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Applied Mathematics

Lectures on Navier-Stokes Equations

Tai-Peng Tsai

American Mathematical Society

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Preface

This book is concerned with the incompressible Navier-Stokes system, which models the motion of incompressible Newtonian fluids. There have been several excellent monographs on Navier-Stokes equations, including Ladyžhenskaya [**121**], Temam [**206**], Constantin-Foias [**36**], Galdi [**57**], Lions [**138**], Sohr [**194**], Lemarié-Rieusset [**128**], Bahouri-Chemin-Danchin [**6**], Seregin [**180**], Lemarié-Rieusset [**129**], and Robinson-Rodrigo-Sadowski [**168**]. The goal of this book is to give a rapid exposition to graduate students on the known theory of the existence, uniqueness, and regularity of its solutions, with a focus on the nonlinear problems. To fit into a course, many auxiliary (e.g., linear) results are described with references but without proof, and it was necessary to omit many important topics.

For a one-semester course for graduate students who have taken an introductory PDE course, I cover Chapters 1 through 6 and Section 8.3. Chapters 7 to 10 are suitable for the second semester or a topic course. In a topic course I gave in Spring 2016 titled "Existence and bifurcation of Navier-Stokes equations", I covered Chapters 1, 2, 7 and Section 8.4.

This book is an outgrowth of lectures I gave at National Taiwan University in Summer 2006, Fall 2013, and Spring 2014; at Harvard University in Fall 2006; and at the University of British Columbia in Spring 2007, Spring 2011, and Spring 2016. The kind support and hospitality of these institutes are gratefully acknowledged. It gives me enormous pleasure to thank all who were involved.

I thank my teacher V. Šverák who introduced me to this subject. It goes without saying that I learned many things in this book from him. I thank K. Kang who was my constant source of advice in the early stages of this book. I warmly thank my collaborators on fluid equations considered in this book: Z. Bradshaw, D. Chae, C.-C. Chen, S. Gustafson, K. Kang, K. Kim, M. Korobkov, H. Kwon, Y. Luo, H. Miura, R. Strain, V. Šverák, and H.-T. Yau. I thank C.-S. Lin who encouraged and supported me while giving the lectures in 2013–2014 that led to this book. I thank N. Ghoussoub for advice on its publication. I thank the AMS team including I. Mette, M. Almeida, and A. O'Sean for the enormous help during the preparation of this book.

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Suggestions for this book are very welcome and appreciated. Errata will be maintained on my personal homepage `www.math.ubc.ca/~ttsai` and at `www.ams.org/bookpages/gsm-192`.

This book is dedicated to my parents, Tong-Pao Tsai and A-Chiao Wu; my parents-in-law, Wei-Min Wang and Hsiu-Ho Kuo; my wife, Hui-Hua Wang; and our children, Nina, Vivian, and Felix, with deep gratitude and love.

Tai-Peng Tsai

Vancouver

May 2017

Notation

Here we summarize the notation used in this book. Some of it is very common and is not given detailed definitions in the main text, to avoid interrupting the flow of thought.

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and let $\mathbb{Z}$ be the set of integers.

Let $\mathbb{R} = (-\infty, \infty)$ be the set of real numbers. Let $\mathbb{R}^n$ be an n-dimensional real Euclidean space of points $x = (x_1, \ldots, x_n)$, with $x_1, \ldots, x_n \in \mathbb{R}$. Let $x^2 = |x|^2$ and $\langle x \rangle = (|x|^2 + 2)^{1/2}$. The half-space $\mathbb{R}^n_+$ is the open subset of points $x \in \mathbb{R}^n$ with $x_n > 0$. In particular, $\mathbb{R}_+ = \mathbb{R}^1_+ = (0, \infty)$.

For a subset $U \subset \mathbb{R}^n$, $|U|$ denotes its Lebesgue measure.

For $f \in L^1(U)$, we denote $(f)_U = \frac{1}{|U|} \int_U f$.

A domain Ω is a connected open subset of $\mathbb{R}^n$, $n \geq 2$. Its boundary is denoted as $\partial\Omega$ and its unit outer normal as $\mathbf{n}$.

We write $U \Subset \Omega$ if $U \subset \overline{U} \subset \Omega$ where $\overline{U}$ is the closure of U in $\mathbb{R}^n$.

For a vector field $v = \{v_j\}_{j=1}^n$, $\operatorname{div} v = \nabla \cdot v = \partial_j v_j$. The summation convention for repeated indices is used. Also, $\operatorname{curl} v = \nabla \times v$ if $n \leq 3$, and $\operatorname{curl} v$ is identified with the 2-tensor $\omega_{ij} = \partial_i v_j - \partial_j v_i$ for all $n \geq 2$.

For a 2-tensor $\sigma = \{\sigma_{ij}\}_{i,j=1}^n$, $(\operatorname{div} v)_i = \partial_j \sigma_{ij}$.

Let δ_0 be the Dirac delta function supported at the origin. The fundamental solution E of the Laplace equation, $-\Delta E = \delta_0$, is given by (2.49). In particular, $E(x) = \frac{1}{4\pi|x|}$ for $n = 3$.

The heat kernel in $\mathbb{R}^n$ is $\Gamma(x,t) = (4\pi t)^{-n/2} \exp(-\frac{|x|^2}{4t})$ for $t > 0$ and $\Gamma(x,t) = 0$ for $t \leq 0$.

The Lorentz tensor U_{ij} is the fundamental solution of the stationary Stokes system, given in Section 2.6, (2.60).

The Oseen tensor S_{ij} is the fundamental solution of the nonstationary Stokes system, given in Section 5.1, (5.3).

The following is a list of function spaces:

$C^k(\Omega)$**:** k-times differentiable functions in $\Omega \subset \mathbb{R}^n$.

$C^{k,\alpha}(\Omega)$**:** Those functions in $C^k(\Omega)$ whose k-th derivatives are Hölder continuous with exponent $\alpha \in (0,1]$.

$C_c^k(\Omega)$**:** k-times differentiable functions with compact support in $\Omega \subset \mathbb{R}^n$. This definition still applies if Ω is not open, for example, $\Omega = \overline{\mathbb{R}^n_+}$.

$C_{c,\sigma}^k(\Omega)$**:** k-times differentiable divergence-free vector fields with compact support in $\Omega \subset \mathbb{R}^n$.

$W^{k,q}(\Omega)$**,** $H^k(\Omega)$**:** Sobolev spaces.

$L^q_\sigma(\Omega)$**,** $W^{k,q}_\sigma(\Omega)$**,** $H^k_\sigma(\Omega)$**:** The completion of $C^\infty_{c,\sigma}(\Omega)$ in Sobolev space norms.

P_q**:** Helmholtz projection from $L^q(\Omega;\mathbb{R}^n)$ to $L^q_\sigma(\Omega)$.

$L^{p,q}(\Omega)$**:** Lorentz spaces.

$L^{p,\infty}(\Omega) = L^p_{wk}(\Omega)$**:** Weak L^p-space.

Chapter 1

Introduction

This book is concerned with the incompressible Navier-Stokes system, which models the motion of incompressible Newtonian fluids and was the historical motivation for many beautiful mathematical theories such as weak solutions, the Leray-Schauder degree, Hopf bifurcation, and strange attractors. There are still many challenging problems unsolved, including the mathematical theory for turbulence, the global-in-time regularity problem, boundary value problem for steady state, the Liouville problem of steady state with finite energy, and the boundary layer description at zero viscosity limit.

The goal of this book is to give an exposition on the known theory on the existence, uniqueness, and regularity of its solutions, with a focus on the regularity problem. To fit into a one-semester course, auxiliary results will be described with references but no proof, and it is necessary to omit many important topics.

In this chapter we will give some background and overview on the incompressible Navier-Stokes equations. We first present the equations and the notation in Section 1.1. We then derive the equations from physical principles in Section 1.2. We describe its scaling property and a priori estimates in Section 1.3. We describe the equations and techniques for the vorticity and pressure in Sections 1.4 and 1.5. We finally consider the Helmholtz decomposition of vector fields and say more about the pressure in Section 1.6. Being an introductory chapter, several properties are stated at a formal level, and several results are quoted without proof.

1.1. Navier-Stokes equations

The Navier-Stokes equations are equations involving the fluid velocity and the pressure for a fluid occupying a spatial domain.

A subset Ω of $\mathbb{R}^n$ is called a *domain* if it is open and connected. Its boundary is denoted as $\partial\Omega$ and its unit outer normal as $\mathbf{n}$. We often consider $n = 3$ or $n = 2$, and Ω is either $\mathbb{R}^n$, $\mathbb{R}^n_+$, a bounded domain, or an *exterior* domain, which is a domain with a bounded complement in $\mathbb{R}^n$.

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $T \in (0, \infty]$. The incompressible *Navier-Stokes equations* for velocity $v(x,t) : \Omega \times (0,T) \to \mathbb{R}^n$ and pressure $p(x,t) : \Omega \times (0,T) \to \mathbb{R}$ are

$$\left.\begin{aligned} \partial_t v - \nu\Delta v + (v\cdot\nabla)v + \nabla p = f \\ \operatorname{div} v = 0 \end{aligned}\right. \quad \text{in } \Omega \times (0,T). \tag{NS}$$

Here f is a given body force and $\nu > 0$ is the viscosity constant. We denote

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_i = \frac{\partial}{\partial x_i}, \tag{1.1}$$

and

$$(v\cdot\nabla)v = v_j\partial_j v = \partial_j(v_j v). \tag{1.2}$$

The summation convention for repeated indices is used throughout the book, except where explicitly noted. The equations are coupled with an initial condition

$$v(x,0) = v_0(x), \quad \operatorname{div} v_0 = 0, \tag{1.3}$$

and, when the boundary $\partial\Omega$ is nonempty, a *noslip* boundary condition

$$v|_{\partial\Omega} = 0. \tag{1.4}$$

Sometimes we also consider nonhomogeneous boundary data, $v|_{\partial\Omega} = v_*$.

A vector field v with $\operatorname{div} v = 0$ is said to be *divergence-free*.[1]

If the nonlinear term $(v\cdot\nabla)v$ is dropped, the linear system is called the *Stokes system*.

The system (NS) is called the *Euler equations* if $\nu = 0$. It models the inviscid ideal fluid. In this case the natural boundary condition is

$$v\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{1.5}$$

It means that the fluid cannot pass through the boundary but is allowed to move tangentially to the boundary. Conditions (1.4) and (1.5) are natural for their corresponding systems mathematically since they both allow local-in-time unique existence theorems. Physically, when $\nu = 0$, there is no friction on the boundary to prevent the fluid from sliding along the boundary.

We will often assume $\nu = 1$ if its value has no significance.

[1]Some texts also call it *solenoidal*. However, in some other texts the word "solenoidal" also requires the boundary condition $v\cdot\mathbf{n}|_{\partial\Omega} = 0$. Hence we will avoid it.

1.2. Derivation of Navier-Stokes equations

In this section we give a very brief derivation of (NS) from physical principles. The derivation helps us to understand the effect of each term in the equations and the behaviour of the solutions.

The derivation is based on the balance of mass and momentum and on the following assumptions:

- The fluid is continuous, with density ρ, velocity v, and pressure p,
- the temperature fluctuation is negligible,
- a constitutive law for the stress tensor (to be defined) for a *Newtonian fluid*, and
- a relation between p and ρ, the so-called "equation of state".

Examples for Newtonian fluids include water and air. *non-Newtonian fluids* include shampoo, blood, and oil, which consist of large molecules.

1.2.1. Equation of continuity (conservation of mass). We first consider the conservation of mass. In any fixed region E with outernormal $\mathbf{n}$ where the fluid occupies, we have

$$\frac{d}{dt}\int_E \rho\, dx = -\int_{\partial E} \rho v \cdot \mathbf{n} dS. \tag{1.6}$$

The left side is the rate of change of total mass in E, and the right side is the gain of mass due to flux through the boundary ∂E. By the divergence theorem and since E is arbitrary, we get

$$\partial_t \rho + \operatorname{div}(\rho v) = 0. \tag{1.7}$$

1.2.2. Equation of motion (balance of momentum). We next consider the i-th component of the momentum. Replacing ρ by ρv_i with i fixed in the above consideration, we get

$$\frac{d}{dt}\int_E \rho v_i\, dx = -\int_{\partial E} \rho v_i v \cdot \mathbf{n} dS + \int_E f_i dx + \int_{\partial E} \sigma_{ij}\mathbf{n}_j dS, \tag{1.8}$$

where

- $\int_E f_i dx$, the volume integral, is the total body force acting on E. It is external and long range. Examples of f include gravity and magnetic fields.
- $\int_{\partial E} \sigma_{ij}\mathbf{n}_j dS$, the surface integral, is the stress force acting on ∂E. It is internal, short range, and due to molecule interaction. The 2-tensor σ_{ij} is called the *stress tensor*, and its explicit form depends on the nature of the fluid.

By the divergence theorem and since E is arbitrary, we get, for $i = 1, 2, 3$,

$$\partial_t(\rho v_i) + \operatorname{div}(\rho v_i v) = f_i + \partial_j \sigma_{ij}. \tag{1.9}$$

In vector form it reads

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) = f + \operatorname{div} \sigma. \tag{1.10}$$

Here $v \otimes v$ is the 2-tensor with $(v \otimes v)_{ij} = v_i v_j$ and $(\operatorname{div} \sigma)_i = \partial_j \sigma_{ij}$.

1.2.3. Stress tensor. When the fluid is at equilibrium, i.e., v is identically zero, the stress force on the boundary ∂E per unit area is $-p\mathbf{n}$, where p is the pressure. Thus the stress tensor is $\sigma_{ij} = -\delta_{ij} p$. In the general non-equilibrium case, $\sigma \mathbf{n}$ may not be parallel to $\mathbf{n}$, and we decompose

$$\sigma_{ij} = \tau_{ij} - \delta_{ij} p, \quad p := -\tfrac{1}{n} \operatorname{tr} \sigma, \tag{1.11}$$

where $\operatorname{tr} \sigma = \sum_{i=1}^n \sigma_{ii}$. The nature of the traceless tensor τ_{ij} depends on the fluid being considered. For a Newtonian fluid, τ_{ij} is expected to be traceless, independent of v (by Galilean invariance) and rotation, symmetric ($\tau_{ij} = \tau_{ji}$), and depending linearly on ∇v. Thus, we are looking for a linear map from the 2-tensor $\partial_j v_i$ to a symmetric, traceless, rotation-invariant 2-tensor. It is a consequence of algebra that, for some constant ν,

$$\tau_{ij} = \nu \left(\partial_i v_j + \partial_j v_i - \frac{2}{n} \delta_{ij} \operatorname{div} v \right). \tag{1.12}$$

From the physical viewpoint, τ_{ij} corresponds to the *viscous stress tensor*, and we require

$$\nu \geq 0. \tag{1.13}$$

For a non-Newtonian fluid, τ_{ij} is expected to depend on ∇v in a nonlinear way.

1.2.4. Equation of state. With the stress tensor given by (1.11) and (1.12), we have four equations (1.7) and (1.9) for five unknowns v_i, ρ, p. We still need a relation between p and ρ. There are two common, opposite choices:

- (Incompressible case) The density $\rho = \rho_0 > 0$ is constant. In this case we get $\operatorname{div} v = 0$ from (1.7). Then, by (1.11) and (1.12),

  $$\partial_j \sigma_{ij} = \nu \Delta v_i - \partial_i p.$$

  Hence we get from (1.9)

  $$\rho_0(\partial_t v_i + v_j \partial_j v_i) = f_i + \nu \Delta v_i - \partial_i p, \tag{1.14}$$

  which is the first equation of (NS) with a factor ρ_0 on the left side. Therefore we get (NS) for incompressible Newtonian fluids.

- (Compressible case) The pressure is a function of the density, $p = p(\rho)$. A common choice is $p(\rho) = \rho^\alpha$, for some $\alpha > 0$. In this case, we get from (1.9), (1.11), and (1.12) with $p(\rho) = \rho^\alpha$ that

$$\partial_t(\rho v_i) + \operatorname{div}(\rho v_i v) = f_i - \partial_i \rho^\alpha + \nu \Delta v_i + \left(1 - \frac{2}{n}\right) \nu \partial_i \operatorname{div} v. \tag{1.15}$$

The system (1.7) and (1.15) describes *compressible Newtonian fluids*.

The equation of state is related to the conservation of energy. A detailed discussion is a subject of thermodynamics.

1.2.5. Lagrangian description. Our previous consideration is based at a fixed location. It is called the *Eulerian description*. In contrast, the *Lagrangian description* follows the trajectory of a fluid particle. If $x(t)$ denotes the location of a fluid particle, then

$$\dot{x}(t) = v(x(t), t). \tag{1.16}$$

If $L(x, t)$ is a quantity carried by the fluid particle, then

$$\frac{d}{dt} L(x(t), t) = \partial_t L + \dot{x} \cdot \nabla L = (\partial_t + v \cdot \nabla) L \tag{1.17}$$

is the derivative of L along a particle trajectory. We thus define the *material derivative* by

$$\frac{D}{Dt} \overset{\text{def}}{=\!=} \partial_t + v \cdot \nabla. \tag{1.18}$$

The equation of motion (1.14) for incompressible fluid can be rewritten as

$$\rho \frac{D}{Dt} v = F, \tag{1.19}$$

where F is the total force field. Equation (1.19) is exactly Newton's second law of motion. In the compressible case, the equation of motion (1.15) can also be written as (1.19) using the equation of continuity (1.7).

If we denote by $E(t)$ the evolution of a region E_0, $E(t) = \{x(t) : x(0) \in E_0\}$, then the conservation of mass implies that (compare with (1.6))

$$\int_{E(t)} \rho \, dx = \text{constant}. \tag{1.20}$$

In the incompressible case it says that the volume of $E(t)$ is constant. Hence an incompressible fluid is also call *volume-preserving*. This property is valid for both incompressible Navier-Stokes equations ($\nu > 0$) and incompressible Euler equations ($\nu = 0$).

Strictly speaking, the word "incompressible" should mean $\frac{D}{Dt}\rho = 0$. It implies $\operatorname{div} v = 0$ by (1.7), but ρ may not be constant. However, it is common to assume ρ is constant when we say a fluid is incompressible.

1.3. Scaling and a priori estimates

In this section we present the two most important properties of (NS), the scaling property and the a priori energy estimates.

The Navier-Stokes equations have the following *scaling property*: If $(v(x,t), p(x,t))$ is a solution with force $f(x,t)$, then for any $\lambda > 0$,

$$v^\lambda(x,t) = \lambda v(\lambda x, \lambda^2 t), \quad p^\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t) \tag{1.21}$$

is also a solution, with force $f^\lambda(x,t) = \lambda^3 f(\lambda x, \lambda^2 t)$. Note that the scaling property is valid for any space dimension. In addition to scaling, the set of solutions of (NS) is also preserved under space-time translations.

The rescaled solutions are useful in the study of the asymptotic behavior of a solution near a singularity or the spatial-temporal infinity. Denote the *balls* and *parabolic cylinders*

$$\begin{aligned} &B_r(x_0) = \{x \in \mathbb{R}^n, |x - x_0| < r\}, \qquad B_r = B_r(0), \\ &Q_r(x_0,t_0) = B_r(x_0) \times (t_0 - r^2, t_0), \quad Q_r = Q_r(0,0), \\ &Q_r^\sharp(x_0,t_0) = B_r(x_0) \times (t_0, t_0 + r^2), \quad Q_r^\sharp = Q_r^\sharp(0,0). \end{aligned} \tag{1.22}$$

The profile of v^λ in Q_1 (or $Q_1^\sharp$) corresponds to that of v in Q_λ (or $Q_\lambda^\sharp$). Thus the behavior of v^λ in Q_1 as $\lambda \to 0$ corresponds to the local behavior of v near $(0,0)$. On the other hand, the behavior of v^λ in $Q_1^\sharp$ as $\lambda \to \infty$ encodes the global behavior of v. The above consideration still applies when we are interested in stationary solutions, for which we replace Q_λ by B_λ.

The second property of (NS) is the a priori energy estimates: Take dot product of (NS) with v itself and then integrate in space-time. Formally the integral $\iint [(v \cdot \nabla)v + \nabla p] \cdot v$ vanishes due to $\operatorname{div} v = 0$ and zero boundary condition. Thus, for suitable f, we have

$$\operatorname*{ess\,sup}_{t>0} \int |v(x,t)|^2 dx + \int_0^\infty \int |\nabla v(x,t)|^2 \, dx \, dt \le C. \tag{1.23}$$

This is the natural *energy estimate* for (NS). It is the same estimate for the Stokes system, as if the nonlinear term is absent. It is called "a priori" because it is valid for any sufficiently regular solution.

Denote the left side of (1.23) by $E(v)$. Assuming the space-time is $\mathbb{R}^n \times \mathbb{R}^+$, we have

$$E(v^\lambda) = \lambda^{2-n} E(v). \tag{1.24}$$

For the regularity theory, the Navier-Stokes system (NS) is *supercritical* for $n = 3$ in the sense that $\lim_{\lambda \to 0} E(v^\lambda) = \infty$. It implies that we lose the energy control of v^λ when $\lambda \to 0$, i.e., when we want to study the local behavior of v. Being supercritical usually implies that the natural energy cannot control the nonlinearity. In contrast (NS) is *critical* for $n = 2$ with

$\lim_{\lambda\to 0} E(v^\lambda) = \text{const}$. We may formally say (NS) is *subcritical* for $n = 1$ since $\lim_{\lambda\to 0} E(v^\lambda) = 0$, although (NS) with $n = 1$ is in fact trivial.

The regularity theory for subcritical PDEs is often standard. That for critical PDEs is quite tricky but sometimes possible. The regularity theory for supercritical equations is usually very difficult, and known examples often require other properties such as a monotonicity formula.

On the other hand, if we are interested in the asymptotic problem, we take $\lambda \to \infty$. We have $\lim_{\lambda\to\infty} E(v^\lambda) = 0$ (subcritical) for $n = 3$ and $\lim_{\lambda\to\infty} E(v^\lambda) = \text{const.}$ (critical) for $n = 2$. This suggests that the 2D asymptotic problem is harder than the 3D problem.

1.4. Vorticity

In this section we consider the vorticity, in particular its physical meaning and its equations.

The most important is the 3-dimensional case $n = 3$, for which the *vorticity* is defined by

$$\omega = \operatorname{curl} v. \tag{1.25}$$

It measures the tendency of a fluid particle to rotate locally at a particular point. This can be seen by the Taylor expansion for small $h \in \mathbb{R}^n$

$$v(x+h) = v(x) + (h\cdot\nabla)v + o(|h|) = v(x) + Sh + Ah + o(|h|) \tag{1.26}$$

where S and A are the symmetric (deformation) and antisymmetric (rotation) parts of ∇v:

$$S_{ij} = \frac{\partial_j v_i + \partial_i v_j}{2}, \quad A_{ij} = \frac{\partial_j v_i - \partial_i v_j}{2}. \tag{1.27}$$

Note that

$$A = \frac{1}{2}\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \quad Ah = \frac{1}{2}\omega \times h. \tag{1.28}$$

The matrix A is rotation about the axis in the direction of ω with angle $|\omega|/2$. The matrix S, being symmetric, can be diagonalized in some orthogonal frame as

$$S = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{1.29}$$

where the eigenvalues λ_j satisfy $\operatorname{tr} S = \lambda_1 + \lambda_2 + \lambda_3 = 0$ if $\operatorname{div} v = 0$. The direction(s) with $\lambda_j > 0$ corresponds to stretching, while the direction(s) with

$\lambda_j < 0$ corresponds to compression. The expansion (1.26) says that the velocity field in a small neighborhood can be, up to leading order, decomposed as the sum of a translation, a deformation, and a rotation.

For 2-dimensional vector fields, we consider the natural imbedding

$$\mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \quad \text{by} \quad (x_1, x_2) \mapsto (x_1, x_2, 0). \tag{1.30}$$

We identify

$$v = (v_1, v_2, 0), \quad \operatorname{curl} \omega = (0, 0, \omega_3) \tag{1.31}$$

where $\omega_3 = \partial_1 v_2 - \partial_2 v_1$.

For any dimension $2 \le n < \infty$, we may consider the vorticity tensor

$$\omega_{ij} = \partial_i v_j - \partial_j v_i \quad (i, j \le n). \tag{1.32}$$

For example, for $n = 3$ we get $\omega_{23} = \partial_2 v_3 - \partial_3 v_2 = \omega_1$.

We should note carefully that the vorticity measures the local rotation of a fluid particle, not the global rotation of the particle trajectory. A simple example is

$$v(x) = g(r)(-x_2, x_1, 0), \quad r = \sqrt{x_1^2 + x_2^2}, \tag{1.33}$$

for some nonnegative function $g(r) \in C_c^1(\mathbb{R}_+)$. Where $g(r) > 0$, all trajectories are counterclockwise circles when viewed from the positive x_3-axis. However,

$$\omega = \operatorname{curl} v = (0, 0, \omega_3), \quad \omega_3 = 2g + rg'. \tag{1.34}$$

The vorticity at a point is in the positive x_3-direction if and only if $\omega_3 > 0$. This is not the case in an interval $I \Subset \mathbb{R}_+$ if $g(r) = \chi(r) r^{-a}$ for some $a > 2$ and $\chi \in C_c^1(\mathbb{R}_+)$ is a cut-off function satisfying $\chi = 1$ on I.

The equations for the vorticity can be derived by taking curl of (NS). Using the identity

$$(v \cdot \nabla) v = \omega \times v + \nabla \left(\frac{1}{2} |v|^2 \right) \tag{1.35}$$

and the *vector identities* $\operatorname{curl} \nabla \varphi = 0$ and

$$\nabla \times (u \times v) = (\nabla \cdot v + v \cdot \nabla) u - (\nabla \cdot u + u \cdot \nabla) v, \tag{1.36}$$

for general function φ and vector fields u and v, we have

$$\operatorname{curl}(v \times \nabla v) = \operatorname{curl}(\omega \times v) = v \cdot \nabla \omega - \omega \cdot \nabla v, \tag{1.37}$$

and hence the *vorticity equations*

$$\begin{gathered} \partial_t \omega - \nu \Delta \omega + v \cdot \nabla \omega - \omega \cdot \nabla v = \operatorname{curl} f, \\ \operatorname{div} \omega = 0. \end{gathered} \tag{1.38}$$

This system is in the form of parabolic PDEs and does not involve the pressure as in (NS). However, the boundary condition of ω is only implicitly given by that of u. The term $v \cdot \nabla\omega$ is the convection of ω by v and usually does not change the properties of ω. On the the other hand, the term $\omega \cdot \nabla v$, usually called the *vortex stretching term*, is a potential term of ω and is complicated to study.

When dimension $n = 2$, the term $\omega \cdot \nabla v$ vanishes by (1.31), and (1.38) becomes the scalar equation

$$\partial_t \omega_3 - \nu\Delta\omega_3 + v \cdot \nabla\omega_3 = \partial_1 f_2 - \partial_2 f_1. \tag{1.39}$$

This equation has no potential term of ω_3 and hence the maximal principle is applicable. It makes the study in 2D much simpler than in 3D.

The velocity v can be determined by the vorticity ω, its boundary condition (1.4), and the elliptic system

$$\operatorname{curl} v = \omega, \quad \operatorname{div} v = 0. \tag{1.40}$$

Explicitly, by the *vector identity*

$$-\Delta v = \operatorname{curl}\operatorname{curl} v - \nabla(\operatorname{div} v), \tag{1.41}$$

we get

$$-\Delta v = \operatorname{curl}\omega, \quad v|_{\partial\Omega} = 0. \tag{1.42}$$

Thus v can be solved from ω. In particular, if $\Omega = \mathbb{R}^3$ and v and ω have suitable decay at ∞, we have

$$v(x) = \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} \operatorname{curl}\omega(y)\, dy. \tag{1.43}$$

Integrating by parts, we get the *Biot-Savart Law*:

$$v(x) = \int_{\mathbb{R}^3} \nabla_x \frac{1}{4\pi|x-y|} \times \omega(y)\, dy. \tag{1.44}$$

Note that the derivation of the Biot-Savart Law only needs (1.40) and does not rely on the evolution equation (1.38).

One can estimate v in terms of w using (1.40) or (1.42) and the usual estimates for elliptic equations. In the case $\Omega = \mathbb{R}^3$ with the Biot-Savart Law, one can use the estimates for Riesz potentials. For $0 < \alpha < n$ and $g \in L^q(\mathbb{R}^n)$, the *Riesz potential* of g is defined as

$$(I_\alpha g)(x) := \int_{\mathbb{R}^n} |y|^{-n+\alpha} g(x-y)\, dy. \tag{1.45}$$

The following are their basic properties.

Theorem 1.1. *If $g \in L^q(\mathbb{R}^n)$, $1 \le q < \infty$, $0 < \alpha < n$, then $I_\alpha g$ converges absolutely almost everywhere. If $1 < q < p < \infty$, $0 < \alpha < n$, and $1/q = 1/p + \alpha/n$, then $I_\alpha g \in L^p(\mathbb{R}^n)$ and*

$$\|I_\alpha g\|_{L^p(\mathbb{R}^n)} \le C\|g\|_{L^q(\mathbb{R}^n)}, \tag{1.46}$$

for a constant $C = C(n,p,q)$ independent of g.

See [**199**, Chapter 5]. Also see [**72**, Lemma 7.12] for the bounded domain setting with more general exponents.

1.5. Pressure

In this section we consider the pressure.

The gradient ∇p of the pressure appears in (NS). We are allowed to add an arbitrary constant c to p since the pair $(v, p - c)$ still solves (NS). The pressure keeps the system consistent: Taking the div of (NS) and using $\operatorname{div} v = 0$, we get

$$-\Delta p = \partial_i\partial_j(v_i v_j) - \operatorname{div} f = (\partial_i v_j)(\partial_j v_i) - \operatorname{div} f. \tag{1.47}$$

Thus in general p is not trivial. In contrast, if we drop the divergence-free condition $\operatorname{div} v = 0$ in (NS), then we may take $p \equiv 0$ and get a usual semilinear parabolic system. Equation (1.47), similar to (1.42), is an elliptic equation and t only serves as a parameter. However, it alone does not determine p even after allowing an adding constant because we do not have a suitable boundary condition for p. The pressure is decided by the full (NS) and the Helmholtz decomposition to be considered in Section 1.6.

In some sense, p is the Lagrange multiplier due to the constraint $\operatorname{div} v = 0$. See the end of Section 1.6.

In vector calculus, a vector field of the form $g = \nabla p$ is called *conservative*, and its line integral along a path depends only on the endpoints of that path, not the particular route taken. This path independence property is equivalent to being conservative. It also satisfies

$$\partial_i g_j = \partial_j g_i, \quad \forall i, j. \tag{1.48}$$

In $\mathbb{R}^3$ this means $\operatorname{curl} g = 0$. Condition (1.48) implies that g is conservative if Ω is simply connected, but it may not be the case when Ω is multiply connected, as shown by the example

$$g(x) = \left(\frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2}\right) \quad \text{in } \Omega = \mathbb{R}^2 \setminus \{0\}. \tag{1.49}$$

In this case $p = \theta$ in polar coordinates and is multivalued in Ω.

Another property of a conservative vector field $g = \nabla p$ is that

$$\int_\Omega g \cdot \zeta = 0, \quad \forall \zeta \in C_c^\infty(\Omega; \mathbb{R}^n) \text{ with } \operatorname{div} \zeta = 0. \tag{1.50}$$

It is related to divergence-free vector fields and is essential to us. The following lemma shows that those g satisfying (1.50) are conservative.

Lemma 1.2. *Let Ω be an arbitrary domain in $\mathbb{R}^n$, $n \geq 2$. If $g \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ satisfies* (1.50)*, then there is a single-valued function $p \in W^{1,1}_{\mathrm{loc}}(\Omega)$ such that $g = \nabla p$.*

Note that (1.50) implies (1.48); see Problem 1.7. However, (1.50) is stronger than (1.48) since the pressure in Lemma 1.2 is single-valued, which excludes example (1.49).

We prove the case $g \in C(\Omega)$ for illustration. It suffices to show that

$$\int_\Gamma g \cdot d\vec{x} = 0, \tag{1.51}$$

for any piecewise smooth, closed curve Γ in Ω. Parametrize Γ by $\gamma(t)$: $[0,1] \to \Omega$, and let $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(x/\varepsilon)$, $0 < \varepsilon < \operatorname{dist}(\Gamma, \partial\Omega)$, be mollifiers with $\eta \in C_c^\infty(\mathbb{R}^n)$, $\int \eta = 1$, and $\operatorname{supp} \eta \subset B_1$. We have

$$\int_\Gamma g * \eta_\varepsilon \cdot d\vec{x} = \int_0^1 \int_{\mathbb{R}^n} g(y)\eta_\varepsilon(\gamma(t) - y) \cdot \gamma'(t) dy\, dt = \int_\Omega g \cdot \zeta_\varepsilon,$$

where $\zeta_\varepsilon(y) = \int_0^1 \eta_\varepsilon(\gamma(t) - y)\gamma'(t)\, dt$, $\zeta_\varepsilon \in C_c^\infty(\Omega; \mathbb{R}^n)$, and

$$\operatorname{div} \zeta_\varepsilon = -\int_0^1 \nabla \eta_\varepsilon(\gamma(t) - y) \cdot \gamma'(t)\, dt = -\int_0^1 \frac{d}{dt}\eta_\varepsilon(\gamma(t) - y)\, dt = 0.$$

By (1.50), we get $\int_\Gamma g * \eta_\varepsilon \cdot d\vec{x} = 0$. Taking limit $\varepsilon \to 0$, we get (1.51). Thus we can fix $x_0 \in \Omega$ and define

$$p(x) = p(x_0) + \int_\Gamma g \cdot d\vec{x}$$

along any Γ connecting x_0 to x. The case $g \in L^1_{\mathrm{loc}}$ is proved by approximation; see [**189**] or [**57**, Lemma III.1.1] for details.

Lemma 1.2 is extended to the case that g is merely a distribution in Lemma 2.2.

In the rest of this section we focus on the whole space case $\Omega = \mathbb{R}^3$. If v, f, and their derivatives have suitable decay at ∞, a solution of (1.47) is given by

$$\tilde{p}(x) = \int_{\mathbb{R}^3} \frac{1}{4\pi|y|}(\partial_i\partial_j(v_iv_j) - \operatorname{div} f)(x - y)\, dy. \tag{1.52}$$

If follows that $p - \tilde{p}$ in harmonic. One can show $p - \tilde{p}$ is a constant if p also has suitable decay at ∞, which may be hard to verify. If we only know the decay of v and f but not their derivatives, we usually integrate (1.52) by parts to get a revised definition of $\tilde{p}$: The contribution from f is

$$p_f(x) = -\int \left(\nabla \frac{1}{4\pi|y|}\right) \cdot f(x-y)dy. \tag{1.53}$$

The contribution from $v_i v_j$ is trickier because the kernels

$$K_{ij}(x) = \partial_i\partial_j \frac{1}{4\pi|x|} = -\frac{\delta_{ij}}{4\pi|x|^3} + \frac{3x_i x_j}{4\pi|x|^5} \tag{1.54}$$

are not locally integrable and we need to take into account the cancellation that the integral of K_{ij} over any sphere centered at the origin is zero. Moreover integration by parts also produces δ-functions at the origin. Taking the integral over $|y| > \varepsilon > 0$, integrating by parts, and then sending $\varepsilon \to 0_+$, we get the following formula (if v, f have suitable decay at infinity):

$$\tilde{p}(x) = -\frac{1}{3}|v|^2(x) + \lim_{\varepsilon\to 0+}\int_{|y|>\varepsilon} K_{ij}(y)(v_i v_j)(x-y)dy + p_f(x). \tag{1.55}$$

See Problem 1.8. An improper integral of the form in (1.55) using the cancellation of the kernel is called a *singular integral*.

In view of the Fourier transform of (1.47) that

$$|\xi|^2\hat{p} = -\xi_i\xi_j\widehat{v_i v_j} - i\xi\cdot\hat{f}, \tag{1.56}$$

one can also define

$$\tilde{p} = R_i R_j(v_i v_j) + p_f, \tag{1.57}$$

where the R_i are Riesz transforms, which are singular integrals with kernel $\frac{c_n x_i}{|x|^{n+1}}$ and Fourier symbol $\frac{i\xi_j}{|\xi|}$; that is, the Fourier transform of $R_j f$ is $\frac{i\xi_j}{|\xi|}\hat{f}(\xi)$.

For dimension $n \geq 2$, $n \neq 3$, formula (1.57) is still valid, and in formula (1.55) we replace $\frac{1}{3}$ by $\frac{1}{n}$ and let $K_{ij} = \partial_i\partial_j E$ where E is the *fundamental solution* of the Laplace equation in $\mathbb{R}^n$, $-\Delta E = \delta_0$; see (2.49).

The kernels mentioned above, K_{ij} and $\frac{c_n x_i}{|x|^{n+1}}$, are all examples of the *Calderon-Zygmund kernels* which include, in particular, any function $K(x)$ in $\mathbb{R}^n$ of the form

$$K(x) = \frac{P(x)}{|x|^{n+d}}, \tag{1.58}$$

where $P(x)$ is a homogeneous polynomial of degree $d \geq 1$ with $\int_{|x|=1} P = 0$. We recall the celebrated theorem of Calderon and Zygmund:

Theorem 1.3. *Suppose $K(x)$ is a Calderon-Zygmund kernel in $\mathbb{R}^n$. For any function $g \in L^q(\mathbb{R}^n)$, $1 < q < \infty$, the limit*

$$(Tg)(x) := \lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} K(y)g(x-y)\,dy \tag{1.59}$$

converges a.e. and in L^q, and

$$\|Tg\|_{L^q} \le C\|g\|_{L^q} \tag{1.60}$$

with a constant $C = C(n,q)$ independent of g.

Thus the singular integral operator T defined above is bounded in $L^q(\mathbb{R}^n)$. See [**199**, Chapter 2] for a treatise of singular integrals, including the proof of Theorem 1.3. Also see [**72**, Theorem 9.9] for the bounded domain setting.

We can estimate $\tilde{p}$ using Theorems 1.1 and 1.3. If $v \in L^q(\mathbb{R}^3)$, $2 < q < \infty$, and f can be ignored (say $f = 0$), then $\tilde{p} \in L^{q/2}(\mathbb{R}^3)$ by Theorem 1.3.

1.6. Helmholtz decomposition

For a given vector field f, we often need to decompose it in the form

$$f = u + \nabla p, \quad \operatorname{div} u = 0. \tag{1.61}$$

This is known as the *Helmholtz decomposition.* We are free to add a constant to p which does not change ∇p. This decomposition is not unique since

$$f = (u + \nabla h) + \nabla(p - h), \quad \forall h \text{ with } \Delta h = 0. \tag{1.62}$$

To make the decomposition unique, we need some boundary condition. In the whole $\mathbb{R}^3$, it is unique if we require u to be bounded, and the decomposition is given by (1.41): Let $\varphi = \frac{1}{4\pi|\cdot|} * f$, then

$$f = -\Delta\varphi = u + \nabla p, \quad u = \operatorname{curl}\operatorname{curl}\varphi, \quad p = -\operatorname{div}\varphi. \tag{1.63}$$

For other domains, the natural boundary condition turns out to be

$$u \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega; \tag{1.64}$$

see Lemma 1.4. Condition (1.64) would not be meaningful if we only require $u \in L^q(\Omega)$ for some q. But the condition $\operatorname{div} u = 0$ saves us: In view of the divergence theorem, we may define for any function $\phi \in C^1(\overline{\Omega})$

$$\int_{\partial\Omega} \phi u \cdot \mathbf{n} = \int_{\Omega} u \cdot \nabla\phi. \tag{1.65}$$

Thus $u \cdot \mathbf{n}|_{\partial\Omega}$ is well-defined in some dual space on $\partial\Omega$. For example, for bounded and smooth Ω we have $H^1(\Omega) \hookrightarrow H^{1/2}(\partial\Omega)$, and $u \cdot \mathbf{n}|_{\partial\Omega}$ can be considered an element of $H^{-1/2}(\partial\Omega)$. In contrast, the tangential part of $u|_{\partial\Omega}$ cannot be defined.

The exact meaning of the decomposition (1.61) depends on the space of f. For this purpose we define the following spaces. For a domain $\Omega \subset \mathbb{R}^n$, $n \ge 2$, define

$$C^\infty_{c,\sigma}(\Omega) = \{\zeta \in C^\infty_c(\Omega;\mathbb{R}^n) : \ \operatorname{div}\zeta = 0\}. \tag{1.66}$$

The subscript σ indicates divergence-free vector fields. For $n = 3$, this set contains all vector fields $\zeta = \operatorname{curl} A$ with $A \in C_c^\infty(\Omega;\mathbb{R}^3)$ and thus has many elements. For $1 < q < \infty$, define

$$L^q_\sigma(\Omega) = \overline{C^\infty_{c,\sigma}(\Omega)}^{L^q(\Omega)}, \tag{1.67}$$

$$G^q(\Omega) = \left\{g \in L^q(\Omega;\mathbb{R}^n) : \ g = \nabla p \text{ for some } p \in L^1_{\mathrm{loc}}\right\}. \tag{1.68}$$

We also have the following characterizations of $L^q_\sigma(\Omega)$; see [**121**, §2.2] and [**57**, III.2].

Lemma 1.4. *Suppose $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a domain with a locally Lipschitz boundary, possibly unbounded. Then for $1 < q < \infty$,*

$$\begin{aligned} L^q_\sigma(\Omega) &= \left\{u \in L^q(\Omega;\mathbb{R}^n) : \quad \int_\Omega u \cdot \nabla p = 0 \quad \forall \nabla p \in G^{q'}(\Omega)\right\} \\ &= \left\{u \in L^q(\Omega;\mathbb{R}^n) : \quad \operatorname{div} u = 0,\ u \cdot \mathbf{n}|_{\partial\Omega} = 0\right\}. \end{aligned} \tag{1.69}$$

The idea is the following. Equation (1.50) remains valid for $u \in L^q_\sigma(\Omega)$ by approximation. We first take those p with compact support to get $\operatorname{div} u = 0$ and then allow the support of p to touch the boundary $\partial\Omega$ to get $u \cdot \mathbf{n}|_{\partial\Omega} = 0$ using (1.65).

The orthogonality relation (1.50) extends to $u \in L^2_\sigma(\Omega)$ and $g \in G^2(\Omega)$. Thus we may rephrase the decomposition problem (1.61) as the unique existence problem of a pair $u \in L^2_\sigma(\Omega)$ and $g \in G^2(\Omega)$ or, equivalently, the validity of

$$L^2(\Omega;\mathbb{R}^n) = L^2_\sigma(\Omega) \oplus G^2(\Omega). \tag{1.70}$$

This is indeed valid for any domain Ω because L^2 is a Hilbert space. We extend this problem to general $1 < q < \infty$ and ask what is the validity of the *Helmholtz decomposition*

$$L^q(\Omega;\mathbb{R}^n) = L^q_\sigma(\Omega) \oplus G^q(\Omega). \tag{1.71}$$

If valid, the associated projection P_q from $L^q(\Omega;\mathbb{R}^n)$ to $L^q_\sigma(\Omega)$ satisfies $P_q = P_q^2$ and is a bounded operator; see [**169**, §5.15, 5.16]. It is called the *Helmholtz projection*. Note that $P_q\zeta$ is independent of $q \in (1,\infty)$ for $\zeta \in C_c^\infty(\Omega;\mathbb{R}^n)$; see Problem 1.9. In general the decomposition may fail, and it is easier to be valid for q closer to 2.

Theorem 1.5 (Helmholtz decomposition in L^q, [**189**]). *Suppose $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is either $\mathbb{R}^n$, $\mathbb{R}^n_+$, a C^1-bounded domain, or a C^1-exterior domain. Then for all $1 < q < \infty$ the decomposition (1.71) holds.*

There are counterexamples if $\partial\Omega$ is unbounded or only Lipschitz. For bounded Lipschitz domains, the validity is proved for $3/2 < q < 3$ in [**46**] and any $1 < q < \infty$ if the domain is convex in [**59**]. But for any $q \in (1, 3/2) \cup (3, \infty)$, a counterexample is found in [**46**]. A counterexample when Ω has unbounded boundary is $\Omega \subset \mathbb{R}^2$ defined by $x > -f(y)$ where $f(y)$ is an even smooth function with $f(y) = cy$ for $|y| \geq 1$, for some $c > 0$. (Ω is the complement of a regularized cone.) See [**13**] and [**57**, III.1].

The Helmholtz decomposition (1.71) is proved by studying the following equivalent problem: For any $f \in L^q(\Omega; \mathbb{R}^n)$, find a unique (up to a constant) function $p : \Omega \to \mathbb{R}$ so that

$$\nabla p \in L^q; \quad \int_\Omega (f - \nabla p) \cdot \nabla \phi = 0, \quad \forall \phi \in L^1_{\mathrm{loc}} \text{ with } \nabla \phi \in L^{q'}. \tag{1.72}$$

It is a weak form of the Neumann problem

$$\nabla p \in L^q; \quad \Delta p = \operatorname{div} f \text{ in } \Omega, \quad \frac{\partial p}{\partial \mathbf{n}} = f \cdot \mathbf{n} \text{ on } \partial\Omega. \tag{1.73}$$

The equivalence is proved in [**57**, Lemma III.1.2]. We already know that there is always a unique solution to (1.72) when $q = 2$. For q away from 2, existence or uniqueness may be lost.

We now consider a density theorem in the $W^{1,q}$ setting. Define

$$W^{1,q}_{0,\sigma}(\Omega) := \overline{C^\infty_{c,\sigma}(\Omega)}^{W^{1,q}(\Omega)}, \tag{1.74}$$

$$\hat{W}^{1,q}_{0,\sigma}(\Omega) := \left\{ v \in W^{1,q}_0(\Omega, \mathbb{R}^n) : \operatorname{div} v = 0 \right\}. \tag{1.75}$$

When $q = 2$, we also denote

$$H^1_{0,\sigma}(\Omega) = W^{1,2}_{0,\sigma}(\Omega). \tag{1.76}$$

We can similarly define $W^{k,q}_\sigma(\Omega)$ and $H^k_\sigma(\Omega)$.

One always has

$$W^{1,q}_{0,\sigma}(\Omega) \subset \hat{W}^{1,q}_{0,\sigma}(\Omega). \tag{1.77}$$

Are they the same? If they differ, we cannot prove things by approximation. This question was first raised by Heywood [**79**], who also pointed out its relevance to the unique solvability problem for (NS). The following theorem is a summary of [**57**, III.4] for domains considered in this book. See Notes in the end of [**57**, III] for references.

Theorem 1.6. *Suppose $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is either $\mathbb{R}^n$, $\mathbb{R}^n_+$, a bounded domain that satisfies an interior cone condition, or an exterior domain that satisfies an interior cone condition. Then for all $1 \leq q < \infty$, $C^\infty_{c,\sigma}(\mathbb{R}^n)$ is dense in $\hat{W}^{1,q}_{0,\sigma}(\Omega)$, and $W^{1,q}_{0,\sigma}(\Omega) = \hat{W}^{1,q}_{0,\sigma}(\Omega)$.*

For such a theorem, the shape of the domain is more a concern than the regularity of the boundary, and the problem for those domains with *unbounded boundary* is the most subtle. The two spaces are the same if Ω is the union of the open unit ball and several disjoint semicylinders originating from the ball. On the other hand, they are different if $n \geq 3$ and Ω is the union of the open unit ball and several "exits" with each exit containing an infinite cone. For example, for the domain inside a hyperboloid

$$\Omega = \left\{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1 + x_3^2\right\}, \tag{1.78}$$

one can show $W^{1,2}_{0,\sigma}(\Omega) \neq \hat{W}^{1,2}_{0,\sigma}(\Omega)$.

We finally explain the comment at the beginning of Section 1.5 that the pressure can be considered as a Lagrange multiplier. Consider the stationary Stokes equations for $v \in W^{1,2}_{0,\sigma}(\Omega)$

$$-\Delta v + \nabla p = f \quad \operatorname{div} v = 0. \tag{1.79}$$

A solution can be found by the variational problem

$$\min_{v \in W^{1,2}_{0,\sigma}(\Omega)} \int_\Omega \frac{1}{2}|\nabla v|^2 - v \cdot f. \tag{1.80}$$

A minimizer should satisfy

$$\int_\Omega \nabla v \cdot \nabla \zeta - \zeta \cdot f = 0, \quad \forall \zeta \in C^1_{c,\sigma}(\Omega). \tag{1.81}$$

If $g = -\Delta v - f \in L^1(\Omega)$, the above equation becomes $\int_\Omega g \cdot \zeta = 0$ for any $\zeta \in C^1_{c,\sigma}(\Omega)$. By Lemma 1.2, we have

$$g = -\Delta v - f = -\nabla p$$

for some scalar function $p \in W^{1,1}_{\mathrm{loc}}$. Thus we recover (1.79).

For an evolution equation of a vector field, we also get the Lagrange multiplier ∇p if the vector field is subject to the divergence-free constraint. This is the case for the incompressible Navier-Stokes equations (NS) and the Euler equations.

The following is a formal geometric view: If the equation is of the form

$$\frac{\partial}{\partial t} v = N(v) + \nabla p, \quad \operatorname{div} v = 0, \tag{1.82}$$

where N is a (linear or nonlinear) differential operator, we may think of it as an equation on the infinite-dimensional manifold $\mathcal{M}$ of divergence-free vector fields, and the Helmholtz projection $P : N(v) \mapsto N(v) + \nabla p$ maps $N(v)$ to the tangent space of $\mathcal{M}$.

1.7. Notes

For more details on the derivation of (NS) from physical principles, see, e.g., [**9**] and [**125**]. Section 1.6 is based on [**57**, III].

Problems

1.1. (Channel flow) Find the stationary solution of (NS) in the strip $(x, y) \in \mathbb{R} \times (0, h)$ of the form $v = f(y)e_x$, with $v(x, 0) = 0$ and $v(x, h) = e_x$. Also find the pressure.

1.2. (Couette flow) Find the stationary solution of (NS) in polar coordinates (r, θ) in the annulus $0 < r_1 < r < r_2$ of the form $v = f(r)e_\theta$, with $v(r_1, 0) = \alpha_1 e_\theta$ and $v(r_2, \theta) = \alpha_2 e_\theta$. Also find the pressure.

1.3. (Poiseuille flow) (i) Find the stationary solution of (NS) in cylindrical coordinates (r, θ, z) in the infinite cylinder $r < R$ of the form $v = f(r)e_z$ with $v(R, z) = 0$ and $v(0, z) = e_z$. Also find the pressure.

(ii) Repeat the problem in the region between two concentric cylinders $r_1 < r < r_2$ with suitable boundary conditions.

1.4. (Lamb-Oseen vortex with circulation α) Verify that

$$v(x,t) = \frac{\alpha}{2\pi} \frac{x^\perp}{|x|^2} \left(1 - e^{-\frac{|x|^2}{4(1+t)}}\right) \tag{1.83}$$

is a solution of (NS) for $x \in \mathbb{R}^2$. Here we denote $x^\perp = (-x_2, x_1)$ if $x = (x_1, x_2)$. Find ∇p and the vorticity ω.

1.5. Denote by $E(t)$ the evolution of a region E_0 under a fluid, $E(t) = \{x(t) : x(0) \in E_0\}$. Denote by σ the density of some physical property carried by the fluid, such as temperature, with no external source. Show that

$$\frac{d}{dt}\int_{E(t)} \sigma\, dx = \int_{E(t)} \partial_t \sigma\, dx + \int_{\partial E(t)} \sigma v \cdot n\, dS = 0. \tag{1.84}$$

1.6. Find a nonzero smooth divergence-free vector field with compact support in $\mathbb{R}^2$. Find another one in $\mathbb{R}^4$.

1.7. For any $g \in W^{1,1}(\Omega; \mathbb{R}^n)$, show that (1.50) implies (1.48).

1.8. Let $\tilde{p}$ be given by (1.52) in $\mathbb{R}^3$, and let v, f, and their derivatives have suitable decay at ∞. Prove (1.55).

1.9. Let Ω be a bounded smooth domain in $\mathbb{R}^n$ and let P_q be the Helmholtz projection on $L^q(\Omega; \mathbb{R}^n)$. Show that $P_q f$ is independent of $q \in (1, \infty)$ for $f \in C_c^\infty(\Omega; \mathbb{R}^n)$.

1.10. Verify that the following is an explicit solution of (NS) with zero force:

$$v = \nabla_x h(x,t), \quad p = -\partial_t h - \frac{1}{2}|\nabla h|^2 \tag{1.85}$$

where $\Delta_x h(x,t) = 0$ for each t. In particular, we may take $h(x,t) = f(t)H(x)$ where $\Delta H = 0$ and f is arbitrary.

Chapter 2

Steady states

In this chapter we study steady states of (NS), i.e., time-independent solutions of

$$\text{(SNS)} \qquad \begin{aligned} -\nu\Delta v + (v\cdot\nabla)v + \nabla p = f \\ \operatorname{div} v = 0 \end{aligned} \qquad \text{in } \Omega,$$

coupled with the boundary condition

$$v|_{\partial\Omega} = v_*. \tag{2.1}$$

Here Ω is a domain in $\mathbb{R}^n$, $n \geq 2$; $\nu > 0$ is the viscosity constant; f is a given volume force; and v_* is the boundary data. The unknown are the velocity v and the pressure p.

This chapter is concerned with the case $v_* = 0$. We will define weak solutions and discuss their existence, uniqueness, and nonuniqueness. In the rest of this chapter we study the regularity theory. We will consider the existence problem for nonzero v_* in Chapter 7.

2.1. Weak solutions

In this section we define weak solutions. Denote

$$\mathbf{V} = \dot{H}^1_{0,\sigma}(\Omega) \tag{2.2}$$

the closure of $C^\infty_{c,\sigma}(\Omega)$ in the norm $\|v\|_{\dot{H}^1_0(\Omega)} = \|\nabla v\|_{L^2(\Omega)}$. Note that $\mathbf{V} = H^1_{0,\sigma}(\Omega)$ when Ω is bounded. Let $\mathbf{V}'$ be the dual of $\mathbf{V}$, which contains $H^{-1}(\Omega, \mathbb{R}^n)$ as a strict subset. Assume $f \in \mathbf{V}'$. Then $|\langle f, \zeta\rangle| \leq \|f\|_{\mathbf{V}'}\|\nabla\zeta\|_{L^2}$. Formally testing (SNS) with $\zeta \in C^\infty_{c,\sigma}(\Omega)$, we get the weak forms:

$$\int \nu \nabla v : \nabla \zeta + (v \cdot \nabla) v \cdot \zeta \, dx = \langle f, \zeta \rangle , \quad \forall \zeta \in C^\infty_{c,\sigma}(\Omega), \tag{2.3}$$

or

$$\int -\nu v \cdot \Delta \zeta - v_i v_j \partial_j \zeta_i \, dx = \langle f, \zeta \rangle , \quad \forall \zeta \in C^\infty_{c,\sigma}(\Omega). \tag{2.4}$$

Here $\langle \, , \, \rangle$ is the pairing between $\mathbf{V}'$ and $\mathbf{V}$, and $\nabla v : \nabla \zeta \overset{\text{def}}{=\!=} (\partial_j v_i) \partial_j \zeta_i$.

Equation (2.4) requires less regularity of v than (2.3). To make sense of (2.4), we only need $v \in L^2_{\text{loc}}$, which is the same for all dimensions. To make sense of (2.3), we need $|v| \cdot |\nabla v| \in L^1_{\text{loc}}$, which is true if $v \in W^{1,3/2}_{\text{loc}}$ for $n = 3$ and $v \in W^{1,\frac{2n}{n+1}}_{\text{loc}}$ for general $n \geq 2$. In particular, (2.4) is meaningful if $v \in \mathbf{V}$ for all dimensions. The weak form of the divergence-free condition is

$$\int v \cdot \nabla \phi = 0, \quad \forall \phi \in C^\infty_c(\Omega). \tag{2.5}$$

If we assume $v \in \mathbf{V} \cap L^4(\Omega)$, one can substitute $\zeta = v$ in (2.3) and use (see Lemma 2.3)

$$\int_\Omega (v \cdot \nabla) v \cdot v \, dx = 0 \tag{2.6}$$

to get

$$\int \nu |\nabla v|^2 = \langle f, v \rangle \leq \|f\|_{\mathbf{V}'} \|\nabla v\|_{L^2}. \tag{2.7}$$

Thus

$$\int |\nabla v|^2 \leq M, \tag{2.8}$$

where $M = \nu^{-2} \|f\|^2_{\mathbf{V}'}$. The same estimate holds for the linear system without the nonlinear term. This a priori estimate will be the key to the construction of weak solutions.

Note that if we only assume $v \in \mathbf{V}$, then $v \in L^4(\Omega)$ for $n \leq 4$ by Sobolev imbedding and we have (2.6). However, for $n \geq 5$, in general $v \notin L^4(\Omega)$ and we cannot conclude (2.6). Nonetheless we can still use (2.8) to construct weak solutions.

We now define three kinds of weak solutions of (SNS).

Definition 2.1. 1) A vector field v defined on Ω is a *very weak solution* of (SNS) with force $f \in \mathbf{V}'$ if $v \in L^2_{\text{loc}}(\Omega)$ and it satisfies the weak form (2.4) and (2.5).

2) Let Ω be a bounded Lipschitz domain. A vector field v is a *weak solution* of (SNS) with force $f \in \mathbf{V}'$ and boundary value $v_* \in W^{1/2,2}(\partial\Omega)$ if

$v \in W^{1,2}(\Omega)$ is a very weak solution with force f and $v|_{\partial\Omega} = v_*$ in the sense of trace.

3) Similarly, a very weak solution v is a *q-weak solution* of (SNS) if $v \in W^{1,q}(\Omega)$.

For a very weak solution, since we only assume it is in L^2_{loc}, its boundary value has no meaning and cannot be specified. Thus it is a local concept. For weak solutions we can specify the boundary values. Note that $W^{1-1/q,q}(\partial\Omega)$ is the space for the boundary trace of functions in $W^{1,q}(\Omega)$ for $1 < q < \infty$ when Ω is locally lipschitzian; see Gagliardo [**55**] and [**57**, Theorem 3.3].

For weak solutions, (2.3) and (2.4) are equivalent. Since $f \in \mathbf{V}'$, by density, (2.3) remains valid for $\zeta \in \mathbf{V}$ for $n \le 4$ and $\zeta \in \mathbf{V} \cap L^n(\Omega)$ for $n \ge 5$. For (2.4) we would need to add boundary integrals involving $\nabla\zeta$ for $\zeta \in \mathbf{V}$.

For q-weak solutions, as mentioned earlier we need $q \ge \frac{2n}{n+1}$. Its existence is less clear since there is no L^q version of the a priori bound (2.8).

The above definitions can also be given for the stationary Stokes system

$$-\Delta v + \nabla p = f, \quad \operatorname{div} v = 0,$$

i.e., (SNS) without the nonlinear term $(v \cdot \nabla)v$.

Finally we note that in the above definitions the pressure is not involved. The pressure can be recovered using the following lemma which addresses the solvability of

$$\nabla p = g$$

for a given $g \in \mathbf{V}'$. For our purpose, g is given by

$$g(\zeta) := \langle f, \zeta \rangle - \int (\nu \nabla v : \nabla\zeta + (v \cdot \nabla)v \cdot \zeta)\, dx.$$

Let $D_0^{1,q}(\Omega)$ be the closure of $C_c^\infty(\Omega)$ in the norm

$$\|v\|_{D^{1,q}(\Omega)} = \|\nabla v\|_{L^q(\Omega)}. \tag{2.9}$$

It agrees with $W_0^{1,q}(\Omega)$ if Ω is bounded, and $D_0^{1,2}(\Omega) = \dot{H}_0^1(\Omega)$. Similarly we let $D_{0,\sigma}^{1,q}(\Omega)$ be the closure of $C_{c,\sigma}^\infty(\Omega)$ in $D^{1,q}(\Omega)$-norm. This space is introduced since we have no control of the L^q-norm of v by the a priori bound (2.8) if Ω is unbounded.

Lemma 2.2 (Solvability of $\nabla p = g$)**.** *Let $\Omega \subset \mathbb{R}^n$, $2 \le n < \infty$, be a Lipschitz bounded or exterior domain, $\Omega = \mathbb{R}^n$, or $\Omega = \mathbb{R}^n_+$. If g is a bounded linear functional on $D_0^{1,q}(\Omega, \mathbb{R}^n)$, $1 < q < \infty$, and $g(\zeta) = 0$ whenever $\operatorname{div}\zeta = 0$, then there is a unique $p \in L^{q'}(\Omega)$ ($\int_\Omega p\,dx = 0$ if Ω is bounded), such that*

$$g(\zeta) = -\int_\Omega p \operatorname{div}\zeta, \quad \forall \zeta \in D_0^{1,q}(\Omega); \quad \|p\|_{L^{q'}(\Omega)} \le C\|g\|_{(D_0^{1,q}(\Omega))'}. \tag{2.10}$$

See [**57**, Theorem III.5.3] or [**194**, II.2] for details. When $g \in L^1_{\mathrm{loc}}(\Omega)$, this follows from Lemma 1.2.

2.2. Small-large uniqueness

In this section we consider the uniqueness problem of (SNS). The general principle for bounded domains is that if there is a small solution, then there is no other. This is so-called *small-large uniqueness*. However, there may be no small solution but two large solutions for the same data. For unbounded domains, the small-large uniqueness principle is delicate and depends on which class of solutions we discuss.

We start with the following lemma.

Lemma 2.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3, 4$. Suppose $u, w \in W_0^{1,2}(\Omega)$ and $v \in W^{1,2}(\Omega)$. Then:*

(i) *For some constant $C_1 > 0$,*

$$\int_\Omega (u \cdot \nabla)v \cdot w \le \|u\|_4 \|\nabla v\|_2 \|w\|_4 \le C_1 \|\nabla u\|_2 \|\nabla v\|_2 \|\nabla w\|_2.$$

(ii) $\int_\Omega (v \cdot \nabla)u \cdot u = 0$ *if* $\operatorname{div} v = 0$. *Here u can be a scalar or a vector.*

Proof. (i) is a consequence of Hölder inequality and Sobolev imbedding. (ii) is true if $u \in C_c^\infty$. For general $u \in W_0^{1,2}$, choose $u_k \in C_c^\infty$ which converges to u in $W^{1,2}$. Using $\int_\Omega (v \cdot \nabla) u_k \cdot u_k = 0$, we have

$$\int_\Omega (v \cdot \nabla)u \cdot u = \int_\Omega (v \cdot \nabla)u \cdot (u - u_k) + \int_\Omega (v \cdot \nabla)(u - u_k) \cdot u_k$$

which is bounded by $\|v\|_4 (\|\nabla u\|_2 \|u - u_k\|_4 + \|\nabla(u - u_k)\|_2 \|u_k\|_4)$. In the limit $k \to \infty$ we get 0. □

For dimension $n \ge 5$, we need to add integrability assumptions. See Problem 2.1.

Theorem 2.4 (Small-large uniqueness)**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3, 4$. Let v and w be two weak solutions of* (SNS) *with the same force f and the same boundary value v_*. If moreover $C_1 \|\nabla v\|_2 < 1$, where C_1 is the constant in Lemma 2.3, then $w = v$.*

Proof. Let $u = w - v \in W_{0,\sigma}^{1,2}(\Omega)$. It satisfies

$$(\nabla u, \nabla \zeta) + (u \cdot \nabla u + u \cdot \nabla v + v \cdot \nabla u, \zeta) = 0, \quad \forall \zeta \in C_{c,\sigma}^\infty. \tag{2.11}$$

We can take $\zeta = u$ by taking $\zeta \in C_{c,\sigma}^\infty$, $\zeta \to u$ in $W^{1,2}$. By Lemma 2.3,

$$\|\nabla u\|_2^2 = -(u \cdot \nabla v, u) \le C_1 \|\nabla u\|_2^2 \|\nabla v\|_2.$$

Thus, if $C_1 \|\nabla v\|_2 < 1$, we must have $\|\nabla u\|_2 = 0$. □

Note that there is no assumption on $\|\nabla w\|_2$, and hence we avoid w in (2.11).

For unbounded domains, the small-large uniqueness principle may not always be valid. Consider the Liouville problem.

Conjecture 2.5 (Liouville problem). *If v is a bounded solution of* (SNS) *in $\mathbb{R}^3$ with zero force and*

$$\int_{\mathbb{R}^3} |\nabla v|^2 dx < \infty, \qquad \lim_{|x|\to\infty} |v(x)| = 0, \tag{2.12}$$

is it necessarily zero?

For the above problem, $v = 0$ is a trivial small solution, and we do not know if it is the only solution. The Liouville problem is important for the study of the hypothetical singularity of the time-dependent system (NS). See for example the study for nonlinear heat equations [**65**–**67**].

2.3. Existence for zero boundary data by the Galerkin method

In this section we give the first existence proof of weak solutions for a given force and zero boundary data, based on the *Galerkin method*. We will give a second proof based on the Leray-Schauder fixed point theorem in Section 2.4. Both methods are based on the a priori bound (2.8). The Galerkin method is very general and works for all dimensions $n \geq 2$, while the Leray-Schauder method only works for $n = 2, 3$. However, when one considers the boundary value problem with nonzero v_* in Chapter 7, the a priori bound (2.8) is unclear for the approximation solutions used by the Galerkin method.

The proof will make use of the following topological lemma, which is a consequence of the Brouwer fixed point theorem (see, e.g., [**42**, §8.1.4]).

Lemma 2.6. *Let $B_R = B(0, R) \subset \mathbb{R}^m$, $m \geq 1$, and let P be a continuous vector field on $\bar{B}_R$ satisfying*

$$P(x) \cdot x > 0, \quad \forall x \in \partial B_R. \tag{2.13}$$

Then there is $x_0 \in B_R$ such that $P(x_0) = 0$.

Proof. If the lemma was incorrect and $P(x)$ never vanishes, one could define

$$\Pi(x) = \frac{-RP(x)}{|P(x)|}.$$

Recall the Brouwer fixed point theorem: If $B_R \subset \mathbb{R}^m$ and $\Pi : \bar{B}_R \to \bar{B}_R$ is continuous, then there is $x_0 \in \bar{B}_R$ so that $\Pi(x_0) = x_0$. For our Π, we conclude $|x_0| = |\Pi(x_0)| = R$ and

$$P(x_0) \cdot x_0 = P(x_0) \cdot \Pi(x_0) = -R|P(x_0)| < 0,$$

contradicting (2.13). This contradiction shows the lemma. $\square$

We now give the existence theorem.

Theorem 2.7 (Existence)**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $2 \le n < \infty$. Let $\mathbf{V} = H^1_{0,\sigma}(\Omega)$, $f \in \mathbf{V}'$, and $v_* = 0$. Then there is a weak solution $v \in \mathbf{V}$ of* (SNS) *satisfying*

$$\int_\Omega |\nabla v|^2 \, dx \le \nu^{-2} \|f\|^2_{\mathbf{V}'}. \tag{2.14}$$

Note that Theorem 2.7 only asserts the existence, not the uniqueness.

Proof by the Galerkin method. We may assume $\nu = 1$. We first choose a basis. Let $\widetilde{\mathbf{V}}$ be the closure of $C^\infty_{c,\sigma}(\Omega)$ in $\mathbf{V} \cap L^n(\Omega)$, equipped with the norm $\|v\|_{H^1_0(\Omega)} + \|v\|_{L^n(\Omega)}$. We have $\widetilde{\mathbf{V}} \subset \mathbf{V}$, and $\widetilde{\mathbf{V}} = \mathbf{V}$ if $n \le 4$ by Sobolev imbedding $L^n \hookrightarrow \mathbf{V}$. Because $\widetilde{\mathbf{V}}$ is separable as a closed subspace of $H^1_0(\Omega)$, we can find a sequence $\{\phi_k\}_{k \in \mathbb{N}} \subset C^\infty_{c,\sigma}(\Omega)$ which is dense in $\widetilde{\mathbf{V}}$. By a Gram-Schmidt process, we can extract from ϕ_k a sequence $\psi_k \in C^\infty_{c,\sigma}(\Omega)$, $k \in \mathbb{N}$, whose linear span is dense in $\widetilde{\mathbf{V}}$ and

$$(\psi_k, \psi_m) = \delta_{km}, \quad \forall k, m \in \mathbb{N}. \tag{2.15}$$

Here $(f, g) = \int_\Omega f \cdot g$.

Step 1. For each integer m, we want to solve an approximation solution v_m in the span of $\{\psi_1, \ldots, \psi_m\}$, which satisfies

$$(\nabla v_m, \nabla \psi_k) + ((v_m \cdot \nabla) v_m, \psi_k) = \langle f, \psi_k \rangle, \quad k = 1, \ldots, m. \tag{2.16}$$

If $v_m = \sum_{j=1}^m x_j \psi_j$, this is a nonlinear algebraic equation for $x \in \mathbb{R}^m$. Define $P : \mathbb{R}^m \to \mathbb{R}^m$ by

$$P(x)_k = (\nabla v, \nabla \psi_k) + ((v \cdot \nabla) v, \psi_k) - \langle f, \psi_k \rangle, \quad v = \sum_{j=1}^m x_j \psi_j,$$

for $k = 1, \ldots, m$. Then P is a continuous vector function on $\mathbb{R}^m$ satisfying

$$\begin{aligned} P(x) \cdot x &= (\nabla v, \nabla v) + 0 - \langle f, v \rangle \\ &\ge \|\nabla v\|_2 \left(\|\nabla v\|_2 - \|f\|_{\mathbf{V}'}\right) \\ &\ge \|\nabla v\|_2 \left(c_m \|v\|_{L^2(K_m)} - \|f\|_{\mathbf{V}'}\right) \end{aligned}$$

where $K_m \Subset \Omega$ is the union of the supports of ψ_k, $1 \le k \le m$, and c_m is the constant of the Sobolev imbedding $H_0^1(K_m) \to L^2(K_m)$. The term $((v \cdot \nabla)v, v)$ is meaningful and vanishes since v is smooth.

Since $\|v\|_{L^2(K_m)} = |x|$ by (2.15), $P(x) \cdot x > 0$ if $|x| = R_m$ is sufficiently large. By Lemma 2.6, there is $x_0 \in \mathbb{R}^m$ with $|x_0| \le R_m$ so that $P(x_0) = 0$. Then $v_m = \sum_{j=1}^m x_{0,j}\psi_j$ solves (2.16). We do not have a uniform bound for R_m but it is fine.

Step 2. It follows from $P(x_0) \cdot x_0 = 0$ that $\int |\nabla v_m|^2 = \langle f, v_m \rangle$. Thus,

$$\int_\Omega |\nabla v_m|^2 \le \|f\|_{\mathbf{V}'}^2. \tag{2.17}$$

With this uniform bound, we can extract a subsequence, still denoted by v_m, which converges weakly in $W^{1,2}$ to some function $\bar v$, and strongly in $L^2(K)$ for any $K \Subset \Omega$. For any fixed k, the limit of (2.16) as $m \to \infty$ gives (since $\psi_k \in C^1$ with compact support)

$$(\nabla \bar v, \nabla \psi_k) + ((\bar v \cdot \nabla)\bar v, \psi_k) = \langle f, \psi_k \rangle, \quad \forall k. \tag{2.18}$$

Note that $((v\cdot\nabla)v, \zeta)$ is defined for $v \in \mathbf{V}$ and $\zeta \in L^n$. Since $\operatorname{span}\{\psi_k : k \in \mathbb{N}\}$ is dense in $\widetilde{\mathbf{V}}$, we conclude that (2.4) is valid for $v = \bar v$ and any $\zeta \in \widetilde{\mathbf{V}}$, in particular for any $\zeta \in C_{c,\sigma}^\infty(\Omega)$. Hence $\bar v$ is a weak solution of (SNS) for the given f. $\square$

The above approach can be extended to exterior domains. On an exterior domain Ω with zero boundary data, the weak solution is sought in the space $Y = D_{0,\sigma}^{1,2}(\Omega)$, defined in (2.9). The force f is in Y', the dual of Y. The space of test functions ζ is $\widetilde{Y}$ which is the closure of $C_{c,\sigma}^\infty(\Omega)$ in the norm $D^{1,2}(\Omega) \cap L^n(\Omega)$. Then the same proof goes through.

2.4. Existence for zero boundary data by the Leray-Schauder theorem

In this section we give the second existence proof of weak solutions using the *Leray-Schauder fixed point theorem*. This method involves compact operators and hence works only for dimension $n \le 3$ but is still applicable for the boundary value problem to be considered in Chapter 7.

Suppose $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $n \le 3$. We first reformulate (SNS) as a nonlinear functional equation in the Hilbert space $\mathbf{V} = W_{0,\sigma}^{1,2}(\Omega)$ with inner product

$$((u, v))_{\mathbf{V}} = \int_\Omega \nabla u : \nabla v, \quad \forall u, v \in \mathbf{V}.$$

Consider the Stokes system with zero boundary condition and $f \in \mathbf{V}'$

$$-\Delta v + \nabla p = f, \quad \operatorname{div} v = 0, \quad v|_{\partial\Omega} = 0. \tag{2.19}$$

Since the map

$$\zeta \mapsto \langle f, \zeta \rangle$$

is a linear functional on $\zeta \in \mathbf{V}$, the Riesz representation theorem gives a unique $T(f) \in \mathbf{V}$ such that

$$((T(f), \zeta))_{\mathbf{V}} = \langle f, \zeta \rangle, \quad \forall \zeta \in \mathbf{V}.$$

Note that $v = T(f)$ satisfies the weak form of (2.19). We check easily that the map

$$T : f \in \mathbf{V}' \to T(f) \in \mathbf{V}$$

is a bounded linear map. It is the *solution operator* of the Stokes system (2.19). We can now rewrite (SNS) as a Stokes system with force

$$N_f(v) = f - (v \cdot \nabla)v. \tag{2.20}$$

Note that $N_f : \mathbf{V} \to \mathbf{V}'$ for $2 \le n \le 4$, because $f \in \mathbf{V}'$ and

$$\left| \int (v \cdot \nabla) v \cdot \zeta \right| \le c \|v\|_4 \|\nabla v\|_2 \|\zeta\|_4 \tag{2.21}$$

and $\|v\|_4 \le c\|\nabla v\|_2$ by the Sobolev imbedding. Applying the operator T, we get

$$v = K(v) \tag{2.22}$$

where $K : \mathbf{V} \to \mathbf{V}$ is given by

$$K(v) \overset{\text{def}}{=\!=} \nu^{-1} T(N_f(v)). \tag{2.23}$$

Thus (SNS) with zero boundary condition is equivalent to (2.22), which is a fixed point problem for the operator K in the Hilbert space $\mathbf{V}$.

Observe that K is compact for $n \le 3$ because the Sobolev imbedding $\mathbf{V} \hookrightarrow L^4(\Omega)$ is compact. This is not so for $n = 4$. Details will be given after (2.26).

When f is sufficiently small in $\mathbf{V}'$, we can find a solution of (2.22) by iteration: Define the sequence

$$v_0 = 0; \quad v_j = K(v_{j-1}), \quad j \in \mathbb{N}. \tag{2.24}$$

Then v_j will converge to some $v \in \mathbf{V}$ which satisfies (2.22) that $v = K(v)$. See Problem 2.3.

For general $f \in \mathbf{V}'$, iteration does not converge. Instead, we will prove that (2.22) has a solution using the following simple version of the *Leray-Schauder fixed point theorem*.

Theorem 2.8 (Leray-Schauder)**.** *Let $K : X \to X$ be a continuous and compact mapping of a Banach space X into itself, such that the set*

$$\{x \in X : x = \sigma K x \text{ for some } \sigma \in [0,1]\} \tag{2.25}$$

is bounded. Then K has a fixed point $x_* = K x_*$.

See [**72**, Theorem 11.3] for a proof. The map $(x, \sigma) \mapsto \sigma K x$ in (2.25) is nonlinear in x but is linear in the parameter σ. See Theorem 7.12 for the more general nonlinear-in-σ version. The intuition is as follows: Since the solutions x are uniformly bounded in σ, the number of solutions x for a fixed σ should be an integer m which is independent of σ. Since $m = 1$ for $\sigma = 0$, we have $m = 1$ for $\sigma = 1$ and hence there is a solution for $\sigma = 1$.

Proof of Theorem 2.7 for $n \le 3$ by the Leray-Schauder theorem. Let the map $K(v)$ be defined by (2.20) and (2.23) in $\mathbf{V} = W^{1,2}_{0,\sigma}(\Omega)$. It maps $\mathbf{V}$ into itself by (2.21). To show its continuity, consider $v_1, v_2 \in \mathbf{V}$ and denote $w = v_2 - v_1$. We have

$$N_f(v_2) - N_f(v_1) = -v_2 \cdot \nabla w - w \cdot \nabla v_1$$

and

$$\int (N_f(v_2) - N_f(v_1)) \cdot \zeta = \int (v_2 \otimes w + w \otimes v_1) : \nabla \zeta$$

for $\zeta \in \mathbf{V}$. Thus, using the boundedness of $T : \mathbf{V}' \to \mathbf{V}$,

$$\begin{aligned} \|K(v_2) - K(v_1)\|_{\mathbf{V}} &\le c\|N_f(v_2) - N_f(v_1)\|_{\mathbf{V}'} \\ &\le c\|v_2 \otimes w + w \otimes v_1\|_{L^2} \\ &\le c(\|v_1\|_4 + \|v_2\|_4)\|w\|_4, \end{aligned} \tag{2.26}$$

which is bounded by $c(\|v_1\|_{\mathbf{V}} + \|v_2\|_{\mathbf{V}})\|v_2 - v_1\|_{\mathbf{V}}$. This shows the continuity of K.

We next show that K is compact. If a sequence $v_k \in \mathbf{V}$, $k \in \mathbb{N}$, is bounded in $\mathbf{V}$, i.e., $\|v_k\|_{H^1(\Omega)} \le C$, by the compactness of the Sololev imbedding $\mathbf{V} \hookrightarrow L^4(\Omega)$ for $n \le 3$, we can extract a subsequence v_{k_l}, $l \in \mathbb{N}$, which converges to some $v \in X$ in L^4-norm: $\|v\|_{H^1(\Omega)} \le C$ and $\|v_{k_l} - v\|_{L^4(\Omega)} \to 0$ as $l \to \infty$. By (2.26), we get $\|K(v_{k_l}) - K(v)\|_X \to 0$ as $l \to \infty$. This shows the compactness of K.

Consider now any solution $v \in \mathbf{V}$ of $v = \sigma K(v)$ for some $\sigma \in [0,1]$. Note that

$$\begin{aligned} ((v, v))_{\mathbf{V}} &= (\sigma K(v), v) = \sigma \nu^{-1} \langle f - v \cdot \nabla v, v \rangle \\ &= \sigma \nu^{-1} \langle f, v \rangle - \sigma \nu^{-1} \langle v \cdot \nabla v, v \rangle . \end{aligned}$$

By Lemma 2.3, we have $\langle v \cdot \nabla v, v \rangle = 0$ for $v \in \mathbf{V}$. Thus

$$\|\nabla v\|_2^2 = \sigma \nu^{-1} \langle f, v \rangle \le \nu^{-1} \|f\|_{\mathbf{V}'} \|\nabla v\|_2.$$

This shows that all solutions $v \in \mathbf{V}$ of $v = \sigma K(v)$ for some $\sigma \in [0,1]$ have the uniform bound

$$\|\nabla v\|_2 \le \nu^{-1} \|f\|_{\mathbf{V}'}. \tag{2.27}$$

We can now apply the Leray-Schauder theorem, Theorem 2.8, to conclude that there is a solution of $v = K(v)$. This finishes the proof of Theorem 2.7 for $n \le 3$. □

When Ω is unbounded, the previous argument does not work directly because we need to replace $\mathbf{V}$ by $Y = D^{1,2}_{0,\sigma}(\Omega)$ for the a priori bound, but N_f is not bounded from Y to Y': The only possible estimate is

$$\left| \int_\Omega (v \cdot \nabla) v \cdot \zeta \right| = \left| - \int_\Omega v_i v_j \partial_j \zeta_i \right| \le \|v\|_4^2 \|\zeta\|_Y,$$

but $\|v\|_4$ is not controlled by $\|v\|_Y$ in an unbounded domain in $\mathbb{R}^n$, $n \le 3$.

