EFFECT OF RUGOSITY ON A FLOW GOVERNED BY STATIONARY NAVIER-STOKES EQUATIONS

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By

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Abstract. We study the stationary flow of a fluid occupying a 3D infinite horizontal domain bounded by a rough wall that is at rest and by a plane that moves with a constant velocity. The rough wall is a plane covered with periodically distributed asperities of size ε. We prove that, outside a neighbourhood of the rough region, the flow behaves asymptotically as a Couette flow, as $\varepsilon \to 0$, up to an exponentially small error.

1. Introduction. Numerical simulation of flows on rough walls, riblets or waves, for example, requires wall laws. These wall laws are boundary conditions that are imposed on a fictive flat wall inside the domain in order to get rid of the region with strong gradient that contains rugosity by taking into account their effect on the flow. Such simulations can be found in [1], [2], [7], for example. Actually, these wall laws come from empirical logarithmic calculations; see, for instance, [9], [12]. This motivates the mathematical study of the effect of rugosity in some simplified situations in order to see how such laws could be rigorously derived.

Here, we consider a viscous incompressible fluid occupying a 3D infinite horizontal domain bounded by two walls, a plane one and a rough one, which is a plane covered with periodically distributed rugosities. The rough wall is assumed to be at rest and the plane one moves with a constant horizontal velocity. We prove the existence of a Couette flow, depending on the size ε of the asperities, which approximates the stationary Navier-Stokes flow outside the rough region, up to an error decreasing exponentially fast with respect to ε. A similar result is established in [5] for a Stokes flow, with an explicitly calculated Couette approximation. The extra difficulty here is that, because of nonlinearity, we are no longer able to perform such an explicit calculation.

More precisely, we prove that the velocity is the sum of three terms: a Couette flow x_3/l_3 where x_3 is the vertical coordinate and l_3 is the thickness of the flow; another Couette flow $\varepsilon(x_3/l_3 - 1)f_\varepsilon$ where f_ε is a constant vector; a remainder w_ε that is bounded outside a neighbourhood of the rough wall by $|w_\varepsilon(x)| \leq c \exp(-c' x_3/\varepsilon)$.

In order to prove this result, we write the equations satisfied by w_ε for an arbitrary given f_ε. Then, we approximate w_ε by a solution Ψ_ε of Navier-Stokes type equations in a half-space. Such a solution has a limit $\overline{\Psi}_{\varepsilon\infty}$ as $x_3 \to \infty$ and the convergence is controlled by $c\exp(-c'x_3/\varepsilon)$. Finally, we prove that f_ε may be chosen such that $\overline{\Psi}_{\varepsilon\infty} = 0$ by using a fixed point argument. This gives the result.

The exponential control of the convergence relies on a Saint-Venant type estimate established in [4]. For similar estimates, see [10], [11], [14], and [15] and references cited therein. For other studies of rugosity in problems governed by various pde's, see [6], [8].

2. Main result. Let us consider a viscous fluid occupying an infinite horizontal domain bounded by two walls, a plane one $\mathcal{P}$ and a rough one $\mathcal{R}_\varepsilon$, covered with periodically distributed asperities of small size ε. The plane $\mathcal{P}$ is moving in a parallel direction to $\mathcal{R}_\varepsilon$ with a constant velocity g. The domain of the flow is

$$\mathcal{O}_\varepsilon = \{x \in \mathbb{R}^3 : x' \in \mathbb{R}^2, \eta_\varepsilon(x') < x_3 < l_3\},$$

where $x' = (x_1, x_2), l_3 > 0, \varepsilon > 0$ and η_ε (representing the profile of asperities) is the function given by

$$\eta_\varepsilon(x') = \varepsilon\eta\left(\frac{x'}{\varepsilon}\right),$$

η being periodic with respect to x_1 and x_2 with periods l_1 and l_2. The velocity $u_\varepsilon = (u_{\varepsilon 1}, u_{\varepsilon 2}, u_{\varepsilon 3})$ and the pressure p_ε satisfy the stationary Navier-Stokes equations

$$-\nu\Delta u_\varepsilon + u_\varepsilon \cdot \nabla u_\varepsilon + \nabla p_\varepsilon = 0, \quad \nabla \cdot u_\varepsilon = 0, \quad u_\varepsilon|_{\mathcal{R}_\varepsilon} = 0, \quad u_\varepsilon|_{\mathcal{P}} = g, \qquad (1)$$

where $\nu > 0$ and $g = (g', 0)$ are constant, and they are assumed to be periodic with respect to x_1 and x_2 with periods εl_1 and εl_2. We assume $1/\varepsilon$ to be an integer so that $\eta_\varepsilon, u_\varepsilon$ and p_ε are also periodic with respect to x_1 and x_2 with periods l_1 and l_2. Setting $S_\varepsilon = (0, \varepsilon l_1) \times (0, \varepsilon l_2)$, we denote by

$$\Omega_\varepsilon = \{x \in \mathbb{R}^3 : x' \in S_\varepsilon, \eta_\varepsilon(x') < x_3 < l_3\}$$

the bounded part of the domain that is bounded by the part of the rough wall $R_\varepsilon = \{x : x' \in S_\varepsilon, x_3 = \eta_\varepsilon(x')\}$, by the part of plane $P = \{x : x' \in S_\varepsilon, x_3 = l_3\}$, and by the immaterial lateral boundary $L_\varepsilon = \{x : x' \in \partial S_\varepsilon, \eta_\varepsilon(x') \leq x_3 \leq l_3\}$. Clearly, $\mathcal{O}_\varepsilon$ is generated by periodic horizontal translations of Ω_ε. We denote, for $m \geq 0$,

$$H_{\text{per}}^m(\Omega_\varepsilon) = \{v \in H_{\text{loc}}^m(\mathcal{O}_\varepsilon) : v \in H^m(\Omega_\varepsilon),$$
$$v(x_1, x_2, x_3) = v(x_1 + \varepsilon l_1, x_2, x_3) = v(x_1, x_2 + \varepsilon l_2, x_3)\},$$

provided with the norm of $H^m(\Omega_\varepsilon)$.

The existence of a velocity u_ε in this space can be proved by using a Hopf divergence free extension of the boundary data, as in Section 7.1, p. 101 of [13]. Here such an extension is explicitly given by $s(x_3)g$ provided that $s(l_3) = 1$ and s vanishes outside a neighbourhood of l_3. The pressure p_ε is then obtained by the de Rham Theorem. Its periodicity may be proved as in Theorem 1 of [4]. In this way, we get the following existence result.

PROPOSITION 1. There exists a solution of (1) such that

$$u_\varepsilon \in (H^1_{\text{per}}(\Omega_\varepsilon))^3, \quad p_\varepsilon \in L^2_{\text{per}}(\Omega_\varepsilon), \quad \int_{\Omega_\varepsilon} p_\varepsilon = 0. \tag{2}$$

Moreover, for ε small enough, such a solution is unique.

We are interested in the asymptotic behaviour, as $\varepsilon \to 0$, of this solution. Our objective is to show that, outside a neighbourhood of the rough wall, the flow behaves asymptotically as a Couette flow and more precisely that

$$u_\varepsilon(x) = \frac{x_3}{l_3} g + \varepsilon \left(\frac{x_3}{l_3} - 1 \right) f_\varepsilon + w_\varepsilon(x), \tag{3}$$

where $f_\varepsilon = (f_{\varepsilon 1}, f_{\varepsilon 2}, 0)$ is a vector of $\mathbb{R}^3$ and $w_\varepsilon(x)$ decays exponentially with respect to ε. Denoting $\overline{\eta} = \sup\{\eta(x') : x' \in \mathbb{R}^2\}$, the main result is the following one.

THEOREM 2. Let $(u_\varepsilon, p_\varepsilon)$ satisfy (1) and (2). There exist two positive numbers c and c' and, for any $\alpha \in \mathbb{N}^3$, a positive number c''_α, such that: for any $\varepsilon < c$, there exists $f_\varepsilon = (f_{\varepsilon 1}, f_{\varepsilon 2}, 0)$ such that the remainder w_ε, defined by (3), and p_ε satisfy

$$|\partial^\alpha w_\varepsilon(x)| + |\partial^\alpha p_\varepsilon(x)| \le c''_\alpha \exp\left(-c' \frac{x_3}{\varepsilon} \right) \tag{4}$$

for any $x \in \Omega_\varepsilon$ such that $x_3 \ge \varepsilon(\overline{\eta} + 1)$.

Remark that f_ε is not unique since the result still holds if an arbitrary vector $f'_\varepsilon = (f'_{\varepsilon 1}, f'_{\varepsilon 2}, 0)$ is such that $|f'_\varepsilon| \le \exp(-c'/\varepsilon)$ is added to f_ε.

