Multi-Physics Phenoma in HTS

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AG Optimal Control of PDE

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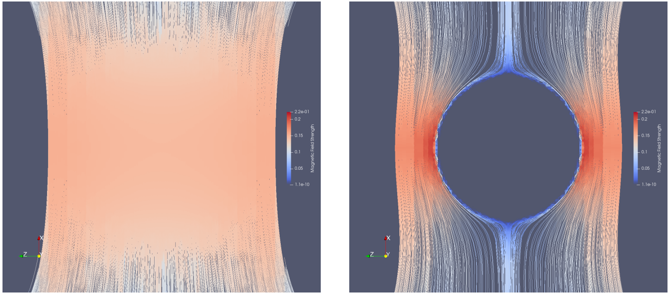
Open-Minded

Introduction

- Stationary Meissner-Ochsenfeld effect[1]



[1] M. Winckler and I. Yousept. Fully discrete scheme for Bean's critical-state model with temperature effects in superconductivity. *SIAM J. Numer. Anal.*, 57(6): 2685–2706, 2019

- Stationary Meissner-Ochsenfeld effect[1]

  ⇝ **Elliptic** curl-curl VI for E-Field:

$$\int_\Omega \epsilon \boldsymbol{E} \cdot (\boldsymbol{v} - \boldsymbol{E})\, dx + \int_\Omega \nu \,\mathbf{curl}\, \boldsymbol{E} \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{E})\, dx \\ + \int_{\Omega_{sc}} j_c |\boldsymbol{v}|\, dx - \int_{\Omega_{sc}} j_c |\boldsymbol{E}|\, dx \geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{E})\, dx \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}).$$

  ⇝ Discrete **Faraday** for M-Field:

$$\boldsymbol{H} = -\nu \,\mathbf{curl}\, \boldsymbol{E}.$$

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$$+ \int_{\Omega_{sc}} j_c |\boldsymbol{v}|\, dx - \int_{\Omega_{sc}} j_c |\boldsymbol{E}|\, dx \geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{E})\, dx \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}).$$

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- **AFEM** with convergence analysis[2]
- **Shape optimization** with numerical implementation[3]

[2] M. Winckler, I. Yousept, and J. Zou. Adaptive edge element approximation for H(curl) elliptic variational inequalities of the second kind. *SIAM J. Numer. Anal*, 58(3): 1941–1964, 2020.

[3] A. Laurain, M. Winckler, and I. Yousept. Shape optimization for superconductors governed by H(curl)-elliptic variational inequalities. *SIAM J. Control Optim.*, to appear, 2021.

Adaptive Finite Element Method

- Establish **error estimate** with **computable quantities**:

$$\eta_{\mathcal{T}}(\boldsymbol{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, \boldsymbol{f}) \lesssim \|\boldsymbol{E} - \boldsymbol{E}_{\mathcal{T}}^{\gamma}\|_{\boldsymbol{H}(\mathbf{curl})} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}\|_{*,a} \lesssim \eta_{\mathcal{T}}(\boldsymbol{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, \boldsymbol{f})$$

Estimator — Computed Approximation — Estimator

- Dual formulation of **stationary** (VI):

$$\begin{cases} a(\boldsymbol{E}, \boldsymbol{v}) + \displaystyle\int_{\Omega} \boldsymbol{\lambda} \cdot \boldsymbol{v}\, dx = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) \\ \\ |\boldsymbol{\lambda}(x)| \leq j_c(x), \quad \boldsymbol{\lambda}(x) \cdot \boldsymbol{E}(x) = j_c(x)|\boldsymbol{E}(x)| \end{cases}$$

Adaptive Finite Element Method

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- Consider **Moreau-Yosida** regularized VI:

$$\begin{cases} a(\boldsymbol{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{v}_h) + \displaystyle\int_\Omega \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma} \cdot \boldsymbol{v}_h \, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \, dx & \forall \boldsymbol{v}_h \in \boldsymbol{V}_{\mathcal{T}} \\ \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}(x) = j_c(x) \dfrac{\gamma \boldsymbol{E}_{\mathcal{T}}^{\gamma}(x)}{\max\{1, \gamma |\boldsymbol{E}_{\mathcal{T}}^{\gamma}(x)|\}} & \text{for a.e. } x \in \Omega. \end{cases}$$

  - Benefits for: **Computation** (SSN method) and the **analysis**.

