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Grégoire Allaire

Shape Optimization by the Homogenization Method

With 54 Illustrations



Springer

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Preface

The topic of this book is homogenization theory and its applications to optimal design in the conductivity and elasticity settings. Its purpose is to give a self-contained account of homogenization theory and explain how it applies to solving optimal design problems, from both a theoretical and a numerical point of view. The application of greatest practical interest targeted by this book is shape and topology optimization in structural design, where this approach is known as the *homogenization method*.

Shape optimization amounts to finding the optimal shape of a domain that, for example, would be of maximal conductivity or rigidity under some specified loading conditions (possibly with a volume or weight constraint). Such a criterion is embodied by an *objective function* and is computed through the solution of a *state equation* that is a partial differential equation (modeling the conductivity or the elasticity of the structure). Apart from those areas where the loads are applied, the shape boundary is always assumed to support Neumann boundary conditions (i.e., isolating or traction-free conditions). In such a setting, shape optimization has a long history and has been studied by many different methods. There is, therefore, a vast literature in this field, and we refer the reader to the following short list of books, and references therein [39], [42], [130], [135], [149], [203], [220], [225], [237], [245], [258]. We shall not consider the case of Dirichlet boundary conditions (for which the homogenization theory is somehow very different), referring the reader only to [62], [63], [64], [267], and references therein. Nor shall we discuss any other types of optimal design problems (like material properties optimization), even in the cases where homogenization could be relevant. We note that optimal design may be seen as a special branch of *optimal control*, where the control is the domain itself or the coefficients in the domain (the material properties). There are also many relevant references in optimal control, and more specifically in controllability, of so-called *distributed systems* (i.e., for which the state equation is a partial differential equation). In addition to the pioneering work of Pontryagin [223], we simply mention the classical books of Lions [164], [165].

In order to shed light on the originality of the homogenization method, we first briefly recall what can be called the *classical method for shape op-*

timization. The main idea goes, at least, as far back as Hadamard [129], and has been further developed by many authors, e.g., [39], [72], [203], [220], [255], [258]. Given an initial guess of the shape, one allows the boundary to vary along its normal (or, more generally, along any smooth velocity field). From both a theoretical and numerical point of view, this approach gives a convenient framework for applying the calculus of variations. Nevertheless, the method suffers from two main drawbacks: It requires a smooth parametrization of the boundary, and it never changes the topology of the original domain (i.e., its number of holes or connected components). In structural design, this is often a serious drawback, since it is widely acknowledged that a porous structure of the same weight as a bulky one may be drastically more efficient. There have been some recent attempts to change the topology in this framework of shape sensitivity (e.g., the bubble method [103], or the topology gradient [68], [257]), but they are either ad hoc methods, or very close in spirit to the homogenization method. Let us mention in passing that topology optimization can also be studied by stochastic approaches, like, for example, genetic algorithms, simulated annealing, or neural networks (e.g., [114], [143]).

The homogenization method was developed in order to remove smoothness and topological constraints inherent in the classical method of shape optimization. Another major difference of the homogenization method is the class of numerical algorithms it yields. As classical numerical methods are *shape-tracking* algorithms (where the shape fits to the mesh, which is deformed during the iterations), the homogenization-based numerical methods are rather *shape-capturing* algorithms (where the shape is captured on a fixed mesh). Before going to numerical issues, we now describe the main features of the homogenization approach.

In a first step, we rephrase a shape optimization problem as a two-phase optimization problem. Thanks to the homogeneous Neumann boundary conditions that they support, the holes in the domain may be filled by a very weak material (a poor conductor or a compliant elastic phase). In the limit where the properties of this ersatz material go to zero, we recover holes with Neumann boundary conditions (note that such a limit process is not always easy to justify rigorously). In other words, shape optimization is just the degenerate limit of two-phase optimal design, where the interface between the two phases replaces the boundary of the shape. A precise definition of two-phase optimization is as follows. In the conductivity setting, given two conducting materials, the goal is to find the best arrangement of the materials in a fixed domain that maximizes, for example, overall conductivity.

Similarly, in the elasticity setting, the goal is to find the best way of mixing two elastic materials yielding the most rigid structure. In both cases, there is a volume constraint on the best component phase, which may be seen as the extra cost on the good conductor or the stiff material (which are expected to be more expensive). Two-phase optimization is also an optimal design problem of interest, and we study it in great detail.

A key feature of two-phase optimal design problems is that they usually do not admit a solution in the absence of any smoothness or topological constraint on the interface or on the shape boundary. The root of this pathology is that it is often more advantageous having many tiny inclusions of one phase into the other than just a few big ones. Therefore, any proposed design can be improved by making small variations of the phases arrangement, and cannot be optimal. This process of making increasingly fine inclusions (or holes) shows that optimality can be achieved only in the limit of infinitely fine scale mixture of the two phases. Such mixtures are called *composite materials* and allow one to define optimal generalized designs. Homogenization is precisely the right mathematical tool for studying composite materials and defining generalized optimal designs.

Therefore, in a second step we introduce generalized designs for which one can prove existence of an optimum and derive necessary conditions of optimality. This process of enlarging the space of admissible designs in order to make the problem well-posed is called *relaxation*. A generalized design is often equivalently called a *relaxed*, or *composite*, design. Homogenization theory is the main ingredient for performing the relaxation of two-phase optimal design problems. As such, homogenization appears as just a trick for proving existence theorems, but its importance goes far beyond that, since it yields new numerical algorithms. Eventually, it remains to justify this relaxation process in the degenerate limit of shape optimization, and to post-process numerically these optimal composite designs to recover classical shapes.

The homogenization method in optimal design was initiated by Murat and Tartar in the late 1970's. Many joined their efforts in the development of that theory, and the first relevant references are the works of Murat and Tartar [205], [206], [269], [271] in France; Cherkaev, Lurie and their coworkers [115], [172], [178], [177], as well as Raitum [227] in Russia; and Kohn and Strang [152] in the United States. Surprisingly enough, the numerical algorithms based on the homogenization method have matured slowly. After the early contributions of Gibiansky and Cherkaev [115], Glowinski [117], Goodman, Kohn, and Reyna [119], and Lavrov, Lurie, and Cherkaev [160],

which were mostly restricted to academic problems, it was the paper of Bendsoe and Kikuchi [47] in the late 1980's, which was the first to demonstrate the efficiency of such methods in shape optimization. After that pioneering contribution, many other works appeared that contributed to make the homogenization method one of the most popular and efficient methods for shape and topology optimization (see Chapter 5 and the References).

Although the main motivation of this book is shape optimization, while homogenization is, to some extent, just a technical tool for implementing optimization methods, we somehow proceed in reverse order. Indeed, the first chapter is concerned solely with homogenization theory. Mathematically, it can be defined as a theory for averaging partial differential equations. Homogenization has many potential applications, but we consider it only as a tool for deriving macroscopic or effective properties of microscopically heterogeneous media. As such it provides a firm basis to the notion of composite material obtained by mixing, on a very fine scale, several phase components. Although this question of averaging and finding effective properties is very old in physics or mechanics, the mathematical theory of homogenization is quite recent, going back to the 1970's. We expose the most general framework, known as the H -convergence, or G -convergence, introduced by Spagnolo [260], [261], and generalized by Tartar and Murat [204], [270].

In the framework of homogenization theory, the second chapter focuses on the study of two-phase composite materials. These composite materials (obtained as fine mixture limits of the two original phases) play a key role in the homogenization method used for shape optimization: Namely, as generalized designs. It is therefore crucial to find the range of all possible effective properties of such composites. This is called the *G-closure problem*, and it was solved in the conductivity setting for two isotropic phases by Murat and Tartar [205], [274], and Lurie and Cherkaev [175], [176]. Unfortunately, a similar answer is yet unknown in the elasticity setting. In this latter case, one can only obtain bounds that must be satisfied by the effective properties. In their most general form such bounds are called *Hashin-Shtrikman energy bounds* since they are derived by using the famous variational principle introduced by Hashin and Shtrikman [133]. They turn out to be optimal, i.e., the values of these bounds are exactly attained by special choices of composites. In particular, optimal composites can be chosen in the class of so-called *sequential laminates*, which will play an important part in shape optimization.

The third chapter is devoted to the application of homogenization to problems of two-phase optimization in conductivity. After introducing a

precise mathematical framework, we give explicit examples of the nonexistence of optimal designs in the original space of “classical” designs. It is therefore necessary to relax the problem by enlarging the space of admissible designs, i.e., by allowing for generalized designs that are made up of composite materials. The key tool of this relaxation process is homogenization theory. As an additional advantage, the relaxed formulation allows one to derive optimality conditions that are at the root of numerical algorithms. Our exposition follows the original work of Murat and Tartar [205].

The fourth chapter is also concerned with the homogenization method for optimal design, but in the elasticity setting. Since the G -closure set of two elastic isotropic phases is still unknown, a rigorous relaxation procedure is available only for special objective functions including compliance (i.e., the work done by the loads). In such a case, the optimization problem being self-adjoint, the relaxation requires only the knowledge of optimal energy bounds instead of the full G -closure. Those bounds (called *Hashin-Shtrikman bounds*) are precisely studied in Chapter 2, where they are shown to be attained by sequential laminates. Therefore, in the relaxation process the unknown full G -closure set can be replaced by its explicit subset of sequential laminates. Eventually, a complete relaxation procedure for shape optimization (i.e., when one of the phases degenerates to holes) is rigorously established. Here we follow mainly our work [15], [21], that of Gibiansky and Cherkaev [115], [116], and that of Kohn and Strang [152].

Finally, the fifth chapter is devoted to numerical issues for the homogenization method in optimal design. We mostly focus on shape optimization for elastic structures, since it is by far the most important application of the homogenization method. Building upon our knowledge of the relaxed, or homogenized, formulation, we discuss the two main types of algorithms, the optimality criteria and gradient methods, as well as several numerical technicalities. In particular, we explain how a numerical procedure, called *penalization*, allow one to post-process the optimal generalized designs in order to recover classical shapes. This chapter is illustrated by several results of two-dimensional and three-dimensional computations. Our exposition is complementary to that of Bendsoe in his book [42], which is a very good introduction to the numerical aspects of the homogenization method for a more practically inclined reader.

In the sequel, we do not point out open problems, but it is clear that the range of applications covered by this book, although very important, is somehow narrow. Of course, there are many other types of optimal design problems, apart from shape optimization, that have not yet been attacked

with the homogenization method (or that cannot be treated by this method). It is our hope that this book can serve as a basis for further developments in new directions. The references are not exhaustive, by any means, and so I apologize to those inadvertently left off of the list.

To conclude this introduction, I want to thank all my coworkers, with whom I learned so much about shape optimization: S. Aubry, Z. Belhachmi, E. Bonnetier, G. Francfort, F. Jouve, R. Kohn, V. Lods, and F. Murat. I benefited from many stimulating discussions with G. Milton and L. Tartar, that I acknowledge with much pleasure. Special thanks are due to F. Jouve, who was the corner stone of the development of a three-dimensional shape optimization code, and to G. Francfort for his comments on an earlier version of the manuscript. Last but not least, I owe a lot to F. Murat, who had so much influence on my understanding of homogenization, and to R. Kohn, who introduced me to optimal design.

*Paris, France
May 2001*

GRÉGOIRE ALLAIRE

Notation

N The space dimension (typically $N = 2$ or 3).

$\mathcal{M}_{p \times N}$ The set of real matrices with p rows and N columns.

\mathcal{M}_N The set of real squared matrices of order N .

\mathcal{M}_N^s The set of real symmetric squared matrices of order N .

\mathcal{M}_N^4 The set of symmetric fourth order tensors acting on symmetric matrices.

I_2 Identity matrix in \mathcal{M}_N .

I_4 Identity tensor in \mathcal{M}_N^4 .

· Scalar product between two vectors: For $a = (a_i)_{1 \leq i \leq N}$ and $b = (b_i)_{1 \leq i \leq N}$ in \mathbb{R}^N , $a \cdot b = \sum_{i=1}^N a_i b_i$.

: Full contraction of two matrices: For $A = (a_{ij})_{1 \leq i,j \leq N}$ and $B = (b_{ij})_{1 \leq i,j \leq N}$ in \mathcal{M}_N , $A : B = \sum_{i,j=1}^N a_{ij} b_{ij}$.

:: Full contraction of fourth order tensors: For $A = (a_{ijkl})_{1 \leq i,j,k,l \leq N}$ and $B = (b_{ijkl})_{1 \leq i,j,k,l \leq N}$ in \mathcal{M}_N^4 , $A :: B = \sum_{i,j,k,l=1}^N a_{ijkl} b_{ijkl}$.

\otimes Tensor product of two vectors or two matrices: If $a = (a_i)_{1 \leq i \leq N}$ and $b = (b_i)_{1 \leq i \leq N}$ are vectors in \mathbb{R}^N , then $a \otimes b$ is the matrix of entries $(a_i b_j)_{1 \leq i,j \leq N}$, while if $A = (a_{ij})_{1 \leq i,j \leq N}$ and $B = (b_{ij})_{1 \leq i,j \leq N}$ are matrices in \mathcal{M}_N , then $A \otimes B$ is the fourth order tensor of entries $(a_{ij} b_{kl})_{1 \leq i,j,k,l \leq N}$.

\odot Symmetrized tensor product of two vectors: If $a = (a_i)_{1 \leq i \leq N}$ and $b = (b_i)_{1 \leq i \leq N}$ are vectors in \mathbb{R}^N , then $a \odot b$ is the symmetric matrix of entries $\left(\frac{1}{2}(a_i b_j + a_j b_i)\right)_{1 \leq i,j \leq N}$.

$\{0, 1\}$ The couple of values 0 and 1.

$[0, 1]$ The closed interval of all values between 0 and 1.

Ω An open set of \mathbb{R}^N .

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Chapter 1

Homogenization

This chapter is a self-contained introduction to the mathematical theory of homogenization. We present all the necessary notions and results of homogenization that will be required for applications in structural optimization. The most general theory in homogenization is that of H -convergence, which was introduced by Spagnolo under the name of G -convergence [260], [261], and further generalized by Tartar [269] and Murat and Tartar [204] as H -convergence. Since it plays a key role in the sequel, it is exposed in great mathematical detail in the context of linear second order elliptic equations (like the conductivity equation or the elasticity system). However, for the less mathematically inclined reader, the first section provides a simpler presentation of what is known as *periodic homogenization*. This first section is sufficient for those who are ready to accept all the mathematical results of the other sections of this chapter. Let us emphasize that we will not discuss all aspects of H -convergence. In particular, we shall not touch its application to hyperbolic, parabolic, or nonlinear partial differential equations. Let us also remark that H -convergence is not the only theory in homogenization, although it is, possibly, the most general. There are also a stochastic or probabilistic theory of homogenization (e.g., [118], [154], [215]), and a variational theory of homogenization, known as the Γ -convergence, due to De Giorgi [93], [94], [95] (see also the books [28], [57], [90]). However, H -convergence is especially well-adapted to applications of homogenization to optimal design problems.

The physical ideas of homogenization, i.e., averaging heterogeneous media in order to derive effective properties, have a very long history going back at least to Maxwell [183], Poisson [221] and Rayleigh [233]. There is a

huge literature in physics and mechanics concerning averaging and effective properties (to quote a few, see [1], [81], [132], [136], [207], [244] and references therein). However, the mathematical theory of homogenization is much younger. Although our goal is definitely not to give an history of homogenization, it is interesting to note that it started in at least three directions. The first direction (and the oldest one) is concerned with a general theory for the convergence of operators (namely the H - or G -convergence). The first contributions are probably those of Spagnolo [260], [261], De Giorgi-Spagnolo [96], Murat and Tartar [204], and Tartar [269], [271], [270]. Further developments were made by Simon [256], Zhikov, Kozlov, and Oleinik [294], [295]. The second direction is the asymptotic study of perforated domains containing many small holes (see, e.g., the early contributions of Marcenko and Khruslov [182], and Rauch and Taylor [232]). The third direction is a systematic study of periodic structures by means of an asymptotic analysis. We shall call such problems “periodic homogenization problems”. Pioneering works in this direction have been done by Babuska [34], [35], Bakhvalov and Panasenko [36], Bensoussan, Lions, and Papanicolaou [50], Keller [148], [158], [159], and Sanchez-Palencia [243]. We remark in passing that the term “homogenization” was not used by the first contributors in this field. Rather, it probably originated from applications to neutron transport and diffusion, where the word was in common use (e.g., [34], [158] and references therein). Then it was adopted for denoting asymptotic analysis in periodic structures, and later on it denoted the whole theory including all contributions cited above. Since then, the mathematical theory of homogenization and its numerous applications have been developed by so many contributors that it is impossible to quote them all. Similarly, we shall not try to tackle the history of the main ideas in homogenization. For simplicity we refer to recent monographs to give an idea of the huge literature in that field (see, e.g., the books [90], [137], [138] and their impressive lists of references).

1.1 Introduction to Periodic Homogenization

Although homogenization is not restricted to periodic problems, it has been used often for the asymptotic analysis of periodic structures. Indeed, in many fields of science and technology one has to solve boundary value problems in periodic media. Quite often the size of the period is small compared to the size of a sample of the medium, and, denoting by ϵ their ratio, an asymptotic analysis, as ϵ goes to zero, allows one to reduce the complexity of

the problem. Starting from a microscopic description of a problem, one seeks a macroscopic, or effective, model. This process of making an asymptotic analysis in order to derive an averaged formulation is precisely the goal of homogenization.

This section is devoted to an elementary introduction to periodic homogenization without any mathematical justification. The interested reader is referred to the classical books [36], [50], [85], [137], [138], [162], [243] for further details. In Section 1.3.4, the reader will also find a rigorous justification of periodic homogenization through H -convergence.

1.1.1 A Model Problem in Conductivity

We consider a model problem of thermal or electrical conductivity in a periodic medium (for example, an heterogeneous domain obtained by mixing periodically two different phases, one being the matrix and the other the inclusions; see Figure 1.1). To fix ideas, the periodic domain is called Ω (a bounded open set in \mathbb{R}^N with $N \geq 1$ the space dimension), its period ϵ (a positive number which is assumed to be very small in comparison with the size of the domain), and the rescaled unit periodic cell $Y = (0, 1)^N$. The conductivity in Ω is not constant, but varies periodically with period ϵ in each direction. It is a matrix (a second order tensor) $A(y)$, where $y = x/\epsilon \in Y$ is the fast periodic variable, while $x \in \Omega$ is the slow variable. Equivalently, x is also called the macroscopic variable, and y the microscopic variable. Since the component conductors do not need to be isotropic, the matrix A can be any second order tensor that is positive definite, i.e., there exists a positive constant $\alpha > 0$ such that, for any vector $\xi \in \mathbb{R}^N$ and at any point $y \in Y$,

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^N A_{ij}(y)\xi_i\xi_j.$$

At this point, the matrix A is not necessarily symmetric (such is the case when some drift is taken into account in the diffusion process). The matrix $A(y)$ is a periodic function of y , with period Y , and it may be discontinuous in y (to model the discontinuity of conductivities from one phase to the other).

Denoting by $f(x)$ the source term (a scalar function defined in Ω), and enforcing a Dirichlet boundary condition (for simplicity), our model problem

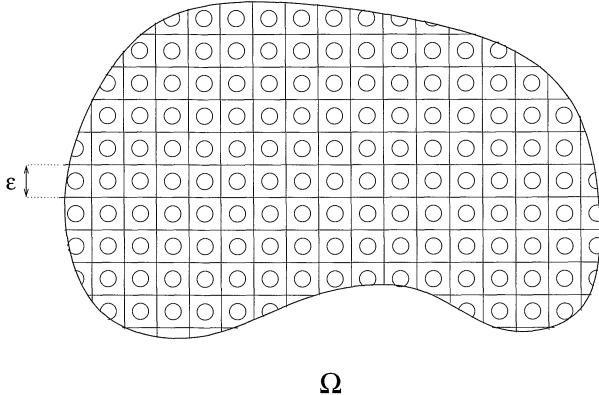


Figure 1.1: A periodic domain.

of conductivity reads

$$\begin{cases} -\operatorname{div}\left(A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon\right) = f & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $u_\epsilon(x)$ is the unknown function, modeling the electrical potential or the temperature.

The domain Ω , with its conductivity $A\left(\frac{x}{\epsilon}\right)$, is highly heterogeneous with periodic heterogeneities of lengthscale ϵ . Usually one does not need the full details of the variations of the potential or temperature u_ϵ , but rather some global or averaged behavior of the domain Ω considered as an homogeneous domain. In other words, an effective or equivalent macroscopic conductivity of Ω is sought. From a numerical point of view, solving equation (1.1) by any method will require too much effort if ϵ is small since the number of elements (or degrees of freedom) for a fixed level of accuracy grows like $1/\epsilon^N$. It is thus preferable to average or homogenize the properties of Ω and compute an approximation of u_ϵ on a coarse mesh. Averaging the solution of (1.1) and finding the effective properties of the domain Ω is what we call homogenization.

There is a difference of methodology between the traditional physical approach of homogenization and the mathematical theory of homogenization. In the mechanical literature, the so-called representative volume element (RVE) method is often used [81], [132], [207] (see also Chapter 1 in

[137]). Roughly speaking, it consists in taking a sample of the heterogeneous medium of size much larger than the heterogeneities, but still much smaller than the medium, and averaging over it the gradient ∇u_ϵ and the flux $A(\frac{x}{\epsilon})\nabla u_\epsilon$. Denoting by ξ the average of the gradient and by σ that of the flux, the effective tensor of conductivity A^* of this sample is defined by the linear relationship $\sigma = A^*\xi$. It turns out that the averaged stored energy $A(\frac{x}{\epsilon})\nabla u_\epsilon \cdot \nabla u_\epsilon$ is also equal to the effective energy $A^*\xi \cdot \xi$. Although this type of definition is very intuitive, it is not clear whether it defines correctly an effective tensor A^* . In particular, it may depend on the choice of source term f , sample size, or boundary conditions.

The mathematical theory of homogenization works completely differently. Rather than considering a single heterogeneous medium with a fixed lengthscale, the problem is first embedded in a sequence of similar problems for which the lengthscale ϵ , becoming increasingly small, goes to zero. Then, an asymptotic analysis is performed as ϵ tends to zero, and the conductivity tensor of the limit problem is said to be the *effective* or *homogenized* conductivity. This seemingly more complex approach has the advantage of defining uniquely the homogenized properties. Further, the approximation made by using effective properties instead of the true microscopic coefficients can be rigorously justified by quantifying the resulting error.

In the case of a periodic medium Ω , this asymptotic analysis of equation (1.1), as the period ϵ goes to zero, is especially simple. The solution u_ϵ is written as a power series in ϵ

$$u_\epsilon = \sum_{i=0}^{+\infty} \epsilon^i u_i.$$

The first term u_0 of this series will be identified with the solution of the so-called homogenized equation whose effective conductivity A^* can be exactly computed. It turns out that A^* is a constant tensor, describing a homogeneous medium, which is independent of f and of the boundary conditions. Therefore, numerical computations on the homogenized equation do not require a fine mesh since the heterogeneities of size ϵ have been averaged out. This homogenized tensor A^* is almost never a usual average (arithmetic or harmonic) of $A(y)$. Various estimates will confirm this asymptotic analysis by telling in which sense u_ϵ is close to u_0 as ϵ tends to zero.

Remark 1.1.1 *From a more theoretical point of view, homogenization can be interpreted as follows. Rather than studying a single problem (1.1) for*

the physically relevant value of ϵ , we consider a sequence of such problems indexed by the period ϵ , which is now regarded as a small parameter going to zero. The question is to find the limit of this sequence of problems. The notion of limit problem is defined by considering the convergence of the sequence $(u_\epsilon)_{\epsilon>0}$ of solutions of (1.1): Denoting by u its limit, the limit problem is defined as the problem for which u is a solution. Of course, u will turn out to coincide with u_0 , the first term in the series defined above, and it is therefore the solution of the homogenized equation. Section 1.2 is devoted to this approach. Clearly the mathematical difficulty is to define an adequate topology for this notion of convergence of problems as ϵ goes to zero.

1.1.2 Two-scale Asymptotic Expansions

The method of two-scale asymptotic expansions is an heuristic method, which allows one to formally homogenize a great variety of models or equations posed in a periodic domain. We present it briefly and refer to the classical books [36], [50], and [243] for more detail. A mathematical justification of what follows is to be found in Subsection 1.3.4. As already stated, the starting point is to consider the following *ansatz*, or *two-scale asymptotic expansion*, for the solution u_ϵ of equation (1.1):

$$u_\epsilon(x) = \sum_{i=0}^{+\infty} \epsilon^i u_i(x, \frac{x}{\epsilon}), \quad (1.2)$$

where each term $u_i(x, y)$ is a function of both variables x and y , periodic in y with period $Y = (0, 1)^N$ (u_i is called a Y -periodic function with respect to y). This series is plugged into the equation, and the following derivation rule is used:

$$\nabla \left(u_i(x, \frac{x}{\epsilon}) \right) = \left(\epsilon^{-1} \nabla_y u_i + \nabla_x u_i \right) \left(x, \frac{x}{\epsilon} \right), \quad (1.3)$$

where ∇_x and ∇_y denote the partial derivative with respect to the first and second variable of $u_i(x, y)$. For example, one has

$$\nabla u_\epsilon(x) = \epsilon^{-1} \nabla_y u_0 \left(x, \frac{x}{\epsilon} \right) + \sum_{i=0}^{+\infty} \epsilon^i (\nabla_y u_{i+1} + \nabla_x u_i) \left(x, \frac{x}{\epsilon} \right).$$

Equation (1.1) becomes a series in ϵ

$$\begin{aligned}
& -\epsilon^{-2} [\operatorname{div}_y A \nabla_y u_0] (x, \frac{x}{\epsilon}) \\
& -\epsilon^{-1} [\operatorname{div}_y A (\nabla_x u_0 + \nabla_y u_1) + \operatorname{div}_x A \nabla_y u_0] (x, \frac{x}{\epsilon}) \\
& -\epsilon^0 [\operatorname{div}_x A (\nabla_x u_0 + \nabla_y u_1) + \operatorname{div}_y A (\nabla_x u_1 + \nabla_y u_2)] (x, \frac{x}{\epsilon}) \\
& - \sum_{i=1}^{+\infty} \epsilon^i [\operatorname{div}_x A (\nabla_x u_i + \nabla_y u_{i+1}) + \operatorname{div}_y A (\nabla_x u_{i+1} + \nabla_y u_{i+2})] (x, \frac{x}{\epsilon}) \\
& = f(x).
\end{aligned} \tag{1.4}$$

Identifying each coefficient of (1.4) as an individual equation yields a cascade of equations (a series of the variable ϵ is zero for all values of ϵ if each coefficient is zero). It turns out that the three first equations are enough for our purpose. The ϵ^{-2} equation is

$$-\operatorname{div}_y A(y) \nabla_y u_0(x, y) = 0,$$

which is nothing else than an equation in the unit cell Y with periodic boundary condition. In this equation, y is the variable, and x plays the role of a parameter. It can be checked (see Lemma 1.3.21 in Subsection 1.3.4) that there exists a unique solution of this equation up to a constant (i.e., a function of x independent of y since x is just a parameter). This implies that u_0 is a function that does not depend on y , i.e., there exists a function $u(x)$ such that

$$u_0(x, y) \equiv u(x).$$

Since $\nabla_y u_0 = 0$, the ϵ^{-1} equation is

$$-\operatorname{div}_y A(y) \nabla_y u_1(x, y) = \operatorname{div}_y A(y) \nabla_x u(x), \tag{1.5}$$

which is an equation for the unknown u_1 in the periodic unit cell Y . Again, it is a well-posed problem, which admits a unique solution up to a constant, as soon as the right hand side is known. Equation (1.5) allows one to compute u_1 in terms of u , and it is easily seen that $u_1(x, y)$ depends linearly on the first derivative $\nabla_x u(x)$.

Finally, the ϵ^0 equation is

$$\begin{aligned} -\operatorname{div}_y A(y) \nabla_y u_2(x, y) &= \operatorname{div}_y A(y) \nabla_x u_1 \\ &\quad + \operatorname{div}_x A(y) (\nabla_y u_1 + \nabla_x u) + f(x), \end{aligned} \quad (1.6)$$

which is an equation for the unknown u_2 in the periodic unit cell Y . Equation (1.6) admits a solution if a compatibility condition is satisfied (the so-called *Fredholm alternative*; see Lemma 1.3.21). Indeed, integrating the left hand side of (1.6) over Y , and using the periodic boundary condition for u_2 , we obtain

$$\int_Y \operatorname{div}_y A(y) \nabla_y u_2(x, y) dy = \int_{\partial Y} [A(y) \nabla_y u_2(x, y)] \cdot n ds = 0,$$

which implies that the right hand side of (1.6) must have zero average over Y , i.e.,

$$\int_Y [\operatorname{div}_y A(y) \nabla_x u_1 + \operatorname{div}_x A(y) (\nabla_y u_1 + \nabla_x u) + f(x)] dy = 0,$$

which simplifies to

$$-\operatorname{div}_x \left(\int_Y A(y) (\nabla_y u_1 + \nabla_x u) dy \right) = f(x) \quad \text{in } \Omega. \quad (1.7)$$

Since $u_1(x, y)$ depends linearly on $\nabla_x u(x)$, equation (1.7) is simply an equation for $u(x)$ involving only the second order derivatives of u .

In order to compute u_1 and to simplify (1.7), we introduce the so-called *cell problems*. We denote by $(e_i)_{1 \leq i \leq N}$ the canonical basis of \mathbb{R}^N . For each unit vector e_i , consider the following conductivity problem in the periodic unit cell:

$$\begin{cases} -\operatorname{div}_y A(y) (e_i + \nabla_y w_i(y)) = 0 & \text{in } Y \\ y \rightarrow w_i(y) & \text{Y-periodic,} \end{cases} \quad (1.8)$$

where $w_i(y)$ is the local variation of potential or temperature created by an averaged (or macroscopic) gradient e_i . By linearity, it is not difficult to compute $u_1(x, y)$, solution of (1.5), in terms of $u(x)$ and $w_i(y)$

$$u_1(x, y) = \sum_{i=1}^N \frac{\partial u}{\partial x_i}(x) w_i(y). \quad (1.9)$$

In truth, $u_1(x, y)$ is merely defined up to the addition of a function $\tilde{u}_1(x)$ (depending only on x), but this does not matter since only its gradient

$\nabla_y u_1(x, y)$ is used in the homogenized equation. Inserting this expression in equation (1.7), we obtain the homogenized equation for u that we supplement with a Dirichlet boundary condition on $\partial\Omega$,

$$\begin{cases} -\operatorname{div}_x A^* \nabla_x u(x) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.10)$$

The homogenized conductivity A^* is defined by its entries

$$A_{ij}^* = \int_Y [(A(y) \nabla_y w_i) \cdot e_j + A_{ij}(y)] dy,$$

or equivalently, after a simple integration by parts in Y ,

$$A_{ij}^* = \int_Y A(y) (e_i + \nabla_y w_i) \cdot (e_j + \nabla_y w_j) dy. \quad (1.11)$$

The constant tensor A^* describes the effective or homogenized properties of the heterogeneous material $A(\frac{x}{\epsilon})$. Note that A^* does not depend on the choice of domain Ω , source term f , or boundary condition on $\partial\Omega$.

Remark 1.1.2 *This method of two-scale asymptotic expansions is unfortunately not rigorous from a mathematical point of view. In other words, it yields heuristically the homogenized equation, but it does not yield a correct proof of the homogenization process. The reason is that the ansatz (1.2) is usually not correct after the two first terms. For example, it does not include possible boundary layers in the vicinity of $\partial\Omega$ (for details, see, e.g., [165]). Nevertheless, it is possible to rigorously justify the above homogenization process (see Subsection 1.3.4).*

1.1.3 Variational Characterizations and Estimates of the Effective Tensor

The homogenized conductivity A^* is defined in terms of the solutions of the cell problems by equation (1.11). When the conductivity tensor $A(y)$ is symmetric, it is convenient to give another definition of A^* involving standard variational principles. From now on we assume that $A(y)$ is indeed symmetric. Therefore, by (1.11), A^* is symmetric too, and is completely determined by the knowledge of the quadratic form $A^* \xi \cdot \xi$ where ξ is any constant vector in \mathbb{R}^N . From definition (1.11) it is not difficult to check that

$$A^* \xi \cdot \xi = \int_Y A(y) (\xi + \nabla_y w_\xi) \cdot (\xi + \nabla_y w_\xi) dy, \quad (1.12)$$

where w_ξ is the solution of the following cell problem:

$$\begin{cases} -\operatorname{div}_y A(y) (\xi + \nabla_y w_\xi(y)) = 0 & \text{in } Y, \\ y \rightarrow w_\xi(y) & \text{Y-periodic.} \end{cases} \quad (1.13)$$

It is well-known that equation (1.13) is the Euler-Lagrange equation of the following variational principle: Find $w(y)$ that minimizes

$$\int_Y A(y) (\xi + \nabla_y w) \cdot (\xi + \nabla_y w) dy$$

over all periodic functions w . In other words, $A^* \xi \cdot \xi$ is given by the minimization of the potential energy

$$A^* \xi \cdot \xi = \min_{w(y) \in H_\#^1(Y)} \int_Y A(y) (\xi + \nabla_y w) \cdot (\xi + \nabla_y w) dy, \quad (1.14)$$

where $H_\#^1(Y)$ is the space of Y -periodic functions w with finite energy, namely,

$$\int_Y (w^2 + |\nabla_y w|^2) dy < +\infty,$$

which is precisely defined in (1.92) as a Sobolev space.

Remark that all the above equivalent definitions of A^* are not simple algebraic formulas, but rather they deliver the value of A^* at the price of a non-explicit computation of the solutions of the cell problems. However, in practice one is not always interested in the precise value of A^* , but rather in lower or upper estimates of its value. In this respect, the variational characterization (1.14) of A^* is useful since it provides an upper bound by choosing a specific test function $w(y)$. The simplest choice is to take $w(y) = 0$, which yields the so-called *arithmetic mean upper bound*

$$A^* \xi \cdot \xi \leq \left(\int_Y A(y) dy \right) \xi \cdot \xi. \quad (1.15)$$

A lower bound can also be obtained from (1.14) if the space of admissible fields in the minimization is enlarged. Indeed, remarking that the gradient $\nabla_y w(y)$ has zero-average over Y because of the periodicity of $w(y)$, this gradient can be replaced by any zero-average vector field

$$A^* \xi \cdot \xi \geq \min_{\substack{\zeta(y) \in L_\#^2(Y)^N \\ \int_Y \zeta(y) dy = 0}} \int_Y A(y) (\xi + \zeta(y)) \cdot (\xi + \zeta(y)) dy, \quad (1.16)$$

where $L^2_\#(Y)$ is the space of square summable Y -periodic functions (see (1.91) for a precise definition of this Lebesgue space). The minimum in the right hand side of (1.16) is easy to compute: The optimal vector $\zeta_\xi(y)$ satisfies the following Euler-Lagrange equation

$$A(y)(\xi + \zeta_\xi(y)) = C,$$

where C is a constant (a Lagrange multiplier for the constraint $\int_Y \zeta_\xi(y) dy = 0$). After some algebra, one can compute explicitly the optimal ζ_ξ , as well as the minimal value that delivers the so-called *harmonic mean lower bound*

$$A^* \xi \cdot \xi \geq \left(\int_Y A^{-1}(y) dy \right)^{-1} \xi \cdot \xi. \quad (1.17)$$

The bounds (1.15) and (1.17) are sometimes called the *Voigt and Reuss bounds* [235], [286]. From a physical point of view, the harmonic mean in (1.17) corresponds to an overall conductivity obtained by assuming that the values of the conductivity $A(y)$ are placed in series, while the arithmetic mean in (1.15) corresponds to an overall conductivity obtained by assuming that the values of the conductivity $A(y)$ are placed in parallel. These estimates hold true in great generality (see Theorem 1.3.14), but usually are not optimal and can be improved (see Chapter 2 in the case of two-phase composites). Actually, improving the harmonic and arithmetic mean bounds is one of the main problems of homogenization theory applied to the modeling of composite materials. It turns out to also have many applications to optimal design problems.

1.1.4 Generalization to the Elasticity System

The asymptotic analysis presented above is not restricted to the conductivity case, but it works for any kind of second order elliptic systems, and in particular for the linearized elasticity system. For the sake of completeness, we briefly indicate the main steps of the asymptotic analysis in this context.

The bounded domain Ω in \mathbb{R}^N is now occupied by a linearly elastic periodic medium with period ϵ (a positive number which is assumed to be very small in comparison with the size of the domain). This models, for example, a composite material obtained by including stiff fibers in a matrix of a more compliant phase. The rescaled unit periodicity cell is denoted by $Y = (0, 1)^N$. The material properties in Ω are represented by a periodic fourth order tensor $A(y)$ (that we shall call *Hooke's law*), with $y = x/\epsilon \in Y$

and $x \in \Omega$. If the Hooke's law A is isotropic, with positive bulk and shear moduli κ and μ , we have

$$A(y) = (\kappa(y) - \frac{2\mu(y)}{N})I_2 \otimes I_2 + 2\mu(y)I_4,$$

but A can be any fourth order tensor satisfying the usual symmetries of linear elasticity, $A_{ijkl} = A_{klji} = A_{jikl} = A_{ijlk}$ for $1 \leq i, j, k, l \leq N$, and which is also positive definite, i.e. there exists a positive constant $\alpha > 0$ such that, for any symmetric matrix ξ with entries ξ_{ij} and at any point $y \in Y$,

$$\alpha|\xi|^2 \leq A(y)\xi : \xi = \sum_{i,j,k,l=1}^N A_{ijkl}(y)\xi_{ij}\xi_{kl}.$$

Denoting by $f(x)$ the external load (or force, a vector function defined in Ω), and enforcing a Dirichlet boundary condition (for simplicity), the model problem of elasticity reads

$$\begin{cases} -\operatorname{div} \left(A\left(\frac{x}{\epsilon}\right) e(u_\epsilon) \right) = f & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.18)$$

where u_ϵ is the unknown displacement (a vector in \mathbb{R}^N), and $e(u_\epsilon)$ is the strain tensor $(\nabla u_\epsilon + (\nabla u_\epsilon)^t)/2$ (a symmetric matrix of order N).

In order to find the homogenized or averaged properties of the heterogeneous domain Ω , an asymptotic analysis of equation (1.18) is performed as the period ϵ goes to zero. The displacement u_ϵ is written as a power series in ϵ

$$u_\epsilon(x) = \sum_{i=0}^{+\infty} \epsilon^i u_i(x, \frac{x}{\epsilon}),$$

where each term $u_i(x, y)$ is a Y -periodic function of y . This series is inserted in equation (1.18), and using the derivation rule (1.3) it yields a cascade of equations when each coefficient in the resulting ϵ series is identified to zero. We denote by $e_y(u_i)(x, y)$ (respectively $e_x(u_i)(x, y)$) the deformation tensor with respect to the microscopic y variable (respectively macroscopic x variable). The ϵ^{-2} equation is

$$-\operatorname{div}_y (A(y)e_y(u_0)) = 0,$$

which is an equation in the unit cell Y with periodic boundary condition. By uniqueness of the solution (up to a translation, see Lemma 1.3.21), it implies that u_0 is a function that depends only on x , namely,

$$u_0(x, y) \equiv u(x).$$

The ϵ^{-1} equation is

$$-\operatorname{div}_y (A(y)e_y(u_1)) = \operatorname{div}_y (A(y)e_x(u)), \quad (1.19)$$

which is an equation for the unknown u_1 in the periodic unit cell Y . Equation (1.19) allows one to compute $u_1(x, y)$ which depends linearly on the strain tensor $e_x(u)$. Denoting by $(e_i)_{1 \leq i \leq N}$ the canonical basis of \mathbb{R}^N , we define a basis of the space of symmetric matrices

$$e_{ij} = \frac{1}{2} (e_i \otimes e_j + e_j \otimes e_i).$$

For each matrix e_{ij} , the cell problem is

$$\begin{cases} -\operatorname{div}_y (A(y)(e_{ij} + e_y(w_{ij}(y)))) = 0 & \text{in } Y \\ y \rightarrow w_{ij}(y) & \text{Y-periodic,} \end{cases} \quad (1.20)$$

where $w_{ij}(y)$ is the displacement created by a mean deformation equal to e_{ij} . By linearity, $u_1(x, y)$ is given by

$$u_1(x, y) = \sum_{i=1}^N \sum_{j=1}^N \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j}(x) + \frac{\partial u_j}{\partial x_i}(x) \right) w_{ij}(y).$$

Finally, the ϵ^0 equation is

$$\begin{aligned} -\operatorname{div}_y (A(y)e_y(u_2)) &= \operatorname{div}_y (A(y)e_x(u_1)) \\ &\quad + \operatorname{div}_x (A(y)(e_y(u_1) + e_x(u))) + f(x), \end{aligned} \quad (1.21)$$

which has a unique solution u_2 (up to a constant displacement) if and only if the compatibility condition (the Fredholm alternative; see Lemma 1.3.21) of zero-average right hand side is satisfied. This condition reads

$$-\operatorname{div}_x \left(\int_Y A(y)(e_y(u_1) + e_x(u)) dy \right) = f(x) \quad \text{in } \Omega.$$

Inserting the value of u_1 in this expression, we obtain the homogenized equation for u , which we supplement with a Dirichlet boundary condition on $\partial\Omega$

$$\begin{cases} -\operatorname{div}_x(A^*e_x(u(x))) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The homogenized Hooke's law A^* is defined by its entries

$$A_{ijkl}^* = \int_Y [(A(y)e_y(w_{ij}))_{kl} + A_{ijkl}(y)] dy,$$

or equivalently, after a simple integration by parts in Y ,

$$A_{ijkl}^* = \int_Y A(y) (e_{ij} + e_y(w_{ij})) : (e_{kl} + e_y(w_{kl})) dy. \quad (1.22)$$

The constant tensor A^* describes the effective or homogenized properties of the heterogeneous material $A(\frac{x}{\epsilon})$, and it does not depend on the choice of domain Ω , force f , or boundary condition on $\partial\Omega$.

As in Subsection 1.1.3, the symmetric Hooke's law A^* is completely determined by the knowledge of the quadratic form $A^*\xi : \xi$ where ξ is any symmetric matrix. A standard variational principle shows that $A^*\xi : \xi$ is equal to the minimum of the elastic energy

$$A^*\xi : \xi = \min_{w(y) \in H_{\#}^1(Y)^N} \int_Y A(y) (\xi + e_y(w)) : (\xi + e_y(w)) dy, \quad (1.23)$$

where $H_{\#}^1(Y)^N$ is the space of Y -periodic displacements with finite energy. The variational characterization (1.23) of A^* is useful since it provides upper and lower estimates by choosing specific test functions $w(y)$. The simple choice $w(y) = 0$ yields the so-called *arithmetic mean upper bound*

$$A^*\xi : \xi \leq \left(\int_Y A(y) dy \right) \xi : \xi. \quad (1.24)$$

Noting that the deformation tensor $e_y(w)$ has zero-average over Y because of the periodicity of $w(y)$, the space of admissible fields in (1.23) can be enlarged to obtain a lower bound

$$A^*\xi : \xi \geq \min_{\substack{\zeta(y) \in L_{\#}^2(Y; \mathcal{M}_N^s) \\ \int_Y \zeta(y) dy = 0}} \int_Y A(y) (\xi + \zeta(y)) : (\xi + \zeta(y)) dy. \quad (1.25)$$

The minimum in the right hand side of (1.25) is easily computed and yields the so-called *harmonic mean lower bound*

$$A^* \xi : \xi \geq \left(\int_Y A^{-1}(y) dy \right)^{-1} \xi : \xi. \quad (1.26)$$

The bounds (1.24) and (1.26) are called the *Paul bounds* [216]. As in the conductivity case, they hold true in great generality (see Theorem 1.3.14), but are usually not optimal and can be improved (see Chapter 2 in the case of two-phase composites).

1.2 Definition of H -convergence

This section is devoted to the general theory of homogenization without any special assumptions on the geometry of the problem (like periodicity or randomness). The most general theory is that of H -convergence, also called G -convergence by a slight misuse of language. Indeed, G -convergence, strictly speaking, is restricted to symmetric operators, while H -convergence is its generalization to possibly unsymmetric operators. G -convergence was introduced by Spagnolo [260], [261], and De Giorgi and Spagnolo [96]. H -convergence is due to Tartar [270], and Murat and Tartar [204]. The latter is sometimes referred to as *strong G -convergence* by Russian authors (see [214], [294], [295], although the book [138] reverts back to the usual terminology G -convergence). A key feature of the H -convergence theory is a systematic use of weak convergence in Sobolev spaces. For the reader unfamiliar with this notion, we recall its main results in the first subsection below.

1.2.1 Some Results on Weak Convergence

In this subsection some classical results on strong and weak convergences in Sobolev spaces are recalled. More details on this topic, including proofs of all statements below, may be found in many textbooks such as [5], [58], [292]. Our choice of presented results is somewhat arbitrary, but it is dictated by the results' application to homogenization theory.

Let Ω be an open set in \mathbb{R}^N . For $1 \leq p \leq +\infty$, the Lebesgue space $L^p(\Omega)$ of all measurable functions u in Ω with finite defined by

$$\|u\|_{L^p(\Omega)} = \left(\int_{\Omega} |u(x)|^p dx \right)^{\frac{1}{p}} \text{ for } 1 \leq p < +\infty$$

and

$$\|u\|_{L^\infty(\Omega)} = \text{ess sup}_{x \in \Omega} |u(x)|,$$

is a Banach space (i.e., a complete normed space). Two types of convergence coexist in this space: the strong and the weak convergence. We shall denote by $(u_\epsilon)_{\epsilon>0}$ a sequence of functions indexed by a sequence of real parameters ϵ going to zero. As usual, the strong convergence is associated to the above norm, i.e., a sequence u_ϵ is said to converge strongly in $L^p(\Omega)$ to a limit u if $\lim_{\epsilon \rightarrow 0} \|u_\epsilon - u\|_{L^p(\Omega)} = 0$. The strong convergence is denoted by an arrow, namely

$$u_\epsilon \rightarrow u \text{ in } L^p(\Omega) \text{ strongly.}$$

On the other hand, the weak convergence in $L^p(\Omega)$ is associated to test functions in its dual space $L^{p'}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. More precisely, for $1 \leq p < +\infty$, a sequence u_ϵ is said to converge weakly in $L^p(\Omega)$ to a limit u if, for any test function $\phi \in L^{p'}(\Omega)$, it satisfies

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} u_\epsilon(x) \phi(x) dx = \int_{\Omega} u(x) \phi(x) dx. \quad (1.27)$$

As usual, the weak convergence is denoted by a harpoon, namely

$$u_\epsilon \rightharpoonup u \text{ in } L^p(\Omega) \text{ weakly.}$$

For $p = +\infty$, there is a subtlety since the dual of $L^\infty(\Omega)$ is not $L^1(\Omega)$. In this latter case, one talks about “weak * convergence”, i.e., a sequence $(u_\epsilon)_{\epsilon>0}$ is said to converge weakly * in $L^\infty(\Omega)$ to a limit u if, for any test function $\phi \in L^1(\Omega)$, it satisfies (1.27). The main interest of the weak convergence is that it is sequentially relatively compact on bounded sets (which is not the case for the strong convergence).

Lemma 1.2.1 *Let $(u_\epsilon)_{\epsilon>0}$ be a bounded sequence in $L^p(\Omega)$ with $1 < p \leq +\infty$, i.e. there exists a positive constant C , which does not depend on ϵ , such that*

$$\|u_\epsilon\|_{L^p(\Omega)} \leq C.$$

*Then, there exist a subsequence $(u_{\epsilon'})_{\epsilon'>0}$ and a limit u such that $(u_{\epsilon'})_{\epsilon'>0}$ converges weakly to u in $L^p(\Omega)$ if $1 < p < +\infty$, or converges weakly * if $p = +\infty$.*

Note that the limit value $p = 1$ is excluded: Indeed, Lemma 1.2.1 is not true for $p = 1$. Strong convergence implies weak convergence but the converse

is false. A simple counterexample is given by the sequence $u_\epsilon(x) = \sin(\frac{x_1}{\epsilon})$ where x_1 is the first coordinate of x . It is easily seen that u_ϵ converges weakly to zero in $L^2(\Omega)$, while $\lim_{\epsilon \rightarrow 0} \|u_\epsilon\|_{L^2(\Omega)}^2 = |\Omega|/2$. Therefore, u_ϵ does not converge strongly to zero. Physically speaking, the strong convergence is more or less the usual pointwise convergence, while the weak convergence is a notion of convergence “in average” (up to fluctuations of zero-mean). The mathematical theory of homogenization uses in an essential way this concept of weak convergence. In this context, taking the weak limit of a sequence can be interpreted as a rule of spatial averaging.

The connection between strong convergence and pointwise convergence is given in the two next lemmas (which are false for weakly converging sequences).

Lemma 1.2.2 *Let Ω be a bounded open set in \mathbb{R}^N . Let u_ϵ be a sequence converging strongly to a limit u in $L^p(\Omega)$, $1 \leq p \leq +\infty$. Then there exist a subsequence, still denoted by ϵ , and a function $h(x) \in L^p(\Omega)$ such that, for this subsequence,*

$$\begin{aligned} u_\epsilon(x) &\rightarrow u(x) \quad \text{almost everywhere in } \Omega \\ |u_\epsilon(x)| &\leq h(x) \quad \text{almost everywhere in } \Omega. \end{aligned}$$

Lemma 1.2.3 *Let Ω be a bounded open set in \mathbb{R}^N . For $1 < p \leq +\infty$, let u_ϵ be a bounded sequence in $L^p(\Omega)$ such that*

$$u_\epsilon(x) \rightarrow u(x) \quad \text{almost everywhere in } \Omega.$$

Then the sequence u_ϵ converges strongly to u in any $L^q(\Omega)$ with $1 \leq q < p$.

As a consequence, the product of two strongly converging sequences does also converge strongly (in a different Lebesgue space) to the product of the two limits. Unfortunately, the same is no longer true for the product of two weakly converging sequences, which has no reason to converge weakly to the product of the two weak limits. A simple counterexample is given again by the sequence $u_\epsilon(x) = \sin(\frac{x_1}{\epsilon})$, which converges weakly to zero in any $L^p(\Omega)$ with $1 \leq p < \infty$, while the product $(u_\epsilon)^2$ converges weakly to $1/2$ in any $L^p(\Omega)$. More generally, if u_ϵ is a sequence converging weakly in $L^p(\Omega)$ to a limit u , and if f is a nonlinear function, then the sequence $f(u_\epsilon)$ usually does not converge weakly (or in any sense) to $f(u)$. This is the main difficulty with weak convergence (for more details, see [272]). Nevertheless, it is possible to pass to the limit in the product of one strongly and one weakly converging sequence.

Lemma 1.2.4 *Let Ω be an open set in \mathbb{R}^N . Let u_ϵ be a sequence converging strongly to a limit u in $L^p(\Omega)$, $1 \leq p \leq +\infty$. Let v_ϵ be a sequence converging weakly to a limit v in $L^{p'}(\Omega)$ (weakly * if $p' = +\infty$), with $\frac{1}{p} + \frac{1}{p'} = 1$. Then the product $u_\epsilon v_\epsilon$ converges to uv in the sense of distributions.*

Thanks to interpolation inequalities, a strong convergence and a weak convergence can be “interpolated” to obtain another strong convergence in an intermediate Lebesgue space.

Lemma 1.2.5 *Let Ω be an open set in \mathbb{R}^N . For $1 \leq p \leq q \leq +\infty$, let u belong to $L^p(\Omega) \cap L^q(\Omega)$. Then, for any $p \leq r \leq q$, $u \in L^r(\Omega)$ and*

$$\|u\|_{L^r(\Omega)} \leq \|u\|_{L^p(\Omega)}^\alpha \|u\|_{L^q(\Omega)}^{1-\alpha} \text{ with } \alpha = \frac{\frac{q}{r}-1}{\frac{q}{p}-1}.$$

Consequently, any sequence u_ϵ bounded in $L^p(\Omega)$ and strongly convergent in $L^q(\Omega)$ also converges strongly in $L^r(\Omega)$ with $q \leq r < p$.

Let us denote by $W^{1,p}(\Omega)$ the Sobolev space of all functions in $L^p(\Omega)$ whose distributional derivative is also in $L^p(\Omega)$, componentwise, for $1 \leq p \leq +\infty$. As for Lebesgue spaces, a strong and a weak convergence can be introduced on Sobolev spaces. Thanks to the celebrated Rellich theorem, it turns out that weak convergence in $W^{1,p}(\Omega)$ implies strong convergence in $L^p(\Omega)$.

Lemma 1.2.6 (Rellich theorem) *Let Ω be a bounded open set in \mathbb{R}^N . Let u_ϵ be a bounded sequence in $W^{1,p}(\Omega)$, $1 \leq p < +\infty$. Then, there exists a subsequence, still denoted by ϵ , and a limit $u \in W^{1,p}(\Omega)$, such that, for this subsequence, u_ϵ converges strongly to u in $L^p(\Omega)$.*

Functions in Sobolev spaces have higher integrability properties, which are summarized in the following Sobolev inequalities (also called *Sobolev embedding properties*).

Lemma 1.2.7 *Let Ω be a bounded open set in \mathbb{R}^N . Let u be a function in $W^{1,p}(\Omega)$, $1 \leq p \leq +\infty$. For any q satisfying $1 \leq q \leq \frac{pN}{N-p}$ if $1 \leq p < N$, or $1 \leq q < +\infty$ if $N \leq p$, there exists a positive constant $C > 0$, depending only on Ω , p and q , such that*

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}.$$

Very often, convergence properties are established for a subsequence of the initial sequence. Therefore, a natural question to ask is When does convergence hold true for the entire sequence? The next lemma furnishes a very simple criterion for answering this question.

Lemma 1.2.8 *Let E be a metric space. Let u_ϵ be a sequence in a fixed compact set of E . If all converging subsequences of u_ϵ converge to the same limit, then the entire sequence converges to this unique limit.*

1.2.2 Problem Statement

For simplicity, we consider the case of a scalar second order elliptic equation, modeling, for example, thermal or electric conduction. However, all the results of this section hold true for any kind of second order elliptic systems (as elasticity; see Section 1.4), and we never use special properties restricted to the scalar equation (like the maximum principle).

Let \mathcal{M}_N be the linear space of square real matrices of order N . For any two positive constants $\alpha > 0$ and $\beta > 0$, we define a subspace of \mathcal{M}_N made of coercive matrices with coercive inverses

$$\mathcal{M}_{\alpha,\beta} = \left\{ M \in \mathcal{M}_N \text{ such that } \begin{array}{l} M\xi \cdot \xi \geq \alpha|\xi|^2 \\ M^{-1}\xi \cdot \xi \geq \beta|\xi|^2 \end{array} \forall \xi \in \mathbb{R}^N \right\}. \quad (1.28)$$

A coercive matrix with coercive inverse is also bounded. Indeed, if $M \in \mathcal{M}_{\alpha,\beta}$, introducing $\eta = M^{-1}\xi$, we deduce from definition (1.28)

$$\beta|M\eta|^2 \leq M\eta \cdot \eta.$$

Applying the Cauchy-Schwarz inequality, we obtain

$$|M\eta| \leq \beta^{-1}|\eta| \quad \forall \eta \in \mathbb{R}^N. \quad (1.29)$$

Similarly, we have

$$|M^{-1}\eta| \leq \alpha^{-1}|\eta| \quad \forall \eta \in \mathbb{R}^N. \quad (1.30)$$

Remark 1.2.9 *From (1.28) and (1.29) we deduce that a necessary condition for a matrix M to belong to $\mathcal{M}_{\alpha,\beta}$ is that $\alpha|\xi|^2 \leq M\xi \cdot \xi \leq \beta^{-1}|\xi|^2$ for any vector ξ . Therefore, the set $\mathcal{M}_{\alpha,\beta}$ is nonempty if and only if the positive constants α and β satisfy $\alpha\beta \leq 1$. In the sequel, we shall always assume that such a condition is satisfied.*

Let Ω be a bounded open set in \mathbb{R}^N . We define a space of admissible coefficients on Ω as $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$. Denoting by $\epsilon > 0$ a sequence of positive reals going to zero, we consider a sequence of matrices $A^\epsilon(x) \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$. For a given source term $f(x) \in H^{-1}(\Omega)$, we consider the following scalar second order elliptic equation with a Dirichlet boundary condition

$$\begin{cases} -\operatorname{div} A^\epsilon(x) \nabla u_\epsilon(x) = f(x) & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.31)$$

Remark 1.2.10 *The parameter ϵ has no specific physical role, but it may be interpreted as a lengthscale or a measure of disorder of the heterogeneities. We emphasize that there is no explicit or implicit assumption on the spatial distribution of A^ϵ . In particular, it is neither assumed to be periodic, random, or ergodic.*

Equation (1.31) admits the following variational formulation, or weak form:

$$\int_{\Omega} A^\epsilon \nabla u_\epsilon \cdot \nabla \phi \, dx = \langle f, \phi \rangle_{H^{-1}, H_0^1(\Omega)} \quad \forall \phi \in H_0^1(\Omega), \quad (1.32)$$

where $\langle \cdot, \cdot \rangle_{H^{-1}, H_0^1(\Omega)}$ is the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Of course, if f is smoother, say $f \in L^2(\Omega)$, then the duality product $\langle f, \phi \rangle_{H^{-1}, H_0^1(\Omega)}$ coincides with the usual integral $\int_{\Omega} f(x) \phi(x) \, dx$. Since A^ϵ is coercive and bounded, by application of the Lax-Milgram lemma (see, e.g., [58], [292]), (1.31) and (1.32) are easily seen to have a unique weak solution $u_\epsilon \in H_0^1(\Omega)$. Replacing ϕ by u_ϵ in the variational formulation (1.32), the norm of the solution u_ϵ can be estimated a priori by using the coercivity of A^ϵ ,

$$\begin{aligned} \alpha \|\nabla u_\epsilon\|_{L^2(\Omega)^N}^2 &\leq \int_{\Omega} A^\epsilon \nabla u_\epsilon \cdot \nabla u_\epsilon \, dx \\ &= \langle f, u_\epsilon \rangle_{H^{-1}, H_0^1(\Omega)} \leq \|f\|_{H^{-1}(\Omega)} \|u_\epsilon\|_{H_0^1(\Omega)}. \end{aligned}$$

Since Ω is a bounded domain, the Poincaré inequality holds (see, e.g., [5], [58], [292]); i.e., there exists a constant C , which depends only on Ω , such that

$$\|\phi\|_{L^2(\Omega)} \leq C \|\nabla \phi\|_{L^2(\Omega)^N} \quad \forall \phi \in H_0^1(\Omega).$$

This implies that $\|\nabla \phi\|_{L^2(\Omega)^N}$ is a norm on $H_0^1(\Omega)$ equivalent to the usual norm $\|\phi\|_{H_0^1(\Omega)}$. Therefore, there exists a positive constant C , which depends

only on Ω , such that

$$\|\nabla u_\epsilon\|_{L^2(\Omega)^N} \leq \frac{C}{\alpha} \|f\|_{H^{-1}(\Omega)}. \quad (1.33)$$

The a priori estimate (1.33) implies that the sequence of solutions u_ϵ is bounded in $H_0^1(\Omega)$ independently of ϵ . By the relative compactness of bounded sets for the weak topology of $H_0^1(\Omega)$ (see Lemma 1.2.1), there exist a subsequence, still denoted by ϵ , and a limit $u \in H_0^1(\Omega)$ such that u_ϵ converges weakly to u in $H_0^1(\Omega)$.

Introducing the flux $\sigma_\epsilon = A^\epsilon \nabla u_\epsilon$, we deduce from the bound (1.29) on A^ϵ and from (1.33) that the sequence σ_ϵ is also bounded in $L^2(\Omega)^N$. By the relative compactness of bounded sets for the weak topology of $L^2(\Omega)^N$, there exist another subsequence, still denoted by ϵ , and a limit $\sigma \in L^2(\Omega)^N$ such that σ_ϵ converges weakly to σ in $L^2(\Omega)^N$. Furthermore, by equation (1.31) σ_ϵ satisfies

$$-\operatorname{div}\sigma_\epsilon = f \quad \text{in } \Omega.$$

Passing to the limit in this equation, we deduce that $-\operatorname{div}\sigma = f$ in Ω . Summarizing these results, up to a subsequence we have

$$\begin{cases} u_\epsilon \rightharpoonup u \text{ weakly in } H_0^1(\Omega) \\ \sigma_\epsilon \rightharpoonup \sigma \text{ weakly in } L^2(\Omega)^N \\ -\operatorname{div}\sigma = f \text{ in } \Omega. \end{cases} \quad (1.34)$$

The main questions are now: What is the relationship between σ and u ? What equation is satisfied by u ? How do the coefficients of this limit equation relate to A^ϵ ? Do the results depend on the domain Ω , on the boundary condition, on the source term f , or on the choice of the subsequence of ϵ ? The goal of H -convergence is precisely to answer all these questions. In other words, the H -convergence theory will establish that weak convergence is precisely the right mathematical tool for describing, rigorously, averaging properties of partial differential equations.

Remark 1.2.11 *As already explained in Subsection 1.1.1 there is a subtle difference between the intuitive physical approach of homogenization and the mathematical theory of H -convergence. While the former amounts to averaging locally in a representative volume element (or RVE; for details see, e.g., [1], [81], [132], [137], [207]) equation (1.31) for a fixed and given value of ϵ , the latter entails considering a sequence of equations indexed by ϵ . Embedding a single physically meaningful problem in a sequence of problems may seem like a complexification of the study, but such an asymptotic*

analysis turns out to deliver a sound and rigorous notion of homogenized properties. Indeed, the “homogenized”, or limit, problem of a sequence of problems is defined as the problem for which the limit of the sequence of solutions is itself a solution. In this sense homogenization is just a notion of convergence for the solution operator of equation (1.31). Finally, the approximation made by using the homogenized problem instead of the original one can be rigorously justified by quantifying the resulting error. For further comment on the physical and mechanical aspects of homogenization in the context of two-phase composite materials, we refer the reader to Section 2.1 in Chapter 2.

To gain some insight into homogenization, it is useful to try first a naive idea for linking u and σ . Since the coefficient matrix A^ϵ belongs to $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$, it is a bounded sequence in $L^\infty(\Omega; \mathcal{M}_N)$. Therefore, extracting another subsequence, still denoted by ϵ , there exists a limit matrix \bar{A} such that A^ϵ converges weakly * to \bar{A} . In view of (1.34), it is tempting to pass to the weak limit in the equality

$$\sigma_\epsilon = A^\epsilon \nabla u_\epsilon, \quad (1.35)$$

since all terms in (1.35) converge weakly. However, the right hand side of (1.35) involves the product of A^ϵ by ∇u_ϵ , and it is not true in general that the product of two weakly converging sequences converges to the product of the weak limits (see the previous subsection). Therefore, the limit of $A^\epsilon \nabla u_\epsilon$ (for any subsequence) is usually not $\bar{A} \nabla u$, and σ is different from $\bar{A} \nabla u$. As we shall see, a more subtle relationship holds between σ and ∇u : We shall prove that there exists another matrix A^* such that $\sigma = A^* \nabla u$ (A^* will be called the homogenized or effective tensor). In general A^* is different from \bar{A} , except if the sequence A^ϵ is known to converge strongly, and not merely weakly * in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ (which is not true in general). In case of strong convergence, it is easy to pass to the limit in the product $A_\epsilon \nabla u_\epsilon$ to obtain $\sigma = \bar{A} \nabla u$ (see Lemma 1.2.22 below).

1.2.3 The One-dimensional Case

We begin by investigating the one-dimensional case $N = 1$, which is easier to handle than higher dimensional cases. The domain is $\Omega = (0, L)$ with $0 < L < +\infty$. Equation (1.31) reduces to

$$\begin{cases} -\frac{d}{dx} \left(A^\epsilon(x) \frac{d}{dx} u_\epsilon(x) \right) = f(x) & \text{for } 0 < x < L \\ u_\epsilon(0) = u_\epsilon(L) = 0. \end{cases} \quad (1.36)$$

From the a priori estimate (1.33), we know that the sequence u_ϵ is bounded in $H_0^1(\Omega)$, which implies that, up to a subsequence still denoted by ϵ , there exists a limit $u \in H_0^1(\Omega)$ such that u_ϵ converges weakly to u in $H_0^1(\Omega)$. Since $f \in H^{-1}(\Omega)$, there exists $g \in L^2(\Omega)$ such that $f = \frac{dg}{dx}$. Thus, from (1.36) we deduce

$$\sigma_\epsilon(x) = A^\epsilon(x) \frac{d}{dx} u_\epsilon(x) = C_\epsilon - g(x),$$

where C_ϵ is a constant, which is bounded in \mathbb{R} since σ_ϵ is a bounded sequence in $L^2(\Omega)$. Extracting a subsequence, still denoted by ϵ , there exists a constant limit C such that C_ϵ converges to C . Therefore, σ_ϵ converges strongly (and not merely weakly) in $L^2(\Omega)$ to a limit $\sigma = C - g$. In the one-dimensional case $N = 1$, A^ϵ is just a positive scalar and $\frac{d}{dx} u_\epsilon$ is given by

$$\frac{d}{dx} u_\epsilon(x) = \frac{\sigma_\epsilon(x)}{A^\epsilon(x)}. \quad (1.37)$$

By definition of $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$, $(A^\epsilon)^{-1}$ is a bounded sequence in $L^\infty(\Omega)$. Extracting another subsequence, still denoted by ϵ , there exists a limit \underline{A} such that $A^{\epsilon-1}$ converges weakly * to \underline{A} in $L^\infty(\Omega)$. Therefore, the right hand side of (1.37) involves the product of a weakly and a strongly convergent sequence, and we can pass to the limit in (1.37) by application of Lemma 1.2.4, which gives

$$\frac{d}{dx} u(x) = \frac{\sigma(x)}{\underline{A}(x)}.$$

Recalling that $-\frac{d}{dx} \sigma = f$, we obtain the equation satisfied by u

$$\begin{cases} -\frac{d}{dx} \left(\underline{A}(x) \frac{d}{dx} u(x) \right) = f(x) & \text{for } 0 < x < L \\ u(0) = u(L) = 0. \end{cases} \quad (1.38)$$

Equation (1.38) is called the *homogenized equation*. This equation is easily seen to admit a unique weak solution in $H_0^1(\Omega)$ since \underline{A} still belongs to $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ and is thus coercive. By uniqueness of the solution of (1.38), if the entire sequence $(A^\epsilon)^{-1}$ converges to the same limit \underline{A} , then all possible weakly converging subsequences of u_ϵ converge to the same limit u . Thus, the entire sequence u_ϵ converges to u without any need to extract subsequences (see Lemma 1.2.8). The coefficient \underline{A} is the *homogenized, or effective, coefficient*. In mathematical terms, it is called the H -limit of the sequence A^ϵ .

Remark 1.2.12 *The homogenized coefficient \underline{A} is called the harmonic mean of the sequence A^ϵ . This form of the homogenized coefficient is always the same, whatever the geometric arrangement of the sequence A^ϵ . This situation of a unique “formula” for the homogenized coefficient is peculiar to the one-dimensional case. It does not hold true in higher dimensions for which things are much more complicated. Indeed, in dimensions $N \geq 2$ the value of the homogenized matrix of coefficients depends on the micro-geometry of A^ϵ and is not always equal to its harmonic mean. Even worse, there is no explicit formula that gives its value in terms of A^ϵ . Indeed, it is a difficult and critical problem to find the possible range of the homogenized matrix given that of the sequence A^ϵ (the so-called G-closure problem).*

It is worth noticing that the harmonic mean \underline{A} (i.e., the inverse of the weak limit of $A^{\epsilon-1}$) has nothing to do, in general (i.e., when A^ϵ converges merely weakly), with the arithmetic mean \overline{A} (i.e., the usual weak limit of A^ϵ).

Lemma 1.2.13 *The harmonic and arithmetic mean of the sequence A^ϵ coincide, i.e., $\underline{A} = \overline{A}$, if and only if A^ϵ converges almost everywhere in Ω , or converges strongly in $L^p(\Omega)$ for some p , $1 \leq p \leq +\infty$.*

Proof. Since A^ϵ belongs to $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$,

$$\alpha^{-1} \int_{\Omega} (A^\epsilon - \overline{A})^2 dx \geq \int_{\Omega} A^{\epsilon-1} (A^\epsilon - \overline{A})^2 dx \geq \beta \int_{\Omega} (A^\epsilon - \overline{A})^2 dx.$$

On the other hand,

$$A^{\epsilon-1} (A^\epsilon - \overline{A})^2 = A^\epsilon - 2\overline{A} + \frac{\overline{A}^2}{A^\epsilon}.$$

Passing to the limit for a subsequence such that $A^{\epsilon-1}$ and A^ϵ converge weakly * in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ to \underline{A}^{-1} and \overline{A} , respectively, yields

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} A^{\epsilon-1} (A^\epsilon - \overline{A})^2 dx = \int_{\Omega} \frac{\overline{A}}{\underline{A}} (\overline{A} - \underline{A}) dx.$$

Thus, $\underline{A} = \overline{A}$ if and only the sequence A^ϵ converges strongly in $L^2(\Omega)$. Since A^ϵ is bounded in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$, by Lemma 1.2.5 its strong convergence in $L^2(\Omega)$ or in $L^p(\Omega)$, $1 \leq p < +\infty$, are equivalent. Eventually, by Lemma 1.2.3 the same result holds true if A^ϵ converges almost everywhere. \square

Remark 1.2.14 The key ingredient in the one-dimensional setting is that the gradient and divergence operators coincide, and therefore the equation $-\operatorname{div}\sigma_\epsilon = f$ implies that $\nabla\sigma_\epsilon$ is a constant sequence in $H^{-1}(\Omega)$; i.e., σ_ϵ converges in $L^2(\Omega)$ not merely weakly but strongly. This does not happen in higher dimensions where the divergence operator does not coincide with the gradient operator.

1.2.4 Main Results

This subsection is devoted to the statement of the main results of the theory of H -convergence. We follow the original exposition of Murat and Tartar [204].

Let Ω be a bounded open set in \mathbb{R}^N . Let $\alpha > 0$ and $\beta > 0$ be two positive constants. Recall definition (1.28) of the space $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ with

$$\mathcal{M}_{\alpha,\beta} = \left\{ M \in \mathcal{M}_N \text{ such that } \begin{array}{l} M\xi \cdot \xi \geq \alpha|\xi|^2 \\ M^{-1}\xi \cdot \xi \geq \beta|\xi|^2 \end{array} \forall \xi \in \mathbb{R}^N \right\}.$$

H -convergence is a notion of convergence for the coefficients of an elliptic partial differential equation, which is defined through some convergence properties of the solution of this equation.

Definition 1.2.15 A sequence of matrices $A^\epsilon(x)$ in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ is said to converge in the sense of homogenization, or simply to H -converge, to an homogenized limit, or H -limit, matrix $A^*(x) \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ if, for any right hand side $f \in H^{-1}(\Omega)$, the sequence u_ϵ of solutions of

$$\begin{cases} -\operatorname{div}A^\epsilon(x)\nabla u_\epsilon(x) = f(x) & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.39)$$

satisfies

$$\begin{cases} u_\epsilon \rightharpoonup u \text{ weakly in } H_0^1(\Omega) \\ A^\epsilon\nabla u_\epsilon \rightharpoonup A^*\nabla u \text{ weakly in } L^2(\Omega)^N, \end{cases} \quad (1.40)$$

where u is the solution of the homogenized equation

$$\begin{cases} -\operatorname{div}A^*(x)\nabla u(x) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.41)$$

This definition makes sense because of the following sequential compactness theorem.

Theorem 1.2.16 *For any sequence $A^\epsilon(x)$ of matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ there exist a subsequence, still denoted by A^ϵ , and an homogenized matrix $A^*(x) \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ such that A^ϵ H -converges to A^* .*

Theorem 1.2.16 is typically a *compactness* result since it proves the existence of an H -limit for a subsequence of a bounded sequence, but it delivers no explicit formula for this limit. This is in sharp contrast to periodic homogenization (see Section 1.1) where the entire sequence is converging to an H -limit defined by an explicit formula (up to solving the cell problems).

Remark 1.2.17 *H -convergence is indirectly defined by the convergence of the problems' solutions and their fluxes. Therefore, it has, a priori, no link with any usual notion of convergence (weak or strong) of the coefficients.*

1. *By definition, the homogenized matrix A^* does not depend on the source term f .*
2. *Theorem 1.2.16 implies that the set $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ is stable (or closed) under H -convergence. In other words, the coercivity constants α, β are the same for the sequence A^ϵ and its H -limit A^* . However, this is not true for the essential upper bound γ defined as the smallest constant such that $|A^\epsilon \xi| \leq \gamma |\xi|$ for any $\xi \in \mathbb{R}^N$. Indeed, it may be possible that $|A^* \xi| \leq \frac{\gamma \beta}{\alpha} |\xi|$, i.e., the essential upper bound has increased of a factor of β/α . This happens only if the matrices A^ϵ are nonsymmetric (see Remark 1.3.12 in the symmetric case).*
3. *It is necessary to extract a subsequence in Theorem 1.2.16 to avoid the possibility of a sequence with different accumulation points. For example, consider the sequence A^ϵ equal to I_2 if $\epsilon = (2n)^{-1}$, and to $2I_2$ if $\epsilon = (2n + 1)^{-1}$; in such a case, both odd and even subsequences of A^ϵ obviously H -converge to a different homogenized limit.*
4. *By definition, if a sequence A^ϵ H -converges to a limit A^* , then any subsequence also H -converges to A^* .*
5. *Since $L^2(\Omega)$ is dense in $H^{-1}(\Omega)$, Definition 1.2.15 is unchanged if the right hand sides f belong to $L^2(\Omega)$ instead of $H^{-1}(\Omega)$.*

H -convergence is “local”, as stated in the next result.

Proposition 1.2.18 *Let $A^\epsilon(x)$ and $B^\epsilon(x)$ be two sequences of matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$, which H -converge to $A^*(x)$ and $B^*(x)$, respectively. Let ω be an open subset compactly embedded in Ω , i.e., $\bar{\omega} \subset \Omega$. If $A^\epsilon(x) = B^\epsilon(x)$ in ω , then $A^*(x) = B^*(x)$ in ω .*

Proposition 1.2.18 has two important consequences. First, it implies that the H -limit of a sequence is unique, i.e., a sequence cannot converge to two different limits (this fact is also a consequence of the constructive proof of Theorem 1.2.16). Second, the value of the homogenized matrix A^* in a region ω does not depend on the values of the sequence A^ϵ outside of this region, which is precisely what we mean by *locality*.

H -convergence is indifferent with respect to the choice of boundary conditions, as stated in the next result.

Proposition 1.2.19 *Let $A^\epsilon(x)$ be a sequence of matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ that H -converges to $A^*(x)$. For any sequence z_ϵ such that*

$$\begin{cases} -\operatorname{div} A^\epsilon \nabla z_\epsilon = f_\epsilon \rightarrow f \text{ strongly in } H_{loc}^{-1}(\Omega) \\ z_\epsilon \rightharpoonup z \text{ weakly in } H_{loc}^1(\Omega), \end{cases} \quad (1.42)$$

A^ϵ satisfies

$$A^\epsilon \nabla z_\epsilon \rightarrow A^* \nabla z \text{ weakly in } L_{loc}^2(\Omega)^N.$$

By Proposition 1.2.19, the convergence result (1.40) in the definition of H -convergence is still valid for a sequence such as z_ϵ . The difference between u_ϵ and z_ϵ is that the latter does not satisfy any precise boundary conditions on $\partial\Omega$, and it is a solution of an equation with a varying right hand side f_ϵ . It implies that the H -convergence can be applied whatever the boundary conditions are (and not merely Dirichlet boundary conditions, as in Definition 1.2.15).

H -convergence implies an energy convergence, as stated in the next result.

Proposition 1.2.20 *Let $A^\epsilon(x)$ be a sequence of matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ that H -converges to $A^*(x)$. For any right hand side $f \in H^{-1}(\Omega)$, the sequence u_ϵ of solutions of*

$$\begin{cases} -\operatorname{div} A^\epsilon(x) \nabla u_\epsilon(x) = f(x) & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies

$$A^\epsilon \nabla u_\epsilon \cdot \nabla u_\epsilon \rightharpoonup A^* \nabla u \cdot \nabla u \text{ in } \mathcal{D}'(\Omega) \quad (1.43)$$

and

$$\int_{\Omega} A^\epsilon \nabla u_\epsilon \cdot \nabla u_\epsilon dx \rightarrow \int_{\Omega} A^* \nabla u \cdot \nabla u dx, \quad (1.44)$$

where u is the weak limit of u_ϵ in $H_0^1(\Omega)$, i.e., the solution of the homogenized equation

$$\begin{cases} -\operatorname{div} A^*(x) \nabla u(x) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Remark 1.2.21 Equation (1.43) is a convergence in the sense of distributions. This means that for any smooth test function ϕ with compact support in Ω (i.e., $\phi \in \mathcal{D}(\Omega)$) we have

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} \phi A^\epsilon \nabla u_\epsilon \cdot \nabla u_\epsilon = \int_{\Omega} \phi A^* \nabla u \cdot \nabla u.$$

Proposition 1.2.20 contains two different results. First, (1.43) is a local convergence of the energy density. It turns out that such a result is also valid for a sequence z_ϵ defined as in Proposition 1.2.19. Second, (1.44) expresses the convergence of the total energy, i.e., the density of energy integrated over the whole domain. This last result holds only in the case of a fixed boundary condition (like the Dirichlet homogeneous one satisfied by u_ϵ). Proposition 1.2.20 can be slightly improved by using Meyers theorem (see Theorem 1.3.41): the convergence (1.43) for the energy density holds true in $L^1(\Omega)$ strongly if the right hand side f belongs to some $W^{-1,p}(\Omega)$ with $p > 2$ (see Remark 1.3.7).

As already said, if the sequence A^ϵ converges strongly, then its H -limit is simply its strong limit, as stated in the next lemma.

Lemma 1.2.22 Let $A^\epsilon(x)$ be a sequence of matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ that either converges strongly to a limit matrix $A^*(x)$ in $L^1(\Omega; \mathcal{M}_N)$, or converges to $A^*(x)$ almost everywhere in Ω . Then, $A^\epsilon(x)$ also H -converges to $A^*(x)$.

Proof. We remark first that the two possible assumptions on the sequence A^ϵ imply that it converges strongly to A^* in any space $L^p(\Omega; \mathcal{M}_N)$ for $1 \leq p < +\infty$. Indeed, the sequence A^ϵ belongs to $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ and is therefore bounded in $L^\infty(\Omega; \mathcal{M}_N)$. Thus, by Lemma 1.2.3 or 1.2.5 it converges strongly to A^* in $L^p(\Omega; \mathcal{M}_N)$ with $1 \leq p < +\infty$. Now, recalling the

notation of the previous subsection, we can pass to the limit in the main relationship

$$\sigma_\epsilon = A^\epsilon \nabla u_\epsilon. \quad (1.45)$$

Up to a subsequence, still denoted by ϵ , there exist limits u and σ such that u_ϵ converges weakly to u in $H_0^1(\Omega)$ and σ_ϵ converges weakly to σ in $L^2(\Omega)^N$. Since A^ϵ converges strongly to A^* in $L^2(\Omega; \mathcal{M}_N)$, (1.45) involves the product of two sequences, one converging strongly, the other converging weakly. By Lemma 1.2.4, the limit of (1.45) is

$$\sigma = A^* \nabla u.$$

Since the homogenized equation (1.41) has a unique solution in $H_0^1(\Omega)$, all possible subsequences of u_ϵ converge to the same limit u . By Lemma 1.2.8, this implies that the entire sequence u_ϵ converges to u . This property is nothing but the definition of the H -convergence of the sequence A^ϵ to A^* . \square

Remark 1.2.23

1. In space dimension $N = 1$, we saw in the previous subsection that a sequence A^ϵ H -converges to a limit A^* if and only if $A^{\epsilon-1}$ converges weakly * in $L^\infty(\Omega)$ to A^{*-1} . Unfortunately, in higher dimensions $N \geq 2$, the H -limit of a sequence usually has no link with any kind of weak limit. In particular, this implies that there is no “general” formula for an H -limit (see however Remark 1.3.5 below).
2. In Subsections 1.3.4 and 1.3.5, some examples of H -convergence are given. Let us point out for the moment that if the sequence of matrix is scalar, i.e., $A^\epsilon = a_\epsilon I_2$ with $a_\epsilon \in \mathbb{R}$, the H -limit A^* is not necessarily so. By the same token, if the matrices A^ϵ are diagonal, the H -limit A^* is not necessarily diagonal.
3. All the above results also hold for elliptic systems (like the elasticity equations), and for some nonlinear equations stemming from convex or monotone operators.

A useful result for the sequel (at least from a theoretical point of view) is that the topology induced by the H -convergence is metrizable.

Proposition 1.2.24 Let $(f_n)_{n \geq 1}$ be a dense countable family in $H^{-1}(\Omega)$. Let A and B be two matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$. Define u_n and v_n the respective unique solutions in $H_0^1(\Omega)$ of

$$\begin{cases} -\operatorname{div} A \nabla u_n = f_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\operatorname{div} B \nabla v_n = f_n & \text{in } \Omega \\ v_n = 0 & \text{on } \partial\Omega. \end{cases}$$

We define a distance function in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ by

$$d(A, B) = \sum_{n \geq 1} 2^{-n} \frac{\|u_n - v_n\|_{L^2(\Omega)} + \|A \nabla u_n - B \nabla v_n\|_{H^{-1}(\Omega)^N}}{\|f_n\|_{H^{-1}(\Omega)}}.$$

Then, $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ is a metric space with this distance d , and the H -convergence is equivalent to the sequential convergence with respect to d .

Remark 1.2.25 One interesting consequence of this metrizability result is the possibility of extracting diagonal subsequences from H -converging sequences. More precisely, we denote by $A^\epsilon(x)$ a sequence of matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ that H -converges to a limit matrix $A^*(x)$ as ϵ goes to zero. Assume that, for any fixed ϵ , there exists another sequence $A^{\epsilon,\eta}(x)$ of matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ that H -converges to $A^\epsilon(x)$ as η goes to zero. Then there exists a diagonal sequence $A^{\epsilon,\eta(\epsilon)}(x)$ that H -converges to $A^*(x)$ as ϵ goes to zero (where $\eta(\epsilon)$ is a function which goes to zero with ϵ). This is a basic consequence of the metrizability of H -convergence. In other words, it justifies the so-called reiterated homogenization procedure.

Proof. Clearly, since matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ are coercive and bounded, we have

$$\|u_n\|_{L^2(\Omega)} + \|A \nabla u_n\|_{H^{-1}(\Omega)^N} \leq C \|f_n\|_{H^{-1}(\Omega)},$$

and the number $d(A, B)$ is therefore always finite. Moreover, the function $d(A, B)$ is symmetric and obviously satisfies the triangle inequality. To conclude that d is a distance, it remains to check that $d(A, B) = 0$ implies that $A = B$. Since the family $(f_n)_{n \geq 1}$ is dense in $H^{-1}(\Omega)$, $d(A, B) = 0$ implies that for any right hand side $f \in H^{-1}(\Omega)$, the solutions u and v of

$$\begin{cases} -\operatorname{div} A \nabla u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\operatorname{div}B\nabla v = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfy $u = v$ and $A\nabla u = B\nabla v$ in Ω . Let $\phi \in \mathcal{D}(\Omega)$ be a smooth function identically equal to 1 on a subset ω compactly embedded in Ω . Let $\lambda \in \mathbb{R}^N$. Taking $f(x) = -\operatorname{div}A\nabla(\phi(x)\lambda \cdot x)$, we deduce $\nabla u = \nabla v = \lambda$ in ω , and therefore $A\lambda = B\lambda$ in ω for any vector λ . This yields $A = B$. Finally, it is not difficult to check that the definition of d is precisely made in order to have an equivalence between H -convergence and convergence with respect to the distance d . To recognize this, we use Rellich theorem (see Lemma 1.2.6) which states that, Ω being a bounded open set, the weak convergence in $H^1(\Omega)$ (respectively, in $L^2(\Omega)$) implies a strong convergence in $L^2(\Omega)$ (respectively, in $H^{-1}(\Omega)$). \square

1.3 Proofs and Further Results

The focus of this section is twofold. First, we give detailed proofs of the main results concerning H -convergence, which were stated in the preceding section. Second, we investigate other properties of H -convergence, including corrector results, a comparison with G -convergence, and its applications to the periodic case and to laminated structures.

1.3.1 Tartar's Method

This subsection is devoted to the proofs of the main results stated in the previous section. The proof of the compactness theorem of H -convergence (Theorem 1.2.16) relies on a method introduced by Tartar (we follow the exposition of Murat and Tartar [204]), which is sometime called the *energy method*. This latter name is not very relevant, and it is probably better to call it the *oscillating test function method* or the *compensated compactness method*. The main interest of this method is that it provides a constructive proof of Theorem 1.2.16. It is based on some ideas of the compensated compactness theory developed by Murat and Tartar [201], [272]. The following lemma of compensated compactness, known as the “div-curl lemma”, plays a crucial role in the sequel.

Lemma 1.3.1 (Div-curl lemma) *Let u_ϵ and v_ϵ be two sequences in $L^2(\Omega)^N$ such that*

$$\begin{cases} u_\epsilon \rightharpoonup u \text{ weakly in } L^2(\Omega)^N, \\ v_\epsilon \rightharpoonup v \text{ weakly in } L^2(\Omega)^N. \end{cases}$$

Define the curl operator by $\operatorname{curl} v = \left(\frac{\partial v^i}{\partial x_j} - \frac{\partial v^j}{\partial x_i} \right)_{1 \leq i, j \leq N}$. Assume furthermore that

$$\begin{cases} \operatorname{div} u_\epsilon \rightarrow \operatorname{div} u & \text{strongly in } H^{-1}(\Omega) \\ \operatorname{curl} v_\epsilon \rightarrow \operatorname{curl} v & \text{strongly in } H^{-1}(\Omega; \mathcal{M}_N). \end{cases}$$

Then

$$u_\epsilon \cdot v_\epsilon \rightharpoonup u \cdot v \text{ in the sense of distributions.} \quad (1.46)$$

Remark 1.3.2 Lemma 1.3.1 is a typical example of compensated compactness [201], [272]. Indeed, except in the one-dimensional case $N = 1$, none of the sequences u_ϵ and v_ϵ is bounded in $H^1(\Omega)^N$, and thus strongly convergent in $L^2(\Omega)^N$ by Rellich theorem (see Lemma 1.2.6). For weakly convergent sequences, it is well known that (1.46) does not hold in general. A simple counterexample is given by the sequences $u_\epsilon(x) = v_\epsilon(x) = \sin(\frac{x_1}{\epsilon})$ for which the weak limits are $u = v = 0$ while the weak limit of $u_\epsilon v_\epsilon$ is $1/2$. However, the "miracle" of compensated compactness is that one can pass to the limit in the product $u_\epsilon \cdot v_\epsilon$ because some derivatives (chosen in such a way that they compensate each other) in each sequence are controlled.

Proof. For simplicity, we prove this lemma in a simple case which is enough for the sequel. Namely, we consider the case where v_ϵ is the product of a fixed function $\psi(x) \in W^{1,\infty}(\Omega)$ and of a sequence of gradients ∇p_ϵ , with p_ϵ a sequence converging weakly to p in $H^1(\Omega)$. In such a case the proof is immediate by integration by parts, while in the general case it is more involved and uses Fourier analysis (see [201]). The curl of $\psi \nabla p_\epsilon$ is

$$\operatorname{curl}(\psi \nabla p_\epsilon) = \left(\frac{\partial p_\epsilon}{\partial x_i} \frac{\partial \psi}{\partial x_j} - \frac{\partial p_\epsilon}{\partial x_j} \frac{\partial \psi}{\partial x_i} \right)_{1 \leq i, j \leq N}$$

which is bounded in $L^2(\Omega; \mathcal{M}_N)$ and thus converges strongly in $H^{-1}(\Omega; \mathcal{M}_N)$ by Rellich theorem (Lemma 1.2.6). Let ϕ be a smooth test function with compact support in Ω , i.e., $\phi \in \mathcal{D}(\Omega)$. Since $v_\epsilon = \psi \nabla p_\epsilon$, integrating by parts leads to

$$\int_{\Omega} \phi u_\epsilon \cdot v_\epsilon dx = - \int_{\Omega} \phi \psi p_\epsilon \operatorname{div} u_\epsilon dx - \int_{\Omega} p_\epsilon u_\epsilon \cdot \nabla(\phi \psi) dx. \quad (1.47)$$

We can pass to the limit in the first term of the right hand side of (1.47) since $\phi\psi p_\epsilon$ is converging weakly in $H_0^1(\Omega)$ and $\operatorname{div} u_\epsilon$ is converging strongly in $H^{-1}(\Omega)$, and in the second term since u_ϵ is converging weakly in $L^2(\Omega)^N$ and p_ϵ is converging strongly in $L^2(\Omega)$ thanks to Rellich theorem. It yields

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} \phi u_\epsilon \cdot v_\epsilon dx = - \int_{\Omega} \phi \psi p \operatorname{div} u dx - \int_{\Omega} p u \cdot \nabla(\phi\psi) dx.$$

Another integration by parts gives the desired result. \square

Proof of Theorem 1.2.16. This proof is decomposed in three steps. The first step gives an abstract convergence result for operators in Banach spaces. The second step is precisely the oscillating test function method. The third step is concerned with the construction of these test functions.

First step:

Lemma 1.3.3 *Let V_1 be a separable Banach space and V_2 a reflexive Banach space. Let S_ϵ be a bounded sequence of linear continuous operators acting from V_1 into V_2 , namely $S_\epsilon \in \mathcal{LC}(V_1, V_2)$ and $\|S_\epsilon\| \leq C$ where $0 < C < +\infty$ is a positive constant independent of ϵ . There exists a subsequence, still denoted by ϵ , and a limit operator $S_0 \in \mathcal{LC}(V_1, V_2)$ such that*

$$\|S_0\| \leq \liminf_{\epsilon \rightarrow 0} \|S_\epsilon\|,$$

and for any $f \in V_1$

$$S_\epsilon f \rightharpoonup S_0 f \text{ weakly in } V_2.$$

Proof. Recall that a separable space is defined as containing a dense countable subset. Let X_1 be such a dense countable subset of V_1 . For any $f \in X_1$, the sequence $S_\epsilon f$ is bounded in V_2 . Recall also that a reflexive space V (i.e., a space that coincides with the dual of its dual, $V = V''$) is such that, from each bounded sequence in V , one can extract a subsequence that is weakly converging (see, e.g., [58], [292]). Since V_2 is reflexive, there exists a subsequence, denoted by ϵ_f (which, a priori, depends on f), and a limit, denoted by $S_0 f$, such that $S_{\epsilon_f} f$ converges weakly to $S_0 f$ in V_2 . Since X_1 is countable, a diagonal subsequence, still denoted by ϵ , can be extracted from the sequences $(\epsilon_f)_{f \in X_1}$ such that, for any $f \in X_1$,

$$S_\epsilon f \rightharpoonup S_0 f \text{ weakly in } V_2.$$

This defines an operator S_0 from X_1 into V_2 which is obviously linear. By the weak lower semicontinuity of the norm, it also satisfies

$$\|S_0 f\|_{V_2} \leq \liminf_{\epsilon \rightarrow 0} \|S_\epsilon f\|_{V_2} \leq C \|f\|_{V_1}, \quad (1.48)$$

which implies that S_0 is also continuous from X_1 into V_2 . Since X_1 is dense in V_1 , S_0 can be extended by continuity to an operator in $\mathcal{LC}(V_1, V_2)$. Then it is not difficult to check that, for any $f \in V_1$ (and not merely in X_1),

$$S_\epsilon f \rightharpoonup S_0 f \text{ weakly in } V_2.$$

From (1.48) we deduce $\|S_0\| \leq \liminf_{\epsilon \rightarrow 0} \|S_\epsilon\|$, which concludes the proof. \square

Lemma 1.3.4 *Let V be a separable and reflexive Banach space. Let $\alpha > 0$ and $\beta > 0$ be two positive constants. Let T_ϵ be a sequence of linear continuous operators acting from V into its dual V' , i.e., $T_\epsilon \in \mathcal{LC}(V, V')$, such that, for any $v \in V$,*

$$\langle T_\epsilon v, v \rangle_{V', V} \geq \alpha \|v\|_V^2 \quad (1.49)$$

and, for any $f \in V'$,

$$\langle T_\epsilon^{-1} f, f \rangle_{V, V'} \geq \beta \|f\|_{V'}^2. \quad (1.50)$$

There exists a subsequence, still denoted by ϵ , and a limit operator $T_0 \in \mathcal{LC}(V, V')$, which satisfies also (1.49)-(1.50) and such that, for any $f \in V'$,

$$T_\epsilon^{-1} f \rightharpoonup T_0^{-1} f \text{ weakly in } V'. \quad (1.51)$$

Proof. We remark first that the coercivity assumption (1.49) and the Lax-Milgram lemma together imply that T_ϵ admits an inverse T_ϵ^{-1} . In (1.49), taking $v = T_\epsilon^{-1} f$ yields

$$\langle f, T_\epsilon^{-1} f \rangle_{V', V} \geq \alpha \|T_\epsilon^{-1} f\|_V^2. \quad (1.52)$$

By the Cauchy-Schwartz inequality, we deduce

$$\|T_\epsilon^{-1} f\|_V \leq \frac{1}{\alpha} \|f\|_{V'}.$$

On the other hand, the dual V' of a separable reflexive Banach space V is also a separable reflexive Banach space (see, e.g., [58], [292]). Therefore, we can apply Lemma 1.3.3 to the sequence T_ϵ^{-1} with $V_1 = V'$ and $V_2 = V$,

which yields (1.51). It remains to prove that T_0^{-1} is invertible and that its inverse T_0 satisfies (1.49) and (1.50). Taking the limit of (1.50) leads to

$$\langle T_0^{-1}f, f \rangle_{V,V'} \geq \beta \|f\|_{V'}^2 \quad \forall f \in V',$$

which, by the Lax-Milgram lemma, proves that T_0^{-1} is invertible. Taking the limit of (1.52) (using the weak lower semicontinuity of the norm) gives

$$\langle f, T_0^{-1}f \rangle_{V',V} \geq \alpha \|T_0^{-1}f\|_V^2.$$

Replacing f by T_0v yields the desired result, i.e. (1.49) for T_0 . \square

Coming back to the proof of Theorem 1.2.16, these abstract lemmas are applied to the following operators:

$$\begin{aligned} T_\epsilon : \quad & H_0^1(\Omega) \rightarrow H^{-1}(\Omega) \\ v \rightarrow & -\operatorname{div} A^\epsilon \nabla v \end{aligned} \tag{1.53}$$

and

$$\begin{aligned} S_\epsilon : \quad & H^{-1}(\Omega) \rightarrow L^2(\Omega)^N \\ f \rightarrow & A^\epsilon \nabla (T_\epsilon^{-1}f), \end{aligned} \tag{1.54}$$

where $A^\epsilon(x)$ is a matrix in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$. As is well known, all spaces $L^2(\Omega)^N$, $H^{-1}(\Omega)$, and $H_0^1(\Omega)$ are separable reflexive Banach spaces. Let us check the assumptions of Lemma 1.3.4 for T_ϵ defined by (1.53). Integrating by parts,

$$\langle T_\epsilon v, v \rangle_{H^{-1}, H_0^1(\Omega)} = \int_\Omega A^\epsilon \nabla v \cdot \nabla v dx \geq \alpha \|\nabla v\|_{L^2(\Omega)^N}^2, \tag{1.55}$$

and

$$\langle T_\epsilon v, v \rangle_{H^{-1}, H_0^1(\Omega)} = \int_\Omega A^{\epsilon-1}(A^\epsilon \nabla v) \cdot (A^\epsilon \nabla v) dx \geq \beta \|A^\epsilon \nabla v\|_{L^2(\Omega)^N}^2. \tag{1.56}$$

By virtue of the Poincaré inequality, $\|\nabla v\|_{L^2(\Omega)^N}$ is a norm in $H_0^1(\Omega)$; thus (1.55) is precisely assumption (1.49). The norm of $T_\epsilon v = -\operatorname{div} A^\epsilon \nabla v$ is

$$\|T_\epsilon v\|_{H^{-1}(\Omega)} = \sup_{\phi \in H_0^1(\Omega)} \frac{\langle T_\epsilon v, \phi \rangle_{H^{-1}, H_0^1(\Omega)}}{\|\phi\|_{H_0^1(\Omega)}},$$

but

$$\langle T_\epsilon v, \phi \rangle_{H^{-1}, H_0^1(\Omega)} = \int_\Omega A^\epsilon \nabla v \cdot \nabla \phi dx,$$

and therefore

$$\|T_\epsilon v\|_{H^{-1}(\Omega)} \leq \|A^\epsilon \nabla v\|_{L^2(\Omega)^N}. \quad (1.57)$$

Combining (1.56) and (1.57), and taking $v = T_\epsilon^{-1} f$, yields assumption (1.50), i.e.,

$$\langle f, T_\epsilon^{-1} f \rangle_{H^{-1}, H_0^1(\Omega)} \geq \beta \|f\|_{H^{-1}(\Omega)}^2 \quad \forall f \in H^{-1}(\Omega).$$

By the same token, S_ϵ satisfies the assumption of Lemma 1.3.3, since

$$\|S_\epsilon f\|_{L^2(\Omega)^N} \leq \beta^{-1} \|\nabla(T_\epsilon^{-1} f)\|_{L^2(\Omega)^N}$$

and

$$\|\nabla(T_\epsilon^{-1} f)\|_{L^2(\Omega)^N} = \|T_\epsilon^{-1} f\|_{H_0^1(\Omega)} \leq \alpha^{-1} \|f\|_{H^{-1}(\Omega)}.$$

From Lemmas 1.3.3 and 1.3.4 we deduce that there exist a subsequence, still denoted by ϵ , and limit operators S_0 and T_0 such that, for any $f \in H^{-1}(\Omega)$,

$$\begin{aligned} u_\epsilon &= T_\epsilon^{-1} f &\rightharpoonup u = T_0^{-1} f \text{ weakly in } H_0^1(\Omega) \\ A^\epsilon \nabla u_\epsilon &= S_\epsilon f &\rightharpoonup S_0 f \text{ weakly in } L^2(\Omega)^N. \end{aligned} \quad (1.58)$$

The above abstract convergence result does not say anything explicit about the limit operators S_0 and T_0 . In particular, it is not yet clear if T_0 has a definition similar to T_ϵ , i.e., if u is solution of an homogenized equation. The main interest of this abstract result is that it proves the existence of a single subsequence for which u_ϵ and $A^\epsilon \nabla u_\epsilon$ converge, whatever the choice of the right hand side f .

Second step:

Let us assume, for the moment, that there exist another subsequence, still denoted by ϵ , and a sequence of N test functions $(w_\epsilon^i)_{1 \leq i \leq N}$ such that

$$\begin{aligned} w_\epsilon^i &\rightharpoonup x_i \text{ weakly in } H^1(\Omega) \\ g_\epsilon^i = -\operatorname{div} A^\epsilon \nabla w_\epsilon^i &\rightarrow g_i \text{ strongly in } H^{-1}(\Omega). \end{aligned} \quad (1.59)$$

The existence of such test functions will be proved in the third step. By assumption (1.59) the N sequences $(A^\epsilon \nabla w_\epsilon^i)_{1 \leq i \leq N}$ are bounded in $L^2(\Omega)^N$. Thus, up to another subsequence, still denoted by ϵ , there exist N limit vectors $a^i(x)$ such that

$$A^\epsilon \nabla w_\epsilon^i \rightharpoonup a^i \text{ weakly in } L^2(\Omega)^N.$$

This allows us to define a matrix $A^*(x) \in L^2(\Omega; \mathcal{M}_N)$ by its columns $(a^i)_{1 \leq i \leq N}$

$$A^*(x) = \left(a^1(x), \dots, a^N(x) \right). \quad (1.60)$$

We shall check later that this matrix A^* belongs to $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$. Let us first prove that A^* is indeed the homogenized matrix. We apply Lemma 1.3.1 (the “div-curl” lemma) to the following nonnegative quantity:

$$(A^\epsilon \nabla u_\epsilon - A^\epsilon \nabla w_\epsilon) \cdot (\nabla u_\epsilon - \nabla w_\epsilon) \geq 0, \quad (1.61)$$

where w_ϵ is a test function defined by

$$w_\epsilon = \sum_{i=1}^N \lambda^i w_\epsilon^i \text{ with } \lambda = (\lambda^i)_{1 \leq i \leq N} \in \mathbb{R}^N. \quad (1.62)$$

The quantity in (1.61) is nonnegative since A^ϵ is coercive. Furthermore, in (1.61), the first term in the parentheses has a divergence that converges strongly in $H^{-1}(\Omega)$ since $-\operatorname{div}(A^\epsilon \nabla u_\epsilon - A^\epsilon \nabla w_\epsilon) = f - g_\epsilon^i$, while the second term is curl-free. Hence, collecting the convergence results (1.58) and (1.59) and applying Lemma 1.3.1, we pass to the limit in (1.61) to obtain

$$(S_0 f - A^* \lambda) \cdot (\nabla u - \lambda) \geq 0. \quad (1.63)$$

Inequality (1.63) holds almost everywhere in Ω . Let x_0 be a point in Ω where (1.63) holds (a so-called Lebesgue point, see Lemma 1.3.26). We choose $\lambda = \nabla u(x_0) - t\mu$ with $t > 0$ and $\mu \in \mathbb{R}^N$. Then (1.63) at x_0 yields

$$(S_0 f(x_0) - A^*(x_0) \nabla u(x_0) + t A^*(x_0) \mu) \cdot t\mu \geq 0.$$

Dividing by t first, and then letting t go to zero leads to

$$(S_0 f(x_0) - A^*(x_0) \nabla u(x_0)) \cdot \mu \geq 0 \quad \forall \mu \in \mathbb{R}^N.$$

Changing μ to $-\mu$ implies that the above inequality is actually an equality. Since μ is any vector in \mathbb{R}^N , this implies that

$$(S_0 f)(x_0) = A^*(x_0) \nabla u(x_0).$$

This equality holds for almost any $x_0 \in \Omega$. Recall that, by definition, $-\operatorname{div} S_\epsilon f = f$, therefore $-\operatorname{div} S_0 f = f$, which yields the following definition of T_0 :

$$\begin{aligned} T_0 : H_0^1(\Omega) &\rightarrow H^{-1}(\Omega) \\ v &\rightarrow -\operatorname{div} A^* \nabla v. \end{aligned}$$

This is nothing other than the conclusion of Theorem 1.2.16, since we have just proved that, up to a subsequence,

$$\begin{cases} u_\epsilon \rightharpoonup u \text{ weakly in } H_0^1(\Omega) \\ A^\epsilon \nabla u_\epsilon \rightharpoonup A^* \nabla u \text{ weakly in } L^2(\Omega)^N, \end{cases}$$

where u is solution of the homogenized equation

$$\begin{cases} -\operatorname{div} A^*(x) \nabla u(x) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

However, it remains to prove that the homogenized matrix A^* does indeed belong to $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$. Let $\phi \in \mathcal{D}(\Omega)$ be a smooth function with compact support in Ω . For w_ϵ defined by (1.62), we have

$$\int_{\Omega} \phi^2 A^\epsilon \nabla w_\epsilon \cdot \nabla w_\epsilon dx \geq \alpha \int_{\Omega} \phi^2 |\nabla w_\epsilon|^2 dx. \quad (1.64)$$

Applying the div-curl Lemma 1.3.1 to the left hand side of (1.64) (the divergence of $A^\epsilon \nabla w_\epsilon$ converges strongly and ∇w_ϵ is curl-free) and using the weak lower semicontinuity of the right hand side, we pass to the limit in (1.64),

$$\int_{\Omega} \phi^2 A^* \lambda \cdot \lambda dx \geq \alpha \int_{\Omega} \phi^2 |\lambda|^2 dx.$$

This implies that $A^* \geq \alpha I_2$ almost everywhere in Ω in the sense of quadratic forms. Since A^ϵ belongs to $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$, it also satisfies

$$\int_{\Omega} \phi^2 A^{\epsilon-1} (A^\epsilon \nabla w_\epsilon) \cdot (A^\epsilon \nabla w_\epsilon) dx \geq \beta \int_{\Omega} \phi^2 |A^\epsilon \nabla w_\epsilon|^2 dx. \quad (1.65)$$

Applying again the div-curl Lemma 1.3.1 to the left hand side of (1.65) and using the weak lower semicontinuity of the right hand side, we pass to the limit in (1.65),

$$\int_{\Omega} \phi^2 A^* \lambda \cdot \lambda dx \geq \beta \int_{\Omega} \phi^2 |A^* \lambda|^2 dx.$$

Replacing λ by $A^{*-1} \mu$ implies that $A^{*-1} \geq \beta I_2$ almost everywhere in Ω in the sense of quadratic forms. Thus, we deduce that the matrix A^* belongs to $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$, although it was merely defined in $L^2(\Omega; \mathcal{M}_N)$.

Third step:

The last step is concerned with the existence of the test functions defined by (1.59). Let $\hat{\Omega}$ be another open bounded set in \mathbb{R}^N such that Ω is compactly embedded in $\hat{\Omega}$, i.e., $\bar{\Omega} \subset \hat{\Omega}$. The coefficient matrix A^ϵ is extended to $\hat{\Omega}$: For example, we define $A^\epsilon = \alpha I_2$ in $\hat{\Omega} \setminus \Omega$. Let $\phi \in \mathcal{D}(\hat{\Omega})$ be a smooth function with compact support in $\hat{\Omega}$ such that $\phi \equiv 1$ in Ω . We apply the results of Step 1 to the operator \hat{T}_ϵ defined as T_ϵ but in $\hat{\Omega}$ rather than in Ω

$$\begin{aligned}\hat{T}_\epsilon : H_0^1(\hat{\Omega}) &\rightarrow H^{-1}(\hat{\Omega}) \\ v &\rightarrow -\operatorname{div} A^\epsilon \nabla v.\end{aligned}$$

From the first step, there exist a subsequence, still denoted by ϵ , and a limit operator \hat{T}_0 such that \hat{T}_ϵ converges to \hat{T}_0 in the sense of (1.51). Introducing $g_i = \hat{T}_0(\phi x_i)$, we define the test functions w_ϵ^i by

$$w_\epsilon^i = \hat{T}_\epsilon^{-1} g_i.$$

Then, it is easy to check the required convergence properties (1.59), which are: w_ϵ^i converges weakly to x_i in $H^1(\Omega)$, and $g_\epsilon^i = -\operatorname{div} A^\epsilon \nabla w_\epsilon^i$ converges strongly to g_i in $H^{-1}(\Omega)$. This finishes the proof of Theorem 1.2.16. We remark that we have introduced a larger domain $\hat{\Omega}$ simply because the test functions w_ϵ^i cannot satisfy an homogeneous Dirichlet boundary condition on $\partial\Omega$, and therefore we are unable to apply the result of Step 1 in Ω itself. \square

Remark 1.3.5 As we shall see below in Lemma 1.3.37, the oscillating test functions $(w_\epsilon^i)_{1 \leq i \leq N}$ are unique in the sense that, if there exists another family $(\tilde{w}_\epsilon^i)_{1 \leq i \leq N}$ of such functions satisfying (1.59), then their difference converges strongly to zero in $H^1(\Omega)$. As a consequence it implies the uniqueness of the H -limit A^* . Indeed, according to formula (1.60), the homogenized matrix A^* is defined by

$$\int_{\Omega} A^*(x) e_i \cdot v(x) dx = \lim_{\epsilon \rightarrow 0} \int_{\Omega} A^\epsilon(x) \nabla w_\epsilon^i(x) \cdot v(x) dx, \quad (1.66)$$

where $(e_i)_{1 \leq i \leq N}$ is the canonical basis of \mathbb{R}^N , and $v(x)$ is any function in $L^2(\Omega)^N$. Since A^* is the weak limit of a sequence in $L^2(\Omega; \mathcal{M}_N)$, and this weak limit does not change by adding a strongly convergent sequence to zero in $L^2(\Omega; \mathcal{M}_N)$, this implies the uniqueness of the H -limit A^* . In other words, an H -converging sequence A^ϵ cannot converge to two different H -limits.

On the other hand, applying the div-curl Lemma 1.3.1 to the left hand side of (1.64), formula (1.66) is also equivalent to

$$\int_{\Omega} \phi^2(x) A^*(x) \lambda \cdot \lambda dx = \lim_{\epsilon \rightarrow 0} \int_{\Omega} \phi^2(x) A^\epsilon(x) \nabla w_\epsilon(x) \cdot \nabla w_\epsilon(x) dx, \quad (1.67)$$

where $\phi \in \mathcal{D}(\Omega)$ is a smooth function with compact support in Ω , λ is any vector in \mathbb{R}^N , and $w_\epsilon = \sum_{i=1}^N \lambda^i w_\epsilon^i$. Formula (1.67) is the analogue of formula (1.11) in the periodic case. Unfortunately, it is of limited practical use since it involves the sequences w_ϵ^i which are abstractly defined with the unknown limit operator \hat{T}_0 .

Proof of Proposition 1.2.18. (Locality of the H -convergence) Since ω is compactly embedded in Ω , there exists a smooth function $\phi \in \mathcal{D}(\Omega)$ with compact support in Ω and such that $\phi \equiv 1$ on ω . For $\lambda \in \mathbb{R}^N$, we set $w(x) = \lambda \cdot x \phi(x)$ and $g = -\operatorname{div} A^* \nabla w$. Let w_ϵ be the solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\operatorname{div} A^\epsilon \nabla w_\epsilon = g & \text{in } \Omega \\ w_\epsilon = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.68)$$

Since A^ϵ H -converges to A^* , the following convergences hold

$$\begin{aligned} w_\epsilon &\rightharpoonup w \text{ weakly in } H_0^1(\Omega) \\ A^\epsilon \nabla w_\epsilon &\rightharpoonup A^* \nabla w \text{ weakly in } L^2(\Omega)^N. \end{aligned}$$

Similarly, for $\mu \in \mathbb{R}^N$, we set $v(x) = \mu \cdot x \phi(x)$ and $f = -\operatorname{div} A^* \nabla v$. Let v_ϵ be the solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\operatorname{div} B^\epsilon \nabla v_\epsilon = f & \text{in } \Omega \\ v_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Since B^ϵ H -converges to B^* , the following convergences hold

$$\begin{aligned} v_\epsilon &\rightharpoonup v \text{ weakly in } H_0^1(\Omega) \\ B^\epsilon \nabla v_\epsilon &\rightharpoonup B^* \nabla v \text{ weakly in } L^2(\Omega)^N. \end{aligned}$$

By application of the div-curl Lemma 1.3.1, we obtain

$$(A^\epsilon \nabla w_\epsilon - B^\epsilon \nabla v_\epsilon) \cdot (\nabla w_\epsilon - \nabla v_\epsilon) \rightharpoonup (A^* \nabla w - B^* \nabla v) \cdot (\nabla w - \nabla v) \quad (1.69)$$

in the sense of distributions. By assumption $A^\epsilon = B^\epsilon$ on ω , thus the left hand side of (1.69) is positive almost everywhere in ω by coerciveness of A^ϵ . This implies that

$$(A^* \nabla w - B^* \nabla v) \cdot (\nabla w - \nabla v) = (A^* \lambda - B^* \mu) \cdot (\lambda - \mu) \geq 0 \text{ a.e. in } \omega. \quad (1.70)$$

Choosing $\mu = \lambda + t\nu$ with $t > 0$ and $\nu \in \mathbb{R}^N$, we first divide by t , then let t go to zero in (1.70). Changing ν in $-\nu$ yields an equality

$$(A^* - B^*) \lambda \cdot \nu = 0 \quad \forall \lambda, \nu \in \mathbb{R}^N.$$

This implies that $A^* = B^*$ in ω . \square

Proof of Proposition 1.2.19. (Irrelevance of the boundary condition) Let ω be a compactly embedded subset of Ω , i.e. $\bar{\omega} \subset \Omega$. By assumption, the sequence z_ϵ is bounded in $H^1(\omega)$. Thus, up to a subsequence still denoted by ϵ , there exists $\sigma \in L^2(\omega)^N$ such that

$$A^\epsilon \nabla z_\epsilon \rightharpoonup \sigma \text{ weakly in } L^2(\omega)^N.$$

In order to recognize this limit σ , we introduce a sequence of test functions w_ϵ defined as in (1.68). Let $\phi \in \mathcal{D}(\Omega)$ be a smooth function with compact support in Ω and such that $\phi \equiv 1$ on ω . For $\lambda \in \mathbb{R}^N$, we set $w(x) = \lambda \cdot x \phi(x)$ and $g = -\operatorname{div} A^* \nabla w$. Then, w_ϵ is defined as the solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\operatorname{div} A^\epsilon \nabla w_\epsilon = g & \text{in } \Omega \\ w_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Since A^ϵ H -converges to A^* , the following convergences hold

$$\begin{aligned} w_\epsilon &\rightharpoonup w \text{ weakly in } H_0^1(\Omega) \\ A^\epsilon \nabla w_\epsilon &\rightharpoonup A^* \nabla w \text{ weakly in } L^2(\Omega)^N. \end{aligned}$$

By coercivity of A^ϵ ,

$$(A^\epsilon \nabla z_\epsilon - A^\epsilon \nabla w_\epsilon) \cdot (\nabla z_\epsilon - \nabla w_\epsilon) \geq 0 \text{ a.e. in } \omega. \quad (1.71)$$

Thanks to the assumption on z_ϵ , we can apply the div-curl Lemma 1.3.1 in ω in order to pass to the limit in (1.71). This yields

$$(\sigma - A^* \lambda) \cdot (\nabla z - \lambda) \geq 0 \text{ a.e. in } \omega. \quad (1.72)$$

We now finish the proof as in Step 2 of the proof of Theorem 1.2.16. Let $x_0 \in \omega$ be a point where inequality (1.72) holds. Choosing $\lambda = \nabla z(x_0) + t\mu$ with $t > 0$ and $\mu \in \mathbb{R}^N$, we first divide by t , then let t go to zero and vary μ . This gives

$$\sigma(x_0) = A^*(x_0)\nabla z(x_0) \text{ a.e. in } \omega. \quad (1.73)$$

Since the weak limit σ is uniquely determined by (1.73) whatever the choice of the subsequence, it implies that for the entire sequence ϵ

$$A^\epsilon \nabla z_\epsilon \rightharpoonup A^* \nabla z \text{ weakly in } L^2(\omega)^N.$$

Since ω is any subset of Ω , this means that $A^\epsilon \nabla z_\epsilon$ converges weakly to $A^* \nabla z$ in $L^2_{loc}(\Omega)^N$. \square

Proof of Proposition 1.2.20. (Energy convergence) By another application of the div-curl Lemma 1.3.1 we get

$$A^\epsilon \nabla u_\epsilon \cdot \nabla u_\epsilon \rightharpoonup A^* \nabla u \cdot \nabla u \quad (1.74)$$

in the sense of distributions in Ω . To obtain the convergence of the total energy integrated over Ω , we cannot use (1.74) since $\phi \equiv 1$ does not belong to $\mathcal{D}(\Omega)!$ Rather, we use the special form of the equation satisfied by u_ϵ , which implies, upon integrating by parts, that

$$\int_{\Omega} A^\epsilon \nabla u_\epsilon \cdot \nabla u_\epsilon dx = \int_{\Omega} f u_\epsilon dx.$$

Since u_ϵ converges weakly to u in $H_0^1(\Omega)$, we have

$$\int_{\Omega} f u_\epsilon dx \rightarrow \int_{\Omega} f u dx.$$

Another integration by parts in the homogenized equation gives

$$\int_{\Omega} A^* \nabla u \cdot \nabla u dx = \int_{\Omega} f u dx.$$

Thus, we finally obtain

$$\int_{\Omega} A^\epsilon \nabla u_\epsilon \cdot \nabla u_\epsilon dx \rightarrow \int_{\Omega} A^* \nabla u \cdot \nabla u dx, \quad (1.75)$$

which is the desired result. \square

Remark 1.3.6 The convergence (1.74) also holds true for a sequence z_ϵ defined as in Proposition 1.2.19, since the div-curl lemma does not require any special boundary condition. On the contrary, the convergence (1.75) does not hold, in general, for a sequence z_ϵ , defined as in Proposition 1.2.19, which does not satisfy any special boundary condition. A simple counterexample is provided by the solution of

$$\begin{cases} -\Delta z_\epsilon + z_\epsilon = 0 & \text{in } \Omega \\ z_\epsilon = g_\epsilon & \text{on } \partial\Omega, \end{cases}$$

where g_ϵ is a sequence that converges weakly, but not strongly, to a limit g in $H^{1/2}(\Omega)$.

Remark 1.3.7 Thanks to Meyers Theorem 1.3.41, the convergence (1.74) of the energy density holds true in a slightly stronger sense. Indeed, let $p > 2$ be the Meyers exponent: The sequence ∇u_ϵ is uniformly bounded in $L^p(\Omega)^N$. Therefore, $A^\epsilon \nabla u_\epsilon \cdot \nabla u_\epsilon$ converges weakly to $A^* \nabla u \cdot \nabla u$ in $L^{p/2}(\Omega)$ with $p/2 > 1$.

1.3.2 G -convergence

In the case of symmetric operators (i.e., when the matrix A^ϵ is symmetric) a previous notion of operator convergence has been introduced by Spagnolo [261] under the name of G -convergence. In the literature both notions of H - and G -convergence are often mixed up, although, strictly speaking, H -convergence is a generalization of G -convergence to the case of unsymmetric operators. This subsection is devoted to a brief presentation of G -convergence, which is a little simpler than H -convergence due to the symmetry hypothesis. From an historical point of view, let us mention that the G stands for “Green”, since the original proof of the compactness theorem for G -convergence used Green functions. There is also another proof relying on the Γ -convergence of De Giorgi, but the simpler proof is that of Tartar, already presented in the previous section for H -convergence.

Let \mathcal{M}_N^s be the linear space of symmetric real matrices of order N . For any two positive constants $\alpha > 0$ and $\beta > 0$, we define a subspace of \mathcal{M}_N^s made of coercive matrices with coercive inverses, namely,

$$\mathcal{M}_{\alpha,\beta}^s = \left\{ M \in \mathcal{M}_N^s \text{ such that } \begin{array}{l} M\xi \cdot \xi \geq \alpha|\xi|^2 \\ M^{-1}\xi \cdot \xi \geq \beta|\xi|^2 \end{array} \forall \xi \in \mathbb{R}^N \right\}. \quad (1.76)$$

Let Ω be a bounded open set in \mathbb{R}^N . We introduce the space $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta}^s)$ of admissible symmetric coefficient matrices.

Definition 1.3.8 A sequence of symmetric matrices A^ϵ in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta}^s)$ is said to *G-converge* to an homogenized, or *G-limit*, matrix $A^* \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta}^s)$ if, for any right hand side $f \in H^{-1}(\Omega)$, the sequence u_ϵ of solutions of

$$\begin{cases} -\operatorname{div} A^\epsilon(x) \nabla u_\epsilon(x) = f(x) & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{cases} \quad (1.77)$$

converges weakly in $H_0^1(\Omega)$ to the solution u of the homogenized equation

$$\begin{cases} -\operatorname{div} A^*(x) \nabla u(x) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This definition makes sense because of the following compactness theorem.

Theorem 1.3.9 For any sequence $A^\epsilon \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ of symmetric matrices, there exist a subsequence, still denoted by ϵ , and an homogenized symmetric matrix $A^* \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta}^s)$ such that A^ϵ G-converges to A^* .

The main difference between *H*- and *G*-convergence is that the latter does not require the convergence of the flux $A^\epsilon \nabla u_\epsilon$. Therefore, *G*-convergence is a weaker notion than *H*-convergence in the sense that if a sequence of symmetric matrices A^ϵ *H*-converges to a symmetric homogenized matrix A^* , then it automatically *G*-converges to the same limit. The converse is not obvious, but it turns out to be true. We shall indeed prove Theorem 1.3.9 by proving this reciprocal result. Let us first prove that the *H*-limit of a symmetric sequence is also symmetric. This is an obvious consequence of the following more general lemma.

Lemma 1.3.10 Let A^ϵ be a sequence of (not necessarily symmetric) matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$. If A^ϵ *H*-converges to a limit A^* in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$, then the adjoint, or transposed, sequence $(A^\epsilon)^t$ *H*-converges to the adjoint limit $(A^*)^t$ in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$.

Proof. For $f \in H^{-1}(\Omega)$, let u_ϵ be the solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\operatorname{div}(A^\epsilon)^t(x) \nabla u_\epsilon(x) = f(x) & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

The usual a priori estimates (see (1.33)) implies that there exist a subsequence, still denoted by ϵ , and limits u and σ such that

$$\begin{aligned} u_\epsilon &\rightharpoonup u \text{ weakly in } H_0^1(\Omega) \\ (A^\epsilon)^t \nabla u_\epsilon &\rightharpoonup \sigma \text{ weakly in } L^2(\Omega)^N. \end{aligned}$$

On the other hand, since A^ϵ H -converges to A^* , for $g \in H^{-1}(\Omega)$, the solution $v_\epsilon \in H_0^1(\Omega)$ of

$$\begin{cases} -\operatorname{div} A^\epsilon(x) \nabla v_\epsilon(x) = g(x) & \text{in } \Omega \\ v_\epsilon = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies

$$\begin{aligned} v_\epsilon &\rightharpoonup v \text{ weakly in } H_0^1(\Omega) \\ A^\epsilon \nabla v_\epsilon &\rightharpoonup A^* \nabla v \text{ weakly in } L^2(\Omega)^N, \end{aligned}$$

where v is the solution of the homogenized equation

$$\begin{cases} -\operatorname{div} A^*(x) \nabla v(x) = g(x) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Applying the div-curl Lemma 1.3.1 to

$$A^\epsilon \nabla v_\epsilon \cdot \nabla u_\epsilon = (A^\epsilon)^t \nabla u_\epsilon \cdot \nabla v_\epsilon$$

leads to

$$A^* \nabla v \cdot \nabla u = \sigma \cdot \nabla v. \quad (1.78)$$

Let ω be any open subset such that $\bar{\omega} \subset \Omega$. Let $\phi \in \mathcal{D}(\Omega)$ such that $\phi = 1$ in ω , and $\lambda \in \mathbb{R}^N$. Choosing $g = -\operatorname{div} A^* \nabla (\phi(x) \lambda \cdot x)$ implies that $v(x) = \lambda \cdot x$ in ω . Then (1.78) becomes

$$A^* \nabla u \cdot \lambda = \sigma \cdot \lambda \text{ a.e. in } \omega,$$

which implies that $\sigma = (A^*)^t \nabla u$ almost everywhere in Ω . By uniqueness of the limit σ , the entire sequence $(A^\epsilon)^t \nabla u_\epsilon$ converges to $(A^*)^t \nabla u$, and not only a subsequence. This proves that $(A^\epsilon)^t$ H -converges to $(A^*)^t$. \square

To establish the compactness of G -convergence, we prove a stronger result, which states the equivalence between H - and G -convergence for symmetric matrices. Then Theorem 1.3.9 is deduced from the compactness of H -convergence.

Proposition 1.3.11 *A sequence A^ϵ of symmetric matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta}^s)$ G -converges to a limit $A^* \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta}^s)$ if and only if it H -converges to A^* .*

Proof. By Lemma 1.3.10, if A^ϵ H -converges to A^* , then automatically A^* is symmetric, and thus A^ϵ also G -converges to A^* . The difficulty is to prove the converse. Let us assume that A^ϵ G -converges to $A^* \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta}^s)$. Then, by Definition 1.3.8 of G -convergence, for any $f \in H^{-1}(\Omega)$, the sequence u_ϵ of solutions of

$$\begin{cases} -\operatorname{div} A^\epsilon(x) \nabla u_\epsilon(x) = f(x) & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{cases} \quad (1.79)$$

converges weakly in $H_0^1(\Omega)$ to the solution u of the homogenized equation

$$\begin{cases} -\operatorname{div} A^*(x) \nabla u(x) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In addition, the sequence of fluxes $A^\epsilon \nabla u_\epsilon$ being bounded, there exist a subsequence and a limit σ such that $A^\epsilon \nabla u_\epsilon$ converges weakly to σ in $L^2(\Omega)^N$. To prove that A^ϵ H -converge to A^* , we have to show that $\sigma = A^* \nabla u$.

By application of Theorem 1.2.16, there exist a subsequence and an H -limit B^* such that, for this subsequence, A^ϵ H -converges to B^* . The matrix B^* is symmetric because of Lemma 1.3.10. Consequently, the limit u of the subsequence of solutions u_ϵ of (1.79) is also the unique solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\operatorname{div} B^*(x) \nabla u(x) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, by H -convergence $A^\epsilon \nabla u_\epsilon$ also converges weakly to $B^* \nabla u$ in $L^2(\Omega)^N$. This implies that $\sigma = B^* \nabla u$. To conclude the proof, it remains for us to check that $B^* = A^*$ (if so, there is no need to extract subsequences, since the H -limit B^* is uniquely determined, and σ is also uniquely given by $\sigma = A^* \nabla u$).

Subtracting the two homogenized equations leads to

$$-\operatorname{div}(A^* - B^*) \nabla u = 0 \text{ in } \Omega. \quad (1.80)$$

Equation (1.80) holds for any right hand side $f \in H^{-1}(\Omega)$, and therefore for any solution $u \in H_0^1(\Omega)$. Provided that u is chosen adequately, (1.80) will

imply that $A^* = B^*$. Let us first choose $u = \lambda \cdot x$ on ω , an open subset of Ω , with $\lambda \in \mathbb{R}^N$. Varying λ yields

$$\sum_{i=1}^N \frac{\partial(A^* - B^*)_{ij}}{\partial x_i} = 0 \text{ in } \omega, \quad \forall j \text{ with } 1 \leq j \leq N,$$

where $(A^* - B^*)_{ij}$ denotes the entries of the matrix $(A^* - B^*)$. This implies that each column of $(A^* - B^*)$ is divergence-free, and (1.80) simplifies to

$$\sum_{i,j=1}^N (A^* - B^*)_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0 \text{ in } \omega.$$

Then, choosing $u = x_i x_j$ in ω leads to

$$(A^* - B^*)_{ij} + (A^* - B^*)_{ji} = 0.$$

Thus, $B^* = A^*$ almost everywhere in Ω since A^* and B^* are symmetric (here, the symmetry is crucial). \square

Remark 1.3.12 As for H -convergence, the set $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta}^s)$ is closed under G -convergence. In the symmetric case, the essential upper bound γ , defined as the smallest constant such that $|A^\epsilon \xi| \leq \gamma |\xi|$ for any $\xi \in \mathbb{R}^N$, is also stable by G -convergence, i.e. $|A^* \xi| \leq \gamma |\xi|$. The reason is that, for a symmetric matrix A^* , the essential upper bound γ is precisely equal to $1/\beta$ where β is the largest coercivity constant for $(A^*)^{-1}$.

Proposition 1.3.11 shows that, for symmetric matrices A^ϵ , the convergence of the flux $A^\epsilon \nabla u_\epsilon$ is a consequence of the convergence of the solutions u_ϵ . If the matrices A^ϵ are not symmetric, this is no longer true. In particular, for nonsymmetric operators, the notion of G -convergence is inconsistent, since it does not guarantee the uniqueness of the G -limit (a highly desirable feature). The following counterexample will convince the reader.

Let A^ϵ be any sequence of matrices which G -converges to an homogenized matrix A^* in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$. Let B be a nonzero constant antisymmetric matrix, i.e.,

$$B = -B^t.$$

If the convergence of the flux is not imposed, then $A^* + B$ is also a G -limit for the sequence A^ϵ . According to Definition 1.3.8, A^ϵ G -converges to A^* if,

for any $f \in H^{-1}(\Omega)$, the sequence of solutions u_ϵ of (1.77) converges weakly in $H_0^1(\Omega)$ to the solution u of the homogenized equation

$$\begin{cases} -\operatorname{div} A^*(x) \nabla u(x) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, the point is that u is also solution of another equation involving $A^* + B$ instead of A^* , namely,

$$\begin{cases} -\operatorname{div} (A^*(x) + B) \nabla u(x) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The reason is that, because B is constant and antisymmetric, it satisfies

$$\sum_{i,j=1}^N B_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0,$$

whatever the value of the function $u(x)$. This implies that A^ϵ G -converges to $A^* + B$. We remark however that A^* and $A^* + B$ cannot be simultaneously symmetric, except if $B = 0$. Of course, if the sequence of fluxes $A^\epsilon \nabla u_\epsilon$ converges to $A^* \nabla u$ for all choices of f (i.e., A^ϵ H -converges to A^*), then $A^\epsilon \nabla u_\epsilon$ cannot converge to $(A^* + B) \nabla u$ for all possible f (i.e., A^ϵ does not H -converge to $A^* + B$).

We conclude this subsection by stating some ordering properties of the G - or H -convergence. In particular, we prove that the homogenized matrix is always bounded below by the harmonic mean and bounded above by the arithmetic mean.