To overcome this difficulty, we can use Leray's method of *invading domains*.

Theorem 2.9 (Existence for unbounded domains)**.** *Let Ω be any domain in $\mathbb{R}^n$, $2 \le n < \infty$, which can be written as the union of increasing subdomains $\Omega_k \subset \Omega_{k+1} \subset \cdots \subset \Omega$, $k \in \mathbb{N}$, with each Ω_k being bounded and Lipschitz. Let $Y = D^{1,2}_{0,\sigma}(\Omega)$. For any $f \in Y'$, there is a weak solution $v \in Y$ of* (SNS) *with zero boundary condition and*

$$\int_\Omega |\nabla v|^2 \, dx \le M = \nu^{-2} \|f\|_{Y'}^2. \tag{2.28}$$

Proof by the method of invading domains. Let $Y_j = D^{1,2}_{0,\sigma}(\Omega_j)$. The restriction of f on Y_j satisfies $f \in Y_j'$ and $\|f\|_{Y_j'} \le \|f\|_{Y'}$. By Theorem 2.7, there is a weak solution $v_j \in Y_j$ of (SNS) with zero boundary condition in Ω_j and

$$\int_{\Omega_j} |\nabla v_j|^2 \, dx \le M.$$

There is a subsequence, still denoted by v_j, and some $\bar v \in Y$, such that

$$\begin{aligned} \nabla v_j &\rightharpoonup \nabla \bar v \quad \text{in } L^2(\Omega), \\ v_j &\to \bar v \quad \text{in } L^2(D), \quad \forall D \Subset \overline{\Omega}. \end{aligned}$$

Above, $\rightharpoonup$ denotes weak convergence. For any $\zeta \in C^\infty_{c,\sigma}(\Omega)$, $\operatorname{supp} \zeta \subset \Omega_j$ for j sufficiently large. Hence (2.3) is valid for $v = v_j$ and ζ for $j \gg 1$. By taking the limit $j \to \infty$ and using that ζ has compact support, (2.3) is also valid for $v = \bar v$ and the same ζ. This shows that $\bar v$ is a weak solution of (SNS) in Ω. □

We finally note that the two methods presented in Sections 2.3 and 2.4 are both based on the a priori bounds (2.8) and (2.27), which are explicit and computable. However, there are situations where an a priori bound exists but is not explicit. For example, for the boundary value problem in Chapter 7, the a priori bound is obtained by a contradiction argument. In that case it is hard to prove a similar bound for the approximation solutions of the Galerkin method, but we may still be able to use the Leray-Schauder fixed point theorem.

2.5. Nonuniqueness

In this section we construct a nonuniqueness example following Yudovich [**224**]. To do so, it is useful to consider the experimental facts about the *Couette-Taylor flow*, first studied analytically and experimentally by Taylor [**205**]. Consider a tank of water between two coaxial cylinders. The height of the tank is so much larger than the radii that it can be considered infinite. The water is at rest if both cylinders are fixed. Keep the outer cylinder fixed and gradually increase the angular velocity α of the inner cylinder. Initially the water has a steady movement rotating about the axis. (Mathematically this planar solution is always there no matter how large the angular velocity is.) After α exceeds a certain threshold α_c, the water gains vertical movement and each fluid particle moves on a torus-like surface. Thus we have two steady states: One does not have vertical movement and the other does. If one keeps increasing α, one will see more complicated structures, and then finally one sees *turbulence*, which is a flow regime characterized by chaotic changes in pressure and flow velocity. For a complete exposition see [**34**]. We first study it rigorously in Section 7.6.

Motivated by the Couette-Taylor flow, we move from the Cartesian to the *cylindrical coordinates* (r, θ, z) via the standard change of variables

$$x = (x_1, x_2, x_3) = (r\cos\theta, r\sin\theta, z) \tag{2.29}$$

and consider *axisymmetric* flows v which are vector fields of the form

$$v = v_r(r, z)e_r + v_\theta(r, z)e_\theta + v_z(r, z)e_z. \tag{2.30}$$

The components v_r, v_θ, v_z do not depend upon θ and the basis vectors e_r, e_θ, e_z are

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \quad e_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \quad e_z = (0, 0, 1). \tag{2.31}$$

In particular, e_r and e_θ are independent of r and z, and

$$\partial_\theta e_r = e_\theta, \quad \partial_\theta e_\theta = -e_r. \tag{2.32}$$

For an axisymmetric flow v, the only dependence on θ is through the basis vectors e_r and e_θ, and we have

$$|\nabla v|^2 = |\partial_r v|^2 + |\partial_z v|^2 + \left|\frac{1}{r}\partial_\theta v\right|^2 = |\partial_r v|^2 + |\partial_z v|^2 + \frac{1}{r^2}(v_r^2 + v_\theta^2) \tag{2.33}$$

and

$$\Delta v = \left(\Delta v_r - \frac{1}{r^2}v_r\right) e_r + \left(\Delta v_\theta - \frac{1}{r^2}v_\theta\right) e_\theta + (\Delta v_z)e_z. \tag{2.34}$$

The divergence free condition $\operatorname{div} v = 0$ is equivalent to

$$(\partial_r + r^{-1})v_r + \partial_z v_z = 0. \tag{2.35}$$

It does not involve v_θ since rotation is always volume-preserving.

The idea of the nonuniqueness example of Yudovich [**224**] is to look for two solutions of the form

$$v_\pm = V \pm u, \quad \operatorname{div} V = \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0 \tag{2.36}$$

where V will be chosen and u is the unknown. The Dirichlet condition for u ensures that $v_\pm$ have the same boundary value $v_* := V|_{\partial\Omega}$. With the above ansatz, we have

$$\begin{aligned} -\nu\Delta v_\pm + v_\pm \cdot \nabla v_\pm &= f_0 \pm f_1, \\ f_0 &= -\nu\Delta V + V\cdot\nabla V + u\cdot\nabla u, \\ f_1 &= -\nu\Delta u + V\cdot\nabla u + u\cdot\nabla V. \end{aligned} \tag{2.37}$$

If we can find V and u so that $f_1 = -\nabla\tau$ for some scalar τ, then $v_\pm = V \pm u$ are two distinct solutions of (SNS) with $v_* = V|_{\partial\Omega}$ and $f = f_0$. The corresponding pressures are $p_\pm = \pm\tau$.

The requirement for u is, for some scalar τ,

$$\begin{aligned} -\nu\Delta u + V\cdot\nabla u + u\cdot\nabla V + \nabla\tau = 0, \\ \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0. \end{aligned} \tag{2.38}$$

It says that u is in the kernel of the linearized Navier-Stokes system around V.

Our strategy is to choose V so that we can solve a nonzero solution u of this linear system. We now set $V = a(r)e_\theta$ and assume u is axisymmetric. We have $\operatorname{div} V = 0$ and

$$V\cdot\nabla u + u\cdot\nabla V = -\frac{2}{r}au_\theta e_r + \left(a' + \frac{1}{r}a\right)u_r e_\theta. \tag{2.39}$$

We now choose $a(r)$ so that $-\frac{2}{r}a = a' + \frac{1}{r}a$ (so that it has a variational structure). Thus $a = \alpha r^{-3}$ for some $0 \neq \alpha \in \mathbb{R}$, and (2.38) becomes

$$-\nu\Delta u - 2\alpha r^{-4}(u_\theta e_r + u_r e_\theta) + \nabla\tau = 0, \quad \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0. \tag{2.40}$$

Let $\Omega \subset \mathbb{R}^3$ be any bounded smooth solid of revolution about the z-axis which does not intersect the z-axis. Let X be the subspace of $H^1_{0,\sigma}(\Omega)$ consisting of axisymmetric vector fields u with

$$\|u\|_X^2 = \int_\Omega |\nabla u|^2 = \int_\Omega |\partial_r u|^2 + |\partial_z u|^2 + \frac{1}{r^2}(u_r^2 + u_\theta^2). \tag{2.41}$$

Consider the constrained variational problem

$$\min_{0 \neq u \in X} J[u], \text{ where } J[u] = \frac{\int_\Omega r^{-4} u_\theta u_r}{\int_\Omega |\nabla u|^2}. \tag{2.42}$$

The constraint is the divergence-free condition $\operatorname{div} u = 0$, or (2.35) which says that $(\partial_r + r^{-1})u_r + \partial_z u_z = 0$. Since our domain is away from the z-axis, $J[u]$ is bounded from below. Also $\min J < 0$ by choosing $u_\theta = -u_r$. Hence there is a minimizer u. From $\frac{d}{d\varepsilon}|_{\varepsilon=0} J[u + \varepsilon\zeta] = 0$ for any $\zeta \in X$, we get

$$\int \left[2J[u]\Delta u + r^{-4}(u_\theta e_r + u_r e_\theta)\right] \cdot \zeta = 0 \tag{2.43}$$

for any $\zeta \in C^\infty_{c,\sigma}(\Omega) \cap X$. By an axisymmetric version of Lemma 2.2,

$$4\alpha J[u]\Delta u + 2\alpha r^{-4}(u_\theta e_r + u_r e_\theta) = \nabla \tau \tag{2.44}$$

for some scalar function $\tau = \tau(r, z)$. Note that $\nabla\tau$ is the Lagrange multiplier due to the constraint $\operatorname{div} u = 0$. We get the first equation of (2.40) if we choose $\alpha = \frac{\nu}{4J[u]} < 0$.

Summarizing, we have shown the following.

Theorem 2.10. *Let $\Omega \subset \mathbb{R}^3$ be any bounded smooth solid of revolution which does not intersect its axis of revolution. Let $\nu > 0$. Then there exist a force f and a boundary value v_* such that the equations*

$$\begin{gathered} -\nu\Delta v + v \cdot \nabla v + \nabla p = f, \quad \operatorname{div} v = 0 \quad \textit{in } \Omega, \\ v|_{\partial\Omega} = v_* \end{gathered} \tag{2.45}$$

has two distinct solutions.

Moreover, for any u satisfying (2.38) for some V, for any $\varepsilon > 0$, $v_{\pm,\varepsilon} = V \pm \varepsilon u$ are two solutions of same boundary value V and body force $f_\varepsilon = -\nu\Delta V + V \cdot \nabla V + \varepsilon^2 (u \cdot \nabla) u$. Thus, the proof of Theorem 2.10 indeed gives a bifurcation family of solutions $v_{\pm,\varepsilon}$, $\varepsilon \geq 0$, for

$$\begin{gathered} -\nu\Delta v + v \cdot \nabla v + \nabla p = f_\varepsilon, \quad \operatorname{div} v = 0 \quad \text{in } \Omega, \\ v|_{\partial\Omega} = v_*. \end{gathered} \tag{2.46}$$

This is an example of *bifurcation*, a subject to be considered again in Sections 7.4–7.6 and 8.4.

Equation (2.38) for the vector field u can be formulated as an eigenvalue problem on the space X. Write $V = \alpha V_1$ and treat $\alpha \in \mathbb{R}$ as a parameter. Let

$$Au = -P\Delta u, \quad Ru = P(V_1 \cdot \nabla u + u \cdot \nabla V_1)$$

where P is the Helmholtz projection on Ω and A is the *Stokes operator* to be considered in Section 5.1. Then (2.38) is equivalent to

$$u \in X, \quad \nu Au + \alpha Ru = 0.$$

With $V_1 = r^{-3}e_\theta$, we can check that R is symmetric,

$$(Ru, v) = \int_\Omega \frac{-2}{r^4}(u_\theta v_r + u_r v_\theta) = (v, Ru), \quad \forall u, v \in X.$$

The fractional power of A, $A^{1/2}$, is well-defined. If we let $w = A^{1/2}u$, then w satisfies

$$Qw = \lambda w, \quad Q = A^{-1/2}RA^{-1/2}, \quad \lambda = -\frac{\nu}{\alpha}.$$

Note that Q is a linear compact selfadjoint operator on X. Thus all eigenvalues are real. Moreover, if (α, u) is an eigenpair, then $(-\alpha, \tilde{u})$ is also an eigenpair, where $\tilde{u} = u_r e_r - u_\theta e_\theta + u_z e_z$ when $u = u_r e_r + u_\theta e_\theta + u_z e_z$. Let λ_1 be the largest eigenvalue of Q, corresponding to $\alpha_1 < 0$. Our functional

$$J[u] = -\frac{(Ru, u)}{4(Au, u)} = -\frac{(Qw, w)}{4(w, w)}.$$

Its infimum is $-\frac{\lambda_1}{4} = \frac{\nu}{4\alpha_1}$. We have a sequence of eigenvalues $\pm\lambda_j$, $j \in \mathbb{N}$, corresponding to $\alpha = \mp\frac{\nu}{\lambda_j}$.

2.6. L^q-theory for the linear system

In the next section we consider the regularity of solutions of (SNS). For the existence results in the previous sections we only need the L^2-theory. For the regularity problem to be studied next, however, we need estimates in L^q-spaces, $1 < q < \infty$, for the linear, stationary Stokes system

$$-\Delta v + \nabla p = f, \quad \operatorname{div} v = 0. \tag{2.47}$$

This is the focus of this section.

2.6.1. Whole space. Let us first find the *Lorentz tensor*, which is the *fundamental solution*, of (2.47) in the entire $\mathbb{R}^n$, $n \geq 2$. Before that, we recall the fundamental solutions for the *Laplace equation* and the *biharmonic equation*,

$$-\Delta E = \delta_0, \quad -\Delta\Phi = E, \quad \Delta^2\Phi = \delta_0, \tag{2.48}$$

where δ_0 is the Dirac delta function supported at the origin. Recall for $r = |x|$, $E(x) = E(r)$ and

$$\partial_r E(r) = -\frac{1}{|S_r|} = -\frac{1}{n\omega_n r^{n-1}} \quad (n \geq 2),$$

where $\omega_n = |B_1^{\mathbb{R}^n}| = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$, and hence

$$E(x) = 2\kappa|x|^{2-n} \quad (n \geq 3); \quad E(x) = -2\kappa \log|x| \quad (n = 2), \tag{2.49}$$

where $\kappa = \frac{1}{2n(n-2)\omega_n}$ if $n \geq 3$ and $\kappa = \frac{1}{4\pi}$ if $n = 2$. We can integrate $\partial_r(r^{n-1}\Phi') = -r^{n-1}E$ to get an explicit formula for Φ:

$$\begin{aligned} \Phi(x) &= \frac{|x|^2}{8\pi}(\log|x| - 1) \quad (n = 2); \qquad \Phi(x) = -\kappa \log|x| \quad (n = 4); \\ \Phi(x) &= \frac{\kappa}{(n-4)}|x|^{4-n} \quad (n = 3 \text{ or } n \geq 5). \end{aligned} \tag{2.50}$$

The Lorentz tensor are U_{ij} and P_j so that, for each fixed j,

$$-\Delta U_{ij} + \partial_i P_j = \delta_{ij}\delta_0, \quad \partial_i U_{ij} = 0, \tag{2.51}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. In vector form with $\vec{U}_j = (U_{ij})_{i=1,\dots,n}$,

$$-\Delta \vec{U}_j + \nabla P_j = \delta_0(x)\, e_j, \quad \nabla \cdot \vec{U}_j = 0. \tag{2.52}$$

Above, e_j is the unit vector in the x_j-direction, $(e_j)_i = \delta_{ij}$.

If $f = (f_i)_{i=1,\dots,n}$ has suitable decay at infinity, a solution of (2.47) is given by

$$v_i = U_{ij} * f_j, \quad p = P_j * f_j,$$

where the convolution $*$ is taken over the entire $\mathbb{R}^n$. Note that if we do not require decay of v and p, we can get many other solutions by adding $(\nabla h, 0)$ to (v, p) where h is any harmonic function in $\mathbb{R}^n$. In particular, $(\nabla h, 0)$ is a solution to (2.47) in $\mathbb{R}^n$ with zero force.

To find U_{ij} and P_j, we can consider their Fourier transforms. We use the following convention of the Fourier transform:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx, \quad f(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} \hat{f}(\xi)e^{ix\cdot\xi}d\xi, \tag{2.53}$$

so that ∂_j corresponds to $\sqrt{-1}\xi_j$ and $\delta_0(x)$ is mapped to the constant function 1. The system (2.51) for U_{ij}, P_j becomes

$$|\xi|^2\hat{U}_{ij} + \sqrt{-1}\xi_i\hat{P}_j = \delta_{ij}, \quad \xi_i\hat{U}_{ij} = 0.$$

Multiplying the first equation by ξ_i and summing in i, we get

$$\hat{P}_j = -\sqrt{-1}\xi_j|\xi|^{-2}. \tag{2.54}$$

Thus we can solve

$$\hat{U}_{ij} = (\delta_{ij}|\xi|^2 - \xi_i\xi_j)|\xi|^{-4}. \tag{2.55}$$

Note that the Fourier transform of (2.48) gives

$$|\xi|^{-2} = \hat{E} \quad \text{and} \quad |\xi|^{-4} = \hat{\Phi}.$$

Inverse Fourier transforms of (2.54) and (2.55) then give

$$U_{ij} = \delta_{ij}E + \partial_i\partial_j\Phi, \quad P_j = -\partial_j E. \tag{2.56}$$

An alternative derivation of (2.56) without using the Fourier transform is as follows. Taking div of the first equation of (2.52), we get $\Delta P_j = \partial_j\delta_0$ in the sense of distributions. In view of (2.48), we can take $P_j = -\partial_j E$. Thus $-\Delta U_{ij} = \delta_{ij}\delta + \partial_i\partial_j E$, and we can take

$$U_{ij} = (\delta_{ij}\delta + \partial_i\partial_j E) * E = \delta_{ij}E + \partial_i\partial_j\Phi, \tag{2.57}$$

recovering (2.56).

We can get explicit formulas for the Lorentz tensor from (2.49), (2.50), and (2.56). For $n = 2$,

$$U_{ij}(x) = \frac{1}{4\pi}\left[-\delta_{ij}\log|x| + \frac{x_i x_j}{|x|^2}\right], \quad P_j(x) = \frac{1}{2\pi}\frac{x_j}{|x|^2}. \tag{2.58}$$

For $n \geq 3$,

$$U_{ij}(x) = \frac{1}{2n\omega_n}\left[\frac{\delta_{ij}}{(n-2)|x|^{n-2}} + \frac{x_i x_j}{|x|^n}\right], \quad P_j(x) = \frac{1}{n\omega_n}\frac{x_j}{|x|^n}. \tag{2.59}$$

Summarizing, for $n \geq 2$,

$$U_{ij}(x) = \frac{1}{2}\delta_{ij}E(x) + \frac{1}{2n\omega_n}\frac{x_i x_j}{|x|^n}, \quad q_j(x) = \frac{x_j}{n\omega_n|x|^n}. \tag{2.60}$$

They satisfy, for any $k \geq 0$ and $x \neq 0$,

$$|\nabla^{k+1}U_{ij}(x)| + |\nabla^k P(x)| \leq C_k|x|^{-n+1-k}. \tag{2.61}$$

2.6.2. Bounded regions. We next consider the Stokes system in a bounded region in $\mathbb{R}^n$, $n \geq 2$. For that purpose we first give an estimate for a vector field v satisfying the elliptic system

$$\operatorname{curl} v = \omega, \quad \operatorname{div} v = g. \tag{2.62}$$

Here for general dimension $n \geq 2$, we identify $\operatorname{curl} v$ with the vorticity tensor $\omega_{ij} = \partial_i v_j - \partial_j v_i$. It is elliptic in view of the *vector identity* when $n = 3$,

$$-\Delta v = \operatorname{curl}\operatorname{curl} v - \nabla(\operatorname{div} v). \tag{2.63}$$

Lemma 2.11 (Interior estimates). *Let $B_r \subset B_R \subset \mathbb{R}^n$, $n \geq 2$, be concentric balls with $0 < r < R$. Let $1 < q < \infty$, $0 < \alpha < 1$. Then there is a constant c depending on r, R, q and α so that, for any vector field v defined in B_R,*

$$\|\nabla v\|_{L^q(B_r)} \leq c\|\operatorname{div} v\|_{L^q(B_R)} + c\|\operatorname{curl} v\|_{L^q(B_R)} + c\|v\|_{L^1(B_R)} \tag{2.64}$$

and

$$\|\nabla v\|_{C^\alpha(B_r)} \leq c\|\operatorname{div} v\|_{C^\alpha(B_R)} + c\|\operatorname{curl} v\|_{C^\alpha(B_R)} + c\|v\|_{L^1(B_R)}. \tag{2.65}$$

Note that $\operatorname{curl} v$ and $\operatorname{div} v$ are part of ∇v. The lemma says that they control the other part of ∇v. We need $\|v\|_{L^1}$ on the right side in view of the example $v = \nabla h$ with a harmonic function h.

Proof. For $\omega_{ij} = \partial_i v_j - \partial_j v_i$, we have $\partial_j \omega_{ij} = \partial_i \operatorname{div} v - \Delta v_i$. Thus

$$\Delta v_i = \partial_j f_{ij}, \quad f_{ij} = \delta_{ij} \operatorname{div} v - \omega_{ij}.$$

By the interior Schauder estimate [**72**, Theorem 4.15],

$$\|\nabla v\|_{C^\alpha(B_r)} \leq c\|f\|_{C^\alpha(B_R)} + c\|v\|_{L^1(B_R)},$$

which gives (2.65). Let

$$w_i(x) = \partial_{x_j} \int_{B_R} E(x-y) f_{ij}(y) dy, \quad u = v - w.$$

We have $\Delta u = 0$ in B_R. By the Calderon-Zygmund estimate and Sobolev imbedding,

$$\|w\|_{W^{1,q}(B_R)} \leq c\|f_{ij}\|_{L^q(B_R)}.$$

By the interior estimate for harmonic functions,

$$\|\nabla u\|_{L^q(B_r)} \leq c\|u\|_{L^1(B_R)} \leq c\|v\|_{L^1(B_R)} + c\|w\|_{L^1(B_R)}.$$

Summing the estimates, we get (2.64). □

We now give interior estimates for the Stokes system. A *very weak solution* of (2.66) below is a divergence-free vector field v in L^1_{loc} that satisfies

$$\int v \cdot \Delta \zeta = \int f_{ij} \partial_j \zeta_i, \quad \forall \zeta \in C^\infty_{c,\sigma}.$$

A very weak solution of (2.68) is defined similarly, replacing the right side by $-\int g \cdot \zeta$.

Lemma 2.12 (Interior estimates for the Stokes system). *Let $B_r \subset B_R \subset \mathbb{R}^n$, $n \geq 2$, be concentric balls with $0 < r < R$. Assume that $v \in L^1(B_R)$ is a very weak solution of the Stokes system*

$$-\Delta v_i + \partial_i p = \partial_j f_{ij}, \quad \operatorname{div} v = 0 \quad \textit{in } B_R, \tag{2.66}$$

where $f_{ij} \in L^q(B_R)$, $1 < q < \infty$. *Then* $v \in W^{1,q}_{\mathrm{loc}}(B_R)$, *and there is* $p \in L^q_{\mathrm{loc}}(B_R)$ *so that the above equation is satisfied in the distributional sense and, for some constant* $c = c(q, r, R)$,

$$\|\nabla v\|_{L^q(B_r)} + \inf_{a\in\mathbb{R}} \|p - a\|_{L^q(B_r)} \le c\|f\|_{L^q(B_R)} + c\|v\|_{L^1(B_R)}. \tag{2.67}$$

Any distributional solution $(v, p) \in L^1_{\mathrm{loc}} \times L^1_{\mathrm{loc}}$ *of* (2.66) *must satisfy* (2.67).

If instead v *is a very weak solution of*

$$-\Delta v_i + \partial_i p = g_i, \quad \operatorname{div} v = 0 \quad \textit{in } B_R, \tag{2.68}$$

with $g_i \in L^q(B_R)$, $1 < q < \infty$, *then* $v \in W^{2,q}_{\mathrm{loc}}$ *and*

$$\|\nabla^2 v\|_{L^q(B_r)} \le c\|g\|_{L^q(B_R)} + c\|v\|_{L^1(B_R)}. \tag{2.69}$$

An important feature of these estimates is that a bound of the pressure p is not needed on the right side. This is desirable if we want to study local properties of weak solutions for which a priori we do not have any estimate of the pressure. One approach to prove such interior estimates is to multiply (v, p) by a cut-off function η and treat $(u, \pi) = (\eta v, \eta p)$ as solutions of the Stokes system in $\mathbb{R}^3$

$$-\Delta u + \nabla \pi = F, \quad \operatorname{div} u = v \cdot \nabla \eta, \tag{2.70}$$

where $F = \eta \operatorname{div} f + p\nabla\eta - 2(\nabla\eta \cdot \nabla)v - v\Delta\eta$. But F contains the term $p\nabla\eta$ and it would be necessary to include $\|p\|_{L^1}$ on the right sides of (2.67) and (2.69).

Proof of Lemma 2.12. Let η be the characteristic function of B_R and

$$\tilde v_i = \partial_k[U_{ij} * (\eta f_{jk})], \quad \tilde p = \partial_k[P_j * (\eta f_{jk})], \tag{2.71}$$

where U_{ij} and P_j are the fundamental solutions for the Stokes system. The L^q-estimates for singular integrals and Riesz potentials give

$$\|\nabla \tilde v\|_{L^q(B_R)} + \|\tilde p\|_{L^q(B_R)} + \|\tilde v\|_{L^q(B_R)} \le c\|f\|_{L^q(B_R)}. \tag{2.72}$$

The difference $u = v - \tilde v$ is a very weak solution of the homogeneous Stokes system

$$-\Delta u + \nabla \pi = 0, \quad \operatorname{div} u = 0, \quad \text{in } B_R, \tag{2.73}$$

i.e.,

$$\int u \cdot \Delta\zeta = 0 \quad \forall \zeta \in C^\infty_{c,\sigma}(B_R), \qquad \int u \cdot \nabla q = 0 \quad \forall q \in C^\infty_c(B_R). \tag{2.74}$$

Let $\delta = \frac{1}{3}(R - r)$. Let $\phi \in C^\infty_c(B_1)$ be a radial function with $\int \phi = 1$, $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$, $0 < \varepsilon \le \delta$, and $u^\varepsilon = u * \phi_\varepsilon$ the mollification of u. The smooth vector field u^ε also satisfies the weak form (2.74) of (2.73) in $B_{R-\delta}$;

thus ω^ε is weakly harmonic, $\Delta\omega^\varepsilon_{ij} = 0$ in $B_{R-\delta}$. The mean value property for harmonic functions gives, for $x \in B_{R-2\delta}$,

$$\begin{aligned} \omega^\varepsilon_{ij}(x) &= \int_{B_{R-\delta}} \phi_\delta(x-y)\omega^\varepsilon_{ij}(y)\,dy \\ &= \int_{B_{R-\delta}} \partial_i\phi_\delta(x-y)u^\varepsilon_j(y) - \partial_j\phi_\delta(x-y)u^\varepsilon_i(y)\,dy. \end{aligned}$$

Since ϕ_δ is smooth, for any $k \in \mathbb{N}$,

$$\|\omega^\varepsilon_{ij}\|_{W^{k,q}(B_{R-2\delta})} \le c_k\|u^\varepsilon\|_{L^1(B_{R-\delta})} \le c_k\|u\|_{L^1(B_R)}.$$

Take $k = 0$. By Lemma 2.11 and $\operatorname{div} u^\varepsilon = 0$, we have

$$\|u^\varepsilon\|_{W^{1,q}(B_r)} \le c\|u\|_{L^1(B_R)} \le c\|v\|_{L^1(B_R)} + c\|\tilde v\|_{L^1(B_R)}. \tag{2.75}$$

This uniform-in-ε bound shows that u has the same bound. Together with (2.72) we get $\|\nabla v\|_{L^q(B_r)}$ bound in (2.67).

We now consider the pressure. By Lemma 1.2, since u^ε satisfies (2.74) in B_R, there is a function $\pi^\varepsilon \in W^{-1,1}_{\mathrm{loc}}$ such that $\nabla\pi^\varepsilon = \Delta u^\varepsilon$, and hence

$$\int \pi^\varepsilon \operatorname{div}\zeta = \int \nabla u^\varepsilon : \nabla\zeta, \quad \forall \zeta \in Z = W^{1,q'}_0(B_r;\mathbb{R}^n).$$

We have

$$\inf_a \|\pi^\varepsilon - a\|_{L^q(B_r)} \le c \sup_{\zeta\in Z, \|\zeta\|_Z \le 1} \int \pi^\varepsilon \operatorname{div}\zeta \le c\|\nabla u^\varepsilon\|_{L^q(B_r)},$$

which is uniformly bounded by (2.75). We may redefine π^ε so that the inf is attained at $a = 0$. This uniform bound allows a subsequence of π^ε which converges weakly to some π in $L^q(B_r)$. The pair (u,π) is a distributional solution of (2.73). Together with (2.72) we get the pressure bound in (2.67).

The proof of (2.69) is similar: One defines

$$\tilde v_i = U_{ij} * (\eta g_j), \quad \tilde p = P_j * (\eta g_j) \tag{2.76}$$

and obtains $\|\tilde v\|_{W^{2,q}(B_R)} \le c\|g\|_{L^q(B_R)}$. One then estimates $\|\nabla^2(v-\tilde v)\|_{L^q(B_r)}$ in a similar way. □

In (2.67), we may replace $\inf_{a\in\mathbb{R}} \|p - a\|_{L^q(B)}$ by $\|p - (p)_B\|_{L^q(B)}$ since $\|p - (p)_B\|_{L^q(B)} = \|p - a - (p-a)_B\|_{L^q(B)} \le c\|p - a\|_{L^q(B)}$.

Finally, we state a global result in a domain.

Theorem 2.13. *Let Ω be a bounded C^2-domain in $\mathbb{R}^n$, $n \ge 2$, and $1 < q < \infty$. For every $f = (f_{ij}) \in L^q(\Omega)$, there is a unique q-weak solution $v \in W^{1,q}_0(\Omega)$ of (2.66) in Ω satisfying*

$$\|v\|_{W^{1,q}_0(\Omega)} \le C\|f\|_{L^q(\Omega)}. \tag{2.77}$$

For every $g = (g_i) \in L^q(\Omega)$, *there is a unique* q*-weak solution* $v \in W^{2,q} \cap W_0^{1,q}(\Omega)$ *of* (2.68) *in* Ω *satisfying*

$$\|v\|_{W^{2,q}(\Omega)} \le C\|g\|_{L^q(\Omega)}. \tag{2.78}$$

There is also pressure p, *and we can insert* $\|p\|_{q/\mathbb{R}} = \inf_a \|p-a\|_{L^q(\Omega)}$ *and* $\|\nabla p\|_{L^q(\Omega)}$ *in the left sides of* (2.77) *and* (2.78), *respectively.*

The case $q = 2$ for weak solutions is treated in Sections 2.3 and 2.4. The proof for the general case is similar to that for the Laplace equation and can be found in [**57**, Theorem IV.6.1]. We give a sketch below.

We first get a formula and estimates for the Green's tensor for the half-space. We next derive estimates for solutions of (2.66) or (2.68) near a flat boundary. We then get estimates in a small region near a curved boundary by a C^2 map and the coefficient freezing technique. Applying a partition of unity argument, we get

$$\begin{aligned} &\|v\|_{W_0^{1,q}(\Omega)} \le c\|f\|_{L^q(\Omega)} + c\|v\|_{L^q(\Omega)} + c\|p\|_{q/\mathbb{R}}, \quad \text{or} \\ &\|v\|_{W_0^{2,q}(\Omega)} \le c\|g\|_{L^q(\Omega)} + c\|v\|_{L^q(\Omega)} + c\|p\|_{q/\mathbb{R}}. \end{aligned} \tag{2.79}$$

The uniqueness of q-weak solutions of (2.66) is immediate for $q \ge 2$ using the a priori bound (2.8). The uniqueness of q-weak solutions for $q < 2$ can be obtained by using (2.79) to improve regularity. With the uniqueness result, we can remove $\|v\|_{L^q(\Omega)} + \|p\|_{q/\mathbb{R}}$ on the right side of (2.79) to get (2.77) and (2.78), by a contradiction argument. Finally, the existence is obtained by using the a priori bounds (2.77) and (2.78) and an approximation argument.

2.7. Regularity

In this section we study the regularity of solutions of the (nonlinear) stationary Navier-Stokes equations (SNS). We may take $\nu = 1$. It is useful to view (SNS) as a *perturbed Stokes system*

$$-\Delta v + b \cdot \nabla v + \nabla p = f, \quad \operatorname{div} v = \operatorname{div} b = 0, \tag{2.80}$$

with $b = v$. A solution v of (SNS) is called *regular* at a point x_0 if it is bounded or Hölder continuous in a neighborhood of x_0, since the nonlinear term $b \cdot \nabla v$, $b \in L^\infty_{\mathrm{loc}}$, can then be treated as a linear lower-order term, and one can show as much regularity as the data f allow, as if the nonlinear term is absent.

To get some insight, consider the following elliptic equation for a scalar function $v(x)$:

$$-\Delta v + b \cdot \nabla v + cv = f, \tag{2.81}$$

where $b(x)$ is a vector function called the *convection* and $c(x)$ is a scalar function called the *potential*. Suppose we have a weak solution v of (2.81) and we want to prove higher regularity for v. A typical strategy is to treat $b\cdot\nabla v+cv$ as a perturbation of the main term Δv. Since $\nabla|x|^{-s}=O(|x|^{-s-1})$, the Laplace operator Δ is of the same order and scaling as $|x|^{-2}$. The critical orders for b and c are

$$b(x)\sim |x|^{-1}\in L^n_{wk},\quad c(x)\sim|x|^{-2}\in L^{n/2}_{wk}. \tag{2.82}$$

Here L^r_{wk} denotes the weak L^r-space. Thus one expects to prove the regularity of v for *subcritical* cases (see [**72**])

$$b(x)\in L^r,\quad c(x)\in L^{r/2},\quad r>n. \tag{2.83}$$

One cannot expect regularity if one only assumes (2.82), as the example $v(x)=|x|^{-s}$ shows.

What if we add some smallness assumptions on b and c? Consider the function $v(x)=(\log|x|)^{-1}$, which is bounded but not Hölder continuous. We have

$$\nabla v=-v^2\frac{x}{|x|^2},\quad \Delta v=\frac{1}{|x|^2}(2v^3+(2-n)v^2).$$

Thus v satisfies in $B_{1/2}$

$$-\Delta v+cv=0,\quad c=\frac{\Delta v}{v}=\frac{1}{|x|^2}(2v^2+(2-n)v). \tag{2.84}$$

Thus $c(x)\in L^{n/2}(B_{1/2})$, $\lim_{r\to 0}\|c\|_{L^{n/2}(B_r)}=0$, and $\lim_{|x|\to 0}|x|^2c(x)=0$ for all $n\ge 2$. Smallness of c in a critical norm does not help.

On the other hand, v also satisfies

$$-\Delta v+b\cdot\nabla v=0,\quad b=-(2v+2-n)\frac{x}{|x|^2}. \tag{2.85}$$

But $b\notin L^n$ and $\lim_{|x|\to 0}|x|b(x)\neq 0$ if $n\ge 3$. (If $n=2$, $b\in L^2$ and $\lim_{|x|\to 0}|x|b(x)=0$.) Thus either $b\in L^n$ or $\lim_{|x|\to 0}|x|b(x)=0$ may prevent singularity for $n\ge 3$. See Problem 2.7.

This example shows that the convection term usually has better properties than the potential term if they are both in critical spaces. That is why we write the nonlinear term in (2.80) as $b\cdot\nabla v$, not cv with $c=(\partial_j v_i)_{ij}$.

When we further assume $\operatorname{div} b=0$, we can relax the integrability condition on b. The idea is that we may shift the gradient to the test function and gain one factor of $|x|$,

$$\int(b\cdot\nabla v)\phi=-\int bv\cdot\nabla\phi.$$

Specifically, we may prove $v\in L^\infty$ if $|b(x)|\le C|x|^{-s}$ for some $s<2$, but we may prove v is Hölder continuous only if $|b(x)|\le C|x|^{-1}$ for arbitrarily large C. See [**106**].

We now prove the regularity of solutions of (SNS) in the 3-dimensional case.

Theorem 2.14 (Interior regularity)**.** *Let Ω be a domain in $\mathbb{R}^3$. Suppose $v \in L^2(\Omega)$ is a very weak solution of* (SNS) *in Ω with smooth f. Then v is smooth if*

(i) *$v \in L^r_{\mathrm{loc}}(\Omega)$, $r > 3$,*

(ii) *$v \in L^3_{\mathrm{loc}}(\Omega)$, or*

(iii) *$v \in L^3_{wk}(\Omega)$ and $\|v\|_{L^3_{wk}(\Omega)}$ is sufficiently small.*

Case (iii) implies (i) and (ii) since $\|v\|_{L^3_{wk}(B_\rho)} \le \|v\|_{L^3(B_\rho)} \to 0$ as $\rho \to 0$. We will prove (i) first and then prove (iii) using (i) as a lemma.

Note that there is no pressure assumption in all cases. Also note that if $|v(x)| \le \frac{C_*}{|x|}$ in B_1, the theorem is applicable only if C_* is small.

Recall that $L^q_{wk} = L^{q,\infty}$ are members of the *Lorentz spaces* $L^{q,r}(\Omega)$. For $1 \le q \le \infty$ and $1 \le r \le \infty$, $L^{q,r}(\Omega)$ is the space of functions with finite quasi-norm

$$\|f\|_{L^{q,r}(\Omega)} = \begin{cases} \left(\int_0^\infty \left(t^{1/q} f^*(t)\right)^r \frac{dt}{t}\right)^{1/r} & \text{if } 1 \le r < \infty, \\ \sup_{t>0} t^{1/q} f^*(t) & \text{if } r = \infty, \end{cases}$$

where $f^* : \mathbb{R}_+ \to \mathbb{R}_+$,

$$f^*(t) = \inf \{\sigma : |\{x \in \Omega : |f(x)| > \sigma\}| \le t\}$$

is nonincreasing and right semicontinuous. An equivalent norm for L^q_{wk}, $1 \le q < \infty$, is

$$\|f\|_{L^q_{wk}} = \sup_{\sigma>0} \sigma |\{x \in \Omega : |f(x)| > \sigma\}|^{1/q}.$$

We have $L^{q,q} = L^q$, $L^{q,r_1} \subset L^{q,r_2}$ if $r_1 \le r_2$, and if $|\Omega|$ is finite, $L^{q_1,r_1} \subset L^{q_2,r_2}$ if $q_1 > q_2$. The function $|x|^{-n/q}$ is in $L^{q,\infty}(\mathbb{R}^n)$ but not in $L^q(\mathbb{R}^n)$. For $q > 1$, it is possible to replace the quasi-norm with a norm, which makes $L^{q,r}$ a Banach space. See [**11**, §1.3].

In view of the elliptic problem (2.81), the assumption $v \in L^3_{\mathrm{loc}}$ is on the borderline. For regularity theory, we often blow up a small neighborhood of the point in question by rescaling. In view of the natural scaling (1.21), the condition $v \in L^r_{\mathrm{loc}}$, $r \ge 3$, ensures that the scaled solution $v^\lambda(x) = \lambda v(\lambda x)$ satisfies

$$\lim_{\lambda \to 0} \|v^\lambda\|_{L^r(B_1)} = 0. \tag{2.86}$$

However, if $v \in L^3_{wk}$, the $\|v^\lambda\|_{L^3_{wk}(B_1)}$ are uniformly bounded in λ but may not go to zero as $\lambda \to 0$. For example, $\|v^\lambda\|_{L^3_{wk}(B_1)}$ is constant for $v(x) = |x|^{-1}$.

We now prove case (i) of Theorem 2.14.

Proof of Theorem 2.14(i). Rewrite (SNS) as

$$-\Delta v + \nabla p = f - \partial_j(v_j v), \quad \operatorname{div} v = 0. \tag{2.87}$$

Let

$$r_0 = r, \quad \frac{1}{r_{k+1}} = \frac{2}{r_k} - \frac{1}{3} \quad (k = 0, 1, \ldots).$$

By induction one has

$$\frac{1}{r_k} - \frac{1}{r_{k+1}} = \frac{1}{3} - \frac{1}{r_k} \geq \frac{1}{3} - \frac{1}{r} > 0.$$

Thus in finite steps, r_{k+1} becomes negative. For any $B(x_0, R_0) \subset \Omega$, let $B_k = B(x_0, 2^{-k}R_0)$. By repeatedly using Sobolev imbedding and (2.67) of Lemma 2.12, we have

$$\begin{aligned} \|v\|_{L^{r_{k+1}}(B_{k+1})} &\leq c\|\nabla v\|_{L^{r_k/2}(B_{k+1})} + c\|v\|_{L^1}, \\ \|\nabla v\|_{L^{r_k/2}(B_{k+1})} &\leq c\|v\|^2_{L^{r_k}(B_k)} + c\|f\|_{-1,r_k/2} + c\|v\|_{L^1}. \end{aligned}$$

In finite steps, $r_{k+1} < 0$, and hence v is Hölder continuous. (If $r_{k+1} = 0$, we can slightly decrease r_0, still greater than 3, so that $r_{k+2} < 0 < r_{k+1}$.)

Now $v \cdot \nabla v \in L^{r_k/2}_{\text{loc}}$ with $r_k/2 > 3$. We have $\nabla^2 v \in L^{r_k/2}_{\text{loc}}$ by (2.69) of Lemma 2.12, and hence ∇v is also Hölder continuous by Sobolev imbedding. We can differentiate (2.87) to get higher derivative estimates using the previous results. Hence we get the regularity of v as far as the regularity of f allows. $\square$

The proof of case (iii) of Theorem 2.14 is based on the following lemma.

Lemma 2.15. *Let $B = B_R$ be a ball in $\mathbb{R}^n$, $n \geq 2$, and $1 < q < n$. Then*

$$\begin{aligned} \|fg\|_{L^q(B)} &\leq C\|f\|_{L^n_{wk}(B)}\|g\|_{L^{\frac{nq}{n-q},q}(B)} \\ &\leq C\|f\|_{L^n_{wk}(B)}\|g\|_{W^{1,q}(B)}. \end{aligned}$$

Above, $C = C(n, q, R)$ and $L^{s,q}$ denotes a Lorentz space.

The first inequality uses Hölder inequality in Lorentz spaces due to O'Neil [**160**]. The second inequality uses Sobolev imbedding of Lorentz spaces due to Kim-Kozono [**101**]. If we have $\|f\|_{L^n(B)}$ bound, then it suffices to use the usual Hölder inequality and Sobolev imbedding.

We will also need the following lemma to find a vector field with a given divergence. Its most general form is due to Bogovskii [**12**]; see [**194**, II.2.1.1] and [**57**, III.3]. Note that Lemma 2.16 is a dual result of Lemma 2.2.

Lemma 2.16 (Solvability of $\operatorname{div} v = g$)**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $2 \le n < \infty$. There is a linear map Φ that maps a scalar $f \in L^q(\Omega)$ with $\int_\Omega f = 0$, $1 < q < \infty$, to a vector field $v = \Phi f \in W_0^{1,q}(\Omega;\mathbb{R}^n)$ and*

$$\operatorname{div} v = f, \quad \|v\|_{W_0^{1,q}(\Omega)} \le c(\Omega, q)\|f\|_{L^q(\Omega)}.$$

The map Φ is independent of q for $f \in C_c^\infty(\Omega)$.

This lemma is often used to localize divergence-free vector fields: If $\operatorname{div} v = 0$ and ϕ is a cut-off function, we define

$$\tilde v = v\phi + w, \quad w = -\Phi(v \cdot \nabla\phi).$$

Then $\tilde v$ has compact support, $\operatorname{div}\tilde v = 0$, and $\|\tilde v\|_{W^{1,q}} \le c_q\|v\|_{W^{1,q}}$.

We postpone the proof of Lemma 2.16 to Section 2.8.

With the help of Lemma 2.15 and Lemma 2.16, we can prove the following unique existence lemma.

Lemma 2.17 (Unique existence for perturbed Stokes system)**.** *Let $B = B_R$ be a ball in $\mathbb{R}^3$. For any $1 < q < 3$, there is a small constant $\varepsilon_* = \varepsilon_*(q, R) > 0$ such that if $\|b\|_{L^3_{wk}(B)} \le \varepsilon_*$ and $\operatorname{div} b = 0$, then for any $f \in W^{-1,q}(B)$ and $g \in L^q(B)$ with $\int g = 0$, there is a unique q-weak solution $u \in W_0^{1,q}(B)$ of*

$$-\Delta u + \operatorname{div}(b \otimes u) + \nabla p = f, \quad \operatorname{div} u = g, \quad \text{in } B. \tag{2.88}$$

Furthermore, it satisfies

$$\|\nabla u\|_{L^q(B)} \le C(q, R)\left(\|f\|_{W^{-1,q}(B)} + \|g\|_{L^q(B)}\right).$$

Above, $W^{-1,q}(B)$ is the dual space of $W_0^{1,q'}(B)$. Any $f \in W^{-1,q}(B)$ can be written in the form

$$f = \operatorname{div} F, \quad \|F\|_{L^q(B)} \sim \|f\|_{W^{-1,q}(B)}.$$

Proof. By Lemma 2.16, we can choose $V \in W_0^{1,q}(B;\mathbb{R}^3)$ so that $\operatorname{div} V = g$ and $\|V\|_{W^{1,q}(B)} \le c_3\|g\|_q$. Decompose $u = w + V$. The difference $w = u - V \in W_0^{1,q}(B;\mathbb{R}^3)$ satisfies

$$-\Delta w + \nabla\pi = F(w), \quad \operatorname{div} w = 0, \quad \text{in } B,$$

where

$$F(w) := f - \operatorname{div}(b \otimes (V + w)).$$

Denote by $\Psi : W^{-1,q}(B;\mathbb{R}^3) \to W_0^{1,q}(B;\mathbb{R}^3)$ the bounded solution operator for the Stokes system given by Theorem 2.13. Then w solves

$$w = K(w) := \Psi(F(w)).$$

By Theorem 2.13 and Lemma 2.15 with $q \in (1,3)$, for $w \in W_0^{1,q}(B;\mathbb{R}^3)$ we have

$$\begin{aligned}\|K(w)\|_{W_0^{1,q}(B)} &\le c\|F(w)\|_{W^{-1,q}(B)}\\ &\le c_1\|f\|_{W^{-1,q}(B)} + c_1\|b\otimes(V+w)\|_{L^q(B)}\\ &\le c_1\|f\|_{W^{-1,q}(B)} + c_1c_2\|b\|_{L^3_{wk}(B)}\cdot\|V+w\|_{W^{1,q}(B)}.\end{aligned}$$

If we choose $\varepsilon_* = (2c_1c_2)^{-1}$ so that $c_1c_2\|b\|_{L^3_{wk}(B)} \le \frac{1}{2}$, then

$$\|K(w)\|_{W_0^{1,q}(B)} \le N + \frac{1}{2}\|w\|_{W_0^{1,q}(B)}$$

where $N = c_1\|f\|_{W^{-1,q}(B)} + \frac{1}{2}c_3\|g\|_{L^q(B)}$. Thus K maps the set $W = \{w \in W^{1,q}_{0,\sigma}(B) : \|w\|_{W_0^{1,q}(B)} \le 2N\}$ into itself. Moreover, for $w, \tilde{w} \in W_0^{1,q}$, $K(w) - K(\tilde{w}) = -\Psi(\operatorname{div}(b\otimes(w-\tilde{w})))$ and hence

$$\|K(w) - K(\tilde{w})\|_{W^{1,q}} \le c_1c_2\|b\|_{L^3_{wk}}\|w-\tilde{w}\|_{W^{1,q}} \le \frac{1}{2}\|w-\tilde{w}\|_{W^{1,q}}. \tag{2.89}$$

Thus K is a contraction mapping in W and has a unique fixed point. This proves the unique existence of the solution for (2.88) in W. Note that (2.89) is valid for all $w, \tilde{w} \in W^{1,q}_{0,\sigma}(B)$ with no size restriction. Thus the uniqueness holds in the entire $W^{1,q}_{0,\sigma}(B)$. □

We can now prove case (iii) of Theorem 2.14.

Proof of Theorem 2.14(iii). Fix $q \in (1, \frac{3}{2})$ and $r = q^* \in (\frac{3}{2}, 3)$. Recall $1/q^* = 1/q - 1/3$. Denote $\varepsilon = \|v\|_{L^3_{wk}(\Omega)} \ll 1$.

For any $B(x_0, 2R) \subset \Omega$, define

$$u(x) = Rv(Rx + x_0), \quad \tilde{f}(x) = R^3 f(Rx + x_0).$$

We have that

$$\|u\|_{L^{2q}(B_2)} \le \|u\|_{L^3_{wk}(B_2)} \le \varepsilon$$

and that u is a very weak solution of

$$-\Delta u + \nabla p = \tilde{f} - \operatorname{div}(u\otimes u), \quad \operatorname{div} u = 0, \tag{2.90}$$

in B_2. By Lemma 2.12, we have $u \in W^{1,q}(B_{3/2})$, and there is $p \in L^q(B_{3/2})$ so that (u,p) is a distributional solution of (2.90) and

$$\begin{aligned}\|\nabla u\|_{L^q(B_{3/2})} + \|p\|_{L^q(B_{3/2})} &\lesssim \|\tilde{f}\|_{-1,q} + \|u\|^2_{L^{2q}(B_2)} + \|u\|_{L^1(B_2)}\\ &\lesssim \|\tilde{f}\|_q + \varepsilon.\end{aligned}$$

Fix a cut-off function $\phi \in C_c^\infty(B_{3/2})$ with $\phi = 1$ in $B_{5/4}$. Let

$$b = u, \quad w = \phi u, \quad \pi = \phi p.$$

They satisfy

$$-\Delta w + \operatorname{div}(b \otimes w) + \nabla \pi = F, \quad \operatorname{div} w = g, \tag{2.91}$$

where

$$F = \phi \tilde{f} - 2\partial_j((\partial_j \phi)u) + (\Delta\phi)u + (\nabla\phi \cdot b)u + p\nabla\phi, \quad g = \nabla\phi \cdot u.$$

We have written carefully the term with derivatives in u in divergence form. Note that

$$\|h\|_{W^{-1,r}(B)} = \sup_{\|u\|_{W_0^{1,r'}(B)} \le 1} \langle h, u \rangle \lesssim \|h\|_{L^q(B)}.$$

Thus

$$\begin{aligned} \|F\|_{W^{-1,r}(B_{3/2})} + \|g\|_{L^r(B_{3/2})} &\lesssim \|\tilde{f}\|_q + \|u\|_r + \|bu\|_q + \|p\|_q \\ &\lesssim \|\tilde{f}\|_q + \varepsilon. \end{aligned}$$

On one hand, by Lemma 2.17 $w = \phi u$ is the unique solution of (2.91) in $W^{1,q}(B_{3/2})$. On the other hand, by Lemma 2.17 with q replaced by r, (2.91) has a solution $\tilde{w} \in W^{1,r}(B_{3/2})$. Thus $\phi u = \tilde{w}$, and $u \in W^{1,r}(B_{5/4})$ which is imbedded in $L^{r^*}(B_{5/4})$. By case (i), we have $u \in L^\infty(B_1)$; i.e., $v \in L^\infty(B(x_0, R))$. Hence v is locally bounded in Ω. □

In the above proof, we only need $f \in L^q_{\text{loc}}$ for some $q > 1$ to get $u \in L^{r^*}$.

The proof gives an explicit estimate when $f = 0$.

Corollary 2.18. *There is $\varepsilon_* > 0$ such that the following hold. If v is a very weak solution of* (SNS) *in $\Omega \subset \mathbb{R}^3$ with zero force and $\varepsilon = \|v\|_{L^3_{wk}(\Omega)} \le \varepsilon_*$, then $v \in L^\infty_{\text{loc}}(\Omega)$ and*

$$|v(x)| \le \frac{C\varepsilon}{\operatorname{dist}(x, \partial\Omega)}.$$

Similar to Theorem 2.14, in a domain $\Omega \subset \mathbb{R}^n$, we can show the regularity of weak solutions of (SNS) in the class

$$v \in H^1 \cap L^n.$$

See Galdi [**57**, Theorem IX.5.1] for the interior case and [**57**, Theorem IX.5.2] for the global case in a C^2 bounded domain.

In the following we list a few open problems.

Conjecture 2.19. *Let $B = B_1 \subset \mathbb{R}^n$, $n \ge 2$, and let v be a very weak solution of* (SNS) *in B with smooth f.*

(i) *If $n = 2$ and $v \in L^2(B)$, is v regular?*
(ii) *If $n = 3$ and $|v(x)| \le C_*|x|^{-1}$ for some large C_*, is v regular at 0?*
(iii) *If $n \ge 5$ and $v \in H^1_0(B)$, is v regular?*

In item (ii), v is large in L^3_{wk}. It is essential to use the fact that v satisfies (SNS) across 0, in view of the *Landau solutions* to be considered in Section 8.2.

For (iii), see Frehse-Růžička [**50–52**] and Struwe [**202**] for partial results.

Conjecture 2.20. *Let $B_2 \subset \mathbb{R}^3$. Suppose $v \in W^{1,2}(B_2)$ is a scalar weak solution of*

$$-\Delta v + b \cdot \nabla v = 0, \quad \text{in } B_2,$$

where $|b(x)| \le C|x|^{-1-\delta}$ for some $0 < \delta \ll 1$, and $\operatorname{div} b = 0$. One can show $v \in L^\infty(B_1)$. Can one show that v is Hölder continuous if $\delta > 0$ is sufficiently small?

2.8. The Bogovskii map

In this section we prove Lemma 2.16. We recall the statement here.

Lemma. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $2 \le n < \infty$. There is a linear map Φ that maps a scalar $f \in L^q(\Omega)$ with $\int_\Omega f = 0$, $1 < q < \infty$, to a vector field $v = \Phi f \in W_0^{1,q}(\Omega;\mathbb{R}^n)$ and*

$$\operatorname{div} v = f, \quad \|v\|_{W_0^{1,q}(\Omega)} \le c(\Omega, q)\|f\|_{L^q(\Omega)}.$$

The map Φ is independent of q for $f \in C_c^\infty(\Omega)$.

The map Φ is not unique. If we do not need $v|_{\partial\Omega} = 0$, we can simply define $v(x) = -\int \nabla E(x-y) f(y)\, dy$ where E is the fundamental solution of $-\Delta E = \delta_0$ and $\nabla E(x) = \frac{-x}{|\partial B_1||x|^n}$. The idea in the special case below is to modify ∇E to get a kernel which gives zero boundary value.

Sketch of proof. Step 1. Special case.

Consider the special case that Ω is star-shaped with respect to any point in a ball B with $\bar{B} \subset \Omega$; i.e., for any $x \in B$ and any $y \in \Omega$, the segment $\overline{xy}$ lies inside Ω. First assume $f \in C_c^\infty(\Omega)$. Choose $\phi \in C_c^\infty(B)$ with $\int \phi = 1$. Define $v(x)$ by the *Bogovskii formula*

$$v(x) = \int_\Omega f(y) N(x,y) dy, \quad N(x,y) = \frac{x-y}{|x-y|^n} \int_{|x-y|}^\infty \phi\left(y + r\frac{x-y}{|x-y|}\right) r^{n-1} dr.$$

We recover $c\nabla E$ if we discard the ϕ factor in the kernel $N(x,y)$ and replace the limit ∞ by 0. The argument inside ϕ for $r \ge |x-y|$ represents the ray from x with the direction $x-y$. Let $\operatorname{spt} f$ denote the support of f. Since ϕ is supported in B, we get $v(x) = 0$ if $x \notin A$, where

$$A = \{tz + (1-t)y : z \in B,\ y \in \operatorname{spt} f,\ 0 \le t \le 1\}.$$

By the star-shape assumption, $\operatorname{spt} v \subset A \subset \Omega$. By the smoothness of f and ϕ, we get $v \in C_c^\infty(\Omega)$. The proof of $\operatorname{div} v = 0$ is similar to that for $\operatorname{div}_x \int \nabla E(x-y) f(y)\, dy = -f(x)$. The proof for $\|v\|_{W^{1,q}} \le c\|f\|_{L^q}$ uses the Calderon-Zygmund theorem. The condition $\int f = 0$ is used in these proofs. The general case $f \in L^q(\Omega)$ follows from approximation. For details see [**57**, pp. 121–125].

Step 2. General case.

Let Ω be a bounded Lipschitz domain with Lipschitz constant L. Choose open rectangles $G_1, \dots, G_m$ which cover $\partial\Omega$ and, for each j, $\partial\Omega \cap G_j$ is a Lipschitz graph $x_n = \gamma(x')$ in a suitable local coordinate system $x = (x', x_n)$ in G_j. For each $x \in \partial\Omega \cap G_j$ the region

$$K_x = \left\{ y = (y', y_n) \in G_j : y_n > x_n + L|y' - x'| \right\}$$

lies inside Ω. By further dividing G_j if necessary, we may assume G_j is so narrow that $\bigcap_{x \in \partial\Omega \cap G_j} K_x$ is nonempty and contains a ball B_j. Then $G_j \cap \Omega$ is star-shaped with respect to this ball. If the difference set $\Omega \setminus \bigcup_{j=1,\dots,m} G_j$ is empty, we set $\ell = m$. Otherwise we cover it by open balls $G_j \Subset \Omega$, $m+1 \le j \le \ell$. These balls are star-shaped with respect to concentric balls B_j with half-radii.

We conclude that $\bigcup_{j=1}^{\ell} G_j$ is an open cover of $\overline{\Omega}$ and each $\Omega_j = \Omega \cap G_j$ is star-shaped with respect to some interior ball. Let $D_j = \bigcup_{k>j} \Omega_j$, $0 \le j < \ell$. We may reorder their indexes so that each D_j is connected. For $1 \le j < \ell$, since D_{j-1} is connected, $E_j = \Omega_j \cap D_j$ is nonempty.

We now define f_j by induction so that $f = \sum_{j=1}^{\ell} f_j$ with $\operatorname{spt} f_j \subset \bar{\Omega}_j$ and $\int f_j = 0$. For a set E denote by 1_E its characteristic function. Let $g_0 = f$. Suppose for some $1 \le j < \ell$, g_{j-1} has been defined with $\operatorname{spt} g_{j-1} \subset \bar{D}_{j-1}$ and $\int g_{j-1} = 0$. Define

$$f_j = 1_{\Omega_j} g_{j-1} - \frac{1_{E_j}}{|E_j|} \int_{\Omega_j} g_{j-1}, \quad g_j = 1_{D_j \setminus \Omega_j} g_{j-1} + \frac{1_{E_j}}{|E_j|} \int_{\Omega_j} g_{j-1}.$$

One verifies that $g_{j-1} = f_j + g_j$ with

$$\int f_j = \int g_j = 0, \quad \operatorname{spt} f_j \subset \bar{\Omega}_j, \quad \operatorname{spt} g_j \subset \bar{D}_j.$$

The above defines f_j for $1 \le j < \ell$. Finally one sets $f_\ell = g_\ell$.

Once the f_j have been defined, we can apply Step 1 to find the $v_j \in W_0^{1,q}(\Omega_j)$ with $\operatorname{div} v_j = f_j$. They can be extended by zero to $W_0^{1,q}(\Omega)$. The function $v = \sum_{j=1}^{\ell} v_j$ is what we look for. $\square$

Remark. It follows from the proof that we can weaken the Lipschitz condition to an interior cone condition. For $f \in C_c^\infty(\Omega)$, Φf is in $C_c^\infty(\Omega)$ when Ω is star-shaped, but not for general Ω.

2.9. Notes

Sections 2.2 and 2.3 are based on [**206**, II] and [**57**, IV, IX].

The proof of Theorem 2.7 in Section 2.3 using the Galerkin method is by Fujita [**53**]. The proof of Theorem 2.7 in Section 2.4 is by Ladyženskaja [**118**] based on the work of Leray [**131**].

In the proof of Theorem 2.7 by the Galerkin method, in fact $\widetilde{\mathbf{V}} = \mathbf{V} \cap L^n(\Omega)$. It follows from the following density lemma; see [**57**, Theorem III.6.2].

Lemma 2.21 (Density)**.** *Let Ω be a bounded or exterior Lipschitz domain in $\mathbb{R}^n$, $2 \le n < \infty$. Assume $v \in D_{0,\sigma}^{1,q}(\Omega) \cap [\bigcap_{i=1}^k L^{r_i}(\Omega)]$ for some $q, r_i \in (1,\infty)$, $i = 1, \dots, k$, $k \in \mathbb{N}$. Then there is a sequence $\{v_j\}_{j\in\mathbb{N}} \subset C_{c,\sigma}^\infty(\Omega)$ such that*

$$\lim_{j\to\infty} \left(\|\nabla(v - v_j)\|_q + \textstyle\sum_{i=1}^k \|v - v_j\|_{r_i} \right) = 0.$$

Section 2.4 is based on [**121**], using the idea of [**131**] which motivated the Leray-Schauder degree theory [**133**] implying Theorem 2.8.

Section 2.5 is based on Yudovich [**224**].

In Section 2.6, the Lorentz tensor is due to Lorentz [**140**]; see [**161**] and [**57**, IV.2]. In a half-space $\mathbb{R}^n_+$, the Poisson kernel is found by Odqvist [**159**, §2], and the Green's tensor is studied in [**146**, Appendix 1] for $n = 2, 3$ and extended in [**57**, IV.3] and [**93**].

Lemma 2.11 is well known; see [**155**, Appendix].

In Lemma 2.12, the estimate (2.67) is due to Šverák-Tsai [**204**]. Similar estimates for the time-dependent Stokes system appeared in [**31**] and include Lemma 2.12 as a special case. Our proof is closer to that in [**31**] and is more direct. Another new proof can be found in [**57**, Remark IV.4.2]. It is extended to the boundary case by Kang [**91**].

In Section 2.7, Lemmas 2.15 and 2.17 are due to Kim-Kozono [**101**].

In Theorem 2.14, case (iii) appears in Luo-Tsai [**141**] and is an improved version of Kim-Kozono [**101**, Theorem 4], in which they assume further the existence of $p \in L^1_{\mathrm{loc}}$ so that (v, p) is a distributional solution of (SNS). Case (i) is well known; see the references in [**141**].

Problems

2.1. For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 5$, show

$$\int_\Omega (v \cdot \nabla)u \cdot u \, dx = 0, \quad \forall v \in W^{1,2}_{0,\sigma}(\Omega), \quad \forall u \in W^{1,2} \cap L^n(\Omega).$$

Here u can be a scalar or a vector. What if Ω is an exterior domain? Compare Lemma 2.3(ii).

2.2. Let Ω be any domain in $\mathbb{R}^n$, $n \geq 2$, $1 \leq p < \infty$, and $\psi \in C^1_c(\Omega)$. If f_k, $k = 1, 2, 3, \ldots$, converges weakly to $\bar{f}$ in $L^p(\Omega)$ and g_k converges strongly to $\bar{g}$ in $L^{p'}(K)$ for any $K \Subset \Omega$, show that

$$\int_\Omega f_k g_k \psi \to \int_\Omega \bar{f} \bar{g} \psi \quad \text{as } k \to \infty.$$

This is related to the convergence of the second term in (2.18).

2.3. Show that there are constants $0 < \varepsilon_0 \ll 1$ and $0 < c_1$ such that if $\varepsilon = \|f\|_{\mathbf{V}'} \leq \varepsilon_0$, then the sequence (2.24) converges to some $v \in \mathbf{V}$ satisfying $v = K(v)$ and $\|v\|_{\mathbf{V}} \leq c_1 \varepsilon$.

2.4. (*Physically reasonable solutions*, [**48**]). Consider (SNS) in $\mathbb{R}^3$ with $f \in X_3$ where for $\alpha > 0$,

$$X_\alpha = \left\{ f : \mathbb{R}^3 \to \mathbb{R} \;\middle|\; \|f\|_{X_\alpha} \stackrel{\text{def}}{=\!=} \||x|^\alpha f(x)\|_{L^\infty(\mathbb{R}^3)} < \infty \right\}.$$

Let $V = \{v \in X_1 \mid |\nabla v| \in X_2\}$ with $\|v\|_V = \|v\|_{X_1} + \|\nabla v\|_{X_2}$. Show that the map K given by (2.23) maps V into itself continuously and that there are constants $0 < \varepsilon_0 \ll 1$ and $0 < c_1$ such that if $\varepsilon = \|f\|_{X_3} \leq \varepsilon_0$, then the sequence (2.24) converges to some $v \in V$ satisfying $v = K(v)$ and $\|v\|_V \leq c_1 \varepsilon$.

Remark. There is a different version: We have $f = \nabla F$ for some 2-tensor $F \in X_2$ small, and we look for a solution $v \in X_1$.

2.5. Let $\Omega = \mathbb{R}^n_+$, $n \geq 2$, and let $\Sigma = \partial\Omega$. Let $x^* = (x', -x_n)$ if $x = (x', x_n)$. Let $\varepsilon_i = 1$ if $i < n$ and $\varepsilon_i = -1$ if $i = n$. The solution of

$$-\Delta v + \nabla p = f, \quad \operatorname{div} v = 0, \quad \text{in } \mathbb{R}^n_+,$$

coupled with the boundary condition $\partial_n v_i|_\Sigma = 0$ for $i < n$ and $v_n|_\Sigma = 0$, is

$$v_i(x) = \sum_{j=1}^n \int_{\mathbb{R}^n_+} [U_{ij}(x-y) + \varepsilon_j U_{ij}(x-y^*)] f_j(y) \, dy.$$

2.6. Let Σ, x^*, and ε_i be as in the last problem. Suppose v, p is a smooth solution of the stationary Stokes system with zero force in $B_1 \cap \mathbb{R}^n_+$ where $B_1 = B(0,1) \subset \mathbb{R}^n$.

(i) (Even extension) If $\partial_n v_i|_\Sigma = 0$ for $i < n$ and $v_n|_\Sigma = \partial_n p|_\Sigma = 0$, is the following extension a solution in B_1:

$$\begin{cases} \bar{v}_i(x) = v_i(x), & \bar{p}(x) = p(x), \qquad \text{if} \quad x_n > 0, \\ \bar{v}_i(x) = \varepsilon_i v_i(x^*), & \bar{p}(x) = p(x^*), \qquad \text{if} \quad x_n < 0? \end{cases}$$

(ii) (Odd extension) If $v_i|_\Sigma = 0$ for $i < n$ and $\partial_n v_n|_\Sigma = p|_\Sigma = 0$, is the following extension a solution in B_1:

$$\begin{cases} \bar{v}_i(x) = v_i(x), & \bar{p}(x) = p(x), \qquad \text{if} \quad x_n > 0, \\ \bar{v}_i(x) = -\varepsilon_i v_i(x^*), & \bar{p}(x) = -p(x^*), \qquad \text{if} \quad x_n < 0? \end{cases}$$

2.7. Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$. If $v \in H^1_{\mathrm{loc}}(\Omega)$ is a weak solution of $-\Delta v + b \cdot \nabla v = 0$ and if $b \in L^n$ or $\lim_{|x| \to 0} |x| b(x) = 0$, show that v is locally Hölder continuous. How about $n = 2$?

Chapter 3

Weak solutions

In this chapter we study the basic properties and existence of weak solutions of the nonstationary Navier-Stokes equations

$$\partial_t v - \Delta v + (v \cdot \nabla)v + \nabla p = f, \quad \operatorname{div} v = 0, \tag{3.1}$$

in $\Omega_T = \Omega \times (0,T)$, where Ω is a domain in $\mathbb{R}^n$, $n = 2, 3$, $0 < T \le \infty$, the force f is given, and the unknown is $(v,p) : \Omega_T \to \mathbb{R}^n \times \mathbb{R}$. The system (3.1) is coupled with the initial and boundary conditions

$$v(x,0) = v_0(x), \quad v(x,t) = 0 \quad \text{if } x \in \partial\Omega. \tag{3.2}$$

Compared to (NS), we have assumed the viscosity constant $\nu = 1$.

As in Chapter 2, the existence proof of weak solutions is based on a priori bounds. In Section 3.1 we define weak solutions and discuss their properties. In Section 3.3 we discuss the existence of solutions of a perturbed Stokes system. In Section 3.4 we prove a compactness lemma. In Section 3.5 we show the existence of suitable weak solutions of (3.1).

The problems of the uniqueness and regularity of weak solutions are very challenging questions and will be discussed in later chapters.

3.1. Weak form, energy inequalities, and definitions

In this section we will first derive several identities and inequalities assuming that (v,p) is a smooth solution. We will then give various definitions of weak solutions and their basic properties.

Denote the space of divergence-free test vector fields

$$C^\infty_{c,\sigma}(\Omega_T) = \{\zeta \in C^\infty_c(\Omega_T; \mathbb{R}^n) : \operatorname{div} \zeta = 0\}.$$

Assume that (v,p) is a smooth solution of (3.1). Testing the first equation in (3.1) with $\zeta \in C^\infty_{c,\sigma}(\Omega_T)$, we get

$$\iint -v\cdot(\partial_t\zeta+\Delta\zeta)+v_iv_j\partial_j\zeta_i - f\cdot\zeta\,dx\,dt=0,\quad \forall\zeta\in C^\infty_{c,\sigma}(\Omega_T). \tag{3.3}$$

To make sense, it suffices that $v\in L^2_{\mathrm{loc}}(\Omega_T)$.

Testing the second equation in (3.1) with $\phi\in C^\infty_c(\Omega)$, we get

$$\int_\Omega v(\cdot,t)\cdot\nabla\phi\,dx=0,\quad \forall\phi\in C^\infty_c(\Omega),\quad \text{a.e. } t. \tag{3.4}$$

The two equations (3.3) and (3.4) are the *weak form* of (3.1).

Testing the first equation in (3.1) with $2v\phi$ where $\phi\in C^\infty$, we get the *local energy identity* using $v|_{\partial\Omega}=0$,

$$\begin{aligned}&\int_{\Omega_t}|v|^2\phi\,dx+2\int_0^t\int_\Omega|\nabla v|^2\phi dx\,dt\\ &=\int_\Omega|v_0|^2\phi\,dx+\int_0^t\int_\Omega|v|^2(\partial_t\phi+\Delta\phi)+(|v|^2+2p)v\cdot\nabla\phi+2v\cdot f\phi\,dx\,dt,\quad\forall t.\end{aligned}\tag{3.5}$$

Here $\Omega_t=\Omega\times\{t\}$. If we take $\phi\equiv1$, we get the *energy identity*,

$$\int_{\Omega_t}|v|^2\,dx+2\int_0^t\int_\Omega|\nabla v|^2dx\,dt=\int_\Omega|v_0|^2\,dx+\int_0^t\int_\Omega 2v\cdot f\,dx\,dt,\quad\forall t. \tag{3.6}$$

We will not be able to get these identities when we construct weak solutions. Instead, we get inequalities: The *energy inequality* is

$$\int_{\Omega_t}|v|^2\,dx+2\int_0^t\int_\Omega|\nabla v|^2dx\,dt\le\int_\Omega|v_0|^2\,dx+\int_0^t\int_\Omega 2v\cdot f\,dx\,dt,\quad\forall t. \tag{3.7}$$

The *local energy inequality* is: For all $\phi\ge0\in C^\infty(\Omega_T)$ supported in $K\times[0,T)$ for some $K\Subset\Omega$, for all t,

$$\begin{aligned}&\int_{\Omega_t}|v|^2\phi\,dx+2\int_0^t\int_\Omega|\nabla v|^2\phi dx\,dt\\ &\quad\le\int_\Omega|v_0|^2\phi\,dx+\int_0^t\int_\Omega|v|^2(\partial_t\phi+\Delta\phi)+(|v|^2+2p)v\cdot\nabla\phi+2v\cdot f\phi\,dx\,dt.\end{aligned}\tag{3.8}$$

If we integrate over (t_0,t_1) instead of $(0,t)$, we get the *strong energy inequality*: For almost all $t_0\in[0,T)$ including $t_0=0$, and for all $t_1\in(t_0,T)$,

$$\int_{\Omega_{t_1}}|v|^2\,dx+2\int_{t_0}^{t_1}\int_\Omega|\nabla v|^2dx\,dt\le\int_{\Omega_{t_0}}|v|^2\,dx+\int_{t_0}^{t_1}\int_\Omega 2v\cdot f\,dx\,dt. \tag{3.9}$$

The energy inequality provides an a priori bound and is the key to the existence theorem for global-in-time weak solutions.

When we only assume $f \in L^2(0,T;\mathbf{V}')$, terms involving f should be understood correspondingly. For example, $\int_{t_0}^{t_1}\int_\Omega v \cdot f\,dxdt$ means $\int_{t_0}^{t_1}\langle f, v\rangle\,dt$.

We now define four different kinds of weak solutions.

Definition 3.1. Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $\mathbf{V} = \dot{H}^1_{0,\sigma}(\Omega)$. Let $0 < T < \infty$ and $\Omega_T = \Omega \times (0,T)$. Assume $v_0 \in L^2_\sigma(\Omega)$ and $f \in L^2(0,T;\mathbf{V}')$.

1) A vector field $v(x,t)$ is a *very weak solution* (VWS) of (3.1) in Ω_T if $v \in L^2_{\mathrm{loc}}(\Omega_T)$ and it satisfies the weak form (3.3) and (3.4).

2) A very weak solution v is a *weak solution* (WS) if $v \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;\mathbf{V})$, $v \in C_{wk}([0,T);L^2_\sigma(\Omega))$, and $v(t) \rightharpoonup v_0$ weakly in L^2 as $t \to 0_+$.

3) A weak solution v is a *Leray-Hopf weak solution* (LHWS) if it satisfies the energy inequality (3.7) and $\lim_{t\to 0_+}\|v(t) - v_0\|_{L^2} = 0$.

4) A Leray-Hopf weak solution is a *suitable weak solution* (SWS) if there is some $p \in L^{3/2}_{\mathrm{loc}}(\overline{\Omega}_T)$ so that (v,p) satisfies (3.1) in the distributional sense (i.e. (3.3) with an extra term $-p\nabla\cdot\zeta$ in the integrand for $\zeta \in C_c^\infty(\Omega_T)$) and also the local energy inequality (3.8).

The vector field v is a weak solution in $\Omega \times (0,\infty)$, up to time of infinity, if it is a weak solution in Ω_T for any $0 < T < \infty$. This is similarly defined for other kinds of solutions.

In the following we give a few remarks.

The above definitions have inclusion relations,

$$\mathrm{SWS} \subset \mathrm{LHWS} \subset \mathrm{WS} \subset \mathrm{VWS}.$$

The weakest kind is a very weak solution, which only requires $v \in L^2_{\mathrm{loc}}(\Omega_T)$. As such, it is a local concept because initial and boundary conditions do not make sense.

Let us explain the notation C_{wk}. For any time interval I, spatial domain Ω, and $p \in [1,\infty]$, a function $v(t) : I \to L^p(\Omega)$ is said to be in $C_{wk}(I;L^p(\Omega))$ if for any $t_0 \in I$ and $w \in L^{p'}(\Omega)$,

$$\int_\Omega v(x,t)\cdot w(x)\,dx \to \int_\Omega v(x,t_0)\cdot w(x)\,dx \quad \text{as } t \to t_0. \tag{3.10}$$

It is defined in the dual sense. Since L^p is reflexive only if $p < \infty$, it is called a *weakly continuous* $L^p(\Omega)$-valued function in t if $1 \leq p < \infty$, and it is *weak star continuous* if $p = \infty$.

We will show in Lemma 3.4 that a very weak solution v in the *energy class*

$$L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;\mathbf{V})$$

can be redefined on a measure zero subset of time so that $v \in C_{wk}([0,T);L^2_\sigma)$. Note, however, that being in the energy class is not sufficient for this purpose; we also need to use the equations. We added the condition $v \in C_{wk}([0,T);L^2_\sigma)$ in the definition for convenience of applications.

A weak solution $v \in C_{wk}([0,T);L^2_\sigma(\Omega))$ that satisfies the energy inequality (3.7) automatically satisfies $\lim_{t\to 0_+} \|v(t)-v_0\|_{L^2} = 0$: We already have $\|v_0\|_2 \le \liminf_{t\to 0_+} \|v(t)\|_2$ from the L^2-weak continuity. By the energy inequality, we also have $\|v_0\|_2 \ge \limsup_{t\to 0_+} \|v(t)\|_2$. Thus we have norm continuity at $t=0$. Together with weak continuity, we get strong L^2-continuity at $t=0$.

The main condition in the definition for suitable weak solutions is the local energy inequality (3.8), which only requires that v be locally in the energy class and $p \in L^{3/2}_{\rm loc}(\Omega_T)$. Thus the suitability is a local condition and can be affiliated to more general solutions, see for example *local Leray solutions* in Section 8.4.

We have limited our test functions in the weak form (3.3) in the smooth class. When the solution is in the energy class, the test function space can be extended to larger classes, for example, $H^1(0,T;\mathbf{V})$ for $n \le 4$.

We now consider implications of Sobolev imbedding. Denote

$$L^s_t L^q_x = L^s_t(0,T;L^q_x(\Omega)), \quad L^q_{t,x} = L^q_t L^q_x.$$

A solution v in the energy class satisfies

$$v \in L^\infty_t L^2_x \cap L^2_t H^1_x.$$

For $n=3$, by imbedding, v is in $L^\infty_t L^2_x \cap L^2_t L^6_x$. Applying Hölder inequality twice, first in space and second in time, we get

$$v \in L^s_t L^q_x, \quad \text{for } \frac{3}{q} + \frac{2}{s} = \frac{3}{2}, \quad 2 \le q \le 6. \tag{3.11}$$

In particular, $v \in L^{10/3}_{t,x}$. For general $n \ge 2$, a similar argument gives

$$v \in L^s_t L^q_x, \quad \text{for } \frac{n}{q} + \frac{2}{s} = \frac{n}{2}, \quad 2 \le q \le 2^*, \quad q < \infty, \tag{3.12}$$

where $2^* = \infty$ for $n=2$ and $2^* = \frac{2n}{n-2}$ for $n>2$.

Finally we comment on the pressure. In the first three definitions, the pressure does not appear. It can be defined as a distribution once v is given.

Consider the nonstationary Stokes system

$$\begin{aligned} \partial_t v - \Delta v + \nabla p = F, \quad \operatorname{div} v = 0, \quad \text{in } \Omega_T, \\ v(\cdot, 0) = v_0, \quad v|_{\partial\Omega} = 0. \end{aligned} \tag{3.13}$$

We may take $F = f - v \cdot \nabla v$. Assume $F \in L^1(0,T;(D_0^{1,2}(\Omega))')$. (Note that $(D_0^{1,2}(\Omega))' \subset H^{-1}$.) Let

$$G(t) := v_0 - v(t) + \int_0^t (f + \Delta v)\, dt'. \tag{3.14}$$

It is the time integral of (3.13) without ∇p and formally $G(t) = \int_0^t \nabla p\, dt'$. We have

$$G(t) \in C([0,T];(D_0^{1,2}(\Omega))') \quad \text{and} \quad \langle G(t), \chi\rangle = 0 \quad \forall \chi \in C_{c,\sigma}^\infty(\Omega). \tag{3.15}$$

By Lemma 2.2, $G(t) = \nabla P(t)$ for a unique $P \in C(0;T;L^2)$, with $\int_\Omega P(t)dx = 0$ if Ω is bounded. We can define $p = \frac{d}{dt}P$, the distributional derivative of P in t. Then the pair v, p is a distributional solution of (3.13).

3.2. Auxiliary results

In this section we give a few auxiliary results on the Banach space-valued function of time. Let $-\infty \le a < b \le \infty$ and let X be a Banach space. A function $u : (a,b) \to X$ is in $L^p(a,b;X)$ if

$$\|u\|_{L^p(a,b;X)} \overset{\text{def}}{=\!=} \big\| \|u\|_X \big\|_{L^p(a,b)} < \infty.$$

If u and v are integrable functions from (a,b) to X, $u, v \in L^1((a,b);X)$, then we say v is a *weak derivative* of u, denoted as $v = \partial_t u$ or $v = u'$, if

$$\int u\eta' = -\int v\eta, \quad \forall \eta \in C_c^1((a,b)).$$

Lemma 3.2. *Let $(a,b) \subset \mathbb{R}$ and let X be a Banach space.*

(i) *Suppose $u \in L^1(a,b;X)$ and*

$$\int_a^b u\theta(t)dt = 0 \quad \forall \theta \in C_c^1((a,b)).$$

Then $u(t) = 0$ for a.e. t.

