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ATTRACTORS FOR THE BÉNARD PROBLEM: EXISTENCE AND PHYSICAL BOUNDS ON THEIR FRACTAL DIMENSION

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INTRODUCTION

Equations governing convection phenomena are dissipative. Therefore, even in the limit of an apparently totally chaotic flow, it is expected that the solutions of the Bénard problem have a finite-dimensional structure. In recent years, the concept of finite dimensionality of a turbulent flow has been clarified and made mathematically rigorous. The latest results include sharp estimates of the number of degrees of freedom for such flows. These estimates are very similar to the bounds obtained with purely physical arguments. Thus, in [8] it was proved for the first time that the attractor for the Navier–Stokes equations is finite-dimensional, while in [4, 5] there were derived sharp estimates of the fractal dimension for nonconvective turbulent flows. For other, earlier articles on this subject the reader is referred to the bibliographies in [4, 5, 8]. In the present article we generalize the idea and methods introduced in those references to the free convection problem. In doing so, we prove and extend a conjecture made in [19].

According to that conjecture, the number of parameters governing two-dimensional convection should be at most proportional to the Grashof number Gr (to be defined later). This was proved partially in [10] using the concept of determining modes to estimate the flow "dimension". However, that result is incomplete because it is limited to a particular boundary condition (space periodicity); and, furthermore, the estimate of the dimension contains a logarithmic factor, i.e. the dimension is the order of $cG\ (\log G)^{1/2}$. Here, instead of the determining modes, we consider the attractor or functional invariant set associated to the flow and we identify the number of degrees of freedom of the flow with the dimension of the attractor. Introducing the Prandtl and the Rayleigh numbers, Pr and Ra (to be defined later), that dimension is found here to be bounded by $cGr(1 + Pr)^2$, or by $c(Ra + Gr)(1 + Pr)$, where c is a constant depending only on the flow geometry. Besides, we consider in this article the three-dimensional case for which partial but significant results are obtained; here the limitations come from the fact that, as is the case for Navier–Stokes equations, the mathematical theory of the three-dimensional convection equations is not settled. As a result, it is not possible to estimate completely the solutions of these equations in terms of the data.

This article is divided into two parts. In the first one we prove the existence of attractors or functional invariant sets describing the long-time behavior (or "permanent regime") of the solutions of the convection problems, while the second one is devoted to the estimate of the fractal and Hausdorff dimensions of these attractors in terms of natural physical quantities. Part I starts in Section 1 by recalling the convection equations and their nondimensional form. In Section 2 we recall the functional setting of the equations and the basic properties of the solution (Sections 2.1 and 2.2); in Section 2.3 we describe other natural boundary conditions to which our results apply as well. In Section 3 we study the two-dimensional case for which we prove the existence of a maximal (universal) attractor, and discuss the three-dimensional case.

Part II is devoted to the estimate of the fractal and Hausdorff dimensions of the attractors. We start in Section 4 with a survey of definitions and results on the Lyapunov exponents and the fractal dimension or capacity of the attractors. Then we proceed in Section 5 with the actual estimate of the dimensions of the attractors for the two-dimensional convection; these estimates rely on the previous results and on a version of the Lieb–Thirring inequality (cf. [16]). The original Lieb–Thirring inequality and its extension presented hear appear as an improved form of the classical Sobolev imbedding theorems. Finally, in Section 6. the three-dimensional case is treated.

I. ATTRACTORS FOR THE BÉNARD PROBLEM

We recall the convection equations and we prove the existence of functional invariant sets in dimensions 2 and 3. In the two-dimensional case we also prove the existence of a universal attractor.

1. *The Bénard problem*

In this section, we recall the equations of the problem and obtain their nondimensional form. Although the nondimensionalization of the equations is not usually detailed in mathematical papers, this is necessary here since our aim is to obtain physically interesting bounds and, therefore, the precise meaning of the different constants will be needed at some point.

We consider in $\mathbb{R}^n$, where $n = 2$ or 3, an orthonormal basis $\{e_1, e_2\}$ or $\{e_1, e_2, e_3\}$, where e_n is parallel to the ascending vertical. A layer of homogeneous incompressible fluid fills the region $0 < x_n < h$, and is limited by rigid surfaces $x_n = 0$ and $x_n = h$. The fluid layer is heated from below in such a way that the lower plate is maintained at a temperature T_0 while the upper one is at $T_1 < T_0$, where T_0 and T_1 are two constants. It is well known that in the Boussinesq approximation the velocity $u = (u_1, u_2)$ or (u_1, u_2, u_3) of the fluid, the pressure p, and the temperature T are governed by

$$\rho_0\left(\frac{\partial}{\partial t} + (u\cdot\nabla)u\right) - \rho_0\nu\Delta u + \nabla p = \rho_0\mathbf{g}[1 + \alpha(T_1 - T)], \tag{1.1}$$

$$\rho_0 C_v\left(\frac{\partial T}{\partial t} + (u\cdot\nabla)T\right) - \rho_0 C_v\kappa\Delta T = 0, \tag{1.2}$$

where $\mathbf{g} = -ge_n$ is the gravity, $\rho_0 > 0$ is the constant mean density, α is the volume expansion coefficient of the fluid, C_v is the specific heat at constant volume, and ν and κ are the (constant) coefficients of kinematic viscosity and thermometric conductivity, respectively.

Dividing by ρ_0 and $\rho_0 C_v$, and replacing p by $(p/\rho_0) + gx_n$, we can rewrite these equations as

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - v\Delta u + \nabla p = g\alpha(T_1 - T), \tag{1.3}$$

$$\frac{\partial T}{\partial t} + (u \cdot \nabla)T - \kappa\Delta T = 0. \tag{1.4}$$

We supplement (1.3), (1.4) with a set of boundary conditions. At $x_n = 0$ and $x_n = h$, we set

$$u = 0 \quad \text{at} \quad x_n = 0 \quad \text{and} \quad x_n = h, \tag{1.5}$$

$$T = T_0 \quad \text{at} \quad x_n = 0, T = T_1 \quad \text{at} \quad x_n = h. \tag{1.6}$$

In the direction x_1 (or x_1, x_2 if $n = 3$), we impose a periodicity condition:

$$p, u, T, \partial u/\partial x_i, \partial T/\partial x_i \quad \text{are periodic in the } x_i \text{ directions}, \quad 1 \leqslant i \leqslant n-1, \tag{1.7}$$

which means

$$\left.\begin{aligned} \varphi|_{x_1=0} &= \varphi|_{x_1=l} && (n = 2, 3) \\ \varphi|_{x_2=0} &= \varphi|_{x_2=L} && (n = 3) \end{aligned}\right\} \tag{1.8}$$

for the corresponding functions φ. Other common boundary conditions will be considered in Section 5.

To make the equations nondimensional, we consider the following change of functions and variables:

$$\left.\begin{aligned} &x = hx', l = hl', L = hL', \\ &t = (h/g\alpha(T_0 - T_1))^{1/2}t', T = (T_0 - T_1)T', \\ &p = hg\alpha(T_0 - T_1)p', \\ &u = (hg\alpha(T_0 - T_1))^{1/2}u', \end{aligned}\right\} \tag{1.9}$$

$$\nu' = \frac{\nu}{(h^3 g\alpha(T_0 - T_1))^{1/2}}. \tag{1.10}$$

$$\kappa' = \frac{\kappa}{(h^3 g\alpha(T_0 - T_1))^{1/2}} = \sqrt{\frac{\kappa}{\nu}}\left(\frac{\kappa\nu}{h^3 g\alpha(T_0 - T_1)}\right)^{1/2}. \tag{1.11}$$

We now introduce the usual nondimensional numbers, i.e. Grashof (Gr), Prandtl (Pr), and Rayleigh (Ra):

$$\left.\begin{aligned} Gr &= (\nu')^{-2}, \\ Pr &= \frac{\nu'}{\kappa'}, \\ Ra &= (\nu'\kappa')^{-1}. \end{aligned}\right\} \tag{1.12}$$

We introduce the variables and quantities (1.9)–(1.11) in (1.3) and (1.8). Omitting the primes, we obtain

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = e_n(T - T_1), \tag{1.13}$$

$$\frac{\partial T}{\partial t} + (u \cdot \nabla)T - \kappa \Delta T = 0, \tag{1.14}$$

with the incompressibility condition

$$\operatorname{div} u = 0. \tag{1.15}$$

The boundary conditions are

$$u = 0 \quad \text{at} \quad x_n = 0 \quad \text{and} \quad x_n = 1, \tag{1.16}$$

$$T = T_0 \quad \text{at} \quad x_n = 0, T = T_1 = T_0 - 1 \quad \text{at} \quad x_n = 1, \tag{1.17}$$

and (1.7).

We can also subtract from T the pure conduction solutions and consider

$$\vartheta = T - T_0 - x_n(T_1 - T_0) = T - T_0 + x_n.$$

Then, again changing p to

$$p - (x_n + x_n^2/2)(T_0 - T_1) = p - (x_n + x_n^2/2),$$

we replace the equations above by

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = e_n \vartheta, \tag{1.18}$$

$$\frac{\partial \vartheta}{\partial t} + (u \cdot \nabla)\vartheta - u_n - \kappa \Delta \vartheta = 0, \tag{1.19}$$

$$\vartheta = 0 \quad \text{at} \quad x_n = 0 \quad \text{and} \quad x_n = 1, \tag{1.20}$$

$$\text{(1.15), (1.16) hold as does (1.7) with } T \text{ replaced by } \vartheta. \tag{1.21}$$

The two sets of equations above constitute the final form of the Bénard problem equations that we will study.

2. *Functional setting and properties of the solution*

We recall here the functional setting of equations (1.18)–(1.21), and the basic results on the existence and uniqueness of solutions (Section 2.1): full details can be found in [11]. In Section 2.2 we derive a useful variant of the maximum principle and in Section 2.3 we discuss briefly other natural boundary conditions and how our results can be adapted to those situations.

2.1. *Functional setting.* We denote by Ω the set $(0, l) \times (0, 1)$ if $n = 2$ and the set $(0, l) \times (0, L) \times (0, 1)$ if $n = 3$, and we introduce the Hilbert spaces $V = V_0 \times V_1$, $H = H_0 \times H_1$, where $V_1 =$ the space of functions in $H^1(\Omega)$, which are vanishing at $x_n = 0$ and

$x_n = 1$ and periodic in the direction x_i, $1 \leq i \leq n-1$. This is a Hilbert space for the scalar product and the norm

$$((u, v)) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x, \qquad \|u\| = \{((u, u))\}^{1/2}.$$

We consider also

$$V_0 = \{v \in V_1^n, \operatorname{div} v = 0\}$$

and $V = V_0 \times V_1$; we denote by $((\cdot, \cdot))$ and $\|\cdot\|$ the canonical scalar product and norm in V_0 or in V; this should not be a source of confusion.

Let $H_1 = L^2(\Omega)$ be endowed with the usual scalar product and norm

$$(u, v) = \int_\Omega u(x) v(x) \, \mathrm{d}x, \qquad |u| = \{(u, u)\}^{1/2},$$

and consider

$$H_0 = \{v \in L^2(\Omega)^n, \quad \operatorname{div} v = 0, \quad v_n|_{x_n=0} = v_n|_{x_n=1} = 0,$$
$$v_i \text{ is periodic in the } x_i \text{ direction}, \quad 1 \leq i \leq n-1\}$$

with $H = H_0 \times H_1$; we denote again by $(\cdot, \cdot)$ and $|\cdot|$ the scalar product and the norm on H_1 and that of H corresponding to the product Hilbert structure.

