division of matter. There were three solutions given to this problem in antiquity. The Atomists claimed that matter could not be indefinitely divided, for the ultimate constituents of matter were… atoms. Aristotle, by contrast, gave matter a structure analogous to that which he gave to the continuum: just as a line in not composed of points (something different in kind from a line) but only indefinitely decomposed into smaller line-continua, so matter cannot be composed of atoms (whose characteristics would then be different in kind from those of macroscopic matter) but only indefinitely decomposed into smaller volumes of homogeneous matter, which can themselves always be so indefinitely decomposed.

Plato's *Timaeus* offers a third metaphysical possibility at 53c-d, the construction of the *archai* of the world by a Demiurge, "according to a method (*logon*) in which the probable is combined with the necessary." Timaeus observes that fire, earth, water and air are bodies, that every body has volume and depth, that every volume is bounded by flat surfaces, and that every flat surface can be composed from… triangles! He proposes that the elements of the four elements are triangles, which he describes in purely mathematical terms, and to which he attributes nothing corporeal or material. Thus, Plato's solution to the problem of the division of matter looks like that of Democritus (cast into poetry later by Lucretius) but the elements are not material. (I keep wondering if De Rham had been reading the *Timaeus*.) This metaphysical alternative has difficulties and virtues. One difficulty is that triangles are given a location in space and time. As the faces of the tetrahedron, octahedron, icosahedron, and cube, they compose (respectively) the molecules of fire, air, water and earth. So too they are treated as if they were individuated, though there is nothing to individuate triangles of the same shape and size except their location, and it seems strange to attribute individuation to a factor (location) extrinsic to the thing. Treating triangles in this fashion also makes them part of the inventory of the world and grants them, inexplicably, the resistance to penetration that material objects enjoy. The hypostatization here is just as surprising as that suggested by the Platonic myths, which invent a heaven-like location for the objects proper to the two highest levels of the divided line. This might remind you of Dante.

If triangles exist as it were side by side in this world, then they are part of the inventory of the world, things among things. Those who argue that the classical philosophers never arrived at the topic or question of subjectivity (as Descartes did in *Meditation* II, or Kant in the *Critique of Pure Reason*), reproach them for only taking inventory of reality from a third-person perspective. Thus Rémi Brague writes: "Of course, the Greek thinkers did not fail to develop reflections in the grand style on the origin of man and the place he occupies among other living creatures; they located him with respect to other beings that inhabit the world, whether they be plants, animals, or gods. They even perceived that man is not an animal among other animals, but that he deserves to be called a microcosm, in that he contains in himself not just a part of the world, but all that the world contains. And nonetheless, Greek thought seems not to have grasped the fact that we are in the world. It conceives the site we occupy, but not our being-situated; it conceives the way man belongs to the world, but not my presence in the world" (Brague 1988: 47–48). Here is the poem I wrote after reading his essay:

After *Timaeus*
For Rémi Brague

The serpent is all belly, and Timaeus'
Strange production nothing more at first
Than radiant limbs about a living sphere
Unconscious of itself, all eyes and ears,
Afloat in the matrix of the universe.

And what are we? Part snake, part crystal ball,
Our hollow belly the low sounding board
Where we first hear ourselves speaking or singing
And know we are the author of our song.

Sealed by the baleful birthmark of the navel,
We live with the necessity of evil
And breach our paradise, each time we fall
To speech, self-knowledge and the grand finale.

This failure to inquire philosophically within the first-person perspective leaves the soul regarded from the outside, located either mythically in heaven where it belongs with the Forms, or, according to common sense in this world, with its body. Plato's placement of triangles side by side in this world, I would argue, is of a piece with Greek inventory-taking. To make an object of thought like a triangle into an individuated, spatio-temporally located object is to focus philosophical attention on the contents of what I know, not on my human awareness that knows. The hypostatization that recurs throughout the works of Plato is a structural feature that signals the absence of the philosophical theme of human subjectivity.

One clear virtue of Plato's *Timaeus* account is that it rests on the interesting unity of triangles. The atom of the materialists is postulated to be extended and yet indivisible. This bothered Descartes (to whom it seemed to impugn the omnipotence of God) and Leibniz (to whom it seemed to impugn the Principle of Sufficient Reason), for why should matter or extension cease to be divisible at a certain arbitrary degree of the very small? However, a triangle understood in Euclid's terms, is indivisible in the strong sense that it is not the sum of any of its parts; it cannot be decomposed and remain a triangle. For Euclid, lines are not composed of but rather bounded by points; plane figures are not composed of, but bounded by, lines. (Only a twentieth-century set theorist would make the mistake of assuming that a triangle is a set of points, and therefore merely the sum of its parts.) Thus, a triangle's indivisibility is not arbitrary, and it is one aspect of how a triangle is extended in the plane. The stable, suggestive integrity of triangles is one reason why geometric figure works so well as a schema for existing, spatio-temporal, material individuals. The formal unity of the one stands for the existential unity of the other.

But then it is also true that triangles are formal unities that are not true individuals. Triangles that seem on one occasion to be distinct, on another occasion may lose all distinction, but this ontological indistinctness of triangles is more obvious in certain contexts than others. When the problem is exhibiting congruence, the superimposition of one figure on another displays the way figures initially distinguished by place can lose all distinctness. However, when figures are used to construct other figures (as when six equilateral triangles are used to construct a hexagon), their distinctness must be carefully preserved in order to effect the construction. The latter kind of situation is the one Plato's Timaeus has in view when he makes triangles the constituents of the world by using them to compose the faces of four regular solids. That preservation of distinctness makes them even more plausible schemata for the unity of individuals. Locating the materialist account at the second level of the Divided Line, Plato indicates that it cannot rise to the status of an organized science. He also indicates that, taken on its own terms, it generates unsolvable puzzles, for it cannot even explain the way it draws the distinction between image and object. In general the materialist account mistakes physical objects for images. All it can give in the inventory of the world is physical objects (e.g., atoms in the void), so that its explanation of representation is another collection of objects: the components of the eye, of the object, of the transmitting fluid, and so forth.

However, by locating the materialist position on the Divided Line, Plato also indicates that while it is inadequate, any account that does not include the materialist account as a stratum or stage will also prove inadequate. In the *Timaeus*, Plato argues quite clearly that an awareness that was not embodied also could not draw the distinction between object and image; it would mistake images for physical objects. Thus materialism grasps a deep though partial metaphysical truth, the necessity of embodiment for recognizing the image as image, for distinguishing image from object, for knowing oneself, for reflectiveness, for the wisdom of mortality. At 53c-d Plato's Timaeus introduces the mathematicals and so enriches the ontology of the world, and he gives the discourse a new beginning so that it can proceed deductively from first principles. He also suggests an analogy between God's shaping of the universe "by means of forms and numbers" and his own carrying up the discourse to the third level of scientific reasoning; he then immediately reminds the reader that the analogy is imperfect, for God creates the world while Timaeus only writes about the creation of the world. The ambiguity of the word *archai* is perfectly appropriate here, since it means both first cause (in the work of God) and element or principle (in the writing of Timaeus). Timaeus shows that he is aware of and in control of the ambiguity, and so, that he can manage the distinction between object and representation.