(ii) *Suppose $u \in W^{1,1}(a,b;X)$. Then there is $\xi \in X$ so that*

$$u(t) = \xi + \int_a^t \partial_t u \quad \textit{for a.e. } t.$$

Proof. The proof of (i) is omitted. We prove (ii). That $u \in W^{1,1}(a,b;X)$ means there is $\partial_t u \in L^1(a,b;X)$ so that

$$\int u\eta' = -\int (\partial_t u)\eta, \quad \forall \eta \in \mathcal{D} := C_c^1((a,b)).$$

The function $\tilde{u}(t) = \int_a^t \partial_t u$ is absolutely continuous and also satisfies the above weak form. Thus $v(t) = u(t) - \tilde{u}(t)$ satisfies

$$\int v\eta' = 0, \quad \forall \eta \in \mathcal{D}.$$

We will show that $v(t)$ is constant for a.e. t. Choose $\phi_0 \in \mathcal{D}$ such that $\int \phi_0 = 1$. Let $\xi = \int \phi_0 v$. Any $\theta \in \mathcal{D}$ can be decomposed as $\theta = \lambda\phi_0 + \eta'$ where

$$\lambda = \int \theta\phi_0, \quad \eta(t) = \int_a^t (\theta - \lambda\phi_0) \in \mathcal{D}.$$

Thus

$$\int_a^b (v(t) - \xi)\theta(t)dt = \int_a^b (v\lambda\phi_0 + v\eta' - \xi\lambda\phi_0 - \xi\eta')dt = \lambda\xi + 0 - \lambda\xi - 0 = 0.$$

Since $\theta \in \mathcal{D}$ is arbitrary, we conclude $v(t) - \xi = 0$ for a.e. t by part (i). $\square$

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain, $0 < T < \infty$, and $k \in \mathbb{N}$.*

(i) *If f_j is a bounded sequence in $L^2(\Omega)$ and $f_j \rightharpoonup g$ in $H^{-k}(\Omega)$ weakly, then $g \in L^2(\Omega)$ and $f_j \rightharpoonup g$ in $L^2(\Omega)$ weakly.*

(ii) *If $u \in L^\infty(0,T;L^2(\Omega))$ and $\partial_t u \in L^1(0,T;H^{-k}(\Omega))$, then there is $\tilde{u} \in C_{wk}([0,T];L^2(\Omega))$ so that $\tilde{u} = u$ a.e. t.*

(iii) *If $u \in L^2(0,T;\mathbf{V})$, $\mathbf{V} = H^1_{0,\sigma}(\Omega)$, and $\partial_t u \in L^2(0,T;\mathbf{V}')$, then there is $\tilde{u} \in C([0,T];L^2(\Omega))$ so that $\tilde{u} = u$ a.e. t, and, as a weak derivative in t,*

$$\frac{d}{dt}\|\tilde{u}\|_2^2 = 2\,\langle \partial_t u, u\rangle\,. \tag{3.16}$$

Recall that $H^k_0(\Omega)$ is the completion of $C^\infty_c(\Omega)$ in the H^k-norm and $H^{-k}(\Omega)$ is the dual space of $H^k_0(\Omega)$. Part (i) will be used to show (ii) with $f_j = u(t_j)$. Part (ii) will be used to show weak L^2-continuity of weak solutions of (3.1). Part (iii) will be used to show strong L^2-continuity of weak solutions of the Stokes system.

Proof. (i) Suppose $\|f_j\|_{L^2} \le M$. For any $z \in C^\infty_c(\Omega)$ we have $|\langle g, z\rangle| = \lim |\langle f_j, z\rangle| \le M\|z\|_{L^2}$. Thus $g \in L^2$ and $\|g\|_{L^2} \le M$. For any $w \in L^2(\Omega)$ and any $\varepsilon > 0$, choose $z \in C^\infty_c(\Omega)$ so that $\|w - z\|_{L^2} \le \varepsilon/4M$. Then

$$\begin{aligned} |(f_j - g, w)| &\le |(f_j - g, w - z)| + |(f_j - g, z)| \\ &\le 2M\|w - z\|_2 + |(f_j - g, z)|. \\ &\le \varepsilon/2 + |(f_j - g, z)|. \end{aligned} \tag{3.17}$$

For j sufficiently large, we have $|(f_j - g, z)| \le \varepsilon/2$ and thus $|(f_j - g, w)| \le \varepsilon$. This shows $|(f_j - g, w)| \to 0$ as $j \to \infty$, for any $w \in L^2$.

(ii) Let $M = \|u\|_{L^\infty(0,T;L^2)}$. By Lemma 3.2, there is $\xi \in H^{-k}(\Omega)$ so that $u(t) = \tilde{u}(t)$ a.e. t, where $\tilde{u}(t) = \xi + \int_0^t \partial_t u\, dt' \in C([0,T]; H^{-k})$. We may now assume $u(t) = \tilde{u}(t)$ for all t.

We first show $u(t_0) \in L^2$ with $\|u(t_0)\|_{L^2} \le M$ for any $t_0 \in [0,T]$: Choose $t_j \in [0,T]$, $t_j \to t_0$, and $u(t_j) \in L^2$ with $\|u(t_j)\|_{L^2} \le M$. Since $u(t_j) \to u(t_0)$ in H^{-k} as $j \to \infty$, by (i), we have $u(t_0) \in L^2$, $u(t_j) \rightharpoonup u(t_0)$ weakly in L^2, and $\|u(t_0)\|_{L^2} \le M$.

We next show weak continuity: For any $t_j \to t_0$, we have $u(t_j) \rightharpoonup u(t_0)$ weakly in H^{-k}. We have $u(t_j) \in L^2$ with $\|u(t_j)\|_{L^2} \le M$ by the previous step. Part (i) again shows $u(t_j) \rightharpoonup u(t_0)$ weakly in L^2.

(iii) Extend u outside $[0,T]$ by zero and denote $u_\varepsilon = u * \eta_\varepsilon$ where η_ε is a mollifier in t. Then the u_ε are smooth in t, $u_\varepsilon \to u$ in $L^2(0,T;\mathbf{V})$, and $\partial_t u_\varepsilon \to \partial_t u$ in $L^2(0,T;\mathbf{V}')$. We have $\frac{d}{dt}\|u_\varepsilon\|_2^2 = 2\langle \partial_t u_\varepsilon, u_\varepsilon\rangle$. In weak form,

$$-\int (u_\varepsilon, u_\varepsilon)\theta'\, dt = \int 2\langle \partial_t u_\varepsilon, u_\varepsilon\rangle\, \theta\, dt, \quad \forall \theta(t) \in C_c^\infty((0,T)). \tag{3.18}$$

Sending $\varepsilon \to 0$, we get $\frac{d}{dt}\|\tilde{u}\|_2^2 = 2\langle \partial_t u, u\rangle$ in the weak sense. Since $\langle \partial_t u, u\rangle \in L_t^1$, $\|\tilde{u}(t)\|_{L^2}^2$ is continuous. Norm continuity plus weak continuity imply strong continuity. $\square$

Compare Problem 3.1.

Lemma 3.4 (Weak L^2-continuity). *Let Ω be any domain in $\mathbb{R}^n$, $2 \le n \le 4$, and $\mathbf{V} = H^1_{0,\sigma}(\Omega)$. If $v \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;\mathbf{V})$ satisfies the weak form* (3.3)*, then it can be redefined at a set of times of measure zero so that $v \in BC_w([0,T);L^2_\sigma)$. Moreover, for this redefined function we have for all $0 \le t_0 < t_1 < T$,*

$$-\int_{t_0}^{t_1} [(v, \varphi_t + \Delta\varphi) + (v, v\cdot\nabla\varphi) - (f,\varphi)]\, dt = (v,\varphi)(t_0) - (v,\varphi)(t_1), \tag{3.19}$$

for all $\varphi \in C^\infty_{c,\sigma}(\Omega_T)$.

Note that (3.19) is a combination of the weak form (3.3) and the weak L^2-continuity of v.

Proof. We have $v \in L^\infty L^2 \cap L^2 L^{2^*}$ by imbedding. By Hölder inequality we get $v \in L^{8/n}L^4$ for $n = 2,3$ (see (3.12)). Choosing $\zeta = \theta(t)\chi(x)$ in the weak form (3.3) with $\theta \in C_c^\infty((0,T))$ and $\chi \in C^\infty_{c,\sigma}(\Omega)$, we get

$$-\int (v,\chi)\theta' dt = \int \{-(\nabla v, \nabla\chi) - (v_i v_j, \partial_j \chi_i) + (f,\chi)\}\, \theta dt.$$

The above can be extended to $\chi \in \mathbf{V}$. Thus $\partial_t v$ exists and

$$\partial_t v \in L^{4/n}(0,T;\mathbf{V}').$$

By Lemma 3.3(ii), there is $\tilde{v}(t) = v(t)$ for a.e. t such that $\tilde{v} \in C_w([0;T], L^2)$.

Fix $\eta(t) \in C^\infty(\mathbb{R})$, $\operatorname{supp}\eta \subset (0,1)$, and $\int \eta = 1$. For $0 \le t_0 < t_1 < T$, replace v by $\tilde{v}$ and ζ by $\theta(t)\varphi(x,t)$ in the weak form (3.3), where $\varphi \in C^\infty_{c,\sigma}(\Omega_T)$ and for small $\varepsilon > 0$

$$\theta(t) = \int_{-\infty}^t \frac{1}{\varepsilon}\left(\eta\left(\frac{s-t_0}{\varepsilon}\right) - \eta\left(\frac{s-t_1}{\varepsilon}\right)\right) ds.$$

We get

$$-\int [(\tilde{v}, \varphi_t + \Delta\varphi) + (\tilde{v}, \tilde{v}\cdot\nabla\varphi) - (f,\varphi)]\,\theta\,dt = \int (\tilde{v},\varphi)\theta'(t)\,dt. \tag{3.20}$$

Sending $\varepsilon \to 0_+$ and using the weak L^2-continuity of $\tilde{v}$, we get (3.19). □

3.3. Existence for the perturbed Stokes system

In this section we study the existence of the perturbed Stokes system which is the nonstationary Stokes system with an additional convection term. Our approach is based on the a priori bound and the Galerkin method. The Galerkin method is also popular for numerical computation. This approach will also be used later in Section 3.5 for Navier-Stokes equations.

Theorem 3.5 (Existence for the perturbed Stokes system)**.** *Suppose $\Omega \subset \mathbb{R}^n$, $n \ge 2$, is either $\mathbb{R}^n$, $\mathbb{R}^n_+$, or a bounded or exterior Lipschitz domain. Let $\mathbf{V} = \dot{H}^1_{0,\sigma}(\Omega)$, and let $0 < T < \infty$. Suppose $f \in L^2(0,T;\mathbf{V}')$, $u_0 \in L^2_\sigma$, and $w \in C^1(\bar{\Omega}\times[0,T];\mathbb{R}^n)$, $\operatorname{div} w = 0$. Then there is a unique weak solution u so that*

$$u \in C([0,T];L^2_\sigma) \cap L^2(0,T;H^1_0), \quad u(0) = u_0, \tag{3.21}$$

$$\partial_t u + (w\cdot\nabla)u - \Delta u + \nabla p = f. \tag{3.22}$$

Furthermore, u satisfies the energy identity (3.6) with v replaced by u, and hence $u \in C([0,T];L^2_\sigma)$.

Proof. 1. We will use the Galerkin method. As in the proof for Theorem 2.7, we can find a sequence $\phi_j \in C^\infty_{c,\sigma}(\Omega)$, $j = 1,2,\ldots$, whose linear span is dense in $\mathbf{V}$. We may normalize it so that $(\phi_j,\phi_k) = \delta_{jk}$ for all $j,k \in \mathbb{N}$. Here $(f,g) = \int_\Omega f\cdot g$.

2. For $m \in \mathbb{N}$, we can solve $u_m(t) = \sum_{j=1}^m g^j_m(t)\phi_j$ in the span of $\phi_1,\ldots,\phi_m$, such that

$$u_m(0) = \sum_{j=1}^m (u_0,\phi_j)\phi_j \tag{3.23}$$

and

$$\int_\Omega \partial_t u_m \cdot \phi_j + \nabla u_m : \nabla \phi_j + (w \cdot \nabla) u_m \cdot \phi_j = \langle f, \phi_j \rangle, \quad \forall j = 1, \ldots, m. \tag{3.24}$$

This is a linear ODE system for $g_m = (g_m^j)_{j=1}^m$ of the form

$$\frac{d}{dt} g + O(g) = F \in L_t^2$$

and hence can be solved for $t \in [0, T]$.

3. Taking the sum $\sum_{j=1}^m g_m^j(t) \cdot (3.24)_j$ (i.e., using u_m itself as a test function), using $\int_\Omega (w \cdot \nabla) u_m \cdot u_m \, dx = 0$, and integrating in t, we get

$$\int_{\Omega_t} |u_m|^2 + \int_0^t \int_\Omega 2 |\nabla u_m|^2 = \int_\Omega |u_0|^2 + \int_0^t 2 \langle f, u_m \rangle. \tag{3.25}$$

Since $| \int_0^t 2 \langle f, u_m \rangle | \le \int_0^t \|\nabla u_m\|_{L_x^2}^2 + \int_0^t \|f\|_{\mathbf{V}'}^2$, we get a uniform bound for $u = u_m$

$$\int_{\Omega_t} |u|^2 + \int_0^t \int_\Omega |\nabla u|^2 \le \int_\Omega |u_0|^2 + \|f\|_{L^2 \mathbf{V}'}^2. \tag{3.26}$$

This implies in particular that $g_m(t) \in L_t^\infty(0, T)$, although it is not uniform in m.

4. Take limits. Due to the above a priori bounds, up to a subsequence,

$$\begin{aligned} u_m &\rightharpoonup \bar{u} && \text{weak * in } L^\infty(0, T; L^2(\Omega)), \\ \nabla u_m &\rightharpoonup \nabla \bar{u} && \text{weakly in } L^2(\Omega_T), \end{aligned}$$

for some $\bar{u}$ which satisfies the same a priori bound (3.26). If we integrate $\int_0^t \sum_{j=1} \theta(t) \cdot (3.24)_j$ for any $\theta \in C_c^1([0, T))$, we get for $\zeta(x, t) = \phi_j(x) \theta(t)$

$$\iint -u \partial_t \zeta + \nabla u : \nabla \zeta + (w \cdot \nabla) u \cdot \zeta = \int \langle f, \zeta \rangle + \int_\Omega u_0 \cdot \zeta(x, 0) dx \tag{3.27}$$

with $u = u_m$. Passing limits $m \to \infty$, we get the same equation with $u = \bar{u}$ on the left side. Since the linear span of such ζ is dense, we have shown that $\bar{u}$ is a weak solution of the perturbed Stokes system (3.22).

5. Taking $\zeta(x, t) = \chi(x) \theta(t)$ for any $\chi \in \mathbf{V}$ and $\theta \in C_c^1([0, T))$, we get

$$- \int \bar{u} \theta' dt = \int \{ -(\nabla \bar{u}, \nabla \chi) + (f - w \cdot \nabla \bar{u}, \chi) \} \, dt + (u_0, \chi) \, \theta(0). \tag{3.28}$$

Thus $\partial_t \bar{u}$ exists in $L^2(0, T; \mathbf{V}')$. Together with $\bar{u} \in L^2(0, T; \mathbf{V})$, we get $\bar{u} \in C([0, T]; L^2)$ after redefinition at a set of times of measure zero by Lemma 3.3(iii). Furthermore, (3.28) implies

$$(\bar{u}(0) - u_0, \chi) \, \theta(0) = 0, \quad \forall \chi \in \mathbf{V}.$$

Choosing $\theta(0) = 1$, we get $\bar{u}(0) = u_0$.

6. By Lemma 3.3(iii), for $u = \bar{u}$ we have

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 + \int_\Omega |\nabla u|^2 = 2\langle f, u\rangle \quad \text{in the weak sense.} \tag{3.29}$$

Together with L^2-continuity, we get the energy identity (3.6) with v replaced by u. The uniqueness follows. This completes the proof. $\square$

We will use the same approach in Section 3.5 for Navier-Stokes equations. Steps 1–3 in the proof are roughly the same. In step 2 we get a nonlinear equation of the form

$$\frac{d}{dt}g + O(g^2) = F \in L_t^2,$$

for which we can only solve a local-in-time solution. But the bound (3.26) in step 3 implies an a priori bound for $\|g\|_{L_t^\infty}$, which guarantees the existence of $g(t)$ for $t \in [0, T]$. The main new difficulty is to show the convergence of the nonlinear term, which is the purpose of the next section.

3.4. Compactness lemma

To construct a weak solution of the nonlinear system (3.1) using approximation solutions, we need to show the convergence of the nonlinear term

$$\int (u_m \cdot \nabla) u_m \cdot \zeta \, dx\, dt. \tag{3.30}$$

Weak convergence of u_m and ∇u_m in $L^2(\Omega_T)$ is not sufficient for this convergence. We will indeed show the *strong* convergence of u_m in $L^2_{\mathrm{loc}}(\Omega_T)$. For this purpose, it suffices to have some very weak control of $\partial_t u_m$, which gives compactness in the time direction.

We will use an auxiliary result.

Lemma 3.6 (Arzela-Ascoli theorem, Banach space version)**.** *Let I be a compact metric space, Y a reflexive Banach space, and $F \subset C(I; Y)$ a subset of continuous maps from I to Y. Then F is relatively compact in $C(I; Y)$ if*

(a) *$\forall t \in I$, the set $\{f(t) : f \in F\} \subset Y$ is relatively compact in Y and*

(b) *F is equicontinuous: $\|f(t) - f(s)\|_Y \to 0$ as $\|t - s\|_I \to 0$, uniformly in $f \in F$.*

The proof is the same as the usual version. For our application I is a closed finite interval.

Here is the main result of this section.

Lemma 3.7 (Compactness). *Let $X \subset Y \subset Z$ be reflexive Banach spaces, with the imbedding $X \subset Y$ being compact.*

(i) *For any $\delta > 0$, there is $C_\delta > 0$ so that*

$$\forall x \in X, \qquad \|x\|_Y \le \delta\|x\|_X + C_\delta\|x\|_Z. \tag{3.31}$$

(ii) *If $0 < T < \infty$, $1 \le p$, $1 < r$, and $f_j \in L^p(0,T;X)$, $j \in \mathbb{N}$, satisfy*

$$\int_0^T \|f_j(t)\|_X^p dt \le C_1, \qquad \int_0^T \left\| \frac{d}{dt} f_j(t) \right\|_Z^r dt \le C_2, \tag{3.32}$$

for some constants C_1, $C_2 < \infty$, then $\{f_j\}_j$ is relatively compact in $L^p(0,T;Y)$.

This lemma is known as the Aubin-Lions lemma [**5**,**137**]. It remains true under more general assumptions. For example, the reflexivity assumption can be removed; see Simon [**190**]. For our application in the next section, $p = 2$, $r = 4/3$, $X = H^1(\Omega_R)$, $Y = L^2(\Omega_R)$, and Z is the dual space of $H^1_{0,\sigma}(\Omega_R)$, $\Omega_R = \Omega \cap B_R$, $1 \ll R < \infty$.

Proof. (i) Suppose the contrary. There is $\delta > 0$ and $x_k \in X$, $k \in \mathbb{N}$, with

$$\|x_k\|_Y \ge \delta\|x_k\|_X + k\|x_k\|_Z.$$

We may normalize $\|x_k\|_X = 1$. Since $X \subset Y$ is compact, a subsequence, still denoted by x_k, converges to some y_* in Y. The above inequality then shows $\|x_k\|_Z \to 0$. Thus $x_k \to 0$ in Z and hence $y_* = 0$. The limit of the above inequality gives the contradiction that $0 \ge \delta$.

(ii) Extend $f_j(t) = 0$ if $t \notin (0,T)$. Fix a mollifier $\phi \in C_c^\infty(\mathbb{R})$, $\operatorname{spt}\phi \in (-1,1)$, and $\int \phi = 1$. For small $\varepsilon > 0$, let $f_j^\varepsilon(t) = f_j * \phi_\varepsilon = \int_{\mathbb{R}} f_j(s)\phi_\varepsilon(t-s)ds$, where $\phi_\varepsilon(t) = \varepsilon^{-1}\phi(t/\varepsilon)$. For fixed $\varepsilon > 0$, we have $f_j^\varepsilon \in C_t^\infty$ and the uniform estimates

$$\int_0^T \|f_j^\varepsilon(t)\|_X^p dt \le 2C_1, \quad \|f_j^\varepsilon(t)\|_X \le C_3(\varepsilon), \quad \left\| \frac{d}{dt} f_j^\varepsilon(t) \right\|_X \le C_4(\varepsilon). \tag{3.33}$$

The above uniform estimates imply

(1) $\forall t \in [0,T]$, $\{f_j^\varepsilon(t) : j \in \mathbb{N}\} \subset \{x \in X : \|x\|_X \le C_3(\varepsilon)\}$ is relatively compact in Y;

(2) $\|f_j^\varepsilon(t) - f_j^\varepsilon(s)\|_Y \le C\|f_j^\varepsilon(t) - f_j^\varepsilon(s)\|_X \le CC_4(\varepsilon)|t-s|$.

By the Banach space version of the Arzela-Ascoli theorem, Lemma 3.6, with $I = [0,T]$ and $F = \{f_j^\varepsilon : j \in \mathbb{N}\}$ for a fixed ε, we conclude that $\{f_j^\varepsilon\}$ is relatively compact in $C([0,T];Y)$, hence also in $L^p(0,T;Y)$.

By part (i), we have

$$\int_0^T \|f_j(t) - f_j^\varepsilon(t)\|_Y^p dt$$
$$\leq \int_0^T c\delta^p \|f_j(t) - f_j^\varepsilon(t)\|_X^p dt + \int_0^T cC_\delta^p \|f_j(t) - f_j^\varepsilon(t)\|_Z^p dt = I_1 + I_2.$$

Note that $I_1 \leq c\delta^p C_1$. For I_2, since $f_j(t) - f_j^\varepsilon(t) = \int_{\mathbb{R}} [f_j(t) - f_j(s)]\phi_\varepsilon(t-s)ds$ and if $t > s$,

$$\|f_j(t) - f_j(s)\|_Z \leq \int_s^t \|f_j'(\tau)\|_Z d\tau$$
$$\leq \left(\int_s^t \|f_j'(\tau)\|_Z^r d\tau\right)^{1/r} |t-s|^\alpha \leq C_2^{1/r} |t-s|^\alpha,$$

where $\alpha = 1 - 1/r > 0$ using $r > 1$, we have

$$I_2 \leq cC_\delta^p \int_0^T \left(\int_{\mathbb{R}} \|f_j(t) - f_j(s)\|_Z \phi_\varepsilon(t-s)ds\right)^p dt$$
$$\leq cC_2^{p/r} C_\delta^p \int_0^T \left(\int_{\mathbb{R}} |t-s|^\alpha \phi_\varepsilon(t-s)ds\right)^p dt = cTC_2^{p/r} C_\delta^p \varepsilon^{\alpha p}.$$

For any $\eta > 0$, we first choose δ so small that $I_1 \leq \eta/2$. We then choose $\varepsilon = \varepsilon(\eta, \delta(\eta))$ so small that $cTC_2^{p/r} C_\delta^p \varepsilon^{\alpha p} \leq \eta/2$. We conclude

$$\forall \eta > 0, \quad \exists \varepsilon > 0, \quad s.t. \quad \sup_j \int_0^T \|f_j(t) - f_j^\varepsilon(t)\|_Y^p dt \leq \eta. \tag{3.34}$$

Finally we take a diagonal argument: Suppose a subsequence $(f_{k-1,j})_{j\in\mathbb{N}}$ of $(f_j)_{j\in\mathbb{N}}$ has been chosen for an integer k. For $\eta = 2^{-k}$, there is $\varepsilon = \varepsilon_k > 0$ so that $(f_{k-1,j})_{j\in\mathbb{N}}$ satisfies (3.34). Since the sequence $(f_{k-1,j}^{\varepsilon_k})_{j\in\mathbb{N}}$ is relatively compact in $L^p(0,T;Y)$, we can choose a subsequence $(f_{k,j})_{j\in\mathbb{N}}$ of $(f_{k-1,j})_{j\in\mathbb{N}}$ so that $(f_{k,j}^{\varepsilon_k})_{j\in\mathbb{N}}$ converges in $L^p(0,T;Y)$ and

$$\int_0^T \|f_{k,j}^{\varepsilon_k}(t) - f_{k,l}^{\varepsilon_k}(t)\|_Y^p dt \leq 2^{-k} \quad \text{for any } j \text{ and } l.$$

Thus

$$\int_0^T \|f_{k,j}(t) - f_{k,l}(t)\|_Y^p dt \leq c2^{-k} \quad \text{for any } j \text{ and } l.$$

The diagonal subsequence $(f_{k,k})_{k\in\mathbb{N}}$ then converges in $L^p(0,T;Y)$. □

3.5. Existence of suitable weak solutions

In this section we construct suitable weak solutions in $\mathbb{R}^3$ and in bounded domains. For suitable weak solutions we need information on the pressure.

For the whole space we can use singular integrals; see (1.52). For bounded domains, we will use the following lemma.

Lemma 3.8 (Global linear estimate). *Let Ω be a bounded $C^{2,\mu}$-domain in $\mathbb{R}^3$, $0 < \mu < 1$. Let $q, s \in (1, \infty)$ and $0 < T \le \infty$. Then for every $a \in H^1_{0,\sigma}(\Omega)$ and $f \in L^s(0,T; L^q(\Omega))$ there exists a unique solution $(u, \nabla p)$ of the Stokes system*

$$\partial_t u - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0, \quad u|_{t=0} = a, \tag{3.35}$$

satisfying $u \in L^s(0, T'; W^{2,q}(\Omega))$ for all finite $T' \le T$ and

$$\|\partial_t u, \nabla^2 u, \nabla p\|_{L^s(0,T;L^q)} \le C\|a\|_{H^1} + C\|f\|_{L^s(0,T;L^q)}, \tag{3.36}$$

with $C = C(q, s, \Omega) > 0$ independent of T. Furthermore, it coincides with any weak solution of (3.35).

The lemma is indeed true for initial data in a larger class and for $\Omega = \mathbb{R}^3$, $\mathbb{R}^3_+$, a bounded domain, or an exterior domain of class $C^{2,\mu}$. For an exterior domain we require $q < 3/2$ because of the existence of a nontrivial kernel. This lemma is not known for general domains with unbounded boundary.

For our purpose, we want

$$\nabla p \in L^{5/3}(0, T; L^{15/14}) \tag{3.37}$$

which implies by Sobolev imbedding $p - p_\Omega(t) \in L^{5/3}(0, T; L^{5/3})$. This is why we require different exponents q and s. See Solonnikov [**197**] when $q = s$, Sohr and von Wahl [**195**] when $q \ne s$ but $C = C(T)$, and Giga and Sohr [**71**] when $q \ne s$ and C is independent of T. For parabolic equations, see [**123**] when $q = s$ and [**218**] when $q \ne s$. Also see [**117**].

The time-independent constant allows the construction of time global solutions. This result is optimal in the exponents and the constant and is proved using the Stokes semigroup, which we will describe briefly in Section 5.1.

We now construct suitable weak solutions.

Theorem 3.9 (Existence of suitable weak solutions). *Let $\Omega = \mathbb{R}^3$ or $\Omega \subset \mathbb{R}^3$ be bounded and $C^{2,\mu}$, $0 < \mu < 1$. Let $0 < T \le \infty$. For any $f \in L^2(0, T; (D^{1,2}_0(\Omega))') \cap L^{5/3}_t L^{15/14}_x(\Omega_T)$ and $v_0 \in L^2_\sigma(\mathbb{R}^3)$ or $v_0 \in H^1_{0,\sigma}(\Omega)$ when Ω is bounded, there exists a suitable Leray-Hopf weak solution (v, p) of* (3.1)–(3.2) *in Ω_T with initial data v_0 and force f. Furthermore, $p \in L^{5/3}(\Omega_T)$.*

Proof. 1. Fix $\psi(x,t) \in C^\infty_c(\mathbb{R}^3 \times \mathbb{R})$ supported in $B_1 \times (1,2)$ with $\iint \psi = 1$. Let $\psi_\delta(x,t) = \delta^{-4}\psi(\frac{x}{\delta}, \frac{t}{\delta})$ for $\delta > 0$. We have $\iint \psi_\delta = 1$.

We first solve v_m for $m = 1, 2, 3, \ldots$ satisfying

$$\begin{gathered} \partial_t v + \Psi_m(v) \cdot \nabla v - \Delta v + \nabla p = f, \quad \operatorname{div} v = 0, \\ v|_{\partial\Omega} = 0, \quad v|_{t=0} = v_0, \end{gathered} \tag{3.38}$$

where

$$\Psi_m(v)(x,t) = \int_0^T \int_\Omega \psi_\delta(x-y, t-s)\, v(y,s)\, dyds, \quad \delta = \frac{T}{m}.$$

If $T = \infty$, we take $\delta = 1/m$. For $k = 0, 1, 2, \ldots$, consider (3.38) for $t \in [k\delta, k\delta + \delta]$, where $\Psi_m(v)$ is defined if using the value of $v(t)$ for $t \in [k\delta - 2\delta, k\delta - \delta]$. Hence $\Psi_m(v) \in C^\infty(\bar\Omega \times [k\delta, k\delta + \delta])$, $\operatorname{div} \Psi_m(v) = 0$. By Theorem 3.5, v_m exists in $C_t L^2_\sigma \cap L^2_t \mathbf{V}$ for $t \in [k\delta, k\delta + \delta]$. By induction on $k \geq 0$, it exists for $t \in [0, T)$.

2. Denote $w_m = \Psi_m(v_m)$. We have $w_m \in C^\infty(\bar\Omega \times [0, T])$ and

$$\|v_m\|_{L^2H^1 \cap L^\infty L^2} \leq C, \quad \|w_m\|_{L^2H^1 \cap L^\infty L^2} \leq C. \tag{3.39}$$

Moreover, since $\partial_t v_m = \operatorname{div}(\nabla v_m - w_m \otimes v_m) + f$ when acting on test functions $\zeta \in C^\infty_{c,\sigma}$, $\|\Delta v_m\|_{L^2\mathbf{V}'} \leq C$, $\|f\|_{L^2\mathbf{V}'} \leq C$, and, for $n = 3$,

$$\left| \iint v_m \cdot (w_m \cdot \nabla)\zeta \right| \leq C \|v_m\|^{1/2}_{L^\infty L^2_x} \|v_m\|^{1/2}_{L^2_t L^6_x} \|w_m\|_{L^2_t L^6_x} \|\nabla\zeta\|_{L^4_t L^2_x} \leq C \|\zeta\|_{L^4_t \mathbf{V}},$$

we have for any $T' \leq T$, $T' < \infty$,

$$\|\partial_t v_m\|_{L^{4/3}(0,T';\mathbf{V}')} \leq C(T'). \tag{3.40}$$

Note that $\partial_t v_m$ may not be in $L^{4/3}(0, T'; H^{-1})$.[1]

3. By these uniform bounds (3.39), up to a subsequence,

$$\begin{aligned} v_m &\rightharpoonup \bar v && \text{weak * in } L^\infty(0,T; L^2(\Omega)), \\ \nabla v_m &\rightharpoonup \nabla \bar v && \text{weakly in } L^2(\Omega_T), \end{aligned}$$

for some $\bar v$ which satisfies the same a priori bound (3.39). By the compactness lemma (Lemma 3.7) and the uniform bound (3.40) for $\partial_t v_m$, a further subsequence $v_m \to \bar v$ strongly in $L^2((\Omega \cap B_R) \times (0, T'))$ for any $R > 0$ and $T' < T$. It is limited to finite regions since we need the imbedding $H^1 \hookrightarrow L^2$ to be compact.

Since $v_m \to \bar v$ strongly in L^2_{loc} and $\|v_m\|_{L^q_{t,x}} \leq C$ for $q \in [2, 10/3]$ by interpolating (3.39), we get that v_m and thus w_m converge strongly to $\bar v$ in $L^q((\Omega \cap B_R) \times (0, T'))$ for any $2 \leq q < 10/3$.

[1] Thus we cannot say $\nabla p_m = -\partial_t v_m + \Delta v_m - \operatorname{div}(\Psi(v_m) \otimes v_m) + f \in L^{4/3}(0, T'; H^{-1})$.

We can now pass to the limit $m \to \infty$ in the weak form of v_m to see that $\bar{v}$ satisfies the weak form (3.3). In particular,

$$\lim_{m\to\infty} \int (w_m \cdot \nabla) v_m \cdot \zeta \, dx \, dt = \int (\bar{v} \cdot \nabla)\bar{v} \cdot \zeta \, dx \, dt, \quad \forall \zeta \in C^\infty_{c,\sigma}(\Omega_T).$$

Similarly, passing to the limit the energy equality (3.6) for v_m, we get the energy *inequality* (3.7) for $\bar{v}$. By Lemma 3.4, $\bar{v}$ can be redefined on a measure zero subset of time so that $\bar{v} \in C_{wk}(0,T;L^2)$ and satisfies (3.19). The corresponding equation for v_m converges to that of $\bar{v}_m$, showing in particular

$$(v_m(0), \zeta) \to (\bar{v}(0), \zeta) \quad \text{as } m \to \infty, \quad \forall \zeta \in C^\infty_{c,\sigma}(\Omega).$$

Since $\bar{v}(0) \in L^2_\sigma(\Omega)$, we get $\bar{v}(0) = v_0$.

4. We now estimate the distribution p_m. We have

$$\partial_t v_m - \Delta v_m + \nabla p_m = F_m := -w_m \cdot \nabla v_m + f. \tag{3.41}$$

Since v_m and w_m are uniformly bounded in $L^{8/3}_t L^4$ by interpolating (3.39), we have $F_m \in L^1(0,T';(D^{1,2}_0(\Omega))')$ for any $T' < T$. By the argument following (3.13), $p_m = \frac{d}{dt} P_m$, where p_m is the distributional derivative in time of a unique $P_m(t) \in C([0,T), L^2(\Omega))$, with $\int_\Omega P(t)dx = 0$ if Ω is bounded.

In case $\Omega = \mathbb{R}^3$, the divergence of (3.41) gives

$$-\Delta p_m = \partial_i \partial_j (w_{m,i} v_{m,j}) - \operatorname{div} f. \tag{3.42}$$

We can define a solution $\tilde{p}_m = R_i R_j (w_{m,i} v_{m,j}) - \partial_j \frac{1}{4\pi|x|} * f_j$ and, by L^q-estimates of singular integrals and Riesz potentials,

$$\|\tilde{p}_m\|^{5/3}_{L^{5/3}_{t,x}} \le C \||w_m| \cdot |v_m|\|^{5/3}_{L^{5/3}_{t,x}} + C\|f\|^{5/3}_{L^{5/3}_t L^{15/14}_x} \le C.$$

Let $h_m = p_m - \tilde{p}_m$. We have $\Delta_x h_m(x,t) = 0$ for a.e. t. The function

$$H(t) = P_m(t) - P_m(0) - \int_0^t \tilde{p}_m(s) ds = \int_0^t h_m(s) ds$$

is harmonic in x and $H(t) \in L^2_x + L^{5/3}_x$ for almost all t. Thus $H(t) = 0$ and hence $h_m = 0$, $p_m = \tilde{p}_m$.

In case Ω is a bounded domain, we cannot solve (3.42) for p_m since it does not have a boundary condition. Instead we use Lemma 3.8. We have

$$\|F_m\|_{L^{5/3}L^{15/14}} \le \|w_m\|_{L^{10}L^{30/13}} \|\nabla v_m\|_{L^2_{t,x}} + \|f\|_{L^{5/3}L^{15/14}} \le C.$$

By Lemma 3.8, the $\|\nabla p_m\|_{L^{5/3}(0,T;L^{15/14})}$ are uniformly bounded. If we replace p_m by $p_m(x,t) - \frac{1}{|\Omega|}\int_\Omega p_m(x,t)dx$ so that $\int_\Omega p_m(x,t)dx = 0$, by Sobolev inequality,

$$\|p_m\|_{L^{5/3}_{t,x}} \le \|\nabla p_m\|_{L^{5/3}(0,T;L^{15/14})} \le C.$$

In either case, we have a uniform bound of p_m in $L^{5/3}(\Omega_T)$. A further subsequence of p_m converges weakly to some $\bar{p}$ in $L^{5/3}(\Omega_T)$. We can show that v_m, p_m solve (3.41) in the distributional sense. Taking the limit $m \to \infty$ of the weak form, $\bar{v}, \bar{p}$ solve (3.1) in the distributional sense.

5. We finally check the convergence of the local energy inequality. Note that v_m, p_m satisfy the local energy *identity*

$$\int_{\Omega_t} |v_m|^2 \phi\, dx + 2\int_0^t \int_\Omega |\nabla v_m|^2 \phi dx\, dt = \int_\Omega |v_0|^2 \phi\, dx$$
$$+ \int_0^t \int_\Omega |v_m|^2(\partial_t \phi + \Delta\phi) + [|v_m|^2 w_m + 2p_m v_m] \cdot \nabla\phi + 2v_m \cdot f\phi\, dx\, dt,$$

for any t and any $0 \le \phi \in C_c^\infty$. The strong convergence of v_m and w_m in $L^3((\Omega \cap B_R) \cap (0,t))$ for any $R > 0$ guarantees the convergence of the right side to that for $(\bar{v}, \bar{p})$, while the left side for $\bar{v}$ is no greater than the lim inf of the left side for v_m. Thus $\bar{v}, \bar{p}$ satisfy the local energy inequality (3.8). □

In the literature, the approximation solutions in step 1 have several choices. In Leray [**132**] in $\mathbb{R}^3$, $\Psi_m(v)$ is a convolution in space only. One can also define $\Psi_m(v)$ by Yosida approximation, $\Psi_m(v) = (I - \frac{1}{m}P\Delta)^{-1}v$, as in [**194**]. In the pressure estimate in step 4, we can also use Lemma 3.8 for the $\mathbb{R}^3$ case, but then we need to assume higher regularity of v_0.

When we only look for a Leray-Hopf weak solution, we do not need the pressure estimate or the local energy inequality, and the assumptions on the domain and the data can be relaxed. The following is an example.

Theorem 3.10 (Existence of the Leray-Hopf weak solution)**.** *Suppose* $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, *is either* $\mathbb{R}^n$, $\mathbb{R}^n_+$, *or a bounded or exterior Lipschitz domain. Let* $\mathbf{V} = \dot{H}^1_{0,\sigma}(\Omega)$, *and let* $0 < T \le \infty$. *For any* $v_0 \in L^2_\sigma(\Omega)$ *and* $f \in L^2(0,T;\mathbf{V}')$, *there exists a Leray-Hopf weak solution* v *of* (3.1)–(3.2).

If Ω *is bounded, the solution above can be further required to satisfy the* strong energy inequality (3.9).

For the first part of the theorem, see, e.g., [**194**, Theorem V.3.1.1] and [**145**]. For the second part on strong energy inequality, see, e.g., [**194**, Theorem V.3.6.2].

It is shown by Masuda [**145**] that any Leray-Hopf weak solution v in $\Omega \times (0,\infty)$ with initial data v_0 and zero force satisfies $v \in C_{\rm loc}((T,\infty); L^2)$ for some $T < \infty$ sufficiently large, and the kinetic energy $k(t) = \int_\Omega |v(x,t)|^2 dx$ decreases to zero as $t \to \infty$.

For a general Leray-Hopf weak solution, it is unknown if it is suitable if it satisfies the energy identity or strong energy inequality or if its L^2-norm as a function of time is continuous.

The energy inequality in the definition of a Leray-Hopf weak solution requires that $k(t) \le k(0)$ for all $t > 0$. It only gives an upper bound for $k(t)$. In contrast, the strong energy inequality says that $k(t)$ is monotone decreasing in $\mathbb{R} \setminus \Sigma$, where Σ is the set of exceptional t_0 for which (3.9) may fail, and has measure zero.

For a Leray-Hopf weak solution, L^2-weak continuity and energy inequality imply L^2-strong continuity at $t = 0$. Similarly, if v satisfies the strong energy inequality, then $v(t_0)$ is right continuous in L^2 at any t_0 where the strong energy inequality holds for all $t_1 > t_0$.

3.6. Notes

For Section 3.1, the definition of weak solution has many variations in the literature; see, e.g., Leray [**132**], Hopf [**80**], Ladyženskaja [**121**], J.-L. Lions [**137**], Temam [**206**], Masuda [**145**], Constantin-Foias [**36**], Sohr [**194**], and Lemarié-Rieusset [**128**]. The concept of suitable weak solutions is introduced by Scheffer [**172**] and Caffarelli-Kohn-Nirenberg [**22**] and revised by F. Lin [**136**] based on the linear estimate of Sohr and von Wahl [**195**].

Section 3.2 is based on [**206**, III].

Section 3.3 is based on [**206**, III] and [**22**].

Section 3.4 is based on Šverák's lecture in 1998. See [**206**, III.2] for other compactness results.

Section 3.5 is based on [**22**].

Construction of suitable weak solutions. Most constructions of SWS regularize (3.1) by either adding hyperviscosity or regularizing the nonlinear term. It is interesting that weak solutions constructed by Leray [**132**] are actually suitable. Guermond [**74**] showed that weak solutions of (3.1) in bounded domains with zero boundary condition obtained by the Galerkin method are suitable for some choices of basis functions ψ_j. See the references in [**74**] for other constructions of SWS.

Strong energy inequality. The solution of Theorem 3.9 can be chosen to satisfy the strong energy inequality (3.9): When Ω is bounded, $v_m \to \bar{v}$ strongly in $L^2(\Omega_T)$; thus

$$\lim_{m\to\infty} \int_0^T g_m(t)dt = 0, \quad g_m(t) = \int_\Omega |v_m - \bar{v}|^2(x,t)\, dx.$$

Since $g_m(t)$ is bounded uniformly in m and t, there is a subset $\Sigma \subset [0,T]$ of measure zero such that a subsequence $g_{m_j}(t) \to 0$ as $m \to \infty$ for every $t \notin \Sigma$. Passing to the limits the energy identity of v_{m_j}, we get (3.9) for all $0 \le t_0 < t_1 < T$ with $t_0, t_1 \notin \Sigma$. By weak continuity, we get

(3.9) for $t_0 \notin \Sigma$ and all $t_1 > t_0$. For $\Omega = \mathbb{R}^3$ we need a localization argument; see [**128**, pp. 138 and 319]. This argument applies to other constructions of weak solutions in bounded domains. A different proof in [**194**, page 340] is based on a time-weighted energy inequality for $\bar{v}$.

Problems

3.1. Show that $u \in L^\infty(0,T;L^2) \cap C([0,T];H^{-1})$, $0 < T < \infty$, is in $C_{wk}([0,T];L^2)$ (up to a redefinition at a set of times of measure zero), but u may not be in $C([0,T];L^2)$.

3.2. If a weak solution u belongs to $L^4_{t,x}$, then it is a suitable Leray-Hopf weak solution satisfying the energy identity.

3.3. Let $u_1 \in C_{wk}([0,T_1],L^2)$ and $u_2 \in C_{wk}([T_1,T_2],L^2)$ be weak solutions with $u_1(T_1) = u_2(T_1)$. Let $u(t) = u_1(t)$ for $t \in [0,T_1]$ and $u(t) = u_2(t)$ for $t \in [T_1,T_2]$. Show that u is a weak solution for $t \in [0,T_2]$. What if u_1 and u_2 are SWS?

3.4. For a bounded smooth domain Ω, can one derive from $u_m \rightharpoonup \bar{u}$ weakly in $L^2(0,T;H^1_0(\Omega))$ that $u_m(t) \to \bar{u}(t)$ strongly in $L^2(\Omega)$ for a.e. $t \in (0,T)$?

3.5. Show that any suitable Leray-Hopf weak solution in $\mathbb{R}^3$ or a bounded smooth domain in $\mathbb{R}^3$ (not necessarily those constructed in the proof of Theorem 3.9) satisfies the strong energy inequality (3.9).

Chapter 4

Strong solutions

In this chapter we consider *strong solutions* of the nonstationary Navier-Stokes equations (NS). Roughly speaking, they are those weak solutions in certain function classes of higher regularity, so that uniqueness and higher regularity of these solutions can be guaranteed. There are many different definitions in the literature.

The existence proof of weak solutions in Chapter 3 is based on the a priori energy bound and only requires the initial data to be in L^2. However, many of their properties are unclear, including uniqueness and regularity. If the data are more regular, one may hope to uniquely solve solutions in function classes with higher regularity. However, so far we can only guarantee their existence for a *short* time interval. One example is the following; see [**206**, Theorem III.3.11, p. 316]. Observe that T is small for large v_0.

Theorem 4.1. *Let Ω be a bounded C^2-domain in $\mathbb{R}^3$. For any $v_0 \in H^1_{0,\sigma}(\Omega)$ and $f \in L^\infty(0,\infty; L^2(\Omega))$, there exist $T > C(\Omega)(\|v_0\|_{H^1} + \|f\|_{L^\infty L^2})^{-4}$ and a unique solution v of* (NS) *in Ω_T such that*

$$v \in L^\infty(0,T; H^1_{0,\sigma}(\Omega)) \cap L^2(0,T; H^2(\Omega)), \quad \partial_t v \in L^2(0,T; L^2_\sigma(\Omega)). \tag{4.1}$$

The proof of Theorem 4.1 is based on energy estimates. See Lemma 10.3 for an H^2 data version in $\mathbb{R}^3$. Another approach, based on Picard iteration, is the subject of Chapter 5.

In this chapter we will refer to *strong solutions* as those weak solutions in the energy class $L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0;T; H^1_{0,\sigma}(\Omega))$ which are also in $L^s(0,T; L^q(\Omega))$ for some q, s satisfying

$$\frac{3}{q} + \frac{2}{s} \le 1, \quad 3 < q < \infty.$$

Sometimes we drop the boundary condition. This class of weak solutions is often called the *Prodi-Serrin class* or the *Prodi-Serrin-Ladyzhenskaya class*. We will study their uniqueness and regularity in this chapter. In Section 4.1 we analyze the dimensions of various quantities, closely related to the scaling property of (NS). In Section 4.2 we considering the uniqueness of strong solutions. In Section 4.3 we considering their existence.

4.1. Dimension analysis

Recall from Section 1.3 that the Navier-Stokes equations have the following *scaling property*: If $(v(x,t),p(x,t))$ is a solution with force $f(x,t)$, then

$$v^\lambda(x,t) = \lambda v(\lambda x, \lambda^2 t), \quad p^\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$$

is also a solution, with force $f^\lambda(x,t) = \lambda^3 f(\lambda x, \lambda^2 t)$. Denote the parabolic cylinders

$$Q_r = B_r \times (-r^2, 0), \quad Q_r^\sharp = B_r \times (0, r^2).$$

The values of v^λ in Q_1 (or $Q_1^\sharp$) correspond to that of v in Q_λ (or $Q_\lambda^\sharp$).

Closely related is the *scaling dimension*. To be preserved under scaling, each term in (NS) should have equal "weight". In particular, since Δv and $v \cdot \nabla v$ have the same weight, v should have the same weight as ∇. Thus we assign the following (scaling) dimensions:

$$x:1, \quad t:2, \quad \partial_x:-1, \quad \partial_t:-2, \quad v:-1, \quad p:-2. \tag{4.2}$$

For example, when $x \in \mathbb{R}^3$,

$$d_1 = \dim \iint |\nabla v|^2 \, dx\, dt = (-1-1)*2+3+2 = 1,$$

and

$$d_2 = \dim \|v\|_{L_t^s L_x^q} = -1 + \frac{2}{s} + \frac{3}{q}.$$

Note that

$$\iint_{Q_1} |\nabla v^\lambda|^2 = \lambda^{-d_1} \iint_{Q_\lambda} |\nabla v|^2, \quad \|v^\lambda\|_{L_t^s L_x^q(Q_1)} = \lambda^{-d_2} \|v\|_{L_t^s L_x^q}(Q_\lambda).$$

In particular, the integral $\|v\|_{L_t^s L_x^q}$ has dimension zero if and only if

$$\frac{3}{q} + \frac{2}{s} = 1. \tag{4.3}$$

The above discussion is based solely on the scaling property of (NS) and is unrelated to its a priori estimates: Since $v \in L_t^\infty L_x^2 \cap L_t^2 L_x^6$ when $n=3$, we have

$$v \in L^s L^q, \quad \frac{3}{q} + \frac{2}{s} = \frac{3}{2}, \quad 2 \le q \le 6. \tag{4.4}$$

These are two parallel line segments on the square $[0,1]^2$ for $(1/q, 1/s)$, denoted as PS (Prodi-Serrin) and EC (energy class) in Figure 4.1.

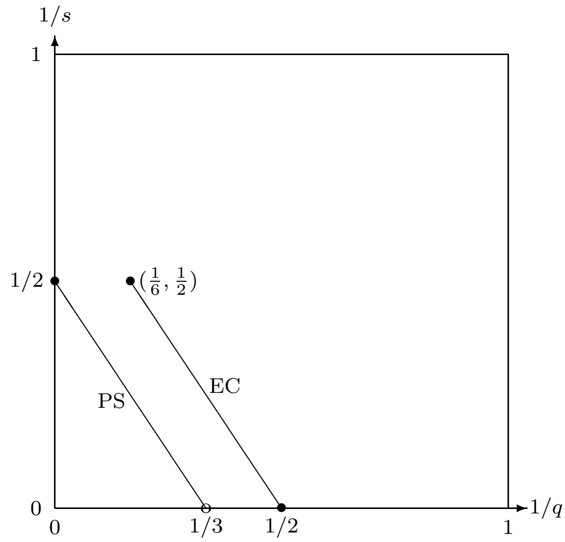


Figure 4.1. Line segments of integral exponents.

A weak solution with a finite 0-dimensional integral has many good properties, including uniqueness and regularity. Roughly speaking, it enables us to estimate the nonlinear term $v \cdot \nabla v$ like a linear term. The endpoint case $(q, s) = (3, \infty)$ is the most difficult and will be discussed separately in Chapter 9. The other cases are simpler because the finiteness of the 0-dimensional integral implies

$$\|v^\lambda\|_{L^s_t L^q_x(Q_1)} = \|v\|_{L^s_t L^q_x(Q_\lambda)} \to 0 \quad \text{as } \lambda \to 0_+. \tag{4.5}$$

This is still true for the endpoint $(q, s) = (\infty, 2)$ but not for $(q, s) = (3, \infty)$. This is why we put a circle, not a solid dot, at $(1/3, 0)$ in Figure 4.1.

4.2. Uniqueness

In this section we prove the uniqueness of strong solutions.

The following lemma involving a 0-dimensional integral will be useful. It is proved by applying Hölder inequality twice, first in x and next in t.

Lemma 4.2. *Let $3/q + 2/s = 1$ with $1 \le q, s \le \infty$. Let $\Omega \subset \mathbb{R}^3$ and $0 < T \le \infty$. For any functions u, v, w defined in $\Omega \times (0, T)$,*

$$\left| \iint uv \nabla w \, dx \, dt \right| \le C \|u\|_{L^s_t L^q_x} \|v\|^{2/s}_{L^\infty_t L^2_x} \|v\|^{3/q}_{L^2_t L^6_x} \|\nabla w\|_{L^2_{t,x}}. \tag{4.6}$$

The following lemma shows that a strong solution has many good properties, including the energy identity.

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^3$ be a locally Lipschitz domain, and let $0 < T \le \infty$. If v is a weak solution of* (NS) *in Ω_T with $v|_{\partial\Omega} = 0$ and $v \in L^s(0,T; L^q(\Omega))$ with $3/q + 2/s = 1$, $q, s \in [2, \infty]$, then v satisfies the energy identity*

$$\|v(t)\|_2^2 + 2\int_0^t \int |\nabla v|^2 \, dx \, dt' = \|v_0\|_2^2 + 2\int_0^t \int v \cdot f \, dx \, dt',$$

for any $t \in (0,T)$. Moreover, $v(t)$ is strongly L^2-continuous, $v \in L^4(\Omega_T)$, and v is a Leray-Hopf weak solution.

Proof. We first show the energy identity. We will take a two-step approximation. Fix $\phi \in C_c^\infty(\mathbb{R})$, $\phi \ge 0$, even, $\int_{-1}^1 \phi = 1$, $\phi(t) = 0$ if $|t| \ge 1$. For fixed $\tau \in (0, T]$, define

$$v_h(x,t) = \int_0^\tau v(x,t')\phi_h(t-t') \, dt' \tag{4.7}$$

where $\phi_h(t) = h^{-1}\phi(t/h)$ for $h > 0$. This is not a convolution in time since the time interval does not contain $(-h, T+h)$. We then use $C_{c,\sigma}^\infty$ approximations v_{hk} of v_h with $k \in \mathbb{N}$ as test functions in the weak form of v and take limits $k \to \infty$ to get

$$\begin{aligned} &\int_0^\tau \int -v \cdot \partial_t v_h + \nabla v : \nabla v_h + (v \cdot \nabla) v \cdot v_h \, dt' \\ &\quad = \int_0^\tau \langle f, v_h \rangle \, dt' - (v(\tau), v_h(\tau)) + (v_0, v_h(0)). \end{aligned} \tag{4.8}$$

Consider now the limits $h \to 0$. The first integral vanishes for small h since

$$\int_0^\tau \int v \cdot \partial_t v_h = \int_0^\tau \int_0^\tau (v(t), v(t'))\partial_t \phi_h(t-t') \, dt' dt$$

and $\partial_t \phi_h$ is odd. The second integral converges to $\iint |\nabla v|^2$. The cubic term

$$= -\iint (v \cdot \nabla) v_h \cdot v = -\iint (v \cdot \nabla) v_h \cdot v_h + \iint (v \cdot \nabla) v_h \cdot (v_h - v)$$

vanishes as $h \to 0$, because the first integral on the right side is zero while the second term is controlled, by Lemma 4.2, by

$$\|v - v_h\|_{L^s L^q} \|v\|_{L^\infty L^2 \cap L^2 L^6} \|\nabla v\|_{L^2_{t,x}} \quad \text{when } s < \infty$$

and by

$$\|v\|_{L^\infty L^3} \|v - v_h\|_{L^2 L^6} \|\nabla v\|_{L^2_{t,x}} \quad \text{when } s = \infty.$$

For the right side of (4.8), by weak L^2-continuity of $v(t)$, $\int_0^\tau \langle f, v_h\rangle$ converges to $\int_0^\tau \langle f, v\rangle$ and

$$(v(\tau), v_h(\tau)) = \int_0^\tau (v(\tau), v(t'))\phi_h(\tau - t')\, dt' \\ = \int_0^\tau (v(\tau), v(\tau) + o(1))\phi_h(\tau - t')\, dt'$$

which converges to $\frac{1}{2}(v(\tau), v(\tau))$ since $\int_{s>0} \phi_h(s)\, ds = 1/2$. Similarly, $(v_0, v_h(0))$ converges to $\frac{1}{2}(v_0, v_0)$. Thus the limit of (4.8) gives the energy identity.

The energy equality implies the continuity of the L^2-norm of $v(t)$. Together with weak L^2-continuity, we get strong L^2-continuity. Hence v is Leray-Hopf. Finally, $v \in L^4_{t,x}$ since

$$\begin{aligned} \|v\|_{L^4_{t,x}} &\le \|v\|^{1/2}_{L^sL^q}\|v\|^{1/2}_{L^rL^p} \\ &\lesssim \|v\|^{1/2}_{L^sL^q}\|v\|^{1/2}_{L^\infty L^2 \cap L^2L^6}, \end{aligned} \tag{4.9}$$

where $(1/p, 1/r) = (1/2, 1/2) - (1/q, 1/s)$ satisfies $3/p + 2/r = 3/2$. □

To visualize (4.9), note that $(1/4, 1/4)$ is the intersection point of the line $x+y = 1/2$ connecting $(0, 1/2)$ to $(1/2, 0)$ and the line $3x+y = 1$ connecting $(1/6, 1/2)$ to $(1/3, 0)$ in Figure 4.1 and is the middle point of $(1/q, 1/s)$ and $(1/p, 1/r)$, which lie on line segments PS and EC, respectively.

We can also start the proof of Lemma 4.3 by (4.9) and then get the rest using Problem 3.2, which further asserts that v is suitable.

We now prove a uniqueness result.

Theorem 4.4 (Weak-strong uniqueness)**.** *Let $\Omega \subset \mathbb{R}^3$ be a locally Lipschitz domain, and let $0 < T \le \infty$. Suppose u and v are both Leray-Hopf weak solutions in $\Omega \times (0,T)$ with the same data v_0 and f, and suppose $u \in L^s(0,T; L^q(\Omega))$, $3/q + 2/s = 1$, $q, s \in [2, \infty]$. If $(q, s) = (3, \infty)$, then $\|u\|_{L^\infty(0,T;L^3(\Omega))}$ is assumed sufficiently small. Then $u \equiv v$.*

The theorem is stronger than the uniqueness in the class of strong solutions. It says that once one can find a strong solution, every other weak solution has to agree with it. It does not exclude the possibility that there could be infinitely many weak solutions without a single strong solution. The case $u \in L^\infty(0,T; L^3(\Omega))$ without smallness assumption is also true; see Section 9.1.

Proof. Let $w = u - v$. Replacing v by $u + w$ in $v \cdot \nabla v$, we get

$$\partial_t w - \Delta w + \nabla \pi = ((u + w)\cdot\nabla)w + (w \cdot \nabla)u, \qquad w|_{t=0} = 0. \tag{4.10}$$

Formally, using w itself as a test function, we get

$$\frac{d}{dt}\int \frac{|w|^2}{2}\,dx + \int |\nabla w|^2\,dx = -\int u\cdot(w\cdot\nabla)w\,dx. \tag{4.11}$$

We have used $\int((u+w)\cdot\nabla)w\cdot w\,dx = 0$ and integration by parts. This computation is not rigorous since for the moment w is in the same space as v and $\int(w\cdot\nabla)w\cdot w$ and $\int(w\cdot\nabla)u\cdot w$ may not be defined. However, the resulting inequality

$$\int \frac{|w|^2(t)}{2}\,dx + \int_0^t\int |\nabla w|^2\,dxdt \le -\int_0^t\int u\cdot(w\cdot\nabla)w\,dxdt \tag{4.12}$$

is meaningful and can be proved as follows. Fix $\tau\in(0,T)$ and let v_{hk} and u_{hk} approximate v and u as in the proof of Lemma 4.3, and put them in the weak forms of u and v to get

(4.13)

$$\int_0^\tau (u,\partial_t v_{hk}) - (\nabla u,\nabla v_{hk}) - (u\cdot\nabla u, v_{hk}) = [(u,v_{hk})]_0^\tau - \int_0^\tau \langle f, v_{hk}\rangle\,,$$

(4.14)

$$\int_0^\tau (v,\partial_t u_{hk}) - (\nabla v,\nabla u_{hk}) - (v\cdot\nabla v, u_{hk}) = [(v,u_{hk})]_0^\tau - \int_0^\tau \langle f, u_{hk}\rangle\,.$$

Thanks to $u\in L^sL^q$, all terms converge in the limits. The sum of the two integrals from nonlinear terms converges to

$$\begin{aligned}\int_0^\tau -(u\cdot\nabla u, v) - (v\cdot\nabla v, u) &= \int_0^\tau (u\cdot\nabla v, u) - (v\cdot\nabla v, u)\\ &= \int_0^\tau (w\cdot\nabla v, u) = \int_0^\tau (w\cdot\nabla w, u).\end{aligned}$$

In the above there is at least one factor u in each term (not counting ∇u), and all steps can be justified. Summing the limit equations of (4.13), (4.14), and the energy inequalities

$$\begin{aligned}\frac12\|u(\tau)\|_2^2 + \int_0^\tau \|\nabla u\|_2^2 &= \frac12\|v_0\|_2^2 + \int_0^\tau \langle f,u\rangle\,,\\ \frac12\|v(\tau)\|_2^2 + \int_0^\tau \|\nabla v\|_2^2 &\le \frac12\|v_0\|_2^2 + \int_0^\tau \langle f,v\rangle\,,\end{aligned}$$

we arrive at (4.12).

Denote $E(t) = \operatorname{ess\,sup}_{s<t}\|w(s)\|_2^2 + \int_0^t \|\nabla w\|_2^2$ and $t_0 = \sup\{t\ge 0 : w(s) = 0 \text{ if } 0<s<t\}$. If $t_0 < T$, by (4.12) and Lemma 4.2,

$$E(t) \le C\|u\|_{L^s((t_0,t);L^q)}\cdot E(t),\quad t_0<t<T. \tag{4.15}$$

We have $C\|u\|_{L^s((t_0,t);L^q)} < 1$ for t sufficiently close to t_0 if $s<\infty$ or if $C\|u\|_{L^\infty((0,T);L^3)} < 1$. Thus $t_0 = T$ and $E(t) = 0$ for all t. □

The theorem remains true if we drop the assumption $u \in L^sL^q$ but assume that both $u, v \in L^4_{t,x}$. See Problem 4.2.

4.3. Regularity

In this section we show the regularity of weak solutions with finite 0-dimensional integral. We state two such results.

Theorem 4.5 (Global regularity). *Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain or $\mathbb{R}^3$ itself, and let $0 < T < \infty$. If a weak solution v of* (NS) *with $f = 0$ in $\Omega \times (0, T)$ satisfies $v \in L^sL^q(\Omega \times (0, T))$ with $\frac{3}{q} + \frac{2}{s} \le 1$, $q > 3$, then $v \in C^\infty_{t,x}(\bar{\Omega} \times (0, T])$.*

The regularity is in both x and t and is up to $t = T$ but not including $t = 0$.

Theorem 4.6 (Interior regularity). *Let $Q_r = B_r \times (-r^2, 0)$, $B_r = \{x \in \mathbb{R}^3, |x| < r\}$. If v is a very weak solution of* (NS) *in Q_1 with smooth f and*

$$v \in L^\infty L^2 \cap L^2H^1 \cap L^sL^q(Q_1) \tag{4.16}$$

with $\frac{3}{q} + \frac{2}{s} \le 1$, $q > 3$, then $v \in L^\infty(Q_{1/2})$.

In this theorem there is no boundary or initial condition. In fact we can improve the *spatial* regularity and get that

$$v \in L^\infty_t C^{k,\mu}_x(Q_{1/2}), \quad \forall k \ge 0,\ 0 < \mu < 1. \tag{4.17}$$

However we cannot improve the *temporal* regularity in view of the explicit solution of (NS) with zero force in the class (4.16),

$$v(x, t) = g(t)\nabla h(x), \quad p = -g'h - \frac{1}{2}g^2|\nabla h|^2, \tag{4.18}$$

where $\Delta h(x) = 0$ and $g(t)$ is bounded but may not be continuous. Therefore the zero boundary condition in Theorem 4.5, which avoids example (4.18), is essential for the temporal regularity.

We will prove Theorem 4.5 in Chapter 5 and Theorem 4.6 in Chapter 6. In this section we prove a special case of Theorem 4.6 with $3/q + 2/s < 1$ and $f = 0$, which is the first such regularity result, proved by Serrin in 1962 [**187**].

Proof of Theorem 4.6 for the case $3/q + 2/s < 1$ and $f = 0$.

Consider the vorticity $\omega = \operatorname{curl} v \in L^2(Q_2)$. It satisfies the weak form of

$$(\partial_t - \Delta)\omega^i = \partial_j g_{ij}, \quad g_{ij} = v_i\omega_j - v_j\omega_i, \tag{4.19}$$

which can be derived by using test functions of the form $\zeta = \operatorname{curl} \eta$ in the weak form of v and noting that $(\chi, \operatorname{curl} \eta) = -(\operatorname{curl} \chi, \eta)$.

We will now proceed by induction (bootstraping). Let $\sigma := 1 - 3/q - 2/s > 0$. Choose $0 < \delta < \sigma/5$ so that $K = \frac{1}{2\delta} \in \mathbb{N}$. Define

$$p_0 = 2, \quad \frac{1}{p_{k+1}} = \frac{1}{p_k} - \delta = \frac{1}{2} - (k+1)\delta \quad (0 \le k \le K-1), \tag{4.20}$$

with $2 = p_0 < \cdots < p_K = \infty$. Choose $r \in (1/2, 1)$ so that $r^{2K+1} = 1/2$. Fix a cut-off function $\phi(x,t) \in C^\infty(\mathbb{R}^4)$, $\phi = 0$ in $\big(\mathbb{R}^3 \times (-\infty, 0)\big) \setminus Q_1$, and $\phi = 1$ in Q_r. Let $\Gamma(x,t) = (4\pi t)^{-3/2} \exp(-\frac{|x|^2}{4t})$ for $t > 0$ and $\Gamma(x,t) = 0$ for $t \le 0$ be the *heat kernel* in $\mathbb{R}^3$.

We have $u \in L^s L^q(Q_1)$ and $\omega \in L^2_{t,x}(Q_1)$. Suppose $\omega \in L^{p_k}_{t,x}(Q_{r^{2k}})$ for some $0 \le k < K$. (We already have the case $k = 0$.) Let

$$\tilde{\omega}_i(x,t) = \int_{-1}^t \int_{\mathbb{R}^3} (\partial_j \Gamma)(x-y, t-s)(\phi_k g_{ij})(y,s)\, dy ds,$$

where $\phi_k(x,t) = \phi(r^{-2k}x, r^{-4k}t)$. We have

$$\phi_k g_{ij} \in L^a_t L^b_x, \quad \frac{1}{a} = \frac{1}{p_k} + \frac{1}{s}, \quad \frac{1}{b} = \frac{1}{p_k} + \frac{1}{q}. \tag{4.21}$$

Note that $a \ge 1$, $b \ge 1$. Also note that for $1 \le \beta < \infty$,

$$\|\nabla\Gamma(\cdot, t)\|_{L^\beta(\mathbb{R}^3)} = Ct^{-\alpha}, \quad \alpha = 2 - \frac{3}{2\beta} > 0.$$

By Young's inequality for convolutions, first in x and then in t, we have

$$\|\partial_j \Gamma(t-s) *_x (\phi_k g_{ij})(s)\|_{L_x^{p_{k+1}}} \le C \|\nabla\Gamma(t-s)\|_{L_x^\beta} \|\phi_k g_{ij}(s)\|_{L_x^b},$$

and

$$\|\tilde{\omega}\|_{L_{t,x}^{p_{k+1}}(\mathbb{R}^3 \times (-1,0))} \le C \|t^{-\alpha}\|_{L^m(0,1)} \|\phi_k g\|_{L^a L^b} < \infty,$$

if β and m satisfy

$$\frac{1}{p_{k+1}} = \frac{1}{\beta} + \frac{1}{b} - 1 = \frac{1}{m} + \frac{1}{a} - 1, \quad \alpha m < 1 \le m. \tag{4.22}$$

That $m \ge 1$ is true because $p_{k+1} \ge p_k \ge a$. That $\alpha m < 1$ is true because of the definitions of β, m, a, b, (4.20), and $5\delta < \sigma = 1 - 3/q - 2/s$. Let

$$B_i = \omega_i - \tilde{\omega}_i \in L^2(Q_1). \tag{4.23}$$

It satisfies

$$(\partial_t - \Delta) B_i = 0 \quad \text{in } Q_{r^{2k+1}}, \tag{4.24}$$

in the weak sense. Thus $B_i \in L^\infty(Q_{r^{2k+2}})$, and $\omega = \tilde{\omega} + B \in L^{p_{k+1}}(Q_{r^{2k+2}})$ for $0 \le k < K$. The last iteration $k = K-1$ shows that $\omega \in L^\infty_{t,x}(Q_{r^{2K}})$.

By the elliptic estimate Lemma 2.11,

$$\|\nabla v\|_{L^\infty L^4(Q_{r2K+1})} \lesssim \|\omega\|_{L^\infty L^4(Q_{r2K})} + \|v\|_{L^\infty L^1(Q_{r2K})} < \infty.$$

By Sobolev imbedding, $v \in L^\infty_{t,x}(Q_{r^{2K+1}})$. $\square$

We can further prove (4.17) using the same approach; see Problem 4.3.

4.4. Notes

Section 4.2 is based on [**188**]. Section 4.3 is based on [**187**].

The study of strong solutions seems to start with Kiselev-Ladyženskaya [**103**], for $n = 2, 3$ and solutions in the class

$$v \in L^\infty L^4, \quad \nabla v, \partial_t v \in L^\infty L^2.$$

For uniqueness, Theorem 4.4 is due to Prodi [**165**]. The proof we presented is due to Sather-Serrin; see [**188**]. Uniqueness in the endpoint class $L^\infty L^3$ is proved by Kozono-Sohr [**115**]; see Chapter 9.

For regularity in the class $L^s L^q$, $3/q + 2/s < 1$, Serrin [**187**] proved the interior regularity, whose proof we presented. It is extended to the boundary by Kaniel and Shinbrot [**94**]. The borderline cases $\frac{3}{q} + \frac{2}{s} = 1$ are also true. When $3 < q \le \infty$, the global case (similar to Theorem 4.5) is proved by Sohr [**193**] based on semigroup methods and Yosida's approximation and independently by Giga [**63**] based on the lower bound of blowup rate in L^q. We will present Giga's argument in Section 5.3. It can also be derived using Ladyženskaja [**119**]; see Problems 4.4–4.6. The interior case theorem, Theorem 4.6, is by Struwe [**201**] based on Moser iteration. Theorem 4.6 will follow from Theorem 6.7 noting that the solution is suitable. Regularity in the endpoint class $L^\infty L^3$ is proved by Escauriaza-Seregin-Šverák [**41**]; see Chapter 9.

The 2-dimensional theory of uniqueness and regularity is much easier; see [**206**, III.3.3]. A 2D weak solution in the energy class belongs to $L^\infty L^2 \cap L^2(\mathrm{BMO})$ since $H^1 \subset \mathrm{BMO}$ (BMO is bounded mean oscillation). By interpolation $u \in L^4_{t,x}$, which has zero dimension in 2D, and hence similar arguments of this chapter are applicable to a general 2D weak solution. To prove their regularity, one may also use the vorticity equation (1.39) which has no potential term in 2D and hence has maximal principle.

Problems

4.1. Let Ω be a bounded C^2-domain in $\mathbb{R}^3$, let $P : L^2(\Omega, \mathbb{R}^3) \to L^2_\sigma(\Omega)$ be its Helmholtz projection, and let $A = -P\Delta$ be the Stokes operator.
(a) Show that $\|u\|_{H^2(\Omega)} \lesssim \|Au\|_{L^2(\Omega)}$ for $u \in \mathbf{V} \cap H^2(\Omega)$. (Hint. Theorem 2.13.)

(b) For $v_0 \in \mathbf{V}$ and $f = 0$, use the test function $\zeta = Av$ in the weak form (3.3) to formally derive

$$\frac{d}{dt}\int |\nabla v|^2 + \int |Av|^2 \le C\left(\int |\nabla v|^2\right)^3.$$

This is the a priori bound used to prove Theorem 4.1. Use it to find the lower bound for T in Theorem 4.1.

4.2. Let $\Omega \subset \mathbb{R}^3$ and $0 < T \le \infty$. Suppose u and v are both weak solutions of (3.1) in Ω_T with same initial data and force, and suppose both $u, v \in L^4_{t,x}$. Show that $u = v$. Compare Problem 3.2.

4.3. Suppose v is a very weak solution of (NS) in Q_1 with $f = 0$ and suppose v satisfies (4.16) with $\frac{3}{q} + \frac{2}{s} < 1$, $q, s \in [2, \infty]$. Show the higher spatial regularity that v satisfies (4.17). (Hint. Use the Hölder estimate for heat potential and (2.65).)

4.4. Let $\Omega \subset \mathbb{R}^3$ be a bounded C^2-domain, and let $0 < T < \infty$. Suppose

$$b \in L^s(0, T; L^q(\Omega, \mathbb{R}^3)), \quad \frac{3}{q} + \frac{2}{s} = 1, \quad q > 3.$$

There is no assumption on $\operatorname{div} b$. For $v_0 \in L^2_\sigma(\Omega)$ and $f \in L^\infty L^2(\Omega_T)$, show the uniqueness of weak solutions in the class $u \in L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; H^1_0(\Omega))$ for the perturbed Stokes system

$$\begin{gathered}\partial_t u - \Delta u + (b\cdot\nabla)u + \nabla p = f, \quad \operatorname{div} u = 0, \quad \text{in } \Omega_T, \\ u|_{t=0} = v_0, \quad u|_{\partial\Omega} = 0.\end{gathered} \tag{4.25}$$

4.5. Use the same assumptions as in Problem 4.4 and further assume $v_0 \in H^1_{0,\sigma}(\Omega)$. Show that there is a weak solution u of (4.25) with

$$\int_0^T \int_\Omega |\partial_t u|^2 + |\nabla^2 u|^2 dx\, dt < \infty. \tag{4.26}$$

4.6. Let $\Omega \subset \mathbb{R}^3$ be a bounded C^2-domain, and let $0 < T < \infty$. Suppose v is a weak solution of (NS) with $v_0 \in H^1_{0,\sigma}(\Omega)$ and $f = 0$, and suppose $v \in L^s(0, T; L^q(\Omega))$, $\frac{3}{q} + \frac{2}{s} = 1$, $q > 3$. Use Problems 4.4 and 4.5 to show that v satisfies (4.26).

Remark. Once v satisfies (4.26), we have

$$\|v\|^2_{L^{10}(\Omega_T)} \le C\int_0^T \int_\Omega |v|^2 + |\partial_t v|^2 + |\nabla^2 v|^2 dx\, dt,$$

by the imbedding lemma, [**123**, Lemma II.3.3]. We can then use Serrin's subcritical result to get the regularity of v.

Chapter 5

Mild solutions

The existence of weak solutions of (NS) is based on the a priori estimate which allows the construction of large, global-in-time weak solutions by taking weak limits. A different approach is to treat the nonlinearity as a perturbation of the linear part and solve the solution by successive approximation, with strong convergence. Solutions constructed in this way are called *mild solutions*. They can be considered a special family of strong solutions since the solutions are unique and more regular, but global-in-time solutions can be constructed only for "small" data.

Due to the scaling property (1.21) of (NS), the size of the data has to be measured in some scaling-invariant (or 0-dimensional) norms. For $\mathbb{R}^3$, the following spaces have been considered:

$$\dot{H}^{1/2} \subset L^3 \subset L^{3,\infty} \subset \dot{B}^{3/p-1}_{p,\infty} \subset \mathrm{BMO}^{-1}, \tag{5.1}$$

with $3 < p < \infty$. Here $L^{p,q}$ denotes the Lorentz spaces with $L^{p,\infty}$ being weak L^p, $\dot{B}^s_{p,q}$ denotes homogeneous Besov spaces, and BMO^{-1} is the set of distributional derivatives of BMO functions.

In this chapter, we will consider mild solutions in a domain $\Omega \subset \mathbb{R}^3$. In Section 5.1 we consider the nonstationary Stokes system. In Section 5.2 we construct mild solutions of (NS) with initial data in $L^q(\Omega)$. In Section 5.3 we use mild solutions to study the regularity of a given weak solution of (NS).

5.1. Nonstationary Stokes system and Stokes semigroup

Let Ω be either $\mathbb{R}^3$ or a bounded smooth domain in $\mathbb{R}^3$. In this section we study the solutions of the nonstationary Stokes system in Ω,

$$\begin{gathered} \partial_t v - \Delta v + \nabla p = f = f_0 + \operatorname{div} F, \quad \operatorname{div} v = 0, \\ v|_{t=0} = v_0, \quad v|_{\partial\Omega} = 0. \end{gathered} \tag{5.2}$$

Above, $F = (F_{ij})$, $(\operatorname{div} F)_i = \partial_j F_{ij}$, and we assume $v_0 = Pv_0$. We already considered the existence of solutions in the energy class of the (perturbed) Stokes system in Section 3.3. In this section we want to treat solutions in other L^q-spaces, $q \neq 2$.

When $\Omega = \mathbb{R}^3$, we use the *Oseen tensor*, which is the *fundamental solution* of the nonstationary Stokes system found by Oseen [**161**] (also see [**196**, page 27])

(5.3)
$$S_{ij}(t,x) = \Gamma(t,x)\delta_{ij} + \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\int_{\mathbb{R}^3}\frac{\Gamma(t,y)}{|x-y|}dy, \quad Q_j(t,x) = \frac{\delta(t)}{4\pi}\frac{x_j}{|x|^3},$$

where Γ is the fundamental solution of the heat equation. It is known [**196**, Theorem 1] that $S = (S_{ij})$ satisfies the following pointwise estimates:

$$\left|\nabla_x^\ell \partial_t^k S(t,x)\right| \le C_{k,l}(|x| + \sqrt{t})^{-3-\ell-2k} \quad (\ell, k \ge 0). \tag{5.4}$$

A solution of the nonstationary Stokes system (5.2) in $\mathbb{R}^3$, if f_0 and F have sufficient decay, is given by

$$\begin{aligned} v_i(x,t) = &\int_{\mathbb{R}^3} S_{ij}(x-y,t)(v_0)_j(y)dy \\ &+ \int_0^t \int_{\mathbb{R}^3} \{S_{ij}(x-y,t-s)f_j(y,s) - \partial_{y_k} S_{ij}(x-y,t-s)F_{jk}(y,s)\}\, dyds. \end{aligned}$$

Hence we can estimate the solution v using (5.4). Note that we do not require $\operatorname{div} f = 0$, and the Helmholtz projection is incorporated in S_{ij}.

When Ω is a general domain, we usually rewrite (5.2) as

$$\partial_t v + A_q v = P_q f, \quad v(0) = v_0, \tag{5.5}$$

where P_q is the Helmholtz projection from $L^q(\Omega;\mathbb{R}^n)$ to $L^q_\sigma(\Omega)$, $1 < q < \infty$, whose restriction on $C_c^\infty(\Omega;\mathbb{R}^n)$ is independent of q (see Problem 1.9), and A_q is the *Stokes operator* in $L^q_\sigma(\Omega)$,

$$A_q = -P_q\Delta, \quad D(A_q) = W^{1,q}_{0,\sigma} \cap W^{2,q}(\Omega). \tag{5.6}$$

The projection P_q is needed because for $v \in D(A_q)$, $\Delta v \in L^q$ is divergence-free but may not satisfy the boundary condition of $L^q_\sigma(\Omega)$.

We now summarize known properties of the Stokes operator.

The Stokes operator A_q defined in (5.6) is a closed, densely defined linear operator in the Banach space $L^q_\sigma = L^q_\sigma(\Omega)$. Its dual operator A_q^* is $A_{q'}$. Its resolvent $(A_q + \lambda)^{-1} : L^q_\sigma(\Omega) \to D(A_q)$ is defined for $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ and satisfies, for any $\omega \in (\pi/2, \pi)$,

$$\|(A_q + \lambda)^{-1} f\|_q \le \frac{C_{\varepsilon,q}}{\lambda}\|f\|_q, \quad \forall f \in L^q_\sigma(\Omega), \quad \forall \lambda \in \Sigma_\omega, \tag{5.7}$$

where Σ_ω denotes the set of nonzero $\lambda \in \mathbb{C}$ with $|\arg \lambda| < \omega$. Consequently, it generates a bounded *analytic semigroup* $\{e^{-tA_q} : t \in \Sigma_\theta \cup \{0\}\}$ in $L^q_\sigma(\Omega)$ for any $\theta \in (0, \pi/2)$.

If we only want to solve the Stokes system (5.5), it is sufficient that $\{e^{-tA_q} : t \geq 0\}$ is a *contraction semigroup*, which is true if and only if (5.7) holds for all $\lambda > 0$, by the Hille-Yosida Theorem. In this case, the solution to (5.5) is formally given by

$$v(t) = e^{-tA}v_0 + \int_0^t e^{-(t-s)A} Pf(s)\, ds, \tag{5.8}$$

where $A = A_q$. It is in $C([0,T], L^q_\sigma)$ if $v_0 \in L^q_\sigma$ and $Pf \in L^1(0,T; L^q_\sigma)$. When $v_0 \in D(A_q)$ and $Pf \in C([0,T]; D(A_q))$, then $v \in C([0,T], D(A_q)) \cap C^1((0,T), L^q_\sigma)$ and (5.5) is valid for every $t > 0$.