Let $D(A) = D(A_0) \times D(A_1)$, where $D(A_0) = V_0 \cap H^2(\Omega)^n$, $D(A_1) = V_1 \cap H^2(\Omega)$ and let A_i be the unbounded linear operator from $D(A_i)$ into H_i defined by

$$(A_i u, v) = ((u, v)), \quad \forall u, v \in D(A_i) \qquad (i = 0, 1). \tag{2.1}$$

Classically A_i is self-adjoint positive and A_i^{-1} is a compact (self-adjoint) linear operator in H_i. We consider also the bilinear operators B_0 and B_1 respectively mapping $D(A_0) \times D(A_0)$ into H_0 and $D(A_0) \times D(A_1)$ into H_1,† defined by

$$(B_0(u, v), w) = \int_\Omega [(u \cdot \nabla) v] w \, \mathrm{d}x, \qquad \forall u, v, w \in D(A_0), \tag{2.2}$$

$$(B_1(u, \varphi), \psi) = \int_\Omega [(u \cdot \nabla) \varphi] \psi \, \mathrm{d}x, \qquad \forall u \in D(A_0), \forall \varphi, \psi \in D(A_1). \tag{2.3}$$

With these notations, the functional setting of (1.18)–(1.21) is

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu A_0 u + B_0(u, u) - e_n \vartheta = 0, \tag{2.4}$$

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}t} + \kappa A_1 \vartheta + B_1(u, \vartheta) - u_n = 0. \tag{2.5}$$

Alternatively, introducing $\varphi = \{u, \vartheta\}$, we write this system as

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} + A\varphi + B(\varphi) + R\varphi = 0, \tag{2.6}$$

† B_0 and B_1 enjoy other continuity properties which will be recalled as needed. These properties are not the same for the space dimensions $n = 2$ or $n = 3$, and actually are the very reason for the difference in the results for $n = 2$ or $n = 3$.

where

$$
\begin{aligned}
A\varphi &= \{\nu A_0 u, \kappa A_1 \vartheta\}, \\
R\varphi &= \{-e_n \vartheta, -u_n\}, \\
B(\varphi) &= B(\varphi, \varphi)
\end{aligned}
\tag{2.7}
$$

with $B(\varphi, \varphi^*) = \{B_0(u, u^*), B_1(u, \vartheta^*)\}$ for all $\varphi = \{u, \vartheta\}$, $\varphi^* = \{u^*, \vartheta^*\}$, and we supplement (2.6) with the initial condition

$$\varphi(0) = \varphi_0, \tag{2.8}$$

$\varphi_0 = \{u_0, \vartheta_0\}$ given.

The results concerning the existence of a solution for (2.6), (2.8) are classical: if $n = 2$, then, for every φ_0 in H, (2.6)–(2.8) possesses a unique solution

$$\varphi \in C([0, \tau]; H) \cap L^2(0, \tau; V) \qquad \forall \tau > 0. \tag{2.9}$$

In fact, φ is more regular for $t > 0$ (regularizing property‡); at least

$$\varphi \in C(]0, \infty[; D(A)). \tag{2.10}$$

The mapping

$$S(t) : \varphi(0) \rightarrow \varphi(t) \tag{2.11}$$

is then well defined from H into $D(A)$ and it enjoys the usual semigroup property

$$S(t + s) = S(t) \cdot S(s), \qquad \forall s, t > 0. \tag{2.12}$$

If $n = 3$, for φ_0 given in V, (2.6) (2.8) possesses a unique solution on some interval $[0, \tau_1]$,

$$\varphi \in C([0, \tau_1], V) \cap L^2(0, \tau_1; D(A)) \tag{2.13}$$

where $\tau_1 = \tau_1(M)$ depends on a bound of the V norm of φ_0:

$$\|\varphi_0\| \leq M. \tag{2.14}$$

We also have

$$\varphi \in C(]0, \tau_1[; D(A)). \tag{2.15}$$

Thus, the mapping $S(t)$ from the balls

$$\{v \in V, \qquad \|v\| \leq M\},$$

of V into V is only defined for $0 \leq t \leq \tau_1(M)$ $(\forall M > 0)$, and enjoys the property (2.12) for $s + t \leq \tau_1(M)$.

All these results extend other results that are well known for the Navier–Stokes equations (see [22]) and the details can be found, for instance, in [11].

2.2. *A variant of the maximum principle.* We now give a variant of the maximum principle for ϑ.

‡ In fact, φ is C^∞ in $\Omega \times]0, \infty[$.

LEMMA 2.1. We assume that ϑ, u satisfy (2.6), (2.8), and (2.9) or (2.13), and that

$$-1 \leqslant \vartheta_0(x) \leqslant 1, \qquad \text{a.e. } x \in \Omega. \tag{2.16}$$

Then

$$-1 \leqslant \vartheta(x, t) \leqslant 1, \qquad \text{a.e. } x \in \Omega, \text{ a.e. } t. \tag{2.17}$$

If $\{\vartheta, u\}$ are defined for all $t > 0$* and (2.16) is not assumed, then

$$\vartheta(\cdot, t) = \tilde{\vartheta}(\cdot, t) + \bar{\vartheta}(\cdot, t), \tag{2.18}$$

where $-1 \leqslant \tilde{\vartheta}(x, t) \leqslant 1$ a.e., and

$$\bar{\vartheta}(\cdot, t) \to 0 \quad \text{in} \quad H_1(=L^2(\Omega)) \quad \text{as} \quad t \to \infty. \tag{2.19}$$

Proof. The results are proved more naturally on the equation for T, i.e. (1.16). The existence and uniqueness results above are easily reinterpreted in terms of u and ϑ. In terms of T, (2.16), (2.17) amount to

$$T_1 \leqslant T(x, 0) \leqslant T_0, \qquad \text{a.e. } x \in \Omega, \tag{2.20}$$

$$T_1 \leqslant T(x, t) \leqslant T_0, \qquad \text{a.e. } x \in \Omega, \quad \text{a.e.} \quad t. \tag{2.21}$$

In order to establish the second inequality in (2.21), we consider the truncated function $(T - T_0)_+$,† which belongs to $L^2(0, \tau; H^1(\Omega))$, vanishes at $x_n = 0$ and $x_n = 1$, and is space-periodic in the directions x_i, $1 \leqslant i \leqslant n-1$.

Multiplying (1.14) by $(T - T_0)_+$ and integrating over Ω, we obtain, after some standard operations,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|(T - T_0)_+|^2 + \kappa \|(T - T_0)_+\|^2 = 0.$$

Hence, using Poincaré's inequality in the x_n direction,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|(T - T_0)_+|^2 + \kappa |(T - T_0)_+|^2 \leqslant 0. \tag{2.22}$$

For (2.21) we observe that $|(T - T_0)_+(t)|$ is a decreasing function of t which vanishes at $t = 0$ and, therefore, it vanishes for all later time $t > 0$; thus $T(\cdot, t) \leqslant T_0$, $\forall t \geqslant 0$. For the proof of the first inequality in (2.21), we consider $(T - T_1)_-$ and proceed similarly.

If we do not assume (2.16), (2.20), we conclude from (2.22) that $|(T - T_0)_+(t)|$ decreases exponentially:

$$|(T - T_0)_+(t)| \leqslant |(T - T_0)_+(0)| \exp(-\kappa t). \tag{2.23}$$

Similarly, one can prove that

$$|(T - T_1)_-(t)| \leqslant |(T - T_0)_-(0)| \exp(-\kappa t). \tag{2.24}$$

Thus, setting

$$T = \tilde{T} + \bar{T}, \qquad \bar{T} = (T - T_0)_+ - (T - T_1),$$

* This is always the case in dimension 2.
† Here $M_+ = \max(M, 0)$, $M_- = \max(-M, 0)$, $\forall M = \text{real}$.

we see that $T_1 \leq \bar{T}(x,t) \leq T_0$ almost everywhere and $\bar{T}(\cdot,t) \to 0$ in $L^2(\Omega)$ as $t \to \infty$:

$$|\bar{T}(\cdot,t)| \leq \{|(T-T_0)_+(0)| + |(T-T_1)_-(0)|\}\exp(-\kappa t). \tag{2.25}$$

Then (2.18), (2.19) is just a rephrasing of this result in terms of ϑ and (2.25) becomes

$$|\bar{\vartheta}(\cdot,t)| \leq \{|(\vartheta-1)_+(0)| + |(\vartheta+1)_-(0)|\}\exp(-\kappa t). \tag{2.26}$$

2.3. *Other boundary conditions.* We conclude this section by showing how one can handle other natural boundary conditions: we describe them and show how the definition of the function spaces should be modified appropriately. All the subsequent results (and the maximum principle of lemma 2.1) are equally valid for these cases.

Case 1. This being the case discussed before, we can consider the following.

Case 2. The top surface is free instead of being rigid.

Then the boundary conditions for T are unchanged (see [3]) and for u we have

$$u = 0 \quad \text{at} \quad x_n = 0 \quad \text{and} \quad u_n = 0 \quad \text{at} \quad x_n = h\,(=1), \tag{2.27}$$

$$\frac{\partial u_i}{\partial x_n} = 0 \quad \text{at} \quad x_n = h, \quad i = 1 \quad \text{if} \quad n = 2, i = 1, 2 \quad \text{if} \quad n = 3, \tag{2.28}$$

(no penetration and no tangential stresses).

In this case the spaces V_1, H_1, $D(A_1)$ are the same while

V_0 = the space of functions $v \in H^1(\Omega)^n$, such that $\operatorname{div} v = 0$ and $v = 0$ at $x_n = 0$, $v_n = 0$ at $x_n = 1$, and which are periodic in the direction x_1 if $n = 2$, or x_1, x_2 if $n = 3$;

$H_0 = \{v \in L^2(\Omega)^n$, $\operatorname{div} v = 0$, $v_n|_{x_n=0} = v_n|_{x_n=1} = 0$, v is periodic in the direction x_1 if $n = 2$, or x_1, x_2 if $n = 3\}$,

$$D(A_0) = \left\{v \in H^2(\Omega)^n \cap V_0, \frac{\partial v_i}{\partial x_n} = 0 \quad \text{at} \quad x_n = 1, \quad 1 \leq i \leq n-1\right\}.$$

We can also take both surfaces (top and bottom) to be free; the modifications are easy.

