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A dynamical approach to large eddy simulation of turbulent flows: existence of weak solutions

Agnieszka Świerczewska*,†

Institute of Applied Mathematics and Mechanics, Warsaw University, Banacha 2, 02-097 Warsaw, Poland

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SUMMARY

We consider a system of equations coming from turbulence models using a large eddy simulation (LES) technique. The idea of this approach bases on decomposing the velocity into a part containing large flow structures and a part consisting of small scales. The equations for large-scale quantities are derived from the Navier–Stokes equations with an additional constitutive relation for the contribution of small eddies. The mathematical difficulties in this paper focus on the non-linear and non-local *turbulent term*. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: non-local operators; large eddy simulation; Smagorinsky model; dynamic Germano model; Young measures

1. INTRODUCTION

Turbulent flows occur in many natural and industrial processes. Describing them requires a good simulation. The wide range of scales of flow structures, which are typical for turbulent flows, prevent us from solving numerically the Navier–Stokes equations. Therefore, turbulence models yield equations which can be numerically approximated thanks to reducing the number of operations needed to compute the solutions. One of the approaches recently very popular is large eddy simulation (LES). The LES technique bases on choosing the scales for which the exact solution is computed directly—the part denoting the large flow structures (large scales, resolved) and the scales for which the solution is modelled (small scales, subgrid). Therefore, the quantity describing the flow, the velocity u, is decomposed into the mean part $\bar{u}$ and turbulent fluctuations u', i.e. $u=\bar{u}+u'$. The fluctuations are first smoothed out and then modelled. The actual selection of scales depends mostly on the computational possibilities of

*Correspondence to: Agnieszka Świerczewska, Institute of Applied Mathematics and Mechanics, Warsaw University, Banacha 2, 02-097 Warsaw, Poland.
†E-mail: aswiercz@mimuw.edu.pl

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the hardware and the discretization scheme related to the choice of a computational mesh. Obviously all flow structures of size smaller than the mesh width will not be seen. Mathematically, the choice of a scale is done by filtering, i.e. convoluting the quantities with some appropriate filter function.

Only the large scales are computed as accurately as possible. In view of real-life applications it seems acceptable to describe turbulent flows by this approach. Usually, the behaviour of large eddies is important and more significant than all the small eddies. However, for determining this flow we also have to consider the interaction between the large and small eddies and the one only among the small eddies. All these interactions influence the behaviour of the big eddies.

Different filters based on convolutions can be used. Usually, the convolution is done with respect to space variables, i.e.

$$\bar{u}(t,x) = u * \varphi_\delta(t,x) = \int_{\mathbb{R}^3} u(t,y)\varphi_\delta(x-y)\,\mathrm{d}y$$

where the index δ denotes the filter width (so-called cut-off length) and φ_δ is the filter. The filter is assumed to be a function of total mass one. In case of a bounded domain $\Omega \subset \mathbb{R}^3$ the problem of filtering near the boundary and of the boundary values of $\bar{u}$ occurs. Choosing periodic boundary conditions in the previously considered case (cf. Reference [1]) eliminated this difficulty. To guarantee that the filtering, i.e. the convolution, is well defined in bounded domains, the functions (u, p) could be extended to the whole $\mathbb{R}^3$. The other possibility, which we choose in the present paper, is to consider a filter with a non-constant width $\delta(x)$ with $\delta(x) \to 0$ when x approaches the boundary. The precise description of the filters is contained in Section 1.1.2. Such choice of the filter is also convenient in view of denoting boundary conditions for $\bar{u}$. Note that when $u = 0$ on $\partial\Omega$, consequently also $\bar{u} = 0$ on $\partial\Omega$, which may fail in case of other kind of filters. For more details on filtering see References [2,3].

The equations of evolution of the filtered quantities are derived from the Navier–Stokes equations. By convoluting them with a filter one obtains

$$\bar{u}_t + \operatorname{div}(\overline{u \otimes u}) - \nu\Delta\bar{u} + \nabla\bar{p} = \bar{f}$$

$$\operatorname{div} \bar{u} = 0$$

where u is a velocity, p a pressure, ν a positive constant of viscosity and f an external force. Because of the non-linearity in the equations the scales cannot be considered separately. Furthermore, looking for solutions representing the resolved scales, the interactions with the subgrid scales have to be taken into consideration. Therefore, we express the convoluted convective term as a difference of the convective term in terms of $\bar{u}$ and of a so-called subgrid stress tensor $\tau = \bar{u} \otimes \bar{u} - \overline{u \otimes u}$ representing the contribution of small scales in the system. There has to be added some constitutive relation closing the system. In LES we find a wide range of closure models for the tensor τ. The most classical one which is still often used is the Smagorinsky model where

$$\tau = (c\delta)^2 |D\bar{u}| D\bar{u}$$

and $c>0$ is constant, Du is the symmetric part of the velocity gradient ∇u, i.e. $Du=(D_{ij}u)_{i,j=1}^3$, $D_{ij}u=\frac{1}{2}((\partial u_i/\partial x_j)+(\partial u_j/\partial x_i))$. This leads to the following initial boundary value problem:

$$\begin{aligned}\bar{u}_t+\operatorname{div}(\bar{u}\otimes\bar{u})-\operatorname{div}(c^2\delta^2|D\bar{u}|D\bar{u})-v\Delta\bar{u}+\nabla\bar{p}&=\bar{f}\\ \operatorname{div}\bar{u}&=0\\ \bar{u}(0,x)=\bar{u}_0(x),\quad \bar{u}_{|\partial\Omega}&=0\end{aligned}\tag{1}$$

Existence and uniqueness to (1) have been shown with use of Galerkin approximation and monotone operator methods. For classical results in this field we refer to References [4,5]. The Smagorinsky model with boundary conditions arising from a boundary-layer model has been studied by Parés [6].

The Smagorinsky model has a lot of disadvantages, see Reference [7] for details. In order to adapt it better to local flow structures a dynamical procedure—the Germano model—is applied, cf. Reference [8], later modified by Lilly [9]. Instead of finding one constant c for the whole flow, we want to find this coefficient dynamically. The idea bases on applying a second filter (test filter) to the Navier–Stokes equations. Denoting the width of the first filter (grid filter) by δ_1, the test filter φ_{δ_2} must have a different width δ_2, with $\delta_2>\delta_1$ usually chosen $\delta_2=2\delta_1$. Applying this second filter extracts a test field from the resolved scales. The idea is the following: The smallest resolved scales are sampled to give information for modelling the subgrid scales (notation: $\tilde{u}=u*\varphi_{\delta_2}$). The next step is to use the so-called *Germano identity*:

$$L=T-\tilde{\tau}\tag{2}$$

where τ and T are the subgrid tensors

$$\tau=\bar{u}\otimes\bar{u}-\overline{u\otimes u}\quad\text{and}\quad T=\tilde{\bar{u}}\otimes\tilde{\bar{u}}-\widetilde{\overline{u\otimes u}}\tag{3}$$

and

$$L=\tilde{\bar{u}}\otimes\tilde{\bar{u}}-\widetilde{\bar{u}\otimes\bar{u}}$$

is a *Leonard tensor*. The Germano identity is simply obtained by applying the test filter to the first identity of (3) and subtracting it from the second. The tensor L can be computed from the resolved field since it is associated with scales of motion between the grid and test scales. In the next step both subgrid tensors are modelled in a similar way as in Smagorinsky's model. The crucial simplification is that they will be modelled with the same $c=c(t,x)$, i.e.

$$\tau=2c\delta_1^2|D\bar{u}|D\bar{u}\quad\text{and}\quad T=2c\delta_2^2|D\tilde{\bar{u}}|D\tilde{\bar{u}}\tag{4}$$

Note that in place of c^2 from Smagorinsky's model we now used c. The goal is to allow for the possibility of negative values corresponding to *backscatter*, i.e. the transfer of energy from subgrid scales to large scales. Substituting (4) into (2)

$$L=2c\delta_2^2|D\tilde{\bar{u}}|D\tilde{\bar{u}}-(2c\widetilde{\delta_1^2|D\bar{u}}|D\bar{u})$$

(the filtering $\sim$ applies to the whole term in brackets) and assuming the additional simplification

$$(c\widetilde{\delta_1^2|D\bar{u}}|D\bar{u})=c(\widetilde{\delta_1^2|D\bar{u}}|D\bar{u})$$

(note that: $c = c(t, x)$ is allowed!) the following equation is obtained:

$$L = 2cM \quad \text{with } M = \delta_2^2 |D\tilde{\bar{u}}| D\tilde{\bar{u}} - \delta_1^2 \widetilde{|D\bar{u}| D\bar{u}}$$

The above equation is in fact an overdetermined system of six equations for the coefficient c. Therefore, the error $Q = |L - 2cM|^2$ is minimized by the least squares method, i.e. $\partial Q / \partial c = 0$, yielding

$$c = \frac{1}{2} \frac{L \cdot M}{M \cdot M} \tag{5}$$

here $L \cdot M = \sum_{i,j=1}^{3} l_{ij} m_{ij}$. This c is substituted into the Smagorinsky system (1). Then $v = \bar{u}$ and $q = \bar{p}$ define a solution to the model equations

$$\begin{aligned} v_t + \operatorname{div}(v \otimes v) - \operatorname{div}(c|Dv|Dv) - \nu \Delta v + \nabla q &= \bar{f} \\ \operatorname{div} v &= 0 \\ v(0, x) = v_0(x),\ v_{|\partial\Omega} &= 0 \end{aligned}$$

For more details on modelling we refer to References [2,7–9].