When e^{-tA_q} is an analytic semigroup, for any $v_0 \in L^q_\sigma$, $e^{-tA_q}v_0$ is infinitely differentiable in $t > 0$, and $(\frac{d}{dt})^k e^{-tA_q} v_0 \in D(A_q)$ for any $k \in \mathbb{N}$. Hence $v(t)$ defined by (5.8) is more regular. More essentially for the application to semilinear equations, being an analytic semigroup allows the definition of the fractional powers A_q^α, $\alpha \in \mathbb{R}$.

Specifically, for $\alpha < 0$, A_q^α is a bounded, injective operator in $L^q_\sigma(\Omega)$, defined by a contour integral of the resolvent of A_q in the complex plane. For $\alpha > 0$, A_q^α is defined as the inverse of $A_q^{-\alpha}$, with domain $D(A_q^\alpha)$ being the range of $A_q^{-\alpha}$. We can prove

$$\|A_q^\alpha e^{-tA_q}\|_{L^q_\sigma \to L^q_\sigma} \leq C_{\alpha,q} t^{-\alpha}, \quad \forall \alpha \geq 0, \quad \forall t > 0. \tag{5.9}$$

For $0 \leq \alpha \leq 1$, the domain $D(A_q^\alpha)$ is the complex interpolation space $[L^q_\sigma, D(A_q)]_\alpha$, which agrees with $L^q_\sigma \cap D((-\Delta_q)^\alpha)$, and is continuously imbedded in the space of Bessel potentials $H^{2\alpha,q}(\Omega; \mathbb{R}^3)$. If $0 < \frac{1}{q} - \frac{2\alpha}{3} < 1$,

$$H^{2\alpha,q}(\Omega) \hookrightarrow L^r(\Omega), \quad \frac{1}{r} = \frac{1}{q} - \frac{2\alpha}{3}. \tag{5.10}$$

For (NS), $A^{1/2}$ plays a particular role. For $v \in D(A_q^{1/2})$, we have the estimate

$$\|A^{1/2}v\|_{L^q} \lesssim \|\nabla v\|_{L^q}, \quad \|\nabla v\|_{L^q} \lesssim \|A^{1/2}v\|_{L^q}. \tag{5.11}$$

(The second inequality is limited to $1 < q < 3$ if Ω is an exterior domain.) Related, we also have for each j, $1 \leq j \leq 3$,

$$\begin{aligned} &\|\partial_{x_j} A_q^{-1/2} Pv\|_{L^q} \leq C\|v\|_{L^q} \\ &\|A_q^{-1/2} P \partial_{x_j} v\|_{L^q} \leq C\|v\|_{L^q} \end{aligned} \quad \forall v \in L^q(\Omega, \mathbb{R}^3). \tag{5.12}$$

Our construction of mild solutions is based on the following two lemmas.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain or $\mathbb{R}^3$ itself.*

(i) *Let $1 < r \le q < \infty$ and $\sigma = \sigma(q, r) = \frac{3}{2}(\frac{1}{r} - \frac{1}{q}) \ge 0$. We have*

$$\begin{aligned} \|e^{-tA}Pa\|_q &\le Ct^{-\sigma}\|a\|_r, \\ \|\nabla e^{-tA}Pa\|_q &\le Ct^{-\sigma-1/2}\|a\|_r, \qquad \forall t > 0. \\ \sup_j \|e^{-tA}P\partial_{x_j}a\|_q &\le Ct^{-\sigma-1/2}\|a\|_r, \end{aligned} \tag{5.13}$$

(ii) *If $a \in L^r$, for $q > r$ we have $t^\sigma\|e^{-tA}Pa\|_q \to 0$ as $t \to 0$ or as $t \to \infty$. We also have $\|e^{-tA}Pa\|_r \to \|Pa\|_r$ as $t \to 0$ and $\|e^{-tA}Pa\|_r \to 0$ as $t \to \infty$.*

Proof. (i) If $\Omega = \mathbb{R}^3$, (5.13) follows from the fact that $P\nabla = \nabla P$, the estimates (5.4) for Oseen's kernel, and Young's inequality for convolution.

If Ω is a bounded smooth domain, by imbedding $D(A_r^\sigma) \subset H^{2\sigma,r} \hookrightarrow L^q$ and (5.9),

$$\|e^{-tA}Pa\|_q \lesssim \|A^\sigma e^{-tA}Pa\|_r \lesssim t^{-\sigma}\|Pa\|_r \lesssim t^{-\sigma}\|a\|_r.$$

Similarly,

$$\|A^{1/2}e^{-tA}Pa\|_{L^q} \lesssim \|A^{\sigma+1/2}e^{-tA}Pa\|_{L^r} \lesssim t^{-\sigma-1/2}\|a\|_r.$$

By (5.11), $\|\nabla e^{-tA}Pa\|_q \lesssim \|A^{1/2}e^{-tA}Pa\|_{L^q}$, giving the second estimate of (5.13). Moreover, by (5.12),

$$\begin{aligned} \|e^{-tA}P\partial_{x_j}a\|_q &= \|A^{1/2}e^{-tA}A^{-1/2}P\partial_{x_j}a\|_q \\ &\lesssim \|A^{1/2}e^{-tA}\|_{L^r_\sigma \to L^q_\sigma} \cdot \|A^{-1/2}P\partial_{x_j}a\|_r \\ &\lesssim t^{-\sigma-1/2}\|a\|_r. \end{aligned}$$

(ii) For $t \to 0$, if $q > r$, for any $\varepsilon > 0$ we can choose $b \in L^r \cap L^q$ with $\|a-b\|_r \le \varepsilon/2C$ where C is the constant in (5.13). Thus $t^\sigma\|e^{-tA}P(a-b)\|_q \le \varepsilon/2$. By (5.13) again,

$$t^\sigma\|e^{-tA}Pb\|_q \le t^\sigma C\|b\|_q \to 0 \quad \text{as } t \to 0_+.$$

This shows the case $q > r$. If $q = r$, we have $e^{-tA}Pa \to Pa$ strongly in L^r.

For $t \to \infty$ and $q \ge r$, the same argument works with $b \in L^r \cap L^{\tilde r}$ for any $\tilde r \in (1, r)$, noting

$$t^\sigma\|e^{-tA}Pb\|_q \le t^\sigma t^{-\tilde\sigma}C\|b\|_{\tilde r}, \quad \tilde\sigma = \frac{3}{2}\left(\frac{1}{\tilde r} - \frac{1}{q}\right) > \sigma. \tag{5.14}$$

□

Lemma 5.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain or $\mathbb{R}^3$ itself. If $3/q + 2/s = 1$ and $3 \le q < 9$, then*

$$\|e^{-tA}Pa\|_{L^s(0,\infty;L^q(\Omega))} \le C_q\|a\|_{L^3(\Omega)}. \tag{5.15}$$

The case $q = 9$ is also true; see [**63**, Acknowledgment]. The cases $q > 9$ seem open. The proof will use the *Marcinkiewicz interpolation* theorem (e.g., [**199**, Appendix B]). Recall that the weak L^s-space, L^s_{wk}, consists of those f with $\|f\|_{L^s_{wk}} = \sup_{\tau>0} \tau |\{|f| > \tau\}|^{1/s} < \infty$ and that a linear map is of *weak type* (r, s) if it maps L^r into L^s_{wk}. The Marcinkiewicz interpolation theorem says that if a linear map U is simultaneously of weak types (r_0, s_0) and (r_1, s_1) with

$$1 \le r_i \le s_i \le \infty \quad (i = 1, 2), \quad r_0 \ne r_1, \quad s_0 \ne s_1,$$

then it is of *strong type* (r, s) if

$$\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \quad \frac{1}{s} = \frac{1-\theta}{s_0} + \frac{\theta}{s_1}, \quad 0 < \theta < 1;$$

i.e., $\|Uf\|_{L^s} \le C\|f\|_{L^r}$ for some constant C independent of f.

Proof of Lemma 5.2. For a fixed q, consider the linear map

$$U : L^r(\Omega, \mathbb{R}^3) \to L^s_{wk}(0, \infty), \quad U(a)(t) = \|e^{-tA}Pa\|_{L^q(\Omega)}. \tag{5.16}$$

By Lemma 5.1(i), our U is of weak type (r, s) for $r, s > 1$ and $2/s = 3(1/r - 1/q)$. If r, s satisfy the previous relation and $1 < r < s < \infty$, one can apply the Marcinkiewicz interpolation theorem to conclude that U is of strong type (r, s); i.e., it maps $L^r(\Omega, \mathbb{R}^3)$ to $L^s(0, \infty)$. Now fix $r = 3$. The restriction $3 < s < \infty$ amounts to $3 < q < 9$. The case $q = 3$ is by Lemma 5.1(i). □

5.2. Existence of mild solutions

In this section we construct mild solutions of (NS), and we write the Stokes operator as A and skip the subscript. The (NS) with $f = 0$ can be written as

$$\partial_t v + Av + Pv \cdot \nabla v = 0, \quad v(0) = v_0. \tag{5.17}$$

It is formally equivalent to the following integral equation:

$$v(t) = e^{-tA}v_0 + B(v, v)(t), \tag{INS}$$

where

$$B(u, v)(t) = -\int_0^t e^{-(t-s)A} P\partial_j(u_j v)(s)\, ds. \tag{5.18}$$

If desirable, we can replace B by $\tilde{B}(u, v) = \frac{1}{2}(B(u, v) + B(v, u))$ to make it symmetric, since it is $B(v, v)$ that occurs in (INS).

Definition 5.3. Let Ω be a domain in $\mathbb{R}^n$ and let $0 < T \le \infty$. Let X be a Banach space of vector functions defined in $\Omega \times (0,T)$, so that $v_1(t) = e^{-tA}v_0 \in X$ for certain v_0 and so that the bilinear map $B(u,v)$ in (5.18) can be interpreted as a bounded bilinear map

$$B : X \times X \to X$$

with a bound C_1. A *mild solution* of (INS) in X with initial data v_0 is some $v \in X$ so that $v = v_1 + B(v,v)$ in X. We also say v is a mild solution in $(0,T)$ if v is a mild solution in $(0,T')$ for any $0 < T' < T$. (This allows the growth of norm $\|v\|_{X_{T'}} \to \infty$ as $T' \to T$.)

For $-\infty < t_0 < t_1 \le \infty$, $u(t)$ is a mild solution of (INS) in $[t_0,t_1)$ with initial data $u(t_0) = v_0$ if $v(t) = u(t+t_0)$ is a mild solution in $[0,t_1-t_0)$ with initial data $v(0) = v_0$.

For mild solutions considered in this chapter, the norm of X is of the form $\|\|v(t)\|_Z\|_{L^s(w(t)dt)}$ for some spatial norm Z and temporal weight $w(t)$, and the integral (5.18) for $B(u,v)$ converges absolutely. However it is not necessary. For example, when $X = BC_w([0,\infty); L^{3,\infty}_\sigma(\Omega))$, the integral is defined by duality [**222**]. Also, for Koch-Tataru solutions [**105**], the X-norm also contains space-time integrals in parabolic cylinders.

Mild solutions are usually constructed by successive approximations (although this is not part of the definition and is unnecessary)

$$v_1(t) = e^{-tA}v_0, \quad v_{k+1} = v_1 + B(v_k, v_k), \quad k = 1,2,\dots. \tag{5.19}$$

Suppose

$$8C_1\|e^{-tA}v_0\|_X \le 1, \tag{5.20}$$

which usually amounts to a smallness condition on either T or v_0; then

$$\begin{aligned} \|v_{k+2} - v_{k+1}\|_X &= \|B(v_{k+1}, v_{k+1} - v_k) + B(v_{k+1} - v_k, v_k)\|_X \\ &\le C_1(\|v_{k+1}\|_X + \|v_k\|_X)\|v_{k+1} - v_k\|_X, \end{aligned} \tag{5.21}$$

and v_k converges to a (unique) solution in the class of $v \in X$ with $\|v\|_X \le 2\|e^{-tA}v_0\|_X$. This is the usual strategy to prove existence for small data.

In the following, BC denotes bounded and continuous functions. The first theorem is concerned with L^q data, $q > 3$.

Theorem 5.4 (L^q data, $q > 3$)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain or $\mathbb{R}^3$ itself. Let $3 < q < \infty$ and $s = \frac{2q}{q-3}$.*

(i) *(Unique existence) There is a constant $C > 0$ so that, for any $v_0 \in L^q_\sigma(\Omega)$, there is a unique mild solution u of* (INS) *satisfying*

$$v \in BC([0,T); L^q_\sigma), \quad T \ge C\|v_0\|_q^{-s}. \tag{5.22}$$

It is unique in the class of all (possibly large) mild solutions in (5.22).

(ii) *(Restriction) For any* $t_1 \in (0,T)$*, the restriction of the solution* v *in part* (i) *to* $[t_1,T)$ *is a mild solution in* $[t_1,T)$ *with initial data* $v(t_1)$*; that is,*

$$v(t) = e^{-(t-t_0)A}v(t_0) - \int_{t_0}^{t} e^{-(t-s)A}P\partial_j(v_j\, v)(s)\, ds, \tag{5.23}$$

for any $t \in [t_0,T)$.

(iii) *(Lower bound of blowup) If* $v \in C((0,T_*); L^q_\sigma)$ *is a mild solution in* $(0,T_*)$ *and* $\|v(t)\|_{L^q} \to \infty$ *as* $t \to T_*- < \infty$*, then*

$$\|v(t)\|_{L^q} \geq C(T_* - t)^{-1/s}. \tag{5.24}$$

Note that $3/q + 2/s = 1$, and we do not need $v(0) \in L^q$ in part (iii).

Also note that this is a *finite time* result, no matter how small $\|v_0\|_q$ is. It is related to the fact that $\|v_0\|_q$ can be changed to any number by rescaling if $q > 3$. Hence a small data global existence result in L^q would imply global existence for *all* data, which is an open question.

Proof. (i) Let $X = L^\infty(0,T; L^q_\sigma(\Omega))$. By Lemma 5.1, we have $\|e^{-tA}v_0\|_q \leq C_q\|v_0\|_q$. For the nonlinear term,

$$\begin{aligned}\|B(u,v)(t)\|_{L^q_x} &\leq \int_0^t \|e^{-(t-\tau)A}P\nabla\|_{(L^{q/2}\to L^q)} \cdot \|u(\tau)\|_q\|v(\tau)\|_q\, d\tau \\ &\leq \int_0^t C|t-\tau|^{-\frac{3}{2q}-\frac{1}{2}}\|u\|_X\|v\|_X\, d\tau = C_0 t^{\frac{1}{2}-\frac{3}{2q}}\|u\|_X\|v\|_X. \end{aligned}\tag{5.25}$$

We have used $\frac{3}{2q} + \frac{1}{2} < 1$ since $q > 3$. Thus $\|B(u,v)\|_X \leq C_1\|u\|_X\|v\|_X$ with $C_1 = C_0T^{\frac{1}{2}-\frac{3}{2q}}$. By (5.21), a mild solution exists for any $v_0 \in L^q_\sigma$ over $(0,T)$ if $C_1 \leq (8C_q\|v_0\|_q)^{-1}$; thus we can take $T = C\|v_0\|_q^{-\frac{2q}{q-3}}$. It also shows uniqueness in the class of mild solutions with $\|v\|_X \leq 2C_q\|v_0\|_q$.

In fact, the solution is unique in the class (5.22) without a smallness assumption. Suppose u is another mild solution in (5.22) with $u(0) = v_0$ and a possibly smaller $T > 0$. Let $w = v-u$ and choose a maximal $t_1 \in [0,T]$ so that $w(t) = 0$ a.e. in $[0,t_1]$. Suppose $t_1 < T$. For $t \in (t_1,T)$ we have

$$w(t) = B(v,v)(t) - B(u,u)(t) = B(v,w)(t) + B(w,u)(t),$$

$$\begin{aligned}\|w(t)\|_{L^q} &\leq \int_{t_1}^t C|t-\tau|^{-\frac{3}{2q}-\frac{1}{2}}(\|u\|_X + \|v\|_X)\|w\|_X\, d\tau \\ &\leq C_0(t-t_1)^{\frac{1}{2}-\frac{3}{2q}}(\|u\|_X + \|v\|_X)\|w\|_X.\end{aligned}$$

It follows that $w(t) = 0$ for $t \in (t_1, t_1+\tau)$ if $C_0\tau^{\frac{1}{2}-\frac{3}{2q}}(\|u\|_X + \|v\|_X) < 1$, contradicting the choice of t_1 unless $t_1 = T$. Thus $u = v$ in $[0,T)$.

For continuity, note that $v_1(t)$ is continuous and that (5.25) shows $B(v,v)(t) \to 0$ as $t \to 0$. This shows continuity at $t = 0$ and the convergence to initial data. For $t_1 > 0$ and $t \to t_1$, we decompose for some r slightly less than 1 and $t > rt_1$,

$$B(v,v)(t) - B(v,v)(t_1) = \int_{rt_1}^{t} e^{-(t-\tau)A} P\nabla F(\tau)d\tau + \int_{rt_1}^{t_1} e^{-(t_1-\tau)A} P\nabla F(\tau)d\tau$$
$$+ \int_0^{rt_1} (e^{-(t-rt_1)A} - e^{-(t_1-rt_1)A}) e^{-(rt_1-\tau)A} P\nabla F(\tau)d\tau =: I_1 + I_2 + I_3.$$

Above, $F = v \otimes v$. Since the integrals converge absolutely, for any $\varepsilon > 0$ we can choose $r < 1$ sufficiently close to 1 so that $|I_1| + |I_2| < \varepsilon/2$, while $I_3 \to 0$ as $t \to t_1$ by the Lebesgue dominated convergence theorem. This shows continuity at t_1.

(ii) As seen in part (i), $B(v,v)$ converges absolutely for $v \in BC([0,T); L^q_\sigma)$. Hence we have (5.23).

(iii) For any $t_1 \in (0, T_*)$, by part (i) there is a mild solution $\tilde{v} \in BC([t_1, t_1 + T(t_1)]; L^q_\sigma)$ with $\tilde{v}(t_1) = v(t_1)$ and $T(t_1) \geq C\|v(t_1)\|_q^{-s(q)}$. By part (ii) and uniqueness of mild solutions in part (i), $v = \tilde{v}$ and $T(t_1) \leq T_* - t_1$. Thus we get (5.24). □

The second theorem is concerned with L^3 data, which has zero dimension and is a member of (5.1).

Theorem 5.5 (L^3 data). *Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain or $\mathbb{R}^3$ itself. Fix any $3 < q_2 < \infty$.*

(i) *For any $v_0 \in L^3_\sigma(\Omega)$, there is $T > 0$ and a mild solution v of* (INS) *in the class*

$$t^{1/s(q)} v(t) \in BC([0,T), L^q), \quad \frac{1}{s(q)} = \frac{1}{2} - \frac{3}{2q}, \quad \forall q \in [3, q_2]. \tag{5.26}$$

The limit $\lim_{t\to 0} t^{1/s(q)} v(t)$ in L^q is 0 for $q > 3$ and v_0 for $q = 3$. This solution is unique in the class (5.26)*, and its norm in* (5.26) *depends continuously on the L^3-norm of the initial data v_0.*

(ii) *Furthermore, for any $q_1 \in (3, 9)$, with a possibly smaller $T > 0$,*

$$v \in L^{s_1}(0,T; L^{q_1}_\sigma), \quad s_1 = s(q_1). \tag{5.27}$$

(iii) *There is a small $\varepsilon(q_1, q_2) > 0$, such that if $\|v_0\|_{L^3} \leq \varepsilon$, then we can take $T = \infty$. In this case,*

$$\|v(t)\|_{L^q} \leq Ct^{-\frac{1}{2}+\frac{3}{2q}} \quad (3 < q \leq q_2), \quad \|v(t)\|_{L^3} \to 0 \text{ as } t \to \infty. \tag{5.28}$$

Note that L^3_x and $L^s_t L^q_x$ with $3/q + 2/s = 1$ are dimension zero. This is essential in obtaining global-in-time existence, and it makes sense to talk about smallness. If $\Omega = \mathbb{R}^3$, we can take $q_2 = \infty$ to get a simpler statement and proof. In this case we also have

$$t^{1-\frac{3}{2q}} \nabla v \in BC([0,T), L^q), \quad \forall q \in [3, \infty). \tag{5.29}$$

Proof. (i) We may take q_2 as large as we want, in particular, $6 < q_2 < \infty$. Fix $T > 0$. For $3 \le q \le \infty$, denote

$$\|v\|_{Y_q} = \|t^{1/s(q)} v(t)\|_{L^\infty(0,T;L^q_\sigma(\Omega))}$$

and

$$X = Y_3 \cap Y_{q_2}, \quad Y = Y_6,$$

with $\|v\|_X = \|v\|_{Y_3} + \|v\|_{Y_{q_2}}$. Let $v \in X_0$ if and only if $\|v\|_X < \infty$ and $\lim_{t\to 0} t^{1/s(q_2)} \|v(t)\|_{L^{q_2}(\Omega)} = 0$, and let $v \in Y_0$ if and only if $\|v\|_Y < \infty$ and $\lim_{t\to 0} t^{1/4} \|v(t)\|_{L^6(\Omega)} = 0$. By Hölder inequality, $\|v\|_Y \le \|v\|_X$, $X \subset Y$, and $X_0 \subset Y_0$. We will show that the mapping $v \mapsto e^{-tA} v_0 + B(v,v)$ maps X_0 into itself and is a contraction if $\|e^{-tA} v_0\|_Y$ is sufficiently small.

By Lemma 5.1(i), $e^{-tA} v_0 \in X_0$ and $\|e^{-tA} v_0\|_X \le C_0 \|v_0\|_{L^3}$.

For $u \in Y$ and $v \in Y_q$, $6/5 < q < \infty$, by Lemma 5.1 again with $1/r = 1/q + 1/6 \in (1/6, 1)$,

$$\begin{aligned} \|B(u,v)(t)\|_{L^q} &\le \int_0^t \|e^{-(t-s)A} P \nabla\|_{(L^r \to L^q)} \|u(s)\|_{L^6} \|v(s)\|_{L^q} \, ds \\ &\le C_u(t) \int_0^t |t-s|^{-3/4} \|u\|_Y s^{-1/4} \|v\|_{Y_q} s^{-1/s(q)} \, ds \\ &\le C_u(t) \|u\|_Y \|v\|_{Y_q} t^{-1/s(q)}. \end{aligned} \tag{5.30}$$

Here $0 \le C_u(t) < C$ and $\lim_{t\to 0} C_u(t) = 0$ if $u \in Y_0$. Since u and v can be switched in the above estimate, we have, in particular, for some C_1,

$$\begin{aligned} &\|B(u,v)\|_X \le C_1 \min(\|u\|_Y \|v\|_X, \ \|u\|_X \|v\|_Y), \\ &\|B(u,v)\|_Y \le C_1 \|u\|_Y \|v\|_Y, \end{aligned}$$

and $B(u,v) \in X_0$ (or Y_0) if $u \in X_0$ (or Y_0) and $v \in Y$.

By Lemma 5.1(ii), for any $v_0 \in L^3$, we can choose $T > 0$ so that $\|e^{-tA} v_0\|_Y \le \delta = (8C_1)^{-1}$. We may choose $T = \infty$ if $C_0 \|v_0\|_{L^3} < \delta$. Let $v_1(t) = e^{-tA} v_0$ and $v_{k+1} = v_1 + B(v_k, v_k)$. One can show by induction using (5.30) that

$$\|v_k\|_X \le 2\|v_1\|_X, \quad \|v_k\|_Y \le 2\delta, \quad v_k \in X_0, \tag{5.31}$$

and, similar to (5.21) but with smallness provided by the Y-norm (note that $\|v_k\|_X \gtrsim \|v_0\|_{L^3}$ is not small)

$$\begin{aligned}\|v_{k+2}-v_{k+1}\|_X &= \|B(v_{k+1},v_{k+1}-v_k)+B(v_{k+1}-v_k,v_k)\|_X\\ &\le C_1(\|v_{k+1}\|_Y+\|v_k\|_Y)\|v_{k+1}-v_k\|_X\\ &\le \tfrac{1}{2}\,\|v_{k+1}-v_k\|_X.\end{aligned}$$

Hence the sequence $\{v_k\}_{k\in\mathbb{N}}$ converges strongly to a mild solution v in X.

We now show $v \in X_0$: For any $\varepsilon > 0$ choose k so that $\|v-v_k\|_{Y_{q_2}} \le \varepsilon/2$. Since $v_k \in X_0$, we can choose $t_0 > 0$ so that $\|t^{1/s(q_2)}v_k(t)\|_{L^{q_2}_x} \le \varepsilon/2$ for $t < t_0$. Thus $\|t^{1/s(q_2)}v(t)\|_{L^{q_2}_x} \le \varepsilon$ for $t < t_0$.

By Hölder inequality, $t^{1/s(q)}v(t) \to 0$ in L^q as $t \to 0$ for any $q \in (3, q_2]$.

The continuity in L^3 at $t = 0$ follows from (5.30); that is, $\|B(v,v)(t)\|_{L^3} \le C_v(t)\|v\|_Y\|v\|_X$ with $C_v(t) \to 0$ as $t \to 0$ since $v \in X_0 \subset Y_0$.

The continuity of $v(t)$ at $t > 0$ can be shown by the same proof for Theorem 5.4.

Uniqueness: If u is another mild solution with $u(0) = v_0$ (possibly large in Y), let $w = v-u$ and choose a maximal $t_1 \in [0,T]$ so that $w(t) = 0$ a.e. in $[0,t_1]$. Suppose $t_1 < T$. For $t \in (t_1,T)$ we have, similar to (5.30),

$$w(t) = B(v,v)(t) - B(u,u)(t) = B(v,w)(t) + B(w,u)(t),$$

$$\begin{aligned}\|w(t)\|_{L^6} &\le C\int_{t_1}^{t} |t-s|^{-3/4}(\|u\|_Y+\|v\|_Y)s^{-1/4}\|w\|_{Y_t}s^{-1/4}\,ds\\ &= C(\|u\|_Y+\|v\|_Y)\|w\|_{Y_t}t^{-1/4}\eta(t_1/t)\end{aligned}$$

where $\|w\|_{Y_t} = \sup_{0<s<t} s^{1/4}\|w(s)\|_{L^6}$ and

$$\eta(\theta) = \int_\theta^1 |1-s|^{-3/4}s^{-1/2}\,ds \le C(1-\theta)^{1/4}, \quad \frac{1}{2} < \theta < 1.$$

It follows that $\|w\|_{Y_t} \le \frac{1}{2}\|w\|_{Y_t}$ and $w(t) = 0$ for $t \in (t_1, t_1+\tau)$ for some $\tau > 0$, contradicting the choice of t_1 unless $t_1 = T$. Thus $u = v$ in $[0,T)$.

Continuous dependence on data: If v is a mild solution on $[0,T]$ with $v(0) = v_0$ and $\|e^{-tA}v_0\|_Y \le \delta$, then for $b \in L^3_\sigma$ with $\|v_0-b\|_{L^3}$ sufficiently small, $\|e^{-tA}b\|_Y \le \|e^{-tA}v_0\|_Y + C_0\|v_0-b\|_{L^3} \le \frac{3}{2}\delta$. One can show that the mild solution u with $u(0) = b$ exists on $[0,T]$ and

$$\begin{aligned}\|u-v\|_X &\le C_0\|v_0-b\|_{L^3} + C_1(\|u\|_Y+\|v\|_Y)\|u-v\|_X\\ &\le C_0\|v_0-b\|_{L^3} + \frac{3}{4}\,\|u-v\|_X.\end{aligned} \tag{5.32}$$

The above finishes the proof of (i).

(ii) To show $v \in L^{s_1}(0,T;L^{q_1})$ for given $q_1 \in (3,9)$, note that

$$\|B(u,v)(t)\|_{L^q} \le C \int_0^t |t-s|^{-\frac{1}{2}-\frac{3}{2q}} \|u(s)\|_{L^q}\|v(s)\|_{L^q}\, ds. \tag{5.33}$$

The Hardy-Littlewood inequality shows

$$\|B(u,v)\|_{L^sL^q} \le C_2\|u\|_{L^sL^q}\|v\|_{L^sL^q} \tag{5.34}$$

for some C_2. By Lemma 5.2, $\|v_0\|_{L^s(0,\infty;L^q)} < \infty$; thus $\|v_0\|_{L^s(0,T';L^q)} \le (8C_2)^{-1}$ for some $T' \in (0,T]$. We can then prove similarly that $\{v_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^s(0,T';L^q)$ with $\|v_k\|_{L^sL^q} \le (4C_2)^{-1}$.

(iii) When $\|v_0\|_3$ is sufficiently small (depending on q_1 if we want $v \in L^{s_1}(\mathbb{R}^+;L^{q_1}_\sigma)$), the previous arguments show that we can take $T = T' = \infty$. By (5.26), we get a decay rate of L^q-norm of $v(t)$ for $q > 3$, but only a uniform bound of L^3-norm of $v(t)$. To prove $v(t) \to 0$ in L^3 as $t \to \infty$, we assume Theorem 5.6. For any $\varepsilon > 0$ choose $b \in L^2_\sigma \cap L^3_\sigma$ with $\|v_0 - b\|_{L^3} \le \varepsilon/(4C_0)$ and let $u(t)$ be the global mild solution with $u(0) = b$. Then $\|v(t) - u(t)\|_{L^3} \le \varepsilon/2$ for all $t > 0$ by (5.32). However, by Theorem 5.6, $\|u(t)\|_{L^3} \to 0$ as $t \to \infty$. Thus $\|u(t)\|_{L^3} \le \varepsilon/2$ and $\|v(t)\|_{L^3} \le \varepsilon$ for t sufficiently large. $\square$

Remark. The existence time T is determined by the smallness of $\|e^{-tA}v_0\|_Y$ (and $\|e^{-tA}v_0\|_{L^{s_1}(0,T;L^{q_1})}$), not by $\|v_0\|_{L^3}$. This allows us to enlarge the space of the initial data v_0 to negative Besov spaces. The choice $Y = Y_6$ is for convenience and can be changed.

If the mild solution in Theorem 5.5 has finite maximal existence time $T_* < \infty$, we do not have an explicit blowup rate as in Theorem 5.4(iii). However we have (see Chapter 9)

$$\lim_{t\to T_*} \|v(t)\|_{L^3(\Omega)} \to \infty.$$

The mild solutions in Theorems 5.4 and 5.5 in fact belong to $C^\infty_{x,t}(\bar\Omega \times (0,T], \mathbb{R}^3)$. See [**68**] and [**63**, Theorem 4].

5.3. Applications to weak solutions

In this section we use mild solutions to study weak solutions. The next theorem relates mild and weak solutions.

Theorem 5.6. *Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain or $\mathbb{R}^3$ itself. If $a \in L^2_\sigma \cap L^3_\sigma(\Omega)$, the mild solution v in Theorem 5.5 is a Leray-Hopf weak solution satisfying the energy identity on $[0,T]$ for the same $T > 0$, and*

$$v \in BC([0,T); L^2 \cap L^3) \cap L^2(0,T;\dot H^1), \tag{5.35}$$

$$t^{1/4}v(t) \in BC([0,T), L^3). \tag{5.36}$$

The additional assumption $a \in L^2_\sigma$ is redundant if Ω is bounded but is necessary for $\Omega = \mathbb{R}^3$ or other unbounded domains. It ensures that v is a weak solution and, when $T = \infty$, it ensures the decay of $\|v(t)\|_{L^3}$ as $t \to \infty$ in (5.36). This theorem is also applicable to solutions of Theorem 5.4 since $L^2 \cap L^q \subset L^2 \cap L^3$ for $q > 3$. In that case we have both (5.22) and (5.35).

Proof. We go back to the approximation sequence $v_{k+1} = v_1 + B(v_k, v_k)$. Since $v_1(t)$ is a solution of the Stokes system, we have the energy identity

$$\|v_1(t)\|_{L^2}^2 + 2\int_0^t \|\nabla v_1\|_{L^2}^2 = \|v_0\|_{L^2}^2, \quad \forall t > 0. \tag{5.37}$$

We first claim that, for every $k \in \mathbb{N}$,

$$\|v_k\|_{L^\infty(0,T';L^2)} \le 2\|v_0\|_2, \tag{5.38}$$

$$\|v_{k+2} - v_{k+1}\|_{L^\infty(0,T';L^2)} \le \frac{1}{2}\|v_{k+1} - v_k(t)\|_{L^\infty(0,T';L^2)}, \tag{5.39}$$

for some $T' \in (0, T]$. These inequalities follow from (5.37),

$$\begin{aligned}\|B(u,v)(t)\|_{L^2} &\le \int_0^t \|e^{-(t-s)A}P\nabla\|_{(L^{3/2}\to L^2)}\|u(s)\|_{L^6}\|v(s)\|_{L^2}\,ds\\ &\le C\int_0^t |t-s|^{-3/4}\|u\|_Y s^{-1/4}\|v\|_{L^\infty L^2}\,ds\\ &\le C\|u\|_Y\|v\|_{L^\infty L^2}, \qquad (5.40)\end{aligned}$$

the same estimate for $B(v,u)$, and (5.31) that $C\|v_k\|_{Y(T')} < 1/8$ for every k for $T' > 0$ sufficiently small. Thus the limit $v \in L^\infty(0, T'; L^2)$.

We next claim that, for every $k \in \mathbb{N}$,

$$\|\nabla v_k\|_{L^2(0,T';L^2)} \le 2\|v_0\|_2, \tag{5.41}$$

$$\|\nabla v_{k+2} - \nabla v_{k+1}\|_{L^2(0,T';L^2)} \le \frac{1}{2}\|\nabla v_{k+1} - \nabla v_k(t)\|_{L^2(0,T';L^2)}, \tag{5.42}$$

for some $T' \in (0, T]$. To show these inequalities, note that

$$\begin{aligned}\|\nabla B(u,v)(t)\|_{L^2} &\le \int_0^t \|\nabla e^{-(t-\tau)A}P\|_{(L^r\to L^2)}\|u(\tau)\|_{L^{q_1}}\|\nabla v(\tau)\|_{L^2}\,d\tau\\ &\le C\int_0^t |t-\tau|^{-1/2-\sigma}\|u(\tau)\|_{L^{q_1}}\|\nabla v(\tau)\|_{L^2}\,d\tau \qquad (5.43)\end{aligned}$$

with $1/r = 1/q_1 + 1/2 \in (1/9, 1/3) + 1/2$ and $\sigma = 3/2(1/r - 1/2) = 1/2 - 1/s_1$. By the Hardy-Littlewood inequality, noting that $1/2 = (1/2 + \sigma) + 1/m - 1$ with $1/m = 1/s_1 + 1/2$,

$$\|\nabla B(u,v)\|_{L^2L^2} \le C\|\|u(t)\|_{L^{q_1}}\|\nabla v(t)\|_{L^2}\|_{L^m_t} \le C\|u\|_{L^{s_1}L^{q_1}}\|\nabla v\|_{L^2L^2}.$$

By the above estimate and the same one for $B(v,u)$, (5.37), and the fact that $C\|v_k\|_{L^{s_1}L^{q_1}(T')} < 1/8$ for every k for $T' > 0$ sufficiently small, we get (5.41) and (5.42). Thus the limit $\nabla v \in L^2(0,T';L^2)$.

Being a weak solution in the class $L^{s_1}L^{q_1}$, Lemma 4.3 shows that v is a Leray-Hopf weak solution satisfying the energy identity in $[0,T']$.

The above argument can be repeated for initial data $v(t)$ for a.e $t \in (0,T')$ since $v \in BC([0,T);L^3_\sigma)$, so eventually we have (5.35) in $(0,T)$.

The continuity can be proved similarly to that in Theorems 5.4 and 5.5.

Suppose $\|v_0\|_{L^3}$ is small and $T = \infty$. Denote $\|v\|_Z = \sup_t t^{1/4}\|v(t)\|_{L^3}$. Equation (5.36) can be shown similarly using $\|e^{-tA}v_0\|_{L^3} \le Ct^{-1/4}\|v_0\|_{L^2}$ and

$$\begin{aligned}\|B(u,v)(t)\|_{L^3} &\le \int_0^t \|e^{-(t-\tau)A}P\nabla\|_{(L^2\to L^3)}\|u(\tau)\|_{L^6}\|v(\tau)\|_{L^3}\,d\tau \\ &\le C\int_0^t |t-\tau|^{-3/4}\|u\|_Y\tau^{-1/4}\|v(s)\|_Z\tau^{-1/4}\,d\tau \\ &\le C\|u\|_Y\|v\|_Z t^{-1/4}.\end{aligned}$$

□

We next give a proof of Theorem 4.5, on the global regularity of strong solutions. We recall its statement:

Theorem. *Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain or $\mathbb{R}^3$ itself, and let $0 < T < \infty$. If a weak solution v of* (NS) *with $f = 0$ in $\Omega \times (0,T)$ satisfies $v \in L^sL^q(\Omega\times(0,T))$ with $\frac{3}{q}+\frac{2}{s} \le 1$, $q > 3$, then $v \in C^\infty_{t,x}(\bar\Omega \times (0,T])$.*

The proof is based on the comparison with the mild solution with the same initial data. Another proof is sketched in Problems 4.4–4.6.

Proof of Theorem 4.5. Since $v \in BC_w([0,T);L^2_\sigma)\cap L^s(0,T;L^q)$, we have $v(t_0) \in L^2 \cap L^q$ for a.e. $t_0 \in (0,T)$.

For such t_0, by Theorem 5.4, there is a mild solution $u \in BC([t_0,T_*),L^q_\sigma)$ for some maximal T_* with $u(t_0) = v(t_0)$. It is regular in (t_0,T_*). Since $u(t_0) \in L^2$, $u(t)$ is a Leray-Hopf weak solution by Theorem 5.6. Thus $u(t) = v(t)$ by the uniqueness Theorem 4.4.

If $T_* \le T < \infty$, then $\|v(t)\|_{L^q} \ge C(T_*-t)^{-1/s}$ for every $t_0 \le t < T_*$ by Theorem 5.4(iii). Hence

$$\int_{t_0}^{T_*} \|v(t)\|^s_{L^q}dt \ge \int_{t_0}^{T_*} C(T_*-t)^{-1}dt = \infty,$$

contradicting $v \in L^s(0,T;L^q)$. Thus $T_* > T$ and v is regular in $(t_0,T]$.

Since such t_0 can be chosen arbitrarily close to 0, v is regular in $(0,T]$. □

5.4. Notes

This chapter is based on Giga-Miyakawa [**68**], Giga [**63**], and Giga-Sohr [**71**].

In Section 5.1, the proof of Lemma 5.1(i) can be found in [**63**, **68**]. Lemma 5.2 is due to [**63**]. For the theory of the Stokes operator, see [**14**, **47**, **61**, **62**, **70**, **71**] and their references. For the semigroup theory, see [**223**, IX] and [**163**]. It seems that the smoothness assumption of $\partial\Omega$ can be weakened to class $C^{2,1}$; see [**47**]. In Sections 5.2 and 5.3, Theorems 5.5 and 5.6 are due to Kato [**95**] and Giga [**63**].

The theory of mild solutions for the Navier-Stokes equations was initiated by Fujita and Kato [**54**, **97**] and Sobolevskiĭ [**192**] via the semigroup theory with initial data in $H^{1/2}$.

For initial data in L^q, $q > 3$, local-in-time solutions in L^sL^q were found by Fabes-Jones-Rivière [**43**] for $\mathbb{R}^n$, Lewis [**134**] for $\mathbb{R}^n_+$, and Fabes-Lewis-Rivière [**44**] for bounded domains, using parabolic singular integral operators. Solutions similar to those in Theorem 5.4 were constructed using semigroups by von Wahl [**217**], Miyakawa [**150**], and [**63**, **68**].

For initial data in L^3, Weissler [**221**] constructed local solutions in $L^3_\sigma(\mathbb{R}^3_+)$. Local solutions and small global solutions are constructed by Kato [**95**] for $\mathbb{R}^3$, by Giga-Miyakawa [**68**] and Giga [**63**] for bounded domains, and by Iwashita [**84**] for exterior domains.

When $\Omega = \mathbb{R}^3$, there is a mild solution theory for initial data in $\dot{B}^{3/p-1}_{p,\infty}$, $3 < p < \infty$, and BMO^{-1}. See [**6**, Chapter 5] and [**105**].

Regularity and analyticity. For $v_0 \in L^3_\sigma(\mathbb{R}^3)$, the mild solution $v \in L^5(\mathbb{R}^3 \times [0,T])$, $T < \infty$, satisfies

$$t^{m+k/2}\partial_t^m \nabla_x^k v(x,t) \in L^5(\mathbb{R}^3 \times (0,T)) \tag{5.44}$$

for any $m, k \in \mathbb{N}$; see [**39**]. Moreover, v is analytic in x and its Taylor expansion in x at (x,t) has radius of convergence at least $Ct^{1/2}$. (It has to shrink to zero as $t \to 0$ since v_0 is only in L^3.) See [**148**].

Problems

5.1. Let Ω be a bounded smooth domain in $\mathbb{R}^3$ and let $0 < t_1 < T$. Let v be a mild solution in the class (5.26) in $(0,T)$. Is the restriction of v to $t \in (t_1, T)$ also a mild solution in the sense of Theorem 5.5?

5.2. Let Ω be a bounded smooth domain in $\mathbb{R}^3$. Let $3 < q < 9$ and $3/q + 2/s = 1$. Show uniqueness of mild solutions with initial data in $L^3_\sigma(\Omega)$ in the class $L^s(0,T;L^q(\Omega))$.

Partial regularity

We proved existence of global-in-time weak solutions of (NS) in Chapter 3. Their regularity and uniqueness are known only for a finite period of time even if the initial data are smooth and have compact supports, by the results of Chapters 4 and 5. It is possible that a weak solution with nice data is regular and unique for $0 < t < T$ for some $T < \infty$ and loses regularity and uniqueness at $t \geq T$. The problems of full regularity and uniqueness seem out of reach at the moment.

The approach of *partial regularity* tries to estimate the size of the (hypothetical) singular set, instead of proving full regularity. It was first carried out for minimal surfaces and quasi-linear elliptic systems (Almgren [**2**], Morrey [**151**]). For the partial regularity of (NS), Scheffer initiated a series of papers [**170**]–[**174**], and Caffarelli-Kohn-Nirenberg [**22**] obtained the best result so far.

The study of partial regularity is usually linked to ε-regularity, which shows regularity of a solution assuming the smallness of certain 0-dimensional (weighted) integral(s) over a small parabolic cylinder.

Let us define the singular points and singular sets. A weak solution v in $\Omega_T = \Omega \times (0,T)$, $\Omega \subset \mathbb{R}^3$, is said to be *regular* at a point $z_0 = (x_0, t_0) \in \bar{\Omega} \times (0,T]$, $t_0 < \infty$, if $v \in L^\infty_{t,x}(Q(z_0,r) \cap \Omega_T)$ for some $r > 0$. Otherwise it is *singular* at z_0. The *set of singular times* of v is

$$\Sigma = \{t \in (0,T] : v \text{ is singular at } (x_0,t) \text{ for some } x_0 \in \bar{\Omega}\}. \tag{6.1}$$

The *sets of singular points* are

$$S = \{z_0 \in \Omega \times (0,T] : v \text{ is singular at } z_0\}, \tag{6.2}$$

$$S^* = \{z_0 \in \bar{\Omega} \times (0,T] : v \text{ is singular at } z_0\}. \tag{6.3}$$

Clearly $S \subset S^*$ and the projection of S^* on the time axis is Σ.

For a set $E \subset \mathbb{R}^{n+1}$ and $\alpha \geq 0$, denote by $\mathcal{H}^\alpha(E)$ its α-dimensional *Hausdorff measure* and by $\mathcal{P}^\alpha(E)$ the *parabolic* version; namely,

$$\mathfrak{H}^\alpha(E) = \lim_{\delta \to 0_+} \inf \left\{ \textstyle\sum_{j=1}^\infty r_j^\alpha : E \subset \bigcup_j B(z_j, r_j),\ r_j \leq \delta \right\}, \tag{6.4}$$

$$\mathcal{P}^\alpha(E) = \lim_{\delta \to 0_+} \inf \left\{ \textstyle\sum_{j=1}^\infty r_j^\alpha : E \subset \bigcup_j Q(z_j, r_j),\ r_j \leq \delta \right\}. \tag{6.5}$$

Recall $Q(z_0, r) = B(x_0, r) \times (t_0 - r^2, t_0)$ for $z_0 = (x_0, t_0)$ and $r > 0$. The balls in (6.4) are $(n+1)$-dimensional. Note that $\mathfrak{H}^\alpha(S) \leq C\mathcal{P}^\alpha(S)$, $\mathcal{H}^\alpha(S) = 0$ if $\alpha > n+1$, and $\mathcal{P}^\alpha(S) = 0$ if $\alpha > n+2$.

We will show that $\mathfrak{H}^{1/2}(\Sigma) = 0$ for any Leray-Hopf weak solution satisfying the strong energy inequality and that $\mathcal{P}^1(S) = 0$ for any suitable Leray-Hopf weak solution. The second result that $\mathcal{P}^1(S) = 0$ does not require boundary conditions. It is also known that $\mathcal{P}^1(S^* \cap (\Gamma \times (0, T])) = 0$ if $v|_\Gamma = 0$ on a part of the boundary $\Gamma \subset \partial\Omega$, if Γ is flat [**177**] and if Γ is C^2 [**181**].

These partial regularity results have an interesting consequence in the context of the experimental study of *turbulence*: We may relate the singular set of a flow to its turbulence region. Since the usual turbulence experiments involve measurements along a prescribed line, it should be very difficult to detect singularities since the singular set has zero 1-dimensional measure.

6.1. The set of singular times

In this section we show $\mathfrak{H}^{1/2}(\Sigma) = 0$ for a Leray-Hopf weak solution satisfying the strong energy inequality (3.9). Although a general weak solution may not satisfy (3.9), for most domains and data we can construct weak solutions satisfying (3.9). Furthermore, suitable Leray-Hopf weak solutions in $\mathbb{R}^3$ and bounded smooth domains satisfy (3.9). See Section 3.6 and Problem 3.5.

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain or $\mathbb{R}^3$ itself, and let $0 < T \leq \infty$. Let v be a Leray-Hopf weak solution in $\Omega \times [0, T)$ with zero force, satisfying the strong energy inequality* (3.9). *Let*

$$\Sigma = \{t \in (0, T] :\ v \text{ is singular at } (x_0, t) \text{ for some } x_0 \in \bar{\Omega}\}.$$

We have $\mathfrak{H}^{1/2}(\Sigma) = 0$.

If $T = \infty$, the set Σ is bounded. This property is called the *eventual regularity*. See Problem 6.2.

Proof. Since $v \in BC_w([0,T); L^2_\sigma)$, $v(t)$ is uniquely determined for every t. Let $g(t) = \|v(t)\|_{L^6}$ and

$$\Sigma_1 = \{t \in (0,T) : g(t) = \infty\}, \quad U_1 = (0,T) \setminus \Sigma_1.$$

Since $\|g\|_{L^2(0,T)} = \|v\|_{L^2(0,T;L^6)} < \infty$, the Lebesgue measure of Σ_1 is zero. Also denote

$$U_2 = \left\{ t_0 \in [0,T) : \begin{array}{l} \text{the strong energy inequality (3.9)} \\ \text{holds with left endpoint } t_0 \end{array} \right\}.$$

By assumption, $|[0,T) \setminus U_2| = 0$, and v is a Leray-Hopf weak solution in $[t_0, T)$ for $t_0 \in U_2$.

For every $t \in U_1$, $v(t) \in L^2_\sigma \cap L^6_\sigma(\Omega)$. By Theorem 5.6 and its remark, there is $T(t) > Cg(t)^{-4}$ and a mild, Leray-Hopf weak solution

$$u \in BC([t, t+T(t)); L^2 \cap L^6) \cap L^2(0,T; \dot{H}^1), \quad u(t) = v(t).$$

Moreover u is regular in $(t, t+T(t))$. By Theorem 4.4, if $t \in U_1 \cap U_2$, then $v = u$ on $I(t) = (t, \min(T, t+T(t)))$. Let

$$U_3 = \bigcup_{t \in U_1 \cap U_2} I(t), \quad \Sigma_3 = (0,T) \setminus U_3.$$

Since each $I(t) \subset U_1$, we have $U_3 \subset U_1$ and $\Sigma_1 \subset \Sigma_3$. Note that $\Sigma_3 \setminus \Sigma_1$ is countable since it consists of the left endpoints of some of the connected components of U_3. Hence $|\Sigma_3| \le |\Sigma_1| = 0$.

We also have $\Sigma \subset \Sigma_3 \cup \{T\}$ since $v(t)$ is regular if $t \in U_3$.

For $0 < \delta_0 \ll 1$, let

$$\Sigma(\delta_0) := \Sigma_3 \cap [\delta_0, \min(1/\delta_0,\ T - \delta_0)].$$

It is compact since U_3 is open. For any $\varepsilon > 0$, choose an open neighborhood V of $\Sigma(\delta_0)$ so that $\int_V g(t)^2 dt < \varepsilon$. For any $0 < \delta \le \delta_0$, choose for every $t \in \Sigma(\delta_0)$ an $r(t) \in (0, \delta/5)$ so that $B(t, r(t)) \subset V$. Recall $B(t,r) = (t-r, t+r)$ for $r > 0$. Note that

$$\Sigma(\delta_0) \subset \bigcup_{t \in \Sigma(\delta_0)} B(t, r(t)).$$

By the Vitali covering lemma, there is a countable subsequence $t_j \in \Sigma(\delta_0)$, $j \in \mathbb{N}$, so that

$$\bigcup_{t \in \Sigma(\delta_0)} B(t, r(t)) \subset \bigcup_{j \in \mathbb{N}} B(t_j, 5r_j), \quad r_j = r(t_j),$$

with $\{B(t_j, r_j)\}_{j \in \mathbb{N}}$ mutually disjoint. For a.e. $t \in (t_j - r_j, t_j)$, $t \notin \Sigma_3$, $t_j \in \Sigma_3$, and hence $g(t) \ge C(t_j - t)^{-1/4}$. Thus

$$\int_{t_j - r_j}^{t_j} g(t)^2 dt \ge \int_{t_j - r_j}^{t_j} C(t_j - t)^{-1/2} dt = C r_j^{1/2}.$$

Being disjoint,

$$\sum_j Cr_j^{1/2} \le \sum_j \int_{t_j - r_j}^{t_j} g(t)^2 dt \le \int_V g(t)^2 dt < \varepsilon.$$

Since $\varepsilon > 0$ and $\delta > 0$ are arbitrary, we get $\mathfrak{H}^{1/2}(\Sigma(\delta_0)) = 0$. Since $\delta_0 > 0$ is arbitrary, we conclude $\mathcal{H}^{1/2}(\Sigma_3) = 0$. ☐

6.2. The set of singular space-time points

In the following two sections, "suitable weak solution" is understood in a local sense. A pair (v, p) is a *suitable weak solution* of (NS) in Ω_T if

$$v \in L^\infty L^2 \cap L^2 \dot{H}^1(R), \quad p \in L^{3/2}(R) \tag{6.6}$$

for any bounded $R \subset \Omega_T$ that satisfies (NS) in the distributional sense and the local energy inequality (3.8). We do not assume any initial or boundary condition or the energy inequality (3.7). It is sufficient for the regularity problem.

In this section we prove the following theorem.

Theorem 6.2. *Let $\Omega \subset \mathbb{R}^3$ be any domain and let $0 < T \le \infty$. Let (v, p) be a suitable weak solution of* (NS) *in $\Omega_T = \Omega \times (0, T)$ with force $f \in L^q(\Omega \times (0, T))$, $q > 5/2$. Let the set of singular points of v be*

$$S = \{z_0 \in \Omega \times (0, T] : v \text{ is singular at } z_0\}. \tag{6.7}$$

Then $\mathcal{P}^1(S) = 0$.

The proof of Theorem 6.2 uses the following ε-regularity criterion, which will be proved in Section 6.3.

Theorem 6.3. *Let (v, p) be a suitable weak solution of* (NS) *in $Q(z_0, r_0) \subset \mathbb{R}^{3+1}$, $r_0 > 0$, with force $f \in L^q(Q(z_0, r_0))$, $q > 5/2$. There is a small $\varepsilon_0 = \varepsilon_0(q) > 0$ such that v is regular at z_0 if*

$$\limsup_{r \to 0_+} \frac{1}{r} \iint_{Q(z_0, r)} |\nabla v|^2 < \varepsilon_0. \tag{6.8}$$

Proof of Theorem 6.2. We will assume a parabolic version of the Vitali covering lemma: Let $J = \{Q_{z_\alpha, r_\alpha}\}_\alpha$ be any collection of parabolic cylinders contained in a bounded subset of $\mathbb{R}^{n+1}$. There exist disjoint $Q_{z_j, r_j} \in J$, $j \in \mathbb{N}$, such that any cylinder in J is contained in $Q_{z_j, 5r_j}$ for some j.

Denote

$$Q^*((x, t), r) = B(x, r) \times (t - \tfrac{7}{8} r^2, t + \tfrac{1}{8} r^2).$$

It is a translation in time of $Q((x, t), r)$, and $Q(z, r/2) \subset Q^*(z, r)$.

Let $S_R = S \cap R$, for any compact $R \subset \Omega \times (0,T]$. By assumption, (6.6) holds for (v,p) in R. Fix any $\delta > 0$ and any open neighborhood $V \subset \Omega \times (0,T+1)$ of S_R. By Theorem 6.3, for any $z \in S_R$, there is $0 < r_z < \delta/5$ so that $Q^*(z,r_z) \subset V$ and

$$\iint_{Q(z,r_z/2)} |\nabla v|^2 > \tfrac{1}{4}\,\varepsilon_0 r_z. \tag{6.9}$$

Thus $S_R \subset \bigcup_{z\in S_R} Q^*(z,r_z)$. (This is why we use Q^*, since $z \in Q^*(z,r)$ but $z \notin Q(z,r)$.) Let $\{Q(z_j,r_j)\}_{j\in\mathbb{N}}$, $r_j = r_{z_j}$, be the countable disjoint subcover guaranteed by the covering lemma, so that

$$S_R \subset \bigcup_{j\in\mathbb{N}} Q^*(z_j, 5r_j), \quad 5r_j < \delta.$$

By (6.9) and noting that $Q(z_j,r_j/2) \subset Q^*(z_j,r_j) \subset V$ are disjoint,

$$\sum 5r_j \le \frac{20}{\varepsilon_0}\sum_j \iint_{Q(z_j,r_j/2)} |\nabla v|^2 \le \frac{20}{\varepsilon_0}\iint_V |\nabla v|^2 < \infty.$$

This shows that S_R has Lebesgue measure 0. Thus one can choose V with arbitrarily small Lebesgue measure, so that the right side is arbitrarily small. Since δ is arbitrary, the above estimate shows $\mathcal{P}^1(S_R) = 0$. Since R is arbitrary, $\mathcal{P}^1(S) = 0$. $\square$

6.3. Regularity criteria in scaled norm

In this section we prove Theorems 6.4 and 6.7, which are regularity criteria in terms of 0-dimensional scaled local integrals. Theorem 6.3 is a corollary of Theorem 6.7.

Theorem 6.4. *For $q > 5/2$ and $A \ge 0$, there are $\varepsilon_1 = \varepsilon_1(q) > 0$, $\theta = \theta(q) \in (0,1/2)$, and $r_1 = r_1(q,A) > 0$ such that if (v,p) is a suitable weak solution of* (NS) *in Q_R for some $0 < R \le r_1$ with force $f \in L^q(Q_R)$ and if*

$$\frac{1}{R^2}\iint_{Q_R} |v|^3 + |p|^{3/2} < \varepsilon_1, \quad \|f\|_{L^q(Q_R)} \le A, \tag{6.10}$$

then $v \in L^\infty(Q_{\theta R})$. The constants ε_1 and θ do not depend on $\|f\|_{L^q}$.

In fact, v is Hölder continuous in $Q_{\theta R}$; see Remark 6.6.

An important feature of (6.10) is that it requires only one R, not infinitely many r as in Theorem 6.3. The smallness of the 0-dimensional scaled L^3-norm of v is a weaker assumption compared to that of the 0-dimensional integrals, e.g., the $L^5(Q_R)$-norm of v.

Instead of $f \in L^q$, $q > 5/2$, we will prove Theorems 6.3 and 6.4 under the weaker assumption that f lies in the Morrey space $M_{2,\gamma}$, $0 < \gamma \le 2$, with

$$\|f\|_{M_{2,\gamma}(E)} = \sup_{Q_{z,r}\subset E} \|f\|_{L^2(Q_{z,r})} r^{-\gamma-1/2} \quad (E \subset \mathbb{R}^{3+1}).$$

Thus

$$\|f\|_{L^2(Q_{z,r})} \le \|f\|_{M_{2,\gamma}(E)} r^{\gamma+1/2} \quad (Q_{z,r} \subset E).$$

The space $M_{2,\gamma}(E)$ contains $L^q(E)$ with

$$\|f\|_{M_{2,\gamma}(E)} \lesssim \|f\|_{L^q(E)}, \quad q = \frac{5}{2-\gamma} > \frac{5}{2}, \quad \gamma = 2 - \frac{5}{q} \in (0,2].$$

The norm $\|f\|_{M_{2,\gamma}}$ has dimension $-1-\gamma$: If we denote $f_r(x,t) = r^3 f(rx, r^2 t)$ according to the scaling (1.21) of (NS), then $f|_{Q_r}$ corresponds to $f_r|_{Q_1}$ and

$$\|f\|_{M_{2,\gamma}(Q_r)} = \|f_r\|_{M_{2,\gamma}(Q_1)} r^{-1-\gamma}. \tag{6.11}$$

The Morrey space $M_{2,\gamma}$ is also denoted as $\mathcal{L}^{2,1+2\gamma}$ in the literature. Our choice of notation is such that γ becomes an upper bound for the Hölder exponent in the proof of Theorem 6.4; see Remark 6.6. Morrey spaces are natural in the partial regularity theory. For example, the condition in Theorem 6.3 is satisfied if

$$\|\nabla v\|^2_{M_{2,0}(Q(z_0,r))} \le \varepsilon_2, \quad \text{for some } r > 0.$$

We will use a parabolic version of the *Morrey-Campanato integral characterization* of Hölder spaces. Denote $(u)_E = \frac{1}{|E|}\int_E u$. The usual (global) version says that, for $p \in [1,\infty)$, $0 < \alpha < 1$, and bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with diameter r_0, a function u is in $C^\alpha(\overline{\Omega})$ if and only if

$$\sup_{x_0\in\Omega, 0<r<r_0} r^{-n-p\alpha} \int_{B(x_0,r)\cap\Omega} |u - (u)_{B(x_0,r)\cap\Omega}|^p \, dx < \infty.$$

See, e.g., [**60**]. For our purpose, we need an interior, parabolic version: A function u is in $C^{\alpha,\alpha/2}_{x,t}(\overline{Q_R})$ if $u \in L^1(Q_{R+r_0})$ for some $r_0 > 0$ and

$$\sup_{z_0\in Q_R, 0<r<r_0} r^{-n-2-p\alpha} \int_{Q(z_0,r)} |u - (u)_{Q(z_0,r)}|^p \, dz < \infty. \tag{6.12}$$

See [**135**, §4.2]. We will use it after (6.24). Recall that the space $C^{\alpha,\alpha/2}_{x,t}(E)$ for $E \subset \mathbb{R}^{n+1}$ consists of those $u \in C(E)$ with

$$[u]_{C^{\alpha,\alpha/2}_{x,t}(E)} = \sup_{(x,t)\ne(y,s)\in E} \frac{|u(x,t)-u(y,s)|}{|x-y|^\alpha + |t-s|^{\alpha/2}} < \infty. \tag{6.13}$$

We start the proof of Theorem 6.4 with a lemma, which will be used for iteration. This approach is due to F. Lin [**136**]. The choice of φ in Lemma 6.5 is from [**122**, Lemma 2.4].

Lemma 6.5. *For any $0 < \theta \le 1/2$ and $0 < \beta < \gamma \le 2$, there are $\varepsilon_2, r_2 \in (0,1]$ such that if (u,p) is a suitable weak solution of* (NS) *in Q_r with force $f \in M_{2,\gamma}(Q_r)$ satisfying*

$$r|u_r| \le 1, \quad \varphi(r,u,p) + \|f\|_{M_{2,\gamma}} r^{\beta+1} \le \varepsilon_2$$

for some $r \le r_2$, where

(6.14)

$$\varphi(r,u,p) = \left(\frac{1}{r^2}\int_{Q_r} |u(x,t)-u_r|^3 dz\right)^{\frac{1}{3}} + \left(\frac{1}{r^2}\int_{Q_r} |p(x,t)-p_r(t)|^{3/2} dz\right)^{\frac{2}{3}},$$

$u_r = \frac{1}{|Q_r|}\int_{Q_r} u\,dz$ and $p_r(t) = \frac{1}{|B_r|}\int_{B_r\times\{t\}} p\,dx$, then

$$\theta r|u_{\theta r}| \le 1, \quad \varphi(\theta r,u,p) \le c_1\theta^{1+\alpha_1}\left(\varphi(r,u,p) + \|f\|_{M_{2,\gamma}} r^{\beta+1}\right), \tag{6.15}$$

for $\alpha_1 = 2/3$ and some constant $c_1 > 0$ independent of θ, β, γ. We may take $r_2 = 1$ if $f = 0$.

Note that φ has zero dimension; $\varphi(r,u_{(\theta)},p_{(\theta)}) = \varphi(\theta r,u,p)$.

The quantity $\|f\|_{M_{2,\gamma}(Q_r)} r^{\beta+1}$ has negative dimension in view of (6.11), and the condition $\|f\|_{M_{2,\gamma}(Q_r)} r^{\beta+1} < \varepsilon_2$ implies f is negligible for small r.

Proof. We first show $\theta r|u_{\theta r}| \le 1$: Since

$$\begin{aligned}|u_{\theta r}| - |u_r| &\le \frac{1}{|Q_{\theta r}|}\int_{Q_{\theta r}} |u-u_r|dz \\ &\le \theta^{-5}\frac{1}{|Q_r|}\int_{Q_r} |u-u_r|dz \le C\theta^{-5}r^{-1}\varphi(r),\end{aligned}$$

we have

$$\theta r|u_{\theta r}| \le \theta r|u_r| + C\theta^{-4}\varepsilon_2 < 1,$$

if $2C\varepsilon_2 < \theta^4$.

We will show the second part of (6.15) by contradiction and suppose it fails for some $\theta \in (0,1/2]$. Then there exist $r_n \searrow 0$ and suitable weak solutions (u_n,p_n) defined on Q_{r_n} with force $f_n \in M_{2,\gamma}(Q_{r_n})$ such that $r_n|(u_n)_{r_n}| \le 1$,

$$\varepsilon_n := \varphi(r_n,u_n,p_n) + \|f_n\|_{M_{2,\gamma}} r_n^{1+\beta} \searrow 0, \quad \varphi(\theta r_n,u_n,p_n) > c_1\theta^{1+\alpha_1}\varepsilon_n,$$

where c_1 will be chosen in the proof. Define

$$\begin{aligned}v_n(x,t) &= \varepsilon_n^{-1} r_n[u_n(r_n x, r_n^2 t) - (u_n)_{Q_{r_n}}],\\ q_n(x,t) &= \varepsilon_n^{-1} r_n^2[p_n(r_n x, r_n^2 t) - (p_n)_{B_{r_n}}(r_n^2 t)].\end{aligned}$$

They are suitable weak solutions of the following system:

$$\left.\begin{aligned}\partial_t v_n - \Delta v_n + (b_n + \varepsilon_n v_n)\cdot\nabla v_n + \nabla q_n &= g_n\\ \operatorname{div} v_n &= 0\end{aligned}\right. \quad \text{in} \quad Q_1, \tag{6.16}$$

where $b_n = r_n (u_n)_{Q_{r_n}}$ and $g_n(x,t) = \varepsilon_n^{-1} r_n^3 f_n(r_n x, r_n^2 t)$. We have

$$|b_n| \le 1, \qquad \int_{Q_1} v_n = 0, \quad (q_n)_{B_1}(t) = 0 \quad \text{for a.e. } t \in (-1,0),$$

and, using (6.11),

$$\begin{aligned} &\|v_n\|_{L^3(Q_1)} + \|q_n\|_{L^{3/2}(Q_1)} \le 1, \quad \varphi(\theta, v_n, q_n) \ge c_1 \theta^{1+\alpha_1}, \\ &\|g_n\|_{M_{2,\gamma}(Q_1)} \le \varepsilon_n^{-1} \|f_n\|_{M_{2,\gamma}(Q_{r_n})} r_n^{1+\gamma} \le r_n^{\gamma-\beta}. \end{aligned} \tag{6.17}$$

By these uniform bounds, (b_n, v_n, q_n) converge, up to a subsequence, to some limit (b, v, q),

$$b_n \to b, \quad v_n \rightharpoonup v \text{ in } L^3(Q_1), \quad q_n \rightharpoonup q \text{ in } L^{3/2}(Q_1).$$

By the local energy inequality for (6.16), derived from that for (u_n, p_n), with a suitable test function $\phi \in C^2(\overline{Q_1})$, $\phi = 1$ on $Q_{7/8}$ and $\phi = 0$ on $Q_1 \setminus Q_{0.99}$, and using (6.17),

$$\|\nabla v_n\|_{L^2(Q_{7/8})} + \|v_n\|_{L_t^\infty L_x^2 \cap L_{t,x}^{10/3}(Q_{7/8})} \le C. \tag{6.18}$$

By the weak form of (6.16) and the bounds (6.17), an argument similar to that for (3.40) gives

$$\|\partial_t v_n\|_{L^{4/3}\left(-\frac{49}{64},0;\ (H^1_{\sigma,0}(B_{7/8}))'\right)} \le C. \tag{6.19}$$

By the compactness lemma, Lemma 3.7, a further subsequence

$$\nabla v_n \rightharpoonup \nabla v \text{ weakly in } L^2(Q_{7/8}), \quad v_n \to v \text{ strongly in } L^q(Q_{7/8}),$$

for $q = 2$. The strong convergence is then improved to all $q < 10/3$ by Hölder inequality and (6.18).

Taking limits $n \to \infty$ of the weak form of (6.16), we find that v and q solve the following linear Stokes system with constant vector b, $|b| \le 1$,

$$\partial_t v - \Delta v + b \cdot \nabla v + \nabla q = 0, \quad \operatorname{div} v = 0 \qquad \text{in} \quad Q_{7/8}. \tag{6.20}$$

Since $(\partial_t - \Delta + b \cdot \nabla) \operatorname{curl} v = 0$, $\operatorname{curl} v \in L^\infty(Q_{5/6})$. Since $\operatorname{div} v = 0$, $\nabla v \in L^\infty L^{100}(Q_{4/5})$ by Lemma 2.11. Since $q \in L^{3/2}_{t,x}$ and $\Delta q = 0$, we get $q \in L_t^{3/2} C^2_{\mathrm{loc}}$. Thus $\partial_t v \in L_t^{3/2} L_x^\infty$. Therefore v is Hölder continuous in $Q_{3/4}$ with

$$[v]_{C^{\alpha_1,\alpha_1/2}_{x,t}(Q_{3/4})} \le C, \quad \alpha_1 = 2/3. \tag{6.21}$$

In particular, since $\theta < 3/4$,

$$\theta^{-5-3\alpha_1} \int_{Q_\theta} |v - v_\theta|^3 \le \tfrac{1}{2} c_2,$$

with c_2 independent of θ, β, γ. By the strong L^3-convergence of v_n to v, we have for n sufficiently large

$$\theta^{-5-3\alpha_1} \int_{Q_\theta} |v_n - (v_n)_\theta|^3 \le c_2. \tag{6.22}$$

We now estimate q_n, which satisfies, by taking div of (6.16),

$$-\Delta q_n = \varepsilon_n \partial_i \partial_j (v_{n,i} v_{n,j}) - \operatorname{div} g_n.$$

Fix $\zeta(x) \in C_c^\infty(B_{7/8})$, $\zeta \ge 0$ and $\zeta = 1$ on $B_{3/4}$. For each fixed t, let

$$\tilde{q}_n(x,t) = \int \partial_{x_i} \frac{1}{4\pi|x-y|} \big\{ \varepsilon_n \partial_j (\zeta v_{n,i} v_{n,j}) - \zeta_i g_n \big\}(y,t)\, dy.$$

By L^q-estimates for singular integrals and Riesz potentials and by integrating in time, we have

$$\|\tilde{q}_n\|_{L^{3/2}(Q_{7/8})} \le C\varepsilon_n \|v_n\|^2_{L^3(Q_{7/8})} + C\|g_n\|_{L_t^{3/2} L_x^1(Q_{7/8})} \le C\varepsilon_n + C r_n^{\gamma-\beta}.$$

Decompose $q_n = \tilde{q}_n + h_n$. Then $\Delta h_n = 0$ on $B_{3/4}$ for all t. By the mean value property of harmonic functions,

$$\int_{B_\theta \times \{t\}} |h_n - h_{n,\theta}(t)|^{3/2} dx \le C\theta^{3+3/2} \int_{B_{3/4}\times\{t\}} |h_n|^{3/2} dx.$$

Integrating in t and if $\varepsilon_n + r_n^{\gamma-\beta} \le \theta^3$, we get

$$\iint_{Q_\theta} |h_n(x,t) - h_{n,\theta}(t)|^{3/2} dz \le C\theta^{9/2} \iint_{Q_{3/4}} |q_n|^{3/2} + |\tilde{q}_n|^{3/2} dz \le C\theta^{9/2}.$$

Summing the estimates for $h_n - h_{n,\theta}$ and $\tilde{q}_n - \tilde{q}_{n,\theta}$, we get

$$\iint_{Q_\theta} |q_n(x,t) - q_{n,\theta}(t)|^{3/2} dz \le c_3 \theta^{9/2} = c_3 \theta^{2+\frac{3}{2}(1+\alpha_1)}, \tag{6.23}$$

if $\varepsilon_n + r_n^{\gamma-\beta} \le \theta^3$, with c_3 independent of θ, β, γ.

Combining (6.22) and (6.23), we get a contradiction to $\varphi(\theta, v_n, q_n) \ge c_1 \theta^{1+\alpha_1}$ if we choose $c_1 > c_2^{1/3} + c_3^{2/3}$, independently of θ, β, γ. This completes the proof. $\square$

Note that if $f = 0$, we can take any $r \in \mathbb{R}$ and $\varepsilon_2 = \varepsilon_2(\theta)$. The dependence of ε_2 and r_2 on θ, γ, β is implicit because it is unclear how large n is needed in (6.22).

We can now prove Theorem 6.4.

Proof of Theorem 6.4. For $q > 5/2$, let $\gamma = 2 - \frac{5}{q} \in (0, 2]$. For any $0 < \alpha < \min(2/3, \gamma)$, let $\beta = \frac{1}{2}(\alpha + \gamma)$, and choose $\theta \in (0, 1/2]$ so that $c_1 \theta^{2/3} + \theta^\beta \le \theta^\alpha$, where c_1 is the constant in Lemma 6.5. Let ε_2 and r_2 (depending on θ, γ, β) be the other constants of Lemma 6.5.

Let $r_1 = \min(r_2, [\frac{\varepsilon_2}{2\|f\|_{M_{2,\gamma}}}]^{\frac{1}{1+\beta}})$. By assumption $R \le r_1$. Let $r_0 = (1-\theta)R$. For $z_0 \in Q_{\theta R}$ and $0 < r \le r_0$, let $\psi(z_0, r)$ be defined similarly to (6.14) but centered at z_0,

$$\psi(z_0, r) = \left(\frac{1}{r^2}\int_Q |v(x,t) - (v)_Q|^3 dz\right)^{\frac{1}{3}} + \left(\frac{1}{r^2}\int_Q |p(x,t) - (p)_B(t)|^{3/2} dz\right)^{\frac{2}{3}} + \|f\|_{M_{2,\gamma}} r^{1+\beta},$$

where $Q = Q(z_0, r)$ and $B = B(x_0, r)$. Note that $R/r < C(\theta)$ for $r \in [\theta r_0, r_0]$. Using $2\|f\|_{M_{2,\gamma}} r_0^{1+\beta} \le \varepsilon_2$, we have

$$r|(v)_Q| \le 1, \quad \psi(z_0, r) \le \varepsilon_2,$$

uniformly for $z_0 \in Q_{\theta R}$ and $r \in [\theta r_0, r_0]$, if ε_1 in (6.10) is sufficiently small, depending on θ and ε_2, but not on $\|f\|_{M_{2,\gamma}}$. Applying Lemma 6.5 recursively, we have

$$\begin{aligned} \psi(z_0, \theta^k r) \le \theta^{1+\alpha}\psi(z_0, \theta^{k-1} r) \le \theta^{(1+\alpha)k}\varepsilon_2, \\ \theta^k r |(v)_{Q(z_0, \theta^k r)}| \le 1, \end{aligned} \qquad \forall k > 0,$$

for $z_0 \in Q_{\theta R}$ and $r \in [\theta r_0, r_0]$. We conclude that, for $c = \varepsilon_2(\theta r_0)^{-1-\alpha}$,

$$\psi(z_0, \rho) \le c\rho^{1+\alpha}, \quad \forall z_0 \in Q_{\theta R}, \quad \forall \rho \in (0, r_0]. \tag{6.24}$$

This implies (6.12) and hence $v \in C^{\alpha, \alpha/2}_{x,t}(Q_{\theta R})$, by the parabolic version of the Morrey-Campanato integral characterization of Hölder spaces. □

Remark 6.6. By the proof, for any $\alpha \in (0, \min(3/2, \gamma))$, we have $v \in C^{\alpha,\alpha/2}_{x,t}(\overline{Q}_{\theta R})$ for $\theta = \theta(\alpha, \gamma)$.

The next theorem contains Theorem 6.3 as the special case $q = s = 2$.

Theorem 6.7 ([**78**]). *For any $1 \le s \le \infty$, there is a small constant $\varepsilon = \varepsilon(s) > 0$ such that any suitable weak solution (u, p) with force $f \in M_{2,\gamma}$, $\gamma > 0$, is regular at z_0 if*

$$\limsup_{r \to 0_+} r^{-(\frac{3}{q^*} + \frac{2}{s} - 1)} \|u - (u)_{B_r}\|_{L^s L^{q^*}(Q_{z_0, r})} \le \varepsilon, \tag{6.25}$$

with $1 \le 3/q^* + 2/s \le 2$, or if

$$\limsup_{r \to 0_+} r^{-(\frac{3}{q} + \frac{2}{s} - 2)} \|\nabla u\|_{L^s L^q(Q_{z_0, r})} \le \varepsilon, \tag{6.26}$$

with $2 \le 3/q + 2/s \le 3$.

These criteria are in terms of 0-dimensional scaled norms. The only assumption on the pressure p is $p \in L^{3/2}_{t,x}$. The size of $\|p\|_{L^{3/2}}$ can be arbitrary. As a result we do not specify the size of the neighbourhood of regularity. We will assume $f = 0$ in the proof for simplicity. For the case of nonzero f, see [**78**].

Theorem 6.7 holds if $u - (u)_{B_r}$ is replaced by u in (6.25) since $\|u-(u)_{B_r}\|_{L^s L^{q^*}} \le c\|u\|_{L^s L^{q^*}}$. In particular, it contains the borderline cases $3/q^* + 2/s = 1$, $s \ne \infty$, of Theorem 4.6, not proved in Section 4.3. When $s = q^*$, it follows from (6.25) that the following is a regularity criterion:

$$\limsup_{r\to 0_+} r^{1-5/s}\|u\|_{L^s(Q_{z_0,r})} \le \varepsilon, \quad s \ge 5/2. \tag{6.27}$$

It is not known if (6.27) is a regularity criterion if $s = 2$.

Proof of Theorem 6.7 with $f = 0$. Without loss of generality, we assume $z = (0,0)$. It is enough to consider $3/q^* + 2/s = 2$ and $3/q + 2/s = 3$ since the other cases follow by Hölder inequality. In this proof, $(u)_r = (u)_{B_r}(t)$. We first define the following 0-dimensional quantities:

$$A(r) = \sup_{-r^2\le s<0} \frac{1}{r}\int_{B_r} |u(y,s)|^2\, dy, \quad E(r) = \frac{1}{r}\int_{Q_r} |\nabla u(y,s)|^2 \,\mathrm{d}y\,\mathrm{d}s,$$

$$C(r) = \frac{1}{r^2}\int_{Q_r} |u(y,s)|^3 \,\mathrm{d}y\,\mathrm{d}s, \quad \tilde{C}(r) = \frac{1}{r^2}\int_{Q_r} |u(y,s)-(u)_{B_r}(s)|^3 \,\mathrm{d}y\,\mathrm{d}s,$$

$$D(r) = \frac{1}{r^2}\int_{Q_r} |p(y,s)|^{\frac{3}{2}} \,\mathrm{d}y\,\mathrm{d}s,$$

$$G_0(r) = \frac{1}{r}\, \|u(y,s)-(u)_{B_r}(s)\|_{L^s L^{q^*}(Q_r)}, \quad G_1(r) = \frac{1}{r}\,\|\nabla u(y,s)\|_{L^s L^q(Q_r)}.$$

The quantity $D(r)$ is meaningful due to the assumption $p \in L^{3/2}$ in the definition of suitable weak solutions.

Our strategy is to show, for some $\theta \in (0, 1/4)$,

$$(C+D)(\theta r) \le \frac{1}{2}(C+D)(r) + \frac{\varepsilon_0}{4}, \quad \forall r < r_0. \tag{6.28}$$

Here ε_0 is the small constant from Theorem 6.4. Once this is shown, by iteration,

$$(C+D)(\theta^k r) \le \frac{1}{2^k}(C+D)(r) + \frac{\varepsilon_0}{2}, \quad \forall r < r_0. \tag{6.29}$$

Thus, for k sufficiently large, depending on the initial size of $(C+D)(r)$, $(C+D)(\theta^k r) \le \varepsilon_0$, from which the origin is a regular point by Theorem 6.4.

To show (6.28), we first show for $0 < 2r \le \rho$

$$C(r) \le N\left(\frac{r}{\rho}\right) C(\rho) + N\left(\frac{\rho}{r}\right)^2 \tilde{C}(\rho) \tag{6.30}$$

and

$$D(r) \le N\left(\frac{r}{\rho}\right) D(\rho) + N\left(\frac{\rho}{r}\right)^2 \tilde{C}(\rho), \tag{6.31}$$

for some constant N which may vary from line to line. The first members on the right sides of (6.30) and (6.31) control the averages of u and p, while the second member controls the oscillation. The first members have a small coefficient Nr/ρ, leading to (6.28). To control the second member, we prove

$$\tilde{C}(r) \le N A^{\frac{1}{s}}(r)\, E^{1-\frac{1}{s}}(r)\, G(r), \tag{6.32}$$

where $G = G_0$ or G_1, and

$$A(r) + E(r) \le N[1 + C(2r) + D(2r)]. \tag{6.33}$$

Equation (6.32) is an imbedding result and all quantities are evaluated in the same cylinder Q_r. Equation (6.33) is a conclusion of the local energy inequality. Assuming (6.30)–(6.33), for $0 < 4r \le \rho$ we have

$$\begin{aligned}(C+D)(r) &\le \frac{Nr}{\rho}(C+D)\left(\frac{\rho}{2}\right) + 2N\left(\frac{\rho}{r}\right)^2 \tilde{C}\left(\frac{\rho}{2}\right) \\ &\le \frac{N_1 r}{\rho}(C+D)(\rho) + N_1\left(\frac{\rho}{r}\right)^2 [1 + C(\rho) + D(\rho)]\, G(\rho).\end{aligned}$$

Choose $\theta \in (0, 1/4)$ so that $N_1\theta < 1/4$. By assumption, there is $r_0 > 0$ such that $G(r) < \frac{\theta^2\varepsilon_0}{1+8N_1}$ for all $r \le r_0$. Replacing r and ρ by θr and r, respectively, we get (6.28).

It remains to show (6.30)–(6.33).