Case 3. In either case 1 or 2, we can change the boundary conditions in the directions x_i, $1 \leq i \leq n-1$ and replace periodicity by fixed walls. For instance, on the walls $x_1 = 0$ or l ($n = 2, 3$) and $x_2 = 0$ or L, we can assume that

$$u = 0, \tag{2.29}$$

$$T = T_0 + x_n = \text{the pure conduction solution}. \tag{2.30}$$

In this case (and assuming that the boundary conditions are those of case 1 at $x_n = 0$ and $x_n = h$ $(=1)$), we set

$$V_1 = H_0^1(\Omega) = \{v \in H^1(\Omega), \quad v = 0 \text{ on } \partial\Omega\},$$

$$V_0 = \{v \in H_0^1(\Omega)^n, \quad \operatorname{div} v = 0\},$$

$$H_1 = L^2(\Omega),$$

$$H_0 = \{v \in L^2(\Omega)^n, \quad \operatorname{div} v = 0, \quad v \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega\}, \ \boldsymbol{\nu} \text{ the unit outward normal on } \partial\Omega,$$

$$D(A_1) = H^2(\Omega) \cap V_1,$$

$$D(A_0) = H^2(\Omega)^n \cap V_0.$$

Case 4. Further, we can consider the thermally driven cavity [21]. We assume that the walls of the cell are rigid like in case 3. Then $u = 0$ on $\partial\Omega$, $T = T_0$ at $x_1 = 0$, $T = T_1$ at $x_1 = l$ and T linearly interpolates these values on the other faces of the cell, $T = T_0 + (x_1/l)(T_1 - T_0)$. The spaces are the same as in case 3, and the difference lies in the definition of ϑ and R: the pure conduction solution is $T = T_0 + (x_1/l)(T_1 - T_0)$, and $\vartheta = T - T_0 - (x_1/l)(T_1 - T_0)$. Then (1.18) is unchanged (x_n is still the vertical direction) but (1.19) is replaced by

$$\frac{\partial\vartheta}{\partial t} + (u \cdot \nabla)\vartheta - \frac{1}{l}u_1 - \kappa\Delta\vartheta = 0. \tag{2.31}$$

Thus, in (2.6), $R\varphi = \{-e_n\vartheta, -(1/l)u_1\}$ for $\varphi = \{u, \vartheta\}$.

3. *The universal attractor*

In the 2-dimensional case we establish some technical time uniform *a priori* estimates. Then we prove the existence of the universal attractor to which every solution converges as $t \to \infty$.

3.1. *Uniform bounds on the solutions.* For the moment $n = 2$ or 3.

First we aim to establish time uniform estimates for the solutions of the convection equations (1.18)–(1.21).

The first estimate for ϑ, uniform in time, follows from (2.18)–(2.19) in lemma 2.1. We denote by $|\vartheta|_\infty$ the norm of ϑ in $L^\infty(0, \infty; H_1)$. Due to (2.18)–(2.26),

$$\begin{aligned} |\vartheta(t)| &\leq |\bar{\vartheta}(t)| + |\bar{\vartheta}(t)| \\ &\leq |\Omega|^{1/2} + \{|(\vartheta - 1)_+(0)| + |(\vartheta + 1)_-(0)|\}\exp(-\kappa t), \end{aligned} \tag{3.1}$$

$$|\vartheta|_\infty \leq |\Omega|^{1/2} + \{|(\vartheta - 1)_+(0)| + |(\vartheta + 1)_-(0)|\}, \tag{3.2}$$

$$\limsup_{t\to\infty} |\vartheta(t)| \leq |\Omega|^{1/2}, \tag{3.3}$$

where $|\Omega|$ = the volume of Ω, which equals l if $n = 2$, lL if $n = 3$.

We obtain other energy type relations on the solutions by taking the scalar product in H_0 of (2.4) with u and the scalar product in H_1 of (2.5) with ϑ. Due to the orthogonality properties [22],

$$(B_0(\varphi, \psi), \psi) = 0 \qquad \text{if } \varphi, \psi \in V_0, \tag{3.4}$$

$$(B_1(\varphi, \vartheta), \vartheta) = 0 \qquad \text{if } \varphi \in V_0, \quad \forall\vartheta \in V_1, \tag{3.5}$$

we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \nu\|u\|^2 - (\vartheta, u_n) = 0, \tag{3.6}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\vartheta|^2 + \kappa\|\vartheta\|^2 - (\vartheta, u_n) = 0. \tag{3.7}$$

Then (3.6) implies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \nu\|u\|^2 \leqslant \int_\Omega |\vartheta|\,|u_n|\,\mathrm{d}x \leqslant |\vartheta|\,|u|. \tag{3.8}$$

Using Poincaré inequality in the x_n direction, we find that

$$|\varphi| \leqslant \|\varphi\|, \qquad \forall\varphi \in V_0 \text{ or } V_1. \tag{3.9}$$

Whence (3.8) implies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \nu\|u\|^2 \leqslant \frac{\nu}{2}\|u\|^2 + \frac{|\vartheta|^2}{2\nu},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \nu\|u\|^2 \leqslant \frac{|\vartheta|^2}{\nu}, \tag{3.10}$$

$$|u(t)|^2 \leqslant |u_0|^2\exp(-\nu t) + \frac{|\vartheta|_\infty^2}{\nu^2}(1-\exp(-\nu t)), \tag{3.11}$$

and, with (3.3),

$$\limsup_{t\to\infty} |u(t)|^2 \leqslant \frac{|\Omega|}{\nu^2}. \tag{3.12}$$

Then, integrating (3.10),

$$|u(t)|^2 + \nu\int_0^t \|u(s)\|^2\,\mathrm{d}s \leqslant |u_0|^2 + \frac{1}{\nu}\int_0^t |\vartheta(s)|^2\,\mathrm{d}s,$$

$$\frac{\nu}{t}\int_0^t \|u(s)\|^2\,\mathrm{d}s \leqslant \frac{1}{\nu t}|u_0|^2 + \frac{1}{\nu t}\int_0^t |\vartheta(s)|^2\,\mathrm{d}s, \quad \limsup_{t\to\infty}\frac{\nu}{t}\int_0^t \|u(s)\|^2\,\mathrm{d}s \leqslant \frac{|\Omega|}{\nu}. \tag{3.13}$$

Then, for ϑ, we infer from (3.7)–(3.9) that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\vartheta|^2 + 2\kappa\|\vartheta\|^2 \leqslant 2|\vartheta||u| \leqslant 2|\vartheta|\,\|u\|, \; \frac{2\kappa}{t}\int_0^t \|\vartheta(s)\|^2\,\mathrm{d}s \leqslant \frac{|\vartheta(0)|^2}{t} + \frac{2}{t}\int_0^t |\vartheta(s)|\,\|u(s)\|\,\mathrm{d}s,$$

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \|\vartheta(s)\|^2\,\mathrm{d}s \leqslant \frac{1}{\kappa}\left\{\limsup_{t\to\infty}|\vartheta(t)|\right\}\left\{\limsup_{t\to\infty}\frac{1}{t}\int_0^t \|u(s)\|^2\,\mathrm{d}s\right\}^{1/2},$$

$$\limsup_{t\to\infty}\frac{\kappa}{t}\int_0^t \|\vartheta(s)\|^2\,\mathrm{d}s \leqslant \frac{|\Omega|}{\nu}. \tag{3.14}$$

Remark 3.1. Note that (3.13) and (3.14) give comparable upper bounds for the dissipation of the mechanical flow energy [i.e. the left-hand side of (3.13)] and the mean entropy production due to heat-dissipation [i.e. the left-hand side of (3.14)].

3.2. *Existence of an absorbing set* ($n = 2$). Unless otherwise stated, it is assumed in the subsections 3.2 and 3.3 that $n = 2$. We recall the following.

Definition. A subset $\mathscr{A}$ of V is said to be absorbing in V, for the semigroup $S(t)$, if for every bounded set $\mathscr{B}$ of V, there exists $t = t(\mathscr{B})$ such that

$$S(t)\mathscr{B} \subset \mathscr{A}, \qquad \forall t \geqslant t(\mathscr{B}). \tag{3.15}$$

Proposition 3.1. There exists a V-bounded absorbing set $\mathscr{A}$ in V, for the semigroup $S(t)$.

The proof relies on technical *a priori* estimates.
We assume that

$$\mathscr{B} \subset \{(u, \vartheta) \in V, \quad \|u\| \leqslant M_0, \|\vartheta\| \leqslant M_1\}$$

and given $\varepsilon > 0$, we infer from (3.1), (3.12) that there exists $t_0 = t_0(\mathscr{B}, \varepsilon)$ such that for $t \geqslant t_0$,

$$|\vartheta(t)| \leqslant |\Omega|^{1/2} + \varepsilon \tag{3.16}$$

$$|u(t)| \leqslant \frac{|\Omega|^{1/2}}{\nu} + \varepsilon. \tag{3.17}$$

We take the scalar product of (2.4) with $A_0 u$ in the space $H_0{}^*$:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \nu|A_o u|^2 &= (\vartheta, u_n) - (B_0(u, u), A_0 u)\\ &\leqslant (\vartheta, u_n) - (B_0(u, u), A_0 u)\\ &\leqslant |\vartheta|\,|u| + |B_0(u, u)|\,|A_0 u|\\ &\leqslant |\vartheta|\,|u| + c_1|u|^{1/2}\|u\|\,|A_0 u|^{3/2}\\ &\leqslant \text{(with Young's inequality)}\\ &\leqslant |\vartheta|\,|u| + \frac{\nu}{2}|A_0 u|^2 + \frac{c_1'}{\nu^3}|u|^2\|u\|^4.\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \nu|A_0 u|^2 &\leqslant 2|\vartheta|\,|u| + \frac{2c_1'}{\nu^3}|u|^2\,\|u\|^4 \quad \leqslant (\text{for } t \geqslant t_0) \leqslant\\ &\leqslant 2(|\Omega|^{1/2} + \varepsilon)\left(\frac{|\Omega|^{1/2}}{\nu} + \varepsilon\right) + \frac{2c_1'}{\nu^3}\left(\frac{|\Omega|^{1/2}}{\nu} + \varepsilon\right)^2\|u\|^4.\end{aligned} \tag{3.18}$$

By integration of (3.8) between t and $t + 1$,

$$|u(t+1)|^2 + 2\nu\int_t^{t+1}\|u(s)\|^2\,\mathrm{d}s \leqslant |u(t)|^2 + 2\int_t^{t+1}|\vartheta(s)|\,|u(s)|\,\mathrm{d}s,$$

and, for $t \geqslant t_0$,

$$\int_t^{t+1}\|u(s)\|^2\,\mathrm{d}s \leqslant \frac{1}{2\nu}\left(\frac{|\Omega|^{1/2}}{\nu} + \varepsilon\right)^2 + \frac{1}{\nu}(|\Omega|^{1/2} + \varepsilon)\left(\frac{|\Omega|^{1/2}}{\nu} + \varepsilon\right). \tag{3.19}$$

* We have (see for instance [23]), $|B_0(\varphi, \varphi)| \leqslant c_1|\varphi|^{1/2}\|\varphi\|\,|A_0\varphi|^{1/2}$, $\forall\varphi \in D(A_0)$.

We then apply the uniform Gronwall lemma (see below) to (3.18), with $\alpha = 1$, $y = \|u\|^2$, $a_3 = a_3(\varepsilon)$, the right-hand side of (3.19),

$$h \equiv a_2(\varepsilon) = 2(|\Omega|^{1/2} + \varepsilon)\left(\frac{|\Omega|^{1/2}}{\nu} + \varepsilon\right),$$

$$g = \frac{2c_1'}{\nu^3}\left(\frac{|\Omega|^{1/2}}{\nu} + \varepsilon\right)\|u\|^2,$$

$$a_1 = a_1(\varepsilon) = \frac{2c_1'}{\nu^3}\left(\frac{|\Omega|^{1/2}}{\nu} + \varepsilon\right) a_3(\varepsilon).$$

We conclude from (3.23) that

$$\|u(t)\|^2 \leqslant (a_2(\varepsilon) + a_3(\varepsilon))\exp(a_1(\varepsilon)) \tag{3.20}$$

for $t \geqslant t_0(\mathscr{B}, \varepsilon) + 1$. Thus we can choose as set $\mathscr{A}$, a set $\mathscr{A}_0 \times \mathscr{A}_1$, where

$$\mathscr{A}_0 = \left\{\varphi \in V_0, \|\varphi\| \leqslant \sqrt{a_2(\varepsilon) + a_3(\varepsilon)}\exp\left(\frac{a_1(\varepsilon)}{2}\right)\right\},$$

and $\varepsilon > 0$ is arbitrarily small.