Lemma 1.3.13 *Let A^ϵ and B^ϵ be two sequences of symmetric matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta}^s)$ that H -converge to the homogenized matrices A^* and B^* , respectively. Assume that, for any ϵ ,*

$$A^\epsilon \xi \cdot \xi \leq B^\epsilon \xi \cdot \xi \quad \forall \xi \in \mathbb{R}^N.$$

Then the homogenized limits are also ordered, that is,

$$A^* \xi \cdot \xi \leq B^* \xi \cdot \xi \quad \forall \xi \in \mathbb{R}^N.$$

Proof. We use the div-curl Lemma 1.3.1 for the usual test functions introduced in the proof of Theorem 1.2.16. Let w_ϵ be the sequence of functions defined by (1.62) that satisfies

$$\begin{aligned} w_\epsilon &\rightharpoonup \lambda \cdot x \text{ weakly in } H^1(\Omega) \\ -\operatorname{div} A^\epsilon \nabla w_\epsilon &\rightarrow f \text{ strongly in } H^{-1}(\Omega), \end{aligned}$$

where λ is any vector in \mathbb{R}^N . Similarly, let v_ϵ be the same sequence with B^ϵ instead of A^ϵ that satisfies

$$\begin{aligned} v_\epsilon &\rightharpoonup \lambda \cdot x \text{ weakly in } H^1(\Omega) \\ -\operatorname{div} B^\epsilon \nabla v_\epsilon &\rightarrow g \text{ strongly in } H^{-1}(\Omega). \end{aligned}$$

By coercivity and symmetry of A^ϵ ,

$$A^\epsilon \nabla w_\epsilon \cdot \nabla w_\epsilon - 2A^\epsilon \nabla w_\epsilon \cdot \nabla v_\epsilon + A^\epsilon \nabla v_\epsilon \cdot \nabla v_\epsilon = A^\epsilon (\nabla w_\epsilon - \nabla v_\epsilon) \cdot (\nabla w_\epsilon - \nabla v_\epsilon) \geq 0.$$

Since $A^\epsilon \leq B^\epsilon$, it implies

$$A^\epsilon \nabla w_\epsilon \cdot \nabla w_\epsilon - 2A^\epsilon \nabla w_\epsilon \cdot \nabla v_\epsilon + B^\epsilon \nabla v_\epsilon \cdot \nabla v_\epsilon \geq 0. \quad (1.81)$$

Applying the div-curl Lemma 1.3.1, we pass to the limit in (1.81) to get

$$A^* \lambda \cdot \lambda - 2A^* \lambda \cdot \lambda + B^* \lambda \cdot \lambda = (B^* - A^*) \lambda \cdot \lambda \geq 0,$$

which is the desired result. \square

Theorem 1.3.14 *Let $A^\epsilon \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ be a sequence of symmetric matrices that H -converges to an homogenized matrix A^* . Assume that*

$$A^\epsilon \rightharpoonup \bar{A} \text{ weakly * in } L^\infty(\Omega; \mathcal{M}_N^s)$$

and

$$(A^\epsilon)^{-1} \rightharpoonup \underline{A}^{-1} \text{ weakly * in } L^\infty(\Omega; \mathcal{M}_N^s).$$

Then the homogenized limit satisfies

$$\underline{A} \xi \cdot \xi \leq A^* \xi \cdot \xi \leq \bar{A} \xi \cdot \xi \quad \forall \xi \in \mathbb{R}^N.$$

Proof. Once again let w_ϵ be the sequence of functions defined by (1.62) that satisfies

$$\begin{aligned} w_\epsilon &\rightharpoonup \lambda \cdot x \text{ weakly in } H^1(\Omega) \\ -\operatorname{div} A^\epsilon \nabla w_\epsilon &\rightarrow f \text{ strongly in } H^{-1}(\Omega), \end{aligned}$$

where λ is any vector in \mathbb{R}^N . By coercivity of A^ϵ

$$A^\epsilon(\nabla w_\epsilon - \lambda) \cdot (\nabla w_\epsilon - \lambda) \geq 0,$$

which implies, by symmetry of A^ϵ ,

$$A^\epsilon \nabla w_\epsilon \cdot \nabla w_\epsilon - 2A^\epsilon \nabla w_\epsilon \cdot \lambda + A^\epsilon \lambda \cdot \lambda \geq 0. \quad (1.82)$$

Applying the div-curl Lemma 1.3.1, we pass to the limit in (1.82) to get

$$A^* \lambda \cdot \lambda - 2A^* \lambda \cdot \lambda + \bar{A} \lambda \cdot \lambda \geq 0,$$

which implies that $\bar{A} \geq A^*$. Similarly, for any $\sigma \in \mathbb{R}^N$,

$$A^{\epsilon-1}(A^\epsilon \nabla w_\epsilon - \sigma) \cdot (A^\epsilon \nabla w_\epsilon - \sigma) \geq 0,$$

which implies, by symmetry,

$$A^\epsilon \nabla w_\epsilon \cdot \nabla w_\epsilon - 2\nabla w_\epsilon \cdot \sigma + A^{\epsilon-1} \sigma \cdot \sigma \geq 0. \quad (1.83)$$

Applying the div-curl Lemma 1.3.1, we pass to the limit in (1.83) to get

$$A^* \lambda \cdot \lambda - 2\lambda \cdot \sigma + \underline{A}^{-1} \sigma \cdot \sigma \geq 0.$$

Choosing $\sigma = \underline{A}\lambda$ implies that $A^* \geq \bar{A}$. □

1.3.3 Homogenization of Eigenvalue Problems

So far homogenization has been defined and applied only for static problems with given source or forcing terms. Actually it turns out that H -convergence is also useful for dynamic problems (parabolic or hyperbolic equations; see, e.g., [50], [138]) and for eigenvalue problems. This latter type of problem is of interest in optimal design. For simplicity, we shall consider only symmetric operators (as in the previous subsection), which is enough for our applications in shape optimization. For a sequence of symmetric matrices A^ϵ in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta}^s)$, H - and G -convergences are equivalent, so we may use either one. As usual, Ω denotes a bounded open set in \mathbb{R}^N . We first recall a classical result in operator theory.

Lemma 1.3.15 *Let $A(x)$ be a symmetric matrix in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta}^s)$, and $\rho(x) \geq C > 0$ be a positive function in $L^\infty(\Omega)$. Then, the spectral problem of finding an eigenvalue $\lambda \in \mathbb{R}$ and a nonzero eigenvector $u \in H_0^1(\Omega)$ satisfying*

$$\begin{cases} -\operatorname{div} A(x) \nabla u(x) = \lambda \rho(x) u(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.84)$$

has a countable infinite number of solution eigenvalues $(\lambda^m)_{m \geq 1}$ which are positive and satisfy $\lim_{m \rightarrow +\infty} \lambda^m = +\infty$. Furthermore, for each eigenvalue λ^m , there exists at least one nonzero eigenvector $u^m \in H_0^1(\Omega)$.

In the sequel we say that an eigenvector u is normalized if it satisfied $\|u\|_{L^2(\Omega)} = 1$. H -convergence implies a spectral convergence, as is proved in the following classical result (see, e.g., [212]).

Theorem 1.3.16 *Let $A^\epsilon \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ be a sequence of symmetric matrices which H -converges to an homogenized matrix A^* . Let ρ_ϵ be a sequence of positive functions, such that*

$$0 < \rho_- \leq \rho_\epsilon(x) \leq \rho_+ < +\infty,$$

*which converges weakly * in $L^\infty(\Omega)$ to a limit $\rho(x)$. Let $(\lambda_\epsilon^m)_{m \geq 1}$ be the eigenvalues, labeled by increasing order, and $(u_\epsilon^m)_{m \geq 1}$ be associated normalized eigenvectors of the spectral problem*

$$\begin{cases} -\operatorname{div} A^\epsilon(x) \nabla u_\epsilon^m(x) = \lambda_\epsilon^m \rho_\epsilon(x) u_\epsilon^m(x) & \text{in } \Omega \\ u_\epsilon^m = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.85)$$

Then, for any fixed $m \geq 1$,

$$\lim_{\epsilon \rightarrow 0} \lambda_\epsilon^m = \lambda^m,$$

and, up to a subsequence, u_ϵ^m converges weakly in $H_0^1(\Omega)$ to u^m , a normalized eigenvector associated to λ^m , which are solutions of the homogenized eigenvalue problem

$$\begin{cases} -\operatorname{div} A^*(x) \nabla u^m(x) = \lambda^m \rho(x) u^m(x) & \text{in } \Omega \\ u^m = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.86)$$

and $(\lambda^m)_{m \geq 1}$ is the complete family of eigenvalues of (1.86), labeled in increasing order.

Remark 1.3.17 Lemma 1.3.15 and Theorem 1.3.16 are not specific to the scalar setting: they also hold true for systems of equations like the elasticity system. In Theorem 1.3.16, the entire sequence of eigenvalues λ_ϵ^m converges to λ^m , but only a subsequence of eigenvectors u_ϵ^m converges to u^m . This is due to the possible multiplicity of the corresponding eigenvalue, which implies that there exist several independent eigenvectors.

Proof of Lemma 1.3.15. Let us define the Green operator of (1.84) by

$$\begin{aligned} S : L^2(\Omega) &\rightarrow L^2(\Omega) \\ f &\rightarrow u \end{aligned}$$

where u is the unique solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\operatorname{div}A(x)\nabla u(x) = \rho(x)f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Introducing the following scalar product on $L^2(\Omega)$

$$\langle f, g \rangle = \int_{\Omega} f(x)g(x)\rho(x)dx$$

makes it easy to check that S is self-adjoint and positive definite. Furthermore, if $(f_n)_{n \geq 1}$ is a sequence converging weakly in $L^2(\Omega)$, the sequence $(u_n = Sf_n)_{n \geq 1}$ converges strongly in $L^2(\Omega)$ by the Rellich theorem (Lemma 1.2.6). Thus, S is a compact operator. Now a classical result of operator theory states that the spectrum of a compact self-adjoint positive definite operator in a Hilbert space is made of a countable sequence of positive eigenvalues converging to zero, and that the eigenvectors can be chosen to be an orthonormal basis of $L^2(\Omega)$ (see, e.g., [58], [292]). The proof is complete upon remarking that the eigenvalues of S are precisely the inverses of those of (1.84). \square

Proof of Theorem 1.3.16. By the min-max principle, the m th eigenvalue λ_ϵ^m is defined by

$$\lambda_\epsilon^m = \min_{E \subset H_0^1(\Omega)} \max_{\substack{u \in E, u \neq 0 \\ \dim E = m}} \frac{\int_{\Omega} A^\epsilon \nabla u \cdot \nabla u dx}{\int_{\Omega} \rho_\epsilon u^2 dx}.$$

By assumptions, A^ϵ and ρ_ϵ are uniformly bounded, which implies

$$\frac{\int_{\Omega} \alpha \nabla u \cdot \nabla u dx}{\int_{\Omega} \rho_+ u^2 dx} \leq \frac{\int_{\Omega} A^\epsilon \nabla u \cdot \nabla u dx}{\int_{\Omega} \rho_\epsilon u^2 dx} \leq \frac{\int_{\Omega} \beta \nabla u \cdot \nabla u dx}{\int_{\Omega} \rho_- u^2 dx},$$

and therefore

$$\lambda_-^m \leq \lambda_\epsilon^m \leq \lambda_+^m,$$

where λ_-^m and λ_+^m are the m th eigenvalues of two elliptic equations similar to (1.85) but with constant coefficients independent of ϵ . Therefore, for fixed m the sequence $(\lambda_\epsilon^m)_{\epsilon>0}$ is bounded in \mathbb{R}^+ . This easily implies that the sequence of corresponding normalized eigenvectors u_ϵ^m is also bounded in $H_0^1(\Omega)$. Therefore, there exist a subsequence, still denoted with index ϵ , and limits such that λ_ϵ^m converges to λ^m and u_ϵ^m converges to u^m weakly in $H_0^1(\Omega)$. By the Rellich theorem (Lemma 1.2.6), u_ϵ^m converges to u^m strongly in $L^2(\Omega)$ and u^m is also normalized by $\|u^m\|_{L^2(\Omega)} = 1$. Lemma 1.2.4 implies that the right hand side of (1.85) converges to $\lambda^m \rho(x) u^m(x)$ weakly in $L^2(\Omega)$ and thus strongly in $H^{-1}(\Omega)$. By definition of H -convergence (see, in particular, Proposition 1.2.19), we deduce that u_ϵ^m converges weakly in $H_0^1(\Omega)$ to the unique solution v of the homogenized problem

$$\begin{cases} -\operatorname{div} A^*(x) \nabla v(x) = \lambda^m \rho(x) u^m(x) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

By uniqueness of the limit of the sequence u_ϵ^m , the two functions v and u^m coincide, and $v = u^m$ is a normalized eigenvector of the homogenized problem. By extracting a diagonal subsequence, still denoted by ϵ , we have just proved that, for any $m \geq 1$,

$$\lim_{\epsilon \rightarrow 0} \lambda_\epsilon^m = \lambda^m, \quad (1.87)$$

with $0 < \lambda^1 \leq \lambda^2 \leq \dots$. It remains to prove that there are no other eigenvalues than these $(\lambda^m)_{m \geq 1}$ for the homogenized problem, and that convergence (1.87) holds for the entire sequence ϵ . Assume that there exists another eigenvalue λ^0 with a normalized eigenvector u^0 . Let $m_0 \geq 1$ be the integer such that $\lambda_{m_0} < \lambda^0 < \lambda_{m_0+1}$. We define w_ϵ as the unique solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\operatorname{div} A^\epsilon(x) \nabla w_\epsilon(x) = \lambda^0 \rho_\epsilon(x) u^0(x) & \text{in } \Omega \\ w_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

By definition of H -convergence (see, again, Proposition 1.2.19), w_ϵ converges weakly in $H_0^1(\Omega)$ to u^0 . On the other hand, for any $m \geq 1$, the equations satisfied by w_ϵ and u_ϵ^m imply

$$\int_{\Omega} A^\epsilon \nabla w_\epsilon \cdot \nabla u_\epsilon^m = \lambda^0 \int_{\Omega} \rho_\epsilon u^0 u_\epsilon^m = \lambda_\epsilon^m \int_{\Omega} \rho_\epsilon w_\epsilon u_\epsilon^m. \quad (1.88)$$

Since w_ϵ and u_ϵ^m converge strongly in $L^2(\Omega)$, passing to the limit in (1.88) yields

$$\lambda^0 \int_{\Omega} \rho u^0 u^m = \lambda^m \int_{\Omega} \rho u^0 u^m,$$

which implies, λ^0 being different from λ^m , that

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} A^\epsilon \nabla w_\epsilon \cdot \nabla u_\epsilon^m = \lim_{\epsilon \rightarrow 0} \int_{\Omega} \rho_\epsilon w_\epsilon u_\epsilon^m = 0. \quad (1.89)$$

Let us show that for any sufficiently small $\epsilon > 0$, the subspace of $H_0^1(\Omega)$ spanned by $(u_\epsilon^1, \dots, u_\epsilon^{m_0}, w_\epsilon)$ has dimension $m_0 + 1$. If this were not true, for a subsequence, w_ϵ would be spanned by $(u_\epsilon^1, \dots, u_\epsilon^{m_0})$ and, upon orthogonality of the eigenvectors u_ϵ^m , it would be given by

$$w_\epsilon = \sum_{m=1}^{m_0} \frac{\int_{\Omega} \rho_\epsilon w_\epsilon u_\epsilon^m}{\int_{\Omega} \rho_\epsilon (u_\epsilon^m)^2} u_\epsilon^m.$$

By (1.89) we would deduce that w_ϵ converges to zero strongly in $L^2(\Omega)$, which is in contradiction with the fact that $u^0 \neq 0$. Thus, $\dim[u_\epsilon^1, \dots, u_\epsilon^{m_0}, w_\epsilon] = m + 1$, and the min-max principle yields

$$\lambda_\epsilon^{m_0+1} \leq \max_{u \in [u_\epsilon^1, \dots, u_\epsilon^{m_0}, w_\epsilon]} \frac{\int_{\Omega} A^\epsilon \nabla u \cdot \nabla u \, dx}{\int_{\Omega} \rho_\epsilon u^2 \, dx}. \quad (1.90)$$

Let us define the maximizer v_ϵ in the right hand side of (1.90) by

$$v_\epsilon(x) = \sum_{m=1}^{m_0} \alpha_\epsilon^m u_\epsilon^m(x) + \alpha_\epsilon w_\epsilon(x),$$

where, by homogeneity in the min-max principle, it can always be assumed that $\sum_{m=1}^{m_0} (\alpha_\epsilon^m)^2 + (\alpha_\epsilon)^2 = 1$. Up to a subsequence, there exist limits such

that each sequence α_ϵ^m and α_ϵ converges to α^m and α , respectively, satisfying also $\sum_{m=1}^{m_0} (\alpha^m)^2 + (\alpha)^2 = 1$. A simple computation using (1.89) shows that

$$\int_{\Omega} A^\epsilon \nabla v_\epsilon \cdot \nabla v_\epsilon = \sum_{m=1}^{m_0} (\alpha_\epsilon^m)^2 \lambda_\epsilon^m \int_{\Omega} \rho_\epsilon(u_\epsilon^m)^2 + (\alpha_\epsilon)^2 \lambda^0 \int_{\Omega} \rho_\epsilon(w_\epsilon)^2 + o(1),$$

and

$$\int_{\Omega} \rho_\epsilon(v_\epsilon)^2 = \sum_{m=1}^{m_0} (\alpha_\epsilon^m)^2 \int_{\Omega} \rho_\epsilon(u_\epsilon^m)^2 + (\alpha_\epsilon)^2 \int_{\Omega} \rho_\epsilon(w_\epsilon)^2 + o(1),$$

where $o(1)$ is a remainder going to zero with ϵ . Passing to the limit in (1.90) leads to

$$\lambda^{m_0+1} \leq \frac{\sum_{m=1}^{m_0} (\alpha^m)^2 \lambda^m \int_{\Omega} \rho(u^m)^2 + (\alpha)^2 \lambda^0 \int_{\Omega} \rho(u^0)^2}{\sum_{m=1}^{m_0} (\alpha^m)^2 \int_{\Omega} \rho(u^m)^2 + (\alpha)^2 \int_{\Omega} \rho(u^0)^2} \leq \lambda^0,$$

which is a contradiction with our hypothesis that $\lambda_{m_0} < \lambda_0 < \lambda_{m_0+1}$. Thus, the $(\lambda^m)_{m \geq 1}$ are all the eigenvalues of the homogenized problem (1.86). This implies that the limit of any converging subsequence of λ_ϵ^m is unique and equal to λ^m , therefore the entire sequence of eigenvalues does converge.

□

1.3.4 A Justification of Periodic Homogenization

This subsection is devoted to the mathematical justification of the heuristic two-scale asymptotic expansion method introduced in Section 1.1. The H -convergence method is applied to periodic homogenization and gives a rigorous proof of convergence to the homogenized limit. Recall that the periodic cell is the unit cube $Y = (0, 1)^N$. To take into account the periodic boundary conditions, we identify Y with the unit N -dimensional torus (this is easily done by gluing together opposite faces of Y). This unit torus is a smooth compact manifold without boundary. In the sequel, a periodic function in Y is actually defined, through this identification, as a function on the unit torus (this has the advantage that the periodicity is somehow built-in). For $1 \leq p \leq +\infty$, we define the Lebesgue space $L_p^\#(Y)$ of measurable and p -summable functions on the unit torus. Identifying the torus with Y , it is equivalently defined as a space of Y -periodic functions in \mathbb{R}^N by

$$L_p^\#(Y) = \left\{ f \in L_p^{\text{loc}}(\mathbb{R}^N) \text{ such that } f \text{ is } Y\text{-periodic} \right\} \quad (1.91)$$

equipped with the norm $\|f\|_{L^p(Y)}$. Similarly, we introduce the Sobolev space $H_\#^1(Y)$ of functions defined on the unit torus, which are, along with their first derivatives, measurable and squared summable. By the same identification, we have

$$H_\#^1(Y) = \left\{ f \in H_{loc}^1(\mathbb{R}^N) \text{ such that } f \text{ is } Y\text{-periodic} \right\} \quad (1.92)$$

equipped with the norm $\|f\|_{H^1(Y)}$. We shall also need the quotient space $H_\#^1(Y)/\mathbb{R}$ defined as the space of classes of functions in $H_\#^1(Y)$ equal up to an additive constant. In the context of H -convergence, we specialize the sequence of matrices $A^\epsilon(x)$ by taking

$$A^\epsilon(x) = A\left(\frac{x}{\epsilon}\right), \quad (1.93)$$

where $A(y)$ is a Y -periodic matrix which belongs to $L_\#^\infty(Y; \mathcal{M}_{\alpha,\beta})$ with $\mathcal{M}_{\alpha,\beta}$ defined by (1.28). As in Section 1.2.2 we study a scalar second order elliptic equation (modeling, for example, a conductivity problem). Let Ω be a bounded open set in \mathbb{R}^N . For any source term $f \in H^{-1}(\Omega)$, we denote by u_ϵ the unique solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\operatorname{div}\left(A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon\right) = f & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.94)$$

The goal of this section is to prove the following theorem.

Theorem 1.3.18 *The sequence A^ϵ defined by (1.93) H -converges to a constant homogenized matrix $A^* \in \mathcal{M}_{\alpha,\beta}$ defined by its entries*

$$A_{ij}^* = \int_Y A(y) (e_i + \nabla w_i) \cdot (e_j + \nabla w_j) dy, \quad (1.95)$$

where $(e_i)_{1 \leq i \leq N}$ is the canonical basis of \mathbb{R}^N , and $(w_i)_{1 \leq i \leq N}$ is the family of unique solutions in $H_\#^1(Y)/\mathbb{R}$ of the cell problems

$$\begin{cases} -\operatorname{div}A(y)(e_i + \nabla w_i(y)) = 0 & \text{in } Y \\ y \rightarrow w_i(y) & Y\text{-periodic.} \end{cases} \quad (1.96)$$

We remark that, in Theorem 1.3.18, the **entire** sequence A^ϵ H -converges to A^* (not merely a subsequence, as is the case in Theorem 1.2.16, which

is concerned with the compactness of H -convergence in full generality). We remark also that (1.95) provides an explicit formula for the homogenized matrix A^* (there is no such formula in the general case). Of course, these two facts are consequences of the periodicity assumption.

To prove Theorem 1.3.18 we need the following lemmas.

Lemma 1.3.19 *Let $f(y) \in L^2_{\#}(Y)$ be a periodic function. The sequence f_ϵ , defined by*

$$f_\epsilon(x) = f\left(\frac{x}{\epsilon}\right),$$

converges weakly in $L^2_{loc}(\mathbb{R}^N)$ to the average $\int_Y f(y)dy$.

Remark 1.3.20 *Lemma 1.3.19 is easily extended to the $L^p_{loc}(\mathbb{R}^N)$ setting for any $1 \leq p \leq +\infty$ (with the weak * convergence if $p = +\infty$).*

Proof. Let Ω be any bounded set in \mathbb{R}^N . Let us first prove that the sequence f_ϵ is bounded in $L^2(\Omega)$. Let $(Y_i^\epsilon)_{1 \leq i \leq n(\epsilon)}$ be a family of nonoverlapping cubes of size ϵ covering Ω . Each cube Y_i^ϵ is obtained from the unit cube Y by rescaling it at size ϵ plus a translation. We denote by x_i^ϵ the origin of the cube Y_i^ϵ . The volume of each cube is $|Y_i^\epsilon| = \epsilon^N$, and their total number is

$$n(\epsilon) = \frac{|\Omega|}{\epsilon^N} (1 + o(1)),$$

where here and below $o(1)$ denotes a remainder term which goes to zero with ϵ . By an obvious change of variables, the periodicity of f yields

$$\int_{Y_i^\epsilon} f_\epsilon^2(x)dx = \epsilon^N \int_Y f^2(y)dy,$$

which implies that

$$\|f_\epsilon\|_{L^2(\Omega)}^2 = \sum_{i=1}^{n(\epsilon)} \int_{Y_i^\epsilon \cap \Omega} f_\epsilon^2(x)dx = |\Omega| \left(\int_Y f^2(y)dy \right) (1 + o(1)).$$

This proves that f_ϵ is a bounded sequence in $L^2(\Omega)$. Let us denote by $m(f)$ the average of $f(y)$ in the unit cell Y , i.e.,

$$m(f) = \int_Y f(y)dy.$$

If we can prove that, for any smooth function ϕ (say, continuous with compact support in Ω), we have

$$\int_{\Omega} f_{\epsilon}(x)\phi(x)dx \rightarrow m(f) \int_{\Omega} \phi(x)dx, \quad (1.97)$$

this implies the desired weak convergence in $L^2(\Omega)$. By periodicity, we also have

$$\int_{Y_i^{\epsilon}} f_{\epsilon}(x)dx = \epsilon^N m(f).$$

Then

$$\left| \int_{Y_i^{\epsilon}} f_{\epsilon}(x)\phi(x)dx - \epsilon^N m(f)\phi(x_i^{\epsilon}) \right| \leq \epsilon^N m(|f|) \max_{x,x' \in Y_i^{\epsilon}} |\phi(x) - \phi(x')|,$$

which implies, upon summing over i ,

$$\left| \int_{\Omega} f_{\epsilon}(x)\phi(x)dx - \epsilon^N m(f) \sum_{i=1}^{n(\epsilon)} \phi(x_i^{\epsilon}) \right| \leq m(|f|) \epsilon^N n(\epsilon) \max_{|x-x'| \leq N\epsilon} |\phi(x) - \phi(x')|.$$

Since ϕ is uniformly continuous on Ω , we obtain

$$\max_{|x-x'| \leq N\epsilon} |\phi(x) - \phi(x')| = o(1),$$

while $\epsilon^N n(\epsilon)$ is uniformly bounded, and the Riemann sum satisfies

$$\epsilon^N \sum_{i=1}^{n(\epsilon)} \phi(x_i^{\epsilon}) = \int_{\Omega} \phi(x)dx + o(1).$$

Therefore we deduce the desired convergence (1.97). \square

Lemma 1.3.21 *Let $f(y) \in L^2_{\#}(Y)$ be a periodic function. There exists a unique solution in $H^1_{\#}(Y)/\mathbb{R}$ of*

$$\begin{cases} -\operatorname{div} A(y) \nabla w(y) = f & \text{in } Y \\ y \rightarrow w(y) & \text{Y-periodic,} \end{cases} \quad (1.98)$$

if and only if $\int_Y f(y)dy = 0$ (this is called the Fredholm alternative).

Remark 1.3.22 By identifying the unit cell Y with the unit torus, equation (1.98) can be seen as posed in the unit torus. Since the torus has no boundary, it has the advantage of requiring no boundary conditions. In other words, the formulation of (1.98) in the unit torus automatically includes the periodicity of the solution.

Proof. Due to the periodic boundary condition, a simple integration by parts yields $\int_Y (\operatorname{div} A(y) \nabla w(y)) dy = 0$ for any function $w \in H_{\#}^1(Y)$. Therefore, $\int_Y f(y) dy = 0$ is a necessary condition of existence of solutions for (1.98). Defining the quotient space $H_{\#}^1(Y)/\mathbb{R}$ of functions defined in $H_{\#}^1(Y)$ up to a constant, it is easily seen that $\|\nabla w\|_{L^2(Y)^N}$ is a norm for this quotient space. We check the assumptions of the Lax-Milgram lemma on the variational formulation of (1.98). Clearly, $\int_Y A(y) \nabla w \cdot \nabla \phi dy$ is a coercive continuous bilinear form on $H_{\#}^1(Y)/\mathbb{R}$. Furthermore, if $\int_Y f(y) dy = 0$, one finds

$$\int_Y f(y) w(y) dy = \int_Y f(y) \left(w(y) - \int_Y w(y) dy \right) dy,$$

which is a continuous linear form on $H_{\#}^1(Y)/\mathbb{R}$ thanks to the following Poincaré-Wirtinger inequality (e.g., [58])

$$\left\| w(y) - \int_Y w(y) dy \right\|_{L^2(Y)} \leq C \|\nabla w\|_{L^2(Y)^N}.$$

This proves that there exists a unique solution $w \in H_{\#}^1(Y)/\mathbb{R}$ of (1.98) if $\int_Y f(y) dy = 0$. \square

Proof of Theorem 1.3.18. The proof is a simple rephrasing of that of Theorem 1.2.16. The two first steps are identical, and only the third step is new. In the periodic setting, the test functions w_{ϵ}^i can be explicitly defined by

$$w_{\epsilon}^i(x) = x_i + \epsilon w_i\left(\frac{x}{\epsilon}\right) \quad \text{for } 1 \leq i \leq N,$$

where $w_i(y)$ is the unique solution of the cell problem (1.96) in $H^1(Y)/\mathbb{R}$ (see Lemma 1.3.21). The sequence w_{ϵ}^i is bounded in $H^1(\Omega)$ since

$$\nabla w_{\epsilon}^i(x) = e_i + (\nabla_y w_i)\left(\frac{x}{\epsilon}\right),$$

and by Lemma 1.3.19 it converges weakly to e_i in $H^1(\Omega)$ since, by periodicity, $\int_Y \nabla w_i(y) dy = 0$. Furthermore, by rescaling the cell problem (1.96), w_{ϵ}^i satisfies

$$-\operatorname{div} A^{\epsilon} \nabla w_{\epsilon}^i = 0 \text{ in } \Omega.$$

Another application of Lemma 1.3.19 implies that the sequence $A^\epsilon \nabla w_\epsilon^i$ converges weakly in $L^2(\Omega)^N$ to $\int_Y A(y) (e_i + \nabla w_i) dy$, which, by integration by parts in (1.95), is easily seen to be equal to $A^* e_i$.

This implies that any H -converging subsequence of A^ϵ H -converges to the same constant homogenized matrix A^* defined by (1.95). Therefore, the entire sequence A^ϵ H -converges to A^* . \square

We now prove a classical result (see, e.g., [91], [138]) which states that there is no loss of generality in the periodic setting of homogenization in the following sense. At each point x , a general H -limit $A^*(x)$ is attained as the limit of a sequence of periodic homogenized matrices.

Theorem 1.3.23 *Let A^ϵ be a sequence of matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ which H -converges to a limit A^* . For any x in Ω and any sufficiently small positive $h > 0$, let $A_{x,\epsilon,h}^*$ be the periodic homogenized matrix defined by its entries*

$$(A_{x,\epsilon,h}^*)_{ij} = \int_Y A^\epsilon(x + hy) (e_i + \nabla w_{x,\epsilon,h}^i) \cdot (e_j + \nabla w_{x,\epsilon,h}^j) dy, \quad (1.99)$$

where $(w_{x,\epsilon,h}^i(y))_{1 \leq i \leq N}$ is the family of unique solutions in $H_\#^1(Y)/\mathbb{R}$ of the cell problems

$$\begin{cases} -\operatorname{div} A^\epsilon(x + hy) (e_i + \nabla w_{x,\epsilon,h}^i(y)) = 0 & \text{in } Y \\ y \rightarrow w_{x,\epsilon,h}^i(y) & \text{Y-periodic.} \end{cases}$$

There exists a subsequence h going to zero such that

$$\lim_{h \rightarrow 0} \lim_{\epsilon \rightarrow 0} A_{x,\epsilon,h}^* = A^*(x)$$

for almost every $x \in \Omega$.

Remark 1.3.24 Theorem 1.3.23 implies that the set of all periodic H -limits (as defined by Theorem 1.3.18) is dense in the set of pointwise values of general H -limits. In other words, the values taken by periodic H -limits are not different from those obtained in the general case.

Proof. We follow the proof of [138]. For sufficiently small h , the cube $Q_{x,h} = x + (0, h)^N$ is included in Ω . By the very definition of H -convergence, it is easy to see that, if $A^\epsilon(x)$ H -converges to $A^*(x)$, then $A^\epsilon(x_0 + hx)$ H -converges to $A^*(x_0 + hx)$ for any $x_0 \in \Omega$ and $h > 0$ small. Therefore, by H -convergence in Y , $A^\epsilon(x + hy)$ H -converges to $A^*(x + hy)$ (recall that, by

virtue of Proposition 1.2.19, H -convergence is also valid for periodic boundary conditions). Passing to the limit in (1.99), we thus obtain

$$\lim_{\epsilon \rightarrow 0} (A_{x,\epsilon,h}^*)_{ij} = (A_{x,h}^*)_{ij} = \int_Y A^*(x + hy) (e_i + \nabla w_{x,h}^i) \cdot (e_j + \nabla w_{x,h}^j) dy,$$

where $(w_{x,h}^i(y))_{1 \leq i \leq N}$ are the unique solutions of

$$\begin{cases} -\operatorname{div} A^*(x + hy) (e_i + \nabla w_{x,h}^i(y)) = 0 & \text{in } Y \\ y \rightarrow w_{x,h}^i(y) & \text{Y-periodic.} \end{cases}$$

Intuitively as h goes to zero, $A^*(x + hy)$ converges to $A^*(x)$ (this is obvious, at least, for continuous functions). This intuition is correct, as shown by Lemma 1.3.26 below: there exists a subsequence $h \rightarrow 0$ such that $A^*(x + hy)$ converges strongly to $A^*(x)$ in $L^2(Y; \mathcal{M}_N)$ for almost every $x \in \Omega$. This implies that $w_{x,h}^i$ converges strongly to zero in $H_\#^1(Y)/\mathbb{R}$ as h goes to zero, and thus

$$\lim_{h \rightarrow 0} (A_{x,h}^*)_{ij} = A_{ij}^*(x),$$

for almost every $x \in \Omega$. Clearly, as a function of x , the periodic homogenized matrix $A_{x,\epsilon,h}^*$ belongs to $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$, and we have just proved that, for a diagonal subsequence in ϵ and h , $A_{x,\epsilon,h}^*$ converges pointwise to A^* in Ω . \square

Remark 1.3.25 *Of special interest for Chapter 2 is the following consequence of Theorem 1.3.23 in the particular case where the tensor takes only two values A and B belonging to $\mathcal{M}_{\alpha,\beta}$. In other words, there exists a sequence of characteristic functions χ^ϵ in $L^\infty(\Omega; \{0, 1\})$ such that $A^\epsilon(x) = \chi^\epsilon(x)A + (1 - \chi^\epsilon(x))B$. Up to a subsequence, there exists a function $\theta \in L^\infty(\Omega; [0, 1])$ such that χ^ϵ converges weakly* to θ in $L^\infty(\Omega; [0, 1])$. This limit θ is the local proportion of A in the H -limit A^* . On the other hand, the proportion of A in the periodic homogenized matrix $A_{x,\epsilon,h}^*$ is $\theta_{x,\epsilon,h} = \int_Y \chi^\epsilon(x + hy) dy$. Following the proof of Theorem 1.3.23, it is easy to check that, not only does $A_{x,\epsilon,h}^*$ converge to $A^*(x)$, but $\theta_{x,\epsilon,h}$ converges to $\theta(x)$ too. From a physical point of view, this means that, in this approximation process, the local proportions of A and B are also preserved in the limit.*

Lemma 1.3.26

1. Let $f(x)$ be a function in $L^1_{loc}(\mathbb{R}^N)$. The sequence f_h , defined by $f_h(x) = \int_Y f(x + hy) dy$ for $h > 0$, converges strongly to f in $L^1_{loc}(\mathbb{R}^N)$.

As a consequence, there exists a subsequence, still denoted by f_h , such that

$$\lim_{h \rightarrow 0} f_h(x) = f(x) \text{ a.e. in } \Omega. \quad (1.100)$$

2. Let $f(x)$ be a function in $L^2_{loc}(\mathbb{R}^N)$. For $h > 0$, define a sequence $f_{h,x}$ by $f_{h,x}(y) = f(x + hy)$. There exists a subsequence, still denoted by f_h , such that, for almost every $x \in \Omega$,

$$f_{x,h}(y) \rightarrow f(x) \text{ strongly in } L^2(Y). \quad (1.101)$$

Remark 1.3.27 Lemma 1.3.26 is sometimes referred to as the Lebesgue points lemma. Indeed, the points x for which convergence (1.100) holds are called Lebesgue points. For such points the value of $f(x)$ is clearly well-defined as the limit average of f on smaller and smaller sets around x . It can be proved (but it is harder) that (1.100) holds for the entire sequence $h > 0$. Note that in (1.101) $f(x)$ is considered a constant function of y in $L^2(Y)$.

Proof. We begin with the proof of (1.100). We check that

$$f_h(x) = \frac{1}{h^N} \int_{x+(0,h)^N} f(y) dy = \int_{\mathbb{R}^N} f(y) \rho_h(y-x) dy = f * \rho_h,$$

where $\rho_h(y) = h^{-N} \chi_Y(h^{-1}y)$ with χ_Y the characteristic function of the unit cube Y . Since smooth functions are dense in $L^1_{loc}(\mathbb{R}^N)$, there exists a sequence f_n of continuous functions such that f_n converges strongly to f in $L^1(\Omega)$ for any bounded set Ω . Clearly, for n fixed, the sequence $f_n * \rho_h$ converges strongly to f_n in $L^1(\Omega)$ as h goes to zero. To prove the same for f , we estimate

$$\int_{\Omega} |f * \rho_h - f| dx \leq \int_{\Omega} (|(f - f_n) * \rho_h| + |f_n * \rho_h - f_n| + |f_n - f|) dx. \quad (1.102)$$

The first term in the right hand side of (1.102) is bounded by $\|\rho_h\|_{L^1(\mathbb{R}^N)} \|f - f_n\|_{L^1_{loc}(\mathbb{R}^N)}$ and $\|\rho_h\|_{L^1(\mathbb{R}^N)} = 1$. Therefore, we deduce from (1.102) that

$$\int_{\Omega} |f * \rho_h - f| dx \leq \int_{\Omega} |f_n * \rho_h - f_n| dx + 2\|f - f_n\|_{L^1_{loc}(\mathbb{R}^N)}.$$

Letting h go to zero first, and then, letting n go to $+\infty$, yields the strong convergence of f_h to f . Eventually, applying Lemma 1.2.2 leads to (1.100).

To prove (1.101), we apply the convergence (1.100) to f_h and g_h with $g = f^2$. It implies that, for a subsequence h ,

$$\int_Y (f(x + hy) - f(x))^2 dy = \int_Y f^2(x + hy) dy + f^2(x) - 2f(x) \int_Y f(x + hy) dy$$

goes to zero with h , for almost every $x \in \Omega$. \square

1.3.5 Homogenization of Laminated Structures

In this subsection we assume that the geometry of the problem varies only in a single direction. In other words, the sequence of matrices A^ϵ depends only on a single space variable, say x_1 , as

$$A^\epsilon(x) \equiv A^\epsilon(x_1). \quad (1.103)$$

This type of assumption is valid when studying laminated composite materials for which the component phases are stacked in slices orthogonal to the e_1 direction. From a mathematical point of view, this case is a generalization of the one-dimensional setting. In particular, we show that H -convergence can be reduced to the usual weak convergence of some combinations of entries of the matrix A^ϵ . In effect, this yields another type of “explicit” formula for the homogenized matrix as in the one-dimensional case. We follow the original exposition of Murat and Tartar [204].

Theorem 1.3.28 *Let A^ϵ be a sequence in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ satisfying assumption (1.103). Then, A^ϵ H -converges to an homogenized matrix A^* if and only if the following convergences hold in $L^\infty(\Omega)$ weak *:*

$$\left\{ \begin{array}{l} \frac{1}{A_{11}^\epsilon} \rightharpoonup \frac{1}{A_{11}^*} \\ \frac{A_{1j}^\epsilon}{A_{11}^\epsilon} \rightharpoonup \frac{A_{1j}^*}{A_{11}^*} \quad 2 \leq j \leq N \\ \frac{A_{i1}^\epsilon}{A_{11}^\epsilon} \rightharpoonup \frac{A_{i1}^*}{A_{11}^*} \quad 2 \leq i \leq N \\ \left(A_{ij}^\epsilon - \frac{A_{1j}^\epsilon A_{i1}^\epsilon}{A_{11}^\epsilon} \right) \rightharpoonup \left(A_{ij}^* - \frac{A_{1j}^* A_{i1}^*}{A_{11}^*} \right) \quad 2 \leq i, j \leq N \end{array} \right. \quad (1.104)$$

where $(A_{ij}^\epsilon)_{1 \leq i,j \leq N}$ and $(A_{ij}^*)_{1 \leq i,j \leq N}$ denote the entries of A^ϵ and A^* , respectively.

Remark 1.3.29 Note that the weak convergences in (1.104) always hold at the price of extracting a subsequence. Note also that the homogenized matrix A^* also depends only on x_1 .

To prove Theorem 1.3.28 we need the following lemma due to Aubin [29].

Lemma 1.3.30 (Aubin's lemma) Let ω be a bounded open set in \mathbb{R}^N . Let $T > 0$ be a positive real number. Introduce the space E of functions $\phi(t, x)$ defined on $(0, T) \times \omega$ as

$$E = \left\{ \phi \in L^2((0, T); L^2(\omega)) \text{ and } \frac{\partial \phi}{\partial t} \in L^2((0, T); H^{-1}(\omega)) \right\}.$$

The space E is compactly embedded in $L^2((0, T); H^{-1}(\omega))$.

Proof of Theorem 1.3.28. Since by Proposition 1.2.18 the H -convergence is local, there is no loss of generality in assuming that Ω is a cylindrical domain $(0, T) \times \omega$ with ω a bounded open set in \mathbb{R}^{N-1} . Aubin's Lemma 1.3.30 will be applied several times with $t = x_1 \in (0, T)$ and $(x_2, \dots, x_N) \in \omega$. Note that we shall not use the compactness of the H -convergence, but rather give a new proof specific to the present setting. As usual, we adopt the notation of Definition 1.2.15 of H -convergence. For $f \in H^{-1}(\Omega)$, u_ϵ is the unique solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\operatorname{div} A^\epsilon(x) \nabla u_\epsilon(x) = f(x) & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Since u_ϵ is a bounded sequence in $H_0^1(\Omega)$, and the flux $\sigma_\epsilon = A^\epsilon \nabla u_\epsilon$ is also a bounded sequence in $L^2(\Omega)^N$, then up to a subsequence there exist limits such that

$$\begin{aligned} u_\epsilon &\rightharpoonup u \quad \text{weakly in } H_0^1(\Omega) \\ \sigma_\epsilon &\rightharpoonup \sigma \quad \text{weakly in } L^2(\Omega)^N. \end{aligned}$$

In particular, the first component σ_ϵ^1 of the flux is bounded in $L^2((0, T); L^2(\omega))$. From the equation satisfied by u_ϵ , we deduce

$$-\frac{\partial \sigma_\epsilon^1}{\partial x_1} = f + \sum_{i=2}^N \frac{\partial \sigma_\epsilon^i}{\partial x_i},$$

which implies that $\frac{\partial \sigma_\epsilon^1}{\partial x_1}$ is also bounded in $L^2((0, T); H^{-1}(\omega))$. By Lemma 1.3.30, σ_ϵ^1 converges strongly in $L^2((0, T); H^{-1}(\omega))$. By definition of σ_ϵ ,

$$\sigma_\epsilon^1 = A_{11}^\epsilon \frac{\partial u_\epsilon}{\partial x_1} + \sum_{i=2}^N A_{1i}^\epsilon \frac{\partial u_\epsilon}{\partial x_i},$$

from which we deduce, since A^ϵ depends only on x_1 ,

$$\frac{\partial u_\epsilon}{\partial x_1} = \frac{\sigma_\epsilon^1}{A_{11}^\epsilon} - \sum_{i=2}^N \frac{\partial}{\partial x_i} \left(\frac{A_{1i}^\epsilon}{A_{11}^\epsilon} u_\epsilon \right). \quad (1.105)$$

The key point is that we can pass to the limit in (1.105). Indeed, (1.104) implies that there exist limits A_{11}^* and A_{1i}^* for $2 \leq i \leq N$, such that

$$\frac{1}{A_{11}^\epsilon} \rightharpoonup \frac{1}{A_{11}^*} \quad \text{weakly } * \text{ in } L^\infty(\Omega)$$

$$\frac{A_{1i}^\epsilon}{A_{11}^\epsilon} \rightharpoonup \frac{A_{1i}^*}{A_{11}^*} \quad \text{weakly } * \text{ in } L^\infty(\Omega).$$

By the Rellich theorem (Lemma 1.2.6), u_ϵ converges also strongly in $L^2(\Omega)$, which implies

$$\frac{\partial}{\partial x_i} \left(\frac{A_{1i}^\epsilon}{A_{11}^\epsilon} u_\epsilon \right) \rightharpoonup \frac{\partial}{\partial x_i} \left(\frac{A_{1i}^*}{A_{11}^*} u \right) \quad \text{in } \mathcal{D}'(\Omega).$$

On the other hand, σ_ϵ^1 converges strongly in $L^2((0, T); H^{-1}(\omega))$, while $(A_{11}^\epsilon)^{-1}$ converges weakly in $L^2((0, T); H^1(\omega))$ (it does not depend on x_2, \dots, x_N), which yields

$$\frac{\sigma_\epsilon^1}{A_{11}^\epsilon} \rightharpoonup \frac{\sigma^1}{A_{11}^*} \quad \text{in } \mathcal{D}'(\Omega).$$

Recall that by virtue of Lemma 1.2.4 we can pass to the limit (in the sense of distributions) in the product of a weakly converging sequence by a strongly convergent one. Finally, as ϵ goes to zero, (1.105) yields

$$\frac{\partial u}{\partial x_1} = \frac{\sigma^1}{A_{11}^*} - \sum_{i=2}^N \frac{\partial}{\partial x_i} \left(\frac{A_{1i}^*}{A_{11}^*} u \right).$$

Since A^* depends only on x_1 , we recover

$$\sigma^1 = A_{11}^* \frac{\partial u}{\partial x_1} + \sum_{i=2}^N A_{1i}^* \frac{\partial u}{\partial x_i}.$$

For each $2 \leq j \leq N$, using (1.105) and the fact that A^ϵ depends only on x_1 , we obtain

$$\begin{aligned}\sigma_\epsilon^j &= A_{j1}^\epsilon \frac{\partial u_\epsilon}{\partial x_1} + \sum_{i=2}^N A_{ji}^\epsilon \frac{\partial u_\epsilon}{\partial x_i} \\ &= \frac{A_{j1}^\epsilon}{A_{11}^\epsilon} \sigma_\epsilon^1 + \sum_{i=2}^N \frac{\partial}{\partial x_i} \left[\left(A_{ji}^\epsilon - \frac{A_{1i}^\epsilon A_{j1}^\epsilon}{A_{11}^\epsilon} \right) u_\epsilon \right].\end{aligned}\tag{1.106}$$

As before, we can pass to the limit in (1.106). Indeed, (1.104) implies that there exist limits A_{j1}^* for $2 \leq j \leq N$, and A_{ij}^* for $2 \leq i, j \leq N$ such that

$$\begin{aligned}\frac{A_{j1}^\epsilon}{A_{11}^\epsilon} &\rightharpoonup \frac{A_{j1}^*}{A_{11}^*} && \text{weakly } * \text{ in } L^\infty(\Omega) \\ \left(A_{ij}^\epsilon - \frac{A_{1j}^\epsilon A_{i1}^\epsilon}{A_{11}^\epsilon} \right) &\rightharpoonup \left(A_{ij}^* - \frac{A_{1j}^* A_{i1}^*}{A_{11}^*} \right) && \text{weakly } * \text{ in } L^\infty(\Omega).\end{aligned}$$

Since u_ϵ and σ_ϵ^1 converge strongly in $L^2(\Omega)$ and $L^2((0, T); H^{-1}(\omega))$, respectively, the limit of (1.106) is

$$\sigma^j = \frac{A_{j1}^*}{A_{11}^*} \sigma^1 + \sum_{i=2}^N \frac{\partial}{\partial x_i} \left[\left(A_{ji}^* - \frac{A_{1i}^* A_{j1}^*}{A_{11}^*} \right) u \right] = \sum_{i=2}^N A_{ji}^* \frac{\partial u}{\partial x_i}.$$

We have thus obtained that $\sigma = A^* \nabla u$. The uniqueness of the limits implies that the entire sequences u_ϵ and σ_ϵ converge. Therefore, we have proved that the convergences (1.104) implies that A^ϵ H -converges to A^* . Conversely, if we assume that A^ϵ H -converges to A^* , then, extracting a subsequence, there exists another matrix B^* for which (1.104) holds true. The same argument as above proves that actually $B^* = A^*$. Therefore, (1.104) holds for A^* . \square

Remark 1.3.31 *From Theorem 1.3.28 we deduce a number of special cases of particular interest. Assume that the matrix A^ϵ is scalar and depends only on x_1*

$$A^\epsilon(x) \equiv a_\epsilon(x_1) I_2.$$

Then, an easy computation shows that the homogenized matrix is

$$A^* = \text{diag } (a(x_1), \bar{a}(x_1), \dots, \bar{a}(x_1)),\tag{1.107}$$

where \underline{a}^{-1} is the weak * limit of a_ϵ^{-1} in $L^\infty(\Omega)$ and \bar{a} is the weak * limit of a_ϵ in $L^\infty(\Omega)$. These limits are also called the **harmonic mean** for \underline{a} and the **arithmetic mean** for \bar{a} . In the context of electrical conductivity, the harmonic mean is the effective conductivity of a mixture of conductors placed in series (in the direction e_1), while the arithmetic mean is the effective conductivity of a mixture of conductors placed in parallel (in any direction orthogonal to e_1). This example shows that the effective conductivity need not be isotropic even if the constituents are isotropic. This result is easily generalized to the case of a diagonal matrix A^ϵ

$$A^\epsilon(x) \equiv \text{diag} \left(a_\epsilon^1(x_1), \dots, a_\epsilon^N(x_1) \right).$$

The corresponding homogenized matrix is

$$A^* = \text{diag} \left(\underline{a}^1(x_1), \bar{a}^2(x_1), \dots, \bar{a}^N(x_1) \right), \quad (1.108)$$

where \underline{a}^1 is the harmonic mean of the sequence a_ϵ^1 , and \bar{a}^i is the arithmetic mean of the sequence a_ϵ^i . We remark that, in the first example (1.107), the matrix A^ϵ is scalar, but, by a simple change of orthogonal basis (a rotation in the physical space), the homogenized matrix A^* may even be nondiagonal.

Another example of application of Theorem 1.3.28 is the celebrated **lamination formula**, which gives the homogenized properties of a simple lamination of two general constituents (possibly nonisotropic and even nonsymmetric).

Lemma 1.3.32 *Let $\chi_\epsilon(x_1)$ be a sequence of characteristic functions that converges to a limit $\theta(x_1)$ in $L^\infty(\Omega; [0, 1])$ weakly * (although χ_ϵ is always equal to 0 or 1, its limit θ can take any value between 0 and 1). Let A and B be two constant matrices in $\mathcal{M}_{\alpha, \beta}$. Define a sequence of matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha, \beta})$*

$$A^\epsilon(x_1) = \chi_\epsilon(x_1)A + (1 - \chi_\epsilon(x_1))B.$$

The sequence A^ϵ H-converges to A^ , which depends only on x_1 and is given by the formula*

$$A^* = \theta A + (1 - \theta)B - \frac{\theta(1 - \theta)}{(1 - \theta)Ae_1 \cdot e_1 + \theta Be_1 \cdot e_1} (A - B)e_1 \otimes (A - B)^t e_1. \quad (1.109)$$

Remark 1.3.33 Recall that the characteristic function of a subset of Ω is defined as the function equal to one in this subset and to zero outside. If A and B correspond to two components, χ_ϵ is therefore the characteristic function of the part of Ω occupied by A . The matrix A^ϵ models the mixture of these two components, and the limit θ is the macroscopic or averaged density of A (see Chapter 2 for more details). Since χ_ϵ depends only on x_1 , this type of mixture of A and B is a simple lamination, i.e., is made of slices of A and B orthogonal to the direction e_1 (see Figure 2.1). There is no loss of generality in assuming that χ_ϵ converges weakly to a limit density θ , since it is always true up to a subsequence. Of course $A^\epsilon(x_1)$ satisfies assumption (1.103) but no other hypothesis is required. In particular, no periodicity or randomness is assumed, which implies that the limit density θ may depend on x_1 as well.

Proof of Lemma 1.3.32. We give two different proofs of Lemma 1.3.32 which are both of interest. The first one relies on Theorem 1.3.28 and on a simple but tedious computation of A^* . The second one does not use Theorem 1.3.28 and is more appealing from a physical point of view.

First proof. From Theorem 1.3.28 we know that A_{11}^* is just the harmonic mean of A_{11}^ϵ , i.e.,

$$A_{11}^* = \left(\frac{\theta}{A_{11}} + \frac{1-\theta}{B_{11}} \right)^{-1}.$$

Similarly, for $2 \leq j \leq N$,

$$A_{1j}^* = A_{11}^* \left(\theta \frac{A_{1j}}{A_{11}} + (1-\theta) \frac{B_{1j}}{B_{11}} \right)$$

with a symmetric formula for A_{ij}^* , $2 \leq i \leq N$. Finally, for $2 \leq j \leq N$,

$$A_{ij}^* = \frac{A_{1j}^* A_{i1}^*}{A_{11}^*} + \left(\theta \left(A_{ij} - \frac{A_{1j} A_{i1}}{A_{11}} \right) + (1-\theta) \left(B_{ij} - \frac{B_{1j} B_{i1}}{B_{11}} \right) \right).$$

Then, a simple computation leads to formula (1.109).

Second proof. Since the H -convergence is local, we can assume with no loss of generality that $\Omega = \mathbb{R}^N$. To compute directly A^* without using Theorem 1.3.28, we construct a sequence u_ϵ of solutions of

$$-\operatorname{div} A^\epsilon \nabla u_\epsilon = 0 \text{ in } \mathbb{R}^N, \quad (1.110)$$

which is piecewise linear in each phase. Let a and b be two vectors in \mathbb{R}^N . We seek a solution u_ϵ such that $\nabla u_\epsilon = a\chi_\epsilon + b(1 - \chi_\epsilon)$, i.e.

$$u_\epsilon = \chi_\epsilon(x_1)a \cdot x + (1 - \chi_\epsilon(x_1))b \cdot x + c_\epsilon(x_1),$$

where $c_\epsilon(x_1)$ is a piecewise constant function such that u_ϵ is continuous throughout \mathbb{R}^N . The continuity of u_ϵ gives a necessary condition on the vectors a and b . At an interface between A and B (which is an hyperplane with normal vector e_1), the function $c_\epsilon(x_1)$ is discontinuous, but constant when moving on each side of this hyperplane. In order to eliminate this constant, we take two points x and y on the same interface, and we subtract the continuity equations for u_ϵ at x and y to get

$$(a - b) \cdot x = (a - b) \cdot y.$$

Thus there exists a real number $t \in \mathbb{R}$ such that

$$b - a = te_1.$$

The expression of u_ϵ simplifies in

$$u_\epsilon = a \cdot x + (1 - \chi_\epsilon(x_1))tx_1 + c_\epsilon(x_1),$$

where $c_\epsilon(x_1)$ is the unique piecewise constant function, ensuring that u_ϵ is continuous and belongs to $H_{loc}^1(\mathbb{R}^N)$ (unique up to a global constant in \mathbb{R}^N).

Then

$$\begin{aligned} \nabla u_\epsilon &= \chi_\epsilon(x_1)a + (1 - \chi_\epsilon(x_1))b \\ \sigma_\epsilon &= \chi_\epsilon(x_1)Aa + (1 - \chi_\epsilon(x_1))Bb. \end{aligned} \tag{1.111}$$

To satisfy $-\operatorname{div}\sigma_\epsilon = 0$ in the sense of distributions in \mathbb{R}^N , since σ_ϵ is piecewise constant, it remains to check the jump condition at the interfaces, i.e.,

$$Aa \cdot e_1 = Bb \cdot e_1.$$

Since $b - a = te_1$, the above yields the following value for t :

$$t = \frac{(A - B)a \cdot e_1}{Be_1 \cdot e_1}.$$

With this value of t , and for any vector a , u_ϵ is a solution of (1.110). From (1.111) we deduce the following convergences:

$$\begin{aligned} \nabla u_\epsilon &\rightharpoonup \nabla u = \theta a + (1 - \theta)b && \text{weakly in } L_{loc}^2(\mathbb{R}^N)^N \\ \sigma_\epsilon &\rightharpoonup \sigma = \theta Aa + (1 - \theta)Bb && \text{weakly in } L_{loc}^2(\mathbb{R}^N)^N. \end{aligned}$$

By Theorem 1.2.16 there exist a subsequence and an homogenized matrix A^* such that A^ϵ H -converges to A^* . Then, by Definition 1.2.15, for any vector $a \in \mathbb{R}^N$

$$\sigma = A^* \nabla u,$$

i.e.,

$$\theta Aa + (1 - \theta)Bb = A^* (\theta a + (1 - \theta)b).$$

Introducing a vector $c = \theta a + (1 - \theta)b$, since $b = a + te_1$ with $t = \frac{(A-B)a \cdot e_1}{Be_1 \cdot e_1}$, we find

$$a = c - (1 - \theta) \frac{(A - B)c \cdot e_1}{(1 - \theta)Ae_1 \cdot e_1 + \theta Be_1 \cdot e_1} e_1.$$

Then, a simple computation gives

$$A^*c = \theta Ac + (1 - \theta)Bc - \frac{\theta(1 - \theta)(A - B)c \cdot e_1}{(1 - \theta)Ae_1 \cdot e_1 + \theta Be_1 \cdot e_1} (A - B)e_1,$$

which is simply the desired formula (1.109). □

Corollary 1.3.34 *When the matrix $(A - B)$ is invertible, formula (1.109) of Lemma 1.3.32 is equivalent to*

$$\theta (A^* - B)^{-1} = (A - B)^{-1} + \frac{(1 - \theta)}{Be_1 \cdot e_1} e_1 \otimes e_1. \quad (1.112)$$

If $(A - B)$ is not invertible but symmetric, then (1.112) holds on $\text{Ker}(A - B)^\perp$ (which is equal to the range of $A - B$), while $A^ = A = B$ on $\text{Ker}(A - B)$.*

Remark 1.3.35 *The lamination formula (1.109) delivered by Lemma 1.3.32 is classical. However the idea of writing it in the equivalent (but strange) form (1.112) is due to Tartar [274]. The advantage of Tartar's formula is that it can be iterated, as will be demonstrated in Chapter 2.*

Proof. The proof is a simple computation based on the fact that, M being an invertible matrix, the inverse of $M + c(Me \otimes M^t e)$ is

$$M^{-1} - \frac{c}{1 + c(Me \cdot e)} e \otimes e.$$

□

1.3.6 Corrector Results

This subsection is devoted to so-called *corrector results*, the goal of which is to improve the convergence of the solution gradients ∇u_ϵ by adding corrector terms. This has the effect of transforming a weak convergence into a strong one.

Let A^ϵ be a sequence of matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ that H -converges to a limit A^* . Define N sequences of oscillating test functions $(w_\epsilon^i)_{1 \leq i \leq N}$ satisfying

$$\begin{aligned} w_\epsilon^i &\rightharpoonup x_i \text{ weakly in } H^1(\Omega) \\ g_\epsilon^i = -\operatorname{div}(A^\epsilon \nabla w_\epsilon^i) &\rightarrow g_i = -\operatorname{div}(A^* e_i) \text{ strongly in } H^{-1}(\Omega). \end{aligned} \quad (1.113)$$

Recall that the third step in the proof of Theorem 1.2.16 (on the compactness of H -convergence) showed that, up to a subsequence, there always exist oscillating test functions satisfying (1.113). The purpose of such functions $(w_\epsilon^i)_{1 \leq i \leq N}$ is to define the homogenized tensor A^* (see Remark 1.3.5). It turns out that these functions can also be seen as correctors for the homogenization process.

Definition 1.3.36 Let A^ϵ be a sequence of matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ that H -converges to a limit A^* . Let $(w_\epsilon^i)_{1 \leq i \leq N}$ be a family of test functions satisfying (1.113). The matrix W^ϵ defined by its columns $(\nabla w_\epsilon^i)_{1 \leq i \leq N}$ is called a *corrector matrix*.

A priori, the oscillating test functions $(w_\epsilon^i)_{1 \leq i \leq N}$ are not uniquely defined. However, we shall now prove that they are unique up to a strongly convergent additive term.

Lemma 1.3.37 Let A^ϵ be a sequence of matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ that H -converges to a limit A^* . The corrector matrix W^ϵ is unique in the sense that, if there exist two corrector matrices W^ϵ and \tilde{W}^ϵ such that their columns satisfy (1.113), then their difference $W^\epsilon - \tilde{W}^\epsilon$ converges strongly to zero in $L^2_{loc}(\Omega; \mathcal{M}_N)$.

Proof. For $1 \leq i \leq N$, let w_ϵ^i and \tilde{w}_ϵ^i be two functions satisfying (1.113). Let $\phi \in \mathcal{D}(\Omega)$ be a smooth function with compact support in Ω . Integrating

by parts, we obtain

$$\begin{aligned}
\alpha \|\phi \nabla(w_\epsilon^i - \tilde{w}_\epsilon^i)\|_{L^2(\Omega)^N}^2 &\leq \int_{\Omega} \phi^2 A^\epsilon \nabla(w_\epsilon^i - \tilde{w}_\epsilon^i) \cdot \nabla(w_\epsilon^i - \tilde{w}_\epsilon^i) dx \\
&= - \int_{\Omega} (\operatorname{div} A^\epsilon \nabla(w_\epsilon^i - \tilde{w}_\epsilon^i)) \phi^2 (w_\epsilon^i - \tilde{w}_\epsilon^i) dx \\
&\quad - \int_{\Omega} (w_\epsilon^i - \tilde{w}_\epsilon^i) A^\epsilon \nabla(w_\epsilon^i - \tilde{w}_\epsilon^i) \cdot \nabla(\phi^2) dx.
\end{aligned} \tag{1.114}$$

Both terms in the right hand side of (1.114) tend to zero with ϵ : the first one because of assumption (1.113), and the second one by the Rellich theorem (Lemma 1.2.6), which implies that $(w_\epsilon^i - \tilde{w}_\epsilon^i)$ converges strongly to zero in $L^2(\Omega)$. We deduce that $\nabla(w_\epsilon^i - \tilde{w}_\epsilon^i)$ converges strongly to zero in $L_{loc}^2(\Omega)^N$. \square

Lemma 1.3.38 *Let A^ϵ be a sequence of matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ that H -converges to a limit A^* . Then*

$$\begin{aligned}
W^\epsilon &\rightharpoonup I_2 \text{ weakly in } L^2(\Omega; \mathcal{M}_N) \\
A^\epsilon W^\epsilon &\rightharpoonup A^* \text{ weakly in } L^2(\Omega; \mathcal{M}_N) \\
(W^\epsilon)^t A^\epsilon W^\epsilon &\rightharpoonup A^* \text{ in } \mathcal{D}'(\Omega; \mathcal{M}_N).
\end{aligned} \tag{1.115}$$

Proof. The first line of (1.115) is just a consequence of property (1.113) satisfied by the columns of W^ϵ . The second line of (1.115) follows by definition (1.60) of the homogenized matrix A^* . Finally the third line is nothing but formula (1.67) which is established in Remark 1.3.5. \square

Theorem 1.3.39 *Let A^ϵ be a sequence of matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ which H -converges to A^* . For $f \in H^{-1}(\Omega)$, let u_ϵ be the solution of*

$$\begin{cases} -\operatorname{div} A^\epsilon(x) \nabla u_\epsilon(x) = f(x) & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Let u be the weak limit of u_ϵ in $H_0^1(\Omega)$, i.e., the solution of the homogenized equation,

$$\begin{cases} -\operatorname{div} A^*(x) \nabla u(x) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then

$$\nabla u_\epsilon = W^\epsilon \nabla u + r_\epsilon, \tag{1.116}$$

where W^ϵ is the corrector matrix and

$$r_\epsilon \rightarrow 0 \text{ strongly in } L^1_{loc}(\Omega)^N. \quad (1.117)$$

Proof. Let $\phi \in \mathcal{D}(\Omega)$ be a smooth function with compact support in Ω . Let $u_n \in \mathcal{D}(\Omega)$ be a sequence of smooth functions that converges strongly in $H_0^1(\Omega)$ to the solution u of the homogenized equation. We now prove that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{\Omega} \phi A^\epsilon (\nabla u_\epsilon - W^\epsilon \nabla u_n) \cdot (\nabla u_\epsilon - W^\epsilon \nabla u_n) dx = \\ \int_{\Omega} \phi A^* (\nabla u - \nabla u_n) \cdot (\nabla u - \nabla u_n) dx. \end{aligned} \quad (1.118)$$

Developing the left hand side of (1.118), we obtain

$$\begin{aligned} & \int_{\Omega} \phi (W^\epsilon)^t A^\epsilon W^\epsilon \nabla u_n \cdot \nabla u_n dx + \int_{\Omega} \phi A^\epsilon \nabla u_\epsilon \cdot \nabla u_\epsilon dx \\ & - \int_{\Omega} \phi A^\epsilon \nabla u_\epsilon \cdot W^\epsilon \nabla u_n dx - \int_{\Omega} \phi A^\epsilon W^\epsilon \nabla u_n \cdot \nabla u_\epsilon dx. \end{aligned}$$

We pass to the limit in the first term by using Lemma 1.3.38, and in the three other terms by application of the div-curl Lemma 1.3.1 (recall that the columns of W^ϵ are gradients), and we get the desired result. Choosing $\phi \equiv 1$ in ω , by coercivity of A^ϵ , (1.118) yields the following bound

$$\lim_{\epsilon \rightarrow 0} \|\nabla u_\epsilon - W^\epsilon \nabla u_n\|_{L^2(\omega)^N}^2 \leq \alpha^{-1} \int_{\Omega} \phi A^* (\nabla u - \nabla u_n) \cdot (\nabla u - \nabla u_n) dx.$$

If u is smooth (i.e. we can choose $u_n = u$), this finishes the proof. If u is not smooth, then we estimate $(u_\epsilon - W^\epsilon \nabla u)$ in the $L^1(\omega)$ norm

$$\begin{aligned} \|\nabla u_\epsilon - W^\epsilon \nabla u\|_{L^1(\omega)^N} & \leq \|\nabla u_\epsilon - W^\epsilon \nabla u_n\|_{L^1(\omega)^N} \\ & + \|W^\epsilon\|_{L^2(\omega; \mathcal{M}_N)} \|\nabla(u - u_n)\|_{L^2(\omega)^N} \\ & \leq C \left(\|\nabla u_\epsilon - W^\epsilon \nabla u_n\|_{L^2(\omega)^N} + \|\nabla(u - u_n)\|_{L^2(\omega)^N} \right) \end{aligned}$$

because W^ϵ is bounded in $L^2(\omega; \mathcal{M}_N)$. Passing to the limit, first as ϵ goes to zero, and second as n goes to infinity, yields (1.117). \square

Remark 1.3.40 In the context of Theorem 1.3.39, we immediately deduce that, if u is smoother, say $u \in H^2(\Omega)$, then

$$u_\epsilon = u + \sum_{i=1}^N (w_\epsilon^i - x_i) \frac{\partial u}{\partial x_i} + r_\epsilon, \quad (1.119)$$

with

$$r_\epsilon \rightarrow 0 \text{ strongly in } W_{loc}^{1,1}(\Omega).$$

In the periodic case, it was proved in Theorem 1.3.18 that the test functions are precisely defined by

$$w_\epsilon^i(x) = x_i + \epsilon w_i\left(\frac{x}{\epsilon}\right),$$

where $w_i(y)$ is the solution of the cell problem (1.96) in the period $Y = (0, 1)^N$. Therefore, (1.119) is a rigorous justification of the two first terms in the two-scale asymptotic expansion (see Section 1.1).

The corrector matrix W^ϵ can also be explicit in the case of laminated structures (see Subsection 1.3.5). Indeed, it is easily checked that

$$\begin{cases} W_{11}^\epsilon = \frac{A_{11}^*}{A_{11}^\epsilon}, \\ W_{1j}^\epsilon = \frac{A_{1j}^* - A_{1j}^\epsilon}{A_{11}^\epsilon} \text{ for } 2 \leq j \leq N, \\ W_{ii}^\epsilon = 1 \text{ for } 2 \leq i \leq N, \\ W_{ij}^\epsilon = 0 \text{ for } i \neq j, 2 \leq i \leq N, 1 \leq j \leq N. \end{cases}$$

The physical interpretation of the corrector matrix is as follows. H -convergence implies the weak convergence of the solutions u_ϵ in $H_0^1(\Omega)$. By the Rellich theorem (Lemma 1.2.6), u_ϵ also converges almost everywhere in Ω (up to a subsequence). However, the flux $\sigma_\epsilon = A^\epsilon \nabla u_\epsilon$ converges only weakly in $L^2(\Omega)^N$, which means that there is no pointwise convergence of σ_ϵ . The interest of the corrector matrix W^ϵ is that it allows to quantify this lack of pointwise convergence. Indeed, a straightforward application of Theorem 1.3.39 shows that

$$\sigma_\epsilon - A^\epsilon W^\epsilon \nabla u \rightarrow 0 \text{ strongly in } L_{loc}^1(\Omega)^N,$$

which implies that this quantity converges almost everywhere (up to a subsequence). The corrector matrix has also been explicitly computed by Briane [60] in the case of a reiterated laminated structure (i.e., laminates of laminates; see Subsection 2.2.1).

The corrector result, Theorem 1.3.39, can be improved if we use the following theorem due to Meyers [184] (its proof is also reproduced in [50]). This theorem implies that the corrector matrix W^ϵ , which is uniformly bounded in $L^2(\Omega; \mathcal{M}_N)$, is actually bounded in better spaces, namely in $L^p(\Omega; \mathcal{M}_N)$ for some $p > 2$.

Theorem 1.3.41 (Meyers theorem) *Let Ω be a bounded open set in \mathbb{R}^N with a smooth boundary of class C^2 . Let $A(x)$ be a matrix in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$. For $f \in H^{-1}(\Omega)$, let u be the unique solution in $H_0^1(\Omega)$ of*

$$\begin{cases} -\operatorname{div}(A(x)\nabla u(x)) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

There exist a number $p > 2$ and a positive constant $C > 0$, which both depend only on α, β , and Ω , such that, if f belongs to $W^{-1,p}(\Omega)$, then the solution u belongs to $W_0^{1,p}(\Omega)$ and satisfies

$$\|u\|_{W_0^{1,p}(\Omega)} \leq C\|f\|_{W^{-1,p}(\Omega)}.$$

Remark 1.3.42 *Meyers theorem is a higher integrability result for solutions of second order elliptic equations. Its interest in the context of homogenization is that the exponent p and the constant C are uniform for the whole class of tensors in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$. The best (i.e., largest) number p in Theorem 1.3.41 is called the Meyers exponent. If Ω is not smooth, then local estimates can still be obtained. We emphasize that Meyers theorem holds true also for systems (in particular that of the linearized elasticity) and for higher order equations.*

Corollary 1.3.43 *Under the same assumptions than Theorem 1.3.39, the remainder $r_\epsilon = \nabla u_\epsilon - W^\epsilon \nabla u$ converges strongly to zero in $L_{loc}^r(\Omega)^N$ with*

$$\frac{1}{r} = \max\left(\frac{1}{2}, \frac{1}{p} + \frac{1}{q}\right)$$

where $2 \leq p \leq +\infty$ is the Meyers exponent such that the corrector matrix W^ϵ is uniformly bounded in $L^p(\Omega; \mathcal{M}_N)$, and $2 \leq q < +\infty$ is an exponent such that ∇u belongs to $L_{loc}^q(\Omega)^N$.

Proof. First of all, we check the assumption of Meyers theorem for the sequence w_ϵ^i , satisfying (1.113), constructed in step 3 of the proof of Theorem 1.2.16. By definition, it satisfies

$$-\operatorname{div}(A^\epsilon \nabla w_\epsilon^i) = -\operatorname{div}(A^* e_i) \text{ in } \Omega,$$

where the right hand side belongs to $W^{-1,\infty}(\Omega)$. Therefore, by Theorem 1.3.41, for each subset ω , compactly embedded in Ω , there exist an exponent $p > 2$ and a positive constant C (independent of ϵ) such that

$$\|\nabla w_\epsilon^i\|_{L^p(\omega)^N} \leq C < +\infty.$$

In the last step of proof of Theorem 1.3.39 (when u is not smooth), we give a different estimate of $(u_\epsilon - W^\epsilon \nabla u)$ in the $L^r(\omega)$ norm. For $1 < m \leq +\infty$ and $\frac{1}{m} + \frac{1}{m'} = 1$, we have

$$\begin{aligned} \|\nabla u_\epsilon - W^\epsilon \nabla u\|_{L^r(\omega)^N} &\leq \|\nabla u_\epsilon - W^\epsilon \nabla u_n\|_{L^r(\omega)^N} \\ &\quad + \|W^\epsilon\|_{L^{mr}(\omega; \mathcal{M}_N)} \|\nabla(u - u_n)\|_{L^{m'r}(\omega)^N} \\ &\leq C \left(\|\nabla u_\epsilon - W^\epsilon \nabla u_n\|_{L^2(\omega)^N} + \|\nabla(u - u_n)\|_{L^q(\omega)^N} \right) \end{aligned}$$

if $1 \leq r \leq 2$, $mr \leq p$, and $m'r \leq q$. Existence of such a m is ensured if $1/r \geq 1/p + 1/q$. We first pass to the limit as ϵ goes to zero, and second as n goes to infinity (since smooth functions are dense in $W_{loc}^{1,q}(\Omega)$ with $q < +\infty$) to get the desired result. \square

Meyers theorem can improve several results of homogenization. For example, Proposition 1.2.18 on the locality of the H -convergence can be generalized by evaluating the distance between two H -limits [100].

Proposition 1.3.44 *There exist a positive constant $C > 0$ and a number $\delta > 1$ such that, for any two sequences of matrices $A^\epsilon(x)$ and $B^\epsilon(x)$ in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$, that H -converge to $A^*(x)$ and $B^*(x)$, respectively, we have*

$$\|A^* - B^*\|_{L^\delta(\Omega; \mathcal{M}_N)} \leq C \liminf_{\epsilon \rightarrow 0} \|A^\epsilon - B^\epsilon\|_{L^\delta(\Omega; \mathcal{M}_N)}.$$

This result is actually true in the $L^p(\Omega; \mathcal{M}_N)$ norm for any p such that $1 \leq p \leq \delta$, and the number δ is one-half the Meyers exponent.

Proof. Let $(w_\epsilon^i)_{1 \leq i \leq N}$ be the family of test functions satisfying (1.113). Let $(v_\epsilon^i)_{1 \leq i \leq N}$ be another family of test functions defined also by (1.113) for the sequence $(B^\epsilon)^t$. By Lemma 1.3.10, $(B^\epsilon)^t$ H -converges to $(B^*)^t$. Applying the div-curl Lemma 1.3.1 yields the convergences

$$A^\epsilon \nabla w_\epsilon^i \cdot \nabla v_\epsilon^j \rightharpoonup A^* e_i \cdot e_j$$

and

$$B^\epsilon \nabla w_\epsilon^i \cdot \nabla v_\epsilon^j = \nabla w_\epsilon^i \cdot (B^\epsilon)^t \nabla v_\epsilon^j \rightharpoonup e_i \cdot (B^*)^t e_j = B^* e_i \cdot e_j$$

in the sense of distributions in Ω . Denoting by W^ϵ the matrix of columns $(\nabla w_\epsilon^i)_{1 \leq i \leq N}$, and V^ϵ that of columns $(\nabla v_\epsilon^i)_{1 \leq i \leq N}$, this result implies that the matrix $(V^\epsilon)^t (A^\epsilon - B^\epsilon) W^\epsilon$ converges to $A^* - B^*$ in the sense of distributions. By construction, the right hand sides of $-\operatorname{div} A^\epsilon \nabla w_\epsilon^i$ and $-\operatorname{div} B^\epsilon \nabla v_\epsilon^i$ belong to $W^{-1,\infty}(\Omega)$. Therefore, by virtue of the Meyers theorem (Theorem 1.3.41), there exists a number $p > 2$ (independent of ϵ) such that ∇w_ϵ^i and ∇v_ϵ^j are uniformly bounded in $L^p(\Omega)^N$ for $1 \leq i, j \leq N$. Therefore, $(V^\epsilon)^t (A^\epsilon - B^\epsilon) W^\epsilon$ is bounded in $L^\delta(\Omega; \mathcal{M}_N)$ with $\delta = p/2$, and it converges weakly to $A^* - B^*$ in $L^\delta(\Omega; \mathcal{M}_N)$. The lower semicontinuity of the weak convergence yields the desired result. \square

1.4 Generalization to the Elasticity System

This section generalizes the previous results of this chapter to the elasticity setting. Most of the proofs are not repeated here since they are easy extensions of their counterparts for the conductivity case.

1.4.1 Problem Statement

Let Ω be a bounded open set in \mathbb{R}^N . Let \mathcal{M}_N^s be the space of symmetric matrices (or symmetric second order tensors). Let \mathcal{M}_N^4 be the space of symmetric fourth order tensors acting on symmetric matrices

$$\mathcal{M}_N^4 = \{A = (a_{ijkl})_{1 \leq i,j,k,l \leq N} \mid a_{ijkl} = a_{klij} = a_{jikl} = a_{ijlk}\}.$$

In the context of linearized elasticity a positive definite fourth order tensor is called a *Hooke's law*. Let $\alpha > 0$ and $\beta > 0$ be two positive constants such that $\alpha\beta \leq 1$. We define a space $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ of admissible Hooke's laws by

$$\mathcal{M}_{\alpha,\beta} = \left\{ A \in \mathcal{M}_N^4 \text{ such that } \begin{array}{l} A\xi : \xi \geq \alpha|\xi|^2 \\ A^{-1}\xi : \xi \geq \beta|\xi|^2 \end{array} \forall \xi \in \mathcal{M}_N^s \right\}. \quad (1.120)$$

Denoting by $\epsilon > 0$ a sequence of positive reals going to zero, we consider a sequence of Hooke's laws $A^\epsilon(x) \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$. For a right hand side $f \in H^{-1}(\Omega)^N$ (a vector-valued forcing term), the linearized elasticity system reads as

$$\begin{cases} -\operatorname{div} A^\epsilon(x) e(u_\epsilon)(x) = f(x) & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.121)$$

where u_ϵ is the unknown displacement vector in $H_0^1(\Omega)^N$, $e(u_\epsilon) = 1/2(\nabla u_\epsilon + (\nabla u_\epsilon)^t)$ is the strain tensor in $L^2(\Omega; \mathcal{M}_N^s)$ (more precisely the entries of $e(u)$ are $e(u)_{ij} = 1/2(\partial u_i / \partial x_j + \partial u_j / \partial x_i)$), and $\sigma_\epsilon = A^\epsilon e(u_\epsilon)$ is the stress tensor in $L^2(\Omega; \mathcal{M}_N^s)$ (the strain and stress tensors are symmetric). Equation (1.121) admits the following variational formulation, or weak form:

$$\int_{\Omega} A^\epsilon e(u_\epsilon) : e(\phi) dx = \langle f, \phi \rangle_{H^{-1}, H_0^1(\Omega)^N} \quad \forall \phi \in H_0^1(\Omega)^N, \quad (1.122)$$

where $\langle \cdot, \cdot \rangle_{H^{-1}, H_0^1(\Omega)^N}$ is the duality product between $H^{-1}(\Omega)^N$ and $H_0^1(\Omega)^N$, and $:$ is the scalar product on \mathcal{M}_N^s . If f is smooth, say $f \in L^2(\Omega)^N$, then the duality product $\langle f, \phi \rangle_{H^{-1}, H_0^1(\Omega)^N}$ coincides with the integral $\int_{\Omega} f(x) \cdot \phi(x) dx$. Since A^ϵ is coercive, replacing ϕ by u_ϵ in the variational formulation (1.122), we obtain

$$\begin{aligned} \alpha \|e(u_\epsilon)\|_{L^2(\Omega; \mathcal{M}_N^s)}^2 &\leq \int_{\Omega} A^\epsilon e(u_\epsilon) : e(u_\epsilon) dx \\ &= \langle f, u_\epsilon \rangle_{H^{-1}, H_0^1(\Omega)^N} \\ &\leq \|f\|_{H^{-1}(\Omega)^N} \|u_\epsilon\|_{H_0^1(\Omega)^N}. \end{aligned}$$

Since Ω is a bounded domain with homogeneous Dirichlet boundary conditions, the Korn inequality holds in Ω (see, e.g., [83]), i.e., there exists a constant C , which depends only on Ω , such that

$$\|\phi\|_{H_0^1(\Omega)^N} \leq C \|e(\phi)\|_{L^2(\Omega; \mathcal{M}_N^s)} \quad \forall \phi \in H_0^1(\Omega)^N.$$

Since the reverse inequality is obvious, it implies that $\|e(\phi)\|_{L^2(\Omega; \mathcal{M}_N^s)}$ is a norm in $H_0^1(\Omega)^N$ equivalent to the usual one. Therefore, by application of the Lax-Milgram lemma, (1.121) and (1.122) have a unique weak solution $u_\epsilon \in H_0^1(\Omega)^N$. Furthermore, this solution satisfies the a priori estimate

$$\|u_\epsilon\|_{H_0^1(\Omega)^N} \leq \frac{C}{\alpha} \|f\|_{H^{-1}(\Omega)^N}. \quad (1.123)$$

Estimate (1.123) implies that the sequence of solutions u_ϵ is bounded in $H_0^1(\Omega)$. By the relative compactness of bounded sets for the weak topology of $H_0^1(\Omega)^N$ (see Lemma 1.2.1), there exist a subsequence, still denoted by u_ϵ , and a limit $u \in H_0^1(\Omega)^N$ such that u_ϵ converges weakly to u in $H_0^1(\Omega)^N$.

Similarly, since the Hooke's laws A^ϵ are uniformly bounded, the sequence of stress tensors $\sigma_\epsilon = A^\epsilon e(u_\epsilon)$ is also bounded in $L^2(\Omega; \mathcal{M}_N^s)$. By the relative compactness of bounded sets for the weak topology of $L^2(\Omega; \mathcal{M}_N^s)$, there exist a subsequence, still denoted by σ_ϵ , and a limit stress σ such that σ_ϵ converges weakly to σ in $L^2(\Omega; \mathcal{M}_N^s)$. The goal of homogenization is to find the relationship between the limit strain and limit stress, $e(u)$ and σ , respectively.

1.4.2 H -convergence

We generalize the definition and the results of H -convergence to the elasticity setting (following the work of [110], [275]).

Definition 1.4.1 *A sequence of Hooke's laws $A^\epsilon(x)$ in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ is said to converge in the sense of homogenization, or simply to H -converge, to an homogenized, or H -limit, Hooke's law $A^*(x) \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ if, for any right hand side $f \in H^{-1}(\Omega)^N$, the sequence u_ϵ of solutions of (1.121) satisfies*

$$\begin{aligned} u_\epsilon &\rightharpoonup u \text{ weakly in } H_0^1(\Omega)^N \\ A^\epsilon e(u_\epsilon) &\rightharpoonup A^* e(u) \text{ weakly in } L^2(\Omega; \mathcal{M}_N^s), \end{aligned} \tag{1.124}$$

where u is the unique solution of the homogenized equation

$$\begin{cases} -\operatorname{div} A^*(x) e(u)(x) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.125}$$

As in the conductivity setting, the definition of H -convergence for Hooke's laws makes sense because of the following compactness theorem.

Theorem 1.4.2 *For any sequence $A^\epsilon(x)$ of Hooke's laws in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ there exist a subsequence, still denoted by A^ϵ , and an homogenized Hooke's law $A^*(x) \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ such that A^ϵ H -converges to A^* .*

The proof of this result is also based on a compensated compactness result which generalizes the div-curl lemma of Murat and Tartar. For the sake of completeness we state this lemma, which can be found in [110].

Lemma 1.4.3 *Let e^ϵ and σ^ϵ be two sequences in $L^2(\Omega; \mathcal{M}_N^s)$ such that*

$$\begin{cases} e^\epsilon \rightharpoonup e \text{ weakly in } L^2(\Omega; \mathcal{M}_N^s) \\ \sigma^\epsilon \rightharpoonup \sigma \text{ weakly in } L^2(\Omega; \mathcal{M}_N^s). \end{cases}$$

Assume furthermore that

$$\operatorname{div} \sigma^\epsilon \rightarrow \operatorname{div} \sigma \text{ strongly in } H^{-1}(\Omega)^N,$$

and, for all $i, j, k, l \in \{1, \dots, N\}$,

$$\begin{aligned} \frac{\partial e_{jl}^\epsilon}{\partial x_i \partial x_k} + \frac{\partial e_{ik}^\epsilon}{\partial x_j \partial x_l} - \frac{\partial e_{jk}^\epsilon}{\partial x_i \partial x_l} - \frac{\partial e_{il}^\epsilon}{\partial x_j \partial x_k} &\rightarrow \\ \frac{\partial e_{jl}}{\partial x_i \partial x_k} + \frac{\partial e_{ik}}{\partial x_j \partial x_l} - \frac{\partial e_{jk}}{\partial x_i \partial x_l} - \frac{\partial e_{il}}{\partial x_j \partial x_k} &\text{strongly in } H^{-2}(\Omega). \end{aligned} \tag{1.126}$$

Then

$$e^\epsilon : \sigma^\epsilon \rightharpoonup e : \sigma \text{ in the sense of distributions.}$$

Remark 1.4.4 *It is easily seen that, if the sequence of symmetric matrices e^ϵ is a strain tensor, i.e. there exists a sequence of displacements u_ϵ such that $e^\epsilon = (\nabla u_\epsilon + (\nabla u_\epsilon)^t)/2$, then assumption (1.126) is automatically satisfied since*

$$\frac{\partial e_{jl}^\epsilon}{\partial x_i \partial x_k} + \frac{\partial e_{ik}^\epsilon}{\partial x_j \partial x_l} - \frac{\partial e_{jk}^\epsilon}{\partial x_i \partial x_l} - \frac{\partial e_{il}^\epsilon}{\partial x_j \partial x_k} = 0.$$

In this sense (1.126) is a condition on the deviation of e^ϵ from strain tensors.

Since Hooke's laws in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ are symmetric by definition, H -convergence turns out to be equivalent to G -convergence in the elasticity setting. Recall first the definition of G -convergence: A sequence of Hooke's laws A^ϵ in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ is said to G -converge to an homogenized tensor $A^* \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ if, for any right hand side $f \in H^{-1}(\Omega)^N$, the sequence u_ϵ of solutions of (1.121) converges weakly in $H_0^1(\Omega)^N$ to the solution u of the homogenized equation (1.125) (no convergence of the stresses is required). Obviously, if a sequence H -converges, it also G -converges since G -convergence is less restrictive. Arguing as in Proposition 1.3.11, it is easy to show that the converse is also true because Hooke's law are symmetric fourth order tensors.

Finally all the other fundamental results on H -convergence in the conductivity setting also hold true for elasticity.

1. Locality of H -convergence

Proposition 1.4.5 *There exist a positive constant $C > 0$ and a number $\delta > 1$ such that, for any two sequences of Hooke's laws $A^\epsilon(x)$ and $B^\epsilon(x)$ in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ that H -converge to $A^*(x)$ and $B^*(x)$, respectively, we have*

$$\|A^* - B^*\|_{L^\delta(\Omega; \mathcal{M}_N^4)} \leq C \liminf_{\epsilon \rightarrow 0} \|A^\epsilon - B^\epsilon\|_{L^\delta(\Omega; \mathcal{M}_N^4)}.$$

In particular, if there exists a compactly embedded open subset ω such that $A^\epsilon(x) = B^\epsilon(x)$ in ω , then $A^(x) = B^*(x)$ in ω .*

This implies that the value of the homogenized Hooke's law A^* at point x depends only on the values of the sequence A^ϵ at the same point x , and not on what happens elsewhere.

2. Irrelevance of the boundary conditions

Proposition 1.4.6 *Let $A^\epsilon(x)$ be a sequence of Hooke's laws in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ that H -converges to $A^*(x)$. Any sequence z_ϵ such that*

$$\begin{cases} -\operatorname{div} A^\epsilon e(z_\epsilon) = f_\epsilon \rightarrow f \text{ strongly in } H_{loc}^{-1}(\Omega)^N \\ z_\epsilon \rightharpoonup z \text{ weakly in } H_{loc}^1(\Omega)^N \end{cases} \quad (1.127)$$

satisfies

$$A^\epsilon e(z_\epsilon) \rightharpoonup A^* e(z) \text{ weakly in } L^2_{loc}(\Omega; \mathcal{M}_N^s).$$

This implies that the main result of H -convergence (i.e., the existence of an homogenized behavior) still holds true for sequences z_ϵ that do not satisfy any precise boundary conditions on $\partial\Omega$.

3. Convergence of the energy

Proposition 1.4.7 *Let $A^\epsilon(x)$ be a sequence of Hooke's laws in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ that H -converges to $A^*(x)$. For any right hand side $f \in H^{-1}(\Omega)^N$, the sequence u_ϵ of solutions of (1.121) satisfies*

$$A^\epsilon e(u_\epsilon) : e(u_\epsilon) \rightharpoonup A^* e(u) : e(u) \text{ in the sense of distributions} \quad (1.128)$$

and

$$\int_{\Omega} A^\epsilon e(u_\epsilon) : e(u_\epsilon) dx \rightarrow \int_{\Omega} A^* e(u) : e(u) dx, \quad (1.129)$$

where u is the weak limit of u_ϵ in $H_0^1(\Omega)^N$, and the solution of the homogenized equation (1.125).

By virtue of this result, the energy convergence is a consequence of H -convergence. Note that the convergence (1.128) of the energy density is also valid for a sequence z_ϵ defined as in (1.127), i.e., satisfying no special boundary conditions. On the contrary, the convergence (1.129) of the total energy is true only for fixed boundary conditions (see Remark 1.3.6).

4. Ordering properties of H -convergence

Proposition 1.4.8 *Let A^ϵ and B^ϵ be two sequences of Hooke's laws in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ that H -converge to A^* and B^* , respectively. Assume that, for any ϵ , almost everywhere in Ω ,*

$$A^\epsilon \xi : \xi \leq B^\epsilon \xi : \xi \quad \forall \xi \in \mathcal{M}_N^s.$$

Then the homogenized limits are also ordered as

$$A^* \xi : \xi \leq B^* \xi : \xi \quad \forall \xi \in \mathcal{M}_N^s.$$

This implies that ordering properties for Hooke's laws are stable under H -convergence. By the same token, the homogenized Hooke's law is always between the harmonic and arithmetic mean bounds as stated below.

Proposition 1.4.9 *Let A^ϵ be a sequence of Hooke's laws in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ that H -converges to A^* . Assume that*

$$A^\epsilon \rightharpoonup \bar{A} \text{ weakly } * \text{ in } L^\infty(\Omega; \mathcal{M}_N^4)$$

and

$$(A^\epsilon)^{-1} \rightharpoonup \underline{A}^{-1} \text{ weakly } * \text{ in } L^\infty(\Omega; \mathcal{M}_N^4).$$

Then the homogenized limit satisfies almost everywhere in Ω

$$\underline{A} \xi : \xi \leq A^* \xi : \xi \leq \bar{A} \xi : \xi \quad \forall \xi \in \mathcal{M}_N^s.$$

The upper bound in Proposition 1.4.9 is called the *arithmetic mean bound*, while the lower bound is called the *harmonic mean bound*.

We end this subsection by various remarks on H -convergence.

1. Let $A^\epsilon(x)$ be a sequence of Hooke's laws in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ that either converges strongly to a limit $A^*(x)$ in $L^1(\Omega; \mathcal{M}_N^4)$, or converges to $A^*(x)$ almost everywhere in Ω . Then, A^ϵ also H -converge to A^* . This is the analogue of Lemma 1.2.22.

2. In space dimension $N = 1$, as in the conductivity setting, a sequence A^ϵ of Hooke's laws in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ H -converges to a limit A^* if and only if $A^{\epsilon^{-1}}$ converges weakly * in $L^\infty(\Omega)$ to A^{*-1} . Unfortunately, in higher dimensions $N \geq 2$, the H -limit of a sequence usually has no link to any kind of weak limit. In particular, this implies that there is no explicit formula for an H -limit.
3. There are also corrector results that improve the weak convergence of the sequence u_ϵ (as in the conductivity case, see Subsection 1.3.6).
4. As in the conductivity case, H -convergence is metrizable in the elasticity setting (see Proposition 1.2.24). Therefore, one can extract diagonal sequences of H -converging sequences. Let $A^\epsilon(x)$ be a sequences of Hooke's laws in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ that H -converges to a limit $A^*(x)$. Assume that, for any fixed ϵ , there exists another sequence $A^{\epsilon,\eta}(x)$ in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ that H -converges to $A^\epsilon(x)$ as η goes to zero. Then there exists a diagonal sequence $A^{\epsilon,\eta(\epsilon)}(x)$ that H -converges to $A^*(x)$ as ϵ goes to zero (where $\eta(\epsilon)$ is a function which goes to zero with ϵ).

1.4.3 Lamination Formulas

In this subsection we revisit the homogenization of laminated structures in the elasticity setting. Although the essence of the convergence theorems does not change from the conductivity to the elasticity setting, the algebra involved in the explicit lamination formula is much more tedious in the latter case. Therefore, we give all necessary details in the proof of the lamination formula.

To simplify further the analysis, we make the following assumption. Rather than considering the general case of Hooke's laws A^ϵ depending only on a single space variable $x_1 \in \mathbb{R}$, namely,

$$A^\epsilon(x) \equiv A^\epsilon(x_1),$$

we focus on the simpler example where A^ϵ takes only two different values A and B , i.e.,

$$A^\epsilon(x_1) = \chi_\epsilon(x_1)A + (1 - \chi_\epsilon(x_1))B,$$

where $\chi_\epsilon(x_1) \in L^\infty(\mathbb{R}; \{0, 1\})$ is a scalar characteristic function. The general case is slightly more complex [275]. This type of assumption is valid when studying laminated composite materials for which there are only two component phases, stacked in slices orthogonal to the e_1 direction. Since it is

a generalization of the one-dimensional case, we find an explicit formula for the H -limits of such sequences A^ϵ . Effective properties of laminated structures have been studied for a long time (see, e.g., [81], [110], [115], [177], [188], [275], and references therein).

The following lemma is concerned with two-phase mixtures and generalizes Lemma 1.3.32.

Lemma 1.4.10 *Let $\chi_\epsilon(x_1)$ be a sequence of characteristic functions that converges to a limit $\theta(x_1)$ in $L^\infty(\mathbb{R}; [0, 1])$ weakly * (although χ_ϵ takes only the values 0 or 1, its limit θ may take any values between 0 and 1). Let A and B be two constant Hooke's laws in $\mathcal{M}_{\alpha,\beta}$. Define a sequence of matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ as*

$$A^\epsilon(x_1) = \chi_\epsilon(x_1)A + (1 - \chi_\epsilon(x_1))B.$$

The sequence A^ϵ H -converges to A^ , which depends only on x_1 , and is given, for any symmetric matrix ξ , by the formula*

$$A^* \xi = (\theta A + (1 - \theta)B) \xi \quad (1.130)$$

$$-\frac{\theta(1 - \theta)}{2}(A - B)(q(e_1)((A - B)\xi)(e_1 \otimes e_1) + (e_1 \otimes e_1)((A - B)\xi)q(e_1)),$$

where $q(e_1)$ is the symmetric matrix defined, for any vector $v \in \mathbb{R}^N$, by the quadratic form

$$q(e_1)^{-1}v \cdot v = (((1 - \theta)A + \theta B)v \otimes e_1) : (v \otimes e_1). \quad (1.131)$$

Remark 1.4.11 *The limit θ can be interpreted as the macroscopic, or averaged, density of phase A in the composite mixture defined by the sequence A^ϵ . Note that the right hand side of (1.131) defines an invertible matrix, i.e., $q(e_1)$ is well defined, since*

$$(((1 - \theta)A + \theta B)v \otimes e_1) : (v \otimes e_1) \geq \alpha|v|^2.$$

We shall frequently use the following notation for the symmetric tensor product of two vectors $u, v \in \mathbb{R}^N$, defined by

$$u \odot v = \frac{1}{2}(u \otimes v + v \otimes u).$$

Proof. Since the H -convergence is local, we can assume with no loss of generality that $\Omega = \mathbb{R}^N$. We construct a sequence of displacements u_ϵ , solutions of

$$-\operatorname{div} A^\epsilon e(u_\epsilon) = 0 \text{ in } \mathbb{R}^N, \quad (1.132)$$

which are piecewise linear in each phase. Let ξ_A and ξ_B be two symmetric matrices in \mathcal{M}_N^s . We seek a solution u_ϵ such that

$$e(u_\epsilon) = \xi_A \chi_\epsilon + \xi_B (1 - \chi_\epsilon). \quad (1.133)$$

This is the case if

$$u_\epsilon(x) = \chi_\epsilon(x_1) \tilde{\xi}_A x + (1 - \chi_\epsilon(x_1)) \tilde{\xi}_B x + c_\epsilon(x_1),$$

where $\tilde{\xi}_A$ and $\tilde{\xi}_B$ are (not necessarily symmetric) matrices such that

$$\xi_A = \frac{1}{2}(\tilde{\xi}_A + \tilde{\xi}_A^t), \quad \xi_B = \frac{1}{2}(\tilde{\xi}_B + \tilde{\xi}_B^t),$$

and $c_\epsilon(x_1)$ is a piecewise constant function. Such a displacement u_ϵ belongs to $H_{loc}^1(\mathbb{R}^N)^N$ if it is continuous at each interface between the regions occupied by phases A and B . This yields a condition on the values of the piecewise constant function $c_\epsilon(x_1)$ and on the matrices $\tilde{\xi}_A$ and $\tilde{\xi}_B$, namely

$$(\tilde{\xi}_A - \tilde{\xi}_B)x = (\tilde{\xi}_A - \tilde{\xi}_B)y \quad \forall x, y \in \text{interface}.$$

Since the interfaces between regions where $\chi_\epsilon = 0$ and $\chi_\epsilon = 1$ are hyperplanes normal to the vector e_1 , we deduce the existence of a constant vector $w \in \mathbb{R}^N$ such that

$$\tilde{\xi}_A - \tilde{\xi}_B = w \otimes e_1,$$

which implies the following condition for the symmetric matrices

$$\xi_A - \xi_B = w \odot e_1. \quad (1.134)$$

On the other hand, we define a piecewise constant stress σ_ϵ by

$$\sigma_\epsilon = \chi_\epsilon(x_1) A \xi_A + (1 - \chi_\epsilon(x_1)) B \xi_B. \quad (1.135)$$

To satisfy $-\operatorname{div} \sigma_\epsilon = 0$ in the sense of distributions in \mathbb{R}^N , it remains to check the jump condition at the interfaces, i.e.,

$$(A \xi_A) e_1 = (B \xi_B) e_1, \quad (1.136)$$

which restricts the possible values of w in (1.134). From (1.133) and (1.135) we deduce the following convergences:

$$\begin{aligned} e(u_\epsilon) &\rightharpoonup e(u) = \theta\xi_A + (1 - \theta)\xi_B \quad \text{weakly in } L^2_{loc}(\mathbb{R}^N; \mathcal{M}_N^s) \\ \sigma_\epsilon &\rightharpoonup \sigma = \theta A\xi_A + (1 - \theta)B\xi_B \quad \text{weakly in } L^2_{loc}(\mathbb{R}^N; \mathcal{M}_N^s). \end{aligned}$$

By Theorem 1.2.16 there exist a subsequence and an homogenized matrix A^* such that A^ϵ H -converges to A^* . Thus

$$\sigma = A^*e(u),$$

i.e.,

$$\theta A\xi_A + (1 - \theta)B\xi_B = A^*(\theta\xi_A + (1 - \theta)\xi_B). \quad (1.137)$$

Introducing a matrix $\xi = \theta\xi_A + (1 - \theta)\xi_B$, we find

$$\xi_A = \xi + (1 - \theta)w \odot e_1, \quad \xi_B = \xi - \theta w \odot e_1.$$

Then, a simple computation gives

$$A^*\xi = \theta A\xi + (1 - \theta)B\xi + \theta(1 - \theta)(A - B)(w \odot e_1).$$

To compute the value of the vector w , we multiply (1.136) by a vector v , and we replace ξ_A and ξ_B by their values to get

$$((A - B)\xi)e_1 \cdot v + (([1 - \theta]A + \theta B)w \odot e_1)e_1 \cdot v = 0.$$

Using the property $(C(w \odot e_1))e_1 \cdot v = (C(w \otimes e_1)) : (v \otimes e_1)$, this yields

$$((A - B)\xi)e_1 \cdot v + q(e_1)^{-1}w \cdot v = 0,$$

or, equivalently,

$$w = -q(e_1)((A - B)\xi)e_1,$$

which establishes (1.130). \square

Lemma 1.3.32 delivers a direct formula for A^* which is not very useful for the sequel. Of course, (1.130) can be simplified if $(A - B)$ is isotropic, but this is not always the case. Rather, we give an equivalent formula for A^* which involves $(A^* - B)^{-1}$. This formula is due to Francfort and Murat [110] and is in the spirit of Tartar's formula in the conductivity setting (see Corollary 1.3.34).

Lemma 1.4.12 *Under the same assumptions as Lemma 1.4.10, and if $(A - B)$ is invertible, the Hooke's law A^* , given by (1.130), is equivalently defined, for any symmetric matrix ξ , by*

$$\begin{aligned} \theta(A^* - B)^{-1}\xi &= (A - B)^{-1}\xi \\ &\quad + \frac{(1 - \theta)}{2}(q_B(e_1)\xi(e_1 \otimes e_1) + (e_1 \otimes e_1)\xi q_B(e_1)), \end{aligned} \quad (1.138)$$

where $q_B(e_1)$ is a symmetric matrix defined, for any vector $v \in \mathbb{R}^N$, by the quadratic form

$$q_B(e_1)^{-1}v \cdot v = (B(v \otimes e_1)) : (v \otimes e_1). \quad (1.139)$$

If $(A - B)$ is not invertible, then (1.138) holds for any matrix $\xi \in \text{Ker}(A - B)^\perp$, and $A^* = A = B$ on $\text{Ker}(A - B)$.

Of course, there is a symmetric formula for A^* obtained from (1.138) by inverting the roles of A and B . Before giving the proof of Lemma 1.4.12, we investigate the special case when B is isotropic. In such a case formula (1.138) simplifies since $q_B(e_1)$ can be computed explicitly.

Corollary 1.4.13 *Assume that B is isotropic, i.e., there exist two positive constants κ_B and μ_B such that*

$$B = 2\mu_B I_4 + \left(\kappa_B - \frac{2\mu_B}{N} \right) I_2 \otimes I_2.$$

Then, the Hooke's law A^ , given by (1.138), is equivalently defined by*

$$\theta(A^* - B)^{-1} = (A - B)^{-1} + (1 - \theta)f_B(e_1), \quad (1.140)$$

where $f_B(e_1)$ is a positive nondefinite fourth order tensor defined (ξ being a symmetric matrix) by

$$f_B(e_1)\xi = \frac{1}{\mu_B}((\xi e_1) \odot e_1 - (\xi e_1 \cdot e_1)e_1 \otimes e_1) + \frac{1}{2\mu_B + \lambda_B}(\xi e_1 \cdot e_1)e_1 \otimes e_1, \quad (1.141)$$

or, equivalently, by the quadratic form

$$f_B(e_1)\xi : \xi = \frac{1}{\mu_B}(|\xi e_1|^2 - (\xi e_1 \cdot e_1)^2) + \frac{1}{2\mu_B + \lambda_B}(\xi e_1 \cdot e_1)^2,$$

with $\lambda_B = \kappa_B - 2\mu_B/N$.

Remark 1.4.14 The lamination formulas are usually written in terms of the Hooke's laws A and B , but they can be equivalently expressed with the compliance tensors A^{-1} and B^{-1} . Doing so is useful when one is working with a dual or complementary formulation of the equations (i.e., with stresses rather than strains). Under the assumptions of Corollary 1.4.13, and reproducing the calculation in the proof of Lemma 1.4.12, one can check that formula (1.140) is equivalent to

$$\theta \left(A^{*-1} - B^{-1} \right)^{-1} = \left(A^{-1} - B^{-1} \right)^{-1} + (1 - \theta) f_B^c(e_1), \quad (1.142)$$

where $f_B^c(e_1)$ is a symmetric positive nondefinite fourth order tensor defined, for any symmetric matrix ξ , by the quadratic form

$$f_B^c(e_1)\xi : \xi = B\xi : \xi - \frac{1}{\mu_B}|B\xi e_1|^2 + \frac{\mu_B + \lambda_B}{\mu_B(2\mu_B + \lambda_B)}((B\xi)e_1 \cdot e_1)^2,$$

with $\lambda_B = \kappa_B - 2\mu_B/N$.

Proof. From (1.139) we compute

$$q_B(e_1)^{-1} = \mu_B I_2 + \left(\kappa_B + \frac{N-2}{N} \mu_B \right) e_1 \otimes e_1,$$

i.e.,

$$q_B(e_1) = \frac{1}{\mu_B} I_2 - \frac{\kappa_B + \frac{N-2}{N} \mu_B}{\mu_B \left(\kappa_B + \frac{2(N-1)}{N} \mu_B \right)} e_1 \otimes e_1.$$

This implies that

$$\begin{aligned} q_B(e_1)\xi(e_1 \otimes e_1) + (e_1 \otimes e_1)\xi q_B(e_1) &= \frac{1}{\mu_B} [(\xi e_1) \otimes e_1 + e_1 \otimes (\xi e_1)] \\ &\quad - \frac{2(\kappa_B + \frac{N-2}{N} \mu_B)}{\mu_B(\kappa_B + \frac{2(N-1)}{N} \mu_B)} (\xi e_1 \cdot e_1) e_1 \otimes e_1, \end{aligned}$$

which is the desired result. \square

Proof of Lemma 1.4.12. The proof starts as that of Lemma 1.4.10. The only differences come in the computations after (1.137). Instead of eliminating ξ_A and ξ_B in terms of ξ , we rather compute ξ and ξ_B in terms of ξ_A . Equation (1.137) then becomes

$$(A^* - B)(\xi_A - (1 - \theta)w \odot e_1) = \theta(A - B)\xi_A,$$

and (1.136) gives

$$((A - B)\xi_A)e_1 = -(Bw \odot e_1)e_1. \quad (1.143)$$

With the change of variables $\eta = (A - B)\xi_A$, multiplying equation (1.143) by a vector v yields

$$q_B(e_1)^{-1}w \cdot v = -(\eta e_1) \cdot v,$$

which gives the value $w = -q_B(e_1)(\eta e_1)$. On the other hand,

$$\theta(A^* - B)^{-1}\eta = (A - B)^{-1}\eta - (1 - \theta)w \odot e_1,$$

which gives the desired result. \square

Chapter 2

The mathematical modeling of composite materials

This chapter is concerned with the application of the homogenization theory to the modeling of composite materials. Composite materials are heterogeneous materials obtained by mixing several phases or constituent materials on a very fine (or microscopic) scale. However, one is usually interested only in the large scale (or macroscopic) properties of such a composite. The main problem with composite materials is, therefore, to determine their effective properties without determining their fine scale structure. There is a huge mechanical literature on this topic, and the reader is referred to [1], [52], [81], [132], [133], [136], [145], [207], [216], [244], [246], [264], [287], [289], [290], and references therein. Mathematicians have been interested in composite materials since the 1970's. Their first main contribution to this field was to give a firm theoretical basis for the notion of effective properties of a composite material. Indeed, homogenization theory permits one to properly define a composite material as a limit, in the sense of homogenization (an H -limit), of a sequence of increasingly finer mixtures of the constituent phases. Effective properties are now defined as *homogenized coefficients*. The application of homogenization to the modeling of composite materials has became a popular subject in applied mathematics. There is by now an extensive mathematical literature on this topic too. A short selection includes [20], [32], [79], [110], [116], [118], [166], [177], [189], [192], [193], [208], [274], and references therein. However, the bibliography of this book, although not possibly exhaustive, contains many more relevant references.

2.1 Homogenized Properties of Composite Materials

2.1.1 Modeling of Composite Materials

We focus on *two-phase composite materials*, that is composites obtained by mixing only two different constituents. Of course there are other types of composites (for example, multiphase composites or polycrystal composites) but, in view of applications to optimal shape design, two-phase composite materials are enough for our purpose. These composites are considered only in the context of electrical or thermal conductivity and elasticity. Other settings are possible (for example, thermo-elasticity or complex dielectric conductivity). We refer the reader to [192] for further information. The equation of state governing the composite behavior is assumed to be linear. For the study of nonlinear composites (like, for example, plastic materials) we refer to, e.g., [53], [222], [264], [291]. A perfect bonding between the two phases is also assumed, which means that, at the interface, perfect transmission conditions hold (namely, continuity of the potential or the displacement, and continuity of the normal current or normal stress). In the sequel the two phases are denoted by A and B , which are also their conductivity or elasticity tensors.

From a physical point of view a two-phase composite material is described by the properties of its two phases A and B , by their volume fraction θ and $(1 - \theta)$ (with $0 \leq \theta \leq 1$), and by their geometric arrangement in the mixture, which we call their *microstructure*. Of course, a detailed knowledge of the microstructure is either impossible or too costly, and one would like to simply characterize the composite material by its effective properties (also called averaged, homogenized, or macroscopic properties), reflected in a conductivity or elasticity tensor A^* . In the mechanical literature the concept of representative volume element (RVE) is often used for computing or estimating A^* . The homogenization theory presented in Chapter 1 offers a more rigorous alternative for defining a composite material. From this mathematical standpoint the fine microstructure of a composite material is represented by a sequence χ_ϵ of characteristic functions of one phase (say A), where ϵ is some parameter going to zero (for example, a lengthscale), and the effective properties of this composite is modeled by the H -limit A^* of the sequence $\chi_\epsilon A + (1 - \chi_\epsilon)B$. The weak limit θ of χ_ϵ turns out to be the local proportion of phase A . In this sense, weak convergence is a rigorous version of the more intuitive notion of averaging (see Subsection 1.2.1 for an introduction

to weak convergence).

Before giving a formal definition of what we call a composite material, we make some fundamental assumptions about the two phases A and B . The tensors A and B are assumed to be symmetric positive definite. We denote by \mathcal{T}_N^s the set of such real symmetric positive definite tensors, which are of second order in the conductivity setting (i.e., $\mathcal{T}_N^s = \mathcal{M}_N^s$), or of fourth order in the elasticity setting (i.e., $\mathcal{T}_N^s = \mathcal{M}_N^4$). We also denote by \cdot the scalar product of elements on which \mathcal{T}_N^s acts (it coincides with the usual dot product for vectors in the conductivity setting, and with the full contraction : of matrices in the elasticity setting). As usual, Ω denotes a bounded open set in \mathbb{R}^N .

Definition 2.1.1 *Let $\chi^\epsilon \in L^\infty(\Omega; \{0, 1\})$ be a sequence of characteristic functions and A^ϵ be a sequence of tensors defined by*

$$A^\epsilon(x) = \chi^\epsilon(x)A + (1 - \chi^\epsilon(x))B. \quad (2.1)$$

Assume that there exist $\theta \in L^\infty(\Omega; [0, 1])$ and $A^ \in L^\infty(\Omega; \mathcal{T}_N^s)$ such that*

$$\chi^\epsilon(x) \rightharpoonup \theta(x) \text{ weakly } * \text{ in } L^\infty(\Omega; [0, 1]) \quad (2.2)$$

and

$$A^\epsilon(x) \text{ } H\text{-converges to } A^*(x) \quad (2.3)$$

in the sense of Definition 1.2.15. The H -limit A^ is said to be the homogenized tensor of a two-phase composite material obtained by mixing A and B in proportions θ and $(1 - \theta)$, respectively, with a microstructure defined by the sequence $(\chi_\epsilon)_{\epsilon > 0}$.*

We remark first that Definition 2.1.1 makes sense because, by virtue of Theorem 1.2.16, for any sequence χ^ϵ there exist a subsequence and limits θ and A^* such that, for this subsequence, the convergences (2.2) and (2.3) hold true. Clearly, a composite material A^* is defined by three ingredients: the data of its two phases A and B , the proportion θ , and the microstructure $(\chi_\epsilon)_{\epsilon > 0}$. Definition 2.1.1 is the most general possible since it involves no restrictive, explicit or implicit, assumptions on the mixture, like periodicity, ergodicity, or randomness. Of course, there may be several composite materials A^* associated with the same proportion θ (they differ by their microstructure). By the same token, it may be possible that different microstructures give rise to the same effective properties A^* .

In this framework, the mathematical theory of composite materials addresses mainly two problems. First, if the components A and B , as well as their proportions θ and $(1 - \theta)$, are specified, what is the set of all possible composite materials A^* attainable in the sense of Definition 2.1.1? In other words, when varying the microstructure, while keeping the volume fractions fixed, what are the possible values of A^* ? A simpler subproblem is to find all composite materials when both the microstructures and the proportions vary. It turns out that this is a very difficult problem, which is solved only in a few cases (for example, for the mixture of two isotropic conductors). Second, when a complete answer to the first question is not available (as is the case for the mixture of two isotropic elastic phases), one can instead try to find bounds or estimates that are satisfied by the effective properties, and to characterize those composites A^* which are optimal for these bounds, i.e., those which attain the value of the bounds. The first question is called the *G-closure problem* and is discussed in the next subsection and in Subsection 2.2.3. The second question has obvious links with optimal design, since it is a problem in finding optimal microstructures. It is therefore crucial to the applications discussed in the remaining chapters of this book, and it is the main focus of the present chapter.

2.1.2 The *G*-closure Problem

We saw in Chapter 1 that, when a sequence of coefficients A^ϵ H -converges to a limit A^* , there is usually no formula for computing A^* (except in some special cases). In the context of Definition 2.1.1 for two-phase composites, there is also no formula for A^* . However, since A^ϵ takes only two values, A and B , the homogenized limit A^* certainly cannot take any possible value. The problem is therefore to find out what the attainable values of A^* are, or at least to find bounds on these values. This problem has a long history, both in the mathematical and in the mechanical communities [20], [32], [79], [81], [110], [116], [118], [132], [133], [136], [145], [152], [177], [189], [192], [205], [207], [246], [287], [289]. The mechanical point of view is to characterize the range of the effective or homogenized properties obtained by varying the microstructure of the mixture (i.e., the sequence χ_ϵ). The mathematical point of view is to find the closure of the set $\{A, B\}$ under G - or H -convergence (the so-called *G-closure* of $\{A, B\}$). This *G*-closure problem can be divided into two different questions. The easiest is to find the *G*-closure without any restrictions on the sequence $\chi_\epsilon(x)$. In particular its

weak limit $\theta(x)$ can take any value between zero and one. The quantities

θ and $(1 - \theta)$ can be interpreted as the averaged proportions or volume fractions of A and B , respectively, in the composite material A^* . A more difficult problem is to find the G -closure of $\{A, B\}$ when the weak limit θ of the sequence χ_ϵ is fixed. In other words, we seek the range of effective properties when the volume fractions of the two phases are given. This latter problem is of interest for optimal design.

We already have some informations on the G -closure. Recall Theorem 1.3.14, which states that A^* is bounded below by the harmonic mean and above by the arithmetic mean, i.e., for almost every $x \in \Omega$

$$\left(\theta(x)A^{-1} + (1 - \theta(x))B^{-1} \right)^{-1} \leq A^*(x) \leq (\theta(x)A + (1 - \theta(x))B), \quad (2.4)$$

in the sense of quadratic forms. These bounds (often called the Voigt [286] and Reuss [235] bounds, Wiener bounds [288], or Paul bounds [216]) do not characterize the G -closure, since there are also tensors satisfying (2.4) that are not the homogenized limit of any sequence A^ϵ taking only the values A and B . They are not even optimal bounds in the sense that there are no effective tensors for which the upper or the lower bound of (2.4) is an equality when evaluated for all possible energies. There are many bounds that improve on (2.4), and which are optimal (we shall establish the most important ones in the next sections). Unfortunately, in most cases, a precise knowledge of the G -closure is still lacking. For example, G -closure is known for the mixture of two isotropic conductors (see Section 2.2.3), but unknown for two isotropic elastic constituents. Therefore, we need to know, a priori, the structure of such a set. This is the purpose of the remainder of this subsection.

Let us settle some notation. For a given density function $\theta \in L^\infty(\Omega; [0, 1])$, we denote by \mathcal{G}_θ the set of all possible H -limits or composite materials, in the sense of Definition 2.1.1, associated to the density θ . In other words,

$$\mathcal{G}_\theta = \left\{ A^* \in L^\infty(\Omega; \mathcal{T}_N^s) \mid \begin{array}{l} \exists \chi^\epsilon \text{ satisfying (2.2)} \\ A^\epsilon, \text{ defined by (2.1), satisfies (2.3)} \end{array} \right\}. \quad (2.5)$$

We want to obtain a more explicit characterization of \mathcal{G}_θ . To this end, for a constant $\theta \in [0, 1]$, we also define a subset P_θ of \mathcal{T}_N^s , made of all constant H -limits obtained by periodic homogenization. More precisely, given a characteristic function $\chi(y)$ such that

$$\int_Y \chi(y) dy = \theta, \quad (2.6)$$

a symmetric tensor A^* in P_θ is defined by its associated quadratic form (see Subsection 1.1)

$$A^* \xi \cdot \xi = \min_{w(y) \in H_{\#}^1(Y)} \int_Y (\chi(y)A + (1 - \chi(y))B)(\xi + \nabla w) \cdot (\xi + \nabla w) dy, \quad (2.7)$$

where, in the conductivity setting, ξ is any constant vector in \mathbb{R}^N , and $H_{\#}^1(Y)$ is the Sobolev space of periodic functions in Y . In the elasticity setting, formula (2.7) must be adequately modified, ξ is now a constant strain (a symmetric matrix of order N), w is now a vector valued function in $H_{\#}^1(Y)^N$ (the elastic displacement), and ∇w is replaced by its symmetric part $e(w) = 1/2(\nabla w + (\nabla w)^t)$. In any event, P_θ is defined by

$$P_\theta = \{A^* \in \mathcal{T}_N^s \text{ defined by (2.7) with } \chi(y) \text{ satisfying (2.6)}\}, \quad (2.8)$$

i.e., P_θ is the set of all constant effective tensors obtained by periodic homogenization of a mixture of A and B in proportions θ and $(1 - \theta)$. We remark that, by virtue of (2.4), the set P_θ is bounded, but not necessarily closed, in \mathcal{T}_N^s . We denote by G_θ its closure in \mathcal{T}_N^s

$$G_\theta = \overline{P}_\theta. \quad (2.9)$$

The main result on the structure of G_θ is the following

Theorem 2.1.2 *For any function $\theta(x)$ in $L^\infty(\Omega; [0, 1])$, the G -closure set G_θ is characterized by*

$$G_\theta = \left\{ A^* \in L^\infty(\Omega; \mathcal{T}_N^s) \mid A^*(x) \in G_{\theta(x)} \text{ a.e. in } \Omega \right\}, \quad (2.10)$$

where, for any real number $\theta \in [0, 1]$, the set $G_\theta \subset \mathcal{T}_N^s$ is defined by (2.9). In particular, (2.10) implies that the set of composites obtained by periodic homogenization is dense in the set of all possible composites.

Remark 2.1.3 Theorem 2.1.2 asserts the local character of the G -closure. A tensor A^* is checked to be an homogenized tensor in G_θ by inspection if its pointwise values $A^*(x)$ belong to the set $G_{\theta(x)}$. In other words, there is no global, but only local, properties of homogenized tensors in G_θ . In full generality (i.e., not necessarily for two-phase composites) this result is due to Dal Maso and Kohn [91] (their proof is yet unpublished) drawing upon ideas of Cabib and Dal Maso [65]. A proof of Theorem 2.1.2, both in a linear and nonlinear setting, can now be found in [231]. However, in the special case

of two-phase conductor composites (for which the set G_θ is explicitly known) Theorem 2.1.2 was first proved by Tartar [274]. Our proof (see below) is a blend of that of [91] and [274], but is restricted to two-phase composites (although it can be extended easily to multiphase composites). It requires Meyers theorem [184], although, as remarked by Tartar in [278], Meyers theorem is no longer necessary if we do not claim that G_θ is the closure of the set P_θ of periodic homogenized tensors.

We remark that Theorem 2.1.2 is in the same spirit as Theorem 1.3.23, since it implies that a general H -limit A^* takes its values precisely in the closure of the set of periodic homogenized tensors.

Remark 2.1.4 In practice, the main importance of Theorem 2.1.2 is that, through it, when studying composite materials, we can safely assume that their effective tensors are constant; even more important, they are obtained by periodic homogenization. In other words, there is no interaction between the spatial variations of A^* and the possible range of its values, at least in the context of two-phase composite materials. We shall rely heavily on this result in the remainder of this chapter since it greatly simplifies the presentation.

The remainder of this subsection is devoted to the proof of Theorem 2.1.2, which requires several technical lemmas. To begin with, we prove that G_θ is actually closed under H -convergence (as a direct consequence of the metrizability properties of H -convergence).

Lemma 2.1.5 Let $\theta(x)$ be a function in $L^\infty(\Omega; [0, 1])$. The G -closure set G_θ is stable under homogenization, i.e., for any sequence A^ϵ in G_θ which H -converges to A^* , the H -limit A^* also belongs to G_θ .

Proof. Let A^ϵ be a sequence in G_θ that H -converges to a limit A^* . For each fixed ϵ , A^ϵ is also an H -limit since it belongs to G_θ . By virtue of Proposition 1.2.24, which states that the topology induced by the H -convergence is metrizable, one can extract a diagonal sequence \tilde{A}^ϵ , which takes only the values A and B , i.e., $\tilde{A}^\epsilon(x) = \chi_\epsilon(x)A + (1 - \chi_\epsilon(x))B$, where $\chi_\epsilon(x)$ is a characteristic function, such that A^* is actually its H -limit. Since each A^ϵ belongs to G_θ , this diagonalization process (as evoked in Remark 1.2.25) guarantees that the weak limit of χ_ϵ is precisely θ . Therefore A^* does indeed belong to G_θ . \square

Lemma 2.1.6 Let $\chi_1(y), \chi_2(y) \in L^\infty_\#(Y; \{0, 1\})$ be two characteristic functions. Let A_1^* and A_2^* be the tensors in \mathcal{T}_N^s obtained by periodic homogenization of $(\chi_1 A + (1 - \chi_1)B)$ and $(\chi_2 A + (1 - \chi_2)B)$, respectively (see formula

(2.7)). There exist two positive constants $C > 0$ and $\delta > 0$, which do not depend on χ_1, χ_2 , such that

$$\|A_1^* - A_2^*\| \leq C \left(\int_Y |\chi_1(y) - \chi_2(y)| dy \right)^\delta, \quad (2.11)$$

where $\|\cdot\|$ denotes any norm in \mathcal{T}_N^s .

Proof. This lemma is valid even when A and B are nonsymmetric tensors. It can be seen as a consequence of Proposition 1.3.44, but for simplicity, we give an alternative simpler proof in the conductivity setting (which is valid, up to obvious changes, in the elasticity setting). In any case, its proof relies on Meyers Theorem 1.3.41. Recall that, for $k = 1, 2$, A_k^* is defined by

$$A_k^* e_i = \int_Y (\chi_k A + (1 - \chi_k) B) (e_i + \nabla w_k^i) dy, \quad (2.12)$$

where $(e_i)_{1 \leq i \leq N}$ is the canonical basis of \mathbb{R}^N , and $(w_k^i)_{1 \leq i \leq N}$ are the unique solutions in $H_\#^1(Y)/\mathbb{R}$ of the cell problems

$$\begin{cases} -\operatorname{div}(\chi_k A + (1 - \chi_k) B)(e_i + \nabla w_k^i) = 0 & \text{in } Y \\ y \rightarrow w_k^i(y) & \text{Y-periodic.} \end{cases}$$

Integrating by parts, we have

$$\begin{aligned} & \int_Y (\chi_1 A + (1 - \chi_1) B) \nabla(w_1^i - w_2^i) \cdot \nabla(w_1^i - w_2^i) dy \\ &= \int_Y (\chi_2 - \chi_1)(A - B) (\nabla w_2^i - e_i) \cdot \nabla(w_1^i - w_2^i) dy, \end{aligned}$$

which yields by coercivity of A and B

$$\|\nabla(w_1^i - w_2^i)\|_{L_\#^2(Y)} \leq C \|(\chi_2 - \chi_1)(\nabla w_2^i - e_i)\|_{L_\#^2(Y)}. \quad (2.13)$$

By Meyers Theorem 1.3.41, there exist a positive constant $C > 0$ and an exponent $p > 2$ (strictly larger than 2) that depend on A and B but not on χ_2 , such that $\|\nabla w_2^i\|_{L_\#^p(Y)} \leq C$. (Note that Meyers theorem is true also for elasticity.) By Young's inequality applied to (2.13), we deduce that

$$\|\nabla(w_1^i - w_2^i)\|_{L_\#^2(Y)} \leq C \left(\|(\chi_2 - \chi_1)\|_{L_\#^q(Y)} \right),$$

with $q = 2p/(p - 2)$. Then, formula (2.12) yields

$$\begin{aligned} |A_1^* e_i - A_2^* e_i| &\leq \int_Y (\chi_1 A + (1 - \chi_1) B) |\nabla w_1^i - \nabla w_2^i| dy \\ &\quad + \int_Y |\chi_1 - \chi_2| (B - A) |e_i + \nabla w_2^i| dy \\ &\leq C \left(\|\nabla(w_1^i - w_2^i)\|_{L_\#^2(Y)} + \|(\chi_2 - \chi_1)\|_{L_\#^2(Y)} \right), \end{aligned}$$

which implies the desired result with $\delta = 1/q$. \square

Lemma 2.1.7 *Let G_θ be the set defined by (2.9) as the closure of the set P_θ of periodic composites. There exist two positive constants $C > 0$ and $\delta > 0$ such that, for any $\theta_1, \theta_2 \in [0, 1]$,*

$$d(G_{\theta_1}, G_{\theta_2}) \leq C|\theta_1 - \theta_2|^\delta, \quad (2.14)$$

where d denotes the Hausdorff distance for subsets of \mathcal{T}_N^s .

Proof. Recall that the Hausdorff distance between two compact sets in a metric space (with distance d , for example, the Euclidean distance) is defined by

$$d(K_1, K_2) = \max_{x_1 \in K_1} \min_{x_2 \in K_2} d(x_1, x_2) + \max_{x_2 \in K_2} \min_{x_1 \in K_1} d(x_2, x_1).$$

It is enough to prove that, for any characteristic function χ_1 with average θ_1 , there exists another characteristic function χ_2 with average θ_2 , i.e.,

$$\int_Y \chi_i(y) dy = \theta_i,$$

such that $|A_1^* - A_2^*| \leq C|\theta_1 - \theta_2|^\delta$ where A_1^* and A_2^* are the tensors obtained by periodic homogenization of $(\chi_1 A + (1 - \chi_1) B)$ and $(\chi_2 A + (1 - \chi_2) B)$ respectively. With no loss of generality we assume $\theta_2 \leq \theta_1$ (the converse case is symmetric). Let E_1 be the measurable subset of Y defined by

$$E_1 = \{y \in Y \mid \chi_1(y) = 1\}.$$

Let $B(r) = \{y \in Y \mid |y| < r\}$ be a ball of radius $r > 0$. Introducing the set $E(r) = E_1 \cap B(r)$, its measure in Y (considered as the unit torus) is a continuous nondecreasing function of r , with $|E(0)| = 0$ and $|E(\sqrt{N})| = \theta_1$.

Therefore, there exists a positive radius r_2 such that $|E(r_2)| = \theta_2$. Defining χ_2 as the characteristic function of $E(r_2)$, it is easily seen that

$$\int_Y (\chi_1(y) - \chi_2(y)) dy = \theta_1 - \theta_2.$$

Eventually, Lemma 2.1.6 leads to the desired result. \square

Proof of Theorem 2.1.2. Let us define a set \mathcal{A}_θ by

$$\mathcal{A}_\theta = \left\{ A^* \in L^\infty(\Omega; \mathcal{T}_N^s) \mid A^*(x) \in G_{\theta(x)} \text{ a.e. in } \Omega \right\}.$$

We first prove that

$$\mathcal{G}_\theta \subset \mathcal{A}_\theta. \quad (2.15)$$

Recall that Theorem 1.3.23 and Remark 1.3.25 state that any H -limit can be locally approximated by a sequence of periodic composites with, asymptotically, the same phase proportions. More precisely, for any $A^* \in \mathcal{G}_\theta$, there exist a sequence $\theta_n(x)$ in $L^\infty(\Omega; [0, 1])$ that converges to $\theta(x)$ almost everywhere in Ω and a sequence A_n^* in $L^\infty(\Omega; \mathcal{T}_N^s)$ such that $A_n^*(x)$ belongs to $P_{\theta_n(x)}$ and converges to $A^*(x)$ for almost every $x \in \Omega$. Since G_θ is the closure of P_θ and varies continuously with respect to θ by virtue of Lemma 2.1.7, we deduce that $A^*(x)$ belongs to $G_{\theta(x)}$ for almost every $x \in \Omega$. This proves (2.15).

To prove the reverse inclusion, we proceed as in [274] by approaching $A^* \in \mathcal{A}_\theta$ by a piecewise constant tensor (with a corresponding piecewise constant density). Of course, the piecewise approximation need not be in the same space \mathcal{A}_θ , but by the locality of H -convergence (see Proposition 1.2.18), it is obviously an H -limit (for example, obtained by periodic homogenization on each subset where it is constant). Passing to the limit when the diameters of these subsets go to zero, we shall prove that A^* does indeed belong to \mathcal{G}_θ . More precisely, let $(\omega_j^n)_{1 \leq j \leq n}$ be a family of disjoint open subsets covering Ω such that the maximal diameter $\max_{1 \leq j \leq n} \text{diam}(\omega_j^n)$ goes to zero when n goes to $+\infty$. Let $\chi_j^n(x)$ denote the characteristic function of ω_j^n . Define a piecewise constant function $\theta^n \in L^\infty(\Omega; [0, 1])$ by

$$\theta^n(x) = \sum_{j=1}^n \theta_j^n \chi_j^n(x) \text{ with } \theta_j^n = \frac{1}{|\omega_j^n|} \int_{\omega_j^n} \theta(x) dx.$$

It is well known that the sequence $\theta^n(x)$ converges to $\theta(x)$ strongly in $L^p(\Omega)$, for any $1 \leq p < +\infty$. In each open set ω_j^n , we project $A^*(x)$ on the set $G_{\theta_j^n}$

to obtain a sequence of matrices $\tilde{A}^n(x)$. From Lemma 2.1.7 we know that, for almost every $x \in \omega_j^n$,

$$|A^*(x) - \tilde{A}^n(x)| \leq C|\theta(x) - \theta_j^n|^\delta, \quad (2.16)$$

and therefore the sequence \tilde{A}^n converges strongly to A^* in any $L^p(\Omega; \mathcal{T}_N^s)$, with $1 \leq p < +\infty$. Each matrix $\tilde{A}^n(x)$ is not yet piecewise constant. Therefore, we define a sequence of piecewise constant matrices

$$\hat{A}^n(x) = \sum_{j=1}^n \hat{A}_j^n \chi_j^n(x) \text{ with } \hat{A}_j^n = \frac{1}{|\omega_j^n|} \int_{\omega_j^n} \tilde{A}^n(x) dx.$$

It is easily seen that

$$\|\hat{A}^n\|_{L^p(\Omega; \mathcal{T}_N^s)} \leq \|\tilde{A}^n\|_{L^p(\Omega; \mathcal{T}_N^s)},$$

which clearly implies that the sequence \hat{A}^n also converges strongly to A^* in $L^p(\Omega; \mathcal{T}_N^s)$, with $1 \leq p < +\infty$. Unfortunately, there is no guarantee that each \hat{A}_j^n still belongs to $G_{\theta_j^n}$. Therefore, we define a constant matrix A_j^n as the projection of \hat{A}_j^n on $G_{\theta_j^n}$. This yields a piecewise constant matrix

$$A^n(x) = \sum_{j=1}^n A_j^n \chi_j^n(x),$$

which, by definition, is an H -limit, i.e., belongs to \mathcal{G}_{θ^n} . Let us prove that the sequence A^n converges to A^* strongly in $L^1(\Omega; \mathcal{T}_N^s)$. Then, by Lemma 1.2.22, A^n also H -converges to A^* . Since each A^n is itself an H -limit, extracting a diagonal sequence (see Lemma 1.2.24) proves that A^* is an H -limit. Furthermore, the strong convergence of θ^n to θ implies that A^* belongs to \mathcal{G}_θ , as desired. By construction, the projection A_j^n satisfies

$$|A_j^n - \hat{A}_j^n| \leq |M - \hat{A}_j^n| \text{ for any } M \in G_{\theta_j^n}.$$

In particular, for almost every $x \in \omega_j^n$,

$$|A^n(x) - \hat{A}^n(x)| = |A_j^n - \hat{A}_j^n| \leq |\tilde{A}^n(x) - \hat{A}^n(x)|.$$

Therefore

$$\begin{aligned} |A^n(x) - A^*(x)| &\leq |A^n(x) - \hat{A}^n(x)| + |\hat{A}^n(x) - \tilde{A}^n(x)| + |\tilde{A}^n(x) - A^*(x)| \\ &\leq 2|\hat{A}^n(x) - \tilde{A}^n(x)| + |\tilde{A}^n(x) - A^*(x)|. \end{aligned} \quad (2.17)$$

We already know that the sequence \tilde{A}^n converges strongly to A^* in $L^p(\Omega; \mathcal{T}_N^s)$, for any $1 \leq p < +\infty$, and that the difference $\hat{A}^n - \tilde{A}^n$ converges strongly to zero in the same space. Eventually, we deduce from (2.17) that A^n converges strongly to A^* in $L^p(\Omega; \mathcal{T}_N^s)$, for any $1 \leq p < +\infty$. \square

2.2 Conductivity

This section is devoted to the study of two-phase composites in the conductivity setting. Thanks to the representation of the G -closure set \mathcal{G}_θ furnished by Theorem 2.1.2, it is enough to consider constant homogenized tensors A^* that belong to \mathcal{G}_θ (i.e., the set of all possible two-phase composite materials at fixed volume fraction θ).