For (6.30), note that by Hölder inequality,

$$|(u)_\rho| \le |B_\rho|^{-1+2/3}\|u\|_{L^3(B_\rho)} \le N\rho^{-1}\|u\|_{L^3(B_\rho)}.$$

Thus

$$C(r) \le \frac{N}{r^2}\int_{-r^2}^{0} |B_r||(u)_\rho|^3 + \frac{N}{r^2}\int_{Q_r} |u-(u)_\rho|^3 \le N\left(\frac{r}{\rho}\right) C(\rho) + N\left(\frac{\rho}{r}\right)^2 \tilde{C}(\rho).$$

For (6.31), let $\phi(x) \ge 0$ be supported in B_ρ with $\phi = 1$ in $B_{\rho/2}$. Recall that p satisfies $-\Delta p = \partial_i\partial_j\,(u_i u_j) = \partial_i\partial_j\left[(u_i - (u_i)_\rho)(u_j - (u_j)_\rho)\right]$ since $\operatorname{div} u = 0$. Let

$$\tilde{p}(x,t) := \int_{\mathbb{R}^3} \partial_{x_i}\frac{1}{4\pi|x-y|}\partial_j\left[(u_i - (u_i)_\rho)(u_j - (u_j)_\rho)\phi\right](y,t)dy$$

and $h(x,t) := p(x,t) - \tilde{p}(x,t)$. Then $\Delta_x h = 0$ in $B_{\rho/2}$ for every t. By Calderon-Zygmund estimates for singular integrals,

$$\int_{B_\rho} |\tilde{p}|^{\frac{3}{2}}\,dx \le N\int_{B_\rho} |u-(u)_\rho|^3\,.$$

By the mean value property of harmonic functions,

$$\frac{1}{r^2}\int_{B_r}|h|^{\frac{3}{2}}\,dx \le \frac{Nr}{\rho^3}\int_{B_{\rho/2}}|h|^{\frac{3}{2}}\,dx \le \frac{Nr}{\rho^3}\int_{B_\rho}|p|^{\frac{3}{2}}dx + \frac{Nr}{\rho^3}\int_{B_\rho}|\tilde{p}|^{\frac{3}{2}}\,dx.$$

Adding these estimates and integrating in time, we get

$$\frac{1}{r^2}\int_{Q_r}|p|^{\frac{3}{2}}\,dz \le \frac{N}{r^2}\int_{Q_r}|\tilde{p}|^{\frac{3}{2}} + |h|^{\frac{3}{2}}\,dz \le \text{RHS of (6.31)}.$$

For (6.32), using Hölder inequality, we get

$$\|u-(u)_r\|^3_{L^3(B_r)} \le N\|u\|^{2/s}_{L^2(B_r)}\,\|u-(u)_r\|^{2-2/s}_{L^6(B_r)}\,\|u-(u)_r\|_{L^{q^*}(B_r)}.$$

Integrating in time, dividing both sides by r^2, and applying Hölder again,

$$\begin{aligned}\tilde{C}(r) &\le \frac{N}{r^2}\int_{-r^2}^0 \|u\|^{2/s}_{L^2(B_r)}\|\nabla u\|^{2-2/s}_{L^2(B_r)}\|u-(u)_r\|_{L^{q^*}(B_r)}dt\\ &\le \frac{N}{r^2}\,\|u\|^{2/s}_{L^\infty L^2}\,\|\nabla u\|^{2-2/s}_{L^2L^2}\,\|u-(u)_r\|_{L^sL^{q^*}}\\ &= NA^{\frac{1}{s}}(r)E^{1-\frac{1}{s}}(r)G_0(r).\end{aligned}$$

If $q>3$, then $G_0(r)\le NG_1(r)$. For the remaining case $(q,s)=(3,1)$ and $G=G_1$, by Gagliardo-Nirenberg and Poincaré inequalities,

$$\begin{aligned}\|u-(u)_r\|^3_{L^3(B_r)} &\le N\|u-(u)_r\|^2_{L^2(B_r)}\|\nabla u\|_{L^3(B_r)} + \frac{N}{r^{3/2}}\|u-(u)_r\|^3_{L^2(B_r)}\\ &\le N\|u\|^2_{L^2(B_r)}\|\nabla u\|_{L^3(B_r)}.\end{aligned}$$

Integrating in time and using Hölder inequality, we get $\tilde{C}(r)\le NA(r)G_1(r)$.

Finally, (6.33) is true because of the local energy inequality (3.8), with suitable localized $\phi\ge 0$, $\phi=1$ in Q_r and $\phi=0$ in $Q_\infty\setminus Q_{2r}$,

$$A(r)+E(r)\le N[C^{\frac{2}{3}}(2r)+C(2r)+r^{-2}\|u\|_{L^3(Q_{2r})}\|p\|_{L^{3/2}(Q_{2r})}],$$

which is bounded by $N[1+C(2r)+D(2r)]$. □

6.4. Notes

Section 6.1 is based on [**128**, §31.1]. It follows from the blowup lower bound of Leray [**132**], as pointed out by Scheffer [**171**] and Foiaş-Temam [**49**].

Section 6.2 is based on Caffarelli-Kohn-Nirenberg [**22**]. Theorem 6.3 is [**22**, Proposition 2].

Section 6.3 is based on F. Lin [**136**], Ladyženskaya and Seregin [**122**], and Gustafson, Kang, and Tsai [**78**]. Theorem 6.4 is essentially [**22**, Proposition 1]. The current revised form is due to Nečas-Růžička-Šverák [**155**] and [**136**] for $f=0$ and [**122**] for nonzero f. Vasseur [**214**] gave another proof using De Giorgi's method. Theorem 6.4 is also true if x_0 lies on a flat boundary with zero boundary condition; see Seregin [**177**]. Theorem 6.7 is due to [**78**]. It contains the special cases $(q, s) = (2, 2)$ of [**22**], $(q^*, s) = (3, 3)$ of [**183**], and $(q^*, s) = (2, \infty)$ of [**208**]. It is also true if ∇u is replaced by ω in (6.26) and $s \neq \infty$. Furthermore,

$$\limsup_{r \to 0_+} r^{-1} \|\nabla \omega\|_{L^s L^{q^\sharp}(Q_{z_0,r})} \le \varepsilon, \tag{6.34}$$

with $2 \le s \le \infty$, and $3/q^\sharp + 2/s = 4$ is also a regularity criterion; see [**78**]. Similar criteria on a flat boundary can be found in [**77**].

There is a rich literature on ε-regularity results for (NS). We refer to the references of [**77**, **78**, **141**] for some of them.

Problems

6.1. In Theorem 6.1, if one further knows that $v \in L^s(0, T; L^q)$ with $q > 3$, $s \ge 1$, and $d = 1 - \frac{s}{2} + \frac{3s}{2q} \in (0, 1/2)$, then $\mathfrak{H}^d(\Sigma_3) = 0$. See [**63**, Theorem 5].

6.2. (*Eventual regularity*) In Theorem 6.1, if $T = \infty$, show that the set Σ is bounded. Hint. Use $v \in L^4(0, \infty; L^3)$ and Theorem 5.6. See [**128**, §31.1].

6.3. For any $q > 5/2$ and $\theta \in (0, 1)$, show that there is $\varepsilon_1 = \varepsilon_1(q, \theta) > 0$ such that the conclusion of Theorem 6.4 holds.

6.4. Formulate the special cases of Theorem 6.7 with

$$(q^*, s) \in \{(3, 3), (3, 2), (2, \infty), (5/2, 5/2)\}.$$

6.5. If v is a suitable weak solution in Q_1 with

$$\sup_{-1<t<0} \|v(t)\|_{L^{3,\infty}(B_1)} \le \varepsilon,$$

for ε sufficiently small, then v is regular at $(0, 0)$. Hint. Theorem 6.7.

Chapter 7

Boundary value problem and bifurcation

In this chapter we return to the study of stationary Navier-Stokes equations

$$-\nu\Delta v + v\cdot\nabla v + \nabla p = f, \quad \operatorname{div} v = 0 \quad \text{in } \Omega,$$
$$v|_{\partial\Omega} = v_*. \tag{7.1}$$

Here Ω is a domain in $\mathbb{R}^n$, $n = 2, 3$, with unit outer normal $\mathbf{n}$. Our first topic is on the *boundary value problem* (BVP), i.e., the existence problem in a domain Ω with nonzero boundary value v_*. When $v_* \neq 0$, there is a compatibility condition which needs to be satisfied:

$$\int_{\partial\Omega} v_*\cdot\mathbf{n} = \int_\Omega \operatorname{div} v = 0. \tag{7.2}$$

Note that we do not expect uniqueness for large data, as shown by the example in Section 2.5. In Sections 7.1 and 7.2 we study two approaches of BVP when the boundary flux across each boundary component is zero (or small). Both approaches are based on a priori bounds, which are obtained in Section 7.1 by a good extension and in Section 7.2 by a contradiction argument. In Section 7.3 we sketch the recent work of Korobkov, Pileckas, and Russo which solves the general 2D boundary value problem.

Our second topic is on the *bifurcation* of solutions of fluid equations. In Section 7.4 we explain the general problem. In Section 7.5 we consider the

Rayleigh-Bénard convection, modeled by the coupled system of the Navier-Stokes equations and the heat equation. In Section 7.6 we consider the Taylor vortex of the Couette-Taylor flows.

7.1. Existence: A priori bound by a good extension

In the following two sections, we present two approaches to the *boundary value problem* (7.1) for given ν, f, and v_*. To apply the same existence argument for Theorem 2.7, we hope to prove an a priori estimate

$$\int_\Omega |\nabla v|^2 \le M. \tag{7.3}$$

To take care of the boundary value, we usually try to choose a suitable extension V and decompose $v = u + V$, with

$$\operatorname{div} V = 0, \quad V|_{\partial\Omega} = v_*. \tag{7.4}$$

The difference u satisfies $u \in H^1_{0,\sigma}(\Omega)$ and

$$\begin{aligned} -\nu\Delta u + (u\cdot\nabla)u + (u\cdot\nabla)V + (V\cdot\nabla)u + \nabla p &= F \\ \operatorname{div} u &= 0 \end{aligned} \tag{7.5}$$

in Ω, where $F := f + \nu\Delta V - (V\cdot\nabla)V$. Testing with u itself, the second, fourth, and fifth terms in (7.5) vanish and we have

$$\int \nu|\nabla u|^2 + \int (u\cdot\nabla)V\cdot u = \int F\cdot u. \tag{7.6}$$

If we can find a vector field V satisfying (7.4) and, for some $\alpha < \nu$,

$$-\int (u\cdot\nabla)V\cdot u \le \alpha\int |\nabla u|^2, \quad \forall u \in H^1_{0,\sigma}(\Omega), \tag{7.7}$$

then we get

$$(\nu-\alpha)\int |\nabla u|^2 \le \int F\cdot u \le \|F\|_{(H^1_{0,\sigma}(\Omega))'}\|\nabla u\|_{L^2},$$

which implies the a priori bound (7.3), and we can construct u in the same way as for Theorem 2.7, using the method in Section 2.3 or 2.4. This is the first approach.

Does such V exist? Not always. It exists if the flux of v_* through each connected component of $\partial\Omega$ is zero (see Theorem 7.3) or sufficiently small (see (7.30), [**57**, §9.4]). It can also exist if Ω and v_* have certain symmetry [**4**]. It does not exist for some explicit Ω and ν; see the example at the end of this section in the annulus (7.17). However, the nonexistence of such V only says that *this proof* of the a priori bound (7.3) fails. It is still possible to prove (7.3) by a contradiction argument. This is the second approach, to be given in Section 7.2.

We prefer a divergence-free V, since otherwise there is an additional term $\int (V\cdot\nabla)u\cdot u = -\frac{1}{2}\int |u|^2 \operatorname{div} V$ in (7.6). One way to get a divergence-free vector field is to find a *vector potential* w such that $V = \operatorname{curl} w$. We have the following lemma.

Lemma 7.1 (Existence of vector potential)**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$, with $\partial\Omega = \bigcup_{i=0}^m \Gamma_i$, $m \ge 0$, where each Γ_i is a connected component of $\partial\Omega$. Let $v_* \in W^{1/2,2}(\partial\Omega)$ satisfy*

$$\mathcal{F}_i \stackrel{\text{def}}{=\!=} \int_{\Gamma_i} v_* \cdot \mathbf{n} = 0 \quad (\forall i = 0, \dots, m). \tag{7.8}$$

There exists a vector field $w \in W^{2,2}(\Omega)$ such that

$$\operatorname{curl} w|_{\partial\Omega} = v_*, \quad \|w\|_{W^{2,2}} \le c\|v_*\|_{W^{1/2,2}(\partial\Omega)}. \tag{7.9}$$

Note that (7.8) implies (7.2). The reverse is false if $m \ge 1$.

For the case $n = 2$, we consider $\mathbb{R}^2 \hookrightarrow \mathbb{R}^3$ and identify (v_1, v_2) with $(v_1, v_2, 0)$. We have $w = (0, 0, w_3)$ and $\operatorname{curl} w = (\partial_2 w_3, -\partial_1 w_3, 0)$.

Proof. We first show the case $n = 3$.

Step 1. By Sobolev space theory, we may extend v_* to $v' \in H^1(\Omega, \mathbb{R}^3)$ with $v'|_{\partial\Omega} = v_*$ and $\|v'\|_{H^1(\Omega)} \le c\|v_*\|_{W^{1/2,2}(\partial\Omega)}$. Since $\int_\Omega \operatorname{div} v' = \int_{\partial\Omega} v_* \cdot \mathbf{n} = 0$, we can apply Lemma 2.16 to get $v'' \in H_0^1(\Omega, \mathbb{R}^3)$ with $\operatorname{div} v'' = \operatorname{div} v'$ and $\|v''\|_{H^1(\Omega)} \le c\|\operatorname{div} v'\|_{L^2(\Omega)}$. Let $v = v' - v''$. We have

$$\operatorname{div} v = 0, \quad v|_{\partial\Omega} = v_*, \quad \|v\|_{H^1(\Omega)} \le c\|v_*\|_{W^{1/2,2}(\partial\Omega)}.$$

Step 2. Choose R large enough so that $\bar\Omega \subset B_R$. Relabel Γ_0 as the outer component of $\partial\Omega$. The set $B_R - \bar\Omega$ can be decomposed as the disjoint union of connected components $\omega_0, \dots, \omega_m$, with $\partial\omega_i = \Gamma_i$ for $i = 1, \dots, m$ and $\partial\omega_0 = \Gamma_0 \cup \partial B_R$. Using the key assumption $\int_{\Gamma_i} v_* \cdot \mathbf{n} = 0$ to apply Lemma 2.16, we can repeat the previous step to find $v_i \in H^1(\omega_i)$ with $v_0|_{\partial B_R} = 0$, and

$$\operatorname{div} v_i = 0, \quad v_i|_{\Gamma_i} = v_*, \quad \|v_i\|_{H^1(\omega_i)} \le c\|v_*\|_{W^{1/2,2}(\partial\Omega)}.$$

Step 3. Define $\bar v = v$ on Ω, $\bar v = v_i$ on ω_i, and $\bar v = 0$ in B_R^c. We have $\bar v \in H_{0,\sigma}^1(\mathbb{R}^3)$. To find w so that $\operatorname{curl} w = \bar v$, we can use the *Biot-Savart Law* (1.44) and define

$$w(x) = \int_{\mathbb{R}^3} \nabla_x \frac{1}{4\pi|x-y|} \times \bar v(y)\, dy. \tag{7.10}$$

Then $\operatorname{curl} w = \bar v$ in $\mathbb{R}^3$ and $\operatorname{curl} w|_{\partial\Omega} = v_*$. The L^2-norm of $\partial_i\partial_j w$ can be estimated using the Calderon-Zygmund theorem. The estimates of $\|\partial_i w\|_2$ and $\|w\|_2$ follow from Sobolev imbedding.

For the case $n=2$, the same construction leads to $\bar{v}=(\bar{v}_1,\bar{v}_2,0)$ defined for $x\in\mathbb{R}^2$. Since $\partial_1\bar{v}_1+\partial_2\bar{v}_2=0$, there is a potential w_3 such that $(\bar{v}_1,\bar{v}_2,0)=(\partial_2 w_3,-\partial_1 w_3,0)$. Thus $\bar{v}=\operatorname{curl}(0,0,w_3)$. □

We will need the following lemma.

Lemma 7.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then there exists $c=c(\Omega,q,n)$ such that for all $u\in W_0^{1,q}(\Omega)$, $1<q<\infty$, we have*

$$\|u\delta^{-1}\|_q\le c\|\nabla u\|_q,$$

where $\delta(x)=\operatorname{dist}(x,\partial\Omega)$.

This is known by Leray [**131**] for $q=2$; see [**57**, Lemma III.6.3]. It can be proved by using the 1D *Hardy inequality* for functions vanishing at 0:

$$\int_0^\delta |h(t)|^q t^{-q}dt\le c_q\int_0^\delta |h'(t)|^q dt,\quad \forall h\in C^1([0,\delta]),\ h(0)=0. \tag{7.11}$$

We can take $c_q=\frac{q}{q-1}$.

We will also need the *regularized distance*. The usual distance function to the boundary, $\delta(x)=\operatorname{dist}(x,\partial\Omega)$, is C^k in a small neighbourhood of $\partial\Omega$ only if $\partial\Omega$ is C^k, $k\ge 2$ (see [**72**, §14.6], [**135**, §IV.5]). However, when $\partial\Omega$ is merely bounded and Lipschitz, there is a regularized distance $\rho(x)\in C^\infty(\Omega)$ which is comparable to $\delta(x)$,

$$\begin{aligned} c_1\delta(x)\le\rho(x)\le c_2\delta(x),\\ |\nabla^k\rho(x)|\le c_k(\delta(x))^{1-k},\quad\forall k\in\mathbb{N}_0. \end{aligned} \tag{7.12}$$

Its definition uses Whitney decomposition of Ω; see [**199**, p. 171].

Here is the main result of this section.

Theorem 7.3 (Existence with zero flux on each boundary component)**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n=2,3$, with $\partial\Omega=\bigcup_{i=0}^m\Gamma_i$, $m\ge 0$, where each Γ_i is a connected component of $\partial\Omega$. Let $v_*\in W^{1/2,2}(\partial\Omega)$ satisfy*

$$\mathcal{F}_i\overset{\text{def}}{=\!=}\int_{\Gamma_i}v_*\cdot\mathbf{n}=0\quad(\forall i=0,\dots,m). \tag{7.13}$$

Let f be in the dual space of $H^1_{0,\sigma}(\Omega)$. Then there is a weak solution $v\in H^1(\Omega;\mathbb{R}^3)$ of (7.1) *with boundary value v_* and force f.*

We will show that (7.13) can be relaxed to $\sum_i|\mathcal{F}_i|\ll 1$; see (7.30).

Proof. We may assume $n=3$. As discussed earlier, it suffices to find $V\in H^1(\Omega;\mathbb{R}^3)$ that satisfies $\operatorname{div}V=0$, $V|_{\partial\Omega}=v_*$, and the inequality (7.7).

Let $w \in H^2(\Omega;\mathbb{R}^3)$ be given by Lemma 7.1 which states that $\operatorname{curl} w|_{\partial\Omega} = v_*$. For small $\varepsilon > 0$ we choose

$$V = \operatorname{curl}(\psi_\varepsilon w), \quad \psi_\varepsilon(x) = \Phi_\varepsilon(\rho(x)), \tag{7.14}$$

where $\rho(x)$ is the regularized distance and $\Phi_\varepsilon(t) \in C_c^\infty(\mathbb{R})$ is essentially given by $\Phi_\varepsilon(t) = 1$ for $t < e^{-2/\varepsilon}$, $\Phi_\varepsilon(t) = 0$ for $t \geq e^{-1/\varepsilon}$, and $\Phi_\varepsilon(t) = (e - e^2 t^\varepsilon)/(e-1)$ for $e^{-2/\varepsilon} < t < e^{-1/\varepsilon}$, and suitably mollified to be smooth. It is defined on a nonlinear scale to gain a small factor ε in the estimate

$$|\Phi'_\varepsilon(t)| \leq C\varepsilon t^{-1}, \quad \forall t > 0. \tag{7.15}$$

Since $\psi_\varepsilon = 1$ near $\partial\Omega$, we get $V|_{\partial\Omega} = v_*$. We also have $V \in H^1$ since $\psi_\varepsilon \in C^2$ and $w \in H^2$. For $u \in H^1_{0,\sigma}(\Omega)$, by integration by parts and the Cauchy-Schwarz inequality,

$$-\int (u\cdot\nabla)V\cdot u = \int u_j V_i \partial_j u_i \leq \|uV\|_{L^2}\|\nabla u\|_{L^2}. \tag{7.16}$$

Note that $|V| \lesssim \psi_\varepsilon|\nabla w| + |\nabla\psi_\varepsilon|\cdot|w|$,

$$\|u\psi_\varepsilon|\nabla w|\|_{L^2} \lesssim \|\psi_\varepsilon\|_\infty \|\nabla u\|_2 \|\nabla w\|_{L^3(\operatorname{spt}\psi_\varepsilon)}$$

(using $\|u\|_6 \lesssim \|\nabla u\|_2$) and

$$\|u|\nabla\psi_\varepsilon||w|\|_{L^2} \lesssim \varepsilon\|w\|_{L^\infty}\|\delta^{-1}u\|_{L^2(\operatorname{spt}\nabla\psi_\varepsilon)} \lesssim \varepsilon\|w\|_{L^\infty}\|\nabla u\|_{L^2},$$

using Lemma 7.2. Summarizing, we get

$$\|uV\|_{L^2} \leq c(\|\nabla w\|_{L^3(\operatorname{spt}\psi_\varepsilon)} + \varepsilon\|w\|_{L^\infty})\|\nabla u\|_{L^2}.$$

Thus for any $\alpha > 0$ we can choose $\varepsilon > 0$ sufficiently small so that $\|uV\|_{L^2} \leq \alpha\|\nabla u\|_{L^2}$ and hence get (7.7) from (7.16). This shows the a priori bound of u, and we can prove Theorem 7.3 in the same way as for Theorem 2.7. $\square$

Remark. If we take a cut-off function of the simple form $\Phi_\varepsilon(t) = \Phi(t/\varepsilon)$, we lose the small factor ε in (7.15). We still have $\|\delta^{-1}u\|_{L^2(\operatorname{spt}\nabla\psi_\varepsilon)} \to 0$ as $\varepsilon \to 0$, but it is not uniform in u with $\|u\|_{H^1_0} \leq 1$.

If we relax the assumption (7.13), the above approach does not always work, as the function V satisfying (7.4) and (7.7) for all $\alpha > 0$ may not exist. Consider the following example of Amick [**4**]. Let

$$\Omega = \left\{x \in \mathbb{R}^2, R_1 < |x| < R_2\right\}, \tag{7.17}$$

where $0 < R_1 < R_2$. Write $V = V_r(r,\theta)e_r + V_\theta(r,\theta)e_\theta$ in polar coordinates. It follows from the divergence-free condition $\partial_r(rV_r) + \partial_\theta V_\theta = 0$ that, for $r \in (R_1, R_2)$,

$$\int_0^{2\pi} V_r(r,\theta) r\,d\theta = C_1, \quad C_1 := \int_{|x|=R_i} v_* \cdot \frac{x}{|x|}, \quad i = 1, 2.$$

C_1 is the flux through the outer boundary $|x| = R_2$. If we take

$$u = a(r)e_\theta, \quad a \in C^1, \quad a(R_1) = a(R_2) = 0, \tag{7.18}$$

we have $\operatorname{div} u = 0$,

$$\begin{aligned} -\int_\Omega (u \cdot \nabla)V \cdot u &= \int_\Omega (u \cdot \nabla)u \cdot V = \int_\Omega ar^{-1}\partial_\theta(ae_\theta) \cdot V \\ &= -\int_\Omega a^2 r^{-1} e_r \cdot V = -\int_{R_1}^{R_2} a^2 r^{-1} \int_0^{2\pi} V_r r d\theta dr \\ &= -C_1 \int_{R_1}^{R_2} a^2 \frac{dr}{r}, \end{aligned}$$

while

$$\int |\nabla u|^2 = 2\pi \int_{R_1}^{R_2} \left((a')^2 + \frac{a^2}{r^2} \right) r dr.$$

Thus $-\int_\Omega (u \cdot \nabla)V \cdot u$ cannot be bounded by $\alpha \int |\nabla u|^2$ for α sufficiently small, if $C_1 < 0$.

7.2. Existence: A priori bound by contradiction

The second approach for the a priori bound (7.3) is based on a contradiction argument. The setup is similar to Section 2.4. When the compatibility condition (7.2) is satisfied, choose any $V \in H^1(\Omega)$ satisfying

$$\operatorname{div} V = 0, \quad V|_{\partial\Omega} = v_*, \quad \|V\|_{H^1(\Omega)} \lesssim \|v_*\|_{W^{1/2,2}(\partial\Omega)}. \tag{7.19}$$

It does not need to satisfy (7.7) and we do not assume all $\mathcal{F}_i = 0$. A vector field satisfying the last two conditions of (7.19) can be found by the Sobelev extension theorem and be made divergence-free using Lemma 2.16 and (7.2). See also Problem 7.2. Decompose $v = V + u$. The difference $u = v - V$ satisfies

$$-\Delta u + \nabla \tilde{p} = F, \quad \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0, \tag{7.20}$$

where

$$F = F(u) = \frac{1}{\nu} f + \Delta V - \frac{1}{\nu}(V + u) \cdot \nabla(V + u). \tag{7.21}$$

Let the Hilbert space $\mathbf{V} = H^1_{0,\sigma}(\Omega)$ be equipped with the inner product $(\!(u, v)\!)_{\mathbf{V}} = \int_\Omega \nabla u : \nabla v$, and let $T : \mathbf{V}' \to \mathbf{V}$ be the solution operator of the Stokes system (7.20), $u = TF$, guaranteed by the Riesz representation theorem. Then the nonlinear system (7.20)–(7.21) is equivalent to the fixed point problem

$$u = K(u), \quad K(u) = T(F(u)). \tag{7.22}$$

As in Section 2.4, $K : \mathbf{V} \to \mathbf{V}$ is compact, and we can apply the *Leray-Schauder theorem* (Theorem 2.8) to get a solution of (7.22) if all solutions of

$$u = \sigma K(u), \quad \sigma \in [0,1], \tag{7.23}$$

have a uniform bound. This is, however, unclear. It was shown in Section 7.1 that we may choose V to satisfy (7.7) if (7.13) holds; in general we may not.

We now try to prove a uniform bound for solutions of (7.23) by a contradiction argument. Suppose the contrary. Then there are $\sigma_j \in [0,1]$ and $\hat{u}_j \in \mathbf{V}$ such that $\hat{u}_j = \sigma_j K(\hat{u}_j)$, $\sigma_j \to \bar{\sigma} \in [0,1]$, and $N_j := \|\hat{u}_j\|_{\mathbf{V}} \to \infty$ as $j \to \infty$. Recall that $\|u\|_{\mathbf{V}} = \|\nabla u\|_{L^2(\Omega)}$. Define $u_j = \hat{u}_j / N_j$ and $\|u_j\|_{\mathbf{V}} = 1$. A subsequence, still denoted as u_j, converges weakly to some w in $\mathbf{V}$, and strongly in $L^r(\Omega)$ for any $r < 2^*$, where $2^* = \infty$ if $n = 2$ and $2^* = 6$ if $n = 3$.

Since $N_j u_j = \sigma_j K(N_j u_j)$, u_j satisfies the weak form of

$$-\Delta u_j + \nabla p_j = \frac{\sigma_j}{N_j} F(N_j u_j), \quad \operatorname{div} u_j = 0; \tag{7.24}$$

that is, for any $\zeta \in \mathbf{V}$

$$\begin{aligned} &\int_\Omega \nabla u_j : \nabla \zeta \\ &\quad = \frac{\sigma_j}{\nu N_j} \int_\Omega (f - (V + N_j u_j) \cdot \nabla (V + N_j u_j)) \cdot \zeta - \frac{\sigma_j}{N_j} \int_\Omega \nabla V : \nabla \zeta. \end{aligned} \tag{7.25}$$

Using u_j itself as a test function, the integral $\int ((V + N_j u_j) \cdot \nabla) N_j u_j \cdot u_j$ vanishes and we get

$$\begin{aligned} 1 &= \int_\Omega |\nabla u_j|^2 \\ &= -\frac{\sigma_j}{\nu} \int_\Omega u_j \cdot \nabla V \cdot u_j + \frac{\sigma_j}{\nu N_j} \int_\Omega \{(f - V \cdot \nabla V) \cdot u_j - \nu \nabla V : \nabla u_j\}. \end{aligned}$$

We now send $j \to \infty$. The last integral vanishes since $N_j \to \infty$. Since $u_j \to w$ strongly in L^4, we get

$$1 = -\frac{\bar{\sigma}}{\nu} \int_\Omega (w \cdot \nabla) V \cdot w. \tag{7.26}$$

In particular, $\bar{\sigma} \neq 0$. For the purpose of comparison, if V satisfies (7.7) for some $\alpha < \nu$, we get a contradiction that

$$\int |\nabla w|^2 \le 1 = -\frac{\bar{\sigma}}{\nu} \int_\Omega (w \cdot \nabla) V \cdot w \le \frac{\alpha}{\nu} \int |\nabla w|^2.$$

Dividing (7.25) by N_j for a fixed $\zeta \in \mathbf{V}$ and taking limit $j \to \infty$, we find that only the nonlinear term survives and

$$0 = -\frac{\bar{\sigma}}{\nu} \int_\Omega (w \cdot \nabla) w \cdot \zeta, \quad \forall \zeta \in \mathbf{V}. \tag{7.27}$$

Thus $w \in \mathbf{V}$ is a weak solution of the stationary *Euler equations*

$$w \cdot \nabla w + \nabla \pi = 0, \quad \operatorname{div} w = 0, \tag{7.28}$$

which further satisfies (7.26). By Lemma 1.2, there is $\pi \in W^{1,1}_{\mathrm{loc}}$ so that the pair w, π solves (7.28) almost everywhere. Since $w \in H^1(\Omega)$,

$$\pi \in W^{1,s}(\Omega), \quad \begin{cases} \forall s < 2, & n = 2, \\ s = 3/2, & n = 3. \end{cases}$$

The following lemma shows that π is almost everywhere a constant π_i on each connected component Γ_i of $\partial\Omega$. The heuristic is that $\nabla \pi = -w \cdot \nabla w$ vanishes on $\partial\Omega$ as $w|_{\partial\Omega} = 0$. However ∇w and $\nabla \pi$ have no trace on $\partial\Omega$.

Lemma 7.4. *The trace of π on each connected component Γ_i of $\partial\Omega$ is a constant π_i $\mathfrak{H}^1$-almost everywhere on Γ_i.*

Proof. For $z_0 \in \Gamma_i$, choose a coordinate system such that z_0 is the origin, and for some small $\varepsilon, \delta > 0$ and some Lipschitz function $h(x')$ defined for $x' \in B_\varepsilon \subset \mathbb{R}^{n-1}$, $x_n = h(x')$ lies on Γ_i and

$$A = A(\varepsilon, \delta) = \left\{ x = (x', x_n) : \ x' \in B_\varepsilon, \ 0 < x_n - h(x') < \delta \right\}$$

is a subset of Ω. By the Cauchy-Schwartz inequality and (7.11) with $q = 2$,

$$\left(\int_A \frac{|w \cdot \nabla w|}{x_n - h(x')} dx \right)^2 \lesssim \|\nabla w\|_2^2 \int_A \frac{|w|^2}{|x_n - h(x')|^2} dx \lesssim \|\nabla w\|_2^2 \int_A |\partial_n w|^2 dx.$$

Thus

$$\int_{A(\varepsilon,\delta)} |\nabla \pi| = o(\delta) \quad \text{as } \delta \to 0.$$

For any $\phi(x') \in C_c^\infty(B_\varepsilon)$ and $j < n$,

$$\begin{aligned} \int_{A(\varepsilon,\delta)} \partial_j \phi(x') \pi(x) dx &= \int_{B_\varepsilon} \int_0^\delta \partial_j \phi(x') \pi(x', h(x') + x_n) dx \\ &= - \int_{B_\varepsilon} \int_0^\delta \phi(x') \left(\partial_j \pi + (\partial_j h) \partial_n \pi \right) dx, \end{aligned}$$

which is $o(\delta)$ as $\delta \to 0$. Since $\int_{B_\varepsilon} \partial_j \phi(x') \pi(x', h(x') + t) dx'$ is a continuous function in $t \in [0, \delta]$,

$$\int_{B_\varepsilon} \partial_j \phi(x') \pi(x', h(x')) dx' = 0, \quad \forall \phi(x') \in C_c^\infty(B_\varepsilon).$$

It shows that π is a constant almost everywhere on Γ_i. $\square$

From (7.26), (7.28), and Lemma 7.4, we have

$$\frac{\nu}{\bar{\sigma}} = \int_\Omega w \cdot \nabla w \cdot V = -\int_\Omega \nabla \pi \cdot V = -\int_{\partial\Omega} \pi v_* \cdot \mathbf{n} = -\sum_{i=0}^{m} \pi_i \mathcal{F}_i. \tag{7.29}$$

Thus we get a contradiction if $\mathcal{F}_i = 0$ for each i. This furnishes a second proof of Theorem 7.3.

We can get more from (7.29). We can normalize π so that $\int_\Omega \pi = 0$. Then $|\pi_i| \le c\|\pi\|_{L^s(\partial\Omega)} \le c\|\nabla \pi\|_{L^s} \le c_1$ is bounded by some computable constant c_1. Then by (7.29)

$$\nu \le \frac{\nu}{\bar{\sigma}} < c_1 \sum_i |\mathcal{F}_i|$$

(strict inequality since $\sum_i \mathcal{F}_i = 0$ by (7.2)). So we get a contradiction if

$$\sum_i |\mathcal{F}_i| \le \nu / c_1. \tag{7.30}$$

This is a weaker assumption than (7.13).

Another application is the following. If $\partial\Omega$ has two components Γ_0 and Γ_1, with $\mathcal{F}_1 = -\mathcal{F}_0$, then

$$(\pi_1 - \pi_0)\mathcal{F}_0 > 0. \tag{7.31}$$

We conclude this section by citing the observation of Amick [**4**]. For the weak solution w of (7.28) studied in this section, Amick emphasized that it is not just any solution of the steady Euler equations, but it is a certain limit of solutions to the steady Navier-Stokes equations. For a solution v, p of the stationary Navier-Stokes equations

$$-\nu \Delta v + v \cdot \nabla v + \nabla p = f, \quad \operatorname{div} v = 0,$$

the *total head pressure*

$$\Phi = \frac{1}{2}|v|^2 + p$$

satisfies

$$-\nu \Delta \Phi + \operatorname{div}(v\Phi) = -\nu \omega^2 + f \cdot v, \tag{7.32}$$

with ω being the vorticity tensor and $\omega^2 = \sum_{i<j} (\partial_j v_i - \partial_i v_j)^2$ for general $n \ge 2$. If $f = 0$, then Φ satisfies the classical *maximal principle* that

$$\operatorname*{ess\,sup}_{\Omega'} \Phi = \operatorname*{ess\,sup}_{\partial\Omega'} \Phi, \quad \forall C^2 \text{ open } \Omega' \Subset \Omega. \tag{7.33}$$

Since this is satisfied by every member of the sequence of solutions of the steady Navier-Stokes equations converging to (w, π), one expects that the corresponding $\Phi = \frac{1}{2}|w|^2 + \pi$ of the limit (w, π) also satisfies some (weak) version of the maximal principle.

Consider Amick's example (7.18) on the annulus (7.17) again. For $w = a(r)e_\theta$ with $a(R_1) = a(R_2) = 0$, we have

$$w \cdot \nabla w = -a^2 r^{-1} e_r, \quad \operatorname{div} w = 0.$$

Thus w is indeed a solution of the Euler equations (7.28) with

$$\pi(r) = \int_{R_1}^{r} a^2 r^{-1} dr,$$

which is increasing in r. In particular, $\pi(R_1) < \pi(R_2)$. For $\Phi = \frac{1}{2}|w|^2 + \pi$, we have $\Phi(R_1) = 0 \le \Phi(r)$ and

$$\frac{d}{dr}\Phi = aa' + a^2/r = \frac{a}{r}(ra)'.$$

If Φ satisfies the maximal principle, then for $R_1 \le s < r \le R_2$

$$\max_{[s,r]} \Phi = \max_{[R_1,r]} \Phi = \Phi(r).$$

Taking $s \to r_-$, we get $a(ra)' \ge 0$ for all r. The function $b(r) = (ra(r))^2$ satisfies $b(R_1) = b(R_2) = 0$ and $b' \ge 0$ and is hence identically zero.

Such a flow cannot arise as the limit of normalized unbounded solutions of the Navier-Stokes equations since it does not satisfy the maximal principle.

7.3. The Korobkov-Pileckas-Russo approach for 2D BVP

In a remarkable series of papers [**108**, **110**, **111**], Korobkov, Pileckas, and Russo solved the general boundary value problem (7.1) in 2D bounded domains. They also obtained results for 2D exterior domains and 3D axisymmetric bounded or exterior domains [**109**, **112**]. In this section we sketch the 2D bounded domain case.

Theorem 7.5 ([**111**]). *Let $\Omega \subset \mathbb{R}^2$ be a bounded C^2-domain, with $\partial\Omega = \bigcup_{i=0}^m \Gamma_i$, $m \ge 0$, where each Γ_i is a connected component of $\partial\Omega$. If $f \in H^1(\Omega)$ and if $v_* \in H^{3/2}(\partial\Omega)$ satisfies $\int_{\partial\Omega} v_* \cdot \mathbf{n} = 0$, then the boundary value problem* (7.1) *admits at least one weak solution.*

We are interested in the multiple boundary components case, $m \ge 1$. The theorem only assumes zero total flux and makes no assumption on individual $\mathcal{F}_i = \int_{\Gamma_i} v_* \cdot \mathbf{n}$.

Its proof is based on an a priori bound of the solutions which is derived from the contradiction argument of the last section. A new ingredient is the use of Bernoulli's Law for Sobolev solutions to the Euler equations, which in turn is based on a Sobolev space version of the Morse-Sard theorem. This version implies that almost all level sets of a function $\psi \in W^{2,1}(\Omega)$ are the finite union of C^1-curves. In the proof, ψ is the stream function of the flow with $w = (-\partial_2\psi, \partial_1\psi)$, and it is important to study the geometry of its level sets. This is why this method is restricted to dimension $n = 2$.

We now give more details.

For a function $w \in L^1_{\rm loc}$, its best representation is

$$w^*(x) = \begin{cases} \lim_{r\to 0} \fint_{B(x,r)} w(z)dz, & \text{if the finite limit exists,} \\ 0, & \text{otherwise.} \end{cases} \tag{7.34}$$

We have $w = w^*$ almost everywhere. In this section we always use the best representation for a Sobolev function and its derivatives.

Recall the α-dimensional Hausdorff measure $\mathfrak{H}^\alpha$ defined in (6.4): For $\alpha > 0$, $0 < \delta \le \infty$, and $F \subset \mathbb{R}^n$,

$$\begin{aligned} &\mathfrak{H}^\alpha_\delta(F) = \inf\left\{\textstyle\sum_{j=1}^\infty (\mathrm{diam} F_j)^\alpha : F \subset \bigcup_{j=1}^\infty F_j,\ \mathrm{diam} F_j < \delta\right\}, \\ &\mathfrak{H}^\alpha(F) = \lim_{\delta\to 0_+} \mathfrak{H}^\alpha_\delta(F). \end{aligned} \tag{7.35}$$

The following lemma will be applied to the stream function ψ and is the starting point of the proof.

Lemma 7.6. *If $\psi \in W^{2,1}(\mathbb{R}^2)$, then ψ is continuous, and there is a set $A_\psi \subset \mathbb{R}^2$ with $\mathfrak{H}^1(A_\psi) = 0$ such that at all $x \in \mathbb{R}^2 \setminus A_\psi$, ψ is differentiable in the classical sense, the classical derivative agrees with (the best representation of) $\nabla\psi$, and $\lim_{r\to 0} \fint_{B_r(x)} |\nabla\psi(z) - \nabla\psi(x)|^2 dz = 0$.*

We now discuss the Morse-Sard theorem. The classical Morse-Sard theorem for scalar functions (there is also a vector version) states that if $\psi \in C^n(\mathbb{R}^n;\mathbb{R})$ and if we let S be its set of critical points, $S = \{x \in \mathbb{R}^n : \nabla\psi(x) = 0\}$, then $\psi(S)$ has Lebesgue measure 0 in $\mathbb{R}$. This means that although the set of critical points may be large, the set of critical values is small. The following theorem is a Sobolev space version.

Theorem 7.7 ([**15**])**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, and let $\psi \in W^{2,1}(\Omega)$. Then:*

(i) $\mathfrak{H}^1(\{\psi(x) : x \in \overline{\Omega} \setminus A_\psi$ *and* $\nabla\psi(x) = 0\}) = 0$.

(ii) *(Lusin property) For any $\varepsilon > 0$, there is $\delta > 0$ so that any $U \subset \overline{\Omega}$ with $\mathfrak{H}^1_\infty(U) < \delta$ satisfies $\mathfrak{H}^1(\psi(U)) < \varepsilon$.*

(iii) *(Small set of singular values) For any $\varepsilon > 0$, there is an open subset $V \subset \mathbb{R}$ with $\mathfrak{H}^1(V) < \varepsilon$ and $g \in C^1(\mathbb{R}^2)$ so that any $x \in \overline{\Omega} \setminus \psi^{-1}(V)$ is not in A_ψ and $\psi(x) = g(x)$, $\nabla\psi(x) = \nabla g(x) \neq 0$.*

(iv) *(Level sets) For $\mathfrak{H}^1$-a.e. $y \in \psi(\overline{\Omega}) \subset \mathbb{R}$, the preimage $\psi^{-1}(y)$ is a finite disjoint union of C^1-curve S_j, $\psi^{-1}(y) = \bigcup_{j=1}^{N(y)} S_j$, and each S_j is either a cycle in Ω (i.e., a closed curve) or a simple arc with endpoints on $\partial\Omega$ and transversal to $\partial\Omega$.*

The fourth property says that most level sets are "nice".

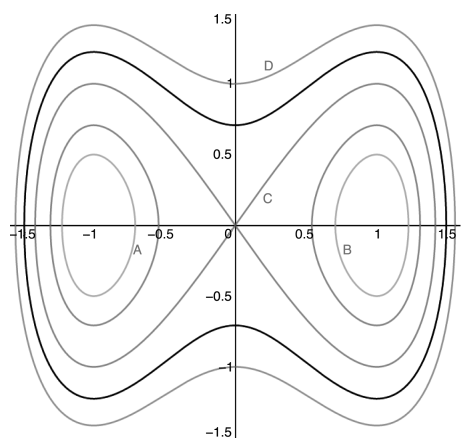
The following lemma is the tree property of level sets of 2D functions.

Lemma 7.8 (Kronrod [**116**]). *Let $Q = [0,1] \times [0,1]$, let $f \in C(Q)$, and let T_f be the family of connected components of level sets of f. For any $A \neq B$ in T_f, there is an injective map $\phi(t) : [0,1] \to T_f$ so that $\phi(0) = A$, $\phi(1) = B$, for any $t_0 \in [0,1]$,*

$$\lim_{t \to t_0} \sup_{x \in \phi(t)} \operatorname{dist}(x, \phi(t_0)) = 0, \tag{7.36}$$

and for any $t_0 \in (0,1)$, the sets A and B lie in different connected components of $Q \setminus \phi(t_0)$.

The set Q can be replaced by any subset of $\mathbb{R}^2$ which is homeomorphic to Q. Property (7.36) says that the function $\phi(t)$ is continuous in the natural topology of T_f. The range $[A,B] = \{\phi(t) : \ 0 \le t \le 1\}$ is called the *Kronrod arc* connecting A and B and can be shown to be unique.



Consider the example $f(x,y) = (x^2 - 1)^2 + y^2$, with Q being the square $|x|, |y| \le 2$. Let A and B be the left and the right components of the level set $f = 1/4$, let C be the level set $f = 1$ (which is the number 8 curve in the figure), and let D be the level set $f = 2$. Let $\phi(t)$ be a parametrization of the arc $[A,B]$. As t increases from $t = 0$, the curve $\phi(t)$ expands continuously from A but remains inside the left portion of C, until at some t_0 it becomes C. For $t > t_0$ it moves to the right plane and shrinks to B as $t \to 1$.

There is a topology change of $\phi(t)$ at $t = t_0$, but (7.36) is still satisfied. Note that $\sup_{x \in \phi(t)} \operatorname{dist}(x, \phi(t_0)) \neq \sup_{x \in \phi(t_0)} \operatorname{dist}(x, \phi(t))$. Similarly we can consider the arcs $[A,D]$ and $[B,D]$. All of them go through C. Hence C is a *branching point* of the tree, which can be characterized by the property that $Q \setminus C$ has more than two components.

We now discuss the *Bernoulli Law*. For given fluid velocity w and pressure p, the *total head pressure* is the scalar function

$$\Phi = \frac{1}{2}|w|^2 + p.$$

The Bernoulli Law states that, for smooth solutions w, p of the stationary Euler equations (7.28), the total head pressure is a constant along any streamline. The constant depends only on the streamline chosen. Indeed, if $q(t)$ denotes a fluid trajectory, then $\frac{d}{dt}q = w(q(t))$ and

$$\frac{d}{dt}\Phi(q(t)) = (w_j\partial_i w_j + \partial_i p)|_{q(t)}\frac{d}{dt}q_i = (w\cdot\nabla w + \nabla p)\cdot w|_{q(t)} = 0.$$

When the dimension is 2, from $\operatorname{div} w = 0$ there is a *stream function* ψ such that

$$w_1 = -\partial_2\psi, \quad w_2 = \partial_1\psi.$$

The stream function is also a constant along any streamline.

The following is a Bernoulli Law for 2D solutions in Sobolev spaces.

Theorem 7.9 (Bernoulli's Law)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain with boundary $\partial\Omega = \bigcup_i \Gamma_i$. Assume $w \in H^1_{0,\sigma}(\Omega)$ and $p \in W^{1,s}(\Omega)$, $1 < s < 2$, satisfies the Euler equations* (7.28) *in Ω, and let Φ and ψ be their total head pressure and stream function. For any compact connected set $K \subset \overline{\Omega}$ such that ψ is constant on K, we have $\Phi(x) = \Phi(y)$ for all $x, y \in K \setminus A_w$.*

We omit the elegant but delicate proof in Korobkov [**107**]. See [**108**] for more details.

We don't know if Φ is continuous on $\overline{\Omega}$. However, the restriction of Φ on any element of T_ψ can be defined by Theorem 7.9, and it can be shown that the restriction of Φ to any Kronrod arc $\phi(t)$, $\Phi(\phi(t))$, is continuous.

With the above preparations, we can now sketch the proof of Theorem 7.5. By the same argument in Section 7.2, we have a sequence of vector fields u_j solving (7.24) and converging to a weak solution w of the Euler equations (7.28) in $H^1_0(\Omega)$.

Let $v_k = \frac{1}{N_k}(V + \hat{u}_k) = \frac{1}{N_k}V + u_k$. It is a weak solution of

$$-\nu_k\Delta v_k + (v_k\cdot\nabla)v_k + \nabla p_k = f_k, \quad \operatorname{div} v_k = 0, \quad v_k|_{\partial\Omega} = \frac{v_*}{N_k}, \tag{7.37}$$

with $\nu_k = \frac{\nu}{\sigma_k N_k}$. Here f_k has the simple form $f_k = \frac{f}{\sigma_k N_k^2}$ if we took V to be the solution of (see Problem 7.2)

$$-\nu V + \nabla q = f, \quad \operatorname{div} V = 0, \quad V|_{\partial\Omega} = v_*.$$

Note that $\|\nabla v_k\|_2 \to 1$. We can solve p_k in (7.37) so that the norms $\|v_k\|_{H^1(\Omega)}$ and $\|p_k\|_{W^{1,s}(\Omega)}$ are uniformly bounded in k for all $s \in [1, 2)$, and

$$v_k \rightharpoonup w \quad \text{in } H^1(\Omega), \quad p_k \rightharpoonup \pi \quad \text{in } W^{1,s}(\Omega).$$

The weak forms of (7.37) converge to (7.27), the weak form of the Euler equations for w as $k \to \infty$.

Define the *total head pressure*

$$\Phi_k = \frac{1}{2}|v_k|^2 + p_k, \quad \Phi = \frac{1}{2}|w|^2 + \pi.$$

Note that
$$\Phi_k \rightharpoonup \Phi \quad \text{in } W^{1,s}(\Omega).$$
As in (7.32), they satisfy
$$-\Delta\Phi_k + \frac{1}{\nu_k}\operatorname{div}(v_k\Phi_k) = -\omega_k^2 + \frac{1}{\nu_k} f_k \cdot v_k \tag{7.38}$$
with $\omega_k = \partial_1 v_{k,2} - \partial_2 v_{k,1}$. If $f = 0$, then the Φ_k satisfy the classical *maximal principle* that
$$\operatorname*{ess\,sup}_{\Omega'} \Phi_k = \operatorname*{ess\,sup}_{\partial\Omega'} \Phi_k, \quad \forall C^2 \text{ open } \Omega' \Subset \Omega. \tag{7.39}$$
A weaker version of this property could be passed to Φ. We will use an "integral version" of the maximal principle which allows nonzero f.

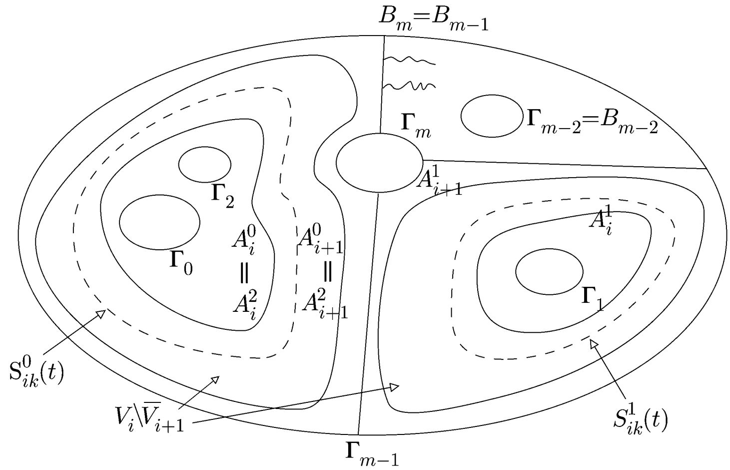
Note that $\Phi \in W^{1,s}(\Omega)$ and $\Phi|_{\Gamma_j} = \pi_j$ $\mathfrak{H}^1$-a.e. by Lemma 7.4. By assumption and (7.29), we have
$$\sum_{i=0}^m \mathcal{F}_i = 0, \quad \sum_{i=0}^m \pi_i \mathcal{F}_i = -\frac{\nu}{\bar{\sigma}}.$$
In particular, $\min \pi_j < \max \pi_j$. There are two cases:

(a) $\operatorname{ess\,sup}_\Omega \Phi = \max \pi_j$.

(b) $\operatorname{ess\,sup}_\Omega \Phi > \max \pi_j$. (In this case $\operatorname{ess\,sup}_\Omega \Phi = \infty$ is allowed.)

In case (a), we may add a constant and change the index so that
$$\max_{j \le \ell} \pi_j < 0 = \pi_{\ell+1} = \cdots = \pi_m = \operatorname*{ess\,sup}_\Omega \Phi,$$
for some $\ell < m$. Denote by B_j the element in T_ψ which contains Γ_j. Choose a small $t_0 > 0$ so that any $C \in \bigcup_{j\le\ell}[B_j, B_m]$ with $\Phi(C) = -t_i$, $t_i = 2^{-i}t_0$, for some $i \in \mathbb{N}$, is a regular cycle. For $j \le \ell$, let A_i^j be a member on the arc $[B_j, B_m]$ that is the closest to B_m with $\Phi(A_i^j) = -t_i$. The level sets A_i^j and $A_i^{j'}$ are either identical or disjoint. Let $A_i = \bigcup_{j=0}^{\ell} A_i^j$ and let V_i be the connected component of $\Omega \setminus A_i$ containing Γ_m. The boundary of V_i contains A_i and part of $\bigcup_{k=\ell+1}^m \Gamma_k$. Changing the index if necessary, there is some $K \in (\ell, m]$ so that $\partial V_i = A_i \cup \Gamma$, $\Gamma = \bigcup_{k=K}^m \Gamma_k$.

Now for $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$ let $W_{ik}(t)$ be the superlevel set for Φ_k,
$$W_{ik}(t) = \left\{x \in V_i \setminus \overline{V}_{i+1} : \Phi_k(x) > -t\right\},$$
and let $S_{ik}(t)$ be its boundary inside $V_i \setminus \overline{V}_{i+1}$,
$$S_{ik}(t) = \partial W_{ik}(t) \setminus \overline{V}_{i+1}.$$
Then $\Phi_k(t) = -t$ $\mathfrak{H}^1$-a.e. on $S_{ik}(t)$. By the Morse-Sard theorem, Theorem 7.7, for a.e. $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$, $S_{ik}(t)$ consists of finitely many C^1-cycles and Φ_k is differentiable in the classical sense at any $x \in S_{ik}(t)$ with $\nabla\Phi_k(x) \neq 0$.



The key lemma now is that, for any $i \in \mathbb{N}$ and $k > k(i)$ sufficiently large,

$$\int_{S_{ik}(t)} |\nabla \Phi_k| ds < C_1 t, \quad a.e.\ t \in \left[\frac{5}{8}t_i, \frac{7}{8}t_i\right], \tag{7.40}$$

for some constant C_1. The idea is to integrate (7.38) over

$$U_h(t) = \{x \in V_i : \Phi_k(x) > -t, \operatorname{dist}(x, \Gamma) > h\}$$

for $0 < h \ll 1$, with $\partial U_h(t) = S_{ik}(t) \cup \Gamma_h$, $\Gamma_h = \{x \in \Omega : \operatorname{dist}(x, \Gamma) = h\}$. Let $\mathbf{n}$ be the outer normal of $U_h(t)$. We get

$$\begin{aligned} -\int_{S_{ik}(t)} \nabla \Phi_k \cdot \mathbf{n}\, ds &= -\frac{1}{\nu_k} \int_{S_{ik}(t)} \Phi_k v_k \cdot \mathbf{n}\, ds - \int_{U_h(t)} |\omega_k|^2 + \int_{U_h(t)} \frac{1}{\nu_k} f_k \cdot v_k \\ &\quad + \int_{\Gamma_h} \left(\nabla \Phi_k - \frac{1}{\nu_k}\Phi_k v_k\right) \cdot \mathbf{n}\, ds \\ &= \textstyle\sum_{j=1}^4 I_j < I_1 + I_3 + I_4, \end{aligned}$$

since evidently $I_2 < 0$. Note that

$$\int_{S_{ik}(t)} |\nabla \Phi_k| ds = -\int_{S_{ik}(t)} \nabla \Phi_k \cdot \mathbf{n}\, ds$$

and

$$I_1 = \frac{t}{\nu_k} \int_{S_{ik}(t)} v_k \cdot \mathbf{n}\, ds = \left(\frac{\sigma_k}{\nu} \textstyle\sum_{j=0}^{\ell} \mathcal{F}_j\right) t.$$

We have $I_2 < 0$ with a uniform-in-k upper bound since $\nabla \Phi = \omega \nabla \psi$ and Φ is not constant in V_{i+1}. We can then use I_2 and the smallness in the boundary conditions (7.37) to prove that I_3 and I_4 are negligible and to get (7.40). (The detailed proof is long.)

Now fix $k > k(i)$. Let $J_i = [\frac{5}{8}t_i, \frac{7}{8}t_i]$ and $E_i = \bigcup_{t\in J_i} S_{ik}(t)$, which is a union of level sets. Recall the *coarea formula*,

$$\int_{E_i} g|\nabla\Phi_k|dx = \int_{J_i}\int_{S_{ik}(t)} g(x)d\mathfrak{H}^1(x)\,dt. \tag{7.41}$$

Taking $g = |\nabla\Phi_k|$, we get using (7.40)

$$\int_{E_i} |\nabla\Phi_k|^2 = \int_{J_i}\int_{S_{ik}(t)} |\nabla\Phi_k|d\mathfrak{H}^1(x)dt \le \int_{J_i} C_1 t dt = \frac{3}{16}C_1 t_i^2.$$

Taking $g = 1$, we get

$$\int_{J_i} \mathfrak{H}^1(S_{ik}(t))dt = \int_{E_i} |\nabla\Phi_k| \le \left(\int_{E_i} |\nabla\Phi_k|^2\right)^{\frac{1}{2}} |E_i|^{\frac{1}{2}} \le Ct_i|E_i|^{\frac{1}{2}}. \tag{7.42}$$

The coarea formula (7.41) requires that g is Lebesgue integrable, the set E_i is measurable, and Φ_k belongs to the Sobolev space $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$; see, e.g., [**144**]. These conditions can be verified. Since $S_{ik}(t)$ separates Γ_j from Γ_m, $j \le \ell$, we have

$$\mathfrak{H}^1(S_{ik}(t)) \ge \min_{j\le m} \operatorname{diam}\Gamma_j.$$

Thus (7.42) gives

$$Ct_i \le Ct_i|E_i|^{1/2},$$

which is a contradiction since $|E_i| \to 0$ as $i \to \infty$.

For case (b) that $\operatorname{ess\,sup}_\Omega \Phi > \max \pi_j$, we can find a regular cycle F with $\Phi(F) > \max \pi_j$. Add a constant to π so that $\Phi(F) = 0$. We can use F to replace the role of Γ_m in the previous argument to derive a similar contradiction.

The contradiction in both cases is due to the assumption that there is no uniform a priori bound for solutions of (7.23) for $\sigma \in [0, 1]$. This proves Theorem 7.5 and resolves the general 2D bounded domain case.

The following 3D case, with no assumption on symmetry or the fluxes $\mathcal{F}_i$, is a significant open problem.

Conjecture 7.10. *Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain. If $f \in H^1(\Omega)$ and if $v_* \in H^{3/2}(\partial\Omega)$ satisfies $\int_{\partial\Omega} v_*\cdot\mathbf{n} = 0$, does the boundary value problem* (7.1) *admits at least one weak solution?*

Recall the observation of Amick at the end of Section 7.2 that Φ satisfies some (weak) version of the maximal principle. This is an essential gradient used in [**108**] and is replaced by a weaker version in [**110**] presented in this section, still using (7.38). It is reasonable to expect that any attempt at the conjecture will involve the use of the maximal principle.

7.4. The bifurcation problem and degree

A *bifurcation* is referred to as the qualitative change of some structure of a family in a system with parameters, as the values of the parameters vary. The values of parameters where this change takes place are called *bifurcation values*. In the context of ODE systems or fluid PDEs, the family under consideration is often the family of fixed points / steady states, or periodic solutions, and we may be interested in the bifurcation of their number and/or their stability property.

Consider the incompressible Navier-Stokes equations in $\Omega \subset \mathbb{R}^n$ for $(v, p) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^n \times \mathbb{R}$

$$\partial_t v - \Delta v + (v \cdot \nabla)v + \nabla p = f^{(\alpha)}, \quad \operatorname{div} v = 0, \tag{7.43}$$

coupled with the boundary condition

$$v|_{\partial\Omega} = v_*^{(\alpha)}, \tag{7.44}$$

for some force $f^{(\alpha)}$ and boundary value $v_*^{(\alpha)}$, depending continuously on a parameter $\alpha \in \mathbb{R}$. We may ask at which value of α the number and stability of steady states change. Consider the special case

$$f^{(\alpha)} = \alpha f, \quad v_*^{(\alpha)} = \alpha v_*, \tag{7.45}$$

where f and v_* are fixed smooth vector fields defined in Ω and $\partial\Omega$, respectively, satisfying the compatibility condition

$$\int_{\partial\Omega} v_* \cdot \mathbf{n} = 0.$$

Suppose $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a bounded domain with smooth boundary $\partial\Omega$ and unit outward normal vector $\mathbf{n}$, and let $\partial\Omega$ be the disjoint union of $m+1$ connected components Γ_j, $j = 0, \ldots, m$. Let $\mathcal{F}_j = \int_{\Gamma_j} v_* \cdot \mathbf{n}$. The following are known:

- If $n = 2$, for every $\alpha \geq 0$ there is at least one stationary solution (v_α, p_α). This is also true for $n = 3$ if one further assumes

$$\mathcal{F}_j = 0 \quad \text{for all } j. \tag{7.46}$$

- For α sufficiently small, $\alpha \in [0, \alpha_0)$ for some $\alpha_0 > 0$, there is a stationary solution, which is unique and stable. This does not assume (7.46). For $\alpha \gg 1$, one does not expect uniqueness.
- The existence problem for $n = 3$ and large α without assuming (7.46) is open.

We may talk about the solution family

$$\alpha \mapsto (v_\alpha, p_\alpha)$$

at least for $\alpha \in [0, \alpha_0)$. We may show that this family is continuous in α and study its stability by considering its *linearized operator*.

Specifically, suppose we already have a steady state $w = v_{\alpha_1}$ of (7.43)–(7.44) with general $v_*^{(\alpha)}$ and $f^{(\alpha)}$ at $\alpha = \alpha_1$ and we want to solve v_α for α near α_1. Let V be the solution of

$$-\Delta V + \nabla p = f^{(\alpha)} - f^{(\alpha_1)}, \quad \operatorname{div} V = 0,$$
$$V|_{\partial\Omega} = v_*^{(\alpha)} - v_*^{(\alpha_1)}.$$

It vanishes as $\alpha \to \alpha_1$. Decompose

$$v = w + V + u.$$

Then u satisfies

$$-\Delta u + \operatorname{div}[w \otimes u + u \otimes w + F(u)] + \nabla \pi = 0, \quad \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0,$$

where

$$F(u) = (V + u) \otimes u + u \otimes V + (w + V) \otimes V + V \otimes w.$$

Let $\mathbf{V} = H^1_{0,\sigma}(\Omega)$ and define $L_w : \mathbf{V} \to \mathbf{V}'$,

$$\langle L_w u, \zeta \rangle = \int_\Omega \nabla u \cdot \nabla \zeta - (w \otimes u + u \otimes w) : \nabla \zeta, \quad \forall \zeta \in \mathbf{V}.$$

Then $u \in \mathbf{V}$ satisfies the weak form

$$\langle L_w u, \zeta \rangle = \int_\Omega F(u) : \nabla \zeta, \quad \forall \zeta \in \mathbf{V}.$$

If L_w is invertible, then for α sufficiently close to α_1 we can solve u.

For a steady state $w \in \mathbf{V}$, let $\sigma(L_w)$ denote the spectrum of L_w and let

$$\gamma(w) = \inf \{\operatorname{Re} z : z \in \sigma(L_w)\}.$$

We say w is *linearly stable* if $\gamma(w) > 0$, which is strong evidence (but not a proof) for the stability of w under the nonlinear equation (7.43).

Now we go back to the special data (7.45). For α small, $\gamma(v_\alpha) > 0$ and hence L_{v_α} is invertible. Since we can solve v_α locally if L_{v_α} is invertible, we can extend the curve $\alpha \mapsto v_\alpha$ to $\alpha \in [0, \alpha_c)$ for some maximal $\alpha_c > \alpha_0$,

$$\alpha_c = \sup \{\alpha_1 > 0 : \ \gamma(v_\alpha) > 0 \text{ for } \alpha \in (0, \alpha_1)\}.$$

If α_c is finite, we must have

$$\lim_{\alpha \to \alpha_c -} \gamma(v_\alpha) = 0.$$

Note that $\{v_\alpha : 0 < \alpha < \alpha_c\}$ is a continuous family although uniqueness of stationary solutions may be lost for $\alpha > \alpha_1$ for some $\alpha_1 \in (\alpha_0, \alpha_c)$.

As $\alpha \to \alpha_c-$, some eigenvalue $\lambda(\alpha)$ of L_{w_α} goes across the imaginary axis. If $\lambda(\alpha_c) = 0$, we may have a *saddle-node bifurcation* and get additional steady states $\tilde{v}_\alpha$ for $\alpha > \alpha_c$ or $\alpha < \alpha_c$. If $\lambda(\alpha_c) = ir$, $r > 0$, we may have a *Hopf bifurcation* and get additional periodic solutions. We also expect a change of stability of v_α as α goes across α_c. See [**90**] for a treatise for general data. The authors made some assumptions on $\lambda(\alpha)$ to ensure bifurcation. It seems very hard to verify those assumptions for the special data (7.45).

The nonuniqueness example of Yudovich [**224**], presented in Section 2.5, gives an example of bifurcation. Let us reformulate it here. The domain Ω is a solid of revolution in $\mathbb{R}^3$, not intersecting the x_3-axis. The parameter is $\alpha \in \mathbb{R}$. The background vector field $V = -cr^{-3}e_\theta$ in cylindrical coordinates is a steady state for $\alpha = 0$ of the system

$$\begin{gathered} -\nu\Delta v + v \cdot \nabla v + \nabla p = f^{(\alpha)}, \quad \operatorname{div} v = 0 \quad \text{in } \Omega, \\ v|_{\partial\Omega} = v_*, \end{gathered} \tag{7.47}$$

where $v_* = V|_{\partial\Omega}$ is independent of α and

$$f^{(\alpha)} = -\nu\Delta V + V \cdot \nabla V + \alpha(u \cdot \nabla)u.$$

The vector field u is a 0-eigenfunction of L_V, $L_V u = 0$ (in $\mathbf{V}'$). In particular, L_V is not invertible. For $0 < \alpha < \infty$, we have two families of solutions of (7.47),

$$v_{\pm,\alpha} = V \pm \sqrt{\alpha}u.$$

There is probably no solution to (7.47) for $\alpha < 0$, but there is no proof of that.

We now quote a theorem on bifurcation from a simple eigenvalue, due to Crandall-Rabinowitz [**37**].

Theorem 7.11. *Let $X \subset Z$ be real Banach spaces, and let $B, A : X \to Z$ be bounded linear operators. Let U be an open subset of X, and let I be an open interval. Let $M \in C^2(U \times I; Z)$ be defined by*

$$\begin{gathered} M(x, \lambda) = Bx - \lambda Ax + N(x, \lambda), \\ N(0, \lambda) = 0, \quad D_x N(0, \lambda) = 0. \end{gathered}$$

Suppose $\lambda_0 \in I$ is a simple eigenvalue of the pair (B, A) in the sense that, for some $y_0 \in X$,

$$\begin{gathered} \dim \ker(B - \lambda_0 A) = 1 = \operatorname{span}\{y_0\}, \\ \dim \operatorname{range}(B - \lambda_0 A) = 1, \quad Ay_0 \notin \operatorname{range}(B - \lambda_0 A). \end{gathered} \tag{7.48}$$

Then there exist C^1-functions

$$x^*(u) = uy_0 + O(u^2), \quad \lambda^*(u) = \lambda_0 + O(u)$$

for small real u, such that $M(x^*(u), \lambda^*(u)) = 0$. They and $\{(0, \lambda) : \lambda \sim \lambda_0\}$ are the only solutions of $M(x, \lambda) = 0$ in a neighbourhood of $(0, \lambda_0)$.

Its proof is by the Liapunov-Schmidt method. See [**37**] or [**35**, §5.5].

The case with a nonsimple eigenvalue is more complicated. Consider the following examples when $X = Z = \mathbb{R}^2$, $\lambda_0 = 0$, and the kernel of $B - \lambda_0 A$ is the entire $\mathbb{R}^2$:

(1) $M(x, \lambda) = (\lambda - x^2)x$. The nonzero solutions of $M(x, \lambda) = 0$ are, for each $\lambda > 0$, the circle $|x| = \sqrt{\lambda}$, i.e., the paraboloid $\lambda = x^2$.

(2) $M(x, \lambda) = \lambda x - |x|^2(x_1, \alpha x_2)$, $\alpha > 1$. There are exactly four nonzero solutions for each $\lambda > 0$:

$$(\pm\sqrt{\lambda}, 0), \quad (0, \pm\sqrt{\lambda/\alpha}).$$

(3) $M(x, \lambda) = \lambda x + (x_2^3, -x_1^3)$. A solution satisfies

$$\lambda x_1 + x_2^3 = 0, \quad \lambda x_2 - x_1^3 = 0.$$

The difference of the first equation multiplied by x_2 and the second by x_1 is $x_1^4 + x_2^4 = 0$. Thus there is no bifurcation.

These examples show that the bifurcation depends on the nonlinearity. See [**35**, Chapter 7] for the case of quadratic and cubic nonlinearities.

For the study of bifurcation, it is useful to consider the *Leray-Schauder degree* of a map in a region and the *index* of an isolated fixed point.

Let ω be a bounded open set in a Banach space X, and let ϕ be a continuous map from $\bar{\omega}$ to X. For $b \in X \setminus \phi(\partial\omega)$, the degree $\deg(\phi, \omega, b)$ is intuitively the (signed) number of solutions of

$$\phi(u) = b, \quad u \in \omega.$$

When X is finite dimensional, we first define

$$\deg(\phi, \omega, b) = \sum_{u \in \omega \cap \phi^{-1} b} \operatorname{sign}(\det \phi'(u))$$

when $\phi \in C^1$ and b is a regular value; i.e., $\phi'(u)$ is invertible for every $u \in \omega \cap \phi^{-1} b$. By *Sard's lemma*, singular values have zero Lebesgue measure. One can hence define $\deg(\phi, \omega, b)$ for general ϕ and b by approximation. This is the *Brouwer degree*. It satisfies the following properties and is uniquely determined by them:

(P1) Normalizaton: $\deg(I, \omega, b) = 1$ for $b \in \omega$.

(P2) Additivity: If ω_1 and ω_2 are disjoint open subsets of ω and $b \notin \phi(\bar{\omega} \setminus (\omega_1 \cup \omega_2))$, then $\deg(\phi, \omega_1, b) + \deg(\phi, \omega_2, b) = \deg(\phi, \omega, b)$.

(P3) Homotopy: Let $H \in C([0, 1] \times \bar{\omega}, \mathbb{R}^N)$ be a homotopy. If $b(t) \in C([0, 1]; \mathbb{R}^N)$, $b(t) \notin H(t, \partial\omega)$ for every t, then $\deg(H(t, \cdot), \omega, b(t))$ is constant in t.

When X is infinite dimensional, we only consider maps of the form $\phi = I - K$ where K is compact. Being compact, on any bounded open set ω we can approximate K by maps K_ε whose ranges are finite dimensional, $\sup_{u\in\omega} \|Ku - K_\varepsilon u\| \le \varepsilon$ and range $K_\varepsilon \hookrightarrow \mathbb{R}^N$. We can then use it to define the degree,

$$\deg(I - K, \omega, b) = \deg(I - K_\varepsilon|_{\omega\cap\mathbb{R}^N}, \omega \cap \mathbb{R}^N, b)$$

for sufficiently small ε. This is the *Leray-Schauder degree*. It satisfied properties (P1) and (P2) above and the revised homotopy property: Suppose $K \in C([0,1]\times\bar{\omega}, X)$ and $K(t,\cdot)$ is compact for every t. If $b(t) \in C([0,1]; \mathbb{R}^N)$, $b(t) \ne u - K(t,u)$ for $(t,u) \in [0,1] \times \partial\omega$, then $\deg(I - K(t,\cdot), \omega, b(t))$ is constant in t.

The following is a corollary.

Theorem 7.12 (Leray-Schauder)**.** *Let X be a Banach space, let $K \in C([0,1] \times \bar{\omega}, X)$, and let $K(t,\cdot)$ be compact for every t. If all solutions of*

$$u = K(t,u), \quad 0 \le t \le 1,$$

satisfy $\|u\|_X \le M$ for some M and if $u = K(0,u)$ has exactly one solution, then $u = K(t,u)$ has at least one solution for every $\sigma \in (0,1]$.

It is shown by taking $\omega = B_{M+\varepsilon}$ for some $\varepsilon > 0$ and

$$\deg(I - K(t,\cdot), \omega, 0) = \deg(I - K(0,\cdot), \omega, 0) = 1.$$

The special case when $K(t,u) = tK_0(u)$ is Theorem 2.8.

We now consider the index. Suppose x_0 is an isolated fixed point of $\phi = I - K$ on a real Banach space with K being compact. We define the index

$$i(\phi, x_0) = \deg(\phi, B_\varepsilon(x_0), 0),$$

where $\varepsilon > 0$ is so small that x_0 is the only zero of ϕ in $\bar{B}_r(x_0)$. If $0 \notin \phi(\partial\omega)$ and $\phi(x) = 0$ has isolated solutions $x_1, \dots, x_k$ in ω, then

$$\deg(\phi, \omega, 0) = \textstyle\sum_{j=1}^k i(\phi, x_j). \tag{7.49}$$

Suppose K is C^1 and $T = K'(x_0)$. Then T is also compact. If in addition $I - T$ is invertible, then zero is the unique zero of $I - T$ and, by taking the homotopy $H(t,x) = x - \frac{1}{t}K(tx)$ and $H(0,x) = x - Tx$, we get

$$i(\phi, x_0) = \deg(I - T, B_\varepsilon(x_0), 0) = i(I - T, 0).$$

Since T is compact, its eigenvalues form a sequence which tends to 0, and for each nonzero eigenvalue λ^{-1}, the generalized eigenspace

$$E_\lambda = \bigcup_{p=1}^{\infty} \ker((I - \lambda T)^p)$$

is finite dimensional. If T is a linear compact operator and $I-\lambda T$ is invertible, then $i(I-\lambda T,0)$ is defined and $i(I-\lambda T,0)=\pm 1$. If $I-\lambda T$ is invertible for λ in an interval J, then $i(I-\lambda T,0)$ is constant for $\lambda\in J$. If $I-\lambda_1 T$ is not invertible, then

$$\lim_{\lambda\to\lambda_1-} i(I-\lambda T,0)=(-1)^\beta \lim_{\lambda\to\lambda_1+} i(I-\lambda T,0),\quad \beta=\dim E_\lambda. \tag{7.50}$$

7.5. Bifurcation of the Rayleigh-Bénard convection

The Rayleigh-Bénard convection is a type of convection of fluid in between two horizontal planes, due to heating from below. When the temperature difference between the top and the bottom is significant enough, one can see a regular pattern of convection cells known as *Bénard cells*. It was obtained experimentally by Henri Bénard in 1900 and analyzed by Lord Rayleigh in 1916. It was later analyzed with the correct boundary conditions by J. Pearson in 1958. The pattern of Bénard cells can be understood as a bifurcation with the temperature difference being the parameter.

Mathematically, it is modeled by the coupled system of the Navier-Stokes equations and the heat equation with convection. There is a trivial solution where the fluid velocity is zero and the temperature depends linearly on the vertical variable x_3. The following *Boussinesq system* is a nondimensionalized form, with the trivial solution subtracted. Let $\Omega=D\times(0,1)$, where D is a 2-dimensional region. For $(u,T):\Omega\times\mathbb{R}_+\to\mathbb{R}^3\times\mathbb{R}$, the equations are

$$\begin{aligned}\partial_t u+u\cdot\nabla u+\nabla p-\Delta u-\lambda T e_3&=0,\\ \operatorname{div} u&=0,\\ \partial_t T+u\cdot\nabla T-\Delta T-\lambda u_3&=0,\\ u|_{\partial\Omega}=0,\qquad T|_{\partial\Omega}&=0,\end{aligned}\tag{7.51}$$

where $\lambda\ge 0$ is a parameter and $e_3=(0,0,1)$. We take the Dirichlet boundary condition, although it is not the most physical.

The system (7.51) has the trivial stationary solution $(u,T)=(0,0)$ for all λ, and we are looking for nonzero stationary solutions. Let

$$\begin{aligned}H&=\left\{(u,T)\in L^2(\Omega;\mathbb{R}^3\times\mathbb{R})\ \middle|\ \operatorname{div} u=0,\ u\cdot\mathbf{n}|_{\partial\Omega}=0\right\},\\ X&=\left\{(u,T)\in H_0^1(\Omega;\mathbb{R}^3\times\mathbb{R})\ \middle|\ \operatorname{div} u=0\right\}.\end{aligned}\tag{7.52}$$

The space H is equipped with the inner product

$$((u,T),(v,\tau))=\int_\Omega u\cdot v+T\tau\,dx.$$

We can take the Helmholtz projection P of the u-equation and reformulate the stationary system (7.51) as

$$M(\phi,\lambda)=L_\lambda\phi+N(\phi)=0,$$

where $M : X \times \mathbb{R} \to X'$, $L_\lambda = B - \lambda A$, and for $\phi = (u, T)$,

$$B\phi = (-P\Delta u, -\Delta T)\,, \quad A\phi = (P(Te_3), u_3),$$

$$N(\phi) = (P(u \cdot \nabla u), u \cdot \nabla T).$$

That is, for $\psi = (v, \tau) \in X$,

$$\begin{aligned}
\langle B\phi, \psi \rangle &= \int_\Omega \nabla u : \nabla v + \nabla T \cdot \nabla \tau, \\
\langle A\phi, \psi \rangle &= \int_\Omega T v_3 + \tau u_3, \\
\langle N(\phi), \psi \rangle &= \int_\Omega (u \cdot \nabla) u \cdot v + (u \cdot \nabla) T \cdot \tau.
\end{aligned}$$

Note that $\langle N(\phi), \phi \rangle = 0$.

Clearly, B and A define symmetric forms in X, and the operator $L_\lambda = B - \lambda A$ is selfadjoint. Since B is positive, the generalized eigenvalue problem

$$B\phi = \lambda A\phi \tag{7.53}$$

has increasing positive eigenvalues. Let $\lambda_1 > 0$ be the smallest eigenvalue and let E_1 be its associated eigenspace. Since L_λ is selfadjoint, the condition (7.48) of λ_1 being simple is reduced to that of E_1 having dimension 1. In that case, Theorem 7.11 guarantees a bifurcation at $\lambda = \lambda_1$.

In fact, for our problem we can identify the leading terms of the bifurcation solutions. Assume λ_1 is a simple eigenvalue of (7.53) with a normalized eigenvector $\xi \in X$, $(\xi, \xi) = 1$. By Theorem 7.11, the bifurcation solutions may be decomposed as

$$\phi = x\xi + \eta, \quad \eta \perp \xi, \quad \lambda = \lambda_1 + t,$$

and for some small $\varepsilon > 0$,

$$-\varepsilon < x < \varepsilon, \quad \|\eta\|_X = O(x^2), \quad t = O(x).$$

Denote $L = L_{\lambda_1}$. Let $\xi^\perp$ be the orthogonal complement of ξ in H, and let Π be the projection from H to $\xi^\perp$,

$$\Pi f = f - (f, \xi)\xi.$$

The equation $0 = M(\phi, \lambda) = L\eta - tA\phi + N(\phi)$ is equivalent to the system

$$L\eta = t\Pi A\phi - \Pi N(\phi), \tag{7.54}$$

$$tx(A\xi, \xi) = (N(\phi), \xi) - t(A\eta, \xi). \tag{7.55}$$

In (7.55), $(A\xi,\xi)=\lambda_1^{-1}(B\xi,\xi)>0$, and we claim that the right side is of order $O(x^3)$. Denote $Q(u,T,v,\tau)=(u\cdot\nabla v,u\cdot\nabla\tau)$. Then $N(\phi)=Q(\phi,\phi)$ and

$$(Q(\phi,\psi),\eta)=-(Q(\phi,\eta),\psi),\quad (Q(\phi,\psi),\psi)=0,$$

for any $\phi,\psi,\eta\in X$. Thus

$$(N(\phi),\xi)=(Q(\phi,x\xi+\eta),\xi)=(Q(\phi,\eta),\xi)=x(Q(\xi,\eta),\xi)+(Q(\eta,\eta),\xi),$$

with leading term $x(Q(\xi,\eta),\xi)=O(x^3)$. In particular, we have the refined bound for t from (7.55) that

$$t=O(x^2).$$

Since L is invertible in $\xi^\perp$, from (7.54) we can solve

$$\eta=-L^{-1}\Pi N(\phi)+L^{-1}t\Pi A\phi=-x^2L^{-1}\Pi N(\xi)+O(x^3).$$

Equation (7.55) divided by x gives

$$\begin{aligned} t(A\xi,\xi)&=(Q(\xi,\eta),\xi)+O(x^3)=-(Q(\xi,\xi),\eta)+O(x^3)\\ &=x^2(N(\xi),L^{-1}\Pi N(\xi))+O(x^3).\end{aligned}\tag{7.56}$$

The coefficient $(N(\xi),L^{-1}\Pi N(\xi))$ is nonnegative and is zero only in the unlikely case $\Pi N(\xi)=0$, which together with $(N(\xi),\xi)=0$ implies $N(\xi)=0$. Assuming it is positive, we get

$$x^2=a^2t+O(x^3),\quad a=\frac{(A\xi,\xi)^{1/2}}{(N(\xi),L^{-1}\Pi N(\xi))^{1/2}}.$$

Thus the solution exists only if $t=\lambda-\lambda_1>0$ and, in this case,

$$x=\pm a\sqrt{t}+O(t),$$

and we have two nonzero solutions

$$\phi_\pm=\pm a(\lambda-\lambda_1)^{1/2}\psi+O(\lambda-\lambda_1).\tag{7.57}$$

This corresponds to a *pitchfork bifurcation.*

We now study the kernel of L_λ. Suppose $\phi=(u,T)$ is in the kernel. It satisfies the generalized eigenvalue problem (7.53); that is,

$$\begin{aligned} -\Delta u_1+\partial_1 p&=0,\\ -\Delta u_2+\partial_2 p&=0,\\ -\Delta u_3+\partial_3 p&=\lambda T,\\ -\Delta T&=\lambda u_3,\\ \operatorname{div} u&=0.\end{aligned}\tag{7.58}$$

To get explicit solutions, we will relax the boundary condition and only require that $u\cdot\mathbf{n}=0$ on $\partial\Omega$ for the moment.

