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APPROXIMATION OF THE LARGER EDDIES IN FLUID MOTIONS II: A MODEL FOR SPACE-FILTERED FLOW

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We present two modifications of continuum models used in large eddy simulation. The first modification is a closure approximation which better attenuates small eddies. The second modification is a change in the boundary conditions for the large eddy model from strict adherence to slip with resistance. For model consistency, the resistance coefficient is a function of the averaging radius so that the model's boundary conditions reduce to a no-slip as the averaging radius decreases to zero.

1. Introduction

We consider the problem of deriving models for the motion of larger eddies in fluid flow. The space filtered approach to large eddy simulation, introduced in 1970 by Deardorff,[5] (see also Refs. 1–3 and 9) begins with the Navier–Stokes equations in a bounded domain Ω in $\mathcal{R}^3$:

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \mathrm{Re}^{-1}\Delta\mathbf{u} = \mathbf{f}\,, \quad \text{in } \Omega\,, \\ \mathbf{u} = \mathbf{0}\,, \quad \text{on } \partial\Omega\,, \quad \displaystyle\int_\Omega p(x,t)\,dx = 0 \\ \nabla\cdot\mathbf{u} = 0\,, \quad \text{in } \Omega\,, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x)\,, \quad \text{in } \Omega\,. \end{cases} \tag{1.1}$$

All dependent variables are extended by zero to $\mathcal{R}^3$ so that a low pass filter can be applied. Define the commonly used Gaussian fitter

$$g_\delta(x) := \left(\frac{\gamma}{\pi}\right)^{3/2} \frac{1}{\delta^3}\exp[-\gamma x_j x_j/\delta^2]\,,$$

where repeated indices are summed from 1 to 3, $\gamma > 0$, and $\delta > 0$ represents an averaging radius. The spatial averaging operator is defined by convolution. Indeed,

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for any $\mathbf{w}(x,t)$

$$\overline{\mathbf{w}}(x,t) = (g_\delta * \mathbf{w})(x,t) = \int_{\mathcal{R}^3} g_\delta(x-x')\mathbf{w}(x',t)\,dx'\,.$$

Applying this averaging operator to the Navier–Stokes equations gives:

$$\begin{cases} \dfrac{\partial \overline{\mathbf{u}}}{\partial t} + \nabla \cdot (\overline{\mathbf{u}\mathbf{u}}) + \nabla \overline{p} - \mathrm{Re}^{-1}\Delta \overline{\mathbf{u}} = \overline{\mathbf{f}}\,, & \text{in } \Omega, 0 < t < T \\ \nabla \cdot \overline{\mathbf{u}} = 0\,, & \text{in } \Omega, 0 < t \le T\,, \\ \overline{\mathbf{u}}(x,0) = \overline{\mathbf{u}}_0(x)\,, & \text{in } \Omega\,, \\ \overline{\mathbf{u}}(x,0) = (g_\delta * \mathbf{u})(x)\,, & \text{on } \mathbf{\Gamma}, 0 < t \le T\,. \end{cases} \tag{1.2}$$

This system is very attractive in that solutions are C^∞ in space and contain no scales of size smaller than $O(\delta)$. However, it cannot be solved because of the well-known problem of closure that $\overline{\mathbf{u}\mathbf{u}} \neq \overline{\mathbf{u}}\,\overline{\mathbf{u}}$. One fundamental question in large eddy simulation is thus to *find a closed system approximating* (1.2) *in the sense that solutions exist*, *are* C^∞ *in space and contain no scales much less than* $O(\delta)$.

With $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$, decompose the nonlinear term in (1.2):

$$\overline{\mathbf{u}\mathbf{u}} = \overline{(\overline{\mathbf{u}} + \mathbf{u}')(\overline{\mathbf{u}} + \mathbf{u}')} = \overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}} + \overline{\overline{\mathbf{u}}\mathbf{u}'} + \overline{\mathbf{u}'\overline{\mathbf{u}}} + \overline{\mathbf{u}'\mathbf{u}'}\,. \tag{1.3}$$

Each term above must be approximated by terms only involving $\overline{\mathbf{u}}$. The first term on the right-hand side of (1.3) is often called the "resolved scales", the second and third the "cross terms" and the last the "subgridscale term".