Obviously, c, c' and c''_α depend on the data g, ν, l and η. In the sequel, c, c' and c'' will still be used to denote various numbers depending on these data. We will denote by c_α or c_M numbers that, moreover, depend on α or M.

3. Decomposition of the remainder.

Let f be an arbitrary vector of $\mathbb{R}^3$ such that

$$f_3 = 0. \tag{5}$$

Replacing in (1) u_ε by the right-hand side of (3) relative to $f_\varepsilon = f$, we get the following equations for $(w_\varepsilon, p_\varepsilon)$:

$$w_\varepsilon \in (H^1_{\text{per}}(\Omega_\varepsilon))^3, \qquad p_\varepsilon \in L^2_{\text{per}}(\Omega_\varepsilon),$$

$$-\nu \Delta w_\varepsilon + w_\varepsilon \cdot \nabla w_\varepsilon + w_{\varepsilon 3} \frac{g + \varepsilon f}{l_3} + \left(\frac{x_3}{l_3}(g + \varepsilon f) - \varepsilon f \right) \cdot \nabla w_\varepsilon + \nabla p_\varepsilon = 0,$$

$$\nabla \cdot w_\varepsilon = 0, \qquad w_\varepsilon|_{\mathcal{P}} = 0, \qquad w_\varepsilon|_{\mathcal{R}_\varepsilon} = -\frac{\eta_\varepsilon}{l_3}(g + \varepsilon f) + \varepsilon f.$$

We will approximate w_ε by a function Ψ_ε defined in the whole rough half-space $\widetilde{\mathcal{O}}_\varepsilon = \{x \in \mathbb{R}^3 : x' \in \mathbb{R}^2, x_3 > \eta_\varepsilon(x')\}$. This will allow us to use a Saint-Venant type property of exponential decay as $x_3 \to \infty$; see (11). The problem on w_ε is close to a problem in such an infinite strip because the aspect ratio length/width of Ω_ε is $l_3/(\varepsilon l_1)$.

To this end, we extend Ω_ε in the vertical direction by

$$\widetilde{\Omega}_\varepsilon = \{x \in \mathbb{R}^3 : x' \in S_\varepsilon, x_3 > \eta_3(x')\},$$

and we consider a corrector $(\Psi_\varepsilon, \xi_\varepsilon)$ in the rough half-space $\widetilde{\mathcal{O}}_\varepsilon$ that satisfies:

$$\Psi_\varepsilon \in (H^1_{\mathrm{per,loc}}(\widetilde{\Omega}_\varepsilon))^3, \qquad \nabla\Psi_\varepsilon \in (L^2_{\mathrm{per}}(\widetilde{\Omega}_\varepsilon))^9, \qquad \xi_\varepsilon \in L^2_{\mathrm{per,loc}}(\widetilde{\Omega}_\varepsilon),$$

$$-\nu\Delta\Psi_\varepsilon + \Psi_\varepsilon \cdot \nabla\Psi_\varepsilon + \Psi_{\varepsilon 3}\frac{g+\varepsilon f}{l_3} + a_\varepsilon \cdot \nabla\Psi_\varepsilon + \nabla\xi_\varepsilon = 0, \tag{6}$$

$$\nabla \cdot \Psi_\varepsilon = 0, \qquad \Psi_\varepsilon|_{\mathcal{R}_\varepsilon} = -\frac{\eta_\varepsilon}{l_3}(g+\varepsilon f) + \varepsilon f,$$

where

$$a_\varepsilon(x_3) = \begin{cases} \dfrac{x_3(g+\varepsilon f)}{l_3} - \varepsilon f & \text{if } \eta_\varepsilon \le x_3 \le l_3, \\[2mm] \dfrac{-gx_3 + g(2l_3 + \varepsilon\bar\eta)}{l_3 + \varepsilon\bar\eta} & \text{if } l_3 < x_3 \le 2l_3 + \varepsilon\bar\eta, \\[2mm] 0 & \text{otherwise.} \end{cases} \tag{7}$$

We use here a function a_ε that vanishes for large enough x_3 in order to get (11); see (16).

More precisely, we define $(\Psi_\varepsilon, \xi_\varepsilon)$ to be the limit as $m \to \infty$ of the periodic solution $(\Psi_\varepsilon^m, \xi_\varepsilon^m)$ that cancels for $x_3 = m$, that is, the solution of:

$$-\nu\Delta\Psi_\varepsilon^m + \Psi_\varepsilon^m \cdot \nabla\Psi_\varepsilon^m + \Psi_{\varepsilon 3}^m\frac{g+\varepsilon f}{l_3} + a_\varepsilon \cdot \nabla\Psi_\varepsilon^m + \nabla\xi_\varepsilon^m = 0,$$

$$\nabla \cdot \Psi_\varepsilon^m = 0, \qquad \Psi_\varepsilon^m|_{\mathcal{R}_\varepsilon} = -\frac{\eta_\varepsilon}{l_3}(g+\varepsilon f) + \varepsilon f, \qquad \Psi_\varepsilon^m|_{x_3=m} = 0. \tag{8}$$

The approximation Ψ_ε of w_ε does not vanish on $\mathcal{P}$. Thus, we add it to the residue $(\Xi_\varepsilon, \sigma_\varepsilon)$ defined in $\mathcal{O}_\varepsilon$ by

$$w_\varepsilon = \Psi_\varepsilon + \Xi_\varepsilon, \qquad p_\varepsilon = \xi_\varepsilon + \sigma_\varepsilon. \tag{9}$$

It satisfies

$$\Xi_\varepsilon \in (H^1_{\mathrm{per}}(\Omega_\varepsilon))^3, \qquad \sigma_\varepsilon \in L^2_{\mathrm{per}}(\Omega_\varepsilon),$$

$$-\nu\Delta\Xi_\varepsilon + \Xi_\varepsilon \cdot \nabla\Xi_\varepsilon + \Xi_\varepsilon \cdot \nabla\Psi_\varepsilon + \Psi_\varepsilon \cdot \nabla\Xi_\varepsilon + \Xi_{\varepsilon 3}\frac{g+\varepsilon f}{l_3}$$
$$+ \left(\frac{x_3}{l_3}(g+\varepsilon f) - \varepsilon f\right) \cdot \nabla\Xi_\varepsilon + \nabla\sigma_\varepsilon = 0, \tag{10}$$

$$\nabla \cdot \Xi_\varepsilon = 0, \qquad \Xi_\varepsilon|_{\mathcal{P}} = -\Psi_\varepsilon|_{\mathcal{P}}, \qquad \Xi_\varepsilon|_{\mathcal{R}_\varepsilon} = 0.$$

There exist, see Lemma 3, $\overline{\Psi}_{\varepsilon\infty} \in \mathbb{R}^3$ and $\overline{\xi}_{\varepsilon\infty} \in \mathbb{R}$ and three positive numbers c, c' and c''_α such that, if $\varepsilon < c$, the solution $(\Psi_\varepsilon, \xi_\varepsilon)$ of (6) satisfies, for all $x \in \mathcal{O}_\varepsilon$ such that $x_3 \ge \varepsilon(\bar\eta + 1)$,

$$|\partial^\alpha(\Psi_\varepsilon - \overline{\Psi}_{\varepsilon\infty})(x)| + |\partial^\alpha(\xi_\varepsilon - \overline{\xi}_{\varepsilon\infty})(x)| \le c''_\alpha \exp\left(-c'\frac{x_3}{\varepsilon}\right). \tag{11}$$

The main point to prove Theorem 1 is to establish that there exists f such that

$$\overline{\Psi}_{\varepsilon\infty} = 0. \tag{12}$$

This will be done by proving that, if ε is small enough, the map $f \mapsto f - \overline{\Psi}_{\varepsilon\infty}/\varepsilon$ is contractive and therefore possesses (at least) one fixed point, which will be denoted f_ε. To prove this, we will define $(\Phi_\varepsilon, \Pi_\varepsilon)$ by

$$\Psi_\varepsilon = \varepsilon(\Phi_\varepsilon + f), \qquad \xi_\varepsilon = \varepsilon\Pi_\varepsilon, \tag{13}$$

and we will prove (Lemma 8) that the map $f \mapsto -\overline{\Phi}'_{\varepsilon\infty} = f - \overline{\Psi}'_{\varepsilon\infty}/\varepsilon$ is contractive. The proof will use preliminary estimates on Ψ_ε (Lemma 4), on $\nabla\Phi_\varepsilon$ (Lemma 5), on $\overline{\Phi}_\varepsilon^2 - \overline{\Phi}_\varepsilon^1$ (Lemma 6) and on its gradient (Lemma 7), where Φ_ε^1 and Φ_ε^2 correspond to distinct data f^1 and f^2.