In the first subsection we describe a useful class of composite materials, namely the sequential laminated composites. The second subsection is concerned with optimal bounds on the effective tensors A^* obtained from two well-ordered phases $A \leq B$. These bounds are obtained by using the *Hashin-Shtrikman variational principle*. Finally, using these preliminary results, the third subsection is devoted to the explicit computation of the set \mathcal{G}_θ when the two phases are isotropic. As a final reminder, all conductivity tensors in this section are symmetric second order tensors.

2.2.1 Laminated Composites

We begin by exploring a special family of two-phase composites, namely the *laminated composites*. Recall from Subsection 1.3.5 that, when the sequence $\chi_\epsilon(x)$, describing the microstructure of a composite material A^* (see Definition 2.1.1), depends on a single space coordinate, then an explicit formula for A^* is available, one which depends only on A , B , θ (the weak limit of χ_ϵ), and the direction of the oscillations of χ_ϵ . More precisely, we recall the result of Corollary 1.3.34. Let e be a unit vector in \mathbb{R}^N ($|e| = 1$) which is the direction of the lamination. Let $\theta(x \cdot e)$ be the weak limit of the sequence $\chi_\epsilon(x \cdot e)$. The homogenized matrix A^* is given by

$$A^* = \theta A + (1 - \theta)B - \frac{\theta(1 - \theta)}{(1 - \theta)Ae \cdot e + \theta Be \cdot e}(A - B)e \otimes (A - B)e, \quad (2.18)$$

or, equivalently,

$$\theta(A^* - B)^{-1} = (A - B)^{-1} + (1 - \theta)\frac{e \otimes e}{Be \cdot e}. \quad (2.19)$$

The composite A^* is said to be a single lamination in the direction e of the two phases A and B in proportions θ and $(1 - \theta)$ (see Figure 2.1). By varying the proportion θ and the direction e , we obtain a whole family of composite materials. But this family can still be enlarged by laminating again these simple laminates. Denoting by A_1^* and A_2^* two simple laminates, we can laminate them in proportions $\tau, 1 - \tau$ to obtain a new composite A^* given by an analogue of (2.18)

$$A^* = \tau A_1^* + (1 - \tau) A_2^* - \frac{\tau(1 - \tau)(A_1^* - A_2^*)e \otimes (A_1^* - A_2^*)e}{(1 - \tau)A_1^*e \cdot e + \tau A_2^*e \cdot e}. \quad (2.20)$$

In fact, formula (2.20) can be interpreted as a mixing rule for composite materials. By Lemma 2.1.5 we know that such laminates of laminates are still two-phase composite materials, i.e., whatever the value of $\tau \in [0, 1]$, A^* still belongs to G_θ (Briane [60] has directly proved that A^* is indeed an homogenized tensor by computing explicitly the correctors). Repeating this process of lamination in any direction and any proportions yields a subset L_θ of G_θ , made of these laminated materials. Equivalently L_θ can be defined as the closure of all matrices A^* delivered by formula (2.18) under the lamination rule (2.20).

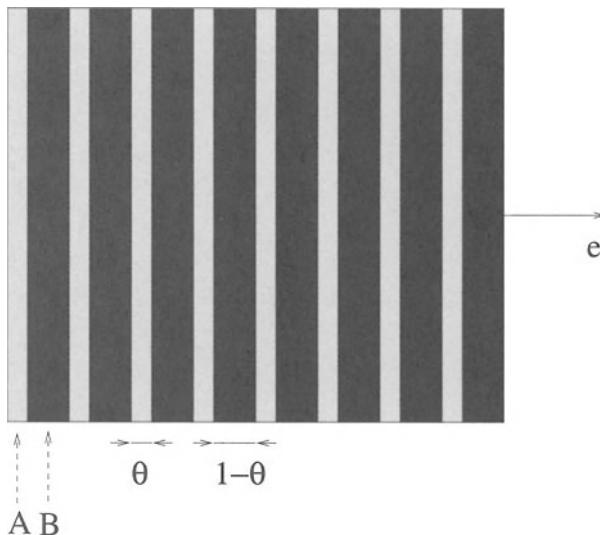


Figure 2.1: A simple laminate defined by (2.19).

Definition 2.2.1 *The subset $L_\theta \subset G_\theta$ of all laminated materials obtained from the phases A and B in proportions θ and $(1 - \theta)$ is the smallest subset of G_θ that contains all simple laminates, as defined by (2.18), and is stable under lamination.*

We shall prove later that in some cases L_θ actually coincides with G_θ (in general, it is unknown whether the inclusion is strict or not). We remark that the definition of L_θ does not furnish an explicit formula for all matrices A^* in L_θ (although in theory it is possible to compute A^* by reiterating formula (2.20), but the algebra soon becomes formidable). However, in many applications full knowledge of L_θ is unnecessary; it is enough to know its subset of *sequential laminates*, which do have an explicit formula.

A sequential laminate is obtained by an iterative process of lamination where the previous laminate is laminated again with a single pure phase (always the same one). By using the special form of (2.19) (which does not deliver directly the value of A^* , contrary to (2.18)), the iterative or sequential laminate can be explicitly characterized. Let $(e_i)_{1 \leq i \leq p}$ be a collection of unit vectors and $(\theta_i)_{1 \leq i \leq p}$ be proportions in $[0, 1]$. By (2.19) a simple laminate A_1^* of A and B , in proportions $\theta_1, (1 - \theta_1)$, is defined by

$$\theta_1 (A_1^* - B)^{-1} = (A - B)^{-1} + (1 - \theta_1) \frac{e_1 \otimes e_1}{B e_1 \cdot e_1}.$$

This simple laminate A_1^* can again be laminated with phase B , in direction e_2 and in proportions $\theta_2, (1 - \theta_2)$ respectively, to obtain a new laminate denoted by A_2^* . By induction, we obtain A_p^* by lamination of A_{p-1}^* and B , in direction e_p and in proportions $\theta_p, (1 - \theta_p)$, respectively. The homogenized tensor A_p^* is defined by

$$\theta_p (A_p^* - B)^{-1} = (A_{p-1}^* - B)^{-1} + (1 - \theta_p) \frac{e_p \otimes e_p}{B e_p \cdot e_p}.$$

Replacing $(A_{p-1}^* - B)^{-1}$ in this formula by its own formula involving $(A_{p-2}^* - B)^{-1}$, and so on up to $A_0^* \equiv A$, yields a formula of the same type as (2.19), namely,

$$\left(\prod_{j=1}^p \theta_j \right) (A_p^* - B)^{-1} = (A - B)^{-1} + \sum_{i=1}^p \left((1 - \theta_i) \prod_{j=1}^{i-1} \theta_j \right) \frac{e_i \otimes e_i}{B e_i \cdot e_i}. \quad (2.21)$$

We remark that we always laminate an intermediate laminate with the same phase B . In other words, the other phase A is coated by several layers of B .

One can say that B plays the role of a matrix phase, and A plays the role of a core phase. Globally, A_p^* can be seen as a mixture of A and B in different layers having a large separation of scales (see Figure 2.2).

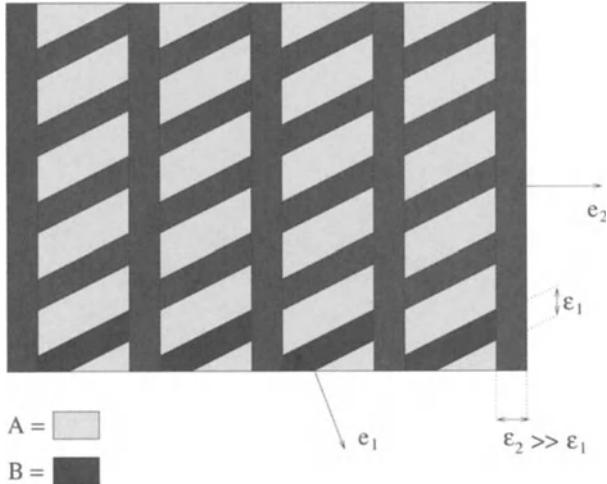


Figure 2.2: A rank-2 sequential laminate defined by (2.21).

Definition 2.2.2 *The composite material A_p^* , defined by formula (2.21), is called a rank- p sequential laminate with matrix B and core A . It is characterized by the lamination directions $(e_i)_{1 \leq i \leq p}$ and the proportions $(\theta_i)_{1 \leq i \leq p}$ at each stage of the process.*

In Definition 2.2.2 the word “sequential” means that this type of laminate is iteratively constructed by successive laminations, always with the same pure phase. In formula (2.21) the overall volume fraction of phase A is

$$\theta = \prod_{i=1}^p \theta_i.$$

If this overall volume fraction θ is fixed, it is interesting to see what the possible rank- p sequential laminates are that can be obtained by varying the proportions $(\theta_i)_{1 \leq i \leq p}$. The result is given by the following lemma.

Lemma 2.2.3 *Let $(e_i)_{1 \leq i \leq p}$ be a collection of unit vectors. Let θ be a volume fraction in $[0, 1]$. For any collection of nonnegative real numbers $(m_i)_{1 \leq i \leq p}$*

satisfying

$$\sum_{i=1}^p m_i = 1 \text{ and } m_i \geq 0, \quad 1 \leq i \leq p,$$

there exists a rank- p sequential laminate A_p^* with matrix B and core A , in proportions $(1 - \theta)$ and θ , respectively, and with lamination directions $(e_i)_{1 \leq i \leq p}$, such that

$$\theta (A_p^* - B)^{-1} = (A - B)^{-1} + (1 - \theta) \sum_{i=1}^p m_i \frac{e_i \otimes e_i}{B e_i \cdot e_i}. \quad (2.22)$$

The numbers $(m_i)_{1 \leq i \leq p}$ are called the lamination parameters.

Proof. Comparing formulas (2.21) and (2.22) gives

$$(1 - \theta)m_i = (1 - \theta_i) \prod_{j=1}^{i-1} \theta_j \quad (2.23)$$

for $1 \leq i \leq p$. Knowing the parameters $(m_i)_{1 \leq i \leq p}$ and the density θ , (2.23) gives sequentially the proportions $(\theta_i)_{1 \leq i \leq p}$ from $i = 1$ to p . Since $\sum_{i=1}^p m_i = 1$, one recovers that $\theta = \prod_{i=1}^p \theta_i$. Conversely, if the proportions $(\theta_i)_{1 \leq i \leq p}$ are known, the parameters $(m_i)_{1 \leq i \leq p}$ are easily computed from (2.23) upon defining $\theta = \prod_{i=1}^p \theta_i$. In other words, equation (2.23) defines a one-to-one map between $(\theta_i)_{1 \leq i \leq p}$ and the couple $(\theta, (m_i)_{1 \leq i \leq p})$. \square

Of course, one can define a symmetric class of sequential laminates by interchanging the roles of A and B , A being now the matrix and B the core. Indeed, formula (2.19) for a simple lamination of A and B in proportions θ and $(1 - \theta)$ is equivalent to its symmetric form,

$$(1 - \theta)(A^* - A)^{-1} = (B - A)^{-1} + \theta \frac{e \otimes e}{A e \cdot e}. \quad (2.24)$$

From (2.24) we obtain a symmetric form of (2.21) for a rank- p sequential laminate with matrix A and core B

$$\left(\prod_{j=1}^p (1 - \theta_j) \right) (A_p^* - A)^{-1} = (B - A)^{-1} + \sum_{i=1}^p \left(\theta_i \prod_{j=1}^{i-1} (1 - \theta_j) \right) \frac{e_i \otimes e_i}{A e_i \cdot e_i},$$

where, at each stage, θ_i is the proportion of phase A and $(1 - \theta_i)$ is that of A_{i-1}^* . Now the overall volume fraction of phase B is

$$1 - \theta = \prod_{i=1}^p (1 - \theta_i).$$

An analogue to Lemma 2.2.3 holds true.

Lemma 2.2.4 *Let $(e_i)_{1 \leq i \leq p}$ be a collection of unit vectors. Let θ be a volume fraction in $[0, 1]$. For any collection of nonnegative real numbers $(m_i)_{1 \leq i \leq p}$ satisfying*

$$\sum_{i=1}^p m_i = 1 \text{ and } m_i \geq 0, \quad 1 \leq i \leq p,$$

there exists a rank- p sequential laminate A_p^ with matrix A and core B , in proportions θ and $(1 - \theta)$ respectively, and with lamination directions $(e_i)_{1 \leq i \leq p}$, such that*

$$(1 - \theta) (A_p^* - A)^{-1} = (B - A)^{-1} + \theta \sum_{i=1}^p m_i \frac{e_i \otimes e_i}{A e_i \cdot e_i}. \quad (2.25)$$

The class of sequential laminates provides a rich parametrized family of composite materials with explicitly computable effective properties. Moreover, it turns out that optimal composites can often be found in this class (see the next subsection for a definition of *optimal composite*). There are, of course, other classes of composites with explicit formulas which may be optimal: For example, the well known concentric spheres construction (see, e.g., [81], [131]), extended by Bergman [51], [52], Milton [187], and Tartar [274] to the confocal ellipsoids construction, or the periodic inclusions of Vigdergauz [284]. But the sequential laminates are probably the easiest ones to work with.

Laminated composites have been known for a long time (see, e.g., [81], [173], [189], [246]), but the class of sequential laminates, with their explicit formula (as delivered by Lemmas 2.2.3 and 2.2.4), was introduced by Tartar [274].

2.2.2 Hashin-Shtrikman Bounds

This section is devoted to the *Hashin-Shtrikman variational principle*, which allows one to obtain bounds or estimates on the effective or homogenized energy of a composite material. If A^* is a composite conductor (in the sense of Definition 2.1.1) and $\xi \in \mathbb{R}^N$ is the local average value of the gradient field (i.e., the gradient of the temperature or electrical potential), the homogenized energy is the quantity $A^* \xi \cdot \xi$. The properties of the composite A^* can be estimated in various situations (i.e., for different average gradient fields), and therefore we focus directly on the more general case of a sum

of homogenized energies $\sum_{i=1}^p A^* \xi_i \cdot \xi_i$, rather than on a single energy (see Remark 2.2.10). Remark that this sum of energies can be rewritten as

$$\sum_{i=1}^p A^* \xi_i \cdot \xi_i = A^* : \left(\sum_{i=1}^p \xi_i \otimes \xi_i \right) = A^* : \xi^t \xi = (\xi A^*) : \xi, \quad (2.26)$$

where ξ is a matrix in $\mathcal{M}_{p \times N}$ with rows (ξ_1, \dots, ξ_p) in \mathbb{R}^N , and $:$ denotes the full contraction of two second order tensors. Since $\xi^t \xi$ is a real symmetric positive matrix of order N , which, by diagonalization, can always be written in the form $\xi^t \xi = \sum_{i=1}^N \tilde{\xi}_i \otimes \tilde{\xi}_i$ with $\tilde{\xi}_i \in \mathbb{R}^N$, there is no loss of generality in assuming that the number of energies is precisely $p = N$.

The Hashin-Shtrikman variational principle was introduced in [133] and has been further developed in the mechanical community (see, e.g., [132], [287], [290]), as well as in the mathematical community (see, e.g., [20], [32], [120], [151], [189], [193]). It works equally well in the elasticity setting (see Section 2.3.2). Although this method is more general, only two-phase composites with well-ordered phases are considered. Namely, we assume that A and B are symmetric positive definite tensors satisfying

$$A\xi \cdot \xi \leq B\xi \cdot \xi \text{ for any } \xi \in \mathbb{R}^N.$$

Part of the results below hold for non well-ordered phases, or for multiphase composites (for more details, see, e.g., [24], [193], [192], [208], [209]).

We begin by defining what we call a *bound*, or an *optimal bound*, on a sum of energies.

Definition 2.2.5 Let ξ be a matrix in \mathcal{M}_N with rows (ξ_1, \dots, ξ_N) in \mathbb{R}^N . Let $\theta \in [0, 1]$ be the volume fraction of phase A and $(1 - \theta)$ be that of phase B . A real-valued function $f^+(\theta, A, B, \xi)$ (respectively, $f^-(\theta, A, B, \xi)$) is called an upper bound (respectively, a lower bound) if, for any homogenized matrix $A^* \in G_\theta$,

$$\sum_{i=1}^N A^* \xi_i \cdot \xi_i = A^* : \xi^t \xi \leq f^+(\theta, A, B, \xi) \quad (\text{respectively, } \geq f^-(\theta, A, B, \xi)).$$

The upper bound $f^+(\theta, A, B, \xi)$ (respectively, lower bound $f^-(\theta, A, B, \xi)$) is said to be optimal if, for any $\xi \in \mathcal{M}_N$, there exists $A_\xi^* \in G_\theta$ (which may depend on ξ) such that

$$\sum_{i=1}^N A_\xi^* \xi_i \cdot \xi_i = f^+(\theta, A, B, \xi) \quad (\text{respectively, } = f^-(\theta, A, B, \xi)).$$

We remark that our definition of optimality requires the bound to be saturated for any choice of the matrix ξ . This is not the case with the trivial harmonic and arithmetic mean bounds, which require that, for any matrix ξ , $A^* \in G_\theta$ satisfies

$$\left(\theta A^{-1} + (1 - \theta)B^{-1} \right)^{-1} : \xi^t \xi \leq A^* : \xi^t \xi \leq (\theta A + (1 - \theta)B) : \xi^t \xi.$$

For certain choices of ξ there are composites A^* that saturate either the upper or the lower bound. But there are other choices of ξ for which none of the bounds is attained by a composite $A^* \in G_\theta$. Therefore, the harmonic and arithmetic mean bounds are not optimal in the sense of Definition 2.2.5. Indeed, the optimal bounds are those of Hashin-Shtrikman given in the following proposition.

Proposition 2.2.6 *For any $\xi \in \mathcal{M}_N$, each homogenized matrix $A^* \in G_\theta$ satisfies*

$$A^* : \xi^t \xi \leq B : \xi^t \xi + \theta \min_{\eta \in \mathcal{M}_N} \left[2\xi : \eta + (B - A)^{-1} : \eta^t \eta - (1 - \theta)h(\eta) \right], \quad (2.27)$$

where $h(\eta)$ is a so-called nonlocal term defined by

$$h(\eta) = \min_{k \in \mathbb{Z}^N, k \neq 0} \frac{|\eta k|^2}{Bk \cdot k}$$

and

$$A^* : \xi^t \xi \geq A : \xi^t \xi + (1 - \theta) \max_{\eta \in \mathcal{M}_N} \left[2\xi : \eta - (B - A)^{-1} : \eta^t \eta - \theta g(\eta) \right], \quad (2.28)$$

where $g(\eta)$ is a so-called nonlocal term defined by

$$g(\eta) = \max_{k \in \mathbb{Z}^N, k \neq 0} \frac{|\eta k|^2}{Ak \cdot k}.$$

Furthermore, these upper and lower bounds are optimal in the sense of Definition 2.2.5, and optimality can always be achieved by a rank- N sequential laminate.

Following [193], our proof of the Hashin-Shtrikman bounds (2.27) and (2.28) relies on the Hashin-Shtrikman variational principle. There are other proofs of these bounds, for example that of Murat and Tartar [205], [274], which use the theory of compensated compactness [201], [272], or that of Lurie and Cherkaev [175], [176], which relies on the null-Lagrangian property of the determinant and on the theory of quasiconvexity (see, e.g., [89], [197]).

Remark 2.2.7 The functions $h(\eta)$ and $g(\eta)$ are called nonlocal terms, since they are the only terms in the Hashin-Shtrikman bounds that contain information on the microstructure of the composite A^* . They are computed through a cell problem by means of Fourier analysis (the vector k is the Fourier variable in the periodicity cell). As a simple matter of algebra, the function $h(\eta)$ is easily seen to be equal to the smallest singular value of the matrix $\eta B^{-1/2}$, while $g(\eta)$ is equal to the largest singular value of the matrix $\eta A^{-1/2}$. Clearly, $g(\eta)$ is convex, and the lower bound (2.28) is therefore given as the result of a finite-dimensional concave maximization. On the other hand,

$$h(\eta) \leq B^{-1} : \eta^t \eta \text{ and } (B - A)^{-1} - (1 - \theta)B^{-1} \geq 0,$$

which implies that the upper bound (2.27) is given as the result of a finite-dimensional convex minimization. It is therefore possible to compute explicitly these bounds. However, their precise values are unimportant in the sequel.

Optimality is always achieved in the class of sequential laminates, but there is no uniqueness, either in this class, or in general. There may well be other microstructures that are optimal: for example, the confocal ellipsoid construction of Tartar [274] (see Remark 3.2.41) or the Vigdergauz distribution of periodic inclusions [284], [122] (see also Remark 2.2.9).

Corollary 2.2.8 Let $A^* \in G_\theta$. The family of upper Hashin-Shtrikman bounds, as ξ runs into \mathcal{M}_N , is equivalent to the following family of bounds, as η runs into \mathcal{M}_N ,

$$\theta (B - A^*)^{-1} : \eta^t \eta \leq (B - A)^{-1} : \eta^t \eta - (1 - \theta)h(\eta). \quad (2.29)$$

The family of lower Hashin-Shtrikman bounds, as ξ runs into \mathcal{M}_N , is equivalent to the following family of bounds, as η runs into \mathcal{M}_N ,

$$(1 - \theta)(A^* - A)^{-1} : \eta^t \eta \leq (B - A)^{-1} : \eta^t \eta + \theta g(\eta). \quad (2.30)$$

Proof. By changing the sign of η , let us rewrite the upper bound (2.27) as

$$\phi_1(\xi) \geq \phi_2(\xi), \quad (2.31)$$

where

$$\phi_1(\xi) = \frac{1}{\theta}(B - A^*) : \xi^t \xi,$$

and

$$\phi_2(\xi) = \max_{\eta \in \mathcal{M}_N} [2\xi : \eta - (B - A)^{-1} : \eta^t \eta + (1 - \theta)h(\eta)].$$

The Legendre, or Fenchel, transform $\phi_1^*(\eta)$ of $\phi_1(\xi)$ is defined by (see, e.g., [58], [86], [102])

$$\phi_1^*(\eta) = \max_{\eta \in \mathcal{M}_N} [2\xi : \eta - \phi(\xi)]. \quad (2.32)$$

As is well known, $\phi_1^*(\eta)$ is a convex function of η . Recall that, for any volume fraction θ , $A \leq A^* \leq B$. This implies that $\phi_1(\xi)$ is a convex function of ξ . Therefore, by convex duality, the Legendre, or Fenchel, transform of $\phi_1^*(\eta)$ is again $\phi_1(\xi)$ (in other words $\phi_1^{**} = \phi_1$). On the other hand, $(B - A)^{-1} : \eta^t \eta - (1 - \theta)h(\eta)$ is also a convex function of η since it is the maximum over the unit sphere of quadratic functions of the type

$$(B - A)^{-1} : \eta^t \eta - (1 - \theta) \frac{|\eta k|^2}{Bk \cdot k},$$

which are obviously positive, as remarked in Remark 2.2.7. This implies that $\phi_2^{**} = \phi_2$. Therefore, by the monotonicity property of the Legendre transform, inequality (2.31) is equivalent to

$$\phi_1^*(\eta) \leq \phi_2^*(\eta),$$

which is nothing other than (2.29). Similarly the lower bound (2.28) can be rewritten as

$$\phi_3(\xi) \geq \phi_4(\xi), \quad (2.33)$$

where

$$\phi_3(\xi) = \frac{1}{1 - \theta}(A^* - A) : \xi^t \xi$$

and

$$\phi_4(\xi) = \max_{\eta \in \mathcal{M}_N} [2\xi : \eta - (B - A)^{-1} : \eta^t \eta - \theta g(\eta)].$$

It is not difficult to check that ϕ_3 and ϕ_4 are convex, as are their Legendre transforms, which implies that (2.33) is equivalent to

$$\phi_3^*(\eta) \leq \phi_4^*(\eta),$$

which is nothing other than (2.30). \square

Proof of Proposition 2.2.6. Since G_θ is the closure of the set P_θ of periodic composites (see (2.9) and Theorem 2.1.2), it is enough to prove

the bounds for a matrix A^* obtained by periodic homogenization (for which an explicit formula is available, see Section 1.1). The proof is divided into two steps: First, the bounds are established with the help of the Hashin-Shtrikman variational principle; second, the bounds are shown to be optimal by exhibiting sequential laminates that achieve them.

First step. We begin with the lower bound. We start from the explicit formula (1.13) for a symmetric homogenized matrix A^* obtained by periodic homogenization. In the present context, it reads

$$A^* \xi_i \cdot \xi_i = \min_{w_i(y) \in H_{\#}^1(Y)} \int_Y (\chi(y)A + (1 - \chi(y))B)(\xi_i + \nabla w_i) \cdot (\xi_i + \nabla w_i) dy$$

with $\chi(y)$ a characteristic function with average $\int_Y \chi(y) dy = \theta$. Summing these definitions for all rows ξ_i of the matrix ξ gives

$$A^* : \xi^t \xi = \min_{w(y) \in H_{\#}^1(Y)^N} \int_Y (\chi(y)A + (1 - \chi(y))B) : (\xi + \nabla w)^t (\xi + \nabla w) dy. \quad (2.34)$$

Note that in (2.34) the minimizations over all components w_i of the vector-valued function w are not coupled. Because of the periodicity condition it is tempting to use Fourier analysis in order to compute A^* . However, there are trilinear terms depending on y in (2.34), while only bilinear terms can be handled by Fourier analysis. Therefore, we first have to simplify (2.34) by making some suitable assumptions.

We subtract from $(\chi A + (1 - \chi)B)$ the *reference material* A :

$$\int_Y (\chi A + (1 - \chi)B) : (\xi + \nabla w)^t (\xi + \nabla w) dy = \quad (2.35)$$

$$\int_Y (1 - \chi)(B - A) : (\xi + \nabla w)^t (\xi + \nabla w) dy + \int_Y A : (\xi + \nabla w)^t (\xi + \nabla w) dy.$$

Since $B - A \geq 0$, by convex duality or Legendre transform, the first term on the right hand side of (2.35) is

$$\int_Y (1 - \chi)(B - A) : (\xi + \nabla w)^t (\xi + \nabla w) dy =$$

$$\max_{\eta(y) \in L_{\#}^2(Y; \mathcal{M}_N)} \int_Y (1 - \chi) \left(2(\xi + \nabla w) : \eta - (B - A)^{-1} : \eta^t \eta \right) dy,$$

which becomes an inequality by specializing the maximization in η to constant matrices in Y :

$$\begin{aligned} & \int_Y (1 - \chi)(B - A) : (\xi + \nabla w)^t (\xi + \nabla w) dy \geq \\ & \geq \max_{\eta \in \mathcal{M}_N} \int_Y (1 - \chi) \left(2(\xi + \nabla w) : \eta - (B - A)^{-1} : \eta^t \eta \right) dy \\ & \geq (1 - \theta) \left(2\xi : \eta - (B - A)^{-1} : \eta^t \eta \right) - 2 \int_Y \chi \nabla w : \eta dy. \end{aligned} \quad (2.36)$$

On the other hand, because of the periodicity, $\int_Y \nabla w dy = 0$ and the second term on the right hand side of (2.35) is

$$\int_Y A : (\xi + \nabla w)^t (\xi + \nabla w) dy = A : \xi^t \xi + \int_Y A : (\nabla w)^t (\nabla w) dy. \quad (2.37)$$

Plugging (2.36) and (2.37) into (2.35) and minimizing in $w(y)$ yields, for any $\eta \in \mathcal{M}_N$,

$$A^* : \xi^t \xi \geq A : \xi^t \xi + (1 - \theta) \left(2\xi : \eta - (B - A)^{-1} : \eta^t \eta \right) - g(\chi, \eta), \quad (2.38)$$

where $g(\chi, \eta)$ is a nonlocal term defined by

$$g(\chi, \eta) = - \min_{w(y) \in H_{\#}^1(Y)^N} \int_Y \left(A : (\nabla w)^t (\nabla w) - 2\chi \nabla w : \eta \right) dy. \quad (2.39)$$

It remains to prove that the nonlocal term is bounded by

$$g(\chi, \eta) \leq \theta(1 - \theta)g(\eta).$$

Then, the Hashin-Shtrikman lower bound (2.28) is deduced from (2.38) by a simple maximization in $\eta \in \mathcal{M}_N$.

The nonlocal term (2.39) can now be computed through the solution of a cell problem by means of Fourier analysis, since it involves only bilinear term depending on y . By periodicity, the characteristic function is written as a Fourier series,

$$\chi(y) = \sum_{k \in \mathbb{Z}^N} \hat{\chi}(k) e^{2i\pi k \cdot y},$$

as well as the test function $w(y)$

$$w(y) = \sum_{k \in \mathbb{Z}^N} \hat{w}(k) e^{2i\pi k \cdot y}.$$

Since χ and w are real-valued, their Fourier coefficients satisfy

$$\overline{\hat{\chi}(k)} = \hat{\chi}(-k) \text{ and } \overline{\hat{w}(k)} = \hat{w}(-k). \quad (2.40)$$

The gradient of w is given by

$$\nabla w(y) = \sum_{k \in \mathbb{Z}^N} 2i\pi e^{2i\pi k \cdot y} \hat{w}(k) \otimes k.$$

The Plancherel formula yields

$$\begin{aligned} & \int_Y \left(A : (\nabla w)^t (\nabla w) - 2\chi \nabla w : \eta \right) dy \\ &= \sum_{k \in \mathbb{Z}^N} \left(4\pi^2 A : (\hat{w}(k) \otimes k)^t (\overline{\hat{w}(k) \otimes k}) - 4i\pi \overline{\hat{\chi}(k)} (\hat{w}(k) \otimes k) : \eta \right) \\ &= \sum_{k \in \mathbb{Z}^N} \left(4\pi^2 (Ak \cdot k) |\hat{w}(k)|^2 + 4\pi \operatorname{Im} \left(\overline{\hat{\chi}(k)} \hat{w}(k) \cdot (\eta k) \right) \right). \end{aligned} \quad (2.41)$$

The minimization over $w(y)$ in $H_\#^1(Y)^N$ is equivalent to the minimizations over all vectors $\hat{w}(k)$ in \mathbb{C}^N , for each frequency k (and the minimization can be performed on each component independently). Note that the frequency $k = 0$ contributes to nothing in the sum (2.41). For all other frequencies the minimum is attained by

$$\hat{w}(k) = -\frac{i\hat{\chi}(k)}{2\pi Ak \cdot k} \eta k,$$

and the minimum value is

$$-\frac{|\hat{\chi}(k)|^2 |\eta k|^2}{Ak \cdot k}.$$

Therefore,

$$\begin{aligned} g(\chi, \eta) &= \sum_{k \in \mathbb{Z}^N, k \neq 0} \frac{|\hat{\chi}(k)|^2 |\eta k|^2}{Ak \cdot k} \\ &\leq \max_{k \in \mathbb{Z}^N, k \neq 0} \frac{|\eta k|^2}{Ak \cdot k} \sum_{k \in \mathbb{Z}^N, k \neq 0} |\hat{\chi}(k)|^2. \end{aligned} \quad (2.42)$$

On the other hand, the Plancherel formula gives

$$\sum_{k \in \mathbb{Z}^N, k \neq 0} |\hat{\chi}(k)|^2 = \int_Y |\chi(y) - \theta|^2 dy = \theta(1 - \theta).$$

Finally, (2.42) yields the desired result,

$$g(\chi, \eta) \leq \theta(1 - \theta)g(\eta).$$

The proof of the upper bound is similar, so we simply outline it. We start from formula (2.34) for the homogenized matrix A^* obtained by periodic homogenization. We subtract from $(\chi A + (1 - \chi)B)$ the other *reference material* B :

$$\begin{aligned} & \int_Y (\chi A + (1 - \chi)B) : (\xi + \nabla w)^t (\xi + \nabla w) dy \\ &= - \int_Y \chi(B - A) : (\xi + \nabla w)^t (\xi + \nabla w) dy \\ & \quad + \int_Y B : (\xi + \nabla w)^t (\xi + \nabla w) dy. \end{aligned} \tag{2.43}$$

Since $B - A \geq 0$, by convex duality or Legendre transform the first term on the right hand side of (2.43) is

$$\begin{aligned} & - \int_Y \chi(B - A) : (\xi + \nabla w)^t (\xi + \nabla w) dy = \\ & - \max_{\eta(y) \in L^2_\#(Y; \mathcal{M}_N)} \int_Y \chi \left(2(\xi + \nabla w) : \eta - (B - A)^{-1} : \eta^t \eta \right) dy, \end{aligned}$$

which becomes an inequality by specializing the maximization in η to constant matrices in Y :

$$\begin{aligned} & - \int_Y \chi(B - A) : (\xi + \nabla w)^t (\xi + \nabla w) dy \leq -\theta \left(2\xi : \eta - (B - A)^{-1} : \eta^t \eta \right) \\ & \quad - 2 \int_Y \chi \nabla w : \eta dy. \end{aligned}$$

On the other hand, the second term on the right hand side of (2.43) is

$$\int_Y B : (\xi + \nabla w)^t (\xi + \nabla w) dy = B : \xi^t \xi + \int_Y B : (\nabla w)^t (\nabla w) dy.$$

Inserting the above informations into (2.43) yields, for any $\eta \in \mathcal{M}_N$,

$$A^* : \xi^t \xi \leq B : \xi^t \xi + \theta \left(-2\xi : \eta + (B - A)^{-1} : \eta^t \eta \right) - h(\chi, \eta), \quad (2.44)$$

where $h(\chi, \eta)$ is another nonlocal term defined by

$$h(\chi, \eta) = - \min_{w(y) \in H_{\#}^1(Y)^N} \int_Y \left(B : (\nabla w)^t (\nabla w) - 2\chi \nabla w : \eta \right) dy. \quad (2.45)$$

As for the lower bound, a simple computation by Fourier analysis shows that

$$h(\chi, \eta) = \sum_{k \in \mathbb{Z}^N, k \neq 0} \frac{|\hat{\chi}(k)|^2 |\eta k|^2}{Bk \cdot k}, \quad (2.46)$$

which is easily bounded by

$$h(\chi, \eta) \leq \theta(1 - \theta)h(\eta).$$

Therefore, (2.44) and (2.46) imply that

$$A^* : \xi^t \xi \leq B : \xi^t \xi + \theta \left[-2\xi : \eta + (B - A)^{-1} : \eta^t \eta - (1 - \theta)h(\eta) \right],$$

which, upon changing the sign of η and minimizing in η , is nothing other than the Hashin-Shtrikman upper bound (2.27).

Second step. Optimality of the bounds (2.27) and (2.28) can be checked by simply exhibiting, for each matrix ξ , a particular composite material A^* , which saturates the inequality and makes it an equality. In this sense it may appear to be a matter of trial and error. However, there is a systematic way to prove the attainability of these bounds by connecting the optimality condition of the optimization in η (which gives the value of the bound) to the lamination formula for an optimal composite material. This idea is due to Avellaneda (see [32] in the elasticity setting).

Let us begin with the lower bound (2.28). The essence of this bound is the concave maximization

$$\max_{\eta \in \mathcal{M}_N} \phi(\eta) = \left[2\xi : \eta - (B - A)^{-1} : \eta^t \eta - \theta g(\eta) \right]. \quad (2.47)$$

Since $\phi(\eta)$ is strictly concave in η (see Remark 2.2.7), there exists a unique extremal η^* that achieves the maximum in (2.47). Therefore, the first order optimality condition is a necessary and sufficient condition for the extremality of η^* . If $\phi(\eta)$ were a smooth function, the optimality condition would

be $\nabla\phi(\eta^*) = 0$. But $g(\eta)$ is usually not a smooth function of η . In this context, the appropriate tool for obtaining first order optimality conditions is the subdifferential calculus or generalized gradient [86]. At the extremal η^* we must have

$$0 \in \partial\phi(\eta^*),$$

where $\partial\phi(\eta^*)$ is the subdifferential of ϕ at η^* . To compute the subdifferential of $g(\eta)$ we recall that

$$g(\eta) = \max_{k \in \mathbb{Z}^N, k \neq 0} \frac{|\eta k|^2}{Ak \cdot k} = \max_{e \in S_{N-1}} |\eta A^{-1/2}e|^2, \quad (2.48)$$

where S_{N-1} is the unit sphere in \mathbb{R}^N . Introducing the set $\mathcal{P}(S_{N-1})$ of probability measures on S_{N-1} (recall that a probability measure is a nonnegative measure of unit mass), the nonlocal term $g(\eta)$ can be rewritten as

$$g(\eta) = \max_{\nu \in \mathcal{P}(S_{N-1})} \int_{S_{N-1}} |\eta A^{-1/2}e|^2 d\nu(e).$$

This implies that (2.47) becomes a min-max optimization problem featuring a convex minimization with respect to ν and concave maximization with respect to η . Therefore, applying the Kuhn-Tucker optimality conditions, the subdifferential of $g(\eta)$ is the set (see Theorem 2.8.2 and Corollary 2, p.87 in [86])

$$\partial g(\eta) = \left\{ 2\eta A^{-1/2} \int_{S_{N-1}} e \otimes e d\nu(e) A^{-1/2} \right\}, \quad (2.49)$$

where ν runs into the subset of probability measures with support concentrated on extremal vectors e , namely on eigenvectors of $(\eta A^{-1/2})^t \eta A^{-1/2}$ associated to the eigenvalue $g(\eta)$. In other words, (2.49) tells us that, since $g(\eta)$ is the supremum of a family of nonnegative quadratic functions, continuously parametrized by e , its subdifferential is the convex hull of the gradients (with respect to η) $2\eta(A^{-1/2}e) \otimes (A^{-1/2}e)$ of those quadratic functions which are extremal in (2.48).

Thus, the optimality condition for η^* states that there exists such a measure ν^* satisfying

$$\xi = \eta^*(B - A)^{-1} + \theta \eta^* A^{-1/2} \int_{S_{N-1}} e \otimes e d\nu^*(e) A^{-1/2}.$$

Remark that the matrix $\int_{S_{N-1}} e \otimes e d\nu^*(e)$ is symmetric and therefore diagonalizable. Since it is also nonnegative and has unit trace, there exist an

orthonormal eigenbasis $(e_i)_{1 \leq i \leq N}$ and nonnegative eigenvalues $(m_i)_{1 \leq i \leq N}$, with $m_i \geq 0$ and $\sum_{i=1}^N m_i = 1$, such that

$$\int_{S_{N-1}} e \otimes e \, d\nu^*(e) = \sum_{i=1}^N m_i e_i \otimes e_i. \quad (2.50)$$

Furthermore, these eigenvectors e_i are also extremal in the definition of $g(\eta)$ since $\partial g(\eta) : \eta = 2g(\eta)$. By virtue of (2.50) the optimality condition for η^* is

$$\xi = \eta^* (B - A)^{-1} + \theta \sum_{i=1}^N m_i \eta^* A^{-1/2} e_i \otimes A^{-1/2} e_i. \quad (2.51)$$

Taking the inner product of (2.51) with η^* gives

$$\begin{aligned} \xi : \eta^* &= (B - A)^{-1} : (\eta^*)^t \eta^* + \theta \sum_{i=1}^N m_i |\eta^* A^{-1/2} e_i|^2 \\ &= (B - A)^{-1} : (\eta^*)^t \eta^* + \theta g(\eta^*). \end{aligned}$$

Therefore, the extremal value of (2.47) is $\phi(\eta^*) = \xi : \eta^*$, and the lower bound (2.28) can be expressed as

$$A^* : \xi^t \xi \geq A : \xi^t \xi + (1 - \theta) \xi : \eta^*. \quad (2.52)$$

To achieve equality in (2.52), consider the sequential laminate A_N^* , furnished by formula (2.25) in Lemma 2.2.4, with the same parameters m_i and the lamination directions $A^{-1/2} e_i$, namely,

$$(1 - \theta) (A_N^* - A)^{-1} = (B - A)^{-1} + \theta \sum_{i=1}^N m_i A^{-1/2} e_i \otimes A^{-1/2} e_i. \quad (2.53)$$

Multiplying (2.53) by η^* and using (2.51) yields

$$(1 - \theta) \eta^* (A_N^* - A)^{-1} = \xi,$$

which becomes

$$\xi A_N^* = \xi A + (1 - \theta) \eta^*,$$

and, upon taking the inner product with ξ ,

$$A_N^* : \xi^t \xi = A : \xi^t \xi + (1 - \theta) \xi : \eta^*,$$

which is nothing but the bound (2.52) saturated by A_N^* . This proves the optimality of the Hashin-Shtrikman lower bound for a sequential laminate of rank at most N .

Optimality of the upper bound (2.27) can be proved in the same manner. The only difference is that the optimal sequential laminate is given by formula (2.22) in Lemma 2.2.3. We safely leave the details to the reader. \square

Remark 2.2.9 *Proposition 2.2.6 delivers two results. First, it gives upper and lower bounds, and second, it exhibits some sequential laminates that saturate these bounds. However, the proof is more precise in the sense that it yields necessary and sufficient conditions for a periodic microstructure to be optimal in these bounds. Actually, two inequalities are made to obtain the lower bound. Thus, for an optimal periodic composite, defined by a characteristic function χ , inequalities (2.36) and (2.42) become equalities. Equality in (2.36) means that the microscopic gradient is constant in phase B , while equality in (2.42) implies that, for those k such that $\hat{\chi}(k) \neq 0$, $|\eta k|^2/(Ak \cdot k)$ is constant, equal to its maximum, where η is optimal in the definition of the bound. Similar conditions are obtained for the upper bound by inverting the role of A and B . For more details, we refer the reader to [25] and [122].*

Remark 2.2.10 *If instead of a sum of N energies we applied the same Hashin-Shtrikman methodology on a single energy, we would have obtained obvious energy bounds, namely the arithmetic (upper) and harmonic (lower) bounds.*

Remark 2.2.11 *In the proof of the optimality of the lower Hashin-Shtrikman bound, the lamination directions of the optimal rank- N sequential laminate are $A^{-1/2}e_i$ with $(e_i)_{1 \leq i \leq N}$ an orthonormal basis. Therefore, if phase A is isotropic, the lamination directions of this optimal sequential laminates are also orthogonal. The same is true for the upper bound if phase B is isotropic.*

Remark 2.2.12 *The lower and upper Hashin-Shtrikman bounds are easily seen to be C^1 functions with respect to θ and ξ . Indeed, they are given as the extremal value of a concave or convex function of η , which is extremized at a unique η^* . Therefore, when differentiating these bounds, the derivative of η^* cancels out due to the optimality condition it satisfies, and the derivatives of the bounds are thus continuous functions. The lower bound (2.28) is also a strictly convex function of θ . It is the maximum of convex quadratic functions of θ and is therefore convex. The strict convexity is a consequence of the fact that $g(\eta) > 0$ for $\eta \neq 0$.*

2.2.3 G -closure of Two Isotropic Phases

In this subsection, we address the problem of the G -closure of the mixture of two isotropic conductors in fixed volume fraction. This problem was first solved by Murat and Tartar [205], [274] and Lurie and Cherkaev [175], [176]. Their method was based either on compensated compactness or on null-lagrangian and quasiconvexity properties. Here we prefer to use the Hashin-Shtrikman bounds derived in the previous subsection. As shown by Milton [189] these two approaches are actually equivalent (see also [20]). From now on, the phases are assumed to be isotropic, i.e.

$$A = \alpha I_2 \text{ and } B = \beta I_2 \text{ with } 0 < \alpha < \beta.$$

The main result is a characterization of the set G_θ defined by (2.9), i.e., the closure of the set P_θ of effective tensors obtained by periodic homogenization of a mixture of α and β in proportions θ and $(1 - \theta)$.

Theorem 2.2.13 *Let G_θ be the set of all constant effective tensors obtained by homogenization of a mixture of α and β in proportions θ and $(1 - \theta)$. The set G_θ is the convex set of all symmetric matrices with eigenvalues $\lambda_1, \dots, \lambda_N$ satisfying*

$$\lambda_\theta^- \leq \lambda_i \leq \lambda_\theta^+ \quad \forall 1 \leq i \leq N \quad (2.54)$$

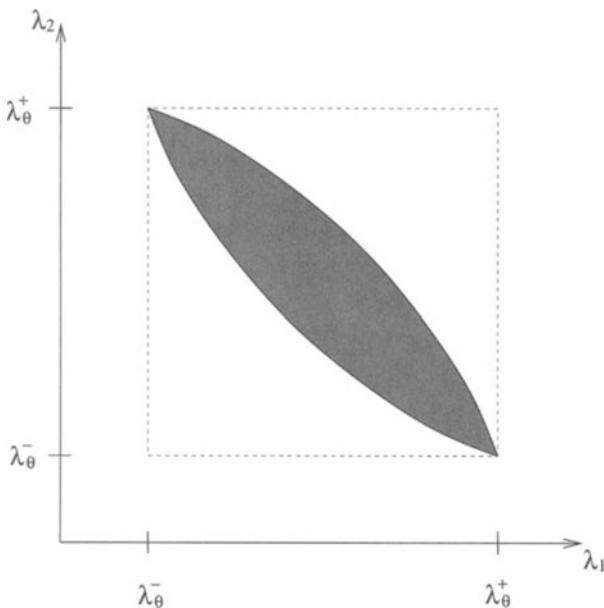
$$\sum_{i=1}^N \frac{1}{\lambda_i - \alpha} \leq \frac{1}{\lambda_\theta^- - \alpha} + \frac{N-1}{\lambda_\theta^+ - \alpha} \quad (2.55)$$

$$\sum_{i=1}^N \frac{1}{\beta - \lambda_i} \leq \frac{1}{\beta - \lambda_\theta^-} + \frac{N-1}{\beta - \lambda_\theta^+}, \quad (2.56)$$

where λ_θ^- and λ_θ^+ are the harmonic and arithmetic means of α and β defined by

$$\lambda_\theta^- = \left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta} \right)^{-1} \quad \text{and} \quad \lambda_\theta^+ = \theta\alpha + (1-\theta)\beta.$$

Remark 2.2.14 *In the characterization of G_θ given by Theorem 2.2.13, the bound $\lambda_i \geq \lambda_\theta^-$ is superfluous. Indeed, from the other bound $\lambda_i \leq \lambda_\theta^+$ and (2.56), it is easily deduced that $\lambda_i \geq \lambda_\theta^-$. A similar computation shows that if an H -limit A^* has one eigenvalue equal to the harmonic mean λ_θ^- , then all its other eigenvalues are equal to the arithmetic mean λ_θ^+ . In two dimensions, the bound $\lambda_i \leq \lambda_\theta^+$ is also superfluous since (2.54) can be deduced from (2.55)*

Figure 2.3: The G -closure set G_θ for $N = 2$.

and (2.56) for $N = 2$ (which is no longer true in higher dimensions). By the same token, it is easily checked that the only tensors that saturate both bounds (2.55) and (2.56) are characterized by one eigenvalue equal to the harmonic mean λ_θ^- , while all other ones are equal to the arithmetic mean λ_θ^+ (which corresponds to rank-1 laminates).

Remark 2.2.15 Since the bounds (2.55) and (2.56) are convex functions of the eigenvalues $\lambda_1, \dots, \lambda_N$, the set G_θ is convex with respect to these eigenvalues. Since a nondecreasing convex function of the eigenvalues is also convex with respect to the associated symmetric matrices (see, e.g., [37], [280]), we deduce that G_θ is a convex subset of the space \mathcal{M}_N^s of symmetric matrices. This result could also be obtained by remarking that G_θ is completely determined by the upper and lower Hashin-Shtrikman bounds on sums of energies, which obviously define a convex set of matrices.

Remark 2.2.16 An equivalent characterization of G_θ is obtained by considering the eigenvalues $\nu_i = 1/\lambda_i$ of the inverse tensor A^* . Defining $\nu_\theta^- = 1/\lambda_\theta^-$ and $\nu_\theta^+ = 1/\lambda_\theta^+$, an easy computation shows that (2.54), (2.55), and (2.56)

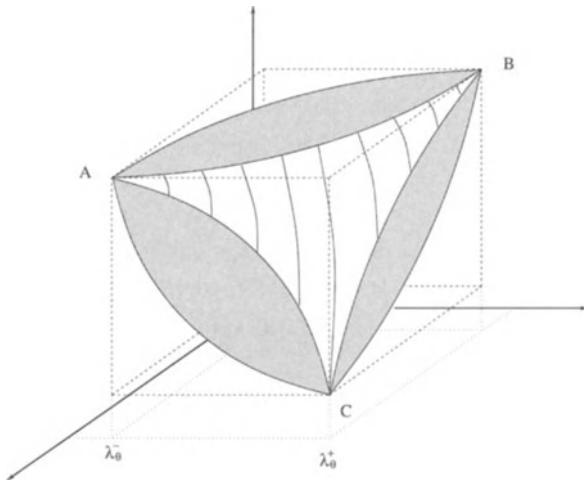


Figure 2.4: The G -closure set G_θ for $N = 3$. The points A, B, C correspond to rank-1 laminates. The three grey lens-shaped surfaces (corresponding to one eigenvalue equal to λ_θ^+) are similar to the two-dimensional G -closure.

are equivalent to

$$\nu_\theta^+ \leq \nu_i \leq \nu_\theta^- \quad \forall 1 \leq i \leq N$$

$$\sum_{i=1}^N \frac{1}{\alpha^{-1} - \nu_i} \leq \frac{1}{\alpha^{-1} - \nu_\theta^-} + \frac{N-1}{\alpha^{-1} - \nu_\theta^+}$$

$$\sum_{i=1}^N \frac{1}{\nu_i - \beta^{-1}} \leq \frac{1}{\nu_\theta^- - \beta^{-1}} + \frac{N-1}{\nu_\theta^+ - \beta^{-1}}.$$

These bounds define a convex set of the eigenvalues ν_1, \dots, ν_N , and therefore G_θ is a convex set of the inverse tensors A^{*-1} .

Remark 2.2.17 There exists yet another equivalent characterization of G_θ which has the advantage of making its dependence with respect to θ even more explicit. It is based on the so-called Y -transform, introduced in [188] (see also [76], [190]; the precise definition of Y may change slightly from article to article), which is defined as the following map on the set of symmetric matrices

$$Y(A^*) = \left(\lambda_\theta^+ I_2 - A^* \right) \left((\lambda_\theta^-)^{-1} A^* - I_2 \right)^{-1}.$$

A simple computation shows that A^* belongs to G_θ if and only if the eigenvalues $(y_i)_{1 \leq i \leq N}$ of $Y(A^*)$ are nonnegative and satisfies

$$\sum_{i=1}^N \frac{\alpha}{\alpha + y_i} \leq N - 1 \leq \sum_{i=1}^N \frac{\beta}{\beta + y_i}. \quad (2.57)$$

The interesting fact is that the set $Y(G_\theta)$, defined by (2.57) and $y_i \geq 0$, does not depend on the proportion θ (it is rather the map Y which varies with θ). This allows a parametrization of G_θ in terms of θ and of matrices independent of θ . We remark, however, that $Y(G_\theta)$ is neither convex nor bounded.

Proof. Let us denote by K_θ the set defined by the inequalities (2.54), (2.55), and (2.56). We remark first that K_θ is a convex set of the eigenvalues $\lambda_1, \dots, \lambda_N$, and thus a convex set of matrices. We recognize in (2.54) the harmonic and arithmetic mean bounds (2.4) which are a straightforward consequence of Theorem 1.3.14. The two other bounds are a consequence of Corollary 2.2.8 on the Hashin-Shtrikman bounds. Indeed, (2.55) and (2.56) are nothing other than (2.30) and (2.29), respectively, specialized to $\eta = I_2$. Therefore, G_θ is included in K_θ .

To prove the reverse inclusion, we proceed by recurrence on the space dimension N . To make clear the dependence on the space dimension, from now on we denote by G_θ^N and K_θ^N the spaces G_θ and K_θ in dimension N . If $N = 1$, then K_θ^1 is reduced to the single scalar matrix λ_θ^- . From Subsection 1.2.3, we know that, in the one-dimensional case, G_θ^1 is also reduced to the unique value λ_θ^- . Therefore, $G_\theta^1 = K_\theta^1$. Assume that the equality $G_\theta^p = K_\theta^p$ holds true for any dimension $p \leq N - 1$. We begin by proving that any point on the boundary of K_θ^N belongs to G_θ^N . The boundary of K_θ^N is defined by one of the inequalities (2.54), (2.55), or (2.56) being saturated. By Remark 2.2.14, it is not necessary to consider the inequalities $\lambda_i \geq \lambda_\theta^-$ (which are never saturated if the other inequalities $\lambda_j \leq \lambda_\theta^+$ for $j \neq i$, are not achieved, either). Let us first consider the case where one eigenvalue is equal to the arithmetic mean. With no loss of generality, we can assume that $\lambda_N = \lambda_\theta^+$. It is immediately seen that $(\lambda_1, \dots, \lambda_{N-1})$ satisfies the $(N-1)$ -dimensional analogues of (2.55) and (2.56), and therefore belongs to K_θ^{N-1} . Applying the recurrence hypothesis $K_\theta^{N-1} = G_\theta^{N-1}$ and Lemma 2.2.19 below shows that $(\lambda_1, \dots, \lambda_N)$ belongs to G_θ^N . Consider now a set of eigenvalues $(\lambda_1, \dots, \lambda_N)$

such that, for $1 \leq i \leq N$, $\lambda_\theta^- < \lambda_i < \lambda_\theta^+$, and (2.56) is an equality, i.e.,

$$\sum_{i=1}^N \frac{1}{\beta - \lambda_i} = \frac{1}{\beta - \lambda_\theta^-} + \frac{N-1}{\beta - \lambda_\theta^+}. \quad (2.58)$$

Let us show that these eigenvalues are those of a sequential laminate of rank- N (see Lemma 2.2.3). Let $(e_i)_{1 \leq i \leq N}$ be an orthonormal basis of eigenvectors associated to the eigenvalues $(\lambda_i)_{1 \leq i \leq N}$. Let $(m_i)_{1 \leq i \leq N}$ be nonnegative real numbers satisfying $m_i \geq 0$ and $\sum_{i=1}^N m_i = 1$. A rank- N sequential laminate (with core α and matrix β) is given by

$$\theta(A^* - \beta I_2)^{-1} = \frac{1}{\alpha - \beta} I_2 + (1 - \theta) \sum_{i=1}^N m_i \frac{e_i \otimes e_i}{\beta}.$$

The tensor A^* has eigenvalues $(\lambda_i)_{1 \leq i \leq N}$ if and only if the m_i can be chosen such that

$$\frac{\theta}{\lambda_i - \beta} = \frac{1}{\alpha - \beta} + \frac{m_i(1 - \theta)}{\beta}. \quad (2.59)$$

It is easily checked that m_i , defined by (2.59), satisfies $0 < m_i < 1$ if and only if $\lambda_\theta^- < \lambda_i < \lambda_\theta^+$. Furthermore, $\sum_{i=1}^N m_i = 1$ is equivalent to (2.58). This proves that such eigenvalues $(\lambda_i)_{1 \leq i \leq N}$ are those of a rank- N sequential laminate in G_θ^N . The case of the last part of the boundary of K_θ^N is completely symmetric. A set of eigenvalues $(\lambda_i)_{1 \leq i \leq N}$ such that, for $1 \leq i \leq N$, $\lambda_\theta^- < \lambda_i < \lambda_\theta^+$ and (2.55) is an equality, i.e.,

$$\sum_{i=1}^N \frac{1}{\lambda_i - \alpha} = \frac{1}{\lambda_\theta^- - \alpha} + \frac{N-1}{\lambda_\theta^+ - \alpha}$$

is attained by a rank- N sequential laminate (with core β and matrix α ; see Lemma 2.2.4). We have thus proved that any point on the boundary of K_θ^N belongs to G_θ^N . To conclude the proof, we now consider a point in the interior of K_θ^N , namely a set of eigenvalues $(\lambda_i)_{1 \leq i \leq N}$ such that (2.55) and (2.56) are strict inequalities, and $\lambda_\theta^- < \lambda_i < \lambda_\theta^+$. We are going to show that such a point is actually a simple lamination of two other points on the boundary of K_θ^N . Since G_θ^N is closed under H -convergence (see Lemma 2.1.5), and therefore stable under lamination, this will prove that such a single laminate is indeed a realizable effective tensor, which implies that K_θ^N is entirely enclosed in G_θ^N . For this purpose, we keep $(\lambda_1, \dots, \lambda_{N-1})$ fixed and increase or decrease λ_N to meet the boundary of K_θ^N : we obtain two

points $(\lambda_1, \dots, \lambda_{N-1}, \lambda_N^+)$ and $(\lambda_1, \dots, \lambda_{N-1}, \lambda_N^-)$ such that λ_N is a convex combination of λ_N^+ and λ_N^- . Applying the lamination formula (1.109) to these two points in the eigendirection e_N of the last eigenvalue λ_N and in proportions $t, (1-t)$ yields an effective tensor with eigenvalues

$$\lambda_i(t) = \lambda_i \text{ for } 1 \leq i \leq N-1$$

$$\lambda_N(t) = t\lambda_N^- + (1-t)\lambda_N^+ - \frac{t(1-t)}{(1-t)\lambda_N^- + t\lambda_N^+} (\lambda_N^- - \lambda_N^+)^2.$$

By choosing adequately $t \in (0, 1)$, we can obtain $\lambda_N(t) = \lambda_N$, which proves that any point inside K_θ^N is a simple lamination of two other points on the boundary of K_θ^N . \square

Remark 2.2.18 Sequential laminates are not the only optimal microstructures in G_θ . The original proof of Tartar [274] used confocal ellipsoids (see Remark 3.2.41). As shown by Grabovsky [122], the so-called Vigdergauz distribution of periodic inclusions [284] is also optimal in two space dimensions.

Lemma 2.2.19 Let a^* be an homogenized matrix of order $N-1$ in G_θ^{N-1} . Then the matrix A^* of order N defined by

$$A^* = \begin{pmatrix} a^* & 0 \\ 0 & \lambda_\theta^+ \end{pmatrix} \quad (2.60)$$

belongs to G_θ^N .

Proof. Since G_θ is the closure of the set P_θ of all periodic composites, it is enough to prove the result for a matrix a^* obtained by periodic homogenization. Let us denote by $y' = (y_1, \dots, y_{N-1})$ a point in $Y' = (0, 1)^{N-1}$, and by $y = (y', y_N)$ a point in $Y = (0, 1)^N$. The matrix $a^* \in P_\theta^{N-1}$ is characterized by a characteristic function $\chi'(y') \in L^\infty(Y')$, with average $\int_{Y'} \chi'(y') dy' = \theta$, such that

$$a^* \zeta' \cdot \zeta' = \min_{w(y') \in H_\#^1(Y')} \int_{Y'} (\chi'(y') \alpha + (1 - \chi'(y')) \beta) |\zeta' + \nabla w|^2 dy',$$

for any $\zeta' \in \mathbb{R}^{N-1}$. Let us extend this characteristic function to Y by defining $\chi(y) = \chi(y', y_N) = \chi'(y')$. To χ is associated a periodic homogenized matrix of order N , $A^* \in P_\theta^N$, defined by

$$A^* \zeta \cdot \zeta = \min_{w(y) \in H_\#^1(Y)} \int_Y (\chi(y) \alpha + (1 - \chi(y)) \beta) |\zeta + \nabla w|^2 dy$$

for any $\zeta \in \mathbb{R}^N$. Since $\chi(y)$ does not depend on y_N , it is easily seen that the minimizer $w(y)$ also does not depend on y_N . Then, a simple computation yields

$$A^* \zeta \cdot \zeta = a^* \zeta' \cdot \zeta' + \lambda_\theta^+ \zeta_N^2,$$

with $\zeta = (\zeta', \zeta_N)$, which is just formula (2.60) for A^* . \square

Remark 2.2.20 *There is a converse to Lemma 2.2.19 in the sense that, if a periodic composite $A^* \in P_\theta$ (defined by a characteristic function χ) has one of its eigenvalues equal to the arithmetic mean, then χ is invariant in the corresponding eigendirection (see [25]).*

2.3 Elasticity

In this section, we consider elastic composite materials obtained by mixing two isotropic phases. Their Hooke's laws A and B are defined by

$$\begin{aligned} A &= 2\mu_A I_4 + \left(\kappa_A - \frac{2\mu_A}{N} \right) I_2 \otimes I_2 \\ B &= 2\mu_B I_4 + \left(\kappa_B - \frac{2\mu_B}{N} \right) I_2 \otimes I_2, \end{aligned} \tag{2.61}$$

where μ_A, μ_B are the shear moduli, and κ_A, κ_B are the bulk moduli. The two phases are assumed to be well-ordered, i.e.,

$$0 < \mu_A \leq \mu_B, \quad 0 < \kappa_A \leq \kappa_B. \tag{2.62}$$

It is convenient to introduce a Lamé coefficient, proportional to the Poisson's ratio, defined by

$$\lambda_A = \kappa_A - \frac{2\mu_A}{N}, \quad \lambda_B = \kappa_B - \frac{2\mu_B}{N}. \tag{2.63}$$

2.3.1 Laminated Composites

As in the conductivity case, we describe a special class of two-phase elastic composites, namely the laminated composites. We know from Subsection 1.3.5 that, when the sequence $\chi_\epsilon(x)$ describing the microstructure of a composite material A^* (see Definition 2.1.1) depends on a single space coordinate, then there exists an explicit formula for the homogenized Hooke's law A^* which depends only on A, B, θ (the weak limit of χ_ϵ), and the direction of the oscillations of χ_ϵ . More precisely, we recall the result of Lemma 1.4.12

and Corollary 1.4.13. Let e be a unit vector in \mathbb{R}^N ($|e| = 1$) that is the direction of the lamination. Let $\theta(x \cdot e)$ be the weak limit of the sequence $\chi_\epsilon(x \cdot e)$. The homogenized Hooke's law A^* is given by

$$(1 - \theta)(A^* - A)^{-1} = (B - A)^{-1} + \theta f_A(e), \quad (2.64)$$

where $f_A(e)$ is a positive nondefinite fourth order tensor defined (ξ being a symmetric matrix) by

$$f_A(e)\xi = \frac{1}{\mu_A} (\xi e \odot e - (\xi e \cdot e)e \otimes e) + \frac{1}{2\mu_A + \lambda_A} (\xi e \cdot e)e \otimes e, \quad (2.65)$$

or, equivalently, by the quadratic form

$$f_A(e)\xi : \xi = \frac{1}{\mu_A} (|\xi e|^2 - (\xi e \cdot e)^2) + \frac{1}{2\mu_A + \lambda_A} (\xi e \cdot e)^2. \quad (2.66)$$

The composite A^* is said to be a single lamination in the direction e of the two phases A and B in proportions θ and $(1 - \theta)$. By varying the proportion θ and the direction e , we obtain a whole family of composite materials. But this family can be enlarged further by laminating again these simple laminates. By Lemma 2.1.5 we know that such iterated laminates are still two-phase composite materials, i.e., they still belong to G_θ . Repeating this lamination process in any directions and any proportion yields a subset L_θ of G_θ , made of these laminated materials.

Definition 2.3.1 *The subset $L_\theta \subset G_\theta$ of all laminated materials obtained from the phases A and B in proportions θ and $(1 - \theta)$ is the smallest subset of G_θ that contains all simple laminates, as defined by (2.64), and is stable under lamination.*

As in Subsection 2.2.1 a smaller subset of L_θ can be defined by considering only sequential laminates. Such laminates are obtained by an iterative process, where, at each stage, the previous laminate is laminated again with a single pure phase (always the same one). This allows one to derive an explicit formula for such homogenized Hooke's law A^* by using the special form of (2.64). A sequential laminate is characterized by a collection of unit vectors $(e_i)_{1 \leq i \leq p}$ and by proportions $(\theta_i)_{1 \leq i \leq p}$ in $[0, 1]$. At the first stage, a simple laminate A_1^* is obtained by laminating B and A in proportions $(1 - \theta_1)$, θ_1 , respectively, and in the direction e_1 . This simple laminate A_1^* can again be laminated with phase A , in direction e_2 , and in proportions

$(1 - \theta_2)$, θ_2 respectively, to obtain a new laminate denoted by A_2^* . By induction, we obtain A_p^* by lamination of A_{p-1}^* and A , in direction e_p and in proportions $(1 - \theta_p)$, θ_p , respectively. By application of formula (2.64), the homogenized tensor A_p^* is defined by

$$(1 - \theta_p) (A_p^* - A)^{-1} = (A_{p-1}^* - A)^{-1} + \theta_p f_A(e_p).$$

By iterating this formula, the term $(A_{p-1}^* - A)^{-1}$ can be replaced by a similar expression involving $(A_{p-2}^* - A)^{-1}$, and so on up to $(A_0^* - A)^{-1}$ with $A_0^* \equiv B$. This yields a formula of the same type as (2.64), namely,

$$\left(\prod_{i=1}^p (1 - \theta_i) \right) (A_p^* - A)^{-1} = (B - A)^{-1} + \sum_{i=1}^p \left(\theta_i \prod_{j=1}^{i-1} (1 - \theta_j) \right) f_A(e_i). \quad (2.67)$$

We emphasize that we always laminate an intermediate laminate with the same phase A . In other words, phase B is coated by several layers of A . One can say that A plays the role of a matrix phase and B that of a core phase. Globally, A_p^* can be seen as a mixture of A and B in different layers having a large separation of scales.

Definition 2.3.2 *The composite material A_p^* defined by formula (2.67) is called a rank- p sequential laminate with matrix phase A and core phase B . It is characterized by the lamination directions $(e_i)_{1 \leq i \leq p}$ and the proportions $(\theta_i)_{1 \leq i \leq p}$ at each stage of the process.*

In Definition 2.3.2 the word “sequential” means that this type of laminate is iteratively constructed by successive laminations always with the same pure phase. In formula (2.67) the overall volume fraction of phase A is

$$\theta = 1 - \prod_{i=1}^p (1 - \theta_i).$$

If this overall volume fraction θ is fixed, by varying the proportions $(\theta_i)_{1 \leq i \leq p}$ we obtain a parametrized family of rank- p sequential laminates.

Lemma 2.3.3 *Let $(e_i)_{1 \leq i \leq p}$ be a collection of unit vectors. Let θ be a volume fraction in $[0, 1]$. For any collection of nonnegative real numbers $(m_i)_{1 \leq i \leq p}$ satisfying*

$$\sum_{i=1}^p m_i = 1 \text{ and } m_i \geq 0, \quad 1 \leq i \leq p,$$

there exists a rank- p sequential laminate A_p^* , with matrix A and core B , in proportions θ and $(1 - \theta)$, respectively, with lamination directions $(e_i)_{1 \leq i \leq p}$ such that

$$(1 - \theta) (A_p^* - A)^{-1} = (B - A)^{-1} + \theta \sum_{i=1}^p m_i f_A(e_i), \quad (2.68)$$

where $f_A(e_i)$ is given by (2.66). The numbers $(m_i)_{1 \leq i \leq p}$ are called the lamination parameters.

By interchanging the roles of A and B , a symmetric result is obtained. There exists another homogenized Hooke's law B_p^* , a rank- p sequential laminate with core A and matrix B in proportions θ and $(1 - \theta)$, respectively, defined by

$$\theta (B_p^* - B)^{-1} = (A - B)^{-1} + (1 - \theta) \sum_{i=1}^p m_i f_B(e_i), \quad (2.69)$$

where $f_B(e_i)$ is given by (2.66) with B replacing A .

Remark 2.3.4 In the elasticity setting, laminated composites have been known for a long time (see, e.g., [115], [116], [188]), but the class of sequential laminates with their explicit formulas (as delivered by Lemma 2.3.3) was introduced by Francfort and Murat [110] following the lead of Tartar [274].

The proof of Lemma 2.3.3 is similar to that of Lemma 2.2.3 (in the conductivity case), so we omit it. There is another way of writing the above lamination formulas, which uses the inverse tensors A^{-1} and B^{-1} (the so-called *compliance tensors*) instead of A and B . Formulas (2.68) and (2.69) are convenient when working with strain tensors, while the new formulas below are of interest when using stress tensors. The Hooke's law A^* defined by (2.64) as a single lamination of material A and B , in proportion θ and $(1 - \theta)$, respectively, in the direction of the unit vector e , is equivalently defined by

$$(1 - \theta) (A^{*-1} - A^{-1})^{-1} = (B^{-1} - A^{-1})^{-1} + \theta f_A^c(e) \quad (2.70)$$

where $f_A^c(e)$ is a symmetric positive nondefinite fourth order tensor defined, for any symmetric matrix ξ , by the quadratic form

$$f_A^c(e)\xi : \xi = A\xi : \xi - \frac{1}{\mu_A} |A\xi e|^2 + \frac{\mu_A + \lambda_A}{\mu_A(2\mu_A + \lambda_A)} ((A\xi)e \cdot e)^2. \quad (2.71)$$

Formula (2.70) and (2.71) are easily obtained by a computation similar to that in Lemma 1.4.12 which gives the “usual” lamination formula (2.64) (see Remark 1.4.14).

Since (2.70) is equivalent to (2.64), we deduce that formula (2.68), defining the rank- p sequential laminate A_p^* , is equivalent to

$$(1 - \theta) (A_p^{*-1} - A^{-1})^{-1} = (B^{-1} - A^{-1})^{-1} + \theta \sum_{i=1}^p m_i f_A^c(e_i). \quad (2.72)$$

By interchanging the roles of A and B , formula (2.69), defining the rank- p sequential laminate B_p^* , is equivalent to

$$\theta (B_p^{*-1} - B^{-1})^{-1} = (A^{-1} - B^{-1})^{-1} + (1 - \theta) \sum_{i=1}^p m_i f_B^c(e_i), \quad (2.73)$$

where $f_B^c(e_i)$ is given by (2.71) with B replacing A .

Definition 2.3.5 We denote by L_θ^- the set of all sequential laminates A_p^* , with matrix A and core B , in proportions θ and $(1 - \theta)$, respectively, defined by formula (2.68) (or, equivalently, (2.72)), obtained by varying the number p of laminations and the lamination directions $(e_i)_{1 \leq i \leq p}$, as well as the lamination parameters $(m_i)_{1 \leq i \leq p}$.

We denote by L_θ^+ the set of all sequential laminates B_p^* , with core A and matrix B , in proportions θ and $(1 - \theta)$, respectively, defined by formula (2.69) (or, equivalently, (2.73)), obtained by varying the number p of laminations and the lamination directions $(e_i)_{1 \leq i \leq p}$, as well as the lamination parameters $(m_i)_{1 \leq i \leq p}$.

In the sequel we shall use a convenient parameterization of the sets L_θ^\pm , which is given in the next result (due to Avellaneda [32]).

Lemma 2.3.6 The set L_θ^- is the bounded closed subset of all symmetric fourth order tensors $A^* \in \mathcal{M}_N^4$ such that there exists a probability measure ν on the unit sphere S_{N-1} satisfying

$$(1 - \theta) (A^* - A)^{-1} = (B - A)^{-1} + \theta \int_{S_{N-1}} f_A(e) d\nu(e), \quad (2.74)$$

where $f_A(e)$ is given by (2.66). Furthermore, any tensor A^* in L_θ^- is the Hooke’s law of a finite-rank sequential laminate defined by (2.68) with rank $p \leq (N + 3)(N + 2)(N + 1)N/24$.

Similarly, L_θ^+ is the bounded closed subset of all symmetric fourth order tensors $B^* \in \mathcal{M}_N^4$ such that there exists a probability measure ν on the unit sphere S_{N-1} satisfying

$$\theta(B^* - B)^{-1} = (A - B)^{-1} + (1 - \theta) \int_{S_{N-1}} f_B(e) d\nu(e), \quad (2.75)$$

where $f_B(e)$ is given by (2.66) with B replacing A . Furthermore, any tensor B^* in L_θ^+ is the Hooke's law of a finite-rank sequential laminate defined by (2.69) with rank $p \leq (N+3)(N+2)(N+1)N/24$.

Remark 2.3.7 In the sequel, we denote by $\mathcal{P}(S_{N-1})$ the space of all probability measures on the unit sphere S_{N-1} . Recall that a probability measure ν is a nonnegative measure with unit mass, i.e.

$$\nu(e) \geq 0 \quad \forall e \in S_{N-1} \text{ and } \int_{S_{N-1}} d\nu(e) = 1. \quad (2.76)$$

Proof. Clearly, any sequential laminate A_p^* defined by (2.68) satisfies (2.74) with a discrete probability measure ν , namely

$$\nu(e) = \sum_{i=1}^p m_i \delta(e - e_i),$$

where δ is the Dirac mass at the origin. Conversely, let us show that any tensor A^* defined by (2.74) with a probability measure ν is indeed a finite-rank sequential laminate. We remark that $f_A(e)$ is an homogeneous polynomial of degree four on the unit sphere. Therefore, there exists a linear map L_A , acting in the space \mathcal{M}_N^4 , such that

$$\int_{S_{N-1}} f_A(e) d\nu(e) = L_A \left(\int_{S_{N-1}} e \otimes e \otimes e \otimes e d\nu(e) \right). \quad (2.77)$$

When ν runs into $\mathcal{P}(S_{N-1})$, the fourth order tensor $\int_{S_{N-1}} e \otimes e \otimes e \otimes e d\nu(e)$ describes a convex subset of the set of fully symmetric fourth order tensors A satisfying

$$A_{ijkl} = A_{kjl} = A_{klj} = A_{jikl},$$

with unit trace $\sum_{i,j=1}^N A_{ijij} = 1$, having dimension $n = (N+3)(N+2)(N+1)N/24 - 1$. By Lemma 2.3.8 below, the extremal points of this subset are precisely of the type $e \otimes e \otimes e \otimes e$. Thus, applying Carathéodory's theorem (see,

e.g., Theorem 2.8 in Chapter 2 of [89], or [161]), we can replace any probability measure ν by a discrete one in (2.74). (Recall that Carathéodory's theorem states that any point of a convex set in a space of finite dimension n can be represented as a convex combination of at most $n+1$ extremal points of this convex set.) Consequently, A^* is the effective tensor of a rank- $(n+1)$ sequential laminate defined by (2.68). This yields the desired characterization of L_θ^- . A symmetric argument works for L_θ^+ . Furthermore, since the set $\mathcal{P}(S_{N-1})$ is sequentially compact for the vague convergence of measures, it implies that L_θ^- and L_θ^+ are closed in \mathcal{M}_N^4 . \square

Lemma 2.3.8 *Let \mathcal{C} denote the convex hull, in the space of fourth order tensors \mathcal{M}_N^4 , of $e \otimes e \otimes e \otimes e$ when e runs into S_{N-1} . The extremal points of \mathcal{C} are exactly the points $e \otimes e \otimes e \otimes e$.*

Proof. This convex hull \mathcal{C} is defined by

$$\mathcal{C} = \left\{ \int_{S_{N-1}} e \otimes e \otimes e \otimes e d\nu(e) \in \mathcal{M}_N^4 \text{ with } \nu \in \mathcal{P}(S_{N-1}) \right\},$$

where $\mathcal{P}(S_{N-1})$ is the space of all probability measures defined in Remark 2.3.7. Let us prove that the extremal points of \mathcal{C} correspond to a measure ν being a Dirac mass, i.e., they are exactly of the type $e \otimes e \otimes e \otimes e$. Let M be such an extremal point defined by

$$M = \int_{S_{N-1}} e \otimes e \otimes e \otimes e d\nu(e).$$

Recall that an extremal point is one that cannot be decomposed as a convex combination of two other points in the same convex set. If we can prove that, for any linear form L , the function $e \rightarrow L(e \otimes e \otimes e \otimes e)$ is constant on the support of the measure ν , then one can replace ν by a Dirac mass supported at any e in the support of ν without changing the value of the matrix M , which proves that $M = e \otimes e \otimes e \otimes e$ as desired.

Let L be a linear form on \mathcal{M}_N^4 . We assume that $L(e \otimes e \otimes e \otimes e)$ is not constant on the support of ν . Upon defining the essential minimum L^- and maximum L^+ of $L(e \otimes e \otimes e \otimes e)$ by

$$L^- = \sup \left(\alpha \in \mathbb{R} \mid \nu\{e \in S_{N-1} \mid L(e \otimes e \otimes e \otimes e) \leq \alpha\} = 0 \right)$$

and

$$L^+ = \inf \left(\beta \in \mathbb{R} \mid \nu\{e \in S_{N-1} \mid L(e \otimes e \otimes e \otimes e) \geq \beta\} = 0 \right),$$

our assumption implies that $L^- < L^+$. Therefore, there exists L^0 such that $L^- < L^0 < L^+$. Introducing

$$S^- = \{e \in S_{N-1} \mid L(e \otimes e \otimes e \otimes e) \leq L^0\}$$

and

$$S^+ = \{e \in S_{N-1} \mid L(e \otimes e \otimes e \otimes e) > L^0\},$$

by construction we have $\theta^- = \nu(S^-) > 0$ and $\theta^+ = \nu(S^+) > 0$ with $\theta^- + \theta^+ = 1$. The measure ν can be decomposed as a convex combination of probability measures

$$\nu = \theta^+ \frac{\nu|_{S^+}}{\theta^+} + \theta^- \frac{\nu|_{S^-}}{\theta^-},$$

where $\nu|_{S^\pm}$ is the restriction of ν to S^\pm . Similarly, the tensor M can be decomposed as a convex combination of two points in \mathcal{C} , namely,

$$M = \theta^+ M^+ + \theta^- M^-,$$

with

$$M^\pm = \int_{S_{N-1}} e \otimes e \otimes e \otimes e \frac{d\nu|_{S^\pm}}{\theta^\pm}.$$

Since $M^+ \neq M^-$ because $L(M^-) < L(M^+)$, this contradicts the extremal character of M . Thus, for any linear form, $L(e \otimes e \otimes e \otimes e)$ is constant on the support of ν , which can therefore be replaced by a Dirac mass. This proves that the extremal points of \mathcal{C} are of the type $e \otimes e \otimes e \otimes e$. \square

Lemma 2.3.6 gives a convenient parametrization of the sets of sequential laminates L_θ^\pm in terms of probability measures. Nevertheless, it leaves open two important questions. First, is there a simple algebraic characterization of L_θ^\pm ? Second, what is the minimal number of laminations necessary to attain any tensor in these sets L_θ^\pm ? By a matter of theory, we know from Lemma 2.3.6 that any tensor in L_θ^\pm corresponds to a rank- p sequential laminate with $p \leq (N+3)(N+2)(N+1)N/24$. However, this number is not optimal and is rather a crude upper bound (it gives $p=5$ in two dimensions and $p=15$ in three dimensions). These questions were addressed in [33] for the two-dimensional case, $N=2$, and in [112] for the three-dimensional case, $N=3$. Their results are summarized in the following lemma.

Lemma 2.3.9 *Any tensor in L_θ^\pm corresponds to a rank- p sequential laminate with $p=3$ if $N=2$, and $p=6$ if $N=3$. Furthermore, if $N=2$ or 3, any tensor A^* in L_θ^- is characterized by*

$$(1-\theta)(A^* - A)^{-1} = (B - A)^{-1} + \theta L_A(M), \quad (2.78)$$

with L_A a linear map on the space of fourth order tensors defined by

$$\begin{aligned} L_A(M)_{ijkl} &= \left(\frac{1}{2\mu_A + \lambda_A} - \frac{1}{\mu_A} \right) M_{ijkl} \\ &+ \frac{1}{4\mu_A} \sum_{m=1}^N (M_{imml}\delta_{jk} + M_{imkm}\delta_{jl} + M_{mjml}\delta_{ik} + M_{mjk}\delta_{il}), \end{aligned} \quad (2.79)$$

where M is any fourth order tensor, being fully symmetric, i.e.,

$$M_{ijkl} = M_{klij} = M_{klji} = M_{jikl}, \quad (2.80)$$

with unit trace

$$\sum_{i,j=1}^N M_{ijij} = 1, \quad (2.81)$$

which is positive in the sense of quadratic forms, i.e., for any symmetric matrix ξ ,

$$\sum_{i,j,k,l=1}^N M_{ijkl}\xi_{ij}\xi_{kl} \geq 0. \quad (2.82)$$

A symmetric result holds for L_θ^+ .

Remark 2.3.10 When $N = 2$ or 3 , the space of fourth order tensors defined by (2.80), (2.81), and (2.82) is precisely equal to the set \mathcal{C} of fourth order moments of probability measures, defined in Lemma 2.3.8. In two dimensions, \mathcal{C} can be further explicitly characterized. According to [112], for each measure ν , there are five different fourth order moments defined by

$$F_j = \int_0^{2\pi} \cos^{4-j} \phi \sin^j \phi d\nu(\phi) \text{ for } 0 \leq j \leq 4,$$

which are characterized by

$$\begin{aligned} F_0, F_2, F_4 &\geq 0, \quad F_0 + 2F_2 + F_4 = 1 \\ F_1^2 &\leq F_0 F_2, \quad F_2^2 \leq F_0 F_4, \quad F_3^2 \leq F_2 F_4 \\ F_0 F_2 F_4 + 2F_1 F_2 F_3 - F_0 F_3^2 - F_1^2 F_4 - F_2^3 &\geq 0. \end{aligned}$$

Another characterization of \mathcal{C} when $N = 2$ is proposed in [33]. When $N = 3$ and in the orthotropic case, a simpler characterization of \mathcal{C} has been found in [169] (see also [167] in the case of transverse isotropy).

We shall prove Lemma 2.3.9 only in the two-dimensional case since the three-dimensional proof is much more involved although in the same spirit (see [112] if necessary). These minimal ranks, $p = 3$ when $N = 2$, and $p = 6$ when $N = 3$, are optimal in the sense that they are the minimal number of laminations needed in order to obtain an isotropic sequential laminate (see [110], [112]).

Proof of Lemma 2.3.9. Following ideas from [33] and [112], we now prove that in two space dimensions ($N = 2$), and for isotropic phases, any tensor in L_θ^\pm corresponds to a rank-3 sequential laminate. We already know that any such tensor is indeed a finite-rank sequential laminate. As is clear from the proof of Lemma 2.3.6 (see in particular (2.77)), the minimal rank p is determined by the minimal number p of Dirac masses whose convex combination yields the same fourth order moments of a probability measure on the unit sphere. In other words, given a probability measure $\nu \in \mathcal{P}(S_1)$, the problem is to find the minimal number p such that there exist unit vectors $(e_i)_{1 \leq i \leq p}$ and positive parameters $(m_i)_{1 \leq i \leq p}$ with $\sum_{i=1}^p m_i = 1$ satisfying

$$\int_{S_1} e \otimes e \otimes e \otimes e d\nu(e) = \sum_{i=1}^p m_i e_i \otimes e_i \otimes e_i \otimes e_i.$$

Let \mathcal{C} be the bounded convex set generated by $\int_{S_1} e \otimes e \otimes e \otimes e d\nu(e)$ when ν runs into $\mathcal{P}(S_1)$. Let us prove that any point M on the boundary of \mathcal{C} corresponds to a measure ν_M , the support of which is restricted to two vectors of S_1 . Since \mathcal{C} is convex, the point M on its boundary admits at least one tangent hyperplane characterized by a fourth order tensor T and a constant γ such that

$$M :: T = \gamma \text{ and } N :: T \geq \gamma \text{ for any } N \in \mathcal{C}. \quad (2.83)$$

Parametrizing any vector $e \in S_1$ by an angle ϕ such that $e = (\cos \phi, \sin \phi)$, the fourth order tensor T defines an homogeneous polynomial p_T of degree four in two variables through the relationship

$$M :: T = \int_{S_1} e \otimes e \otimes e \otimes e :: T d\nu_M(e) = \int_0^{2\pi} p_T(\cos \phi, \sin \phi) d\nu_M(\phi).$$

Remark that on S_1 the constant $1 = (\cos^2 \phi + \sin^2 \phi)^2$ is also an homogeneous polynomial of degree four in the two variables $(\cos \phi, \sin \phi)$, we deduce

that (2.83) can be rewritten as

$$\begin{aligned} \int_0^{2\pi} p(\cos \phi, \sin \phi) d\nu_M(\phi) &= 0 \\ \int_0^{2\pi} p(\cos \phi, \sin \phi) d\nu(\phi) &\geq 0 \text{ for any } \nu \in \mathcal{P}(S_1), \end{aligned} \quad (2.84)$$

where $p(x_1, x_2) = p_T(x_1, x_2) - \gamma(x_1^2 + x_2^2)^2$ is a homogeneous polynomial of degree four in two variables. By choosing ν to be a Dirac mass at any angle ϕ_0 , we obtain

$$p(\cos \phi_0, \sin \phi_0) \geq 0.$$

By homogeneity of p , dividing this inequality by $\cos^4 \phi_0$ yields

$$p(1, \tan \phi_0) \geq 0, \quad (2.85)$$

where $p(1, t) \equiv q(t)$ is a polynomial of degree four. When ϕ_0 runs into $(-\pi/2, +\pi/2)$, (2.84) implies that $q(t)$ is a nonnegative polynomial. Clearly, the real roots of a nonnegative polynomial of degree four must have even multiplicity. Therefore, $q(t)$ has at most two distinct real roots. On the other hand,

$$\int_0^{2\pi} p(\cos \phi, \sin \phi) d\nu_M(\phi) = \int_0^{2\pi} \cos^4 \phi q(\tan \phi) d\nu_M(\phi) = 0,$$

implies that the support of $\nu_M(\phi)$ is concentrated on those ϕ such that $\tan \phi$ are the roots of $q(t)$. There are at most two such values of $\tan \phi$, and thus two values of ϕ in the interval $(-\pi/2, +\pi/2)$. Since the fourth order moments $e \otimes e \otimes e \otimes e$ are invariant by changing ϕ into $\phi + \pi$, there is no loss of generality in assuming that the support of the measure $\nu_M(\phi)$ is included in $[-\pi/2, +\pi/2]$. We can then conclude that the support of $\nu_M(\phi)$ is restricted to at most two vectors in S_1 . Therefore, in view of (2.74) and (2.77), any point M on the boundary of \mathcal{C} corresponds to a rank-2 laminate.

Let us now show that any point M inside \mathcal{C} may be attained by a measure ν_M , the support of which is restricted to three vectors of S_1 . Let us choose any Dirac mass δ_{e_0} supported at $e_0 \in S_1$. By definition, its fourth order moment tensor M_0 belongs to \mathcal{C} too. If it coincides with M , we are done. If not, the line passing through the two points M and M_0 intersects the boundary $\partial\mathcal{C}$ in two other points, since \mathcal{C} is a bounded convex set. Choosing one of these points M_1 on $\partial\mathcal{C}$ such that M is a convex combination of M_0

and M_1 , we deduce that the measure ν_M is also a convex combination of ν_{M_0} and ν_{M_1} . Since the support of ν_{M_0} is concentrated in one direction e_0 and that of ν_{M_1} in two, it follows that the support of ν_M is restricted to three vectors of S_1 . Therefore, any point inside \mathcal{C} corresponds to a rank-3 laminate.

To finish the proof, it remains to show that \mathcal{C} is characterized by (2.80), (2.81), (2.82). Clearly, \mathcal{C} is included in the set defined by these conditions. Let M be a tensor on the boundary of \mathcal{C} . We have already shown that any tangent hyperplane to \mathcal{C} at M is characterized by an homogeneous positive polynomial p of degree four in two variables satisfying (2.84). Since p is also positive, it can be written as a sum of squares of homogeneous polynomial q_k of degree two, that is,

$$p(x_1, x_2) = \sum_{k=1}^{k^*} q_k(x_1, x_2)^2.$$

Thus, (2.84) implies that for each q_k we have

$$\begin{aligned} \int_0^{2\pi} q_k(\cos \phi, \sin \phi)^2 d\nu_M(\phi) &= 0 \\ \int_0^{2\pi} q_k(\cos \phi, \sin \phi)^2 d\nu(\phi) &\geq 0 \text{ for any } \nu \in \mathcal{P}(S_1). \end{aligned} \tag{2.86}$$

Since any homogeneous polynomial q_k of degree two is of the type

$$q_k(x_1, x_2) = A_{11}x_1^2 + 2A_{12}x_1x_2 + A_{22}x_2^2, \tag{2.87}$$

where $A = (A_{ij})_{1 \leq i, j \leq 2}$ is a symmetric 2×2 matrix, (2.86) is equivalent to

$$MA : A = 0$$

$$NA : A \geq 0 \text{ for any } N \in \mathcal{C}.$$

Conversely, for any symmetric two by two matrix A , one can define a polynomial q_k by (2.87) and find a probability measure ν_A , the support of which is included in the zero set of q_k . This measure ν_A yields a tensor M on the boundary of \mathcal{C} , and the polynomial q_k yields a tangent hyperplane satisfying (2.86). We have thus established a one-to-one map between the tangent hyperplanes to \mathcal{C} and the symmetric matrices A . Since \mathcal{C} is convex, \mathcal{C} is the intersection of the closed half spaces defined by its tangent hyperplanes. Therefore, \mathcal{C} coincides with the set defined by (2.80), (2.81), and (2.82). \square

2.3.2 Hashin-Shtrikman Energy Bounds

This section is devoted to the Hashin-Shtrikman variational principle in the elasticity setting. As in the conductivity setting, this method allows one to derive optimal bounds on the effective or homogenized energy of a composite material. Denoting by A^* the homogenized Hooke's law, and by ξ a symmetric matrix in \mathcal{M}_N^s , the homogenized elastic energy evaluated at the strain tensor ξ is the quantity $A^* \xi : \xi$. The method, introduced by Hashin and Shtrikman in [133], was further developed by many individuals, e.g., [20], [32], [132], [151], [189], [193], [287], [290]. Recall that we assumed that A and B are isotropic and well-ordered, i.e.,

$$A \leq B, \quad (2.88)$$

in the sense of quadratic forms on \mathcal{M}_N^s . Of course, any homogenized Hooke's law $A^* \in G_\theta$ satisfies the trivial harmonic and arithmetic mean bounds, i.e.,

$$\left(\theta A^{-1} + (1 - \theta)B^{-1} \right)^{-1} \leq A^* \leq (\theta A + (1 - \theta)B).$$

For certain choices of the strain ξ there are composites A^* such that their effective energy $A^* \xi : \xi$ saturates either one of these upper and lower bounds. But there are also other choices of ξ for which none of these bounds is attained by a composite $A^* \in G_\theta$. In this sense, the harmonic and arithmetic mean bounds are not optimal. As was shown by Avellaneda [32], the Hashin-Shtrikman variational principle allows one to improve considerably the harmonic and arithmetic mean bounds by bounding any homogenized Hooke's law between two effective tensors of sequential laminates.

Theorem 2.3.11 *For any effective Hooke's law A^* in G_θ there exist two sequential laminates $A^+ \in L_\theta^+$ (i.e., with core A and matrix B) and $A^- \in L_\theta^-$ (i.e., with matrix A and core B), such that*

$$A^- \leq A^* \leq A^+$$

in the sense of quadratic forms (of course, the sequential laminates A^+, A^- vary with A^).*

Remark 2.3.12 *Theorem 2.3.11 holds true in the conductivity setting too. As we shall see below, it allows one to obtain optimal bounds on sums of energies, but it is of course a more precise result than these optimal bounds. We therefore proceed slightly differently than in the previous section, devoted to the conductivity setting.*

Proof of Theorem 2.3.11. The proof relies on the Hashin-Shtrikman variational principle, as in the conductivity setting (see Proposition 2.2.6). Since the set P_θ of periodic composites is dense in the set G_θ of all possible composites (see Theorem 2.1.2), and since the sets L_θ^\pm are closed (see Lemma 2.3.6), we may restrict ourselves to a periodic composite $A^* \in P_\theta$. Such an effective tensor A^* is defined through a characteristic function $\chi(y)$ in the unit cell Y , which satisfies $\int_Y \chi(y) dy = \theta$. By periodic homogenization, A^* is given by the formula

$$A^* \xi : \xi = \min_{\phi \in H_\#^1(Y)^N} \int_Y (\chi A + (1 - \chi)B) (\xi + e(\phi)) : (\xi + e(\phi)) dy, \quad (2.89)$$

where ξ is any symmetric matrix in \mathcal{M}_N^s . We first establish a lower bound to this effective energy (2.89). By adding and subtracting a reference energy $A(\xi + e(\phi)) : (\xi + e(\phi))$, we obtain

$$\begin{aligned} A^* \xi : \xi &= \min_{\phi \in H_\#^1(Y)^N} \left[\int_Y (1 - \chi)(B - A)(\xi + e(\phi)) : (\xi + e(\phi)) dy \right. \\ &\quad \left. + \int_Y A(\xi + e(\phi)) : (\xi + e(\phi)) dy \right]. \end{aligned} \quad (2.90)$$

Using the positivity of $B - A$ and convex duality, the first integral on the right hand side of (2.90) is rewritten as

$$\sup_{\eta(y) \in L_\#^2(Y; \mathcal{M}_N^s)} \int_Y (1 - \chi) \left(2\eta : (\xi + e(\phi)) - (B - A)^{-1}\eta : \eta \right) dy. \quad (2.91)$$

One can get a lower bound in (2.91) by specializing to constant matrices η . Since $\int_Y \chi(y) dy = \theta$, doing so yields the following lower bound for (2.91),

$$2(1 - \theta)\eta : \xi - (1 - \theta)(B - A)^{-1}\eta : \eta + \int_Y 2(1 - \chi)\eta : e(\phi) dy.$$

On the other hand, since by periodicity $\int_Y e(\phi) dy = 0$, we have

$$\int_Y A(\xi + e(\phi)) : (\xi + e(\phi)) dy = A\xi : \xi + \int_Y Ae(\phi) : e(\phi) dy.$$

Therefore, (2.90) leads to

$$A^* \xi : \xi \geq A\xi : \xi + 2(1 - \theta)\xi : \eta - (1 - \theta)(B - A)^{-1}\eta : \eta - g(\chi, \eta) \quad (2.92)$$

where $g(\chi, \eta)$ is a nonlocal term defined, through a cell problem, by

$$g(\chi, \eta) = - \min_{\phi(y) \in H_{\#}^1(Y)^N} \int_Y [Ae(\phi) : e(\phi) + 2(1 - \chi)\eta : e(\phi)] dy. \quad (2.93)$$

The infimum over ϕ in (2.93) is easily computed by Fourier analysis, since it involves only bilinear terms depending on y . By periodicity, the characteristic function is written as a Fourier series,

$$\chi(y) = \sum_{k \in \mathbb{Z}^N} \hat{\chi}(k) e^{2i\pi k \cdot y},$$

as well as the test function $\phi(y)$

$$\phi(y) = \sum_{k \in \mathbb{Z}^N} \hat{\phi}(k) e^{2i\pi k \cdot y}.$$

Since χ and ϕ are real-valued, the relationships

$$\overline{\hat{\chi}(k)} = \hat{\chi}(-k) \text{ and } \overline{\hat{\phi}(k)} = \hat{\phi}(-k) \quad (2.94)$$

hold true. The strain tensor of ϕ is given by

$$e(\phi)(y) = \sum_{k \in \mathbb{Z}^N} 2i\pi e^{2i\pi k \cdot y} \hat{\phi}(k) \odot k.$$

Plancherel's formula yields

$$\begin{aligned} & - \int_Y (Ae(\phi) : e(\phi) - 2\chi\eta : e(\phi)) dy \\ &= - \sum_{k \in \mathbb{Z}^N} \left[4\pi^2 A(\hat{\phi}(k) \odot k) : (\hat{\phi}(k) \odot k) - 4i\pi\eta : (\hat{\phi}(k) \odot k) \right] \quad (2.95) \\ &= - \sum_{k \in \mathbb{Z}^N} \left[4\pi^2 \left(2\mu_A |k|^2 |\hat{\phi}(k)|^2 + \lambda_A (\hat{\phi}(k) \cdot k)^2 \right) + 4\pi \mathcal{I}m \left(\overline{\hat{\chi}(k)} (\eta k) \cdot \hat{\phi}(k) \right) \right]. \end{aligned}$$

The minimization over $\phi(y)$ in $H_{\#}^1(Y)^N$ is equivalent to the minimizations over all vectors $\hat{\phi}(k)$ in \mathbb{C}^N , for each frequency k (and the minimization can be performed on each component independently). We remark that the frequency $k = 0$ contributes for nothing in the sum (2.95). For all other frequencies the minimum is attained by

$$\hat{\phi}(k) = - \frac{i\hat{\chi}(k)}{4\pi\mu_A |k|^2} \left(\eta k - \frac{\lambda_A}{2\mu_A + \lambda_A} \frac{\eta k \cdot k}{|k|^2} k \right),$$

and the minimum value is

$$|\hat{\chi}(k)|^2 f_A\left(\frac{k}{|k|}\right) \eta : \eta, \quad (2.96)$$

where $f_A(e)\eta : \eta$ is defined by

$$f_A(e)\eta : \eta = \frac{1}{\mu_A} \left(|\eta e|^2 - (\eta e \cdot e)^2 \right) + \frac{1}{2\mu_A + \lambda_A} (\eta e \cdot e)^2. \quad (2.97)$$

Therefore,

$$g(\chi, \eta) = \sum_{k \in \mathbb{Z}^N, k \neq 0} |\hat{\chi}(k)|^2 f_A\left(\frac{k}{|k|}\right) \eta : \eta. \quad (2.98)$$

On the other hand, Plancherel's formula gives

$$\sum_{k \in \mathbb{Z}^N, k \neq 0} |\hat{\chi}(k)|^2 = \int_Y |\chi(y) - \theta|^2 dy = \theta(1 - \theta).$$

Thus, introducing a probability measure ν on S_{N-1} , the nonlocal term (2.98) can be rewritten

$$g(\chi, \eta) = \theta(1 - \theta) \int_{S_{N-1}} f_A(e)\eta : \eta d\nu(e),$$

where S_{N-1} is the unit sphere and ν is defined by

$$\theta(1 - \theta) \int_{S_{N-1}} f(e) d\nu(e) = \sum_{k \in \mathbb{Z}^N, k \neq 0} |\hat{\chi}(k)|^2 f\left(\frac{k}{|k|}\right) \quad (2.99)$$

for any continuous function f , defined from S_{N-1} into \mathbb{R} . The probability measure ν is called the *H-measure* of χ (it contains two-point correlations of χ). The notion of *H-measure* was introduced, in a much more general context, by Gerard [113] and Tartar [276].

Recall that, by its definition (2.97), $f_A(e)\eta : \eta$ is an homogeneous polynomial of degree four as a function of e . Therefore, the value of the nonlocal term $g(\chi, \eta)$ depends only on the fourth order moments of the *H-measure* ν . By Carathéodory's theorem and Lemma 2.3.8, any probability measure can be replaced by a convex combination of Dirac masses as far as their fourth order moments are concerned. In other words, there exist an integer $p \geq 1$, unit vectors $(e_i)_{1 \leq i \leq p}$, and positive parameters $(m_i)_{1 \leq i \leq p}$ such that for $m_i \geq 0$ and $\sum_{i=1}^p m_i = 1$,

$$\int_{S_{N-1}} e \otimes e \otimes e \otimes e d\nu(e) = \sum_{i=1}^p m_i e_i \otimes e_i \otimes e_i \otimes e_i.$$

Thus, we have

$$g(\chi, \eta) = \theta(1 - \theta) \left(\sum_{i=1}^p m_i f_A(e_i) \right) \eta : \eta,$$

where the vectors e_i and parameters m_i do not depend on η . Consequently, the right hand side of (2.92) is a simple Legendre transform of a quadratic function, which yields

$$A^* \xi : \xi \geq A \xi : \xi + (1 - \theta) \left((B - A)^{-1} + \theta \sum_{i=1}^p m_i f_A(e_i) \right)^{-1} \xi : \xi. \quad (2.100)$$

Comparing (2.100) with the lamination formula (2.68) immediately implies that there exists a sequential laminate A^- , with matrix A and core B , such that

$$A^* \xi : \xi \geq A^- \xi : \xi. \quad (2.101)$$

Since A^- is independent of ξ (it depends only on the H -measure of the characteristic function χ), (2.101) gives the desired inequality $A^* \geq A^-$.

The other inequality $A^* \leq A^+$ is derived similarly by establishing an upper bound on the effective energy. We subtract from (2.89) the other reference material B :

$$\begin{aligned} A^* \xi : \xi &= \min_{\substack{\phi \in H^1_\#(Y)^N \\ \#}} \left[- \int_Y \chi(B - A)(\xi + e(\phi)) : (\xi + e(\phi)) dy \right. \\ &\quad \left. + \int_Y B(\xi + e(\phi)) : (\xi + e(\phi)) dy \right]. \end{aligned} \quad (2.102)$$

Using the positivity of $B - A$ and convex duality, the first integral on the right hand side of (2.102) is rewritten as

$$- \sup_{\eta(y) \in L^2_\#(Y; \mathcal{M}_N^s)} \int_Y \chi \left(2\eta : (\xi + e(\phi)) - (B - A)^{-1}\eta : \eta \right) dy. \quad (2.103)$$

One can get an upper bound in (2.103) by specializing the maximization in η to constant matrices in Y :

$$- \theta \left(2\xi : \eta - (B - A)^{-1}\eta : \eta \right) - 2 \int_Y \chi e(\phi) : \eta dy.$$

On the other hand, by periodicity we have

$$\int_Y B(\xi + e(\phi)) : (\xi + e(\phi)) dy = B\xi : \xi + \int_Y Be(\phi) : e(\phi) dy.$$

Therefore, (2.102) yields, for any $\eta \in \mathcal{M}_N^s$,

$$A^*\xi : \xi \leq B\xi : \xi + \theta \left(-2\xi : \eta + (B - A)^{-1}\eta : \eta \right) - h(\chi, \eta), \quad (2.104)$$

where $h(\chi, \eta)$ is another nonlocal term defined by

$$h(\chi, \eta) = - \min_{\phi(y) \in H_{\#}^1(Y)^N} \int_Y (Be(\phi) : e(\phi) - 2\chi e(\phi) : \eta) dy. \quad (2.105)$$

As for the lower bound, a simple computation by Fourier analysis shows that

$$h(\chi, \eta) = \sum_{k \in \mathbb{Z}^N, k \neq 0} |\hat{\chi}(k)|^2 f_B\left(\frac{k}{|k|}\right) \eta : \eta, \quad (2.106)$$

which, using the same probability measure ν (the H -measure of χ), can be rewritten as

$$h(\chi, \eta) = \theta(1 - \theta) \int_{S_{N-1}} f_B(e) \eta : \eta d\nu(e).$$

Replacing the measure ν by the same convex combination of Dirac masses, we obtain

$$h(\chi, \eta) = \theta(1 - \theta) \left(\sum_{i=1}^p m_i f_B(e_i) \right) \eta : \eta.$$

Consequently, the right hand side of (2.104) is the Legendre transform of a quadratic function, and this yields

$$A^*\xi : \xi \leq B\xi : \xi - \theta \left((B - A)^{-1} - (1 - \theta) \sum_{i=1}^p m_i f_B(e_i) \right)^{-1} \xi : \xi. \quad (2.107)$$

Comparing (2.107) with the lamination formula (2.69) immediately implies that there exists a sequential laminate A^+ , with core A and matrix B , such that $A^* \leq A^+$. \square

As an immediate application of Theorem 2.3.11, we can recover the famous original Hashin-Shtrikman bounds [133] on the bulk and shear moduli of an isotropic two-phase composite material (which have been shown to be attainable by finite-rank sequential laminates in [110]). Recall that, by assumption (2.88), the phases are well-ordered.

Theorem 2.3.13 Let A^* be an isotropic homogenized tensor in G_θ , i.e.,

$$A^* = 2\mu_* I_4 + \left(\kappa_* - \frac{2\mu_*}{N} \right) I_2 \otimes I_2.$$

Its bulk and shear moduli κ_* and μ_* satisfy

$$\frac{1-\theta}{\kappa_* - \kappa_A} \leq \frac{1}{\kappa_B - \kappa_A} + \frac{\theta}{2\mu_A + \lambda_A} \quad (2.108)$$

$$\frac{\theta}{\kappa_B - \kappa_*} \leq \frac{1}{\kappa_B - \kappa_A} + \frac{1-\theta}{2\mu_B + \lambda_B} \quad (2.109)$$

and

$$\frac{1-\theta}{2(\mu_* - \mu_A)} \leq \frac{1}{2(\mu_B - \mu_A)} + \frac{\theta(N-1)(\kappa_A + 2\mu_A)}{(N^2 + N - 2)\mu_A(2\mu_A + \lambda_A)} \quad (2.110)$$

$$\frac{\theta}{2(\mu_B - \mu_*)} \leq \frac{1}{2(\mu_B - \mu_A)} + \frac{(1-\theta)(N-1)(\kappa_B + 2\mu_B)}{(N^2 + N - 2)\mu_B(2\mu_B + \lambda_B)}. \quad (2.111)$$

Furthermore, the lower bounds (2.108) and (2.110), as well as the upper bounds (2.109) and (2.111), are simultaneously attained by an isotropic rank- p sequential laminate with $p \leq (N+3)(N+2)(N+1)N/24$. Optimality is achieved with $p=3$ in $N=2$, and $p=6$ in $N=3$ dimensions.

Remark 2.3.14 The bounds (2.108), (2.110), (2.109) and (2.111) do not characterize all possible isotropic homogenized tensors A^* in G_θ . In other words, there exist isotropic elasticity tensors with moduli satisfying these bounds that are not composite materials obtained by mixing phases A and B in proportions θ , $(1-\theta)$, respectively. Better bounds (i.e., those which define a smaller region of the (κ_*, μ_*) -plane) have been obtained in [194], and further improved in [77].

Proof. In view of (2.100) the lower bound $A^* \geq A^-$ can be rewritten

$$(B - A)^{-1} + \theta \sum_{i=1}^p m_i f_A(e_i) \geq (1-\theta)(A^* - A)^{-1}. \quad (2.112)$$

Similarly, (2.107) implies that the upper bound $A^* \leq A^+$ is equivalent to

$$\theta(B - A^*)^{-1} \leq (B - A)^{-1} - (1-\theta) \sum_{i=1}^p m_i f_B(e_i). \quad (2.113)$$

Contracting these fourth order tensors with $I_2 \otimes I_2$ yields the desired bounds for κ_* since $f_A(e)I_2 : I_2$, as well as $f_B(e)I_2 : I_2$, are independent of e , i.e.,

$$f_A(e)I_2 : I_2 = \frac{1}{2\mu_A + \lambda_A}.$$

On the other hand, introducing the positive fourth order tensor P_s defined by

$$P_s = I_4 - \frac{1}{N}I_2 \otimes I_2,$$

which is the orthogonal projection operator on trace-free matrices (i.e., pure shear tensors), and contracting (2.112) and (2.113) with P_s , we obtain the desired bounds on μ_* , since

$$(B - A)^{-1} :: P_s = \text{tr} \left((B - A)^{-1} P_s \right) = \frac{\text{tr} P_s}{2(\mu_B - \mu_A)},$$

with $\text{tr} P_s = (N^2 + N - 2)/2$, and

$$f_A(e) :: P_s = \text{tr} (f_A(e)P_s) = \frac{(N - 1)(\kappa_A + 2\mu_A)}{2\mu_A(2\mu_A + \lambda_A)},$$

which is independent of e . The lower bounds on κ_* and μ_* are optimal if there exists a finite-rank sequential laminate A^- for which equality holds. This obviously happens if one can find lamination parameters $(m_i)_{1 \leq i \leq p}$ such that the fourth order tensor

$$T = \sum_{i=1}^p m_i f_A(e_i)$$

is isotropic. In view of formula (2.77), $f_A(e_i)$ depends only on $e_i \otimes e_i \otimes e_i \otimes e_i$ in such a way that T is isotropic if

$$\sum_{i=1}^p m_i e_i \otimes e_i \otimes e_i \otimes e_i$$

is isotropic. Let us denote by de the constant probability measure on S_{N-1} . For symmetry reasons the tensor

$$\int_{S_{N-1}} e \otimes e \otimes e \otimes e de = \frac{2}{N+2} \left(\frac{1}{N} I_4 + \frac{1}{2N} I_2 \otimes I_2 \right)$$

is isotropic. By the same argument of Carathéodory's theorem and Lemma 2.3.8 (as in Theorem 2.3.11), there exist an integer $p \geq 1$, unit vectors $(e_i)_{1 \leq i \leq p}$, positive parameters $(m_i)_{1 \leq i \leq p}$ such that, for $m_i \geq 0$ and $\sum_{i=1}^p m_i = 1$,

$$\int_{S_{N-1}} e \otimes e \otimes e \otimes e \, de = \sum_{i=1}^p m_i e_i \otimes e_i \otimes e_i \otimes e_i.$$

These parameters $(m_i)_{1 \leq i \leq p}$ are therefore those of an isotropic sequential laminate, which saturates the lower bounds on κ_* and μ_* . A similar argument works for the upper bounds.

A more direct way of checking the optimality of these bounds in dimension $N = 2$ or 3 is, following [110], to exhibit a particular isotropic and optimal sequential laminate given by one of the two formulas (2.68) and (2.69) (symmetric by interchanging the roles of A and B). When $N = 2$, we choose a rank-3 sequential laminate with lamination parameters $m_1 = m_2 = m_3 = 1/3$ and lamination directions

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} \cos \frac{\pi}{3} \\ \sin \frac{\pi}{3} \end{pmatrix}, e_3 = \begin{pmatrix} \cos \frac{2\pi}{3} \\ \sin \frac{2\pi}{3} \end{pmatrix}.$$

A simple but tedious computation shows that this laminate is isotropic and saturates the two lower bounds (2.108) and (2.110) for formula (2.68) (with a matrix of A and a core of B), or the two upper bounds (2.109) and (2.111) for the symmetric formula (2.69) (with a matrix of B and a core of A).

When $N = 3$, we choose a rank-6 sequential laminate with lamination parameters $m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = 1/6$ and lamination directions corresponding to the six northern hemisphere vertices of the regular icosahedron

$$e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, e_2 = \begin{pmatrix} \sin 2\beta \\ 0 \\ \cos 2\beta \end{pmatrix}, e_3 = \begin{pmatrix} \sin 2\beta \cos 2\alpha \\ \sin 2\beta \sin 2\alpha \\ \cos 2\beta \end{pmatrix},$$

$$e_4 = \begin{pmatrix} \sin 2\beta \cos 4\alpha \\ \sin 2\beta \sin 4\alpha \\ \cos 2\beta \end{pmatrix}, e_5 = \begin{pmatrix} \sin 2\beta \cos 6\alpha \\ \sin 2\beta \sin 6\alpha \\ \cos 2\beta \end{pmatrix}, e_6 = \begin{pmatrix} \sin 2\beta \cos 8\alpha \\ \sin 2\beta \sin 8\alpha \\ \cos 2\beta \end{pmatrix},$$

with $\alpha = \pi/5$ and $\cos 2\beta = \sqrt{5}/5$. A tedious computations shows that this laminate is isotropic and saturates the two lower bounds (2.108) and (2.110) for formula (2.68) (with a matrix of A and a core of B), or the two upper

bounds (2.109) and (2.111) for the symmetric formula (2.69) (with a matrix of B and a core of A). \square

Another application of Theorem 2.3.11 is the derivation of optimal bounds on the effective energy of a composite material. We recall our definition of an optimal bound (see Definition 2.2.5).

Definition 2.3.15 Let ξ be a symmetric matrix in \mathcal{M}_N^s . A real-valued function $f^+(\theta, A, B, \xi)$ (respectively, $f^-(\theta, A, B, \xi)$) is called an upper bound (respectively, a lower bound) if, for any homogenized tensor $A^* \in G_\theta$,

$$A^* \xi : \xi \leq f^+(\theta, A, B, \xi) \quad (\text{respectively, } \geq f^-(\theta, A, B, \xi)).$$

The upper bound f^+ (respectively, lower bound f^-) is said to be optimal if, for any $\xi \in \mathcal{M}_N^s$, there exists $A_\xi^* \in G_\theta$ (which may depend on ξ) such that

$$A_\xi^* \xi : \xi = f^+(\theta, A, B, \xi) \quad (\text{respectively, } = f^-(\theta, A, B, \xi)).$$

There is no special reason for restricting our attention to a single effective energy, i.e., to a unique strain tensor ξ , and Definition 2.3.15 extends easily to the sum of energies

$$\sum_{i=1}^p A^* \xi_i : \xi_i = A^* :: \left(\sum_{i=1}^p \xi_i \otimes \xi_i \right),$$

where $(\xi_i)_{1 \leq i \leq p}$ is a collection of strain tensors. The sum $\sum_{i=1}^p \xi_i \otimes \xi_i$ is a symmetric fourth order tensor, and any symmetric fourth order tensor M can always be diagonalized as

$$M = \sum_{i=1}^{N(N+1)/2} \tilde{\xi}_i \otimes \tilde{\xi}_i,$$

where each $\tilde{\xi}_i \in \mathcal{M}_N^s$, and $N(N + 1)/2$ is the dimension of the space \mathcal{M}_N^s . Therefore, there is no loss of generality in assuming that the number of energies is exactly $p = N(N + 1)/2$. Such energy bounds are also called trace bounds [189] since they can be written

$$\sum_{i=1}^p A^* \xi_i : \xi_i = \text{tr}(A^* M) = A^* :: M \quad \text{with} \quad M = \sum_{i=1}^p \xi_i \otimes \xi_i.$$

Proposition 2.3.16 Let ξ_1, \dots, ξ_p be symmetric matrices in \mathcal{M}_N^s . Any homogenized tensor $A^* \in G_\theta$ satisfies

$$\sum_{i=1}^p A^* \xi_i : \xi_i \geq \sum_{i=1}^p A \xi_i : \xi_i + (1 - \theta) \max_{(\eta_i)_{1 \leq i \leq p} \in \mathcal{M}_N^s} \quad (2.114)$$

$$\left[\sum_{i=1}^p \left(2\xi_i : \eta_i - (B - A)^{-1} \eta_i : \eta_i \right) - \theta g(\eta_1, \dots, \eta_p) \right],$$

where the nonlocal term g is given by

$$g(\eta_1, \dots, \eta_p) = \max_{e \in S_{N-1}} \sum_{i=1}^p f_A(e) \eta_i : \eta_i \quad (2.115)$$

and

$$\sum_{i=1}^p A^* \xi_i : \xi_i \leq \sum_{i=1}^p B \xi_i : \xi_i + \theta \min_{(\eta_i)_{1 \leq i \leq p} \in \mathcal{M}_N^s} \quad (2.116)$$

$$\left[\sum_{i=1}^p \left(2\xi_i : \eta_i + (B - A)^{-1} \eta_i : \eta_i \right) - (1 - \theta) h(\eta_1, \dots, \eta_p) \right],$$

where the nonlocal term h is given by

$$h(\eta_1, \dots, \eta_p) = \min_{e \in S_{N-1}} \sum_{i=1}^p f_B(e) \eta_i : \eta_i \quad (2.117)$$

and $f_{A,B}(e)$ is defined by

$$f_{A,B}(e) \eta : \eta = \frac{1}{\mu_{A,B}} \left(|\eta e|^2 - (\eta e \cdot e)^2 \right) + \frac{1}{2\mu_{A,B} + \lambda_{A,B}} (\eta e \cdot e)^2. \quad (2.118)$$

Furthermore, these upper and lower bounds are optimal in the sense of Definition 2.3.15 and optimality is always achieved by a finite-rank sequential laminate, of rank 3 in two dimensions and rank 6 in three dimensions.

Remark 2.3.17 The functions g and h are called nonlocal terms since they are the only terms in the Hashin-Shtrikman bounds that contain information on the microstructure of the homogenized Hooke's law A^* . Clearly, g is convex, and the lower bound (2.114) is therefore given as the result of a finite-dimensional concave maximization. On the other hand, one can check that

$$h(\eta_1, \dots, \eta_p) \leq \sum_{i=1}^p B^{-1} \eta_i : \eta_i,$$

and since $(B - A)^{-1} - (1 - \theta)B^{-1} \geq 0$, the upper bound (2.116) is given as the result of a finite-dimensional convex minimization.

In principle, the bounds (2.114) and (2.116) can be computed explicitly since they reduce to a finite-dimensional optimization. In practice, this is almost impossible because, even for isotropic phases, there are no simple formulas for the nonlocal terms g and h . Indeed, the optimization with respect to $e \in S_{N-1}$ in (2.115) and (2.117) is intractable (the optimal e 's are usually not eigenvectors of any matrix η_i).

We remark that the positive nondefinite fourth order tensors $f_A(e)$ and $f_B(e)$ defined by (2.118) are exactly the same as those appearing in the lamination formula (see (2.66)). This is at the root of the optimality of these bounds. Theorem 2.3.11 and Proposition 2.3.16 can be generalized easily to the case of nonisotropic Hooke's laws A and B .

Deriving the Hashin-Shtrikman bounds given by Proposition 2.3.16 is a classical matter: It goes back to the original work of Hashin and Shtrikman [133] (modern expositions may be found in [20], [189], or in [115], [116] where the translation method is used, instead of the Hashin-Shtrikman variational principle).

Proof of Proposition 2.3.16. We focus on the lower bound (the case of the upper bound is completely symmetric). Applying Theorem 2.3.11 yields

$$\sum_{i=1}^p A^* \xi_i : \xi_i \geq \min_{A^- \in L_\theta^-} \sum_{i=1}^p A^- \xi_i : \xi_i. \quad (2.119)$$

The closed character of L_θ^- immediately implies that (2.119) is an optimal lower bound. It remains to evaluate the right hand side of (2.119) with the help of characterization (2.74) of L_θ^- which associates to each tensor A^- a probability measure ν . Applying the Legendre transform to $(A^- - A)$ leads to

$$\begin{aligned} \sum_{i=1}^p A^- \xi_i : \xi_i &= \sum_{i=1}^p A \xi_i : \xi_i + (1 - \theta) \max_{(\eta_i)_{1 \leq i \leq p} \in \mathcal{M}_N^s} \\ &\quad \left[\sum_{i=1}^p \left(2\xi_i : \eta_i - (B - A)^{-1} \eta_i : \eta_i \right) - \theta g_\nu(\eta_1, \dots, \eta_p) \right], \end{aligned}$$

where g_ν is defined by

$$g_\nu(\eta_1, \dots, \eta_p) = \sum_{i=1}^p \left(\int_{S_{N-1}} f_A(e) d\nu(e) \right) \eta_i : \eta_i. \quad (2.120)$$

To obtain a lower bound, we minimize the above expression with respect to ν . Since the expression is linear with respect to ν and concave with respect to η_i , we can exchange the minimization in ν and the maximization in η_i . We thus have to maximize g_ν over all probability measures ν . This is easily realized by taking a Dirac mass concentrated on the unit vectors where $\sum_{i=1}^p f_A(e)\eta_i : \eta_i$ is maximal, i.e., $\max_\nu g_\nu = g$. This yields the lower bound (2.114). \square

Remark 2.3.18 As in the conductivity case (see Remark 2.2.9), the proof of Proposition 2.3.16 also delivers necessary and sufficient conditions for a periodic microstructure to be optimal in the Hashin-Shtrikman bounds. Actually, the lower bound is obtained by making two inequalities. The first one is made in (2.91) by restricting the tensors η_i to constants. Therefore, the microscopic strain of an optimal tensor must be constant in phase B. The second inequality is made in (2.120) when maximizing with respect to the H-measure ν of the microstructure χ . By definition (2.99) of ν , this yields restrictions on the Fourier coefficients of χ . For more details, we refer the reader to [25] and [122].

Remark 2.3.19 In all the previous bounds, optimality has been proved by exhibiting sequential laminates that saturate these bounds. Laminates are not the only possible optimal microstructures, but they are the more general ones. Indeed, since the celebrated work of Hashin [131], the optimal bounds (2.108) and (2.109) on the bulk modulus are known to be achieved by a concentric sphere assemblage of the two phases (see also [81] and Remark 4.1.37). However, for the other optimal bounds (2.110) and (2.111) on the shear modulus, the only available optimal microstructure is the sequential laminated one. In the case of a single energy bound, there are also other optimal microstructures, at least in some restricted regimes of these bounds. Grabovsky [122] has shown that the confocal ellipsoid construction is optimal under some conditions (see Remark 4.1.37). Vigdergauz's distribution of periodic inclusions [284] is also conditionally optimal in an energy bound (see [125]). The interested reader is referred to Subsection 2.3.4 for further details on this topic.

In the case of a single energy, many simplifications arise. In particular the nonlocal terms can now be computed explicitly. To simplify the exposition, we rephrase Proposition 2.3.16 for $p = 1$.

Proposition 2.3.20 Let ξ be a symmetric matrix in \mathcal{M}_N^s . Any homogenized Hooke's law $A^* \in G_\theta$ satisfies

$$A^* \xi : \xi \geq A \xi : \xi + (1 - \theta) \max_{\eta \in \mathcal{M}_N^s} [2\xi : \eta - (B - A)^{-1}\eta : \eta - \theta g(\eta)], \quad (2.121)$$

where $g(\eta)$ is a nonlocal term defined by

$$g(\eta) = \max_{e \in S_{N-1}} f_A(e)\eta : \eta,$$

and

$$A^* \xi : \xi \leq B \xi : \xi + \theta \min_{\eta \in \mathcal{M}_N^s} [2\xi : \eta + (B - A)^{-1}\eta : \eta - (1 - \theta)h(\eta)], \quad (2.122)$$

where $h(\eta)$ is another nonlocal term defined by

$$h(\eta) = \min_{e \in S_{N-1}} f_B(e)\eta : \eta,$$

where $f_A(e), f_B(e)$ are defined by (2.118). Furthermore, these upper and lower bounds are optimal in the sense of Definition 2.3.15 and optimality is always achieved by a finite-rank sequential laminate.

For a single energy, we obtain explicit formulas for the nonlocal terms $g(\eta)$ and $h(\eta)$.

Lemma 2.3.21 Let $\eta_1 \leq \dots \leq \eta_N$ be the eigenvalues of the symmetric matrix η . Then, the nonlocal term $h(\eta)$ is equal to

$$h(\eta) = \frac{1}{2\mu_B + \lambda_B} \min \left(\eta_1^2, \eta_N^2 \right).$$

If $N = 2$ or $\lambda_A \geq 0$, then the nonlocal term $g(\eta)$ is equal to

$$g(\eta) = \begin{cases} \frac{(\eta_1 - \eta_N)^2}{4\mu_A} + \frac{(\eta_1 + \eta_N)^2}{4(\lambda_A + \mu_A)} & \text{if } \eta_N \geq \frac{2\mu_A + \lambda_A}{2(\mu_A + \lambda_A)}(\eta_1 + \eta_N) \geq \eta_1 \\ \frac{\eta_1^2}{2\mu_A + \lambda_A} & \text{if } \eta_1 > \frac{2\mu_A + \lambda_A}{2(\mu_A + \lambda_A)}(\eta_1 + \eta_N) \\ \frac{\eta_N^2}{2\mu_A + \lambda_A} & \text{if } \eta_N < \frac{2\mu_A + \lambda_A}{2(\mu_A + \lambda_A)}(\eta_1 + \eta_N). \end{cases}$$

Thanks to Lemma 2.3.21 the minimal rank of the optimal sequential laminates in Proposition 2.3.20 can be further decreased.

Proposition 2.3.22 *Optimality in the Hashin-Shtrikman bounds (2.121) and (2.122) can be achieved by a rank- N sequential laminate with lamination directions that are extremal in the definition of the nonlocal terms $g(\eta)$ and $h(\eta)$. In particular, the optimal rank- N sequential laminate for the upper bound (2.122) is aligned with the eigendirections of ξ .*

Remark 2.3.23 *Lemma 2.3.21 and Proposition 2.3.22 are specific to the case of isotropic phases A and B. These results first appeared in [151] for incompressible elasticity and in [20] for general elasticity. The precise values of the bounds (2.121) and (2.122) are not important in the sequel, but they can be computed explicitly, at least in two dimensions, or in three dimensions when one phase is degenerate (see [7], [22], [115], [116]).*

Proof of Lemma 2.3.21. We are looking for the extremal values of $f_A(e)\eta : \eta$ (or similarly $f_B(e)\eta : \eta$), defined by (2.118), when e runs into the unit sphere S_{N-1} . For simplicity, we drop the index A or B , and we recall that

$$f(e)\eta : \eta = \frac{1}{\mu} \left(|\eta e|^2 - (\eta e \cdot e)^2 \right) + \frac{1}{2\mu + \lambda} (\eta e \cdot e)^2.$$

By the method of Lagrange multipliers, an extremal e must satisfy the optimality condition

$$\frac{1}{\mu} \left(\eta^2 e - (\eta e \cdot e) \eta e \right) + \frac{1}{2\mu + \lambda} (\eta e \cdot e) \eta e = \ell e, \quad (2.123)$$

where ℓ is a real constant (due to the constraint $|e| = 1$). Equation (2.123) shows that $\eta^2 e$ is a linear combination of e and ηe . Therefore, the subspace of \mathbb{R}^N spanned by $e, \eta e$ is stable under the action of the symmetric matrix η . Consequently, η is diagonalizable on this subspace, and there exist two orthogonal unit eigenvectors e_i, e_j of η , corresponding to the eigenvalues η_i, η_j , such that the extremal e satisfies

$$e = c_i e_i + c_j e_j,$$

where c_i, c_j are real constants satisfying $c_i^2 + c_j^2 = 1$. The optimality condition (2.123) becomes

$$\frac{(\eta_p^2 c_p - 2(\eta_i c_i^2 + \eta_j c_j^2) \eta_p c_p)}{\mu} + \frac{(\eta_i c_i^2 + \eta_j c_j^2) \eta_p c_p}{2\mu + \lambda} = \ell c_p, \quad p = 1, 2. \quad (2.124)$$

If $c_i = 0$, then e is an eigenvector of η associated to the eigenvalue η_j , and the corresponding value of $f(e)\eta : \eta$ is

$$f(e)\eta : \eta = \frac{\eta_j^2}{2\mu + \lambda}. \quad (2.125)$$

Interchanging the role of i and j , a similar situation arises if $c_j = 0$. If $\eta_i = \eta_j$, then e is once again an eigenvector of η and (2.125) holds. Assume now that $c_i \neq 0, c_j \neq 0$ and $\eta_i \neq \eta_j$. Then, after simplification by c_p , subtraction of the two components of (2.124) yields

$$\eta_i c_i^2 + \eta_j c_j^2 = \frac{2\mu + \lambda}{2(\mu + \lambda)} (\eta_i + \eta_j).$$

Together with the normalization $c_i^2 + c_j^2 = 1$, this is a linear system of two equations, the solution of which is

$$c_i^2 = \frac{(2\mu + \lambda)\eta_i - \lambda\eta_j}{2(\mu + \lambda)(\eta_i - \eta_j)}, \quad c_j^2 = \frac{(2\mu + \lambda)\eta_j - \lambda\eta_i}{2(\mu + \lambda)(\eta_j - \eta_i)}. \quad (2.126)$$

Of course, these values of c_i^2 and c_j^2 must be nonnegative, which is equivalent to the condition

$$\eta_j \geq \frac{2\mu + \lambda}{2(\mu + \lambda)} (\eta_i + \eta_j) \geq \eta_i, \quad (2.127)$$

where we assumed, with no loss of generality, that $\eta_j > \eta_i$. The associated value of $f(e)\eta : \eta$ is

$$f(e)\eta : \eta = \frac{(\eta_i - \eta_j)^2}{4\mu} + \frac{(\eta_i + \eta_j)^2}{4(\mu + \lambda)}. \quad (2.128)$$

It is easily checked that (2.128) is always larger than both values of (2.125) for i and j . Therefore, the minimum of $f(e)\eta : \eta$ is always achieved by an eigenvector of η , which gives the desired formula for $h(\eta)$.