There are many proposals for this approximation, well presented in Aldama.[1] For example, for the first "resolved scale" term, two natural possibilities are:

$$\text{direct calculation} : \overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}} = g_\delta * \overline{\mathbf{u}}\,\overline{\mathbf{u}}\,, \tag{1.4}$$

$$\text{Taylor series in } \delta : \overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}} \cong \overline{\mathbf{u}}\,\overline{\mathbf{u}} + \frac{\delta^2}{4\gamma}\frac{\partial \overline{\mathbf{u}}}{\partial x_i}\frac{\partial \overline{\mathbf{u}}}{\partial x_i}\,. \tag{1.5}$$

The approximation (1.5) is an example of the basic method developed by Leonard[9] and Clark, Ferziger and Reynolds[3] for treating the first three terms on the right-hand side of (1.3) (often referred to as the "Leonard terms" as a result).

The "subgridscale" terms $\overline{\mathbf{u}'\mathbf{u}'}$ are often treated by a different technique which adds extra stabilization to algorithms used to discretize the resulting continuum model, see, e.g., Refs. 2, 3, 7 and 8. We will not discuss this possibility herein.

2. Closure Approximations

The key properties of the averaging process $\mathbf{w} \longmapsto g_{\delta *}\mathbf{w}$ are that it is smoothing, converges to $\mathbf{u}$ as $\delta \to 0$ and eliminates high frequency oscillations in $\mathbf{u}$. In this section, we review one path to the approximation (1.5) and, motivated by this approach, give a modification which better preserves these essential features (whereas the approximation (1.5) *increases* those high frequency components in the approximation).

Extending all variables by zero outside Ω, (1.3) can be approached via Fourier transforms. If $\mathbf{k}$ denotes the dual variable, recall that

$$\mathcal{F}(g_\delta(x)) = \hat{g}_\delta(\mathbf{k}) = \exp\left[\left(-\frac{\boldsymbol{\delta^2}}{\mathbf{4}\gamma}\right)\mathbf{k}_\ell\mathbf{k}_\ell\right] \tag{2.1}$$

and

$$\hat{\overline{\mathbf{u}}}(\mathbf{k}) = \mathcal{F}(\mathbf{g}_\delta * \mathbf{u}) = \hat{\mathbf{g}}_\delta(\mathbf{k})\hat{\mathbf{u}}(\mathbf{k})\,.$$

Elimination of high frequencies in $\overline{\mathbf{u}}$ corresponds to the exponential decay at infinity of $\hat{g}(\mathbf{k})$.

Since $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}', g_\delta * \mathbf{u} = g_\delta * (\overline{\mathbf{u}} + \mathbf{u}')$ and hence $\overline{\mathbf{u}} = g_\delta * \overline{\mathbf{u}} + g_\delta * \mathbf{u}'$. Fourier transform of this yields

$$\hat{\overline{\mathbf{u}}}(\mathbf{k}) = \hat{g}_\delta(\mathbf{k})\hat{\overline{\mathbf{u}}}(\mathbf{k}) + \hat{g}_\delta(\mathbf{k})\hat{\mathbf{u}}'(\mathbf{k})\,.$$

Thus, $\hat{\mathbf{u}}'$ is given exactly by

$$\hat{\mathbf{u}}'(\mathbf{k}) = \left(\frac{1}{\hat{g}_\delta(\mathbf{k})} - 1\right)\hat{\overline{\mathbf{u}}}(\mathbf{k})\,.$$