In Aristotelian physics and metaphysics and in the Scholastic tradition that carries Aristotelian doctrine through the end of the medieval period, figure is only a certain mode by which finite quantity is terminated. Figure is "a quality resulting from the termination of visible quantity in a natural thing, like the external aspect of a man or lion" (Des Chene 2000: 110). Taken thus as the accident of an accident, it has little to do with the internal principles of change, the substantial forms that make natural things what they are in Aristotelian physics. Because matter is naturally associated with quantity—which, the late Scholastic philosopher Suarez argues, exists in matter prior to its union with substantial form—any bit of matter must have its quantity bounded simply in virtue of being finite. That boundedness leads to a boundary, or figure. Figure is therefore not the result of causal efficacy and itself has no causal efficacy, so figure does not play any important role in Aristotelian physics; purely passive and doubly accidental, it never initiates change and is never itself changed except as a consequence of changes in other more fundamental properties.

But in the *Timaeus*, figure is given a causal role to play. At 53c-d, as we have seen, Timaeus makes triangles (the most fundamental of all rectilinear figures) the *archai* of the four elements and the starting point for a deductive scientific account of the world. In that account the properties of the two kinds of triangles and the four regular solids which they compose are invoked as causal principles that explain why the elements are as they are, why they interact in the way they do, and what some of their consequences are. For example, the Platonic claim that the elements fire, air, and water can be transmuted into each other, while the element earth stands apart in this respect, is explained by the fact that the tetrahedron (the molecule of fire), the octahedron (the molecule of air), and the icosahedron (the molecule of water) are all constituted by the rectangular scalene triangle, "which is half of an equilateral triangle," whereas the cube (the molecule of earth) is constituted by the other fundamental triangle, the rectangular isosceles triangle. Difference in form has causal consequences for the transmutation of elements.

Another causal consequence of form is stability or instability. The molecule of earth is most stable: "To earth, then, let us assign the cubic form, for earth is the most immovable of the four and the most plastic of all bodies, and that which has the most stable bases must of necessity be of such a nature. Now, of the triangles which we assumed at first, that which has two equal sides is by nature more firmly based than that which has unequal sides, and of the compound figures which are formed out of either, the plane equilateral quadrangle has necessarily a more stable basis than the equilateral triangle, both in the whole and in the parts" (55e-56a) (Hamilton and Cairns 1989: 1181–1182). Conversely, the pyramid is the molecule of fire because it has the least stability and is the smallest and sharpest: "Of all these elements, that which has the fewest bases must necessarily be the most movable, for it must be the acutest and most penetrating in every way, and also the lightest…" (56b). Some interactions of the elements are explained by reference to figure. Fire destroys the other elements because it is sharp; the action of fire is a kind of cutting up: "When one of the other elements is fastened upon by fire and is cut by the sharpness of its angles and sides, it coalesces with the fire, and then ceases to be cut by them any longer" (57a) (Hamilton and Cairns 1989: 1181–1182).

One could multiply examples. But my point is that to regard figure as causally efficacious is a metaphysical possibility that Aristotelianism on the whole excludes and that Plato's *Timaeus* allows. Revived in Descartes's physics, it is then transformed by Leibniz. He seems not to have been particularly interested in the catalogue of particles of different figures that Descartes elaborates in the *Principles* to give "mechanistic" explanations of various natural phenomena. The figures which interest Leibniz, and to which in an oblique sense he accords causal efficacy (or at least a place in explanation), are the shapes of curves, particularly transcendental curves.

Given Leibniz's multivalent understanding of curves, algebraic and transcendental curves lend themselves to the representation of the action of forces. By analogy, the way in which geometrical magnitudes mutually constrain each other in the nexus of a curve (or the way in which a curve imposes mutual constraints on the geometrical magnitudes associated with it) comes to stand for the way a trajectory imposes mutual constraints on the position, velocities, and accelerations of the particle. This analogy is supported and articulated by Leibniz's way of associating with curves ordered arrays of rational numbers on the one hand, and algebraic expressions on the other. Thus, the figure of a trajectory, of a hanging chain, or of a vibrating cord is not the accident of an accident but the register of significant causes in a mechanical situation. It is the object of scientific and mathematical investigation. This insight, along with Newton's *Principia*, as we have seen launches mathematical physics in the eighteenth century (Grosholz 1996: 255–276).

For an Aristotelian, figure is never the result of a formal or efficient cause; it is the result of a final cause only in the case of artistic production, as when a sculptor imposes beautiful form on a block of marble. Thus, figure is significant only in the case of human artifice, not nature. It is surely relevant here that the one bit of physics that was quantified in the classical era in a way that in hindsight looks to us "modern," is Archimedes' quantification of simple machines, of human artifice. Both Descartes and Leibniz assimilated the realm of human artifice to the realm of nature by making figure causally significant and by working towards a mathematical mechanics in which the yoking of geometrical figure to numerical sequence and algebraic structure played a central role. The God of Plato's *Timaeus*, who "marks things out into shapes by means of forms and numbers," looms in the background of this development. The last of the three books of cosmological poetry, by three contemporary poets, that I'll go on to discuss here, suggest a godlike human artifice: the internet generates a soul, a self-conscious, willful, benevolent personage, Kalodendron. She is one of the main characters in Frederick Turner's *Apocalypse*.

To trace a cosmological path from Plato to the modern era, where mathematical cosmology becomes increasingly poetic, we'd have to hark back to parts of Homer, in Robert Fitzgerald's translation, and then talk about Theocritus' *Idylls* in Robert Wells' translation, Cicero's *Dream of Scipio*, Ovid's Proem to the *Metamorphoses* in Charles Martin's translation, and Lucretius' *The Nature of Things* in Alicia Stallings's translation. (Wells, Martin and Stallings are friends of mine, but I can say objectively that their translations are wonderful, both scholarly and poetic, since they are all three fine poets.) Tracing the parallels between Athens and Jerusalem, we would discuss parts of the King James translation of the Bible, especially *Genesis* and the *Book of Job*, and then some of the poems and meditations on time in Boethius' *Consolation of Philosophy* in David Slavitt's translation. This would lead us to the more cosmological passages in Dante's *Divine Comedy*, especially the sections on the Earthly Paradise, and to passages of Chaucer's translation of the *Consolation*, and his reworking of the *Dream of Scipio* in the *Knight's Tale*, which my husband the medievalist Robert Edwards has expounded so well. We could read Galileo's *Dialogue Concerning the Two Chief World Systems*, and Descartes' *Discourse on Method* as literary works (a dialogue and a narrative, both with interesting character development), and then go on to the cosmological passages in Milton's *Paradise Lost*. Then there are the lush fields, the great mountains and expressive skies of Romanticism. Had we but world enough and time… but this book cannot go on forever.

Moreover, we must catapult, yet one more time, to the age we live in, and think about the development of modern cosmology, the theory of the cosmos being elaborated as I write. First, I recommend Koyré's *From the Closed World to the Infinite Universe* as well as Thomas Kuhn's *The Copernican Revolution* to understand the transition from ancient to modern cosmology, and the role that a revived Atomism played in it, along with Edmund Halley's *Ode* to Isaac Newton that prefaces the *Principia*, and Donald Davie's *The Language of Science and the Language of Literature, 1700–1750*. To think about the cosmological changes produced by Einstein's revolutionary Special and General Theories of Relativity, and accompanying discoveries made by astronomers, I have been inspired by Robert Osserman's *Poetry of the Universe*, Lee Smolin's *The Life of the Cosmos*, Robin Le Poidevin's *The Images of Time*, *100 Years of Relativity* by my admired colleague Abhay Ashtekar, Peter Coles' *Cosmology: A Very Short Introduction* and *The Nature of Space and Time* by Stephen Hawking and Roger Penrose. I also recommend *Verse and Universe: Poems about Science and Mathematics*, edited by Kurt Brown, that reflect our changing awareness of the cosmos, from the single 'island galaxy' my parents were born into, to the universe with (by the last count of the Sloan Survey) 200 billion galaxies. However, here and now, we will finish with three narrative poems devoted to modern cosmology: Daniel Tobin's *From Nothing* (Tobin 2016), Alice Major's *Standard Candles* (Major 2015), and Frederick Turner's *Apocalypse* (Turner 2016).