A maximum of $f(e)\eta : \eta$ is either equal to (2.125), or to (2.128) if assumption (2.127) is met. To simplify the exposition, we first discuss the two-dimensional case, $N = 2$. Labeling the eigenvalues such that $\eta_1 \leq \eta_2$, if condition (2.127) is not satisfied, then either

$$\eta_1 > \frac{2\mu + \lambda}{2(\mu + \lambda)} (\eta_1 + \eta_2), \quad (2.129)$$

or

$$\eta_2 < \frac{2\mu + \lambda}{2(\mu + \lambda)} (\eta_1 + \eta_2). \quad (2.130)$$

Since $\mu + \lambda > 0$, it is easily seen that (2.129) yields $0 > \eta_2 > \eta_1$, i.e., $\eta_1^2 > \eta_2^2$, and (2.130) yields $\eta_2 > \eta_1 > 0$, i.e., $\eta_2^2 > \eta_1^2$. This gives the desired formula for $g(\eta)$. In higher dimensions, $N \geq 3$, we assume that $\lambda \geq 0$ which greatly simplifies the analysis. In particular, if a couple of eigenvalues (η_i, η_j) satisfy condition (2.127), then the same is true for the extreme eigenvalues (η_1, η_N) (the eigenvalues are labeled in such way that $\eta_1 \leq \dots \leq \eta_N$), and the corresponding values of (2.128) are ordered

$$\frac{(\eta_i - \eta_j)^2}{4\mu} + \frac{(\eta_i + \eta_j)^2}{4(\mu + \lambda)} \leq \frac{(\eta_1 - \eta_N)^2}{4\mu} + \frac{(\eta_1 + \eta_N)^2}{4(\mu + \lambda)}.$$

Therefore, a maximum of (2.128) is attained for a combination of the extreme eigenvalues. Eventually, a discussion similar to that in the two-dimensional case yields the desired formula for $g(\eta)$. \square

Proof of Proposition 2.3.22. We follow the argument of [151], [20]. The attainability of the Hashin-Shtrikman bound is checked by inspecting the optimality condition with respect to the matrix η . We begin with the upper bound (2.122) which reads

$$A^* \xi : \xi \leq B \xi : \xi + \theta \min_{\eta \in \mathcal{M}_N^s} \phi(\eta),$$

where

$$\phi(\eta) = 2\xi : \eta + (B - A)^{-1}\eta : \eta - (1 - \theta)h(\eta).$$

Since $\phi(\eta)$ is strictly convex in η (see Remark 2.3.17), there exists a unique extremal η^* , which achieves the minimum of ϕ . Therefore, the first order optimality condition is a necessary and sufficient condition for the extremality of η^* . If $\phi(\eta)$ were a smooth function, the optimality condition would be $\nabla\phi(\eta^*) = 0$. But $h(\eta)$ is usually not a smooth function of η . In this context, the appropriate tool for obtaining first order optimality conditions is the subdifferential calculus or generalized gradients (see, e.g., [86]). At the extremal η^* we must have

$$0 \in \partial\phi(\eta^*),$$

where $\partial\phi(\eta^*)$ is the subdifferential of ϕ at η^* . To compute the subdifferential of $h(\eta)$ we recall that

$$h(\eta) = \min_{e \in S_{N-1}} \left[f_B(e)\eta : \eta = \frac{|\eta e|^2 - (\eta e \cdot e)^2}{\mu_B} + \frac{(\eta e \cdot e)^2}{2\mu_B + \lambda_B} \right], \quad (2.131)$$

where S_{N-1} is the unit sphere in \mathbb{R}^N . Introducing the set $\mathcal{P}(S_{N-1})$ of probability measures on S_{N-1} , defined by (2.76), the nonlocal term $h(\eta)$ can be rewritten as

$$h(\eta) = \min_{\nu \in \mathcal{P}(S_{N-1})} \int_{S_{N-1}} f_B(e) \eta : \eta d\nu(e).$$

This implies that the minimization of $\phi(\eta)$ becomes a min-max optimization problem featuring a concave maximization with respect to ν and convex minimization with respect to η . Therefore, applying the Kuhn-Tucker optimality conditions, the subdifferential of $h(\eta)$ is the set (see Theorem 2.8.2 and Corollary 2, p.87 in [86])

$$\partial h(\eta) = \left\{ 2 \int_{S_{N-1}} f_B(e) \eta d\nu(e) \right\}, \quad (2.132)$$

where ν runs into the subset of probability measures with support concentrated on extremal vectors e . In other words, (2.132) tells us that, since $h(\eta)$ is the minimum of a family of nonnegative quadratic functions, continuously parametrized by e , its subdifferential is the convex hull of the gradients (with respect to η) of those quadratic functions which are extremal in (2.131). Since (2.132) involves the fourth order moments of probability measures, invoking again Carathéodory's theorem, as in Lemma 2.3.6 or Theorem 2.3.11, the subdifferential of $h(\eta)$ is equivalently defined by

$$\partial h(\eta) = \left\{ 2 \sum_{i=1}^p m_i f_B(e_i) \eta \right\},$$

where each unit vector e_i is extremal for (2.131), and the m_i are any parameters such that $m_i \geq 0$, $\sum_{i=1}^p m_i = 1$, and $p = N(N+1)/2 + 1$. Thus, the optimality condition

$$0 \in \partial \phi(\eta^*)$$

is equivalent to

$$\xi = -(B - A)^{-1} \eta^* + (1 - \theta) \sum_{i=1}^p m_i f_B(e_i) \eta^*, \quad (2.133)$$

using the same notation. Since phase B is isotropic, Lemma 2.3.21 implies that these extremal e_i are all eigenvectors of η^* for the same eigenvalue η_1^* or η_N^* , where we have ordered the eigenvalues $\eta_1^* \leq \dots \leq \eta_N^*$ of η^* . Therefore,

$$f_B(e_i) \eta^* = \frac{\eta_q^*}{2\mu_B + \lambda_B} e_i \otimes e_i,$$

with $q = 1$ or $q = N$, and (2.133) becomes

$$\xi = -(B - A)^{-1}\eta^* + (1 - \theta)\frac{\eta_q^*}{2\mu_B + \lambda_B} \sum_{i=1}^p m_i e_i \otimes e_i.$$

The following symmetric matrix

$$M = \sum_{i=1}^p m_i e_i \otimes e_i$$

is diagonalizable and, since it is nonnegative with unit trace, it can be diagonalized as

$$M = \sum_{i=1}^N \tilde{m}_i \tilde{e}_i \otimes \tilde{e}_i,$$

with its unit eigenvectors \tilde{e}_i , and its eigenvalues \tilde{m}_i such that $\tilde{m}_i \geq 0$, $\sum_{i=1}^N \tilde{m}_i = 1$. Since the vectors e_i belong to the same eigenspace of η^* , so do the vectors \tilde{e}_i , which are thus still extremal for (2.131). Therefore, there is no need to consider p larger than N in the optimality condition (2.133). To achieve equality in the upper Hashin-Shtrikman bound (2.122), consider the sequential laminate B_N^* (with core A and matrix B) provided by formula (2.69) with the same parameters \tilde{m}_i and directions \tilde{e}_i :

$$\theta(B_N^* - B)^{-1} = (A - B)^{-1} + (1 - \theta) \sum_{i=1}^N \tilde{m}_i f_B(\tilde{e}_i). \quad (2.134)$$

Multiplying (2.134) by η^* and using the optimality condition (2.133) yields

$$\theta(B_N^* - B)^{-1} \eta^* = \xi,$$

which becomes

$$B_N^* \xi = B \xi + \theta \eta^*,$$

and, upon taking the inner product with ξ ,

$$B_N^* \xi : \xi = B \xi : \xi + \theta \eta^* : \xi.$$

Since $\phi(\eta)$ is the Legendre transform of a quadratic function, it satisfies

$$\min_{\eta \in \mathcal{M}_N^s} \phi(\eta) = \phi(\eta^*) = \eta^* : \xi,$$

and therefore B_N^* achieves equality in the upper Hashin-Shtrikman bound (2.122).

We now turn to the case of the lower Hashin-Shtrikman bound (2.121) which is slightly more complicated because of the more involved formula defining $g(\eta)$. This lower bound is

$$A^* \xi : \xi \geq A \xi : \xi + (1 - \theta) \max_{\eta \in \mathcal{M}_N^s} \phi(\eta),$$

where

$$\phi(\eta) = 2\xi : \eta - (B - A)^{-1}\eta : \eta - \theta g(\eta).$$

Its necessary and sufficient optimality condition with respect to η is

$$\xi = (B - A)^{-1}\eta^* + \theta \sum_{i=1}^p m_i f_A(e_i)\eta^*, \quad (2.135)$$

where the unit vectors e_i are optimal in the definition of $g(\eta)$, and $m_i \geq 0$, $\sum_{i=1}^p m_i = 1$. Lemma 2.3.21 gives three possible regimes for the value of $g(\eta)$. Denoting by $\eta_1 \leq \dots \leq \eta_N$ the eigenvalues of η^* , the two last regimes are

$$g(\eta^*) = \frac{\eta_1^2}{2\mu_A + \lambda_A} \quad \text{or} \quad \frac{\eta_N^2}{2\mu_A + \lambda_A}.$$

In such cases, the optimal e_i in (2.135) are again eigenvectors of η^* associated to the same eigenvalue. A discussion similar to that given above for the upper bound shows that one can take $p \leq N$. The first regime is

$$g(\eta^*) = \frac{(\eta_1 - \eta_N)^2}{4\mu_A} + \frac{(\eta_1 + \eta_N)^2}{4(\lambda_A + \mu_A)}.$$

In such a case, the optimal e_i in (2.135) are of the type

$$e_i = c_1 e_{1,i} + c_N e_{N,i},$$

where c_1 and c_N are given by (2.126) (independently of i), $e_{1,i}$ is an eigenvector associated to the first eigenvalue η_1 , and $e_{N,i}$ is an eigenvector associated to the last eigenvalue η_N . For such vectors e_i , formula (2.118) yields

$$f_A(e_i)\eta^* = \alpha_1 e_{1,i} \otimes e_{1,i} + \alpha_N e_{N,i} \otimes e_{N,i},$$

with

$$\alpha_1 = \frac{(2\mu_A + \lambda_A)\eta_1 - \lambda_A\eta_N}{4\mu_A(\mu_A + \lambda_A)} \quad \text{and} \quad \alpha_N = \frac{(2\mu_A + \lambda_A)\eta_N - \lambda_A\eta_1}{4\mu_A(\mu_A + \lambda_A)}.$$

Therefore,

$$M = \sum_{i=1}^p m_i f_A(e_i) \eta^* = \sum_{i=1}^p m_i (\alpha_1 e_{1,i} \otimes e_{1,i} + \alpha_N e_{N,i} \otimes e_{N,i}). \quad (2.136)$$

By virtue of (2.136), η^* and M commute and are therefore diagonalizable in the same eigenbasis. Because of (2.135), ξ is also diagonalizable in the same eigenbasis as η^* and M . In particular, this implies that in such a basis M can be written as in (2.136) with $p \leq N$. Then, using the lamination formula (2.68), a similar argument as that for the upper bound yields that the lower Hashin-Shtrikman bound (2.121) is also attained by a sequential laminate of rank at most N . \square

So far, we have concentrated on bounds for the elastic energy written in terms of strain, i.e., the *primal energy*. The same can be done for the *complementary or dual energy*, i.e., the elastic energy written in terms of stress. Since we shall use complementary energy in shape optimization, we now give the corresponding Hashin-Shtrikman bounds without proofs.

Proposition 2.3.24 *Let $\sigma_1, \dots, \sigma_p$ be symmetric matrices in \mathcal{M}_N^s . Any homogenized Hooke's law $A^* \in G_\theta$ satisfies*

$$\sum_{i=1}^p A^{*-1} \sigma_i : \sigma_i \geq \sum_{i=1}^p B^{-1} \sigma_i : \sigma_i + \theta \max_{(\eta_i)_{1 \leq i \leq p} \in \mathcal{M}_N^s} \quad (2.137)$$

$$\left[\sum_{i=1}^p \left(2\sigma_i : \eta_i - (A^{-1} - B^{-1})^{-1} \eta_i : \eta_i \right) - (1 - \theta) g^c(\eta_1, \dots, \eta_p) \right],$$

where the nonlocal term g^c is given by

$$g^c(\eta_1, \dots, \eta_p) = \max_{e \in S_{N-1}} \sum_{i=1}^p f_B^c(e) \eta_i : \eta_i, \quad (2.138)$$

and

$$\sum_{i=1}^p A^{*-1} \sigma_i : \sigma_i \leq \sum_{i=1}^p A^{-1} \sigma_i : \sigma_i + (1 - \theta) \min_{(\eta_i)_{1 \leq i \leq p} \in \mathcal{M}_N^s} \quad (2.139)$$

$$\left[\sum_{i=1}^p \left(2\sigma_i : \eta_i + (A^{-1} - B^{-1})^{-1} \eta_i : \eta_i \right) - \theta h^c(\eta_1, \dots, \eta_p) \right],$$

where the nonlocal term h^c is given by

$$h^c(\eta_1, \dots, \eta_p) = \min_{e \in S_{N-1}} \sum_{i=1}^p f_A^c(e) \eta_i : \eta_i, \quad (2.140)$$

and, for $M = A, B$, $f_M^c(e)$ is defined by

$$f_M^c(e) \eta : \eta = M\eta : \eta - \frac{1}{\mu_M} |M\eta e|^2 + \frac{\mu_M + \lambda_M}{\mu_M(2\mu_M + \lambda_M)} ((M\eta)e \cdot e)^2. \quad (2.141)$$

Furthermore, these upper and lower bounds are optimal in the sense of Definition 2.3.15, and optimality is always achieved by a finite-rank sequential laminate of rank 3 in two dimensions, and rank 6 in three dimensions.

For a single energy, Proposition 2.3.24 is simplified.

Proposition 2.3.25 *Let σ be a symmetric matrix. Any homogenized Hooke's law $A^* \in G_\theta$ satisfies*

$$A^{*-1}\sigma : \sigma \geq B^{-1}\sigma : \sigma + \theta \max_{\eta \in \mathcal{M}_N^s} \left[2\sigma : \eta - \left(A^{-1} - B^{-1} \right)^{-1} \eta : \eta - (1-\theta)g^c(\eta) \right], \quad (2.142)$$

where $g^c(\eta)$ is a nonlocal term given by

$$g^c(\eta) = \max_{e \in S_{N-1}} \left(f_B^c(e)\eta : \eta \right) = B\eta : \eta - h(B\eta),$$

and

$$A^{*-1}\sigma : \sigma \leq A^{-1}\sigma : \sigma + (1-\theta) \min_{\eta \in \mathcal{M}_N^s} \left[2\sigma : \eta + \left(A^{-1} - B^{-1} \right)^{-1} \eta : \eta - \theta h^c(\eta) \right], \quad (2.143)$$

where $h^c(\eta)$ is a nonlocal term given by

$$h^c(\eta) = \min_{e \in S_{N-1}} \left(f_A^c(e)\eta : \eta \right) = A\eta : \eta - g(A\eta),$$

where $f_A^c(e)$ and $f_B^c(e)$ are defined by (2.141). Furthermore, these upper and lower bounds are optimal in the sense of Definition 2.3.15 and optimality is achieved by a rank- N sequential laminate with lamination directions given by the extremal vectors e in the definition of the nonlocal terms $g^c(\eta)$ and $h^c(\eta)$. In particular, the optimal rank- N sequential laminate for the lower bound (2.142) is aligned with the eigendirections of σ .

Remark 2.3.26 The fourth order tensors $f_A^c(e)$ and $f_B^c(e)$ defined by (2.141) are exactly those appearing in the lamination formula written in terms of stress (see (2.71)). This is at the root of the attainability of these bounds by sequential laminates.

Remark 2.3.27 The lower and upper dual Hashin-Shtrikman bounds (2.142) and (2.143) are actually C^1 functions with respect to θ and σ . Indeed, they are given as the extremal value of a strictly concave or convex function of η which is extremal at a unique η^* . Therefore, when differentiating these bounds, the derivative of η^* cancels out due to its optimality condition, and the derivatives of the bounds are thus continuous functions. The lower bound (2.142) is also convex in θ since it is the maximum of convex quadratic functions of θ . It is further strictly convex. To show this, let us denote by $F(\theta)$ the value of the lower bound, i.e.,

$$F(\theta) = B^{-1}\sigma : \sigma + \max_{\eta \in \mathcal{M}_N^s} Q(\eta, \theta),$$

with

$$Q(\eta, \theta) = \theta^2 g^c(\eta) + \theta \left[2\sigma : \eta - \left(A^{-1} - B^{-1} \right)^{-1} \eta : \eta - g^c(\eta) \right].$$

For any $\theta, \theta' \in [0, 1]$ and any $t \in [0, 1]$ we have

$$tQ(\eta, \theta) + (1-t)Q(\eta, \theta') - Q(\eta, t\theta + (1-t)\theta') = g^c(\eta)t(1-t)(\theta - \theta')^2,$$

which implies

$$tF(\theta) + (1-t)F(\theta') - \left(B^{-1}\sigma : \sigma + Q(\eta, t\theta + (1-t)\theta') \right) \geq g^c(\eta)t(1-t)(\theta - \theta')^2.$$

Taking η to be the maximizer in the definition of $F(t\theta + (1-t)\theta')$ yields

$$tF(\theta) + (1-t)F(\theta') - F(t\theta + (1-t)\theta') \geq g^c(\eta)t(1-t)(\theta - \theta')^2.$$

Since for any value of $t\theta + (1-t)\theta'$ the maximizer η is nonzero if $\sigma \neq 0$, the coefficient $g^c(\eta)$ is strictly positive, which implies the strict convexity. The same is true for the lower bound (2.137) on several energies.

Remark 2.3.28 The lower bound (2.142) is also a strictly convex function of σ since it is equal to the sum of a quadratic term and a maximum of affine

functions of σ . It turns out that the upper bound (2.143) is also convex in σ , but for a different reason. Indeed, by the Legendre transform, we have

$$A^{*-1}\sigma : \sigma = \max_{\xi \in \mathcal{M}_N^s} (2\sigma : \xi - A^*\xi : \xi),$$

and thus, switching the two maximizations,

$$\max_{A^* \in G_\theta} A^{*-1}\sigma : \sigma = \max_{\xi \in \mathcal{M}_N^s} \left(2\sigma : \xi - \min_{A^* \in G_\theta} A^*\xi : \xi \right). \quad (2.144)$$

Formula (2.144) proves two facts. First, the dual upper bound (2.143) is a Legendre transform, which implies it is a convex function of σ . Second, since $\min_{A^* \in G_\theta} A^*\xi : \xi$ is just the primal lower bound (2.121), which is convex in ξ because it is the sum of a quadratic term and a maximum of affine functions of ξ , it shows that these two bounds (2.121) and (2.143) are conjugate functions through the Legendre transform. In other words, the collection of dual upper bounds (2.143) (when σ runs into \mathcal{M}_N^s) is equivalent to the collection of primal lower bounds (2.121) (when ξ runs into \mathcal{M}_N^s). This can be further checked in the algebraic formula by using the following identity

$$(B - A)^{-1} = A^{-1} \left(A^{-1} - B^{-1} \right)^{-1} A^{-1} - A^{-1}. \quad (2.145)$$

The same is true between the dual lower bound (2.142) and the primal upper bound (2.122), and also between corresponding bounds on sums of energies.

Proof of Proposition 2.3.25. The proof is identical to a combination of those of Proposition 2.3.20 and Proposition 2.3.22, so we confidently leave it to the reader. The only statement to prove is that

$$g^c(\eta) = B\eta : \eta - h(B\eta) \quad (2.146)$$

(the same is true with $h^c(\eta) = A\eta : \eta - g(A\eta)$). To do so, we remark that, for any $e \in S_{N-1}$,

$$f_B^c(e)\eta : \eta + f_B(e)(B\eta) : (B\eta) = B\eta : \eta. \quad (2.147)$$

In particular, since B is isotropic, (2.146) implies that the extremal vectors e in the definition of the nonlocal terms $g^c(\eta)$ are the same as those for $h(B\eta)$, and that they are the eigenvectors of η , which is simultaneously diagonalizable with σ . This implies that optimality in the lower bound (2.142) is attained by a rank- N sequential laminate with lamination directions given by the eigenvectors of σ . (This fact could also be deduced from Remark 2.3.28, which states the equivalence between the dual lower bound and the primal upper bound.) \square

2.3.3 Toward G -closure

Contrary to the conductivity case, the G -closure of two isotropic elastic phases in prescribed proportions is unknown. Specifically, there is no explicit algebraic characterization of the set G_θ defined as the set of all constant effective Hooke's laws obtained by homogenization of phases A and B in volume fractions θ and $(1 - \theta)$, respectively. Even if we restrict ourselves to isotropic composites, we do not know the subset of isotropic tensors in G_θ (see Remark 2.3.14). Even worse, as the volume fraction θ varies in the range $[0, 1]$, we do not know the union of G -closure sets G_θ . As we shall see in Chapter 4, this fact has important consequences in optimal design of elastic structures. Due to our lack of knowledge of G_θ , homogenization can be applied only to a restricted class of optimal design problems. Any progress made in this G -closure problem would also benefit structural design.

The main reason for this lack of knowledge concerning G -closure is that the Hashin-Shtrikman energy bounds given by Proposition 2.3.16, or Avellaneda's Theorem 2.3.11, are insufficient for characterizing all homogenized tensors. This is in sharp contrast to the conductivity setting, where energy bounds are enough for explicitly computing G_θ . Indeed, as was first remarked by Cherkaev and Gibiansky [76], [77], the usual energy bounds can be improved by considering coupled bounds involving both strain and stress (i.e., a combination of primal and dual energies). Subsequently, Francfort and Milton proved that determining the G -closure set G_θ is equivalent to computing all optimal coupled bounds on sums of primal and dual energies [109], [191]. Unfortunately, optimal coupled bounds are unknown, and it is possible that for such bounds optimal composites are not laminates, but, instead, they exhibit more complex microstructures.

The goal of this subsection is therefore to establish a result due to Tartar [276] and Francfort, Murat, and Tartar [112], which is a first step toward the solution of the G -closure problem. Their result is concerned with a characterization of the G -closure in the low contrast, or small amplitude, limit. Indeed, if we assume that the two phases A and B have very close moduli, then the full G -closure is known in two and three dimensions up to second order terms in the contrast $(B - A)$. More precisely, we assume that there exist a small positive parameter γ and a positive isotropic fourth order tensor D such that

$$B = A + \gamma D. \tag{2.148}$$

The next lemma gives a formula for a periodic composite A^* in terms of its H -measure, up to second order in γ .

Lemma 2.3.29 *Let A^* be any periodic homogenized tensor in P_θ associated to a characteristic function χ . Introducing the H -measure ν of χ , defined as a probability measure on S_{N-1} by*

$$\theta(1-\theta) \int_{S_{N-1}} f(e) d\nu(e) = \sum_{k \in \mathbb{Z}^N, k \neq 0} |\hat{\chi}(k)|^2 f\left(\frac{k}{|k|}\right)$$

for any continuous function f on S_{N-1} , we have

$$A^* = \theta A + (1-\theta)B - \gamma^2 \theta(1-\theta) D \left(\int_{S_{N-1}} f_A(e) d\nu(e) \right) D + \mathcal{O}(\gamma^3),$$

where, for any $\eta \in \mathcal{M}_N^s$,

$$f_A(e)\eta : \eta = \frac{1}{\mu_A} \left(|\eta e|^2 - (\eta e \cdot e)^2 \right) + \frac{1}{2\mu_A + \lambda_A} (\eta e \cdot e)^2,$$

and the remainder is uniform with respect to A^* in the sense that there exists a positive constant C , independent of γ and χ , such that

$$\|\mathcal{O}(\gamma^3)\| \leq C\gamma^3.$$

Remark 2.3.30 We already introduced the notion of H -measure for a periodic characteristic function χ in (2.99). Let us recall that it is a kind of two-point correlation function. The notion of H -measure was introduced in a much more general context by Gerard [113] and Tartar [276].

Upon recalling Lemma 2.3.9 (on the characterization of fourth order moments of probability measures; see [112]), we immediately obtain the following theorem.

Theorem 2.3.31 *Assume that the space dimension is $N = 2$ or 3 . Then, any tensor A^* in G_θ is of the form*

$$A^* = \theta A + (1-\theta)B - \gamma^2 \theta(1-\theta) DL_A(M)D + \mathcal{O}(\gamma^3)$$

where M is any fully symmetric fourth order tensor, i.e.,

$$M_{ijkl} = M_{klij} = M_{klji} = M_{jikl},$$

with unit trace $\sum_{i,j=1}^N M_{ijij} = 1$, and nonnegative in the sense of quadratic forms, i.e., for any symmetric matrix ξ

$$\sum_{i,j,k,l=1}^N M_{ijkl} \xi_{ij} \xi_{kl} \geq 0,$$

and L_A is the linear map on the space of fourth order tensors,

$$\begin{aligned} L_A(M)_{ijkl} &= \left(\frac{1}{2\mu_A + \lambda_A} - \frac{1}{\mu_A} \right) M_{ijkl} \\ &+ \frac{1}{4\mu_A} \sum_{m=1}^N (M_{imml}\delta_{jk} + M_{imkm}\delta_{jl} + M_{mjml}\delta_{ik} + M_{mjk}\delta_{il}). \end{aligned}$$

Remark 2.3.32 Theorem 2.3.31 could also be deduced from the well known fact that the upper and lower Hashin-Shtrikman bounds coincide up to second order in the contrast $(B - A)$. More precisely, in Theorem 2.3.11, the two sequential laminates A^- and A^+ that bound the composite tensor A^* do coincide up to a remainder of order γ^3 .

Proof of Lemma 2.3.29. Recall from Section 1.1 that a periodic composite A^* is defined through the solution of a cell problem. Namely, for any tensor $\xi \in \mathcal{M}_N^s$,

$$A^*\xi : \xi = \int_Y A(y)(\xi + e(w)) : \xi \, dy,$$

where $A(y) = \chi(y)A + (1 - \chi(y))B$ and w is the unique solution in $H_\#^1(Y)^N$ (up to a rigid body displacement) of

$$\begin{cases} -\operatorname{div}_y(A(y)(\xi + e(w(y)))) = 0 & \text{in } Y \\ y \rightarrow w(y) & \text{Y-periodic.} \end{cases}$$

We remark that $A(y) = A + \gamma(1 - \chi(y))D$, and therefore the solution w depends analytically on the small contrast parameter γ . We shall not use this fact, but rather expand w as

$$w(y) = \gamma v(y) + \gamma^2 r(y),$$

where v is the unique solution in $H_\#^1(Y)^N$ (up to a rigid body displacement) of

$$\begin{cases} -\operatorname{div}_y(Ae(v) + (1 - \chi(y))D\xi) = 0 & \text{in } Y \\ y \rightarrow v(y) & \text{Y-periodic.} \end{cases}$$

By subtraction, it is easily checked that r is the solution of

$$\begin{cases} -\operatorname{div}_y(A(y)e(r) + (1 - \chi(y))De(v)) = 0 & \text{in } Y \\ y \rightarrow r(y) & \text{Y-periodic.} \end{cases}$$

Note that v is independent of γ , but not r . Nevertheless, since $A(y) \geq A$, r satisfies the a priori estimate

$$\|e(r)\|_{L^2(\Omega; \mathcal{M}_N^s)} \leq C|\xi|,$$

where C is a positive constant that does not depend on γ nor χ . Plugging the ansatz for w into the definition of A^* and integrating by parts leads to

$$\begin{aligned} A^* \xi : \xi &= A\xi : \xi + \gamma(1-\theta)D\xi : \xi \\ &\quad - \gamma^2 \int_Y Ae(v) : e(v) dy + \gamma^3 \int_Y De(r) : \xi dy. \end{aligned} \tag{2.149}$$

Thanks to the a priori estimate on r , the last term in (2.149) is a uniform remainder term of order γ^3 . The term of order γ^2 in (2.149) can be computed by Fourier analysis as in the proof of Theorem 2.3.11. Indeed, it is the same computation as that of the nonlocal term defined by (2.93). More precisely, we have

$$\int_Y Ae(v) : e(v) dy = g(\chi, D\xi) = \sum_{k \in \mathbb{Z}^N, k \neq 0} |\hat{\chi}(k)|^2 f_A\left(\frac{k}{|k|}\right) (D\xi) : (D\xi),$$

where $f_A(\cdot)$ is defined by (2.97). Introducing the H -measure ν and remarking that $\sum_{k \in \mathbb{Z}^N, k \neq 0} |\hat{\chi}(k)|^2 = \theta(1-\theta)$ yields the desired result. \square

Proof of Theorem 2.3.31. By density of the set P_θ of periodic composites in the set G_θ of all possible composites (see Theorem 2.1.2), and by the closed character of the set \mathcal{C} of fourth order tensors of probability measures, the small amplitude formula of Lemma 2.3.29 is easily extended to any composite tensor $A^* \in G_\theta$ (this can be obtained directly by using the proper notion of H -measure; see [276] for details). In dimensions $N = 2$ or 3 , Lemma 2.3.9 furnishes a characterization of this set \mathcal{C} , which yields the desired result. \square

2.3.4 An Explicit Optimal Bound for Shape Optimization

This subsection is devoted to the explicit computation in dimension two or three of the lower Hashin-Shtrikman bound on complementary energy which is crucial for shape optimization (as will be discussed in Chapters 4 and 5). In shape optimization, the weak material A is actually degenerate, i.e., it corresponds to holes cut in material B . The fact that $A = 0$ also greatly simplifies the algebra. Thus, from now on we assume that

$$A = 0.$$

Under such an assumption, a two-phase mixture is now a perforated or porous media supporting Neumann boundary conditions. Let us remark that homogenization theory does not work in full generality in such a situation. Although for periodically distributed holes the homogenization results are the same as for two-phase mixtures (see [84]), there are examples of complicated porous media where the homogenized equation is of a different type than the original one (see [61], [156]). Therefore, all the results in this subsection are purely algebraic in nature. In other words, we study the limits of two-phase energy bounds when A goes to zero, but we do not claim that these limits are rigorous bounds for the effective properties of porous composites. With this word of caution, we begin with an obvious corollary of Proposition 2.3.24 valid in any space dimension.

Corollary 2.3.33 *Let $\sigma_1, \dots, \sigma_p$ be symmetric matrices in \mathcal{M}_N^s and $A^* \in G_\theta$. When $A = 0$ the Hashin-Shtrikman lower bound on complementary energies (2.137) simplifies as*

$$\sum_{i=1}^p A^{*-1} \sigma_i : \sigma_i \geq \sum_{i=1}^p B^{-1} \sigma_i : \sigma_i + \frac{\theta}{(1-\theta)} g^*(\sigma_1, \dots, \sigma_p), \quad (2.150)$$

where g^* is the conjugate function of the nonlocal term g^c defined by (2.138), i.e.,

$$g^*(\sigma_1, \dots, \sigma_p) = \max_{(\eta_i)_{1 \leq i \leq p} \in \mathcal{M}_N^s} \left(\sum_{i=1}^p 2\sigma_i : \eta_i - g^c(\eta_1, \dots, \eta_p) \right).$$

Furthermore, g^* is continuous, convex, positive (except if all σ_i are zero), and homogeneous of degree two.

Proof. Formula (2.150) is easily obtained by remarking that $A = 0$ implies $(A^{-1} - B^{-1})^{-1} = 0$. The properties of g^* are a consequence of the fact that it is the conjugate function (or Legendre transform) of g^c , which, by definition, is a positive function homogeneous of degree 2. \square

We now restrict ourselves to the lower bound (2.142) on a single energy (see Proposition 2.3.24) that we shall compute explicitly in two and three dimensions. To simplify the notation, and since there is only one material, we drop the index B and denote by κ and μ its bulk and shear moduli, respectively, i.e.,

$$B = 2\mu I_4 + \left(\kappa - \frac{2\mu}{N} \right) I_2 \otimes I_2.$$

We also define $\lambda = \kappa - 2\mu/N$, which is proportional to the Poisson ratio of B . The value of the bound is of interest, but so are the precise parameters of the sequential laminate that achieves optimality in the bound. Therefore, we exhibit a finite-rank sequential laminate that saturates (2.142) (it is the most rigid perforated composite capable of sustaining a given stress; note, however, that it is not unique). We recall the suitable lamination formula in the case at hand, $A = 0$ (see (2.73)).

Definition 2.3.34 *Let A^* be a rank- p sequential laminate of material B around a core of void in proportions $(1 - \theta)$ and θ , respectively, with lamination directions $(e_i)_{1 \leq i \leq p}$ and lamination parameters $(m_i)_{1 \leq i \leq p}$ satisfying $0 \leq m_i \leq 1$ and $\sum_{i=1}^p m_i = 1$. Then the Hooke's law A^* is given by*

$$\theta [A^{*-1} - B^{-1}]^{-1} = (1 - \theta) \sum_{i=1}^p m_i f_B^c(e_i), \quad (2.151)$$

where $f_B^c(e_i)$ is the fourth order tensor defined for any symmetric matrix ξ by its quadratic form

$$f_B^c(e_i) \xi : \xi = B\xi : \xi - \frac{1}{\mu} |B\xi e_i|^2 + \frac{\mu + \lambda}{\mu(2\mu + \lambda)} ((B\xi)e_i \cdot e_i)^2. \quad (2.152)$$

We begin with the two dimensional result, which was derived in [115] and [21].

Theorem 2.3.35 *In two dimensions, the bound (2.150) takes the form*

$$A^{*-1} \sigma : \sigma \geq B^{-1} \sigma : \sigma + \frac{(\kappa + \mu)\theta}{4\kappa\mu(1 - \theta)} (|\sigma_1| + |\sigma_2|)^2 \quad (2.153)$$

where σ_1 and σ_2 are the eigenvalues of the stress σ (a 2×2 matrix in two dimensions). Furthermore, an optimal rank-2 sequential laminate is obtained by taking the eigenvectors of σ as lamination directions and choosing the lamination parameters

$$m_1 = \frac{|\sigma_2|}{|\sigma_1| + |\sigma_2|}, \quad m_2 = \frac{|\sigma_1|}{|\sigma_1| + |\sigma_2|}. \quad (2.154)$$

In three dimensions, we make another hypothesis, which greatly simplifies the computations. Namely, we assume that B has a nonnegative Poisson ratio, i.e., $\lambda \geq 0$. Recall that, in order to ensure coerciveness of the Hooke's

law (the so-called *very strong ellipticity*), the shear modulus μ and the bulk modulus $\kappa = \lambda + 2\mu/N$ are always assumed to be positive. In general, there is no reason that λ be nonnegative, although this is the case for most actual materials. Therefore our hypothesis is not too restrictive. The result in three dimensions was found in [116] and [7].

Theorem 2.3.36 *In three dimensions, assume the material B satisfies $\lambda \geq 0$. Then, the bound (2.150) takes the form*

$$A^{*-1}\sigma : \sigma \geq B^{-1}\sigma : \sigma + \frac{\theta}{(1-\theta)}g^*(\sigma), \quad (2.155)$$

where, labeling the eigenvalues of σ such that $\sigma_1 \leq \sigma_2 \leq \sigma_3$, $g^*(\sigma)$ is defined according to the following regimes:

(i) Assume $0 \leq \sigma_1 \leq \sigma_2 \leq \sigma_3$. Then

$$g^*(\sigma) = \frac{2\mu + \lambda}{4\mu(2\mu + 3\lambda)} (\sigma_1 + \sigma_2 + \sigma_3)^2 \quad (2.156)$$

if $\sigma_3 \leq \sigma_1 + \sigma_2$

$$g^*(\sigma) = \frac{1}{2\mu} \left((\sigma_1 + \sigma_2)^2 + \sigma_3^2 \right) - \frac{\lambda(\sigma_1 + \sigma_2 + \sigma_3)^2}{2\mu(2\mu + 3\lambda)} \quad (2.157)$$

if $\sigma_3 \geq \sigma_1 + \sigma_2$.

(ii) Assume $\sigma_1 \leq 0 \leq \sigma_2 \leq \sigma_3$. Then

$$g^*(\sigma) = \frac{2\mu + \lambda}{4\mu(2\mu + 3\lambda)} \left(\sigma_3 + \sigma_2 - \frac{\mu + 2\lambda}{\mu + \lambda} \sigma_1 \right)^2$$

if
$$\begin{cases} \sigma_3 + \sigma_2 \geq \frac{-\mu}{\mu + \lambda} \sigma_1 \\ \sigma_3 - \sigma_2 \leq \frac{-\mu}{\mu + \lambda} \sigma_1 \end{cases}$$
 (2.158)

$$g^*(\sigma) = \frac{1}{2\mu} \left((\sigma_3 + \sigma_2)^2 + \sigma_1^2 \right) - \frac{\lambda(\sigma_1 + \sigma_2 + \sigma_3)^2}{2\mu(2\mu + 3\lambda)} \quad (2.159)$$

if $\sigma_3 + \sigma_2 \leq \frac{-\mu}{\mu + \lambda} \sigma_1$

$$g^*(\sigma) = \frac{1}{2\mu} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{\sigma_1\sigma_2}{\mu + \lambda} - \frac{\lambda(\sigma_1 + \sigma_2 + \sigma_3)^2}{2\mu(2\mu + 3\lambda)} \quad (2.160)$$

if $\sigma_3 - \sigma_2 \geq \frac{-\mu}{\mu + \lambda} \sigma_1$.

(iii) The other cases are obtained from (A) and (B) by symmetry, changing σ in $-\sigma$.

Furthermore, optimality is always achieved by a sequential laminate with lamination directions given by the eigenvectors of σ . In regime (2.156) it is a rank-3 sequential laminate with parameters

$$m_1 = \frac{\sigma_3 + \sigma_2 - \sigma_1}{\sigma_1 + \sigma_2 + \sigma_3}, \quad m_2 = \frac{\sigma_1 - \sigma_2 + \sigma_3}{\sigma_1 + \sigma_2 + \sigma_3}, \quad m_3 = \frac{\sigma_1 + \sigma_2 - \sigma_3}{\sigma_1 + \sigma_2 + \sigma_3}. \quad (2.161)$$

In regime (2.157) it is a rank-2 sequential laminate with parameters

$$m_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2}, \quad m_2 = \frac{\sigma_1}{\sigma_1 + \sigma_2}, \quad m_3 = 0. \quad (2.162)$$

In regime (2.158) it is a rank-3 sequential laminate with parameters

$$m_1 = \frac{\sigma_3 + \sigma_2 + \frac{\mu}{\mu+\lambda}\sigma_1}{\sigma_3 + \sigma_2 - \frac{\mu+2\lambda}{\mu+\lambda}\sigma_1}, \quad m_2 = \frac{\mu + \lambda}{\mu} \frac{\sigma_3 - \sigma_2 - \frac{\mu}{\mu+\lambda}\sigma_1}{\sigma_3 + \sigma_2 - \frac{\mu+2\lambda}{\mu+\lambda}\sigma_1}, \quad (2.163)$$

$$m_3 = -\frac{\mu + \lambda}{\mu} \frac{\sigma_3 - \sigma_2 + \frac{\mu}{\mu+\lambda}\sigma_1}{\sigma_3 + \sigma_2 - \frac{\mu+2\lambda}{\mu+\lambda}\sigma_1}.$$

In regime (2.159) it is a rank-2 sequential laminate with parameters

$$m_1 = 0, \quad m_2 = \frac{\sigma_3}{\sigma_2 + \sigma_3}, \quad m_3 = \frac{\sigma_2}{\sigma_2 + \sigma_3}. \quad (2.164)$$

In regime (2.160) it is a rank-2 sequential laminate with parameters

$$m_1 = \frac{\sigma_2}{\sigma_2 - \sigma_1}, \quad m_2 = \frac{-\sigma_1}{\sigma_2 - \sigma_1}, \quad m_3 = 0. \quad (2.165)$$

Remark 2.3.37 In two dimensions only one type of optimal laminate, namely, rank-2 laminates, are used (although they can degenerate to rank-1 when one of the eigenvalues vanishes). On the contrary, in three dimensions there are two distinct regimes of optimal laminates: Rank-3 or rank-2 (which in turn can degenerate to rank-1). This can be explained easily as follows. The conditions defining regimes (2.156) or (2.158), where a rank-3 laminate is

optimal, imply that the three principal stresses are of the same order of magnitude. This means that the material can be optimally layered in the three principal directions, creating a microstructure made of plate-like holes in the matrix of material. On the other hand, the remaining regimes (2.157), (2.159), or (2.160), where a rank-2 laminate is optimal, correspond to a setting where one of the principal stresses is large compared to the other two. In this case, it is more economical not to layer in the direction of the largest principal stress, but simply to translate in this direction a planar optimal microstructure, which allows the available material to sustain the largest stress in the direction of translation. The corresponding microstructure looks like an array of tubes or channels of holes aligned in the direction of the largest principal stress.

When material B has zero Poisson ratio, Theorem 2.3.36 simplifies considerably (as well as its proof; see [21]).

Corollary 2.3.38 *In three dimensions, assume the material satisfies $\lambda = 0$. Then, labeling the eigenvalues of σ such that $|\sigma_1| \leq |\sigma_2| \leq |\sigma_3|$, the bound (2.150) takes the form*

$$A^{*-1}\sigma : \sigma \geq B^{-1}\sigma : \sigma + \frac{\theta}{4\mu(1-\theta)} (|\sigma_1| + |\sigma_2| + |\sigma_3|)^2 \quad (2.166)$$

if $|\sigma_3| \leq |\sigma_1| + |\sigma_2|$, and

$$A^{*-1}\sigma : \sigma \geq B^{-1}\sigma : \sigma + \frac{\theta}{2\mu(1-\theta)} ((|\sigma_1| + |\sigma_2|)^2 + |\sigma_3|^2) \quad (2.167)$$

if $|\sigma_3| \geq |\sigma_1| + |\sigma_2|$.

Furthermore, optimality in the first regime (2.166) is achieved by a rank-3 sequential laminate with parameters

$$m_1 = \frac{|\sigma_3| + |\sigma_2| - |\sigma_1|}{|\sigma_1| + |\sigma_2| + |\sigma_3|}, \quad m_2 = \frac{|\sigma_1| - |\sigma_2| + |\sigma_3|}{|\sigma_1| + |\sigma_2| + |\sigma_3|}, \quad m_3 = \frac{|\sigma_1| + |\sigma_2| - |\sigma_3|}{|\sigma_1| + |\sigma_2| + |\sigma_3|},$$

while optimality in the second regime (2.167) is achieved by a rank-2 sequential laminate with parameters

$$m_1 = \frac{|\sigma_2|}{|\sigma_1| + |\sigma_2|}, \quad m_2 = \frac{|\sigma_1|}{|\sigma_1| + |\sigma_2|}, \quad m_3 = 0.$$

It would be tempting to assume that the three-dimensional result degenerates into the two-dimensional result in a plane stress situation, i.e., when one of the principal stresses is equal to zero. That this outcome does not occur is the object of the following lemma.

Lemma 2.3.39 *In three dimensions, a plane stress σ with eigenvalues denoted by $\sigma_1, \sigma_2, \sigma_3$, such that*

$$\sigma_1 = 0, \quad \sigma_2 \neq 0, \quad \text{and} \quad \sigma_3 \neq 0,$$

is considered. Then,

$$g^*(\sigma) = \frac{1}{2\mu} (\sigma_2^2 + \sigma_3^2) - \frac{\lambda}{2\mu(2\mu + 3\lambda)} (\sigma_2 + \sigma_3)^2, \quad (2.168)$$

and optimality is achieved by a rank-1 sequential laminate in the direction of the eigenvector corresponding to σ_1 , i.e.,

$$m_1 = 1, \quad m_2 = 0, \quad m_3 = 0.$$

The proof of Lemma 2.3.39 is immediate through inspection of the formulas. From a practical standpoint, it has the consequence that, if we can use three-dimensional microstructures for optimizing a two-dimensional energy bound, then it is preferable to use a “varying thickness plate” approach (corresponding to the optimal rank-1 laminate) than a “plane Michell trusses” approach (corresponding to the rank-2 laminates, optimal only in two-dimensions). We refer the interested reader to Subsection 5.2.5 for more details on this interpretation. Another interesting limit case is that of a uniaxial stress. In a uniaxial stress setting, the optimal microstructure looks like an array of fibers aligned with the stress, and any type of cross-sectional arrangement is admissible. This is the purpose of the next lemma (the proof of which is also immediate).

Lemma 2.3.40 *In three dimensions, a uniaxial stress σ with eigenvalues denoted by $\sigma_1, \sigma_2, \sigma_3$, such that*

$$\sigma_1 = 0, \quad \sigma_2 = 0, \quad \text{and} \quad \sigma_3 \neq 0,$$

is considered. Then,

$$g^*(\sigma) = \frac{\mu + \lambda}{\mu(2\mu + 3\lambda)} \sigma_3^2, \quad (2.169)$$

and optimality is achieved by any rank-2 sequential laminate in the directions of the two first eigenvectors of σ , i.e., any triplet m_1, m_2, m_3 with

$$m_3 = 0, \quad m_1 \geq 0, \quad m_2 \geq 0, \quad m_1 + m_2 = 1.$$

Let us explain briefly how we are going to prove Theorems 2.3.35 and 2.3.36. By virtue of Proposition 2.3.24, the lower bound (2.142) is achieved by a rank- N sequential laminate (B being the matrix) with lamination directions given by the eigendirections of the stress σ . Therefore, to compute the value of the lower bound (2.142), it is sufficient to minimize $A^{*-1}\sigma : \sigma$ among all possible rank- N laminates A^* with lamination directions corresponding to the eigendirections of σ . In view of formula (2.151) for A^* , this minimization takes place over the lamination parameters $(m_i)_{1 \leq i \leq N}$. Since $\sum_{i=1}^N m_i = 1$, the minimization involves only one degree of freedom in two dimensions, and two in three dimensions, but the inversion of the lamination formula (2.151) is, unfortunately, tedious.

By definition, a rank- N sequential laminate is said to be *orthogonal* if its lamination directions $(e_i)_{1 \leq i \leq N}$ form an orthonormal basis of \mathbb{R}^N . The next lemma inverts the lamination formula (2.151), i.e., computes A^{*-1} for such an orthogonal rank- N sequential laminate. Recall that its lamination parameters $(m_i)_{1 \leq i \leq N}$ satisfy

$$0 \leq m_i \leq 1, \quad \sum_{i=1}^N m_i = 1. \quad (2.170)$$

Let us introduce new parameters $(\alpha_i)_{1 \leq i \leq N}$ defined by

$$\alpha_i = \left(1 - \frac{2\mu m_i}{2\mu + \lambda}\right)^{-1}. \quad (2.171)$$

Throughout the sequel, we assume that the material B has positive Poisson's ratio, i.e.,

$$\lambda \geq 0. \quad (2.172)$$

Under assumption (2.172) (which is not necessary in space dimension $N = 2$), it is easy to see that (2.170) is equivalent to

$$1 \leq \alpha_i \leq \frac{2\mu + \lambda}{\lambda}, \quad \sum_{i=1}^N \frac{1}{\alpha_i} = N - \frac{2\mu}{2\mu + \lambda}. \quad (2.173)$$

Lemma 2.3.41 *The inverse Hooke's law A^{*-1} of an orthogonal rank- N sequential laminate is defined by the quadratic form*

$$A^{*-1}\sigma : \sigma = B^{-1}\sigma : \sigma + \frac{\theta}{2\mu(1-\theta)}G(\alpha_i, \sigma) \quad (2.174)$$

with

$$\begin{aligned} G(\alpha_i, \sigma) = & \sum_{i,j=1, i \neq j}^N \frac{\sigma_{ij}^2}{1 - m_i - m_j} + \sum_{i=1}^N \alpha_i \sigma_{ii}^2 \\ & - \frac{\lambda}{N\kappa} \left(\sum_{i=1}^N \sigma_{ii} \right)^2 + \frac{\lambda}{N\kappa} \frac{\left(\sum_{i=1}^N (\alpha_i - 1) \sigma_{ii} \right)^2}{1 - \frac{\lambda}{N\kappa} \sum_{i=1}^N \alpha_i}, \end{aligned}$$

where σ_{ij} denote the entries of a symmetric matrix σ in the orthonormal basis of lamination directions.

Remark 2.3.42 *The quadratic form (2.174) defines a coercive Hooke's law A^* in dimension $N \geq 3$ as soon as none of the parameters m_i is zero, that is, if the material is effectively laminated in all N directions e_i . (Indeed, $m_i > 0$ for all $1 \leq i \leq N$ implies that $1 - m_i - m_j > 0$ for all $1 \leq i, j \leq N$ and $i \neq j$.) Thus, in three dimensions an orthogonal rank-3 laminate is a realistic composite material. On the contrary, in two dimensions, we always have $1 - m_i - m_j = 0$. Thus, formula (2.174) is valid only for stresses σ that are diagonal in the basis of lamination directions (i.e., such that $\sigma_{ij} = 0$). In other words, in two dimensions, an orthogonal rank-2 laminate cannot support a stress whose eigendirections are not aligned with the lamination directions. This is an inconvenience shared with all other optimal microstructures (see Theorem 2.3.44 below). Note also that an orthogonal rank- N laminate is not isotropic even if all lamination parameters m_i are set equal to $1/N$.*

Remark 2.3.43 *In view of (2.170) and (2.172), it is easily checked that the denominator $1 - \lambda(N\kappa)^{-1} \sum_{i=1}^N \alpha_i$ in formula (2.174) is always nonnegative. Furthermore, it is equal to zero if and only if it corresponds to a rank-1 laminate (i.e., all m_i but one are equal to zero). In other words, an orthogonal laminate of rank at least 2 can support any stress that is aligned with its lamination directions (in any spatial dimension), while a rank-1 laminate can support only stresses orthogonal to its single lamination direction.*

Proof. The starting point is the lamination formula (2.151), which gives for any symmetric matrix ϵ

$$\theta \left[A^{*-1} - B^{-1} \right]^{-1} \epsilon = (1 - \theta) \sum_{i=1}^N m_i f_B^c(e_i) \epsilon. \quad (2.175)$$

Let us define a matrix σ by

$$\sigma = \sum_{i=1}^N m_i f_B^c(e_i) \epsilon.$$

With this definition (2.175) becomes

$$A^{*-1} \sigma = B^{-1} \sigma + \frac{\theta}{(1 - \theta)} \epsilon. \quad (2.176)$$

Thus, it remains to compute ϵ in terms of σ . The degenerate Hooke's law $f_B^c(e_i)$ is defined by (2.152). This yields

$$\sigma = B\epsilon - \frac{1}{\mu} B \sum_{i=1}^N m_i \left[((B\epsilon)e_i) \odot e_i - \frac{\mu + \lambda}{2\mu + \lambda} ((B\epsilon)e_i \cdot e_i) e_i \odot e_i \right]. \quad (2.177)$$

Since $B\epsilon = 2\mu\epsilon + \lambda(\text{tr}\epsilon)I_2$, formula (2.177) can be developed as follows:

$$\begin{aligned} \sigma = & 2\mu \sum_{\substack{i,j=1 \\ i \neq j}}^N (1 - m_i - m_j) \epsilon_{ij} e_i \odot e_j + 2\mu \sum_{i=1}^N \left[\left(1 - \frac{2\mu m_i}{2\mu + \lambda} \right) \epsilon_{ii} \right. \\ & \left. + \frac{\lambda}{2\mu + \lambda} (\text{tr}\epsilon)(1 - m_i) - \frac{\lambda}{2\mu + \lambda} \left(\sum_{j=1}^N m_j \epsilon_{jj} \right) \right] e_i \odot e_i. \end{aligned} \quad (2.178)$$

From formula (2.178), inverting the off-diagonal terms is easy

$$\epsilon_{ij} = \frac{\sigma_{ij}}{2\mu(1 - m_i - m_j)} \quad \text{if } i \neq j.$$

Using definition (2.171) of the parameters α_i , the diagonal terms are solutions of an $N \times N$ linear system

$$\epsilon_{ii} + \frac{\lambda}{2\mu + \lambda} \text{tr}\epsilon(1 - m_i) \alpha_i - \frac{\lambda}{2\mu + \lambda} \left(\sum_{j=1}^N m_j \epsilon_{jj} \right) \alpha_i = \frac{\alpha_i \sigma_{ii}}{2\mu}. \quad (2.179)$$

To invert system (2.179), we compute $\text{tr}\epsilon$ and $\sum_{j=1}^N m_j \epsilon_{jj}$ by summing adequately weighted lines of (2.179). This gives the following simple 2×2 system:

$$\begin{cases} \left(1 + \frac{\lambda \sum_{i=1}^N (1 - m_i) \alpha_i}{2\mu + \lambda}\right) \text{tr}\epsilon - \frac{\lambda \sum_{i=1}^N \alpha_i}{2\mu + \lambda} \sum_{j=1}^N m_j \epsilon_{jj} = \sum_{i=1}^N \frac{\alpha_i \sigma_{ii}}{2\mu} \\ \frac{\lambda \sum_{i=1}^N (1 - m_i) m_i \alpha_i}{2\mu + \lambda} \text{tr}\epsilon + \left(1 - \frac{\lambda \sum_{i=1}^N m_i \alpha_i}{2\mu + \lambda}\right) \sum_{j=1}^N m_j \epsilon_{jj} = \sum_{i=1}^N \frac{\alpha_i \sigma_{ii} m_i}{2\mu}. \end{cases}$$

A routine calculation leads to its solution

$$\begin{cases} \text{tr}\epsilon = \sum_{i=1}^N \frac{\alpha_i \sigma_{ii}}{N\kappa} + \frac{2\mu\lambda}{N\kappa(2\mu + \lambda)} \frac{(\sum_{i=1}^N \alpha_i)(\sum_{i=1}^N \alpha_i m_i \sigma_{ii})}{N\kappa - \lambda \sum_{i=1}^N \alpha_i} \\ \sum_{i=1}^N m_i \epsilon_{ii} = \frac{\lambda}{N\kappa(2\mu + \lambda)} \sum_{i=1}^N \alpha_i m_i \sigma_{ii} - \frac{\lambda}{N\kappa 2\mu} \sum_{i=1}^N \alpha_i \sigma_{ii} \\ \quad + \frac{2\mu}{2\mu + \lambda} \frac{\sum_{i=1}^N \alpha_i m_i \sigma_{ii}}{N\kappa - \lambda \sum_{i=1}^N \alpha_i}, \end{cases} \quad (2.180)$$

which has been simplified with the help of the following identities

$$\sum_{i=1}^N m_i^2 \alpha_i = \frac{2\mu + \lambda}{2\mu} \left(\sum_{i=1}^N m_i \alpha_i - 1 \right), \quad \sum_{i=1}^N m_i \alpha_i = \frac{2\mu + \lambda}{2\mu} \left(\sum_{i=1}^N \alpha_i - N \right).$$

Combining (2.179) and (2.180) gives the diagonal terms of ϵ in terms of those of σ . Finally, multiplying equation (2.176) by σ and replacing ϵ by its value in terms of σ yields the desired result (2.174). \square

Proof of Theorem 2.3.35. In space dimension $N = 2$ the computation is easier with the variable m_i instead of the parameter α_i . A simple computation shows that $G(\alpha_i, \sigma)$ defined by (2.174) is equal to

$$G(\alpha_i, \sigma) = \frac{2\mu + \lambda}{2\kappa} \left(\frac{\sigma_1^2}{m_2} + \frac{\sigma_2^2}{m_1} \right).$$

The optimal values of m_1, m_2 are then easily found to be those given by (2.154). \square

Proof of Theorem 2.3.36. To compute the lower optimal bound (2.142), it is enough to perform the following minimization

$$g^*(\sigma) = \min_{\alpha_i} G(\alpha_i, \sigma) \quad (2.181)$$

with

$$G(\alpha_i, \sigma) = \sum_{i=1}^3 \alpha_i \sigma_i^2 - \frac{\lambda}{3\kappa} \left(\sum_{i=1}^3 \sigma_i \right)^2 + \frac{\lambda}{3\kappa} \frac{\left(\sum_{i=1}^3 (\alpha_i - 1) \sigma_i \right)^2}{1 - \frac{\lambda}{3\kappa} \sum_{i=1}^3 \alpha_i} \quad (2.182)$$

where σ_i denote the eigenvalues of σ . Note that there is no contribution from the off-diagonal entries of σ since, by definition, σ is diagonal in the basis of the lamination directions. The minimization in (2.181) is subject to the constraint (2.173), i.e.,

$$1 \leq \alpha_i \leq \frac{2\mu + \lambda}{\lambda}, \quad (2.183)$$

which is equivalent to $0 \leq m_i \leq 1$, and

$$\sum_{i=1}^3 \frac{1}{\alpha_i} = 3 - \frac{2\mu}{2\mu + \lambda}, \quad (2.184)$$

which comes from $\sum_{i=1}^3 m_i = 1$.

Let us briefly explain our strategy for minimizing (2.182). First, by ignoring the constraint (2.183) (but not (2.184)), optimality conditions are easily obtained, which yield the values of the optimal parameters α_i in terms of σ . In a second step, the constraint (2.183) will be tested for those optimal parameters, and according to the value of σ there will be two cases. If the constraint is satisfied, then the minimum value of (2.182) is attained for a rank-3 sequential laminate corresponding to those parameters; if not, then one of the α_i is set equal to 1 (i.e., $m_i = 0$), and (2.182) will be minimized over two parameters only (corresponding to rank-2 sequential laminates).

Let us show that, in the minimization of (2.182) under the sole constraint (2.184), the optimal parameters α_i (if any) satisfy

$$\alpha_i = \frac{C}{|\sigma_i + D|}, \quad (2.185)$$

where the constant C is given in terms of D by

$$C = \left(3 - \frac{2\mu}{2\mu + \lambda} \right)^{-1} \sum_{i=1}^3 |\sigma_i + D|$$

and D is solution of the piecewise linear equation

$$D + \frac{\lambda}{3\kappa} \operatorname{tr}\sigma = \frac{(2\mu + \lambda)\lambda}{(2\mu + 3\lambda)(4\mu + 3\lambda)} \left(\sum_{i=1}^3 |\sigma_i + D| \right) \left(\sum_{i=1}^3 \frac{\sigma_i + D}{|\sigma_i + D|} \right). \quad (2.186)$$

The optimality condition, with the constraint that $\sum_{i=1}^3 \alpha_i^{-1}$ is fixed, is nothing other than

$$\frac{\partial G(\alpha_i, \sigma)}{\partial \alpha_k} = \frac{C^2}{\alpha_k^2}$$

for some positive constant C . Differentiating (2.182) gives

$$\frac{\partial G(\alpha_i, \sigma)}{\partial \alpha_k} = \left[\sigma_k + \frac{\lambda}{3\kappa} \frac{\sum_{i=1}^3 (\alpha_i - 1)\sigma_i}{1 - \frac{\lambda}{3\kappa} \sum_{i=1}^3 \alpha_i} \right]^2.$$

This yields (2.185) with the following value of D

$$D = \frac{\lambda}{3\kappa} \frac{\sum_{i=1}^3 (\alpha_i - 1)\sigma_i}{1 - \frac{\lambda}{3\kappa} \sum_{i=1}^3 \alpha_i}.$$

The constraint (2.184) gives the value of C in terms of D , while equation (2.186) is obtained from the above formula for D by replacing α_i by its value (2.185). One can also check that for the optimal α_i , defined by (2.185), the function $G(\alpha_i, \sigma)$ takes the value

$$\frac{2\mu + \lambda}{4\mu + 3\lambda} \left(\sum_{i=1}^3 |\sigma_i + D| \right) \left(\sum_{i=1}^3 \frac{\sigma_i + D}{|\sigma_i + D|} \sigma_i \right) - D \sum_{i=1}^3 \sigma_i - \frac{\lambda}{3\kappa} \left(\sum_{i=1}^3 \sigma_i \right)^2. \quad (2.187)$$

The next step is to solve equation (2.186) to compute the constant D . For this we use the assumption $\lambda \geq 0$. In three dimensions, labeling the eigenvalues of the stress σ such that

$$\sigma_1 \leq \sigma_2 \leq \sigma_3, \quad (2.188)$$

there are two basic cases to investigate for solving (2.186): the first one corresponds to $\sigma_1 + D \geq 0$, and the second one to $\sigma_2 + D \geq 0 \geq \sigma_1 + D$ (the two remaining cases $\sigma_3 + D \geq 0 \geq \sigma_2 + D$ and $0 \geq \sigma_3 + D$ are obtained from the previous cases by symmetry, changing σ in $-\sigma$).

(1) Assume $\sigma_1 + D \geq 0$.

Then equation (2.186) reduces to

$$D + \frac{\lambda}{3\kappa} \operatorname{tr}\sigma - \frac{(2\mu + \lambda)3\lambda}{(2\mu + 3\lambda)(4\mu + 3\lambda)} (\operatorname{tr}\sigma + 3D) = 0,$$

which gives the following value for D :

$$D = \frac{\lambda}{4\mu} (\sigma_1 + \sigma_2 + \sigma_3).$$

This yields

$$C = \frac{2\mu + \lambda}{4\mu} \operatorname{tr}\sigma \text{ and } \alpha_i = \frac{(2\mu + \lambda)\operatorname{tr}\sigma}{4\mu\sigma_i + \lambda\operatorname{tr}\sigma}. \quad (2.189)$$

However, the constraint (2.183) on the lamination parameters is

$$1 \leq \alpha_3 \leq \alpha_1 \leq \frac{2\mu + \lambda}{\lambda},$$

which, using (2.189), is easily seen to be equivalent to

$$\sigma_1 \geq 0 \text{ and } \sigma_3 \leq \sigma_1 + \sigma_2. \quad (2.190)$$

We remark that condition (2.190) automatically implies the assumption $\sigma_1 + D \geq 0$. In view of (2.187), the extremal value of (2.182) corresponding to (2.189) is

$$G(\alpha_i, \sigma) = \frac{2\mu + \lambda}{2(2\mu + 3\lambda)} (\sigma_1 + \sigma_2 + \sigma_3)^2. \quad (2.191)$$

Together with the admissibility condition (2.190), this is nothing other than regime (2.156) with (2.161) in Theorem 2.3.36.

(2) Assume $\sigma_2 + D \geq 0 \geq \sigma_1 + D$.

Then equation (2.186) reduces to

$$D + \frac{\lambda}{3\kappa} \operatorname{tr}\sigma - \frac{(2\mu + \lambda)\lambda}{(2\mu + 3\lambda)(4\mu + 3\lambda)} (\sigma_3 + \sigma_2 - \sigma_1 + D) = 0,$$

which gives the following value for D

$$D = \frac{-\lambda}{4(\mu + \lambda)^2} \left((\mu + \lambda)(\sigma_3 + \sigma_2) + (3\mu + 2\lambda)\sigma_1 \right).$$

This yields

$$C = \frac{2\mu + \lambda}{4(\mu + \lambda)^2} \left((\mu + \lambda)(\sigma_3 + \sigma_2) - (\mu + 2\lambda)\sigma_1 \right),$$

and

$$\left\{ \begin{array}{l} \alpha_1 = (2\mu + \lambda) \frac{(\mu + \lambda)(\sigma_3 + \sigma_2) - (\mu + 2\lambda)\sigma_1}{\lambda(\mu + \lambda)(\sigma_3 + \sigma_2) - (4\mu^2 + 5\mu\lambda + 2\lambda^2)\sigma_1} \\ \alpha_2 = (2\mu + \lambda) \frac{(\mu + \lambda)(\sigma_3 + \sigma_2) - (\mu + 2\lambda)\sigma_1}{-\lambda(\mu + \lambda)\sigma_3 + (\mu + \lambda)(4\mu + 3\lambda)\sigma_2 - \lambda(3\mu + 2\lambda)\sigma_1} \\ \alpha_3 = (2\mu + \lambda) \frac{(\mu + \lambda)(\sigma_3 + \sigma_2) - (\mu + 2\lambda)\sigma_1}{(\mu + \lambda)(4\mu + 3\lambda)\sigma_3 - \lambda(\mu + \lambda)\sigma_2 - \lambda(3\mu + 2\lambda)\sigma_1}. \end{array} \right. \quad (2.192)$$

Now the constraint (2.183) on the lamination parameters takes the form

$$1 \leq \alpha_3 \leq \alpha_2 \leq \frac{2\mu + \lambda}{\lambda} \text{ and } 1 \leq \alpha_1 \leq \frac{2\mu + \lambda}{\lambda},$$

which, combined with (2.192), and after a few lines of calculation (it helps to note that the denominator of each α_i is positive), leads to

$$\left\{ \begin{array}{l} \sigma_2 \geq 0 \geq \sigma_1 \\ \sigma_3 - \sigma_2 \leq \frac{-\mu}{\mu + \lambda}\sigma_1 \\ \sigma_3 + \sigma_2 \geq \frac{-\mu}{\mu + \lambda}\sigma_1. \end{array} \right. \quad (2.193)$$

A tedious, but simple, computation shows that (2.193) automatically implies the assumption $\sigma_2 + D \geq 0 \geq \sigma_1 + D$. In view of (2.187), the extremal value of (2.182) corresponding to (2.192) is

$$G(\alpha_i, \sigma) = \frac{2\mu + \lambda}{2(2\mu + 3\lambda)} \left(\sigma_3 + \sigma_2 - \frac{\mu + 2\lambda}{\mu + \lambda}\sigma_1 \right)^2. \quad (2.194)$$

Together with the admissibility condition (2.193), this is nothing other than regime (2.158) with (2.163) in Theorem 2.3.36.

If condition (2.190) ((2.193) resp.) is not satisfied in case (1) (case (2), respectively), then $G(\alpha_i, \sigma)$ does not attain its extrema inside the domain defined by the constraints (2.183), (2.184), but rather on the boundaries of that domain, which are made of rank-2 laminates. Let us consider the case of rank-2 laminates in the directions e_1 and e_2 , i.e., $m_3 = 0$, (the other cases

will be obtained by symmetry). Taking into account that $\alpha_3 = 1$, we now have to minimize the simplified expression of $G(\alpha_i, \sigma)$:

$$\alpha_1\sigma_1^2 + \alpha_2\sigma_2^2 + \sigma_3^2 - \frac{\lambda}{3\kappa} \left(\sum_{i=1}^3 \sigma_i \right)^2 + \frac{\lambda}{3\kappa} \frac{((\alpha_1 - 1)\sigma_1 + (\alpha_2 - 1)\sigma_2)^2}{1 - \frac{\lambda}{3\kappa}(\alpha_1 + \alpha_2 + 1)}$$

with the new constraints

$$1 \leq \alpha_1 \leq \frac{2\mu + \lambda}{\lambda}, \quad 1 \leq \alpha_2 \leq \frac{2\mu + \lambda}{\lambda}, \quad (2.195)$$

and

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = \frac{2(\mu + \lambda)}{2\mu + \lambda}. \quad (2.196)$$

The optimality conditions under the sole constraint (2.196) are still of the same type as in (2.185), and the optimal parameters are given by

$$\alpha_1 = \frac{E}{|\sigma_1 + F|}, \quad \alpha_2 = \frac{E}{|\sigma_2 + F|},$$

where the constant E is given in terms of F by

$$E = \frac{2\mu + \lambda}{2(\mu + \lambda)} (|\sigma_1 + F| + |\sigma_2 + F|)$$

and F is solution of the piecewise linear equation

$$F + \frac{\lambda}{2(\lambda + \mu)} (\sigma_1 + \sigma_2) - \frac{(2\mu + \lambda)\lambda}{4(\mu + \lambda)^2} (|\sigma_1 + F| + |\sigma_2 + F|) \left(\frac{\sigma_1 + F}{|\sigma_1 + F|} + \frac{\sigma_2 + F}{|\sigma_2 + F|} \right) = 0. \quad (2.197)$$

The solution of (2.197) is similar to that of (2.186), but a lot simpler since the corresponding value of the optimal parameters always satisfy the remaining constraints (2.195). Keeping in mind the labeling convention (2.188), there are again two basic cases (the remaining ones being obtained by symmetry, changing σ in $-\sigma$).

(A) Assume $\sigma_1 + F \geq 0$.

Then the solution of (2.197) is $F = \lambda(\sigma_1 + \sigma_2)/(2\mu)$, and the corresponding parameters are

$$\alpha_1 = \frac{(2\mu + \lambda)(\sigma_1 + \sigma_2)}{\lambda\sigma_2 + (2\mu + \lambda)\sigma_1}, \quad \alpha_2 = \frac{(2\mu + \lambda)(\sigma_1 + \sigma_2)}{(2\mu + \lambda)\sigma_2 + \lambda\sigma_1}.$$

The constraint (2.195) is equivalent to

$$\sigma_1 \geq 0,$$

while

$$G(\alpha_i, \sigma) = (\sigma_1 + \sigma_2)^2 + \sigma_3^2 - \frac{\lambda}{2\mu + 3\lambda} (\text{tr}\sigma)^2. \quad (2.198)$$

(B) Assume $\sigma_2 + F \geq \sigma_1 + F$.

Then the solution of (2.197) is $F = -\lambda(\sigma_1 + \sigma_2)/(2\mu + 2\lambda)$, and the corresponding parameters are

$$\alpha_1 = \frac{(2\mu + \lambda)(\sigma_2 - \sigma_1)}{\lambda\sigma_2 - (2\mu + \lambda)\sigma_1}, \quad \alpha_2 = \frac{(2\mu + \lambda)(\sigma_2 - \sigma_1)}{(2\mu + \lambda)\sigma_2 - \lambda\sigma_1}.$$

The constraint (2.195) is equivalent to

$$\sigma_2 \geq 0 \geq \sigma_1,$$

while

$$G(\alpha_i, \sigma) = (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{2\mu}{\mu + \lambda} \sigma_1 \sigma_2 - \frac{\lambda}{2\mu + 3\lambda} (\text{tr}\sigma)^2. \quad (2.199)$$

The cases of rank-2 laminates in other directions (i.e., $m_1 = 0$ or $m_2 = 0$) are obtained by simply permuting the indexes 1, 2, 3 in the above formulas. As a matter of fact, the extremal values (2.198) and (2.199) of $G(\alpha_i, \sigma)$ are minimum values among rank-2 laminates, since, in the limit of rank-1 laminates, $G(\alpha_i, \sigma)$ goes to infinity (see Remark 2.3.43). Furthermore, these minimum values are easily checked to be always larger than the extremal values (2.191) and (2.194) for rank-3 laminates, which are therefore minimum values themselves.

The proof of Theorem 2.3.36 can now be completed by simply seeking the best rank-2 laminates in the event that the admissibility conditions for the existence of an optimal rank-3 laminate are not satisfied in cases (1) and (2) above. We can safely leave to the reader the task of comparing the different optimal rank-2 laminates (i.e., $m_1 = 0$, $m_2 = 0$, or $m_3 = 0$). We simply indicate the final result. If the compatibility condition (2.190) in case (1) is not satisfied, then the minimum of $G(\alpha_i, \sigma)$ is attained for a rank-2 laminate corresponding to case (A) above (this is regime (2.157) with (2.162) in Theorem 2.3.36). If the compatibility condition (2.193) in case (2) does not hold, then the minimum of $G(\alpha_i, \sigma)$ is attained for one of

the following two rank-2 laminates: if $\sigma_3 - \sigma_2 \geq -\mu(\mu + \lambda)^{-1}\sigma_1$, case (B) above is optimal (this is regime (2.160) with (2.165) in Theorem 2.3.36), and if $\sigma_3 + \sigma_2 \leq -\mu(\mu + \lambda)^{-1}\sigma_1$, interchanging directions 1 and 3 in case (A) (i.e., $m_1 = 0$) gives the optimal result (this is regime (2.159) with (2.164) in Theorem 2.3.36). \square

As already noted in Remark 2.3.42, an orthogonal rank-2 sequential laminate is degenerate in two space dimensions, namely it cannot sustain a stress not aligned with its lamination directions. As a result of Lemma 2.3.41, its Hooke's law has a zero component, $A_{1212}^* = 0$. In other words, this rank-2 sequential laminate has no shear rigidity. Therefore, the proposed optimal microstructure in Theorem 2.3.35 has a singular (noncoercive) Hooke's law. Although it does not mean that this structure is unstable for any load condition, it is generically impossible to solve the corresponding elasticity equations. This is an undesirable feature, in view of the numerical algorithms discussed in Chapter 5, since they require iteratively the solution of elasticity problems with the microstructure that is optimal for the previous iteration. Such a difficulty is peculiar to the two-dimensional case because an orthogonal rank-3 sequential laminate in three dimensions does not suffer from this degeneracy property if all lamination parameters $(m_i)_{1 \leq i \leq 3}$ are nonzero.

It is therefore an interesting question to ask whether there exists or not other extremal microstructures that achieve optimality in the two-dimensional lower Hashin-Shtrikman dual bound (2.153). There are several examples of other optimal microstructures in the literature. Mainly, they are the concentric sphere assemblages of Hashin [131], the confocal ellipsoid assemblages of Bergman [51], [52], Milton [187], and Tartar [274] (extended to the elasticity setting by Grabovsky [122], [124]), and the Vigdergauz periodic inclusions [285], [125]. Unfortunately, all these microstructures are conditionally optimal, i.e., they are optimal only in some regimes of the bound, namely when $\det \sigma > 0$.

In [10] it is proved that, if $\det \sigma \leq 0$, then any optimal Hooke's law in the two-dimensional lower Hashin-Shtrikman dual bound (2.153) is actually degenerate. In the other regime, $\det \sigma > 0$, the authors of [10] exhibit a higher order sequential laminate, which is both optimal and nondegenerate. In other words, the numerical problem of dealing with noncoercive Hooke's law cannot be alleviated if $\det \sigma \leq 0$, while a simple remedy is available if $\det \sigma > 0$.

Theorem 2.3.44 Let $A^* \in G_\theta$ be the Hooke's law of an optimal microstructure for the lower Hashin-Shtrikman bound (2.153), i.e.,

$$A^{*-1}\sigma : \sigma = B^{-1}\sigma : \sigma + \frac{(\kappa + \mu)\theta}{4\kappa\mu(1-\theta)} (|\sigma_1| + |\sigma_2|)^2.$$

If $\det \sigma \leq 0$, then, A^* is degenerate, as the rank-2 sequential laminate, in the sense that, in the eigenbasis of σ , it satisfies

$$A_{1212}^* = 0 \text{ and } (A^{*-1})_{1212} = +\infty,$$

i.e., it cannot sustain a nonaligned shear stress.

Remark 2.3.45 In a recent paper [78] Cherkaev, Grabovsky, Movchan, and Serkov found the optimal shape of a simply connected hole in an infinite elastic plane submitted to a shear stress at infinity (i.e., $\det \sigma < 0$). This problem is equivalent, in the low volume limit (i.e., $\theta \rightarrow 0$), to that of finding the optimal periodic microstructure under the restriction that the hole is simply connected (which is not the case for the rank-2 laminate). Of course, the homogenized properties of such a perforated microstructure is not degenerate. However, they remarked (see Section 1.2 in [78]) that the energy of their microstructure is significantly higher than that of the rank-2 laminate, which is consistent with Theorem 2.3.44.

Lemma 2.3.46 Assume that $\det \sigma > 0$. Denoting by σ_1, σ_2 the eigenvalues of σ , and by n_1, n_2 the corresponding unit orthogonal eigenvectors, there exists a nondegenerate rank-4 laminate achieving optimality in the lower Hashin-Shtrikman bound (2.153), which is defined by the lamination directions

$$e_1 = n_1, \quad e_2 = n_2, \quad e_3 = \frac{n_1 + n_2}{|n_1 + n_2|}, \quad e_4 = \frac{n_1 - n_2}{|n_1 - n_2|},$$

and the lamination parameters

$$m_1 = \frac{|\sigma_2|}{2(|\sigma_1| + |\sigma_2|)}, \quad m_2 = \frac{2|\sigma_1| - |\sigma_2|}{2(|\sigma_1| + |\sigma_2|)}, \quad m_3 = m_4 = m_1,$$

where, without loss of generality, we have assumed that $|\sigma_1| \geq |\sigma_2|$.

Remark 2.3.47 The condition $\det \sigma > 0$ for having optimal higher rank sequential laminate is not a surprise. This is precisely the assumption used

by Grabovsky and Kohn in [124], [125] to show that the so-called confocal ellipsoid assemblages in [51], [52], [187], [274] and the periodic constructions of Vigdergauz [285] are two other types of optimal microstructures for the Hashin-Shtrikman bound (2.153). Of course, such composites are not optimal when $\det \sigma < 0$, and when $\det \sigma = 0$ they all degenerate to the optimal rank-1 sequential laminate.

Proof of Theorem 2.3.44. Although we already have an explicit formula for the lower bound (2.153), we briefly recall how it was derived in order to shed light on its optimality condition. Recall that, from Theorem 2.3.11, we know that for any tensor A^* in G_θ there exists a sequential laminate $A^+ \in L_\theta^+$ such that

$$A^* \leq A^+. \quad (2.200)$$

In truth, (2.200) holds true for a two-phase composite where neither of the phases is degenerate. But passing to the limit $A \rightarrow 0$ is always possible, and A^+ has to be understood as given by Definition 2.3.34. This procedure is compatible with our implicit definition of the lower Hashin-Shtrikman bound (2.153) which is defined as the limit when A goes to zero of the nondegenerate two-phase bound. Therefore, (2.200) yields

$$A^{*-1} \geq A^{+-1}. \quad (2.201)$$

On the other hand, by definition of L_θ^+ (see Lemma 2.3.6),

$$A^{+-1} = B^{-1} + \frac{\theta}{(1-\theta)} \left(\int_{S_{N-1}} f_B^c(e) d\nu(e) \right)^{-1} \quad (2.202)$$

where ν is a probability measure (the H -measure of the characteristic function of one phase). Thus, using the Legendre transform, we obtain

$$A^{*-1} \sigma : \sigma \geq B^{-1} \sigma : \sigma + \frac{\theta}{(1-\theta)} \max_{\eta \in \mathcal{M}_N^s} (2\sigma : \eta - g_\nu^c(\eta)),$$

where

$$g_\nu^c(\eta) = \int_{S_{N-1}} f_B^c(e) \eta : \eta d\nu(e).$$

Finally, minimizing with respect to the probability measure ν yields the lower bound

$$A^{*-1} \sigma : \sigma \geq B^{-1} \sigma : \sigma + \frac{\theta}{(1-\theta)} \max_{\eta \in \mathcal{M}_N^s} (2\sigma : \eta - g^c(\eta)) \quad (2.203)$$

where g^c defined by (2.138) is equivalently given by

$$g^c(\eta) = \max_{\nu \in \mathcal{P}(S_{N-1})} g_\nu^c(\eta). \quad (2.204)$$

From (2.141) an easy computation shows that

$$f_B^c(e)\eta : \eta = \frac{4\kappa\mu}{\kappa + \mu} \left(|\eta|^2 - 2|\eta e|^2 + (\eta e \cdot e)^2 \right), \quad (2.205)$$

which allows one to check that the bound (2.203) is indeed equal to (2.153). To evaluate $g^c(\eta)$, we maximize (2.205) on the unit sphere. Decomposing the vector e on the eigenbasis of the symmetric matrix η , an easy maximization yields

$$g^c(\eta) = \frac{4\kappa\mu}{\kappa + \mu} \max(\eta_1^2, \eta_2^2),$$

where η_1 and η_2 are the two eigenvalues of η , and the corresponding optimal vector e are eigenvectors of η associated to the eigenvalues of largest absolute value. The main new fact here is that the optimal probability measures in (2.204) have their support included in the eigenspaces of those eigenvalues of largest absolute value.

We now compute these η 's which are optimal in the bound (2.203). Since $g^c(\eta)$ depends only on the eigenvalues of η , the optimal η in (2.203) must be simultaneously diagonal with σ because this is so for the inner product $\sigma : \eta$. For $\sigma_1\sigma_2 \neq 0$, the unique optimal choice is easily seen to be $\eta_1 = \text{sgn}(\sigma_1)t$ and $\eta_2 = \text{sgn}(\sigma_2)t$ with

$$t = \frac{\kappa + \mu}{4\kappa\mu}(|\sigma_1| + |\sigma_2|).$$

When one of the eigenvalues of σ is zero, say $\sigma_2 = 0$, an optimal η is only constrained by

$$\eta_1 = \frac{\kappa + \mu}{4\kappa\mu}\sigma_1, \text{ and } |\eta_2| \leq |\eta_1|.$$

When $\det \sigma < 0$, the optimal η in (2.203) is thus

$$\eta = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix},$$

which has two distinct eigenvalues. In this case, e maximizes (2.205) if and only if it is one of the two eigenvectors of σ . Therefore, the support of the optimal measure ν in (2.204) is concentrated on at most two directions.

When $\det \sigma > 0$, the optimal η in (2.203) is $\eta = tI_2$. Then (2.205) is a constant and every e is a maximizer. In this case, there are no restrictions on the support of the optimal measure ν . Finally, when $\det \sigma = 0$ (and $\sigma \neq 0$), the optimal η is not unique, but it can be chosen to be proportional to σ . Then e maximizes (2.205) if and only if it is an eigenvector of σ for its unique nonzero eigenvalue. In such a case, the optimal measure ν is a Dirac mass supported in only one direction.

Assume now that A^* is an optimal tensor in the Hashin-Shtrikman bound (2.153) or (2.203). By inequality (2.201), the sequential laminate A^+ is also optimal. Its associated measure ν , through (2.202), is thus optimal in (2.204), and, since we assumed that $\det \sigma \leq 0$, it is supported on at most the two eigendirections of σ . Therefore, A^+ is an orthogonal sequential laminate of rank at most 2, and as noted in Remark 2.3.42, it is degenerate, i.e., $A_{1212}^+ = 0$ and $A_{1212}^{+-1} = \infty$. By virtue of inequality (2.201), this is also the case for A^* . \square

Proof of Lemma 2.3.46. We are seeking a rank-4 sequential laminate composite $A_L^* \in L_\theta^+$ that saturates the Hashin-Shtrikman bound (2.153), i.e.,

$$A_L^{*-1}\sigma : \sigma = B^{-1}\sigma : \sigma + \frac{\theta}{(1-\theta)} \frac{\kappa + \mu}{4\kappa\mu} (|\sigma_1| + |\sigma_2|)^2. \quad (2.206)$$

By Definition 2.3.34, A_L^* satisfies

$$A_L^{*-1}\sigma : \sigma = B^{-1}\sigma : \sigma + \frac{\theta}{(1-\theta)} \left[\sum_{i=1}^4 m_i f_B^c(e_i) \right]^{-1} \sigma : \sigma$$

with $f_B^c(e_i)$ defined by (2.152) and the lamination directions denoted by $(e_i)_{i=1,\dots,4}$. Denoting by n_1, n_2 an orthonormal basis of eigenvectors of σ , we fix the unit lamination directions to be equal to

$$e_1 = n_1, \quad e_2 = n_2, \quad e_3 = \frac{n_1 + n_2}{\sqrt{2}}, \quad e_4 = \frac{n_1 - n_2}{\sqrt{2}}.$$

Our goal is to compute the parameters $(m_i)_{i=1,\dots,4}$ such that (2.206) holds true. The main task is to invert the (possibly degenerate) fourth order tensor $\sum_{i=1}^4 m_i f_B^c(e_i)$. For a unit vector v , with components v_1, v_2 such that $v_1^2 + v_2^2 = 1$, and a symmetric matrix η , we have

$$f_B^c(v)\eta = \frac{4\kappa\mu}{\kappa + \mu} (v_2^2 \eta_{11} - 2v_1 v_2 \eta_{12} + v_1^2 \eta_{22}) \begin{bmatrix} v_2^2 & -v_1 v_2 \\ -v_1 v_2 & v_1^2 \end{bmatrix}.$$

Applying it to each e_i and imposing $m_3 = m_4$ gives

$$\sum_{i=1}^4 m_i f_B^c(e_i) \eta = \frac{2\kappa\mu}{\kappa + \mu} \begin{bmatrix} 2m_2\eta_{11} + m_3(\eta_{11} + \eta_{22}) & m_3\eta_{12} \\ m_3\eta_{12} & 2m_1\eta_{22} + m_3(\eta_{11} + \eta_{22}) \end{bmatrix}.$$

Inverting the relationship

$$\tau = \left[\sum_{i=1}^4 m_i f_B^c(e_i) \right] \eta,$$

yields

$$\eta = \frac{\kappa + \mu}{4\kappa\mu} \begin{bmatrix} \frac{2m_1\tau_{11} - m_3(\tau_{11} + \tau_{22})}{2m_1m_2 + m_3(m_1 + m_2)} & \frac{\tau_{12}}{m_3} \\ \frac{\tau_{12}}{m_3} & \frac{2m_2\tau_{22} - m_3(\tau_{11} + \tau_{22})}{2m_1m_2 + m_3(m_1 + m_2)} \end{bmatrix}.$$

This gives explicitly the value of $A_L^{*-1}\tau$ for any symmetric matrix τ , namely,

$$A_L^{*-1}\tau = B^{-1}\tau + \frac{\theta}{1-\theta}\eta.$$

We now apply this formula to σ , which is diagonal in the (n_1, n_2) basis, and restrict ourselves to the case where $\det \sigma > 0$. Equalizing both sides of (2.206) leads to

$$m_1 = m_2 + \frac{|\sigma_2| - |\sigma_1|}{|\sigma_2| + |\sigma_1|}. \quad (2.207)$$

Together with the constraints $m_4 = m_3$ and $m_1 + m_2 + m_3 + m_4 = 1$, (2.207) characterizes many possible choices of optimal rank-4 sequential laminates. For simplicity, assuming that $|\sigma_1| \geq |\sigma_2|$, we choose

$$m_1 = \frac{|\sigma_2|}{2(|\sigma_1| + |\sigma_2|)}, \quad m_2 = \frac{2|\sigma_1| - |\sigma_2|}{2(|\sigma_1| + |\sigma_2|)}, \quad m_3 = m_4 = m_1.$$

For this special choice, we check that A_L^* is a nondegenerate rank-4 laminate. Introducing $m_0 = m_1(m_1 + 3m_2)$, A_L^* is given by

$$\begin{aligned} \left(A_L^{*-1}\right)_{1111} &= \frac{(\kappa + \mu)(m_0(1 - \theta) + 3m_1\theta)}{4\kappa\mu m_0(1 - \theta)}, \\ \left(A_L^{*-1}\right)_{1122} = \left(A_L^{*-1}\right)_{2211} &= \frac{(\mu - \kappa)m_0(1 - \theta) - (\kappa + \mu)m_1\theta}{4\kappa\mu m_0}, \\ \left(A_L^{*-1}\right)_{2222} &= \frac{(\kappa + \mu)(m_0(1 - \theta) + (2m_2 + m_1)\theta)}{4\kappa\mu m_0(1 - \theta)}, \\ \left(A_L^{*-1}\right)_{1212} &= \frac{2\kappa m_1(1 - \theta) + (\kappa + \mu)\theta}{4\kappa\mu m_1}. \end{aligned}$$

As $m_0, m_1 \neq 0$, all entries of A_L^{*-1} , and in particular $\left(A_L^{*-1}\right)_{1212}$, are finite, i.e., the Hooke's law A_L^* is nondegenerate when $\det \sigma > 0$. \square

Chapter 3

Optimal Design in Conductivity

This chapter is concerned with optimal design problems in the conductivity setting that we shall treat by the homogenization method. There is a huge variety of optimal design problems, but we focus only on *two-phase optimization problems*. Such problems are defined as one in which we seek the optimal distribution of two components in a given domain that minimizes a criterion (also called an *objective function*), computed through the solution of a partial differential equation modeling the conductivity of the whole domain (the *state equation*). This type of problem includes, as a limit case, *shape optimization problems*. These are defined as ones in which we seek the shape of a domain (filled with a single material) that minimizes a criterion, computed again through the solution of a state equation. Indeed, if in a two-phase problem the conductivity of one of the components is allowed to go to zero, then, in the limit, this weak component mimics void or holes in the domain, supporting homogeneous Neumann boundary conditions. Therefore, a two-phase problem yields a shape optimization problem, where the boundaries, as well as the topology of the holes (i.e., their number and connectivity), are subject to optimization. In other words, the weak phase with very small conductivity properties mimics holes with current-free boundary conditions. From a physical point of view, this limit procedure is clear, but its mathematical justification is delicate and not always known.

The conductivity setting is also relevant to a few peculiar problems in structural mechanics. For example, the torsion of an elastic cylindrical bar reduces to a plane conductivity problem in which the unknown is the axial

displacement. Likewise, an antiplane shear elasticity problem degenerates into a plane conductivity equation.

There are other classes of optimal design problems that can be dealt with by the homogenization method (not to mention those that cannot). For example, multiphase optimization, orientation optimization for a non-isotropic material, and thickness optimization for plates (see, e.g., [42], [54], [248], [249]). However, for the sake of brevity, and since in this case the theory is more complete, we content ourselves in dealing only with two-phase optimization problems. In principle the main ideas of the homogenization method are the same for all these problems, but the technical difficulties may greatly vary according to the context. There are also several other relevant techniques for studying optimal design problems. We shall not discuss them, but simply refer the reader to the introduction for additional references.

The content of this chapter is the following. In a first section we introduce a setting for two-phase optimal design in conductivity. We explain how shape optimization can be obtained in the limit of one degenerate phase. We proceed to a first mathematical analysis and exhibit counterexamples for which no optimal design exists in the proposed class of admissible designs. To overcome this generic difficulty of nonexistence of solutions, we introduce in a second section the homogenization method that allows one to define generalized designs (actually composite materials, as studied in Chapter 2) and to relax the problem, i.e., to prove the existence of generalized optimal designs. The homogenization method also yields optimality conditions, which are at the root of new numerical algorithms. Our exposition is greatly influenced by the pioneering works and ideas of Murat and Tartar [205], [206], [269]. Let us also mention the independent pioneering research of Lurie, Cherkaev, and their coworkers [177], [178], and that of Raitum [227].

3.1 Setting of Optimal Shape Design

In this section we introduce a framework for the optimal design of a mixture of two isotropic conductors in a fixed domain that minimizes a given objective function. We show that this problem is generically ill-posed, i.e., that it usually does not admit a solution in the class of admissible designs. Finally, we indicate how this two-phase optimization problem is linked with a shape optimization problem when one of the phases is degenerate, i.e., when it has a vanishing conductivity.

3.1.1 Definition of a Model Problem

Let Ω be a fixed domain (a bounded open set in \mathbb{R}^N). This domain is occupied by two constituent media (see Figure 3.1) with constant isotropic conductivity α and β such that

$$0 < \alpha < \beta < +\infty.$$

We denote by $\chi(x)$ the characteristic function of that part of the domain Ω occupied by phase α , i.e., $\chi(x) = 1$ if phase α is present at point x , and $\chi(x) = 0$ if, on the contrary, phase β is present there. The overall conductivity in Ω is denoted by a_χ , defined as

$$a_\chi(x) = \alpha\chi(x) + \beta(1 - \chi(x)). \quad (3.1)$$

The governing equation for this mixture of α and β (characterized by χ) is the partial differential equation modeling thermal or electrical conductivity in the domain Ω . Both materials α and β are assumed to be linear, and a perfect bonding (i.e., perfect transmission conditions) is supposed to hold between the phases.

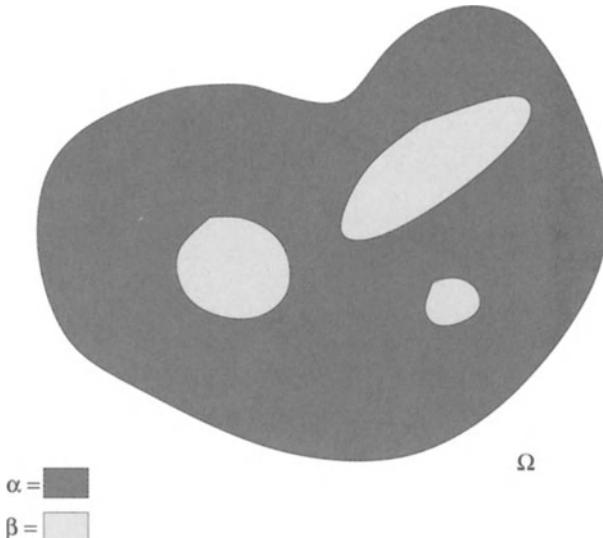


Figure 3.1: A two-phase domain.

A generic model problem is obtained by taking a given source term $f(x)$ in Ω (a scalar function independent of χ) and enforcing a Dirichlet boundary

condition (for simplicity) on $\partial\Omega$. Then the state equation reads

$$\begin{cases} -\operatorname{div}(a_\chi \nabla u_\chi) = f & \text{in } \Omega \\ u_\chi = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.2)$$

where $u_\chi(x)$ is the unknown function, modeling electrical potential or temperature. The performance of the mixture is measured by an objective function J , defined as

$$J(\chi) = \int_{\Omega} [\chi(x)g_\alpha(x, u_\chi(x)) + (1 - \chi(x))g_\beta(x, u_\chi(x))] dx, \quad (3.3)$$

where g_α and g_β are sufficiently smooth functions, depending only on the point x and on the value of the unknown $u_\chi(x)$ at the same point. An optimal design is a characteristic function χ that minimizes J . Typical examples of such objective functions J are

$$J(\chi) = \int_{\Omega} f(x)u_\chi(x)dx,$$

which is the energy dissipated in Ω , or

$$J(\chi) = \int_{\Omega} [\chi(x)|u_\chi(x) - u_\alpha(x)|^2 + (1 - \chi(x))|u_\chi(x) - u_\beta(x)|^2] dx,$$

where u_α and u_β are prescribed target functions for the unknown u_χ .

To give a precise mathematical statement of this optimization problem, we now introduce the appropriate function spaces. First of all, the *space of admissible designs*, i.e., the space over which the functional $J(\chi)$ is minimized, is defined as the set of all measurable characteristic functions in Ω having a prescribed volume V_α (with $0 \leq V_\alpha \leq |\Omega|$, where $|\Omega|$ is the measure or volume of Ω). In other words, the amount of material α is supposed to be given by V_α . An admissible design is therefore a function $\chi(x)$ such that

$$\chi \in L^\infty(\Omega; \{0, 1\}), \text{ and } \int_{\Omega} \chi(x)dx = V_\alpha.$$

(Recall that $L^\infty(\Omega; \{0, 1\})$ is the space of measurable functions in Ω taking only the values zero and one.) This implies that the conductivity $a_\chi(x)$ is a coercive function in $L^\infty(\Omega)$, and assuming that the source term $f(x)$ belongs to $L^2(\Omega)$, it yields that (3.2) admits a unique solution u_χ in $H_0^1(\Omega)$. Finally, we require the objective function $J(\chi)$ to be well-defined on $L^\infty(\Omega)$, and even more to be continuous for the weak topologies of χ and u_χ . More

precisely, we ask that the functions $u(x) \rightarrow g_\alpha(x, u(x)), g_\beta(x, u(x))$ be continuous from $H_0^1(\Omega)$, endowed with its weak topology, into $L^1(\Omega)$, with its strong topology. Such a criterion is met if $g_{\alpha,\beta}(x, \lambda)$, defined from $\Omega \times \mathbb{R}$ into \mathbb{R} , are Carathéodory functions and satisfy a suitable growth condition at infinity in λ . For example, we assume

$$\begin{cases} x \rightarrow g_{\alpha,\beta}(x, \lambda) \text{ measurable } \forall \lambda \in \mathbb{R} \\ \lambda \rightarrow g_{\alpha,\beta}(x, \lambda) \text{ continuous a.e. } x \in \Omega \\ |g_{\alpha,\beta}(x, \lambda)| \leq k(x) + C\lambda^m \text{ with } k(x) \in L^1(\Omega), 1 \leq m < \frac{2N}{N-2}. \end{cases} \quad (3.4)$$

(In dimensions $N = 1$ or 2 , the above condition on the exponent m has to be understood in the sense that $1 \leq m < +\infty$.) By the usual Sobolev embedding theorem (see Lemma 1.2.7 or, e.g., [5], [58]), (3.4) implies that $u \rightarrow g_{\alpha,\beta}(x, u)$ is indeed continuous from $H_0^1(\Omega)$ -weak into $L^1(\Omega)$ -strong. Our optimal design problem is the minimization

$$\inf_{\substack{\chi \in L^\infty(\Omega; \{0,1\}), \\ \int_\Omega \chi(x) dx = V_\alpha}} J(\chi), \quad (3.5)$$

where J is defined by (3.3) and the state equation is (3.2). The motivation of (3.5) is the design of a macroscopic conducting device in Ω with desired properties under the action of the source term f . If, for example, the objective function J measures an overall resistivity in Ω , its global minimum would certainly be reached by using only the best conductor β . However, since there is a volume constraint on the poor conductor α , we are forced to use not only β but also a fixed proportion of α , which makes (3.5) a highly nontrivial problem.

The minimization in (3.5) is constrained by the prescribed volume fraction of material α . Such a constraint is routinely handled in elementary calculus of variations through the introduction of a Lagrange multiplier $\ell \in \mathbb{R}$. Therefore, a variant of (3.5) is the following:

$$\inf_{\chi \in L^\infty(\Omega; \{0,1\})} J(\chi) + \ell \int_\Omega \chi(x) dx. \quad (3.6)$$

For any value of ℓ there exists a volume constraint V_α such that (3.6) is equivalent to (3.5). However, the converse is not so clear in general (depending on J), and for each value of V_α one has to check whether there exists a Lagrange multiplier ℓ such that (3.5) is equivalent to (3.6). In the sequel, we shall consider (3.6) an optimization problem of interest by itself, and we shall not dwell on its equivalence with (3.5).

An essential feature of (3.5) and (3.6) is that the space of admissible designs contains no smoothness or topology restrictions. Indeed, apart from being measurable the characteristic function χ does not satisfy any further property. In other words, the region occupied by α is not necessarily smooth or connected, and may have a wild boundary. From a physical point of view, it means that we do not take into account any feasibility constraint on the mixture of α and β . Such a constraint could be that a minimal lengthscale, a number of connected components, or bounds on the interface curvature are prescribed. From a mathematical point of view, this is a crucial point for being able to apply the homogenization method. Indeed, if additional assumptions on the admissible design χ are made, then, as we shall see, homogenization may be irrelevant and other methods could be applied.

We propose another model problem, very close to the first one. We keep the same objective function, but we change slightly the state equation. Indeed, we replace the source term by nonhomogeneous boundary conditions. This approach is often more adapted to real-life problems where the body forces are negligible relative to the applied transmission conditions on the boundary. Instead of (3.2) we consider the following equation for u_χ :

$$\begin{cases} -\operatorname{div}(a_\chi \nabla u_\chi) = 0 & \text{in } \Omega \\ u_\chi = u_0 & \text{on } \Gamma_D \\ a_\chi \nabla u_\chi \cdot n = \sigma_0 \cdot n & \text{on } \Gamma_N, \end{cases} \quad (3.7)$$

where Γ_D and Γ_N make a disjoint partition of $\partial\Omega$. If u_0 and σ_0 are smooth enough (namely, there exists an extension $\tilde{u}_0 \in H^1(\Omega)$ such that $\tilde{u}_0 = u_0$ on Γ_D and $a_\chi \nabla u_0 \cdot n = \sigma_0 \cdot n$ on Γ_N), then there exists a unique solution u_χ of (3.7) in $H^1(\Omega)$. Of course, if Γ_D is empty, then σ_0 must satisfy the compatibility condition $\int_{\partial\Omega} \sigma_0 \cdot n = 0$, which expresses the global equilibrium of the imposed normal flux. The motivation for (3.7) is that there are often no source terms in conducting devices, the purpose of which is to transmit an electric current or a heat flux under the action of a potential difference or temperature gradient on the boundary. Another motivation of (3.7) comes from the fact that, in this problem, it is possible to let the poor conductor α degenerate to zero, i.e., to replace it by holes. This is not possible with (3.5), since there could be no source term f in a hole.

Remark 3.1.1 *The objective function J depends on only the space variable x , the design parameter χ , and the value of the potential or temperature $u_\chi(x)$. It is perfectly legitimate to imagine that J could also depend on the gradient ∇u_χ or on higher derivatives. For example, J could depend on the*

heat flux or current $a_\chi \nabla u_\chi$ in order to optimize its distribution. However, in such a case, the homogenization method is inoperative. More precisely, except in a very few special cases (see [230], [277]), the application of the homogenization method (as described in this book) to such objective functions does not do enough to analyze the problem correctly. Indeed, further arguments (still pertaining to homogenization) are necessary to understand completely the relaxation process.

Remark 3.1.2 We assumed that the objective function J depends on u continuously in Ω . However, as a straightforward generalization, we can consider objective functions depending on u only on an hypersurface in Ω . More precisely, denoting by Γ such an $(N - 1)$ -dimensional surface, our analysis also applies to

$$J(\chi) = \int_{\Gamma} [\chi(x)g_\alpha(x, u_\chi(x)) + (1 - \chi(x))g_\beta(x, u_\chi(x))] ds(x),$$

where $ds(x)$ is the surfacic measure on Γ , and $g_{\alpha,\beta}$ satisfy suitable growth conditions.

Remark 3.1.3 As already said, the objective function J cannot depend on smoothness or topological measures of the design parameter χ , otherwise the homogenization method could not be applied. Since χ is a characteristic function, it does not have smooth derivatives, but one can introduce its gradient in the sense of distributions. In this framework, $\nabla \chi$ is a measure, and $\int_{\Omega} |\nabla \chi| dx$ is its total variation, which corresponds to the perimeter of the region occupied by α . This is, of course, a measure of the smoothness of the design. If, for example, the objective function J depends on this perimeter, it is known that homogenization does not occur in this problem (see [26]). Therefore, we purposely ask that J depend on the value $\chi(x)$ of the design parameter, but not on its higher “derivatives”, in order for it to fit into the framework of the homogenization method.

3.1.2 A first Mathematical Analysis

The mathematical analysis of the optimal design problems (3.5) and (3.7) amounts mainly to ask two questions. First, are these problems well-posed, i.e., do they admit a (possibly unique) solution in the class of admissible designs? Second, what are the optimality conditions (or Euler equations) associated with these optimal designs (solutions of (3.5) or (3.7))? The first

question is clearly unavoidable: One of the goals of mathematics is to assert the existence and uniqueness of solutions. The least one can ask of a mathematical problem is that it admit a solution, otherwise it must be modified appropriately (for example, by changing the class of admissible solutions, or the objective function). The second question turns out to be a characterization of the possible solutions in terms of the Euler-Lagrange equations arising from the stationarity of the objective function. Roughly speaking, it amounts to saying that the gradient ∇J of the objective function must vanish for an optimal design. It gives additional information on the optimal designs, but most of all it is at the root of many numerical methods for computing optimal designs, such as gradient descent methods or optimality criteria methods. There are of course other interesting questions to study: For example, the continuity of the solutions with respect to the data, or the qualitative properties of a solution, such as its smoothness or topology. Very little is known on these latter problems, therefore we focus on the two first questions (which are also the most important ones).

We first address the question of existence of optimal design. The classical method for obtaining the existence of solutions is the *direct method of the calculus of variations* (see [89], [102], [293]), which we now describe in our context. Since our optimal design problems (3.5) and (3.7) are minimization problems, we consider a minimizing sequence $(\chi_n)_{n \geq 1}$ of nearly optimal designs approaching the infimum value of J , namely

$$\lim_{n \rightarrow \infty} J(\chi_n) = \inf J(\chi) \quad (3.8)$$

where the infimum is taken on the space of admissible designs. The goal is to try to pass to the limit, at least for a subsequence of $(\chi_n)_{n \geq 1}$, and prove that there exists a limit χ_∞ in a suitable topology such that

$$\begin{cases} \lim_{n \rightarrow \infty} \chi_n = \chi_\infty \\ \lim_{n \rightarrow \infty} J(\chi_n) \geq J(\chi_\infty). \end{cases} \quad (3.9)$$

Then, combining (3.8) and (3.9) yields that χ_∞ is indeed an optimal design, i.e., a minimizer of J . The principle of the direct method of the calculus of variations is very simple, but the difficulty is finding a correct topology for the sequence $(\chi_n)_{n \geq 1}$ such that, on the one hand, it be compact and, on the other hand, the function $J(\chi)$ be continuous (or at least lower semi-continuous). Unfortunately, these two requirements are often contradictory. In the space of admissible designs $L^\infty(\Omega; \{0, 1\})$, two different topologies or convergences are available: The strong (or pointwise) convergence, and the

weak * convergence (see Subsection 1.2.1). For the strong convergence the continuity of J is obvious but the minimizing sequences are not compact, while for the weak * convergence J is not continuous (and even not lower semicontinuous), albeit minimizing sequences are relatively compact since they are bounded.

Actually, the only a priori estimate for the minimizing sequence $(\chi_n)_{n \geq 1}$ comes from its very definition: Since they are characteristic functions, they are uniformly bounded, that is,

$$\|\chi_n\|_{L^\infty(\Omega)} \leq 1. \quad (3.10)$$

There is no other information on χ_n ; in particular, the state equation does not furnish a priori estimate on χ_n , but rather on the potential u_{χ_n} . With the sole estimate (3.10), there is no hope that a subsequence of $(\chi_n)_{n \geq 1}$ converges strongly in $L^\infty(\Omega)$. Consequently, the strong convergence is not the appropriate tool for studying the sequence $(\chi_n)_{n \geq 1}$. However, by the sequential compactness of bounded sets for the weak * topology of $L^\infty(\Omega)$ (see Lemma 1.2.1), there exists a subsequence of $(\chi_n)_{n \geq 1}$ that converges to a limit χ_∞ weakly * in $L^\infty(\Omega)$. There is a subtle point here: Although each χ_n is a characteristic function (taking only the values zero and one), the limit χ_∞ is usually not a characteristic function, since it can take any value between zero and one. This is not a surprise since the weak convergence can be interpreted as a convergence “in average”. More precisely, we have the following lemma (see [272]).

Lemma 3.1.4 *Let K be a subset of \mathbb{R}^p . The weak * closure of the space $L^\infty(\Omega; K)$ is $L^\infty(\Omega; \mathcal{K})$, where \mathcal{K} is the closed convex hull of K .*

In other words, if a sequence takes its values only in $K \subset \mathbb{R}^p$, its weak limit may span the entire closure (in \mathbb{R}^p) of the convex hull of K (the convex combinations of elements of K). Applying this result to $K = \{0, 1\}$, which admits $\mathcal{K} = [0, 1]$ as its closed convex hull, implies that the weak limit χ_∞ , rather than being the characteristic function of the region occupied by the phase α , is the density, or proportion, of α in the domain Ω . Here, we think of $L^\infty(\Omega; [0, 1])$ as a space of density functions. Weak convergence seems therefore to be irrelevant for studying the sequence $(\chi_n)_{n \geq 1}$, since its limit escapes from the space of admissible designs; it is not even worth studying the continuity of the objective function J .

Nevertheless, we shall see in Subsection 3.1.5 that weak convergence is not responsible for this trouble. Rather, the choice of the space of admissible designs is the culprit. Indeed, it is precisely the generic behavior of

minimizing sequences $(\chi_n)_{n \geq 1}$ to converge, not to a classical design (i.e., a characteristic function), but to a generalized design (i.e., a density function). As a conclusion, the optimal design problems (3.5) and (3.7) are not well-posed, specifically they usually do not admit solutions (classical designs) in $L^\infty(\Omega; \{0, 1\})$. Subsection 3.1.5 provides an explicit counterexample of existence. The first counterexamples were obtained by Murat [198], [199], [200], Murat and Tartar [205], and Lurie [172], [178]. Numerical evidence of this fact was obtained by [73] in a slightly different context of plate thickness optimization: The computed optimal designs were not stable under mesh refinement, a fact that is typical of non well-posed problems (similar difficulties were also reported by Armand [27]).

To obtain the existence of solutions to the optimal design problems (3.5) and (3.7), we therefore need to change the definition of the space of admissible designs. There are two possible routes to follow. First, we can make further restrictions on the characteristic function χ , such as a bound on its perimeter or on the curvature of its associated region. This was done, e.g., in [26], [71], [72], where existence results were proved under one of these additional assumptions. (In a different context, namely for Dirichlet boundary conditions, this route was further explored by [62], [267].) However, the question of the dependence of the optimal designs with respect to the value of these bounds is open, and this approach does not furnish a new perspective for numerical methods. Second, and this is our approach in this book, we can find the closure of the space of admissible designs and extend the objective function to this closure in order that the direct method of the calculus of variations succeed with this new space and objective function. This process is called *relaxation* and it goes back at least to the work of Young [293] (for a modern exposition see, e.g., [89], [102] and references therein). In the next section we show that homogenization is a key tool for obtaining this relaxation, i.e., to devise a notion of generalized design that makes the optimization problem well-posed without changing its physical relevance.

The question of finding optimality conditions for the minimization problems (3.5) and (3.7) is also not obvious in the present setting. As is well known the Euler-Lagrange equations of the calculus of variations are obtained by perturbing a stationary point. This is possible if the space of admissible designs is stable under some linear combination of its elements. This is clearly not the case for $L^\infty(\Omega; \{0, 1\})$, since a convex combination of different characteristic functions is never a characteristic function. This difficulty in making variations of a domain was recognized a long time ago. Hadamard [129] proposed to vary a domain along the normal to its boundary.

This yields only a strict subclass of possible variations which, in particular, never change the topology of the domain. Nevertheless, many authors have developed and improved this method of shape sensitivity (see, e.g., [203], [220], [258]), yielding numerical algorithms that we shall call “classical”. In particular, more general domain variations can be obtained by applying smooth diffeomorphisms. However, these methods give only partial necessary optimality conditions, which are by no means sufficient, since the domain topology is fixed. As a consequence, another motivation for introducing homogenization and relaxation is to find new optimality conditions that take into account topology changes. In practice, this objective yields new numerical algorithms for computing optimal designs, which explains the interest and success of this approach, both from a theoretical and a numerical standpoint.

3.1.3 Multiple State Equations

In the previous subsection our model problem involved a single state equation, which means that the design is optimized in a single configuration (this is a so-called *single load* problem). If we change the source term or the boundary conditions, the design will no longer be optimal. In particular, such an optimal design is often unstable upon perturbation of the data, which is clearly unacceptable. In real practice, an electrical device is often not optimized for just one task, but for several ones. Motivated by this remark, we modify our model problem in order to introduce several state equations that allow one to optimize a design concurrently in several configurations. This type of problem is also called a *multiple loads* optimization problem.

The design is still defined as a characteristic function χ in $L^\infty(\Omega; \{0, 1\})$, which yields an overall conductivity, defined by (3.1) as $a_\chi(x) = \alpha\chi(x) + \beta(1 - \chi(x))$. The number of configurations is denoted by the integer $n \geq 1$. Each configuration is characterized by a source term $f_i(x)$ in Ω (a scalar function in $L^2(\Omega)$ with $1 \leq i \leq n$) and a state equation

$$\begin{cases} -\operatorname{div}(a_\chi \nabla u_{i,\chi}) = f_i & \text{in } \Omega \\ u_{i,\chi} = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.11)$$

where the unknown $u_{i,\chi}(x)$ models the electrical potential or the temperature in the i th configuration. Under our assumptions on the conductivity a_χ and the source term f_i , equation (3.11) admits a unique solution $u_{i,\chi}$ in $H_0^1(\Omega)$.

The performance of χ is measured by an objective function J , taking into account all state equations, defined as

$$\begin{aligned} J(\chi) = & \int_{\Omega} [\chi(x)g_{\alpha}(x, u_{1,\chi}(x), \dots, u_{n,\chi}(x)) \\ & + (1 - \chi(x))g_{\beta}(x, u_{1,\chi}(x), \dots, u_{n,\chi}(x))] dx, \end{aligned} \quad (3.12)$$

where g_{α} and g_{β} are functions depending only on x and the values of the unknowns $u_{i,\chi}(x)$. As before, in order that $J(\chi)$ be well-defined on $L^{\infty}(\Omega)$, we assume that $g_{\alpha,\beta}(x, \lambda)$, defined from $\Omega \times \mathbb{R}^n$ into \mathbb{R} , are Carathéodory functions and satisfy a suitable growth condition at infinity in λ . More precisely,

$$\left\{ \begin{array}{l} x \rightarrow g_{\alpha,\beta}(x, \lambda) \text{ measurable } \forall \lambda \in \mathbb{R}^n, \\ \lambda \rightarrow g_{\alpha,\beta}(x, \lambda) \text{ continuous a.e. } x \in \Omega, \\ |g_{\alpha,\beta}(x, \lambda)| \leq k(x) + C|\lambda|^m \text{ with } k(x) \in L^1(\Omega), 1 \leq m < \frac{2N}{N-2}. \end{array} \right. \quad (3.13)$$

Finally, the optimal design problem is the minimization

$$\inf_{\chi \in L^{\infty}(\Omega; \{0,1\})} J(\chi) + \ell \int_{\Omega} \chi(x) dx, \quad (3.14)$$

where ℓ is a fixed Lagrange multiplier for a volume constraint on material α , J is defined by (3.12), and the state equations are (3.11) for $1 \leq i \leq n$.

Typical examples of such a multicriterion objective function include a sum of individual objective functions associated to a single configuration, i.e.,

$$J(\chi) = \sum_{i=1}^n \int_{\Omega} [\chi(x)g_{\alpha}^i(x, u_{i,\chi}(x)) + (1 - \chi(x))g_{\beta}^i(x, u_{i,\chi}(x))] dx,$$

and the maximum of these individual objective functions, i.e.,

$$J(\chi) = \max_{1 \leq i \leq n} \left(\int_{\Omega} [\chi(x)g_{\alpha}^i(x, u_{i,\chi}(x)) + (1 - \chi(x))g_{\beta}^i(x, u_{i,\chi}(x))] dx \right),$$

although this latter one, being not differentiable, requires extra care in the general framework defined by (3.12).

As we did in the previous subsection, the state equation (3.11) can be slightly modified by canceling the source term and introducing nonhomogeneous boundary conditions, as in (3.7), in order to obtain a variant of the

optimal design problem (3.14). This variant is well adapted to the degenerate limit when α goes to zero.

The mathematical analysis of the multiple loads optimization problem (3.14) is completely similar to that in case of a single state equation. Therefore, as already claimed in the previous subsection, (3.14) is not a well-posed problem in the sense that it usually does not admit any solution in the space of characteristic functions (or classical design) $L^\infty(\Omega; \{0, 1\})$. By the same token, general optimality conditions cannot be obtained in this same space. Our goal is thus to relax the problem by introducing generalized designs using the homogenization method.

In the context of optimal design, the main difference between one and several state equations lies in the consequences of the optimality conditions for the relaxed problem. Indeed, these conditions are going to be much more complicated for several state equations than for a single one.

3.1.4 Shape Optimization as a Degeneracy Limit

So far we have focused on two-phase optimization problems. In truth, our main motivation is shape optimization problems that can formally be seen as the limit of two-phase problems, where one of the phases degenerates so as to become holes. Usually, the goal of shape optimization is to find the best design included in a fixed working space or ground structure. That part of the working space not occupied by the design appears as holes. It is actually a common practice in the shape optimization community to replace these holes by an almost degenerate phase (a very poor conductor). This changes a free-boundary problem into an interface problem, and it allows for easier numerical computations in the fixed working space. However, a mathematical justification for this approximation process is still lacking in full generality. In this subsection we simply pose the problem and explain its difficulty. We refer the reader to Remark 3.2.32 and Section 4.2 for a partial answer.

We begin by introducing a model problem in shape optimization, and in a next step approximate it by a two-phase problem (as described in the previous subsections). Let Ω be a given domain (a bounded open set in \mathbb{R}^N) that plays the role of a working space or ground structure. An admissible shape is a smooth open subset ω entirely included in Ω . In other words the working space is now partly occupied by a single constituent material filling the shape ω , while the remaining part $\Omega \setminus \omega$ corresponds to void or holes (see Figure 3.2). The material is assumed to have a constant isotropic

conductivity

$$0 < \beta < +\infty.$$

The state equation associated to the shape ω is that modeling thermal or electrical conductivity in ω with prescribed boundary conditions on the working domain's boundary $\partial\Omega$ and homogeneous Neumann condition (zero normal flux) on the holes' boundaries $\partial\omega \setminus \partial\Omega$. We assume that there is no volumic source terms, and we introduce the following disjoint partition of the boundary $\partial\Omega$:

$$\partial\Omega = \Gamma_D \cup \Gamma_N, \quad (3.15)$$

where Γ_D corresponds to a Dirichlet boundary condition, and Γ_N to a Neumann boundary condition. The state equation reads

$$\begin{cases} -\operatorname{div}(\beta \nabla u_\omega) = 0 & \text{in } \omega \\ u_\omega = u_0 & \text{on } \Gamma_D \\ \beta \frac{\partial u_\omega}{\partial n} = \sigma_0 \cdot n & \text{on } \Gamma_N \\ \beta \frac{\partial u_\omega}{\partial n} = 0 & \text{on } \partial\omega \setminus \{\Gamma_D \cup \Gamma_N\}, \end{cases} \quad (3.16)$$

where u_0 and σ_0 are smooth enough data (namely there exists an extension $\tilde{u}_0 \in H^1(\Omega)$ such that $\tilde{u}_0 = u_0$ on Γ_D and $\beta \nabla \tilde{u}_0 \cdot n = \sigma_0 \cdot n$ on Γ_N). Then there exists a unique solution u_ω of (3.16) in $H^1(\Omega)$.

To simplify the analysis, we assume that the boundary data u_0 and σ_0 do not vanish on subsets of $\partial\Omega$ with positive surface measure. In such a case, there is no restriction on assuming that an admissible shape ω is further required to satisfy

$$\partial\Omega \subset \partial\omega, \quad (3.17)$$

which ensures that (3.16) makes sense. If the boundary data were allowed to vanish on part of $\partial\Omega$, then an admissible ω may well not contain that part of the boundary $\partial\Omega$ where the data vanish. This latter case can be treated in much the same way but its presentation is slightly more complicated. Of course, if Γ_D is empty, then σ_0 must satisfy the compatibility condition $\int_{\Gamma_N} \sigma_0 \cdot n = 0$, which expresses the global equilibrium of the imposed normal flux. By our smoothness assumption on the boundary data, if ω is a smooth open set, there exists a unique solution u_ω of (3.16) in $H^1(\omega)$. Introducing a cost function J defined by

$$J(\omega) = \int_{\omega} g(x, u_\omega(x)) dx + \ell \int_{\omega} dx, \quad (3.18)$$

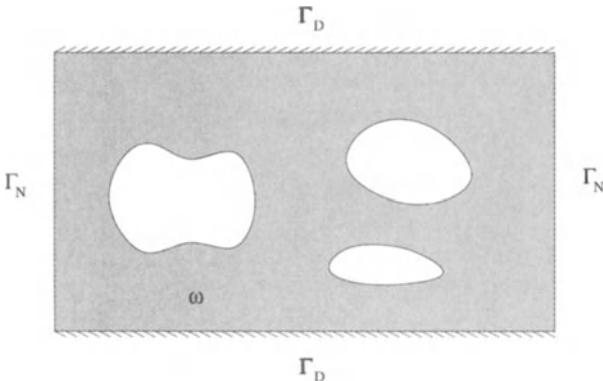


Figure 3.2: Rectangular working domain Ω and shape ω .

where $g(x, u)$ satisfies the usual assumptions (3.4), and $\ell \in \mathbb{R}$ is a fixed Lagrange multiplier for a volume constraint on the shape ω , a model problem in shape optimization is

$$\inf_{\omega \subset \Omega} J(\omega), \quad (3.19)$$

where the minimization takes place on admissible shapes ω that are smooth (Lipschitz continuous) open sets and satisfy (3.17).

As already stated, in numerical practice the holes $\Omega \setminus \omega$ are often filled with a very poor conductor α . This has the effect of approximating the minimization problem (3.19) by a two-phase optimization problem as introduced in Subsection 3.1.1. Denoting by $\chi(x)$ the characteristic function of the holes in Ω , i.e., $\chi(x) = 1$ if $x \in \Omega \setminus \omega$, and $\chi(x) = 0$ if $x \in \omega$, for any value of α such that $\beta > \alpha > 0$, we introduce an overall conductivity

$$a_\chi(x) = \alpha\chi(x) + \beta(1 - \chi(x)), \quad (3.20)$$

which, as α goes to zero, mimics the conductivity of ω and the holes in the working space Ω . We consider an approximate cost function J_α , defined on the space of characteristic functions by

$$J_\alpha(\chi) = \int_{\Omega} (1 - \chi(x))g(x, u_\chi^\alpha(x))dx + \ell \int_{\Omega} (1 - \chi(x))dx, \quad (3.21)$$

where u_χ^α is the unique solution in $H^1(\Omega)$ of (3.7), namely

$$\begin{cases} -\operatorname{div}(a_\chi \nabla u_\chi^\alpha) = 0 & \text{in } \Omega \\ u_\chi^\alpha = u_0 & \text{on } \Gamma_D \\ a_\chi \nabla u_\chi^\alpha \cdot n = \sigma_0 \cdot n & \text{on } \Gamma_N. \end{cases} \quad (3.22)$$

Recall that the variational character of (3.22) includes the usual transmission conditions at the interfaces between the two phases. To emphasize its dependence on α the solution of (3.22) is indexed by α . The functional J_α is minimized on the space of characteristic functions, i.e.,

$$\inf_{\chi \in L^\infty(\Omega; \{0,1\})} J_\alpha(\chi). \quad (3.23)$$

A justification of using (3.23) with small α instead of the true shape optimization problem (3.19) would amount to proving that the infimum value of (3.23) converges to that of (3.19) when α goes to zero, and that the minimizers of (3.23) (if any) converge to minimizers of (3.19). Such a result, pertaining to Γ -convergence theory [90], [94], is not obvious at all. A first difficulty arises since (3.19) and (3.23) usually admit no minimizers in the original class of admissible designs. One must therefore replace problems (3.19) and (3.23) by their relaxed formulations, and ask the same question for these relaxations. Even so, the answer is unknown except in a few special cases (see Remark 3.2.32). For the moment we content ourselves with proving a consistency result when using (3.23) in place of (3.19): For any given smooth open subset ω , the shape optimization cost function is the limit of its two-phase counterpart.

Lemma 3.1.5 *Let ω be a smooth open subset of Ω satisfying (3.17). Let χ be the characteristic function of $\Omega \setminus \omega$. Then*

$$\lim_{\alpha \rightarrow 0} J_\alpha(\chi) = J(\omega).$$

Remark 3.1.6 *Clearly, the fact that $(1 - \chi)$ is the characteristic function of a smooth open set ω is an additional restriction on the admissible designs. There are many functions $(1 - \chi) \in L^\infty(\Omega; \{0,1\})$ that are not characteristic functions of an open set, but merely of a measurable set. Also, the convergence in Lemma 3.1.5 is usually not uniform with respect to ω . Therefore, it is not clear that the infimum value of J_α converges to that of J .*

Proof. By virtue of assumption (3.17) on ω , the conductivity a_χ is equal to β on $\Gamma_N \cup \Gamma_D$. Moreover, since ω is smooth and its boundary contains $\partial\Omega$, there exists a smooth cut-off function equal to one on $\partial\Omega$ and to zero on $\partial\omega \setminus \partial\Omega$. Combined with our assumption on the boundary data, it implies that there exists a function v_0 in $H^1(\Omega)$, which does not depend on α and vanishes on the holes $\Omega \setminus \omega$, such that

$$v_0 = u_0 \text{ on } \Gamma_D, \quad \text{and } \beta \nabla v_0 \cdot n = \sigma_0 \cdot n \text{ on } \Gamma_N.$$

This allows one to define a function $v_\chi^\alpha = (u_\chi^\alpha - v_0)$ that satisfies homogeneous boundary conditions. More precisely, since v_0 has its support in ω where $a_\chi = \beta$, v_χ^α is the solution of

$$\begin{cases} -\operatorname{div}(a_\chi \nabla v_\chi^\alpha) = \operatorname{div}(\beta \nabla v_0) & \text{in } \Omega \\ v_\chi^\alpha = 0 & \text{on } \Gamma_D \\ a_\chi \nabla v_\chi^\alpha \cdot n = 0 & \text{on } \Gamma_N. \end{cases} \quad (3.24)$$

Multiplying (3.24) by v_χ^α implies

$$\int_{\Omega} a_\chi |\nabla v_\chi^\alpha|^2 dx = - \int_{\Omega} \beta \nabla v_\chi^\alpha \cdot \nabla v_0 dx,$$

which yields the estimate

$$\|\nabla v_\chi^\alpha\|_{L^2(\omega)^N} + \alpha^{1/2} \|\nabla v_\chi^\alpha\|_{L^2(\Omega \setminus \omega)^N} \leq C, \quad (3.25)$$

where the constant C does not depend on α . If Γ_D is not empty, applying Poincaré's inequality in ω to v_χ^α leads to

$$\|v_\chi^\alpha\|_{L^2(\omega)} \leq C, \quad (3.26)$$

where the constant C does not depend on α (if Γ_D is empty, then the solutions of (3.22) and (3.24) are defined up to a constant, and Poincaré's inequality is not necessary). Thanks to estimates (3.25) and (3.26), the sequence v_χ^α is bounded in $H^1(\omega)$ as α goes to zero. Therefore, up to a subsequence it converges weakly to a limit denoted by v_χ^0 , which has zero trace on Γ_D too. On the other hand, by (3.25), the sequence $\alpha \nabla v_\chi^\alpha$ converges strongly to zero in $L^2(\Omega \setminus \omega)^N$. Multiplying (3.24) by a fixed test function $\phi \in H^1(\Omega)$ gives

$$\int_{\Omega} a_\chi \nabla v_\chi^\alpha \cdot \nabla \phi dx = - \int_{\omega} \beta \nabla v_0 \cdot \nabla \phi dx.$$

Passing to the limit as α goes to zero yields

$$\int_{\omega} \beta \nabla v_\chi^0 \cdot \nabla \phi dx = - \int_{\omega} \beta \nabla v_0 \cdot \nabla \phi dx,$$

which is nothing other than the variational formulation for $(u_\omega - v_0)$ with u_ω the solution of (3.16). Since (3.16) has a unique solution, the entire sequence $u_\chi^\alpha = v_\chi^\alpha + v_0$ converges to $u_\omega = v_\chi^0 + v_0$ in $H^1(\omega)$. By the Rellich theorem (see Lemma 1.2.6), u_χ^α converges strongly in $L^2(\omega)$, which easily implies that $J_\alpha(\chi)$ converges to $J(\omega)$. \square

3.1.5 Counterexample to the Existence of Optimal Designs

This subsection is devoted to a simple example of a two-phase optimal design problem in the framework of Subsection 3.1.1, for which there is no optimal solution in the class of admissible designs. The first counterexamples to the existence of solutions for similar control problems, where the control acts in the coefficients of the state equation, are due to Murat [198], [199], [200], and Lurie [172], [178]. In the context of optimal design, more explicit counterexamples have been devised by Murat and Tartar [205], and recently in [18], [25]. Inspired by these latter works, we propose a simple counterexample. Although it is stated in any space dimension $N \geq 2$, its spirit is truly one-dimensional.

We consider a bounded open set Ω in \mathbb{R}^N , $N \geq 2$, with a smooth boundary $\partial\Omega$ (say, Lipschitz continuous). This domain Ω is filled by a mixture of two isotropic conductors $\beta > \alpha > 0$, corresponding to a characteristic function χ . Denoting by $a_\chi = \alpha\chi + \beta(1 - \chi)$ the conductivity in Ω , the state equation is

$$\begin{cases} -\operatorname{div}(a_\chi \nabla u_\chi) = 0 & \text{in } \Omega \\ a_\chi \nabla u_\chi \cdot n = \sigma_0 \cdot n & \text{on } \partial\Omega, \end{cases} \quad (3.27)$$

where the imposed flux σ_0 on the boundary is defined as a constant vector in the direction of the N th vector e_N of the canonical basis

$$\sigma_0 = |\sigma_0|e_N. \quad (3.28)$$

Since the boundary data are compatible with the absence of source term in Ω , namely $\int_{\partial\Omega} \sigma_0 \cdot n = 0$, (3.27) admits a unique solution u_χ in $H^1(\Omega)/\mathbb{R}$ (i.e., up to an additive constant). As an objective function we choose a weighted sum of the energy dissipated in Ω and of the volume constraint on the best conductor β , i.e.,

$$J(\chi) = \int_{\partial\Omega} (\sigma_0 \cdot n) u_\chi ds + \ell \int_{\Omega} (1 - \chi) dx, \quad (3.29)$$

where $\ell \geq 0$ is a fixed positive Lagrange multiplier, and ds is the surfacic measure on $\partial\Omega$. The optimal design problem is then to minimize J in the space $L^\infty(\Omega; \{0, 1\})$, that is,

$$\inf_{\chi \in L^\infty(\Omega; \{0, 1\})} J(\chi). \quad (3.30)$$

If we assume that the good conductor β is more expensive than the poor conductor α , the optimal design problem (3.30) amounts to finding the best

arrangement of these two phases, one that produces a domain Ω of maximal conductivity and minimal price all together. The overall conductivity of Ω is measured by the stored energy (which is equal to the dissipated energy), and ℓ is the price versus efficiency ratio for β , which measures the trade-off between a low price and a good overall conductivity. Our main result is the following nonexistence theorem.

Theorem 3.1.7 *Assume $N \geq 2$. Let us define*

$$\ell^- = \frac{|\sigma_0|^2(\beta - \alpha)}{\beta^2} \leq \ell^+ = \frac{|\sigma_0|^2(\beta - \alpha)}{\alpha^2}.$$

If $\ell^+ > \ell > \ell^-$, there exists no minimizer of the optimal design problem (3.30) in the space of admissible designs $L^\infty(\Omega; \{0, 1\})$. If $\ell \leq \ell^-$, then the constant function $\chi \equiv 0$ is the unique minimizer of (3.30). If $\ell \geq \ell^+$, then the constant function $\chi \equiv 1$ is the unique minimizer of (3.30).