For the component ϑ, we take the scalar product of (2.5) with $A_1\vartheta$ in H_1 and proceed in a totally parallel way; we omit the details.

LEMMA 3.1 (UNIFORM GRONWALL LEMMA). Let g, h, y be three positive locally integrable functions on $]t_0, +\infty[$ which satisfy

$$\frac{\mathrm{d}y}{\mathrm{d}t} \leqslant gy + h \qquad \text{for} \quad t \geqslant t_0. \tag{3.21}$$

$$\left.\begin{aligned} &\int_t^{t+\alpha} g(s)\,\mathrm{d}s \leqslant a_1, \\ &\int_t^{t+\alpha} h(s)\,\mathrm{d}s \leqslant a_2, \quad \forall t \geqslant t_0. \\ &\int_t^{t+\alpha} y(s)\,\mathrm{d}s \leqslant a_3, \end{aligned}\right\} \tag{3.22}$$

where α, a_1, a_2, a_3, are positive constants. Then

$$y(t+\alpha) \leqslant \left(\frac{a_3}{\alpha} + a_2\right)\exp(a_1), \qquad \forall t \geqslant t_0. \tag{3.23}$$

Proof. Assume that $t_0 \leqslant t \leqslant s < t + \alpha$. We deduce from (3.21) that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(y(s)\exp\left(-\int_t^s g(\tau)\,\mathrm{d}\tau\right)\right) \leqslant h(s)\exp\left(-\int_t^s g(\tau)\,\mathrm{d}\tau\right) \leqslant h(s).$$

Then by integration between s and $t+\alpha$,

$$y(t+\alpha) \leq y(s)\exp\left(\int_s^{t+\alpha} g(\tau)\,\mathrm{d}\tau\right) + \left(\int_s^{t+\alpha} h(\tau)\,\mathrm{d}\tau\right)\exp\left(\int_s^{t+\alpha} g(\tau)\,\mathrm{d}\tau\right)$$
$$\leq (y(s)+a_2)\exp(a_1).$$

Integration of this last inequality with respect to s between t and $t+\alpha$ gives precisely (3.23).

Remark 3.2. The proof of proposition 3.1 shows actually that $\mathcal{A}$ is also absorbing in H: for every bounded set $\mathcal{B} \subset H$, $S_t\mathcal{B}$ is included and bounded in V, $\forall t>0$, whence $S_t\mathcal{B} \subset \mathcal{A}$ for $t>t(\mathcal{B})$.

Remark 3.3. We can prove (as in [8, 12, 19] for the Navier–Stokes equations), that the solution $u(t)$, $\vartheta(t)$ of the two-dimensional convection problem is analytic in time with values in $D(A)$, and that all the H^m norms of $u(t)$, $\vartheta(t)$ remain uniformly bounded in time for $t \geq \delta > 0$. For the complete proofs see [11].

3.3. *The two-dimensional case.* We recall the following.

Definition. A functional invariant set for the semigroup $\{S(t)\}_t$ is a set $X \subset H$ such that

$$S(t)X = X, \qquad \forall t>0. \tag{3.24}$$

Such a set is said to be attracting in H (or V or $D(A)$), if it possesses a neighborhood $\mathcal{O}$ in H (or V or $D(A)$), such that

$$\operatorname{dist}(S(t)\varphi_0, X) \to 0 \qquad t\to\infty, \tag{3.25}$$

$\forall\varphi_0 \in \mathcal{O}$, where the distance is respectively taken in H or V or $D(A)$.

Note that if X is a functional invariant set, then $S(t)\varphi_0$ exists $\forall t \in \mathbb{R}$, $\forall \varphi_0 \in X$, i.e. the initial value problem for the convection equation is well posed for all time forward and backward, $\forall \varphi_0 \in X$. Indeed, because of the backward uniqueness of solutions of (2.6) (see Ghidaglia [11], Bardos and Tartar [2]), the semigroup operators $S(t)$ are *one-to-one*, and (3.24) implies

$$S(t)X = X \qquad \forall t \in \mathbb{R}, \tag{3.26}$$

for any functional invariant set X. Note also that, due to the regularizing properties of the semigroup $S(t)$, a functional set X bounded in H is included and bounded in V (proposition 3.1).

In fact, from the regularity results for the conduction equations, we infer the following (see [11] for the proof and [13], [23] for the Navier–Stokes equations).

Proposition 3.2. Any functional invariant set for the semigroup $\{S(t)\}_t$, which is bounded in H if $n=2$ (respectively in V if $n=3$) is in fact included in $C^\infty(\bar{\Omega})^n \times C^\infty(\bar{\Omega})$ and is bounded for all the norms H^m, $m \geq 0$.

Returning to the absorbing set $\mathscr{A}$ introduced in proposition 3.1, we consider the ω-limit set of $\mathscr{A}$, $\bar{X} = \omega(\mathscr{A})$, that is to say,

$$\bar{X} = \bigcap_{s \geq 0} \bigcup_{t \geq s} \overline{S(t)\mathscr{A}}, \tag{3.27}$$

where the closures are taken in H. It is easy to see that $\psi \in \bar{X}$ if and only if there exists a sequence $t_m \to \infty$, and a sequence φ_m of elements of $\mathscr{A}$ such that

$$S(t_m)\varphi_m \to \psi \quad \text{in} \quad H, \qquad \text{as} \quad m \to \infty. \tag{3.28}$$

It is classical to show that $\bar{X}$ is a functional invariant set $S(t)\bar{X} = \bar{X} \quad \forall t > 0$ (and thus $\forall t \in \mathbb{R}$) that attracts all the trajectories and the bounded sets of V (or H; see remark 3.1). Also $\bar{X}$ contains all the functional invariant sets for $S(t)$ and, by proposition 3.2, $\bar{X}$ is included in $C^\infty(\bar{\Omega})^2 = C^\infty(\bar{\Omega})$. Finally, we have proved the main result of this section.

THEOREM 3.1. When the dimension of space is $n = 2$, there exists a closed bounded set $\bar{X}$ of V which is an attractor for the semigroup. This set contains all the functional invariant sets for $S(t)$ which are bounded in V (or H). It lies in $C^\infty(\bar{\Omega})^2 \times C^\infty(\bar{\Omega})$.

This set $\bar{X}$ will be called the *maximal (or universal) attractor*: For the corresponding data it is the largest attractor, and it contains all the trajectories describing all the permanent flows that can be observed.

Remark 3.4. From the proof of lemma 2.1 (see especially (2.25)) and from the definition (3.27) of $\bar{X}$ we readily infer that

$$|\vartheta(x)| \leq 1 \quad \text{a.e. in } \Omega$$

for all $\varphi = \{u, \vartheta\} \in \bar{X}$. This fact is also valid for any functional invariant set and is actually used in the proof of proposition 3.2.

3.4. *The three-dimensional case.* In the three-dimensional case we are not able to prove the existence of an absorbing set or a maximal attractor in V, since the initial value problem for (2.6) is not solved completely. However, some useful partial results can be proved.

In dimension 3, the definition of a functional invariant set itself must be modified. According to (2.13)–(2.15), the mappings $S(t)$: $\varphi(0) \mapsto \varphi(t)$ from V into itself are not defined for any $\varphi_0 \in V$ and any $t > 0$. We say that $S(t)\varphi_0$ exists for $t \in [0, \tau]$ if there exists φ which satisfies (2.13) with τ_1 replaced by τ, (2.6) (on $]0, \tau[$) and (2.8). It follows from (2.13)–(2.15) that for every φ_0 in the ball $\{\psi \in V, \|\psi\| \leq M\}$, $S(t)\varphi_0$ is at least defined for $t \in [0, \tau_1(M)]$. We then have the following.

Definition. A functional invariant set for the semigroup $\{S(t)\}_t$ is a set $X \subset V$ such that

$$S(t)\varphi_0 \quad \text{is defined} \quad \forall \varphi_0 \in X, \qquad \forall t > 0, \tag{3.29}$$

$$S(t)X = X, \qquad \forall t > 0. \tag{3.30}$$

Note that if X is bounded in V (which we will assume most often), then $X \subset \{\varphi_0 \in V, \|\varphi_0\| \leq M\}$ for some M and $S(t)X$ is well defined for every $t \in [0, \tau_1(M)]$. As observed in the two-dimensional case, and since $S(t)$ is one-to-one for $t \in [0, \tau_1(M)]$, (3.30) implies that $S(t)X = X$, $\forall t < 0$ as well.

We are not able to prove the existence of functional invariant sets (and *a fortiori* attractors) in the most general three-dimensional case. Nevertheless, some interesting results can be obtained if we consider a particular solution $\varphi = \{u, \vartheta\}$ of (2.6), (2.8), which enjoys the boundedness property

$$\sup_{t>0} \|u(t)\| < \infty, \tag{3.31}$$

assuming that such a solution exists. Indeed, one can prove as in the two-dimensional case (see the proof of proposition 3.1) that

$$\sup_{t>0} \|\vartheta(t)\| < \infty, \tag{3.32}$$

whence

$$\sup_{t>0} \|\varphi(t)\| \leq M < +\infty, \tag{3.33}$$

and we have the following.

THEOREM 3.2. Let $\varphi = \{u, \vartheta\}$ be a solution of (2.6), (2.8), which satisfies (3.31). Then there exists a functional invariant set $X = X(\varphi_0)$ for $\{S_t\}_t$, which is bounded in V, and $\varphi(t)$ converges in H to X, as $t \to \infty$.

The set $X(\varphi_0)$ is actually included in $C^\infty(\bar{\Omega})^3 \times C^\infty(\bar{\Omega})$.

Proof. The set $X(\varphi_0)$ is the ω-limit set of φ_0,

$$X(\varphi_0) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \{S(t)\varphi_0\}},$$

where the closures are taken in H. This set is bounded in V by (3.33) and closed in H, hence it is compact in H. Exactly as in [8], it is easy to see that $X(\varphi_0)$ is invariant with respect to $S(t)$: the assertion is proved first for $0 \leq t \leq \tau_1(M)$, and then (3.30) follows by iteration. The convergence of $\varphi(t)$ to $X(\varphi_0)$ as $t \to \infty$ is shown by contradiction—in the contrary case we would have a sequence $t_n \to \infty$ with dist $(\varphi(t_n), X(\varphi_0)) \geq \eta$ for some $\eta > 0$. But $\{\varphi(t_n)\}$ is bounded in V and compact in H; hence it contains a subsequence t_{n_i} such that $\varphi(t_{n_i}) \to \psi$ as $n_i \to \infty$. By a characterization of $X(\varphi_0)$ similar to (3.28), we conclude that $\psi \in X(\varphi_0)$ and the contradiction follows.

Finally, the regularity result $X(\varphi_0) \subset C^\infty(\bar{\Omega})$ follows from proposition 3.2.

Remark 3.5. When $n = 3$, if we assume (or *if we know*) that the initial value problem for (2.6) is well posed for all time and all initial values $\varphi_0 \in V$, (i.e., $\tau_1 = \tau_1(M) = +\infty\, \forall M$, in (2.13)–(2.15)), then we can establish a result stronger than theorem 3.2 and similar to theorem 3.1.