Using (12), we will prove, in Lemma 9, that the H^1 norm of the residue Ξ_ε goes exponentially fast to 0 as ε goes to 0. This will imply the pointwise decay

$$|\partial^\alpha \Xi_\varepsilon(x)| + |\partial^\alpha \sigma_\varepsilon(x)| \leq c''_\alpha \exp\left(\frac{-c'}{\varepsilon}\right) \tag{14}$$

for all $x \in \Omega_\varepsilon$ such that $x_3 \geq \varepsilon(\overline{\eta} + 1)$. Together with the pointwise decay (11) and with (12), this will provide the announced decay (4).

4. Saint-Venant estimate. Let (Ψ', ξ') be a couple of functions in $\mathbb{R}^3_+$ that are periodic with respect to x_i with period k_i, $i = 1$ or 2, and that satisfy

$$\Psi' \in (H^1_{\text{loc}}(\mathbb{R}^3_+))^3, \qquad \xi' \in L^2_{\text{loc}}(\mathbb{R}^3_+), \qquad \int_{\Lambda^0} |\nabla\Psi'|^2 \leq E,$$

$$-\nu\Delta\Psi' + \Psi' \cdot \nabla\Psi' + \Psi'_3 b + a \cdot \nabla\Psi' + \nabla\xi' = 0, \tag{15}$$

$$\nabla \cdot \Psi' = 0, \qquad \int_\Sigma \Psi'_3(x', 0)\, dx' = 0, \qquad |\overline{\Psi}'(0)| \leq E,$$

where $\Sigma = (0, k_1) \times (0, k_2)$, $\Lambda^t = \{x \in \mathbb{R}^3 : x' \in \Sigma, x_3 > t\}$, $b \in \mathbb{R}^3$, $b_3 = 0$ and

$$a = a(x_3), \quad a_3 = 0, \quad a = 0 \text{ in } [2l_3, +\infty), \quad a \in (W^{1,\infty}(0, \infty))^3. \tag{16}$$

LEMMA 3. There exists $\overline{\Psi}'_\infty \in \mathbb{R}^3$ such that $\overline{\Psi}'_{\infty 3} = 0$ and

$$|\overline{\Psi}'(t) - \overline{\Psi}'_\infty| \leq \frac{\kappa K^2}{\nu k_1 k_2} \int_{\Lambda^t} |\nabla\Psi'|^2 \tag{17}$$

where κ is a universal constant and $K = \max(k_1, k_2)$.

If $K \leq \frac{1}{2\kappa}\sqrt{\nu/|b|}$, one has, for all $t \geq K$,

$$\|\nabla\Psi'\|_{(L^2(\Lambda^t))^9} \leq \mu \left(\int_{\Lambda^0} |\nabla\Psi'|^2\right)^{1/2} \exp\left(-\frac{\lambda t}{K}\right) \tag{18}$$

where $\lambda > 0$. For all $\alpha \in \mathbb{N}^3$ and $t \geq K$, one has

$$\|\Psi' - \Psi'_\infty\|_{(W^{\alpha,\infty}(\Lambda^t))^3} + \|\xi'\|_{W^{\alpha,\infty}(\Lambda^t)} \leq \sigma_\alpha K^{-(m+2)} \|\Psi' - \Psi'_\infty\|_{(H^1(\Lambda^{t-K}))^3}. \tag{19}$$

Given a positive real number c, one can choose λ, μ and σ_α such that (18) and (19) hold for all a, b, k_1 and k_2 such that $|a| \leq c, |b| \leq c, k_1/k_2 \leq c$ and $k_2/k_1 \leq c$.

For the reader's convenience, let us give hints about the proof of Lemma 3, which is rather complicated. In the much simpler case of Laplace's equation, a proof based on an explicit decomposition on a special basis is given in [5], Lemma 4. In the case of stationary Navier-Stokes equations in a semi-infinite channel with lateral Dirichlet boundary condition, a similar result is proved in [10], Vol. 2, Theorem 4.1, p. 261; see also references therein.

Here, the lateral Dirichlet boundary condition is replaced by a periodicity condition, and therefore the mean value $\overline{\Psi}'(t) = (1/|\Sigma|) \int_\Sigma \Psi'(x', t')\, dx'$ may no longer be bounded

by using the Poincaré inequality as in [10]. However, a similar method may be used, using the Poincaré-Wirtinger inequality and the fact that $\overline{\Psi}'(t)$ remains bounded. This is done in [4], Theorem 2.

Here, Ψ' is not governed by stationary Navier-Stokes equations; it is governed by (15), which contains the extra term $\Psi'_3 b + a \cdot \nabla \Psi'$. However, the proof of [4] extends to this case, the extra term being bounded by similar calculus; this is done in [3], Theorem 2 and its proof. Outlines are as follows.

In a first step, using the fact that $\nabla \Psi' \in (L^2(\Lambda^0))^9$, we prove that, for $i = 1$ or 2,

$$\frac{d\overline{\Psi}'_i}{dt} = \frac{1}{\nu k_1 k_2} \int_\Sigma (\Psi'_3 \Psi'_i)(x', t) \, dx'.$$

It follows that $\overline{\Psi}'(t)$ goes to a limit $\overline{\Psi}'_\infty$ as $t \to \infty$ and that (17) is satisfied.

In a second step, we prove that, if $K \leq \sqrt{\nu/|b|}/(2\kappa)$, then $y(t) = \int_{\Lambda^t} |\nabla \Psi'|^2 \, dx'$ satisfies the following ordinary differential inequality, with c_i independent of K:

$$\frac{dy}{dt}(t) + \frac{c_1}{K} \int_t^\infty y(\tau) \, d\tau \leq \frac{c_2}{K} y(t).$$

This inequality implies $|y(t)| \leq c_3 y(0) \exp(-c_4 t/K)$ and thus (18).

Finally, the pointwise decay (19) follows from regularity properties of solutions of Stokes equations, as in [4], proof of Theorem 2, pp. 363–364.

5. Properties of Ψ_ε.

Let us define, for $t \geq \varepsilon \overline{\eta}$,

$$\overline{\Psi}_\varepsilon(t) = \frac{1}{l_1 l_2} \int_S \Psi_\varepsilon(x', t) \, dx'.$$

Remark that (11) implies that

$$\overline{\Psi}_{\varepsilon\infty} = \lim_{t \to +\infty} \overline{\Psi}_\varepsilon(t).$$

LEMMA 4. Given $M > 0$, there exists a positive number c_M such that, if $|f| \leq M$ and $\varepsilon < c_M$, then

$$\int_{\tilde{\Omega}_\varepsilon} |\nabla \Psi_\varepsilon|^2 \leq c\varepsilon^3(|g| + \varepsilon M), \tag{20}$$

$$|\overline{\Psi}_\varepsilon(\varepsilon\overline{\eta})| \leq c\varepsilon(|g| + \varepsilon M), \tag{21}$$

where c does not depend on M. For all $t \geq \varepsilon \overline{\eta}$,

$$\overline{\Psi}_{\varepsilon 3}(t) = 0, \tag{22}$$

$$|\overline{\Psi}_\varepsilon(t) - \overline{\Psi}_{\varepsilon\infty}| \leq \frac{\kappa c_l}{\nu} \int_{\Theta^t} |\nabla \Psi_\varepsilon|^2, \tag{23}$$

where $c_l = \max(l_1/l_2, l_2/l_1)$.

We denote by Θ^t the semi-infinite vertical strip defined by

$$\Theta^t = \{x \in \mathbb{R}^3 : x' \in S_\varepsilon, x_3 > t\}.$$

The dependence on ε of Θ^t and of other quantities defined later on is not pointed out by a subscript in order to reduce notation.