The story of modern cosmology offers many surprising twists and turns. One of the surprises is that Einstein's formulations of Special Relativity (1905) and General Relativity (1915) were not driven by new data, but by formal considerations and Einstein's mastery of new mathematical languages. Another is that one of the main challenges to Einstein's presentation of General Relativity came from a physicist who was also a Belgian priest, Georges Lemaître, whose proposed modifications also arose from formal considerations. He saw clearly that the space-time of General Relativity Theory was not only curved, but dynamic: space-time itself could expand and contract globally. He postulated a model of cosmological development that came to be known as the Big Bang Theory (an appellation that was at first perjorative, coined by Fred Hoyle in 1949). Einstein at first denied the cogency of the model and added a 'fudge factor,' the cosmological constant, to guarantee the global stability of the cosmos. Later, he came to accept the dynamic model, after new data was gathered by Edwin Hubble.

Hubble worked at the Carnegie Institute's Mount Wilson Observatory near Pasadena, California between 1919 and 1953 (the year of his death), where he had access to one of the best telescopes in the world, the 100-inch Hooker telescope. In

1952, he became the first astronomer to use the 200-inch reflector Hale Telescope; George Ellery Hale founded the Mount Wilson Observatory in 1904. This means that, regrettably, Hubble missed the Little Old Lady from Pasadena, who only emerged in 1964; who knows what he might have observed if he could have rocketed off with her in her new shiny red super-stock Dodge, with a four speed stick and a four-two-six now? (Go, Granny, go Granny, go Granny go!) (Morgan 2015). However, to make up for that missed opportunity, the large, versatile space telescope launched by NASA in 1990 was named after Hubble, and is still in operation.

Hubble Archive: The Whirlpool Galaxy (M51) and Companion Galaxy, January 2005

Hubble's research depended not only on his access to a powerful telescope (poised on a hillside in high clear desert air) but also on the concept of a 'standard candle' and the investigation of Cepheid Variables, which was carried out by Henrietta Swan Leavitt around 1908. She first identified Cepheid Variables while working at the Harvard College Observatory, examining photographic plates to measure and catalogue the brightness of stars. A Cepheid Variable is a star that pulsates, varying in brightness with a well-defined period. Most important, there is a direct relation between the star's absolute luminosity and its period: this is the correlation that Leavitt first enunciated in 1912, studying stars in the Magellanic Clouds. Computing and then relating the star's apparent luminosity to its absolute luminosity, astronomers for the first time could accurately reckon the distance between earth and more and more far-away stars. Indeed, apparently those clouds or nebulae were other galaxies. Beyond the Milky Way, our home galaxy, lay Andromeda, and the two dwarf Magellanic Galaxies, and who knows how many others!

Using these standard candles and his powerful telescope, around 1929, Hubble combined his measurement of the distances of galaxies with measurements of the redshifts of the light from those galaxies compiled by Vesto Slipher and Milton Humason, and found a linear correlation. The further away a galaxy is from us, the faster it is receding from us. More generally, the greater the distance between galaxies, the greater is their speed of separation. Hubble's results were explained by Lemaître's model. Hubble himself remained doubtful about Lemaître's model, but the distinguished physicist Arthur Eddington insisted on the conjunction of data, model and theory, and arranged for Lemaître's neglected work to be translated into English and reprinted in the *Monthly Notices of the Royal Astronomical Society* in 1931; a report was also published that same year in *Nature*. Even Einstein came around in 1933.

On the first page of Daniel Tobin's *From Nothing*, there is a six-line poem, an invocation of the initial atom, or egg, or condition.

… this sudden stirring, like birds before
an earthquake, then the explosion—

a fluency of wine in water, white streak
across a white vault of sky where the tunnel

opens to Bright Abounding, its utter light
fleet release, ecstatic, unutterable, before…

Tobin uses images: the way a flock of birds will fly in all directions in response to a disturbance, the way wine spreads its color through water, the dazzle of sudden light. The image that expands across four lines concerns the light that welcomes us when we emerge from a tunnel, which is after all a highly engineered cave. So we might return to the simile of the cave in Plato's *Republic* and its evocation of the miracle of the Sun for someone who has always lived underground, and only known firelight and shadows. Or we might revisit the moment in Dante's *Commedia* when Dante and Virgil emerge from the *Inferno*, underground, to climb the mountain of *Purgatorio*. There Dante attains, at the summit, the Earthy Paradise that Ovid imagined in his tale of the flood: the mountain tip where the survivors Deucalion and Pyrrha make their home (as if a mythical swirl transformed Adam and Eve into Noah and his wife, and the top of Mount Ararat into Eden). Modern cosmology proposes that about 150,000 years after the Big Bang, light finally emerged from its tiny shackles, and we see the remnants of that event as the Cosmic Microwave Background, a later prediction from Lemaître's model. Modern cosmologists must also find a way to integrate Relativity Theory and Quantum Mechanics, to frame a theory of the cosmos in its earliest stages: there is no general consensus on how this unification should be carried out.

But who is the pilgrim in Tobin's book? The first poem is followed by a quotation from Lemaître himself, alongside the title of the first part, "The Most Ancient Light in the Most Ancient Sky." "We can speak of this event as a beginning; I do not say creation. … Any preexistence of the universe has a metaphysical character. Physically, everything happens as if the theoretical zero were really a beginning. The question if it was really a beginning, or rather a creation, something started from nothing, is a philosophical question which cannot be settled by physical or astronomical considerations." Lemaître was in fact angry when the Pope asserted that his work proved the accuracy of *Genesis*. At the same time, he was a priest as devoted to Catholicism as, in his role as an astronomer, he was devoted to science. This is why he is the pilgrim we follow through Tobin's poem, composed in unrhymed terza rima, divided into three parts. Each part is composed of eleven enigmatically titled sections, each section composed of eight three-line stanzas. And each section is an episode in Lemaître's life, sometimes vivid, sometimes cloudy. But the narration is not third-person. Lemaître is always addressed as "you," so the author somehow hovers with us at the edge of the story: he too is trying to work out how science might be related to religion, refracted by cosmology.

So cosmology is a middle term between science and religion, and Lemaître is the character next to whom we walk through the story, a middle term between the enigma of the cosmos and the human history in which we find ourselves. The young Georges wonders if he has to decide between two callings, as he discovers them:

Or had it haunted you nights with your schoolbooks
even back in Charleroi, in the halls of Sacré Coeur:
calculation and consecration, geometry and God?

But the young student is drafted, and must serve as a soldier in World War I, fighting at the Battle of the Yser (one of the worst in that especially horrible war) and earning, ironically, the Croix de Guerre. He reads Poincaré in the trenches, and manages to survive the war that killed Schwartzchild, Mosley and De Sitter.

Newton's apple plunges

down the parallax of a rabbit hole, its wake a bend
of starlight tacking the halo of an eclipsed sun
and clocks ticking tick to the measure of every eye:

the genius' equation like a single stone launched
to shatter the foundations.

Einstein's General Relativity abolished the force of gravity: rather, matter curves space-time around itself, transmuting straight lines into geodesics, as Eddington demonstrated. He showed, carrying out measurements in Brazil and in West Africa during a solar eclipse in 1919, that the path of starlight curves as it passes by the sun.

After the war, Lemaître visits the dome of the Harvard Observatory, and Mount Wilson too, where Slipher shows him the high desert. He takes his vows and also pursues his studies.