From the first two equations of (7.58), there is a potential ϕ so that

$$u_1 = \partial_1 \phi, \quad u_2 = \partial_2 \phi, \quad p = \Delta \phi.$$

From the last equation of (7.58), we get

$$\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2 = -\Delta_2 \phi, \quad \Delta_2 = \partial_1^2 + \partial_2^2. \tag{7.59}$$

We use separation of variables and assume

$$u_3 = f(x')h(x_3), \quad \phi = F(x')H(x_3).$$

Then (7.59) becomes

$$fh' = -(\Delta_2 F)H.$$

Thus we can take $H = h'$ and $f = -\Delta_2 F$. We now choose $f(x')$ to be eigenfunctions of Δ_2 in D,

$$-\Delta_2 f = a^2 f, \quad F = a^{-2} f. \tag{7.60}$$

We get

$$u_3 = fh, \quad \phi = a^{-2} fh', \quad (u_1, u_2) = a^{-2} h' \nabla' f,$$

where $\nabla' = (\partial_1, \partial_2)$. The condition $u \cdot \mathbf{n} = 0$ on Γ implies $\partial_{\mathbf{n}} f = 0$; i.e., f satisfies the Neumann boundary condition.

Note that, for $\theta = \theta(x_3)$, we have

$$\Delta(f\theta) = (\Delta_2 f)\theta + f\theta'' = fS\theta, \qquad \text{where } S = \partial_3^2 - a^2.$$

Thus $p = \Delta\phi = a^{-2} fSh'$. In view of the fourth equation of (7.58), we may assume

$$T = f\theta(x_3), \quad -S\theta = \lambda h.$$

Dividing the third equation of (7.58) by f, we get

$$-Sh + a^{-2} Sh'' = \lambda\theta; \quad i.e., \quad S^2 h = \lambda a^2 \theta.$$

Applying S to the last equation and using $S\theta = -\lambda h$, we get

$$S^3 h = \lambda a^2 S\theta = -\lambda^2 a^2 h. \tag{7.61}$$

By the zero boundary condition of u_3 at $x_3 = 0, 1$, we have

$$h(x_3) = \sin(k\pi x_3), \quad k \in \mathbb{N}. \tag{7.62}$$

Thus $Sh = (-k^2\pi^2 - a^2)h$, and (7.61) gives

$$(k^2\pi^2 + a^2)^3 = \lambda^2 a^2.$$

From here we can solve the eigenvalue λ of (7.58),

$$\lambda(a, k) = \frac{1}{a}(k^2\pi^2 + a^2)^{3/2}.$$

Its eigenfunction (u, T) for f, h given by (7.60) and (7.62) is

$$u = (a^{-2}h'\nabla' f, fh), \quad T = \frac{1}{a}(k^2\pi^2 + a^2)^{1/2} fh. \tag{7.63}$$

Now assume D is a rectangle; $D = [0, L_1] \times [0, L_2]$. Eigenfunctions of Δ_2 in D with Neumann boundary condition are of the form

$$f(x_1, x_2) = \cos(a_1 x_1) \cdot \cos(a_2 x_2), \quad a_j = \frac{k_j \pi}{L_j}, \quad k_j \in \mathbb{N}_0, \quad j = 1, 2. \tag{7.64}$$

In this case, the boundary conditions satisfied by (7.63) are

$$T = 0 \quad \text{on } x_3 = 0, 1; \quad \frac{\partial T}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma,$$

and the *Navier boundary condition* for u

$$u \cdot \mathbf{n} = 0, \quad \frac{\partial u}{\partial \mathbf{n}} \times \mathbf{n} = 0.$$

Moreover, $a^2 = a_1^2 + a_2^2 = \pi^2(\frac{k_1^2}{L_1^2} + \frac{k_2^2}{L_2^2})$, and

$$\lambda(a(k_1, k_2), k) = g_k(\alpha) := \pi^2 \frac{(k^2 + \alpha)^{3/2}}{\sqrt{\alpha}}, \quad \alpha = \frac{k_1^2}{L_1^2} + \frac{k_2^2}{L_2^2}.$$

The smallest eigenvalue λ is

$$\lambda_1 = \min_{k_1, k_2 \in \mathbb{N}_0} g_1(\alpha). \tag{7.65}$$

The function $g_1(\alpha)$ is unbounded as $\alpha \to 0_+$ and as $\alpha \to \infty$. Its minimum over $\alpha \in \mathbb{R}_+$ is attained at $\alpha = 1/2$. For general L_1 and L_2 we can estimate the dimension of the eigenspace E_1 of λ_1. In the special case that $L_1 = L_2 = 2p$ and p is an integer, λ_1 is attained at $k_1 = k_2 = p$ with $\alpha = 1/2$. Hence $\lambda_1 = g_1(1/2)$ is an eigenvalue with (u, T) given by (7.63) and

$$f(x') = \cos \frac{\pi x_1}{2} \cdot \cos \frac{\pi x_2}{2}, \quad h(x_3) = \sin(\pi x_3).$$

Is λ_1 simple? The answer is yes if p is a prime and $p \equiv 3 \bmod 4$, because in this case $(k_1, k_2) = (p, p)$ is the unique positive integer solution of

$$k_1^2 + k_2^2 = 2p^2. \tag{7.66}$$

See Problem 7.3. Because λ_1 is simple, Theorem 7.11 suggests a bifurcation at $\lambda = \lambda_1$. We do not have a proof yet because we have different boundary conditions and need to change the function spaces.

For other choices of L_1 and L_2, λ_1 may not be simple and one may get sophisticated bifurcation patterns. See [**142**, Chapter 4].

We can repeat the above calculation for a disk D instead of a rectangle. In that case, we use Bessel functions instead of (7.64) to write eigenfunctions. See Problem 7.4.

7.6. Bifurcation of Couette-Taylor flows

We briefly explained the experimental facts about the *Couette-Taylor flow* in Section 2.5. A fluid is inside two coaxial cylinders. The outer cylinder is fixed while the inner cylinder has angular velocity α, which serves as the parameter of the system. There is a planar steady flow for all α, but it is stable only for small α. After α exceeds a certain threshold, we get a second steady solution which has vertical movement. This is the bifurcation to be considered in this section.

For flows in two concentric cylinders, it is natural to use the cylindrical coordinates (r, θ, z) with

$$x = (x_1, x_2, x_3) = (r\cos\theta, r\sin\theta, z),$$

and we consider *axisymmetric* flows v of the form

$$v = v_r(r, z)e_r + v_\theta(r, z)e_\theta + v_z(r, z)e_z,$$

where the components v_r, v_θ, v_z do not depend upon θ and the basis vectors e_r, e_θ, e_z are

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \quad e_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \quad e_z = (0, 0, 1).$$

For axisymmetric flows, the stationary Navier-Stokes system (7.1) with zero force becomes the following system using (2.32) and (2.34):

$$\begin{aligned} \left(-\nu\Delta + \frac{\nu}{r^2} + \tilde{v}\cdot\nabla\right)v_r - \frac{1}{r}v_\theta^2 + \partial_r p &= 0, \\ \left(-\nu\Delta + \frac{\nu}{r^2} + \tilde{v}\cdot\nabla\right)v_\theta + \frac{1}{r}v_r v_\theta &= 0, \\ (-\nu\Delta + \tilde{v}\cdot\nabla)\,v_z + \partial_z p &= 0, \\ \partial_r(rv_r) + \partial_z(rv_z) &= 0, \end{aligned} \tag{7.67}$$

where $p = p(r, z)$,

$$\tilde{v} = v_r e_r + v_z e_z, \quad \Delta = \partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2.$$

Note that the v_θ-equation does not involve p.

For the Couette-Taylor flow, we consider the system (7.67) in $r_1 < r < r_2$ for some $0 < r_1 < r_2 < \infty$, with

$$v|_{r=r_1} = \alpha e_\theta, \quad v|_{r=r_1} = 0, \tag{7.68}$$

and $\alpha \ge 0$. A planar solution is $(\alpha\bar{u}, \alpha^2\bar{p})$ where

$$\bar{u} = \bar{u}_\theta e_\theta, \quad \bar{u}_\theta(r) = \frac{r_1}{r_2^2 - r_1^2}\left(\frac{r_2^2}{r} - r\right), \quad \bar{p}(r) = \int_{r_1}^r \bar{u}_\theta(s)^2\frac{ds}{s}. \tag{7.69}$$

The ansatz for $\bar{u}_\theta$ is because all solutions $f(r)$ of

$$0 = f'' + \frac{1}{r}f' - \frac{1}{r^2}f = \left(\frac{1}{r}(rf)'\right)'$$

are of the form $f(r) = ar + b/r$.

Consider a perturbed solution $v = \alpha(\bar{u} + u)$, $p = \alpha^2(\bar{p} + q)$ with $u = u_r e_r + u_\theta e_\theta + u_z e_z$ being small. Let $\tilde{u} = u_r e_r + u_z e_z$ and $g = u_\theta$. Let

$$A = \partial_z^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}.$$

Let $\lambda = \alpha/\nu$. We get from (7.67)

$$\left(-\frac{1}{\lambda}A + \tilde{u}\cdot\nabla\right)u_r - \frac{1}{r}(2\bar{u}_\theta g + g^2) + \partial_r q = 0, \tag{7.70}$$

$$\left(-\frac{1}{\lambda}A + \tilde{u}\cdot\nabla\right)g + \frac{1}{r}u_r(r\bar{u}_\theta' + \bar{u}_\theta + g) = 0, \tag{7.71}$$

$$\left(-\frac{1}{\lambda}\Delta + \tilde{u}\cdot\nabla\right)u_z + \partial_z q = 0, \tag{7.72}$$

$$\partial_r(ru_r) + \partial_z(ru_z) = 0. \tag{7.73}$$

For (7.71) we have used $\tilde{u}\cdot\nabla\bar{u}_\theta = u_r\partial_r\bar{u}_\theta$. We require $u|_{r=r_1,r_2} = 0$.

By (7.72), there is a stream function f such that

$$u_r = \frac{1}{r}\partial_z(rf) = \partial_z f, \quad u_z = -\frac{1}{r}\partial_r(rf).$$

The product rf is unique up to a constant. It is single-valued since the rz-domain is simply connected.

Eliminating q by $\partial_z(7.70) - \partial_r(7.72)$, we get the system for f, g:

$$\begin{aligned} A^2 f + \lambda(a\partial_z g + M(f,g)) &= 0,\\ Ag + \lambda(b\partial_z f + N(f,g)) &= 0, \end{aligned} \tag{7.74}$$

where

$$\begin{aligned} M(f,g) &= -\partial_r((\partial_z f)Af) + \partial_z\left[\frac{1}{r}(\partial_r(rf))Af\right] + \frac{1}{r}\partial_z(g^2),\\ a(r) &= 2\bar{u}_\theta/r = \frac{r_1}{r_2^2 - r_1^2}\left(\frac{r_2^2}{r^2} - 1\right),\\ N(f,g) &= -\frac{1}{r}\partial_r(rg)\cdot\partial_z f + \frac{1}{r}\partial_r(rf)\cdot\partial_z g,\\ b(r) &= -\frac{1}{r}\partial_r(r\bar{u}_\theta) = \frac{2r_1}{r_2^2 - r_1^2}, \end{aligned} \tag{7.75}$$

with the boundary conditions

$$f = \partial_r f = g = 0 \quad \text{for } r = r_1 \text{ and } r = r_2.$$

We are looking for periodic-in-z solutions. Let $L > 0$ be the (unknown) z-period. We set the domain for (r, z) to be

$$\mathcal{O}_L = (r_1, r_2) \times (0, L)_{\text{per}}$$

where $(0, L)_{\text{per}}$ is a periodic interval, i.e., a circle, of length L. Let $L^2(\mathcal{O}_L)$ be equipped with the induced inner product

$$(u, v) = \int_{\mathcal{O}_L} uv\, rdr\, dz. \tag{7.76}$$

Let

$$H_0^k(\mathcal{O}_L) = \left\{ f \in H^k(\mathcal{O}_L) : \ \partial_r^j f = 0 \text{ at } r = r_1 \text{ and } r = r_2, \ \forall j < k \right\}.$$

The operator A is selfadjoint in $L^2(\mathcal{O}_L)$ with respect to (7.76) and

$$-(u, Av) = \int_{\mathcal{O}_L} \left(\partial_z u \cdot \partial_z v + \partial_r u \cdot \partial_r v + \frac{1}{r^2} uv \right) rdr\, dz. \tag{7.77}$$

Thus $\|u\|_{H_0^1}^2 \sim -(u, Au)$ for $u \in H_0^1(\mathcal{O}_L)$. For given $f \in L^2(\mathcal{O}_L)$, there is a unique solution $u \in H_0^1(\mathcal{O}_L)$ of

$$Au = f$$

by (7.77) and the Lax-Milgram theorem. Moreover, $\|u\|_{H^2} \lesssim \|f\|_{L^2}$ by the regularity theorem for elliptic equations; see [**1**, Chapter 8; **72**]. This shows that there is a unique bounded map

$$A^{-1} : \ L^2(\mathcal{O}_L) \to H_0^1 \cap H^2(\mathcal{O}_L).$$

Similarly, since the bilinear form $B_2(u, v) = \int_{\mathcal{O}_L} Au \cdot Av \ rdr\, dz$ is bounded and positive, there is a unique weak solution $u \in H_0^2(\mathcal{O}_L)$ of

$$A^2 u = f$$

by the Lax-Milgram theorem, which belongs to H^4 by the regularity theorem for elliptic equations. Thus there is a unique bounded map

$$(A^2)^{-1} : \ L^2(\mathcal{O}_L) \to H_0^2 \cap H^4(\mathcal{O}_L).$$

Note that $(A^2)^{-1} \neq (A^{-1})^2$.

Consider now the space for solutions (f, g) of (7.74),

$$V = (H^3 \cap H_0^2)(\mathcal{O}_L) \times (L^\infty \cap H_0^1)(\mathcal{O}_L).$$

Note that if $(f, g) \in V$, then

$$a\partial_z g + M(f, g),\ b\partial_z f + N(f, g) \in L^2(\mathcal{O}_L).$$

Thus the nonlinear map $K : V \to V$,

$$K(f,g) = \left(-(A^2)^{-1}(a\partial_z g + M(f,g)),\ -(A)^{-1}(b\partial_z f + N(f,g)\right),$$

is well-defined. Moreover it is a continuous and compact operator in V. The problem (7.74) is equivalent to

$$\phi = \lambda K\phi, \quad \phi \in V. \tag{7.78}$$

The zero function $\phi = (0,0)$ is a solution for all λ and is the unique solution for small λ since we have, for $\phi_i = (f_i, g_i) \in V$, $i = 1, 2$,

$$\|K(\phi_1) - K(\phi_2)\|_V \le C(1 + \|\phi_1\|_V + \|\phi_2\|_V)\|\phi_1 - \phi_2\|_V.$$

We hope to get a bifurcation at some critical λ.

It can be shown that the Fréchet differential of K at 0 is the operator

$$B : (f,g) \to (-(A^2)^{-1}(a\partial_z g),\ -A^{-1}(b\partial_z f)).$$

That is, for $\phi = (f,g) \in V$,

$$\frac{\|K\phi - B\phi\|_V}{\|\phi\|_V} \to 0, \quad \text{as} \quad \|\phi\|_V \to 0.$$

The operator B is a linear compact operator in V. To find bifurcation, we want to first find a solution to the eigenvalue problem

$$\phi = \lambda B\phi, \quad \phi \in V. \tag{7.79}$$

Consider a closed subspace $\tilde{V}$ of V containing those (f,g) with f odd in z and g even in z. This class is preserved by both B and K. Such a pair of functions has the Fourier expansion

$$f(r,z) = \sum_{n=1}^{\infty} f_n(r)\sin(n\sigma z), \quad g(r,z) = \sum_{n=0}^{\infty} g_n(r)\cos(n\sigma z)$$

with $\sigma = 2\pi/L$. The restriction of B on each Fourier subspace maps to itself and is denoted as B_n, with $(f_n', g_n') = B_n(f_n, g_n)$ solving

$$\begin{aligned} M_n^2 f_n'(r) &= n\sigma a g_n(r), \\ M_n g_n'(r) &= n\sigma b f_n(r), \\ M_n &= -\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{r^2} + (n\sigma)^2 \end{aligned} \tag{7.80}$$

with boundary conditions

$$f_n' = \partial_r f_n' = g_n' = 0 \quad \text{at } r = r_1 \text{ and } r = r_2.$$

The space is

$$(f_n, g_n) \in H_0^2(r_1, r_2) \times H_0^1(r_1, r_2).$$

Since the case $n = 0$ corresponds to constants in z, we may assume $n > 0$.

We note the following result.

Lemma 7.13. *The Green's functions $G_n(r,s)$ and $H_n(r,s)$ of M_n and M_n^2 with the above boundary conditions are positive for $(r,s) \in (r_1,r_2)^2$.*

It follows from Kirchgässner [**102**], where a more general result holds for $M = \alpha(r)\frac{d^2}{dr^2} + \beta(r)\frac{d}{dr} + \gamma(r)$ with $\alpha(r) < 0$ and $\gamma(r) > 0$.

The restriction of the eigenvalue problem (7.79) becomes

$$\begin{aligned} M_n^2 f_n(r) &= \lambda n\sigma a g_n(r), \\ M_n g_n(r) &= \lambda n\sigma b f_n(r), \end{aligned} \tag{7.81}$$

or

$$\begin{aligned} f_n(r) &= \lambda n\sigma \int_{r_1}^{r_2} G_n(r,s)(ag_n)(s)\,ds, \\ g_n(r) &= \lambda n\sigma \int_{r_1}^{r_2} H_n(r,s)(bf_n)(s)\,ds. \end{aligned} \tag{7.82}$$

Eliminating g_n,

$$f_n(r) = \mu \int_{r_1}^{r_2} T_n(r,s) f_n(s)\,ds, \tag{7.83}$$

where $\mu = \lambda^2$ and $T_n(r,s) = (n\sigma)^2 \int_{r_1}^{r_2} G_n(r,t)H_n(t,s)a(t)b\,dt$ is continuous on $[r_1,r_2]^2$ and positive in $(r_1,r_2)^2$. By the Krein-Rutman theorem, an extension of the Perron-Frobenius theorem (for square matrices with positive entries) to linear compact operators on Banach spaces, (7.83) has a simple eigenvalue $\mu_n > 0$, corresponding to an eigenfunction f_n which is continuous in $[r_1,r_2]$ and positive in (r_1,r_2). Moreover, any other eigenvalue is strictly larger than μ_n in absolute value. Let $\lambda_n = \sqrt{\mu_n}$ and let g_n be defined by (7.82). We now have an eigenfunction (f_n,g_n) of (7.82) with eigenvalue λ_n.

The following gives a lower bound for λ_n and shows $\lambda_n \to \infty$ as $n \to \infty$.

Lemma 7.14. $\lambda_n \geq \frac{2}{\max|a+b|} n^2\sigma^2$.

Proof. Let (f_n,g_n) be the eigenfunction of the n-th eigenvalue problem (7.81) with eigenvalue λ_n. Skipping the subscript $_n$, we have $M^2 f = \lambda n\sigma a g$ and $Mg = \lambda n\sigma b f$. Denote $(f,g) = \int_{r_1}^{r_2} fg\,rdr$ and $J = (M^2 f, f) + (Mg, g)$. On one hand,

$$\begin{aligned} J &= \|M_0 f\|^2 + 2(M_0 f, n^2\sigma^2 f) + n^4\sigma^4\|f\|^2 + (M_0 g, g) + n^2\sigma^2\|g\|^2 \\ &\geq n^4\sigma^4\|f\|^2 + n^2\sigma^2\|g\|^2. \end{aligned}$$

On the other hand,

$$I = \lambda n\sigma\left[(ag,f)+(bf,g)\right] \le \lambda n\sigma \max|a+b|(f,g)$$
$$\le \frac{1}{2}\lambda \max|a+b|(n^2\sigma^2\|f\|^2+\|g\|^2).$$

This proves the lemma. □

Thus there is a largest integer m so that $\lambda_m = \inf_{n\ge 1}\lambda_n$.

By replacing L by L/m, we may assume $\lambda_1 < \lambda_n$ for any $n > 1$. (The choice of L is otherwise arbitrary. We can choose any initial $L > 0$ and modify it as this step.)

Lemma 7.15. *λ_1 is an eigenvalue of B whose generalized eigenspace in $\tilde{V}$ has dimension* 1.

The generalized eigenspace contains those ϕ such that $(I-\lambda_1 B)^p\phi = 0$ for some $p \in \mathbb{N}$. Such a ϕ is an eigenfunction if $p = 1$.

Proof. Let $\phi_1 = (f_1(r)\sin(\sigma z), g_1(r)\cos(\sigma z))$ be the eigenfunction of eigenvalue λ_1, $f_1(r) > 0$ and $g_1(r) > 0$ for $r_1 < r < r_2$. We have

$$f_1 = Ig_1, \quad g_1 = Jf_1,$$

where I and J denote the integral operators on the right sides of (7.82). We want to show there is no ϕ such that $(I-\lambda_1 B)\phi = \phi_1$. Since λ_1 is an eigenvalue only in the first eigenmode, other modes of ϕ must vanish and $\phi = (f(r)\sin(\sigma z),\ g(r)\cos(\sigma z))$ with

$$f = Ig + f_1, \quad g = Jf + g_1.$$

Thus $f = IJf + Ig_1 + f_1 = Tf + 2f_1$, where T is the right side of (7.83) and $Tf_1 = f_1$. Since T is compact, by the Fredholm alternative its adjoint T^* with integral kernel $\mu_1 T_1^*(r,s) = \mu_1 T_1(s,r)$ also has an eigenfunction f_1^* with $T^* f_1^* = f_1^*$ and $f_1^*(r) > 0$ in (r_1, r_2). However,

$$0 < (2f_1, f_1^*) = ((I-T)f, f_1^*) = (f, (I-T^*)f_1^*) = 0.$$

Thus f_1 does not exist. □

We now return to the nonlinear problem (7.78). We want to find a nonzero solution in $\tilde{V}$. It is uniquely solvable for small λ. We have chosen L such that λ_1 is a simple eigenvalue of B in $\tilde{V}$ and is the minimum in absolute value. We may assume $\phi = 0$ is the only solution for any $\lambda \in [0, \lambda_1]$ since otherwise we are done.

Fix any $\varepsilon > 0$ and let ω be the open ball in $\tilde{V}$ centered at 0 with radius ε. We claim there is $\delta > 0$ so that (7.78) has no solution on the boundary $\partial\omega$ of ω for $\lambda \in [\lambda_1, \lambda_1+\delta]$: Otherwise there is a sequence $\phi_j = \lambda_j K\phi_j$ with $\|\phi_j\|_V = \varepsilon$ and $\lambda_j \to \lambda_1+$. Since K is compact, a subsequence, still denoted as ϕ_j, satisfies $K\phi_j \to g$ strongly for some g. Let $\phi := \lambda_1 g$. We have $\|\phi\|_V = \lambda_1 \lim_j \|K\phi_j\|_V = \lim_j \|\phi_j\|_V = \varepsilon$, and

$$\phi = \lambda_1 g = \lim_j \lambda_j K\phi_j = \lim \phi_j.$$

Thus both sides of $\phi_j = \lambda_j K\phi_j$ converge strongly and we get $\phi = \lambda_1 K\phi \neq 0$. It is a contradiction to the uniqueness at $\lambda = \lambda_1$.

Thus $\deg(I - \lambda K, \omega, 0)$ is well-defined and constant for $0 \leq \lambda \leq \lambda_1 + \delta$. Thus $\deg(I - \lambda K, \omega, 0) = \deg(I, \omega, 0) = 1$.

By possibly taking $\delta > 0$ smaller, λ_1 is the only value of $\lambda \in [0, \lambda_1 + \delta)$ for which $I - \lambda B$ is not inveritible. The index $i(I - \lambda B, 0)$ is thus defined for $\lambda \in [0, \lambda_1) \cup (\lambda_1, \lambda_1 + \delta)$ and is constant in both intervals. We have $i(I - \lambda B, 0) = 1$ for $0 \leq \lambda < \lambda_1$ by uniqueness. By (7.50) and Lemma 7.15, $i(I - \lambda B, 0) = -1$ in $(\lambda_1, \lambda_1 + \delta)$.

When $\lambda_1 < \lambda < \lambda_1 + \delta$, since $\deg(I - \lambda K, \omega, 0) \neq i(I - \lambda B, 0)$, there are nonzero solutions of $x - \lambda K x = 0$ by (7.49).

Summarizing, we have proved the following theorem.

Theorem 7.16. *Fix $0 < r_1 < r_2 < \infty$. For any $L_1 > 0$, there is a z-periodic solution of the stationary axisymmetric Navier-Stokes equations* (7.67) *with period $L = L_1/m$ for some $m \in \mathbb{N}$ and the boundary condition* (7.68) *for some $\alpha > 0$. It is different from the planar solution given by* (7.69).

The Taylor problem is a special case of a general fixed point problem in a Banach space,

$$x = \Phi(x, \lambda), \quad \Phi(0, \lambda) = 0,$$

where $\lambda \in (a, b)$ is the parameter and we want to find a bifurcation by studying the eigenvalue of $D_x\Phi(0, \lambda)$. For the Taylor problem, the nonlinear map $\Phi(x, \lambda) = \lambda K(x)$ is linear in λ and $D_x\Phi(0, \lambda) = \lambda B$. Hence it suffices to study the spectrum of the linear operator B. The situation is similar for the Yudovich example in Section 7.4 and the Bénard problem in Section 7.5. For general problems such as (7.43)–(7.44), it is much harder since we need to study the spectra of a family of linearized operators $D_x\Phi(0, \lambda)$, $\lambda \in (a, b)$.

7.7. Notes

Sections 7.1 and 7.2 are based on [**57**, IX]. Both approaches of proving the a priori bound (7.3) are initiated by Leray [**131**]. Example (7.18) on the annulus (7.17) and Lemma 7.4 are due to Amick [**4**].

Section 7.3 is based on [**108**] and [**111**].

In Section 7.4, Theorem 7.11 is due to Crandall-Rabinowitz [**37**]. Also see Chow-Hale [**35**, §5.5]. The part on degree theory is based on Nirenberg [**158**] and Ambrosetti-Arcoya [**3**].

Section 7.5 is based on Rayleigh [**167**], Ma-Wang [**142**, §4.1], and [**35**, §5.5]. See also Ukhovskii-Yudovich [**212**], Velte [**215**], and Rabinowitz [**166**].

Section 7.6 is due to Velte [**216**], and I followed Temam [**206**, II.4].

Problems

7.1. Prove the 1D Hardy inequality (7.11).

7.2. Let Ω be a bounded Lipshitz domain in $\mathbb{R}^n$, $n = 2, 3$. For any $\nu > 0$, $f \in H^{-1}(\Omega; \mathbb{R}^n)$, and $v_* \in W^{1/2,2}(\partial\Omega; \mathbb{R}^n)$ with $\int_{\partial\Omega} v_* \cdot \mathbf{n} = 0$, show that there is a *unique* weak solution $V \in H^1(\Omega; \mathbb{R}^n)$ of the Stokes system

$$-\nu\Delta V + \nabla\pi = f, \quad \operatorname{div} V = 0, \quad V|_{\partial\Omega} = v_*$$

satisfying $\|V\|_{H^1(\Omega)} \lesssim \|f\|_{H^{-1}} + \|v_*\|_{W^{1/2,2}(\partial\Omega)}$.

7.3. Show that if p is a prime and $p \equiv 3 \bmod 4$, then $(k_1, k_2) = (p, p)$ is the unique positive integer solution of

$$k_1^2 + k_2^2 = 2p^2.$$

7.4. Solve the eigenvalue problem (7.58) when D is a disk of radius L.

Chapter 8

Self-similar solutions

In this chapter we discuss self-similar and discretely self-similar solutions and their variants of the Navier-Stokes system (NS). Depending on their time dependence, they are called *stationary, backward,* and *forward*. The stationary ones give the asymptotics of stationary solutions near a singularity or the spatial infinity. The backward ones are candidates for possible singularities. The forward ones give typical asymptotics as the time approaches infinity. They will be discussed in this order in Sections 8.2–8.4, after the introduction in Section 8.1.

8.1. Self-similar solutions and similarity transform

In this section we introduce self-similar and discretely self-similar solutions and their variants. We will also present the similarity transform and Leray equations.

Recall the *scaling property* (1.21) of (NS): If $(v(x,t), p(x,t))$ is a solution with force $f(x,t)$, then for any $\lambda > 0$,

$$v^\lambda(x,t) = \lambda v(\lambda x, \lambda^2 t), \quad p^\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t) \tag{8.1}$$

is also a solution, with force $f^\lambda(x,t) = \lambda^3 f(\lambda x, \lambda^2 t)$. This is valid in the following three cases:

(1) stationary case: $x \in \mathbb{R}^3$, $v = v(x)$ is time independent,

(2) backward case: $(x,t) \in \mathbb{R}^3 \times (-\infty, 0)$,

(3) forward case: $(x,t) \in \mathbb{R}^3 \times (0, \infty)$.

The spatial domain $\mathbb{R}^3$ can be replaced by the half-space $\mathbb{R}^3_+$ or a cone-like domain.

We say v is *self-similar* (SS) if $v = v^\lambda$ for every $\lambda > 0$. For the stationary case, the value of $v(x)$ is decided by its value on the unit sphere, and

$$v(x) = \lambda(x)V(\lambda(x)x), \quad \lambda(x) = \frac{1}{|x|}, \tag{8.2}$$

with $V(x) = v(x/|x|)$, which can be considered either as a vector field defined on the unit sphere or a zero homogeneous vector field defined in $\mathbb{R}^3 \setminus \{0\}$. For the two time-dependent cases, the value of $v(x,t)$ is decided by its value at time ± 1, and we have

$$v(x,t) = \lambda(t)V(\lambda(t)x), \quad \lambda(t) = \frac{1}{\sqrt{\mu t}}, \tag{8.3}$$

where $V(x) = v(x, 1/\mu)$ and $\mu = \operatorname{sgn} t$; i.e., we take the parameter $\mu = 0$ for the stationary case, $\mu = -1$ for the backward case, and $\mu = 1$ for the forward case. We may also consider (8.3) for $\mu \in \mathbb{R}$; then μ carries a physical dimension. In all three cases, with zero force, the profile $V(x)$ satisfies the *stationary Leray equations*

$$-\Delta V - \frac{1}{2}\mu V - \frac{1}{2}\mu x \cdot \nabla V + (V \cdot \nabla)V + \nabla P = 0, \quad \operatorname{div} V = 0, \tag{8.4}$$

in $\mathbb{R}^3$ if $\mu \neq 0$ and in $\mathbb{R}^3 \setminus \{0\}$ if $\mu = 0$.

More generally, if $v = v^\lambda$ only for *one* particular $\lambda > 1$, we say v is *discretely self-similar* (DSS) with *factor* λ, or λ-DSS. When $\mu = 0$, the value of $v(x)$ in $\mathbb{R}^3 \setminus \{0\}$ is decided by its value in the annulus $1 \le |x| < \lambda$. When $\mu \neq 0$, the value of $v(x,t)$ in $\mathbb{R}^4_\pm$ is decided by its value in the strip $x \in \mathbb{R}^3$ and $1 \le |t| < \lambda^2$. Clearly, if v is λ-DSS, then it is also λ^k-DSS for any $k \in \mathbb{N}$.

Motivated by (8.3) and (8.4), for the time-dependent cases ($\mu \neq 0$) consider the *similarity transform*

$$v(x,t) = \frac{1}{\sqrt{\mu t}}\, V(y,s), \quad p(x,t) = \frac{1}{\mu t}\, P(y,s), \tag{8.5}$$

where the *similarity variables* y and s are

$$y = \frac{x}{\sqrt{\mu t}}, \quad s = \frac{1}{\mu}\log(\mu t). \tag{8.6}$$

If (v,p) satisfies (NS) with zero force, then (V,P) satisfies the *time-dependent Leray equations*

$$\begin{gathered} \partial_s V - \Delta V - \tfrac{1}{2}\mu V - \tfrac{1}{2}\mu y \cdot \nabla V + (V \cdot \nabla)V + \nabla P = 0, \\ \operatorname{div} V = 0. \end{gathered} \tag{8.7}$$

The Navier-Stokes system (NS) with $(x,t) \in \mathbb{R}^3 \times \mathbb{R}_\pm$ is equivalent to (8.7) with $(y,s) \in \mathbb{R}^3 \times \mathbb{R}$. In the backward case, the limit $s \to +\infty$ corresponds to $t \to 0_-$ and V magnifies the local behavior of v near a potential singularity.

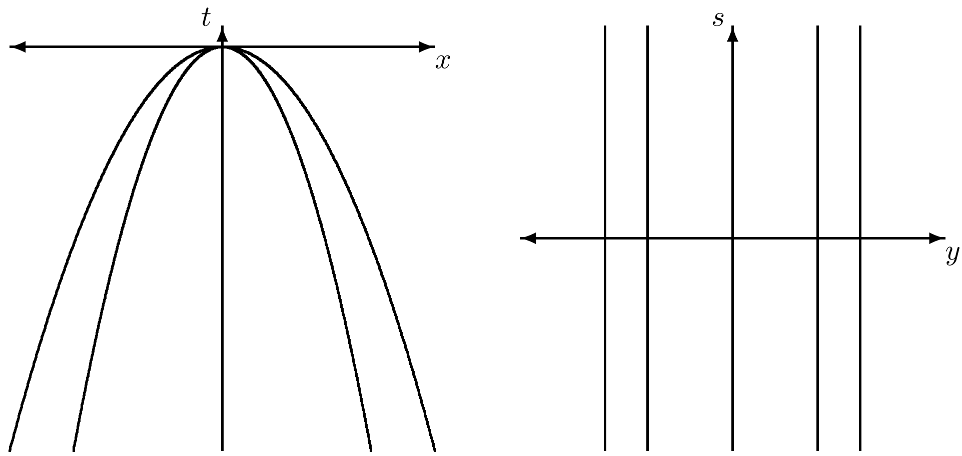


Figure 8.1. Backward physical and similarity variable space-time.

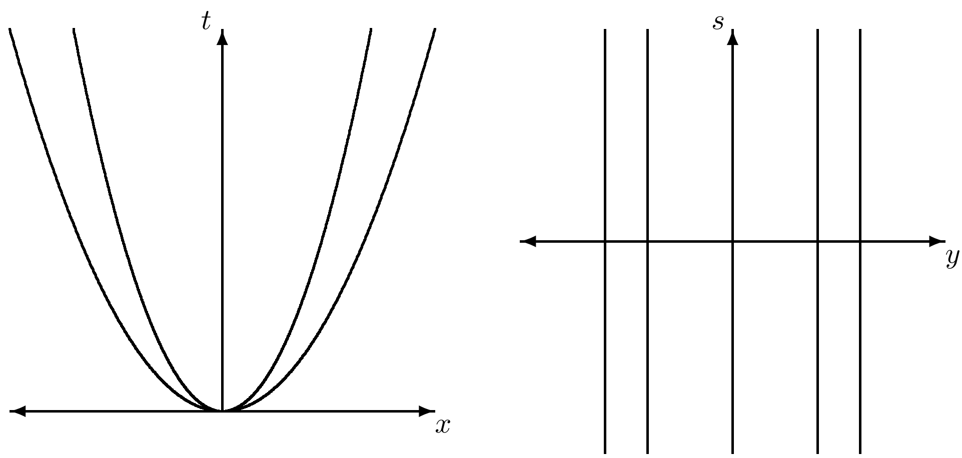


Figure 8.2. Forward physical and similarity variable space-time.

In the forward case, the limit $s \to +\infty$ corresponds to $t \to +\infty$ and V localizes the spatial-temporal asymptotic behavior of v. In either case, self-similar solutions of (NS) correspond to stationary solutions of (8.7), while DSS solutions of (NS) correspond to time-periodic solutions of (8.7) with period $\frac{2}{|\mu|} \log \lambda$.

For the stationary case, see Problem 8.2.

Giga and Kohn [**65**–**67**] were aware of the correspondence between (NS) and (8.7) and used the corresponding similarity transform to study the singularity of nonlinear heat equations $\partial_t u = \Delta u + u^p$. This study is further refined by many authors; see, e.g., the survey in [**99**].

We can incorporate rotations into the similarity transform to get special solutions with more complicated structures. For ease of notation, we only consider rotations around the x_3-axis with matrices

$$R(s) = \begin{bmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{8.8}$$

Note that $R(s)R(\tau) = R(\tau)R(s)$ for any $s, \tau \in \mathbb{R}$, and

$$\frac{d}{ds}R(s) = JR(s) = R(s)J, \quad J = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A vector field $v(x,t)$ is said to be *rotated self-similar* (RSS) if, for some fixed $\alpha \in \mathbb{R}$ and for *all* $\lambda > 0$, $x \in \mathbb{R}^3$, and $\mu t > 0$,

$$v(x,t) = \lambda R(-2\alpha \log \lambda)\, v\big(\lambda R(2\alpha \log \lambda)x, \lambda^2 t\big)\,. \tag{8.9}$$

An RSS vector field is always DSS with any factor $\lambda > 1$ such that $2\alpha \log \lambda \in 2\pi\mathbb{Z}$. When $\alpha = 0$, it becomes SS. The choice $\theta(\lambda) = 2\alpha \log \lambda$ in the argument of $R(\cdot)$ is natural, because $\lambda > 0$ is arbitrary and hence $\theta(\lambda) + \theta(\nu) = \theta(\lambda\nu)$ for all $\lambda, \nu > 0$.

A vector field $v(x,t)$ is said to be *rotated discretely self-similar* (RDSS) if, for *some* $\lambda > 1$ and some $\phi \in \mathbb{R}$,

$$v(x,t) = \lambda R(-\phi)\, v\big(\lambda R(\phi)x, \lambda^2 t\big)\,, \tag{8.10}$$

for all $x \in \mathbb{R}^3$ and $\mu t > 0$. If $n\phi = 2\pi m$ for some integers $n > 0$ and m, then v is DSS with factor λ^n. If $\frac{\phi}{2\pi}$ is irrational, in general v is not DSS.

Similarly to SS and DSS functions, the value of v is determined by its value at any fixed time if v is RSS, and by its value in the strip $1 \le \mu t < \lambda^2$ if v is RDSS. The inclusion property between these classes is

$$\text{SS} \subsetneq \text{RSS} \subsetneq \text{DSS} \subsetneq \text{RDSS}.$$

These solutions are better understood after the similarity transform. If we first take the similarity transform (8.5)–(8.6) and then assume

$$V(y,s) = R(\alpha s)u(z,s), \quad P(y,s) = \pi(z,s), \quad z = R(-\alpha s)y, \tag{8.11}$$

then (8.7) is equivalent to

$$\begin{gathered} \partial_s u + \alpha(Ju - Jz\cdot\nabla u) - \tfrac{\mu}{2}(u + z\cdot\nabla u) - \Delta u + u\cdot\nabla u + \nabla\pi = 0, \\ \nabla\cdot u = 0, \end{gathered} \tag{8.12}$$

where $\nabla = \nabla_z$ and $\Delta = \Delta_z$. If $v(x,t)$ is an RSS solution of (NS) satisfying (8.9), then $u(z,s)$ is a stationary solution of (8.12). If $v(x,t)$ is an RDSS solution with parameters $\lambda > 1$ and $\phi \in \mathbb{R}$, then $u(z,s)$ is a periodic solution of (8.12) with period $T = 2\log\lambda$ and $\alpha = \frac{1}{T}(2k\pi + \phi)$ for any integer k.

A stationary vector field v is called RSS or RDSS if it satisfies (8.9) or (8.10) with the time dependence ignored.

All special solutions in this section can also be considered in the half-space $\mathbb{R}^3_+$, with the additional Dirichlet boundary condition on $\partial\mathbb{R}^3_+$.

8.2. Stationary self-similar solutions

In this section we consider solutions of the stationary Navier-Stokes equations

$$-\Delta v + (v \cdot \nabla)v + \nabla p = 0, \quad \operatorname{div} v = 0 \tag{8.13}$$

in $\Omega = \mathbb{R}^3 \setminus \{0\}$ that satisfies

$$|v(x)| \le \frac{C_0}{|x|} \tag{8.14}$$

for some $C_0 > 0$. The regularity theory in Section 2.7 implies that v is smooth. In fact, by [**204**],

$$|\nabla^k v(x)| \le \frac{C_k}{|x|^{k+1}} \quad (x \in \Omega), \tag{8.15}$$

for some $C_k = C_k(C_0)$, for all $k \ge 0$. It can be shown that they satisfy the Navier-Stokes equations with delta force at the origin,

$$-\Delta v + (v \cdot \nabla)v + \nabla p = b\delta_0, \quad \operatorname{div} v = 0 \tag{8.16}$$

in $\mathbb{R}^3$ for some $b \in \mathbb{R}^3$, where δ_0 is the delta function at the origin.

A special family of solutions of (8.13)–(8.14) is the *Landau solutions* or *Slezkin-Landau solutions*. They are self-similar (i.e., minus one homogeneous) and axisymmetric with no swirl. For such vector fields it is convenient to choose the axis of symmetry as the x_3-axis and to use *spherical coordinates* ρ, θ, ϕ with

$$(x_1, x_2, x_3) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\phi)$$

and basis vectors

$$e_\rho = \frac{x}{\rho}, \quad e_\theta = (-\sin\theta, \cos\theta, 0), \quad e_\phi = e_\theta \times e_\rho.$$

Thus $\rho = |x|$, θ is the azimuthal angle, and ϕ is the polar angle, the angle between x and e_3.

A vector field v is called *axisymmetric* if it is of the form

$$v = v_\rho(\rho, \phi)e_\rho + v_\theta(\rho, \phi)e_\theta + v_\phi(\rho, \phi)e_\phi, \tag{8.17}$$

with v_ρ, v_θ, and v_ϕ independent of θ. It is the same as being axisymmetric in *cylindrical coordinates*; see Section 2.5. It has no swirl if the *swirl* component v_θ is zero.

The Landau solutions are denoted as U^b with parameter $b \in \mathbb{R}^3$. In spherical coordinates, they are

$$U^b = \frac{1}{\rho}\left(\frac{2(a^2-1)}{(a-\cos\phi)^2} - 2\right)e_\rho + 0e_\theta + \frac{-2\sin\phi}{\rho(a-\cos\phi)}e_\phi, \tag{8.18}$$
$$P^b = \frac{4(a\cos\phi-1)}{\rho^2(a-\cos\phi)^2},$$

where $a \in (1,\infty]$. They solve (8.16) in the distributional sense with $b = \beta(a)e_3$, where

$$\beta(a) := 16\pi\left(a + \frac{1}{2}a^2\log\frac{a-1}{a+1} + \frac{4a}{3(a^2-1)}\right). \tag{8.19}$$

Note that $\beta(a) \in [0,\infty)$ is strictly decreasing in $a \in (1,\infty]$; thus $a = \beta^{-1}(|b|)$ is defined. Any axisymmetric vector field v with no swirl has a *stream function* Ψ so that

$$v = \operatorname{curl}(\Psi e_\theta) = \frac{1}{\rho\sin\phi}\partial_\phi(\Psi\sin\phi)e_\rho - \frac{1}{\rho}\partial_\rho(\rho\Psi)e_\phi. \tag{8.20}$$

We have

$$U^b = \operatorname{curl}(\Psi e_\theta), \quad \Psi = \frac{2\sin\phi}{a-\cos\phi}. \tag{8.21}$$

We first show that any axisymmetric solution v of (8.13) is one of the Landau solutions. The fact that v has no swirl is a conclusion, not an assumption.

Theorem 8.1. *If v is an axisymmetric self-similar solution of the stationary Navier-Stokes equations* (8.13) *in $\mathbb{R}^3 \setminus \{0\}$, then $v = U^b$ for some $b \in \mathbb{R}^3$.*

Statitionary Navier-Stokes equations (8.13) for axisymmetric flows in spherical coordinates (ρ,θ,ϕ) read (see [**9**, Appendix 2])

$$\begin{aligned}
(\tilde v\cdot\nabla)v_\rho - \frac{v_\phi^2}{\rho} - \frac{v_\theta^2}{\rho} &= \Delta v_\rho - \frac{2v_\rho}{\rho^2} - \frac{2}{\rho^2\sin\phi}\partial_\phi(\sin\phi\, v_\phi) - \partial_\rho p,\\
(\tilde v\cdot\nabla)v_\phi + \frac{v_\rho v_\phi}{\rho} - \frac{v_\theta^2\cot\phi}{\rho} &= \Delta v_\phi + \frac{2}{\rho^2}\partial_\phi v_\rho - \frac{v_\phi}{\rho^2\sin^2\phi} - \frac{1}{\rho}\partial_\phi p,\\
(\tilde v\cdot\nabla)v_\theta + \frac{v_\theta v_\rho}{\rho} + \frac{v_\phi v_\theta\cot\phi}{\rho} &= \Delta v_\theta - \frac{v_\theta}{\rho^2\sin^2\phi},\\
\partial_\rho(\rho^2\sin\phi\, v_\rho) + \partial_\phi(\rho\sin\phi\, v_\phi) &= 0.
\end{aligned} \tag{8.22}$$

Above, $\tilde v = v_\rho e_\rho + v_\phi e_\phi$. The variables are

$$\rho \in (0,\infty), \quad \phi \in (0,\pi).$$

The natural boundary conditions for a smooth vector field v when $\rho > 0$ are

$$\partial_\phi v_\rho = v_\phi = v_\theta = \partial_\phi p = 0, \quad \text{when } \phi = 0,\pi.$$

Also note that

$$\Delta K = \frac{1}{\rho^2}\partial_\rho(\rho^2\partial_\rho K) + \frac{1}{\rho^2\sin\phi}\partial_\phi(\sin\phi\partial_\phi K) + \frac{1}{\rho^2\sin^2\phi}\partial_\theta^2 K.$$

Thus

$$\Delta\frac{k(\phi)}{\rho} = 0 + \frac{1}{\rho^3}(k'' + k'\cot\phi) + 0.$$

Proof of Theorem 8.1. In the spherical coordinate, an axisymmetric self-similar solution (v, p) is of the form

$$v = \frac{1}{\rho}f(\phi)e_\rho + \frac{1}{\rho}g(\phi)e_\phi + \frac{1}{\rho}h(\phi)e_\theta, \quad p = \frac{1}{\rho^2}P(\phi).$$

Due to symmetry and regularity of v,

$$f'(0) = g(0) = h(0) = f'(\pi) = g(\pi) = h(\pi) = 0. \tag{8.23}$$

The system (8.22) becomes

$$f'' + f'\cot\phi = gf' - (f^2 + g^2 + h^2) - 2P, \tag{8.24}$$

$$f' = gg' - h^2\cot\phi + P', \tag{8.25}$$

$$(h' + h\cot\phi)' = g(h' + h\cot\phi), \tag{8.26}$$

$$f + g' + g\cot\phi = 0. \tag{8.27}$$

Setting $H(\phi) := h' + h\cot\phi = (h\sin\phi)'/\sin\phi$, (8.26) becomes $H' = gH$. Since

$$\int_0^\pi H(\phi)\sin\phi d\phi = h(\phi)\sin\phi\big|_0^\pi = 0,$$

$H(\phi_1) = 0$ at some ϕ_1. By uniqueness of the ODE $H' = gH$, H is identically zero. Hence $h(\phi)\sin\phi$ is a constant, which must be 0 by the boundary condition. We conclude $h(\phi) \equiv 0$.

Integrating (8.25), we have

$$f = \frac{g^2}{2} + P + C_1 \tag{8.28}$$

for some constant C_1. Substituting (8.28) into (8.24), we obtain

$$f'' + f'\cot\phi = gf' - f^2 - 2f + 2C_1. \tag{8.29}$$

Set

$$A = f(\phi)\sin\phi, \quad B = g(\phi)\sin\phi.$$

Note that $A = -B'$ by (8.27). Thus (8.29) becomes

$$(f'\sin\phi)' = (Bf)' + 2B' + 2C_1\sin\phi.$$

Integrating, we get

$$f'\sin\phi = Bf + 2B - 2C_1\cos\phi + C_2.$$

By the boundary condition (8.23), we get $0 = \mp 2C_1 + C_2$. Thus $C_1 = C_2 = 0$ and

$$f' \sin\phi = Bf + 2B. \tag{8.30}$$

Let $L(t) = B(\phi)$ with $t = \cos\phi$. Noting that $B' = -L'\sin\phi$ and $B'' = L''\sin^2\phi - L'\cos\phi$, we get from (8.30) using $f = -B'/\sin\phi$

$$(1-t^2)L'' + 2L + LL' = 0, \quad \text{for } t \in (-1,1),$$
$$L(-1) = L(1) = 0.$$

The linearized equation $(1-t^2)L'' + 2L = 0$ has a solution $L = 1 - t^2$. By variation of parameters, we set $L(t) = (1-t^2)u(t)$. Then u solves

$$u' + \frac{u^2}{2} = 0.$$

Thus $u(t) = \frac{2}{t-a}$ for some $a \in \mathbb{R}$, $|a| > 1$, and

$$L(t) = \frac{2(1-t^2)}{t-a} = B(\phi) = \frac{2\sin^2\phi}{\cos\phi - a}.$$

We conclude

$$h(\phi) = 0, \quad g(\phi) = \frac{2\sin\phi}{\cos\phi - a}, \quad f(\phi) = \frac{2(a^2-1)}{(a-\cos\phi)^2} - 2.$$

By (8.28), $P = f - g^2/2$. These formulas give the Landau solutions (8.18) with parameter a. We may assume $a > 1$ since a negative value of a corresponds to the same solution with opposite direction of the x_3-axis, $U_{-a}(\rho,\phi,\theta) = U_a(\rho, \pi - \phi, \theta)$. $\square$

Lemma 8.2. *The Landau solution given by* (8.18) *with parameter* $a \in (1,\infty]$ *satisfies* (8.16) *with* $b = \beta(a)e_3$ *and* $\beta(a)$ *given by* (8.19).

Proof. Let

$$\ell[\phi] = \int -U\cdot\Delta\phi - U_jU_i\partial_j\phi_i - P\,\mathrm{div}\,\phi$$

for $\phi \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$. We have

$$\begin{aligned}\ell[\phi] &= \lim_{\varepsilon\to 0_+}\int_{B_\varepsilon^c} (\text{same integrand}) \\ &= \lim_{\varepsilon\to 0_+}\int_{\partial B_\varepsilon}\left(\phi\cdot\frac{\partial U}{\partial \mathbf{n}} - U\cdot\frac{\partial\phi}{\partial\mathbf{n}} - U_jU_i\phi_i\mathbf{n}_j - P\phi\cdot\mathbf{n}\right),\end{aligned}$$

where $\mathbf{n} = -x/\rho$. Since U is self-similar, $\frac{\partial U}{\partial \mathbf{n}} = \rho^{-1}U$. We get

$$\ell[\phi] = \phi(0)\cdot\int_{S^2}[U + 0 + (U\cdot x)U + Px]\,dS_x.$$

Using (8.18) and symmetry, the integral is $(0, 0, \beta)$ with

$$\begin{aligned}\beta &= \int_{S^2} (1+f)(f\cos\phi - g\sin\phi) + P\cos\phi\, dS_x \\ &= 2\pi \int_0^\pi [(1+f)(f\cos\phi - g\sin\phi) + P\cos\phi]\sin\phi\, d\phi \\ &= 2\pi \int_0^\pi \left(f(f+3) - \frac{1}{2}g^2\right)\sin\phi\cos\phi\, d\phi,\end{aligned}$$

where we have used $P = f - g^2/2$, $(g\sin\phi)' = -f\sin\phi$, and hence

$$\int_0^\pi fg\sin^2\phi\, d\phi = 0, \quad \int_0^\pi -g\sin^2\phi\, d\phi = \int_0^\pi f\sin\phi\cos\phi\, d\phi.$$

Changing variable $t = \cos\phi$, we get

$$\beta = 2\pi \int_{-1}^1 \left(\frac{4(a^2-1)^2 t}{(a-t)^4} + \frac{2t(a+t)}{t-a} - 2t\right) dt.$$

Evaluating the integral, we get (8.19). □

The following theorem of Šverák [**203**] shows that a self-similar solution, even without assuming symmetry or smallness, must be a Landau solution.

Theorem 8.3 (Rigidity). *If a minus one homogeneous vector field v is a solution of the stationary Navier-Stokes equations in $\mathbb{R}^3 \setminus \{0\}$, then $v = U^b$ for some $b \in \mathbb{R}^3$.*

The Landau solutions can serve as the asymptotic profile of a stationary solution in the class (8.14) either at spatial infinity in an exterior domain or at a singularity in a bounded domain. Actually, they also serve as the spatial asymptotic profile of small time-periodic solutions in [**92**].

Theorem 8.4 (Asymptotes of small solutions [**114, 149**]). *For any $0 < \alpha < 1$, there is an $\varepsilon_0(\alpha) > 0$ such that the following hold:*

(i) *If $v(x)$ is a solution of the stationary Navier-Stokes equations in $1 < |x| < \infty$ with $|v(x)| \le \frac{\varepsilon}{|x|}$, $0 \le \varepsilon \le \varepsilon_0(\alpha)$, then there is $b \in \mathbb{R}^3$, $|b| \le C\varepsilon$, such that*

$$|v(x) - U^b(x)| \le \frac{C\varepsilon}{|x|^{1+\alpha}}, \quad 1 < |x| < \infty.$$

(ii) *If $v(x)$ is a solution of the stationary Navier-Stokes equations in $0 < |x| < 1$ with $|v(x)| \le \frac{\varepsilon}{|x|}$, $0 \le \varepsilon \le \varepsilon_0(\alpha)$, then there is $b \in \mathbb{R}^3$, $|b| \le C\varepsilon$, such that*

$$|v(x) - U^b(x)| \le \frac{C\varepsilon}{|x|^{\alpha}}, \quad 0 < |x| < 1.$$

It is unknown whether there are other solutions of stationary (NS) in $\mathbb{R}^3 \setminus \{0\}$ in the class (8.14). It is a corollary of Theorem 8.4 that strictly DSS/RSS/RDSS solutions do not exist if C_0 in (8.14) is sufficiently small, but their existence for large C_0 is open.

8.3. Backward self-similar solutions

Backward self-similar solutions are candidates for singularities of (NS). Recall the lower bounds for blowup rate: Suppose a solution $v(x,t)$ of (NS) is defined for $t \in (-1,0)$ and blows up at $z_0 = (0,0)$. Leray [**132**] proves that (see Theorem 5.4)

$$\|v(t)\|_{L^q_x} \geq C_q\, |t|^{-\frac{q-3}{2q}}, \quad 3 < q \leq \infty. \tag{8.31}$$

Caffarelli-Kohn-Nirenberg [**22**] shows that (see Theorem 6.3)

$$\left(\frac{1}{|Q_r|}\int_{Q_r} |v|^3 + |p|^{3/2} dz\right)^{1/3} \geq \frac{C}{r}. \tag{8.32}$$

A natural rate for blowup at $(0,0)$ is thus

$$|v(x,t)| \geq \frac{C}{|x| + \sqrt{-t}}. \tag{8.33}$$

Most regularity criteria say that v is regular at z_0 if $|v| \leq \frac{\varepsilon}{|x|+\sqrt{-t}}$, in some average sense, for some small ε. The first significant candidate for blowup has the rate

$$|v(x,t)| \sim \frac{C}{|x| + \sqrt{-t}}, \quad \text{with } C = O(1). \tag{8.34}$$

A singularity of such a rate is called a *Type I* singularity. Singularities of other rates are called *Type II*. Type I singularity has been excluded in two cases: either if v is self-similar (Theorem 8.6) or if v is axisymmetric [**31**] (see Chapter 10).

Backward self-similar solutions of the form (8.3) (with $\mu < 0$) are suggested by Leray [**132**] as the first candidate for a (Type I) singularity. His blowup conjecture was formulated over an interval $[0,T)$ where T was the singular time,

$$\tilde{v}(x,t) = \frac{1}{\sqrt{T-t}} V\left(\frac{x}{\sqrt{T-t}}\right).$$

A backward self-similar solution $\tilde{v}$ defined on $[0,T)$ can be shifted to give a solution on $[-T,0)$, $v(x,t) = \tilde{v}(x,t-T)$, which can then be extended by self-similarity to a solution on $(-\infty,0)$. In addition to being an explicit singular solution, such solutions may occur as limit functions, as in the study of nonlinear heat equations (see Giga-Kohn [**65–67**]).

We need to require that some certain natural energy of v is finite; otherwise there are nontrivial examples.

Example 8.5 ([**209**, Remark 5.4]). Let Φ be an arbitrary harmonic function in $\mathbb{R}^3$. Let $V = \nabla\Phi$ and $P = -\frac{1}{2}|V|^2 + \frac{1}{2}\mu y \cdot V$. Then (V, P) solves Leray's equations (8.4), for any $a \in \mathbb{R}$.

This example is motivated by the Bernoulli quantity Π to be defined in (8.36). It satisfies $\Pi = 0$.

We now state the main theorem of this section, the nonexistence of backward self-similar singularity with finite local energy.

Theorem 8.6 ([**209**]). *Suppose v is a self-similar very weak solution of* (NS) *with zero force of the form* (8.3) *in the cylinder $Q_1 = B_1 \times (-1, 0)$. It is zero if it has finite local energy*

$$\operatorname*{ess\,sup}_{-1<t<0} \int_{B_1} |v(x,t)|^2 dx + \int_{-1}^{0} \int_{B_1} |\nabla v(x,t)|^2 dx\, dt < \infty \tag{8.35}$$

or if $V \in L^q(\mathbb{R}^3)$ for some $q \in [3, \infty)$. Also, V is constant if $V \in L^\infty(\mathbb{R}^3)$.

The original question of Leray [**132**] is the existence of self-similar singularity with finite *global energy*; i.e., the quantity in (8.35) with B_1 replaced by $\mathbb{R}^3$ is finite. Such a global estimate is stronger that (8.35) and implies $V \in L^3(\mathbb{R}^3)$, whose nonexistence is shown by Nečas, Růžička, and Šverák [**155**]. The condition $V \in L^3$ is scaling invariant and implies $v \in L^\infty L^3_x$ since $\|v(t)\|_{L^3(\mathbb{R}^3)} = C\|V\|_{L^3(\mathbb{R}^3)}$. The $L^\infty L^3$ regularity criteria of [**41**] gives another proof for this case; see Chapter 9.

On the other hand, if $|V(y)| \sim \frac{C}{1+|y|}$ and $\mu = -1$, then for $\lambda = (-t)^{-1/2}$

$$|v(x,t)| = \lambda |V(\lambda x)| \sim \lambda \frac{C}{1 + \lambda|x|} = \frac{C}{|x| + \sqrt{-t}},$$

which is exactly the Type I singularity in (8.34). In this case

$$\int_{B_1} |v(x,t)|^3 dx = \int_{|y|<\lambda(t)} |V(y)|^3\, dy \sim -\log(-t),$$

which is unbounded as $t \to 0_-$.

Weak solutions of Leray equations in $L^3_{\mathrm{loc}}(\mathbb{R}^3)$ are smooth, by the same proof for stationary Navier-Stokes equations (see Section 2.7). The main issue is the existence of solutions in the entire $\mathbb{R}^3$ with finite L^q-norm. Theorem 8.6 says that there are no nontrivial solutions of (8.4) in $\mathbb{R}^3$ with $\mu < 0$ if $V \in L^q(\mathbb{R}^3)$, $3 \le q \le \infty$. In contrast, we will show in Section 8.4 an abundance of *forward* self-similar solutions, i.e., many nontrivial solutions of the stationary Leray equations (8.4) in $\mathbb{R}^3$ with $\mu > 0$. For solutions of (8.4) with $\mu = 0$ and no singularity at the origin, i.e., solutions of (NS) in the entire $\mathbb{R}^3$, one knows that $V = 0$ if $V \in L^q \cap L^3_{\mathrm{loc}}(\mathbb{R}^3)$ with $q \le 9/2$ [**57**, X.9]. The cases $q > 9/2$ are open; see the *Liouville problem* stated in Conjecture 2.5.

To be consistent with the notation in [**155**, **209**], we take $\mu = -2a$ in (8.3), so that $a > 0$.

The starting point of the proof of Theorem 8.6 is the observation that the Bernoulli quantity, or the *total head pressure*,

$$\Pi(y) = \frac{1}{2}|V(y)|^2 + P(y) + ay \cdot V(y) \tag{8.36}$$

satisfies the equation

$$-\Delta\Pi(y) + (V(y) + ay) \cdot \nabla\Pi(y) = -|\operatorname{curl} V(y)|^2 \le 0 \tag{8.37}$$

and hence the maximal principle. This property for the case $a = 0$ is well known ([**186**, p. 261], [**73**], [**111**]) and has played an important role in the study of stationary (NS). It is the quantity in Bernoulli's Law for inviscid flow (Euler equations). It also plays an important role in Section 7.3 for 2-dimensional boundary value problems. Equation (8.37) is derived in [**155**].

The term $ay \cdot \nabla\Pi$ in (8.37) has an unbounded coefficient and seems to be a bad term. However, it can be thought of as a "magnifying force", in view of the ODE,

$$v'' = xv'.$$

Therefore, a solution of (8.37) should either be constant or grow unboundedly. If the coefficient $V(y)$ can be ignored for large y, the exact radial solution of $\Delta\phi(y) = ay \cdot \nabla\phi(y)$ has exponential growth at ∞ and can be used as a comparison function.

Lemma 8.7 ([**209**]). *Let $\Pi : \mathbb{R}^3 \to \mathbb{R}$ and let $V : \mathbb{R}^3 \to \mathbb{R}^3$ be smooth and satisfy for some $a > 0$*

$$-\Delta\Pi(y) + (V(y) + ay) \cdot \nabla\Pi(y) \le 0 \quad \text{in } \mathbb{R}^3. \tag{8.38}$$

If $|V(y)| \le b|y|$ for some $b \in (0, a)$ and y sufficiently large and if

$$\Pi(y) \le o(\phi(|y|)) \text{ as } |y| \to \infty, \quad \phi(|y|) = \int^{|y|} s^{-2} e^{(a-b)s^2/2} ds, \tag{8.39}$$

then Π is constant.

Note that ϕ is a radial solution of $\Delta\phi = (a - b)y \cdot \nabla\phi$.

Proof. Let $M(r) = \max_{|y|=r} \Pi(y)$. It is nondecreasing in r by the maximal principle for (8.38). Suppose $|V(y)| \le b|y|$ for $|y| \ge r_0 > 0$. The function $\phi(|y|)$ satisfies

$$-\Delta\phi(|y|) + (V(y) + ay) \cdot \nabla\phi(|y|) \ge 0 \qquad \text{for } |y| > r_0. \tag{8.40}$$

That is, it is a supersolution. Let

$$\psi_\epsilon(y) = M(r_0) + \varepsilon\,[\phi(|y|) - \phi(r_0)],$$

for $\epsilon > 0$. Then all ψ_ϵ are supersolutions for $|y| > r_0$. It is clear that $\psi_\epsilon(y) \geq \Pi(y)$ for $|y| = r_0$ and for $|y|$ near ∞ by the growth of Π. By the comparison principle we have

$$\Pi(y) \leq M(r_0) + \varepsilon\,[\phi(|y|) - \phi(r_0)] \qquad \text{for } |y| \geq r_0.$$

Now letting ϵ go to zero, we get $\Pi(y) \leq M(r_0)$ for all $|y| \geq r_0$, and hence $\max \Pi$ is attained at some y, $|y| = r_0$. By the strong maximal principle (note that our coefficients are locally bounded), Π must be constant. $\square$

If Π is constant, then $\operatorname{curl} V = 0$ by (8.37). Together with $\operatorname{div} V = 0$, we get $\Delta V = 0$, which implies V is constant if $V(y) = o(|y|)$ as $|y| \to \infty$. Thus, to verify the assumptions of Lemma 8.7 and conclude the theorem, it suffices to show that

$$|V(y)| = o(|y|), \quad |P(y)| = O(|y|^N) \quad \text{as } |y| \to \infty \tag{8.41}$$

for some large N.

Proof of Theorem 8.6. We will assume (8.35). For the other case $V \in L^q(\mathbb{R}^3)$, see [**209**].

Step 1, V bound. The estimate (8.35) implies $u \in L^{10/3}(Q_1)$, which is equivalent to

$$\int_{\mathbb{R}^3} |V|^{10/3}\, w(y)\, dy < \infty, \tag{8.42}$$

where $w(y) = (1 + |y|)^{-5/3}$. Since V is smooth, we may replace $w(y)$ by

$$w(y) = |y|^{-5/3}. \tag{8.43}$$

With this bound, we can show $V \in C^\infty_{\mathrm{loc}}$ by the same proof for steady-state (NS) (Section 2.7).

Step 2, P bound. A priori we may define P locally to satisfy (8.4) but we do not have any global bound. Taking the divergence of (8.4), we get

$$-\Delta P = \partial_i \partial_j (V_i V_j). \tag{8.44}$$

We may define a solution of (8.44) by

$$\tilde{P}(y) = \int_{\mathbb{R}^3} \frac{1}{4\pi|y - z|}\, \partial_i \partial_j (V_i V_j)(z)\, dz = R_i R_j (V_i V_j)(y), \tag{8.45}$$

where the R_i are the Riesz transforms. We need a weighted version of the Calderon-Zygmund theorem, the A_q weight theory ([**200**, V, pp. 204–211]): A positive function $w(y)$ defined for $y \in \mathbb{R}^d$ is an A_q weight, $1 < q < \infty$, if

$$\|w\|_{A_q} := \sup \left(\frac{1}{|B|} \int_B w(y)dy \right) \cdot \left(\frac{1}{|B|} \int_B w(y)^{-\frac{1}{q-1}} dy \right)^{q-1} < \infty, \tag{8.46}$$

where the sup is taken over all balls $B \subset \mathbb{R}^d$. An equivalent definition is that

$$\|Mf\|_{w,q} \le C\|f\|_{w,q}, \quad \|f\|_{w,q}^q := \int_{\mathbb{R}^d} |f(y)|^q w(y)dy, \tag{8.47}$$

for all functions f and a constant C independent of f, where Mf is the maximal function of f. It can be shown that a singular integral T whose kernel has suitable decay (including Riesz transforms) is bounded in this w-weighted L^q-norm,

$$\|Tf\|_{w,q} \le C\|f\|_{w,q}. \tag{8.48}$$

It is known that $|y|^\alpha$ is an A_q weight if $-d < \alpha < d(q-1)$. In particular, our $w(y) = |y|^{-5/3}$ is an $A_{5/3}$ weight and thus, by (8.42),

$$\|\tilde{P}\|_{w,5/3} \le C \sum \|V_i V_j\|_{w,5/3} < \infty. \tag{8.49}$$

We only know $\Delta \tilde{P} = \Delta P$ and we need to show $\nabla \tilde{P} = \nabla P$. Let

$$H = -\Delta V + aV + a\,(y \cdot \nabla)\, V + (V \cdot \nabla)\, V + \nabla \tilde{P} = \nabla \tilde{P} - \nabla P.$$

We have $\operatorname{div} H = 0$ and $\operatorname{curl} H = 0$. Thus $\Delta H = 0$. Using the analyticity of H, one can show $H \equiv 0$ by showing for all multi-index α,

$$\begin{aligned} D^\alpha H(0) &= \int_{\mathbb{R}^3} D^\alpha H(y) \varepsilon^3 \phi(\varepsilon y)\, dy \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^3} H(y) \varepsilon^{3+|\alpha|} (D^\alpha \phi)(\varepsilon y)\, dy \underset{\varepsilon \to 0_+}{\longrightarrow} 0, \end{aligned}$$

where $\phi \in C_c^\infty$ is a radial function with $\int \phi = 1$. The first equality is by the mean value property of harmonic functions, and the second is by integration by parts. We consider the term involving $y \cdot \nabla V$ as an example: We have

$$I_\varepsilon = \left| \int_{\mathbb{R}^3} (y \cdot \nabla V) \varepsilon^{3+|\alpha|} (D^\alpha \phi)(\varepsilon y)\, dy \right| = \left| \varepsilon^{3+|\alpha|} \int_{\mathbb{R}^3} V(y) \Psi(\varepsilon y)\, dy \right|,$$

where $\Psi(\varepsilon y) = 3(D^\alpha \phi)(\varepsilon y) + \varepsilon y^j (\partial_j D^\alpha \phi)(\varepsilon y)$. Then

$$\begin{aligned} I_\varepsilon &\le \varepsilon^{3+|\alpha|} \left(\int (|V(y)||y|^{-\frac{1}{2}})^{\frac{10}{3}} dy \right)^{\frac{3}{10}} \left(\int (|\Psi(\varepsilon y)||\varepsilon y|^{\frac{1}{2}})^{\frac{10}{7}} \varepsilon^3 dy \right)^{\frac{7}{10}} \varepsilon^{-\frac{13}{5}} \\ &= C\varepsilon^{|\alpha|+2/5}, \end{aligned}$$

which goes to 0 as $\varepsilon \to 0_+$. Thus $H \equiv 0$ and we may set $P = \tilde{P}$. The estimate (8.49) implies $p(x,t) = \lambda^2(t)P(\lambda(t)x) \in L^{5/3}(Q_1)$, better than $L^{3/2}$. Since (V,P) is regular, (v,p) is a suitable weak solution in Q_1.

Step 3. Local L^∞-bound of V and P. Rewriting Leray's equations (8.4) as the Stokes system with force, one can get the L^∞-bound of V and P in any unit ball $B(y_0,1)$ by bootstrapping. Because the force contains an unbounded term $y \cdot \nabla V$, the best one can show is a polynomial growth for both V and P.

To prove $|V(y)| = o(|y|)$, we go back to (NS) and use the partial regularity result for suitable weak solutions (see [**128**, p. 370] for a different argument). We claim that v is regular at any $z_0 = (x_0, 0)$ with $0 < |x_0| < 1$, i.e., $v \in L^\infty(Q(z_0,r))$ for some $r > 0$. Otherwise, v is singular at sz_0 for $0 < s < 1$ by self-similarity. The singular set of v thus contains a line segment, contradicting the partial regularity result that its 1-dimensional Hausdorff measure is 0. Since the sphere $|x| = 1/2$ is compact, we can find $r > 0$ such that $v \in L^\infty((B_{1/2+r} \setminus B_{1/2-r}) \times (-r,0))$, which implies $V(y) = O(|y|^{-1})$, establishing (8.41) and the theorem. $\square$

8.3.1. Variants of self-similar solutions. Having excluded backward self-similar solutions, we should next study the existence of backward DSS, RSS, and RDSS solutions.

If $V(y,s)$ in (8.5) of the DSS case, or $u(z,s)$ in (8.11) of the RSS/RDSS cases, is in $L^\infty(\mathbb{R}, L^3(\mathbb{R}^3))$, then $v \in L^\infty((-\infty,0), L^3(\mathbb{R}^3))$. It follows from the result of Escauriaza, Seregin, and Šverák [**41**] (see Section 9.3) that v must be regular, and thus V must be zero. This result is extended by Seregin [**178**] to the half-space.

Thus one is concerned with their existence if V only has the bound

$$|V(y,s)| \le \frac{C}{1+|y|}, \quad \forall (y,s) \in \mathbb{R}^{3+1}, \tag{8.50}$$

for some large constant C. Self-similar solutions with this bound are excluded by Theorem 8.6, but the other cases remain open.

Conjecture 8.8. *Suppose $v(x,t)$ is a backward RSS/DSS/RDSS solution of* (NS) *in $\mathbb{R}^3$ with*

$$|v(x,t)| \le \frac{C_*}{|x|+\sqrt{-t}} \quad \text{in } \mathbb{R}^3 \times (-1,0). \tag{8.51}$$

Does v remain bounded up to time 0?

A very recent work of Chae-Wolf [**30**] shows that if v is a λ-DSS solution satisfying (8.51) and $0 < \lambda - 1 \le C(C_*)$ sufficiently small, then v is zero. The proof uses a contradiction argument and reduces the problem to the self-similar case.

The ansatz of RSS solutions was originally proposed by Grigori Perelman for backward solutions defined for $-\infty < t < 0$ to Seregin around a decade ago (private communication of G. Seregin). It can be formulated as follows.

Conjecture 8.9. *Suppose $v(x,t)$ is a solution of* (NS) *in $\mathbb{R}^3$ for $t < 0$ with*

$$v(x,t) = \frac{1}{\sqrt{-t}} R(\alpha s) u\left(R(-\alpha s)\frac{x}{\sqrt{-t}}\right), \quad s = -\log(-t), \tag{8.52}$$

for some $\alpha \ne 0$, and

$$|u(y)| \le \frac{C}{|y|+1} \quad \text{in } \mathbb{R}^3. \tag{8.53}$$

Are v and u identically zero?

One can study $u(y)$ directly, which satisfies (8.53) and

$$\begin{gathered} \alpha J u - \alpha J y \cdot \nabla u + \frac{1}{2}u + \frac{y}{2}\cdot\nabla u - \Delta_y u + u\cdot\nabla u + \nabla p = 0, \\ \operatorname{div} u = 0, \end{gathered} \tag{8.54}$$

in $\mathbb{R}^3$. Note that, for the above system with $\alpha \ne 0$, there does not seem to be an analogue quantity of Π that satisfies the maximal principle.

Regarding the half-space, generalized self-similar solutions in the class $v \in L^\infty(-1,0;L^3(\mathbb{R}^3_+))$ do not exist by [**178**], and nothing is known under the assumption (8.51). We formulate a problem.

Conjecture 8.10. *Suppose $v(x,t)$ is a solution of* (NS) *in $\mathbb{R}^3_+ \times (-\infty, 0)$ with zero boundary condition and*

$$v(x,t) = \frac{1}{\sqrt{-t}} u\left(\frac{x}{\sqrt{-t}}\right), \tag{8.55}$$

with $u(y)$ satisfying (8.53). *Are v and u identically zero?*

8.3.2. Scheffer's examples. Recall that Caffarelli-Kohn-Nirenberg [**22**] proved $\mathcal{P}^1(S) = 0$. Their original proof only uses the local energy inequality, not the (NS) itself. (Compare to the fact that F. Lin's proof for Theorem 6.4 uses (NS).) Scheffer [**175, 176**] constructed examples showing that if one only uses the local energy inequality and not the (NS) itself, then the estimate $\mathcal{P}^1(S) = 0$ is optimal. He constructed in [**175**] one such vector field with one singular point and then used it as a building block in [**176**] to construct a vector field whose singular set at the first singular time has Hausdorff dimension arbitrarily close to one.

Theorem 8.11 ([**175**]). *There are functions $(v,p):\mathbb{R}^3\times[0,\infty)\to\mathbb{R}^3\times\mathbb{R}$ satisfying the following:*

$$\begin{cases} \text{there is a compact } K\subset\mathbb{R}^3 \text{ such that } v(x,t)=0 \text{ if } x\notin K,\\ v \text{ is smooth in } x \text{ and } \operatorname{div} v=0,\\ p(x,t)=\int_{\mathbb{R}^3}(\partial_i v_j)(\partial_j v_i)(y,t)\,\frac{1}{4\pi|x-y|}\,dy,\\ v\in L^\infty_t L^2_x\cap L^3_{t,x}, \quad \nabla v\in L^2_{x,t}, \quad \text{and } p\in L^{3/2}_{x,t}, \end{cases}$$

the local energy inequality: for all $\phi\in C^\infty_c(\mathbb{R}^3\times(0,\infty))$ with $\phi\ge 0$,

$$\int_0^\infty\int_{\mathbb{R}^3}|\nabla v|^2\phi\le\int_0^\infty\int_{\mathbb{R}^3}\frac12|v|^2(\partial_t\phi+\Delta\phi)+\left(\frac12|v|^2+p\right)v\cdot\nabla\phi,$$

and

$$\text{for some } z_0\in\mathbb{R}^3\times(0,\infty),\quad v\notin L^\infty(Q(z_0,r)),\quad \forall r>0.$$

In this example, v does not need to satisfy any equation and can be discontinuous in t. One thinks of (v,p) as a solution of (NS) with "speed-reducing" force

$$\partial_t v-\Delta v+v\cdot\nabla v+\nabla p=f,\quad \operatorname{div} v=0, \tag{8.56}$$

where

$$f\cdot v\le 0.$$

Taking the dot product of the equation with v itself, we have

$$\frac{\partial}{\partial t}\frac12|v|^2-v\cdot\Delta v+v\cdot\nabla\left(\frac12|v|^2+p\right)=f\cdot v; \tag{8.57}$$

thus f reduces the speed. It does, however, change the direction of the velocity rapidly. The blowup is due to the nonlocal effect of the pressure.

Scheffer's example is in fact a DSS solution of (8.56) with factor $\lambda>1$ and a speed-reducing force which is also DSS, $f=f^\lambda$; i.e.,

$$f(x,t)=\lambda^3 f(\lambda x,\lambda^2 t),\quad \forall (x,t)\in\mathbb{R}^4_-$$

for the same DSS-factor λ. The similarity transform (8.5) and (8.6) with $\mu=-2a<0$ gives a time-periodic solution of the time-dependent *Leray equations* with a time-periodic forcing term,

$$\partial_s V-\Delta V+(V+ay)\cdot\nabla V+aV+\nabla P=F,\quad \operatorname{div} V=0, \tag{8.58}$$

for $(y,s)\in\mathbb{R}^3\times\mathbb{R}$, where $F\cdot V\le 0$ in the distributional sense. Note that F is not exactly a function and may contain delta functions in time at discrete times. There is a compact set $\tilde K\subset\mathbb{R}^3$ so that $V(y,s)=0$ whenever $y\notin\tilde K$.

8.4. Forward self-similar solutions

In this section we consider forward self-similar solutions and their variants in $\Omega \times (0, \infty)$, where $\Omega = \mathbb{R}^3$ or $\Omega = \mathbb{R}^3_+$. In the forward case the Navier-Stokes system is coupled with an initial condition

$$v(0, x) = v_0(x), \quad \operatorname{div} v_0 = 0, \tag{8.59}$$

and v_0 should be either SS, DSS, RSS, or RDSS. When $\Omega = \mathbb{R}^3_+$, we also include a Dirichlet boundary condition on $\partial\mathbb{R}^3_+$. Our main concern is their existence for given initial data.