Remark 3.1.8 *If $\ell \geq \ell^+$, then the price of the good conductor β is so high that it is better to use α only. On the contrary if $\ell \leq \ell^-$, the good conductor β is so cheap that the cost function is minimized by using only β . In the remaining case, there is a trade-off between price and efficiency that leads to an intricate mixture of both phases. As we shall see in the next section, there exists an optimal generalized design, which is a composite material (a simple laminate) obtained by mixing α and β .*

Proof. By the principle of minimum complementary energy, we rewrite the dissipated energy as the result of a minimization, i.e.,

$$\int_{\partial\Omega} (\sigma_0 \cdot n) u_\chi ds = \min_{\sigma \in H_0} \int_{\Omega} a_\chi^{-1} \sigma \cdot \sigma dx, \quad (3.31)$$

where the affine space H_0 of admissible fluxes is defined by

$$H_0 = \left\{ \sigma \in L^2(\Omega)^N \text{ such that } \begin{array}{l} \operatorname{div} \sigma = 0 \text{ in } \Omega \\ \sigma \cdot n = \sigma_0 \cdot n \text{ on } \partial\Omega \end{array} \right\}.$$

We remark that, in the definition of H_0 , the boundary condition makes sense in $H^{-1/2}(\partial\Omega)$. Obviously the minimum in (3.31) is attained by the flux $\sigma_\chi = a_\chi \nabla u_\chi$ where u_χ is the solution of (3.27). The optimal design problem (3.30) can thus be rewritten as

$$\inf_{\chi \in L^\infty(\Omega; \{0, 1\})} \inf_{\sigma \in H_0} \int_{\Omega} (a_\chi^{-1} \sigma \cdot \sigma + \ell(1 - \chi)) dx, \quad (3.32)$$

where the order of the two minimizations is irrelevant. For a given χ , we define the averages

$$\theta = \frac{1}{|\Omega|} \int_{\Omega} \chi(x) dx \quad \text{and} \quad a_0 = \frac{1}{|\Omega|} \int_{\Omega} a_{\chi}(x) dx = \alpha\theta + \beta(1 - \theta).$$

Next, we remark that the constant boundary data σ_0 is also the average of any σ in H_0 , i.e.,

$$\sigma_0 = \frac{1}{|\Omega|} \int_{\Omega} \sigma(x) dx.$$

Indeed, for any $\sigma \in H_0$ since $(\sigma - \sigma_0) \cdot n = 0$ on $\partial\Omega$, an integration by parts yields

$$0 = \int_{\Omega} \operatorname{div}(\sigma(x) - \sigma_0) x_i dx = - \int_{\Omega} (\sigma(x) - \sigma_0) \cdot e_i dx,$$

where x_i for $1 \leq i \leq N$ is the i th component of x . By virtue of Lemma 3.1.9 below, for any $x \in \Omega$, we have

$$a_{\chi}(x)^{-1} |\sigma(x)|^2 \geq a_0^{-1} |\sigma_0|^2 - \frac{(a_{\chi}(x) - a_0)}{a_0^2} |\sigma_0|^2 + \frac{2}{a_0} \sigma_0 \cdot (\sigma(x) - \sigma_0). \quad (3.33)$$

Integrating over Ω , the two last terms of (3.33) cancel out since they have a zero average. This yields

$$\int_{\Omega} (a_{\chi}^{-1} \sigma \cdot \sigma + \ell(1 - \chi)) dx \geq |\Omega| (a_0^{-1} \sigma_0 \cdot \sigma_0 + \ell(1 - \theta)). \quad (3.34)$$

Since a_0 depends linearly on θ , the right hand side of (3.34) is a strictly convex function of θ in $[0, 1]$. Its minimum is thus a lower bound for (3.32), namely,

$$\inf_{\chi \in L^{\infty}(\Omega; \{0, 1\})} \inf_{\sigma \in H_0} \int_{\Omega} (a_{\chi}^{-1} \sigma \cdot \sigma + \ell(1 - \chi)) dx \geq I_{\ell}$$

with

$$I_{\ell} = |\Omega| \begin{cases} \frac{|\sigma_0|^2}{\alpha} & \text{if } \ell \geq \ell^+ = \frac{|\sigma_0|^2(\beta - \alpha)}{\alpha^2} \\ \frac{|\sigma_0|^2}{\beta} + \ell & \text{if } \ell \leq \ell^- = \frac{|\sigma_0|^2(\beta - \alpha)}{\beta^2} \\ 2|\sigma_0| \sqrt{\frac{\ell}{\beta - \alpha}} - \frac{\alpha\ell}{\beta - \alpha} & \text{if } \ell^+ > \ell > \ell^-. \end{cases}$$

We now examine whether this lower bound can be attained. In its derivation, the only inequality comes from (3.33) for which we know the exact remainder by virtue of Lemma 3.1.9. Assume that there exists $\chi \in L^\infty(\Omega; \{0, 1\})$, and therefore $\sigma_\chi \in H_0$, such that

$$J(\chi) = \int_{\Omega} \left(a_\chi^{-1} \sigma_\chi \cdot \sigma_\chi + \ell(1 - \chi) \right) dx = I_\ell.$$

From (3.33) and (3.36), we deduce that, for any $x \in \Omega$,

$$a_\chi(x)^{-1} \left| \sigma_\chi(x) - \frac{a_\chi(x)}{a_0} \sigma_0 \right|^2 = 0,$$

which implies

$$\sigma_\chi(x) = \frac{a_\chi(x)}{a_0} \sigma_0 \text{ in } \Omega.$$

If χ is a constant function, namely $\chi \equiv 0$ or $\chi \equiv 1$ in Ω , then σ_χ is also constant and equal to σ_0 . Then it is easy to check that $J(\chi)$ coincides with the lower bound I_ℓ if and only if ℓ is outside of the open interval (ℓ^-, ℓ^+) . If χ is not a constant function, then σ_χ takes only two values, $\frac{\alpha}{a_0} \sigma_0$ and $\frac{\beta}{a_0} \sigma_0$, which are both different from the boundary data σ_0 , since $\alpha < a_0 < \beta$. Thus, σ_χ cannot satisfy the boundary condition. Consequently, $J(\chi)$ is always strictly greater than the bound I_ℓ if ℓ lies inside the open interval (ℓ^-, ℓ^+) .

To finish the proof, it remains to prove that the lower bound I_ℓ is precisely the infimum of $J(\chi)$. When $\ell \geq \ell^+$ we already know that $J(1) = I_\ell$, and similarly, when $\ell \leq \ell^-$ we know that $J(0) = I_\ell$. For all other values of ℓ , it turns out that the right hand side of (3.34) is minimal for a unique value, denoted by θ^* ,

$$\theta^* = \frac{1}{\beta - \alpha} \left(\beta - |\sigma_0| \sqrt{\frac{\ell}{\beta - \alpha}} \right),$$

which satisfies $0 < \theta^* < 1$. We build a minimizing sequence $(\chi_n)_{n \geq 1}$ of characteristic functions such that χ_n converges to the constant limit θ^* weakly * in $L^\infty(\Omega; [0, 1])$ and $\lim_{n \rightarrow +\infty} J(\chi_n) = I_\ell$. Introducing a 1-periodic function $\chi(x_1)$ given by

$$\chi(x_1) = \begin{cases} 1 & \text{if } 0 \leq x_1 < \theta^* \\ 0 & \text{if } \theta^* \leq x_1 < 1 \end{cases},$$

we define χ_n by

$$\chi_n(x) = \chi(nx_1),$$

which, upon recalling Lemma 1.3.19 on periodically oscillating sequences, converges weakly * in $L^\infty(\Omega; [0, 1])$ to θ^* . Such characteristic functions χ_n corresponds to both periodic and laminated microstructures. Since $N \geq 2$, the lamination direction e_1 is orthogonal to the boundary data σ_0 . Therefore, applying Theorem 1.3.18 and Lemma 1.3.32 we deduce that the sequence of conductivity tensors $A_n = a_{\chi_n} I_2$ H -converges to the constant homogenized tensor A^* defined by

$$A^* = \text{diag} \left((\theta^* \alpha^{-1} + (1 - \theta^*) \beta^{-1})^{-1}, a_0, \dots, a_0 \right) \quad \text{with } a_0 = \theta^* \alpha + (1 - \theta^*) \beta.$$

Furthermore, as a consequence of Proposition 1.2.20 on the energy convergence, we also have

$$\lim_{n \rightarrow +\infty} \min_{\sigma \in H_0} \int_{\Omega} a_{\chi_n}^{-1} \sigma \cdot \sigma dx = \min_{\sigma \in H_0} \int_{\Omega} (A^*)^{-1} \sigma \cdot \sigma dx. \quad (3.35)$$

Since A^* is a constant tensor and σ_0 a constant vector, it is easy to see that the minimizer in the right hand side of (3.35) is precisely $\sigma(x) \equiv \sigma_0$. Thus, we deduce that

$$\lim_{n \rightarrow +\infty} J(\chi_n) = |\Omega| \left(\frac{|\sigma_0|^2}{a_0} + \ell(1 - \theta^*) \right) = I_\ell,$$

which is the desired result. \square

Lemma 3.1.9 *The function $\phi(a, \sigma)$, defined from $\mathbb{R}^+ \times \mathbb{R}^N$ into \mathbb{R} by*

$$\phi(a, \sigma) = a^{-1} |\sigma|^2,$$

is convex and satisfies

$$\phi(a, \sigma) = \phi(a_0, \sigma_0) + D\phi(a_0, \sigma_0) \cdot (a - a_0, \sigma - \sigma_0) + \phi(a, \sigma - \frac{a}{a_0} \sigma_0), \quad (3.36)$$

where the differential $D\phi$ is given by

$$D\phi(a_0, \sigma_0) \cdot (b, \tau) = -\frac{b}{a_0^2} |\sigma_0|^2 + \frac{2}{a_0} \sigma_0 \cdot \tau.$$

Proof. Formula (3.36) is easily obtained by a simple computation. It is a Taylor development at first order with an exact remainder. Since $\phi(a, \sigma - \frac{a}{a_0} \sigma_0)$ is nonnegative by definition, we deduce that ϕ , being always above its tangent hyperplane, is convex. \square

Remark 3.1.10 The counterexample provided by Theorem 3.1.7 works only in space dimension $N \geq 2$. In a one-dimensional setting, i.e., $N = 1$, the situation is completely different. In such a case, for any χ the flux σ_χ is constant, equal to the boundary data σ_0 . Therefore, $J(\chi)$ is explicitly given in terms of the average θ of χ

$$J(\chi) = |\Omega| \left(a_0^{-1} |\sigma_0|^2 + \ell(1 - \theta) \right).$$

Minimizing in θ the right hand side gives the unique value θ^* for which $J(\chi)$ is minimum, and such a minimum is attained by any characteristic function χ with average θ^* . Thus, if $0 < \theta^* < 1$, there exist infinitely many minimizers in one dimension.

Remark 3.1.11 A complete family of counterexamples for similar problems in which the (possibly multiple) state equations have linear boundary conditions may be found in [18] and [25]. In these references, counterexamples are also established if one of the phases is degenerate, i.e., $\alpha = 0$ or $\beta = +\infty$.

3.2 Relaxation by the Homogenization Method

In this section, generalized admissible designs are introduced; the above shape optimization problems can now admit solutions in this enlarged class of designs. These generalized designs are actually composite materials, as defined in Chapter 2, that are studied by means of the homogenization theory. This process of transforming an ill-posed problem into a well-posed one by defining generalized solutions is called *relaxation* in the calculus of variations (for more details on relaxation, see, e.g., [89], [102], [152], [293]). Then, we compute the optimality conditions that allow one to characterize the optimal composite materials.

3.2.1 Existence of Generalized Designs

We focus on the case of a single state equation (the multiple state case is completely similar). Recall that our problem of optimal design is

$$\inf_{\chi \in L^\infty(\Omega; \{0,1\})} J(\chi), \quad (3.37)$$

i.e., to find a characteristic function χ that minimizes in $L^\infty(\Omega; \{0,1\})$ the objective function

$$J(\chi) = \int_{\Omega} [\chi(x) g_\alpha(x, u_\chi(x)) + (1 - \chi(x)) g_\beta(x, u_\chi(x))] dx + \ell \int_{\Omega} \chi(x) dx,$$

where ℓ is a given real parameter and $u_\chi(x)$ is the unique solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\operatorname{div}(a_\chi \nabla u_\chi) = f & \text{in } \Omega \\ u_\chi = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.38)$$

with a given right hand side $f \in L^2(\Omega)$ and a conductivity $a_\chi = \alpha\chi + \beta(1-\chi)$. The functions g_α and g_β satisfy (3.4), which implies, at least, that $J(\chi)$ is well defined on $L^\infty(\Omega)$.

As proved in Subsection 3.1.5, there is usually no minimizer for J in the space of admissible “classical” designs $L^\infty(\Omega; \{0, 1\})$. To overcome this difficulty, rather than changing completely the definition of admissible designs or enforcing additional constraints for a design to be admissible, we proceed to the relaxation of the minimization of the objective function J . The principle of the relaxation process is to study the behavior of minimizing sequences and to define generalized admissible designs that include the possible limits of these minimizing sequences (limits that may escape from the original space of admissible designs). In Subsection 3.1.1 we showed that the weak limit of a sequence of characteristic functions is usually a density function θ which may take its values in the whole range $[0, 1]$. According to Chapter 2, the corresponding mixture of conductors α and β is a composite material of conductivity tensor A^* .

Following the work of Murat and Tartar [205] our concept of generalized admissible designs is precisely these couples (θ, A^*) made of a density function and a composite conductivity tensor (or an H -limit). They are called *composite or generalized designs*, as opposed to the previous *classical designs* that are characteristic functions. In view of Theorems 2.1.2 and 2.2.13 we know precisely the set of such composite designs, which we denote by \mathcal{CD} , as follows:

$$\mathcal{CD} = \left\{ (\theta, A^*) \in L^\infty(\Omega; [0, 1] \times \mathcal{M}_N^s) \mid A^*(x) \in G_{\theta(x)} \text{ a.e. in } \Omega \right\}, \quad (3.39)$$

where, for each constant value $0 \leq \theta \leq 1$, G_θ is the set of all symmetric matrices with eigenvalues $\lambda_1, \dots, \lambda_N$ satisfying

$$\lambda_\theta^- \leq \lambda_i \leq \lambda_\theta^+ \quad \forall 1 \leq i \leq N$$

$$\sum_{i=1}^N \frac{1}{\lambda_i - \alpha} \leq \frac{1}{\lambda_\theta^- - \alpha} + \frac{N-1}{\lambda_\theta^+ - \alpha}$$

$$\sum_{i=1}^N \frac{1}{\beta - \lambda_i} \leq \frac{1}{\beta - \lambda_\theta^-} + \frac{N-1}{\beta - \lambda_\theta^+},$$

with $\lambda_\theta^- = \left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\right)^{-1}$ and $\lambda_\theta^+ = \theta\alpha + (1-\theta)\beta$. We remark that, when θ takes the value zero or one, then A^* is just phase α or β . Therefore, our generalized designs include, as a special case, classical two-phase designs. The objective function and the state equation are also generalized as follows

$$\begin{aligned} J^*(\theta, A^*) = & \int_{\Omega} [\theta(x)g_{\alpha}(x, u(x)) + (1 - \theta(x))g_{\beta}(x, u(x))] dx \\ & + \ell \int_{\Omega} \theta(x) dx, \end{aligned} \quad (3.40)$$

where $u(x)$ is the unique solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\operatorname{div}(A^* \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.41)$$

The proposed relaxed problem is then the minimization

$$\min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*). \quad (3.42)$$

The main result of this section is the following theorem, which proves that (3.42) is indeed the correct relaxed problem, i.e., composite designs are a pertinent notion of generalized designs.

Theorem 3.2.1 *The minimization problem (3.42) is a true relaxation of the original problem (3.37) in the sense that*

1. *there exists at least one minimizer of J^* in \mathcal{CD} ;*
2. *up to a subsequence, every minimizing sequence of classical designs χ_n for J converges weakly * in $L^\infty(\Omega; [0, 1])$ to a density function θ , and the associated conductivity $a_{\chi_n} = \alpha\chi_n + \beta(1 - \chi_n)$ H -converges to an homogenized conductivity tensor A^* such that (θ, A^*) is a minimizer of J^* in \mathcal{CD} ;*
3. *conversely, every minimizer (θ, A^*) of J^* in \mathcal{CD} is attained by a minimizing sequence, for J , of classical designs χ_n , namely θ is the weak * limit of χ_n in $L^\infty(\Omega; [0, 1])$ and A^* is the H -limit of $a_{\chi_n} = \alpha\chi_n + \beta(1 - \chi_n)$.*

Remark 3.2.2 *Theorem 3.2.1 is due to Murat and Tartar [205]. It is the key result of this chapter and it justifies the homogenization method for two-phase optimal design. Indeed, when a notion of generalized solutions is introduced, one must check that it does not modify the problem, i.e., that there still exists a connection between classical and generalized solutions. The three conditions stated in Theorem 3.2.1 insure that the relaxation of the original problem does not physically change the problem. In particular, the minimum values of the original and relaxed functional are the same, and every optimal composite design can be approximated, as close as we want, by a nearly optimal classical design. In this respect, “homogenization” and “composite materials” are just the correct language for expressing what is a minimizing sequence of classical designs.*

Proof. Let $(\chi_n)_{n \geq 1}$ be a minimizing sequence of characteristic functions for the original problem (3.37). As already said, the only available a priori estimate is

$$\|\chi_n\|_{L^\infty(\Omega)} \leq 1. \quad (3.43)$$

Thanks to the uniform bound (3.43), there exists a subsequence, still denoted by χ_n , and a limit θ_∞ such that χ_n converges weakly * to θ_∞ in $L^\infty(\Omega; [0, 1])$. By virtue of Theorem 1.2.16 on the compactness of H -convergence, there exists another subsequence, still denoted by χ_n , and an H -limit A_∞^* such that $a_{\chi_n} = \alpha\chi_n + \beta(1 - \chi_n)$ H -converges to A_∞^* . By Definition 1.2.15 of H -convergence, the sequence u_{χ_n} of solutions of the state equation (3.38) converges weakly in $H_0^1(\Omega)$ to the solution u_∞ of (3.41) with the conductivity tensor A_∞^* . Then, the Rellich theorem (see Lemma 1.2.6) implies that u_{χ_n} converges strongly in $L^2(\Omega)$ to u_∞ , and Lemma 1.2.2 yields that u_{χ_n} converges to u_∞ almost everywhere in Ω , up to another subsequence. Therefore, due to the growth properties (3.4) of the functions $g_{\alpha,\beta}$, the Lebesgue dominated convergence theorem implies that the sequence $g_{\alpha,\beta}(x, u_{\chi_n}(x))$ converges strongly in $L^1(\Omega)$ to $g_{\alpha,\beta}(x, u_\infty(x))$. Since one can pass to the limit in the product of a weakly convergent sequence and a strongly convergent one (see Lemma 1.2.4), we deduce that

$$\lim_{n \rightarrow \infty} J(\chi_n) = J^*(\theta_\infty, A_\infty^*). \quad (3.44)$$

Since χ_n is a minimizing sequence, we obtain

$$J^*(\theta_\infty, A_\infty^*) = \inf_{\chi \in L^\infty(\Omega; \{0, 1\})} J(\chi). \quad (3.45)$$

Note that, even if χ_n were not a minimizing sequence, we would have obtained, at least, the continuity relation (3.44) for the objective function. Furthermore, by the very definition of the G -closure set G_θ (see in particular Theorem 2.1.2), any generalized composite design (θ, A^*) in \mathcal{CD} is attained as the limit of a sequence of classical designs χ_n , in the sense that θ is the weak * limit of χ_n in $L^\infty(\Omega; [0, 1])$ and A^* is the H -limit of $a_{\chi_n} = \alpha\chi_n + \beta(1 - \chi_n)$. Thus, for any choice (θ, A^*) in \mathcal{CD} , there exists a sequence $\chi_n \in L^\infty(\Omega; \{0, 1\})$ such that

$$J^*(\theta, A^*) = \lim_{n \rightarrow \infty} J(\chi_n) \geq \inf_{\chi \in L^\infty(\Omega; \{0, 1\})} J(\chi). \quad (3.46)$$

Combining (3.45) and (3.46) we deduce that $(\theta_\infty, A_\infty^*)$ is a minimizer of J^* in \mathcal{CD} . This proves the two first points in the statement of Theorem 3.2.1. Finally, the third point is a special case of the fact that any generalized composite design $(\theta, A^*) \in \mathcal{CD}$ is attained as the limit of a sequence $\chi_n \in L^\infty(\Omega; \{0, 1\})$, which is obviously minimizing if (θ, A^*) is a relaxed minimizer.

□

Remark 3.2.3 *Theorem 3.2.1 extends easily to the other model problems defined previously in Subsection 3.1.1. Its proof is very simple as far as homogenization theory is known. Indeed, it is precisely for this reason that the theory of H -convergence was developed by Murat and Tartar [205], [206], [269]. We remark also that the assumptions (3.4) on the functions $g_{\alpha, \beta}$ are used to obtain the continuity property (3.44) for the objective function.*

The relaxation process using the homogenization method allows us to prove the existence of relaxed optimal designs, but it leaves open two important questions: Is the relaxed optimal design unique? If not, are there local minima for the relaxed formulation? It turns out that the answer to the first question is “no” (there are examples of problems with infinitely many solutions), while the second one is still an open problem (numerically, only global minima have been observed). The nonuniqueness of optimal designs has already been obtained in one-dimensional examples (see Remark 3.1.10), and Subsection 3.2.6 gives other examples in higher dimensions. Concerning the possibility of existence of local minima that are not global minima, a negative answer is given by Remark 3.2.26 in the case of affine boundary conditions and by Remark 3.2.28 for a simple example with any boundary conditions (in both cases, all local minima are actually global minima). We do not know if such a result is true generically. Of course, it is a fundamental

issue from a numerical point of view, since we do not want minimization algorithms to get stuck in local minima (for a positive result in this direction, see Theorem 5.1.5).

3.2.2 Optimality Conditions

The relaxed formulation (3.39), (3.40) of the original optimization problem was devised for obtaining the existence of generalized or composite optimal designs. It also has another advantage: While it was impossible to get optimality conditions in the original formulation without further restrictions, we are going to derive such conditions for the relaxed formulation since it is now possible to perform variations in the space \mathcal{CD} of composite designs.

We need further assumptions on the functions $g_{\alpha,\beta}(x, \lambda)$ on top of (3.4). To differentiate the objective function $J^*(\theta, A^*)$, these functions must be differentiable with respect to $\lambda \in \mathbb{R}$, and the combinations of their partial derivatives,

$$\theta(x) \frac{\partial g_\alpha}{\partial \lambda}(x, u(x)) + (1 - \theta(x)) \frac{\partial g_\beta}{\partial \lambda}(x, u(x)),$$

must belong to $H^{-1}(\Omega)$. This is the case, for example, if the partial derivatives $\partial g_{\alpha,\beta}/\partial \lambda(x, \lambda)$, defined from $\Omega \times \mathbb{R}$ into \mathbb{R} , are Carathéodory functions and satisfy a suitable growth condition at infinity, i.e.,

$$\left\{ \begin{array}{l} x \rightarrow \frac{\partial g_{\alpha,\beta}}{\partial \lambda}(x, \lambda) \text{ measurable } \forall \lambda \in \mathbb{R} \\ \lambda \rightarrow \frac{\partial g_{\alpha,\beta}}{\partial \lambda}(x, \lambda) \text{ continuous, a.e. } x \in \Omega \\ \left| \frac{\partial g_{\alpha,\beta}}{\partial \lambda}(x, \lambda) \right| \leq k'(x) + C'' \lambda^{m-1} \text{ for } 1 \leq m < \frac{2N}{N-2} \\ \text{with } k'(x) \in L^q(\Omega), q > \frac{2N}{N+2}. \end{array} \right. \quad (3.47)$$

As is usual in optimization problems involving a state equation, the derivative of the objective function is easier to write down by introducing the solution of the so-called *adjoint state equation*. We denote by p this adjoint state, which is the unique solution in $H_0^1(\Omega)$ of

$$\left\{ \begin{array}{l} -\operatorname{div}(A^* \nabla p) = \theta \frac{\partial g_\alpha}{\partial \lambda}(x, u) + (1 - \theta) \frac{\partial g_\beta}{\partial \lambda}(x, u) \quad \text{in } \Omega \\ p = 0 \quad \text{on } \partial\Omega. \end{array} \right. \quad (3.48)$$

We remark that assumption (3.47) implies that, for any $\theta \in L^\infty(\Omega)$ and any $u \in H_0^1(\Omega)$, the right hand side of (3.48) belongs to $L^q(\Omega)$, for some $q > \frac{2N}{N+2}$, which is included in $H^{-1}(\Omega)$ by duality and the Sobolev embedding theorem for $N \geq 2$ (see Lemma 1.2.7).

Theorem 3.2.4 *The objective function $J^*(\theta, A^*)$ is Gâteaux differentiable on the space of admissible composite designs \mathcal{CD} , defined by (3.39), and its directional derivative is*

$$\begin{aligned}\delta J^*(\theta, A^*) = & \int_{\Omega} \delta\theta(x) [g_\alpha(x, u(x)) - g_\beta(x, u(x)) + \ell] dx \\ & - \int_{\Omega} \delta A^*(x) \nabla u(x) \cdot \nabla p(x) dx,\end{aligned}\tag{3.49}$$

where $\delta\theta$ and δA^* are admissible increments in \mathcal{CD} , u is the solution of the state equation (3.41), and p that of the adjoint state equation (3.48). Consequently, if (θ, A^*) is a minimizer of the objective function J^* , it must satisfy $\delta J^*(\theta, A^*) \geq 0$ for any admissible increments $\delta\theta, \delta A^*$.

Remark 3.2.5 Recall that, in the framework of Gâteaux differentiability (see, e.g., [66]), an admissible increment $(\delta\theta, \delta A^*)$ is the derivative with respect to a scalar parameter $t \in [0, 1]$ at $t = 0$ of a smooth (continuously differentiable) path $(\theta(t), A^*(t))$ in \mathcal{CD} which coincides for $t = 0$ with the point (θ, A^*) where the derivative of J^* is computed, i.e.,

$$(\delta\theta, \delta A^*) = \left(\frac{d\theta}{dt}(0), \frac{dA^*}{dt}(0) \right).$$

Then, introducing the corresponding path of state variable $u(t)$, which is the solution of equation (3.41) with the design parameters $(\theta(t), A^*(t))$, we define the state increment

$$\delta u = \frac{du}{dt}(0)$$

and the objective increment

$$\delta J^* = \frac{d}{dt} [J^*(\theta(t), A^*(t))] (0).$$

Proof. A straightforward derivation shows that

$$\begin{aligned}\delta J^*(\theta, A^*) = & \int_{\Omega} \delta\theta(x) [g_\alpha(x, u(x)) - g_\beta(x, u(x)) + \ell] dx \\ & + \int_{\Omega} \delta u(x) \left[\theta \frac{\partial g_\alpha}{\partial \lambda}(x, u) + (1 - \theta) \frac{\partial g_\beta}{\partial \lambda}(x, u) \right] dx,\end{aligned}$$

where δu is the increment in the solution u caused by the increments $\delta\theta$ and δA^* . By differentiating the state equation (3.41), we obtain that δu is the unique solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\operatorname{div}(A^*\nabla\delta u) = \operatorname{div}(\delta A^*\nabla u) & \text{in } \Omega \\ \delta u = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.50)$$

The introduction of the adjoint state allows one to eliminate the increment δu . Indeed, multiplying equations (3.50) by p and (3.48) by δu , then integrating by parts, yields

$$\int_{\Omega} \delta u(x) \left[\theta \frac{\partial g_{\alpha}}{\partial \lambda}(x, u) + (1 - \theta) \frac{\partial g_{\beta}}{\partial \lambda}(x, u) \right] dx = - \int_{\Omega} \delta A^*(x) \nabla u(x) \cdot \nabla p(x) dx,$$

which completes the proof. We implicitly used assumption (3.47) to give a meaning to the left hand side in the above equality. \square

The precise set of admissible increments $\delta\theta, \delta A^*$ is not easy to characterize if one varies simultaneously the density θ and the tensor A^* . Of course, there are obvious restrictions on $\delta\theta$ at the points x where the proportion $\theta(x)$ is equal to zero or one, but they are not the only ones. If θ is kept fixed, then one can achieve a large class of admissible increments δA^* by using the convexity of the set G_θ (as proved in Theorem 2.2.13). However, if θ varies, so does the set G_θ , which yields complicated restrictions on the possible δA^* . This is also a reason for preferring Gâteaux differentiability instead of the usual Fréchet differentiability.

We obtain a first necessary condition of optimality by keeping θ fixed, i.e., $\delta\theta = 0$. Since G_θ is a convex set, one can choose a straight path $A^*(t) = tA^0 + (1 - t)A^*$ connecting two tensors A^* and A^0 in G_θ , which yields $\delta A^* = A^0 - A^*$.

Theorem 3.2.6 *If (θ, A^*) is a minimizer of the objective function J^* , then except at the points where $\nabla u = 0$ or $\nabla p = 0$ we have*

$$\begin{aligned} 2A^*\nabla u &= (\lambda_{\theta}^+ + \lambda_{\theta}^-) \nabla u + (\lambda_{\theta}^+ - \lambda_{\theta}^-) \frac{|\nabla u|}{|\nabla p|} \nabla p \\ 2A^*\nabla p &= (\lambda_{\theta}^+ + \lambda_{\theta}^-) \nabla p + (\lambda_{\theta}^+ - \lambda_{\theta}^-) \frac{|\nabla p|}{|\nabla u|} \nabla u, \end{aligned} \quad (3.51)$$

where u is the solution of the state equation (3.41), p that of the adjoint state equation (3.48), and $\lambda_{\theta}^- = \left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\right)^{-1}$, $\lambda_{\theta}^+ = \theta\alpha + (1 - \theta)\beta$. At

points where $\nabla u/|\nabla u| \neq \nabla p/|\nabla p|$, the value of the optimal tensor $A^* \in G_\theta$ satisfying (3.51) is unique and corresponds to a rank-1 laminate.

Furthermore, there exists another minimizer $(\tilde{\theta}, \tilde{A}^*)$, which admits the same state u and adjoint state p as (θ, A^*) , and is such that, almost everywhere in Ω , the optimal tensor \tilde{A}^* is a rank-1 laminate.

Remark 3.2.7 By Remark 2.2.16, G_θ is also a convex set of the inverse tensors $(A^*)^{-1}$. This suggests another path connecting A^* and A^0 in G_θ , namely,

$$A^*(t) = \left(t(A^0)^{-1} + (1-t)(A^*)^{-1} \right)^{-1},$$

which corresponds to $\delta A^* = A^*((A^*)^{-1} - (A^0)^{-1})A^*$. Introducing the notation

$$\sigma = A^* \nabla u, \quad \tau = A^* \nabla p,$$

where $\sigma \neq 0$ and $\tau \neq 0$, this approach yields the following optimality conditions

$$\begin{aligned} 2(A^*)^{-1}\sigma &= (\nu_\theta^+ + \nu_\theta^-)\sigma + (\nu_\theta^+ - \nu_\theta^-)\frac{|\sigma|}{|\tau|}\tau \\ 2(A^*)^{-1}\tau &= (\nu_\theta^+ + \nu_\theta^-)\tau + (\nu_\theta^+ - \nu_\theta^-)\frac{|\tau|}{|\sigma|}\sigma, \end{aligned} \tag{3.52}$$

with $\nu_\theta^\pm = (\lambda_\theta^\pm)^{-1}$. These conditions are actually equivalent to (3.51).

To give a second necessary condition of optimality, we now take an optimal tensor A^* , as defined by (3.51), and we vary the density θ . We introduce the angle ϕ between ∇u and ∇p , defined by

$$\nabla u \cdot \nabla p = |\nabla u| |\nabla p| \cos \phi. \tag{3.53}$$

Note that the angle ϕ is not determined if ∇u or ∇p vanish.

Theorem 3.2.8 If (θ, A^*) is a minimizer of the objective function J^* , then, defining the quantity

$$Q(x) = g_\alpha(x, u) - g_\beta(x, u) + \ell + \frac{\beta - \alpha}{\alpha\beta} |\nabla u| |\nabla p| \left(\alpha\beta \cos^2 \frac{\phi}{2} - (\lambda_\theta^-)^2 \sin^2 \frac{\phi}{2} \right),$$

the optimal density satisfies almost everywhere in Ω

$$\begin{aligned} \theta(x) &= 0 && \text{if } Q(x) > 0 \\ \theta(x) &= 1 && \text{if } Q(x) < 0 \\ 0 \leq \theta(x) &\leq 1 && \text{if } Q(x) = 0, \end{aligned} \tag{3.54}$$

and conversely

$$\begin{aligned} Q(x) &\geq 0 & \text{if } \theta(x) = 0 \\ Q(x) &\leq 0 & \text{if } \theta(x) = 1 \\ Q(x) &= 0 & \text{if } 0 < \theta(x) < 1. \end{aligned} \quad (3.55)$$

Remark 3.2.9 By using Theorems 3.2.6 and 3.2.8, we can devise a numerical algorithm for computing relaxed optimal designs. Its principle is to compute the state u and the adjoint state p , to update the design parameters (A^* can be chosen as a rank-1 laminate oriented by ∇u and ∇p , while $\theta(x)$ is the root of the equation $Q(x) = 0$), and to iterate this process. Such an algorithm pertains to the class of optimality criteria methods, which are very successful in shape optimization [42], [237], [239], [296], and Subsection 5.1.1. We remark that, assuming u and p to be known, the density θ can be deduced from the optimality condition $Q(x) = 0$ if $\sin(\phi/2) \neq 0$. In the other case $\sin(\phi/2) = 0$, we refer to Remark 3.2.11, where another form of the optimality condition is used.

Proof of Theorem 3.2.6. We fix the density θ in $L^\infty(\Omega; [0, 1])$, and we choose $A^*(t) = tA^0 + (1 - t)A^*$, with A^0 any other tensor in $L^\infty(\Omega; \mathcal{M}_N^s)$ such that $A^0(x) \in G_{\theta(x)}$ a.e. $x \in \Omega$. For any $0 \leq t \leq 1$ and almost every $x \in \Omega$, $A^*(t)$ stays in G_θ since it is a convex set by virtue of Theorem 2.2.13. Therefore, $\delta\theta = 0$, $\delta A^* = A^0 - A^*$, and, as a consequence of Theorem 3.2.6, the optimality condition $\delta J^*(\theta, A^*) \geq 0$ becomes

$$-\int_\Omega (A^0 - A^*) \nabla u \cdot \nabla p dx \geq 0.$$

This is equivalent to the pointwise constraint almost everywhere in Ω ,

$$A^* \nabla u \cdot \nabla p = \max_{A^0 \in G_\theta} (A^0 \nabla u \cdot \nabla p). \quad (3.56)$$

If ∇u or ∇p vanishes, then any A^* in G_θ is optimal in (3.56). Otherwise, defining two unit vectors

$$e = \frac{\nabla u}{|\nabla u|} \text{ and } e' = \frac{\nabla p}{|\nabla p|},$$

(3.56) is equivalent to finding all maximizers A^* of

$$4A^0 e \cdot e' = A^0(e + e') \cdot (e + e') - A^0(e - e') \cdot (e - e').$$

An upper bound is easily seen to be

$$\begin{aligned} \max_{A^0 \in G_\theta} 4A^0 e \cdot e' &\leq \max_{A^0 \in G_\theta} A^0(e + e') \cdot (e + e') - \min_{A^0 \in G_\theta} A^0(e - e') \cdot (e - e') \\ &= \lambda_\theta^+ |e + e'|^2 - \lambda_\theta^- |e - e'|^2. \end{aligned} \quad (3.57)$$

It turns out that (3.57) is indeed attained. Assume first that $e \neq \pm e'$ (remark that $e + e'$ is orthogonal to $e - e'$). If $A^* \in G_\theta$ achieves equality in (3.57), it admits $e + e'$ and $e - e'$ as eigenvectors with λ_θ^+ and λ_θ^- as respective eigenvalues. There exists a unique tensor having this property, which corresponds to a rank-1 laminate in the direction $e - e'$. Indeed, as noticed in Remark 2.2.14, an effective tensor in G_θ that has one eigenvalue equal to the harmonic mean λ_θ^- has all its other eigenvalues equal to the arithmetic mean λ_θ^+ , and it corresponds to a rank-1 laminate. If $e = -e'$, the same is true. However, if $e = e'$, then any tensor with eigenvalue λ_θ^+ in the direction $e + e'$ is optimal. This is the case for a rank-1 laminate in any direction orthogonal to $e + e'$, but also for any microstructure which is invariant in the direction $e + e'$ (see Lemma 2.2.19 and Remark 2.2.20). Uniqueness of the optimal tensor A^* is lost when $e = e'$, except in two dimensions, where again the unique optimal A^* is the rank-1 laminate in the single direction orthogonal to $e + e'$. In any case, we obtain that any optimal A^* satisfies

$$A^*(e + e') = \lambda_\theta^+(e + e') \text{ and } A^*(e - e') = \lambda_\theta^-(e - e'). \quad (3.58)$$

Linear combinations of (3.58) lead to (3.51).

Following Raitum [227] (see also [205]), let us show that if (θ, A^*) is a minimizer, then there exists another minimizer $(\tilde{\theta}, \tilde{A}^*)$ with the same state u and adjoint state p such that \tilde{A}^* is a rank-1 laminate. We already know that, if $\nabla u \neq 0$ and $\nabla p \neq 0$, then, taking $\tilde{\theta} = \theta$, A^* can be replaced by a rank-1 laminate \tilde{A}^* without changing the values of $A^* \nabla u$ and $A^* \nabla p$, thus without changing u and p . If $\nabla u = 0$, we can clearly take $\tilde{\theta} = \theta$ and replace A^* by any rank-1 laminate \tilde{A}^* : It does not change u although p may change. When $\nabla p = 0$, we change both θ and A^* while keeping the same state u and adjoint state p (a symmetric argument would also work in the case $\nabla u = 0$). Indeed, by Lemma 3.2.10 below, there exist two densities θ^-, θ^+ and two rank-1 laminates $A^- \in G_{\theta^-}$ and $A^+ \in G_{\theta^+}$ such that $A^* \nabla u = A^- \nabla u = A^+ \nabla u$ satisfying either $0 \leq \theta^- < \theta < \theta^+ \leq 1$ or $\theta = \theta^-$ or θ^+ (in this latter case it is not necessary to change the density). Denoting by ω the measurable

subset of Ω where $\nabla p = 0$, we build a function $\tilde{\theta}$ in ω , taking only the values θ^- and θ^+ (depending on ∇u and $A^* \nabla u$), which minimizes the “restriction” of the cost function to ω , i.e., the affine function

$$\theta \rightarrow \int_{\omega} [\theta(x)g_{\alpha}(x, u(x)) + (1 - \theta(x))g_{\beta}(x, u(x))] dx + \ell \int_{\omega} \theta(x)dx.$$

Taking \tilde{A}^* to be the corresponding rank-1 laminate, equal almost everywhere to A^- or A^+ , we have thus obtained another minimizer $(\tilde{\theta}, \tilde{A}^*)$ having the desired properties. \square

Lemma 3.2.10 *Let B be a symmetric matrix, with eigenvalues $(\lambda_i)_{1 \leq i \leq N}$ satisfying $\lambda_{\theta}^- \leq \lambda_i \leq \lambda_{\theta}^+$. Let e and \bar{e} be two vectors in \mathbb{R}^N such that $\bar{e} = Be$. Then we have*

$$\|\bar{e} - \frac{1}{2}(\lambda_{\theta}^+ + \lambda_{\theta}^-)e\| \leq \frac{1}{2}(\lambda_{\theta}^+ - \lambda_{\theta}^-)\|e\|.$$

Furthermore, there exist two values θ^- and θ^+ , with $0 \leq \theta^- \leq \theta \leq \theta^+ \leq 1$, depending only on e and \bar{e} , such that, for $0 \leq t \leq 1$,

$$\|\bar{e} - \frac{1}{2}(\lambda_t^+ + \lambda_t^-)e\| \leq \frac{1}{2}(\lambda_t^+ - \lambda_t^-)\|e\| \quad (3.59)$$

if and only if $t \in [\theta^-, \theta^+]$, and there exist two corresponding rank-1 laminates $A^- \in G_{\theta^-}$, $A^+ \in G_{\theta^+}$ such that

$$\bar{e} = Be = A^-e = A^+e.$$

Proof. By definition, the eigenvalues of $(B - \frac{1}{2}(\lambda_{\theta}^+ + \lambda_{\theta}^-)I_2)$ belong to the interval $[-\frac{1}{2}(\lambda_{\theta}^+ - \lambda_{\theta}^-), \frac{1}{2}(\lambda_{\theta}^+ - \lambda_{\theta}^-)]$. Thus

$$\|\bar{e} - \frac{1}{2}(\lambda_{\theta}^+ + \lambda_{\theta}^-)e\| = \|(B - \frac{1}{2}(\lambda_{\theta}^+ + \lambda_{\theta}^-)I_2)e\| \leq \frac{1}{2}(\lambda_{\theta}^+ - \lambda_{\theta}^-)\|e\|.$$

Let us introduce a function $f(t)$, which is nonnegative if and only if (3.59) holds,

$$f(t) = \frac{1}{4}(\lambda_t^+ - \lambda_t^-)^2\|e\|^2 - \|\bar{e} - \frac{1}{2}(\lambda_t^+ + \lambda_t^-)e\|^2 = -(\lambda_t^+e - \bar{e}) \cdot (\lambda_t^-e - \bar{e}).$$

An easy computation shows that $f(0) = -\|\bar{e} - \beta e\|^2$ and $f(1) = -\|\bar{e} - \alpha e\|^2$, while $f(t)/\lambda_t^-$ is a polynomial of degree two in t , whose coefficient of t^2 is

$$-\frac{(\beta - \alpha)^2 Be \cdot e}{\alpha \beta},$$

which is negative since the matrix B is positive. Therefore, if (3.59) holds true for some $\theta \in [0, 1]$, then there exist two values $0 \leq \theta^- \leq \theta \leq \theta^+ \leq 1$ such that $f(t)$ is nonnegative with $0 \leq t \leq 1$ if and only if $t \in [\theta^-, \theta^+]$. Note that $\theta^- < \theta < \theta^+$ if $f(\theta) > 0$, while $\theta = \theta^-$ or $\theta = \theta^+$ if $f(\theta) = 0$. For $\theta_0 = \theta^-$ or θ^+ , we have

$$f(\theta_0) = -(\lambda_{\theta_0}^+ e - \bar{e}) \cdot (\lambda_{\theta_0}^- e - \bar{e}) = 0.$$

If $0 < \theta_0 < 1$, this equation is equivalent to

$$f \cdot f' = 0 \text{ with } f = \frac{\lambda_{\theta_0}^+ e - \bar{e}}{\lambda_{\theta_0}^+ - \lambda_{\theta_0}^-} \text{ and } f' = \frac{\lambda_{\theta_0}^- e - \bar{e}}{\lambda_{\theta_0}^+ - \lambda_{\theta_0}^-}.$$

Then, defining a rank-1 laminate $A^0 \in G_{\theta_0}$ in the direction f if $f \neq 0$, or in any direction orthogonal to f' if $f = 0$, we have

$$A^0 f = \lambda_{\theta_0}^- f \text{ and } A^0 f' = \lambda_{\theta_0}^+ f'.$$

Since $e = f - f'$ and $\bar{e} = \lambda_{\theta_0}^- f - \lambda_{\theta_0}^+ f'$, we thus deduce that $A^0 e = \bar{e}$, as desired. If $\theta_0 = 0$ or 1 , i.e., if $\theta^- = 0$ or $\theta^+ = 1$, we obtain either $\bar{e} = \beta e = A^- e$, with $A^- = \beta I_2 \in G_0$, or $\bar{e} = \alpha e = A^+ e$, with $A^+ = \alpha I_2 \in G_1$. \square

Proof of Theorem 3.2.8. We now consider a smooth (continuously differentiable) path $(\theta(t), A^*(t))$ in \mathcal{CD} that takes the value (θ, A^*) for $t = 0$. We denote by (u, p) the state and adjoint state computed at $t = 0$. By virtue of Theorem 3.2.6, one can exchange A^* with the effective tensor of a rank-1 laminate without changing u and p and without changing θ where ∇u and ∇p are nonzero. Therefore, without loss of generality we can assume that A^* is a rank-1 laminate where $\nabla u \neq 0$ and $\nabla p \neq 0$. There is no restriction on the smooth path $\theta(t)$ in $L^\infty(\Omega; [0, 1])$, but $A^*(t)$ is required to belong to $G_{\theta(t)}$ (almost everywhere in Ω) for any $t \in [0, 1]$. We ask the following further restriction on $A^*(t)$ where $\nabla u \neq 0$ and $\nabla p \neq 0$:

$$\begin{aligned} 2A^*(t)\nabla u &= (\lambda_{\theta(t)}^+ + \lambda_{\theta(t)}^-) \nabla u + (\lambda_{\theta(t)}^+ - \lambda_{\theta(t)}^-) \frac{|\nabla u|}{|\nabla p|} \nabla p \\ 2A^*(t)\nabla p &= (\lambda_{\theta(t)}^+ + \lambda_{\theta(t)}^-) \nabla p + (\lambda_{\theta(t)}^+ - \lambda_{\theta(t)}^-) \frac{|\nabla p|}{|\nabla u|} \nabla u. \end{aligned}$$

(No requirements on $A^*(t)$ are made where $\nabla u = 0$ or $\nabla p = 0$.) There exists such a smooth path $A^*(t)$; indeed A^* is itself a rank-1 laminate where

$\nabla u \neq 0$ and $\nabla p \neq 0$, and $A^*(t)$ is taken as a rank-1 laminate in the same direction with just a smooth variation of the proportion. This implies that

$$2A^*(t)\nabla u \cdot \nabla p = \left[(\lambda_{\theta(t)}^+ + \lambda_{\theta(t)}^-) \cos \phi + (\lambda_{\theta(t)}^+ - \lambda_{\theta(t)}^-) \right] |\nabla u| |\nabla p|,$$

where ϕ is the angle defined by (3.53). Therefore, differentiating with respect to t (and recalling that u and p do not depend on t), the admissible increment δA^* satisfies

$$2\delta A^*\nabla u \cdot \nabla p = \frac{\alpha - \beta}{\alpha\beta} \left[(\alpha\beta + (\lambda_\theta^-)^2) \cos \phi + (\alpha\beta - (\lambda_\theta^-)^2) \right] |\nabla u| |\nabla p| \delta\theta,$$

which, upon noticing that $1 + \cos \phi = 2 \cos^2(\phi/2)$ and $1 - \cos \phi = 2 \sin^2(\phi/2)$, yields the optimality condition

$$\delta J^*(\theta, A^*) = \int_{\Omega} Q(x) \delta\theta(x) dx \geq 0$$

for any admissible increment $\delta\theta$ in $L^\infty(\Omega)$. Such an admissible increment must satisfy the constraints $\delta\theta(x) \geq 0$ if $\theta(x) = 0$ and $\delta\theta(x) \leq 0$ if $\theta(x) = 1$. By choosing a localized increment around each Lebesgue point of θ , we easily obtain the desired result. \square

Remark 3.2.11 In the proof of Theorem 3.2.8 we used the optimal tensor A^* defined by Theorem 3.2.6. Equivalently, we can use the optimal inverse tensor A^{*-1} defined by (3.52). Since $\delta A^* = A^* \delta(A^{*-1}) A^*$, this formulation leads to

$$\delta A^* \nabla u \cdot \nabla p = \delta(A^{*-1}) \sigma \cdot \tau,$$

with the notation $\sigma = A^* \nabla u$ and $\tau = A^* \nabla p$. Introducing an angle ψ defined by

$$\sigma \cdot \tau = |\sigma| |\tau| \cos \psi,$$

a similar computation yields

$$2\delta(A^{*-1}) \sigma \cdot \tau = (\beta - \alpha) \left[((\nu_\theta^+)^2 + (\alpha\beta)^{-1}) \cos \psi + ((\nu_\theta^+)^2 - (\alpha\beta)^{-1}) \right] |\sigma| |\tau| \delta\theta,$$

where $\nu_\theta^\pm = (\lambda_\theta^\pm)^{-1}$. Finally, we obtain

$$\begin{aligned} \theta(x) &= 0 && \text{if } \tilde{Q}(x) > 0 \\ \theta(x) &= 1 && \text{if } \tilde{Q}(x) < 0 \\ 0 \leq \theta(x) \leq 1 & && \text{if } \tilde{Q}(x) = 0, \end{aligned} \tag{3.60}$$

and conversely

$$\begin{aligned}\tilde{Q}(x) &\geq 0 & \text{if } \theta(x) = 0 \\ \tilde{Q}(x) &\leq 0 & \text{if } \theta(x) = 1 \\ \tilde{Q}(x) &= 0 & \text{if } 0 < \theta(x) < 1,\end{aligned}\tag{3.61}$$

where \tilde{Q} is defined by

$$\tilde{Q}(x) = g_\alpha(x, u) - g_\beta(x, u) + \ell + (\beta - \alpha)|\sigma| |\tau| \left((\nu_\theta^+)^2 \cos^2 \frac{\psi}{2} - (\alpha\beta)^{-1} \sin^2 \frac{\psi}{2} \right).$$

These conditions are actually equivalent to (3.54) and (3.55) since one can check that $\tilde{Q}(x) = Q(x)$. However, in the context of an optimality criteria numerical method, (3.60) and (3.61) may be sometimes more convenient than (3.54) and (3.55). Indeed, if $\phi = 0$, then one cannot extract the root θ of the optimality condition $Q(x) = 0$ (assuming that u and p are known). But a simple computation shows that $\phi = 0$ corresponds to $\psi = 0$, which allows one to extract the root θ of the optimality condition $\tilde{Q}(x) = 0$ (see Subsection 5.1.1).

3.2.3 Multiple State Equations

In the case of $n \geq 2$ state equations, the relaxation procedure of the optimization problem (3.14) is very similar to the case of a single state equation. The only differences appear in the optimality conditions.

As before, a generalized admissible design is a couple (θ, A^*) of a density function and a composite conductivity tensor. The set \mathcal{CD} of such relaxed or composite designs is still defined by (3.39). The relaxed objective function becomes

$$\begin{aligned}J^*(\theta, A^*) = & \int_{\Omega} [\theta(x)g_\alpha(x, u(x)) + (1 - \theta(x))g_\beta(x, u(x))] dx \\ & + \ell \int_{\Omega} \theta(x) dx,\end{aligned}\tag{3.62}$$

where for simplicity we denote by u the vector of components $(u_i)_{1 \leq i \leq n}$, each $u_i(x)$ being the unique solution in $H_0^1(\Omega)$ of the homogenized state equation

$$\begin{cases} -\operatorname{div}(A^* \nabla u_i) = f_i & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega. \end{cases}\tag{3.63}$$

The relaxed problem is the minimization

$$\min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*),\tag{3.64}$$

as precisely stated in the following theorem.

Theorem 3.2.12 *The minimization problem (3.64) is a true relaxation of the original minimization problem (3.14) in the sense that*

1. *there exists at least one relaxed minimizer of (3.64);*
2. *up to a subsequence, every minimizing sequence χ_n in (3.14) converges weakly * in $L^\infty(\Omega; [0, 1])$ to a density θ , and the associated conductivity $a_{\chi_n} = \alpha\chi_n + \beta(1 - \chi_n)$ H -converges to a homogenized tensor A^* such that (θ, A^*) is a minimizer of (3.64);*
3. *conversely, every minimizer of (3.64) is attained by a minimizing sequence of (3.14) in the above sense.*

The proof of Theorem 3.2.12 is identical to that of Theorem 3.2.1, so we omit it. In order to differentiate $J^*(\theta, A^*)$ and obtain optimality conditions, we add further assumptions on the functions $g_{\alpha,\beta}(x, \lambda)$. For example, we assume that the partial derivatives $\partial g_{\alpha,\beta}/\partial\lambda_i(x, \lambda)$, defined from $\Omega \times \mathbb{R}^n$ into \mathbb{R} , with $\lambda = (\lambda_i)_{1 \leq i \leq n}$, are Carathéodory functions and satisfy a suitable growth condition at infinity, i.e., for any $1 \leq i \leq n$

$$\left\{ \begin{array}{l} x \rightarrow \frac{\partial g_{\alpha,\beta}}{\partial \lambda_i}(x, \lambda) \text{ measurable } \forall \lambda \in \mathbb{R}^n \\ \lambda \rightarrow \frac{\partial g_{\alpha,\beta}}{\partial \lambda_i}(x, \lambda) \text{ continuous a.e. } x \in \Omega \\ \left| \frac{\partial g_{\alpha,\beta}}{\partial \lambda_i}(x, \lambda) \right| \leq k'(x) + C' |\lambda|^{m-1} \text{ for } 1 \leq m < \frac{2N}{N-2} \\ \text{with } k'(x) \in L^q(\Omega), q > \frac{2N}{N+2}. \end{array} \right. \quad (3.65)$$

To compute the derivative of $J^*(\theta, A^*)$, we introduce the adjoint state p , a vector of components $(p_i)_{1 \leq i \leq n}$, where each $p_i(x)$ is the unique solution in $H_0^1(\Omega)$ of the adjoint state equation

$$\left\{ \begin{array}{ll} -\operatorname{div}(A^* \nabla p_i) = \theta \frac{\partial g_\alpha}{\partial \lambda_i}(x, u) + (1-\theta) \frac{\partial g_\beta}{\partial \lambda_i}(x, u) & \text{in } \Omega \\ p_i = 0 & \text{on } \partial\Omega. \end{array} \right. \quad (3.66)$$

Theorem 3.2.13 *The objective function $J^*(\theta, A^*)$ is Gâteaux differentiable on the space of admissible composite designs \mathcal{CD} defined by (3.39), and its directional derivative is*

$$\begin{aligned}\delta J^*(\theta, A^*) = & \int_{\Omega} \delta\theta(x) [g_{\alpha}(x, u(x)) - g_{\beta}(x, u(x)) + \ell] dx \\ & - \sum_{i=1}^n \int_{\Omega} \delta A^*(x) \nabla u_i(x) \cdot \nabla p_i(x) dx,\end{aligned}\quad (3.67)$$

where $\delta\theta$ and δA^* are admissible increments in \mathcal{CD} , u_i is the solution of the state equation (3.63), and p_i that of the adjoint state equation (3.66) for $1 \leq i \leq n$. Consequently, if (θ, A^*) is a minimizer of the objective function J^* , it must satisfy $\delta J^*(\theta, A^*) \geq 0$ for any admissible increments $\delta\theta$, δA^* .

The proof of Theorem 3.2.13 is identical to that of Theorem 3.2.4, so we omit it. Its consequences are, however, different.

Theorem 3.2.14 *Let (θ, A^*) be a minimizer of the objective function J^* . Introducing a symmetric matrix*

$$M = \sum_{i=1}^n \nabla u_i \odot \nabla p_i, \quad (3.68)$$

we define a function $f(\theta, M)$ by

$$f(\theta, M) = \max_{A^0 \in G_{\theta}} (A^0 : M). \quad (3.69)$$

Then, we have $A^* : M = f(\theta, M)$ and, except at the points where M vanishes, A^* is a tensor corresponding to a sequential laminate of rank at most N with lamination directions given by the eigenvectors of M . Furthermore, the function $\theta \rightarrow f(\theta, M)$ is C^1 on $[0, 1]$, and defining the quantity

$$Q(x) = g_{\alpha}(x, u) - g_{\beta}(x, u) + \ell - \frac{\partial f}{\partial \theta}(\theta, M), \quad (3.70)$$

the optimal density satisfies

$$\begin{aligned}\theta(x) &= 0 && \text{if } Q(x) > 0 \\ \theta(x) &= 1 && \text{if } Q(x) < 0 \\ 0 \leq \theta(x) &\leq 1 && \text{if } Q(x) = 0,\end{aligned}$$

and conversely

$$\begin{aligned}Q(x) &\geq 0 && \text{if } \theta(x) = 0 \\ Q(x) &\leq 0 && \text{if } \theta(x) = 1 \\ Q(x) &= 0 && \text{if } 0 < \theta(x) < 1.\end{aligned}$$

Remark 3.2.15 In Theorem 3.2.14 it is, in principle, possible to compute explicitly the function $f(\theta, M)$ and the optimal tensor A^* in terms of θ and the eigenvalues of M (a similar computation is done in [25]; see also Lemma 3.2.17). However, the algebra is tedious and the resulting formula very complicated. Theorem 3.2.14 paves the way for numerical algorithms computing relaxed optimal designs. They fall into the category of optimality criteria methods, which are popular in shape optimization [42], [237], [239], [296]. As before, the strategy is to compute iteratively the states u_i and the adjoint states p_i , and then to update the design parameters $A^*(x)$ and $\theta(x)$ by using the optimality conditions.

Proof of Theorem 3.2.14. We fix the density θ in $L^\infty(\Omega; [0, 1])$ and, for any tensor $A^0 \in L^\infty(\Omega; \mathcal{M}_N^s)$ such that $A^0(x) \in G_{\theta(x)}$ a.e. $x \in \Omega$, we take $A^*(t) = tA^0 + (1-t)A^*$, the values of which remain in the convex set G_θ . Therefore, $\delta\theta = 0$ and $\delta A^* = A^0 - A^*$ imply that the optimality condition $\delta J^*(\theta, A^*) \geq 0$ becomes

$$-\sum_{i=1}^n \int_{\Omega} (A^0 - A^*) \nabla u_i \cdot \nabla p_i dx \geq 0. \quad (3.71)$$

Since $(A^0 - A^*)$ is symmetric, introducing the symmetric matrix M defined by (3.68), (3.71) is equivalent to the pointwise constraint, almost everywhere in Ω ,

$$A^* : M = \max_{A^0 \in G_\theta} (A^0 : M). \quad (3.72)$$

If $M = 0$, then any tensor $A^* \in G_\theta$ is optimal. Otherwise, by a classical result of von Neumann (see, e.g., [195]), an optimal A^* in (3.72) must be simultaneously diagonalizable with M . Denoting by $(e_i)_{1 \leq i \leq N}$ an orthonormal basis of eigenvectors of M with associated eigenvalues $(\mu_i)_{1 \leq i \leq N}$, (3.72) is equivalent to

$$\max_{(\lambda_i) \in G_\theta} \sum_{i=1}^N \lambda_i \mu_i, \quad (3.73)$$

where the eigenvalues λ_i of A^* are subject to the constraints defining G_θ (see Theorem 2.2.13). This computation can be done explicitly [25], but, in any case, the optimal A^* belongs to the boundary of G_θ since we are maximizing a linear function on a convex set. Let us now prove that any tensor A^* in the boundary of G_θ corresponds to a sequential laminate of rank at most N with orthogonal lamination directions. Indeed, the boundary of G_θ is included in

one of the hypersurfaces

$$\sum_{i=1}^N \frac{1}{\lambda_i - \alpha} = \frac{1}{\lambda_\theta^- - \alpha} + \frac{N-1}{\lambda_\theta^+ - \alpha} \quad (3.74)$$

or

$$\sum_{i=1}^N \frac{1}{\beta - \lambda_i} = \frac{1}{\beta - \lambda_\theta^-} + \frac{N-1}{\beta - \lambda_\theta^+} \quad (3.75)$$

or

$$\lambda_i = \lambda_\theta^+ \text{ for some } 1 \leq i \leq N. \quad (3.76)$$

Recall from Remark 2.2.14 that it is not necessary to consider the case $\lambda_i = \lambda_\theta^-$. If A^* satisfies (3.74) then it corresponds to a rank- N sequential laminate with core α and matrix β . If A^* satisfies (3.75), then it corresponds to a rank- N sequential laminate with core β and matrix α . Finally, if A^* satisfies (3.76), we assume with no loss of generality that the $(N-m)$ last eigenvalues, with $1 \leq m \leq N-1$, of A^* are equal to λ_θ^+ and that the m first are strictly smaller. Then, the tensor $a^* = \text{diag}(\lambda_1, \dots, \lambda_m)$ belongs to the m -dimensional equivalent of G_θ , and (3.73) is equivalent to

$$\max_{a^* \in G_\theta} \sum_{i=1}^m \lambda_i \mu_i.$$

This problem is of the same type as (3.73) but in a smaller space dimension $m < N$. Thus, by a recurrence on the space dimension (as in the proof of Theorem 2.2.13), we obtain that a^* is a rank- m sequential laminate with orthogonal lamination directions. By Lemma 2.2.19 A^* is also a rank- m sequential laminate (with no lamination in the $(N-m)$ last directions) which is a particular case of a rank- N sequential laminate with orthogonal lamination directions.

We now consider a smooth path $(\theta(t), A^*(t))$ in \mathcal{CD} , which takes the value (θ, A^*) for $t = 0$, such that $0 \leq \theta(t) \leq 1$ and $A^*(t) \in G_{\theta(t)}$ almost everywhere in Ω , for any $t \in [0, 1]$. We require that $A^*(t)$ satisfy

$$A^*(t) : M = f(\theta(t), M),$$

where M is defined by (3.68) at $t = 0$. This is not a restriction where $M = 0$, and where $M \neq 0$ we know that A^* is a sequential laminate, so a smooth path $A^*(t)$ is easily obtained by taking $A^*(t)$ as a laminate in the same directions with just a smooth variation of the proportions. Assume for

the moment that $\theta \rightarrow f(\theta, M)$ is differentiable. Then, differentiating with respect to t , we obtain

$$\delta A^* : M = \frac{\partial f}{\partial \theta}(\theta, M) \delta \theta,$$

which, combined with (3.67), yields the following optimality condition

$$\delta J^*(\theta, A^*) = \int_{\Omega} Q(x) \delta \theta(x) dx \geq 0$$

for any admissible increment $\delta \theta$ in $L^\infty(\Omega)$. It remains to prove the differentiability of $f(\theta, M)$. If M is nonnegative, then $f(\theta, M)$ is just the upper Hashin-Shtrikman bound, defined by (2.27), while, if M is nonpositive, it is minus the lower Hashin-Shtrikman bound, defined by (2.28). These bounds are easily seen to be C^1 functions of θ (see Remark 2.2.12). If M has both positive and negative eigenvalues (with no loss of generality we assume that the $(N - m)$ last ones are positive), then a maximizer A^* in (3.72) must have its largest eigenvalues corresponding to the positive ones of M . This is easily established by taking A^* with its last $(N - m)$ eigenvalues equal to the arithmetic mean λ_θ^+ , while the first m are determined by computing an m -dimensional lower Hashin-Shtrikman bound. This proves again that $f(\theta, M)$, being a combination of the arithmetic mean and a lower-dimensional Hashin-Shtrikman bound, is a C^1 function of θ . \square

Remark 3.2.16 In dimension $N = 2$, one can prove that, if (θ, A^*) is a minimizer, there exists another minimizer $(\tilde{\theta}, \tilde{A}^*)$, which admits the same state u and adjoint state p as (θ, A^*) , such that, everywhere in Ω , the optimal tensor \tilde{A}^* is a rank-2 sequential laminate. The reason is that either A^* belongs to the boundary of G_θ and thus is itself a rank-2 sequential laminate, or A^* belongs to the interior of G_θ and then there exist two densities θ^- and θ^+ with $0 \leq \theta^- < \theta < \theta^+ \leq 1$ and two rank-2 sequential laminates $A^- \in G_{\theta^-}$ and $A^+ \in G_{\theta^+}$ such that $A^* = A^- = A^+$ (this is not true in higher dimensions). Then, an argument similar to that in the proof of Theorem 3.2.6 allows one to change θ where $M = 0$ without changing u and p (where $M \neq 0$, A^* is already a rank-2 sequential laminate).

In $N = 2$ space dimensions we can easily compute $f(\theta, M)$ and characterize the optimal tensors A^* .

Lemma 3.2.17 Let $\mu_1 \leq \mu_2$ be the eigenvalues of M in dimension $N = 2$. The function $f(\theta, M)$ defined by (3.69) is equal to

$$f(\theta, M) = \begin{cases} \beta(\mu_1 + \mu_2) - \frac{(\sqrt{\mu_1} + \sqrt{\mu_2})^2}{(\beta - \lambda_\theta^+)^{-1} + (\beta - \lambda_\theta^-)^{-1}} \\ \quad \text{if } \mu_1 \geq 0 \text{ and } \sqrt{\mu_1} \geq \frac{\beta - \lambda_\theta^+}{\beta - \lambda_\theta^-} \sqrt{\mu_2} \\ \alpha(\mu_1 + \mu_2) + \frac{(\sqrt{-\mu_1} + \sqrt{-\mu_2})^2}{(\lambda_\theta^+ - \alpha)^{-1} + (\lambda_\theta^- - \alpha)^{-1}} \\ \quad \text{if } \mu_2 \leq 0 \text{ and } \sqrt{-\mu_1} \leq \frac{\lambda_\theta^+ - \alpha}{\lambda_\theta^- - \alpha} \sqrt{-\mu_2} \\ \mu_1 \lambda_\theta^- + \mu_2 \lambda_\theta^+ \text{ otherwise.} \end{cases}$$

Proof. From the proof of Theorem 3.2.14 we already know that A^* and M must have the same eigenvectors. Therefore, denoting by μ_1, μ_2 the eigenvalues of M , we have

$$A^* : M = \max_{(\lambda_1, \lambda_2) \in G_\theta} \sum_{i=1}^2 \lambda_i \mu_i,$$

where the only constraints on (λ_1, λ_2) are

$$\sum_{i=1}^2 \frac{1}{\lambda_i - \alpha} \leq \frac{1}{\lambda_\theta^- - \alpha} + \frac{1}{\lambda_\theta^+ - \alpha}, \quad \sum_{i=1}^2 \frac{1}{\beta - \lambda_i} \leq \frac{1}{\beta - \lambda_\theta^-} + \frac{1}{\beta - \lambda_\theta^+}.$$

The bound $\lambda_i \leq \lambda_\theta^+$ is superfluous when $N = 2$ as noticed in Remark 2.2.14. A simple computation leads to formula (3.69). Let us remark that the first regime of (3.69) corresponds to a rank-2 sequential laminate with matrix β , the second regime corresponds to a rank-2 sequential laminate with matrix α , while the third regime corresponds to a rank-1 sequential laminate. \square

3.2.4 Gradient of the Objective Function

In the previous subsections we computed the derivative of the objective function with respect to the design parameters (θ, A^*) . This was useful for deriving optimality conditions, but it turns out to be inadequate in building a gradient method for numerical purposes. Indeed, as already said the variations of A^* are coupled with those of θ since one must have $A^* \in G_\theta$ almost everywhere in Ω . We would then have to resort to some intricate version of

a projected gradient method in order to stay in the admissible set of design parameters. To avoid this inconvenience, the main idea of this subsection is to introduce a new parametrization of the admissible designs and compute the gradient of the objective function in this new set of variables.

We keep the proportion $\theta \in L^\infty(\Omega; [0, 1])$, but we replace the tensor A^* by its so-called *Y-transform*, introduced by Milton [188], [190], and Cherkaev and Gibianski [76]. As recalled in Remark 2.2.17, the *Y*-transform is the map on the set of symmetric matrices defined by

$$Y(A^*) = \left(\lambda_\theta^+ I_2 - A^* \right) \left((\lambda_\theta^-)^{-1} A^* - I_2 \right)^{-1}.$$

Denoting by $(y_i)_{1 \leq i \leq N}$ the eigenvalues of $Y(A^*)$, A^* belongs to G_θ if and only if

$$\begin{cases} y_i \geq 0 \text{ for all } 1 \leq i \leq N \\ \sum_{i=1}^N \frac{\alpha}{\alpha + y_i} \leq N - 1 \leq \sum_{i=1}^N \frac{\beta}{\beta + y_i}. \end{cases} \quad (3.77)$$

In two space dimensions, i.e., $N = 2$, the set $Y(G_\theta)$, defined by (3.77), has a simpler expression, namely

$$\alpha^2 \leq y_1 y_2 \leq \beta^2, \quad y_1 \geq 0, \quad y_2 \geq 0. \quad (3.78)$$

The advantage of the *Y*-transform is that the set $Y(G_\theta)$, defined by (3.77), does not depend on θ . Its inverse mapping is

$$A^*(Y) = \left(\lambda_\theta^+ I_2 + Y \right) \left((\lambda_\theta^-)^{-1} Y + I_2 \right)^{-1}.$$

We now parametrize a composite design by (θ, Y^*) with $Y^* = Y(A^*)$ for some $A^* \in G_\theta$. This has the advantage that the constraints on θ and Y^* are now uncoupled, making easier the implementation of a projected gradient method for minimizing the objective function.

To simplify the exposition, we work in the case of a single state equation (an obvious generalization yields the same results in the case of several state equations). We keep the notation of Subsection 3.2.2, and we denote by $J^*(\theta, Y^*)$ the objective function $J^*(\theta, A^*(Y^*))$ defined by (3.40). From Theorem 3.2.4 we easily deduce

Theorem 3.2.18 *The objective function $J^*(\theta, Y^*)$ is Gâteaux differentiable in $L^\infty(\Omega; [0, 1] \times \mathcal{M}_N^s)$, and its directional derivative is*

$$\begin{aligned} \delta J^*(\theta, Y^*) = & \int_{\Omega} \delta\theta [g_\alpha(x, u) - g_\beta(x, u) + \ell] dx \\ & + \frac{\beta - \alpha}{\alpha\beta} \int_{\Omega} \delta\theta Z^* (\alpha I_2 + Y^*) (\beta I_2 + Y^*) Z^* \nabla u \cdot \nabla p dx \\ & + \frac{(\beta - \alpha)^2}{\alpha\beta} \int_{\Omega} \theta(1 - \theta) Z^* \delta Y^* Z^* \nabla u \cdot \nabla p dx, \end{aligned} \quad (3.79)$$

where

$$Z^* = \left((\lambda_\theta^-)^{-1} Y^* + I_2 \right)^{-1}$$

and $\delta\theta$ and δY^* are increments in $L^\infty(\Omega; [0, 1] \times \mathcal{M}_N^s)$, u is the solution of the state equation (3.41), and p that of the adjoint state equation (3.48).

Theorem 3.2.18 gives the gradient of the objective function J^* with respect to the two independent variables θ and Y^* . It is therefore possible to compute numerically minimizers of J^* by using a projected gradient method with respect to the design parameters (θ, Y^*) . Note, however, that the set $Y(G_\theta)$ is neither convex nor bounded, which can make the projection step quite difficult. This can be corrected, at least in two dimensions, where, introducing the new variables

$$t_1 = \log y_1, \quad t_2 = \log y_2,$$

(3.78) is equivalent to the simpler (convex and rectangular) set

$$2 \log \alpha \leq t_1 + t_2 \leq 2 \log \beta, \quad t_1 \in \mathbb{R}, \quad t_2 \in \mathbb{R}.$$

3.2.5 Self-adjoint Problems

A special case of Subsection 3.2.3 is obtained when the problem is self-adjoint, i.e., when each adjoint state p_i is equal to the state u_i (or, symmetrically, when each adjoint p_i is equal to $-u_i$). This corresponds to the following choice of objective functions:

$$J^*(\theta, A^*) = \sum_{i=1}^n \int_{\Omega} f_i(x) u_i(x) dx + \ell \int_{\Omega} \theta(x) dx \quad (3.80)$$

or

$$J^*(\theta, A^*) = - \sum_{i=1}^n \int_{\Omega} f_i(x) u_i(x) dx + \ell \int_{\Omega} \theta(x) dx, \quad (3.81)$$

i.e., $g_\alpha(x, u) = g_\beta(x, u) = \pm \sum_{i=1}^n f_i(x) u_i$. It is easily seen that (3.80) yields $p_i = u_i$, while (3.81) yields $p_i = -u_i$. Since for each $1 \leq i \leq n$

$$\int_{\Omega} f_i u_i dx = \int_{\Omega} A^* \nabla u_i \cdot \nabla u_i dx,$$

(3.80) is the sum of the dissipated thermal energies plus a volume constraint. Such a sum can be seen as a global measure of the conductivity in Ω . Therefore, minimizing (3.80) is equivalent to finding the best conducting design made of α and β in Ω , while, on the contrary, minimizing (3.81) amounts to find the worst conducting design. The analysis of these two problems is completely parallel, and, in some sense, dual (see Remark 3.2.32). For the moment we focus on problem (3.81). The results of the previous subsection are still valid but the optimality conditions can be made more precise. We begin first with an easy corollary of Theorem 3.2.14.

Corollary 3.2.19 *If (θ, A^*) is a minimizer of the objective function (3.81), then, introducing the $n \times N$ matrix ξ defined by its rows $(\nabla u_i)_{1 \leq i \leq n}$, we have*

$$A^* : \xi^t \xi = g(\theta, \xi)$$

where $g(\theta, \xi)$ is the lower Hashin-Shtrikman bound defined by (2.28). In other words, A^* is optimal for the lower Hashin-Shtrikman bound. At the points x where ξ does not vanish, it implies that A^* is a tensor corresponding to a sequential laminate of rank at most N (with a matrix of α and a core of β) having orthogonal lamination directions given by the eigenvectors of $\xi^t \xi$. Furthermore, the function $\theta \rightarrow g(\theta, \xi)$ is C^1 , and $Q(x)$ defined by (3.70) reduces to

$$Q(x) = \ell + \frac{\partial g}{\partial \theta}(\theta, \xi).$$

Proof. Coming back to the proof of Theorem 3.2.14, we simply remark that the matrix M , defined by (3.68), i.e.,

$$M = - \sum_{i=1}^n \nabla u_i \otimes \nabla u_i, \quad (3.82)$$

is just equal to $-\xi^t \xi$ and is nonpositive. Therefore, the maximization problem (3.72) is equivalent to the computation of the lower Hashin-Shtrikman

bound in Proposition 2.2.6. In such a case, we know that optimality is achieved by a sequential laminate of rank at most N where α is the matrix and β the core. Even more, if $\xi \neq 0$, then any optimal A^* is the effective tensor of such a sequential laminate. This is an easy consequence of the fact that an optimal A^* belongs necessarily to the hypersurfaces (3.74) or (3.76) of the boundary of G_θ , combined with an argument similar to that at the end of the proof of Theorem 3.2.14. Finally, $g_\alpha = g_\beta$ and $f(\theta, M) = -g(\theta, \xi)$ yields the simplified expression of $Q(x)$. \square

Remark 3.2.20 *The advantage of Corollary 3.2.19 upon Theorem 3.2.14 is that the same type of sequential laminate is always optimal, whatever the value of ξ . Namely, we can restrict our space of admissible designs to sequential laminates for which α is the matrix and β the core. Of course, since $\beta > \alpha > 0$, such laminates are worse conductors than the opposite construction where α is the core and β the matrix. Corollary 3.2.19 indicates also that the knowledge of the full G -closure is not necessary, but only the lower Hashin-Shtrikman bound (2.28) is required in the optimality condition.*

One advantage of self-adjoint problems is that the objective function can itself be written as a minimum of the stored thermal energy. Indeed, by the principle of minimal energy, we have

$$-\int_{\Omega} f_i u_i dx = \min_{v_i \in H_0^1(\Omega)} \left(\int_{\Omega} A^* \nabla v_i \cdot \nabla v_i dx - 2 \int_{\Omega} f_i v_i dx \right), \quad (3.83)$$

and the minimum on the right hand side of (3.83) is achieved by $v_i = u_i$. Introducing vectors f and v of components $(f_i)_{1 \leq i \leq n}$ and $(v_i)_{1 \leq i \leq n}$, respectively, we may rewrite the objective functional as

$$J^*(\theta, A^*) = \min_{v \in H_0^1(\Omega)^n} \left(\int_{\Omega} A^* : \nabla v^t \nabla v dx - 2 \int_{\Omega} f \cdot v dx \right) + \ell \int_{\Omega} \theta dx. \quad (3.84)$$

Since the order of minimization is irrelevant, we obtain

$$\begin{aligned} \min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*) &= \min_{v \in H_0^1(\Omega)^n} \left\{ \min_{(\theta, A^*) \in \mathcal{CD}} \left(\int_{\Omega} A^* : \nabla v^t \nabla v dx \right. \right. \\ &\quad \left. \left. + \ell \int_{\Omega} \theta dx \right) - 2 \int_{\Omega} f \cdot v dx \right\}. \end{aligned} \quad (3.85)$$

The next lemma gives a representation formula for the right hand side of (3.85).

Lemma 3.2.21 *For $v \in H_0^1(\Omega)^n$, we have*

$$\min_{(\theta, A^*) \in \mathcal{CD}} \left(\int_{\Omega} A^* : \nabla v^t \nabla v \, dx + \ell \int_{\Omega} \theta \, dx \right) = \int_{\Omega} QF(\nabla v) \, dx,$$

where $QF(\nabla v)$ is a continuous function with quadratic growth defined by

$$QF(\nabla v) = \min_{0 \leq \theta \leq 1} (g(\theta, \nabla v) + \ell\theta), \quad (3.86)$$

where $g(\theta, \nabla v)$ is the lower Hashin-Shtrikman bound (2.28). Furthermore, there exists a unique minimizer θ in the right hand side of (3.86).

Proof. By the local character of the G -closure (see Theorem 2.1.2), we have

$$\min_{(\theta, A^*) \in \mathcal{CD}} \int_{\Omega} (A^* : \nabla v^t \nabla v + \ell\theta) \, dx = \int_{\Omega} \min_{0 \leq \theta \leq 1} \left(\min_{A^* \in G_\theta} (A^* : \nabla v^t \nabla v) + \ell\theta \right) \, dx.$$

As already said in Corollary 3.2.19, we have

$$\min_{A^* \in G_\theta} (A^* : \nabla v^t \nabla v) = g(\theta, \nabla v),$$

which establishes formula (3.86). By virtue of Remark 2.2.12, $g(\theta, \nabla v)$ is a strictly convex function of θ . Thus, the function of θ on the right hand side of (3.86) admits a unique minimizer on $[0, 1]$. Since g is also a C^1 function, if the minimizer θ^* satisfies $0 < \theta^* < 1$, it necessarily implies that

$$\ell + \frac{\partial g}{\partial \theta}(\theta^*, \nabla v) = 0. \quad (3.87)$$

The properties of $QF(\nabla v)$ are obvious from the definition (2.28) of $g(\theta, \nabla v)$.

□

Thanks to Lemma 3.2.21 we have established an equivalent formulation of the relaxed optimal design problem.

Theorem 3.2.22 *For the self-adjoint objective functional (3.81), the relaxed optimal design problem*

$$\min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*) \quad (3.88)$$

is equivalent to its nonlinear conductivity formulation

$$\min_{v \in H_0^1(\Omega)^n} \left(\int_{\Omega} QF(\nabla v) \, dx - 2 \int_{\Omega} f \cdot v \, dx \right), \quad (3.89)$$

where QF is defined by (3.86), in the sense that a minimizer of one yields a minimizer for the other by using the optimality conditions in v or in (θ, A^*) .

Remark 3.2.23 It is possible to compute more explicitly $QF(\nabla v)$ but we shall not need it here (see [18], [25]). Lemma 3.2.21 together with (3.85) gives a new insight into the optimality criteria algorithm. Recall that the principle of such a numerical method (see Remark 3.2.9) is to iteratively compute the state u and the design parameters (θ, A^*) thanks to the optimality conditions. By virtue of (3.85), the optimal design problem appears as a double minimization with respect to u and (θ, A^*) . As proposed in [15], [17], this double minimization can be performed by an alternate direction algorithm, which amounts to minimizing iteratively and separately in u and (θ, A^*) . Minimizing in (θ, A^*) for a given u is precisely the computation that is done in the proof of Lemma 3.2.21. Remarkably, the condition (3.87) for a minimizer in θ corresponds exactly to the optimality condition furnished by Theorem 3.2.14 and Corollary 3.2.19. Therefore, with formulation (3.85) of the problem, an optimality criteria method is simply a descent method, and this method will always converge to a stationary point of the objective function (see Remark 5.1.3 for more details).

One can also write the objective functional as the minimum of the energy for the original, unrelaxed, problem. Recalling that

$$J(\chi) = - \sum_{i=1}^n \int_{\Omega} f_i(x) u_{i,\chi}(x) dx + \ell \int_{\Omega} \chi(x) dx,$$

with $u_{i,\chi}$ the solution of (3.11), and taking into account

$$- \int_{\Omega} f_i u_{i,\chi} dx = \min_{v_i \in H_0^1(\Omega)} \left(\int_{\Omega} a_{\chi} |\nabla v_i|^2 dx - 2 \int_{\Omega} f_i v_i dx \right),$$

by switching the two minimizations we have

$$\inf_{\chi \in L^\infty(\Omega; \{0,1\})} J(\chi) = \inf_{v \in H_0^1(\Omega)^n} \left\{ \int_{\Omega} \min_{\chi=0,1} (a_{\chi} |\nabla v|^2 dx + \ell \chi) dx - 2 \int_{\Omega} f \cdot v dx \right\}.$$

The pointwise minimization in χ is easy and yields

$$\inf_{\chi \in L^\infty(\Omega; \{0,1\})} J(\chi) = \inf_{v \in H_0^1(\Omega)^n} \left\{ \int_{\Omega} F(\nabla v) dx - 2 \int_{\Omega} f \cdot v dx \right\}, \quad (3.90)$$

with

$$F(\lambda) = \min (\alpha |\lambda|^2 + \ell, \beta |\lambda|^2).$$

The minimization problem on the right hand side of (3.90) appears as a nonlinear conductivity problem. It is equivalent to the original optimization problem in the sense that, if there exists a minimizer for one problem, then the optimality condition in χ or that in v yields a minimizer for the other one. Kohn and Strang [152] recognized that relaxing the optimal design problem is actually equivalent to relaxing the nonlinear conductivity problem. In the former case, relaxation is obtained through homogenization, while in the latter case it is obtained through quasiconvexification (see Remark 3.2.25, for details on this notion). This is a deep link that we shall use later in Chapter 4.

Theorem 3.2.24 *The relaxation of (3.90) is given by (3.89), and $QF(\lambda)$ is the quasiconvex envelope of $F(\lambda)$ defined by (3.91) or (3.93). In other words,*

1. *there exists at least one solution of the relaxed conductivity problem (3.89);*
2. *up to a subsequence, any minimizing sequence of (3.90) converges weakly in $H_0^1(\Omega)^n$ to a minimizer of (3.89);*
3. *for any minimizer v of (3.89), there exists a minimizing sequence of (3.90) that converges to v weakly in $H_0^1(\Omega)^n$, and*

$$\inf_{v \in H_0^1(\Omega)^n} \int_{\Omega} (F(\nabla v) - 2f \cdot v) dx = \min_{v \in H_0^1(\Omega)^n} \int_{\Omega} (QF(\nabla v) - 2f \cdot v) dx.$$

Remark 3.2.25 *We recall the notion of quasiconvexification in the sense of Morrey (see, e.g., [37], [89], [197], [218], [234]) for a real-valued functional F defined on \mathbb{R}^{nN} . Assume that F is continuous and satisfies*

$$c(|\lambda|^2 - 1) \leq F(\lambda) \leq C(|\lambda|^2 + 1), \quad \lambda \in \mathbb{R}^{nN},$$

for some positive constants $0 < c \leq C$. The quasiconvex envelope of F (or its quasiconvexification) is defined for every λ in \mathbb{R}^{nN} as

$$QF(\lambda) = \inf_{\phi \in H_0^1(Y)^n} \int_Y F(\lambda + \nabla \phi(x)) dx, \quad (3.91)$$

where $Y = (0, 1)^N$ is the unit cube of \mathbb{R}^N . The function QF is a locally Lipschitz function, as is easily seen upon noting that QF is rank-1 convex and

satisfies the same growth assumption as F (see, e.g., [181]). The main property of the quasiconvexification is its link with relaxation. For any bounded open set Ω of \mathbb{R}^N , the lower semicontinuous envelope in $H_0^1(\Omega)^n$ of

$$I(u) = \int_{\Omega} F(\nabla u) dx$$

for the sequential weak topology of $H_0^1(\Omega)^n$, i.e., the functional

$$RI(u) = \inf_{\substack{u_\epsilon \in H_0^1(\Omega)^n \\ u_\epsilon \rightharpoonup u \text{ weakly in } H_0^1(\Omega)^n}} \liminf I(u_\epsilon)$$

is given by

$$RI(u) = \int_{\Omega} QF(\nabla u) dx. \quad (3.92)$$

According to [38] and [89], in the definition (3.91) of QF it is possible to replace the space $H_0^1(Y)^n$ of test fields by that of periodic fields $H_{\#}^1(Y)^n$ (at least when F is bounded below, continuous, and grows, at most, quadratically). Thus, QF is equivalently defined by

$$QF(\lambda) = \inf_{\phi \in H_{\#}^1(Y)^n} \int_Y F(\lambda + \nabla \phi(x)) dx, \quad (3.93)$$

where $H_{\#}^1(Y)^n$ is the Sobolev space of Y -periodic vector-valued functions. A proof of (3.92) may be found in [89], Theorems 1.1 and 2.1 in Chapter 5, and in [2], Statement 3.7.

Proof. According to [38], [89], the quasiconvexification of F is defined as the right hand side of (3.93). Let us denote this quasiconvexification by \tilde{F} . We only have to prove that \tilde{F} is indeed equal to QF as defined by (3.86). Since F is defined as a minimum, a simple switch in the minimizations leads to

$$\tilde{F}(\lambda) = \inf_{\chi \in L^\infty(Y; \{0,1\})} \left(A^* : \lambda^t \lambda + \ell \int_Y \chi dy \right),$$

where, denoting by $(e_i)_{1 \leq i \leq N}$ the canonical basis of \mathbb{R}^N , A^* is the homogenized tensor defined by its entries

$$A^* e_i \cdot e_j = \min_{\phi \in H_{\#}^1(Y)} \int_Y (\chi \alpha + (1 - \chi) \beta)(e_i + \nabla \phi) \cdot (e_j + \nabla \phi) dy.$$

Denoting by θ the average of χ in Y , by definition (2.8) A^* belongs to P_θ , the closure of which is G_θ . Thus, we have

$$\tilde{F}(\lambda) = \inf_{0 \leq \theta \leq 1} \left(\inf_{A^* \in G_\theta} (A^* : \lambda^t \lambda) + \ell\theta \right). \quad (3.94)$$

According to Corollary 3.2.19, $\inf_{A^* \in G_\theta} (A^* : \lambda^t \lambda)$ is equal to the lower Hashin-Shtrikman bound $g(\theta, \lambda)$. Therefore, (3.94), combined with Lemma 3.2.21, implies that $\tilde{F}(\lambda) = QF(\lambda)$. \square

Remark 3.2.26 *A useful feature of minimizers of convex functionals is that they are always global minimizers. It is not known if such a property holds true also for minimizers of quasiconvex functionals. It is a fundamental issue from a numerical point of view since most minimization algorithms cannot escape from local minima. If all minima are global, then these algorithms can safely be used, since they will converge to global minima. Numerical evidence (see Chapter 5) suggests that, for the quasiconvex functionals arising from self-adjoint shape optimization problems, there are no local minima that are not global. We conjecture this property to be true for the quasiconvex function (3.86). More precisely, any local minimizer in $H^1(\Omega)^n$ of (3.89) would be a global minimizer (at least under some geometrical condition on the domain). Surprisingly, this result is true for any quasiconvex function if the boundary conditions are affine (with no force, i.e., $f = 0$) and the domain is star-shaped. The (classical) argument goes as follows. For the sake of simplicity, we consider a quasiconvex function QF with quadratic growth, and we assume that the domain Ω is star-shaped with respect to the origin (this means that, for any point in Ω , the segment connecting the origin to this point is included in Ω). For a given $n \times N$ matrix ξ (the affine boundary condition), the problem*

$$\min_{(v(x)-\xi x) \in H_0^1(\Omega)^n} \int_{\Omega} QF(\nabla v) dx \quad (3.95)$$

admits ξx as a global minimizer (by definition of quasiconvexity). Let u be a local minimizer of (3.95). For any $0 < t \leq 1$, define the function

$$u_t(x) = \begin{cases} t u(t^{-1}x) & \text{in } t\Omega \\ \xi x & \text{in } \Omega \setminus t\Omega \end{cases}$$

where $t\Omega \subset \Omega$ thanks to the star-shaped character of Ω . It is easy to check that $(u_t(x) - \xi x) \in H_0^1(\Omega)^n$ and the sequence u_t converges strongly to u as t

goes to one. Since, by rescaling,

$$\int_{\Omega} QF(\nabla u_t) dx = t^N \int_{\Omega} QF(\nabla u) dx + (1 - t^N)|\Omega|QF(\xi),$$

if u were not a global minimizer, we would have for any $0 < t < 1$

$$\int_{\Omega} QF(\nabla u_t) dx < \int_{\Omega} QF(\nabla u) dx,$$

which would contradict the fact that u is a local minimizer in $H^1(\Omega)^n$. Thus u is indeed a global minimizer. Eventually, we emphasize that there must be a geometrical constraint on the domain Ω since it is known [224] that for the two-dimensional annulus with affine boundary conditions there exist local minimizers that are not global ones.

In the case of a single state equation, i.e., $n = 1$, relaxation simplifies considerably, since quasiconvexification reduces to convexification.

Theorem 3.2.27 Assume that $n = 1$. Then, the quasiconvex envelope $QF(\lambda)$ of $F(\lambda)$ coincides with its convex envelope $CF(\lambda)$ and, for $\lambda \in \mathbb{R}^N$, if $\ell > 0$ we have

$$QF(\lambda) = CF(\lambda) = \begin{cases} \alpha|\lambda|^2 + \ell & \text{if } \sqrt{\frac{\ell\beta}{\alpha(\beta-\alpha)}} \leq |\lambda| \\ 2\sqrt{\frac{\ell\alpha\beta}{\beta-\alpha}}|\lambda| - \frac{\ell\alpha}{\beta-\alpha} & \text{if } \sqrt{\frac{\ell\alpha}{\beta(\beta-\alpha)}} < |\lambda| < \sqrt{\frac{\ell\beta}{\alpha(\beta-\alpha)}} \\ \beta|\lambda|^2 & \text{if } |\lambda| \leq \sqrt{\frac{\ell\alpha}{\beta(\beta-\alpha)}} \end{cases}$$

while, if $\ell \leq 0$, $QF(\lambda) = CF(\lambda) = F(\lambda) = \alpha|\lambda|^2 + \ell$.

Remark 3.2.28 In the context of Theorem 3.2.27, i.e., for a single state equation, the nonlinear conductivity formulation of the relaxed optimal design problem is a convex minimization problem. Therefore, (3.89) has only global minimizers, and thus the relaxed formulation (3.86) has only global minimizers too. This is a simple example for which it is possible to prove that any local minimizer of the relaxed formulation is actually a global minimizer.

Proof. According to Lemma 3.2.21, in order to compute QF we first compute explicitly the lower Hashin-Shtrikman bound $g(\theta, \nabla v)$, defined by (2.28), i.e.,

$$g(\theta, \nabla v) = \alpha |\nabla v|^2 + (1 - \theta) \max_{\eta \in \mathbb{R}^N} \left[2\nabla v \cdot \eta - (\beta - \alpha)^{-1} |\eta|^2 - \theta g(\eta) \right],$$

where $g(\eta)$ is the nonlocal term defined by

$$g(\eta) = \max_{k \in \mathbb{Z}^N, k \neq 0} \frac{|\eta \cdot k|^2}{\alpha |k|^2}.$$

Since $\eta \in \mathbb{R}^N$, it is easy to see that $g(\eta) = \alpha^{-1} |\eta|^2$ and thus

$$g(\theta, \nabla v) = \frac{\alpha \beta}{\theta \beta + (1 - \theta) \alpha} |\nabla v|^2.$$

Minimizing with respect to θ in (3.86) leads to the desired result. \square

Instead of problem (3.81), we can consider the physically more relevant problem (3.80). In this case, too, the objective function can be written as the result of a minimization. However, we need to use the principle of minimal complementary energy, instead of primal energy as in (3.83), which gives

$$\int_{\Omega} f_i u_i dx = \min_{\substack{\tau_i \in L^2(\Omega)^N \\ -\operatorname{div} \tau_i = f_i \text{ in } \Omega}} \int_{\Omega} A^{*-1} \tau_i \cdot \tau_i dx, \quad (3.96)$$

and the minimum on the right hand side of (3.96) is achieved by $\sigma_i = A^* \nabla u_i$. Introducing a vector f of components $(f_i)_{1 \leq i \leq n}$ and a matrix τ , defined by its rows $(\tau_i)_{1 \leq i \leq n}$, the objective functional (3.80) can be rewritten

$$J^*(\theta, A^*) = \min_{\substack{\tau \in L^2(\Omega)^{nN} \\ -\operatorname{div} \tau = f \text{ in } \Omega}} \int_{\Omega} A^{*-1} : \tau^t \tau dx + \ell \int_{\Omega} \theta dx. \quad (3.97)$$

Since the order of minimization is irrelevant, we can switch the minimization with respect to $(\theta, A^*) \in \mathcal{CD}$ and that with respect to $\tau \in L^2(\Omega)^{nN}$. For a fixed τ the minimization in (θ, A^*) is performed in the next lemma.

Lemma 3.2.29 *For $\tau \in L^2(\Omega)^{nN}$, we have*

$$\min_{(\theta, A^*) \in \mathcal{CD}} \left(\int_{\Omega} A^{*-1} : \tau^t \tau dx + \ell \int_{\Omega} \theta dx \right) = \int_{\Omega} QF(\tau) dx,$$

where $QF(\tau)$ is a continuous function with quadratic growth, defined by

$$QF(\tau) = \min_{0 \leq \theta \leq 1} (g(\theta, \tau) + \ell\theta), \quad (3.98)$$

with $g(\theta, \tau)$ the lower Hashin-Shtrikman bound on complementary energies.

Proof. By the local character of the G -closure (see Theorem 2.1.2), we have

$$\min_{(\theta, A^*) \in \mathcal{CD}} \int_{\Omega} (A^{*-1} : \tau^t \tau + \ell\theta) dx = \int_{\Omega} \min_{0 \leq \theta \leq 1} \left(\min_{A^* \in G_\theta} A^{*-1} : \tau^t \tau + \ell\theta \right) dx.$$

A result similar to Proposition 2.2.6 in the context of complementary energies gives the value of the Hashin-Shtrikman lower bound

$$g(\theta, \tau) = \min_{A^* \in G_\theta} A^{*-1} (:\tau^t \tau),$$

which is again a strictly convex and C^1 function of θ . Thus, the function of θ on the right hand side of (3.98) admits a unique minimizer in $[0, 1]$. \square

Lemma 3.2.29 leads to an equivalent formulation of the relaxed optimal design problem.

Theorem 3.2.30 *For the self-adjoint objective functional (3.80), the relaxed optimal design problem*

$$\min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*) \quad (3.99)$$

is equivalent to its nonlinear dual conductivity formulation

$$\min_{\substack{\tau \in L^2(\Omega)^{nN} \\ -\operatorname{div} \tau = f \text{ in } \Omega}} \int_{\Omega} QF(\tau) dx, \quad (3.100)$$

where QF is defined by (3.98), in the sense that a minimizer of one yields a minimizer for the other by using the optimality conditions in τ or in (θ, A^) .*

The original objective functional can also be rewritten as the minimum of the complementary energy, and switching the two minimizations yields

$$\inf_{\chi \in L^\infty(\Omega; \{0,1\})} J(\chi) = \inf_{\substack{\tau \in L^2(\Omega)^{nN} \\ -\operatorname{div} \tau = f \text{ in } \Omega}} \int_{\Omega} F(\tau) dx, \quad (3.101)$$

with

$$F(\tau) = \min \left(\alpha^{-1} |\tau|^2 + \ell, \beta^{-1} |\tau|^2 \right).$$

The minimization problem on the right hand side of (3.101) appears as a nonlinear dual conductivity problem, which is equivalent to the original optimization problem in the sense that minimizers (if any) are related through the optimality conditions in χ or in τ . As before, there is a link between the relaxations of the two problems in (3.101), i.e., between homogenization theory and quasiconvexification.

Theorem 3.2.31 *The relaxation of the right hand side of (3.101) is given by (3.100), and $QF(\tau)$ is the quasiconvex envelope of $F(\tau)$ defined by*

$$QF(\tau) = \inf_{\substack{\eta \in L^2_\#(Y; \mathbf{R}^{nN}) \\ -\operatorname{div} \eta = 0 \text{ in } Y \text{ and } \int_Y \eta dx = 0}} \int_Y F(\tau + \eta(x)) dx, \quad (3.102)$$

where $Y = (0, 1)^N$ is the unit cube.

The proof of Theorem 3.2.31 is in the same spirit as that of Theorem 3.2.24 (although in a dual setting), so we omit it (its counterpart for elasticity is explored in greater details; see Theorem 4.1.15). The general theory of quasiconvexification was originally developed for functionals depending on gradients (see, e.g., [89], or Remark 3.2.25). However, it has been further extended to the case at hand, namely, fields with prescribed divergence (see, e.g., [107], or Remark 4.1.16).

Remark 3.2.32 *The link between the optimal design problem (3.80) and its nonlinear dual conductivity formulation is of paramount importance when the weak conductor α is allowed to degenerate to zero. Recall that in this limit we recover a shape optimization problem (see Subsection 3.1.4). Indeed, homogenization theory is inoperative when one of the phases is degenerate, while the quasiconvexification approach is still useful. More precisely, Theorem 3.2.31 is still valid when $\alpha = 0$, although Theorem 3.2.30 makes no sense. We shall give a complete proof of this fact in the elasticity setting (see Section 4.2), which works equally well in the present conductivity framework.*

3.2.6 Counterexample to the Uniqueness of Optimal Designs

In Subsection 3.2.1 the existence of generalized optimal designs was proved and the question was raised about their possible uniqueness. We now give

an explicit example where there is an infinite number of optimal designs, all of which are all global minimizers. Another striking feature of this example is that the optimal designs are actually classical designs, and not only generalized ones. This example is based on the so-called *coated confocal ellipsoid construction* of Tartar [274] (also introduced by Bergman [51], [52] and Milton [187] in the context of complex dielectric composites), which generalizes the well known coated concentric sphere construction [81], [131].

Let Ω be a smooth bounded open set of \mathbb{R}^N . Let $(\xi_i)_{1 \leq i \leq n}$ be a collection of n vectors in \mathbb{R}^N . We consider a mixture of the two isotropic conducting phases α and β , with $0 < \alpha < \beta$, i.e., a characteristic function χ in $L^\infty(\Omega; \{0, 1\})$ that yields an overall conductivity, defined by

$$a_\chi(x) = \alpha\chi(x) + \beta(1 - \chi(x)). \quad (3.103)$$

This design χ is tested in n conductivity problems (the state equations) with no source terms and with affine boundary conditions for the electrical potential or the temperature $u_{i,\chi}(x)$, the unique solution in $H^1(\Omega)$ of

$$\begin{cases} -\operatorname{div}(a_\chi \nabla u_{i,\chi}) = 0 & \text{in } \Omega \\ u_{i,\chi} = \xi_i \cdot x & \text{on } \partial\Omega. \end{cases} \quad (3.104)$$

The goal of this optimal design problem is to find the best isolating mixture (or the worst conducting mixture) in Ω . More precisely, the performance of χ is measured by the objective function

$$J(\chi) = \sum_{i=1}^n \int_{\partial\Omega} a_\chi \nabla u_{i,\chi} \cdot n \, ds + \ell \int_{\Omega} \chi \, dx, \quad (3.105)$$

where ds is the surface measure on $\partial\Omega$ and ℓ is a fixed Lagrange multiplier for a volume constraint on material α . Finally, the optimal design problem is the minimization

$$\inf_{\chi \in L^\infty(\Omega; \{0, 1\})} J(\chi). \quad (3.106)$$

An easy integration by parts in the state equations shows that the objective function is also equal to

$$J(\chi) = \sum_{i=1}^n \int_{\Omega} a_\chi \nabla u_{i,\chi} \cdot \nabla u_{i,\chi} \, dx + \ell \int_{\Omega} \chi \, dx,$$

which can be rewritten as

$$J(\chi) = \min_{u \in D_\xi} \int_{\Omega} a_\chi \nabla u : \nabla u \, dx + \ell \int_{\Omega} \chi \, dx, \quad (3.107)$$

where ξ is the $n \times N$ matrix of lines $(\xi_i)_{1 \leq i \leq n}$, and D_ξ denotes the space of vector-valued functions

$$D_\xi = \left\{ \xi x + H_0^1(\Omega; \mathbb{R}^n) \right\}.$$

Problem (3.106) is self-adjoint, and as in Subsection 3.2.5, the two minimizations with respect to χ and u can be switched to yield an equivalent formulation which is

$$\inf_{u \in D_\xi} \int_{\Omega} F(\nabla u) dx, \quad (3.108)$$

with

$$F(\nabla u) = \min \left(\alpha |\nabla u|^2 + \ell, \beta |\nabla u|^2 \right). \quad (3.109)$$

As already seen in Theorem 3.2.24, the integrand F is not quasiconvex and thus the functional (3.108) must be relaxed by quasiconvexification. The resulting relaxed functional was proved to be

$$\min_{u \in D_\xi} \int_{\Omega} QF(\nabla u) dx, \quad (3.110)$$

where QF is the quasiconvex envelope of F defined by (3.91). Formula (3.86) in Lemma 3.2.21 furnishes an alternative definition of this quasiconvex envelope

$$QF(\xi) = \min_{0 \leq \theta \leq 1} (g(\theta, \xi) + \ell\theta), \quad (3.111)$$

where $g(\theta, \xi)$ is the lower Hashin-Shtrikman bound defined by (2.28)

$$g(\theta, \xi) = \min_{A^* \in G_\theta} (A^* : \xi^t \xi). \quad (3.112)$$

To make this example fully explicit, the choice of affine boundary conditions is crucial. Indeed, because of the affine character of the boundary conditions and by definition (3.93) of QF , the simple function $u_0(x) = \xi x$ is actually a global minimizer of the relaxed functional (3.110), and the value of the minimum is

$$|\Omega| QF(\xi) = \inf_{u \in D_\xi} \int_{\Omega} F(\nabla u) dx. \quad (3.113)$$

After addressing these preliminaries, we can state the main result of this subsection, Theorem 3.2.33, which was first established in [18], [25].

Theorem 3.2.33 Let $0 \leq \nu_1 \leq \dots \leq \nu_N$ be the singular values of the matrix ξ . Assume that the affine boundary conditions are such that

$$\nu_1 > \frac{\sum_{i=1}^N \nu_i}{\beta/\alpha + N - 1}, \quad (3.114)$$

and that the Lagrange multiplier ℓ satisfies

$$\frac{\left(\sum_{i=1}^N \nu_i\right)^2}{\beta/\alpha + N - 1} < \frac{\ell}{\beta - \alpha} < (\beta/\alpha + N - 1)\nu_1^2. \quad (3.115)$$

Then there exist infinitely many classical minimizers of (3.106) in $L^\infty(\Omega; \{0, 1\})$, which are all global minimizers.

The optimal designs χ that minimize (3.106) are ellipsoidal inclusions of all sizes of phase β into a matrix of phase α (see Figure 3.3).

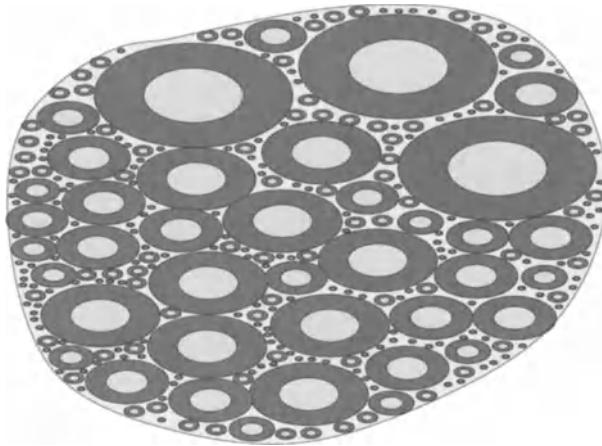


Figure 3.3: Optimal design obtained by filling the domain Ω with confocal ellipsoids.

Remark 3.2.34 Recall that the singular values of ξ are the positive square roots of the eigenvalues of the symmetric matrix $\xi^t \xi \in \mathcal{M}_N^s$. Condition (3.114) is satisfied for $\xi = I_2$ and, more generally, for matrices ξ such that $\xi^t \xi$ is close to I_2 . In particular, Theorem 3.2.33 applies only if there are at least $n \geq N$ state equations.

In order to prove this result, we revisit the explicit construction of coated confocal ellipsoids, due to Tartar [274], which will turn out to be optimal designs for (3.106) (in the slightly different context of complex dielectric composites, such confocal ellipsoids were already introduced by Bergman [51], [52] and Milton [187]). Let $m = (m_1, \dots, m_N) \in \mathbb{R}^N$ be given numbers. For any $\rho \in \mathbb{R}$ such that

$$\rho + \min_{1 \leq i \leq N} m_i > 0,$$

we define an ellipsoid B_ρ by its boundary

$$\sum_{j=1}^N \frac{x_j^2}{\rho + m_j} = 1. \quad (3.116)$$

Actually, (3.116) defines an implicit function $\rho(x)$ from \mathbb{R}^N into \mathbb{R} , called the ellipsoidal coordinate (see pp.20-27 in [157]). We first define a class of harmonic functions defined in terms of ρ .

Lemma 3.2.35 *A function $u(x) = x_j f(\rho(x))$, defined from \mathbb{R}^N into \mathbb{R} , is harmonic, i.e., $\Delta u = 0$, if and only if the derivative of f satisfies*

$$f'(\rho) = \frac{C}{(\rho + m_j)g_m(\rho)}, \quad (3.117)$$

where C is a constant and $g_m(\rho)$ is proportional to the volume of B_ρ , namely,

$$g_m(\rho) = \prod_{k=1}^N (\rho + m_k)^{1/2}. \quad (3.118)$$

Proof. A straightforward derivation yields

$$\Delta u = 2f'(\rho) \frac{\partial \rho}{\partial x_j} + x_j f'(\rho) \Delta \rho + x_j f''(\rho) |\nabla \rho|^2. \quad (3.119)$$

To compute the partial derivatives of ρ , we differentiate (3.116) to get

$$\frac{2x_j}{\rho + m_j} - \frac{\partial \rho}{\partial x_j} \sum_{k=1}^N \frac{x_k^2}{(\rho + m_k)^2} = 0,$$

which yields

$$\frac{\partial \rho}{\partial x_j} = \frac{x_j}{\rho + m_j} \sigma_m \quad (3.120)$$

with

$$\frac{1}{\sigma_m} = \frac{1}{2} \sum_{k=1}^N \frac{x_k^2}{(\rho + m_k)^2}. \quad (3.121)$$

In particular, (3.120) implies that

$$|\nabla \rho|^2 = 2\sigma_m.$$

Differentiating (3.120) and (3.121) leads to

$$\frac{\partial^2 \rho}{\partial x_j^2} = \frac{\sigma_m}{\rho + m_j} - \frac{x_j \sigma_m}{(\rho + m_j)^2} \frac{\partial \rho}{\partial x_j} + \frac{x_j}{\rho + m_j} \frac{\partial \sigma_m}{\partial x_j}$$

and

$$\frac{\partial \sigma_m}{\partial x_j} = -\sigma_m^2 \left(\frac{x_j}{(\rho + m_j)^2} - \frac{\partial \rho}{\partial x_j} \sum_{k=1}^N \frac{x_k^2}{(\rho + m_k)^3} \right),$$

which, after combining terms, gives

$$\frac{\partial^2 \rho}{\partial x_j^2} = \frac{\sigma_m}{\rho + m_j} - 2 \frac{x_j^2 \sigma_m^2}{(\rho + m_j)^3} + \frac{x_j^2 \sigma_m^3}{(\rho + m_j)^2} \sum_{k=1}^N \frac{x_k^2}{(\rho + m_k)^3}.$$

Eventually, by summation we get

$$\Delta \rho = \sigma_m \sum_{k=1}^N \frac{1}{\rho + m_k}.$$

Plugging these derivatives of ρ into (3.119) and simplifying by $x_j \sigma_m$ yields

$$f'(\rho) \left(\frac{2}{\rho + m_j} + \sum_{k=1}^N \frac{1}{\rho + m_k} \right) + 2f''(\rho) = 0,$$

which by integration gives the only possible solutions (3.117). \square

We use Lemma 3.2.35 to compute special solutions of the conductivity equation in an homogeneous anisotropic medium containing an inclusion of the two phases. Let $\rho^+ > \rho^- > -\min_i m_i$ be two numbers defining two confocal ellipsoids B_{ρ^-} and B_{ρ^+} of equation (3.116). We define a conductivity tensor A by

$$A(x) = \begin{cases} \beta I_2 & \text{in } B_{\rho^-} \\ \alpha I_2 & \text{in } B_{\rho^+} \setminus B_{\rho^-} \\ \sum_{i=1}^N \lambda_i e_i \otimes e_i & \text{in } \mathbb{R}^N \setminus B_{\rho^+} \end{cases}$$

with $(\lambda_1, \dots, \lambda_N) \in \mathbb{R}_+^N$.

Lemma 3.2.36 Define a function $u_j(x)$ by

$$\begin{cases} u_j(x) = x_j & \text{in } B_{\rho^-} \\ u_j(x) = x_j \left(1 + \frac{\beta - \alpha}{2\alpha} g_m(\rho^-) \int_{\rho^-}^{\rho(x)} \frac{dr}{(r + m_j)g_m(r)} \right) & \text{in } B_{\rho^+} \setminus B_{\rho^-} \\ u_j(x) = x_j \left(1 + \frac{\beta - \alpha}{2\alpha} g_m(\rho^-) \int_{\rho^-}^{\rho^+} \frac{dr}{(r + m_j)g_m(r)} \right) & \text{in } \mathbb{R}^N \setminus B_{\rho^+}. \end{cases} \quad (3.122)$$

Then $u_j(x)$ satisfies

$$-\operatorname{div}(A(x)\nabla u_j(x)) = 0 \text{ in } \mathbb{R}^N \quad (3.123)$$

if we have

$$\frac{1}{\lambda_j - \alpha} = \frac{1}{\beta - \alpha} \frac{g_m(\rho^+)}{g_m(\rho^-)} + \frac{g_m(\rho^+)}{2\alpha} \int_{\rho^-}^{\rho^+} \frac{dr}{(r + m_j)g_m(r)}. \quad (3.124)$$

Remark 3.2.37 The main interest of the above solution u_j is that its gradient is constant in $\mathbb{R}^N \setminus B_{\rho^+}$ (it is also constant in B_{ρ^-}). It has the important consequence that the two ellipsoids, filled with α and β , could be replaced by the anisotropic conductor surrounding them without changing the solution of (3.123) in $\mathbb{R}^N \setminus B_{\rho^+}$ (the last line of (3.122) would be the solution everywhere). In other words, including these two confocal ellipsoids in this anisotropic material does not change the field outside them.

Remark 3.2.38 The ellipsoids B_{ρ^\pm} can always be rescaled through multiplication of ρ^+ , ρ^- and the m_i by a positive number. This does not change the result of Lemma 3.2.36 which holds true for the same anisotropic medium $(\lambda_1, \dots, \lambda_N)$.

Proof. By Lemma 3.2.35 each line of (3.122) is a solution of the equation in its domain of definition. It remains to check that they match at the interfaces ∂B_{ρ^-} and ∂B_{ρ^+} . The normal to an ellipsoid B_ρ is $n = (n_1, \dots, n_N)$ with

$$n_k = \sqrt{\frac{\sigma_m}{2}} \frac{x_k}{\rho + m_k}.$$

It is easily seen that u_j as well as $(A\nabla u_j) \cdot n$ are continuous through ∂B_{ρ^-} . Similarly, u_j is continuous through ∂B_{ρ^+} , but $(A\nabla u_j) \cdot n$ is continuous if and only if condition (3.124) holds true. \square

The next step is to find the range of anisotropic conductivity tensors $\text{diag}(\lambda_1, \dots, \lambda_N)$ such that, for every $1 \leq j \leq N$, (3.123) admits a solution of the type (3.122) with the same ellipsoids.

Lemma 3.2.39 *Define a constant $0 < \theta < 1$ by*

$$1 - \theta = \frac{g_m(\rho^-)}{g_m(\rho^+)} = \prod_{k=1}^N \sqrt{\frac{\rho^- + m_k}{\rho^+ + m_k}}. \quad (3.125)$$

Then, $(\lambda_1, \dots, \lambda_N)$ satisfies

$$\lambda_\theta^- = \left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta} \right)^{-1} < \lambda_j < \lambda_\theta^+ = \alpha\theta + \beta(1-\theta) \quad \text{for } 1 \leq j \leq N, \quad (3.126)$$

and

$$\sum_{j=1}^N \frac{1}{\lambda_j - \alpha} = \frac{N-1}{\lambda_\theta^+ - \alpha} + \frac{1}{\lambda_\theta^- - \alpha}. \quad (3.127)$$

Conversely, for any $(\lambda_1, \dots, \lambda_N)$ satisfying (3.126) and (3.127), there exists a family of positive numbers $m_k > 0$ and two radii ρ^- and ρ^+ such that (3.125) and (3.124), for any $1 \leq j \leq N$, hold true.

Remark 3.2.40 *Since $g_m(\rho)$ is, up to a constant multiplicative factor, the volume of the ellipsoid B_ρ , the constant θ defined by (3.125) is the proportion of phase α in B_{ρ^+} .*

Proof. Summing up (3.124) for j between 1 and N leads to

$$\sum_{j=1}^N \frac{1}{\lambda_j - \alpha} = \frac{N}{(\beta - \alpha)} \frac{g_m(\rho^+)}{g_m(\rho^-)} + \frac{g_m(\rho^+)}{2\alpha} \int_{\rho^-}^{\rho^+} \sum_{j=1}^N \frac{dr}{(r + m_j) g_m(r)}.$$

But differentiating (3.118) gives

$$\frac{1}{2} \sum_{j=1}^N \frac{1}{r + m_j} = \frac{g'_m(r)}{g_m(r)},$$

and thus

$$\sum_{j=1}^N \frac{1}{\lambda_j - \alpha} = \frac{N}{(\beta - \alpha)} \frac{g_m(\rho^+)}{g_m(\rho^-)} - \frac{1}{\alpha} \left(1 - \frac{g_m(\rho^+)}{g_m(\rho^-)} \right),$$

which, combined with (3.125), yields (3.127). On the other hand, since the last term in (3.124) is positive, we have

$$\frac{1}{\lambda_j - \alpha} > \frac{1}{(\beta - \alpha)} \frac{g_m(\rho^+)}{g_m(\rho^-)},$$

which, with (3.125), implies that $\lambda_j < \lambda_\theta^+$. Eventually, this last inequality, for all $1 \leq j \leq N$, and (3.127) implies that $\lambda_\theta^- < \lambda_j$ (see Remark 2.2.14).

To prove the converse, we first make some simplifications. It is easily seen that the result of Lemma 3.2.36 is invariant by rescaling the ellipsoids. Therefore, with no loss of generality we can assume that

$$\rho^- = 0, \quad \rho^+ = 1, \quad \text{and } m_k > 0.$$

We then use the Y -transformation (introduced in Remark 2.2.17), which is the following change of variables

$$y_j = \frac{\lambda_\theta^+ - \lambda_j}{(\lambda_\theta^-)^{-1}\lambda_j - 1}.$$

By Remark 2.2.17, this transforms the set of $(\lambda_1, \dots, \lambda_N)$ satisfying (3.126) and (3.127) into

$$\sum_{j=1}^N \frac{\beta}{\beta + y_j} = N - 1 \quad \text{and} \quad 0 < y_j < +\infty \text{ for } 1 \leq j \leq N.$$

We make a further change of variables by

$$z_j = \frac{2\theta}{(1-\theta)\beta + y_j},$$

such that the set defined by (3.126) and (3.127) is now equivalent to

$$\sum_{j=1}^N z_j = \frac{2\theta}{1-\theta} \quad \text{and} \quad 0 < z_j < +\infty \text{ for } 1 \leq j \leq N.$$

From (3.124) we eventually get

$$z_j = \prod_{k=1}^N \sqrt{1+m_k} \int_0^1 \frac{dr}{(r+m_j) \prod_{k=1}^N \sqrt{r+m_k}}.$$

Following Tartar (see Proposition 6 of [274]), we now prove by a degree argument that, when m spans \mathbb{R}_+^N , $z = (z_j)_{1 \leq j \leq N}$ also spans \mathbb{R}_+^N . This proves that all volume fractions θ can be attained, as can all points $(\lambda_1, \dots, \lambda_N)$ satisfying (3.126) and (3.127). Defining $t = (1/m_j)_{1 \leq j \leq N}$ the function $t \rightarrow z(t)$ is continuous from \mathbb{R}_+^N into \mathbb{R}_+^N . It maps each boundary $t_j = 0$ to itself (i.e., $z_j = 0$), and it sends infinity to infinity. The invariance by homotopy of the topological degree of the maps $\epsilon I_2 + (1 - \epsilon)z$ for $0 \leq \epsilon \leq 1$ (see, e.g., Theorem 2.6 in [106]) shows that, for any point $p \in \mathbb{R}_+^N$, $\deg(\mathbb{R}_+^N, z, p) = 1$, and thus there exists $t \in \mathbb{R}_+^N$ such that $z(t) = p$. \square

Proof of Theorem 3.2.33. We start from the value $QF(\xi)$ of the minimum, and remark that in formula (3.112) the optimal tensor A^* must be simultaneously diagonalizable with $\xi^t \xi$ by a result of von Neumann (see, e.g., [195]). From now on we assume, with no loss of generality, that the canonical basis $(e_i)_{1 \leq i \leq N}$ is an orthonormal basis generated by the common eigendirections of $\xi^t \xi$ and the optimal A^* . Thus, (3.112) reduces to

$$g(\theta, \xi) = \min_{\lambda_i} \sum_{i=1}^N \lambda_i \nu_i^2, \quad (3.128)$$

where the ν_i^2 are the eigenvalues of $\xi^t \xi$, and the λ_i are constrained by (2.54) and (2.56), i.e., $A^* \in G_\theta$. The minimization (3.128) can be performed explicitly (see Proposition 3.3 in [25]). Condition (3.114) is precisely a necessary and sufficient condition for the unique minimizer λ_i to satisfy (3.126) and (3.127). In such a case, we obtain that

$$g(\theta, \xi) = \alpha |\xi|^2 + \left(\sum_{i=1}^N \nu_i \right)^2 \left(\frac{N-1}{\lambda_\theta^+ - \alpha} + \frac{1}{\lambda_\theta^- - \alpha} \right)^{-1}.$$

To obtain $QF(\xi)$, it remains for us to minimize with respect to θ in (3.111). Assumption (3.115) is exactly a necessary and sufficient condition for the unique minimizer θ to satisfy $0 < \theta < 1$. For such minimizers θ and A^* , by Lemma 3.2.39 there exists $\rho^- = 0$, $\rho^+ = 1$, and a family of parameters $(m_j)_{1 \leq j \leq N}$ such that θ is given by (3.125) and, for the corresponding ellipsoids B_0 and B_1 , Lemma 3.2.36 holds true for any $1 \leq j \leq N$. Vitali's covering theorem implies the existence of a countable family \mathcal{B} of disjoint homothetics of B_1 that fill Ω , i.e., such that

$$\left| \Omega - \bigcup_{B^+ \in \mathcal{B}} B^+ \right| = 0. \quad (3.129)$$

For any $B^+ \in \mathcal{B}$ we define an internal confocal ellipsoid B^- , which is homothetic to B_0 with the same ratio as B^+ with respect to B_1 . Then, for $x \in \Omega$ and $\zeta \in \mathbb{R}^N$, we also define

$$u_\zeta(x) = \begin{cases} \sum_{j=1}^N \frac{\zeta_j}{f_j(\rho^+)} u_j(x) & \text{if } x \in B^+, B^+ \in \mathcal{B} \\ \zeta \cdot x & \text{otherwise,} \end{cases} \quad (3.130)$$

where u_j is defined by Lemma 3.2.36 for the ellipsoids B^+ and B^- , and

$$f_j(\rho) = 1 + \frac{\beta - \alpha}{2\alpha} g_m(\rho^-) \int_{\rho^-}^\rho \frac{dr}{(r + m_j) g_m(r)}.$$

By construction, u_ζ is a continuous function. Thus, by integration by parts for a smooth bounded domain Q including B^+ , we obtain

$$\begin{aligned} \int_{B^+} A(x) \nabla u_\zeta \cdot \nabla u_\zeta dx &= \int_{\partial B^+} u_\zeta (A(x) \nabla u_\zeta \cdot n) ds \\ &= \int_{\partial Q} u_\zeta (A(x) \nabla u_\zeta \cdot n) ds - \int_{Q \setminus B^+} A(x) \nabla u_\zeta \cdot \nabla u_\zeta dx \end{aligned}$$

where, in the last two equalities, n represents the outward normal to the hypersurface over which integration is performed. But, according to Lemma 3.2.36, $u_\zeta = \zeta \cdot x$ on $Q \setminus B^+$ so that

$$\begin{cases} \int_{\partial Q} u_\zeta (A(x) \nabla u_\zeta \cdot n) ds &= |Q| \sum_{j=1}^N \lambda_j \zeta_j^2 \\ \int_{Q \setminus B^+} A(x) \nabla u_\zeta \cdot \nabla u_\zeta dx &= (|Q| - |B^+|) \sum_{j=1}^N \lambda_j \zeta_j^2, \end{cases}$$

and eventually

$$\int_{B^+} A(x) \nabla u_\zeta \cdot \nabla u_\zeta dx = |B^+| \sum_{j=1}^N \lambda_j \zeta_j^2.$$

Thus, by virtue of (3.129),

$$\int_{\Omega} A(x) \nabla u_\zeta \cdot \nabla u_\zeta dx = |\Omega| \sum_{j=1}^N \lambda_j \zeta_j^2. \quad (3.131)$$

Choose ζ to be successively ξ_1, \dots, ξ_n (the lines of the matrix ξ), and define

$$u_\xi = (u_{\xi_1}, \dots, u_{\xi_n}). \quad (3.132)$$

By definition u_ξ belongs to D_ξ , and

$$\begin{aligned} \int_{\Omega} F(\nabla u_\xi) dx &= |\Omega| \sum_{i=1}^n \sum_{j=1}^N \lambda_j \xi_{ji}^2 + \ell \sum_{B^+ \in \mathcal{B}} |B^+ \setminus B^-| \\ &= |\Omega| \left(A^* : \xi^t \xi + \ell \theta \right) \\ &= |\Omega| QF(\xi). \end{aligned} \quad (3.133)$$

Combining (3.113) and (3.133) permits us to conclude that u_ξ is a minimizer for (3.108) over D_ξ . By virtue of the equivalence between (3.108) and (3.106), the characteristic function χ of the set $\bigcup_{B^+ \in \mathcal{B}} (B^+ \setminus B^-)$ is a minimizer of (3.106). Of course, there are as many such minimizers χ as there are possible Vitali's covering of Ω by ellipsoids B^+ ; thus, there are an infinite number of them. \square

Remark 3.2.41 *An interesting consequence of Lemma 3.2.39 and of the proof of Theorem 3.2.33 is that this confocal ellipsoids construction achieves optimality in the Hashin-Shtrikman bound (2.55). This provides another optimal microstructure different from the previously studied orthogonal rank- N sequential laminate.*

Remark 3.2.42 *As in Subsection 3.2.5 we can maximize the objective function (3.105) instead of minimizing it, i.e., we can seek the best conducting design rather than the worst (the latter of which is more interesting in usual practice). The above analysis extends to this latter case upon replacing the primal variational formulation with a dual one (the boundary condition will involve the flux rather than the potential). The presentation is therefore a little more complicated (although there is no new conceptual difficulty), and for simplicity we restrict our exposition to the minimization of (3.105). In the case where (3.105) is maximized, the coated confocal ellipsoid construction still furnishes infinitely many classical optimal designs, with the only change being that the best conductor β is now the coating, while the worst α is the core of the ellipsoids. For further details on the best conducting design case, we refer the reader to [18].*

We conclude this subsection with a lemma on the minimizing function u_ξ , built by the confocal ellipsoids construction in the proof of Theorem

3.2.33, which will be useful in Chapter 4 for extending this construction to the linearized elasticity setting.

Lemma 3.2.43 *Let $n = N$ and ξ be a symmetric positive definite matrix with eigenvalues $0 < \nu_1 \leq \dots \leq \nu_N$. Assume that these eigenvalues satisfy (3.114) and (3.115). Then, the minimizer u_ξ of (3.108), defined by (3.132), has the property that its gradient matrix is symmetric, i.e.,*

$$\nabla u_\xi = \nabla^t u_\xi \text{ a.e. in } \Omega.$$

Proof. From the proof of Theorem 3.2.33 we know that the ellipsoids are aligned with the eigenvectors of $\xi^t \xi$, and thus of ξ . Without loss of generality, we assume that the canonical basis $(e_i)_{1 \leq i \leq N}$ is the eigenbasis of $\xi = \text{diag}(\nu_1, \dots, \nu_N)$. Then, u_ξ is defined by its components $(u_{\xi_i})_{1 \leq i \leq N}$ which are

$$u_{\xi_i}(x) = \begin{cases} \frac{\nu_i}{f_i(\rho^+)} u_i(x) & \text{if } x \in B_{\rho^+}, B_{\rho^+} \in \mathcal{B} \\ \nu_i x_i & \text{otherwise,} \end{cases} \quad (3.134)$$

where u_i is defined by Lemma 3.2.36 and

$$f_i(\rho) = 1 + \frac{\beta - \alpha}{2\alpha} g_m(\rho^-) \int_{\rho^-}^\rho \frac{dr}{(r + m_i)g_m(r)}.$$

From (3.124) and (3.125) we deduce

$$\frac{1}{\lambda_i - \alpha} = \frac{f_i(\rho^+)}{(1 - \theta)(\beta - \alpha)},$$

where the λ_i are the eigenvalues of the homogenized tensor A^* which is optimal in the Hashin-Shtrikman bound $g(\theta, \xi)$. The optimality condition for A^* (or equivalently for its eigenvalues λ_i) in (3.128) is

$$\nu_i^2 = \mu^2 \frac{1}{(\lambda_i - \alpha)^2},$$

where μ^2 is a Lagrange multiplier for the active constraint

$$\sum_{i=1}^N \frac{1}{\lambda_i - \alpha} = \frac{N-1}{\lambda_\theta^+ - \alpha} + \frac{1}{\lambda_\theta^- - \alpha},$$

which is the only active constraint according to the assumptions on ξ (for details, see Proposition 3.3 in [25]). The value of the Lagrange multiplier is easily found to be

$$\mu = \frac{\text{tr}(\xi)}{\frac{N-1}{\lambda_\theta^+ - \alpha} + \frac{1}{\lambda_\theta^- - \alpha}},$$

and thus

$$\frac{\nu_i}{f_i(\rho^+)} = \frac{\alpha \text{tr}(\xi)}{N\alpha + \theta(\beta - \alpha)}.$$

Therefore,

$$\nabla u_{\xi_i}(x) = \begin{cases} \frac{\alpha \text{tr}(\xi)}{N\alpha + \theta(\beta - \alpha)} e_i & \text{in } B_{\rho^-} \\ \frac{\alpha \text{tr}(\xi)}{N\alpha + \theta(\beta - \alpha)} f_i(\rho) e_i + \sum_{j=1}^N \frac{(\beta - \alpha) \text{tr}(\xi)}{N\alpha + \theta(\beta - \alpha)} \\ \frac{g_m(\rho^-) \sigma_m}{2g_m(\rho)} \frac{x_i x_j}{(\rho + m_i)(\rho + m_j)} e_j & \text{in } B_{\rho^+} \setminus B_{\rho^-}, B_{\rho^+} \in \mathcal{B} \\ \nu_i e_i & \text{otherwise,} \end{cases} \quad (3.135)$$

which proves that ∇u_ξ is a symmetric matrix almost everywhere in Ω . In particular, we find that

$$\begin{cases} u_\xi(x) = \frac{\alpha \text{tr}(\xi)}{N\alpha + \theta(\beta - \alpha)} x & \text{in } B_{\rho^-} \\ \text{tr}(\nabla u_\xi)(x) = \text{tr}(\xi) \frac{N\alpha + (\beta - \alpha)}{N\alpha + \theta(\beta - \alpha)} & \text{in } B_{\rho^+} \setminus B_{\rho^-}, B_{\rho^+} \in \mathcal{B} \\ u_\xi(x) = \xi x & \text{otherwise,} \end{cases} \quad (3.136)$$

which will be useful in Subsection 4.1.5. □

Chapter 4

Optimal Design in Elasticity

This chapter is devoted to the application of the homogenization method in structural optimization. Namely, we generalize the approach introduced in Chapter 3 for conductivity problems to optimal design problems in the setting of linearized elasticity. We first focus on two-phase optimization problems, which amount to finding the optimal distribution of two elastic components in a fixed domain that minimizes an objective function. This objective function depends on the solution of a state equation, which is here the linearized elasticity system. Like in the conductivity case, this type of problem is generically ill-posed, i.e., it admits no optimal solution in the proposed class of admissible designs. Homogenization theory still provides a notion of generalized designs (which are composite materials in the sense of Chapter 2), which makes the problem well-posed and allows one to derive optimality conditions and new numerical algorithms. There is however a serious additional difficulty in the elasticity setting compared to the conductivity one. Since the homogenization method defines generalized designs as composite materials, it is necessary to know the full set of such two-phase composites. Unfortunately, this set is yet unknown for the mixture of two elastic isotropic components. Therefore, in full generality the homogenization method is useless in the elasticity setting since no explicit characterization of the space of composite admissible designs is available. Nevertheless, it is possible in some important cases (mainly, for self-adjoint problems) to overcome this difficulty by restricting the space of admissible composite designs to its explicit subset of sequential laminated composites.

Two-phase optimization problems include, as a limit case, shape optimization problems, the examination of which is our main motivation here.

The goal of shape optimization is to find the shape of a domain (filled with a single material), which best minimizes a criterion computed through the solution of the elasticity system in that domain. This limit is obtained in a two-phase problem, when the elasticity properties of one component go to zero. In such a degenerate limit, this weak component becomes void or holes in the domain, supporting homogeneous Neumann (no-traction) boundary conditions. Therefore, at least from an heuristic point of view, a two-phase problem yields a shape optimization problem, where the boundaries, as well as the topology of the holes (i.e., its number and connectivity), are subject to optimization. Shape optimization is actually a major issue in structural design. One of the most challenging aspects of shape optimization is what structural engineers refer to as the *layout, or topology, optimization*. Classical methods of shape optimization, based on boundary motion (see, e.g., [39], [149], [202], [220], [225], [258]), are ill-equipped to capture the possible topological complexity of the shapes because the required smooth motions of the boundary do not permit hole-punching, although it is widely acknowledged that creating holes (i.e., changing the topology) may drastically improve the performance of a candidate optimal shape. On the contrary, the homogenization method provides an effective way of optimizing the domain topology without having to keep track of complex hole boundaries.

From a theoretical point of view, bona fide shape optimization imposes an additional hurdle. One material perforated by holes may be seen as a two-phase mixture, where one of the phases is actually degenerate. Unfortunately, homogenization theory is crippled by the presence of material voids, and, although formal computations suggest as a placebo the filling of holes with a very compliant material, a full mathematical justification is only available in a few special cases. The importance of the homogenization method for shape optimization goes far beyond theoretical issues (like existence of optimal designs or necessary conditions of optimality), since it is at the root of a new class of numerical algorithms that are viewed as “topology optimization” algorithms (see Chapter 5 for more details).

The content of this chapter is the following. The first section is devoted to two-phase optimal design problems. We mainly address *self-adjoint problems* (including compliance optimization) for which the homogenization method is fully operative. The second section focuses on shape optimization problems with an emphasis on rigorous results (the numerical point of view is developed in Chapter 5). Besides our own work [6], [13], [15], [21], our exposition owes much to the pioneering contributions of Kohn and Strang [152], Lurie, Cherkaev, and their coworkers [115], [116], [177], [178], and

Murat and Tartar [205], [206].

4.1 Two-phase Optimal Design

In this section we introduce a framework for the optimal design of a mixture of two isotropic elastic materials in a given domain that minimizes some objective function. We show that this problem is generically ill-posed, namely it usually does not admit a solution in the class of admissible designs. Introducing generalized designs that are composite materials, we compute the relaxation of the optimization problem by the homogenization method.