Proof of (22). Let Ψ_ε^m be the approximation of Ψ_ε defined by (8). Let us first prove that, for all $t \geq \varepsilon\bar{\eta}$,

$$\overline{\Psi}_{\varepsilon 3}^m(t) = 0. \tag{24}$$

If $m \leq t$, it is obvious since Ψ_ε^m cancels if $x_3 \geq m$. Else, let $\Lambda = \{x : x' \in S_\varepsilon, t < x_3 < m\}$ and $\Sigma^t = \{x : x' \in S, x_3 = t\}$. The incompressibility condition gives

$$0 = \int_\Lambda \nabla \cdot \Psi_\varepsilon = \int_{\Sigma^t \cup \Sigma^m \cup L} \Psi_\varepsilon \cdot n \, ds,$$

where L is the lateral part of the boundary of Λ and n is the unit outward vector field. The integral over L cancels since Ψ_ε is periodic with respect to x_1 and x_2, and the integral over S^m cancels since Ψ_ε^m vanishes on it. Then,

$$\int_{S^t} \Psi_{\varepsilon 3}^m \, ds = 0.$$

This gives (24), and (22) follows by passing to the limit as $m \to \infty$. □

Proof of (20). Let Ψ_ε^m be again the approximation of Ψ_ε defined by (8). Let us consider the extension in $\widetilde{\mathcal{O}}_\varepsilon$ of the trace of Ψ_ε on $\mathcal{R}_\varepsilon$ given by

$$h(x', x_3) = \begin{cases} -\dfrac{x_3}{l_3}(g + \varepsilon f) + \varepsilon f & \text{if } \eta_\varepsilon \leq x_3 < \varepsilon\bar{\eta}, \\[2mm] \left(-\dfrac{\varepsilon\bar{\eta}}{l_3}(g + \varepsilon f) + \varepsilon f\right) \dfrac{2x_3 - l_3}{2\varepsilon\bar{\eta} - l_3} & \text{if } \varepsilon\bar{\eta} \leq x_3 < l_3/2, \\[2mm] 0 & \text{otherwise.} \end{cases} \tag{25}$$

Let $\psi = \Psi_\varepsilon^m - h$ and $\xi = \xi_\varepsilon^m$ where $m \geq l_3/2$, and let $\Omega_\varepsilon^m = \{x \in \widetilde{\Omega}_\varepsilon : x_3 < m\}$. In Ω_ε^m, the pair (ψ, ξ) satisfies

$$-\nu\Delta\psi - \nu\Delta h + \psi \cdot \nabla\psi + h \cdot \nabla\psi + \psi \cdot \nabla h + \psi_3 \frac{g + \varepsilon f}{l_3} + a_\varepsilon \cdot \nabla\psi + \nabla\xi = 0, \tag{26}$$

$$\nabla \cdot \psi = 0, \qquad \psi|_{\mathcal{R}_\varepsilon} = 0, \qquad \psi|_{x_3 = m} = 0.$$

Multiplying the first equation in (26) by ψ and integrating by parts over Ω_ε^m, we obtain

$$\nu \int_{\Omega_\varepsilon^m} |\nabla\psi|^2 + \nu \int_{\Omega_\varepsilon^m} \nabla\psi \cdot \nabla h + \int_{\Omega_\varepsilon^m} \psi \cdot \nabla h \cdot \psi + \int_{\Omega_\varepsilon^m} \psi_3 \frac{g + \varepsilon f}{l_3} \cdot \psi = 0. \tag{27}$$

Let us estimate the last three integrals. Using the definition of h and (24), we get

$$\left| \int_{\Omega_\varepsilon^m} \nabla\psi \cdot \nabla h \right| \leq \frac{1}{2} \int_{\Omega_\varepsilon^m} |\nabla\psi|^2 + c\varepsilon^3(|g| + \varepsilon M),$$

$$\left| \int_{\Omega_\varepsilon^m} \psi \cdot \nabla h \cdot \psi_3 \right| \leq c\varepsilon^2(|g| + M) \int_{\Omega_\varepsilon^m} |\nabla\psi|^2,$$

and

$$\left| \int_{\Omega_\varepsilon^m} \psi_3 \frac{g + \varepsilon f}{l_3} \cdot \psi \right| \leq c\varepsilon^2(|g| + M) \int_{\Omega_\varepsilon^m} |\nabla\psi|^2.$$

Inequality (20) comes from (27) with these three estimates by using the decomposition $\Psi_\varepsilon^m = \psi + h$ and by passing to the limit over m. □

Proof of (21). Let us notice that

$$\Psi_\varepsilon(x', \varepsilon\overline\eta) = \int_{\eta_\varepsilon(x')}^{\varepsilon\overline\eta} \partial_z \Psi_\varepsilon(x', z)\, dz - \frac{\eta_\varepsilon(x')}{l_3}(g + \varepsilon f) + \varepsilon f.$$

Integrating with respect to x', we get

$$|\overline\Psi_\varepsilon(\varepsilon\overline\eta)| \leq \frac{1}{\varepsilon^2 l_1 l_2} \int_{S_\varepsilon} \int_{\eta_\varepsilon(x')}^{\varepsilon\overline\eta} |\partial_z \Psi_\varepsilon(x', z)|\, dz\, dx' + \varepsilon\left(1 + \frac{\overline\eta}{l_3}\right)(|g| + \varepsilon M).$$

Using the Cauchy-Schwarz inequality and (20), we get (21).   □

Proof of (23). Thanks to Lemma 3, (23) follows from the momentum equation in (21), from (5) and from (22), since $\nabla\Psi_\varepsilon \in (L^2(\Theta^t))^9$ by (20).   □

6. Exponential decay of $\nabla\Phi_\varepsilon$.
Notice that, by (13), $\nabla\Phi_\varepsilon = \nabla\Psi_\varepsilon/\varepsilon$.

LEMMA 5. *Given $M > 0$, there exists a positive number c_M such that, if $|f| \leq M$, $\varepsilon < c_M$ and $x_3 \geq \varepsilon(\overline\eta + 1)$, then*

$$|\nabla\Phi_\varepsilon(x', x_3)| \leq c'_M \sqrt\varepsilon \exp\left(-c''_M \frac{x_3}{\varepsilon}\right). \tag{28}$$

Proof. Let us shift $(\Psi_\varepsilon, \xi_\varepsilon)$ by $\varepsilon\overline\eta$ in the vertical direction, that is, let

$$(\Psi', \xi')(x', x_3) = (\Psi_\varepsilon, \xi_\varepsilon)(x', x_3 + \varepsilon\overline\eta)$$

for all $(x', x_3) \in \mathbb{R}^3_+$. By (6), (7), (20), and (21), the pair (Ψ', ξ') satisfies (15) with $k_1 = \varepsilon l_1, k_2 = \varepsilon l_2$ and $E = \varepsilon^3 c'''_M(|g| + c_M M)$. Then Lemma 3 gives

$$|\nabla\Psi'(x', x_3)| \leq c'_M \varepsilon^{3/2} \exp\left(-c''_M \frac{x_3}{\varepsilon}\right)$$

for all $x_3 \geq \varepsilon$. This implies (28) since, for all $x_3 \geq \varepsilon(\overline\eta + 1)$,

$$\nabla\Phi_\varepsilon(x', x_3) = \frac{1}{\varepsilon}\nabla\Psi'(x', x_3 - \varepsilon\overline\eta).   □$$

7. Estimate of $\Phi^2_\varepsilon - \Phi^1_\varepsilon$.
Let $(\Psi^1_\varepsilon, \xi^1_\varepsilon)$ and $(\Psi^2_\varepsilon, \xi^2_\varepsilon)$ be the solutions of (6) associated, respectively, with two different vectors f^1 and f^2 such that $f^1_3 = f^2_3 = 0$. More precisely, they are the limits as $m \to \infty$ of the corresponding approximate solutions Ψ^{m1}_ε and Ψ^{m2}_ε of (8). We denote

$$\Phi^1_\varepsilon = \frac{\Psi^1_\varepsilon}{\varepsilon} - f^1, \qquad \Phi^2_\varepsilon = \frac{\Psi^2_\varepsilon}{\varepsilon} - f^2.$$

LEMMA 6. *If $|f^1| \leq M$ and $|f^2| \leq M$, then, for any $t > \varepsilon\overline\eta$,*

$$|(\overline{\Phi^2_\varepsilon} - \overline{\Phi^1_\varepsilon})(t)| \leq c_M\left(\left(1 + \frac{1}{\sqrt\varepsilon}\right)\left(\int_{\tilde\Omega_\varepsilon} |\nabla(\Phi^2_\varepsilon - \Phi^1_\varepsilon)|^2\right)^{1/2} + \varepsilon|f^2 - f^1|\right). \tag{29}$$

Proof. Let us denote

$$f = f^1 - f^2, \qquad \Phi = \Phi^1_\varepsilon - \Phi^2_\varepsilon, \qquad \pi = \pi^1_\varepsilon - \pi^2_\varepsilon.$$