For initial data v_0 in these symmetry classes, it is natural to assume

$$|v_0(x)| \le \frac{C_*}{|x|}, \quad \text{or} \quad \|v_0\|_X \le C_*, \tag{8.60}$$

for some constant $C_* > 0$, and to look for solutions satisfying

$$|v(x, t)| \le \frac{C(C_*)}{|x|}, \quad \text{or} \quad \|v(\cdot, t)\|_X \le C(C_*). \tag{8.61}$$

Above, X is a scaling-invariant functional space containing $|x|^{-1}$, e.g., one of $L^{3,\infty} \subset \dot{B}^{3/p-1}_{p,\infty} \subset \mathrm{BMO}^{-1}$ with $3 < p < \infty$. Note that we cannot take $\dot{H}^{1/2}$ or L^3 (the other members of (5.1)), since they do not contain $|x|^{-1}$.

For *small* initial data, the existence of SS/DSS/RSS/RDSS solutions follows from the unique existence theory of *mild solutions* in various scaling-invariant functional spaces. Indeed, for small data in such spaces, the unique existence of small mild solutions has been obtained by Giga-Miyakawa [**69**] and Kato [**96**] in Morrey spaces, Barraza [**8**] in $L^{3,\infty}$, Cannone-Planchon [**23**] in Besov spaces, and Koch-Tataru [**105**] in BMO^{-1}. Suppose $v_0(x)$ is DSS and satisfies (8.60) with small C_* and $v(x, t)$ is the corresponding small mild solution of (NS). Since $v_0^\lambda = v_0$, by the uniqueness property we also have $v^\lambda = v$; hence $v(x, t)$ is also DSS. The same can be said when v_0 is SS, RSS, or RDSS. This gives us the existence in these symmetry classes for small data.

For *large* data, mild solutions still make sense but there is no existence theory since perturbative methods like the contraction mapping no longer work. The usual theory for weak solutions is not applicable since $v_0 \notin L^2(\Omega)$.

Jia and Šverák [**87**] established the first large data existence result for self-similar data which is Hölder continuous in $\mathbb{R}^3 \setminus \{0\}$. It is based on a priori Hölder estimates near the initial time for the *local Leray solutions* introduced by Lemarié-Rieusset in [**128**] (see also [**100**]). This was extended by Tsai [**211**] to construct λ-DSS solutions under the assumption that λ is close to 1 or if the data is axisymmetric with no swirl. A second construction was obtained by Korobkov and Tsai [**113**], which is valid in the half-space as well as the whole space and is based on the a priori H^1-estimate obtained by Leray's method of contradiction, presented in Section 7.2. Note that the first construction [**87**, **211**] does not work in the half-space, while the second construction [**113**] does not work for DSS solutions. A third construction of Bradshaw-Tsai [**16**, **17**] constructs SS/DSS/RSS/RDSS solutions for any data in $L^{3,\infty}(\mathbb{R}^3)$ and, in the DSS/RDSS cases, any $\lambda > 1$. It is based on new a priori energy estimates which are particular to the Leray equations (see (8.7) and (8.12)) and are not available for the Navier-Stokes equations (NS). These estimates give a "weak" solution using the Galerkin approximation. In contrast, [**87**, **113**, **211**] use the Leray-Schauder theorem to construct "strong" solutions. The method of [**87**] is extended to initial data in L^2_{loc}, [**19**, **28**, **129**]. The method of [**16**, **17**] is extended to data in Besov spaces [**18**].

Note that these existence theorems do not assert uniqueness. In fact, nonuniqueness is conjectured; see [**87**]. Moreover, a nonuniqueness example of such generalized self-similar solutions would imply nonuniqueness of Leray-Hopf weak solutions by suitable cut-off; see [**88**].

How do we find a nonuniqueness example? A possible approach is to look for bifurcation of self-similar solutions of the Navier-Stokes equations in the whole space, with size increasing initial data

$$v|_{t=0} = \alpha v_0,$$

v_0 being self-similar and $\alpha \to \infty$. It is more natural to consider this problem in the similarity variables. In the similarity variables (8.5)–(8.6), self-simlar solutions become stationary solutions of the Leray equations (8.7), and DSS solutions become time-periodic solutions. The initial condition for v becomes the boundary condition for V at spatial infinity; see (8.72). If there is a saddle-node bifurcation at $\alpha = \alpha_c$, we get two self-simlar solutions for the same initial data. If there is a Hopf bifurcation, we get a DSS solution. The bifurcation problem is, however, very difficult analytically compared to those in Sections 7.4–7.6, since the linearized operator has a nonlinear dependence on the parameter α. See [**76**] for a numerical investigation.

In this section we will present the Leray-Schauder theorem approach of [**87**] for SS data in $\mathbb{R}^3$ and the Galerkin method approach of [**16**, **17**] for SS/DSS data in $\mathbb{R}^3$ and $\mathbb{R}^3_+$.

8.4.1. Leray-Schauder theorem approach. In this approach, we construct the solutions using the Leray-Schauder theorem, Theorem 7.12. We need to first reformulate the problem as a fixed point problem in a Banach space with a suitable parameter.

The solutions will be sought in the class of *local Leray solutions* introduced by Lemarié-Rieusset in [**128**] (see also [**100**]). Denote for $q \in [1, \infty)$

$$\|f\|_{L^q_{\text{uloc}}} = \sup_{x_0 \in \mathbb{R}^3} \left(\int_{B_1(x_0)} |f|^q(x)\, dx \right)^{\frac{1}{q}} < \infty.$$

Here "uloc" stands for "uniform local". Suppose

$$v_0 \in L^2_{\text{uloc}}(\mathbb{R}^3; \mathbb{R}^3), \quad \operatorname{div} v_0 = 0.$$

A *local Leray solution* of (NS) with initial data v_0 (and zero force) is a pair

$$(v, p) \in L^2_{\text{loc}} \times L^{3/2}_{\text{loc}}(\mathbb{R}^3 \times [0, \infty))$$

that satisfies the following:

(i) For any $R > 0$,

$$\operatorname*{ess\,sup}_{0 \le t < R^2} \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} |v(x,t)|^2 dx + \sup_{x_0 \in \mathbb{R}^3} \int_0^{R^2} \int_{B_R(x_0)} |\nabla v(x,t)|^2 dx\, dt < \infty.$$

(ii) For any $R > 0$,

$$\lim_{|x_0| \to \infty} \int_0^{R^2} \int_{B_R(x_0)} |v(x,t)|^2 dx\, dt = 0.$$

(iii) (v, p) satisfies (NS) in $\mathbb{R}^4_+$ in the sense of distributions.

(iv) $\lim_{t \to 0+} \|v(\cdot, t) - v_0\|_{L^2(K)} = 0$ for any compact set $K \subset \mathbb{R}^3$.

(v) (v, p) satisfies the *local energy inequality*: For all smooth $\phi \ge 0$ with $\operatorname{spt} \phi \Subset \mathbb{R}^3 \times (0, \infty)$,

$$2 \int_{\mathbb{R}^4_+} |\nabla v|^2 \phi\, dx\, dt \le \int_{\mathbb{R}^4_+} |v|^2 (\partial_t \phi + \Delta \phi) + (|v|^2 + 2p) v \cdot \nabla \phi\, dx\, dt. \tag{8.62}$$

Condition (ii) is a weak spatial decay. It allows one to derive an integral formula of the pressure and the a priori bounds in Lemma 8.13 below.

Condition (v) implies (3.8) with $f = 0$, i.e., allowing $\phi|_{t=0} \ne 0$, by replacing ϕ by $\phi \eta_\varepsilon(t)$ and taking $\varepsilon \to 0_+$, where $\eta_\varepsilon(t) = \eta(t/\varepsilon)$, $\eta \in C^1(\mathbb{R})$, $\eta(t) = 1$ for $t \ge 1$, and $\eta(t) = 0$ for $t \le 0$.

The class of local Leray solutions contains both Leray-Hopf weak solutions with $v_0 \in L^2(\mathbb{R}^3)$ and mild solutions with v_0 in L^3 or $L^{3,\infty}(\mathbb{R}^3)$. In particular, it allows self-similar initial data.

Let E^q be the closure of $C_c^\infty(\mathbb{R}^3)$ under L^q_{uloc}-norm. It consists of those $f \in L^q_{\text{uloc}}(\mathbb{R}^3)$ with $\lim_{|x_0| \to \infty} \|f\|_{L^q(B_1(x_0))} = 0$.

We have the following global-in-time existence theorem.

Theorem 8.12. *For any $v_0 \in E^2$ with $\operatorname{div} v_0 = 0$, there is at least one local Leray solution to* (NS) *in $\mathbb{R}^3 \times (0,\infty)$ with initial data v_0.*

We do not expect uniqueness.

We have the following a priori estimates for local Leray solutions, first proved in [**128**], given in the current form in [**86**]. It is the basis for the proof of existence, Theorem 8.12.

Lemma 8.13. *There are constants $0 < C_1 < 1 < C_2$ such that the following holds. Suppose $\operatorname{div} v_0 = 0$, $A = \sup_{x_0\in\mathbb{R}^3} \int_{B_R(x_0)} |v_0(x)|^2 dx < \infty$ for some $R > 0$, and v is a local Leray solution with initial data v_0. Then for $\lambda = C_1 \min\left(A^{-2}R^2, 1\right)$,*

$$\operatorname*{ess\,sup}_{0<t<\lambda R^2} \sup_{x_0\in\mathbb{R}^3} \int_{B_R(x_0)} |v(x,t)|^2 dx + \sup_{x_0\in\mathbb{R}^3} \int_0^{\lambda R^2} \int_{B_R(x_0)} |\nabla v(x,t)|^2 dx dt \le C_2 A.$$

For some $p_{x_0,R}(t)$,

$$\sup_{x_0\in\mathbb{R}^3} \int_0^{\lambda R^2} \int_{B_R(x_0)} |p(x,t) - p_{x_0,R}(t)|^{3/2} dx dt \le C_2 A^{3/2} R^{1/2}.$$

The following theorem states that if we have better local-in-space estimates for the initial data, we also have better local-in-space near initial time estimates for a local Leray solution.

Theorem 8.14. *Suppose $\operatorname{div} v_0 = 0$, $\|v_0\|^2_{L^2_{\mathrm{uloc}}} \le A < \infty$, and $\|v_0\|_{C^\gamma(B_2)} \le M < \infty$ for some $\gamma \in (0,1)$. Then there is $T = T(A,\gamma,M) > 0$ such that any local Leray solution v with initial data v_0 satisfies*

$$v \in C^\gamma_{\mathrm{par}}(\overline{B_{1/4}} \times [0,T]) \quad \textit{and} \quad \|v\|_{C^\gamma_{\mathrm{par}}(\overline{B_{1/4}}\times[0,T])} \le C(A,\gamma,M).$$

This is [**87**, Theorem 3.2], and $C^\gamma_{\mathrm{par}} = C^{\gamma,\gamma/2}_{x,t}$ is as in (6.13). It enables us to prove the following a priori estimates for self-similar local Leray solutions. Denote $\langle x\rangle = (|x|^2+2)^{1/2}$.

Lemma 8.15. *Suppose v is a forward self-similar local Leray solution of* (NS) *in $\mathbb{R}^4_+$ with self-similar initial data v_0, $\operatorname{div} v_0 = 0$. Suppose $v_0 \in C^\gamma_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$, $0 < \gamma < 1$, and $\|v_0\|_{C^\gamma(\overline{B}_2\setminus B_1)} \le C_*$. Let $u(t) = v(t) - e^{t\Delta}v_0$. Then for some $C = C(\gamma, C_*)$, for any $(x,t) \in \mathbb{R}^4_+$,*

$$|v(x,t)| < \frac{C}{|x|+\sqrt{t}}, \quad |u(x,t)| < \frac{C}{\sqrt{t}} \left\langle \frac{x}{\sqrt{t}} \right\rangle^{-2}. \tag{8.63}$$

Proof. Since $|(e^{t\Delta}v_0)(x)| < C(|x|+\sqrt{t})^{-1}$, it suffices to show the u-estimate in (8.63). We will first show a weaker estimate

$$|u(x,t)| < \frac{C(\gamma, C_*)}{\sqrt{t}} \left\langle \frac{x}{\sqrt{t}} \right\rangle^{-1-\gamma}, \quad (x,t) \in \mathbb{R}^4_+. \tag{8.64}$$

By Theorem 8.14, there exists $T_1 = T_1(\gamma, C_*) > 0$ such that for any $x_0 \in \mathbb{R}^3$ with $|x_0| = 1$

$$\|u, v\|_{C^\gamma_{\text{par}}(\overline{B_{1/9}(x_0)}\times[0,T_1])} \le C(\gamma, C_*),$$

where we have used that $e^{t\Delta}v_0$ satisfies the same Hölder estimate. Since $u(x,0) = 0$, we get

$$|u(x,t)| \le Ct^{\gamma/2}, \quad \frac{8}{9} \le |x| \le \frac{10}{9}, \quad 0 \le t \le T_1.$$

Since u is self-similar, $u(x,t) = \frac{1}{|x|}u(\frac{x}{|x|}, \frac{t}{|x|^2})$ and

$$|u(x,t)| \le \frac{C}{|x|}\left(\frac{t}{|x|^2}\right)^{\gamma/2} \le \frac{Ct^{\gamma/2}}{(\sqrt{t}+|x|)^{1+\gamma}}, \quad \text{if } t \le T_1|x|^2.$$

For the complement $t > T_1|x|^2$, note that $V(x) = v(x,1)$ satisfies the stationary Leray equations (8.4) with $\mu = 1$ in $\mathbb{R}^3$. By Lemma 8.13 with $R = 1$ and using the self-similarity,

$$\int_{|x|<8T_1^{-1/2}} |V(x)|^2 + |\nabla V(x)|^2 dx \le C(\gamma, C_*).$$

By regularity theory for perturbed stationary Navier-Stokes equations,

$$\|V\|_{C^2(B(4T_1^{-1/2}))} \le C(\gamma, C_*). \tag{8.65}$$

Thus $|v(x,t)| + |u(x,t)| \le Ct^{-1/2}$ for $t > T_1|x|^2$. Combining estimates in both regions, we get the weaker estimate (8.64).

We now show (8.63). Let $f = -u \otimes u$. By an estimate of $e^{t\Delta}v_0$ and (8.64),

$$|f(x,t)| \lesssim \frac{1}{x^2+t} \quad (x,t) \in \mathbb{R}^4_+.$$

Define a vector field Φf by

$$(\Phi f)_i(x,t) = -\int_0^t \int_{\mathbb{R}^3} \partial_{y_k} S_{ij}(x-y, t-s) f_{jk}(y,s) dyds, \tag{8.66}$$

where S_{ij} is the Oseen tensor given in (5.3). By (5.4), $|\nabla_x S(t,x)| \le C(|x|^2+t)^{-2}$. Thus

$$|\tilde{u}(x,t)| \le \int_0^t \int_{\mathbb{R}^3} \frac{C}{(|x-y|^2+t-s)^2} \frac{1}{y^2+s} dyds \le \frac{C}{\sqrt{t}} \left\langle \frac{x}{\sqrt{t}} \right\rangle^{-2}.$$

(See [**211**, Lemma 2.1].) Note that $w = u - \Phi f$ satisfies the bound (8.64) and the linear Stokes system in $\mathbb{R}^4_+$ with zero initial data and zero source. Hence $w \equiv 0$ (see Problem 8.4). Thus we have $u = \Phi f$ and u satisfies (8.63). □

We will need the following uniqueness result for small data, for which there is a small mild solution. This lemma shows that there is no other local Leray solution.

Lemma 8.16 (Small-large uniqueness)**.** *Suppose v_0 satisfies the assumption of Lemma* 8.15*. If C_* is sufficiently small, then the local Leray solution is unique. Hence it is self-similar.*

This lemma is a special case of [**211**, Lemma 4.1] for DSS solutions, proved using the local energy inequality and Lemma 8.15. It also follows from a general uniqueness result of local Leray solutions with small initial data in $L^{3,\infty}(\mathbb{R}^3)$ of Jia [**85**]. We skip the proof.

We can now prove the existence of self-similar solutions.

Theorem 8.17. *Suppose $v_0 \in C^\gamma_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\}; \mathbb{R}^3)$, $0 < \gamma < 1$, is self-similar and* $\operatorname{div} v_0 = 0$*. Then there is a self-similar local Leray solution v of* (NS) *in $\mathbb{R}^4_+$ with initial data v_0.*

Proof. Let $C_* = \|v_0\|_{C^\gamma(\bar{B}_2 \setminus B_1)}$ and $v_1(x,t) = (e^{t\Delta} v_0)(x)$. By the assumption on v_0, v_1 is self-similar and $|v_1(x,t)| \le CC_*(x^2+t)^{-1/2}$.

Introduce a parameter $\sigma \in [0,1]$. We look for a self-similar solution $v(x,t)$ of (NS) with initial data σv_0. Hence v is of the form

$$v(x,t) = \sigma v_1(x,t) + u(x,t), \quad v(x,0) = \sigma v_0(x).$$

The difference u satisfies the nonhomogeneous Stokes system with zero initial data and force

$$f = -(\sigma v_1 + u) \otimes (\sigma v_1 + u).$$

We expect that u is self-similar and

$$f(x,t) = \lambda^3 f(\lambda x, \lambda^2 t), \quad \forall (x,t) \in \mathbb{R}^4_+. \tag{8.67}$$

In view of Lemma 8.15, we also expect

$$|u(x,t)| \lesssim \frac{1}{\sqrt{t}} \left\langle \frac{x}{\sqrt{t}} \right\rangle^{-1-\gamma}, \quad |f(x,t)| \lesssim \frac{1}{x^2+t}. \tag{8.68}$$

We now set up the framework for the application of the Leray-Schauder theorem. Let

$$X = \left\{ u \in C_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^3) : \quad \operatorname{div} u = 0, \ \|u\|_X < \infty \right\},$$

where

$$\|u\|_X := \sup_{x\in\mathbb{R}^3} \langle x\rangle^{1+\gamma} |u(x)|.$$

For each $u \in X$, we define its self-similar extension by

$$Eu(x,t) = t^{-1/2}u(t^{-1/2}x), \quad (x,t) \in \mathbb{R}^4_+.$$

We now define an operator $K : X \times [0,1] \to X$ by

$$K(u,\sigma) := -\Phi[(\sigma v_1 + Eu) \otimes (\sigma v_1 + Eu)]\big|_{t=1}. \tag{8.69}$$

Above, Φ is the solution operator of the nonhomogeneous Stokes system defined by (8.66).

Note that for $u \in X$ with $\|u\|_X < M$ and $0 \le \sigma \le 1$, the force $f = -(\sigma v_1 + Eu) \otimes (\sigma v_1 + Eu)$ satisfies (8.67) and (8.68), and Φf is self-similar. Thus its restriction at $t = 1$ is inside X and $\|K(u,\sigma)\|_X \le C(C_* + M)^2$. Thus K indeed maps bounded sets in $X \times [0,1]$ into bounded sets in X.

Furthermore, K is compact because its main term $\Phi(\sigma v_1 \otimes \sigma v_1)|_Q$ is 1-dimensional while the other terms of K have extra decay by Lemma 8.15 and are Hölder continuous in x by (8.65).

We have now a fixed point problem

$$u = K(u,\sigma) \quad \text{in } X \tag{8.70}$$

that satisfies the following:

(1) K is continuous; $K(\cdot,\sigma)$ is compact for each σ.
(2) It is uniquely solvable in X for small σ by Lemma 8.16.
(3) We have a priori estimates in X for solutions $(u,\sigma) \in X \times [0,1]$ by Lemma 8.15.

By the Leray-Schauder theorem, Theorem 7.12, there is a solution $u \in X$ of (8.70) with $\sigma = 1$. It follows that Eu satisfies the nonhomogeneous Stokes system with $f = -(v_1 + Eu) \otimes (v_1 + Eu)$, and hence $v = v_1 + Eu$ is a self-similar local Leray solution of (NS) with initial data v_0. $\square$

Note the nonlinear dependence of $K(u,\sigma)$ on σ in (8.69), unlike Sections 7.5–7.6.

The above method is extended to DSS solutions in [**211**] when either the DSS factor λ is sufficiently close to 1, $\lambda - 1 \le \varepsilon(C_*)$, or when v_0 is axisymmetric with no swirl.

8.4.2. Galerkin method approach. In this approach, instead of constructing solutions of (NS) directly, we construct the solutions of the Leray equations (8.7) with $\mu = 1$ using the Galerkin method. Moreover, the method is also valid in the half-space. From now on we assume

$$\Omega \in \{\mathbb{R}^3,\ \mathbb{R}^3_+\}.$$

When $\Omega = \mathbb{R}^3_+$, we add the zero boundary condition.

Recall the similarity transform (8.5)–(8.6). Let $v_1(x,t) = (e^{-tA}v_0)(x)$ be the solution of the Stokes system in Ω with initial data v_0, where A is the Stokes operator in Ω. Define V_0 by

$$V_0(y,s) = \sqrt{t}\, v_1(x,t), \quad y = \frac{x}{\sqrt{t}}, \quad s = \log t. \tag{8.71}$$

In view of Lemma 8.15 (which is also valid for DSS solutions if $\lambda - 1$ is sufficiently small by [**211**]), we expect more spatial decay of $V - V_0$ than V_0. In other words, V_0 can be regarded as the boundary value of V at spatial infinity, and the initial value problem of λ-DSS solutions of (NS) is equivalent to the *boundary value problem* of periodic solutions of (8.7) with period $T = 2\log\lambda$. One way to give this boundary condition a concrete sense is to require

$$\int_0^T \int_{\mathbb{R}^3} |V - V_0|^2(y,s)\, dyds < \infty. \tag{8.72}$$

Note that $V_0 \notin L^2(\mathbb{R}^3 \times (0,T))$ since we expect

$$|V_0(y,s)| \sim \frac{C}{1+|y|}. \tag{8.73}$$

Let $L^3_{w,\sigma}(\Omega) = \big\{v \in L^3_w(\Omega;\mathbb{R}^3) :\ \operatorname{div} v = 0,\ v\cdot \mathbf{n}|_{\partial\Omega} = 0\big\}$. Since the theory of local Leray solutions is not available in $\mathbb{R}^3_+$, we make the following definition.

Definition 8.18 (EP-solutions)**.** Let Ω be a domain in $\mathbb{R}^3$. A vector field v defined on $\Omega \times (0,\infty)$ is an *energy perturbed solution to* (NS) — i.e., an *EP-solution* — with initial data $v_0 \in L^3_{w,\sigma}(\Omega)$ if

$$\int_0^\infty \big((v,\partial_s f) - (\nabla v, \nabla f) - (v\cdot\nabla v, f)\big)\, ds = 0, \tag{8.74}$$

for all $f \in \{f \in C^\infty_c(\Omega \times \mathbb{R}_+) : \nabla\cdot f = 0\}$, if

$$v - v_1 \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),$$

for any $T > 0$, and if

$$\lim_{t\to 0^+} \|v(t) - v_1(t)\|_{L^2(\Omega)} = 0,$$

where $v_1(t) \in L^\infty(0,\infty; L^3_{w,\sigma}(\Omega))$ is the solution to the time-dependent Stokes system with initial data v_0 and zero boundary value.

The name "energy perturbed solution" means that the difference $v - v_1$ is in the energy class, although v_1 is not. The pressure is not mentioned in the definition.

We have the following:

Theorem 8.19. *Let $\Omega \in \{\mathbb{R}^3, \mathbb{R}^3_+\}$ and assume v_0 is in $L^3_{w,\sigma}(\Omega)$.*

(i) *(SS) If v_0 is SS, then there exists a self-similar EP-solution v on $\Omega \times [0,\infty)$ with initial data v_0.*

(ii) *(DSS) If v_0 is DSS with factor $\lambda > 1$, then there exists a λ-DSS EP-solution v on $\Omega \times [0,\infty)$ with initial data v_0.*

In both cases, $v|_{\partial\Omega} = 0$ if $\Omega = \mathbb{R}^3_+$.

We will prove case (ii) by showing the existence of the corresponding time-periodic solutions of the Leray equations (8.7) with $\mu = 1$. We seek a solution of (8.7) of the form $V = V_0 + u$ as this homogenizes the boundary condition at spatial infinity. The difference u satisfies a perturbed Leray system,

$$\begin{gathered} Lu + (u\cdot\nabla)u + (V_0\cdot\nabla)u + (u\cdot\nabla)V_0 + \nabla p = F(V_0), \\ \operatorname{div} u = 0, \end{gathered} \tag{8.75}$$

in Ω, where $F(V_0) = -LV_0 - (V_0\cdot\nabla)V_0$ and

$$Lu = \partial_s u - \Delta u - \frac{1}{2}u - \frac{1}{2}y\cdot\nabla u. \tag{8.76}$$

Testing with u itself, several terms in (8.75) vanish and we formally have, similar to (7.6),

$$\frac{d}{2ds}\int |u|^2 + \int \left(|\nabla u|^2 + \frac{1}{4}|u|^2\right) = \int (u\cdot\nabla)u\cdot V_0 + \langle F, u\rangle\,. \tag{8.77}$$

The integral $\int (u\cdot\nabla)u\cdot V_0$ is not necessarily small and prevents us from proving an a priori bound. This difficulty is similar to that in Section 7.1.

To get around this, we replace V_0 by W which eliminates the possibly large behavior of V_0 near the origin, with the correction $W - V_0$ being compactly supported. This will give us the crucial bound,

$$\int (f\cdot\nabla f)\cdot W\,dy \le \delta(||\nabla f||^2_{L^2} + ||f||^2_{L^2}), \tag{8.78}$$

where δ is a given small parameter. This is not possible for V satisfying (8.73) with a large constant. The integral $\iint |u|^2$ on the left side of (8.77), which is absent in (7.6), helps to control the right side of (8.78).

We start with the properties of the profile V_0.

Lemma 8.20. *Let $\Omega \in \{\mathbb{R}^3, \mathbb{R}^3_+\}$ and assume $v_0 \in L^3_{w,\sigma}(\Omega)$ is DSS with factor $\lambda > 1$. The vector field $V_0(y,s) : \Omega \times \mathbb{R} \to \mathbb{R}^3$ defined by (8.71) is periodic-in-s with period $T = 2\log\lambda$, is divergence-free, vanishes on $\partial\Omega$, and satisfies, for all $q \in (3,\infty]$,*

- *for all divergence-free $f \in C_c^\infty(\Omega \times \mathbb{R}; \mathbb{R}^3)$, $\int_{\mathbb{R}}\int_\Omega (LV_0)f = 0$,*
- *the inclusions:*

$$V_0 \in L^\infty(0,T; L^4 \cap L^q(\Omega)),$$

$$\partial_s V_0 \in L^\infty(0,T; L^{6/5}_{\mathrm{loc}}(\overline{\Omega})), \quad \nabla V_0 \in L^2(0,T; L^2_{\mathrm{loc}}(\overline{\Omega})),$$

- *the decay estimate:*

$$\sup_{s\in[0,T]} \|V_0\|_{L^q(\Omega\setminus B_R)} \le \Theta(R), \tag{8.79}$$

for some $\Theta : [0,\infty) \to [0,\infty)$ such that $\Theta(R) \to 0$ as $R \to \infty$.

The proof is based on a characterization of DSS functions in $L^{3,\infty}$ and the solution formula for $v_1(x,t)$ using the Green's tensor in Ω. See [**17**, §3] for the proof. Note that $q > 3$ because of (8.73). Also note that $LV_0 = 0$ if $\Omega = \mathbb{R}^3$ but $LV_0 = \nabla P$ for some P if $\Omega = \mathbb{R}^3_+$.

We now define the revised profile W. Fix $Z \in C^\infty(\Omega)$ with $0 \le Z \le 1$, $Z(y) = 1$ for $|y| > 1$, and $Z(y) = 0$ for $|y| < 1/2$. Let $\Omega_r = \Omega \cap B_r$ and let Φ be the Bogovskii map in Ω_1 given in Lemma 2.16. Then

$$\Phi_r f(y) = (\Phi f_r)(y/r), \quad f_r(y) = f(ry),$$

is a Bogovskii map in Ω_r for $f \in L^q(\Omega_r)$, $\int_{\Omega_r} f = 0$, with $\Phi_r f \in W_0^{1,q}(\Omega_r)$,

$$\operatorname{div}\Phi_r f = f, \quad \|\nabla(\Phi_r f)\|_{L^q(\Omega_r)} \le C\|f\|_{L^q(\Omega_r)},$$

where C is independent of r. For $r > 1$, let

$$W(y,s) = V_0(y,s)Z(y/r) - \hat{W}(y,s), \quad \operatorname{div} W = 0, \tag{8.80}$$

where

$$\hat{W}(\cdot,s) = \Phi_r\left(V_0 \cdot r^{-1}\nabla Z(\cdot/r)\right),$$

noting that $\int_{\Omega_r} V_0 \cdot r^{-1}\nabla Z(\cdot/r) = \int_{\Omega_r} \operatorname{div}(V_0 Z(\cdot/r)) = \int_{\partial\Omega_r} V_0 Z(\cdot/r)\cdot \mathbf{n} = 0$.

Note that $\hat{W}$ is determined by V_0 in $\Omega_r \setminus \Omega_{r/2}$, and we can show

$$\|\hat{W}(\cdot,s)\|_{L^q(\Omega_r)} \le C\|V_0(\cdot,s)\|_{L^q(\Omega_r\setminus\Omega_{r/2})}.$$

Hence it is small for large r by (8.79).

Lemma 8.21 (Revised asymptotic profile)**.** *Let $\Omega \in \{\mathbb{R}^3, \mathbb{R}^3_+\}$. Fix $q \in (3,\infty]$ and suppose V_0 satisfies the conclusion of Lemma 8.20 for this q. For any small $\delta > 0$, let W be defined by (8.80) with $r = r_0(\delta)$ sufficiently large.*

Then W is T-periodic and divergence-free,

$$V_0 - W \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),$$
$$\|W\|_{L^\infty(0,T;L^q(\Omega))} \le \delta,$$

and

$$\|W\|_{L^\infty(0,T;L^4(\Omega))} + \|LW\|_{L^\infty(0,T;H^{-1}(\Omega))} \le c(r_0, V_0).$$

We now decompose

$$V = W + u.$$

Then u satisfies (8.75) with V_0 replaced by W, that is, the *perturbed Leray equations*

$$Lu + (u\cdot\nabla)u + (W\cdot\nabla)u + (u\cdot\nabla)W + \nabla p = F(W), \qquad \operatorname{div} u = 0, \tag{8.81}$$

where $F(W) = -LW - W\cdot\nabla W$.

We can now derive a priori bounds for $u = V - W$ for a suitable W. Fix $q \in (3,\infty]$. Let $s = 2q/(q-2) \in [2,6)$, and let C_1 be the constant of $\|\nabla u\|_{L^2}\|u\|_{L^s} \le C_1 E(u)$, where

$$E(u) = ||\nabla u||_{L^2}^2 + \frac{1}{4}||u||_{L^2}^2.$$

Let W be as in Lemma 8.21 with $\|W\|_{L^\infty L^q} \le \delta = 1/(4C_1)$. Since

$$\int (u\cdot\nabla u)\cdot W\,dy \le \|W\|_{L^q}\|\nabla u\|_{L^2}\|u\|_{L^s} \le \delta C_1 E(u) = \frac{1}{4}E(u) \tag{8.82}$$

and

$$\langle F, u\rangle \le \frac{1}{4}E(u) + C_2\|F\|_{H^{-1}}^2,$$

from (8.77) (with V_0 replaced by W) we get

$$\frac{d}{2ds}\int |u|^2 + E(u) \le \frac{1}{4}E(u) + \frac{1}{4}E(u) + C_2\|F\|_{H^{-1}}^2. \tag{8.83}$$

Integrating (8.83) in $s \in [0,T]$, we get

$$\int_0^T E(u(s))ds \le M, \quad M = 2C_2\int_0^T \|F(W(s))\|_{H^{-1}}^2 ds. \tag{8.84}$$

Choose s_1 so that $E(u(s_1)) \le M/T$. Integrating (8.83) from $s = s_1$, we get

$$\operatorname*{ess\,sup}_{s\in[0,T]} \int |u(s)|^2 \le CM\left(\frac{1}{T} + 1\right). \tag{8.85}$$

The a priori bounds (8.84)–(8.85) enable us to use the *Galerkin method* to construct a periodic weak solution u of (8.81) in the class

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \tag{8.86}$$

Choose $\{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{C}^\infty_{c,\sigma}(\Omega)$ whose span is dense in $H^1_{0,\sigma}(\Omega)$ and which is orthonormal in $L^2_\sigma(\Omega)$; see the proof of Theorem 2.7. For a fixed k, we look for an approximation solution of the form

$$u_k(y,s) = \textstyle\sum_{i=1}^k b_{ki}(s)\psi_i(y). \tag{8.87}$$

We first prove the existence of and a priori bounds for T-periodic solutions $b_k = (b_{k1},\dots,b_{kk})$ to the system of ODEs

$$\frac{d}{ds}b_{kj} = \sum_{i=1}^k A_{ij}b_{ki} + \sum_{i,l=1}^k B_{ilj}b_{ki}b_{kl} + C_j, \tag{8.88}$$

for $j \in \{1,\dots,k\}$, where

$$\begin{aligned} A_{ij} &= -(\nabla\psi_i,\nabla\psi_j) + \left(\frac{1}{2}\psi_i + \frac{y}{2}\cdot\nabla\psi_i - \psi_i\cdot\nabla W - W\cdot\nabla\psi_i,\psi_j\right), \\ B_{ilj} &= -(\psi_l\cdot\nabla\psi_i,\psi_j), \\ C_j &= \langle F(W),\psi_j\rangle. \end{aligned}$$

Lemma 8.22. *For any $k\in\mathbb{N}$, the system of ODEs* (8.88) *has a T-periodic solution $b_k \in H^1(0,T)$. Moreover, for u_k defined by* (8.87)*, we have*

$$||u_k||_{L^\infty(0,T;L^2(\Omega))} + ||u_k||_{L^2(0,T;H^1(\Omega))} < C, \tag{8.89}$$

where C is independent of k.

Proof. For any $u^0 \in \mathrm{span}(\psi_1,\dots,\psi_k)$, there exist $b_{kj}(s)$ uniquely solving (8.88) with initial value $b_{kj}(0) = (u^0,\psi_j)$ and belonging to $H^1(0,\tilde{T})$ for some time $0 < \tilde{T} \le T$. Multiply the j-th equation of (8.88) by b_{kj} and sum to obtain

$$\frac{1}{2}\frac{d}{ds}||u_k||^2_{L^2} + E(u_k) = -(u_k\cdot\nabla W,u_k) + \langle F(W),u_k\rangle.$$

Above, we have used $((u_k+W)\cdot\nabla u_k,u_k) = 0$. This equality implies the linear bound $\frac{d}{ds}||u_k||^2_{L^2} \le C||u_k||^2_{L^2} + C$. Hence the solution is global and $\tilde{T} = T$.

We now bound the right-hand side of the above equality. By (8.82),

$$\big|(u_k\cdot\nabla W,u_k)\big| \le \frac{1}{4}E(u_k).$$

For $F(W) = -LW - \mathrm{div}(W\otimes W)$,

$$|\langle F(W),u_k\rangle| \le (\|LW\|_{H^{-1}} + \|W\|^2_{L^4})\|u_k\|_{H^1} \le C_2 + \frac{1}{4}E(u_k),$$

where $C_2 = C(\|LW\|_{L^\infty H^{-1}} + \|W\|^2_{L^\infty L^4})^2$ is bounded by Lemma 8.21 and is independent of k. The above estimates imply

$$\frac{d}{ds}||u_k||^2_{L^2} + E(u_k) \le 2C_2. \tag{8.90}$$

Recall that $E(u_k) = ||\nabla u_k||_{L^2}^2 + \frac{1}{4}||u_k||_{L^2}^2$. The Gronwall inequality implies

$$
\begin{aligned}
e^{s/4}||u_k(s)||_{L^2}^2 &\le ||u^0||_{L^2}^2 + \int_0^T e^{\tau/4} 2C_2 \, d\tau \\
&\le ||u^0||_{L^2}^2 + e^{T/4} 2C_2 T
\end{aligned}
\tag{8.91}
$$

for all $s \in [0, T]$. By (8.91) we can choose $\rho > 0$ (independent of k) so that

$$||u^0||_{L^2} \le \rho \Rightarrow ||u_k(T)||_{L^2} \le \rho.$$

The mapping $\mathcal{T} : \bar{B}_\rho^k \to \bar{B}_\rho^k$ given by $\mathcal{T}(b_k(0)) = b_k(T)$, where $\bar{B}_\rho^k$ is the closed ball of radius ρ in $\mathbb{R}^k$, is continuous. Thus $\mathcal{T}$ has a fixed point by the Brouwer fixed point theorem; i.e., there exists some $u^0 \in \operatorname{span}(\psi_1, \ldots, \psi_k)$ so that $b_k(0) = b_k(T)$.

It remains to check that (8.89) holds. The $L^\infty L^2$-bound follows from (8.91) since $\|u^0\|_{L^2} \le \rho$, which is independent of k. Integrating (8.90) over $[0, T]$ and using $u_k(0) = u_k(T)$, we get $\int_0^T E(u_k(s)) ds \le 2C_2 T$, which gives a uniform-in-k upper bound for $\|u_k\|_{L^2(0,T;H^1)}$. □

We can now construct a periodic weak solution u of (8.81) in the class (8.86). Since the u_k satisfy the uniform bound (8.89), an argument similar to the proof of Theorem 3.9 shows that there exists u satisfying the same bound and a subsequence of $\{u_k\}$ (which we still index with k) so that

$$
\begin{aligned}
&u_k \to u \text{ weakly in } L^2(0, T; H^1), \\
&u_k \to u \text{ strongly in } L^2(0, T; L^2(K)) \text{ for all compact sets } K \subset \bar{\Omega}, \\
&u_k(s) \to u(s) \text{ weakly in } L^2 \text{ for all } s \in [0, T].
\end{aligned}
$$

The weak convergence guarantees that $u(0) = u(T)$ and that u is a periodic weak solution of (8.81).

Let $V = u + W$. It satisfies the weak form of (8.7) with $\mu = 1$. Since W and u are T-periodic, so is V. The a priori bounds for $V - W$ extend to bounds for $V - V_0$ since $V_0 - W \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$.

We now define $v(x, t) = t^{-1/2} V(y, s)$ according to the similarity transform (8.5)–(8.6). It is a weak solution of (NS). Let $v_2 = v - v_1$, $v_2(x, t) = t^{-1/2}(V - V_0)(y, s)$. The a priori bound for $V - V_0$ implies

$$v_2 \in L^\infty(1, \lambda^2; L^2(\Omega)) \cap L^2(1, \lambda^2; H^1(\Omega)).$$

The λ-DSS scaling property implies that, for all $t > 0$,

$$||v_2(t)||_{L^2(\Omega)}^2 \lesssim t^{1/2} \sup_{1 \le \tau \le \lambda^2} ||v_2(\tau)||_{L^2(\Omega)}^2$$

(thus $\lim_{t\to 0}||v_2(t)||_{L^2(\Omega)} = 0$) and

$$\int_0^{\lambda^2} ||\nabla v_2(t)||_{L^2}^2\, dt \lesssim \left(\sum_{k=0}^{\infty} \lambda^{-k}\right) \int_1^{\lambda^2} ||\nabla v_2(t)||_{L^2}^2\, dt.$$

It follows that

$$v_2 \in L^\infty(0,\lambda^2; L^2(\Omega)) \cap L^2(0,\lambda^2; H^1(\Omega)).$$

By rescaling, these bounds hold up to any finite time. Thus, v is an EP-solution and we have proved case (ii) of Theorem 8.19.

8.5. Notes

Concerning Section 8.2, stationary self-similar solutions were first found by Slezkin [**191**] (see [**56**, Appendix] for English translation), then independently by Landau [**124**] (see also [**125**, §23]) and by Squire [**198**]. See also Batchelor [**9**, §4.6], Tian-Xin [**207**], and Cannone-Karch [**24**]. Our proofs of Theorem 8.1 and Lemma 8.2 are due to Šverák and Tsai [**210**, §4.3].

Section 8.3 is based on Nečas-Růžička-Šverák [**155**], Tsai [**209**], and Scheffer [**175**]. The triviality of the profile V has also been shown under the assumption $V \in C^\infty_{\mathrm{loc}} \cap L^{q,\infty}(\mathbb{R}^3)$, $\frac{3}{2} < q < 3$ by [**29, 75**]. I learned of the DSS interpretation of Scheffer's example from R. Kohn in 1998. There are also results on "asymptotically" or "locally" self-similar solutions; see [**26, 83**].

Section 8.4 is based on Jia and Šverák [**87**], Tsai [**211**], Bradshaw-Tsai [**16, 17**]. See also extensions of Lemarié-Rieusset [**129**, Chapter 16], Chae-Wolf [**30**], and Bradshaw-Tsai [**18, 19**].

Problems

8.1. For a solution v of (8.13)–(8.14), prove (8.15) using Lemma 2.11. We may assume $|x| = 1$ by a scaling argument.

8.2. For stationary solutions (v, p) of (NS) satisfying the bound $|v(x)| \lesssim |x|^{-1}$ in $\mathbb{R}^3 \setminus \{0\}$, introduce the similarity variables

$$x = \rho\theta, \quad \rho = |x|, \quad \theta \in \mathbb{S}^2, \quad \tau = \log\rho,$$

$$v(\rho,\theta) = \rho^{-1}V(\tau,\theta), \quad p(\rho,\theta) = \rho^{-2}P(\tau,\theta).$$

Find the equations satisfied by $(V(\tau,\theta), P(\tau,\theta))$ for $\tau \in \mathbb{R}$ and $\theta \in \mathbb{S}^2$. If v is DSS with factor $\lambda > 1$, then $V(\tau,\theta)$ is periodic in τ with period $\log\lambda$.

8.3. It is shown by Brandolese [**20**] that small forward self-similar solutions with initial data $v_0 \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$ satisfy for any $T \geq 1$

$$\sup_{1 \leq t \leq T} |v(x,t) - v_0(x)| \leq \frac{C_T}{|x|^2} \quad (|x| > 1).$$

What if v is DSS?

8.4. Suppose a very weak solution w of the linear Stokes system in $\mathbb{R}^4_+$ with zero initial data and zero source satisfies the bound (8.64). Show that $w \equiv 0$.

Chapter 9

The uniform L^3 class

In this chapter we consider the uniqueness and regularity for *weak* solutions of (NS) in the class

$$v \in L^\infty(0,T; L^3_\sigma(\Omega)). \tag{9.1}$$

Recall that a Leray-Hopf weak solution is a vector field

$$v \in C_{wk}([0,T); L^2_\sigma(\Omega)) \cap L^2(0,T; H^1_0(\Omega))$$

that satisfies the energy inequality and the Navier-Stokes equations (NS) in the weak sense. We have shown in Chapter 4 the uniqueness and regularity of Leray-Hopf weak solutions in the classes

$$v \in L^s(0,T; L^q(\Omega)), \quad 3/q + 2/s \le 1, \quad 2 < s < \infty. \tag{9.2}$$

Equation (9.1) is an endpoint case of (9.2). Regularity results in the classes (9.2) can be considered as ε-regularity results, because if v satisfies (9.2), then

$$\lim_{\varepsilon \to 0_+} \|v\|_{L^s(T-\varepsilon,T;L^q(\Omega))} = 0. \tag{9.3}$$

This is not the case if $v \in L^\infty_t L^3_x$. With (9.1), we do have certain smallness because $\|v(\cdot,t)\|_{L^3(B_R)} \to 0$ as $R \to 0$ for each t; however, this convergence may not be uniform in time. A possible profile in the class (9.1) is

$$v(x,t) \sim \lambda(t) V(\lambda(t)x), \quad V \in L^3(\mathbb{R}^3),$$

where

$$\lambda(t) \to \infty, \quad \text{as } t \to T_-.$$

The choice of $\lambda(t)$ could be quite arbitrary.

A hypothetical *Type I* singularity

$$|v(x,t)| \sim \frac{C}{|x-x_0|+\sqrt{T-t}} \tag{9.4}$$

does not satisfy either (9.2) or (9.1). It belongs to the class $L_t^\infty L_x^{3,\infty}$ and is excluded only if C is small; see Problem 6.5.

9.1. Uniqueness

Our goal in this section is to prove Theorem 9.3, the *uniqueness* of *weak* solutions in the $L^\infty L^3$ class. It is due to Kozono-Sohr [**115**], which improves the earlier result of Masuda [**145**]. In this theorem, only one of the two solutions needs to be in $L^\infty L^3$. This is another instance of weak-strong uniqueness.

Theorem 9.1 (Masuda [**145**]). *Let Ω be any domain in $\mathbb{R}^3$. Let both u, v be Leray-Hopf weak solutions of* (NS) *in $\Omega \times [0,T)$ with same initial data $a \in L^2_\sigma \cap L^3_\sigma(\Omega)$ and zero force. Suppose also that $u \in L^\infty(0,T;L^3(\Omega))$ and that u is right continuous on $[0,T)$ with values in L^3. Then $u \equiv v$ in $[0,T)$.*

The zero boundary condition is part of the definition for Leray-Hopf weak solutions; see Definition 3.1.

Proof. Let $w = u - v$. Recall (4.12) from the proof of Theorem 4.4,

$$\int |w|^2(t)\,dx + 2\int_0^t\int |\nabla w|^2\,dxdt \le I(t) := -2\int_0^t\int u\cdot(w\cdot\nabla)w\,dxdt. \tag{9.5}$$

The idea was to show that RHS(t) is bounded by $\varepsilon \sup_{s<t}$ LHS(s) using (9.3), which fails for $s=\infty$. The new idea is to show

$$|I(t)| \le \int_0^t\int |\nabla w|^2\,dxdt + N\int_0^t\int |w|^2\,dxdt \tag{9.6}$$

for some large N depending on u (not v), for $t \in [0,\delta]$ for some $\delta > 0$. If this is true, by (9.5) and the Gronwall inequality we get $w(t) = 0$ in $[0,\delta]$. To show that $w(t) = 0$ in $[0,T)$, let $t_0 = \sup\{t \ge 0 : w(s) = 0 \text{ if } 0 \le s \le t\}$. If $t_0 < T$, we can apply the above argument with initial time t_0 to get a contradiction.

To prove (9.6), we decompose $u = u_1 + u_2$ where

$$u_1 = \phi_q(|u|)\,u, \quad u_2 = [1-\phi_q(|u|)]\,u,$$

$\phi_q(s) = \phi(s/q)$, $\phi(s) \ge 0$ is a fixed C^1-function on $\mathbb{R}$ with $\phi(s) = 1$ if $s < 1/2$, and $\phi(s) = 0$ if $s > 1$. For any $0 < \varepsilon \ll 1$, we can choose q so large, depending on $u(0)$ and ε, that

$$\|u_2(0)\|_{L^3} \le \varepsilon.$$

Using the inequality for two vectors ξ and η,

$$\big|\phi_q(|\xi|)\xi - \phi_q(|\eta|)\eta\big| \le C|\xi - \eta|, \quad C = \sup_s(\phi(s) + s|\phi'(s)|),$$

which can be proved by integrating $\frac{d}{dt}(\phi_q(|x(t)|)x(t))$, $x(t) = \eta + t(\xi - \eta)$, we have $|u_2(t) - u_2(0)| \le C|u(t) - u(0)|$. Hence

$$\|u_2(t) - u_2(0)\|_{L^3} \le C\|u(t) - u(0)\|_{L^3} \le \varepsilon$$

for $t \in (0, \delta)$ and $\delta = \delta(u) > 0$ sufficiently small, using the assumption that $u(t)$ is right continuous. Thus $\|u_2(t)\|_{L^3} \le \|u_2(0)\|_{L^3} + \|u_2(t) - u_2(0)\|_{L^3} \le 2\varepsilon$. We now have $\|u_1\|_{L^\infty_{t,x}} \le q$ and, for $t < \delta$,

$$\begin{aligned} |I(t)| &\le 2[\|u_1\|_{L^\infty_{t,x}}\|w\|_{L^2_{t,x}} + \|u_2\|_{L^\infty L^3}\|w\|_{L^2L^6}] \cdot \|\nabla w\|_{L^2_{t,x}} \\ &\le 2q^2\|w\|^2_{L^2_{t,x}} + \left(\frac{1}{2} + 2\varepsilon C_1\right)\|\nabla w\|^2_{L^2_{t,x}}. \end{aligned}$$

We have used Hölder inequality, Schwartz inequality, and Sobolev inequality $\|w\|_{L^6} \le C_1\|\nabla w\|_{L^2}$. The time domain is $[0, t]$ in the norms. This shows (9.6) with $\varepsilon = 1/(4C_1)$ and $N = 2q^2$. $\square$

Masuda's theorem is valid for higher dimensions for a refined definition of weak solutions.

The following lemma shows that the right continuity can be proved for domains allowing strong mild solutions in $L^2 \cap L^3$.

Lemma 9.2 (Kozono-Sohr [**115**]). *Let $\Omega \subset \mathbb{R}^3$ be $\mathbb{R}^3$ or a smooth bounded domain. If $u \in L^\infty(0, T; L^3(\Omega))$ is a Leray-Hopf weak solution of* (NS) *in $\Omega \times (0, T)$ with zero force, then $u \in C_{wk}([0, T); L^3(\Omega))$ and u is continuous from the right in the L^3-norm.*

Proof. *Step 1.* $u(t) \in L^3$ for all t, and $u \in C_{wk}([0, T), L^3)$.

Let $M = \|u\|_{L^\infty L^3}$. For any $t_0 \in [0, T)$, there is a sequence $t_k \to t_0$ with $\|u(t_k)\|_{L^3} \le M$. A subsequence converges weakly to some w in L^3 as $t_k \to t$. Being a Leray-Hopf weak solution, $u(t_k)$ converges weakly to $u(t_0)$ in L^2. Thus $u(t_0) = w \in L^2 \cap L^3$ and $\|u(t_0)\|_{L^3} \le M$. Together with $u \in C_{wk}([0, T), L^2)$, we get $u \in C_{wk}([0, T), L^3)$: For any $\phi \in L^{3/2}(\Omega)$ and any $\varepsilon > 0$, choose $\psi \in L^2 \cap L^{3/2}$ with $\|\phi - \psi\|_{L^{3/2}} \le \varepsilon/(4M)$. Then

$$\begin{aligned} (u(t) - u(t_0), \phi) &= (u(t) - u(t_0), \psi) + (u(t) - u(t_0), \phi - \psi) \\ &=: I + J. \end{aligned}$$

We have $|J| \le \varepsilon/2$ for any t, while $I \to 0$ as $t \to t_0$.

Step 2. Right continuity of $u(t)$ in L^3.

(In this step we use the assumption on the domain.)

For any $t_0 \in [0,T)$, $u(t_0) \in L^2 \cap L^3(\Omega)$. By Theorem 5.6, for our domain Ω, there is $\delta > 0$ and a mild solution $\tilde{u} \in BC([t_0, t_0+\delta), L^2 \cap L^3(\Omega))$ with $\tilde{u}(t_0) = u(t_0)$, which is also a Leray-Hopf weak solution.

Since $\tilde{u}$ is continuous, we have $u(t) = \tilde{u}(t)$ in $[t_0, t_0+\delta)$ by Theorem 9.1. Thus $u(t)$ is right continuous at t_0 in L^3. □

As a corollary of Theorem 9.1 and Lemma 9.2, we have the following.

Theorem 9.3 ([**115**])**.** *Let $\Omega \subset \mathbb{R}^3$ be as in Lemma 9.2. Any two Leray-Hopf weak solutions of* (NS) *in $\Omega \times [0,T)$ with same initial data and zero force must agree with each other if one of them is in $L^\infty(0,T;L^3(\Omega))$.*

Theorem 9.3 implies that, starting from an initial data $v_0 \in L^2_\sigma \cap L^3(\Omega)$, the weak solution remains unique as long as it is in L^3.

9.2. Auxiliary results for regularity

In this section we describe a few auxiliary results to be used in the next section, when we prove the regularity of weak solutions in the $L^\infty L^3$ class. The proof, due to Escauriaza, Seregin, and Šverák [**41**], uses in an essential way the vorticity equations, which are a parabolic system of the form

$$\partial_t w_i - \Delta w_i = b_{ijk}\partial_k w_j + c_{ij} w_j. \tag{9.7}$$

We will need two results for such systems: *backward uniqueness* and *unique continuation*. The proof of the latter will be sketched in Section 9.4. For solutions of the heat equation, both results follow from their analyticity. Solutions of (9.7) may not be analytic, but both results are still valid and can be proved using Carleman-type inequalities (see Lemmas 9.9 and 9.11). These results and Carleman-type inequalities are basic in the study of control theory and inverse problems.

We will also give an interior estimate for the time-dependent Stokes system.

We first describe the backward uniqueness result.

Theorem 9.4 (Backward uniqueness)**.** *Let $n, m \in \mathbb{N}$ and $D = \mathbb{R}^n_+ \times (-1,0)$. Suppose $w : D \to \mathbb{R}^m$, $w, \partial_t w, \nabla^2 w \in L^2(D \cap Q_R)$ for any $R > 0$, and*

$$w(x,0) = 0, \quad x \in \mathbb{R}^n_+, \tag{9.8}$$

$$|\partial_t w - \Delta w| \le c_1(|w| + |\nabla w|), \quad |w(x,t)| \le e^{M|x|^2} \quad \text{in } D, \tag{9.9}$$

for some constants c_1 and M. Then $w \equiv 0$ in D.

It says that if w satisfies (9.9) and w vanishes at the final time $t = 0$, then it is identically zero. It is important to note that no assumption is made on the boundary value of w on $\partial\mathbb{R}^n_+ \times (-1, 0)$. The first condition in (9.9) is implied by (9.7) if the coefficients are bounded. The uniform upper bound $|w(x,t)| \le e^{M|x|^2}$ is necessary in view of the Tychonoff examples [**89**, §7.1].

We next describe the unique continuation result. An example in the elliptic PDE setting is for $u : B_1 \subset \mathbb{R}^n \to \mathbb{R}$,

$$\Delta u = b_i(x)\partial_i u + c(x)u, \quad b_i, c \in C(B_1). \tag{9.10}$$

If $|u(x)| \le C_k|x|^k$ for any $k \in \mathbb{N}$, then $u \equiv 0$ in B_1. This result is well known if $b_i(x) = c(x) = 0$ by analyticity of $u(x)$ and can be proved using Carleman-type inequality. It is the time-independent case of Theorem 9.5 below, to be proved in Section 9.4.

Theorem 9.5 (Unique continuation)**.** *Let* $D = B_R \times (0,T) \subset \mathbb{R}^n \times \mathbb{R}$ *with* $R, T > 0$. *Suppose* $w : D \to \mathbb{R}^m$ *and* $w, \partial_t w, \nabla w, \nabla^2 w \in L^2(D)$. *If* w *satisfies for some constant* c_1

$$|\partial_t w + \Delta w| \le c_1(|w| + |\nabla w|) \quad \textit{a.e. in } D \tag{9.11}$$

and

$$w(x,t)(x^2 + t)^{-k} \in L^\infty(D), \quad \forall k \in \mathbb{N}, \tag{9.12}$$

then $w(x,0) = 0$ *for all* $x \in B_{R/20}$.

Above, we have changed the sign of t. The second condition (9.12) means that w vanishes at the origin to arbitrary order. This theorem does not address $w(x,t)$ for $t \ne 0$.

We will also need an interior estimate for the Stokes system.

Lemma 9.6 (Interior estimate for the Stokes system)**.** *Let* $1 < s, q < \infty$. *If* u *is a weak solution of the time-dependent Stokes system,*

$$\partial_t u - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0$$

in $D = B_R \times (0,T)$, $B_R \subset \mathbb{R}^n$, *then for any small* $\varepsilon > 0$,

$$\| |\partial_t u| + |\nabla^2 u| + |\nabla p| \|_{L^s(\varepsilon,T;L^q(B_{R-\varepsilon}))} \le c\|f\|_{L^sL^q(D)} + c\| |u| + |p| \|_{L^sL^1(D)},$$

where $c = c(R, T, \varepsilon, s, q)$.

This lemma can be proved using cut-off and Lemma 3.8. Since it is on the entire $\mathbb{R}^n$ after cut-off, instead of Lemma 3.8 we can use the fundamental solution of the Stokes system and the boundedness of parabolic singular integrals in $L^sL^q(\mathbb{R}^{n+1}_+)$. See [**31**, Lemma A.2] for similar estimates.

9.3. Regularity

We next prove regularity of weak solutions in the $L^\infty L^3$ class. We state two theorems: Theorem 9.7 is a global result while Theorem 9.8 is local.

Theorem 9.7 (Global regularity). *Suppose v is a Leray-Hopf weak solution of* (NS) *in $D = \mathbb{R}^3 \times (0,T)$ with $f = 0$ and $v \in L^\infty L^3(D)$. Then $v \in L^5(D)$.*

It then follows that $v \in L^\infty(\mathbb{R}^3 \times (\varepsilon, T))$ for any $\varepsilon > 0$ by the results in Chapter 5.

Theorem 9.8 (Interior regularity). *Suppose (v,p) satisfies $v \in L^\infty L^3(Q_1)$, $\nabla v \in L^2(Q_1)$, $p \in L^{3/2}(Q_1)$, and* (NS) *with $f = 0$ in the sense of distributions. Then $v \in L^\infty(Q_\sigma)$ for any $\sigma < 1$.*

We will first prove Theorem 9.8 and then use it to prove Theorem 9.7.

Proof of Theorem 9.8, local version. The proof is by contradiction and consists of several steps.

Step 1. Preliminary estimates.

By assumption, $v \in L^\infty L^3(Q_1)$, $\nabla v \in L^2(Q_1)$, $p \in L^{3/2}(Q_1)$.

By imbedding, $v \in L^2 L^6 \cap L^\infty L^3(Q_1) \subset L^s L^q(Q_1)$, $3/q + 1/s = 1$.

In particular, $v \in L^4(Q_1)$.

With $v \in L^4$, $\int((v\cdot\nabla)v, v\phi)dt$ is defined and equals $-\frac{1}{2}\int(|v|^2 v, \nabla\phi)dt$, and one can show that (v,p) satisfies the local energy equality in Q_1.

By Hölder,

$$(v \cdot \nabla)v \in L^{4/3}(Q_1).$$

By interior estimate for the Stokes system (Lemma 9.6),

$$\partial_t v, \nabla^2 v, \nabla p \in L^{4/3}(Q_L), \quad \forall L < 1.$$

Thus, after a redefinition on a set of time of measure zero, which does not change the conclusion of Theorem 9.8,

$$v \in C([-L^2, 0], L^{4/3}(B_L)), \quad \forall L < 1.$$

Together with the bound $\|v\|_{L^\infty L^3}(Q_1)$ and Step 1 of the proof of Lemma 9.2, for any $L < 1$,

$$\begin{cases} v \in C_{wk}([-L^2, 0]; L^3(B_L)), \\ \|v(\cdot,t)\|_{L^3(B_L)} \le \|v\|_{L^\infty L^3(Q_1)}, \quad \forall t \in [-L^2, 0]. \end{cases} \tag{9.13}$$

Step 2. Contradiction assumption.

Suppose $z_0 = (x_0, t_0) \in \overline{Q_\sigma}$, $0 < \sigma < 1$, is a singular point of v. By Theorem 6.7 (regularity criteria), there is a small $\varepsilon > 0$, independent of v, and $r_k \to 0_+$ so that

$$\frac{1}{r_k^2} \int_{Q(z_0, r_k)} |v|^3 \geq \varepsilon. \tag{9.14}$$

Choose a smooth cut-off function $\zeta(x) \geq 0$ which equals 1 for $|x| < \frac{1}{4}(\sigma + 3)$ and 0 for $|x| > 1$. Decompose $p = \tilde{p} + h$ where

$$\tilde{p}(x,t) = \int \partial_i \partial_j (v_i v_j \zeta)(y,t) \frac{1}{4\pi |x-y|}\, dy. \tag{9.15}$$

One has $\Delta h(x,t) = 0$ for $|x| < \frac{1}{4}(\sigma + 3)$ and

$$\|\tilde{p}\|_{L^\infty(-1,0;L^{3/2}(\mathbb{R}^3))} \leq C \|v\|^2_{L^\infty L^3(Q_1)} \leq C, \tag{9.16}$$

$$\begin{aligned} \|h\|_{L^{3/2}(-1,0;L^\infty(B_{\frac{\sigma+1}{2}}))} &\leq C \|h\|_{L^{3/2}(-1,0;L^1(B_{\frac{\sigma+3}{4}}))} \\ &\leq C\|p\|_{L^{3/2}(Q_1)} + C\|\tilde{p}\|_{L^{3/2}(Q_1)} \leq C. \end{aligned} \tag{9.17}$$

Extend (v, p) outside Q_1 to $\mathbb{R}^3 \times \mathbb{R}$ by 0, define

$$\begin{aligned} v^k(x,t) &= r_k v(x_0 + r_k x, t_0 + r_k^2 t), \\ p^k(x,t) &= r_k^2 p(x_0 + r_k x, t_0 + r_k^2 t), \end{aligned}$$

and define $\tilde{p}^k$ and h^k similarly. The pair (v^k, p^k) satisfies (NS) and the local energy inequality in Q_L, $L \leq \frac{1-\sigma}{r_k}$, and

$$\int_{Q_1} |v^k|^3 \geq \varepsilon, \tag{9.18}$$

$$\|v^k\|_{L^\infty L^3(\mathbb{R}^{3+1})} + \|\tilde{p}^k\|_{L^\infty L^{3/2}(\mathbb{R}^{3+1})} \leq C,$$

$$\|h^k\|_{L^{3/2} L^\infty(Q_L)} \leq C r_k^{2/3}, \quad \forall L \leq \frac{1-\sigma}{2 r_k}.$$

Thus, for any finite $L > 0$ and k large enough,

$$\|v^k\|_{L^3(Q_L)} + \|p^k\|_{L^{3/2}(Q_L)} \leq C_L, \tag{9.19}$$

and hence, by local energy inequality,

$$\operatorname*{ess\,sup}_{-L^2 < t < 0} \int_{B_L} |v^k(x,t)|^2 dx + \int_{Q_L} |\nabla v^k|^2 \leq C_L. \tag{9.20}$$

By imbedding and Hölder,

$$\|v^k\|_{L^4(Q_L)} + \|(v^k \cdot \nabla) v^k\|_{L^{4/3}(Q_L)} \leq C_L. \tag{9.21}$$

By Lemma 9.6 (interior estimate for the Stokes system),

$$\|\partial_t v^k\|_{L^{4/3}(Q_L)} \le C\||(v^k\cdot\nabla)v^k| + |v^k| + |p^k|\|_{L^{4/3}(Q_{L+1})} \le C_L. \tag{9.22}$$

Thus $v_k \in C([-L^2,0], L^{4/3}(B_L))$ with uniform bound in k.

Step 3. Scaling limits.

From the sequence (v^k, p^k) we can extract a subsequence, still denoted by (v^k, p^k), so that (v^k, p^k) weakly converges to some limit function $(\bar v, \bar p)$,

$$\begin{aligned} &v^k \overset{*}{\rightharpoonup} \bar v \quad \text{in } L^\infty L^3(Q_L), \quad \nabla v^k \rightharpoonup \nabla \bar v \quad \text{in } L^2(Q_L),\\ &\tilde p^k \overset{*}{\rightharpoonup} \bar p \quad \text{in } L^\infty L^{3/2}(Q_L), \quad h^k \to 0 \quad \text{in } L^{3/2}L^\infty(Q_L),\\ &v^k \to \bar v \quad \text{strongly in } C([-L^2,0], L^{4/3}(B_L)), \end{aligned} \tag{9.23}$$

for any $L > 1$. Since v^k is uniformly bounded in $L^\infty L^3$ and $L^4(Q_L)$, by Hölder inequality, for any $0 < \delta \ll 1$,

$$v^k \to \bar v \quad \text{strongly in } C([-L^2,0], L^{3-\delta}(B_L)) \cap L^{4-\delta}(Q_L). \tag{9.24}$$

The above ensure the convergence of the weak form of (NS), in particular, the nonlinear term by (9.24). We conclude that $(\bar v, \bar p)$ solves (NS) in Q_L for any $L > 1$, that

$$\|\bar v\|_{L^\infty L^3(\mathbb{R}^{3+1})} + \|\bar p\|_{L^\infty L^{3/2}(\mathbb{R}^{3+1})} \le C, \tag{9.25}$$

that $(\bar v, \bar p)$ satisfies the local energy inequality, and that, by (9.18) and (9.24),

$$\int_{Q_1} |\bar v|^3 \ge \varepsilon. \tag{9.26}$$

Step 4. Zero velocity at final time.

We claim $\bar v(x,0) = 0$ for a.e. $x \in \mathbb{R}^3$. For any $x \in \mathbb{R}^3$ and any $\delta > 0$, with $B = B(x,1)$,

$$\int_B |\bar v(y,0)|^2 dy \le 2\int_B |\bar v(y,0) - v^k(y,0)|^2 dy + 2\int_B |v^k(y,0)|^2 dy.$$

Since $v^k \to \bar v$ strongly in $BC([-L^2,0], L^2(B))$, the middle integral is bounded by $\delta/2$ for k sufficiently large. The last integral equals

$$\frac{2}{r_k}\int_{B(x_0+r_k x, r_k)} |v(t_0)|^2$$

and is bounded, for $L_k = (1+|x|)r_k$, by

$$\le \frac{2}{r_k}\int_{B(x_0,L_k)} |v(t_0)|^2 \le C(1+|x|)\|v(t_0)\|^2_{L^3(B(x_0,L_k))},$$

which vanishes as $k \to \infty$ by (9.13). This shows the claim.

Step 5. Zero vorticity for large x.

Due to (9.25), for some R sufficiently large,

$$\int_{-4}^{0}\int_{|x|>R-1}|\bar v|^3+|\bar p|^{3/2}dxdt\le\varepsilon_0, \tag{9.27}$$

where $\varepsilon_0>0$ is the small constant in Theorem 6.4. By Theorem 6.4, $\bar v$ is regular in $U=(\mathbb{R}^3\setminus B_R)\times(-3,0)$, and

$$\nabla_x^k\bar v\in L^\infty(U),\quad \forall k\in\mathbb{N}.$$

The vorticity $\bar\omega=\operatorname{curl}\bar v$ satisfies

$$\partial_t\bar\omega_i-\Delta\bar\omega_i+\bar v_j\partial_j\bar\omega_i-(\partial_j\bar v_i)\bar\omega_j=0,\quad \operatorname{div}\bar\omega=0. \tag{9.28}$$

In particular, it satisfies

$$(\partial_t-\Delta)\bar\omega\le M(|\bar\omega|+|\nabla\bar\omega|),\quad |\bar\omega|\le M,\quad \text{in } U;$$

$$\bar\omega(x,0)=0,\quad \text{if } |x|>R.$$

By backward uniqueness (Theorem 9.4), applied for each $\xi\in\mathbb{S}^2$ to the half-space $H_\xi=\{x\in\mathbb{R}^3, x\cdot\xi>R\}$, we get $\bar\omega=0$ in U.

In U we have $\operatorname{div}\bar v=\operatorname{curl}\bar v=0$ and hence $\Delta\bar v=0$. By $\bar v\in L^\infty L^3$, $\bar p\in L^\infty L^{3/2}$, and

$$\Delta\bar v=0,\quad -\Delta\bar p=\partial_i\partial_j\bar v_i\bar v_j,\quad \partial_t\bar v=-\bar v\cdot\nabla\bar v-\nabla\bar p\quad \text{in } U,$$

we get for all $k\in\mathbb{N}$, for some $c_k>0$,

$$\max_{B_{R+1}^C\times(-3,0)}\Big(|\nabla^k\bar v|+|\nabla^k\bar p|+|\nabla^k\partial_t\bar v|\Big)\le c_k<\infty. \tag{9.29}$$

Step 6. Zero velocity for almost all time.

We want to apply Theorem 9.5 (unique continuation) to show $\bar\omega=0$ in B_R (hence in $\mathbb{R}^3$) for almost all $t\in(-2,0)$, for which the vanishing condition (9.12) is satisfied at (x_0,t), $|x_0|>R$, and we need to show that the coeffients $\bar v$ and $\nabla\bar v$ in (9.28) are uniformly bounded in $B_{2R}\times(-2,0)$, hence in $\mathbb{R}^3\times(-2,0)$. We will consider the equations for

$$u=\varphi\bar v-\tilde w,\quad q=\varphi\bar p-\pi,$$

in $Q_*=B_{4R}\times(-2,0)$, where $\varphi\in C^\infty(\mathbb{R}^3)$ is a fixed cut-off function with $\varphi(x)=1$ if $x\in B_{2R}$ and $\varphi(x)=0$ if $|x|>3R$; and $\tilde w$ is introduced to make $\operatorname{div}u=0$: Let $\tilde w,\pi$ solve the stationary Stokes system in B_{4R} for all $t\in(-2,0)$,

$$-\Delta\tilde w+\nabla\pi=0,\quad \operatorname{div}\tilde w=\bar v\cdot\nabla\varphi\quad \text{in } B_{4R},$$

$$\tilde w(x,t)=0\quad \text{if } |x|=4R.$$

By (9.29) and estimates for the stationary Stokes system[1], we have

$$\max_{B_{4R}\times(-2,0)} \Big(|\nabla^k \tilde{w}| + |\nabla^k \pi| + |\nabla^k \partial_t \tilde{w}|\Big) \le c_k < \infty. \tag{9.30}$$

We have

$$\partial_t u - \Delta u + (u+\tilde{w})\cdot\nabla u + u\cdot\nabla\tilde{w} + \nabla q = g, \quad \operatorname{div} u = 0,$$

in Q_*, where

$$g = (\varphi^2-\varphi)\bar{v}\cdot\nabla\bar{v} + (\bar{v}\cdot\nabla\varphi^2)\bar{v} - \bar{v}\Delta\varphi - 2\nabla\bar{v}\cdot\nabla\varphi + \bar{p}\nabla\varphi - \tilde{w}\cdot\nabla\tilde{w} - \partial_t\tilde{w}.$$

By (9.29) and (9.30), we have $\nabla^k g \in L^\infty(Q_*)$ for all $k\in\mathbb{N}$. We also have

$$u \in L^\infty L^3 \cap CL^2 \cap L^2 H^1(Q_*), \quad \partial_t u, \nabla^2 u, \nabla q \in L^{4/3}(Q_*).$$

Since $\nabla u \in L^2(Q_*)$, $\nabla u(\cdot,t_0)\in L^2(B_{4R})$ for almost all $t_0\in(-2,0)$. Fix one such t_0. By (the nonzero force version of) Theorem 5.6 on the short time solvability for the Navier-Stokes equations with $L^2\cap L^3(B_{4R})$ initial data and by weak-strong uniqueness (Theorems 4.4 or 9.3), we can find a number $\delta_0=\delta_0(t_0)>0$ such that

$$u \in L^5(B_{4R}\times(t_0,t_0+\delta_0)) \cap L^\infty(B_{4R}\times I_\varepsilon), \quad I_\varepsilon = (t_0+\varepsilon, t_0+\delta_0),$$

for any $\varepsilon\in(0,\delta_0)$. Since $\bar{v}=u$ in $B_{2R}\times(-2,0)$, together with (9.29) we get

$$\sup_{B_{4R}\times I_\varepsilon} |\bar{v}| + |\nabla\bar{v}| \le C_\varepsilon$$

with C_ε depending on ε. Thus

$$|\partial_t\bar{\omega} - \Delta\bar{\omega}| \le C_\varepsilon(|\bar{\omega}|+|\nabla\bar{\omega}|), \quad |\bar{\omega}|\le C_\varepsilon \quad \text{in } B_{4R}\times I_\varepsilon.$$

Since $\bar{\omega}=0$ in $(B_{4R}\setminus B_R)\times(-2,0)$, by unique continuation (Theorem 9.5), we get $\bar{\omega}=0$ in $B_{4R}\times I_\varepsilon$. It implies that $\bar{v}(x,t)$ is harmonic in $x\in\mathbb{R}^3$ with finite L^3-norm for each $t\in I_\varepsilon$, and thus $\bar{v}=0$ in $\mathbb{R}^3\times I_\varepsilon$. Since ε is arbitrary, $\bar{v}=0$ in $\mathbb{R}^3\times(t_0,t_0+\delta_0)$.

Since $\nabla\bar{v}(\cdot,t_0)\in L^2(B_{4R})$ for almost every t_0, by the following lemma we have $\bar{v}(\cdot,t)=0$ for almost every t, which contradicts (9.26) and proves Theorem 9.8. □

In the last paragraph we have used the following lemma.

Lemma. *Let I be an open interval and let Z be a measure zero subset of I. Suppose for any $t\in I-Z$ it is assigned an open interval $I_t\subset I$ with $\inf I_t = t$ and let $V=\bigcup_{t\in I-Z} I_t$. Then $|I-V|=0$. (Note that $t\notin I_t$.)*

[1]We need the $W^{k,q}$-version, $k\in\mathbb{N}$, of Theorem 2.13 and Lemma 2.16 in a ball. That of Theorem 2.13 is by the same proof as [**57**, Theorem IV.6.1]. That of Lemma 2.16 follows from Step 1 of its proof in Section 2.8 since a ball is convex and the same formula works for $W^{k,q}$-estimates.

Proof. The open set V can be written as a countable, disjoint union of open intervals $V = \bigcup_{k\in\mathbb{Z}} J_k$ with $J_k = (a_k, b_k)$. For any $t \notin Z \cup V$, there is $k(t)$ so that $I_t \subset J_{k(t)}$. It follows that $t = a_{k(t)}$. Thus $I - V$ is a subset of $Z \cup \{a_k\}_{k\in\mathbb{Z}}$ and has zero measure. $\square$

Proof of Theorem 9.7, global version. Recall $D = \mathbb{R}^3 \times (0, T)$. By assumption,

$$v \in L^\infty L^2 \cap L^\infty L^3 \cap L^2 L^6(D), \quad \nabla v \in L^2(D).$$

Since $v(0) \in L^2_\sigma(\mathbb{R}^3)$, by the existence theorem (Theorem 3.9) and uniqueness theorem (Theorem 9.3), there is an associated pressure $p \in L^{5/3}(D)$ such that (v, p) is a suitable weak solution in D and

$$p(\cdot, t) = R_i R_j (v_i v_j(\cdot, t))$$

for each t, where R_i is the Riesz transform. Then

$$p \in L^\infty L^{3/2}(D).$$

As in Step 1 of the proof of Theorem 9.8, after a redefinition on a set of time of measure zero, we have

$$v \in C_{wk}([0, T]; L^3(\mathbb{R}^3)), \qquad \sup_{0\le t\le T} \|v(\cdot, t)\|_{L^3(\mathbb{R}^3)} \le \|v\|_{L^\infty L^3(D)}.$$

We can now apply (rescaled) Theorem 9.8 to get

$$v \in L^\infty(B_R \times (\delta, T)), \quad \forall R > 0, \quad \forall \delta \in (0, T).$$

Since $\lim_{R\to\infty} \int_{B_R^c \times (0,T)} |v|^3 + |p|^{3/2} = 0$, by Theorem 6.4, $v(x, t)$ is uniformly bounded in $B_R^c \times (\delta, T)$ for sufficiently large R. Thus

$$v \in L^\infty(\delta, T; L^2 \cap L^\infty(\mathbb{R}^3)) \subset L^5(\mathbb{R}^3 \times (\delta, T)),$$

for any $\delta > 0$. On the other hand, since $v(0) \in L^2_\sigma \cap L^3_\sigma(\mathbb{R}^3)$, the combination of Theorems 5.6 and 9.3 shows $v \in L^5(\mathbb{R}^3 \times (0, \delta))$ for some $\delta > 0$. $\square$

Theorem 9.7 implies that if $v(\cdot, t) \in L^2 \cap L^3(\mathbb{R}^3)$ is a Leray-Hopf weak solution in $0 \le t < T$ and $T < \infty$ is a singular time, then

$$\limsup_{t\to T_-} \|v(t)\|_{L^3(\mathbb{R}^3)} = \infty. \tag{9.31}$$

It has been shown by Seregin [**179**] that, in fact,

$$\lim_{t\to T_-} \|v(t)\|_{L^3(\mathbb{R}^3)} = \infty. \tag{9.32}$$

9.4. Backward uniqueness and unique continuation

Both Theorems 9.4 and 9.5 are proved in [**41**] using Carleman-type inequalities, Lemmas 9.9 and 9.11 below. To illustrate, we will first prove Lemma 9.9 and then use it to prove Theorem 9.5. See [**41**] for the proofs of Lemma 9.11 and Theorem 9.4.

Lemma 9.9. *Let $D = \mathbb{R}^n \times (0,2)$. For any function $u \in C_c^\infty(D;\mathbb{R}^n)$ and any $a > 0$,*

(9.33)
$$\int_D h^{-2a}(t)e^{-\frac{|x|^2}{4t}}\left(\frac{a+1}{t}|u|^2 + |\nabla u|^2\right)dz \le c_0 \int_D h^{-2a}(t)e^{-\frac{|x|^2}{4t}}|\partial_t u + \Delta u|^2 dz.$$

Here $h(t) = te^{\frac{1-t}{3}}$ and c_0 is a constant independent of a and u.

This weighted estimate bounds lower-order terms $|u|+|\nabla u|$ by $\partial_t u + \Delta u$. For application, one chooses a sufficiently large a. The weight $h(t) \sim t$ emphasizes the behavior near $t = 0$. Its specific choice will be determined in the proof.

Proof. We first explain the idea. Write the weight in (9.33) as $e^{2\phi}$ where

$$\phi(x,t) = -\frac{x^2}{8t} - a\log h(t).$$

The usual approach is to try to write the right side of (9.33) as $\int_D |Lv|^2$, where

$$v = e^\phi u, \quad Lv = e^\phi Pu = e^\phi P e^{-\phi} v \quad (Pu = \partial_t u + \Delta u). \tag{9.34}$$

One decomposes L as the sum of its symmetric and antisymmetric parts,

$$L = S + A, \quad S = \frac{1}{2}(L + L^*), \quad A = \frac{1}{2}(L - L^*),$$

where the adjoint L^* is defined with respect to

$$(f,g) = \int_D f \cdot g \, dxdt,$$

so that

$$(Lf,g) = (f, L^*g), \quad (Sf,g) = (f,Sg), \quad (Af,g) = -(f,Ag).$$

One has

$$(Lv,Lv) = (Sv,Sv) + (Av,Av) + ([S,A]v,v) \ge ([S,A]v,v) \tag{9.35}$$

where the commutator term $([S,A]v,v) = ((SA - AS)v,v)$ consists of lower-order derivatives of v. If the commutator term has a sign,

$$([S,A]v,v) \ge c_1(v,v) + c_2(\nabla v, \nabla v), \quad c_1 > 0,$$

then using $(\nabla v, \nabla v) \lesssim (v,v) + (Lv,Lv)$ and (9.35), one may prove (9.33).

For our specific $P = \partial_t + \Delta$ and a general $\phi(x,t)$, we have

$$(9.36)\qquad \begin{aligned} L &= e^{\phi}(\partial_t + \Delta)e^{-\phi} = S + A, \\ Sv &= \Delta v + (|\nabla\phi|^2 - \partial_t\phi)v, \\ Av &= \partial_t v - A_1 v, \\ A_1 v &= \partial_j((\partial_j\phi)v) + (\partial_j\phi)\partial_j v, \\ ([S,A]v,v) &= (4\phi_{,kl}v_{,k}, v_{,l}) + (Wv,v), \\ W &= 2\nabla\phi\cdot\nabla|\nabla\phi|^2 - \Delta^2\phi + \partial_t^2\phi - 2\partial_t|\nabla\phi|^2. \end{aligned}$$

Above, $\phi_{,k} = \partial_k\phi$ and $\phi_{,kl} = \partial_k\partial_l\phi$. If we substitute $\phi(x,t) = -\frac{x^2}{8t} - a\log h(t)$ with $h(t)\sim t$ to be determined, we have

$$(9.37)\qquad \nabla\phi = -\frac{x}{4t}, \quad |\nabla\phi|^2 = \frac{x^2}{16t^2}, \quad \phi_{,kl} = -\frac{\delta_{kl}}{4t}, \quad \Delta^2\phi = 0,$$

$$W = -\frac{x}{2t}\cdot\frac{x}{8t^2} - 0 - \frac{x^2}{4t^3} - a\partial_t\left(\frac{h'}{h}\right) + \frac{x^2}{4t^3} = -\frac{x^2}{16t^3} - a\partial_t\left(\frac{h'}{h}\right).$$

The sign of $\frac{x^2}{16t^3}$ in W is wrong and the choice of $h(t)$ does not help since it would not contain x^2.