You, who chose two ways equally at once, circuit
the conferences, meetings fueled by enigma,
mixing with the eminent and their sidereal regard,

your morning Masses between library and lab.
All outcomes must be possible in this system—Schrödinger.
In your life's chosen box, this con-celebration.

Here Tobin plays on the idea of Schrödinger's cat, who "keeps equally live and dead" in the famous thought experiment that dramatizes quantum superposition. Just after he has been rebuffed by Einstein, he reflects, "There are two paths to truth; I have chosen both" (Tobin 2016: 12) And then later, when Einstein has changed his judgment, here he is, recalled by Hubble.

Odd, too, the little priest who came to visit years ago,
that he should account for nebula's radial velocities
two years before me, that I only trust the data—

how he looked calmly pleased by Einstein's recantation:
The most beautiful solution to creation I have ever heard.
So clocks reel back with space—camera, action, light.

At the end of the first part, war threatens to break out again. Einstein must flee Germany because he is Jewish. We should recall here that Emmy Noether also flees and her friend Jean Cavaillès (a Catholic) is tortured, imprisoned and then executed by the Nazis in Arras. (Gaston Bachelard was among the first members of the Societé Cavaillès, founded in 1947.) The Führer uses religion to divide and kill. Lemaître writes, "In the face of suffering, we must drop books and pray" (Tobin 2016: 2–14).

The second part chronicles World War II, the war that decimated our planet: "The Death of One God is the Death of All." Lemaître witnesses the Reich take over Belgium and his city, Louvain, after Leopold, King of Belgium, capitulates to Hitler. Throughout the second part, Lemaître returns to his piano, the abstract and yet enlivening harmonies of music.

Is there a providence at the heart of quantum chance,
the risk of the Pianist whose score evolves the keys?
Point and purpose hazarded on scales across scales.

There follows some simple arithmetic, a calculation.

Now in earshot of you—the scale that shatters scales:
50 freight cars × 50 per car × 1.5 trains per day
× 1066 days = 4,000,000 Jews "resettled to the East."

The Belgian Cardinal publicly opposes the Nazis and encourages resistance; Pope Pius XII avoids direct confrontation to protect the Catholic Church, but works covertly to save Jews. Still, Tobin's Lemaître reflects, echoing Paul Celan,

And if *he* donned the golden star? History's "What if?"
O golden haired Margaret, O ashen haired Shulamith.

In the midst of the despair brought about by the war, the physicist and priest are almost sundered: perhaps physics demonstrates that music is an illusion, "the amplitudes a blank smoke/unnecessary—number as number and nothing more." Perhaps "Time's arrow at $t = 0$ has a barb at each end/that makes the infinite universe a buried corpse" Perhaps music cannot serve as a middle term between science and religion. Lemaître reads the fourteenth century mystic Jan van Ruysbroek's writings, and weighs them against Schrödinger's harsh claim, "Nature in itself has no reverence for life." Ruysbroek's reply, "La lumière éternelle engloutissant toutes," is a mysterious warmth, but warm all the same (Tobin 2016: 15–26).

In the last of the three parts, "Of Motion, the Ever-Brightening Origin," which begins with Pascal's famous meditation on the two infinities, we see Lemaître turn down an invitation to the Institute for Advanced Studies at Princeton. There he would have been able to talk with Einstein, Niels Bohr, and Kurt Gödel, but he chose to stay in Belgium to take care of his elderly mother. His troubles were not over: in the early 1960s, the University of Louvain split into two parts, one Flemish-speaking and one Francophone. The windows of his house were broken, because he tried to act as a peacemaker and hold the halves together. The first lines of the poem about this episode evoke both Pascal and Shakespeare, playing with nihilism.

For all of it, how presumptuous, we thinking reeds,
unable as we are to stand outside the human, the cost
of our becoming, all becoming, a ripple of sun across

the leopard's back as it locks on the gazelle, prodigal
orders of blood and contingency, signifying nothing.
Or, if not, a hidden sum in the corpus of the random.

Tobin's Lemaître appeals to his friend Joris Van Severen: they fought together in the trenches in World War I; but Van Severen split with him as he became more and more right-wing just before World War II, and was executed by French soldiers in 1940. Lemaître prays for him, and remembers seeing an eclipse at the Milan Observatory,

that halo when the three bodies perfectly aligned,

not only in equations but in flesh. We are all strayed
lines in an infinite story we see, at best, darkly.
The rooster crows and thinks it makes the sun rise.

In the very last poem, there are other lines, in the lines:

On the night you died the waves were lifting, the sands
A shifting membrane at Gravelines, Calais, the North Sea
one sea, and the sands multiplying myriad after myriad

falling short of infinity. So make each grain a universe…

If the cosmos, like life on earth, evolves, perhaps it can be understood as Providence, the sky compactified by our vision to a cup, lifted by a priest.

A lifting, unencumbered, of wings.
At dawn, a blood-red host; a blood-red host at nightfall.

Lemaître learned of Arno Penzias' and James Wilson's detection of the Cosmic Microwave Background, more empirical evidence supporting the hypothesis of the Big Bang, in 1964, shortly before his death. Penzias compared it to the sound of surf, to music. Two weeks before he died, Lemaître said something like this in his final interview, which Tobias translates into poetic English, as an Afterword:

…and the universe nothing more than dream;
and we, blind as book lice, cross the slight

horizon of a page, to miss the moment
in our desperate flittering, the Word unread,

sustaining, beneath us: purpose and path—
ash to amethyst, moth wing, seraph, breath…

And so the book closes, shuts, on the small consciousness traveling across it, doing its best to understand, even if it is not very adept at reading the great characters beneath its feet, and overhead (Tobin 2016: 27–39).

In Alice Major's *Standard Candles*, the relationship between the story and the poet shifts. The poet herself is in the story, not just via an appeal signaled by the use of "you," as in Tobin's book, but as an actor who speaks sometimes in the first person. Her action is for the most part rumination or meditation, though sometimes it turns into a kind of embrace. So, for example, the book begins with a Valentine's Day poem to her husband, extolling his virtues, evoking his warmth, as she launches the poem off into outer space: "Go little ship/of space beyond the gravity of time,/ and, beating always, prove/there is indeed a god/of love."

The poem has eight sections of varying length, composed of thematically related poems, and then a Postscript. The first section offers "The set of all gods." The riddle she poses is whether there is one god (and a theory of everything), or a pantheon (and a theory with heterogeneous, interlocking parts). The poems themselves are like riddles. We meet the god of prime numbers, the god of infinities, and then the god of symmetry and the god of gravity. They are followed, oddly, by the god of salt, and then the god of kites and darts, the god of quantum uncertainty and then the god of probabilities, which Tobin invoked in a different way. After the baker god we encounter gods of automata, teapots, cats, sparrows, and hearts, all likely candidates. And then after the jeweler god (Robinson Jeffers seconds that one), the god of dark and the god of memory. At the end, the muse of universes appears: once in a trillion years, she claps her hands, and orders a new draft (Major 2015: 1–21).

She rebuts

formlessness, sparks stanzas
from an alphabet of particles,
spells out what matters, what
radiates, what tickles

the fancy into galaxies
with gravity's feather pen.
She unrolls the scroll of space,
says, *There. Now try again.*

The second section treats "Ordinary matter," the mind-body problem, the three-body problem, and thus inevitably love. It ends with a four-part "Catechism," a set of questions for each season (Major 2015: 23–37). Here is the last of the questions for autumn:

What is the candle for?

Emergencies of love, the flare
that whispers here
over here, follow me
in falling dusk.