From (6), (Φ, π) satisfies

$$\Phi \in (H^1_{\mathrm{per,loc}}(\widetilde{\Omega}_\varepsilon))^3, \qquad \pi \in L^2_{\mathrm{per,loc}}(\widetilde{\Omega}_\varepsilon), \qquad \int_{\widetilde{\Omega}_\varepsilon} |\nabla \Phi|^2 < \infty,$$

$$-\nu \Delta \Phi + \varepsilon \Phi^1_\varepsilon \cdot \nabla \Phi + \varepsilon \Phi \cdot \nabla \Phi^2_\varepsilon + \varepsilon \Phi^1_3 \frac{f}{l_3}$$

$$+ \Phi_3 \frac{g + \varepsilon f^2}{l_3} + (a^1_\varepsilon + \varepsilon f^1) \cdot \nabla \Phi + (a^1_\varepsilon - a^2_\varepsilon + \varepsilon f) \cdot \nabla \Phi^2_\varepsilon + \nabla \pi = 0, \tag{30}$$

$$\nabla \cdot \Phi = 0, \qquad \Phi|_{\mathcal{R}_\varepsilon} = -\frac{\eta_\varepsilon f}{l_3}.$$

Obviously,

$$|\overline{\Phi}(t)| \leq |\overline{\Phi}(t) - \overline{\Phi}(\varepsilon(\overline{\eta}+1))| + |\overline{\Phi}(\varepsilon(\overline{\eta}+1))|. \tag{31}$$

In a similar way to (23), integrating on S^ε the horizontal momentum equations given in (30), for $t \geq \varepsilon(\overline{\eta}+1)$ fixed, and using the periodicity of the solution, we get

$$|\overline{\Phi}(t) - \overline{\Phi}(\varepsilon(\overline{\eta}+1))| \leq \frac{c}{\varepsilon}(\|\nabla \Phi^1_\varepsilon\|_{(L^2(\Theta^{\varepsilon(\overline{\eta}+1)}))^9} + \|\nabla \Phi^2_\varepsilon\|_{(L^2(\Theta^{\varepsilon(\overline{\eta}+1)}))^9})\|\nabla \Phi\|_{(L^2(\Theta^{\varepsilon(\overline{\eta}+1)}))^9}.$$

Therefore, by (28),

$$|\overline{\Phi}(t) - \overline{\Phi}(\varepsilon(\overline{\eta}+1))| \leq c_M \|\nabla \Phi\|_{(L^2(\Theta^{\varepsilon(\overline{\eta}+1)}))^9}. \tag{32}$$

On the other hand,

$$\Phi(x', z) = \int_{\eta_\varepsilon(x')}^z \partial_z \Phi(x', \zeta)\, d\zeta - \frac{\eta_\varepsilon(x')}{l_3} f. \tag{33}$$

Therefore, choosing $z = \varepsilon(\overline{\eta}+1)$ and integrating with respect to x', we get

$$|\overline{\Phi}(\varepsilon(\overline{\eta}+1))| \leq \frac{c}{\sqrt{\varepsilon}} \|\partial_z \Phi\|_{(L^2(\widetilde{\Omega}_\varepsilon))^3} + c\varepsilon|f|. \tag{34}$$

Using (31), (32), and (34), we get (29). $\square$

In order to bound the right-hand side of (29) in terms of $c|f^2 - f^1|$, let us prove the following result.

LEMMA 7. There exist positive numbers c_M and c'_M such that, if $|f^1| \leq M$ and $|f^2| \leq M$ and $\varepsilon < c_M$,

$$\|\nabla(\Phi^2_\varepsilon - \Phi^1_\varepsilon)\|_{(L^2(\widetilde{\Omega}_\varepsilon))^9} \leq c'_M \varepsilon^{3/2} |f^2 - f^1|. \tag{35}$$

Proof. In order to get a homogeneous boundary condition on $\mathcal{R}_\varepsilon$, let k be the extension of the trace of Φ on $\mathcal{R}_\varepsilon$ defined by

$$k(x', x_3) = \begin{cases} -\dfrac{x_3}{l_3} f & \text{if } \eta_\varepsilon \leq x_3 < \varepsilon\overline{\eta}, \\[2mm] -\varepsilon\dfrac{\overline{\eta} f(2x_3 - l_3)}{l_3(2\varepsilon\overline{\eta} - l_3)} & \text{if } \varepsilon\overline{\eta} \leq x_3 < l_3/2, \\[2mm] 0 & \text{otherwise.} \end{cases}$$

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Let us set $\phi = \Phi - k$. Then (ϕ, π) satisfies

$$\phi \in (H^1_{\mathrm{per,loc}}(\widetilde{\Omega}_\varepsilon))^3, \qquad \pi \in L^2_{\mathrm{per,loc}}(\widetilde{\Omega}_\varepsilon), \qquad \int_{\widetilde{\Omega}_\varepsilon} |\nabla \phi|^2 < \infty,$$

$$-\nu \Delta \phi + \varepsilon \Phi^1_\varepsilon \cdot \nabla \phi + \varepsilon \phi \cdot \nabla \Phi^2_\varepsilon + \varepsilon \Phi^1_{\varepsilon 3} \frac{f}{l_3} + \phi_3 \frac{g + \varepsilon f^2}{l_3} + (a^1_\varepsilon + \varepsilon f^1) \cdot \nabla \phi \qquad (36)$$

$$+ (a^1_\varepsilon - a^2_\varepsilon + \varepsilon f) \cdot \nabla \Phi^2_\varepsilon - \nu \Delta k + \varepsilon \Phi^1_\varepsilon \cdot \nabla k + \varepsilon k \cdot \nabla \Phi^2_\varepsilon + \nabla \pi = 0,$$

$$\nabla \cdot \phi = 0, \qquad \phi|_{\mathcal{R}_\varepsilon} = 0,$$

where a^1_ε and a^2_ε are the functions associated to f^1 and f^2 by (7). Multiplying by ϕ the momentum equation in (36), we get

$$\nu \int_{\widetilde{\Omega}_\varepsilon} |\nabla \phi|^2 = -\varepsilon \int_{\widetilde{\Omega}_\varepsilon} \phi \cdot \nabla \Phi^2_\varepsilon \cdot \phi - \varepsilon \int_{\widetilde{\Omega}_\varepsilon} \Phi^1_{\varepsilon 3} \frac{f}{l_3} \cdot \phi - \int_{\widetilde{\Omega}_\varepsilon} \phi_3 \frac{g + \varepsilon f^2}{l_3} \cdot \phi$$

$$- \int_{\widetilde{\Omega}_\varepsilon} (a^1_\varepsilon - a^2_\varepsilon + \varepsilon f) \cdot \nabla \Phi^2_\varepsilon \cdot \phi - \nu \int_{\widetilde{\Omega}_\varepsilon} \nabla k \cdot \nabla \phi \qquad (37)$$

$$- \varepsilon \int_{\widetilde{\Omega}_\varepsilon} \Phi^1_\varepsilon \cdot \nabla k \cdot \phi - \varepsilon \int_{\widetilde{\Omega}_\varepsilon} k \cdot \nabla \Phi^2_\varepsilon \cdot \phi$$

$$= I_1 + \cdots + I_7.$$

This integration by parts may be justified by passing to the limit on the approximate solutions Ψ_m. It may also be justified by remarking that $-\nu \Delta \phi + \nabla \pi = \nabla \cdot (-\nu \nabla \phi + \pi \, \mathrm{Id})$, which is in L^1 and which is in $L^{3/2}$ in any bounded set thanks to (28), (36) and the fact that the Φ^i_ε are bounded outside a neighbourhood of the boundary.

Let us estimate $I_1, I_2, \ldots, I_7$ in terms of $\int_{\widetilde{\Omega}_\varepsilon} |\nabla \phi|^2$. We will decompose the integral over $\widetilde{\Omega}_\varepsilon$ into an integral over the infinite strip with flat bottom

$$\Theta' = \{x \in \mathbb{R}^3 : x' \in S_\varepsilon, \quad x_3 > 2l_3 + \varepsilon \overline{\eta}\}$$

and an integral over the bounded domain

$$\Lambda' = \{x \in \mathbb{R}^3 : x' \in S_\varepsilon, \quad \eta_\varepsilon(x') < x_3 < 2l_3 + 2\varepsilon \overline{\eta}\}.$$

We will use the Poincaré-Wirtinger inequality in Θ', and the Poincaré inequality in Λ'.