The above procedure can produce negative values of c. This has been conceived as an advantage, allowing to describe the backscatter. Nevertheless, the negative values of c may lead to numerical instabilities. Also numerical tests show that c can vary strongly. In practice, the nominator and denominator of c, cf. (5), are averaged to compute a smoother function (see Reference [2] for details).

We have analysed the behaviour of the function c more precisely. To define c at those points where the denominator becomes zero, it must be possible to estimate somehow the matrix L with help of the matrix M. However, we have found a counterexample, which presents the situation, when $M = 0$ but $L \neq 0$. This was a motivation to some necessary modifications of the turbulent term for the subsequent mathematical analysis. Its properties are assembled in Section 1.1.1. We will not propose any new formula for c, but describe the mathematical assumptions we used in this paper.

1.1. Filtering and properties of the turbulent term

In the following the subset of symmetric matrices in $\mathbb{R}^{n \times n}$ will be denoted by $\mathbb{S}^n$. Let $\mathscr{D}(\Omega)$ be the space of all C^∞-functions with compact support in Ω. By $\mathscr{D}(-\infty, T; \mathscr{V})$ we mean the space of all C^∞-functions with compact support from $(-\infty, T)$ to some function space $\mathscr{V}$. We will also work in spaces of divergence-free functions. Given $\mathscr{V} = \{u : u \in \mathscr{D}(\Omega), \operatorname{div} u = 0\}$, let V denote the closure of $\mathscr{V}$ w.r.t. the norm $\|u\|_V = \left(\int_\Omega |\nabla u|^3 \, \mathrm{d}x\right)^{1/3}$ and let H be the closure of $\mathscr{V}$ w.r.t. the standard L^2-norm. To simplify the notation, function spaces for vector valued functions are denoted in the same way as function spaces for scalar functions. Moreover, we use (throughout the whole paper) Einstein's summation convention, i.e. $a_i b_i := \sum_{i=1}^{3} a_i b_i$ and k for a generic constant.

1.1.1. Properties of the turbulent term. By the *turbulent term* we mean the operator

$$c(y)|Dv|Dv$$

with the notation for non-local (filtered) variables

$$y = (\tilde{v}, \widetilde{vv}, \widetilde{Dv}, \widetilde{|Dv|Dv})$$

The properties of the operator c are the following:

(C1) $c : \mathbb{R}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \longrightarrow \mathbb{R}$ is a continuous function w.r.t. y.
(C2) c satisfies the condition

$$0 < \alpha \leqslant c(y) \leqslant \beta < \infty \tag{6}$$

For later use we assemble also the properties of the operator $\eta \mapsto |\eta|\eta$ for $\eta \in \mathbb{S}^3$. There exists a scalar function $U \in C^2(\mathbb{S}^3)$, $U(\eta) = \frac{1}{3}|\eta|^3$ such that for all $\eta, \xi \in \mathbb{S}^3$ and $i, j = 1, 2, 3$

$$\frac{\partial U(\eta)}{\partial \eta_{ij}} = |\eta|\eta_{ij} \tag{7}$$

$$\frac{\partial^2 U(\eta)}{\partial \eta_{mn} \partial \eta_{rs}} \xi_{mn} \xi_{rs} \geqslant |\eta||\xi|^2 \tag{8}$$

Moreover, $|\eta|\eta$ is *strongly monotone*, i.e. there exists a positive constant K_1 such that

$$(|\eta|\eta_{ij} - |\xi|\xi_{ij}) \cdot (\eta_{ij} - \xi_{ij}) \geqslant K_1 |\eta - \xi|^3 \tag{9}$$

for all $\eta, \xi \in \mathbb{S}^3$. Obviously, the strong monotonicity implies the *strict monotonicity*, i.e.

$$(|\eta|\eta_{ij} - |\xi|\xi_{ij}) \cdot (\eta_{ij} - \xi_{ij}) > 0 \tag{10}$$

for all $\eta, \xi \in \mathbb{S}^3$, $\eta \neq \xi$.

1.1.2. Filtering technique. In bounded domains the definition of the filtering is rather delicate. Filters are non-negative C^∞-functions of compact support contained in Ω. The support shrinks to a one-point set near the boundary. Nevertheless, the mass of the filter remains one; thus the filters tend to Dirac δ-distributions on the boundary. To be more precise, let $\varphi \in C_0^\infty(\Omega)$ with $\operatorname{supp} \varphi \subset B_1$ be non-negative such that $\int_\Omega \varphi(y)\,\mathrm{d}y = 1$, $\varphi(x) = \varphi(-x)$. Let $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. Then we define the filter $\varphi_{\delta(x)}$ by

$$\varphi_{\delta(x)}(y) = \frac{1}{\delta(x)^3} \varphi\left(\frac{y}{\delta(x)}\right) \tag{11}$$

For a description of the application of filters with non-uniform filter width in numerical analysis we refer to Reference [10].

In LES for time-dependent equations the filtering is usually done only w.r.t. space variables. Nevertheless, the general definition of the filter (cf. Reference [2, p. 9]) admits also space–time filtering. In that case, also the problem of filtering near the initial value occurs. We will solve it in a similar way to the filtering near the boundary. However, to find the solution in time τ, we only want to consider times $0 \leqslant t \leqslant \tau$. Therefore, let $\varphi^t \in L^\infty((0,T))$ be a non-negative function with $\int_0^T \varphi^t(\tau)\,\mathrm{d}\tau = 1$. Moreover, let $\varphi^t(\tau)$ have compact support in $[0,1)$. The time- and space-dependent filter $\varphi_{\delta(t,x)}$ is defined by

$$\varphi_{\delta(t,x)}(\tau,y) = \varphi^t_{\delta(t)}(\tau)\varphi^x_{\delta(x)}(y), \quad \varphi^t_{\delta(t)}(\tau) = \frac{1}{\delta(t)}\varphi^t\left(\frac{\tau}{\delta(t)}\right), \quad \delta(t) = t$$

and $\varphi^x_{\delta(x)}$ corresponds to $\varphi_{\delta(x)}$ defined by (11). Given the space–time cylinder $Q_T = (0,T)\times\Omega$ we understand by filtering the process

$$\tilde{v}(t,x) = \int_{Q_T} v(\tau,y)\varphi_{\delta(t,x)}(t-\tau, x-y)\,\mathrm{d}\tau\,\mathrm{d}y$$

Remark
On the level of modelling, the commutation of convolution and differentiation is assumed. This property obviously holds for filters with constant width. For the case of non-uniform filters used here this may fail. For a wider study of the so-called *commutation error* we refer to References [11–13]. In the following the commutation error will be neglected.

1.2. Main results

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a sufficiently smooth boundary $\partial\Omega$. We are looking for a velocity $v : Q_T \longrightarrow \mathbb{R}^3$ and a pressure $q : Q_T \longrightarrow \mathbb{R}$ solving in Ω the system

$$\begin{aligned} v_t + v\cdot\nabla v - \operatorname{div}\,[c(y)|Dv|Dv] - \nu\Delta v + \nabla q &= f \\ \operatorname{div}\, v &= 0 \\ v(0,x) &= v_0(x) \end{aligned} \tag{12}$$

with boundary conditions

$$v(t,x) = 0 \quad \text{on } (0,T)\times\partial\Omega \tag{13}$$

As before, $y = (\tilde{v}, \widetilde{vv}, \widetilde{Dv}, \widetilde{|Dv|Dv})$.

Definition 1.1
Given $f \in L^{3/2}(0,T;V')$ and $v_0 \in H$ a function

$$v \in L^3(0,T;V) \cap L^\infty(0,T;H)$$

is a weak solution to problem (12), (13) if the equation

$$\begin{aligned} &\int_\Omega \int_0^T (-v\phi_t + v\cdot\nabla v\,\phi + c(y)|Dv|Dv\cdot D\phi + \nu\nabla v\cdot\nabla\phi)\,\mathrm{d}t\,\mathrm{d}x \\ &\quad = \int_\Omega v_0\phi\,\mathrm{d}x + \int_0^T \langle f,\phi\rangle\,\mathrm{d}t \end{aligned}$$

is satisfied for all $\phi \in \mathscr{D}(-\infty,T;\mathscr{V})$.

Theorem 1.1 (*Existence*)
Let $v_0 \in H$, $f \in L^{3/2}(0,T;V')$ and let the function c satisfy conditions (C1)–(C2). Then, for all $T>0$, there exists a weak solution in the sense of Definition 1.1 to problem (12), (13).

Moreover, we will show, that in the Galerkin approximation scheme below the sequence of approximate solutions converges strongly in $L^3(0,T;V)$. This result will be formulated in Theorem 3.1.