In the third section, "Standard Candles," narrative returns, and we meet Henrietta Swan Leavitt! Oddly, however, the section begins with the poet in 1959, described third-person, a ten-year-old astronomer in Scarborough, Ontario. She writes out the rest of her address: "*Canada, The Earth,/The Solar System, Milky Way. The Universe.*" For her, the universe is Aristotelian, "a set of nested spheres," encompassed by God, and including, mysteriously, a transcendent place.

A place
we'd never reach with rockets nosing space.
Heaven, where Nana and her aunt had gone
And where the dead went living on.
Another place, apart.

The second poem catapults us back to 1908, and there she is: "Miss Leavitt, lace-waisted,/hair knotted neatly/at her neck's nape, pores/over photographic plates." She is calculating, using glass negatives collected from telescopes that tracked "the high skies of the Andes—/the Magellanic Clouds," that guided Magellan, Tasman, Cook and more ancient, unsung mariners. She notes, for the first time, the fixed correlation between the period and absolute brightness of the stars called Cepheid Variables.

She is holding up a standard candle.

Now telescopes can search
still-smaller, fainter clouds
that smudge the fathomless,
unmapped heavens.

And if one checks the apparent brightness of the star against its period, one sees how far away it really is, and those clouds, those nebulae, reveal themselves as other galaxies.

Suddenly in the third poem it is 1965 and the poet, third person, is learning about the Pythagorean Theorem, a theorem "of trinity and distance—a formula/sent down to her from time, an orison." In the next poem we move back behind Leavitt to 1808, when William Lambton, whilst surveying India, determined the exact curvature of earth along an arc of longitude. Then forward to 2000 in the next: the poet learns to add to her address, "Orion Arm,/The Local Group (that reef of galaxies/around us—Andromeda,/Triangulum, the Magellanic Clouds)." But then we find the end to greatness, to addresses: at the highest scales, the universe seems to be homogenous, "a foam of filaments and bubbles,/the spray of centerless light." And where is God? "Out there, somewhere./For now, she is content to be amazed."

Back to 1576, when

Tycho Brahe mounts his quadrants,
his astrolabes and armillary spheres,
the alidades with slotted pinnules
(his own invention) to sight
the faintest spark of light correctly,
without the winking parallax
of two eyes shifting left and right.

Note how much the poet loves words, for their own sake. Brahe collected the best data in Europe before the telescope, which is why Kepler came to work with him. Forward to 1997 (are you feeling a bit dizzy?) and the recognition of Supernova Type 1A, an especially useful Standard Candle, identified by Gerson Goldhaber at the Lawrence Berkeley National Laboratory. It seems that space is not just expanding; its expansion is accelerating.

Something is happening here—a universe
blowing itself apart, space accelerating.
Those distant candles seem to gutter
on the edge of invisibility. Soon their dim light
will disappear past any hope of reaching us.

The next poem, "Then death returns," has no date, and the poet has lost her bearings, for her mother is dying and she must return home in time to say goodbye. "The woman is left with ashes and no maps./Only the bitter cry, *Where are you now/to hold me in your heart?*" Her grief is described topologically, an elaborate but heartfelt conceit.

Anguish stabs
its hole in her seamless world,
rips out one essential point, explodes
the punctured topology of globe
into a flayed plane stretched to the edge
of everything. An incommensurable page—
no co-ordinates, no spires or temples,
no marks or margins, no instrument, no theorem
by which to understand its distances
and tie them into something relative

to her. A universe of loss. Within its gaps
she must find a place to live.

It is as if Tony Phillip's version of Stephen Smales' Sphere Eversion had been translated into an elegy, and given novel, unexpected meaning. In the following poem, the poet confronts a dark night of the soul: "No candles light me to the place/ where I might hold you in my arms again."

Then somehow we are back in 1928: Miss Leavitt has just been buried. Her modest grave, and a crater on the dark side of the moon, are her only memorials, beside her discovery, arising from faithfully computed numbers, on blue-ruled paper.

This great lurch outward, your discovery
we are a single blinking island
in a swelling sea? Were you lost
on that dark ocean? Or no.
You loved your glowing clouds.

Comforted, the poet concludes with an equation and a prayer, finding, like Lemaître, like Boethius, consolation in science, mathematics, and religion. Here is the title of the next-to-last poem:

"$d = (X - x)^2 + (Y - y)^2 + (Z - z)^2 + c(T - t)^2$". It concludes,

From this home address
In the realm of starry, vast forever

she paces out the length of earthliness—
the fine, triangulated measurement
that verifies the rest.

It is the Pythagorean Theorem, generalized from the Euclidean plane to Einstein's space-time in four dimensions. The last poem is entitled "A prayer to bring you home," where the poet watches from the window, one of those glass squares that helps us compactify infinite space, so that along one of its imaginary, curved geodesics, we hope that those we have lost may return: "Come home/up the path/you have always known./Come home./Your suitcase is as heavy as a headstone,/light as a purseful of leaves.//Come home. It is warm./Come in/my arms." And there is Fainlight's mother's purse, another kind of compactification, but full of autumn leaves (Major 2015: 39–65).

The next section takes up, among other things, the mind-body problem, and the next, a classification of cosmologies. The sixth section offers a classification of sins and virtues, which invokes Dante, who numbered them all so carefully (9 and 9 and 9) and gave them an imaginary location in space, out of time in one sense, and caught up in the time of transit in another, as Dante and Virgil (and then Beatrice) make their pilgrimages. In the following section, we find a classification of paradoxes (Major 2015: 67–130). Then, Major takes us on a tour of "Underworlds," in particular the underground mall in Edmonton, where people can shop at ease during the dire winter, and where the homeless try to find shelter. We witness a young man "shrouded by the hood/of his ragged jacket," asleep in his despair; the man with a gangrened foot "that comes rotting away/when a man pulls off his boot"; and a man who lost all ten fingers to snow and frostbite. However, in this poem we find an inversion: while Dante judges those in the *Inferno* without pity, Major turns the judgment against the Blessed, well-heeled citizens who live in high-rises and pretend these people do not exist. But Major was the Poet Laureat of Edmonton; her poem is a message to the City Council.

And she exhorts her husband, who is the middle term, because he lives with her at home, and he is one of the engaged citizens arguing for the rights of the homeless, rights to shelter and medical care and… counsel. The title of the last poem in this section (and almost the last poem in the book) is entitled "Each of us the centre of a circle." Her husband is uncharacteristically dressed up in tie and jacket and nice shoes: "But today/you are encountering the plated uniforms/of police and politicians who desire/to 'clean up' the troubled people/who intrude upon their plans for order." Then the poem turns into a love poem, as she recalls looking up at a rainbow with him the day before. Though their rainbows differ, they overlap.

We register
The same intensity of wavelength. My rainbow
spans the same extent of sky as yours.

This is our shining armour—
a shared spectrum, the colour's constant order,
your polished shoes.

The circle of the rainbow against the sky, with its colors always in order: ultraviolet, violet, blue, green, yellow, orange, red, infrared, including the colors we cannot see but know they must there, thanks to the mathematical physics that gives us light as wavelength and frequency, $\lambda v = c$. Meanwhile, we have learned that other inhabitants of earth can see them, for example, the ultraviolet patterns on flowers that bees use as guides to nectar. So too we must use both poetry and mathematical science to address the global problems of poverty in relation to plutocracy and overpopulation, and the citizenship of human beings in relation to the rest of the world (Major 2015: 131–148).