Indeed, we first prove in this case that there exists an absorbing set $\mathcal{A}$ bounded in V, $\mathcal{A} = \mathcal{A}_0 \times \mathcal{A}_1$. In order to prove the existence of $\mathcal{A}_0$ we proceed essentially as for theorem 1.1 of [4]* (see also [4, remark 3.2] and [23, p. 71]), and use the calculations of Section 3.1 (which

* The authors wish to thank J. Hale for pointing out to them that, in the case of the 3-dimensional Navier–Stokes equations, and after a minor modification in the proof, theorem 1.1 of [4] implies the existence of an absorbing set (and thus a maximal attractor), if the initial value problem is well posed for all initial data in H^1, i.e. if there is no singularity in the sense of Leray [15]. The existence for any given equation of an absorbing set is referred to as a (bounded) dissipativity property of the equation in Hale [14] and other articles quoted therein.

are still valid), as well as (3.16), (3.17), and (3.19) (which do not depend on the rest of proposition 3.1). Once we have established the existence of $\mathscr{A}_0$, we finish as in proposition 3.1. It is then clear that the ω-limit set of $\mathscr{A}$,

$$\tilde{X} = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)\mathscr{A}}$$

is a maximal attractor in V, bounded in V and included in $C^\infty(\bar{\Omega})^3 \times C^\infty(\bar{\Omega})$, whence the following result.

THEOREM 3.3. If the dimension of space is $n = 3$ and if the initial value problem for (2.6) is well posed for all initial data φ_0 in V, then there exists a closed bounded set $\tilde{X}$ of V which is an attractor for the semigroup $\{S(t)\}_t$. This set contains all the functional invariant sets for $S(t)$ which are bounded in V. It lies in $C^\infty(\bar{\Omega})^3 \times C^\infty(\bar{\Omega})$.

II. PHYSICAL BOUNDS ON THE FRACTAL DIMENSION OF THE ATTRACTORS

Our aim is now to estimate in terms of the physical data the dimension of the attractors and functional invariant sets introduced before. We start by recapitulating a few results borrowed mostly from Constantin, Foias and Temam [4]; we study then the two-dimensional and three-dimensional cases.

4. *Lyapunov exponents and fractal dimensions of attractors: survey of results*

Let X be a functional invariant set bounded in V for the semigroup $\{S(t)\}_t$. Then

$$S(t)X = X, \qquad \forall t > 0. \tag{4.1}$$

One can prove as in [4] that for all $t > 0$, the mapping $\varphi_0 \to S(t)\varphi_0$ is differentiable from H into V if $n = 2$, while for $n = 3$ this mapping is "uniformly differentiable on X" in the following sense: $\forall \varphi_0 \in X$, $\exists L(t, \varphi_0)$ linear continuous on H and

$$\sup_{\substack{\varphi_0, \varphi_1 \in X \\ 0 < |\varphi_0 - \varphi_1| \leq \varepsilon}} \frac{|S(t)\varphi_1 - S(t)\varphi_0 - L(t, \varphi_0) \cdot (\varphi_1 - \varphi_0)|}{|\varphi_1 - \varphi_0|} \to 0 \tag{4.2}$$

as $\varepsilon \to 0$.

For both $n = 2$ and 3, $L(t, \varphi_0)$ is defined as follows: For every $\xi \in H$, $L(t, \varphi_0) \cdot \xi$ is equal to $\Phi(t)$, where Φ is the solution of the linearized form of (2.6) (first variation equation) with initial value ξ (see (2.6), (2.7)):

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} + A\Phi + B(\varphi, \Phi) + B(\Phi, \varphi) + R\Phi = 0, \tag{4.3}$$

$$\Phi(0) = \xi. \tag{4.4}$$

Here $\varphi = \varphi(t) = S(t)\varphi_0$, and, as part of the proof of differentiability in [4], it is shown that in the interval of time $[0, \tau]$ under consideration ($t \in [0, \tau]$) (4.3), (4.4) possess a unique solution in $C([0, \tau], V)$.

For $L \in L(H)$ and $j \in N$, we denote by $\omega_j(L)$ the norm of the exterior product $\wedge^j L$ in $\wedge^j H$, thus if $L = L(t, \varphi_0)$,

$$\omega_j(L(t, \varphi_0)) = \sup_{\substack{\xi_1, \ldots, \xi_j \in H \\ |\xi_1|, \ldots, |\xi_j| \leq 1}} |\Phi_1(t) \wedge \ldots \wedge \Phi_j(t)|_{\wedge^m H}$$

where $\Phi_1, \ldots, \Phi_j$, are j solutions of (4.3), (4.4) corresponding respectively to $\xi = \xi_1, \ldots, \xi = \xi_j$. The number $\omega_j(L)$ gives a bound on how the jth dimensional volumes are expanded by L.

We then set

$$\omega_j(t) = \sup_{\varphi_0 \in X} \omega_j(L(t, \varphi_0))$$

and observe that these numbers are subexponential with respect to t:

$$\omega_j(t + t') \leqslant \omega_j(t) \cdot \omega_j(t'), \qquad \forall t, t' \geqslant 0. \tag{4.5}$$

Due to (4.5), the limit

$$\lim_{t \to \infty} \omega_j(t)^{1/t} = \Pi_j \tag{4.6}$$

exists for every j. The *uniform Lyapunov numbers for* X are then defined by the formula $\lambda_1 = \Pi_1$, $\Pi_j = \lambda_1 \ldots \lambda_j$, $j \geqslant 2$, i.e.

$$\lambda_1 = \Pi_1, \qquad \lambda_j = \Pi_j / \Pi_{j-1}, \qquad j \geqslant 2. \tag{4.7}$$

The *uniform Lyapunov exponents* are the numbers $\mu_j = \log \lambda_j$, $j \geqslant 1$.

A general result on attractors in [4] (see theorem 3.3 and remark 3.1, iii) states that

> If for some integer $m \geqslant 1$, $\mu_1 + \ldots + \mu_m < 0$, then the Hausdorff dimension of X is less than or equal to m, and the fractal dimension of X is less than or equal to
>
> $$m \left\{ \max_{1 \leqslant k \leqslant m-1} \left(1 + \frac{\mu_1 + \ldots + \mu_k}{|\mu_1 + \ldots + \mu_m|} \right) \right\}. \tag{4.8}$$

In Sections 5 and 6 we will apply this result to obtain a bound on the dimension of the attractors in the two- and three-dimensional cases. We conclude this section by recalling briefly the definition of the Hausdorff and fractal dimensions (capacity) (Federer [7], Mandelbrot [18]).

The d-dimensional Hausdorff measure of X is the number

$$\mu_H(X, d) = \lim_{\varepsilon \searrow 0} \mu_H(X, d, \varepsilon) = \sup_{\varepsilon > 0} \mu_H(X, d, \varepsilon),$$

where

$$\mu_H(X, d, \varepsilon) = \inf \sum_i r_i^d,$$

the infimum being taken for all the coverings of X by balls of radii $r_i \leqslant \varepsilon$. There is a number $d = d_H(X) \in [0, +\infty]$ such that $\mu_H(X, d) = 0$ for $d > d_H(X)$ and $= \infty$ for $d < d_H(X)$: $d_H(X)$ is the Hausdorff dimension of X.

The fractal dimension of X is

$$d_M(X) = \limsup_{\varepsilon \to 0} \frac{\log n_X(\varepsilon)}{\log 1/\varepsilon},$$

where $n_X(\varepsilon)$ is the minimum number of balls of radius $\leqslant \varepsilon$ which is necessary to cover X. It is known [18] that

$$d_M(X) = \inf\{d > 0, \mu_M(X, d) = 0\},$$

where

$$\mu_M(X, d) = \limsup_{\varepsilon \to 0} \varepsilon^d n_X(\varepsilon).$$

Since $\mu_M(X, d) \geq \mu_H(X, d)$, it is clear that the fractal dimension of a set is larger than or equal to its Hausdorff dimension.

5. *Estimate of the dimension of the universal attractor* ($n = 2$)

As mentioned in the Introduction, our sharp estimates in the fractal dimension of the attractors for Bénard convection rely on the general results of [4] on attractors and on Lieb–Thirring–Sobolev inequalities which we now recall.

5.1. *Lieb–Thirring–Sobolev inequalities.* If $\mathbb{O}$ is an open set of $\mathbb{R}^n$ with a sufficiently regular boundary there, the Sobolev imbedding theorem (see [1, 17]) implies that $H^1(\mathbb{O}) \subset L^p(\mathbb{O})$, where $p = 6$ if $n = 3$, $p < \infty$ arbitrary if $n = 2$. A related inequality of Gagliardo–Nirenberg–Ladyzenskaya is that

$$\int_{\mathbb{R}^2} u^4 \,\mathrm{d}x \leq 2 \left(\int_{\mathbb{R}^2} u^2 \,\mathrm{d}x \right) \left(\int_{\mathbb{R}^2} |\mathrm{grad}\, u|^2 \,\mathrm{d}x \right), \qquad \forall u \in H^1(\mathbb{R}^2) \tag{5.1}$$

if $n = 2$. For $n = 3$, we have

$$\int_{\mathbb{R}^3} u^4 \,\mathrm{d}x \leq 48 \left(\int_{\mathbb{R}^3} u^2 \,\mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}^3} |\mathrm{grad}\, u|^2 \,\mathrm{d}x \right)^{3/2}, \qquad \forall u \in H^1(\mathbb{R}^3). \tag{5.2}$$

which implies in particular, by interpolation,

$$\int_{\mathbb{R}^3} |u|^{10/3} \,\mathrm{d}x \leq c_1 \left(\int_{\mathbb{R}^3} u^2 \,\mathrm{d}x \right)^{2/3} \int_{\mathbb{R}^3} |\mathrm{grad}\, u|^2 \,\mathrm{d}x, \qquad \forall u \in H^1(\mathbb{R}^3). \tag{5.3}$$

The Lieb–Thirring inequalities [16] are improvements of the above (Sobolev-type inequalities). They state that if $\mathbb{O}$ is an open set of $\mathbb{R}^n$, $n = 2$, and if u_j is an orthonormal family of $L^2(\mathbb{O})$, with $u_j \in H_0^1(0)$, $\forall j$, then

$$\int_{\mathbb{O}} \rho(x)^2 \,\mathrm{d}x \leq c_2 \sum_{j=1}^{N} \int_{\mathbb{O}} |\mathrm{grad}\, u_j|^2 \,\mathrm{d}x, \tag{5.4}$$

where N is an arbitrary integer,

$$\rho(x) = \sum_{j=1}^{N} |u_j(x)|^2, \tag{5.5}$$

and the constant c_2 is *independent* of N. If $n = 3$, the same is true with (5.3) replaced by

$$\int_{\mathbb{O}} \rho(x)^{5/3} \,\mathrm{d}x \leq c_2 \sum_{j=1}^{N} \int_{\mathbb{O}} |\mathrm{grad}\, u_j|^2 \,\mathrm{d}x. \tag{5.6}$$

It is obvious that (5.1) and (5.3) imply (5.4) and (5.6) with constants depending on N; the fact that c_2 does not depend on N is the essential point.