Using these formulas appropriately gives:

$$\mathcal{F}(\overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}}) = \hat{g}_\delta(\mathbf{k})\hat{\overline{\mathbf{u}}}(\mathbf{k}) * \hat{\overline{\mathbf{u}}}(\mathbf{k})\,, \tag{2.2a}$$

$$\mathcal{F}(\overline{\overline{\mathbf{u}}\mathbf{u}'}) = \hat{g}_\delta(\mathbf{k})\hat{\overline{\mathbf{u}}} * \left[\left(\frac{1}{\hat{g}_\delta(\mathbf{k})} - 1\right)\hat{\overline{\mathbf{u}}}(\mathbf{k})\right]\,, \tag{2.2b}$$

$$\mathcal{F}(\overline{\mathbf{u}'\overline{\mathbf{u}}}) = \hat{g}_\delta(\mathbf{k})\left[\left(\frac{1}{\hat{g}_\delta(\mathbf{k})} - 1\right)\hat{\overline{\mathbf{u}}}(\mathbf{k})\right] * \hat{\overline{\mathbf{u}}}(\mathbf{k})\,, \quad \text{and} \tag{2.2c}$$

$$\mathcal{F}(\overline{\mathbf{u}'\mathbf{u}'}) = \hat{g}_\delta(\mathbf{k})\left[\left(\frac{1}{\hat{g}_\delta(\mathbf{k})} - 1\right)\hat{\overline{\mathbf{u}}}\right] * \left[\left(\frac{1}{\hat{g}_\delta(\mathbf{k})} - 1\right)\hat{\overline{\mathbf{u}}}\right]\,. \tag{2.2d}$$

Some conventional large eddy models can arise by expanding $\hat{g}_\delta(\mathbf{k})$ and $(\frac{1}{g_\delta(\mathbf{k})} - 1)$ in Taylor series in δ:

$$\hat{g}_\delta(\mathbf{k}) \doteq 1 - \frac{\delta^2}{4\gamma}k_\ell k_\ell(+O(\delta^4))\,, \tag{2.3a}$$

$$\frac{1}{\hat{g}_\delta(\mathbf{k})} - 1 \doteq \frac{\delta^2}{4\gamma}k_\ell k_\ell(+O(\delta^4))\,. \tag{2.3b}$$

Substitution of these expansions into (2.2) and taking inverse Fourier transforms gives the *usual* approximations:

$$\overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}} \cong \overline{\mathbf{u}}\,\overline{\mathbf{u}} + \frac{\delta^2}{4\gamma}\frac{\partial\overline{\mathbf{u}}}{\partial x_\ell}\frac{\partial\overline{\mathbf{u}}}{\partial x_\ell}(+O(\delta^4),\ \text{formally})\,, \tag{2.4a}$$

$$\overline{\overline{\mathbf{u}}\mathbf{u}'} \cong -\frac{\delta^2}{4\gamma}\frac{\partial\overline{\mathbf{u}}}{\partial x_\ell}\frac{\partial\overline{\mathbf{u}}}{\partial x_\ell}(+O(\delta^4),\ \text{formally})\,, \tag{2.4b}$$

$$\overline{\mathbf{u}'\mathbf{u}'} \cong \frac{\delta^4}{16\gamma^2}\frac{\partial^2\overline{\mathbf{u}}}{\partial x_\ell\partial x_\ell}\frac{\partial^2\overline{\mathbf{u}}}{\partial x_\ell\partial x_\ell}(+O(\delta^6),\ \text{formally})\,. \tag{2.4c}$$

One usual model for used large eddy simulation arises by substitution of the approximation (2.4) into (1.2), and ignoring the terms of formal order $O(\delta^4)$. This results in the problem of finding $(\mathbf{w}, q)$ satisfying

$$\begin{cases} \dfrac{\partial \mathbf{w}}{\partial t} + \nabla(\mathbf{w}\mathbf{w}) + \nabla(q) \\ -\mathrm{Re}^{-1}\Delta\mathbf{w} + \nabla\left(\dfrac{\delta^2}{2\gamma}\dfrac{\partial \mathbf{w}}{\partial x_\ell}\dfrac{\partial \mathbf{w}}{\partial x_\ell}\right) = \overline{\mathbf{f}}\,, \quad \text{in } \Omega\,, \\ \nabla(\mathbf{w}) = 0\,, \quad \text{in } \Omega, \mathbf{w}(x,0) = \overline{\mathbf{u}}_0(x)\,, \quad \text{in } \Omega, \mathbf{w}(x,t) = 0\,, \quad \text{on } \mathbf{\Gamma}\,. \end{cases} \tag{2.5}$$

Note that $\mathbf{w}$ is the solution to the system of partial differential equation (2.5), (*not* the average of the true flow field) which approximates $\overline{\mathbf{u}}$.