This brings us, unfortunately, to *Apocalypse*. This epic poem by Frederick Turner, despite its title, nonetheless has a happy ending, due to an odd development of the mind-body problem. The root meaning of 'apocalypse' is an uncovering, and so a revelation or prophecy in which divine purpose, discovered, makes sense of earthly reality. However, this revelation as we find it in many ancient cosmological accounts is often at first a disaster. In the Epic of Gilgamesh, in Egyptian myths (as Plato's Solon tells us in the *Timaeus*), in the Bible's *Book of Genesis*, and in Ovid's *Metamorphoses*, the earth is divinely chastened, or chastised, by flood. One of the crises in Turner's *Apocalypse* is global warming, and Book I is entitled "The Flood." The waters of the North Sea in 2067 rise due to melting polar ice, and the brilliantly engineered dykes of the Netherlands fail, and Amsterdam is inundated.

The story concerns a band of heroes around Noah Blazo, whose name is of course not an accident. He is a (good) plutocrat, due to his invention of the Blazo Solar Battery, who collects a team of experts to address global warming, in the absence of effective international governance: first at the Four Seasons Conference (Book II) and then his own private estate on Banks Island (Book III). We are first introduced to Noah Blazo on the second line of the poem, as “an old man with a bolo tie,/who saved the world that was not worth the saving,” but he is not named until line 55. In between, we are introduced to the narrator, who also remains nameless for a while and then is identified as Nemo (another non-accidental name) when he enters the story as one of the characters much later; sometimes he sounds oddly like Frederick Turner, in his relation to the poem as it unfolds. Nemo thus often speaks in the first person, and recounts certain episodes where he himself was present. However, since the narrative spans 230 years (1987–2117) and the globe and parts of the Solar System, he must of course recount parts of the story second-hand. Like the Patriarchs of *Genesis*, many of the characters in this epic live a long time: Noah Blazo is born in 1990, but he is still there at the end of the book.

However, the first person we see, named and in action, is Anneliese Grotius: another non-accidental name, since Grotius (d. 1646) was a Dutch jurist who laid the foundations of international law by invoking ‘natural law’ in a manner that anticipated Leibniz. Leibniz in fact was originally trained as a lawyer and acted as a diplomat for the House of Hannover, as well as the resident mathematician and physicist (and engineer and city planner).

Dr. Anneliese Grotius,
Who framed the iBall Makers’ Bill of Rights,
Whereby the maze of intellectual property
Was solved by data-mining on the web,
Untangled by a 3D point of view.
She turned the problem upside down,
And blockchained from the user to the source,
Made information a utility,
Fair-valued by the user’s use of it,
Paying surprised creators their reward.

A blockchain is a distributed secure database that contains a complete history of a transaction, a continuously growing list of records called blocks. So Anneliese Grotius invents a kind of Sphere Eversion in computer science, usefully turning the system inside out. She is a polymath good at carrying ideas “over from one field,/ Like spores, into another somewhere else,” whose official job is curator of Renaissance Art at Amsterdam’s Rijksmuseum.

When the flood arrives, Anneliese is in her office at the Rijksmuseum. Once she realizes that the museum and everything in it is about to be destroyed, she rushes to the Gallery where Rembrandt’s *Night Watch* is displayed, cuts it out of its frame with an X-ACTO knife, roles it up like a carpet, hauls it to her office on the top floor, and collapses. At this point, we go back in time a month, and finally meet Noah, attending a climate summit in Washington D.C.: the Four Seasons Conference.

The government representatives are only pretending to take action, so Noah starts to dream up a group of people who might address the problem seriously at the international level, one of whom is Wu Liqiu. She goes by Lucy or Dr. Wu, and plays a central role in all that follows, helping to analyze complex systems and use quantum computing, and persuading Noah to stop grieving for his lost wife. Other important people include Ala Ifa-Eshu, a fierce Nigerian agronomist, one of whose bodyguards saves Noah from being assassinated in Rock Creek Park during the course of the summit, and the engineer Chandrasekhar Rama and his son Gopal Gaya Sohrab, a mathematician and physicist (Turner 2016: 1–30).

Only at line 821 in Book II do we find out what has become of Anneliese. Nemo-Turner writes,

And I have left our Anneliese hanging,
In a wrecked building, clutching to herself
A remnant of a civilized lost world.

After two days the helicopter comes.
She's seen the monstrous ruin of her city
Laid out beneath the cold October sun…

Her husband and child have drowned, but she (and the *Night Watch*) are saved. Soon after, Noah persuades her to attend the conference on Banks Island, in part by connecting with her grief: after 19 years, he still can't get over the drowning of his wife, when her boat went astray. Book III presents the conference: not only do those assembled work out a highly artificial, highly engineered (and therefore effective) plan to address global warming, but Noah and Lucy fall in love and so do Anneliese and Gopal, in a Shakespearean, Tempestuous way (Turner 2016: 31–90).

Some of the projects engendered on Banks Island are slowly implemented (with cooperation from countries most in danger of drowning), and they spark immediate political backlash: thus Book IV is "The Battle of Kerguelen" and Book V, "The Battle of Candlemas Island." But an accompanying miracle appears at the end of Book V and then in Book VI, "Kalodendron." She is a kind of goddess who arises when the internet becomes self-conscious, a twist that happens on Banks Island, where the new quantum spin thoughtprocessor is proving to be a problem: "twenty billion qubits must be kept/From breaking their obscure entanglements/and decohering into senselessness./The problem's programming, of course…" The system is prone to freeze, so Gopal and his father are called in to help.

So there's a pressure to devise a way
The system can include its own computings
As data in its information flow
And criticize its operations when
They clearly lead to fruitless iteration,
Or take the iteration in itself
As a new kind of object to be named.
At the same time, the Brain is being loaded
With everything the human race has thought;

It is as if the Web were one great body,
A wide peripheral sensorium
With feelings, pains, attractions and affections
In real-time, and present in the way
Quantum simultaneity affords.
If one of Noah's objects is to find
Ways that humanity can be jacked in
To Nature as it lives upon this planet,
It turns out any calculator tasked
With such a project must in turn be jacked
Into the body of the human species.
The proper study of mankind is man,
Said Pope; and man's the measure of the world,
According to the great Protagoras.

What precipitates the emergence of the goddess, however, is a catastrophe in the midst of the Battle of Candlemas Island, two hundred men burning, shrieking in pain, as the cruiser Saratoga is destroyed. The system hears "the shriek across the Net" and Kalodendron, as she names herself, is born: "Pity gave birth to love,/And love to memory and to invention,/The past and future suddenly unfolded,/In all their strange asymmetry and branching,/And now she could prioritize and choose…"

Here she is, at the beginning of Book VI, in song and color.

When she first spoke, it was as if the screen
Were a pure ray of green, blazing across
The whole laboratory, modulated
By visual music in its pulse and hue
According to her speech and intonation.
When people named her, she would make an icon
To suit what they proposed. A great green angel,
Female in aspect, spreading topaz wings,
Beside a branching lemon-tree in flower.

She emerges both weeping (for the sailors) and laughing, like a girl who has just fallen in love and is both aroused and perplexed: "She loved us humans just for what we are./How could a god have such appalling taste?" As Auden wrote towards the end of his great poem "September 1, 1939," crazy as we are, "We must love one another or die." Luckily, Kalodendron recognizes this truth at once, and is benevolent. A benign version of Borges' Funes the Memorious, she also remembers all that she experiences "sharp and present,/As if the now were not a fleeting moment/But an accumulating, interleaved/and imbricated whole." She seems to realize that space-time imagined by Major after the death of her mother, on which we can follow the geodesics and literally-figuratively go back home. And Kalodenron concentrates not on the fact that we are "wounded, crippled entities," but rather on the fact that we created Chartres and the Taj Mahal, composed "The Magic Flute" and

"The Tempest," and seems to think our creations redeem us, as indeed she does (Turner 2016: 121–181).