A Posteriori Error Estimators

- Residual for $T \in \mathcal{T}$ and jumps for $F \in \mathcal{F}_{\mathcal{T}}$

$$\mathbf{R}_T := \boldsymbol{f}|_T - \epsilon \boldsymbol{E}_{\mathcal{T}}^{\gamma}|_T - \mathbf{curl}\, \nu\, \mathbf{curl}\, \boldsymbol{E}_{\mathcal{T}}^{\gamma}|_T - \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}|_T$$

$$\mathbf{J}_{F,1} := [\nu\, \mathbf{curl}\, \boldsymbol{E}_{\mathcal{T}}^{\gamma} \times \boldsymbol{n}_F] \quad \text{and} \quad J_{F,2} := [(\boldsymbol{\lambda}_{\mathcal{T}}^{\gamma} + \epsilon \boldsymbol{E}_{\mathcal{T}}^{\gamma}) \cdot \boldsymbol{n}_F].$$

- Estimators: For $\mathcal{T} \in \mathcal{T}$

$$\eta_{\mathcal{T},1}^2(\boldsymbol{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, \boldsymbol{f}, T) := h_T^2 \|\mathbf{R}_T\|_{\boldsymbol{L}^2(T)}^2 + \sum_{F \in \partial T \cap \Omega} h_F \|\mathbf{J}_{F,1}\|_{\boldsymbol{L}^2(F)}^2$$

$$\eta_{\mathcal{T},2}^2(\boldsymbol{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, T) := h_T^2 \|\operatorname{div} \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}\|_{\boldsymbol{L}^2(T)}^2 + \sum_{F \in \partial T \cap \Omega} h_F \|J_{F,2}\|_{\boldsymbol{L}^2(F)}^2.$$

A Posteriori Error Estimators

- Residual for $T \in \mathcal{T}$ and jumps for $F \in \mathcal{F}_{\mathcal{T}}$

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$\rightsquigarrow$ Necessary due to Schöberl interpolation: $\boldsymbol{v} - \boldsymbol{\Pi}_{\mathcal{T}}^{s} \boldsymbol{v} = \boldsymbol{\phi} + \nabla \varphi$

Reliability

Theorem

There exists a constant $C > 0$, such that the solutions $(\boldsymbol{E}, \boldsymbol{\lambda})$ and $(\boldsymbol{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma})$ satisfy

$$\|\boldsymbol{E} - \boldsymbol{E}_{\mathcal{T}}^{\gamma}\|_a^2 + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}\|_{*,a}^2 \leq C \left(\eta_{\mathcal{T}}^2(\boldsymbol{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, \boldsymbol{f}) + \frac{1}{\gamma} \right).$$

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- Problem: **No Galerkin-Orthogonality** of $\boldsymbol{E}$ and $\boldsymbol{E}_{\mathcal{T}}^{\gamma}$!

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- Problem: **No Galerkin-Orthogonality** of $\boldsymbol{E}$ and $\boldsymbol{E}_{\mathcal{T}}^{\gamma}$!
- Crucial ingredient: Auxiliary **linear** problem. Let $\boldsymbol{z} \in \boldsymbol{H}_0(\mathbf{curl})$ be the solution of

$$a(\boldsymbol{z}, \boldsymbol{v}) + \int_{\Omega} \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma} \cdot \boldsymbol{v}\, dx = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl})$$

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- It holds that $\|\boldsymbol{E} - \boldsymbol{E}_{\mathcal{T}}^{\gamma}\|_a^2 + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}\|_{*,a}^2 \leq C \left(\|\boldsymbol{E}_{\mathcal{T}}^{\gamma} - \boldsymbol{z}\|_a^2 + \frac{1}{\gamma} \right)$

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$$\|\boldsymbol{E} - \boldsymbol{E}_\mathcal{T}^\gamma\|_a^2 + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_\mathcal{T}^\gamma\|_{*,a}^2 \leq C \left(\eta_\mathcal{T}^2(\boldsymbol{E}_\mathcal{T}^\gamma, \boldsymbol{\lambda}_\mathcal{T}^\gamma, \boldsymbol{f}) + \frac{1}{\gamma} \right).$$

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- It holds that $\|\boldsymbol{E} - \boldsymbol{E}_\mathcal{T}^\gamma\|_a^2 + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_\mathcal{T}^\gamma\|_{*,a}^2 \leq C \left(\|\boldsymbol{E}_\mathcal{T}^\gamma - \boldsymbol{z}\|_a^2 + \frac{1}{\gamma} \right)$

Efficiency

Theorem

There exists a constant $C > 0$, such that the solutions $(\boldsymbol{E}, \boldsymbol{\lambda})$ and $(\boldsymbol{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma})$ satisfy

$$
\begin{aligned}
C\eta_{\mathcal{T}}^2(\boldsymbol{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, \boldsymbol{f}, T) \leq \|\boldsymbol{E} - \boldsymbol{E}_{\mathcal{T}}^{\gamma}\|_{a,\omega_T}^2 + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}\|_{*,a,\omega_T}^2 \\
+ \operatorname{osc}_{\mathcal{T}}{}^2(\boldsymbol{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, \boldsymbol{f}, \omega_T) \quad \forall T \in \mathcal{T},
\end{aligned}
$$

The proof is based on the well-known **bubble functions**.