There are several possible difficulties with (2.5). First, the approximation (2.3a–b) does *not* preserve the essential feature that $\overline{\mathbf{u}} = g_\delta * \mathbf{u}$ is smoother than $\mathbf{u}$. In fact, the approximation (2.3a–b), while valid for small wave numbers (i.e. for $|\delta\mathbf{k}|$ small) actually increases the noise at high wave numbers. It is thus not unexpected that mathematical support for (2.5) for larger Re seems intractable. The kinetic energy of solutions to the Navier–Stokes equations is non-increasing yet there is no reason not to expect the kinetic energy in solutions to (2.5) to increase or even blow up in finite time.

With these considerations in mind, the modification we propose seems obvious; We shall replace (2.3a–b) with an approximation of the same order accuracy for large eddies (small $|\mathbf{k}|$) but which attenuates the small eddies (decays as $|\mathbf{k}| \to \infty$).

To this end, consider the rational approximation $\overline{e}^{\beta x} \doteq \frac{1}{1+\beta x} + O(\beta^2 x^2)$.

Using this subdiagonal Padé approximation to $\hat{g}_\delta(\mathbf{k})$ gives:

$$\hat{g}_\delta(\mathbf{k}) \doteq \frac{1}{1 + \dfrac{\delta^2}{4\gamma}|\mathbf{k}|^2} + O\left(\frac{\delta^4}{16\gamma^2}|\mathbf{k}|^4\right). \tag{2.6}$$

The formal accuracy of (2.6) is equal to that of (2.3a). The approximation

$$\left[1 + \frac{\delta^2}{4\gamma}|\mathbf{k}|^2\right]^{-1}$$

also preserves some of the decay at infinity of $\hat{g}_\delta(\mathbf{k})$, i.e. some of the attenuation of small eddies of $g_\delta * \mathbf{u}$.

From (2.6) we also approximate:

$$\left(\frac{1}{\hat{g}_\delta(\mathbf{k})} - 1\right) \doteq \frac{\delta^2}{4\gamma}|\mathbf{k}|^2\,. \tag{2.7}$$

Using (2.6) and (2.7) in (2.2a–d) gives

$$\mathcal{F}(\overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}}) \doteq \left(1 + \frac{\delta^2}{4\gamma}|\mathbf{k}|^2\right)^{-1} \hat{\overline{\mathbf{u}}} * \hat{\overline{\mathbf{u}}}\,, \tag{2.8a}$$

$$\mathcal{F}(\overline{\overline{\mathbf{u}}\mathbf{u}'}) \doteq \left(1 + \frac{\delta^2}{4\gamma}|\mathbf{k}|^2\right)^{-1} \left[\hat{\overline{\mathbf{u}}} * \left(\frac{\delta^2}{4\gamma}|\mathbf{k}|^2\hat{\overline{\mathbf{u}}}\right)\right], \tag{2.8b}$$

$$\mathcal{F}(\overline{\mathbf{u}'\overline{\mathbf{u}}}) \doteq \left(1+\frac{\delta^2}{4\gamma}|\mathbf{k}|^2\right)^{-1}\left[\left(\frac{\delta^2}{4\gamma}|\mathbf{k}|^2\hat{\overline{\mathbf{u}}}\right) * \hat{\overline{\mathbf{u}}}\right], \tag{2.8c}$$

$$\mathcal{F}(\overline{\mathbf{u}'\mathbf{u}'}) \doteq \left(1+\frac{\delta^2}{4\gamma}|\mathbf{k}|^2\right)^{-1}\left[\frac{\delta^4}{16\gamma^2}|\mathbf{k}|^2\hat{\overline{\mathbf{u}}} * |\mathbf{k}|^2\hat{\overline{\mathbf{u}}}\right]. \tag{2.8d}$$