2. PROOF OF THEOREM 1.1

Let $y^n = (\widetilde{v^n}, \widetilde{v^n v^n}, \widetilde{Dv^n}, |\widetilde{Dv^n|Dv^n})$ and let $\{\omega_r\}_{r=1}^{\infty}$ be an orthonormal basis of H consisting of eigenvectors of the Stokes operator. Let $V^n = \mathrm{span}\{\omega_1, \ldots, \omega_n\}$. On H define the projection

$$P^n : H \to V^n, \quad P^n u = \sum_{r=1}^{n} (u, \omega_r)\omega_r$$

Note that there exists $k = k(\Omega) > 0$ such that (cf. References [14,15])

$$\|P^n u\|_{W^{2,2}(\Omega)} \leqslant k\|u\|_{W^{2,2}(\Omega)}$$

We define $v^n(t) = \sum_{r=1}^{n} \lambda_r^n(t)\omega_r$, $v^n \in V^n$ as the solution to

$$\begin{aligned} \left(\frac{\mathrm{d}}{\mathrm{d}t} v^n, \omega_r\right) + \langle c(y^n)|Dv^n|Dv^n, D\omega_r\rangle + \nu(\nabla v^n, \nabla\omega_r) + b(v^n, v^n, \omega_r) &= \langle f, \omega_r\rangle \\ v^n(0) &= P^n v_0 \end{aligned} \tag{14}$$

for all $1 \leqslant r \leqslant n$. Here b denotes the trilinear form

$$b(u, v, w) := \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i \,\mathrm{d}x$$

Note that for divergence-free functions $b(u, v, v) = 0$.

Before establishing existence of solutions to the approximated problem let us prove some *a priori* estimates. Multiplying equations (14) by λ_r^n and summing over r we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v^n\|_H^2 + \int_\Omega c(y^n)|Dv^n|^3 \,\mathrm{d}x + \nu\|\nabla v^n\|_{L^2(\Omega)}^2 = \langle f, v^n\rangle$$

Estimating the l.h.s. with the help of Korn's inequality (cf. Reference [16]) and (6) yields

$$\int_\Omega c(y^n)|Dv^n|^3 \,\mathrm{d}x \geqslant \alpha \int_\Omega |Dv^n|^3 \,\mathrm{d}x \geqslant k_\alpha \|v^n\|_{W^{1,3}(\Omega)}^3 \geqslant k_\alpha \|v^n\|_V^3$$

We estimate the r.h.s. with Young's inequality

$$|\langle f, v^n\rangle| \leqslant \|f\|_{V'}\|v^n\|_V \leqslant \frac{k_\alpha}{2}\|v^n\|_V^3 + \frac{k}{2}\|f\|_{V'}^{3/2}$$

to obtain after integrating over $(0,s), s \in (0,T)$

$$\|v^n(s)\|_H^2 + k_\alpha \int_0^s \|v^n\|_V^3 \,\mathrm{d}t + \nu \int_0^s \|\nabla v^n\|_{L^2(\Omega)}^2 \,\mathrm{d}t \leqslant k \int_0^T \|f\|_{V'}^{3/2} \,\mathrm{d}t + \|v_0^n\|_H^2 \tag{15}$$

This allows to conclude that

$$v^n \quad \text{is bounded in} \quad L^\infty(0,T;H) \cap L^3(0,T;V)$$

Let us now analyse v_t^n. Due to Equation (14) we obtain after estimating all the other terms of the equation that

$$v_t^n \text{ is bounded in } L^{3/2}(0,T;(W^{2,2}(\Omega)\cap V)')$$

For its proof take an arbitrary $\phi \in L^3(0,T;W^{2,2}(\Omega)\cap V)$ with $\|\phi\|_{L^3(0,T;W^{2,2}(\Omega)\cap V)} \leqslant 1$ and estimate (v_t^n, ϕ). Note that $(v_t^n,\phi) = (v_t^n, P^n\phi)$. Hence, due to Equation (14), the four integrals below are finite. First,

$$\begin{aligned}
\int_0^T \int_\Omega |v^n \cdot \nabla v^n\, P^n\phi| \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_\Omega |v^n \otimes v^n \cdot \nabla P^n\phi| \,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant \int_0^T \|v^n\|_{L^4(\Omega)}^2 \|\nabla P^n\phi\|_{L^2(\Omega)} \,\mathrm{d}t \leqslant k \int_0^T \|v^n\|_V^2 \|\nabla P^n\phi\|_{W^{1,2}(\Omega)} \,\mathrm{d}t \\
&\leqslant k \int_0^T \|v^n\|_V^2 \|P^n\phi\|_{W^{2,2}(\Omega)} \,\mathrm{d}t \leqslant k \int_0^T \|v^n\|_V^2 \|\phi\|_{W^{2,2}(\Omega)} \,\mathrm{d}t \\
&\leqslant \|v^n\|_{L^3(0,T;V)}^2 \|\phi\|_{L^3(0,T;W^{2,2}(\Omega))} \leqslant k
\end{aligned}$$

and

$$\begin{aligned}
\int_0^T \int_\Omega |\nabla v^n \cdot \nabla P^n\phi| \,\mathrm{d}x\,\mathrm{d}t &\leqslant \int_0^T \|\nabla v^n\|_{L^3(\Omega)} \|\nabla P^n\phi\|_{L^{3/2}(\Omega)} \,\mathrm{d}t \\
&\leqslant k \int_0^T \|\nabla v^n\|_{L^3(\Omega)} \|P^n\phi\|_{W^{2,2}(\Omega)} \,\mathrm{d}t \leqslant k \int_0^T \|\nabla v^n\|_{L^3(\Omega)} \|\phi\|_{W^{2,2}(\Omega)} \,\mathrm{d}t \\
&\leqslant k\|v^n\|_{L^{3/2}(0,T;V)} \|\phi\|_{L^3(0,T;W^{2,2}(\Omega))} \leqslant k\|v^n\|_{L^3(0,T;V)} \leqslant k
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_0^T |\langle f, P^n\phi\rangle| \,\mathrm{d}t &\leqslant \int_0^T \|f\|_{V'} \|P^n\phi\|_V \,\mathrm{d}t \leqslant k \int_0^T \|f\|_{V'} \|P^n\phi\|_{W^{2,2}(\Omega)} \,\mathrm{d}t \\
&\leqslant k \int_0^T \|f\|_{V'} \|\phi\|_{W^{2,2}(\Omega)} \,\mathrm{d}t \leqslant k\|f\|_{L^{3/2}(0,T;V')} \|\phi\|_{L^3(0,T;W^{2,2}(\Omega))} \leqslant k
\end{aligned}$$

and, finally

$$
\begin{aligned}
&\int_0^T \int_\Omega |c(y^n)|Dv^n|Dv^n \cdot \nabla P^n \phi| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant \beta \int_0^T \int_\Omega |Dv^n|^2 |\nabla P^n \phi| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant k \int_0^T \int_\Omega |\nabla v^n|^2 |\nabla P^n \phi| \,\mathrm{d}x\,\mathrm{d}t \leqslant k \int_0^T \|\nabla v^n\|^2_{L^3(\Omega)} \|\nabla P^n \phi\|_{L^3(\Omega)} \,\mathrm{d}t \\
&\quad \leqslant k \|v^n\|^2_{L^3(0,T;V)} \|\phi\|_{L^3(0,T;W^{2,2}(\Omega))} \leqslant \|v^n\|^2_{L^3(0,T;V)} \leqslant k
\end{aligned}
$$

Theorem 2.1

For given $f \in L^{3/2}(0,T;V')$ and $v_0 \in H$ Equation (14) possesses an absolutely continuous solution v^n on $(0,T)$.

Proof

Let $\lambda^n = (\lambda_1^n, \ldots, \lambda_r^n)$ and let n be fixed. We can rewrite the system (14) in the form

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\lambda_r^n(t) &= F_r(t, \lambda^n(t), y^n) \\
\lambda_r^n(0) &= (u_0, \omega_r)
\end{aligned} \tag{16}
$$

where $1 \leqslant r \leqslant n$, $F(\cdot) = (F_1(\cdot), \ldots, F_n(\cdot))$ and

$$
\begin{aligned}
F_r(t, \lambda^n(t), y^n) = (f, \omega_r) - \lambda_i^n(t)\lambda_k^n(t) \int_\Omega \omega_i^j \frac{\partial \omega_k^l}{\partial x_j} \omega_r^l \,\mathrm{d}x - v\lambda_r^n(t) \|\nabla \omega_r\|^2_{L^2} \\
-\lambda_i^n(t) \int_\Omega c(y^n) |\lambda_k^n(t) D\omega_k|\, D_{lm}\omega_i\, D_{lm}\omega_r \,\mathrm{d}x
\end{aligned}
$$

with

$$
y^n = \left(\widetilde{\sum_{i=1}^n \lambda_i^n \omega_i},\ \widetilde{\sum_{i=1}^n \lambda_i^n \omega_i \sum_{j=1}^n \lambda_j^n \omega_j},\ \widetilde{\sum_{i=1}^n \lambda_i^n D\omega_i},\ \widetilde{\left|\sum_{i=1}^n \lambda_i^n D\omega_i\right| \sum_{j=1}^n \lambda_j^n D\omega_j} \right)
$$

Remembering that $\delta(t) = t$, let us rewrite all filtered terms by changing the variables in the time-filtering, i.e.