The methods used to prove (5.4), (5.6) can be applied to other boundary conditions (i.e. $H_0^1(\mathbb{O})$ replaced by another subspace of $H^1(\mathbb{O})$), to vector functions and to subspaces of vector functions related to the condition $\operatorname{div} u = 0$, which appears in the Navier–Stokes equations. The details are given in [24, Appendix]; here we only mention the inequalities to be used in the sequel.

$$\left.\begin{array}{l}
\text{If } \psi_j = \{\omega_j, \tau_j\},\ 1 \leq j \leq N, \text{ is a family of elements of } V \text{ which is orthonormal in } H \\
(V, H \text{ defined in Section 2}), \text{ then, if } n = 2, \\
\text{(i)} \qquad \displaystyle\int_\Omega \rho(x)^2 \, \mathrm{d}x \leq c_3 \sum_{j=1}^{N} \int_\Omega (|\operatorname{grad} \omega_j|^2 + |\operatorname{grad} \tau_j|^2) \, \mathrm{d}x, \\
\text{where } N \text{ is arbitrary, } c_s \text{ is independent of } N, \text{ and} \\
\qquad \displaystyle\rho(x) = \sum_{j=1}^{N} \{|\omega_j(x)|^2 + |\tau_j(x)|^2\}. \\
\text{If } n = 3, \text{ (i) is replaced by} \\
\text{(ii)} \qquad \displaystyle\int_\Omega \rho(x)^{5/3} \, \mathrm{d}x \leq c_4 \int \sum_{j=1}^{N} (|\operatorname{grad} \omega_j|^2 + |\operatorname{grad} \tau_j|^2) \, \mathrm{d}x.
\end{array}\right\} \qquad (5.7)$$

5.2 *The main result on dimensions.* We now intend to apply (4.8) to the universal attractor for the two-dimensional Bénard problem (theorem 3.1). We will prove the following.

THEOREM 5.1. The fractal and the Hausdorff dimensions of the universal attractor for the two-dimensional Bénard problem are both bounded by

$$c_5 l(1 + Gr)(1 + Pr)^2 \quad \text{and} \quad c_5 l(1 + Ra + Gr)(1 + Pr), \qquad (5.8)$$

where c_5 is a universal (and nondimensional) constant and l, Pr, Gr, Ra, were defined in Section 1.

Proof. (i) We return to the notations of Section 4 and let $\Phi_1, \ldots, \Phi_m$ be m solutions of the first variation equation (4.3) with initial data, in (4.4), $\Phi_j(0) = \xi_j$, $\xi_j \in H$, $|\xi_j| \leq 1$, $1 \leq j \leq m$. The estimation of the Liapunov numbers and $\omega_j(t)$ depends on the estimation of $\Phi_1(t) \wedge \ldots \wedge \Phi_m(t)$. Exactly as in [4], it can be shown that

$$\frac{\mathrm{d}}{\mathrm{d}t} |\Phi_1 \wedge \ldots \wedge \Phi_m|^2 + 2\, Tr(\mathscr{A}(\varphi) \cdot Q_m) \cdot |\Phi_1 \wedge \ldots \wedge \Phi_m|^2 = 0, \qquad (5.9)$$

and thus

$$|\Phi_1(t) \wedge \ldots \wedge \Phi_m(t)| \leq |\xi_1 \wedge \ldots \wedge \xi_m| \exp\left(-\int_0^t \operatorname{Tr}(\mathscr{A}(\varphi) \cdot Q_m) \, \mathrm{d}s\right), \qquad (5.10)$$

where $\mathscr{A}(\varphi) = \mathscr{A}(\varphi(s))$ is the linear mapping

$$\Psi \mapsto A\Psi + B(\varphi(s), \Psi) + B(\Psi, \varphi(s)) + R\Psi,$$

and $Q_m = Q_m(s) = Q_m(s, \xi_1, \ldots, \xi_m)$ is the projector in H onto the space spanned by $\Phi_1(s), \ldots, \Phi_m(s)$.

Whence [4],

$$\omega_m(t) \leq \sup_{\varphi_0 \in X} \exp\left(-\int_0^t \inf_{\xi_j} (\operatorname{Tr} \mathscr{A}(\varphi) \cdot Q_m)\, \mathrm{d}s\right) \tag{5.11}$$

and

$$\Pi_m = \lim_{t\to\infty} \omega_m(t)^{1/t} \leq \exp(-q_m),$$

where

$$q_m = \limsup_{t\to\infty} \left\{ \inf_{\varphi_0\in X} \frac{1}{t} \int_0^t \inf_{\xi_1,\ldots,\xi_m \in H} (\operatorname{Tr} \mathscr{A}(\varphi)\cdot Q_m)\, \mathrm{d}s \right\}. \tag{5.12}$$

We then infer from (4.8) that if $q_m > 0$ for some m, then the Hausdorff dimension of X is less than or equal to m and its fractal dimension is bounded by

$$m\left(1 + \max_{1\leq k\leq m} \frac{q_k}{-q_m}\right). \tag{5.13}$$

(ii) In order to estimate the q_e and $\operatorname{Tr}\mathscr{A}(\varphi)\cdot Q_m$, let for (almost) every s, $\Psi_j = \Psi_j(s) = \{\omega_j(s), \tau_j(s)\}$, $j \geq 1$, be an orthonormal basis of H with $\Psi_j(s) \in V$, $\forall j$, $\forall s$, and such that $\Psi_1, \ldots, \Psi_m$ span $Q_m(s)H$. By the continuity of $\Phi_1(s), \ldots, \Phi_m(s)$ with respect to s in H (or even V), the existence of the Ψ_j depending measurably (or even continuously) on s is easy to obtain.

Then

$$\begin{aligned}
\operatorname{Tr}\mathscr{A}(\varphi)\cdot Q_m &= \sum_{j=1}^{\infty} (\mathscr{A}(\varphi)\Psi_j, \Psi_j) \\
&= \sum_{j=1}^{m} (\mathscr{A}(\varphi)\Psi_j, \Psi_j) \quad \text{(by (2.7), (3.4), (3.5))} \\
&= \sum_{j=1}^{m} \{(A\Psi_j, \Psi_j) + (B(\Psi_j, \varphi), \Psi_j) + (R\Psi_j, \Psi_j)\} \\
&= \sum_{j=1}^{m} \{\nu\|\omega_j\|^2 + \kappa\|\tau_j\|^2 - 2\int_\Omega \tau_j (w_j)_n \,\mathrm{d}x \\
&\quad + \int_\Omega [((w_j\cdot\nabla)u)w_j + ((w_j\cdot\nabla)\vartheta)\tau_j]\,\mathrm{d}x\}.
\end{aligned}$$

The last integrals are integrated by part and majorized as follows:

$$\begin{aligned}
\left|\int_\Omega (((w_j\cdot\nabla)\vartheta)\tau_j)\,\mathrm{d}x\right| &= \left|\int_\Omega \vartheta w_j(\nabla\tau_j)\,\mathrm{d}x\right| \\
&\leq \text{(by remark 3.4 and Schwarz's inequality)} \\
&\leq |w_j|\|\nabla\tau_j| \\
&\leq \|\tau_j\|;
\end{aligned}$$

the last inequality, $|w_j| \leqslant 1$, holds since $|w_j| \leqslant (|w_j|^2 + |\tau_j|^2)^{1/2} = |\Psi_j|$ and $|\Psi_j| = 1$, the family $\{\Psi_j\}_j$ being orthonormal in H. Thus,

$$\left|\sum_{j=1}^{m} \int_\Omega (((w_j \cdot \nabla)\vartheta)\tau_j)\,\mathrm{d}x\right| \leqslant \sum_{j=1}^{m} \|\tau_j\| \leqslant \frac{\kappa}{4}\sum_{j=1}^{m} \|\tau_j\|^2 + \frac{m}{\kappa}. \tag{5.14}$$

Thanks to the Schwarz inequality we then write the pointwise inequality

$$|[((w_j \cdot \nabla) \cdot u)w_j](x)| \leqslant |\nabla u(x)|\,|w_j(x)|^2,$$

and using the function $\rho = \rho(x)$ in (5.7), we write

$$\begin{aligned}\left|\sum_{j=1}^{m} \int_\Omega ((w_j \cdot \nabla)u)w_j\,\mathrm{d}x\right| &\leqslant \int_\Omega |\nabla u(x)|\rho(x)\,\mathrm{d}x \\ &\leqslant \text{(by Cauchy–Schwarz's inequality)} \\ &\leqslant \|u\|\left(\int_\Omega \rho^2\,\mathrm{d}x\right)^{1/2}.\end{aligned} \tag{5.15}$$

We now use (5.7) and majorize this expression by

$$\begin{aligned} c_3^{1/2}\|u\|\left(\sum_{j=1}^{m} (\|w_j\|^2 + \|\tau_j\|^2)\right)^{1/2} &\leqslant c_3^{1/2}\|u\|\left\{\left(\sum_{j=1}^{m}\|w_j\|^2\right)^{1/2} + \left(\sum_{j=1}^{m}\|\tau_j\|^2\right)^{1/2}\right\} \\ &\leqslant \frac{\nu}{4}\sum_{j=1}^{m}\|w_j\|^2 + \frac{\kappa}{4}\sum_{j=1}^{m}\|\tau_j\|^2 + c_3\left(\frac{1}{\nu} + \frac{1}{\kappa}\right)\|u\|^2. \end{aligned}$$

The sum

$$2\sum_{j=1}^{m}\int_\Omega \tau_j (w_j)_n\,\mathrm{d}x$$

is bounded in absolute value by

$$2\sum_{j=1}^{m}|\tau_j|\,|w_j|, \qquad \text{(recall that } |\cdot| = \text{the } L^2 \text{ norm)},$$

and this expression is majorized by

$$\sum_{j=1}^{m}(|w_j|^2 + |\tau_j|^2),$$

which is precisely m since the family Ψ_j is orthonormal in H and $1 = |\Psi_j|^2 = |w_j|^2 + |\tau_j|^2$. Finally,

$$\operatorname{Tr}\mathscr{A}(\varphi)\cdot Q_m \geqslant \sum_{j=1}^{m}\left\{\frac{\nu}{2}\|w_j\|^2 + \frac{\kappa}{2}\|\tau_j\|^2\right\} - m\left(1 + \frac{1}{\kappa}\right) - c_3\left(\frac{\nu+\kappa}{\nu\kappa}\right)\|u\|^2. \tag{5.16}$$

Recall that in (5.16) several quantities depend on s, although for the sake of simplicity s does not appear explicitly. We will integrate (5.16) with respect to s, but before doing that we recall the following: $\sum_{j=1}^{m} \|w_j\|^2 + \sum_{j=1}^{m} \|\tau_j\|^2$ is equal to*

$$(A_{0m}(w_1 \wedge \cdots \wedge w_m), w_1 \wedge \cdots \wedge w_m) + (A_{1m}(\tau_1 \wedge \cdots \wedge \tau_m), \tau_1 \wedge \cdots \wedge \tau_m),$$

and each of these terms is respectively bounded from below by the first eigenvalue of A_{0m} and of A_{1m}; and these eigenvalues are, respectively, the sum of the first m eigenvalues of A_0 and A_1. The asymptotic behavior of these eigenvalues is known (see Courant–Hilbert for Laplace's equation [6] and Métivier [20] for the Stokes operator with other boundary conditions): $\nu_m \sim cm/|\Omega|$ as $m \to \infty$, and thus there exists a constant c_2' such that

$$2\frac{c_2' m^2}{|\Omega|} \leqslant \sum_{j=1}^{m} (\|w_j\|^2 + \|\tau_j\|^2). \tag{5.17}$$