Algorithm

Algorithm 1 AFEM Algorithm

1: Set $k = 0$ and choose an initial conforming mesh $\mathcal{T}_0$
2: (SOLVE) Compute $(\boldsymbol{E}_{\mathcal{T}_k}^{\gamma_k}, \boldsymbol{\lambda}_{\mathcal{T}_k}^{\gamma_k}) =: (\boldsymbol{E}_k, \boldsymbol{\lambda}_k)$
3: (ESTIMATE) Compute $\eta_{\mathcal{T}}(\boldsymbol{E}_k, \boldsymbol{\lambda}_k, \boldsymbol{f}, T)$ for each $T \in \mathcal{T}_k$
4: (MARK) Mark $\mathcal{M}_k \subset \mathcal{T}_k$ containing $\tilde{T} \in \mathcal{T}_k$ with

$$\eta_k(\boldsymbol{E}_k, \boldsymbol{\lambda}_k, \boldsymbol{f}, \tilde{T}) = \max_{T \in \mathcal{T}_k} \eta_k(\boldsymbol{E}_k, \boldsymbol{\lambda}_k, \boldsymbol{f}, T).$$

5: (REFINE) Refine each $T \in \mathcal{M}_k$ by bisection to obtain $\mathcal{T}_{k+1}$
6: Set $k = k + 1$ and go to step 2.

Strong Convergence

Theorem

Let $\{(\boldsymbol{E}_k, \boldsymbol{\lambda}_k, \mathcal{T}_k, \boldsymbol{V}_k, \gamma_k)\}_{k\in\mathbb{N}_0}$ *be the sequence generated by Algorithm 1 and* $(\boldsymbol{E}, \boldsymbol{\lambda})$ *be the solution of (VI). Under the assumption that* $\gamma_k h_{\tilde{T}_k} \leq C$*, it holds that*

$$\lim_{k\to\infty} \|\boldsymbol{E}_k - \boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl})} = 0.$$

Strong Convergence

Theorem

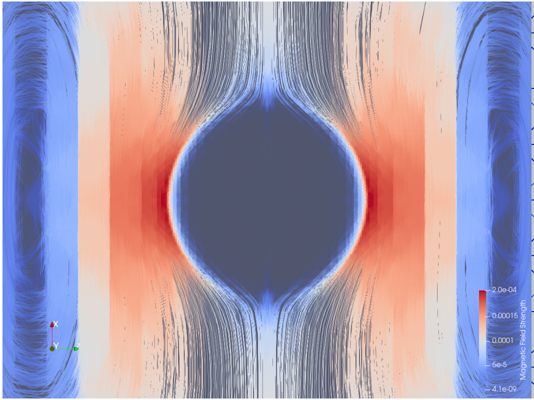
Let $\{(\boldsymbol{E}_k, \boldsymbol{\lambda}_k, \mathcal{T}_k, \boldsymbol{V}_k, \gamma_k)\}_{k\in\mathbb{N}_0}$ be the sequence generated by Algorithm 1 and $(\boldsymbol{E}, \boldsymbol{\lambda})$ be the solution of (VI). Under the assumption that $\gamma_k h_{\tilde{T}_k} \leq C$, it holds that

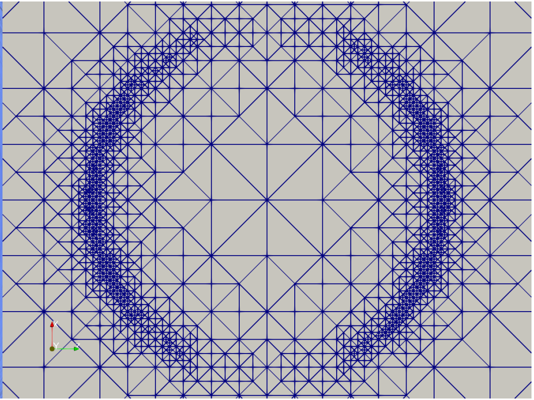
$$\lim_{k\to\infty} \|\boldsymbol{E}_k - \boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl})} = 0.$$