$$
\begin{aligned}
\widetilde{\lambda_i^n \omega_i}(t,x) &= \int_0^1 \varphi^t(s) \lambda_i^n(t - ts) \,\mathrm{d}s \int_\Omega \varphi^x_{\delta(x)}(x - y)\omega_i(y) \,\mathrm{d}y \\
\widetilde{\lambda_i^n \omega_i \lambda_j^n \omega_j}(t,x) &= \int_0^1 \varphi^t(s) \lambda_i^n(t - ts) \lambda_j^n(t - ts) \,\mathrm{d}s \int_\Omega \varphi^x_{\delta(x)}(x - y)\omega_i(y)\omega_j(y) \,\mathrm{d}y \\
\widetilde{\lambda_i^n D\omega_i}(t,x) &= \int_0^1 \varphi^t(s) \lambda_i^n(t - ts) \,\mathrm{d}s \int_\Omega \varphi^x_{\delta(x)}(x - y) D\omega_i(y) \,\mathrm{d}y \\
\widetilde{|\lambda_i^n D\omega_i| \lambda_j^n D\omega_j}(t,x) &= \int_0^1 \int_\Omega \varphi^t(s) \varphi^x_{\delta(x)}(x - y) |\lambda_i^n(t - ts) D\omega_i(y)| \lambda_j^n(t - ts) D\omega_j(y) \,\mathrm{d}y\,\mathrm{d}s
\end{aligned}
$$

To find the value of λ^n at time $t = t_1$ we need the information on the values of λ^n in all $0 \leqslant t \leqslant t_1$. Let $\lambda_t \in C([0,1]; \mathbb{R}^n)$ be defined by $\lambda_t(s) = \lambda(t(1-s)), 0 \leqslant s \leqslant 1$. Taking into account all filtered terms it will be more convenient to specify the dependence of F on λ^n as

$$F(t, \lambda^n(t), y^n) =: \mathscr{F}(t, \lambda^n(t), \lambda_t^n)$$

Therefore, let us describe the dependence on filtered terms with help of some function $\mathscr{C}$, namely $\mathscr{C}(\lambda_t^n) = c(y^n)$ to get

$$\mathscr{F}_r(t, \lambda^n(t), \lambda_t^n) = (f, \omega_r) - \lambda_i^n(t)\lambda_k^n(t) \int_\Omega \omega_i^j \frac{\partial \omega_k^l}{\partial x_j} \omega_r^l \, \mathrm{d}x - v\lambda_r^n(t) \|\nabla \omega_r\|_{L^2}^2$$

$$-\lambda_i^n(t) \int_\Omega \mathscr{C}(\lambda_t^n) |\lambda_k^n(t) D\omega_k| \, D_{lm}\omega_i \, D_{lm}\omega_r \, \mathrm{d}x$$

First step: *Local existence*
Let there be given $t_0 \in [0, T)$ and define the constant

$$a = \min\left\{\frac{1}{(2K_1+1)^3}, \frac{1}{2(K_2+K_3+K_4)}\right\}$$

where the constants K_i will be explained in estimates (17)–(20). Note that the K_i's depend on n and on the initial data $\lambda^n(t_0)$ and are independent of t. We replace (16) by the integral equation

$$\lambda^n(t) = \lambda^n(t_0) + \int_{t_0}^t \mathscr{F}(\tau, \lambda^n(\tau), \lambda_\tau^n) \, \mathrm{d}\tau$$

and define the operator S by

$$S(\lambda^n) = \lambda^n(t_0) + \int_{t_0}^t \mathscr{F}(\tau, \lambda^n(\tau), \lambda_\tau^n) \, \mathrm{d}\tau$$

Then (16) is equivalent to the fixed point problem

$$\lambda^n = S(\lambda^n), \quad \lambda^n \in B \subseteq X$$

where

$$X = C([t_0, t_0 + a]), \quad \|\lambda\|_X = \max_{t \in [t_0, t_0+a]} |\lambda(t)|$$

$$B = \{\lambda^n \in X : \|\lambda^n - \lambda^n(t_0)\|_X \leqslant 1\}$$

Assume that $|\lambda^n(t) - \lambda^n(t_0)| \leqslant 1$ for all $t \in [t_0, t_0 + a]$, where for $t_0 = 0$ we defined $\lambda^n(t_0)$ in (16). Observe for each $1 \leqslant r \leqslant n$ the following estimates:

$$\int_{t_0}^{t_0+a} |(f, \omega_r)| \, \mathrm{d}\tau \leqslant \left(\int_{t_0}^{t_0+a} |(f, \omega_r)|^{3/2} \, \mathrm{d}\tau\right)^{2/3} \left(\int_{t_0}^{t_0+a} 1 \, \mathrm{d}\tau\right)^{1/3}$$

$$\leqslant \|f\|_{L^{3/2}(0,T;V')} \|\omega_r\|_V a^{1/3} = K_1 a^{1/3} \tag{17}$$

and

$$\int_{t_0}^{t_0+a} \left| \lambda_i^n(\tau)\lambda_k^n(\tau) \int_\Omega \omega_i^j \frac{\partial \omega_k^l}{\partial x_j} \omega_r^l \, \mathrm{d}x \right| \mathrm{d}\tau \leqslant k \max_{1\leqslant i\leqslant n} \|\nabla\omega_i\|_{L^2}^3 \int_{t_0}^{t_0+a} |\lambda^n|^2 \, \mathrm{d}\tau$$

$$\leqslant k(|\lambda^n(t_0)|+1)^2 \max_{1\leqslant i\leqslant n} \|\nabla\omega_i\|_{L^2}^3 a = K_2 a \tag{18}$$

and

$$v \int_{t_0}^{t_0+a} \left|\lambda^n(\tau)\|\nabla\omega_r\|_{L^2}^2\right| \, \mathrm{d}\tau \leqslant v(|\lambda^n(t_0)|+1)\|\nabla\omega_r\|_{L^2}^2 \int_{t_0}^{t_0+a} 1 \, \mathrm{d}\tau = K_3 a \tag{19}$$

Moreover, since $\mathscr{C}$ is bounded from above by β,

$$\int_{t_0}^{t_0+a} \left| \lambda_i^n(\tau) \int_\Omega \mathscr{C}(\lambda_\tau^n)|\lambda_k^n(\tau) D\omega_k| \, D_{lm}\omega_i \, D_{lm}\omega_r \, \mathrm{d}x \right| \, \mathrm{d}\tau$$

$$\leqslant \beta \int_{t_0}^{t_0+a} |\lambda^n(\tau)|^2 \int_\Omega |D\omega_r|^3 \, \mathrm{d}x \, \mathrm{d}\tau \leqslant \beta(|\lambda^n(t_0)|+1)^2 \|D\omega_r\|_{L^3}^3 \int_{t_0}^{t_0+a} 1 \, \mathrm{d}\tau$$

$$\leqslant \beta \, (|\lambda^n(t_0)|+1)^2 \|D\omega_r\|_{L^3}^3 a = K_4 a \tag{20}$$

Thus, we can conclude that

$$\int_{t_0}^{t_0+a} |\mathscr{F}(\tau, \lambda^n(\tau), \lambda_\tau^n)| \, \mathrm{d}\tau \leqslant 1 \tag{21}$$

First, see that $S(\lambda^n) \subseteq B$ for $\lambda^n \in B$, namely

$$|S(\lambda^n) - \lambda^n(t_0)| \leqslant \int_{t_0}^{t_0+a} |\mathscr{F}(\tau, \lambda^n(\tau), \lambda_\tau^n)| \, \mathrm{d}\tau \overset{(21)}{\leqslant} 1$$

To prove compactness of the operator S, note that $S(\lambda^n)$ is uniformly bounded on B, since for all $t \in [t_0, t_0+a]$ and $\lambda^n \in B$

$$|S(\lambda^n(t))| \leqslant |\lambda^n(t_0)| + \int_{t_0}^{t} |\mathscr{F}(\tau, \lambda^n(\tau), \lambda_\tau^n)| \, \mathrm{d}\tau \leqslant |\lambda^n(t_0)| + 1$$

Moreover, $S(B)$ is equicontinuous; namely, with a slight generalization of (17)–(20), for all

$$t_1, t_2 \in [t_0, t_0+a] \text{ and } \lambda^n \in B \text{ if } |t_1 - t_2| \leqslant \min\left\{ \left(\frac{\varepsilon}{2K_1+1}\right)^3, \frac{\varepsilon}{2(K_2+K_3+K_4)} \right\}$$

we get that

$$|S(\lambda^n(t_1)) - S(\lambda^n(t_2))| = \left| \int_{t_0}^{t_1} \mathscr{F}(\tau, \lambda^n(\tau), \lambda_\tau^n) \, \mathrm{d}\tau - \int_{t_0}^{t_2} \mathscr{F}(\tau, \lambda^n(\tau), \lambda_\tau^n) \, \mathrm{d}\tau \right|$$

$$\leqslant \left| \int_{t_1}^{t_2} \mathscr{F}(\tau, \lambda^n(\tau), \lambda_\tau^n) \, \mathrm{d}\tau \right| \leqslant K_1|t_1-t_2|^{1/3} + (K_2+K_3+K_4)|t_1-t_2| \leqslant \varepsilon \tag{22}$$

Hence by the Ascoli-Arzelà theorem the set $S(B)$ is relatively compact in X. To conclude the compactness of the operator S we only have to note that S is continuous. Therefore, let $\lambda_j^n \to \lambda^n$ uniformly in $[t_0, t_0+a]$ as $j \to \infty$. Note, since $\mathscr{C}$ is a continuous function of λ_t^n, that $\mathscr{F}$ is also continuous w.r.t. λ^n and λ_t^n. Hence we can conclude with help of the dominated convergence theorem that

$$S(\lambda_j^n(t)) - S(\lambda^n(t)) = \int_{t_0}^{t} [\mathscr{F}(\tau, \lambda_j^n(\tau), \lambda_{\tau,j}^n) - \mathscr{F}(\tau, \lambda^n(\tau), \lambda_\tau^n)]\,\mathrm{d}\tau$$

converges pointwise to 0. Moreover, (22) provides the uniform convergence; thus, S is continuous. Finally, as B is a non-empty, closed, bounded and convex subset of X and the operator S is compact, due to the Schauder fixed point theorem there exists a solution to the equation $\lambda^n = S(\lambda^n)$ for $t \in [t_0, t_0+a]$.