4.1.1 The Original Problem

Let Ω be a given domain (a bounded open set in \mathbb{R}^N). This domain is occupied by two isotropic elastic phases with constant Hooke's laws A and B defined by

$$\begin{aligned} A &= 2\mu_A I_4 + \left(\kappa_A - \frac{2\mu_A}{N} \right) I_2 \otimes I_2 \\ B &= 2\mu_B I_4 + \left(\kappa_B - \frac{2\mu_B}{N} \right) I_2 \otimes I_2, \end{aligned}$$

where μ_A, μ_B are the shear moduli and κ_A, κ_B are the bulk moduli. The two phases are assumed to be wellordered, i.e.,

$$0 < \mu_A \leq \mu_B, \quad 0 < \kappa_A \leq \kappa_B. \quad (4.1)$$

It is convenient to introduce a Lamé coefficient, proportional to the Poisson's ratio, defined by

$$\lambda_A = \kappa_A - \frac{2\mu_A}{N}, \quad \lambda_B = \kappa_B - \frac{2\mu_B}{N}.$$

We denote by $\chi(x)$ the characteristic function of that part of the domain Ω occupied by phase A , i.e., $\chi(x) = 1$ whenever phase A is present at point x , and $\chi(x) = 0$ when, on the contrary, B is present. The overall Hooke's law in Ω is denoted by A_χ defined as

$$A_\chi(x) = \chi(x)A + (1 - \chi(x))B. \quad (4.2)$$

Both materials A and B are assumed to be linear and a perfect bonding is supposed to hold between them (i.e., the displacement, as well as the normal stress, is continuous through the interface). Under these assumptions, the

state equation in Ω is the linearized elasticity system of partial differential equations modeling the mechanical behavior of the domain Ω .

A simple model problem is obtained by taking a volumic force term $f(x)$ (a vector-valued function in $L^2(\Omega)^N$), and enforcing a zero displacement boundary condition on $\partial\Omega$. Then the state equation reads

$$\begin{cases} -\operatorname{div}(A_\chi e(u_\chi)) = f & \text{in } \Omega \\ u_\chi = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.3)$$

where $u_\chi(x)$ is the unknown displacement (a vector-valued function), and $e(u_\chi)$ is the strain tensor defined as $(\nabla u_\chi + (\nabla u_\chi)^t)/2$. Since the Hooke's law $A_\chi(x)$ is coercive in $L^\infty(\Omega; \mathcal{M}_N^4)$, equation (4.3) admits a unique solution u_χ in $H_0^1(\Omega)^N$ (see Subsection 1.4.1 or, e.g., [83]).

Remark 4.1.1 In (4.3) we assume, for simplicity, that the displacement u_χ satisfies an homogeneous Dirichlet (clamped) boundary condition on the whole boundary $\partial\Omega$. Of course, everything in this section holds true also with other standard boundary conditions, such as homogeneous Neumann (no-traction) conditions, mixed conditions, and nonhomogeneous conditions. In the case of pure Neumann conditions, one must check a compatibility condition of equilibrium between the body forces and the surface loadings.

The performance of the structure is measured by an objective function J defined as

$$J(\chi) = \int_{\Omega} [\chi(x)g_A(x, u_\chi(x)) + (1 - \chi(x))g_B(x, u_\chi(x))] dx, \quad (4.4)$$

where g_A and g_B are smooth enough functions depending only on the point x and on the value of the displacement $u_\chi(x)$ at the same point. An optimal design is a characteristic function χ that minimizes J . A typical example is the compliance of Ω , which is the work done by the load

$$J(\chi) = \int_{\Omega} f(x) \cdot u_\chi(x) dx,$$

which we shall study in great detail later on. In general, we assume that $g_{A,B}(x, \lambda)$, defined from $\Omega \times \mathbb{R}^N$ into \mathbb{R} , are Carathéodory functions and satisfy a suitable growth condition at infinity, i.e.,

$$\begin{cases} x \rightarrow g_{A,B}(x, \lambda) \text{ measurable } \forall \lambda \in \mathbb{R}^N \\ \lambda \rightarrow g_{A,B}(x, \lambda) \text{ continuous a.e. } x \in \Omega \\ |g_{A,B}(x, \lambda)| \leq k(x) + C|\lambda|^m \text{ with } k(x) \in L^1(\Omega), 1 \leq m < \frac{2N}{N-2}. \end{cases} \quad (4.5)$$

(In dimensions $N = 1$ or 2 , the above condition on the exponent m has to be understood in the sense that $1 \leq m < +\infty$.) By the usual Sobolev embedding theorem (see Lemma 1.2.7 or, e.g., [5], [58]), (4.5) implies that $u \rightarrow g_{A,B}(x, u)$ is indeed continuous from $H_0^1(\Omega)^N$ -weak into $L^1(\Omega)$ -strong. This implies that $J(\chi)$ is well-defined and continuous in some weak sense.

We now define the space of admissible designs. It is the space of all characteristic functions χ in $L^\infty(\Omega; \{0, 1\})$ with prescribed integral V_A (with $0 \leq V_A \leq |\Omega|$). In other words, the amount of material A is supposed to be given by V_A . Our optimal design problem is to minimize

$$\inf_{\substack{\chi \in L^\infty(\Omega; \{0, 1\}), \\ \int_\Omega \chi(x) dx = V_A}} J(\chi), \quad (4.6)$$

where J is defined by (4.4) and the state equation is (4.3). The motivation of (4.6) is the design of an elastic structure in Ω with desired properties under the action of the loading term f . If, for example, the objective function J is the compliance (which measures an overall rigidity of Ω), its global minimum would certainly be reached by using only the strongest phase B . However, since there is a volume constraint on the weak phase A , we are forced to use, not only B , but also a fixed proportion of A , which makes (4.6) a highly nontrivial problem.

The minimization in (4.6) is constrained by the prescribed volume fraction of material A . Such a constraint is routinely handled in elementary calculus of variations through the introduction of a Lagrange multiplier $\ell \in \mathbb{R}$. Therefore, a variant of (4.6) is

$$\inf_{\chi \in L^\infty(\Omega; \{0, 1\})} J(\chi) + \ell \int_\Omega \chi(x) dx. \quad (4.7)$$

For any value of ℓ there exists a volume constraint V_A such that (4.7) is equivalent to (4.6). However, the converse is not obvious in general, and for each value of V_A one has to check whether there exists a Lagrange multiplier ℓ such that (4.6) is equivalent to (4.7). In the sequel, we shall consider (4.7) as an optimization problem of interest by itself, and we shall not dwell on its equivalence with (4.6) (see, however, the constrained problem (4.169) and Lemma 4.2.11).

An essential feature of (4.6) and (4.7) is that the space of admissible designs does not contain any smoothness or topology restrictions. Indeed, apart from being measurable, the characteristic function χ does not satisfy

any further property. In other words, the region occupied by A is not necessarily smooth or connected, and may have a wild boundary. From a physical point of view, it means that we do not take into account any feasibility constraint on the mixture of A and B . Such constraints could be modeled by surface energies, bounds on the interface perimeter or on its curvature, minimal lengthscales, or some type of topological restrictions. From a mathematical point of view, it is crucial to avoid any such constraints in order to apply the homogenization method, since, as we shall see below, they yield existence of classical optimal designs without any need of homogenization.

The integrand in the objective function J depends only on the space variable x , the design parameter χ , and the displacement u_χ . It is perfectly legitimate to imagine that it could also depend on the strain $e(u_\chi)$ or stress tensor $\sigma_\chi = A_\chi e(u_\chi)$. However, in such a case, the precise relaxed formulation is unknown (except for a simple model problem in conductivity, see [277]), since it involves deeper results in homogenization theory.

The model problems (4.6) and (4.7) involve a single state equation, which means that the structure is optimized for a single loading configuration. If we change the loads or the boundary conditions, the structure will not be optimal any longer. In real practice, for stability reasons a structure often has to be optimized for multiple loads. We therefore introduce another model problem having several state equations, which allow one to optimize a structure concurrently in $n \geq 1$ configurations. Each one is characterized by a loading force $f_i(x)$ in $L^2(\Omega)^N$ and a corresponding state equation

$$\begin{cases} -\operatorname{div}(A_\chi e(u_{i,\chi})) = f_i & \text{in } \Omega \\ u_{i,\chi} = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.8)$$

where the displacement $u_{i,\chi}(x)$ is the unique solution in $H_0^1(\Omega)^N$ of (4.8). The performance of the structure defined by χ is measured by an objective function J , taking into account all state equations, defined as

$$\begin{aligned} J(\chi) = & \int_{\Omega} \left(\chi(x) g_A(x, u_{1,\chi}(x), \dots, u_{n,\chi}(x)) \right. \\ & \left. + (1 - \chi(x)) g_B(x, u_{1,\chi}(x), \dots, u_{n,\chi}(x)) \right) dx, \end{aligned} \quad (4.9)$$

where g_A and g_B are now functions depending on x and on the values of the displacements $u_{i,\chi}(x)$. As before, we assume that $g_{A,B}(x, \lambda)$, defined from $\Omega \times \mathbb{R}^{Nn}$ into \mathbb{R} , are Carathéodory functions and satisfy a suitable growth

condition at infinity in λ , namely (4.5) with λ in \mathbb{R}^{Nn} instead of \mathbb{R}^n . The optimal design problem is still the minimization problem (4.7) where $J(\chi)$ is now defined by (4.9). Typical examples of such a multiple loads objective function are the sum of all compliances

$$J(\chi) = \sum_{i=1}^n \int_{\Omega} f_i(x) \cdot u_{i,\chi}(x) dx, \quad (4.10)$$

or the maximum of all compliances

$$J(\chi) = \max_{1 \leq i \leq n} \int_{\Omega} f_i(x) \cdot u_{i,\chi}(x) dx. \quad (4.11)$$

As in the conductivity setting (see Chapter 3), the mathematical analysis of the optimal design problems (4.6) or (4.7) mainly amounts to two questions. First, is it a well-posed problem, i.e., does it admit a (possibly unique) solution in the class of admissible designs? Second, what are the optimality conditions (or Euler equations) associated with the solutions of (4.7)? The first question is obviously a typically mathematical question (although it also has important practical and numerical consequences). If its answer is negative, then the optimal design problem must be generalized suitably by relaxation, i.e., by enlarging the class of admissible designs. The second question is interesting, not only from a mathematical point of view, but also from a practical standpoint. Indeed, the Euler-Lagrange equations of stationarity of the objective function J provides a convenient characterization of the optimal designs, which can be used in many numerical methods, such as gradient descent methods or optimality criteria methods. There are of course other interesting questions to study: for example, the continuity of the solutions with respect to the data, or the qualitative properties of the optimal designs, but we focus on the two previous main questions.

As in Chapter 3, the direct method of the calculus of variations [89], [102], [293] is inoperative for showing the existence of optimal designs for problem (4.7). Indeed, the only a priori estimate for a minimizing sequence $(\chi_\epsilon)_{\epsilon>0}$ of (4.7) is

$$\|\chi_\epsilon\|_{L^\infty(\Omega)} \leq 1, \quad (4.12)$$

which is an obvious consequence of the definition of characteristic functions. By the sequential compactness of bounded sets for the weak * topology of $L^\infty(\Omega)$ (see Lemma 1.2.1), there exists a subsequence of $(\chi_\epsilon)_{\epsilon>0}$ which converges to a limit χ_∞ weakly * in $L^\infty(\Omega)$. Unfortunately, although each

χ_ϵ is a characteristic function (taking only the values zero and one), the limit χ_∞ is usually not a characteristic function since it can take any value between zero and one (see Lemma 3.1.4). This is not a surprise since the weak convergence can be interpreted as a convergence “in average”. More precisely, the limit χ_∞ , rather than being the characteristic function of some region occupied by phase A , is the density of A in the domain Ω . In other words, the space of admissible designs is not closed under the “natural” type of convergence for minimizing sequences, a fact that indicates a failure of the direct method of the calculus of variations. This is not an accident, as we shall see in the next subsection, where it is shown on one example that the optimal design problem (4.7) admits no classical solution in $L^\infty(\Omega; \{0, 1\})$. This is confirmed by numerical evidence as obtained by [73] in the context of plate thickness optimization (the computed optimal designs were not stable under mesh refinement, which is typical of non well-posed problems).

To obtain the existence of solutions to the optimal design problem (4.7), we therefore need to change the definition of the space of admissible designs. There are two possible routes for this. First, we can add further restrictions on the characteristic function χ , such as a bound on its perimeter or on the curvature of its associated region. This has been done, e.g., in [26], [71], [72], where existence results are proved under one of these additional assumptions. However, the question of the dependence of the optimal designs with respect to the value of these bounds is open, and this approach does not furnish a new perspective for numerical methods. Second, and this is our approach in this book, we can find the closure of the space of admissible designs and extend the objective function to this closure in order that the direct method of the calculus of variations succeed with this larger space and extended objective function. This process is called *relaxation* (for further details, see, e.g., [89], [102], [293], and Section 3.2). It heavily relies on homogenization, which allows one to define a suitable notion of generalized designs, making the optimization problem well-posed without changing its physical significance.

The question of finding optimality conditions for the minimization problem (4.7) is also not obvious in the present setting. As is well known, the Euler-Lagrange equations of the calculus of variations are obtained by perturbing a stationary point. This is possible if the space of admissible designs is stable under some linear combination of its elements. This is clearly not the case for $L^\infty(\Omega; \{0, 1\})$, since a convex combination of different characteristic functions is never a characteristic function. This difficulty in making variations of a domain was recognized a long time ago. Hadamard [129] proposed varying a domain along the normal to its boundary (see Figure 4.1).

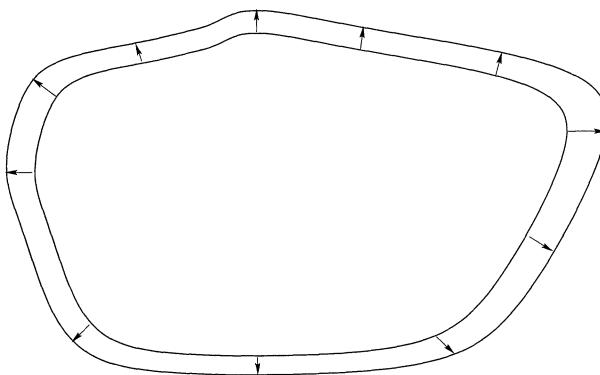


Figure 4.1: Variation of a domain along its normal.

This yields only a strict subclass of possible variations which, in particular, never change the topology of the domain. Nevertheless, many authors have developed and improved this method of shape sensitivity, e.g., [203] (note that this fundamental reference is summarized in [186], [202], which are more accessible), and in [220], [255], [258]. In particular, more general domain variations can be obtained by applying smooth diffeomorphisms. However, these methods give only some necessary optimality conditions, which are by no means sufficient, since the domain topology is fixed. Another motivation for relaxing these optimal design problems by the homogenization method is to find new optimality conditions that are able to take into account topology changes. In practice, the knowledge of these relaxed optimality conditions yields new numerical algorithms for computing optimal designs. This explains the interest in this approach, from both a theoretical and a numerical standpoint.

4.1.2 Counterexample to the Existence of Optimal Designs

This subsection is devoted to a simple example of a two-phase optimal design problem in the framework of Subsection 4.1.1, for which there is no optimal solution in the class of admissible designs. Counterexamples to the existence of solutions for similar control problems, in which the control acts in the coefficients of the state equation, were first found by Murat [198], [199], [200], and Lurie [172], [178]. Following our recent work [25], we propose a simple explicit counterexample, stated in any space dimension $N \geq 2$, although its spirit is truly one-dimensional.

We consider a bounded open set Ω in \mathbb{R}^N with a smooth boundary $\partial\Omega$ (say, Lipschitz continuous). This domain Ω is filled by two isotropic elastic materials A and B with wellordered elastic moduli, $0 < \mu_A < \mu_B$ and $0 < \kappa_A < \kappa_B$. In order to simplify the exposition we assume that they both have a zero Poisson ratio, i.e.,

$$\lambda_A = \kappa_A - \frac{2\mu_A}{N} = 0, \quad \lambda_B = \kappa_B - \frac{2\mu_B}{N} = 0.$$

This assumption is not essential (see Remark 4.1.6), but it simplifies greatly the algebra in this subsection. The Hooke's law in Ω is

$$A_\chi(x) = \chi(x)A + (1 - \chi(x))B,$$

where χ is the characteristic function of the part occupied by phase A . The state equation is

$$\begin{cases} -\operatorname{div}\sigma_\chi = 0 & \text{in } \Omega \\ \sigma_\chi = A_\chi e(u_\chi) & \text{in } \Omega \\ \sigma_\chi n = \sigma_0 n & \text{on } \partial\Omega, \end{cases} \quad (4.13)$$

where the imposed stress σ_0 on the boundary is defined as a constant rank-1 matrix corresponding to a uniaxial traction in the direction of the N th vector e_N of the canonical basis

$$\sigma_0 = |\sigma_0|e_N \otimes e_N. \quad (4.14)$$

Since the boundary datum σ_0 is constant, the domain is in equilibrium, and (4.13) admits a unique solution u_χ in $H^1(\Omega)^N$ (unique up to a rigid body displacement, see, e.g., [83]). As an objective function we choose a weighted sum of the compliance and of the volume constraint on the strongest material B , i.e.,

$$J(\chi) = \int_{\partial\Omega} (\sigma_0 n) \cdot u_\chi ds + \ell \int_{\Omega} (1 - \chi) dx, \quad (4.15)$$

where $\ell \geq 0$ is a fixed positive Lagrange multiplier, and ds is the surfacic measure on $\partial\Omega$. The optimal design problem is then to minimize J in the space $L^\infty(\Omega; \{0, 1\})$, i.e.,

$$\inf_{\chi \in L^\infty(\Omega; \{0, 1\})} J(\chi). \quad (4.16)$$

If we assume that the strong material B is more expensive than the weak material A , the optimal design problem (4.16) amounts to finding the best

arrangement of these two phases, the one that gives the structure Ω maximal rigidity and minimal price all together. The rigidity of Ω is measured by the compliance (which is equal to the stored elastic energy), and ℓ is a price versus efficiency ratio for B that measures the trade-off between a low price and a good rigidity. Our main result is the following nonexistence theorem.

Theorem 4.1.2 *Assume $N \geq 2$. Let us define*

$$\ell^- = \frac{|\sigma_0|^2(\mu_B - \mu_A)}{2\mu_B^2} \leq \ell^+ = \frac{|\sigma_0|^2(\mu_B - \mu_A)}{2\mu_A^2}.$$

If $\ell^+ > \ell > \ell^-$, there exists no minimizer of the optimal design problem (4.16) in the space of admissible designs $L^\infty(\Omega; \{0, 1\})$. If $\ell \leq \ell^-$, then the constant function $\chi = 0$ is the unique minimizer of (4.16). If $\ell \geq \ell^+$, then the constant function $\chi = 1$ is the unique minimizer of (4.16).

Remark 4.1.3 *If $\ell \geq \ell^+$, then the price of the strong material B is so high that it is better to use A only. On the contrary if $\ell \leq \ell^-$, the strong phase B is so cheap that the cost function is minimized by using B only.*

Remark 4.1.4 *The counterexample provided by Theorem 4.1.2 works only in space dimension $N \geq 2$. In a one-dimensional setting, i.e., $N = 1$, the situation is completely different. In such a case, for any χ the stress σ_χ is constant, equal to the boundary datum σ_0 . Therefore, $J(\chi)$ is explicitly given in terms of the average θ of χ*

$$J(\chi) = |\Omega| \left(\frac{|\sigma_0|^2}{2\mu_A\theta + 2\mu_B(1-\theta)} + \ell(1-\theta) \right).$$

Minimizing in θ the right hand side gives the unique value θ^ for which $J(\chi)$ is minimum, and such a minimum is attained by any characteristic function χ with average θ^* . Thus, if $0 < \theta^* < 1$, there exist infinitely many minimizers in one dimension.*

Proof. By the principle of minimum complementary energy, we rewrite the compliance as the result of a minimization, i.e.,

$$\int_{\partial\Omega} (\sigma_0 n) \cdot u_\chi ds = \min_{\sigma \in H_0} \int_{\Omega} A_\chi^{-1} \sigma : \sigma dx, \quad (4.17)$$

where the affine space H_0 of statically admissible stresses is defined by

$$H_0 = \left\{ \sigma \in L^2(\Omega; \mathcal{M}_N^s) \text{ such that } \begin{array}{l} \operatorname{div} \sigma = 0 \text{ in } \Omega \\ \sigma n = \sigma_0 n \text{ on } \partial\Omega \end{array} \right\}.$$

Note that, in the definition of H_0 , the boundary condition makes sense in $H^{-1/2}(\partial\Omega)^N$. Obviously the minimum in (4.17) is attained by the stress $\sigma_\chi = A_\chi e(u_\chi)$ where u_χ is the unique solution of (4.13). The optimal design problem (4.16) can thus be rewritten as

$$\inf_{\chi \in L^\infty(\Omega; \{0,1\})} \inf_{\sigma \in H_0} \int_{\Omega} (A_\chi^{-1} \sigma : \sigma + \ell(1 - \chi)) dx, \quad (4.18)$$

where the order of the two minimizations is irrelevant. For a given χ , we define the averages

$$\theta = \frac{1}{|\Omega|} \int_{\Omega} \chi(x) dx \text{ and } A_0 = \frac{1}{|\Omega|} \int_{\Omega} A_\chi(x) dx = \theta A + (1 - \theta) B.$$

Next, we remark that the constant boundary datum σ_0 is also the average of any stress σ in H_0 , i.e.,

$$\sigma_0 = \frac{1}{|\Omega|} \int_{\Omega} \sigma(x) dx.$$

Indeed, for any $\sigma \in H_0$, an integration by parts yields

$$0 = \int_{\Omega} \operatorname{div}(\sigma(x) - \sigma_0) \cdot (x_i e_j) dx = - \int_{\Omega} (\sigma(x) - \sigma_0) : (e_i \otimes e_j) dx,$$

where $(e_i)_{1 \leq i \leq N}$ is the canonical basis of \mathbb{R}^N . By virtue of Lemma 4.1.5 below, for any $x \in \Omega$, we have

$$\begin{aligned} A_\chi(x)^{-1} \sigma(x) : \sigma(x) &\geq A_0^{-1} \sigma_0 : \sigma_0 - A_0^{-1} (A_\chi(x) - A_0) A_0^{-1} \sigma_0 : \sigma_0 \\ &\quad + 2 A_0^{-1} \sigma_0 : (\sigma(x) - \sigma_0). \end{aligned} \quad (4.19)$$

Integrating over Ω , the two last terms of (4.19) cancel out since they have zero average. This yields

$$\int_{\Omega} (A_\chi^{-1} \sigma : \sigma + \ell(1 - \chi)) dx \geq |\Omega| (A_0^{-1} \sigma_0 : \sigma_0 + \ell(1 - \theta)). \quad (4.20)$$

Since A_0 depends linearly on θ , the right hand side of (4.20) is a strictly convex function of θ in $[0, 1]$. An easy computation shows that its minimum

I_ℓ is given by

$$I_\ell = |\Omega| \begin{cases} A^{-1}\sigma_0 : \sigma_0 & \text{if } \ell \geq \ell^+ = \frac{|\sigma_0|^2(\mu_B - \mu_A)}{2\mu_A^2} \\ B^{-1}\sigma_0 : \sigma_0 + \ell & \text{if } \ell \leq \ell^- = \frac{|\sigma_0|^2(\mu_B - \mu_A)}{2\mu_B^2} \\ 2|\sigma_0| \sqrt{\frac{\ell}{2\mu_B - 2\mu_A}} - \frac{\mu_A \ell}{\mu_B - \mu_A} & \text{if } \ell^+ > \ell > \ell^-. \end{cases}$$

Therefore, (4.20) leads to

$$\inf_{\chi \in L^\infty(\Omega; \{0,1\})} \inf_{\sigma \in H_0} \int_{\Omega} (A_\chi^{-1} \sigma : \sigma + \ell(1-\chi)) dx \geq I_\ell.$$

We now examine whether this lower bound can be attained. In its derivation, the only inequality comes from (4.19), for which we know the exact remainder by virtue of Lemma 4.1.5. Assume that there exists $\chi \in L^\infty(\Omega; \{0,1\})$, and therefore $\sigma_\chi \in H_0$, such that

$$J(\chi) = \int_{\Omega} (A_\chi^{-1} \sigma_\chi : \sigma_\chi + \ell(1-\chi)) dx = I_\ell.$$

From (4.19) and (4.23), we deduce that, for a.e. $x \in \Omega$,

$$A_\chi(x)^{-1} (\sigma_\chi(x) - A_\chi(x)A_0^{-1}\sigma_0) : (\sigma_\chi(x) - A_\chi(x)A_0^{-1}\sigma_0) = 0,$$

which implies

$$\sigma_\chi(x) = A_\chi(x)A_0^{-1}\sigma_0 \text{ in } \Omega.$$

If χ is a constant function, namely $\chi \equiv 0$ or $\chi \equiv 1$ in Ω , then σ_χ is also constant and equal to σ_0 . In such a case, it is easy to check that $J(\chi)$ coincides with the lower bound I_ℓ if and only if either $\ell \leq \ell^-$ or $\ell \geq \ell^+$. If χ is not a constant function, then σ_χ takes only two values, $AA_0^{-1}\sigma_0$ and $BA_0^{-1}\sigma_0$, which are both different from the boundary datum σ_0 since $A < A_0 < B$ in the sense of quadratic forms. Thus, σ_χ cannot satisfy the boundary condition. Consequently, $J(\chi)$ is always strictly greater than the bound I_ℓ if $\ell^- < \ell < \ell^+$.

To finish the proof it remains to prove that the lower bound I_ℓ is precisely the infimum of $J(\chi)$. When $\ell \geq \ell^+$ we already know that $J(1) = I_\ell$, and

similarly, when $\ell \leq \ell^-$ we know that $J(0) = I_\ell$. For all other values of ℓ , it turns out that the right hand side of (4.20) is minimal for a unique value, denoted by θ^* , and given by

$$\theta^* = \frac{1}{\mu_B - \mu_A} \left(\mu_B - |\sigma_0| \sqrt{\frac{\ell}{\mu_B - \mu_A}} \right),$$

which satisfies $0 < \theta^* < 1$. We build a minimizing sequence $(\chi_n)_{n \geq 1}$ of characteristic functions such that χ_n converges to the constant limit θ^* weakly * in $L^\infty(\Omega; [0, 1])$ and $\lim_{n \rightarrow +\infty} J(\chi_n) = I_\ell$. Introducing a 1-periodic function $\chi(x_1)$ given by

$$\chi(x_1) = \begin{cases} 1 & \text{if } 0 \leq x_1 < \theta^* \\ 0 & \text{if } \theta^* \leq x_1 < 1 \end{cases},$$

we define χ_n by

$$\chi_n(x) = \chi(nx_1),$$

which, upon recalling Lemma 1.3.19 on periodically oscillating sequences, converges weakly * in $L^\infty(\Omega; [0, 1])$ to θ^* . Such characteristic functions χ_n correspond to both periodic and laminated microstructures. Therefore, applying Theorem 1.3.18 and recalling formula (2.70) we deduce that the sequence of Hooke's laws A_{χ_n} H -converges to the constant homogenized tensor A^* defined by

$$(1 - \theta) \left(A^{*-1} - A^{-1} \right)^{-1} = \left(B^{-1} - A^{-1} \right)^{-1} + \theta f_A^c(e_1)$$

where $f_A^c(e_1)$ is a symmetric positive nondefinite fourth order tensor defined, for any symmetric matrix ξ , by the quadratic form

$$f_A^c(e_1)\xi : \xi = A\xi : \xi - \frac{1}{\mu_A} |A\xi e_1|^2 + \frac{1}{2\mu_A} ((A\xi)e_1 \cdot e_1)^2.$$

Furthermore, as a consequence of Proposition 1.2.20 on the energy convergence, we also have

$$\lim_{n \rightarrow +\infty} \min_{\sigma \in H_0} \int_{\Omega} A_{\chi_n}^{-1} \sigma : \sigma dx = \min_{\sigma \in H_0} \int_{\Omega} A^{*-1} \sigma : \sigma dx. \quad (4.21)$$

Since A^* is a constant Hooke's law and σ_0 a constant tensor, it is easy to see that the minimizer in the right hand side of (4.21) is precisely $\sigma(x) \equiv \sigma_0$. Thus, we have proved

$$\lim_{n \rightarrow +\infty} J(\chi_n) = |\Omega| \left(A^{*-1} \sigma_0 : \sigma_0 + \ell(1 - \theta^*) \right).$$

The final step is to show that A^* behaves like A_0 under the stress σ_0 , i.e., $A^{*-1}\sigma_0 : \sigma_0 = A_0^{-1}\sigma_0 : \sigma_0$. At this point, our assumption on the phases, having zero Poisson ratio, simplifies greatly an otherwise tedious computation. Indeed by formula (1.130) in Lemma 1.4.10 (which gives the homogenized tensor of a simple laminate), we have

$$A^*\sigma_0 = (2\mu_A\theta + 2\mu_B(1 - \theta))\sigma_0 \quad (4.22)$$

since σ_0 , being a uniaxial stress along e_N , satisfies $\sigma_0 e_1 = 0$ (when $N \geq 2$). Inverting (4.22) yields

$$A^{*-1}\sigma_0 : \sigma_0 = A_0^{-1}\sigma_0 : \sigma_0,$$

and thus

$$\lim_{n \rightarrow +\infty} J(\chi_n) = I_\ell,$$

which is the desired result. \square

Lemma 4.1.5 *The function $\phi(A, \sigma)$, defined from $\mathcal{M}_N^4 \times \mathcal{M}_N^s$ into \mathbb{R} by*

$$\phi(A, \sigma) = A^{-1}\sigma : \sigma,$$

is convex when A is positive definite, and satisfies

$$\phi(A, \sigma) = \phi(A_0, \sigma_0) + D\phi(A_0, \sigma_0) \cdot (A - A_0, \sigma - \sigma_0) + \phi(A, \sigma - AA_0^{-1}\sigma_0), \quad (4.23)$$

where the differential $D\phi$ is given by

$$D\phi(A_0, \sigma_0) \cdot (B, \tau) = -A_0^{-1}BA_0^{-1}\sigma_0 : \sigma_0 + 2A_0^{-1}\sigma_0 : \tau.$$

Proof. Formula (4.23) is easily obtained by a simple computation. It is a Taylor development at first order with an exact remainder. Since ϕ is nonnegative by definition, we deduce that ϕ , being always above its tangent hyperplanes, is convex. \square

Remark 4.1.6 *A complete family of counterexamples for similar problems in which the state equation has linear boundary conditions may be found in [25]. In particular, one can remove the assumption on the Poisson ratio of both materials being zero.*

4.1.3 Relaxed Formulation of the Problem

In this subsection we generalize the relaxation process by homogenization already introduced in Section 3.2 for the conductivity setting (following the pioneering work of Murat and Tartar [205]). We focus on the case of a single load (the multiple loads case is completely similar). Recall that our problem of optimal design is

$$\inf_{\chi \in L^\infty(\Omega; \{0,1\})} J(\chi), \quad (4.24)$$

with the objective function J defined by

$$J(\chi) = \int_{\Omega} [\chi(x)g_A(x, u_{\chi}(x)) + (1 - \chi(x))g_B(x, u_{\chi}(x))] dx + \ell \int_{\Omega} \chi(x) dx,$$

where ℓ is a given real parameter and $u_{\chi}(x)$ is the unique solution in $H_0^1(\Omega)^N$ of

$$\begin{cases} -\operatorname{div}(A_{\chi}e(u_{\chi})) = f & \text{in } \Omega \\ u_{\chi} = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.25)$$

with a given body force $f \in L^2(\Omega)^N$ and a Hooke's law $A_{\chi} = \chi A + (1 - \chi)B$. The functions g_A and g_B satisfy (4.5), which implies, at least, that $J(\chi)$ is well defined on $L^\infty(\Omega)$.

As proved in Subsection 4.1.2, there is usually no minimizer for J in the space of admissible designs $L^\infty(\Omega; \{0,1\})$. To overcome this difficulty, rather than changing completely the definition of admissible designs or enforcing additional constraints for a design to be admissible, we proceed to the relaxation of the minimization of the objective function J . The principle of the relaxation process is to study the behavior of minimizing sequences and to define generalized admissible designs that include the possible limits of these minimizing sequences (that may escape from the original space of admissible designs). By Lemma 3.1.4, the weak limit of a sequence of characteristic functions is usually a density function θ , which may take its values in the whole range $[0, 1]$. According to Chapters 1 and 2, the corresponding mixture of conductors A and B is a composite material with Hooke's law A^* .

The concept of generalized admissible designs entails precisely these couples (θ, A^*) made of a density function and a composite Hooke's law (or an H -limit). In view of Theorem 2.1.2 the set of such generalized or composite designs, denoted by \mathcal{CD} , is

$$\mathcal{CD} = \left\{ (\theta, A^*) \in L^\infty(\Omega; [0, 1] \times \mathcal{M}_N^4) \mid A^*(x) \in G_{\theta(x)} \text{ a.e. in } \Omega \right\}, \quad (4.26)$$

where, for each constant value $0 \leq \theta \leq 1$, G_θ is the set of all homogenized Hooke's laws obtained by mixing the phases A and B in proportions θ , $(1-\theta)$ (see Theorem 2.1.2). When θ takes the value zero or one, then $G_0 = \{B\}$ and $G_1 = \{A\}$, namely, A^* is just phase B or A . Therefore, our set of generalized designs include, as a special case, the classical two-phase designs. Unfortunately in all other cases, we lack a precise characterization of G_θ . The objective function and the state equation are modified as follows:

$$\begin{aligned} J^*(\theta, A^*) = & \int_{\Omega} [\theta(x)g_A(x, u(x)) + (1 - \theta(x))g_B(x, u(x))] dx \\ & + \ell \int_{\Omega} \theta(x) dx, \end{aligned} \quad (4.27)$$

where $u(x)$ is the unique solution in $H_0^1(\Omega)^N$ of

$$\begin{cases} -\operatorname{div}(A^*e(u)) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.28)$$

The proposed relaxed problem is therefore the minimization

$$\min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*). \quad (4.29)$$

The main result is the following theorem, which proves, as in the conductivity setting (see Chapter 3), that (4.29) is indeed the correct relaxed problem, i.e., composite designs are suitable generalized designs.

Theorem 4.1.7 *The minimization problem (4.29) is a true relaxation of the original problem (4.24), in the sense that*

1. *there exists at least one minimizer of the relaxed problem (4.29);*
2. *up to a subsequence, every minimizing sequence of classical designs χ_n for J converges weakly * in $L^\infty(\Omega; [0, 1])$ to a density function θ , and the associated Hooke's law $A_{\chi_n} = \chi_n A + (1 - \chi_n)B$ H -converges to a composite Hooke's law A^* such that (θ, A^*) is a minimizer of J^* in \mathcal{CD} ;*
3. *conversely, every minimizer (θ, A^*) of J^* in \mathcal{CD} is attained by a minimizing sequence for J of classical designs χ_n , namely θ is the weak * limit of χ_n in $L^\infty(\Omega; [0, 1])$ and A^* is the H -limit of $A_{\chi_n} = \chi_n A + (1 - \chi_n)B$.*

The proof of Theorem 4.1.7 is identical to that of Theorem 3.2.1 in the conductivity setting, so we skip it. Similarly, we can compute the derivative of the objective function J^* upon further regularity assumptions on the partial derivatives $(\partial/\partial\lambda)g_{A,B}(x, \lambda)$, which are assumed to be Caratheodory functions satisfying a suitable growth condition, similar to (3.47) and (3.65). In order to properly define the derivative of J^* we introduce the adjoint state p , which is the unique solution in $H_0^1(\Omega)^N$ of

$$\begin{cases} -\operatorname{div}(A^*e(p)) = \theta \frac{\partial g_A}{\partial \lambda}(x, u) + (1 - \theta) \frac{\partial g_B}{\partial \lambda}(x, u) & \text{in } \Omega \\ p = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.30)$$

Note that p is well-defined since, by assumption on the partial derivatives $(\partial/\partial\lambda)g_{A,B}(x, \lambda)$, the right hand side of (4.30) belongs to $H^{-1}(\Omega)^N$.

Theorem 4.1.8 *The objective function $J^*(\theta, A^*)$ is Gâteaux differentiable in the space of admissible composite designs \mathcal{CD} , defined by (4.26), and its directional derivative is*

$$\delta J^*(\theta, A^*) = \int_{\Omega} \delta\theta(x) [g_A(x, u(x)) - g_B(x, u(x)) + \ell] dx - \int_{\Omega} \delta A^*(x)e(u) : e(p) dx, \quad (4.31)$$

where $\delta\theta$ and δA^* are admissible increments in \mathcal{CD} , u is the solution of the state equation (4.28), and p that of the adjoint state equation (4.30). Consequently, if (θ, A^*) is a minimizer of the objective function J^* , it must satisfies $\delta J^*(\theta, A^*) \geq 0$ for any admissible increments $\delta\theta, \delta A^*$.

The proof of Theorem 4.1.8 is also completely similar to that of its counterpart in the conductivity setting, Theorem 3.2.4, so we skip it. Theorems 4.1.7 and 4.1.8 are indeed a justification of the homogenization method for two-phase optimal design. The class of admissible designs has been enlarged in order to obtain existence of an optimum, but not by too much, in order to keep a connection between classical and generalized (composite) designs. The three conditions, stated in Theorem 4.1.7, insure that our relaxation does not physically change the problem. In particular, the minimum values of the original and relaxed function are the same, and every optimal composite design can be approximated, as close as we want, by a nearly optimal classical design. In this respect, homogenization and composite materials are

just the correct notions for characterizing minimizing sequences of classical designs.

There is however one big disadvantage of the relaxed formulation (4.26): the set G_θ of all homogenized Hooke's laws is unknown. This is in sharp contrast with the conductivity setting, where the set of homogenized conductivity tensors is explicitly characterized by Theorem 2.2.13. Of course, we know, at least, that G_θ is the closure of the set P_θ of Hooke's laws obtained by periodic homogenization (see Theorem 2.1.2), and that each Hooke's law in P_θ is characterized by a formula involving the knowledge of the microstructure (see (2.7) or (1.23)), but this fact does not help to explicitly characterize G_θ . In particular, it is not known if the set G_θ is convex, which drastically restricts the choice of explicit smooth paths inside G_θ , and thus of admissible increments δA^* (in practice the only fully explicit paths in G_θ are those obtained by lamination, which yields complicated expressions of δA^*). This is really a serious drawback of the homogenization method, since it implies that the set of generalized admissible designs \mathcal{CD} , as well as the possible increments in \mathcal{CD} , are unknown. Consequently, the optimality condition furnished by Theorem 4.1.8 is useless.

Fortunately enough, in a few important cases described below it is possible to reduce the full set G_θ to a smaller subset made of sequential laminates, which now is explicitly known. Therefore, the homogenization method is fully operative in these situations, which are studied in the next subsections. In all other cases, for practical purposes the full relaxation (4.26) may be replaced by a partial relaxation, i.e., the set G_θ is reduced to a smaller, yet explicit, subset (see Subsection 5.2.8 in Chapter 5).

4.1.4 Compliance Optimization

A very important special case of Subsection 4.1.3 is obtained when the problem is self-adjoint, i.e., when the adjoint state p is equal to the state u (or, symmetrically, when p is equal to $-u$). This corresponds to the following choice of objective functions:

$$J^*(\theta, A^*) = \int_{\Omega} f(x) \cdot u(x) dx + \ell \int_{\Omega} \theta(x) dx \quad (4.32)$$

or

$$J^*(\theta, A^*) = - \int_{\Omega} f(x) \cdot u(x) dx + \ell \int_{\Omega} \theta(x) dx, \quad (4.33)$$

i.e., $g_\alpha(x, u) = g_\beta(x, u) = \pm f(x) \cdot u$. It is easily seen that (4.32) yields $p = u$, while (4.33) yields $p = -u$. The quantity $\int_{\Omega} f \cdot u dx$ is called the *compliance*.

It is the work done by the load, and it is also equal to the stored elastic energy, since

$$\int_{\Omega} f \cdot u dx = \int_{\Omega} A^* e(u) : e(u) dx.$$

Therefore, minimizing (4.32) amounts to finding the most rigid structural design made of A and B in Ω , while, on the contrary, minimizing (4.33) corresponds to finding the most compliant design. The analysis of these two problems is, in some sense, dual (see Remark 4.1.21). For the moment we focus on the physically more relevant problem (4.32). The results of the previous subsection are still valid but we perform a new analysis in order to replace the unknown set G_θ by its subset of sequential laminates.

One advantage of a self-adjoint problem is that the objective function can itself be written as the result of a minimization. By the principle of minimal complementary energy we have

$$\int_{\Omega} f \cdot u dx = \min_{\substack{\tau \in L^2(\Omega; \mathcal{M}_N^s) \\ -\operatorname{div} \tau = f \text{ in } \Omega}} \int_{\Omega} A^{*-1} \tau : \tau dx, \quad (4.34)$$

and the minimum on the right hand side of (4.34) is achieved by $\sigma = A^* e(u)$. We may rewrite the objective function as

$$J^*(\theta, A^*) = \min_{\substack{\tau \in L^2(\Omega; \mathcal{M}_N^s) \\ -\operatorname{div} \tau = f \text{ in } \Omega}} \int_{\Omega} A^{*-1} \tau : \tau dx + \ell \int_{\Omega} \theta dx. \quad (4.35)$$

Since the order of minimization is irrelevant, we obtain

$$\min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*) = \min_{\substack{\tau \in L^2(\Omega; \mathcal{M}_N^s) \\ -\operatorname{div} \tau = f \text{ in } \Omega}} \min_{(\theta, A^*) \in \mathcal{CD}} \int_{\Omega} (A^{*-1} \tau : \tau + \ell \theta) dx. \quad (4.36)$$

By virtue of (4.36), the optimal design problem appears as a double minimization for which optimality conditions are easily derived.

Theorem 4.1.9 *If (θ, A^*) is a minimizer of the objective function (4.32), and if σ is the unique corresponding minimizer of (4.35), then σ satisfies*

$$\begin{cases} \sigma = A^* e(u) & \text{in } \Omega \\ -\operatorname{div} \sigma = f & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega, \end{cases} \quad (4.37)$$

with $u \in H_0^1(\Omega)^N$, and (θ, A^*) satisfies, almost everywhere in Ω ,

$$A^{*-1} \sigma : \sigma = g(\theta, \sigma) \quad (4.38)$$

where $g(\theta, \sigma)$ is the lower Hashin-Shtrikman bound on complementary energy defined by (2.142), and θ is the unique minimizer of the convex minimization

$$\min_{0 \leq \theta \leq 1} (g(\theta, \sigma) + \ell\theta). \quad (4.39)$$

Remark 4.1.10 The optimality conditions furnished by Theorem 4.1.9 are necessary but it is not known if they are sufficient. However, if an optimal stress tensor σ is known, then (4.38) and (4.39) are necessary and sufficient for (θ, A^*) . Conversely, if an optimal (θ, A^*) is known, then (4.37) is obviously necessary and sufficient for σ .

Proof. Equation (4.37) is just the state equation which is the Euler-Lagrange equation of the principle of minimal complementary energy. In view of (4.36), (θ, A^*) is also a minimizer of

$$\min_{(\theta, A^*) \in \mathcal{CD}} \int_{\Omega} (A^{*-1} \sigma : \sigma + \ell\theta) dx.$$

By the local character of the G -closure (see Theorem 2.1.2), the minimization and the integration can be switched:

$$\min_{(\theta, A^*) \in \mathcal{CD}} \int_{\Omega} (A^{*-1} \sigma : \sigma + \ell\theta) dx = \int_{\Omega} \min_{0 \leq \theta \leq 1} \left(\min_{A^* \in G_\theta} A^{*-1} \sigma : \sigma + \ell\theta \right) dx.$$

By Proposition 2.3.25, we have

$$\min_{A^* \in G_\theta} (A^{*-1} \sigma : \sigma) = g(\theta, \sigma),$$

where $g(\theta, \sigma)$ is the lower Hashin-Shtrikman bound on dual energy defined by (2.142). This yields condition (4.38). Furthermore, by its very definition, $g(\theta, \sigma)$ is a C^1 and strictly convex function of θ (see Remark 2.3.27). Thus, the function

$$\theta \rightarrow g(\theta, \sigma) + \ell\theta$$

admits a unique minimizer in $[0, 1]$, which implies (4.39). \square

Remark 4.1.11 One might wonder if the above optimality conditions could be obtained by using a primal variational principle for the compliance (i.e., using strains rather than stresses). Actually, the compliance is delivered as the maximum of some energy of the displacement u . Therefore, problem (4.32) of compliance minimization becomes a min-max problem, which is

unfortunately not a saddle point problem, i.e., the order of the minimization in (θ, A^*) and the maximization in u cannot be exchanged. We remark however that Lipton [168] proved that, at least, the minimization in A^* and the maximization in u can be switched (but not that in θ). It explains why a stress formulation is necessary at this stage. As we shall see below (Theorem 4.1.19), it is rather problem (4.33) of compliance maximization that requires a primal variational formulation.

From Theorem 4.1.9 can be deduced a simplified relaxed problem, one which does not require the knowledge of G_θ but uses only its subset L_θ^+ of sequential laminates, with core A and matrix B , in proportions θ and $(1-\theta)$ respectively, introduced in Definition 2.3.5. Let us define the set \mathcal{LD}^+ of sequentially laminated designs as follows:

$$\mathcal{LD}^+ = \left\{ (\theta, A^*) \in L^\infty(\Omega; [0, 1] \times \mathcal{M}_N^4) \mid A^*(x) \in L_{\theta(x)}^+ \text{ a.e. in } \Omega \right\}, \quad (4.40)$$

which is a strict subset of the set of all composite designs \mathcal{CD} , defined by (4.26). The advantage of \mathcal{LD}^+ versus \mathcal{CD} is its explicit character as stated in Lemma 2.3.6.

Theorem 4.1.12 *For the objective function (4.32) we have*

$$\min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*) = \min_{(\theta, A^*) \in \mathcal{LD}^+} J^*(\theta, A^*).$$

If (θ, A^) is a minimizer of J^* in \mathcal{CD} , and if σ is its associated stress tensor which minimizes (4.35), then there exists a sequential laminate \tilde{A}^* such that (θ, \tilde{A}^*) is a minimizer of J^* in \mathcal{LD}^+ , σ is again its associated stress tensor, and $A^{*-1}\sigma = \tilde{A}^{*-1}\sigma$. Furthermore, \tilde{A}^* can be chosen as a rank- N sequential laminate (with a core of A and a matrix of B) having orthogonal lamination directions given by the eigenvectors of σ .*

Remark 4.1.13 *Theorem 4.1.12 is very important since it gives sense to the homogenization method for compliance optimization. As we already said, the general theory of the previous subsection is useless because the G -closure set G_θ of all homogenized Hooke's laws is unknown. Fortunately, when the objective function is the compliance, Theorem 4.1.12 states that the optimization can be performed solely on the smaller and explicit subset L_θ^+ of sequential laminates (with a core of A and a matrix of B).*

Remark 4.1.14 It is possible to compute explicitly $g(\theta, \sigma)$ and an associated optimal sequential laminate \tilde{A}^* in two space dimensions (see [22], [115]). The same computation is in principle achievable in three space dimensions but, since the algebra is formidable, it has only been done in the case $A = 0$ (see Subsection 2.3.4, [116], [7]). Theorems 4.1.9 and 4.1.12 give a new insight into optimality criteria algorithms. The principle of these numerical methods (see [42], [237], [239], [296], and Chapter 5) is to iteratively compute the state u (or its associated stress σ) and the design parameters (θ, A^*) thanks to the optimality conditions. By virtue of (4.36), the optimal design problem is a double minimization with respect to τ and (θ, A^*) . As proposed in [15], [17] (see Subsection 5.2.1 for details), this double minimization can be performed by an alternate directions algorithm, which amounts to minimizing iteratively and separately in τ and (θ, A^*) . Minimizing in (θ, A^*) for given τ can be done explicitly by taking A^* to be the optimal rank- N sequential laminate furnished by Theorem 4.1.12 and by taking θ the unique minimizer of (4.39) (with τ replacing σ). Therefore, in view of formulation (4.36) of the problem, an optimality criteria method is simply a descent method, and it will always converge to a stationary point of the objective function.

Proof. Let (θ, A^*) be a minimizer of J^* in \mathcal{CD} , and σ be its associated stress tensor which minimizes (4.35). By Theorem 4.1.9 A^* is optimal for the lower dual Hashin-Shtrikman bound (2.142), i.e.,

$$A^{*-1}\sigma : \sigma = \min_{B^* \in G_\theta} (B^{*-1}\sigma : \sigma).$$

By Proposition 2.3.25, there exists a rank- N sequential laminate \tilde{A}^* , with lamination directions given by the eigenvectors of σ , which is also optimal for the bound (2.142), i.e.,

$$\tilde{A}^{*-1}\sigma : \sigma = \min_{B^* \in G_\theta} (B^{*-1}\sigma : \sigma).$$

In view of (4.35), this implies that

$$J^*(\theta, A^*) = \int_{\Omega} \tilde{A}^{*-1}\sigma : \sigma dx + \ell \int_{\Omega} \theta(x) dx \geq J^*(\theta, \tilde{A}^*),$$

and the last inequality is actually an equality since (θ, A^*) is a minimizer of J^* . We deduce first that (θ, \tilde{A}^*) is another minimizer of J^* , which belongs to

\mathcal{LD} , and second that σ is also the stress tensor associated to (θ, \tilde{A}^*) , which is a minimizer of

$$\min_{\substack{\tau \in L^2(\Omega; \mathcal{M}_N^s) \\ -\operatorname{div} \tau = \text{fin } \Omega}} \int_{\Omega} \tilde{A}^{*-1} \tau : \tau dx.$$

To prove that $\tilde{A}^{*-1} \sigma = A^{*-1} \sigma$, we simply observe that the function

$$\tau \rightarrow (A^{*-1} \tau : \tau - g(\theta, \tau))$$

is C^1 , nonnegative, and minimum at $\tau = \sigma$ (since it vanishes for this value), and thus its derivative vanishes also at $\tau = \sigma$. Thus, for any optimal A^* we have

$$A^{*-1} \sigma = \frac{\partial g}{\partial \tau}(\theta, \sigma),$$

which gives the desired equality since \tilde{A}^* is also optimal. \square

One can also write the objective function as the minimum of the energy in the original, unrelaxed, problem. Recalling that

$$J(\chi) = \int_{\Omega} f(x) \cdot u_{\chi}(x) dx + \ell \int_{\Omega} \chi(x) dx,$$

with u_{χ} the solution of (4.3). Taking into account

$$\int_{\Omega} f \cdot u_{\chi} dx = \min_{\substack{\tau \in L^2(\Omega; \mathcal{M}_N^s) \\ -\operatorname{div} \tau = \text{fin } \Omega}} \int_{\Omega} A_{\chi}^{-1} \tau : \tau dx, \quad (4.41)$$

switching the two minimizations and performing a pointwise minimization with respect to χ leads to

$$\inf_{\chi \in L^{\infty}(\Omega; \{0,1\})} J(\chi) = \inf_{\substack{\tau \in L^2(\Omega; \mathcal{M}_N^s) \\ -\operatorname{div} \tau = \text{fin } \Omega}} \int_{\Omega} \min_{\chi=0,1} (A_{\chi}^{-1} \tau : \tau + \ell \chi) dx.$$

The pointwise minimization in χ is easy and yields

$$\inf_{\chi \in L^{\infty}(\Omega; \{0,1\})} J(\chi) = \inf_{\substack{\tau \in L^2(\Omega; \mathcal{M}_N^s) \\ -\operatorname{div} \tau = \text{fin } \Omega}} \int_{\Omega} F(\tau) dx, \quad (4.42)$$

with

$$F(\tau) = \min (A^{-1} \tau : \tau + \ell, B^{-1} \tau : \tau). \quad (4.43)$$

The minimization problem on the right hand side of (4.42) appears as a nonlinear elasticity problem. We call it the stress formulation of the original optimal design problem with which it is equivalent in the following sense. If there exists a minimizer χ of J , then the optimal τ in (4.41) is also a minimizer of (4.42), while, if τ is a minimizer of (4.42), then the characteristic function χ , defined pointwise by,

$$\chi = \begin{cases} 1 & \text{if } A^{-1}\tau : \tau + \ell \leq B^{-1}\tau : \tau \\ 0 & \text{otherwise} \end{cases}$$

is a minimizer of $J(\chi)$. Kohn and Strang [152] recognized that relaxing the optimal design problem is actually equivalent to relaxing the nonlinear elasticity problem. In the former case, relaxation is obtained through homogenization, while in the latter case it is obtained through quasiconvexification. We briefly recall some basic definitions and results on this notion in Remark 4.1.16 below.

Theorem 4.1.15 *The relaxation of the stress formulation (4.42) is*

$$\min_{\substack{\tau \in L^2(\Omega; \mathcal{M}_N^s) \\ -\operatorname{div} \tau = f \text{ in } \Omega}} \int_{\Omega} QF(\tau) dx, \quad (4.44)$$

where $QF(\tau)$ is the quasiconvex envelope of $F(\tau)$, a continuous function with quadratic growth, explicitly defined by

$$QF(\tau) = \min_{0 \leq \theta \leq 1} (g(\theta, \tau) + \ell\theta)$$

with $g(\theta, \tau)$ the lower dual Hashin-Shtrikman bound (2.142), i.e.,

$$g(\theta, \tau) = \min_{A^* \in G_\theta} (A^{*-1}\tau : \tau).$$

Remark 4.1.16 Recall that quasiconvexification in the sense of Morrey for functionals of gradients was defined in Remark 3.2.25. Here we introduce the appropriate generalization of this notion for a real-valued functional W defined on divergence-free symmetric tensors. Assume that W is a measurable locally bounded function that satisfies, for some constants $0 < c < C$,

$$c(|\tau|^2 - 1) \leq W(\tau) \leq C(|\tau|^2 + 1), \quad \tau \in \mathcal{M}_N^s. \quad (4.45)$$

According to Fonseca and Müller [107] the quasiconvex envelope of W (or its quasiconvexification) is defined for every τ in \mathcal{M}_N^s as

$$QW(\tau) = \inf_{\substack{\eta \in L^2_{\#}(Y; \mathcal{M}_N^s) \\ -\operatorname{div} \eta = 0 \text{ in } Y \text{ and } \int_Y \eta dx = 0}} \int_Y W(\tau + \eta(x)) dx, \quad (4.46)$$

where $Y = (0, 1)^N$ is the unit cube of \mathbb{R}^N . Dacorogna [88] previously introduced a different definition of QW , namely

$$QW(\tau) = \inf_{\substack{\eta \in L^2(Y; \mathcal{M}_N^s) \\ -\operatorname{div} \eta = 0 \text{ in } Y \text{ and } \int_Y \eta dx = 0}} \int_Y W(\tau + \eta(x)) dx, \quad (4.47)$$

where the test fields $\eta(x)$ are not periodic. However, unlike in the gradient case (see Remark 3.2.25), it is not clear that these two definitions coincide, especially concerning relaxation results. The function QW is a locally Lipschitz function, as is easily seen upon noting that QW is rank- $(N-1)$ convex and satisfies the same growth assumption as W [181]. The main property of the quasiconvexification is its link with relaxation. For any bounded open set Ω of \mathbb{R}^N and force $f \in L^2(\Omega)^N$, define the following space of given divergence stress fields

$$\Sigma_f(\Omega) = \left\{ \tau \in L^2(\Omega; \mathcal{M}_N^s) \mid \operatorname{div} \tau = f \text{ in } \Omega \right\}.$$

Then, the lower semicontinuous envelope in $\Sigma_f(\Omega)$ of

$$I(\tau) = \int_{\Omega} W(\tau) dx$$

for the sequential weak topology of $L^2(\Omega; \mathcal{M}_N^s)$, i.e., the functional

$$RI(\tau) = \liminf_{\substack{\tau_n \in \Sigma_f(\Omega) \\ \tau_n \rightharpoonup \tau \text{ weakly in } L^2(\Omega; \mathcal{M}_N^s)}} I(\tau_n)$$

is given by

$$RI(\tau) = \int_{\Omega} QW(\tau) dx. \quad (4.48)$$

Until very recently, a proof of (4.48) was hard to locate in the literature for at least two reasons. First, attention had been focused primarily on functionals that depend on the gradient of a vector-valued field rather than on fields

with prescribed divergence. Second, the integrand W is often assumed to be continuous, whereas we have in mind a discontinuous function W (at least in Section 4.2). The last point is the easiest to deal with: it is well known among experts that the continuity of W is not an issue, at least when W satisfies a growth condition like (4.45). In the gradient setting a proof of the relaxation result for homogeneous measurable integrands W can be found, e.g., in [57] by combining Propositions 6.7 and 9.2, and Theorem 12.5 (for W continuous, other references are [89], Theorems 1.1 and 2.1 in Chapter 5, and [2], Statement 3.7). The first point is more subtle. A general study of quasiconvexity for functionals of vector fields that satisfy some first order differential constraints in the spirit of compensated compactness has appeared only recently in [107] (see also [88]). In this work it is proved that quasiconvexity is a necessary and sufficient condition for weak lower semicontinuity, but no relaxation theorem is stated, although it is clear for the experts that such a result holds true.

Proof. Let QF be the quasiconvexification of F defined by (4.46). Since F is defined as a minimum by (4.43), denoting by χ the characteristic function of phase A , a simple switch in the minimizations leads to

$$QF(\tau) = \inf_{\chi \in L^\infty(Y; \{0,1\})} \left(A^{*-1} \tau : \tau + \ell \int_Y \chi \, dy \right),$$

where A^* is the periodic homogenized Hooke's law defined by (1.23), i.e.,

$$A^{*-1} \tau : \tau = \min_{\substack{\eta \in L^2_\#(Y; \mathcal{M}_N^s) \\ -\operatorname{div} \eta = 0 \text{ in } Y \text{ and } \int_Y \eta \, dy = 0}} \int_Y (\chi A^{-1} + (1-\chi)B^{-1})(\tau + \eta) : (\tau + \eta) \, dy.$$

Denoting by θ the average of χ in Y , by definition (2.8) A^* belongs to P_θ . As χ varies, A^* runs into the entire set P_θ , the closure of which is G_θ . Thus, we have

$$QF(\tau) = \inf_{0 \leq \theta \leq 1} \left(\inf_{A^* \in G_\theta} A^{*-1} \tau : \tau + \ell \theta \right). \quad (4.49)$$

We already saw that the two infima in (4.49) are attained, and that the minimum over G_θ of the complementary energy is just the lower Hashin-Shtrikman dual bound $g(\theta, \tau)$. Therefore, (4.49) is the desired result. The properties of $QF(\tau)$ are obvious from the definition (2.142) of $g(\theta, \tau)$. \square

Remark 4.1.17 Of course, the relaxed stress formulation (4.44) is equivalent to the relaxed optimal design problem (4.32) in the sense that, if the

design (θ, A^) is a minimizer in CD of (4.32), then the associated stress σ , which minimizes its compliance (4.35), is actually a minimizer of (4.44), and conversely a minimizer σ of (4.44) yields an optimal design (θ, A^*) through the optimality conditions of Theorem 4.1.9.*

Remark 4.1.18 *So far we proved the existence of relaxed optimal designs, obtained necessary conditions of optimality, and established various equivalent formulations of the problem. The next obvious question is thus to investigate the potential uniqueness of the solution, at least in the context of compliance optimization, and possibly using the notion of quasiconvexity. Unfortunately, as we shall see in Subsection 4.1.5, it turns out that infinitely many optimal designs do exist in some simple examples of compliance optimization. However, numerical evidence (see Subsection 5.2.3, Chapter 5) suggests that, for the quasiconvex functional arising from compliance minimization, there exist only global minima, since the optimal designs are insensitive to the initialization of the numerical algorithm (the algorithm never get trapped into a local suboptimal minima). This is of course a fundamental issue from a numerical point of view since most minimization algorithms cannot escape from local minima. As we already said in Remark 3.2.26, we conjecture that (4.44) admits only global minimizers in $L^2(\Omega; \mathcal{M}_N^s)$. This turns out to be the case for the example of Subsection 4.1.5, and more generally for any quasiconvex function when the boundary condition is affine (see Remark 3.2.26). Another fact supporting this conjecture is that formally, in the limit where ℓ goes to infinity, the functional becomes convex and corresponds to the so-called Michell truss problem (see Subsection 4.2.3).*

Instead of problem (4.32), we can consider the other objective function (4.33). The task of finding the most compliant design is not interesting in structural optimization, but it makes sense as a model in damage mechanics (as proposed by Francfort and Marigo [108]). In this case too the objective function can be written as the minimum of some energy. However, it is the primal elastic energy (using strains) instead of the complementary elastic energy (using stresses) that is pertinent. Indeed we have

$$-\int_{\Omega} f \cdot u dx = \min_{v \in H_0^1(\Omega)^N} \left(\int_{\Omega} A^* e(v) : e(v) dx - 2 \int_{\Omega} f \cdot v dx \right), \quad (4.50)$$

and the minimum on the right hand side of (4.50) is achieved by $v = u$. The

objective function is rewritten

$$J^*(\theta, A^*) = \min_{v \in H_0^1(\Omega)^N} \int_{\Omega} A^* e(v) : e(v) dx - 2 \int_{\Omega} f \cdot v dx + \ell \int_{\Omega} \theta(x) dx. \quad (4.51)$$

Since the order of minimization is irrelevant, we obtain

$$\begin{aligned} \min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*) &= \min_{v \in H_0^1(\Omega)^N} \left\{ \min_{(\theta, A^*) \in \mathcal{CD}} \left(\int_{\Omega} A^* e(v) : e(v) dx \right. \right. \\ &\quad \left. \left. + \ell \int_{\Omega} \theta(x) dx \right) - 2 \int_{\Omega} f \cdot v dx \right\}. \end{aligned} \quad (4.52)$$

Problem (4.33) of compliance maximization appears as a double minimization for which optimality conditions are easily derived. All the following results are in some sense parallel or dual to the previous ones established for problem (4.32) of compliance minimization. We therefore confidently leave their proofs to the reader. They were first obtained by Francfort and Marigo [108].

Theorem 4.1.19 *If (θ, A^*) is a minimizer of the objective function (4.33), and if u is the unique corresponding minimizer of (4.51), then u satisfies*

$$\begin{cases} -\operatorname{div} A^* e(u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and (θ, A^*) satisfies, almost everywhere in Ω ,

$$A^* e(u) : e(u) = g(\theta, e(u)),$$

where $g(\theta, e(u))$ is the lower Hashin-Shtrikman energy bound defined by (2.121), and θ is the unique minimizer of the convex minimization problem

$$\min_{0 \leq \theta \leq 1} (g(\theta, e(u)) + \ell\theta).$$

From Theorem 4.1.19 can be deduced a simplified relaxed problem which does not require the knowledge of G_θ , but uses only its subset L_θ^- of sequential laminates, with core B and matrix A , in proportions $(1 - \theta)$ and θ , respectively, defined in Definition 2.3.5 (see also Lemma 2.3.6). Let us introduce the set \mathcal{LD}^- of sequentially laminated designs, defined by

$$\mathcal{LD}^- = \left\{ (\theta, A^*) \in L^\infty(\Omega; [0, 1] \times \mathcal{M}_N^4) \text{ such that } A^*(x) \in L_\theta^-(x) \text{ a.e. in } \Omega \right\},$$

which is a strict subset of the set of all composite designs \mathcal{CD} , defined by (4.26). Note that L_θ^- (respectively \mathcal{LD}^-) is different from the previously used set L_θ^+ (respectively \mathcal{LD}^+).

Theorem 4.1.20 *For the objective function (4.33) we have*

$$\min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*) = \min_{(\theta, A^*) \in \mathcal{LD}^-} J^*(\theta, A^*).$$

If (θ, A^) is a minimizer of J^* in \mathcal{CD} , and if u is its associated displacement which minimizes (4.51), then there exists a sequential laminate \tilde{A}^* such that (θ, \tilde{A}^*) is a minimizer of J^* in \mathcal{LD}^- , u is again its associated displacement, and $A^*e(u) = \tilde{A}^*e(u)$. Furthermore, \tilde{A}^* can be chosen as a rank- N sequential laminate (with a core of B and a matrix of A).*

Remark 4.1.21 *The lamination directions of the optimal rank- N sequential laminate \tilde{A}^* in Theorem 4.1.20 are usually not orthogonal (this is its main difference with respect to Theorem 4.1.12 due to the fact that laminates belong to L_θ^- and not to L_θ^+). However, these lamination directions can still be explicitly computed as the optimal vectors in the definition of the nonlocal term $g(\eta)$ in the lower Hashin-Shtrikman energy bound of Proposition 2.3.20 (see the proof of Lemma 2.3.21 for details).*

Theorem 4.1.15 can also be generalized to the objective function (4.33) in this primal setting. It appeals to the usual notion of quasiconvexification as recalled in Remark 3.2.25 (the situation is indeed very similar to the conductivity setting as treated in Subsection 3.2.5). By writing the negative of the compliance as the minimum of some elastic energy, and exchanging the orders of minimization, the optimal design problem (4.33) is equivalent to its strain formulation

$$\inf_{v \in H_0^1(\Omega)^N} \int_{\Omega} (F(e(v)) - 2f \cdot v) dx, \quad (4.53)$$

where F is defined by

$$F(\eta) = \min(A\eta : \eta + \ell, B\eta : \eta).$$

Theorem 4.1.22 *The relaxation of the strain formulation (4.53) is*

$$\min_{v \in H_0^1(\Omega)^N} \int_{\Omega} (QF(e(v)) - 2f \cdot v) dx, \quad (4.54)$$

where $QF(\eta)$ is the quasiconvex envelope of $F(\eta)$, a continuous function with quadratic growth, explicitly defined by

$$QF(\eta) = \min_{0 \leq \theta \leq 1} (g(\theta, \eta) + \ell\theta)$$

with $g(\theta, \eta)$ the lower Hashin-Shtrikman primal bound (2.121), i.e.,

$$g(\theta, \eta) = \min_{A^* \in G_\theta} (A^* \eta : \eta).$$

Remark 4.1.23 In this entire subsection we used, in an essential way, the assumption that the two phases A and B are wellordered, i.e., they satisfy (4.1). Indeed, it is a necessary assumption for establishing the optimal Hashin-Shtrikman energy bounds that are used for building the relaxed formulation.

Instead of the “single load” objective functions (4.32) and (4.33), we could have considered equally well their “multiple loads” equivalents. We focus on compliance minimization and recall the relaxed formulation of (4.10)

$$\min_{(\theta, A^*) \in \mathcal{CD}} \left\{ J^*(\theta, A^*) = \sum_{i=1}^n \int_{\Omega} f_i \cdot u_i dx + \ell \int_{\Omega} \theta dx \right\}, \quad (4.55)$$

where each displacement u_i is solution in $H_0^1(\Omega)^N$ of

$$\begin{cases} -\operatorname{div} A^* e(u_i) = f_i & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.56)$$

Each compliance is the minimum of the complementary energy

$$\int_{\Omega} f_i \cdot u_i dx = \min_{\substack{\tau_i \in L^2(\Omega; \mathcal{M}_N^s) \\ -\operatorname{div} \tau_i = f_i \text{ in } \Omega}} \int_{\Omega} A^{*-1} \tau_i : \tau_i dx, \quad (4.57)$$

and the minimum on the right hand side of (4.57) is achieved by $\sigma_i = A^* e(u_i)$. We may rewrite the objective function as

$$J^*(\theta, A^*) = \sum_{i=1}^n \min_{\substack{\tau_i \in L^2(\Omega; \mathcal{M}_N^s) \\ -\operatorname{div} \tau_i = f_i \text{ in } \Omega}} \int_{\Omega} A^{*-1} \tau_i : \tau_i dx + \ell \int_{\Omega} \theta dx, \quad (4.58)$$

and, switching the order of minimization, (4.55) is equivalent to

$$\min_{\substack{\{\tau_i\} \in L^2(\Omega; \mathcal{M}_N^s) \\ -\operatorname{div} \tau_i = f_i \text{ in } \Omega}} \min_{(\theta, A^*) \in \mathcal{CD}} \int_{\Omega} \left(\sum_{i=1}^n A^{*-1} \tau_i : \tau_i + \ell \theta \right) dx. \quad (4.59)$$

In view of (4.59), the optimal design problem is a double minimization for which optimality conditions are derived in the following obvious generalization of Theorem 4.1.9.

Theorem 4.1.24 If (θ, A^*) is a minimizer of the objective function (4.55), and if $(\sigma_i)_{1 \leq i \leq n}$ are the unique corresponding minimizers of (4.58), then each σ_i is given by $\sigma_i = A^{*-1}e(u_i)$ where u_i is the unique solution of (4.56), and (θ, A^*) satisfies, almost everywhere in Ω ,

$$\sum_{i=1}^n A^{*-1}\sigma_i : \sigma_i = g^n(\theta, \{\sigma_i\}), \quad (4.60)$$

where $g^n(\theta, \{\sigma_i\})$ is the lower Hashin-Shtrikman bound on complementary energies defined by (2.137), and θ is the unique minimizer of the convex minimization

$$\min_{0 \leq \theta \leq 1} (g^n(\theta, \{\sigma_i\}) + \ell\theta). \quad (4.61)$$

Remark 4.1.25 As explained in Subsection 2.3.2, by diagonalizing the fourth order tensor $\sum_{i=1}^n \sigma_i \otimes \sigma_i$, there is no need to consider n larger than $N(N+1)/2$ in the Hashin-Shtrikman bound on sums of energies. Therefore, if the number of loading configurations n is larger than $N(N+1)/2$, we nevertheless use the bound g^{n_0} for $n_0 = N(N+1)/2$ stresses that diagonalize the tensor $\sum_{i=1}^n \sigma_i \otimes \sigma_i$.

As in the single load case, a simplified relaxed formulation is obtained by replacing the unknown G -closure set G_θ by its explicit subset L_θ^+ of sequential laminates, with core A and matrix B , in proportions θ and $(1-\theta)$, respectively.

Theorem 4.1.26 Let \mathcal{LD}^+ be the space of sequentially laminated designs, defined by (4.40). For the objective function (4.55) we have

$$\min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*) = \min_{(\theta, A^*) \in \mathcal{LD}^+} J^*(\theta, A^*).$$

If (θ, A^*) is a minimizer of J^* in \mathcal{CD} , and if $(\sigma_i)_{1 \leq i \leq n}$ are its associated stress tensors that minimize (4.58), then there exists a sequential laminate \tilde{A}^* such that (θ, \tilde{A}^*) is a minimizer of J^* in \mathcal{LD}^+ , $(\sigma_i)_{1 \leq i \leq n}$ are again its associated stress tensors, and $A^{*-1}\sigma_i = \tilde{A}^{*-1}\sigma_i$ for any $1 \leq i \leq n$.

Remark 4.1.27 As Theorem 4.1.12 in the single load case, Theorem 4.1.26 gives sense to the homogenization method since, for compliance optimization, it is legitimate to restrict the full G -closure set G_θ of all homogenized Hooke's laws to the smaller and explicit subset L_θ^+ of sequential laminates (with a core of A and a matrix of B). The double minimization in (4.59), as well as Theorems 4.1.24 and 4.1.26, are at the basis of numerical algorithms described in Subsection 5.2.6 of Chapter 5.

Theorem 4.1.26 does not say anything concerning the rank of the optimal sequential laminate \tilde{A}^* . In the single load case, it was proved that \tilde{A}^* can be chosen as a rank- N sequential laminate with orthogonal lamination directions furnished by the eigenvectors of the stress tensor σ . No such result is known in the multiple loads case. We lack an explicit characterization of the lower Hashin-Shtrikman bound on complementary energies (2.137). By a matter of theory (see Lemma 2.3.9), we know that the rank of the optimal laminate is always smaller or equal to 3 in two dimensions and 6 in three dimensions. But we have no idea of the lamination directions. In practice, the optimal laminate is found by a numerical optimization process (see Subsection 5.2.6).

In the multiple loads case, there is also a link between relaxation by homogenization and quasiconvexification. Recall that the original objective function is

$$J(\chi) = \sum_{i=1}^n \int_{\Omega} f_i \cdot u_{i,\chi} dx + \ell \int_{\Omega} \chi dx,$$

with $u_{i,\chi}$ the solution of (4.8). Using the principle of minimal complementary energy for each compliance, and switching the two minimizations leads to

$$\inf_{\chi \in L^\infty(\Omega; \{0,1\})} J(\chi) = \inf_{\substack{\{\tau_i\} \in L^2(\Omega; \mathcal{M}_N^s) \\ -\operatorname{div} \tau_i = f_i \text{ in } \Omega}} \int_{\Omega} \min_{\chi=0,1} \left(\sum_{i=1}^n A_\chi^{-1} \tau_i : \tau_i + \ell \chi \right) dx.$$

The pointwise minimization in χ is easy and yields

$$\inf_{\chi \in L^\infty(\Omega; \{0,1\})} J(\chi) = \inf_{\substack{\{\tau_i\} \in L^2(\Omega; \mathcal{M}_N^s) \\ -\operatorname{div} \tau_i = f_i \text{ in } \Omega}} \int_{\Omega} F^n(\{\tau_i\}) dx, \quad (4.62)$$

with

$$F^n(\{\tau_i\}) = \min \left(\ell + \sum_{i=1}^n A^{-1} \tau_i : \tau_i, \sum_{i=1}^n B^{-1} \tau_i : \tau_i \right). \quad (4.63)$$

The minimization problem on the right hand side of (4.62) is called the *stress formulation* of the optimal design problem (it is a nonlinear elasticity problem). Both sides of (4.62) are equivalent in the sense that, if there exists a minimizer χ of J , then the optimal $\{\tau_i\}$ in the principle of minimal complementary energy is also a minimizer of (4.62), while, if $\{\tau_i\}$ is a minimizer of (4.62), then the characteristic function χ defined pointwise by

$$\chi = \begin{cases} 1 & \text{if } \ell + \sum_{i=1}^n A^{-1} \tau_i : \tau_i \leq \sum_{i=1}^n B^{-1} \tau_i : \tau_i \\ 0 & \text{otherwise} \end{cases}$$

is a minimizer of J . As recognized by Kohn and Strang [152], relaxing the optimal design problem by the homogenization method is actually equivalent to relaxing its stress formulation by quasiconvexification.

Theorem 4.1.28 *The relaxation of the stress formulation (4.62) is*

$$\min_{\substack{\{\tau_i\} \in L^2(\Omega; \mathcal{M}_N^s) \\ -\operatorname{div} \tau_i = f_i \text{ in } \Omega}} \int_{\Omega} QF^n(\{\tau_i\}) dx, \quad (4.64)$$

where $QF^n(\{\tau_i\})$ is the quasiconvex envelope of $F^n(\{\tau_i\})$, a continuous function with quadratic growth, explicitly defined by

$$QF^n(\{\tau_i\}) = \min_{0 \leq \theta \leq 1} (g^n(\theta, \{\tau_i\}) + \ell\theta)$$

with $g^n(\theta, \{\tau_i\})$ the lower dual Hashin-Shtrikman bound (2.137), i.e.,

$$g(\theta, \{\tau_i\}) = \min_{A^* \in G_\theta} \sum_{i=1}^n A^{*-1} \tau_i : \tau_i.$$

The proofs of Theorems 4.1.24, 4.1.26, and 4.1.28 are identical, respectively, to those of their counterparts in the single load case, so we omit them.

So far, we considered multiple loads optimization as minimizing the (possibly weighted) sum of compliances as described in (4.55). However, in many applications it is preferable to minimize the worst case compliance, i.e., to minimize the maximum of the compliances. The relaxed formulation of such a problem (see (4.11)) is easily seen to be

$$\min_{(\theta, A^*) \in \mathcal{CD}} \left\{ J^*(\theta, A^*) = \max_{1 \leq i \leq n} c_i(\theta, A^*) + \ell \int_{\Omega} \theta dx \right\}, \quad (4.65)$$

where \mathcal{CD} is the space of homogenized admissible designs and each compliance c_i is defined by

$$c_i(\theta, A^*) = \min_{\substack{\tau_i \in L^2(\Omega; \mathcal{M}_N^s) \\ -\operatorname{div} \tau_i = f_i \text{ in } \Omega}} \int_{\Omega} A^{*-1} \tau_i : \tau_i dx. \quad (4.66)$$

As before, a simplified relaxed formulation can be obtained by replacing the unknown G -closure set G_θ by its explicit subset L_θ^+ of sequential laminates.

Theorem 4.1.29 Let \mathcal{LD}^+ be the space of sequentially laminated designs, defined by (4.40). For the “worst case design” objective function (4.65) we have

$$\min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*) = \min_{(\theta, A^*) \in \mathcal{LD}^+} J^*(\theta, A^*).$$

If (θ, A^*) is a minimizer of J^* in \mathcal{CD} , and if $(\sigma_i)_{1 \leq i \leq n}$ are its associated stress tensors that minimize (4.66), then there exists a sequential laminate \tilde{A}^* such that (θ, \tilde{A}^*) is a minimizer of J^* in \mathcal{LD}^+ , $(\sigma_i)_{1 \leq i \leq n}$ are again its associated stress tensors, and $A^{*-1}\sigma_i = \tilde{A}^{*-1}\sigma_i$ for any $1 \leq i \leq n$.

Proof. To begin with, we rewrite (4.65) as an equivalent constrained problem

$$\min_{\substack{(\theta, A^*) \in \mathcal{CD} \\ c_i(\theta, A^*) \leq c_0}} \left\{ c_0 + \ell \int_{\Omega} \theta dx \right\}.$$

Introducing Lagrange multipliers ℓ_i , this problem is also equivalent to

$$\min_{(\theta, A^*) \in \mathcal{CD}} \max_{\ell_i \geq 0} \min_{\substack{\tau_i \in L^2(\Omega; \mathcal{M}_N^s) \\ -\operatorname{div} \tau_i = f_i \text{ in } \Omega}} F(\theta, A^*, \ell_i, \tau_i)$$

with

$$F(\theta, A^*, \ell_i, \tau_i) = c_0 + \sum_{i=1}^n \ell_i \left(\int_{\Omega} A^{*-1} \tau_i : \tau_i dx - c_0 \right) + \ell \int_{\Omega} \theta dx.$$

For the sake of brevity in the sequel we no longer write the admissibility constraint on the stress τ_i . Using a standard min-max theorem, one can exchange, in a first step, the maximization in ℓ_i with the minimization in τ_i , and then switch the minimizations in A^* and in τ_i to obtain another equivalent formulation

$$\min_{\theta} \min_{\tau_i} \min_{A^* \in G_{\theta}} \max_{\ell_i \geq 0} F(\theta, A^*, \ell_i, \tau_i).$$

Since a min-max is always larger than a max-min, we have

$$\min_{A^* \in G_{\theta}} \max_{\ell_i \geq 0} F(\theta, A^*, \ell_i, \tau_i) \geq \max_{\ell_i \geq 0} \min_{A^* \in G_{\theta}} F(\theta, A^*, \ell_i, \tau_i). \quad (4.67)$$

However, by Proposition 2.3.24 on the Hashin-Shtrikman energy bounds, we have

$$\min_{A^* \in G_{\theta}} F(\theta, A^*, \ell_i, \tau_i) = \min_{A^* \in L_{\theta}^+} F(\theta, A^*, \ell_i, \tau_i). \quad (4.68)$$

Parametrizing the effective tensor A^* by its H -measure ν (see the lamination formula (2.73) and Lemma 2.3.6), Lipton [168] proved that the minimization with respect to ν is actually convex, and thus the following min-max property holds (note that the maximization in ℓ_i is concave) :

$$\max_{\ell_i \geq 0} \min_{A^* \in L_\theta^+} F(\theta, A^*, \ell_i, \tau_i) = \min_{A^* \in L_\theta^+} \max_{\ell_i \geq 0} F(\theta, A^*, \ell_i, \tau_i). \quad (4.69)$$

Since L_θ^+ is a subset of G_θ we also have

$$\min_{A^* \in L_\theta^+} \max_{\ell_i \geq 0} F(\theta, A^*, \ell_i, \tau_i) \geq \min_{A^* \in G_\theta} \max_{\ell_i \geq 0} F(\theta, A^*, \ell_i, \tau_i),$$

which, combined with (4.67), (4.68), and (4.69) yields

$$\min_{A^* \in G_\theta} \max_{\ell_i \geq 0} F(\theta, A^*, \ell_i, \tau_i) = \min_{A^* \in L_\theta^+} \max_{\ell_i \geq 0} F(\theta, A^*, \ell_i, \tau_i).$$

Going backward from here (i.e., switching the minimizations in A^* and in τ_i , then exchanging the maximization in ℓ_i with the minimization in τ_i), we deduce the desired result

$$\begin{aligned} \min_{(\theta, A^*) \in \mathcal{CD}} \max_{\ell_i \geq 0} \min_{\substack{\tau_i \in L^2(\Omega; \mathcal{M}_N^s) \\ -\operatorname{div} \tau_i = f_i \text{ in } \Omega}} F(\theta, A^*, \ell_i, \tau_i) = \\ \min_{(\theta, A^*) \in \mathcal{LD}^+} \max_{\ell_i \geq 0} \min_{\substack{\tau_i \in L^2(\Omega; \mathcal{M}_N^s) \\ -\operatorname{div} \tau_i = f_i \text{ in } \Omega}} F(\theta, A^*, \ell_i, \tau_i). \end{aligned} \quad (4.70)$$

Formula (4.70) shows that this worst case optimization problem is just a weighted compliance sum problem as (4.55), except that the weights ℓ_i have to be adjusted iteratively. \square

Following the classical idea of robust control in control theory, one can modify the multiple loads optimization problem in order to take into account a possible uncertainty in the applied loads. Indeed, very often the precise loads to which a structure is submitted and has to be optimized are not precisely known. Therefore, the applied load is decomposed in the sum of two parts: first a known force $f \in L^2(\Omega)^N$, and second an unknown and varying force $g \in L^2(\Omega)^N$ (the magnitude of which is assumed to be bounded by a given constant $m > 0$). The relaxation process for this problem works as before, and we directly state the relaxed problem

$$\min_{(\theta, A^*) \in \mathcal{CD}} \left\{ J^*(\theta, A^*) = \max_{\|g\|_{L^2(\Omega)^N} \leq m} c(g) \right\}, \quad (4.71)$$

where $c(g)$ is the compliance for the load $f + g$, i.e.,

$$c(g) = \int_{\Omega} (f + g) \cdot u \, dx \quad (4.72)$$

with u the solution of

$$\begin{cases} -\operatorname{div} A^* e(u) = f + g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

There are many variants of this problem: One can consider surface loads, the unknown load g may satisfy further constraints, or it may be applied only on a subdomain. Under a slightly different form, this robust optimization problem was previously studied in [75]. Another motivation for this type of problem is the stabilization of unstable designs that are optimized for a single load. Quite often, single load optimal designs are very compliant for other loads; this would not be the case with such robust optimal designs. Once again, a simplified relaxed formulation can be obtained by replacing the unknown G -closure set G_θ by its explicit subset L_θ^+ of sequential laminates.

Theorem 4.1.30 *Let \mathcal{LD}^+ be the space of sequentially laminated designs, defined by (4.40). For the “robust optimization” objective function (4.71) we have*

$$\min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*) = \min_{(\theta, A^*) \in \mathcal{LD}^+} J^*(\theta, A^*).$$

Proof. We rewrite the compliance as a maximization with respect to the displacement

$$c(g) = \max_{v \in H_0^1(\Omega)^N} \left(- \int_{\Omega} A^* e(v) : e(v) \, dx + 2 \int_{\Omega} (f + g) \cdot v \, dx \right). \quad (4.73)$$

Exchanging the two maximizations, the objective function becomes

$$J^*(\theta, A^*) = \max_{v \in H_0^1(\Omega)^N} \max_{\|g\|_{L^2(\Omega)^N} \leq m} - \int_{\Omega} A^* e(v) : e(v) \, dx + 2 \int_{\Omega} (f + g) \cdot v \, dx$$

and

$$J^*(\theta, A^*) = \max_{v \in H_0^1(\Omega)^N} - \int_{\Omega} A^* e(v) : e(v) \, dx + 2m \left(\int_{\Omega} |v|^2 \, dx \right)^{1/2} + 2 \int_{\Omega} f \cdot v \, dx. \quad (4.74)$$

Note that, if $f = 0$, then (4.74) is just the so-called *Auchmuy variational principle* for computing the first eigenvalue in Ω [31]. In any case, there always exists at least a maximizer on the right hand side of (4.74). Now, generalizing a saddle point result of Lipton [168], we can prove that, for a fixed density $\theta \in L^\infty(\Omega; [0, 1])$, we have

$$\begin{aligned} \min_{A^*(x) \in G_{\theta(x)} \text{ a.e. in } \Omega} J^*(\theta, A^*) &= \max_{v \in H_0^1(\Omega)^N} - \int_{\Omega} \left(\max_{A^* \in G_\theta} A^* e(v) : e(v) \right) dx \\ &\quad + 2m \left(\int_{\Omega} |v|^2 dx \right)^{1/2} + 2 \int_{\Omega} f \cdot v dx. \end{aligned} \tag{4.75}$$

The proof of (4.75) is completely similar to that of Theorem 4.1.43 below, so we omit it for the sake of brevity. By Proposition 2.3.24 on the Hashin-Shtrikman energy bounds, we can replace G_θ by L_θ^+ on the right hand side of (4.75). Therefore, we have proved that

$$\min_{A^*(x) \in G_{\theta(x)} \text{ a.e. in } \Omega} J^*(\theta, A^*) = \min_{A^*(x) \in L_{\theta(x)}^+ \text{ a.e. in } \Omega} J^*(\theta, A^*)$$

which yields the desired result. \square

Remark 4.1.31 As in the usual multiple loads case, it is not possible to specify the lamination parameters of the optimal sequential laminate for the worst case design or the robust optimization problems (4.65), (4.71). In particular, the rank of an optimal sequential laminate may be higher than N .

4.1.5 Counterexample to the Uniqueness of Optimal Designs

As in the conductivity case (see Subsection 3.2.6) there is usually no uniqueness of the generalized or relaxed optimal designs. Based on the coated confocal ellipsoid construction of Bergman [51], [52], Milton [187], and Tartar [274] (which already yields nonuniqueness of optimal designs in the conductivity setting), we give an explicit example of compliance optimization, where there is a an infinite number of optimal designs, which are, furthermore, global minimizers and classical designs. The coated concentric sphere construction is well known in elasticity [81], [131], but it is only recently that Grabovsky [122], [124] showed that confocal ellipsoids also work in the elasticity setting.

Let Ω be a smooth bounded open set of \mathbb{R}^N . Let ξ be a symmetric matrix in \mathcal{M}_N^s . We consider a mixture of two isotropic wellordered elastic phases A and B satisfying assumption (4.1). For a characteristic function χ in $L^\infty(\Omega; \{0, 1\})$ the overall Hooke's law is

$$A_\chi(x) = \chi(x)A + (1 - \chi(x))B. \quad (4.76)$$

This design χ is tested on a single load problem with no source term and affine boundary conditions for the displacement $u_\chi(x)$, which is the unique solution in $H^1(\Omega)^N$ of

$$\begin{cases} -\operatorname{div}(A_\chi e(u_\chi)) = 0 & \text{in } \Omega \\ u_\chi = \xi x & \text{on } \partial\Omega. \end{cases} \quad (4.77)$$

The goal of this optimal design problem is to find the weakest structure in Ω that has an average strain ξ (there is a similar dual version of this problem, which amounts to conversely finding the strongest structure; see below). More precisely, the performance of χ is measured by the following objective function

$$J(\chi) = \int_{\partial\Omega} (A_\chi e(u_\chi)n) \cdot u_\chi \, ds + \ell \int_{\Omega} \chi \, dx, \quad (4.78)$$

where ds is the surface measure on $\partial\Omega$ and ℓ is a fixed Lagrange multiplier for a volume constraint on material A . Finally, the optimal design problem is the minimization

$$\inf_{\chi \in L^\infty(\Omega; \{0, 1\})} J(\chi). \quad (4.79)$$

An easy integration by parts in the state equation shows that the objective function is also equal to

$$J(\chi) = \int_{\Omega} A_\chi e(u_\chi) : e(u_\chi) \, dx + \ell \int_{\Omega} \chi \, dx,$$

which can be rewritten as

$$J(\chi) = \min_{u \in D_\xi} \int_{\Omega} A_\chi e(u) : e(u) \, dx + \ell \int_{\Omega} \chi \, dx, \quad (4.80)$$

where D_ξ denotes the space of vector-valued functions

$$D_\xi = \left\{ \xi x + H_0^1(\Omega)^N \right\}.$$

Problem (4.79) is self-adjoint, and as in Subsection 4.1.4, the two minimizations with respect to χ and u can be switched to yield an equivalent formulation, which is

$$\inf_{u \in D_\xi} \int_{\Omega} F(e(u)) dx, \quad (4.81)$$

with

$$F(\eta) = \min(A\eta : \eta + \ell, B\eta : \eta). \quad (4.82)$$

Remark 4.1.32 We claim that (4.79) corresponds to finding the weakest design in Ω , although we are minimizing the elastic energy. This seemingly contradiction disappears after one recognizes that a prescribed displacement is imposed on the boundary. Thus, a weak design requires very little elastic energy to achieve this deformation, while a strong design would require more energy.

As already seen, the integrand F is not quasiconvex, and thus the functional (4.81) must be relaxed by quasiconvexification. According to Theorem 4.1.22 the relaxed functional is

$$\min_{u \in D_\xi} \int_{\Omega} QF(e(u)) dx, \quad (4.83)$$

where QF is the quasiconvex envelope of F defined by (3.91), which is also equal to

$$QF(\eta) = \min_{0 \leq \theta \leq 1} (g(\theta, \eta) + \ell\theta), \quad (4.84)$$

where $g(\theta, \eta)$ is the lower Hashin-Shtrikman primal bound defined by (2.121), namely,

$$g(\theta, \eta) = \min_{A^* \in G_\theta} (A^* \eta : \eta). \quad (4.85)$$

To make this example fully explicit, the choice of affine boundary conditions is crucial. Indeed, because of the affine character of the boundary conditions and by definition (3.93) of QF , the simple function $u_0(x) = \xi x$ is actually a minimizer of the relaxed functional (4.83), and the value of the minimum is

$$|\Omega| QF(\xi) = \inf_{u \in D_\xi} \int_{\Omega} F(e(u)) dx. \quad (4.86)$$

After these preliminaries we can state the main result of this subsection which was established in [25], [122].

Theorem 4.1.33 Let $\xi_1 \leq \dots \leq \xi_N$ be the eigenvalues of the symmetric matrix ξ . If ξ or $-\xi$ is a positive definite matrix such that, for any $1 \leq i \leq N$,

$$\frac{N\xi_i}{\text{tr}(\xi)} > \frac{\kappa_A + \frac{2(N-1)}{N}\mu_A}{\kappa_B + \frac{2(N-1)}{N}\mu_A} \quad (4.87)$$

and

$$(\kappa_B - \kappa_A) \frac{\kappa_A + 2\frac{N-1}{N}\mu_A}{\kappa_B + 2\frac{N-1}{N}\mu_A} < \frac{\ell}{(\text{tr}(\xi))^2} < (\kappa_B - \kappa_A) \frac{\kappa_B + 2\frac{N-1}{N}\mu_A}{\kappa_A + 2\frac{N-1}{N}\mu_A}, \quad (4.88)$$

there exist infinitely many classical minimizers of (4.79) in $L^\infty(\Omega; \{0, 1\})$, which are all global minimizers.

The optimal designs χ that minimize (4.79) are ellipsoidal inclusions of all sizes of phase B into a matrix of phase A (see Figure 3.3).

Remark 4.1.34 Condition (4.87) is satisfied for $\xi = I_2$ and more generally for matrices ξ close to I_2 or $-I_2$. Condition (4.88) is actually equivalent to the fact that the optimal volume fraction θ^* in (4.84) satisfies $0 < \theta^* < 1$. This condition is also equivalent to $QF(\xi) < F(\xi)$, since $\theta^* = 0$ or 1 implies that $QF(\xi) = F(\xi)$, while all other values of θ^* imply that there is a strict inequality. In the case of an hydrostatic strain $\xi = I_2$, we recover the well known Hashin-Shtrikman bound on the bulk modulus (see Theorem 2.3.13)

$$g(\theta, I_2) = \kappa_* \text{tr}(I_2)^2 = N^2 \left(\kappa_A + (1 - \theta) \frac{(\kappa_B - \kappa_A)(2\mu_A + \lambda_A)}{2\mu_A + \lambda_A + \theta(\kappa_B - \kappa_A)} \right),$$

and the confocal ellipsoid construction reduces to the well known concentric sphere assemblage.

Proof. By virtue of (4.86), the value of the minimum is $|\Omega|QF(\xi)$. The lower Hashin-Shtrikman bound $g(\theta, \xi)$, which appears in definition (4.84) of $QF(\xi)$, has been computed in Chapter 2 and its value is given by (2.121) as

$$g(\theta, \xi) = A\xi : \xi + (1 - \theta) \max_{\eta \in \mathcal{M}_N^s} [2\xi : \eta - (B - A)^{-1}\eta : \eta - \theta g(\eta)], \quad (4.89)$$

where $g(\eta)$ is a nonlocal term defined by

$$g(\eta) = \max_{e \in S_{N-1}} f_A(e)\eta : \eta,$$

with $f_A(e)$ defined by (2.118). The right hand side of (4.89) is a concave maximization, which admits a unique maximizer η^* . Writing its optimality condition (see (2.133)) yields that assumption (4.87) is precisely a necessary and sufficient condition for having η^* proportional to I_2 (see Proposition 5.2 in [25]). In such a case the value of $g(\theta, \xi)$ is easily computed, and we have

$$g(\theta, \xi) = 2\mu_A \left(|\xi|^2 - \frac{1}{N} (\text{tr}(\xi))^2 \right) + \kappa_* (\text{tr}(\xi))^2$$

whenever (4.87) is satisfied, with κ_* the lower Hashin-Shtrikman bulk modulus

$$\kappa_* = \kappa_A + (1 - \theta) \frac{(\kappa_B - \kappa_A)(2\mu_A + \lambda_A)}{2\mu_A + \lambda_A + \theta(\kappa_B - \kappa_A)}.$$

Eventually, for such ξ , the minimization with respect to θ in (4.84) yields an optimal proportion satisfying $0 < \theta < 1$ if and only if (4.88) holds true. Furthermore, denoting by A^* any homogenized Hooke's law in G_θ that is optimal for (4.85), an argument similar to that in the proof of Theorem 4.1.12 shows that

$$A^* \xi = \frac{\partial g}{\partial \xi}(\theta, \xi) = A \xi + 2(1 - \theta) \eta^*(\xi),$$

where $\eta^*(\xi)$ is the optimal matrix in (4.89). Since, for such ξ 's, $\eta^*(\xi)$ is proportional to I_2 , this last result implies that ξ and $A^* \xi$ are diagonalizable in the same basis. In other words, any optimal A^* in (4.85) is characterized by N values $(\lambda_i)_{1 \leq i \leq N}$ such that

$$\xi = \text{diag}(\xi_1, \dots, \xi_N)$$

implies that

$$A^* \xi = \text{diag}(\lambda_1 \xi_1, \dots, \lambda_N \xi_N). \quad (4.90)$$

For such strain ξ , proportion θ , and Hooke's law A^* , by Lemma 4.1.35 there exist two confocal ellipsoids B_{ρ^+}, B_{ρ^-} , (aligned with the eigendirections of ξ , with volume ratio θ and eccentricity parameters (m_i) determined by λ_i) such that the vector-valued function u_ξ , defined by (4.94), is a solution of the elasticity problem (4.95). By Vitali's covering theorem, there exists a countable family \mathcal{B} of disjoint homothetics of B_{ρ^+} which fill Ω

$$\left| \Omega - \bigcup_{B^+ \in \mathcal{B}} B^+ \right| = 0.$$

For any $B^+ \in \mathcal{B}$ we define an internal confocal ellipsoid B^- , which is homothetic to B_{ρ^-} with the same ratio as B^+ with respect to B_{ρ^+} . In the domain Ω we build a function, still denoted by u_ξ , which is equal to a scaled copy of the original u_ξ in each B^+ , and is equal to ξx elsewhere. By definition u_ξ is continuous in Ω and thus belongs to D_ξ . Furthermore,

$$\begin{aligned} \int_{\Omega} F(e(u_\xi)) dx &= |\Omega| \sum_{i=1}^N \lambda_i \xi_i^2 + \ell \sum_{B^+ \in \mathcal{B}} |B^+ \setminus B^-| \\ &= |\Omega| (A^* \xi : \xi + \ell \theta) \\ &= |\Omega| QF(\xi). \end{aligned} \quad (4.91)$$

Combining (4.86) and (4.91) shows that u_ξ is a minimizer for (4.81) over D_ξ . By virtue of the equivalence between (4.81) and (4.79), the characteristic function χ of the set $\bigcup_{B^+ \in \mathcal{B}} (B^+ \setminus B^-)$ is a minimizer of (4.79). There are as many such minimizers χ as there are possible Vitali's covering of Ω by ellipsoids B^+ , thus there are an infinite number of them. \square

Before stating Lemma 4.1.35, used in the proof of Theorem 4.1.33, we recall some notation on the confocal ellipsoids construction (see Subsection 3.2.6). Let ξ be a positive definite symmetric matrix satisfying (4.87) with eigenvalues $0 \leq \xi_1 \leq \dots \leq \xi_N$. Up to a rotation, we can assume that ξ is diagonal, i.e.,

$$\xi = \text{diag} (\xi_1, \dots, \xi_N).$$

Let $\theta \in (0, 1)$ be the optimal volume fraction in (4.84). Let A^* be any homogenized Hooke's law in G_θ that is optimal in (4.85), and thus, by virtue of (4.90), satisfies

$$A^* \xi = \text{diag} (\lambda_1 \xi_1, \dots, \lambda_N \xi_N).$$

Working in the eigenbasis of ξ , an ellipsoid B_ρ is defined by its boundary

$$\sum_{j=1}^N \frac{x_j^2}{\rho + m_j} = 1. \quad (4.92)$$

By Lemma 3.2.39 there exist two numbers $\rho^+ > \rho^- > -\min_i(m_i)$ and a family of parameters $(m_i)_{1 \leq i \leq N}$ such that

$$1 - \theta = \frac{g_m(\rho^-)}{g_m(\rho^+)} = \prod_{k=1}^N \sqrt{\frac{\rho^- + m_k}{\rho^+ + m_k}}$$

and

$$\begin{aligned} \frac{1}{\lambda_j - (2\mu_A + \lambda_A)} &= \frac{1}{N(\kappa_B - \kappa_A)} \frac{g_m(\rho^+)}{g_m(\rho^-)} \\ &\quad + \frac{g_m(\rho^+)}{2(2\mu_A + \lambda_A)} \int_{\rho^-}^{\rho^+} \frac{dr}{(r + m_j)g_m(r)}, \end{aligned} \quad (4.93)$$

with $g_m(\rho) = \prod_{k=1}^N (\rho + m_k)^{1/2}$. Introducing two confocal ellipsoids B_{ρ^-} and B_{ρ^+} defined by equation (4.92) with radii ρ^-, ρ^+ , we define an elasticity tensor C by

$$C(x) = \begin{cases} 2\mu_B I_4 + \left(\kappa_B - \frac{2\mu_B}{N}\right) I_2 \otimes I_2 & \text{in } B_{\rho^-} \\ 2\mu_A I_4 + \left(\kappa_A - \frac{2\mu_A}{N}\right) I_2 \otimes I_2 & \text{in } B_{\rho^+} \setminus B_{\rho^-} \\ A^* & \text{in } \mathbb{R}^N \setminus B_{\rho^+}. \end{cases}.$$

Lemma 4.1.35 *Let ξ be a positive definite symmetric matrix satisfying (4.87). Let θ , A^* , B_{ρ^+} and B_{ρ^-} be defined as above. Define a vector-valued function u_ξ by*

$$u_\xi(x) = \begin{cases} \frac{(2\mu_A + \lambda_A)\text{tr}(\xi)}{N(2\mu_A + \lambda_A) + \theta N(\kappa_B - \kappa_A)} x & \text{in } B_{\rho^-} \\ \sum_{i=1}^N \frac{\xi_i x_i f_i(\rho)}{f_i(\rho^+)} e_i & \text{in } B_{\rho^+} \setminus B_{\rho^-} \\ \xi x & \text{in } \mathbb{R}^N \setminus B_{\rho^+}, \end{cases} \quad (4.94)$$

where $f_i(\rho)$ is given by

$$f_i(\rho) = 1 + \frac{N(\kappa_B - \kappa_A)}{2(2\mu_A + \lambda_A)} g_m(\rho^-) \int_{\rho^-}^{\rho} \frac{dr}{(r + m_i)g_m(r)}.$$

Then, $u_\xi(x)$ satisfies

$$-\operatorname{div}(C(x)e(u_\xi)(x)) = 0 \text{ in } \mathbb{R}^N, \quad (4.95)$$

and

$$\frac{1}{|B_{\rho^+}|} \int_{B_{\rho^+}} C(x)e(u_\xi) : e(u_\xi) dx = \sum_{i=1}^N \lambda_i \xi_i^2 = A^* \xi : \xi. \quad (4.96)$$

Remark 4.1.36 Lemma 4.1.35 is due to Grabosky [122]. His original proof is different, being based on the compensated compactness method (also called the translation method). Although his proof is shorter, here we prefer our more tedious proof, based on an explicit computation, since it does not require any knowledge of the compensated compactness method.

Proof. We start from the function u_ξ defined in the conductivity setting by Lemma 3.2.43. For any elasticity tensors A and B , with wellordered moduli $0 < \kappa_A < \kappa_B$ and $0 < \mu_A < \mu_B$, we introduce isotropic conductivities

$$\begin{cases} \alpha = 2\mu_A + \lambda_A \\ \beta = 2\mu_A + \lambda_A + N(\kappa_B - \kappa_A), \end{cases}$$

which satisfies $0 < \alpha < \beta$. Replacing α, β by these values in definition (3.134) of u_ξ , taking into account that $\nabla u_\xi = e(u_\xi)$ and that $\text{tr}(\nabla u_\xi)$ is constant in $B_{\rho^+} \setminus B_{\rho^-}$, allows one to show that u_ξ is actually a solution of (4.95). A similar tedious computation leads to (4.96). \square

Remark 4.1.37 Theorem 4.1.33 shows that $g(\theta, \xi)$ coincides with the average elastic energy of the confocal ellipsoids construction, at least for some range of ξ . This proves that this microstructure is optimal in the Hashin-Shtrikman bound (2.121) for such ξ . In particular, the concentric spheres assemblage is optimal if $\xi = I_2$, i.e., for the bound on the effective bulk modulus.

Remark 4.1.38 Note that this type of classical optimal design would be very difficult to compute numerically. Indeed, its boundary is very complex since it involves an infinite number of connected components on various length scales. Therefore, even in this case, the relaxed formulation is more practical from a numerical standpoint. Indeed, it is a desperate task to try to compute numerically an optimal design built from infinitely many small inclusions with very different sizes. The mesh size would act as a threshold for the size of the smallest ellipsoids, thus preventing optimality in a discrete computation.

In the case we want to maximize (4.79), i.e., we seek the strongest design instead of the weakest, a dual variational formulation is required in order to exchange the two minimizations. We also slightly change the boundary condition, which is now a constant traction $\xi \in \mathcal{M}_N^s$. In other words the

state equation is

$$\begin{cases} \sigma_\chi = A_\chi e(u_\chi) & \text{in } \Omega \\ -\operatorname{div} \sigma_\chi = 0 & \text{in } \Omega \\ \sigma_\chi n = \xi n & \text{on } \partial\Omega. \end{cases}$$

The objective function becomes

$$J(\chi) = \int_{\partial\Omega} \sigma_\chi n \cdot u_\chi \, ds + \ell \int_{\Omega} \chi \, dx, \quad (4.97)$$

which is the usual compliance of the design, equivalently defined by

$$J(\chi) = \min_{\tau \in \Sigma_\xi} \int_{\Omega} A_\chi^{-1} \tau : \tau \, dx + \ell \int_{\Omega} \chi \, dx, \quad (4.98)$$

where Σ_ξ denotes the space of statically admissible stress fields

$$\Sigma_\xi = \left\{ \tau \in L^2(\Omega; \mathcal{M}_N^s) \mid \operatorname{div} \tau = 0 \text{ in } \Omega, \tau n = \xi n \text{ on } \partial\Omega \right\}.$$

The minimization

$$\inf_{\chi \in L^\infty(\Omega; \{0,1\})} J(\chi) \quad (4.99)$$

is an optimal design problem where the strongest structure in Ω is sought under the uniform surface loading ξ . Switching the two minimizations with respect to χ and τ yields an equivalent formulation, which is

$$\inf_{\tau \in \Sigma_\xi} \int_{\Omega} F(\tau) \, dx, \quad (4.100)$$

with

$$F(\tau) = \min \left(A^{-1} \tau : \tau + \ell, B^{-1} \tau : \tau \right). \quad (4.101)$$

According to Theorem 4.1.15, the integrand F is not quasiconvex and the functional (4.100) must be relaxed by quasiconvexification. The relaxed functional is

$$\min_{\tau \in \Sigma_\xi} \int_{\Omega} QF(\tau) \, dx, \quad (4.102)$$

where QF is the quasiconvex envelope of F defined by (4.148), which is also equal to

$$QF(\tau) = \min_{0 \leq \theta \leq 1} (g(\theta, \tau) + \ell\theta), \quad (4.103)$$

where $g(\theta, \tau)$ is the lower Hashin-Shtrikman dual bound defined by (2.142). Since the surface loading is constant, the constant stress $\sigma = \xi$ is actually a

minimizer of the relaxed functional (4.102), and the value of the minimum is

$$|\Omega| QF(\xi) = \inf_{\tau \in \Sigma_\xi} \int_{\Omega} F(\tau) dx. \quad (4.104)$$

It is easy to generalize Theorem 4.1.33 to the present context.

Theorem 4.1.39 *Let $\xi_1 \leq \dots \leq \xi_N$ be the eigenvalues of the symmetric matrix ξ . If ξ or $-\xi$ is a positive definite matrix such that, for any $1 \leq i \leq N$,*

$$\frac{N\xi_i}{\text{tr}(\xi)} < \frac{\kappa_B + \frac{2(N-1)}{N}\mu_B}{\kappa_A + \frac{2(N-1)}{N}\mu_B} \quad (4.105)$$

and

$$\frac{\kappa_B - \kappa_A}{\kappa_B^2} \frac{\kappa_B + 2\frac{N-1}{N}\mu_B}{\kappa_A + 2\frac{N-1}{N}\mu_B} < -\frac{N^2\ell}{(\text{tr}(\xi))^2} < \frac{\kappa_B - \kappa_A}{\kappa_A^2} \frac{\kappa_A + 2\frac{N-1}{N}\mu_B}{\kappa_B + 2\frac{N-1}{N}\mu_B}, \quad (4.106)$$

then, there exists infinitely many classical global minimizers of (4.99) in $L^\infty(\Omega; \{0, 1\})$ which are confocal ellipsoid assemblages of the two phases.

Remark 4.1.40 Condition (4.105) is satisfied for $\xi = I_2$ and, more generally, for matrices ξ close to I_2 or $-I_2$. Condition (4.106) is actually equivalent to the fact that the optimal volume fraction θ^* in (4.103) satisfies $0 < \theta^* < 1$. This condition is also equivalent to $QF(\xi) < F(\xi)$. In the case of an hydrostatic stress $\xi = I_2$, the lower Hashin-Shtrikman dual bound $g(\theta, \xi)$ reduces to the upper bound on the effective bulk modulus (see Theorem 2.3.13)

$$g(\theta, I_2) = \frac{N^2}{\kappa^*} = N^2 \left(\kappa_B - \theta \frac{(\kappa_B - \kappa_A)(2\mu_B + \lambda_B)}{2\mu_B + \lambda_B + (1-\theta)(\kappa_B - \kappa_A)} \right)^{-1},$$

and the confocal ellipsoid construction reduces to the well known concentric sphere assemblage.

The proof of Theorem 4.1.39 is completely similar to that of Theorem 4.1.33. The only difference is that the weak phase A plays now the role of inclusions, and the strong phase B that of the matrix, in the optimal confocal ellipsoid assemblage. The fact that phase A appears as isolated inclusions in a matrix of B allows one to pass to the degenerate limit $A \rightarrow 0$ (see [18] for details on the justification of this procedure). In such a limit, Theorem 4.1.39 also furnishes counterexamples to the uniqueness of optimal shapes as studied in Section 4.2.

4.1.6 Eigenfrequency Optimization

This subsection is devoted to a problem that is not covered by the general framework introduced previously, and yet can be solved completely by the homogenization method. It is a problem of eigenfrequency maximization for two-phase designs. It was studied in [13], and in [87] for the conductivity setting. The physical motivation is to reduce noise produced by vibrations or to avoid possible resonance with external low frequencies. The converse problem of eigenfrequency minimization, although physically less interesting, can be treated in the same way (it is even easier).

As usual, a bounded open set Ω is filled by two isotropic elastic materials. Let A and B denote the materials' Hooke's laws, which are assumed to be ordered in the sense of quadratic forms, i.e., A is the most compliant and B the most rigid (see assumption (4.1)). Let $\chi(x)$ denote the characteristic function of the most compliant material A . The heterogeneous Hooke's law in Ω is therefore

$$A_\chi = \chi(x)A + (1 - \chi(x))B.$$

Both materials also have a positive density $\rho_A, \rho_B > 0$ (their ordering is irrelevant). The heterogeneous density in Ω is

$$\rho_\chi = \chi(x)\rho_A + (1 - \chi(x))\rho_B.$$

The boundary $\partial\Omega$ is divided in two disjoint parts Γ_D and Γ_N supporting, respectively, Dirichlet boundary condition (zero displacement) and Neumann boundary condition (zero traction). We assume that the surface measure of Γ_D is nonzero.

The vibration frequencies ω of the heterogeneous domain Ω , filled by A and B , are the square roots of the eigenvalues of the following problem

$$\begin{cases} -\operatorname{div} A_\chi e(u) = \omega^2 \rho_\chi u & \text{in } \Omega \\ A_\chi e(u)n = 0 & \text{on } \Gamma_N \\ u = 0 & \text{on } \Gamma_D. \end{cases} \quad (4.107)$$

where $u \in H^1(\Omega)^N$ is the displacement vector, and $e(u) = 1/2 (\nabla u + (\nabla u)^t)$ is the strain tensor. By Lemma 1.3.15, problem (4.107) admits a countably infinite family of positive eigenvalues

$$0 < \omega_1^2(\chi) \leq \omega_2^2(\chi) \leq \dots \text{ with } \lim_{k \rightarrow +\infty} \omega_k^2(\chi) = +\infty,$$

characterized by the min-max principle

$$\omega_k^2(\chi) = \min_{\substack{u_1, \dots, u_k \in \mathcal{H}, \\ \dim[u_1, \dots, u_k] = k}} \max_{u \in \text{span}[u_1, \dots, u_k]} \frac{\int_{\Omega} A_{\chi} e(u) : e(u) dx}{\int_{\Omega} \rho_{\chi} |u|^2 dx}, \quad (4.108)$$

where \mathcal{H} is the subspace of $H^1(\Omega)^N$ made of functions vanishing on Γ_D , namely

$$\mathcal{H} = \left\{ u \in H^1(\Omega)^N \text{ such that } u = 0 \text{ on } \Gamma_D \right\}. \quad (4.109)$$

An important problem in structural design is to find out the arrangement of A and B in Ω that would best maximize the first eigenvalue, or a linear combination of the first ones. However, without further restriction on the proportions of A and B , this problem is often trivial. For example, if $\rho_A = \rho_B$, then the optimal solution is to fill Ω with the most rigid material B only. Therefore, as usual, we add a constraint on the volume of A . Introducing a Lagrange multiplier $\ell \in \mathbb{R}$, our objective functional is

$$\sup_{\chi \in L^{\infty}(\Omega; \{0,1\})} J(\chi) = \left\{ \omega_1^2(\chi) + \ell \int_{\Omega} \chi(x) dx \right\}, \quad (4.110)$$

or more generally, denoting nonnegative coefficients by $\alpha_k \geq 0$, $1 \leq k \leq n$,

$$\sup_{\chi \in L^{\infty}(\Omega; \{0,1\})} J(\chi) = \left\{ \sum_{k=1}^n \alpha_k \omega_k^2(\chi) + \ell \int_{\Omega} \chi(x) dx \right\}. \quad (4.111)$$

Remark 4.1.41 *More complicated problems fit into our framework. In particular, all our results hold if part of Ω is not subject to any optimization (such a situation is usually called a reinforcement problem). It is even possible that Ω contains a third material, which is fixed and not subject to optimization. We could also consider more complex objective functions involving special nonlinear expressions of the eigenfrequencies, or we could mix an eigenfrequency optimization with a usual compliance optimization problem.*

To simplify the exposition, we focus on problem (4.110), but our approach works equally well for (4.111). As for the model problem in Subsection 4.1.1, these eigenfrequencies optimization problems are known to be generically ill-posed, i.e., they usually admit no optimal solutions. The remedy is still to enlarge the class of admissible designs by allowing for fine mixtures of the two materials and to derive a relaxed formulation by the homogenization

method. It turns out that, as for compliance optimization, the knowledge of the full set of all composite materials is not necessary, but sequential laminates are optimal microstructures for this problem too.

To define the relaxed formulation, we first recall definition (4.26) of the space of generalized or composite designs

$$\mathcal{CD} = \left\{ (\theta, A^*) \in L^\infty \left(\Omega; [0, 1] \times \mathcal{M}_N^4 \right) \mid A^*(x) \in G_{\theta(x)} \text{ a.e. in } \Omega \right\},$$

where, for each constant density $0 \leq \theta \leq 1$, G_θ is the set of all homogenized Hooke's laws obtained by mixing the phases A and B in proportions θ and $(1 - \theta)$ (see Theorem 2.1.2). Then, we introduce the homogenized eigenvalue problem

$$\begin{cases} -\operatorname{div} A^* e(u) = \omega^2 \bar{\rho} u & \text{in } \Omega \\ A^* e(u)n = 0 & \text{on } \Gamma_N \\ u = 0 & \text{on } \Gamma_D, \end{cases} \quad (4.112)$$

where $\bar{\rho}(x)$ is the homogenized material density, namely,

$$\bar{\rho}(x) = \theta(x)\rho_A + (1 - \theta(x))\rho_B.$$

As the original problem (4.107), the homogenized problem admits a countably infinite number of positive eigenvalues

$$0 < \omega_1^2(\theta, A^*) \leq \omega_2^2(\theta, A^*) \leq \dots \text{ with } \lim_{k \rightarrow +\infty} \omega_k^2(\theta, A^*) = +\infty.$$

The relaxed objective function is

$$\max_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*) = \left\{ \omega_1^2(\theta, A^*) + \ell \int_{\Omega} \theta(x) dx \right\}. \quad (4.113)$$

Theorem 4.1.42 *The homogenized formulation (4.113) is a true relaxation of the original problem (4.110) in the sense that*

1. *there exists at least one maximizer (θ, A^*) of (4.113);*
2. *up to a subsequence, every maximizing sequence χ_n of (4.110) converges weakly * in $L^\infty(\Omega; [0, 1])$ to a density function θ , and the associated Hooke's law A_{χ_n} H -converges to a composite Hooke's law A^* such that (θ, A^*) is a maximizer of (4.113);*
3. *conversely, every maximizer (θ, A^*) of (4.113) is attained by a maximizing sequence χ_n of (4.110), namely θ is the weak * limit of χ_n in $L^\infty(\Omega; [0, 1])$ and A^* is the H -limit of A_{χ_n} .*

The outcome of Theorem 4.1.42 is that enlarging the space of admissible designs by allowing for two-phase composite materials makes the problem well-posed without changing its physical signification. Unfortunately, the relaxed formulation (4.113) is not yet very useful, since we lack a precise knowledge of the set G_θ of all effective Hooke's laws. It turns out, as for compliance optimization, that the set \mathcal{CD} of all homogenized designs can be replaced by its subset \mathcal{LD}^+ of sequentially laminated designs defined by (4.40), namely,

$$\mathcal{LD}^+ = \left\{ (\theta, A^*) \in L^\infty \left(\Omega; [0, 1] \times \mathcal{M}_N^4 \right) \mid A^*(x) \in L_{\theta(x)}^+ \text{ a.e. in } \Omega \right\},$$

where L_θ^+ is the set of sequentially laminated composites with core A and matrix B , in proportions θ and $(1 - \theta)$, respectively, introduced in Definition 2.3.5.

Actually, we shall prove a stronger result. Recall that the first eigenvalue of (4.112) is defined by

$$\min_{u \in \mathcal{H}} \frac{\int_{\Omega} A^* e(u) : e(u) dx}{\int_{\Omega} \bar{\rho} |u|^2 dx},$$

where \mathcal{H} is defined by (4.109). Therefore, (4.113) is equivalent to the min-max problem

$$\max_{(\theta, A^*) \in \mathcal{CD}} \min_{u \in \mathcal{H}} \frac{\int_{\Omega} A^* e(u) : e(u) dx}{\int_{\Omega} \bar{\rho} |u|^2 dx} + \ell \int_{\Omega} \theta(x) dx. \quad (4.114)$$

As is well known, in full generality it is not possible to interchange a minimization and a maximization. However, the proof of the next result shows that (4.114) is equal to

$$\max_{\theta \in L^\infty(\Omega; [0, 1])} \left\{ \min_{u \in \mathcal{H}} \frac{\int_{\Omega} \max_{A^* \in G_\theta} (A^* e(u) : e(u)) dx}{\int_{\Omega} \bar{\rho} |u|^2 dx} + \ell \int_{\Omega} \theta(x) dx \right\}, \quad (4.115)$$

i.e., it is legitimate to interchange the minimization with respect to u and the maximization with respect to A^* .

Theorem 4.1.43 *For the objective function (4.113) we have*

$$\max_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*) = \max_{(\theta, A^*) \in \mathcal{LD}^+} J^*(\theta, A^*).$$

Remark 4.1.44 *Theorem 4.1.43 is very important since it gives meaning to the homogenization method for eigenfrequency optimization. As we already said, the homogenization theory is useless if the unknown set G_θ of all homogenized Hooke's laws cannot be replaced by an explicit subset. Fortunately, like for compliance optimization, Theorem 4.1.43 states that the optimization can be performed solely on the smaller and explicit subset L_θ^+ of sequential laminates. Theorem 4.1.43 (due to [13]) is not completely standard, since the Rayleigh quotient, which gives the first eigenvalue, is not a convex function of u . It is similar to a previous saddle point theorem, proved by Lipton in [168]. In the conductivity setting, Cox and Lipton [87] proved, as in Theorem 4.1.43, that the full set G_θ can be restricted to the set of sequential laminates. Let us emphasize that Theorems 4.1.42 and 4.1.43 hold true also for the objective function (4.111) involving several eigenvalues.*

Remark 4.1.45 *In the relaxed formulation (4.113), it is not possible to exchange the maximization with respect to θ and the minimization with respect to u since, for fixed u , the right hand side of (4.115) has no concavity properties. Even more so, for any $A^* \in L_\theta^+$, the energy $A^*e(u) : e(u)$ is a convex function of θ (see formula (2.75)), and thus the upper bound $\max_{A^* \in G_\theta} A^*e(u) : e(u)$ is also convex in θ (this was first remarked in [142]).*

We now give necessary conditions of optimality deduced from Theorem 4.1.43.

Theorem 4.1.46 *If (θ, A^*) is a maximizer of the relaxed formulation (4.113), and if u is a corresponding first eigenvector of (4.112), then A^* satisfies, almost everywhere in Ω ,*

$$A^*e(u) : e(u) = g(\theta, e(u)) = \max_{B^* \in G_\theta} (B^*e(u) : e(u)), \quad (4.116)$$

where $g(\theta, e(u))$ is the upper Hashin-Shtrikman bound defined by (2.122), and θ is a maximizer in $L^\infty(\Omega; [0, 1])$ of

$$\theta \rightarrow \frac{\int_\Omega g(\theta, e(u)) dx}{\int_\Omega \bar{\rho}(\theta) |u|^2 dx} + \ell \int_\Omega \theta dx. \quad (4.117)$$

Furthermore, there exists a sequential laminate \tilde{A}^* such that (θ, \tilde{A}^*) is also a maximizer of (4.113) with the same first eigenvector u . If the first eigenvalue is simple, then \tilde{A}^* can be chosen as a rank- N sequential laminate with lamination directions given by the eigenvectors of the strain tensor $e(u)$.

Remark 4.1.47 The last statement of Theorem 4.1.46 holds true only when the first eigenvalue that we maximize is simple (i.e., its multiplicity is one). In the case of a first eigenvalue of multiplicity larger than one, or in the case of a sum of several eigenvalues, like in (4.111), there still exists an optimal sequential laminate \tilde{A}^* , which may be of rank higher than N , but with no explicit formula for its lamination directions. As a consequence of Theorem 4.1.46, the maximization of a simple first eigenvalue is completely similar to compliance optimization for a single load (see Theorem 4.1.12) since one can further restrict L_θ^+ to its subspace of rank- N sequential laminates with lamination directions aligned with the principal directions of the strain tensor $e(u)$. The optimality condition (4.117) may be of little interest for computing the optimal density θ . For example, if $\rho_A = \rho_B$, then, according to Remark 4.1.45, the function defined by (4.117) is convex in θ , which is an undesired property for maximization.

Proof of Theorem 4.1.42. The proof is very similar to that for the model problem of Subsection 4.1.1 (see also Section 3.2). We repeat it in brief and point out the new arguments. The starting point is a maximizing sequence $(\chi_n)_{n \geq 1} \in L^\infty(\Omega; \{0, 1\})$ for (4.110). Since χ_n is bounded in $L^\infty(\Omega)$, one can extract a subsequence, still denoted by χ_n , such that it converges weakly * in $L^\infty(\Omega; [0, 1])$ to a limit θ . According to Theorem 1.4.2 there exists a subsequence of $A_{\chi_n} = \chi_n(x)A + (1 - \chi_n(x))B$ which H -converges to an homogenized tensor A^* as n goes to infinity. As a consequence of Theorem 1.3.16, the eigenvalues $(\omega_k^2(\chi_n))_{k \geq 1}$ (labeled in increasing order) and the corresponding normalized eigenvectors $(u_k^n)_{k \geq 1}$, with $\|u_k^n\|_{L^2(\Omega)^N} = 1$, solutions of

$$\begin{cases} -\operatorname{div} A_{\chi_n} e(u_k^n) = \omega_k^2(\chi_n) \rho_{\chi_n} u_k^n & \text{in } \Omega \\ A_{\chi_n} e(u_k^n) n = 0 & \text{on } \Gamma_N \\ u_k^n = 0 & \text{on } \Gamma_D, \end{cases} \quad (4.118)$$

satisfy, for any fixed k ,

$$\lim_{n \rightarrow +\infty} \omega_k^2(\chi_n) = \omega_k^2(\theta, A^*), \quad (4.119)$$

and the sequence of eigenvectors u_k^n converges as n goes to infinity (up to a subsequence) weakly in $H^1(\Omega)^N$ and strongly in $L^2(\Omega)^N$ to an eigenvector u_k of the homogenized problem (4.112) associated to the eigenvalue

$\omega_k^2(\theta, A^*)$. Furthermore, all the eigenvalues of (4.112) are exactly the limits $(\omega_k^2(\theta, A^*))_{k \geq 1}$ (there are no other eigenvalues of (4.112)). Of course, (θ, A^*) is an admissible generalized design, i.e., $A^*(x) \in G_{\theta(x)}$ almost everywhere in Ω , where G_θ is the set of effective Hooke's laws obtained by homogenization of a mixture of A and B in proportions θ and $(1 - \theta)$, respectively (see Theorem 2.1.2). The convergence (4.119) allows one to pass to the limit in the objective function (4.110)

$$\lim_{n \rightarrow +\infty} \left\{ \omega_1^2(\chi_n) + \ell \int_{\Omega} \chi_n(x) dx \right\} = \left\{ \omega_1^2(\theta, A^*) + \ell \int_{\Omega} \theta(x) dx \right\}. \quad (4.120)$$

Since this argument is valid for any sequence (χ_n, A_{χ_n}) converging in the sense of homogenization to (θ, A^*) (not necessarily a maximizing sequence), it gives the desired result. \square

Proof of Theorem 4.1.43. For any given $\theta(x)$ in $L^\infty(\Omega; [0, 1])$, we have to prove that

$$\max_{A^* \in G_\theta} \min_{u \in \mathcal{H}} \frac{\int_{\Omega} A^* e(u) : e(u) dx}{\int_{\Omega} \bar{\rho}|u|^2 dx} = \min_{u \in \mathcal{H}} \frac{\int_{\Omega} \max_{A^* \in G_\theta} (A^* e(u) : e(u)) dx}{\int_{\Omega} \bar{\rho}|u|^2 dx},$$

and that G_θ can be replaced by L_θ^+ on the right hand side. Recall that \mathcal{G}_θ is defined by

$$\mathcal{G}_\theta = \left\{ A(x) \in L^\infty(\Omega; \mathcal{M}_N^4) \mid A(x) \in G_{\theta(x)} \text{ a.e. in } \Omega \right\}.$$

Similarly, we define

$$\mathcal{L}_\theta^+ = \left\{ A(x) \in L^\infty(\Omega; \mathcal{M}_N^4) \mid A(x) \in L_{\theta(x)}^+ \text{ a.e. in } \Omega \right\},$$

which satisfies $\mathcal{L}_\theta^+ \subset \mathcal{G}_\theta$ since, for any $\theta_0 \in [0, 1]$, $L_{\theta_0}^+ \subset G_{\theta_0}$. Thus, we have

$$\max_{A^* \in \mathcal{G}_\theta} \min_{u \in \mathcal{H}} \frac{\int_{\Omega} A^* e(u) : e(u) dx}{\int_{\Omega} \bar{\rho}|u|^2 dx} \geq \sup_{A^* \in \mathcal{L}_\theta^+} \min_{u \in \mathcal{H}} \frac{\int_{\Omega} A^* e(u) : e(u) dx}{\int_{\Omega} \bar{\rho}|u|^2 dx}. \quad (4.121)$$

By Theorem 2.3.11, for each tensor $A_0^* \in G_{\theta_0}$ there exists another tensor $B_0^* \in L_{\theta_0}^+$ such that $A_0^* \leq B_0^*$ in the sense of quadratic forms on the space of symmetric matrices. This result easily extends to a tensor-valued function

$A^* \in \mathcal{G}_\theta$ for which there exists another tensor-valued function $B^* \in \mathcal{L}_\theta^+$ such that

$$A^*(x) \leq B^*(x) \text{ a.e. } x \in \Omega. \quad (4.122)$$

(This is obvious for piecewise constant functions A^* , and since such tensors are dense in \mathcal{G}_θ for the strong $L^p(\Omega)$ topology with $1 \leq p < +\infty$, a density argument yields the desired result once the closed character of \mathcal{G}_θ and \mathcal{L}_θ^+ is noticed.) Clearly, (4.122) implies the converse inequality of (4.121) and the supremum in \mathcal{L}_θ^+ is attained, i.e.,

$$\max_{A^* \in \mathcal{G}_\theta} \min_{u \in \mathcal{H}} \frac{\int_\Omega A^* e(u) : e(u) dx}{\int_\Omega \bar{\rho}|u|^2 dx} = \max_{A^* \in \mathcal{L}_\theta^+} \min_{u \in \mathcal{H}} \frac{\int_\Omega A^* e(u) : e(u) dx}{\int_\Omega \bar{\rho}|u|^2 dx}. \quad (4.123)$$

On the other hand, another application of Theorem 2.3.11 shows that locally the set \mathcal{G}_θ can also be replaced by the set \mathcal{L}_θ^+ of sequential laminates, i.e.,

$$\max_{A^* \in \mathcal{G}_\theta} (A^* e(u) : e(u)) = \max_{A^* \in \mathcal{L}_\theta^+} (A^* e(u) : e(u)),$$

which is nothing other than the upper Hashin-Shtrikman bound (see Proposition 2.3.20). Remarking that, in view of the definition of \mathcal{G}_θ , the constraint $A^* \in \mathcal{G}_\theta$ is local, for a given function $u \in \mathcal{H}$ we deduce

$$\begin{aligned} \max_{A^* \in \mathcal{G}_\theta} \frac{\int_\Omega A^* e(u) : e(u) dx}{\int_\Omega \bar{\rho}|u|^2 dx} &= \frac{\int_\Omega \max_{A^* \in \mathcal{G}_\theta} (A^* e(u) : e(u)) dx}{\int_\Omega \bar{\rho}|u|^2 dx} \\ &= \frac{\int_\Omega \max_{A^* \in \mathcal{L}_\theta^+} (A^* e(u) : e(u)) dx}{\int_\Omega \bar{\rho}|u|^2 dx} = \max_{A^* \in \mathcal{L}_\theta^+} \frac{\int_\Omega A^* e(u) : e(u) dx}{\int_\Omega \bar{\rho}|u|^2 dx}. \end{aligned}$$

Therefore, the min-max equality (4.115) is proved if we can show that

$$\max_{A^* \in \mathcal{L}_\theta^+} \min_{u \in \mathcal{H}} \frac{\int_\Omega A^* e(u) : e(u) dx}{\int_\Omega \bar{\rho}|u|^2 dx} = \min_{u \in \mathcal{H}} \max_{A^* \in \mathcal{L}_\theta^+} \frac{\int_\Omega A^* e(u) : e(u) dx}{\int_\Omega \bar{\rho}|u|^2 dx}, \quad (4.124)$$

where \mathcal{H} is defined by (4.109).