Since $\int_{\Theta'} \overline{\phi} \cdot \nabla \Phi^2_\varepsilon \cdot \overline{\phi} = 0$, we can write

$$I_1 = -\varepsilon \int_{\Theta'} (\phi - \overline{\phi}) \cdot \nabla \Phi^2_\varepsilon \cdot (\phi - \overline{\phi}) + \overline{\phi} \cdot \nabla \Phi^2_\varepsilon \cdot (\phi - \overline{\phi}) + (\phi - \overline{\phi}) \cdot \nabla \Phi^2_\varepsilon \cdot \overline{\phi}$$

$$-\varepsilon \int_{\Lambda'} \phi \cdot \nabla \Phi^2_\varepsilon \cdot \phi.$$

Using the Hölder inequality it follows that

$$|I_1| \leq \varepsilon \|\nabla \Phi^2_\varepsilon\|_{(L^\infty(\Theta'))^9} (\|\phi - \overline{\phi}\|_{(L^2(\Theta'))^3})^2$$

$$+ 2\varepsilon \|\overline{\phi}\|_{(L^\infty(\Theta'))^3} \|\nabla \Phi^2_\varepsilon\|_{(L^2(\Theta'))^9} (\|\phi - \overline{\phi}\|_{(L^2(\Theta'))^3})^2 \qquad (38)$$

$$+ c\varepsilon^{4/3} (\|\phi\|_{(L^6(\Lambda'))^3})^2 \|\nabla \Phi^2_\varepsilon\|_{(L^2(\Lambda'))^9}.$$

From (28) we have

$$\|\nabla \Phi^2_\varepsilon\|_{(L^\infty(\Theta'))^9} \leq c_M \sqrt{\varepsilon} \qquad (39)$$

and from the Poincaré-Wirtinger inequality,

$$\|\phi - \overline{\phi}\|_{(L^2(\Theta'))^3} \leq c\varepsilon \|\nabla_{x'}\phi\|_{(L^2(\Theta'))^6}. \tag{40}$$

Thanks to (28),

$$\|\nabla\Phi_\varepsilon^2\|_{(L^2(\Theta'))^9} \leq c_M \varepsilon^{3/2}. \tag{41}$$

Since $k = 0$ in Θ', (29) and (34) give

$$\|\overline{\phi}\|_{(L^\infty(\Theta'))^3} \leq \frac{c_M}{\sqrt{\varepsilon}} \|\nabla\phi\|_{(L^2(\Theta'))^9} + \varepsilon|f|. \tag{42}$$

Using the change of variable $x' \mapsto x'/\varepsilon$, which maps the domain Λ' onto a domain of section $(0, l_1) \times (0, l_2)$, and the fact that $\phi|_{\mathcal{R}_\varepsilon} = 0$, we get

$$\|\phi\|_{(L^6(\Lambda'))^3} \leq c\varepsilon^{-2/3} \|\nabla\phi\|_{(L^2(\Lambda'))^9}. \tag{43}$$

Using (39), (40), (41), (42), and (43), the inequality (38) gives

$$|I_1| \leq c_M \varepsilon^{1/2} \|\nabla\phi\|_{(L^2(\widetilde{\Omega}_\varepsilon))^9} (\|\nabla\phi\|_{(L^2(\widetilde{\Omega}_\varepsilon))^9} + \varepsilon^3|f|). \tag{44}$$

Let us now bound I_2. There we decompose the integral over $\widetilde{\Omega}_\varepsilon$ into integrals over

$$\Theta'' = \{x \in \mathbb{R}^3 : x' \in S_\varepsilon, \quad x_3 > \varepsilon\overline{\eta}\},$$
$$\Lambda'' = \{x \in \mathbb{R}^3 : x' \in S_\varepsilon, \quad \eta_\varepsilon(x') < x_3 < \varepsilon\overline{\eta}\}.$$

We use a domain Λ'' of vertical size $c\varepsilon$ to get the coefficient ε in (45). Using (22),

$$I_2 = -\varepsilon \int_{\Theta''} (\Phi_{\varepsilon 3}^1 - \overline{\Phi}_{\varepsilon 3}^1) \frac{f}{l_3}(\phi - \overline{\phi}) - \varepsilon \int_{\Lambda''} \Phi_{3\varepsilon}^1 \frac{f}{l_3} \phi.$$

We bound the first integral using (40) for ϕ and Φ^1 (with Θ'' instead of Θ'), (13) and (20). To bound the second integral, we use the following Poincaré inequality:

$$\|\phi\|_{(L^2(\Lambda''))^3} \leq c\varepsilon \|\nabla\phi\|_{(L^2(\Lambda''))^9} \tag{45}$$

and

$$\|\Phi_{\varepsilon 3}^1\|_{(L^2(\Lambda''))^3} \leq c_M \varepsilon^{3/2}.$$

This last inequality comes from (33) by using (13) and (20). Thus, we get

$$|I_2| \leq c_M \varepsilon^{7/2} |f| \|\nabla\phi\|_{(L^2(\widetilde{\Omega}_\varepsilon))^9}. \tag{46}$$

Similarly, we have

$$|I_3| \leq c_M \varepsilon^{7/2} (\|\nabla\phi\|_{(L^2(\widetilde{\Omega}_\varepsilon))^9})^2. \tag{47}$$

Since $a_\varepsilon^1 = a_\varepsilon^2 = 0$ on $[2l_3 + \varepsilon\overline{\eta}, +\infty)$, we can write

$$I_4 = -\int_{\Theta'} (\varepsilon f \cdot \nabla\Phi_\varepsilon^2 \cdot (\phi - \overline{\phi}) + \varepsilon f \cdot \nabla\Phi_\varepsilon^2 \cdot \overline{\phi}) - \int_{\Lambda'} (a_\varepsilon^1 - a_\varepsilon^2 + \varepsilon f) \cdot \nabla\Phi_\varepsilon^2 \cdot \phi.$$

It follows that

$$|I_4| \leq \varepsilon|f|(\|\phi - \overline{\phi}\|_{(L^2(\Theta'))^3} \|\nabla\Phi_\varepsilon^2\|_{(L^2(\Theta'))^9} + \|\nabla\Phi_\varepsilon^2\|_{(L^1(\Theta'))^9} \|\overline{\phi}\|_{(L^\infty(\Theta'))^3})$$
$$+ (\varepsilon|f| + \|a_\varepsilon^1 - a_\varepsilon^2\|_{(L^\infty(\Lambda'))^3}) \|\nabla\Phi_\varepsilon^2\|_{(L^2(\Lambda'))^9} \|\phi\|_{(L^2(\Lambda'))^3}.$$

Thanks to (28),

$$\|\nabla \Phi_\varepsilon^2\|_{(L^1(\Theta'))^9} \leq c_M \varepsilon^{3/2}. \tag{48}$$

Then using (40), (41), (42), (48) and

$$\|a_\varepsilon^1 - a_\varepsilon^2\|_{(L^\infty(\Lambda'))^3} \leq c\varepsilon|f|$$

and using the Poincaré inequality

$$\|\phi\|_{(L^2(\Lambda'))^3} \leq c\|\nabla\phi\|_{(L^2(\Lambda'))^9},$$

we get

$$|I_4| \leq c_M \varepsilon^{5/2}|f|(\|\nabla\phi\|_{(L^2(\tilde{\Omega}_\varepsilon))^9} + \varepsilon|f|). \tag{49}$$

Using similar arguments and the fact that k vanishes for $x_3 \geq l_3/2$, we get

$$|I_5| \leq c\varepsilon^{3/2}|f|\,\|\nabla\phi\|_{((L^2(\tilde{\Omega}_\varepsilon))^9}, \tag{50}$$

$$|I_6| \leq c\varepsilon^{7/2}|f|\,\|\nabla\phi\|_{((L^2(\tilde{\Omega}_\varepsilon))^9}, \tag{51}$$

$$|I_7| \leq c_M \varepsilon^{5/2}|f|\,\|\nabla\phi\|_{((L^2(\tilde{\Omega}_\varepsilon))^9}. \tag{52}$$

From (37), the estimates (44), (46), (47), (49), (50), (51), and (52) give, for ε small enough,

$$\int_{\tilde{\Omega}_\varepsilon} |\nabla\phi|^2 \leq c_M \varepsilon^{3/2}|f|(\|\nabla\phi\|_{((L^2(\tilde{\Omega}_\varepsilon))^9} + \varepsilon^2|f|).$$

This implies (35) if ε is small enough.   □

8. Fixed point argument.

It follows from (11) and (13) that $\overline{\Phi}_\varepsilon(t)$ possesses a limit $\overline{\Phi}_{\varepsilon\infty}$ as $t \to \infty$.