Second step: *Global existence of solutions*
To obtain the global existence of solutions we will repeat the above procedure in further time intervals. Note that the construction of solutions in the interval (t_0, t_0+a) for $t_0 \neq 0$ uses also the values of λ^n from the interval $(0, t_0)$. These quantities do not influence estimates (17)–(19). They only appear in estimate (20) as arguments of the function $\mathscr{C}$. But since $\mathscr{C}$ is uniformly bounded by β, the proof follows the same lines.

Due to the orthonormality of $\{\omega_r\}$ in H the *a priori* estimates, cf. (15), assure that $\lambda^n(t)$ is uniformly bounded. Thus, also the initial data for further existence problems are bounded implying that the value of the constants K_i will not increase; consequently, the length of existence intervals a will not decrease. Hence the proof can be done in a finite number of steps.

Equation (16) implies that for $t \in (0, T)$ the solution is absolutely continuous. □

Using the information on the boundedness of the sequence (v^n) we can extract a subsequence, still denoted by v^n, such that

$$v^n \rightharpoonup v \quad \text{in } L^3(0,T;V) \tag{23}$$

$$v^n \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(0,T;H) \tag{24}$$

$$v_t^n \rightharpoonup v_t \quad \text{in } L^{3/2}(0,T;(W^{2,2}(\Omega)\cap V)') \tag{25}$$

Since $V \subset\subset H \subset (W^{2,2}(\Omega)\cap V)'$, due to (23) and (25), using the Aubin–Lions Lemma (cf. Reference [15]) we conclude that

$$v^n \longrightarrow v \quad \text{in } L^3(0,T;H) \quad \text{and} \quad \text{a.e. in } Q_T \tag{26}$$

This strong convergence is needed to show that

$$\int_0^T b(v^n, v^n, \phi)\,\mathrm{d}t \longrightarrow \int_0^T b(v, v, \phi)\,\mathrm{d}t$$

It is obtained as follows:

$$\int_0^T \int_\Omega (v^n \nabla v^n - v\nabla v)\phi\,\mathrm{d}x\,\mathrm{d}t$$

$$= \int_0^T \int_\Omega (v^n - v)\nabla v^n \phi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega v(\nabla v^n - \nabla v)\phi\,\mathrm{d}x\,\mathrm{d}t$$

According to Hölder's inequality the first integral can be estimated by

$$\left|\int_0^T \int_\Omega (v^n - v)\nabla v^n \phi \,\mathrm{d}x\,\mathrm{d}t\right| \leqslant \int_0^T \|v^n - v\|_{L^2(\Omega)} \|\nabla v^n\|_{L^3(\Omega)} \|\phi\|_{L^6(\Omega)}\,\mathrm{d}t$$
$$\leqslant \|v^n - v\|_{L^3(0,T;H)} \|v^n\|_{L^3(0,T;V)} \|\phi\|_{L^3(0,T;L^6(\Omega))}$$

And due to the strong convergence (26) this integral converges to zero. The convergence of the second integral to zero is achieved by the weak convergence of gradients. Finally, due to (23), there exist $\bar{A}, \chi \in L^{3/2}(Q_T)$ such that

$$c(y^n)|Dv^n|Dv^n \rightharpoonup \bar{A} \quad \text{in } L^{3/2}(Q_T) \tag{27}$$

and

$$|Dv^n|Dv^n \rightharpoonup \chi \quad \text{in } L^{3/2}(Q_T) \tag{28}$$

Hence, we can state the limit identity

$$\int_0^T \int_\Omega (v_t \cdot \phi + v \cdot \nabla v \cdot \phi + \bar{A} \cdot D\phi + v\nabla v \cdot \nabla\phi)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle f, \phi\rangle\,\mathrm{d}t \tag{29}$$

for all $\phi \in \mathscr{D}(-\infty, T; \mathscr{V})$.

For later use we will show that the strong energy equality, see (38) below, holds. To this aim we need to show that (29) holds for all $\phi \in L^3(0,T;V)$. We observe the following estimates:

$$\int_0^T \int_\Omega |v \cdot \nabla v \cdot \phi|\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_0^T \|v\|_{L^3(\Omega)} \|\nabla v\|_{L^3(\Omega)} \|\phi\|_{L^3(\Omega)}\,\mathrm{d}t \leqslant k \int_0^T \|v\|_V^2 \|\phi\|_V\,\mathrm{d}t$$
$$\leqslant k\|v\|^2_{L^3(0,T;V)} \|\phi\|_{L^3(0,T;V)} \tag{30}$$

and

$$\int_0^T \int_\Omega |\bar{A} \cdot D\phi|\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_0^T \|\bar{A}\|_{L^{3/2}(\Omega)} \|D\phi\|_{L^3(\Omega)}\,\mathrm{d}t \leqslant k\|\bar{A}\|_{L^{3/2}(Q_T)} \|\phi\|_{L^3(0,T;V)} \tag{31}$$

Moreover,

$$\int_0^T \int_\Omega |\nabla v \cdot \nabla\phi|\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_0^T \|\nabla v\|_{L^{3/2}(\Omega)} \|\nabla\phi\|_{L^3(\Omega)}\,\mathrm{d}t \leqslant k \int_0^T \|v\|_V \|\phi\|_V\,\mathrm{d}t$$
$$\leqslant k\|\nabla v\|_{L^3(0,T;V)} \|\phi\|_{L^3(0,T;V)} \tag{32}$$

and

$$\int_0^T |\langle f, \phi\rangle|\,\mathrm{d}t \leqslant \int_0^T \|f\|_{V'} \|\phi\|_V\,\mathrm{d}t \leqslant \|f\|_{L^{3/2}(0,T;V')} \|\phi\|_{L^3(0,T;V)} \tag{33}$$

Collecting (30)–(33) allows to conclude that

$$\mathscr{F}(\phi) \equiv \int_0^T \left(b(v,v,\phi) + \int_\Omega \bar{A} \cdot D\phi \, \mathrm{d}x + \nu(\nabla v, \nabla \phi) - \langle f, \phi \rangle \right) \mathrm{d}t \tag{34}$$

is a linear bounded functional on $L^3(0,T;V)$. From (29) it holds

$$\mathscr{F}(\phi) = \int_0^T \int_\Omega v_t \phi \, \mathrm{d}x \, \mathrm{d}t \tag{35}$$

Thus, v_t belongs to $L^{3/2}(0,T;V') = (L^3(0,T;V))'$, which provides that (29) holds for all $\phi \in L^3(0,T;V)$. This allows to test (29) against the solution v to obtain

$$\int_0^T \int_\Omega (v_t \cdot v + \bar{A} \cdot Dv + \nu \nabla v \cdot \nabla v) \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \langle f, v \rangle \, \mathrm{d}t \tag{36}$$

Finally, due to Proposition A.9, since $v \in L^3(0,T;V)$ and $v_t \in L^{3/2}(0,T;V')$ then for all $0 \leqslant s \leqslant t \leqslant T$ it holds

$$\int_s^t \langle v_t(\tau), v(\tau) \rangle \, \mathrm{d}\tau = \frac{1}{2} \|v(t)\|_H^2 - \frac{1}{2} \|v(s)\|_H^2 \tag{37}$$

and hence

$$\frac{1}{2} \|v(t)\|_H^2 + \int_s^t \int_\Omega \bar{A} \cdot Dv \, \mathrm{d}x \, \mathrm{d}\tau + \nu \int_s^t \|\nabla v\|_H^2 \, \mathrm{d}\tau = \frac{1}{2} \|v(s)\|_H^2 + \int_s^t \langle f, v \rangle \, \mathrm{d}\tau \tag{38}$$

Next, we will formulate a lemma concerning convergence of filtered terms.