Taking (5.17) into account in (5.16), we obtain:

$$\operatorname{Tr} \mathscr{A}(\varphi) \cdot Q_m \geqslant \left(\frac{c_2' m^2}{|\Omega|}\right)\left(\frac{\nu\kappa}{\nu+\kappa}\right) - m\left(\frac{\kappa+1}{\kappa}\right) - c_3\left(\frac{\nu+\kappa}{\nu\kappa}\right)\|u\|^2,$$

and since the right-hand side of this inequality does not depend on $\xi_1, \ldots, \xi_m$, we write after integration:

$$\begin{aligned} &\frac{1}{t}\int_0^t \inf_{\xi_j} (\operatorname{Tr} \mathscr{A}(\varphi) \cdot Q_m)\, \mathrm{d}s \\ &\geqslant \left(\frac{c_2' m^2}{|\Omega|}\right)\left(\frac{\nu\kappa}{\nu+\kappa}\right) - m\left(\frac{\kappa+1}{\kappa}\right) - c_3\left(\frac{\nu+\kappa}{\nu\kappa}\right)\frac{1}{t}\int_0^t \|u\|^2\, \mathrm{d}s. \end{aligned} \tag{5.18}$$

Due to (3.13),

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \|u\|^2\, \mathrm{d}s \leqslant \frac{|\Omega|}{\nu^2},$$

and thus,

$$q_m \geqslant \alpha m^2 - \beta m - \gamma \geqslant \frac{\alpha m^2}{2} - \gamma - \frac{\beta^2}{2\alpha}, \tag{5.19}$$

$$\alpha = \left(\frac{c_2'}{|\Omega|}\right)\left(\frac{\nu\kappa}{\nu+\kappa}\right), \quad \beta = \frac{\kappa+1}{\kappa}, \quad \gamma = c_3|\Omega|\left(\frac{\nu+\kappa}{\nu^3\kappa}\right).$$

(iii) Let now $\bar{m}$ be the first integer such that

$$m > 2\left(\frac{\beta}{\alpha} + \sqrt{\frac{\gamma}{\alpha}}\right). \tag{5.20}$$

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* If T is a linear operator in H, we denote by T_m the linear operator in $\wedge^m H$ defined by $T_m = T\wedge I \wedge \ldots \wedge I + I\wedge T\wedge I\wedge \ldots \wedge I + \ldots + I\wedge \ldots \wedge I \wedge T$, where I denotes the identity operator in H.

Then $q_m \geqslant 0$ and, as observed above, (4.8) implies that the Hausdorff dimension of the Bénard attractor is majorized by $\bar{m}$. For the fractal dimension we write (5.19) for $m = k \leqslant \bar{m} - 1$:

$$\frac{-q_k}{q_m} \leqslant \frac{-\alpha k^2 + \gamma + (\beta^2/2\alpha)}{(\alpha \bar{m}^2/2) - \gamma - (\beta^2/2\alpha)} \leqslant \frac{\gamma + (\beta^2/2\alpha)}{\gamma + (\beta^2/2\alpha)} = 1.$$

Whence (5.13) with m replaced by $\bar{m}$ shows that the fractal dimension of the attractor is bounded by $2\bar{m}$.

To conclude, we express the bound on $\bar{m}$ given by (5.20) in terms of the physical non-dimensional numbers:

$$\begin{aligned}
\bar{m} &\leqslant 1 + 2\left(\frac{\beta}{\alpha} + \sqrt{\frac{\gamma}{\alpha}}\right) \\
&\sim c_3'|\Omega| \left(\frac{\kappa + 1}{\kappa}\right)\left(\frac{\nu + \kappa}{\nu\kappa}\right) + c_3'|\Omega| \left(\frac{\nu + \kappa}{\nu^2 \kappa}\right) \\
&\sim c_3'|\Omega|(Ra + Gr^{1/2})(1 + Pr) + c_3'|\Omega| Gr(1 + Pr) \\
&\sim c_3'|\Omega|\{Gr^{1/2}(1 + Pr) + Gr(1 + Pr)^2\}.
\end{aligned}$$

We can write $|\Omega|$ = the nondimensional surface of the domain which is equal to l = the shape factor of the domain. Theorem 5.1 follows now easily.

Remark 5.1. For reasonably large Grashof and Prandtl numbers a physical insight into this result may be gained as follows. Under such conditions, most of the temperature drop occurs near the top and bottom boundaries in thin layers, say of thickness $\delta \ll h$. In the remainder of the layer, the temperature is nearly uniform (and $\sim \Delta T$) with the heat flux, q, being carried vertically by the moving fluid. Since the mean vertical heat flux is the same at all points within the convecting layer, we have

$$q \sim u_2 \Delta T \sim \kappa \Delta T / \delta.$$

Now, recall that $u_2 \sim (\alpha \Delta T g h)^{1/2}$ [19] (α being here the coefficient of volume expansion), whence

$$h/\delta \sim Gr^{1/2} Pr.$$

Arguing that the smallest wavelength to be resolved must be comparable with δ, we see that the number of degrees of freedom, m, needed to describe this two-dimensional flow, is simply

$$m \sim \left(\frac{h}{\delta}\right)^2 \sim Gr\, Pr^2.$$

Remark 5.2. We can give for $\bar{m}$ an alternate expression which is similar to the estimate that will be derived in the three-dimensional case. We introduce the dissipation rate of energy per unit mass and time averaged on the attractor: in terms of the *dimensional variables* this is

$$\varepsilon = \nu \limsup_{t \to \infty} \left\{ \sup_{\varphi_0 \in X} \frac{1}{t} \int_0^t \int_\Omega |\nabla u(x, s)|^2 \frac{\mathrm{d}x}{|\Omega|} \mathrm{d}s \right\}, \tag{5.21}$$

and we associate to ε the Kolmogorov dissipation length

$$l_d = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}.$$

Denoting by l_0 a (dimensional) typical length for Ω so that $|\Omega|$ = the (dimensional) area $\sim l_0^2$, we have

$$\left(\frac{l_0}{l_d}\right)^4 = \frac{l_0^4}{|\Omega|\nu^2} \limsup_{t\to\infty} \left\{ \sup_{\varphi_0\in\bar{X}} \frac{1}{t}\int_0^t \|u(s)\|^2 \, ds \right\}.$$

Returning to the nondimensionalized variables, l_0 = the height = 1, $|\Omega| = l$, and we define the Kolmogorov dissipation length by

$$\left(\frac{l_0}{l_d}\right)^4 = \frac{1}{|\Omega|\nu^2} \limsup_{t\to\infty} \left\{ \sup_{\varphi_0\in\bar{X}} \frac{1}{t}\int_0^t \|u(s)\|^2 \, ds \right\} \qquad \text{(by (3.13))} \tag{5.22}$$

$$\leqslant \frac{1}{\nu^4} = Gr^2.$$

It is easy to see from (5.18) that the term γ in (5.19) can be chosen equal to

$$c_3 \frac{\nu + \kappa}{\nu\kappa} |\Omega| \nu^2 \left(\frac{l_0}{l_d}\right)^4,$$

and (5.20) then yields the alternative expression:

$$\bar{m} \sim c_4' l \left\{ (Ra + Gr^{1/2}) + \left(\frac{l_0}{l_d}\right)^2 \right\} (1 + Pr).$$

6. *The three-dimensional case*

There are some differences between the two- and the three-dimensional cases which are due, on one hand, to the fact that we do not know if the initial value problem is well posed in $\mathbb{R}^3$, and, on the other hand, to some differences in the Sobolev–Lieb–Thirring inequalities. We are able, however, to derive physical bounds for the fractal dimension of *functional invariant sets* (or attractors) corresponding to three-dimensional Bénard convection, when it is assumed that these attractors are bounded in V (H^1-norm).

We explain now how the computations of Section 5 can be adapted to this case.

With the definition (5.12) of q_m, the statement (5.13) is valid: if $q_m \geqslant 0$ for some integer m, then the Hausdorff dimension of the set X is $\leqslant m$ and its fractal dimension is bounded by (5.13). For the estimation of the q_m's we proceed as in theorem 5.1 up to (5.15), which is replaced by

$$\left|\sum_{j=1}^m \int_\Omega ((\omega_j \cdot \nabla)u)w_j \, ds\right| \leqslant \int_\Omega |\nabla u(x)|\rho(x)\, dx \leqslant \text{(by Hölder's inequality)}$$

$$\leqslant \left(\int_\Omega |\nabla u|^{5/2}\, dx\right)^{2/5} \left(\int_\Omega \rho^{5/3}\, dx\right)^{3/5}. \tag{6.1}$$

We now use (5.7) (ii) and bound this expression by

$$c_4'|\nabla u|_{5/2}\left(\sum_{j=1}^{m}(\|w_j\|^2+\|\tau_j\|^2)\right)^{3/5}\leqslant\frac{3}{5}\left(\frac{\nu\kappa}{\nu+\kappa}\right)$$

$$\sum_{j=1}^{m}(\|w_j\|^2+\|\tau_j\|^2)+c_5'\left(\frac{\nu+\kappa}{\nu\kappa}\right)^{3/2}|\nabla u|_{5/2}^{5/2}\quad\text{(by Younger's inequality).}$$

Then (5.16) is replaced by

$$\operatorname{Tr}\mathscr{A}(\varphi)\cdot Q_m\geqslant\frac{2}{5}\left(\frac{\nu\kappa}{\nu+\kappa}\right)\sum_{j=1}^{m}(\|w_j\|^2+\|\tau_j\|^2)$$

$$-m\left(\frac{1+\kappa}{\kappa}\right)-c_5'\left(\frac{\nu+\kappa}{\nu\kappa}\right)^{3/2}|\nabla u|_{5/2}^{5/2}. \tag{6.2}$$

In dimension 3 the eigenvalues of the Stokes and Laplace operators behave like $\nu_m\sim c(m/|\Omega|)^{2/3}$, and instead of (5.17) we write [3, 6, 20]:

$$\frac{c_6'm^{5/3}}{|\Omega|^{2/3}}\leqslant\sum_{j=1}^{m}(\|w_j\|^2+\|\tau_j\|^2), \tag{6.3}$$

whence

$$\operatorname{Tr}\mathscr{A}(\varphi)\cdot Q_m\geqslant\left(\frac{c_7'}{|\Omega|^{2/3}}\right)\left(\frac{\nu\kappa}{\nu+\kappa}\right)m^{5/3}-m\left(\frac{1+\kappa}{\kappa}\right)-c_5'\left(\frac{\nu+\kappa}{\nu\kappa}\right)^{3/2}|\nabla u|_{5/2}^{5/2}, \tag{6.4}$$

and, by integration (and minimization with respect to the ξ's), we obtain the analogue of (5.18) and (5.19):

$$\frac{1}{t}\int_0^t\inf_{\xi_j}(\operatorname{Tr}\mathscr{A}(\varphi)\cdot Q_m)\,ds\geqslant\left(\frac{c_7'}{|\Omega|^{2/3}}\right)\left(\frac{\nu\kappa}{\nu+\kappa}\right)m^{5/3}-m\left(\frac{1+\kappa}{\kappa}\right)-c_5'\left(\frac{\nu+\kappa}{\nu\kappa}\right)^{3/2}\frac{1}{t}\int_0^t|\nabla u|_{5/2}^{5/2}\,dx,$$

$$q_m\geqslant\alpha m^{5/3}-\beta m-\gamma,\qquad\forall m, \tag{6.5}$$

$$\alpha=\left(\frac{c_7'}{|\Omega|^{2/3}}\right)\left(\frac{\nu\kappa}{\nu+\kappa}\right),\qquad\beta=\frac{1+\kappa}{\kappa}, \tag{6.6}$$

$$\gamma=c_5'\left(\frac{\nu+\kappa}{\nu\kappa}\right)^{3/2}\limsup_{t\to\infty}\left\{\sup_{\varphi_0\in X}\frac{1}{t}\int_0^t|\nabla u|_{5/2}^{5/2}\,ds\right\}. \tag{6.7}$$