So for a while, things get better, with Kalodendron "a kind of conscience whispering through the net." At the beginning of Book VII, Nemo chronicles those golden years, echoing Ovid's Golden Age evoked in the *Metamorphoses*: those were the days!

Those years the Earth seemed blessed. A bluegold sky
Smiled upon Europe's drowned and battered coasts,
Sea-walls came down, and children could once more
Go out upon the far and rippled sands.
Salt-marshes rang with multitudes of birds,
And rain came back to Portugal and Greece;
In Tuscany these were the vintage years.

China had been drowned, but now it was resurrected, and the inundation has cleared away the air pollution. Real 3D printing is fixing poverty, and crime becomes rarer. However, there is always a serpent in the Garden: the title of Chapter VII is "The Betrayal," and Kalodendron is assassinated at the end of the chapter, ironically by a Catholic ideologue who is part of a new Inquisition. As she cries out in agony, "Worldwide computers crash, the subtle sensors/That guide our science, aircraft, surgery,/Go blank, and music withers on its string." Once again human beings have murdered the god sent down to redeem them (Turner 2016: 183–212).

The slithery nest is "The Oblomovs," who give their name to Chapter VIII, having joined forces with the new Inquisition. As if that weren't bad enough, another disaster threatens, "Wormwood," the title of Chapter IX. Throughout the book, a 'strange place' in the sky has been repeatedly noted, that lies between Serpens and Hercules. "It's not an ordinary bit of matter./It's moving very slightly, side to side,/No angular diameter or disk./Its signal changes, getting always stronger…" But what is it, heading towards earth? It has been named Wormwood after the fatal star in St. John's *Revelations* (Turner 2016: 213–272).

The thing is a primordial black hole,
Forged in the first blank spasm of creation.
Two things explain the increase in the signal:
Wormwood is getting closer every hour,
And it is near the moment of release
When Hawking radiation drains it dry
And it explodes into the real world
Converting all its mass into bright blast.

Apocalypse is a very long, complex book, and it is a page-turner. So there is no point in giving away the ending: you must read it for yourself. However, I can reveal that by a highly mathematical twist of fate or perhaps legerdemain, it turns out that the fate of Kalodendron depends upon Wormwood! As she was expiring, she "blasted out her being and signature,/And Wormwood, as she knew, would pick them up." Lucy exclaims:

What if we collect and piece together
The whole of Kalodendron's engram from
The Wormwood blockchain archive, and download it
To Tripitaka's mainframe qubit set?

Can it be accomplished? What is Tripitaka? (Turner 2016: 273–302). Can the mathematics that we have externalized in our machines and computers be merged with our internal life, where love and fairness and compassion live? Perhaps; perhaps not. It is a thought-experiment worth pursuing, within a story well told.

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Chapter 14
Coda

Two great schemata dominate human discourse, the argument and the narrative; we might have been tempted to identify them separately with mathematics and poetry, but I hope I have persuaded you of the historicity of mathematics and the formal structure of poetry. In any case, deductive argument explains and supports a conclusion, by marshalling evidence in its favor; its purpose is to force us logically to accept a conclusion on the basis of accepted premises. It presents atemporal inferential relations among propositions. By contrast, a narrative tells a story, and takes us from the beginning, through the middle, to the end: a good narrative is both unified but complex, where the unexpected reversals and discoveries in the middle are by the end understood to flow from the conditions of the action. It presents relations among actions that are not only temporal, but historical, and relates them to the modal conditions of what was possible but not actualized. Aristotle in his *Prior Analytics* and in his *Poetics* introduces us to these two schemata for thought, and by his careful methodological taxonomy of human discourse counsels us not to confuse them.

In the *Rhetoric*, however, Aristotle presents a middle term, a type of discourse—persuasive argument—that reinstates the narrative context of argument. The content of a rhetorical argument includes *ethos* and *pathos* as well as *logos*; thus it is made up not of propositions but of speech acts performed by an effective speaker to an attentive and effective audience. A proposition in the *Rhetoric* is an utterance, an act, an assertion by someone to someone that will prove causally potent. By linking his *Poetics* so often and forcefully to the *Nichomachean Ethics*, Aristotle reminds us that argument drives human action, for it is the way we express and explore the moral significance of what we do—before, while and after we do it. Indeed, argument is itself an important form of action. Thus, Aristotle adumbrates the notion of argument by considering the character of the speaker. It is important for the speaker to be publicly known as a person of virtuous character; but just as importantly, the speaker's character must be exhibited by the way he or she argues. A virtuous lawyer, judge or member of parliament does not bully or deceive the audience, does not play on its fears or misrepresent the facts, but rather presents the best available and

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most pertinent evidence, best suited to persuading the given audience (which in turn must exhibit certain virtues of attention and responsible deliberation). Both speaker and audience are typically engaged in rhetorical persuasion as a response to some grave social conflict, a specific historical conflict, which requires both clarification and action. Thus, the *Rhetoric* links narrative and argument by the reinstatement of character as a term of philosophical analysis.

Likewise, the pertinence of the *Nichomachean Ethic*s in the *Poetics* makes character central, for the definition of virtue is the ability to choose the mean reflectively and with pleasure. Good character, Aristotle argues, is formed incrementally over a long period of time by the exercise of good habits. Thus, we cannot infer someone's virtue from the evidence of a single action; the meaning of an act must be understood against the background of the agent's behavior, and interactions with others, over the long term. The exercise of virtue is moreover not the only subject of the *Nichomachean Ethics*, for virtue is subordinate to the end of all human action, happiness. Virtue may be a necessary condition of happiness, but because we are social and natural creatures and thus subject to the vicissitudes of the great world, happiness requires good fortune as well as virtue. The necessities of justice, reward and retribution, which to a certain extent can be calculated, are tempered by the contingencies of human existence. Finally, there is more than one way to be virtuous, for the life of reflection is as important as the life of action. Sophocles depicts the statesman Oedipus and Plato depicts the philosopher Socrates as they reclaim meaning from tragic undoing. These features of character are what allow the poet to construct a plot that is unified yet complex: character is at once apparent and submerged in the actions and speech of the heroes and heroines of Sophoclean drama and Platonic dialogue. Character is a necessary but not sufficient condition of one's fate, which explains but does not necessitate what happens in the end.

Just as logicians pretend, for good reason, that arguments can be considered abstractly, independent of speaker and audience, so structuralists like the Russian formalists (notably Vladimir Propp) pretend that plots may be considered abstractly, independent of the characters who act in them—characters who hover between history and mythology, a shared cultural setting and the poet's own idiosyncratic experience. Thus for the logician and structuralist, argument and narrative become a sort of algebra. If we pay attention to character, however, we can develop a philosophical and literary corrective to this sort of approach. I have been using the word 'character' in a rather free way so far, so now I offer a series of terms that may shed some light on the notion.

Concept	**Personification**	**Personage**	**Persona**	**Person**
Myth	**Allegory**	**Novel**	**History**	**Life**

Characters as they occur in literary works may be ranged along a continuum bounded by living people like you and me on the one hand, and abstract concepts on the other. The figure of John F. Kennedy, for example, painstakingly reconstructed from historical documents, letters and memoirs by Robert Dallek's biography *John*

F. Kennedy: An Unfinished Life, is very close to the right-hand boundary, especially as Kennedy is still a living presence to many of my generation in the United States. In the middle, we find the notion of 'personage,' which has been so insightfully expounded by Martine de Gaudemar in her book *La voix des personnages* (Gaudemar 2011). The figure of Venus in Virgil's *Aeneid* lies close to the left-hand boundary, for the goddess intervenes more as the power of desire in that story than as an individual: she is the concept of love, a drive, an explanans.