Lemma 2.2
Let the sequence $(v^n)_{n \in \mathbb{N}}$ converge weakly to v in $L^3(0,T;V)$ and let $\chi \in L^{3/2}(Q_T)$ be as in (28). Then, for $n \to \infty$, the following sequences converge almost everywhere in Q_T:

$$\widetilde{v^n} \longrightarrow \widetilde{v}$$

$$\widetilde{v^n v^n} \longrightarrow \widetilde{vv}$$

$$\widetilde{Dv^n} \longrightarrow \widetilde{Dv}$$

We can extract a further subsequence of (v^n) such that

$$\widetilde{|Dv^n| Dv^n} \longrightarrow \widetilde{\chi} \quad \text{a.e. in } Q_T$$

Proof
Since v^n is bounded in $L^3(0,T;V)$, then also, for a subsequence, $Dv^n \rightharpoonup Dv$ in $L^3(Q_T)$, and $v^n \rightharpoonup v$ in $L^3(Q_T)$; hence

$$\int_{Q_T} v^n \phi \, \mathrm{d}y \, \mathrm{d}\tau \to \int_{Q_T} v\phi \, \mathrm{d}y \, \mathrm{d}\tau \quad \forall \phi \in L^{3/2}(Q_T)$$

We choose as a test function $\phi(\tau, y) = \varphi_{\delta(t,x)}(t-\tau, x-y)$ with parameters $(t,x) \in Q_T$, where $\varphi_{\delta(t,x)}$ is a filter. The filters are obviously in $L^{3/2}(Q_T)$ except for the points $x \in \partial\Omega$ or $t=0$. However, since Q_T is open,

$$\int_{Q_T} v^n(\tau, y)\varphi(t-\tau, x-y)\,\mathrm{d}y\,\mathrm{d}\tau \to \int_{Q_T} v(\tau, y)\varphi(t-\tau, x-y)\,\mathrm{d}y\,\mathrm{d}\tau \quad \text{for a.a.}\,(t,x) \in Q_T$$

which is equivalent to

$$\widetilde{v^n} \to \tilde{v} \quad \text{a.e. in } Q_T \tag{39}$$

In the same way from the information on the symmetric part of the gradients we conclude that

$$\widetilde{Dv^n} \to \widetilde{Dv} \quad \text{a.e. in } Q_T \tag{40}$$

To analyse the limit of the sequence $\widetilde{v^n v^n}$ we deduce from the strong convergence of the sequence v^n in $L^2(Q_T)$ also the strong convergence of $v^n v^n$ to vv in $L^1(Q_T)$. Of course the strong convergence implies the weak convergence. Thus, following analogous arguments as above, we get that

$$\widetilde{v^n v^n} \to \widetilde{vv} \quad \text{a.e. in } Q_T \tag{41}$$

Convergence (28) implies for the filtered terms

$$\widetilde{|Dv^n|Dv^n} \to \tilde{\chi} \quad \text{a.e. in } Q_T$$

which completes the proof of Lemma 2.2. □

For the passage to the limit in the turbulent term we apply Lemma A.1 to the operator

$$A(y,z) = c(y)|z|z : (\mathbb{R}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3) \times \mathbb{S}^3 \to \mathbb{S}^3$$

Again let

$$y = (\tilde{v}, \widetilde{vv}, \widetilde{Dv}, \widetilde{|Dv|Dv}), \quad y^n = (\widetilde{v^n}, \widetilde{v^n v^n}, \widetilde{Dv^n}, \widetilde{|Dv^n|Dv^n}), \quad z^n = Dv^n$$

The function A does not depend directly on (t,x) and is continuous w.r.t. all other variables, which provides that assumption (i) of Lemma A.1 is fulfilled. Next, assumption (ii); for all $s \in \mathbb{R}^{21}$ and $\xi_1, \xi_2 \in \mathbb{S}^3$, $\xi_1 \neq \xi_2$, by (10)

$$(c(s)|\xi_1|\xi_1 - c(s)|\xi_2|\xi_2) \cdot (\xi_1 - \xi_2) = c(s)(|\xi_1|\xi_1 - |\xi_2|\xi_2) \cdot (\xi_1 - \xi_2) > 0$$

Then, assumption (iii); from assumptions (C1)–(C2) it holds

$$c(s)|\xi|\xi \cdot \xi \geqslant \alpha|\xi|^3$$

and

$$|c(s)|\xi|\xi| \leqslant \beta|\xi|^2$$

Assumption (iv) holds by Lemma 2.2, with $\bar{y} = (\tilde{v}, \widetilde{vv}, \widetilde{Dv}, \tilde{\chi})$, namely,

$$y^n \to \bar{y} \quad \text{a.e. in } Q_T$$

Due to (23) and (27) assumption (v) is satisfied. We only have to check assumption (vi). To this aim we will prove the following claim.

Claim

$$v^n(t) \rightharpoonup v(t) \text{ in } H \text{ for all } t \in [0, T] \tag{42}$$

Proof
From (26) it holds

$$v^n(t) \to v(t) \text{ in } H \text{ for a.a. } t \in [0, T] \tag{43}$$

in particular,

$$v^n(t) \rightharpoonup v(t) \text{ in } H \text{ for all } t \in [0, T] \setminus E \tag{44}$$

where E is a set of measure zero. Let us first show that

$$v^n(t) \rightharpoonup v(t) \text{ in } (W^{2,2}(\Omega) \cap V)' \text{ for all } t \in [0, T] \tag{45}$$

Thus, consider $t \in E$. For each such t choose $(t_k) \subset (0, T) \setminus E$ such that $t_k \to t$ as $k \to \infty$. Then for all $\phi \in W^{2,2}(\Omega) \cap V$

$$\begin{aligned} |\langle v^n(t) - v(t), \phi\rangle_{(W^{2,2}\cap V)', W^{2,2}\cap V}| &\leqslant |\langle v^n(t) - v^n(t_k), \phi\rangle_{(W^{2,2}\cap V)', W^{2,2}\cap V}| \\ &\quad + |\langle v^n(t_k) - v(t_k), \phi\rangle_{(W^{2,2}\cap V)', W^{2,2}\cap V}| \\ &\quad + |\langle v(t_k) - v(t), \phi\rangle_{(W^{2,2}\cap V)', W^{2,2}\cap V}| \\ &= I_1 + I_2 + I_3 \end{aligned} \tag{46}$$

Consider first the term I_1. Since v^n is bounded in $L^3(0, T; V)$ and v_t^n is bounded in $L^{3/2}(0, T; (W^{2,2}(\Omega) \cap V)')$, the sequence v^n is bounded in $W^{1,3/2}(0, T; (W^{2,2}(\Omega) \cap V)')$. According to Morrey's theorem (cf. Reference [17, p. 266]) $W^{1,3/2} \subset C^{0,1/3}$; thus,

$$\|v^n(t_1) - v^n(t_2)\|_{(W^{2,2}\cap V)'} \leqslant m|t_1 - t_2|^{1/3} \quad \text{for all } t_1, t_2 \in [0, T]$$

This assures that (v^n) is an equicontinuous family of functions. Obviously the same estimate holds for v itself. Thus,

$$I_1 \leqslant m|t - t_k|^{1/3} \quad \text{and} \quad I_3 \leqslant m|t - t_k|^{1/3}$$

Moreover, (44) with the embedding $L^2(\Omega) \subset (W^{2,2}(\Omega) \cap V)'$ implies that for $n \to \infty$ and all $t_k \in (0, T) \setminus E$

$$v^n(t_k) \rightharpoonup v(t_k) \text{ in } (W^{2,2}(\Omega) \cap V)'$$

and hence $\lim_{n\to\infty} I_2 = 0$. Thus, letting $n \to \infty$ in (46) yields

$$\lim_{n\to\infty} \langle v^n(t) - v(t), \phi\rangle_{(W^{2,2}\cap V)', W^{2,2}\cap V} = 0 \tag{47}$$

which proves (45).

Since the embedding $(W^{2,2}(\Omega) \cap V) \subset H$ is dense and (v^n) is bounded in $L^\infty(0, T; H)$, we conclude from Lemma III 1.4 in Reference [18] that

$$(v^n(t), \phi) \to (v(t), \phi) \quad \text{for all } \phi \in H, t \in [0, T]$$

hence (42) is proved.

From (14) it holds

$$\int_{Q_T} c(y^n)|Dv^n|Dv^n \cdot Dv^n \,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\|v_0^n\|_H^2 - \frac{1}{2}\|v^n(T)\|_H^2 - v\|\nabla v^n\|_{L^2(Q_T)}^2 + \int_0^T \langle f, v^n\rangle \,\mathrm{d}t$$

Letting $n \to \infty$ and using the lower semicontinuity of the norm w.r.t. the weak convergence (42) we obtain

$$\limsup_{n\to\infty} \int_{Q_T} c(y^n)|Dv^n|Dv^n \cdot Dv^n \,\mathrm{d}x\,\mathrm{d}t$$

$$\leqslant \frac{1}{2}\|v_0\|_H^2 - \frac{1}{2}\|v(T)\|_H^2 - v\|\nabla v\|_{L^2(Q_T)}^2 + \int_0^T \langle f, v\rangle \,\mathrm{d}t$$

Inserting the energy equality (38) into the r.h.s. yields

$$\limsup_{n\to\infty} \int_{Q_T} c(y^n)|Dv^n|Dv^n \cdot Dv^n \,\mathrm{d}x\,\mathrm{d}t \leqslant \int_{Q_T} \bar{A} \cdot Dv \,\mathrm{d}x\,\mathrm{d}t$$

which is exactly the desired inequality for assumption (vi). Now Lemma A.1 implies that $Dv^n \to Dv$ in measure, and thus for a subsequence

$$Dv^n \to Dv \quad \text{a.e. in } Q_T$$

Hence $|Dv^n|Dv^n \to |Dv|Dv$ a.e. in Q_T which together with (28) implies that $\chi = |Dv|Dv$ a.e. in Q_T. Thus,

$$\bar{y} = y \quad \text{and} \quad y^n \to y \quad \text{a.e. in } Q_T$$

Concerning the turbulent term we conclude that

$$c(y^n)|Dv^n|Dv^n \to c(y)|Dv|Dv \quad \text{a.e. in } Q_T$$

As $c(y^n)|Dv^n|Dv^n$ is bounded in $L^{3/2}(Q_T)$ we apply Lemma A.8 and get that

$$c(y^n)|Dv^n|Dv^n \rightharpoonup c(y)|Dv|Dv \quad \text{in } L^{3/2}(Q_T)$$

This convergence completes the proof of the theorem. □

3. COMPACTNESS OF SOLUTIONS

In this short section we will observe an additional property of solutions, which is formulated in the forthcoming theorem.