- Assumption $\gamma_k h_{\tilde{T}_k} \leq C$ due to stability estimate for $\operatorname{div} \boldsymbol{\lambda}_T^{\gamma}$ ($\tilde{\mathcal{T}}_k$ element with maximal error estimator)

$$h_T \|\operatorname{div} \boldsymbol{\lambda}_k\|_{\boldsymbol{L}^2(T)} \leq \frac{\gamma_k h_T \|j_c\|_{L^\infty(\Omega)}}{\sqrt{2}} \|\mathbf{curl}\, \boldsymbol{E}_k\|_{\boldsymbol{L}^2(T)} \quad \forall T \in \mathcal{T}_k$$

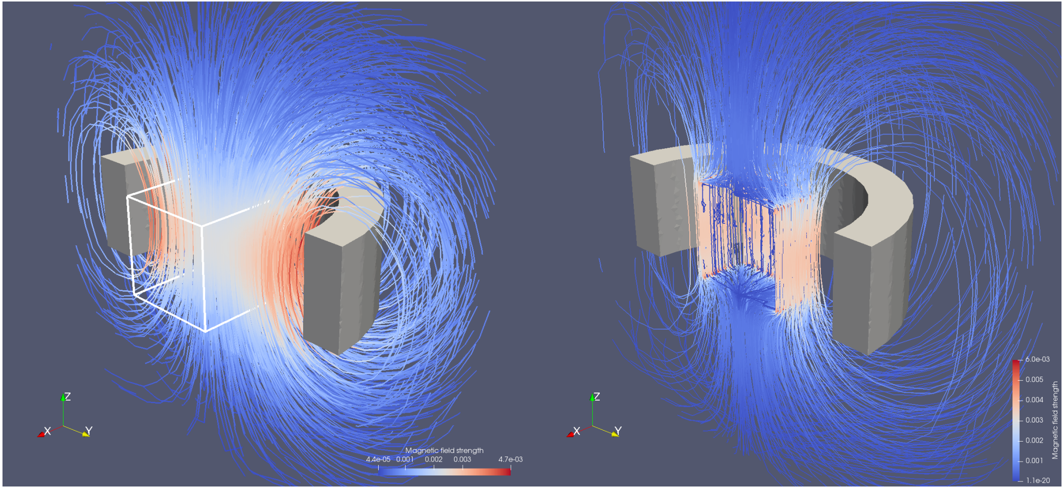
Numerical Experiment – high j_c





Shape Optimization

Shape Optimization – Motivation



Shape optimization problem

The minimization problem for some $B \subset \Omega$

(P) $$\min_{\omega \in \mathcal{O}} J(\omega) := \frac{1}{2} \int_B \kappa |\boldsymbol{E}(\omega) - \boldsymbol{E}_d|^2 \, dx + \int_\omega dx$$

Admissible shapes $\mathcal{O} = \{\omega \subset B \mid \omega \text{ is open, } L\text{-Lipschitz}\}$ where $\boldsymbol{E}(\omega)$ solves

(VI$_\omega$) $$\begin{aligned} a(\boldsymbol{E}(\omega), \boldsymbol{v} - \boldsymbol{E}(\omega)) + \int_\omega j_c |\boldsymbol{v}| \, dx - \int_\omega j_c |\boldsymbol{E}(\omega)| \, dx \\ \geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{E}(\omega)) \, dx \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) \end{aligned}$$

with

$$a(\boldsymbol{v}, \boldsymbol{w}) := \int_\Omega \varepsilon \boldsymbol{v} \cdot \boldsymbol{w} \, dx + \int_\Omega \nu \, \mathbf{curl}\, \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{w} \, dx,$$

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The minimization problem for some $B \subset \Omega$

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Admissible shapes $\mathcal{O} = \{\omega \subset B \mid \omega \text{ is open, } L\text{-Lipschitz}\}$ where $\boldsymbol{E}(\omega)$ solves

$$\begin{aligned} (\mathrm{VI}_\omega) \quad a(\boldsymbol{E}(\omega), \boldsymbol{v} - \boldsymbol{E}(\omega)) + \int_\omega j_c |\boldsymbol{v}| \, dx - \int_\omega j_c |\boldsymbol{E}(\omega)| \, dx \\ \geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{E}(\omega)) \, dx \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) \end{aligned}$$