Taking the inverse transform of (2.8) yields the following approximations valid to all $O(\delta^4)$,

$$\overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}} \doteq \left(-\frac{\delta^2}{4\gamma}\Delta + 1\right)^{-1}\overline{\mathbf{u}}, \overline{\mathbf{u}}, \tag{2.9a}$$

$$\overline{\overline{\mathbf{u}}\mathbf{u}'} \doteq -\frac{\delta^2}{4\gamma}\left[-\frac{\delta^2}{4\gamma}\Delta + I\right]^{-1}\overline{\mathbf{u}}(\Delta\overline{\mathbf{u}}), \tag{2.9b}$$

$$\overline{\mathbf{u}'\overline{\mathbf{u}}} \doteq -\frac{\delta^2}{4\gamma}\left[-\frac{\delta^2}{4\gamma}\Delta + I\right](\Delta\overline{\mathbf{u}})\overline{\mathbf{u}}, \quad \text{and} \tag{2.9c}$$

$$\overline{\mathbf{u}'\,\mathbf{u}'} \doteq \frac{\delta^4}{16\gamma^2}\left[-\frac{\delta^2}{4\gamma}\Delta + I\right]^{-1}\Delta\overline{\mathbf{u}}\Delta\overline{\mathbf{u}}. \tag{2.9d}$$

At this point, the most accurate boundary conditions upon $[-\frac{\delta^2}{4\gamma}\Delta + I]^{-1}$ are yet unknown. For initial tests, Neumann seem reasonable (see also Sec. 3).

If the approximations (2.9) are used in (1.3) we arrive at an approximation to the quadratic terms:

$$\begin{aligned}\overline{(\overline{\mathbf{u}}+\mathbf{u}')(\overline{\mathbf{u}}+\mathbf{u}')} &\doteq \left(-\frac{\delta^2}{4\gamma}\Delta + I\right)^{-1}\left[\overline{\mathbf{u}}\,\overline{\mathbf{u}} - \frac{\delta^2}{4\gamma}(\overline{\mathbf{u}}\Delta\overline{\mathbf{u}} + \Delta\overline{\mathbf{u}}\,\overline{\mathbf{u}})\right] \\ &= \left(-\frac{\delta^2}{4\gamma}\Delta + I\right)^{-1}\left[\overline{\mathbf{u}}\,\overline{\mathbf{u}} - \frac{\delta^2}{4\gamma}\Delta(\overline{\mathbf{u}}\,\overline{\mathbf{u}}) - \frac{\delta^2}{2\gamma}\nabla\overline{\mathbf{u}}\nabla\overline{\mathbf{u}}\right] \\ &= \overline{\mathbf{u}}\,\overline{\mathbf{u}} - \left(-\frac{\delta^2}{4\gamma}\Delta + I\right)^{-1}\left(\frac{\delta^2}{2\gamma}\nabla\overline{\mathbf{u}}\nabla\overline{\mathbf{u}}\right).\end{aligned}$$

Inserting this approximation into (1.2) leads to a set of equations for $\mathbf{w}$ approximating the large eddy motion $\overline{\mathbf{u}}$ to $O(\delta^4)$:

$$\begin{cases} \dfrac{\partial \mathbf{w}}{\partial t} + \nabla(q) - \nabla\cdot\,[\mathrm{Re}^{-1}(\nabla\mathbf{w} + \nabla\mathbf{w}^{tr})] + \nabla\cdot(\mathbf{w}\mathbf{w}) \\ \qquad - \nabla\cdot\left(-\dfrac{\delta^2}{4\gamma}\Delta + I\right)^{-1}\left(\dfrac{\delta^2}{2\gamma}\nabla\mathbf{w}\nabla\mathbf{w}\right) = \overline{\mathbf{f}}, \quad \text{in } \Omega, \\ \nabla\cdot \quad \mathbf{w} = 0, \quad \text{in } \Omega, \mathbf{w}(x,0) = \overline{\mathbf{u}}_0(x), \quad \text{in } \Omega. \end{cases} \tag{2.10}$$

Again, once boundary conditions are imposed, the solution $\mathbf{w}$ to (2.10) is proposed as an approximation to $\overline{\mathbf{u}}$.

