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A DIAGONALLY DOMINANT SECOND-ORDER ACCURATE IMPLICIT SCHEME*

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Abstract—An unconditionally stable, second order accurate, implicit, finite difference method is described. The coefficient matrix is tridiagonal and always diagonally dominant. As an illustrative example it is used to solve Burgers' equation.

INTRODUCTION

In this note, a second order spatially accurate implicit finite-difference method is described. The coefficient matrix is tridiagonal and always diagonally dominant so that a cell Reynolds number limitation, usually found with second-order accurate differencing, does not occur. The scheme is unconditionally stable.

In order to illustrate the main idea of this technique, the following linear equation is considered:

$$\frac{\partial u}{\partial t} + C\frac{\partial u}{\partial x} = \sigma\frac{\partial^2 u}{\partial x^2}; \qquad C \geq 0 \tag{1}$$

The derivatives in (1) are differenced as follows:

$$u_t = \frac{u_j^{n+1} - u_j^n}{\Delta t}:$$ Forward time

$$u_x = \frac{u_j^{n+1} - u_{j-1}^{n+1}}{\Delta x} + \frac{u_{j+1}^n - 2u_j^n + u_{j+1}^n}{2\,\Delta x}:$$ Modified Central Space, which for a converged solution and $u_j^n = u_j^{n+1}$ reduces to central spacing.

$$u_{xx} = \frac{u^{n+1} - 2u_j{}^{n+1} + u^{n+1}}{(\Delta x)^2}:$$ Central Space

Substituting in equation (1), we find

$$-A_j u_{j+1}^{n+1} + B_j u_j^{n+1} - C_j u_{j-1}^{n+1} = D_j \tag{2}$$

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where

$$A_j = \frac{\sigma\,\Delta t}{(\Delta x)^2}, \qquad C_j = \frac{C\,\Delta t}{\Delta x} + \frac{\sigma\,\Delta t}{(\Delta x)^2}$$

$$B_j = 1 + \frac{C\,\Delta t}{\Delta x} + 2\frac{\sigma\,\Delta t}{(\Delta x)^2},$$

A_j, B_j and $C_j > 0$ and $B_j > A_j + C_j$. The system (2) is always diagonally dominant and therefore can be reduced to the form[1]:

$$u_j^{n+1} = E_j u_{j+1}^{n+1} + F_j \tag{3}$$

where

$$E_j = \frac{A_j}{B_j - C_j E_{j-1}}; \qquad |E_j| \leq 1$$

$$F_j = \frac{D_j + C_j F_{j-1}}{B_j - C_j E_{j-1}}$$

If central space differencing was applied for u_x, diagonal dominance would result only if $\frac{C\,\Delta x}{\sigma} \leq 2$, or $\frac{C\,\Delta x}{\sigma} \geq 2$ and $\frac{C\,\Delta t}{\sigma} \leq 1$. If these conditions were not satisfied, the reduction procedure (3) could lead to the amplification of round-off or boundary errors[1].

It is significant that many implicit methods[2] that have been used to date have been 'unstable' for values of $\frac{C\,\Delta x}{\sigma} \gg 1$ even though the procedure would appear to be unconditionally stable. One method which has been successful for boundary layer flows uses a marching procedure with upwind differencing for one coordinate direction and leads to a diagonally dominant system even when the cell Reynolds number exceeds 2.

STABILITY AND CONSISTENCY

Let $u_j^n = \xi^n e^{im\,\Delta x j}$. Substituting in equation (2), we get

$$\xi = \frac{1 + \frac{C\,\Delta t}{\Delta x}(1 - \cos m\,\Delta x)}{1 + \left\{\frac{C\,\Delta t}{\Delta x} + \frac{2\sigma\,\Delta t}{(\Delta x)^2}\right\}(1 - \cos m\,\Delta x) + i\frac{C\,\Delta t}{\Delta x}\sin m\,\Delta x}$$

As $|\xi| < 1$ for all $m\,\Delta x$, $\frac{C\,\Delta t}{\Delta x}$, $\frac{\sigma\,\Delta t}{\Delta x^2}$, the scheme is unconditionally stable. Expanding u_{j+l}^{n+l} by a Taylor series, it can be shown that the scheme (2) is consistent; i.e., as Δt, $\Delta x \to 0$, (1) is recovered with second-order spatial accuracy. The truncation error is $0(\Delta t, \Delta x^2, \Delta t\,\Delta x)$.

As an illustrative example, we have solved Burgers' equation,

$$\frac{\partial u}{\partial \bar{t}} + u\frac{\partial u}{\partial \bar{x}} = \sigma\frac{\partial^2 u}{\partial \bar{x}^2} \tag{4}$$

$$u(\pm\infty) = \pm 1$$

With $x = \frac{\bar{x}}{\sigma}$, $t = \frac{\bar{t}}{\sigma}$, equation (3) with $\sigma = 1$ is recovered. In order to satisfy the boundary conditions exactly, a transformation of the independent variable of the form

$$\xi = \tanh x$$

is used. Various values of Δt and Δx were used; no restriction on the step size was found. For $\Delta\xi = 0{\cdot}01$ and $\Delta t = 0{\cdot}04$, a steady state solution is achieved in about 50 time steps. Note that $\Delta x = \dfrac{\Delta\xi}{1-\xi^2}$ so that $0{\cdot}01 \leq \Delta x \leq \infty$. The cell Reynolds number is $u/\Delta x$.

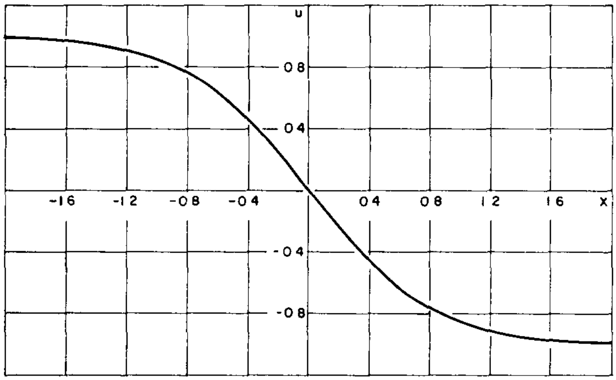

Fig. 1. Velocity distribution of the weak shock.

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