Theorem 3.1
Let all the assumptions of Theorem 1.1 be satisfied and let (v^n) be a sequence of solutions to the approximate problem (14) and v the solution to (12). Then

$$v^n \to v \quad \text{in } L^3(0,T;V) \tag{48}$$

Proof
Since in the proof of Theorem 1.1 we showed that all the assumptions of Lemma A.1 are satisfied, we can apply Lemma A.2, which proves (48). □

APPENDIX A

The current section contains two important lemmata, which recall results shown in Reference [19]. Nevertheless, for completeness of the paper, we provide also their proofs.

In the following, $C_0(\mathbb{R}^d)$ denotes the closure of the space of continuous functions on $\mathbb{R}^d$ with compact support w.r.t. the $\|\cdot\|_\infty$-norm. Its dual space can be identified with $\mathscr{M}(\mathbb{R}^d)$, the space of signed Radon measures with finite mass. The related duality pairing is given by

$$\langle \mu, f\rangle = \int_{\mathbb{R}^d} f(\xi)\,\mathrm{d}\mu(\xi)$$

Lemma A.1
Let $\Omega\subset\mathbb{R}^{d'}$ be a measurable set of finite measure and let $A(x,s,\xi):\Omega\times\mathbb{R}^m\times\mathbb{R}^d\longrightarrow\mathbb{R}^d$ be an operator satisfying the following conditions:

(i) $A(x,s,\xi)$ is a Carathéodory function (measurable w.r.t. x and continuous w.r.t. (s,ξ)).
(ii) For all $x\in\Omega$, $s\in\mathbb{R}^m$ and $\xi_1,\xi_2\in\mathbb{R}^d$, $\xi_1\neq\xi_2$,

$$[A(x,s,\xi_1)-A(x,s,\xi_2)]\cdot[\xi_1-\xi_2]>0$$

(iii) There exist positive constants c_1,c_2 such that for $p>1$ it holds

$$A(x,s,\xi)\cdot\xi\geqslant c_1|\xi|^p \quad\text{and}\quad |A(x,s,\xi)|\leqslant c_2|\xi|^{p-1}$$

Let $y^n:\Omega\to\mathbb{R}^m$ and $z^n:\Omega\to\mathbb{R}^d$ be sequences of measurable functions such that
(iv) $y^n\to\bar{y}$ a.e. in Ω,
(v) $z^n\rightharpoonup z$ in $L^p(\Omega)$ and $A(x,y^n,z^n)\rightharpoonup\bar{A}$ in $L^{p/(p-1)}(\Omega)$,
(vi)

$$\limsup_{n\to\infty}\int_\Omega A(x,y^n,z^n)\cdot z^n\,\mathrm{d}x\leqslant\int_\Omega\bar{A}\cdot z\,\mathrm{d}x$$

Then there exists a subsequence of (z^n) such that $z^n\to z$ in measure.

Proof
We apply Lemma A.5 to the function $A(x,y^n,z^n)\cdot z^n$. The coercivity condition from assumption (iii) assures that the negative part of this function is equal to zero; thus, it is certainly weakly relatively compact in $L^1(\Omega)$. This allows to conclude that

$$\limsup_{n\to\infty}\int_\Omega A(x,y^n,z^n)\cdot z^n\,\mathrm{d}x\geqslant\int_\Omega\int_{\mathbb{R}^m\times\mathbb{R}^d}A(x,s,\xi)\cdot\xi\,\mathrm{d}\mu_x(s,\xi)\,\mathrm{d}x \tag{A1}$$

where μ_x is the Young measure generated by the sequence (y^n,z^n). However, according to Lemma A.6, we are able to characterize this Young measure more precisely. The sequence y^n

converges to $\bar{y}$ a.e., and a subsequence of z^n generates a Young measure v_x. Then the Young measure μ_x generated by this pair satisfies $\mu_x = \delta_{\bar{y}(x)} \otimes v_x$. Therefore, due to Fubini's theorem

$$\int_\Omega \int_{\mathbb{R}^m \times \mathbb{R}^d} A(x,s,\xi) \cdot \xi \,\mathrm{d}\mu_x(s,\xi)\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^d} A(x,\bar{y}(x),\xi) \cdot \xi \,\mathrm{d}v_x(\xi)\,\mathrm{d}x \tag{A2}$$

In the same way we obtain

$$\int_\Omega \int_{\mathbb{R}^m \times \mathbb{R}^d} A(x,s,\xi) \,\mathrm{d}\mu_x(s,\xi)\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^d} A(x,\bar{y}(x),\xi) \,\mathrm{d}v_x(\xi)\,\mathrm{d}x \tag{A3}$$

Since the sequence $|A(x,y^n,z^n)|$ is bounded in $L^{p/(p-1)}(\Omega)$, it is weakly relatively compact in $L^1(\Omega)$. Thus, we can use Lemma A.5 again, which allows to conclude that the weak limit $\bar{A}(x) = \int_{\mathbb{R}^d} A(x,s,\xi)\,\mathrm{d}\mu_x(s,\xi)$. From Corollary A.4, taking $q=1$, $g=\mathrm{id}$, we can conclude that $z^n \rightharpoonup z = \int_{\mathbb{R}^d} \xi \,\mathrm{d}v_x(\xi)$ in $L^p(\Omega)$. Then assumption (vi) can be formulated as follows:

$$\limsup_{n\to\infty} A(x,y^n,z^n)z^n\,\mathrm{d}x \leqslant \int_\Omega \int_{\mathbb{R}^d} A(x,\bar{y}(x),\xi)\,\mathrm{d}v_x(\xi) \int_{\mathbb{R}^d} \xi'\,\mathrm{d}v_x(\xi')\,\mathrm{d}x \tag{A4}$$

Thus, from (A1), (A2) and (A4), the following inequality holds:

$$\int_\Omega \int_{\mathbb{R}^d} A(x,\bar{y}(x),\xi)\,\mathrm{d}v_x(\xi) \cdot \int_{\mathbb{R}^d} \xi'\,\mathrm{d}v_x(\xi')\,\mathrm{d}x \geqslant \int_\Omega \int_{\mathbb{R}^d} A(x,\bar{y}(x),\xi)\cdot\xi\,\mathrm{d}v_x(\xi)\,\mathrm{d}x \tag{A5}$$

Next, we can deduce from the monotonicity of A w.r.t. the last variable that

$$\int_\Omega \int_{\mathbb{R}^d} h(x,\xi)\,\mathrm{d}v_x(\xi)\,\mathrm{d}x \geqslant 0 \tag{A6}$$

where h is defined by

$$h(x,\xi) := \left[A(x,\bar{y}(x),\xi) - A(x,\bar{y}(x),\int_{\mathbb{R}^d} \xi'\,\mathrm{d}v_x(\xi'))\right]\left[\xi - \int_{\mathbb{R}^d} \xi'\,\mathrm{d}v_x(\xi')\right]$$

Since the sequence (z^n) is bounded in L^p, the tightness condition is satisfied and $\|v_x\|_{\mathscr{M}(\mathbb{R}^d)} = 1$. Simple calculations imply that

$$\begin{aligned}\int_\Omega \int_{\mathbb{R}^d} & h(x,\xi)\,\mathrm{d}v_x(\xi)\,\mathrm{d}x \\ &= \int_\Omega \int_{\mathbb{R}^d} A(x,\bar{y}(x),\xi)\cdot\xi\,\mathrm{d}v_x(\xi)\,\mathrm{d}x - \int_\Omega \int_{\mathbb{R}^d} A(x,\bar{y}(x),\xi)\,\mathrm{d}v_x(\xi) \cdot \int_{\mathbb{R}^d} \xi'\,\mathrm{d}v_x(\xi')\,\mathrm{d}x\end{aligned}$$

which, together with (A5), assures that

$$\int_\Omega \int_{\mathbb{R}^d} h(x,\xi)\,\mathrm{d}v_x(\xi)\,\mathrm{d}x \leqslant 0 \tag{A7}$$

Then, (A6) and (A7) imply that $\int_{\mathbb{R}^d} h(x,\xi)\,\mathrm{d}v_x(\xi) = 0$ for a.a. $x \in \Omega$. Moreover, since $v_x \geqslant 0$ is a probability measure and $A(x,s,\cdot)$ is strongly monotone, we conclude that

$$\operatorname{supp}\{v_x\} \overset{\text{a.e.}}{=} \left\{\int_{\mathbb{R}^d} \xi'\,\mathrm{d}v_x(\xi')\right\}$$

where the r.h.s. is equal to $z(x)$, which is the weak limit of the sequence (z^n). Finally, we conclude that $\nu_x = \delta_{z(x)}$ a.e.. A direct application of Lemma A.7 implies that $z^n \to z$ in measure. □

Lemma A.2
With the assumptions of Lemma A.1 there exists a subsequence of (z^n) such that $z^n \to z$ in $L^p(\Omega)$.