Problem: **Not differentiable!** $\rightsquigarrow$ Regularization

Shape optimization problem

The minimization problem for some $B \subset \Omega$

$$(\mathsf{P}_\gamma) \qquad \min_{\omega \in \mathcal{O}} J_\gamma(\omega) := \frac{1}{2} \int_B \kappa |\boldsymbol{E}^\gamma(\omega) - \boldsymbol{E}_d|^2 \, dx + \int_\omega dx$$

Admissible shapes $\mathcal{O} = \{\omega \subset B \mid \omega \text{ is open, } L\text{-Lipschitz}\}$ where $\boldsymbol{E}^\gamma(\omega)$ solves

$$\begin{cases} a(\boldsymbol{E}^\gamma, \boldsymbol{v}) + \displaystyle\int_\omega \boldsymbol{\Lambda}_\gamma(\boldsymbol{E}^\gamma) \cdot \boldsymbol{v} \, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) \\ \boldsymbol{\Lambda}_\gamma(\boldsymbol{E}^\gamma)(x) = \dfrac{j_c \gamma \boldsymbol{E}^\gamma(x)}{\max\{1, \gamma |\boldsymbol{E}^\gamma(x)|\}} \text{ for a.e. } x \in \omega. \end{cases}$$

Problem: Mapping $\boldsymbol{v} \mapsto \boldsymbol{\Lambda}_\gamma(\boldsymbol{v})$ **still not differentiable!** $\rightsquigarrow$ Regularization

Shape optimization problem

The minimization problem for some $B \subset \Omega$

$$(\mathrm{P}_\gamma) \qquad \min_{\omega \in \mathcal{O}} J_\gamma(\omega) := \frac{1}{2} \int_B \kappa |\boldsymbol{E}^\gamma(\omega) - \boldsymbol{E}_d|^2 \, dx + \int_\omega dx$$

Admissible shapes $\mathcal{O} = \{\omega \subset B \mid \omega \text{ is open, } L\text{-Lipschitz}\}$ where $\boldsymbol{E}^\gamma(\omega)$ solves

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Finally: Mapping $\boldsymbol{v} \mapsto \boldsymbol{\Lambda}_\gamma(\boldsymbol{v})$ **is Gâteaux-differentiable**[4]

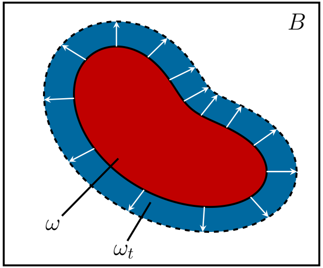
[4] J. C. De Los Reyes. Optimal control of a class of variational inequalities of the second kind. *SIAM Journal on Control and Optimization*, 49(4):1629–1658, 2011.

Shape sensitivity analysis

- Shape derivative at $\omega \in \mathcal{O}$ in direction $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}_c^{0,1}(B, \mathbb{R}^3)$:

$$dJ(\omega)(\boldsymbol{\theta}) := \lim_{t \searrow 0} \frac{J(\omega_t) - J(\omega)}{t},$$

where $\omega_t = \boldsymbol{T}_t(\omega)$ with $\boldsymbol{T}_t : \Omega \to \Omega$ the flow of $\boldsymbol{\theta}$



- Lagrangian approach:

$$\mathcal{L}(\omega, \boldsymbol{e}, \boldsymbol{v}) := \frac{1}{2} \int_\Omega \kappa |\boldsymbol{e} - \boldsymbol{E}_d|^2 \, dx + \int_\omega \, dx + a(\boldsymbol{e}, \boldsymbol{v}) + \int_\omega \boldsymbol{\Lambda}_\gamma(\boldsymbol{e}) \cdot \boldsymbol{v} \, dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx$$

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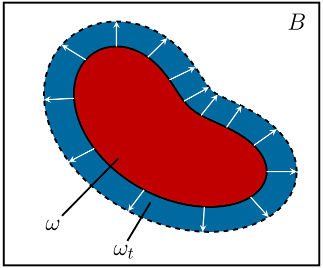

- Lagrangian approach:

$$\mathcal{L}(\omega_t, \boldsymbol{e}, \boldsymbol{v}) := \frac{1}{2} \int_B \kappa |\boldsymbol{e} - \boldsymbol{E}_d|^2 \, dx + \int_{\omega_t} dx + a(\boldsymbol{e}, \boldsymbol{v}) + \int_{\omega_t} \boldsymbol{\Lambda}_\gamma(\boldsymbol{e}) \cdot \boldsymbol{v} \, dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx$$