LEMMA 8. *There exists a positive number c such that, for $\varepsilon < c$, the map $f \mapsto -\overline{\Phi}_{\varepsilon\infty}$ possesses a fixed point.*

Proof. Given $M > 0$, let us denote $B_M = \{f \in \mathbb{R}^3 : f_3 = 0 \text{ and } |f| \leq M\}$.   □
Choice of M such that $f \mapsto -\overline{\Phi}_{\varepsilon\infty}$ maps B_M into itself. Let $f \in B_M$. By (13), $\overline{\Psi}_\varepsilon = \varepsilon(\overline{\Phi}_\varepsilon + f)$. Therefore,

$$|\overline{\Phi}_\varepsilon(t)| \leq \frac{1}{\varepsilon}|\overline{\Psi}_\varepsilon(t) - \overline{\Psi}_\varepsilon(\varepsilon\overline{\eta})| + |\overline{\Phi}_\varepsilon(\varepsilon\overline{\eta})|. \tag{53}$$

By (20) and (23),

$$|\overline{\Psi}_\varepsilon(t) - \overline{\Psi}_\varepsilon(\varepsilon\overline{\eta})| \leq c\varepsilon^3(|g| + \varepsilon M). \tag{54}$$

On the other hand, by (6), $\Phi_\varepsilon(x', \eta_\varepsilon(x')) = -\frac{1}{l_3}\eta(x'/\varepsilon)(g + \varepsilon f)$; thus

$$\Phi_\varepsilon(x', \varepsilon\overline{\eta}) = \int_{\eta_\varepsilon(x')}^{\varepsilon\overline{\eta}} \partial_z \Phi_\varepsilon(x', z)\, dz - \frac{1}{l_3}\eta\left(\frac{x'}{\varepsilon}\right)(g + \varepsilon f).$$

Therefore,

$$|\overline{\Phi}_\varepsilon(\varepsilon\overline{\eta})| \leq \frac{1}{\varepsilon^2 l_1 l_2} \int_{S_\varepsilon} \int_{\eta_\varepsilon(x')}^{\varepsilon\overline{\eta}} |\partial_z \Phi_\varepsilon(x', z)|\, dz\, dx' + \frac{\overline{\eta}}{l_3}(g + \varepsilon M).$$

Using the Cauchy-Schwarz inequality, $\partial_z \Phi_\varepsilon = \partial_z \Psi_\varepsilon / \varepsilon$, and (20), we get

$$|\overline{\Phi}_\varepsilon(\varepsilon \overline{\eta})| \le c((|g| + \varepsilon M)^{1/2} + (|g| + \varepsilon M)). \tag{55}$$

Using (53), (54), and (55) and passing to the limit as $t \to \infty$, we obtain

$$|\overline{\Phi}_{\varepsilon\infty}| \le c((|g| + \varepsilon M)^{1/2} + (|g| + \varepsilon M)).$$

Therefore, taking $M = 2c(|g| + |g|^{1/2})$, there exists a positive number c_M such that for $\varepsilon < c_M$, $|\overline{\Phi}_{\varepsilon\infty}| \le M$. Then $-\overline{\Phi}_{\varepsilon\infty}$ belongs to B_M, since its third component vanishes thanks to (13) and (22).

   Choice of ε such that $f \mapsto -\overline{\Phi}_{\varepsilon\infty}$ is a contractive map on B_M. Let $(\Phi^1_\varepsilon, \pi^1_\varepsilon)$ and $(\Phi^2_\varepsilon, \pi^2_\varepsilon)$ be the solutions, respectively, associated by (6) and (13) with two vectors f^1 and f^2 of B_M. From (29) and (35) there exists a positive number c_M such that, for any t,

$$|(\overline{\Phi^2_\varepsilon} - \overline{\Phi^1_\varepsilon})(t)| < c_M \varepsilon |f^2 - f^1|.$$

Therefore, if $\varepsilon < 1/c_M$,

$$|\overline{\Phi^2_{\varepsilon\infty}} - \overline{\Phi^1_{\varepsilon\infty}}| < \alpha |f^2 - f^1|, \quad \alpha < 1.$$

Then the map $f \mapsto -\overline{\Phi}_{\varepsilon\infty}$ possesses a fixed point.

   9. Exponential decay of the residue. Let us now choose $f = f_\varepsilon$ where f_ε denotes the fixed point obtained in Lemma 8. Thanks to (13), we get $\overline{\Psi}_{\varepsilon\infty} = 0$. Then (11) gives, for all $\alpha \in \mathbb{N}^3$ and $x_3 \ge \varepsilon(\overline{\eta} + 1)$,

$$|\partial^\alpha \Psi_\varepsilon(x)| + |\partial^\alpha \xi_\varepsilon(x)| \le c'_M \exp\left(-c''_M \frac{x_3}{\varepsilon}\right). \tag{56}$$

To complete the proof of Theorem 1, according to the decomposition (9), it remains to get the exponential decay (14) of the residue $(\Xi_\varepsilon, \sigma_\varepsilon)$. At first, let us bound its H^1 norm, as follows.

LEMMA 9. *There exist positive numbers c_M, c'_M and c''_M such that if $\varepsilon < c_M$,*

$$\|\Xi_\varepsilon\|_{(H^1(\Omega_\varepsilon))^3} \le c'_M \exp\left(-\frac{c''_M}{\varepsilon}\right). \tag{57}$$

   Proof. In order to get a homogeneous boundary condition on the plane $\mathcal{P}$, let us define a divergence free extension r of the trace of $-\Psi_\varepsilon$ on $\mathcal{P}$. At first, we define an extension r^1 in

$$\omega = \left\{ x : x' \in S_\varepsilon, \frac{l_3}{2} < x_3 < l_3 \right\}$$

by

$$r^1(x', x_3) = -\Psi_\varepsilon(x', l_3) \frac{2x_3 - l_3}{l_3}.$$

Clearly $r^1 \in (H^1_{\text{per}}(\omega))^3$ and $\int_\omega \nabla \cdot r^1 = 0$. Then, see for instance [10], Theorem 3.1, p. 127, there exists $r^2 \in (H^1_0(\omega))^3$ such that

$$\nabla \cdot r^2 = -\nabla \cdot r^1 \text{ in } \omega, \qquad \|r^2\|_{(H^1(\omega))^3} \le c\|\nabla \cdot r^1\|_{(L^2(\omega))}.$$

Let us extend r^2 periodically in the x' variable. Then the function

$$r = \begin{cases} r^1 + r^2 & \text{in } \omega, \\ 0 & \text{in } \Omega_\varepsilon \setminus \overline{\omega} \end{cases}$$

belongs to $(H^1_{\text{per}}(\Omega_\varepsilon))^3$ and satisfies $\nabla \cdot r = 0$ and

$$\|r\|_{(H^1(\Omega_\varepsilon))^3} \le c\|\Psi_\varepsilon(x', l_3)\|_{(H^1(S_\varepsilon))^3}. \tag{58}$$

Now let $\xi = \Xi_\varepsilon - r$ and $\sigma = \sigma_\varepsilon$. From (10), we get

$$\xi \in (H^1_{\text{per}}(\Omega_\varepsilon))^3, \qquad \sigma \in L^2_{\text{per}}(\Omega_\varepsilon),$$
$$-\nu\Delta\xi - \nu\Delta r + \xi \cdot \nabla\xi + r \cdot \nabla\xi + \xi \cdot \nabla r + r \cdot \nabla r$$
$$+ \Psi_\varepsilon \cdot \nabla\xi + \Psi_\varepsilon \cdot \nabla r + \xi \cdot \nabla\Psi_\varepsilon + r \cdot \nabla\Psi_\varepsilon + \xi_3\frac{g + \varepsilon f_\varepsilon}{l_3} + r_3\frac{g + \varepsilon f_\varepsilon}{l_3}$$
$$+ \left(\frac{x_3}{l_3}(g + \varepsilon f_\varepsilon) - \varepsilon f_\varepsilon\right) \cdot \nabla\xi + \left(\frac{x_3}{l_3}(g + \varepsilon f_\varepsilon) - \varepsilon f_\varepsilon\right) \cdot \nabla r + \nabla\sigma = 0,$$
$$\nabla \cdot \xi = 0, \qquad \xi|_{\mathcal{P}} = 0, \qquad \xi|_{\mathcal{R}_\varepsilon} = 0.$$

Multiplying the equation by ξ and integrating by parts, we get

$$\nu\int_{\Omega_\varepsilon}|\nabla\xi|^2 = -\nu\int_{\Omega_\varepsilon}\nabla r \cdot \nabla\xi - \int_{\Omega_\varepsilon}\xi \cdot \nabla r \cdot \xi - \int_{\Omega_\varepsilon}r \cdot \nabla r \cdot \xi - \int_{\Omega_\varepsilon}\Psi_\varepsilon \cdot \nabla r \cdot \xi$$
$$- \int_{\Omega_\varepsilon}\xi \cdot \nabla\Psi_\varepsilon \cdot \xi - \int_{\Omega_\varepsilon}r \cdot \nabla\Psi_\varepsilon \cdot \xi - \int_{\Omega_\varepsilon}\xi_3\frac{g + \varepsilon f_\varepsilon}{l_3} \cdot \xi$$
$$- \int_{\Omega_\varepsilon}r_3\frac{g + \varepsilon f_\varepsilon}{l_3} \cdot \xi - \int_{\Omega_\varepsilon}\left(\frac{x_3}{l_3}(g + \varepsilon f_\varepsilon) - \varepsilon f_\varepsilon\right) \cdot \nabla r \cdot \xi \tag{59}$$
$$= J_1 + \cdots + J_9.$$