Proof
Since z^n converges in measure, then at least for a subsequence $z^n \to z$ a.e.. Using the information that $\nu_x = \delta_{z(x)}$ together with Lemma A.5 and assumption (vi) yields

$$\limsup_{n\to\infty} \int_\Omega A(x, y^n, z^n) \cdot z^n \, \mathrm{d}x \leqslant \int_\Omega A(x, \bar{y}, z) z \, \mathrm{d}x \leqslant \liminf_{n\to\infty} \int_\Omega A(x, y^n, z^n) \cdot z^n \, \mathrm{d}x$$

Hence the limit exists and

$$\lim_{n\to\infty} \int_\Omega A(x, y^n, z^n) \cdot z^n \, \mathrm{d}x = \int_\Omega A(x, \bar{y}, z) z \, \mathrm{d}x$$

We can set $a^n = A(x, y^n, z^n) \cdot z^n$, $a = A(x, \bar{y}, z) z$ and claim that

$$a^n \geqslant 0, \quad a \in L^1(\Omega), \quad \int_\Omega a^n \, \mathrm{d}x \to \int_\Omega a \, \mathrm{d}x, \quad a^n \to a \quad \text{a.e. in } \Omega$$

Noticing that

$$\int_\Omega |a^n - a| \, \mathrm{d}x = \int_\Omega (a^n - a) \, \mathrm{d}x + 2 \int_{\{x: a^n \leqslant a\}} (a - a^n) \, \mathrm{d}x$$

we conclude by Lebesgue's dominated convergence theorem that

$$A(x, y^n, z^n) z^n \to A(x, \bar{y}, z) z \quad \text{in } L^1(\Omega)$$

Thus, by Vitali's theorem, the sequence $A(x, y^n, z^n) z^n$ is uniformly integrable. Due to the coercivity condition also the sequence $|z^n|^p$ is uniformly integrable. Using again Vitali's theorem yields that $z^n \to z$ in $L^p(\Omega)$, which completes the proof. □

For the proof of the fundamental theorem on Young measures we refer the reader to References [20,21].

Theorem A.3 (*Fundamental theorem on Young measures*)
Let $\Omega \subset \mathbb{R}^d$ be a measurable set of finite measure and let $z^j : \Omega \to \mathbb{R}^d$ be a sequence of measurable functions. Then there exists a subsequence z^{j_k} and a weakly* measurable map $\nu : \Omega \to \mathscr{M}(\mathbb{R}^d)$ such that the following holds:

(i) $\nu_x \geqslant 0$, $\|\nu_x\|_{\mathscr{M}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \mathrm{d}\nu_x \leqslant 1$ for a.a. $x \in \Omega$.
(ii) For $g \in C_0(\mathbb{R}^d)$ let $\bar{g}(x) = \langle \nu_x, g \rangle$. Then $g(z^{j_k}) \overset{*}{\rightharpoonup} \bar{g}$ in $L^\infty(\Omega)$.
(iii) Let $K \subset \mathbb{R}^d$ be compact. Then

$$\operatorname{supp} \nu_x \subset K \text{ if } \operatorname{dist}(z^{j_k}, K) \to 0 \text{ in measure}$$

(iv) Additionally, $\|\nu_x\|_{\mathscr{M}(\mathbb{R}^d)} = 1$ for a.a. $x \in \Omega$ if and only if the 'tightness condition' is satisfied, i.e.

$$\lim_{M\to\infty} \sup_k |\{|z^{j_k}| \geqslant M\}| = 0$$

(v) If the tightness condition is satisfied, and moreover, if $A \subset \Omega$ is measurable, $g \in C(\mathbb{R}^d)$ and $g(z^{j_k})$ is relatively weakly compact in $L^1(A)$, then

$$g(z^{j_k}) \rightharpoonup \bar{g} \quad \text{in } L^1(A), \quad \bar{g}(x) = \langle \nu_x, g \rangle$$

(vi) If the tightness condition is satisfied, then in (iii) one can replace 'if' by 'if and only if'.

Remark
The map $\nu : \Omega \to \mathscr{M}(\mathbb{R}^d)$ is called the Young measure generated by the sequence z^{j_k}. Every (weakly* measurable map) $\nu : \Omega \to \mathscr{M}(\mathbb{R}^d)$ that satisfies (i) is generated by some sequence z^k.

Remark
If for some $s > 0$ and all $j \in \mathbb{N}$ it holds $\int_\Omega |z^j|^s \leqslant k$, then the tightness condition is satisfied.

The straightforward consequence of the assertion (v) is the following corollary.

Corollary A.4 (*Müller* [*21, Remark 5, p. 33*])
Let Ω, z^{j_k}, ν be as in Theorem A.3, with (z^j) bounded in $L^p(\Omega)$. Then for all $g \in C(\mathbb{R}^d)$ satisfying the growth condition

$$|g(\xi)| \leqslant k(1 + |\xi|)^q \quad \forall \xi \in \mathbb{R}^d \quad \text{for some } 0 < q < p$$

it holds $g(z^{j_k}) \rightharpoonup \bar{g}$ in $L^{p/q}(\Omega)$ and $\bar{g}(x) \overset{\text{a.e}}{=} \langle \nu_x, g \rangle$.

Lemma A.5 (*Müller* [*21, Corollary 3.3*])
Suppose that the sequence of maps $z^j : \Omega \to \mathbb{R}^d$ generates the Young measure ν. Let $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a Carathéodory function such that the negative part $f^-(x, z^j(x))$ is weakly relatively compact in $L^1(\Omega)$. Then

$$\liminf_{j\to\infty} \int_\Omega f(x, z^j(x))\, \mathrm{d}x \geqslant \int_\Omega \int_{\mathbb{R}^d} f(x, \lambda)\, \mathrm{d}\nu_x(\lambda)$$

If, in addition, the sequence of functions $x \mapsto |f|(x, z^j(x))$ is weakly relatively compact in $L^1(\Omega)$, then

$$f(\cdot, z^j(\cdot)) \rightharpoonup \int_{\mathbb{R}^d} f(x, \lambda)\, \mathrm{d}\nu_x(\lambda) \quad \text{in } L^1(\Omega)$$

Remark
In an obvious way the second part of the above theorem can be extended to vector valued functions f.

Lemma A.6 (*Müller* [*21, Corollary 3.4*])
Let $u^j: \Omega \to \mathbb{R}^d$, $v^j: \Omega \to \mathbb{R}^{d'}$ be measurable and suppose that $u^j \to u$ a.e. while v^j generates the Young measure ν. Then the sequence of pairs $(u^j, v^j): \Omega \to \mathbb{R}^{d+d'}$ generates the Young measure $x \mapsto \delta_{u(x)} \otimes \nu_x$.

Lemma A.7 (*Müller* [*21, Corollary 3.2*])
Suppose that a sequence z^j of measurable functions from Ω to R^d generates the Young measure $\nu: \Omega \to \mathscr{M}(\mathbb{R}^d)$. Then

$$z^j \to z \text{ in measure if and only if } \nu_x = \delta_{z(x)} \quad \text{a.e.}$$

Finally, we mention two well-known results already used in Section 2.

Lemma A.8
Let Ω be an open bounded subset of $\mathbb{R}^d$, let g^n, g be the functions from $L^p(\Omega)$, with $1 < p < \infty$, such that $\|g^n\|_{L^p(\Omega)} \leqslant c$, $g^n \to g$ a.e. in Ω. Then

$$g^n \rightharpoonup g \quad \text{in } L^p(\Omega)$$

For the proof see Reference [4, Lemma 1.3, p. 12]. The assertion of Lemma A.8 is also true if the sequence (g^n) converges locally in measure, see Reference [22, p. 264].

Before stating the next proposition (cf. Reference [23, Proposition 23.23, p. 422]) we introduce the notion of an *evolution triple* '$V \subseteq H \subseteq V'$' as follows: V is a real, separable, and reflexive Banach space, H is a real, separable Hilbert space with the dense and continuous embedding $V \subset H$. Then set $W_p^1(0,T;V,H) = \{u \in L^p(0,T;V) : u_t \in L^q(0,T;V')\}$, where $1 < p < \infty$, $p^{-1} + q^{-1} = 1$. By $(\cdot,\cdot)_H$ we mean the scalar product in H and by $\langle \cdot,\cdot \rangle_V$ the dual pairing between V and V'.

Proposition A.9
Let '$V \subseteq H \subseteq V'$' be an evolution triple, and let $1 < p < \infty$, $p^{-1} + q^{-1} = 1$, $0 < T < \infty$. Then the following hold:

(i) The set of all functions $u \in L^p(0,T;V)$ that have generalized derivative $u_t \in L^q(0,T;V')$ forms a real Banach space with the norm

$$\|u\|_{W_p^1} = \|u\|_{L^p(0,T;V)} + \|u_t\|_{L^q(0,T;V')}$$

(ii) The embedding

$$W_p^1(0,T;V,H) \subseteq C([0,T];H)$$

is continuous.

(iii) For all $u, v \in W_p^1(0,T;V,H)$ and arbitrary t, s, $0 \leqslant s \leqslant t \leqslant T$, the following generalized integration by parts formula holds:

$$(u(t), v(t))_H - (u(s), v(s))_H = \int_s^t (\langle u_t(\tau), v(\tau) \rangle_V + \langle v_t(\tau), u(\tau) \rangle_V) \, d\tau \tag{A8}$$

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