The proof of equality (4.124) is based on Lemma 2.3.6, which furnishes the following characterization of L_θ^+ . Denoting by S^{N-1} the unit sphere in \mathbb{R}^N , L_θ^+ is equivalently defined as the set of all fourth order tensors $A^*(\nu)$ given by

$$\theta(A^*(\nu) - B)^{-1} = (A - B)^{-1} + (1 - \theta) \int_{S^{N-1}} f_B(e) d\nu(e), \quad (4.125)$$

where $\nu(e)$ is any probability measure on S^{N-1} (see (2.76)). We denote by $\mathcal{P}(S^{N-1})$ the convex set of such probability measures on S^{N-1} . Before going on, we simplify a little the notation. Let us define \mathcal{B} by

$$\mathcal{B} = \left\{ u \in \mathcal{H} \mid \int_{\Omega} \bar{\rho}|u|^2 dx = 1 \right\},$$

which, by the Rellich theorem (see Lemma 1.2.6), is a closed set for the weak topology in \mathcal{H} . Let us also define a function $g(u, \nu)$ from $\mathcal{B} \times L^\infty(\Omega; \mathcal{P}(S^{N-1}))$ into \mathbb{R}^+ by

$$g(u, \nu) = \int_{\Omega} A^*(\nu)e(u) : e(u) dx,$$

where $A^*(\nu)$ is given by (4.125). With the preceding notation, (4.124) is equivalent to

$$\max_{\nu \in L^\infty(\Omega; \mathcal{P}(S^{N-1}))} \min_{u \in \mathcal{B}} g(u, \nu) = \min_{u \in \mathcal{B}} \max_{\nu \in L^\infty(\Omega; \mathcal{P}(S^{N-1}))} g(u, \nu). \quad (4.126)$$

As was first proved in [168], for a given $u \in \mathcal{H}$, the function $\nu \rightarrow g(u, \nu)$ is concave in $L^\infty(\Omega; \mathcal{P}(S^{N-1}))$ since formula (4.125) depends linearly on ν and $M \rightarrow M^{-1}e(u) : e(u)$ is convex on \mathcal{M}_N^4 . Note that $u \rightarrow g(u, \nu)$ is also convex in \mathcal{H} , but unfortunately u must belong to the nonconvex set \mathcal{B} . Note that, for fixed ν , there always exists a minimizer (possibly nonunique) $u \in \mathcal{B}$ of $g(u, \nu)$: It is just a first eigenvector of the Hooke's law $A^*(\nu)$ corresponding to the first eigenvalue $\min_{u \in \mathcal{B}} g(u, \nu)$ (which may have a multiplicity larger than one). The function $\nu \rightarrow \min_{u \in \mathcal{B}} g(u, \nu)$ is also concave (as the minimum of concave functions) on the convex set $L^\infty(\Omega; \mathcal{P}(S^{N-1}))$, and it admits at least one maximizer ν^* because the maxima in (4.123) are attained. Of course, for this measure ν^* , there exists a nonunique minimizer u^* in \mathcal{B} of $g(u, \nu^*)$. We shall prove that there exists a choice of minimizer $u^* \in \mathcal{B}$ of $g(u, \nu^*)$ such that (u^*, ν^*) is a saddle point of $g(u, \nu)$, i.e., for any $u \in \mathcal{B}$ and any $\nu \in L^\infty(\Omega; \mathcal{P}(S^{N-1}))$,

$$g(u^*, \nu) \leq g(u^*, \nu^*) \leq g(u, \nu^*). \quad (4.127)$$

Then, it is a classical result in the calculus of variations to show that (4.127) implies the desired result (4.126) (see, e.g., Chapter VI in [102]). The second inequality of (4.127) is obvious since it is just the definition of u^* as a minimizer of $g(u, \nu^*)$. To prove the first inequality of (4.127), we introduce, for any measure $\nu \in L^\infty(\Omega; \mathcal{P}(S^{N-1}))$ and for any $t \in [0, 1]$, a measure $\nu(t) = t\nu + (1-t)\nu^*$, which, by convexity, also belongs to $L^\infty(\Omega; \mathcal{P}(S^{N-1}))$. Let $u(t)$ denote a minimizer in \mathcal{B} of $g(u, \nu(t))$. Since ν^* is a maximizer of $\min_{u \in \mathcal{B}} g(u, \nu)$, we have

$$g(u^*, \nu^*) \geq g(u(t), \nu(t)), \quad (4.128)$$

which implies that, as t goes to zero, the sequence $u(t)$ is bounded in $H^1(\Omega)^N$. Therefore, there exists a limit \tilde{u} such that, up to a subsequence $u(t)$ converges to \tilde{u} weakly in $H^1(\Omega)^N$ and strongly in $L^2(\Omega)^N$ by the Rellich theorem (see Lemma 1.2.6). Thus, \tilde{u} belongs to \mathcal{B} too. Since $\nu(t)$ converges strongly to ν^* in $L^\infty(\Omega; P)$, as t goes to zero, the convexity of $g(u, \nu)$ with respect to u in $H^1(\Omega)^N$ implies

$$\lim_{t \rightarrow 0} g(u(t), \nu(t)) \geq g(\tilde{u}, \nu^*),$$

which, in view of (4.128), proves that \tilde{u} is also a minimizer of $g(u, \nu^*)$ in \mathcal{B} . Hence, from now on our choice of minimizer u^* is $u^* = \tilde{u}$. By concavity of $g(u, \nu)$ with respect to ν , we have

$$\begin{aligned} g(u(t), \nu(t)) &\geq tg(u(t), \nu) + (1-t)g(u(t), \nu^*) \\ &\geq tg(u(t), \nu) + (1-t)g(u^*, \nu^*). \end{aligned} \quad (4.129)$$

Combining (4.128) and (4.129), for $t > 0$ we deduce that

$$g(u^*, \nu^*) \geq g(u(t), \nu).$$

Then, letting t go to zero, by convexity of $g(u, \nu)$ with respect to u in $H^1(\Omega)^N$, we obtain

$$g(u^*, \nu^*) \geq g(u^*, \nu)$$

which completes the proof of the saddle point result (4.127). \square

Remark 4.1.48 *The key ingredients in the proof of the saddle point inequality (4.127) are the convexity of $u \rightarrow g(u, \nu)$ in $H^1(\Omega)^N$ (although u runs into the nonconvex set \mathcal{B}), and the closeness of \mathcal{B} for the weak topology of $H^1(\Omega)^N$. These two arguments are still valid for more general objective functions involving a sum of eigenvalues. Therefore, Theorem 4.1.43 holds true also in this latter case.*

Proof of Theorem 4.1.46. On the right hand side of (4.115) we recognize the upper Hashin-Shtrikman bound

$$g(\theta, e(u)) = \max_{B^* \in G_\theta} B^* e(u) : e(u) \quad (4.130)$$

established in Proposition 2.3.20, and this yields (4.116). Eliminating therefore the variable A^* , and since the derivative of u with respect to θ cancels out because u is a minimizer, we get (4.117), where u , a first eigenvector of (4.112), is considered independent of θ . Note that g is a C^1 function of the two variables θ and $e(u)$ (see Remark 2.3.27). By Proposition 2.3.22 the bound (4.130) is achieved by a rank- N sequential laminate with lamination directions given by the eigenvectors of the strain tensor $e(u)$. Therefore, if the first eigenvalue is simple, arguing as in the proof of Theorem 4.1.12 (in the setting of single load compliance minimization), one can replace any optimal A^* by such a laminate \tilde{A}^* without changing θ and u . \square

Remark 4.1.49 *Theorems 4.1.43 and 4.1.46 give a firm theoretical basis for a numerical algorithm based on alternate directions optimization (similar to that proposed for compliance problems). In Subsection 5.2.7 of Chapter 5 numerical experiments demonstrate the efficiency of such an algorithm, and several technical issues are discussed at length. Numerical algorithms inspired by the homogenization method have been developed for frequency optimization problems by several authors [13], [43], [98], [155], [180], [210] (see also Chapter 8 in [42]). Since the main motivation of the homogenization method is shape optimization, it would be interesting to generalize the results of this subsection to the case where the most compliant material A is void (all its material parameters are zero). Unfortunately, we are unable to pass to the limit in the previous proofs when A goes to zero.*

4.2 Shape Optimization

This section is devoted to the generalization of the previous one in the event that the weak material degenerates and is replaced by holes. We propose a relaxed formulation, which is formally obtained by letting the weak material properties go to zero in the two-phase relaxation. The main difficulty is to justify this limiting process in the relaxation. Indeed, homogenization and singular limits do not easily commute, and we have to restrict ourselves to compliance optimization problems for proving a complete relaxation theorem.

4.2.1 Compliance Shape Optimization

Since the weak material A is now degenerate, i.e., it is replaced by void or holes, it cannot sustain body forces. Therefore, we slightly change the model problem by replacing body forces with surface loadings on the boundary. The bounded domain Ω is assumed to have a smooth boundary $\partial\Omega$ which is subjected to a “smooth enough” surface loading f (e.g., $f \in L^2(\partial\Omega)^N$, although $f \in H^{-1/2}(\partial\Omega)^N$ would be enough). The loading is assumed to let Ω under equilibrium, i.e., for any rigid body displacement $v(x) = Mx + b$ with $M = -M^t \in \mathcal{M}_N$ and $b \in \mathbb{R}^N$,

$$\int_{\partial\Omega} f \cdot v \, ds = 0.$$

Part of the domain is occupied by the isotropic, linearly elastic material B , while the remaining part of Ω is void, i.e., $A = 0$. Let χ denote the characteristic function of that part of Ω occupied by holes, and ω denote the complementary part of Ω occupied by the elastic material B . Whenever ω is a smooth enough open subdomain of Ω such that $\partial\omega$ contains the subset of $\partial\Omega$ where f is nonzero, the elasticity problem in ω is well-posed, i.e., the set of equations

$$\begin{cases} \sigma = Be(u) & \text{with } e(u) = 1/2 (\nabla u + (\nabla u)^t) \\ \operatorname{div} \sigma = 0 & \text{in } \omega \\ \sigma n = f & \text{on } \partial\omega \cap \partial\Omega \\ \sigma n = 0 & \text{on } \partial\omega \setminus \partial\Omega \end{cases} \quad (4.131)$$

has a unique solution $u \in H^1(\omega)^N$ (up to a rigid displacement field). Here, u is the displacement vector and σ is the associated stress tensor uniquely defined in $L^2(\omega; \mathcal{M}_N^s)$. The compliance of the structure ω is defined as

$$c(\omega) = \int_{\partial\omega \cap \partial\Omega} f \cdot u \, ds,$$

where u is the solution of (4.131). By the principle of minimum complementary energy, the compliance satisfies

$$c(\omega) = \int_{\omega} B^{-1} \sigma : \sigma \, dx = \min_{\tau \in \Sigma(\omega)} \int_{\Omega} B^{-1} \tau : \tau \, dx \quad (4.132)$$

where the set $\Sigma(\omega)$ is defined by

$$\Sigma(\omega) = \left\{ \tau \in L^2(\omega; \mathcal{M}_N^s) \mid \begin{array}{ll} \operatorname{div} \tau = 0 & \text{in } \omega \\ \tau n = f & \text{on } \partial\omega \cap \partial\Omega \\ \tau n = 0 & \text{on } \partial\omega \setminus \partial\Omega \end{array} \right\}. \quad (4.133)$$

The goal of shape optimization is to devise the least compliant structure compatible with the loads for a given weight of the structure (or equivalently to minimize its weight with a given constraint on its compliance). Introducing a Lagrange multiplier $\ell \in \mathbb{R}$ for the constraint, the objective function is

$$J(\omega) = c(\omega) + \ell |\Omega \setminus \omega| \quad (4.134)$$

where $|\Omega \setminus \omega|$ is the volume occupied by the holes. Since χ is the characteristic function of the holes, the functional J defined above is similar to that defined by (4.4), where the holes are filled with a weak material A .

The space of admissible designs for (4.134) has so far implicitly been defined as the class of all smooth open subsets of Ω , the boundary of which contains the part of $\partial\Omega$ where the surface load is applied. It turns out that this class is too restrictive and not easy to work with (in particular the constraint on ω to be open is not obvious). We therefore introduce a weaker version of (4.134). Let $\chi(x)$ be any function in $L^\infty(\Omega; \{0, 1\})$ that is the characteristic function of the holes. We still denote by ω the subset of Ω where $\chi(x) = 0$, i.e., the subdomain occupied by the material. However, ω is no longer an open set (it is just a measurable Borel set), and its boundary does not necessarily contain the surface where the loads are applied. Nevertheless, we can still define the compliance of ω . We remark that, for a smooth ω , any tensor field τ in $\Sigma(\omega)$ can be extended by zero outside of ω to give an element of $L^2(\Omega; \mathcal{M}_N^s)$, still denoted by τ , such that

$$\operatorname{div} \tau = 0 \text{ in } \Omega.$$

This equality holds in the distributional sense because of the boundary condition $\tau n = 0$ satisfied on $\partial\omega \setminus \partial\Omega$. Therefore, for any $\chi \in L^\infty(\Omega; \{0, 1\})$, we introduce a space of statically admissible stress fields, constrained to vanish on the holes, defined by

$$\Sigma(\chi) = \left\{ \tau \in L^2(\Omega; \mathcal{M}_N^s) \mid \begin{array}{l} \operatorname{div} \tau = 0 \text{ in } \Omega \\ \tau n = f \text{ on } \partial\Omega \\ \tau(x) = 0 \text{ a.e. where } \chi(x) = 1 \end{array} \right\}. \quad (4.135)$$

For a general measurable set ω (not necessarily open), system (4.131) does not admit necessarily a solution. We may, however, define a compliance

$$c(\chi) = \inf_{\tau \in \Sigma(\chi)} \int_{\Omega} B^{-1} \tau : \tau dx. \quad (4.136)$$

The constraint that the boundary $\partial\omega$ must include that part of $\partial\Omega$ where the loads are applied is automatically taken into account in the definition of $\Sigma(\chi)$. Indeed, if this is not the case, the set $\Sigma(\chi)$ is empty (there is no statically admissible stress) and, as is usual in the calculus of variations, the infimum in (4.136) is $+\infty$.

To analyze further the problem, and in particular to prove the existence of a stress minimizing (4.136), we rewrite the compliance in a slightly different way. Instead of (4.135), we introduce an unconstrained space of statically admissible stress fields

$$\Sigma(\Omega) = \left\{ \tau \in L^2(\Omega; \mathcal{M}_N^s) \mid \operatorname{div} \tau = 0 \text{ in } \Omega, \quad \tau n = f \text{ on } \partial\Omega \right\}. \quad (4.137)$$

The constraint that the stress vanishes inside the holes is now put into the integrand, and the compliance is defined by

$$c(\chi) = \min_{\tau \in \Sigma(\Omega)} \int_{\Omega} [(1 - \chi(x))B]^{-1} \tau : \tau dx. \quad (4.138)$$

Shape optimization is thus the following minimization problem:

$$\inf_{\chi \in L^\infty(\Omega; \{0,1\})} \left\{ J(\chi) = c(\chi) + \ell \int_{\Omega} \chi(x) dx \right\}, \quad (4.139)$$

where the space of admissible designs is just $L^\infty(\Omega; \{0,1\})$ without any further constraints.

Remark 4.2.1 *The form of the integrand in (4.138) forces the stress to vanish at points where $\chi(x) = 1$ if the infimum is not $+\infty$ (i.e., if $\Sigma(\chi)$ is not empty). Note that it is a normal integrand in the sense of Definition 1.1 in Chapter VIII of [102]. Thus, by virtue of Proposition 1.4 in Chapter VIII of [102], there exists a minimizer on the right hand side of (4.138). Furthermore, in the case where $c(\chi) < +\infty$, this minimizer $\sigma \in \Sigma(\Omega)$ is actually unique by the strict convexity of $B^{-1}\tau : \tau$, and we shall call it the stress associated to the design χ . It is not clear, however, if it admits a displacement field.*

Problem (4.139) can be further analyzed by exchanging the minimization with respect to χ and τ (a trick already applied for compliance optimization in Subsection 4.1.4). We remark first that, if $\ell \geq 0$, the problem admits the obvious minimizer $\chi \equiv 0$, which minimizes each term defining $J(\chi)$. We

shall, therefore, assume from now on that $\ell < 0$. Then, (4.139) is equivalent to

$$\inf_{\tau \in \Sigma(\Omega)} \int_{\Omega} \min_{\chi=0,1} \left([(1-\chi)B]^{-1} \tau : \tau + \ell \chi \right) dx.$$

For a given stress tensor $\tau \in \Sigma(\Omega)$, and since $\ell < 0$, the minimum is attained by $\chi = 0$ if $\tau \neq 0$ and by $\chi = 1$ if $\tau = 0$ (there is actually no other choice). Thus, (4.139) is equivalent to

$$\inf_{\tau \in \Sigma(\Omega)} \int_{\Omega} F(\tau) dx, \quad (4.140)$$

where

$$F(\tau) = \begin{cases} \ell & \text{if } \tau = 0 \\ B^{-1}\tau : \tau & \text{if } \tau \neq 0. \end{cases} \quad (4.141)$$

Problem (4.140) is called the *stress formulation* of the shape optimization problem (4.139). They are equivalent in the sense that, if it exists, a minimizer σ of (4.140) yields a minimizer χ of (4.139), defined as the characteristic function of the subset $\{x \in \Omega \mid \sigma(x) = 0\}$, and conversely, if χ is a minimizer of (4.139), then the minimizer σ of the compliance $c(\chi)$ is also a minimizer of (4.140).

Remark 4.2.2 *For the sake of simplicity we consider only the case where surface loads are applied on the whole boundary $\partial\Omega$. A straightforward modification of the model would, however, permit the consideration of the clamping of part of the boundary $\partial\Omega$, namely the enforcement of a Dirichlet boundary condition $u = 0$. The reader is referred to the numerical examples presented in Chapter 5 that include different types of boundary conditions.*

The above optimization problem is usually referred to as a “single load” problem. This means that the elastic structure is optimized for a single configuration of loading forces and may well be totally inadequate for other loads. In practice this is an undesirable feature and it is quite often more realistic to investigate the so-called “multiple loads” problem which amounts to an optimization of the structure for several configurations. Specifically, various surface loadings f_1, \dots, f_n are given and we consider the minimization problem

$$\inf_{\chi \in L^\infty(\bar{\Omega}; \{0,1\})} \left\{ \sum_{i=1}^n c_i(\chi) + \ell \int_{\Omega} \chi(x) dx \right\} \quad (4.142)$$

where $c_i(\chi)$ is the compliance defined by (4.138) for the boundary condition f_i . As in the single load case, (4.142) admits a stress formulation

$$\inf_{\{\tau_i\} \in \Sigma^n(\Omega)} \int_{\Omega} F^n(\{\tau_i\}) dx, \quad (4.143)$$

where

$$\Sigma^n(\Omega) = \left\{ (\tau_i)_{1 \leq i \leq n} \in L^2(\Omega; \mathcal{M}_N^s)^n \mid \begin{array}{ll} \operatorname{div} \tau_i = 0 & \text{in } \Omega \\ \tau_i n = f_i & \text{on } \partial\Omega \end{array} \right\}, \quad (4.144)$$

and

$$F^n(\{\tau_i\}) = \begin{cases} \ell & \text{if all } \tau_i = 0 \\ \sum_{i=1}^n B^{-1} \tau_i : \tau_i & \text{if at least one } \tau_i \neq 0. \end{cases} \quad (4.145)$$

4.2.2 The Relaxation Process

This subsection derives the relaxed formulation for the shape optimization problem of least compliance and weight introduced in (4.139). The output is very similar to that obtained in Section 4.1.4 except that the weak material is now degenerate, i.e., $A = 0$. However, the proof of the relaxation theorem is delicate since homogenization theory does not work in full generality with degenerate materials or holes. We shall rely heavily on the *stress formulation* (4.140) of our shape optimization problem. A complete relaxation theorem will be given for the stress formulation, and only a partial result will be deduced for the original formulation in terms of characteristic functions.

Let us first explain formally how the relaxed formulation is obtained. It is physically reasonable and, as will be seen later, mathematically sound to fill holes with a very compliant material A without disturbing the problem too much. In other words, for small values of the tensor A , F is approximated by F_A , defined by (4.43), i.e.,

$$F_A(\tau) = \min \left\{ A^{-1} \tau : \tau + \ell, B^{-1} \tau : \tau \right\}. \quad (4.146)$$

Note that $0 \leq A \leq A'$ (in the sense of quadratic forms) implies

$$F_{A'}(\tau) \leq F_A(\tau) \quad \text{for } \tau \in \mathcal{M}_N^s,$$

so, as A (the stiffness of the ersatz material) tends to zero, the sequence of functions F_A monotonically converges to F . In Subsection 4.1.4, we performed the relaxation process at fixed nonzero A of

$$\inf_{\tau \in \Sigma(\Omega)} \int_{\Omega} F_A(\tau) dx. \quad (4.147)$$

In truth, Subsection 4.1.4 deals with the case of body forces instead of surface loads, but it works equally well in this latter case. According to Theorem 4.1.15 the relaxation of (4.147) is

$$\min_{\tau \in \Sigma(\Omega)} \int_{\Omega} QF_A(\tau) dx, \quad (4.148)$$

where

$$QF_A(\tau) = \min_{0 \leq \theta \leq 1} (g_A(\tau, \theta) + \ell\theta), \quad (4.149)$$

and $g_A(\tau, \theta)$ is defined as the lower dual Hashin-Shtrikman bound (2.142)

$$g_A(\tau, \theta) = \min_{A^* \in G_\theta} (A^{*-1}\tau : \tau). \quad (4.150)$$

Note that, by its definition (2.142),

$$g_{A'}(\theta, \tau) \leq g_A(\theta, \tau)$$

if $0 \leq A \leq A'$ (in the sense of quadratic forms). Allowing the weak material A to degenerate to holes, i.e., as A decreases to zero, we define the monotone limit

$$g(\tau, \theta) = \lim_{A \rightarrow 0} g_A(\tau, \theta), \quad (4.151)$$

which is the lower Hashin-Shtrikman bound on complementary energy for a perforated composite material (see Corollary 2.3.33). Similarly, we define the monotone limit

$$QF(\tau) = \lim_{A \rightarrow 0} QF_A(\tau) = \min_{0 \leq \theta \leq 1} (g(\tau, \theta) + \ell\theta). \quad (4.152)$$

We shall prove that the function QF is indeed the quasiconvexification of F . The relaxation of the stress formulation (4.140) is therefore proposed to be

$$\min_{\tau \in \Sigma(\Omega)} \int_{\Omega} QF(\tau) dx, \quad (4.153)$$

where $\Sigma(\Omega)$ is defined by (4.137).

Theorem 4.2.3 *The stress formulation (4.140), (4.141) of the optimal shape problem admits (4.152), (4.153) as a relaxed formulation. In other words, QF is the quasiconvexification of F , and*

1. *there exists at least one solution in $\Sigma(\Omega)$ of the relaxed minimization problem (4.153);*

2. up to a subsequence, any minimizing sequence of (4.140) converges weakly in $L^2(\Omega; \mathcal{M}_N^s)$ to a minimizer of (4.153);
3. for any minimizer τ of (4.153) there exists a minimizing sequence of (4.140) which converges to τ weakly in $L^2(\Omega; \mathcal{M}_N^s)$, and

$$\inf_{\tau \in \Sigma(\Omega)} \int_{\Omega} F(\tau) dx = \min_{\tau \in \Sigma(\Omega)} \int_{\Omega} QF(\tau) dx. \quad (4.154)$$

The above result is concerned only with the stress formulation of shape optimization. It does not say anything on relaxed shapes and their link to minimizing sequences of classical shapes. We first introduce a relaxed formulation of the shape optimization problem (4.139). Defining a set G_θ^0 as the limit of G_θ when A goes to zero, we have

$$QF(\tau) = \min_{0 \leq \theta \leq 1} \min_{A^* \in G_\theta^0} (A^{*-1} \tau : \tau + \ell \theta).$$

Inverting backward the two minimizations with respect to τ and (θ, A^*) in (4.153), we define an homogenized compliance

$$c(\theta, A^*) = \min_{\tau \in \Sigma(\Omega)} \int_{\Omega} A^{*-1} \tau : \tau dx, \quad (4.155)$$

and we deduce the following relaxation

$$\min_{(\theta, A^*) \in \mathcal{CD}} \left\{ J^*(\theta, A^*) = c(\theta, A^*) + \ell \int_{\Omega} \theta dx \right\}, \quad (4.156)$$

where \mathcal{CD} is the space of relaxed admissible designs defined as

$$\mathcal{CD} = \left\{ (\theta, A^*) \in L^\infty(\Omega; [0, 1] \times \mathcal{M}_N^s) \mid A^*(x) \in G_{\theta(x)}^0 \text{ a.e. in } \Omega \right\}. \quad (4.157)$$

For $(\theta, A^*) \in \mathcal{CD}$, the existence of a unique stress minimizing the compliance (4.155) follows from the same arguments exposed in Remark 4.2.1.

Remark 4.2.4 Let us explain more precisely how G_θ^0 is defined. For a positive tensor A , the G -closure set G_θ is defined by (2.9). It obviously depends on A , and, when A goes to zero, we define its limit G_θ^0 in the sense of Kuratowski. In other words, G_θ^0 is the set of all cluster points of sequences in G_θ indexed by A . It is important to remark that G_θ^0 is defined through a limit process and not by homogenization of perforated microstructures. In other words, we do not know if the tensors in G_θ^0 are actually the result of the homogenization of material B perforated by holes. There are only a few classes of microstructures for which such a result is known (see, e.g., [84]).

To recover a link between the minimizing sequences of characteristic functions in the original problem (4.139) and the optimal densities in the relaxed formulation (4.156) we must recall now some properties of the lower Hashin-Shtrikman bound on complementary energy $g(\tau, \theta)$, defined as a monotone limit by (4.151). This bound has been computed explicitly in Theorems 2.3.35 and 2.3.36 (in two and three dimensions, respectively). Let us simply point out those properties that will be of use in the sequel and that are valid in any space dimension.

Lemma 4.2.5 *The function $g(\tau, \theta)$ is given by*

$$g(\tau, \theta) = B^{-1}\tau : \tau + \frac{\theta}{1-\theta}g^*(\tau), \quad (4.158)$$

where $g^*(\tau)$ is a continuous and convex function of τ only, homogeneous of degree two, and strictly positive for any $\tau \neq 0$. Therefore, g is continuous in (τ, θ) with values in $\mathbb{R}^+ \cup \{+\infty\}$ and strictly convex separately in τ and in θ . For any $\tau \in \mathcal{M}_N^s$, there exists a unique minimizer θ_τ in (4.152) given by

$$\theta_\tau = \max \left(0, 1 - \sqrt{\frac{g^*(\tau)}{-\ell}} \right). \quad (4.159)$$

Lemma 4.2.5 is a direct consequence of Corollary 2.3.33. It implies that, for any minimizer τ of the relaxed formulation (4.153), a unique density function θ_τ , which we call a *relaxed (or generalized, or composite) optimal shape* is associated through (4.159). In truth, θ_τ is the local proportion of holes, and it is rather the proportion $(1 - \theta_\tau)$ of material B that is the material density of the composite shape. Nevertheless, to simplify the exposition, we shall call θ_τ a composite shape. It remains to understand in which sense the minimizing sequences of characteristic functions are related to this (possibly nonunique) optimal density. This is the subject of the following theorem.

Theorem 4.2.6 *There exists at least one relaxed optimal shape, i.e., a density θ and a tensor A^* such that $(\theta, A^*) \in \mathcal{CD}$, that minimizes (4.156). Furthermore, for any minimizing sequence of characteristic functions $\chi_n \in L^\infty(\Omega; \{0, 1\})$ for (4.139), there exist a subsequence and a limit density θ such that this subsequence converges to θ weakly * in $L^\infty(\Omega; [0, 1])$ and θ is a relaxed optimal shape, i.e., there exists a tensor A^* such that $(\theta, A^*) \in \mathcal{CD}$ minimizes (4.156).*

Remark 4.2.7 Theorem 4.2.6 is a weak version of the desired result of relaxation. It lacks the attainability of any relaxed optimal shape θ by a minimizing sequence χ_n of characteristic functions. Furthermore, it does not say anything about an optimal tensor A^* . We do not know if it is attained as the homogenized Hooke's law of a perforated structure $\chi_n B$, and we do not even know if, for a minimizing sequence, $\chi_n B$ H -converges, in some suitable sense, to an homogenized Hooke's law.

Remark 4.2.8 Since by Theorem 4.2.3 we have proved the existence of at least one minimizer τ for the relaxed stress formulation (4.153), and by Theorem 4.2.6 that of an optimal relaxed design (θ, A^*) , the next obvious question concerns the uniqueness of such minimizers. Note first that uniqueness in τ implies uniqueness in θ , but the converse is *a priori* false (there may well be different minimum stress fields τ that yield the same optimal density θ). It turns out that there is, in general, no uniqueness of either the minimizer τ or the optimal density θ . This nonuniqueness of an optimal shape is completely similar to that of an optimal two-phase design as studied in Subsection 4.1.5. Indeed, as remarked in [18], the confocal ellipsoid construction of Subsections 3.2.6 and 4.1.5 is still valid when the core phase degenerates to zero. Therefore, Theorem 4.1.33 still provides a specific example for which there is an infinite number of minimizers τ and optimal densities θ , which are furthermore "classical" shapes, namely taking only the values zero or one.

As in Subsection 4.1.4 a simplified relaxed problem is available by replacing the unknown set G_θ^0 with its explicit subset L_θ^{0+} of perforated sequential laminates, (i.e., a matrix of B with holes in proportions $(1 - \theta)$ and θ , respectively), defined by taking the limit $A \rightarrow 0$ in Definition 2.3.5. Note that the limit set L_θ^{0+} is well defined and characterized (see Definition 2.3.34). Let us introduce the set \mathcal{CD}^+ of perforated sequentially laminated designs, defined by

$$\mathcal{CD}^+ = \left\{ (\theta, A^*) \in L^\infty(\Omega; [0, 1] \times \mathcal{M}_N^4) \mid A^*(x) \in L_{\theta(x)}^{0+} \text{ a.e. in } \Omega \right\}, \quad (4.160)$$

which is a strict subset of the set of all composite designs \mathcal{CD} , defined by (4.157).

Theorem 4.2.9 For the objective function (4.156) we have

$$\min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*) = \min_{(\theta, A^*) \in \mathcal{CD}^+} J^*(\theta, A^*).$$

If (θ, A^*) is a minimizer of J^* in \mathcal{CD} , and if σ is the stress tensor that minimizes its associated compliance (4.155), then there exists a sequential laminate \tilde{A}^* such that (θ, \tilde{A}^*) is a minimizer of J^* in \mathcal{CD}^+ , σ is again its associated stress tensor, and $A^{*-1}\sigma = \tilde{A}^{*-1}\sigma$. Furthermore, \tilde{A}^* can be chosen as a rank- N perforated sequential laminate having orthogonal lamination directions given by the eigenvectors of σ .

Remark 4.2.10 In Theorem 4.2.9 the optimal rank- N perforated sequential laminate \tilde{A}^* can be computed explicitly in terms of the optimal σ (this has been done in Theorems 2.3.35 and 2.3.36, in two and three dimensions, respectively). Recall that the optimal θ is also explicitly given in terms of σ by (4.159). This is at the root of a numerical algorithm that will be described in Chapter 5.

Proof of Theorem 4.2.3. According to Remark 4.1.16 on the notion of quasiconvexity, it is a straightforward consequence of the fact that QF defined in (4.152) is the quasiconvexification of F defined in (4.141). We prove this statement.

Since F_A monotonically increases to F as A tends to zero, we have $F_A \leq F$ and thus

$$QF_A(\tau) \leq QF(\tau), \quad \tau \in \mathcal{M}_N^s,$$

where QF_A and QF denote the quasiconvex envelope of F_A and F , respectively. But QF_A monotonically increases, as A tends to zero, to a limit that, for the moment, we denote by \tilde{F} (we want to prove that $QF = \tilde{F}$). Thus,

$$\tilde{F}(\tau) \leq QF(\tau), \quad \tau \in \mathcal{M}_N^s.$$

By definition the integrand QF is quasiconvex and it has quadratic growth at infinity. Thus, it is easily seen in the spirit of part 3 of the proof of Theorem 1.1 in Section 4.1 of [89] that QF is rank- $(N - 1)$ convex, hence continuous. Define

$$G_A(\tau) = \min(F_A(\tau), QF(\tau)).$$

It is a monotone sequence of continuous functions that converges, as A tends to zero, to a continuous function, namely QF , and Dini's theorem implies the uniform convergence of G_A to QF over any compact subset of \mathcal{M}_N^s . Furthermore,

$$QG_A(\tau) \leq QF_A(\tau),$$

because $G_A \leq F_A$. Thus,

$$\lim_{A \rightarrow 0} QG_A(\tau) \leq \tilde{F}(\tau) \leq QF(\tau).$$

We now prove that

$$QF(\tau) \leq \lim_{A \rightarrow 0} QG_A(\tau), \quad (4.161)$$

which, in view of the previous inequality, establishes that $\tilde{F} = QF$, i.e., \tilde{F} is the quasiconvexification of F . Note that, as soon as

$$B^{-1}\tau : \tau \leq A^{-1}\tau : \tau + \ell,$$

i.e., for τ large enough (uniformly with respect to A going to zero), we have

$$F_A(\tau) = B^{-1}\tau : \tau = F(\tau) \geq QF(\tau),$$

and thus

$$G_A(\tau) = QF(\tau) \text{ for large } \tau.$$

This implies that G_A differs from QF only on a compact subset of \mathcal{M}_N^s , and consequently G_A converges uniformly to QF on the whole \mathcal{M}_N^s . This property allows to pass to the limit $A \rightarrow 0$ in the definition of QG_A . Recall that QG_A is defined by (4.148), i.e.

$$QG_A(\tau) = \inf_{\substack{\eta \in L^2_\#(Y; \mathcal{M}_N^s) \\ -\operatorname{div} \eta = 0 \text{ in } Y \text{ and } \int_Y \eta(x) dx = 0}} \int_Y G_A(\tau + \eta) dx, \quad (4.162)$$

where Y is the unit cube of \mathbb{R}^N . For any τ in \mathcal{M}_N^s and any positive fourth order tensor A , there exists a statically admissible test function η_A which approaches the infimum in (4.162)

$$\begin{aligned} QG_A(\tau) &\geq \int_Y G_A(\tau + \eta_A) dx - |A| \\ &\geq \int_Y QF(\tau + \eta_A) dx - |A| - \int_Y |G_A - QF|(\tau + \eta_A) dx. \end{aligned}$$

But G_A converges uniformly to QF , thus, for any positive ϵ ,

$$\int_Y |G_A - QF|(\tau + \eta_A) dx \leq \epsilon,$$

whenever A is small enough. Eventually, for any positive ϵ and for A small enough, we have

$$QG_A(\tau) \geq \int_Y QF(\tau + \eta_A) dx - |A| - \epsilon \geq QF(\tau) - |A| - \epsilon,$$

by definition (4.148) of QF . This proves (4.161) and completes the proof of Theorem 4.2.3. \square

Proof of Theorem 4.2.6. The existence of a minimizer $(\theta, A^*) \in \mathcal{CD}$ of (4.156) follows from the existence of a minimizer σ of (4.153): Then θ is deduced from formula (4.159) and A^* is any optimal tensor in the following definition of the Hashin-Shtrikman lower bound:

$$g(\sigma, \theta) = \min_{A^* \in G_\theta^0} A^{*-1} \sigma : \sigma.$$

(There exists at least one optimal tensor by virtue of Theorems 2.3.35 and 2.3.36.)

Let us now consider a minimizing sequence $(\chi_n)_{n \geq 1}$ in $L^\infty(\Omega; \{0, 1\})$ of the original problem (4.139) which converges weakly * to some limit θ in $L^\infty(\Omega; [0, 1])$. Let us define the mixture Hooke's law

$$A_{\chi_n} = \chi_n A + (1 - \chi_n) B.$$

For any statically admissible stress $\tau \in \Sigma(\Omega)$ such that $\tau = 0$ where $\chi_n = 1$, we have

$$\begin{aligned} \int_\Omega B^{-1} \tau : \tau dx &= \int_\Omega A_{\chi_n}^{-1} \tau : \tau dx \\ &\geq \min_{\sigma \in \Sigma(\Omega)} \int_\Omega A_{\chi_n}^{-1} \sigma : \sigma dx. \end{aligned} \tag{4.163}$$

The minimum on the right hand side of (4.163) is attained by a unique stress τ_n^A . Thus, minimizing in τ and recalling that χ_n is a minimizing sequence yields

$$\inf_\chi J(\chi) \geq \limsup_{n \rightarrow +\infty} \left\{ \int_\Omega A_{\chi_n}^{-1} \tau_n^A : \tau_n^A dx + \ell \int_\Omega \chi_n dx \right\}. \tag{4.164}$$

According to Theorem 1.4.2, a subsequence of A_{χ_n} H -converges to a tensor A_A^* as n goes to infinity. Thus, the corresponding τ_n^A weakly converges in $L^2(\Omega; \mathcal{M}_N^s)$ to a limit τ^A , and by Proposition 1.2.20 on the energy convergence, for that subsequence,

$$\int_\Omega A_{\chi_n}^{-1} \tau_n^A : \tau_n^A dx \rightarrow \int_\Omega (A_A^*)^{-1} \tau^A : \tau^A dx. \tag{4.165}$$

Consequently, (4.164) and (4.165) imply that

$$\inf_{\chi} J(\chi) \geq \int_{\Omega} (A_A^*)^{-1} \tau^A : \tau^A dx + \ell \int_{\Omega} \theta dx. \quad (4.166)$$

Since $\inf_{\chi} J(\chi)$ is finite (take $\chi \equiv 0$ in Ω) and $A_A^* \leq B$, the sequence τ^A is bounded in $L^2(\Omega; \mathcal{M}_N^s)$ independently of A . A subsequence, still indexed by A , is such that, as A goes to zero, τ^A converges weakly to some τ in $L^2(\Omega; \mathcal{M}_N^s)$. Furthermore, since

$$\|(\chi_n A + (1 - \chi_n)B) - (\chi_n A' + (1 - \chi_n)B)\|_{L^\infty(\Omega)} \leq |A - A'|,$$

Proposition 1.3.44 implies that the same holds true for the H -limits as n tends to infinity:

$$\|A_A^* - A_{A'}^*\|_{L^\infty(\Omega)} \leq |A - A'|.$$

Hence A_A^* converges uniformly to a limit A^* as A goes to zero. The convex character of the mapping (see Lemma 4.1.5)

$$(A^*, \tau) \rightarrow A^{*-1} \tau : \tau$$

yields

$$\liminf_{A \rightarrow 0} \int_{\Omega} (A_A^*)^{-1} \tau^A : \tau^A dx \geq \int_{\Omega} (A^*)^{-1} \tau : \tau dx,$$

for which it is deduced, upon recalling (4.166), that

$$\inf_{\chi} J(\chi) \geq \int_{\Omega} (A^*)^{-1} \tau : \tau dx + \ell \int_{\Omega} \theta dx. \quad (4.167)$$

By Theorem 2.1.2, the homogenized Hooke's law $A_A^*(x)$ belongs to $G_{\theta(x)}$ a.e. $x \in \Omega$. According to the definition of the limit set G_θ^0 as A goes to zero (see Remark 4.2.4), the limit $A^*(x)$ of the sequence $A_A^*(x)$ belongs to $G_{\theta(x)}^0$ a.e. $x \in \Omega$.

By definition (4.150) of $g_A(\tau, \theta)$, i.e.,

$$g_A(\tau, \theta) = \min_{A^* \in G_\theta} A^{*-1} \tau : \tau,$$

its monotone limit $g(\tau, \theta)$, as A tends to zero, is given by

$$g(\tau, \theta) = \min_{A^* \in G_\theta^0} A^{*-1} \tau : \tau. \quad (4.168)$$

Since $A^*(x) \in G_{\theta(x)}^0$ a.e. in Ω , we deduce from (4.167) and (4.168) that

$$\begin{aligned} \inf_{\chi} J(\chi) &\geq \int_{\Omega} g(\tau, \theta) dx + \ell \int_{\Omega} \theta dx \\ &\geq \int_{\Omega} \min_{0 \leq \theta \leq 1} (g(\tau, \theta) + \ell \theta) dx \\ &\geq \int_{\Omega} QF(\tau) dx. \end{aligned}$$

By Theorem 4.2.3 τ is a minimizer of (4.153), and the above inequalities become equalities. Furthermore, by virtue of the strict convexity in θ of $g(\tau, \theta)$, the limit θ of the minimizing sequence χ_n is the unique density associated to τ through minimization in θ (see Lemma 4.2.5). This concludes the proof of Theorem 4.2.6. \square

Proof of Theorem 4.2.9. Let σ be a minimizer of (4.153). It yields a minimizer $(\theta, A^*) \in \mathcal{CD}$ of (4.156) by taking θ defined by formula (4.159) and A^* an optimal tensor in the lower Hashin-Shtrikman dual bound $g(\sigma, \theta)$. By virtue of Theorems 2.3.35 and 2.3.36 (see also Proposition 2.3.25), there exists a rank- N perforated sequential laminate that attains the bound $g(\sigma, \theta)$ and has orthogonal lamination directions given by the eigenvectors of σ . Thus, A^* can be chosen such that the minimizer (θ, A^*) of (4.156) actually belongs to \mathcal{LD}^+ .

If σ is the stress tensor that minimizes the homogenized compliance (4.155), then the same argument as in the proof of Theorem 4.1.12 shows that any optimal A^* in the bound g satisfies

$$A^{*-1}\sigma = \frac{\partial g}{\partial \tau}(\sigma, \theta),$$

which gives the desired equality since \tilde{A}^* is also optimal. \square

We have so far been working with an unconstrained formulation of the optimal shape design problem. We could have considered instead the constrained version,

$$\inf_{\substack{\chi \in L^\infty(\Omega; \{0,1\}) \\ \int_{\Omega} (1-\chi(x)) dx = \Theta}} c(\chi), \quad (4.169)$$

where $0 < \Theta < |\Omega|$ is the allowed volume of material B . For any value of the Lagrange multiplier ℓ , there exists a volume constraint $\Theta(\ell)$ such that the

unconstrained problem (4.139) is equivalent to the constrained one, (4.169). The converse (i.e., the existence of a value of $\ell(\Theta)$ for each volume constraint Θ) is believed to be true, but a rigorous proof is still lacking. The relaxation process for (4.169) is completely similar to that for (4.139). The relaxed formulation of (4.169) is still (4.153), or equivalently (4.156), with $\ell = 0$ and with the volume constraint

$$\Theta = \int_{\Omega} (1 - \theta(x)) dx.$$

A rigorous proof of such a relaxation holds true only for those values of Θ such that there exists a Lagrange multiplier ℓ for which a minimizer θ_ℓ of (4.156) satisfies $\int_{\Omega} (1 - \theta_\ell) dx = \Theta$. Once again, a complete relaxation proof would amount to showing the existence, for each $0 < \Theta < |\Omega|$, of a multiplier ℓ for which the relaxed problem (4.156) admits a solution θ_ℓ satisfying the desired volume constraint. We are unable to prove such a result. It would be true if we could prove that the function

$$\Theta(\ell) = \int_{\Omega} (1 - \theta_\ell) dx$$

is continuous in ℓ . In fact, $\Theta(\ell)$ may well be a multivalued function, since the optimal density θ_ℓ is not necessarily unique. We will merely establish in Lemma 4.2.11 below that it is an increasing function of ℓ and that it goes to zero as ℓ goes to minus infinity, provided that the surface loadings are smooth enough. The continuity of $\Theta(\ell)$ is at present an open question.

Lemma 4.2.11 *Define on \mathbb{R}^- the (possibly multivalued) function*

$$\Theta(\ell) = \int_{\Omega} (1 - \theta_\ell) dx,$$

where θ_ℓ is the optimal density associated to any (possibly nonunique) minimizer τ_ℓ of the relaxed problem (4.153). It is an increasing function of ℓ . Furthermore, $\Theta(\ell)$ goes to zero as ℓ goes to minus infinity, provided that the surface loading f is such that there exists at least one admissible test field $\sigma \in \Sigma(\Omega)$ that is uniformly bounded in Ω .

Proof. Assume that $\ell < \ell' \leq 0$, and take any minimizer τ_ℓ (respectively, $\tau_{\ell'}$) of (4.153) for the Lagrange multiplier ℓ (respectively, ℓ') and its associated optimal density θ_ℓ (respectively, $\theta_{\ell'}$). We start from the optimality of $(\tau_{\ell'}, \theta_{\ell'})$:

$$\int_{\Omega} g(\tau_{\ell'}, \theta_{\ell'}) dx + \ell' (|\Omega| - \Theta(\ell')) \leq \int_{\Omega} g(\tau_\ell, \theta_\ell) dx + \ell' (|\Omega| - \Theta(\ell)),$$

which is equivalent to

$$\begin{aligned} \int_{\Omega} g(\tau_{\ell'}, \theta_{\ell'}) dx + \ell(|\Omega| - \Theta(\ell')) &\leq \int_{\Omega} g(\tau_{\ell}, \theta_{\ell}) dx + \ell(|\Omega| - \Theta(\ell)) \\ &\quad + (\ell' - \ell)(\Theta(\ell') - \Theta(\ell)). \end{aligned} \tag{4.170}$$

Since the sum of the two first terms on the right hand side of (4.170) is precisely the minimum value of

$$\int_{\Omega} (g(\tau, \theta) + \ell\theta) dx,$$

the last term of (4.170) must be positive. This yields

$$0 \leq \Theta(\ell) \leq \Theta(\ell').$$

Now, assume that the surface load f is such that there exists at least one statically admissible stress field $\sigma_0 \in \Sigma(\Omega)$ that is uniformly bounded in Ω . Then, the density θ_0 , associated through formula (4.159), satisfies

$$1 - \theta_0 \leq \frac{C}{\sqrt{-\ell}}$$

for $-\ell$ large enough. Consequently,

$$\min_{0 \leq \theta \leq 1} (g(\sigma_0, \theta) + \ell(\theta - 1)) \leq C(1 + \sqrt{-\ell}) \text{ a.e. in } \Omega.$$

Thus, any minimizing pair $(\tau_{\ell}, \theta_{\ell})$ satisfies

$$\int_{\Omega} g(\tau_{\ell}, \theta_{\ell}) dx - \ell\Theta(\ell) \leq C(1 + \sqrt{-\ell}),$$

which proves that $\sqrt{-\ell}\Theta(\ell)$ is bounded and thus

$$\lim_{\ell \rightarrow -\infty} \Theta(\ell) = 0,$$

as claimed. \square

The relaxation results, Theorems 4.2.3 and 4.2.6, still hold true for a problem with multiple loads, as introduced in (4.142). The relaxed formulation is formally obtained as in the case of a single load. Introducing a weak material A , we approximate F^n , defined in (4.145), by F_A^n , defined in (4.63),

$$F_A^n(\{\tau_i\}) = \min \left(\sum_{i=1}^n A^{-1} \tau_i : \tau_i + \ell, \sum_{i=1}^n B^{-1} \tau_i : \tau_i \right).$$

Once again, as A tends to zero, the sequence of functions F_A^n monotonically converges to F^n . The relaxation of the functional

$$\inf_{\{\tau_i\} \in \Sigma^n(\Omega)} \int_{\Omega} F_A^n(\{\tau_i\}) dx,$$

where $\Sigma^n(\Omega)$ is defined in (4.144), has been done in Theorem 4.1.28, and the result is similar to the single load case, namely

$$\min_{\{\tau_i\} \in \Sigma^n(\Omega)} \int_{\Omega} QF_A^n(\{\tau_i\}) dx,$$

where

$$QF_A^n(\{\tau_i\}) = \min_{0 \leq \theta \leq 1} \left(g_A^n(\{\tau_i\}, \theta) + \ell\theta \right), \quad (4.171)$$

and g_A^n is the lower Hashin-Shtrikman dual bound on the sum of n effective complementary energies (see Proposition 2.3.24), namely,

$$g_A^n(\{\tau_i\}, \theta) = \min_{A^* \in G_\theta} \sum_{i=1}^n A^{*-1} \tau_i : \tau_i. \quad (4.172)$$

Then, as in (4.151), we define g^n as the monotone limit of g_A^n when A tends to zero. Introducing

$$QF^n(\{\tau_i\}) = \lim_{A \rightarrow 0} QF_A^n(\{\tau_i\}) = \min_{0 \leq \theta \leq 1} \left(g^n(\{\tau_i\}, \theta) + \ell\theta \right), \quad (4.173)$$

the relaxation of the multiple loads problem (4.143) is given by

$$\min_{\{\tau_i\} \in \Sigma^n(\Omega)} \int_{\Omega} QF^n(\{\tau_i\}) dx. \quad (4.174)$$

Theorem 4.2.12 *The stress formulation (4.143), (4.145) of the multiple loads shape optimization problem admits (4.174), (4.173) as a relaxed formulation. In other words,*

1. *there exists at least one solution in $\Sigma^n(\Omega)$ of the relaxed minimization problem (4.174);*
2. *up to a subsequence, any minimizing sequence of (4.143) converges weakly in $L^2(\Omega; \mathcal{M}_N^s)^n$ to a minimizer of (4.174);*

3. for any minimizer $\{\tau_i\}$ of (4.174) there exists a minimizing sequence of (4.143), which converges to $\{\tau_i\}$ weakly in $L^2(\Omega; \mathcal{M}_N^s)^n$, and

$$\inf_{\{\tau_i\} \in \Sigma^n(\Omega)} \int_{\Omega} F^n(\{\tau_i\}) dx = \min_{\{\tau_i\} \in \Sigma^n(\Omega)} \int_{\Omega} QF^n(\{\tau_i\}) dx.$$

The proof of Theorem 4.2.12 is very similar to that of Theorem 4.2.3 in the single load case, because the functions $QF^n(\{\tau_i\}, \theta)$ and $g^n(\tau, \theta)$ enjoy similar properties. In particular, Corollary 2.3.33 yields the following result on the lower Hashin-Shtrikman dual bound g^n .

Lemma 4.2.13 *The function $g^n(\{\tau_i\}, \theta)$ is given by*

$$g^n(\{\tau_i\}, \theta) = \sum_{i=1}^n B^{-1} \tau_i : \tau_i + \frac{\theta}{1-\theta} g_n^*(\{\tau_i\}), \quad (4.175)$$

where $g_n^*(\{\tau_i\})$ is a continuous and convex function, homogeneous of degree two, and positive whenever $\{\tau_i\} \neq 0$. Therefore, g^n is continuous in $(\{\tau_i\}, \theta)$ and strictly convex separately in $\{\tau_i\}$ and in θ . For any $\{\tau_i\} \in (\mathcal{M}_N^s)^n$, there exists a unique minimizer $\theta_{\{\tau_i\}}$ in (4.173) given by

$$\theta_{\{\tau_i\}} = \max \left(0, 1 - \sqrt{\frac{g_n^*(\{\tau_i\})}{-\ell}} \right). \quad (4.176)$$

Lemma 4.2.13 implies that a unique optimal density (called a relaxed, or generalized, or composite, optimal shape) is associated through (4.176) to any minimizer of the relaxed stress formulation (4.174). Theorem 4.2.12 is concerned only with the stress formulation, so we now address the relaxation of the original multiple loads shape optimization problem (4.142). By (4.173), we have

$$QF^n(\{\tau_i\}) = \min_{0 \leq \theta \leq 1} \min_{A^* \in G_\theta^0} \left(\sum_{i=1}^n A^{*-1} \tau_i : \tau_i + \ell \theta \right),$$

where G_θ^0 is the limit of G_θ as A goes to zero (see Remark 4.2.4). Inverting the minimizations, we introduce homogenized compliances

$$c_i(\theta, A^*) = \min_{\substack{\tau_i \in L^2(\Omega; \mathcal{M}_N^s) \\ \operatorname{div} \tau_i = 0 \text{ in } \Omega, \tau_i n = f_i \text{ on } \partial\Omega}} \int_{\Omega} A^{*-1} \tau_i : \tau_i dx, \quad 1 \leq i \leq n, \quad (4.177)$$

and we deduce the relaxation

$$\min_{(\theta, A^*) \in \mathcal{CD}} \left\{ J^*(\theta, A^*) = \sum_{i=1}^n c_i(\theta, A^*) + \ell \int_{\Omega} \theta dx \right\}, \quad (4.178)$$

where \mathcal{CD} is the space of relaxed admissible designs defined by (4.157).

Theorem 4.2.14 *There exists at least one relaxed optimal shape, i.e., a density θ and a tensor A^* such that $(\theta, A^*) \in \mathcal{CD}$, which minimizes (4.178). Furthermore, for any minimizing sequence of characteristic functions $\chi_n \in L^\infty(\Omega; \{0, 1\})$ for (4.142), there exists a subsequence and a limit density θ such that this subsequence converges to θ weakly * in $L^\infty(\Omega; [0, 1])$ and θ is a relaxed optimal shape, i.e., there exists a tensor A^* such that $(\theta, A^*) \in \mathcal{CD}$ minimizes (4.178).*

The proof of Theorem 4.2.14 is completely similar to that of Theorem 4.2.6, so we omit it. Eventually, a simplified relaxed formulation is available by replacing the unknown set G_θ^0 with its explicit subset L_θ^{0+} of perforated sequential laminates.

Theorem 4.2.15 *For the multiple loads objective function (4.178) we have*

$$\min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*) = \min_{(\theta, A^*) \in \mathcal{LD}^+} J^*(\theta, A^*),$$

where \mathcal{CD} is the set of all composite designs (4.157) and \mathcal{LD}^+ that of perforated sequentially laminated designs (4.160). If (θ, A^*) is a minimizer of J^* in \mathcal{CD} , and if $\{\sigma_i\}$ are the stress tensors that minimize the associated compliances (4.177), then there exists a sequential laminate \tilde{A}^* such that (θ, \tilde{A}^*) is a minimizer of J^* in \mathcal{LD}^+ , $\{\sigma_i\}$ are again the associated stress tensors, and $A^{*-1}\sigma_i = \tilde{A}^{*-1}\sigma_i$.

Theorem 4.2.15 is similar to Theorem 4.2.9 except on one important point concerning the number of laminations and the orientation of these laminations for the optimal sequential laminate \tilde{A}^* . Unfortunately, in the multiple loads case no explicit formulas are available as of yet for the computation of the optimal laminate. In particular, the lamination directions are not necessarily aligned with the principal stress directions and the number of laminations, although independent of the number of loads n , is higher than for a single load. By Lemma 2.3.9, we know that the rank of the optimal laminate in the optimal lower Hashin-Shtrikman bound g^n , defined by (4.175), is always smaller or equal to 3 in two dimensions and 6 in three dimensions. But we have no idea of the lamination directions. In practice, the optimal laminate is found by a numerical optimization process (see Subsection 5.2.6).

4.2.3 Link with the Michell Truss Theory

The goal of this subsection is to establish a link between the homogenized relaxed formulation of shape optimization and the Michell truss theory. This theory was introduced by Michell [185] a century ago, and has been developed by many authors since then, both from a theoretical point of view and from a numerical one (for a review, see, e.g., [130], [135], [149], [225], [226], [237], [239]). Michell trusses are very popular in the community of mechanical engineering and structural optimization, where they are in constant use. Indeed, the Michell truss theory already features some of the nicest ingredients of the homogenization method for shape optimization. It uses a special class of generalized designs, and it is a layout, or topology, optimization approach. This suffices to motivate our interest in that theory, but even more, we shall show that it is an asymptotic theory of homogenized compliance optimization in the low volume limit.

We briefly summarize this theory of Michell trusses in plane elasticity for a single loading configuration (which has an equivalent interpretation as grillage-like continua for plate theory). A Michell structure is made up entirely of infinitely many, and infinitely thin, linear truss elements (or bars), each capable of withstanding a maximal tensile or compressive stress. The possibility of element buckling is ignored. By mechanical principles, for a given statically admissible stress field τ , the truss elements should be oriented with the eigendirections of τ (the so-called *directions of principal stress*). Each member of the truss adapts itself to the stress τ by choosing its cross-section to be such that it reaches its limit load as dictated by plasticity theory (the smaller the stress, the smaller the cross-section). Therefore, a bar aligned with the i th eigendirection of τ has a cross-section proportional to the i th eigenvalue τ_i . This is the well known principle of material economy due to Michell. Since the elements are infinitesimal, the intersections of the truss bars that run into orthogonal eigendirections of τ are neglected. Therefore, the density of material required to withstand τ is proportional to $|\tau_1| + |\tau_2|$. If the truss is confined to a plane domain Ω , its total weight is

$$\int_{\Omega} (|\tau_1| + |\tau_2|) dx.$$

The optimal Michell truss is then obtained by minimizing its weight as τ varies among statically admissible stress fields τ . In other words, the Michell

truss optimization problem is

$$\inf_{\substack{\operatorname{div} \tau = 0 \text{ in } \Omega \\ \tau n = f \text{ on } \partial\Omega}} \int_{\Omega} (|\tau_1| + |\tau_2|) dx, \quad (4.179)$$

where τ_1, τ_2 are the eigenvalues of the stress τ , and f is some vector-valued surface loading. Note that (4.179) does not say in which functional space minimizers are sought. Actually, the mathematical theory of this problem is not yet complete, and we write (4.179) as an infimum rather than a minimum. The reason is that the integrand has linear growth, which makes $L^1(\Omega; \mathcal{M}_N^s)$ the natural space for admissible stresses. Minimizing sequences will be bounded in that space, but unfortunately not precompact for the weak convergence (see Lemma 1.2.1). In general, minimizers of (4.179) would be measures, instead of functions, which implies that an optimal truss can be singular, i.e., the stress can concentrate along lines or curves. By the same token, it is difficult to interpret the boundary conditions for functions that are measures. This situation is somehow similar to that of plastic limit analysis (see, e.g., [263], [279]). Nevertheless, (4.179) is a convex problem for which one can use duality theory and optimality conditions to find exact minimizers in many special cases. There is a rich theory of explicit solutions (see the references above). The dual problem is easily seen to be

$$\sup_{|e_1(u)| \leq 1, |e_2(u)| \leq 1 \text{ in } \Omega} \int_{\partial\Omega} f \cdot u ds, \quad (4.180)$$

where u is a vector-valued displacement, and $e(u) = (\nabla u + (\nabla u)^t)/2$ is its strain tensor with eigenvalues $e_1(u)$ and $e_2(u)$ (see, e.g., [262]). Of course, we recognize in (4.180) the compliance which is the objective function. The optimality conditions for the optimal dual pairing (τ, u) are

$$\begin{cases} e(u) = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}, & \tau = \begin{pmatrix} t_1 e_1 & 0 \\ 0 & t_2 e_2 \end{pmatrix}, \\ t_1 \geq 0, t_2 \geq 0, t_1 = 0 \text{ if } |e_1| < 1, & t_2 = 0 \text{ if } |e_2| < 1. \end{cases} \quad (4.181)$$

Remark 4.2.16 *There has been some recent progress, made by Bouchitté, Buttazzo, and Seppecher [55], [56], on the mathematical theory of Michell trusses. Indeed, they embed it in the theory of Monge-Kantorovitch equations, which allows one to view the truss design problem as a mass allocation problem. This is of course consistent with the mechanical derivation of the Michell truss problem.*

We now show formally that, in two dimensions, when the Lagrange multiplier ℓ goes to minus infinity (namely, when the volume of material B goes to zero), the relaxed problem of compliance minimization is asymptotically equivalent to the Michell truss problem. This was first established in [21] (see also [46], [240]). We begin by recalling the relaxed formulation of optimal design of minimum compliance (see Subsection 4.2.2). For a bounded domain Ω in \mathbb{R}^2 and a surface loading f , it reads

$$\min_{\tau \in \Sigma(\Omega)} \int_{\Omega} QF(\tau) dx, \quad (4.182)$$

with $\Sigma(\Omega)$ the space of statically admissible stresses

$$\Sigma(\Omega) = \left\{ \tau \in L^2(\Omega; \mathcal{M}_N^s) \mid \operatorname{div} \tau = 0 \text{ in } \Omega, \quad \tau n = f \text{ on } \partial\Omega \right\},$$

and

$$QF(\tau) = \min_{0 \leq \theta \leq 1} \{g(\tau, \theta) + \ell\theta\}, \quad (4.183)$$

where g is the lower Hashin-Shtrikman bound on complementary energy for a perforated composite material. This bound has been explicitly computed in Theorem 2.3.35 as

$$g(\tau, \theta) = B^{-1}\tau : \tau + \frac{(\kappa + \mu)\theta}{4\kappa\mu(1 - \theta)} (|\tau_1| + |\tau_2|)^2 \quad (4.184)$$

where τ_1 and τ_2 are the eigenvalues of the stress τ . Furthermore, the optimal density in (4.183) is

$$\theta_\tau = \max \left(0, 1 - \sqrt{\frac{\kappa + \mu}{-4\kappa\mu\ell}} (|\tau_1| + |\tau_2|) \right).$$

Let us denote by τ^ℓ a minimizer of (4.182). In the limit of ℓ going to $-\infty$ we assume that everywhere in Ω

$$\lim_{\ell \rightarrow -\infty} \frac{|\tau_1^\ell| + |\tau_2^\ell|}{\sqrt{-\ell}} = 0.$$

(Here is the formal step, since it is not clear that this happens everywhere for the minimizers of (4.182).) Then, θ_{τ^ℓ} is always positive and the relaxed integrand simplifies to

$$QF(\tau^\ell) = B^{-1}\tau^\ell : \tau^\ell + \ell\rho(\tau^\ell)(2 - \rho(\tau^\ell)) + \ell, \quad (4.185)$$

where $\rho(\tau)$ is the material density $1 - \theta_\tau$, i.e.,

$$\rho(\tau) = \sqrt{\frac{\kappa + \mu}{-4\kappa\mu\ell}} (|\tau_1| + |\tau_2|).$$

Furthermore,

$$\lim_{\ell \rightarrow -\infty} \frac{QF(\tau^\ell) - \ell}{\sqrt{-\ell}} = \sqrt{\frac{\kappa + \mu}{\kappa\mu}} (|\tau_1^\ell| + |\tau_2^\ell|)$$

which is nothing but the integrand of the Michell truss problem (4.179).

Remark 4.2.17 *The above asymptotic derivation of the Michell truss problem as ℓ goes to $-\infty$ is only formal. A rigorous statement of this result would involve some kind of Γ -convergence argument [90]. There are additional difficulties due to the fact that the limit problem (4.179) would admit in general only measured-value minimizers. Similarly, the sequence of minimizers τ_ℓ in (4.182) is merely bounded in $L^1(\Omega; \mathcal{M}_N^s)$ which implies that convergence will hold in the weak sense of measures. In such a setting, it is difficult to keep track of the boundary conditions, making the rigorous justification of this limit process quite technical (although not impossible).*

The same formal asymptotic process can be performed in three dimensions; the result is somehow very different. To simplify the exposition, we assume that material B has zero Poisson ratio. Then, Corollary 2.3.38 gives the following value for the lower Hashin-Shtrikman bound (labeling the eigenvalues of τ such that $|\tau_1| \leq |\tau_2| \leq |\tau_3|$):

$$g(\tau, \theta) = B^{-1}\tau : \tau + \frac{\theta}{4\mu(1-\theta)} \rho(\tau)^2$$

with

$$\rho(\tau) = \begin{cases} |\tau_1| + |\tau_2| + |\tau_3| & \text{if } |\tau_3| \leq |\tau_1| + |\tau_2| \\ \sqrt{(|\tau_1| + |\tau_2|)^2 + |\tau_3|^2} & \text{if } |\tau_3| > |\tau_1| + |\tau_2|. \end{cases}$$

The same argument shows that, as ℓ goes to $-\infty$, the relaxed formulation (4.182) admits the asymptotic limit

$$\inf_{\substack{\operatorname{div}\tau=0 \text{ in } \Omega \\ \tau n=f \text{ on } \partial\Omega}} \int_{\Omega} \rho(\tau) dx,$$

which is different from the straightforward generalization of the two-dimensional Michell truss problem, since it has two regimes. The first regime does indeed correspond to what one would expect, and the corresponding optimal homogenized microstructures are rank-3 sequential laminates. The second regime has different optimal homogenized microstructures, namely rank-2 sequential laminates, which look like an array of orthogonal plates aligned in the third eigendirection. This plate-like microstructure is thus not equivalent to a three-dimensional truss, which is sub-optimal in this second regime.

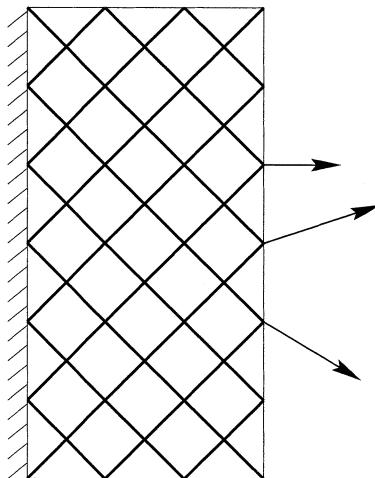


Figure 4.2: Ground structure of a Michell truss.

Remark 4.2.18 *The Michell integrand, $(|\tau_1| + |\tau_2|)$, which, in its mechanical derivation, may seem a little mysterious for the mathematician, is now well justified by the above asymptotic process. In particular, the fact that the truss elements shall be aligned with the principal stress directions is now a direct consequence of homogenization theory. To be more precise, Proposition 2.3.25 states that an optimal microstructure in the lower Hashin-Shtrikman bound on complementary energy (4.184) is aligned with the principal stress directions, and this property carries over in the limit when ℓ goes to $-\infty$. This self-adaptation of microstructure to the stress that it should sustain is a rigorous consequence of homogenization theory and not a postulate. It confirms the basic principles of Michell theory for material economy in frame structures.*

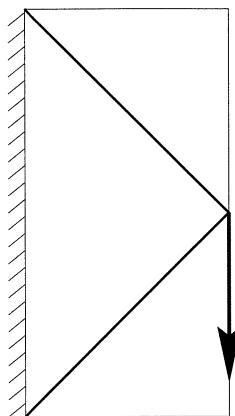


Figure 4.3: An optimal Michell truss: the short cantilever.

Remark 4.2.19 *The fact that Michell truss is a limiting case of the relaxed formulation (4.182) may explain the success of numerical computations based on the homogenization method (see Chapter 5). More specifically, many of the obtained optimal structures look like a network of trusses, or bars, at least in two dimensions, a fact that is consistent with the above asymptotic analysis.*

Remark 4.2.20 *The theory of continuous Michell trusses is very useful in deriving explicit optimal designs, but it has a discrete analogue which is also very useful, since the latter amounts to solving a linear programming problem. We therefore briefly describe the framework of discrete Michell trusses (see, e.g., [42], [130], [135], [149]). The computational domain is discretized in a so-called ground structure made of a finite number of nodes connected by truss members or bars (see Figure 4.2). One has to decide if all nodes are a priori connected or if only some connections are allowed. In other words, a graph of possible connectivity is set up. The intersection of bars is ignored, and at each node the member forces are supposed to be in equilibrium with the external forces. The area or cross-section of the bars is supposed to be proportional to the force it withstands. This is again a criterion reminiscent of limit load analysis in plasticity. Then the total weight of the truss is minimized among all statically admissible configurations. It turns out that this is a linear convex minimization problem (see, e.g., Chapter 4 in [42]). At the minimum, some bars may have zero area (i.e., they are not part of the optimal truss), and the minimizing structure satisfies kinematic equilibrium*

(for example, see Figure 4.3 where the optimal truss is a cantilever made of two bars meeting at 90 degrees when the ground structure has an aspect ratio of 2 over 1). The fact that this is a linear programming problem makes it very easy (in principle) to solve, and Michell truss theory is actually a very popular approach for structural optimization among mechanical or civil engineers. There are, of course, many generalizations of this problem (described here in the single load case) and many numerical subtleties for which we refer the reader to [3], [4], [40], [42], [163], [239], [241].

Chapter 5

Numerical Algorithms

This chapter is devoted to algorithmic and numerical issues in the application of the homogenization method to optimal design problems. We describe how the results of Chapters 3 and 4 can be used to build new numerical algorithms. Although homogenization theory has been devised as a tool for shape optimization by some of its first contributors (see the works of Murat and Tartar [205], [269], [271], [273]), the first significant numerical applications appeared well after these theoretical breakthroughs. There were some early contributions by Gibiansky and Cherkaev [115], in the context of plate optimization (equivalent to two-dimensional elasticity), Glowinski [117], Goodman, Kohn, and Reyna [119], and Lavrov, Lurie, and Cherkaev [160] in the conductivity setting (equivalent to a torsion problem for an elastic bar). Note that these works consider only two-phase optimization problems with nondegenerate components. The first study of shape optimization by the homogenization method in a general elasticity setting (including the fact that one phase was degenerating to holes) is due to Bendsoe and Kikuchi [47]. Their work was really pioneering in the sense that they treated convincing real life examples educating the whole community of structural optimization to the homogenization approach. They have been followed by many others, notably [15], [17], [21], [41], [97], [98], [127], [142], [249], [265]. Although the field of homogenization-based algorithms for shape optimization is increasingly popular, there remains much to be done.

There are already a few good surveys or conference proceedings concerning numerical methods for shape optimization in the framework of the homogenization method [42], [48], [74], [213], [238], [239]. In particular the book of Bendsoe [42] is highly recommended for the less theoretically, but

more practically inclined, reader. Nevertheless, we believe that efficient numerical algorithms take root in a clear and deep understanding of the theory, and it is the purpose of this book to cover all aspects of the homogenization method in optimal design, leaving no gaps between theory and numerical practice.

The first section is concerned with numerical algorithms in the two-phase conductivity setting. There are mainly two types of iterative methods in optimal design, both of which alternate evaluations of the state (and possibly of the adjoint state) and updating of the design parameters. The first one is the so-called *optimality criteria method*. Its principle is to use optimality conditions to update the design parameters. This is a very popular method for which there is a vast body of literature in the structural engineering community (see, e.g., [42], [149], [225], [237], [239], [245], [296], and references therein). The second method is the more classical descent method based on gradient information. It includes several algorithms, like gradient descent with fixed step, steepest descent, conjugate gradient (not to mention Newton-type methods, which also use the Hessian matrix of second derivatives), but we call all these variants, without distinction, *gradient methods*. There is an even more enormous literature on this topic, regarding which we simply indicate a few relevant references in optimal design: [39], [66], [202], [220], [258], and references therein.

The second section, by far the most important one, is devoted to structural optimization in the linearized elasticity setting. We focus essentially on the shape optimization problem, rather than the two-phase one, since shape optimization is truly the main application of the homogenization method. Shape optimization is a major issue in many industrial fields including aerospace industry, car manufacturing, and civil engineering. To emphasize the importance of the homogenization method in shape optimization, let us simply recall that there exist several commercial softwares based on this method (in addition to many research and academic codes). As in the conductivity setting, there are two types of numerical algorithms: optimality criteria and gradient methods.

There is an additional difficulty in shape optimization, compared to two-phase optimal design, which is the presence of holes or voids in the material domain. Numerically (as well as theoretically) the difficulty is alleviated by filling the holes with a very compliant material. This has the effect of transforming a free-boundary problem into a free-interface one. This is a deep and crucial step, which allows one to develop new algorithms that may be called *shape-capturing* instead of *shape-tracking* (we borrow this terminology

from the computer fluid dynamics community where it applies to shocks). Indeed, with homogenization-based algorithms, the computation takes place on a fixed mesh, and the contours of the shape are detected where strong gradients of a material density occur on this given background. One essential feature of such algorithms is their absence of any a priori, or a posteriori, restrictions on the topology of the design (i.e., the number of holes or connected components). For this very reason, homogenization-based numerical algorithms are often viewed as topology optimizers. This is their main advantage over classical methods of shape optimization based on boundary motion, which are unable to change the topology of the initial design because smooth motions of the boundary do not permit the creation of holes. Since it is widely acknowledged that creating holes (i.e., changing the topology) may drastically improve the performance of a candidate optimal shape, this explains the success of the homogenization method. Of course, the shapes, captured by the homogenization method on a fixed mesh, may not have the required smoothness (due to some feasibility constraints). This is not a problem if the homogenization method is used as a pre-processor for topology optimization before applying a more classical method of boundary motion.

We conclude this introduction by recognizing that the homogenization method is not the only one useful for topology optimization. There are also the stochastic methods (genetic algorithms, simulated annealing, etc.) [114], [143], [144], the material distribution method [41], [42], [69], the so-called topology gradient method [104], [103], [68], [257], or the more classical Michell truss layout problem [3], [4], [40], [130], [149], [163], [237]. We hasten to add that the above list of references is far from complete and we urge the reader to look at other references inside these works.

5.1 Algorithms for Optimal Design in Conductivity

This section is devoted to two different types of numerical algorithms for solving optimal design problems in the conductivity setting as discussed in Chapter 3. They both rely on detailed knowledge of the optimality conditions. For simplicity, we treat only single load problems, i.e., with a single state equation, although the proposed algorithms could be extended to several loads as well. We also discuss only two-phase optimal design problems (with isotropic phases), and not shape optimization.

5.1.1 Optimality Criteria Method

A very popular algorithm in structural design is the *optimality criteria method* (see, e.g., [42], [149], [225], [237], [239], [245], [296], and references therein). Its principle is to solve iteratively the optimality conditions of the relaxed formulation. To explain the method in more detail, we first recall some of the notation of Section 3.2.

Introducing a space \mathcal{CD} of homogenized (or composite) designs

$$\mathcal{CD} = \left\{ (\theta, A^*) \in L^\infty(\Omega; [0, 1] \times \mathcal{M}_N^s) \mid A^*(x) \in G_{\theta(x)} \text{ a.e. in } \Omega \right\}, \quad (5.1)$$

where G_θ is defined by Theorem 2.2.13, the relaxed two-phase optimal design problem is the following minimization

$$\min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*), \quad (5.2)$$

where the objective function is

$$\begin{aligned} J^*(\theta, A^*) = & \int_{\Omega} [\theta(x)g_{\alpha}(x, u(x)) + (1 - \theta(x))g_{\beta}(x, u(x))] dx \\ & + \ell \int_{\Omega} \theta(x)dx, \end{aligned} \quad (5.3)$$

with $u(x)$, the unique solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\operatorname{div}(A^* \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.4)$$

We shall also need the adjoint state p defined as the unique solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\operatorname{div}(A^* \nabla p) = \theta \frac{\partial g_{\alpha}}{\partial \lambda}(x, u) + (1 - \theta) \frac{\partial g_{\beta}}{\partial \lambda}(x, u) & \text{in } \Omega \\ p = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.5)$$

The precise knowledge of the set G_θ is not necessary since, as a result of Theorem 3.2.6, we can restrict the class of admissible tensors A^* to that of rank-1 laminates. Recall from Subsection 2.2.1 that, in the orthonormal basis for which the first vector is the lamination direction, a rank-1 laminate A^* in G_θ is defined by

$$A^* = \operatorname{diag} \left(\lambda_{\theta}^-, \lambda_{\theta}^+, \dots, \lambda_{\theta}^+ \right), \quad (5.6)$$

with

$$\lambda_\theta^- = \left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta} \right)^{-1} \text{ and } \lambda_\theta^+ = \theta\alpha + (1-\theta)\beta.$$

Any other rank-1 laminate in G_θ is simply obtained by rotation of (5.6). As a further consequence of Theorem 3.2.6, the lamination direction e of an optimal A^* , i.e., the first eigenbasis vector in (5.6) (up to normalization), is given, at each point of Ω where ∇u and ∇p are nonzero, by

$$e = \frac{\nabla u}{|\nabla u|} - \frac{\nabla p}{|\nabla p|} \quad (5.7)$$

if it different from zero, and otherwise by any direction orthogonal to $\frac{\nabla u}{|\nabla u|} + \frac{\nabla p}{|\nabla p|}$. On the other hand, introducing the angle ϕ between ∇u and ∇p , defined by

$$\nabla u \cdot \nabla p = |\nabla u| |\nabla p| \cos \phi, \quad (5.8)$$

by virtue of Theorem 3.2.8 an optimal density θ satisfies at each point of Ω where $0 < \theta < 1$

$$g_\alpha(x, u) - g_\beta(x, u) + \ell + \frac{\beta - \alpha}{\alpha\beta} |\nabla u| |\nabla p| \left(\alpha\beta \cos^2 \frac{\phi}{2} - (\lambda_\theta^-)^2 \sin^2 \frac{\phi}{2} \right) = 0. \quad (5.9)$$

Formulas (5.7) and (5.9) are the informations that we select in the optimality conditions for use in the optimality criteria method.

We now describe the principle of the optimality criteria method for solving (5.2). It is an iterative method where we successively compute approximations of the fields (u, p) and of the design parameters (θ, A^*) .

1. We *initialize* the design parameters θ_0 and A_0^* (for example, we take a constant proportion θ_0 and a layered material $A_0^* \in G_{\theta_0}$ with a constant lamination direction).
2. Until convergence, for $k \geq 0$ we *iteratively* compute the state u_k and the adjoint state p_k , solutions of (5.4) and (5.5) respectively, with the previous design parameters (θ_k, A_k^*) , and then *update* these parameters by using the optimality conditions (5.7) and (5.9) to obtain new values $(\theta_{k+1}, A_{k+1}^*)$.

Let us describe more precisely how we use the optimality conditions. The tensor A^* of a rank-1 laminate depends actually on the density θ and

on the lamination direction e . There are therefore two design parameters to update: θ and e . If ∇u_k and ∇p_k are nonzero, we take

$$e_{k+1} = \frac{\nabla u_k}{|\nabla u_k|} - \frac{\nabla p_k}{|\nabla p_k|},$$

if e_{k+1} is nonzero, otherwise we take $e_{k+1} = e_k$. The possibility that ∇u_k or ∇p_k are equal to zero almost never occurs in numerical practice, and in any case the value of the tensor A^* is not important at these points. On the contrary, it may be possible that $\frac{\nabla u_k}{|\nabla u_k|} = \frac{\nabla p_k}{|\nabla p_k|}$, or at least that equality holds up to a small remainder (which produces large error in the evaluation of the direction, i.e., when normalizing e_{k+1}). For example, it is always the case for the self-adjoint problem where $u = p$. In such a case, we switch to the other possibility in the optimality condition (5.7), which gives e_{k+1} as any orthogonal direction to the vector $\frac{\nabla u_k}{|\nabla u_k|} + \frac{\nabla p_k}{|\nabla p_k|}$, which is certainly not zero if $u = p$.

To compute the value θ_{k+1} , we look for a root of the following function of θ :

$$\begin{aligned} F(\theta) = & g_\alpha(x, u_k) - g_\beta(x, u_k) + \ell \\ & + \frac{\beta - \alpha}{\alpha \beta} |\nabla u_k| |\nabla p_k| \left(\alpha \beta \cos^2 \frac{\phi_k}{2} - (\lambda_\theta^-)^2 \sin^2 \frac{\phi_k}{2} \right), \end{aligned}$$

which is a strictly increasing function of θ on $[0, 1]$ if $\sin \frac{\phi_k}{2} \neq 0$. Therefore, $F(\theta)$ admits at most one root $\theta_{k+1} \in [0, 1]$ such that $F(\theta_{k+1}) = 0$. If $F(1) < 0$, we take $\theta_{k+1} = 1$, while if $F(0) > 0$ we take $\theta_{k+1} = 0$ (which agrees with the optimality conditions of Theorem 3.2.8). The only trouble comes if $\sin \frac{\phi_k}{2} = 0$, or at least is very small. For example, this is always the case for the self-adjoint problem where $u = p$ and thus $\phi = 0$. In such a case, one cannot compute θ_{k+1} as a root of $F(\theta)$. However, according to Remark 3.2.11, the optimality condition (5.9) can be rewritten in terms of the fluxes $\sigma = A^* \nabla u$ and $\tau = A^* \nabla p$ and reads, for $0 < \theta < 1$,

$$g_\alpha(x, u) - g_\beta(x, u) + \ell + (\beta - \alpha) |\sigma| |\tau| \left((\lambda_\theta^+)^{-2} \cos^2 \frac{\psi}{2} - (\alpha \beta)^{-1} \sin^2 \frac{\psi}{2} \right) = 0. \quad (5.10)$$

The angle ψ is defined by

$$\sigma \cdot \tau = |\sigma| |\tau| \cos \psi,$$

and it is clear that $\phi = 0$ implies $\psi = 0$. Thus, if $\phi = 0$, one can extract the root θ_{k+1} of (5.10), instead of (5.9), by the same procedure.

Remark 5.1.1 *If, instead of working with a fixed Lagrange multiplier ℓ , a volume constraint is enforced, the above algorithm must be coupled with an iterative procedure that updates the value ℓ_k at each iteration. Since the total volume of phase α is a nonincreasing function of ℓ , a simple dichotomy for adjusting ℓ is enough.*

Remark 5.1.2 *The above algorithm is easily extended to the case of several state equations. We remark, however, that the optimal tensor A^* would then be a sequential laminate of rank- N (see Theorem 3.2.14).*

Remark 5.1.3 *It is important to note that the above algorithm is just one possible version of an optimality criteria method. We already saw that either (5.9) or (5.10) can be used for updating the density, but other updating schemes are available as well. The only requirement is that, upon convergence, the scheme leads to a design that satisfies the optimality condition. The reason for selecting the above algorithm comes from its interpretation as a descent method (thus guaranteeing convergence) in the case of self adjoint problems. As noted in Remark 3.2.23, if $p = -u$, then computing θ and e by (5.9) and (5.7) amounts to minimizing the cost function. In other words, if $p = -u$, the above algorithm decreases the objective function at each iteration. The same is true in the case $p = u$ if (5.10) is used instead of (5.9).*

The first advantage of this optimality criteria method is its simplicity. The criteria for updating the design parameters is local and simple. However, except in a few special cases, there is no guarantee that this algorithm converges. Indeed, in some cases it fails to converge, or it requires a relaxation procedure (i.e., the design parameters are not updated to their new value, but to a carefully tuned convex combination of the old and new ones). Nevertheless, when it converges (which happens most of the time), it is a very fast and efficient algorithm. We remark that the optimality conditions are necessary, but it is not known if they are sufficient. So there is no theoretical guarantee that a design satisfying the optimality conditions is actually optimal.

5.1.2 Gradient Method

For the more mathematically inclined reader, optimality criteria methods are not very satisfactory since they do not rely on a firm theory of convergence. Therefore, it is appropriate to propose a second class of numerical

algorithms for which convergence (to a local minimum) is always guaranteed, although these algorithms may be slower in some cases. We therefore discuss a gradient method for minimizing the objective function (5.3) (we keep the notation of the previous subsection). There are several type of gradient algorithms, and for simplicity we choose to describe a fixed step-size descent method (other algorithms, like steepest descent or conjugate gradient, are easy generalizations). As seen in Subsection 3.2.4, computing the gradient of the objective function is a task very similar to that of deriving optimality conditions. In the case of a single state equation, we choose as design parameters the density θ and the rotation for the rank-1 laminate tensor A^* . In 2-D a rotation is parametrized by a single angle $\phi \in [0, \pi]$, while in 3-D two angles are required. For simplicity we describe the 2-D algorithm which is trivially extended to the 3-D case.

We therefore work with the following class of homogenized tensors

$$A^*(\theta, \phi) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \lambda_\theta^+ & 0 \\ 0 & \lambda_\theta^- \end{pmatrix} \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}.$$

In other words, the objective function is now a function of the two scalar design variables θ and ϕ , and we still denote it by $J^*(\theta, \phi)$. The computation of its gradient with respect to (θ, ϕ) is very similar to that carried out in Subsection 3.2.4. For an admissible increment $(\delta\theta, \delta\phi)$, we find

$$\delta J^*(\theta, \phi) = \int_{\Omega} \frac{\partial A^*}{\partial \phi} \nabla u \cdot \nabla p \delta\phi dx + \int_{\Omega} \bar{Q}(x) \delta\theta dx, \quad (5.11)$$

where $\bar{Q}(x)$, being very similar to $Q(x)$ (introduced in Theorem 3.2.8), is defined by

$$\begin{aligned} \bar{Q}(x) &= g_\alpha(x, u) - g_\beta(x, u) + \ell \\ &+ \frac{d\lambda_\theta^-}{d\theta} \left[\sin^2 \phi \frac{\partial u}{\partial x_1} \frac{\partial p}{\partial x_1} + \cos^2 \phi \frac{\partial u}{\partial x_2} \frac{\partial p}{\partial x_2} + \cos \phi \sin \phi \left(\frac{\partial u}{\partial x_1} \frac{\partial p}{\partial x_2} + \frac{\partial u}{\partial x_2} \frac{\partial p}{\partial x_1} \right) \right] \\ &+ \frac{d\lambda_\theta^+}{d\theta} \left[\cos^2 \phi \frac{\partial u}{\partial x_1} \frac{\partial p}{\partial x_1} + \sin^2 \phi \frac{\partial u}{\partial x_2} \frac{\partial p}{\partial x_2} - \cos \phi \sin \phi \left(\frac{\partial u}{\partial x_1} \frac{\partial p}{\partial x_2} + \frac{\partial u}{\partial x_2} \frac{\partial p}{\partial x_1} \right) \right]. \end{aligned}$$

Formula (5.11) yields the partial derivative of $J^*(\theta, \phi)$. There is no restriction on the range of the angle ϕ , but the density θ must stay between zero and one. Therefore, applying a projection step to the density, the gradient method is defined as follows.

1. We *initialize* the design parameters θ_0 and ϕ_0 (for example, we take a constant angle ϕ_0 and proportion θ_0).
2. Until convergence, for $k \geq 0$ we *iteratively* compute the state u_k and the adjoint state p_k , solutions of (5.4) and (5.5) respectively with the previous design parameters (θ_k, ϕ_k) , and then *update* these parameters by

$$\theta_{k+1} = \max \left(0, \min \left(1, \theta_k - t_k \bar{Q}_k \right) \right)$$

and

$$\phi_{k+1} = \phi_k - t_k \frac{\partial A^*}{\partial \phi}(\theta_k, \phi_k) \nabla u_k \cdot \nabla p_k$$

where $t_k > 0$ is a small step such that $J^*(\theta_{k+1}, \phi_{k+1}) < J^*(\theta_k, \phi_k)$.

Such a gradient method always converges to a (local) minimum, and its speed of convergence is partly governed by the efficiency of the line search for finding a good step t_k . This method was implemented in [117] (see also [16]). This method can be accelerated by a conjugate gradient algorithm or by a Newton type method.

Remark 5.1.4 *The above gradient algorithm can be extended to the case of several state equations. Since the optimal tensor A^* would be a sequential laminate of rank- N instead of rank-1, we parametrize A^* in terms of the Y -transform introduced in Subsection 3.2.4. In such a case the design parameters would be (θ, Y) ; the gradient has been computed in Theorem 3.2.18.*

5.1.3 A Convergence Proof

This subsection is devoted to a convergence proof of the above optimality criteria method in the case of a self-adjoint optimization problem. This result is due to Toader [281], [282]. In the framework of Subsection 5.1.1, we specialize problem (5.2) to the case

$$J^*(\theta, A^*) = \int_{\Omega} f(x)u(x)dx + \ell \int_{\Omega} \theta(x)dx,$$

namely $g_{\alpha}(x, u) = g_{\beta}(x, u) = f(x)$, which implies that the adjoint state is $p = u$. This choice of objective function amounts to minimizing a weighted sum of the stored potential energy and of the volume of phase α (i.e., to finding a best conducting design). If $\ell \geq 0$, the optimum is easily seen to be achieved by using only phase β , i.e., $\theta = 0$. Therefore, in the sequel we

assume that $\ell < 0$. All the results of this subsection are easily generalized to the other self-adjoint problem obtained by taking $g_\alpha(x, u) = g_\beta(x, u) = -f(x)$, which corresponds to $p = -u$, i.e., one wants to maximize (instead of minimize) the stored potential energy.

The optimality criteria method amounts to iteratively computing $u = p$, the solution of the state equation (5.4), and the design parameters (θ, A^*) , which are given by the optimality conditions. The self-adjointness of the problem allows for several simplifications. The first term in the objective function can be rewritten (see Subsection 3.2.5) as the minimum of the complementary energy, i.e.,

$$\int_{\Omega} f u dx = \min_{\substack{\tau \in L^2(\Omega)^N \\ -\operatorname{div} \tau = f \text{ in } \Omega}} \int_{\Omega} A^{*-1} \tau \cdot \tau dx.$$

Thus, the optimal design problem of minimizing J^* is equivalent to

$$\min_{(\theta, A^*) \in \mathcal{CD}} \min_{\substack{\tau \in L^2(\Omega)^N \\ -\operatorname{div} \tau = f \text{ in } \Omega}} \int_{\Omega} (A^{*-1} \tau \cdot \tau + \ell \theta) dx. \quad (5.12)$$

Since in (5.12) the optimization is expressed in terms of the flux τ rather than the potential u , we shall use optimality conditions written with τ instead of u , namely we rely on Remarks 3.2.7 and 3.2.11 rather than Theorems 3.2.6 and 3.2.8. As noted in Remark 5.1.3 the optimality criteria method turns out to be an alternate minimization algorithm for the double minimization (5.12). The optimality conditions used in the optimality criteria method are nothing other than the necessary conditions for τ, θ, A^* to be minimizers of (5.12). Indeed, for given θ and τ , an optimal tensor A^* is a rank-1 laminate in a direction orthogonal to τ , and the lower dual energy bound reads

$$\frac{|\tau|^2}{\lambda_\theta^+} = \min_{A^* \in G_\theta} A^{*-1} \tau \cdot \tau. \quad (5.13)$$

Furthermore, the function

$$\theta \rightarrow \frac{|\tau|^2}{\lambda_\theta^+} + \ell \theta \quad (5.14)$$

is strictly convex and thus admits a unique minimizer in $[0, 1]$. Taking into account these facts, the algorithm is structured as follows:

1. We initialize the design parameters θ_0 and A_0^* (for example, we take a constant proportion θ_0 and a layered material $A_0^* \in G_{\theta_0}$ with a constant lamination direction).
2. Until convergence, for $k \geq 0$ we iteratively compute $u_k = p_k$, solution of (5.4), with the previous design parameters (θ_k, A_k^*) , and then update these parameters by taking A_{k+1}^* to be a rank-1 laminate in a direction e_{k+1} orthogonal to $\sigma_k = A_k^* \nabla u_k$ and θ_{k+1} as the root of (5.10), namely

$$\theta_{k+1} = \begin{cases} 0 & \text{if } \ell + \frac{(\beta-\alpha)|\sigma_k|^2}{\beta^2} \geq 0 \\ (\beta - \alpha)^{-1} \left(\beta - \sqrt{\frac{(\beta-\alpha)|\sigma_k|^2}{-\ell}} \right) & \text{if } \frac{(\beta-\alpha)|\sigma_k|^2}{\beta^2} < -\ell < \frac{(\beta-\alpha)|\sigma_k|^2}{\alpha^2} \\ 1 & \text{if } \ell + \frac{(\beta-\alpha)|\sigma_k|^2}{\alpha^2} \leq 0 \end{cases} \quad (5.15)$$

The optimality conditions used in this optimality criteria algorithm are nothing but necessary conditions for τ, θ, A^* to be minimizers of (5.12). More precisely, $\sigma_k = A_k^* \nabla u_k$ is a minimizer of the complementary energy for the design (θ_k, A_k^*) , i.e.,

$$\int_{\Omega} A_k^{*-1} \sigma_k \cdot \sigma_k dx = \min_{\substack{\tau \in L^2(\Omega)^N \\ -\operatorname{div} \tau = f \text{ in } \Omega}} \int_{\Omega} A_k^{*-1} \tau \cdot \tau dx,$$

while A_{k+1}^* is a rank-1 laminate minimizing (5.13) and θ_{k+1} is the minimizer of (5.14) for the given flux σ_k , i.e.,

$$\int_{\Omega} \left(A_{k+1}^{*-1} \sigma_k \cdot \sigma_k + \ell \theta_{k+1} \right) dx = \min_{(\theta, A^*) \in \mathcal{CD}} \int_{\Omega} \left(A^{*-1} \sigma_k \cdot \sigma_k + \ell \theta \right) dx.$$

As a consequence, the above optimality criteria algorithm always decreases the value of the objective function

$$\begin{aligned} \int_{\Omega} \left((A_k^*)^{-1} \sigma_{k-1} \cdot \sigma_{k-1} + \ell \theta_k \right) dx &\geq \int_{\Omega} \left((A_k^*)^{-1} \sigma_k \cdot \sigma_k + \ell \theta_k \right) dx \\ &\geq \int_{\Omega} \left((A_{k+1}^*)^{-1} \sigma_k \cdot \sigma_k + \ell \theta_{k+1} \right) dx. \end{aligned}$$

Introducing the quantity

$$I_k = \int_{\Omega} \left((A_{k+1}^*)^{-1} \sigma_k \cdot \sigma_k + \ell \theta_{k+1} \right) dx = \int_{\Omega} \left(\frac{|\sigma_k|^2}{\lambda_{\theta_{k+1}}^+} + \ell \theta_{k+1} \right) dx,$$

we have established that

$$I_{k-1} \geq J^*(\theta_k, A_k^*) \geq I_k.$$

Theorem 5.1.5 *Let $(\theta_k, A_k^*, \sigma_k)_{k \geq 1}$ be the sequence of iterates of the above optimality criteria algorithm. Up to a subsequence, it converges to a limit $(\tilde{\theta}, \tilde{A}^*, \tilde{\sigma})$ weakly * in $L^\infty(\Omega; [0, 1]) \times L^\infty(\Omega; \mathcal{M}_N^s) \times L^2(\Omega; \mathbb{R}^N)$ which is a global minimizer of J^* and*

$$\lim_{k \rightarrow +\infty} J^*(\theta_k, A_k^*) = J^*(\tilde{\theta}, \tilde{A}^*) = \min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*).$$

Proof. Since $0 \leq \theta(x) \leq 1$ and $\alpha I_2 \leq A^*(x) \leq \beta I_2$ a.e. in Ω , the sequence (θ_k, A_k^*) is bounded in $L^\infty(\Omega; [0, 1] \times \mathcal{M}_N^s)$. By the uniform boundedness of A_k^* , the sequence σ_k is bounded in $L^2(\Omega; \mathbb{R}^N)$. Therefore, up to a subsequence, they converge weakly * (or weakly for σ_k) in these spaces to some limit $(\tilde{\theta}, \tilde{A}^*, \tilde{\sigma})$. Let $(\hat{\theta}, \hat{A}^*)$ be a minimizer of J^* and $\hat{\sigma}$ a corresponding minimizer of the complementary energy in (5.12). We use repeatedly Lemma 3.1.9 on the strict convexity of the function $\phi(a, \sigma) = a^{-1}|\sigma|^2$ defined on $\mathbb{R}^+ \times \mathbb{R}^N$. Formula (3.36) yields

$$\begin{aligned} J^*(\hat{\theta}, \hat{A}^*) &= \int_{\Omega} \left(\frac{|\sigma_k|^2}{\lambda_{\theta_{k+1}}^+} + \ell \theta_{k+1} \right) dx + \ell \int_{\Omega} (\hat{\theta} - \theta_{k+1}) dx \\ &\quad + \int_{\Omega} \left(\frac{2}{\lambda_{\theta_{k+1}}^+} \sigma_k \cdot (\hat{\sigma} - \sigma_k) + \frac{(\beta - \alpha)(\hat{\theta} - \theta_{k+1})}{(\lambda_{\theta_{k+1}}^+)^2} |\sigma_k|^2 \right) dx \\ &\quad + \int_{\Omega} \frac{1}{\lambda_{\theta_{k+1}}^+} |\hat{\sigma} - \frac{\lambda_{\hat{\theta}}^+}{\lambda_{\theta_{k+1}}^+} \sigma_k|^2 dx. \end{aligned}$$

But the optimality condition for θ_{k+1} implies that

$$(\hat{\theta} - \theta_{k+1}) \left(\ell + \frac{(\beta - \alpha)}{(\lambda_{\theta_{k+1}}^+)^2} |\sigma_k|^2 \right) \geq 0,$$

so that

$$J^*(\hat{\theta}, \hat{A}^*) \geq I_k + \int_{\Omega} \frac{2}{\lambda_{\theta_{k+1}}^+} \sigma_k \cdot (\hat{\sigma} - \sigma_k) dx. \quad (5.16)$$

On the other hand, introducing $\sigma(t) = (1-t)\sigma_k + t\hat{\sigma}$ for $t \geq 0$, we have

$$\begin{aligned} \int_{\Omega} (A_{k+1}^*)^{-1} \sigma(t) \cdot \sigma(t) &= \int_{\Omega} (A_{k+1}^*)^{-1} \sigma_k \cdot \sigma_k + 2t \int_{\Omega} (A_{k+1}^*)^{-1} \sigma_k \cdot (\hat{\sigma} - \sigma_k) \\ &\quad + t^2 \int_{\Omega} (A_{k+1}^*)^{-1} (\hat{\sigma} - \sigma_k) \cdot (\hat{\sigma} - \sigma_k), \end{aligned}$$

which implies, since $(A_{k+1}^*)^{-1} \sigma_k = (\lambda_{\theta_{k+1}}^+)^{-1} \sigma_k$, and σ_{k+1} minimizes the complementary energy associated to A_{k+1}^* , that

$$J^*(\theta_{k+1}, A_{k+1}^*) \leq I_k + t \int_{\Omega} \frac{2}{\lambda_{\theta_{k+1}}^+} \sigma_k \cdot (\hat{\sigma} - \sigma_k) + \alpha^{-1} t^2 \int_{\Omega} |\hat{\sigma} - \sigma_k|^2. \quad (5.17)$$

Combining (5.16) and (5.17), and recalling that the sequence σ_k is bounded, leads to

$$J^*(\theta_{k+1}, A_{k+1}^*) - I_k \leq t \left(J^*(\hat{\theta}, \hat{A}^*) - I_k \right) + Ct^2.$$

Minimizing over $t \geq 0$ implies

$$\frac{(J^*(\hat{\theta}, \hat{A}^*) - I_k)^2}{4C} \leq I_k - J^*(\theta_{k+1}, A_{k+1}^*) \leq I_k - I_{k+1}. \quad (5.18)$$

Since I_k is a decreasing sequence, bounded by below, it converges and (5.18) shows that its limit is just the minimum value of J^* . Applying again the convexity of $\phi(a, \sigma)$, formula (3.36) gives

$$\begin{aligned} I_k &= \int_{\Omega} \left(\frac{|\tilde{\sigma}|^2}{\lambda_{\tilde{\theta}}^+} + \ell \tilde{\theta} \right) dx + \ell \int_{\Omega} (\theta_{k+1} - \tilde{\theta}) dx \\ &\quad + \int_{\Omega} \left(\frac{2}{\lambda_{\tilde{\theta}}^+} \tilde{\sigma} \cdot (\sigma_k - \tilde{\sigma}) + \frac{(\beta - \alpha)(\theta_{k+1} - \tilde{\theta})}{(\lambda_{\tilde{\theta}}^+)^2} |\tilde{\sigma}|^2 \right) dx \\ &\quad + \int_{\Omega} \frac{1}{\lambda_{\tilde{\theta}}^+} |\sigma_k - \frac{\lambda_{\theta_{k+1}}^+}{\lambda_{\tilde{\theta}}^+} \tilde{\sigma}|^2 dx \end{aligned} \quad (5.19)$$

Since

$$\int_{\Omega} \left(\frac{|\tilde{\sigma}|^2}{\lambda_{\tilde{\theta}}^+} + \ell \tilde{\theta} \right) dx \geq J^*(\tilde{\theta}, \tilde{A}^*),$$

passing to the weak limit in (5.19) implies

$$\min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*) \geq J^*(\tilde{\theta}, \tilde{A}^*) + \lim_{k \rightarrow +\infty} \int_{\Omega} \frac{1}{\lambda_{\tilde{\theta}}^+} |\sigma_k - \frac{\lambda_{\theta_{k+1}}^+}{\lambda_{\tilde{\theta}}^+} \tilde{\sigma}|^2 dx.$$

Thus,

$$\min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*) = J^*(\tilde{\theta}, \tilde{A}^*),$$

and $(\lambda_{\theta_{k+1}}^+)^{-1} \sigma_k$ converges strongly to $(\lambda_{\tilde{\theta}}^+)^{-1} \tilde{\sigma}$ in $L^2(\Omega; \mathbb{R}^N)$. \square

Remark 5.1.6 *The proof of Theorem 5.1.5 uses in an essential way the convexity of the optimal design problem, besides its self-adjoint character. Therefore, it cannot be straightforwardly generalized to the case of several state equations or to the elasticity setting for which the relaxed problem, although self-adjoint, is quasiconvex instead of convex.*

5.1.4 Numerical Examples

We now present some numerical computations to illustrate the above algorithms on a simple self-adjoint optimization problem. Specifically, we take $g_{\alpha}(x, u) = g_{\beta}(x, u) = -1$, and we seek minimizers of

$$\min_{(\theta, A^*) \in \mathcal{CD}} \left\{ J^*(\theta, A^*) = - \int_{\Omega} u(x) dx + \ell \int_{\Omega} \theta(x) dx \right\}, \quad (5.20)$$

where u is the solution of

$$\begin{cases} -\operatorname{div}(A^* \nabla u) = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (5.21)$$

and the adjoint state is $p = -u$. This example was first numerically studied by Glowinski [117] (following a suggestion of Tartar and with the help of Reinhart for the numerical computations), Goodman, Kohn, and Reyna [119], Kawhol, Stará, and Wittum [147], and Lavrov, Lurie, and Cherkaev [160]. In the conductivity setting, we seek the best isolating (or worst conducting) designs submitted to a uniform source term. In two space dimensions, (5.20) is equivalent to maximizing the torsional rigidity of an elastic bar under all possible two-phase designs of its cross-section Ω . Another interpretation of (5.20) is the maximization of the dissipated viscous energy of a Poiseuille flow in a pipe of cross-section Ω .

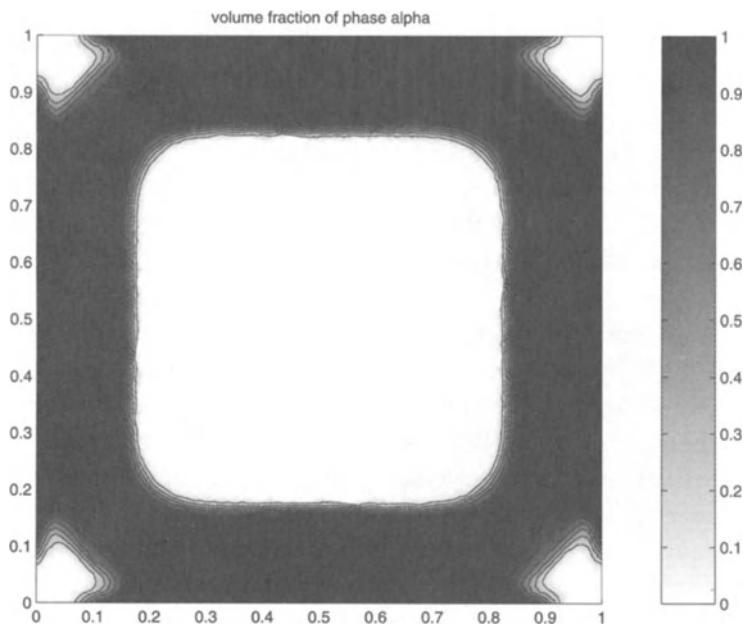
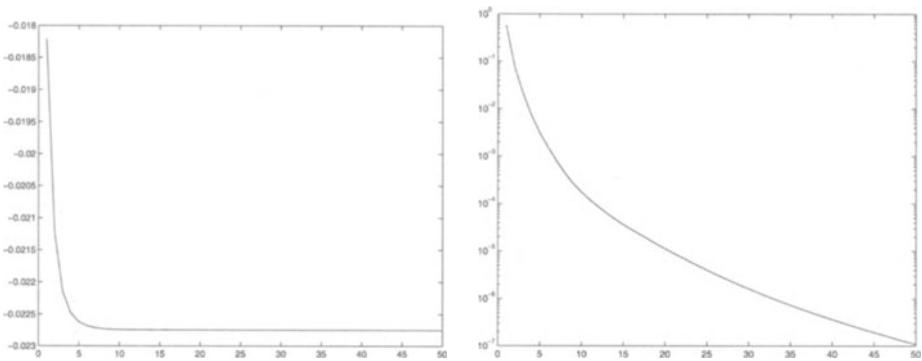
Figure 5.1: Volume fraction θ .

Figure 5.2: Convergence history: objective function (left), and residual (right), in terms of the iteration number.

We implement the optimality criteria algorithm which, as explained in the previous subsection, turns out to be a double minimization algorithm for this self-adjoint problem. This method yields the same results and is faster than the gradient method (which was used in [117]) or than the approach of [119], which amounts to solving the nonlinear conductivity problem obtained in Theorem 3.2.22 by eliminating the design parameters. Following these authors, we solve (5.20) on a square domain $\Omega = (0, 1)^2$ for two phases $\alpha = 1$ and $\beta = 2$. We work with a volume constraint of 50% of phase α , and the Lagrange multiplier ℓ is iteratively determined to satisfy this constraint (at the optimum, the value of ℓ is approximately 0.0165). In such a case, the objective function reduces to the first term on the right hand side of (5.20). We initialize with a constant $\theta = 0.5$ and a constant zero angle of lamination. The computations have been performed with $P1$ finite elements on a triangular mesh of 14912 elements in the Matlab software. Figure 5.1 displays the optimal volume fraction θ , Figure 5.3 the lamination direction of the optimal rank-1 laminate A^* , and Figure 5.4 the corresponding state u , the solution of (5.21). These results are obtained after 50 iterations of the algorithm, which has quickly converged as can be checked on Figure 5.2 (the residual is the L^2 -norm of the difference $\theta_k - \theta_{k-1}$ of two successive iterations). The results show that the worst conductor is placed along the edges of Ω , while the best one occupies the middle and the corners of Ω (in the context of torsional rigidity maximization, the smallest conductivity corresponds to the largest shear modulus and vice versa). There is only a small volume fraction of composite materials between the corners and the middle zone occupied by the good conductor.

Remark 5.1.7 *The same computation when Ω is a disk yields a classical optimal design, i.e., θ takes only the values zero and one (phase β occupies an inner concentric disk, while phase α lies in the remaining corona), as predicted theoretically in [205] and [273].*

5.2 Algorithms for Structural Optimization

This section describes the proposed numerical algorithms for shape optimization, which are based on the homogenization method. The key idea is to compute generalized (homogenized) optimal shapes for the relaxed formulation, rather than classical shapes, which are merely approximately optimal for the original formulation. We mainly follow our work [13], [14], [15], [17], which treats one choice of algorithm, and make references throughout the

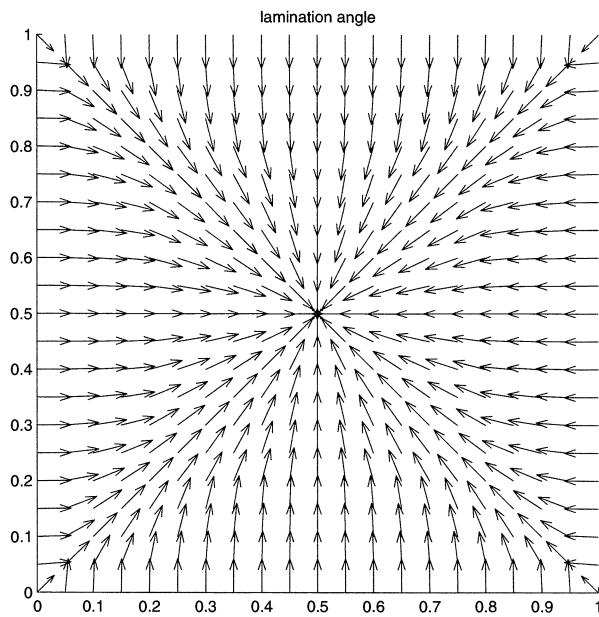
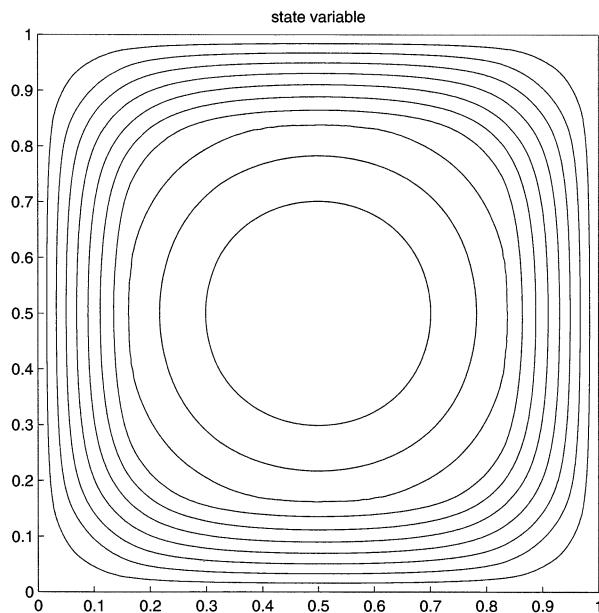


Figure 5.3: Direction of lamination.

Figure 5.4: Optimal state solution u .

exposition to other possible implementations of the homogenization method. As a final reminder, in this section we always assume that the weak phase is degenerate or void, i.e., we study shape optimization and not two-phase optimization (except where otherwise stated).

5.2.1 Compliance Optimization

We begin with the single load compliance optimization. We recall its relaxed formulation as computed in Section 4.2 (see also Subsection 2.3.4). According to Theorems 4.2.3 and 4.2.6 the relaxed objective function is

$$\min_{\tau \in \Sigma(\Omega)} \int_{\Omega} QF(\tau) dx, \quad (5.22)$$

where

$$QF(\tau) = \min_{0 \leq \theta \leq 1} \left(\min_{A^* \in L_{\theta}^{0+}} A^{*-1} \tau : \tau + \ell \theta \right), \quad (5.23)$$

with $\Sigma(\Omega)$ the set of statically admissible stresses defined by

$$\Sigma(\Omega) = \left\{ \tau \in L^2(\Omega; \mathcal{M}_N^s) \mid \operatorname{div} \tau = 0 \text{ in } \Omega, \quad \tau n = f \text{ on } \partial\Omega \right\}, \quad (5.24)$$

and L_{θ}^{0+} the set of all sequential laminated composites of material B around a core of void, in proportion $(1-\theta)$ and θ , respectively (see Definition 2.3.34). In other words, A^* belongs to L_{θ}^{0+} if and only if there exist an integer p , lamination directions $(e_i)_{1 \leq i \leq p}$, and lamination parameters $(m_i)_{1 \leq i \leq p}$ satisfying $0 \leq m_i \leq 1$ and $\sum_{i=1}^p m_i = 1$, such that

$$\theta \left[A^{*-1} - B^{-1} \right]^{-1} = (1-\theta) \sum_{i=1}^p m_i f_B^c(e_i) \quad (5.25)$$

where $f_B^c(e_i)$ is the fourth order tensor defined, for any symmetric matrix ξ , by its quadratic form

$$f_B^c(e_i) \xi : \xi = B \xi : \xi - \frac{1}{\mu} |B \xi e_i|^2 + \frac{\mu + \lambda}{\mu(2\mu + \lambda)} ((B \xi) e_i \cdot e_i)^2. \quad (5.26)$$

Recall that λ and μ are the Lamé coefficients of phase B as defined by (4.1). The stress relaxed formulation (5.22) is also equivalent to a minimization in terms of the design parameters

$$\min_{(\theta, A^*) \in \mathcal{LD}^+} \left\{ J^*(\theta, A^*) = c(\theta, A^*) + \ell \int_{\Omega} \theta dx \right\}, \quad (5.27)$$

where c is the design compliance

$$c(\theta, A^*) = \min_{\tau \in \Sigma(\Omega)} \int_{\Omega} A^{*-1} \tau : \tau dx, \quad (5.28)$$

and \mathcal{LD}^+ is the space of homogenized admissible designs, which are sequentially laminated composites

$$\mathcal{LD}^+ = \{(\theta, A^*) \in L^\infty(\Omega; [0, 1] \times \mathcal{M}_N^s) \mid A^*(x) \in L_{\theta(x)}^{0+} \text{ a.e. in } \Omega\}. \quad (5.29)$$

Furthermore, in Subsection 2.3.4 we performed an explicit computation of a minimizer A^* on the right hand side of (5.23), in two or three dimensions. In other words, we found a microstructure, with homogenized Hooke's law A^* , which is the most rigid composite of material density $(1 - \theta)$ under the stress τ . This optimal A^* is a sequential laminate of rank N with orthogonal lamination directions aligned with the eigenvectors of the stress τ (this optimal orientation, which is a consequence of Proposition 2.3.25, can also be found by using a result of Pedersen [217]). The minimal value is called the *lower Hashin-Shtrikman dual bound* and is denoted by

$$g(\tau, \theta) = \min_{A^* \in L_{\theta}^{0+}} A^{*-1} \tau : \tau.$$

We first recall Theorem 2.3.35 in 2-D

$$g(\tau, \theta) = B^{-1} \tau : \tau + \frac{\theta}{(1 - \theta)} g^*(\tau)$$

with

$$g^*(\tau) = \frac{2\mu + \lambda}{4\mu(\mu + \lambda)} (|\tau_1| + |\tau_2|)^2 \quad (5.30)$$

where τ_1 and τ_2 are the eigenvalues of the stress τ (a 2×2 symmetric matrix in 2-D). Furthermore, the associated optimal rank-2 sequential laminate is characterized by (5.25) with $p = 2$, e_1, e_2 the eigenvectors of τ , and its parameters

$$m_1 = \frac{|\tau_2|}{|\tau_1| + |\tau_2|}, \quad m_2 = \frac{|\tau_1|}{|\tau_1| + |\tau_2|}. \quad (5.31)$$

In 3-D Theorem 2.3.36 yields, under the assumption that $\lambda \geq 0$, the following result. If $\tau_1 \leq \tau_2 \leq \tau_3$ are the eigenvalues of τ , then

$$g(\tau, \theta) = B^{-1} \tau : \tau + \frac{\theta}{(1 - \theta)} g^*(\tau)$$

with $g^*(\tau)$ as given in the following cases:

(1) In the case where $0 \leq \tau_1 \leq \tau_2 \leq \tau_3$

$$g^*(\tau) = \frac{2\mu + \lambda}{4\mu(2\mu + 3\lambda)} (\tau_1 + \tau_2 + \tau_3)^2 \quad (5.32)$$

if $\tau_3 \leq \tau_1 + \tau_2$

$$g^*(\tau) = \frac{1}{2\mu} \left((\tau_1 + \tau_2)^2 + \tau_3^2 \right) - \frac{\lambda(\tau_1 + \tau_2 + \tau_3)^2}{2\mu(2\mu + 3\lambda)} \quad (5.33)$$

if $\tau_3 \geq \tau_1 + \tau_2$.

(2) In the case where $\tau_1 \leq 0 \leq \tau_2 \leq \tau_3$

$$g^*(\tau) = \frac{2\mu + \lambda}{4\mu(2\mu + 3\lambda)} \left(\tau_3 + \tau_2 - \frac{\mu + 2\lambda}{\mu + \lambda} \tau_1 \right)^2 \quad (5.34)$$

if $\tau_3 + \tau_2 \geq -\frac{\mu}{\mu + \lambda} \tau_1$, and $\tau_3 - \tau_2 \leq -\frac{\mu}{\mu + \lambda} \tau_1$

$$g^*(\tau) = \frac{1}{2\mu} \left((\tau_3 + \tau_2)^2 + \tau_1^2 \right) - \frac{\lambda(\tau_1 + \tau_2 + \tau_3)^2}{2\mu(2\mu + 3\lambda)} \quad (5.35)$$

if $\tau_3 + \tau_2 \leq -\frac{\mu}{\mu + \lambda} \tau_1$

$$g^*(\tau) = \frac{1}{2\mu} \left(\tau_1^2 + \tau_2^2 + \tau_3^2 \right) - \frac{2\mu\tau_1\tau_2}{2\mu(\mu + \lambda)} - \frac{\lambda(\tau_1 + \tau_2 + \tau_3)^2}{2\mu(2\mu + 3\lambda)} \quad (5.36)$$

if $\tau_3 - \tau_2 \geq -\frac{\mu}{\mu + \lambda} \tau_1$.

(3) The remaining cases are obtained from (1) and (2) by symmetry, changing τ into $-\tau$.

Furthermore, optimality in the regime (5.32) is achieved by a rank-3 sequential laminate with parameters

$$m_1 = \frac{\tau_3 + \tau_2 - \tau_1}{\tau_1 + \tau_2 + \tau_3}, \quad m_2 = \frac{\tau_1 - \tau_2 + \tau_3}{\tau_1 + \tau_2 + \tau_3}, \quad m_3 = \frac{\tau_1 + \tau_2 - \tau_3}{\tau_1 + \tau_2 + \tau_3}; \quad (5.37)$$

in the regime (5.33) it is achieved by a rank-2 sequential laminate with parameters

$$m_1 = \frac{\tau_2}{\tau_1 + \tau_2}, \quad m_2 = \frac{\tau_1}{\tau_1 + \tau_2}, \quad m_3 = 0; \quad (5.38)$$

in the regime (5.34) it is achieved by a rank-3 sequential laminate with parameters

$$m_1 = \frac{\tau_3 + \tau_2 + \frac{\mu}{\mu + \lambda} \tau_1}{\tau_3 + \tau_2 - \frac{\mu + 2\lambda}{\mu + \lambda} \tau_1}, \quad m_2 = \frac{\mu + \lambda}{\mu} \frac{\tau_3 - \tau_2 - \frac{\mu}{\mu + \lambda} \tau_1}{\tau_3 + \tau_2 - \frac{\mu + 2\lambda}{\mu + \lambda} \tau_1}, \quad (5.39)$$

$$m_3 = -\frac{\mu + \lambda}{\mu} \frac{\tau_3 - \tau_2 + \frac{\mu}{\mu + \lambda} \tau_1}{\tau_3 + \tau_2 - \frac{\mu + 2\lambda}{\mu + \lambda} \tau_1}; \quad (5.40)$$

in the regime (5.35) it is achieved by a rank-2 sequential laminate with parameters

$$m_1 = 0, \quad m_2 = \frac{\tau_3}{\tau_2 + \tau_3}, \quad m_3 = \frac{\tau_2}{\tau_2 + \tau_3}; \quad (5.41)$$

in the regime (5.36) it is achieved by a rank-2 sequential laminate with parameters

$$m_1 = \frac{\tau_2}{\tau_2 - \tau_1}, \quad m_2 = \frac{-\tau_1}{\tau_2 - \tau_1}, \quad m_3 = 0. \quad (5.42)$$

Eventually, according to Lemma 4.2.5, the minimizer θ_τ on the right hand side of (5.23) is given by

$$\theta_\tau = \max \left(0, 1 - \sqrt{\frac{g^*(\tau)}{-\ell}} \right). \quad (5.43)$$

Note that (5.43) implies that $\theta = 1$ if and only if $\tau = 0$ (since, by virtue of Corollary 2.3.33, $g^*(\tau) = 0$ if and only if $\tau = 0$), which means that holes are created only where the stress vanishes.

The relaxed formulation (5.22)–(5.24), or equivalently (5.27)–(5.29), is actually a double minimization with respect to the stress τ and the design parameters (θ, A^*) . If one of these variables is eliminated, we obtain a highly nonlinear functional, which is not easy to minimize (see Remarks 5.2.1 and 5.2.2). It is much easier (and it turns out to be more efficient) to keep these two independent sets of variables during the minimization process. Furthermore, it is also easier not to minimize directly in the triplet of variables (τ, θ, A^*) , but rather to minimize separately and successively in the design variables (θ, A^*) and in the field variable τ . These two key ideas are at the root of the *alternate directions algorithm*, which was introduced in [15], [17].

The first advantage of this alternate directions algorithm is its simplicity. Indeed, minimizing in τ for a fixed design (θ, A^*) amounts to solving a linear elasticity problem. This can even been performed with a primal finite element method (i.e., a displacement formulation), and the resulting stress τ is just computed afterward (this is the way our computations are done in the sequel). On the other hand, the minimization in (θ, A^*) for a given stress field τ is explicit in view of the previous formulas. The optimal density θ_τ is given by (5.43), while the optimal A^* is a rank- N sequential laminate given by (5.25) with $p = N$, the lamination directions being the eigenvectors of τ , and the lamination parameters $(m_i)_{1 \leq i \leq N}$ furnished by one of the above formulas (depending on the value of τ). These optimality conditions are applied locally (i.e., in each mesh cell or at each quadrature point) and they simply require the diagonalization of the stress tensor τ . There is no gradient method to apply, since the optimality conditions yield directly the minimizer (θ, A^*) . This is the second advantage of this alternate directions algorithm, which is, consequently, very cheap in terms of computing time. Eventually, the alternate directions algorithm is an iterative method, structured as follows:

1. Initialization of the design parameters (θ_0, A_0^*) (for example, taking $\theta_0 = 0$ and $A_0^* = B$ everywhere in the domain).
2. Iteration until convergence, for $k \geq 0$:
 - (a) Computation of τ_k through a problem of linear elasticity with (θ_k, A_k^*) as design variables.
 - (b) Updating of the design variables $(\theta_{k+1}, A_{k+1}^*)$ by using the stress τ_k in the explicit optimality formulas.

The third advantage of this alternate directions algorithm is its quick and guaranteed convergence to a stationary point of the objective function. Notice that the above iterative process always decreases the value of the objective function at each iteration. Indeed, since θ_{k+1} and A_{k+1}^* minimize the compliance under the stress field τ_k , and since τ_{k+1} minimizes the elastic complementary energy corresponding to the Hooke's law A_{k+1}^* , it follows that

$$\begin{aligned} \int_{\Omega} (A_k^*)^{-1} \tau_k : \tau_k dx + \ell \int_{\Omega} \theta_k dx &\geq \int_{\Omega} (A_{k+1}^*)^{-1} \tau_k : \tau_k dx + \ell \int_{\Omega} \theta_{k+1} dx \\ &\geq \int_{\Omega} (A_{k+1}^*)^{-1} \tau_{k+1} : \tau_{k+1} dx + \ell \int_{\Omega} \theta_{k+1} dx . \end{aligned}$$

Convergence is detected when the objective function becomes stationary, or when the change in the design variables becomes smaller than some preset threshold. Although in theory the algorithm converges only to a stationary point, which may be a saddle point rather than a local minimizer, this outcome never happens in numerical practice because of the rounding errors, which always select a stable local minima. As for all nonconvex minimization problems, the minimizer obtained by such an iterative numerical procedure may be a local (and not global) minimizer and may depend on the initialization. It turns out that in all our numerical experiments the resulting minimizer is independent of the initialization, suggesting that the method always converges to a global minimizer. We already know from Subsection 4.1.5 that there is usually no uniqueness of the minimizer. Therefore, our numerical algorithm just selects a special minimizer, but we do not know on which criterion it is selected. However, this experimental fact suggests that problem (5.22), although not convex (but quasiconvex, see Theorem 4.2.3), admits only global minimizers.

Remark 5.2.1 *As already said, the alternate directions algorithm is simpler and more efficient than a standard gradient algorithm applied to the functional $J^*(\theta, A^*)$ defined by (5.27). The computation of the gradient of J^* with respect to the design parameters (θ, A^*) is similar to the derivation of the optimality conditions as described above. However, a gradient method requires a line search, which may be quite expensive. On the contrary, the alternate directions algorithm always decreases the objective functional without a line search. As we shall see in Subsection 5.2.8, gradient methods are nevertheless useful for other objective functions, more general than the compliance.*

Remark 5.2.2 *The relaxed stress formulation (5.22)–(5.24) is actually a problem of nonlinear elasticity. The optimal density (a generalized shape) is recovered by the optimality condition (5.43) on θ . It was proposed in [21] to solve (in a first step) this nonlinear minimization problem in the stress τ , by using, for example, a conjugate gradient method, and to recover (in a second step) an optimal density θ through the optimality condition. Such an approach was successfully implemented in [21], for the 2-D case only. Unfortunately, it is not completely satisfactory for the following reasons. As in all computations involving complementary energies, high degree finite elements have to be used for stress accuracy. For example, Clough-Tocher C^1 conforming finite elements were used for the Airy stress potential (which*

is possible only in 2-D). In 3-D, one would have to resort to some type of mixed finite elements. Furthermore, the highly nontrivial energy QF is not smooth at $\tau = 0$ since it is continuous but not C^1 . It calls for special care in the gradient method. Convergence to the minimum is usually fairly slow. The resulting computations are thus costly and limited, in practice, to a two dimensional setting.

Remark 5.2.3 The alternate directions algorithm is related to the optimality criteria method used by Bendsoe and Kikuchi [47], Suzuki and Kikuchi [265], as well as many others. These authors first transform the minimization over statically admissible stresses into a maximization over kinematically admissible displacements, and the compliance (5.28) becomes

$$c(\theta, A^*) = \max_{u \in H^1(\Omega)^N} \left(2 \int_{\partial\Omega} f \cdot u ds - \int_{\Omega} A^* e(u) : e(u) dx \right), \quad (5.44)$$

where $e(u)$ is the strain tensor $(\nabla u + (\nabla u)^t)/2$. The ensuing numerical scheme is based on the first order optimality conditions at the saddle point of the functional (5.27)–(5.44), namely

$$\min_{(\theta, A^*) \in \mathcal{LD}^+} \max_{u \in H^1(\Omega)^N} \left(2 \int_{\partial\Omega} f \cdot u ds - \int_{\Omega} A^* e(u) : e(u) dx + \ell \int_{\Omega} \theta dx \right). \quad (5.45)$$

This leads to a rather intricate updating process for the design variables, which is also performed alternately: First, the solution u of a problem of linear elasticity where all design variables are fixed is obtained, and second, the design variables are updated using an optimality criteria method. To be more specific, let us describe the optimality criteria for the density θ (following the review of Bendsoe [42]), the case of the other design variables being similar. The stationarity of (5.45) with respect to θ gives

$$\frac{1}{\ell} \frac{\partial A^*}{\partial \theta} e(u) : e(u) = 1.$$

Recall that $\ell < 0$ and $\theta \rightarrow A^*$ is decreasing. Furthermore, for mechanical reasons it is expected that large values of the strain $e(u)$ takes place where the rigidity is small, i.e., where θ is closer to one than to zero. Therefore, a heuristic updating of the density θ is

$$1 - \theta_{k+1} = (1 - \theta_k) \left(\frac{1}{\ell} \frac{\partial A^*}{\partial \theta} (\theta_k) e(u_k) : e(u_k) \right)^q, \quad (5.46)$$

where $q \in (0, 1)$ is a tuning parameter. Furthermore, θ_{k+1} is truncated to stay within the range $[0, 1]$ and its increment with respect to θ_k is also limited by some move limit parameter. If the sequence of densities θ_k converges, then (5.46) converges to the desired optimality condition. Nevertheless, the convergence of the updating scheme (5.46) is unclear. The existence of a saddle point for

$$\min_{A^* \in L_\theta^{0+}} \max_{u \in H^1(\Omega)^N} \left\{ 2 \int_{\partial\Omega} f \cdot u ds - \int_{\Omega} A^* e(u) : e(u) dx + \ell \int_{\Omega} \theta dx \right\}$$

is established by Lipton in Theorem 4.1 of [168]. However, the minimization in θ cannot be exchanged with the maximization in u since the integrand does not satisfy any concave-convex type condition. Even more, the function

$$\theta \rightarrow \min_{A^* \in L_\theta^{0+}} -A^* e(u) : e(u)$$

is concave, instead of convex (see Remark 4.1.45). Therefore, convergence of the algorithm is *a priori* not guaranteed, although in practice it works fine. Another reason for preferring our alternate directions algorithm over this version of the optimality criteria method is that, when using a displacement-based formulation, the optimal microstructure has different regimes according to the value of the strain tensor. This inconvenience is not shared by our stress-based formulation, at least in 2-D. We emphasize again that our numerical algorithm can use a displacement finite element code, even though the design parameters are updated in terms of the stress tensor.

From a practical point of view, the advantage of the alternate directions algorithm (as well as any other algorithm based on the relaxed homogenized formulation) is that it transforms the original layout optimization problem in a sizing optimization problem, i.e., shapes are replaced by densities. The latter problem is much easier to solve numerically than the former one. In other words, a discrete 0 – 1 problem (material or void) has been changed in a continuous problem (material density between 0 and 1).

Remark 5.2.4 The above algorithm is easily transposed for the problem of maximizing the compliance (see the objective function (4.33)). As proposed by Francfort and Marigo [108], this problem is a model of brittle damage in elastic materials. Some computations were performed with this algorithm in [11] (see also Chapter 5.5 in [42]).

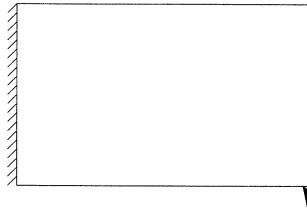


Figure 5.5: Boundary conditions for the cantilever problem.

5.2.2 Numerical Examples

The purpose of this subsection is to illustrate various aspects of the above algorithm of shape optimization. We use the code of Jouve [15], where our algorithm has been implemented in 2-D and 3-D (see its description in the next subsection). All test cases have the following data, unless otherwise specified. The Young modulus E of material B is normalized to 1 and the Poisson ratio ν is fixed to 0.3. Recall that the Young's modulus E and the Poisson ratio ν are related to the bulk and shear moduli (κ, μ) by

$$\mu = \frac{E}{2(1 + \nu)}, \quad \kappa - \frac{2}{N}\mu = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}.$$

The applied forces are also normalized to have unit norm. The algorithm is initialized with a working domain full of material ($\theta_0 = 0$). The lowest admissible value of the material density ($1 - \theta$) is 10^{-3} . Other numerical details are described next, in Subsection 5.2.3. In all examples we perform 100 iterations of our algorithm. In truth a good convergence is already reached in about 20 iterations, but we let the algorithm iterate more just to check its smooth convergence. The resulting optimal designs are called *composite designs*, since they usually include large regions of composite materials. As we shall explain later, in Subsection 5.2.4, we recover classical designs (i.e., those with only pure material or void) by a penalization procedure (which is also iterative) applied to the optimal composite designs. We always perform 20 penalized iterations, although less are needed in practice. The convergence history is displayed for these 120 iterations, but there is, of course, a change of algorithm at the hundredth iteration. We plot the material density ($1 - \theta$) on a grey scale: white is void, black is pure material, and grey is composite (or porous) media (the figures are drawn with 10 isovalue contours of the density constant by element, i.e., with no smoothing).

In the two-dimensional setting $N = 2$ we first study cantilever problems.

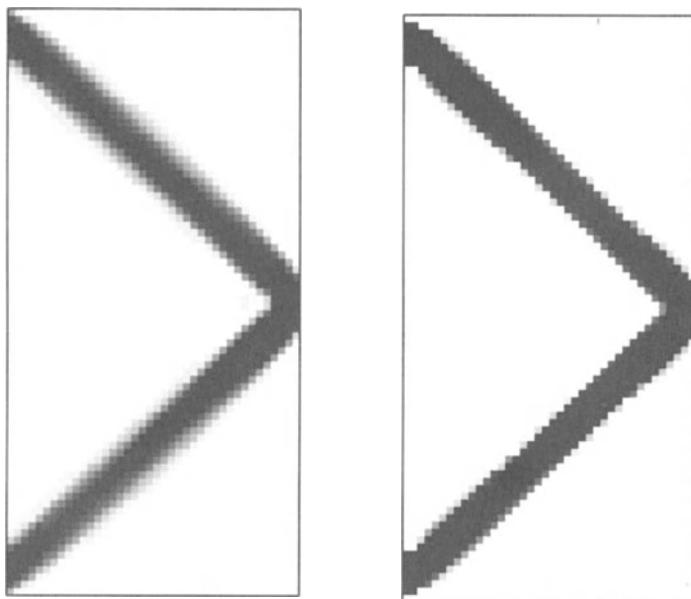


Figure 5.6: Optimal shape of the short cantilever: composite (left) and penalized (right).