- Pull back: $\int_{\omega_t} \to \int_\omega$ with $x \mapsto \boldsymbol{T}_t(x)$ $\quad\rightsquigarrow\quad$ terms like $\boldsymbol{e} \circ \boldsymbol{T}_t$ with $\boldsymbol{e} \in \boldsymbol{H}_0(\mathbf{curl})$

- **Covariant transformation**

$$\Psi_t\colon \boldsymbol{H}_0(\mathbf{curl}) \to \boldsymbol{H}_0(\mathbf{curl}), \qquad \Psi_t(\boldsymbol{e}) := (D\boldsymbol{T}_t^{-\mathsf{T}}\boldsymbol{e}) \circ \boldsymbol{T}_t^{-1}.$$

with important identity

$$\big(\mathbf{curl}\,\Psi_t(\boldsymbol{e})\big) \circ \boldsymbol{T}_t = \xi(t)^{-1} D\boldsymbol{T}_t\,\mathbf{curl}\,\boldsymbol{e},$$

- Shape-Lagrangian

$$\begin{aligned} G(t,\boldsymbol{e},\boldsymbol{v}) := \mathcal{L}(\omega_t, \Psi_t(\boldsymbol{e}), \Psi_t(\boldsymbol{v})) &= \frac{1}{2}\int_B \kappa|\Psi_t(\boldsymbol{e}) - \boldsymbol{E}_d|^2\,dx + \int_{\omega_t} dx \\ &+ a(\Psi_t(\boldsymbol{e}), \Psi_t(\boldsymbol{v})) + \int_{\omega_t} \boldsymbol{\Lambda}_\gamma(\Psi_t(\boldsymbol{e})) \cdot \Psi_t(\boldsymbol{v})\,dx - \int_\Omega \boldsymbol{f} \cdot \Psi_t(\boldsymbol{v})\,dx. \end{aligned}$$

Averaged adjoint method

Shape-Lagrangian after change of variables $x \mapsto \boldsymbol{T}_t(x)$:

$$
\begin{aligned}
G(t,\boldsymbol{e},\boldsymbol{v}) = \frac{1}{2}\int_B \kappa\circ\boldsymbol{T}_t|D\boldsymbol{T}_t^{-\mathsf{T}}\boldsymbol{e}-\boldsymbol{E}_d\circ\boldsymbol{T}_t|^2\xi(t)\,dx + \int_\omega \xi(t)\,dx \\
+ \int_\Omega \mathbb{M}_1(t)\,\mathbf{curl}\,\boldsymbol{e}\cdot\mathbf{curl}\,\boldsymbol{v} + \mathbb{M}_2(t)\boldsymbol{e}\cdot\boldsymbol{v}\,dx \\
+ \int_\omega \mathbb{M}_3(t,\boldsymbol{e})\cdot\boldsymbol{v}\,dx - \int_\Omega (\boldsymbol{f}\circ\boldsymbol{T}_t)\cdot(D\boldsymbol{T}_t^{-\mathsf{T}}\boldsymbol{v})\xi(t)\,dx.
\end{aligned}
$$

with the notations

$$
\begin{aligned}
\mathbb{M}_1(t) &:= \xi(t)^{-1}D\boldsymbol{T}_t^{\mathsf{T}}(\nu\circ\boldsymbol{T}_t)D\boldsymbol{T}_t,\\
\mathbb{M}_2(t) &:= \xi(t)D\boldsymbol{T}_t^{-1}(\varepsilon\circ\boldsymbol{T}_t)D\boldsymbol{T}_t^{-\mathsf{T}},\\
\mathbb{M}_3(t,\boldsymbol{e}) &:= \xi(t)D\boldsymbol{T}_t^{-1}\boldsymbol{\Lambda}_\gamma(D\boldsymbol{T}_t^{-\mathsf{T}}\boldsymbol{e}).
\end{aligned}
$$

Averaged adjoint method

Shape-Lagrangian after change of variables $x \mapsto \boldsymbol{T}_t(x)$:

$$G(t, \boldsymbol{e}, \boldsymbol{v}) = \frac{1}{2} \int_B \kappa \circ \boldsymbol{T}_t |D\boldsymbol{T}_t^{-\mathsf{T}} \boldsymbol{e} - \boldsymbol{E}_d \circ \boldsymbol{T}_t|^2 \xi(t)\, dx + \int_\omega \xi(t)\, dx$$
$$+ \int_\Omega \mathbb{M}_1(t)\, \mathbf{curl}\, \boldsymbol{e} \cdot \mathbf{curl}\, \boldsymbol{v} + \mathbb{M}_2(t) \boldsymbol{e} \cdot \boldsymbol{v}\, dx$$
$$+ \int_\omega \mathbb{M}_3(t, \boldsymbol{e}) \cdot \boldsymbol{v}\, dx - \int_\Omega (\boldsymbol{f} \circ \boldsymbol{T}_t) \cdot (D\boldsymbol{T}_t^{-\mathsf{T}} \boldsymbol{v}) \xi(t)\, dx.$$