Let us now estimate $J_1, \ldots, J_9$ in terms of $\exp(-c''_M/\varepsilon)\|\nabla\xi\|_{(L^2(\Omega_\varepsilon))^9}$ or of $o(\varepsilon)(\|\nabla\xi\|_{(L^2(\Omega_\varepsilon))^9})^2$. First

$$|J_1| \le \nu\|\nabla r\|_{(L^2(\Omega_\varepsilon))^9}\|\nabla\xi\|_{(L^2(\Omega_\varepsilon))^9}.$$

By (56) and (58),

$$\|r\|_{(H^1(\Omega_\varepsilon))^3} \le c'_M \exp\left(-\frac{c''_M}{\varepsilon}\right) \tag{60}$$

and thus

$$|J_1| \le c'_M \exp\left(-\frac{c''_M}{\varepsilon}\right)\|\nabla\xi\|_{(L^2(\Omega_\varepsilon))^9}. \tag{61}$$

For J_2, we remark that

$$J_2 = \int_{\Omega_\varepsilon}\xi \cdot \nabla r \cdot \xi = -\int_{\Omega_\varepsilon}\xi \cdot \nabla\xi \cdot r.$$

The Hölder inequality gives

$$|J_2| \le c\varepsilon^{1/3}\|r\|_{(L^6(\Omega_\varepsilon))^3}\|\xi\|_{(L^6(\Omega_\varepsilon))^3}\|\nabla\xi\|_{(L^2(\Omega_\varepsilon))^9}.$$

Any function $\varphi \in (H^1(\Omega_\varepsilon))^3$ that cancels on R_ε satisfies

$$\|\varphi\|_{(L^6(\Omega_\varepsilon))^3} \le c\varepsilon^{-2/3}\|\nabla\varphi\|_{(L^2(\Omega_\varepsilon))^9}. \tag{62}$$

This is proved in the same way as (43). Using in addition (60), we get

$$|J_2| \leq c'_M \exp\left(-\frac{c''_M}{\varepsilon}\right) \|\nabla\xi\|^2_{(L^2(\Omega_\varepsilon))^9}. \tag{63}$$

The same arguments lead to

$$|J_3| \leq c'_M \exp\left(-\frac{c''_M}{\varepsilon}\right) \|\nabla\xi\|_{(L^2(\Omega_\varepsilon))^9}. \tag{64}$$

For J_4, we remark that

$$J_4 = \int_{\Omega_\varepsilon} \Psi_\varepsilon \cdot \nabla r \cdot \xi = -\int_{\Omega_\varepsilon} \Psi_\varepsilon \cdot \nabla\xi \cdot r.$$

The Hölder inequality gives

$$|J_4| \leq c\varepsilon^{1/3} \|r\|_{(L^6(\Omega_\varepsilon))^3} \|\Psi_\varepsilon\|_{(L^6(\Omega_\varepsilon))^3} \|\nabla\xi\|_{(L^2(\Omega_\varepsilon))^9}.$$

Using the extension h of Ψ_ε defined by (25) and (62), we get

$$\|\Psi_\varepsilon\|_{(L^6(\Omega_\varepsilon))^3} \leq c\varepsilon^{-2/3} \|\nabla(\Psi_\varepsilon - h)\|_{(L^2(\Omega_\varepsilon))^9} + \|h\|_{(L^6(\Omega_\varepsilon))^3}$$

and then

$$\|\Psi_\varepsilon\|_{(L^6(\Omega_\varepsilon))^3} \leq c\varepsilon^{-2/3} \|\nabla\Psi_\varepsilon\|_{(L^2(\Omega_\varepsilon))^9} + c_M \varepsilon^{1/2}.$$

Therefore, (20) and (60) give

$$|J_4| \leq c'_M \exp\left(-\frac{c''_M}{\varepsilon}\right) \|\nabla\xi\|_{(L^2(\Omega_\varepsilon))^9}. \tag{65}$$

The same arguments lead to

$$|J_5| \leq c'_M \varepsilon^{1/2} \|\nabla\xi\|^2_{(L^2(\Omega_\varepsilon))^9} \tag{66}$$

and

$$|J_6| \leq c'_M \exp\left(-\frac{c''_M}{\varepsilon}\right) \|\nabla\xi\|_{(L^2(\Omega_\varepsilon))^9}. \tag{67}$$

For J_7, we decompose the integral over Ω_ε into integrals over the domains

$$\omega' = \{x : x' \in S_\varepsilon, \quad \eta_\varepsilon(x') < x_3 < \varepsilon\bar{\eta}\},$$
$$\omega'' = \{x : x' \in S_\varepsilon, \quad \varepsilon\bar{\eta} < x_3 < l_3\}.$$

Since ξ is divergence free and periodic, $\bar{\xi} = 0$ in ω'' and therefore

$$J_7 = \int_{\omega'} \xi_3 \frac{g + \varepsilon f_\varepsilon}{l_3} \cdot \xi + \int_{\omega''} (\xi_3 - \bar{\xi}_3) \frac{g + \varepsilon f_\varepsilon}{l_3} \cdot (\xi - \bar{\xi}).$$

Using the Poincaré inequality (45) in ω' (instead of Λ'') and the Poincaré-Wirtinger inequality (40) in ω'' (instead of Θ'), we obtain

$$|J_7| \leq c'_M \varepsilon^2 (\|\nabla\xi\|_{(L^2(\Omega_\varepsilon))^9})^2. \tag{68}$$

Similarly,

$$|J_8| \leq c'_M \varepsilon^2 \|\nabla r\|_{(L^2(\Omega_\varepsilon))^9} \|\nabla\xi\|_{(L^2(\Omega_\varepsilon))^9}.$$

Using (60), this gives

$$|J_8| \leq c'_M \exp\left(-\frac{c''_M}{\varepsilon}\right) \|\nabla\xi\|_{(L^2(\Omega_\varepsilon))^9}. \tag{69}$$

For the last term in (59), we remark that

$$J_9 = \int_{\Omega_\varepsilon} \left(\frac{x_3}{l_3}(g + \varepsilon f_\varepsilon) - \varepsilon f_\varepsilon\right) \cdot \nabla r \cdot \xi$$

$$= -\int_{\Omega_\varepsilon} \left(\frac{x_3}{l_3}(g + \varepsilon f_\varepsilon) - \varepsilon f_\varepsilon\right) \cdot \nabla\xi \cdot r.$$

Thus

$$|J_9| \leq c'_M \|r\|_{(L^2(\Omega_\varepsilon))^3} \|\nabla\xi\|_{(L^2(\Omega_\varepsilon))^9},$$

and (60) gives

$$|J_9| \leq c'_M \exp\left(-\frac{c''_M}{\varepsilon}\right) \|\nabla\xi\|_{(L^2(\Omega_\varepsilon))^9}. \tag{70}$$

Now bounding the right-hand side of (59) with (61), (63), (64), (65), (66), (67), (68), (69), and (70) we prove the existence of a positive number c_M such that for $\varepsilon < c_M$,

$$\|\nabla\xi\|_{(L^2(\Omega_\varepsilon))^9} \leq c'_M \exp\left(-\frac{c''_M}{\varepsilon}\right).$$

Since $\Xi_\varepsilon = \xi + r$, using (60), we get

$$\|\Xi_\varepsilon\|_{(H^1(\Omega_\varepsilon))^3} \leq c'_M \exp\left(-\frac{c''_M}{\varepsilon}\right).$$

The proof of Lemma 9 is therefore complete.   □

Finally, the exponential decay (57) of the norm implies the pointwise decay (14) outside of a neighbourhood of the rough wall, that is, for $x_3 \geq \varepsilon(\bar{\eta} + 1)$. This has been done in Theorem 2 of [4] in the case of a half space. The adaptation of the proof to the present case of a strip $\{x : x' \in \mathbb{R}^2, \varepsilon(\bar{\eta} + 1) < x_3 < l_3\}$ is elementary. Then we got f_ε such that (11), (12), and (14) are satisfied. Thanks to the decomposition (9), this proves (4) and therefore Theorem 2.   □

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