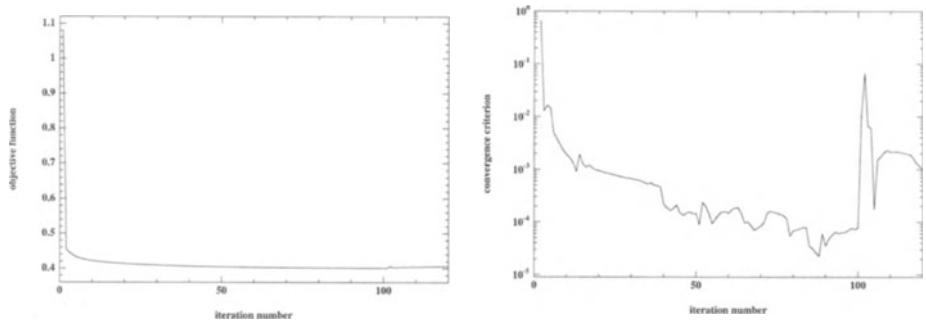


Figure 5.7: Convergence history of the short cantilever: objective function (left) and convergence criterion (right).

The working domain is a rectangle with zero displacement boundary condition on the left side and a unit vertical point load at the middle of the right side (see Figure 5.5). A first example is a *short cantilever*: The domain size is 10×20 , discretized with a rectangular 40×80 mesh, and the Lagrange multiplier ℓ is fixed to 1. After 100 iterations the optimal homogenized design is displayed on the left of Figure 5.6. A classical design is recovered after another 20 penalized iterations on the right of Figure 5.6. The weight of the composite optimal shape is 18.6% of that of the full working domain, and that of the penalized shape is 17.2%. The convergence is smooth and very quick, as can be checked in Figure 5.7 (the convergence criterion is defined in (5.47)). The algorithm has truly converged much before the 100 iterations. We remark that the bump of the convergence history at the hundredth iteration corresponds to the beginning of the penalization process. In this example there is no real need to penalize the optimal composite shape to obtain a classical shape. The obtained optimal design corresponds to the classical Michell truss solution, i.e., two bars meeting at 90 degrees (see, e.g., [237]).

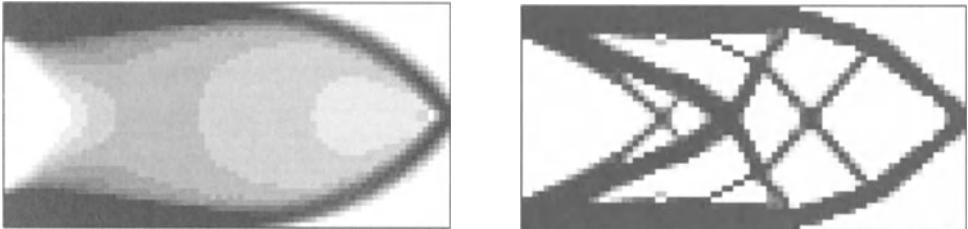


Figure 5.8: Optimal shape of the cantilever: composite (left) and penalized (right).

A second example is a *medium cantilever*: The domain size is 20×10 discretized with a rectangular 80×40 mesh, and the Lagrange multiplier ℓ is iteratively adjusted so that the weight of the structure is constrained to be 40% of that of the full working domain (the objective function is therefore the compliance). The optimal homogenized design (obtained after 100 iterations) and the penalized resulting shape (after another 20 penalized iterations) are displayed in Figure 5.8. Again the convergence is smooth and quick (see Figure 5.9). Note that the initialization does not respect the volume constraint, which explains the increase in the objective function between the first and second iteration. We remark that the penalized classical shape is very different from the optimal composite one, although the

objective function has only increased of about 5%.

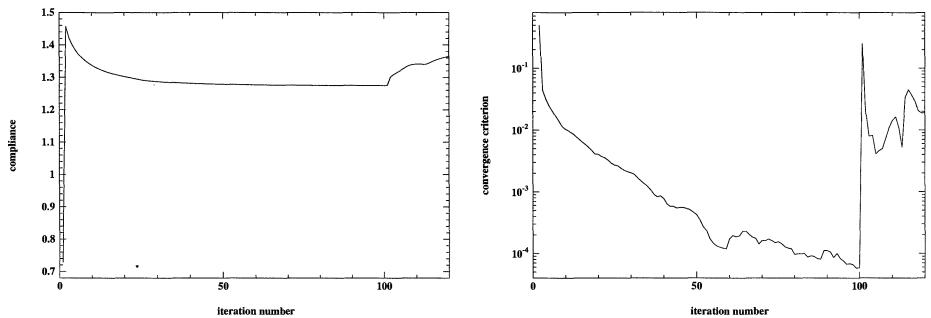


Figure 5.9: Convergence history of the cantilever: objective function (left) and convergence criterion (right).

A third example is a *bridge problem*: The domain size is 20×12 , at the two lower corners the vertical displacement is zero, and a unit vertical force is applied at the middle of the bottom side (see Figure 5.10). The volume constraint is fixed at 20% of the full working domain, and the mesh is rectangular with 80×48 elements. The optimal homogenized design (obtained after 100 iterations) and the penalized shape (after another 20 penalized iterations) are displayed in Figure 5.11.

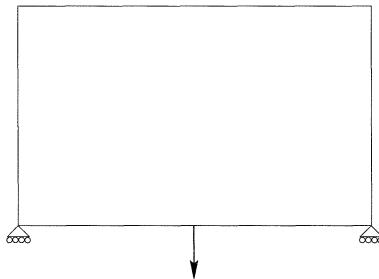


Figure 5.10: Boundary conditions for the bridge problem.

A fourth example is a *three-dimensional cantilever*: The domain size is $10 \times 5 \times 15$, the left side of the box is fixed (Dirichlet boundary condition) and a unit force in the x -direction is applied at the middle of the right side. The volume constraint is fixed at 40% of the full working domain. By symmetry only half of the domain is discretized by a $40 \times 10 \times 60$ mesh. Figures 5.12 and 5.13 display the isosurfaces of the material density. We remark

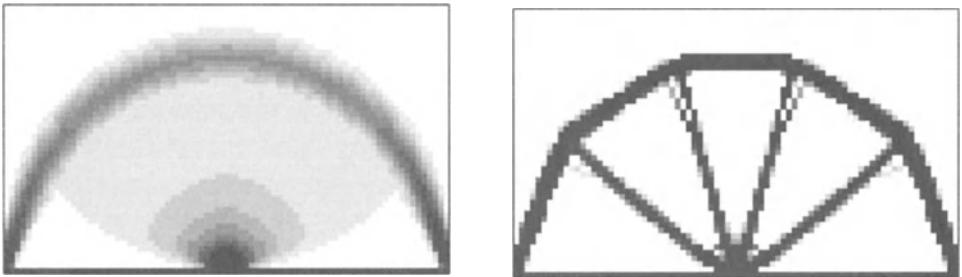


Figure 5.11: Optimal shape of the bridge: composite (left) and penalized (right).

that they are different for the composite design (indicating that composite material is really used), while they are all almost the same for the penalized design (showing that the density takes only the values zero and one). The convergence is as good as in 2-D, see Figure 5.14.

Finally a last example is a *three-dimensional bridge* optimized for two different working domains as shown on Figure 5.15. It demonstrates the strong influence of the choice of working domain on the final design. The boundaries of the working domain are indicated by solid lines on the figures.

5.2.3 Technical Algorithmic Issues

Finite elements. The most popular choice of finite elements for solving the successive problems of linear elasticity is the quadrangular Q_1 elements for the displacements (as in [15], [47], [142], [265]). Then, the stress τ_k , its principal directions and principal values, are computed at the center of each element. The parameters for the optimal laminate are also computed in each cell using the optimality conditions recalled in Subsection 5.2.1. All the computations that illustrate this section are performed with such a scheme (we use the code, mainly developed by Jouve, which is described in [15]).

An alternate choice is to use a stress-based variational formulation. In two dimensions when only surface loads are applied, a divergence-free stress derives from a scalar potential, the Airy stress function, and the elasticity problem becomes a fourth order partial differential equation (like a plate problem). The inconvenience of this approach is that it requires high degree finite elements, but it yields very precise results on the stress field. The Airy stress function is discretized on a triangular mesh using the Clough-Tocher finite element (see, e.g., [82]). Each triangle being divided into three sub-

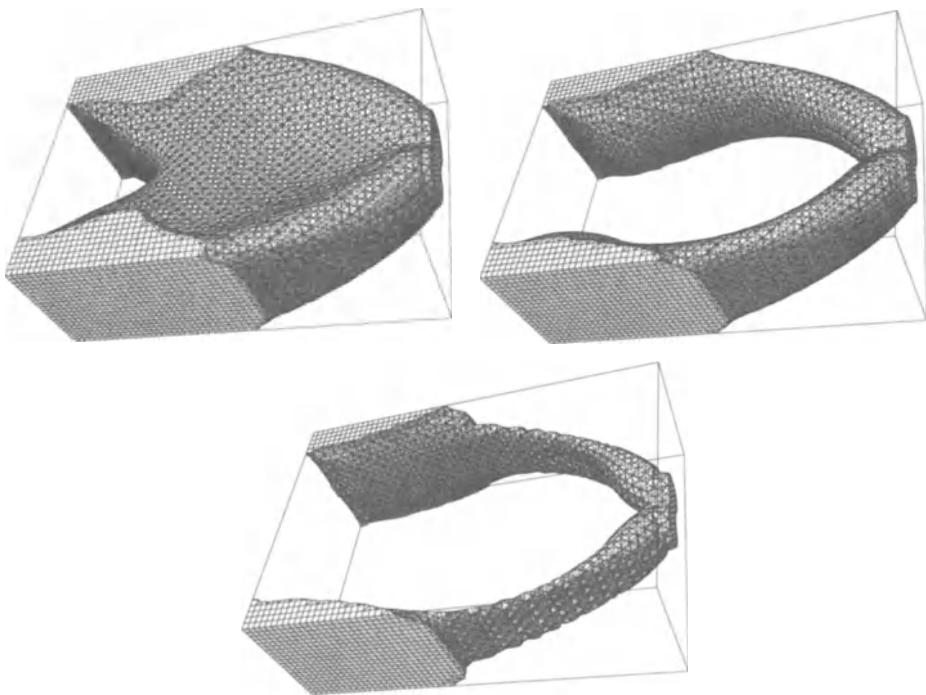


Figure 5.12: Optimal composite design of the 3-D cantilever: isosurfaces of the material density 0.3 (left), 0.5 (right), 0.9 (bottom).

triangles, it is piecewise cubic on subtriangles, and globally C^1 . The design parameters are computed at each Gauss quadrature point. This approach was introduced in [21] (see also [17], [23]), but is not especially recommended because of its complexity.

Convergence criterion. The alternate directions algorithm is iterated until the quantity

$$\max \left(\max_i \left(|\theta_i^{k+1} - \theta_i^k| \right), 1 - \frac{\int_{\Omega} (A_{k+1}^*)^{-1} \tau^k : \tau^k + \ell \int_{\Omega} \theta^{k+1}}{\int_{\Omega} (A_k^*)^{-1} \tau^{k-1} : \tau^{k-1} + \ell \int_{\Omega} \theta^k} \right) \quad (5.47)$$

becomes smaller than a preset threshold (here the index i refers to the cell number). In general, about 100 iterations are required to reach a criterion of 10^{-5} . However, when plotting the density map, a “visual” convergence is

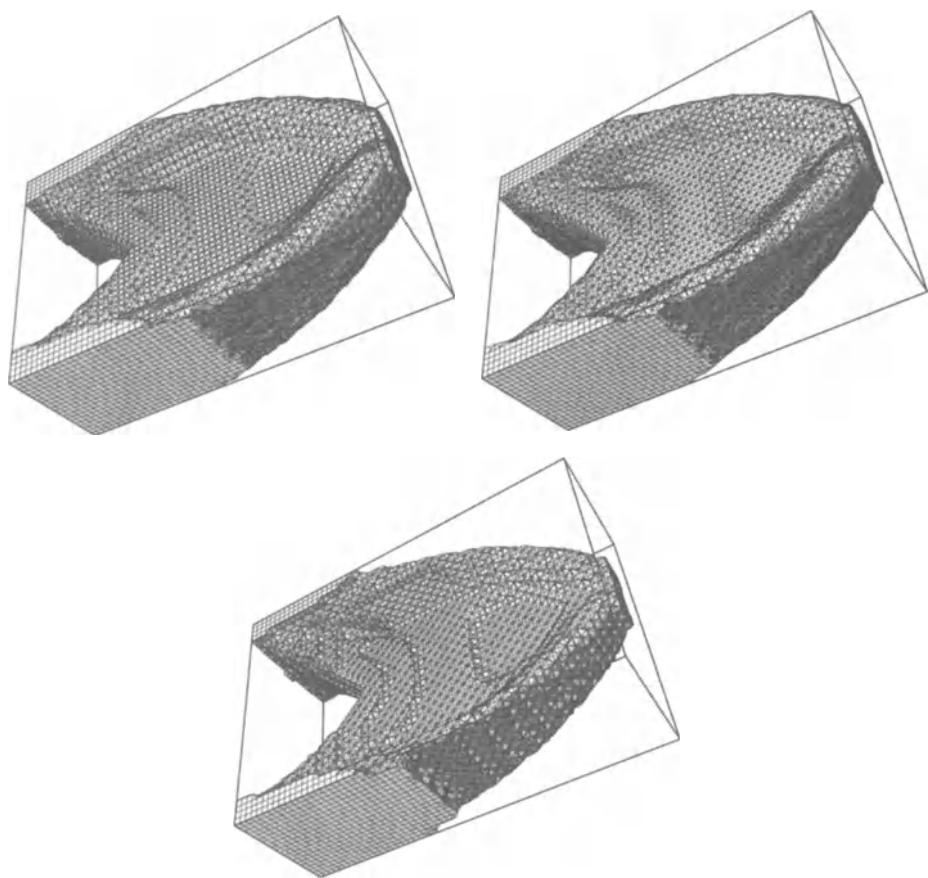


Figure 5.13: Optimal penalized design of the 3-D cantilever: isosurfaces of the material density 0.3 (left), 0.5 (right), 0.9 (bottom).

reached very quickly in about 20 iterations on average. Other convergence criteria could be used, for instance, the L^2 norm of $\tau^{k+1} - \tau^k$. For the cantilever problem, Figures 5.16 shows the design for various iteration numbers (to be compared with Figure 5.8 for 100 iterations).

Volume constraint. Our formulation of the shape optimization problem has always involved a fixed Lagrange multiplier ℓ for the volume constraint. The short cantilever example (see Figure 5.6) has been computed likewise with a fixed value of ℓ in order to check the convergence properties of the algorithm. However, in real practice one would rather like to work with a

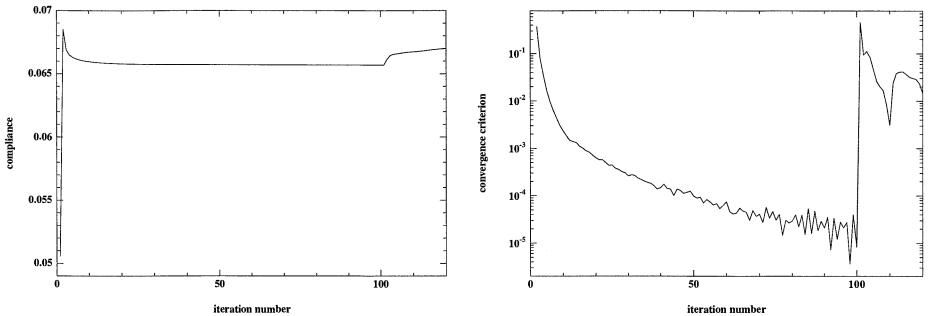


Figure 5.14: Convergence history of the 3-D cantilever: objective function (left) and convergence criterion (right).

fixed value of the volume constraint (or, similarly, with a fixed value of the compliance, thus minimizing the weight). Although, as a matter of theory, we do not know if for each volume constraint Θ there exists a Lagrange multiplier ℓ , it is proved in Lemma 4.2.11 that, at least, the total volume Θ varies monotonically with ℓ . Therefore, it is possible during the numerical computations to adjust the value of ℓ at each iteration, so that the corresponding value of the optimal density satisfies the volume constraint. All other computations have been done this way. The following updating procedure works very well: Once the stress τ_{k-1} is computed, we determine θ_k and ℓ_k by solving

$$\theta_k = \max \left(0, 1 - \sqrt{\frac{g^*(\tau_{k-1})}{-\ell_k}} \right), \quad \int_{\Omega} \theta_k = \Theta$$

through a simple iterative process. Recall from Lemma 4.2.11 that $\Theta(\ell)$ is a monotone function of ℓ , and thus a simple dichotomy allows one to find the correct value of ℓ_k . We remark that, if the proposed initialization procedure is used ($\theta_0 = 0$ and $A_0^* = B$ everywhere), the volume constraint is enforced only after the first iteration. However, a simple change in the initialization allows one to satisfy the volume constraint from the start. For example, we take the holes' proportion $\theta_0 = \Theta/|\Omega|$ constant in Ω . Concerning A_0^* , either we take it to be a rank- N sequential laminate with directions given by the canonical basis and parameters $m_i = 1/N$, $1 \leq i \leq N$ (but it is not an isotropic Hooke's law, and it therefore induces privileged directions in the design), or we choose the explicit isotropic sequential laminate, which achieves the Hashin-Shtrikman bounds in Theorem 2.3.13.

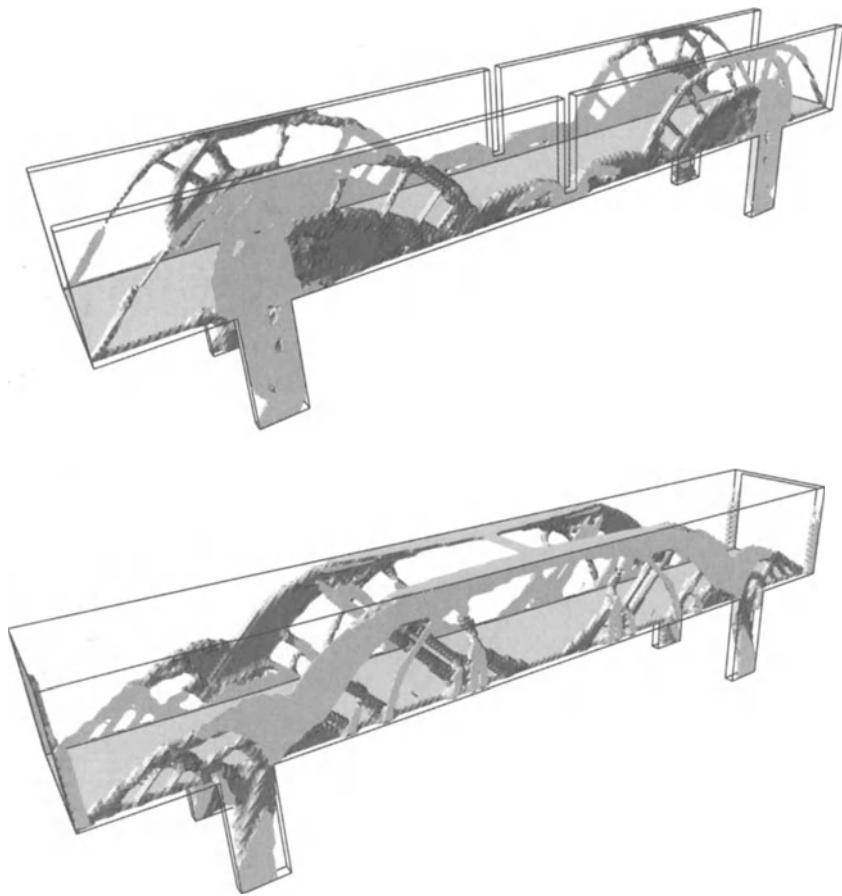


Figure 5.15: Optimal penalized design of the 3-D bridge for two different working domains.

optimal microstructure when $\det \tau \leq 0$; see Theorem 2.3.44). The singularity is avoided by adding a small correction term to the composite Hooke's law.

We describe three possible regularizations of rank-2 laminates in 2-D proposed in [15]. Up to a rotation, we assume that the principal directions of τ coincide with the canonical basis of the plane. The elasticity tensor of the corresponding rank-2 laminate only has the following nonzero coefficients

$$\begin{aligned} A_{1111}^* &= \frac{4\kappa\mu(\kappa + \mu)(1 - \theta)(m_1 + \theta m_2)m_2}{4\kappa\mu m_1 m_2(1 - \theta)^2 - (\kappa + \mu)^2\theta} \\ A_{1122}^* = A_{2211}^* &= \frac{4\kappa\mu(\kappa - \mu)(1 - \theta)^2 m_1 m_2}{4\kappa\mu m_1 m_2(1 - \theta)^2 - (\kappa + \mu)^2\theta} \\ A_{2222}^* &= \frac{4\kappa\mu(\kappa + \mu)(1 - \theta)(m_2 + \theta m_1)m_1}{4\kappa\mu m_1 m_2(1 - \theta)^2 - (\kappa + \mu)^2\theta}, \end{aligned}$$

while $A_{1212}^* = A_{1221}^* = A_{2112}^* = A_{2121}^* = 0$.

The first method of correction simply amounts to replacing $A_{1212}^* = A_{1221}^* = A_{2112}^* = A_{2121}^*$ by $2\mu\epsilon$, with a small value of ϵ (but not too small). The second method consists in replacing A^* by the optimal laminate corresponding to a two-phase mixture of A and B with $A = \epsilon B$, i.e., by a configuration that achieves the minimum of the Hashin-Shtrikman bound $g_{\epsilon B}(\tau, \theta)$, defined in (4.150). The soft phase A is chosen proportional to B , which simplifies the computations of the optimal Hooke's law that is now nonsingular. The third regularization corresponds to the two-dimensional projection (along e_3) of a rank-3 laminate, with laminations (e_1, e_2, e_3) (the canonical basis of \mathbb{R}^3) in proportions $(1-\epsilon)m_1, (1-\epsilon)m_2$, and ϵ , respectively. The resulting elasticity tensors are easily computed with the formulas from Subsection 5.2.1.

Computational runs suggest that the three corrections give comparable results, although the second correction is slightly better and converges faster. Numerical experiments show that, when the coefficient ϵ is too small, the algorithm might select a wrong solution, possibly a local minimum. In practice we use $\epsilon = 10^{-2}$. We remark that, upon convergence, the stress is aligned with the microstructure and therefore the value of the correction for A_{1212}^* , A_{1221}^* , A_{2112}^* , A_{2121}^* is irrelevant.

Checkerboard instabilities. The alternate directions algorithm, like any other algorithm based on the homogenization method, is subject to so-called

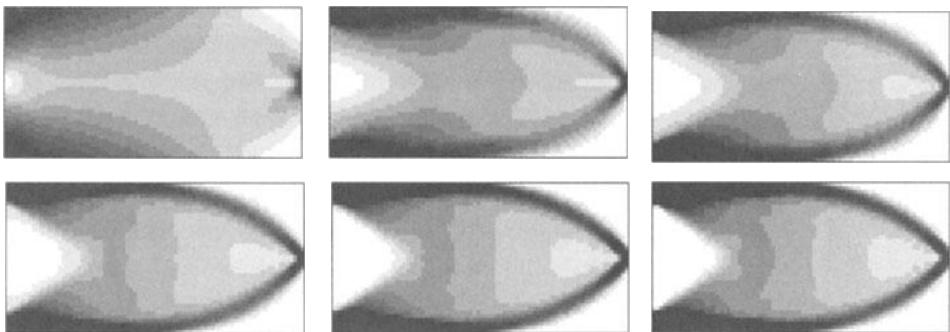


Figure 5.16: Composite designs after 1, 10, 20, 30, 40, and 50 iterations (from left to right and top to bottom).

Singularities in the composite Hooke's law. The homogenized Hooke's laws computed at each iteration may become singular, an undesired feature when solving problems of linear elasticity. This degeneracy has several sources.

First, we note that the effective tensor is equal to zero when the material density vanishes, i.e., when $\theta = 1$. Implicitly, the corresponding stress field should vanish simultaneously. This problem, which occurs in 2-D and 3-D, is easily circumvented by imposing a positive threshold on the material density. In practice, the largest admissible value of θ is fixed at $(1 - \epsilon)$ with $\epsilon = 10^{-3}$. Numerical experiments suggest that the choice of $\epsilon = 10^{-3}$ is not important: Any other value between 10^{-2} and 10^{-6} works equally well. Note, however, that, if the volume constraint Θ is too close to the volume of Ω (i.e., we are working with a very small volume fraction of material), then the threshold ϵ may be not small enough to meet this volume constraint, and in this case it is better to take a normalized threshold $\epsilon = 10^{-3}(|\Omega| - |\Theta|)/|\Omega|$. Of course, the smaller ϵ is, the more singular the finite element matrix turns out to be. One cannot decrease too much the value of ϵ , especially if an iterative solver is used for inverting the linear system issued from the finite element method.

We also remark that rank-1 and rank-2 laminates produce degenerate Hooke's laws. In 3-D, the proportions m_i are forced to be greater than a positive threshold ϵ (typically $\epsilon = 10^{-2}$). Consequently, the algorithm only uses rank-3 laminates, which are nonsingular (see Remark 2.3.42). In 2-D, rank-1 laminates are eliminated, as in the 3-D case. However, the algorithm uses rank-2 laminates as optimal microstructures, which are always degenerate according to Remark 2.3.42 (a feature which is shared by any other

checkerboard instabilities for the density θ . This phenomenon has been reported by many authors (see, e.g., [15], [44], [140], [141], [142]). It appears in 2-D computations (it is more infrequent in 3-D) when using quadrangular $Q1$ finite elements for the displacement and piecewise constant elements for the density. In regions where the density should be almost uniform, the algorithm instead exhibits checkerboard patterns of the density. For example, Figure 5.17 shows the same cantilever problem as in Figure 5.8 (after 100 and 200 iterations, respectively) when no special care is taken to eliminate these checkerboard patterns.

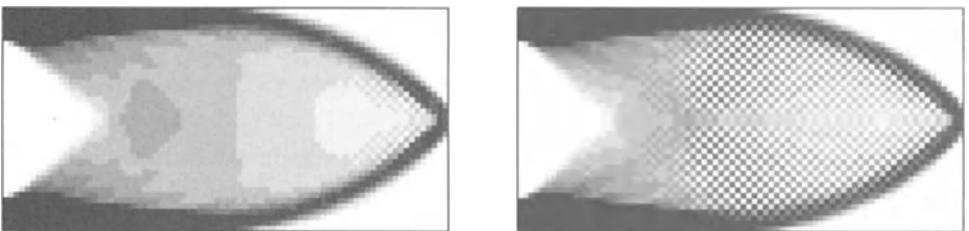


Figure 5.17: Without checkerboard control: optimal composite shape after 100 iterations (left) and 200 (right).

These instabilities never appear when using higher order elements. For example, they are not observed with the stress formulation of [21], or in a primal setting if the displacements are computed with quadrangular $Q2$ elements, while the design parameters are still computed with only piecewise elements. Note, however, that $Q2$ elements are expensive for very fine meshes and for 3-D calculations.

Such checkerboard instabilities are reminiscent of pressure instabilities in a Stokes flow computation when the discrete finite elements spaces for the velocity and the pressure do not satisfy the required inf-sup condition (see, e.g., [59]). Their numerical onset is still mysterious, although, in practice, such instabilities only appear after a large number of iterations, when the convergence criterion is very tight. The reason why checkerboard patterns are more favorable than uniform density regions is however clear: For some values of a uniform plane stress, a checkerboard-like patch of four $Q1$ finite elements has lower energy than a uniform density patch with the same overall volume fraction. A thorough investigation of this phenomenon was done in [99], [140], and [251], to which we refer the reader for further details.

An inexpensive way of eliminating these instabilities in 2-D calculations is to use the same method of pressure filtering in Stokes equations for the den-

sity [59]. Once the piecewise constant optimal densities θ_i^k are determined, we project them onto superelements, which are clusters of four adjacent elements, so as to eliminate the checkerboard mode and preserve the overall density. Unless otherwise mentioned, all the presented 2-D computations have been performed this way. Such a correction is not necessary in practice for 3-D calculations.

Convergence under mesh refinement. The classical methods of shape optimization, based on boundary motion, are well known to produce different solutions for different mesh sizes (see, e.g., the seminal work of Cheng and Olhoff [73]). The finer the mesh, the more oscillations appear in the optimal shape boundary (at least, if the number of control nodes increase with the inverse of the mesh size). In other words, classical algorithms usually do not converge under mesh refinement. This is the numerical manifestation of the mathematical nonexistence of solutions for the original formulation of shape optimization. On the contrary, the homogenization-based algorithms for computing relaxed optimal shape converge under mesh refinement. This is the case with our alternate directions algorithm. This numerical good behavior corresponds to the mathematical fact that relaxation smooths out oscillations in the original objective functional.

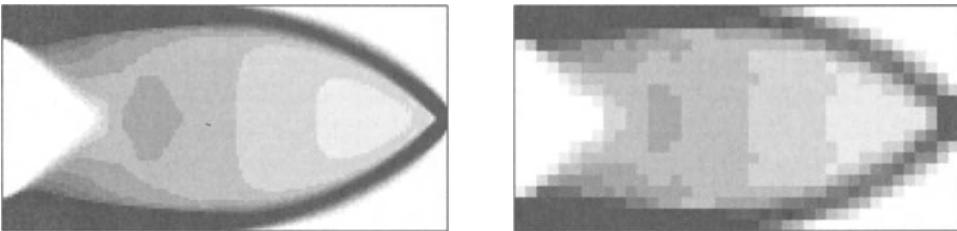


Figure 5.18: Cantilever problem: 160×80 mesh (left) and 40×20 mesh (right).

In Figure 5.18 we display the optimal composite shape for the same cantilever problem as in Figure 5.8, but with a mesh having twice as many or half as many elements in each direction. One can clearly check the convergence of the optimal material density under mesh refinement.

Insensitivity to the initialization. As described in Subsection 5.2.1, the alternate directions algorithm produces an optimal relaxed shape that

depends, *a priori*, on the initialization procedure (i.e., the initial guess). The dependence of the solution on this initial guess is a very important issue. As already stated, the relaxed problem, although well-posed, is not convex (but quasiconvex, see Theorem 4.2.3), and therefore any numerical algorithm may fall into local minimizers (if any) instead of global ones. The theory is unfortunately helpless in the matter. However, numerically it is possible to check whether an iterative algorithm converges to local minimizers by simply running the same computation with several different initial guesses. We have performed such computational experiments, which all yield the same conclusion: The computed relaxed optimal shape is insensitive to the initial guess (even if we try to lure the algorithm by proposing as an initial guess a reasonable, close to optimal, shape). As an example, we start the cantilever problem from the penalized design displayed in Figure 5.8, but the boundary conditions are reversed, as in a mirror (the right side is fixed now and the force is pulling on the middle of the left side). Figure 5.19 shows the evolution of the composite design as the number of iterations increases (it still converges to the same composite shape). Of course, the number of iterations required for the shape to converge is highly dependent on the initial guess. Thus, initialization calls for special care if good CPU time performances are to be achieved. This very good behavior of the numerical algorithm suggests that the relaxed problem (5.22), although not convex, admits only global minimizers (see Remarks 3.2.26 and 4.1.18).

5.2.4 Penalization of Intermediate Densities

As explained in Subsection 5.2.1, the proposed numerical algorithm for computing optimal design is based on the relaxed, or homogenized, formulation introduced in Chapter 4. The numerical computations deliver optimal relaxed, or generalized, or composite, shapes (i.e., a density of material) rather than optimal classical shapes for the original formulation (i.e., a characteristic function of the material domain). In other words, the homogenization method produces a layout of material, which, as expected, includes large regions of composite materials with intermediate density. From a practical standpoint, this is an undesirable feature since the primary goal is to find a real shape, i.e., a density taking only the values zero or one! This drawback is avoided through a post-processing technique that “penalizes” composite regions. This penalization technique has been introduced by many authors including [15], [17], [23], [41], [296]. The goal is to deduce, from the optimal densities, a quasioptimal shape. In loose terms, the solution of the relaxed

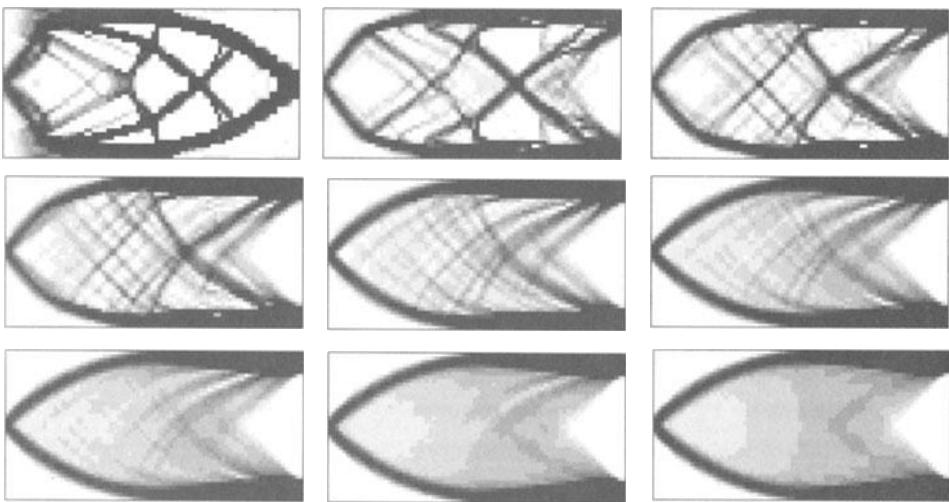


Figure 5.19: Designs after 1, 10, 30, 50, 100, 150, 200, 300, and 500 iterations (from left to right and top to bottom).

problem is projected onto the set of classical solutions of the original problem, in the hope that the value of the objective functional will not increase too much in the process.

The strategy is as follows. Upon convergence to an optimal density, we run a few more iterations of the alternate directions algorithm, where we force the density to take values close to zero or one. This changes the optimal density and produces a quasi-optimal shape. Of course, the procedure is purely numerical and mesh dependent. The finer the mesh, the more detailed the resulting structure will appear at the outset of the penalization process. The method works well because the relaxed design is characterized not only by a density θ but also by a microstructure A^* , which is hidden at the submesh level. The penalization tends to reproduce the microstructure at the mesh level. Eventually, the complete algorithm of shape optimization is:

1. Computation of a relaxed or homogenized optimal shape by the alternate directions algorithm.
2. Derivation of a quasioptimal classical shape, close to the previous homogenized optimal shape, by the following iterative penalization technique:
 - (a) Initialization with the optimal homogenized design parameters

$$(\theta, A^*),$$

- (b) Computation of a few more iterations of the alternate directions algorithm where the design variables (θ_k, A_k^*) are updated using penalized formulas instead of the true optimality formulas.

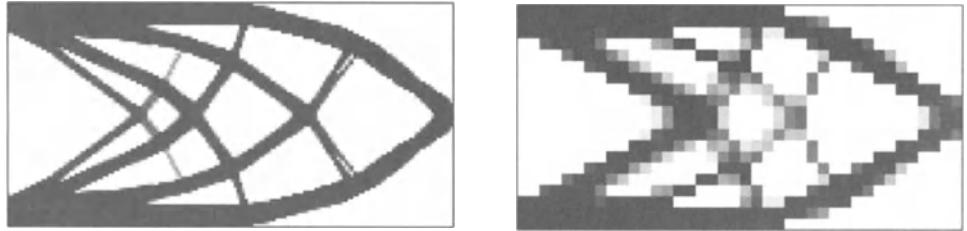


Figure 5.20: Cantilever problem: 160×80 mesh (left) and 40×20 mesh (right).

In Figure 5.20 we display the penalized shape for the same cantilever problem as in Figure 5.8, but with a mesh with twice as many or half as many elements in each direction. It is clear on these pictures that penalization is mesh-dependent.

Two penalization techniques for the intermediate composite densities are of current use. Both amount to a modification of the explicit formula (5.43) that expresses the optimal density in terms of the stress (the optimality formula for A^* remains unchanged). Specifically, instead of updating the density with the true optimal density θ_{opt} , a value θ_{pen} is used. A first choice for θ_{pen} is

$$\theta_{pen} = \frac{1 - \cos(\pi\theta_{opt})}{2}.$$

It implies that, if $0 < \theta_{opt} < 1/2$, then $\theta_{pen} < \theta_{opt}$, while, if $1/2 < \theta_{opt} < 1$, then $\theta_{pen} > \theta_{opt}$. Thus, θ_{pen} is closer to 0 or 1 than θ_{opt} . The choice of a cosine function for the penalized density is arbitrary. Any other function having the same properties would do the job. We remark however that, if θ_{pen} remains too close to θ_{opt} , the scheme is insensitive to the proposed penalization, while if θ_{pen} is forced too close to zero or one, the penalization procedure is brutal and destroys the fine patterns of the shape.

Originally proposed in the context of plate thickness optimization, a

second technique consists in setting

$$1 - \theta_{pen} = \left(\frac{(1 - \theta_{opt})^2}{p} \right)^{1/(1+p)} \quad \text{with } 0 < p \leq 1.$$

This alternate choice also gives good results. It corresponds to the optimal value of θ for a modified integrand, namely

$$PF(\tau) = \min_{0 \leq \theta \leq 1} (g(\tau, \theta) - \ell(1 - \theta)^p), \quad (5.48)$$

which is supposed to take into account “manufacturing costs” of perforated materials (the “cost” of intermediate densities increases in (5.48) as p decreases from one to zero).

Remark 5.2.5 *A different approach for penalizing composites is the so-called perimeter method, introduced in [128], and analyzed in [219]. It combines the above pointwise penalization technique with the introduction of a new term in the objective function that models the perimeter (in 2-D) of the shape. The perimeter is taken to be the total variation of the density function θ (as in [26]). Minimizing the perimeter also has the effect that, upon a careful choice of the different numerical parameters, the penalization procedure becomes mesh independent. In other words, the classical shape produced by this coupled penalization procedure does not depend on the mesh size.*

The convergence criterion of the penalization procedure is based on the convergence of the density function θ_{pen} to a characteristic function taking only the values zero and one. It has, of course, nothing to do with the stationarity of the objective function which slightly increases during the process. Usually, on the order of 10 penalized iterations are necessary to get a classical shape out of an optimal composite design. For the cantilever problem, Figure 5.21 shows intermediate designs during the penalization process (to be compared with Figure 5.8, where the penalized shape is obtained after 20 iterations).

The success of the penalization technique can be explained by recalling property 3 of Theorem 4.2.3. It says that any minimizer of the relaxed formulation is attained by a minimizing sequence of classical shapes. In other words, an optimal generalized design is close to almost optimal classical designs. Penalization is just one way of constructing a nearby classical shape. This is confirmed by the relatively small increase of the objective function (on the order of 5% in average) upon penalization.

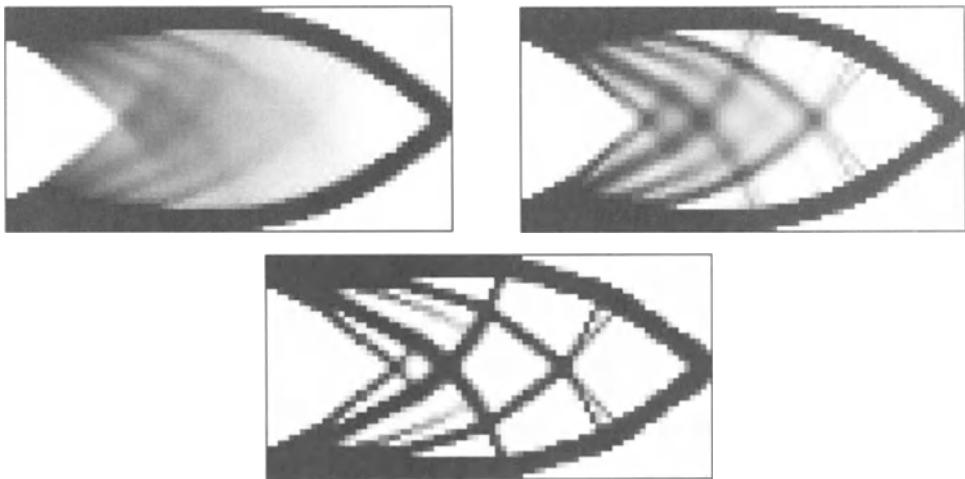


Figure 5.21: Penalized designs after 5 (left), 10 (right), or 15 iterations (bottom).

We remark that it is important to start the penalization process from the previously obtained optimal relaxed design. Indeed, if the penalization procedure is applied too early, it may slow down the convergence toward a global minimum, and numerical practice shows that the algorithm often falls into a local minimizer.

5.2.5 Quasiconvexification versus Convexification

This subsection is devoted to a comparison of the relaxed or homogenized formulation introduced in Chapter 4 (which corresponds to the quasiconvexification of the original stress formulation; see Theorem 4.2.3) with the convexification of the original problem. Such a comparative study is motivated by the occasional use of the convexified formulation for the computation of optimal shapes, under the name of “fictitious material approach” (see, e.g., [8], [41], [196], [241]). This method was pioneered in the work of [69], [70].

Let us briefly describe the argument. When combined with the penalization procedure, the numerical algorithms for computing optimal shapes (based on the relaxed formulation) may seem self-defeating. Homogenization theory is introduced, proper optimal microstructures and complicated formulas for updating the design variables are laboriously derived, and yet, in the end, this wealth of information is seemingly wasted through the penal-

ization process! A natural idea is thus to propose a simpler approach based on the convexification of the original problem (which is easily computed; see below), coupled with the same penalization procedure as described in Sub-section 5.2.4. The advantages of this approach are the following: The layout optimization problem still becomes a sizing optimization problem (shapes are replaced by densities); existence of global minimizers is guaranteed by the convexity of the functional; implementation is straightforward since the convexified formulation is very simple; no knowledge of homogenization or composite materials is required; and, in 2-D, the method relies on the physical notion of “variable thickness” plates. The drawbacks are the lack of connection with the original formulation (a convexified design is not the limit of a sequence of classical designs), an outcome that yields worse numerical results. The convexification method is much more sensitive to the penalization and, although comparable results are produced in a few test cases, it usually produces worse designs than those of the homogenization (or relaxed) method. Therefore, we do not recommend this technique except if reasonable (although not optimal) results are required in a very short time.

There are two points of view for deriving this convexified formulation. The first one starts from the original formulation (4.139) in terms of characteristic functions, namely,

$$\inf_{\chi \in L^\infty(\Omega; \{0,1\})} \left\{ J(\chi) = c(\chi) + \ell \int_\Omega \chi(x) dx \right\},$$

with $c(\chi)$ the compliance defined by (4.136). The space $L^\infty(\Omega; \{0,1\})$ is not convex, and we simply replace it by its convex hull $L^\infty(\Omega; [0, 1])$. We thus obtain the convexified formulation

$$\min_{\theta \in L^\infty(\Omega; [0, 1])} \left\{ J(\theta) = c(\theta) + \ell \int_\Omega \theta(x) dx \right\}, \quad (5.49)$$

with the convexified compliance

$$c(\theta) = \min_{\tau \in \Sigma(\Omega)} \int_\Omega ((1 - \theta(x))B)^{-1} \tau : \tau dx, \quad (5.50)$$

where $\Sigma(\Omega)$ is the space of statically admissible stresses defined by (4.137). Equivalently, one can begin with the original stress formulation (4.140)

$$\inf_{\tau \in \Sigma(\Omega)} \int_\Omega F(\tau) dx,$$

where

$$F(\tau) = \begin{cases} \ell & \text{if } \tau = 0 \\ B^{-1}\tau : \tau & \text{if } \tau \neq 0. \end{cases}$$

The space $\Sigma(\Omega)$ is affine and thus convex, but $F(\tau)$ is not convex (recall that $\ell \leq 0$). Therefore, according to, e.g., [102], the convexification of (4.140) is

$$\min_{\tau \in \Sigma(\Omega)} \int_{\Omega} CF(\tau) dx, \quad (5.51)$$

where CF is the convex envelope of the function F (i.e., the largest convex function below F), which is easily computed as

$$CF(\tau) = \begin{cases} B^{-1}\tau : \tau & \text{if } B^{-1}\tau : \tau \geq -\ell \\ 2\sqrt{-\ell B^{-1}\tau : \tau} + \ell & \text{if } B^{-1}\tau : \tau \leq -\ell. \end{cases} \quad (5.52)$$

Of course, (5.49) and (5.51) are equivalent, as is readily checked by inverting the minimization in θ and τ .

The second approach for deriving (5.49) and (5.51) follows the point of view of the fictitious material method. The starting point is the state equation

$$\begin{cases} \sigma = (1 - \theta(x))Be(u) & \text{with } e(u) = (\nabla u + (\nabla u)^t)/2 \\ \operatorname{div} \sigma = 0 & \text{in } \Omega \\ \sigma n = f & \text{on } \partial\Omega, \end{cases} \quad (5.53)$$

where $\theta(x)$ is a density function in $L^\infty(\Omega; [0, 1])$. This is clearly an extension of the original state equation (4.131) since, if θ takes only the values zero and one, we recover the usual elasticity equations in the domain $\{x \in \Omega \mid \theta(x) = 0\}$. However, for a general density θ , the Hooke's law $(1-\theta)B$ corresponds to a fictitious material that is not a composite material obtained by perforating microscopically the pure phase B with holes. As soon as θ is uniformly bounded away from one (i.e., there are no holes), there exists a unique solution (up to rigid motions) of (5.53). For a general $\theta \in L^\infty(\Omega; [0, 1])$ which may be equal to one in some places (corresponding to holes), (5.53) may be not a well-posed problem, but we can still defined a compliance $c(\theta)$ by formula (5.50) (see Remark 4.2.1). The original objective function (4.139) is extended in a natural way: The goal is to minimize, over all possible densities, the weighted sum of the compliance and of the holes' volume, which is nothing other than (5.49).

In 2-D, a justification of the fictitious material approach is possible. It amounts to viewing (5.53) as a variable thickness plate problem, where $(1-\theta)$

is the relative thickness of the plate (obtained by integration along the plate normal direction). In other words, (5.53) derives from a 3-D problem for a very thin body whose midsurface is planar and which is subjected to plane loads only. It turns out to be a correct model if relaxation is first applied to the 3-D domain and then reduction to 2-D is performed. This is the outcome of the following lemma, which claims that, in a plane stress situation, the three-dimensional relaxed formulation coincides with the two-dimensional convexified formulation.

Lemma 5.2.6 *In three dimensions, a plane stress τ with eigenvalues denoted by τ_1, τ_2, τ_3 , such that*

$$\tau_1 = 0, \quad \tau_2 \neq 0, \quad \text{and} \quad \tau_3 \neq 0,$$

is considered. Then, the convex envelope of F coincides with its quasiconvex envelope at τ

$$QF(\tau) = CF(\tau),$$

and the 2-D convexified formulation (5.51) is the restriction to plane stress of the 3-D true relaxed formulation (5.22).

Proof. Recall from (4.152) that

$$QF(\tau) = \min_{0 \leq \theta \leq 1} (g(\tau, \theta) + \ell\theta),$$

and that, for a plane stress τ , Lemma 2.3.39 gives the value of the function $g(\tau, \theta)$, i.e.,

$$g(\tau, \theta) = B^{-1}\tau : \tau + \frac{\theta}{(1 - \theta)}g^*(\tau)$$

with

$$g^*(\tau) = \frac{1}{2\mu} (\tau_2^2 + \tau_3^2) - \frac{\lambda}{2\mu(2\mu + 3\lambda)} (\tau_2 + \tau_3)^2.$$

Since $\tau_1 = 0$, a simple computation shows that

$$g(\tau, \theta) = \frac{1}{1 - \theta} B^{-1}\tau : \tau,$$

and minimizing in θ implies that the integrand QF coincides with the convex envelope CF of the original integrand F when evaluated at τ . \square

Remark 5.2.7 Lemma 5.2.6 states that the 2-D convexified formulation is the same as the 3-D “in-plane” relaxed formulation, but of course it is different from the 2-D relaxed formulation. From a practical standpoint, it means that, if we can use 3-D microstructures for solving a 2-D problem (which seems like cheating), then this is preferable to using a “varying thickness plate” approach (corresponding to an optimal rank-1 laminate in the direction normal to the plate; see Lemma 2.3.39), rather than a “plane stress” relaxed approach (for which rank-2 laminates are optimal).

Due to the convex character of the integrand in (5.51), it is fairly easy to prove the following result.

Theorem 5.2.8 *There exists at least one solution of the convexified formulation (5.49) and (5.51), and all local minimizers are actually global minimizers.*

As is well known (see, e.g., [89]) the convex envelope is below the quasiconvex envelope, therefore

$$F(\tau) \geq QF(\tau) \geq CF(\tau),$$

where the inequalities are strict for most choices of the stress τ in view of the explicit formulas for QF , (5.23), and for CF , (5.52). If we denote by I the infimum of the original formulation,

$$I = \inf_{\tau \in \Sigma(\Omega)} \int_{\Omega} F(\tau) dx,$$

by QI the minimum of the relaxed (or homogenized) formulation,

$$QI = \min_{\tau \in \Sigma(\Omega)} \int_{\Omega} QF(\tau) dx,$$

and by CI the minimum of the convexified formulation,

$$CI = \min_{\tau \in \Sigma(\Omega)} \int_{\Omega} CF(\tau) dx,$$

we have

$$I = QI \geq CI,$$

where the last inequality is strict for most choices of the loading conditions (for explicit examples, see, e.g., [18], [25]). This last (strict) inequality implies that solving the convexified problem has nothing to do with solving the original problem. In particular, minimizers of CI are usually not attained by minimizing sequences of I , contrary to QI (see Theorem 4.2.3).

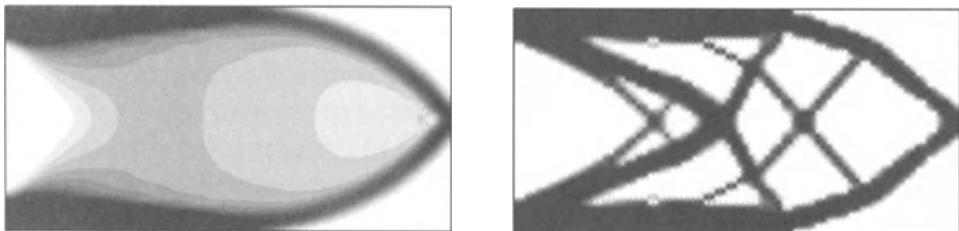


Figure 5.22: Convexification method for the cantilever: optimal fictitious shape (left) and after penalization (right).

Remark 5.2.9 *The mathematical fact that $QF(\tau) \geq CF(\tau)$ is equivalent to saying that the class of fictitious materials $(1 - \theta)B$ is much larger than that of composite materials A^* . But, of course, only the latter ones have a physical meaning. This has drastic consequences on the resulting numerically computed optimal shapes. Indeed, since a fictitious material $(1 - \theta)B$ is not the limit of a perforated design, there is usually no classical design that is energetically close to a minimizer of the convexified formulation (contrary to what happens for relaxed minimizers). It is therefore expected that a numerical penalization technique, when applied to fictitious materials, will produce classical shapes with much worse performances than in the case of homogenized composite designs.*

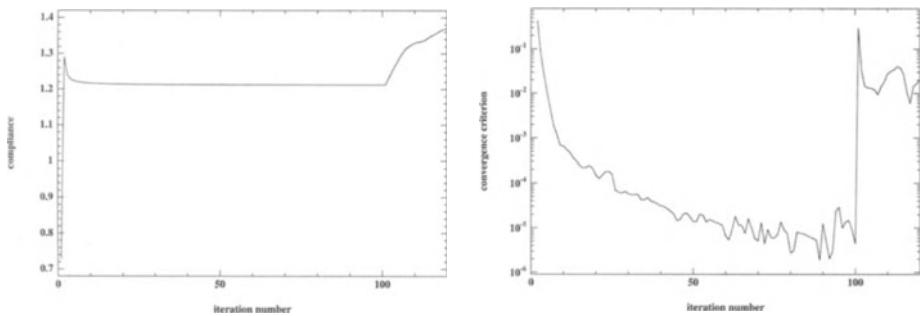


Figure 5.23: Convergence history of the convexification method for the cantilever: objective function (left) and convergence criterion (right).

We have numerically implemented the convex formulation with the alternate directions strategy described in Subsection 5.2.1. For a given density θ , we compute the stress τ solution to the linear elasticity state equation,

then update the design variable θ using the optimality relation

$$\theta = \begin{cases} 0 & \text{if } B^{-1}\tau \cdot \tau \geq -\ell \\ 1 - \sqrt{\frac{B^{-1}\tau : \tau}{\ell}} & \text{if } B^{-1}\tau \cdot \tau \leq -\ell. \end{cases}$$

The results are displayed on Figure 5.22 for the same cantilever problem as in Figure 5.8. The algorithm converges quickly and smoothly (see Figure 5.23), and we supplement it with the penalization procedure of Subsection 5.2.4. For the cantilever problem, the fictitious penalized design is qualitatively comparable to its homogenized counterpart (see Figure 5.8). However, for the bridge problem, the differences are much more sensitive (compare Figures 5.24 and 5.11). In general, the fictitious penalized designs lack the complexity and pattern details of the homogenized ones. We numerically checked this fact for all examples $QI > CI$, but, after penalization, the inequality is always reversed, i.e., $QI_{pen} < CI_{pen}$ (see Figure 5.25 where the objective function, the compliance, is plotted for the two methods), a situation that is consistent with our intuition in Remark 5.2.9.

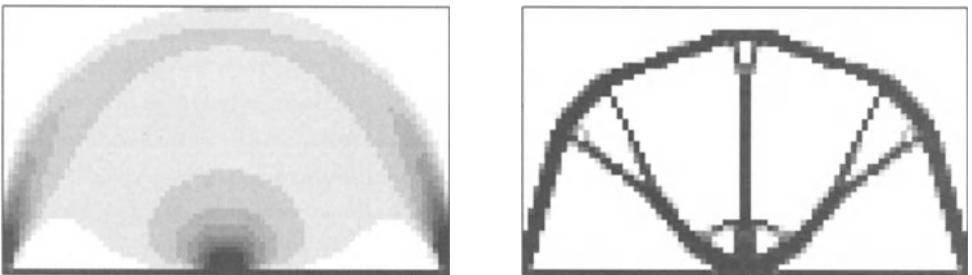


Figure 5.24: Optimal shape of the bridge: convexified (left) and penalized (right).

The absence of implicit submesh microstructures explains the lower performance of the fictitious material approach. Loosely speaking, the convex formulation has a single free design parameter (the density θ) while the homogenized, or relaxed, formulation has more parameters (θ and the microstructure A^*), allowing for greater flexibility in the design of optimal shapes. In other words, while the minimizers of the relaxed formulation are close to classical shapes (by virtue of property 3 of Theorem 4.2.3), this is not the case for the minimizers of the convexified formulation.

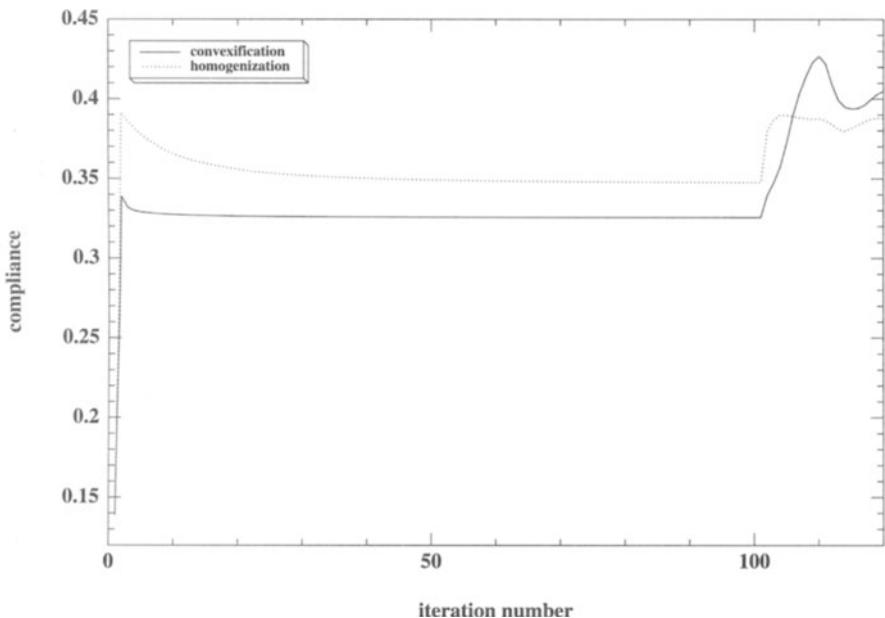


Figure 5.25: Convergence history of the objective function for the bridge problem.

Remark 5.2.10 We used so far fictitious material of the type $(1 - \theta)B$, which corresponds to the convexified formulation of shape optimization. Some authors [42], [249], [296] propose to use other fictitious materials like $(1 - \theta)^p B$ with $p \geq 1$ (typically $p = 3$ or $p = 5$). This method is often called SIMP (simplified isotropic material with penalization). The advantage of materials in this class is that they are self-penalizing, i.e., the resulting designs are free of composite zones and no penalization procedure is required. However, the designs often converge to local minima, except if some tricky filtering technique is used [250]. Of course, the SIMP method for shape optimization has nothing to do with either convexification or quasiconvexification methods.

Remark 5.2.11 If one insists in using, instead of the relaxed formulation (5.22), an approximate formulation where there is only one design parameter, namely the holes' density θ , then we propose to use a so-called “isotropic composite” formulation, rather than the convexification (5.51). This approach (studied in [30], [49]) is based on the isotropic composite that achieves simultaneously the optimal upper Hashin-Shtrikman bounds for the bulk and shear moduli. According to Theorem 2.3.13, such a composite, which can be obtained by finite rank sequential lamination, is defined by

$$A^* = 2\mu_* I_4 + \left(\kappa_* - \frac{2\mu_*}{N} \right) I_2 \otimes I_2 \quad (5.54)$$

with

$$\frac{\theta}{\kappa - \kappa_*} = \frac{1}{\kappa} + \frac{1 - \theta}{2\mu + \lambda},$$

and

$$\frac{\theta}{2(\mu - \mu_*)} = \frac{1}{2\mu} + \frac{(1 - \theta)(N - 1)(\kappa + 2\mu)}{(N^2 + N - 2)\mu(2\mu + \lambda)}.$$

Such a tensor A^* , which depends only on θ , is the most rigid isotropic composite obtained by homogenization of B and holes in proportions $(1 - \theta)$ and θ . The proposed isotropic composite formulation is

$$\inf_{\theta \in L^\infty(\Omega; [0,1])} \left\{ J(\theta) = c(\theta) + \ell \int_{\Omega} \theta(x) dx \right\}, \quad (5.55)$$

where $c(\theta)$ is the compliance of A^* , defined by (5.54), i.e.,

$$c(\theta) = \min_{\tau \in \Sigma(\Omega)} \int_{\Omega} (A^*)^{-1} \tau : \tau dx,$$

with $\Sigma(\Omega)$ the space of statically admissible stresses. Formulation (5.55) is a so-called partial relaxation of the original problem (see Subsection 5.2.8 for more details). It does not necessarily admit a minimizer, but, at least, any such isotropic design (θ, A^*) is attained by a limit of classical designs. Therefore, the penalization process for (5.55) will be much more satisfactory than for the convexified formulation. On the other hand, the isotropic composite formulation (5.55) retains the simplicity of the convexified formulation since it involves the density as single design parameter.

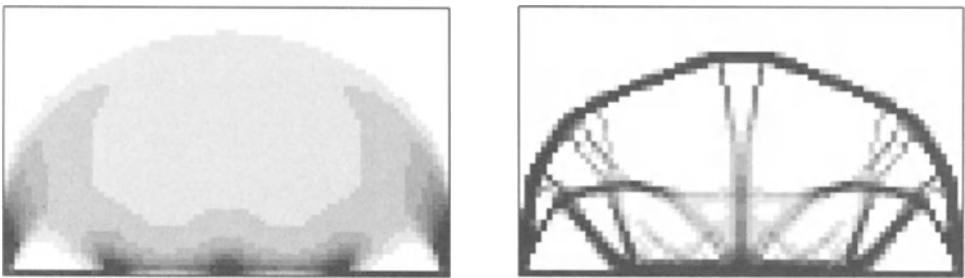


Figure 5.26: Multiple loads bridge: convexified (left) and penalized shape (right).

Remark 5.2.12 The fictitious material approach can also be used in the framework of multiple loads optimization (see Subsection 5.2.6). Its implementation is very easy, since there is a single design parameter θ which is optimized for several loading configurations as described in Subsection 5.2.6. However, the approach yields even poorer results in this case than the homogenization method. For example, the convexified solution of the multiple loads bridge problem is shown in Figure 5.26. This result is worse than that shown in Figure 5.29, resulting from the homogenization method.

5.2.6 Multiple Loads Optimization

In this subsection, we extend the previous alternate directions algorithm from the single load to the multiple loads case. We follow our work [14], [30] (there are many other contributions on this problem; see, e.g., [42], [97], [171], and references therein). Recall first some notation of Chapter 4. For

n surface loads $(f_i)_{1 \leq i \leq n}$, the relaxed formulation is

$$\min_{\substack{\tau_i \in L^2(\Omega; \mathcal{M}_N^s) \\ \operatorname{div} \tau_i = 0 \text{ in } \Omega, \tau_i n = f_i \text{ on } \partial\Omega}} \int_{\Omega} \min_{0 \leq \theta \leq 1} \min_{A^* \in L_\theta^{0+}} \left(\sum_{i=1}^n A^{*-1} \tau_i : \tau_i + \ell \theta \right) dx \quad (5.56)$$

where L_θ^{0+} is the set of all sequential laminated composites of material B around a core of void, in proportions $(1 - \theta)$ and θ , respectively, defined by (5.25). Of course, in the above sum of compliances each term can be weighted by a constant nonnegative factor, which balances the importance of different loads. These factors can also be adjusted iteratively in order to minimize the maximal compliance (see the worst case design problem in Theorem 4.1.29).

As proved in Theorem 4.2.15, an optimal tensor can still be found in the class L_θ^{0+} of sequential laminates. The major difference between a single load and multiple ones is the absence of an explicit algebraic formula for computing the lamination variables (both directions and proportions) in the multiple case. Although we know from theoretical results (see Lemma 2.3.9) that any sequential laminate in L_θ^{0+} can be attained by no more than 3 laminations in 2-D, and 6 in 3-D, we are not able to compute explicitly the optimal directions and lamination parameters. Therefore, we have to perform numerically this computation using some minimization algorithm. We keep the general structure of the alternate directions algorithms of Subsection 5.2.1, but at each iteration, and for each cell of the mesh, the optimal lamination parameters are now found numerically instead of by an explicit formula.

To be more specific, we first recall the lamination formula (5.25),

$$\theta \left(A^{*-1} - B^{-1} \right)^{-1} = (1 - \theta) \sum_{j=1}^p m_j f_B^c(e_j), \quad (5.57)$$

where the lamination parameters $(m_j)_{1 \leq j \leq p}$ satisfy

$$0 \leq m_j \leq 1 \text{ and } \sum_{j=1}^p m_j = 1,$$

and f_B^c is defined by (5.26). Given n fixed stress tensors $(\tau_i)_{1 \leq i \leq n}$, we want to minimize over L_θ^{0+} the quantity

$$\sum_{i=1}^n A^{*-1} \tau_i : \tau_i. \quad (5.58)$$

A first possibility would be to fix the number of laminations in (5.57) to $p = 3$ in 2-D, or $p = 6$ in 3-D, and minimize (5.58) with respect to the lamination parameters $(m_j)_{1 \leq j \leq p}$ and the directions $(e_j)_{1 \leq j \leq p}$. Since a unit vector is characterized by one angle in 2-D and two in 3-D, this gives rise to 5 parameters in 2-D and 17 in 3-D (eliminating the constraint $\sum_{j=1}^p m_j = 1$). Although it is not a desperate optimization task, (5.58) as a function of the directions is expected to be nonconvex and even degenerate when two or more directions become parallel. On the other hand, for fixed directions $(e_j)_{1 \leq j \leq p}$, the function

$$(m_j)_{1 \leq j \leq p} \rightarrow \sum_{i=1}^n A^{*-1} \tau_i : \tau_i$$

is convex as is easily seen from (5.57). Therefore, it seems preferable to fix the directions and to minimize only with respect to the parameters (m_j) . However, to avoid a too fine discretization of the unit sphere S_{N-1} , i.e., a large number p of directions (which is necessary for good precision), we add another design parameter, which is a rotation of the whole microstructure (as already introduced in Subsection 5.1.2). Combining a moderate number of directions p (between 4 and 8) and a rotation (parametrized by one angle in 2-D and two in 3-D) turns out to be the more efficient and economical solution in numerical practice. In other words, we now assume that an homogenized Hooke's law is given by

$$\theta \left(A^{*-1} - B^{-1} \right)^{-1} = (1 - \theta) \sum_{j=1}^p m_j R^t f_B^c(e_j) R, \quad (5.59)$$

where R is the fourth order tensor corresponding to a rotation $Q \in \mathcal{M}_N$ in the physical space (with $Q^{-1} = Q^t$), i.e., for any symmetric matrix $\xi \in \mathcal{M}_N^s$

$$R\xi = Q^{-1}\xi Q.$$

We remark that formula (5.59) is simply obtained by rotation of (5.57).

Hence, in 2-D, we discretize the unit sphere uniformly by setting

$$e_j = \left(\cos \frac{j\pi}{p}, \sin \frac{j\pi}{p} \right), \quad 0 \leq j \leq p-1$$

and in 3-D, in a less uniform way (but this can be easily improved), with $p = p_1 p_2$,

$$e_j = \left(\cos \frac{j_1 \pi}{p_1} \cos \frac{j_2 \pi}{p_2}, \sin \frac{j_1 \pi}{p_1} \cos \frac{j_2 \pi}{p_2}, \sin \frac{j_2 \pi}{p_2} \right), \quad \begin{cases} 0 \leq j_1 \leq p_1 - 1 \\ 0 \leq j_2 \leq p_2 - 1. \end{cases}$$

Denoting by ϕ one angle in 2-D or two angles in 3-D that parametrize a rotation $Q(\phi)$ in the physical space, the resulting approximate optimization problem amounts to finding minimizers of

$$W(\theta, \phi, \{m_j\}) = \ell\theta + \sum_{i=1}^n \left(B^{-1}\tau_i : \tau_i + \frac{\theta}{1-\theta} \left(\sum_{j=1}^p m_j f_B^c(e_j) \right)^{-1} \eta_i : \eta_i \right), \quad (5.60)$$

with

$$\eta_i = Q^{-1}(\phi) \tau_i Q(\phi)$$

and the constraints $0 \leq \theta \leq 1$, $m_j \geq 0$ and $\sum_{j=1}^p m_j = 1$. Eventually, the alternate directions algorithm for the multiple loads case is structured as follows:

1. Initialization of the design parameters $(\theta_0, \phi_0, \{m_j^0\})$ (for example, taking $\theta_0 = 0$, $\phi_0 = 0$, and $m_j^0 = 1/p$ everywhere in the domain).
2. Iteration until convergence, for $k \geq 0$:
 - (a) Computation of $\{\tau_i^k\}_{1 \leq i \leq n}$ through n independent problems of linear elasticity with $(\theta_k, \phi_k, \{m_j^k\})$ as design variables.
 - (b) Updating of the design variables $(\theta_{k+1}, \phi_{k+1}, \{m_j^{k+1}\})$ as minimizers of (5.60) with the current stresses τ_i^k .

The minimization of (5.60) is a coupled optimization problem for the variables $(\theta, \phi, \{m_j\})$ which may be quite complicated to solve. One advantage of the double minimization aspect of the relaxed formulation (5.56) is that, for decreasing the cost function, it is not necessary to find an absolute minimizer of (5.60); it is only necessary to find a smaller value than the previous one. This trick allows one to decouple the minimization of (5.60) in three successive minimizations, which all decrease the cost function as desired. First, we minimize with respect to the angle ϕ only and find the new value ϕ_{k+1} in terms of τ_i^k , and $(\theta_k, \{m_j^k\})$. Second, we find minimizers $\{m_j^{k+1}\}$ in terms of τ_i^k , and (θ_k, ϕ_{k+1}) . Third, we compute the minimizer θ_{k+1} , which is explicitly given by formula (4.176), i.e.,

$$\theta_{k+1} = \max \left(0, 1 - \sqrt{\frac{g_k}{-\ell}} \right), \quad (5.61)$$

with

$$g_k = \sum_{i=1}^n \left(\sum_{j=1}^p m_j^{k+1} f_B^c(e_j) \right)^{-1} \left(Q^{-1}(\phi_{k+1}) \tau_i^k Q(\phi_{k+1}) \right) : \left(Q^{-1}(\phi_{k+1}) \tau_i^k Q(\phi_{k+1}) \right).$$

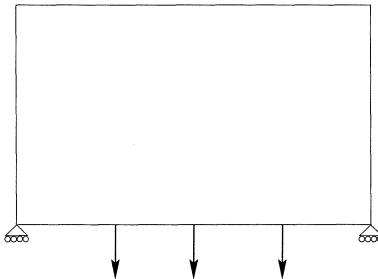


Figure 5.27: Boundary conditions for the multiple loads bridge problem.

The minimization problem over the $\{m_j\}$ can be handled by different classical algorithms. Since the minimization process is performed locally for each cell of the mesh, and for each iteration of the alternate directions method, the chosen algorithm must combine a good rate of convergence with minimal cost in terms of computation time. Special care must be taken for satisfying the constraints $m_j \geq 0$ and $\sum_{j=1}^p m_j = 1$. Newton-like methods (like the interior points method) involving inversions of the Hessian matrix have been excluded for their high CPU cost due to the computation of the Hessian and the $p \times p$ matrix inversion at each iteration. Numerical tests in [14] demonstrated that a good strategy is to eliminate the equality constraint $\sum_{j=1}^p m_j = 1$ by taking for example $m_1 = 1 - \sum_{j=2}^p m_j$ (the choice of the eliminated variable does not bias the solution). After this transformation, the problem becomes an optimization problem with only inequalities constraints. It is treated by a classical projected gradient algorithm.

The minimization over the angle ϕ is different in 2-D and in 3-D. In 3-D we perform it numerically and, since the changes are usually small from one iteration to the next one, it is not necessary to spend too much time on finding the absolute minimizer ϕ_k , and so we stop the minimization after a few iterations of a standard gradient method. In 2-D, as remarked in [42], the optimal angle can be determined analytically. Indeed, taking

$$Q(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix},$$

we have

$$Q^{-1}(\phi) \tau Q(\phi) = \begin{pmatrix} \tau_{12} \sin(2\phi) + \frac{\tau_{11}-\tau_{22}}{2} \cos(2\phi) + \frac{\tau_{11}+\tau_{22}}{2} & \frac{\tau_{22}-\tau_{11}}{2} \sin(2\phi) + \tau_{12} \cos(2\phi) \\ \frac{\tau_{22}-\tau_{11}}{2} \sin(2\phi) + \tau_{12} \cos(2\phi) & -\tau_{12} \sin(2\phi) + \frac{\tau_{22}-\tau_{11}}{2} \cos(2\phi) + \frac{\tau_{11}+\tau_{22}}{2} \end{pmatrix},$$

and thus $W(\theta, \phi, \{m_j\})$ is a polynomial of degree two in $\sin(2\phi)$ and $\cos(2\phi)$. We rewrite it as

$$W(\phi) = a_5 \sin^2(2\phi) + a_4 \cos^2(2\phi) + a_3 \sin(2\phi) \cos(2\phi) + a_2 \sin(2\phi) + a_1 \cos(2\phi) + a_0.$$

A minimizer of $W(\phi)$ has to be found among the real roots of

$$W'(\phi) = 0.$$

Since $W(\phi)$ is periodic, there are at least two real roots of its derivative. A simple computation with the change of variables $t = \tan \phi$ shows that $W'(\phi) = 0$ if and only if

$$(a_3 - a_2)t^4 - 2(2a_5 - 2a_4 + a_1)t^3 - 6a_3t^2 + 2(2a_5 - 2a_4 - a_1)t + (a_3 + a_2) = 0.$$

The roots of this fourth order polynomial in t can be found analytically, thus simplifying considerably the minimization of $W(\phi)$.

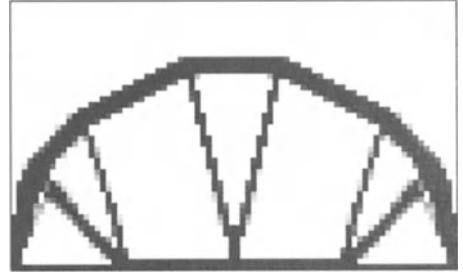
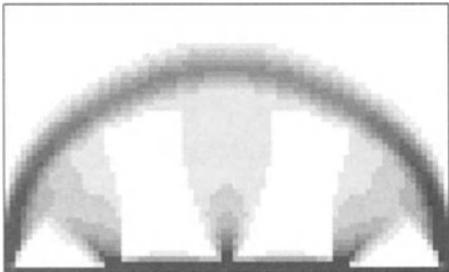


Figure 5.28: Single load: composite (left) and penalized designs (right).

To illustrate this subsection, we come back to the bridge problem where we now apply three unit vertical forces equidistributed along the bottom (see Figure 5.27). The domain size 20×12 , the mesh of 80×48 elements, the boundary conditions at the lower corners, and the 20% volume constraint are unchanged. For the sake of comparison, we first apply the three forces together in a single load optimization. The optimal homogenized design (obtained after 100 iterations) and the penalized shape (after another 20

penalized iterations) are displayed in Figure 5.28. On the other hand, applying the above optimization algorithm for the three distinct loads (with equal weight of the compliances), we obtain the results displayed in Figure 5.29 (which are, of course, much more stable than the previous ones upon changing the loading conditions). Only 4 lamination directions (plus a rotation angle) have been used; more laminations are not necessary. The convergence is displayed on Figure 5.30. A 3-D example is shown in Figure 5.31: It is a car suspension triangle optimized for two different loads corresponding to acceleration or breaking.

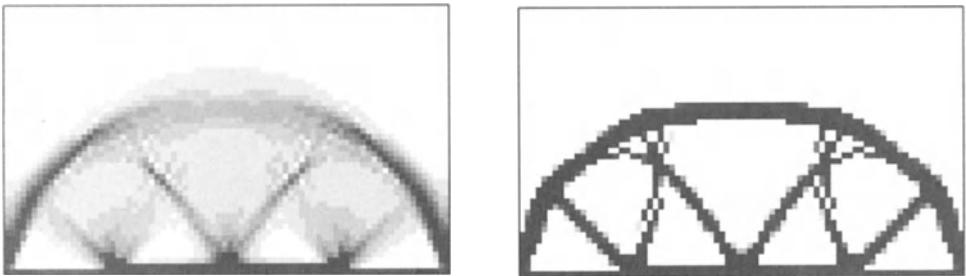


Figure 5.29: Multiple loads: composite (left) and penalized designs (right).

Remark 5.2.13 *A different approach of multiple loads optimization was advocated in [170], [171]. It is based on Lemma 2.3.6, which states that any sequential laminate is parametrized by its H -measure, i.e., the positive measure on the unit sphere giving the proportions of material in each direction of lamination. Furthermore, its Hooke's law depends only on the fourth order moments of its H -measure. The idea is thus to optimize in terms of these fourth order moments, rather than in terms of the original H -measure. This approach is probably more efficient than ours, but is unfortunately restricted to the 2-D case, since this set of fourth order moments of H -measures has an explicit simple form only in this case, and not in 3-D (see Remark 2.3.10, and [33], [112]).*

Remark 5.2.14 *The multiple loads algorithm is a little more costly than its single load counterpart because of the higher cost of design parameters optimization (not taking into account the fact that several elasticity problems have to be solved at each iteration). There is, however, a simpler and faster alternative, which is called the orthogonal directions method and explored in*

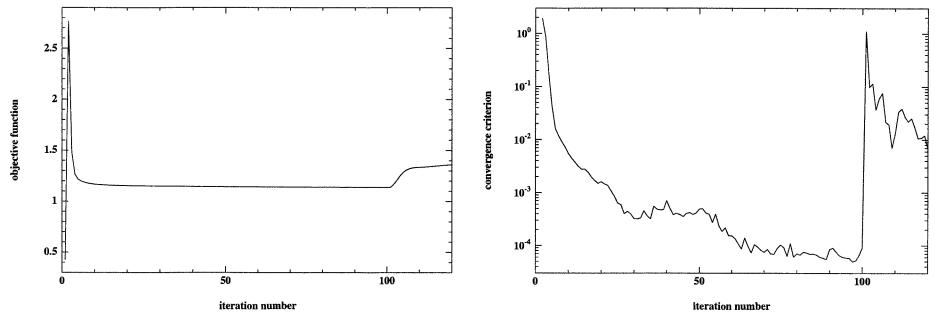


Figure 5.30: Convergence history for the multiple loads bridge: objective function (left) and convergence criterion (right).

[14]. It is less accurate, but faster, and sometimes, at least when one of the loads is preeminent, gives good results. The method is based on the following key idea: Use rank- N laminates in N dimensions (as has been rigorously established in the single load case), and align the lamination directions along the principal directions of the stronger stress tensor in each mesh cell. Hence, the minimization of (5.58) reduces to a minimization over one variable in 2-D (the parameter m_1 , since $m_1 + m_2 = 1$) and two variables in 3-D. When the lamination directions are fixed, the optimization in m_1 can be done explicitly in 2-D, and is easy to perform numerically in 3-D. The density θ is updated with the same formula (5.61). Since suboptimal microstructures are used, the orthogonal directions algorithm is a method of partial relaxation (as explained in Subsection 5.2.8).

5.2.7 Eigenfrequency Optimization

This subsection is devoted to a numerical method for eigenfrequency maximization introduced in [13]. Others numerical algorithms inspired by the homogenization method have been developed for frequency optimization problems by several authors including [43], [98], [155], [179], [180] (see also Chapter 8 in [42]). The so-called *buckling load criterion* is another problem of eigenvalue maximization for which the homogenization method is also relevant (see, e.g., [105], [210]). We rely on the previous analysis of Subsection 4.1.6 in Chapter 4, where all the results are derived for two nondegenerate phases A and B . In truth, we are interested in shape optimization, namely in the case when the most compliant material A is void (all its material

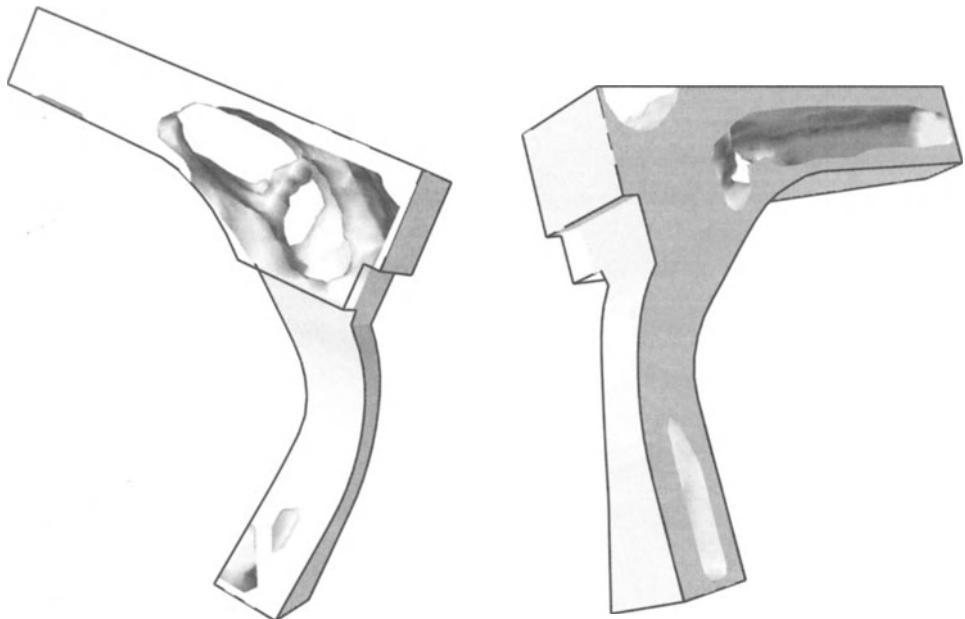


Figure 5.31: Two views of the penalized optimal design for the multiple loads design of an automobile suspension triangle.

parameters are zero). Unfortunately, we do not know if the homogenization method rigorously holds true in the limit when A goes to zero. Nevertheless we act as if it were all right, and we discuss this issue again later in this subsection.

For simplicity, our exposition is restricted to the case where the first eigenfrequency is maximized (indications about the case of several eigenfrequencies are given as remarks). Recall that, by Theorem 4.1.43, the relaxed formulation is

$$\max_{\theta \in L^\infty(\Omega; [0,1])} \left\{ \min_{u \in \mathcal{H}} \frac{\int_\Omega \left(\max_{A^* \in L_\theta^+} A^* e(u) : e(u) \right) dx}{\int_\Omega \bar{\rho}|u|^2 dx} + \ell \int_\Omega \theta(x) dx \right\}, \quad (5.62)$$

with $\bar{\rho} = \theta\rho_A + (1 - \theta)\rho_B$, L_θ^+ the set of sequential laminates, and \mathcal{H} the displacement space, defined by

$$\mathcal{H} = \left\{ u \in H^1(\Omega)^N \text{ such that } u = 0 \text{ on } \Gamma_D \right\},$$

where Γ_D is a subset of $\partial\Omega$ (see Subsection 4.1.6 for further details on the notation). Furthermore, the minimization in u and the maximization in A^* can be exchanged, which allows one to recover the first eigenfrequency

$$\omega_1^2(\theta, A^*) = \min_{u \in \mathcal{H}} \frac{\int_\Omega A^* e(u) : e(u) dx}{\int_\Omega \bar{\rho}|u|^2 dx}.$$

Recall that in the elasticity setting the first eigenvalue may have a multiplicity greater than one. Even if the maximization process starts with a simple first eigenvalue, there is no reason for it to remain so after a few iterations. Nevertheless, we shall often make the simplifying assumption that the first eigenvalue is simple (which is the case for most practical problems, including the ones presented here).

A first possible numerical algorithm is a generalization of the previous alternate directions algorithm introduced for compliance minimization. The main idea is still to compute iteratively the solution of the elasticity problem and to update the design parameters. However, in the present case this algorithm cannot be seen as a double minimization, and it belongs rather to the class of optimality criteria methods. This iterative method is structured as follows:

1. Initialization of the design parameters (θ_0, A_0^*) (for example, taking $\theta_0 = 0$ and $A_0^* = B$ everywhere in the domain).
2. Iteration until convergence, for $k \geq 0$:
 - (a) Computation of u_k as a first eigenfunction for a problem of linear elasticity with design variables (θ_k, A_k^*) .
 - (b) Updating of the design variables $(\theta_{k+1}, A_{k+1}^*)$ by using the displacement u_k in the optimality conditions.

Necessary conditions of optimality were derived in Theorem 4.1.46. It remains to explain how we use them to compute the design variables $(\theta_{k+1}, A_{k+1}^*)$. The optimality condition for A^* is easily used: at each iteration, A_{k+1}^* is the rank- N sequential laminate, with lamination directions given by the eigenvectors of the matrix $e(u_k)$, which is optimal in the upper Hashin-Shtrikman bound (2.122) (see Proposition 2.3.20). One could compute explicitly the optimal lamination parameters of A_{k+1}^* in terms of $e(u_k)$ (this was done in 2-D in [142]). There is an alternative way, however, which amounts to reusing the optimal formulas already computed (in 2-D and in 3-D) for compliance optimization (see Subsection 5.2.1). It is based on Remark 2.3.28, which claims that the primal upper bound

$$g^+(\theta, e(u)) = \max_{A^* \in G_\theta} A^* e(u) : e(u) \quad (5.63)$$

is equivalent to the dual lower bound

$$g^-(\theta, \sigma) = \min_{A^* \in G_\theta} A^{*-1} \sigma : \sigma, \quad (5.64)$$

since they are conjugate functions through the Legendre transform. In other words, a tensor A^* , which is optimal in (5.63), is also optimal in (5.64) for the choice $\sigma = A^* e(u)$, and conversely, the same is true. Therefore, we compute A_{k+1}^* as the optimal rank- N sequential laminate for (5.64) with $\sigma_k = A_k^* e(u_k)$, using the explicit formulas of Subsection 5.2.1.

As already noted in Remark 4.1.47, the optimality condition with respect to θ (furnished by Theorem 4.1.46) is more troublesome. Recall that it yields θ as a maximizer of the function

$$\theta \rightarrow \frac{\int_{\Omega} g^+(\theta, e(u)) dx}{\int_{\Omega} \bar{\rho}(\theta) |u|^2 dx} + \ell \int_{\Omega} \theta dx, \quad (5.65)$$

where $g^+(\theta, e(u))$ is defined by (5.63). According to Remarks 4.1.45 and 4.1.47, for fixed u , the function $\theta \rightarrow g^+(\theta, e(u))$ is convex. Similarly, the function

$$\theta \rightarrow \left(\int_{\Omega} \bar{\rho}|u|^2 \right)^{-1}$$

is also convex. Therefore, in the two extreme cases where either $\rho_A = \rho_B$, or $A = B$, the function (5.65) is convex, an undesirable property when maximizing a function. It indicates that, if we use this optimality condition to update θ , it will have a tendency to produce 0-1 values of θ , and will be unable to predict precise intermediate values. Since we already updated the tensor A^* based on the value of the stress σ , we propose a similar correction for the density θ . By the conjugacy of g^- and g^+ , we rewrite the optimal elastic energy

$$\int_{\Omega} g^+(\theta, e(u)) dx = \int_{\Omega} \max_{\sigma \in \mathcal{M}_N^s} (2e(u) : \sigma - g^-(\theta, \sigma)) dx.$$

Therefore, by a change of sign, maximizing (5.65) is equivalent to minimizing

$$\theta \rightarrow \frac{\int_{\Omega} (g^-(\theta, \sigma) - 2e(u) : \sigma) dx}{\int_{\Omega} \bar{\rho}(\theta)|u|^2 dx} - \ell \int_{\Omega} \theta dx, \quad (5.66)$$

with $\sigma = A^*e(u)$ and A^* , the optimal tensor in (5.63), (5.64). This has the advantage that, for given σ , the function $g^-(\theta, \sigma)$ is convex with respect to θ . Consequently, we could compute θ_{k+1} as the minimizer of the function

$$F(\theta) = \frac{\int_{\Omega} (A_{k+1}^{*-1}(\theta)\sigma_k : \sigma_k - 2e(u_k) : \sigma_k) dx}{\int_{\Omega} \bar{\rho}(\theta)|u_k|^2 dx} - \ell \int_{\Omega} \theta dx.$$

However, due to the fact that θ appears both in the denominator and in the numerator, this yields a non-local condition on θ_{k+1} , and so we make a further simplification to get a tractable condition. For simplicity, we do not take into account the denominator, which is thus evaluated at the previous iteration. More precisely, we compute θ_{k+1} as the minimizer of

$$F(\theta) = \frac{\int_{\Omega} (A_{k+1}^{*-1}(\theta)\sigma_k : \sigma_k) dx}{\int_{\Omega} \bar{\rho}(\theta_k)|u_k|^2 dx} - \ell \int_{\Omega} \theta dx,$$

where A_{k+1}^* is an orthogonal rank- N sequential laminate with lamination directions $(e_i)_{1 \leq i \leq N}$ which are the principal directions of $\sigma_k = A_k^* e(u_k)$, and lamination parameters $(m_i)_{1 \leq i \leq N}$ optimized in terms of the eigenvalues of σ_k (see formula (5.25)). When the weak phase is $A = 0$, the optimal value of θ_{k+1} is found locally and, from (5.43), is given by

$$\theta_{k+1} = \max \left(0, 1 - \sqrt{\frac{g^*(\sigma_k)}{\ell \int_{\Omega} \bar{\rho}(\theta_k) |u_k|^2 dx}} \right). \quad (5.67)$$

The explicit formulas for updating the effective tensor A_{k+1}^* and the density θ_{k+1} are therefore the same as those used for compliance optimization problems (see Subsection 5.2.1). We use an additional procedure adjusting at each iteration the value of the Lagrange multiplier ℓ to keep constant the total amount of material.

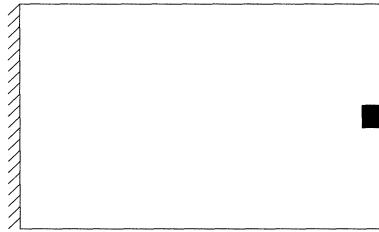


Figure 5.32: Boundary conditions for the eigenvalue maximization of the cantilever.

Remark 5.2.15 Even in the case of a simple first eigenvalue, there is no guarantee that the above optimality criteria algorithm converges (or even that it increases the first eigenvalue). The first reason is that the relaxed formulation (5.62) is not a saddle point problem, namely the maximization with respect to θ cannot be exchanged with the minimization with respect to u . Another reason comes from the simplifications made in the optimality conditions with respect to θ . Note however that, if the algorithm converges, it does so to a design that actually satisfies the necessary conditions of optimality of Theorem 4.1.46. As usual with optimality criteria methods, if it converges, it is a very fast algorithm, but it lacks the robustness of other methods like gradient or descent algorithms.

We test this algorithm on the same cantilever as that described in Subsection 5.2.2 (with a degenerate weak phase $A = 0$). In order to get a

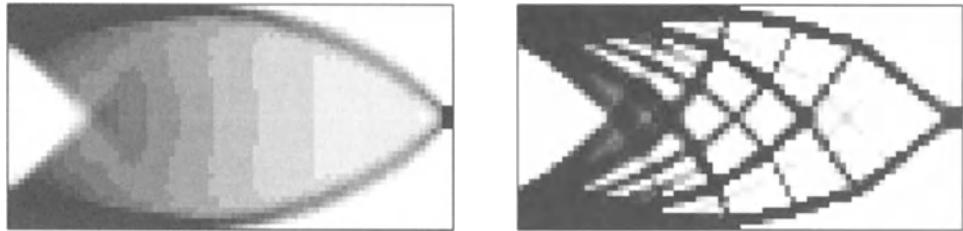


Figure 5.33: Maximization of the first eigenfrequency: composite (left) and penalized designs (right).

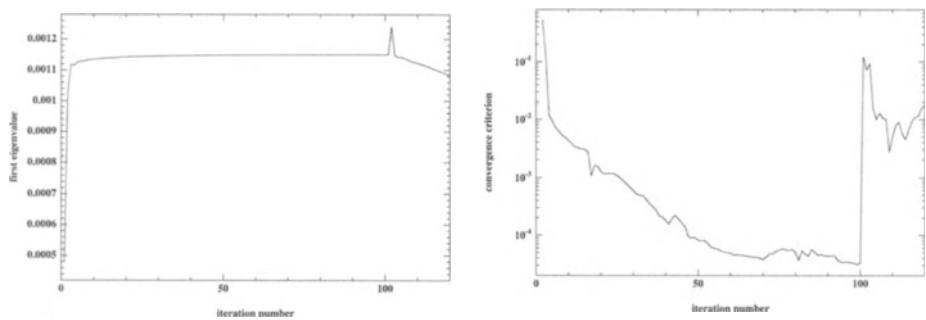


Figure 5.34: Convergence history: first eigenfrequency (left) and convergence criterion (right).

non-trivial solution (there are no more external forces), we perform a reinforcement problem: we put in the middle of the right side (see Figure 5.32) a small subdomain containing an heavy material that is not subject to optimization (its Lamé moduli are the same as those of B , but its material volumic density is 10 times greater). The results are displayed on Figure 5.33. For this problem the convergence is smooth and monotone as can be checked on Figure 5.34 (the little peak on the objective function history is caused by a temporary violation of the volume constraint at the onset of the penalization procedure). The algorithm works also in 3-D, and Figure 5.35 shows the resulting penalized design when the first eigenfrequency is maximized for the same 3-D cantilever as in Figure 5.13 (as in 2-D, there is a small heavy subdomain at the center of the right side of the working domain).

Maximization of a sum of eigenvalues can be dealt with in the same manner as for multiple loads optimization. For example, in Figure 5.36 we maximize the sum of the three first eigenvalues of the cantilever with the following weights

$$10\omega_1^2(\theta, A^*) + \omega_2^2(\theta, A^*) + \frac{1}{2}\omega_3^2(\theta, A^*).$$

The iterative method just described above does not always converge smoothly, even if the first eigenvalue remains simple. For example, it does not converge for the maximization of the first (simple) eigenfrequency of the short cantilever (see Subsection 5.2.2). As pointed out by some authors [13], [43], when used for shape optimization, i.e., in the limit of A going to zero, this method of optimality criteria may exhibit oscillations in some test cases, leading to nonconvergence of the numerical method. Indeed, when the most compliant phase A is almost degenerate, mimicking void, the first eigenvalue does not necessarily correspond to an eigenvector supported mainly in the solid phase B . In other words, spurious eigenvalues appear, which correspond to vibrational modes of the weak phase only and have nothing to do with the optimized structure. This phenomenon of “spectral pollution” is due to the process of replacing void by a very compliant phase, which is probably not correct in such a case. Spectral pollution was rigorously analyzed in a similar context in Section VII.1 of [242]. This phenomenon has nothing to do with a possible crossing of different modes, and the first eigenvalue remains simple in the considered test cases. It is just that the algorithm, having produced a large enough zone of void, suddenly finds smaller eigenvalues corresponding to modes concentrated in the void area.

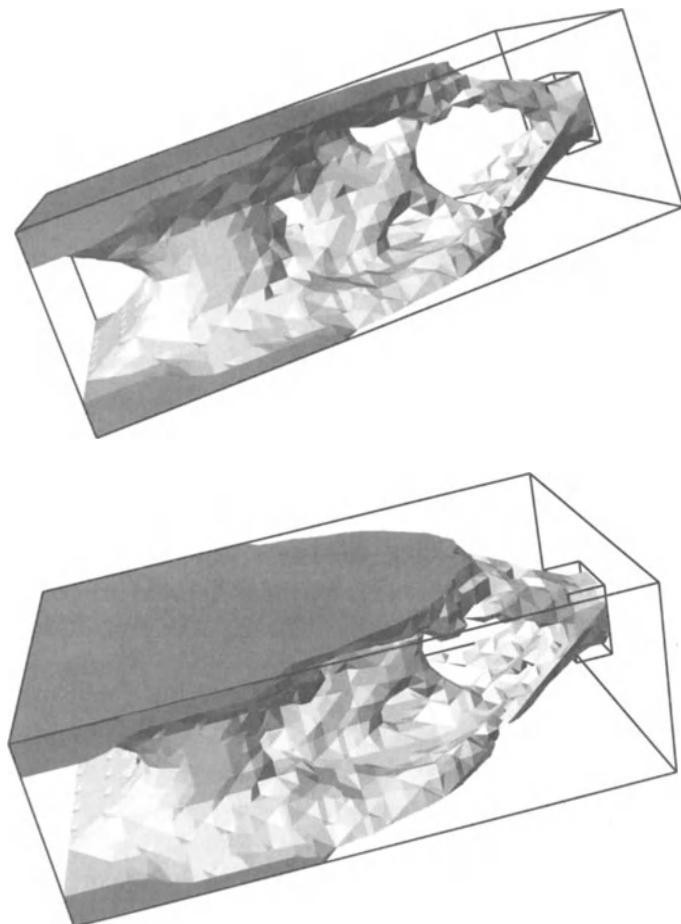


Figure 5.35: Maximization of the first eigenfrequency: two views of the penalized design.

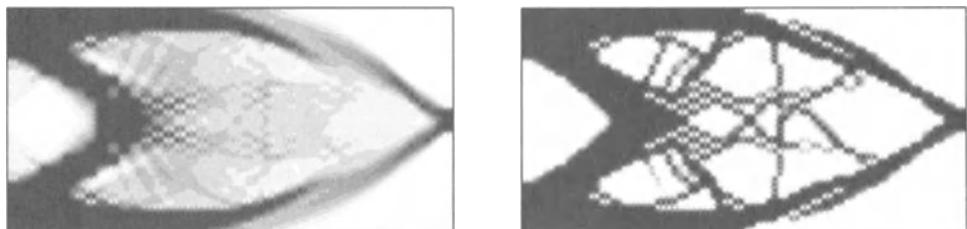


Figure 5.36: Maximization of the three first eigenfrequencies: composite (left) and penalized designs (right).

There are well known remedies for stabilizing this type of computation (filtering high strains in void region, following the right mode by orthogonality of successive eigenvectors), but they are purely numerical tricks with no firm rigorous justification and no systematic success. Actually, these numerical results suggest that the usual practice of filling the holes with a compliant material A is not correct in the present situation. In other words, relaxation by homogenization and the limit $A \rightarrow 0$ probably do not commute for eigenfrequency maximization.

We therefore propose another more robust algorithm, which is a gradient method. Since the basis of the method is to differentiate the first eigenvalue with respect to the design parameters, we still assume that the first eigenvalue is simple, and thus differentiable. In the case of a multiple first eigenvalue, the proposed gradient method should be generalized adequately, replacing the gradient by the subdifferential of the first eigenvalue (we shall not dwell on this issue, but simply refer the reader to, e.g., [155], [247]). A gradient method for maximizing the first eigenvalue ensures that it will always increase through the iterations (although it can fall into a local maximum).

We describe its 2-D implementation (there is no conceptual difficulty in extending it to 3-D). By Theorem 4.1.46 we can restrict the set G_θ of homogenized Hooke's law A^* to the subset of rank-2 laminates with orthogonal lamination directions. Such a laminated composite is parametrized by three variables: The holes density $\theta \in [0, 1]$, the lamination parameter $m \in [0, 1]$, and an angle of rotation $\phi \in [0, \pi]$. More precisely, by formula (2.69) its homogenized Hooke's law $A^*(\theta, \phi, m)$ is given locally by

$$\theta (A^* - B)^{-1} = (A - B)^{-1} + (1 - \theta) R(\phi)^t (m f_B(e_1) + (1 - m) f_B(e_2)) R(\phi), \quad (5.68)$$

where $R(\phi)$ is the fourth order tensor corresponding to a rotation $Q(\phi)$ of angle ϕ in the physical space \mathbb{R}^2 , (e_1, e_2) is the canonical basis of \mathbb{R}^2 , and $f_B(e)$ is defined, for any symmetric matrix ξ , by

$$f_B(e)\xi : \xi = \frac{1}{\mu_B} \left(|\xi e|^2 - (\xi e \cdot e)^2 \right) + \frac{1}{2\mu_B + \lambda_B} (\xi e \cdot e)^2. \quad (5.69)$$

As new design parameters, we choose the density $\theta \in [0, 1]$, the angle of rotation $\phi \in [0, \pi]$, and the proportion $m \in [0, 1]$. The relaxed optimal design problem (5.62) is therefore equivalent to

$$\max_{(\theta, \phi, m) \in L^\infty(\Omega; [0, 1] \times [0, \pi] \times [0, 1])} \left\{ J^*(\theta, \phi, m) = \omega_1^2(\theta, \phi, m) + \ell \int_\Omega \theta dx \right\},$$

where ω_1^2 is the first eigenvalue of

$$\begin{cases} -\operatorname{div} A^* e(u) = \omega^2 \bar{\rho} u & \text{in } \Omega \\ A^* e(u)n = 0 & \text{on } \Gamma_N \\ u = 0 & \text{on } \Gamma_D. \end{cases} \quad (5.70)$$

As is well known, the computation of the partial derivatives with respect to these parameters is possible when the first eigenvalue is simple. To avoid any difficulties, we assume that the parameters $(\theta, \phi, m) \in L^\infty(\Omega; [0, 1] \times [0, \pi] \times [0, 1])$ are such that the first eigenvalue $\omega_1^2(\theta, \phi, m)$ is simple. Then, for a given direction $(\delta\theta, \delta\phi, \delta m) \in L^\infty(\Omega)^3$, we define a function of t in the neighborhood of zero by

$$F(t) = J^*(\theta + t\delta\theta, \phi + t\delta\phi, m + t\delta m). \quad (5.71)$$

By a classical result of spectral perturbation (see [146], or Theorem 9.10 in [242]), for sufficiently small positive values of t the first eigenvalue in (5.71) is simple and $F(t)$ is differentiable. The computation of $F'(0)$ is just a matter of algebra and the result is given in the following lemma.

Lemma 5.2.16 *Assume that the first eigenvalue $\omega_1^2(\theta, \phi, m)$ is simple at $(\theta, \phi, m) \in L^\infty(\Omega; [0, 1] \times [0, \pi] \times [0, 1])$. Then, J^* is Gâteaux differentiable at this point,*

$$F'(0) = \delta J^*(\theta, \phi, m) = \int_{\Omega} \nabla_{\theta} J^* \delta\theta dx + \int_{\Omega} \nabla_{\phi} J^* \delta\phi dx + \int_{\Omega} \nabla_m J^* \delta m dx,$$

with partial derivatives given by

$$\nabla_{\theta} J^* = \frac{\frac{\partial A^*}{\partial \theta} e(u) : e(u) - \omega_1^2(\rho_A - \rho_B)|u|^2}{\int_{\Omega} \bar{\rho}|u|^2} + \ell$$

$$\nabla_m J^* = \frac{\frac{\partial A^*}{\partial m} e(u) : e(u)}{\int_{\Omega} \bar{\rho}|u|^2}$$

$$\nabla_{\phi} J^* = \frac{\frac{\partial A^*}{\partial \phi} e(u) : e(u)}{\int_{\Omega} \bar{\rho}|u|^2}$$

where u is a first eigenvector for (5.70) associated to the first eigenvalue $\omega_1^2(\theta, \phi, m)$ and

$$\frac{\partial A^*}{\partial \theta} = T^{-1} \left((A - B)^{-1} + M \right) T^{-1}$$

$$\frac{\partial A^*}{\partial m} = -\theta(1 - \theta)T^{-1} (f_B(e_1) - f_B(e_2)) T^{-1}$$

$$\frac{\partial A^*}{\partial \phi} = -\theta(1 - \theta)T^{-1} \frac{\partial M}{\partial \phi} T^{-1}$$

with

$$M(m, \phi) = R^t(\phi) (m f_B(e_1) + (1 - m) f_B(e_2)) R(\phi)$$

and

$$T(\theta, m, \phi) = (A - B)^{-1} + (1 - \theta)M(m, \phi).$$

Of course, since there are constraints on the parameters m and θ (which must stay both in the range $[0, 1]$), the formulas of Lemma 5.2.16 are combined with a projection step in order to satisfy the constraints. The projected gradient algorithm is thus:

1. Initialization of the design parameters θ_0, ϕ_0, m_0 (for example, we take them to be constant).
2. Iteration until convergence, for $k \geq 0$:
 - (a) Computation of the first eigenvector u_k with the previous design parameters θ_k, ϕ_k, m_k .
 - (b) Updating of these parameters by

$$\theta_{k+1} = \max (0, \min (1, \theta_k + t_k \nabla_\theta J_k^*)) ,$$

$$m_{k+1} = \max (0, \min (1, m_k + t_k \nabla_m J_k^*)) ,$$

$$\phi_{k+1} = \phi_k + t_k \nabla_\phi J_k^*$$

where $t_k > 0$ is a small step such that

$$J^*(\theta_{k+1}, \phi_{k+1}, m_{k+1}) > J^*(\theta_k, \phi_k, m_k).$$

A good step t_k is computed through a line search that may be expensive, especially if there is also a total volume constraint. (Recall that each evaluation of the objective function requires the solution of an eigenvalue problem.) The gradient method is usually more expensive than the optimality criteria method. Therefore, in practice the best strategy is to start with the optimality criteria method and to switch to the gradient method only when the computed first eigenvalue becomes lower than the previous one.

5.2.8 Partial Relaxation

As we have seen in Chapter 4, the homogenization method for structural optimization is fully operative in only a small number of cases, yet very important in practice, namely compliance optimization (for single or multiple loads) and eigenfrequency optimization. The reason for this restricted range of application is our lack of knowledge of the G -closure set G_θ that describes all possible homogenized elasticity tensors A^* . This is actually the only obstacle and it is a pity that it prevents the use of all the other deep results of homogenization that could be very useful in optimal design for more general objective functions. Examples of other objective functions include the design of compliant mechanisms [211], [250], [253], the design of thermoelastic or piezoelectric materials [139], [252], [254], the design of optimal periodic microstructures [134], [248], [249]. In Subsection 4.1.3 we established all the required results of the homogenization method: A space of generalized designs was found for a large class of objective functions, a relaxation theorem was proved which gives the existence of optimal generalized designs and connects them to minimizing sequences of classical designs. All these ingredients are available, except a precise characterization of optimal homogenized tensors A^* . From a numerical standpoint, the same analysis can be made: Many new ideas have been introduced (computing generalized designs, using optimality conditions, using a penalization procedure), which could be applied to other objective functions, except that we have no explicit characterization of G_θ . In order to circumvent this difficulty and to proceed, we give up the requirement of working with the full set G_θ and decide to restrict ourselves to its known subclass of sequential laminates L_θ^+ . In full generality, restricting G_θ to one of its subsets leads to a so-called *partial relaxation*. This allows one to derive a workable numerical method, but the price paid is that this subclass of G_θ may contain only suboptimal composite tensors A^* . This simple idea has been studied, e.g., in [12], [30], [54], [153]. There are, of course, other possible choices of explicit subclasses:

For example, the so-called *isotropic composite tensor*, introduced in Remark 5.2.11 (the entire set G_θ is replaced by the unique isotropic tensor in L_θ^+), the class of rank- N orthogonal sequential laminates, as discussed in Remark 5.2.14, or, as in the early work of Bendsoe and Kikuchi [47], we could use tensors arising from periodic homogenization of a squared cell perforated by a rectangular hole.

We briefly recall some notation of Chapter 4 before giving a precise definition of the partial relaxation. For the moment we consider two-phase optimal design problems rather than shape optimization. The original problem of optimal design is

$$\inf_{\chi \in L^\infty(\Omega; \{0,1\})} J(\chi), \quad (5.72)$$

with an objective function J defined by

$$J(\chi) = \int_{\Omega} [\chi(x)g_A(x, u_\chi(x)) + (1 - \chi(x))g_B(x, u_\chi(x))] dx + \ell \int_{\Omega} \chi(x) dx,$$

where u_χ is the unique solution in $H_0^1(\Omega)^N$ of

$$\begin{cases} -\operatorname{div}(A_\chi e(u_\chi)) = f & \text{in } \Omega \\ u_\chi = 0 & \text{on } \partial\Omega, \end{cases}$$

with a given body force $f \in L^2(\Omega)^N$ and a Hooke's law

$$A_\chi = \chi A + (1 - \chi)B.$$

Having in mind the design of structures with prescribed response, a possible example of an objective function is

$$J(\chi) = \int_{\Omega} [\chi|u_\chi - u_A|^2 + (1 - \chi)|u_\chi - u_B|^2] dx + \ell \int_{\Omega} \chi dx,$$

where $u_A(x)$ and $u_B(x)$ are given target displacements for the structure. Upon some conditions on g_A and g_B , the relaxed formulation of (5.72) is

$$\min_{(\theta, A^*) \in \mathcal{CD}} J^*(\theta, A^*), \quad (5.73)$$

with an extended objective function

$$J^*(\theta, A^*) = \int_{\Omega} (\theta g_A(x, u) + (1 - \theta)g_B(x, u) + \ell\theta) dx, \quad (5.74)$$

where $u(x)$ is the unique solution in $H_0^1(\Omega)^N$ of the homogenized problem

$$\begin{cases} -\operatorname{div}(A^*e(u)) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (5.75)$$

and \mathcal{CD} is the space of generalized, or composite, designs

$$\mathcal{CD} = \left\{ (\theta, A^*) \in L^\infty(\Omega; [0, 1] \times \mathcal{M}_N^4) \mid A^*(x) \in G_{\theta(x)} \text{ a.e. in } \Omega \right\}, \quad (5.76)$$

where, for each constant value $0 \leq \theta \leq 1$, G_θ is the set of all homogenized Hooke's laws obtained by mixing the phases A and B in proportions θ , $(1-\theta)$, respectively (see Theorem 2.1.2).

By restricting G_θ to its subset L_θ^+ of sequential laminates, with core A and matrix B in proportions θ and $(1-\theta)$, respectively, (see Definition 2.3.5), we introduce a set \mathcal{LD}^+ of sequentially laminated designs, defined by

$$\mathcal{LD}^+ = \left\{ (\theta, A^*) \in L^\infty(\Omega; [0, 1] \times \mathcal{M}_N^4) \mid A^*(x) \in L_{\theta(x)}^+ \text{ a.e. in } \Omega \right\}. \quad (5.77)$$

The proposed partial relaxation is

$$\inf_{(\theta, A^*) \in \mathcal{LD}^+} J^*(\theta, A^*), \quad (5.78)$$

with the same objective function J^* defined by (5.74).

Theorem 5.2.17 *Formulation (5.78) is a partial relaxation of the original problem (5.72) in the sense that*

1. *any classical design (χ, A_χ) belongs to \mathcal{LD}^+ ;*
2. *conversely, every generalized design (θ, A^*) in \mathcal{LD}^+ is attained by a sequence of classical designs χ_n , namely θ is the weak * limit of χ_n in $L^\infty(\Omega; [0, 1])$, A^* is the H -limit of A_{χ_n} , and*

$$\lim_{n \rightarrow +\infty} J(\chi_n) = J^*(\theta, A^*).$$

The proof of Theorem 5.2.17 is an immediate consequence of the definition of the homogenized properties A^* and of the assumptions on the integrands $g_{A,B}$, so we omit it. It clearly implies that the infima of J and J^* coincide, i.e.,

$$\inf_{\chi \in L^\infty(\Omega; \{0,1\})} J(\chi) = \inf_{(\theta, A^*) \in \mathcal{LD}^+} J^*(\theta, A^*).$$

Remark 5.2.18 We remark that Theorem 5.2.17 does not state the existence of a minimizer of the partial relaxation (5.78); this is the main difference between a partial relaxation and a full relaxation, as in Theorem 4.1.7. It seems that we have gained very little in replacing the ill-posed problem (5.72) with another ill-posed problem (5.78). Nevertheless, loosely speaking the latter is less ill-posed than the former since its integrand has been smoothed or averaged, at least partially, leading to better convexity properties. The question of how much qualitatively the partial relaxation improves on the original formulation is linked to the question of how far from optimal are the microstructures in L_θ^+ . As a possible justification of this partial relaxation (5.78), let us simply recall that in the cases of compliance or eigenfrequency optimization, it coincides with the complete relaxation.

The advantage of dealing with generalized designs in \mathcal{LD}^+ , instead of \mathcal{CD} , is that we can work out explicitly the optimality conditions derived in Theorem 4.1.8. As such, these necessary conditions of optimality may be of little use since we do not expect, in general, to find optimal designs in \mathcal{LD}^+ , but they furnish the derivative of the objective function, which allows one to build numerical gradient methods. Recall from Lemma 2.3.6 that a sequential laminate A^* in L_θ^+ is parametrized by the proportion θ of phase A and by a probability measure ν (its H -measure). More precisely, L_θ^+ is the set of all symmetric fourth order tensors A^* such that there exists a probability measure $\nu \in \mathcal{P}(S_{N-1})$ (see Remark 2.3.7) on the unit sphere S_{N-1} satisfying

$$\theta (A^* - B)^{-1} = (A - B)^{-1} + (1 - \theta) \int_{S_{N-1}} f_B(e) d\nu(e), \quad (5.79)$$

with $f_B(e)$ defined by (5.69). We therefore view θ and ν as the true independent design parameters in \mathcal{LD}^+ . In other words, the partial relaxation (5.78) is equivalent to

$$\inf_{(\theta,\nu) \in L^\infty(\Omega; [0,1] \times \mathcal{P}(S_{N-1}))} J^*(\theta, \nu),$$

where the objective function $J^*(\theta, \nu)$ is just $J^*(\theta, A^*(\theta, \nu))$. Following [30], one can compute the partial derivatives of J^* from Theorem 4.1.8. We recall the definition of the adjoint state p which is the solution of

$$\begin{cases} -\operatorname{div}(A^* e(p)) = \theta \frac{\partial g_A}{\partial \lambda}(x, u) + (1 - \theta) \frac{\partial g_B}{\partial \lambda}(x, u) & \text{in } \Omega \\ p = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.80)$$

Lemma 5.2.19 *The objective function $J^*(\theta, \nu)$ is Gâteaux differentiable in $L^\infty(\Omega; [0, 1] \times \mathcal{P}(S_{N-1}))$, and denoting by $\delta\theta$ and $\delta\nu$ admissible increments, its directional derivative is*

$$\delta J^*(\theta, \nu) = \int_{\Omega} \nabla_{\theta} J^* \delta\theta \, dx + \int_{\Omega} \int_{S_{N-1}} \nabla_{\nu} J^* d(\delta\nu) \, dx, \quad (5.81)$$

with

$$\nabla_{\theta} J^*(x) = g_A(x, u(x)) - g_B(x, u(x)) + \ell - \frac{\partial A^*}{\partial \theta} e(u) : e(p)$$

$$\nabla_{\nu} J^*(x, e) = -\frac{\partial A^*}{\partial \nu}(x, e) e(u) : e(p)$$

and

$$\frac{\partial A^*}{\partial \theta}(x) = T^{-1} \left((A - B)^{-1} + \int_{S_{N-1}} f_B(e) d\nu(e) \right) T^{-1}$$

$$\frac{\partial A^*}{\partial \nu}(x, e) = -\theta(1 - \theta) T^{-1} f_B(e) T^{-1}$$

$$T = (A - B)^{-1} + (1 - \theta) \int_{S_{N-1}} f_B(e) d\nu(e)$$

where u is the solution of the state equation (5.75), and p that of the adjoint state equation (5.80).

Proof. As for Lemma 5.2.16, we define a function of $t \in \mathbb{R}^+$ in the neighborhood of zero by

$$F(t) = J^*(\theta + t\delta\theta, \nu + t\delta\nu), \quad (5.82)$$

for a given direction $(\delta\theta, \delta\nu)$ in $L^\infty(\Omega)$ with values in \mathbb{R} and in the space of Radon measures on S_{N-1} , respectively. If $(\delta\theta, \delta\nu)$ is an admissible increment, for sufficiently small $t > 0$, $(\theta + t\delta\theta, \nu + t\delta\nu)$ belongs to $L^\infty(\Omega; [0, 1] \times \mathcal{P}(S_{N-1}))$ and $F(t)$ is differentiable. The computation of $F'(0) = \delta J^*(\theta, \nu)$ is easy. Let us simply explain our notation. The partial derivative $\frac{\partial A^*}{\partial \theta}$ is a function of x only, which belongs to $L^\infty(\Omega)$. On the other hand, the other partial derivative $\frac{\partial A^*}{\partial \nu}$ is a function of x and $e \in S_{N-1}$, which is continuous on S_{N-1} , namely it belongs to $L^\infty(\Omega; C(S_{N-1}))$. Therefore, in the last term of (5.81), integration has to be performed with respect to the measure $d\nu$ on S_{N-1} and to the Lebesgue measure dx in Ω . We remark eventually that the lamination formula (5.79) is linear in ν and therefore its derivative with respect to ν is just $f_B(e)$. \square

Lemma 5.2.19 gives the basis for a numerical gradient method. Of course, since (θ, ν) is constrained to be locally in $[0, 1] \times \mathcal{P}(S_{N-1})$, the gradient method must be combined with a projection step to satisfy these constraints. Recall that a probability measure $\nu \in \mathcal{P}(S_{N-1})$ must be nonnegative, $\nu(e) \geq 0$, and of unit mass, $\int_{S_{N-1}} d\nu(e) = 1$. For numerical computations we must discretize the unit sphere S_{N-1} as we did in Subsection 5.2.6 for multiple loads optimization. In other words, upon fixing a finite number of directions $(e_j)_{1 \leq j \leq q}$, we replace the “continuous” lamination formula (5.79) by the “discrete” one

$$\theta (A^* - B)^{-1} = (A - B)^{-1} + (1 - \theta) \sum_{j=1}^q m_j f_B(e_j). \quad (5.83)$$

In order to keep a small number of directions q and yet have good results, we can also introduce a global rotation of the microstructure, namely use, instead of (5.83), the rotated lamination formula

$$\theta (A^* - B)^{-1} = (A - B)^{-1} + (1 - \theta) \sum_{j=1}^q m_j R^t f_B(e_j) R \quad (5.84)$$

where R is the fourth order tensor corresponding to a rotation $Q \in \mathcal{M}_N$ in the physical space (with $Q^{-1} = Q^t$), i.e., for any symmetric matrix $\xi \in \mathcal{M}_N^s$,

$$R\xi = Q^{-1}\xi Q.$$

This introduces another design parameter, denoted by ϕ , which corresponds to one angle in 2-D or two angles in 3-D necessary to parametrize a rotation $Q(\phi)$ in the physical space. Adding this rotation parameter ϕ requires the computation of another partial derivative of J^* , which was already done in Lemma 5.2.16 for the case of the first eigenfrequency, and which is easily extended to the present setting. The result is

$$\nabla_\phi J^*(x) = - \frac{\partial A^*}{\partial \phi}(x) e(u) : e(p)$$

with

$$\begin{aligned}\frac{\partial A^*}{\partial \phi} &= -\theta(1-\theta)T^{-1}\frac{\partial M}{\partial \phi}T^{-1} \\ T(\phi) &= (A-B)^{-1} + (1-\theta)\sum_{j=1}^q m_j R^t(\phi) f_B(e_j) R(\phi) \\ M(\phi) &= R^t(\phi) \left(\sum_{j=1}^q m_j f_B(e_j) \right) R(\phi).\end{aligned}$$

The resulting gradient algorithm is very similar to a combination of the same algorithm for eigenfrequency optimization (see Subsection 5.2.7) and of the alternate directions algorithm for multiple loads optimization (see Subsection 5.2.6).

Upon discretization of the unit sphere S_{N-1} , a measure ν is now completely determined by the vector $m = (m_j)_{1 \leq j \leq q}$ such that

$$\nu(e) = \sum_{j=1}^q m_j \delta(e - e_j)$$

where δ is the Dirac mass at the origin. Therefore, the partial derivative $\nabla_\nu J^*$ is replaced by

$$\nabla_m J^*(x) = -\frac{\partial A^*}{\partial m}(x) e(u) : e(p) \quad (5.85)$$

with

$$\frac{\partial A^*}{\partial m} = \left(\frac{\partial A^*}{\partial m_j} \right)_{1 \leq j \leq q} = \left(-\theta(1-\theta)T^{-1} f_B(e_j) T^{-1} \right)_{1 \leq j \leq q}.$$

Recall that the above discrete ν is a probability measure if the vector m satisfies $m_j \geq 0$ and $\sum_{j=1}^q m_j = 1$. This gives the required projection for the gradient algorithm. We now have all the ingredients to define the proposed numerical algorithm:

1. Initialization of the design parameters θ_0, ϕ_0, m_0 (for example, we take them constant satisfying the constraints).
2. Iteration until convergence, for $k \geq 0$:

- (a) Computation of the state u_k and the adjoint state p_k , solutions of (5.75) and (5.80), respectively, with the previous design parameters θ_k, ϕ_k, m_k .
- (b) Updating of these parameters by

$$\theta_{k+1} = \max(0, \min(1, \theta_k - t_k \nabla_\theta J_k^*))$$

$$\phi_{k+1} = \phi_k - t_k \nabla_\phi J_k^*$$

$$m_{j,k+1} = \max\left(0, m_{j,k} - t_k \nabla_{m_j} J_k^* + \ell_k\right).$$

where ℓ_k is a Lagrange multiplier (iteratively adjusted) for the constraint $\sum_{j=1}^q m_{j,k} = 1$, and $t_k > 0$ is a small step such that

$$J^*(\theta_{k+1}, \phi_{k+1}, m_{k+1}) < J^*(\theta_k, \phi_k, m_k).$$

A good descent step t_k is computed through a line search that may be expensive, since each evaluation of the objective function requires the solution of the direct and adjoint equation. In practice, we stop as soon as $J_{k+1}^* \leq J_k^*$, and we divide the step by two if not. In the case of further constraints on the design parameters or on some structural response parameters, it may be difficult to work out an efficient projected gradient algorithm. In such a case, a popular descent method is the method of moving asymptotes proposed by Svanberg [266], which is reported to be very efficient.

Remark 5.2.20 For simplicity we focused on the case of a single load optimization problem. There is obviously no difficulty in extending the previous analysis to multiple loads problems. In particular, we never used in an essential manner the fact that there is only one state equation (we did not reduce the number of laminations as we had done for compliance optimization).

Remark 5.2.21 Our choice of partial relaxation, obtained by working in the set L_θ^+ of sequential laminates, can also easily be extended to shape optimization problems, i.e., in the limit when A becomes holes or void. Indeed, when A goes to zero, the lamination formula (5.79) converges to a well-defined formula (see (5.25) or Subsection 2.3.4). This implies that L_θ^+ has a limit L_θ^{0+} when $A \rightarrow 0$ on which we can define a partial relaxation for shape optimization.

To illustrate this numerical method in 2-D (3-D would work as well), we first choose an academic example, namely the minimization of the norm of the displacement field. To stay in the context of shape and topology optimization, phase A is assumed to be void, while phase B has material properties as defined in Subsection 5.2.2. In order to avoid the effects of rounding errors for large values of the displacement, the objective function is rescaled to be exactly the $L^m(\Omega)$ -norm of the displacement, with $2 \leq m < +\infty$ in 2-D, i.e.,

$$J^*(\theta, A^*) = \left(\int_{\Omega} (1 - \theta) |u|^m dx \right)^{1/m} + \ell \int_{\Omega} \theta dx.$$

We study a medium cantilever problem (as described in Subsection 5.2.2) with a rectangular 80×40 mesh, and the Lagrange multiplier ℓ is iteratively adjusted so that the weight of the structure is constrained to be 40% of that of the full working domain.

As expected, the composite solutions (i.e., the numerical output of the partial relaxation) exhibit large areas of intermediate densities (which indicates that in practice the laminated composites are often optimal for this problem). To recover classical designs (i.e., with pure material and void) we apply a penalization procedure as described in Subsection 5.2.4. In the context of a gradient method, this process is slightly modified as follows. A first possibility amounts to adding a penalizing term to the standard objective function $J^*(\theta, A^*)$ of the type

$$c_{pen} \int_{\Omega} (1 - \theta)^r \theta^r dx$$

where c_{pen} is a positive constant and r is an exponent larger or equal to 1. We prefer a second more efficient procedure, which changes the lamination formula giving the value of the homogenized tensor A^* . Instead of (5.83) (or its rotated version (5.84)) we use

$$\theta^r (A^* - B)^{-1} = (A - B)^{-1} + (1 - \theta^r) \sum_{i=1}^q m_i f_B(e_i), \quad (5.86)$$

where $r > 1$ is typically 3 or 5. The effect of (5.86) is that the resulting fictitious composite A^* is much weaker than the usual laminate. Therefore, it is not advantageous to use any such composite of intermediate density. Using the modified formula (5.86) results in a very effective penalization scheme: Almost all grey areas in the homogenized design disappear to yield

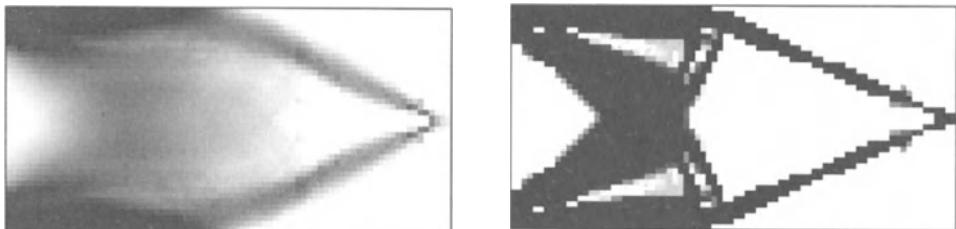


Figure 5.37: Optimal shape of the cantilever for $m = 2$: composite (left) and penalized (right).

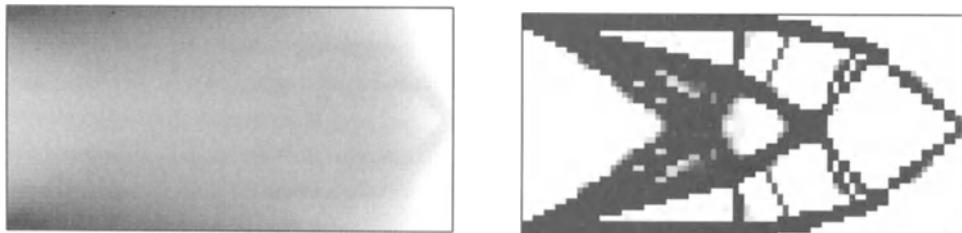


Figure 5.38: Optimal shape of the cantilever for $m = 10$: composite (left) and penalized (right).

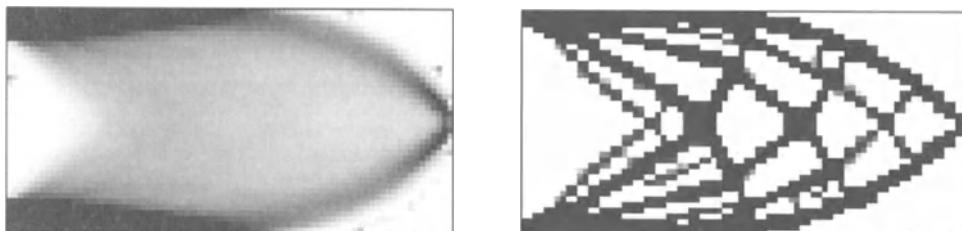


Figure 5.39: Optimal shape of the cantilever for $m = 100$: composite (left) and penalized (right).

a black and white penalized design, as can be seen in Figures 5.37, 5.38, and 5.39.

In all computations we fix the number of lamination directions to 12 without rotation. Figure 5.37 displays the results for $m = 2$, Figure 5.38 for $m = 10$, and Figure 5.39 for $m = 100$. We remark that the optimal designs for $m = 100$ are very close to those of compliance optimization (Figure 5.8), as it should be since for such a point load, minimizing the compliance or the maximal displacement is the same.

To conclude this section we present two examples of micromechanism design inspired from the work of Sigmund [250] and obtained in collaboration with Jouve [19]. The first one is a force inverter (see Figure 5.40). The goal is to produce the largest possible output force when imposing a given input force. This output force is in the opposite direction of the input force. In order to obtain the result displayed on Figure 5.41, we choose a different objective function, namely, we maximize the displacement to the left at the output port.

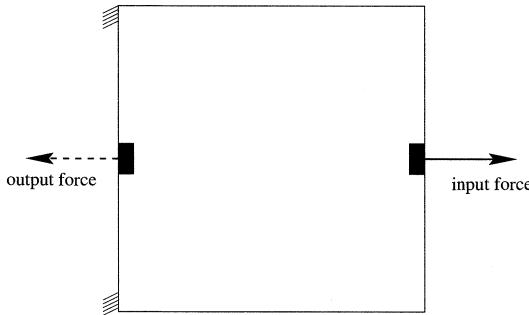


Figure 5.40: Boundary conditions for the force inverter.

The second example is a gripping mechanism (see Figure 5.42). The goal is again to produce the largest possible output force when imposing a given input force (like a nutcracker). The result displayed in Figure 5.43 was also obtained by maximizing the displacement at the output port. In both examples, it is not allowed to remove material from the input and output ports (the black areas in Figures 5.40 and 5.42). We remark also that only the input force is imposed: The output force is the result of the optimization.

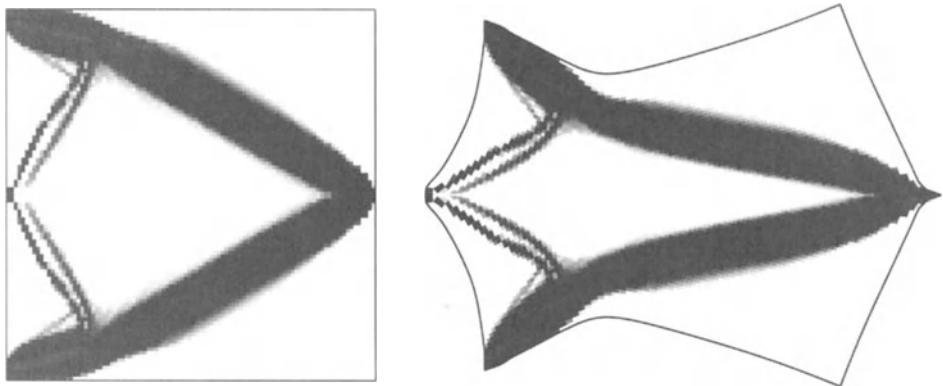


Figure 5.41: Optimal shape of a force inverter (left) and deformed configuration (right).

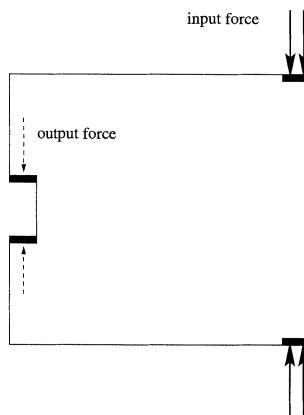


Figure 5.42: Boundary conditions for the gripping mechanism.

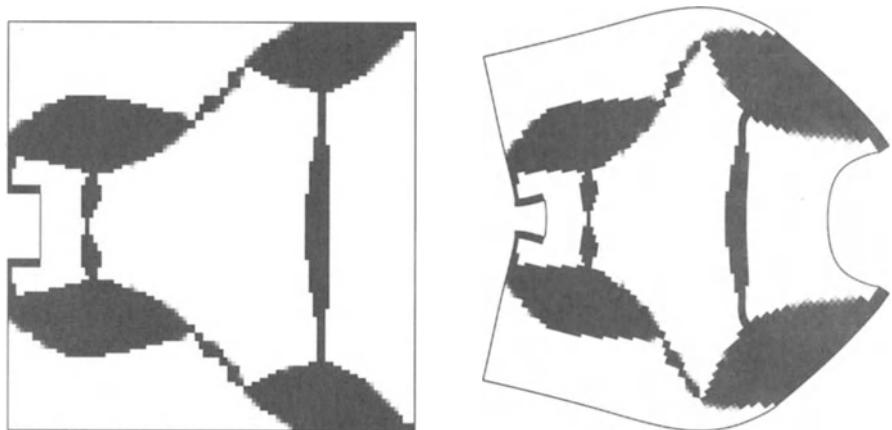


Figure 5.43: Optimal shape of a gripping mechanism (left) and deformed configuration (right).

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