$\rightsquigarrow$ Compute $dJ_\gamma(\omega)(\boldsymbol{\theta}) = \partial_t G(t, \boldsymbol{E}^\gamma, \boldsymbol{P}^\gamma)|_{t=0}$ via **Averaged Adjoint Method**[5]

[5] A. Laurain and K. Sturm. Distributed shape derivative *via* averaged adjoint method and applications. *ESAIM Math. Model. Numer. Anal.*, 50(4):1241–1267, 2016.

Shape Derivative of J_γ

Theorem

The functional J_γ in (P_γ) is shape differentiable with

$$dJ_\gamma(\omega)(\boldsymbol{\theta}) = \partial_t G(0, \boldsymbol{E}^\gamma, \boldsymbol{P}^\gamma) = \int_B S_1^\gamma : D\boldsymbol{\theta} + \boldsymbol{S}_0^\gamma \cdot \boldsymbol{\theta}\, dx,$$

where $\boldsymbol{E}^\gamma \in \boldsymbol{H}_0(\mathbf{curl})$ is the solution to (VI_γ) and $\boldsymbol{P}^\gamma \in \boldsymbol{H}_0(\mathbf{curl})$ solves

$$a(\hat{\boldsymbol{e}}, \boldsymbol{P}^\gamma) + \int_\omega \boldsymbol{\Lambda}_\gamma'(\boldsymbol{E}^\gamma)\hat{\boldsymbol{e}} \cdot \boldsymbol{P}^\gamma\, dx = -\int_B \kappa(\boldsymbol{E}^\gamma - \boldsymbol{E}_d) \cdot \hat{\boldsymbol{e}}\, dx \quad \forall \hat{\boldsymbol{e}} \in \boldsymbol{H}_0(\mathbf{curl}).$$

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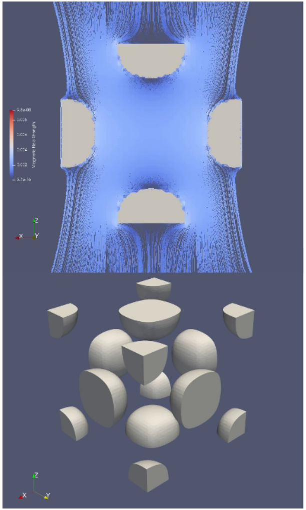
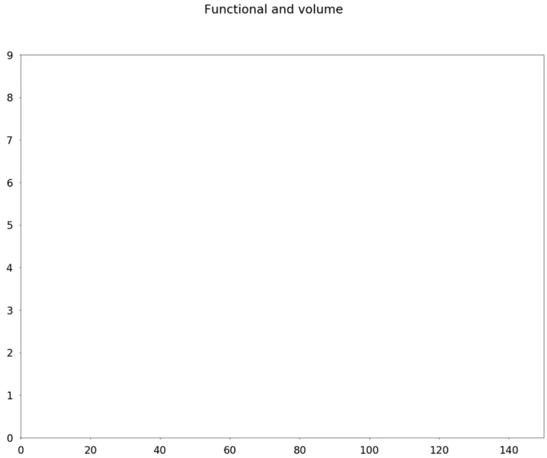
Shape Derivative of J_γ