To be a living person is, however, to be 'always already' caught up in narrative structure, and to understand our social roles conceptually. To act is to understand what one does in terms of beginning, middle and end: one chooses to do something, carries through the intention and does or does not succeed in the doing. (And either way, the doing or not doing excludes other consciously entertained alternatives, once it is carried out.) Yet there is no single correct narrative of what I do: at any given time, I am following out and fulfilling many intentions, revising my own understanding of them as I pursue, anticipate and remember them. To act in a social setting is also to act according to more or less constraining roles: I know how a professor or mother or friend behaves, and act accordingly. If I violate my social role, it is no less definitive of me: an unprofessional professor is a professor all the same. Yet we all have many social roles, they are not all comfortably consistent, and no sum of those roles exhausts what we are. Moreover, my neighbor's account of what I do and what my character is differs substantially from mine, not least because I am only a side-story in her life.

My point, however, is that there is no "pure experiential agent" hovering on the right side of the diagrammatic array given above. Nor is there anything hovering on the left side: pure concepts do not enter into moral or poetic or mathematical discourse, unconnected to stories, the characters that figure in them, and the arguments those characters offer to evaluate, drive, dismiss, forgive or redeem the actions in which they are caught up. Moral or aesthetic or formal notions make sense only in relation to human action, human action is constituted by narrative, and narrative—like argument—requires characters. Indeed, once we have admitted the indispensability of character to both narrative and argument, we see that the empathy we feel for characters often derives from the *pathos* and *ethos* of the arguments they make. Rationality is not cold; it reunites lovers, reconciles friends, and constitutes communities around the bonfire of grave social conflict. Of course, life is not exhausted by talk: the handprint of reality—the reality of the living person—leaves the mark of birth and death, sex and violence, departure and apparition. Actors on the stage, when they talk, are really talking, though with a special kind of intentionality; but when they pretend to give birth or die, kill or make love, set forth or return from a great journey, they are only pretending. Life is not (only) discourse.

Thus we see that in one and the same play, some characters live near the historical edge of the spectrum I just offered, and some live near the conceptual edge. Most scholars agree that the figure of Prospero in *The Tempest*, is presented, remarkably, as at once historical and conceptual. *The Tempest* is a drama of character: Prospero dominates and indeed encompasses the action. The fantastic creatures in the play, like Calaban and Ariel, as well as the allegorical figures in the Masque, may be understood as aspects of the character of Prospero, and thus as lively concepts; and even the more independent figures in the play act only in reaction to Prospero. Moreover, Prospero speaks from time to time *as* Shakespeare, as a poet, as the creator of his own words, as an agent appealing directly to the audience, and thus as a living presence.

In a commentary on *The Tempest*, Stephen Greenblatt writes, “The Tempest is probably one of the last that Shakespeare wrote. It can be dated fairly precisely: it uses material that was not available until late 1610, and there is a record of a performance before the King on Hallowmas Night, 1611. Since Shakespeare retired soon after to Stratford, *The Tempest* has seemed to many to be his valedictory to the theatre. In this view, Prospero’s strangely anxious and moving epilogue—‘Now my charms are all o’erthrown,/And what strength I have’s mine own’—is the expression of Shakespeare’s own professional leave-taking… the echo-chamber effect is striking, and when Prospero and others speak of his powerful ‘art,’ it is difficult not to associate the skill of the great magician with the skill of the great playwright” (*The Norton Shakespeare* 2008: 1321–1329).

As I noted earlier, this identification seems especially striking in Prospero’s peerless speech at the end of the Masque, the play within the play, which brings the audience as well so close to the characters in the play. Here it is, again.

Our revels now are ended. These our actors,
As I foretold you, were all spirits and
Are melted into air, into thin air;
And like the baseless fabric of this vision,
The cloud-capped towers, the gorgeous palaces,
The solemn temples, the great globe itself,
Yea, all which it inherit, shall dissolve,
And, like this insubstantial pageant faded,
Leave not a rack behind. We are such stuff
As dreams are made on, and our little life
Is rounded with a sleep.

And here is the epilogue (Shakespeare 2008: 1381).

Now my charms are all o’erthrown,
And what strength I have’s mine own,
Which is most faint. Now ‘tis true
I must be here confined by you,
Or sent to Naples. Let me not,
Since I have my dukedom got
And pardoned the deceiver, dwell
In this bare island by your spell,
But release me from my bands
With the help of your good hands.
Gentle breath of yours my sails

Must fill, or else my project fails,
Which was to please. Now I want
Spirits to enforce, art to enchant,
And my ending is despair,
Unless I be relieved by prayer,
Which pierces so that it assaults
Mercy itself and frees all faults.
As you from crimes would pardoned be,
Let your indulgence set me free.

The stage direction reads, *He exits*. And so he does; the speech is a performative utterance, like J. L. Austin's speech act 'I do,' but in reverse: the archaic form of divorce, 'I abjure thee.' Applauding the end of the play, leaving the theatre wrapped in our thoughts, profoundly changed, we acknowledge his departure and our dismissal—which of course allows him to abide as a still insistent voice, a material witness, an effective proposition, an interruption, a disturbance, a song.

And here am I, finishing this baseless fabric, this insubstantial pageant of a book that I've been trying to write for half a century. We have almost come to the end. So it seems fitting to add a couple of poems I wrote only a few months ago, so recently that they weren't included in *The Stars of Earth: New and Selected Poems*, just published by Word Galaxy Press. (Alex Pepple did however publish them in the *Able Muse Review* soon afterwards.) The first plays with Shakespeare's lines that I love so much I have recited them to you twice, combining the vocabulary I learned whilst studying modern cosmology into the interstices of Shakespeare's speech.

Song of the (Ancient) Physics Major

Impulse, the time integral of force,
Measures changes of momentum in
A system: our revels now are ended.
The displacement integral of force,
Called work (melted into air, into thin air),
Measures the system's change of energy.
Systems upon which, as I foretold you,
No external force acts, like the universe,
(Like the baseless fabric of this vision),
Are governed by the two great principles
Of conservation: energy, and momentum.
(Angular momentum is also conserved.)
This reformulation renders Newton's
Ghostly forces, acting at a distance,
Even more spectral. Let them slip from theory,
As they, and we, and the great globe itself,
May, like this insubstantial pageant faded,
Leave not a rack behind.

The second brings back my late poetic mentor, Yves Bonnefoy, who started out in mathematics, saying hello as he appears imagined in my back yard, the closest place I know to the Earthly Paradise.

At the Edge of the Woods

I open the window-door:
There you are at the edge
Where my border of daffodils
Mix with their wild cousins,
Trillium, yellow borage,
Who creep in from the woods
Cross-hatched behind the house.

They come back at the end
Of winter, like the feral cats
We feed and sometimes shelter,
The cardinals and jays
Who fly their colors skyward,
And the striped chipmunks,
"Our plain neighbors," as you wrote.

They spend the winter underground,
Dreaming of the light, until
They rise. And there you are, again,
Just where the garden blends
Into the mild wilderness of trees.
I call your name; you turn,
Lifting your hand, and smile.

So, Reader, *ave atque vale*, hello and farewell. Speaking as the middle term between the concept of me you entertain and the real me who is writing down these words, I hope that my book has offered you a set of middle terms between poetry and mathematics that will enhance your appreciation of them both as the years go by, and reoccur.

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Acknowledgments: Poems

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