Instead, we consider a modified weighted form. With $h(t)\sim t$ and v and Lv still defined by (9.34), the right side of (9.33) is comparable to $\int_D |t^k Lv|^2$ if we choose

$$(9.38)\qquad \phi(x,t) = -\frac{x^2}{8t} - (a+k)\log h(t).$$

It turns out that $k=1$ is a good choice. We now decompose

$$\begin{aligned} tL &= te^{\phi}(\partial_t + \Delta)e^{-\phi} = S + A, \\ Sv &= t\left(\Delta v + (|\nabla\phi|^2 - \partial_t\phi)v\right) - \frac{1}{2}v, \\ Av &= \frac{1}{2}(\partial_t(tv) + t\partial_t v) - tA_1 v, \end{aligned}$$

where $A_1 v = \partial_j((\partial_j\phi)v) + (\partial_j\phi)\partial_j v$ as before. For general $\phi(x,t)$, after some manipulation,

$$\begin{aligned} I := ([S,A]v,v) &= (4t^2\phi_{,kl}v_{,k}, v_{,l}) + \int t|\nabla v|^2 + (Wv,v), \\ W &= 4t^2\phi_{,kl}\phi_{,k},\phi_{,l} + t^2(\partial_t^2\phi - 2\partial_t|\nabla\phi|^2 - \Delta^2\phi) + t(\partial_t\phi - |\nabla\phi|^2). \end{aligned}$$

For $\phi(x,t)$ given by (9.38) with $k=1$, we still have (9.37) and

$$I = (Wv,v), \quad W = -(a+1)t^2\left[\partial_t\left(\frac{h'}{h}\right) + \frac{h'}{th}\right].$$

We now choose $h(t) = te^{(1-t)/3}$ and get $W = \frac{a+1}{3}t > 0$,

$$I = \frac{a+1}{3}(tv, v) \le \|tLv\|^2. \tag{9.39}$$

By (9.36),

$$|\nabla v|^2 = \frac{1}{2}P|v|^2 - v \cdot Pv, \quad P = \partial_t + \Delta = L + V_1 + A_1$$

where $V_1 = \partial_t \phi - |\nabla \phi|^2 = |\nabla \phi|^2 - (a+1)(\frac{1}{t} - \frac{1}{3})$. Thus

$$\int_D t^2 |\nabla v|^2 = -\int_D t|v|^2 - (t^2 v, Lv) - (t^2 v, V_1 v) - (t^2 v, A_1 v).$$

Note that $(t^2 v, A_1 v) = 0$,

$$\begin{aligned}\int_D t^2 \left(|\nabla v|^2 + |v|^2 |\nabla \phi|^2\right) &= -\int_D t|v|^2 - (t^2 v, Lv) + (a+1)\int_D \left(t - \frac{t^2}{3}\right)|v|^2 \\ &\le 3I + |(t^2 v, Lv)| \lesssim \|tLv\|^2. \end{aligned} \tag{9.40}$$

Since

$$e^{\phi} u = v, \quad e^{\phi}|\nabla u| \lesssim |v| + |\nabla v||\nabla \phi|,$$

we get (9.33) from (9.39) and (9.40). $\square$

We now use Lemma 9.9 to prove the following lemma, which implies Theorem 9.5.

Lemma 9.10. *Assume the same assumptions as in Theorem* 9.5*. For some* $\gamma = \gamma(c_1) \in (0, 3/16)$ *and* $\beta = \frac{1}{64}$,

$$|w(x,t)| \le cA_0 e^{-\frac{x^2}{4t}}, \tag{9.41}$$

where $A_0 = \max_D(|u| + |\nabla u|)$*, for any* $(x,t) \in D$ *such that*

$$0 < t \le \gamma \min(T, 1), \quad \frac{\sqrt{t}}{2} \le |x| \le \frac{R}{20}.$$

Sketch of Proof. Denote $Q(R,T) = B_R \times (0,T)$. For any $|x_0| \le \frac{3}{8}R$ and $8t_0 \le |x_0|^2$, let $\lambda = \sqrt{2t_0}$ and $\rho = 2|x_0|/\lambda \ge 4$. Rescale

$$v(y,s) = u(\lambda y, \lambda^2 s), \quad (y,s) \in Q(\rho, 2).$$

Let $w_\varepsilon = v\varphi\theta_\varepsilon$ where the cut-off functions $\varphi(y,s)$ and $\theta_\varepsilon(s)$ satisfy $\varphi(y,s) = 1$ for $(y,s) \in Q(\rho - 1, \frac{3}{2})$, $\varphi(y,s) = 0$ for $|y| > \rho$ or $s > 2$, $\theta_\varepsilon(s) = 1$ for $2\varepsilon < s \le 2$, and $\theta_\varepsilon(s) = 0$ for $s < \varepsilon$. By Lemma 9.9 we have (9.33) for w_ε. Note that

$$\begin{aligned}|\partial_s w_\varepsilon + \Delta w_\varepsilon| &\le c_1 \lambda (|w_\varepsilon| + |\nabla w_\varepsilon|) \\ &\quad + c\left(|\nabla \varphi||\nabla v| + (|\nabla \varphi| + |\Delta \varphi| + |\partial_s \varphi|)|v|\right) + c|\theta_\varepsilon'||v|.\end{aligned}$$

If $\gamma \le (40c_0c_1^2)^{-1}$, then $c_1\lambda$ is sufficiently small that the first term on the right side of (9.33) can be absorbed into the left. The contribution from the last term $|\theta'_\varepsilon||v|$ vanishes as $\varepsilon \to 0$ using $|v(y,s)| \le C_k\lambda^k(|y|+\sqrt{s})^k$ for $k \sim [a]+2$. The remaining terms of (9.33) are boundary terms. Taking $a = C\beta\rho^2$ with $\beta \le 2^{-6}$, the function $g(s) = h^{-2a}(s)\exp(-\frac{\rho^2}{16s})$ satisfies $g' \ge 0$ for $s \in (0,2)$, and one can estimate the boundary terms as $\varepsilon \to 0$ to get

$$\int_{Q(\rho-1,\frac{3}{2})} h^{-2a}(s)e^{-\frac{|y|^2}{4s}}(|\nabla v|+|v|)^2 dyds \le cA_0^2 e^{-\beta\rho^2}.$$

The left side of the above dominates $\int_{B(x/\lambda,1)\times(\frac{1}{2},1)} e^{-\frac{|y|^2}{2}}|v|^2 dyds$, $x=\sqrt{2\beta}x_0$, which in turn controls $v(x/\lambda,1/2)$ by regularity theory. Tracking the factors and scaling back, one gets (9.41). □

See [**41**, §4] for the details of the proof.

9.5. Notes

Section 9.1 is based on Masuda [**145**] and Kozono-Sohr [**115**]. There are many earlier results for the uniqueness of weak solutions in $C([0,T);L^n)$, assuming time continuity in addition to boundedness. See [**128**, Chapter 27] and its references.

Sections 9.2–9.4 are based on the expository article [**41**]. The content of Section 9.3 was given in Seregin and Šverák [**182**] assuming Theorem 9.4, which was later proved by Escauriaza-Seregin-Šverák [**40**]. Earlier partial results include Neustupa [**156**] and Beirão da Veiga [**10**]. The $L_t^\infty L_x^3$ assumption has been relaxed by Seregin and Šverák [**184**] and Seregin [**179**]. It has been extended to the half-space case by Seregin [**178**] and Barker-Seregin [**7**] and to the bounded domain case by Mikhailov-Shilkin [**147**]. There are also results in Lorentz and Besov spaces [**58, 98, 164, 219**].

For classical results on unique continuation and Carleman estimates, see Hörmander [**81**]. The proof of Theorem 9.4 in [**41**, §6] uses, in addition to Lemma 9.9, the following Carleman-type inequality in the half-space.

Lemma 9.11. *Fix $\alpha \in (1/2,1)$. Let $D = (\mathbb{R}^n_+ + e_n)\times(0,1)$ and*

$$\phi_a(x,t) = -\frac{|x'|^2}{8t} + a(1-t)\frac{x_n^{2\alpha}}{t^\alpha}, \quad x=(x',x_n).$$

There are $a_\alpha, c_\alpha > 0$ so that, for any function $u \in C_c^\infty(D;\mathbb{R}^n)$ and any $a > a_\alpha$,

$$\int_D t^2 e^{2\phi_a(x,t)}\left(\frac{a|u|^2}{t^2}+\frac{|\nabla u|^2}{t}\right)dxdt \le c_\alpha \int_D t^2 e^{2\phi_a(x,t)}|\partial_t u + \Delta u|^2 dxdt.$$

Axisymmetric flows

In this chapter we study the regularity of axisymmetric flows. The class of axisymmetric flows is preserved under the Navier-Stokes equations. It is particular in several aspects: First, its components do not depend on θ, and hence there is one less space variable. Second, the equation of its angular component does not depend on the pressure and allows maximal principle. Third, the coupled system of the angular components of the velocity and the vorticity is "in principle" self-contained. These make global-in-time regularity in this class more probable, although it has not been established.

The chapter is organized as follows. We will first discuss the equations and a priori estimates for axisymmetric flows in Section 10.1. We then show the global-in-time regularity of axisymmetric solutions in the no-swirl case in Section 10.2. We show two proofs in Sections 10.3 and 10.4 that there is no Type I singularity in this class, even if the swirl component is present. We discuss their relation in Section 10.5.

10.1. Axisymmetric Navier-Stokes equations

Recall that the *cylindrical coordinates* of $\mathbb{R}^3$ are r, θ, z with

$$(x_1, x_2, x_3) = (r\cos\theta, r\sin\theta, z)$$

and the basis vectors e_r, e_θ, e_z are

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \quad e_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \quad e_z = (0, 0, 1).$$

A vector field v in cylindrical coordinates is written in the form

$$v = v_r e_r + v_\theta e_\theta + v_z e_z, \tag{10.1}$$

with scalar components v_r, v_θ, and v_z, and we call v_θ its *swirl* component. We say v has *no swirl* if $v_\theta = 0$.

A scalar function φ is called *axisymmetric* if it is independent of θ in cylindrical coordinates, $\varphi = \varphi(r, z, t)$. A vector field v is called *axisymmetric* if its components v_r, v_θ, and v_z in (10.1) do not depend upon θ. In other words, they are invariant under rotations about the z-axis: If we denote by $R(\alpha)$ the rotation about the z-axis by angle α, defined in (8.8), then axisymmetric φ and v satisfy

$$\varphi(x) = \varphi(R(-\alpha)x), \quad v(x) = R(\alpha)\, v(R(-\alpha)x), \quad \forall \alpha \in \mathbb{R}. \tag{10.2}$$

The definition using (10.2) is independent of the choice of cylindrical or spherical coordinates. Compare (8.17) and Landau solutions in Section 8.2.

The Navier-Stokes equations (NS) preserve the class of axisymmetric flows and also the subclass of axisymmetric flows with zero swirl. The Navier-Stokes equations (NS) for axisymmetric vector fields in cylindrical coordinates become

$$\left(\partial_t + b \cdot \nabla - \Delta + \frac{1}{r^2}\right) v_r - \frac{1}{r} v_\theta^2 + \partial_r p = 0, \tag{10.3}$$

$$\left(\partial_t + b \cdot \nabla - \Delta + \frac{1}{r^2}\right) v_\theta + \frac{1}{r} v_r v_\theta = 0, \tag{10.4}$$

$$(\partial_t + b \cdot \nabla - \Delta)\, v_z + \partial_z p = 0, \tag{10.5}$$

$$\partial_r (r v_r) + \partial_z (r v_z) = 0, \tag{10.6}$$

where $p = p(r, z, t)$ is axisymmetric and

$$b = v_r e_r + v_z e_z.$$

We may replace Δ in the above equations by

$$\Delta_{\text{axisym}} = \partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2.$$

Note that the v_θ-equation does not involve p. The no-swirl subclass $v_\theta = 0$ is preserved since there is no source term in the v_θ-equation.

The vorticity $\omega = \operatorname{curl} v$ for axisymmetric v is

$$\omega(x, t) = \omega_r e_r + \omega_\theta e_\theta + \omega_z e_z, \tag{10.7}$$

where

$$\omega_r = -\partial_z v_\theta, \quad \omega_\theta = \partial_z v_r - \partial_r v_z, \quad \omega_z = (\partial_r + r^{-1}) v_\theta.$$

The *vorticity equations* (1.38) restricted to this class become

$$\left(\partial_t + b\cdot\nabla - \Delta + \frac{1}{r^2}\right)\omega_r - \omega_r\partial_r v_r - \omega_z\partial_z v_r = 0, \tag{10.8}$$

$$\left(\partial_t + b\cdot\nabla - \Delta + \frac{1}{r^2}\right)\omega_\theta - \frac{\partial_z v_\theta^2}{r} - \frac{v_r}{r}\omega_\theta = 0, \tag{10.9}$$

$$\left(\partial_t + b\cdot\nabla - \Delta\right)\omega_z - \omega_r\partial_r v_z - \omega_z\partial_z v_z = 0. \tag{10.10}$$

Since

$$\operatorname{curl} b = \omega_\theta e_\theta, \quad \operatorname{div} b = 0,$$

we have $-\Delta b = \operatorname{curl}(\omega_\theta e_\theta)$ and, if considered in the entire $\mathbb{R}^3$, we can recover

$$b = (-\Delta)^{-1}\operatorname{curl}(\omega_\theta e_\theta). \tag{10.11}$$

Moreover, ∇b is a singular integral of $\omega_\theta e_\theta$, and $\nabla(v-b)$ is a singular integral of $\omega - \omega_\theta e_\theta$.

It is important to note that the system (10.4), (10.9), and (10.11) for v_θ and ω_θ does not depend on the pressure and is self-contained in $\mathbb{R}^3$.

In view of the *vector identity* (2.63) that

$$-\Delta\psi = \operatorname{curl}\operatorname{curl}\psi - \nabla(\operatorname{div}\psi),$$

we may introduced a vector-valued *stream function* ψ so that

$$-\Delta\psi = \omega, \quad \operatorname{curl}\psi = v, \quad \operatorname{div}\psi = 0.$$

We may then replace (10.11) by

$$\left(-\Delta + \frac{1}{r^2}\right)\psi_\theta = \omega_\theta, \quad b = \operatorname{curl}(\psi_\theta e_\theta) = (-\partial_z\psi_\theta)e_r + \frac{\partial_r(r\psi_\theta)}{r}\,e_z \tag{10.12}$$

and consider the self-contained coupled system (10.4), (10.9), and (10.12) for v_θ, ω_θ, and ψ_θ.

For the regularity problem, by the partial regularity theorem $\mathcal{P}^1(S) = 0$ (Theorem 6.2), a singularity of an axisymmetric suitable weak solution can only occur on the z-axis, since otherwise there is a circle of singularity whose 1-dimensional measure is positive.

At the z-axis, a regular solution v satisfies

$$v_\theta = v_r = \omega_\theta = \omega_z = \psi_\theta = \psi_z = 0, \quad \text{at } r = 0. \tag{10.13}$$

However, we do not have explicit control on how fast they go to zero as $r \to 0$. To study the behavior near $r = 0$, we may consider the change of variables $\omega_\theta = r^\alpha \Omega$ and $v_\theta = r^\beta \Gamma$. They satisfy

$$
\begin{aligned}
&\left(\partial_t + b\cdot\nabla - \Delta + \frac{1}{r^2} + \frac{\beta+1}{r}v_r - \frac{\beta^2}{r^2} - \frac{2\beta}{r}\partial_r\right)\Gamma = 0,\\
&\left(\partial_t + b\cdot\nabla - \Delta + \frac{1}{r^2} + \frac{\alpha-1}{r}v_r - \frac{\alpha^2}{r^2} - \frac{2\alpha}{r}\partial_r\right)\Omega = r^{-1-\alpha}\partial_z v_\theta^2.
\end{aligned}
\tag{10.14}
$$

To study their behavior near $r=0$, one may factor out r from v_θ, ω_θ, and ψ_θ and consider

$$
v_1 = \frac{v_\theta}{r},\quad \omega_1 = \frac{\omega_\theta}{r},\quad \psi_1 = \frac{\psi_\theta}{r}.
$$

Their equations correspond to (10.14) with $\alpha=\beta=1$. This is the approach used in Liu-Wang [**139**] and Hou-Li [**82**].

From the analytical point of view, for the borderline regularity problem we try to avoid potential terms (see the discussion for the model equation (2.81)). Thus we choose $\alpha=1$, $\beta=-1$, and (10.14) becomes

$$
\left(\partial_t + b\cdot\nabla - \Delta + \frac{2}{r}\partial_r\right)\Gamma = 0, \qquad \Gamma = r v_\theta, \tag{10.15}
$$

$$
\left(\partial_t + b\cdot\nabla - \Delta - \frac{2}{r}\partial_r\right)\Omega = r^{-2}\partial_z v_\theta^2, \qquad \Omega = \frac{\omega_\theta}{r}. \tag{10.16}
$$

This system looks symmetric; however, Γ has dimension 0 while Ω has dimension -3. As will be seen in the proof, $-\frac{2}{r}\partial_r\Omega$ has a good sign, while $\frac{2}{r}\partial_r\Gamma$ has a bad sign, which is fortunately taken care of by $\Gamma|_{r=0}=0$ (we don't know $\Omega|_{r=0}$). When $v_\theta=0$, the Ω-equation is very good and allows easy estimates. When $v_\theta\neq 0$, however, the Ω-equation has a very singular source term $r^{-2}\partial_z v_\theta^2$ of dimension -5. This is one of the main reasons why the global-in-time regularity of axisymmetric flows has not been shown.

We next consider a priori bounds. For a weak solution v we always have the energy bound,

$$
v\in L^\infty(0,\infty;L^2(\mathbb{R}^3)),\quad \nabla v\in L^2(\mathbb{R}^3\times\mathbb{R}_+).
$$

For axisymmetric solutions we also have the following.

Lemma 10.1. *Suppose v is an axisymmetric Leray-Hopf weak solution of* (NS) *with zero force in $\mathbb{R}^3\times[0,T)$ and $v\in L^\infty(0,t_1;L^3)$ for any $t_1\in(0,T)$. If $rv_\theta(0)\in L^p(\mathbb{R}^3)$ for some $p\in[2,\infty]$, then $rv_\theta\in L^\infty(0,T;L^p)$ and $|rv_\theta|^{p/2}\in L^2(0,T;W^{1,2})$ if $p<\infty$.*

In particular, with $p=\infty$, we get

$$
|v_\theta(r,z,t)| \le \frac{C}{r}\quad (r>0,\ z\in\mathbb{R},\ 0<t<T).
$$

Proof. Let $\Gamma = rv_\theta$. Suppose $2 \le p < \infty$ and assume v has sufficient regularity and spatial decay. Testing the Γ-equation (10.15) with $|\Gamma|^{p-2}\Gamma$, we get

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3}|\Gamma|^p + \frac{4(p-1)}{p^2}\int_{\mathbb{R}^3}|\nabla|\Gamma|^{p/2}|^2 \\
&\quad = -\int_{\mathbb{R}^3}\frac{2}{pr}\partial_r|\Gamma|^p = -\frac{4\pi}{p}\int_{\mathbb{R}}\int_0^\infty \partial_r|\Gamma|^p drdz \\
&\quad = -\frac{4\pi}{p}\int_{\mathbb{R}}[|\Gamma|^p]_{r=0}^\infty\, dz = 0,
\end{aligned}$$

thanks to the decay and $\Gamma|_{r=0} = 0$. Thus

$$\max_t \int_{\mathbb{R}^3}|\Gamma(t)|^p + 2\int_0^T\int_{\mathbb{R}^3}|\nabla|\Gamma|^{p/2}|^2 \le \int_{\mathbb{R}^3}|\Gamma(0)|^p \tag{10.17}$$

and $\|rv_\theta(t)\|_{L^p} \le \|rv_\theta(0)\|_{L^p}$. Taking $p \to \infty$, we get $\|rv_\theta(t)\|_{L^\infty} \le \|rv_{\theta,0}\|_{L^\infty}$. The L^∞-bound also follows from the maximal principle for (10.15).

To justify the formal computation assuming the regularity and decay of v, note that a solution satisfying the a priori bound (10.17) can be constructed by the Galerkin method, which has to agree with v by weak-strong uniqueness (Theorem 9.3). □

We actually know v is smooth in $\mathbb{R}^3 \times (0,T)$ by Theorem 9.7.

Other known a priori estimates include, e.g.,

$$r^3\omega_\theta \in L^\infty_{\rm loc}([0,T);L^2) \cap L^2_{\rm loc}([0,T);H^1), \tag{10.18}$$

from [**27**], and

$$|\omega_\theta| \le \frac{C}{r^{7/2}}, \quad |\omega| \le \frac{C}{r^{10}}, \quad |v| \le \frac{C}{r^2}, \tag{10.19}$$

from [**21**, **126**].

We now summarize existence theorems for general solutions in H^2, not necessarily axisymmetric, as the base for the rest of this chapter.

Lemma 10.2. *Let $k \in \mathbb{N}$, $f \in L^2(0,\infty;H^{k-1}(\mathbb{R}^3))$, and $v_0 \in H^k(\mathbb{R}^3)$, $\operatorname{div} v_0 = 0$. For the Stokes system, for any $T \in (0,\infty)$, there is a unique solution (v,p) with*

$$\begin{gathered}
v \in L^\infty(0,T;H^k(\mathbb{R}^3)) \cap L^2(0,T;H^{k+1}(\mathbb{R}^3)), \\
\partial_t v \in L^2(0,T;H^{k-1}(\mathbb{R}^3)).
\end{gathered} \tag{10.20}$$

The version for $k = 0$ is contained in Theorem 3.5. Since the derivatives of v also satisfy the Stokes system in $\mathbb{R}^3$, we get the lemma.

Lemma 10.3. *Let $f \in L^2(0,\infty; H^1(\mathbb{R}^3))$ and $v_0 \in H^2(\mathbb{R}^3)$, $\operatorname{div} v_0 = 0$. For the Navier-Stokes equations* (NS)*, there is $T \in (0,\infty)$ and a unique solution (v,p) with*

$$v \in L^\infty(0,T; H^2(\mathbb{R}^3)) \cap L^2(0,T; H^3(\mathbb{R}^3)),\quad \partial_t v \in L^2(0,T; H^1(\mathbb{R}^3)). \tag{10.21}$$

Here $T \geq C(\|v_0\|_{H^2} + \|f\|_{L^2(\mathbb{R}_+;H^1)})^{-2}$. Let $T_{\max}$ be the sup of such T. If $T_{\max} < \infty$, then $v \notin L^\infty(0,T_{\max}; H^1)$.

The uniqueness guarantees that if v_0 is axisymmetric, so is v.

We can prove the $\mathbb{R}^3$ counterpart of Theorem 4.1 in $L^\infty H^1$ similarly.

Proof. For $0 < T < \infty$, set

$$X = X_T = L^\infty(0,T; H^2(\mathbb{R}^3)) \cap L^2(0,T; H^3(\mathbb{R}^3)).$$

Let $v_1(t) \in X$ be the solution of the Stokes system with $v_1(0) = v_0$ and force f, given by Lemma 10.2. Also let

$$G : L^2(0,T; H^1(\mathbb{R}^3)) \to X$$

be the solution operator given by Lemma 10.2 that maps f to v with zero initial data, $v = Gf$. Consider the map

$$Sv = v_1 - G(v \cdot \nabla v), \quad v \in X.$$

A fixed point $v = Sv$ is a solution of (NS) in X. Let $R = 2\|v_1\|_X$, $R \leq C\|v_0\|_{H^2} + C\|f\|_{L^2(\mathbb{R}_+;H^1)}$, and $X(R) = \{v \in X : \|v\|_X \leq R\}$. We claim that S is a contraction map in $X(R)$ if T is sufficiently small. Indeed, with the help of the *Agmon inequalities*,

$$\|v\|_\infty \leq C\|v\|_2^{\frac{1}{4}}\|\Delta v\|_2^{\frac{3}{4}}, \quad \|v\|_\infty \leq C\|Dv\|_2^{\frac{1}{2}}\|D^2 v\|_2^{\frac{1}{2}}, \tag{10.22}$$

or simply using that $H^2(\mathbb{R}^3)$ is an algebra, we can show

$$\int_0^T \|v \cdot \nabla v - w \cdot \nabla w\|_{H^1}^2 \leq CT\|v - w\|_{L^\infty H^2}^2(\|v\|_{L^\infty H^2}^2 + \|w\|_{L^\infty H^2}^2).$$

Thus, if $v, w \in X(R)$,

$$\|Sv - Sw\|_X \leq CRT^{\frac{1}{2}}\|v - w\|_X \leq \tfrac{1}{4}\|v - w\|_X$$

if $T \leq (4CR)^{-2}$, and

$$\|Sv\|_X \leq \|S0\|_X + \|Sv - S0\|_X \leq \frac{1}{2}R + \frac{1}{4}R,$$

which implies $Sv \in X(R)$. This shows the existence of a unique solution $v \in X(R)$ if $T = (4CR)^{-2}$. The estimate for $\partial_t v$ follows from Lemma 10.2.

If $T_{\max} < \infty$ and $v \in L^\infty(0, T_{\max}; H^1)$, we can take $t_j \nearrow T_{\max}$ as $j \to \infty$ with $\|v(t_j)\|_{H^1} \le M := \|v\|_{L^\infty(0,T_{\max};H^1)} + \|f\|_{L^2(0,\infty;H^1)}$. By an $\mathbb{R}^3$-version of Theorem 4.1, there is a strong solution u of (NS) in $L^\infty(I_j; H^1) \cap L^2(I_j; H^2)$, $I_j = (t_j, t_j + \tau)$, with $u(t_j) = v(t_j)$, $\|u\|_{L^\infty(I_j;H^1)\cap L^2(I_j;H^2)} \le C(M)$, and $\tau = C(M) > 0$. This solution agrees with v by weak-strong uniqueness.

Thus we can find $t'_j \in (t_j, T_{\max})$ so that $\|v(t'_j)\|_{H^2} \le C(M)$. By the first part of this theorem, there is a strong solution u of (NS) in $L^\infty(I'_j; H^2) \cap L^2(I'_j; H^3)$, $I'_j = (t'_j, t'_j + \tau')$, with $u(t'_j) = v(t'_j)$ and $\tau' = C(M) > 0$. This solution agrees with v by weak-strong uniqueness. Since $t'_j \to T_{\max}$, this shows v can be extended beyond $T_{\max}$ in the class $L^\infty H^2 \cap L^2 H^3$, a contradiction. Thus $v \notin L^\infty(0, T_{\max}; H^1)$. □

10.2. No swirl case

In this section we consider axisymmetric flows with no swirl, $v_\theta = 0$.

Lemma 10.4. *For axisymmetric v with $v_\theta = 0$, $\omega = \operatorname{curl} v = \omega_\theta e_\theta$ and, for $1 < p < \infty$,*

$$\|Dv\|_{L^p(\mathbb{R}^3)} \sim \|\omega_\theta\|_{L^p(\mathbb{R}^3)}, \quad \|D^2 v\|_{L^p(\mathbb{R}^3)} \sim \|\nabla \omega_\theta\|_{L^p(\mathbb{R}^3)} + \|\frac{\omega_\theta}{r}\|_{L^p(\mathbb{R}^3)}.$$

Above, $D = (\partial_1, \partial_2, \partial_3)$ and $\nabla = (\partial_r, \partial_z)$.

Proof. Note that $Dv = D(-\Delta)^{-1} \operatorname{curl} \omega$ is given by singular integrals of ω. Thus

$$\|Dv\|_{L^p(\mathbb{R}^3)} \sim \|\omega\|_{L^p(\mathbb{R}^3)}, \quad \|D^2 v\|_{L^p(\mathbb{R}^3)} \sim \|D\omega\|_{L^p(\mathbb{R}^3)}.$$

When $v_\theta = 0$, we have $\omega = \omega_\theta e_\theta$ and thus the pointwise bounds

$$|\omega| = |\omega_\theta|, \quad |D\omega| \sim |\nabla \omega_\theta| + \frac{1}{r}|\omega_\theta|.$$

The lemma follows. □

Theorem 10.5. *If $v_0 \in L^2_\sigma \cap L^3_\sigma(\mathbb{R}^3)$ is axisymmetric with $v_{0,\theta} = 0$, then there is a global-in-time axisymmetric Leray-Hopf weak solution $v(t)$ of (NS) with zero force and $v_\theta(t) = 0$ and, for any $0 < t_1 < \infty$,*

$$v \in L^\infty(t_1, \infty; H^2) \cap L^2(t_1, \infty; H^3), \quad \partial_t v \in L^2(t_1, \infty; H^1).$$

It is unique in the class of Leray-Hopf weak solutions.

Proof. Step 1. By Theorem 5.6, there is a Leray-Hopf weak solution

$$v \in BC([0, t_b); L^2 \cap L^3) \cap L^2(0, t_b; H^1)$$

for some $t_b \in (0, \infty)$. For any $t_1 \in (0, t_b)$, there is some $t_a \in (0, t_1)$ so that $v(t_a) \in H^1$. The unique existence theorem with H^1 data (Theorem 4.1) shows $v \in L^\infty(t_a, t_b; H^1) \cap L^2(t_a, t_b; H^2)$ with a possibly smaller t_b. There is some $t_0 \in (t_a, t_1)$ so that $v(t_0) \in H^2$. Denote

$$C_0 = \|v_0\|_{L^2}, \quad C_1 = \|\omega_\theta(t_0)\|_{L^2} + \left\| \frac{1}{r}\omega_\theta(t_0) \right\|_{L^2}.$$

Note that $C_1 \lesssim \|v(t_0)\|_{H^2} < \infty$ by Lemma 10.4. By Lemma 10.3, we may extend v to (t_0, T) for some maximal $T \in (t_0, \infty]$ such that

$$v \in L^\infty(t_0, T'; H^2(\mathbb{R}^3)) \cap L^2(t_0, T'; H^3(\mathbb{R}^3)),$$

for any $T' \in (t_0, T)$.

Step 2. Let $\Omega = \frac{1}{r}\omega_\theta$. Testing the Ω-equation (10.16) with Ω itself (noting that the source term $r^{-2}\partial_z v_\theta^2$ is zero), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \Omega^2 + \int_{\mathbb{R}^3} |\nabla\Omega|^2 &= \int_{\mathbb{R}^3} \frac{1}{r}\partial_r \Omega^2 \\ &= 2\pi \int_{\mathbb{R}} \int_0^\infty \partial_r \Omega^2 dr dz = 2\pi \int_{\mathbb{R}} \left[\Omega^2\right]_{r=0}^\infty dz \le 0, \end{aligned}$$

thanks to the decay and the good sign at $r = 0$. Thus

$$\sup_{t_0<t<T} \int_{\mathbb{R}^3} |\Omega(t)|^2 + 2\int_{t_0}^T \int_{\mathbb{R}^3} |\nabla\Omega|^2 \le \int_{\mathbb{R}^3} |\Omega(t_0)|^2.$$

In particular, $\sup_{t_0<t<T} \|\frac{\omega_\theta(t)}{r}\|_{L^2} \le C_1$.

Step 3. Testing the ω_θ-equation (10.9) with ω_θ itself and using $v_\theta = 0$, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \omega_\theta^2 + \int_{\mathbb{R}^3} \left(|\nabla\omega_\theta|^2 + \frac{\omega_\theta^2}{r^2}\right) = \int_{\mathbb{R}^3} \frac{1}{r} v_r \omega_\theta^2.$$

Thus for $t_0 < t_2 < T$ and $J = (t_0, t_2)$,

$$\int_{\mathbb{R}^3} \omega_\theta^2(t_2) + 2\int_J \int_{\mathbb{R}^3} \left(|\nabla\omega_\theta|^2 + \frac{\omega_\theta^2}{r^2}\right) \le \int_{\mathbb{R}^3} \omega_\theta^2(t_0) + \int_J \int_{\mathbb{R}^3} \frac{2}{r}|v_r|\omega_\theta^2.$$

Note that

$$\begin{aligned} \int_J \int_{\mathbb{R}^3} \frac{2}{r}|v_r|\omega_\theta^2 &\le C\int_J \|v\|_\infty \|\omega_\theta\|_2 \left\|\frac{1}{r}\omega_\theta\right\|_2 \\ &\le CC_1 \int_J \|Dv\|_2^{3/2} \|D^2 v\|_2^{1/2}, \end{aligned} \tag{10.23}$$

where we have used Step 2, Lemma 10.4, and (10.22). By Young's inequality, for any $\varepsilon > 0$,

$$\begin{aligned} (10.23) &\le \varepsilon \int_J \|D^2 v\|_2^2 + CC_1^4 \varepsilon^{-3} \int_J \|Dv\|_2^2 \\ &\le C\varepsilon \int_J \int_{\mathbb{R}^3} \left(|\nabla\omega_\theta|^2 + \frac{\omega_\theta^2}{r^2}\right) + CC_1^4 \varepsilon^{-3} C_0^2, \end{aligned}$$

where we used Lemma 10.4 for the second inequality. By taking $\varepsilon = 1/C$, we have shown

$$\int_{\mathbb{R}^3} \omega_\theta^2(t_2) + \int_J \int_{\mathbb{R}^3} \left(|\nabla \omega_\theta|^2 + \frac{\omega_\theta^2}{r^2} \right) \le C_1^2 + C C_1^4 C_0^2.$$

In particular, we may take $t_2 \to T$ and $\omega_\theta \in L^\infty(t_0, T; L^2)$.

Step 4. By Lemma 10.4 again, we have

$$\|v\|_{L^\infty(t_0,T;H^1)} + \|D^2 v\|_{L^2(t_0,T;L^2)} \le C \tag{10.24}$$

where C is independent of T. By Lemma 10.3, $T = \infty$ and the solution is global in time. We have the estimate (10.24) with $T = \infty$. □

10.3. Type I singularity: De Giorgi-Nash-Moser approach

In this section and the next, we will show the nonexistence of Type I singularity for axisymmetric flows. Recall from Section 8.3 that a *Type I* singularity for a solution $v(x,t)$ of (NS) at $(x,t) = (0,0)$ satisfies the scaling invariant blowup rate

$$|v(x,t)| \sim \frac{C}{|x| + \sqrt{-t}}, \quad C = O(1). \tag{10.25}$$

It is at the borderline of typical *ε-regularity criteria* which guarantee the regularity assuming

$$|v(x,t)| \le \frac{\varepsilon}{|x| + \sqrt{-t}}, \quad \varepsilon \ll 1,$$

in some weak sense. Examples include Theorems 6.3, 6.4, and 6.7.

For axisymmetric flows, there are some other possible scaling invariant bounds which are weaker than (10.25), e.g.,

$$|v(x,t)| \le \begin{cases} C_*(r^2 - t)^{-1/2} & \textit{or} \\ C_* r^{-1} & \textit{or} \\ C_*(-t)^{-1/2}. \end{cases} \tag{10.26}$$

Theorem 10.6. *Let (v,p) be an axisymmetric weak solution of the Navier-Stokes equations (NS) in $D = \mathbb{R}^3 \times (-T_0, 0)$ with $v(-T_0) = v_0 \in L^2_\sigma \cap L^3(\mathbb{R}^3)$, $r v_{0,\theta} \in L^\infty(\mathbb{R}^3)$, $p \in L^{5/3}(D)$, and let v be pointwise bounded as, for some $0 \le \varepsilon \le 1/2$,*

$$|v(x,t)| \le \frac{C_*}{r^{1-2\varepsilon}(-t)^\varepsilon}, \quad (x,t) \in D. \tag{10.27}$$

The constant $C_* < \infty$ is allowed to be large. Then $v \in L^\infty(B_R \times [-T_0, 0])$ for any $R > 0$.

The scaling invariant condition (10.27) contains all cases of (10.26). This theorem was first proved assuming $|v| \le C_*(r^2 - t)^{-1/2}$ by [**31**]. It was then extended to other cases by [**32**] and [**104**] independently.

The exponent 5/3 for the norm of p can be replaced. However, it is the natural exponent occurring in the existence theory for weak solutions; see Theorem 3.9.

In this section and the next, we will sketch the proof for the case assuming $|v| \le C_*/r$. In this section we follow the approach of [**31**] and [**32**] which use the methods of De Giorgi, Nash, and Moser. In next section we present the approach of [**104**] using the Liouville theorem.

The main idea of the first approach is as follows. We may assume v is regular for $t < 0$. For $t \in (-T_0, 0)$ the swirl component v_θ gains a modicum of regularity using the Γ-equation (10.15): For some small $\alpha = \alpha(C_*) > 0$, (10.27) enables us to conclude that

$$|v_\theta(t, r, z)| \le C r^{\alpha - 1}. \tag{10.28}$$

This estimate breaks the scaling, thereby transforming the problem from order one to ϵ-regularity.

Equation (10.28) is the Hölder estimate at $r = 0$ for $\Gamma = r v_\theta$ which satisfies (10.15). In the classical setting for the parabolic PDE

$$\partial_t u = \partial_i(a_{ij}(x, t)\partial_j u)$$

where the a_{ij} are measurable, uniformly elliptic, and allowed to be discontinuous, the methods of De Giorgi, Nash, and Moser [**38, 152–154**] show that u is bounded and Hölder continuous. Our system (10.15)–(10.16) shifts the focus from discontinuous coefficients of highest-order terms to those of the lower-order terms. One such example is Zhang [**225**] on the Hölder continuity of weak solutions of

$$\partial_t u - \Delta u + b \cdot \nabla u = 0, \quad \operatorname{div} b = 0, \tag{10.29}$$

if $b = b(x)$ is independent of time and b satisfies an integral condition, which is fulfilled if, say, b is controlled by $1/|x|$. His proof makes use of Moser iteration and Gaussian bounds.

If one assumes a weaker condition $|b| \le C|x|^{-1-\varepsilon}$, $0 < \varepsilon \ll 1$, one can show the L^∞-bound of u (see [**106**] for the elliptic case), but its Hölder continuity is an open question for both the elliptic and parabolic cases. This is one of the difficulties of excluding Type II singularity of axisymmetric (NS).

To show (10.28) we prove the following.

Theorem 10.7. *Suppose that* $\Gamma(x, t)$ *is a smooth bounded solution of* (10.15) *in* Q_2 *with smooth* $b(x, t)$*, that both may depend on* θ *and* t*, and that*

$$\Gamma|_{r=0} = 0, \quad \operatorname{div} b = 0, \quad |b| \le C_*/r \quad \text{in } Q_2.$$

Then there exist constants C and $\alpha > 0$ which depend only upon C_* such that

$$|\Gamma(x,t)| \le C\|\Gamma\|_{L^\infty_{t,x}(Q_2)} r^\alpha \quad \text{in } Q_1. \tag{10.30}$$

Comparing (10.15) to (10.29), the bound $|b| \le C_*/r$ is pointwise, time dependent, and more singular than $1/|x|$. The condition $\Gamma|_{r=0} = 0$ compensates for the bad sign of the term $\frac{2}{r}\partial_r \Gamma$ but also precludes a direct lower bound on the fundamental solution. We will give a proof of Theorem 10.7 in this section using the methods of De Giorgi and Moser. A second proof will be given in Section 10.5.

Proof of Theorem 10.7. The proof consists of several steps.

Step 1. Reformulation. Let $X = (x,t)$. Define the modified parabolic cylinder at the origin

$$Q(R,\tau) = \{X : |x| < R, -\tau R^2 < t < 0\}.$$

Here $R > 0$ and $\tau \in (0,1]$. Also denote $Q_R = Q(R) = Q(R,1)$. For fixed $0 < R < 1$, let

$$m_2 \equiv \inf_{Q(2R)} \Gamma, \quad M_2 \equiv \sup_{Q(2R)} \Gamma, \quad M \equiv M_2 - m_2 > 0.$$

Notice that $m_2 \le 0 \le M_2$ since $\Gamma|_{r=0} = 0$. Define

$$u \equiv \begin{cases} 2(\Gamma - m_2)/M & \text{if} \quad -m_2 > M_2, \\ 2(M_2 - \Gamma)/M & \text{otherwise.} \end{cases} \tag{10.31}$$

In either case u solves (10.15), $0 \le u \le 2$ in $Q(2R)$, and

$$a \equiv u|_{r=0} \ge 1. \tag{10.32}$$

In view of (10.32), we want to show that $\inf_{Q(cR)} u > \delta$ for some small constants $c, \delta > 0$ independent of u and R. It would imply the oscillation estimate (10.45), which implies (10.30) by iteration.

Step 2. Energy estimates. Define $v_\pm = (u-k)_\pm$ with $k \ge 0$. Consider a radial test function $0 \le \zeta(x,t) \le 1$ for which $\zeta = 0$ on $\partial B_R \times [-R^2, 0]$ and $\frac{\partial \zeta}{\partial r} \le 0$. We multiply (10.15) for $u - k$ with $\zeta^2 v_\pm$ and integrate over $\mathbb{R}^3 \times [t_0, t]$ to obtain

$$\begin{aligned} &\frac{1}{2}\left[\int_{\mathbb{R}^3} |\zeta v_\pm|^2\right]_{t_0}^t + \int_{t_0}^t \int_{\mathbb{R}^3} |\nabla(\zeta v_\pm)|^2 \\ &\quad = \int_{t_0}^t \int_{\mathbb{R}^3} v_\pm^2 \left(b\zeta \cdot \nabla\zeta + \zeta\frac{\partial \zeta}{\partial t} + |\nabla \zeta|^2 + \frac{2\zeta}{r}\frac{\partial \zeta}{\partial r}\right) \\ &\qquad + 2\pi[(a-k)_\pm]^2 \int_{t_0}^t \int_{\mathbb{R}} dz\ \zeta^2|_{r=0}. \end{aligned} \tag{10.33}$$

To control the term involving b, we first apply Young's inequality,

(10.34)
$$\begin{aligned}\int_{\mathbb{R}^3} v_\pm^2 b\zeta\cdot\nabla\zeta &= R^{-2}\int_{\mathbb{R}^3} v_\pm^2\zeta^2 Rb\frac{R\nabla\zeta}{\zeta}\\ &\le R^{-2}\int_{\mathbb{R}^3} v_\pm^2\zeta^2\delta|Rb|^{1+\varepsilon} + R^{-2}\int_{\mathbb{R}^3} v_\pm^2\zeta^2 C_\delta\left(\frac{R\nabla\zeta}{\zeta}\right)^{\frac{1+\varepsilon}{\varepsilon}},\end{aligned}$$

which separates the singularities of b and $|\nabla\zeta|/\zeta$. We then choose $0 < \varepsilon < 1/3$ so that $\||Rb|^{1+\varepsilon}\|_{L^{3/2}(B_R)} \lesssim R^2$, and we choose ζ to decay like $(1-|x|/R)^n$ near ∂B_R with n large enough, so that $R|\nabla\zeta|/\zeta \lesssim \zeta^{-1/n}$. By Hölder and Sobolev inequalities and finally choosing δ small enough,

$$\int_{\mathbb{R}^3} v_\pm^2 b\zeta\cdot\nabla\zeta \le \frac{1}{4}\int_{\mathbb{R}^3}|\nabla(v_\pm\zeta)|^2 + CR^{-2}\int_{B_R} v_\pm^2. \tag{10.35}$$

Choose $\sigma\in(1/4,1)$ and require that $\zeta\equiv 1$ on $Q(\sigma R,\tau)$. If we further choose $\zeta(x,t_0)=0$ with $t_0=-\tau R^2$, then, using (10.35), we estimate (10.33) as

$$\begin{aligned}&\sup_{-\tau\sigma^2R^2<t<0}\int_{B(\sigma R)\times\{t\}} v_\pm^2 + \int_{Q(\sigma R,\tau)}|\nabla v_\pm|^2\\ &\qquad\le \frac{C_{**}}{\tau(1-\sigma)^2R^2}\int_{Q(R,\tau)} v_\pm^2 + C\tau R^3[(a-k)_\pm]^2.\end{aligned} \tag{10.36}$$

If we alternatively choose $\zeta=\zeta(x)$, then (10.33) takes the form

$$\begin{aligned}&\sup_{t_0<s<t}\int_{B(\sigma R)\times\{s\}} v_\pm^2 + \int_{t_0}^t\int_{B(\sigma R)}|\nabla v_\pm|^2 - \int_{B_R\times\{t_0\}} v_\pm^2\\ &\qquad\le \frac{C_{**}}{(1-\sigma)^2R^2}\int_{t_0}^t\int_{B_R} v_\pm^2 + C\tau R^3[(a-k)_\pm]^2.\end{aligned} \tag{10.37}$$

Notice that there is no τ^{-1} in (10.37) compared to (10.36). The energy estimates (10.36) and (10.37) are the standard parabolic De Giorgi classes except for the last term, which vanishes for v_- if $0\le k\le 1$.

The following three steps give three lemmas.

Step 3. Time continuity of Lebesgue measure of sublevel sets.

Lemma 10.8. *Suppose there exists a $t_0\in[-\tau R^2,0]$, $K\in(0,1)$, and $\gamma\in(0,1)$ so that*

$$|\{x\in B_R : u(x,t_0)\le K\}|\le\gamma|B_R|.$$

Further suppose that u satisfies (10.37) for v_-. Then for all $\eta\in(0,1-\sqrt{\gamma})$ and $\mu\in(\gamma/(1-\eta)^2,1)$ there exists $\theta\in(0,1)$ such that

$$|\{x\in B_R : u(x,t)\le\eta K\}|\le\mu|B_R|,\quad \forall t\in[t_0,t_0+(\tau\wedge\theta)R^2].$$

It suffices to take $\theta > 0$ so that, for some $\sigma < 1$ close to 1,

$$(1-\eta)^{-2}\left(\gamma + \frac{C_{**}(\tau \wedge \theta)}{(1-\sigma)^2}\right) + (1-\sigma^3) \le \mu, \tag{10.38}$$

where C_{**} is the constant in (10.37).

It is proved using (10.37) for $v_- = (u-K)_-$ and the Chebyshev inequality. Since $(a-K)_- = 0$, it is similar to the classical case.

Step 4. Arbitrarily small density of small value subset on a time slice.

Lemma 10.9. *Suppose that $u(x,t)$ satisfies (10.36) for v_-. In addition*

$$|\{x \in B_R : u(x,t) \le K\}| \le \gamma |B_R|, \quad \forall t \in [t_0, t_0 + \theta R^2] = I,$$

where $\theta > 0$, $K, \gamma \in (0,1)$, and $B_R \times I \subset Q(R,\tau)$. Then for all $\epsilon \in (0,1)$ there exists a $\delta \in (0,1)$ such that

$$|\{X \in B_R \times I : u(X) \le \delta\}| \le \epsilon |B_R \times I|.$$

Sketch of proof. Denote, for $n = 0, 1, 2, 3, \ldots$,

$$A_n(t) = \left\{x \in B_R : u(x,t) \le 2^{-n}K\right\}, \quad A_n = \{(x,t) : t \in I, x \in A_n(t)\}.$$

Invoking the following inequality that for any $v \in W^{1,1}(B_R)$ and $\alpha < \beta$

$$|\{x \in B_R : v(x) \le \alpha\}| \le \frac{CR^{3+1}/(\beta-\alpha)}{|\{x \in B_R : v(x) > \beta\}|} \int_{B_R \cap \{\alpha < v \le \beta\}} |\nabla v|,$$

we get

$$\begin{aligned} |A_{n+1}(t)| &\le \frac{C2^{n+1}R}{K(1-\gamma)} \int_{A_n(t) - A_{n+1}(t)} |\nabla u| \\ &= \frac{C2^{n+1}R}{K(1-\gamma)} \int_{A_n(t) - A_{n+1}(t)} |\nabla (u-\beta)_-| \end{aligned}$$

where $\beta = 2^{-n}K$. Integrating in time, using the Cauchy-Schwartz inequality and (10.36), we get

$$|A_{n+1}| \le \frac{CR^{5/2}}{K(1-\gamma)} |A_n - A_{n+1}|^{1/2}. \tag{10.39}$$

Denote $p_n = \frac{|A_n|}{|B_R \times I|}$. It follows from (10.39) that

$$p_n^2 \le C_1(p_{n-1} - p_n), \quad C_1 = \frac{C}{\theta K^2 (1-\gamma)^2},$$

$$n p_n^2 \le \sum_{j=1}^n p_j^2 \le \sum_{j=1}^n C_1(p_{j-1} - p_j) = C_1(p_0 - p_n) \le C_1.$$

We complete the proof by choosing n sufficiently large. □

Step 5. Small value subset has density less than one. The following lemma provides an initial estimate to start with.

Lemma 10.10. *Let* $0 \le \tau \le 1/8$*. There exists* $\kappa \in (0,1)$ *such that* $0 < \lambda < \kappa\tau$ *implies*

$$\left|\left\{X \in Q(R,\tau) : u(X) \le \lambda^2\right\}\right| \le (1-4\lambda)|Q(R,\tau)|.$$

Suppose the opposite, that

$$\left|\left\{X \in Q(R,\tau) : u(X) > \lambda^2\right\}\right| < 4\lambda|Q(R,\tau)|. \tag{10.40}$$

Testing the equation (10.15) with $(u+\epsilon)^{-1/2}\zeta^2$ for $\zeta \ge 0$ and sending $\epsilon \downarrow 0$, one gets an upper bound for a from the boundary term at $r = 0$ and a contradiction with $a \ge 1$ for κ sufficiently small.

Step 6. Arbitrarily small density of small value subset up to time zero. Here we assemble the previous steps. At the beginning we have Lemma 10.10 with the parameter $\tau \in (0, 1/8)$ to be decided. By Lemma 10.10, there is a $t_1 \in [-\tau R^2, -2\lambda\tau R^2]$ so that

$$|\left\{x \in B_R : u(x,t_1) \le \lambda^2\right\}| \le (1-2\lambda)|B_R|. \tag{10.41}$$

Apply Lemma 10.8 to (10.41) with $K = \lambda^2$, $\gamma = 1-2\lambda$, $\eta = \lambda$, and $\mu = 1-\lambda^2$ to get

$$|\left\{x \in B_R : u(x,t) \le \lambda^3\right\}| \le (1-\lambda^2)|B_R|, \quad \forall t \in [t_1, t_1+\theta_* R^2] \equiv I_*.$$

Here $\theta_* = \theta \wedge \lambda\tau$ and θ is the constant chosen in Lemma 10.8. From here Lemma 10.9 gives

$$|\{X \in B_R \times I_* : u(X) \le \delta_*\}| \le \frac{\epsilon_*}{2}|B_R \times I_*|,$$

where $\epsilon_* > 0$ is as small as we want and $\delta_* = \delta_*(\epsilon_*)$.

Then, as in (10.41), there exists a $t_2 \in I_*$ (so that $t_2 \le -\lambda\tau R^2$) such that

$$|\{x \in B_R : u(x,t_2) \le \delta_*\}| \le \epsilon_*|B_R|. \tag{10.42}$$

Up until now all the small parameters that we have chosen depend upon τ. But, above, ϵ_* can be taken arbitrarily small independent of the size of τ.

Now first choose $1-\sigma^3 = 1/4$ and $\tau < 1/8$ so that $C_{**}\tau/(1-\sigma)^2 \le 1/4$. Then take δ_* from (10.42) with $\epsilon_* < 1/16$ above playing the role of γ in Lemma 10.8. Also take $\eta < 1/4$. With all this, from Lemma 10.8, we can choose $\mu < 1$ so that $\theta = \tau$ satisfies (10.38) and

$$|\{x \in B_R : u(x,t) \le \eta\delta_*\}| \le \mu|B_R|, \quad \forall t \in [t_2, t_2+\tau R^2] \supset [-\lambda\tau R^2, 0].$$

Finally apply Lemma 10.9 again to obtain

$$|\{X \in Q(R,\lambda\tau) : u(X) \leq \delta\}| \leq \epsilon |Q(R,\lambda\tau)|, \tag{10.43}$$

with $\epsilon > 0$ arbitrarily small.

The above step has two iterations starting from (10.41) and (10.42). The latter (10.42) gains more smallness than (10.41).

Step 7. Lower bound. Let $U = \delta - u$, where δ is the constant from (10.43). Clearly U is a solution of (10.15) and $U|_{r=0} = \delta - a < 0$. We apply (10.36) to U on $Q(2d)$ (with $\tau = 1$) to get

$$\begin{aligned}\sup_{-\sigma^2 d^2 < t < 0} \int_{B(\sigma d)\times\{t\}} |(U-k)^+|^2 + \int_{Q(\sigma d)} |\nabla (U-k)^+|^2 \\ \leq \frac{C_{**}}{(1-\sigma)^2 d^2} \int_{Q(d)} |(U-k)^+|^2.\end{aligned}$$

This holds for all $k > 0$ and $\sigma \in (0,1)$. The classical De Giorgi estimate gives

$$\sup_{Q(d/2)} (\delta - u) \leq \left(\frac{C}{|Q(d)|} \int_{Q(d)} |(\delta-u)^+|^2 \right)^{1/2}. \tag{10.44}$$

Let $d = \sqrt{\lambda\tau} R$ so that $Q(d) \subset Q(R,\lambda\tau)$. By (10.44) and (10.43),

$$\delta - \inf_{Q(d/2)} u \leq \left(\frac{C\delta^2 \epsilon |Q(R,\lambda\tau)|}{|Q(d)|} \right)^{1/2} = C\delta\epsilon^{1/2} (\lambda\tau)^{-3/4},$$

which is less than $\frac{\delta}{2}$ if ϵ is chosen sufficiently small. We conclude $\inf_{Q(d/2)} u \geq \frac{\delta}{2}$. It follows that

$$\operatorname{osc}(\Gamma, d/2) \leq \left(1 - \frac{\delta}{4}\right) \operatorname{osc}(\Gamma, 2R). \tag{10.45}$$

Step 8. Iteration. Iterating (10.45), we get

$$\operatorname{osc}(\Gamma, R) \leq C \|\Gamma\|_{L^\infty(Q(1))} R^\alpha, \quad \forall R \in (0,1), \tag{10.46}$$

for $\alpha = 2\log(1-\frac{\delta}{4})/\log(\lambda\tau/16) > 0$.

An arbitrary point $X = (r,z,t) \in Q_1$ belongs to $Q(X_0, r)$ where $X_0 = (0,z,t)$. Applying (10.46) to Γ on $Q(X_0,1)$, we get

$$|\Gamma(X)| \leq |\Gamma(X_0)| + \operatorname{osc}(\Gamma, Q(X_0,r)) \leq 0 + C\|\Gamma\|_{L^\infty(Q(2))} r^\alpha.$$

We have proved Theorem 10.7. $\square$

Remark. The above proof uses methods of De Giorgi and Moser; see [**31**, §3]. It is also possible to prove it using Nash's idea for a lower bound, applied directly to a solution and not the fundamental solution; see [**32**, §5].

We next prove Theorem 10.6 using Theorem 10.7 with $\varepsilon = 0$ in assumption (10.27); i.e., we assume $|v| \le C_*/r$. Our proof does not take scaling limits and all bounds are computable. We will give another proof in the next section.

Proof of Theorem 10.6 assuming $|v| \le C_*/r$. We may assume $T_0 \ge 4$. Let M be the maximum of $|v|$ up to a fixed time $t_1 < 0$. We will derive an upper bound of M in terms of C_* and independent of t_1. We may assume $M > 1$. Define

$$v^M(X,T) = M^{-1} v(X/M, T/M^2), \quad X = (X_1, X_2, Z).$$

For $x = (x_1, x_2, z)$ and $X = (X_1, X_2, Z)$, let $r = (x_1^2 + x_2^2)^{1/2}$ and $R = (X_1^2 + X_2^2)^{1/2}$. We have the following estimates for all r and R for time $t \le t_1$ and $T \le M^2 t_1$:

$$|v(x,t)| \le C/r, \quad |v^M(X,T)| \le C/R, \quad |\nabla^k v^M| \le C_k. \tag{10.47}$$

The last inequality follows from $\|v^M\|_{L^\infty} \le 1$ for $t < t_1$ and the regularity theorem of Navier-Stokes equations. Its angular component (we omit the time dependence below) $v_\theta^M(R,Z)$ satisfies $v_\theta^M(0,Z) = 0 = \partial_Z v_\theta^M(0,Z)$ for all Z. By the mean value theorem and (10.47), $|v_\theta^M(R,Z)| \le CR$ and $|\partial_Z v_\theta^M(R,Z)| \le CR$ for $R \le 1$. Together with (10.47) for $R \ge 1$, we get

$$|v_\theta^M| \le C \min(R, R^{-1}), \quad |\partial_Z v_\theta^M| \le C \min(R, 1), \quad \text{for } R > 0.$$

By Theorem 10.7, $\Gamma = r v_\theta$ satisfies $|\Gamma(r,z,t)| \le C r^\alpha$ in Q_1, for some C and small $\alpha > 0$ depending on C_*. Thus, $|v_\theta^M(R,Z)| \le C R^{-1+\alpha} M^{-\alpha}$ for $R > 0$. From these estimates we have

$$\begin{aligned} \frac{|\partial_Z (v_\theta^M)^2(R)|}{R^2} &\le \frac{C \min(R, R^{-1+\alpha} M^{-\alpha}) \cdot \min(R,1)}{R^2} \\ &\le \frac{C}{R^{3-\alpha} M^\alpha + 1}. \end{aligned} \tag{10.48}$$

Consider now the angular component of the rescaled vorticity. Recall $\Omega = \omega_\theta / r$. Let

$$f = \Omega^M(X,T) = \Omega(X/M, T/M^2) M^{-3} = \omega_\theta^M(X,T)/R.$$

Since ω_θ^M and $\nabla \omega_\theta^M$ are bounded by (10.47) and $\omega_\theta^M|_{R=0} = 0$, we have $|f| \le C(1+R)^{-1}$. From the equation of ω_θ (see (10.9)), we have

$$(\partial_T - L) f = g, \quad L = \Delta + \frac{2}{R}\partial_R - b^M \cdot \nabla_X,$$

where $g = R^{-2}\partial_Z (v_\theta^M)^2$ and $b^M = v_r^M e_R + v_z^M e_Z$. Let $P(T,X;S,Y)$ be the evolution kernel for $\partial_T - L$. By Duhamel's formula

$$
\begin{aligned}
f(X,T) &= \int P(T,X;S,Y) f(Y,S)dY + \int_S^T \int P(T,X;\tau,Y) g(Y,\tau) dY d\tau \\
&=: I + II.
\end{aligned}
$$

By Carlen and Loss [**25**], in particular, equation (2.5) in [**25**], the kernel P satisfies $P \geq 0$, $\int P(T,X;S,Y)dY = 1$, and, using $\|b^M\|_\infty < 1$,

$$
P(T,X;S,Y) \leq C(T-S)^{-3/2} e^{-h(|X_3 - Y_3|, T-S)}, \quad h(a,\tau) = \frac{1}{4\tau}(a-\tau)_+^2.
$$

Here we only assert the spatial decay in the X_3-direction so that the proof of [**25**], where the term $R^{-1}\partial_R$ in L is not present, needs no revision. With these bounds, coordinate $Y = (R,Z)$, and Hölder inequality, we get for $T - S > 1$ (using $e^{-h(a,\tau)} \leq Ce^{-Ca/\tau}$ when $\tau > 1$)

$$
\begin{aligned}
|I| &\leq \left[\int P(T,X;S,Y) |f(Y,S)|^3 dY \right]^{\frac{1}{3}} \\
&\leq \left[C(T-S)^{-\frac{3}{2}} \int e^{-C\frac{|X_3 - Y_3|}{T-S}} \frac{RdR}{(1+R)^3} dY_3 \right]^{\frac{1}{3}} \leq C(T-S)^{-1/6},
\end{aligned} \tag{10.49}
$$

and

$$
\begin{aligned}
|II| &\leq \int_S^T \left[\int P(T,X;S,Y) |g(Y,S)|^2 dY \right]^{\frac{1}{2}} d\tau \\
&\leq C \int_0^{T-S} \left[\int \tau^{-3/2} e^{-C\frac{(a-\tau)_+^2}{\tau}} \frac{R\,dR\,da}{(R^{3-\alpha}M^\alpha + 1)^2} \right]^{1/2} d\tau \\
&\leq C \int_0^{T-S} \left[(\tau^{-1} + \tau^{-1/2}) M^{-\frac{2\alpha}{3-\alpha}} \right]^{1/2} d\tau \\
&\leq C(T-S)^{3/4} M^{-\frac{\alpha}{3}}.
\end{aligned} \tag{10.50}
$$

Combining these two estimates and choosing $S = T - M^{\frac{4}{11}\alpha} > -T_0 M^2$ (hence f is defined), we have $|f(X,T)| \leq CM^{-\frac{2\alpha}{33}}$. Thus

$$
\begin{aligned}
|\omega_\theta(x,t)| &\leq |\omega_\theta^M(rM, zM, tM^2)| M^2 \\
&\leq |\Omega^M(rM, zM, tM^2)| M^2 rM \leq CM^{3-\frac{2\alpha}{33}} r.
\end{aligned}
$$

Therefore, we have

$$
|\omega_\theta(x,t)| \leq CM^{2-\frac{\alpha}{33}}, \quad \text{for } r \leq M^{-1+\frac{\alpha}{33}}. \tag{10.51}
$$

Let $b = v_r e_r + v_z e_z$ and $B_\rho(x_0) = \{x : |x - x_0| < \rho\}$. It satisfies $-\Delta b = \operatorname{curl}(\omega_\theta e_\theta)$ and hence the following estimate with $p > 1$ (see, e.g., [**72**, Theorem 8.17]):

$$\sup_{B_\rho(x_0)} |b| \le C(\rho^{-\frac{3}{p}} ||b||_{L^p(B_{2\rho}(x_0))} + \rho \sup_{B_{2\rho}(x_0)} |\omega_\theta|).$$

Let $x_0 \in \{(r,\theta,z) : r < \rho\}$ and $1 < p < 2$. By the assumption $|v| \le C/r$,

$$\rho^{-\frac{3}{p}} ||b||_{L^p(B_{2\rho}(x_0))} \le C\rho^{-\frac{3}{p}} ||1/r||_{L^p(B_{2\rho}(x_0))} \le C\rho^{-1}.$$

Hence

$$\sup_{B_\rho(x_0)} |b| \le C(\rho^{-1} + \rho^2 M^{3-\frac{2\alpha}{33}}) \le CM^{1-\frac{2\alpha}{99}} \le C\rho^{-1}$$

for $\rho \le M^{-1+\frac{2\alpha}{99}}$. This together with the fact that

$$|v_\theta| = M|v_\theta^M| \le MC\min(R, R^{-1+\alpha}M^{-\alpha}) \le CM^{1-\frac{\alpha}{2-\alpha}}$$

implies

$$|v(x,t)| \le CM^{1-\frac{2\alpha}{99}}, \quad \text{for } r \le M^{-1+\frac{2\alpha}{99}}.$$

On the other hand, the assumption $|v| \le C/r$ implies $|v| \le CM^{1-\frac{2\alpha}{99}}$ for $r \ge M^{-1+\frac{2\alpha}{99}}$. Since M is the maximum of v, this gives its upper bound. $\square$

10.4. Type I singularity: Liouville theorem approach

In this section we give another proof of Theorem 10.6 under the assumption $|v| \le \frac{C_*}{r}$, based on the following Liouville theorem. An *ancient solution* is a solution defined for $t \in (-\infty, 0)$.

Theorem 10.11 (Liouville theorem for ancient solution)**.** *Let* $D = \mathbb{R}^3 \times (-\infty, 0)$*. If* $v(x,t) : D \to \mathbb{R}^3$ *is an axisymmetric solution of* (NS) *with zero force satisfying*

$$|v| \le \frac{C_*}{1+r} \quad \text{in } D,$$

then $v \equiv 0$ *in* D*.*

Second proof of Theorem 10.6 using Theorem 10.11. We can show that v is an L^∞-valued mild solution in the sense of Giga-Inui-Matsui [**64**]. Suppose $v \in L^\infty(\mathbb{R}^3 \times (-T_0, t_0))$ for all $t_0 < 0$ and v develops a singularity at $t = 0$. Then by [**64**],

$$\infty > \|v(t)\|_{L^\infty(\mathbb{R}^3)} \ge \frac{\varepsilon_1}{\sqrt{-t}}, \quad -T_0 < \forall t < 0.$$

Let $M(t) = \sup_{-T_0 \le s \le t} \|v(s)\|_{L^\infty(\mathbb{R}^3)}$. There is a sequence (r_k, z_k, t_k) with $t_k \nearrow 0$ such that $M_k := |v(r_k, z_k, t_k)| \ge \frac{k}{k+1} M(t_k)$. By assumption

$|v| \le \frac{C_*}{r}$, we have $r_k \le C_*/M_k$. Set

$$v^{(k)}(r,z,t) = \frac{1}{M_k} v\left(\frac{r}{M_k}, z_k + \frac{z}{M_k}, t_k + \frac{t}{M_k^2}\right).$$

They are axisymmetric mild solutions in $\mathbb{R}^3 \times (-\frac{M_k^2 T_0}{2}, 0]$, satisfying

$$|v^{(k)}| \le \min\left(\frac{k+1}{k}, \frac{C_*}{r}\right) \quad \text{and} \quad |v^{(k)}(M_k r_k, 0, 0)| = 1.$$

By regularity of L^∞ mild solutions, all derivatives are bounded;

$$\|\nabla_x^l \partial_t^m v^{(k)}\|_{L^\infty(\mathbb{R}^n \times (-T,0))} \le C(l,m,T), \tag{10.52}$$

for all $l, m, T \in \mathbb{N}$, if k is sufficiently large.

By the above bounds, there is a subsequence $v^{(k_j)}$ which converges to an ancient axisymmetric mild solution $\bar v$ defined in D satisfying

$$|\bar v| \le \min\left(1, \frac{C_*}{r}\right) \ \text{ in } D \quad \text{and} \quad |\bar v(\bar r, 0, 0)| = 1,$$

where $\bar r = \lim_{j\to\infty} M_k r_{k_j} \le C_*$. By Liouville's theorem (Theorem 10.11), $\bar v \equiv 0$, contradicting $|\bar v(\bar r, 0, 0)| = 1$. □

To prove Theorem 10.11, we prepare a lemma. Let U be a domain in $\mathbb{R}^n$ and let $T > 0$. Consider the equation

$$\partial_t u - \Delta u + b \cdot \nabla u = 0 \quad \text{in } U_T = U \times (0,T), \tag{10.53}$$

with $b \in L^\infty(U_T)$. There is no assumption on $\operatorname{div} b$. The classical *strong maximal principle* says that any bounded solution u in U_T with $u(\bar x, T) = \sup_{U_T} u$ for some $\bar x \in U$ ($\Rightarrow \bar x \notin \partial U$) is necessarily constant. The following is a refined version.

Lemma 10.12 (Stability of strong maximal principle / Harnack inequality)**.** *Let K be a compact subset of a domain $U \subset \mathbb{R}^n$, let $T > 0$, and let $b \in L^\infty(U_T; \mathbb{R}^n)$. For any $\varepsilon, \tau > 0$, there is $\delta = \delta(U, K, T, \|b\|_{L^\infty}, \varepsilon, \tau) > 0$ such that if u is a bounded solution of* (10.53) *with $\sup_{U_T} |u| = M > 0$ and $\sup_{x\in K} u(x,T) > (1-\delta)M$, then $\inf_{K\times(\tau,T)} u > (1-\varepsilon)M$.*

Proof. We can take $M = 1$ without loss of generality. Assuming the statement fails for some $\varepsilon > 0$, there must exist a sequence of coefficients b^k, solutions u^k of (10.53) with $b = b^k$, points $x_k \in K$, and $(y_k, t_k) \in K \times (\tau, T)$ such that $|b^k| \le C$, $|u^k| \le 1$, $u^k(x_k, T) \to 1$, and $u^k(y_k, t_k) \le 1 - \varepsilon$. After passing to a subsequence, b^k converges weakly* in $L^\infty_{x,t}$ to $\bar b$, u^k converges locally uniformly to $\bar u$, $x_k \to \bar x \in K$, and $(y_k, t_k) \to (\bar y, \bar t) \in K \times [\tau, T]$. The regularity properties of solutions of (10.53) imply that $\bar u$ solves (10.53) with $b = \bar b$, $|\bar u| \le 1$ in U_T, $\bar u(\bar x, T) = 1$, and $\bar u(\bar y, \bar t) \le 1 - \varepsilon$. This, however, is impossible due to the strong maximal principle. □

Proof of Theorem 10.11.

Step 1. Zero swirl. Let $\Gamma = rv_\theta$, $M = \sup_D \Gamma$, and $m = \inf_D \Gamma$. We have $m \le 0 \le M$ since $\Gamma|_{r=0} = 0$. We will show a contradiction if $M \ge |m|$ and $M > 0$, and similarly if $M < |m|$.

For some $0 < \sigma \ll 1$ and $0 < \varepsilon \ll 1$ to be chosen, let

$$K = \{(r,z)|\ \sigma \le r \le 1,\ |z| \le 1\}, \quad U = \left\{(r,z)|\ \frac{\sigma}{2} < r < 2,\ |z| < 2\right\},$$

and $\delta = \delta(U, K, 2, \frac{2C_*}{\sigma}, \varepsilon/2, 1) > 0$ be the constant of Lemma 10.12 with $T = 2$ and $\tau = 1$.

There is $(\bar r, \bar z, \bar t) \in D$ so that $\Gamma(\bar r, \bar z, \bar t) > M(1-\delta)$. Let $\lambda = \bar r$ and

$$\Gamma^\lambda(r,z,t) = \Gamma(\lambda r, \bar z + \lambda z, \bar t + \lambda^2 t), \quad b^\lambda(r,z,t) = \lambda b(\lambda r, \bar z + \lambda z, \bar t + \lambda^2 t). \tag{10.54}$$

Then Γ^λ satisfies the same equation (10.15) in $U \times (-2,0)$ with

$$\sup_{U\times(-2,0)} \Gamma^\lambda \in [M(1-\delta), M], \quad \sup_K \Gamma^\lambda(\cdot, 0) \ge M(1-\delta).$$

By Lemma 10.12,

$$\inf_{K\times(-1,0)} \Gamma^\lambda \ge M(1-\delta/2)(1-\varepsilon/2) \ge M(1-\varepsilon).$$

Let $\varphi(r,z,t) = \xi(r)\eta(z)\zeta(t)$ be a smooth cut-off function with $\xi, \eta, \zeta \ge 0$, $\xi(r) = 1$ if $0 \le r \le 0.9$ and $\xi(r) = 0$ for $r \ge 1$, $\eta(z) = 1$ if $|z| \le 0.9$ and $\eta(z) = 0$ for $|z| \ge 1$, $\zeta(t) = 1$ for $-0.9 \le t \le -0.1$ and $\zeta(t) = 0$ for $t \le -1$ or $t \ge 0$. Multiplying the equation (10.15) for $\Gamma^\lambda - M$ and integrating over space-time, we obtain

$$\begin{aligned}\int (\Gamma - M)(\partial_t \varphi + \Delta\varphi + b\cdot\nabla\varphi)dxdt &= -\int \frac{2}{r}\partial_r(\Gamma - M)\varphi 2\pi r drdzdt \\ &= 4\pi \int (\Gamma - M)(\partial_r\varphi)dzdt + C_1 M\end{aligned}$$

where we used $\Gamma|_{r=0} = 0$ and $C_1 = 4\pi \int \varphi|_{r=0} dzdt$. In the support of φ, we have $|\Gamma - M| \le M\varepsilon$ if $r \ge \sigma$ and $|\Gamma - M| \le M$ if $r < \sigma$. Thus

$$C_1 M \le CM \int_{\operatorname{supp}\varphi} (1_{r<\sigma} + \varepsilon 1_{r>\sigma})\left(C + \frac{C}{r}\right) \le CM(\sigma + \varepsilon).$$

We get a contradiction if we take ε, σ sufficiently small. Thus $\Gamma \equiv 0$.

Step 2. Zero vorticity. Since v is regular in D, $\Omega = \omega_\theta/r \in L^\infty(D)$ and it satisfies (10.16) in D with no source term. Let $M_1 = \sup_D \Omega \ge 0$ and $m_1 = \inf_D \Omega \le 0$. Suppose $M_1 \ge |m_1|$ and $M_1 > 0$. For any $L > 0$, let $\delta = \delta(B_{2L}, \bar B_L, 4L, \|b\|_{L^\infty(D)}, \frac{1}{2}, L) > 0$ be the constant of Lemma 10.12, with $B_L \subset \mathbb{R}^5$. For some $P \in D$, $\Omega(P) \ge M_1(1-\delta)$. Applying Lemma 10.12

to the solution Ω of (10.16), considered as an equation in $\mathbb{R}^5 \times (-\infty, 0)$ so that $\Delta + \frac{2}{r}\partial_r = \Delta_{\mathbb{R}^5}$ on axisymmetric functions, we get $\Omega \ge M_1/2$ in $Q(P, L)$. However, this would mean that $\omega_\theta = \Omega r$ is unbounded, a contradiction. Therefore $M_1 \le 0$. In the same way we can show $m_1 \ge 0$, and hence ω_θ vanishes identically.

Step 3. Zero velocity. Since $\operatorname{div} v = 0$ and $\operatorname{curl} v = \omega_\theta e_\theta = 0$, v is harmonic in x. Since $|v| \le C/(1+r)$, $v \equiv 0$. ☐

10.5. Connections between the two approaches

In both approaches of Sections 10.3 and 10.4, the key is to get better estimate of the swirl component v_θ; see Theorem 10.7 and Step 1 of the proof of Theorem 10.11.

In this respect, we note their connections as follows. First note that Step 1 of the proof of Theorem 10.11 follows from Theorem 10.7: For any λ, we have $|v_\theta^\lambda(r, z, t)| \le C r^{\alpha-1}$ in Q_1, or

$$|v_\theta(R, Z, T)| \le C\lambda^{-1}(R/\lambda)^{\alpha-1} = C R^{\alpha-1}\lambda^{-\alpha} \quad \text{in } Q_\lambda.$$

Sending $\lambda \to \infty$, we get $v_\theta \equiv 0$ in D.

Next note that we can prove Theorem 10.7 using the argument of Step 1 of the proof of Theorem 10.11.

Second proof of Theorem 10.7. Redefine $Q(R) = \{(r, z, t) : r, |z| < R, -R^2 < t < 0\}$. Let $M = \sup_{Q_2} \Gamma$ and $m = \sup_{Q_2} \Gamma_-$. Assume $M \ge m$. We claim $\Gamma \le (1-\delta)M$ in Q_σ for $\sigma > 0$ and $\delta > 0$ sufficiently small. Suppose to the contrary that $\Gamma(P) > (1-\delta)M$ for some $P = (\bar r, \bar z, \bar t) \in Q_\sigma$. Then $0 < \bar r \le \sigma$. Define Γ^λ and b^λ by (10.54) with $\lambda = \bar r/\sigma$. The same argument after (10.54) gives a contradiction if σ and ε are chosen sufficiently small. Thus

$$\operatorname{osc}_{Q(\sigma)} \Gamma \le (1-\delta)M + m \le \left(1 - \frac{\delta}{2}\right)(M+m) = \left(1 - \frac{\delta}{2}\right) \operatorname{osc}_{Q(2)} \Gamma.$$

We get the same estimate if $M < m$. By rescaling we get $\operatorname{osc}_{Q(P,\sigma R)} \Gamma \le (1 - \frac{\delta}{2}) \operatorname{osc}_{Q(P,2R)} \Gamma$ for all $R < 1$ and $P \in Q_1 \cap z$-axis. This implies Theorem 10.7 by iteration. ☐

The second approach has a cleaner proof since its estimates stay away from the singularity $r = 0$ as much as possible and it utilizes weak limits. The first approach has the advantage that it does not take scaling limit and hence can be used to derive a priori estimates, e.g., (10.19) by Lei, Navas, and Zhang [**126**]), and can be adapted to study *Type II* singularity, e.g., Pan [**162**].

Specifically, Pan [**162**] excludes the existence of axisymmetric *very weak* Type II singularity: An axisymmetric solution is regular up to time T if, for some $\alpha \in [0, 0.028]$, for all $t < T$,

$$|b(r,z,t)| \le \begin{cases} \dfrac{C}{r}\left(\ln\left|\ln\dfrac{r}{3}\right|\right)^{\alpha} & \text{if } r \le 1, \\ \dfrac{C}{r} & \text{if } r > 1. \end{cases} \tag{10.55}$$

10.6. Notes

In Section 10.1, the equations for axisymmetric flows can be found in [**9**, Appendix]. For Lemma 10.1 see [**27**] and [**157**].

In Section 10.2, Theorem 10.5 (with H^2 initial data) was proved by Ukhovskii and Yudovich [**213**] and Ladyženskaja [**120**] independently. Our proof is based on [**130**].

Section 10.3 is based on [**31, 32**]. For Hölder estimates of parabolic equations, see [**38, 135, 152, 153**] for the method of De Giorgi and Moser and [**45, 154**] for the method of Nash.

Section 10.4 is based on [**104**]; see also [**185**]. Step 1 of the proof of Theorem 10.11 is reformulated for easy adaption in Section 10.5.

Section 10.5 contains some observations of mine.

Very recently, exciting new results were derived using the coupled system of $\Omega = \omega_\theta / r$ and $J = \omega_r / r$. It can be written in the form

$$\begin{cases} L_1 \Omega = \dfrac{\partial_z[(v_\theta)^2]}{r^2} = -\dfrac{2v_\theta}{r} J, \\ L_1 J = \operatorname{div}\left\{ v_\theta e_\theta \times \nabla\left(\dfrac{v_r}{r}\right)\right\} \sim \operatorname{div}\left\{ v_\theta e_\theta \Omega \right\}, \end{cases} \tag{10.56}$$

where $L_1 f = \left(\partial_t + b\cdot\nabla - \Delta - \frac{2}{r}\partial_r\right) f$. Recall that we bound b by Ω using (10.11). Because v_θ has the 0-dimensional a priori bound $r v_\theta \le C$, the nonlinear terms on the right sides of (10.56) can be treated as potential terms with the potentials at critical spaces, and we can bound Ω and J with very weak assumptions. The coupled system for Ω and J was first studied by Chen-Fang-Zhang [**33**], who used it to derive various regularity criteria. In Lei-Zhang [**127**], the global regularity was obtained if we assume $|\Gamma| \le C|\ln r|^{-2}$. This was later improved to $|\Gamma| \le C|\ln r|^{-3/2}$ in Wei [**220**].

Helical symmetry. Besides axisymmetry, we may consider the *helical symmetry* under the group $(T_\alpha\phi)(r,\theta,z) = \varphi(r, \theta-\alpha, z - c\alpha)$; see, e.g., [**143**]. Functions and vector fields invariant under the group $\{T_\alpha : \alpha \in \mathbb{R}\}$ do not decay in z and do not belong to $L^2(\mathbb{R}^3)$. To study them in the L^2-setting, instead of $\mathbb{R}^3$, we may consider a domain periodic in z, e.g., $\mathbb{R}^2 \times (0, L)_{\text{per}}$.

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This book is a graduate text on the incompressible Navier-Stokes system, which is of fundamental importance in mathematical fluid mechanics as well as in engineering applications. The goal is to give a rapid exposition on the existence, uniqueness, and regularity of its solutions, with a focus on the regularity problem. To fit into a one-year course for students who have already mastered the basics of PDE theory, many auxiliary results have been described with references but without proofs, and several topics were omitted. Most chapters end with a selection of problems for the reader.

After an introduction and a careful study of weak, strong, and mild solutions, the reader is introduced to partial regularity. The coverage of boundary value problems, self-similar solutions, the uniform L^3 class including the celebrated Escauriaza-Seregin-Šverák Theorem, and axisymmetric flows in later chapters are unique features of this book that are less explored in other texts.

The book can serve as a textbook for a course, as a self-study source for people who already know some PDE theory and wish to learn more about Navier-Stokes equations, or as a reference for some of the important recent developments in the area.

The book is an excellent contribution to the literature concerning the mathematical analysis of the incompressible Navier-Stokes equations. It provides a very good introduction to the subject, covering several important directions, and also presents a number of recent results, with an emphasis on non-perturbative regimes. The book is well written, and both beginners and experts will benefit from it. It can also provide great material for a graduate course.

—Vladimir Šverák, University of Minnesota

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