The terms S_1^γ and $\boldsymbol{S}_0^\gamma$ are given by

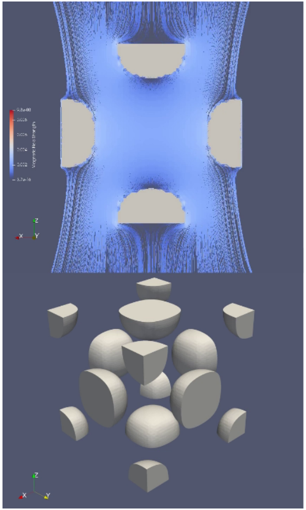
$$
\begin{aligned}
S_1^\gamma = {} & \big[\frac{\kappa}{2}|\boldsymbol{E}^\gamma - \boldsymbol{E}_d|^2 + \chi_\omega - \nu\, \mathbf{curl}\, \boldsymbol{E}^\gamma \cdot \mathbf{curl}\, \boldsymbol{P}^\gamma + \varepsilon \boldsymbol{E}^\gamma \cdot \boldsymbol{P}^\gamma + \chi_\omega \boldsymbol{\Lambda}_\gamma(\boldsymbol{E}^\gamma) \cdot \boldsymbol{P}^\gamma \\
& - \boldsymbol{f} \cdot \boldsymbol{P}^\gamma\big] \boldsymbol{I}_3 - \kappa \boldsymbol{E}^\gamma \otimes (\boldsymbol{E}^\gamma - \boldsymbol{E}_d) + \nu\, \mathbf{curl}\, \boldsymbol{E}^\gamma \otimes \mathbf{curl}\, \boldsymbol{P}^\gamma \\
& + \nu^\mathsf{T}\, \mathbf{curl}\, \boldsymbol{P}^\gamma \otimes \mathbf{curl}\, \boldsymbol{E}^\gamma - \boldsymbol{P}^\gamma \otimes \varepsilon \boldsymbol{E}^\gamma - \boldsymbol{E}^\gamma \otimes \varepsilon^\mathsf{T} \boldsymbol{P}^\gamma + \boldsymbol{P}^\gamma \otimes \boldsymbol{f} \\
& - \chi_\omega \boldsymbol{\Lambda}_\gamma(\boldsymbol{E}^\gamma) \otimes \boldsymbol{P}^\gamma - \boldsymbol{E}^\gamma \otimes \boldsymbol{\psi}^\gamma(\boldsymbol{E}^\gamma) \boldsymbol{P}^\gamma, \\
\boldsymbol{S}_0^\gamma = {} & \frac{\nabla \kappa}{2}|\boldsymbol{E}^\gamma - \boldsymbol{E}_d|^2 - \kappa D \boldsymbol{E}_d^\mathsf{T}(\boldsymbol{E}^\gamma - \boldsymbol{E}_d) + (D\nu^\mathsf{T}\, \mathbf{curl}\, \boldsymbol{E}^\gamma)\, \mathbf{curl}\, \boldsymbol{P}^\gamma \\
& + (D\epsilon^\mathsf{T} \boldsymbol{E}^\gamma) \boldsymbol{P}^\gamma - D\boldsymbol{f}^\mathsf{T} \boldsymbol{P}^\gamma.
\end{aligned}
$$

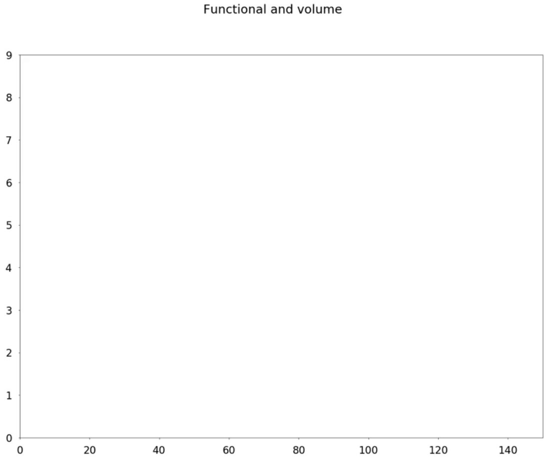
Numerical Experiments





Numerical Experiments





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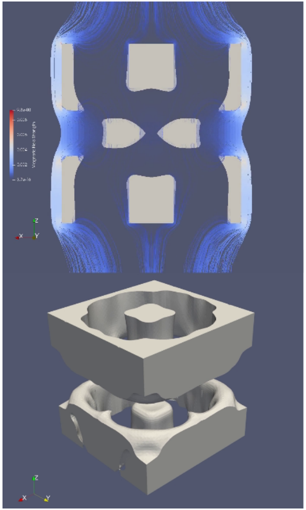

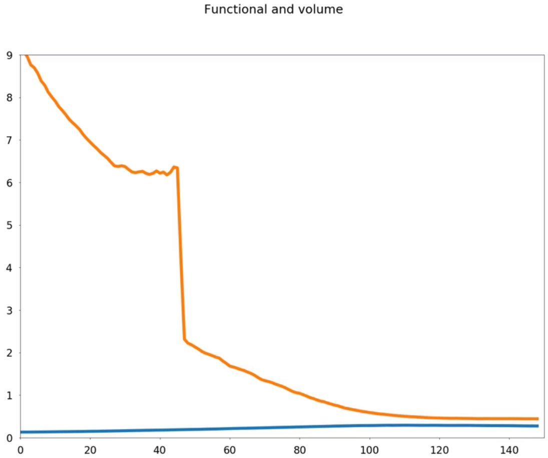

Thank you!