3. Boundary Conditions for Large Eddy Models

The system (2.10) is similar to the more usual large eddy model (2.5); the principal difference lies in the smoothing properties preserved in (2.10) through $[-\frac{\delta^2}{4\gamma}\Delta + I]^{-1}$ and absent in (2.5). Another difference in the model proposed herein is the treatment of boundary conditions for (2.10). Navier in 1827 suggested that fluids in motion satisfy the boundary conditions of no penetration and limited slip with resistance

$$\mathbf{w}\cdot\hat{\mathbf{n}} = 0 \quad \text{and} \quad (1-\alpha)\mathbf{w}\cdot\hat{\boldsymbol{\tau}}_j + \alpha\mathbf{T}(\mathbf{w})\cdot\hat{\mathbf{n}}\cdot\hat{\boldsymbol{\tau}}_j = 0\,, \quad \text{on } \boldsymbol{\Gamma} = \partial\Omega, j = 1, 2\,, \tag{3.1}$$

where $\mathbf{T}(\mathbf{w})$ is the stress tensor, $\hat{\mathbf{n}}$ is the outward unit normal to $\boldsymbol{\Gamma}$ and $\hat{\boldsymbol{\tau}}_1, \hat{\boldsymbol{\tau}}_2$ are unit tangent vectors to $\boldsymbol{\Gamma}$. Serrin[10] also cited experimental evidence for (3.1) (where α depends on the ratio of tangential and normal stresses) for high Reynolds number flows of gases. Nevertheless, the usual no-slip boundary condition $\mathbf{u} = 0$, on $\boldsymbol{\Gamma}$, is accepted in mathematical fluid dynamics.

In this section we propose that *even when the fluid satisfies the no-slip boundary condition on* $\boldsymbol{\Gamma}$ *the approximation* $\mathbf{w}$ *to the large eddy models* (2.5) *or* (2.10) *should satisfy a condition of the form* (3.1) *on* $\boldsymbol{\Gamma}$.

Indeed, suppose $\mathbf{u} = 0$ on $\boldsymbol{\Gamma}$ and that there is (for the simplest case) a laminar boundary layer region as described by Prandtl's boundary layer theory (and depicted in the next figure). Keeping in mind that $\mathbf{u}$ is extended by zero outside Ω, it is clear that

$$\mathbf{u} = 0 \text{ on } \boldsymbol{\Gamma} \text{ but } \overline{\mathbf{u}} = g_\delta * \mathbf{u} \neq 0 \text{ on } \boldsymbol{\Gamma}\,.$$

From Fig. 3.1, it is a reasonable approximation to set

$$\mathbf{w}\cdot\hat{\mathbf{n}} = 0\,, \quad \text{on } \boldsymbol{\Gamma}\,,$$

as $(g_\delta * \mathbf{u})\cdot\hat{\mathbf{n}} \doteq 0$ on $\boldsymbol{\Gamma}$. Equally clear, $(g_\delta * \mathbf{u})\cdot\hat{\boldsymbol{\tau}}_j \neq 0$ on $\boldsymbol{\Gamma}$.

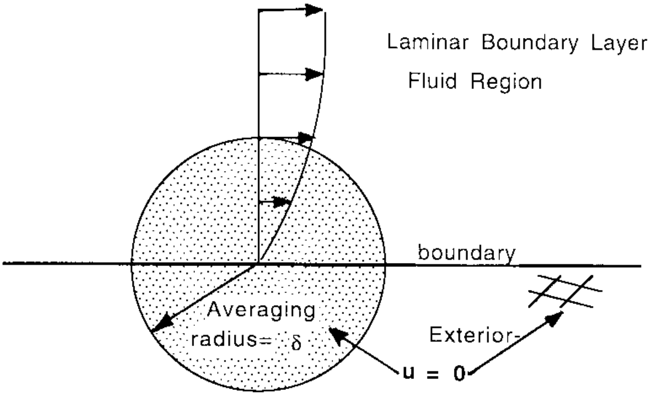

Fig. 3.1.