The major difference with the two-dimensional case is that we do not know how to estimate γ in terms of the data, particularly ν and κ. However, we can proceed as in remark 5.2 and bound γ by expressions which are well-defined on the attractor: We introduce the dissipation length $\bar{l}_d$ defined by:

$$\left(\frac{l_0}{\bar{l}_d}\right)^4=\frac{1}{\nu^2|\Omega|^{4/5}}\limsup_{t\to\infty}\left\{\sup_{\varphi_0\in X}\left(\frac{1}{t}\int_0^t\int_\Omega|\nabla u(x,s)|^{5/2}\,dx\,ds\right)^{4/5}\right\}. \tag{6.8}$$

This length $\bar{l}_d$ plays the same role as the Kolmogorov dissipation length. Indeed, if the powers 5/2 and 4/5 above would be replaced by 2 and 1/2 respectively, the length defined by the above formula would be the Kolgomorov dissipation length l_d associated to X (see (5.22)). At present we cannot estimate $\bar{l}_d$ in terms of the data, nor give a physical argument in favor of $\bar{l}_d$ vis-à-vis l_d. We observe also that as in the two-dimensional case (remark 5.2), the typical macroscopic length l_0 is presently ~ 1. From (6.7) and the definition of $\bar{l}_d$, it follows readily

$$\gamma \leqslant \gamma' = c_5' \left(\frac{\nu + \kappa}{\nu\kappa}\right)^{3/2} |\Omega| \nu^{5/2} \left(\frac{l_0}{\bar{l}_d}\right)^5, \tag{6.9}$$

and

$$\begin{aligned} q_m &\geqslant \alpha m^{5/3} - \beta_m - \gamma' \qquad \text{(by Young's inequality)} \\ &\geqslant \frac{2}{5}\alpha m^{5/3} - \frac{2}{5} + \frac{\beta^{5/2}}{\alpha^{3/2}} - \gamma', \forall m. \end{aligned} \tag{6.10}$$

Let now $\bar{m}$ be the first integer m such that

$$m \geqslant 4^{3/5} \left(\frac{\beta}{\alpha}\right)^{3/2} + \left(\frac{10\gamma}{\alpha}\right)^{3/5}. \tag{6.11}$$

Then

$$q\bar{m} \geqslant \frac{2}{5}\left(\frac{\beta^{5/2}}{\alpha^{3/2}}\right) + \gamma' > 0$$

and the Hausdorff dimension of X is $\leqslant \bar{m}$. For the fractal dimension we estimate the ratio $-q_k/q_m$, $1 \leqslant k \leqslant \bar{m} - 1$:

$$-\frac{q_k}{q_m} \leqslant \frac{-(2/5)(\alpha k^{5/3}) + (2/5)(\beta^{5/2}/\alpha^{3/2}) + \gamma'}{(2/5)(\beta^{5/2}/(\alpha^{3/2} + \gamma'))} \leqslant 1.$$

We infer from this that the fractal dimension of X is $\leqslant 2\bar{m}$.

We conclude now by expressing the bound on $\bar{m}$ given by (6.11) in terms of the physical nondimensional numbers:

$$\begin{aligned} \bar{m} &\leqslant 1 + 4^{3/5} \left(\frac{\beta}{\alpha}\right)^{3/2} + \left(\frac{10\gamma'}{\alpha}\right)^{3/5} \\ &\sim c_8'|\Omega| \left(\frac{1+\kappa}{\kappa}\right)\left(\frac{\nu+\kappa}{\nu\kappa}\right)^{3/2} + c_8'|\Omega| \left(\frac{\nu+\kappa}{\kappa}\right)^{3/2} \left(\frac{l_0}{\bar{l}_d}\right)^3 \\ &\sim c_8'|\Omega| \left\{(Ra + Gr^{1/2})^{3/2} + \left(\frac{l_0}{\bar{l}_d}\right)^3\right\}(1 + Pr)^{3/2}. \end{aligned}$$

Finally, we have proved the following.

THEOREM 6.1. For the three-dimensional Bénard problem, any functional invariant set bounded in V and the maximal attractor if it exists (cf. theorems 3.2, 3.3) have a finite Hausdorff and fractal dimensions which are bounded by

$$c_0' \left\{(Ra + Gr^{1/2})^{3/2} + \left(\frac{l_0}{\bar{l}_d}\right)^3\right\}(1 + Pr)^{3/2}, \tag{6.12}$$

where c_9' is a (nondimensional) constant.

Remark 6.1. The constant c_9' depends on the shape of Ω in a way given by the formula before theorem 6.1: $c_9' = c_8'|\Omega| = c_8' lL$, where c_8' is an absolute constant. Notice that (6.12) and (5.23) can be written as a single expression:

$$c|\Omega|\left\{(Ra + Gr^{1/2})^{n/2} + \left(\frac{l_0}{l_d}\right)^n\right\}(1 + Pr)^{n/2},$$

where $n = 2, 3$ is the space dimension and we ignore (for $n = 2$) the difference between $\bar{l}_d$ and l_d.

Remark 6.2. On carrying out an argument similar to that presented in remark 5.1, we find easily that for this three-dimensional flow,

$$m \sim \left(\frac{h}{\delta}\right)^3 \sim (Ra\, Pr)^{3/2},$$

thus accounting for at least some of the degrees of freedom mandated by (6.12). It appears then that the additional term, $(l_0/\bar{l}_d)^3(1 + Pr)^{3/2}$ accounts for the degrees of freedom needed to describe the fluid motion on length scale between δ and $\bar{l}_d$.

Remark 6.3. In the definition of l_d (see Section 5)

$$\underline{l}_d = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}, \qquad \varepsilon = \nu \limsup_{t\to\infty}\left\{\sup_{\varphi_0\in X}\left(\frac{1}{t}\int_0^t \frac{1}{|\Omega|}\int_\Omega |\nabla u(x,s)|^2\, dx\, ds\right)\right\} \tag{6.13}$$

and of $\bar{l}_d$

$$\bar{l}_d = \left(\frac{\nu^3}{\bar{\varepsilon}}\right)^{1/4}, \qquad \bar{\varepsilon} = \nu \limsup_{t\to\infty}\left\{\sup_{\varphi_0\in X}\left(\frac{1}{t}\int_0^t \frac{1}{|\Omega|}\int_\Omega |\nabla u(x,s)|^{5/2}\, dx\, dt\right)^{4/5}\right\}, \tag{6.14}$$

the effects of the space intermittency are smeared out by the space averaging. If these effects are to be taken into account, then a natural dissipation length is

$$\underline{l}_d = \left(\frac{\nu^3}{\bar{\bar{\varepsilon}}}\right)^{1/4}, \qquad \bar{\bar{\varepsilon}} = \nu \limsup_{t\to\infty}\left\{\sup_{\varphi_0\in X}\left(\sup_{x\in\Omega}|\nabla u(x,t)|^2\right)\right\}. \tag{6.15}$$

It is obvious that

$$\varepsilon \leqslant \bar{\varepsilon} \leqslant \varepsilon^{4/5}\bar{\bar{\varepsilon}}^{1/5}$$

and thus

$$l_d \geqslant \bar{l}_d \geqslant l_d^{4/5}\underline{l}_d^{1/5} = l_d(\underline{l}_d/l_d)^{1/5}. \tag{6.16}$$

Therefore, from theorem 6.1, we infer directly that the Hausdorff and the fractal dimensions of any functional invariant set bounded in V are bounded above by

$$c_9'\left\{(Ra + Gr^{1/2})^{3/2} + \left(\frac{l_0}{l_d}\right)^3\left(\frac{l_d}{\underline{l}_d}\right)^{3/5}\right\}(1 + Pr)^{3/2}. \tag{6.17}$$

The supplementary factor $(l_d/\underline{L}_d)^{3/5}$ besides the natural term $(l_0/l_d)^3$ can be possibly interpreted as due to the occurrence of spatial intermittency. Notice that unlike in (6.12) the dissipative length l_d is the natural Kolmogorov length associated to the set X.

Remark 6.4. It is known that for the Bénard convection, entropy is related to the relative enhancement in the heat transfer coefficient Nu, i.e. the Nusselt number (see [26] for the case of a stationary solution). Indeed, for an arbitrary regular solution $\varphi = \{u, \vartheta\}$ of (2.6), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}|u|^2 - \int_\Omega x_n \vartheta\,\mathrm{d}x\right] = -\nu\|u\|^2 - \int_\Omega x_n u_n\,\mathrm{d}x - \kappa\int_{\Omega'} \frac{\partial\vartheta}{\partial x_n}\bigg|_{x_n=1} \mathrm{d}x', \tag{6.18}$$

where $\Omega = \Omega' \times [0, 1]$. Note that

$$\int_\Omega x_n u_n\,\mathrm{d}x = \frac{1}{2}\int_\Omega (u\cdot\nabla)x_n^2\,\mathrm{d}x = \frac{1}{2}\int_\Omega \nabla\cdot(ux_n^2)\,\mathrm{d}x = 0$$

(by (1.16)). The Nusselt number $Nu(\varphi)$, which is the ratio of the heat transferred by both conduction and convection to the heat transferred by conduction alone, is defined by

$$Nu(\varphi) = -\frac{1}{|\Omega'|}\int_{\Omega'} \frac{\partial}{\partial x_n}\vartheta\,\mathrm{d}x' + 1,$$

and therefore the Nusselt number associated to a functional invariant set X bounded in V should be defined by

$$Nu = \limsup_{t\to\infty}\left\{\sup_{\varphi_0\in X}\left(\frac{1}{t}\int_0^t Nu(\varphi(s))\,\mathrm{d}s\right)\right\}. \tag{6.19}$$

Integrating (6.18) with respect to t and using (3.12), we obtain easily after some elementary computation that

$$Pr\left(\frac{l_0}{l_d}\right)^4 = lL(Nu-1)Gr, \tag{6.20}$$

where c is a constant depending only on the geometry of Ω. Plainly (6.17) and the first relation (6.20) show that the Hausdorff and fractal dimensions of any functional invariant set, bounded in V, are bounded above by

$$c'_{10}\left\{(Ra + Gr^{1/2})^{3/2} + \left(\frac{Nu-1}{Pr}Gr\right)^{3/4}\cdot\left(\frac{l_d}{\underline{l}_d}\right)^{3/5}(1+Pr)^{3/2}\right\}. \tag{6.21}$$

It should be noted here that measurements of convective heat transfer by different fluids with a wide range of Prandtl numbers and over Rayleigh numbers $10^4 \leq Ra \leq 10^7$ show that $Nu - 1 \leq 0.16\,Ra^{0.3}$ [27]. In this range the expression (6.21) is $\sim Ra^{3/2}$.

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