Due to the importance of fluid behavior near $\mathbf{\Gamma}$ to eddy formation, it is thus critical to obtain a more accurate boundary condition for $\mathbf{w}$ on $\mathbf{\Gamma}$ than simply $w = 0$. Consider the conditions

$$\mathbf{w} \cdot \hat{\mathbf{n}} = 0\,, \quad \text{on } \mathbf{\Gamma}\,, \quad \text{and } \mathbf{w} \cdot \hat{\boldsymbol{\tau}}_j - \beta_j(\delta, \mathrm{Re})\mathbf{T}(\mathbf{w}) \cdot \hat{\mathbf{n}} \cdot \hat{\boldsymbol{\tau}}_j = 0\,, \quad \text{on } \mathbf{\Gamma}\,, \tag{3.2}$$

where, for consistency with the continuous problem, we must clearly have $\beta_j(\delta) \to 0$ as $\delta \to 0$.

The determination of β in (3.2) becomes ultimately a parameter identification problem. Some insight is possibly obtainable via asymptotic analysis.

For example, in the case illustrated in Fig. 3.1, in which the ball of radius δ about boundary points is contained in a laminar boundary layer.

In this case, an asymptotic approximation to $\overline{\mathbf{u}} = g_\delta * \mathbf{u}$ and $\overline{\mathbf{T}} = g_\delta * \mathbf{T}(\mathbf{u})$ can be calculated from the boundary layer equations and β (motivated (3.2)) by

$$\beta_j(\delta, \mathrm{Re}) := \overline{\mathbf{u}} \cdot \hat{\boldsymbol{\tau}}_j / \overline{\mathbf{T}} \cdot \hat{\mathbf{n}} \cdot \hat{\boldsymbol{\tau}}_j\,.$$

The same procedure can be carried out using other boundary layer formulas, e.g. the law of the wall, as future experiments show to be appropriate.

4. Remarks on the Modelling Error

The fundamental open question for any large eddy model or turbulence model is to estimate the modelling error: $\|\overline{\mathbf{u}} - \mathbf{w}\|$. Traditionally, this is done with careful experiments comparing approximations to $\mathbf{w}$ to averages of direct numerical simulations of $\mathbf{u}$ (necessarily at lower Reynolds numbers). The authors aim (in work in progress) to give a rigorous mathematical analysis of the modelling error in (2.10).

We believe that this is possible for the following reason. Let the model (2.10) be written abstractly as:

$$P_{\mathrm{div}}\left[\frac{\partial w}{\partial t} - \nabla \cdot (\mathrm{Re}^{-1}(\nabla \mathbf{w} + \nabla \mathbf{w}^t)) + \nabla \cdot (N(\mathbf{w})) - \overline{f}\right] = 0\,, \tag{4.1}$$

where P_{div} is the usual projection onto the divergence free functions (appropriately defined). Then, a comparable equation for $\overline{\mathbf{u}}$ can be written:

$$P_{\mathrm{div}}\left[\frac{\partial \overline{\mathbf{u}}}{\partial t} - \nabla \cdot (\mathrm{Re}^{-1}(\nabla \overline{\mathbf{u}} + \nabla \overline{\mathbf{u}}^t)) + \nabla \cdot N(\overline{\mathbf{u}}) - \overline{f}\right] = P_{\mathrm{div}}[\nabla \cdot (N(\overline{\mathbf{u}}) - \overline{\mathbf{u}\mathbf{u}})]\,. \tag{4.2}$$

Subtracting (4.1) and (4.2) gives an equation for $\mathbf{w} - \overline{\mathbf{u}}$. Note that this equation is driven by the closure approximation term $\nabla \cdot (N(\overline{\mathbf{u}}) - \overline{\mathbf{u}\mathbf{u}})$, which we believe can be agreeably bounded for any reasonable closure approximation. It would then remain to show that the deviation in trajectories in the large eddy model (2.10) can be bounded by the difference in their forcing functions. At precisely this step, we believe the enhanced stability properties of (2.10) versus (2.5) will allow an analysis of the modelling error.

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